The Equivalence between Uniqueness and Continuous Dependence of Solution for BDSDEs

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May 17, 2010

Abstract

In this paper, we prove that, if the coefficient $f = f(t, y, z)$ of backward doubly stochastic differential equations (BDSDEs for short) is assumed to be continuous and linear growth in $(y, z)$, then the uniqueness of solution and continuous dependence with respect to the coefficients $f$, $g$ and the terminal value $\xi$ are equivalent.

\textit{keywords}: backward doubly stochastic differential equations, uniqueness, continuous dependence

1 Introduction

Nonlinear backward stochastic differential equations (BSDEs in short) have been independently introduced by Pardoux and Peng \cite{Pardoux1992} and Duffie and Epstein \cite{Duffie1992}. Since then, BSDEs have been studied intensively. In particular, many efforts have been made to relax the assumption on the generator. For instance, Lepeltier and San Martin \cite{Lepeltier1995} have proved the existence of a solution for the case when the generator is only continuous with linear growth, and Jia and Peng \cite{Jia2003} obtained that BSDE has either one or uncountably many solutions, if the generator satisfies the conditions given in \cite{Lepeltier1995}. Jia and Yu \cite{Jia2004} studied the equivalence between uniqueness and continuous dependence of solution for BSDEs with continuous coefficient. Another main reason is due to their enormous range of applications in such diverse fields as mathematical finance (see \cite{ElKaroui1997} and El Karoui et al. \cite{ElKaroui1997}), partial differential equations (see Peng \cite{Peng2000}), stochastic control (see Ji and Wu \cite{Ji2003}), nonlinear mathematical expectations (see Jiang \cite{Jiang2005} and Fan \cite{Fan2007}), and so on.

A class of backward doubly stochastic differential equations (BDSDEs in short) was introduced by Pardoux and Peng \cite{Pardoux1994} in 1994, in order to provide

\[\textsuperscript{*}\text{This work is supported by National Natural Science Foundation of China Grant 10771122, Natural Science Foundation of Shandong Province of China Grant Y2006A08 and National Basic Research Program of China (973 Program, No.2007CB814900)}\]

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a probabilistic interpretation for the solutions of a class of semilinear stochastic partial differential equations (SPDEs in short). They have proved the existence and uniqueness of solutions for BDSDEs under uniformly Lipschitz conditions. Since then, Shi et al. [15] have relaxed the Lipschitz assumptions to linear growth conditions. Bally and Matoussi [1] have given a probabilistic interpretation of the solutions in Sobolev spaces for semilinear parabolic SPDEs in terms of BDSDEs. Zhang and Zhao [17] have proved the existence and uniqueness of solutions for BDSDEs on infinite horizons, and described the stationary solutions of SPDEs by virtue of the solutions of BDSDEs on infinite horizons. Recently, Ren et al. [14] and Hu and Ren [5] considered the BDSDEs driven by Levy process with Lipschitz coefficient and applications in SPDEs.

Because of their important significance to SPDEs, it is necessary to give intensive investigation to the theory of BDSDEs. In this paper we will prove that if the coefficient $f$ satisfies the conditions given in [15], then the uniqueness of solution and continuous dependence with respect to $f$, $g$ and $\xi$ are equivalent.

We consider the following 1-dimesional backward doubly stochastic differential equations:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T,$$

where $\{W_t; 0 \leq t \leq T\}$ and $\{B_t; 0 \leq t \leq T\}$ are two mutually independent standard Brownian Motions with values in $\mathbb{R}^d$ and $\mathbb{R}^l$, respectively, defined on $(\Omega, \mathcal{F}, P)$. The terminal condition $\xi$ and the coefficients $f = f(t, y, z)$ and $g = g(t, y, z)$ are given. The solution $(Y_t, Z_t)_{t \in [0,T]}$ is a pair of square integrable processes. An interesting problem is: what is the relationship between uniqueness of solution and continuous dependence with respect to $f$, $g$ and $\xi$? In the standard situation where $f$ satisfies Lipschitz condition in $(y, z)$, it was proved by Pardoux and Peng [12] that there exists a unique solution. In this case, the continuous dependence with respect to $f$ and $\xi$ is an obvious result. However in the case where $f$ is only continuous in $(y, z)$, in place of the Lipschitz condition, Shi et al. [15] have proved that there is at least one solution. In fact, there is either one or uncountable many solutions in this situation (see Shi and Zhu [16]). Does the uniqueness of solution of BDSDEs also imply the continuous dependence with respect to $f$, $g$ and $\xi$?

This paper is organized as follows. In Section 2 we formulate the problem accurately and give some preliminary results. Section 3 is devoted to proving the equivalence of uniqueness and continuous dependence with respect to terminal value $\xi$. Finally, in Section 4 we will prove the equivalence of uniqueness and continuous dependence with respect to parameters $f$, $g$ and $\xi$.

### 2 Preliminary

**Notation** The Euclidean norm of a vector $x \in \mathbb{R}^k$ will be denoted by $|x|$, and for a $d \times k$ matrix $A$, we define $|A| = \sqrt{Tr AA^*}$, where $A^*$ is the transpose of $A$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space, and $T > 0$ be an arbitrarily fixed constant throughout this paper. Let $\{W_t; 0 \leq t \leq T\}$ and $\{B_t; 0 \leq t \leq T\}$ be two
mutually independent standard Brownian Motions with values in \( \mathbb{R}^d \) and \( \mathbb{R}^f \), respectively, defined on \((\Omega, \mathcal{F}, \mathcal{P})\). Let \( \mathcal{N} \) denote the class of \( \mathcal{P} \)-null sets of \( \mathcal{F} \). For each \( t \in [0, T] \), we define \( \mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^R \), where for any process \( \{ \eta_t \} \), \( \mathcal{F}_{s,t}^\eta = \sigma \{ \eta_r - \eta_s; s \leq r \leq t \} \lor \mathcal{N} \), \( \mathcal{F}_0^\eta = \mathcal{F}_{0,t}^\eta \). Note that the collection \( \{ \mathcal{F}_t; t \in [0,T] \} \) is neither increasing nor decreasing, so it does not constitute a filtration.

We introduce the following notations:

\[
S^2([0,T];\mathbb{R}^n) = \{ v_t, 0 \leq t \leq T, \text{ is an } \mathbb{R}^n\text{-valued, } \mathcal{F}_t\text{-measurable process such that } E(\sup_{0 \leq t \leq T}|v_t|^2) < \infty \},
\]

\[
M^2(0,T;\mathbb{R}^n) = \{ v_t, 0 \leq t \leq T, \text{ is an } \mathbb{R}^n\text{-valued, } \mathcal{F}_t\text{-measurable process such that } E \int_0^T |v_t|^2 dt < \infty \}.
\]

Let

\[
f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, \quad g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^f,
\]

be jointly measurable such that for any \( (y,z) \in \mathbb{R} \times \mathbb{R}^d \),

\[
f(\cdot, y, z) \in M^2(0,T;\mathbb{R}), \quad g(\cdot, y, z) \in M^2(0,T;\mathbb{R}^f).
\]

and satisfy the following conditions:

(H1) linear growth: \( \exists 0 < K < \infty \), such that

\[
|f(\omega, t, y, z)| \leq K(1 + |y| + |z|), \quad \forall (\omega, t, y, z) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d;
\]

(H2) For fixed \( \omega \) and \( t \), \( f(\omega, t, \cdot, \cdot) \) is continuous;

(H3) there exist constants \( c > 0 \) and \( 0 < \alpha < 1 \) such that

\[
|g(\omega, t, y^1, z^1) - g(\omega, t, y^2, z^2)|^2 \leq c|y^1 - y^2|^2 + \alpha|z^1 - z^2|^2,
\]

for all \( (\omega, t) \in \Omega \times [0,T], \ (y^1, z^1) \in \mathbb{R} \times \mathbb{R}^d, \ (y^2, z^2) \in \mathbb{R} \times \mathbb{R}^d \).

**Remark 2.1** In fact, (H1) can be replaced by the following condition:

(H4) there exist a constant \( 0 < K < \infty \), such that

\[
|f(\omega, t, y, z) - f(\omega, 0, 0, 0)| \leq K(1 + |y| + |z|), \quad \forall (\omega, t, y, z) \in \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d.
\]

In the sequel, it is not hard to check that all results in this paper also hold under Assumptions (H2)-(H4).

By Theorem 4.1 in [15], under (H1)-(H3) and for each given \( \xi \in L^2(\Omega, \mathcal{F}_T, P) \), there exists at least one solution \( (Y_t, Z_t)_{t \in [0,T]} \in S^2 \times M^2 \) of BDSDE (1). [15] gives also the existence of the minimal solution \( (Y_t, Z_t)_{t \in [0,T]} \) of BDSDE (1) and [16] gives the maximal solution \( (\overline{Y}_t, \overline{Z}_t)_{t \in [0,T]} \) of BDSDE (1) in the sense that any solution \( (Y_t, Z_t)_{t \in [0,T]} \in S^2 \times M^2 \) of BDSDE (1) must satisfy \( Y_t \leq \overline{Y}_t \leq \overline{Y}_t, \) a.s., for all \( t \in [0,T] \).

It is well known that under the standard assumptions where \( f \) is Lipschitz continuous in \( (y,z) \), for any random variable \( \xi \in L^2(\Omega, \mathcal{F}_T, P) \), the BDSDE (1) has a unique adapted solution, say \( (Y_t, Z_t)_{t \in [0,T]} \) such that \( Y \in S^2 \) and \( Z \in M^2 \) (see [12]). And we have the following estimate for solution of BDSDEs with Lipschitz continuous generator \( f \) comes from [12].
Lemma 2.2 If \( \xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_T, P) \), \( f \) is Lipschitz continuous in \((Y, Z)\) and \( g \) satisfies (H3). Then, for the solutions \((Y^1_t, Z^1_t)_{t \in [0, T]}\) and \((Y^2_t, Z^2_t)_{t \in [0, T]}\) of the BDSDEs \((f, g, T, \xi^1)\) and \((f, g, T, \xi^2)\) respectively, we have
\[
E[\sup_{0 \leq t \leq T} |Y^1_t - Y^2_t|^2] \leq CE|\xi^1 - \xi^2|^2,
\]
where \( C \) is a positive constant only depending on Lipschitz constants of \( f \) and \( g \).

Now, we recall some properties and associated approximation about BDSDEs with \( f \) and \( g \) satisfying Assumptions (H1)-(H3) (see [12] for details).

Lemma 2.3 If \( f \) satisfies Assumptions (H1) and (H2), and we set
\[
f_m(\omega, t, y, z) = \inf_{(y', z') \in Q^{1+d}} \{f(\omega, t, y', z') + m(|y - y'| + |z - z'|)\}
\]
and
\[
\overline{f}_m(\omega, t, y, z) = \sup_{(y', z') \in Q^{1+d}} \{f(\omega, t, y', z') - m(|y - y'| + |z - z'|)\}
\]
then for any \( m \geq K \), we have
(i) linear growth: \( \forall (y, z) \in \mathbb{R} \times \mathbb{R}^d \) and \( t \in [0, T] \),
\[
|f_m(t, y, z)| \leq K(1 + |y| + |z|), \quad \text{and} \quad |\overline{f}_m(t, y, z)| \leq K(1 + |y| + |z|).
\]
(ii) monotonicity in \( m \): \( \forall (y, z) \in \mathbb{R} \times \mathbb{R}^d \) and \( t \in [0, T] \), \( f_m(t, y, z) \) is non-decreasing in \( m \) and \( \overline{f}_m(t, y, z) \) is non-increasing in \( m \).
(iii) Lipschitz condition: \( \forall y_1, y_2, z_1, z_2 \in \mathbb{R} \times \mathbb{R}^d \) and \( t \in [0, T] \),
\[
|f_m(t, y, z) - f_m(t, y', z')| \leq m(|y - y'| + |z - z'|),
\]
and
\[
|\overline{f}_m(t, y, z) - \overline{f}_m(t, y', z')| \leq m(|y - y'| + |z - z'|).
\]
(iv) strong convergence: if \((y_m, z_m) \to (y, z)\) then
\[
f_m(t, y_m, z_m) \to f(t, y, z), \quad \text{and} \quad \overline{f}_m(t, y_m, z_m) \to f(t, y, z), \quad \text{as} \ m \to \infty.
\]

Lemma 2.4 We assume \((Y^m_t, Z^m_t) \in S^2 \times M^2\) and \((\overline{Y}^m_t, \overline{Z}^m_t) \in S^2 \times M^2\) are the unique solutions of the BDSDEs \((f_m, g, T, \xi)\) and \((\overline{f}_m, g, T, \xi)\) respectively. Then
\[
(Y^m_t, Z^m_t)_{t \in [0, T]} \to (Y_t, Z_t)_{t \in [0, T]},
\]
and
\[
(\overline{Y}^m_t, \overline{Z}^m_t)_{t \in [0, T]} \to (\overline{Y}_t, \overline{Z}_t)_{t \in [0, T]}, \quad (m \to \infty)
\]
in \( S^2 \times M^2 \), where \((Y_t, Z_t)_{t \in [0, T]}\) and \((\overline{Y}_t, \overline{Z}_t)_{t \in [0, T]}\) are the minimal solution and maximal solution of BDSDE \((\xi)\).
3 A simple case: continuous dependence with respect to terminal condition

This section is devoted to the equivalence of unique solution and continuous dependence with respect to terminal value \( \xi \). Our main result is:

**Theorem 3.1** If Assume (H1)-(H3) hold for \( f \) and \( g \), then the following two statements are equivalent.

(i) Uniqueness: The equation \( f \) has a unique solution.

(ii) Continuous dependence with respect to \( \xi \): For any \( \{\xi_n\}_{n=1}^{\infty}, \xi \in L^2(\Omega, \mathcal{F}_T, P) \), if \( \xi_n \to \xi \) in \( L^2(\Omega, \mathcal{F}_T, P) \) as \( n \to \infty \), then

\[
\lim_{n \to \infty} E\left[ \sup_{t \in [0,T]} |Y^\xi_t - Y_t^\xi|^2 \right] = 0, \tag{2}
\]

where \((Y_t^\xi, Z_t^\xi)_{t \in [0,T]}\) is any solution of BDSDEs (1) and \((Y_n^\xi, Z_n^\xi)_{t \in [0,T]}\) are any solutions of BDSDEs \((f, g, T, \xi_n)\).

**Proof.** Firstly, we prove that (i) implies (ii). Given \( n \), we note that for any solution \((Y_n^\xi, Z_n^\xi)_{t \in [0,T]}\) of BDSDEs \((f, g, T, \xi_n)\), we have

\[
Y_n^\xi_t \leq Y_t^\xi \leq \bar{Y}_t^\xi, \text{ P-a.s. } t \in [0,T], \tag{3}
\]

where \(Y_n^\xi\) and \(\bar{Y}_t^\xi\) are the minimal and maximal solutions of BDSDE \((f, g, T, \xi_n)\), respectively.

Now, we consider the following equations:

\[
\begin{align*}
Y^m_n t &= \xi_n + \int_t^T f_m(s, Y^m_s, Z^m_s, \xi_n) \, ds + \int_t^T g(s, Y^m_s, Z^m_s, \xi_n) \, dB_s - \int_t^T Z^m_s \, dW_s \tag{4}
\end{align*}
\]

and

\[
\begin{align*}
\bar{Y}^m_n t &= \xi_n + \int_t^T f_m(s, \bar{Y}^m_s, \bar{Z}^m_s, \xi_n) \, ds + \int_t^T g(s, \bar{Y}^m_s, \bar{Z}^m_s, \xi_n) \, dB_s - \int_t^T \bar{Z}^m_s \, dW_s \tag{5}
\end{align*}
\]

where \((Y^m_n, Z^m_n)_{t \in [0,T]}\) and \((\bar{Y}^m_n, \bar{Z}^m_n)_{t \in [0,T]}\) are unique solutions of (4) and (3) respectively.

Thanks to Lemma 2.3, we know that

\[
(Y^m_n, Z^m_n) \to (Y_n^\xi, Z_n^\xi) \quad \text{and} \quad (\bar{Y}^m_n, \bar{Z}^m_n) \to (\bar{Y}^\xi_n, \bar{Z}^\xi_n), \quad t \in [0,T]
\]

in \( S^2 \times M^2 \) as \( m \to \infty \), and from Comparison Theorem 3.1 of [16] get the following inequalities

\[
Y^m_n t \leq Y^\xi_n t \leq \bar{Y}^\xi_n t \leq \bar{Y}^m_n t, \quad \text{for any } n, t \in [0,T] \text{ and } m \geq K. \tag{6}
\]

From (3), we have

\[
Y^\xi_n t - Y^\xi t = Y^\xi_n t - \bar{Y}^m_n t + \bar{Y}^m_n t - Y^m_n t + Y^m_n t - Y^\xi t \leq (\bar{Y}^m_n t - Y^m_n t) + (Y^m_n t - Y^\xi t),
\]
and
\[ Y_t^{\xi_n} - Y_t^\xi = Y_t^{m,\xi_n} - Y_t^{m,\xi} + Y_t^{m,\xi_n} - Y_t^{m,\xi} + Y_t^{m,\xi} - Y_t^\xi \]
\[ \geq (Y_t^{m,\xi_n} - Y_t^{m,\xi}) + (Y_t^{m,\xi} - Y_t^\xi). \]

Thus
\[ E[\sup_{t \in [0,T]} |Y_t^{\xi_n} - Y_t^\xi|^2] \leq 2E[\sup_{t \in [0,T]} |Y_t^{m,\xi_n} - Y_t^{m,\xi}|^2] + 2E[\sup_{t \in [0,T]} |Y_t^{m,\xi} - Y_t^\xi|^2] \]
\[ + 2E[\sup_{t \in [0,T]} |Y_t^{m,\xi_n} - Y_t^{m,\xi}|^2] + 2E[\sup_{t \in [0,T]} |Y_t^{m,\xi} - Y_t^\xi|^2], \]
where \((Y_t^{m,\xi_n}, Z_t^{m,\xi_n})_{t \in [0,T]}\) and \((Y_t^{m,\xi_n}, Z_t^{m,\xi_n})_{t \in [0,T]}\) are unique solutions of BDSDEs \((f_{m}, g, T, \xi)\) and \((f_{m}, g, T, \xi)\) respectively.

By Lemma 2.2 and Lemma 2.3, as \(n \to \infty\), we have
\[ E[\sup_{t \in [0,T]} |Y_t^{m,\xi_n} - Y_t^{m,\xi}|^2] \to 0, \quad \text{and} \quad E[\sup_{t \in [0,T]} |Y_t^{m,\xi_n} - Y_t^{m,\xi}|^2] \to 0, \quad \text{for any } m. \]

By Lemma 2.3 and the uniqueness of solution for BDSDEs (1), we get
\[ E[\sup_{t \in [0,T]} |Y_t^{m,\xi} - Y_t^\xi|^2] \to 0 \quad \text{and} \quad E[\sup_{t \in [0,T]} |Y_t^{m,\xi} - Y_t^\xi|^2] \to 0 \]
as \(m \to \infty\). That is (ii).

Now, we prove that (ii) implies (i). We take \(\xi_n = \xi\). For equation \((f, g, T, \xi_n)\), we set \(Y_t^{\xi_n} = \overline{Y}_t^{\xi_n} = \overline{Y}_t^\xi\). For the equation \((1)\), we set \(Y_t^\xi = \overline{Y}_t^\xi\). For (ii), we have \(\overline{Y}_t^\xi = \overline{Y}_t^\xi\). \(\square\)

**Remark 3.2** In fact, when the solution of (1) is not unique, the continuous dependence may not hold true in general. For example, we take \(f(t, y, z) = 3y^{2/3}, \quad \xi = 0\) and \(g\) such that \(g(t, y, 0) = 0\) for all \(t \in [0, T]\), \((y, z) \in \mathbb{R} \times \mathbb{R}^d\). It is easy to know that \((y_t, z_t)_{t \in [0,T]} = (0,0)_{t \in [0,T]}\) and \((Y_t, Z_t)_{t \in [0,T]} = ((T - t)^3, 0)_{t \in [0,T]}\) both are solutions of BDSDE
\[ Y_t = \int_t^T 3Y_s^{2/3} ds + \int_t^T g(s, Y_s, Z_s) dB_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \]

Set \(\xi_n = 1/n\), the BDSDEs
\[ Y_t = \frac{1}{n} + \int_t^T 3Y_s^{2/3} ds + \int_t^T g(s, Y_s, Z_s) dB_s - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad n = 1, 2, \ldots \]
have unique solutions \((y_t^n, z_t^n) = ((T - t + \frac{1}{\sqrt{n}})^3, 0)\) for \(n = 1, 2, \ldots\). But
\[ \lim_{n \to \infty} E[\sup_{t \in [0,T]} |y_t^n - y_t|^2] = T^6 \neq 0 = \lim_{n \to \infty} E[\sup_{t \in [0,T]} |y_t^n - y_t|^2]. \]
4 The general case

In this section, we will deal with the more general case, that is, the relationship between uniqueness of solution and the continuous dependence with respect not only to $\xi$ but also to $f$ and $g$. Now, we consider the following BDSDEs:

$$Y_t^\lambda = \xi^\lambda + \int_t^T f^\lambda(s, Y_s^\lambda, Z_s^\lambda)ds + \int_t^T g^\lambda(s, Y_s^\lambda, Z_s^\lambda)dB_s - \int_t^T Z_s^\lambda dW_s,$$

(7)

where $\lambda$ belongs to a nonempty set $D \subset \mathbb{R}^n$. The coefficients $f^\lambda(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, and $g^\lambda(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$, satisfying the following conditions:

(H1') linear growth: $\exists 0 < K < \infty$, such that $|f^\lambda(\omega, t, y, z)| \leq K(1 + |y| + |z|), \quad \forall \lambda, \omega, t, y, z \in D \times \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d$.

(H2') For fixed $\lambda$, $\omega$ and $t$, $f^\lambda(\omega, t, \cdot, \cdot, \cdot)$ is continuous.

(H3') uniform continuity: $f^\lambda$ and $g^\lambda$ are continuous in $\lambda = \lambda_0$ uniformly with respect to $(y, z)$.

(H4') For fixed $\lambda$, there exist constants $c > 0$ and $0 < \alpha < 1$ such that $|g^\lambda(\omega, t, y^1, z^1) - g^\lambda(\omega, t, y^2, z^2)|^2 \leq c|y^1 - y^2|^2 + \alpha|z^1 - z^2|^2,$

for all $(\omega, t) \in \Omega \times [0, T]$, $(y^1, z^1) \in \mathbb{R} \times \mathbb{R}^d$, $(y^2, z^2) \in \mathbb{R} \times \mathbb{R}^d$.

(H5') Lipschitz condition: $\exists 0 < c < \infty$, such that $|f^\lambda(\omega, t, y^1, z^1) - f^\lambda(\omega, t, y^2, z^2)|^2 \leq c(\|y^1 - y^2\|^2 + |z^1 - z^2|^2),$ for all $(\omega, t) \in \Omega \times [0, T]$, $(y^1, z^1) \in \mathbb{R} \times \mathbb{R}^d$, $(y^2, z^2) \in \mathbb{R} \times \mathbb{R}^d$.

Under (H3')-(H5'), the BDSDE (7) has a unique adapted solution for any $\lambda \in D$. And we have the following property:

Lemma 4.1 If $\xi^\lambda \rightarrow \xi^\lambda_0$ in $L^2(\Omega, \mathcal{F}_T, P)$ as $\lambda \rightarrow \lambda_0$, assumptions (H3')-(H5') hold for $f^\lambda$ and $g^\lambda$. Moreover, $(Y^\lambda_t, Z^\lambda_t)_{t\in[0,T]}$ and $(Y^{\lambda_0}_t, Z^{\lambda_0}_t)_{t\in[0,T]}$ are the solutions of the BDSDEs $(f^\lambda, g^\lambda, T, \xi^\lambda)$ and $(f^{\lambda_0}, g^{\lambda_0}, T, \xi^{\lambda_0})$ respectively, then

$$E\left[ \sup_{t \in [0,T]} |Y^\lambda_t - Y^{\lambda_0}_t|^2 \right] \leq C E|\xi^\lambda - \xi^{\lambda_0}|^2 + CE \int_0^T |f^\lambda(t, Y^\lambda_t, Z^\lambda_t) - f^{\lambda_0}(t, Y^{\lambda_0}_t, Z^{\lambda_0}_t)|^2dt$$

$$+ CE \int_0^T |g^\lambda(t, Y^\lambda_t, Z^\lambda_t) - g^{\lambda_0}(t, Y^{\lambda_0}_t, Z^{\lambda_0}_t)|^2dt,$$

(8)

where $C$ is a positive constant only depending on Lipschitz constant $c$ and $\alpha$. Moreover, we have

$$\lim_{\lambda \rightarrow \lambda_0} E\left[ \sup_{t \in [0,T]} |Y^\lambda_t - Y^{\lambda_0}_t|^2 \right] = 0.$$  

(9)

Proof. By the usual techniques of BDSDEs we can get inequality (8) (see [12] for detail). Because of the continuity of $f^\lambda$ and $g^\lambda$ in $\lambda = \lambda_0$ and Lebesgue dominated convergence theorem we take limit both sides of (8) and get equation (9). □

Now, we introduce the approximation sequences of $f^\lambda$ as follows:

$$f_m^\lambda(\omega, t, y, z) = \inf_{(y', z') \in Q^{1+d}} \{f^\lambda(\omega, t, y', z') + m(|y - y'| + |z - z'|) \}$$
and
\[ \overline{f}_m^\lambda(\omega, t, y, z) = \sup_{(y', z') \in Q^{1+4}} \{ f^\lambda(\omega, t, y', z') - m(|y - y'| + |z - z'|) \} \]

**Lemma 4.2** If \( f^\lambda \) satisfies Assumptions (H1')-(H3'), then for any \( m \geq K \), we have

(i) linear growth: \( \forall (y, z) \in \mathbb{R} \times \mathbb{R}^d \) and \( t \in [0, T] \),
\[ |f^\lambda_m(t, y, z)| \leq K(1 + |y| + |z|), \quad \text{and} \quad |\overline{f}_m^\lambda(t, y, z)| \leq K(1 + |y| + |z|). \]

(ii) monotonicity in \( m \): \( \forall (y, z) \in \mathbb{R} \times \mathbb{R}^d \) and \( t \in [0, T] \), \( f^\lambda_m(t, y, z) \) is non-decreasing in \( m \) and \( \overline{f}_m^\lambda(t, y, z) \) is non-increasing in \( m \).

(iii) Lipschitz condition: \( \forall y_1, y_2 \in \mathbb{R}, z_1, z_2 \in \mathbb{R}^d \) and \( t \in [0, T] \),
\[ |f^\lambda_m(t, y, z) - f^\lambda_m(t, y', z')| \leq m(|y - y'| + |z - z'|), \]
and
\[ |\overline{f}_m^\lambda(t, y, z) - \overline{f}_m^\lambda(t, y', z')| \leq m(|y - y'| + |z - z'|). \]

(iv) strong convergence: if \( (y_m, z_m) \to (y, z) \) as \( m \to \infty \), then
\[ f^\lambda_m(t, y_m, z_m) \to f^\lambda(t, y, z), \quad \text{and} \quad \overline{f}_m^\lambda(t, y_m, z_m) \to \overline{f}^\lambda(t, y, z) \quad \text{as} \quad m \to \infty. \]

(v) Both \( f^\lambda_m \) and \( \overline{f}_m^\lambda \) are continuous in \( \lambda = \lambda_0 \).

**Proof.** It is easy to check (i)-(iv) (see [15]). Now, we prove (v). For any \( \varepsilon > 0 \), by the definition of \( f^\lambda_m \), there exist \( (y^{\varepsilon, \lambda}, z^{\varepsilon, \lambda}) \) and \( (y^{\varepsilon, \lambda_0}, z^{\varepsilon, \lambda_0}) \) such that
\[ f^\lambda(t, y^{\varepsilon, \lambda}, z^{\varepsilon, \lambda}) + m(|y - y^{\varepsilon, \lambda}| + |z - z^{\varepsilon, \lambda}|) - \varepsilon \leq f^\lambda_m(t, y, z) \]
\[ \leq f^\lambda(t, y^{\varepsilon, \lambda_0}, z^{\varepsilon, \lambda_0}) + m(|y - y^{\varepsilon, \lambda_0}| + |z - z^{\varepsilon, \lambda_0}|), \]
and
\[ f^\lambda_0(t, y^{\varepsilon, \lambda_0}, z^{\varepsilon, \lambda_0}) + m(|y - y^{\varepsilon, \lambda_0}| + |z - z^{\varepsilon, \lambda_0}|) - \varepsilon \leq f^\lambda_0(t, y, z) \]
\[ \leq f^\lambda_0(t, y^{\varepsilon, \lambda}, z^{\varepsilon, \lambda}) + m(|y - y^{\varepsilon, \lambda}| + |z - z^{\varepsilon, \lambda}|), \]
thus
\[ f^\lambda(t, y^{\varepsilon, \lambda}, z^{\varepsilon, \lambda}) - f^\lambda_0(t, y^{\varepsilon, \lambda}, z^{\varepsilon, \lambda}) - \varepsilon \leq f^\lambda(t, y, z) - f^\lambda_0(t, y, z) \]
\[ \leq f^\lambda(t, y^{\varepsilon, \lambda_0}, z^{\varepsilon, \lambda_0}) - f^\lambda_0(t, y^{\varepsilon, \lambda_0}, z^{\varepsilon, \lambda_0}) + \varepsilon. \]
Because \( f^\lambda \) is continuous in \( \lambda = \lambda_0 \) uniformly with respect to \( (y, z) \), we obtain the continuity of \( f^\lambda_m \) and \( \overline{f}_m^\lambda \) in \( \lambda = \lambda_0 \). \( \square \)
Lemma 4.3 If \( f^\lambda \) and \( g^\lambda \) satisfy (H1')-(H4'), and the processes \( (Y_{t}^{\lambda,m}, Z_{t}^{\lambda,m})_{t\in[0,T]} \) and \( (\bar{Y}_{t}^{\lambda,m}, \bar{Z}_{t}^{\lambda,m})_{t\in[0,T]} \) are the unique solutions of the BDSDEs \( (f_{\lambda,m}^\lambda, g^\lambda, T, \xi^\lambda) \) and \( (\bar{f}_{\lambda,m}^\lambda, g^\lambda, T, \xi^\lambda) \) respectively, then, for any \( \lambda \in D \), we have

\[
(Y_{t}^{\lambda,m}, Z_{t}^{\lambda,m})_{t\in[0,T]} \rightarrow (Y_{t}^{\lambda}, Z_{t}^{\lambda})_{t\in[0,T]},
\]

and

\[
(\bar{Y}_{t}^{\lambda,m}, \bar{Z}_{t}^{\lambda,m})_{t\in[0,T]} \rightarrow (\bar{Y}_{t}^{\lambda}, \bar{Z}_{t}^{\lambda})_{t\in[0,T]}
\]
in \( S^2 \times M^2 \) as \( m \rightarrow \infty \), where \( (Y_{t}^{\lambda}, Z_{t}^{\lambda})_{t\in[0,T]} \) and \( (\bar{Y}_{t}^{\lambda}, \bar{Z}_{t}^{\lambda})_{t\in[0,T]} \) are the minimal solution and maximal solution of BDSDE (7).

Now, we give our result for the general case.

Theorem 4.4 If \( f^\lambda \) and \( g^\lambda \) satisfy (H1')-(H4'), then the following statements are equivalent.

(iii) Uniqueness: there exists a unique solution of BDSDE (7) with \( \lambda = \lambda_0 \), that is, the solution of \( (f^{\lambda_0}, g^{\lambda_0}, T, \xi^{\lambda_0}) \) is unique.

(iv) Continuous dependence with respect to \( f, g \) and \( \xi \): for any \( \xi^\lambda, \xi^{\lambda_0} \in L^2(\Omega, \mathcal{F}_T, P) \), if \( \xi^\lambda \rightarrow \xi^{\lambda_0} \) in \( L^2(\Omega, \mathcal{F}_T, P) \) as \( \lambda \rightarrow \lambda_0 \), \( (Y_{t}^{\lambda}, Z_{t}^{\lambda})_{t\in[0,T]} \) are any solutions of BDSDE (7), \( (Y_{t}^{\lambda_0}, Z_{t}^{\lambda_0})_{t\in[0,T]} \) is any solution of BDSDE (7) with \( \lambda = \lambda_0 \), then

\[
\lim_{\lambda \rightarrow \lambda_0} E[\sup_{t\in[0,T]} |Y_{t}^{\lambda} - Y_{t}^{\lambda_0}|^2] = 0.
\]

Proof. This proof is similar to that of Theorem 3.1. For the sake of completeness, we give the sketch of proof. Firstly, we prove (iii) implies (iv). We can get the inequalities similarly to (6), that is, from Comparison Theorem 3.1 of [16] get the following inequalities

\[
Y_{t}^{\lambda,m} \leq Y_{t}^{\lambda} \leq \bar{Y}_{t}^{\lambda} \leq \bar{Y}_{t}^{\lambda,m}, \text{ for any } t \in [0, T] \text{ and } m \geq K.
\]

So

\[
E[\sup_{t\in[0,T]} |Y_{t}^{\lambda} - Y_{t}^{\lambda_0}|^2] \leq 2E[\sup_{t\in[0,T]} |Y_{t}^{\lambda,m} - Y_{t}^{\lambda_0,m}|^2] + 2E[\sup_{t\in[0,T]} |Y_{t}^{\lambda_0,m} - Y_{t}^{\lambda_0}|^2] + 2E[\sup_{t\in[0,T]} |\bar{Y}_{t}^{\lambda,m} - \bar{Y}_{t}^{\lambda_0,m}|^2] + 2E[\sup_{t\in[0,T]} |\bar{Y}_{t}^{\lambda_0,m} - \bar{Y}_{t}^{\lambda_0}|^2].
\]

Fixed \( m \), by Lemma 4.1 and Lemma 4.2 and the continuity of \( f_{\lambda,m} \) and \( \bar{f}_{\lambda,m} \) in \( \lambda = \lambda_0 \), we have

\[
E[\sup_{t\in[0,T]} |Y_{t}^{\lambda,m} - Y_{t}^{\lambda_0,m}|^2] \rightarrow 0 \text{ and } E[\sup_{t\in[0,T]} |\bar{Y}_{t}^{\lambda,m} - \bar{Y}_{t}^{\lambda_0,m}|^2] \rightarrow 0
\]
as $\lambda \to \lambda_0$, for any $m \geq K$. By Lemma 4.3 and the uniqueness of solution for BDSDEs $(f^{\lambda_0}, g^{\lambda_0}, T, \xi^{\lambda_0})$ (Condition (iii)), we get, as $m \to \infty$,

$$E[\sup_{t \in [0,T]} |Y_t^{\lambda_0,m} - Y_t^{\lambda_0}|^2] \to 0 \text{ and } E[\sup_{t \in [0,T]} |Y_t^{\lambda_0,m} - Y_t^{\lambda_0}|^2] \to 0.$$  

This implies (iv).

Now, we prove that (iv) implies (iii). We take $\xi^\lambda = \xi^{\lambda_0}, f^\lambda = f^{\lambda_0}, g^\lambda = g^{\lambda_0}$. For equation (7), set $Y_t^\lambda = Y_t^{\lambda_0}$. For the equation $(f^{\lambda_0}, g^{\lambda_0}, T, \xi^{\lambda_0})$, we set $Y_t^{\lambda_0} = Y_t^\lambda$. For (iv), we have $\xi_t^\lambda = \xi_t^{\lambda_0}$. □

**Remark 4.5** In the standard situation where $f$ satisfies linear growth condition and Lipschitz condition in $(y, z)$, it has been proved by Pardoux and Peng [12] that there exists a unique solution. In this case, the continuous dependence with respect to $f$, $g$ and $\xi$ is described by the inequality (8) (see [15]). Our result in this paper, which can be regarded as the analog of the inequality (8) in some sense, provides a useful method to study BSDEs with continuous coefficient.

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