Topics on Quantum Locality

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January 28, 2020

Abstract

It has been 20 years since Deutsch and Hayden demonstrated that quantum systems can be completely described locally — notwithstanding Bell’s theorem. More recently, Raymond-Robichaud proposed another approach to the same conclusion. Here, these means of describing quantum systems are shown to be equivalent. Then, they have their cost of description quantified by the dimensionality of their space: The dimension of a single qubit grows exponentially with the size of the total system considered. Finally, the methods are generalized to continuous systems.

But to admit things not visible to the gross creatures that we are is, in my opinion, to show a decent humility, and not just a lamentable addiction to metaphysics.

— John S. Bell [4] —

1 Introduction

It is still a widespread belief that a complete description of a composite entangled quantum system cannot be obtained by descriptions of the parts, if those are expressed independently of what happens to other parts. This apparently holistic feature of entangled quantum states entails violation of Bell inequalities [3][1] and quantum teleportation [5], which are repeatedly invoked to sanctify the “nonlocal” character of quantum theory. But
this widespread belief has been proven false more than twenty years ago by Deutsch and Hayden [10], who by the same token, provided an entirely local explanation of Bell-inequality violations and teleportation.

Descriptions of dynamically isolated — but possibly entangled — systems A and B are local[1] if that of A is unaffected by any process system B may undergo, and vice versa. The descriptions are complete if they can predict the distributions of any measurement performed on the whole system AB. For instance, if AB is in a pure entangled state \( |\Psi\rangle^{AB} \), the reduced density matrices

\[
\rho^A = \text{tr}_B |\Psi\rangle\langle \Psi| \quad \text{and} \quad \rho^B = \text{tr}_A |\Psi\rangle\langle \Psi|
\]

are local but incomplete descriptions. If instead the descriptions of A and B are both taken to be the global wave function \( |\Psi\rangle^{AB} \), then one finds a complete but nonlocal account.

Following Gottesman’s [15] quantum computation in the Heisenberg picture, Deutsch and Hayden define so-called descriptors for individual qubits, which can be intuited to encode the quantum information of a qubit in a Heisenberg-picture-inspired object. Such a mode of description is showed to be both local and complete, hence vindicating the locality of quantum theory. More recently, Raymond-Robichaud has shown that any non-signalling theory with reversible operations can be reformulated in terms of so-called noumenal states, which also satisfy the desirable properties [6]. As a special case of such a non-signalling theory, quantum mechanics also finds noumenal states, as prescribed by Raymond-Robichaud in Ref. [18, Chapter 4].

| Mode of description | Local | Complete |
|---------------------|-------|----------|
| Reduced density matrices \( \rho^A \) and \( \rho^B \) | Yes | No |
| Global wave function \( |\Psi\rangle^{AB} \) | No | Yes |
| DH’s descriptors & RR’s noumenal states | Yes | Yes |

In this paper, equivalences between DH’s descriptors, RR’s abstract noumenal states and their quantum prescription are established (§3). An

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1 After Bell, it has become conventional wisdom to equate locality with a possible explanation by a local hidden variable theory. However, local hidden variables are only one way in which locality can be instantiated [7]. Here, locality is taken in the spirit of Einstein: “the real factual situation of the system \( S_2 \) is independent of what is done with the system \( S_1 \), which is spatially separated from the former” [19].
important drawback of such local descriptions is demonstrated: The di-

dimensionality of the state space of a system as tiny as a qubit scales expon-

entially with the whole system considered (§4). Finally, the formalism is

extended to continuous degrees of freedom (§5).

2 Preliminaries

The DH formalism [10], as well as RR’s abstract [6] formalism and its

quantum instantiation [18, Chapter 4] are briefly covered in this section.

For a more elementary and more detailed introduction to the DH formal-

ism, see the Appendix A.

2.1 Deutsch-Hayden’s Formalism

Let $\Omega$ be a computational network of $n$ qubits. At time 0 the descriptor of

qubit $i$ is given by

$$q_i(0) = 1^{i-1} \otimes (\sigma_x, \sigma_z) \otimes 1^{n-i},$$

where $\sigma_x$ and $\sigma_z$ are the corresponding Pauli matrices. The descriptor is

therefore a vector of two components, each of which being an operator

on the whole network. Suppose that between the discrete times $s - 1$ and

$s$, only one gate is performed, whose matrix representation is denoted $G_s$.

Let $U = G_t \ldots G_2 G_1$. The descriptor of qubit $i$ at time $t$ is given by

$$q_i(t) = U^\dagger q_i(0) U.$$

The object of $n$ components that encodes the descriptor of each qubit is

noted $q(t)$. Alternatively, $q_i(t)$ can be expressed as

$$q_i(t) = U_{G_t}^\dagger (q(t)) q_i(t-1) U_{G_t} (q(t-1)),$$

$^2$Deutsch and Hayden originally defined the descriptor with a third component, namely, with $\sigma_y$. It is however redundant.
where \( U_{G_t}(\cdot) \) is a fixed operator valued function of some components of \( q(t) \) such that \( U_{G_t}(q(0)) = G_t \). In fact, if \( U = G_t V \), then

\[
q_i(t) = V^\dagger G_t^\dagger q_i(0) G_t V
= V^\dagger U_{G_t}^\dagger(q(0)) V V^\dagger q_i(0) V V^\dagger U_{G_t}(q(0)) V
= U_{G_t}^\dagger(V^\dagger q(0) V) q_i(t-1) U_{G_t}(V^\dagger q(0) V)
= U_{G_t}^\dagger(q(t-1)) q_i(t-1) U_{G_t}(q(t-1)).
\]

The locality of the descriptors is recognized by the following. If the gate \( G_t \) acts only on qubits of the subset \( I \subset \{1,2,\ldots,n\} \), then its functional representation \( U_{G_t} \) shall only depend on components of \( q_k(t-1) \), for \( k \in I \). For \( j \not\in I \), the descriptor \( q_j(t-1) \) shall then commute with \( U_{G_t}(q(t-1)) \), so it will remain unchanged between times \( t-1 \) and \( t \).

Deutsch and Hayden's descriptors are also complete, in that the expectation value of any observable \( O(t) \) that concerns only qubits of \( I \) can be determined by the descriptors \( q_k(t) \), with \( k \in I \). This can be seen more clearly at time 0, where an observable on the qubits of \( I \) is a linear (hermitian) operator that acts non-trivially only on the qubits of \( I \). Since any such operator can be generated additively and multiplicatively by the components of \( q_k(0) \), with \( k \in I \),

\[
O(0) = f_O(|q_k(0)|_{k \in I}), \quad \text{so} \quad O(t) = U^\dagger O(0) U = f_O(|q_k(t)|_{k \in I}).
\]

### 2.2 Abstract Formalism of Parallel Lives

Systems form a boolean algebra. Specifically, the union and the intersection of systems are systems, and there exist a whole system \( S \) and an empty system \( \emptyset \) with respect to which systems can be complemented, i.e., \( \bar{A} \) satisfies \( \bar{A} \cup A = S \) and \( \bar{A} \cap A = \emptyset \).

To each system \( A \) is associated a noumenal state \( N^A \), a “real state of affairs”, from which a phenomenal state \( \rho^A \) can be determined by an injective function, \( \varphi(N^A) = \rho^A \). The phenomenal state encompasses all that can be observed, which may be informationally coarser than the noumenal state. In quantum theory, the phenomenal state boils down to the density matrix of the system, justifying the notation.

To system \( A \) is also associated a group of transformations \( \text{Op}(A) \) whose
elements have an action on both noumenal\textsuperscript{3} and phenomenal states. The function $\varphi$ is promoted to a morphism, since it preserves the group action, namely, for any $V \in \text{Op}(A)$,

$$\varphi(V \cdot N^A) = V \ast \rho^A,$$

where $\cdot$ and $\ast$ denote the actions on noumenal and phenomenal states, respectively. The morphism $\varphi$ also preserves the \textit{tracing out} of systems,

$$\varphi(\text{tr}_B N^{AB}) = \text{tr}_B \rho^{AB},$$

where $\text{tr}_B(\cdot)$ returns a state of system $A$ from that of system $AB$.

Evolution and tracing out are merely paralleled by noumenal and phenomenal states, but the whole relevance of introducing noumenal states is to impose that these must be described locally. Raymond-Robichaud makes this locality explicit, in that they impose the existence of a \textit{join product}, noted $\odot$, such that any noumenal state of a joint system $AB$ can be obtained by merging the local descriptions of $A$ and of $B$,

$$N^{AB} = N^A \odot N^B.$$

If $V \in \text{Op}(A)$ and $W \in \text{Op}(B)$, then the direct product $V \times W$, defined by its action on local noumenal states as

$$(V \times W) \cdot N^{AB} = (V \cdot N^A) \odot (W \cdot N^B),$$

is required to be a valid operation on $AB$. Transformations $U$ and $U'$ on the whole system $S$ are \textit{equivalent with respect to $A$}, noted $U \sim^A U'$, if they are connected by a transformation that acts trivially on $A$,

$$U \sim^A U' \iff \exists W \in \text{Op}(A): U' = (\mathbb{1}^A \times W)U.$$

In the abstract formalism of Raymond-Robichaud, the noumenal state space associated to system $A$ is defined as the set of equivalence classes, and a particular noumenal state is then

$$N^A = [U]^A.$$

\textsuperscript{3}The action is faithful on noumenal states, which means that if $V \cdot N^A = \bar{V} \cdot N^A$ for all $N^A$, then $V = \bar{V}$. 

5
This equivalence class \([U]^A\) encodes what has happened to the whole system \(S\) since the beginning, up to evolutions that do not causally concern system \(A\). From such a definition of the noumenal states, evolution by \(V \in \text{Op}(A)\), tracing out and merging are defined as

\[
V \cdot [U]^A \overset{\text{df}}{=} [V \times 1^A U]^A, \quad \text{tr}_B[U]^{AB} \overset{\text{df}}{=} [U]^A \quad \text{and} \quad [U]^A \circ [U]^B \overset{\text{df}}{=} [U]^{AB}.
\]  

Finally, the morphism \(\varphi\) depends upon a reference phenomenal state \(\rho_0\) on system \(S\), and is defined as

\[
\varphi([U]^A) \overset{\text{df}}{=} \text{tr}_{\neg A}(U \star \rho_0).
\]

### 2.3 Quantum Formalism of Parallel Lives

Let \(A\) be a subsystem of the whole system \(S\), and let \(\mathcal{H}^A\) be its Hilbert space, with some basis \(\{|i\rangle^A\}\). In the quantum formalism, the noumenal state of system \(A\) is defined, not as an equivalence class; rather as an evolution matrix,

\[
N^A = \|U\|^A, \quad \text{whose matrix elements are} \quad \|U\|^A_{ij} = U^\dagger(j|i\rangle^A \otimes 1^A)U.
\]

As in the abstract case, \(U\) is the operation that occurred on \(S\) between time 0 and time \(t\). The dependence of the evolution matrix on \(U\) is only up to the \(\sim^A\) equivalence relation, which is defined analogously as in Eq. (1). Indeed, if \(U' = (1^A \otimes V)U\),

\[
\|U'\|^A_{ij} = U'^\dagger(j|i\rangle^A \otimes 1^A)U' = U^\dagger(1^A \otimes V^\dagger)(|j\rangle\langle i|) \otimes 1^A)(1^A \otimes V)U = U^\dagger(|j\rangle\langle i|) \otimes 1^A)U = \|U\|^A_{ij},
\]

and one finds the same evolution matrix. The invariance of the evolution matrix within the equivalence class \([\cdot]^A\) is necessary but insufficient to identify the evolution matrix with the equivalence class, defined as the noumenal state in the abstract formalism. But Theorem 3.1 justifies the identification by proving that the equivalence class is uniquely determined by the evolution matrix.
In quantum theory, \( \text{Op}(A) \) is the group of unitary transformations \( U(H^A) \). Let \( A \) and \( B \) be disjoint systems. Then evolution by \( V \in U(H^A) \), tracing out and merging are defined as

\[
(V\|U\|^{A})_{ij} \overset{\text{df}}{=} \sum_{mn} V_{im} \|U\|^{A}_{mn} V^+_{nj} \\
(\text{tr}_B\|U\|^{AB})_{ij} \overset{\text{df}}{=} \sum_k \|U\|^{AB}_{ikjk} \\
(\|U\|^{A}\odot\|U\|^{B})_{ikjl} \overset{\text{df}}{=} \|U\|^{A}_{ij} \|U\|^{B}_{kl}.
\]

The above definitions are quite different from those of the abstract formalism, displayed in Eqns (2). Remarkably, these relations instead find their analogues as theorems, derived from the above definitions.

**Theorem 2.1** (Raymond-Robichaud). Let \( A \) and \( B \) be disjoint systems and let \( V \in U(H^A) \). Then

\[
V\|U\|^{A} = (V \otimes 1^A)U^{A}, \quad \text{tr}_B\|U\|^{AB} = \|U\|^{A} \quad \text{and} \quad \|U\|^{A}\odot\|U\|^{B} = \|U\|^{AB}.
\]

The morphism \( \varphi \) is defined from a fixed reference density matrix \( \rho_0 \) as

\[
(\varphi\|U\|^{A})_{ij} \overset{\text{df}}{=} \text{tr}\left(\|U\|^{A}_{ij}\rho_0\right).
\]

Notice that this definition differs from its abstract counterpart, Eq. (3), which will again be derived as a theorem. Moreover, the following theorem verifies that the morphism \( \varphi \) intertwines evolution and tracing out, so that these relations are in fact paralleled by noumenal and phenomenal states.

**Theorem 2.2** (Raymond-Robichaud). Let \( A \) and \( B \) be disjoint systems and let \( V \in U(H^A) \). Then

\[
\varphi\|U\|^{A} = \text{tr}_\mathbb{A}(U\ast\rho_0), \quad V\ast\varphi\|U\|^{A} = \varphi(V\cdot\|U\|^{A}) \quad \text{and} \quad \text{tr}_B\varphi\|U\|^{A} = \varphi\text{tr}_B\|U\|^{A}.
\]

One must recall that in quantum theory, the action \( \ast \) of operations on phenomenal states is given by \( U \ast \rho = U \rho U^+ \).
3 Equivalences

The three approaches to quantum locality presented in §2 are equivalent in many respects. First the descriptors and the evolution matrices are related by a mere change of operator basis. Second, the quantum formalism of parallel lives can be seen as the instantiation of the abstract one, because the evolution matrices are identified to the equivalence class, at least for qubits.

3.1 DH’s Descriptor ↔ RR’s Evolution Matrix

To establish the equivalence between descriptors and evolution matrices, consider an \( n \)-qubit computational network \( \mathcal{N} \), and let \( Q_k \) denote the \( k \)-th qubit. The apparent lack of generality to restrict the considered quantum system to a network of qubits is lifted by their ability to simulate any other quantum system with arbitrary accuracy [8]. At time \( t \), the descriptor of \( Q_k \) is given by

\[
q_k(t) = U^\dagger (1^{k-1} \otimes \sigma_x \otimes 1^{n-k}, 1^{k-1} \otimes \sigma_z \otimes 1^{n-k}) U,
\]

while its evolution matrix is given by

\[
\left[ U \right]_{Q_k}^{ij} = U^\dagger (|j\rangle \langle i| \otimes 1^{Q_k}) U.
\]

In both cases, \( U \) is the unitary operator according to which the network has so far evolved, and notwithstanding the different notation, the identity operators are applied on the same subspaces. They can be seen to be informationally equivalent, namely, \( \left[ U \right]_{Q_k}^{ij} \) can be computed from \( q_k(t) \) and vice versa. In fact, they differ only by a change of operator basis; descriptors are expressed in the Pauli basis, and evolution matrices, in the canonical matrix basis. One should keep in mind that while the descriptor is composed of only two operators, \( q_{kx}(t) \) and \( q_{kz}(t) \), their multiplicative
abilities permit the reconstruction of \( q_{kx}(t) = i q_{kx}(t) q_{kz}(t) \). Therefore,

\[
q_{kx}(t) = \|U\|_{12}^{Q_k} + \|U\|_{21}^{Q_k} \\
q_{kz}(t) = \|U\|_{11}^{Q_k} - \|U\|_{22}^{Q_k}
\]

The connection to observations is also equivalent in both formalisms. Without loss of generality, the reference density matrix \( \rho_0 \) can be fixed to \( |0\rangle \langle 0| \). In fact, purity can be consecrated in the Church of the larger Hilbert space and from there, altering the global evolution \( U \) permits to fix the reference state. The reduced density matrix \( \rho(t) = \text{tr}_{Q_k}(U|0\rangle\langle 0|U^\dagger) \) of qubit \( Q_k \) at time \( t \) can be expressed in the Pauli basis as

\[
\rho(t) = \frac{1}{2} \left( I + \sum_{w \in \{x,y,z\}} p_w(t) \sigma_w \right).
\]

From the trace relations of Pauli matrices, the components \( p_w(t) \) are

\[
p_w(t) = \text{tr}(\rho(t) \sigma_w) = \text{tr} \left( U|0\rangle\langle 0|U^\dagger(1^{k-1} \otimes \sigma_w \otimes 1^{n-k}) \right) = \langle 0|q_{kw}(t)|0\rangle.
\]

The second equality from the left comes from that \( \rho^A \mapsto \rho^A \otimes 1^B \) is, as a super-operator, the adjoint of \( \rho^{AB} \mapsto \text{tr}_B(\rho^{AB}) \), and the rightmost equality follows from cyclicity of the trace.

In the evolution matrices framework, one can instead expand the reduced density matrix in its canonical representation \( \rho(t) = \sum_{ij} \rho_{ij}(t) |i\rangle \langle j| \).

The matrix elements can be obtained as

\[
\rho_{ij}(t) = \text{tr}(\rho(t)|j\rangle\langle i|) = \text{tr} \left( U|0\rangle\langle 0|U^\dagger(|i\rangle \langle j| \otimes 1^{\overline{Q_k}}) \right) = \text{tr} \left( \|U\|_{ij}^{Q_k}|0\rangle\langle 0| \right),
\]

consistently with the definition of the morphism, Eq. (5).

### 3.2 RR: Abstract → Quantum

The following theorem permits to identify equivalence classes with evolution matrices in the case of qubits.
Theorem 3.1. Let \( \mathcal{N} \) be an \( n \)-qubit computational network, and let \( Q_k \) denote the \( k \)-th qubit. For all possible evolutions \( U \) and \( U' \) of \( \mathcal{N} \),

\[
[U]^{Q_k} = [U']^{Q_k} \iff \llbracket U \rrbracket^{Q_k} = \llbracket U' \rrbracket^{Q_k}.
\]

Proof. The “\( \Rightarrow \)” has already been established by Raymond-Robichaud, and is presented in eq. (4) of § 2.

Thanks to the DH-RR equivalence, \( \llbracket U \rrbracket^{Q_k} \) can be equivalently represented by

\[
q_k(t) = U^\dagger q_k(0) U.
\]

To prove the “\( \Leftarrow \)”, assume \( [U]^{Q_k} \neq [U']^{Q_k} \) and therefore, \( U' \neq (\mathbb{I}^{Q_k} \otimes V)U \), for some \( V \) acting on qubit \( k \). Hence, \( U' = MU \), for some global operator \( M \), whose functional form \( U_M(q(0)) \) depends explicitly on terms of \( q_k(0) \).

But then, if \( M \) is thought to occur between time \( t \) and \( t' \),

\[
q_k(t') = U^\dagger M^\dagger q_k(0) MU
\]

\[
= U^\dagger M^\dagger UU^\dagger q_k(0) U U^\dagger M U
\]

\[
= U_M^\dagger(q(t))q_k(t)U_M(q(t)).
\]

But because of its dependence on \( q_k(t) \), \( U_M(q(t)) \) acts nontrivially on \( q_k(t) \) which changes it to a \( q_k(t') \neq q_k(t) \), i.e., \( \llbracket U \rrbracket^{Q_k} \neq \llbracket U' \rrbracket^{Q_k} \).

The previous theorem allows, at least for qubits\(^4\), to unify the definitions of the abstract and the quantum formalisms of parallel lives. The abstract notion of a noumenal state, defined as the equivalence class, can now be realized by the evolution matrix in the quantum setting.

4 The proof could be extended to more general systems, but the analysis for qubits was eased by the DH formalism. For a system \( A \) of arbitrary dimension, one can generalize the methods of the DH formalism by constructing a generating set of traceless operators acting on \( A \) and \( \overline{A} \). This can be achieved with a generalization of Pauli matrices.
4.1 Density-Matrix Space of a Qubit

Consider first the well-known example of the density-matrix space of a single qubit $Q_k$ within an $n$-qubit network $N$. In RR’s terminology, this is the phenomenal space. The geometric object that characterizes such a state space, notwithstanding the size of the total system to which it belongs, is a unit ball in $\mathbb{R}^3$. This comes from the one-to-one correspondence between the density matrices over a qubit and the points on and inside the Bloch sphere, i.e.,

$$\rho = \frac{1}{2}(1 + p \cdot \sigma),$$

where the polarisation vector $p = (p_x, p_y, p_z)$ is constrained by $|p| \leq 1$. The space of density matrices of a qubit ranges along with the range of $p$, which is the unit ball in $\mathbb{R}^3$,

$$\text{Density}^{Q_k} \simeq D^3 = \{p \in \mathbb{R}^3 : |p| \leq 1\}.$$

In particular, $\text{Dim}(\text{Density}^{Q_k}) = 3$.

4.2 Descriptor Space of a Qubit in an $n$-Qubit Network

How big is the descriptor space — or equivalently, the noumenal space — of a qubit then? Unlike the density-matrix space, the dimension of the descriptor space of a qubit scales (exponentially!) with the size of the whole system $N$ to which it belongs. The proof of this assertion involves basic notions of Lie groups, which, in the present context, can be simplified to the special case of a regular hypersurface endowed with a group structure. For more on the topic, see, for instance, Ref. [17].

From the equivalences established in §3, a descriptor space of a qubit can be identified to the space of equivalence classes. Define $H \subset U(2^n)$ the set of operations of the form $1^{Q_k} \otimes V$, where $V \in U(2^{n-1})$ acts on $Q_k$. It is a closed subgroup of $U(2^n)$, so also a Lie group. Therefore,

$$\text{Descriptor}^{Q_k \subset N} \simeq U(2^n)/H.$$

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5A hypersurface of dimension $n$ is an object defined by $m$ independent constraints in $\mathbb{R}^{n+m}$, $\{y \in \mathbb{R}^{n+m} : F^a(y) = 0, \ a = 1, \ldots, m\}$. It is regular if the $m \times (n + m)$ matrix with elements $\frac{\partial F^a}{\partial y^i}$ has full rank in all points.
Let $\tilde{k} \neq k$. Denote $\text{CNOT}_{\tilde{k} \rightarrow k}$ the controlled-not gate in which qubit $\tilde{k}$ controls qubit $k$ and denote $N^k$ the negation gate applied to $Q^k$. Then

$$\text{CNOT}_{\tilde{k} \rightarrow k}(1^{Q_{\tilde{k}}} \otimes N^k)\text{CNOT}_{\tilde{k} \rightarrow k} \notin H,$$

because it does not act trivially on $Q_k$, in particular, it changes $|00\rangle^{k\tilde{k}}$ to $|11\rangle^{k\tilde{k}}$. Because CNOT is self-inverse, the above means that $H$ is not a normal subgroup of $U(2^n)$, and so the quotient $U(2^n)/H$ is not a group. However, the quotient of Lie groups remains a differential manifold, whose dimension is the difference of the dimensions of the Lie groups involved in the quotient. The group $U(N)$ has (real) dimension $N^2$, because it is a hypersurface in $\mathbb{C}^{N^2} \cong \mathbb{R}^{2N^2}$ subject to the $N^2$ independent (real) constraints $\sum_j u^*_{ji} u_{jk} = \delta_{ik}$. Since, $H \cong U(2^{n-1})$, one finds

$$\dim(\text{Descriptor}^{Q_k \subseteq N}) = \dim U(2^n) - \dim U(2^{n-1})$$

$$= 2^{2n} - 2^{2n-2}$$

$$= \frac{3}{4} \cdot 2^{2n},$$

and in particular, the dimension of the descriptor space scales exponentially with the size of the whole system $\Omega$.

Compared to describing the 3-dimensional reduced density matrix of a qubit, if one instead faces the task of describing the descriptor of the same qubit, then she must feel like she has the Universe to describe. This is in contradiction with the analysis by Hewitt-Horsman and Vedral [16, §3], who claim (in bold font omitted here) that “in general a given state defined by a density matrix has a unique representation in terms of Deutsch-Hayden operators”. This statement hinges on a flaw in their analysis: In a nutshell, the number of constraints that determine a descriptor from a density matrix is over counted, so the descriptor should be left under-determined by the density matrix.

Notice that for such an $n$-qubit network $\Omega$ as a whole system, the universal wave function $|\Psi\rangle$, i.e., the Shrödinger state of the whole network, has dimensionality $2^{n+1} - 2$. Indeed, the amplitudes are fixed by $2 \cdot 2^n$ real parameters, and the normalization and the irrelevance of a global phase cut down two parameters. Therefore, the descriptor of a single qubit has larger dimensionality than the Shrödinger state of the whole network — or of the Universe!
Although the previous statement is surprising, one should not be astounded nor desperate by the exponential scaling of descriptors for single qubits, since it was to be expected. Indeed, the most economical local repartition of the necessary $2^{n+1} - 2$ parameters of a complete description of $n$ qubits must still leave $\sim 2^n/n$ parameters in each qubit!

### 4.3 The Universal Descriptor

If the descriptor of a single qubit has larger dimensionality than that of the universal wave function, then how big is the space of universal descriptors? It turns out that it is not much bigger than the qubit descriptor space. The previous analysis can be paralleled, with $\Omega$ as the considered system, whose complement is the empty system $\emptyset$. Hence, the subgroup $H$ are the operation of the form $1^\Omega \otimes e^{i\phi}$, which can be identified to $U(1)$. Consequently,

$$\text{Descriptor}^\Omega \simeq U(2^n)/U(1) \quad \text{and} \quad \text{Dim(Descriptor}^\Omega) = 2^{2^n} - 1.$$

Therefore, the universal descriptor is, up to a phase, the unitary operator that occurred on the whole system from time 0 to now. In this case, $H$ is a normal subgroup of $U(2^n)$, so $\text{Descriptor}^\Omega$ keeps a group structure, namely, that of $SU(2^n)$.

A more pedestrian approach can also be used to establish that a complete description of the whole system entails the knowledge of the evolution $U$, up to a phase. Indeed, from the descriptors or evolution matrices of each qubit of the network, one can multiplicatively and additively reconstruct $U^\dagger |j\rangle \langle i|U$ for all $i$ and $j$, where $\{|i\rangle\}_{i=0}^{2^n-1}$ is a basis of $\mathcal{H}^\Omega$. The matrix element $\ell,k$ of $U^\dagger|j\rangle \langle i|U$ is given by

$$\langle \ell|U^\dagger|j\rangle \langle i|U|k \rangle = u^*_{j\ell} u_{ik}.$$

By setting $i = j = k = \ell = 0$, one finds $|u_{00}|^2$ and by setting $j = \ell = 0$, but leaving $i$ and $k$ free, one finds $u^*_{00} u_{ik}$ for all $i$ and $k$. Therefore, up to a phase, $U$ can be computed from $U^\dagger|j\rangle \langle i|U$ for all $i$ and $j$, which can be computed from $q_i(t)$ or $[U]^Q_i$ for all $i$.

If the initial state is denoted $|0\rangle$, the universal wave function is obtained (up to a phase), by

$$|\Psi\rangle = U|0\rangle.$$
This corresponds to only one column of the universal descriptor, which is (up to a phase) $U$, so $U|\phi_0\rangle$ for all possible initial state $|\phi_0\rangle$. If the multiplicity of classical-like terms in Everett’s universal wave function has prompted some to coin a Many Worlds Interpretation, then the multiplicity of Everett’s states in a universal descriptor could be thought as many many worlds; namely, as many “many worlds” as there are dimensions in the whole Hilbert space.

4.4 What More than the Universal Wave Function?

The many-to-one correspondence between the universal descriptor and the global Schrödinger state (or global density operator) has already been pointed out by Wallace and Timpson [22]. They argued that since the descriptors corresponding to the same Schrödinger state lead to the same observations, they should be equated by some “quantum gauge equivalence”. In such a case the description left out boils down again to the usual Schrödinger state, retrieving non-locality. In response, Deutsch [9] attacks the premise and argues that the dynamics that has lead to such an actual Schrödinger state, too, may manifest in observations. Indeed, in §5 of his paper, he proposes a way in which one can tell apart different descriptors that yield the same Schrödinger state. Consistently with our identification of the universal descriptor to the evolution operator, his proposal inevitably sums up to network tomography. Raymond-Robichaud, also aware of the injectivity of the morphism $\phi$ between noumenal and phenomenal states, hold an intermediate standpoint that crops up in their nomenclature. The whole point of their work is to oppose to the Wallace-Timpson identification and authorize — in the name of locality — the existence of noumenal states as elements of reality. They however recognize that different noumenal states may lead to the same observations, encompassed by the phenomenal state.

But what is the extra information that the universal descriptor $q(t)$ gives, that is unobtainable from the universal wave function $|\Psi\rangle$ alone? It can be thought to encode the universal wave function for any possible initial state. In fact, $|\Psi\rangle = U|0\rangle$ is of no use to determine $|\Psi'\rangle = U|0'\rangle$ for a different initial state, with $\langle 0|0'\rangle = 0$. However, $q(t)$ can be used to compute this alternative universal Shrödinger state $|\Psi'\rangle$, or, more in hand with

---

6In his work [13,14], Everett never referred to “Many Worlds”. 
the Heisenberg picture, the expectation \( \langle \Psi' | O | \Psi' \rangle = \langle 0' | U^\dagger O U | 0' \rangle \) of any observable. A computation as such can be done by first defining a unitary operator \( V \) such that \( V | 0 \rangle = | 0' \rangle \). Recalling that \( O \) can be reconstructed from \( q(0) \),

\[
\langle 0' | U^\dagger q(0) U | 0' \rangle = \langle 0 | V^\dagger U^\dagger q(0) U V | 0 \rangle \\
= \langle 0 | U^\dagger_U (q(0')) U^\dagger_V (q(0)) q(0) U_V (q(0)) U_U (q(0')) | 0 \rangle \\
= \langle 0 | U^\dagger_U (q(0')) q(0') U_U (q(0')) | 0 \rangle,
\]

where \( 0' \) can be thought as an intermediary time delimiting, together with time \( 0 \), the application of \( V \). Therefore, since \( q(t) = U^\dagger q(0) U \) can be determined by a fixed function of \( q(0) \), then \( V^\dagger U^\dagger q(0) U V \) is determined by the same function, but instead evaluated on argument \( q(0') \).

This puts in evidence a particular feature of the DH formalism, namely, it enables the evolution of the descriptors in both directions in time, simultaneously. On the one hand, adding a gate at the end of the network affects the outer shell, that is to say, the function that determines \( q(t + 1) \) from \( q(0) \) will differ from that of \( q(t) \). On the other hand, supplementing a gate at the beginning of the network changes the inner shell: The defining function of \( q(t) \) remains the same, but it is instead applied to the argument \( q(0') \).

5 Continuous Systems

Evolution matrices can naturally be extended to locally describe quantum systems of continuous degrees of freedom. The mathematical structures required to formalize the approach are those of Dirac calculus, once made mathematically meaningful by Schwartz’ distribution theory [20]. For a concise presentation, see Ref. [2, p.28].

Consider a system \( A \) with a continuous one dimensional observable (e.g., the position of a particle). Associated to this system is a rigged Hilbert space admitting a Dirac-orthonormal basis \( \{|x\rangle \}_{x \in \mathbb{R}} \), where

\[
\langle x | x' \rangle = \delta(x - x') \quad \text{and} \quad \int_{\mathbb{R}} |x\rangle \langle x| = 1.
\]

The wave function can then be represented spatially by \( \psi(x) = \langle x | \psi \rangle \).
evolution matrix associated to the system \( A \) is a “continuous matrix\(^7\)” whose matrix elements are given by

\[
\langle y | x \rangle \otimes 1^A
\]

Here again, \( U \) is the evolution that the whole system has undergone, which could have been represented by any other \( (1^A \otimes V)U \). Let \( A \) and \( B \) be disjoint systems of a continuous one-dimensional observable. Analogously as in §2.3 evolution by \( V \in U(\mathcal{H}^A) \), tracing out and merging are defined as

\[
(W \| U \|)^A_{xy} \overset{df}{=} \int_{\mathbb{R}^2} dx'dy' V_{xx'} (W \| U \|)^A_{x'y'} V_{y'y}^* \]

\[
(\text{tr}_B (W \| U \|)^A)_{xy} \overset{df}{=} \int_{\mathbb{R}} dz (W \| U \|)^A_{xz} (W \| U \|)^B_{yz} \]

\[
(W \| U \|)^A \otimes (W \| U \|)^B_{x_y x_A y_B} \overset{df}{=} (W \| U \|)^A_{x_A x_B y_A y_B} \]

With those definitions in hand, the analogue of Theorem 2.1 holds.

\textbf{Theorem 5.1.} Let \( A \) and \( B \) be disjoint systems of a continuous observable and let \( V \in U(\mathcal{H}^A) \).

\[
V \langle W \| U \| A = \langle (W \otimes 1^A)U \| A \\
\text{tr}_B (W \| U \|)^A \otimes (W \| U \|)^B_{x_y x_A y_B} = \langle W \| U \|^A \otimes (W \| U \|)^B_{x_y x_A y_B} \\
\]

For a fixed reference density matrix \( \rho_0 \), the morphism \( \phi \) is defined as

\[
\phi \langle W \| U \| A_{xy} \overset{df}{=} \text{tr} \langle W \| U \|_{xy} \rho_0 \)

and Theorem 2.2 also generalizes to continuous systems.

\textbf{Theorem 5.2.} Let \( A \) and \( B \) be disjoint systems of a continuous variable and and let \( V \in U(\mathcal{H}^A) \). Then

\[
\phi \langle W \| U \| A = \text{tr}_A (U \rho_0) \quad V \ast \phi \langle W \| U \| A = \phi (V \langle W \| U \| A) \quad \text{and} \quad \text{tr}_B \phi \langle W \| U \| A = \phi \text{tr}_B (W \| U \| A)

The proofs of theorems 5.1 and 5.2 are relegated to Appendix B.

\footnote{An object \( M \) as such is in fact a sesquilinear form on test functions, which maps \( f \) and \( g \) to \( \int_{\mathbb{R}^2} dx dy M_{xy} f^*(x) g(y) \).}
6 Conclusions

Deutsch and Hayden conclude their paper with a beautiful analogy that compares their descriptor obtained in the Heisenberg picture to the usual representation of a quantum state framed in the Schrödinger picture:

The relationship between the two pictures is somewhat analogous to that between any descriptive piece of information, such as a text or a digitized image, and an algorithmically compressed version of the same information that eliminates redundancy to achieve a more compact representation. If the compression algorithm used is not lossy, then, considered as a description of the original data, the two versions are mathematically equivalent. However, the elimination of redundancy results in strong interdependence between the elements of the compressed description so that, for instance, a localized change in the original data can result in changes all over the compressed version, so that a particular character or pixel from the original is not necessarily located at any particular position in the compressed version. Nevertheless, it would be a serious error to conclude that this holistic property of the compressed description expresses any analogous property in the original text or image, or of course in the reality that they refer to.

The underdetermination of the descriptor by the Schrödinger state renders the “compression algorithm” lossy. But the analogy does not collapse; the usual representation of a quantum state may now exhibit holistic features because of its compactness or because of its lost information.

As discussed in §4.4, the lost information is about the various other dynamics of the network, would it have been initialized differently. In quantum information theory, qubits initialized in a state $|0\rangle$ are taken as a free entity; but how does one really get such an initialized qubit in a unitary quantum realm? This may be referred to as the preparation problem, dual to the measurement problem. A parsimonious solution should provide a mechanism that explains, from within unitary quantum theory, why computations can be done as if the state really was $|0\rangle$. Such an explanation would rely on decoherence arguments, and in the larger unitary scheme, not only $|0\rangle$ should go through the whole network, perhaps justifying the need for more dynamics.
The complexity of the descriptor was investigated in §4 through the dimensionality of its space, well motivated in physics. However, a computer theoretic approach may regard as the complexity cost of a descriptor its difficulty in time, in space or in program size to produce it. An investigation as such should be hand in hand with circuit complexity, since the whole descriptor is but a compact representation of the operator representing its generating circuit. Perhaps, also, a new insight into quantum Kolmogorov complexity could be provided in the DH formalism.

If one is willing to pay Everett’s price, and accepts that the $n$-qubit universe is encoded in a point $|\Psi\rangle$ moving in $2^{n+1} - 2$ dimensions, then one should without regrets square this number up to $2^{2n} - 1$ and instead use the universal descriptor for an entirely local story. One then faces the surprising consequence that more than $3/4$ of the whole dimensionality resides in each qubit. Most of this information is locally inaccessible; it accounts for common histories among qubits, keeping track of whom is entangled with whom. The consequence becomes more digestible when one appreciates how entangled the universe really is. And before backing off from the implications of a well-motivated paradigm shift, one reminds Wallace’s advice [21]: “The moral is clear: our intuitions as to what is unreasonable or absurd were formed to aid our ancestors scratching a living on the savannahs of Africa, and the Universe is not obliged to conform to them”.

Looping the loop with whom we started, Bell also stated [4] that “Either the wave function, as given by the Shrödinger equation, is not everything, or it is not right”. It is so far right, but not everything; and completing it by the universal descriptor is perhaps what Einstein Podolsky and Rosen [12] were looking for.
A  Introduction to the DH Formalism

When bold foundational statements such as those established by Deutsch and Hayden (cf. Section 1) collect a mere 165 citations in 20 years, it is perhaps because a large portion of the community of quantum foundations is unaware of their contribution, or does not understand it properly. This appendix is an elementary introduction to the DH formalism. It covers Sections 2 and 3 of Ref. [10] in much more length, providing examples of calculations and explanations from different standpoints. It is aimed both for experts and non-experts in quantum theory: A reader with introductory knowledge in quantum information theory, with or without a physics background, should understand this text.

A.1  A Question of Picture

In quantum theory, computations leading to measurable quantities all take the same form: They are expected values of some observables. An observable $O$ is represented by a hermitian operator which admits a spectral decomposition

$$O = \sum_{i} \lambda_i \Pi_i,$$

where $\lambda_i \in \mathbb{R}$ are the eigenvalues corresponding to the measurement outcomes and the $\Pi_i$ are the corresponding projectors on the eigensubspaces. If the system is in state $|\psi\rangle$, the expected value of such an observable is given by $\langle \psi | O | \psi \rangle$, since

$$\langle \psi | O | \psi \rangle = \langle \psi | \sum_{i} \lambda_i \Pi_i | \psi \rangle = \sum_{i} \langle \psi | \Pi_i | \psi \rangle \lambda_i = \sum_{i} p_i \lambda_i,$$

where $p_i$ is the probability of measuring outcome $\lambda_i$. This type of computation is routine for physicists, but quantum information scientists usually compute probabilities of measurement outcomes. An $n$-qubit network in the state

$$\sum_{j=0}^{2^n-1} \alpha_j |j\rangle$$

has a probability $|\alpha_l|^2$ to return the classical value “$l$”. But

$$|\alpha_l|^2 = \langle \psi | l \rangle \langle l | \psi \rangle$$
is nothing but the expectation value of the observable $|l\rangle\langle l|$.

In general $|\psi\rangle$ could be a complex state that comes from a large network applied to the initial state $|0\rangle$, in some fixed basis. Hence, if $U$ is the unitary operator representing the network,

$$|\psi\rangle = U|0\rangle.$$ 

Therefore, the computations carried to predict statistical properties of the quantities measured in the laboratory all have the form

$$\langle 0|U^\dagger OU|0\rangle,$$  

(6)

where $|0\rangle$ is the initial state, $U$ is the unitary evolution and $O$ is the observable.

The Schrödinger picture is about viewing the sandwich Equation (6) as if the bread evolves and the meat stays constant, namely,

$$\left(\langle 0|U^\dagger \right) O \left(U|0\rangle\right).$$

With such a viewpoint, the initial state $|0\rangle$ evolves to the final state $|\psi\rangle = U|0\rangle$ and the observable $O$ remains constant.

The Heisenberg picture is about regarding the sandwich equation as if the meat evolves but the bread remains constant,

$$\langle 0|\left(U^\dagger OU\right)|0\rangle.$$  

(7)

In this picture, the state vector remains fixed to $|0\rangle$ but the observable $O$ evolves to $U^\dagger OU$. Therefore, in the Heisenberg picture, the term ‘state’, which refers to a quantity that is fixed to $|0\rangle$ becomes a misnomer. For this reason, it will be referred to as the reference vector. Deutsch and Hayden's descriptors come from encoding the information of the quantum system into evolving observables, as if one tries to define a “Heisenberg state”.

A.2 Tracking Observables

In the Heisenberg picture, a quantum system shall no longer be described by its Schrödinger state, but rather by an object that encodes the information about all the evolved observables on the system. Luckily, observables are linear operators and so form a vector space. Since the evolution $O \rightarrow U^\dagger OU$ is linear, one does not need to track the evolution of infinitely many observables: Only a basis of the linear operators suffices. Indeed, if $O = \sum_j a_j B_j$, then $U^\dagger OU = \sum_j a_j U^\dagger B_j U$. 

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A.2.1 The Descriptor of a 1-Qubit Network

In the case of a single qubit, the Pauli matrices together with the identity,
\[ \sigma = (\sigma_x, \sigma_y, \sigma_z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \] and \( \sigma_0 = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)
form a basis of any 2 × 2 matrices, if the linear combinations are taken over complex numbers. Following the evolution of \( 1 \) is trivial, \( U^\dagger U = 1 \), so it can be neglected and one only follows the evolution of \( \sigma \). Hence, in the Heisenberg picture, a qubit is represented by a descriptor \( q(t) = U^\dagger \sigma U \), where \( U \) is the unitary operator that represents the evolution undergone by quantum network between time 0 and time \( t \).

**Exercise 1.** Describe \(|+\rangle\) in the Heisenberg picture.

**Solution:** One takes the initial reference vector to be fixed to \(|0\rangle\). In the Shrödinger picture, \(|+\rangle = H|0\rangle\), where \( H \) is the Hadamard gate so the descriptor is given by
\[ H^\dagger \sigma H = H(\sigma_x, \sigma_y, \sigma_z)H = (\sigma_z, -\sigma_y, \sigma_x). \]

The descriptor is not uniquely determined by the Shrödinger state \(|+\rangle\), since any other unitary transformation \( U \) such that \(|+\rangle = U|0\rangle\) can be taken instead of \( H \). This underdetermination is explored in more details in Section 4.

A.2.2 Descriptors of an \( n \)-Qubit Network

A natural basis to the space of all operators on \( n \) qubits is the product of Pauli operators, namely
\[ B^n = \{ \sigma_{\mu_1} \otimes \sigma_{\mu_2} \otimes \ldots \otimes \sigma_{\mu_n} : \mu_i \in \{0, x, y, z\} \}. \]

There are \( 4^n \) such matrices, which are linearly independent and hence they form a basis of the \( 2^n \times 2^n = 4^n \) dimensional complex vector space of linear operators on \( n \)-qubits.

This means that if one knows how each observable of the basis evolves by the action of some unitary operator \( U \),
\[ \sigma_{\mu_1} \otimes \sigma_{\mu_2} \otimes \ldots \otimes \sigma_{\mu_n} \rightarrow U^\dagger \sigma_{\mu_1} \otimes \sigma_{\mu_2} \otimes \ldots \otimes \sigma_{\mu_n} U, \quad \mu_i \in \{0, x, y, z\}, \]
then we know, by linearity, how each observable evolves.
A.2.3 DH’s Shortcut

In the case of \( n \) interacting qubits of some quantum computational network \( \mathcal{D} \), Deutsch and Hayden suggest to track the set of observables

\[
q_i(0) = 1^{i-1} \otimes \sigma \otimes 1^{n-i}, \quad i = 1, \ldots, n,
\]

(8)

where \( 1^k \) stands for the tensor product of \( k \) copies of the identity. Note that for each \( i \), \( q_i(0) \) has 3 components. The \( n \)-tuple whose components are the \( q_i(0) \) is noted \( q(0) \). Bold quantities are vectors, so one writes \( q_i(0) \), but \( q_{ix}(0) \). The vector \( q(0) \) represents the initial observables, namely, those at time \( t = 0 \), whence the notation.

Importantly, note that \( q(0) \) contains much fewer components than \( \mathcal{B}^n \) contains elements. In fact, instead of tracking the \( 4^n \) operators of \( \mathcal{B}^n \) only \( 3^n \) are suggested here. The reason is that these \( 3^n \) operators have a multiplicative structure that allows to generate any of the \( 4^n \) basis operators. Moreover, this multiplicative structure is preserved by the evolution \( U \), namely, if \( q \) and \( \bar{q} \) are any operator,

\[
q \bar{q} \rightarrow (q \bar{q})' = U^\dagger q \bar{q} U = U^\dagger q U U^\dagger \bar{q} U = q' \bar{q}'.
\]

Remark A.1. The operators of \( q(0) \) satisfy the \( \text{su}(2)^{\otimes n} \) algebra, namely

\[
[q_{iw}(0), q_{jw'}(0)] = 0 \quad (i \neq j \text{ and } \forall w, w')
\]

\[
q_{ix}(0)q_{iy}(0) = iq_{iz}(0) \quad \text{(and cyclic permutations)}
\]

(9)

\[
q_{iw}(0)^2 = 1 \quad (\forall w).
\]

A.2.4 One more Shortcut

Following Gottesman [15], the generating tuple \( q(0) \) could be reduced to \( 2n \) elements by noticing a redundancy due to the \( \text{su}(2)^{\otimes n} \) algebra. In fact, only two of the three \( (q_{ix}(0), q_{iy}(0), q_{iz}(0)) \) operators are required, for any \( i \), since the case operator is obtained by the product of the selected two. In what follows, the notation will not be modified, but one will happily use this shortcut to avoid tracking the observables \( q_{iy}(0) \), since \( q_{iy}(0) = -iq_{ix}(0)q_{iz}(0) \).

Summing this up, knowing the evolution of the \( 2n \) observables of \( q(0) \) (without the \( q_{iy}(0) \)) allows to infer, by group multiplication, the evolution of the \( 4^n \) observables of \( \mathcal{B}_n \), which allows to infer, by linearity, the evolution of any observable.
In this case, the descriptor of qubit $i$ at time $t$ is given by

$$q_i(t) = U^\dagger q_i(0) U,$$  \hspace{1cm} (EVO 1)

where $U$ is the unitary operator that represents the evolution undergone by quantum network between time 0 and time $t$.

### A.3 Evolution from the future?!

Although $O \to U^\dagger O U$ looks like a completely fine way in which observables should evolve, when $U$ is broken down into different gates, for instance $U = WV$, one finds that the observables evolve in the wrong order! In fact, $WV$ means that $V$ is done before $W$, or diagrammatically,

$$\begin{aligned} 
\hline 
V \\
\hline 
W 
\end{aligned},
$$

but the observable evolves as

$$O \to V^\dagger W^\dagger OWV,$$  \hspace{1cm} (10)

$i.e.$, $W$ is applied first, then $V$. In a computational network, the evolution of observables then occurs from the last gate of the network to the first, which is completely unnatural and in most cases inconvenient, since the network needs to be final before computing anything.

The way out of this conundrum is to notice that inasmuch as observables $O$ are linear operators generated by some set $q(0)$ of operators, the evolution operators $U$ are too. They are generated multiplicatively and linearly by the same set $q(0)$, since questions of hermicity versus unitarity did not arise.

For a fixed gate with matrix representation $G$, its generation by $q(0)$ defines a function $U_G(\cdot)$ through

$$G = U_G(q(0)).$$  \hspace{1cm} (11)

The function $U_G(\cdot)$ takes value in unitary operators and will be referred to as the functional representation of the gate $G$. Its functionality encodes the multiplicative and linear generation of $G$ by the elements of $q(0)$. For instance, the familiar negation and Hadamard gates are described by

$$N = \begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix} = \sigma_x = q_x(0)$$

and

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix}1 & 1 \\ 1 & -1\end{pmatrix} = \frac{q_x(0) + q_z(0)}{\sqrt{2}},$$
so their functional representations are

\[ U_N(q(0)) = q_x(0) \quad \text{and} \quad U_H(q(0)) = \frac{q_x(0) + q_z(0)}{\sqrt{2}}. \]

The clockwise rotation of a state vector in the \(|0\rangle \& |1\rangle\) plane\(^8\) is described by

\[ R_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \cos \theta \mathbb{1} + i \sin \theta \sigma_y = \cos \theta q_x(0)^2 - \sin \theta q_x(0)q_z(0), \]

which defines its functional representation \( U_{R_\theta}(\cdot) \).

Now, when \( q(t) \) varies with \( t \), the matrix representation \( U_G(q(t)) \) also varies, but it is the fixed functionality that plays a role in Heisenberg computations.

### A.3.1 Back in order!

Since the usual matrix representation of a gate \( V \) is expressed by \( U_V(q(0)) \), then if \( V \) is the first gate of the quantum network, by Equation (EVO 1),

\[ q_i(1) = U_V^\dagger(q(0))q_i(0)U_V(q(0)). \tag{12} \]

The apparently reversed ordered evolution of Equation (10) can then be transformed back in the right order:

\[
V^\dagger W^\dagger OWV = U_V^\dagger(q(0))U_W^\dagger(q(0))U_W^\dagger(q(0))U_V(q(0))
\]
\[ \quad \quad = U_V^\dagger(q(0))U_W^\dagger(q(0))U_V(q(0))U_W^\dagger(q(0))U_V^\dagger(q(0))U_W(q(0))
\]
\[ \quad \quad = U_W^\dagger(q(1))U_V^\dagger(q(0))U_V(q(0))U_W(q(1)). \]

Where the last equation, namely that \( U_W(q(1)) = U_V^\dagger(q(0))U_W(q(0))U_V(q(0)) \), comes from the following. Since \( U_W(q(0)) \) is some function of the components of \( q(0) \), when it is sandwiched between \( U_V^\dagger(q(0)) \) and \( U_V(q(0)) \), every

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\(^8\)Note that this operation represents the rotation of a polarized photon, but not exactly that of the spin of an electron. The reason for this is that a \( \pi/2 \) rotation of a photon takes the horizontal polarization \(|\leftrightarrow\rangle \equiv |0\rangle\) to the vertical polarization \(|\uparrow\rangle \equiv |1\rangle\). However, the spin of an electron needs a \( \pi \) rotation to take the \(|\uparrow_z\rangle \equiv |0\rangle\) to \(|\downarrow_z\rangle \equiv |1\rangle\). Such a rotation is better represented on the Bloch sphere and shall be discussed in section A.7.1.
term containing some \( q_{iw}(0) \) gets transformed to its corresponding \( q_{iw}(1) \). Terms of \( U_W(q(0)) \) that contain products \( q_{iw}(0)q_{jw'}(0) \) need extra bread in the middle of the sandwich, i.e.

\[
U_Y^\dagger(q(0))q_{iw}(0)U_Y(q(0))U_Y^\dagger(q(0))q_{jw'}(0)U_Y(q(0)),
\]

yielding \( q_{iw}(1)q_{jw'}(1) \).

Iterating the argument, a quantum network of many gates \( G_1, G_2, \ldots, G_N \) has its observables tracked in two possible ways.

- With the usual fixed matrix representation of unitary operators that act in the wrong order on observables (but in the right order if they were to act in the Shrödinger picture),

\[
G_1^\dagger G_2^\dagger \cdots G_N^\dagger O G_N \cdots G_2 G_1
\]

- With the operators defined as a fixed function of the generating set \( q(t) \) which act in the right order,

\[
U_{G_N}^\dagger(q(N-1)) \cdots U_{G_2}^\dagger(q(1)) U_{G_1}^\dagger(q(0)) O U_{G_1}^\dagger(q(0)) U_{G_2}^\dagger(q(1)) \cdots U_{G_N}^\dagger(q(N-1))
\]

The later approach is preferred to perform computations in the Heisenberg picture.

### A.4 Another Evolution Equation

Deutsch and Hayden do not pass by Equation \( [\text{EVO 1}] \) to evolve the descriptor from time 0 to \( t \). Instead, the descriptor is claimed to evolve iteratively as

\[
q_i(t + 1) = U_W^\dagger(q(t))q_i(t)U_W^\dagger(q(t)), \tag{EVO 2}
\]

where \( W \) is the gate performed on the network between time \( t \) and \( t + 1 \). However, such an iterative evolution is equivalent to the one prescribed by Equation \( [\text{EVO 1}] \).

\( [\text{EVO 1}] \Rightarrow [\text{EVO 2}] \). Let \( V \) be the unitary operator representing the evolution of the network between time 0 and time \( t \).

\[
q_i(t + 1) = (WV)^\dagger q_i(0)WV = V^\dagger U_W^\dagger(q(0))VV^\dagger q_i(0)V V^\dagger U_W(q(0))V = U_W(q(t))q_i(t)U_W(q(t)).
\]
Let $G_s$ be the gate that occurs between time $s-1$ and $s$. The base of the induction is easily verified

$$q_i(1) = U_G^\dagger(q(0))q_i(0)U_G(q(0))$$

$$= G_1^\dagger q_i(0) G_1,$$

and with induction hypothesis

$$q_i(t-1) = G_1^\dagger G_2^\dagger ... G_{t-1}^\dagger q_i(0) G_{t-1} ... G_2 G_1,$$

one finds

$$q_i(t) = U_G^\dagger(q(t-1))q_i(t-1)U_G^\dagger(q(t-1))$$

$$= \ldots q_i(0) G_{t-1} ... G_2 G_1 U_G^\dagger(q(t-1))$$

$$= \ldots q_i(0) G_{t-1} ... G_2 G_1 U_G^\dagger \left( G_1^\dagger G_2^\dagger ... G_{t-1}^\dagger q(0) G_{t-1} ... G_2 G_1 \right)$$

$$= \ldots q_i(0) G_{t-1} ... G_2 G_1 G_1^\dagger G_2^\dagger ... G_{t-1}^\dagger U_G^\dagger(q(0)) G_{t-1} ... G_2 G_1$$

$$= \ldots q_i(0) G_t G_{t-1} ... G_2 G_1.$$

The fourth line is obtained from the third by a similar argument as in Eq. (13). For conciseness, the left of $q_i(0)$ has been omitted since it has a symmetric behaviour as what happens to the right of it.

### A.5 Not its matrix rep, but its action!

In the Shrödinger picture, the state $|\psi(t)\rangle$ at time $t$ can be computed by the action of the gates of the network on $|\psi(0)\rangle$. The computation of the descriptor $q(t)$ at time $t$ can also conveniently be computed form the action of the gates. However, it is not achieved by matrix multiplication, rather, through the functional representation of the gates and the relations $su(2)^\otimes n$ algebra.

**Remark A.2.** Even if $q(t)$ loses its initial tensor product form of Equation (8), it still satisfies the $su(2)^\otimes n$ algebra (Relations (9)):

$$[q_{iw}(t), q_{jw'}(t)] = q_{iw}(t)q_{jw'}(t) - q_{jw'}(t)q_{iw}(t)$$

$$= U^\dagger q_{iw}(0) U U^\dagger q_{jw'}(0) U - U^\dagger q_{jw'}(0) U U^\dagger q_{iw}(0) U$$

$$= U^\dagger q_{iw}(0) q_{jw'}(0) U - U^\dagger q_{jw'}(0) q_{iw}(0) U$$

$$= U^\dagger [q_{iw}(0), q_{jw'}(0)] U$$

$$= 0 \quad (i \neq j \text{ and } \forall w, w')$$
\[ q_{ix}(t)q_{iy}(t) = U^\dagger q_{ix}(0) U U^\dagger q_{iy}(0) U \]
\[ = U^\dagger q_{ix}(0) q_{iy}(0) U \]
\[ = U^\dagger i q_{iz}(0) U \]
\[ = i q_{iz}(t) \quad \text{(and cyclic permutations)} \]

\[ q_{iw}(t)^2 = U^\dagger q_{iw}(0) U U^\dagger q_{iw}(0) U \]
\[ = U^\dagger q_{iw}(0) q_{iw}(0) U \]
\[ = U^\dagger 1 U \]
\[ = 1 \quad \forall w \].

Let \( W \) be the gate performed between time \( t \) and time \( t + 1 \). For each \( i \), its action on \( q_i(t) \) is

\[ W: q_i(t) \rightarrow q_i(t + 1) = U_W^\dagger(q(t)) q_i(t) U_W(q(t)) \]
\[ = U_W^\dagger(q(t))(q_{ix}(t), q_{iz}(t)) U_W(q(t)). \]

For a generic gate, the updating of \( q_i(t) \) to \( q_i(t + 1) \) requires \( 2n \) sandwich-like calculations. However, if \( W \) acts on only two qubits (e.g., qubits \( j \) and \( k \), it reduces to only 4 such calculations. Indeed, the linear transformation \( W \) (one can think of its matrix representation) acts as the identity on all product spaces that concerns not qubits \( j \) and \( k \). Therefore, the functional representation of the gate, defined by \( U_W(q(0)) = W \), can only depend on \( q_{jx}(t), q_{jz}(t), q_{kx}(t) \) and \( q_{kz}(t) \). Because of the preserved algebraic relations, particularly \([q_l(t), q_j(t)] = 0 = [q_l(t), q_k(t)]\), the update of the descriptor \( q_i(t) \) is trivial for any qubit different than qubit \( j \) or \( k \). The computation is therefore majorly enlightened, and noted

\[ W: \begin{cases} q_j(t + 1) \\ q_k(t + 1) \end{cases} = U_W^\dagger(q(t)) \begin{cases} q_j(t) \\ q_k(t) \end{cases} U_W(q(t)) \]
\[ = \begin{cases} U_W^\dagger(q(t))(q_{jx}(t), q_{jz}(t)) U_W(q(t)) \\ U_W^\dagger(q(t))(q_{kx}(t), q_{kz}(t)) U_W(q(t)) \end{cases} \].

\( ^9 \text{Universal gate sets can be formed from gates acting on no more than two qubits, for instance, the CNOT supplemented by arbitrary unary gates.} \)
A.6 Examples

Let $H_i$ denote the Hadamard gate $H$ performed on the $i$-th qubit.

$$U_{H_i}(q(t)) = \frac{q_{ix}(t) + q_{iz}(t)}{\sqrt{2}}.$$ \hspace{1cm}

The action on descriptor $q_i$ is then

$$H_i : (q_{ix}(t), q_{iz}(t)) \rightarrow (q_{ix}(t + 1), q_{iz}(t + 1))$$

$$= \frac{q_{ix}(t) + q_{iz}(t)}{\sqrt{2}} (q_{ix}(t), q_{iz}(t)) \frac{q_{ix}(t) + q_{iz}(t)}{\sqrt{2}}$$

$$= \frac{1}{2} (q_{ix} + q_{iz} + q_{iz} - q_{ix}, -q_{iz} + q_{ix} + q_{ix} + q_{iz})$$

$$= (q_{iz}, q_{ix}).$$

When the context does not require it, “$(t)$” can be omitted and one notes $H_i : (q_{ix}, q_{iz}) \rightarrow (q_{iz}, q_{ix})$. And when not specified, all the other $q_k$ with $k \neq i$ remain unchanged by the action by $H_i$.

The negation gate $N$ on qubit $i$ has $U_{N_i}(q(t)) = q_{ix}(t)$ and so

$$N_i : (q_{ix}, q_{iz}) \rightarrow (q_{ix}, -q_{iz}).$$

The rotation $R_\theta$ has $U_{R_\theta}(q(t)) = \cos \theta \mathbb{1} - \sin \theta q_x(t)q_z(t)$, so

$$R_\theta : (q_x, q_z) \rightarrow (\cos \theta + \sin \theta q_x q_z)(q_x, q_z)(\cos \theta - \sin \theta q_x q_z)$$

$$= ((\cos^2 \theta - \sin^2 \theta) q_x - 2 \cos \theta \sin \theta q_z, (\cos^2 \theta - \sin^2 \theta) q_z + 2 \cos \theta \sin \theta q_x)$$

$$= (\cos 2\theta q_x - \sin 2\theta q_z, \cos 2\theta q_z + \sin 2\theta q_x).$$

A.6.1 The CNOT

Consider a CNOT gate where the qubit $c$ controls the target qubit $t$. Restricting to the subspace acted upon, the linear transformation is represented by

$$\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$
The functional representation is established by \( U_{\text{CNOT}}(q(0)) = \text{CNOT} \), which can be found by decomposing the above matrix.

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} + \frac{1}{2} \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} = \frac{1}{2}(I \otimes I + \sigma_z \otimes I)
\]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{bmatrix} = \frac{1}{2} \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix} + \frac{1}{2} \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} = \frac{1}{2}(I \otimes \sigma_x - \sigma_z \otimes \sigma_x),
\]

so

\[
\text{CNOT} = \frac{1}{2}(I + q_{cz}(0) + q_{tx}(0) - q_{cz}(0)q_{tx}(0)).
\]

The functional form of CNOT (c controls t) is hence given by

\[
U_{\text{CNOT}}(q(t)) = \frac{1}{2}(I + q_{cz}(t) + q_{tx}(t) - q_{cz}(t)q_{tx}(t)).
\]

The action of a CNOT is given by

\[
\text{CNOT}: \left\{ \begin{array}{c}
(q_{cx}(t + 1), q_{cz}(t + 1)) \\
(q_{tx}(t + 1), q_{tz}(t + 1))
\end{array} \right\} = \left\{ \begin{array}{c}
(q_{cx}(t)q_{tx}(t), q_{cz}(t)) \\
(q_{tx}(t), q_{cz}(t)q_{tz}(t))
\end{array} \right\}.
\]

The calculation of \( q_{cx}(t + 1) \) can be done as follows.

\[
q_{cx}(t + 1) = \frac{1}{4}(I + q_{cz} + q_{tx} - q_{cz}q_{tx})q_{cx}(I + q_{cz} + q_{tx} - q_{cz}q_{tx})
\]

\[
= \frac{1}{4}(q_{cx} + q_{cx}q_{cz} + q_{cx}q_{tx} - q_{cx}q_{cz}q_{tx} + q_{cz}q_{cx} + q_{cx}q_{cx}q_{cz} - q_{cz}q_{cx}q_{cz}q_{tx} + q_{tx}q_{cx} + q_{tx}q_{cx}q_{cz} + q_{tx}q_{tx}q_{cx} - q_{tx}q_{cx}q_{cz}q_{tx} - q_{cz}q_{tx}q_{cx}q_{cz} - q_{cz}q_{tx}q_{cx}q_{tx}q_{cx}q_{cz}q_{tx} + q_{cx}q_{tx}q_{cx}q_{cz}q_{tx})
\]

\[
= \frac{1}{4}(q_{cx} + q_{cx}q_{cz} + q_{cx}q_{tx} - q_{cx}q_{cx}q_{tx} - q_{cz}q_{cz} - q_{cx}q_{cx}q_{tx} + q_{cx}q_{tx} + q_{cx}q_{cx}q_{tx} + q_{cx} - q_{cx}q_{cx} + q_{cx}q_{tx}q_{cx}q_{cz}q_{tx} + q_{cx}q_{tx}q_{cx}q_{cz}q_{tx} + q_{cx}q_{cx}q_{tx} + q_{cx}q_{tx} + q_{cx}q_{cx}q_{tx} - q_{cx})
\]

\[
= q_{cx}q_{tx}.
\]
where, the dependency on $t$ has been discarded.

The action of a gate on a descriptor can be found directly from the matrix representation of the gate, without the detour by its functional representation and the gymnastic of the $\mathfrak{su}(2)\otimes^n$ algebra. Let’s exemplify the method with the case of the CNOT, which in this case consists of calculating

$$\text{CNOT}\left\{ \begin{array}{c} q_c(0) \\ q_t(0) \end{array} \right\} \text{CNOT}.$$  

For the $q_{cx}$ element, this yields

$$\text{CNOT}(\sigma_x \otimes 1)\text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \sigma_x \otimes \sigma_x = q_{cx}(0)q_{tx}(0),$$

consistently with the previous approach. But why does this work?

In fact what has been computed is

$$q_{cx}(1) = U^{\dagger}_{\text{CNOT}}(q(0))q_{cx}(0)U_{\text{CNOT}}(q(0)) = q_{cx}(0)q_{tx}(0).$$

The leap to the general case, i.e., to have $t+1$ and $t$ instead of 1 and 0, follows from observing that the above equation could have been obtained by replacing $U_{\text{CNOT}}(q(0))$ by its functional representation, use the $\mathfrak{su}(2)\otimes^n$ algebraic relations. But since the algebraic relations are preserved, $q(0)$ can invariably be changed to $q(t)$.

A.7 A Note to the Reader

At this stage, the reader who is curious to unravel the mystery of Bell inequality violations and of quantum teleportation is directed to §4 and §5 of the article by Deutsch and Hayden.
In fact, the explanation that the developed formalism provides to the two most famous “nonlocal” manifestations of quantum theory reaches far more than mystery breaking. It roots back quantum theory together with all other scientific theories: the act of measurement needs not to be treated as fundamentally different evolution, and it is completely local. It explores core concepts of the theory — invisible from the Shrödinger picture — that are key to good explanations. Therefore, it changes our vision of reality, making it clearer.

For the reader who is about to jump into Deutsch and Hayden’s article, what follows will be useful. However, it is not needed in the present paper.

A.7.1 Rotation on the Bloch Sphere

Rotating a qubit on the Bloch sphere is described by a rotation of angle \( \theta \) around the unit vector \( \hat{n} \). To distinguish this type of rotation with the rotation in the \(|0\rangle \& |1\rangle\) plane, we denote it \( \tilde{R}_{\hat{n}; \theta} \).

The function representing a general rotation on the Bloch Sphere is given by

\[
U_{\tilde{R}_{\hat{n}; \theta}}(q(t)) = e^{i(\theta/2)\hat{n} \cdot q(t)}.
\]

(14)

The differences with the rotation \( R_{\theta} \) in the \(|0\rangle \& |1\rangle\) plane are two-fold. First, instead of exponentiating the \( q_y(t) \) operator, a more general operator \( \hat{n} \cdot q(t) \) is exponentiated. Second, the parameter becomes \( \theta/2 \). This is because rotating a state in the \(|0\rangle \& |1\rangle\) plane can be seen as a rotation in the Bloch sphere with \( \hat{n} = (0, 1, 0) \), i.e., fixed pointing in the \( y \) direction. However, when seen this way, a rotation of 180° on the Bloch sphere corresponds to a rotation of 90° in the plane, whence the factor of 1/2.
B  Appendix: Proofs of §5

Proof of Theorem 5.1.

\begin{align*}
(V[U]^A)_{xy} &= \int dx'dy'V_{xx'}[U]_{x'y'}^A V_{y'y}^+
= \int dx'dy'(x|V|x') \left( U^+(y'|x') \otimes 1^{\overline{A}} U \right) (y'|V^+|y)
= \int dx'dy' \left( U^+(y'|y') V^+|y \right) (x|V|x') \otimes 1^{\overline{A}} U
= U^+(V^+|y)(x|V \otimes 1^{\overline{A}}) U
= U^+(V^+ \otimes 1^{\overline{A}})(|x| \otimes 1^{\overline{A}})(V \otimes 1^{\overline{A}}) U
= \langle (V \otimes 1^{\overline{A}}) U \rangle^A.
\end{align*}

\begin{align*}
(tr_B[U]^{AB})_{xy} &= \int dz[U]_{xz, yz}^{AB}
= \int dz U^+(|y,z) (x,z) \otimes 1^{\overline{AB}} U
= \int dz U^+(|y) (x^A \otimes z^B) (z^B \otimes 1^{\overline{AB}}) U
= U^+(V^+ \otimes 1^{\overline{A}})(|x| \otimes 1^{\overline{AB}}) U
= \langle (V \otimes 1^{\overline{A}}) U \rangle^A.
\end{align*}

\begin{align*}
([U]^A \otimes [U]^B)_{x_A x_B; y_A y_B} &= [U]_{x_A y_A}^A [U]_{x_B y_B}^B
= U^+ |y_A) (x_A) \otimes 1^B \otimes 1^{\overline{AB}} U U^+ \left( 1^A \otimes |y_B) (x_B) \otimes 1^{\overline{AB}} \right) U
= U^+ |y_A) (x_A) \otimes |y_B) (x_B) \otimes 1^{\overline{AB}} U
= [U]_{x_A x_B; y_A y_B}^{AB}.
\end{align*}

\[\square\]
Proof of Theorem 5.2

\[
\begin{align*}
(\varphi[|U|^A])_{xy} &= \text{tr}\left([|U|^A\rho_0]\right) \\
&= \text{tr}\left(U^\dagger(|y\rangle\langle x| \otimes 1^A)U\rho_0\right) \\
&= \text{tr}\left(|y\rangle\langle x| \otimes \int dz|z\rangle\langle z|U\rho_0U^\dagger\right) \\
&= \int dz(x,z|U\rho_0U^\dagger|y,z) \\
&= \left(\text{tr}_A(U \ast \rho_0)\right)_{xy}.
\end{align*}
\]

\[
V \ast \varphi[|U|^A] &= V\text{tr}_A(U\rho_0U^\dagger)V^\dagger \\
&= \text{tr}_A(1^A)(V \otimes 1^A)U\rho_0U^\dagger(V \otimes 1^A)^\dagger \\
&= \text{tr}_A(V \otimes 1^A)U \ast \rho_0 \\
&= \varphi[(V \otimes 1^A)U]^A \\
&= \varphi(V[|U|^A]).
\]

\[
\text{tr}_B\varphi\left([|U|^A]^B\right) = \text{tr}_B\left(\text{tr}_A(U \ast \rho_0)\right) \\
&= \text{tr}_A(U \ast \rho_0) \\
&= \varphi[|U|^A] \\
&= \varphi\text{tr}_B[|U|^A]^B.
\]

\[\square\]

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