SMALL CLASS NUMBER FIELDS IN THE FAMILY $\mathbb{Q}(\sqrt{9m^2 + 4})$

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Abstract. We study the class number one problem for real quadratic fields $\mathbb{Q}(\sqrt{9m^2 + 4})$, where $m$ is an odd integer. We show that for $m \equiv 1 \pmod{3}$ there is only one such field with class number one and only one such field with class number two.

1. Introduction

The size of the class group of a number field $K$, called the class number of $K$, is an important object of study in algebraic number theory. The ring $\mathcal{O}_K$, of algebraic integers in $K$, is a unique factorization domain if and only if the class group of $K$ is trivial. It is known that there are only nine imaginary quadratic fields with class number one (see [1, 19]). On the other hand, it was conjectured by Gauss that there are infinitely many real quadratic fields with class number one (see [9]). This conjecture is still open. However, there are several partial/prelim results (see [2, 3, 11, 16, 5]). In particular, we mention some results showing finiteness of class number one real quadratic fields in special families of real quadratic fields.

In [2], proving a conjecture of Yokoi, Biro showed that, for any positive integer $m$ such that $m^2 + 4$ is square-free, the real quadratic field $\mathbb{Q}(\sqrt{m^2 + 4})$ has class number one precisely when $m \in \{1, 3, 5, 7, 13, 17\}$. Along the similar lines, in [3] Biro proved a conjecture of Chowla; if $m$ is a positive integer such that $4m^2 + 1$ is square free then the real quadratic field $\mathbb{Q}(\sqrt{4m^2 + 1})$ has class number one if and only if $m \in \{1, 2, 3, 5, 7, 13\}$.

Another line of work, in these contexts, is finding necessary, and sufficient conditions for the class number of a real quadratic field to be one. We mention the work of Mollin [16] and Loboutin [15]. In [5], Byeon and Kim obtained one such criterion for real quadratic fields of Richaud-Degert type (R-D type); these are fields of the form $\mathbb{Q}(\sqrt{d})$, where $d = n^2 + r$ is square-free with $n, r \in \mathbb{Z}, -n < r \leq n$ and $r | 4n$ (for definitions of R-D type fields in narrow sense and R-D type fields in wide sense

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we refer [20, 21]). There are many results for small class number of R-D type fields (see [5, 6, 7, 14]) as the fundamental units of such fields are explicitly known [8].

In this article we obtain lower bound on class number of fields which are not of R-D type. We study fields of the form \( \mathbb{Q}(\sqrt{9m^2 + 4m}) \), where \( m \) is an odd integer.

For any odd integer \( m \), we put \( D = 9m^2 + 4m \) and \( K = \mathbb{Q}(\sqrt{D}) \). Let \( \zeta_K(s) \) denote the Dedekind zeta function of the number field \( K \) and \( h_K \) denote its class number. We use the notation \( N \) to denote the number of distinct prime divisors of \( m \) which are greater than 3. We prove the following theorem in this article.

**Theorem 1.1.** Let \( m \) be any odd integer such that \( m \not\equiv 2 \pmod{3} \) and \( D \) is square-free. Then \( h_K = 1 \) if and only if \( m \in \{-3, 1, 3\} \).

We remark that the case \( m > 0 \) of Theorem 1.1 follows from the main theorem proved in [4]. However, our approach differs from the approach of Biro and Lapkova. From our approach, we obtain better lower bound on class number of \( \mathbb{Q}(\sqrt{9m^2 + 4m}) \) in the case \( m \equiv 1 \pmod{3} \). To this effect we prove the following theorem.

**Theorem 1.2.** Let \( m \equiv 1 \pmod{3} \) be such that \( D = 9m^2 + 4m \) is square-free. If \( m \not\in \{-5, 1\} \) then \( h_K \geq 3 \).

In case \( m \) has large number of prime factors then the class number \( h_K \) is large. We record this in the following theorem.

**Theorem 1.3.** If \( m \) has three or more prime factors and \( D \) is square-free then

\[
h_K \geq \begin{cases} 
1 + N & \text{if } m \equiv 2 \pmod{3}, \\
2 + N & \text{otherwise}, 
\end{cases}
\]

where \( N \) is number of prime factors of \( m \) bigger than 3.

For integers \( d \) and \( N \) the Diophantine equation \( x^2 - dy^2 = N \) is widely studied, under the popular name ‘Pell’s equations’. The necessary and sufficient conditions for the solvability of such equations are known for long. We refer the article of Mollin [17] and the references therein. Employing the techniques used in the study of the above problems, we present a family of Pell’s equations and prove that they are always soluble.

**Theorem 1.4.** Let \( m \) be an odd positive square-free integer such that \( q = 9m + 4 \) is a prime. Then for \( D = m(9m + 4) \), the Diophantine equation \( x^2 - Dy^2 = 4q \) is always soluble.
In Section 2 we recall some results on special values of Dedekind zeta function and develop preliminaries required for the proofs. In Section 3 we give all the proofs. Section 4 presents some computations performed by us with the help of Pari.

2. Preliminaries

We begin this section by mentioning two methods to compute special values of Dedekind zeta function associated to a real quadratic field. By specializing Siegel’s formula (see [18]) for \( \zeta_K(1-2n) \), in [22], Zagier obtained the value of \( \zeta_K(-1) \) for any real quadratic field. We mention this in the following theorem.

**Theorem 2.1.** (Zagier [22]) Let \( K \) be a real quadratic field with discriminant \( D \). Then

\[
\zeta_K(-1) = \frac{1}{60} \sum_{\substack{|t| < \sqrt{D} \\ t^2 \equiv D \pmod{4}}} \sigma \left( \frac{D - t^2}{4} \right),
\]

where \( \sigma(n) \) denotes the sum of divisors of \( n \).

However, there is another method, due to H. Lang (see [13]), for computing special values of partial Dedekind zeta function of an ideal class in real quadratic fields. Let \( K = \mathbb{Q}(\sqrt{d}) \) be a real quadratic field with discriminant \( D \) and fundamental unit \( \epsilon_d \). Let \( \mathfrak{A} \) be an ideal class in the class group of \( K \), \( \mathfrak{a} \) be an integral ideal in the class \( \mathfrak{A}^{-1} \) with an integral basis \( \{r_1, r_2\} \). Let \( r'_1, r'_2 \) denote the conjugate of \( r_1, r_2 \) respectively and let

\[
\delta(\mathfrak{a}) = r_1r'_2 - r'_1r_2.
\]

Let \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) be an integral matrix satisfying

\[
\epsilon \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = M \begin{bmatrix} r'_1 \\ r'_2 \end{bmatrix}.
\]

Then the partial zeta value \( \zeta_K(-1, \mathfrak{A}) \), associated to the ideal class \( \mathfrak{A} \), was obtained by H. Lang (see [13]) in terms of the parameters \( \delta(\mathfrak{a}), a, b, c, d \) and generalized Dedekind sums.

**Theorem 2.2.** (Lang [13]) With the above notations, we have

\[
(1) \quad \zeta_K(-1, \mathfrak{A}) = \frac{\text{sgn}\delta(\mathfrak{a})r_2r'_2}{360N(\mathfrak{a})c^3}\{(a + d)^3 - 6(a + d)N(\epsilon) - 240c^3(\text{sgnc})S^3(a, c) \\
+ 180ac^3(\text{sgnc})S^2(a, c) - 240c^3(\text{sgnc})S^3(d, c) + 180dc^3(\text{sgnc})S^2(d, c)\},
\]
where \( N(a) \) is the norm of the ideal \( a \), and \( S^i(a, c) = S_i^1(a, c) \) denote the generalised Dedekind sum.

Now we mention some special values of generalized Dedekind sum as obtained in [12].

Lemma 2.3. For any positive integer \( m \), we have

1. \( S^3(\pm 1, m) = \pm(-m^4 + 5m^2 - 4)/(120m^3) \)
2. \( S^2(\pm 1, m) = (m^4 + 10m^2 - 6)/(180m^3) \)

For other values of generalized Dedekind sum we recall the formula obtained by Lang. For \( n = 2, 3, 4, \ldots \), and \( r = 0, 1, \ldots, 2n \), Lang [13] obtained following formula

\[
S_{2n}^r(a, c) = \sum_{j \pmod{c}} P_{2n-r}(\frac{j}{c}) P_r(\frac{aj}{c}).
\]

Here \( P_t(x) = \sum_{s=0}^{t} \binom{t}{s} B_s(\{x\}) (t-s) \), \( B_s \) is the \( s \)th Bernoulli number and \( \{x\} \) denotes the fractional part for any real number \( x \). In particular we have \( P_1(x) = \{x\} - \frac{1}{2} \), \( P_2(x) = \{x\}^2 - \frac{1}{2} \{x\} + \frac{1}{6} \) and \( P_3(x) = \{x\}^3 - \frac{3}{2} \{x\}^2 + \frac{1}{2} \{x\} \). The generalized Dedekind sums we are concerned with are recorded in the following lemma.

Lemma 2.4. We have

\[
S^2(a, c) = \sum_{j \pmod{c}} P_2(\frac{j}{c}) P_2(\frac{aj}{c}) \quad \text{and} \quad S^3(a, c) = \sum_{j \pmod{c}} P_1(\frac{j}{c}) P_3(\frac{aj}{c}).
\]

We now mention a result of Yokoi that facilitates us with the fundamental unit of the fields of the form \( \mathbb{Q}(\sqrt{0m^2 + 4m}) \).

Theorem 2.5. (Lemma 3 in Yokoi [20]) Let \( p \) be any prime congruent to \(-1 \) mod 4, and assume that an unit \( \epsilon \) of a real quadratic field \( \mathbb{Q}(\sqrt{D}) \) (\( D > 0 \) square-free) is of the form

\[
\epsilon = \frac{1}{2}(t + pv\sqrt{D}) \quad \text{or} \quad t + p\sqrt{D} \quad (t > 0)
\]

Then, the real quadratic field \( \mathbb{Q}(\sqrt{D}) \) is of R-D type in narrow sense or the unit \( \epsilon \) is the fundamental unit of \( \mathbb{Q}(\sqrt{D}) \) satisfying \( N(\epsilon) = 1 \).

Theorem 2.6. (Yokoi [20]) For any prime \( p \) congruent to \(-1 \) modulo 4, there exists an integer \( D_0 = D_0(p) \) such that if \( D = p^2m^2 \pm 4m \) has no square factor except \( 4 \) and is bigger than \( D_0 \), then the fundamental unit \( \epsilon_D \) of the real quadratic field \( \mathbb{Q}(\sqrt{D}) \) is of the following form:
We end this section with a result of Hasse (see [10]).

**Theorem 2.7.** (Hasse [10]) If a real quadratic field \( K = \mathbb{Q}(\sqrt{D}) \) has \( h_K = 1 \) then 
\( D = p, 2p \) or \( qr \) where \( p, q \) and \( r \) are primes and \( p \equiv 1 \pmod{4}, q \equiv r \equiv 3 \pmod{4} \).

3. Proofs

We note that \( D = 9m^2 + 4m \) is square-free and \( D \equiv 1 \pmod{4} \). So the discriminant of the field \( K = \mathbb{Q}(\sqrt{D}) \) is \( D \). We provide proofs for \( m > 0 \). Proof for \( m < 0 \) goes along the same line, however it is not identical. This difference creeps in because of the difference of fundamental units in two cases. As in Theorem 2.6, let \( \epsilon_D = \frac{1}{2}[(9m + 2) + 3\sqrt{D}] \). Then \( N(\epsilon_D) = 1 \) and \( \epsilon_D \) is a unit of \( K \). It is easily observed that there are no integers \( n, r \) with \(-n < r \leq n \) and \( r \mid 4n \) for which the equality \( 9m^2 + 4m = n^2 + r \) holds. By Theorem 2.5, \( \epsilon_D \) is the fundamental unit of \( K \). When \( m < 0 \) then \( D = 9m'^2 - 4m' \) for \( m' = -m > 0 \). By Theorem 2.5 the fundamental unit in this case is given by \( \epsilon_D = \frac{1}{2}[(9m' - 2) + 3\sqrt{D}] \).

Once again, we emphasize that now onwards \( m > 0 \). We shall use notations \( \mathfrak{C}, \mathfrak{U} \) respectively to denote the trivial ideal class and the ideal class containing the prime ideal above 3. If \( p > 3 \) is a prime divisor of \( m \), then \( \mathfrak{P} \) shall denote the ideal class containing the prime ideal above \( p \). In case we are having two distinct prime divisors \( p_1, p_2 > 3 \) of \( m \) then the corresponding ideal classes are denoted by \( \mathfrak{P}_1, \mathfrak{P}_2 \).

The following lemma is an easy consequence of the factorization of rational primes in quadratic fields.

**Lemma 3.1.** (i) If \( m \equiv 0 \pmod{3} \) then \( 3\mathfrak{O}_K = < 3, \frac{3+\sqrt{D}}{2} >^2 \) and \( \{ 3, \frac{3+\sqrt{D}}{2} \} \) is an integral basis of the ideal \( < 3, \frac{3+\sqrt{D}}{2} > \).

(ii) If \( m \equiv 1 \pmod{3} \) then \( 3\mathfrak{O}_K = < 3, \frac{1+\sqrt{D}}{2} > < 3, \frac{1-\sqrt{D}}{2} > \) and \( \{ 3, \frac{1+\sqrt{D}}{2} \} \) is an integral basis of the ideal \( < 3, \frac{1+\sqrt{D}}{2} > \).

(iii) If \( m \equiv 2 \pmod{3} \) then \( 3\mathfrak{O}_K = < 3, \frac{3+3\sqrt{D}}{2} > \) and \( \{ 3, \frac{3+3\sqrt{D}}{2} \} \) is an integral basis of the ideal \( < 3, \frac{3+3\sqrt{D}}{2} > \).

(iv) If \( p > 3 \) is a prime divisor of \( m \) then \( p\mathfrak{O}_K = < p, \frac{p+\sqrt{D}}{2} >^2 \) and \( \{ p, \frac{p+\sqrt{D}}{2} \} \) is an integral basis of the ideal \( < p, \frac{p+\sqrt{D}}{2} > \).

(v) If \( q > 3 \) is a prime divisor of \( D \) then \( q\mathfrak{O}_K = < q, \frac{q+\sqrt{D}}{2} >^2 \) and \( \{ q, \frac{q+\sqrt{D}}{2} \} \) is an
integral basis of the ideal $< q, \frac{q + \sqrt{D}}{2} >$.

**Proposition 3.2.** Let $\mathfrak{C}$ be the trivial ideal class in the class group of $K$. Then

$$\zeta_K(-1, \mathfrak{C}) = \frac{1}{120} (9m^3 + 6m^2 + 19m + 6).$$

**Proof.** We consider the integral ideal $\mathcal{O}_K$ in the ideal class $\mathfrak{C}^{-1}$. Then $r_1 = \frac{1 + \sqrt{D}}{2}, r_2 = 1$ is an integral basis of $\mathcal{O}_K$. If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integral matrix satisfying

$$\epsilon_D \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = M \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}.$$  

Then we obtain

$$a = \frac{1}{2}[(9m + 2) + 3], b = \frac{3}{4}(9m^2 + 4m - 1), c = 3 \text{ and } d = \frac{1}{2}[(9m + 2) - 3].$$

Clearly $a \equiv 1 \pmod{c}$ and $d \equiv 1 \pmod{c}$.

From the Lemma 2.3 we get

$$S^3(a, c) = S^3(d, c) = S^3(1, 3) = -\frac{1}{81}$$

and

$$S^2(a, c) = S^2(d, c) = S^2(1, 3) = \frac{11}{324}.$$ 

Substituting these in Theorem 2.2 we obtain the proposition. □

**Proposition 3.3.** If $3 | m$ and $\mathfrak{U}$ is an ideal class in the class group of the field $K$ such that $< 3, \frac{3 + \sqrt{D}}{2} > \in \mathfrak{U}^{-1}$ then

$$\zeta_K(-1, \mathfrak{U}) = \frac{1}{360} (3m^3 + 2m^2 + 273m + 162).$$

**Proof.** By Lemma 3.1, $r_1 = \frac{3 + \sqrt{D}}{2}, r_2 = 3$ is an integral basis of the ideal $\mathfrak{U} =< 3, \frac{3 + \sqrt{D}}{2} >$. We have $\delta(\mathfrak{U}) = 3\sqrt{D}$.

If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an integral matrix satisfying

$$\epsilon_D \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = M \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}.$$  

Then we obtain $a = \frac{9m + 11}{2}, b = \frac{9m^2 + 4m - 9}{4}, c = 9, d = \frac{9m - 7}{2}$. From this it is clear that $a \equiv 1 \pmod{c}, d \equiv 1 \pmod{c}$. Hence, by definition, we have $S^3(a, c) = S^3(d, c) = S^3(1, c)$ and $S^2(a, c) = S^2(d, c) = S^2(1, c)$. By Lemma 2.3 we obtain

$$S^3(a, c) = S^3(d, c) = \frac{-154}{3.9^3} \quad \text{and} \quad S^2(a, c) = S^2(d, c) = \frac{491}{12.9^3}.$$
Substituting these in (1) gives
\[ \zeta_K(-1, \mathfrak{U}) = \frac{1}{360}(3m^3 + 2m^2 + 273m + 162). \]
\[ \square \]

**Proposition 3.4.** If \( m \equiv 1 \pmod{3} \) and \( \mathfrak{U} \) is an ideal class in the class group of the field \( K \) such that \( < 3, \frac{1+\sqrt{D}}{2} > \in \mathfrak{U}^{-1} \) then
\[ \zeta_K(-1, \mathfrak{U}) = \frac{1}{360}(3m^3 + 2m^2 + 113m + 2). \]

**Proof.** By Lemma 3.1, \( r_1 = \frac{1+\sqrt{D}}{2}, r_2 = 3 \) is an integral basis of the ideal \( \mathfrak{u} = < 3, \frac{1+\sqrt{D}}{2} > \). We have \( \delta(\mathfrak{u}) = 3\sqrt{D} \).

If \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is an integral matrix satisfying
\[ \epsilon_D \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = M \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \]

Then we obtain \( a = \frac{9m+5}{2}, b = \frac{9m^2+4m-1}{4}, c = 9, d = \frac{9m-1}{2} \). From this it is clear that \( a \equiv 7 \pmod{c}, d \equiv 4 \pmod{c} \). Hence, by definition, we have \( S^3(a, c) = S^3(7, 9), S^3(d, c) = S^3(4, 9) \) and \( S^2(a, c) = S^2(7, 9), S^2(d, c) = S^2(4, 9) \). By Lemma 2.4, we obtain
\[ S^3(7, 9) = \frac{62}{3.9^3}, \quad S^3(4, 9) = \frac{8}{3.9^3}, \quad S^2(7, 9) = \frac{203}{12.9^3} \text{ and } S^2(4, 9) = \frac{203}{12.9^3}. \]

Substituting these in (1) gives
\[ \zeta_K(-1, \mathfrak{U}) = \frac{1}{360}(3m^3 + 2m^2 + 113m + 2). \]
\[ \square \]

**Proposition 3.5.** Let \( p \) be a prime dividing \( m \). If \( \mathfrak{P} \) is an ideal class in the class group of the field \( K \) such that \( < p, \frac{p+\sqrt{D}}{2} > \in \mathfrak{P}^{-1} \) then
\[ \zeta_K(-1, \mathfrak{P}) = \frac{1}{120 \times p^2}(9m^3 + 6m^2 + 9mp^4 + 10mp^2 + 6p^4). \]

**Proof.** By Lemma 3.1, \( r_1 = \frac{p+\sqrt{D}}{2}, r_2 = p \) is an integral basis of the ideal \( \mathfrak{p} = < p, \frac{p+\sqrt{D}}{2} > \). We have \( \delta(\mathfrak{p}) = p\sqrt{D} \).

If \( M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) is an integral matrix satisfying
\[ \epsilon_D \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = M \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}. \]
Then we obtain $a = \frac{9m+2+3p}{2}, b = \frac{3D-3p^2}{4p}, c = 3p, d = \frac{9m+2-3p}{2}$. Clearly $a \equiv 1 \pmod{c}$ and $d \equiv 1 \pmod{c}$. Consequently,

$$S^3(a, c) = S^3(d, c) = S^3(1, 3p) = \frac{-(3p)^4 + 5(3p)^2 - 4}{120(3p)^3}$$

and

$$S^2(a, c) = S^2(d, c) = S^2(1, 3p) = \frac{(3p)^4 + 10(3p)^2 - 6}{180(3p)^3}.$$

Substituting these in (1) gives

$$\zeta_K(-1, \Psi) = \frac{1}{120 \times p^2}(9m^3 + 6m^2 + 9mp^4 + 10mp^2 + 6p^4).$$

□

Proof. (Theorem 1.1) If $m = 1$ then $K = \mathbb{Q}(\sqrt{13})$ and its class number is one. Now we assume that $m > 1$. If $m$ is not a prime then by Theorem 2.7 it follows that $h_K > 1$. Thus we can assume that $m = p$ for some odd prime $p$.

From Proposition 3.2, Proposition 3.3, Proposition 3.4, we see that $\zeta_K(-1, C) \neq \zeta_K(-1, U)$, unless $m = 3$.

This proves that $h_K \geq 2$ whenever $m > 3$. For $m = 3$, we have $K = \mathbb{Q}(\sqrt{93})$ and $h_K = 1$. This proves the theorem. □

Proof. (Theorem 1.2) We have already seen that the ideal classes $C$ and $U$ are different elements in the class group of $K$. To establish Theorem 1.2, we obtain a lower bound on

$$\sum_{|t| < \sqrt{D}} \sigma \left( \frac{D - t^2}{4} \right).$$

Since

$$(3m)^2 < D < (3m + 1)^2,$$

we have

$$3m = [\sqrt{D}] < \sqrt{D} < (3m + 1).$$

Further $t^2 \equiv D \pmod{4}$ holds if and only if $t$ is odd. Thus, we conclude that $t$ runs over integers

$$t = 3m - 2s, \text{ for } s = 0, 1, 2, ..., \left(\frac{3m-1}{2}\right).$$
For \( t = 3m - 2s \) we have \( \left( \frac{D - t^2}{4} \right) = -s^2 + 3ms + m \). Thus

\[
\left( D - t^2 \right) \prod_{|t| < \sqrt{D}} \sigma \left( \frac{D - t^2}{4} \right) = 2 \left( \sum_{s=0}^{(3m-1)/2} \sigma(-s^2 + 3ms + m) \right).
\]

For each \( s \) considering the trivial divisors of \(-s^2 + 3ms + m\) we obtain following contribution in the right side of (2)

\[
2 \left( \sum_{s=0}^{3m-1/2} \frac{1 + m(1 + 3s) - s^2}{3} \right) = \frac{1}{2}(9m^3 + 6m^2 + 7m + 2).
\]

Since \( m \equiv 1 \mod 3 \), we see that 3 is a divisor of \(-s^2 + 3ms + m\) whenever \( s \equiv 1 \mod 3 \) or \( s \equiv 2 \mod 3 \). Thus, for each \( t = 3m - 2s \) with \( s \equiv 1 \mod 3 \) or \( s \equiv 2 \mod 3 \) we get extra contribution of \( \frac{3m^2 + 3ms + m}{3} \) in the right side of (2). The total contribution due to these factors, in the right side of (2) is

\[
2 \left( \sum_{s=0}^{3m-1/2} \frac{3 + m(1 + 3s) - s^2}{3} \right) = m^3 + \frac{2m^2}{3} + \frac{19m}{3}.
\]

Further, for \( s = m \) and \( t = m \) we get extra contribution of \( m + (2m + 1) \) in the right side of (2).

Taking in account all these contributions we get

\[
\sum_{|t| < \sqrt{D}} \sigma \left( \frac{D - t^2}{4} \right) \geq \frac{1}{2}(9m^3 + 6m^2 + 7m + 2) + (m^3 + \frac{2m^2}{3} + \frac{19m}{3}) + 2(3m + 1).
\]

From Theorem 2.1 we get

\[
(6) \quad \zeta_K(-1) \geq \frac{1}{60} \left( \frac{1}{2}(9m^3 + 6m^2 + 13m + 4) + (m^3 + \frac{2m^2}{3} + \frac{19m}{3}) + 2(3m + 1) \right)
\]

Using Proposition 3.2 and Proposition 3.4 we see that

\[
\zeta_K(-1) > \zeta_K(-1, \mathcal{C}) + \zeta_K(-1, \mathcal{U}),
\]

whenever \( m > 4 \). Thus \( h_K \geq 3 \) whenever \( m > 4 \). \( \square \)

Proof. (Theorem 1.3) As noted in the proof of Theorem 1.1 the ideal classes \( \mathcal{C} \) and \( \mathcal{U} \) are two distinct elements in the class group of \( K \). Let \( p > 3 \) be a prime divisor of \( m \). Using Proposition 3.2 and Proposition 3.3 we note that \( \zeta_K(-1, \mathcal{C}) = \zeta_K(-1, \mathcal{U}) \) gives

\[
p^2(9m^3 + 6m^2 + 19m + 6) = (9m^3 + 6m^2 + 9mp^4 + 10mp^2 + 6p^4).
\]
Simplifying this we obtain
\[(9m^3 + 6m^2 - 9mp^2 - 6p^2)(p^2 - 1) = 0.\]

This gives \(9m^3 + 6m^2 - 9mp^2 - 6p^2 = 0\) which leads to \(m = p\). But this is not possible as \(m\) has at least three distinct prime factors. Consequently the ideal classes \(C\) and \(\mathfrak{P}\) are different.

Similarly we see that \(\zeta_K(-1, \mathfrak{U}) \neq \zeta_K(-1, \mathfrak{B})\). Further, if \(p_1, p_2\) are two distinct primes dividing \(m\) then we see that \(\zeta_K(-1, \mathfrak{P}_1) \neq \zeta_K(-1, \mathfrak{P}_2)\).

Thus we get a different ideal class in the class group of \(K\) for each prime \(p > 3\) dividing \(m\) in addition to ideal classes \(C\) and \(\mathfrak{U}\). This proves the theorem.

Before proving Theorem 1.4, we need one more proposition about zeta values. We recall that \(D = m(9m + 4)\) and consider the case when \(9m + 4 = q\) is an odd prime. Let \(q\) denote the prime ideal in \(K\) lying above \(q\) and \(\mathfrak{Q}\) be an ideal class such in the class group of \(K\) such that \(q\) lies in \(\mathfrak{Q}^{-1}\).

**Proposition 3.6.** Let \(D = mq\), where \(q = 9m + 4\) is an odd prime. With above notations,
\[
\zeta_K(-1, \mathfrak{Q}) = \frac{1}{120}(9m^3 + 6m^2 + 19m + 6).
\]

**Proof.** By Lemma 3.1, \(r_1 = \frac{q + \sqrt{D}}{2}, r_2 = q\) is an integral basis of the ideal \(q = <q, \frac{q + \sqrt{D}}{2}>\). We have \(\delta(q) = q\sqrt{D}\).
If \(M = \begin{bmatrix}a & b \\c & d\end{bmatrix}\) is an integral matrix satisfying
\[
\epsilon_D \begin{bmatrix}r_1 \\r_2\end{bmatrix} = M \begin{bmatrix}r_1 \\r_2\end{bmatrix}.
\]
Then we obtain \(a = \frac{9m+2+3q}{2}, b = \frac{3D-3q^2}{4q}, c = 3q, d = \frac{9m+2-3q}{2}\). Clearly \(a \equiv 2q - 1\) (mod \(c\)) and \(d \equiv 2q - 1\) (mod \(c\)). Consequently,
\[
S^3(a, c) = S^3(d, c) = S^3(2q - 1, 3q), \quad S^2(a, c) = S^2(d, c) = S^2(2q - 1, 3q).
\]
Now we compute \(S^2(2q - 1, 3q)\) using Lemma 2.4.

\[
S^2(2q - 1, 3q) = \sum_{j=0}^{3q-1} P_2 \left(\frac{j}{3q}\right) P_2 \left(\frac{(2q - 1)j}{3q}\right)
\]
\[
= \sum_{k=0}^{q-1} P_2 \left(\frac{3k}{3q}\right) P_2 \left(\frac{(2q - 1)(3k)}{3q}\right) + \sum_{k=0}^{q-1} P_2 \left(\frac{3k + 1}{3q}\right) P_2 \left(\frac{(2q - 1)(3k + 1)}{3q}\right)
\]
Now, we evaluate each sum individually.

\[ + \sum_{k=0}^{q-1} P_2 \left( \frac{3k+2}{3q} \right) P_2 \left( \frac{2q-1)(3k+2)}{3q} \right) \]

Since \( q \equiv 1 \pmod{3} \), we observe that

\[ \{(3k)(2q-1)(3q)\} = (3q-3k)/3q \text{ for } k > 0, \quad \{(3k+1)(2q-1)/3q\} = (-3k+2q-1)/3q \]
for \( k \leq (2q-2)/3 \), \( \{(3k+1)(2q-1)/3q\} = (-3k+5q-1)/3q \text{ for } k \geq (2q+1)/3, \)
\[ \{(3k+2)(2q-1)/3q\} = (-3k+q-2)/3q \text{ for } k \leq (q-4)/3 \]
and \( \{(3k+2)(2q-1)/3q\} = (-3k+4q-2)/3q \text{ for } k \geq (q-1)/3. \]

Recall that \( P_i(x) = P_i(\{x\}) \) for \( i = 1, 2, 3 \) and \( x \in \mathbb{R} \). Hence we obtain

\[ S^2(2q-1, 3q) = \sum_{k=0}^{q-1} P_2 \left( \frac{3k}{3q} \right) P_2 \left( \frac{3q-3k}{3q} \right) + \sum_{k=0}^{q-2} P_2 \left( \frac{3k+1}{3q} \right) P_2 \left( \frac{3q-3k+1}{3q} \right) \]
\[ + \sum_{k=2q+1}^{q-1} P_2 \left( \frac{3k+1}{3q} \right) P_2 \left( \frac{3q-3k-1}{3q} \right) + \sum_{k=0}^{q-4} P_2 \left( \frac{3k+2}{3q} \right) P_2 \left( \frac{3q-3k-2}{3q} \right) \]
\[ + \sum_{k=2q+1}^{q-4} P_2 \left( \frac{3k+2}{3q} \right) P_2 \left( \frac{3q-3k-2}{3q} \right). \]

Now, we evaluate each sum individually.

\[ \sum_{k=0}^{q-1} P_2 \left( \frac{3k}{3q} \right) P_2 \left( \frac{3q-3k}{3q} \right) = \sum_{k=0}^{q-1} \left( \frac{q^2-6kq+6k^2}{6q^2} \right) \left( \frac{q^2-6kq+6k^2}{6q^2} \right) \]
\[ = \sum_{k=0}^{q-1} \left( \frac{q^4-12kq^3+48k^2q^2-72k^3q+36k^4}{36q^4} \right). \]

Now we use the formula for sum of fixed powers of consecutive numbers to obtain

\[ \sum_{k=0}^{q-1} P_2 \left( \frac{3k}{3q} \right) P_2 \left( \frac{3q-3k}{3q} \right) = \frac{q^5+10q^3-6q}{180q^4}. \]

Next, we have

\[ \sum_{k=0}^{q-2} P_2 \left( \frac{3k+1}{3q} \right) P_2 \left( \frac{-3k+2q-1}{3q} \right) \]
\[ = \sum_{k=0}^{q-2} \left( \frac{18k^2+(2-18q)k+2-6q+3q^2}{18q^2} \right) \left( \frac{18k^2+(12-6q)k+2-2q-q^2}{18q^2} \right). \]
\[
\begin{align*}
&= \sum_{k=0}^{2q-2} \left( \frac{324k^4 + (432 - 432q)k^3 + (144q^2 - 432q + 216)k^2}{324q^4} \right) \\
&\quad + \left( \frac{96q^2 - 144q + 48)k + (4 - 16q + 16q^2 - 3q^4)}{324q^4} \right).
\end{align*}
\]

Using the formula for sum of fixed powers of consecutive numbers we obtain

(10)

\[
\sum_{k=0}^{2q-2} P_2 \left( \frac{3k + 1}{3q} \right) P_2 \left( \frac{-3k + 2q - 1}{3q} \right) = \frac{-26q^5 - 45q^4 - 160q^3 + 156q + 120}{324 \times 45 \times q^4}.
\]

Now we evaluate the sum

\[
\sum_{k=\frac{2q+1}{3}}^{q-1} P_2 \left( \frac{3k + 1}{3q} \right) P_2 \left( \frac{-3k + 5q - 1}{3q} \right)
\]

\[
= \sum_{k=\frac{2q+1}{3}}^{q-1} \left( \frac{18k^2 + (12 - 18q)k + 2 - 6q + 3q^2}{18q^2} \right) \left( \frac{18k^2 + (12 - 42q)k + 2 - 14q + 23q^2}{18q^2} \right)
\]

\[
= \sum_{k=\frac{2q+1}{3}}^{q-1} \left( \frac{324k^4 + (432 - 1080q)k^3 + (1224q^2 - 1080q + 216)k^2}{324q^4} \right) \\
+ \sum_{k=\frac{2q+1}{3}}^{q-1} \left( \frac{(816q^2 - 540q^3 - 360q + 48)k + (69q^4 - 180q^3 + 136q^2 - 40q + 4)}{324q^4} \right).
\]

From the formulas for sum of fixed powers of consecutive numbers we get

(11)

\[
\sum_{k=\frac{2q+1}{3}}^{q-1} P_2 \left( \frac{3k + 1}{3q} \right) P_2 \left( \frac{-3k + 5q - 1}{3q} \right) = \frac{-13q^5 + 45q^4 + 90q^3 - 80q^2 + 78q - 120}{324 \times 45 \times q^4}.
\]

Next we look at the sum

\[
\sum_{k=0}^{q-4} P_2 \left( \frac{3k + 2}{3q} \right) P_2 \left( \frac{-3k + q - 2}{3q} \right)
\]

\[
= \sum_{k=0}^{q-4} \left( \frac{18k^2 + (24 - 18q)k + 8 - 12q + 3q^2}{18q^2} \right) \left( \frac{18k^2 + (24 + 6q)k + 8 + 4q - q^2}{18q^2} \right)
\]

\[
= \sum_{k=0}^{q-4} \left( \frac{324k^4 + (864 - 216q)k^3 + (864 - 432q - 72q^2)k^2}{324q^4} \right)
\]
Using the formula for sum of fixed powers of consecutive numbers we get

\[ \sum_{k=0}^{q-1} \sum_{\frac{k}{q}} P_2 \left( \frac{3k + 2}{3q} \right) P_2 \left( \frac{-3k + q - 2}{3q} \right) = \frac{-26q^5 + 90q^4 + 180q^3 - 160q^2 + 156q - 240}{324 \times q^4}. \]

The last sum to be handled is

\[
\sum_{k=\frac{q-1}{3}}^{q-1} P_2 \left( \frac{3k + 2}{3q} \right) P_2 \left( \frac{-3k + 4q - 2}{3q} \right) = \frac{18k^2 + (24 - 18q)k + 8 - 12q + 3q^2}{18q^2} \left( \frac{18k^2 + (24 - 30q)k + 8 - 20q + 11q^2}{18q^2} \right)
\]

\[
= \sum_{k=\frac{q-1}{3}}^{q-1} \left( \frac{324k^4 + 864(1-q)k^3 + (729q^2 - 1728q + 864)k^2}{324q^4} \right) \left( \frac{384 - 1152q + 1056q^2 - 288q^3}{324q^4} \right) + \left( \frac{33q^4 - 192q^3 + 352q^2 - 256q + 64}{324q^4} \right).
\]

This leads to

\[ \sum_{k=\frac{q-1}{3}}^{q-1} P_2 \left( \frac{3k + 2}{3q} \right) P_2 \left( \frac{-3k + 4q - 2}{3q} \right) = \frac{-26q^5 - 45q^3 - 160q^2 + 156q + 120}{324 \times 45 \times q^4}. \]

Using equations (16), (17), (18), (19) and (20) in equation (15) we get

\[ S^2(2q - 1, 3q) = \frac{q^4 + 330q^2 - 160q^2 - 6}{324 \times 15 \times q^3}. \]

Similar computations yield

\[ S^3(2q - 1, 3q) = \frac{q^4 + 40q^3 - 160q^2 + 80q + 4}{324 \times 10 \times q^3}. \]

Using these values in (11) gives

\[ \zeta_K(-1, \Omega) = \frac{1}{3^3 \times 360} (q^3 - 6q^2 + 171q - 166). \]

Since \( q = 9m + 4 \) we get

\[ \zeta_K(-1, \Omega) = \frac{1}{120} (9m^3 + 6m^2 + 19m + 6). \]
Proof. (Theorem 1.4) We consider the quadratic field $K = \mathbb{Q}(\sqrt{9m^2 + 4m})$. Let $\mathfrak{C}$ and $\mathfrak{Q}$ denote the principle ideal class and the ideal class containing prime ideal $\mathfrak{q}$ respectively. By Proposition 3.2 and Proposition 3.6 we have

$$\zeta_K(-1, \mathfrak{C}) = \zeta_K(-1, \mathfrak{Q}).$$

From a result of Lang (see section 4 of [13]) we conclude that $\mathfrak{C} = \mathfrak{Q}$. Consequently the ideal $\mathfrak{q}$ is principal. This gives

$$\mathfrak{q} = (x + \sqrt{D}y)/2$$

for some integers $x, y$.

Taking norm of both the sides gives

$$4q = x^2 - Dy^2.$$

This proves the theorem. □

We end with tables of real quadratic fields in the family $\mathbb{Q}(\sqrt{9m^2 + 4m})$ and their class number. We consider $m$ in the range $[-160, 160]$ and divide them in cases depending on congruency class of $m$ modulo 3 and consider only those $D$ which are square-free.
### Table 1
For $m \equiv 0 \pmod{3}$

| $m$  | $D$         | $h_K$ |
|------|-------------|-------|
| -159 | 226893      | 22    |
| -141 | 178365      | 16    |
| -129 | 149253      | 12    |
| -123 | 135669      | 30    |
| -111 | 110445      | 20    |
| -105 | 98805       | 16    |
| -87  | 67773       | 8     |
| -69  | 42573       | 6     |
| -57  | 29013       | 6     |
| -51  | 23205       | 8     |
| -39  | 13533       | 8     |
| -33  | 9669        | 10    |
| -21  | 3885        | 4     |
| -15  | 1965        | 2     |
| -3   | 69          | 1     |
| 3    | 93          | 1     |
| 15   | 2085        | 2     |
| 21   | 4053        | 4     |
| 33   | 9933        | 4     |
| 39   | 13845       | 4     |
| 51   | 23613       | 6     |
| 57   | 29469       | 12    |
| 87   | 68469       | 18    |
| 105  | 99645       | 16    |
| 111  | 111333      | 12    |
| 123  | 136653      | 16    |
| 129  | 150285      | 24    |
| 141  | 179493      | 12    |
| 159  | 228165      | 24    |

### Table 2
For $m \equiv 1 \pmod{3}$

| $m$  | $D$         | $h_K$ |
|------|-------------|-------|
| -155 | 215605      | 28    |
| -149 | 199213      | 28    |
| -143 | 183469      | 30    |
| -137 | 168373      | 24    |
| -119 | 126973      | 20    |
| -113 | 114469      | 32    |
| -107 | 102613      | 24    |
| -101 | 91405       | 16    |
| -95  | 80845       | 20    |
| -89  | 70933       | 24    |
| -83  | 61669       | 23    |
| -77  | 53053       | 16    |
| -71  | 45085       | 14    |
| -65  | 37765       | 16    |
| -59  | 31093       | 10    |
| -53  | 25069       | 16    |
| -47  | 19693       | 11    |
| -41  | 14965       | 8     |
| -35  | 10885       | 10    |
| -29  | 7453        | 6     |
| -23  | 4669        | 8     |
| -17  | 2533        | 4     |
| -11  | 1045        | 4     |
| -5   | 205         | 2     |
| 1    | 13          | 1     |
| 7    | 469         | 3     |
| 31   | 8773        | 7     |
| 37   | 12469       | 14    |
| 43   | 16813       | 10    |
| 55   | 27445       | 12    |
| 61   | 33733       | 10    |
| 67   | 40669       | 13    |
| 73   | 48253       | 12    |
| 79   | 56485       | 20    |
| 85   | 65365       | 20    |
| 91   | 74893       | 18    |
| 97   | 85069       | 24    |
| 109  | 107365      | 20    |
| 115  | 119485      | 30    |
| 127  | 145669      | 30    |
| 133  | 159733      | 24    |
| 139  | 174445      | 28    |

### Table 3
For $m \equiv 2 \pmod{3}$

| $m$  | $D$         | $h_K$ |
|------|-------------|-------|
| -157 | 221213      | 14    |
| -151 | 204605      | 16    |
| -145 | 188645      | 16    |
| -139 | 173333      | 10    |
| -133 | 158669      | 18    |
| -127 | 144653      | 12    |
| -115 | 118565      | 10    |
| -109 | 106493      | 14    |
| -103 | 95069       | 14    |
| -97  | 84293       | 8     |
| -91  | 74165       | 8     |
| -85  | 64853       | 12    |
| -79  | 55853       | 6     |
| -73  | 47669       | 8     |
| -67  | 40133       | 9     |
| -61  | 33245       | 8     |
| -55  | 27005       | 8     |
| -43  | 16469       | 5     |
| -37  | 12173       | 4     |
| -19  | 3173        | 3     |
| -13  | 1469        | 2     |
| -7   | 413         | 1     |
| -1   | 5           | 1     |
| 11   | 1133        | 1     |
| 17   | 2669        | 4     |
| 23   | 4853        | 3     |
| 29   | 7685        | 4     |
| 35   | 11165       | 4     |
| 41   | 15293       | 4     |
| 47   | 20069       | 10    |
| 53   | 25493       | 4     |
| 59   | 31565       | 10    |
| 65   | 38285       | 12    |
| 71   | 45653       | 7     |
| 77   | 53669       | 8     |
| 83   | 62333       | 5     |
| 89   | 71645       | 8     |
| 95   | 81605       | 8     |
| 101  | 92213       | 14    |
| 107  | 103469      | 13    |
| 113  | 115373      | 8     |
| 137  | 169469      | 20    |
Appendix

Her we compute $S^3(2q - 1, 3q)$ using Lemma 2.4.

(14) \[ S^3(2q - 1, 3q) = \sum_{j=0}^{3q-1} P_1(j/c)P_3((2q - 1)j/c) \]

\[ = \sum_{k=0}^{q-1} P_1((3k)/c)P_3((2q - 1)(3k)/c) + \sum_{k=0}^{q-1} P_1((3k + 1)/c)P_3((2q - 1)(3k + 1)/c) \]

\[ + \sum_{k=0}^{q-1} P_1((3k + 2)/c)P_3((2q - 1)(3k + 2)/c) \]

Since \( q \equiv 1 \pmod{3} \), we observe that

\[ \{(3k)(2q-1)/3\} = (3q-3k)/3q \text{ for } k > 0, \]

\[ \{(3k+1)(2q-1)/3\} = (-3k+2q-1)/3q \]

for \( k \leq (2q - 2)/3 \), \( \{(3k + 1)(2q - 1)/3\} = (-3k + 5q - 1)/3q \text{ for } k \geq (2q + 1)/3, \)

\[ \{(3k + 2)(2q - 1)/3\} = (-3k + q - 2)/3q \text{ for } k \leq (q - 4)/3 \]

and \( \{(3k + 2)(2q - 1)/3\} = (-3k + 4q - 2)/3q \text{ for } k \geq (q - 1)/3. \)

Recall that \( P_i(x) = P_i(\{x\}) \) for \( i = 1, 2, 3 \) and \( x \in \mathbb{R} \). Hence we obtain

(15)

\[ S^3(2q-1, 3q) = \sum_{k=0}^{q-1} P_1 \left( \frac{3k}{3q} \right) P_3 \left( \frac{3q - 3k}{3q} \right) + \sum_{k=0}^{q-1} P_1 \left( \frac{3k + 1}{3q} \right) P_3 \left( \frac{-3k + 2q - 1}{3q} \right) \]

\[ + \sum_{k=2q+1}^{q-1} P_1 \left( \frac{3k + 1}{3q} \right) P_3 \left( \frac{-3k + 5q - 1}{3q} \right) + \sum_{k=0}^{q-1} P_1 \left( \frac{3k + 2}{3q} \right) P_3 \left( \frac{-3k + q - 2}{3q} \right) \]

\[ + \sum_{k=2q+1}^{q-1} P_1 \left( \frac{3k + 2}{3q} \right) P_3 \left( \frac{-3k + 4q - 2}{3q} \right). \]

Now, we evaluate each sum individually.

\[ \sum_{k=0}^{q-1} P_1 \left( \frac{3k}{3q} \right) P_3 \left( \frac{3q - 3k}{3q} \right) = \sum_{k=0}^{q-1} \left( \frac{-3q + 6k}{6q} \right) \left( \frac{-q^2k + 3qk^2 - 2k^3}{2q^3} \right) \]

\[ = \sum_{k=0}^{q-1} \left( \frac{q^3k - 5q^2k^2 + 8qk^3 - 4k^4}{4q^4} \right). \]
Now we use the formula for sum of fixed powers of consecutive numbers to obtain

\[
\sum_{k=0}^{q-1} P_1 \left( \frac{3k}{3q} \right) P_3 \left( \frac{3q - 3k}{3q} \right) = \frac{q^4 - 5q^2 + 4}{120q^3}.
\]

Next, we have

\[
\sum_{k=0}^{2q-2} P_1 \left( \frac{3k + 1}{3q} \right) P_3 \left( \frac{-3k + 2q - 1}{3q} \right) = \sum_{k=0}^{2q-2} \left( \frac{6k + 2 - 3q}{6q} \right) \times \left( -54k^3 + (27q - 54)k^2 + (9q^2 + 18q - 18)k + (-2q^3 + 3q^2 + 3q - 2) \right) \\
= \sum_{k=0}^{2q-2} \left( -324k^4 + (324q - 432)k^3 + (-27q^2 + 324q - 216)k^2 \right) \\
\quad + \left( -3q^3 - 18q^2 + 108q - 48 \right) \times \left( 6q^4 - 13q^3 - 3q^2 + 12q - 4 \right)
\]

Using the formula for sum of fixed powers of consecutive numbers we obtain

\[
\sum_{k=0}^{2q-2} P_1 \left( \frac{3k + 1}{3q} \right) P_3 \left( \frac{-3k + 2q - 1}{3q} \right) = \frac{2q^5 + 15q^4 - 5q^3 + 50q^2 - 52q - 40}{4860q^4}.
\]

Now we evaluate the sum

\[
\sum_{k=2q+1}^{q-1} P_1 \left( \frac{3k + 1}{3q} \right) P_3 \left( \frac{-3k + 5q - 1}{3q} \right) = \sum_{k=2q+1}^{q-1} \left( \frac{6k + 2 - 3q}{6q} \right) \times \left( -54k^3 + (189q - 54)k^2 + (-207q^2 + 126q - 18)k + (70q^3 - 69q^2 + 21q - 2) \right) \\
= \sum_{k=2q+1}^{q-1} \left( -324k^4 + (1296q - 432)k^3 + (-1809q^2 + 1296q - 216)k^2 \right) \\
\quad + \left( 1041q^3 - 1206q^2 + 432q - 48 \right) \times \left( 6q^4 - 13q^3 - 3q^2 + 12q - 4 \right) \\
\quad \times \left( 6q^4 - 13q^3 - 3q^2 + 12q - 4 \right)
\]
From the formulas for sum of fixed powers of consecutive numbers we get

\[
(18) \quad \sum_{k=2q+1}^{q-1} P_1 \left( \frac{3k + 1}{3q} \right) P_3 \left( \frac{-3k + 5q - 1}{3q} \right) = \frac{-43q^5 + 30q^4 - 35q^3 + 20q^2 - 52q + 80}{9720q^4}.
\]

Next we look at the sum

\[
(19) \quad \sum_{k=0}^{q-4} P_1 \left( \frac{3k + 2}{3q} \right) P_3 \left( \frac{-3k + q - 2}{3q} \right) = \frac{-43q^5 + 30q^4 - 35q^3 + 20q^2 - 52q + 80}{9720q^4}.
\]

Using the formula for sum of fixed powers of consecutive numbers we get

\[
(19) \quad \sum_{k=0}^{q-4} P_1 \left( \frac{3k + 2}{3q} \right) P_3 \left( \frac{-3k + q - 2}{3q} \right) = \frac{-43q^5 + 30q^4 - 35q^3 + 20q^2 - 52q + 80}{9720q^4}.
\]

The last sum to be handled is

\[
(19) \quad \sum_{k=q+1}^{q-1} P_1 \left( \frac{3k + 2}{3q} \right) P_3 \left( \frac{-3k + 4q - 2}{3q} \right) = \frac{-43q^5 + 30q^4 - 35q^3 + 20q^2 - 52q + 80}{9720q^4}.
\]

\[
(19) \quad \sum_{k=0}^{q-4} P_1 \left( \frac{3k + 2}{3q} \right) P_3 \left( \frac{-3k + q - 2}{3q} \right) = \frac{-43q^5 + 30q^4 - 35q^3 + 20q^2 - 52q + 80}{9720q^4}.
\]

Using the formula for sum of fixed powers of consecutive numbers we get

\[
(19) \quad \sum_{k=0}^{q-4} P_1 \left( \frac{3k + 2}{3q} \right) P_3 \left( \frac{-3k + q - 2}{3q} \right) = \frac{-43q^5 + 30q^4 - 35q^3 + 20q^2 - 52q + 80}{9720q^4}.
\]
This leads to
\[
\sum_{k=1}^{q-1} F_1 \left( \frac{3k + 2}{3q} \right) F_3 \left( \frac{-3k + 4q - 2}{3q} \right) = \frac{2q^5 + 15q^4 - 5q^3 + 50q^2 - 52q - 40}{4860q^3}.
\]

Using equations (16), (17), (18), (19) and (20) in equation (15) we get
\[
S^2(2q-1, 3q) = \frac{q^4 + 40q^3 - 165q^2 + 80q + 4}{3240q^3}.
\]

REFERENCES

[1] A. Baker, Linear forms in the logarithms of algebraic numbers. I, II, III, Mathematika 13 (1966), 204-216; ibid. 14 (1967), 102-107; ibid 14 (1967), 220-228.
[2] A. Biro, Yokoi’s conjecture, Acta Arith. 106 (2003), no. 1, 85-104.
[3] A. Biro, Chowla’s conjecture, Acta Arith. 107 (2003), no. 2, 179-194.
[4] A. Biro, K. Lapkova The class number one problem for real quadratic fields \( \mathbb{Q}(\sqrt{(an)^2 + 4a}) \), Acta Arith. 172 (2016), no. 2, 117-131.
[5] D. Byeon, H. K. Kim, Class number 1 criteria for real quadratic fields of Richaud-Degert type, J. Number Theory, 57 (1996), no. 2, 328-339.
[6] D. Byeon and H. K. Kim, Class number 2 criteria for real quadratic fields of Richaud-Degert type, J. Number Theory 62 (1997), 257–272.
[7] K. Chakraborty, A. Hoque and M. Mishra, A note on certain real quadratic fields with class number up to three, Kyushu Journal of Mathematics (To appear).
[8] G. Degert, Über die Bestimmung der Grundeinheit gewisser reell-quadratischer Zahlkörper, Abh. Math. Sem. Univ. Hamburg 22 (1958), 92-97.
[9] C. F. Gauss, Disquisitiones arithmaticae. Translated into English by Arthur A. Clarke, S. J. Yale University Press, New Haven, Conn., London.
[10] H. Hasse, Über mehrklassige, aber eigen schlechtige reell-quadratische Zahlkörper, Elem. Math. 20 (1965), 49–59. corrigendum, ibid. 52 (2010), 207–208.
[11] F. Kawamoto, Y. Kishi, H. Suzuki, K. Tomita, Real quadratic fields, continued fractions, and a construction of primary symmetric parts of ELE type, Kyushu J. Math. 73 (2019), 165-187.
[12] H. K. Kim, "A Conjecture of S. Chowla and Related Topics in Analytic Number Theory", Thesis, The Johns Hopkins Univ., Baltimore, 1988.
[13] H. Lang, Über eine Gattung elementar-arithmetischer Klassen invarianten reell-quadratischer Zahlkörper, Angew. Math. 223 (1968), 123–175.
[14] Lee, Jungyun, The complete determination of wide Richaud-Degert types which are not 5 modulo 8 with class number one, Acta Arith. 140 (2009), no. 1, 1-29.
[15] S. R. Louboutin, Continued fractions and real quadratic field. J. Number Theory 30 (1988), 167-176.
[16] R. A. Mollin, Class number one criteria for real quadratic fields, Proc. Japan Acad. Ser. A Math. Sci. 63 1987.
[17] R. A. Mollin, A simple criterion for solvability of both \( x^2 - dy^2 = c \) and \( x^2 - dy^2 = -c \), New York J. Math. 7 (2001), 87-97.
[18] C. L. Siegel, Berechnung von Zetafunktionen an ganzzahligen Stellen, Nachr. Akad. Wiss. Göttingen, Math.-Phys. Klasse 10 (1969), 87–102.
[19] H. M. Stark, A complete determination of the complex quadratic fields of class number one, Michigan Math. J. 14 (1967), 1-27.
[20] H. Yokoi, On the Fundamental Unit of Real Quadratic Fields with Norm 1, J. Number Theory 2 (1970), 106–115.
[21] H. Yokoi, On real quadratic fields containing units with norm $-1$, *Nagoya Math. J.* **Vol. 33** (1968), 139-152.
[22] D. B. Zagier, On the values at negative integers of the zeta function of a real quadratic field, *Enseig. Math.* **19** (1976), 55–95.

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