An intermediate value theorem
for sequences with terms in a finite set

Mihai Caragiu
Department of Mathematics, Ohio Northern University
m-caragiu1@onu.edu

Laurence D. Robinson
Department of Mathematics, Ohio Northern University
l-robinson.1@onu.edu

Abstract
We prove an intermediate value theorem of an arithmetical flavor, involving the consecutive averages \( \{ \bar{x}_n \} \) of sequences with terms in a given finite set \( \{ a_1, ..., a_r \} \). For every such set we completely characterize the numbers \( \Pi \) ("intermediate values") with the property that the consecutive averages \( \{ \bar{x}_n \} \) of every sequence \( \{ x_n \} \) with terms in \( \{ a_1, ..., a_r \} \) cannot increase from a value \( \bar{x}_k < \Pi \) to a value \( \bar{x}_l > \Pi \) without taking the value \( \bar{x}_s = \Pi \) for some \( s \) with \( k < s < l \).

(2000) Mathematics Subject Classification: 11B99, 26D15
Keywords: sequences, averages, intermediate values
1 INTRODUCTION

Let $r \geq 1$ be an integer and let $a_1, a_2, ..., a_r$ be real numbers with $a_1 < a_2 < ... < a_r$

Define $SEQ(a_1, ..., a_r)$ to be the set of all sequences $\{x_n\}_{n \geq 1}$ such that $x_n \in \{a_1, a_2, ..., a_r\}$ for all $n \geq 1$. For example $SEQ(0, 1)$ is the set of all binary sequences.

To each sequence $\{x_n\}_{n \geq 1}$ we associate the sequence of consecutive averages $\{\overline{x}_n\}_{n \geq 1}$ defined by

$$\overline{x}_n = \frac{x_1 + ... + x_n}{n}$$

Clearly, if $\{x_n\}_{n \geq 1} \in SEQ(a_1, ..., a_r)$ then

$$a_1 \leq \overline{x}_n \leq a_r$$

for all $n \geq 1$.

We are now in a position to define the sets which will be studied in the current article.

DEFINITION. For $a_1 < a_2 < ... < a_r$ let us define

$$IV(a_1, ..., a_r)$$

to be the set of all numbers $\Pi \in (a_1, a_r)$ with the following "intermediate value property": if $\{x_n\}_{n \geq 1} \in SEQ(a_1, ..., a_r)$ and if $\overline{x}_k < \Pi < \overline{x}_l$ for some integers $k < l$ then there exists an integer $s$ with $k < s < l$ such that $\overline{x}_s = \Pi$.

A Putnam Exam problem [1] asks whether $\frac{4}{5}$ is in $IV(0, 1)$. Indeed the answer turns out to be affirmative. More generally, we can make the following statement:

THEOREM 1. $IV(0, 1) = \left\{ \frac{k}{k+1} : k \geq 1 \right\} \subset (0, 1)$. 
In the present paper we will fully generalize the above Theorem 1, providing a complete description of all "sets of intermediate values" \( IV(a_1, ..., a_r) \). In particular we will determine necessary and sufficient conditions under which \( IV(a_1, ..., a_r) \neq \emptyset \).

**NOTE.** By definition, the numbers \( \Pi \in IV(a_1, ..., a_r) \) are precisely those which cannot be "skipped" or "jumped over" by increasing averages. In the last section we will discuss the case of intermediate values which cannot be skipped by decreasing averages. This being said, in the next three sections, the term "skipped" will signify "skipped" by averages going up (e.g. \( \Pi = 0.7 \) being skipped at the step between the third and the fourth averages of the sequence 0,1,1,1,...).

### 2 CASE OF BINARY SEQUENCES

To prove **THEOREM 1**, we first show that

\[
IV(0,1) \subseteq \left\{ \frac{k}{k+1} : k \geq 1 \right\}.
\]

Indeed, if

\[
\Pi \in \left(0, \frac{1}{2}\right) \cup \left(\frac{1}{2}, \frac{2}{3}\right) \cup \left(\frac{2}{3}, \frac{3}{4}\right) \cup \left(\frac{3}{4}, \frac{4}{5}\right) \cup ...\]

then the consecutive averages of the sequence

\[
0, 1, 1, 1, 1, ...
\]

will skip \( \Pi \).

We now prove the reverse inclusion, namely,

\[
\left\{ \frac{k}{k+1} : k \geq 1 \right\} \subseteq IV(0,1).
\]

That is, we prove that if \( \Pi = \frac{k}{k+1}, k \geq 1 \) then \( \Pi \) cannot be skipped by a sequence of consecutive averages \( \{\bar{x}_n\}_{n \geq 1} \). We will proceed by contradiction. Assuming \( \{\bar{x}_n\}_{n \geq 1} \) skips \( \Pi = \frac{k}{k+1} \) it follows that

\[
(1) \quad \bar{x}_n < \frac{k}{k+1} < \bar{x}_{n+1}
\]

3
for some \( n \geq 1 \). First note that \( x_{n+1} \) must be 1, because if \( x_{n+1} = 0 \) then the average \( \bar{x}_{n+1} \) cannot be larger than \( \bar{x}_n \). Thus, if we denote \( S = x_1 + \ldots + x_n \), (1) can be rewritten as follows:

\[
\frac{S}{n} < \frac{k}{k+1} < \frac{S+1}{n+1}.
\]

By cross-multiplying (2) is equivalent with the system of the following two inequalities:

(3) \((k+1)S < nk\),

and

(4) \((n+1)k < (k+1)(S+1)\).

From (3) and (4) it follows that \( nk - 1 < (k+1)S < nk \), which is impossible, as all three terms are integers, and as there can be no integer falling between consecutive integers. This concludes the proof of THEOREM 1.

From THEOREM 1, a simple linearity argument leads us to the following result.

**THEOREM 2.** If \( a < b \), then

\[
IV(a, b) = \left\{ \frac{1}{k+1}a + \frac{k}{k+1}b : k \geq 1 \right\}.
\]

In the next section we will consider sequences with terms in a set with three elements.

### 3 CASE OF TERNARY SEQUENCES

Let \( 0 < \mu < 1 \). In order to find \( IV(0, \mu, 1) \) we distinguish between the case of an irrational \( \mu \) and the case of a rational \( \mu \). The easier case is the case of
an irrational $\mu$. Then there are no intermediate values for the consecutive averages of sequences with terms in the set $\{0, \mu, 1\}$. In other words:

**THEOREM 3.** If $0 < \mu < 1$ is irrational then $IV(0, \mu, 1) = \emptyset$.

**PROOF.** We already know that every $\Pi \in (0, 1)$ that is not of the form $\frac{k}{k+1}$ can be skipped by the averages of some sequence in $SEQ(0, 1) \subset SEQ(0, \mu, 1)$. It will be enough to show that if $\mu$ is irrational then every $\Pi \in (0, 1)$ that is of the form $\frac{k}{k+1}$ can be skipped by the averages of some sequence in $SEQ(0, \mu, 1)$. Indeed, every $\Pi = \frac{k}{k+1} < \mu$ will be skipped by the consecutive averages of the sequence

$0, \mu, \mu, \mu, ...$

(the averages form an increasing sequence of irrationals with limit $\mu$), while every $\Pi = \frac{k}{k+1} > \mu$ will be skipped by the consecutive averages of the sequence

$\mu, 1, 1, 1, ...$

(here, the averages form an increasing sequence of irrationals with limit 1). This concludes the proof of THEOREM 3.

Next we consider the case $0 < \mu = \frac{p}{q} < 1$ with $p, q$ relatively prime positive integers. We will prove the following result:

**THEOREM 4.** If $0 < \frac{p}{q} < 1$ with $p, q$ relatively prime, then

$IV(0, \frac{p}{q}, 1) = \left\{ 1 - \frac{1}{qt} : t = 1, 2, 3, ... \right\}$.

**PROOF.** We know that every $\Pi \in (0, 1)$ that is not of the form $\frac{k}{k+1}$ can be skipped. The question that remains is which numbers of the form $\Pi = \frac{k}{k+1}$ can be skipped by the consecutive averages of some sequence in $SEQ(0, \mu, 1)$.

First note that the sequence

$x_1 = \frac{p}{q}, x_2 = x_3 = ... = 1$

has consecutive averages of the form

$$\frac{p + l}{l + 1} = \frac{p + ql}{q + ql}$$

(5)
where \( l = 0, 1, 2, \ldots \). Then every \( \Pi = \frac{k}{k+1} > \frac{p}{q} \) that is not of the form \( \frac{p+ql}{q+ql} \) can be skipped by the sequence of increasing averages (5), so it cannot be in \( IV(0, \frac{p}{q}, 1) \). We now need to determine which fractions of the form \( \frac{p+ql}{q+ql} \) are also of the form \( \frac{k}{k+1} \). The equality

\[
\frac{p+ql}{q+ql} = \frac{k}{k+1}
\]

can be rewritten in the following equivalent form:

(6) \[ p(k+1) = q(k-l). \]

From (6), keeping in mind that \( p, q \) are relatively prime, we get

\[ k + 1 = qt, \]

and

\[ k - l = pt \]

for some integer \( t \). In this case, \( \frac{k}{k+1} \) is of the form \( \frac{qt-1}{qt} = 1 - \frac{1}{qt} \). As a consequence, if a \( \Pi = \frac{k}{k+1} > \frac{p}{q} \) is in the set of intermediate values \( IV(0, \frac{p}{q}, 1) \) then \( q \) divides \( k + 1 \).

We will now prove that there is no \( \Pi \) in \( IV(0, \frac{p}{q}, 1) \) that is of the form \( \frac{k}{k+1} \) and is less than \( \frac{p}{q} \). To do this, we will show that if

\[
\Pi = \frac{k}{k+1} < \frac{p}{q}
\]

then there exists a sequence in \( SEQ(0, \frac{p}{q}, 1) \) whose averages skip \( \Pi \). Indeed, let us consider the sequence

(7) \[ x_1 = 0, x_2 = x_3 = \ldots = \frac{p}{q}. \]

The averages of the sequence (7) form an increasing sequence approaching \( \frac{p}{q} \). If no \( \bar{x}_k \) equals \( \Pi \), then \( \Pi \) will be skipped. If \( \bar{x}_n = \Pi \) for some \( n \), then we may consider the sequence

(8) \[ x_1 = 0, x_2 = x_3 = \ldots = x_{n-1} = \frac{p}{q}, x_n = 1 \]
obtained by changing the $n$-th term of (7) into a one. Clearly, $\Pi$ will be skipped at the transition between the $n-1$-th and the $n$-th averages of the sequence (8).

At this point we know that every $\Pi \in (0, 1)$ that is not of the form $1 - \frac{1}{qt}$ can be skipped by the consecutive averages of some sequence in $SEQ(0, \frac{p}{q}, 1)$, in other words,

$$IV(0, \frac{p}{q}, 1) \subseteq \left\{ 1 - \frac{1}{qt} : t \geq 1 \right\}.$$  

The reverse inclusion

$$\left\{ 1 - \frac{1}{qt} : t \geq 1 \right\} \subseteq IV(0, \frac{p}{q}, 1).$$

will be proved by contradiction. Assume that $\frac{qt-1}{qt}$ can be skipped by the consecutive averages of some sequence in $SEQ(0, \frac{p}{q}, 1)$. Without loss of generality we may assume that $\frac{qt-1}{qt}$ is in between the average

$$\bar{x}_n = \frac{x_1 + ... + x_n}{n}$$

with $u$ of the $x_1, ..., x_n$ being zeros, $v$ being $\frac{p}{q}$ and $w$ being ones ($u + v + w = n$) and the average

$$\bar{x}_{n+1} = \frac{x_1 + ... + x_{n+1}}{n + 1}$$

with $u$ of the $x_1, ..., x_n$ being zeros, $v$ being $\frac{p}{q}$ and $w+1$ being ones ($x_{n+1} = 1$):

$$\frac{v\frac{p}{q} + w}{n} < \frac{qt - 1}{qt} < \frac{v\frac{p}{q} + w + 1}{n + 1}.$$  

Equivalently, (11) can be rewritten as follows:

$$\frac{pv + qw}{n} < \frac{qt - 1}{t} < \frac{pv + qw + q}{n + 1},$$

which is equivalent with the system consisting of the following two inequalities:

$$pvt + qwt < nqt - n$$

and

$$nqt - n + qt - 1 < pvt + qwt + qt$$
By using $n = u + v + w$ and after simplifying, the above two inequalities will be equivalent to the following system:

(12) \[ pvt < qut + qvt - n \]

and

(13) \[ qut + qvt - n - 1 < pvt. \]

From (12) and (13) it follows that

\[ pvt < qut + qvt - n < pvt + 1, \]

which is again a contradiction (as there can be no integer falling between consecutive integers). This concludes the proof of the reverse inclusion (10). From (9) and (10), THEOREM 4 follows.

A straightforward linearity argument based on the previous two theorems leads us to the following intermediate value theorem characterizing all sets $IV(a, b, c)$.

**THEOREM 5.** Let $a < b < c$ and let $\mu := \frac{b-a}{c-a}$. If $\mu$ is irrational then $IV(a, b, c) = \emptyset$.

If $\mu = \frac{p}{q}$ with $p, q$ relatively prime positive integers, then

\[ IV(a, b, c) = \left\{ \left(1 - \frac{1}{qt}\right)c + \frac{1}{qt}a : t = 1, 2, 3, \ldots \right\} \]

4 THE GENERAL INTERMEDIATE VALUE THEOREM

We will now completely characterize the intermediate value sets of the form $IV(0, \mu_1, \ldots, \mu_r, 1)$ where $0 < \mu_1 < \ldots < \mu_r < 1$. First, note we can immediately obtain the following result.
THEOREM 6. If $\mu_i$ is irrational for some $i = 1, 2, ..., r$, then

$$IV(0, \mu_1, ..., \mu_r, 1) = \emptyset.$$  

PROOF. Follows from THEOREM 3, since if $\mu_i$ is rational then every $\Pi \in (0, 1)$ can be skipped by the averages of some sequence in

$$SEQ(0, \mu_i, 1) \subset SEQ(0, \mu_1, ..., \mu_r, 1).$$

Now let us assume that all $\mu_i$’s are rational:

$$\mu_i = \frac{p_i}{q_i}, \ i = 1, .., r,$$

with $gcd(p_i, q_i) = 1$ for all $i = 1, ..., r$.

Let $M$ be the least common multiple of the denominators of the reduced fractions $\frac{p_i}{q_i}, i = 1, ..., r$. We will prove that the following result holds true.

THEOREM 7. With the above notations, we have

$$(14) \quad IV(0, \mu_1, ..., \mu_r, 1) = \left\{ 1 - \frac{1}{Mt} : t = 1, 2, 3, ... \right\}$$

PROOF. Let $i \in \{1, 2, ..., r\}$. From THEOREM 4 it follows that for $A \geq 2$, the element

$$\Pi = 1 - \frac{1}{A}$$

will be skipped by the averages of some sequence in

$$SEQ(0, \frac{p_i}{q_i}, 1) \subset SEQ(0, \mu_1, ..., \mu_r, 1),$$

as long as $q_i$ does not divide $A$. Thus, if $\Pi = 1 - \frac{1}{A}$ cannot be skipped by the averages of the sequences in $SEQ(0, \mu_1, ..., \mu_r, 1)$ then $q_1|A, q_2|A, ..., q_r|A$, that is, $M|A$, or

$$\Pi = 1 - \frac{1}{Mt}$$
for some \( t \geq 1 \) (the number theory background necessary for the current paper can be found, for example, in [2], Chapter 1). Thus we have proved that

\[(15) \quad IV(0, \mu_1, ..., \mu_r, 1) \subseteq \left\{ 1 - \frac{1}{Mt} : t = 1, 2, 3, \ldots \right\} .\]

To complete the proof we will prove the reverse inclusion:

\[(16) \quad \left\{ 1 - \frac{1}{Mt} : t = 1, 2, 3, \ldots \right\} \subseteq IV(0, \mu_1, ..., \mu_r, 1).\]

We proceed by contradiction. Assume that \( \frac{Mt-1}{Mt} \) can be skipped by the consecutive averages of some sequence in \( SEQ(0, \mu_1, ..., \mu_r, 1) \). Without loss of generality we may assume that \( \frac{Mt-1}{Mt} \) is in between the average

\[\bar{x}_n = \frac{x_1 + \ldots + x_n}{n}\]

with \( u \) of the \( x_1, ..., x_n \) being zeros, \( v_1 \) being \( \frac{p_1}{q_1} \), \( v_2 \) being \( \frac{p_2}{q_2} \), ..., \( v_r \) being \( \frac{p_r}{q_r} \), and \( w \) being ones \( (u + v_1 + \ldots + v_r + w = n) \), and the average

\[\bar{x}_{n+1} = \frac{x_1 + \ldots + x_{n+1}}{n+1}\]

with \( u \) of the \( x_1, ..., x_n, x_{n+1} = 1 \) being zeros, \( v_1 \) being \( \frac{p_1}{q_1} \), \( v_2 \) being \( \frac{p_2}{q_2} \), ..., \( v_r \) being \( \frac{p_r}{q_r} \), and \( w+1 \) being ones (we took \( x_{n+1} = 1 \) which leads to the greatest possible increase in the average):

\[(17) \quad \frac{v_1 \frac{p_1}{q_1} + \ldots + v_r \frac{p_r}{q_r} + w}{n} < \frac{Mt-1}{Mt} < \frac{v_1 \frac{p_1}{q_1} + \ldots + v_r \frac{p_r}{q_r} + w + 1}{n+1} .\]

For every \( i = 1, ..., r \) let us define

\[Q_i := \frac{lcm(q_1, \ldots, q_r)}{q_i} = \frac{M}{q_i}\]

With this notation, a multiplication of all terms in (17) by \( M \) gives

\[\frac{p_1 Q_1 v_1 + \ldots + p_r Q_r v_r + Mw}{n} < \frac{Mt-1}{t} < \frac{p_1 Q_1 v_1 + \ldots + p_r Q_r v_r + Mw + M}{n+1},\]
which, by cross-multiplications turns out to be equivalent to the following system of inequalities:

\[(18) \quad p_1 Q_1 v_1 t + ... + p_r Q_r v_r t + Mwt < Mnt - n,\]

and

\[(19) \quad Mnt + Mt - n - 1 < p_1 Q_1 v_1 t + ... + p_r Q_r v_r t + Mwt + Mt\]

Finally, from (18) and (19) it follows that

\[p_1 Q_1 v_1 t + ... + p_r Q_r v_r t + Mwt < Mnt - n < p_1 Q_1 v_1 t + ... + p_r Q_r v_r t + Mwt + 1\]

which is, again, a contradiction (as there can be no integer falling between consecutive integers). This shows that (16) is true. From (15) and (16), (14) follows. This concludes the proof of THEOREM 7.

From the above result, a linearity argument leads us to the following arithmetic intermediate value theorem.

**THEOREM 8.** Let \( a_1 < a_2 < ... < a_r \) (\( r \geq 3 \)). For \( i = 2, ..., r - 1 \), let

\[\mu_i := \frac{a_i - a_1}{a_r - a_1}.\]

Then the following hold true.

a) If for some \( i = 2, ..., r - 1 \) the number \( \mu_i \) is irrational, then \( IV(a_1, ..., a_r) = \emptyset \).

b) If \( \mu_2, ..., \mu_{r-1} \) are all rational numbers, \( \mu_i = \frac{p_i}{q_i} \), with \( \gcd(p_i, q_i) = 1 \) for \( i = 2, ..., r - 1 \) and \( M = \text{lcm}(q_2, ..., q_{r-1}) \), then

\[IV(a_1, ..., a_r) = \left\{ \frac{1}{Mt} a_1 + \left(1 - \frac{1}{Mt}\right) a_r : t = 1, 2, 3, ... \right\}\]
5 FURTHER COMMENTS

Note that for \( a_1 < a_2 < \ldots < a_r \) the sets \( IV(a_1, \ldots, a_r) \) represent the values \( \Pi \) with the property that the consecutive averages \( \{\bar{x}_n\} \) of every sequence \( \{x_n\}_{n \geq 1} \in SEQ(a_1, \ldots, a_r) \) cannot increase from a value \( \bar{x}_k < \Pi \) to a value \( \bar{x}_l > \Pi \) without taking the value \( \bar{x}_s = \Pi \) for some \( s \) with \( k < s < l \). Similarly we can define the sets

\[
DV(a_1, \ldots, a_r)
\]

representing the the values \( \Pi \) with the property that the consecutive averages \( \{\bar{x}_n\} \) of every sequence \( \{x_n\}_{n \geq 1} \in SEQ(a_1, \ldots, a_r) \) cannot decrease from a value \( \bar{x}_k > \Pi \) to a value \( \bar{x}_l < \Pi \) without taking the value \( \bar{x}_s = \Pi \) for some \( s \) with \( k < s < l \).

The connection between the sets \( IV(a_1, \ldots, a_r) \) and \( DV(a_1, \ldots, a_r) \) can be expressed in a simple way as follows:

\[
DV(a_1, \ldots, a_r) = -IV(-a_r, \ldots, -a_1).
\]

The proof of (20) is straightforward if we use the transformation

\[
\{x_n\}_{n \geq 1} \mapsto \{-x_n\}_{n \geq 1}.
\]

Clearly, (21) is a one-to-one correspondence between \( SEQ(a_1, \ldots, a_r) \) and \( SEQ(-a_r, \ldots, -a_1) \). Under this correspondence, the sequence of averages of \( \{x_n\}_{n \geq 1} \) skips (going up) \( \Pi \in (a_1, a_r) \) if and only if the sequence of averages of \( \{-x_n\}_{n \geq 1} \) skips (going down) \( -\Pi \in (-a_r, -a_1) \).

We can use (21) to translate the Theorems 2 and 8 for decreasing trends. Thus, we obtain

**THEOREM 9.** If \( a < b \), then

\[
DV(a, b) = \left\{ \frac{k}{k+1}a + \frac{1}{k+1}b : k \geq 1 \right\},
\]

and
THEOREM 10. Let $a_1 < a_2 < \ldots < a_r$ $(r \geq 3)$. For $i = 2, \ldots, r - 1$, let

$$\mu_i := \frac{a_i - a_1}{a_r - a_1}.$$ 

Then the following hold true.

a) If for some $i = 2, \ldots, r - 1$ the number $\mu_i$ is irrational, then $DV(a_1, \ldots, a_r) = \emptyset$.

b) If $\mu_2, \ldots, \mu_{r-1}$ are all rational numbers, $\mu_i = \frac{p_i}{q_i}$, with $\gcd(p_i, q_i) = 1$ for $i = 2, \ldots, r - 1$ and $M = \text{lcm}(q_2, \ldots, q_{r-1})$, then

$$DV(a_1, \ldots, a_r) = \left\{ \left( 1 - \frac{1}{Mt} \right) a_1 + \frac{1}{Mt} a_r : t = 1, 2, 3, \ldots \right\}$$

REFERENCES

[1] 2004 Putnam Exam, Problem A1
[2] I. Niven, H.S. Zuckerman and H.L. Montgomery, An Introduction to the Theory of Numbers, 5-th edition, Wiley 1991.