Theoretical Developments in SUSY

M. Shifman

William I. Fine Theoretical Physics Institute, University of Minnesota, Minneapolis, MN 55455

Abstract

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Main topics:

- Heterotic strings from $\mathcal{N} = 1$ gauge theories;
- Planar equivalence and emergent center symmetry in QCD-like theories;
- Exact result for gluon scattering amplitudes in $\mathcal{N} = 4$ (dual conformality).
1 Non-Abelian heterotic strings in $\mathcal{N} = 1$: setting the stage

Seiberg and Witten presented [1] the first ever demonstration of the dual Meissner effect in non-Abelian theory, a celebrated analytic proof of linear confinement, which caused much excitement in the community. The Seiberg–Witten flux tubes are essentially Abelian (of the Abrikosov–Nielsen–Olesen type), so that the hadrons they create are not alike those in QCD [2].

What do we mean when we speak of Abelian versus non-Abelian flux tubes? In the former case, gauge dynamics relevant to distances where the tubes are formed is that of an Abelian theory (although short-distance dynamics can well be non-Abelian, as in the Seiberg–Witten case). In the latter case, in the infrared, at distances relevant to the tube formation, dynamics is determined by non-Abelian theory, with all gauge bosons equally operative. Correspondingly, we can speak of Abelian versus non-Abelian confinement. There are reasons to believe that no phase transition occurs between these two regimes in the Seiberg–Witten solution.\(^1\) However, in the limit of large-$\mu$ deformations, when a non-Abelian regime presumably sets in and non-Abelian strings develop in the model considered by Seiberg and Witten, theoretical control is completely lost. What was badly needed and sought for was a model in which non-Abelian strings develop in a fully controllable manner, i.e. at weak coupling.

Ever since, searches for non-Abelian flux tubes and non-Abelian monopoles continued, with a decisive breakthrough in 2003-04 [3, 4]. By that time the program of finding field-theoretical analogs of all basic constructions of string/D-brane theory was in full swing. BPS domain walls, analogs of D branes, had been identified in supersymmetric Yang–Mills theory [5]. It had been demonstrated that such walls support gauge fields localized on them. BPS saturated string-wall junctions had been constructed [6].

1.1 Non-Abelian flux tubes and monopoles

Non-Abelian strings were first found in $\mathcal{N} = 2$ super-Yang–Mills theories with $U(2)_{\text{gauge}}$ and two matter hypermultiplets [3, 4]. The $\mathcal{N} = 2$ vector multiplet consists of the $U(1)$ gauge field $A_\mu$ and the $SU(2)$ gauge field $A^a_\mu$, (here $a = 1, 2, 3$), and their Weyl fermion superpartners ($\lambda^1$, $\lambda^2$) and ($\lambda^{1a}$, $\lambda^{2a}$), plus complex scalar fields $a$, and $a^a$. The global $SU(2)_R$ symmetry inherent to $\mathcal{N} = 2$ models manifests itself through rotations $\lambda^1 \leftrightarrow \lambda^2$.

The quark multiplets consist of the complex scalar fields $q^{kA}$ and $\bar{q}_{Ak}$ (squarks)

\(^1\)It was argued [7] that, under certain conditions, transition from Abelian to non-Abelian confinement is smooth in non-supersymmetric QCD compactified on $S_1 \times R_3$. 

2
The 't Hooft–Polyakov monopole

Almost free monopole

Confined monopole, quasiclassical regime

Confined monopole, highly quantum regime

Figure 1: Various regimes for monopoles and strings.

and the Weyl fermions $\psi^{kA}$ and $\tilde{\psi}_{Ak}$, all in the fundamental representation of the SU(2) gauge group ($k = 1, 2$ is the color index while $A$ is the flavor index, $A = 1, 2$). The scalars $q^{kA}$ and $\tilde{q}^{kA}$ form a doublet under the action of the global SU(2)$_R$ group. The quarks and squarks have a U(1) charge too.

If one introduces a non-vanishing Fayet–Iliopoulos parameter $\xi$ the theory develops isolated quark vacua, in which the gauge symmetry is fully Higgsed, and all elementary excitations are massive. In the general case, two matter mass terms allowed by $\mathcal{N} = 2$ are unequal, $m_1 \neq m_2$. There are free parameters whose interplay determines dynamics of the theory: the Fayet–Iliopoulos parameter $\xi$, the mass difference $\Delta m$ and a dynamical scale parameter $\Lambda$, an analog of the QCD scale $\Lambda_{\text{QCD}}$ (Fig. 1). Extended supersymmetry guarantees that some crucial dependences are holomorphic, and there is no phase transition.

Both the gauge and flavor symmetries of the model are broken by the squark condensation. All gauge bosons acquire the same masses (which are of the order of inverse string thickness). A global diagonal combination of color and flavor groups, SU(2)$_{C+F}$, survives the breaking (the subscript $C+F$ means a combination of global color and flavor groups).

While SU(2)$_{C+F}$ is the symmetry of the vacuum, the flux tube solutions break it spontaneously. This gives rise to orientational moduli on the string world sheet.

The bulk theory is characterized by three parameters of dimension of mass: $\xi$, $\Delta m$, and $\Lambda$. As various parameters vary, the theory under consideration evolves in a very graphic way, see Fig. 1. At $\xi = 0$ but $\Delta m \neq 0$ (and $\Delta m \gg \Lambda$) it presents a very clear-cut example of a model with the standard 't Hooft–Polyakov monopole.
The monopole is unconfined — the flux tubes are not yet formed.

Switching on $\xi \neq 0$ traps the magnetic fields inside the flux tubes, which are weak as long as $\xi \ll \Delta m$. The flux tubes change the shape of the monopole far away from its core, leaving the core essentially intact. Orientation of the chromomagnetic field inside the flux tube is essentially fixed. The flux tubes are Abelian.

With $|\Delta m|$ decreasing, fluctuations in the orientation of the chromomagnetic field inside the flux tubes grow. Simultaneously, the monopole which no longer resembles the 't Hooft–Polyakov monopole, is seen as a string junction.

Finally, in the limit $\Delta m \to 0$ the transformation is complete. A global SU(2) symmetry restores in the bulk. Orientational moduli develop on the string worldsheet making it non-Abelian. The string worldsheet theory is CP(1) (CP($N - 1$) for generic values of $N$). Two-dimensional CP($N - 1$) models with four supercharges are asymptotically free. They have $N$ distinct vacuum states.

Each vacuum state of the worldsheet CP($N - 1$) theory presents a distinct string from the standpoint of the bulk theory. There are $N$ species of such strings; they have degenerate tensions $T_{st} = 2\pi\xi$. (The ANO string tension is $N$ times larger.)

Two different strings can form a stable junction. Figure 2 shows this junction in the limit

$$\Lambda_{\text{CP}(1)} \ll |\Delta m| \ll \sqrt{\xi}$$

(1)

corresponding to the lower left corner in Fig. 1. The magnetic fluxes of the U(1) and SU(2) gauge groups are oriented along the $z$ axis. In the limit (1) the SU(2) flux is oriented along the third axis in the internal space. However, as $|\Delta m|$ decreases, fluctuations of $B^a_z$ in the internal space grow, and at $\Delta m \to 0$ it has no particular orientation in SU(2) (the lower right corner of Fig. 1). In the language of the worldsheet theory this phenomenon is due to restoration of the O(3) symmetry in the quantum vacuum of the CP(1) model.

The junctions of degenerate strings present what remains of the monopoles in this highly quantum regime [8, 9]. It is remarkable that, despite the fact we are deep inside the highly quantum regime, holomorphy allows one to exactly calculate the mass of these monopoles. This mass is given by the expectation value of the kink...
central charge in the worldsheet CP$(N-1)$ model (including the anomaly term),
$M_M \sim N^{-1} \langle R \psi_L^\dagger \psi_R \rangle$.

1.2 Towards $\mathcal{N} = 1$

The unwanted feature of $\mathcal{N} = 2$ theory, making it less similar to QCD, is the presence of the adjoint chiral superfields $A$ and $A^a$. One can get rid of them making them heavy. To this end we can endow the adjoint superfield with a mass term of the type $\mu A^2$. More exactly, we will consider the $\mathcal{N} = 1$ preserving deformation superpotential

$$W = \frac{\mu}{2} \left[ A^2 + (A^a)^2 \right] \quad (2)$$

where $A$ and $A^a$ are adjoint chiral superfields. Now, supersymmetry of the bulk model becomes $\mathcal{N} = 1$. At large $\mu$ the adjoint fields decouple.

With the deformation superpotential (2) the 1/2 BPS classical flux tube solution stays the same as in the absence of this superpotential [10]. Moreover, the number of the boson and fermion zero modes, which become moduli fields on the string worldsheet, does not change either. For the fermion zero modes this statement follows from an index theorem proved in [11]. If the string solution and the number of zero modes remain the same, what can one say about the string worldsheet theory?

2 Worldsheet theory on strings in $\mathcal{N} = 1$ bulk theories

The discovery of non-Abelian strings in $\mathcal{N} = 1$ bulk theories [10] was a crucial step on the way to the desired $\mathcal{N} = 0$ theories. It turns out that these strings are quite remarkable. One can call them heterotic non-Abelian strings: the corresponding worldsheet theory is a chiral $\mathcal{N} = (0,2)$ extension of the bosonic CP$(N-1)$ model which was unknown previously!

$\mathcal{N} = 2$ SUSY Yang–Mills theories which support non-Abelian flux tubes have eight supercharges. The flux tube solutions are 1/2 BPS-saturated. Hence, the effective low-energy theory of the moduli fields on the string worldsheet must have four supercharges. The bosonic moduli consist of two groups: two translational moduli $(x_0)_{1,2}$ corresponding to translations in the plane perpendicular to the string axis, and two orientational moduli whose interaction is described by CP(1). The fermion moduli also split in two groups: four supertranslational moduli $\zeta_L, \zeta_L^\dagger, \zeta_R, \zeta_R^\dagger$ plus four superorientational moduli. $\mathcal{N} = 2$ supersymmetry in the bulk and on the worldsheet guarantees that $(x_0)_{1,2}$ and $\zeta_{L,R}$ form a free field theory on the worldsheet completely decoupling from (super)orientational moduli, which in turn form $\mathcal{N} = (2,2)$ supersymmetric CP(1) model.
2.1 Heterotic models on the worldsheet

What happens when one deforms the bulk theory to break $\mathcal{N} = 2$ down to $\mathcal{N} = 1$? Now we have four supercharges in the bulk and expect two supercharges on the worldsheet. If supertranslational sector continued to be decoupled from the superorientational one (which seemed to be a reasonable assumption) supersymmetrization of the orientational and translational modes would occur separately. It is well known that the requirement of two supercharges in CP(1) automatically leads to a nonchiral model with extended supersymmetry, $\mathcal{N} = (2, 2)$, with four supercharges (for a review see e.g. [12]). This was the line of reasoning Yung and I followed in 2005 [10] in arguing that non-Abelian strings obtained in the $\mathcal{N} = 1$ bulk theories have an “accidentally” enhanced supersymmetry. As we will see shortly, the assumed decoupling does not take place.

Edalati and Tong noted [13] that, with two supercharges on the worldsheet, only $\zeta_L, \zeta_L^\dagger$ remain protected. At the same time, $\zeta_R$ can and does mix with the superorientational moduli. Edalati and Tong outlined a general structure of the chiral $\mathcal{N} = (0, 2)$ generalization of CP(1). Derivation of the heterotic CP(1) model from the bulk theory was carried out in Ref. [14]. In this model the right- and left-moving fermions acquire different interactions; hence, the flux tube becomes heterotic!

The Lagrangian of the heterotic CP(1) model is

$$L_{\text{heterotic}} = \bar{\zeta}_R^i i \partial_L \zeta_R + \left[ \gamma \zeta_R R (i \partial_L \phi^i) \psi_R + \text{H.c.} \right] - g_0^2 |\gamma|^2 \left( \bar{\zeta}_R^i \zeta_R \right) \left( R \psi_L^i \psi_L \right)$$

$$+ G \left\{ \partial_\mu \phi^i \partial^\mu \phi + \frac{i}{2} (\psi_L^i \partial_R \psi_L + \psi_R^i \partial_L \psi_R) \right.$$ 

$$- \frac{i}{\chi} [\psi_L^i \psi_L (\phi^i \partial_L \phi)] + \psi_R^i \psi_R (\phi^i \partial_L \phi)] - \frac{2(1 - g_0^2|\gamma|^2)}{\chi^2} \psi_L^i \psi_L \psi_R^i \psi_R \right\}, \quad (3)$$

where $G$ is the Fubini–Study metric,

$$G = \frac{2}{g_0^2} \frac{1}{(1 + |\phi|^2)^2}, \quad (4)$$

$R$ stands for the Ricci tensor, and

$$\partial_L = \frac{\partial}{\partial t} + \frac{\partial}{\partial z}, \quad \partial_R = \frac{\partial}{\partial t} - \frac{\partial}{\partial z}. \quad (5)$$

The constant $\gamma$ in Eq. (3) is the parameter which determines the “strength” of the heterotic deformation, and the left-right asymmetry in the fermion sector. It is related to the parameter $\mu$ in Eq. (2) as follows:

$$\gamma = \frac{1}{g_0} \frac{\delta}{\sqrt{1 + 2|\delta|^2}} \quad (6)$$
where $\delta$ is known in two limits \cite{14},

$$\delta = \begin{cases} \text{const. } \mu, & \mu \text{ small}, \\ \text{const. } \sqrt{\ln \mu}, & \mu \text{ large}. \end{cases} \tag{7}$$

The second and third lines in Eq. (3) are the same as in the conventional $\mathcal{N} = (2, 2)$ CP(1) model, except the last coefficient.

Generalization for arbitrary $N$ (i.e. the $\mathcal{N} = (0, 2)$ deformed CP$(N - 1)$ model) is as follows \cite{14}:

$$L_{\text{heterotic}} = \zeta_R^\dagger i \partial_L \zeta_R + \left[ \gamma g_0^2 \zeta_R G_{ij} (i \partial_L \psi^+_j \psi^+_R) + \text{H.c.} \right]$$

$$- g_0^2 |\gamma|^2 \left( \zeta_R^\dagger \zeta_R \right) \left( G_{ij} \psi^+_L \psi^+_L \right)$$

$$+ G_{ij} [\partial_{\mu} \phi^+_j \partial_{\mu} \phi^+_i + i \bar{\psi}^+_i \gamma^\mu D_{\mu} \psi^+_j]$$

$$- \frac{g_0^2}{2} \left( G_{ij} \psi^+_R \psi^+_i \right) \left( G_{km} \psi^+_L \psi^+_k \right)$$

$$+ \frac{g_0^2}{2} \left( 1 - 2g_0^2 |\gamma|^2 \right) \left( G_{ij} \psi^+_R \psi^+_i \right) \left( G_{km} \psi^+_L \psi^+_k \right). \tag{8}$$

Introduction of a seemingly rather insignificant heterotic deformation drastically changes dynamics of the CP(1) model, leading to spontaneous SUSY breaking. This is rather obvious at small $\mu$. Indeed, the supercurrent of the deformed model acquires extra terms proportional to $\gamma \{ R \psi^+_R \psi^+_L \} \zeta_R$ at small $\mu$. In this limit the expression in the braces can be evaluated in the undeformed CP(1) model. As well known (see e.g. \cite{12}), a nonvanishing bifermion condensate $\langle R \psi^+_R \psi^+_L \rangle \sim \pm \Lambda$ develops in this model ($\Lambda$ is the scale parameter) labeling two distinct vacua. Thus, the additional terms in the supercurrent emerging in the deformed theory (at small $\gamma$) have the form

$$\Delta J_{sc} = \gamma \langle R \psi^+_R \psi^+_L \rangle \zeta_R^\dagger, \quad \Delta J^\dagger_{sc} = \gamma \langle R \psi^+_L \psi^+_R \rangle \zeta_R. \tag{9}$$

Since $\zeta_R$ is strictly massless, Eq. (9) clearly demonstrates that $\zeta_R$ is a Goldstino, with the residue $\langle R \psi^+_R \psi^+_L \rangle$. Supersymmetry is spontaneously broken, with the vacuum energy

$$\mathcal{E}_{\text{vac}} = |\gamma|^2 \left| \langle R \psi^+_R \psi^+_L \rangle \right|^2 \tag{10}$$

times a numerical factor, one and the same for both vacua. A nonvanishing $\mathcal{E}_{\text{vac}}$ for arbitrary values of $\gamma$ in heterotically deformed $CP(N - 1)$ models was obtained in \cite{15} using large $N$ expansion. The very possibility of the spontaneous supersymmetry breaking is due to the fact that Witten's index $I_W$ of the deformed theory vanishes, in
sharp contradistinction with the undeformed conventional \( \mathcal{N} = (2, 2) \) model where \( I_W = N \). Spontaneous breaking of SUSY in heterotic \( CP(N - 1) \) was anticipated in [16].

### 2.2 Large-\(N\) solution of the heterotic \( CP(N - 1) \) models

Solution of the heterotic model (8) in the large-\(N\) limit was found in [15] basing on the pattern suggested by Witten long ago [17]. Here I will briefly describe a general structure of the solution which depends on a single scaling variable

\[
u = \frac{16\pi}{N} \frac{1}{g_0^2} |\delta|^2,
\]

where the deformation parameter \( \delta \) was introduced in Sect. 2.1 while \( g_0^2 \) is the coupling of the \( CP(N - 1) \) model related to the bulk gauge coupling as \( g_0^{-2} = 2\pi g_2^{-2} \), see e. g. the review paper [18].

For arbitrary values of the deformation parameter the chiral condensate takes the form

\[
\langle R\psi_R^\dagger R\psi_L \rangle = \Lambda \exp \left( -\frac{u}{2} + \frac{2\pi ik}{N} \right)
\]

where the integer \( k \) \((k = 1, 2, ..., N)\) labels \( N \) distinct degenerate vacua of the theory corresponding to the spontaneous breaking of an axial \( Z_N \) symmetry. At large \( u \) the above order parameter becomes small. The \( \mathcal{N} = (0, 2) \) supersymmetry is always spontaneously broken as long as \( u \neq 0 \),

\[
\mathcal{E}_{\text{vac}} = \frac{N}{4\pi} \Lambda^2 \left( 1 - e^{-u} \right) \sim \begin{cases} uN^{-1} |\langle R\psi_R^\dagger\psi_L \rangle|^2, & u \to 0 \\ \frac{N}{4\pi} \Lambda^2, & u \to \infty. \end{cases}
\]

The theory has a massless Goldstino. At small \( u \) its role is played by \( \zeta_R \), while in the large-\( u \) limit \( \psi_R \) becomes massless. In the large-\( u \) limit, when \( \langle R\psi_R^\dagger\psi_L \rangle \) is small, the low-energy effective theory contains, apart from bosonic states, a single fermion: the massless Goldstino.

From the \( \mathcal{N} = 1 \) bulk theory standpoint, the spontaneous breaking of the \( \mathcal{N} = (0, 2) \) supersymmetry on the worldsheet means the loss of BPS saturation of the heterotic strings due to nonperturbative effects.

### 3 Planar Equivalence

Planar equivalence is equivalence in the large-\(N\) limit of distinct QCD-like theories in their common sectors (see [19]). Most attention received equivalence between
SUSY gluodynamics and its orientifold and $Z_2$ orbifold daughters. The Lagrangian of the parent theory is

$$\mathcal{L} = -\frac{1}{4g_P^2} G_{\mu\nu}^a G_a^{\mu\nu} + i \frac{g^2}{g_P^2} \lambda^{a\alpha} D_{\alpha\dot{\beta}} \bar{\lambda}^{a\dot{\beta}}$$

Equation (14)

where $\lambda^{a\alpha}$ is the gluino (Weyl) field in the adjoint representation of SU($N$), and $g_P^2$ stands for the coupling constant in the parent theory. The orientifold daughter is obtained by replacing $\lambda^{a\alpha}$ by the Dirac spinor in the two-index (symmetric or antisymmetric) representation (to be referred to as orienti-S or orienti-AS). The gauge coupling stays intact. To obtain the $Z_2$ orbifold daughter (to be referred to as orbi) we must pass to the gauge group SU($N/2$)$\times$SU($N/2$), replace $\lambda^{a\alpha}$ by a bifundamental Dirac spinor, and rescale the gauge coupling, $g_D^2 = 2g_P^2$.

### 3.1 Brief history

Genesis of planar equivalence can be traced to string theory. In 1998 Kachru and Silverstein studied [20] various orbifolds of $R^6$ within the AdS/CFT correspondence, of which I will speak later. Starting from $\mathcal{N} = 4$, they obtained distinct — but equivalent in the infinite-$N$ limit — four-dimensional daughter gauge field theories with matter, with varying degree of supersymmetry, all with vanishing $\beta$ functions.\(^2\)

The next step was made by Bershadsky et al. [21]. These authors eventually abandoned AdS/CFT, and string methods at large. Analyzing gauge field theories per se they proved that an infinite set of amplitudes in the orbifold daughters of the parent $\mathcal{N} = 4$ theory in the large-$N$ limit coincide with those of the parent theory, order by order in the gauge coupling. Thus, explicitly different theories have the same planar limit, at least perturbatively.

After a few years of relative oblivion, interest in the issue of planar equivalence was revived by Strassler [22]. He shifted the emphasis away from the search for supersymmetric daughters, towards engineering QCD-like daughters. Strassler considered $Z_N$ orbifolds. In 2003 an orientifold daughter of SUSY gluodynamics was suggested as a prime candidate for nonperturbative equivalence [23, 19]. At $N = 3$ this orientifold daughter identically reduces to one-flavor QCD! Thus, one-flavor QCD is planar-equivalent to SUSY gluodynamics. This remarkable circumstance allows one to copy results of these theories from one to another. For instance, color confinement of one-flavor QCD to supersymmetric Yang–Mills, and the exact gluino condensate in the opposite direction. This is how the quark condensate was calculated, for the first time analytically, in one-flavor QCD [24].

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\(^2\)This statement is slightly inaccurate; I do not want to dwell on subtleties.
3.2 Recent Developments

Kovtun, Ünsal and Yaffe formulated (and derived) [25, 26] the necessary and sufficient conditions for nonperturbative planar equivalence to be valid. This condition is nonbreaking of discrete symmetries: interchange $Z_2$ invariance for the $Z_2$ orbifold daughter, and $C$ invariance for the orientifold daughter. Although at first glance it does not seem to be a hard problem to prove that spontaneous breaking of the discrete symmetries does not occur, in fact, this is a challenging problem which defies exhaustive solution so far.

The question of the discrete symmetry nonbreaking would be automatically solved if one could prove that the expansion in fermion loops (say, for the vacuum energy) is convergent in some sense [27].

To be more exact, let us give a mass term $m$ to the fermions, and assume at first this mass term to be large, $m \gg \Lambda$. Then the $N_f$ expansion is certainly convergent. The question is “is there a singularity, so that at small $m$ the convergence is lost?” I believe that there is no such singularity. If so, both $Z_2$ orbifolds and orienti-S/AS are nonperturbatively equivalent to supersymmetric gluodynamics. Note that this statement does not refer to $Z_N$ orbifold with $N > 2$. In this case no mass term is possible in the orbifold theory, it is chiral.

On what I base my belief? Consider supersymmetric gluodynamics with SUSY slightly broken by a small mass term of gluino. At $N = \infty$ the vacuum structure of this theory is exactly the same as the vacuum structure of pure Yang–Mills (the latter was derived by Witten [28], see also [29, 30]). Thus, I would say that the expansion in the number of fermion loops should work. This is of course not a mathematical theorem, but rather a physics argument.

Since for given number of fermion loops and given $m \neq 0$ each expansion term in supersymmetric gluodynamics is exactly the same as the corresponding expansion term in $Z_2$ orbifold and orienti-S/AS, the fermion loop expansions in all three theories must be convergent.

Since in pure gauge theory, with no fermions, the vacuum is unique [28], then so is the case for $Z_2$ orbifold and orienti-S/AS at $m \neq 0$. The uniqueness of the vacuum state (for $\theta = 0$) implies the absence of the spontaneous breaking of the discrete symmetries in the above daughter theories.

If the statement is valid for small $m \neq 0$ extrapolation to $m = 0$ must be smooth since none of these theories has massless particles in the limit $m = 0$. They all have a mass gap $\sim \Lambda$.

3.3 Center-group symmetry and the limit $N \to \infty$

The planar equivalence between the parent and daughter theories described in the beginning of Sect. 3 holds not only on $R_4$ but in arbitrary geometry. Therefore,
one can compare phase diagrams and, in particular, temperature dependences. This topic was open by Sannino [31], a thorough discussion was presented by Ünsal [32].

There is a famous Polyakov criterion regarding confinement/deconfinement in SU($N$) Yang–Mills theories. If one compactifies $R^4$ into $R^3 \times S^1$ and considers the Polyakov line along the compactified direction, its expectation value may or may not vanish. If it does not vanish, the $Z_N$ symmetry — the center of the gauge SU($N$) group — is broken. On the other hand, if the Polyakov line vanishes the $Z_N$ symmetry is unbroken. The former case corresponds to deconfinement, the latter to confinement.

Introducing quarks in the Yang–Mills Lagrangian brings in a problem with this criterion, since now there is no apparent $Z_N$ center symmetry at the Lagrangian level. This is in one-to-one correspondence with the fact that there are no genuine long strings in QCD. They break through quark-antiquark pair creation.

How can this apparent absence of the $Z_N$ symmetry be compatible with planar equivalence? In [33] I argued that the center symmetry is dynamically restored in the $N = \infty$ limit. In SU($N$) Yang–Mills theory with quarks in the fundamental representation ($N \to \infty$) the fundamental quarks decouple, and we are left with pure Yang–Mills which does have the $Z_N$ center group. Once it is unbroken, the theory is in the confinement phase. The Polyakov line is a good order parameter. The same is valid with regards to supersymmetric gluodynamics even at finite $N$. Gluinos do not decouple at large $N$, but they do not ruin the center-group symmetry.

Now, comes a nontrivial remark. Consider, for instance, the AS orientifold daughter of supersymmetric gluodynamics. At $N = \infty$ two-index antisymmetric fermions do not decouple. There is no apparent center symmetry in this theory. At the Lagrangian level the orientifold theories have at most a $Z_2$ center.

And yet, the full $Z_N$ center symmetry dynamically emerges in the orientifold theories [34] in the limit $N \to \infty$. In the confining phase the manifestation of this enhancement is the existence of stable $k$-strings in the large-$N$ limit of the orientifold theories. These strings are identical to those of supersymmetric Yang–Mills theories. The critical temperatures of the confinement-deconfinement phase transitions are the same in the orientifold daughters and their supersymmetric parent up to $1/N$ corrections.

The Lagrangian of the orientifold theories has the form

$$\mathcal{L} = -\frac{1}{4g^2} G^{\alpha \mu \nu} G_{\alpha \mu \nu} + \frac{i}{g^2} \psi_{ij}^{\alpha} D_{\alpha \beta} \bar{\psi}^{\beta ij}$$

where $\psi_{ij}$ is the Dirac spinor in the two-index antisymmetric or symmetric representation. The center symmetry is $Z_2$ for even $N$ and none for odd $N$.

Integrating out the two-index antisymmetric fermion yields

$$\log \det \left( i \gamma_{\mu} D_{\mu}^{\text{AS}} - m \right)$$
\[
= N^2 \sum_{n \in \mathbb{Z}} \sum_{C_n} \frac{\alpha(C_n)}{2} \left\{ \left[ \frac{\text{Tr}}{N} U(C_n) \right]^2 - \frac{1}{N} \frac{\text{Tr}}{N} U^2(C_n) \right\}. \quad (16)
\]

In the large-$N$ limit we can ignore the single-trace terms since they are suppressed by $1/N$ compared to the $O(N^2)$ double-trace term. The single-trace term contribution scales as that of the fundamental fermions, and is quenched in the same manner.

A typical double-trace term $(\text{Tr} U(C_n))^2$ is $O(N^2)$ and is a part of the leading large-$N$ dynamics. Thus, the impact of the two-index antisymmetric fermions on dynamics is as important as that of the glue sector of the theory.

The action of the pure glue sector is local and manifestly invariant under the $Z_N$ center. Integrating out fermions, induces a nonlocal sum (16) over gluonic observables. This sum includes both topologically trivial loops with no net winding around the compact direction (the $n = 0$ term) and nontrivial loops with non-vanishing winding numbers. The topologically trivial loops are singlet under the $Z_N$ center symmetry by construction, while the loops with non-vanishing windings are non-invariant.

Let us inspect the $N$ dependence of the effective action more carefully. If we expand the fermion action in the given gluon background we get

\[
\langle \exp \left\{ -N^2 \sum_{n \neq 0} \sum_{C_n} \frac{\alpha(C_n)}{2} \left[ \frac{\text{Tr}}{N} U(C_n) \right]^2 \right\} \rangle,
\]

where $\langle \ldots \rangle$ means averaging with the exponent combining the gluon Lagrangian with the zero winding number term. This weight function is obviously center-symmetric. If $h$ is an element of the SU($N$) center, a typical term in the sum (17) transforms as

\[
\langle \frac{\text{Tr}}{N} U(C_n) \frac{\text{Tr}}{N} U(C_n) \rangle \rightarrow h^{2n} \langle \frac{\text{Tr}}{N} U(C_n) \frac{\text{Tr}}{N} U(C_n) \rangle
\]

\[
= h^{2n} \left[ \langle \frac{\text{Tr}}{N} U(C_n) \rangle \langle \frac{\text{Tr}}{N} U(C_n) \rangle + \langle \frac{\text{Tr}}{N} U(C_n) \frac{\text{Tr}}{N} U(C_n) \rangle_{\text{con}} \right],
\]

where I picked up a quadratic term as an example. The connected term in the expression above is suppressed relative to the leading factorized part by $1/N^2$, as follows from the standard $N$ counting, and can be neglected at large $N$. As for the factorized part, planar equivalence implies that all expectation values of multi-winding Polyakov loops are suppressed in the large $N$ limit by $1/N$,

\[
\langle \frac{1}{N} \text{Tr} U(C_n) \rangle^\text{SYM} = 0,
\]

\[
\langle \frac{\text{Tr}}{N} U(C_n) \rangle^\text{AS} = O \left( \frac{1}{N} \right) \rightarrow 0, \quad n \in \mathbb{Z} - \{0\},
\]

(19)
where the first relation follows from unbroken center symmetry in the SYM theory and the latter is a result of planar equivalence (in the C-unbroken, confining phase of orienti-AS).

Consequently, the non-invariance of the expectation value of the action under a global center transformation is

\[ \langle \delta S \rangle = \langle S(h^n \text{Tr } U(C_n)) - S(\text{Tr } U(C_n)) \rangle = O \left( \frac{1}{N} \right) \langle S \rangle, \quad (20) \]

which implies, in turn, dynamical emergence of center symmetry in orientifold theories in the large-\(N\) limit. Let us emphasize again that the fermion part of the Lagrangian which explicitly breaks the \(Z_N\) symmetry is not sub-leading in large \(N\). However, the effect of the \(Z_N\) breaking on physical observables is suppressed at \(N \to \infty\).

This remarkable phenomenon is a natural (and straightforward) consequence of the large-\(N\) equivalence between \(N = 1\) SYM theory and orienti-AS. Despite the fact that the center symmetry in the orienti-AS Lagrangian is at most \(Z_2\), in the \(N = \infty\) limit all observables behave as if they are under the protection of the \(Z_N\) center symmetry. We refer to this emergent symmetry of the orienti-AS vacuum as the custodial symmetry. The custodial symmetry becomes exact in the \(N = \infty\) limit, and is approximate at large \(N\).

Thus, we do have a \(Z_N\) center-group symmetry in large-\(N\) Yang–Mills theories with quarks in one and two-index representations of \(SU(N)\) after all! Conceptually, this is a non-trivial statement which invalidates some statements in the literature; in particular, it restores “equal-rights” status for even and odd values of \(N\).

4 Exact perturbative calculations with gluons in \(\mathcal{N} = 4\)

Obtaining high orders in the perturbative expansion (multiparton scattering amplitudes at tree level and with loops) is an immense technical challenge. Due to the gauge nature of interactions, the final expressions for such amplitudes are orders of magnitude simpler than intermediate expressions.

In the 1990’s Bern, Dixon and Kosower pioneered applying string methods to obtain loop amplitudes in supersymmetric theories. The observed simplicity of these results (generalizing the elegant structure of the Parke–Taylor amplitude [35]) led to an even more powerful approach based on unitarity. Their work resulted in an advanced helicity formalism exhibiting a feature of the amplitudes, not apparent from the Feynman rules, an astonishing simplicity. In 2003 Witten uncovered [36] a hidden and elegant mathematical structure in terms of algebraic curves in terms
of twistor variables in gluon scattering amplitudes: he argued that the unexpected simplicity could be understood in terms of twistor string theory.

This observation created a diverse and thriving community of theorists advancing perturbative calculations at tree level and beyond, as it became clear that loop diagrams in gauge theories have their own hidden symmetry structure. Most of these results do not directly rely on twistors and twistor string theory, except for some crucial inspiration. So far, there is no good name for this subject. Marcus Spradlin noted that an unusually large fraction of contributors’ names start with the letter B.\textsuperscript{3} Therefore, perhaps, we should call it $B$ theory, with $B$ standing for beautiful, much in the same way as $M$ in $M$ theory stands for magic. I could mention a third reason for “$B$ theory”: Witten linked the scattering amplitudes to a topological string known as the “$B$ model.”

$B$ theory revived, at a new level, many methods of the pre-QCD era, when $S$-matrix ideas ruled the world. For instance, in a powerful paper due to Britto, Cachazo, Feng and Witten (BCFW) \cite{37}, tree-level on-shell amplitudes were shown in a very simple and general way to obey recursion relations. Their proof was based only on Cauchy’s theorem and general (factorization) properties of tree-level scattering! The BCFW recursion relations gave us a way to calculate scattering amplitudes without using any gauge fixing or unphysical intermediate states.

Although the ultimate goal of the $B$ theory is calculating QCD amplitudes, the concept design of various ideas and methods is carried out in supersymmetric theories, which provide an excellent testing ground. Looking at super-Yang–Mills offers a lot of insight into how one can deal with the problems in QCD.

Of all supersymmetric theories probably the most remarkable is $\mathcal{N} = 4$ Yang–Mills. Its special status is due to the fact that (a) it is conformal, and (b) in the planar strong coupling limit it is dual to string theory on AdS$_5 \times$ S$^5$.

I would like to briefly review a remarkable progress that has been achieved in understanding the gluon “scattering amplitudes” in this theory. (For more detailed reviews, with exhaustive lists of references, see e.g. \cite{38}.) I refer to the scattering amplitudes in quotation marks because, strictly speaking, the notion of the $S$ matrix is ambiguous in conformal theories. This problem is resolved by regulating the theory in the infrared. The standard regularization in this range of questions is dimensional reduction to $D = 4 - 2\epsilon$ dimensions.

\textsuperscript{3}E.g. Badger, Bedford, Berger, Bern, Bidder, Bjerrum-Bohr, Brandhuber, Britto, Buchbinder, ... (Of course, one should not forget about Cachazo, Dixon, Feng, Forde, Khoze, Kosower, Roiban, Spradlin, Svrček, Travaglini, Vaman, Volovich, ...). This reminds me of a joke of a proof given by a physicist that almost all numbers are prime: one is prime, two is prime, three is prime, five is prime, while four is an exception just supporting the general rule.
4.1 Bern–Dixon–Smirnov hypothesis

In 2005 Bern, Dixon and Smirnov calculated, in $\mathcal{N} = 4$ theory, the 2 gluons $\to$ 2 gluons amplitude up to three loops [39]. Based on this and earlier results with Anastasiou and Kosower [40] they suggested an ansatz for the maximally helicity violating (MHV) $n$-point amplitudes to all loop orders in perturbation theory in the planar limit. For 2 gluons $\to$ 2 gluons amplitude the Bern–Dixon–Smirnov (BDS) hypothesis takes the form

$$A(2 \text{ gluons} \to 2 \text{ gluons}) = A(2 \text{ gluons} \to 2 \text{ gluons})_{\text{tree}} \times$$

$$\exp \left[ (\text{IR divergent}) + \frac{\Gamma_{\text{cusp}}(\lambda)}{4} \left( \ln \frac{s}{t} \right)^2 + \text{const.} \right]$$

(21)

where the infrared divergent $1/\epsilon^2$, $1/\epsilon$ part in the above expression is separated by virtue of the $\epsilon$ expansion, $\lambda$ is the ’t Hooft coupling, and the function $f(\lambda)$ is directly related with the cusp anomalous dimension,\(^4\)

$$\Gamma_{\text{cusp}}(\lambda) = \begin{cases} 
    \lambda - \frac{\pi^2}{12} \lambda^2 + \ldots, & \lambda \to 0, \\
    \sqrt{\lambda}, & \lambda \to \infty.
\end{cases}$$

(22)

Then Alday and Maldacena [44] in a tour-de-force work performed the strong coupling computation of the same amplitude by using the gauge theory/gravity duality that relates $\mathcal{N} = 4$ Yang–Mills to string theory on AdS$_5 \times$ S$^5$. They found that the leading order result at large values of the ’t Hooft coupling $\lambda$ is given by a single classical string configuration. The classical string solution corresponds to a minimal area $A_{\text{min}}$ which, in turn, depends on the momenta $k_i^\mu$ of the final and initial gluons ($i = 1, 2, ..., n$),

$$A_n / (A_n)_{\text{tree}} \propto \exp \left\{ -\sqrt{\lambda} A_{\text{min}} \right\}.$$  

(23)

The Alday–Maldacena strong coupling result for the finite part of the amplitude (21) turned out to be exactly the same, $\exp \left\{ (\Gamma_{\text{cusp}}/2) \ln^2(s/t) \right\}$. It should be noted that the cusp anomalous dimension $\Gamma_{\text{cusp}}$ which fully determines the finite part of the above amplitude is believed to be known exactly [45], through a maximal transcendentality hypothesis.

The successful matching of the small and large-$\lambda$ expansions for the four-point amplitude was an inspirational event, after which it was natural to assume that the BDS representation holds for arbitrary $n$-point amplitudes, $n = 4, 5, 6, ...$, i.e. the

\(^4\)The cusp anomalous dimension was introduced in QCD in connection with the renormalization-group equation for Wilson loops with cusps [41], see also [42]. It also controls the large-spin limit of the anomalous dimensions of twist-2 operators [42, 43].
finite parts of all $n$-gluon amplitudes factorize into a $\lambda$ dependent factor represented by $\Gamma_{\text{cusp}}$ and a momentum dependent factor, say, for $n = 5$

$$A_n / (A_n)_{\text{tree}} \propto \exp \left\{ -\frac{\Gamma_{\text{cusp}}(\lambda)}{8} \sum_i \ln \frac{s_{i,i+1}}{s_{i,i+2}} \ln \frac{s_{i+1,i+2}}{s_{i+2,i+3}} \right\},$$

(24)

where $s_{i,i+1}$ are appropriate kinematic invariants. Is this remarkably simple formula true?

Many people contributed to the solution of this question. Today it is known that the answer is negative, beginning from $n = 6$, and it is known why. From the weak-coupling side, the six-gluon amplitude calculated at two loops [46] reveals the occurrence of extra terms absent in Eq. (24) which depend on conformally invariant ratios of various $s_{i,j}$'s (see Sect. 4.2). Such ratios cannot be built for $n = 4$ and $5$. From the strong-coupling side, the classical string solution with appropriate boundary conditions found in the limit $n \to \infty$ [47] also leads to a representation of the $n$-gluon amplitude incompatible with the BDS ansatz.

### 4.2 A magical correspondence between the coordinate and momentum spaces

The four-gluon amplitudes reveal an intriguing iterative structure both at weak and strong couplings (i.e. in the Bern–Dixon–Smirnov ansatz and the Alday–Maldacena approach, respectively). One may wonder where does this structure come from? The answer to this question was found in a series of papers by Drummond, Henn, Korchemsky and Sokatchev (DHKS) [48, 49, 50, 51, 52] who found that the planar gluon scattering amplitudes in $\mathcal{N} = 4$ Yang–Mills possess a hidden symmetry, the so-called dual conformal symmetry. This symmetry becomes manifest after one passes from on-shell gluon momenta $p_i^\mu$ to dual four-dimensional “coordinates” $x_i^\mu$,

$$p_i^\mu = x_i^\mu - x_{i+1}^\mu, \quad \sum_i p_i^\mu = 0,$$

(25)

and considers the $n$-gluon scattering amplitude as a function of the dual coordinates $x_i$ (with the periodicity condition $x_{i+n} = x_i$). In this way, one discovers that, quite surprisingly, the Feynman integrals contributing to four-gluon amplitudes up to four loops (!) are invariant under the conformal $\text{SO}(2,4)$ transformations of the dual coordinates $x_i^\mu$. If $x_i$ were “normal” coordinates in the configurational space, one could expect this symmetry to be related with the conformal symmetry of the underlying $\mathcal{N} = 4$ theory. However, $x_i$’s belong to the momentum rather than configurational space!

In actuality, the dual conformal symmetry has a different origin. Indeed, the dual coordinates are intrinsically related to momenta. Similar to conventional conformal
symmetry, the dual conformal symmetry imposes severe constraints on the possible form of the planar gluon amplitudes. Namely, as was shown by DHKS, if the dual conformal symmetry survived to all loops, it would allow one to determine the four- and five-gluon amplitudes to all orders.

The dual conformality is slightly broken by infrared regularization and, as a consequence, the scattering amplitudes satisfy anomalous conformal Ward identities. It is remarkable that for the four- and five-gluon amplitudes the solution to these Ward identities is unique: it coincides with the Bern–Dixon–Smirnov ansatz. For \( n \geq 6 \) the Bern–Dixon–Smirnov ansatz for the MHV amplitudes goes through the same Ward identities, but its general solution is determined up to an arbitrary function of (dual) conformal ratios. If one assumes that the gluon scattering amplitudes enjoy the dual conformal symmetry to all orders, then the failure of the BDS ansatz would imply that this function does not vanish.

Using the dual conformal symmetry as a guiding principle, DHKS suggested that the MHV scattering amplitudes in planar \( \mathcal{N} = 4 \) theory are equal to the Wilson loops evaluated along a closed polygon-like contour in the Minkowski space-time built from light-like segments defined by on-shell gluon momenta (Fig. 3).

![Figure 3: The \( n \)-gluon MHV amplitude \( \langle 0 | S | 1^{-2^{-3^+}} \ldots n^+ \rangle \) depicted on the left is equivalent to the Wilson loop \( \langle 0 | \text{Tr} \ P \exp (i \int_{C} dx \cdot A(x)) \ | 0 \rangle \) with \( n \) cusps depicted on the right.](image)

This relation is very surprising and counter-intuitive because, firstly, it relates two quantities of a different nature (an on-shell \( S \)-matrix element and the vacuum expectation value of nonlocal functional of gauge fields) and, secondly, defining the integration contour in terms of gluon momenta, one assigns a wrong engineering dimensions to the points in the configuration space. Nevertheless, the two objects share the same symmetry (the dual conformal symmetry of the scattering amplitudes versus conventional conformal symmetry of the Wilson loops in \( \mathcal{N} = 4 \) theory). The duality between the \( n = 4 \) and \( n = 5 \) gluon amplitudes and the Wilson loop is explicitly checked up to two loops in Refs. [49, 50, 53]. The all-order proof presented in [50] is based on the conformal Ward identities.
For $n \geq 6$ the situation changes dramatically, since as was already mentioned, the conformal symmetry allows for the existence of an arbitrary function of harmonic ratios. The dual conformal symmetry alone is not sufficient to fix this function. It is natural to ask whether the duality with the Wilson loops will still be valid.

This question was addressed and clarified (for $n = 6$ in two loops) in Refs. [46, 52]. We recall that the Bern–Dixon–Smirnov ansatz provides a definite prediction for this amplitude. It was found that the BDS formula fails to describe the $n = 6$ amplitude starting from two loops, and the discrepancy function depends only on the (dual) conformal ratios, in agreement with the dual conformal symmetry.

Although the BDS fails for $n = 6$, nevertheless one can ask about duality between the amplitude and the light-like Wilson loop, as in Fig. 3. Comparing the two-loop expression for the hexagon Wilson loop with the $n = 6$ gluon amplitudes, one finds a perfect agreement! This strongly suggests that the Wilson loops/scattering amplitudes duality should hold in $\mathcal{N} = 4$ theory to all orders and all values of $n$.

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