Chiral anomaly and transport in Weyl metals

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Abstract
We present an overview of our recent work on transport phenomena in Weyl metals, which may be connected to their nontrivial topological properties, particularly to chiral anomaly. We argue that there are two basic phenomena, which are related to chiral anomaly in Weyl metals: anomalous Hall effect (AHE) and chiral magnetic effect (CME). While AHE is in principle present in any ferromagnetic metal, we demonstrate that a magnetic Weyl metal is distinguished from an ordinary ferromagnetic metal by the absence of the extrinsic and the Fermi surface part of the intrinsic contributions to the AHE, as long as the Fermi energy is sufficiently close to the Weyl nodes. The AHE in a Weyl metal is thus shown to be a purely intrinsic, universal property, fully determined by the location of the Weyl nodes in the first Brillouin zone. In other words, a ferromagnetic Weyl metal may be thought of as the only example of a ferromagnetic metal with a purely intrinsic AHE. We further develop a fully microscopic theory of diffusive magnetotransport in Weyl metals. We derive coupled diffusion equations for the total and axial (i.e. node-antisymmetric) charge densities and show that chiral anomaly manifests as a magnetic-field-induced coupling between them. We demonstrate that an experimentally-observable consequence of CME in magnetotransport in Weyl metals is a quadratic negative magnetoresistance, which will dominate all other contributions to magnetoresistance under certain conditions and may be regarded as a smoking-gun transport characteristic, unique to Weyl metals.

Keywords: Weyl semimetal, Dirac semimetal, topological insulator
(Some figures may appear in colour only in the online journal)

1. Introduction

The last decade has seen an explosion of research in topologically-nontrivial states of matter, following the remarkable discovery of topological insulators (TI) [1–8]. One normally associates topologically nontrivial properties with insulators, the spectral gap providing the rigidity and insensitivity to fluctuations, which are characteristic of these states of matter. One of the most recent notable developments in this field, however, has been the realization that even gapless metallic and semimetallic states may be topologically nontrivial in much the same sense as gapped insulators [9–12]. These recent developments have partly been anticipated in the earlier pioneering work of Volovik [13], promoting topological classification of all possible fermionic ground states, which shed new light on such wide-ranging phenomena as the robustness of the Fermi liquid and the hierarchy problem in particle physics.

The recent work focussed on Weyl semimetals [9–12], which, in Volovik’s classification, belong to the Fermi point universality class of fermionic vacua. In the condensed matter context a Weyl point or node is a point of contact between two nondegenerate bands in the first Brillouin zone (BZ). Such electronic structure features have in fact been noticed and studied since the earliest days of the theory of solids [14–16], but only recently have their topological properties come into focus. Two important ingredients in the appearance of Weyl node features in the electronic structure are broken time reversal (TR) or inversion (I) symmetries and dimensionality of space. The broken symmetry requirement is a consequence of the well-known Kramers theorem. If both TR and I symmetries are present, the band eigenstates in a solid must be (at least)
The minimal momentum to be at the BZ centre, i.e. at the Invariant crystal momentum. For concreteness, we will this paper, except in some of the final results. The point of band degeneracy occurs when all three components of the crystal momentum vanish. This is why three-dimensionality is important: we must have three momentum components available as ‘tuning parameters’ to create a point of degeneracy between the two bands. A naively analogous degeneracy point in 2D graphene in fact does not exist, as was pointed out in the seminal work of Kane and Mele [1], which started the field of TI. It only looks like a degeneracy point due the smallness of the spin–orbit (SO) interactions in graphene, which always open up a small gap and destroy the degeneracy.

The presence of Weyl degeneracy nodes is, in principle, a common feature of the electronic structure of all 3D magnetic or noncentrosymmetric materials. In most cases, however, their presence is of no consequence, since they will generically occur far away from the Fermi energy [18]. Weyl semimetal refers to a situation in which the Fermi level exactly coincides with the Weyl nodes and no other states are present at the Fermi energy. This situation is not generic and occurs only under special, but not entirely uncommon circumstances.

To identify these, it is convenient to start from the situation in which both TR and I symmetries are present. TR symmetry requires that if a Weyl node is present at momentum k, another node with the same chirality must be present at −k. I symmetry, on the other hand, requires that if a Weyl node is present at momentum kz, a node of opposite chirality must be present at −kz. This implies that in the presence of both TR and I, there may exist a pair of opposite-chirality Weyl nodes at the same TR and I-invariant crystal momentum. For concreteness, we will take this momentum to be at the BZ centre, i.e. at the Γ point. The minimal k · p Hamiltonian, describing this situation, has the form

\[
H = \pm v_F \mathbf{\sigma} \cdot \mathbf{k} + \Delta \tau^z,
\]

where the triplet of Pauli matrices \( \mathbf{\sigma} \) describes the pair of touching bands, the sign in front refers to two possible chiralities of the band contact point and we have subsumed any possible spatial anisotropies into the definition of the crystal momentum \( \mathbf{k} \). We will use \( \hbar = c = 1 \) units throughout this paper.

The latter situation, i.e. with \( \Delta \) fine-tuned to zero, occurs at the transition point between an inversion-symmetric 3D TI and a normal insulator (NI) [11, 28–30]. In the present review we will focus exclusively on this case. Our results, however, do not depend on this choice and are applicable to all realizations of Dirac and Weyl semimetals. As mentioned above, Weyl semimetal may be obtained from Dirac semimetal by breaking either TR or I. We will focus on the TR-breaking case here, since it produces the simplest possible kind of Weyl semimetal, with only a single pair of Weyl nodes. Generalization to many pairs is in most cases straightforward, but it does complicate the technical details of course. Formally, this type of Weyl semimetal is obtained by adding a Zeeman spin-splitting term to the Dirac Hamiltonian equation (2)

\[
H = v_F \mathbf{\tau} \cdot \mathbf{\sigma} \cdot \mathbf{k} + \Delta \tau^z + \hbar \sigma^z,
\]

where the spin-splitting term may arise either from an external magnetic field, or from an intrinsic magnetic ordering and we have chosen the z-direction to be the magnetization direction. To analyze this Hamiltonian, it is convenient to perform the following canonical (i.e. commutation relation preserving) transformation of the \( \mathbf{\sigma} \) and \( \mathbf{\tau} \) operators

\[
\sigma^\pm \rightarrow \tau^z \sigma^\pm, \quad \tau^z \rightarrow \sigma^z \tau^z.
\]

The Hamiltonian equation (3) then transforms to

\[
H = v_F (\sigma^z k_z + \sigma^z k_z) + (b + v_F k_z \tau^z + \Delta \tau^z) \sigma^z,
\]

where

\[
m_{\pm}(k_z) = b \pm \sqrt{v_F^2 k_z^2 + \Delta^2}.
\]

Equation (6) looks like the Hamiltonian of a pair of 2D Dirac fermions with ‘masses’ \( m_{\pm}(k_z) \), which depend on the z-component of the crystal momentum as a parameter. The mass \( m_{\pm}(k_z) \) is always positive, but \( m_{-}(k_z) \) may vanish and change sign if \( b > \Delta \). If this is the case, \( m_{-}(k_z) \) vanishes at \( k_z^\pm = \pm k_0 \), where

\[
k_0 = \frac{1}{v_F} \sqrt{b^2 - \Delta^2}.
\]

These points along the z-axis in momentum space, where the mass \( m_{-}(k_z) \) vanishes, are the locations of the Weyl nodes. The two bands, which are the eigenstates of the \( H_{\pm} \) block of the Hamiltonian, are degenerate at those points. Since massive 2D Dirac fermions are associated with half-quantized Hall conductivity [31]

\[
\sigma_{xy}^{2D} = \frac{e^2}{2\hbar} \text{sign}(m),
\]
it follows that the Weyl semimetal at \( b > \Delta \) has a Hall conductivity given by

\[
\sigma_{xy} = \frac{e^2}{h} 2k_0, \quad (10)
\]

i.e., equal to the distance between the Weyl nodes in units of \( e^2 / h [32] \).

It is easy to show that this Hall conductivity may be associated with chiral edge states. Indeed, suppose the sample is finite in the \( y \)-direction and we will take the sample to occupy the \( y < 0 \) half-space. Replacing \( k_y \rightarrow -i\partial / \partial y \), the \( H_- \) block of the Hamiltonian becomes

\[
H_- = -i v_F \frac{\partial}{\partial y} \sigma^y + v_F \sigma^x k_x + m_- (k_z, y) \sigma^z. \quad (11)
\]

Let us first set \( k_z = 0 \) and look for a zero-energy solution of the Schrödinger equation (11)

\[
H_- \Psi(k_z, y) = 0, \quad (12)
\]

localized at the sample boundary \( y = 0 \). We look for a solution in the form

\[
\Psi(k_z, y) = i \sigma^y e^{i f(k_z, y)} \phi, \quad (13)
\]

where \( f(k_z, y) \) is a scalar function, while \( \phi \) is a two-component spinor. Plugging this into the SE equation (12), we obtain

\[
\left[ \frac{\partial f(k_z, y)}{\partial y} + m_- (k_z, y) \sigma^x \right] \phi = 0. \quad (14)
\]

The solution of equation (14), satisfying our requirements is

\[
f(k_z, y) = \frac{1}{v_F} \int_0^y \sigma^x m_- (k_z, y') dy', \quad (15)
\]

and \( \phi = |\sigma^x = -1 \), i.e., the eigenstate of \( \sigma^x \) with eigenvalue \(-1\). Thus, we finally obtain the following result for the solution of equation (14), localized at the sample boundary

\[
\Psi(k_z, y) = e^{i \frac{1}{v_F} \int_0^y \sigma^x m_- (k_z, y') dy'} |\sigma^x = 1 \rangle. \quad (16)
\]

This solution exists as a localized edge state as long as \( m_- (k_z, y \rightarrow -\infty) \geq 0 \), i.e., as long as \(-k_0 \leq k_z \leq k_0 \). For this reason, it is called a Fermi arc [9].

At this point, we will wrap up the introductory part of this review and move on to a more detailed account of the transport properties of Weyl semimetals and metals, based on a more realistic microscopic model of Weyl semimetal. The rest of the paper is organized as follows. In section 2 we introduce a realistic microscopic model of a Weyl semimetal, based on a magnetically-doped TI-NI multilayer heterostructure. In section 3 we discuss basic electromagnetic response properties of this model Weyl semimetal and introduce two distinct components of the response, which are directly related to the nontrivial topology of Weyl nodes: the anomalous Hall effect (AHE) [33, 34] and the chiral magnetic effect (CME). In section 4 we extend the theory of electromagnetic response of Weyl metals to diffusive transport regime. We demonstrate that magnetic Weyl metals are distinguished from ordinary ferromagnetic metals by a lack of impurity-scattering and Fermi-surface contributions to their anomalous Hall conductivity. Instead their anomalous Hall conductivity is determined by the distance between the Weyl nodes in momentum space and nothing else. We also show that CME manifests in the diffusive regime as a new kind of weak-field magnetoresistance, which is negative and quadratic in the field. This novel magnetoresistance is expected to occur in all types of Weyl metals and may be regarded as their smoking-gun transport characteristic. We conclude in section 5 with a brief overview of the main results.

2. Heterostructure model of Weyl semimetal

As discussed in the Introduction, perhaps the most straightforward (at least theoretically) way to realize a Weyl semimetal phase is to break TR symmetry in a material, which in its nonmagnetic state is naturally poised near the phase transition between a TI and NI. One way to achieve this is to engineer a composite material, made of alternating thin layers of TI and NI, as shown in figure 1. This system may be regarded as a ‘hydrogen atom’ of Weyl semimetals: the most elementary yet realistic system, the description of which is simple enough that purely analytical theory of many phenomena is possible, as will be seen below.

This system may be described as a chain of 2D Dirac surface states of the TI layers, which are coupled by tunnelling matrix elements \( \Delta_S \) for a pair of surface states, belonging to the same TI layer and \( \Delta_D \) for a pair belonging to nearest-neighbor TI layers. We will assume, for concreteness, that \( \Delta_{S,D} > 0 \). The corresponding Hamiltonian has the form

\[
H = v_F \tau^z (\hat{z} \times \sigma) \cdot k + \Delta_D \tau^z \delta_{i,j} + \Delta_S \tau^z \delta_{i,j+1} + \frac{1}{2} \Delta_D (\tau^z \delta_{i,j+1} + \tau^- \delta_{i,j-1}), \quad (17)
\]

where \( i, j \) label the unit cells of the superlattice in the growth (\( z \)) direction, \( k_{\perp} = (k_x, k_y) \) are the crystal momentum components, transverse to the growth direction, Pauli matrices \( \tau \) act on the which surface pseudospin degree of freedom, while \( \sigma \) act in the spin degree of freedom. Diagonalizing the growth direction hopping part of the Hamiltonian by Fourier transform and performing the canonical transformation of equation (4), we obtain

\[
H = v_F (\hat{z} \times \sigma) \cdot k_{\perp} + \hat{A} (k_z) \sigma^z, \quad (18)
\]
where

$$\tilde{\Delta}(k_z) = \Delta_S \tau^z + \frac{\Delta_D}{2} \left( \tau^x e^{ik_z \Delta_1} + \text{h.c.} \right),$$

(19)
d being the period of the superlattice in the growth direction. $\tilde{\Delta}(k_z)$ may now be diagonalized separately, as it commutes with the Hamiltonian. Its eigenvalues are given by $\pm \Delta(k_z)$, where

$$\Delta(k_z) = \sqrt{\Delta_S^2 + \Delta_D^2 + 2 \Delta_S \Delta_D \cos(k_z d)}.$$  

(20)
The corresponding two-component spinor wavefunctions are

$$|u'(k_z)\rangle = \frac{1}{\sqrt{2}} \left( 1, i \frac{\Delta_S + \Delta_D e^{-ik_z d}}{\Delta(k_z)} \right),$$

(21)
where $t = \pm$ labels the two eigenvalues. The spin block of the Hamiltonian may now be diagonalized as well, giving rise to the following eigenvalues

$$\epsilon_{st}(k) = s \epsilon_t(k) = s \sqrt{\frac{1}{\Delta_S^2} (k_x^2 + k_y^2) + m_t^2(k_z)},$$

(22)
where $s = \pm$ and

$$m_t(k_z) = t \Delta(k_z),$$

(23)
The double degeneracy of the band eigenvalues at every $k$ in equation (22) is the Kramers degeneracy, arising due to the presence of both TR and I symmetries. The corresponding eigenvectors are given by

$$|v'^{st}(k)\rangle = \frac{1}{\sqrt{2}} \left( 1 + s \frac{m_t(k_z)}{\epsilon_t(k)} \right),$$

(24)
where $e^{i\phi} = \frac{k_x + ik_y}{\sqrt{k_x^2 + k_y^2}}$. The full four-component eigenvectors of the Hamiltonian equation (18) may be represented as a tensor product $|u'(k_z)\rangle$ and $|v'^{st}(k)\rangle$:

$$|v^{st}(k)\rangle = |v'^{st}(k)\rangle \otimes |u'(k_z)\rangle.$$

(25)
The heterostructure, described by equation (17), can exist in two distinct phases: strong 3D TI and an NI. This can be easily seen by computing the parity operator $\tau^z \sigma^z$ (or $\tau^z \sigma^- \sigma^-$ after the canonical transformation of equation (4)) at two TR-invariant momenta in the first BZ: $\Gamma_1 = (0, 0, 0)$ and $\Gamma_2 = (0, 0, \pi/d)$. Taking the product of the parity eigenvalues over all the occupied bands and over $\Gamma_{1,2}$, the $Z_2$ index is found to be

$$(-1)^{3} = \text{sign}(\Delta_S - \Delta_D).$$

(26)
Thus the multilayer is a strong 3D TI when $\Delta_D > \Delta_S$ and an NI otherwise. The point $\Delta_S = \Delta_D$ marks the transition point between the 3D TI and NI. At this point, the gap at $\Gamma_2$ closes and the heterostructure is a Dirac semimetal.

To obtain a Weyl semimetal, we break TR symmetry in the heterostructure, assuming that it is tuned close to the TI-NI transition point and the band gap $|\Delta_S - \Delta_D|$ is not large. The TR breaking may be accomplished by doping the heterostructure with magnetic impurities. At sufficient concentration, the impurities will form a ferromagnetic (FM) state and we will assume that the magnetization points along the growth direction of the heterostructure (distinct, non-Weyl nodal states are obtained if the magnetization is in the $xy$-plane [36]). We will describe the FM state of the heterostructure by adding a term $b \sigma^z \delta_{ij}$ to the Hamiltonian equation (17). This splits the degeneracy of the $t = \pm$ Kramers doublet, modifying equation (23) as

$$m_t(k_z) = b + t \Delta(k_z),$$

(27)
Taking the spin splitting $b$ to be always positive, the 'mass' $m_+(k_z)$ is then always positive, while $m_-(k_z)$ may vanish and change sign as a function of $k_z$, if the spin splitting is sufficiently large. This happens when $b \geq b_D = |\Delta_S - \Delta_D|$. The Weyl node locations along the $k_z = k_y = 0$ line in the crystal momentum space are given by the solutions of the equation

$$m_+(k_z) = b - \Delta(k_z) = 0,$$

(28)
which gives $k_{z+}^2 = \pi/d \pm k_0$, where

$$k_0 = \frac{1}{d} \arccos \left( \frac{\Delta_S^2 + \Delta_D^2 - b^2}{2 \Delta_S \Delta_D} \right).$$

(29)
When the spin splitting reaches the upper critical value $b_{c2} = \Delta_S + \Delta_D$, $k_0 = \pi/d$ and the Weyl nodes are annihilated again at the BZ edge. The resulting state is a 3D quantum anomalous Hall insulator with a quantized Hall conductivity [37]

$$\sigma_{xy} = \frac{e^2}{hd}.$$  

(30)

3. Electromagnetic response of Weyl semimetals

3.1. General remarks

Topologically-nontrivial states of matter, such as 3D TI and Weyl semimetals, have a distinguishing spectroscopic feature: the presence of metallic edge states. These surface metallic states are unusual and may only exist on the surface of a bulk 3D topological phase. As discussed in the Introduction, in the Weyl semimetal state, of interest to us here, the Fermi surface of this metal is not a closed 2D curve, as it must be in any regular 2D metal, but instead forms an open Fermi arc, with the end points at the locations of the bulk Weyl nodes, projected onto the surface BZ [9, 38].

However, apart from such spectroscopic distinguishing features, topological phases often have unusual electromagnetic response characteristics, which are always of particular interest, as these are the manifestations of the unique quantum physics of these phases on macroscopic scales. In 3D TI, this unusual electromagnetic response may be described by the so-called $\theta$-term in the action of the electromagnetic field [8]

$$S_\theta = \frac{e^2 \theta}{32 \pi^2} \int dt \, d^3r e^{i \mu \nu \alpha \beta} F_{\mu \nu} F_{\alpha \beta} = \frac{e^2 \theta}{4 \pi} \int dt \, d^3r \mathbf{E} \cdot \mathbf{B}.$$  

(31)

The parameter $\theta$ is equal to $\pi$ in 3D TI, the only nonzero value, compatible with TR symmetry [8]. The $\theta$-term in
equation (31), however, is a full derivative and thus has no effect on Maxwell’s equations in the bulk of the TI. Its only real effect is to generate the half-quantized AHE on the sample surfaces in the presence of surface magnetization (e.g., due to magnetic impurities).

When TR is broken in the bulk, as it is in the Weyl semimetal state, a \( \theta \)-term with non-constant \( \theta \) is now allowed by symmetry

\[
S_\theta = \frac{e^2}{32\pi^2} \int \mathrm{d}^4r \theta(r) e^{\mu \nu \alpha \beta} F_{\mu \alpha} F_{\nu \beta}.
\]  

(32)

The only functional form, compatible with translational symmetry, is linear, i.e.

\[
\theta(r) = 2b \cdot r,
\]  

(33)

where \( b \) is a vector, which is odd under TR and even under I and coincides, as will be seen explicitly below, with the bulk spin splitting. If, in addition to TR, I symmetry is violated as well, the ‘axon field’ \( \theta \) may also acquire a linear time dependence

\[
\theta(r, t) = 2b \cdot r - 2b_0 t,
\]  

(34)

where \( b_0 \) is the energy difference between the Weyl nodes, allowed if I is violated. Unlike the \( \theta \)-term in 3D TI, equation (32) does modify Maxwell’s equations in the bulk of the Weyl semimetal and thus has observable consequences, namely the already mentioned AHE and the Chiral Magnetic Effect (CME) [39], both of which will be discussed in detail below.

The action of equation (32) may be described as being a consequence of chiral anomaly, an important concept in relativistic field theory [40–42], which has recently found its way into condensed matter physics and has been realized to play an important role in the theory of topologically-nontrivial states of matter [43–45]. To gain a basic understanding of the system in terms of an imaginary time action, including possible coupling of the electrons to an electromagnetic field

\[
S = \int \mathrm{d}^3r \psi^\dagger [\partial_+ + i e A_0 + b_0 \tau^z + \tau^z \sigma \cdot (i \nabla + e A)] \psi,
\]  

(35)

where we have included an energy difference between the Weyl nodes, described by the \( b_0 \tau^z \) term, which is allowed when the I symmetry is violated and oriented the TR-breaking vector \( b \) along an arbitrary direction. It is convenient to represent the system in terms of an imaginary time action, including possible coupling of the electrons to an electromagnetic field

\[
S = \int \mathrm{d}^3r \psi^\dagger [\partial_+ + i e A_0 + b_0 \tau^z + \tau^z \sigma \cdot (i \nabla + e A)] \psi,
\]  

(36)

where \( A_\mu = (A_0, A) \) is the electromagnetic gauge potential and \( \psi^\dagger, \psi \) are the 4-component spinor Grassman fields. We have suppressed all explicit spinor indices in the Grassmann fields for brevity. We now observe that in addition to the charge conservation symmetry, the imaginary time action equation (36) possesses an extra chiral symmetry

\[
\psi \to e^{-i\tau^z/2} \psi,
\]  

(37)

which expresses an apparent separate conservation of the number of fermions of left and right chirality. This suggests that the terms \( b_0 \tau^z \) and \( b \cdot \sigma \) in equation (36) could be eliminated by a gauge transformation

\[
\psi \to e^{-i\tau^z/(\tau r)}/2) \psi, \quad \psi^\dagger \to \psi^\dagger e^{i\tau^z/(\tau r)}/2,
\]  

(38)

where \( \theta(r, \tau) = 2b \cdot r - 2ib_0 \tau \) and one should keep in mind that \( \psi \) and \( \psi^\dagger \) are not complex conjugates of each other, but are independent variables in the fermion path integral. The imaginary time action then becomes

\[
S = \int \mathrm{d}r \psi^\dagger \left[ \partial_+ + ieA_0 + \tau^z \sigma \cdot (i \nabla + e A) \right] \psi.
\]  

(39)

which describes two Weyl nodes of opposite chirality, existing at the same point in momentum space and in energy. This argument then leads one to the conclusion that the system of Weyl nodes, separated in energy and momentum, is equivalent to the system of two degenerate Weyl nodes and thus does not possess any special transport properties, which is incorrect. What is missing in the above argument is exactly the chiral anomaly: while the imaginary time action equation (36) does indeed possess the chiral symmetry, the gauge transformation of equation (38) changes not only the action itself, but also the measure of the path integral. This change of the path integral measure is what gives rise to the \( \theta \)-term equation (32). This may be shown explicitly using the Fujikawa’s method [46,47]. We refer the reader to [47] for technical details.

To see the physical consequences of the \( \theta \)-term, we integrate equation (32) by parts and eliminate a total derivative term. This leads to the following action, which resembles the Chern–Simons term in 2 + 1-dimensions

\[
S_\theta = -\frac{e^2}{8\pi^2} \int \mathrm{d}^3r \partial_\mu \theta e^{\mu \nu \alpha \beta} A_\nu \partial_\alpha A_\beta.
\]  

(40)

Varying equation (40) with respect to the vector potential, we obtain the following expression for the current

\[
j_v = \frac{e^2}{2\pi^2} b_\mu e^{\mu \nu \alpha \beta} \partial_\alpha A_\beta, \quad \mu = 1, 2, 3,
\]  

(41)

and

\[
j_v = \frac{e^2}{2\pi^2} b_0 e^{0 \nu \alpha \beta} \partial_\nu A_\beta.
\]  

(42)

Equation (41) clearly represents the AHE, while equation (42) represents another effect, related to chiral anomaly in Weyl semimetals, the Chiral Magnetic Effect [39], whose physical meaning is somewhat subtle and will be discussed in detail later.

However, the picture of chiral anomaly and related transport phenomena, presented above, while simple and appealing, is not fully satisfactory, for the following reasons. Chiral anomaly is a sharply-defined concept in the context of relativistic field theory, where massless fermions in unbounded
momentum space possess exact chiral symmetry equation (37), which may be violated by the anomaly. In the condensed matter context, however, the situation is less clear. Even though chiral symmetry may be formally defined in a low-energy model of a Weyl semi-metal equation (35), in which the band dispersion is approximated as being exactly linear and unbounded, no real microscopic model of Weyl semi-metal actually possesses such an exact symmetry. It may only appear as an approximate low-energy symmetry. Since the chiral symmetry is not present to begin with, it is then unclear to what extent it is meaningful to speak of its violation by chiral anomaly. The role of impurity scattering, present in any real condensed matter system and important even conceptually in any discussion of transport phenomena; as well as the role of finite electron or hole density, present when the Fermi energy does not exactly coincide with the location of the Weyl nodes, are also completely unclear in the above field-theory discussion. In the remainder of this paper we will thus develop a fully microscopic theory of chiral anomaly and its manifestations, including the contribution of impurity scattering and finite charge carrier density, based on the TI-NI heterostructure model, described in section 2, which does not possess chiral symmetry, yet exhibits both the AHE and CME, described above.

3.2. Microscopic theory: anomalous Hall effect

We will now go back to our microscopic model of a Weyl semi-metal, described in section 2. We want to explicitly derive the topological term in the action of the electromagnetic field by integrating over electron variables, coupled to the field. We then start from the imaginary time action of electrons in the Weyl semi-metal, coupled to electromagnetic field

$$S = \int d\mathbf{r} d\tau \left\{ \Psi^\dagger(\mathbf{r}, \tau) \left[ \hat{\partial}_\tau - \mu + i e A_0(\mathbf{r}, \tau) + \hat{H} \right] \Psi(\mathbf{r}, \tau) \right\},$$

where $A_0(\mathbf{r}, \tau)$ is the scalar potential and

$$\hat{H} = \frac{e}{v_F} \mathbf{r} \times \mathbf{A} \cdot (-i \nabla + e\mathbf{A}) + \hat{\Delta} + b\sigma^z,$$

is the Hamiltonian of non-interacting electrons in Weyl semi-metal, minimally coupled to the vector potential $\mathbf{A}$. We will ignore the $z$-component of the vector potential as it will not play any role in what follows.

Using the results of section 2 for the eigenvalues and eigenvectors of the Weyl semi-metal Hamiltonian in the absence of the electromagnetic field, we can now integrate over electron variables in equation (43) and obtain an effective action for the electromagnetic field, induced by coupling to the electrons. This action will contain two distinct kinds of contributions. The first kind will contain terms, proportional to $\mathbf{E}^2$ and $\mathbf{B}^2$, where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields. These terms describe the ordinary electric and magnetic polarizability of the material and we will not discuss this part of the response here. The second kind contains the ‘topological’ contribution, which has the form of the ‘3D Chern–Simons’ term of equation (40).

Adopting the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$, we obtain

$$S = \sum_{\mathbf{q}, \mathbf{i} \Omega} \epsilon_{0\mathbf{q}\mathbf{i} \Omega} \Pi(\mathbf{q}, \mathbf{i} \Omega) A_0(\mathbf{q}, \mathbf{i} \Omega) A_\beta(\mathbf{q}, \mathbf{i} \Omega),$$

where $\hat{q}_\alpha = q_\alpha/q$ and summation over repeated indices is implicit. The $z$-direction in equation (45) is picked out by the magnetization $\mathbf{b}$. The response function $\Pi(\mathbf{q}, \mathbf{i} \Omega)$ is given by

$$\Pi(\mathbf{q}, \mathbf{i} \Omega) = \frac{i e^2 v_F}{V} \sum_k n_F [\xi_{\mathbf{q} \mathbf{i} \Omega}(\mathbf{k})] + \frac{n_F [\xi_{\mathbf{q} \mathbf{i} \Omega}(\mathbf{k} + \mathbf{q})]}{\mathbf{i} \Omega + \xi_{\mathbf{q} \mathbf{i} \Omega}(\mathbf{k}) - \xi_{\mathbf{q} \mathbf{i} \Omega}(\mathbf{k} + \mathbf{q})} \times (\mathbf{z}_{k-q}^* \sigma^y \mathbf{z}_{q}^y) (\mathbf{z}_{k}^y \sigma^y \mathbf{z}_{k+q}^y),$$

and

$$\frac{\Theta(\mathbf{q}, \mathbf{i} \Omega)}{\pi} = -\frac{i e^2 q}{\delta \mathbf{q}^2} \sum_{\mathbf{k}, \mathbf{r}} \int_{-\pi/\mathbf{d}}^{\pi/\mathbf{d}} dk_z \frac{m_i(k_z)}{\epsilon_F} \Theta(\mathbf{q}, \mathbf{i} \Omega) \frac{\mathbf{m}_i(k_z)}{\epsilon_F}.$$
i.e. the intraband contribution to $\Pi(q, i\omega)$ is equal to the second term in the square brackets in equation (47) in magnitude, but opposite in sign. Combining the inter- and intraband contributions to $\Pi(q, i\omega)$ we thus obtain

$$
\Pi(q, i\omega) = \frac{e^2q}{8\pi^2} \sum_i \int_{-\pi/d}^{\pi/d} dk_z \text{sign}[m_i(k_z)]
\times \left[ 1 - \Theta(\epsilon_F - |m_i(k_z)|) \right],
$$

(50)
i.e. the last term in equation (47) cancels out when the low-frequency, long-wavelength limit is taken in such a way that $\Omega/v_F q \to 0$. On the other hand, when $\Omega/v_F q \to \infty$, the intraband contribution vanishes and

$$
\Pi(q, i\omega) = \Pi^{\text{inter}}(q, i\omega).
$$

(51)

This physical difference in the kind of response the system exhibits is the basis of Streda’s separation of contributions to the Hall conductivity into $\sigma_{xy}^I$ and $\sigma_{xy}^{II}$ [48]. Our analysis makes it clear that this is the most physically-meaningful separation of contributions to the intrinsic anomalous Hall conductivity, since it corresponds to distinct and, at least in principle, separately measurable, contributions to the response function $\Pi(q, i\omega)$.

Generalizing the above results to arbitrary sign of $\epsilon_F$ we finally obtain

$$
S = -ie^{2\omega_0}\sigma_{xy} \int d^3r d\tau A_0(r, \tau) \partial_\alpha A_\beta(r, \tau),
$$

(52)
where, if the low-frequency long-wavelength limit is taken so that $\Omega/v_F q \to \infty$:

$$
\sigma_{xy} = \frac{e^2}{8\pi^2} \sum_i \int_{-\pi/d}^{\pi/d} dk_z \left\{ \text{sign}[m_i(k_z)]
\times \left[ \Theta(\epsilon_F + |m_i(k_z)|) - \Theta(\epsilon_F - |m_i(k_z)|) \right]
+ \frac{m_i(k_z)}{\epsilon_F} \Theta(|\epsilon_F| - |m_i(k_z)|) \right\}.
$$

(53)
This expression corresponds to the full DC anomalous Hall conductivity. On the other hand, when the low-frequency, long-wavelength limit is taken so that $\Omega/v_F q \to 0$, we obtain

$$
\sigma_{xy} = \sigma_{xy}^{II} = \frac{e^2}{8\pi^2} \sum_i \int_{-\pi/d}^{\pi/d} dk_z \text{sign}[m_i(k_z)]
\times \left[ \Theta(\epsilon_F + |m_i(k_z)|) - \Theta(\epsilon_F - |m_i(k_z)|) \right].
$$

(54)
This is precisely the Streda’s $\sigma_{xy}^{II}$ contribution to the Hall conductivity, which is a thermodynamic equilibrium quantity, equal to

$$
\sigma_{xy}^{II} = e \left( \frac{\partial N}{\partial \mu} \right)_\epsilon,
$$

(55)
where $N$ is the total electron number. This relation follows immediately from equation (52) and the order of limits $\Omega/v_F q \to 0$, which corresponds to thermodynamic equilibrium. Correspondingly, the $\sigma_{xy}^I$ contribution is given by

$$
\sigma_{xy}^I = \sigma_{xy} - \sigma_{xy}^{II} = \frac{e^2}{8\pi^2} \sum_i \int_{-\pi/d}^{\pi/d} dk_z \frac{m_i(k_z)}{\epsilon_F} \Theta(|\epsilon_F| - |m_i(k_z)|).
$$

(56)
As clear from the above analysis, $\sigma_{xy}^I$ is the contribution to $\sigma_{xy}$ that can be associated with states on the Fermi surface. This contribution is nonuniversal, i.e. it depends on details of the electronic structure and is a continuous function of the Fermi energy. $\sigma_{xy}^{II}$, on the other hand, is the contribution of all states below the Fermi energy and is a thermodynamic equilibrium property of the ferromagnet. It attains a universal value, which depends only on the distance between the Weyl nodes, when the Fermi energy coincides with the nodes, i.e. when $\epsilon_F = 0$

$$
\sigma_{xy}^{II} = \frac{e^2K}{4\pi^2},
$$

(57)
where

$$
K = \frac{2}{d} \arccos \left( \frac{\Delta_D^2 + \Delta_D^2 - b^2}{2\Delta_D^2} \right),
$$

(58)
is the distance between the Weyl nodes. When $b > b_c$, the Weyl nodes annihilate at the edges of the Brillouin zone and a gap opens up. In this case $K = 2\pi/d$, i.e. a reciprocal lattice vector and $\sigma_{xy}^{II}$ is quantized as long as the Fermi level is in the gap [37]. Both contributions, along with the total anomalous Hall conductivity $\sigma_{xy}$, are plotted as a function of the Fermi energy in figures 2 and 3 in two different cases: when Weyl nodes are present and when they are not. The former occurs when $b_1 < b < b_{2c}$. As can be seen from figure 2, Weyl nodes provide the dominant contribution to $\sigma_{xy}^{II}$ and to the total Hall conductivity $\sigma_{xy}$, if the Fermi level is not too far from the nodes.

It is interesting to note the following property, which is evident from figure 2. Both the total anomalous Hall conductivity $\sigma_{xy}$ and the two distinct contributions to it, $\sigma_{xy}^I$ and $\sigma_{xy}^{II}$, appear to exhibit a quasi-plateau behavior when $\epsilon_F$ is not too far from the Weyl nodes. To understand the origin of this behavior, consider the derivative of $\sigma_{xy}^{II}$ with respect to the Fermi energy

$$
\frac{\partial \sigma_{xy}^{II}}{\partial \epsilon_F} = -\frac{e^2}{8\pi^2} \sum_i \int_{-\pi/d}^{\pi/d} dk_z \Theta(\epsilon_F + |m_i(k_z)|) \delta(\epsilon_F - |m_i(k_z)|).
$$

(59)
This is straightforward to evaluate analytically and we obtain

$$
\frac{\partial \sigma_{xy}^{II}}{\partial \epsilon_F} = -\frac{e^2}{8\pi^2} \int_{-\pi/d}^{\pi/d} dk_z \left[ \delta(\Delta(k_z)) - \theta(\Delta(k_z) - b - \epsilon_F) \right]
- b + \epsilon_F - \theta(\Delta(k_z) - b - \epsilon_F)
= \frac{e^2}{4\pi^2} (1/\bar{v}_{F+} - 1/\bar{v}_{F-}),
$$

(60)
where we have assumed that $\epsilon_F$ is sufficiently close to zero, so that only the $t = +$ bands contribute to the integral and

$$
\bar{v}_{F\pm} = \frac{d}{2(b \pm \epsilon_F)} \sqrt{[(b \pm \epsilon_F)^2 - b_{c1}^2][b_{c2}^2 - (b \pm \epsilon_F)^2]},
$$

(61)
are the two Fermi velocities, corresponding to two pairs of solutions of the equation $|b - \Delta(k_z)| = \epsilon_F$, which arise from the Fermi level crossing the $s = +, t = -$ band on the two sides of each Weyl node along the $z$-axis in momentum space. The two Fermi velocities are nearly equal when the band dispersion near the nodes is almost perfectly linear, but start to differ significantly when deviations from linearity become
Figure 2. (a) Plot of the band edges along the z-direction in momentum space. The parameters are such that two Weyl nodes are present. (b) Total intrinsic anomalous Hall conductivity (solid line), $\sigma_{Iy}$ (dashed line) and $\sigma_{IIy}$ (dotted line). Note that the van Hove-like singularities in $\sigma_{Iy}$ and $\sigma_{IIy}$, associated with band edges, mutually cancel and the total Hall conductivity $\sigma_{xy}$ is a smooth function of the Fermi energy.

Figure 3. (a) Plot of the band edges along the z-direction in momentum space. The spin splitting is not large enough for the Weyl nodes to appear (i.e. $b < b_{c1}$) and the spectrum has a full gap. (b) Total intrinsic anomalous Hall conductivity (solid line), $\sigma_{Iy}$ (dashed line) and $\sigma_{IIy}$ (dotted line).

noticeable. Explicitly, as long as $b_{c1} \ll b \pm \epsilon_F \ll b_{c2}$, both Fermi velocities are independent of the Fermi energy and thus $\partial \sigma_{IIy} / \partial \epsilon_F$ vanishes. A useful way to think about this is in terms of an approximate chiral symmetry, which emerges in this system in the limit of small $\epsilon_F$ and which manifests in an almost perfectly linear band dispersion near the Weyl nodes.

By a nearly identical calculation it is easy to show that $\partial \sigma_{Iy} / \partial \epsilon_F$ also vanishes when $\epsilon_F$ is sufficiently close to zero. This is the origin of the quasi-plateau behavior in figure 2. This result implies that the intrinsic anomalous Hall conductivity is equal to its thermodynamic equilibrium part, $\sigma_{IIy}$, not just when the Fermi energy coincides with the Weyl nodes, but even away from them as long as the band dispersion may be assumed to be linear [49]. This property will be discussed in more detail in section 4.

When attempting to understand the physical meaning of the two contributions to the anomalous Hall conductivity, $\sigma_{Iy}$ and $\sigma_{IIy}$, it may be tempting to say that, while $\sigma_{Iy}$ is clearly associated with states on the Fermi surface, $\sigma_{IIy}$ might perhaps be associated with the chiral Fermi arc edge states. Unfortunately, such an interpretation is clearly incorrect, at least in the context of the present model, since, as can be seen in figure 3, $\sigma_{IIy}$ may be nonzero even when Weyl nodes and thus the Fermi arc edge states are absent. More work is thus needed to fully understand the relation between $\sigma_{IIy}$ and the edge states.

3.3. Microscopic theory: chiral magnetic effect

In this subsection we consider the second part of the topological response in Weyl semimetals, namely the CME. This arises due to the presence of an energy difference between the Weyl nodes, as in equation (34). To induce such an energy difference in our microscopic model, we need to find an operator that acts as an ‘axial chemical potential’ term, shifting the nodes with different chirality in opposite directions in energy. The expression for this operator may be found based on symmetry considerations. We define an ‘axial charge density’ operator, $\hat{n}_a$, as a local operator, which is odd under inversion and $z \rightarrow -z$ reflections, even under time reversal, but odd under time reversal combined with rotation of the spin quantization axis by $\pi$ around either $x$ or $y$ axis. This leads to the following expression for the axial charge density operator

$$\hat{n}_a = \tau^y \sigma^z.$$

Adding a term $-\mu_a \hat{n}_a$, where $\mu_a$ is the ‘axial chemical potential’, to the multilayer Hamiltonian, introduces an energy
difference between the Weyl nodes of magnitude
\[ \Delta \epsilon = \frac{2 \mu_a \tilde{v}_F}{\Delta \delta d}, \]  
(63)

where \( \tilde{v}_F = \frac{d}{2B} \sqrt{(b^2 - b_1^2)(b_2^2 - b^2)} \),

(64)
is the \( z \)-component of the Fermi velocity at the location of the Weyl nodes.

Similarly to the previous subsection, we now couple the electrons to electromagnetic field and the axial chemical potential and integrate out the electron variables to obtain an induced action for the electromagnetic field. We will assume, without loss of generality, that the electromagnetic field consists of a magnetic field in the \( z \)-direction and a vector potential \( \mathbf{A}_z \), whose time derivative gives the \( z \)-component of the electric field \( E_z = -\partial_t A_z \). We will allow for a time and \( z \)-coordinate dependence of the vector potential \( \mathbf{A}_z \) of the axial chemical potential \( \mu_a \), but assume that the magnetic field is time-independent and uniform.

For this calculation it is convenient to use the Landau level (LL) basis of the multilayer placed in a uniform external magnetic field along the growth direction, i.e. the basis of the eigenvectors, i.e. 
\[ \phi_n^{\sigma,\tau}(k) = \sum |\mathbf{k}\rangle |\sigma\rangle |\tau\rangle \],

(65)

Adopting Landau gauge for the vector potential \( \mathbf{A} = x \hat{B} \hat{y} \), we have the following expressions LL eigenstates, which have the form, typical for LLs in Dirac systems
\[ |\mathbf{k}\rangle = \sum_v e^{i k_z z} |v\rangle |\mathbf{k}\rangle, \]

(66)

Here 
\[ \phi_{nk}(\mathbf{r}) \] is the Landau-gauge orbital wavefunctions and \( \sigma \) and \( \tau \) are the spin and pseudospin indices respectively. Finally, the four-component eigenvector \( |\mathbf{z}_{n}^{\alpha}(k)\rangle \) may be written as a tensor product of the two-component spin and pseudospin eigenvectors, i.e. \( |\mathbf{z}_{n}^{\alpha}(k)\rangle = |\mathbf{z}_{n}^{\alpha}(k)\rangle \otimes |\sigma^{\alpha}(k)\rangle \), where
\[ |v_{nk}^{\sigma}(k)\rangle = \frac{1}{\sqrt{2}} \left( 1 + s \frac{m_n(k)}{\epsilon_n(k)} - i \frac{s m_{n+1}(k)}{\epsilon_n(k)} \right), \]
\[ |u_{nk}^{\sigma}(k)\rangle = \frac{1}{\sqrt{2}} \left( 1 + i \frac{\Delta + \Delta \partial e^{-ikdz}}{\Delta(k)} \right), \]

(68)

and the eigenstate energies are given by
\[ \epsilon_n(k) = s \sqrt{2m_B^2 n + m_B^2 (k)} \equiv s\epsilon_n(k). \]

(69)

where \( \omega_B = v_F/\ell_B \) is the Dirac cyclotron frequency and \( \ell_B = 1/\sqrt{\epsilon_B} \) is the magnetic length.

As in all Dirac systems, the lowest \( n = 0 \) LL is special and needs to be considered separately. The \( s \) quantum number is absent in this case and taking \( B > 0 \) for concreteness, we have \( \epsilon_0(k) = -m_1(k) \) and \( |v_0^{\sigma}(k)\rangle = (0, 1) \).

The topological term, of interest to us, is proportional to the product of \( \mu_a \) and \( A_z \). Integrating out the electron variables and leaving only this term in the imaginary time action, we obtain
\[ S = B \sum_{q, i\Omega} \Pi(q, i\Omega) A_z(q, i\Omega) \mu_a(-q, -i\Omega) \]

(70)

where the response function \( \Pi(q, i\Omega) \) is given by:
\[ \Pi(q, i\Omega) = \frac{e}{2\pi \ell_c} \sum_{n,kz} n \eta [\xi_{n\sigma}(kz) - n \eta [\xi_{n\sigma}(kz + q)] \]
\[ + \langle v_{nk}^{\sigma}|v_{nk}^{\tau}\rangle \langle v_{nk}^{\sigma}|v_{nk}^{\tau}\rangle \]

\[ \times \langle v_{nk}^{\sigma}|v_{nk}^{\tau}\rangle \langle v_{nk}^{\sigma}|v_{nk}^{\tau}\rangle \]

(71)

Here \( n \eta \) is the Fermi–Dirac distribution function, \( \xi_{n\sigma}(kz) = \epsilon_{n\sigma}(kz) - \epsilon_F \) and the magnetic field \( B \) in equation (70) arises from the Landau level orbital degeneracy. We have also ignored the \( q \)-dependence of the matrix elements in equation (71), which is not important for small \( q \).

At this point we will specialize to the case of an undoped Weyl semimetal, i.e. set \( \epsilon_F = 0 \). Then it is clear from equation (71) that for Landau levels with \( n \geq 1 \), only terms with \( s \neq s' \) contribute due to the difference of Fermi factors in the numerator. We are interested ultimately in the zero frequency and zero wavevector limit of the response function \( \Pi(q, i\Omega) \). As already seen in the previous section in the context of AHE, the value of \( \Pi(0, 0) \) depends on the order in which the momentum integrals \( \Delta \)

Before we proceed with an explicit evaluation of the \( q \to 0 \) and \( \Omega \to 0 \) limit, let us note an important property of the response function \( \Pi(q, \Omega) \). If we take into account the following symmetry properties of the matrix elements in equation (71)
\[ \langle v_{nk}^{\sigma}|v_{nk}^{\tau}\rangle = -\langle v_{nk}^{\sigma}|v_{nk}^{\tau}\rangle, \]
\[ \langle v_{nk}^{\sigma}|v_{nk}^{\tau}\rangle = \langle v_{nk}^{\sigma}|v_{nk}^{\tau}\rangle. \]

(72)

it is easy to see that when the limit \( \Omega \to 0 \) is taken, independently of the value of \( q \), the \( n \geq 1 \) Landau levels in fact do not contribute at all, mutually cancelling due to equation (72). It follows that \( \Pi(q, i\Omega) \) at small \( \Omega \) is determined completely by the contribution of the two \( n = 0 \) Landau levels, whose energy eigenvalues and the corresponding eigenvectors are independent of the magnetic field. This means that in the small \( \Omega \) limit \( \Pi(q, i\Omega) \) becomes independent of the magnetic field and the effective action in equation (70) then depends linearly on both \( \mu_a \) and \( B \), independently of the magnitude of \( B \).

Let us now proceed to explicitly evaluate \( \Pi(0, 0) \), which determines the low-frequency and long-wavelength response of our system. Let us first look at the situation when we send \( q \) to zero before sending \( \Omega \) to zero. Explicitly evaluating the matrix elements in equation (71) and the momentum integrals we obtain the following simple expression for \( \Pi(0, 0) \)
\[ \Pi(0, 0) = e^2 \frac{2\tilde{v}_F}{4\pi^2 \Delta \delta d}. \]

(73)
Thus, after Wick’s rotation $\tau \rightarrow i\tau$, $\Delta \epsilon \rightarrow -i\Delta \epsilon$, we finally obtain the following result for the electromagnetic field action

$$S = -\frac{\epsilon^2}{4\pi^2} \frac{2\tilde{\gamma}_F}{\Delta_{sd}} B \int d^3r dt \, A_z(r, t) \mu_q(r, t),$$  

(74)

which, taking into account equation (63), has precisely the form of equation (40). Functional derivative of equation (74) with respect to $A_z$ gives the current that flows in response to magnetic field and axial chemical potential, i.e., the CME

$$j_z = -\frac{\epsilon^2}{4\pi^2} B.$$

(75)

The physical interpretation of equation (75) requires some care. The issue is again the order of limits when calculating $\Pi(0, 0)$, which always arises when calculating response functions in gapless systems. Equations (74) and (75) was obtained by sending $q \to 0$ before $\Omega \to 0$. Let us now see what happens when $\Omega$ is sent to zero before taking the limit $q \to 0$. In this case, in addition to the contribution to $\Pi(0, 0)$, given by equation (73), which arises due to transitions between the $t = +$ and $t = -$ lowest ($n = 0$) Landau levels, there is an extra contribution due to the intra-Landau-level processes within the $t = -$ Landau level, which crosses the Fermi energy at the location of the Weyl nodes. This extra contribution is given by

$$\tilde{\Pi}(0, 0) = \frac{\epsilon}{2\pi L_z} \sum_{k_z} \frac{dn_F(\epsilon)}{d\epsilon} |\epsilon_m-\epsilon_{k_z}|^2 \langle \tilde{z}_{\Omega k} | j_z(k_z) \tilde{z}_{\Omega k}^\dagger \rangle \times |\tilde{z}_{\Omega k}^\dagger | \tau_z | \tilde{z}_{\Omega k}^\dagger \rangle,$$

(76)

and is easily shown to be equal to equation (73) in magnitude, but opposite in sign, which means that in this case $\Pi(0, 0)$ vanishes. Thus, the final result for $\Pi(0, 0)$ depends on the order in which the $q \to 0$ and $\Omega \to 0$ limits are taken

$$\lim_{\Omega \to 0} \lim_{q \to 0} \Pi(q, \Omega) = -\frac{\epsilon^2}{4\pi^2} \frac{2\tilde{\gamma}_F}{\Delta_{sd}},$$

$$\lim_{q \to 0} \lim_{\Omega \to 0} \Pi(q, \Omega) = 0.$$

(77)

What is the physical meaning of these two distinct orders of limits in calculating $\Pi(0, 0)$? This is again identical to the AHE case, discussed in the previous section. When $q$ is taken to zero first, one is calculating the low-frequency limit of response to a time-dependent external field, in our case magnetic field along the $z$-direction. This response is finite and represents CME, described by equation (75). If one takes $\Omega$ to zero first, however, one is calculating a thermodynamic property, in our case change of the ground state energy of the system in the presence of an additional static vector potential in the $z$-direction

$$j_z = \frac{1}{V} \frac{\partial E(A_z)}{\partial A_z} = 0.$$

(78)

This could be nonzero in, for example, a current-carrying state of a superconductor, which possesses phase rigidity, but vanishes identically in our case [50, 51].

4. Diffusive transport in Weyl metals

In the previous section we have focused on the basic properties of electromagnetic response of clean Weyl semimetals and metals (doped Weyl semimetals), neglecting the effect of impurity scattering. It is, however, always present and while probably not of significant importance in an undoped Weyl semimetal with density of states at the Fermi energy vanishing as $\sim e_\alpha^2$, it may be expected to play an important role in a Weyl metal with $e_F$ significantly different from zero. If nothing else, impurity scattering is crucial for establishing a steady state under applied electric field and thus must be included in any serious discussion of low-frequency transport phenomena. In this section we will thus generalize the theory of electromagnetic response, presented in section 3, to the case of diffusive transport in dirty Weyl metals. As will be demonstrated below, this generalization leads to two important new results.

First, we will demonstrate that the anomalous Hall conductivity of a Weyl metal is purely intrinsic and universal, lacking both the so-called extrinsic contribution due to the impurity scattering and also the part of the intrinsic electronic-structure contribution, coming from incompletely filled bands. Only the purely intrinsic semi-quantized contribution, $\sigma_{xy}^{\text{II}}$, arising from completely filled bands and proportional to the separation between the Weyl nodes, survives in the Weyl metal, as long as the Fermi energy is close enough to the Weyl nodes, such that the band dispersion at the Fermi energy may be assumed to be linear.

Second, we show that generalization of the chiral magnetic effect to the case of diffusive transport leads to a novel weak-field magneto-resistance effect, negative and quadratic in the magnetic field. We argue that this effect may be regarded as a universal smoking-gun transport characteristic of all Weyl metals.

4.1. Anomalous Hall effect in dirty Weyl metals

We start again from the heterostructure model of Weyl semimetal described in detail in the previous sections. We would like to evaluate the anomalous Hall conductivity of this model ferromagnetic Weyl metal, in the presence of impurity potential $V(r) = V_0 \sum_\delta (\delta(r - r_i))$, which we will assume for simplicity to be gaussian, with only second order correlators present: $\langle V(r) V(r') \rangle = \gamma^2 \delta(r - r')$, where $\gamma^2 = n_i V_0^2$ and $n_i$ is the impurity density. The higher-order correlators, which are known to be important for AHE in principle, as they give rise to the so-called skew-scattering contribution to the Hall conductivity, do not in fact affect our results in a significant way. We will also assume that the impurity potential is diagonal in both the pseudospin $\tau$ and the $t = \pm$ index, which labels the eigenstates of the $\hat{\Delta}(k_z)$ operator equation (19). Again, this assumption is used only for computational simplicity and does not affect the essence of our results. To find the anomalous Hall conductivity, we will use the method of section 3.2, which we find to be the most convenient one for our purposes, as it allows to most clearly separate physically distinct contributions to the AHE. Namely,
we imagine coupling electromagnetic field to the electrons and integrating the electron degrees of freedom out to obtain an effective action for the electromagnetic field, which, at quadratic order, describes the linear response of the system. The part of this action we are interested in has the appearance of a Chern–Simons term, which, adopting the Coulomb gauge for the electromagnetic vector potential $\mathbf{A} = 0$, is given by
\begin{equation}
S = \sum_{q, i \Omega} e^{i q a \beta} \Pi(q, i \Omega) A_0(-q, -i \Omega) \hat{a}_q A_\beta(q, i \Omega), \tag{79}
\end{equation}
where $\hat{a}_q = q_a / q$ and summation over repeated indices is implicit. The $z$-direction in equation (79) is again picked out by the magnetization $b$. As we will be interested specifically in the zero frequency and zero wavevector limits of the response function $\Pi(q, i \Omega)$, we will assume henceforth that $q = q \hat{x}$, which does not lead to any loss of generality due to full rotational symmetry in the $xy$-plane. The anomalous Hall conductivity is given by the zero frequency and zero wave vector limit of the response function $\Pi(q, i \Omega)$ as
\begin{equation}
\sigma_{xy} = \lim_{\Omega \to 0} \lim_{q \to 0} \frac{1}{q^2} \Pi(q, i \Omega). \tag{80}
\end{equation}
A significant advantage of equation (80), compared to the more standard Kubo formula for the anomalous Hall conductivity, which relates it to the current–current correlation function, is that equation (80) ties the Hall conductivity to the response of a conserved quantity, i.e., the particle density. This means, in particular, that the response function $\Pi(q, i \Omega)$ must satisfy exact Ward identities, which follow from charge conservation, providing a useful correctness check on the results.

The impurity average of the response function $\Pi(q, i \Omega)$ may be evaluated by the standard methods of diagrammatic perturbation theory. Due to our assumption that the impurity potential is diagonal in the band index $i$, we can do this calculation separately for each pair of bands, labeled by $i$ and then simply sum the individual contributions. We will thus omit the $t$ index in what follows, until we come to the final results. The retarded impurity averaged one-particle Green’s functions have the following general form
\begin{equation}
G^R_{\sigma \sigma_0}(k, \epsilon) = \frac{\delta(k_x - k_x', k_y + k_y') \delta(\epsilon - \epsilon_F)}{\epsilon - \xi^k_{\sigma} + i/2 \tau_e}. \tag{81}
\end{equation}
Here $s = \pm$, as before, labels the positive and negative energy pairs of bands (the sum over $s$ is made implicit above)
\begin{equation}
\xi^k_{\sigma} = s \epsilon_k - \epsilon_F = s \sqrt{\frac{\epsilon^2}{2} (k_x^2 + k_y^2) + m^2(k_z) - \epsilon_F}, \tag{82}
\end{equation}
are the band energies, counted from the Fermi energy $\epsilon_F$ and
\begin{equation}
|\xi^k_{\sigma}| = \frac{1}{\sqrt{2}} \left( 1 + s \frac{m(k_z)}{\epsilon_k}, -i s e^{i \phi} \left( 1 - s \frac{m(k_z)}{\epsilon_k} \right) \right), \tag{83}
\end{equation}
is the corresponding eigenvector. In what follows we will assume, for concreteness, that $\epsilon_F > 0$, i.e. the Weyl metal is electron-doped. The impurity scattering rates $1/\tau_{\sigma \sigma_0}$ are given, in the Born approximation, by
\begin{equation}
\frac{1}{\tau_{\sigma}(k_z)} = \frac{1}{\tau} \left[ 1 + s \frac{m(k_z)}{\epsilon_F} \right], \tag{84}
\end{equation}
where $1/\tau = \pi y^2 g(\epsilon_F)$ and
\begin{equation}
g(\epsilon_F) = \int \frac{d^3k}{(2\pi)^3} \delta(\epsilon_k - \epsilon_F) = \frac{\epsilon_F}{4\pi^2 v_F} \int_{-\pi/d}^{\pi/d} d\zeta \Theta(\epsilon_F - |m(k_z)|), \tag{85}
\end{equation}
is the density of states at Fermi energy. We have also defined the average of $m(k_z)$ over the Fermi surface as
\begin{equation}
\langle m \rangle = \frac{1}{g(\epsilon_F)} \int \frac{d^3k}{(2\pi)^3} m(k_z) \delta(\epsilon_k - \epsilon_F). \tag{86}
\end{equation}
The impurity averaged response function, analytically continued to real frequency as $\Pi(q, \Omega) = \Pi(q, i \Omega - \Omega + i\eta)$, is given, in the self-consistent non-crossing approximation, by the sum of ladder diagrams, as illustrated in figure 4
\begin{equation}
\Pi(q, \Omega) = \Pi^I(q, \Omega) + \Pi^{II}(q, \Omega), \tag{87}
\end{equation}
where
\begin{equation}
\Pi^I(q, \Omega) = 2e^2 v_F \Omega \int_{-\infty}^{\infty} \frac{d\eta}{2\pi i} P_{\Omega x}(q, \epsilon - i\eta, \epsilon + \Omega + i\eta), \tag{88}
\end{equation}
and
\begin{equation}
\Pi^{II}(q, \Omega) = 4ie^2 v_F \int_{-\infty}^{\infty} \frac{d\eta}{2\pi i} P_{\Omega x}(q, \epsilon + i\eta, \epsilon + \Omega + i\eta). \tag{89}
\end{equation}
The $4 \times 4$ matrix $P$, whose $0x$ component we are interested in, is given by
\begin{equation}
P(q, -i\eta, \Omega + i\eta) = g^{-2} I^{RA}(q, \Omega) D(q, \Omega), \tag{90}
\end{equation}
where
\begin{equation}
D = (1 - I^{RA})^{-1}, \tag{91}
\end{equation}
is the diffusion propagator and
\begin{equation}
I^{RA}_{\alpha\beta}(q, \Omega) = \frac{\alpha^2}{2} R_{\sigma_0\sigma_1} R_{\sigma_0\sigma_1} \times \int \frac{d^3k}{(2\pi)^3} G^R_{\sigma_0\sigma_1}(k + q, \Omega) G^A_{\sigma_0\sigma_1}(k, 0), \tag{92}
\end{equation}
is given by
\begin{equation}
I^{RA}_{\alpha\beta}(\epsilon, q, \Omega) = \frac{1}{2} \epsilon^\alpha_{\sigma_0\sigma_1} \epsilon^\beta_{\sigma_0\sigma_1} \times \int \frac{d^3k}{(2\pi)^3} G^R_{\sigma_0\sigma_1}(k + q, \epsilon + \Omega) G^R_{\sigma_0\sigma_1}(k, \epsilon). \tag{92}
\end{equation}
The physical meaning of the two distinct contributions to the response function $\Pi_{AA}'(q, \Omega)$ is clear from equations (88) and (89). $\Pi_{AA}'(q, \Omega)$ describes the non-equilibrium part of the response that happens at the Fermi surface, as clear from the appearance of the derivative of the Fermi–Dirac distribution function in equation (88). This response is diffusive when $\Omega \tau \ll 1$ and ballistic in the opposite limit and is generally affected significantly by the impurity scattering. We will discuss this in more detail below. In contrast, $\Pi_{AA}^\text{dif}(q, \Omega)$ is an equilibrium, nondissipative contribution to the overall response, to which all states below the Fermi energy contribute and which will be seen below.

We will start by evaluating the nonequilibrium part of the response function, $\Pi_{AA}'(q, \Omega)$. Computation of the matrix elements $I_{AA}^R(q, \Omega)$ is easily done in the standard way, assuming $eF \tau \gg 1$. One obtains

$$\Pi_{AA}'(q, \Omega) = i e^2 v_F e_F \tau g(e_F)[I_{00}^R D_{00} + I_{0z}^R D_{0z} + I_{zz}^R D_{zz}],$$

(93)

where we have taken into account that $D_{yz} = 0$ by symmetry. The relevant matrix elements of the diffusion propagator can be found analytically to first order in $q$. One obtains

$$\Pi_{AA}'(q, \Omega) = e^2 v_F e_F \tau g(e_F)\frac{I_{RA}^R}{\Gamma(1 - I_{RA}^R)} = \frac{m}{2eF} \frac{\tau_{x}/\tau - 1}{1 - i\Omega \tau_{x}}, \quad I_{RA}^R = i v_F q \left( \frac{m}{eF} \frac{\tau_{x}/\tau}{1 - i\Omega \tau_{x}} \right),$$

(94)

where $\Gamma(q, \Omega) = (1 - I_{RA}^R)(1 - I_{RA}^R) - I_{RA}^R I_{RA}^R$ is the determinant of the $0z$ block of the diffusion propagator, which corresponds to the diffusion of the charge density, a conserved quantity (this block decouples from the rest of the diffusion when $q \to 0$). This means, in particular, that $\Gamma$ must satisfy an exact Ward identity $\Gamma(0, 0) = 0$, which is easily checked to be true. Explicitly, the relevant matrix elements of $I_{AA}^R(q, \Omega)$ to first order in $q$ are given by

$$I_{00}^R = \frac{\tau_{x}/\tau}{1 - i\Omega \tau_{x}}, \quad I_{0z}^R = i v_F q \frac{m}{eF} \frac{\tau_{x}/\tau}{1 - i\Omega \tau_{x}},$$

$$I_{zz}^R = \frac{m}{eF} \frac{\tau_{x}/\tau}{1 - i\Omega \tau_{x}}, \quad I_{zz}^R = \frac{m}{eF} \frac{\tau_{x}/\tau}{1 - i\Omega \tau_{x}},$$

(96)

$$I_{xx}^R = \frac{1}{2} \left( \frac{m}{eF} \frac{\tau_{x}/\tau}{1 - i\Omega \tau_{x}} \right),$$

where the average over the Fermi surface is defined in the same way as in equation (86).

Expanding $\Gamma(0, \Omega)$ to first order in $\Omega$ and taking the limit $\Omega \to 0$ at fixed $\tau$, which corresponds to the diffusive limit, we obtain

$$\Pi_{AA}'(q, 0) = i q e^2 v_F^2 \frac{g(e_F)}{2eF} \frac{m}{eF \tau} F[m],$$

(97)

where

$$F[m] = \frac{1 + \frac{1}{\tau} \left( \frac{1 - m^2 / e_F^2}{\tau} \right)}{1 - \frac{1}{\tau} \left( \frac{1 - m^2 / e_F^2}{\tau} \right)} \left[ \frac{\partial \Gamma(0, \Omega)}{\partial (\Omega \tau)} \right]_{\Omega = 0}^{-1}.$$

(98)

The explicit form of the functional $F[m]$ is in fact not that important for our purposes, except for the evenness property, easily seen from equation (98): $F[-m] = F[m]$. As a consequence, $\Pi_{AA}^\text{dif}(q, 0)$ is an odd functional of $m$, which will play an important role below. It is important to note that the charge conservation, whose mathematical consequence is the presence of the diffusion pole in $\Pi_{AA}'(q, \Omega)$, is crucial in obtaining a nonzero result in the diffusive limit in equation (97). The analogous quantity in the calculation of the spin Hall conductivity, for example, would vanish in the diffusive limit [52–55].

It is also of interest to examine the ballistic limit of $\Pi_{AA}'$, which corresponds to the case of a clean Weyl metal. In this case we send both $\Omega$ and $1/\tau$ to zero in such a way that $\Omega \tau \to \infty$. In this case we obtain

$$\Pi_{AA}^\text{bal}(q, 0) = \lim_{\Omega \to 0, 1/\tau \to 0} \Pi_{AA}'(q, \Omega) = \lim_{\Omega \to 0, 1/\tau \to 0} i q e^2 v_F^2 g(e_F) I_{00}^R = -\frac{i q e^2 v_F^2 g(e_F)}{2eF} \frac{m}{eF \tau},$$

(99)

which agrees with the clean Weyl metal result equation (56). A nice feature of the present calculation, compared with the calculation in a clean Weyl metal in section 3.2, is that here it is particularly clear that $\Pi_{AA}'(q, \Omega)$ is associated with states on the Fermi surface. As seen from equations (97) and (99), the difference between $\Pi_{AA}^\text{diff}$ and $\Pi_{AA}^\text{bal}$ is only quantitative. In the AHE literature, this difference is said to arise from the so-called side-jump processes [55–59]. The final step of the calculation is to evaluate the equilibrium part of the response function, $\Pi_{AA}^\text{eq}(q, \Omega)$. In the limit $eF \tau \gg 1$ one finds that this part of the response function is unaffected by the impurity scattering and is given by

$$\Pi_{AA}^\text{eq}(q, \Omega) = e^2 v_F \int \frac{d^3k}{(2\pi)^3} \langle \xi_s^x | z^x_k | \xi_s^y \rangle \langle z^y_k | v^x | z^x_k \rangle,$$

(100)

where summation over the band indices $s, s'$ is again implicit. Evaluating equation (100) in the small $\Omega$ and $q$ limit gives

$$\Pi_{AA}^\text{eq}(q, 0) = -\frac{i q e^2 v_F^2}{8\pi^2} \int_{-\pi/4}^{\pi/4} dk \text{sign}(m(k_c)) \times [1 - \Theta[|m(k_c)| - |m(k)|]]$$

(101)

The first term in equation (101) arises from the completely filled bands, while the second is the contribution of the incompletely filled bands.

We can now finally evaluate the anomalous Hall conductivity. We will focus on the diffusive limit results, as ballistic limit has already been discussed in section 3. At this point we will also explicitly include the contribution of both
t = ± pairs of bands, which simply amounts to restoring the index t in $m_t$ and summing over t. Using equations (80), (97) and (101) and remembering that $A_0 \rightarrow i A_0$ upon Wick rotation to the real time, we obtain

$$\sigma_{xy}^I = \epsilon^2 v_F^2 \sum_i g_i(\epsilon_F) \left( \frac{m_t \tau_t}{\epsilon_F \tau_F} \right) \Sigma[m_t],$$

and

$$\sigma_{xy}^{II} = \frac{\epsilon^2}{8\pi^2} \sum_i \int_{-\pi/d}^{\pi/d} dk_z \text{sign}[m_t(k_z)]$$

$$\times \{1 - \Theta[\epsilon_F - |m_t(k_z)|]\}.$$

Since $m_+(k_z)$ is positive throughout the first BZ, while $m_-(k_z)$ changes sign at the Weyl nodes, the first term in equation (103), which comes from completely filled bands, gives a universal (almost) quantized contribution

$$\sigma_{xy}^{\text{quant}} = \frac{\epsilon^2 K}{4\pi^2},$$

where $K$ is the distance between the Weyl nodes, which is the same as the clean well metal result, equation (57). This equation also describes the cases when the Weyl nodes are absent, in which case $\sigma_{xy}^{\text{quant}}$ is truly quantized since $K = 0$, $G$, where $G = 2\pi/d$ is a reciprocal lattice vector.

The most important new result of this subsection comes from examining the remaining, non-quantized parts of $\sigma_{xy}$. Suppose we have a situation when the Weyl nodes are present and $\epsilon_F$, while not zero, is not too far from it, as shown in figure 5. Recall that at the location of the Weyl nodes $m_-(k_z) = b - \Delta(k_z)$ changes sign. This implies that, as long as $K \frac{d\sigma}{dk_z} \bigg|_{k_z = 0} \gg \epsilon_F$, where $k_0$ is the location of a given Weyl node, the average of any odd function of $m_-(k_z)$ over the Fermi surface will vanish. This means that in such a situation, which we call Weyl metal, all contributions to the anomalous Hall conductivity, associated with incompletely filled bands, will vanish and $\sigma_{xy}$ attains a universal value, characteristic of Weyl semimetal $\sigma_{xy} = \sigma_{xy}^{\text{quant}}$, where $\sigma_{xy}^{\text{quant}}$ is given by equation (104). This may be seen explicitly in figure 6. Note that the linear dispersion sufficiently close to Weyl nodes is a topological property, in the sense that it follows directly and exclusively from the existence of a nonzero topological charge by the so-called Atiyah–Bott–Shapiro construction [17].

To understand this result physically, recall that the Weyl nodes are monopole sources of the Berry curvature $\Omega_k$. In a clean metal, the anomalous Hall conductivity $\sigma_{xy}$ is given by the integral of the z-component of the Berry curvature over all occupied states $\sigma_{xy} = \epsilon^2 \int \frac{d\Omega}{(2\pi)^3} m_F(\epsilon_k) \Omega_k$. However, as clear from figure 6, when the Fermi surface breaks up into disconnected sheets, enclosing individual nodes, the contribution of the states, enclosed by the Fermi surface, to this integral will always be very small, vanishing exactly in the limit when the band dispersion away from the nodes may be taken to be exactly linear. A useful analogy here is with the electric field of a dipole. A pair of Weyl nodes is like a dipole of two topological charges. Its field has a well-defined and nonzero on average z-component at large distances from the dipole. At short distances, however, the field is that of individual charges, which winds around the location of each charge and thus any particular component of it averages to zero.

We have thus demonstrated that the AHE in Weyl metals has a purely intrinsic origin and can be associated entirely with the Weyl nodes, just as in the case of a Weyl semimetal, when the Fermi energy coincides with the nodes and the Fermi surface is absent [60]. This is in contrast to an ordinary ferromagnetic metal, in which the anomalous Hall conductivity always has both a significant Fermi surface contribution and an extrinsic contribution. This property of magnetic Weyl metals may be thought of as being a consequence of emergent chiral symmetry, as discussed in section 3.2.
1. The near-conservation of the axial charge density relevant relaxation time may in fact still be long, even when it is not conserved. However, as will be shown below, the magnetic field in the z direction $B = B z$ and scalar impurity potential $V(r)$, whose precise form will be specified later. Adopting Landau gauge for the vector potential $A = \lambda B \hat{y}$, the second-quantized Hamiltonian of our system may be written as

$$H = \sum_{n a k z} \epsilon_{n a}(k z) c_{n a k z}^\dagger c_{n a k z} + \sum_{n a k z, n a' k' z} \langle n, a, k z | V | n', a', k' z \rangle c_{n a k z}^\dagger c_{n a' k' z}.$$

Here $\epsilon_{n a}(k z)$ are Landau-level (LL) eigenstate energies of a clean multilayer in magnetic field, $n = 0, 1, 2, \ldots$ is the main LL index, $k_z$ is the Landau-gauge intra-LL orbital quantum number, $k_z$ is the conserved component of the crystal momentum along the z-direction and $a = (s, t)$ is a composite index (introduced for compactness of notation), consisting of $s = \pm$, which labels the electron- ($s = +$) and hole- ($s = -$) like sets of Landau levels and $t = \pm$, which labels the two components of a Kramers doublet of LLs, degenerate at $b = 0$. The LL eigenstate energies and the corresponding eigenvectors have been derived in section 3.3.

To proceed with the derivation, we add a constant uniform magnetic field in the $\hat{z}$ direction $B = B z$ and scalar impurity potential $V(r)$, whose precise form will be specified later. Adopting Landau gauge for the vector potential $A = \lambda B \hat{y}$, the second-quantized Hamiltonian of our system may be written as

$$H = \sum_{n a k z} \epsilon_{n a}(k z) c_{n a k z}^\dagger c_{n a k z} + \sum_{n a k z, n a' k' z} \langle n, a, k z | V | n', a', k' z \rangle c_{n a k z}^\dagger c_{n a' k' z}.$$

Here $\epsilon_{n a}(k z)$ are Landau-level (LL) eigenstate energies of a clean multilayer in magnetic field, $n = 0, 1, 2, \ldots$ is the main LL index, $k_z$ is the Landau-gauge intra-LL orbital quantum number, $k_z$ is the conserved component of the crystal momentum along the z-direction and $a = (s, t)$ is a composite index (introduced for compactness of notation), consisting of $s = \pm$, which labels the electron- ($s = +$) and hole- ($s = -$) like sets of Landau levels and $t = \pm$, which labels the two components of a Kramers doublet of LLs, degenerate at $b = 0$. The LL eigenstate energies and the corresponding eigenvectors have been derived in section 3.3.

To proceed, we will make the standard assumption, which we also made in the previous subsection, that the impurity potential obeys Gaussian distribution, with $\langle V(r) V(r') \rangle = \gamma^2 \delta(r - r')$. To simplify calculations further we will also assume that the momentum transfer due to the impurity scattering is smaller than the size of the BZ, i.e. $|k_z - k'_z|d \ll 1$. In this case $(u'(k_z)|u'(k'_z)) \approx b_{n a t}$, i.e. the $t$ quantum number may be assumed to be approximately preserved during the impurity scattering. We treat the impurity scattering in the standard self-consistent Born approximation (SCBA), illustrated in figure 7. The retarded SCBA self-energy satisfies the equation

$$\Sigma^R_{n a k z}(\omega) = \frac{1}{L} \sum_{n a' k' z} \langle |n, a, k z | V | n', a', k' z \rangle^2 \delta^{\text{str}}(\omega, \omega' - \epsilon_{n a}(k z)).$$

We will assume that the Fermi energy $\epsilon_F$ is positive, i.e. the Weyl semimetal is electron-doped and large enough that the impurity-scattering-induced broadening of the density of states is small on the scale of the Fermi energy $\epsilon_F$.

1 For a theory of transport in Weyl semimetals in the limit $\epsilon_F \to 0$ see [61].
functions $G^{R,A}$ are given by

$$G_{aa'}^{R,A}(\bf r, \bf r' | \Omega) = \sum_{nk, k'} \langle \bf r | n, t, k, z_n \rangle \langle t, k, z_{n'} | \bf r' \rangle \frac{\delta(\Omega - \xi_{nt}(k_z) \pm i/2\tau(k_z))}{\Omega - \xi_{nt}(k_z) \pm i/2\tau(k_z)}, \tag{113}$$

where $\xi_{nt}(k_z) = \epsilon_{nt}(k_z) - \epsilon_F$.

In general, the evaluation of equation (112) is a rather complicated task, primarily due to the fact that the impurity scattering will mix different LLs. At this point we will thus specialize to the case of transport along the $z$-direction only, as this is where we can expect chiral anomaly to be manifest. In this case the contributions of different LLs to equation (112) decouple. Setting $\bf q = q \hat{z}$, we obtain

$$I_{a_1 a_2, a_2 a_3}(q, \Omega) = \frac{\gamma^2}{2\pi \varepsilon_F^2} \int_{-\pi/d}^{\pi/d} \frac{dk_z}{2\pi} \sum_{nt} m_n(k_z) \delta[\epsilon_{nt}(k_z) - \epsilon_F]. \tag{114}$$

which is much easier to evaluate.

As mentioned above, $I$ and $D^{-1}$ are large $16 \times 16$ matrices, which contain a lot of information of no interest to us. We are interested only in hydrodynamic physical quantities, with long relaxation times. All such quantities need to be identified, if they are expected to be coupled to each other. One such quantity is obviously the total charge density $n(r, t)$, which has an infinite relaxation time due to the exact conservation of particle number. Another is the axial charge density $\sum_{n,t} m_n(k_z)$, which, as discussed above, may be almost conserved under certain conditions. On physical grounds, we expect no other hydrodynamic quantities to be present in our case. We are thus only interested in the $2 \times 2$ block of the matrix $D^{-1}$, which corresponds to the coupled evolution of the total and the axial charge densities. To separate out this block, we apply the following transformation to the inverse diffusion matrix

$$D^{-1}_{a_1 b_1, a_2 b_2} = \frac{1}{2} (\sigma^{a_1} \sigma^{b_1})_{a_1 a_2} D^{-1}_{a_1 a_2, a_1 a_3} (\sigma^{a_1} \sigma^{b_2})_{a_2 a_3}, \tag{115}$$

where $a_1 b_2 = x, y, z$. The components of interest to us are $a_1 = b_1 = 0$, which corresponds to the total charge density, $a_1 = 0, b_2 = y$, which corresponds to the axial charge density and the corresponding cross-terms.

We will be interested, as mentioned above, in the hydrodynamic regime, which corresponds to low frequencies and long wavelengths, i.e. $\Omega \tau \ll 1$ and $v_F q \tau \ll 1$. We will also assume that the magnetic field is weak, so that $v_B \ll v_F$. Finally, we will assume that the Fermi energy is close enough to the Weyl nodes, so that only the $t = -$ states participate in transport and $m_{-}(k_z) \approx 0$, since $m_{-}(k_z)$ changes sign at the nodes [49].

In accordance with the above assumptions, we expand the inverse diffusion propagator to leading order in $\Omega \tau$, $v_F q \tau$ and $v_B / \epsilon_F$ and obtain after a straightforward but lengthy calculation

$$D^{-1}(q, \Omega) = \begin{pmatrix} -\Omega \tau + D q^2 \tau & -i q \Gamma \tau \\ -i q \Gamma \tau & -\Omega \tau + D q^2 \tau + \tau / \tau_a \end{pmatrix}, \tag{116}$$

or
Here $D = \tilde{v}_F^2 \tau (m^*_{\perp}/\epsilon_F^2)$ is the charge diffusion constant, associated with the diffusion in the $z$-direction, $\Gamma = eB/2\pi^2 g(\epsilon_F)$ is the total charge-axial charge coupling coefficient and

$$1/\tau_a = 1 - (\tilde{v}_F/\Delta s^2)^2 \frac{(v_F/\Delta s^2)^2}{\tau},$$

(117)

is the axial charge relaxation rate. Several comments are in order here. First, an important thing to note is that the off-diagonal matrix elements in $D^{-1}$, proportional to $B$ and responsible for the total charge to axial charge coupling, come entirely from the contribution of the $n = 0$ LL. The remaining matrix elements arise from the contribution of all the $n \geq 1$ Landau levels and we have taken the limit $B \rightarrow 0$ after summing over the LLs, i.e. left only the leading term in the $\omega_B/\epsilon_F$ expansion. The next-to-leading term results in a negative correction to the diffusion coefficient, proportional to $B^2$, which corresponds to the well-known classical positive magnetoresistance. We have ignored this correction here, but will comment on its effects later. Second, we note that the axial charge relaxation rate $1/\tau_a \geq 0$, as it should be and vanishes when $\tilde{v}_F = \Delta s d$. It is easy to see that this is identical to the condition of the vanishing of the commutator of the axial charge operator with the Hamiltonian equation (105), again as it should be. Henceforth we will assume that this condition is nearly satisfied so that $\tau_a \gg \tau$. Finally, the situation when $\tilde{v}_F = \Delta s d$ and thus $1/\tau_a$ appears to vanish, actually needs to be treated with some care. Namely, the condition $\tilde{v}_F = \Delta s d$ may be satisfied exactly only in the limit $\epsilon_F \rightarrow 0$. The Fermi velocity depends on the Fermi energy as [71]

$$\tilde{v}_F(\epsilon_F) = \frac{d}{2(b + \epsilon_F)} \sqrt{[b(\epsilon_F)]^2 - b^2_c} \sqrt{[(b + \epsilon_F)^2 - b^2_c] - (b + \epsilon_F)^2}.$$

(118)

When $b = \sqrt{b_c b_c}$ and thus $\tilde{v}_F(0) = \Delta s d$, the Fermi energy dependence of $\tilde{v}_F$ needs to be taken into account. Expanding to leading non-vanishing order in $\epsilon_F$ we obtain in this case

$$1/\tau_a = \frac{\epsilon_F^2}{\Delta s^2},$$

(119)

i.e. $1/\tau_a$ is in fact always finite, but may be very small. We can estimate the minimal value of the axial charge relaxation rate by setting $\epsilon_F \approx 1/\tau$ in equation (119), which gives $(\tau/\tau_a)_{\min} \approx 1/(\Delta s^2)^2$.

We may now write down the coupled diffusion equations for the total and axial charge densities, which correspond to the propagator equation (116). These equations read

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial z^2} + \frac{\Gamma}{4} \frac{\partial n_a}{\partial z},$$

$$\frac{\partial n_a}{\partial t} = D \frac{\partial^2 n_a}{\partial z^2} - n_a + \frac{\Gamma}{4} \frac{\partial n}{\partial z},$$

(120)

Manifestation of chiral anomaly in these equations is the coupling between the total and the axial charge densities, proportional to the applied magnetic field. Since the total particle number is conserved, the right-hand side of the first of equations (120) must be equal to minus the divergence of the total particle current. Then we obtain the following expression for the density of the charge current in the $z$-direction

$$j = \sigma_0 \frac{\partial \mu}{\partial z} \frac{\epsilon_F^2}{2\pi^2 \mu_a},$$

(121)

where $\sigma_0 = \epsilon_F^2 g(\epsilon_F) D$ is the zero-field diagonal charge conductivity, $\mu$ and $\mu_a$ are the total and axial electrochemical potentials and we have used $\partial n = g(\epsilon_F) \partial \mu$, $\partial n_a = g(\epsilon_F) \partial \mu_a$. The last relation is valid when $\tilde{v}_F/\Delta s d$ is close to unity, as seen from equation (63). Thus chiral anomaly manifests in an extra contribution to the charge current density, proportional to the magnetic field and the axial electrochemical potential. This is known as chiral magnetic effect (CME) in the literature [39]. It is important to realize that the second term in equation (121) does not by any means imply that an equilibrium current may be driven by an applied magnetic field, despite appearances. The axial chemical potential $\mu_a$, appearing in equation (121), is a purely nonequilibrium quantity. If an equilibrium energy difference, $\mu_{a0}$, exists between the Weyl nodes due to explicitly broken inversion symmetry [66], then it is the difference $\mu_a - \mu_{a0}$ that enters in equation (121). We have explicitly considered an inversion-symmetric Weyl metal here, in which case $\mu_{a0} = 0$.

To find measurable consequences of the CME contribution to the charge current, we consider a steady-state situation, with a fixed current density $j$ flowing through the sample in the $z$-direction. We want to find the corresponding electrochemical potential drop and thus the conductivity. Assuming the current density is uniform, we obtain from the second of equation (120)

$$\frac{n_a}{\sigma_0} = \Gamma \tau_a \frac{\partial n}{\partial z},$$

(122)

which is the nonequilibrium axial charge density, induced by the current and the corresponding electrochemical potential gradient. Substituting this into the expression for the charge current density equation (121), we finally obtain the following result for the conductivity

$$\sigma = \sigma_0 + \frac{e^2 B^2 \tau_a}{4\pi^2 \epsilon_F},$$

(123)

In the limit when $\epsilon_F$ is not far from the Weyl nodes, such that the dispersion may be assumed to be linear, we have $g(\epsilon_F) = \epsilon_F^2/\pi^2 \tilde{v}_F^2$, which gives

$$\Delta \sigma = \sigma - \sigma_0 \approx -\frac{e^2 \tilde{v}_F \tau_a}{2\pi^2 \epsilon_F} \left( \frac{e^2 \tilde{v}_F^2 B}{\epsilon_F} \right)^2,$$

(124)

which agrees with the Son and Spivak result [62,63]. Thus we see that a measurable consequence of CME is a positive magnetoconductivity, proportional to $B^2$ in the limit of a weak magnetic field. This of course needs to be compared with the classical negative magnetoconductivity, which is always present and arises from the $B^2$ corrections to the diffusion constant $D$, which we have neglected

$$\frac{\Delta \sigma}{\sigma_0} \sim -(\omega_\tau)^2,$$

(125)
where $\omega_c = eB/\hbar$ is the cyclotron frequency. This gives

$$\frac{\Delta \sigma}{\Delta \sigma_{cl}} \sim \frac{\tau_a}{\tau} \left(\frac{eB}{\hbar}\right)^2. \tag{126}$$

Thus the CME-related positive magnetoconductivity will dominate the classical negative magnetoconductivity, provided $\tau_a$ is long enough.

We have so far ignored the Zeeman effect due to the applied magnetic field. Its effect is to modify the spin-splitting parameter $b$ as $b \rightarrow b + g\mu_B B/2$. In principle, the dependence on $b$ does enter into our final results through the dependence of the Fermi velocity $\tilde{v}_F$ on $b$. Naively, this will then generate an additional linear magnetoconductivity, which may be expected to dominate the quadratic one at small fields. However, the condition of large $\tau_a$, which is the same as $\tilde{v}_F/\Delta s \approx 1$, is equivalent to the condition $b_{1,1} \ll b \ll b_{2,2}$, in which case the dependence of $\tilde{v}_F$ on $b$ becomes negligible. Thus, in the regime in which the positive magnetoconductivity dominates the negative classical one and is thus observable, one may also expect a negligible linear magnetoconductivity in any type of Weyl metal.

## 5. Conclusions

In this paper we have provided an overview of transport phenomena in Weyl metals, which may be attributed to chiral anomaly, in particular the semi-quantized intrinsic Anomalous Hall Effect and the Chiral Magnetic Effect, which manifests in negative quadratic in the magnetic field longitudinal magnetoresistance.

The AHE is generally present in any ferromagnetic metal. In a generic FM metal this is a complicated phenomenon, with many physically distinct sources contributing to the final observed effect. In particular, extrinsic contribution to AHE, arising from impurity scattering, is always at least of the same order of magnitude and often significantly larger, than the more theoretically appealing and universal intrinsic one. In a Weyl metal, however, as we have demonstrated in this paper, AHE has a purely intrinsic origin. As long as the Fermi energy is not too far from the Weyl nodes, such that the band dispersion may be taken to be linear and the chiral symmetry is present to a good approximation, the anomalous Hall conductivity of a Weyl metal turns out to be given exactly by the semi-quantized expression of a pure undoped Weyl semimetal, where it simply becomes linear and the chiral symmetry is present to a good approximation, the anomalous Hall conductivity of a Weyl metal, however, as we have demonstrated in this paper, AHE is always at least of the same order of magnitude and often significantly larger, than the more theoretically appealing and universal intrinsic one. In a Weyl metal, however, as we have demonstrated in this paper, AHE has a purely intrinsic origin.

Perhaps even more important in the context of Weyl metals is the Chiral Magnetic Effect and the associated negative quadratic magnetoresistance. This may be expected to occur in any type of Weyl metal, either characterized by broken time reversal symmetry or broken inversion (or both). It is thus of particular importance for the experimental characterization of Weyl metals. The negative quadratic longitudinal magnetoresistance should in fact be observed even in Dirac semimetals, which have already been realized experimentally. The theory, presented in section 4.2 is directly applicable to this case, if $\Delta S = \Delta D$ limit is taken and the spin-splitting $b$ is identified with the Zeeman splitting $b = g\mu_B B/2$. In fact, this effect appears to have been observed very recently in a 3D Dirac semimetal material ZrTe5 [67].

What has been presented in this paper is by no means a complete story of the possible observable manifestations of chiral anomaly in Weyl metals. Other related phenomena, such as linear high-field magnetoresistance [68], plasmon-magnon coupling [69], nonlocal transport [70], anomalous density response [71], anomalous thermoelectric response [72] and others [73–84], have been discussed in the literature. One of the possible promising directions for future research in this area is the interplay of chiral anomaly and superconductivity in Weyl metals [85–87].

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