Finite-dimensional Hopf $C^*$-bimodules and $C^*$-pseudo-multiplicative unitaries

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February 2, 2008

Abstract

Finite quantum groupoids can be described in many equivalent ways [8, 11, 16]: In terms of the weak Hopf $C^*$-algebras of Böhm, Nill, and Szlachányi [2] or the finite-dimensional Hopf-von Neumann bimodules of Vallin [14], and in terms of finite-dimensional multiplicative partial isometries [4] or the finite-dimensional pseudo-multiplicative unitaries of Vallin [15].

In this note, we show that in finite dimensions, the notions of a Hopf-von Neumann bimodule and of a pseudo-multiplicative unitary coincide with the notions of a concrete Hopf-$C^*$-bimodule and of a $C^*$-pseudo-multiplicative unitary, respectively.

1 Introduction

The theory of quantum groupoids is very well understood in the finite and in the measurable case, that is, in the setting of finite-dimensional $C^*$-algebras [9, 11, 16] and in the setting of von Neumann algebras [7]. The basic objects in this theory are the weak Hopf $C^*$-algebras and the multiplicative partial isometries of Böhm, Nill, and Szlachányi [2, 1, 3, 4] on one side and the Hopf-von Neumann bimodules and the pseudo-multiplicative unitaries of Vallin [15] on the other side. For finite quantum groupoids, both approaches are well known to be equivalent [8, 16].

To extend the theory of quantum groupoids to the locally compact case, that is, to the setting of $C^*$-algebras, we introduced the notion of a concrete Hopf $C^*$-bimodule and of a $C^*$-pseudo-multiplicative unitary [13]. In this short note, we show that in the finite-dimensional case, these concepts coincide with the notion of a Hopf-von Neumann bimodule and of a pseudo-multiplicative unitary, respectively. This note is of expository nature and the results contained in it are straightforward.

This work was supported by the SFB 478 “Geometrische Strukturen in der Mathematik” which is funded by the Deutsche Forschungsgemeinschaft (DFG).

Organization We proceed as follows:

In Section 2, we show that every $C^*$-factorization of a finite-dimensional Hilbert space is uniquely determined by the associated representation, and that the $C^*$-relative tensor product of finite-dimensional Hilbert spaces introduced in [13] coincides with the usual relative tensor product.

In Section 3, we show that in the finite-dimensional case, the spatial fiber product of $C^*$-algebras introduced in [13] coincides with the usual fiber product of von Neumann algebras,
and that the notion of a finite-dimensional concrete Hopf $C^\ast$-bimodule and of a finite-dimensional Hopf-von Neumann bimodule are equivalent.

In Section 4, we show that for finite-dimensional Hilbert spaces, the notion of a $C^\ast$-pseudo-multiplicative unitary \cite{11} and of a pseudo-multiplicative unitary are equivalent, and remark that the associated concrete Hopf $C^\ast$-bimodules and Hopf-von Neumann bimodules coincide.

**Preliminaries** Given a subset $Y$ of a normed space $X$, we denote by $[Y] \subseteq X$ the closed linear span of $Y$.

Given a Hilbert space $H$ and a subset $X \subseteq \mathcal{L}(H)$, we denote by $X'$ the commutant of $X$. Given Hilbert spaces $H, K$, a $C^\ast$-subalgebra $A \subseteq \mathcal{L}(H)$, and a $*$-homomorphism $\pi: A \rightarrow \mathcal{L}(K)$, we put

$$\mathcal{L}'(H, K) := \{T \in \mathcal{L}(H, K) \mid Ta = \pi(a)T \text{ for all } a \in A\};$$

thus, for example, $A' = \mathcal{L}^{\text{op}}(A)$. We shall make extensive use of (right) $C^\ast$-modules, also known as Hilbert $C^\ast$-modules or Hilbert modules. A standard reference is \cite{6}.

All sesquilinear maps like inner products of Hilbert spaces or $C^\ast$-modules are assumed to be conjugate-linear in the first component and linear in the second one.

Let $A$ and $B$ be $C^\ast$-algebras. Given $C^\ast$-modules $E$ and $F$ over $B$, we denote the space of all adjointable operators $E \rightarrow F$ by $\mathcal{L}_B(E, F)$.

Let $E$ and $F$ be $C^\ast$-modules over $A$ and $B$, respectively, and let $\pi: A \rightarrow \mathcal{L}_B(F)$ be a $*$-homomorphism. Then one can form the internal tensor product $E \otimes_A F$, which is a $C^\ast$-module over $B$ [5 Chapter 4]. This $C^\ast$-module is the closed linear span of elements $\eta \otimes_A \xi$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle \eta \otimes_A \xi | \eta' \otimes_A \xi' \rangle = \langle \xi | \pi(\eta' \otimes_A \xi') \rangle$ and $\langle (\eta \otimes_A \xi) b - \eta \otimes_A \xi b \rangle$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, and $b \in B$. We denote the internal tensor product by “$\otimes_A$”; thus, for example, $E \otimes_A F = E \otimes_A F$. If the representation $\pi$ or both $\pi$ and $A$ are understood, we write “$\otimes_A$” or “$\otimes$”, respectively, instead of “$\otimes_A$”.

Given $E, F$ and $\pi$ as above, we define a flipped internal tensor product $F \ast \otimes E$ as follows. We equip the algebraic tensor product $F \otimes E$ with the structure maps $\langle \xi \otimes \eta \xi' \otimes \eta' \rangle := \langle \xi | \pi(\eta' \otimes \eta) \rangle$, $\langle \xi \otimes \eta \xi' \rangle := \langle \eta \otimes \xi' \rangle$, and by factoring out the null-space of the semi-norm $\zeta \mapsto \|\zeta\|^{1/2}$ and taking completion, we obtain a $C^\ast$-$B$-module $F \ast \otimes E$. This is the closed linear span of elements $\xi \otimes \eta$, where $\eta \in E$ and $\xi \in F$ are arbitrary, and $\langle (\xi \otimes \eta) b - \xi \otimes \eta \rangle = \langle \xi | \pi(\eta' \otimes \eta) \rangle$ and $\langle (\xi \otimes \eta) b - \xi \otimes \eta \rangle$ for all $\eta, \eta' \in E$, $\xi, \xi' \in F$, and $b \in B$. As above, we write “$\ast$” or simply “$\otimes$” instead of “$\ast \otimes$” if the representation $\pi$ or both $\pi$ and $A$ are understood, respectively.

Evidently, the usual and the flipped internal tensor product are related by a unitary map $\Sigma: F \otimes E \rightarrow F \ast \otimes E$, $\eta \otimes \xi \mapsto \xi \otimes \eta$.

Given a state $\mu$ on a finite-dimensional $C^\ast$-algebra $B$, we denote by $(H_\mu, \pi_\mu, \zeta_\mu)$ a GNS-representation for $\mu$, by $J_\mu: H_\mu \rightarrow H_\mu$ the modular conjugation (an antilinear isometry), and by $\pi_\mu^{\text{op}}: B^{\text{op}} \rightarrow \mathcal{L}(H_\mu)$ the representation given by $b^{\text{op}} \mapsto J_\mu \pi_\mu(b) J_\mu$.

## 2 The relative tensor product of finite-dimensional Hilbert spaces

In the finite-dimensional case, the $C^\ast$-relative tensor product and the usual fiber product of Hilbert spaces coincide. Before we can prove this assertion, we need to recall the notion of a $C^\ast$-base and of a $C^\ast$-factorization.

### $C^\ast$-bases
Recall that a $C^\ast$-base is a triple $(\mathfrak{B}, \mathfrak{B}^\dagger)$, shortly written $\mathfrak{B}_B$, consisting of a Hilbert space $\mathfrak{B}$ and two commuting nondegenerate $C^\ast$-algebras $\mathfrak{B}, \mathfrak{B}^\dagger \subseteq \mathcal{L}(\mathfrak{B})$. We say
that two $C^*$-bases $(\mathfrak{h}, \mathfrak{B}, \mathfrak{B}^\dagger)$ and $(\mathfrak{h}, \mathfrak{C}, \mathfrak{C}^\dagger)$ are equivalent if there exists a unitary $U : H \to K$ such that $\mathfrak{C} = U \mathfrak{B}$ and $\mathfrak{C}^\dagger = U \mathfrak{B}^\dagger$; in that case, we write $(\mathfrak{h}, \mathfrak{B}, \mathfrak{B}^\dagger) \sim (\mathfrak{h}, \mathfrak{C}, \mathfrak{C}^\dagger).

**Definition 2.1.** Let $\mathfrak{H}_{\mathfrak{B}}$, be a $C^*$-base. We call a vector $\zeta \in \mathfrak{H}$ bicyclic if it is cyclic for $\mathfrak{B}$ and for $\mathfrak{B}^\dagger$. We call $\mathfrak{H}_{\mathfrak{B}}$, standard if there exists a bicyclic vector $\zeta \in \mathfrak{H}$, and finite-dimensional if $\mathfrak{H}$ has finite dimension.

**Example 2.2.** If $\mu$ is a KMS-state on a $C^*$-algebra $N$, then the triple $(H_\mu, \pi_\mu(N), \pi_\mu^\text{op}(N^\text{op}))$ is a standard $C^*$-base, called the $C^*$-base associated to $\mu$, and $\zeta_\mu \in H_\mu$ is bicyclic.

**Lemma 2.3.** If $\mathfrak{H}_{\mathfrak{B}}$, is standard and finite-dimensional, then $\mathfrak{B}^\dagger = \mathfrak{B}'$ and $\mathfrak{B} = (\mathfrak{B}^\dagger)'$.

**Proof.** By definition, $\mathfrak{B}^\dagger \subseteq \mathfrak{B}'$. If $\zeta \in \mathfrak{H}$ is bicyclic, then the map $j : \mathfrak{B}' \to H$ given by $T \mapsto T \zeta$ is injective and $j(\mathfrak{B}^\dagger) = \mathfrak{H} - j(\mathfrak{B}')$. Therefore $\mathfrak{B}^\dagger = \mathfrak{B}'$. $\square$

Using standard results on GNS-representations and the lemma above, one finds:

**Lemma 2.4.** If $\mathfrak{H}_{\mathfrak{B}}$, is a finite-dimensional standard $C^*$-base and $\zeta \in \mathfrak{H}_{\mathfrak{B}}$, is bicyclic, then the state $\mu := \langle \cdot, \zeta \rangle$ on $\mathfrak{B}$ is faithful and there exists a unique unitary $U : \mathfrak{H} \to H_\mu$ such that $U \zeta = \zeta_\mu$ and $\pi_\mu(\mathfrak{B}) = U \pi_\mathfrak{B}(\mathfrak{B}) U^*$. Moreover, then $\pi_\mu^\text{op}(\mathfrak{B}^\text{op}) = \mathfrak{B}^\dagger$. $\square$

**Remark 2.5.** Let $\mathfrak{H}_{\mathfrak{B}}$, be a finite-dimensional standard $C^*$-base, $\zeta \in \mathfrak{H}_{\mathfrak{B}}$, bicyclic, and $U : \mathfrak{H} \to H_\mu$ as above. Then we can identify $\mathfrak{B}^\dagger$ with $\mathfrak{B}^\text{op}$ via $(\pi_\mu^\text{op})^{-1} \circ U \pi_\mathfrak{B}$. More concretely, if $J : \mathfrak{H} \to \mathfrak{H}$ denotes the antiunitary part in the polar decomposition of the map $\mathfrak{H} \to \mathfrak{H}$, $b_\zeta \mapsto b^* \zeta$, then $J = U^* J_\mu U$, and the map $b^\mu \mapsto Jb^\mu J$ is an isomorphism $\mathfrak{B}^\text{op} \cong \mathfrak{B}^\dagger$.

**C$^*$-factorizations** Let $\mathfrak{H}_{\mathfrak{B}}$, be a standard $C^*$-base with bicyclic vector $\zeta \in \mathfrak{H}$ and let $H$ be a Hilbert space.

Recall that a $C^*$-factorization of $H$ with respect to $\mathfrak{H}_{\mathfrak{B}}$, is a closed subspace $\alpha \subseteq \mathcal{L}(\mathfrak{H})$ satisfying $[\alpha^* \alpha] = \mathfrak{B}$, $[\alpha \mathfrak{B}] = \alpha$, and $[\alpha \mathfrak{H}] = H$. We denote by $\text{C$^*$-fact}(H; \mathfrak{H}_{\mathfrak{B}})$ the set of all $C^*$-factorizations of $H$ with respect to $\mathfrak{H}_{\mathfrak{B}}$.

**Lemma 2.6.** For each $\alpha \in \text{C$^*$-fact}(H; \mathfrak{H}_{\mathfrak{B}})$, the map $\alpha \mapsto H$, $\xi \mapsto \xi$, is injective and has dense image.

**Proof.** If $\xi \in \alpha$ and $\xi \zeta = 0$, then $\xi \zeta = [\xi \mathfrak{B}] = [\rho_\alpha(\mathfrak{B}^\dagger) \xi \zeta] = 0$ and hence $\xi = 0$. Therefore, the map $\xi \mapsto \xi$ is injective. It has dense image because $[\alpha \zeta] = [\alpha \mathfrak{B}] = \alpha$. $\square$

From now on, we assume that $H$ has finite dimension. Let $\rho : \mathfrak{B}^\dagger \to \mathcal{L}(H)$ be a nondegenerate faithful representation and put

$$\mathcal{L}^\rho(\mathfrak{H}, H) := \{ T \in \mathcal{L}(\mathfrak{H}, H) | T b^\dagger = \rho(b^\dagger) T \text{ for all } b^\dagger \in \mathfrak{B}^\dagger \}.$$

**Lemma 2.7.**

i) For each $\xi \in H$, there exists a unique $R^\rho(\xi) \in \mathcal{L}^\rho(\mathfrak{H}, H)$ such that $R^\rho(\xi) \zeta = \xi$.

ii) For each $T \in \mathcal{L}^\rho(\mathfrak{H}, H)$, one has $T = R^\rho(T \zeta)$.

**Proof.** Straightforward. $\square$

The $C^*$-factorizations of $H$ are uniquely determined by their associated representations:

**Proposition 2.8.** Let $\rho : \mathfrak{B}^\dagger \to \mathcal{L}(H)$ be a nondegenerate faithful representation and $\alpha \in \text{C$^*$-fact}(H; \mathfrak{H}_{\mathfrak{B}})$.

i) There exists a unique nondegenerate faithful representation $\rho_\alpha : \mathfrak{B}^\dagger \to \mathcal{L}(H)$ such that $\rho_\alpha(b^\dagger) \xi \zeta = b^\dagger \xi \zeta$ for all $b^\dagger \in \mathfrak{B}^\dagger$, $\xi \in \alpha$, $\zeta \in \mathfrak{H}$.

ii) $\mathcal{L}^\rho(\mathfrak{H}, H) \subseteq \text{C$^*$-fact}(H; \mathfrak{H}_{\mathfrak{B}})$.

iii) $\rho_\alpha = \rho$ if and only if $\alpha = \mathcal{L}^\rho(\mathfrak{H}, H)$.
Proof. i) The representation \( \rho_\alpha \) is well-defined because for all \( \xi, \xi' \in \alpha, \zeta, \zeta' \in \mathcal{H} \), and \( b^1 \in \mathcal{B}^1 \),

\[
\langle \xi \xi^* b^1 \xi' \rangle - \langle \xi \xi^* \xi' \rangle - \langle \xi b^1 \xi' \rangle - \langle \xi (b^1)^* \xi' \rangle;
\]

here, we used \( \alpha^* \alpha \subseteq \mathcal{B} \equiv (\mathcal{B}^1)' \). Combining this calculation with the relation \([\alpha^* \alpha \delta] = \delta \), we find that \( \rho_\alpha \) is faithful. It is nondegenerate because \([\rho_\alpha (\mathcal{B}^1)H] - [\alpha^* \mathcal{B} \delta] - [\alpha \delta] - H \).

ii) Put \( \beta := \mathcal{L}^\omega (\mathcal{H}, \mathcal{H}) \). Lemma \( \ref{prop:2.7} \) ii) implies \([\beta \delta] = H \), and a short calculation shows \([\beta^* \beta] \subseteq (\mathcal{B}^1)' - \mathcal{B} \). We prove that this inclusion is an equality. Choose a bicyclic vector \( \zeta \in \mathcal{H} \) and consider the map \( j: \mathcal{B} \to \mathcal{L}(\mathcal{B}^1, \mathcal{C}) \) given by \( c \mapsto \langle \xi | c \cdot \zeta \rangle \). Since \( \zeta \) is cyclic for \( \mathcal{B}^1 \) and \( \mathcal{B}^1 \) commutes with \( \mathcal{B} \), this map is injective. Moreover, since \( \rho \) is faithful and

\[
j(\mathcal{R}_\xi (\xi^* \mathcal{R}_\zeta (\zeta^*))) = \langle \mathcal{R}_\xi (\xi^*) \mathcal{R}_\zeta (\zeta^*) \rangle - \langle \xi \rangle \in H,
\]

we have \( j([\beta \beta^*]) = \mathcal{L}(\mathcal{B}^1, \mathcal{C}) \cong j(\mathcal{B}) \). Consequently, \([\beta \beta^*] = \mathcal{B} \). Finally, we prove \([\beta^* \beta] - \beta \). Short calculations show that \([\beta \beta^*] \subseteq [\mathcal{R}_\xi (\xi^*) \mathcal{R}_\zeta (\zeta^*)] \mathcal{R}_\xi (\xi^*) \mathcal{R}_\zeta (\zeta^*) \mathcal{R}_\xi (\xi^*) \mathcal{R}_\zeta (\zeta^*)\) for each \( T \in \rho(\mathcal{B}^1)' \), \( \xi \in H \), and therefore, \([\beta \beta^*] - [\beta^* \beta] \subseteq [\mathcal{R}_\xi (\xi^*) \mathcal{R}_\zeta (\zeta^*)] \mathcal{R}_\xi (\xi^*) \mathcal{R}_\zeta (\zeta^*) \mathcal{R}_\xi (\xi^*) \mathcal{R}_\zeta (\zeta^*)\). Conversely, \( \beta - \beta \mathcal{R}_\delta \mathcal{R}_\delta \subseteq [\beta \beta^*] \)."
Define a sesquilinar form \( \langle \cdot , \cdot \rangle_\mu \) on \( H \otimes K \) by

\[
\langle \xi \otimes \eta | \xi' \otimes \eta' \rangle_\mu := \langle R^\mu_{\mu\sigma} (\xi)R^\sigma_{\mu\sigma} (\eta) | R^\mu_{\mu\sigma} (\xi')R^\sigma_{\mu\sigma} (\eta') \rangle_\mu
\]

for all \( \xi, \eta, \xi', \eta' \in K \).

Factoring out the null space of the associated seminorm, we obtain a Hilbert space \( H_\mu \otimes K \).

For all \( \xi \in H \) and \( \eta \in K \), we denote by \( \xi_\mu \otimes \eta \) the image of \( \xi \otimes \eta \) in \( H_\mu \otimes K \).

The \( C^*\)-relative tensor product of \( H \) and \( K \) with respect to \( \mathfrak{B}(\mathfrak{H}) \), \( \alpha, \beta \) is the internal tensor product \( H_\mu \otimes K := \alpha \otimes \mathfrak{B} \otimes \beta \) [13, Section 2].

**Proposition 2.10.** There exists a unitary

\[
\Phi^{U,\xi}_{\alpha,\beta} : H_\mu \otimes \xi, K \rightarrow H_\sigma \otimes \eta, K,
\]

\( \xi_\mu \otimes \eta \mapsto R^\mu_{\sigma\eta}(\xi)U \otimes \xi_\sigma \otimes R^\sigma_{\sigma\eta}(\eta)U \).

**Proof of Proposition 2.10.** By Proposition 2.8 and Lemma 2.7 ii), \( \alpha = \{ R^\mu_{\mu\sigma}(\xi) U \mid \xi \in H \} \) and \( \beta = \{ R^\sigma_{\sigma\eta}(\eta) U \mid \eta \in K \} \), and by definition,

\[
\langle R^\mu_{\mu\sigma}(\xi) U \otimes \eta | R^\sigma_{\mu\sigma}(\xi') U \otimes \eta \rangle \rightarrow \langle R^\mu_{\mu\sigma}(\xi) R^\mu_{\mu\sigma}(\xi') \rangle_\mu \eta \rightarrow \langle \xi_\mu \otimes \eta | \xi_\mu \otimes \eta \rangle
\]

for all \( \xi, \xi' \in H, \eta \in K \). Therefore, \( \Phi^{U,\xi}_{\alpha,\beta} \) is a well-defined isometry. It is surjective because \( \zeta \) is cyclic for \( \mathfrak{B} \) (and for \( \mathfrak{B}^1 \)). \( \square \)

### 3 Finite-dimensional Hopf \( C^* \)-bimodules

In the finite-dimensional case, the notion of a Hopf \( C^* \)-bimodule and of a Hopf-von Neumann bimodule coincide. To prove this view, we first review the fiber product of finite-dimensional \( C^* \)-algebras and the fiber product of morphisms.

**The fiber product of finite-dimensional \( C^* \)-algebras** The spatial fiber product of finite-dimensional \( C^* \)-algebras [13] coincides with the usual fiber product of \( C^* \)-algebras. To make this statement precise, we briefly recall the two constructions. Let

i) \( H \) and \( K \) be finite-dimensional Hilbert spaces,

ii) \( A \subseteq \mathcal{L}(H) \) and \( B \subseteq \mathcal{L}(K) \) be nondegenerate \( C^* \)-subalgebras,

iii) \( N \) be a finite-dimensional \( C^* \)-algebra with faithful state \( \mu \) and injective unital \( * \)-homomorphisms \( \rho : N^\mu \rightarrow A, \sigma : N \rightarrow B \),

iv) \( \mathfrak{B}(\mathfrak{N}) \), be a finite-dimensional standard \( C^* \)-base with bicyclic vector \( \zeta \in \mathfrak{B} \) and \( C^* \)-factorizations \( \alpha \in C^* \)-fact\( (A; \mathfrak{B}(\mathfrak{N})) \), \( \beta \in C^* \)-fact\( (B; \mathfrak{B}(\mathfrak{N})) \), where

\[
C^* \text{-fact}(A; \mathfrak{B}(\mathfrak{N})) = \{ \alpha \in C^* \text{-fact}(A; \mathfrak{B}(\mathfrak{N})) \mid \rho_\alpha(\mathfrak{B}) \subseteq A \},
\]

\[
C^* \text{-fact}(B; \mathfrak{B}(\mathfrak{N})) = \{ \beta \in C^* \text{-fact}(B; \mathfrak{B}(\mathfrak{N})) \mid \rho_\beta(\mathfrak{B}) \subseteq B \},
\]

such that (i) holds.

The fiber product of \( A \) and \( B \) with respect to \( \mu, \rho, \sigma \) is defined as follows. For each \( S \in \mathcal{N} \subseteq \rho(N^\mu) \) and \( T \in \mathcal{B} \subseteq \sigma(N^\mu) \), there exists a well-defined operator

\[
S_\mu \otimes \eta, T_\mu \otimes \sigma : \mathcal{N} \otimes \mathfrak{B} \rightarrow \mathcal{M} \otimes \mathfrak{B},
\]

The fiber product of \( A \) and \( B \) is the commutant \( A_\mu \otimes \mathfrak{B} B := (A \mathcal{N} \otimes \mathfrak{B} B)^\prime \subseteq \mathcal{L}(H_\mu \otimes K) \).
The spatial $C^*$-fiber product of $A$ and $B$ with respect to $\mathfrak{B}_\Sigma$, $\alpha, \beta$ is defined as follows. Using the isomorphisms

$$\alpha \otimes_{\rho, \beta} K \cong H_{\alpha \otimes \beta} K \cong H_{\rho, \beta} \cong \xi \otimes \eta \cong \xi \otimes \eta \equiv \xi \otimes \eta,$$  

(2)

(see [13 Section 2]), one defines for each $\xi \in \alpha$ and $\eta \in \beta$ operators

$$|\xi|_{1}: K \rightarrow H_{\alpha \otimes \beta} K, \quad \xi \mapsto \xi \otimes \eta, \quad \langle \xi \rangle_{1}: |\xi|_{1} : \xi \otimes \eta \mapsto \rho_{\beta}(\langle \xi \rangle_{1} \eta).$$

$$|\eta|_{2}: H \rightarrow H_{\alpha \otimes \beta} K, \quad \eta \mapsto \xi \otimes \eta, \quad \langle \eta \rangle_{2}: |\eta|_{2} : \xi \otimes \eta \mapsto \rho_{\alpha}(\langle \eta \rangle_{2} \xi).$$

(3)

Put $[\alpha]_{1} := \{ |\xi|_{1} | \xi \in \alpha \}$ and similarly define $[\beta]_{2}, [\beta]_{3}$. Then

$$A_{\alpha \otimes \beta} B = \{ T \in L(H_{\alpha \otimes \beta} K) \} T[\alpha]_{1} T^{*}[\alpha]_{1} B \subseteq [\alpha]_{1} B \text{ and } T|\beta|_{2} T^{*}[\beta]_{2} A \subseteq [\beta]_{2} A \}.$$

The two constructions described above coincide in the following sense:

**Proposition 3.1.** Conjugation by $\Phi_{\alpha \otimes \beta}: H_{\rho, \beta} \rightarrow H_{\alpha \otimes \beta} K$ induces an isomorphism

$$\phi_{\alpha \otimes \beta} : A_{\alpha \otimes \beta} B \cong A_{\alpha \otimes \beta} B.$$

**Proof.** Put $\Phi := \Phi_{\alpha \otimes \beta}$ and let $T \in L(H_{\alpha \otimes \beta} K)$. By definition, $T \in \text{Ad}_{\alpha}(A_{\rho, \beta} B)$ if and only if $[T, A \circ \text{id}_{\beta}] = 0 \iff [T, \text{id}_{\alpha} \circ \beta]$, that is, if and only if for all $\eta, \eta' \in \beta$ and $\xi, \xi' \in \alpha,$

$$\langle \eta \rangle_{2}|T, A \circ \text{id}_{\beta}|[\eta|_{2}| = 0 \quad \text{and} \quad \langle \xi \rangle_{1}|T, \text{id}_{\alpha} \circ \beta|[\xi|_{1}| = 0,$$

or, equivalently, if and only if $\langle \beta \rangle_{2}|T[\beta]|_{2} A \subseteq A^{\beta} - A$ and $\langle \alpha \rangle_{1}|T[\alpha]|_{1} \subseteq B^{\alpha} - B$.

If $T \in A_{\alpha \otimes \beta} B$, then $T \in \text{Ad}_{\alpha}(A_{\rho, \beta} B)$ because

$$[\alpha]_{1} T[\alpha]_{1} \subseteq \langle [\alpha]_{1} B \rangle \subseteq [\alpha]_{1} B,$$

and similarly $T^{*}[\alpha]_{1} \subseteq [\alpha]_{1} B$ and $T[\beta]_{2} T^{*}[\beta]_{2} A \subseteq [\beta]_{2} A$. 

**Morphisms of finite-dimensional $C^*$-algebras** Let $\mathfrak{B}_\Sigma$, be a $C^*$-base. A nondegenerate finite-dimensional concrete (shortly nfc.) $C^*$-base $\mathfrak{B}_\Sigma$-algebra $(H, A, \alpha)$ consists of a finite-dimensional Hilbert space $H$, a nondegenerate $C^*$-algebra $A \subseteq L(H)$, and a $C^*$-factorization $\alpha \in C^*$-fact$(A, \mathfrak{B}_\Sigma)$.

A morphism of nfc. $C^*$-base $\mathfrak{B}_\Sigma$-algebras $(H, A, \alpha)$ and $(K, B, \beta)$ is a $\alpha$-homomorphism $\pi: A \rightarrow B$ such that $\pi = [I, \alpha]$, where $I_{\alpha} := \{ V \in L(H, K) \} \alpha \subseteq \beta$, $V^{*} \pi \subseteq \alpha \}$. We denote the set of such morphisms by Mor$(A, B, \beta)$.

**Lemma 3.2.** Let $\mathfrak{B}_\Sigma$, be a standard $C^*$-base, $(H, A, \alpha), (K, B, \beta)$ nfc. $C^*$-base $\mathfrak{B}_\Sigma$-algebras, and $\pi: A \rightarrow B$ a unital $\alpha$-homomorphism. Then $\pi \in \text{Mor}(A, B, \beta)$ if and only if

$$\pi(p_{\alpha}(b)) = p_{\beta}(b) \quad \text{for all } b \in \mathfrak{B}.$$

(4)

**Proof.** If $\pi$ is a morphism, then [12] holds by [12] Lemma 2.2. Conversely, assume [11]. By Lemma 2.7 it suffices to prove $[\mathcal{L}^{\pi}(H, K)H] = K$. Since $A$ has finite dimension, there exist $n \in \mathbb{N}$ and central projections $p_{1}, \ldots, p_{n} \in A$ such that $\sum p_{i} = 1_{A}$ and such that each $p_{i} A$ is a matrix algebra. Since $\sum \pi(p_{i}) K = K$, it suffices to show that $[\mathcal{L}^{\pi}(p_{i} H, K)] = \pi(p_{i}) K$ for each $i = 1, \ldots, n$; here, $p_{i}: p_{i} A \rightarrow L(p_{i} K)$ denotes the restriction of $\pi$. But both the identity representation and the representation $\pi_{i}$ of $p_{i} A$ are direct sums of the irreducible representation of the matrix algebra $p_{i} A$ which is unique up to unitary equivalence, and therefore, $[\mathcal{L}^{\pi}(p_{i} H, K)] = \pi(p_{i}).$ 

\[ \square \]
The fiber product of morphisms  In the finite-dimensional case, the classical fiber product coincides with the spatial fiber product also on the level of morphisms. More precisely, let

\[ i) \quad H, K, L, M \text{ be finite-dimensional Hilbert spaces,} \]
\[ ii) \quad A \subseteq \mathcal{L}(H), B \subseteq \mathcal{L}(K), C \subseteq \mathcal{L}(L), D \subseteq \mathcal{L}(M) \text{ be nondegenerate } C^*\text{-algebras,} \]
\[ iii) \quad \phi : A \to C \text{ and } \psi : B \to D \text{ be unital } \ast\text{-homomorphisms,} \]
\[ iv) \quad N \text{ be a } C^*\text{-algebra, } \mu \text{ a faithful state on } N, \text{ and } \rho : N^{\text{op}} \to A, \sigma : N \to B, \nu : N^{\text{op}} \to C, \omega : N \to D \text{ injective unital } \ast\text{-homomorphisms,} \]
\[ v) \quad \mathfrak{g}, \mathfrak{h}, \text{ be a standard } C^*\text{-base, } \zeta \in \mathfrak{g} \text{ a bicyclic vector, and } \alpha \in C^*\text{-fact}(A; \mathfrak{g}, \mathfrak{h}), \beta \in C^*\text{-fact}(B; \mathfrak{g}, \mathfrak{h}), \gamma \in C^*\text{-fact}(C; \mathfrak{g}, \mathfrak{h}), \delta \in C^*\text{-fact}(D; \mathfrak{g}, \mathfrak{h}), \text{ and assume} \]
\[
    \left( \begin{smallmatrix} \mathfrak{g}, \mathfrak{h}, \mathfrak{k} \\ \mathfrak{g}', \mathfrak{h}', \mathfrak{k}' \end{smallmatrix} \right) \sim \left( \begin{smallmatrix} H_\mu, \pi_\mu(N), \pi_\mu^{\text{op}}(N^{\text{op}}) \\ U \zeta = \zeta_\mu \end{smallmatrix} \right), \quad \rho = \rho_\alpha \circ \text{Ad}_{U\mu} \circ \pi_\mu^{\text{op}}, \quad \sigma = \rho_\beta \circ \text{Ad}_{U\mu} \circ \pi_\mu, \quad \nu = \rho_\gamma \circ \text{Ad}_{U\mu} \circ \pi_\mu^{\text{op}}, \quad \omega = \rho_\delta \circ \text{Ad}_{U\mu} \circ \pi_\mu.
\]
\[ (5) \]

Note that by Example 2.2 and Proposition 2.3 (given the data listed in iv), we can construct the data listed in v) such that (5) is satisfied, and vice versa.

By Lemma 3.2 the following conditions are equivalent:

\[ i) \quad \phi \circ \rho = \nu, \quad \psi \circ \sigma = \omega, \quad \quad ii) \quad \phi \in \text{Mor}(A, \mathfrak{a}, \mathfrak{c}), \quad \psi \in \text{Mor}(B, \mathfrak{b}, \mathfrak{d}). \]
\[ (6) \]

Assume that these conditions hold. Then by [10] Proof of 1.2.4 and [13] Proposition 3.13, respectively, there exist unique \( \ast\)-homomorphisms
\[
\phi \ast_\mu \psi : A_\mu \ast_\mu B \to C_\mu \ast_\mu D \quad \text{ and } \quad \phi \ast_\mu \psi : A_\mu \ast_\mu B \to C_\mu \ast_\mu D
\]
such that for all \( X \in \mathcal{L}^\phi(H, L), Y \in \mathcal{L}^\phi(K, M), S \in A_\mu \ast_\mu B, T \in A_\mu \ast_\mu B,
\[
    \left( \phi \ast_\mu \psi \right)(S) \cdot (X \otimes Y) - (X \otimes Y) \cdot S \quad \text{ and } \quad \left( \phi \ast_\mu \psi \right)(T) \cdot (X \otimes Y) - (X \otimes Y) \cdot T.
\]

Proposition 3.3. If condition (6) holds, then the following diagram commutes:

\[ A_\mu \ast_\mu B \xrightarrow{\phi \ast_\mu \psi} C_\mu \ast_\mu D \]
\[ \phi = \frac{\phi \ast_\mu \psi}{\phi \ast_\mu \psi} \xrightarrow{\phi \ast_\mu \psi} U \zeta = \frac{\phi \ast_\mu \psi}{\phi \ast_\mu \psi} \]
\[ A_\mu \ast_\mu B \xrightarrow{\phi \ast_\mu \psi} C_\mu \ast_\mu D. \]

Proof. This follows from the definition of \( \phi \ast_\mu \psi \) and \( \phi \ast_\mu \psi \) and the fact that \( \Phi_{U, \zeta}^\phi(X \otimes Y) = (X \otimes Y) \Phi_{U, \zeta}^\phi \) for all \( X \in \mathcal{L}^\phi(H, L), Y \in \mathcal{L}^\phi(K, M) \). □

Finite-dimensional concrete Hopf \( C^*\)-bimodules and Hopf-von Neumann bimodules  Let us briefly recall the definition of a concrete Hopf \( C^*\)-bimodule and of a concrete Hopf-von Neumann bimodule. Suppose that

\[ i) \quad H \text{ is a finite-dimensional Hilbert space, } A \subseteq \mathcal{L}(H) \text{ is a nondegenerate } C^*\text{-subalgebra,} \]
\[ ii) \quad N \text{ is a finite-dimensional } C^*\text{-algebra with a faithful state } \mu \text{ and injective unital } \ast\text{-homomorphisms} \rho : N^{\text{op}} \to A \text{ and } \sigma : N \to A \text{ such that } \rho(N^{\text{op}}) \text{ and } \sigma(N) \text{ commute,} \]
\[ iii) \quad \mathfrak{g}, \mathfrak{h}, \text{ is a finite-dimensional standard } C^*\text{-base with bicyclic vector } \zeta \in \mathfrak{g} \text{ and compatible } C^*\text{-factorizations } \alpha \in C^*\text{-fact}(A; \mathfrak{g}, \mathfrak{h}) \text{ and } \beta \in C^*\text{-fact}(A; \mathfrak{g}, \mathfrak{h}). \]
and assume that condition (1) holds. Note that by Example 2.2, Lemma 2.4, and Proposition 2.8 (given the data listed in ii), we can construct the data listed in iii) such that (1) is satisfied, and vice versa.

We can form the classical fiber product $A_{\mu \Lambda} A$ and define representations

$$\rho_{[A]} : N^{op} \to A_{\mu \Lambda} A, \ x \mapsto 1 \otimes \rho(x), \quad \sigma_{[A]} : N \to A_{\mu \Lambda} A, \ y \mapsto \sigma(y) \otimes 1.$$  

A finite-dimensional Hopf-von Neumann bimodule is a tuple $(N, \mu, A, \rho, \sigma, \Delta)$, where $N, \mu, A, \rho, \sigma$ are as above and $\Delta : A \to A_{\mu \Lambda} A$ is a $\sigma$-homomorphism that satisfies $\Delta \circ \rho = \rho_{[A]}$ and $\Delta \circ \sigma = \sigma_{[A]}$ and makes the following diagram commute:

We can also form the spatial $C^*$-fiber product $A_{\alpha \beta} A$ and define $C^*$-factorizations

$$\alpha \cdot \beta := [\alpha][\beta] \in C^*$-fact$(A_{\alpha \beta} A; \beta \beta \beta \beta), \quad \beta \cdot \beta := [\beta][\beta] \in C^*$-fact$(A_{\alpha \beta} A; \alpha \alpha \alpha \alpha),$  

where $[\alpha][\beta]$ and $[\beta][\beta]$ were defined below Equation (6). The associated representations are given by

$$\rho_{[\alpha \cdot \beta]} (b^1) - 1 \otimes \rho_{\alpha} (b^1), \quad \rho_{[\beta \cdot \beta]} (b) - \rho_{\beta} (b) \otimes 1$$

for all $b^1 \in \mathfrak{g}, \ b \in \mathfrak{h}$ [13, Proposition 2.7]. A finite-dimensional concrete Hopf $C^*$-bimodule is a tuple $(\mathfrak{g} \mathfrak{h}, H, A, \alpha \beta, \Delta)$, where $\mathfrak{g} \mathfrak{h}, H, A, \alpha, \beta$ are as above and $\Delta \in \text{Mor}(A_{\alpha \beta} (A_{\alpha \beta} A; \alpha \alpha \alpha \alpha) \cap \text{Mor}(A_{\beta \beta} (A_{\alpha \beta} A; \beta \beta \beta \beta))$ makes the following diagram commute:

Combining the results obtained so far, we find:

**Proposition 3.4.** Let $\Delta_{\mu} : A \to A_{\mu \Lambda} A$ and $\Delta_{\beta} : A \to A_{\beta \beta} A$ be $\sigma$-homomorphisms such that $\Delta_{\mu} = \phi \otimes \phi \circ \Delta_{\mu}$. Then $(N, \mu, A, \rho, \sigma, \Delta)$ is a Hopf-von Neumann bimodule if and only if $(\mathfrak{g} \mathfrak{h}, H, A, \alpha \beta, \Delta)$ is a concrete Hopf-$C^*$-bimodule.

Thus, in the finite-dimensional case, Hopf-$C^*$-bimodules and Hopf-von Neumann bimodules are equivalent descriptions of the same objects.

## 4 Finite-dimensional pseudo-multiplicative unitaries

In the finite-dimensional case, the notion of a pseudo-multiplicative unitary and of a $C^*$-pseudo-multiplicative unitary coincide. To make this statement precise, we recall the necessary definitions. Let

i) $H$ be a finite-dimensional Hilbert space,

ii) $N$ be a finite-dimensional $C^*$-algebra with a faithful state $\mu$ and nondegenerate faithful representations $\rho : N^{op} \to \mathcal{L}(H)$ and $\sigma, \tilde{\sigma} : N \to \mathcal{L}(H)$ such that $\rho(N^{op}), \sigma(N), \tilde{\sigma}(N)$ commute pairwise,
iii) be a finite-dimensional standard $C^*$-base with bicyclic vector $\zeta \in \mathcal{H}$ and $C^*$-factorizations $\alpha \in C^*\text{-fact}(H; \mathfrak{A})$, $\beta, \tilde{\beta} \in C^*\text{-fact}(H; \mathfrak{B})$ such that $\alpha, \beta, \tilde{\beta}$ are pairwise compatible.

Assume that
\[
(\mathcal{H}, \mathfrak{A}, \mathfrak{B}^1) \sim_U (H_\mu, \pi_\mu(N), \pi^{op}_\mu(N^{op})), \quad U\zeta = \zeta_\mu, \tag{7}
\]
\[
\rho - \rho_\alpha \circ \text{Ad}_{\mu} \circ \pi^{op}_\mu, \quad \sigma - \rho_\beta \circ \text{Ad}_{\mu} \circ \pi^{op}_\mu, \quad \tilde{\sigma} - \rho_{\tilde{\beta}} \circ \text{Ad}_{\mu} \circ \pi^{op}_\mu.
\]

Similarly as before, we can, given the data listed in ii), construct the data listed in iii) such that (7) is satisfied, and vice versa.

By Proposition 2.10, we can identify $H_\mu \otimes_\mu H \cong H_\mu \otimes_\mu H \otimes \alpha H$ and $H_\mu \otimes_\mu H \cong H_\mu \otimes_\mu H$.

Let
\[
V : H_\mu \otimes_\mu H \cong H_\mu \otimes_\mu H \to H_\mu \otimes_\mu H \cong H_\mu \otimes_\mu H
\]
be a unitary. Recall that $V$ is a pseudo-multiplicative unitary [15] if for all $x \in N, y \in N^{op}$,
\[
V(\rho(y) \otimes_\mu 1) - (1 \otimes_\mu \rho(y))V, \quad V(\sigma(x) \otimes_\mu 1) - (\sigma(x) \otimes_\mu 1)V, \quad V(1 \otimes_\mu \sigma(x)) - (1 \otimes_\mu \sigma(x))V,
\]
and if the following diagram commutes,
\[
\begin{array}{c}
\xymatrix{
H_\mu \otimes_\mu H \ar[r]^{\text{id} \otimes \text{id}} & H_\mu \otimes_\mu H \\
H_\mu \otimes_\mu H \ar[u]^{V \otimes \text{id}} & H_\mu \otimes_\mu H \ar[l]_{\text{id} \otimes \text{id}} \\
H_\mu \otimes_\mu H \ar[u]_{\text{id} \otimes \text{id}} & H_\mu \otimes_\mu H \ar[l]^{V \otimes \text{id}}
\end{array}
\]
\[
\hspace{1cm}
\begin{array}{c}
\xymatrix{
H_\mu \otimes_\mu H \ar[r]^{\text{id} \otimes \text{id}} & H_\mu \otimes_\mu H \\
H_\mu \otimes_\mu H \ar[u]^{V \otimes \text{id}} & H_\mu \otimes_\mu H \ar[l]_{\text{id} \otimes \text{id}} \\
H_\mu \otimes_\mu H \ar[u]_{\text{id} \otimes \text{id}} & H_\mu \otimes_\mu H \ar[l]^{V \otimes \text{id}}
\end{array}
\tag{9}
\]

where \text{id} \otimes \Sigma^{\mu}$ and $\Sigma^{[2]} \otimes \mu$ flip the second and the third component in the respective relative tensor product.

On the other hand, $V$ is a $C^*$-pseudo-multiplicative unitary [13] if it satisfies
\[
V(\alpha \otimes_\mu \alpha) - \alpha \otimes_\mu \alpha, \quad V(\tilde{\beta} \otimes_\mu \beta) - \tilde{\beta} \otimes_\mu \beta, \quad V(\beta \otimes_\mu \beta) - \beta \otimes_\mu \beta
\]
and if the following diagram commutes,
\[
\begin{array}{c}
\xymatrix{
H_\mu \otimes_\mu H \ar[r]^{\text{id} \otimes \text{id}} & H_\mu \otimes_\mu H \\
H_\mu \otimes_\mu H \ar[u]^{V \otimes \text{id}} & H_\mu \otimes_\mu H \ar[l]_{\text{id} \otimes \text{id}} \\
H_\mu \otimes_\mu H \ar[u]_{\text{id} \otimes \text{id}} & H_\mu \otimes_\mu H \ar[l]^{V \otimes \text{id}}
\end{array}
\tag{11}
\]

where $\text{id} \otimes \Sigma^{[0]}$ and $\Sigma^{[2]} \otimes \mu$ flip the second and the third component in the respective $C^*$-relative tensor product.

Combining the results of the preceding section, we find:
Proposition 4.1. \( V : H_\beta \otimes_\delta H \cong H_\delta \otimes_\mu H \rightarrow H_\mu \otimes_\rho H \cong H_\rho \otimes_{\mu_\rho} H \) is a pseudo-multiplicative unitary if and only if it is a \( C^* \)-pseudo-multiplicative unitary.

Proof. By Proposition 2.8, \( V \) satisfies (8) if and only if it satisfies (10). Moreover, using the explicit formula for the identifications \( H_\delta \otimes_\mu H \cong H_\beta \otimes_\delta H \) and \( H_\delta \otimes_\mu H \cong H_\rho \otimes_{\mu_\rho} H \) given in Proposition 2.10, one easily verifies that diagram (9) commutes if and only if diagram (11) commutes.

Remark 4.2. If the unitary \( V \) above is a sufficiently well-behaved (\( C^* \)-)pseudo-multiplicative unitary, one can associate to it two finite-dimensional Hopf-von Neumann bimodules \([5]\) and two finite-dimensional concrete Hopf \( C^* \)-bimodules \([13]\). One easily verifies that these bimodules coincide in the sense of Proposition 3.4.

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