Yield Criterion for Concrete Formed From the Deviatoric and Meridional Shape Functions

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Abstract. Design of yield criterion for concrete using the deviatoric and meridional section shape functions is presented. For the deviatoric section, Podgórski type shape functions are applied and discussed, while for the meridians Cassini’s oval is considered. Derivation of the shape functions with determination of free parameters using typical experimental test is shown. Comparison with empirical data for concrete is given. The proposed approach is a quite general tool in development of failure criteria for frictional and brittle materials.

1. Introduction

Yield criteria and plastic potentials for frictional materials and concrete are still of interest of the research society. Experimental results show that many materials exhibit isotropic behaviour at failure. Limit state criteria are then primary dependent on the stress tensor. Therefore, the yield conditions are scalar-valued functions of three invariants of the stress tensor. Experimental data on failure of the mentioned materials in triaxial stress states show several challenging features which should be reproduced by theoretical models [3,6,8]. For example, for some materials plasticity surfaces can be closed, while for other they are open in the principal stresses space. It is recognised that continuous, differentiable and convex functions are the proper choice to define failure criteria and/or plastic potentials [2,4,5,9,10,12].

In this paper, an attempt is being made on formulation of yield criteria based on the shape functions applied to the deviatoric and meridional sections of the plasticity surface. To model deviatoric cross-section Podgórski’s shape function [9] is used, which is a generalization of other well-known criteria of Ottosen [8], Lade [7] and Coulomb-Mohr [11]. Using polynomial representation of a scalar-valued function a simple derivation of those shape functions is presented [11,12]. In case of the meridional shape function Cassini’s oval is used to generate a smooth bounded surface in the principal stresses space. In the following sections we derive the shape functions, neglecting convexity discussion referring the reader to [2,4]. The determination of free parameters is extensively presented. A comparison of the criterion’s prediction with selected empirical data for concrete is given.

2. Deviatoric shape functions

In this paper, we make use of the Haigh-Westergard space of principal stresses to illustrate representations of isotropic scalar-valued functions of the stress tensor. Therefore, we consider the cylindrical invariants of the stress tensor $\sigma : \zeta, r$ and $\Theta$, defined as follows:
\[ \xi = \frac{1}{\sqrt{3}} \text{tr} \sigma, \quad r = \sqrt{\text{tr} s^2}, \quad \Theta = \frac{1}{3} \arccos \left( \frac{\sqrt{6} \text{tr}s^3}{\sqrt{\text{tr}s^2}} \right) \quad (1) \]

and \( s = \sigma - \xi \mathbf{k}, \mathbf{k} = \mathbf{I}/\sqrt{3} \), where \( \mathbf{I} \) is the second order unit tensor. A yield criterion is then expressed via invariants (1), i.e.

\[ f(\sigma) = 0 \quad \rightarrow \quad f(\xi, r, \Theta) = 0 \quad \rightarrow \quad f(\xi, r, \cos 3\Theta) = 0. \quad (2) \]

A curve defined by (2) for fixed \( \xi \) located on the octahedral plane has to be periodic of \( 2\pi/3 \) period with the axes of symmetry at \( \Theta = k \pi/3 \) for \( k = 0, 1, 2 \). To meet this property invariants \( \cos 3\Theta \) or \( \sin 2\Theta \) can be used in the definition of the deviatoric shape function.

Let us first consider a function of only two invariants \( r \) and \( \Theta \) fulfilling required symmetry for the deviatoric cross-section. Appropriate polynomial function up to the sixth degree of the stress invariants \( \text{tr}s^2 \) and \( \text{tr}s^3 \) can be assumed as:

\[ f_0(\Theta, r) = d_0 + d_1 r^2 + d_2 r^3 \cos 3\Theta + d_4 r^4 + d_5 r^5 \cos 3\Theta + e_6 r^6 \sin^2 3\Theta, \quad (3) \]

where \( d_i \) and \( e_6 \) are constants, but they can be regarded as functions of the \( \xi \) invariant [11,12] as well.

In the following a special case of function (3) is discussed in details. The simplest function with noncircular shape can be assumed in the form [12]:

\[ f_0(\Theta, r) = d_0 + d_1 r^2 + d_3 r^3 \cos 3\Theta = 0, \quad (4) \]

where only two of three parameters are independent. Consider then \( r = R_r \) being a root of the equation (4) located on the tensile meridian \( (\Theta = 0) \), and \( r = R_c \) another root located on the compressive meridian \( (\Theta = \pi/3) \). Then equation (4) can be rewritten as:

\[ R_c^2 R_r^2 - (R_c^2 - R_c R_r + R_r^2) r^2 - (R_c^2 - R_r^2) r^3 \cos 3\Theta = 0. \quad (5) \]

It is convenient to introduce dimensionless shape parameter defined as:

\[ p = \frac{R_c}{R_r} \quad \text{with range} \quad \frac{1}{2} \leq p \leq 2. \quad (6) \]

As a result, equation (5) is transformed to the following form:

\[ \left( \frac{R_c}{r} \right)^3 - (p^2 - p + 1) \left( \frac{R_c}{r} \right) - p(p-1) \cos 3\Theta = 0. \quad (7) \]

This equation can be solved for \( R_c/r \) using trigonometric substitution. Equation (7) has three real roots for the range of \( p \) given in (6), where one root is always negative and remaining two are always positive. The smallest positive root is expressed as:

\[ \frac{R_c}{r} = \frac{1}{Q} \cos \left[ \frac{1}{3} \arccos \left( \frac{1}{Q} \right) \right] = \frac{g(\Theta)}{Q}, \quad (8) \]

where
\[ Q = \frac{\sqrt{3}}{2\sqrt{p^2 - p + 1}}, \quad \gamma = \frac{3\sqrt{3}(p-1)p}{2\sqrt{(p^2 - p + 1)^3}}. \]  

Function \( g(\Theta) \) introduced by Ottosen [8] and denoted as a shape function by Podgórski [9] describes shape of the deviatoric cross-section of the plasticity surface. It is worth mentioning that equation (7) defines three curves, while equation (8) specifies only one curve, which uniquely separates safe from unsafe stress states. This issue is extensively analysed in [5,10,11,12] among others, while in [1] representations of the shape functions for several failure criteria are presented.

More sophisticated shape function can be obtained based on general function (3). Making use of another root \( r = R_s \) located on the shear meridian \( (\Theta = \pi / 6) \) along with roots used for derivation of (5) the following results can be obtained:

\[ R_s \frac{r}{Q} = \frac{1}{Q} \cos \left[ \frac{1}{3} \arccos (\gamma \cos 3\Theta) - \frac{1}{3} \arccos \delta \right] = \frac{g(\Theta)}{Q}, \]  

where now

\[ Q = \sqrt{\frac{4p^2 - k^2(1 + p)^2}{4p(p - k^2)}}, \quad \gamma = \frac{k^2(1 + p)^2}{p^2} - 1 \quad \sqrt{1 - \frac{k^2(1 + p)^2}{4p^2}}, \]

\[ \delta = \frac{k(p - 1)[3p^2 - k^2(p^2 + 1)]}{2\sqrt{p(p - k^2)^3}} \quad \text{for} \quad 1 \leq p \leq 2, \]  

and with new dimensionless shape parameter \( k \) defined as:

\[ k = \frac{R_s}{R_r} \quad \text{of range} \quad \frac{\sqrt{3}p}{1 + p} \leq k \leq \frac{p}{\sqrt{p^2 - p + 1}}. \]  

The function \( g(\Theta) \) in (10) was introduced by Podgórski [9] as a generalization of the earlier proposals of Ottosen [8] and Lade [7]. Formulas for parameters (11) with appropriate limits were determined in [11]. In case of the upper limit of \( k \) Ottosen’s criterion is retrieved, for which \( \delta = 1 \). For lower limit in (12) one can obtain solution for the shape function being a generalization of classical Coulomb-Mohr criterion [11] with:

\[ \gamma = 1, \quad Q = \frac{\sqrt{3}}{2\sqrt{p^2 - p + 1}}, \quad \delta = \frac{3\sqrt{3}(p-1)p}{2\sqrt{(p^2 - p + 1)^3}}. \]  

In figure 1a curves defined by equation (8) are shown. The equilateral triangles are received for \( p = 1/2 \) and \( p = 2 \), while the shaded region reflects possible locations of curves for \( 1/2 \leq p \leq 2 \). Graphs of curves (10) are presented in figure 1b for several values of \( k \) parameter and fixed \( p = 1.5 \). The admissible region for the deviatoric sections is bound by the external curve of Ottosen and internal curve of Coulomb-Mohr.
3. Meridional shape functions

In this section characteristic meridians are proposed based on requirement that plastic potential has to be an even function of the second invariant $r$. The following equation of Cassini’s oval is one of possible choices [11]:

$$\left[ a^2 (\xi_v - \xi)^2 + r^2 \right]^2 - 2c^2 \left[ a^2 (\xi_v - \xi)^2 - r^2 \right] + c^4 \left( 1-b^2 \right) = 0,$$

where $a$, $c$, $b$, $\xi_v$ are free parameters, which are assumed to meet conditions: $a > 0$, $c > 0$, $\xi_v > 0$ and $0 < b \leq 1$. The proposed meridian is smooth with exception of the case $b = 1$, when equation (14) defines Bernoulli’s lemniscate with one singular point. In order to derive functions for the characteristic meridians, equation (14) is represented in the explicit form with $r \geq 0$:

$$r(\xi) = \frac{c}{\sqrt{4a^2 (\xi_v - \xi)^2 + b^2c^2 - a^2 (\xi_v - \xi)^2 + c^2}}$$

for $\xi \leq \xi_v$.  

(15)

Functions of the characteristic meridians $R_r(\xi)$, $R_c(\xi)$, and $R_s(\xi)$ describe the shape of the plasticity surface in the meridional cross-sections.

To define material parameters $a$, $b$, $c$, and $\xi_v$ involved in definition of the meridians typical experimental tests are used, being listed in table 1 [7-9]. Two of them are located on the tensile meridian ($\Theta = 0$), the next two on the compressive meridian ($\Theta = \pi / 3$) and the last one on the shear meridian ($\Theta = \pi / 6$).

Shift parameter $\xi_v$ can be determined by investigation of the limit case $b = 1$ and with $c \to \infty$. Then (15) is reduced to the linear function:

$$r(\xi) = a (\xi_v - \xi)$$

for $\xi \leq \xi_v$.  

(16)

Selecting tests located on the tensile meridian the following value of the parameter is obtained:
Next, roots of equation (15) for $r = 0$ are expressed as:

$$
\xi_{0}^\text{min} = \xi_{V} - \frac{c}{a} \sqrt{1+b} < 0 \quad \text{and} \quad \xi_{0}^\text{max} = \xi_{V} - \frac{c}{a} \sqrt{1-b} > 0 \quad \text{for} \quad \xi \leq \xi_{V}.
$$

In this study we assume $\xi_{0}^\text{max} = \xi_{0}$ to be a controlling parameter determining location of the surface apex for triaxial tension, within limits $\xi_{T} < \xi_{0} < \xi_{V}$.

### Table 1. Experimental tests used for calibration

| Principal stresses, invariants | Uniaxial tension | Uniaxial compression | Biaxial compression | Triaxial compression | Biplane shear |
|-------------------------------|------------------|----------------------|---------------------|----------------------|---------------|
| $\sigma_{T}$                  | $\sigma_{T}$     | 0                    | $-\sigma_{BC}$      | $\sigma_{TC}$, $\eta > 1$ | $-\sigma_{BC}$ |
| $\sigma_{C}$                  | 0                | 0                    | $-\sigma_{BC}$      | $-\sigma_{BC}$       | $-\sigma_{RS}$ |
| $\sigma_{R}$                  | $-\sigma_{C}$    | $-\sigma_{BC}$       | $-\sigma_{BC}$      | $-\eta \sigma_{TC}$   | $-\sigma_{RS}$ |

3.1. Calibration of the tensile meridian

Function of the tensile meridian is defined as:

$$
R_{T}(\xi) = \sqrt{c \sqrt{4a^{2}(\xi_{V} - \xi)^{2} + b^{2}c^{2} - \left[a^{2}(\xi_{V} - \xi)^{2} + c^{2}\right]}} ,
$$

where constants $a$, $b$, $c$ are determined using uniaxial tension and biaxial compression tests with assumed $\xi_{0}$. From the apex condition one can obtain:

$$
R_{T}(\xi_{0}) = 0 \quad \rightarrow \quad b = 1 - \frac{a^{2}}{c^{2}}(\xi_{V} - \xi_{0})^{2}.
$$

Introducing notations:

$$
h_{0} = \xi_{V} - \xi_{0} > 0, \quad h_{T} = \xi_{V} - \xi_{T} > 0, \quad h_{BC} = \xi_{V} - \xi_{BC} > 0,
$$

calibration equations based on (14) can be expressed as:

$$
\begin{align*}
(a^{2}h_{T}^{2} + r_{T}^{2}) - 2c^{2}\left(a^{2}h_{T}^{2} - r_{T}^{2}\right) + c^{4}(1-b^{2}) &= 0, \\
(a^{2}h_{BC}^{2} + r_{BC}^{2}) - 2c^{2}\left(a^{2}h_{BC}^{2} - r_{BC}^{2}\right) + c^{4}(1-b^{2}) &= 0.
\end{align*}
$$
Solution to system of equations (22) using result (20) leads to:

\[
2c^2 = \frac{(a^2 h_{BC}^2 + r_{BC}^2)^2 - (a^2 h_0^2 + r_0^2)^2}{(a^2 h_{BC}^2 - r_{BC}^2)^2 - (a^2 h_0^2 - r_0^2)^2}, \quad a^2 = \beta_1 + 2\sqrt{\beta_1} \cosh \left( \frac{1}{3} \arccosh \left( \frac{\beta_1}{\sqrt{\beta_1}} \right) \right),
\]

where additional notations are introduced:

\[
\beta_1 = -\frac{\alpha_1}{3\alpha_1}, \quad \beta_2 = \frac{\alpha_2}{9\alpha_1^2} - \frac{\alpha_3}{3\alpha_1}, \quad \beta_3 = -\frac{\alpha_1^3}{27\alpha_1^4} + \frac{\alpha_2\alpha_4}{6\alpha_1} - \frac{\alpha_3}{2\alpha_1},
\]

\[
\alpha_1 = (h_{BC}^2 - h_0^2)(h_{BC}^2 - h_0^2)(h_0^2 - h_0^2), \quad \alpha_2 = r_{BC}^2(r_{BC}^2 - r_{BC}^2),
\]

\[
\alpha_3 = r_{BC}^4(h_{BC}^2 - h_0^2)(h_0^2 + h_0^2 + 2h_{BC}^2 - r_{BC}^2 - h_{BC}^2)(h_0^2 + h_{BC}^2 - 2h_0^2),
\]

\[
\alpha_4 = r_{BC}^4(h_{BC}^2 - h_0^2)(h_{BC}^2 - h_0^2)(h_{BC}^2 - h_0^2) - 2r_{BC}^2 h_{BC}^2 - h_{BC}^2.
\]

3.2. Calibration of the compressive and shear meridians

The same type of meridian functions are assumed for the compressive meridian and the shear meridian:

\[
R_c(\xi) = \sqrt{c_c\sqrt{4a_c^2(\xi_v^2 - \xi^2)^2 + b_c^2 c_c^2 - \left[ a_c^2(\xi_v^2 - \xi^2)^2 + c_c^2 \right]}},
\]

\[
R_s(\xi) = \sqrt{c_s\sqrt{4a_s^2(\xi_v^2 - \xi^2)^2 + b_s^2 c_s^2 - \left[ a_s^2(\xi_v^2 - \xi^2)^2 + c_s^2 \right]}},
\]

with new set of material parameters, \(a_c, b_c, c_c\), and \(a_s, b_s, c_s\). Meridians (25) and (26) have the same apexes \(\xi_{v_{\min}}^0\) and \(\xi_{v_{\max}}^0\) as the tensile meridian, so one can find the following relations:

\[
b_c = b_s = b \quad \text{and} \quad c_c^2 = c_s^2 = \frac{a_c^2}{a_s^2}.
\]

Two remaining parameters are calculated from two experimental tests, one located on the compressive meridian and another on the shear meridian:

\[
\left( a_c^2 h_c^2 + r_c^2 \right)^2 - 2c_c^2 \left( a_c^2 h_c^2 - r_c^2 \right) + c_c^2 \left( 1 - b_c^2 \right) = 0,
\]

\[
\left( a_s^2 h_{rs}^2 + r_{rs}^2 \right)^2 - 2c_s^2 \left( a_s^2 h_{rs}^2 - r_{rs}^2 \right) + c_s^2 \left( 1 - b_s^2 \right) = 0.
\]

With additional notations:

\[
h_c = \xi_v - \xi_c > 0, \quad h_{rs} = \xi_v - \xi_{rs} > 0,
\]

solution to equations (29) is of the form:

\[
a_c^2 = a_c^2 r_c^2 \frac{a_c^4 h_c^2 + c^2 + c_c \sqrt{4a_c^4 h_c^2 + c_c^4 b_c^2}}{c_c^4 b_c^2 - (c_c^2 - a_c^2 h_c^2)^2}, \quad a_s^2 = a_s^2 r_{rs}^2 \frac{a_s^4 h_{rs}^2 + c^2 + c_s \sqrt{4a_s^4 h_{rs}^2 + c_s^4 b_s^2}}{c_s^4 b_s^2 - (c_s^2 - a_s^2 h_{rs}^2)^2}.
\]

It can be shown that for the proposed meridians the following conditions are met:
\[
\frac{R_c}{R_r} = \frac{a_c}{a} = p, \quad \frac{R_s}{R_r} = \frac{a_s}{a} = k. \tag{32}
\]

4. Results and discussions

For calibration of the surface typical strength ratios for concrete are taken from the literature: \( \sigma_y = 0.1 \sigma_c, \sigma_{bc} = 1.16 \sigma_c, \sigma_{bs} = 1.28 \sigma_c \) \[3,6\]. Calculated values of parameters are the following: \( a = 0.6355, \ b = 0.99999, \ c = 6.0293 \sigma_c, \ a_c = 1.0724, \ c_c = 10.175 \sigma_c, \) and \( a_s = 0.7104, \ c_s = 6.7402 \sigma_c \), which result in the shape parameters \( p = 1.6876, \ k = 1.1179 \). The shift parameter is \( \xi_0 = 0.1895 \sigma_c \), and apex was estimated to be \( \xi_0 = 0.8455 \xi_0 \) in order to pass through the test point of triaxial compression \( \sigma_{tc} = 2.00 \sigma_c \) with \( \eta = 4.06 \) (table 1).

Yield limit in the uniaxial compression test \( \sigma_c \) is treated as a scaling factor in the following presentation of the obtained results. In figure 2 characteristic meridians are plotted with marked tests used in calibration. Smoothness of the plasticity surface in the apex zone is shown. In figure 3 the deviatoric sections are shown for \( \xi = \xi_r, \xi = \xi_c, \xi = \xi_{bc}, \) and \( \xi = \xi_{bc} \). Three graphs are presented for the shape functions of Coulomb-Mohr, Podgórski and Ottosen respectively, showing differences in deviatoric shape of the analysed criterion.

![Figure 2](image)

**Figure 2.** The characteristic meridians of the plasticity surface with enlarged apex zone

![Figure 3](image)

**Figure 3.** Deviatoric sections for hydrostatic pressure set for calibration points in case of three types of the shape functions of Coulomb-Mohr, Podgórski and Ottosen, \( \sigma_i = \sigma_r \sqrt{2/3} \)

Three curves according to the shape functions of Coulomb-Mohr, Podgórski and Ottosen for the plane stress conditions are shown in figure 4. In the same figure comparison between Podgórski’ shape function prediction and experimental data of Kupfer et al. \[6\] is shown. Generally, the prediction reflects well empirical results. Noticeable difference is observed in the zone between the uniaxial compression test \( \sigma_c \) and the biplane shear test \( \sigma_{bs} \).
5. Conclusions
The proposed approach allows for convenient formulation of the failure criteria using split of the description onto definition of the shape functions for deviatoric sections and characteristic meridians. Meridians can be defined by variety of functions leading to the continuous, convex and closed or open plasticity surfaces.

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