Stochastic quantization of an interacting classical particle system

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Abstract. Starting from a many-body classical system governed by a trace-form entropy we derive, in the stochastic quantization picture, a family of nonlinear and non-Hermitian Schrödinger equations describing, in the mean field approximation, a quantum system of interacting particles. The time evolution of the main physical observables is analysed by means of the Ehrenfest equations, showing that, in general, this family of equations takes into account dissipative and damped effects due to the interaction of the system with the background. We explore the presence of steady states by means of solitons, describing conservative solutions. The results are specialized to the case of a system governed by the Boltzmann–Gibbs entropy.

Keywords: dissipative systems (theory), stochastic particle dynamics (theory), nonlinear dynamics, stationary states (theory)
1. Introduction

The problem of the quantization of a many-body system attracted the attention of the physics community immediately in the first years after the introduction of quantum mechanics [1]. Different methods have been introduced to study this problem. Among the many we recall the Hartree–Fock method, the Thomas–Fermi approach, the Bohm–Madelung quantization, the second quantization of Heisenberg, the stochastic quantization of Nelson and others [2,3].

As regards a non-relativistic system, non-linear Schrödinger equations (NSEs) are generally employed to take into account, through the non-linearity, the interactions among the many particles. As a paradigm, the NSE with the cubic non-linearity [4,5] has been widely used in the literature to study, for instance, the Bose–Einstein condensation of alkali atoms like $^7\text{Li}$, $^{23}\text{Na}$ and $^{87}\text{Rb}$ [6].

In [7], Bialynicki-Birula and Mycielski (BBM) introduced the non-linearity $-b \ln(|\psi|^2)\psi$ which was selected from the assumption of a separability condition among composed systems. Subsequently, in order to save only partially the superposition principle, valid in the linear theory, Weinberg [8] suggested a very general class of NSEs with a homogeneous non-linearity.

Motivated by these works, some experimental tests were carried out based on neutron interferometry [9,10] and $^{9}\text{Be}^+$ transition frequency spectroscopy [11]. All of them fail to point out the existence of non-linear effects, suggesting, in this way, that there are no real grounds, at a fundamental level, for such non-linearities in the Schrödinger equation.

Notwithstanding, NSEs can be usefully applied in the description of extended objects. In fact, depending on the form of the non-linearity, some NSEs admit soliton solutions, i.e. non-dispersive wavepackets identifiable with quasi-classical particles.

For instance, in the BBM equation a soliton solution named the *gausson* has been studied [7,12]. This solution (see [13] and references therein) represents the experimentally measurable charge or mass density $\rho = |\psi|^2$ of an atomic nucleus, when it is considered as an elementary object (neglecting, in this way, its internal quark structure).
Remarkably, the BBM equation is closely connected with the Boltzmann–Gibbs (BG) entropy. In [13], it has been shown that the binding energy released in the splitting of an arbitrary wavefunction into non-overlapping parts of the same form $\psi(x) \rightarrow \sum_{i} p_{i}^{1/2} \psi(x - x_{i})$, with $p_{i}$ normalized according to $\sum_{i} p_{i} = 1$, is given by $\Delta E = b S^{BG}(p)$, where $S^{BG}(p) = -\sum_{i} p_{i} \ln p_{i}$. This suggests that the BBM equation can be used in the thermodynamical description of spatially extended or self-interacting quantum mechanical objects.

Also the BG entropy finds several applications in the study of quantum systems. For instance, in [14]–[16] this entropy has been applied to generalize the Heisenberg uncertainty relation for a pair of conjugate observables. In [17] a worthwhile discussion concerning the BG entropy and the Fisher information measure has been advanced. The author clarifies the role of these entropies in the Smoluchowski diffusion process which describes the underlying quantum kinetics of the linear Schrödinger equation.

Moreover, in recent years, an intensive research activity has been focused on the study of complex quantum systems which are intimately related to generalized versions of the BG entropy [18]–[20], although the link between the microscopic properties of these systems and their statistical mechanics features is still an open question. In this respect, in [21] a microscopic derivation of a statistical system described by a general entropy has been considered. A microcanonical approach for a quantum system, at equilibrium, is studied in [22, 23] (and references therein) whereas, more recently in [24], the author deals with the canonical quantization of a classical system governed by an arbitrary entropy whose kinetics is described by a very general non-linear Fokker–Planck equation (NFPE). He obtained a wide family of NSEs where the structure of the complex non-linearity is related to the form of the entropy of the ancestor classical system. The method has the drawback that, in consequence of the canonical quantization, this family of NSEs describes the time evolution of isolated quantum systems conserving energy and momentum.

The purpose of the present paper is to accomplish the quantization of a many-body system of interacting particles in thermal contact with a reservoir (the background). This can be performed by using a generalized version of the stochastic quantization method initially proposed by Nelson [25]–[27]. In this way, we derive a family of NSEs with a complex non-linearity, describing damped and dissipative processes. We remark that complex non-linearities arise whenever the quantum system obeys a diffusive continuity equation. This, for example, occurs for the family of Schrödinger equations derived by Doebner and Goldin [28] as the most general class compatible with the linear Fokker–Planck equation, or, in [29], where a NSE has been derived from a generalized Pauli exclusion–inclusion principle accounted for through a suitable modification (in a non-linear manner) of the particle quantum current. Also in [30], a class of NSEs with a complex non-linearity has been obtained in the stochastic quantization framework, starting from the most general kinetics containing a non-linear drift term and compatible with a linear diffusion term. Quite generally, all these NSEs can be transformed into other ones, with a purely real non-linearity, by means of a gauge transformation of third kind [28, 31, 32].

Although sometimes criticized [33], the Nelson method is, at present, an argument of intensive research activity [34]–[37]. It merges together both the Newtonian mechanics and the diffusion process with the prospect of understanding the quantum mechanics in a purely classical language. Typically, one assumes a quantum system undergoing an irreducible Brownian motion in the configuration space. Notwithstanding, the stochastic
mechanics is not merely a mathematical description of a diffusion process. Two main aspects make the difference from the usual treatment of a standard diffusive process: the prescription associated with the reverse of the time axis in the diffusion process and the introduction of a stochastic acceleration, whose definition has been generalized in several papers [38,39].

In the following, we generalize the original Nelson method according to the following two steps:

(i) by assuming a non-linear kinetics underlying the stochastic diffusion motion, so that one can replace the forward and backward linear Fokker–Planck equations with a couple of non-linear kinetic equations, in a manner compatible with the original NFPE describing the kinetics of a classical system;

(ii) by generalizing the definition of the mean acceleration in the most general quadratic form, compatible with a local expression of an Euler equation.

In the next section 2, we recall the relationship between an arbitrary entropy and the corresponding NFPE describing the kinetics of a classical system. In section 3, we start from this NFPE and derive the family of NSEs describing a quantum many-body system. In section 4, we investigate the time evolution of the main observables (Ehrenfest relations) whilst, in section 5, we inquire into the existence of a special class of solutions, the solitons, which describe an isentropic and non-dissipative process. Finally, in section 6, as an example, we discuss the quantization of a classical system described by the BG entropy. The conclusions are reported in the final section 7.

2. Non-linear kinetic equation

In the literature, there have been presented many different approaches for relating, in a classical system, the production of entropy to a NFPE [40].

In the following, we consider a many-body classical system described with the field $\rho(x,t)$, labelled by the vector $x = (x_1, x_2, x_3)$, and normalized in

$$\int \rho(x,t) \, dx = 1.$$  \hspace{1cm} (2.1)

Quite generally, the time evolution of this field can be obtained, in the configuration space, starting from the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0,$$  \hspace{1cm} (2.2)

and assuming for the current the expression

$$J = -\rho F(\rho).$$  \hspace{1cm} (2.3)

The thermodynamic force $F(\rho)$, defined through the relation

$$F(\rho) = \nabla \left( \frac{\delta \mathcal{L}}{\delta \rho} \right),$$  \hspace{1cm} (2.4)

is related to the functional

$$\mathcal{L}(\rho) = U - D S(\rho),$$  \hspace{1cm} (2.5)
where
\[U(t) = \int E(x) \rho(x, t) \, dx,\] (2.6)
is the mean energy of the system, \(E(x)\) is the total energy for particle and \(D\) is the constant diffusion coefficient.

Let us assume for the entropy \(S(\rho)\) the very general trace-form expression
\[S(\rho) = -\int dx \int d\rho \ln \kappa(\rho),\] (2.7)
(throughout this paper we use units with the Boltzmann constant \(k_B = 1\)).

In equation (2.7), the quantity \(\kappa(\rho)\) is a smooth function of the density field \(\rho(x, t)\). Depending on the expression of \(\kappa(\rho)\), equation (2.7) encompasses different entropies already introduced in the literature [41].

From (2.2)–(2.7) we obtain the following continuity equation:
\[
\frac{\partial \rho}{\partial t} + \nabla \cdot \left[ \rho \mathbf{v}_{\text{drift}} - D f_0(\rho) \nabla \rho \right] = 0,\] (2.8)
where we have introduced the drift velocity
\[\mathbf{v}_{\text{drift}} = -\nabla E(x),\] (2.9)
whilst the function \(f_0(\rho)\) is given by
\[f_0(\rho) = \rho \frac{\partial}{\partial \rho} \ln \kappa(\rho).\] (2.10)

Equation (2.8) is a NFPE in the Smoluchowski picture since it describes the kinetics process in the configuration space rather than in the momentum space.

We observe that in equation (2.8) the total current \(\mathbf{J} = \mathbf{J}_{\text{drift}} + \mathbf{J}_{\text{diff}}\) is the sum of the standard linear drift current \(\mathbf{J}_{\text{drift}} = \rho \mathbf{v}_{\text{drift}}\) and the non-linear diffusion current \(\mathbf{J}_{\text{diff}} = -D f_0(\rho) \nabla \rho\) which reduces to the standard Fick form \(\mathbf{J}_{\text{Fick}} = -D \nabla \rho\) when \(\kappa(\rho) = \rho\).

Since \(D\) is constant, equation (2.8) describes an isothermal diffusive process where the system is in thermal contact with a reservoir. In fact, it is known that for an isothermal process the diffusion coefficient plays the role of the inverse of a kind of temperature [40]. In this case, we easily recognize that equation (2.5) defines a free energy for the system we are looking at. Actually, we can easily verify that the quantity \(\mathcal{L}(\rho)\) is a Lyapunov function for the given problem since, according to equation (2.8), we have
\[\frac{d\mathcal{L}}{dt} = -\int \rho \left[ \nabla E(x) - D \nabla \left( \frac{\delta S}{\delta \rho} \right) \right]^2 \, dx \leq 0,\] (2.11)
where equality holds at equilibrium.

It is worth recalling that recently, different NFPEs have been related to generalized entropies of the kind (2.7) by using several methods substantially equivalent to the one described above. In [42], generalized Kramers and Smoluchowski equations were derived by means of a variational principle, which minimizes the dissipative ratio of a generalized free energy equivalent to equation (2.5). The derivation of a Smoluchowski equation starting from a Kramers equation has been rigorously derived in [43] by mean
of the Chapman–Enskog expansion in the high friction limit. Finally, in [41], a kinetic interaction principle has been proposed for deriving a family of NFPEs starting from a classical Markovian process describing the kinetics of a system of $N$-body particles. The kinetic interaction principle is realized through a suitable factorization of the transition probability $\pi(t, x \rightarrow y) = r(t, x, y) a(\rho) b(\rho') c(\rho, \rho')$ in terms of the population of the initial site $x$ and the final site $y$, with $\rho = \rho(t, x)$ and $\rho' = \rho(t, y)$, where the functions $a(\rho)$, $b(\rho')$ and $c(\rho, \rho')$ fix the non-linearities in the NFPE and impose the form of the entropy associated with the system.

3. Stochastic quantization of the many-body system

In the stochastic quantization framework, particles are assumed to be undergoing a time-asymmetric Markovian random process. The space coordinate $x$ satisfies a couple of stochastic differential equations
\[ \mathrm{d}x(t) = v^{(\pm)}(x, t) \, \mathrm{d}t + \mathrm{d}W^{(\pm)}(t), \quad (3.1) \]
describing forward ($+$) and backward ($-$) processes, respectively. They can be seen as a system of Langevin equations in the configuration space, where $v^{(\pm)}(x, t)$ are the forward and backward velocities, two vector valued functions depending on space and time, whilst the functions $W^{(\pm)}(t)$ describe a Wiener process.

According for equation (3.1), $x(t)$ is not a differentiable function. This means that there is no velocity as a standard time derivative. Thus, one introduces a forward and a backward time derivative given by
\[ \left( \frac{\mathrm{d}}{\mathrm{d}t} \right)^{(\pm)} = \frac{\partial}{\partial t} + v^{(\pm)} \cdot \nabla \pm D \Delta, \quad (3.2) \]
so that
\[ v^{(\pm)}(x, t) = \left( \frac{\mathrm{d}}{\mathrm{d}t} \right)^{(\pm)} x(t). \quad (3.3) \]
Let us identify the field $\rho(x, t)$ with the probability density of a quantum system at the position $x(t)$. We assume that it fulfils the following forward and backward NFPEs:
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot [v^{(\pm)} \rho \mp D f^{(\pm)}(\rho) \nabla \rho] = 0, \quad (3.4) \]
where, without lost of generality, we impose
\[ f^{(\pm)}(\rho) = 2 \rho \frac{\partial}{\partial \rho} \ln \kappa^{(\pm)}(\rho), \quad (3.5) \]
with $\kappa^{(\pm)}(\rho)$ two arbitrary functions. Clearly, the form of equations (3.4) is reminiscent of the expression (2.8) which describes the kinetic of a classical system.

From equations (3.4), we see that forward and backward velocities are related to each other by
\[ v^{(+)} = v^{(-)} + 2 D \nabla g(\rho), \quad (3.6) \]
where
\[ g(\rho) = \ln \left( \kappa^{(+)}(\rho) \kappa^{(-)}(\rho) \right). \] (3.7)

As is customary, we introduce the mean velocity and the osmotic velocity according to the relations
\[ \mathbf{v} = \frac{1}{2} \left( \mathbf{v}^{(+)} + \mathbf{v}^{(-)} \right), \quad \mathbf{u} = \frac{1}{2} \left( \mathbf{v}^{(+)} - \mathbf{v}^{(-)} \right). \] (3.8)
The former quantity represents the global velocity of the density shape of the quantum fluid, whilst the latter, which has the form
\[ \mathbf{u} = D \nabla g(\rho), \] (3.9)
has an intrinsic stochastic origin and is related to the spatial variation of the density.

Endowed with the definitions (3.8), by averaging equations (3.4), we obtain
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot \left[ \rho \mathbf{v} - D f_0(\rho) \nabla \rho \right] = 0, \] (3.10)
a non-linear continuity equation for the field \( \rho(x, t) \), where
\[ f_0(\rho) = \frac{1}{2} \left[ f^{(+)}(\rho) - f^{(-)}(\rho) \right] = \rho \frac{\partial}{\partial \rho} \ln \kappa(\rho), \] (3.11)
and
\[ \kappa(\rho) = \frac{\kappa^{(+)}(\rho)}{\kappa^{(-)}(\rho)}. \] (3.12)
Equation (3.10) is formally equivalent to the NFPE (2.8). According to this correspondence, the expression for the function \( \kappa(\rho) \) is fixed through the entropy of the ancestor classical system although the same entropy does not determine unequivocally the functions \( \kappa^{(\pm)}(\rho) \). From a mathematical point of view, this means that there are an infinite number of possible choices for the functions \( \kappa^{(\pm)}(\rho) \), all compatible with the form of the entropy (2.7), according to equation (3.12). Note that equation (3.10) is consistent with the kinetic interaction principle proposed in [41], where a very general NFPE, whose equation (3.10) is a particular case with the linear drift, has been derived starting from a master Pauli equation describing a Markovian process.

Let us now introduce the stochastic acceleration. In the original Nelson work it is defined by
\[ \mathbf{a} = \frac{1}{2} \left[ \left( \frac{d}{dt} \right)^{(+)} \mathbf{v}^{(-)} + \left( \frac{d}{dt} \right)^{(-)} \mathbf{v}^{(+)} \right], \] (3.13)
i.e. by means of the algebraic mean of the quadratic operators \( (d/dt)^{(+)}(d/dt)^{(-)} \) and \( (d/dt)^{(-)}(d/dt)^{(+)} \) acting on the vector \( \mathbf{x}(t) \). The non-uniqueness of the definition of \( \mathbf{a} \) has been noted in [38] and alternative definitions have been proposed in [38, 39]. Hereinafter, we assume the most general quadratic form, in the forward and backward time derivatives, compatible with a local expression for an Euler equation describing the dynamics of \( \mathbf{v} \).
Thus, we impose
\[
\mathbf{a} = \tau \left( \frac{d}{dt} \right)^{(+)\mathbf{v}} + \mu \left[ \left( \frac{d}{dt} \right)^{(+)\mathbf{v}} + \left( \frac{d}{dt} \right)^{(-)\mathbf{v}} \right] + \nu \left( \frac{d}{dt} \right)^{(-)\mathbf{v}},
\]
(3.14)
with the condition \( \tau + 2\mu + \nu = 1 \).

We observe that the original Nelson definition is recovered from equation (3.14) by imposing \( \tau = \nu = 0 \) and \( \mu = 1/2 \) whilst the definition advanced in [38] is obtained by imposing \( \tau = \nu = 1/8 \) and \( \mu = 3/8 \). Moreover, in the framework of a stochastic Lagrangian formulation [39], compatible with the variational stochastic calculus introduced in [44], another definition of \( \mathbf{a} \) has been advanced. It follows from equation (3.14) by imposing \( \tau = \nu = 1/2 \) and \( \mu = 0 \).

As is customary, let us introduce the potential field \( \Sigma(x,t) \) related to the mean velocity through the relation
\[
m \mathbf{v} = \nabla \Sigma.
\]
(3.15)
Remarkably, this equation is formally equivalent to the relation (2.9) between the classical drift velocity and the particle energy spectrum although the fields \( \Sigma(x,t) \) and \( E(x) \) represent two different physical quantities. In this sense, the quantum mean velocity \( \mathbf{v} \) assumes a different meaning, in the quantum picture, with respect to the meaning that the classical drift velocity \( \nu_{\text{drift}} \) has in the classical picture.

Taking into account that both fields \( \mathbf{v} \) and \( \mathbf{u} \) are given through gradients, of the scalar functions \( g(\rho) \) and \( \Sigma(x,t) \), respectively, and recalling Newton’s law, from the definition (3.14) one obtains
\[
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + a_1 \frac{\partial \mathbf{u}}{\partial t} + a_1 \nabla (\mathbf{u} \cdot \mathbf{v}) + D a_1 \Delta \mathbf{v} + a_2 \mathbf{u} \cdot \nabla \mathbf{u} + D a_2 \Delta \mathbf{u} + \frac{1}{m} \nabla V = 0,
\]
(3.16)
where \( a_1 = \tau - \nu \) and \( a_2 = \tau - 2\mu + \nu \).

In effect, (3.16) is an Euler equation for the mean velocity field \( \mathbf{v}(x,t) \). We remark that, since the osmotic velocity is a function of the field \( \rho(x,t) \), the quantity \( \partial \mathbf{u} / \partial t \) can be easily handled by using the continuity equation (3.10).

On employing the definitions (3.9) and (3.15), the Euler equation takes the form
\[
\frac{\partial \Sigma}{\partial t} + \frac{(\nabla \Sigma)^2}{2m} + f_1(\rho) \Delta \Sigma + f_2(\rho) (\nabla \rho)^2 + f_3(\rho) \Delta \rho + V = 0,
\]
(3.17)
where
\[
\begin{align*}
f_1(\rho) &= a_1 D \left( 1 - \rho \frac{\partial g(\rho)}{\partial \rho} \right), \\
f_2(\rho) &= m D^2 \left[ a_1 \frac{\partial f_0(\rho)}{\partial \rho} \frac{\partial g(\rho)}{\partial \rho} + a_2 \left( \frac{\partial g(\rho)}{\partial \rho} \right)^2 + a_2 \frac{\partial^2 g(\rho)}{\partial \rho^2} \right], \\
f_3(\rho) &= m D^2 (a_1 f_0(\rho) + a_2) \frac{\partial g(\rho)}{\partial \rho}.
\end{align*}
\]
(3.18)
As is known, in the hydrodynamic representation of a quantum system [45, 46], the quantum effects are taken into account through the introduction of the quantum potential.
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\[ U_q(\rho) = -(\hbar^2 / 2m) \Delta \sqrt{\rho} / \sqrt{\rho}. \]

In the original Nelson formulation, the potential \( U_q(\rho) \) arises from the particular form assumed by the osmotic velocity that, in that case, is given by

\[ u = D \ln \rho. \tag{3.19} \]

In the non-linear case studied in this paper, the same form \( (3.19) \) can be recovered by imposing

\[ \kappa^+(\rho) = \sqrt{\rho \kappa(\rho)}, \quad \kappa^-(\rho) = \sqrt{\rho / \kappa(\rho)}, \tag{3.20} \]

although, according to the non-uniqueness of \( \kappa^{(\pm)}(\rho) \), the choice \( (3.20) \) is not unique.

In this way, equation \( (3.17) \) simplifies to

\[ \frac{\partial \Sigma}{\partial t} + \frac{(\nabla \Sigma)^2}{2m} + U_q(\rho) + W(\rho) + V = 0, \tag{3.21} \]

where

\[ W(\rho) = mD^2 \frac{a_1}{\rho} \nabla \cdot \left( f_0(\rho) \nabla \rho \right), \tag{3.22} \]

and \( a_2 = -(\hbar/2mD)^2 \).

Equations \((3.10)\) and \((3.21)\) govern the time evolution of the fields \( \rho(x,t) \) and \( \Sigma(x,t) \).

Actually, \( (3.10) \) is a quantum continuity equation for the field \( \rho(x,t) \) describing the kinetics of the quantum fluid \( \text{à la} \) Bohm and Madelung, where the quantum current, according to the position \( (3.15) \), assumes the form

\[ j = \rho \frac{\nabla \Sigma}{m} - D f_0(\rho) \nabla \rho. \tag{3.23} \]

In contrast, \( (3.21) \) can be interpreted as a non-linear \textit{Hamilton–Jacobi} equation for the potential field \( \Sigma(x,t) \) whose dynamics is ruled through the quantum potential, the extra non-linear potential \( W(\rho) \) and the external potential \( V \).

Finally, with the position

\[ \psi(x,t) = \sqrt{\rho(x,t)} \exp \left( \frac{i}{\hbar} \Sigma(x,t) \right), \tag{3.24} \]

equations \((3.10)\) and \((3.21)\) can be joined to obtain the following NSE:

\[ i \hbar \frac{\partial \psi}{\partial t} = H \psi, \tag{3.25} \]

where \( H \) is the non-Hermitian operator

\[ H = -\frac{\hbar^2}{2m} \Delta + \Lambda(\rho) + V, \tag{3.26} \]

with a non-linearity \( \Lambda(\rho) \) depending only on the field \( \rho(x,t) \). It is given by

\[ \Lambda(\rho) = \frac{\lambda}{\rho} \nabla \cdot \left( f_0(\rho) \nabla \rho \right), \tag{3.27} \]
where the complex coupling constant takes the form
\[
\lambda = m D^2 a_1 + i \frac{\hbar}{2} D. \tag{3.28}
\]

The family of NSEs (3.25) describes, in the mean field approximation, the dynamics of a quantum system of identical particles interacting with a thermal reservoir. The non-linearity \( \Lambda(\rho) \) is fixed through the entropy (2.7) which governs the kinetics of the ancestor classical system and consequently determines also the continuity equation of the quantum system.

In conclusion, let us detail a few considerations:

- First, the original Nelson derivation is recovered by imposing \( \tau = \nu = 0 \) and \( \mu = 1/2 \). This implies \( a_2 = -1 \) and we recover the well known relation \( D = \hbar/2m \) between the diffusion coefficient and the mass of the particles of the system [25]. Moreover, according to the original proposal, we must impose \( \kappa^+(\rho) = \kappa^-(\rho) = \sqrt{\rho} \). As a consequence \( f_0(\rho) \) vanishes and equation (3.25) reduces to the ordinary linear Schrödinger equation.

- Second, when \( a_1 \) vanishes, equation (3.25) describes a family of NSEs with a purely imaginary non-linearity. In this case only the continuity equation acquires an extra non-linear term (the diffusive one) whilst the Euler equation maintains the same expression as the Bohm–Madelung theory.

- Finally, by means of a non-linear gauge transformation [28, 31, 32], the family of NSEs (3.25) can be transformed into another one with a purely real non-linearity.

In fact, by performing the transformation
\[
\psi(x,t) \rightarrow \phi(x,t) = \psi(x,t) \exp\left(-\frac{i}{\hbar} m D \int \frac{f_0(\rho)}{\rho} \, d\rho \right), \tag{3.29}
\]
the non-linearity (3.27) changes according to
\[
\Lambda(\rho) \rightarrow \tilde{\Lambda}(\rho, \sigma) = g_1(\rho) \Delta \sigma + g_2(\rho) (\nabla \rho)^2 + g_3(\rho) \Delta \rho, \tag{3.30}
\]
with
\[
g_1(\rho) = -D f_0(\rho), \quad g_2(\rho) = m D^2 \left[ \frac{1}{2} \left( \frac{f_0(\rho)}{\rho} \right)^2 + \frac{a_1}{\rho} \frac{\partial f_0(\rho)}{\partial \rho} \right], \tag{3.31}
\]
\[
g_3(\rho) = m D^2 a_1 \frac{f_0(\rho)}{\rho}.
\]

In equation (3.30), \( \sigma(x,t) \) is the phase of the new field \( \phi(x,t) \), related to the phase of the old field \( \psi(x,t) \) through the relation
\[
\sigma(x,t) = \Sigma(x,t) - m D \int \frac{f_0(\rho)}{\rho} \, d\rho. \tag{3.32}
\]

We observe that the new non-linearity \( \tilde{\Lambda}(\rho, \sigma) \), albeit purely real, now depends on both fields \( \rho(x,t) \) and \( \sigma(x,t) \).

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4. Time evolution of observables

The family of NSEs (3.25) describes dissipative and damped processes. To show this we derive the Ehrenfest relations for the main observables of the system.

In passing, we notice that, as a consequence of the continuity equation (3.10), the normalization of the wavefunction

$$\int \rho(x, t) \, dx = 1,$$

is preserved in time. We recall that non-linear evolution equations, in general, are not ray invariants. This means that, given a solution \( \psi \) of equation (3.25), \( A \psi \) with \( A \) a constant does not obey the same equation. Therefore, in this case the normalization condition (4.1) can be accomplished by fixing appropriately one of the free parameters of the model. As discussed in [8], the ray invariance can be restored when equation (3.25) is homogeneous in \( \rho(x, t) \).

Let us now introduce the linear momentum \( P = \int \rho \nabla \Sigma \, dx \). By using equations (3.10) and (3.21), we can derive the following continuity equation:

$$\frac{\partial p}{\partial t} + \nabla \cdot \mathbf{T} = \mathbf{R}_P,$$

for the density \( p = \rho \nabla \Sigma \), where the momentum–stress tensor \( \mathbf{T} \) has entries

$$T_{ij} = \frac{\rho}{m} \partial_i \Sigma \partial_j \Sigma + \frac{\hbar^2}{2m \rho} \partial_i \rho \partial_j \rho + \delta_{ij} G_1(\rho),$$

with

$$G_1(\rho) = -\frac{\hbar^2}{4m} \Delta \rho + a_1 m D^2 \int \rho \frac{d}{d\rho} \left[ \frac{1}{\rho} \nabla \cdot \left( f_0(\rho) \nabla \rho \right) \right] \, d\rho.$$

Clearly, the momentum source term \( \mathbf{R}_P \) is responsible for the non-conservation of \( P \). Its expression is given as

$$\mathbf{R}_P = G_2(\rho) \nabla \Sigma - \rho \nabla V,$$

with

$$G_2(\rho) = D \nabla \cdot \left( f_0(\rho) \nabla \rho \right).$$

In the same way, by defining the energy of the system \( E = -\int \rho (\partial \Sigma / \partial t) \, dx \), we can derive the evolution equation for the density \( \epsilon = -\rho \partial \Sigma / \partial t \), which reads

$$\frac{\partial \epsilon}{\partial t} + \nabla \cdot \mathbf{J}_E = \rho \mathbf{R}_E,$$

where the energy current \( \mathbf{J}_E \) is given by

$$\mathbf{J}_E = -\frac{\rho}{m} \nabla \Sigma \frac{\partial \Sigma}{\partial t} - \frac{\hbar^2}{4m} \nabla \rho \frac{\partial \rho}{\partial t}.$$

The energy source term takes the form

$$\mathbf{R}_E = -G_2(\rho) \frac{\partial \Sigma}{\partial t} + G_3(\rho) \frac{\partial \rho}{\partial t} + \rho \frac{\partial V}{\partial t},$$

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with

\[ G_3(\rho) = a_1 m D^2 \rho \frac{\partial}{\partial \rho} \left[ \frac{1}{\rho} \nabla \cdot \left( \frac{f_0(\rho)}{\rho} \nabla \rho \right) \right], \tag{4.10} \]

and the quantities \( \partial \rho / \partial t \) and \( \partial \Sigma / \partial t \) are computed through equations (3.10) and (3.21), respectively.

By integrating equations (4.2) and (4.7) on the whole configuration space and assuming uniform boundary conditions on the field \( \rho(x, t) \) we obtain the relations

\[ \frac{dP}{dt} = \langle R_P \rangle, \quad \frac{dE}{dt} = \langle R_E \rangle, \tag{4.11} \]

where \( \langle R_x \rangle = \int R_x \, dx \).

A further couple of relations easily derivable from equations (4.2) and (3.10) are given by

\[ \frac{dL}{dt} = \langle x \times R_P \rangle, \quad \frac{dx_{mc}}{dt} = \frac{P}{m}, \tag{4.12} \]

with \( L = \int x \times p \, dx \) the total angular momentum and \( x_{mc} = \int \rho x \, dx \) the mass centre of the system.

Thus, as stated by relations (4.11) and (4.12), equation (3.25) describes a quantum dissipative process due to the presence of the sources \( R_P \) and \( R_E \). Notwithstanding, depending on the form of \( f_0(\rho) \), there could exist special classes of solutions describing conservative systems.

5. Soliton solutions

Among the many solutions of a NSE, solitons are of particular interest in physics. As is known, soliton solutions arise when the dispersive effects due to the term \( \Delta \psi \) are balanced by the non-linear term \( \Lambda(\rho) \). In this case, the many-particle system moves coherently giving rise to an extended object with a particle-like behaviour preserving its shape in time.

When the system is constrained in one-dimensional space, without the external potential \( V = 0 \), a soliton solution can be obtained through the following ansatz:

\[ \psi(x, t) = \sqrt{\rho(\xi)} \exp \left[ \frac{i}{\hbar} \left( s(\xi) - \frac{1}{2} m v^2 t \right) \right], \tag{5.1} \]

where \( \xi = x - vt \) and \( v \) is the soliton velocity.

Taking into account the relations \( \partial / \partial t = -v \partial / \partial \xi \) and \( \partial / \partial x = \partial / \partial \xi \), equations (3.10) and (3.21) form a couple of ordinary differential equations for the scalar fields \( s(\xi) \) and \( \rho(\xi) \):

\[ \rho \frac{ds}{d\xi} - m D f_0(\rho) \frac{d\rho}{d\xi} - m v \rho = 0, \tag{5.2} \]

\[ v \frac{ds}{d\xi} - \frac{1}{2} m \left( \frac{ds}{d\xi} \right)^2 - U_q(\rho) - a_1 m D^2 \frac{1}{\rho} \frac{d}{d\xi} \left( f_0(\rho) \frac{d\rho}{d\xi} \right) + \frac{1}{2} m v^2 = 0. \tag{5.3} \]
By introducing the function
\[ y(\rho) = \left(\frac{1}{\rho} \frac{d\rho}{d\xi}\right)^2, \]  
we can obtain the following first-order differential equation:
\[ \frac{dy(\rho)}{d\rho} + F_1(\rho) y(\rho) + F_2(\rho) = 0, \]  
where
\[ F_1(\rho) = \frac{1}{\rho} \frac{f_0(\rho)}{a_1 f_0(\rho) - a_0} \left[ f_0(\rho) + 2 a_1 \rho (\rho f_0(\rho) + 2 a_1) - a_0 \right], \]  
\[ F_2(\rho) = \frac{1}{\rho} \frac{2 (v/D)^2}{a_1 f_0(\rho) - a_0}, \]  
and \( a_0 = (\hbar/2 m D)^2. \)

Equation (5.5) is solved by quadrature so that, taking into account equation (5.4), we get
\[ \xi = \int_1^{\rho} \left[ C \exp \left( - \int F_1(\rho') d\rho' \right) - \int F_2(\rho') \exp \left( \int F_1(\rho'') d\rho'' \right) d\rho' \right]^{-1/2} d\rho, \]  
where \( C \) is the integration constant.

This equation defines implicitly the shape of the soliton. Physically relevant solutions arise when \( \rho(\xi) \) vanishes in a sufficiently fast way for \( \xi \to \pm \infty \), so that the integral \( \int_{-\infty}^{\infty} \rho(\xi) d\xi \) converges. Clearly, this last condition selects the form of \( f_0(\rho) \) and consequently the possible NSEs admitting soliton solutions.

We remark that this special class of solutions is not dissipative or damped. They are steady states for the system under inspection. In fact, we easily verify that
\[ P = \int_{-\infty}^{\rho} \rho \frac{d\Sigma}{dx} \frac{dx}{d\xi} = \int_{-\infty}^{+\infty} \rho \frac{ds}{d\xi} d\xi, \]  
and according to the boundary condition \( \rho(\pm \infty, t) \to 0 \) we obtain that
\[ \frac{\partial P}{\partial t} = -v \frac{dP}{d\xi} = 0. \]  
In the same way
\[ E = \int_{-\infty}^{+\infty} \rho \frac{d\Sigma}{dx} \frac{dx}{d\xi} = -v \int_{-\infty}^{+\infty} \rho \frac{ds}{d\xi} d\xi - \frac{1}{2} m v^2, \]  
so that
\[ \frac{\partial E}{\partial t} = -v \frac{dE}{d\xi} = 0. \]  
Actually, the solution (5.8) describes a particle-like object travelling with constant velocity \( v \), momentum \( P = m v \) and energy \( E = m v^2/2 \). It is also easy to verify that this solution describes an isentropic process. This is consistent with the meaning of the entropy as a measure of the information. In fact, since the soliton does not change its shape in time, it is reasonable that the information (entropy) contained in it does not change in time, according to the relation \( dS(\rho)/dt = 0 \).
6. An example

As an example, let us consider the quantization of a classical system described by the Boltzmann–Gibbs entropy

$$S^{BG}(\rho) = -\int \rho \ln \rho \, d\mathbf{x}, \quad (6.1)$$

which is obtainable from equation (2.7) by imposing $\kappa(\rho) = \rho/e$. This implies $f_0(\rho) = 1$, so that the kinetics of the classical system is governed by the linear Fokker–Planck equation.

The quantum system is described by the following homogeneous NSE:

$$i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + \lambda \frac{\Delta \rho}{\rho} \psi + \mathbf{V} \psi, \quad (6.2)$$

where $\lambda$ is given in equation (3.28). This evolution equation belongs to the family of NSEs studied in [28] and derived on physical grounds as the most general class of non-linear quantum evolution equations compatible with the linear Fokker–Planck equation.

On employing the gauge transformation (3.29), equation (6.2) reduces to a NSE with a purely real non-linearity

$$i \hbar \frac{\partial \phi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \phi + W(\rho, \sigma) \phi + \mathbf{V} \phi, \quad (6.3)$$

where

$$W(\rho, \sigma) = -D \Delta \sigma + \frac{1}{2} m D^2 \left( \nabla \rho \right)^2 + a_1 m D^2 \frac{\Delta \rho}{\rho}, \quad (6.4)$$

and

$$\sigma = \Sigma - m D \ln \rho. \quad (6.5)$$

On imposing $a_1 = (a_0 - 1)/2$ and $V = 0$, equation (6.2) admits a physical solution whose density and phase read

$$\rho(\mathbf{x}, t) = A \left( \frac{m c_1}{4 \pi} \right)^{3/2} \exp \left( -\frac{1}{4} m c_1 \left( \frac{\mathbf{x}}{t} \right)^2 \right), \quad (6.6)$$

$$\Sigma(\mathbf{x}, t) = c_2 + \frac{1}{t} \left[ \frac{3}{16} \left( 4 m^2 D^2 + \hbar^2 \right) + \frac{1}{2} m \mathbf{x}^2 \right] - \frac{c_1}{4} m^2 D \left( \frac{\mathbf{x}}{t} \right)^2, \quad (6.7)$$

respectively, where $A, c_1 > 0$ and $c_2$ are constants. Accounting for the homogeneity of equation (6.2) we can accomplish the normalization of $\rho(\mathbf{x}, t)$ by imposing $A = (m c_1/4 \pi)^{3/2}$.

Solution (6.6) and (6.7) describes a dispersive wavefunction spreading out for $t \to +\infty$, since $\rho(\mathbf{x}, +\infty) \to 0$. Correspondingly, we also have $\mathbf{P}(+\infty) \to 0$ and $E(+\infty) \to 0$, indicating that this solution describes an irreversible process, damped and dissipative. It is also possible to verify that the corresponding entropy is given by

$$S^{BG} = \ln t + C t - \ln A, \quad (6.8)$$

with $C = 12 (\pi/m c_1)^{3/2}$, which grows in time.

With the same choice for the constant $a_1$ and for the potential $V = 0$, equation (6.2) admits a soliton solution. Its shape has the form

$$\rho(\xi) = A e^{-\left( a/4 \right) \xi^2}, \quad (6.9)$$

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where $\alpha = (4m v^2)/(4m^2 D^2 + \hbar^2)$ and $A = \sqrt{\alpha/4\pi}$. The phase of the wavefunction is given by equation (5.2) and takes the form

$$\Sigma(\xi, t) = -\frac{1}{4} m D \alpha \xi^2 + m v \xi - \frac{1}{2} m v^2 t.$$  \hspace{1cm} (6.10)

The Gaussian shape (6.9) is known in the literature as a gausson [7, 12] and it was derived firstly as the soliton solution of the BBM equation, although, in that case, the phase $\Sigma(\xi, t)$ assumes a different form. Solution (6.9) and (6.10) describes a non-dissipative reversible process with a constant entropy

$$S_{BG} = \sqrt{\frac{\pi}{\alpha}} + \ln \sqrt{\frac{\pi}{\alpha}} + \ln 2.$$ \hspace{1cm} (6.11)

7. Conclusions

In the stochastic quantization framework, we have approached the problem of the quantization of a many-body classical system whose kinetics is described by a general NFPE compatible with an arbitrary trace-form entropy. By a suitable modification of the original Nelson method we have derived a family of NSEs describing, in the mean field approximation, a system of collectively interacting particles. The time evolution of the mean field $\psi(x, t)$ is given by means of a non-Hermitian operator $H$ whose non-linearity, describing dissipative and damped effects, takes into account the collective interactions among the many particles and the background. This is a substantially different situation to that addressed by the canonical approach studied in [24], which deals only with conservative systems.

We have investigated the existence of a special class of solutions representing a localized object with particle-like properties originating from the collective motion of the many particles of the system. This solution describes a reversible and non-dissipative process without production of entropy.

As an explicit example, we have studied the quantization of a classical system governed by the BG entropy, obtaining, in this way, a NSE with a complex non-linearity. This example emphasizes the difference between the quantization method employed in the present paper and the canonical one used in [24]. There, starting from a classical system governed by the BG entropy, the resulting quantum evolution equation is given by the linear Schrödinger equation which describes the dynamics of an isolated system. This is consistent with the widely accepted opinion that the BG entropy governs systems with weakly and local interactions. In contrast, in the present case, although the particles in the system must still be considered weakly interacting (due to the presence of the BG entropy), there is a further interaction between the system and the background which is accounted for by the non-linearity in the evolution equation.

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