A generalized method for scattering from wide cavities with specified wave functions

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Abstract
This study developed a generalized solution based on modal expansion for scattering by large cavities with known wave functions placed in an infinite perfect electric plane. Under the assumption of a large cavity, to reduce simulation time and simplify expressions, the half-space above cavity with a strong singular Green’s function is substituted by an arbitrary semi-waveguide. The fields inside the cavity are expanded by the semi-waveguide eigenfunctions. The corresponding modes are matched to create a system of linear equations for unknown expansion coefficients. To demonstrate the validity and ability of this method, it is applied to several grooves with different shapes (triangular, circular, and elliptical grooves) and then their scattering signatures are compared with each other. The results are also verified by the results obtained by the method of moment. The measured simulation time for both methods shows that this scheme can be an appropriate candidate for analysing the scattered fields by large cavities.

INTRODUCTION

The evaluating of scattered waves from cavities is an important subject in many applications such as the radar cross-section (RCS) reduction, non-destructive testing, optical communications, spectroscopy, and nanotechnology. In the electrically large cavities widely used in optic structures, when the size of the cavities increases, as is known, the computational efficiency decreases. In this case, we have a significant problem especially with meshing methods including method of moment (MoM) and the finite element method (FEM). Some efficient methods were recently developed to compensate for the inefficiency and inaccuracy of common full-wave solvers [1–12].

In this regard, for high-frequency incident waves, Hao used the shooting-and-bouncing ray (SBR) approach to analyse scattering from open-ended cavities with rectangular and circular cross sections [1]. Morgan applied mode-matching method to a large rectangular groove and obtained an analytic solution for the coefficients of the expansions [2]. Xiang, in [3], solved a magnetic field integral equation on a large arbitrary-shaped cavity by a numerical hybrid technique that uses boundary-element and wavelet-transform methods. Bao and Sun presented a fast algorithm based on the Fourier transform for solving electromagnetic scattering from a large rectangular cavity embedded in a ground plane [4]. They also use an asymptotic technique for a large rectangular groove to improved mode-matching method [5]. Zhao introduced an efficient fourth-order finite difference scheme for large rectangular grooves [6]. Basha presented a general scattering matrix for scattering from a finite arbitrarily shaped groove [7]. He divided the groove into L layers and combined the boundary conditions between fields in the groove and upper half-space. This method is able to analyse scattering waves from grooves with arbitrary shapes. However, in large cavities, a serious difficulty is encountered when using this technique. Ching-Yen investigated scattering signals of monochromatic light incident by rectangular channels with sizes from micrometres to sub-micrometres [8]. Huiyuan employed a Legendre spectral Galerkin method to predict transverse electric (TE) backscattering from large cavities [9]. For scattering from an arbitrarily shaped, large cavity, Lai presented a fast direct solver [10]. They reformulated the Maxwell equations as a well-conditioned second-kind integral equation and used a hierarchical matrix factorization technique to solve the resulting linear system. In [11], a mode-matching solution was proposed to obtain the scattered fields by a large 2-D isosceles triangular groove. For scattering from a large rectangular groove, Zhaolou utilized geometrical optics approximation to obtain the electric field on...
the groove. He divided the incident wave into various parts during reflection and refraction and determined the effect of every part [12].

To extend the mode expansion technique presented for a rectangular and right isosceles groove in [2,11], we suggest a generalized manner that covers large cavities with specified wave functions. Here, instead of solving complicated singular integrals originated from half-space Green’s functions, we compute some non-singular integrals. However, for a large cavity, we will encounter high oscillatory integrals that some of them have an analytic solution. Thus, we should utilize advanced mathematical techniques to compute high oscillatory integrals accurately and rapidly. The scheme proposed here is as follows. First, the half-space above a large groove is substituted by an arbitrary semi-waveguide with the known wave functions. The tangential fields in both regions are expanded by semi-waveguide eigenfunctions selected as the basis functions of the expansion. Given an electrical large groove, the equivalent current on the cavity can be approximated by a physical optic induced electric current. By applying boundary conditions on the groove surface and matching the corresponding modes, two independent equations are obtained for TE and transverse electric (TM) modes. Using the orthogonality property of the wave functions and first-order derivatives of eigenfunctions of the Sturm–Liouville operator, a system of linear equation is constructed for unknown coefficients. In Section 4, this procedure is applied to different groove shapes (triangular, circular and elliptical groove) and verified by electromagnetic simulators FEKO which uses MoM. To compare the computation efficiency of the proposed method with MoM, we measure the simulation time of each method. An inspection of the obtained results demonstrates that the suggested method is faster than MoM. It is also noted that this approach has been developed for an electrically wide cavity and consequently is not valid for a groove size smaller than the wavelength. However, when the aperture width increases, the results become more accurate. We also used the suggested method to determine the scattering signature of several different grooves and compare them with each other.

2 | PROBLEM DESCRIPTION

In Figure 1, a normalized TE or TM plane wave

$$H_z^1, E_z^1 = \frac{jk_0}{\varepsilon_0} \left[ x - \frac{w}{\pi} \cos \phi_0 \right], \quad \phi_0 \in (0, \pi)$$

illuminates a 2-D wide arbitrarily shaped groove located in an infinite ground plane and filled with a material \( \varepsilon_2 \) and \( \mu_2 \), where \( k_1, \phi_0 \) and \( W \) are the free-space propagation constant, the incidence angle, and the width of the groove, respectively.

To avoid solving complex singular integrals (logarithmic and hypersingular integrals), we replace the half-space above the groove by an arbitrary semi-infinite waveguide as shown in Figure 2 that has a non-singular Green's function. This substitution makes significant error for narrow grooves but is accurate for large cavities.

Assume the scalar eigenfunctions \( \psi_n^{(1)} \) and \( \psi_n^{(2)} \) given for the arbitrary semi-infinite waveguide and open cavity, respectively, satisfy Maxwell’s equation simplified to the following Helmholtz equation and the boundary conditions as

$$\nabla^2 \psi_n^{(1)} + k_2^2 \psi_n^{(1)} = 0,$$

$$\begin{cases}
\psi_n^{(1)} = 0 & \text{on } \Gamma_1, \ TE \ mode \\
\frac{\partial \psi_n^{(1)}}{\partial n} = 0 & \text{on } \Gamma_1, \ TM \ mode
\end{cases} \quad (2)$$

Also, for the arbitrary open cavity depicted in Figure 2, we have

$$\nabla^2 \psi_n^{(2)} + k_2^2 \psi_n^{(2)} = 0,$$

$$\begin{cases}
\psi_n^{(2)} = 0 & \text{on } \Gamma_2, \ TE \ mode \\
\frac{\partial \psi_n^{(2)}}{\partial n} = 0 & \text{on } \Gamma_2, \ TM \ mode
\end{cases} \quad (3)$$

where the curved walls \( \Gamma_1 \) and \( \Gamma_2 \) surround the semi-infinite waveguide and the cavity, respectively. The tangential electric and magnetic fields \( E_z^{(1)}, E_x^{(1)}, H_z^{(1)}, \) and \( H_x^{(1)} \) in region 1 can be expanded as the sum of infinite \( n \) eigenfunctions \( \psi_n^{(1)TM} \) and \( \psi_n^{(1)TE} \) as follows:

$$E_z^{(1)}(x,y) = \sum_{n=0}^\infty a_n^{TM} \psi_n^{(1)TM}(x,y) \quad (4)$$

FIGURE 1. The geometry of a 2-D arbitrarily shaped, large groove filled with a material \( \varepsilon_2, \mu_2 \)

FIGURE 2. Substituting the upper half-space with an arbitrary semi-waveguide
\( H_x^{(1)}(x, y) = \frac{-1}{j \omega \mu_1} \sum_{n=0}^{\infty} a_n^{TM} \psi_n^{(1)TM}(x, y) \) \text{ TM mode} \quad (5) \\
and
\( H_x^{(1)}(x, y) = \sum_{n=0}^{\infty} a_n^{TE} \psi_n^{(1)TE}(x, y) \) \text{ TE mode} \quad (6) \\
\( E_x^{(1)}(x, y) = \frac{1}{j \omega e_1} \sum_{n=0}^{\infty} a_n^{TM} \psi_n^{(1)TM}(x, y). \) \text{ TE mode.} \quad (7) \\
Also, the tangential fields in region 2 are given as
\( E_x^{(2)}(x, y) = \sum_{n=0}^{\infty} b_n^{TM} \psi_n^{(2)TM}(x, y) \) \text{ TM mode} \quad (8) \\
\( H_x^{(2)}(x, y) = \frac{-1}{j \omega \mu_2} \sum_{n=0}^{\infty} b_n^{TM} \psi_n^{(2)TM}(x, y) \) \text{ TM mode} \quad (9) \\
and
\( H_x^{(2)}(x, y) = \sum_{n=0}^{\infty} b_n^{TE} \psi_n^{(2)TE}(x, y) \) \text{ TE mode} \quad (10) \\
\( E_x^{(2)}(x, y) = \frac{1}{j \omega e_2} \sum_{n=0}^{\infty} b_n^{TE} \psi_n^{(2)TE}(x, y) \) \text{ TE mode,} \quad (11)

where \( \psi_n^{'} \) denotes the first-order derivative of eigenfunction with respect to \( y \). Here, we suppose the groove has a wide opening, and consequently, we can consider a physical optic current across the aperture as
\( \overrightarrow{j}_{op} = 2\gamma \times \overrightarrow{H}_i \quad (12) \)

where \( \overrightarrow{H}_i \) denotes the incident tangential magnetic field over the aperture including \( \overrightarrow{H}_x \) and \( \overrightarrow{H}_z \) for TM and TE modes, respectively. By applying the continuity of tangential fields on the groove, we construct two set equations for both modes as
\( E_x^{(1)}(x, 0) = E_x^{(2)}(x, 0) = \frac{1}{j \omega e_1} \sum_{n=0}^{\infty} b_n^{TE} \psi_n^{(1)TE}(x, 0) \)
\( = \frac{1}{j \omega e_2} \sum_{n=0}^{\infty} b_n^{TE} \psi_n^{(2)TE}(x, 0) \) \quad (13)
\( H_x^{(1)}(x, 0) - H_x^{(2)}(x, 0) = \overrightarrow{j}_{op} = \sum_{n=0}^{\infty} a_n^{TE} \psi_n^{(1)TE}(x, 0) \)
\( - \sum_{n=0}^{\infty} b_n^{TE} \psi_n^{(2)TE}(x, 0) \)
\( = 2\epsilon \left( x - \frac{a}{2} \right) \cos \varphi_o \) \quad (14)

and for TM modes,
\( E_x^{(1)}(x, 0) = E_x^{(2)}(x, 0) = \sum_{n=0}^{\infty} a_n^{TM} \psi_n^{(1)TM}(x, 0) \)
\( = \sum_{n=0}^{\infty} b_n^{TM} \psi_n^{(2)TM}(x, 0) \) \quad (15)
\( H_x^{(1)}(x, 0) - H_x^{(2)}(x, 0) = \overrightarrow{j}_{op} = \frac{-1}{j \omega \mu_1} \sum_{n=0}^{\infty} a_n^{TM} \psi_n^{(1)TM}(x, 0) \)
\( + \frac{1}{j \omega \mu_2} \sum_{n=0}^{\infty} b_n^{TM} \psi_n^{(2)TM}(x, 0) \)
\( = \frac{-2 \sin \varphi_o}{\eta} j_k \left( x - \frac{a}{2} \right) \cos \varphi_o \) \quad (16)

We know that the first-order derivatives of the eigenfunctions of the Sturm–Liouville operator are orthogonal, too. Therefore, by multiplying both sides of Equations (13)/(14)/(15) (16) by \( \psi_n^{(2)TE}, \psi_n^{(2)TE}, \psi_n^{(2)TM}, \) and \( \psi_n^{(2)TM} \), respectively, and integration across the groove opening, for each mode number \( n \), Equations (13)–(16) can be simplified as follows:
\( \frac{1}{j \omega e_1} \sum_{n=0}^{\infty} a_n^{TE} A_{mn} = b_n^{TM} \) \quad (17)
\( \sum_{n=0}^{\infty} a_n^{TE} B_{mn} + b_n^{TM} = C_n^{TE} \) \quad (18)

and for TM mode,
\( \sum_{m=0}^{\infty} a_n^{TM} A_{mn} = b_n^{TM} \) \quad (19)
\( \frac{-1}{j \omega \mu_1} \sum_{m=0}^{\infty} a_n^{TM} B_{mn} + \frac{b_n^{TM}}{j \omega \mu_2} = C_n^{TM} \) \quad (20)

The known coefficients \( A_{mn}, B_{mn}, C_n^{TE}, A_n, B_n^{TM} \), and \( C_n^{TM} \) are given by
\( A_{mn} = \int_0^W \psi_n^{(2)TE} \psi_n^{(1)TE} dx, \) \quad (21)
\( B_{mn} = \int_0^W \psi_n^{(2)TE} \psi_n^{(1)TE} dx, \) \quad (22)
\( C_n^{TE} = 2 \int_0^W \psi_n^{(2)TE} j_k \left( x - \frac{a}{2} \right) \cos \varphi_o dx, \) \quad (23)
\( A_n = \int_0^W \psi_n^{(2)TM} \psi_n^{(1)TM} dx, \) \quad (24)
### Table 1: Some eigenfunctions and eigenvalues in regular cavities

| Groove Shape          | TM modes                                                                 | TE modes                                                                 | Eigenvalues |
|-----------------------|---------------------------------------------------------------------------|---------------------------------------------------------------------------|-------------|
| Triangular isoceles   | \( \psi_{n}^{TM} = \sin(k_{x}x)\sin(k_{y}y) - \sin(k_{y}x)\sin(k_{x}y) \) | \( \psi_{n}^{TE} = \cos(k_{x}x)\cos(k_{y}y) + \cos(k_{y}x)\cos(k_{x}y) \) | \( k_{1} = \sqrt{k_{1}^2 - k_{2}^2} \) |
| Right [13]            | \( \psi_{n}^{TM} = \cos(k_{x}x)\cos(k_{y}y) + \cos(k_{y}x)\cos(k_{x}y) \) | \( \psi_{n}^{TE} = \cos(k_{x}x)\cos(k_{y}y) + \cos(k_{y}x)\cos(k_{x}y) \) | \( k_{2} = \sqrt{k_{1}^2 - k_{2}^2} \) |
| Triangular 30-60 right| \( \psi_{n}^{TM} = \cos(k_{x}x)\cos(k_{y}y) + \sin(k_{x}x)\sin(k_{y}y) \) | \( \psi_{n}^{TE} = \cos(k_{x}x)\cos(k_{y}y) + \cos(k_{y}x)\cos(k_{x}y) \) | \( k_{3} = \sqrt{k_{1}^2 - k_{2}^2} \) |
| Triangular equilateral| \( \psi_{n}^{TM} = \sin(k_{x}x)\sin(k_{y}y) + \sin(k_{y}x)\sin(k_{x}y) \) | \( \psi_{n}^{TE} = \cos(k_{x}x)\cos(k_{y}y) + \cos(k_{y}x)\cos(k_{x}y) \) | \( k_{4} = \sqrt{k_{1}^2 - k_{2}^2} \) |
| Triangular 120 isoceles| \( \psi_{n}^{TM} = \sin(k_{x}x)\sin(k_{y}y) + \sin(k_{y}x)\sin(k_{x}y) \) | \( \psi_{n}^{TE} = \cos(k_{x}x)\cos(k_{y}y) + \cos(k_{y}x)\cos(k_{x}y) \) | \( k_{5} = \sqrt{k_{1}^2 - k_{2}^2} \) |
| Rectangular [3]       | \( \psi_{n}^{TM} = \cos(k_{x}x)\cos(k_{y}y) \)                            | \( \psi_{n}^{TE} = \cos(k_{x}x)\cos(k_{y}y) \)                            | \( k_{6} = \sqrt{k_{1}^2 - k_{2}^2} \) |
| Sectoral (the central angle \( \beta \)) [14] | \( \psi_{n}^{TM} = f_{n}(k_{x}x)\sin(m\varphi + \beta) \) | \( \psi_{n}^{TE} = f_{n}(k_{x}x)\cos(m\varphi + \beta) \) | \( \nu = \frac{2\pi}{\beta}, r = \sqrt{x^2 + y^2} \) |
| Elliptical [15]       | \( \psi_{n}^{TM} = f_{n}(k_{x}x)\sin(m\varphi + \beta) \) | \( \psi_{n}^{TE} = f_{n}(k_{x}x)\cos(m\varphi + \beta) \) | \( \nu = \frac{2\pi}{\beta}, r = \sqrt{x^2 + y^2} \) |

\( L_{mn} \), \( B_{mn} \), \( C_{mn} \), and \( Q_{mn} \) are the elements of the matrix \( L_{mn} \), \( B_{mn} \), \( C_{mn} \), and \( Q_{mn} \), respectively.\( \psi_{n}^{TM} \) and \( \psi_{n}^{TE} \) are the TM and TE modes, respectively.\( k_{n} \) is the \( n \)-th TM mode.\( \eta \) and \( \psi \) are the radial and angular eigenfunctions, respectively. \( k_{1}, k_{2}, k_{3}, \ldots \) are the eigenvalues.

### Figure 3: Geometry of a 2-D isocles triangular groove with \( 90^\circ \) vertex angle fins with \( \alpha = \beta \).

By considering enough accuracy, we can truncate Equations (27) and (28) at \( m = n = M = N \) to obtain a finite system of algebraic equations for the unknown coefficients \( a_{mn}^{TM} \) and \( a_{mn}^{TE} \). The coefficients \( a_{mn}^{TM} \) and \( a_{mn}^{TE} \) can be eliminated, and consequently the four-equation systems are reduced to two-equation systems as shown in the equations below:

\[
B_{mn}^{TM} = 2\sin(k_{n}x)\psi_{n}^{TM}(x, y)\cos(n\varphi),
\]

\[
B_{mn}^{TE} = 2f_{n}(k_{n}x)\psi_{n}^{TE}(x, y)\cos(n\varphi),
\]

\[
C_{mn}^{TM} = \frac{\partial^{2}f_{n}(k_{n}x)\psi_{n}^{TM}(x, y)\cos(n\varphi)}{\partial x^{2}},
\]

\[
C_{mn}^{TE} = \frac{\partial^{2}f_{n}(k_{n}x)\psi_{n}^{TE}(x, y)\cos(n\varphi)}{\partial x^{2}}.
\]

Finally, the unknown coefficients \( a_{mn}^{TM} \) and \( a_{mn}^{TE} \) can be determined as\( [Q] = [C]^{-1}[B] \).

\[ L_{mn} = B_{mn}^{TM} - C_{mn}^{TM}Q_{mn} = C_{mn}^{TM}, \quad K_{mn} = a_{mn}^{TM} = L_{mn}^{-1} \]

Equations (25) and (26) are two sets of infinite systems of algebraic equations for the unknown coefficients \( a_{mn}^{TM} \) and \( a_{mn}^{TE} \). The coefficients \( a_{mn}^{TM} \) and \( a_{mn}^{TE} \) can be eliminated, and consequently the four-equation systems are reduced to two-equation systems as shown in the equations above.
FIGURE 4 (a) Rotating the groove in 45° about the origin and translating the x'y'-coordinate system to xy-coordinate system. (b) An isosceles triangular groove with 90° vertex angle in xy-coordinate system. (c) Simulation of half-space above the groove by a semi-infinite waveguide.

\[ \sigma_{TE, TM}^{2D} = \lim_{\rho \to \infty} 2\pi \rho \frac{|H_z, E_y|^2}{|H_z, E_x|^2}. \quad (33) \]

3 | STUDY OF SPECIAL CASES

In Figure 2, the semi-infinite waveguide can be assumed as two parallel walls on the opening of cavity like those given in [2,11]. In this case, the eigenfunctions in region 2 can be written as

\[ \psi_n^{(TE)}(x,y) = \cos(\alpha_n x) e^{-\gamma x}, \quad (34) \]

\[ \psi_n^{(TM)}(x,y) = \sin(\alpha_n x) e^{-\gamma x}, \quad (35) \]

where \( \alpha_n = \frac{n \pi}{W} \) and \( \beta_n = \sqrt{k^2 - \alpha_n^2} \). A look at the integrals in Equations (21)–(22), (23)–(24), (25)–(26), we deduce that the coefficients \( A_{TM}, B_{TM}, A_{TE}, B_{TE}, C_{TM}, C_{TE} \), and \( C_{TM} \) for this case are the coefficients of the sine and cosine Fourier series. In other words, to calculate the coefficients in Equations (21)–(26), it is necessary to calculate the Fourier series coefficients of the eigenfunctions inside the open cavity and the incident wave. There are some numerical methods that compute Fourier series coefficients rapidly. However, in some grooves with sine and cosine eigenfunctions, these coefficients can analytically be determined. Some eigenfunctions and eigenvalues in well-known cavities with rectangular, triangular, sectorial, and elliptical shapes are shown in Table 1 [13–15].

For example, the above procedure is utilized for a large isosceles triangular groove with a 90° vertex angle shown in Figure 3. The eigenfunctions for this configuration are not mentioned in Table 1. However, they can be extracted from the eigenfunctions of an isosceles triangular groove with a 45° vertex angle shown in Figure 4b.

For TM and TE polarizations, the transverse fields \( E_z^{(TE)}(x', y') \) and \( H_z^{(TM)}(x', y') \) in the groove shown in Figure 4b can be represented by the sum of infinite mode functions as follows [13]:

\[ E_z^{(TE)}(x', y') = \sum_{n=1}^{\infty} b_n \left[ \sin(\alpha_n x') \sin(\beta_n y') - \sin(\beta_n x') \sin(\alpha_n y') \right] \quad (36) \]
FIGURE 6  The amplitude of the tangential electric fields $E_z$ and $E_x$ on the filled grooves of Figure 3 ($y = 0$) as a function of x-position for transverse electric and transverse magnetic modes with $\epsilon_r = 2.5 - j0.2$, $\mu_r = 1.8 - j0.1$ for (a) triangular isosceles groove with $W = 2\lambda$ and vertex angle $\alpha = 120^\circ$, (b) triangular isosceles groove with $W = 2\lambda$ and vertex angle $\alpha = 90^\circ$, (c) semi-circular groove with radius $\frac{W}{2} = \lambda$, and (d) semi-elliptical groove with $a = \frac{W}{2} = \lambda$ and $b = 0.5\lambda$ are semi-major and semi-minor axes, respectively.
In this case, the upper half-space is replaced by a semi-infinite parallel-plate waveguide shown in Figure 4c. We assume that the groove opening is large enough to use the physical optics (PO) approximation. As the groove width increases, this approximation becomes more accurate. By substituting the above tangential fields in the boundary conditions (13)–(16) at y = 0, two independent equations are constructed for each polarization. For TM modes, we have

\[
\frac{1}{j\omega\mu_0} \sum_{n=1}^{\infty} a_n^{TM} [\sin(\alpha_n x) - \frac{j}{\omega\mu_2} \sum_{n=1}^{\infty} b_n^{TM} [\cos(\alpha_n x) \\
\sin(\beta_n (a - x)) + \beta_n \sin(\alpha_n x) \cos(\beta_n (a - x)) \\
- \beta_n \cos(\beta_n x) \sin(\alpha_n (a - x)) \\
- \alpha_n \sin(\beta_n x) \cos(\alpha_n (a - x))] = 2 \frac{\sin \varphi_0}{\eta} e^{j k_1 (x - \frac{a}{2}) \cos \varphi_0}
\]

(41)

and

\[
\sum_{n=1}^{\infty} b_n^{TM} [\sin(\alpha_n x) \sin(\beta_n (a - x))] - \sin(\beta_n x) \sin(\alpha_n (a - x))] = \sum_{n=1}^{\infty} a_n^{TM} \cos(\alpha_n x) 
\]

(42)

Also, for the TE mode,

\[
\sum_{n=0}^{\infty} a_n^{TE} \cos(\alpha_n x) - \sum_{n=0}^{\infty} b_n^{TE} \left[ \cos(\alpha_n x) \cos(\beta_n (a - x)) + \cos(\beta_n x) \cos(\alpha_n (a - x)) \right] = 2 e^{j k_1 (x - \frac{a}{2}) \cos \varphi_0}
\]

(43)

and

\[
H_z^{(2)}(x, y) = \sum_{n=0}^{\infty} b_n^{TE} [\cos(\alpha_n x) \cos(\beta_n y) + \cos(\beta_n y) \cos(\alpha_n y)] 
\]

(40)

and

\[
E_z^{(2)}(x, y) = \sum_{n=1}^{\infty} a_n^{TM}[\sin(\alpha_n x) \sin(\beta_n y) - \sin(\beta_n y) \sin(\alpha_n y)] 
\]

(39)

Substituting \(x\)' and \(y\)' into Equations (36) and (37), the fields \(E_z^{(2)}\) and \(H_z^{(2)}\) can be represented in \(xy\)-coordinate system as

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
\cos(45^\circ) & \sin(45^\circ) \\
-\sin(45^\circ) & \cos(45^\circ)
\end{bmatrix} \begin{bmatrix}
x - a/2 \\
y + a/2
\end{bmatrix}. \quad (38)
\]

and

\[
H_z^{(2)}(x', y') = \sum_{n=0}^{\infty} b_n^{TE} [\cos(\alpha_n x') \cos(\beta_n y') + \cos(\beta_n y') \cos(\alpha_n y')] 
\]

(37)

where \(a_n = a_n^2 + \beta_n^2\). Equations (36) and (37) are valid for all values \(a_n\) and \(\beta_n\). Here, the eigenvalues of this problem are selected similar to an isosceles right triangle waveguide, that is, \(a_n^2 = \sqrt{2n \pi / a}\) and \(\beta_n^2 = k_0^2 - a_n^2\). No limitation is placed on the value of \(n\), but it must be integers. The Groove depicted in Figure 4b is a 45° counter-clockwise rotation of Figure 4b around the origin. Now, if the \(xy\)-coordinate system in Figure 4a is translated a distance \(a/2\) to the left and a distance \(a/2\) upward, we have the \(xy\)-coordinate system. Therefore, we use the following transfer matrix to represent Equations (36) and (37) in the \(xy\)-coordinate system:

\[
\begin{bmatrix}
x' \\
y'
\end{bmatrix} = \begin{bmatrix}
\cos(45^\circ) & \sin(45^\circ) \\
-\sin(45^\circ) & \cos(45^\circ)
\end{bmatrix} \begin{bmatrix}
x - a/2 \\
y + a/2
\end{bmatrix}. \quad (38)
\]
Figure 8  Bistatic echo-width patterns at two different incidence angles $\varphi_0 = 90^\circ$ and $\varphi_0 = 60^\circ$ for the various groove shapes and widths ($W = 2\lambda$ and $W = 4\lambda$): (a) transverse electric mode, (b) transverse magnetic mode
| Shape                  | $\varphi_0 = 90^0$                  | $\varphi_0 = 60^0$                  |
|------------------------|------------------------------------|------------------------------------|
| Triangular (120°)      | ![Triangular 90°](image1)          | ![Triangular 60°](image2)          |
| (90°) $W = 2a$         | ![Triangular 90°](image3)          | ![Triangular 60°](image4)          |
| Circular               | ![Circular 90°](image5)            | ![Circular 60°](image6)            |
| Elliptical             | ![Elliptical 90°](image7)          | ![Elliptical 60°](image8)          |
| Triangular (120°)      | ![Triangular 90°](image9)          | ![Triangular 60°](image10)         |
| (90°) $W = 4a$         | ![Triangular 90°](image11)         | ![Triangular 60°](image12)         |
| Elliptical             | ![Elliptical 90°](image13)         | ![Elliptical 60°](image14)         |

(b) TM mode

FIGURE 8 (Continued)
\[-1 \sum_{n=0}^{\infty} \frac{a_n^{TE}}{j \omega_1} r_n \cos(\alpha_n x) \sin(\phi n) \sin(\alpha_n x)]\]
\[
\cos(\beta_n(a-x)) = \beta_n \cos(\alpha_n x) \sin(\beta_n(a-x)) \sin(\alpha_n x) \sin(\beta_n x) \sin(\alpha_n x) (a-x))
\]

(44)

In order to determine coefficients \(A_{mn}^{TE}, B_{mn}^{TE}, C_{n}^{TM}, A_{mn}^{TM}, B_{mn}^{TM}, \) and \(C_{n}^{TM}, \) we take \(\psi_n^{(1)TM} = \sin(\alpha_n x)\) and \(\psi_n^{(1)TM} = \cos(\alpha_n x)\). Also, the eigenfunctions in the region 2 are given as

\[
\psi_n^{(2)TM} = \sin(\alpha_n x) \sin(\beta_n(a-x)) - \sin(\beta_n x) \sin(\alpha_n(a-x))
\]

(45)

\[
\psi_n^{(TE)} = \cos(\alpha_n x) \cos(\beta_n(a-x)) + \cos(\beta_n x) \cos(\alpha_n(a-x))
\]

(46)

Therefore, the integrals (21)–(26) consist of multiplying three triangular functions \(\cos(\cdot)\) and \(\sin(\cdot)\). They can be solved analytically by some formulae given in [16].

4 | RESULTS

For different cavity shapes, several examples are presented to demonstrate the validity and accuracy of the expressions presented in Section 3. Here, the tangential electric fields \(E_x\) and \(E_z\) on the cavities \((y = 0)\) are compared with the results obtained by MoM used in FEKO software. To simulate 2-D cavities in FEKO, we used an infinite trough with 1-D periodic boundary condition. We considered four different filled grooves (isosceles triangular grooves with a 90° and 120° vertex angles, semi-circular and semi-elliptical) as shown in Figure 5. In this section, the results were obtained with the assumption that half-space above groove has been substituted with a semi-waveguide with two parallel walls.

The first grooves to be investigated are two filled isosceles grooves shown in Figure 5a,b with \(W = 2\lambda, \, c_1 = c_0, \mu_1 = \mu_0, \) and vertex angle \(\alpha = 120^\circ\) and \(90^\circ\). The specification of the filling dielectric in all examples is \(\epsilon_0 = 2.5 - j0.2, \mu_0 = 1.8 - j0.1.\) For the isosceles groove with the vertex angle 120°, we used the wave functions given in Table 1 to handle expressions (13)–(16). The coefficients \(A_{mn}^{TE}, B_{mn}^{TE}, A_{mn}^{TM}, B_{mn}^{TM}, C_{n}^{TM}, \) and \(C_{n}^{TM}\) for this groove can be determined analytically by using practical formulae in [16]. Also, for isosceles groove with the vertex angle 90°, we derived expressions (41)–(44).

Moreover, we applied our procedure to a semi-circular groove shown in Figure 5c with radius \(\frac{W}{2} = \lambda\) and used the cylindrical wave functions given in Table 1 for a sectorial groove when \(\beta = 180^\circ.\) In this case, to compute the coefficients (21)–(26), we should compute several high oscillatory integrals, numerically. These integrals consist of the trigonometric and Bessel functions with large values in the argument. Thus we need to use efficient numerical methods such as the Filon method or other methods given by [17,18] to calculate these integrals accurately and rapidly. Finally, to compute the tangential electric fields on a semi-elliptical groove, we used radial and angular Mathieu eigenfunctions defined in Table 1 [15]. Radial and angular Mathieu functions can be expanded as a Bessel and a trigonometric series. In this example, the coefficients (21)–(26) can be computed by [16,19,20].

Figure 6 represents the amplitude of electric fields \(E_x\) and \(E_z\) on four different grooves \((y = 0)\) as a function of \(x\)-position for both modes. As shown in Figure 6, the results are in good agreement with full numerical methods MoM. By examining the results, we found that the proper value of \(M = N\) for truncating the infinite series in Equations (13)(14)(15)(16) are the number of propagating modes in the cavity. When the size of the groove increases, we also should increase \(M.\) However, the results show that this method is just valid for the groove with the aperture width greater than the wavelength. Instead for smaller ones, we can use traditional meshing methods such as MoM and FEM or other methods developed for scattering from different cavities [7,9,14,15]. In another study, we use this method to investigate the effects of different cavity shapes and their size on TE and TM bistatic echo-width patterns. For this purpose, we first compare the bistatic echo width of a triangular isosceles groove with \(W = 2\lambda\) and the vertex angle \(\alpha = 120^\circ\) with the results obtained by MoM (Figure 7). This figure exhibits that there is a good agreement between the result of this method and MoM. Then, we plotted the bistatic echo-width patterns in the polar coordinate for different grooves when \(W = 4\lambda, 2\lambda,\) and \(\varphi_0 = 60^\circ, 90^\circ\) (Figure 8). As seen in Figure 8, for all types of groove shapes, when the groove width increases, the number of side lobes increases, and the breadths of the main lobes decrease. The pattern lobes of the triangular grooves are wider than others. In other words, electromagnetic fields incident on a triangular groove are scattered in most directions. Unlike a circular groove, the scattered fields by a triangular groove for the observation angles close to the ground surface are relatively strong. However, as seen in Figure 8, the blind zones in circular grooves are broader than others. For all grooves, at the incidence angle \(\varphi_0 = 60^\circ,\) we see that the backscattered fields are weaker than the fields in the reflection direction.

To demonstrate the time efficiency of the proposed method, we measured CPU time for different groove shapes with \(W = 4\lambda\) and compare them with the simulation time of FEKO software (Table 2). An inspection of the results in

| Table 2 | The CPU time measured for different groove shapes \((W = 4\lambda)\) |
|---------|-----------------|-------------------|
|         | FEKO            | This method       |
| Triangular | 49 min          | 300 ms           |
| Circular  | 45 min          | 35.23 s          |
| Elliptical | 42 min          | 45.01 s          |


Table 2 demonstrates that this procedure for a large groove is efficient. Thus, it can be a proper procedure to analyse the electromagnetic scattered waves from large grooves.

5 | CONCLUSION

A generalized solution for scattering by large cavities in a PEC with specified wave functions was suggested. To decrease simulation time and simplify expressions, we replaced half-space above the cavity by an arbitrary waveguide with known wave functions. To match modes inside and outside of the cavity, the wave functions of the upper waveguide was selected as the basis functions of the expansion. An infinite system of linear equations was constructed for unknown expansion coefficients. If we use a semi-waveguide with two parallels walls, the expansion coefficients become the Fourier series coefficients. The proposed method was applied to different groove shapes. The results were verified with the results obtained by MoM used in electromagnetic simulators FEKO. For both modes, the effects of groove shape and size on bistatic echo-width patterns were investigated. We also measured the CPU time of our method and MoM to compare the computational efficiency of two different techniques. This efficient method has just been developed for wide grooves.

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REFERENCES
1. Ling, H., Lee, S., Chou, R.: High-frequency RCS of open cavities with rectangular and circular cross sections. IEEE Trans. Antenn. Propag., 37(5), 648–654 (1989)
2. Morgan, M.A.: Mode expansion solution for scattering by a material filled rectangular groove. Progr. Electromagn. Res., PIER 18 (1998)
3. Xiang, Z., Chia, T.-T.: A hybrid BEM/WTM approach for analysis of the EM scattering from large open-ended cavities. IEEE Trans. Antenn. Propag., 49(2), 165–173 (2001)
4. Bao, G., Sun, W.: A fast algorithm for the electromagnetic scattering from a large cavity. SIAM J. Sci. Comput., 27(2), 553–574 (2005)
5. Bao, G., Zhang, W.: An improved mode-matching method for large cavities. IEEE Antenn. Wirel. Propag. Lett., 4, 393–396 (2005)
6. Zhao, M., Qiao, Z., Tang, T.: A fast high order method for electromagnetic scattering by large open cavities. J. Comput. Math., 29, 287–304 (2011)
7. Basha, M.A., Chaudhuri, S., Safavi-Naeini, S.: Generalised formulation for electromagnetic scattering from finite arbitrarily shaped grooves in a perfect conducting plane. IET Microw. Antenn. Propag., 516(12), 1455–1462 (2011)
8. Ho, C.-Y., Chen, B.-C., Tsai, Y.-H.: Scattering signals of monochromatic light incident on a rectangular microchannel. Comput. Math. Appl., 64(5), 1514–1521 (2012)
9. Li, H., Ma, H., Sun, W.: Legendre spectral Galerkin method for electromagnetic scattering from large cavities. SIAM J. Numer. Anal., 51(1), 353–376 (2013)
10. Ambikasaran, S., Lai, J., Greengard, L.: A fast direct solver for high frequency scattering from a large cavity in two dimensions. SIAM J. Sci. Comput., 36(6), 887–903 (2015)
11. Bozorgi, M.: A mode-matching solution for scattering from a large isosceles right triangle groove in PEC plane. Adv. Electromagn. 8(5), 37–43 (2019)
12. Cao, Z., et al.: Geometrical optics approximation for plane-wave scattering by a rectangular groove on a surface. Appl. Opt., 59, 2600–2605 (2020)
13. Overfelt, L., White, D.J.: TE and TM modes of some triangular cross-section waveguides using superposition of plane waves. IEEE Trans. Microw. Theory Tech., 34(1), 161–167 (1986)
14. Taaur, D.-H., Chang, K.-H.: Electromagnetic plane-wave scattering from a sectorial groove in a perfectly conducting plane. Open Phys., 7(1), 160–167 (2009)
15. Uslenghi, P.L.E.: Exact penetration, radiation, and scattering for a slotted semielliptical channel filled with isorefractive material. IEEE Trans. Antenn. Propag., 52(6), 1473–1480 (2004)
16. Abramowitz, M., Stegun, I.A.: Handbook of mathematical functions: With formulas, graphs, and mathematical tables. Wiley, New Jersey (1972)
17. Xiang, S., Gui, W.: On generalized quadrature rules for fast oscillatory integrals. Appl. Math. Comput., 197, 60–75 (2008)
18. Milovanovic, G.V., Rassias, M.Th.: Number Theory, Analytic, Theory, Approximation, & Special Functions, pp. 613–649. Springer, New York (2014)
19. Lowan, A., et al.: Tables relating to mathieu functions: Characteristic values, coefficients, and joining factors. Columbia University Press, New York (1951)
20. Wolf, G.: On the asymptotic behavior of the Fourier coefficients of Mathieu functions. Journal of research of the National Institute of Standards and Technology. 113(1), 11–15 (2008)

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