GEOMETRIC LIMITS OF JULIA SETS FOR SUMS OF POWER MAPS AND POLYNOMIALS

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Abstract. For maps of one complex variable, \( f \), given as the sum of a degree \( n \) power map and a degree \( d \) polynomial, we provide necessary and sufficient conditions that the geometric limit as \( n \) approaches infinity of the set of points that remain bounded under iteration by \( f \) is the closed unit disk or the unit circle. We also provide a general description, for many cases, of the limiting set when it is neither the disk nor the circle.

1. Introduction

Let \( q \) be a degree \( d \geq 2 \) polynomial; define \( f_{n,q} : \mathbb{C} \to \mathbb{C} \) by
\[
f_{n,q}(z) = z^n + q(z),
\]
and note that \( f_{n,q} \) is the sum of a power map (whose power we increase in the limit) and a fixed degree \( d \) polynomial, \( q \). For a map \( f : \mathbb{C} \to \mathbb{C} \), the filled Julia set for \( f \), \( K(f) \), is the set of points that remain bounded under iteration by \( f \). We use the notation \( S_0 = \{ z \in \mathbb{C} : |z| = 1 \} \) for the unit circle and \( \mathbb{D} = \{ z \in \mathbb{C} : |z| \leq 1 \} \) for the closed unit disk. The purpose of this study is to describe the limit of \( K(f_{n,q}) \) in the Hausdorff topology as \( n \to \infty \).

This work was inspired by a 2012 study by Boyd and Schulz [4] that included a result for the family \( f_{n,q} \) with \( \deg q = 0 \); that is, \( q(z) = c \in \mathbb{C} \). Among many other things, they proved

**Theorem 1.1** (Boyd-Shulz, 2012). If \( q(z) = c \), then under the Hausdorff metric,
\[
\text{for any } |c| < 1, \quad \lim_{n \to \infty} K(f_{n,c}) = \{ z \in \mathbb{C} : |z| \leq 1 \};
\]
\[
\text{for any } |c| > 1, \quad \lim_{n \to \infty} K(f_{n,c}) = \{ z \in \mathbb{C} : |z| = 1 \}.
\]

It comes as little surprise that this phenomena is easily disrupted. It was shown in [12] that when \( |c| = 1 \), the limiting behavior of \( K(f_{n,c}) \) depends on number-theoretic properties of \( c \) and the limit almost always fails to exist. In another study by Alves [1], it was show that for maps of the form \( f_{n,c}(z) = z^n + cz^k \) for a fixed positive integer \( k \), if \( |c| < 1 \), then the limit of \( K(f_{n,c}) \) as \( n \to \infty \) is \( S_0 \).

Returning to the more general case in which \( \deg q = d > 1 \), some results from the \( \deg q = 0 \) cases still hold. If \( |z| > 1 \), we can still expect the image of \( z \) under \( f_{n,q} \) to have large modulus for large enough \( n \). Guided by this intuition, we find the following generalization of a lemma from [4]. We adopt the notation
\[
\mathbb{D}_r = \{ z \in \mathbb{C} : |z| < r \} \text{ and } \mathbb{D}_r = \{ z \in \mathbb{C} : |z| \leq r \}.
\]

**Lemma 1.2.** For any polynomial \( q \) and any \( \epsilon > 0 \), there is an \( N \geq 2 \) such that for all \( n \geq N \),
\[
K(f_{n,q}) \subset \mathbb{D}_{1+\epsilon}.
\]

As in the \( \deg q = 0 \) cases, one can prove that \( S_0 \) is always a subset of the \( \liminf_{n \to \infty} K(f_{n,q}) \), and, in particular, a subset of the limit if the limit exists. Evidence for this fact (proved in Section 3) comes by noting that when \( n \) is much larger than the degree of \( q \), the \( n \) fixed points of \( f_{n,q} \) are roughly equidistributed around the unit circle, a result connected to the work of Erdös, Turan, et al. [6] [10] [11] on distribution of zeros for sequences of complex polynomials.
There are still simple conditions that describe precisely when we can expect the closed unit disk, $D$, or the unit circle, $S_0$.

**Theorem 1.3.** Suppose $\deg q \geq 2$ and $q$ has no fixed points in $S_0$. Under the Hausdorff metric,

\[
\begin{align*}
(1) \quad \lim_{n \to \infty} K(f_{n,q}) = D \text{ if and only if } q(D) \subset D, \\
(2) \quad \lim_{n \to \infty} K(f_{n,q}) = S_0 \text{ if and only if } q(D) \cap D = \emptyset.
\end{align*}
\]

At the heart of this result is the idea that if the degree $n$ of $f_{n,q}$ is large enough, then $|z^n|$ will be very large or very small, depending on whether $z$ is inside or outside of the unit disk, respectively. The condition $q(D) \subset D$ in Theorem 1.3 allows a disk of radius $1 - \epsilon$ centered at the origin to be forward invariant for all $n$ sufficiently large. The condition $q(D) \cap D = \emptyset$ allows a disk of radius $1 - \epsilon$ centered at the origin to be in the basin of infinity for all $f_{n,q}$ with $n$ sufficiently large.

Outside of these two cases, the situation is substantially more interesting. See Figure 1 for examples of filled Julia sets for $f_{n,q}$, where $q(z) = z^2 + c$ and $|c| < 1$, that very clearly fail to limit to the closed unit disk or the unit circle. The color gradation in the pictures indicates the number of iterates required to exceed a fixed modulus bound.

**Figure 1.** $K(f_{200,q_i})$ with $q_1(z) = z^2 + 0.25 + 0.25i$ (left) and $q_2(z) = z^2 + 0.45 + 0.25i$ (right)

Some of the dependence of the limiting behavior of $K(f_{n,q})$ on $q$ is obvious; by Lemma 1.2, one should expect any point whose orbit by $q$ leaves the unit disk to not be in $K(f_{n,q})$ for all $n$ sufficiently large. Thus, one should expect the limit of $K(f_{n,q})$ to contain the set

\[
\{ z \in \mathbb{C} : |q^k(z)| \leq 1 \text{ for all } k \}.
\]

While this ends up being the case, we have also mentioned that the unit circle $\{ z \in \mathbb{C} : |z| = 1 \}$ is also always contained in limit (when it exists) of $K(f_{n,q})$. An interesting byproduct of this phenomena arises when $K(q) \setminus D \neq \emptyset$. In this situation, an open neighborhood of a point on the unit circle intersects $K(q)$. The $n$ preimages of this neighborhood are also roughly equidistributed around the unit circle; this can be seen in the form of small components of the filled Julia set, appearing as small black spots, around the unit circle in Figure 1. Since the unit circle is contained in the limit of $K(f_{n,q})$, then one should also expect the preimages of the unit circle by $q$ (that still have modulus less than or equal to one) to be contained in the limit of $K(f_{n,q})$. These preimages
also contain the preimages of intersections of $K(q)$ with the unit circle, yielding similar small components of the filled Julia set (appearing as black spots) along the preimages as along the unit circle; again, see Figure 1. These ideas and the preceding lemmas lead to the next definition and theorem. We now define the limit of the sets $K(f_{n,q})$.

**Definition 1.4.** Let

$$K_\infty := K_q \cup \bigcup_{j \geq 0} S_j,$$

where $K_q := \{ z \in \mathbb{C} : |q^k(z)| < 1 \text{ for all } k \}$, $S_0$ is the unit circle $\{ z \in \mathbb{C} : |z| = 1 \}$, and for any integer $j \geq 0$,

$$S_j := \{ z \in \mathbb{C} : |q^j(z)| = 1 \text{ and } |q^i(z)| < 1 \text{ for all } 0 \leq i < j \}.$$

$K_\infty$ is the set of points in $K(q)$ whose orbits by $q$ remain in $\overline{D}$ (the set $K_q$), the unit circle ($S_0$), and the iterated preimages of $S_0$ that remain in $\overline{D}$ at each step (the sets $S_j$ for $j \geq 1$). See Figure 2 for an example of $K(f_{n,q})$ with $q(z) = z^2 - 0.1 + 0.75i$ and several different values of $n$ compared to a sketch of $K_\infty$ for this polynomial $q$.

![Figure 2](image-url)

**Figure 2.** Top, left to right: $K(f_{n,-0.1+0.75i})$ with $q(z) = z^2 - 0.1 + 0.75i$ and $n = 6, 12, 25, 50$. Bottom left: $K(f_{1800,-0.1+0.75i})$. Bottom right: Sketch of $K_\infty$, where $K_q$ is green, $S_0$ is red, and the sets $S_j$ are magenta.
Theorem 1.5. If \( \deg q \geq 2 \) and \( q \) is hyperbolic with no attracting periodic points on \( S_0 \), then under the Hausdorff metric
\[
\lim_{n \to \infty} K(f_{n,q}) = K_\infty.
\]

What is happening here heuristically is that as long as the orbit of \( z \) remains in \( \mathbb{D} \), the polynomial \( q(z) \) dominates the dynamics; if the orbit of \( z \) leaves \( \mathbb{D} \), then the power map \( z^n \) dominates. When the orbit hits \( S_0 \), it is not clear whether \( q(z) \) or \( z^n \) should win, so you get a point in the Julia set. We reserve all proofs for Section \( \ref{sec:proofs} \) following a brief tour of background information and examples in Section \( \ref{sec:background} \).

There is not yet any evidence that \( K_\infty \) is not the limit of \( K(f_{n,q}) \) when \( \deg q \geq 2 \); the following question remains open.

**Question.** Is it true that for any polynomial \( q \) with \( \deg q \geq 2 \) that \( \lim K(f_{n,q}) = K_\infty \), without any additional assumptions on \( q \)?

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2. Background and Examples

2.1. Notation and Terminology. The main results in this note rely on the convergence of sets in the Riemann sphere, \( \hat{\mathbb{C}} \), where the convergence is with respect to the Hausdorff metric. Given two sets \( A, B \) in a metric space \( (X,d) \), the Hausdorff distance \( d_H(A,B) \) between the sets is defined as

\[
d_H(A,B) = \max \left\{ \sup_{a \in A} d(a,B), \sup_{b \in B} d(b,A) \right\} = \max \left\{ \sup_{a \in A} \inf_{b \in B} d(a,b), \sup_{b \in B} \inf_{a \in A} d(a,b) \right\}.
\]

The distance from each point in \( A \) to \( B \) has a least upper bound, and the same it true for each point from \( B \) to \( A \). The Hausdorff distance is the supremum over all of these distances. As an example, consider a regular hexagon \( A \) with sides of length \( r \) inscribed in a circle \( B \) of radius \( r \). In this case, \( d_H(A,B) = r(1 - \sqrt{3}/2) \), the shortest distance from the circle to the midpoint of any of the sides of the hexagon.

Filled Julia sets \( K(f_{n,q}) \) are compact \( \ref{comp} \) and contained in the compact space \( \hat{\mathbb{C}} \). Moreover, with the Hausdorff metric \( d_H \), the space of all subsets of \( \hat{\mathbb{C}} \) is complete \( \ref{comp} \). Suppose \( S_n \) and \( S \) are compact subsets of \( \mathbb{C} \). We say \( S_n \) converges to \( S \) and write \( \lim_{n \to \infty} S_n = S \) if for all \( \epsilon > 0 \), there is \( N > 0 \) such that for all \( n \geq N \), we have \( d_H(S_n, S) < \epsilon \).

2.2. Fatou Components and Hyperbolicity. We provide here the fine details relevant to this paper from a important subject, the description of the Fatou set and classification of its components. Thorough explorations of this subject and proof of all the facts below can be found in \( \ref{fatou} \). The Fatou set of rational map \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \), denoted \( F(f) \), is the set of points for which the iterates form a normal family; the Julia set of \( f \), denoted \( J(f) \), is the complement of \( F(f) \) in \( \hat{\mathbb{C}} \). When \( f \) is a polynomial map, the Julia set of \( f \) is the boundary of the filled Julia set; that is, \( J(f) = \partial K(f) \).

We say a point \( z \in \hat{\mathbb{C}} \) is periodic for \( f \) with period \( k \) if \( f^k(z) = z \) and the points \( z, f(z), \ldots, f^{k-1}(z) \) are all distinct. The multiplier \( \lambda \) of a periodic point \( z_0 \) of period \( k \) is defined as

\[
\lambda = (f^k)'(z_0) = \prod_{i=0}^{k-1} f'(f^i(z)).
\]
If $|\lambda| < 1$, then $z_0$ is attracting; if $\lambda > 1$, $z_0$ is repelling; if $\lambda = 1$, $z_0$ is indifferent. Repelling periodic points are contained in $J(f)$; in fact, repelling periodic points are dense in $J(f)$. Attracting periodic points, on the other hand, are contained in $\mathcal{F}(f)$. Moreover, for every attracting periodic point $z_0$ of period $k$, there is an open neighborhood $B(z_0, \epsilon)$ such that $f^k(B(z_0, \epsilon)) \subset B(z_0, \epsilon)$ and the orbit by $f^k$ of any point in $B(z_0, \epsilon)$ converges to $z_0$. The set of all points whose orbits by $f^k$ converge to $z_0$ is called the basin of attraction for $z_0$.

A component of the Fatou set of a rational map that is mapped to itself must be one of the following: a basin of an attracting periodic point, a basin for a petal of an indifferent periodic cycle, a cycle of Siegel disks, or a cycle of Herman rings. Only the first of these four categories will be relevant for this paper. The Sullivan Nonwandering Theorem [15], states that every Fatou component for $f$ is eventually periodic, and from this it follows that the four possible types of Fatou components listed above is comprehensive. For this reason, this is referred to as Sullivan’s classification of Fatou components.

A rational map $f$ is called hyperbolic if there is some conformal metric $\mu$ defined in a neighborhood of $J(f)$ such that for every $z \in J(f)$, we have

$$\|Df_z(v)\|_{\mu} > \|v\|_{\mu}$$

for every nonzero $v \in T\hat{C}_z$. A rational map with this property is said to be expanding on $J(f)$. Hyperbolic maps are well-behaved dynamically in the following ways. If a rational map is hyperbolic, then all rational maps sufficiently close will also be hyperbolic, and the Julia sets of these maps deform continuously through these sufficiently close maps. Hyperbolic maps are also very common amongst rational maps; however, while it has been long conjectured that hyperbolic maps are dense in the set of rational maps, it has yet to be proved in even the quadratic polynomial case.

For our purposes, we will require in the proof of Theorem 1.5 that every point in $\mathcal{F}(f)$ converges to an attracting periodic cycle. This result follows from hyperbolicity by way of Sullivan’s classification [13, Theorem 3.13].

2.3. Examples. Let $q(z) = 0.75z^2 + c_j$ with $c_1 = 0.21 + 0.017i$, $c_2 = 0.41 + 0.047i$, and $c_3 = 1.41 + 1.17i$. For $c_1$, we have $q(D) \subset \mathbb{D}$, so in this case, $K_\infty = \mathbb{D}$. For $c_3$, we have $q(D) \cap \mathbb{D} = \emptyset$, so in this case, $K_\infty = S_0$. Both of these cases follow from Theorem 1.3. The more interesting case is $c_2$ in which $q(D) \setminus \mathbb{D} \neq \emptyset$. See Figure 3. The limit, should it exist, of $K(f_{n,q})$ is the set $K_\infty$, which is now significantly more complicated, neither the closed unit disc nor the unit circle.

3. Proof of Main Results

Proof of Lemma 1.2. Let $z \in \mathbb{C} \setminus \overline{\mathbb{D}}_{1+c}$. We prove $|f_{n,q}(z)| \geq B^m$ for all $m \geq 1$ using induction. Let $a_i$ be the coefficients of $q$, and pick $M > d(\max |a_i|)$. Then for any $|z| > 1$, we have $|q(z)| \leq M|z|^d$. Choose $B > \max \{1, M\}$ and $N > d$ large enough that $|z|^N > \max\{4B, 2M|z|^d\}$. Let $n \geq N$.

Observe that

$$|f_{n,q}(z)| \geq |z|^n - |q(z)| \geq |z|^n - M|z|^d \geq |z|^n - \frac{1}{2}|z|^n \geq 2B > B.$$ 

Now suppose for some $m \geq 1$, we know $|f_{n,q}(z)| \geq B^m$. Let $z_m = f_{n,q}(z)$, and note that $|q(z_m)| \leq M|z_m|^d < |z_m|^N$. Then for any $n \geq N$,

$$|f_{n,q}(z)| \geq |z_m^n| - |q(z_m)| \geq |z_m|^n - M|z_m|^d \geq B^{mn} - B^{md}|z_m|^d \geq B^m - B^{md}B = B^{m+1}

\geq B^{m+1}.$$ 

It follows that $|f_{n,q}(z)| \geq B^m$ for all $m \geq 1$. Since $B > 1$, the orbit of $z$ under $f_{n,q}$ escapes to infinity. Thus, $z \notin K(f_{n,q})$. \qed

Before proving Theorems 1.5 we need a couple more lemmas.
Figure 3. Each row, left to right: $q(\overline{D})$ for $q(z) = 0.75z^2 + c_j$, $K(q)$, and $K(f_{1800})$. The first row is for $c_1 = 0.21 + 0.017i$, the second row is $c_2 = 0.41 + 0.047i$, and the third row is $c_3 = 1.41 + 1.17i$. Scale is the same in pictures in the last two columns.

Lemma 3.1. If $\{q^i(z_0)\}_{i=0}^{k-1} \subset \mathbb{D}$ and $|q^k(z_0)| = 1$, then for any positive integer $m < k$, there is an $N$ such that for all $n \geq N$,

$$\{f_i^{n,q}(z_0)\}_{i=0}^{m} \subset \mathbb{D}.$$ 

Moreover, for all $\epsilon > 0$ and any positive integer $m \leq k$, there is an $N$ such that for all $n \geq N$,

$$\max_{0 \leq i \leq m} |f_i^{n,q}(z_0) - q^i(z_0)| < \epsilon.$$

Proof. This proof follows by continuity and is left to the reader. \qed
Lemma 3.2. For all $\epsilon > 0$, there is an $N$ such that for any $n \geq N$, any $j \geq 0$, and all $z \in S_j$, 
\[ d(z, K(f_{n,q})) < \epsilon. \]

There is a body of work on the distribution of polynomial roots begun by Erdős and Turán in [6]. There are specific results in [10, 11] dealing with the accumulation of polynomial roots around the unit circle, and these could be applied to the polynomials $f_{n,q}(z) - z$ to find fixed points. However, the case here is simpler in the sense that $n - d - 1$ of the coefficients of $f_{n,q}$ are all zero, so we present a concise argument using the following potential theory lemma.

Lemma 3.3. For any fixed degree $d$ polynomial, $q$, the zeros of the polynomial $f_{n,q}(z) = z^n + q(z)$ cluster uniformly around the unit circle as $n \to \infty$. More specifically, for each $n$, let 
\[ \mu_n = \frac{1}{n} \sum_{f_{n,q}(z) = 0} \delta_z, \]
where $\delta_z$ is a point mass at $z$, and the roots of $f_{n,q}$ are counted with multiplicity. Then $\mu_n \to \mu$ weakly as $n \to \infty$, where $\mu$ is normalized Lebesgue measure on $S_0$.

Proof. Note that 
\[ \mu = dd^c \log_+ |z|, \quad \text{where} \quad \log_+ |z| = \begin{cases} \log |z|, & \text{if } |z| \geq 1 \\ 0, & \text{if } |z| < 1. \end{cases} \]

Let $Z_q$ be the zero set of $q$ and $A$ be the maximum of $|q(z)|$ on $S$. Let $K$ be a compact subset of $\mathbb{C} \setminus (S_0 \cup Z_q)$; then there is an $\epsilon > 0$ such that for any $z \in K$, we have $|q(z)| > \epsilon$ and either $|z| \geq 1 + \epsilon$ or $|z| \leq 1 - \epsilon$.

If $|z| \geq 1 + \epsilon$, then
\begin{equation}
\frac{1}{n} \log |f_{n,q}(z)| = \frac{1}{n} \log \left| z^n \left( 1 + \frac{q(z)}{z^n} \right) \right| \\
= \frac{1}{n} \log |z^n| + \frac{1}{n} \log \left| 1 + \frac{q(z)}{z^n} \right| \leq \log_+ |z| + \frac{C}{n},
\end{equation}
where $C = \log(1 + A/(1 + \epsilon))$.

If $|z| < 1 - \epsilon$, then there is an $N$ such that for all $n \geq N$, we have $z^n \leq \max\{\epsilon/2, A\}$. Then
\[ \frac{\epsilon}{2} \leq |q(z)| - |z^n| \leq |f_{n,q}(z)| \leq |z^n| + |q(z)| \leq 2A. \]

Noting that $\log_+ |z| = 0$ when $|z| \leq 1 - \epsilon$, we have for all $|z| \leq 1 + \epsilon$ that
\begin{equation}
\frac{1}{n} \log |f_{n,q}(z)| \leq \frac{1}{n} \log(2A) = \log_+ |z| + \frac{\log(2A)}{n}.
\end{equation}

Using Equations (1) and (2), we have $\frac{1}{n} \log |f_{n,q}(z)| \to \log_+ |z|$ uniformly on $K$ as $n \to \infty$; by the compactness theorem for families of subharmonic functions [9, Theorem 4.1.9], it follows that $\frac{1}{n} \log |f_{n,q}(z)| \to \log_+ |z|$ in $L^1_{loc}(\mathbb{C})$. Note that $dd^c \log_+ |z| = \mu$, and we have from the Poincaré-Lelong formula [7] that $\frac{1}{n} dd^c \log |f_{n,q}(z)| = \mu_n$. Thus, we have
\[ \mu_n = \frac{1}{n} dd^c \log |f_{n,q}(z)| \to dd^c \log_+ |z| = \mu \]
weakly as $n \to \infty$. \hfill \Box

Proof of Lemma 3.2. Let $z \in S_0$ and $\epsilon > 0$. Define 
\[ g_n(z) := f_{n,q}(z) - z. \]
Then the zeros of $g_{n,z_0}$ are fixed points of $f_{n,q}$. By Lemma 3.3, the fixed points of $f_{n,q}$ cluster uniformly near the unit circle. If any of the fixed points are repelling, then they are contained in
Otherwise, they are attracting or indifferent, in which case they must be $\epsilon$ close to $J(f_{n,q})$ because $K(f_{n,q}) \subseteq D_{1+\epsilon}$.

If $S_j$ is empty for all $j > 0$, we are done, so suppose $S_j$ is nonempty for some $j > 0$. By construction, either $S_j$ is nonempty for all $j$ or there is some $J$ such that $S_j$ is nonempty if and only if $1 \leq j \leq J$. Suppose the latter. Each of these sets $S_j$ has compact closure, so for each $j \leq J$, there is a set of points $\{z_1, \ldots, z_{\ell}\} \subseteq S_j$ such that $S_j \subseteq \bigcup_{i=1}^{\ell} B(z_i, \epsilon/2)$. By construction, we have for each $1 \leq i \leq \ell$ that $q^i(B(z_i, \epsilon/2))$ is an open neighborhood (by the open mapping theorem) of $q^i(z_i) \in S_0$. By Lemma 3.2, there is an $N_j$ such that for all $n \geq N_j$, there are $w_{i,n} \in K(f_{n,q})$ for all $1 \leq i \leq \ell$ such that $w_{i,n} \in q^i(B(z_i, \epsilon/2))$, so in particular, $d(q^i(z_i), w_{i,n}) < \epsilon/2$. By Lemma 3.1, we may make $N_j$ large enough that for all $1 \leq i \leq \ell$, all $0 \leq k \leq j$, and all $n \geq N_j$,

$$|q^k(z_i) - f_{n,q}^k(z_i)| < \epsilon/2.$$ 

By construction, there is some $z_{i,n} \in B(z_i, \epsilon/2)$ such that $q^i(z_{i,n}) = w_{i,n}$, so there is an $\hat{w}_{i,n} \in B(z_{i,n}, \epsilon/2)$ such that $f_{n,q}^i(\hat{w}_{i,n}) = w_{i,n}$, so $|z_i - \hat{w}_{i,n}| < \epsilon$. Since $K(F_{n,q})$ is backward invariant, we have that $\hat{w}_{i,n} \in K(f_{n,q})$. Thus, for any $z \in S_j$, we have $d(z, K(f_{n,q})) < \epsilon$. We may choose $N = \max_{j=1}^{\ell} N_j$, and the result holds for any $z \in \bigcup_{j=1}^{J} S_j$.

We turn our attention to the case in which $S_j$ is nonempty for all $j$. The preceding argument can be applied to $\bigcup_{j=1}^{J} S_j$ for any fixed $J$, so we need only prove the lemma for $\bigcup_{j>0} S_j$. Note first that for large enough $j$, the sets $S_j$ are arbitrarily close to $K_q$. To see this, define

$$\mathcal{K}_j = \bigcap_{i=0}^{J} \{z \in C : |q^i(z)| < 1\},$$

and note that for any $j > J$, we have from construction that $S_j \subseteq \mathcal{K}_j$ and $\lim_{J \to \infty} \mathcal{K}_j = K_q$. Since each $S_j$ is nonempty, we also have that $\mathcal{K}_j \setminus K_q \neq \emptyset$. Moreover, for any $J$, the set $\mathcal{K}_j \setminus K_q$ has compact closure. Then for any $\epsilon > 0$, there is a $J$ and points $\{z_1, \ldots, z_\ell\} \subseteq \bigcup_{j>0} S_j$ such that

$$\bigcup_{j>0} S_j \subseteq \bigcup_{i=1}^{\ell} B(z_i, \epsilon/2);$$

that is, for all $z \in \bigcup_{j>0} S_j$, there is an integer $1 \leq i \leq \ell$ such that $|z - z_i| < \epsilon/2$. Using the same argument as in the previous case, one can find an $N$ such that for all $n \geq N$ and all $z_i \in \{z_1, \ldots, z_\ell\}$, we have $d(z_i, K(f_{n,q})) < \epsilon/2$. Thus, for all $z \in \bigcup_{j>0} S_j$, we have

$$d(z, K(f_{n,q})) \leq d(z, z_i) + d(z_i, K(f_{n,q})) < \epsilon.$$

Lemma 3.4. For any $\epsilon > 0$, there is an $N$ such that for any $n \geq N$ 

$$z_0 \in K(f_{n,q}) \text{ implies } d(z_0, K_\infty) < \epsilon.$$

Proof. We prove the contrapositive,

$$d(z_0, K_\infty) \geq \epsilon \text{ implies } z_0 \notin K(f_{n,z}).$$

If $|z_0| > 1 + \epsilon$, then by Lemma 1.2 there is a large enough $N$ such that for all $n \geq N$, we have $z_0 \notin K(f_{n,z})$. Now suppose $z_0 \notin D_{1+\epsilon} \setminus K_\infty$. Since $d(z_0, K_\infty) \geq \epsilon$ and $S_0 \subseteq K_\infty$, we have that $z_0 \in D_{1-\epsilon} \setminus K_\infty$. Note that $\{z \in C : d(z, K_\infty) \geq \epsilon\} \cap D_{1-\epsilon}$ is a compact set. Then there is some $j$ such that for any $z_0 \in \{z \in C : d(z, K_\infty) \geq \epsilon\} \cap D_{1-\epsilon}$, we have $|q^j(z_0)| > 1$, and by Lemma 3.1, $|f_{n,q}^j(z_0)| > 1$ for some large enough $n$. 

\begin{flushright}
$\square$
\end{flushright}
Lemma 3.5. For any periodic orbit \( \{z_i\}_{i=0}^{k-1} \) of \( q \) contained in \( \mathbb{D} \) and any \( \epsilon > 0 \), there is an \( N \) such that for all \( n \geq N \), \( f_{n,q} \) has a periodic orbit \( \{z_{i,n}\}_{i=0}^{k-1} \) also contained \( \mathbb{D} \) such that
\[
\max_{0 \leq i < k-1} |z_i - z_{i,n}| < \epsilon.
\]
Moreover, if \( \{z_i\}_{i=0}^{k-1} \) is attracting (repelling) for \( q \), then each cycle \( \{z_{i,n}\}_{i=0}^{k-1} \) is attracting (repelling) for each corresponding \( f_{n,q} \).

While zeros of non-constant polynomials depend continuously on the coefficients of the polynomial, the set of polynomials \( \{f_{n,q}\} \) is discrete. Nevertheless, Lemma 3.5 still follows quickly from Rouche’s theorem and the fact that on any compact subset \( K \) of \( \mathbb{D} \), we have \( f_{n,q}|_K \to q \) uniformly and \( f'_{n,q}|_K \to q' \) uniformly. We include the details of the proof below for the sake of completeness.

Proof. For each \( 0 \leq i \leq k - 1 \), define \( \hat{q}_i(z) = q(z) - z_{i+1} \). Then for \( 0 \leq i \leq k - 1 \), \( z_i \) is a zero of \( \hat{q}_i \), and the zeros of each \( \hat{q}_i \) are isolated. Then we can choose \( r_i > 0 \) small enough that for each \( 0 \leq i \leq k - 1 \), the point \( z_i \) is the only zero of \( \hat{q}_i \) in \( B(z_i, r_i) \). Since the boundary of each \( B(z_i, r_i) \) is compact, there is an \( \epsilon > 0 \) such that for all \( 0 \leq i \leq k - 1 \), we have
\[
\epsilon < \inf \{ |\hat{q}_i(B(z_i, r_i))| \}.
\]
Choose \( N \) large enough that for any \( n \geq N \), we have \( |z|^n < \epsilon/2 \). Then by Rouche’s theorem, \( f \) and \( q \) have the same number of periodic points in \( \bigcup_{i=0}^{k-1} B(z_i, r_i) \).

If \( \{z_i\}_{i=0}^{k-1} \) is attracting for \( q \), we know that its multiplier is less than one: \( \lambda = \prod_{i=0}^{k-1} |q'(z_i)| < 1 \). For all \( \delta > 0 \), there is an \( N \) such that for any \( n \geq N \) and any \( 0 \leq i \leq k - 1 \), we have \( n|z_i|^n < \delta \). We may choose \( \delta \) small enough that
\[
\prod_{i=0}^{k-1} |f'(z_i)| = \prod_{i=0}^{k-1} |nz_i^n + q(z_i)| \leq \prod_{i=0}^{k-1} (n|z_i|^n + |q(z_i)|) < 1.
\]
A similar argument can be made for repelling cycles. \( \square \)

Proof of Theorem 1.3. Let \( \epsilon > 0 \). We must show there is an \( N \) such that for all \( n \geq N \),
\[
(3) \quad \text{for all } z \in K(f_{n,q}), \quad d(z, K_{n}) < \epsilon \quad \text{and}
\]
\[
(4) \quad \text{for all } z \in K_{n}, \quad d(z, K(f_{n,q})) < \epsilon.
\]
The inequality (3) follows from Lemma 3.4. Much of the work for inequality (4) is also done; by Lemma 3.2, we know that for all \( \epsilon > 0 \), there is an \( N \) such that for all \( n \geq N \), any \( j \geq 0 \), and all \( z \in S_j \), we have \( d(z, K(f_{n,q})) < \epsilon \). Since \( K_{\infty} \) is the union of these \( S_j \)’s and \( K_q \), it remains only to deal with \( K_q \). That is, we need only now show that for any \( \epsilon > 0 \) there is an \( N \) such that for all \( n \geq N \) and all \( z \in K_q \), \( d(z, K(f_{n,q})) < \epsilon \). If \( K_q \) is empty, we are done; we proceed with the assumption that \( K_q \) is nonempty.

Let \( F(q) \) and \( J(q) \) be the Fatou and Julia sets, respectively, for \( q \), and let \( K_q^0 \) and \( \partial K_q \) be the interior and boundary, respectively, of \( K_q \). Note that \( J(q) \) is the boundary of \( K(q) \) \([14]\). Since \( K_q \subset K(q) \), any subset of \( K_q^0 \) must be disjoint from \( J(q) \).

Suppose that \( K_q^0 \) is nonempty, so \( K_q \cap F(q) \) is nonempty. Since \( q \) is hyperbolic, we know by Sullivan’s classification \([15, 14, 5]\) that the orbit of every point in \( F(q) \) converges to an attracting periodic cycle. Suppose \( \{z_i\}_{i=0}^{k-1} \) is such an attracting cycle and \( |z_i| > 1 \) for some \( i \). Then any \( z \) whose orbit converges to this cycle must eventually leave \( \mathbb{D} \). Since we have assumed \( q \) has no attracting periodic points on \( S_0 \), it follows that the orbit of every point in \( K_q \cap F(q) \) converges to an attracting cycle for \( q \) contained in \( \mathbb{D} \). Then for each \( z_i \), there is a neighborhood \( B(z_i, r_i) \) such that \( q^k(B(z_i, r_i)) \subset B(z_i, r_i) \subset \mathbb{D} \). Let \( r = \min r_i \).
By Lemma 3.5, there is an $N_0$ such that for all $n \geq N_0$, $f_{n,q}$ has an attracting periodic orbit 
\( \{z_i,n\}_{i=0}^{k-1} \) where $z_i,n \in B(z_i,r)$. By Lemma 3.1, there is an $N_1$ such that for all $n \geq N_1$ we also have 
\( f_{n,q}^k(B(z_i,r)) \subset B(z_i,r) \).

Let $K_{\epsilon/2}$ be a compact subset of $K_q^0$ such that for all $z \in K_{\epsilon/2}$, there is some $z_0 \in \partial K_q$ such that 
\( d(z,z_0) < \epsilon/2 \).

There is an $m$ such that 
\[ q^m(K_{\epsilon/2}) \subset \bigcup_{i=0}^{k-1} B(z_i,r) \].

Again by Lemma 3.1, there is an $N_2$ such that for all $m \geq N_2$ we also have 
\[ f_{n,q}^m(K_{\epsilon/2}) \subset \bigcup_{i=0}^{k-1} B(z_i,r) \].

Let $N = \max_{0 \leq i \leq 2} N_i$. For all $n \geq N$, it follows that the orbit by $f_{n,q}$ of any point $z \in K_{\epsilon/2}$ converges to an attracting periodic cycle of $f_{n,q}$ contained in $\mathbb{D}$; that is, $K_{\epsilon/2} \subset K(f_{n,q})$. Then for any point $z \in K_q^0$, there is a $\hat{z} \in K_{\epsilon/2} \subset K(f_{n,q})$ such that $d(z,\hat{z}) < \epsilon/2$.

Now we address $\partial K_q$. In the proof of Lemma 3.2, we showed that the sets $S_j$ get arbitrarily close to $K_q$ as $j$ increases, so choose $j$ large enough that for any point $z_0 \in \partial K_q$, we have $d(z_0,S_j) < \epsilon/2$. By Lemma 3.2, we may choose $N$ such that for all $n \geq N$ and all $z \in S_j$, we have $d(z,K(f_{n,q})) < \epsilon/2$. Pick $z_1 \in S_j$ such that $d(z_0,z_1) < \epsilon/2$. Then 
\[ d(z_0,K(f_{n,q})) \leq d(z_0,z_1) + d(z_1,K(f_{n,q})) < \epsilon/2 + \epsilon/2 = \epsilon. \]

Thus, for any point $z \in K_q = K_q^0 \cup \partial K_q$, it follows that $d(z,K(f_{n,q})) < \epsilon$. \( \square \)

**Proof of Theorem 1.3.** We prove the first part of the theorem. Suppose first that the image of $\overline{\mathbb{D}}$ under $q$ is contained in $\overline{\mathbb{D}}$ and let $0 < \epsilon < 1$. By the open mapping theorem, we know $q(\mathbb{D})$ is an open set in $\overline{\mathbb{D}}$, so $q(\mathbb{D}) \subset \mathbb{D}$. Since $\deg q \geq 2$, we have from the Denjoy-Wolff Theorem [14] that $q$ has a fixed point $z_0$ in $\mathbb{D}$ to which orbits of all points in any compact subset of $\mathbb{D}$ converge. We have assumed $q$ has no fixed points in $S_0$, so $z_0 \in \mathbb{D}$, and in this case, we also have from the Denjoy-Wolff Theorem that $z_0$ is the unique fixed point in $\mathbb{D}$. From Lemma 3.5, we have that for all $\epsilon > 0$, there is an $N$ such that for all $n \geq N$, $f_{n,q}$ has an attracting fixed point $z_n$ such that $|z_0 - z_n| < \epsilon$ and no other fixed points in $\overline{\mathbb{D}}_{1-\epsilon}$. Thus, we have $\mathbb{D}_{1-\epsilon} \subset K(f_{n,q})$. Combining this with Lemma 1.2, for any $\epsilon > 0$, we may choose $N$ large enough such that 
\[ \mathbb{D}_{1-\epsilon} \subset K(f_{n,q}) \subset \mathbb{D}_{1+\epsilon}. \]

Now suppose the image of $\overline{\mathbb{D}}$ under $q$ is not contained in $\overline{\mathbb{D}}$, so $q(\mathbb{D}) \setminus \overline{\mathbb{D}}$ is nonempty. Since $\mathbb{D}$ is compact, $q(\mathbb{D})$ is also compact, and since $\mathbb{C} \setminus \overline{\mathbb{D}}$ is open and $q(\mathbb{D}) \setminus \overline{\mathbb{D}}$ is nonempty, there is some $z_0 \in \mathbb{D}$ and $r > 0$ such that $B(z_0,r) \subset \mathbb{D}$ and for any $z \in B(z_0,r)$, we have $|q(z)| > 1$. Then one can pick $N$ large enough that for any $n \geq N$ and any $z \in B(z_0,r)$, we have $|f_{n,q}(z)| > 1$. Then for all $n \geq N$, $z_0 \notin K(f_n)$.

For the second part of the theorem, assume $\overline{\mathbb{D}}$ under $q$ does not intersect $\mathbb{D}$, so $q(\mathbb{D}) \subset (\mathbb{C} \setminus \mathbb{D})$. That is, for all $z_0 \in \mathbb{D}$, we have $|q(z_0)| \geq 1$. Let 
\[ s = \min_{z \in \mathbb{D}_{1-\epsilon}} \{|q(z)|\}. \]
so $1 < s$. Then $(s - 1)/2 > 0$. Since $\mathbb{D}_{1-\epsilon}$ is compact, we may choose this $N$ so that for any $z \in \mathbb{D}_{1-\epsilon}$, we have $|z|^n < (s - 1)/2$. Then for any $z \in \mathbb{D}_{1-\epsilon}$, we also have
\[
|f_{n,q}(z)| \geq ||z|^n - \left|q(z)\right|| = \left|q(z)\right| - |z|^n > s - (s - 1)/2 > 1 + (s - 1)/2.
\]
By Lemma 1.2, it follows that $\mathbb{D}_{1-\epsilon}$ is in the basin of infinity of $f_{n,q}$ for all $n \geq N$. The result then follows from this fact and Lemma 3.2.

Lastly, suppose the image $\mathbb{D}$ under $q$ does intersect $\mathbb{D}$. Then by Theorem 1.5 if the limit exists, $\bigcup_{j>1} S_j \subset \mathbb{D}$ is nonempty, so the limit cannot be $S_0$. \qed

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