Analysis of stability and bearing capacity of reinforced panels made of composite materials under shear

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Abstract. When performing stress verification analysis of wing spar supported composite walls it is essential to estimate local stability of smooth wall and general stability of orthotropic panel under shear forces. Almost square fragment of spar wall supported with ribs (between two ribs) is examined in this paper. In order to estimate general stability and load-bearing capacity of orthotropic panel geometrically nonlinear problem is examined. When examining linear problem compact expression for critical shear flow definition is explicitly written in order to estimate general stability with ultimate loading. When examining nonlinear problem a system of two equations in deflection amplitude is obtained. Besides nonlinear problem of smooth wall (skin) local stability and postbuckling state estimation with possible geometrically nonlinear behavior is examined. Analytical solution of orthotropic rectangular panel with shear geometrically nonlinear problem is obtained. Based on method of design by postbuckling state with critical shear stress provided with buckling a method of skin thickness definition is suggested. In order to apply obtained solutions to design activities method of orthotropic panels design by stability and postbuckling state with corresponding load level is shown.

1. Introduction

Let us examine the composite reinforced wall of the wing spar of light aircraft. It is assumed that the overall dimensions of the wall fragment (figure 1) between the ribs are commensurable with one another (L≈B). The design of a spar wall structure generally requires analysis of local and overall shear stability under limit loads, as well as an assessment of the load-bearing capacity with ultimate loading.

It should be noted that the actual task is to determine the stability and load-carrying capacity of the square structural-orthotropic panel with shear taking the peculiarities of the buckling shape into account provided the commensurability of the sides L≈B (figure 1).

In addition, for thin wall structures of small aircraft, the local form of buckling is acceptable for load-bearing panels (including thin walls of spars) with loads close to ultimate level. In this case, geometrically nonlinear analysis of orthotropic rectangular panels with L>b (figure 1) in the refined formulation is necessary. Considering the fragment of a reinforced wall (figure 2), it should be noted that when assessing the postbuckling behavior and local buckling of this case (figure 2), it is correct to assume rigid support conditions along the long sides. In this case, it is also necessary to use analysis methods for stability and evaluation of possible postbuckling behavior taking into account the selection of an appropriate function of deflection due to shear. In addition, the methodology for design of orthotropic rectangular panels based on postbuckling state is presented further on. In this case the panel thickness is determined provided critical stress values with buckling.
It should be noted that the considered problems of stability and postbuckling behavior are essential, and weight characteristics of aircraft structures depend on the efficiency of their solution. Contemporary theories for composite structures design and analysis taking strength, stability and postbuckling behavior problems into account are shown in fundamental studies [1-3]. Reviews [4-5] as well as numerical and analytical solutions of composite panels stability problems with shear [6-7] are of certain interest. The results of studies of the behavior of composite panels are given in [8-12]. Composite panels design method based on postbuckling state using analytical solutions of geometrically nonlinear problems is shown in papers [13-14].

2. Problem statement and basic equations
To solve the stability and load-bearing capacity problems, we use geometrically nonlinear basic equations. Let us write down the equation of strain compatibility for the orthotropic panel in accordance with the theory of composite structures in the following form [3]

$$ L_1(F) - L_2(W) = 0, $$

(1)
with

\[ L_1(F) = c_1 \frac{\partial^4 F}{\partial x^4} + (c-2c_{xy}) \frac{\partial^4 F}{\partial x^2 \partial y^2} + c_2 \frac{\partial^4 F}{\partial y^4}, \]

\[ L_2(W) = \left( \frac{\partial^2 W}{\partial x \partial y} \right)^2 - \left( \frac{\partial^2 W}{\partial x^2} \right) \left( \frac{\partial^2 W}{\partial y^2} \right). \]

Hereinafter the generalized stiffness of orthotropic panels according to [3] is determined by the relations

\[ B_{mn} = I_{mn}^{(0)} * D_{mn} = I_{mn}^{(2)} - 2eI_{mn}^{(1)} + e^2 I_{mn}^{(0)}, \]

\[ I_{mn}^{(r)} = \frac{1}{r+1} \sum_{k=1}^{K} A_{mn}^{(i)} (t^{r+1} - t^{r-1}), \quad (r = 0, 1, 2), \quad E_{11,12} = \frac{E_{12}}{1 - \mu_{12} \mu_{21}}, \]

with

\[ A_{11}^{(i)} = \bar{E}_{1}^{(i)} \sin^2 \varphi_i + \bar{E}_{2}^{(i)} \cos^2 \varphi_i + 2(\bar{E}_{1}^{(i)} \mu_{12}^{(i)} + 2G_{12}^{(i)}) \sin^2 \varphi_i \cos^2 \varphi_i, \]

\[ A_{12}^{(i)} = \bar{E}_{1}^{(i)} \mu_{12}^{(i)} + \left[ \bar{E}_{1}^{(i)} + \bar{E}_{2}^{(i)} - 2(\bar{E}_{1}^{(i)} \mu_{12}^{(i)} + 2G_{12}^{(i)}) \right] \sin^2 \varphi_i \cos^2 \varphi_i, \]

\[ A_{11}^{(i)} = [\bar{E}_{1}^{(i)} + \bar{E}_{2}^{(i)} - 2\bar{E}_{1}^{(i)} \mu_{12}^{(i)}] \sin^2 \varphi_i \cos^2 \varphi_i + G_{12}^{(i)} \cos^2 2\varphi_i, \]

Also we have

\[ c_x = \frac{B_{22}}{B}, \quad c_y = \frac{B_{12}}{B}, \quad c_{xy} = c_{yx} = \frac{B_{12}}{B}, \quad c_{yy} = c_{xx} = \frac{B_{12}}{B}, \quad B = B_{11}B_{22} - B_{12}^2 \]

The second nonlinear equation of Karman type for orthotropic panels of following form

\[ L_3 (F,W) - L_4 (W) = 0, \]

with

\[ L_3 (F,W) = \frac{\partial^2 F}{\partial y^2} \frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 F}{\partial x^2} \frac{\partial^2 W}{\partial y^2} - 2 \frac{\partial^2 F}{\partial x \partial y} \frac{\partial^2 W}{\partial x \partial y}, \]

\[ L_4 (W) = D_{11} \frac{\partial^4 W}{\partial x^4} + 2(D_{12} + 2D_{33}) \frac{\partial^4 W}{\partial x^2 \partial y^2} + D_{22} \frac{\partial^4 W}{\partial y^4}. \]

For smooth orthotropic skins the bending stiffness determined by the following formulas

\[ D_{11} = \frac{E_s \delta^3}{12(1-\nu_{xy} \mu_{yx})}, \quad D_{22} = \frac{E_s \delta^3}{12(1-\nu_{xy} \mu_{yx})}, \quad D_{12} = D_{11} + 2D_{33} = \mu_{xy} D_{11} + \frac{G_{xy} \delta^3}{6}. \]

For a structurally orthotropic panel the bending stiffness \( D_{11} \) is determined with consideration of reinforcing elements by relations (2) taking the "smearing" hypothesis of stringers into account, and the remaining bending stiffnesses \( D_{22}, D_{12} \) and \( D_{33} \) refer to the skin and are calculated by relations (5).

To solve the geometrically nonlinear problem using the Bubnov-Galerkin method we will use following equations

\[ \int_0^a \int_0^b \left[ (L_3 (F,W) - L_4 (W)) W_k \right] dx dy = 0. \]

with \( W_k \) being deflection function.

Let us also write down the equations determining stress function [3]
So, next we consider two geometrically nonlinear problems. First one for a structurally orthotropic (stiffened) square panel, and second one for a smooth orthotropic rectangular panel.

3. Applied methods for calculating stability and postbuckling state

Let us first examine the geometrically nonlinear problem of the structural-orthotropic panel taking into account the geometric parameters corresponding in the general case to a square shape with \( a \approx b \), where \( a \) and \( b \) are the length and width of the panel.

Under the boundary conditions corresponding to the hinge support with tangential forces \( q_{xy} \) acting, let us represent the deflection as

\[ W = f_1 \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} + f_2 \sin \frac{2\pi x}{a} \sin \frac{2\pi y}{b}. \]  

After substituting expression (8) into the equation of strain compatibility (1), we can obtain equation

\[ L(F) = \left[ \left( \frac{\pi^2}{ab} \right)^2 f_1^2 \left( \cos \frac{2\pi x}{a} + \cos \frac{2\pi y}{b} \right) + \right. \]

\[ + 4f_1f_2 \left( \cos \frac{3\pi x}{a} \cos \frac{\pi x}{a} + \cos \frac{\pi y}{b} \cos \frac{3\pi y}{b} \right) + f_2^2 \left( \cos \frac{4\pi x}{a} + \cos \frac{4\pi y}{b} \right) \]

the solution of which is a stress function of the following form

\[ F = A_1 f_1^2 \cos \frac{2\pi x}{a} + A_2 f_1^2 \cos \frac{2\pi y}{b} + A_3 f_1f_2 \cos \frac{3\pi x}{a} \cos \frac{\pi y}{b} + \]

\[ + A_4 f_1f_2 \cos \frac{\pi x}{a} \cos \frac{3\pi y}{b} + A_5 f_1^2 \cos \frac{4\pi x}{a} + A_6 f_2^2 \cos \frac{4\pi y}{b} + \]

\[ + \frac{q_{xy}^2}{2} + \frac{q_{xy}^2}{2} + q_{xy} xy, \]

with

\[ A_1 = \frac{1}{32c_y} \frac{a^2}{b^2}, \quad A_2 = \frac{1}{32c_y} \frac{b^2}{a^2}, \quad A_3 = \frac{4a^2b^2}{81c_yb^4 + 9(c - 2c_y)a^2b^2 + c_ya^4}, \]

\[ A_4 = \frac{1}{32c_y} \frac{b^2}{a^2}, \quad A_5 = \frac{4a^2b^2}{c_yb^4 + 9(c - 2c_y)a^2b^2 + 81c_ya^4}, \]

\[ A_6 = \frac{1}{32c_y} \frac{a^2}{b^2}, \]

To solve the geometrically nonlinear problem using the Bubnov-Galerkin method (6) we will use the equations

\[ \int_0^a \int_0^b [L_4(W) - L_3(F,W)] \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} dx dy = 0, \]  

\[ \int_0^a \int_0^b [L_4(W) - L_3(F,W)] \sin \frac{2\pi x}{a} \sin \frac{2\pi y}{b} dx dy = 0. \]
After substituting expressions (1) and (9) into equations (10)-(11) and some cumbersome transformations, we can obtain a nonlinear system of equations in deflection amplitudes (with \( f_1 \neq 0, f_2 \neq 0 \))

\[
\frac{\pi^4 f_1}{4a^4 b^4} \left\{ D_1 b^4 + 2D_3 a^2 b^4 + D_{22} a^4 \right\} + \frac{\pi^4}{4ab} \left[ 2A_{02} f_1^3 + A_{03} f_1 f_2^2 + 9A_{04} f_1 f_2^2 \right] + \\
+ \frac{\pi^4}{4ab} \left[ 2A_{01} f_1^3 + 9A_{03} f_1 f_2^2 + A_{04} f_2^2 \right] + \frac{3\pi^4}{2ab} \left[ 2A_{03} f_1 f_2^2 + A_{04} f_1 f_2^2 \right] = (12)
\]

\[
= \left[ \frac{\pi^2 q_x f_1 b}{a} + \frac{\pi^2 q_y f_1 a}{b} + \frac{32 f_1 q_{xy}}{9} \right]
\]

\[
\frac{4\pi^4 f_1}{a^4 b^3} \left\{ D_1 b^4 + 2D_3 a^2 b^4 + D_{22} a^4 \right\} + \frac{\pi^4}{16ab} \left[ A_{03} f_1^2 f_2 + 128A_{06} f_1^3 + 9A_{04} f_1^3 f_2 \right] + \\
+ \frac{\pi^4}{16ab} \left[ A_{04} f_1^2 f_2 + 9A_{03} f_1^2 f_2 + 128A_{06} f_1^3 \right] + \frac{3\pi^4}{8ab} \left[ A_{03} f_1^2 f_2 + A_{04} f_1^2 f_2 \right] = (13)
\]

\[
= \left[ \frac{\pi^2 q_x f_1 b}{a} + \frac{\pi^2 q_y f_1 a}{b} + \frac{32 f_1 q_{xy}}{9} \right]
\]

To assess the stability of the square panel, next we need to examine the linear system of equations (12)-(13) using the procedure of nonlinear terms truncation \( f_1 \rightarrow 0, f_2 \rightarrow 0 \). Then let us write down the determinant of the linear system and you can find the expression for the critical buckling force of the square panel in shear

\[
q_{xy} = \frac{9\pi^4}{324} \left\{ D_1 b + 2D_3 a + b^2 \right\}. (14)
\]

It should be noted that the obtained nonlinear system (12)-(13) at a given load numerically allows us to determine the deflection amplitudes \( f_1 \) and \( f_2 \), and to find the stress state of the structural-orthotropic panel based on the definition of the stress function (7).

Now let us examine the second problem concerning the stability and load-bearing capacity of a smooth orthotropic rectangular panel with rigid support along the long sides. It should be noted that the solution of the verification analysis for this case is shown in paper [14]. Here we also present the problem of design based on postbuckling state provided the critical stress \( \tau_{xy} \) is reached with loads close to ultimate level.

In this case, the deflection of the orthotropic skin can be written as follows

\[
W = f \left( \sin \frac{\pi y}{b} \right)^2 \sin \frac{\pi (x-\alpha y)}{s}. (15)
\]

Based on the solution of the geometrically nonlinear problem we obtain the stress function of Airy [14]

\[
F = f^2 \Delta_1 - p_x \delta y^2 - p_x \delta x^2 + p_{xy} \delta xy, \quad (16)
\]

with

\[
\Delta_1 = \begin{cases} 
K_1 \cos \frac{2\pi y}{b} + K_2 \cos \frac{2\pi (x-\alpha y)}{s} - K_3 \cos \frac{4\pi y}{b} - \\
-K_4 \left\{ \frac{2\pi (x-\alpha y)}{s} - \frac{2\pi y}{b} \right\} - K_4 \left\{ \frac{2\pi (x-\alpha y)}{s} - \frac{2\pi y}{b} \right\},
\end{cases}
\]
The tangential stress is obtained from the above mentioned function $F$. Under the action of shear flows $q_{xy} = \tau_{xy} \delta$ and provided the strength critical values are reached, we have the following expression

$$
\tau_{xy} = \frac{\partial^2 F}{\partial x \partial y} = f^2 \Delta_2 + \frac{q_{xy}}{\delta},
$$

with

$$
\Delta_2 = \pi^4 \left[ \frac{4K_4}{s^2} \cos \left( \frac{2\pi(x - \alpha y)}{s} \right) \right] + \frac{4K_4}{sb} \cos \left( \frac{2\pi(x - \alpha y)}{s} \right) - \frac{4K_5}{sb} \cos \left( \frac{2\pi(x - \alpha y)}{b} \right).
$$

When a combination of flows acts, solution by Bubnov-Galerkin method leads to the following nonlinear equation

$$
D_{as} \delta^3 + f^2 \delta \Delta_3 = \frac{1}{16s} \left[ \frac{p}{b} \delta \left( 3\alpha^2 b^2 + 4s^2 \right) + 3b_{xs} \delta + 3a b p_{xy} \delta \right],
$$

with

$$
\Delta_3 = \pi^4 \left[ \frac{3K_4}{8s^3} + \frac{3K_2}{8sb} + \frac{K_3}{2sb} + \frac{K_4 (s + \alpha b)^2}{4s^3b} + \frac{K_5 (s - \alpha b)^2}{s^3b} + \frac{K_1 (4s^3 + 3\alpha^2 b^2)}{8s^3b} + \frac{K_4 (\alpha^2 b^2 + 2abs + 2s^2)}{4s^3b} + \frac{K_5 (\alpha^2 b^2 - 2abs + 2s^2)}{4s^3b} + \frac{3K_4}{2s^3b} + \frac{K_4 (s + \alpha b)}{2s^3b} + \frac{K_4 \alpha (s + \alpha b)}{s^3} + \frac{K_5 \alpha (s - \alpha b)}{s^3} + \frac{2K_5 \alpha (s - \alpha b)}{bs^3} \right],
$$

$$
D_{as} = \frac{3\pi^4 b}{16s^3} 12 \left( 1 - \mu_{xy} \mu_{yx} \right) + \frac{\pi^4}{16s^3} \left( 3\alpha^2 b^2 + 4s^2 \right) \left[ \frac{\mu_{xy} E_x}{12 \left( 1 - \mu_{xy} \mu_{yx} \right)} + \frac{G_{xy}}{6} \right] + \frac{1}{16s^3} \left[ 3\alpha^4 b^4 + 24\alpha^2 b^2 s^2 + 16s^4 \right] \frac{E_x}{12 \left( 1 - \mu_{xy} \mu_{yx} \right)}.
$$

Further on to solve the problem of shear stability of an orthotropic rectangular panel from (18) we can also write down a linear expression for small deflections (under the condition $f^2 \rightarrow 0$)
Galerkin method solutions of following problems having practical significance are obtained in order to estimate stability and strength is satisfied. To determine the critical parameters of wave formation, we can transform expression (19) into

\[ \frac{\alpha q_{sy}}{D_{22}} = \frac{\pi^2}{b^2} \left( \frac{\lambda_{y}}{3} \left( 3\alpha^2 \gamma + 4 \right) + \left[ \alpha^4 + 8\alpha^2 + \frac{16}{3\gamma} \right] \right), \]

with \( \gamma = \frac{b^2}{s^2} \), \( \lambda = \frac{E_s \mu_{xy} + 2G_{xy}}{E_s} \), \( \lambda_i = \frac{E_s}{E_y} \), and then use the traditional minimization procedure for the parameters \( \alpha \) and \( \gamma \) [1]. After some transformations, we obtain nonlinear equations determining critical parameters of wave formation

\[ 2\alpha^2 \sqrt{3(\alpha^4 + \lambda \alpha^2 + \lambda_i)} - 2\lambda_i - \alpha^4 = 0, \quad \frac{1}{\gamma^2} = \frac{3}{16} (\alpha^4 + \lambda \alpha^2 + \lambda_i), \]

which have only numerical solutions.

It should be noted that the critical parameters of wave formation depend on the ratios of stiffness parameters and are determined by the composite layout.

Further on it is assumed that the tangential flow is given, not the stress. Then expressing the deflection amplitude from (17) provided the critical tangential stress is reached, after substitution to (18) we can obtain the required nonlinear equation determining the optimal thickness of the orthotropic panel

\[ \delta^3 D_{ax} + \delta \frac{\tau_{sy} \Delta_s}{\Delta_2} = \left( \frac{3\alpha b}{16s} + \frac{1}{\Delta_2} \right) q_{sy}. \]

4. Applied design of stiffened wall skin taking into account possibility of postbuckling state of the skin with shear

It is noted that based on the obtained relations we can propose a methodology for design of stiffened wall skin (figure 1) based on two conditions. Firstly, it is necessary to ensure stability under limit loads and secondly, it is necessary to ensure strength in the convex skin with ultimate level loading.

So, the critical parameters of wave formation (\( \alpha, \gamma \)) are numerically determined by the known composite layout using nonlinear relations (21). Then the thickness \( \delta_{lim} \) with which local stability is ensured under limit loads is determined from equation (19) with the known pitch of stiffening elements b. Then we calculate the thickness \( \delta_{ul} \) from relation (18), this thickness provides the minimum allowable margin of safety with postbuckling behavior at the ultimate level of loading.

Let us analyze two situations. If \( \delta_{lim} > \delta_{ul} \), it is possible to choose the thickness \( \delta_{lim} \), but it is also possible to decrease pitch b and the analysis up to the condition of \( \delta_{lim} = \delta_{ul} \). Provided that \( \delta_{lim} = \delta_{ul} \), the thickness \( \delta_{ul} \) should be finally chosen, but it is also possible to increase pitch b and also the analysis up to the condition \( \delta_{lim} = \delta_{ul} \). It should be noted that it is only possible to achieve the condition \( \delta_{lim} = \delta_{ul} \) when using an analytical solution. In this case, under the action of limit loads \( q_{lim}^{ul} \), equations (19) must be transformed with respect to the thickness, which is then substituted into relation (22), written under the action of ultimate loads \( q_{sy}^{ul} \). As a result, a nonlinear equation can be obtained with respect to the panel width b, at which the equality \( \delta_{lim} = \delta_{ul} \) with all the above mentioned conditions for stability and strength is satisfied.

5. Conclusion

In order to estimate stability and postbuckling state of supported wall under shear forces analytical solutions of following problems having practical significance are obtained in this paper using Bubnov-Galerkin method:

- estimation of orthotropic wall stability and postbuckling behavior with hinge support,
- estimation of orthotropic smooth wall (skin) stability and postbuckling behavior with rigid support along long sides.

Besides a design method providing stability with limit loading with allowable postbuckling behavior due to ultimate tangential flows is suggested on the basis of obtained solution of orthotropic panel geometrically nonlinear problem.

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