Reanalysis of the $X(3915), X(4500)$ and $X(4700)$ with QCD Sum Rules

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Abstract

In this article, we study the $C\gamma_5 \otimes \gamma_5 C$ type and $C \otimes C$ type scalar $c\bar{s}\bar{c}\bar{s}$ tetraquark states with the QCD sum rules by calculating the contributions of the vacuum condensates up to dimension 10 in a consistent way. The ground state masses $M_{C\gamma_5 \otimes \gamma_5 C} = 3.89 \pm 0.05$ GeV and $M_{C \otimes C} = 5.48 \pm 0.10$ GeV support assigning the $X(3915)$ to be the ground state $C\gamma_5 \otimes \gamma_5 C$ type tetraquark state with $J^{PC} = 0^{++}$, but do not support assigning the $X(4700)$ to be the ground state $C \otimes C$ type $c\bar{s}\bar{c}\bar{s}$ tetraquark state with $J^{PC} = 0^{++}$. Then we tentatively assign the $X(3915)$ and $X(4500)$ to the $1S$ and $2S$ $C\gamma_5 \otimes \gamma_5 C$ type scalar $c\bar{s}\bar{c}\bar{s}$ tetraquark states respectively, and obtain the $1S$ mass $M_{1S} = 3.85^{+0.38}_{-0.17}$ GeV and $2S$ mass $M_{2S} = 4.35^{+0.10}_{-0.11}$ GeV from the QCD sum rules, which support assigning the $X(3915)$ to be the $1S$ $C\gamma_5 \otimes \gamma_5 C$ type tetraquark state, but do not support assigning the $X(4500)$ to be the $2S$ $C\gamma_5 \otimes \gamma_5 C$ type tetraquark state.

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1 Introduction

Recently, the LHCb collaboration performed the first full amplitude analysis of the decays $B^+ \rightarrow J/\psi\phi K^+$ with $J/\psi \rightarrow \mu^+\mu^-$, $\phi \rightarrow K^+K^-$ with a data sample of 3 fb$^{-1}$ of $pp$ collision data collected at $\sqrt{s} = 7$ and 8 TeV with the LHCb detector, confirmed the two old particles $X(4140)$ and $X(4274)$ in the $J/\psi\phi$ mass spectrum with statistical significance 8.4$\sigma$ and 6.0$\sigma$, respectively, determined the quantum numbers to be $J^{PC} = 1^{++}$ with statistical significance 5.7$\sigma$ and 5.8$\sigma$, respectively [1]. Moreover, the LHCb collaboration observed two new particles $X(4500)$ and $X(4700)$ in the $J/\psi\phi$ mass spectrum with statistical significance 6.1$\sigma$ and 5.6$\sigma$, respectively, determined the quantum numbers to be $J^{PC} = 0^{++}$ with statistical significance 4.0$\sigma$ and 4.5$\sigma$, respectively [1]. The measured masses and widths are

$$
X(4140) : M = 4146.5 \pm 4.5^{+4.6}_{-2.8} \text{ MeV}, \Gamma = 83 \pm 21^{+21}_{-14} \text{ MeV},
$$

$$
X(4274) : M = 4273.3 \pm 8.3^{+17.2}_{-3.6} \text{ MeV}, \Gamma = 56 \pm 11^{+8}_{-11} \text{ MeV},
$$

$$
X(4500) : M = 4506 \pm 11^{+12}_{-15} \text{ MeV}, \Gamma = 92 \pm 21^{+21}_{-20} \text{ MeV},
$$

$$
X(4700) : M = 4704 \pm 10^{+14}_{-24} \text{ MeV}, \Gamma = 120 \pm 31^{+42}_{-33} \text{ MeV}.
$$

There have been several possible assignments for the two new particles $X(4500)$ and $X(4700)$.

In Ref.[2], Chen et al study the newly observed $X(4500)$ and $X(4700)$ based on the diquark-antidiquark configuration within the framework of QCD sum rules, and interpret them as the D-wave $c\bar{s}\bar{c}\bar{s}$ tetraquark states with $J^P = 0^+$. In Ref.[3], Liu studies the possible rescattering effects contribute to the process $B^+ \rightarrow J/\psi\phi K^+$, and observes that the $D_1^{*+}\bar{D}_s^-$ rescattering via the open-charmed meson loops and the $\psi'\phi$ rescattering via the $\psi'K_1$ loops may simulate the structures of the $X(4140)$ and $X(4700)$, respectively, and it is hard to attribute the $X(4274)$ and $X(4500)$ to the P-wave threshold rescattering effects.

In Ref.[4], Maiani, Polosa and Riquer assign the $X(4500)$ and $X(4700)$ to be the 2S tetraquark states based on the constituent diquark model. Also in Ref.[5], Zhu assigns the $X(4500)$ and $X(4700)$ to be the 2S tetraquark states based on the constituent diquark model.

In Ref.[6], Lebed and Polosa propose that the $X(3915)$ is the ground state $c\bar{s}\bar{c}\bar{s}$ state based on lacking of the observed $D\bar{D}$ and $D^*\bar{D}^*$ decays, and attribute the single known decay mode $J/\psi\omega$ to the $\omega - \phi$ mixing effect.

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The diquarks $\varepsilon^{ijk} q_i^T C \gamma q_k$ in color antitriplet have five structures in Dirac spinor space, where $CT = C\gamma_5$, $C$, $C\gamma_\mu \gamma_5$, $C\gamma_\mu$, and $C\gamma_\mu \gamma_\nu$ for the scalar, pseudoscalar, vector, axialvector, and tensor diquarks, respectively. In Ref. [7], we study the masses and pole residues of the $X(3915)$, $X(4500)$ and $X(4700)$ in the scenario of tetraquark states with the QCD sum rules by calculating the contributions of the vacuum condensates up to dimension 10. The theoretical calculations support assigning the $X(3915)$ and $X(4500)$ to be the 1$S$ and 2$S$ $C\gamma_\mu \otimes \gamma_\mu C$ type scalar $c\bar{c}s\bar{s}$ tetraquark states, respectively, and assigning the $X(4700)$ to be the 1$S$ $C\gamma_\mu \gamma_5 \otimes \gamma_5 \gamma_\mu C$ type scalar $c\bar{c}s\bar{s}$ tetraquark state. In subsequent work, we take the $X(4140)$ as the diquark-antidiquark type $c\bar{c}s\bar{s}$ tetraquark state with $J^{PC} = 1^{++}$, and study the mass and pole residue with the QCD sum rules in details by constructing two types interpolating currents. The numerical results $M_{X_{L^+}} = 3.95 \pm 0.09$ GeV and $M_{X_{R^+}} = 5.00 \pm 0.10$ GeV disfavor assigning the $X(4140)$ to be the $C\gamma_5 \otimes \gamma_\mu C + C\gamma_\mu \otimes \gamma_5 C$ type or $C \otimes \gamma_5 \gamma_\mu C + C\gamma_\mu \gamma_5 \otimes C$ type tetraquark state with $J^{PC} = 1^{++}$. [8].

The attractive interactions of one-gluon exchange favor formation of the diquarks in color antitriplet, respectively. In Ref. [7], we study the masses and pole residues of the hidden charm (bottom) tetraquark states in details for the first time, and suggest a formula,

$$\mu = \sqrt{M_{X/Y/Z}^2 - (2M_Q)^2},$$

with the effective heavy quark mass $M_Q$ to determine the energy scales of the QCD spectral densities in the QCD sum rules, which works well.

Now we take a short digression to discuss the energy scale dependence of the QCD sum rules for the hidden charm or hidden bottom tetraquark states. The correlation functions $\Pi(p)$ can be written as

$$\Pi(p) = \int_{4m_Q^2(\mu)}^{s_0} ds \frac{\rho_{QCD}(s, \mu)}{s - p^2} + \int_{s_0}^{\infty} ds \frac{\rho_{QCD}(s, \mu)}{s - p^2},$$

through dispersion relation at the QCD side, where the $s_0$ are continuum threshold parameters. The $\Pi(p)$ are energy scale independent,

$$\frac{d}{d\mu} \Pi(p) = 0,$$

which does not mean

$$\frac{d}{d\mu} \int_{4m_Q^2(\mu)}^{s_0} ds \frac{\rho_{QCD}(s, \mu)}{s - p^2} \rightarrow 0,$$

due to the following two reasons inherited from the QCD sum rules:

- Perturbative corrections are neglected, the higher dimensional vacuum condensates are factorized into lower dimensional ones therefore the energy scale dependence of the higher dimensional vacuum condensates is modified;
- Truncations $s_0$ set in, the correlation between the threshold $4m_Q^2(\mu)$ and continuum threshold $s_0$ is unknown.
In the QCD sum rules for the hidden charm or hidden bottom tetraquark states, the integrals
\[ \int_{4m_Q^2}^{M^2} ds \rho_{QCD}(s,\mu) \exp \left(-\frac{s}{T^2}\right), \]
are sensitive to the heavy quark masses \( m_Q(\mu) \) or the energy scales \( \mu \), where the \( T^2 \) denotes the Borel parameters. Variations of the heavy quark masses or the energy scales \( \mu \) lead to changes of integral ranges \( 4m_Q^2(\mu) - s_0 \) of the variable \( ds \) besides the QCD spectral densities \( \rho_{QCD}(s,\mu) \), therefore changes of the Borel windows and predicted masses and pole residues. We cannot obtain energy scale independent QCD sum rules, but we have an energy scale formula to determine the energy scales consistently.

According to the formula, the energy scale \( \mu = 1 \) GeV is too low to result in robust predictions [12]. In Ref.[7], we use the energy scale formula in Eq.(2) to determine the energy scales of the QCD spectral densities in the QCD sum rules, and observe that the \( X(3915) \) and \( X(4500) \) can be assigned to be the 1S and 2S \( C_\gamma \otimes \gamma^C \) type scalar \( cs\bar{s} \) tetraquark states, respectively, and the \( X(4700) \) can be assigned to be the 1S \( C_\gamma \otimes \gamma^C \) type scalar \( cs\bar{s} \) tetraquark state. In Ref.[7], we obtain the values \( M_X^{(3915)} = 3.91^{+0.06}_{-0.17} \) GeV, \( M_X^{(4500)} = 4.50^{+0.08}_{-0.09} \) GeV and \( M_X^{(4700)} = 4.70^{+0.08}_{-0.09} \) GeV. The value \( M_X^{(3915)} = 3.91^{+0.06}_{-0.17} \) GeV is much smaller than the value 4.44 GeV obtained in Ref.[12]. If the \( C_\gamma \otimes \gamma^C \) type and \( C_\gamma \otimes \gamma^C \) type tetraquark states have degenerate masses, then the \( X(3915) \) and \( X(4500) \) can have another diquark-antidiquark structure, \( C_\gamma \otimes \gamma^C \).

In this article, we study the \( C_\gamma \otimes \gamma^C \) type and \( C \otimes C \) type scalar \( cs\bar{s} \) tetraquark states with the QCD sum rules by calculating the contributions of the vacuum condensates up to dimension 10 in a consistent way. In calculations, we use the energy scale formula to determine the optimal energy scales of the QCD spectral densities to extract to tetraquark masses to identify the \( X(3915), X(4500) \) and \( X(4700) \).

The article is arranged as follows: we derive the QCD sum rules for the masses and pole residues of the 1S \( C_\gamma \otimes \gamma^C \) type and \( C \otimes C \) type \( cs\bar{s} \) tetraquark states in section 2; in section 3, we derive the QCD sum rules for the masses and pole residues of the 1S and 2S \( C_\gamma \otimes \gamma^C \) type \( cs\bar{s} \) tetraquark states; section 4 is reserved for our conclusion.

## 2 QCD sum rules for the scalar tetraquark states \( X_L \) and \( X_H \)

In the following, we write down the two-point correlation functions \( \Pi_{L/H}(p) \) in the QCD sum rules,
\[ \Pi_{L/H}(p) = i \int d^4xe^{ip\cdot x} \langle 0 | T \left\{ J_{L/H}(x)J_{L/H}^+(0) \right\} | 0 \rangle, \]
where
\[ J_L(x) = \epsilon^{ijk}\epsilon^{imn}s^i(x)C\gamma^\mu(x)c^j(x)s^m(x)\gamma_5C\bar{\epsilon}^\mu(x), \]
\[ J_H(x) = \epsilon^{ijk}\epsilon^{imn}s^i(x)C\gamma^\mu(x)c^j(x)s^m(x)C\bar{\epsilon}^\mu(x), \]
where the \( i, j, k, m, n \) are color indexes, the \( C \) is the charge conjugation matrix. The scalar diquark states are more stable than the pseudoscalar diquark states, so we expect that the \( C_\gamma \otimes \gamma^C \) type tetraquark states have much smaller masses than the corresponding \( C \otimes C \) type tetraquark states, and add the subscripts \( L \) and \( H \) to denote the light and heavy tetraquark states, respectively.

At the phenomenological side, we can insert a complete set of intermediate hadronic states with the same quantum numbers as the current operators \( J_{L/H}(x) \) into the correlation functions \( \Pi_{L/H}(p) \) to obtain the hadronic representation [16] [17]. After isolating the ground state contributions of
the scalar $cs\bar{c}\bar{s}$ tetraquark states $X_{L/H}$, we get the results,

$$\Pi_L(p) = \frac{\lambda_{X_L}^2}{M_{X_L}^2 - p^2} + \cdots,$$

$$\Pi_H(p) = \frac{\lambda_{X_H}^2}{M_{X_H}^2 - p^2} + \cdots,$$

(9)

where the pole residues $\lambda_{X_{L/H}}$ are defined by $\langle 0|J_{L/H}(0)|X_{L/H}(p)\rangle = \lambda_{X_{L/H}}$.

In the following, we briefly outline the operator product expansion for the correlation functions $\Pi_{L/H}(p)$ in perturbative QCD. We contract the $s$ and $c$ quark fields in the correlation functions $\Pi_{L/H}(p)$ with Wick theorem, and obtain the results:

$$\Pi_L(p) = \int d^4xe^{ip.x}\left[\text{Tr}\left[\gamma_5\Pi^{kk'}(x)\gamma_5CS^{j'T}(x)C\right] + \text{Tr}\left[\gamma_5C^m'n(-x)\gamma_5CS^{m'T}(-x)C\right]\right],$$

$$\Pi_H(p) = \int d^4xe^{ip.x}\left[\text{Tr}\left[C^{kk'}(x)CS^{j'T}(x)C\right] + \text{Tr}\left[C^{m'n}(-x)CS^{m'T}(-x)C\right]\right],$$

(10)

where the $S_{ij}(x)$ and $C_{ij}(x)$ are the full $s$ and $c$ quark propagators, respectively [17, 18],

$$S^{ij}(x) = \frac{i\delta_{ij}}{2\pi^2x^4} - \frac{\delta_{ij}m_s}{4\pi^2x^2} - \frac{\delta_{ij}\langle ss\rangle}{12} + \frac{i\delta_{ij}\langle ss\rangle}{48} - \frac{\delta_{ij}x^2\langle ss\rangle}{192} + \frac{i\delta_{ij}x^2\langle ss\rangle}{1152\pi^2x^2} - \frac{1}{2\pi^2x^2}\frac{1}{7776} - \frac{(\bar{s}_j\gamma^\mu s_i)\gamma_\mu}{27648},$$

$$C_{ij}(x) = \frac{i}{(2\pi)^4}\int d^dp e^{-ik.x}\left\{\frac{\delta_{ij}}{k - m_c} - \frac{g_s\lambda}{4} \frac{t^n}{(k^2 - m_c^2)^2} \frac{\sigma^\alpha(\lambda + \mu)}{(k^2 - m_c^2)^2} + \frac{g_sD_{ab}\lambda}{3(k^2 - m_c^2)^4} \frac{t^n}{(k^2 - m_c^2)^2} + \cdots\right\},$$

(11)

$$f^{\lambda\alpha\beta} = (k + m_c)\gamma^\lambda(k + m_c)\gamma^\alpha(k + m_c)\gamma^\beta(k + m_c),$$

$$f^{\alpha\beta\mu} = (k + m_c)\gamma^\alpha(k + m_c)\gamma^\beta(k + m_c)\gamma^\mu(k + m_c),$$

(13)

and $t^n = \lambda^n$, the $\lambda^n$ is the Gell-Mann matrix. $D_{ab} = \partial_v \lambda^n_i G^a_{\alpha\beta x} G^b_{\mu\nu}$. Then we compute the integrals both in coordinate space and in momentum space, and obtain the correlation functions $\Pi_{L/H}(p)$, therefore the QCD spectral densities through dispersion relation. In this article, we take into account the vacuum condensates which are vacuum expectations of the operators of the orders $\mathcal{O}(\alpha_s^k)$ with $k \leq 1$ consistently. For the technical details, one can consult Ref.[19]. We neglect the radiative $\mathcal{O}(\alpha_s)$ corrections for the perturbative contributions, it is a challenging or formidable work to calculate the radiative $\mathcal{O}(\alpha_s)$ corrections in the QCD sum rules for hidden charm or hidden bottom tetraquark states, though the corrections may be large in the presence of two heavy quarks, just like in the QCD sum rules for the vector and axialvector $B_c$ mesons [20].

Once the analytical QCD spectral densities are obtained, we take the quark-hadron duality below the continuum thresholds $s_{L/H}^0$ and perform Borel transform with respect to the variable
\[ P^2 = -p^2 \] to obtain the QCD sum rules:

\[
\lambda^2_{X_L} \exp \left( -\frac{M^2_{X_L}}{T^2} \right) = \int_{4m^2}^{s_0} ds \, \rho_L(s) \exp \left( -\frac{s}{T^2} \right),
\]

\[
\lambda^2_{X_H} \exp \left( -\frac{M^2_{X_H}}{T^2} \right) = \int_{4m^2}^{s_0} ds \, \rho_H(s) \exp \left( -\frac{s}{T^2} \right),
\]

where

\[
\rho_L(s) = \rho_0(s) + \rho_3(s) + \rho_4(s) + \rho_5(s) + \rho_6(s) + \rho_7(s) + \rho_8(s) + \rho_{10}(s),
\]

\[
\rho_H(s) = \rho_L(s) \mid_{m_c \to -m_c},
\]

\[
\rho_0(s) = \frac{1}{512 \pi^6} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \, y z (1-y-z)^3 (s-m_c^2)^2 (7s^2 - 6sm_c^2 + m_c^4)
+ \frac{m_c m_{\rho}}{256 \pi^6} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \, (y+z) (1-y-z)^2 (s-m_c^2)^2 (5s-2m_c^2),
\]

\[
\rho_3(s) = -\frac{m_c (\bar{s}s)}{16 \pi^4} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \, y (1-y-z) (s-m_c^2) (2s-m_c^2)
+ \frac{m_c (\bar{s}s)}{16 \pi^4} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \, yz (1-y-z) (10s^2 - 12sm_c^2 + 3m_c^4)
- \frac{m_c m_{\rho}}{8 \pi^4} \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \, (s-m_c^2),
\]

\[
\rho_4(s) = -\frac{m_c^2}{384 \pi^4} \left( \frac{\alpha_{sGG}}{\pi} \right) \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \, \left( \frac{z}{y^2} + \frac{y}{z^2} \right) (1-y-z)^3 \left\{ 2s-m_c^2 + \frac{s^2}{6} \delta (s-m_c^2) \right\}
+ \frac{1}{512 \pi^4} \left( \frac{\alpha_{sGG}}{\pi} \right) \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \, (y+z) (1-y-z)^2 (10s^2 - 12sm_c^2 + 3m_c^4)
+ \frac{m_c m_{\rho}}{384 \pi^4} \left( \frac{\alpha_{sGG}}{\pi} \right) \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \, \left( \frac{1}{y^2} + \frac{1}{z^2} \right) (y+z) (1-y-z)^2 \left\{ 1 + \frac{s}{2} \delta (s-m_c^2) \right\}
+ \frac{m_c m_{\rho}}{256 \pi^4} \left( \frac{\alpha_{sGG}}{\pi} \right) \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \, \left( \frac{z}{y^2} + \frac{y}{z^2} \right) (1-y-z)^2 (3s-2m_c^2)
+ \frac{m_c m_{\rho}}{128 \pi^4} \left( \frac{\alpha_{sGG}}{\pi} \right) \int_{y_1}^{y_f} dy \int_{z_1}^{1-y} dz \, (1-y-z) (3s-2m_c^2),
\]
\[
\rho_5(s) = \frac{m_c \langle \bar{s} g_s \sigma_{G_5} \rangle}{64 \pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (y + z) (3s - 2m_c^2) \\
- \frac{m_c \langle \bar{s} g_s \sigma_{G_5} \rangle}{64 \pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (y + z) (1 - y - z) (3s - 2m_c^2) \\
- \frac{m_s \langle \bar{s} g_s \sigma_{G_5} \rangle}{16 \pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz yz \left\{ 2s - m_c^2 + \frac{s^2}{6} \delta (s - m_c^2) \right\} \\
+ \frac{m_s m_c^2 \langle \bar{s} g_s \sigma_{G_5} \rangle}{32 \pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{1}{y} + \frac{1}{z} \right), \\
\]

\[
\rho_6(s) = \frac{m_c^2 \langle \bar{s} s \rangle^2}{12 \pi^2} \int_{y_i}^{y_f} dy + \frac{g_s^2 \langle \bar{s} s \rangle^2}{108 \pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz yz \left\{ 2s - m_c^2 + \frac{s^2}{6} \delta (s - m_c^2) \right\} \\
- \frac{g_s^2 \langle \bar{s} s \rangle^2}{512 \pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (1 - y - z) \left\{ 2 \left( \frac{z}{y} + \frac{y}{z} \right) (3s - 2m_c^2) + \left( \frac{z}{y^2} + \frac{y}{z^2} \right) m_c^2 \left[ 2 + s \delta (s - m_c^2) \right] \right\} \\
- \frac{g_s^2 \langle \bar{s} s \rangle^2}{3888 \pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (1 - y - z) \left\{ 3 \left( \frac{z}{y} + \frac{y}{z} \right) (3s - 2m_c^2) + \left( \frac{z}{y^2} + \frac{y}{z^2} \right) m_c^2 \left[ 2 + s \delta (s - m_c^2) \right] \right\} \\
- \frac{m_s m_c^2 \langle \bar{s} s \rangle^2}{12 \pi^2} \int_{y_i}^{y_f} dy \left\{ 1 + \frac{s}{2} \delta (s - m_c^2) \right\} \\
+ \frac{m_s m_c^2 \langle \bar{s} s \rangle^2}{64 \pi^4} \int_{y_i}^{y_f} dy \left\{ 1 + \frac{s}{2} \delta (s - m_c^2) \right\} \\
- \frac{m_s m_c^2 \langle \bar{s} s \rangle^2}{216 \pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left\{ \frac{1}{y} + \frac{1}{z} + \left( \frac{1}{y^2} + \frac{1}{z^2} \right) \frac{m_c^2}{2} \delta (s - m_c^2) \right\} \\
- \frac{m_s m_c^2 \langle \bar{s} s \rangle^2}{64 \pi^4} \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{y}{z} + \frac{z}{y} \right) \left\{ 1 + \frac{s}{2} \delta (s - m_c^2) \right\},
\]

(20)

(21)
\[
\rho_\tau(s) = \frac{m_c^3 \langle \bar{s}s \rangle}{288\pi^2} \left( \frac{\alpha_s}{\pi} \right) \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{1}{y^3} + \frac{1}{z^3} \right) (y + z)(1 - y - z) \left( 1 + \frac{s}{T^2} \right) \delta (s - m_c^2)
\]

\[
- \frac{m_c \langle \bar{s}s \rangle}{96\pi^2} \left( \frac{\alpha_s}{\pi} \right) \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz \left( \frac{y}{z^2} + \frac{z}{y^2} \right) (1 - y - z) \{ 2 + s \delta (s - m_c^2) \}
\]

\[
- \frac{m_c \langle \bar{s}s \rangle}{576\pi^2} \left( \frac{\alpha_s}{\pi} \right) \int_{y_i}^{y_f} dy \left\{ 2 + s \delta (s - m_c^2) \right\}
\]

\[
- \frac{m_s m_c^2 \langle \bar{s}s \rangle}{144\pi^2} \left( \frac{\alpha_s}{\pi} \right) \int_0^1 dy \int_0^{1-y} dz \left( \frac{z}{y^2} + \frac{y}{z^2} \right) (1 - y - z) \left( 1 + \frac{s}{T^2} + \frac{2s^2}{T^4} \right) \delta (s - m_c^2)
\]

\[
+ \frac{m_s m_c^2 \langle \bar{s}s \rangle}{144\pi^2 T^2} \left( \frac{\alpha_s}{\pi} \right) \int_0^1 dy \int_0^{1-y} dz \left( \frac{1}{y^3} + \frac{1}{z^3} \right) \delta (s - m_c^2)
\]

\[
- \frac{m_s m_c^2 \langle \bar{s}s \rangle}{48\pi^2} \left( \frac{\alpha_s}{\pi} \right) \int_0^1 dy \int_0^{1-y} dz \left( \frac{1}{y^2} + \frac{1}{z^2} \right) \delta (s - m_c^2)
\]

\[
+ \frac{m_s \langle \bar{s}s \rangle}{64\pi^2} \left( \frac{\alpha_s}{\pi} \right) \int_{y_i}^{y_f} dy \int_{z_i}^{1-y} dz (y + z) \left\{ 1 + \left( \frac{2s}{3} + \frac{s^2}{6T^2} \right) \delta (s - m_c^2) \right\}
\]

\[
- \frac{m_s m_c^2 \langle \bar{s}s \rangle}{288\pi^2} \left( \frac{\alpha_s}{\pi} \right) \int_0^1 dy \left( 1 + \frac{s}{T^2} \right) \delta (s - m_c^2),
\]  \hspace{1cm} (22)

\[
\rho_s(s) = - \frac{m_s^2 \langle \bar{g}s\sigma Gs \rangle}{24\pi^2} \int_0^1 dy \left( 1 + \frac{s}{T^2} \right) \delta (s - m_c^2)
\]

\[
+ \frac{\langle \bar{s}s \rangle \langle \bar{g}s\sigma Gs \rangle}{48\pi^2} \int_0^1 dy s \delta (s - m_c^2)
\]

\[
+ \frac{5m_s m_c \langle \bar{s}s \rangle \langle \bar{g}s\sigma Gs \rangle}{144\pi^2} \int_0^1 dy \left( 1 + \frac{s}{T^2} + \frac{s^2}{2T^4} \right) \delta (s - m_c^2)
\]

\[
- \frac{m_s m_c \langle \bar{s}s \rangle \langle \bar{g}s\sigma Gs \rangle}{96\pi^2} \int_0^1 dy \left( \frac{1 - y}{y} + \frac{y}{1 - y} \right) \left( 1 + \frac{s}{T^2} \right) \delta (s - m_c^2),
\]  \hspace{1cm} (23)
\[
\rho_{10}(s) = \frac{m_c^2 \langle \bar{s}g_s \sigma Gs \rangle^2}{192 \pi^2 T^6} \int_0^1 dy \, s^2 \delta (s - \tilde{m}_c^2) \\
- \frac{m_c^4 \langle \bar{s}s \rangle^2}{216 T^4} \left( \frac{\alpha_s GG}{\pi} \right) \int_0^1 dy \left\{ \frac{1}{y^3} + \frac{1}{(1-y)^4} \right\} \delta (s - \tilde{m}_c^2) \\
+ \frac{m_c^2 \langle \bar{s}s \rangle^2}{72 T^2} \left( \frac{\alpha_s GG}{\pi} \right) \int_0^1 dy \left\{ \frac{1}{y^2} + \frac{1}{(1-y)^2} \right\} \delta (s - \tilde{m}_c^2) \\
- \frac{\langle \bar{s}g_s \sigma Gs \rangle^2}{192 \pi^2 T^4} \int_0^1 dy \, s^2 \delta (s - \tilde{m}_c^2) \\
+ \frac{\langle \bar{s}g_s \sigma Gs \rangle^2}{128 \pi^2 T^2} \int_0^1 dy \, s \delta (s - \tilde{m}_c^2) \\
+ \frac{m_c^2 \langle \bar{s}s \rangle^2}{216 T^6} \left( \frac{\alpha_s GG}{\pi} \right) \int_0^1 dy \, s^2 \delta (s - \tilde{m}_c^2) \\
- \frac{m_c m_c \langle \bar{s}g_s \sigma Gs \rangle^2}{576 \pi^2 T^8} \int_0^1 dy \, s^3 \delta (s - \tilde{m}_c^2) \\
- \frac{m_c m_c \langle \bar{s}s \rangle^2}{432 T^4} \left( \frac{\alpha_s GG}{\pi} \right) \int_0^1 dy \left\{ \frac{1}{y^3} + \frac{1}{(1-y)^4} \right\} \left( 1 - \frac{s}{T^2} \right) \delta (s - \tilde{m}_c^2) \\
- \frac{m_c m_c \langle \bar{s}s \rangle^2}{144 T^4} \left( \frac{\alpha_s GG}{\pi} \right) \int_0^1 dy \left\{ \frac{1}{y^2} + \frac{y}{(1-y)^2} \right\} s \delta (s - \tilde{m}_c^2) \\
+ \frac{m_c m_c \langle \bar{s}g_s \sigma Gs \rangle^2}{576 \pi^2 T^6} \int_0^1 dy \left\{ \frac{1}{y} - \frac{y}{1-y} \right\} s^2 \delta (s - \tilde{m}_c^2) \\
- \frac{m_c m_c \langle \bar{s}s \rangle^2}{864 T^8} \left( \frac{\alpha_s GG}{\pi} \right) \int_0^1 dy \, s^3 \delta (s - \tilde{m}_c^2),
\]

(24)
the subscripts 0, 3, 4, 5, 6, 7, 8, 10 denote the dimensions of the vacuum condensates, \( y_f = \sqrt{1 - 4m_c^2/s} \), \( y_i = \sqrt{1 - 4m_c^2/2} \), \( z_i = \frac{y_i m_c^2}{y_i - m_c^2} \), \( m_c^2 = \frac{(y + z)^2}{y^2} \), \( \tilde{m}_c^2 = \frac{m_c^2}{y_i - y_i} \). \( \int_{z_i}^{1-y} dy \to \int_0^1 dy \), \( \int_{z_i}^{1-y} dz \to \int_0^1 dz \), when the \( \delta \) functions \( \delta (s - \tilde{m}_c^2) \) and \( \delta (s - \tilde{m}_c^2) \) appear.

We derive Eqs.(14-15) with respect to \( \frac{d}{ds} \), then eliminate the pole residues \( \lambda_{X_{L/H}} \), and obtain the QCD sum rules for the masses \( M_{X_{L/H}} \) of the scalar tetraquark states,

\[
M^2_{X_{L/H}} = \frac{\int_{4m_c^2}^{\infty} ds \frac{d}{ds} \rho_{L/H}(s) \exp \left(-\frac{s}{T^2}\right)}{\int_{4m_c^2}^{\infty} ds \rho_{L/H}(s) \exp \left(-\frac{s}{T^2}\right)}. \tag{25}
\]

The vacuum condensates are taken to be the standard values \( \langle \bar{q}q \rangle = -(0.24 \pm 0.01) \text{GeV}^3 \), \( \langle \bar{s}s \rangle = (0.8 \pm 0.1) \text{GeV}^3 \), \( \langle \bar{s}g_s \sigma Gs \rangle = m_c^2 \langle \bar{s}s \rangle \), \( m_c^2 = (0.8 \pm 0.1) \text{GeV}^2 \), \( \langle \alpha_s GG \rangle = (0.33 \text{GeV})^4 \) at the energy scale \( \mu = 1 \text{ GeV} \). The quark condensates and mixed quark condensates evolve with the renormalization group equation, \( \langle \bar{s}s \rangle(\mu) = \langle \bar{s}s \rangle(Q) \left[ \frac{\alpha_s(Q)}{\alpha_s(\mu)} \right]^{\frac{1}{2}} \), and \( \langle \bar{s}g_s \sigma Gs \rangle(\mu) = \langle \bar{s}g_s \sigma Gs \rangle(Q) \left[ \frac{\alpha_s(Q)}{\alpha_s(\mu)} \right]^{\frac{1}{2}} \).

We take the \( MS \) masses \( m_c(m_c) = (1.275 \pm 0.025) \text{GeV} \) and \( m_c(\mu = 2 \text{GeV}) = (0.095 \pm 0.005) \text{GeV} \) from the Particle Data Group, and take into account the energy-scale dependence.
of the \( \overline{MS} \) masses from the renormalization group equation,

\[
m_c(\mu) = m_c(\mu) \left[ \frac{\alpha_s(\mu)}{\alpha_s(m_c)} \right]^{\frac{42}{12}},
\]

\[
m_s(\mu) = m_s(2\text{GeV}) \left[ \frac{\alpha_s(\mu)}{\alpha_s(2\text{GeV})} \right]^{\frac{4}{5}},
\]

\[
\alpha_s(\mu) = \frac{1}{b_0 t} \left[ 1 - \frac{b_1 \log t}{b_0^2} + \frac{b_2^2 (\log^2 t - \log t - 1) + b_0 b_2}{b_0^2 t^2} \right],
\]

where \( t = \log \frac{\mu^2}{\Lambda^2} \), \( b_0 = \frac{33 - 2n_f}{12\pi} \), \( b_1 = \frac{153 - 19n_f}{24\pi} \), \( b_2 = \frac{2857 - 5033n_f + 255n_f^2}{128\pi^2} \), \( \Lambda = 213 \text{ MeV} \), 296 MeV and 339 MeV for the flavors \( n_f = 5, 4 \) and \( 3 \), respectively [29].

In the four-quark system \( q\bar{q}QQ \), the \( Q \)-quark serves as a static well potential and attracts the light quark \( q \) to form a heavy diquark \( D \) in color antitriplet, while the \( \bar{Q} \)-quark serves as another static well potential and attracts the light antiquark \( q' \) to form a heavy antidiquark \( \bar{D} \) in color triplet [13, 14, 15, 19]. Then the \( D \) and \( \bar{D} \) attract each other to form a compact tetraquark state [13, 14, 15, 19]. The two heavy quarks \( Q \) and \( \bar{Q} \) stabilize the tetraquark state [24]. The tetraquark states \( D\bar{D} \) are characterized by the effective heavy quark masses \( m_Q \) and the virtuality \( V = \sqrt{M^2_{X/Y/Z} - (2M_Q)^2} \). It is natural to take the energy scale \( \mu = V \). We cannot obtain energy scale independent QCD sum rules, but we have an energy scale formula to determine the energy scales consistently. We fit the effective \( Q \)-quark mass \( m_Q \) to reproduce the experimental value \( M_{Z_b(3900)/Z_b(10610)} \), the empirical effective \( Q \)-quark mass \( m_Q \) is universal in the QCD sum rules for the hidden-charm or hidden bottom tetraquark states and embodies the net effects of the complex dynamics. We take the empirical energy scale formula and reproduce the experimental values of the masses of the \( X(3872), Z_c(3900), Z_c(4020), Z_c(4025), Z_c(4430), Y(4660), Z_b(10610) \) and \( Z_b(10650) \) in the scenario of tetraquark states [13, 14, 15, 19, 23, 24]. In this article, we take the updated value \( m_c = 1.82 \text{ GeV} \) [25].

We search for the Borel parameters \( T^2 \) and continuum threshold parameters \( s^0_{L/H} \) according to the four criteria:

1. Pole dominance at the phenomenological side;
2. Convergence of the operator product expansion;
3. Appearance of the Borel platforms;
4. Satisfying the energy scale formula.

Now we take a short digression to discuss how to choose the Borel parameters. At the phenomenological side of the QCD sum rules, we prefer smaller Borel parameters so as to depress the contributions of the higher excited states and continuum states and determine the upper bound of the Borel parameters \( T^2_{max} \). At the QCD side of the QCD sum rules, we prefer larger Borel parameters so as to warrant the convergence of the operator product expansion and determine the lower bound of the Borel parameters \( T^2_{min} \). In the QCD sum rules for the tetraquark states, the operator product expansion converges slowly, the \( T^2_{min} \) is postponed to large value, the Borel window \( T^2_{max} - T^2_{min} \) is rather small. However, the small Borel window does exist, the Borel parameter is just a free parameter, the predicted masses and pole residues should be independent on this parameter, in other words, there appears Borel platform.

The resulting Borel parameters \( T^2 \) and threshold parameters \( s^0_{L/H} \) are

\[
X_L : \quad T^2 = (2.7 - 3.1) \text{ GeV}^2, \quad s^0_L = (4.4 \pm 0.1 \text{ GeV})^2,
\]

\[
X_H : \quad T^2 = (5.2 - 5.6) \text{ GeV}^2, \quad s^0_H = (6.0 \pm 0.1 \text{ GeV})^2.
\]

The pole contributions are

\[
X_L : \quad \text{pole} = (39 - 62)\% \quad \text{at} \quad \mu = 1.40 \text{ GeV},
\]

\[
X_H : \quad \text{pole} = (43 - 58)\% \quad \text{at} \quad \mu = 4.10 \text{ GeV}.
\]
the pole dominance condition is satisfied.

The contributions come from the vacuum condensates of dimension \( i D_i \) are

\[
\begin{align*}
X_L & : \quad D_0 = (37 - 39)\%, \quad D_3 = (65 - 67)\%, \quad D_4 = 1\%, \quad D_5 = -(14 - 17)\%, \quad D_6 = (9 - 11)\%, \quad D_7 = 2\%, \quad D_8 = -(1 - 2)\%, \quad D_{10} \ll 1\% \text{ at } \mu = 1.40 \text{ GeV}, \\
X_H & : \quad D_0 = (165 - 174)\%, \quad D_3 = -(69 - 78)\%, \quad D_4 = 1\%, \quad D_5 = 1\%, \quad D_6 = (2 - 3)\%, \quad D_7 = -1\%, \quad D_8 \ll 1\% \text{ at } \mu = 4.10 \text{ GeV},
\end{align*}
\]

where \( i = 0, 3, 4, 5, 6, 7, 8, 10 \), the operator product expansion is well convergent.

Now we take into account uncertainties of all the input parameters, and obtain the values of the masses and pole residues of the \( X_L \) and \( X_H \),

\[
\begin{align*}
M_{X_L} &= 3.89 \pm 0.05 \text{ GeV}, \quad \text{Experimental value } 3918.4 \pm 1.9 \text{ MeV} \quad [23], \\
M_{X_H} &= 5.48 \pm 0.10 \text{ GeV}, \quad \text{Experimental value } 4704 \pm 10^{+14}_{-24} \text{ GeV} \quad [1], \\
\lambda_{X_L} &= (2.21 \pm 0.22) \times 10^{-2} \text{ GeV}^5, \\
\lambda_{X_H} &= (1.98 \pm 0.08) \times 10^{-1} \text{ GeV}^5,
\end{align*}
\]

which are also shown in Figs.1-2. In Figs.1-2, we plot the masses and pole residues with variations of the Borel parameters at much larger intervals than the Borel windows shown in Eqs.(27-28). From Eqs.(33-34) and Figs.1-2, we can see that the energy scale formula is satisfied and there appear platforms in the Borel windows, the uncertainties originate from the Borel parameters in the Borel windows are very small, \( \delta M_{X_L}/M_{X_L}, \delta M_{X_H}/M_{X_H} \ll 1\% \), while the uncertainties originate from the Borel parameters outside of the Borel windows are rather large compared to the central values, where the criterion 1 or the criterion 2 is not satisfied. In the Borel windows, the four criteria are all satisfied, we expect to make reliable predictions. In calculations, we observe that the predicted masses \( M_{X_L} \) and \( M_{X_H} \) decrease monotonously with increase of the energy scales, the uncertainty \( \delta \mu = \pm 0.1 \text{ GeV} \) can lead to uncertainties \( \delta M_{X_L} = \pm 20 \text{ MeV} \) and \( \delta M_{X_H} = \pm 3 \text{ MeV} \).

The present prediction \( M_{X_L} = 3.89 \pm 0.05 \text{ GeV} \) is compatible with the experimental value \( M_{X(3915)} = 3918.4 \pm 1.9 \text{ MeV} \quad [23] \), so it is reasonable to assign the \( X_L \) to be the \( X(3915) \). The predicted mass \( M_{X_H} = 5.48 \pm 0.10 \text{ GeV} \) lies above the upper bound of the experimental value \( M_{X(4700)} = 4704 \pm 10^{+14}_{-24} \text{ GeV} \), it is impossible to assign the \( X(4700) \) to be the \( C - C \) type scalar tetraquark state. In Ref. [1], we obtain the values \( M_{X(3915)} = 3.91^{+0.21}_{-0.17} \text{ GeV}, \quad M_{X(4500)} = 4.50^{+0.08}_{-0.09} \text{ GeV}, \quad M_{X(4700)} = 4.70^{+0.08}_{-0.09} \text{ GeV} \), which are in excellent agreement with the experimental data, and support assigning the \( X(3915) \) and \( X(4500) \) to be the 1S and 2S \( C \gamma_\mu \gamma_5 \otimes \gamma^\mu C \) type tetraquark states, and support assigning the \( X(4700) \) to be the 1S \( C \gamma_\mu \gamma_5 \otimes \gamma_\mu C \) type tetraquark state.

For the hidden-charm mesons, the energy gaps between the ground states and the first radial excited states are \( M_{\psi'} - M_{J/\psi} = 589 \text{ MeV}, \quad M_{\eta'} - M_{\eta_c} = 656 \text{ MeV} \quad [23], \quad M_{Z(4430)} - M_{Z(3900)} = 576 \text{ MeV} \quad [24] \). In this article, we choose the continuum threshold parameters as \( \sqrt{s_{X_L}} - M_{X_L} = 0.4 - 0.6 \text{ GeV} \) and \( \sqrt{s_{X_H}} - M_{X_H} = 0.4 - 0.6 \text{ GeV} \), the contaminations of the radial excited states can be neglected. On the other hand, the currents \( J_{L/H}(x) \) couple potentially to the scattering states \( J/\psi \omega, J/\psi \phi, D_s^+ D_s^-, \cdots \), we can take into account the contributions of the intermediate meson-loops. All the renormalized self-energies contribute a finite imaginary part to modify the dispersion relation, we can take into account the finite width effects by the simple replacement of the hadronic spectral densities,

\[
\delta \left( s - M_{L/H}^2 \right) \rightarrow \frac{1}{\pi} \frac{\sqrt{s} \Gamma_{L/H}(s)}{(s - M_{L/H}^2)^2 + s \Gamma_{L/H}^2(s)}.
\]

The widths \( \Gamma_{X(3915)} = 20 \pm 5 \text{ MeV} \quad [23] \) and \( \Gamma_{X(4700)} = 120 \pm 31^{+42}_{-33} \text{ MeV} \quad [1] \) are not broad, the effects of the finite widths can be absorbed safely into the pole residues \( [7, 8, 28] \), the present predictions of the masses are reasonable.
Figure 1: The masses $M_{X_L}$ and $M_{X_H}$ with variations of the Borel parameters $T^2$, where the experimental value denotes the experimental value of the mass $M_{X(3915)}$.

Figure 2: The pole residues $\lambda_{X_L}$ and $\lambda_{X_H}$ with variations of the Borel parameters $T^2$.

Now we can obtain the conclusion tentatively that the $X(3915)$ maybe have both $C\gamma_\mu \otimes \gamma^\mu C$ type and $C\gamma_5 \otimes C\gamma_5 C$ type tetraquark components.

In the following, we perform Fierz re-arrangement to the currents $J_L$ and $J_H$ both in the color and Dirac-spinor spaces to obtain the results,

$$ J_L = \frac{1}{4} \left\{ -\bar{c}\sigma^{\mu\nu}s + \bar{c}\gamma_\mu \gamma_5 c \bar{s}\gamma_\mu s - \bar{c}\gamma^\mu c \bar{s}\gamma_5 s - \bar{c}\gamma^\mu c \bar{s}\gamma_5 s \gamma_5 c \bar{s}\gamma_\mu s + \frac{1}{2} \bar{c}\sigma_{\mu\nu}s \bar{s}\sigma^{\mu\nu}s + \bar{s}\bar{c}\gamma_\mu \gamma_5 s \bar{s}\gamma_\mu \gamma_5 s \gamma_5 c \bar{s}\gamma_\mu \gamma_5 s - \frac{1}{2} \bar{c}\sigma_{\mu\nu}s \bar{s}\sigma^{\mu\nu}c \right\}, \quad (37) $$

$$ J_H = \frac{1}{4} \left\{ \bar{c}\gamma_\mu c \bar{s}\gamma_\mu s - \bar{c}\sigma^{\mu\nu}s + \bar{c}\gamma_\mu c \bar{s}\gamma_5 s \gamma_5 c \bar{s}\gamma_\mu \gamma_5 s + \frac{1}{2} \bar{c}\sigma_{\mu\nu}c \bar{s}\sigma^{\mu\nu}c \bar{s}\sigma^{\mu\nu}s + \bar{s}\bar{c}\gamma_\mu \gamma_5 s \bar{s}\gamma_\mu \gamma_5 s \gamma_5 c \bar{s}\gamma_\mu \gamma_5 s - \frac{1}{2} \bar{c}\sigma_{\mu\nu}c \bar{s}\sigma^{\mu\nu}c \right\}, \quad (38) $$

the components couple to the meson pairs, for example, $J/\psi\phi$, $D_s \bar{D}_s$, $D^*_s \bar{D}^*_s$. The two-body strong
decays

\[
X_L \rightarrow J/\psi \phi \rightarrow J/\psi \omega \ (\phi - \omega \text{ mixing}), \\
X_H \rightarrow J/\psi \phi + D_s \bar{D}_s, \quad D_s^* \bar{D}_s^*, \quad D_{s0} \bar{D}_{s0}, \quad D_{s1} \bar{D}_{s1}
\]

are Okubo-Zweig-Iizuka super-allowed. We can search for the \(X_L\) and \(X_H\) in those decays in the future. The diquark-antidiquark type tetraquark state can be taken as a special superposition of a series of meson-meson pairs, and embodies the net effects. The decays to its components, for example, \(J/\psi \phi\), are Okubo-Zweig-Iizuka super-allowed, but the re-arrangements in the color-space are non-trivial.

3 QCD sum rules for the \(X(3915)\) and \(X(4500)\) as the \(C\gamma_5 \otimes \gamma_5 C\) type tetraquark states

Now we tentatively assign the \(X(3915)\) and \(X(4500)\) to be the 1S and 2S \(C\gamma_5 \otimes \gamma_5 C\) type tetraquark states, respectively, and study their masses and pole residues with the QCD sum rules. At the phenomenological side, we isolate the 1S and 2S scalar \(c\bar{s}\)\(\bar{s}\) tetraquark states in the correlation function \(\Pi_L(p)\), and get the following result,

\[
\Pi_L(p) = \frac{\lambda_{X_L(1S)}^2}{M_{X_L(1S)}^2 - p^2} + \frac{\lambda_{X_L(2S)}^2}{M_{X_L(2S)}^2 - p^2} + \cdots,
\]

where the pole residues \(\lambda_{X_L(1S,2S)}\) are defined by \(\langle 0|J_L(0)|X_L(1S, 2S)(p)\rangle = \lambda_{X_L(1S,2S)}\). As the current \(J_L(x)\) couples potentially to the scattering states \(J/\psi\omega, J/\psi\phi, D_s^{*+}D_s^{*-}, \cdots\), which contribute a finite imaginary part to modify the dispersion relation, we can take into account the finite width effects according to Eq.(36), and absorb the finite width effects into the pole residues \([7,8,28]\). The contributions of the scattering states \(J/\psi\omega, J/\psi\phi, D_s^{*+}D_s^{*-}, \cdots\) can be safely neglected if only the predicted masses \(M_{X_L(1S,2S)}\) are concerned.

We take into account the contribution of the \(X(4500)\) and postpone the continuum threshold \(s_L^0\) to the large value \(s_{X(4500)}^0\) in the QCD sum rule in Eq.(14) to obtain the QCD sum rule,

\[
\lambda_{X_L(1S)}^2 \exp \left( -\frac{M_{X_L(1S)}^2}{T^2} \right) + \lambda_{X_L(2S)}^2 \exp \left( -\frac{M_{X_L(2S)}^2}{T^2} \right) = \int_{m_{\gamma_5}^2}^{s_{X(4500)}^0} ds \rho_L(s) \exp \left( -\frac{s}{T^2} \right),
\]

then we introduce the notations \(\tau = \frac{s}{T^2}, D^n = (-\frac{\tau s}{m_2^4})^n\), and use the subscripts 1 and 2 to denote the 1S state \(X(3915)\) and the 2S state \(X(4500)\) respectively for simplicity. The QCD sum rule can be rewritten as

\[
\lambda_i^2 \exp(-\tau M_i^2) + \lambda_j^2 \exp(-\tau M_j^2) = \Pi_{QCD}(\tau),
\]

here we add the subscript \(QCD\) to denote the QCD side of the correlation function \(\Pi_L(\tau)\). We derive the QCD sum rule in Eq.(42) with respect to \(\tau\) to obtain

\[
\lambda_i^2 M_i^2 \exp(-\tau M_i^2) + \lambda_j^2 M_j^2 \exp(-\tau M_j^2) = D \Pi_{QCD}(\tau).
\]

Then we have two equations, and obtain the QCD sum rules,

\[
\lambda_i^2 \exp(-\tau M_i^2) = \frac{(D - M_i^2) \Pi_{QCD}(\tau)}{M_i^2 - M_{i,j}^2},
\]

where \(i \neq j\). Now we derive the QCD sum rules in Eq.(44) with respect to \(\tau\) to obtain

\[
M_i^2 = \frac{(D^2 - M_i^2 D) \Pi_{QCD}(\tau)}{(D - M_i^2) \Pi_{QCD}(\tau)},
\]

\[
M_i^4 = \frac{(D^3 - M_i^2 D^2) \Pi_{QCD}(\tau)}{(D - M_i^2) \Pi_{QCD}(\tau)}.
\]
The squared masses $M^2_i$ satisfy the following equation,

$$M_i^4 - b M_i^2 + c = 0,$$  \hspace{1cm} (46)

where

$$b = \frac{D^3 \otimes D^0 - D^2 \otimes D}{D^2 \otimes D^0 - D \otimes D},$$

$$c = \frac{D^3 \otimes D - D^2 \otimes D^2}{D^2 \otimes D^0 - D \otimes D},$$

$$D^j \otimes D^k = D^j \Pi_{QCD}(\tau) D^k \Pi_{QCD}(\tau),$$  \hspace{1cm} (47)

$i = 1, 2, j, k = 0, 1, 2, 3$. We solve the equation in Eq.(46) and obtain the solutions

$$M_1^2 = \frac{b - \sqrt{b^2 - 4c}}{2},$$  \hspace{1cm} (48)

$$M_2^2 = \frac{b + \sqrt{b^2 - 4c}}{2}.$$  \hspace{1cm} (49)

In Ref.\cite{27}, M. S. Maior de Sousa and R. Rodrigues da Silva study the masses and decay constants of the $\rho(1S, 2S), \psi(1S, 2S), T(1S, 2S)$ using Eqs.(48-49), and observe that the ground state masses are (much) smaller than the experimental values. In Ref.\cite{26}, we apply this approach to study the $Z_c(3900)$ and $Z(4430)$ as the 1S and 2S axialvector hidden-charm tetraquark states, respectively, and use the energy scale formula $\mu = \sqrt{\frac{M_{X/Y/Z}^2}{(2M_c)^2}}$ to overcome the shortcoming \cite{27}, and reproduce the experimental values of the masses $M_{Z_c(3900)}$ and $M_{Z(4430)}$. In Ref.\cite{7}, we use the same approach to study the $X(3915)$ and $X(4500)$ as the 1S and 2S $C\gamma_5 \otimes \gamma^\mu C$ type scalar $cs\bar{c}\bar{s}$ tetraquark states, respectively, and reproduce the experimental values of the masses $M_{X(3915)}$ and $M_{X(4500)}$. Now we tentatively assign the $X(3915)$ and $X(4500)$ to be the 1S and 2S $C\gamma_5 \otimes \gamma_5 C$ type tetraquark states, respectively, and resort to the same approach study the masses and pole residues with the QCD sum rules.

Again, we search for the Borel parameter $T^2$ and continuum threshold parameter $s^0_{X(4500)}$ according to the four criteria:

1. Pole dominance at the phenomenological side;
2. Convergence of the operator product expansion;
3. Appearance of the Borel platforms;
4. Satisfying the energy scale formula.

The resulting Borel parameter and threshold parameter are

$$X(3890/4350) : T^2 = (1.8 - 2.2) \text{ GeV}^2, \hspace{0.5cm} s^0_{X(4500)} = (4.9 \pm 0.1 \text{ GeV})^2,$$ \hspace{1cm} (50)

here we use the notations 3890 and 4350 in stead of 3915 and 4500 according to the masses extracted from the QCD sum rules. The pole contributions are

$$X(3890) + X(4350) : \text{pole} = (87 - 98)\% \text{ at } \mu = 1.25 \text{ GeV},$$  \hspace{1cm} (51)

$$X(3890) + X(4350) : \text{pole} = (93 - 99)\% \text{ at } \mu = 2.40 \text{ GeV},$$  \hspace{1cm} (52)

the pole dominance condition is well satisfied. The contributions come from the vacuum condensates of dimension 10 $D_{10}$ are

$$X(3890) + X(4350) : D_{10} = (1 - 3)\% \text{ at } \mu = 1.25 \text{ GeV},$$  \hspace{1cm} (53)

$$X(3890) + X(4350) : D_{10} = (0 - 1)\% \text{ at } \mu = 2.40 \text{ GeV},$$  \hspace{1cm} (54)

the operator product expansion is well convergent.
Now we take into account uncertainties of all the input parameters, and obtain the masses and pole residues of the $X(3890)$ and $X(4350)$, 

\begin{align*}
M_{X(3890)} &= 3.85^{+0.18}_{-0.15} \text{ GeV}, \quad \text{Experimental value } 3918.4 \pm 1.9 \text{ MeV} \ \ [29], \\
M_{X(4350)} &= 4.83 \text{ GeV}, \\
\lambda_{X(3890)} &= 1.84^{+0.76}_{-0.54} \times 10^{-2} \text{ GeV}^5, \\
\lambda_{X(4350)} &= 5.82 \times 10^{-2} \text{ GeV}^5, \\
\end{align*}

(55) at the energy scale $\mu = 1.25 \text{ GeV},$

\begin{align*}
M_{X(3890)} &= 3.29 \text{ GeV}, \\
M_{X(4350)} &= 4.35^{+0.10}_{-0.11} \text{ GeV}, \quad \text{Experimental value } 4506 \pm 11^{+12}_{-15} \text{ MeV} \ \ [1], \\
\lambda_{X(3890)} &= 1.16 \times 10^{-2} \text{ GeV}^5, \\
\lambda_{X(4350)} &= 6.01^{+0.93}_{-0.80} \times 10^{-2} \text{ GeV}^5, \\
\end{align*}

(56) at the energy scale $\mu = 2.40 \text{ GeV}$, where we have neglected the uncertainties out of control.

Then we take the central values of the masses and pole residues as the input parameters, and obtain the pole contributions of the $X(3890)$ and $X(4350)$ respectively,

\begin{align*}
\text{pole}_{X(3890)} &= (72 - 90)\%, \\
\text{pole}_{X(4350)} &= (8 - 15)\%, \\
\end{align*}

(57) at the energy scale $\mu = 1.25 \text{ GeV and}$

\begin{align*}
\text{pole}_{X(3890)} &= (55 - 76)\%, \\
\text{pole}_{X(4350)} &= (23 - 38)\%, \\
\end{align*}

(58)
at the energy scale $\mu = 2.40$ GeV.

The pole contribution of the $X(3890/4350)$ at $\mu = 1.25/2.40$ GeV is larger than that at $\mu = 2.40/1.25$ GeV, we prefer to extract the mass and pole residue of the $X(3890/4350)$ from the QCD spectral density at $\mu = 1.25/2.40$ GeV and neglect the ones at $\mu = 2.40/1.25$ GeV. The neglected values of the masses do not satisfy the energy scale formula in Eq.(2). In this article, we choose the values

$$\lambda_{X(3890)} = 1.84^{+0.76}_{-0.54} \times 10^{-2} \text{GeV}^5,$$
$$\lambda_{X(4350)} = 6.01^{+0.93}_{-0.86} \times 10^{-2} \text{GeV}^5,$$

which are also shown in Figs.3-4. In Figs.3-4, we plot the masses and pole residues with variations of the Borel parameter $T^2$ at much larger interval than the Borel window shown in Eq.(50). There appear platforms not very flat in the Borel window, which result in uncertainties $\delta M_{X,L}/M_{X,L}$, $\delta M_{X,H}/M_{X,H} = \pm 1\%$, while the uncertainties originate from the Borel parameter outside of the Borel window are rather large compared to the central values. The predicted masses $M_{X(3890)} = 3.85^{+0.18}_{-0.17}$ GeV and $M_{X(4350)} = 4.35^{+0.10}_{-0.11}$ GeV satisfy the energy scale formula. From Fig.3, we can see that the predicted mass $M_{X(3890)} = 3.85^{+0.18}_{-0.17}$ GeV is compatible with the experimental value $M_{X(3915)} = 3918.4 \pm 1.9$ MeV [28], so it is reasonable to assign the $X(3890)$ to be the $X(3915)$. The predicted mass $M_{X(4350)} = 4.35^{+0.10}_{-0.11}$ GeV lies below the lower bound of the experimental value $M_{X(4500)} = 4506 \pm 11^{+12}_{-15}$ MeV [1], it is not favored to assign the $X(4500)$ to be the $2S C\gamma_5 \otimes \gamma_5 C$ type scalar tetraquark state. Now we can draw the conclusion tentatively that the $X(3915)$ maybe have both $C\gamma_5 \otimes \gamma^a C$ type and $C\gamma_5 \otimes \gamma_5 C$ type tetraquark components, while the $X(4500)$ can be assigned to the $2S C\gamma_5 \otimes \gamma^a C$ type tetraquark state [7].

In the Borel or Laplace QCD sum rules, we introduce a new parameter $T^2$ therefore an expo-
ential factor $\exp \left(-\frac{s}{T^2} \right)$ to suppress the experimentally unknown higher resonances. The Borel parameter $T^2$ has no physical significance other than being a mathematical artifact, leads to rather narrow stability window for the hidden charm or hidden bottom tetraquark states. On the other hand, the continuum threshold $s_0$, which has a clear physical interpretation, is often exponentially suppressed $\exp \left(-\frac{s}{s_0 T^2} \right)$, or at best reduced in importance [29]. While in the finite energy QCD sum rules or Hilbert moment QCD sum rules, the threshold $s_0$ shows a power-like welcome feature, it is interesting to study the hidden charm or hidden bottom tetraquark states with the finite energy QCD sum rules or Hilbert moment QCD sum rules, this may be our next work.

4 Conclusion

In this article, we study the $C\gamma_5 \otimes \gamma_5 C$ type and $C \otimes C$ type scalar $cs\bar{c}s$ tetraquark states with the QCD sum rules by calculating the contributions of the vacuum condensates up to dimension 10 in a consistent way. In calculations, we use the energy scale formula $\mu = \sqrt{M^2_{X/Y/Z} - (2M_c)^2}$ to determine the optimal energy scales of the QCD spectral densities. The ground state masses $M_{C\gamma_5 \otimes \gamma_5 C} = 3.89 \pm 0.05 \text{ GeV}$ and $M_{C \otimes C} = 5.48 \pm 0.10 \text{ GeV}$ support assigning the $X(3915)$ to be the 1S $C\gamma_5 \otimes \gamma_5 C$ type tetraquark state with $J^{PC} = 0^{++}$, but do not support assigning the $X(4700)$ to be the 1S $C \otimes C$ type $cs\bar{c}s$ tetraquark state with $J^{PC} = 0^{++}$. Then we tentatively assign the $X(3915)$ and $X(4500)$ to be the 1S and 2S $C\gamma_5 \otimes \gamma_5 C$ type scalar $cs\bar{c}s$ tetraquark states respectively, and obtain the 1S mass $M_{1S} = 3.85^{+0.18}_{-0.17} \text{ GeV}$ and 2S mass $M_{2S} = 4.35^{+0.10}_{-0.11} \text{ GeV}$ from the QCD sum rules, which support assigning the $X(3915)$ to be the 1S $C\gamma_5 \otimes \gamma_5 C$ type tetraquark state, but do not support assigning the $X(4500)$ to be the 2S $C\gamma_5 \otimes \gamma_5 C$ type tetraquark state. The $X(3915)$ maybe have both $C\gamma_5 \otimes \gamma_5 C$ type and $C \otimes C$ type scalar tetraquark components, while the $X(4500)$ can be assigned to the 2S $C\gamma_5 \otimes \gamma_5 C$ type tetraquark state.

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