DEFORMATION QUANTIZATION OF VERTEX POISSON ALGEBRAS

SHINTAROU YANAGIDA

Abstract. We introduce dg Lie algebras controlling the deformations of vertex algebras and vertex Poisson algebras, utilizing the notion of operadic dg Lie algebra and the theory of chiral algebra. In terms of those dg Lie algebras, we formulate the deformation quantization problem of vertex Poisson algebras to vertex algebras.

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0. Introduction

0.1. Vertex Poisson algebra and deformation problem. The motivation of this note is a formulation of deformation quantization problem of vertex Poisson algebra. In order to state our deformation problem, let us begin with the recollection of vertex Poisson algebra.

0.1.1. Vertex Poisson algebra. Let us recall the notion of vertex Poisson algebra following [FBZ04, Chap. 16]. We will use the notation \((V,|0\rangle,T,Y)\) for a vertex algebra following [FBZ04, Chap. 1], and work over \(\mathbb{C}\) in this introduction.

A **vertex Lie algebra** is a triple \((L,T,Y_-)\) consisting of
- a vector space \(L\),
- a linear endomorphism \(T \in \text{End}(L)\)
- a series of endomorphisms \(Y_-(A,z) = \sum_{n \geq 0} A(z^n) z^{-n-1} \in \text{End}(L) \otimes z^{-1} \mathbb{C}[[z^{-1}]]\) for each \(A \in L\) satisfying some conditions which we omit.

The axiom of vertex Lie algebra is built so that the following statement holds. For any vertex algebra \((V,|0\rangle,T,Y)\), its polar part \((V,T,Y_-)\) with \(Y_-(A,z) := Y(A,z_-)\) is a vertex Lie algebra. Here we set \(f(z) := \sum_{n<0} f_n z^n\) for a given series \(f(z) = \sum_{n \geq 0} f_n z^n\).

A vertex algebra \((V,|0\rangle,T,Y)\) is called **commutative** if for any \(A,B \in V\) \([Y(A,z),Y(B,w)] = 0\). It is known [B86], [FBZ04, Chap. 1] that a commutative vertex algebra \((V,|0\rangle,T,Y)\) is equivalent to a commutative \(\mathbb{C}\)-algebra \((V,\circ)\).
with unit \( |0\rangle \) and a derivation \( T \). The equivalence is given by
\[
A \circ B = A(-1)B; \quad Y(A,z)B = e^{zT} A \circ B.
\]

A *vertex Poisson algebra* is a quintuple \( (V, |0\rangle , T, Y_+, Y_-) \) such that

1. \( (V, |0\rangle , T, Y_+, Y_-) \) is a commutative vertex algebra,
2. \( (V,T,Y_-) \) is a vertex Lie algebra,
3. all the coefficients of \( Y_-(A,z) \) are derivations of the the commutative product on \( V \) induced by \( Y_+ \).

There is a natural construction of vertex Poisson algebras from vertex algebras, which we call the *limit construction*. Assume that \( (V^h,Y^h) \) is a flat family of vertex algebras over \( \mathbb{C}[\hbar] \), and that \( V^0 := V^h/hV^h \) is a commutative vertex algebras over \( \mathbb{C} \) with \( Y^0 := Y \pmod{\hbar} \). Then \( V^0 \) is a vertex Poisson algebras with
\[
Y_-(A,z) := \frac{1}{\hbar} Y^h(\overline{A},z)_- \pmod{\hbar},
\]
where \( \overline{A} \in V^h \) is a lift of \( A \in V^0 \).

Let us give two examples of limit construction of vertex Poisson algebra. The first is \( V_\infty(g) := V_K(g)/K^{-1}V_K(g) \), the limit of the affine vertex algebra \( V_K(g) \) for a finite Lie algebra \( g \). Here we consider the level to be an indeterminate \( K \), and \( V_K(g) \) to be defined over \( \mathbb{C}[K^{\pm 1}] \). It describes the Poisson structure on the space of connections on the trivial \( G \)-bundles on the punctured disc. The second one is \( W_\infty(g) := W_K(g,e_{reg})/K^{-1}W_K(g,e_{reg}) \), the limit of the \( W \) algebra \( W_K(g,e_{reg}) \) associated to a finite dimensional \( g \) and the regular nilpotent element \( e_{reg} \). It presents the Poisson structure on the space of opers.

0.1.2. *Ad-hoc formulation of our deformation problem.* Once one knows the limit construction of vertex Poisson algebra, it is natural to ask the following question. Given a vertex Poisson algebra \( (V^0, |0\rangle , T, Y_+, Y_-) \), classify vertex algebras \( (V^h, |0\rangle , T, Y_+, Y_-) \) flat over \( \mathbb{C}[\hbar] \) such that
\[
Y = Y_+ + hY_- + \hbar^2 Y_2 + \cdots.
\]

This problem looks similar to the *deformation quantization problem* of (usual) Poisson algebras. Namely, given a Poisson algebra \( (A,\circ, \{ \}) \) over \( \mathbb{C} \), classify associative algebras \( (A,*) \) over \( \mathbb{C}[\hbar] \) such that
\[
a \ast b = a \circ b + h\{a,b\} + \hbar^2 \alpha_2(a,b) + \cdots = \sum_{n \geq 0} b^n \alpha_n(a,b)
\]
with \( \alpha_n \in \text{Hom}(A^{\otimes 2}, A) \). Hereafter Hom means the space of \( \mathbb{C} \)-linear maps.

0.1.3. *Deformation quantization.* Now let us briefly recall the usual deformation quantization (see [K03, §1, §3] for a detailed explanation).

Given a Poisson algebra \( (A,\circ, \{ \}) \), two deformations \( (A,\ast_1) \) and \( (A,\ast_2) \) are called *equivalent* if there exists \( \varphi = \sum_{n \geq 0} h^n \varphi_n \in \text{End}(A[[\hbar]]) \) such that \( \varphi(a \ast_1 b) = \varphi(a) \ast_2 \varphi(b) \).

The equivalent class of deformations is described by the *Hochschild complex* \( C^\bullet(A,A) = (\oplus_{n \geq 0} C^n(A,A), d) \), and the *Hochschild cohomology* \( H^\bullet(A,A) \). Recall that the Hochschild complex is given by
\[
C^n(A,A) := \text{Hom}(A^{\otimes n}, A),
\]
\[
df(a_0, \ldots, a_n) := a_0 \circ f(a_1, \ldots, a_n) + \sum_{i=1}^{n} (-1)^i f(a_0, \ldots, a_{i-1} \circ a_i, \ldots, a_n) + (-1)^{n+1} f(a_0, \ldots, a_{n-1}) \circ a_n.
\]

\( H^\bullet(A,A) \) is the cohomology of this complex. As for the deformations of \( (A,\circ, \{ \}) \), we have

- the equivalence class of \( \alpha_1 = \{ \} \) is an element of \( H^2(A,A) \),
- using the Gerstenhaber bracket \( [\ ] \) given by
\[
\frac{1}{2} [\alpha_i, \alpha_j] (a,b,c) := \alpha_i(\alpha_j(a,b),c) - \alpha_i(a,\alpha_j(b,c))
\]
the associativity of \( * \) is rewritten as
\[
d\alpha + \frac{1}{2} \sum_{i+j=m} [\alpha_i, \alpha_j]' = 0.
\]

These properties can be expressed in terms of *dg Lie algebra*. The equations \( d\alpha + \frac{1}{2} \sum_{i+j=m} [\alpha_i, \alpha_j]' = 0 \) can be rewritten as the *Maurer-Cartan equation*
\[
d\alpha + [\alpha, \alpha] = 0, \quad \alpha = \sum_{n} \alpha_n \in \mathfrak{g}^1
\]
of the dg Lie algebra \( \mathfrak{g} = (C^\bullet(A,A), [\ ], d) \) associated to the Hochschild complex. Here the grading is
\[
\mathfrak{g} = \oplus_{n \geq -1} \mathfrak{g}^n, \quad \mathfrak{g}^n := C^{n+1}(A,A) = \text{Hom}(A^{\otimes (n+1)}, A).
\]
The Lie bracket \( [\ ] \) is given by
\[
[\alpha, \beta] := \alpha \circ \beta - (-1)^{|\alpha||\beta|} \beta \circ \alpha,
\]
Thus a deformation quantization of a Poisson algebra is the problem to find and classify solutions $\alpha = \sum_{n \geq 0} \alpha_n$ of
Maurer-Cartan equation of the dg Lie algebra $C^*(A, A)$ with $\alpha_0$ and $\alpha_1$ equal to the given $\circ$ and $\{ \}$.  

0.1.4. Main problem: the dg Lie algebra for our deformation problem. Now going back to our situation, it is natural to ask what the dg Lie algebra controlling deformations of vertex Poisson algebra is. One may find two hints in the literature.

- According to the theory of operad, one can construct a dg Lie algebra controlling the deformations of $P$-algebras for any Koszul operad $P$.
- According to the theory of chiral algebra by Beilinson and Drinfeld, vertex algebras and vertex Poisson algebras can be formulated in terms of operads.

0.2. Recollection of Operads. Here we briefly explain notations and some basic facts of operads. We refer [LV12, Chap. 5–6] for the fundamentals of operads. We denote by $S_n$ the $n$-th symmetric group.

0.2.1. Basic notions. An $S$-module is a series $M = \{ M(n) \}_{n \geq 0}$ of right $S_n$-modules. We also denote $M$ as $M = \bigoplus_{n \geq 0} M(n)$.

The first example of $S$-module is the identity $S$-module $I = 0 \oplus C \oplus 0 \oplus 0 \oplus \cdots$ for two $S$-modules $M$ and $N$, define another $S$-module $M \circ N$ by $M \circ N := \bigoplus_n M(n) \otimes_{S_n} N^\otimes n$.

A morphism of $S$-modules is a series of $S_n$-module homomorphisms.

An $S$-module $P$ with $P(0) = 0$ is called reduced. A reduced $S$-module is sometimes denoted as $P = \bigoplus_{n \geq 1} P(n)$. An operad is a triple $P = (P, \gamma, \eta)$ of

- an $S$-module $P = \bigoplus_{n \geq 0} P(n)$ consisting of spaces of $n$-ary operations,
- an $S$-module morphism $\gamma : P \circ P \to P$ called the composition map,
- an $S$-module morphism $\eta : I \to P$ called the unit

satisfying some compatibility conditions. A morphism of operads is defined to be a morphism of underlying $S$-modules which are compatible with $\gamma$’s and $\eta$’s.

Let us denote by $Assoc$, $Com$ and $Lie$ the operads of commutative algebras, (non-commutative) associative algebras and Lie algebras respectively. For each of these operads, denoted as $P$, we have an element $\mu \in P(2)$ corresponding to the binary operation (associative product, commutative product and Lie bracket).

We also have the operad of endomorphisms on a vector space $V$. Set the $S$-module $End_V$ by

$$End_V := \bigoplus_{n \geq 0} End_V(n), \quad End_V(n) := \text{Hom}_C(V^\otimes n, V).$$

Then the composition of linear endomorphisms gives $End_V$ a natural structure of operad.

Now for an operad $P$ and a vector space $V$, a $P$-algebra structure on $V$ is an operad morphism $P \longrightarrow End_V$.

After a moment thought, one finds that for $P = Assoc$, $Com$ and $Lie$, an $P$-algebra is nothing but the usual associative, commutative and Lie algebra respectively.

We will also need cooperads, dually defined as operads. A cooperad is a triple $\mathcal{C} = (\mathcal{C}, \Delta, \varepsilon)$ consisting of a $S$-module $\mathcal{C}$, the decomposition map $\Delta : \mathcal{C} \to \mathcal{C} \circ \mathcal{C}$ and the counit map $\varepsilon : \mathcal{C} \to I$.

0.2.2. Koszul dual (co)operads. Let us denote by $\mathcal{F}(E)$ the free cooperad of an $S$-module $E$. We will give a brief account in §2.2.1 and §2.2.2. It has a weight grading $\mathcal{F}(E) = \bigoplus_{d \geq 0} \mathcal{F}(E)^{(d)}$. In a dual way, we have the free cooperad $\mathcal{F}^* (E)$ of an $S$-module $E$.

Recall the notion of quadratic operad. A quadratic data $(E, R)$ is a pair of an $S$-module $E$ and a sub-$S$-module $R \subset \mathcal{F}(E)^{(2)}$. Now the quotient

$$P(E, R) := \mathcal{F}(E)/(R)$$

has a structure of operad, and called the quadratic operad for $(E, R)$. The operads $Assoc$, $Com$ and $Lie$ are standard examples. In a dual way, we have the quadratic cooperad $\mathcal{C}(E, R)$ of the quadratic data $(E, R)$.

The notions mentioned so far have dg version. Namely, a dg $S$-module is a series of complexes with right action of $S_n$. Similarly, one can define a dg (co)operad, a dg (co)operad of endomorphisms and and a dg free (co)operad.

Let us denote by $s$ the shift of a complex in the following way.

$$(sM)_p = M_{p-1}.$$
Then the **Koszul dual cooperad** of a quadratic operad $\mathcal{P} = \mathcal{P}(E, R)$ is given by

$$\mathcal{P}^{\text{cl}} := C(sE, s^2R).$$

$\mathcal{P}^{\text{cl}}$ has a weight grading similarly as the free operad $\mathcal{P}(E)$.

0.2.3. **Convolution dg Lie algebra.** We recall the convolution dg Lie algebra following [LV12, Chap. 6].

There is a good class of quadratic operads called **Koszul operads**. The operads $\mathcal{Assoc}$, $\mathcal{Com}$ and $\mathcal{Lie}$ are Koszul.

**Fact 0.3** ([LV12, Chap. 6]). For a Koszul operad $\mathcal{P}$ and a vector space $V$, the vector space

$$\mathfrak{g} \equiv g_{\mathcal{P}, V} \equiv \text{Hom}_{\mathcal{P}}(\mathcal{P}^{\text{cl}}, \mathcal{End}_V) := \oplus_{n \geq 0} \text{Hom}_{\mathcal{P}}(\mathcal{P}^{\text{cl}}(n), \mathcal{End}_V(n))$$

has a natural structure of dg Lie algebra $(\mathfrak{g}, [\; , \;], \partial)$.

It is called the **convolution dg Lie algebra**. Note that it has a double grading. One is the grading $\mathfrak{g} = \oplus_n \mathfrak{g}^n$ as a complex, and the other is the weight grading $\mathfrak{g} = \oplus_{d \geq 0} \mathfrak{g}^{(d)}$ induced by that on $\mathcal{P}^{\text{cl}}$.

Now recall the Maurer-Cartan equation (0.2) which is defined for any dg Lie algebra $\mathfrak{g} = \oplus_n \mathfrak{g}^n$. Define

$$\text{Tw}(\mathfrak{g}) := \{\text{degree } n = -1 \text{ solutions of the Maurer-Cartan equation of } \mathfrak{g}\}.$$

**Fact 0.2** ([LV12, Proposition 10.1.4]). For a Koszul operad $\mathcal{P}$ and a vector space $V$,

$$\{ \text{$\mathcal{P}$-algebra structures on $V$} \} \xleftarrow{1:1} \text{MC}(g_{\mathcal{P}, V}) := \{\text{weight } d = 1 \text{ elements in } \text{Tw}(g_{\mathcal{P}, V})\}$$

Once a weight 1 solution $\mu$ of the Maurer-Cartan equation is given, we have a twisted dg Lie algebra

$$g_{\mathcal{P}, V}^{\mu} := (\text{Hom}(\mathcal{P}^{\text{cl}}, \mathcal{End}_V), [\; , \;], \partial^\mu := \partial + [\mu, -]).$$

and it encodes the deformation of $\mathcal{P}$-algebra structure $\mu$.

$$\text{MC}(g_{\mathcal{P}, V}^{\mu}) \xleftarrow{1:1} \{\text{$\mathcal{P}$-algebra structures deforming } \mu\}.$$

For example, if $\mathcal{P} = \mathcal{Assoc}$, then $\mu \in \text{MC}(g_{\mathcal{Assoc}, V})$ is an associative product on $V$, and $g_{\mathcal{Assoc}, V}^{\mu}$ coincides with the Hochschild complex (0.1) for $A = (V, \mu)$.

0.3. **Chiral dg Lie algebra.** The theory of **chiral algebras** by Beilinson and Drinfeld [BD04] is an operadic formulation of vertex algebras. One can apply to it the general construction of convolution dg Lie algebra in the previous subsection. The obtained dg Lie algebra is what we look for.

0.3.1. **Chiral algebras.** Here we briefly spell out what a chiral algebra is, and defer a detailed explanation to §1.

Let $X$ be a smooth curve. $\mathcal{M}(X)$ denotes the category of right $\mathcal{D}_X$-modules (quasi-coherent as $\mathcal{O}_X$-modules) on $X$. For $n \in \mathbb{Z}_{\geq 1}$ denote by $\Delta^{(n)} : X \longrightarrow X^n$ the diagonal embedding and $j^{(n)} : U^{(n)} := \{(x_i) \in X^n \mid x_i \neq x_j \; (\forall i \neq j)\} \longrightarrow X^n$ the complement of diagonal divisors.

For $M \in \mathcal{M}(X)$, consider a $\mathcal{O}_X$-module $\mathcal{End}_M^{\text{ch}} = \oplus_n \mathcal{End}_M^{\text{ch}}(n)$ given by

$$\mathcal{End}_M^{\text{ch}}(n) := \text{Hom}_{\mathcal{M}(X)}(j^{(n)}(\varphi^{-1}M^\otimes n), \Delta^{(n)}(M)).$$

It has an operad structure, which we call the **chiral operad** on $M$.

A **chiral algebra** structure (without unit) on $M \in \mathcal{M}(X)$ is an operad morphism

$$\varphi : \mathcal{Lie} \longrightarrow \mathcal{End}_M^{\text{ch}}.$$ 

Denoting $\mu_{\text{lie}} \in \mathcal{Lie}(2)$ the binary operation corresponding to the Lie bracket, we call Its image $\varphi(\mu_{\text{lie}}) \in \mathcal{End}_M^{\text{ch}}(2)$ the **chiral bracket**.

Now we recall the relation between vertex algebras and chiral algebras. By [FBZ04, Chap. 5], one can construct from a vertex algebra $V$ a locally free sheaf $\mathcal{F}$ on a smooth curve $X$. If $V$ is quasi-conformal, then $\mathcal{F}$ is a left $\mathcal{D}_X$-module. Denote by

$$\varphi^2 : j^{(2)}(j^{(2)\ast} \varphi^2) \longrightarrow \Delta^{(2)}(V)$$

the morphism of left $\mathcal{D}$-modules induced by $\varphi$. Locally it is given by

$$\varphi^2(f(\cdot, w)^A \otimes B) = f(\cdot, w)Y(A, z - w)B \pmod{V[[z, w]]}.$$ 

Also denote by $\omega_X$ the canonical sheaf on $X$. Given a left $\mathcal{D}_X$-module $M$, one has a right $\mathcal{D}_X$-module

$$M^\ast := M \otimes_{\mathcal{O}_X} \omega_X.$$

**Fact 0.3** ([BD04], [FBZ04, Chap. 19]). For a quasi-conformal vertex algebra $V$, the right $\mathcal{D}$-module $V^\ast := V \otimes \omega_X$ has a structure of chiral algebra. The chiral bracket $\mu \in \mathcal{End}_{V^\ast}^{\text{ch}}(2)$ is given by $\mu = (\varphi^2)^\ast$. 
0.3.2. Coisson algebras. In [BD04, §2.6], an operad structure corresponding to a vertex Poisson algebra is defined. We call it the compound operad and denote it by $\mathcal{E}nd_{M}^{S}$ for $M \in \mathcal{M}(X)$. Its underlying $\mathcal{S}$-module is given by
\[
\mathcal{E}nd_{M}^{S}(n) = \oplus_{S \in Q([n])} \mathcal{E}nd_{M}^{S}(n)_{S},
\]
and hence we have a map $\mathcal{Q}(X)^{\mathcal{S}} \otimes (\mathcal{E}nd_{M}^{S}(1) \otimes \mathcal{E}nd_{M}^{S}(1)).$
\[
\mathcal{E}nd_{M}^{S}(n) = \text{Hom}_{\mathcal{M}(X)}(M^{\mathcal{S}}, \Delta^{(S)} M) \otimes (\mathcal{E}nd_{M}^{S}(1) \otimes \mathcal{E}nd_{M}^{S}(1)).
\]
Here $Q([n])$ denotes the set of equivalence classes of surjections $[n] \rightarrow S$ from the set $[n] = \{1, \ldots, n\}$, and $\Delta^{(S)} : X \mapsto X^{S}$ denotes the diagonal embedding.

A coisson algebra structure on $M \in \mathcal{M}(X)$ is an operad morphism
\[
\mathcal{L}ie \longrightarrow \mathcal{E}nd_{M}^{S}.
\]
Let us explain the relation between coisson algebras and chiral algebras. $\mathcal{E}nd_{M}^{S}$ has a filtration $W^{\ast}$ such that
\[
\mathcal{E}nd_{M}^{S}(n) = W^{0} \supset W^{-1} \supset \cdots \supset W^{-n} \supset W^{-n-1} = 0,
\]
and we have an inclusion of operads
\[
\mathcal{L}ie_{W} \mathcal{E}nd_{M}^{S} \longrightarrow \mathcal{E}nd_{M}^{S}.
\]
The filtration comes from the Cousin complex of $\omega_{X}$.

As a corollary, given a family $A_{t}$ of chiral algebras flat over $\mathbb{C}[t]$, $A_{0} := A_{t}/tA_{t}$ has a structure of coisson algebra. Since the bundle of the vertex Poisson algebra yields a coisson algebra, the limit construction of vertex Poisson algebra from a quasi-conformal vertex algebra, as mentioned previously, is a special case of coisson to chiral limit.

0.4. Chiral dg Lie algebra. Finally we can introduce our main object. As in the previous subsection, let $X$ denote a smooth curve.

0.4.1. Definition of our dg Lie algebra. Similarly as in Fact 0.1, we have the following construction.

**Proposition.** For a right $D_{X}$-module $M$, the $\mathcal{S}$-module
\[
\mathcal{E}nd_{M}^{S}(n) = \mathcal{L}ie^{cl} \mathcal{E}nd_{M}^{S}(n),
\]
has a structure of dg Lie algebra $(\mathcal{E}nd_{M}^{S}, [\cdot, \cdot], \partial = 0)$.

**Definition.** We call $\mathcal{E}nd_{M}^{S}$ the chiral Lie algebra.

Then similarly as in Fact 0.2, we have the following description of chiral algebra structure.

**Proposition.** For a right $D_{X}$-module $M$,
\[
\{\text{chiral algebra structures on } M\} \xleftarrow{1:1} \mathcal{MC}(\mathcal{E}nd_{M}^{S}).
\]

Given $\alpha \in \mathcal{MC}(\mathcal{E}nd_{M}^{S})$, we can twist the graded Lie algebra $\mathcal{E}nd_{M}^{S}$ as follows.

**Definition.** The following has a structure of dg Lie algebra, which we call the chiral dg Lie algebra.
\[
\mathcal{E}nd_{M}^{S}(\mathcal{E}nd_{M}^{S}, [\cdot, \cdot], \partial_{\alpha} := \partial + [\alpha, -] = [\alpha, -].
\]

The standard deformation theory says

**Proposition.** Given $\alpha \in \mathcal{MC}(\mathcal{E}nd_{M}^{S})$, we have
\[
\{\text{chiral algebra structures on } M \text{ deforming } \alpha\} \xleftarrow{1:1} \mathcal{MC}(\mathcal{E}nd_{M}^{S}).
\]

We have a similar argument for coisson algebras. One can construct the coisson dg Lie algebra whose underlying $\mathcal{S}$-module is given by
\[
\mathcal{E}nd_{M}^{S}(n) = \mathcal{L}ie^{cl} \mathcal{E}nd_{M}^{S}(n),
\]
A coisson algebra structure corresponds to a weight 1 element of $Tw(\mathcal{E}nd_{M}^{S})$ bijectively.

0.4.2. Deformation problem revisited. By the arguments in §0.3.2 of coisson algebras, we have morphisms of operads
\[
\mathcal{E}nd_{M}^{S} \longrightarrow \mathcal{L}ie_{W} \mathcal{E}nd_{M}^{S} \longrightarrow \mathcal{E}nd_{M}^{S}.
\]

The first theorem in this note is

**Theorem.** The above induces a morphism of dg Lie algebras
\[
\mathcal{E}nd_{M}^{S} \longrightarrow \mathcal{E}nd_{M}^{S}
\]
and hence we have a map
\[
\psi : \mathcal{MC}(\mathcal{E}nd_{M}^{S}) \longrightarrow \mathcal{MC}(\mathcal{E}nd_{M}^{S}).
\]

See Remark 3.12 for the difference between our situation and the usual deformation quantization of associative algebras.

**Definition.** We call $\mu \in \mathcal{MC}(\mathcal{E}nd_{M}^{S})$ a chiral deformation quantization of $\mu^{c} \in \mathcal{MC}(\mathcal{E}nd_{M}^{S})$ if $\psi(\mu) = \mu^{c}$.

Obviously, if the map $\psi$ is surjective, then a chiral deformation quantization exists. Our main theorem is
Theorem. \( \psi \) is always injective.

Thus, a vertex Poisson algebras arising from the limit construction, like \( V_\infty(\mathfrak{g}) \) and \( W_\infty(\mathfrak{g}, \epsilon_{\text{reg}}) \), has a unique chiral deformation quantization.

0.5. Organization of this note. In §1 we give a recollection of chiral algebras following [BD04]. We start §1.1 with the \( \ast \)-pseudo-tensor structure and \( ! \)-tensor structure on the category of \( \mathcal{D} \)-modules. Using these structures, we introduce coisson algebras in §1.2 and chiral algebras in §1.3. In §1.4 and §1.5 we explain the relation between chiral and coisson algebras.

§2 is the recollection of operad theory and the deformation theory of algebras over an operad following [LV12]. In §2.1 we introduce the operadic convolution dg Lie algebra. In §2.3 the dg Lie algebra controlling operadic algebra structure is introduced. §2.4 gives a brief recollection of deformation theory using dg Lie algebras.

§3 is the main part of this note, and we will use all the notions prepared so far. §3.1 introduces and studies the dg Lie algebras controlling the deformations of chiral and coisson algebras. In §3.2 we cast our deformation quantization problem to the dg Lie algebras and prove the main theorems.

General Notations. For a category \( \mathcal{M} \), \( A \in \mathcal{M} \) means that \( A \) is an object of \( \mathcal{M} \).

A tensor category means a monoidal category in the sense of [MT1].

Sets denotes the category of arbitrary sets and maps. For a set \( I \), \( |I| \) denotes its cardinality.

In the main text we will work over a fixed field \( \mathbb{K} \) of characteristic 0. \( \otimes \) denotes the tensor product \( \otimes_{\mathbb{K}} \) of \( \mathbb{K} \)-vector spaces unless otherwise stated.

1. Recollection of chiral algebras and coisson algebras

This section gives a recollection of chiral algebras and coisson algebras following [BD04].

Beilinson and Drinfeld started [BD04] with a large account of the general theory of pseudo-tensor category. It is equivalent to the notion of colored operad, and roughly speaking, it is a category whose sets of morphisms have composition rules. A pseudo-tensor structure with one object is nothing but an operad.

We avoid copying their argument and specialize it to the case of \( \mathcal{D} \)-modules. The main purpose of this section is to introduce two operads, which we name chiral operad and coisson operad. These are restricted versions of chiral and coisson pseudo-tensor structures originally introduced in [BD04, §1.4, §2.2, §3.1], and encode the operadic structure of vertex algebras and vertex Poisson algebras.

1.1. \( \mathcal{D} \)-module category and (pseudo)-tensor structures. We follow [BD04, §2.2].

Let \( X \) be a smooth scheme over a fixed field \( \mathbb{K} \) of characteristic 0. \( \mathcal{O}_X \), \( \mathcal{D}_X \) and \( \Theta_X \) denote the structure sheaf, the sheaf of differential operators, and the sheaf of vector fields on \( X \) respectively. \( \mathcal{M}_\mathcal{O}(X) \) denotes the category of quasi-coherent \( \mathcal{O}_X \)-modules. We will also use the simplified symbols \( \mathcal{O} := \mathcal{O}_X \) and \( \mathcal{D} := \mathcal{D}_X \). Denote by \( \mathcal{M}(X) \) the category of right \( \mathcal{D} \)-modules on \( X \) (more precisely, sheaves of right \( \mathcal{D}_X \)-modules which are quasi-coherent as \( \mathcal{O}_X \)-modules). Similarly denote by \( \mathcal{M}^\ell(X) \) the category of left \( \mathcal{D} \)-modules.

1.1.1. The \( ! \)-tensor structure. For \( L_1, L_2 \in \mathcal{M}^\ell(X) \), \( L_1 \otimes_{\mathcal{O}_X} L_2 \) is naturally a left \( \mathcal{D} \)-module. So \( \mathcal{M}^\ell(X) \) is a tensor category with a unit object \( \mathcal{O}_X \).

For a right \( \mathcal{D} \)-module \( M \) and a left \( \mathcal{D} \)-module \( L \), the sheaf \( M \otimes_{\mathcal{O}_X} L \) is naturally a right \( \mathcal{D} \)-module by

\[
(m \otimes l)_{\tau} := m \tau \otimes l - m \otimes l_{\tau}
\]

for \( \tau \in \Theta_X \subset \mathcal{D}_X \). We will denote this right \( \mathcal{D} \)-module by \( M \otimes L \).

The canonical sheaf \( \omega_X := \mathcal{O}_X^{\dim X} \) has a canonical right \( \mathcal{D} \)-module structure

\[
\nu_{\tau} = - \text{Lie}_{\tau}(\nu)
\]

for \( \nu \in \omega_X \), where \( \text{Lie}_{\tau} \) denotes the Lie derivative. Then we have the standard equivalence

\[
\mathcal{M}^\ell(X) \to \mathcal{M}(X), \quad L \mapsto L^\tau := \omega_X \otimes L.
\]

The inverse is give by

\[
M \mapsto M^\ell := M \otimes \omega_X^{-1}.
\]

Pulling back the tensor structure on \( \mathcal{M}^\ell(X) \) by this equivalence, we have a tensor structure on \( \mathcal{M}(X) \). Namely,

\[
M \otimes^! N := M^\ell \otimes_{\mathcal{O}_X} N.
\]

The canonical sheaf \( \omega_X \) is a unit object for \( \otimes^! \).
1.1.2. The *-pseudo-tensor structure. In order to introduce the *-pseudo-tensor structure, we need to recall some functors of $\mathcal{D}$-modules. Let $DM(X)$ denote the derived category of right $\mathcal{D}$-modules on $X$, and $DM^\ell(X)$ the derived category of left $\mathcal{D}$-modules. We have an equivalence

$$DM^\ell(X) \to DM(X), \quad L \mapsto \omega_X \otimes L[\dim X].$$

Thus $DM(X)$ has the original $t$-structure $\mathcal{M}(X)$ and the other $t$-structure $\mathcal{M}^\ell(X)$, differing shifts by $\dim X$.

For a morphism $f : X \to Y$ of smooth schemes, we have the standard derived functors

$$f_* : DM(Y) \to DM(X), \quad f^! : DM(Y) \to DM(X).$$

If $f$ is a closed embedded, then $f_*$ is exact with respect to the $t$-structure $\mathcal{M}(X)$, and its right adjoint $f^!$ is left exact. They define the equivalence (Kashiwara’s lemma)

$$f_* : M(X) \leftrightarrow M(Y)_X : f^!.$$

Here $M(Y)_X \subset M(Y)$ is the full subcategory of $\mathcal{D}_Y$-modules vanishing on $Y \setminus X$.

For a finite collection $\{X_i\}_{i \in I}$ of smooth schemes, denote by $\boxtimes$ the exterior tensor product. Namely we have

$$\prod_{i \in I} M(X) \to M(\prod_{i \in I} X_i), \quad (M_i) \mapsto \boxtimes_{i \in I} M_i.$$

The functors $f_*$ and $f^!$ are pseudo-tensor functors on $\mathcal{M}(X)$. Let $\mathcal{S}$ be the category of finite non-empty sets and surjective maps. For a morphism $\pi : J \to I$ and $i \in I$, we set $J_i := \pi^{-1}(i) \subset J$. We will also denote $n := \{1, 2, \ldots, n\} \in \mathcal{S}$ if confusion may not occur. For $I \in \mathcal{S}$, we denote the diagonal embedding by

$$\Delta^{(I)} : X \hookrightarrow X^I.$$

For $I \in \mathcal{S}$ and $L_i, M \in \mathcal{M}(X)$ with $i \in I$, set

$$P^\pi_I(\{L_i\}_{i \in I}, M) := \text{Hom}_{\mathcal{M}(X)}\left(\boxtimes_{i \in I} L_i, \Delta^{(I)}_* M\right).$$

Then for a surjection $\pi : J \twoheadrightarrow I$, one can define a multi-linear map

$$\gamma^*_\pi : P^\pi_I(\{L_i\}_{i \in I}, M) \otimes \bigotimes_{i \in I} P^\pi_{J_i}(\{K_j\}_{j \in J_i}, L_i) \to P^\pi_J(\{K_j\}_{j \in J}, M)$$

(1.1) by composing morphisms of $\mathcal{D}$-modules. Here $\otimes := \otimes_K$ is the tensor product of $K$-vector spaces. Explicitly, $\gamma^*_\pi(\varphi, (\psi_i)_{i \in I})$ is given by

$$\Box_{j \in J} K_j = \Box_{i \in I} \Box_{j \in J_i} K_j \to \Box_{i \in I} \Delta^{(J)}_i L_i = \Delta^{(\pi)}_i \left(\Box_{i \in I} L_i\right) \xrightarrow{\Delta^{(\pi)}_i(\varphi)} \Delta^{(\pi)}_i \Delta^{(I)}_i M = \Delta^{(J)}_i M,$$

where we used

$$\Delta^{(\pi)} := \prod_{i \in I} \Delta^{(I)}_i : X^I \hookrightarrow X^J$$

and the natural identification $\Delta^{(\pi)} \Delta^{(I)} = \Delta^{(J)}$. Let us call $\gamma^*_\pi$ the composition.

The compositions are associative in the following sense. Let us use a simplified symbol

$$\varphi(\psi_i) := \gamma^*_\pi(\varphi_i, (\psi_i)_{i \in I})$$

for a composition of morphisms. If $\rho : H \to J$ is another surjective map, $\{F_h\}_{h \in H}$ an $H$-family, and if $\chi_j \in P^\rho_{H_j}(\{F_h\}_{h \in H_j}, K_j)$, then in $P^\rho_H(\{F_h\}_{h \in H}, M)$ one has

$$\varphi(\psi(\chi_j)) = (\varphi(\psi_i))\chi_j$$

(1.2)

We also have $\eta_M := \text{id}_M \in \text{Hom}_{\mathcal{M}(X)}(M, M) = P^1(\{M\}, M)$. Then for any $\varphi \in P^\pi_I(\{L_i\}_{i \in I}, M)$ we have

$$\eta_M(\varphi) = \varphi(\eta_L) = \varphi.$$

(1.3)

Although we don’t need the following definition in a full generality, let us recall

**Definition** ([BD04, §1.1]). A pseudo-tensor category $(\mathcal{M}, P, \gamma, \eta)$ consists of the following data.

- A class $\mathcal{M}$ of objects,
- A set
  $$P^M_I(\{L_i\}_{i \in I}, M) \equiv P_I(\{L_i\}, M)$$
  for any $I \in \mathcal{S}$, any $I$-family of objects $\{L_i\}_{i \in I}$ in $\mathcal{M}$, and any $M \in \mathcal{M}$. It is called the set of $I$-operations.
- A map
  $$\gamma_\pi : P_I(\{L_i\}_{i \in I}, M) \times \prod_{i \in I} P_{J_i}(\{K_j\}_{j \in J_i}, M) \to P_J(\{K_j\}_{j \in J}, M)$$
  for any morphism $\pi : J \to I$ in $\mathcal{S}$, any families $\{L_i\}_{i \in I}$ and $\{K_j\}_{j \in J}$, and any $M \in \mathcal{M}$. It is called the composition map.
- An element $\eta_M \in P^{\{1\}}(\{M\}, M)$ for any $M \in \mathcal{M}$ called the unit operation.
These should satisfy the following conditions.

1. The composition map is associative in the sense (1.2).
2. The equality (1.3) holds for any \( \varphi \in P_I((L_i)_i, M) \).

A pseudo-tensor category \( M \) is naturally a category in the usual sense by setting the set of morphisms \( \text{Hom}_M(M, N) \) to be \( P_I([M], N) \) and the identity functor to be \( \eta_M \).

**Remark 1.1.** One can modify the definition of pseudo-tensor category by requiring \( P_I \) to be an object of a fixed tensor category \( \mathcal{A} \). Such a structure will be called a pseudo-tensor \( \mathcal{A} \)-category. If \( \mathcal{A} \) is the tensor category of \( R \)-modules, where \( R \) is a commutative ring, then it is called a pseudo-tensor \( R \)-category. Hereafter we will always work on a pseudo-tensor \( \mathcal{K} \)-category, and suppress the phrase ‘\( \mathcal{K} \)-’.

Using these notions, we can state

**Definition 1.2.** For any smooth scheme \( X \) over \( \mathbb{K} \), we have a pseudo-tensor category

\[
M(X)^* := (M(X), P^*, \gamma^*, \eta^*)
\]

and call it the \( * \)-pseudo-tensor structure on \( M(X) \).

Note also that a tensor category \( (M, \otimes) \) has a pseudo-tensor structure by \( P_I((L_i)_i, M) := \text{Hom}_M(\otimes_{i \in I} L_i, M) \). In particular, we have a pseudo-tensor category

\[
M'(X) := (M(X), \otimes^I).
\]

A pseudo-tensor category having only one object is nothing but a reduced operad. In other words, for a pseudo-tensor category \( M \), one can define an operad for each \( M \in M \) by setting

\[
\text{End}_M^M := \oplus_{n \geq 1} P_n^M(M), \quad P_n^M(M) := P_n^M([M, \ldots, M], M).
\]

The \( \mathfrak{S}_n \)-action on each factor is given by the compositions with respect to bijections from \( [n] = \{1, \ldots, n\} \) to itself. The compositions and \( \eta_M \) yield an operad structure on the \( \mathfrak{S} \)-module \( \text{End}_M^M \).

**Definition.** For an operad \( B \) and a pseudo-tensor category \( M \), a \( B \)-algebra in \( M \) is an object \( L \in M \) with an operad morphism \( B \to \text{End}_{L}^{M} \). Denote by \( B(M) \) the category of \( B \)-algebras in \( M \).

For later use, we define the following symbols for an operad \( B \).

\[
\mathcal{B}^*(X) := B(M(X)^*), \quad \mathcal{B}^!(X) := B(M(X)^!).
\]

Let us close this subsection by recalling

**Definition ([BD04, §1.1]).** A pseudo-tensor functor \( M \to N \) of two pseudo-tensor categories \( M \) and \( N \) is a functor

\[
\tau : M \to N
\]

of categories together with a map

\[
\tau_I : P_I^M ((L_i)_{i \in I}, M) \to P_I^N ([\tau(L_i)]_{i \in I}, \tau(M))
\]

for any \( I \in \mathcal{S} \) so that \( \tau_I \) are compatible with compositions and \( \tau_I(\text{id}_M) = \text{id}_{\tau(M)} \) for any \( M \in M \).

Thus a pseudo-tensor functor between operads (seen as pseudo-tensor categories with one object) is nothing but a morphism of operads.

1.2. **Coisson algebras.** We follow [BD04, §1.4] to introduce coisson algebras. Our definition is due to [BD04, §1.4.28. Lemma] and different from the original definition.

Recall that \( \mathcal{S} \) denotes the category of finite non-empty sets and surjections. An operad \( B \) can be enlarged to a functor over \( \mathcal{S} \) by

\[
I \mapsto B_I := (\oplus f B(n))_{\mathfrak{S}_n}.
\]

Here we set \( n := |I| \) and \( f \) runs over the bijections \( f : I \to \{1, \ldots, n\} \). The last term means the module of \( \mathfrak{S}_n \)-coinvariants. We use this construction for \( B = \mathcal{L}ie \) below.

Let \( X \) be a smooth scheme over \( \mathbb{K} \) as before. We introduce a new pseudo-tensor structure

\[
M(X)^c = (M(X), P_c^c, \gamma^c, \eta^c)
\]

on \( M(X) \) by setting

\[
P_I^c ((L_i)_{i \in I}, M) := \oplus_{S \subseteq Q(I)} P_I^c (\{L_i\}_S, M)_S, \quad P_I^c (\{L_i\}_{i \in I}, M)_S := P_S^c (\{\otimes_{i \in I} L_i\}_{s \in S}, M) \otimes (\otimes_{s \in S} \mathcal{L}ie_{L_i}).
\]

for \( I \in \mathcal{S} \). Here \( Q(I) \) is the set of all the surjections \( I \to S \). The composition \( \gamma^c \) is given by the tensor product of \( \gamma^* \) and \( \mathcal{L}ie \) operations.

As before, one has an operad for each \( M \in M(X) \) induced by \( M(X)^c \).

**Definition 1.3.** For \( M \in M(X) \), the reduced operad \( \text{End}_M^c \) arising from \( M(X)^c \) is called the coisson operad on \( M \).
Thus the underlying $\mathfrak{S}$-module of $\text{End}_{M}^\varepsilon$ is given by $\text{End}_{M}^\varepsilon(n) = P^\varepsilon_{n}(M)$.

**Definition 1.4 ([BD04, §1.4.28. Lemma]).** A coisson algebra (without unit) on $X$ is a $\text{Lie}$-algebra in $M(X)^\varepsilon$.

In other words, a coisson algebra structure on $M \in M(X)$ is an operad morphism $\text{Lie} \to \text{End}_{M}^\varepsilon$.

**Remark.** Let us say a few words on the original definition. Our category $M(X)$ with the $\varepsilon$-pseudo tensor structure $P^\varepsilon$ and the tensor structure $\otimes^\varepsilon$ has a compound pseudo-tensor structure [BD04, §1.3]. A coisson algebra is originally defined for any abelian augmented compound tensor category $M^{\varepsilon}$.

A compound tensor category $M^{\varepsilon} = (M, P^\varepsilon, \otimes^\varepsilon)$ consists of a pseudo-tensor structure $M^\varepsilon = (M^\varepsilon, P^\varepsilon)$ on a category $M$ and a (usual) tensor structure $M^{\varepsilon!} = (M^{\varepsilon}, \otimes)$ on the dual category $M^{\varepsilon!}$ satisfying some duality relation. We also skip the explanation of abelian property and augmentation. Denote by $M^\varepsilon$ the tensor category dual to $M^{\varepsilon!}$.

For an operad $B$, we call $B$-algebras in $M^{\varepsilon}$ simply $B^\varepsilon$-algebras, and those in $M^\varepsilon$ simply $B^\varepsilon$-algebras. (Here we consider the tensor category $M^\varepsilon$ as a pseudo-tensor category.)

Let us denote by $\text{Comu}$ the operad of commutative associative algebra with unit. Finally, a coisson algebra in $M^{\varepsilon!}$ is a pair $(A, \{\})$ of $\text{Comu}^\varepsilon$-algebra $A$ which is also a $\text{Lie}^\varepsilon$-algebra, such that the Lie bracket $\{\}$ is in $P^\varepsilon_{2\varepsilon}(\{A, A\}, A)$ satisfies the Leibniz rule with respect to commutative product.

**1.3. Chiral algebras.** We follow [BD04, §3.1] to introduce chiral algebras.

Recall that $S$ denotes the category of finite non-empty sets and surjections. For $I \in S$ and a smooth scheme $X$, denote by $U(I)$ the complement of diagonal divisors. Namely

$$U(I) := \{(x_i) \in X^I \mid x_i \neq x_j \text{ for any } i \neq j\}.$$  

The open embedding is denoted by

$$j(I) : U(I) \hookrightarrow X^I.$$ 

For $L_i, M \in M(X)$ ($i \in I$), we set

$$P^ch_I(L_i, M) := \text{Hom}^{M(X^I)}(j(I)\ast j(I)^*\mathfrak{I}(\mathfrak{I}(L_i), \Delta(I)^*M)).$$

Elements of this set are called chiral operations.

**Remark.** Here is the explanation of the functor $f^*$ used in the definition. For a morphism $f : X \to Y$ of smooth schemes, the standard derived functor $f^*: D^b(M)(Y) \to D^b(M)(X)$ is compatible with the standard pull-back functors $Rf^! \circ Lf^* : D^b(M)(Y) \to D^b(M)(X)$ as $\mathfrak{O}$-modules. Thus $f^!$ is right exact with respect to the $t$-structure $M^\varepsilon(X)$. We denote the corresponding functor as $f^*: M^\varepsilon(Y) \to M^\varepsilon(X)$. Now by the standard equivalence, we read this functor as $f^*: M(Y) \to M(X)$.

**Fact.** ([BD04, §3.1.2]). There is a pseudo-tensor structure $(M(X), P^ch\varepsilon, \gamma, \eta)$ on $M(X)$.

The resulting pseudo-tensor category is denoted by $M(X)^{\varepsilon!}$ and called the chiral pseudo-tensor structure. For an operad $B$, the category of $B$-algebras in $M(X)^{\varepsilon!}$ will be denoted by $B^{\varepsilon!}(X)$.

**Definition 1.5.** The operad structure induced by $M(X)^{\varepsilon!}$ on

$$\text{End}_{M}^{ch} := \oplus_{n \geq 1} \text{End}_{M}^{ch}(n), \quad \text{End}_{M}^{ch}(n) \equiv P^{ch}_{n}(M) := P^{ch}_{\{M, \ldots, M\}}(M)$$

with $\{n\} = \{1, \ldots, n\}$ is called the chiral operad.

Roughly speaking, a chiral algebra is a $\text{Lie}$-algebra in $M(X)^{\varepsilon!}$, but we require it to have a unit. In order to state the precise definition, we need a few more notions.

For $M \in M(X)$, the unit operation $\varepsilon_M \in P^{ch}_{\{2\}}(\{\omega_X, M\}, M)$ is defined to be the composition

$$j^*\omega_X \otimes M \to (j_+j^*\omega_X \boxtimes M)/\omega_X \boxtimes M \to \Delta_{+}M,$$

where the last arrow comes from the canonical isomorphism $\omega_X \otimes^\varepsilon M \cong M$.

Consider a $\text{Lie}$-algebra $A$ in $M(X)^{\varepsilon!}$. It has an operation $\mu_{A} \in P^{ch}_{\{2\}}(\{A, A\}, A)$ coming from the bracket in $\text{Lie}$. A unit in $A$ is a morphism of $D$-modules $1_A : \omega_X \to A$ such that $\mu_{A}(1_A, \text{id}_A) \in P^{ch}_{\{2\}}(\omega_X, A)$ coincides with the unit operation $\varepsilon_A$.

**Definition 1.6.** A chiral algebra on $X$ is a $\text{Lie}^{ch}$-algebra in $M(X)^{\varepsilon!}$ with unit. Denote by $\mathfrak{C}A(X)$ the category of chiral algebras and morphisms preserving units.

**1.4. Classical limit.** The next goal is to explain the limit construction of coisson algebras from chiral algebras. For a preparation, we introduce the classical limit pseudo-tensor structure following [BD04, §3.1–3.2].

Hereafter we assume dim $X = 1$. 


1.4.1. The Lie operad in terms of chiral operations. Consider the chiral operad \( \mathcal{E}nd_{\omega_X}^{ch} \) on \( \omega_X \). Set

\[
\lambda_I := (\mathbb{K}[1])^{\otimes |I|}/[-|I|].
\]  

(1.5)

The group \( \text{Aut}(I) \) acts on it by the sign character (see Remark 2.1 for our treatment of tensor product of complexes). If \( |I| = 2 \), then the residue morphism

\[
\text{Res} : j^*(I) j^*(I) \omega_{X^I} \longrightarrow \Delta^*(I) \omega_X
\]

yields a map

\[
r_I : \lambda_I \longrightarrow P_I^{ch}(\omega_X).
\]

Fact 1.7 ([BD04, §3.1.5, Theorem]). For \( \dim X = 1 \), there is a unique isomorphism of operads

\[
\kappa : \mathcal{L}ie \overset{\sim}{\longrightarrow} \mathcal{E}nd_{\omega_X}^{ch}
\]

which coincides with \( r_I \) for \( |I| = 2 \).

Remark. In [BD04, §3.1.11] this fact is called Cohen’s theorem, which originally states that the Gerstenhaber operad is isomorphic to the homology of the little cube operad of dimension 2. We refer [BD04, §3.1.11] for the reason of this naming and the relation of \( P^{ch} \) to the homology of configuration spaces.

We will not repeat the proof of this fact, but recall some notions for later use. For a \( \mathcal{D}_{X^I} \)-module \( M \), a special filtration on \( M \) is a finite increasing filtration \( W_* \) such that every graded component \( \text{gr}_n^W M := W_n M/W_{n-1} M \) is a finite sum of copies of \( \Delta^{(1)} \omega_{X^I} \) for some \( T \in Q(I, -m) \) with

\[
Q(I, m) := \{ S \in Q(I) \mid |S| = m \}.
\]

We set \( W_0 M := M \), so that a special filtration \( W_* \) looks like

\[
W_{-|I|} M \subset W_{-|I|+1} M \subset \cdots \subset W_{-1} M \subset W_0 M = M.
\]

An important step in the proof of the above fact is to show that \( j^*(I) j^*(I) \omega_{X^I} \) has a special filtration \( W_* \) [BD04, §3.1.7, Lemma]. It comes from the Cousin complex of \( \omega_{X^I} \), but we skip the argument and cite an explicit description of its graded component from [BD04, §3.1.10] for later use. For \( T \in Q(I) \), set the vector space

\[
\mathcal{L}ie_{I/T} := \otimes_{t \in T} \mathcal{L}ie_{I_t}.
\]

Then there is a canonical isomorphism

\[
\text{gr}_n^W j^*(I) j^*(I) \omega_{X^I} \overset{\sim}{\longrightarrow} \bigoplus_{T \in Q(I, m)} \Delta^{(1/T)} \omega_{X^T} \otimes \mathcal{L}ie_{I/T}
\]

(1.6)

Here \( \mathcal{L}ie_{I/T} \) is the linear dual of \( \mathcal{L}ie_{I/T} \), and \( \Delta^{(1/T)} \) denotes the embedding associated to the surjection \( I \twoheadrightarrow T \):

\[
\Delta^{(1/T)} := \prod_{t \in T} \Delta^{(t)} : X^T \longrightarrow X^I.
\]

1.4.2. The classical limit pseudo-tensor structure. Now we will introduce a new pseudo-tensor structure \( \mathcal{M}(X)^{cl} \) and explain the following sequence of pseudo-tensor functors.

\[
\mathcal{M}(X)^{ch} \longrightarrow \mathcal{M}(X)^{cl} \longrightarrow \mathcal{M}(X)^c.
\]

For the introduction of \( \mathcal{M}(X)^{cl} \), let us recall the special filtration \( W_* \) on \( j^*(I) j^*(I) \omega_{X^I} \). Since

\[
j^*(I) j^*(I) \omega_{X^I} \sim (j^*(I) j^*(I) \omega_{X^I}) \otimes (\otimes_{t \in L_1^I} L^I_t),
\]

(1.7)

the special filtration induces another finite filtration on \( j^*(I) j^*(I) \omega_{X^I} \), which we denote by the same symbol \( W_* \). It yields a canonical filtration on the space of chiral operations as

\[
P_I^{ch}((L_i), M) := \text{Hom}_{\mathcal{M}(X^I)}(j^*(I) j^*(I) \omega_{X^I}, W_{n-1-|I|} j^*(I) j^*(I) \omega_{X^I}, \Delta^{(1)} M)
\]

(1.8)

Note that it is an increasing filtration. We denote by \( \text{gr}^n := P^{ch, n} / P^{ch, n+1} \) its \( n \)-th graded component.

This filtration is compatible with the composition of chiral operations, hence we have

Definition. The graded components

\[
P_I^{cl} := \text{gr}^* P_I^{ch}
\]

give a pseudo-tensor structure on \( \mathcal{M}(X) \) called the classical limit of the chiral structure. We denote it by \( \mathcal{M}(X)^{cl} \).
Next we explain that there is an embedding $M(X)^{cl} \hookrightarrow M(X)^{c}$. The graded component (1.6) of $W_\bullet$ and the isomorphism (1.7) induce the canonical surjection
\[ \phi_I : \bigoplus_{T \in Q(I,m)} \Delta_{s}^{(I/T)} \otimes_{\mathbb{K}} \text{Lie}_{I}^e \to \text{gr}^{W}_{m} j_0^{(I)} \ast (\otimes_{i \in I} L_i). \]

It is induced by the canonical map $\Delta_{s}^{(I/T)} (\otimes_{i \in I} L_i^I) \to \otimes_{i \in I} L_i^I$, where $\otimes$ over $L_i^I$ is the tensor structure on $M^e(X)$. So $\phi_I$ is an isomorphism if $L_i$ are $\mathbb{K}$-flat. $\phi_I$ induces the following canonical embedding
\[ \text{gr}^{n} P_{I}^{P}(\{L_i\}, M) \hookrightarrow \bigoplus_{T \in Q(I,m)} P_{T}^{\ast} (\{\otimes_{i \in I} L_i \} \otimes_{\mathbb{K}} \text{Lie}_{I}^e). \tag{1.9} \]

Let us write down this embedding explicitly. For simplicity, we denote
\[ L := \otimes_{i \in I} L_i, \quad N := \otimes_{i \in I} (\otimes_{i \in I} L_i) \otimes_{\mathbb{K}} \text{Lie}_{I}^e. \]

Given $\varphi \in P_{I}^{P}(\{L_i\}, M)^n$, we have a composition of morphisms
\[ \oplus_{T \in Q(I,m-1)} \Delta_{s}^{(I/T)} N \to \text{gr}^{W}_{m-1} L = W_{-(m-1)} L/W_{m} L \to L/W_{m} L \to \Delta_{s}^{(I)} M. \]

It is an element of the direct sum over $T$ of the spaces
\[ \text{Hom}_{M(X^I)}(\Delta_{s}^{(I/T)} N, \Delta_{s}^{(T)} M) \simeq \text{Hom}_{M(X^I)}(\Delta_{s}^{(I/T)} N, \Delta_{s}^{(I/T)} \Delta_{s}^{(T)} M) \simeq \text{Hom}_{M(X^T)}(N, \Delta_{s}^{(T)} M) \]
\[ \simeq \text{Hom}_{M(X^T)}(\otimes_{i \in I} L_i^I, \Delta_{s}^{(T)} M) \otimes_{\mathbb{K}} \otimes_{i \in I} \text{Lie}_{I}^e = P_{T}^{\ast}(N, M) \otimes \text{Lie}_{I}^e. \]

Here the first isomorphism comes from the equality $\Delta_{s}^{(I)} = \Delta_{s}^{(I/T)} \Delta_{s}^{(T)}$, and the second one is by the exactness of the functor $\Delta_{s}^{(T)}$. This construction is independent of the choice of $\varphi$, and we have the desired embedding.

Now recalling the pseudo-tensor structure $M(X)^{c}$ in (1.4), we see that the embedding (1.9) gives a canonical fully faithful embedding of pseudo-tensor categories
\[ M(X)^{cl} \hookrightarrow M(X)^{c} \]
which extends the identity functor on $M(X)$.

1.5. Coisson algebra as the classical limit of chiral algebra. Now we argue that a coisson algebra may be obtained as a classical limit of chiral algebras. As explained in the introduction, it is an analogy of the limit construction of vertex Poisson algebras from vertex algebras.

1.5.1. Chiral structure and the compound tensor structure. Now we explain the following pseudo-tensor functors
\[ M(X)^{\dagger} \otimes \text{Lie} \to \text{Hom}_{M(X)}(\otimes_{i \in I} L_i, M) \otimes_{\mathbb{K}} \text{Lie}_{I}. \tag{1.10} \]

As for $\beta$, first note that the natural morphism $\otimes L_i \to \otimes_{i \in I} L_i$ yields a map
\[ \beta_I : P_{I}^{P}(\{L_i\}, M) \to P_{I}^{\ast}(\{L_i\}, M), \]
which is compatible with the composition. So it further yields a pseudo-tensor functor
\[ \beta : M(X)^{ch} \to M(X)^{c}. \]

extending the identity functor on $M(X)$. This functor respects the augmentations.

As for $\alpha$, recall that $M(X)^{\dagger}$ is a tensor category and $\text{Lie}$ is a pseudo-tensor category associated to the operad structure. The tensor product $M(X)^{\dagger} \otimes \text{Lie}$ is a pseudo-tensor category whose space of operations is given by
\[ P_{I}(\{L_i\}, M) = \text{Hom}_{M(X)}(\otimes_{i \in I} L_i, M) \otimes_{\mathbb{K}} \text{Lie}_{I}. \]

The pseudo-tensor functor $\alpha$ is given by
\[ \alpha_I : \text{Hom}_{M(X)}(\otimes_{i \in I} L_i, M) \otimes_{\mathbb{K}} \text{Lie}_{I} \to P_{I}^{P}(\{L_i\}, M) \]
which sends $\varphi \otimes \mu$ to the chiral operation
\[ j_0^{(I)} \ast (\otimes_{i \in I} L_i) \simeq (\otimes_{i \in I} L_i) \otimes j_0^{(I)} \ast (\otimes_{i \in I} L_i) \otimes \Delta_{s}^{(I)} \text{Lie}_{I} \simeq \Delta_{s}^{(I)} (\otimes_{i \in I} L_i) \]
\[ \simeq \Delta_{s}^{(I)} (\otimes_{i \in I} L_i) \to \Delta_{s}^{(I)} (\otimes_{i \in I} L_i) \]
where $\kappa$ is the isomorphism in Fact 1.7. $\alpha_I$ is injective for any $I$, so that $\alpha$ is a faithful pseudo-tensor functor.
1.5.2. Commutative chiral algebras. For an operad $\mathcal{B}$, consider $\mathcal{B}$-algebras on the pseudo-tensor categories in the sequence (1.10). Then we have the sequence of functors
\[ \mathcal{B}(M(X)^1 \otimes \text{Lie}) \xrightarrow{\alpha} \mathcal{B}^{ch}(X) \xrightarrow{\beta} \mathcal{B}^e(X). \]

Applying this argument to $\mathcal{B} = \text{Lie}$, we have
\[ \mathcal{C}om^1(X) \xrightarrow{\alpha_{\text{Lie}}} \mathcal{L}ie^{ch}(X) \xrightarrow{\beta_{\text{Lie}}} \mathcal{L}ie^{e}(X). \]

Here $\mathcal{C}om$ is the operad of commutative algebras (without unit), and we used the following fact.

**Fact** ([BD04, 1.1.10. Lemma]). For any pseudo-tensor category $\mathcal{M}$, $\mathcal{L}ie(\mathcal{M} \otimes \text{Lie}) \simeq \mathcal{C}om(\mathcal{M})$.

Recall that a chiral algebra (or a $\mathcal{L}ie^{ch}$-algebra in general) is equipped with the binary chiral operation $\mu_A \in P_2^e(\{A,A\},A)$.

**Definition.** A $\mathcal{L}ie^{ch}$-algebra $A$ is said to be commutative if $[\ ]_A = 0$, where
\[ [\ ]_A := \beta_{\text{Lie}}(\mu_A) \in P_2^e(\{A,A\},A). \]

Denote by $\mathcal{L}ie^{ch}_{\text{com}}(X) \subset \mathcal{L}ie^{ch}(X)$ the full subcategory of commutative $\mathcal{L}ie^{ch}$-algebras, and by $\mathcal{E}A(X)_{\text{com}} \subset \mathcal{E}A(X)$ the full subcategory of commutative chiral algebras. We call $[\ ]_A$ the $*$-bracket of $A$.

Finally we remark the equivalences
\[ \alpha_{\text{Lie}} : \mathcal{C}om^1(X) \xrightarrow{\sim} \mathcal{L}ie^{ch}_{\text{com}}(X), \quad \mathcal{C}omu^1(X) \xrightarrow{\sim} \mathcal{E}A(X)_{\text{com}}, \]
where $\mathcal{C}omu$ is the operad of commutative algebra with unit. One can obtain these from the the sequence (1.10) and the observation that the composition $\beta_{\text{Lie}} \circ \alpha_{\text{Lie}} : P_2^e \otimes \text{Lie} \rightarrow P_2^e$ vanishes for $|I| \geq 2$ and the sequence $0 \rightarrow P_2^e \otimes \text{Lie}_2 \rightarrow P_2^e$ is exact.

1.5.3. The deformation problem. A coisson algebra can be considered as classical limits of chiral algebras on $X$, as indicated in [BD04, §3.3.11].

Let $A_t$ be a flat family of chiral algebras over $\mathbb{K}[t]$, namely it is a chiral algebra defined over $\mathbb{K}[t]$ which is flat as a $\mathbb{K}[t]$-module. Assume that $A_0 := A_t/tA_t$ is a commutative chiral algebra, namely, the $*$-bracket $[\ ]_t$ of $A_t$ vanishes modulo $t$.

**Fact.** Under the assumption, $A_0$ has a structure of coisson algebra.

In fact, $\{\} := t^{-1}[\ ]_t$ is a $\mathcal{L}ie^*$-bracket on $A_1$, and the corresponding $\mathcal{L}ie^*$-algebra acts on the chiral algebra $A_t$ by adjoint. Modulo $t$, we see that $A_0 \in \mathcal{C}omu^1(X)$ and $\{\} := \{\}$ (mod $t$) is a coisson bracket.

We call $A_t$ a chiral deformation quantization of the coisson algebra $(A_0,\{\})$ (although [BD04] simply called it quantization).

Using the sequence $\mathcal{M}^{ch} \rightarrow \mathcal{M}^{cl} \rightarrow \mathcal{M}^{e}$ of (1.10) and Definition 1.4 that a coisson algebra is equivalent to a $\mathcal{L}ie$-algebra in $\mathcal{M}^{e}$, one can restate our deformation problem as follows. For an object $A \in \mathcal{M}(X)$, we have a series of operad morphisms
\[ \mathcal{E}nd_A^{ch} \rightarrow \mathcal{E}nd_A^{cl} \rightarrow \mathcal{E}nd_A^{e} \]
(1.11)
between the operads induced by the pseudo-tensor structures $\mathcal{M}(X)^{ch}$, $\mathcal{M}(X)^{cl}$ and $\mathcal{M}(X)^{e}$. Then a chiral and a coisson algebra structure on $A$ are given by operad morphisms
\[ \mathcal{L}ie \rightarrow \mathcal{E}nd_A^{ch}, \quad \mathcal{L}ie \rightarrow \mathcal{E}nd_A^{cl}, \quad \mathcal{L}ie \rightarrow \mathcal{E}nd_A^{e} \]
respectively. Then a deformation of a coisson algebra $A$ is to lift the operad morphism from $\mathcal{E}nd_A^{e}$ to $\mathcal{E}nd_A^{ch}$.

2. Convolution dg Lie algebra in operad formalism

In §0.2.3 we give a brief explanation of the convolution dg Lie algebra for $\mathcal{P}$-algebra structures. In this section we give a more detailed discussion following [LV12].

2.1. Convolution dg Lie algebra.
2.1.1. Convolution pseudo-tensor category. We will introduce a convolution pseudo-tensor category, mimicking the discussion in [LV12, §6.4] for operads.

One may define the notion of co-pseudo-tensor category in a dual way. It is the data \((\mathcal{L}, C, \Delta, \varepsilon)\) consisting of the followings.

- A class \(\mathcal{L}\) of objects.
- A \(\mathbb{K}\)-vector space of cooperations \(C^C_I (L, \{M_i\}, \{N_j\}) \equiv \text{Hom}_K (\text{Ext}_I (L, \{M_i\}), \text{Hom}_R (\{N_j\}))\) for any \(I \in S\) and objects \(L, M_i, N_j \in \mathcal{L}\).
- A decomposition map

\[
\Delta : C_J (L, \{N_j\}) \longrightarrow \bigoplus_{I \in Q(J)} \bigoplus_{\{M_i\}} C_I (L, \{M_i\}) \otimes \mathbb{K} \bigotimes_{\varepsilon \in I} C_{J_i} (M_i, \{N_j\}_{J_j})
\]

for any \(J \in S\), where the second summation is over all the \(I\)-families of objects \(\{M_i\}\) in \(\mathcal{L}\).
- An element \(\varepsilon_N \in C_I (\mathcal{L}, \{M_i\}, \{N_j\})\) for any \(N \in \mathcal{L}\) called the counit cooperation.

These should satisfy coassociativity and counit axiom like in the pseudo-tensor structure.

Similarly as in the operad case, a co-pseudo-tensor category with one object coincides with a reduced cooperad. Also a co-pseudo-tensor category has a structure of the category in the usual sense.

Assume that we are given a pseudo-tensor category \(\mathcal{M} = (\mathcal{M}, P, \gamma, \eta)\) and co-pseudo-tensor category \(\mathcal{L} = (\mathcal{L}, C, \Delta, \varepsilon)\). Fix \(I \in S\) and consider the \(\mathbb{K}\)-module

\[
P^I_{\text{Hom}} (K, \{L_i\}; \{M_i\}, N) := \text{Hom}_K (C_I (K, \{L_i\}), P_I (\{M_i\}, N)).
\]

for \(K, L_i \in \mathcal{L}\) and \(M_i, N \in \mathcal{M}\) \((i \in I)\). We claim that there is a pseudo-tensor category structure over the product category \(\mathcal{L} \otimes \mathcal{M}\) (in the usual sense) encoded by the above \(P^I_{\text{Hom}}\). Suppressing the symbols of objects such as \(K\) and \(L_i\)'s, the composition map is written as

\[
\gamma^\text{Hom} : P^I_{\text{Hom}} \otimes \bigotimes_{i \in I} P^J_{\text{Hom}} \longrightarrow P^J_{\text{Hom}}
\]

for \(I \in Q(J)\). We define it by sending \(\varphi \otimes (\otimes_i \psi_i)\) to

\[
C_J \xrightarrow{\Delta} \bigoplus_{I \in Q(J)} \bigoplus_{\{M_i\}} C_I \otimes \bigotimes_{\varepsilon \in I} C_{J_i} \xrightarrow{\varphi \otimes (\otimes_i \psi_i)} P_I \otimes \bigotimes_{i \in I} P_{J_i} \xrightarrow{\gamma} P_J.
\]

One can easily find the unit operation \(\eta^\text{Hom}\) using \(\eta\) and \(\varepsilon\). We denote the resulting pseudo-tensor category as

\[
\text{Hom}_{\text{PT}} (\mathcal{L}, \mathcal{M}) := (\mathcal{L} \otimes \mathcal{M}, P^I_{\text{Hom}}, \gamma^\text{Hom}, \eta^\text{Hom}).
\]

We will call it the convolution pseudo-tensor category.

Now let us introduce the dg version.

**Definition.** A dg pseudo-tensor category is a pseudo-tensor A-category (see Remark 1.1) where \(A\) is taken to be the abelian category of complexes \((\mathbb{Z}\text{-graded vector spaces with endomorphisms} \ d \text{ of degree 1 with} \ d^2 = 0)\) over \(\mathbb{K}\).

Thus \(P_I\) in a dg pseudo-tensor category \(\mathcal{M} = (\mathcal{M}, P, \gamma, \eta)\) is a complex over \(\mathbb{K}\). A dg co-pseudo-tensor category is similarly defined.

**Remark 2.1.** We understand complexes are sign-graded, namely the tensor product \(V \otimes W\) is equipped with the symmetry

\[
\tau : V \otimes W \longrightarrow W \otimes V, \quad v \otimes w \longmapsto (-1)^{|v||w|} w \otimes v.
\]

Let \(\mathcal{M}\) be a dg pseudo-tensor category and \(\mathcal{L}\) be a dg co-pseudo-tensor category. Then the space \(P^I_{\text{Hom}}\) has a grading induced by the grading structures in \(\mathcal{M}\) and \(\mathcal{L}\).

**Definition 2.2.** For a homogeneous element \(f \in P^I_{\text{Hom}} = \text{Hom}_K (C_I, P_I)\), define

\[
\partial (f) := d_P f - (-1)^{|f|} f d_C,
\]

where \(|f|\) is the grading of \(f\) and \(d_P, d_C\) are the differentials of the complex \(P_I, C_I\) respectively.

Then we immediately have \(\partial^2 = 0\). Thus we have

**Lemma.** \((\text{Hom}_{\text{PT}} (\mathcal{L}, \mathcal{M}), \partial)\) is a dg pseudo-tensor category.
2.1.2. Infinitesimal composition maps. The next goal is to introduce a dg Lie algebra associated to each object of the convolution pseudo-tensor category. As a preliminary, we introduce infinitesimal composition maps for operads and cooperads, following [LV12, §6.1]. We also introduce several basic notions of (co)operads for later purpose.

First let us recall the composite of \( \mathcal{S} \)-modules. Given \( \mathcal{S} \)-modules \( M = \oplus_{n \geq 0} M(n) \) and \( N = \oplus_{n \geq 0} N(n) \), we set

\[
M \circ N := \bigoplus_{n \geq 0} M(n) \otimes_{\mathcal{S}_n} N^{\otimes n},
\]

where \( \mathcal{S}_n \) acts on \( M(n) \) by the given right action and on \( N^{\otimes n} \) by permutation of factors. Let us denote an element of \((M \circ N)(n) = M(n) \otimes_{\mathcal{S}_n} N^{\otimes n}\) by

\[
(\mu; \nu_1, \ldots, \nu_n)
\]

with \( \mu \in M(n) \) and \( \nu_i \in N \).

Using the composite, we can consider the composition map \( \gamma \) of an operad \( \mathcal{P} = (\mathcal{P}, \gamma, \eta) \) as an \( \mathcal{S} \)-module morphism

\[
\gamma : \mathcal{P} \circ \mathcal{P} \rightarrow \mathcal{P}.\]

Similarly the decomposition map \( \Delta \) of a cooperad \( \mathcal{C} = (\mathcal{C}, \Delta, \varepsilon) \) is a morphism \( \mathcal{C} \rightarrow \mathcal{C} \circ \mathcal{C} \). The image \( \Delta(\mu) \) is written as

\[
\Delta(\mu) = \sum (\nu; \nu_1, \ldots, \nu_n), \quad \nu \in \mathcal{C}(n), \nu_i \in \mathcal{C}.
\]

For two morphisms \( f : M_1 \rightarrow M_2 \) and \( g : N_1 \rightarrow N_2 \) of \( \mathcal{S} \)-modules, we have a natural composite

\[
f \circ g : M_1 \circ N_1 \rightarrow M_2 \circ N_2.
\]

It can be written down as

\[
(f \circ g)(\mu; \nu_1, \ldots, \nu_n) := (f(\mu); g(\nu_1), \ldots, g(\nu_n)).
\]

For \( \mathcal{S} \)-modules \( L, M \) and \( N \), we set

\[
L \circ (M; N) := \bigoplus_{n \geq 0} L(n) \otimes_{\mathcal{S}_n} \left( \bigoplus_{i=1}^{n} M^{\otimes(i-1)} \otimes N \otimes M^{\otimes(n-i)} \right),
\]

where \( N \) sits in the \( i \)-th position, and the \( \mathcal{S}_n \)-actions are similar as in the composite \( M \circ N \). Clearly it is a sub \( \mathcal{S} \)-module of the composite \( L \circ (M \otimes N) \). We also set

\[
M \circ (1)_N := M \circ (I; N),
\]

where \( I \) denotes the identity \( \mathcal{S} \)-module given by

\[
I = \oplus_{n \geq 0} I(n), \quad I(0) = 0, \quad I(1) = \mathbb{K}, \quad I(2) = I(3) = \cdots = 0.
\]

Thus an element of \((M \circ (1)_N)(n) = M(n) \otimes_{\mathcal{S}_n} (\oplus_{i=1}^{n} I^{\otimes(i-1)} \otimes N \otimes I^{\otimes(n-i)})\) can be written as

\[
(\mu; k_1, \ldots, k_{i-1}, \nu, k_{i+1}, \ldots, k_n)
\]

with \( \mu \in M(n), \nu \in N \) and \( k_j \in I \simeq \mathbb{K} \). Now we have

**Definition 2.3.** For an operad \( \mathcal{P} = (\mathcal{P}, \gamma, \eta) \), we define the infinitesimal composition map \( \gamma_{(1)} \) to be

\[
\gamma_{(1)} : \mathcal{P} \circ (1) \mathcal{P} \hookrightarrow \mathcal{P} \circ (I \oplus \mathcal{P}) \xrightarrow{id_{\mathcal{P}} \circ (\eta + id_{\mathcal{P}})} \mathcal{P} \circ \mathcal{P} \xrightarrow{\gamma} \mathcal{P}.
\]

By definition the map \( \gamma_{(1)} \) is the restriction of \( \gamma \) where we only compose two operations.

**Definition 2.4.** For two morphisms \( f : M_1 \rightarrow M_2 \) and \( g : N_1 \rightarrow N_2 \) of \( \mathcal{S} \)-modules, we define the morphism

\[
f \circ (1)_1 : M_1 \circ (1) N_1 \rightarrow M_2 \circ (1) N_2, \quad (\mu; k_1, \ldots, \nu, \ldots, k_n) \mapsto (f(\mu); k_1, \ldots, g(\nu), \ldots, k_n).
\]

Next we recall the infinitesimal composition map of cooperad. Let us introduce another composite \( \circ' \) of two \( \mathcal{S} \)-module morphisms \( f : M_1 \rightarrow M_2 \) and \( g : N_1 \rightarrow N_2 \). It is given by

\[
f \circ' g := \sum_i f \otimes (id_{N_1}^{\otimes(i-1)} \otimes g \otimes id_{N_1}^{\otimes(n-i)}): M_1 \circ N_1 \rightarrow M_2 \circ (N_1; N_2),
\]

where \( g \) sits in the \( i \)-th position. Then we have

**Definition 2.5.** For a cooperad \( \mathcal{C} = (\mathcal{C}, \Delta, \varepsilon) \), the infinitesimal decomposition map \( \Delta_{(1)} \) of \( \mathcal{C} \) is defined to be

\[
\Delta_{(1)} : \mathcal{C} \rightarrow \mathcal{C} \circ (1) \mathcal{C} \xrightarrow{id_{\mathcal{C}} \circ \varepsilon + id_{\mathcal{C}}} \mathcal{C} \circ (\mathcal{C}; \mathcal{C}) \xrightarrow{id_{\mathcal{C}} \circ (\varepsilon \circ id_{\mathcal{C}})} \mathcal{C} \circ (I; \mathcal{C}) = \mathcal{C} \circ (1) \mathcal{C}.
\]

It is a decomposition of an element of \( \mathcal{C} \) into two parts. Namely, using (2.1) we have

\[
\Delta_{(1)}(\mu) = \sum_{i=1}^{n} (\nu; \varepsilon(\nu_1), \ldots, \varepsilon(\nu_{i-1}), \nu_i, \varepsilon(\nu_{i+1}), \ldots, \varepsilon(\nu_n))
\]
2.1.3. Convolution Lie algebra. Now we will now define a Lie bracket on the convolution pseudo-tensor category \( \text{Hom}_{\mathcal{PT}}(\mathcal{L}, \mathcal{M}) \) following \cite[§6.4]{LV12}.

Fix an object \((L, M) \in \mathcal{L} \otimes_{\mathbb{K}} \mathcal{M}\). Then we have a reduced operad \( \text{End}_M = (\text{End}_M, \gamma_M, \eta_M) \) with the underlying \( \Sigma \)-module

\[
\text{End}_M(n) := P_n^M(M) = P_n^M(\{M, \ldots, M\}, M).
\]

The composition map \( \gamma_M \) comes from \( \gamma \) of \( M \). Similarly we have a reduced cooperad \( \co\text{End}_L = (\co\text{End}_L, \Delta_L, \varepsilon_L) \) with

\[
\text{coEnd}_L(n) := C_n^L(L) = C_n^L(\{L, \ldots, L\}).
\]

We also have a reduced operad \( \text{Hom}_{L,M} = (\text{Hom}_{L,M}, \gamma_{L,M}, \eta_{L,M}) \) with

\[
\text{Hom}_{L,M}(n) := \text{Hom}_\mathbb{K}((\text{coEnd}_L(n), \text{End}_M(n))).
\]

Below we simply denote

\[
P := \text{End}_M, \quad \varepsilon := \text{coEnd}_L, \quad \text{Hom} \equiv \text{Hom}_\mathbb{K}(\varepsilon, P) := \text{Hom}_{L,M}.
\]

Using Definitions 2.3–2.5 we introduce

**Definition.** For \( f, g \in \text{Hom} \), define \( f \star g \) to be the following composition of maps.

\[
f \star g := (\varepsilon \Delta_{(1)} \circ \varepsilon \delta_{(1)} g \circ \varepsilon \delta_{(1)} P \circ \nu_{(1)} \Delta_{(1)} \circ \varepsilon \delta_{(1)} P) \rho_{(1)} \rho_{(1)} \rho_{(1)} \rho_{(1)} \rho_{(1)}.
\]

**Remark.** Precisely speaking, this definition is due to \cite[Proposition 6.4.3]{LV12} and not the original definition in \cite[§5.4.3]{LV12} where partial composition is used.

Using (2.1) for \( \mu \in \varepsilon \) and denoting \( \text{id} = \eta(1) \in \mathcal{P}(1) \), one can write down the definition as

\[
(f \star g)(\mu) = \sum_{i=1}^n \varepsilon(\nu_1) \cdots \varepsilon(\nu_{i-1}) \cdots \varepsilon(\nu_{i+1}) \cdots \varepsilon(\nu_n) \cdot \gamma((f(\nu); \text{id}, \ldots, \text{id}, g(\nu_i), \text{id}, \ldots, \text{id})). \tag{2.3}
\]

**Proposition 2.6** \cite[Proposition 6.4.3]{LV12}. The product \( \star \) is a pre-Lie product, namely it satisfies

\[
(f \star g) \star h - f \star (g \star h) = (f \star h) \star g - f \star (h \star g).
\]

Hence the anti-symmetrization

\[
[f, g] := f \star g - g \star f
\]

satisfies the Jacobi rule, and \( \text{Hom} \) has a structure of Lie algebra.

**Proof.** Although a proof is given in \cite[§6.4]{LV12}, let us give another proof using the explicit formula (2.3). The coassociativity of \( \Delta \) in \( \varepsilon \) is given by

\[
(\Delta \circ \text{id}_\varepsilon) \Delta = (\text{id}_\varepsilon \circ \Delta) \Delta : \varepsilon \rightarrow (\varepsilon \circ \varepsilon) \circ \varepsilon \simeq \varepsilon \circ (\varepsilon \circ \varepsilon),
\]

and on an element \( \mu \in \varepsilon \) with (2.1) it means

\[
\sum \left( (\omega; o_1, o_2, \ldots, o_j) ; \nu_1, \ldots, 1, \ldots, \nu_n \right) = \sum \left( o_1; o_1, o_2, \ldots, o_j, o_1, \ldots, o_2, \ldots, o_j, \ldots, \nu_n \right),
\]

with \( \Delta(\nu) = (\omega; o_1, 1, \ldots, o_j, 1, \ldots, o_j, \ldots, \nu_n, o_1, \ldots, o_j, \ldots, \nu_n, 1, \ldots, o_j, \ldots, \nu_n, o_1, \ldots, o_j, \ldots, \nu_n) \) and \( \Delta(\nu_i) = (\omega_i; o_1, 1, \ldots, o_j, 1, \ldots, o_j, \ldots, \nu_n, o_1, \ldots, o_j, \ldots, \nu_n, 1, \ldots, o_j, \ldots, \nu_n, o_1, \ldots, o_j, \ldots, \nu_n) \).

The formula (2.3) yields

\[
(f \star g) \star h(\mu) = \sum \varepsilon(\tilde{\omega}) \cdot \varepsilon(\tilde{\nu}) \times \gamma((f(\omega); \text{id}, \ldots, g(\omega_j), \ldots, \text{id}); \text{id}, \ldots, h(\mu), \ldots, \text{id}),
\]

with \( \varepsilon(\tilde{\omega}) := \prod_{k \neq j} \varepsilon(\omega_k) \) and \( \varepsilon(\tilde{\nu}) := \prod_k \varepsilon(\nu_k) \).

Similarly we have

\[
(f \star g) \star h(\mu) = \sum \varepsilon(\tilde{\omega}) \cdot \varepsilon(\tilde{\nu}) \times \gamma((f(\omega); \text{id}, \ldots, g(\omega_j), \ldots, \text{id}); \text{id}, \ldots, h(\mu), \ldots, \text{id}, \ldots, \text{id}),
\]

with \( \tilde{\omega}_j := \prod_{k \neq j} \varepsilon(\omega_k) \).

We also note that the counit property of \( \varepsilon \) in \( \varepsilon \) is given by

\[
(\varepsilon \circ \varepsilon) \Delta = (\varepsilon \circ \varepsilon) \Delta = \text{id}_\varepsilon : \varepsilon \rightarrow \varepsilon \circ I \simeq I \circ \varepsilon \simeq \varepsilon.
\]

On the element \( \nu_i \) it means \( \sum \left( \prod_k \varepsilon(\omega_k) \cdot (\omega_i; 1, 1, 1, 1) ; \nu_1, \ldots, 1, \ldots, \nu_n \right) = \nu_i \), where \( (\omega_i; 1, 1, 1, 1) \in \varepsilon \circ I \simeq \varepsilon \). In particular, we have

\[
\prod_k \varepsilon(\omega_k)(g(\omega_i)) = g(\nu_i) \quad \text{for } g \in \text{Hom}_{L,M}.
\]

This formula and the associativity of \( \gamma \) implies

\[
\begin{align*}
((f \star g) \star h)(\mu) &= \sum \varepsilon(\tilde{\omega}) \cdot \gamma((f(\omega); \text{id}, \ldots, g(\omega_j), \ldots, \text{id}); \text{id}, \ldots, h(\mu), \ldots, \text{id}),
\end{align*}
\]

\[
\begin{align*}
((f \star g) \star h - f \star (g \star h))(\mu) &= \sum_{i \neq j, k \neq i, j} \varepsilon(\nu_k) \cdot \gamma((f(\omega); \text{id}, \ldots, g(\nu_i), \ldots, \text{id}); \text{id}, \ldots, h(\mu), \ldots, \text{id}),
\end{align*}
\]

This expression is symmetric for \( g \) and \( h \), so that we have the result. \( \square \)

**Definition.** Denote the subspace of \( \Sigma \)-equivariant morphisms in \( \text{Hom} \) by

\[
\text{Hom}_{\Sigma}(\varepsilon, P) := \bigoplus_n \text{Hom}_{\Sigma_n}(\varepsilon(n), P(n)).
\]
The proof of Proposition 2.6 implies

**Lemma ([LV12, Lemma 6.4.4])**. \(\text{Hom}_\mathcal{O}(\mathcal{C}, \mathcal{P})\) is stable under the pre-Lie product \(*\), so that it is a Lie algebra.

We also see that the differential \(\partial\) in \(\text{Hom}_{\mathcal{P}^T}(\mathcal{L}, \mathcal{M})\) in Definition 2.2 is compatible with this Lie bracket. Hence

**Fact ([LV12, Proposition 6.4.5])**. For an operad \(\mathcal{P}\) and a cooperad \(\mathcal{C}\) we have a dg Lie algebra

\[
(\text{Hom}_\mathcal{O}(\mathcal{C}, \mathcal{P}), [,], \partial).
\]

In particular, going back to our original situation, we have

**Corollary 2.7.** Let \(M\) be a dg pseudo-category, \(M \in \mathcal{M}\) and \(\text{End}_M^{\mathcal{M}}\) the associated dg operad. Also let \(\mathcal{C}\) be a cooperad. Then we have a dg Lie algebra

\[
(\text{Hom}_\mathcal{O}(\mathcal{C}, \mathcal{P}), [,], \partial).
\]

For any dg Lie algebra \(g = (g, [,], \partial)\) one can consider the Maurer-Cartan equation in it.

\[
\partial(\alpha) + \frac{1}{2}[\alpha, \alpha] = 0, \quad \alpha \in g.
\]

In the next subsection we recall the meaning of the solution of this Maurer-Cartan equation in \(\text{Hom}_\mathcal{O}(\mathcal{C}, \mathcal{P})\) following [LV12, §6.5]. We close this subsection by

**Definition.** We denote by \(\text{Tw}(\mathcal{C}, \mathcal{P})\) the space of homogeneous solutions of the Maurer-Cartan equation of degree \(-1\).

\[
\text{Tw}(\mathcal{C}, \mathcal{P}) := \{ \alpha \in \text{Hom}_\mathcal{O}(\mathcal{C}, \mathcal{P}) \mid |\alpha| = -1, \ \partial(\alpha) + \frac{1}{2}[\alpha, \alpha] = 0 \}.
\]

We call its element a twisted morphism from \(\mathcal{C}\) to \(\mathcal{P}\).

2.2. **Koszul operads.**

2.2.1. **Cobar construction and twisting morphisms.** Let us recall the bar construction of an augmented dg operad.

First we recall the free operad \(\mathcal{F}(M)\) and the cofree cooperad \(\mathcal{F}^r(M)\) of an \(\mathcal{S}\)-module \(M\). We refer [LV12, §5.6, §5.8] for a full account.

Let \(M\) be an \(\mathcal{S}\)-module. Define inductively \(\mathcal{T}_n M\) by

\[
\mathcal{T}_0 M := I, \quad \mathcal{T}_1 M := I \oplus M, \quad \mathcal{T}_n M := I \oplus (M \circ \mathcal{T}_{n-1} M) \quad (n \geq 2).
\]

Using the inclusion \(\mathcal{T}_n M \hookrightarrow \mathcal{T}_{n+1} M\), we get an \(\mathcal{S}\)-module \(\mathcal{T}M\) given by

\[
\mathcal{T}M := \cup_{n \geq 0} \mathcal{T}_n M = \text{colim}_{n \geq 0} \mathcal{T}_n M.
\]

Then \(\mathcal{T}M\) has a structure of an operad such that any \(\mathcal{S}\)-module morphism \(f : M \rightarrow \mathcal{P}\) to an operad extends uniquely to an operad morphism \(\mathcal{T}M \rightarrow \mathcal{P}\). We will denote the resulting operad by \(\mathcal{T}(M)\).

A dual construction also exists. Recall that a coaugmented cooperad \(\mathcal{C} = (\mathcal{C}, \Delta, \varepsilon, \eta)\) is a cooperad \((\mathcal{C}, \Delta, \varepsilon)\) with a cooperad morphism \(\eta : I \rightarrow \mathcal{C}\) such that \(\varepsilon \eta = \text{id}_I\). Here \(I\) denotes the identity cooperad whose underlying \(\mathcal{S}\)-module is given by (2.2). \(\eta\) is called the coaugmented morphism of \(\mathcal{C}\). The image of \(1 \in I(1) = \mathbb{K} \subset I\) is denoted by \(\text{id} \in \mathcal{C}(1)\).

The \(\mathcal{S}\)-module \(\mathcal{T}M\) explained above has a structure of cooperad such that any \(\mathcal{S}\)-module morphism \(g : \mathcal{C} \rightarrow M\) sending \(\text{id}\) from \(I\) to \(0\) from a conilpotent cooperad \(\mathcal{C}\) factors through \(\mathcal{T}M\) (we have the projection \(\mathcal{T}M \rightarrow M\)). We will denote by \(\mathcal{F}^r(M)\) this universal cooperad, and call it the cofree cooperad of \(M\).

Let us introduce the notation for the cobar construction. For a coaugmented dg cooperad \(\mathcal{C} = (\mathcal{C}, \Delta, \varepsilon, \eta)\), there is an augmented dg operad \(\Omega \mathcal{C}\) whose operad structure is given by

\[
\Omega \mathcal{C} := \mathcal{F}(s^{-1}\mathcal{C}).
\]

Here \(\mathcal{C}\) is the coaugmentation coideal \(\text{Coker}(\eta : I \rightarrow \mathcal{C})\) of the coaugmented cooperad \(\mathcal{C}\). We skip the explanation of the dg structure and refer [LV12, §6.5] for the full account. \(\Omega \mathcal{C}\) is called the cobar construction.

Now we can state the following representability of the functor given by \(\text{Tw}\).

**Fact 2.8 ([LV12, Theorem 6.5.7])**. For every augmented dg operad \(\mathcal{P}\) and conilpotent dg cooperad \(\mathcal{C}\), there exist natural isomorphisms

\[
\text{Hom}_{dgOp}(\Omega \mathcal{C}, \mathcal{P}) \simeq \text{Tw}(\mathcal{C}, \mathcal{P}).
\]

Here \(dgOp\) denotes the category of dg operads.
2.2.2. Koszul dual. Let us recall the Koszul dual cooperad of a quadratic operad. We follow the description in [LV12, §§7.1–7.2].

Recall the free operad $\mathcal{F}(E)$ associated to an $\mathcal{S}$-module $E$. $\mathcal{F}(E)$ has a weight grading $w$ defined by
\[
w(1) = 0, \quad w(\mu) = 1 \text{ for } \mu \in E(n), n > 0
\]
where 1 is the basis of the first part $I(1) = \mathbb{K}$ of the identity $\mathcal{S}$-module $I$, and
\[
w(\mu; \nu_1, \ldots, \nu_n) := w(\mu) + w(\nu_1) + \cdots + w(\nu_n)
\]
for $(\mu; \nu_1, \ldots, \nu_n) \in \mathcal{F}_nE$. Denote by $\mathcal{F}^{(d)}(E)$ the weight $d$ subspace.

Let $(E, R)$ be a pair of a graded $\mathcal{S}$-module $E$ and a graded sub-$\mathcal{S}$-module $R \subseteq \mathcal{F}(E)^{(2)}$. Such a pair is called a quadratic operad. The quadratic operad associated to $(E, R)$ is defined to be
\[
\mathcal{P}(E, R) := \mathcal{F}(E)/(R)
\]
where $(R)$ is the (operadic) ideal generated by $R$. In terms of the universality, it is universal among the quotient operads $\mathcal{P}$ of $\mathcal{F}(E)$ such that the composite $R \hookrightarrow \mathcal{F}(E) \twoheadrightarrow \mathcal{P}$ vanishes. $\mathcal{P}(E, R)$ has a weight grading induced by that on $\mathcal{F}(E)$, and the weight $d$ subspace is denoted by $\mathcal{P}(E, R)^{(d)}$.

Similarly, the quadratic cooperad $\mathcal{C}(E, R)$ associated to $(E, R)$ is the sub-cooperad of the cofree cooperad $\mathcal{F}^c(E)$ which is among the sub-cooperads $\mathcal{C}$ of $\mathcal{F}^c(E)$ such that the composite $\mathcal{C} \hookrightarrow \mathcal{F}^c(E) \twoheadrightarrow \mathcal{F}^c(E)/(R)$ vanishes. The underlying $\mathcal{S}$-module of $\mathcal{C}(E, R)$ is $\mathcal{F}_E$, and $\mathcal{C}(E, R)$ has a weight grading. The weight $d$ subspace is denoted by $\mathcal{P}(E, R)^{(d)}$.

Now for a quadratic operad $\mathcal{P} = \mathcal{P}(E, R)$, its Koszul dual cooperad is defined to be the quadratic cooperad
\[
\mathcal{P}^{\text{cl}} := \mathcal{C}(sE, s^2R)
\]

2.2.3. Koszul operad. A Koszul operad is a quadratic operad whose Koszul complex is acyclic. We will not give the definition of Koszul complex for a quadratic operad, and refer [LV12, §7.4].

Let us mention following criterion of Koszul-ness. By the definition of the cobar construction, for a quadratic operad $\mathcal{P} = \mathcal{P}(E, R)$ we have a natural projection
\[
p : \Omega\mathcal{P}^{\text{cl}} \longrightarrow \mathcal{P}.
\]

**Fact 2.9** ([LV12, Theorem 7.4.2]). A quadratic operad $\mathcal{P}$ is Koszul if and only if the natural projection $p$ is a quasi-isomorphism of dg operads.

2.3. Twisted morphism and algebra structure. Fix a pseudo-tensor category by $\mathcal{M}$. As before, for an object $M \in \mathcal{M}$, we denote the associated operad by
\[
\mathcal{E}nd_M \equiv \mathcal{E}nd^M := \oplus_{n \geq 1} \mathcal{P}_n^M(M).
\]

For a Koszul operad $\mathcal{B}$, a homotopy $\mathcal{B}$-algebra, or a $\mathcal{B}_\infty$-algebra, in $\mathcal{M}$ means an $\Omega\mathcal{B}^{\text{cl}}$-algebra in $\mathcal{M}$. Thus a $\mathcal{B}_\infty$-algebra structure on $M \in \mathcal{M}$ is a morphism of dg operads
\[
\Omega\mathcal{B}^{\text{cl}} \longrightarrow \mathcal{E}nd_M.
\]

A $\mathcal{B}$-algebra is an example of $\mathcal{B}_\infty$-algebra, since the structure morphism gives
\[
\Omega\mathcal{B}^{\text{cl}} \xrightarrow{q_{\mathcal{B}}} \mathcal{B} \longrightarrow \mathcal{E}nd_M.
\]

Here $q_{\mathcal{B}}$ is the quasi-isomorphism given by Fact 2.9.

By Fact 2.8 and Corollary 2.7, a morphism of dg operads $\Omega\mathcal{B}^{\text{cl}} \rightarrow \mathcal{E}nd_M$ is equivalent to a twisted morphism in the dg Lie algebra.

**Definition 2.10.**
\[
\mathfrak{g} \equiv \mathfrak{g}^M_{\mathcal{B}, M} := \mathcal{H}(\mathcal{B}^{\text{cl}}_\mathfrak{s}, \mathcal{E}nd^M_M).
\]

It has a weight grading $\mathfrak{g} = \oplus_{n \geq 0} \mathfrak{g}^{(n)}$ given by
\[
\mathfrak{g}^{(n)} := \mathcal{H}(\mathcal{B}^{\text{cl}}_\mathfrak{s}^{(n)}, \mathcal{E}nd^M_M),
\]
where $\mathcal{B}^{\text{cl}} = \oplus_{n \geq 0} (\mathcal{B}^{\text{cl}})^{(n)}$ denotes the weight decomposition of the Koszul dg cooperad $\mathcal{B}^{\text{cl}}$.

Now writing down the differential of $\mathfrak{g}$, we have

**Fact 2.11** ([LV12, Proposition 10.1.4]). For a Koszul operad $\mathcal{B}$, a $\mathcal{B}_\infty$-algebra in a pseudo-tensor category $\mathcal{M}$ is a $\mathcal{B}$-algebra if and only if its twisting morphism is concentrated in weight 1.

Thus, as explained in the introduction, we have
\[
\{ \text{$\mathcal{B}$-algebra structures on $M$} \} \xrightarrow{1:1} \mathcal{M}(\mathfrak{g}_{\mathcal{B}, M}) := \{ \text{weight 1 elements in $\mathcal{T}(\mathfrak{g}_{\mathcal{B}, M})$} \}.
\]
2.4. Deformation theory. Let us briefly recall the deformation theory using dg Lie algebras following [GM88]. For a full account using operadic language, see [LV12, §12.2].

For a dg Lie algebra $g = (g, [], d)$, let us denote the space of solutions of the Maurer-Cartan equation with (weight) grading $1$ by $MC(g)$. On this space the subspace $g^0$ acts infinitesimally on $MC(g)$. Namely the map

$$g^0 \ni \lambda \mapsto d\lambda + [-, \lambda] \in \Gamma(MC(g), TMC(g))$$

is a morphism of Lie algebras, where the target is the Lie algebra of vector fields on $MC(g)$. The exponentiated action can be written as

$$e^t\lambda \alpha = e^{t \text{ad}(\lambda)}(\alpha) + \frac{id - e^{t \text{ad}(\lambda)}}{\text{ad}(\lambda)}(d\lambda).$$

Let $G$ be the adjoint group of $g^0$. We denote by

$$MC(g) := MC(g)/G$$

the moduli space of Maurer-Cartan elements of $g$, which is considered as a coset, or a groupoid (called the Deligne groupoid).

Let $\mathcal{B}$ be a Koszul operad, $M$ be an object of a pseudo-tensor category $M$. Hereafter we write

$$g := g^M_{\mathcal{B}, M} = (\text{Hom}_\mathcal{B}(\mathcal{B}^{\text{cl}}, \text{End}_M), [\cdot, \cdot], \partial).$$

A solution $\varphi \in MC(g^M_{\mathcal{B}, M})$ gives a $\mathcal{B}$-algebra structure on $M$. Once $\varphi$ is chosen, one can twist the dg Lie algebra $g$ as

**Definition 2.12.** For $\varphi \in MC(g)$, the twisted dg Lie algebra $g^\varphi$ is defined to be

$$g^\varphi := (\text{Hom}_\mathcal{B}(\mathcal{B}^{\text{cl}}, \text{End}_M), [\cdot, \cdot], \partial_\varphi), \quad \partial_\varphi := \partial + [\varphi, -].$$

Consider a local Artin $\mathbb{K}$-algebra $R$ with $\mathfrak{m}$ the maximal ideal. An $R$-deformation of $\varphi$ is an $R$-linear $\mathcal{B} \otimes \mathbb{K}$-$R$-algebra structure on $M \otimes \mathbb{K} R$ which is equal to $\varphi$ modulo $\mathfrak{m}$. The equivalence of $R$-deformations can be defined naturally, and we denote it by $\sim$. We define

$$\text{Def}_\varphi(R) := \{ R\text{-deformations of } \varphi \}, \quad \text{Def}_\varphi(R) := \text{Def}_\varphi(R)/\sim.$$

The latter one can be considered as a set or a groupoid. The standard deformation theory says

**Fact** ([LV12, Proposition 12.2.6]). We have a natural bijections (or equivalence of groupoids)

$$\text{Def}_\varphi(R) \simeq MC(g^\varphi \otimes \mathfrak{m}), \quad \text{Def}_\varphi(R) \simeq MC(g^\varphi \otimes \mathfrak{m}).$$

We also have the standard descriptions of rigidity and obstructions. For $R = \mathbb{K}[\hbar]$, the condition $H^1(g^\varphi) = 0$ implies that any $R$-deformation of $\varphi$ is trivial, and the condition $H^2(g^\varphi) = 0$ implies that any infinitesimal deformation of $g^\varphi$ extends to an $R$-deformation.

3. Chiral dg Lie algebra

We now apply the construction of convolution dg Lie algebras in the operad theory [LV12] explained in §2 to the chiral or coisson operad explained in §1.

3.1. Description of our dg Lie algebras. Let $X$ be a smooth curve and $M(X)$ be the category of right $\mathcal{D}$-modules over $X$. Consider the pseudo-tensor categories $M(X)^c$ and $M(X)^{ch}$. Applying Definition 2.10 of $g^M_{\mathcal{B}, M}$ to the case $\mathcal{B} = Lie$ and $M = M^{ch}, M^c$ we have following two Lie algebras.

**Definition 3.1.** For an object $M \in M(X)$, set

$$g^{ch}_{M} := g^{M^{ch}}_{Lie, M} = \text{Hom}_{\mathcal{B}}(Lie^{cl}, \text{End}^{M^{ch}}_M), \quad g^c_M := g^{M^c}_{Lie, M} = \text{Hom}_{\mathcal{B}}(Lie^{cl}, \text{End}^{M^c}_M).$$

We call them the **chiral** and **coisson Lie algebras** of $M$ respectively.

By the definitions of chiral and coisson Lie algebras and the discussion in §§2.3–§3.2, these objects control the structures of chiral algebra and coisson algebra respectively.

Recall Definition 2.2 of the differential $\partial$. For $g^{ch}_M$ and $g^c_M$, since $Lie^{cl}, \text{End}^{ch}_M$ and $\text{End}^c_M$ have null differentials, we have $\partial = 0$ on $g^{ch}_M$ and $g^c_M$. Hence we called them (graded but with trivial differential) Lie algebras.

Let us write down these Lie algebras explicitly. First we recall an explicit description of $Lie^{cl}$.

**Lemma 3.2.** The graded $\mathcal{S}$-module structure of $Lie^{cl}$ is given by

$$Lie^{cl}(n) \simeq s^{1-n} \text{sgn}_n,$$

where $\text{sgn}_n$ is the sign representation of $\mathcal{S}_n$. 

Proof. The shortest argument is to use the fact \( \mathcal{L}ie^1 \simeq \mathcal{C}om \). For a quadratic operad \( \mathcal{P} = \mathcal{P}(E, R) \), its Koszul dual operad \( \mathcal{P}^! \) is defined to be

\[ \mathcal{P}^! := (coEnd_{sK} \otimes_H \mathcal{P}^*)^\ast. \]

Here \( coEnd_{sV} \) denotes the cooperad of endomorphisms on \( V \) with \( coEnd_{sV} = \oplus_{n \geq 0} \text{Hom}_K(V, V^{\otimes n}) \). The Hadamard tensor product \( \otimes_H \) of graded vector spaces is defined by \( (M \otimes_H N)_n := M_n \otimes N_n \). Finally \( \ast \) denotes the linear dual. As a corollary, we have

\[ \mathcal{P}^! \simeq coEnd_{s^1 - 1K} \otimes_H (\mathcal{P}^!)^\ast \]

Now for \( \mathcal{P} = \mathcal{L}ie \), since \( \mathcal{P}^!(n) = \text{Com}(n) \) is the trivial \( \mathfrak{S}_n \)-module and \( coEnd_{s^1 - 1K}(n) \simeq s^{-1} \text{sgn}_n \) as graded \( \mathfrak{S} \)-module (recall Remark 2.1 that we are considering sign-graded complexes), we have the result. \( \square \)

Next recall that the chiral operad \( End_{\mathfrak{M}}^{ch} \) is given by

\[ End_{\mathfrak{M}}^{ch}(n) = \text{Hom}_{\mathfrak{M}(X^\ast)}(j_\ast j^* M^{\otimes n}, \Delta(M)) \]

where \( j : j(n) : X \hookrightarrow X^n \) is the diagonal embedding and \( \Delta := \Delta(n) : U(n) \hookrightarrow X^n \) is the complement of diagonal divisors. Then by Lemma 3.2 we have

\[ g_M^{ch}(n) \simeq s^{-1} \text{sgn}_n \otimes_{s\mathfrak{E}_n} \text{Hom}_{\mathfrak{M}(X^\ast)}(j_\ast j^* M^{\otimes n}, \Delta(M)) \simeq s^{-1} \text{sgn}_n \text{Hom}_{\mathfrak{M}(X^\ast)}(j_\ast j^* (\wedge^n M), \Delta(M)). \]

Let us restate this formula as

**Lemma 3.3.** The graded \( \mathfrak{S} \)-module structure of \( g_M^{ch} \) is given by

\[ g_M^{ch}(n) \simeq s^{-1} \text{C}^{ch,n}(M), \quad C^{ch,n}(M) := \text{Hom}_{\mathfrak{M}(X^\ast)}(j_\ast j^* (\wedge^n M), \Delta(M)). \]

**Remark 3.4.** \( g_M^{ch} \) looks quite similar to the Chevalley-Eilenberg complex \( C^*(L, L) = \text{Hom}(L^\ast L, L) \) of a Lie algebra \( L \) over \( K \).

Next we study the pre-Lie structure \( \ast \) on \( g_M^{ch}(n) \). As expected from the above remark, the result has the same form as the Nijenhuis-Richardson product [LV12, §13.2.9] on the Chevalley-Eilenberg complex.

For an operad \( (\mathfrak{P}, \gamma, \eta) \), we denote by \( \circ_i \) the \( i \)-th partial composition. It is given by

\[ \mu \circ_i \nu := \gamma(\mu; \text{id}, \ldots, \text{id}, \nu, \text{id}, \ldots, \text{id}) \quad (3.1) \]

with \( \nu \) sitting at the \( i \)-th position and \( \eta := \eta(1) \in \mathfrak{P}(1) \) as before. For \( \mu \in \mathfrak{P}(m) \) and \( \nu \in \mathfrak{P}(n), \mu \circ_i \nu \) is defined for \( 1 \leq i \leq m \) and \( \mu \circ_i \nu \in \mathfrak{P}(m + n - 1) \).

Let us also recall that a \((p, q)\)-shuffle is a permutation

\[ \sigma = \begin{pmatrix} 1 & \cdots & p & p + 1 & \cdots & p + q \end{pmatrix} \]

such that \( i_1 < \cdots < i_p \) and \( j_1 < \cdots < j_q \). The inverse of a \((p, q)\)-shuffle is called \((p, q)\)-unshuffle. Denote by \( \mathfrak{S}_{p+q}^{1\bullet} \subset \mathfrak{S}_{p+q} \) the subset of \((p, q)\)-unshuffles.

**Lemma.** For \( f \in C^{ch,p}(M) \) and \( g \in C^{ch,q}(M) \), we have

\[ f \ast g = \sum_{\sigma \in \mathfrak{S}_{p+q}^{1\bullet}} \text{sgn}(\sigma)(-1)^{(p-1)(q-1)}(f \circ_1 g)_{\sigma}. \quad (3.2) \]

Here \( \circ_1 \) is given by (3.1) with \( \gamma = \gamma^{ch} \) the composition on the chiral operad \( End_{\mathfrak{M}}^{ch} \).

**Proof.** Let us denote by \( \tilde{f} \in \tilde{g}_M^{ch} \) the element corresponding to \( f \) under the isomorphism in Lemma 3.3, and similarly by \( \tilde{g} \) the one corresponding to \( g \). Recall the expression (2.3) of \( \ast \). For \( \mu \in \mathcal{L}ie^{cl} \) with \( \Delta(\mu) \) given by (2.1) we have

\[ (\tilde{f} \ast \tilde{g})(\mu) = \sum_{i} \gamma^{ch}(\tilde{f}; \varepsilon(\nu_1), \ldots, \varepsilon(\nu_{i-1}), g(\nu_i), \varepsilon(\nu_{i+1}), \ldots, \varepsilon(\nu_n)). \quad (3.3) \]

Note that \( \Delta \) means the decomposition in the cooperad \( \mathcal{L}ie^{cl} \), and \( \gamma^{ch} \) is the composition in the operad \( End_{\mathfrak{M}}^{ch} \). By the above expression, it is enough to consider the infinitesimal decomposition \( \Delta_{(1)} \) (see Definition 2.5). By Lemma 3.2 \( \dim \mathcal{L}ie^{cl}(n) = 1 \) and denote the basis by \( \ell_n \). Also by the same lemma \( \Delta \) is induced by the decompositions on \( coEnd \) and \( \mathfrak{C}om^\ast \). Now one can find

\[ \Delta_{(1)}(\ell_n) = \sum_{p+q=n+1} \sum_{\sigma \in \mathfrak{S}_{p+q}^{1\bullet}} (-1)^{(p-1)(q-1)}(\ell_p, \ell_q, \text{id}, \ldots, \text{id})_{\sigma}. \quad (3.4) \]

Here \( \text{id} = \eta(1) \in \mathcal{L}ie^{cl}(1) \) is the image of \( 1 \in K \) under the coaugmentation \( \eta \) of \( \mathcal{L}ie^{cl} \). Going back to \( \ast \), we note that \( \tilde{f} \) can be seen as the map \( \ell_p \mapsto f \). Then (3.3) and (3.4) give the result. \( \square \)

We summarize the argument so far in
Proposition 3.5. The chiral Lie algebra $\mathfrak{g}_M^{ch}$ is described as
\[ \mathfrak{g}_M^{ch}(n) \simeq s^{-n}C^{ch,n}(M), \quad C^{ch,n}(M) := \text{Hom}_{\text{End}_M^n}(j_*(\wedge^n M), \Delta_*(M)). \]
The Lie bracket is given by $[f, g] = f \star g - g \star f$, where the pre-Lie product $\star$ is (3.2) with $\circ_1$ corresponding to the composition map $\gamma = \gamma^{ch}$ of the chiral operad $\text{End}^{ch}_M$.

By the same argument we have

Proposition 3.6. The coisson Lie algebra $\mathfrak{g}_M^{ch}$ is described as
\[ \mathfrak{g}_M^{ch}(n) \simeq s^{-n}C^{ch,n}(M), \quad C^{ch,n}(M) := \bigoplus_{\sigma \in Q(n)} \text{Hom}_{\text{End}_M^n}(\wedge_\sigma s(M^\otimes [n]), \Delta_*^{(S)}(M)) \otimes (\otimes s \in S_{\text{Lie}}([n]),), \]
where $[n] = \{1, \ldots, n\}$ and $[n]_\sigma := \{x \in [n] \mid \pi(x) = s\}$ for $\pi : [n] \to S$ and $s \in S$. The Lie bracket is similarly described as in Proposition 3.5 where we replace $\gamma^{ch}$ with the composition $\gamma^c$ of the cooperad $\mathfrak{g}_M^{ch}$.

Next assume that we are given $\mu \in \text{MC}(\mathfrak{g}_M^{ch})$ or $\text{MC}(\mathfrak{g}_M^{ch})$ and consider the corresponding twisted dg Lie algebra (see Definition 2.12).

Definition 3.7. The twisted dg Lie algebra
\[ \mathfrak{g}_M^{ch,\mu} := (\text{Hom}_\Theta(\mathcal{Lie}^{ch}, \text{End}^{ch}_M), [], \partial_\mu), \quad \partial_\mu := \partial + [\mu, -] = [\mu, -], \]
is called the chiral dg Lie algebra. Similarly,
\[ \mathfrak{g}_M^{ch,\mu} := (\text{Hom}_\Theta(\mathcal{Lie}^{ch}, \text{End}^{ch}_M), [], \partial_\mu) \]
is called the coisson dg Lie algebra.

Recall that $\mu \in \text{MC}(\mathfrak{g}_M^{ch}) \subset \mathfrak{g}_M^{ch}(2)$ is a binary operation. Then from the description of $\mathfrak{g}_M^{ch}$, we find

Proposition 3.8. The differential $\partial_\mu$ of $\mathfrak{g}_M^{ch,\mu}$ is described by
\[ \partial_\mu(f)(x_1 \wedge \cdots \wedge x_{n+1}) = \sum_{i=0}^{n} (-1)^i \mu(x_i, f(x_0 \wedge \cdots \widehat{x}_i \cdots x_n)) \]
\[ + \sum_{0 \leq i < j \leq n} (-1)^{i+j-1} f(\mu(x_i, x_j) \wedge x_0 \wedge \cdots \widehat{x}_i \cdots \widehat{x}_j \cdots x_n) \]
for $f \in C^{ch,n}(M) \simeq s^{-n}\mathfrak{g}_M^{ch}(n)$ and $x_0 \wedge \cdots \wedge x_n \in j_*j^* (\wedge^n M)$. $\widehat{x}_i$ denotes skipping the term $x_i$. The same expression holds for $\mathfrak{g}_M^{ch,\mu}$ with $\mu \in \text{MC}(\mathfrak{g}_M^{ch})$.

Remark. Continuing Remark 3.4, the differential obtained coincides with that of the Chevalley-Eilenberg complex $C^*(L, L)$. Indeed, the convolution dg Lie algebra $(\text{Hom}_\Theta(\mathcal{Lie}^{ch}, \text{End}_L^n), [], \partial_\mu)$ with $\text{End}_L^n = \oplus_{n \geq 1} \text{Hom}_{\text{End}_L}^{\otimes n}(L \otimes L)$ and $\mu$ the given Lie bracket on $L$ is nothing but $C^*(L, L)$ up to shift, as explained in [LV12, §13.2.7].

Let us close this section by translating our dg Lie algebra in the language of vertex algebras. Recall Fact 0.3 of the correspondence between vertex and chiral algebras. Let $\mu \in \text{MC}(\mathfrak{g}_V^{ch})$ be the element corresponding to a quasiconformal vertex algebra $(V, T, [0], Y)$, where $V^*$ is the right $D_X$-module attached to $V$. Then the differential $\partial_\mu$ in Proposition 3.8 reads
\[ \partial_\mu(f(a_0, \ldots, a_n) = \sum_{r=0}^{n-1} (-1)^r Y(a_r, z)f(a_0, \ldots, \widehat{a_r}, \ldots, a_n) \]
\[ + \sum_{0 \leq r \leq s \leq n-1} (-1)^{n+r+s} f(a_0, \ldots, \widehat{a_r}, \ldots, \widehat{a_s}, \ldots, a_n, Y(a_r, z)a_s) \]
with $a_s \in V$ and $f \in C^{ch,n}(V^*)$.

In the coisson dg Lie algebra for a coisson algebra structure $\mu \in \text{MC}(\mathfrak{g}_V^{ch})$ corresponding to a vertex Poisson algebra $(V, Y_+, Y_-)$, we have a similar formula as (3.5) replacing $Y$ with $Y_-$.

Remark. (1) One can apply the construction of convolution dg Lie algebra to the $*$-pseudo tensor structure on $\text{M}(X)$ (see Definition 1.2). Namely, we replace $\text{End}^{ch}_M$ by the operad $\text{End}^{ch}_M := \oplus_n P_n^*(M)$. As explained in [FBZ04, Chap. 19] and [BBD04], the corresponding algebra structure, called Lie$^*$-algebra, is equivalent to a vertex Lie algebra. Thus the resulting dg Lie algebra $\mathfrak{g}_M^{ch,\mu}$ controls deformations of a vertex Lie algebra structure corresponding to $\mu$. Lie conformal algebra cohomology [DSK09] by De Sole and Kac is a cohomology theory of vertex Lie algebras, and from the expressions of our construction (or, similarly from the simplicity of the coisson to the Chevalley-Eilenberg complex), $\mathfrak{g}_M^{ch,\mu}$ coincides with ours.

(2) Tamarkin+’s dg Lie algebra for chiral algebras [T02] seems to be almost equivalent to ours. However it considers a pro-finite limit of pseudo-tensor structure. At present we don’t know the role of the pro-finite limit.
(3) Yi-Zhi Huang introduced in [H14a, H14b] a cohomology theory for graded vertex algebras, where a certain condition on convergence is required to the coefficients in the cohomology complex. Except for this convergence problem, his construction seems to coincide with ours. Since our construction requires quasi-conformal property to vertex algebras, Huang’s construction is not covered by ours.

3.2. Deformation problem. Let us recall the sequence (1.11) of operads. On the binary part it yields

$$\text{End}_{M}^b(2) \longrightarrow \text{gr} \text{End}_{M}^b(2) \longrightarrow \text{End}_{M}^c(2).$$

(3.6)

The construction of convolution dg Lie algebra immediately implies

**Proposition 3.9.** The above sequence induces the following morphism of dg Lie algebras preserving weight gradings.

$$\tilde{\psi} : \mathfrak{g}^b_M \longrightarrow \mathfrak{g}^c_M.$$

Since the Maurer-Cartan equation is given universally for dg Lie algebras, the existence of \( \tilde{\psi} \) yields

**Theorem 3.10.** The morphism \( \tilde{\psi} \) induces a map

$$\psi : \text{MC}(\mathfrak{g}^b_M) \longrightarrow \text{MC}(\mathfrak{g}^c_M).$$

**Definition 3.11.** We call \( \mu \in \text{MC}(\mathfrak{g}^b_M) \) a **chiral deformation quantization** of \( \mu^c \in \text{MC}(\mathfrak{g}^c_M) \) if \( \psi(\mu) = \mu^c \).

**Remark 3.12.** Here is the totally different feature of our quantization problem from the usual deformation quantization of associative algebras. The dg Lie algebra of a Poisson algebra \( (A, \circ, \{ \}) \) is given by \( \Lambda_A^* \text{Der}(A) \) with the Gerstenhaber bracket. We have the Hochschild-Kostant-Rosenberg quasi-isomorphism \( f : \Lambda_A^* \text{Der}(A) \xrightarrow{\sim} H(C(A, A)) \). The difficult point is that the linear map \( \tilde{f} : \Lambda_A^* \text{Der}(A) \to C^*(A, A) \) obtained naturally from \( f \) is not a morphism of dg Lie algebras. [K03] succeeded to deform \( \tilde{f} \) to an \( L_\infty \)-morphism \( f_\infty \), and obtained a dg Lie algebra morphism. In our situation, we have a natural morphism of dg Lie algebras from the beginning.

One may ask why we don’t treat the coset \( \text{MC}(\mathfrak{g}) = \text{MC}(\mathfrak{g})/G \) where \( G \) is the adjoint group of \( \mathfrak{g}^0 \). By Propositions 3.5 and 3.6 we have \( \mathfrak{g}^0 = \mathfrak{g}(1) = \text{Hom}_{\text{M}(X)}(M, M) \) for \( \mathfrak{g} = \mathfrak{g}^b_M \) and \( \mathfrak{g}^c_M \), so that it is enough to treat \( \text{MC}(\mathfrak{g}) \).

Obviously, if the map \( \psi \) is surjective, then a chiral deformation quantization exists.

**Proposition 13.1.** If \( M \) is a projective \( \mathcal{D} \)-module, then \( \psi \) is surjective, so that for any \( \mu^c \in \text{MC}(\mathfrak{g}^c_M) \) has a chiral deformation quantization.

**Proof.** If \( M \) is projective then \( \text{gr} \text{End}_{M}^c(2) \to \text{End}_{M}^c(2) \) in (3.6) is an isomorphism, as remarked in [BD04, §3.2.4]. \( \square \)

Now our main theorem is

**Theorem 3.14.** \( \psi \) is always an injection.

**Proof.** By Propositions 3.5 and 3.6, we have

$$\text{MC}(\mathfrak{g}^b_M) \simeq \{ \alpha \in C^{\text{ch},2}(M) \mid \alpha \star \alpha = 0 \}, \quad \text{MC}(\mathfrak{g}^c_M) \simeq \{ \alpha \in C^{c,2}(M) \mid \alpha \star \alpha = 0 \}$$

with

$$C^{\text{ch},2}(M) := \text{Hom}_{\text{M}(X^2)}(j_*j^*(\lambda^2 M), \Delta_*(M)) \quad \text{and} \quad C^{c,2}(M) := \text{Hom}_{\text{M}(X^2)}(\lambda^2 M, \Delta_*(M)) \bigoplus \text{Hom}_{\text{M}(X)}(M^{\otimes 2}, M) \otimes \text{Lie}(2).$$

Now recall the special filtration \( W^* \) on \( \text{End}_{M}^c(2) \) given by §1.4.2, (1.8). On \( C^{\text{ch},2} \) it yields a decreasing filtration

$$0 = W^0C^{\text{ch},2} \subset W^{-1}C^{\text{ch},2} \subset W^{-2}C^{\text{ch},2} \subset W^{-3}C^{\text{ch},2} = C^{\text{ch},2}.$$  

We want to write it down this sequence explicitly. Recall that we have an explicit description (1.6) of the graded components \( \text{gr}^W \text{End}_{M}^c(2) \). For \( I = [2] \) it reads

$$j_*j^*\omega_X = W^0j_*j^*\omega_X \supset W^{-1}j_*j^*\omega_X \supset W^{-2}j_*j^*\omega_X \supset W^{-3}j_*j^*\omega_X = 0,$$

$$\text{gr}^W_0 = 0, \quad \text{gr}^W_1 = \Delta_*\omega_X, \quad \text{gr}^W_2 = \omega_X^{\otimes 2} \otimes \text{Lie}(2)^*.$$  

We also have the following Cousin complex.

$$0 \longrightarrow \omega_X^{\otimes 2} \longrightarrow j_*j^*\omega_X \longrightarrow \Delta_*\omega_X \longrightarrow 0.$$

Thus the exact sequence splits. Now going back to \( W^*C^{\text{ch},2} \), we find

$$C^{\text{ch},2} \simeq \text{gr}^W_1 C^{\text{ch},2} \simeq \text{gr} \text{End}_{M}(2).$$

Thus the first arrow in (3.6) is an isomorphism, so that (3.6) is an injection in total. Thus the induced map \( \psi \) is an injection. \( \square \)

**Corollary 3.15.** A chiral deformation quantization is unique if it exists.
Remark. As shown in [BD04, §2.6] and mentioned in the introduction, we have two standard examples of coisson algebras. The first one corresponds to the vertex Poisson algebra $V_\infty(\mathfrak{g})$ arising from the affine vertex algebra, and the second one $W_\infty(\mathfrak{g}, e_{\text{reg}})$ arising from the $W$-algebra. By Corollary 3.15, their chiral deformation quantizations are unique. We know the existence, namely $V_k(\mathfrak{g})$ and $W_k(\mathfrak{g}, e_{\text{reg}})$, so that all the chiral deformation quantizations are isomorphic to these standard vertex algebras.

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References

[BD04] Beilinson, A., Drinfeld, V., Chiral algebras, American Mathematical Society Colloquium Publications, 51, American Mathematical Society, Providence, RI, 2004.

[B86] Borel, A., Vertex algebras, Kac-Moody algebras, and the Monster, Proc. Nat. Acad. Sci. U.S.A., Vol. 83 (1986), No. 10, 3068–3071.

[DSK09] De Sole, A., Kac, V., Lie conformal algebra cohomology and the variational complex, Comm. Math. Phys. 292 (2009), no. 3, 667–719.

[FBZ04] Frenkel, E., Ben-Zvi, D., Vertex algebras and algebraic curves, Second edition, Mathematical Surveys and Monographs, 88, American Mathematical Society, Providence, RI, 2004.

[GM88] Goldman, W. M., Millson, J. J., The deformation theory of representations of fundamental groups of compact Kähler manifolds, Inst. Hautes Études Sci. Publ. Math. (1988), no. 67, 43–96.

[H14a] Huang, Y., First and second cohomologies of grading-restricted vertex algebras, Comm. Math. Phys. 327 (2014), no. 1, 261–278.

[H14b] Huang, Y., A cohomology theory of grading-restricted vertex algebras, Comm. Math. Phys. 327 (2014), no. 1, 279–307.

[K03] Kontsevich, M., Deformation quantization of Poisson manifolds, Lett. Math. Phys. 66 (2003), no. 3, 157–216.

[LV12] Loday, J., Vallette, B., Algebraic operads, Grundlehren der Mathematischen Wissenschaften, 346. Springer, Heidelberg, 2012.

[M71] MacLane, S., Categories for the working mathematician, Graduate Texts in Mathematics, 5, Springer-Verlag, New York, 1971.

[T02] Tamarkin, D., Deformations of chiral algebras, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), 105–116, Higher Ed. Press, Beijing, 2002.

Graduate School of Mathematics, Nagoya University Furocho, Chikusaku, Nagoya, Japan, 464-8602.

E-mail address: yanagida@math.nagoya-u.ac.jp