Covariant Quantization of “Massive” Spin-$\frac{3}{2}$ Fields in the de Sitter Space

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Abstract

We present a covariant quantization of the free “massive” spin-$\frac{3}{2}$ fields in four-dimensional de Sitter space-time based on analyticity in the complexified pseudo-Riemannian manifold. The field equation is obtained as an eigenvalue equation of the Casimir operator of the de Sitter group. The solutions are calculated in terms of coordinate-independent de Sitter plane-waves in tube domains and the null curvature limit is discussed. We give the group theoretical content of the field equation. The Wightman two-point function $S_{\alpha\alpha}'(x,x')$ is calculated. We introduce the spinor-vector field operator $\Psi_\alpha(f)$ and the Hilbert space structure. A coordinate-independent formula for the field operator $\Psi_\alpha(x)$ is also presented.

1 Introduction

The recent observational data indicated that our universe in first approximation might be the de Sitter space-time. Quantum field theory in de Sitter space is a very important subject
and it was studied extensively in the 1980s due to the inflationary model and linear quantum gravity. This space is the simplest possible generalization of the Minkowski space-time. The quantization of various fields (scalar, spinor and vector fields) in the de Sitter space have been studied by several authors [1-3]. It has been shown that the massive and massless conformally coupled scalar fields in the de Sitter space correspond to the principal and complementary series representation of the de Sitter group, respectively [4, 5]. The massive vector field in the de Sitter space has been associated with the principal series, whereas massless field corresponds to the lowest representation of the vector discrete series in the de Sitter group [6, 7]. The massive and massless spin-2 fields in the de Sitter space have been also associated with the principal series and the lowest representation of the rank-2 tensor discrete series of the de Sitter group representation, respectively [8-14]. The importance of the massless spin-2 field in the de Sitter space is due to the fact that it plays the central role in quantum gravity and quantum cosmology.

Supergravity was proposed in 1970 for describing both gravity and the other interactions and based upon the principle of general relativity and quantum mechanics to “grand unified schemes”[15]. Supersymmetry is a global symmetry between boson and fermion and in local form (supergravity) includes gravitational field. In this framework, the fermionic partner of the gravitational field is a spin-$\frac{3}{2}$ field, which is called gravitino. Supersymmetry breaking in supergravity leads to massive gravitinos, and the gravitino gets mass by the super Higgs mechanism, so massive spin-$\frac{3}{2}$ fields are essential for understanding the effective description of supergravity processes in the de Sitter space [16]. The spin-$\frac{3}{2}$ field in flat space-time was studied by Rarita-Schwinger [17] and recently it has been considered in [18, 19]. The gravitino propagator has been derived in anti-de Sitter space [20].

It is instructive to perform a covariant quantization of spinor-vector field in the de Sitter space. The spinor field in the de Sitter space has been treated in many of the papers [21-24]. In this paper spinor-vector field in the de Sitter space is considered. For simplicity the following units are used:

\[ c = \hbar = 1, \quad [x^0/H] = 1, \quad [M] = H, \]

where \( c, \hbar \) and \( H \) are light velocity, Planck constant and Hubble parameter respectively. In section 2, we fix the de Sitter space notations and introduce the two independent Casimir operators. In section 3, spin-$\frac{3}{2}$ field equation is obtained as eigenvalue equation of the Casimir
operator. The classification of the unitary irreducible representation of the de Sitter group in terms of the two parameters $p$ and $q$ is discussed. In the Minkowskian limit these parameters represent a spin ($s$) and a mass ($m$) that classify the unitary irreducible representation of the Poincaré group. Then, we derive the de Sitter field equation from the second order Casimir operator.

Section 4 is devoted to solve the field equation. The solution is written in terms of a polarization spinor-vector part and a de Sitter-plane wave

\[ \Psi^i_\alpha(x) = \mathcal{U}^i_\alpha(x, \xi)(Hx, \xi)^\sigma : i = 1, 2, 3, 4; \alpha = 0, 1, 2, 3, 4, \]

where $i, \alpha, \sigma$ are a spinor index, a space-time index and a complex number respectively. A five-vector $\xi_\alpha$ is a future directed null vector in ambient space notation:

\[ \xi \in \mathbb{C}^+ = \{ \xi \in \mathbb{R}^5; \quad \eta_{\alpha\beta} \xi^\alpha \xi^\beta = (\xi^0)^2 - \vec{\xi}.\vec{\xi} - (\xi^4)^2 = 0, \quad \xi^0 > 0 \}. \]

The solution, $(Hx, \xi)^\sigma$, had been introduced in the context of harmonic analysis on de Sitter space in the framework of the $SO(1, 4)$ representation theory by Molchanov [25] and it has been completely developed by Bros et al. [4, 5]. The plane wave in curved space, such as three-dimensional Lobachevsky and Riemann models, were given by some authors [26]. In contrast to Minkowski space, $\mathcal{U}^i_\alpha(x, \xi)$ is a function of the space-time point $x^\alpha$, because the momentum operators acquire a spin part [24, 27]. The spinor-vector $\mathcal{U}^i_\alpha(x, \xi)$ can be fixed such that in the null curvature limit one obtain the spinor-vector in the Minkowskian space. These solutions are not globally defined due to the ambiguity concerning to the phase factor. For solving this problem the solution is considered in the complex de Sitter space [5]. This notation permits us to define the solution globally on de Sitter hyperboloid and independent of the choice of the metrics.

In section 5, the Wightman two-point function $S(x, y)$ is calculated. This function satisfies the conditions of: a) positivity, b) locality, c) covariance, and d) normal analyticity. Normal analyticity allows one to define the Wightman two-point function $S(x, y)$ as the boundary value of the analytic function $S(z_1, z_2)$ from the tube domains. The normal analyticity is related to the Hadamard condition, which selects an unique vacuum state [5, 10, 24]. $S(z_1, z_2)$ is defined in terms of spinor de Sitter plane-waves in their tube domains. In section 6, we introduce
the Hilbert space structure and define the field operators $\Psi_\alpha(f)$. We also give a coordinate-independent formula for the field operator $\Psi_\alpha(x)$. Finally, a brief conclusion and an outlook are given in section 7.

2 The de Sitter space notations

The de Sitter space is conveniently seen as a hyperboloid embedded in a five-dimensional Minkowski space:

$$X_H = \{ x \in \mathbb{R}^5; x^2 = \eta_{\alpha\beta}x^\alpha x^\beta = -H^{-2} = -\frac{3}{\Lambda}, \quad \eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1) \},$$

where $\Lambda$ is a positive cosmological constant. The metric is

$$ds^2 = \eta_{\alpha\beta}dx^\alpha dx^\beta|_{x^2=-H^{-2}} = g_{\mu\nu}dX^\mu dX^\nu, \quad \mu = 0, 1, 2, 3,$$

where the $X^\mu$ is space-time intrinsic coordinates on the de Sitter hyperboloid. A spinor-tensor field $\Psi_{\alpha_1..\alpha_l}(x)$ on $X_H$ can be viewed as an homogeneous function on $\mathbb{R}^5$ with an arbitrary degree of homogeneity $\sigma$ and the transversality condition [21]:

$$x \cdot \partial \Psi = \sigma \Psi,$$

$$x \cdot \Psi(x) = 0.$$

The tangential or transverse derivative on the de Sitter space is defined as

$$\partial^\top_\alpha = \theta_{\alpha\beta} \partial^\beta = \partial_\alpha + H^2 x_\alpha x \cdot \partial, \quad x \cdot \partial^\top = 0,$$

(2.1)

where $\theta_{\alpha\beta} = \eta_{\alpha\beta} + H^2 x_\alpha x_\beta$ is transverse projection tensor ($\theta_{\alpha\beta} x^\alpha = \theta_{\alpha\beta} x^\beta = 0$).

The kinematical group of the de Sitter space is the 10-parameter group $\text{SO}_0(1, 4)$ (connected component of the identity), which is one of the two possible deformations of the Poincaré group (the other one being $\text{SO}_0(2, 3)$ ). The unitary irreducible representations of $\text{SO}_0(1, 4)$ are characterized by the eigenvalues of the two Casimir operators $Q^{(1)}$ and $Q^{(2)}$. These operators commute with the group generators and they are constant in each unitary irreducible representation. They read

$$Q^{(1)} = -\frac{1}{2}L_{\alpha\beta}L^{\alpha\beta}, \quad Q^{(2)} = -W_\alpha W^\alpha, \quad W_\alpha = \frac{1}{8}\epsilon_{\alpha\beta\gamma\delta\eta}L^{\beta\gamma}L^{\delta\eta},$$

(2.2)
where $\epsilon_{\alpha\beta\gamma\delta\eta}$ is the usual antisymmetrical tensor in $\mathbb{R}^5$ and $L_{\alpha\beta} = M_{\alpha\beta} + S_{\alpha\beta}$ is an infinitesimal generator. The orbital part $M_{\alpha\beta}$ is

$$M_{\alpha\beta} = -i(x_\alpha \partial_\beta - x_\beta \partial_\alpha) = -i(x_\alpha \partial^\dagger_\beta - x_\beta \partial^\dagger_\alpha). \quad (2.3)$$

In order to precise the action of the spinorial part $S_{\alpha\beta}$ on a tensor field or spinor-tensor field, one must treat separately the integer and half-integer cases. Integer spin fields can be represented by tensor fields of rank $l$, $\Psi_{\gamma_1...\gamma_l}(x)$, and the spinorial action reads [28]

$$S_{\alpha\beta}^{(l)} \Psi_{\gamma_1...\gamma_l} = -i \sum_{i=1}^{l} \left( \eta_{\alpha\gamma_i} \Psi_{\gamma_1...\gamma_{i-1}\beta\gamma_i...\gamma_l} - \eta_{\beta\gamma_i} \Psi_{\gamma_1...\gamma_{i-1}\alpha\gamma_i...\gamma_l} \right), \quad (2.4)$$

where $(\gamma_i \rightarrow \beta)$ means $\gamma_i$ index replaced with $\beta$. Half-integer spin fields with spin $s = l + \frac{1}{2}$ are represented by four component spinor-tensor $\Psi_{i\gamma_1...\gamma_l}$ with $i = 1, 2, 3, 4$. In this case, the spinorial part is

$$S_{\alpha\beta}^{(s)} = S_{\alpha\beta}^{(l)} + S_{\alpha\beta}^{(\frac{1}{2})} \quad \text{with} \quad S_{\alpha\beta}^{(\frac{1}{2})} = -\frac{i}{4} [\gamma_\alpha, \gamma_\beta].$$

The five matrices $\gamma_\alpha$ are determined by the relations $[8, 24, 29]$

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\eta^{\alpha\beta}, \quad \gamma^0 = \gamma^\alpha \gamma^\alpha = \gamma^0 \gamma^0,$$

$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma^4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$

$$\gamma^1 = \begin{pmatrix} 0 & i \sigma^1 \\ i \sigma^1 & 0 \end{pmatrix}, \quad \gamma^2 = \begin{pmatrix} 0 & -i \sigma^2 \\ -i \sigma^2 & 0 \end{pmatrix}, \quad \gamma^3 = \begin{pmatrix} 0 & i \sigma^3 \\ i \sigma^3 & 0 \end{pmatrix}, \quad (2.5)$$

where $\sigma_i$ are Pauli matrices and $I$ is a $2 \times 2$ unit matrix. The Casimir operators are simple to manipulate in ambient space notation. Since $Q^{(1)}$ is a second order derivative operator, it is used for obtaining the field equation in this paper. In particular, it is easy to show that for a $l$-rank tensor field $\Psi_{\gamma_1...\gamma_l}(x)$ one has

$$Q^{(1)}_l \Psi = Q^{(1)}_0 \Psi - 2\Sigma_1 \partial x. \Psi + 2\Sigma_1 x \partial \Psi + 2\Sigma_2 \eta \Psi' - l(l + 1) \Psi, \quad (2.6)$$

where

$$Q^{(1)}_l = -\frac{1}{2} L_{\alpha\beta}^{(l)} L^{(l)}_{\alpha\beta} = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} - \frac{1}{2} S_{\alpha\beta}^{(l)} S^{\alpha\beta(l)} - M_{\alpha\beta} S^{\alpha\beta(l)}, \quad (2.7)$$

$$M_{\alpha\beta} S^{\alpha\beta(l)} \Psi(x) = 2\Sigma_1 \partial x. \Psi - 2\Sigma_1 x \partial \Psi - 2l \Psi. \quad (2.8)$$
\[ \frac{1}{2} S_{\alpha\beta}^{(l)} S^{\alpha\beta(l)} \Psi = l(l + 3) \Psi - 2\Sigma_2 \eta \Psi', \quad (2.9) \]

\[ Q_0^{(1)} = -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta}. \quad (2.10) \]

\( \Psi' \) is the trace of the \( l \)-rank tensor \( \Psi(x) \) viewed as a homogeneous function of the variables \( x^\alpha \) and \( \Sigma_p \) is the non-normalized symmetrization operator:

\[ \Psi'_{\alpha_1...\alpha_{l-2}} = \eta^{\alpha_l-1\alpha_l} \Psi_{\alpha_1...\alpha_{l-2} \alpha_l-1\alpha_l}, \quad (2.11) \]

\[ (\Sigma_p AB)_{\alpha_1...\alpha_l} = \sum_{i_1 < i_2 < ... < i_p} A_{\alpha_1...\alpha_{i_p}} B_{\alpha_{i_1}...\alpha_{i_p} \alpha_{i_{p+1}}...\alpha_l}. \quad (2.12) \]

For half-integer spin fields with spin \( s = l + \frac{1}{2} \), the \( S_{\alpha\beta}^{(1/2)} \) acts only upon the index \( i \), and we have [24, 30]

\[ S_{\alpha\beta}^{(1/2)} S^{\alpha\beta(l)} \Psi(x) = l\Psi(x) - \Sigma_1 \gamma(\gamma \cdot \Psi(x)). \]

In this case, the Casimir operator is

\[ Q_s^{(1)} = -\frac{1}{2} \left( M_{\alpha\beta} + S_{\alpha\beta}^{(l)} + S_{\alpha\beta}^{(1/2)} \right) \left( M^{\alpha\beta} + S^{\alpha\beta(l)} + S^{\alpha\beta(1/2)} \right) \]

\[ = Q_l^{(1)} - \frac{5}{2} + \frac{i}{2} \gamma_\alpha \gamma_\beta M^{\alpha\beta} - S_{\alpha\beta}^{(1/2)} S^{\alpha\beta(l)}. \quad (2.13) \]

Then we obtain

\[ Q_s^{(1)} \Psi(x) = \left( Q_l^{(1)} - l - \frac{5}{2} + \frac{i}{2} \gamma_\alpha \gamma_\beta M^{\alpha\beta} \right) \Psi(x) + \Sigma_1 \gamma(\gamma \cdot \Psi(x)), \quad (2.14) \]

or

\[ Q_s^{(1)} \Psi(x) = \left( -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} + \frac{i}{2} \gamma_\alpha \gamma_\beta M^{\alpha\beta} - l(l + 2) - \frac{5}{2} \right) \Psi(x) \]

\[ -2\Sigma_1 \partial x. \Psi(x) + 2\Sigma_1 x \partial x. \Psi(x) + 2\Sigma_2 \eta \Psi'(x) + \Sigma_1 \gamma(\gamma \cdot \Psi(x)). \quad (2.15) \]

The spin-\( \frac{3}{2} \) field equation can be written in terms of the Casimir operator \( Q^{(1)} \), which will be done in the next section.
3 Field equation

The operator $Q^{(1)}_{\frac{3}{2}}$ commutes with the group generators and consequently it is constant on each unitary irreducible representation. In fact, the spinor-vector unitary irreducible representations can be classified by using the eigenvalues of $Q^{(1)}$ and the field equation can be written as

$$\left(Q^{(1)}_{\frac{3}{2}} - <Q^{(1)}_{\frac{3}{2}}>\right)\Psi(x) = 0.$$  \hfill (3.1)

From Takahashi [31] and Dixmier [32], we get a general classification scheme for all the unitary irreducible representations of the de Sitter group, which may be labeled by a pair of parameters $(p, q)$ with $2p \in N$ and $q \in C$. In terms of which the eigenvalues of $Q^{(1)}$ and $Q^{(2)}$ are expressed as follows:

$$<Q^{(1)}> = [-p(p + 1) - (q + 1)(q - 2)], \quad <Q^{(2)}> = [-p(p + 1)q(q - 1)].$$  \hfill (3.2)

For spin-$\frac{3}{2}$ field, according to the possible values of $p$ and $q$, two types of unitary irreducible representation are distinguished for the de Sitter group $SO(1, 4)$ namely, the principal and the discrete series. The flat limit indicates that for the principal series the value of $p$ has the meaning of spin. For the discrete series case, the only representation which has a physically meaningful Minkowskian counterpart is $p = q = s$ case. Mathematical details of the group contraction and the relationship between the de Sitter and the Poincaré groups can be found in [33, 34]. The spin-$\frac{3}{2}$ field representations relevant to the present work are as follows:

i) The unitary irreducible representations $U^{\frac{3}{2}, \nu}_{\pm}$ in the principal series where $p = s = \frac{3}{2}$ and $q = \frac{1}{2} + i\nu$ correspond to the Casimir spectral values:

$$<Q^{(1)}_{\frac{3}{2}}> = \nu^2 - \frac{3}{2}, \quad \nu \in \mathbb{R}, \quad \nu > \frac{3}{2}.$$  \hfill (3.3)

Note that $U^{\frac{3}{2}, \nu}_{\pm}$ and $U^{\frac{3}{2}, -\nu}_{\pm}$ are equivalent.

ii) The unitary irreducible representations $\Pi^{\pm}_{\frac{3}{2}, q}$ of the discrete series, where $p = s = \frac{3}{2}$, correspond to

$$<Q^{(1)}_{\frac{3}{2}}> = -\frac{5}{2}, \quad q = \frac{3}{2}, \quad \Pi^{\pm}_{\frac{3}{2}, \frac{3}{2}},$$

$$<Q^{(1)}_{\frac{3}{2}}> = -\frac{3}{2}, \quad q = \frac{1}{2}, \quad \Pi^{\pm}_{\frac{3}{2}, \frac{1}{2}}.$$  \hfill (3.4)
Let us recall at this point the physical content of the principal series representation from the point of view of a Minkowskian observer at the limit $H = 0$. The principal series unitary irreducible representation $U^{3,\nu}_{\frac{3}{2}}$, $\nu > \frac{3}{2}$, contracts toward the direct sum of two massive spinor-vector unitary irreducible representations of the Poincaré group $\mathcal{P}^{<}(m, \frac{3}{2})$ and $\mathcal{P}^{>}(m, \frac{3}{2})$, with negative and positive energies, respectively:

$$U^{3,\nu}_{\frac{3}{2}}H \to 0 \quad \nu \to \infty \quad \to \mathcal{P}^{<}(m, \frac{3}{2}) \bigoplus \mathcal{P}^{>}(m, \frac{3}{2}).$$

The contraction limit has to be understood through the constraint $m_H = H\nu$. The quantity $m_H$, supposed to depend on $H$, goes to the Minkowskian mass $m$ when the curvature goes to zero.

The spin-$\frac{3}{2}$ field in discrete series corresponds to $\Pi^{\pm}_{\frac{3}{2}}$ and $\Pi^{\pm}_{\frac{3}{2}}$, in which the sign $\pm$ stands for the helicity. In these cases, the two representations $\Pi^{\pm}_{\frac{3}{2}}$, with $p = q = \frac{3}{2}$, have a Minkowskian interpretation. The representation $\Pi^{\pm}_{\frac{3}{2}}$ has a unique extension to a direct sum of two unitary irreducible representations $C(\frac{5}{2}, \frac{3}{2}, 0)$ and $C(-\frac{5}{2}, \frac{3}{2}, 0)$ of the conformal group $SO(2, 4)$ with positive and negative energies respectively [33, 35]. The latter restricts to the massless unitary irreducible representations of Poincaré group $\mathcal{P}^{>}(0, \frac{3}{2})$ and $\mathcal{P}^{<}(0, \frac{3}{2})$ with positive and negative energies respectively and positive helicity. The following diagrams illustrate these relations:

$$\Pi^{+}_{\frac{3}{2}} \quad \leftrightarrow \quad \bigoplus \quad \mathcal{C}(\frac{5}{2}, \frac{3}{2}, 0) \quad \bigoplus \quad \mathcal{C}(\frac{5}{2}, \frac{3}{2}, 0) \quad \leftrightarrow \quad \mathcal{P}^{>}(0, \frac{3}{2})$$

$$\Pi^{-}_{\frac{3}{2}} \quad \leftrightarrow \quad \bigoplus \quad \mathcal{C}(\frac{5}{2}, 0, \frac{3}{2}) \quad \bigoplus \quad \mathcal{C}(\frac{5}{2}, 0, \frac{3}{2}) \quad \leftrightarrow \quad \mathcal{P}^{>}(0, -\frac{3}{2})$$

$$\Pi^{+}_{\frac{3}{2}} \quad \leftrightarrow \quad \bigoplus \quad \mathcal{C}(\frac{5}{2}, 0, \frac{3}{2}) \quad \bigoplus \quad \mathcal{C}(\frac{5}{2}, 0, \frac{3}{2}) \quad \leftrightarrow \quad \mathcal{P}^{<}(0, -\frac{3}{2})$$

(3.5)

where the arrows $\leftrightarrow$ designate unique extension. It is important to note that the representations $\Pi^{\pm}_{\frac{3}{2}}$ do not have a corresponding flat limit.

Now let us consider the unitary irreducible representations of the principal series. For spin-$\frac{3}{2}$ field, we obtain the following field equation:

$$\left[Q^{(1)}_{\frac{3}{2}} - \left(\nu^2 - \frac{3}{2}\right)\right] \Psi(x) = 0,$$

(3.7)
where

\[ Q^{(1)}_\frac{3}{2} \Psi(x) = \left( -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} + \frac{i}{2} \gamma_{\alpha} \gamma_{\beta} M^{\alpha\beta} - 3 - \frac{5}{2} \right) \Psi(x) \]

\[-2\partial x. \Psi(x) + 2x \partial. \Psi(x) + \gamma(\gamma. \Psi(x)). \]

If the spin-$\frac{3}{2}$ field satisfies the following subsidiary conditions:

i) transversality, \( x. \Psi(x) = 0 \),

ii) divergencelessness, \( \partial^\top . \Psi(x) = 0 \),

iii) and \( \gamma. \Psi(x) = 0 \),

it can be associated with an unitary irreducible representation of the de Sitter group. Then the action of \( Q^{(1)}_\frac{3}{2} \) on \( \Psi(x) \) gives

\[ Q^{(1)}_\frac{3}{2} \Psi(x) = \left( -\frac{1}{2} M_{\alpha\beta} M^{\alpha\beta} + \frac{i}{2} \gamma_{\alpha} \gamma_{\beta} M^{\alpha\beta} - \frac{5}{2} - 3 \right) \Psi(x). \]  

(3.8)

Now by using (3.7) and (3.8) we obtain first order spin-$\frac{3}{2}$ field equation such as [8, 24, 29]

\[ \left( -i \not{x} - \partial^\top + 2i + \nu \right) \Psi_\alpha(x) = 0, \]

(3.9)

where \( \not{x} = \gamma_{\alpha} x^\alpha \) and \( \partial^\top = \gamma^\alpha \partial^\top_\alpha \). This equation is exactly the same as the de Sitter-Dirac spin-$\frac{1}{2}$ field equation. This procedure is similar to obtain the Dirac equation from the Klein-Fock-Gordon equation. It reduces to the usual Rarita-Schwinger equation in the Minkowski space-time in the null curvature limit and mass can be find by

\[ \lim_{H \to 0, \nu \to \infty} H \nu = m. \]

Similar to the spin-$\frac{1}{2}$ field in the de Sitter space [24], due to the orthogonality of the solutions and obtaining the Minkowskian solution in the null curvature limit, the adjoint spinor field \( \Psi_\alpha(x) \) in ambient space notation is defined as follows:

\[ \Psi_\alpha(x) \equiv \Psi_\alpha^\dagger(x) \gamma^0 \gamma^4. \]

It satisfies the following field equation:

\[ \Psi_\alpha(x) \left[ \gamma^4 \widehat{Q}^{(1)}_\frac{3}{2} \gamma^4 - \left( \nu^2 - \frac{3}{2} \right) \right] = 0, \]

(3.10)
or equivalently
\[ \Psi_{\alpha}(x) \gamma^4 \left( i \beta^\top \mathbf{k} - 2i + \nu \right) \gamma^4 = 0. \] (3.11)

The derivative acts to the left in the usual notation. In the next section the second order field equation (3.7) will be solved and in the appendix we solve the first order equation (3.9).

4 The de Sitter spin-\(\frac{3}{2}\) plane waves

By using the de Sitter plane waves, which were presented by Bros et al. [4], we calculated the de Sitter-Dirac plane wave for spinor field [24]. The spinor-vector solution can be written in terms of the spinor fields (for simplicity from now on we set \(H = 1\)):
\[ \Psi_{\alpha}(x) = Z_{\alpha}^\top \psi_1 + D_{\frac{3}{2}\alpha} \psi_2 + \gamma_{\alpha}^\top \psi_3, \] (4.1)
where \(\psi_1, \psi_2,\) and \(\psi_3\) are spinor fields. \(\gamma^\top\) is the transverse projection of \(\gamma, (\gamma^\top_{\alpha} = \theta_{\alpha\beta} \gamma^\beta)\) and \(D_{\frac{3}{2}} = -\partial^\alpha - \gamma^\top_{\alpha} \mathbf{k}. Z\) is an arbitrary five-component constant vector field:
\[ Z_{\alpha}^\top = \theta_{\alpha\beta} Z^\beta = Z_{\alpha} + x_\alpha x \cdot Z, \quad x \cdot Z^\top = 0. \]

Putting \(\Psi_{\alpha}\) in equation (3.7) and using the following identities:
\[ Q_{\frac{3}{2}} D_{\frac{3}{2}} = D_{\frac{3}{2}} Q_{\frac{3}{2}}, \] (4.2)
\[ Q_{\frac{3}{2}} \gamma^\top \psi = \gamma^\top (Q_0 - \frac{5}{2} + \mathbf{k} \cdot \beta^\top) \psi(x), \] (4.3)
\[ \mathbf{k} \cdot \beta^\top Z^\top \psi = Z^\top \mathbf{k} \cdot \beta^\top \psi - \gamma^\top \mathbf{k} (Z x) \psi + x_\alpha \mathbf{k} \cdot \gamma^\top \psi, \] (4.4)
\[ Q_{\frac{3}{2}} Z^\top \psi = Z^\top (Q_{\frac{3}{2}} - 3) \psi - 2D_1 (Z x) \psi - \gamma^\top \mathbf{k} (Z x) \psi + \gamma^\top (Z \gamma^\top) \psi, \] (4.5)
we find that the spinor fields \(\psi_1, \psi_2,\) and \(\psi_3\) must obey the following equations:
\[ \left( Q_{\frac{3}{2}} - (\nu^2 + \frac{3}{2}) \right) \psi_1 = 0 \quad \text{or} \quad \left( -i \mathbf{k} \gamma \beta + 2i + \nu \right) \psi_1(x) = 0, \] (4.6)
\[ \left( Q_{\frac{3}{2}} - (\nu^2 - \frac{3}{2}) \right) \psi_2 = 2(x Z) \psi_1 = 0, \] (4.7)
\[ \left( Q_0 - \frac{5}{2} + \mathbf{k} \cdot \beta - (\nu^2 - \frac{3}{2}) \right) \psi_3 - 3 \mathbf{k} (x Z) \psi_1 + (Z \gamma^\top) \psi_1 = 0. \] (4.8)
It is clear that $\psi_1$ is a “massive” spinor field associated to the principal series $U_{1^+2}$. The divergencelessness condition, $\partial^\top \Psi(x) = 0$, results in

$$(Z^\top \partial + 4Z.x)\psi_1 + (Q_0 + \not\partial^\top)\psi_2 + (4 \not\partial^\top + \gamma^\top \cdot \partial^\top)\psi_3 = 0. \quad (4.9)$$

The subsidiary condition $\gamma^\top \Psi(x) = 0$ gives

$$\psi_3 = -\frac{1}{4} (\gamma.Z^\top \psi_1 - (\gamma.\partial^\top + 4 \not\partial^\top)\psi_2). \quad (4.10)$$

By using equations (4.9) and (4.10), the spinor field $\psi_2$ and $\psi_3$ can be written in terms of spinor field $\psi_1$:

$$\psi_2 = \frac{1}{(\nu^2 + 1)} \left(2x.Z + \frac{2}{3} Z.\partial^\top + \frac{1}{3}(i\nu + 1) Z^\top \not\partial^\top \psi_1, \right.$$

$$\psi_3 = -\frac{1}{4} \left[\gamma.Z^\top - \frac{\gamma.\partial^\top + 4 \not\partial^\top}{(\nu^2 + 1)} \left(2x.Z + \frac{2}{3} Z.\partial^\top + \frac{1}{3}(i\nu + 1) Z^\top \not\partial^\top \right) \psi_1. \quad (4.11)$$

Then the spinor-vector solution is obtained in the following form:

$$\Psi_\alpha(x) = \left(Z^\top_\alpha - \frac{1}{4} \gamma^\top_\alpha Z^\top \right) \psi_1 - \left(\partial^\top_\alpha - \frac{1}{4} \gamma^\top_\alpha \not\partial^\top \right) \psi_2$$

$$= \left[Z^\top_\alpha - \frac{1}{4} \gamma^\top_\alpha Z^\top - \frac{\partial^\top_\alpha - \frac{1}{4} \gamma^\top_\alpha \not\partial^\top}{(\nu^2 + 1)} \left(2x.Z + \frac{2}{3} Z.\partial^\top + \frac{1}{3}(i\nu + 1) Z^\top \not\partial^\top \right) \psi_1. \quad (4.12)$$

This solution can be written in the following compact form:

$$\Psi_\alpha(x) = \mathcal{D}_\alpha \psi_1, \quad (4.13)$$

where

$$\mathcal{D}_\alpha = Z^\top_\alpha - \frac{1}{4} \gamma^\top_\alpha Z^\top - \frac{1}{\nu^2 + 1} \left(-\frac{1}{4}(1 - i\nu) \gamma^\top_\alpha Z^\top + 2\partial^\top_\alpha x.Z \right.$$ 

$$+ \frac{2}{3} \partial^\top_\alpha Z.\partial^\top + \frac{1}{3}(i\nu + 1)\partial^\top_\alpha Z^\top \not\partial^\top - \frac{1}{3}(i\nu + 1)\gamma^\top_\alpha Z.x \not\partial^\top - \frac{2}{3} \gamma^\top_\alpha Z.x \not\partial^\top \not\partial^\top$$

$$- \frac{1}{6} \gamma^\top_\alpha Z.\partial^\top \not\partial^\top - \frac{1}{6} i\nu \gamma^\top_\alpha \not\partial^\top + \frac{1}{12}(i\nu + 1)\gamma^\top_\alpha Z^\top \not\partial^\top \not\partial^\top \not\partial^\top \not\partial^\top$$

and $\psi_1$ is the solution of de Sitter-Dirac field equation. In the previous paper, the spinor field $\psi_1$ was explicitly calculated and the solutions are given by [8, 24]

$$(\psi_1)_1 = (x.\xi)^{-2 + i\nu} V(x, \xi),$$

$$(\psi_1)_2 = (x.\xi)^{-2 - i\nu} U(\xi), \quad (4.14)$$
where \( \mathcal{V}(x, \xi) = \mathcal{U}(\xi) \) and \( \xi \in \mathbb{C}^+ \). The two spinors \( \mathcal{V}(\xi) \) and \( \mathcal{U}(\xi) \) are

\[
\mathcal{U}^a(\xi) = \frac{\xi^0 - \xi \tilde{\gamma}^0 + 1}{\sqrt{2(\xi^0 + 1)}} \mathcal{U}^a(\xi^+), \quad \mathcal{V}^a(\xi) = \frac{1}{\sqrt{2(\xi^0 + 1)}} \mathcal{U}^a(\xi^-), \quad a = 1, 2,
\]

(4.15)

where

\[
\mathcal{U}^1(\xi^+) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \alpha \\ \alpha \end{array} \right), \quad \mathcal{U}^2(\xi^+) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \beta \\ \beta \end{array} \right),
\]

(4.16)

\[
\mathcal{U}^1(\xi^-) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \alpha \\ -\alpha \end{array} \right), \quad \mathcal{U}^2(\xi^-) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \beta \\ -\beta \end{array} \right),
\]

(4.17)

with \( \alpha = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \), \( \beta = \left( \begin{array}{c} 0 \\ 1 \end{array} \right) \) and \( \xi = \xi \equiv (1, \tilde{\xi}, \pm 1) \).

Finally the two possible solutions for \( \Psi_\alpha(x) \) are

\[
\Psi^a_1(\xi) = \left[ Z^\top_a - \frac{1}{4} \gamma^a_\alpha \mathcal{E}^\top - \frac{\partial^\top_a - \frac{1}{4} \gamma^a_\alpha \mathcal{D}^\top}{\nu^2 + 1} \left( 2x.Z + \frac{2}{3} Z.\partial^\top + \frac{1}{3}(i\nu + 1) \mathcal{E}^\top \mathcal{F} \right) \right] \mathcal{V}^a(\xi)(x.\xi)^{-2+i\nu} = \mathcal{V}^a_\alpha(x, \xi, Z)(x.\xi)^{-2+i\nu},
\]

(4.18)

and

\[
\Psi^a_2(x) = \left[ Z^\top_a - \frac{1}{4} \gamma^a_\alpha \mathcal{E}^\top - \frac{\partial^\top_a - \frac{1}{4} \gamma^a_\alpha \mathcal{D}^\top}{\nu^2 + 1} \left( 2x.Z + \frac{2}{3} Z.\partial^\top + \frac{1}{3}(i\nu + 1) \mathcal{E}^\top \mathcal{F} \right) \right] \mathcal{U}^a(\xi)(x.\xi)^{-2-i\nu} = \mathcal{U}^a_\alpha(x, \xi, Z)(x.\xi)^{-2-i\nu},
\]

(4.19)

where \( a \) is two spinor states equation (4.16) or (4.17). In the following we see that the sign of \( \nu \) in the plane wave play the role of the sign of energy in the Minkowskian limit. By taking the derivative of the plane wave \( (x.\xi)^\sigma \), the explicit forms of \( \mathcal{U}_\alpha \) and \( \mathcal{V}_\alpha \) are obtained in terms of \( \xi \) [appendix A.2].

The arbitrary constant vector \( Z \), which defines the polarization states in de Sitter space, is fixed in the null curvature limit \( H = 0 \) (in this part we add \( H \)). In this limit, \( (Hx \cdot \xi)^{-2-i\nu} \) becomes the plane wave \( e^{ik \cdot X} \) and \( \mathcal{V}(\xi) \) and \( \mathcal{U}(\xi) \) become the spinors \( \mathcal{V}(k) \) and \( \mathcal{U}(k) \) in the Minkowski space [24], and \( \mathcal{D}_\alpha \) becomes the vector polarization \( \varepsilon^{(\lambda)}(\mu)(k) \) in the Minkowskian space-time [6]:

\[
\lim_{H \to 0, \nu \to \infty} \mathcal{U}_\alpha(x, \xi, Z) (Hx(X) \cdot \xi)^{-2-i\nu} \equiv \varepsilon^{(\lambda)}(\mu)(k) \mathcal{U}(k) e^{-ik \cdot X},
\]

12
\[
\lim_{H \to 0, \nu \to \infty} \mathcal{V}_\alpha(x, \xi, Z)[Hx(X) \cdot \xi]^{-2+i\nu} \equiv \varepsilon^{(\lambda)*}(k)\mathcal{V}(k)e^{ikX},
\]
where \(\lambda\) is taking the three values for three polarization states of a massive vector field [6], and \(\xi\) is parameterized in terms of the four-momentum \(k\):

\[
\xi = \left(\frac{k^0}{mc}, \sqrt{\frac{k^2}{m^2c^2} + 1}, \frac{k}{mc}, -1\right).
\]

The four-vector \(\varepsilon^{(\lambda)}(k)\) is the three possible polarization vectors, which satisfies the following relations [36]:

\[
\varepsilon^{(\lambda)} \cdot k = 0, \quad \varepsilon^{(\lambda)} \cdot \varepsilon^{(\lambda')} = \delta_{\lambda\lambda'},
\]
\[
\sum_{\lambda=1}^{3} \varepsilon^{(\lambda)}(k)\varepsilon^{(\lambda)}(k) = -(\eta_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{m^2}),
\]
and \(\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)\). For simplicity we choose three five-component vectors \(Z^{(\lambda)}\) which obey the transverse constraints:

\[
Z^{(\lambda)} \cdot \xi = 0,
\]
\[
Z^{(\lambda)} = (\varepsilon^{(\lambda)}(k), Z^{(\lambda)}_4 = 0).
\]

Here, the de Sitter point \(x \equiv x_H(X)\) has been expressed in terms of the intrinsic coordinate \(X^\mu = (X_0 = ct, \vec{X})\) measured in units of the de Sitter radius \(H^{-1}\):

\[
x_H(X) = \left(x^0 = \frac{\sinh HX^0}{H}, \bar{x} = \frac{\vec{X}}{HX^0 \sin H \parallel \vec{X} \parallel}, \cosh HX^0 \sin H \parallel \vec{X} \parallel,\right.\]
\[
\left. x^4 = \frac{\cosh HX^0}{H} \cos H \parallel \vec{X} \parallel \right).
\]

Note that \((X^0, \vec{X})\) are global coordinates. The compact spherical nature of spatial part of de Sitter space-time at fixed \(X^0\) is apparent in (4.24).

These solutions are singular at \(x \cdot \xi = 0\), and they are not globally defined due to the ambiguity concerning to the phase factor. In contrast with the Minkowskian exponentials plan wave, these waves are singular on three-dimensional light-like manifolds and can at first be defined only on suitable halves of \(X_H\). We will need an appropriate \(i\epsilon\)-prescription (indicated below) to obtain global waves, for detail see [5]. For a complete determination, one may
consider the solution in the complex the de Sitter space-time \( X_H^{(c)} \) [4, 5]. The complex de Sitter space-time is defined by

\[
X_H^{(c)} = \left\{ z = x + iy \in \mathbb{C}^5; \quad \eta_{\alpha \beta} z^\alpha z^\beta = (z^0)^2 - \vec{z} \cdot \vec{z} - (z^4)^2 = -H^{-2} \right\}
\]

\[
= \left\{ (x, y) \in \mathbb{R}^5 \times \mathbb{R}^5; \quad x^2 - y^2 = -H^{-2}, \; x \cdot y = 0 \right\}.
\]  

(4.25)

Let \( T^\pm = \mathbb{R}^5 + iV^\pm \) be the forward and backward tubes in \( \mathbb{C}^5 \). The domain \( V^+ \) (resp. \( V^- \)) stems from the causal structure on \( X_H \):

\[
V^\pm = \left\{ x \in \mathbb{R}^5; \quad x^0 > \sqrt{\|\vec{x}\|^2 + (x^4)^2} \right\}.
\]  

(4.26)

The forward and backward tubes of the complex de Sitter space-time \( X_H^{(c)} \) are defined by their respective intersections with \( X_H^{(c)} \),

\[
T^\pm = T^\pm \cap X_H^{(c)}.
\]  

(4.27)

Finally, the “tuboid” is defined above \( X_H^{(c)} \times X_H^{(c)} \) by

\[
T_{12} = \{(z, z'); \; z \in T^+, \; z' \in T^-\}.
\]  

(4.28)

Details are given in [5]. When \( z \) varies in \( T^+ \) (or \( T^- \)) and \( \xi \) lies in the positive cone \( C^+ \), the plane wave solutions are globally defined because the imaginary part of \( (z, \xi) \) has a fixed sign. The phase is chosen such that

\[
\text{boundary value of } (Hz, \xi)^\sigma \big|_{x, \xi > 0} > 0.
\]  

(4.29)

Therefore we have

\[
\Psi_{1\alpha}(z) = U^{(\Lambda)}_{\alpha}(z, \xi)(z \cdot \xi)^{-2+i\nu},
\]

(4.30)

\[
\Psi_{2\alpha}(z) = V^{(\Lambda)}_{\alpha}(z, \xi)(z \cdot \xi)^{-2-i\nu},
\]  

(4.31)

in which \( z \in X_H^{(c)} \) and \( \xi \in C^+ \). The boundary value of the complexified solution is

\[
bv(z \cdot \xi)^{-2+i\nu} = (x \cdot \xi)^{-2+i\nu} + e^{-i\pi(-2+i\nu)}(x \cdot \xi)^{-2+i\nu},
\]

where \( (x \cdot \xi)_+ = \begin{cases} 0 & \text{for } x \cdot \xi \leq 0 \\ (x \cdot \xi) & \text{for } x \cdot \xi > 0 \end{cases} \) [37]. These solutions are globally defined and they are independent of the choice of the coordinate system in the de Sitter hyperboloid, i.e. they are independent of the choice of the metrics of the de Sitter space.
In the same way as in the Minkowskian space, it is seen that for the scalar and vector fields the two solutions are complex conjugate of each other, but for the spinor field, there is no such relation between them [24].

5 The Wightman two-point function

The Wightman two-point function of spin-$\frac{3}{2}$ field is defined as

$$S^{ij}_{\alpha\alpha'}(x,x') = \langle \Omega | \Psi^i_\alpha(x) \overline{\Psi}_{\alpha'}(x') | \Omega \rangle,$$  \hspace{1cm} (5.1)

where $x, x' \in X_H$. This function is a solution of equations (3.7) and (3.10) with respect to $x$ and $x'$ respectively. It can be found in terms of the Wightman two-point function of spinor field, which was calculated in the previous paper [24].

By using the recurrence formula (4.1), we define

$$S_{\alpha\alpha'}(x,x') = \theta_\alpha \theta_{\alpha'} S_1(x,x') - D_{\frac{3}{2}} \gamma^4 D_{\frac{3}{2}} \gamma_{\alpha'} - \gamma^4 S_3(x,x')\gamma^4 \gamma_{\alpha'} \gamma^4. \hspace{1cm} (5.2)$$

By imposing the two-point function $S_{\alpha\alpha'}$ to obey equation (3.7) and by using the identities of equations (4.2)-(4.5), $S_1$, $S_2$, and $S_3$ must be satisfied by the following equations:

$$[Q_{\frac{3}{2}} - (\nu^2 + \frac{3}{2})] S_1(x,x') = 0,$$  \hspace{1cm} (5.3)

$$[Q_{\frac{3}{2}} - (\nu^2 - \frac{3}{2})] S_2(x,x')\gamma^4 \overline{D}_{\frac{3}{2}} \gamma^4 - 2(x.\theta') (-S_1(x,x')) = 0,$$  \hspace{1cm} (5.4)

$$((Q_1 - \frac{7}{2} + \beta \beta^\top) - (\nu^2 - \frac{3}{2})) S_3(x,x')\gamma^4 \gamma_{\alpha'} \gamma^4 + 3 \beta (\theta'.x) S_1(x,x') + (\theta'.\gamma_{\alpha'}) S_1(x,x') = 0.$$  \hspace{1cm} (5.5)

By using the conditions

$$x^\alpha S_{\alpha\alpha'}(x,x') = \gamma^\alpha S_{\alpha\alpha'}(x,x') = \partial^\alpha S_{\alpha\alpha'}(x,x') = \partial^{\top\alpha} S_{\alpha\alpha'}(x,x') = 0,$$  

we find that $S_2$ and $S_3$ are given in terms of $S_1$ as

$$-S_2(x,x')\gamma^4 \overline{D}_{\frac{3}{2}} \gamma^4 = \frac{1}{(\nu^2 + 1)} \left( \frac{2}{3} \theta_{\alpha'} \beta^\top + 2x.\theta_{\alpha'} + \frac{1}{3} (i\nu + 1) \gamma^\top \theta_{\alpha'} \beta \right) S_1,$$

and

$$-S_3(x,x')\gamma^4 \gamma_{\alpha'} \gamma^4 = -\frac{1}{4} \left( \gamma^\top \theta_{\alpha'} \right) S_1 + \frac{1}{4} \left( \beta^\top + 4 \beta \right) S_2.$$
The Wightman function can then be written in the form

\[ S_{\alpha\alpha'}(x, x') = D_{\alpha\alpha'}(x, \theta^\top; x', \theta'^\top)S_1(x, x'), \quad (5.6) \]

where

\[
D_{\alpha\alpha'} = \theta_\alpha \cdot \theta'_{\alpha'} - \frac{1}{4} \gamma^\top_\alpha \gamma^\top \cdot \theta'_{\alpha'} - \frac{1}{\nu^2 + 1} \left( \frac{(i\nu + 1)}{4} \gamma^\top_\alpha \gamma^\top \cdot \theta'_{\alpha'} + 2\theta^\top_\alpha x \cdot \theta'_{\alpha'} 
+ \frac{2}{3} \theta^\top_\alpha \theta'_{\alpha'} \cdot \theta^+ - \frac{2}{3} \gamma^\top_\alpha \theta'_{\alpha'} \cdot x \rho^+ + \frac{(i\nu + 1)}{3} \theta^\top_\alpha \gamma^\top \cdot \theta'_{\alpha'} \rho^+ - \frac{(i\nu + 1)}{3} \gamma^\top_\alpha \theta'_{\alpha'} \cdot x \rho^+
- \frac{1}{6} \gamma^\top_\alpha \theta'_{\alpha'} \cdot \theta^+ - \frac{1}{6} i\nu \gamma^\top_\alpha \rho^+ \rho^+ \rho^+ - \frac{(i\nu + 1)}{12} \gamma^\top_\alpha \gamma^\top \cdot \theta'_{\alpha'} \rho^+ \rho^+ \rho^+ - \frac{1}{6} \gamma^\top_\alpha \theta'_{\alpha'} \cdot \theta^+ - \frac{1}{6} i\nu \gamma^\top_\alpha \rho^+ \rho^+ \rho^+ - \frac{(i\nu + 1)}{12} \gamma^\top_\alpha \gamma^\top \cdot \theta'_{\alpha'} \rho^+ \rho^+ \rho^+ \right),
\quad (5.7)
\]

and \( S_1 \) is solution to (5.3), which is given by [24]

\[
S_1 = i \frac{\nu(1 + \nu^2)}{64\pi \sinh(\pi \nu)} \left[ (2 - i\nu)P_{2-2\nu}(x, x') \rho^+ - (2 + i\nu)P_{2+2\nu}(x, x') \rho^+ \right] \gamma^4.
\quad (5.8)
\]

Similar to the spinor case [24], it is easy to show that this Wightman two-point function satisfies the following conditions.

a) **Positiveness**: for any spinor-vector test function \( f_\alpha \in \mathcal{D}(X_H) \), we have

\[
\int_{X_H \times X_H} \bar{f}_i(x) S_{\alpha\alpha'}^{ij}(x, x') f_j(x') d\sigma(x) d\sigma(x') \geq 0,
\quad (5.9)
\]

where \( \bar{f} \) is the adjoint of \( f \) and \( d\sigma(x) \) denotes the de Sitter-invariant measure on \( X_H \). \( \mathcal{D}(X_H) \) is the space of spinor-vector test function \( C^\infty \) with compact support in \( X_H \).

b) **Locality**: for every space-like separated pair \( (x, x') \), i.e. \( x \cdot x' > -H^{-2} \),

\[
S_{\alpha\alpha'}^{ji}(x, x') = -S_{\alpha'\alpha}^{ji}(x', x),
\quad (5.10)
\]

where \( S_{\alpha'\alpha}^{ji}(x', x) = < \Omega \mid \Psi_{\alpha}(x') \Psi_{\alpha}(x) \mid \Omega >. \)

c) **Covariance**:

\[
\Lambda_\alpha^\gamma \Lambda_{\beta'}^{\gamma'} g^{-1} S_{\alpha\alpha'}(\Lambda(g)x, \Lambda(g)x') i(g) = S_{\beta\beta'}(x, x'),
\quad (5.11)
\]

where \( \Lambda \in SO(1, 4) \), \( g \in Sp(2, 2) \) and \( g \gamma^\alpha g^{-1} = \Lambda_\beta^\alpha \gamma^\beta \). \( i(g) \) is the group involution defined by

\[
i(g) = -\gamma^4 g \gamma^4.
\quad (5.12)\]
d) **Transversality:**

\[ x \cdot S(x, x') = 0 = x' \cdot S(x, x'), \quad (5.13) \]

e) **Divergencelessness:**

\[ \partial_x \cdot S(x, x') = 0 = \partial_{x'} \cdot S(x, x'), \quad (5.14) \]

f) **Normal analyticity:** \( S_{\alpha\alpha'}(x, x') \) is the boundary value (in the distributional sense) of an analytic function \( S_{\alpha\alpha'}(z, z') \).

\( S_{\alpha\alpha'}(z, z') \) is maximally analytic, i.e., can be analytically continued to the “cut domain” [5, 24]:

\[ \Delta = \{ (z, z') \in X_H^c \times X_H^c : (z - z')^2 \leq 0 \}. \]

The Wightman two-point function \( S_{\alpha\alpha'}(x, x') \) is the boundary value of \( S_{\alpha\alpha'}(z, z') \) from \( T_{12} \) and the “permuted Wightman function” \( S_{\alpha'\alpha}(x', x) \) is the boundary value of \( S_{\alpha\alpha'}(z, z') \) from the domain:

\[ T_{21} = \{ (z, z') ; z' \in T^+, z \in T^- \}. \]

### 6 Quantum field

The existence of a two-point function with the above-mentioned properties allows us to define (via the reconstruction theorem [38]) a massive spin-\( \frac{3}{2} \) field operator \( \Psi_{\alpha} \), satisfying the field equation, as an operator-valued distribution on \( X_H \) defined on (a dense domain in) a separable Hilbert space \( \mathcal{H} \). The Hilbert space \( \mathcal{H} \) can be described as the Hilbertian sum [39]:

\[ \mathcal{H} = \mathcal{H}_0 \oplus \bigoplus_{n=1}^{\infty} A(\otimes^n \mathcal{H}_1), \quad (6.1) \]

where \( A \) denotes the antisymmetrization operation and

\[ \mathcal{H}_0 = \{ \lambda \Omega, \lambda \in \mathbb{C} \}. \quad (6.2) \]

\( \mathcal{H}_1 \) is defined as follows (given the positive definite inner product):

\[ \langle h, f \rangle = \int_{X_H \times X_H} \overline{h}(x_1) S(x_1, x_2) f(x_2) d\sigma(x_1) d\sigma(x_2). \quad (6.3) \]

A regular element \( h^i_{\alpha} \in \mathcal{H}_1 \) is a class of spinorial-vector test functions \( h(x) \) (i.e., in \( \mathcal{D}(X_H) \)) modulo the functions \( g \) such that the corresponding seminorm vanishes. The full Hilbert space
$\mathcal{H}_1$ is the completion with this norm of the space of regular elements. In terms of creation and annihilation operators the smeared field operators $\Psi_\alpha(f)$ are realized as

$$(\Psi(f)h)^{(n)}(x_1, i_1, \alpha_1; x_2, i_2, \alpha_2; \ldots; x_n, i_n, \alpha_n) =$$

$$\sqrt{n+1} \int_{X_H \times X_H} f^i(x)(S^{\alpha\beta})_i(x, y)h^{(n+1)}(y, \beta; x_1, i_1, \alpha_1; \ldots; x_n, i_n, \alpha_n) d\sigma(x)d\sigma(y)$$

$$+ \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (-1)^{k+1} f^i_{ik}(x_k)h^{(n-1)}(x_1, i_1, \alpha_1; \ldots; \hat{x}_k, \hat{i}_k, \hat{\alpha}_k; \ldots; x_n, i_n, \alpha_n). \quad (6.4)$$

The field operator $\Psi_\alpha$ in terms of plane waves and creation and annihilation operators reads as

$$\Psi_\alpha(x) = \int_\Gamma \sum_{a, \lambda} (c_\lambda(x)U^a_{\alpha}(x, \xi))(Hx \cdot \xi)^{2+i\nu} + e^{-i\pi(2+i\nu)}(Hx \cdot \xi)^{-2+i\nu}$$

$$+ (d^\dagger_\lambda(x)\nu^a_{\alpha}(x, \xi))(Hx \cdot \xi)^{2+i\nu} + e^{-i\pi(2+i\nu)}(Hx \cdot \xi)^{-2+i\nu}) d\mu_\Gamma(\xi),$$

$$\equiv \Psi^{(-)}_\alpha(x) + \Psi^{(+)}_\alpha(x). \quad (6.5)$$

By using this field operator, we have the two-point function (5.6) for any given value of the mass parameter $\nu$. Here $\Gamma$ denotes an orbital basis of $C^+$. $d\mu_\Gamma(\xi)$ is an invariant measure defined by

$$d\mu_\Gamma(\xi) = i\Xi w_{C^+}\big|_\Gamma,$$

where $i\Xi w_{C^+}$ denotes the 3-form on $C^+$ obtained from the contraction of the vector field $\Xi$ with the volume form

$$w_{C^+} = \frac{d\xi^0 \wedge \cdots \wedge d\xi^4}{d(x \cdot \xi)}. \quad (6.7)$$

The operators $c$ and $d$ annihilate the fundamental state and $c^\dagger$ and $d^\dagger$ create “one particle” states:

$$c_\lambda(\xi, \nu)|\Omega> = 0, \quad d_\lambda(\xi, \nu)|\Omega> = 0,$$

$$c^\dagger_\lambda(\xi, \nu)|\Omega> = |(\xi, \lambda, a)^{(c)}>, \quad d^\dagger_\lambda(\xi, \nu)|\Omega> = |(\xi, \lambda, a)^{(d)}>.$$

By using the above conditions one can show that $\Psi_\alpha(x)$ satisfies the locality properties [24]:

$$\{\Psi^i_\alpha(x_1), \Psi^j_\beta(x_2)\} = 0,$$

for every space-like separated pair $(x_1, x_2)$ in $X_H$. 18
7 Conclusions

In this paper, we have considered the “massive” field associated to the principal series of the de Sitter group $SO_0(1,4)$ with $<Q_\nu> = \nu^2 - \frac{3}{2}$, $\nu > \frac{3}{2}$. For the discrete series

$$<Q_{\frac{3}{2}}^\pm> = -\frac{15}{4} - (q + 1)(q - 2), \quad q = \frac{1}{2}, \frac{3}{2},$$

we can replace $\nu$ by $\nu = 0$ and $\nu = \pm i$ for $q = \frac{1}{2}, \frac{3}{2}$, respectively. For the discrete series only the representations $\Pi_{\frac{3}{2}}^\pm$ have a physically meaningful Poincaré limit. These are precisely the “massless” spinor-vector field and $\nu$ must be replaced by $\pm i$ in the previous formulas. But the solutions of equations (4.18) and (4.19) are divergent at this limit ($\nu = \pm i$). This type of singularity is actually due to the auxiliary conditions $(\partial \cdot \Psi = \gamma \cdot \Psi = 0)$ imposed in order to associate this field with a specific unitary irreducible representation of the de Sitter group. It can be as well understood from equation (4.11), allowing one to determine $\psi_2$ and $\psi_3$ in terms of $\psi_1$. To solve this problem, the auxiliary conditions must be dropped out and then the field equation becomes gauge invariant, which had been studied in Riemannian space-time in the intrinsic coordinate [40]. This situation will be considered in the ambient space notation and from a group theoretical point of view in a forthcoming paper [41].

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A Appendix

A.1 The solution of the first order field equation

In this appendix, we would like to find the solution of the first order equation (3.9). By substituting $\Psi_\alpha(x)$, which is given by equation (4.1), we have

$$\left(-i \not\! k \not\! \partial^T + 2i + \nu\right) \left(Z^\alpha_\alpha^T \psi_1 + D^\alpha_\alpha \psi_2 + \gamma^\alpha_\alpha^T \psi_3\right) = 0. \quad (A.1)$$

By using the following identities:

$$\not\! k \not\! \partial^T Z^\alpha_\alpha^T \psi_1 = -\gamma^\alpha_\alpha^T \not\! k x Z \psi_1 + x_\alpha \not\! k \not\! \partial^T \psi_1 + Z^\alpha_\alpha^T \not\! k \not\! \partial^T \psi_1, \quad (A.2)$$
\[
\begin{align*}
&\partial_t \beta^T \partial_x^T \psi_2 = \partial_t \beta^T \psi_2 + x_a \beta^T \psi_2 + \partial_x^T \psi_2, \quad (A.3) \\
&\partial_x^T \gamma_a \partial_x^T \psi_2 = 5 \gamma_a \beta \psi_2 - 4 x_a + \partial_x^T \beta \psi_2 + 2 \partial_x^T \beta^T \psi_2, \quad (A.4) \\
&\partial_x^T \gamma_a \beta^T \psi_3 = \gamma_a \beta^T \psi_3 + 4 x_a \beta \psi_3 + 2 \partial_x^T \beta \psi_3 + \gamma_a \beta^T \psi_3, \quad (A.5)
\end{align*}
\]

and by putting equation (4.10) in equation (A.5) and using of equations (A.2 - 4) we have

\[
\begin{align*}
&(\beta^T - 2 + i\nu)\psi_1 = 0, \quad (A.6) \\
&(i\nu - 1) \beta \psi_2 + \frac{i\nu - 1}{4} \beta^T \psi_2 + \frac{1}{4} \beta \beta^T \beta^T \psi_2 \\
&= \beta x.\beta^T \psi_1 - \frac{1}{4} \beta^T x.\beta^T \psi_1 + \frac{1}{4} \beta \beta^T \beta^T \psi_1 + \frac{i\nu}{4} \beta \beta^T \beta^T \psi_1 \\
&\beta^T \psi_2 = 2(i\nu - 1) \beta \psi_2 - \beta^T \psi_1. \quad (A.7)
\end{align*}
\]

By substituting equation (A.8) in equation (A.7), \(\psi_2\) and \(\psi_3\) are obtained as follows:

\[
\begin{align*}
\psi_2 &= \frac{1}{(\nu^2 + 1)} \left( 2 x.\beta^T + \frac{2}{3} Z.\beta^T + \frac{1}{3} (i\nu + 1) Z^T \beta \right) \psi_1, \quad (A.9) \\
\psi_3 &= -\frac{1}{4} \left[ \gamma Z^T - \frac{\gamma \beta^T + 4}{(\nu^2 + 1)} \left( 2 x.\beta^T + \frac{2}{3} Z.\beta^T + \frac{1}{3} (i\nu + 1) Z^T \beta \right) \right] \psi_1. \quad (A.10)
\end{align*}
\]

### A.2 Spinor-vector \(U\) and \(V\)

By taking the derivative of plane wave,

\[
\partial_x (H \beta x)^\sigma = \sigma \xi (H \beta x)^\sigma - 1,
\]

we obtain

\[
\begin{align*}
V_{a}(x, \xi, Z) &= Z_a^T - \frac{1}{4} \gamma_a \beta^T - \frac{1}{\nu^2 + 1} \left[ \frac{i\nu + 3}{3} Z_a^T \\
&+ \xi_a^T \left( \frac{(8i\nu - 16) x.\beta x.\beta^T}{3} + \frac{(i\nu - 1)(i\nu - 2)}{3} Z^T \beta \right. \\
&\left. + \frac{2(i\nu - 2)(i\nu - 3) Z.\xi^T}{(x.\xi)^2} \right) \right], \quad (A.11) \\
U_{a}(x, \xi, Z) &= Z_a^T - \frac{1}{4} \gamma_a \beta^T - \frac{1}{\nu^2 + 1} \left[ -\frac{i\nu + 3}{3} Z_a^T \\
&+ \xi_a^T \left( -\frac{(8i\nu + 16) x.\beta x.\beta^T}{3} - \frac{(i\nu + 1)(i\nu + 2)}{3} Z^T \beta \\
&\right. + \frac{2(i\nu + 2)(i\nu + 3) Z.\xi^T}{(x.\xi)^2} \right) \right], \quad (A.12)
\end{align*}
\]
### A.3 Two-point function

Here, the two-point function is calculated with respect to $x'$, which satisfies equation (3.10). Putting equation (5.2) in equation (3.10), we obtain

$$S_1(x, x') \gamma^4 \left[ \frac{\tilde{Q}_1}{2} - (\nu^2 + \frac{3}{2}) \right] \gamma^4 = 0, \tag{A.13}$$

$$D_{\frac{2}{4} \alpha} S_2(x, x') \gamma^4 \left[ \frac{\tilde{Q}_1}{2} - (\nu^2 - \frac{3}{2}) \right] \gamma^4 - 2S_1(x, x') \gamma_4^4 = 0, \tag{A.14}$$

$$\gamma_\alpha^\top S_3(x, x') \gamma^4 \left( \frac{\tilde{Q}_1}{2} - \frac{7}{2} + \frac{\gamma^\top \gamma'}{\theta} \right) - (\nu^2 - \frac{3}{2}) \gamma^4$$

$$-3S_1(x, x') \gamma^4 \gamma'(\theta, x') \gamma^4 - S_1(x, x') \gamma^4 \left( \theta, \gamma_\alpha^\top \gamma' \right) \gamma^4 = 0. \tag{A.15}$$

By using the subsidiary conditions:

$$x^\alpha S_{\alpha\alpha'}(x, x') = S_{\alpha\alpha'}(x, x') \gamma^4 \gamma_{\alpha'} \gamma^4 = S_{\alpha\alpha'}(x, x') \frac{\hat{\gamma} \gamma'}{\theta} = S_{\alpha\alpha'}(x, x') \frac{\hat{\gamma} \gamma'}{\theta} = 0,$$

we can write $S_2$ and $S_3$ in terms of $S_1$:

$$D_{\frac{2}{4} \alpha} S_2(x, x') = \frac{1}{(\nu^2 + 1)} S_1 \left( 2x' \cdot \theta_\alpha + \frac{2}{3} \theta_\alpha \frac{\hat{\gamma} \gamma'}{\theta} - \frac{1}{3} (i\nu + 1) \gamma_4^4 \gamma_\alpha^\top \cdot \theta_\alpha \gamma' \gamma^4 \right),$$

and

$$\gamma_\alpha^\top S_3(x, x') = -\frac{1}{4} S_1 \gamma^4 \left( \gamma_\alpha^\top \cdot \theta_\alpha \right) \gamma^4 - \frac{1}{4} S_2 \gamma^4 \left( \frac{\hat{\gamma} \gamma'}{\theta} + 4 \frac{\hat{\gamma} \gamma'}{\theta} \right) \gamma^4.$$
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