Effective action of $\beta$-deformed $\mathcal{N} = 4$ SYM theory and AdS/CFT

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Abstract

We compute the one-loop effective action in $\mathcal{N} = 1$ superconformal $SU(N)$ gauge theory which is an exactly marginal deformation of the $\mathcal{N} = 4$ SYM. We consider an abelian background of constant $\mathcal{N} = 1$ gauge field and single chiral scalar. While for finite $N$ the effective action depends non-trivially on the deformation parameter $q = e^{i\pi \beta}$, this dependence disappears in the large $N$ limit if the parameter $\beta$ is real. This conclusion matches the strong-coupling prediction coming from the form of a D3-brane probe action in the dual supergravity background: for the simplest choice of the D3-brane position the probe action happens to be the same as for a D3-brane in $AdS_5 \times S^5$ placed parallel to the boundary of $AdS_5$. This suggests that in the real $\beta$ deformation case there exists a large $N$ non-renormalization theorem for the $F^4$ term in the action.

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1 Introduction

The study of AdS/CFT duality for less supersymmetric cases was recently boosted by the discovery of the supergravity background \[1\] dual to the exactly marginal \(\mathcal{N} = 1\) superconformal \(\beta\)-deformation \[2\] of the maximally supersymmetric \(SU(N)\) SYM theory (earlier work on this gauge theory and its supergravity dual appeared in \[3\] and \[4, 5\]). The most immediate implication of the large \(N\) AdS/CFT duality is the matching between the anomalous dimensions of single-trace composite operators and the corresponding spectrum of string energies. This matching was demonstrated in a certain “semiclassical” limit in \[6, 7\] (using, in particular, gauge-theory results of \[8\]).

Some properties of anomalous dimensions and correlation functions of the \(\beta\)-deformed gauge theory were recently studied in \[9, 10, 11, 12\]. Here we complement this work by considering the one-loop low-energy effective action for a simple gauge field \(F\) and scalar \(\Phi\) background on a Coulomb branch \[13, 14\] of the \(\beta\)-deformed theory.\(^1\) This allows us to compare the \(F^4/|\Phi|^4\) term in this action with the corresponding term in the action of a D3-brane probe placed in the deformed \((AdS_5 \times S^5)_\beta\) background.\(^2\) As is well known, in the case of undeformed \(\mathcal{N} = 4\) SYM theory the two terms agree \[15\] (for a review and extensions see \[16\]), and this may be interpreted as a manifestation of a non-renormalization theorem \[17\].

For the simplest abelian gauge theory background we shall consider below only one of the three chiral scalar fields will be chosen to have a non-zero value. In the dual supergravity picture this translates into the position of the D3-brane probe on \((S^5)_\beta\) being at \(\mu_1 = 1, \mu_2 = \mu_3 = 0\) (in the notation of \[11\]) with the three other isometric angles being trivial. In this case the inspection of the deformed background in \[11\] shows that all the dependence on the (in general, complex) deformation parameter \(\beta\) drops out, i.e. the D3-brane probe action happens to be the same as in the \(AdS_5 \times S^5\) case.

The coefficient of the \(F^4/|\Phi|^4\) term in the one-loop \(SU(N)\) gauge theory effective action we shall compute below has, in general, a non-trivial dependence on \(\beta\). However, this dependence completely drops out in the large \(N\) limit, provided \(\beta\) is real. In fact, the large \(N\) limit of the 1-loop effective action in the real deformation case turns out to be the same as in the undeformed SYM theory. This agrees with the strong-coupling prediction

\(^1\)Second-derivative term in the low-energy effective action at a generic point of the Coulomb branch was considered in \[14\]. Here we will be interested in 4-derivative \(F^4\), etc., terms.

\(^2\)Matching of constant scalar potential term in D3-brane probe action with 1-loop correction in the general deformed gauge theory away from superconformal point was observed earlier in \[4\].
coming from the D3-brane probe action, with a plausible explanation of this matching being the existence of a non-trivial large \( N \) non-renormalization theorem.

The case of complex \( \beta \) is different: here the dependence of the one-loop gauge theory effective action on the deformation parameter survives the large \( N \) limit and thus disagrees with the form of the D3-brane action. This provides another indication that the complex \( \beta \) deformation case is more complicated than the real \( \beta \) one, and that the implications of the AdS/CFT duality here are much harder to uncover (other complications of complex \( \beta \) case are lack of integrability on both gauge theory and string theory sides, need to use S-duality \[1\] to construct the string background implying lack of useful perturbative definition of the corresponding string theory, etc., \[6\]).

Below in section 2 we shall review the structure of the \( \beta \)-deformed gauge theory and write down the general expression for its 1-loop effective action in an abelian background. In section 3 we shall find the explicit form of the term quartic in the gauge field strength and analyze its dependence on the deformation parameter \( q = e^{i \pi \beta} \) and \( N \). Appendices A, B and C contain some technical details while in Appendix D we present the form of the second-derivative term in the effective action in the case of a more general diagonal abelian background.

2 One-loop effective action in the \( \beta \)-deformed \( \mathcal{N} = 4 \) SYM theory

Following the \( \mathcal{N} = 1 \) superspace conventions of \[18, 20\] the \( \beta \)-deformed \( \mathcal{N} = 4 \) \( SU(N) \) SYM theory is described by the action

\[
S = \int d^8z \text{Tr} (\Phi_i^\dagger \Phi_i) + \frac{1}{g^2} \int d^6z \text{Tr}(\mathcal{W}_\alpha \mathcal{W}_\alpha) \\
+ \left\{ h \int d^6z \text{Tr}(q \Phi_1 \Phi_2 \Phi_3 - q^{-1} \Phi_1 \Phi_3 \Phi_2) + \text{c.c.} \right\}, \quad q \equiv e^{i \pi \beta}, \quad (2.1)
\]

where \( q \) is the deformation parameter, \( g \) is the gauge coupling constant, and \( h \) is related to \( g \) and \( q \) by the conformal invariance condition (\( h = g \) in the undeformed theory when \( q = 1 \)). Here \( \Phi_i = \Phi_i^\mu(z)T_\mu \quad (i = 1, 2, 3) \) are the covariantly chiral superfields, \( \mathcal{D}_a \Phi_i = 0 \).

The covariantly chiral field strength \( \mathcal{W}_\alpha, \mathcal{D}_\alpha \mathcal{W}_\alpha = 0 \), is associated with the gauge covariant derivatives

\[
\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha, \mathcal{D}_A^\alpha) = D_A + i \Gamma_A, \quad \Gamma_A = \Gamma_A^\mu(z)T_\mu, \quad (2.2)
\]

\[3\] The \( SU(N) \) generators \( T_\mu = (T_\mu)^\dagger \) are normalized so that \( \text{Tr} (T_\mu T_\nu) = \delta_{\mu\nu} \).
where $D_A$ are the flat covariant derivatives. The gauge covariant derivatives satisfy the following algebra:

\[
\{D_\alpha, D_\beta\} = \{\bar{D}_\dot{\alpha}, \bar{D}_\dot{\beta}\} = 0, \quad \{D_\alpha, \bar{D}_\dot{\beta}\} = -2i D_{\alpha\beta},
\]

\[
[D_\alpha, D_\beta\dot{\gamma}] = 2i \varepsilon_{\alpha\beta} \bar{W}_{\dot{\gamma}}, \quad [\bar{D}_\dot{\alpha}, D_\beta\dot{\beta}] = 2i \varepsilon_{\dot{\alpha}\dot{\beta}} W_{\beta},
\]

\[
[D_{\alpha\dot{\alpha}}, D_{\beta\dot{\beta}}] = i F_{\alpha\dot{\alpha},\beta\dot{\beta}} = -\varepsilon_{\alpha\beta} \bar{D}_{\dot{\alpha}} \bar{W}_{\dot{\beta}} - \varepsilon_{\dot{\alpha}\dot{\beta}} D_\alpha W_\beta.
\]

(2.3)

The spinor field strengths $W_\alpha$ and $\bar{W}_{\dot{\alpha}}$ obey the Bianchi identities $D_\alpha W_\alpha = \bar{D}_{\dot{\alpha}} \bar{W}_{\dot{\alpha}}$.

The extrema of the scalar potential (the Coulomb branch) are described by the equations (here $\Phi_i$ are the first components of the chiral superfields)

\[
\sum_i [\Phi_i, \Phi_i^\dagger] = 0, \quad q \Phi_i \Phi_{i+1} - q^{-1} \Phi_{i+1} \Phi_i = \frac{1}{N} (q - q^{-1}) \text{Tr}(\Phi_2 \Phi_{i+1}) 1.
\]

(2.4)

In what follows, we shall consider the simplest special solution

\[
\Phi_1 \equiv \Phi, \quad \Phi_2 = \Phi_3 = 0,
\]

(2.5)

where $\Phi$ is a diagonal traceless $N \times N$ matrix. For such special background one is able to study only a limited class of gauge-invariant quantities like effective action and anomalous dimensions of certain scalar operators.

To quantize the theory, we use the $\mathcal{N} = 1$ background field formulation \[19\] and split the dynamical variables into the background and quantum ones (for a summary and the gauge conditions chosen see Appendix A)

\[
\Phi_i \rightarrow \Phi_i + \varphi_i, \quad D_\alpha \rightarrow e^{-g v} D_\alpha e^{g v}, \quad \bar{D}_{\dot{\alpha}} \rightarrow \bar{D}_{\dot{\alpha}},
\]

(2.6)

with lower-case letters used for the quantum superfields. Choosing $\Phi_2 = \Phi_3 = 0$ and $\Phi_1 \equiv \Phi \neq 0$, the quadratic part of the gauge-fixed action is $S + S_{gf}$ is

\[
S^{(2)} + S_{gf} = -\frac{1}{2} \int d^8 z \text{Tr} \left( v \Box v - g^2 v [\Phi^\dagger, [\Phi, v]] \right)
\]

\[
+ \int d^8 z \text{Tr} \left( \varphi^\dagger_1 \varphi_1 - g^2 [\Phi^\dagger, [\Phi, \varphi^\dagger_1]] (\Box^+)^{-1} \varphi_1 \right) + \ldots
\]

\[
+ \int d^8 z \text{Tr} \left( \varphi^\dagger_2 \varphi_2 + \varphi^\dagger_3 \varphi_3 \right) + \left\{ h \int d^6 z \text{Tr} \left( q \Phi_2 \varphi_3 - \frac{1}{q} \Phi_3 \varphi_2 \right) + c.c. \right\},
\]

(2.7)

where the dots stand for the terms with derivatives of the background (anti)chiral superfields $\Phi^\dagger$ and $\Phi$. The vector d’Alembertian, $\Box_v$, is defined by

\[
\Box_v = D^\alpha D_\alpha - W^\alpha D_\alpha + \bar{W}_\dot{\alpha} \bar{D}_{\dot{\alpha}}.
\]

(2.8)
We shall choose the background superfields (i) to be covariantly constant and on-shell, and (ii) to take their values in the Cartan subalgebra of $SU(N)$. In particular, they will satisfy the conditions ($\mathcal{D}_\alpha \mathcal{W}_\beta \neq 0$):

$$[\Phi, \Phi^\dagger] = 0, \quad \mathcal{D}_\alpha \Phi = 0, \quad \mathcal{D}_\alpha \mathcal{W}_\beta = 0, \quad \mathcal{D}^\alpha \mathcal{W}_\alpha = 0.$$ 

Let us introduce the mass operator

$$M_{(h,q)} \Sigma = d \left( q \Phi \Sigma - \frac{1}{q} \Sigma \Phi \right) - \frac{h}{N} (q - \frac{1}{q}) \text{Tr}(\Phi \Sigma) \mathbf{1},$$

(2.10)

The commutator $\left[ M_{(h,q)}, M_{(h,q)}^\dagger \right]$ does not vanish, for $q \neq \pm 1$, only when acting on special vectors in the Cartan subalgebra. Then (2.7) becomes

$$S^{(2)} + S_{gf} = \int d^8 z \text{Tr} \left[ \varphi^1 (\Box^+)^{-1} (\varphi^1 - |M_{(g,1)}|^2) \varphi_1 - \frac{1}{2} v (\Box v - |M_{(g,1)}|^2) v \right] + \int d^8 z \left( \varphi_2 \varphi_2 + \varphi_3 \varphi_3 \right) + \left\{ \int d^6 z \text{Tr} \left( \varphi_3 M_{(h,q)} \varphi_2 \right) + \text{c.c.} \right\}.$$ 

(2.11)

Similarly, the quadratic part of the Faddeev-Popov ghost action takes the form

$$S_{gh}^{(2)} = \int d^8 z \text{Tr} \left[ c^\dagger (\Box^+)^{-1} (\Box^+ - |M_{(g,1)}|^2) c^\dagger (\Box^+)^{-1} (\Box^+ - |M_{(g,1)}|^2) c \right].$$

(2.12)

One should also take into account the Nielsen-Kallosh ghost action (A.7).

The one-loop effective action can then be shown to be

$$\Gamma_{\text{1-loop}} = \frac{1}{2} \text{Tr} \ln (\Box v - |M_{(g,1)}|^2) + i \text{Tr} \ln (\Box^+ - |M_{(h,q)}|^2) - i \text{Tr} \ln (\Box^+ - |M_{(g,1)}|^2).$$

(2.13)

In the case of $\mathcal{N} = 4$ SYM, we have $h = g$ and $q = 1$, and then only the contribution in the first line survives. As follows from the discussion in section 3 and Appendix C, $\Gamma_{\text{1-loop}}$ is finite.

### 3 Evaluation of the effective action

More specifically, we shall choose the background scalar and vector superfields as

$$\Phi = g^{-1} \phi H_0, \quad \mathcal{W}_\alpha = W_\alpha H_0.$$ 

(3.1)
where $\phi$ and $W_\alpha$ are singlet fields and $H_0$ is a special generator in the Cartan subalgebra of $SU(N)$. The characteristic feature of this field configuration is that it leaves the subgroup $U(1) \times SU(N-1) \subset SU(N)$ unbroken, where $U(1)$ is associated with $H_0$ and $SU(N-1)$ is generated by $\{H_L, E_{ij}\}$ (see Appendix B for the notation and explicit form of the Cartan-Weyl basis).

Not all components $u^\mu$ of a quantum superfield $u$ of the form (3.1) couple to the background vector multiplet. As follows from the identity

$$[H_0, E_{ij}] = e \left( \delta_{0i} E_{0j} - \delta_{0j} E_{0i} \right),$$

$$e \equiv \sqrt{\frac{N}{N-1}},$$

there are $(N-1)$ superfields $u^{0\pm}$ of charge $+e$, and $(N-1)$ superfields $u^{\pm0}$ of charge $-e$. The rest of components of $u^\mu$ are neutral. The mass operator (2.10) acts on the generators associated with the charged components as

$$\mathcal{M}_{(h,q)} E_{0\pm} = e\phi g^{-1} h \left[ q - \frac{1}{N} \left( q - \frac{1}{q} \right) \right] E_{0\pm} \equiv \mu^\pm_{(h,q)} E_{0\pm},$$

$$\mathcal{M}_{(h,q)} E_{\pm0} = -e\phi g^{-1} h \left[ \frac{1}{q} + \frac{1}{N} \left( q - \frac{1}{q} \right) \right] E_{\pm0} \equiv -\mu^-_{(h,q)} E_{\pm0}. \quad (3.4)$$

The two eigenvalues in (3.4) have the same norm if $|q| = 1$.

Let us now recall the condition that guarantees the one-loop anomalous dimension matrix for chiral superfields vanishes, which is the same as the UV finiteness condition up to 2 loops [9, 10, 23]:

$$|h|^2 \left[ \frac{1}{2} \left( |q|^2 + \frac{1}{|q|^2} \right) - \frac{1}{N^2} \left| q - \frac{1}{q} \right|^2 \right] = g^2. \quad (3.5)$$

For real $\beta$ deformation or $|q| = 1$, eq. (3.5) reduces to

$$|h|^2 \left( 1 - \frac{1}{N^2} \left| q - \frac{1}{q} \right|^2 \right) = g^2, \quad |q| = 1. \quad (3.6)$$

As was argued in [12] using the analogy [1] with the non-commutative theory, in the large $N$ limit, the condition of finiteness of the real deformation (3.6) or $|h| = g$, is actually
the exact condition for conformal invariance to all loops. The condition of finiteness for complex deformation case \([3, 5]\) is actually true to three-loop order \([9, 11]\) but is likely to receive higher-loop corrections.

Then it follows from (3.4) that in the real \(\beta\)-deformation case

\[
P(\mathcal{M}^{+\dagger}_{(h,q)} \mathcal{M}_{(h,q)} - |\mathcal{M}_{(g,1)}|^2) = O\left(\frac{1}{\sqrt{N}}\right), \tag{3.7}\]

where \(P\) is an orthogonal projector on the subspace of charged states,

\[
P E_{0\pm} = E_{0\pm}, \quad P E_{\pm} = E_{\pm}, \quad P E_{+\pm} = P H_I = 0. \tag{3.8}\]

One concludes that in the real \(\beta\) case the deformation-dependent contribution coming from the second line of (2.13) is subleading in the large \(N\) limit: one is left then with the contribution of the first line of (2.13), which is just the effective action in the \(N = 4\) SYM case. Equivalently, the planar limit of the one-loop effective action in the real deformation case does not depend on \(\beta\). This will not be true in the complex \(\beta\) case.

Let us first consider the case of finite \(N\). The effective action (2.13) is given by the contributions from several \(U(1)\)-charged superfields

\[
\Gamma_{1\text{-loop}} = i (N - 1) \text{tr}^{(e)} \ln (\Box_v - |\mu_{(g,1)}|^2)
+ i (N - 1) \text{tr}^{(e)} \ln \left[ (\Box_{+} - |\mu^+_{(h,q)}|^2) (\Box_{+} - |\mu^-_{(h,q)}|^2) (\Box_{+} - |\mu_{(g,1)}|^2)^{-2} \right], \tag{3.9}\]

with \(\mu^\pm_{(h,q)}\) defined in (3.4). The notation \(\text{tr}^{(e)}\) (or \(\text{tr}^{(e)}_{+}\)) indicates that the corresponding operator acts on the space of unconstrained (or chiral) superfields of charge \(e\).

The transformations needed to put (3.9) into an explicit proper-time representation form are described in Appendix C. The result is

\[
\Gamma_{1\text{-loop}} = -i (N - 1) \int_0^\infty ds \left( \int d^8 z K(z, z) - is \right) e^{-s|\mu_{(g,1)}|^2}
+ \int d^6 z K_+(z, z) - is \left[ e^{-s|\mu^+_{(h,q)}|^2} + e^{-s|\mu^-_{(h,q)}|^2} - 2 e^{-s|\mu_{(g,1)}|^2} \right], \tag{3.10}\]

where we have Wick-rotated the proper-time integrals. The background-dependent heat kernels \(K\) and \(K_+\) are defined in Appendix C. As is seen from (C.8), there is no need to introduce a regularization – the effective action is finite. Following [16] and introducing the functions

\[
\omega(x, y) = \omega(y, x) \equiv \frac{\cosh x - 1}{x^2} \frac{\cosh y - 1}{y^2} \frac{x^2 - y^2}{\cosh x - \cosh y},
\]

\[
\zeta(x, y) = \zeta(y, x) \equiv -\frac{1}{y^2} \left\{ \frac{\cosh x - 1}{x^2} \frac{x^2 - y^2}{\cosh x - \cosh y} - 1 \right\}, \tag{3.11}\]
one finally obtains

\[
\Gamma^{\text{1-loop}} = \frac{N}{16\pi^2} \int d^6z W^2 \ln \frac{e^2}{\sigma^+_{(h,q)} \sigma^-_{(h,q)}} + \frac{(N-1)}{8\pi^2} \int d^8z \frac{\overline{W}^2 W^2}{\phi^2 \phi^2} \int_0^\infty ds s e^{-s} \left[ \omega(s \Psi/e, s \overline{\Psi}/e) + \zeta(s \Psi/\sigma^+_{(h,q)}, s \overline{\Psi}/\sigma^+_{(h,q)}) \right. \\
+ \frac{\zeta(s \Psi/\sigma^-_{(h,q)}, s \overline{\Psi}/\sigma^-_{(h,q)})}{2(\sigma^+_{(h,q)}/e)^2} - \left. \zeta(s \Psi/e, s \overline{\Psi}/e) \right] .
\] (3.12)

Here we have used the following notation:

\[
\bar{\Psi}^2 = \frac{1}{4} D^2 \left( \frac{W^2}{\phi^2 \phi^2} \right) , \quad \Psi^2 = \frac{1}{4} D^2 \left( \frac{\overline{W}^2}{\phi^2 \phi^2} \right) ,
\] (3.13)

and defined

\[
\sigma^+_{(h,q)} \equiv e \left| g^{-1} h \left[ q - \frac{1}{N} (q - \frac{1}{q}) \right] \right|^2 , \quad \sigma^-_{(h,q)} \equiv e \left| g^{-1} h \left[ \frac{1}{q} + \frac{1}{N} (q - \frac{1}{q}) \right] \right|^2 .
\] (3.14)

The superfields \( \Psi \) and \( \overline{\Psi} \) are superconformal scalars \([16]\) so that the functional (3.12) is invariant under the \( N = 1 \) superconformal group. In (3.12), \( \phi \) and \( W_a \) may no longer be assumed to obey the constant field approximation.

If the deformation parameter \( \beta \) is real, i.e. \( |q| = 1 \), then \( \sigma^+_{(h,q)} = \sigma^-_{(h,q)} \equiv \sigma_{(h,q)} \). The condition of conformal invariance (3.10) implies \( \sigma_{(h,q)} = 1 + O(1/N) \), so that, as already mentioned above, in the large \( N \) limit the effective action for the real deformation reduces to that for the \( N = 4 \) SYM theory, i.e. to \([16]\)

\[
\Gamma^{\text{1-loop}} = \frac{N}{8\pi^2} \int d^8z \frac{\overline{W}^2 W^2}{\phi^2 \phi^2} \int_0^\infty ds s e^{-s} \left[ \omega(s \Psi, s \overline{\Psi}) + O(\frac{1}{N}) \right] .
\] (3.15)

In the case of general complex \( \beta \) deformation, i.e. \( |q| \neq 1 \), there is no simple relationship between \( \Gamma^{\text{1-loop}} \) and the effective action for \( N = 4 \) SYM. Keeping only the two- and four-derivative terms, the finite \( N \) effective action takes the form

\[
\Gamma^{\text{1-loop}} \approx \frac{N}{16\pi^2} C_2 \int d^6z W^2 + \frac{(N-1)}{16\pi^2} C_4 \int d^8z \frac{\overline{W}^2 W^2}{\phi^2 \phi^2} ,
\] (3.16)

\[
C_2 = \ln \frac{e^2}{\sigma^+_{(h,q)} \sigma^-_{(h,q)}}, \quad C_4 = 1 + \frac{1}{12} \left[ \frac{e^2}{(\sigma^+_{(h,q)})^2} + \frac{e^2}{(\sigma^-_{(h,q)})^2} - 2 \right] .
\] (3.17)
Using the finiteness relation between $h, g$ and $q$ (3.5), one finds in the $N \to \infty$ limit
\[
\frac{e^2}{\sigma_{(h,q)}^+} = \frac{1}{4} \left( \frac{|q|^2}{|q|^2} + \frac{1}{|q|^2} \right)^2 + O\left( \frac{1}{N} \right),
\]
(3.18)
\[
\frac{e^2}{(\sigma_{(h,q)}^+)^2} + \frac{e^2}{(\sigma_{(h,q)}^-)^2} = \frac{1}{4} \left( |q|^2 + \frac{1}{|q|^2} \right)^4 - \frac{1}{2} \left( |q|^2 + \frac{1}{|q|^2} \right)^2 + O\left( \frac{1}{N} \right).
\]
(3.19)
Thus $\Gamma_{1\text{-loop}}$ depends on $|q|$ even in the large $N$ limit.

4 Conclusions

The above computation illustrates the difference between the real and complex $\beta$ deformation cases. The real deformation is obviously much closer to the $\mathcal{N} = 4$ SYM theory. Its simplicity should have its origin in the possibility to give a noncommutative theory interpretation to the real $\beta$ deformation case [1], given that the noncommutative theories are known to simplify in the large $N$ limit.

In particular, as discussed in the Introduction, the matching between the leading terms in the D3-brane probe action in the dual geometry of [1] and in the above 1-loop large $N$ effective action for the real deformation case suggests that the corresponding $F^2$ and $F^4$ non-renormalization theorems of undeformed theory continue to hold in the large $N$ real deformation case, despite the reduction in the amount of supersymmetry from $\mathcal{N} = 4$ to $\mathcal{N} = 1$.

It would obviously be interesting to generalize the above computation of the effective action to more complicated backgrounds for which the real $\beta$ deformation dependence remains in the large $N$ limit. This would allow one to probe the dual supergravity background of [1] in a more non-trivial way. In Appendix D we present the expression for the leading $W^2$ term in $\Gamma_{1\text{-loop}}$ in a more general diagonal background that should correspond to several separated brane probes. Another open problem is to find the effective action in the presence of the second exactly marginal deformation $h' \sum_i \Phi_i^3$ of [2] for which the exact dual supergravity background is not known at present. Another direction is generalization to deformations of nonconformal theories, cf. [1, 25].
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A Background-field quantization

To quantize the $\beta$-deformed $\mathcal{N} = 4$ SYM theory (2.1) we use the $\mathcal{N} = 1$ background field formulation [19]. The first step is to implement the background-quantum splitting (2.6). Then the action becomes

$$S = \int d^8z \text{Tr} \left( (\Phi_i + \phi_i)^\dagger e^{gv} (\Phi_i + \phi_i) e^{-gv} \right) + \frac{1}{g^2} \int d^8z \text{Tr} \left( W^\alpha W_\alpha \right)$$

$$+ \left\{ \int d^6z \mathcal{L}_c(\Phi_i + \phi_i) + \text{c.c.} \right\},$$

where $L_c(\Phi_i)$ stands for the superpotential in (2.1), and

$$W_\alpha = -\frac{1}{8} \tilde{D}^2 \left( e^{-gv} D_\alpha e^{gv} \cdot 1 \right) = W_\alpha - \frac{1}{8} \tilde{D}^2 \left( g D_\alpha v - \frac{1}{2} g^2[v, D_\alpha v] \right) + O(v^3).$$

Since both the gauge and matter background superfields are non-zero, it is convenient to use the $\mathcal{N} = 1$ supersymmetric ’t Hooft gauge (a special case of the supersymmetric $R_\zeta$-gauge introduced in [21] and further developed in [22]) which is specified by the nonlocal gauge condition

$$-4\chi = \tilde{D}^2 v + g \left[ \Phi_i, (\square_+)^{-1} \tilde{D}^2 \phi_i \right] = \tilde{D}^2 v + g \left[ \Phi_i, \tilde{D}^2 (\square_-)^{-1} \phi_i \right].$$

Here $\square_+$ and $\square_-$ stand for the covariantly chiral and antichiral d’Alembertians,

$$\square_+ = \mathcal{D}^\alpha \mathcal{D}_a - W^\alpha W_\alpha - \frac{1}{2} (\mathcal{D}^\alpha W_\alpha) , \quad \square_- = \mathcal{D}^\alpha \mathcal{D}_a + \tilde{W}_\alpha \tilde{D}^\alpha + \frac{1}{2} (\tilde{D}_a \tilde{W}^a).$$

The gauge conditions chosen lead to the following Faddeev-Popov ghost action

$$S_{gh} = \text{Tr} \int d^8z \left( \tilde{c} - \tilde{c}^\dagger \right) \left\{ L_{gv/2} (c + c^\dagger) + L_{gv/2} \coth(L_{gv/2})(c - c^\dagger) \right\}$$

$$- \text{Tr} \int d^8z \left\{ g^2 \left[ \tilde{c}, \Phi_i \right] (\square_-)^{-1} [c^\dagger, \Phi_i^\dagger + \phi_i^\dagger] + g^2 [\tilde{c}, \Phi_i^\dagger] (\square_+)^{-1} [c, \Phi_i + \phi_i] \right\},$$
with $L_X Y = [X,Y]$. Here the ghost (anti-commuting) superfields $c$ and $\tilde{c}$ are background covariantly chiral. One should add also the standard gauge-fixing functional

$$S_{gf} = - \int d^8 z \text{Tr} (\chi^\dagger \chi) , \quad (A.6)$$

which is also accompanied by the Nielsen-Kallosh ghost action

$$S_{NK} = \int d^8 z \text{Tr} (b^\dagger b) , \quad (A.7)$$

where the third (anti-commuting) ghost superfield $b$ is background-covariantly chiral. The Nielsen-Kallosh ghosts lead to a one-loop contribution only.

**B Group-theoretical relations**

Let us describe the $SU(N)$ conventions adopted in this paper. Lower-case Latin letters from the middle of the alphabet, $i,j,\ldots$, are used to denote the matrix elements in the fundamental representation. We also set $i = (0,i) = 0,1,\ldots,N-1$. A generic element of the Lie algebra $su(N)$ is

$$u = u^I H_I + u^{ij} E_{ij} \equiv u^\mu T_\mu , \quad i \neq j . \quad (B.1)$$

We choose a Cartan-Weyl basis to consist of the elements:

$$H_I = \{ H_0, H_L \} , \quad L = 1,\ldots,N-2 , \quad E_{ij} , \quad i \neq j . \quad (B.2)$$

The basis elements defined as matrices in the fundamental representation are $[20]$,

$$(E_{ij})_{kl} = \delta_{ik} \delta_{jl} , \quad (H_I)_{kl} = \frac{1}{\sqrt{(N-I)(N-I-1)}} \left\{ (N-I) \delta_{kl} \delta_{li} - \sum_{i=I}^{N-1} \delta_{ki} \delta_{li} \right\} . \quad (B.3)$$

They satisfy

$$\text{Tr}(H_I H_J) = \delta_{IJ} , \quad \text{Tr}(E_{ij} E_{kl}) = \delta_{il} \delta_{jk} , \quad \text{Tr}(H_I E_{kl}) = 0 . \quad (B.4)$$

**C Proper-time representation for the effective action**

Here we present some technical details relevant for the evaluation of the effective action $(3.9)$; we follow $[24]$ where references to earlier work on covariant proper-time techniques in supersymmetric theories can be found, see also $[18]$. 
The effective action (3.9) can be expressed in terms of two different types of Green's functions in a background of a $U(1)$ vector multiplet described by the gauge covariant derivatives (2.2) with $\Gamma^{\mu}_A = \Gamma^{\mu}_A(z) e$. Here $e$ is a charge operator, $e = \pm e$. The corresponding gauge-invariant chiral field strength is $W_\alpha = W_\alpha e$. Associated with $\Box_v$ in (3.9) is the Green's function $G(z, z')$

$$\left( \Box_v - |\mu|^2 \right) G(z, z') = -\delta^8(z - z'), \quad G(z, z') = i \int_0^\infty ds \, K(z, z'|s) \, e^{-i|\mu|^2s}. \quad (C.1)$$

Associated with the chiral d’Alembertian $\Box_+$ in (3.9) is the Green’s function $G_+(z, z'|s)$ which is covariantly chiral in both arguments, $\bar{D}_\dot{\alpha}G_+(z, z') = \bar{D}'_\dot{\alpha}G_+(z, z') = 0$; it satisfies

$$\left( \Box_+ - |\mu|^2 \right) G_+(z, z') = -\delta_+(z, z'), \quad \delta_+(z, z') = -\frac{1}{4} \bar{D}^2 \delta^8(z - z'). \quad (C.2)$$

This Green’s function is generated by the chiral heat kernel $K_+(z, z'|s)$ which is introduced similarly to how this is done in (C.1).

The heat kernel in (C.1) has the following explicit form [24]

$$K(z, z'|s) = -\frac{i}{(4\pi s)^2} \sqrt{\det \left( \frac{2s \mathcal{F}}{e^{2s\mathcal{F}} - 1} \right)} \, U(s) \, \zeta^2 \bar{\zeta}^2 \, e^{\frac{i}{4} \rho \mathcal{F} \coth(s\mathcal{F})} \, I(z, z'). \quad (C.3)$$

Here

$$U(s) = \exp \left\{ -is(W^\alpha \mathcal{D}_\alpha + \bar{W}^{\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\alpha}}) \right\}, \quad I(z, z') = \exp \left\{ -i \int_{z'}^z dt \, \zeta^A \Gamma_A(z(t)) \right\}, \quad (C.4)$$

where the integration is carried out along the straight line connecting the points $z'$ and $z$. The variables $\rho$ and $\zeta$ are components of the supersymmetric two-point function $\zeta^A(z, z') \equiv (\rho^a, \zeta^\alpha, \bar{\zeta}_{\dot{\alpha}})$ defined as

$$\rho^a = (x - x')^a - i(\theta - \theta')\sigma^a \bar{\theta}' + i\theta' \sigma^a (\bar{\theta} - \bar{\theta}') \, , \quad (C.5)$$

$$\zeta^\alpha = (\theta - \theta')^\alpha \, , \quad \bar{\zeta}_{\dot{\alpha}} = (\bar{\theta} - \bar{\theta'})_{\dot{\alpha}} \, . \quad (C.6)$$

The chiral heat kernel $K_+(z, z'|s)$ is given by [24]

$$K_+(z, z'|s) = -\frac{1}{4} \bar{D}^2 K(z, z'|s) = -\frac{i}{(4\pi s)^2} \sqrt{\det \left( \frac{2s \mathcal{F}}{e^{2s\mathcal{F}} - 1} \right)} \, U(s) \times \zeta^2 \exp \left[ \frac{i}{4} \rho \mathcal{F} \coth(s\mathcal{F}) \rho - \frac{i}{2} \rho^a \mathcal{W}_{\sigma^a} \bar{\zeta} \right] \, I(z, z'). \quad (C.7)$$
For the values of the heat kernels at coincident points one obtains

\[-i K(z, z - is) = \frac{s^2}{(4\pi)^2} W^2 \bar{W}^2 \frac{\sinh^2(sB/2)}{(sB/2)^2} \frac{\sinh^2(s\bar{B}/2)}{(s\bar{B}/2)^2} \sqrt{\det \left( \frac{sF}{\sin(sF)} \right)},\]

\[-i K_+(z, z - is) = \frac{1}{(4\pi)^2} W^2 \frac{\sinh^2(sB/2)}{(sB/2)^2} \sqrt{\det \left( \frac{sF}{\sin(sF)} \right)},\]  

(C.8)

where we have introduced the notation

\[B^2 = \frac{1}{2} \text{tr} N^2, \quad \mathcal{N}_\alpha^\beta = D_\alpha W^\beta, \quad \bar{B}^2 = \frac{1}{2} \text{tr} \bar{N}^2, \quad \bar{N}_{\dot{\alpha}}^{\dot{\beta}} = \bar{D}_{\dot{\alpha}} \bar{W}^{\dot{\beta}}.\]  

(C.9)

For the background superfields under consideration, \(B^2 = \frac{1}{4} D^2 W^2\) and \(\bar{B}^2 = \frac{1}{4} \bar{D}^2 \bar{W}^2\). One also finds that

\[\sqrt{\det \left( \frac{sF}{\sin(sF)} \right)} = \frac{1}{2} s^2 (B^2 - \bar{B}^2) \cosh(sB) - \cosh(s\bar{B}).\]  

(C.10)

The resulting proper-time representation for the effective action is then given by (3.10).

D \quad \textbf{Leading term in the effective action for a more general diagonal background}

Here we present the expression for the effective action for a more general abelian background than the one studied in section 3: we shall allow the diagonal entries of the background fields to be different, i.e.

\[\Phi = \frac{1}{g} \text{diag} (\phi^1, \ldots, \phi^N), \quad W_\alpha = \text{diag} (W^1_\alpha, \ldots, W^N_\alpha), \quad \sum_{i=1}^N \phi^i = \sum_{i=1}^N W^i_\alpha = 0.\]  

(D.1)

The quantum superfields that couple to the background vector multiplet are associated with the off-diagonal generators \(E_{ij}\),

\[W_\alpha E_{ij} = (W^i_\alpha - W^j_\alpha) E_{ij} \equiv W^{[ij]}_\alpha E_{ij}.\]  

(D.2)

The mass operator (2.10) acts on the generators associated with charged components as

\[\mathcal{M}_{(h,q)} E_{ij} = g^{-1} h \left( q^i \phi^i - q^{-1} \phi^j \right) E_{ij} \equiv \mu^{[ij]}(h,q) E_{ij}.\]  

(D.3)
The effective action (2.13) is given by the sum of contributions from several $U(1)$-charged superfields

$$
\Gamma_{1-\text{loop}} = i \sum_{i<j} \text{tr}^{[ij]} \ln (\Box_v - |\mu^{[ij]}_{(g,1)}|^2)
$$

(D.4)

$$
+ i \sum_{i<j} \text{tr}^{[ij]}_+ \ln \left[ (\Box_+ - |\mu^{[ij]}_{(h,q)}|^2) (\Box_+ - |\mu^{[ji]}_{(h,q)}|^2) (\Box_+ - |\mu^{[ij]}_{(g,1)}|^2)^{-2} \right],
$$

with $\mu^{[ij]}_{(h,q)}$ defined in (D.3). The notation $\text{tr}^{[ij]}$ (or $\text{tr}^{[ij]}_+$) indicates that the corresponding operator $\Box_v$ (or $\Box_+$) is associated with the $U(1)$ vector multiplet of field strength $W_{\alpha} = W^{[ij]}_{\alpha}$. Eq. (D.4) leads to the following proper-time representation:

$$
\Gamma_{1-\text{loop}} = -i \sum_{i<j} \int_0^\infty \frac{ds}{s} \left( \int d^8z K^{[ij]}(z, z) e^{-s|\mu^{[ij]}_{(g,1)}|^2} \right)
$$

(D.5)

$$
+ \int d^6z K^{[ij]}_+(z, z) \left[ e^{-s|\mu^{[ij]}_{(h,q)}|^2} + e^{-s|\mu^{[ji]}_{(h,q)}|^2} - 2 e^{-s|\mu^{[ij]}_{(g,1)}|^2} \right]
$$

Here the four- and higher-derivative contributions can be written in a form similar to the second and third lines in (3.12). Compared to the first term in (3.12), the two-derivative part of (D.5) has non-trivial dependence on the scalar field background

$$
\Gamma_{1-\text{loop}} \approx \frac{1}{16\pi^2} \int d^6z \sum_{i<j} (W^{[ij]}_{\alpha})^2 \ln \left[ \frac{g^2(\phi^i - \phi^j)^2}{h^2(q \phi^i - q^{-1} \phi^i)(q^{-1} \phi^i - q \phi^j)} \right] + \text{c.c.} \quad \text{(D.6)}
$$

A similar expression for the 1-loop effective action on the Coulomb branch appeared in [14].

For the simplest background (3.1), the right-hand side in (D.6) reduces to the two-derivative part of (3.10). Indeed, non-vanishing contributions occur only if $i = 0$ and $j = j$, and then $W^{[0j]}_{\alpha} = e W_{\alpha}$, where $e = \sqrt{N/(N-1)}$.

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