APERIODIC CHAIN RECURRENCE CLASSES OF C¹-GENERIC DIFFEOMORPHISMS

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ABSTRACT. We consider the space of C¹-diffeomorphisms equipped with the C¹-topology on a three dimensional closed manifold. It is known that there are open sets in which C¹-generic diffeomorphisms display uncountably many chain recurrences classes, while only countably many of them may contain periodic orbits. The classes without periodic orbits, called aperiodic classes, are the main subject of this paper. The aim of the paper is to show that aperiodic classes of C¹-generic diffeomorphisms can exhibit a variety of topological properties. More specifically, there are C¹-generic diffeomorphisms with (1) minimal expansive aperiodic classes, (2) minimal but non-uniquely ergodic aperiodic classes, (3) transitive but non-minimal aperiodic classes, (4) non-transitive, uniquely ergodic aperiodic classes.

Keywords: Partially hyperbolic diffeomorphisms, wild dynamical systems.

2020 Mathematics Subject Classification: 37C20-37D30-57M30

September 28, 2022

1. INTRODUCTION

1.1. General setting: Wild dynamical systems. The notion of chaotic dynamical systems goes back to Poincaré [Po]: He noticed that certain deterministic behavior, governed by simple equations, presents a very complicated behavior if the stable and the unstable manifold of a periodic saddle point intersect transversely. This phenomenon, called a homoclinic intersection according to his terminology, not only implies that the system is chaotic, but also every system described by slightly perturbed equations are as such. The behavior is robustly chaotic.

In the middle of the twentieth century, Anosov and Smale develop a theory of a geometric structure called (uniform) hyperbolicity. Hyperbolic systems may be chaotic but they are stable: The systems close to a hyperbolic system are obtained by looking the system through a small continuous change of coordinates. More precisely, the theory of structural stability (see [Ro1, Ro2, Sa, Pa, Ma1, Ha]) shows that the notions of stability called C¹-structural stability (on the all space) or C¹-Ω-stability (restricted to the non-wandering set) are characterized by geometric structures having loquacious names: the Axiom A + the strong transversality property and the Axiom A + the no cycle condition respectively. These geometric structures allow a very precise description of the dynamics from topological and ergodic viewpoints.
However, these structures are so rigid that they can only describe specific regions in the space of dynamical systems. It has been noticed in the late sixties by Newhouse \cite{Ne1} (for the \(C^2\)-topology on surface diffeomorphisms) and Abraham and Smale \cite{AS} (for the \(C^1\)-topology, in dimension \(\geq 3\)) that there are open sets of non-hyperbolic dynamical systems. These systems are robustly unstable: Perturbations of the systems change the qualitative behavior of dynamics.

How can we describe these robustly unstable systems? In the non-conservative setting, the global dynamics split into several somehow independent pieces (we refer \cite{Bo} for a global overview of the space of \(C^1\) dynamical systems). The first global description as such is given (in the very general setting of homeomorphisms of compact metric spaces) by Conley \cite{Co} using the notion of attracting/repelling sets: Two recurrent points are not in the same \emph{chain recurrence class} if and only if there is an attracting/repelling set containing one and only one of these two points.

Hyperbolic dynamical systems have only finitely many chain recurrence classes, and these chain recurrence classes are the \emph{homoclinic classes} (closure of the transverse homoclinic intersections) and are the maximal transitive sets. For robustly non-hyperbolic generic systems, there are two kinds of typical behavior.

- Either they have finitely many classes whose number remains locally constant in the space of all systems (see \cite{Sh, Ma1} for examples). They are called \emph{tame dynamical systems}.
- Or, locally generically (i.e., on a residual subset of a non-empty open set of all the systems) they have infinitely many classes (see \cite{Ne2, BD2, BD1}). They are called \emph{wild dynamical systems}.

Indeed, in the space of \(C^1\)-diffeomorphisms, by \cite{BC} we know that every \(C^1\)-generic diffeomorphism can be classified one of above kinds of dynamical systems. This paper focuses on \(C^1\)-generic wild diffeomorphisms on 3-manifolds. Recall that for \(C^1\)-generic dynamical systems, Kupka-Smale theorem \cite{Ku, Sm} implies that the periodic orbits are all hyperbolic and Pugh’s closing lemma \cite{Pu}, the connecting lemma by \cite{Ha} and \cite{BC} imply that the set of hyperbolic periodic orbits is dense in the chain recurrent set. According to \cite{BC}, for \(C^1\)-generic diffeomorphisms, a chain recurrence class containing a periodic orbit coincides with its homoclinic class and is a maximal transitive set. This property is an important starting point for understanding the dynamics inside these classes.

While the cited results imply that for \(C^1\)-generic tame systems we can always find a hyperbolic periodic point, \(C^1\)-generic wild diffeomorphisms may have \emph{aperiodic classes}, i.e., chain recurrence classes with no periodic orbit. As far as we know, very few are known about the dynamics inside aperiodic classes. Up to now, the unique known example have the structure called adding machines or odometers. However, there are no theoretical result which asserts it should be always the case.

The aim of this paper is to provide a great variety of topological behaviors in aperiodic classes of \(C^1\)-generic diffeomorphisms on 3-manifolds. For instance, some of these classes are not transitive, breaking for the first time the equivalence between the notion of chain recurrence classes and maximal transitive sets for \(C^1\)-generic diffeomorphisms. The following is the results we give in this paper:

**Theorem 1.** Let \(M\) be a closed (compact and without boundary) 3-manifold. Then there is a non-empty \(C^1\)-open set \(\mathcal{O}\) of \(\text{Diff}^1(M)\) such that there is a residual subset \(\mathcal{G} \subset \mathcal{O}\) in which every \(f \in \mathcal{G}\) has...
(1) an uncountable set of chain recurrence classes which are all minimal and expansive;
(2) an uncountable set of chain recurrence classes which are all minimal but support infinitely many ergodic measures;
(3) an uncountable set of chain recurrence classes which are all transitive but contain at least two minimal sets;
(4) an uncountable set of chain recurrence classes which are all uniquely ergodic but not transitive.

1.2. Presentation of the main results. Let us see the strategy of the construction. In the series of papers [BS1, BS2, BS3], we develop a technique for expelling a non-trivial hyperbolic set from a partially hyperbolic chain recurrence class satisfying the condition called property (ℓ), by an arbitrarily $C^1$-small perturbation. This means that, before the perturbation the set was contained in the class, but after the perturbation its continuation is separated from the class by a filtrating set. Conley theory tells us that the separation is $C^0$-robust, hence $C^1$-robust. In [BS3] this expulsion process is carefully examined and we prove that a small perturbation supported in the filtrating set allow us to recover, in the new class, exactly the same properties which allow us to perform the expulsion, that is, property (ℓ).

In the language introduced in [BS1] this means that this property (ℓ) is a viral property. Namely, once we have a chain recurrence class which satisfies the property (ℓ), then by adding an arbitrarily small perturbation we can produce a new chain recurrence class, and the same is true for the newly created class. This enables us to repeat the creation of distinct chain recurrence classes. As a result, we can prove that nearby $C^1$-generic systems possess nested infinite sequence of filtrating sets which has branches at each depth. Each infinite sequence corresponds to an aperiodic class. As a result, we have an uncountable family of aperiodic classes. This structure appears among all known examples of aperiodic classes for $C^1$-generic diffeomorphisms, even for those built before the notion of viral property being invented. However, up to now, all the known aperiodic classes are adding machines which are minimal and uniquely ergodic. They are forced to have such simple behavior since the filtrating sets at each level has simple combinatorial structure (periodic attracting/repelling balls) which prohibits the complexity of the limit dynamics.

The aim of the series of papers [BS1, BS2, BS3] is to establish techniques to expel chain recurrence classes, keeping the complexity of each depth and furthermore controlling the combinatorics of nested layers. In our construction, each level is no longer a sequence of attracting/repelling balls but a collection of disjoint union of finitely many cylinders which behave in a Markovian way under the iteration of maps. The structure is called partially hyperbolic filtrating Markov partitions. Their behavior can be modeled by means of subshifts of finite type. The techniques which we develop enables us to continue the expulsion process holding some control over combinatorics of each level. By choosing convenient sequence of combinatorics, we can prove the creation of aperiodic classes having prescribed properties.

As we will see, the dynamics inside aperiodic classes of $C^1$-generic diffeomorphism can be very rich. Our aim is not to produce an exhaustive catalog of the possible dynamics but just to illustrate the great diversity which co-exist in a single
generic diffeomorphism. We know a few more possible properties that the technology presented here may produce, but we would like to leave them as problems to be discussed in the future.

Our main result is Theorem 1. Let us restate it clarifying the prerequisite of the open set $O$. Let $f$ be a diffeomorphism on a closed 3-manifold. In [BS] we introduced the notion of partially hyperbolic filtrating Markov partitions, which is a package of information involving the partial hyperbolicity and the recurrence of the points. It is a disjoint union of finitely many $C^1$-cylinders in $M$ such that $f$ maps them Markovian ways. At the same time, the union of the cylinders is a filtrating set. Thus, the information of the chain recurrence is localized there. For more precise information, see Section 2.

Let $p$ be a hyperbolic periodic saddle point of $f$. In [BS2], we discuss a property called property $(\ell)$ about a chain recurrence class contained in a partially hyperbolic filtrating Markov partition, say $R$, which is related to the partial hyperbolicity around the class and the topological conditions about the dynamics in the neighborhood. For the precise definition of the property $(\ell)$, see Section 2.2. Property $(\ell)$ is a $C^1$-robust property and in [BS3] we prove that the property $(\ell)$ is a viral property: Arbitrarily small $C^1$-perturbation $g$ of $f$ provides new filtrating Markov partition $R_1 \subset R$ disjoint from the class of $p_g$ (continuation of $p$ for $g$), and a periodic point $q \in R_1$ such that $g, R_1$ and $q$ satisfies $(\ell)$. The aperiodic classes we build are the intersection of a nested sequences of such filtrating Markov partitions. However, in order to get a control of the dynamical behavior inside the aperiodic classes, we need to prescribe some topological relation about how the Markov partition $R_1$ are nested inside $R$. By deliberately investigating this relation, we can check the creation of the desired classes.

In summary, we can restate Theorem 1 as follows. Below, we say that a filtrating Markov partition is transitive if given any two rectangles $U$ and $V$ we can find a sequence of rectangles $(W_i)_{i=0,\ldots,n}$ such that $W_0 = U$, $W_n = V$ and $f(W_i) \cap W_{i+1} \neq \emptyset$ holds for $i = 0,\ldots,n-1$.

**Theorem 2 (Restatement of Theorem 1).** Let $O$ be a $C^1$-open set of diffeomorphisms on a closed 3-manifold $M$ admitting a transitive, partially hyperbolic filtrating Markov partition $U$ containing a hyperbolic periodic point $p$. Suppose that for every $g \in O$, $U$ is a partially hyperbolic filtrating Markov partition and we can define a continuation $p_g$ of $p$ contained in $U$. If the chain recurrence class $[p_g]$ satisfies the property $(\ell)$ for every $g$, then there is a $C^1$-residual subset $\mathcal{G} \subset O$ such that every $f \in \mathcal{G}$ has uncountably many chain recurrence classes in (1–4) of Theorem 1.

### 1.3. Questions.

In this subsection, we discuss a few questions related to Theorem 1.

1.3.1. Which subshifts can be aperiodic classes? In Theorem 2 all the properties announced are realized as aperiodic chain recurrent classes which are projective limits of subshifts of finite type. Our method consists of expelling hyperbolic subsets repeatedly. At each step, we expel a part of hyperbolic subset in the previous level. By adjusting the combinatorics of each level, we obtain the desired condition. While we have some control over the choice of the hyperbolic subset we will expel, we do not have full freedom, and we do know if all aperiodic chain recurrent compact subsets of a subshift of finite type can be realized as an aperiodic chain recurrence class of a $C^1$-generic diffeomorphism. For instance, the full shift of finitely many
symbols contains minimal invariant sets with positive topological entropy (see for instance [Gr]), and it is not difficult to realize it as a chain recurrence class of a (non-generic) $C\infty$-diffeomorphism. We suspect that our method cannot produce such aperiodic classes for $C^1$-generic diffeomorphisms, since at each step we need to abandon certain amount of complexity which the previous level has. Thus, the following question is interesting to investigate.

**Question 1.1.** For $C^1$-generic diffeomorphisms, what are the topological entropies of aperiodic classes? Are they always equal to zero?

Let us give one more possible direction of future research. It would be interesting to ask what kind of minimal Cantor sets can be realized as aperiodic classes of $C^1$-generic diffeomorphisms. Let us propose a concrete question. Theorem 1 produces minimal, expansive aperiodic classes. We do not know if our example are homeomorphic to some known examples. A typical example of such dynamical systems is obtained by considering Denjoy’s minimal sets (see [By] for instance for the investigation of Denjoy’s minimal sets as subshifts). Then the following would be interesting to consider:

**Question 1.2.** Can one construct aperiodic classes in (1) of Theorem 1 in such a way that they are conjugate to Denjoy’s minimal sets?

See also [GM], in which some invariants of minimal Cantor sets are proposed.

1.3.2. **Adding machines?** Theorem 1 builds aperiodic classes for locally $C^1$-generic diffeomorphisms with several different dynamics. Note that none of these classes are adding machines, because they violate either minimality, non-expansivity or unique ergodicity. However, we do not know if our example is indeed free from an adding machine or not. Let us discuss the possibility of the existence of adding machine in our setting. First, if it exists in the setting of partially hyperbolic filtrating Markov partitions, then it must be contained in finitely many periodic contracting center-stable discs: Any point in an adding machine admit a neighborhood whose orbits remain close under iterations. However, points in different center stable discs have their positive iterates which separate one for the other by a uniform distance. Thus, the problem is reduced to the $C^1$-locally generic existence of adding machines in dimension two, which is not known until now. Note that this problem is tightly related to Smale’s conjecture, which asserts that Axiom A diffeomorphisms may be $C^1$-dense on surfaces. A positive answer to the conjecture implies the non-existence of locally generic production of aperiodic classes. In other words, to have locally generic existence of adding machines we need to have negative answer to Smale’s conjecture.

**Question 1.3.** Do $C^1$-generic diffeomorphisms satisfying the assumption of Theorem 1 exhibit aperiodic classes which are conjugated to adding machines?

Since the solution of Smale’s conjecture seems to be out of the reach of our current state of art, it may be difficult to have some progress in the $C^1$-topology. Meanwhile, one may ask a similar question in a higher regularity setting. One may well expect that the diffeomorphisms in the assumption of Theorem 1 display homoclinic tangencies in restriction to some periodic center stable discs. Thus $C^2$-generic diffeomorphisms in this setting would present adding machine in periodic discs (see for instance [BDV page 33]). Thus, under an extra assumption, for
instance assuming the existence of a periodic orbit which is volume expansive in the center stable direction, it is very likely that one can prove the $C^r$-locally generic existence of adding machines. So far, we do not have any conjecture about the general case, and the investigation of such dynamical systems looks very interesting. Even though there are lacks of perturbation techniques which are available in the $C^1$-topology, the following problem would be interesting to pursue:

**Question 1.4.** Let $r \geq 2$. Do $C^r$-generic diffeomorphisms satisfying the assumption of Theorem 2 exhibit aperiodic classes which are conjugated to adding machines?

Clearly, the following question is also interesting.

**Question 1.5.** Let $r \geq 2$. Is Theorem 2 true in the $C^r$-setting?

1.3.3. The topology of aperiodic classes. Our construction leads to aperiodic classes which are totally disconnected and contain a Cantor set. The aperiodic classes presented here are intersection of filtrating sets whose connected components have diameters tending to 0. Recall that the set of periodic orbits are dense in the chain recurrent set of any $C^1$-generic diffeomorphism (see [BC]). Thus, an aperiodic class has empty interior. It is the unique topological property we know about these aperiodic classes and it would be interesting what kind of topology aperiodic classes can have. For instance:

**Question 1.6.** Are there locally $C^1$-generic diffeomorphisms having aperiodic classes containing a non-degenerate continuum (i.e., a continuum which is not a point)?

Note that Mañé showed that a compact metric space which admits minimal, expansive homeomorphisms is homeomorphic to a totally disconnected set [Ma3], see also [Ar]. Thus this question makes sense only for the case (2-4) in Theorem 2.

1.4. Organization of the paper. Finally, let us see the organization of this paper. In Section 2 we give some review about fundamental notations and the results of papers [BS1, BS2, BS3]. We keep the review not to be comprehensive in order that the readers can understand the whole structure of the proof of Theorem 1 without too much meddled with the technical parts of previous results, which are not mandatory for the understanding of the proof of this paper. In Section 3 we give several topological criterion for the confirmation of properties claimed in Theorem 1. The results presented here are purely topological. Thus, we present our results assuming that the ambient space is just a compact metric space. In Section 4 we provide the method to construct a nested sequence of filtrating sets satisfying the conditions given in Section 3 by adding an arbitrarily $C^1$-small perturbations to the systems in the assumption of Theorem 2.

**Acknowledgment.** This work is supported by the JSPS KAKENHI Grant Numbers 21K03320. KS is grateful for the hospitality of Institut de Mathématiques de Bourgogne of Université de Bourgogne during his visit.

2. Preliminaries

In this subsection, we give some fundamental notions of dynamical systems and reviews of results of [BS1, BS2, BS3] which are used in this paper.
2.1. Filtrating sets and chain recurrence. Let $f$ be a homeomorphism of a compact metric space $(X,d)$. A compact set $A \subset X$ is called an attracting set if

$$f(A) \subset \hat{A},$$

where $\hat{A}$ is the interior of $A$. A repelling set is an attracting set for $f^{-1}$.

**Remark 1.**

1. If $A$ is an attracting set for $f$, then any compact subset $B \subset A$ containing $f(A)$ in its interior (i.e., $f(A) \subset B$) is an attracting set for $f$ and the maximal invariant sets in $A$ and $B$ coincide.

2. If $A$ is an attracting set for $f$, then $f(A)$ is also an attracting set.

3. If $A$ is an attracting set for $f$, then the complement $X \setminus \hat{A}$ is a repelling set.

**Definition 2.1.** A filtrating set $U$ is an intersection of an attracting set $A$ and a repelling set $R$.

Let $f,g \in \text{Homeo}(X)$, where $\text{Homeo}(X)$ denotes the group of homeomorphisms of $X$. By $\text{supp}(g,f)$ we denote the closure of the set $\{x \in M \mid f(x) \neq g(x)\}$ and call it the support of $g$ with respect to $f$.

**Lemma 2.1.** Let $f \in \text{Homeo}(X)$, $O \subset X$ be a compact subset and $g \in \text{Homeo}(X)$ such that $\text{supp}(g,f) \subset O$. Then, for $O$, being an attracting set for $f$ or for $g$ are equivalent. The same is true for a repelling set and a filtrating set.

**Proof.** By definition, $f(X \setminus O) = g(X \setminus O)$ and therefore $f(O) = g(O)$. Thus, $g(O) \subset \hat{O}$ if and only if $f(O) \subset \hat{O}$, in other words, $O$ is an attracting set for $g$ if and only if it is so for $f$.

For the assertion about the repelling set, note that in general, $\text{supp}(g,f) \subset O$ does not imply $\text{supp}(g^{-1},f^{-1}) \subset O$. Instead, we know $\text{supp}(g^{-1},f^{-1}) \subset f(O) = g(O)$. Thus we know that $f(O) = g(O)$ is an attracting set for $f^{-1}$ if and only if it is so for $g^{-1}$. Then, we can deduce the conclusion by Remark 1 (2).

Assume now that $O = A \cap R$ where $A$ and $R$ are an attracting and a repelling set for $f$. As $g$ coincides with $f$ outside $A$ and outside $R$, the previous arguments imply that $A$ is still both attracting and repelling for $g$. Thus $O$ is a filtrating set for $g$.

One of the main properties of a filtrating set $U$ is that, if $x \in U$ and $f(x) \notin U$ then for any $n > 0$ one has $f^n(x) \notin U$ and the similar result holds for $f^{-1}$. In other words, for any $x \in X$ the set of $n \in \mathbb{Z}$ such that $f^n(x) \in U$ is an interval.

This property remains true for pseudo orbits. Recall that a (finite or infinite) sequence $(x_n) \subset X$ is called an $\varepsilon$-pseudo orbit if for every $i$ we have $d(f(x_i), x_{i+1}) < \varepsilon$ holds, whenever it makes sense. For a filtrating set $U$, there is $\varepsilon > 0$ such that if $x_i, i \in \mathbb{Z}$ is an $\varepsilon$-pseudo orbit satisfying $x_0 \in U$ and $x_1 \notin U$, then $x_i \notin U$ for $i > 0$. A similar result holds for $f^{-1}$.

Recall that a point $x \in X$ is called chain recurrent if for any $\varepsilon > 0$ we can find an $\varepsilon$-pseudo orbit $(x_i)_{i=0,\ldots,n}$ ($n > 0$) such that $x_0 = x_n = x$. For a chain recurrent point $x$, the chain recurrence class $[x]$ is the set of points $y \in X$ for which we can find $\varepsilon$-pseudo orbits starting from $x$ and ending at $y$, and vice versa for any $\varepsilon > 0$. One can prove that $[x]$ is a compact $f$-invariant set of $X$.

An important consequence is that $U$ is saturated for the chain recurrence classes, that is, chain recurrence class is either disjoint or contained in $U$, if $U$ is a filtrating set.
2.2. Partially hyperbolic filtrating Markov partitions and the property (ℓ). In this subsection, \( f \) denotes a \( C^1 \)-diffeomorphism on a three dimensional closed manifold \( M \). In [BS²], we introduced the notion of partially hyperbolic filtrating Markov partitions, which we just call filtrating Markov partitions in the sequel for simplicity. It is a set of information about a filtrating set having some partial hyperbolicity adapted to its shape. It enables us to conclude several properties about the isolation of chain recurrence classes.

Let us review the concept of filtrating Markov partitions. Since in this paper we do not need the precise definition of them, we only describe some rough ideas of what they are. Those who are interested in precise definition of them, see [BS², Definition 2.5].

By a cylinder we mean a compact subset \( C \) of a three dimensional manifold which is \( C^1 \)-diffeomorphic to a cylinder \( \mathbb{D}^2 \times I \), where \( \mathbb{D}^2 \subset \mathbb{R}^2 \) is a round disk and \( I \subset \mathbb{R} \) is an interval. For a cylinder, by the lid boundary we mean the subset of the boundary which corresponds to \( \mathbb{D}^2 \times (\partial I) \) and the side boundary is to \( (\partial \mathbb{D}^2) \times I \).

A (partially hyperbolic) filtrating Markov partition (of saddle type) is a filtrating set \( U = A \cap R \) of a diffeomorphism \( f \) having additional properties such as:

- \( U \) is a disjoint union of finitely many cylinders: \( U = \bigcup C_i \) where \( C_i \) is a cylinder. Each \( C_i \) is referred as a rectangle.
- For each rectangle, the lid boundary is contained in \( \partial R \) and the side boundary is in \( \partial A \).
- For each rectangle \( C_i \), the set \( f(U) \cap C_i \) consists of finitely many cylinders which properly crosses \( C_i \) in the axial direction.
- There is a partially hyperbolic structure of the form \( E^u \oplus E^{cs} \) in the neighborhood of \( U \) satisfying the following:
  - \( E^u \) is one dimensional and uniformly expanding. It is almost parallel to the axial direction of rectangles.
  - \( E^{cs} \) is two dimensional and \( E^u \oplus E^{cs} \) forms a dominated splitting. It is almost parallel to the lid boundary of rectangles.

For filtrating Markov partitions, we can deduce the following:

- Being a Markov partition is a \( C^1 \)-robust property (see [BS², Lemma 2.9, (1)]).
- For each rectangle \( C_i \), the set \( f^{-1}(U) \cap C_i \) consists of finitely many cylinders which properly crosses \( C_i \) in the horizontal direction (see [BS², Proposition 2.10]).
- For each \( m, n \geq 0 \), the set \( f^{-m}(U) \cap f^n(U) \) is a filtrating Markov partition, too. We call it the \((m, n)\)-refinement of \( U \) and denote it by \( U_{(m, n)} \) (see [BS², Corollary 2.13]).
- \( \cap_{m \geq 0} f^m(U) \) is a disjoint union of \( C^1 \)-lamination of curves which coincides with the unstable manifold of the locally maximal invariant set of \( U \) (see [BS², Lemma 2.15]).
- \( \cap_{m \geq 0} f^{-m}(U) \) is a continuous family of \( C^1 \)-discs such that each disc cuts the rectangle it belongs to (see [BS², Lemma 2.15]).

A useful property of a filtrating Markov partition is that we can determine the isolation of chain recurrence classes by local information. Let us explain this. For a Markov partition \( U \), the backward invariant set \( \cap_{k \geq 0} f^{-k}(U) \) is a continuous family of two dimensional center stable manifolds. Suppose that we have a hyperbolic periodic point \( p \) in \( U \) of stable index two. We denote the connected component
of $\cap_{k \geq 0} f^{-k}(\mathcal{U})$ containing $p$ by $W_{\text{loc}}^{+}(p)$. As explained above one can prove that $W_{\text{loc}}^{+}(p)$ is $C^1$-diffeomorphic to a two dimensional disc which cuts the rectangle containing $p$ horizontally. We say that $p$ has a large stable manifold if $W^s(p)$ contains $W_{\text{loc}}^{+}(p)$. For periodic points having a large stable manifold, we can determine the information of the size of the chain recurrence class of $p$ just by looking its fundamental domain, since the image of the fundamental domain under $f^{-1}$ goes out from the filtrating set and never comes back (see [BS2 Proposition 2.23]).

This information enabled us to conclude the abundance of the isolation of saddles (see [BS2 Corollary 1.2]), together with the notion of $\varepsilon$-flexible points. We will not review the precise definition of it (see [BS2 Section 3.1] for the definition). It is a hyperbolic periodic point of stable index two whose derivative cocycle in the center stable direction admits very peculiar deformation of size $\varepsilon$ in the $C^1$-distance: By a continuous deformation of the cocycle of size $\varepsilon$ it can be deformed into a stable index one periodic point, and it also can be deformed into a stable index two periodic point having non-real eigenvalues in the center stable direction.

By combining these two kinds of deformations, we proved in [BS1] that, for an $\varepsilon$-flexible point, by performing a perturbation whose $C^1$-size is less than $\varepsilon$, we can deform the point to be a stable index one periodic point having a neutral eigenvalue such that in the fundamental domain of the center stable manifold the position of the strong stable manifold is given by any prescribed $C^1$-curve subject to a topological limitation (see [BS1 Theorem 1.1]).

Thus, if we have a filtrating Markov partition which contains $\varepsilon$-flexible points with large stable manifolds, then by performing a $C^1$-$\varepsilon$-small perturbation we can make their chain recurrence classes to be the periodic orbit themselves. In [BS2], we have shown that the existence such periodic points for an arbitrarily small $\varepsilon > 0$ in filtrating Markov partitions which satisfy a condition called property ($\ell$) (see [BS2 Proposition 5.1], [BS3 Section 3.8], [BS2 Proposition 5.1]). As a result, we proved that such a diffeomorphism is wild, that is, $C^1$-generically it has infinitely many chain recurrence classes.

Let us review the definition of the property $(\ell)$ for a chain recurrence class, which guarantees the existence of $\varepsilon$-flexible points with large stable manifolds for small $\varepsilon > 0$. In this article by $\text{Diff}^1(M)$ we denote the set of $C^1$-diffeomorphisms of $M$ with the $C^1$-topology.

**Definition 2.2** (Definition 2, [BS3]). Let $f \in \text{Diff}^1(M)$ and $p$ be a hyperbolic periodic point of stable index two. We say that $[p]$, the chain recurrence class of $p$, satisfies property ($\ell$) if the following holds:

1. $[p]$ is contained in a filtrating Markov partition $\mathcal{U}$ such that $p$ has a large stable manifold in $\mathcal{U}$.
2. $p$ is homoclinically related to a hyperbolic periodic point $p_1$ of stable index two such that $Df^{\pi_1}_{p_1}$ has non-real eigenvalues (where $\pi_1$ is the period of $p_1$).
3. There are two hyperbolic sets $K$ and $L$ such that
   - $K$ has stable index two and contains $p$.
   - $L$ has stable index one.
   - $K$ and $L$ form a robust heterodimensional cycle (see Proposition 5.1 of [BS1]).

In the following, we refer this property as $p$ is in a robust heterodimensional cycle.
Note that, in the above definition, the fact that $\mathcal{U}$ is filtrating forces $p_1$ and the heteroclinic points between $p$ and $p_1$ to be contained in the interior of $\mathcal{U}$. A similar property holds for the robust heterodimensional cycle, too.

In this article, we also consider a local version of property $(\ell)$, see also [BS3, Definition 4.1]. Let $p$ be a hyperbolic periodic point of stable index two contained in a filtrating Markov partition $\mathcal{U}$. By a sub Markov partition of $\mathcal{U}$ or a non-filtrating Markov partition we mean the union of the collection of rectangles of $\mathcal{U}$. Note that in general a sub Markov partition is not a filtrating set. Let $W$ be a sub Markov partition of $\mathcal{U}$ which contains the orbit of $p$. By a relative homoclinic class $H(p, W)$ we mean the closure of the transverse homoclinic intersections of $W^u(p)$ and $W^s(p)$ whose orbit is in $W$. Notice that, due to the filtrating property of $\mathcal{U}$, $H(p, W)$ is a compact $f$-invariant set which is contained in the interior of $W$.

**Definition 2.3.** Let $p, f, W$ be as above. We say that $[p]$ satisfies property $(\ell_W)$ if the following holds:

- $p$ has a large stable manifold in $\mathcal{U}$.
- There is a hyperbolic saddle $p_1$ of stable index two such that $p$ and $p_1$ are homoclinically related in $W$.
- $p$ has a robust heterodimensional cycle in $W$, that is, there are hyperbolic sets $K, L$ of different indices contained in $W$ such that $K$ contains $p$ and $K, L$ form a robust heterodimensional cycle in $W$.

One can see that having property $(\ell_W)$ is a $C^1$-robust property. If $[p]$ has property $(\ell_W)$ then it implies that it has property $(\ell)$ for the Markov partition which contains $W$. In this sense, $(\ell_W)$ is a condition which is stronger than $(\ell)$. The advantage of the condition $(\ell_W)$ is that it guarantees the existence of the flexible point whose orbits are localized in $W$, as we will see in the next subsection.

### 2.3. Tools from [BS3]

We cite several results from [BS3] which are used to construct aperiodic classes. To give the results, we introduce a few definitions.

Let $\mathcal{U}$ be a filtrating Markov partition of $f$ and $g \in \text{Diff}^1(M)$ is so $C^1$-close to $f$ that $\mathcal{U}$ is still a filtrating Markov partition for $g$. Then by $\mathcal{U}_{(m,n)}$ we denote the $(m,n)$-refinement of $\mathcal{U}$ with respect to $g$. When we do not need to indicate with which map we take the refinement, we also use the notation $\mathcal{U}_{(m,n)}$.

Let $\mathcal{U}$ be a filtrating Markov partition and $S \subset \mathcal{U}$. By $\mathcal{U}(S)$ we denote the sub Markov partition of $\mathcal{U}$ which consists of the rectangles having non-empty intersection with $S$.

A *circuit* of points of $f$ is a set of finitely many hyperbolic periodic orbits $\{O(q_i)\}$ of stable-index 2 and transverse homo/heteroclinic orbits $\{O(Q_j)\}$ among them, see [BS3, Section 1.3]. Given a circuit, we can obtain a directed graph by setting vertices to be the periodic orbits and edges to be the homo/heteroclinic orbits. We say that a circuit is transitive if the directed graph is transitive. In this article, we only treat transitive circuit and we always assume so without mentioning it. Note that $S$ is a uniformly hyperbolic set.

Suppose that we have circuits of points $K$ of $f$ and $L$ of $g$. We say that $K$ and $L$ are *similar* if the following holds:

- There is a continuous bijection $h: K \to L$ which conjugates $f$ and $g$: $f \circ h = h \circ g$ holds on $K$.

We say that they are $\delta$-close if the $C^0$-norm of $h$ can be chosen smaller than $\delta$, more precisely, $d(x, h(x)) < \delta$ holds for all $x \in K$. 

2.3.1. **Expulsion result.** The first two results will be used to obtain a filtrating set by a small perturbation of the diffeomorphism in such a way that it contains a prescribed circuit. The expulsion process consists of two steps. For expelling a new filtrating set, we need to have that the rectangles containing the circuit be affine, which guarantees the regularity of the shape of the rectangles. Since the precise definition of it is not used in this paper, we do not state it here. The reader interested in can find it in Section 2 of [BS].

The second result claims that if a sub Markov partition is affine and all the periodic orbits involved are ε-flexible points with large stable manifolds, then by adding 2ε-perturbation and taking forward refinement we have the expulsion of a chain recurrence class, keeping the combinatorial information of the Markov partitions. Recall that for a filtrating Markov partition we have defined the notion of α-robustness (see Section 2.7 of [BS]). We do not review the precise definition of it here. Roughly speaking, α-robustness implies that the filtrating Markov partition persists against perturbations of C¹-size α. One property which we use about the robustness is that taking refinements does not decrease the robustness (see Section 2.7 of [BS]).

Let us give the first result (see Theorem 1.4 of [BS]):

**Theorem 3.** Let \( f \in \text{Diff}^1(M) \) having an α-robust filtrating Markov partition \( U \) containing a circuit (of points) \( S \). Assume that every periodic orbit of \( S \) has a large stable manifold in \( U \). Then for any neighborhood \( U \) of \( S \) and any sufficiently small C¹-neighborhood \( O \) of \( f \) there is \( g \in O \) such that the following holds:

- The support \( \text{supp}(g, f) \) is contained in \( U \).
- For \( g \) we have a continuation of \( S \) which we denote \( S_g \). Then, there exists \( m_0, n_0 > 0 \) such that for every \( m \geq m_0, n \geq n_0 \), the Markov partition \( U_{(m,n)}(S_g) \) is affine.
- The conjugate periodic points in \( S \) and \( S_g \) have the same orbits and the same derivatives along them.
- \( U_{(m,n,g)} \) is α-robust, too.

To give the second result, we prepare a definition.

**Definition 2.4.** Let \( U \) be a (possibly non-filtrating) Markov partition and \( V \) be another (possibly non-filtrating) Markov partition such that

- \( V \subset U \),
- each rectangle of \( U \) contains one and only one rectangle of \( V \).

We say that \( V \) is matching to \( U \) if these properties hold.

The information of being matching implies that the two Markov partitions have similar combinatorial information. This enables us to determine the properties of new Markov partitions we produce.

Then we have the following (see Theorem 1.5 of [BS]):

**Theorem 4.** Let \( f \in \text{Diff}^1(M) \) having an α-robust filtrating Markov partition \( U \) containing a circuit \( S \) whose periodic orbits are all ε-flexible, have large stable manifolds in \( U \) and \( U(S) \) is affine. Suppose \( \alpha > 2\epsilon \). Then there exists \( n_0 > 0 \) such that for every \( n \geq n_0 \) there is a C¹-diffeomorphism \( h_n \) which is 2\epsilon-C¹-close to \( f \) and satisfying the following:

- \( \text{supp}(h_n, f) \) is contained in the interior of \( U(S) \).
• $h_n$ has a transitive filtrating Markov partition $V$ containing a circuit $S_n$ which is similar to $S$.

• For $S_n$ we have the following:
  - We have $U(S_n) = U(S)$ and we can require that the points which are conjugate belong to the same rectangle and the conjugate periodic orbits of $S_n$ has the same orbit as in $S$.
  - Every periodic orbit of $S_n$ has a large stable manifold in $V$ and it is $\varepsilon$-flexible.

• For $V$ we have the following:
  - Each rectangle of $V$ is a vertical sub rectangle of a rectangle of $U_{(0,n;h_n)}(S_n)$.
  - $V$ is matching to $U_{(0,n,h_n)}(S_n) = U_{(0,n,h_n)}(S)$.
  - $V$ is $(\alpha - 2\varepsilon)$-robust, too.

2.3.2. Recovering condition $(\ell)$ from flexibility. Our second tool will allow us to get a circuit satisfying convenient dynamical conditions (see Section 4.1.2 of [BS3]).

**Theorem 5.** Let $f \in \text{Diff}^1(M)$, $p$ be a periodic point of $f$, $U = \cup_i U_i$ be a $\alpha$-robust filtrating Markov partition and $W$ be a sub Markov partition of $U$. Assume that $p$ is $\varepsilon$-flexible, has a large stable manifold, the orbit of $p$ is contained in $W$, the relative homoclinic class $H(p,W)$ is non-trivial, and $\alpha > 4\varepsilon$ holds.

Then, there is a diffeomorphism $g$ which is $4\varepsilon$-$C^1$-close to $f$ such that $p$ satisfies the condition $(\ell_W)$. Furthermore, we may assume the following:

• $\text{supp}(g, f)$ is contained in the interior of $W$.

• $U$ is still a filtrating Markov partition for $g$.

• Suppose that $p$ is contained in a circuit $K$. Then for any $\delta > 0$ by choosing appropriate $g$ we may assume there is a circuit which is $\delta$-close to $K$ containing $p_g$.

2.3.3. Abundance of flexible points. The last result shows that given a periodic orbit satisfying condition $(\ell_W)$ for some sub Markov partition $W$, up to an arbitrarily small local perturbation one can obtain a lot of flexible points with large periods, see Section 4.1.1 of [BS3].

**Theorem 6.** Let $f \in \text{Diff}^1(M)$ and $U$ be a filtrating Markov partition. Let $W$ be a sub Markov partition of $U$ and $p$ a hyperbolic periodic point whose orbit is contained in $W$. Assume that $p$ satisfies the condition $(\ell_W)$. Then given $\varepsilon > 0$, $N > 0$ and a $C^1$-neighborhood $O$ of $f$, there is a diffeomorphism $g \in O$ such that the following holds:

• $\text{supp}(g, f)$ is contained in the interior of $W$.

• There is a periodic point $x \in H(p_g,W;g)$ homoclinically related to $p_g$ in $W$ which is $\varepsilon$-flexible, having a large stable manifold in $U$, $\varepsilon$-dense in $H(p_g,W;g)$ and whose period is larger than $N$.

3. Topological principles for the genericity of certain aperiodic classes

Throughout this section $(X,d)$ denotes a compact metric space, $G \subset \text{Homeo}(X)$ is a subgroup of homeomorphisms endowed with a topology finer than or equal to the $C^0$-topology such that $G$ is a Baire space with it. By $f$ we denote a homeomorphism in $G$. In the next section (Section 4), $X$ will be a compact manifold $M$ and $G$ will be the space of $C^1$-diffeomorphism on $M$ endowed with the $C^1$-topology.
In this section we discuss several properties of invariant sets defined as a limit of a nested sequence of filtrating sets. Our arguments in this section are purely topological and we prefer to give them in a topological setting for avoiding the implicit use of properties resulting from the differentiability assumptions.

### 3.1. Stability for local perturbation
Suppose that $f$ has a compact set $K \subset X$ on which $f$ satisfies a certain property. We are interested in if the property holds for $g$ which differs only inside $K$. In order to discuss it we introduce a few definitions.

**Definition 3.1.** Let $(X, d)$, $f$, $g$ and $K$ be as above. We say that the support of $g$ (with respect to $f$) is strictly contained in $K$ if the following holds:

1. $\text{supp}(g, f) \subset K$.
2. For each connected component $K$ of $K$, we have $K \setminus \text{supp}(g, f) \neq \emptyset$.

A property is called stable under local perturbation if whenever it holds for $f$ and $K$ then it holds for $g$ and $K$ where the support of $g$ is strictly contained in $K$.

By the discussion of Section 2.1 we know that being an attracting set, a repelling set or a filtrating set is a property which are stable under local perturbations. Let us see an important property of homeomorphisms which is preserved by local perturbations.

**Lemma 3.1.** Let $K \subset X$ be a compact set. Let $f, g \in G$ such that $\text{supp}(g, f)$ is strictly contained in $K$. Then for every connected component $L$ of $K$ one has $f(L) = g(L)$.

**Proof.** By assumption we know $f(K) = g(K)$. Thus, the sets of connected components of $f(K)$ and $g(K)$ are the same. Consider a connected component $L$ of $K$. Then $f(L) \cap f(K) \neq \emptyset$. Since $f$ is a homeomorphism, $f(L)$ is one of the connected components of $f(K)$ and the same holds for $g(L)$. Consider a point $x \in L$ for which we have $f(x) = g(x)$. Note that the existence of such $x$ is guaranteed by the assumption. Then $f(L)$ and $g(L)$ are connected components of $f(K) = g(K)$ which contains the common point $f(x) = g(x)$. Thus, they coincide, that is, $f(L) = g(L)$. □

**Remark 2.** A natural way of stating “the support is contained in $K$” would be “$f \equiv g$ outside $K$.” However, this condition is not enough to obtain the conclusion in Lemma 3.1 in general. For instance, consider a compact metric space $Y$ which consists of two homeomorphic connected components. Let $K = Y$ and consider $f = \text{id}_Y$ and some map which exchanges two connected components which we denote by $g$. Then for an obvious reason $f \equiv g$ outside $K$ but in the level of the connected components they are different.

Meanwhile, under a mild extra condition we have the same conclusion for these two assumptions. Let us see it.

**Lemma 3.2.** Suppose that $X$ is connected and $K$ is compact, has finitely many connected components and $K, X \setminus K$ are non-empty. If $f \equiv g$ on $X \setminus K$, then the support of $g$ is strictly contained in $K$ (thus the conclusion of Lemma 3.1 holds).

**Proof.** As $f$ and $g$ coincides on $X \setminus K$ we have $g(K) = f(K)$, and even $f \equiv g$ on $\partial_X K$, where $\partial_X K$ denotes the boundary of $K$ in $X$. Thus the conclusion is the direct consequence of the following:

**Claim 1.** For any connected component $L$ of $K$ we have $L \cap \partial_X K \neq \emptyset$. 

Let us prove the Claim. Take a small open neighborhood \( O \) of the connected component \( L \) in \( X \). Since \( K \) is compact and has finitely many connected components, we may assume \( K \cap O = L \). We have \( O \setminus L \neq \emptyset \), otherwise it contradicts the connectedness of \( X \). Thus \( O \) contains a point \( x_O \in X \setminus K \). By shrinking \( O \), we obtain an accumulation of such points in \( K \cap \partial_X K \). \( \square \)

Remark 3. In Section 4, we consider the case where \( X \) is a compact connected manifold and \( K \) has finitely many connected components and is not equal to neither \( \emptyset \) nor \( X \). Thus Lemma 3.2 is always applicable.

In this article, we discuss two kinds of robustness of properties: The robustness with respect to the G-topology, that is, the topology of the space of dynamical systems we are interested in, and the robustness with respect to the location of the support of the perturbation. For the proof of our main theorem, the second kind of robustness is not necessary, but we will discuss it. Let us see why.

The first reason is that it enables us to make some of the proofs simpler. In the construction of aperiodic classes, we produce filtrating sets by small perturbations using the techniques in Section 2.3. In the course of the construction, we use them successively and we need to consider their interference. Then using only G-robustness brings some non-trivial complication of the proof. One advantage of stability under the local perturbation is it makes some arguments about the interference of the perturbations much simpler, see Remark 17.

Another reason is that the stability by the local perturbations enables us to construct a concrete example of dynamical systems which exhibits the chain recurrence class we announced. We prove a perturbation result which produces a filtrating set by a local perturbation. Doing it successively, we obtain a sequence of maps \( f_n \) which converges to a map \( f \) having the desired chain recurrence set. This construction is more concrete than the genericity approach, because for this \( f \) we certainly know the behavior outside the support of the perturbation. See Remark 5.

3.2. A criterion for chain recurrence. In this subsection, we discuss a sufficient condition which guarantees the chain transitivity for an invariant set obtained as the limit of nested filtrating sets.

For \( x \in K \), we denote the connected component of \( K \) containing \( x \) by \( K(x) \).

Definition 3.2. Let \( U \subset X \) be a compact set and \( f \in G \).

- \( U \) is said to be regular if for any pair of connected components \( U_1, U_2 \) of \( U \), \( f(U_1) \cap U_2 \neq \emptyset \) implies \( f(U_1) \cap U_2 \neq \emptyset \).
- We say that a (finite or infinite) sequence \((x_i) \subset U \) is a \( U \)-chain of points or \( U \)-pseudo orbit if for any \( i \) we have \( f(U(x_i)) \cap U(x_{i+1}) \neq \emptyset \).
- Also, we say that a finite or infinite sequence \((U_i)\), where \( U_i \) is a connected component of \( U \), is a \( U \)-chain of connected components if for any \( i \) we have \( f(U_i) \cap U_{i+1} \neq \emptyset \).
- We say that \( U \) is \( U \)-chain transitive if given any two connected components \( V_0, V_1 \) of \( U \), there is a finite \( U \)-chain of connected components \((U_i)\) starting from \( V_0 \) and ending at \( V_1 \).
- The minimum \( U \)-period of \( U \) is the smallest length of a periodic \( U \)-chain of connected components, see Remark 4.

For \( K \subset X \), by \( \text{cdiam}(K) \) we denote the supremum of the diameter of the connected components of \( K \).
Remark 4. For $\mathcal{U}$-chains, we have the following.

1. A $\mathcal{U}$-chain of points $(x_i)$ defines a unique $\mathcal{U}$-chain of connected components $(U_i)$ such that $x_i \in U_i$ (set $U_i = \mathcal{U}(x_i)$).
2. Given a $\mathcal{U}$-chain of connected components $(V_i)$ we can find a $\mathcal{U}$-chain of points $(y_i)$ such that $y_i \in V_i$ by choosing a point from each $V_i$. Due to these correspondences, the definition of the chain transitivity and minimum $\mathcal{U}$-period can be given in terms of $\mathcal{U}$-chain of points.
3. If $\text{cdiam}(\mathcal{U}), \text{cdiam}(f(\mathcal{U})) < \varepsilon$ then $\mathcal{U}$-chain of points $(x_i)$ defines a $2\varepsilon$-pseudo orbit.

Let us discuss basic properties of $\mathcal{U}$-chain transitivity.

Lemma 3.3. Suppose that $f \in G$ has a compact set $\mathcal{U}$ which is regular, has finitely many connected components, and is $\mathcal{U}$-chain transitive for $f$. Then, for $g$ which is $C^0$-close to $f$, $\mathcal{U}$ is $\mathcal{U}$-chain transitive for $g$ as well.

Proof. The regularity condition guarantees that once we have $f(U_1) \cap U_2 \neq \emptyset$ then the $g(U_1) \cap U_2 \neq \emptyset$ holds for $g$ which is sufficiently $C^0$-close to $f$. Thus, if the number of the connected components is finite, all the condition as such holds for nearby homeomorphisms, which implies the $\mathcal{U}$-chain transitivity. \qed

Lemma 3.4. Suppose that $f$ has a compact set $\mathcal{U}$ which is $\mathcal{U}$-chain transitive. If the support $\text{supp}(f, g)$ is strictly contained in $\mathcal{U}$, then $\mathcal{U}$ is $\mathcal{U}$-chain recurrent for $g$, too.

Proof. The hypothesis ensures that the set of $\mathcal{U}$-chains of connected components are the same for $f$ and $g$, see Lemma 5.1. Thus, the $\mathcal{U}$-chain transitivity of $f$ and that of $g$ are equivalent. \qed

Proposition 3.1. Consider a sequence $\{(f_n, \mathcal{U}_n)\}_{n \geq 1}$ where $f_n \in G$ and $\mathcal{U}_n$ is a filtrating set for $f_n$ such that:

- $\mathcal{U}_{n+1} \subset \mathcal{U}_n$,
- $\text{supp}(f_{n+1}, f_n)$ is strictly contained in $\mathcal{U}_n$,
- $\text{cdiam}(\mathcal{U}_n), \text{cdiam}(f_n(\mathcal{U}_n)) \to 0$ as $n \to 0$,
- $\mathcal{U}_n$ is $\mathcal{U}_n$-chain transitive for $f_n$,
- The minimum period of $\mathcal{U}_n$ tends to $\infty$ as $n \to \infty$.

Then the sequence $f_n$ converges to a homeomorphism $f$ in the $C^0$-topology for which $\Lambda = \bigcap_{n \geq 1} \mathcal{U}_n$ is an aperiodic chain recurrence class (especially, $\Lambda$ is $f$-invariant).

Proof. For each $n$, consider the sequence $(f_{n+k})_{k \geq 0}$. Each $f_{n+k}$ is equal to $f_n$ on the open set $X \setminus \mathcal{U}_n$. This implies that the limit $f$ is well defined on $X \setminus \Lambda$. Also, for each connected component $U$ of $\mathcal{U}_n$, the image $f(U)$ is well defined and coincides with $f_n(U)$. Note that this implies $\mathcal{U}_n$ is $\mathcal{U}_n$-chain transitive for $f$, too.

The hypothesis on the diameters ensures that for $x \in \Lambda$, $\bigcap_n f(\mathcal{U}_n(x))$ is a singleton. We denote it by $y$. We extend $f$ over $\Lambda$ by mapping $x \in \Lambda$ to $y$. Then one can check that the map $f: X \to X$ is a homeomorphism. The continuity and the injectivity of $f$ on $X$ are easy to see. The surjectivity of $f$ on $X \setminus \Lambda$ is also easy. Let assume $\Lambda \subset f(\Lambda)$. Take $x \in \Lambda$. By the $\mathcal{U}_n$-chain transivity of $f$, for every $n$ there is a $\mathcal{U}_n$-chain of connected components for $f$ starting from and ending at $\mathcal{U}_n(x)$. For each $n$, we consider the second last connected component of the chain and denote it by $V_n$. Let us choose a sequence $(z_n) \subset X$ such that $z_n \in V_n$ and take
its accumulation point \( z \). Then, the compactness of \( U_n \) and the nested property of \((U_n)\) imply that \( z \in \cap_{n \geq 1} U_n = \Lambda \). Also, the assumption on the diameters ensures that \( f(z) = x \). Accordingly, we have \( \Lambda \subset f(\Lambda) \).

Let us show that \( \Lambda \) is a chain recurrence class for \( f \). First, the condition that the support of \( f_{n+1} \) is strictly contained in \( U_n \) tells us that \( U_n \) is \( U_n \)-chain transitive for \( f \). It implies that every point in \( \Lambda \) is a chain recurrent point (in the usual sense) and every pair of points in \( \Lambda \) is chain equivalent, see Remark 4. As a result, we see that \( \Lambda \) is contained in a chain recurrence class. Furthermore, since \((U_n)\) is a decreasing sequence of filtrating sets, the points outside \( \Lambda \) cannot be chain equivalent to points in \( \Lambda \). Thus, \( \Lambda \) itself forms an entire chain recurrence class.

If \( \Lambda \) contains a periodic point \( p \) of \( f \), then the \( U_n \)-period of the \( U_n \) for \( f \) is bounded by the period of \( p \) since its orbit for \( f \) is a \( U_n \)-chain of points for every \( f_n \). This confirms the assertion about the aperiodicity.

\[ \text{Definition 3.3.} \quad \text{We say that a sequence } \{(f_n, U_n)\}_{n \geq 1} \text{ is a nested sequence for an aperiodic class if it satisfies the hypothesis of Proposition 3.4.} \]

3.3. Minimality of aperiodic classes. In this subsection we discuss the minimality of the chain recurrence classes obtained as a limit of nested filtrating sets. Recall that a compact invariant set \( K \subset X \) is called minimal if every point of \( K \) has a dense orbit in \( K \), that is, for every \( x \in K \) the set \( \mathcal{O}(x) := \{f^i(x)\}_{i \in \mathbb{Z}} \) is dense in \( K \).

\[ \text{Definition 3.4.} \quad \text{Let } f \in G \text{ and } U, V \subset X \text{ be compact filtrating sets of } f \text{ satisfying } V \subset U. \text{ We say that } V \text{ is } U \text{-minimal if given any pair of connected components } U_0 \text{ of } U \text{ and } V_0 \text{ of } V \text{ there is } n > 0 \text{ such that } f^n(U_0) \subset U_0 \text{ holds. We denote by } n(U_0, V_0) > 0 \text{ the smallest positive integer } n \text{ satisfying } f^n(U_0) \subset U_0. \]

We give an important observation.

\[ \text{Lemma 3.5.} \quad \text{If } V \text{ is } U \text{-minimal, then for any connected components } U_0 \text{ of } U \text{ and } V_0 \text{ of } V \text{ we have the following equality:} \]

\[ n(U_0, V_0) = \min\{i > 0 \mid f^i(V_0) \cap U_0 \neq \emptyset\}. \]

\[ \text{Proof.} \quad \text{For } i > 0 \text{ suppose } f^i(V_0) \cap U_0 \neq \emptyset \text{ but } f^i(V_0) \not\subset U_0. \text{ Then there is } x \in V_0 \text{ such that } f^i(x) \notin U. \text{ As } U \text{ is a filtrating set, if a point } y \in U \text{ satisfies } f(y) \notin U \text{ then } f^k(y) \notin U \text{ for all } k \geq 0. \text{ Applying this property to the orbit of } x \text{ we have } f^j(x) \notin U \text{ for any } j > i, \text{ which implies } n(U_0, V_0) \leq i. \]

The following lemma tells us that the \( U \)-minimality is a \( C^0 \)-robust property under a mild assumption.

\[ \text{Lemma 3.6.} \quad \text{Let } f \in G \text{ and } U, V \text{ be filtrating sets of } f. \text{ We assume that both } U \text{ and } V \text{ have finitely many connected components. If } V \text{ is } U \text{-minimal, then there is a neighborhood } O \text{ of } f \text{ in } G \text{ such that for any } g \in O \text{ the compact set } V \text{ is } U \text{-minimal.} \]

\[ \text{Proof.} \quad \text{For any pair of connected components of } U_0 \text{ of } U \text{ and } V_0 \text{ of } V, \text{ we have } f^n(U_0, V_0)(V_0) \subset U_0. \text{ This condition is a } C^0 \text{-robust, hence } G \text{-robust condition. Since the number of connected components of } U \text{ and } V \text{ are finite, having this condition for every pair of } U_0 \text{ and } V_0 \text{ is also } G \text{-robust.} \]
Lemma 3.7. Let $f \in G$ and $U, V$ be compact filtrating sets of $f$ such that $V$ is $U$-minimal. Let $g$ be a homeomorphism whose support is strictly contained in $V$. Then $U, V$ are compact filtrating sets of $g$ and $V$ is $U$-minimal for $g$, too.

Proof. The fact that $U, V$ are filtrating sets for $g$ is given by Lemma 2.1. We just need to prove the $U$-minimality of $V$ for $g$.

Recall that we have defined the number $n(U_0, V_0)$ for any pair of connected components of $U_0$ of $U$ and $V_0$ of $V$, see Definition 3.4. By definition, we have

$$f^{n(U_0, V_0)}(V_0) \subset U_0$$

for every pair of $U_0$ and $V_0$. We prove that the same relation holds for $g$, that is, we have

$$g^{n(U_0, V_0)}(V_0) \subset U_0,$$

which implies the $U$-minimality of $V$.

Assume that this does not hold for some $U_0$ and $V_0$. We take the minimum number of $n(U_0, V_0)$ among such $U_0$ and $V_0$ and denote it by $N$. Namely, consider

$$(1) \quad N = \min \{i > 0 \mid \exists U_0, \exists V_0 \text{ such that } i = n(U_0, V_0) \text{ and } g^i(V_0) \not\subset U_0\}.$$

Note that we have $N > 1$ because $g(V_0) = f(V_0)$.

Consider the set $g(V_0) = f(V_0)$. By assumption, there is $x \in V_0$ such that $g^N(x) \not\in U_0$. We take $y \in V_0$ such that $g^N(x) = f^N(y)$ holds. Then we have the following:

Claim 2. $g(x) \in \hat{V}$.

Proof. If $g(x) \not\in \hat{V}$ then we have $g^2(x) = g(g(x)) = f(g(x)) = f(f(y)) = f^2(y)$, where in the second equality we used the fact that $f \equiv g$ outside $V$. Furthermore, since $V$ is filtrating for both $f$ and $g$, we know that the point $g^2(x) = f^2(y)$ does not belong to $\hat{V}$, either. It enables us to repeat the argument for the point $g^2(x) = f^2(y)$. Thus, arguing inductively, we have $g^k(x) = f^k(y)$ for any $k > 0$.

Now this implies $g^N(x) = f^N(y) \in U_0$ which contradicts the choice of $x$. $\square$

Thus $g(x) = f(y)$ belongs to $\hat{V}$ and therefore to some connected component of $V$, which we denote by $V_1$. Then consider the integer $n(U_0, V_1)$. Since $f^N(y) = f^{N-1}(f(y)) \in U_0$, together with Lemma 3.5 and the definition of $n(U_0, V_1)$, we know that $n(U_0, V_1) = N - 1$. To be precise, $f^{N-1}(f(y)) \in U_0$ and Lemma 3.5 shows $n(U_0, V_1) \le N - 1$, and it cannot be smaller because if so, it contradicts $n(U_0, V_0) = N$.

On the other hand, note that we have $g^{N-1}(f(y)) = g^{N-1}(g(x)) = g^N(x) \not\in U_0$. Thus $g^{N-1}(V_1) \not\in U_0$. However, this contradicts the minimality of $N$, see (1). Thus the proof is completed. $\square$

We prepare a result which enables us to construct a concrete example of an aperiodic class.

Proposition 3.2 (A criterion for minimality). Let $\{(f_n, U_n)\}$ be a nested sequence for an aperiodic class such that $U_{n+1}$ is $U_n$-minimal for $f_{n+1}$ for every $n$. Then the sequence $(f_n)$ converges in the $C^0$-topology to a homeomorphism $f: X \to X$ for which $\Lambda = \bigcap U_n$ is an aperiodic chain recurrence class which is minimal and has $U_n$ as a basis of (filtrating) neighborhoods.
Proof. The fact that $\Lambda$ is an aperiodic class is confirmed in Proposition 3.1. Thus we only need to check the minimality.

As $U_{n+1}$ is $U_n$-minimal for $f$, given $n$ and $x, y \in \Lambda$ there is $k > 0$ such that $f^k(U_{n+1}(x)) \subset U_n(y)$. As the diameter of $U_n(y)$ tends to zero, this shows that the orbit of $x$ passes arbitrarily close to $y$. As $x, y$ are any points in $\Lambda$ this shows that the minimality of $\Lambda$. $\square$

Remark 5. For the proof of Theorem 4, we only need the case $f_n = f$. For instance, in the proof of Proposition 3.4 we use Proposition 3.2 letting $f_n = f$. Meanwhile, this version has an advantage. It enables us to construct examples of homeomorphisms satisfying the desired condition in a concrete way. See Proposition 4.

3.4. A criterion for expansiveness. In this subsection, we describe a sufficient condition for the expansiveness of the limit invariant set of a nested sequence of filtrating set. Recall that a compact set $K \subset X$ is expansive if there is $\delta > 0$ such that for any $x, y \in X$ ($x \neq y$) there is $n = n(x, y)$ satisfying $d(f^n(x), f^n(y)) > \delta$.

Definition 3.5. Let $f \in G$ having a compact filtrating set $U$ and $K \subset U$ be a compact set. We say that $K$ is $U$-expansive for chains (with respect to $f$) if given any two bi-infinite $K$-chains $(x_i)$ and $(y_i)$ ($i \in \mathbb{Z}$) of points we have the following dichotomy:

- Either for any $i \in \mathbb{Z}$ the connected components $K(x_i)$ and $K(y_i)$ are equal,
- or, there is $i \in \mathbb{Z}$ such that $U(x_i) \neq U(y_i)$.

Lemma 3.8. Let $f$, $U$ and $K$ be as above. If the support of $g$ is strictly contained in $K$, then $K$ is $U$-expansive for chains with respect to $g$, too.

Proof. Suppose that we have bi-infinite $K$-chains of points $(x_i)$ and $(y_i)$ with respect to $g$. We need to show if there is $i_0$ such that $K(x_{i_0}) \neq K(y_{i_0})$ then there is $i_1$ such that $U(x_{i_1}) \neq U(y_{i_1})$.

Lemma 3.1 shows that $(K(x_i))$, $(K(y_i))$ are $K$-chains of connected components with respect to $f$, too. Since for $f$ we have $U$-expansiveness of $K$, we know there is $i_1$ such that $U(x_{i_1}) \neq U(y_{i_1})$.

For $i_1$ above we have $U(x_{i_1}) \neq U(y_{i_1})$, which shows the $U$-expansiveness of $K$ with respect to $g$. $\square$

In the same manner, we can prove the following:

Lemma 3.9. Let $f$, $U$ and $K$ be as above. Assume that the number of connected components of $K$ is finite. Suppose that $K$ is $U$-expansive for chains with respect to $f$. Then, being $U$-expansive is a $C^0$-robust property. More precisely, there exists $\varepsilon > 0$ such that if $g \in \text{Homeo}(X)$ satisfies $d(f(x), g(x)) < \varepsilon$ for every $x \in X$ then $K$ is $U$-expansive for $K$-chains with respect to $g$, too.

Proof. In order to check the $U$-expansiveness for chains for $K$ with respect to $g$, the confirmation of the following information is enough:

If $g(K_i) \cap K_j \neq \emptyset$ then $f(K_i) \cap K_j \neq \emptyset$, equivalently, if $f(K_i) \cap K_j = \emptyset$ then $g(K_i) \cap K_j = \emptyset$. 


Then, repeating the argument in the proof of Lemma 3.8 we obtain the condition. Note that, by the compactness of \( K_i \) and \( K_j \), the condition above for fixed \( K_i \) and \( K_j \) is a \( C^0 \)-robust property. Then the finitude of the number of the connected components enables us to obtain the conclusion. \( \square \)

Now we can state a criterion to have expansive aperiodic classes.

**Proposition 3.3.** Let \( \{ (f_n, \mathcal{U}_n) \}_{n \geq 1} \) be a nested sequence for an aperiodic class. Assume that

- \( \mathcal{U}_{n+1} \) is \( \mathcal{U}_n \)-expansive for chains with respect to \( f_{n+1} \) for \( n \geq 1 \).
- \( \mathcal{U}_1 \) has finitely many connected components.

Then the sequence \( (f_n) \) converges to a homeomorphism \( f \) such that \( \Lambda = \bigcap_{n \geq 1} \mathcal{U}_n \) is an expansive chain recurrence class.

**Proof.** We only need to prove the expansiveness of \( \Lambda \). Take two different points \( x, y \in \Lambda \) and consider the orbits \( \{ f^i(x) \} \) and \( \{ f^i(y) \}, i \in \mathbb{Z} \). There is \( n \) such that the diameter of \( \mathcal{U}_n(x) \) is smaller than \( d(x, y) \). Thus \( \mathcal{U}_n(x) \neq \mathcal{U}_n(y) \), and the \( f \)-orbit of \( x \) and \( y \) define distinct \( \mathcal{U}_n \)-chains of \( f_n \). As \( \mathcal{U}_n \) is \( \mathcal{U}_{n-1} \)-expansive with respect to \( f_n \) and \( f_n \equiv f \) outside \( \mathcal{U}_n \), by Lemma 3.8 we know that the orbits of \( x \) and \( y \) are distinct \( \mathcal{U}_{n-1} \)-chains of points for \( f \).

Arguing inductively, we know that they are distinct \( \mathcal{U}_1 \)-pseudo orbits of \( f \). In other words, there is \( i_0 \) such that \( f^{i_0}(x) \) and \( f^{i_0}(y) \) belong to different components of \( \mathcal{U}_1 \). As \( \mathcal{U}_1 \) is assumed to have at most finitely many connected components, there is \( \delta > 0 \) such that two distinct components of \( \mathcal{U}_1 \) are at distance larger than \( \delta \). Thus we know \( d(f^{i_0}(x), f^{i_0}(y)) > \delta \) and this shows the expansiveness of \( \Lambda \) for \( f \). \( \square \)

Let us state a result which enables us to obtain the generic existence of minimal, expansive aperiodic classes. We prepare some definitions.

**Definition 3.6.** We say that a \( G \)-robust property \( \mathcal{P} \) for \( f \) on \( \mathcal{U} \) is type \( \mathcal{P}_{\mathcal{MEx}} \), where \( f \in G \) and \( \mathcal{U} \) is a filtrating set of \( f \), if the following holds: If \( f \) satisfies \( \mathcal{P} \) then for any \( \delta > 0, N > 0 \) and any \( G \)-neighborhood \( O \) of \( f \in G \) there is \( g \in O \) satisfying the following:

- The support of \( g \) is strictly contained in \( \mathcal{U} \) with respect to \( f \).
- There are disjoint, regular filtrating sets \( \mathcal{U}_0, \mathcal{U}_1 \subset \mathcal{U} \) for \( g \) which are \( \mathcal{U} \)-minimal and \( \mathcal{U} \)-expansive for chains.
- \( \mathcal{U}_i \) is \( \mathcal{U}_i \)-chain transitive for \( i = 1, 2 \).
- The number of connected components of \( \mathcal{U}_1, \mathcal{U}_2 \) are both finite,
- The minimum period of \( \mathcal{U}_i \) is larger than \( N \) for \( i = 1, 2 \),
- \( \text{cdiam}(\mathcal{U}_i), \text{cdiam}(g(\mathcal{U}_i)) < \delta \) for \( i = 1, 2 \),
- The restrictions \( g \) satisfies the property \( \mathcal{P} \) on \( \mathcal{U}_i \) for \( i = 1, 2 \).

If a dynamics have a robust property which is type \( \mathcal{P}_{\mathcal{MEx}} \), it implies the locally generic coexistence of uncountably many aperiodic classes which are minimal and expansive. To explain this, we prepare some notation which will be used throughout this paper. Consider the sets of finite or infinite sequences \( \{1,2\}^k, k \in \mathbb{N} \) and \( \{1,2\}^\mathbb{N} \). For \( \omega = (\omega_1, \ldots, \omega_k) \) and any \( j \leq k \), or \( \omega = (\omega_j)_{j \in \mathbb{N}} \) and any \( j \in \mathbb{N} \) we write

\[
[\omega]_j = (\omega_1, \ldots, \omega_j),
\]

that is, \([\omega]_j \) is the restriction of \( \omega \) to its first \( j \) terms.
Now we can state a principle for the locally generic coexistence of minimal, expansive aperiodic classes.

**Proposition 3.4.** If \( f \) has a filtrating set \( U \) such that \( f \) satisfies \( G \)-robust property \( P \) on a filtrating set \( U \) which is type \( \mathcal{P}_{\text{MEx}} \), then there is a neighborhood \( O_1 \) of \( f \) in \( G \) and a \( G \)-residual subset \( R \subset O_1 \) such that every \( g \in R \) admits an uncountable family of aperiodic chain recurrence classes which are minimal and expansive.

**Proof.** As \( P \) is \( G \)-robust, there is a non-empty open neighborhood \( O_1 \) of \( f \) such that every \( g \in O_1 \) satisfies \( P \).

Then the property \( P \) being type \( \mathcal{P}_{\text{MEx}} \) allows us to build by induction, a sequence of open subsets \( O_n \subset O_1 \) and a sequence \( \delta_n > 0 \) tending to 0 as \( n \to \infty \) with the following property (this kind of constructions appear in many papers, for instance in Section 3.3 of [BD1] or Section 4.6 of [BS3], so we omit the detail):

- \( O_{n+1} \subset O_n \) and \( O_{n+1} \) is dense in \( O_n \).
- Every \( f \in O_n \) has \( 2^n \) disjoint compact regular filtrating sets \( U_\omega, \omega \in \{1,2\}^n \) such that the maps \( f \rightarrow U_\omega \) are locally constant (see Section 4.6 of [BS3]),
- For any \( f \in O_{n+1} \) and \( \omega \in \{1,2\}^{n+1} \) one has \( U_\omega \subset U_{\omega|n} \), and \( U_\omega \) is \( U_{\omega|n} \)-minimal and \( U_{\omega|n} \)-expansive.
- \( U_\omega \) is \( U_{\omega|n} \)-chain transitive for every \( \omega \in \{1,2\}^n \).
- The minimum \( U_\omega \) period is larger than \( |\omega| \) for every \( \omega \in \{1,2\}^n \),
- \( \max(\text{cdiam}(U_\omega), \text{cdiam}(f(U_\omega))) \) \( < \delta_n \) for every \( \omega \in \{1,2\}^n \).

Now \( R = \cap_{n \geq 1} O_n \) is a \( G \)-residual subset of \( O_1 \), and Proposition 3.2 and 3.3 imply that for any \( f \in R \) and any \( \omega \in \{1,2\}^n \) the compact set \( \Lambda_\omega = \cap_{n \geq 1} U_{\omega|n} \) is a minimal expansive chain recurrence class of \( f \) (apply Proposition 3.2 and 3.3 letting \( f_n = f \)).

Finally, we prepare a technical lemma which enables us to choose a convenient neighborhood for constructing nested sequences.

**Lemma 3.10.** Let \( f : X \to X \) be a homeomorphism of a locally connected compact metric space \( X \) and \( U \subset X \) be a compact set having finite number of connected components. Assume that \( K \subset U \) is a totally disconnected compact set such that given any \( x \in K \) and any connected component \( C \) of \( U \) there is \( n > 0 \) such that \( f^n(x) \in C \).

Then, there is a compact neighborhood \( O \) of \( K \) having finitely many connected components with the following property: Let \( g \) be a homeomorphism of \( X \) such that the support \( \text{supp}(g,f) \) is strictly contained in \( O \). Then given any connected components \( A \) of \( O \) and \( C \) of \( U \), there is \( n > 0 \) such that \( g^n(A) \subset C \).

**Proof.** Due to the finiteness of the number of connected components of \( U \), we first construct the desired neighborhood for a fixed connected component \( C \) of \( U \), see the last comment for the general case.

First, let us see that there is \( N > 0 \) such that for any \( x \in K \) there is \( 0 < n \leq N \) satisfying \( f^n(x) \in C \). To see this, for each \( x \in K \) we take \( n > 0 \) such that \( f^n(x) \in C \). Then the same holds in a neighborhood of \( x \). The compactness of \( K \) allows us to cover \( K \) by finitely many open sets where the number \( n \) is constant. Thus, the number \( N \) for the component \( C \) is uniformly bounded on \( K \).

By hypothesis, \( K \) and \( X \setminus U \) has positive distance. We denote it by \( \delta > 0 \). Now let us choose the neighborhood of \( K \). For each \( x \in X \), by \( B(\delta,x) \) we denote the closed ball of radius \( \delta \) centered at \( x \).
We choose \( \varepsilon_1 \) and \( \varepsilon_2 \) as follows: First, \( \varepsilon_1 \) is a positive real number which is so small that for every \( x \in K, i = 0, \ldots, N \), we have \( f^n(B(\varepsilon_1, x)) < \delta/5 \). To define \( \varepsilon_2 \), we fix an auxiliary number \( \eta > 0 \) satisfying the following: If \( (z_i)_{i=0,\ldots,N} \) is an \( f \)-\( \eta \)-pseudo orbit then for any \( i = 0, \ldots, N \) we have
\[
d(f(z_0), z_i) < \delta/5.
\]
Such an \( \eta \) can be chosen by the uniform continuity of \( f \).

Now we choose \( \varepsilon_2 \). Since we assume that \( X \) is locally connected, for every \( \varepsilon > 0 \) and \( x \in X \) we can choose a compact connected neighborhood of \( x \) whose diameter is less than \( \varepsilon \). We denote such a neighborhood by \( V(\varepsilon, x) \). For each \( \varepsilon > 0 \), consider the covering \( \bigcup_{x \in K} V(\varepsilon, x) \). Note that we can choose finite sub-covering, which we denote by \( \{ V(\varepsilon, x_i) \} \).

Then, \( \varepsilon_2 \) is a positive real number which is so small that if we consider the finite covering \( O := \bigcup_{i} V(\varepsilon_2, x_i) \), then every connected component of \( O \) has diameter less than \( \eta \) and \( \varepsilon_1/2 \). Note that the existence of \( \varepsilon_2 \) is guaranteed by the totally disconnectedness of \( K \) (if such an \( \varepsilon_2 \) does not exist, then \( K \) must contain a non-degenerate continuum) and \( O \) has finitely many connected components (the number is bounded by the number of \( \{ V(\varepsilon_2, x_i) \} \)). We assume that \( \varepsilon_2 < \varepsilon_1 \).

Consider the neighborhood \( O \) of \( K \) defined as above. Take a connected component \( A \) of \( O \) and \( x_0 \in A \cap K \). Consider \( 0 \leq n \leq N \) such that \( f^n(x_0) \in C \). Let us show \( g^n(A) \subset C \). Let \( g \) be a homeomorphism whose support is strictly contained in \( O \). By Lemma 3.1, we have \( f(A) = g(A) \) for every connected component of \( O \). Hence, the \( C^0 \)-distance between \( f \) and \( g \) is less than \( \eta \). Thus, for any \( x \in A \) the sequence \( \{ g^n(x) \}_{n=0,\ldots,N} \) defines an \( f \)-\( \eta \)-pseudo orbit. As a result, we know that \( d(f^n(x), g^n(x)) < \delta/5 \) for any \( x \in A \).

Since the diameter of \( A \) is less than \( \varepsilon_1/2 \), we know \( A \subset B(\varepsilon_1, x_0) \). Thus we know the diameter of \( f^n(A) \) is less than \( \delta/5 \) by the choice of \( \varepsilon_1 \). By the above observation, we know \( g^n(A) \) is in the \( \delta/5 \)-neighborhood of \( f^n(A) \). Thus \( g^n(A) \) is contained in the \( 2\delta/5 \)-neighborhood of \( f^n(x_0) \). As a result, we have \( g^n(A) \subset C \).

For obtaining \( O \), we construct \( A \) for each \( C \). For each \( C \) we have \( \varepsilon_2 = \varepsilon_2(C) \). Then we choose the minimum \( \varepsilon_2 > 0 \) and take \( O \). It gives us the desired neighborhood.

3.5. Infinitely many ergodic measures for a limit set. In this subsection, we consider a sufficient condition which guarantees the existence of infinitely many ergodic measures on the limit of nested sequence of filterizing sets.

Let \( (X, d) \) be a compact metric space and \( \mathcal{P}(X) \) be the space of probability measures on \( X \) endowed with the distance
\[
\delta(\mu_1, \mu_2) = \sum_{n=1}^{\infty} \frac{1}{2^n} \left| \int \phi_n d\mu_1 - \int \phi_n d\mu_2 \right|,
\]
where \( \mu_1, \mu_2 \in \mathcal{P}(X) \) and \( (\phi_n) \) is a sequence of continuous functions bounded by 1 such that \( (\phi_n) \) is dense in the unit ball of the set of continuous functions equipped with the supremum norm.

We say that a finite set of measures is linearly independent if they are independent as vectors in the space of continuous functionals on the space of continuous functions on \( X \).

**Definition 3.7.** Let \( k \geq 2 \) and \( M = \{ \mu_i \}_{i=1,\ldots,k} \) be a finite set of linearly independent probability measures. We define the independence radius of \( M \), denoted by
\[ \rho(\mathcal{M}), \text{ as the supremum of the set of real numbers for which we have the following property:} \]

Any \( k \)-ple of probability measures \( \mathcal{N} = \{\nu_i\}_{i=1}^{k} \) satisfying \( \mathcal{d}(\mu_i, \nu_i) < r \) for every \( i = 1, \ldots, k \) is linearly independent.

\textbf{Remark 6.} For any finite set \( \mathcal{M} \) of \( k \) linearly independent probability measures, the independence radius is strictly positive. It varies continuously with \( \mathcal{M} \). More precisely, for \( \tilde{\mathcal{M}} = \{\tilde{\mu}_i\}_{i=1}^{k} \) satisfying \( \mathcal{d}(\tilde{\mu}_i, \mu_i) < \varepsilon \) for every \( i = 1, \ldots, k \), we have

\[ \rho(\mathcal{M}) + \varepsilon \geq \rho(\tilde{\mathcal{M}}) \geq \rho(\mathcal{M}) - \varepsilon. \]

\textbf{Proof of the inequality.} It is mere an application of triangular inequality: Consider \( \mathcal{N} = \{\nu_i\}_{i=1}^{k} \). Then for every \( i \) we have

\[ \mathcal{d}(\tilde{\mu}_i, \nu_i) + \mathcal{d}(\tilde{\mu}_i, \mu_i) \geq \mathcal{d}(\mu_i, \nu_i) \geq \mathcal{d}(\tilde{\mu}_i, \nu_i) - \mathcal{d}(\tilde{\mu}_i, \mu_i) \]

and thus

\[ \mathcal{d}(\tilde{\mu}_i, \nu_i) + \varepsilon > \mathcal{d}(\mu_i, \nu_i) > \mathcal{d}(\tilde{\mu}_i, \nu_i) - \varepsilon. \]

We now consider the infimum of these quantities when \( \mathcal{N} \) is not independent and we obtain the announced inequality. \( \square \)

\textbf{Lemma 3.11.} Consider a triangular sequence of probability measures \( \mathcal{M}_n = \{\mu_i^n\}_{i=1}^{n} \subset \mathcal{P}(X), n \geq 1 \), such that \( \mathcal{M}_n \) is linearly independent for every \( n \). Suppose that for any \( n \), \( i \in \{1, \ldots, n\} \) and \( k \geq 1 \) one has

\[ \mathcal{d}(\mu_i^n, \mu_i^{n+k}) < \frac{1}{2} \rho(\mathcal{M}_n). \]

For \( i \geq 1 \), let \( \mu_i \) be an accumulation point of the sequence \( \{\mu_i^m\}_{m \geq 1} \). Then for any finite \( N > 0 \), the set of measures \( \{\mu_i\}_{i=1}^{N} \) is linearly independent.

\textbf{Proof.} Fix any \( N > 0 \). Then \( \mathcal{d}(\mu_i^N, \mu_i) \leq \frac{1}{2} \rho(\mathcal{M}_N) \) for \( i \in \{1, \ldots, N\} \). Thus \( \{\mu_1, \ldots, \mu_N\} \) is independent by definition of \( \rho(\mathcal{M}_N) \) and Remark \( \text{6} \). \( \square \)

Hereafter, \( \lambda_0 \) denotes a fixed positive real number in \((0, 1)\) satisfying

\[ \prod_{i=1}^{+\infty}(1 + \lambda_i^0) \cdot \left(\sum_{i=1}^{+\infty} \lambda_i^0\right) \leq \frac{1}{2}. \]

Let us give a criterion to have infinitely many distinct ergodic measures on an aperiodic class.

\textbf{Proposition 3.5.} Let \( \{(f_n, \mathcal{U}_n)\}_{n \geq 1} \) be a nested sequence for an aperiodic class (see \text{Proposition 3.2}). Recall that the sequence homeomorphisms \( f_n \) converges to a homeomorphism \( f \) for which \( \Lambda = \bigcap_n \mathcal{U}_n \) is a chain recurrence class.

Assume that for every \( n \geq 1 \) the filtrating set \( \mathcal{U}_n \) contains a family of mutually disjoint periodic orbits \( \Gamma_n := \{\gamma^n_1, \ldots, \gamma^n_n\} \) of \( f_n \). We denote by \( \mu^n_i \) the ergodic probability Dirac measure associated to \( \gamma^n_i \). Note that the set of measures \( \mathcal{M}_n := \{\mu_i\}_{i=1}^{n} \) is linearly independent.

Suppose that for any \( f_{n+k}, k \geq 1, \Gamma_n \) is still a family of periodic orbits and that for any \( n \geq 1 \) and \( i \in \{1, \ldots, n\} \) we have

\[ \mathcal{d}(\mu_i^n, \mu_i^{n+1}) < \lambda_0^{n+1} \cdot \rho(\mathcal{M}_n). \]

Then there are infinitely many \( f \)-invariant ergodic probability measures on \( \Lambda \).

We prepare a lemma.
Lemma 3.12. Under the assumption of Proposition 3.4 for any \( n, i \in \{1, \ldots, n\} \) and \( k > 0 \) one has

\[
\mathcal{d}(\mu_i^n, \mu_i^{n+k}) < \frac{1}{2} \rho(\mathcal{M}_n).
\]

Proof. The assumption \( \mathcal{d}(\mu_i^n, \mu_i^{n+1}) < \lambda_0^{n+1} \cdot \rho(\mathcal{M}_n) \) implies \( \rho(\mathcal{M}_{n+1}) < (1 + \lambda_0^{n+1}) \rho(\mathcal{M}_n) \), and thus, by induction

\[
\rho(\mathcal{M}_{n+k}) < \rho(\mathcal{M}_n) \cdot \prod_{j=1}^k (1 + \lambda_0^{n+j}).
\]

Then, for every \( n, k > 0 \) and \( i \in \{1, \ldots, n\} \) the distance \( \mathcal{d}(\mu_i^n, \mu_i^{n+k}) \) is bounded by

\[
\mathcal{d}(\mu_i^n, \mu_i^{n+k}) \leq \sum_{j=0}^{k-1} \mathcal{d}(\mu_i^{n+j}, \mu_i^{n+j+1}) \leq \sum_{j=0}^{k-1} \lambda_0^{n+j+1} \cdot \rho(\mathcal{M}_{n+j})
\]

\[
\leq \left( \sum_{j=1}^{+\infty} \lambda_0^{n+j} \right) \cdot \rho(\mathcal{M}_n) \cdot \left( \prod_{j=1}^{+\infty} (1 + \lambda_0^{n+j}) \right) \leq \frac{\lambda_0^n}{2} \rho(\mathcal{M}_n).
\]

Note that the last inequality follows by the choice of \( \lambda_0 \).

\[ \square \]

Proof of Proposition 3.4. Fix some \( N > 0 \). Consider \( \{\mu_1, \ldots, \mu_N\} \) where \( \mu_i \) is an accumulation point of the sequence \( \{\mu_i^n\}_{m \geq i} \). Note that \( \{\mu_i\} \) are probability measures supported on \( \Lambda \) and they are \( f \)-invariant as they are limits of \( f_n \)-invariant probability measures.

According to Lemma 3.12 one has

\[
\mathcal{d}(\mu_i^n, \mu_i) \leq \frac{1}{2} \cdot \rho(\mathcal{M}_N).
\]

Then Lemma 3.11 implies that \( \mu_1, \ldots, \mu_N \) are linearly independent. For each of them, consider its decomposition into ergodic measures for \( f \). The measures \( \{\mu_i\} \) belong to the vector space generated by these ergodic measures. This implies that the vector space has dimension at least \( N \). Thus there are at least \( N \) ergodic measures supported on \( \Lambda \).

As \( N \) is any positive number, one deduces that \( \Lambda \) supports infinitely many ergodic measures for \( f \).

\[ \square \]

3.6. A criterion and a principle for minimal non ergodic classes. In this subsection give a criterion to have minimal chain recurrence classes with infinitely many ergodic measures.

Proposition 3.6. Let \( \{(f_n, \mathcal{U}_n, \Gamma_n)\}_{n \geq 1} \) be a sequence of triples such that

- \( \{(f_n, \mathcal{U}_n)\} \) is a nested sequence for an aperiodic class \( \Lambda = \cap \mathcal{U}_n \).
- \( \mathcal{U}_{n+1} \) is \( \mathcal{U}_n \)-minimal for every \( n \geq 1 \).
- \( \Gamma_n = \{\gamma_i^n\}_{i=1, \ldots, n} \) is a set of \( n \) distinct periodic orbits in \( \mathcal{U}_n \).
- Each member of \( \Gamma_n \) is outside \( \mathcal{U}_{n+1} \) and they are still the periodic orbit of the same orbit for \( f_{n+1} \) (thus for \( f_{n+k} \) for any \( k \geq 1 \)).
- For any \( i \in \{1, \ldots, n\} \) one has
  \[
  \mathcal{d}(\mu_i^n, \mu_i^{n+1}) < \lambda_0^{n+1} \rho(\{\mu_i^n \mid i = 1, \ldots, n\}),
  \]
  where \( \mu_i^n \) is the Dirac probability measure associated to \( \gamma_i^n \).
Then \( (f_n) \) converges to a homeomorphism such that \( \Lambda \) is minimal and supports infinitely many ergodic measures.

The proof is a direct consequence of Proposition 3.2 and Proposition 3.5.

Now we state a version of Proposition 3.6 which assures the locally generic coexistence of minimal aperiodic classes with infinitely many distinct ergodic measures. To state it, we prepare a definition.

Let \( f \in G \) and \( \gamma \) be a periodic orbit of \( f \). We say that \( \gamma \) has a continuation in \( G \) if there are a neighborhood \( O \) of \( f \) in \( G \) and a map \( T: g \mapsto \gamma_g \) defined over \( O \) such that \( T(f) = \gamma \). \( T \) is continuous with respect to the Hausdorff distance and \( \gamma_g \) is a periodic orbit for \( g \).

**Definition 3.8.** Consider a triple \((f, U, \Gamma_n)\), where \( f \in G \), \( U \) is a filtrating set of \( f \) and \( \Gamma_n = \{\gamma_1, \ldots, \gamma_n\} \) is a set of \( n \) distinct periodic orbits of \( f \) in \( U \) such that each \( \gamma_i \) has a continuation in \( G \).

A family of \( G \)-robust properties \((Q^n)_{n \geq 1} \) for \((f, U, \Gamma_n)\) is said to be type \( \mathfrak{F}_{M,\infty} \) if the following holds: If \((f, U, \{\gamma_1, \ldots, \gamma_n\})\), satisfies \( Q^n \), then for any \( \delta > 0, N > 0 \) and any neighborhood \( O \) of \( f \) in which the continuations of orbits of \( \Gamma_n \) are defined, there is \( g \in O \) such that the following holds:

- The support \( \text{supp}(g, f) \) is strictly contained in \( U \).
- There are disjoint regular filtrating sets \( U_1, U_2 \subset U \) for \( g \) which are both \( U \)-minimal for chains.
- For \( j = 1, 2 \), \( U_j \) has finitely many connected components and \( U_j \)-chain transitive.
- \( \text{cdiam}(U_j), \text{cdiam}(g(U_j)) < \delta \) for \( j \in \{1, 2\} \).
- For \( j = 1, 2 \), the minimum period of \( U_j \) is larger than \( N \).
- There are periodic orbits \( \gamma_i^j \subset U_j, j \in \{1, 2\}, i \in \{1, \ldots, n, n+1\} \) such that \((g, U_j, \{\gamma_1^j, \ldots, \gamma_n^j\})\) satisfies the property \( Q^{n+1} \) for \( j \in \{1, 2\} \).
- For \( j \in \{1, 2\} \) and \( i \in \{1, \ldots, n\} \) one has
  \[
  \delta(\mu_i^0, \mu_i^1) < \lambda_i^{n+1} \rho(\{\mu_i^0 \mid i = 1, \ldots, n\})
  \]
  where \( \mu_i^0, \mu_i^1 \) are the Dirac probabilities associated to \( \gamma_i, g \) (the continuation of \( \gamma_i \in \Gamma_n \)) and \( \gamma_i^j \), respectively.

In the next section, we give an example of a family of local properties \((Q^n)\) which is type \( \mathfrak{F}_{M,\infty} \), see Proposition 3.3. Let us see the consequence of such a property.

**Proposition 3.7.** Let \((Q^n)_{n \geq 1} \) a family of \( G \)-robust properties which is type \( \mathfrak{F}_{M,\infty} \). Assume that \((f, U, \gamma_1)\) satisfies property \( Q^1 \) and it persists over a \( G \)-open neighborhood \( O \) of \( f \). Then there is a \( G \)-residual subset \( R \) of \( O \) such that every \( g \in R \) has uncountably many aperiodic classes which are minimal and supports infinitely many ergodic probability measures.

The proof is similar to the proof of Proposition 3.4. Let us give it.

**Proof.** The fact that \((Q^n)\) is type \( \mathfrak{F}_{M,\infty} \) allows us to build by induction, a decreasing sequence of open subsets \((O_n)\) and a sequence \( \delta_n > 0 \) tending to 0 as \( n \to \infty \) with the following property:

- \( O_n \) is dense in \( O \) for every \( n \).
- Every \( f \in O_n \) has \( 2^n \) disjoint regular filtrating sets \( U_\omega, \omega \in \{1, 2\}^n \).
Proof.\] an uniquely ergodic, aperiodic chain recurrence class.\] Then, the sequence \(\{\nu_n\}_{n \geq 1}\) is a well defined object.

Now \(R = \bigcap_{n \geq 1} O_n\) is a residual subset of \(O\) and it has uncountably many aperiodic minimal chain recurrence classes. By proposition 3.5 we know that they support infinitely many ergodic measures. \(\square\)

3.7. A criterion for unique ergodicity. Suppose that we have a compact set \(\mathcal{U}\) of \(X\). The set of Dirac measures of \(\mathcal{U}\)-pseudo orbits, denoted by \(\mathcal{P}_{k,\text{pseudo}}(\mathcal{U})\) is the collection of measures\

\[
\left\{ \frac{1}{K} \sum_{i=1}^{k} \delta_{x_i} \left| (x_i)_{i=1,...,k} \text{ is a } \mathcal{U}\text{-chain of points of length } k. \right\}
\]

where \(\delta_x\) denotes the Dirac probability measure supported at \(x\). By \(\mathcal{P}_{\infty,\text{pseudo}}(\mathcal{U})\), we denote the set of accumulation points of sequences \((\nu_i)\) where \(\nu_i \in \mathcal{P}_{1,\text{pseudo}}(\mathcal{U})\).

Lemma 3.13. Let \(\{\langle f_n, \mathcal{U}_n \rangle\}_{n \geq 1}\) be a nested sequence for an aperiodic class. Assume that there is a sequence of positive real numbers \((\varepsilon_n)\) tending to 0 as \(n \to \infty\) such that, for every \(n\) one has

if \(\mu_1\) and \(\mu_2\) are probability measures which are accumulated by the Dirac probabilities along \(\mathcal{U}_n\)-pseudo orbits (that is, if \(\mu_1, \mu_2 \in \mathcal{P}_{\infty,\text{pseudo}}(\mathcal{U}_n)\) then

\[
\Theta(\mu_1, \mu_2) < \varepsilon_n.
\]

Then, the sequence \(\langle f_n \rangle\) converges to a homeomorphism \(f\) such that \(\Lambda = \bigcap \mathcal{U}_n\) is an uniquely ergodic, aperiodic chain recurrence class.

Proof. The proof of the fact that the sequence \(\langle f_n \rangle\) converges to a homeomorphisms for which \(\Lambda\) is an aperiodic class is due to Proposition 3.2. Let us prove the unique ergodicity.

The \(f\)-orbits in \(\Lambda\) are \(\mathcal{U}_n\)-pseudo orbits for \(f_n\), for every \(n\). Thus the \(\delta\)-distance between any two probability measures accumulated by Birkhoff averages of the Dirac measure along \(f\)-orbits in \(\Lambda\) is less than \(\varepsilon_n\) for every \(n\). In other words, they are equal. Therefore, there is a unique measure accumulated by these Birkhoff averages along \(f\) orbits in \(\Lambda\). It implies the unique ergodicity of \(\Lambda\). \(\square\)

Remark 7. The hypothesis for each fixed \(n\) is robust in the \(C^0\)-topology, if the number of connected components is finite. Also, it is stable for local perturbations, since local perturbations do not change the set of \(\mathcal{U}_n\)-chains.
3.8. A criterion for transitivity. In this subsection, we discuss conditions which guarantees transitivity of limit aperiodic classes. We say that a compact $f$-invariant set $K \subset X$ is transitive if it has a point having a dense orbit in $K$.

**Definition 3.9.** Let $f$ be a homeomorphisms and $\mathcal{U}_0$ and $\mathcal{U}_1$ be two compact filtrating sets such that $\mathcal{U}_1 \subset \mathcal{U}_0$ and $\mathcal{U}_i$ is $\mathcal{U}_i$-chain transitive for $i = 0, 1$. Let $C$ be a connected component of $\mathcal{U}_1$. We say that $(\mathcal{U}_1, C)$ is $\mathcal{U}_0$-transitive (or just $\mathcal{U}_1$ is $\mathcal{U}_0$-transitive) if the following conditions hold:

- For any component $A$ of $\mathcal{U}_0$ there is an $i^+(A)$, $i^-(A) \geq 0$ such that $f^{i^+(A)}(C) \subset A$ and $f^{-i^-(A)}(C) \subset A$.
- For every connected component $C_0$ of $\mathcal{U}_0$, there is a connected component $C_1$ of $\mathcal{U}_1$ such that $C_1 \subset C_0$.

In the following, $i^+(A), i^-(A)$ denote the smallest non-negative integers satisfying the condition above.

**Lemma 3.14.** If the number of the connected components of $\mathcal{U}_0$ is finite, then $\mathcal{U}_0$-transitivity for some $C$ is a $C^0$-robust property: Any homeomorphism $g$ which is enough $C^0$-close to $f$ has $\mathcal{U}_0$ and $\mathcal{U}_1$ as filtrating sets and $\mathcal{U}_1$ is $\mathcal{U}_0$-transitive.

**Proof.** Recall that being a filtrating set is a $C^0$-robust property. Thus we know that $\mathcal{U}_1$, $\mathcal{U}_0$ are filtrating sets for homeomorphisms sufficiently close. To see the $C^0$-robustness of the transitivity, there are only finitely many conditions which we need to require and they are obviously $C^0$-robust. Thus we have the conclusion.

**Remark 8.** If the number of connected components of $\mathcal{U}_0$ is finite, then there is $N^+, N^- > 0$ such that $\{f^i(C)\}_{i=0,...,N^+}$, $\{f^i(C)\}_{i=-N^-,...,0}$ passes through all the connected components of $\mathcal{U}_0$.

**Proof.** Set $N^\pm$ to be the maximum of $i^\pm(A)$ respectively.

While $\mathcal{U}_0$-transitivity is a robust property, it is not stable by perturbations supported on $\mathcal{U}_1$. Below we give a sufficient condition for stable $\mathcal{U}$-transitivity.

**Lemma 3.15.** For $\mathcal{U}_1 \subset \mathcal{U}_0$ and $f$ as above, assume that $(\mathcal{U}_1, C)$ is $\mathcal{U}_0$-transitive. Furthermore, assume that the following condition holds:

- For any $\mathcal{U}_1$-chain of connected components $(U_k)_{k=0,...,i_0}$ satisfying $U_0 = C$ where $0 \leq i_0 \leq i^+(A)$, we have $f^{i^+(A)-i_0}(U_{i_0}) \subset A$.
- The same holds for $f^{-1}$ and $i^-(A)$: For any $\mathcal{U}_1$-chain of connected components $(U_k)_{k=-i_0,...,0}$ satisfying $U_0 = C$ where $0 \leq i_0 \leq i^-(A)$, we have $f^{-(i^-(A)-i_0)}(U_{i_0}) \subset A$.

Then $(\mathcal{U}_1, C)$ is stably $\mathcal{U}_0$-transitive for local perturbations whose supports are strictly contained in $\mathcal{U}_1$.

**Proof.** Consider $g$ whose support is strictly contained in $\mathcal{U}_1$. First consider the condition about the forward iteration. Note that the condition about the support guarantees that the set of $\mathcal{U}_1$-chains of connected components are the same for $f$ and $g$. Given $x \in C$ and a connected component $A$ of $\mathcal{U}_0$, consider the orbit $\{g^i(x)\}_{i=0,...,i^+(A)}$. If the whole orbit is contained in $\mathcal{U}_1$, it defines a $\mathcal{U}_1$-chain of connected components of $g$, which appears in the set of chains connected components of $f$. Since the sets of the chains of connected components are the same for $f$ and $g$, we know $g^i(x) \in A$ (just consider the case $i_0 = 0$). If not, there is $g^k(x)$...
such that \( g^k(x) \not\in U_1 \). We choose \( k \) to be the minimum positive integer among such numbers. By definition we know there is \( y \in U_1(g^{k-1}(x)) \) such that \( f(y) = g^k(x) \). Since \( U_1 \) is a filtrating set, we know that \( g^{k+i}(x) \not\in U_1 \) for \( i > 0 \). Then, since outside \( U_1 \) we have \( g \equiv f \), we conclude \( g^{i(A)-k}(g^k(x)) = f^{i(A)-k}(f(y)) \). By definition, we know that this point belongs to \( \tilde{A} \). Thus, we are done.

Now, let us consider the case for the backward iterations. One point which differs from the forward case is the domain of the support. For \( g^{-1} \), we only have \( \text{supp}(g^{-1}, f^{-1}) \subset f(U_1) = g(U_1) \). Take \( x \in C \). If \( \{g^{-i}(x)\}_{i=0}^{+\infty} \) is contained in \( U_1 \), then we can conclude \( g^{-i(\cdot)(A)}(x) \in \tilde{A} \) as in the forward case. If not, then we choose minimum \( k \geq 1 \) such that \( g^{-k}(x) \not\in U_1 \). Now, note that \( g^{-k+1}(x) \in U_1 \setminus (U_1 \cap g(U_1)) = U_1 \setminus (U_1 \cap f(U_1)) \). This means \( \{g^{-k-i}(x)\}_{i=0}^{+\infty} \) is outside \( f(U_1) \), due to the filtrating property of \( f(U_1) \) with respect to \( g \). Thus we have \( g^{-k-l}(x) = f^{-l}(g^{-k}(x)) \) for \( l \geq 0 \) and the conclusion follows by considering \( l = i(\cdot)(A) - k \).

\[ \square \]

**Remark 9.** The assumption of Lemma 3.15 implies the first condition of \( U \)-transitivity in Definition 3.9 (considering the case \( i_0 = 0 \)). Thus, the second condition of Definition 3.9 and the assumption of Lemma 3.15 imply the stable \( U \)-transitivity for local perturbations.

### 3.9. A criterion for non-minimality.

The next lemma is a criterion for building transitive aperiodic classes admitting several minimal sets:

**Lemma 3.16.** Let \( \{(f_n, U_n)\}_{n \geq 1} \) be a nested sequence for an aperiodic class. Assume that for every \( n \) one has

- \( (U_{n+1}, C_{n+1}) \) is stably \( U_n \)-transitive for \( f_{n+1} \) where \( C_{n+1} \) is a connected component of \( U_{n+1} \).
- The number of connected components of \( U_n \) is finite.

Then the sequence \( (f_n) \) converges to a homeomorphism \( f \) for which \( \Lambda = \bigcap U_n \) is a transitive chain recurrence class.

If, furthermore, there are two disjoint compact sets \( A, B \) such that for every \( n \), there are two closed \( U_n \)-pseudo periodic orbits for \( f_n \) contained in \( A \) and \( B \) respectively, then \( \Lambda \) contains at least two minimal sets.

**Proof.** According to Proposition 3.1 the sequence \( (f_n) \) converges to \( f \) and \( \Lambda \) is an aperiodic chain recurrence class of \( f \). Let us prove the transitivity. Consider \( (U_n, C_n) \) for \( f_n \) and we fix \( N^+ > 0 \) by Remark 8. Note that by definition of \( U_n \)-transitivity, \( C_n \) contains at least one connected component of \( U_{n+1} \). Inductively, we know that \( C_n \cap \Lambda \neq \emptyset \). Take \( x \in C_n \cap \Lambda \). By definition and the assumption about the support, \( \{x, f(x), \ldots, f^{N^+}(x)\} \) meets every component of \( U_{n+1} \). If \( n \) is large enough these components are arbitrarily small, say, smaller that \( \varepsilon > 0 \) so that the positive orbit of \( x \) is \( \varepsilon \)-dense in \( \Lambda \). The same holds for its backward orbits.

Thus given any two open sets \( U, V \) of \( \Lambda \) there is \( x \in X \) whose positive orbit meets both \( U \) and \( V \) and so does its negative orbit. This means that the positive iteration of \( U \) meets \( V \) and negative iteration of \( U \) meets \( V \), too. This implies the topological transitivity of \( f \), see Remark 10.

Let us show that \( A \) and \( B \) contain at least one minimal set. By \( (x_k^n) \) we denote the pseudo periodic orbit in \( \tilde{A} \) in \( U_k \). Consider the set of accumulation points of these orbits (that is, the accumulation of points of sequences \( (y_n) \) where \( y_n \) is one
of $(x^n)$. We denote it by $\Lambda_A$. One can check that it is compact, $f$-invariant and contained in $A$. Thus, there is a minimal set which is strictly contained in $A$. The same holds for $B$, which concludes the proof. \hfill \Box

Remark 10. The condition about $U$ and $V$ in the proof implies a property stronger than the transitivity: It implies the existence of a point whose $\alpha, \omega$-limit sets coincide with $\Lambda$.

Remark 11. If $U_n$ is regular, then having a pseudo periodic orbit inside a fixed compact set is a $C^0$-robust property.

We can now express our principle for the local genericity of transitive non minimal aperiodic classes.

Let $V$ be a possibly non-filtrating set and $A$ a compact set such that every connected component of $V$ is either being contained in $\tilde{\mathcal{A}}$ or disjoint of $\mathcal{A}$. Then, by the restriction of $V$ to $A$, denoted by $V|_A$, we mean the compact set $V \cap A$.

Definition 3.10. Let $A, B$ be compact disjoint subsets of $X$, $f \in G$ and $\mathcal{U}$ a filtrating set of $f$. A property $\mathcal{P}$ on $(f, \mathcal{U})$ is called type $\mathcal{P}_{TnM,A,B}$ if the following holds: For any $N > 0$, $\delta > 0$, $G$-neighborhood $O$ of $f$ there is $g \in O$ such that:

• $\text{supp}(g, f)$ is strictly contained in $\mathcal{U}$.
• There are disjoint regular filtrating sets $\mathcal{U}_1, \mathcal{U}_2 \subset \tilde{\mathcal{U}}$ for $g$.
• The number of connected components of $\mathcal{U}_1, \mathcal{U}_2$ are finite.
• $\mathcal{U}_i$ is $\mathcal{U}_i$-chain transitive for $i = 1, 2$.
• $\max\{\text{cdiam}(\mathcal{U}_i), \text{cdiam}(\mathcal{U}_i(g))\} < \delta$ for $i = 1, 2$.
• Minimum periods of $\mathcal{U}_i$ are larger than $N$ for $i = 1, 2$.
• $(\mathcal{U}_i, C_i)$ is $\mathcal{U}$-transitive for $i = 1, 2$ (where $C_i$ is some connected component of $\mathcal{U}_i$).
• $\mathcal{U}_i|_A$ (resp. $\mathcal{U}_i|_B$) can be defined and $A$ (resp. $B$) contains a $\mathcal{U}_i|_A$-pseudo periodic orbit for $i = 1, 2$.
• $(g, \mathcal{U}_i)$ satisfies property $\mathcal{P}$ for $i = 1, 2$.

Then the following is a direct corollary of Lemma 3.16.

Proposition 3.8. Suppose that $(f, \mathcal{U})$ satisfies a $G$-robust property $\mathcal{P}$ which is type $\mathcal{P}_{TnM,A,B}$ for some two disjoint compact sets $A$ and $B$. Then there is a neighborhood $O_0$ of $f$ in $G$ and a residual subset $R \subset O_0$ such that every $g \in R$ admits an uncountable family of aperiodic transitive classes containing at least two minimal sets.

The proof is almost the same as Proposition 3.4 or Proposition 3.7. We need to construct a branching family of nested sequences for aperiodic classes satisfying the conditions of Lemma 3.16. So we omit it.

3.10. A criterion for non-transitive class. Let us discuss a condition which guarantees the non-transitivity of limit aperiodic classes.

Lemma 3.17. Let $\{(f_n, \mathcal{U}_n)\}_{n \geq 1}$ be a nested sequence for an aperiodic class. Assume that there are two sequences $(A_n)$ and $(B_n)$ of connected components of $\mathcal{U}_n$ such that for every $n$ one has

• $A_{n+1} = A_n \cap \mathcal{U}_{n+1}$, $B_{n+1} = B_n \cap \mathcal{U}_{n+1}$.
• any $U_n$-chain of connected components of length $n + 1$ starting from or ending at $A_n$ do not contain $B_n$.

Then the sequence $(f_n)$ converges to a homeomorphism $f$ such that $\Lambda = \bigcap U_n$ is a non-transitive, aperiodic chain recurrence class.

Proof. First, by Proposition 3.1 we know that $\Lambda := \bigcap U_n$ is an aperiodic chain recurrence class. We will prove that $\Lambda$ contains two points $a, b$ such that they are isolated in $\Lambda$ and have distinct orbits. This implies the non-existence of the point $p \in \Lambda$ such that $\bar{O}(p) = \Lambda$. Indeed, if $p \in \Lambda$ satisfies $a \in \bar{O}(p)$ then the assumption that $a$ is isolated implies $\bar{O}(p) = O(a)$. Similarly, $b \in \bar{O}(p)$ implies $\bar{O}(p) = O(b)$. However, these cannot hold simultaneously if the orbits of $a$ and $b$ are different.

Now, let us prove above assertion. The nestedness of $(A_n)$ and $(B_n)$, together with $\text{cdiam}(U_n) \to 0$ imply that $\bigcap_n A_n, \bigcap_n B_n$ exist and they are singletons. We denote them by $a$ and $b$ respectively. The definitions of $(A_n)$ and $(B_n)$ imply that $a, b$ are isolated in $\Lambda$.

Recall that $f_n$ and $f_{n+k}$ coincides outside $U_n$ for every $k \geq 0$. This implies that for every $n$ and $k$ the sets of $U_n$-chains for $f_n$ and $f_{n+k}$ are the same. In particular, they are same for $f_n$ and $f$. Suppose $O(a) \cap O(b) \neq \emptyset$ for $f$. Then, it implies there is $K \in \mathbb{Z}$ such that $f^K(a) = b$, but this contradicts the fact that in $U_K$ we do not have $U_K$-chain of length $K$ between $A_K$ and $B_K$. 

\textbf{Remark 12.} The non-existence of a certain $U_k$-chain of finite length is a $C^0$-robust property, if the number of the connected components of $U_k$ is finite. The condition is stable under a local perturbation whose support is strictly contained in $U_k$, because such a perturbation does not change the set of the $U_k$-chains.

\textbf{Remark 13.} By carefully repeating the above proof one can see that $f$ has no point whose $\omega$-limit set (resp. $\alpha$-limit set) coincides with $\Lambda$.

### 3.11. A principle for the local genericity of non-transitive uniquely ergodic classes.

Now we can give a principle which ensures the generic existence of non-transitive uniquely ergodic aperiodic classes.

\textbf{Definition 3.11.} Let $A, B$ be compact disjoint sets of $X$. Let $P$ be a $G$-robust property on $(f, U)$ where $f \in G$, $U$ is a compact filtrating set of $f$ such that $U$ has unique connected components contained in $A$ and $B$ respectively. We say that a property $P$ is type $P_{nT, A, B}$ if the following holds: For any $\delta_1 > 0, \delta_2 > 0, N > 0, M > 0$ and any neighborhood $O$ of $f \in G$ there is $g \in G$ such that:

- $\text{supp}(g, f)$ is strictly contained in $U$.
- There are disjoint regular filtrating sets $U_1, U_2 \subset U$ for $g$ which are $U_1$, $U_2$-chain transitive respectively. Their minimal periods are larger than $N$. Furthermore, the number of connected components of $U_i$ is finite for $i = 1, 2$.
- $\max(\text{cdiam}(U_i), \text{cdiam}(U_i)) < \delta_1$ for $i = 1, 2$.
- The ergodic diameter of $U_i$ is less than $\delta_2$ for $i = 1, 2$ (see Section 3.7): If $\mu_i$ and $\mu'_i$ are probability measures which are accumulated by the Dirac probabilities along $U_i$-pseudo orbits, then
  \[ \mathfrak{d}(\mu_i, \mu'_i) < \delta_2. \]
For $i = 1, 2$, $U_i$ has two connected components $A_i, B_i$ which are unique components contained in the connected component $A_0, B_0$ of $U$ contained in $A, B$ respectively.

- Any $U_i$-chain of connected components of length $M$ starting from or ending at $A_i$ do not contain $B_i$ for $i = 1, 2$.
- $g$ satisfies property $P$ on $U_i$ for $i = 1, 2$.

We have the following principle.

**Proposition 3.9.** Suppose that $(f, U)$ satisfies a $G$-robust property $P$ which is of type $\mathcal{P}_{n,T,A,B}$ for some compact disjoint sets $A$ and $B$. Then, there is a neighborhood $O_0$ of $f$ in $G$ and a residual subset $R \subset O_0$ such that every $g \in R$ admits an uncountable family of aperiodic chain-recurrent classes which are not transitive but uniquely ergodic.

**Proof.** As $\mathcal{P}_{n,T,A,B}$ is $G$-robust, there is a non-empty open neighborhood $O_1$ of $f$ such that every $f_1 \in O_1$ satisfies $\mathcal{P}_{n,T,A,B}$ on $U$.

Then the assumption of Proposition 3.9 allows us to build a sequence of open subsets $O_n \subset O_1$ and sequences $\delta_n > 0, \varepsilon_n > 0$ tending to 0 as $n \to \infty$ with the following property (see also Remark 7 and Remark 12):

- $O_{n+1} \subset O_n$ and $O_n$ is dense in $O_n$.
- Any $f \in O_n$ has $2^n$ disjoint regular filtrating sets $U_\omega$ ($\omega \in \{1, 2\}^n$) having finitely many connected components such that the map $f \mapsto U_\omega$ is locally constant.
- $\max \{\text{diam}(U_\omega), \text{diam}(g(U_\omega))\} < \delta_n$ for every $\omega \in \{1, 2\}^n$.
- Each $U_\omega$ is $U_\omega$-chain transitive and has its minimum period larger than $|\omega|$.
- For any $f \in O_{n+1}$ and $\omega \in \{1, 2\}^{n+1}$ one has $U_\omega \subset U_{\omega_\omega}$.
- For any $f \in O_{n+1}$ and $\omega \in \{1, 2\}^{n+1}$ one has two connected components $A_\omega$ and $B_\omega$ of $U_\omega$ such that
  $$A_\omega = U_\omega \cap A_{\omega_\omega}, \text{ and } B_\omega = U_\omega \cap A_{\omega_\omega}.$$ 
- For any $f \in O_n$ and $\omega \in \{1, 2\}^n$, every $U_\omega$-chain of connected components of length $|\omega|$ starting from or ending at $A_\omega$ does not contain $B_\omega$.
- If $\mu_1, \omega$ and $\mu_2, \omega$ are probability measures which are accumulated by the Dirac probabilities along $U_\omega$-pseudo orbits ($\omega \in \{1, 2\}^n$), then
  $$\mathfrak{d}(\mu_1, \omega, \mu_2, \omega) < \varepsilon_n.$$ 

Now $R = \bigcap O_n$ is a residual subset of $O_0$ and Lemma 3.12 implies that for any $f \in R$ and any $\omega \in \{1, 2\}^N$ the compact set $A_\omega = \bigcap_n U_{\omega_n}$ is a uniquely ergodic chain recurrence class of $f$, and Lemma 3.17 implies that it is not transitive. Thus the proof is completed. \hfill \Box

4. **Principles for the local $C^1$-genericity of aperiodic classes with prescribed dynamics**

In this section, we discuss the existence of properties proposed in the previous section and complete the proof of Theorem 11. In this section $M$ denotes a smooth closed connected three dimensional manifold and $G = \text{Diff}^1(M)$ with the $C^1$-topology.
4.1. Basic properties. In this subsection we investigate several basic properties of filtrating Markov partitions.

In the course of the construction we use properties of finiteness of the number of connected components of Markov partitions and regularity. Note that by definition of the filtrating Markov partition (of saddle type) we have the following properties (see [BS$_2$, Definition 2.5]):

- Filtrating Markov partitions have at most finitely many connected components.
- Filtrating Markov partitions are regular: Given two connected components $U_1, U_2$ of a Markov partition $U$ of a diffeomorphism $f$, if $f(U_1) \cap U_2 \neq \emptyset$ then each connected component of the intersection is a vertical cylinder. Thus their interiors also have non-empty intersection.

We prepare some fundamental results about filtrating Markov partitions. The following lemma is given in [BS$_3$, Section 2.2]. So we omit the proof. For the definition of $U(m,n)$ and $U(m,n)(K)$, see Section 2.2 and 2.3.

Lemma 4.1. Let $U \subset M$ be a filtrating Markov partition of $f \in \text{Diff}^1(M)$ and $K$ be a circuit of points such that every periodic point has a large stable manifold. Then the Markov partition $U(m,n)(K)$ converges to $K$ as $m, n \to \infty$. More precisely, for any neighborhood $O$ of $K$ there exists $m_0, n_0 > 0$ such that for every $m \geq m_0, n \geq n_0$ we have $U(m,n)(K) \subset O$.

As a consequence, we have the following (see also [BS$_3$, Section 4.4]).

Lemma 4.2. Let $U$ be a filtrating Markov partition and $K$ be a circuit of points such that every periodic point has its period larger than $\pi$ and a large stable manifold. Then, for sufficiently large $m$ and $n$ the Markov partition $U(m,n)(K)$ has minimum period larger than $\pi$ as well.

Proof. When $m$ and $n$ are large, $U(m,n)(K)$ is close to $K$. Consider a pseudo periodic orbit in $U(m,n)(K)$. If it follows the periodic orbit, then the period must be larger than $\pi$. If not, then it must follows one of a homo/heteroclinic orbit, but when $m$ and $n$ are large, then the number of iterations needed to follow the homo/heteroclinic orbits to arrive at near the initial point must be large, in particular larger than $\pi$. Thus in both cases we have the conclusion. $\square$

Finally, let us observe the following.

Remark 14. Let $U$ be a filtrating Markov partition and $\mathcal{V}$ be a filtrating Markov partition matching to $U$ (see Definition 2.4). If the minimum period of $U$ is larger than $\pi$, then so is for $\mathcal{V}$. In other words, the property “having minimum period larger than $\pi$” inherits to matching Markov partitions.

Proof. Consider a pseudo periodic orbit of $\mathcal{V}$. Then the fact that $\mathcal{V}$ is matching to $U$ enables us to take the lift of the pseudo periodic orbit to $U$ with the same period. Hence, the pseudo orbit of $\mathcal{V}$ must have the period larger than $\pi$. $\square$

The following lemma enables us to find a homo/heteroclinic point.

Lemma 4.3. Let $U$ be a sub Markov partition of some filtrating Markov partition and let $p$ be a periodic point whose orbit is contained in $U$ having a large stable manifold. If we have a finite sequence of $U$-chains of points $(x_i)_{i=0,\ldots,n}$ such that $x_0$ and $x_n$ belong to the orbit of $p$, it is $U$-shadowed by an orbit of the point of
transverse intersection between $W^s(p)$ and $W^u(p)$, that is, there is a point $y$ of the transverse intersection between $W^s(p)$ and $W^u(p)$ such that $x_i$ and $f^i(y)$ belong to the same component of $U$ for every $i = 0, \ldots, n$.

Proof. By the existence of the chain $(x_i)$, we know that there is a segment of the local unstable manifold $\sigma \subset W^u_{\text{loc}}(p)$ such that $f^i(\sigma)$ belongs to the rectangle $U(x_i)$ and $f^n(\sigma)$ is a segment which properly crosses $U(p)$ in the vertical direction (see also Lemma 2.15 of [BS2]). Thus the largeness of the stable manifold of $p$ assures there exists a point $y \in W^s(p) \cap \sigma \neq \emptyset$. This gives the desired point. □

Remark 15. It may be that the point obtained is $p$ itself. However, if we know that one of the rectangles $U(x_i)$ does not contain the orbit of $p$, then we can conclude that it is a homoclinic point. In that case, $y$, $f^n(y)$ belong to $W^u_{\text{loc}}(p)$, $W^s_{\text{loc}}(p)$ respectively.

As a simple consequence of Lemma 4.3 we obtain the following:

Corollary 4.1. Let $U$ be a transitive filtrating Markov partition containing a periodic point $p$ with a large stable manifold. Then the relative homoclinic class $H(p, U)$ has non-empty intersection with every rectangle of $U$.

4.2. Expansiveness of sub Markov partitions.

4.2.1. Expansiveness and circuits. Given a possibly non-filtrating Markov partition $U$ we say that $U$ is generating if given any components $C_1, C_2$ of $U$, $f(C_1) \cap C_2$ is either empty or connected. The following is given in Section 2.3 of [BS3].

Remark 16. Given a filtrating Markov partition $U$ and $(m, n) \neq (0, 0)$, its $(m, n)$-refinement is generating.

Recall that we defined the notion of $U$-expansiveness, see Definition 3.5. In this subsection, we prove the following:

Lemma 4.4. Let $U$ be a generating filtrating Markov partition and $K \subset U$ be a circuit of points such that all of its periodic points have large stable manifolds. For every sufficiently large $m, n > 0$, the Markov partition $U_{(m,n)}(K)$ is $U$-expansive for chains.

Given a filtrating Markov partition $U$ and a $f$-invariant set $K \subset U$ we say that $K$ is $U$-expansive if given any $x \neq y$ in $K$ there is $n \in \mathbb{Z}$ such that $f^n(x)$ and $f^n(y)$ belongs to different components of $U$. Note that this is different from $U$-expansiveness for chains we discussed in Section 3.4.

Let us give some preparatory results for the proof of Lemma 4.4.

Lemma 4.5. Let $U$ be a filtrating Markov partition and $K \subset U$ be a circuit of points whose periodic orbits have large stable manifolds. Then every point in $K$ has a large stable manifold (that is, the center stable manifold which contains the point is contained in the stable manifold of the point).

Proof. The property of having a large stable manifold is invariant by negative iteration (if $x$ has a large stable manifold, $f^{-1}(x)$ has a large stable manifold, too). If a periodic point has a large stable manifold, so does every point in its orbit, and also every point in its stable manifold. In a circuit every point belongs to the stable manifold of a periodic point which is assumed to have a large stable manifold. Thus we have the desired property. □
Lemma 4.6. Let \( \mathcal{U} \) be a filtrating Markov partition and \( K \subset \mathcal{U} \) be a circuit of points whose periodic orbits have large stable manifolds. Then any neighborhood of \( K \) contains a hyperbolic basic set \( \Lambda \) such that

- \( K \subset \Lambda \), and
- every point in \( \Lambda \) has a large stable manifold in \( \mathcal{U} \).

Proof. Any non-trivial circuit of points is contained in a hyperbolic basic set \( \Lambda_0 \) and \( \Lambda_0 \) can be chosen arbitrarily close to \( K \). Recall that in \( \Lambda_0 \) the local stable manifolds vary continuously with respect to the point. As every point of the circuit has a large stable manifold in \( \mathcal{U} \), every point in \( \Lambda_0 \) close enough to \( K \) has a large stable manifold in \( \mathcal{U} \), too. Thus, by choosing \( \Lambda_0 \) very close to \( K \) we obtain the conclusion. \( \square \)

Lemma 4.7. Let \( \mathcal{U} \) be a generating filtrating Markov partition and \( \Lambda_0 \subset \mathcal{U} \) be a hyperbolic basic set such that every point of \( \Lambda_0 \) has a large stable manifold. Then \( \Lambda_0 \) is \( \mathcal{U} \)-expansive.

Proof. Suppose that \( x, y \in \Lambda_0 \) belong to the same rectangle \( U_0 \) of \( \mathcal{U} \) and \( f(x), f(y) \) belong to the same rectangle \( U_1 \). Then \( x, y \) belong to \( U_0 \cap f^{-1}(U_1) \) which is connected as \( \mathcal{U} \) is generating. Arguing by induction, one gets that, if \( f'(x), f'(y) \) belong to the same rectangles for \( i \in \{−m, \ldots, n\} \) then \( x \) and \( y \) belong to the same connected component of \( \mathcal{U}(m,n) \).

As the points in \( \Lambda_0 \) have large stable manifolds in \( \mathcal{U} \), the diameter of the components of \( \mathcal{U}(m,n)(\Lambda_0) \) tends to 0. Thus letting \( m, n \to \infty \) we obtain the conclusion. \( \square \)

Now let us complete the proof of Lemma 4.4.

Proof of Lemma 4.4. Note that \( K \) is a hyperbolic set and \( \mathcal{U}(m,n)(K) \) is very close to \( K \) by Lemma 4.1 when \( m, n \) are sufficiently large. Hence, for \( m, n \) large enough, the maximal invariant set in \( \mathcal{U}(m,n)(K) \) is a hyperbolic basic set \( \Lambda(m,n)(K) \) whose points have large stable manifolds in \( \mathcal{U} \).

By an argument similar to the proof of Lemma 4.7 given any \( \mathcal{U}(m,n)(K) \)-pseudo orbit of points \( (x_i)_{i \in \mathbb{Z}} \), there is unique point \( y \in \Lambda(m,n)(K) \) such that \( f^i(y) \) and \( x_i \) belong to the same rectangle of \( \mathcal{U}(m,n)(K) \). More precisely, for \( x_i \) consider the rectangle in \( \mathcal{U}(m+m',n+n')(K) \) which has the same itinerary as \( (x_i) \) and let \( m', n' \to \infty \): Then we obtain a point in a locally maximal invariant set of \( \mathcal{U}(m,n)(K) \) which has the same itinerary as \( (x_i) \). We say that the point \( y \) \( \mathcal{U}(m,n)(K) \)-shadows \( (x_i)_{i \in \mathbb{Z}} \).

Assume that \( (x_i^j)_{i \in \mathbb{Z}}, j = 1, 2 \) are two \( \mathcal{U}(m,n)(K) \)-pseudo orbits of points such that \( x_i^1 \) and \( x_i^2 \) belong to the same rectangle of \( \mathcal{U} \) for \( i \in \mathbb{Z} \). Let \( y^1 \) and \( y^2 \) be the points of \( \Lambda(m,n)(K) \) which \( \mathcal{U}(m,n)(K) \)-shadow \( (x_i^1)_{i \in \mathbb{Z}} \) and \( (x_i^2)_{i \in \mathbb{Z}} \), respectively. This means that \( f^i(y^1) \) and \( f^i(y^2) \) belong to the same rectangle of \( \mathcal{U} \) for every \( i \in \mathbb{Z} \).

According to Lemma 4.7, this implies \( y_1 = y_2 \). As a consequence, \( x_i^1 \) and \( x_i^2 \) belong to the same rectangle of \( \mathcal{U}(m,n)(K) \) for every \( i \in \mathbb{Z} \). \( \square \)

4.2.2. Expansiveness for refinements and sub Markov partitions. In this subsection, we discuss two auxiliary results related to expansiveness.
Lemma 4.8. Let $\mathcal{U}$ be a (possibly non-filtrating) Markov partition of a diffeomorphism $f$, $\mathcal{V}$ is a generating filtrating Markov partition such that $\mathcal{V} \subset \mathcal{U}$ holds. Let $S$ be a compact invariant subset of $\mathcal{V}$ and assume $\mathcal{V}(S)$ is $\mathcal{U}$-expansive. Then for every $n > 0$ the refinement $\mathcal{V}_{(0,n)}(S)$ is $\mathcal{U}$-expansive.

Proof. We only prove that $\mathcal{V}_{(0,1)}(S)$ is $\mathcal{U}$-expansive. The general case can be done by induction. This is a direct consequence of generating property of $\mathcal{V}$. Suppose we have bi-infinite chains $(V_i), (W_i)$ of connected components of $\mathcal{V}_{(0,1)}(S)$. For each $V_i$, we denote by $V_i'$ the rectangle of $\mathcal{V}(S)$ which contains $V_i$. Similarly, we construct $(W_i')$.

Assume that there is $j$ such that $V_j \neq W_j$. If $V_j$ and $W_j$ are contained in different components of $\mathcal{V}(S)$, then it means that $(V_j') \neq (W_j')$ as a chain. Hence by assumption we know there is $k$ such that $V_k'$ and $W_k'$ belong to the different rectangles of $\mathcal{U}$, and the same holds for $V_k$ and $W_k$. If $V_j$ and $W_j$ are contained in the same component of $\mathcal{V}(S)$, as $\mathcal{V}$ is generating, we know that $f^{-1}(V_j)$ and $f^{-1}(W_j)$ are contained in a different component of $\mathcal{V}$. Thus we have the conclusion by considering $i = j - 1$. \qed

The following is a direct consequence of the definition of the $\mathcal{U}$-expansiveness, so we omit the proof.

Lemma 4.9. Let $\mathcal{U}$ be a filtrating Markov partition, $\mathcal{U}'$ be its sub Markov partition and $\mathcal{V}$ be another Markov partition contained in $\mathcal{U}'$ which is $\mathcal{U}'$-expansive. Suppose $\mathcal{W}$ is a filtrating Markov partition which is matching to $\mathcal{V}$. Then $\mathcal{W}$ is $\mathcal{U}'$-expansive (thus $\mathcal{U}$-expansive), too.

4.3. Local genericity of minimal expansive aperiodic classes. Now we are ready to identify a property which is type $\mathcal{P}_{\text{Min}}$.

Proposition 4.1. Let $M$ be a smooth closed connected three dimensional manifold. Let us consider the following $C^1$-robust property $\mathcal{P}_1$ on a filtrating set $\mathcal{U}$:

- $\mathcal{U}$ is a generating, transitive filtrating Markov partition.
- There is a periodic point $p \in \mathcal{U}$ such that the chain recurrence class $[p] \subset \mathcal{U}$ satisfies property (\ell).

Property $\mathcal{P}_1$ is type $\mathcal{P}_{\text{Min}}$. In other words, if $(f, \mathcal{U}, p)$ satisfies the condition above then for any $\delta > 0$, $N > 0$ and any $C^1$-neighborhood $O$ of $f$ there is $g \in O$ such that the following holds:

- $\text{supp}(g, f)$ is strictly contained in $\mathcal{U}$.
- There are disjoint transitive filtrating Markov partitions $\mathcal{U}_1, \mathcal{U}_2 \subset \mathcal{U}$ for $g$ which are both $\mathcal{U}$-minimal and $\mathcal{U}$-expansive.
- The minimum period of $\mathcal{U}_i$ is greater than $N$ for $i = 1, 2$.
- $\max\{|\text{diam}(\mathcal{U}_i), \text{diam}(g(\mathcal{U}_i))\} < \delta$ for $i = 1, 2$.
- There are periodic orbits $\mathcal{O}(p_i) \subset \mathcal{U}_i$ such that $[p_i] \subset \mathcal{U}_i$ satisfies (\ell) for $i = 1, 2$ (hence $(g, \mathcal{U}_i, p_i)$ satisfies property $\mathcal{P}_1$).

Proof. The proof consists of several steps.

Step 1. Preparation.

First, we fix $\varepsilon > 0$ such that every $10\varepsilon$-$C^1$-small perturbation of $f$ belongs to $O$. By letting $\varepsilon$ small, we may assume that the filtrating Markov partition $\mathcal{U}$ is $10\varepsilon$-robust. We also choose $D > 0$ such that every diffeomorphisms which is $10\varepsilon$-$C^1$-close to $f$ is $D$-Lipschitz continuous.
As the chain recurrence class \([p]\) satisfies property \((\ell)\), Theorem \ref{thm:expulsion} asserts that, by performing an arbitrarily small \(C^1\)-perturbation of \(f\) supported in \(\mathcal{U}\), we can produce two \(\varepsilon\)-flexible points \(p_1, p_2\) (which are not equal to \(p\)), with large stable manifolds in \(\mathcal{U}\) and such that the orbits of the points \(p_i, i = 1, 2\) meet every rectangle of \(\mathcal{U}\) (for the last property apply Corollary \ref{cor:largetime} and let \(\varepsilon\) very small). Note that we may assume that the period of \(p_i\) is larger than \(N\).

For \(i = 1, 2\), as \(p_i\) has a large stable manifold, it admits homoclinic orbits, and thus there are hyperbolic circuits \(K_i\) which consists of a the unique periodic orbit \(O(p_i)\) and a homoclinic orbit of \(p_i\). Now, we apply Theorem \ref{thm:expulsion} to \(K_1\) and \(K_2\). Thus up to an arbitrarily small perturbation we have that for sufficiently large \(m\) and \(n\) the refinement \(\mathcal{U}_{(m,n)}(K_i)\) is affine. By abuse of notation, we denote the perturbed diffeomorphism by \(f\) as well. Note that we can assume all of the perturbations has support which are strictly contained in \(\mathcal{U}\). Recall that in the setting of filtrating Markov partitions, the two notions “coincides outside \(\mathcal{U}\)” and “the support is strictly contained in \(\mathcal{U}\)” are synonymous, see Remark \ref{rem:coincideoutside}. Thus in the following we use these two phrases interchangeably.

**Step 2: Expulsion.**

Consider the (possibly non-filarting) transitive Markov partition \(\mathcal{U}_{(m,n)}(K_i)\). According to Lemma \ref{lem:minperiod} for \(m, n\) large enough the sub Markov partition \(\mathcal{U}_{(m,n)}(K_i)\) is \(\mathcal{U}\)-expansive for \(\mathcal{U}_{(m,n)}(K_i)\)-chains. Lemma \ref{lem:affinechains} guarantees that for large \(m, n\) the minimum period of \(\mathcal{U}_{(m,n)}(K_i)\) are larger than \(N\). Lemma \ref{lem:affinechains} shows that each rectangle of \(\mathcal{U}_{(m,n)}(K_i)\) has diameter less than \(\delta/D\). Note that by the choice of \(D\) we know that for every diffeomorphism \(f_0\) which is \(10\varepsilon\)-close to \(f\) and every rectangles \(U_i\) of \(\mathcal{U}_{(m,n)}(K_i)\), the diameter of its image under \(f_0\) is less than \((\delta/D) \cdot D = \delta\). Also, by Lemma \ref{lem:neighborhood} we choose a small neighborhood \(O_i\) of \(K_i\) and choose \(m, n\) such that \(\mathcal{U}_{(m,n)}(K_i) \subset O_i\), which guarantees the \(\mathcal{U}\)-minimality of \(\mathcal{U}_{(m,n)}(K_i)\). Note that \(\mathcal{U}_{(m,n)}\) is \(10\varepsilon\)-robust.

Now Theorem \ref{thm:expulsion} allows us to obtain a \(2\varepsilon\)-perturbation \(g = g_\nu\) of the diffeomorphism with circuits \(K_i\) similar to \(K_i\), supported in \(\mathcal{U}_{(m,n)}(K_1) \cup \mathcal{U}_{(m,n)}(K_2)\) such that \(g\) admits transitive, filtrating Markov partitions \(\mathcal{U} \subset \mathcal{U}_{(m,n)}(K_i) = \mathcal{U}_{(m,n)}(K'_i)\), \(i = 1, 2\) matching to \(\mathcal{U}_{(m,n+\nu)}(K'_i)\) (where \(\nu\) is some positive integer).

Remark \ref{rem:largeness} guarantees that we have the largeness of minimum period for \(\mathcal{U}\). The fact that the diameter of \(\mathcal{U}_{(m,n)}(K_i)\) is less than \(\delta/D\) shows that the same holds for \(\mathcal{U}\). Lemma \ref{lem:expansiveness} guarantees the \(\mathcal{U}\)-minimality of \(\mathcal{U}\).

Let us confirm the \(\mathcal{U}\)-expansiveness of \(\mathcal{U}\). Recall that \(\mathcal{U}_{(m,n)}(K_i) = \mathcal{U}_{(m,n)}(K'_i)\) is \(\mathcal{U}\)-expansive for both \(f\) and \(g\) (the \(\mathcal{U}\)-expansiveness for \(g\) is a consequence of the support, see Lemma \ref{lem:expansiveness}). By Lemma \ref{lem:expansiveness} we know that \(\mathcal{U}_{(m,n+\nu)}(K'_i)\) is \(\mathcal{U}\)-expansive for \(g\). Finally, Lemma \ref{lem:matching} concludes that \(\mathcal{U}\), which is a matching Markov partition of \(\mathcal{U}_{(m,n+\nu)}(K'_i)\), is \(\mathcal{U}\)-expansive for \(g\). Also, note that the matching property implies the generating property of \(\mathcal{U}\).

**Step 3: Recovery.**

The filtrating Markov partitions \(\mathcal{U}\) satisfies all the desired condition except property \((\ell)\). Let us perform the final perturbation to recover it. Recall that \(\mathcal{U}\) is \(10\varepsilon - 2\varepsilon = 8\varepsilon\)-robust (this is a consequence of Theorem \ref{thm:expulsion} and the fact \(\mathcal{U}_{(m,n)}\) is \(10\varepsilon\)-robust for \(f\)) and contains a periodic point \(p_i\) which is still \(\varepsilon\)-flexible and having a large stable manifold in \(\mathcal{U}\) (this is a consequence of Theorem \ref{thm:expulsion}). Furthermore, the homoclinic class of \(p_i\) in \(\mathcal{U}\) is not trivial, because it contains the circuit \(S_{i,r}\).

Now we apply Theorem \ref{thm:expulsion} to \(g\): Then we take a diffeomorphism \(h\) which is \(4\varepsilon\)-close
to $g$ such that the relative homoclinic class of $p_{i,h}$ in $\mathcal{U}_i$ satisfies property ($\ell_{\mathcal{U}_i}$). In particular, the chain recurrence class of $p_{i,h}$ in $\mathcal{U}_i$ satisfies property ($\ell$).

Since the supports of the perturbations for $\mathcal{U}_1$ and $\mathcal{U}_2$ are disjoint, we can perform the perturbation without any interference. Note that the structure of $\mathcal{U}_i$ as a filtrating Markov partition is preserved, so we can keep the transitivity, $\mathcal{U}$-expansiveness, smallness of the diameters and the largeness of the minimum period. Also, $\mathcal{U}$-minimality is preserved for the perturbation is local, see Lemma 3.6. Thus we constructed the desired filtrating Markov partitions $\mathcal{U}_i$ up to $6\varepsilon$-perturbations whose support is contained in $\mathcal{U}$. □

Remark 17. Let us see the usefulness of the concept of the stability under local perturbations. Consider the perturbation we made by Theorem 4 in Step 2 of above proof. We want to produce a new filtrating Markov partition keeping the minimality. We know that $\mathcal{U}_{(m,n)}(K_i)$ is $\mathcal{U}$-minimal and this property is $C^1$-robust. However, the size of the perturbation by Theorem 4 is $2\varepsilon$ and we are not sure if the $C^0$-robustness for the minimality is greater than that. We can shrink $\varepsilon$ as small as we want, but it would change the $C^1$-robustness of the minimality.

This brings a non-trivial problem, but if we know that the $\mathcal{U}$-minimality is stable under the local perturbations, we can circumvent it.

Note that if $(\mathcal{U}, p)$ satisfies property ($\ell$), then by taking refinements we obtain the assumption of Proposition 4.1. Thus Proposition 4.1 together with Proposition 3.4 implies item 1 of Theorem 1.

Theorem 7. Let $(\mathcal{O}, \mathcal{R}, p_f)$ be a $C^1$-open set of diffeomorphisms on a closed 3-manifold $M$ admitting a transitive filtrating Markov partition $\mathcal{R}$ and a periodic point $p_f$ varying continuously with $f$ such that $(f, \mathcal{R}, [p_f])$ satisfies the property ($\ell$).

Then there is a residual subset $\mathcal{G} \subset \mathcal{O}$ such that every $f \in \mathcal{G}$ has an uncountable set of chain recurrence classes which are all minimal and expansive.

4.4. Constructing an example. Theorem 4 gives an example of $C^1$-locally generic existence of $C^1$-diffeomorphisms having minimal, expansive aperiodic classes. This is a consequence of Baire’s category theorem and because of that there is a lack of concrete information about diffeomorphisms we obtained.

On the other hand, it is possible to construct a concrete example of such diffeomorphism by applying Proposition 4.1 successively. Let us see this. First, by applying Proposition 4.1 repeatedly one can prove the following (we omit the proof for it is straightforward):

Proposition 4.2. Let $M$ be a smooth closed connected three dimensional manifold. Let us consider $f \in \text{Diff}^1(M)$ which satisfies the assumption of Proposition 4.1. Then, one can construct a Cauchy sequence (in the $C^1$-distance) $(f_n)_{n \geq 1}$ and a nested sequence of transitive, generating filtrating Markov partitions $(\mathcal{U}_n)$ satisfying the following:

- $f_1 = f$.
- $\{(f_n, \mathcal{U}_n)\}_{n \geq 1}$ is a nested sequence for an aperiodic class.
- $\mathcal{U}_{n+1}$ is $\mathcal{U}_n$-expansive for chains with respect to $f_{n+1}$.
- $\mathcal{U}_{n+1}$ is $\mathcal{U}_n$-minimal for $f_{n+1}$.

Now, take a $C^1$-diffeomorphism $f_\infty = \lim_{n \to \infty} f_n$. By Proposition 3.2 and Proposition 4.3, we know that $\Lambda := \cap_{n \geq 1} \mathcal{U}_n$ is an expansive, minimal aperiodic
class. This construction is more concrete than the result of Theorem 7 since we have some information on $f_\infty$, for instance it coincides with $f$ outside $U_1$. This method would be useful, if one wishes to construct some peculiar example keeping control for some of the part of the diffeomorphism. Note that we can perform such a concrete construction since we have Proposition 3.2 and Proposition 3.3, which are stated in terms of sequence of maps $(f_n)$.

We can obtain the similar results in the later examples, but for the sake of simplicity we will not state them explicitly.

4.5. Ergodic diameters of sub Markov partitions. In this subsection, we discuss the size of the space of invariant measures supported on a special kind of sub Markov partition. We begin with a definition.

**Definition 4.1.** Given a (possibly non-filtrating) Markov partition $V$, its **ergodic diameter** $\delta_{\text{erg}}(V)$ is the diameter of the set of probability measures obtained as accumulation points of the $V$-pseudo orbits of points $(x_i) \subset V$, that is, the diameter of the closed set $P_\infty,\text{pseudo}(V)$, see Section 3.7.

**Lemma 4.10.** Consider a circuit of points $K$ of a diffeomorphism $f$ consisting of a unique periodic orbit $\gamma$ and a finite set of homoclinic orbits of $\gamma$. Let $U$ be a filtrating Markov partition containing $K$ such that $\gamma$ has a large stable manifold in $U$.

Then for every $n, m \geq 0$, $U_{(m,n)}(K)$ is a transitive Markov partition. Moreover, for any $\eta > 0$ there is $m_0, n_0 > 0$ such that for any $m \geq m_0, n \geq n_0$, the ergodic diameter of $U_{(m,n)}(K)$ is smaller than $\eta$:

$$\delta_{\text{erg}}(U_{(m,n)}(K)) < \eta.$$

**Proof.** $U_{(m,n)}(K)$ is a sub Markov partition of $U_{(m,n)}$. Let us see the transitivity of it. Every rectangle contains at least a point of $K$ which is chain recurrent. Therefore, for any $\varepsilon > 0$ one can find an $\varepsilon$-pseudo orbit in $K$ connecting any pair of connected components of $U_{(m,n)}(K)$. This defines a $U_{(m,n)}(K)$-chain of connected components between any pair of connected components.

Now, let us discuss the ergodic diameter. Recall that $U_{(m,n)}(K)$ converges to $K$ as $m, n \to \infty$, see Lemma 4.11. Now, for $m, n$ large enough, $U_{(m,n)}(K)$-pseudo orbits are $\varepsilon$-close to some $\varepsilon$-pseudo orbits of points in $K$, for $\varepsilon$ arbitrarily small. Note that a pseudo-orbit in $K$ has two types of orbit segments: One which follows $\gamma$ or the one which follows the homoclinic orbits. When $m, n$ are large and $\varepsilon$ is small, the former one tends to $\gamma$. Also, for the latter one the parts of the orbit which follows $\gamma$ will have an arbitrarily large portion. Thus, when $m, n$ are large then for every sufficiently large $k$ the measures $P_{k,\text{pseudo}}(U_{(m,n)}(K))$ are all very close to the Dirac measures supported on $\gamma$.

Therefore the ergodic diameter of $U_{(m,n)}(K)$ is arbitrarily small for every sufficiently large $m$ and $n$. □

**Remark 18.** If $U, V$ are possibly non-filtrating Markov partitions such that $V$ is matching to $U$, then for every $n$ we have

$$P_{k,\text{pseudo}}(V) \subset P_{k,\text{pseudo}}(U).$$

If $V$ contains a periodic orbit, then we know that $P_{k,\text{pseudo}}(V) \neq \emptyset$. Thus we have

$$\delta_{\text{erg}}(V) \leq \delta_{\text{erg}}(U).$$
4.6. Local genericity of minimal aperiodic classes supporting infinitely many ergodic measures. Recall that we have fixed \( \lambda_0 > 0 \) satisfying \( \prod_{i=1}^{\infty} (1 + \lambda_0^i) > \frac{1}{2} \), see Section 3.5. Also, recall that \((\ell_V)\) is a version of property \((\ell)\) for a relative homoclinic class in \(V\), see Section 2.2.

The aim of this section is to prove the following.

Proposition 4.3. Let us consider the following family of \(C^1\)-robust properties \((P^n)_{n\geq 1}\) for a \(C^1\)-diffeomorphism \(f\) having a transitive filtrating Markov partition \(U\) containing periodic orbits \(\Gamma_n := \{\gamma_i\}_{i=1,\ldots,n}\). There is \((m_0,n_0)\) such that \(U_{(m_0,n_0)}\) has \(n\) mutually distinct sub Markov partitions \((W_i)_{i=1,\ldots,n}\) such that

(A) \(\gamma_i \subset W_i\) and \((f,W_i,p_i)\) satisfies property \((\ell_{W_i})\), where \(p_i\) is a point of \(\gamma_i\).
(B) \(\delta_{\text{erg}}(W_i) < \lambda_0^{n+1} \rho(M_n)\) where \(M_n = \{\mu_1,\ldots,\mu_n\}\) is the set of Dirac measures supported on the orbits of \(\Gamma_n\) and \(\rho(\cdot)\) denotes the independence radius, see Section 3.5.
(C) For any connected component \(C\) of \(U\) one has \(C \cap \mathcal{W}_i \neq \emptyset\).

This family of properties \((P^n)\) is type \(\mathcal{P}_{\mathbb{M},\infty}\) (see Definition 3.8), that is, given any \(C^1\)-neighborhood \(O\) of \(f\) and for any \(\delta > 0\) and \(N > 0\) there is \(g \in O\) such that the following holds:

1. The support \(\text{supp}(g,f)\) is strictly contained in \(U\).
2. There are disjoint, transitive filtrating Markov partitions \(U_1, U_2 \subset U\) which are both \(U\)-minimal for chains (with respect to \(g\)).
3. For \(g\), the continuations \(\{\gamma^g_i\}\) are defined and \(\{\gamma^g_i\}\) are outside \(U_1 \cup U_2\).
4. \(\text{cdiam}(U_j), \text{cdiam}(g(U_j)) < \delta\) and the minimum period of \(U_j\) is larger than \(N\) for \(j = 1, 2\).
5. There are periodic orbits \(\gamma^g_i \subset U_j, j \in \{1, 2\}, i \in \{1, \ldots, n + 1\}\).
6. For any \(j \in \{1, 2\}\) and \(i \in \{1, \ldots, n\}\) one has
   \[\delta(\mu^g_i, \mu^f_i) < \lambda_0^{n+1} \rho(\{\mu_i^n \mid i = 1, \ldots, n\})\]
   where \(\mu^g_i, \mu^f_i\) are the Dirac probabilities associated to \(\gamma^g_i\) and \(\gamma^f_i\), respectively.
7. For each \(j = 1, 2\), there is a pair of integers \((m_j, n_j)\) such that in the refinements \(U_{(m_j,n_j)}\) there are disjoint transitive sub Markov partitions \(W^g_j\) of \(U_{(m_j,n_j)}\), \((i = 1, \ldots, n+1)\) which satisfies the following:
   - There are periodic orbits \((p^g_i) \subset W^g_j\) \((i = 1, \ldots, n + 1)\) such that \((g,W^g_j,p^g_i)\) satisfies the property \((\ell_{W^g_j})\).
   - \(\delta_{\text{erg}}(W^g_j) < \lambda_0^{n+2} \rho(M^g_{n+1})\) where \(M^g_{n+1} = \{\mu^g_1,\ldots,\mu^g_{n+1}\}\) is the set of Dirac measures supported on the orbits of \(p^g_i\).
   - For any connected component \(C\) of \(U_j\) one has \(C \cap W^g_j \neq \emptyset\) for every \(j = 1, 2\) and \(i = 1, \ldots, n + 1\).

Note that condition (7) guarantees that \((g,U_j,\{\gamma^g_i\}_{1 \leq i \leq n+1})\) satisfies property \(P^{n+1}\).

Proof. We choose \(\varepsilon > 0\) such that any \(10\varepsilon\)-perturbation of \(f\) belongs to \(O\) and the filtrating Markov partition \(U\) is \(10\varepsilon\)-robust.

In the course of the proof, the confirmations of conditions (1, 2, 4) are easy or similar in the proof of Proposition 4.1. Thus we avoid the detailed explanation of them and concentrate on how to obtain conditions (3, 5, 6, 7).

**Step 1: Preparation.**
As \( f \) satisfies \((\ell_\mathcal{W})\) in \( \mathcal{W}_i \), according to Theorem \[\text{[6]}\] there is an arbitrarily \( C^1\)-small perturbation \( f_0 \) of \( f \) supported in \( \bigcup \mathcal{W}_i \) such that there are \( \varepsilon \)-flexible points whose orbits are \( \varepsilon \)-dense in the relative homoclinic class of \( p_i^{f_0} \) in \( \mathcal{W}_i \), \( i = 1, \ldots, n \). We choose \( f_0 \) such that assumptions (A), (B) and (C) still hold (note that a local perturbation on a filtrating Markov partition does not change the ergodic diameter, see Remark \[\text{[7]}\]).

This allows us to choose a family \( p_i^{f_j}, j = 1, 2, i = 1, \ldots, n + 1 \) with the following properties.

- All the points \( p_i^{f_1}, j = 1, 2, i = 1, \ldots, n \) are \( \varepsilon \)-flexible and have large stable manifolds in \( \mathcal{W}_i \) (thus in \( \mathcal{U} \)).
- \( p_i^{f_{n+1}} \) is \( \varepsilon \)-flexible and has a large stable manifold in \( \mathcal{U} \) (for that we only need to choose one from one of \( \mathcal{W}_i \)).
- As the consequence of assumption (C), \( p_i^{f_1} \) are pairwise homoclinically related in \( \mathcal{U} \). Also, each \( p_i^{f_j} \) has homoclinic point to itself.
- For \( i = 1, \ldots, n \), the orbits of \( p_i^{f_j} \) are contained in \( \mathcal{W}_i \) and meet every connected component of \( \mathcal{U} \).

Thus we chose two circuits \( K_j \) \( (j = 1, 2) \) consisting of the following objects:

- Periodic orbits \( \{O(p_i^{f_j})\}, i = 1, \ldots, n + 1 \).
- Heteroclinic orbits between \( p_i^{f_1} \) and \( p_i^{f_{n+1}} \) for \( i \neq i' \).
- Homoclinic orbits for each \( p_i^{f_j} \).

We apply Theorem \[\text{[8]}\] to \( K_j \) such that for sufficiently fine refinement \( \mathcal{U}' \), the sub Markov partition \( \mathcal{U}'(K_j) \) are affine for \( j = 1, 2 \). We denote the perturbed map by \( f_1 \).

Then we apply Lemma \[\text{[3, 10]}\] for \( K_j \): We choose a small neighborhood \( O_j \) of \( K_j \) satisfying the conclusion for \( j = 1, 2 \). Then for every sufficiently large \( m \) and \( n \) we have \( \mathcal{U}'_{(m,n)}(K_j) \) is a subset of \( O_j \). Note that we may assume every connected component of them has a small diameter by letting \( m, n \) large. In particular, we may assume that \( \mathcal{U}'_{(m,n)}(K_1), \mathcal{U}'_{(m,n)}(K_2) \) are disjoint.

**Step 2. Expulsion of circuits.**

According to Theorem \[\text{[3]}\] there is \( g \) which is \( 2\varepsilon \)-\( C^1 \)-close to \( f_1 \) such that the support of \( g \) is strictly contained in \( \mathcal{U}'_{(m,n)}(K_1) \cup \mathcal{U}'_{(m,n)}(K_2) \), there are transitive filtrating Markov partitions \( \mathcal{U}_j \subset \mathcal{U}'_{(m,n+\nu)}(K_j) \) (where \( \nu \) is some positive integer) and circuits \( K'_j \subset \mathcal{U}_j \) \( (j = 1, 2) \) with the following properties:

- The periodic \( g \)-orbits of the circuit \( K'_j \) are the \( f_0 \)-periodic orbits \( p_i^{f_1}, j = 1, 2, i = 1, \ldots, n + 1 \).
- \( p_i^{f_1} \) \( (j = 1, 2, i = 1, \ldots, n + 1) \) are still \( \varepsilon \)-flexible and have large stable manifolds in \( \mathcal{U}_j \).
- \( \mathcal{U}_j \subset \mathcal{U}'_{(m,n+\nu)}(K'_j) \) are matching and they are \( 8\varepsilon \)-robust.

Notice that by the choice of \( \mathcal{U}'_{(m,n)}(K'_j) \), the filtrating Markov partitions \( \mathcal{U}_1 \) and \( \mathcal{U}_2 \) are both \( \mathcal{U} \)-minimal.

As the periodic orbits \( p_i^{f_j} \) for \( f_1 \) and \( g \) coincide and for \( i = 1, \ldots, n \) these orbits are contained in \( \mathcal{W}_i \), we have

\[
\mathcal{O}(\mu_i^{f_1}, \mu_i^{g}) < \lambda_0^{n+1} \rho(\{p_i^{f_i} \mid i = 1, \ldots, n\}) \text{ for } i = 1, \ldots, n \text{ and } j = 1, 2
\]

where \( \mu_i^{f_1} \) is the Dirac measure supported on the orbit of \( p_i^{f_1} \).
Step 3: Recovery.

It remains to recover properties in (7). For each $p_t^i$, since $U_j$ is transitive, we can find a homoclinic orbit of it such that it visits every rectangle of $U_j$, see Lemma 4.3. Consider a circuit $S_{i,j}$ which consists of the periodic orbit $O(p_t^i)$ and such a homoclinic orbit. Then consider $U_{j,(m,n)}(S_{i,j})$. Since $p_t^i$ has a large stable manifold in $U_j$, by Lemma 4.10 for every sufficiently large $m,n$ we have

$$\delta_{\text{erg}}(U_{j,(m,n)}(S_{i,j})) < \lambda_0^{n+2}\rho(M_{n+1}).$$

Furthermore, by definition of $S_{i,j}$ for any connected component $C$ of $U_j$ one has

$$U_{j,(m,n)}(S_{i,j}) \cap C \neq \emptyset.$$

We fix sufficiently large $m,n$ and put $W_i^j := U_{j,(m,n)}(S_{i,j})$. Now Theorem 5 asserts that there is a $4\varepsilon$-perturbation $\tilde{h}$ of $g$ with support strictly contained in $W_i^j$ having property $(\ell_{W_i^j})$. Note that they are disjoint if $m,n$ are sufficiently large, considering the fact that $p_t^i$ has large stable manifold. Also, this perturbation keeps the structure of the Markov partitions $U_j$ and $\{W_i^j\}$, and does not destroy the properties we obtained. Thus we have the conclusion. \qed

To obtain item 2 of Theorem 1 we are left to show that property $(\ell)$ implies $P^1$. This can be observed as follows:

- If $(U,p)$ satisfy property $(\ell)$, then by adding arbitrarily small perturbation we may assume that there is a hyperbolic periodic point $p_0$ homoclinically related to $p$, having large stable manifold, $\varepsilon$-flexible for very small $\varepsilon$ and visits every rectangle of $U$. 
- Consider a circuit $K_0$ consisting of $p_0$ and a homoclinic point. By Theorem 5 we may assume that $(p_0,U_{(m,n)}(K_0))$ satisfy property $\ell_{U_{(m,n)}(K_0)}$ up to some small perturbation.

Thus, by Proposition 4.3 and Proposition 5.7 we have the following:

**Theorem 8.** Let $O$ be a $C^1$-open set of Diff$^1(M)$ (where $M$ is a closed 3-manifold) such that every $f \in O$ admits a transitive filtrating Markov partition having a hyperbolic periodic point $p_f$ varying continuously with $f$ over $O$ and $(f,U,p_f)$ satisfies the property $(\ell)$. Then there is a residual subset $R \subset O$ such that every $f \in R$ has an uncountable set of aperiodic chain recurrence classes which are all minimal but are supporting infinitely many ergodic measures.

### 4.7. $C^1$-genericity of transitive non-minimal aperiodic classes.

In this subsection, we give an example of a property of dynamical systems which is type $\mathcal{P}_{\text{TAM,AB}}$. We begin with a result about filtrating Markov partitions.

**Lemma 4.11.** Let $U$ be a filtrating Markov partition and $K \subset \bar{U}$ be a circuit such that every periodic point has a large stable manifold. Assume that there is a point $p \in K$ such that for every connected component $C$ of $U$ there are $i < 0 < j$ such that $f^i(p), f^j(p) \in C$. Then, for every sufficiently large $m,n$, $(U_{(m,n)},U_{(m,n)}(p))$ is stably $U$-transitive for local perturbations.

**Proof.** Let us confirm the second condition of Definition 3.9 and the assumption of Lemma 3.15 see Remark 9.
Since \( \mathcal{U} \) is transitive, we deduce that every rectangle of \( \mathcal{U} \) contains at least one rectangle of \( \mathcal{U}_{(m,n)} \) and it is in its interior if \( m, n \geq 1 \). This implies the second condition of Definition 3.9. Now, let us confirm the assumption of Lemma 3.15. Consider a connected component \( A \) of \( \mathcal{U} \). By definition, there is \( i^+(A) \) such that \( f^{i^+(A)}(p) \in \hat{A} \).

We denote the distance between \( \{ f^{i^+(A)}(p) \} \) and \( M \setminus \hat{A} \) by \( \delta_j \), where \( M \) is the ambient manifold. For each \( f^j(p) \), \( j = 0, \ldots, i^+(A) \), we choose a sufficiently small compact neighborhood \( O_j \) such that the distance between \( f^{i^+(A)-j}(O_j) \) and \( M \setminus \hat{A} \) is larger than \( \delta/2 \).

The assumption that every periodic orbit of \( K \) has a large stable manifold implies that as \( m, n \to \infty \) the Markov partition \( \mathcal{U}_{(m,n)}(K) \) satisfies \( \text{diam} \mathcal{U}_{(m,n)}(K) \to 0 \), see Lemma 4.11. Thus, when \( m, n \) are large then for every \( \mathcal{U}_{(m,n)}(K) \)-chain of connected components \( (U_i)_{i=0,\ldots,i_0} \) where \( i_0 \leq i^+(A) \), we have \( U_{i_0} \subset O_{i_0} \), since when each connected component is small then the chain of connected components is almost equal to the true orbit. Accordingly we have \( f^{i^+(A)-i_0}(U_{i_0}) \) and \( \mathcal{U}_{(m,n)} \mathcal{U}_{(m,n)}(K) \to 0 \).

We can obtain the sufficient condition for the stable \( \mathcal{U} \)-transitivity in Lemma 3.15 for a fixed connected component \( A \) by taking sufficiently fine refinement.

The same argument holds for backward iterations. Since the number of the connected components of \( \mathcal{U} \) is finite, by choosing \( m, n \) for each connected component and letting \( m, n \) larger than all of them, we obtain the desired refinement. \( \square \)

**Lemma 4.12.** Let \( \mathcal{U}, \mathcal{V} \) be a filtrating Markov partition of \( f \in \text{Diff}^1(M) \) such that \( \mathcal{V} \subset \mathcal{U} \) and \( (\mathcal{V}, V_0) \) is stably \( \mathcal{U} \)-transitive. If \( \mathcal{W} \) is a matching Markov partition of \( \mathcal{V} \), then \( (\mathcal{W}, W_0) \) is stably \( \mathcal{U} \)-transitive, too, where \( W_0 \) is the unique rectangle of \( \mathcal{W} \) contained in \( V_0 \).

**Proof.** Let \( A \) be any connected component of \( \mathcal{U} \). Then there is a connected component \( C \) of \( \mathcal{V} \) such that \( C \subset \hat{A} \). Since \( \mathcal{W} \) is matching to \( \mathcal{V} \), there is a connected component \( C' \) of \( \mathcal{W} \) contained in \( C \). Thus, \( C' \subset \hat{A} \) and this shows the second condition of Definition 3.9.

Let us check the first condition for \( g \) whose support is contained in \( \mathcal{W} \). Since \( \mathcal{V} \) is stably \( \mathcal{U} \)-transitive, there is \( i^+(A) \) such that \( g^{i^+(A)}(V_0) \subset \hat{A} \), which implies \( g^{i^+(A)}(W_0) \subset \hat{A} \). The confirmation for the backward iteration is similar. \( \square \)

Let us prove the main result.

**Proposition 4.4.** Let \( f \) be a diffeomorphism of a closed 3-manifold \( M \). Let \( A, B \subset M \) be two disjoint compact subsets.

Consider the following \( C^1 \)-robust property \( \mathcal{P}_3 \) for a filtrating set \( \mathcal{U} \):

- \( \mathcal{U} \) is a transitive filtrating Markov partition whose rectangles are either disjoint from \( A \) (resp. \( B \)) or included in \( \hat{A} \) (resp. \( \hat{B} \)). Thus we can define the restriction of Markov partition \( \mathcal{U}|_A \) (resp. \( \mathcal{U}|_B \)), which consists of rectangles contained in \( A \) (resp. \( B \)), see Definition 3.10.

- There is a periodic point \( p \in \mathcal{U} \) such that \( (f, \mathcal{U}, p) \) satisfies property (f).

- The restriction of \( \mathcal{U} \) to \( A \) (resp. \( B \)) contains a transitive sub Markov partition and it satisfies property \( (t_{\mathcal{U}|_A}) \) (resp. \( (t_{\mathcal{U}|_B}) \)).

This property is type \( \mathcal{P}_{\text{Trans}}, \) that is, if \( f \) satisfies these conditions for \( (\mathcal{U}, p) \), then, for any \( \delta > 0, N > 0 \) and any \( C^1 \)-neighborhood \( O \) of \( f \) there is \( g \in O \) such that:

- \( \text{supp}(g, f) \) is strictly contained in \( \mathcal{U} \).
• There are disjoint transitive filtrating Markov partitions $U_1, U_2$ whose minimum periods are larger than $N$.
• $U_1, U_2 \subset U$ are both stably $U$-transitive for local perturbations with respect to $g$.
• $\text{cdiam}(U_i), \text{cdiam}(U_i) < \delta$ for $i = 1, 2$.
• $U_1, U_2$ satisfy the property $P_3$, that is:
  – The restrictions of $U_i$ ($i = 1, 2$) to $A$ (resp. $B$) is well defined.
  – $U_i$ satisfies property $(\ell)$.
  – $\ell_{U_i|A}$ (resp. $\ell_{U_i|B}$) contains a transitive sub Markov partition satisfying property $(\ell_{U_i|A})$ (resp. $(\ell_{U_i|B})$).

Proof. Similar to the proofs of Proposition 4.1 and Proposition 4.3, we divide the proof into three steps. Again the arguments for obtaining the smallness of the diameter of new Markov partitions and the largeness of the minimum periods are almost the same, so we keep the explanation of them short.

Step 1: Preparation.
We fix $\varepsilon > 0$ such that every diffeomorphism which is $10\varepsilon$-$C^1$-close to $f$ is contained in $O$. Also, we assume that the filtrating Markov partition $U$ is $10\varepsilon$-robust.

First, Theorem 6 allows us to perform an arbitrarily small perturbation such that there are six $\varepsilon$-flexible points $p_0^i, p_A^i, p_B^i$ ($i = 1, 2$) with large stable manifolds in $U$ satisfying the following:

• The orbit of $p_0^i$ meets every rectangles of $U$.
• The orbit of $p_A^i$ is contained in $A$. There is a homoclinic orbit $x_A^i$ of $p_A^i$ whose orbit is contained in $A$.
• The same condition holds for $p_B^i, B$ and a homoclinic orbit $x_B^i$.

Let us see how to take such points. To take $p_A^i$, we apply Theorem 9 to the transitive Markov partition satisfying property $(\ell_{U|A})$ in the hypothesis. Then using the largeness of the stable manifold of $p_A^i$ we can choose a homoclinic point $x_A^i$. The construction of $p_B^i$ and $x_B^i$ can be done similarly. For the choice of $p_0^i$ apply Theorem 8 to the transitive Markov partition $U$. By abuse of notation we denote the perturbed map by $f$. Note that we may assume that these six periodic points have large periods.

Note that the largeness of the stable manifolds of $p_0^i$ and $p_A^i, p_B^i$ implies that there are heteroclinic orbits connecting among them. We choose such heteroclinic orbits and denote them by $y_{0A}^i, y_{A0}^i, y_{0B}^i$ and $y_{B0}^i$ (where $y_{0A}^i$ is a heteroclinic orbit from $p_0^i$ to $p_A^i$, etc...). Then we take two disjoint circuits $K_i$ ($i = 1, 2$) such that:

• Periodic points of $K_i$ are $p_0^i, p_A^i, p_B^i$.
• Homoclinic orbits are $x_A^i$ and $x_B^i$.
• Heteroclinic orbits are $y_{0A}^i, y_{A0}^i, y_{0B}^i, y_{B0}^i$.

Note that we have sub circuits $K_A^i$ (resp. $K_B^i$) of $K_i$ which consists of a periodic point $p_A^i$ (resp. $p_B^i$) and a homoclinic orbit $x_A^i$ (resp. $x_B^i$) which are contained in $A$ (resp. $B$).

For each $K_i$, we apply Theorem 8. Then by an arbitrarily small perturbation we may assume that for every large $m$ and $n$ the Markov partition $U_{(m,n)}(K_i)$ is affine.
Note that $p_0^i \in \mathcal{U}_{(m,n)}(K_i)$ for every $m$ and $n$. Thus for every sufficiently large $m$ and $n$ we can apply Lemma 4.11. We fix such large $m$ and $n$. We also assume that $\text{cdiam}(\mathcal{U}_{(m,n)}(K_i))$ is very small.

**Step 2: Expulsion.**

Now Theorem 4.1 allows us to take a $4\varepsilon$-perturbation $g$ of the diffeomorphism supported in $\mathcal{U}_{(m,n)}(K_1) \cup \mathcal{U}_{(m,n)}(K_2)$ such that $g$ admits transitive filtrating Markov partitions $\mathcal{U}_i \subset \mathcal{U}_{(m+n)}(K_i)$ (where $\nu$ is some positive integer) with the following properties:

- $\mathcal{U}_i$ is matching to $\mathcal{U}_{(m,n+\nu)}(K_i)$.
- There are circuits $K'_i \subset \mathcal{U}_i$ which are similar to $K_i$ for $i = 1, 2$.
- The periodic orbits of $K'_i$ coincide with the those of $K_i$.
- Every periodic point of $K'_i$ is $\varepsilon$-flexible and has a large stable manifold in $\mathcal{U}_i$.
- $\mathcal{U}_i$ is stably $\mathcal{U}$-transitive for local perturbations.

**Step 3: Recovery.**

Note that for $\mathcal{U}_i$ we can define restrictions to $A$ and $B$, for it is a subset of $\mathcal{U}_{(m,n)}$. We apply Theorem 5 to $\mathcal{U}_i|_A$ and $\mathcal{U}_i|_B$ (note that these perturbations do not interfere, for the supports are disjoint): As $\mathcal{U}_i \cap A$ and $\mathcal{U}_i \cap B$ contain sub circuit containing a periodic point and a homoclinic point, one deduces that $\mathcal{U}_i \cap A$ and $\mathcal{U}_i \cap B$ contain a non-trivial transitive sub Markov partition satisfying property $(\ell_{u_i}|_A)$ and $(\ell_{u_i}|_B)$ up to a $4\varepsilon$-perturbation. Note that it implies that $\mathcal{U}_i$ satisfies property $(\ell)$.

By the choice of $m$ and $n$, Lemma 4.12 and Lemma 4.13 imply that $\mathcal{U}_i$ is stably $\mathcal{U}$-transitive. Thus, $\mathcal{U}_i$ satisfy all the announced properties, and it completes the proof.

Now to conclude the item 3 of Theorem 1 we only need to prove the following:

**Lemma 4.13.** Suppose that a chain recurrence class $[p]$, where $p$ is a hyperbolic periodic point of a $C^1$-diffeomorphism $f$ satisfies property $(\ell)$. Then, there is a $C^1$-open neighborhood $O$ of $f$ such that there is a transitive filtrating Markov partition $\mathcal{U}$ and disjoint compact set $A$, $B$ such that $(f, \mathcal{U})$ satisfies the assumption of Proposition 4.4.

**Proof.** For the proof, we only need to repeat the proof of Proposition 4.4. Let us explain this.

First, by assumption we know that $[p]$ contains a non-trivial homoclinic class in a filtrating Markov partition which we denote by $\mathcal{U}$. Thus by Theorem 5 up to an arbitrarily small perturbation there are three different $\varepsilon$-flexible periodic points $p_0, p_A, p_B$ with large stable manifolds where $\varepsilon$ can be chosen arbitrarily small, in particular, we may assume that $\mathcal{U}$ is $10\varepsilon$-robust. Since they are homoclinically related and their homoclinic class is non-trivial, we may assume that

- $p_A, p_B$ has a homoclinic point $x_A, x_B$ respectively.
- $p_A, p_0$ are homoclinically related with the heteroclinic orbits $y_{A0}, y_{0A}$.
- $p_B, p_0$ are homoclinically related with the heteroclinic orbits $y_{B0}, y_{0B}$.

Then consider the circuit with periodic points $p_A, p_B, p_0$ and homo/heteroclinic orbits $x_A, x_B, y_{A0}, y_{0A}, y_{BA}$ and $y_{0B}$ and denote it by $K$. We denote the sub circuit consisting of $p_A$ and $x_A$ by $K_A$, and $p_B$ and $x_B$ by $K_B$. Since every periodic point of $K$ has a large stable manifold, by Lemma 4.1 we have $\mathcal{U}_{(m,n)}(K)$ is very close
to $K$. Thus choosing $m$ and $n$ very large we may assume that $U_{(m,n)}(K_A)$ and $U_{(m,n)}(K_B)$ are disjoint.

Now we apply Theorem 4 to $K$ and $U_{(m,n)}(K)$: Then by $2\varepsilon$-perturbation we obtain a diffeomorphism $g$ such that there is a transitive filtrating Markov partition $U'$ containing $K_g$ (the continuation of $K$). Note that $U'(K_A), U'(K_B)$ are disjoint sub Markov partitions. Thus we may choose two disjoint compact sets $A, B$ such that $U'(K_A) \subset A$ and $U'(K_B) \subset B$.

Now we apply Theorem 4 to $U'(K_A)$ and $U'(K_B)$: Up to a $4\varepsilon$-perturbation we have that $U'(K_A)$ and $U'(K_B)$ satisfy property $(\ell_{U'(K_A)})$ and $(\ell_{U'(K_B)})$ respectively, having almost the same circuit $K_g$. Note that at this moment $(U', p_0)$ satisfies property $(\ell)$. Thus letting $A$ and $B$ as above we obtain the conclusion. □

4.8. $C^1$-genericity of non-transitive aperiodic classes. In this subsection, we complete the proof of the locally generic existence of non-transitive uniquely ergodic aperiodic classes.

Proposition 4.5. Let $f$ be a diffeomorphism of a closed three manifold $M$ and $A, B$ be disjoint compact subsets of $M$. Consider the following $C^1$-robust property $P_4$ for a filtrating set $U$:

- $U$ is a transitive filtrating Markov partition having unique rectangles contained in $A$ and $B$ which we denote by $A_0$ and $B_0$ respectively.
- There is a circuit of points $L$ which consists of one periodic orbit $O(p)$ with a large stable manifold in $U$ and three homoclinic orbits of $p$ which we denote by $\gamma_j, j \in \{0, 1, 2\}$.
- $L \cap A_0, L \cap B_0$ are singletons. We denote them by $x_1$ and $x_2$ respectively. We have $x_i \in \gamma_i$ for $i = 1, 2$.
- Let $L_0$ be the sub circuit of $L$ consisting of $O(p)$ and $\gamma_0$. Then $(f, U(L_0))$ satisfies property $(\ell_{U(L_0)})$.

This property $P_4$ is type $P_{2T, A, B}$ (see Proposition 3.9). Namely, for any $\delta_1 > 0, \delta_2 > 0, N > 0, M > 0$ and any $C^1$-neighborhood $O$ of $f$ there is $g \in O$ such that:

- supp$(g, f)$ is strictly contained in $U$.
- There are disjoint, transitive filtrating Markov partitions $U_i \subset U$ $(i = 1, 2)$ whose minimum periods are larger than $N$.
- $\max\{|\text{diam}(U_i), \text{diam}(g(U_i))| < \delta_1$ for $i = 1, 2$.
- The ergodic diameter of $U_i$ (see Section 4.5) is less than $\delta_2$ for $i = 1, 2$.
- For $i = 1, 2$, the rectangles $A_0$, $B_0$ of $U$ contain exactly one rectangle of $U_i$, which we denote by $A_i^0$ and $B_i^0$ respectively.
- Any $U_i$-chain of components of length $M$ starting from or ending at $A_i^0$ does not contain $B_i^0$ for $i = 1, 2$.
- $U_i$ satisfies the property $P_4$ for $i = 1, 2$. Namely,
  - There is a circuit $L_i$ consisting of one periodic orbit $O(p^i)$ and three homoclinic orbits $\gamma_j^i (j = 0, 1, 2)$ of $O(p^i)$.
  - $L_i \cap A_0, L_i \cap B_0$ are singletons. We denote them by $x_1^i$ and $x_2^i$. We have $x_i^j \in \gamma_j^i$ for $j = 1, 2$.
  - Let $L_0^i$ be the sub circuit of $L^i$ consisting of $O(p^i)$ and $\gamma_0^i$. Then $(g, U_i(L_0^i))$ satisfies property $(\ell_{U_i(L_0^i)})$.

Proof. We choose $\varepsilon > 0$ such that any $10\varepsilon$-perturbation of $f$ belongs to $O$ and the filtrating Markov partition $U$ is $10\varepsilon$-robust.
Step 1: Preparation.

As $f$ satisfies property $(f_{\mathcal{U}(L_0)})$, by Theorem 6 we know there is an arbitrarily small perturbation of $f$ (still denoted by $f$) having two distinct $\varepsilon$-flexible periodic points $p^i$ ($i = 1, 2$) whose orbits are contained in $\mathcal{U}(L_0)$, with large stable manifolds in $\mathcal{U}$. Note that by assumption, $\mathcal{U}(L_0)$ does not contain $A_0$ and $B_0$.

Let us show that there is a circuit $K_i$ consisting of the orbit of $p^i$ and 3 homoclinic orbits of $p^i$ as follows:

- $\gamma^i_0$: whose orbit is disjoint from $A_0$ and $B_0$,
- $\gamma^i_1$: whose orbit is disjoint from $B_0$ and passes $A_0$,
- $\gamma^i_2$: whose orbit is disjoint from $A_0$ and passes $B_0$.

The existence of $\gamma^i_0$ is just a consequence of the non-triviality of the relative homoclinic class of $p^i$ in $\mathcal{U}(L_0)$.

Let us see how to find $\gamma^i_1$. The existence of the homoclinic orbit $\gamma_1$ in the assumption tells us there is a $\mathcal{U}$-chain of points starting from $\mathcal{U}(p^i)$, passing $A_0$ only once, ending at $\mathcal{U}(p^i)$, and do not pass $B_0$. Thus by Lemma 4.3 and Remark 15 (note that the orbit of $p^i$ does not pass $A_0$), there is a homoclinic orbit $\gamma^i_1$ of $p^i$ which has such an itinerary. This gives us the desired homoclinic orbit. Similarly, we can find $\gamma^i_2$. We put $A_0 \cap \gamma^i_1 = x^i_1$ and $B_0 \cap \gamma^i_2 = x^i_2$ for $i = 1, 2$.

Now we apply Theorem 3 to $K_i$. Up to an arbitrarily small perturbation, for every sufficiently large $m$ and $n$ we have that $\mathcal{U}_{(m,n)}(K_i)$ are affine. By abuse of notation, we denote the perturbed diffeomorphism by $f$, too. Since $\text{cdiam}(\mathcal{U}_{(m,n)}(K_i)) \to 0$ as $m, n \to \infty$ by Lemma 4.11 we know that given $N > 0, M > 0, \delta_1 > 0$ and $\delta_2 > 0$ by choosing $m$ and $n$ large we have the following conditions:

- $\text{cdiam}(\mathcal{U}_{(m,n)}(K_i)), \text{cdiam}(g(\mathcal{U}_{(m,n)}(K_i))) < \delta_1$ for every $g$ which is $10\varepsilon$-$C^1$-close to $f$.
- The minimum period of $\mathcal{U}_{(m,n)}(K_i)$ is larger than $N$ for $i = 1, 2$.
- Each $\mathcal{U}_{(m,n)}(K_i)$-chain of components of length $M$ starting from or ending at $\mathcal{U}_{(m,n)}(x^i_1)$ does not contain the rectangle $\mathcal{U}_{(m,n)}(x^i_2)$.
- $\delta_{\text{erg}}(\text{cdiam}(\mathcal{U}_{(m,n)}(K_i))) < \delta_2$ (see Lemma 4.10).

Let us explain how to obtain the third condition. If $m, n$ are large then $\mathcal{U}_{(m,n)}(K_i)$ is very close to $K_i$. Thus every $\mathcal{U}_{(m,n)}(K_i)$-chain of connected components starting from $\mathcal{U}_{(m,n)}(x^i_1)$ needs to be long to reach $\mathcal{U}_{(m,n)}(O(p^i))$. Especially, it should be long to reach $\mathcal{U}_{(m,n)}(x^i_2)$ as well.

Step 2: Expulsion.

Now Theorem 4 allows us to perform a $2\varepsilon$-perturbation $g$ of $f$ supported in $\mathcal{U}_{(m,n)}(K_i) \cup \mathcal{U}_{(m,n)}(K_2)$ such that $g$ admits filtrating transitive Markov partitions $\mathcal{U}_i \subset \mathcal{U}_{(m,n+\nu)}(K_i)$, where $i = 1, 2$ and $\nu$ is some positive integer, with the following properties:

- $\mathcal{U}_i$ is matching to $\mathcal{U}_{(m,n+\nu)}(K_i)$ and $8\varepsilon$-robust.
- There are circuits $K'_i \subset \mathcal{U}_i$ which is similar to $K_i$ and whose periodic orbits coincide with $O(p^i)$ for $i = 1, 2$.
- $p^i$ is $\varepsilon$-flexible and has a large stable manifold in $\mathcal{U}_i$ for $i = 1, 2$.

Note that the three conditions for $\mathcal{U}_{(m,n)}(K_i)$ (smallness of the connected components, non-existence of chains and the smallness of the ergodic diameters) inherits to $\mathcal{U}_i$.

Since $\mathcal{U}_i$ is matching to $\mathcal{U}_{(m,n+\nu)}(K_i)$ and $\mathcal{U}_{(m,n+\nu)}(K_i)$ contains unique rectangle in $A_0$ (resp. $B_0$), there is a unique rectangle of $\mathcal{U}_i$ contained in $A_0$ (resp. $B_0$) which
we denote by $A_i^1$ (resp. $B_i^1$). Note that $K'_i \cap A_i^1$ (resp. $K'_i \cap B_i^1$) is a singleton and it is contained in the homoclinic orbits which conjugates to $\gamma^+_i$ (resp. $\gamma^-_i$).

**Step 3: Recovery.**
Recall that $p^j$ has the third homoclinic orbit in $K'_i$ which is disjoint from $A_i^1 \cup B_i^1$. Let $K'_{i,o}$ denote the circuit consisting of $O(p^j)$ and the homoclinic orbit which conjugates to $\gamma^+_0$. Applying Theorem 5 to $U'(K'_{i,o})$, we know that up to a $4\varepsilon$--perturbation $U_i(K'_{i,o})$ satisfies the condition $(\ell_{U_i(K'_{i,o})})$.

Note that this last perturbation’s support is contained in $U_i$, thus it does not change the set of $U_i$-chains, and as a result keeps the ergodic diameter small. □

Let us complete the proof of the existence of non-transitive classes.

**Theorem 9.** Let $O$ be a $C^1$-open set of diffeomorphisms on a closed 3-manifold $M$ admitting a transitive filtrating Markov partition $U$ containing a periodic point $p_f$ varying continuously with $f$ such that $(f, U, p_f)$ satisfies the property $(\ell)$. Then there is a $C^1$-residual subset $R \subset O$ such that every $f \in G$ has an uncountable set of chain recurrence classes which are not transitive and uniquely ergodic.

**Proof.** We only need to confirm that property $(\ell)$ implies the existence of transitive Markov partition satisfying the condition required for the “first-level” of the induction process. Then Proposition 4.5 guarantees that $C^1$-generically we have uncountably many nested sequences of filtrating Markov partitions satisfying the assumption of Lemma 3.17, and the result holds.

Suppose that $(f, U)$ satisfies condition $(\ell)$. First, by Theorem 6 we know that up to an arbitrarily small perturbation and for any arbitrarily small $\varepsilon > 0$ there is a periodic point $p$ which is $\varepsilon$-flexible with a large stable manifold, having a non-trivial homoclinic class. Since the homoclinic class is non-trivial, $p$ has infinitely many homoclinic points whose orbits are mutually distinct. We choose three of them and call them $x_i$ ($i = 0, 1, 2$).

Now, using the largeness of the stable manifold of $p$, we choose a sufficiently fine refinement of $U$ such that there is a unique rectangles $A$ and $B$ containing $x_1$ and $x_2$ respectively. Then consider the circuit $L$ consisting of $O(p)$ and $O(x_i)$ ($i = 0, 1, 2$). We denote the sub-circuit $L_0$ consisting of $O(p)$ and $O(x_0)$. Then, apply Theorem 5 to $U(L_0)$ to obtain the property $(\ell_{U(L_0)})$. □

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