LIMITING ABSORPTION PRINCIPLE ON \(L^p\)-SPACES AND SCATTERING THEORY

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Abstract. In this paper, we study the mapping property from \(L^p\) to \(L^q\) of the resolvent of the Fourier multiplier operators and scattering theory of generalized Schrödinger operators. Though the first half of the subject is studied in [4], we extend their result to away from the duality line and we also study the Hölder continuity of the resolvent.

1. Introduction

In this note, we study \(L^p\)-estimates for resolvents of the Fourier multipliers and the scattering theory of the discrete Schrödinger operator, the fractional Schrödinger operators, the Dirac operators.

One of the interest in the scattering theory of the Schrödinger operator is to prove the asymptotic completeness of the wave operators:

\[
W_\pm = s - \lim_{t \to \pm \infty} e^{it(-\Delta + V)} e^{-it(-\Delta)},
\]

i.e. that \(W_\pm\) are surjections onto the absolutely continuous subspace of \(L^2(\mathbb{R}^d)\). Through the Kato’s smooth perturbation theory, the asymptotic completeness of the wave operators is closely related to the limit absorption principle:

\[
\sup_{z \in I_\pm \setminus I} \| |V|^{\frac{1}{2}} (-\Delta - z)^{-1} |V|^{\frac{1}{2}} \|_{B(L^2(\mathbb{R}^d))} < \infty, \tag{1}
\]

\[
\sup_{z \in I_\pm \setminus I} \| |V|^{\frac{1}{2}} (-\Delta + V - z)^{-1} |V|^{\frac{1}{2}} \|_{B(L^2(\mathbb{R}^d))} < \infty, \tag{2}
\]

where \(I \subset (0, \infty)\) is an interval and \(I_\pm = \{ z \in \mathbb{C} \mid \pm \text{Im} z \geq 0 \}\) and \(V\) is a real-valued function. A strong tool for proving (1) and (2) is the Mourre theory [23], which gives sufficient conditions that (1) and (2) hold.

On the other hands, Kenig, Ruiz and Sogge [19] establish the \(L^p\)-type limiting absorption principle for the free Schrödinger operator:

\[
\|(-\Delta - z)^{-1}\|_{B(L^p(\mathbb{R}^d), L^q(\mathbb{R}^d))} \leq C_{p,q} |z|^{\frac{d}{2}(\frac{1}{p} - \frac{1}{q}) - 1}, \quad z \in \mathbb{C} \setminus [0, \infty), \quad d \geq 3 \tag{3}
\]

where \(C_{p,q} > 0\) is independent of \(z \in \mathbb{C} \setminus [0, \infty)\) and \((1/p, 1/q) \in (0, 1) \times (0, 1)\) satisfies \(2/(d + 1) \leq 1/p - 1/q \leq 2/d, (d + 1)/2d < 1/p\) and \(1/q < (d - 1)/(2d)\).

(3) is also proved by Kato and Yajima [18] independently when \(1/p + 1/q = 1\), and applied to the scattering theory of the Schrödinger operator \(-\Delta + V\), where \(V \in L^p(\mathbb{R}^d), d/2 \leq p < (d + 1)/2\) is real-valued. Note that (1) for \(V \in L^p(\mathbb{R}^d)\) for \(d/2 \leq p \leq (d+1)/2\) follow from (3) and Hölder’s inequality. Goldberg and Schlag [7]

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proved the $L^p$-type limiting absorption principle for Schrödinger operator $-\Delta + V$ with a real-valued potential $V \in L^r(\mathbb{R}^d) \cap L^{3/2}(\mathbb{R}^d)$, $r > 3/2$:

$$\sup_{\Re z \geq \lambda_0, 0 < -\Im z \leq 1} \|(-\Delta + V - z)^{-1}\|_{B(L^p(\mathbb{R}^d), L^q(\mathbb{R}^d))} \leq C(\Re z)^{\frac{d}{2(p-q)\lambda_0}}^{-1},$$

where $\lambda_0 > 0$, $d = 3$, $p = 4/3$ and $q = 4$. The strategy of the proof in [7] is to replace the $L^2$-trace theorem in the classical Agmon-Kato-Kuroda theorem [24, Theorem XIII. 33] by Stein-Tomas $L^p$-restriction theorem for the sphere [30]. Ionescu and Schlag [13] extends the result of [7] to a large class of potentials $V$, which contains $L^p(\mathbb{R}^d)$, $d/2 \leq p \leq (d+1)/2$, the global Kato class potentials and some perturbations of first order operators. See also the recent works by Huang, Yao, Zheng [11] and Mizutani [23]. Moreover, in [13], it is also proved that existence and asymptotic completeness of the wave operators. We note that there are no positive eigenvalues of $-\Delta + V$ when $V \in L^p(\mathbb{R}^d)$, $d/2 \leq p \leq (d+1)/2$ and it is false if $p > (d+1)/2$ ([12] and [20]).

In this paper, for a large class of operators $T(D)$ on $X^d$, we study the uniform resolvent estimates

$$\|\chi(D)(T(D) - z)^{-1}\|_{B(L^p(X^d), L^q(X^d))} \leq N(z).$$

where $X = \mathbb{R}$ or $X = \mathbb{Z}$, $T(D)$ is a Fourier multiplier on $X^d$ and $\chi(D)$ is a Fourier multiplier on $X^d$ which plays a role of a cut-off function in $\hat{X}^d$. The estimate like (4) is investigated in [4] and [5] when $1/p + 1/q = 1$ and $X = \mathbb{R}$ in order to study the Lieb-Thirring type bounds of fractional Schrödinger operators and Dirac operators. One of the purpose is to prove (4) away form the duality line and to extend to the case of $X = \mathbb{Z}$. We follow the argument in [9, Appendix] for the Laplacian on the Euclidean space, however, the argument in [9] does not cover the general case since in the proof of [9, Theorem 6], the spherical symmetry and the Stein-Tomas theorem for the sphere are crucial. Moreover, we study the scattering theory of the discrete Schrödinger operator, the fractional Schrödinger operators and the Dirac operators. We note that the limiting absorption principle for free discrete Schrödinger operators is studied in [14], [21] and [29]. In [21], the scattering theory of the discrete Schrödinger operators perturbed by $L^p$-potentials are studied for a range of $p$. In [29], it is proved that the range of $(p, q)$ which the uniform resolvent estimate holds for the discrete Schrödinger operators differs from the one for the continuous Schrödinger operators when $d \geq 5$.

We remark that almost all results in this paper can be extended to the Lorentz space $L^{p, r}$ by real interpolation, however, we do not mention this below for simplicity.

Throughout this paper, we denote $X^d = \mathbb{Z}^d$ or $\mathbb{R}^d$ for an integer $d \geq 2$. We denote $\mu$ by the Lebesgue measure if $X^d = \mathbb{R}^d$ by the counting measure if $X^d = \mathbb{Z}^d$. Moreover, we write $\hat{X}^d = \mathbb{R}^d$ if $X^d = \mathbb{R}^d$ and $\hat{X}^d = \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$ if $X^d = \mathbb{Z}^d$. We often use $[-1/2, 1/2]^d \subset \mathbb{R}^d$ as a fundamental domain of $\mathbb{T}^d$.

Let $T \in C^\infty(\hat{X}^d, \mathbb{R})$. Moreover, we assume $T \in \mathcal{S}'(\mathbb{R}^d)$ if $X = \mathbb{R}$. We denote the set of all critical values of $T$ by $\Lambda(T)$ and set $M_\lambda = \{\xi \in \hat{X}^d \mid T(\xi) = \lambda\}$ for $\lambda \in \mathbb{R}$. We denote the induced surface measure by $\mu_\lambda$ away from the critical points of $T$. Moreover, for $I \subset \mathbb{R}$, we write $I_\pm = \{z \in \mathbb{C} \mid \Re z \in I, \pm \Im z \geq 0\}$. 


Set
\[ S_k = \left\{ \left( \frac{1}{p}, \frac{1}{q} \right) \in [0,1] \times [0,1] \mid \frac{1}{q} \leq \frac{k}{1 + 2k}, \frac{1 + k}{1 + 2k} < \frac{1}{p}, \frac{k}{1 + 2k} < \frac{k}{p} \right\}. \]

**Assumption A.** Let \( U \subset \mathbb{R}^d \) be a relativity compact open set and an compact interval \( I \subset \mathbb{R} \). Suppose \( \partial_k T(\xi) \neq 0 \) for \( \xi \in U \). The Fourier transform of the induced surface measure satisfies the following estimate: For any \( a \in C_c^\infty(X^d) \) supported in \( U \), there exists \( C > 0 \) such that
\[ \int_{M_\lambda} e^{2\pi i x \cdot \xi} a(\xi) d\mu_\lambda(\xi) \leq C(1 + |x|)^{-k}, \quad x \in X^d, \lambda \in I. \]

**Example 1.** Suppose that \( M_\lambda \cap \text{supp} \chi \) has at least \( m \) nonvanishing principal curvature curvature at every point, then (6) holds for \( k = m/2 \).

Set \( R_0^\pm(z) = (T(D) - z)^{-1} \) for \( z \in \{ z \in \mathbb{C} \mid \pm \text{Im} z > 0 \} \). Moreover, for a signature \( \pm \), we define \( \chi(D)R_0^\pm(\lambda \pm i0) \) if \( \partial_k T \neq 0 \) on \( \text{supp} \chi \) by the Fourier multiplier with its symbol \( \chi(\xi)(T(\xi) - \lambda \pm i0)^{-1} \). For \( 1 \leq p \leq \infty \), \( L^p(X^d) \) denotes the Lebesgue space with the Lebesgue measure if \( X = \mathbb{R} \) and with the counting measure if \( X = \mathbb{Z} \).

Our first result is the following:

**Theorem 1.1.** Let \( T \in C^\infty(X^d, \mathbb{R}) \) and let \( I \) be a compact interval of \( \mathbb{R} \). Fix a signature \( \pm \). Let \( \chi \in C_c^\infty(X^d) \). Suppose that (6) holds for \( \lambda \in I \) and \( \text{supp} \chi \subset U \).

(i) There exists such that
\[ \sup_{z \in I_\pm} \| \chi(D)R_0^\pm(z) \|_{B(L^p(X^d), L^q(X^d))} < \infty, \]
for \((1/p, 1/q) \in S_k \).

(ii) Set \( k_\delta = k - \delta \) for \( 0 < \delta \leq 1 \) and \( \beta_\delta = (2/p - 1)\delta \). Then
\[ \sup_{z,w \in I_\pm, |z - w| \leq 1} |z - w|^{-\beta_\delta} \| \chi(D)(R_0^\pm(z) - R_0^\pm(w)) \|_{B(L^p(X^d), L^q(X^d))} < \infty, \]
for \((1/p, 1/q^*) \in S_k \), where \( p^* = p/(p - 1) \).

1.1. **Applications to the discrete Schrödinger operators.** The scattering theory of the discrete Schrödinger operators is studied in [21] for the potential \( V \in L^p(\mathbb{Z}^d) \), \( 1 \leq p < 6/5 \) if \( d = 3 \) and \( 1 \leq p < 3d/(2d + 1) \) if \( d \geq 4 \). In this subsection, we extend their results to when \( V \in L^p(\mathbb{Z}^d) \) for \( 1 \leq p \leq d/3 \) at the cost of the restriction of the dimension: \( d \geq 4 \). We define the discrete Schrödinger operator:
\[ H_0u(x) = -\sum_{|x - y| = 1}^d (u(x) - u(y)), \quad x \in \mathbb{Z}^d. \]

Note that \( H_0 \) is a bounded self-adjoint operator on \( L^2(\mathbb{Z}^d) \). If we define \( h_0(\xi) = 4\sum_{j=1}^d \sin^2 \pi \xi_j \) for \( \xi \in \mathbb{T}^d \), then
\[ H_0 = h_0(D) \]
and hence the spectrum \( \sigma(H_0) \) of \( H_0 \) is equal to \([0, 4d]\). Moreover, \( \sigma_{ac}(H_0) = [0, 4d] \), where \( \sigma_{ac}(H_0) \) is the absolutely continuous spectrum of \( H_0 \). Set \( R_0^\pm(z) = (H_0 - z)^{-1} \) for \( \pm \text{Im} z > 0 \). Note that \( \Lambda_\pm(h_0(D)) = (4k)^d \sum_{k=0}^\infty \), where we recall that \( \Lambda_\pm(h_0(D)) \) is the set of all critical values of \( h_0(\xi) \). Moreover, if \( V \in L^p(\mathbb{Z}^d, \mathbb{R}) \) for some
1 \leq p < \infty$, $H = H_0 + V$ is a bounded self-adjoint operator and $\sigma_{ess}(H) = [0, 4d]$ since $V \in L^p(\mathbb{Z}^d) \subset L^\infty(\mathbb{Z}^d)$ and $V(x) \to \infty$ as $|x| \to \infty$. Here $\sigma_{ess}(H)$ denotes the essential spectrum of $H$.

**Theorem 1.2.** Fix a signature $\pm$ and let $d \geq 4$. Let $I \subset \mathbb{R} \setminus \{4k\}_{k=0}^d$ be a compact interval. We define $R_0^\pm(\lambda)$ for $\lambda \in I$ by the Fourier multiplier of the distribution $(h_0(\xi) - (\lambda \pm i0))^{-1}$, where this distribution is well-defined since $h_0$ has no critical points in $h_0^{-1}(I)$.

(i) We have
\[
\sup_{z \in I_+} \| R_0^\pm(z) \|_{B(L^p(\mathbb{Z}^d), L^q(\mathbb{Z}^d))} < \infty,
\]
for $(1/p, 1/q) \in S_{(d-3)/3}$, where $S_{(d-3)/3}$ is as in (5).

(ii) Let $0 < \delta \leq 1$ and $\beta_\delta = (2/p - 1)\delta$. Then
\[
\sup_{z,w \in I_+ | z-w| \leq 1} |z - w|^{-\beta_\delta} \| (R_0^\pm(z) - R_0^\pm(w)) \|_{B(L^p(\mathbb{Z}^d), L^q(\mathbb{Z}^d))} < \infty,
\]
for $(1/p, 1/q) \in S_{(d-3)/3 - \delta}$.

(iii) Let $V \in L^{d/3}(\mathbb{Z}^d, \mathbb{R})$ and let $H = H_0 + V$. Then the wave operators
\[
W_\pm = s - \lim_{t \to \pm \infty} e^{itH} e^{-itH_0}
\]
exist and are complete, i.e. the ranges of $W_\pm$ are the absolutely continuous subspace $\mathcal{H}_{ac}(H)$ of $H$.

**Remark 1.3.** We note that $L^{p_1}(\mathbb{Z}^d) \subset L^{p_2}(\mathbb{Z}^d)$ if $1 \leq p_1 \leq p_2 \leq \infty$. In view of a duality argument, (i) with $q = p/(p-1)$ is essentially proved if $1 \leq p < 12/11$ with $d = 3$, $1 \leq p < 6d/(5d + 1)$ with $d = 4$ in [21, Theorem 1.3] and if $p = 2d/(d + 3)$ with $d \geq 4$ in [29, Proposition 3.3]. We notice that $(1/p, (p-1)/p) \in S_{(d-3)/3}$ if and only if $p = 2d/(d + 3)$ and $6d/(5d + 1) \geq 2d/(d + 3)$. However, both in [21] and [29], one can take $I = \mathbb{R}$.

1.2. Applications to the fractional Schrödinger operators and the Dirac operators. Let $n = 2^{d/2}$ if $d$ is even and $n = 2^{(d+1)/2}$ if $d$ is odd. We define the Dirac operators on $\mathbb{R}^d$:
\[
D_0 = \sum_{j=1}^d \alpha_j D_j, \ D_1 = \sum_{j=1}^d \alpha_j D_j + \alpha_{d+1},
\]
where $\alpha_j$ are $n \times n$ Hermitian matrix and satisfy the Clifford relations:
\[
\alpha_j \alpha_k + \alpha_k \alpha_j = -2\delta_{jk} I_{n \times n}
\]
and $D_j = \partial_{x_j}/(2\pi i)$. Note that if we define $D_{d+1} = mI_{n \times n}$, then
\[
D_0^2 = -\sum_{j=1}^d I_{n \times n} D_j^2 = -\Delta \cdot I_{n \times n}, \ D_1^2 = (-\Delta + 1) \cdot I_{n \times n},
\]
where we denote $\Delta = (\sum_{j=1}^d \partial_{x_j}^2)/(4\pi^2)$. In this subsection, we suppose that $T(D)$ is the one of the following operators:
\[
T(D) = (-\Delta)^{s/2}, \ T(D) = (-\Delta + 1)^{s/2} - 1, \ T(D) = D_0, \ T(D) = D_1,
\]
where $0 < s \leq d$. We use the convention that $s = 1$ when $T(D) = D_0$ or $T(D) = D_1$. Moreover, we denote the product space $Z^n$ for a function space $Z$ by simply $Z$ when $T(D) = D_0$ or $T(D) = D_1$. As is noted in [4, §2],

$$
\Lambda_\varepsilon((-\Delta)^{s/2}) = \begin{cases} 
\{0\} & \text{if } s > 1, \\
\emptyset & \text{if } s \leq 1,
\end{cases} \quad \Lambda_\varepsilon((-\Delta + 1)^{s/2} - 1) = \{0\},
$$

and

$$
\Lambda_\varepsilon(D_0) = \{0\}, \quad \Lambda_\varepsilon(D_1) = \{-1, 1\}.
$$

Moreover, $T(D)$ is self-adjoint on its domain $H^s(\mathbb{R}^d)$ by the elliptic regularity.

Let $Y_1, Y_2$ be Banach spaces such that

(7) \[(Y_1, Y_2) \in \bigcup_{(\frac{\alpha}{q}, \frac{\beta}{q}) \in S_{\frac{1}{q} - \frac{1}{2}}} \{L^p(\mathbb{R}^d) \times \{L^q(\mathbb{R}^d)\}, \]

if $2d/(d + 1) \leq s \leq d$ and

(8) \[(Y_1, Y_2) \in \bigcup_{(\frac{\alpha}{p_1}, \frac{\beta}{p_2}) \in S_{\frac{1}{p_1} - \frac{1}{2}}, \frac{\alpha}{p_2} - \frac{\beta}{p_2} = \frac{1}{2}} \{L^{p_1}(\mathbb{R}^d) + L^{p_2}(\mathbb{R}^d) \times \{L^{q_1}(\mathbb{R}^d) \cap L^{q_2}(\mathbb{R}^d)\}, \]

if $0 < s < \frac{2d}{d+1}$.

A part of the following estimate is a generalization of [4, Theorem 3.1].

**Theorem 1.4.** Let $I \subset \mathbb{R} \setminus \Lambda_c(T(D))$ be a compact interval. We define $R^+_0(\lambda)$ for $\lambda \in I$ by the Fourier multiplier of the distribution $(T(\xi) - (\lambda \pm i0))^{-1}$, where this distribution is well-defined since $T(\xi)$ has no critical points in $T^{-1}(I)$.

(i) We have

$$
\sup_{z \in I_\pm} \|R^+_0(z)\|_{B(Y_1, Y_2)} < \infty.
$$

(ii) Let $(Y_1, Y_2)$ be satisfying $p = q$ in (7) if $2d/(d + 1) \leq s \leq d$ and $p_1 = q_1$ in (8) if $0 < s < 2d/(d + 1)$. Let $0 < \delta \leq 1$ and $\delta_\beta = (2/p - 1)\delta$. Then

$$
\sup_{z, w \in I_\pm, \|z - w\| \leq 1} |z - w|^{-\delta_\beta} \|\|R^+_0(z) - R^+_0(w)\|_{B(Y_1, Y_2)} < \infty.
$$

(iii) Let $V \in L^{(d+1)/2}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d, \mathbb{R})$. Set $H = H_0 + V$. Then the wave operators

$$
W_\pm = s - \lim_{t \to \pm \infty} e^{itH}e^{-itH_0}
$$

exist and are complete, i.e. the ranges of $W_\pm$ are the absolutely continuous subspace $\mathcal{H}_{ac}(H)$ of $H$.

**Remark 1.5.** In (iii), the condition $V \in L^\infty(\mathbb{R}^d)$ is expected to be removed if we consider the appropriate self-adjoint extension of $T(D) + V$. However, in order to avoid the technical difficulty, we assume $V \in L^\infty(\mathbb{R}^d)$.

**Remark 1.6.** (i) is proved in [4] if $1/p + 1/q = 1$. In [11], (i) is proved when $T(D) = (-\Delta)^{s/2}$ for $2d/(d + 1) \leq s < d$.

We fix some notations. For an integer $k \geq 1$, $C^\infty_c(X^k)$ denotes $C^\infty_c(\mathbb{R}^k)$ if $X = \mathbb{R}$ and the set of all finitely supported functions if $X = \mathbb{Z}$. Moreover, for an integer $n > 0$, $*_n$ denotes the convolution in $X^k$: $f*_ng = \int_{X^k} f(x-y)g(y)dy$ for measurable
functions $f, g : X^k \to \mathbb{C}$. For $1 \leq p \leq \infty$, we write $p^* = p/(p-1)$. We denote $t_+ = \max(t, 0)$ for $t \in \mathbb{R}$. We define the Bezov space $\mathcal{B}$ and $\mathcal{B}^*$ by

$$
\|u\|_{\mathcal{B}} = \|u\|_{L^2(|x| \leq 1)} + \sum_{j=1}^{\infty} 2^{j/2} \|u\|_{L^2(2^{j-1} \leq |x| < 2^j)},
$$

$$
\|u\|_{\mathcal{B}^*} = \|u\|_{L^2(|x| \leq 1)} + \sup_{j\geq 1} 2^{-j/2} \|u\|_{L^2(2^{j-1} \leq |x| < 2^j)},
$$

$$
\mathcal{B} = \{u \in L^2_{loc}(X^d) \mid \|u\|_{\mathcal{B}} < \infty\}, \quad \mathcal{B}^* = \{u \in L^2_{loc}(X^d) \mid \|u\|_{\mathcal{B}^*} < \infty\}.
$$

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2. Abstract Theorem

Let $K \in L^\infty(X^d)$. We consider the following assumptions:

**Assumption B.** There exists $C_0, C_1 > 0$ such that for any $x_d \in X$

$$
\| \int_{X^{d-1}} K(x', x_d) e^{-2\pi i x' \cdot \xi} dx' \|_{L^\infty((X_d')^\perp)} \leq C_0,
$$

$$
\sup_{x' \in X^{d-1}} |K(x', x_d)| \leq C_1 (1 + |x_d|)^{-k}.
$$

Note that Assumption B implies that

$$
\|K(\cdot, x_d) *_{d-1} g\|_{L^2(X^{d-1})} \leq C_0 \|g\|_{L^2(X^{d-1})},
$$

$$
\|K(\cdot, x_d) *_{d-1} g\|_{L^\infty(X^{d-1})} \leq C_1 (1 + |x_d|)^{-k} \|g\|_{L^1(X^{d-1})},
$$

where we recall $*_{d-1}$ denotes the convolution operator on $X^{d-1}$. Moreover, by the Riesz-Thorin interpolation theorem, (9) and (10) imply

$$
\|K(\cdot, x_d) *_{d-1} g\|_{L^{p^*}(X^{d-1})} \leq C_0^{2 - \frac{2}{p^*}} C_1^{\frac{2}{p^*} - 1} (1 + |x_d|)^{-k(\frac{2}{p^*} - 1)} \|g\|_{L^p(X^{d-1})},
$$

for $1 \leq p \leq 2$.

**Proposition 2.1.** Suppose Assumption B. Then there exists a universal constant $M_d > 0$ and $M_{p, k} > 0$ such that

$$
\left(\sup_{R > 0, x_0 \in X^d} \frac{1}{R} \int_{|x - x_0| \leq R} |K *_{d} f(x)|^2 dx\right)^{\frac{1}{2}} \leq M_d C_0 \|f\|_{\mathcal{B}}, \quad f \in \mathcal{B},
$$

$$
\|K *_{d} f\|_{L^{p^*}(X^d)} \leq M_{p, k} C_0^{2 - \frac{2}{p^*}} C_1^{\frac{2}{p^*} - 1} \|f\|_{L^p(X^d)}, \quad f \in L^p(X^d)
$$

for $1 \leq p \leq 2(k+1)/(k+2)$.

**Proof.** By the density argument, we may assume $f \in C_0^\infty(X^d)$. We observe that

$$
\sup_{R > 0, x_0 \in X^d} \frac{1}{R} \int_{|x - x_0| < R} |K *_{d} f(x)|^2 dx \leq \sup_{x_d \in \mathbb{R}} \|K *_{d} f(\cdot, x_d)\|_{L^2(\mathbb{R}^{d-1})}^2 \leq \|f\|_{B},
$$

$$
\int_{\mathbb{R}} \|f(\cdot, y_d)\|_{L^2(\mathbb{R}^{d-1})} dy_d \leq M_d \|f\|_{B},
$$
with some universal constant $M_d > 0$. Using the Minkowski inequality and (9), we obtain (12).

Next, we prove (13). By the Minkowski inequality and (11), we have

$$\|K * f\|_{L^p(R^d)} = \int_{\mathbb{R}} K(\cdot, x_d - y_d) \ast f(y) \, dy \leq C^2 \frac{\sqrt[2]{p}}{C^1} \|f\|_{L^p(R^d)}$$

where we use the fractional integration theorem in the last line. This gives (13). \qed

For $x_d \in X$, we define $T_{x_d}$ and $T_{x_d}^*$ by

$$T_{x_d} f(x') = \int_{X_d} K(x - y) f(y) \, dy, \quad T_{x_d}^* g(y) = \int_{X_d}^* K(x - y) g(x') \, dx'.$$

We define

$$S_{x_d}(y_d, z_d) g(y') = \int_{X_d} \int_{X_d} K(x - y)K(x - z) g(z') \, dz' \, dx'.$$

Note that

$$T_{x_d}^* T_{x_d} f(y) = \int_X (S_{x_d}(y_d, z_d) f(\cdot, z_d))(y') \, dz_d.$$

Next, we consider the following assumption.

**Assumption C.** There exists $C_0, C_1 > 0$ such that for any $x_d, y_d, z_d \in X$

$$\|\int_{\mathbb{R}} \int_{\mathbb{R}} e^{2\pi i y' \cdot z'} K(x', x_d - y_d) K(x' - y', x_d - z_d) \, dx' \, dy'\|_{L^\infty(\mathbb{R}^d)} \leq C^2_2,$$

$$\sup_{y', z' \in X_d} \left| \int_{\mathbb{R}} K(x - y) K(x - z) \, dx' \right| \leq C^2_3 (1 + |y_d - z_d|)^{-k}.$$

Assumption C implies

$$\|S_{x_d}(y_d, z_d) g\|_{L^2(\mathbb{R}^d)} \leq C^2_2 \|g\|_{L^2(\mathbb{R}^d)},$$

$$\|S_{x_d}(y_d, z_d) g\|_{L^\infty(\mathbb{R}^d)} \leq C^2_3 (1 + |y_d - z_d|)^{-k} \|g\|_{L^1(\mathbb{R}^d)}.$$

By the Riesz-Thorin interpolation theorem, (14) and (15) imply

$$\|S_{x_d}(y_d, z_d) g\|_{L^p(\mathbb{R}^d)} \leq \left( C^2_2 \frac{2}{p} C^2_3 \frac{2}{q} - 1 \right)^2 (1 + |y_d - z_d|)^{-k} \|g\|_{L^p(\mathbb{R}^d)},$$

for $1 \leq p \leq 2$.

**Proposition 2.2.** Suppose Assumption C. Then there exists a universal constant $M'_{p, k} > 0$ such that

$$\|S_{x_d}(y_d, z_d) g\|_{L^p(\mathbb{R}^d)} \leq (C^2_2 \frac{2}{p} C^2_3 \frac{2}{q} - 1)^2 (1 + |y_d - z_d|)^{-k} \|g\|_{L^p(\mathbb{R}^d)},$$

for $1 \leq p \leq 2(k + 1)/(k + 2)$. 

$$K * f \|_{L^p(\mathbb{R}^d)} = \int_{\mathbb{R}} K(\cdot, x_d - y_d) \ast f(y) \, dy \leq M'_{p, k} C^2_2 \frac{2}{p} C^2_3 \frac{2}{q} - 1 \|f\|_{L^p(\mathbb{R}^d)},$$

for $1 \leq p \leq 2(k + 1)/(k + 2)$.
Proof. By a density argument, we may assume \( f \in C_c^\infty(X^d) \). First, we prove (17). As in the proof of Proposition (2.1), it suffices to prove that
\[
\|T_{x,d}f\|_{L^p(X^{d-1})} \leq M'_{p,k}C_2^{2-\frac{2}{p}}C_3^{\frac{2}{p}-1}\|f\|_{L^p(X^d)}, \quad f \in C_c^\infty(X^d).
\]
By the standard \( T^*T \) argument, this estimate is equivalent to
\[
\|T^*_{x,d}T_{x,d}f\|_{L^{p^*}(X^d)} \leq (M'_{p,k}C_2^{2-\frac{2}{p}}C_3^{\frac{2}{p}-1})^2\|f\|_{L^p(X^d)}.
\]
By the Minkowski inequality and (11), we have
\[
\|T^*_{x,d}T_{x,d}f\|_{L^{p^*}(X^d)} = \|\left(\int_X (S_{x,d}(y_d,z_d)f(\cdot,z_d))(y')dz_d\right)\|_{L^{p^*}(X^{d-1})} \|f\|_{L^{p^*}(X_{z_d})}
\]
\[
\leq (C_2^{2-\frac{2}{p}}C_3^{\frac{2}{p}-1})^2\|f\|_{L^p(X^d)}
\]
\[
\times \left|\int_X (1 + |y_d - z_d|)^{-k(\frac{2}{p} - 1)}\|f(\cdot,z_d)\|_{L^{p^*}(X^{d-1})}dy_d\right|_{L^{p^*}(X_{z_d})}
\]
\[
\leq (M'_{p,k}C_2^{2-\frac{2}{p}}C_3^{\frac{2}{p}-1})^2\|f\|_{L^p(X^d)},
\]
where we use the fractional integration theorem in the last line. This proves (17). \( \square \)

We impose the additional assumption.

Assumption D. There exists \( C_4 > 0 \) such that
\[ |K(x)| \leq C_4(1 + |x|)^{-k}, \quad x \in X^d. \]

Remark 2.3. If \( K \) satisfies Assumption C and D, then \( \tilde{K} \) also satisfies Assumption B and D.

Under Assumption C and D, we obtain the estimates similar to (13) away from the H"older exponent.

Proposition 2.4. Suppose Assumption C and D. Then there exists a universal constant \( L'_{p,q,k} > 0 \) such that
\[
\|K * f\|_{L^q(X^d)} \leq L'_{p,q,k} C_{p,q,k,l} \|f\|_{L^p(X^d)}, \quad f \in L^p(X^d),
\]
where \( 1/p - 1/q = 1/l \) and
\[
C_{p,q,k,l} = \begin{cases} 
C_2^{\frac{2}{k+1}}C_3^{1-\frac{2}{p}}C_4^{1-\frac{2}{q}}, & \text{if } 1 \leq l < \frac{(k+1)(2k+1)}{k^2+3k+1} \text{ and } q > \frac{1+2k}{k}\frac{k+1}{p',q} \leq 1, \\
C_2^{\frac{2}{k+1}}C_3^{1-\frac{2}{p}}C_4^{1-\frac{2}{q}}, & \text{if } 1 \leq l < k + 1, \quad \frac{k+1}{p'} < \frac{k+1}{kq} < \frac{1+2k}{k}, \\
C_2^{\frac{2}{k+1}}C_3^{1-\frac{2}{p}}C_4^{1-\frac{2}{q}}, & \text{if } 1 \leq l < \frac{1+2k}{k}, \quad q \geq \frac{(2k+1)(k+1)}{k^2+3k+1}, \quad \frac{k+1}{kq} \leq \frac{1}{p'}.
\end{cases}
\]

We prove this proposition by a series of lemmas.

Lemma 2.5. Suppose Assumption C. Let \( \psi \in C_c^\infty(\mathbb{R}^2) \). Define \( K^j(x,y) = \psi((2x_d - z_d)/2^j+1, (2y_d - z_d)/2^{j+1})K(x,y) \) for \( j \) and \( z_d \in X \). Then for \( 1 \leq p \leq 2(k+1)/(k+1) \)
\[
\int_{X^d} \int_{X^d} K^j(x,y)f(y)dy \| \leq 2(M'_{p,k}C_2^{2-\frac{2}{p}}C_3^{\frac{2}{p}-1})^{2j}2^{j2\|\psi\|_{L^\infty(X^2)}}\|f\|_{L^p(X^d)},
\]
with \( C > 0 \) independent of \( j \) and \( z_d \in X \).

Proof. The proof is same as in the proof of Proposition 2.2. \( \square \)
Lemma 2.6. Let $F \in C_c^\infty(\mathbb{R})$. Then there exists $\psi \in C_c^\infty(\mathbb{R}^2)$ such that

$$F(\frac{x_d - y_d}{2^j}) = L_j \int_X \psi(\frac{2x_d - z_d}{2^j}, \frac{2y_d - z_d}{2^j})dz_d, \quad x_d, y_d \in \mathbb{R},$$

where $L_j = 2^{-j}$ if $X = \mathbb{R}$ and $2^{-j-2} \leq L_j \leq 2^{-j}$ if $X = \mathbb{Z}$.

**Proof.** We define $\psi \in C_c^\infty(\mathbb{R}^2)$ as follows: Take $\chi_2 \in C_c^\infty(\mathbb{R}, [0, 1])$ such that $\int_x \chi_2(x)dx = 2$ and supp $\chi_2 \subset (-1/2, 1/2)$ if $X = \mathbb{R}$ and such that $\chi_2(t) = 1$ on $|t| \leq 1$ and $\chi(t) = 0$ on $|t| \geq 2$ if $X = \mathbb{Z}$. We define $\psi(z, z') = F(z - z')\chi_2(z + z')$, then $F(x_d) = \int_X \psi(x_d + z, z)dz$ if $X = \mathbb{R}$ and

$$\int_X \psi(\frac{2x_d - z_d}{2^{j+1}}, \frac{2y_d - z_d}{2^{j+1}})dz_d = \sum_{z_d \in \mathbb{Z}} \psi(\frac{2x_d - z_d}{2^{j+1}}, \frac{2y_d - z_d}{2^{j+1}})$$

$$= F(\frac{x_d - y_d}{2^j}) \sum_{z_d \in \mathbb{Z}} \chi_2(\frac{z_d}{2^j}).$$

if $X = \mathbb{Z}$. Note that

$$2^j \leq \sum_{z_d \in \mathbb{Z}} \chi_2(\frac{z_d}{2^j}) \leq 2^{j+2}.$$

Thus we set $L_j = 1$ if $X = \mathbb{R}$ and $L_j = \sum_{z_d \in \mathbb{Z}} \chi_2(\frac{z_d}{2^j})$ if $X = \mathbb{Z}$, we are done. \(\square\)

The following lemma is a consequence of Lemma 2.5, however its proof is a bit technical due to the convolution type cut-off. The conclusion of the following lemma is same as [9, Lemma 1] where the uniform resolvent estimate of the Laplacian is studied. However, since their proof strongly depends on the spherical symmetry of the Laplacian and the Stein-Tomas theorem for the sphere, we cannot directly apply their argument to our cases. In order to overcome this difficulty, we borrow an idea from the proof of the Carleson-Sjölin theorem [10, Theorem 2.1].

**Lemma 2.7.** Suppose Assumption C. Let $F \in C_c^\infty(\mathbb{R})$. Define $K^{j, \text{con}}(x, y) = F((x_d - y_d)/2^j)K(x - y)$ for $j \geq 0$. Then for $1 \leq p \leq 2(k+1)/(k+2)$, there exists a universal constant $M''_{p,k}$ such that

$$\|K^{j, \text{con}} \ast f(x)\|_{L^p(\mathbb{R}^d)} \leq M''_{p,k} C_2^{2-\frac{2}{p}} C_3^{\frac{2}{p} - 1} 2^{\frac{2}{j}} \|f\|_{L^p(\mathbb{R}^d)}$$

**Proof.** By Lemma 2.6, we have

$$\int_{\mathbb{R}^d} |K^{j, \text{con}}f(x)|^2 dx$$

$$\leq 2^{-2j} \int_{\mathbb{R}^d} |\int_{\mathbb{R}^d} K(x - y)\psi(\frac{2x_d - z_d}{2^{j+1}}, \frac{2y_d - z_d}{2^{j+1}})f(y)dydz_d|^2 dx.$$

Take $\varphi \in C_c^\infty(\mathbb{R})$ such that $\psi(x_d, y_d) = \psi(x_d, y_d)\varphi(y_d)$. Using the Cauchy-Schwarz inequality and Lemma 2.5, we have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x - y)\psi(\frac{2x_d - z_d}{2^{j+1}}, \frac{2y_d - z_d}{2^{j+1}})f(y)dydz_d^2 dx$$

$$\leq L2^j \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} K(x - y)\psi(\frac{2x_d - z_d}{2^{j+1}}, \frac{2y_d - z_d}{2^{j+1}})f(y)dy|^2 dx dz_d$$

$$\leq 2L(M'_{p,k} C_2^{2-\frac{2}{p}} C_3^{\frac{2}{p} - 1}) 2^{2j} \int_{\mathbb{R}^d} \|\varphi(\frac{2 \cdot - z_d}{2^{j+1}})f\|^2_{L^p(\mathbb{R}^d)} dx dz_d$$
with a constant $L > 0$ which depends only on $\psi$. Since $p \leq 2$, by using the Minkowski inequality, we have

$$\int_X \|\varphi(\frac{2^j \cdot x}{2^{j+1}})f\|^2_{L^p(X^d)} dx \leq 2^{j+1} \|\varphi\|_{L^2(X)}^2 \|f\|_{L^p(X^d)}^2.$$  

Thus we obtain

$$\int_X \int_{X^d} K(x-y)F(\frac{x_d - y_d}{2^j}) f(y) dy \|f\|_{L^p(X^d)}^2 \leq (M^n_{p,k} 2^{-\frac{d}{p}} C_2^2 C_3^{\frac{2}{3} - 1})^2 \|f\|_{L^p(X^d)},$$

where $(M^n_{p,k})^2 = 4L(M^n_{p,k})^2 \|\varphi\|_{L^2(X)}^2$. \hfill \Box

**Corollary 2.8.** Suppose Assumption D. Then there exists a constant $L_1 > 0$ which depends only on $F$, $d$ and $k$ such that

$$\|K^j,\text{conv} * f\|_{L^\infty(X^d)} \leq L_1 C_4 2^{-jk} \|f\|_{L^1(X^d)}.$$  

In addition, we suppose Assumption C. Set $1/p_1 = 1 - q/2p$ and $L_{2,p,q} = (M^n_{p,k})^{2/q} L_1^{1-2/q}$. Then

$$\|K^j,\text{conv} * f\|_{L^q(X^d)} \leq L_{2,p,q} C_2 C_2^{\frac{2}{3} - 1} C_4^{1 - \frac{1}{p} - \frac{1}{q}} 2^{(d+2j)/(d+1) - jk} \|f\|_{L^p(X^d)}$$

if $q \geq 2$ and $(k+1)(1-1/p)/k \leq 1/q$ and

$$\|K^j,\text{conv} * f\|_{L^q(X^d)} \leq L_{2,q',p'} C_2 C_2^{\frac{2}{3} - 1} C_4^{1 - \frac{1}{p'} - \frac{1}{q'}} 2^{(d+2j)/(d+1) - jk} \|f\|_{L^p(X^d)}$$

if $p \leq 2$ and $(k+1)/(kq) \leq 1 - 1/p$.

**Proof.** (19) follows from

$$\|K^j,\text{conv} * f\|_{L^\infty(X^d)} \leq \|F(\cdot/2^j)K\|_{L^\infty(X^d)} \|f\|_{L^1(X^d)}$$

with some constant $L_1 > 0$ by Assumption D. By complex interpolating (18) and (19), we obtain (20). Since $\hat{K}$ also satisfies Assumption C and D, by duality, (21) holds. \hfill \Box

**Proof of Proposition 2.4.** Take $\eta \in C_c^\infty(\mathbb{R}, [0, 1])$ such that $\eta(t) = 1$ on $0 \leq t \leq 1$ and $\eta = 0$ on $t \geq 2$. Set $F(x) = \eta(|x|) - \eta(|x|/2)$. By Corollary 2.8, if $(k+1)(1-1/p)/k \leq 1/q$, $q > (1 + 2k)/k$, then

$$\|K * f\|_{L^q(X^d)} = \sum_{j=0}^\infty \|K^j,\text{conv} * f\|_{L^q(X^d)} \leq \sum_{j=0}^\infty \|K^j,\text{conv} * f\|_{L^q(X^d)}$$

$$\leq L_{2,p,q} C_2 C_2^{\frac{2}{3} - 1} C_4^{1 - \frac{1}{p} - \frac{1}{q}} \sum_{j=0}^\infty 2^{j/2 + j(d+1)/q - 1/2} \|f\|_{L^p(X^d)}$$

$$\leq L_{2,p,q} C_2 C_2^{\frac{2}{3} - 1} C_4^{1 - \frac{1}{p}} \|f\|_{L^p(X^d)},$$

where $L_{2,p,q} = L_{2,p,q} \sum_{j=0}^\infty 2^{j/2 + j(d+1)/q - 1/2}$. Similarly, if $(k+1)/(kq) \leq 1 - 1/p$, $p < (1 + 2k)/(1 + k)$, then

$$\|K * f\|_{L^q(X^d)} \leq L_{2,q',p'} C_2 C_2^{\frac{2}{3} - 1} C_4^{1 - \frac{1}{p'}} \|f\|_{L^p(X^d)}.$$  

In order to prove the end point estimates, we use Bourgain’s interpolation trick ([2], [3, §6.2], [16, Lemma 3.3]). This trick is also used in [1] for the Stein-Tomas
theorem for a large class of measures in Euclidean space. See also [6] and [9]. We denote the Lorentz space for index $1 \leq p \leq \infty$ and $1 \leq r \leq \infty$ by $L^{p,r}(\mathbb{R}^d)$:

$$
\|f\|_{L^{p,r}(\mathbb{R}^d)} = \begin{cases} 
\sup_{\alpha > 0} \alpha \mu\{ x \in \mathbb{R}^d \mid |f(x)| > \alpha \}^{\frac{1}{p}} \|f\|_{L^r}, & r < \infty, \\
\|f\|_{L^{p,1}(\mathbb{R}^d)}, & r = \infty,
\end{cases}
$$

$L^{p,r}(\mathbb{R}^d) = \{ f : \mathbb{R}^d \to \mathbb{C} \mid f : \text{measurable}, \|f\|_{L^{p,r}(\mathbb{R}^d)} < \infty \}$.

Bourgain’s interpolation trick with (20) and (21) implies that if $1 \leq p \leq (k + 1)(2k + 1)/(k^2 + 3k + 1), q = (1 + 2k)/k$, then

$$
\|K * f\|_{L^{p,\infty}(\mathbb{R}^d)} \leq L^{'L}_{2,p,q} C_2^{\frac{p}{2}} C_3^{\frac{p}{2} - 1} C_4^{\frac{p}{2} - \frac{p}{q}} \|f\|_{L^{p,1}(\mathbb{R}^d)}
$$

with a universal constant $L^{'L}_{2,p,q}$. Similarly if $p = (1 + 2k)/(1 + k), q \geq (2k + 1)(k + 1)/k^2$, then

$$
\|K * f\|_{L^{p,\infty}(\mathbb{R}^d)} \leq L^{'L}_{2,q'-p} C_2^{\frac{p}{2}} C_3^{\frac{p}{2} - 1} C_4^{\frac{p}{2} - \frac{p}{q'}} \|f\|_{L^{p,1}(\mathbb{R}^d)}.
$$

By real interpolating above estimates, we complete the proof.

\[ \square \]

3. PROOF OF THEOREM 1.1

Proof of Theorem 1.1. We follows the argument as in [4, Lemma 3.3]. By using a partition of unity and a linear coordinate change, we may assume that

$\lim_{\xi \to 0} \int_0^\infty \alpha \mu\{ x \in \mathbb{R}^d \mid |f(x)| > \alpha \}^{\frac{1}{p}} \|f\|_{L^r}, \|f\|_{L^{p,1}(\mathbb{R}^d)}, \|f\|_{L^{p,\infty}(\mathbb{R}^d)}$.

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$\lim_{\xi \to 0} \int_0^\infty \alpha \mu\{ x \in \mathbb{R}^d \mid |f(x)| > \alpha \}^{\frac{1}{p}} \|f\|_{L^r}, \|f\|_{L^{p,1}(\mathbb{R}^d)}, \|f\|_{L^{p,\infty}(\mathbb{R}^d)}$.
Proof. Note that \[ \int_{X^{d-1}} K_{z,\pm}(y', x_d) e^{-2\pi i y' \cdot \xi'} dy' = \int_X \frac{e^{2\pi i x_d \xi}}{T(\xi-z)} d\xi. \] If necessary we take supp \( \chi \) is small, it suffices to replace the integration region by \( \mathbb{R} \). Thus by (22), we have
\[
\int_{\mathbb{R}} \frac{e^{2\pi i x \xi - \xi}}{T(\xi-z)} d\xi = \int_{\mathbb{R}} \frac{e^{2\pi i x \xi + h_{\lambda}(\xi)}}{T(\xi-z)} d\xi = e^{2\pi i x \xi + h_{\lambda}(\xi)} \gamma_{z,\pm}(\xi, x_d).
\]
By using [4, (3.10)] for \( \pm \text{Im} \, z > 0 \) and [5, (A.6)] for \( \pm \text{Im} \, z = 0 \), we have
\[
|\partial^\alpha \gamma_{z,\pm}(\xi', x_d)| \leq C_{\alpha}
\]
for \( \alpha \in \mathbb{N}^{d-1} \). We will prove (23) in Lemma A.3.. Thus the first inequality holds. Moreover, we note that
\[
K_{z,\pm}(x) = \int_{X^{d-1}} \gamma_{z,\pm}(\xi') e^{2\pi i x' \cdot \xi + h_{\lambda}(\xi')} d\xi'.
\]
Since \( \gamma_{z,\pm} \) is compactly supported in \( \xi' \)-variable, then (6) and (23) imply the second inequality. The estimates for \( K_{z,\pm}(x) \) follow from the estimates
\[
|\partial^\alpha \gamma_{z,\pm}(\xi', x_d)| \leq C_{\alpha} |z - w|^\delta (1 + |x_d|)^{\delta},
\]
which is also proved after Lemma A.3: (39).

\[\square\]

Lemma 3.2. There exists \( C_3 > 0 \) such that
\[
|\int_{X^{d-1}} \int_{X^{d-1}} e^{2\pi i x' \cdot \xi'} K_{z,\pm}(x', x_d - y_d) K_{z,\pm}(x' - y', x_d - z_d) dx' dy' | \leq C_0^2
\]
\[
|\int_{X^{d-1}} K_{z,\pm}(x' - y', x_d - y_d) K_{z,\pm}(x' - z', x_d - z_d) dx' | \leq C_3^2 (1 + |y_d - z_d|)^{-k}
\]
where \( C_0 > 0 \) is as in the proof of Lemma 3.1.

Proof. Note that
\[
\int_{X^{d-1}} \int_{X^{d-1}} e^{2\pi i y' \cdot \xi'} K_{z,\pm}(x', x_d - y_d) K_{z,\pm}(x' - y', x_d - z_d) dx' dy' = e^{2\pi i [y_d - z_d] h_{\lambda}(\xi') \gamma_{z,\pm}(\xi', x_d - y_d)},
\]
where \( \gamma_{z,\pm} \) is as in the proof of Lemma 3.1. Moreover, we have
\[
\int_{X^{d-1}} K_{z,\pm}(x' - y', x_d - y_d) K_{z,\pm}(x' - z', x_d - z_d) dx' = \int_{X^{d-1}} e^{2\pi i (y' - z') \xi' + 2\pi i (y_d - z_d) h_{\lambda}(\xi') \gamma_{z,\pm}(\xi', x_d - y_d)} d\xi'.
\]
Thus (6) and (23) imply the conclusion.

\[\square\]

Lemma 3.1 and 3.2 imply that \( K_{z,\pm} \) satisfies Assumption B, C and D. This completes the proof of Theorem 1.1.
Remark 3.3. In order to prove (i), it is sufficient to prove (i) for $\pm \text{Im} z = 0$ by using the Phragmén-Lindelöf principle as in [26, Section 5.3]. See also [4, Appendix A] for the estimates of the Shatten norm of the resolvent. Here we avoid using the Phragmén-Lindelöf principle.

**Corollary 3.4.** Let $r_1, r_2 \in (1, 4k + 2]$ satisfying $1/r_1 + 1/r_2 \geq 1/(k + 1)$. Then
\begin{equation}
\sup_{z \in I_\pm} \| W_1 \chi(D) R_0^\pm (z) W_2 \|_{B(L^2(X^d))} \leq C \| W_1 \|_{L^{r_1}(X^d)} \| W_2 \|_{L^{r_2}(X^d)}
\end{equation}
for $W_1 \in L^{r_1}(X^d)$ and $W_2 \in L^{r_2}(X^d)$. Moreover, if $W_1 \in L^{r_1}(X^d)$ and $W_2 \in L^{r_2}(X^d)$, then $W_1 \chi(D) R_0^\pm (z) W_2 \in B_\infty (L^2(X^d))$ and $W_1 \chi(D) R_0^\pm (z) W_2$ is continuous in $z \in I_\pm$. In addition, if $r = r_1 = r_2 \in (1, 4k + 2]$, then
\begin{equation}
\| W_1 \chi(D) (R_0^\pm (z) - R_0^\pm (w)) W_2 \|_{B(L^2(X^d))} \leq C |z - w|^\beta \| W_1 \|_{L^{r_1}(X^d)} \| W_2 \|_{L^{r_2}(X^d)}
\end{equation}
for $z, w \in I_\pm, |z - w| \leq 1$.

**Proof.** (24) and (25) follow from Theorem 1.1. For proving the other statements, we may assume $W_1, W_2 \in C^\infty_c (X^d)$ by $\epsilon/3$-argument and (24). Since $W_1$ and $W_2$ are compactly supported and since the integral kernel of $\chi(D) R_0^\pm$ is in $L^\infty$ by Lemma 3.1, then the integral kernel of $W_1 \chi(D) R_0^\pm (z) W_2$ is square integrable and hence Hilbert-Schmidt. Thus $W_1 \chi(D) R_0^\pm (z) W_2$ is compact. Moreover, by (25), $W_1 \chi(D) R_0^\pm (z) W_2$ is continuous in $z \in I_\pm$. Next, we suppose $W_1 \in L^{r_1}(X^d)$ and $W_2 \in L^{r_2}(X^d)$.

4. Applications

4.1. Discrete Schrödinger operator. In this subsection, we consider the case of $X = \mathbb{Z}$ and consider the discrete Schrödinger operators. By using
\[
\delta(E) = \int_{\mathbb{R}} e^{2\pi i \tau E} d\tau, \quad E \in \mathbb{R},
\]
we have
\begin{equation}
\chi(D) \delta (H_0 - \lambda) (x) = \int_{\mathbb{R}} \int_{\mathbb{T}^d} e^{2\pi i (x \cdot \xi + \tau (h_0(\xi) - \lambda))} \chi(\xi) d\xi d\tau
\end{equation}
for $\chi \in C^\infty_c (\mathbb{T}^d)$ supported away from the critical points of $h_0$, where $\chi(D) \delta (H_0 - \lambda) (x)$ is the convolution kernel of $\chi(D) \delta (H_0 - \lambda)$. On the other hands, it is well-known that
\begin{equation}
\chi(D) \delta (H_0 - \lambda) (x) = \int_{M_\lambda} e^{2\pi i x \cdot \xi} \chi(\xi) \frac{d\mu_\lambda(\xi)}{\|
abla h_0(\xi)\|},
\end{equation}
where we recall $M_\lambda = \{ h_0(\xi) = \lambda \}$ and $d\mu_\lambda$ is the induced surface measure. Now we use the following estimate:

**Lemma 4.1.** Let $d \geq 4$ and $\chi \in C^\infty_c (\mathbb{T}^d)$ be supported away from the critical point of $h_0$. Then there exist an integer $M > 0$ and $C > 0$ independent of $z$ such that
\begin{equation}
| \int_{M_\lambda} e^{2\pi i x \cdot \xi} \chi(\xi) d\mu_\lambda(\xi) | \leq C \sum_{|\alpha| \leq M} | \partial^\alpha_\xi \chi(\xi) | (1 + |x|)^{-\frac{d-4}{2}}.
\end{equation}
Proof. Note that $\partial_{\xi_j}^2 h_0(\xi) + \partial_{\xi_j}^2 h_0(\xi) \neq 0$ for $\xi \in \mathbb{T}^d$ and $j = 1, \ldots, d$. Using the van der Corput lemma as in [28], we have
\[
| \int_{\mathbb{T}^d} e^{2\pi i(x \cdot \xi + r(h_0(\xi) - \lambda))} \chi_\epsilon(\xi) d\xi| \leq C_\epsilon \sum_{|\alpha| \leq M} |\partial^\alpha \chi(\xi)|(1 + |x| + |r|)^{-\frac{d}{4}}.
\]
Thus we obtain the desired bound (28) by the representation (26), (27) and the fact that $\nabla h_0(\xi) \neq 0$ on supp $\chi$. \hfill \Box

Lemma 4.2. Let $1 \leq p \leq q \leq \infty$ and $m \in C^\infty(\mathbb{T}^d)$. Then $m(D) \in B(L^p(\mathbb{Z}^d), L^q(\mathbb{Z}^d))$.

Proof. We denote the convolution kernel of $m(D)$ by $m(D)(x)$. Since $m \in C^\infty(\mathbb{T}^d)$, then $|m(D)(x)| \leq C_N(1 + |x|)^{-N}$ for each $N > 0$. By Schur’s lemma, we have $m(D) \in B(L^p(\mathbb{Z}^d))$ for any $1 \leq p \leq \infty$. Since $L^p(\mathbb{Z}^d) \subset L^q(\mathbb{Z}^d)$ for $1 \leq p \leq q \leq \infty$, we have $m(D) \in B(L^p(\mathbb{Z}^d), L^q(\mathbb{Z}^d))$. Real interpolating this, we have $m(D) \in B(L^{p,r}(\mathbb{Z}^d), L^{q,r}(\mathbb{Z}^d))$ for $1 \leq r \leq \infty$. \hfill \Box

Proof of Theorem 1.2. By virtue of Theorem 1.1 and Lemma 4.1, we obtain Theorem 1.2 (i) and (ii) since $(1 - \chi(D))R^\pm_0(z)$ is smooth and bounded in $z \in I_\pm$ by virtue of Lemma 4.2, where we recall $I \subset \mathbb{R} \setminus \Lambda_{\epsilon}(h_0(D))$ is a compact interval. Thus it remains to prove (iii). We need the following lemma.

Lemma 4.3. Let $r_1, r_2 \in (1, 4d/3 - 2]$ satisfying $1/r_1 + 1/r_2 \geq 3/d$. Then
\[
(29) \quad \sup_{z \in I_\pm} \|W_1 \chi(D) R^\pm_0(z) W_2 \|^2_{B(L^2(X^d))} \leq C \|W_1\|_{L^r(\mathbb{X}^d)} \|W_2\|^2_{L^2(X^d)}
\]
for $W_1 \in L^{r_1}(\mathbb{X}^d)$ and $W_2 \in L^{r_2}(\mathbb{X}^d)$. Moreover, if $W_1 \in L^{r_1}(\mathbb{X}^d)$ and $W_2 \in L^{r_2}(\mathbb{X}^d)$, then $W_1 \chi(D) R^\pm_0(z) W_2 \in B_\infty(L^2(\mathbb{X}^d))$ and $W_1 \chi(D) R^\pm_0(z) W_2$ is continuous in $z \in I_\pm$. In addition, if $r = r_1 = r_2 \in (1, 4d/3 - 2 - \delta)$, then
\[
(30) \quad \|W_1 \chi(D)(R^\pm_0(z) - R^\pm_0(w)) W_2 \|^2_{B(L^2(X^d))} \leq C \|z - w\|^2 \|W_1\|^2_{L^r(\mathbb{X}^d)} \|W_2\|^2_{L^r(\mathbb{X}^d)}
\]
for $z, w \in I_\pm, |z - w| \leq 1$.

Proof. The proof is same as in the proof of 3.4 by using the compactness argument. \hfill \Box

We follow the argument as in [18] and [21]. Let $V \in L^{4/3}(\mathbb{Z}^d)$ be a real-valued function. Set $W_1 = (\text{sgn} V)|V|^{1/2} \in L^{2d/3}(\mathbb{Z}^d)$, $W_2 = |V|^{1/2} \in L^{2d/3}(\mathbb{Z}^d)$, $H = H_0 + V$ and $R(z) = (H - z)^{-1}$ for $z \in \mathbb{C} \setminus \mathbb{R}$. We note that for $\pm \text{Im} z > 0$
\[
(31) \quad W_2 R^\pm_0(z) W_2 - W_2 R(z) W_2 W_1 R^\pm_0(z) W_2.
\]
Note that $W_1 R^\pm_0(z) W_2$ is compact and continuous in $z \in I_\pm$. In addition, $I + W_1 R^\pm_0(z) W_2$ is invertible in $B(L^2(\mathbb{Z}^d))$ for $z \in \mathbb{C} \setminus \mathbb{R}$ due to the Birman-Schwinger principle. In fact, if $I + W_1 R^\pm_0(z) W_2$ is not invertible at $z \in \mathbb{C} \setminus \mathbb{R}$, then the compactness of $W_1 R^\pm_0(z) W_2$ implies that $I + W_1 R^\pm_0(z) W_2$ has a non-trivial kernel. Then it follows that $R(z)$ has a non-trivial kernel by the Birman-Schwinger principle. However, this contradicts to the self-adjointness of $H_0 + V$. Moreover, if we set
\[
\sigma_{BS}(H) = \sigma_{BS}^+(H) = \{ \lambda \in \mathbb{R} \mid \ker_{L^2(\mathbb{Z}^d)}(I + W_1 R^\pm_0(z) W_2) \neq 0 \},
\]
then by Proposition B.3, $\sigma_{BS}(H)$ is a closed set with Lebesgue measure zero. Since $W_1 R^0(z) W_2 \in B_\infty(L^2(\mathbb{Z}^d))$ for $z \in I_\pm$, $I + W_1 R^0(z) W_2$ is a Fredholm operator with index 0. Thus (31) gives

$$W_2 R(z) W_2 = W_2 R^0(z) W_2 (I + W_1 R^0(z) W_2)^{-1}, \ z \in I_\pm \setminus \sigma_{BS}(H_0).$$

Let $[a, b] \subset I \setminus \sigma_{BS}(H_0)$ with $a < b$. Then since $(I + W_1 R^0(z) W_2)^{-1}$ is continuous in $z \in [a, b]_\pm$, then

$$\sup_{z \in [a, b]_\pm} \|(I + W_1 R^0(z) W_2)^{-1}\|_{B(L^2(\mathbb{Z}^d))} < \infty.$$ Combining this with the part $(i)$ and Hölder’s inequality, we obtain

$$\sup_{z \in [a, b]_\pm} \|W_2 R(z) W_2\|_{B(L^2(\mathbb{Z}^d))} < \infty.$$ Since $|W_1| = |W_2|$, then

$$\sup_{z \in [a, b]_\pm} \|W_i R(z) W_i\|_{B(L^2(\mathbb{Z}^d))} < \infty.$$ for $i_1, i_2 = 1, 2$. By [24, Theorem XIII. 30, 31], the local wave operators $s - \lim_{t \to \pm \infty} e^{itH} e^{-itH_0} E_{H_0}(\xi, (a, b))$ exist and are complete, where $E_{H_0}(J)$ is the spectral projection to the interval $J \subset \mathbb{R}$ associated with $H_0$. Since $[0, 4d] \setminus \Lambda_c(H_0) \cup \sigma_{BS}(H)$ is a countable union of such interval $(a, b)$, the wave operators $W_\pm = s - \lim_{t \to \pm \infty} e^{itH} e^{-itH_0}$ exist and are complete.

Finally, we mention the further estimates of the form (6) for the discrete Schrödinger operators.

**Lemma 4.4.** Suppose $I \subset (0, 4) \cap (4(d - 1), 4d)$ if $d = 2$ and $I \subset (0, 2) \cap (4d - 2, 4d)$ if $d \geq 3$. If $\sup \chi \subset h^{-1}_0(I)$, then (6) holds with $k = (d - 1)/2$.

**Proof.** As is proved in [15, Lemma 4.3], the all principal curvature of $M_\lambda = \{ h = \lambda \}$ is positive, in particular non-vanishing if $\lambda \in I$. By Example 1, we obtain the desired result.

**Lemma 4.5.** Suppose that $\chi$ is supported away from the critical points of $h_0$ and max$\xi \in \text{supp} \chi \# \{ j \in \{1, \ldots, d \} \mid \xi_j = \pi/4 \text{ or } \xi_j = -\pi/4 \} \leq l$. Then (6) holds with $k = l/3 + (d - l)/2 - 1$.

**Proof.** We note that max$\xi \in \text{supp} \chi \# \{ j \in \{1, \ldots, d \} \mid \partial^2 \xi_j h_0(\xi) \neq 0 \} \leq l$. We repeat the proof of Lemma 4.1 and obtain the desired result.

**4.2. Fractional Schrödinger operators and Dirac operators.** In this subsection, we suppose that $T(D)$ is the one of the following operators:

$$T(D) = (-\Delta)^{s/2}, \ T(D) = (-\Delta + 1)^{s/2} - 1, \ T(D) = \mathcal{D}_0, \ T(D) = \mathcal{D}_1,$$

where $0 < s \leq d$.

**Proof of Theorem 1.4.** First, we consider the case when $T(D) = (-\Delta)^{s/2}$ or $T(D) = (1 - \Delta)^{s/2}$. We take a real-valued function $\chi \in C_c^\infty(\mathbb{R}^d, [0, 1])$ such that $\chi = 1$ on $T^{-1}(I)$ and $\sup \chi \subset \mathbb{R} \setminus \Lambda_c(T(D))$. Note that $M_\lambda = \{ T(\xi) = \lambda \}$ is sphere and
hence has non vanishing Gaussian curvature. If \( \lambda \in \sigma(T(D)) \setminus \Lambda_\epsilon(T(D)) \). Then we apply Theorem 1.1 with \( k = (d - 1)/2 \) (see [27, Theorem 1.2.1]) and obtain

\[
\sup_{z \in I_{\pm}} \| \chi(D)R_0^{\pm}(z)\|_{B(L^p(\mathbb{R}^d), L^q(\mathbb{R}^d))} < \infty
\]

for \((p, q) \in S_{d/2}^+\). On the other hand, by the support property of \( \chi \) and the Hardy-Littlewood-Sobolev inequality, we have

\[
\sup_{z \in I_{\pm}} \| (1 - \chi(D))R_0(z)\|_{B(L^p(\mathbb{R}^d), L^q(\mathbb{R}^d))} < \infty
\]

if \( 1/p - 1/q \leq s/d \). In fact, if \( 2\alpha = -d/2 + d/p \) and \( 2\beta = -d/q + d/2 \), then

\[
\| (1 - \chi(D))R_0(z)\|_{B(L^p(\mathbb{R}^d), L^q(\mathbb{R}^d))}
\leq \| (I - \Delta) - \alpha \|_{B(L^p(\mathbb{R}^d), L^q(\mathbb{R}^d))} \| (1 - \chi(D))(I - \Delta)^{\alpha+\beta} R_0(z)\|_{B(L^q(\mathbb{R}^d))}
\times \| (I - \Delta)^{-\beta}\|_{B(L^q(\mathbb{R}^d), L^q(\mathbb{R}^d))}.
\]

Thus (33) follows from the the Hardy-Littlewood-Sobolev inequality. Combining (32) with (33), we obtain (i), (ii) is similarly proved.

Existence and completeness of the wave operators are similarly proved as in the proof of Theorem 1.2 (iii) by using Corollary 4.6 below.

The case when \( T(D) = D_0 \) or \( T(D) = D_1 \) is similarly proved if we notice that

\[
D_0^2 = -\Delta I_{n \times n}, \quad D_0^2 = (-\Delta + 1)I_{n \times n}
\]

as in the proof of [4, Theorem 3.1].

\[\square\]

**Corollary 4.6.** If \( 2d/(d + 1) \leq s \leq d \) and \( 0 < \delta \leq 1 \), then

\[
\sup_{z \in I_{\pm}} \| W_1 R_0^{\pm}(z)W_2\|_{B(L^q(\mathbb{R}^d))} \leq C \| W_1\|_{L^r(\mathbb{R}^d)} \| W_2\|_{L^{r^\prime}(\mathbb{R}^d)}
\]

\[
\| W_3(W_0^{\pm}(z) - R_0^{\pm}(w))W_4\|_{B(L^q(\mathbb{R}^d))} \leq C |z - w|^{\beta}\| W_3\|_{L^r(\mathbb{R}^d)} \| W_4\|_{L^{r^\prime}(\mathbb{R}^d)}
\]

for \( z, w \in I_{\pm} \) with \( |z - w| \leq 1 \) and \( W_1 \in L^{r_1}(\mathbb{R}^d) \), \( W_2 \in L^{r_2}(\mathbb{R}^d) \), \( W_3, W_4 \in L^r(\mathbb{R}^d) \) with \( r_1, r_2 \in (1, 2(d + 1)] \) and \( r \in (2d/s, 2(d + 1) - 4\delta) \) satisfying \( 2/(d + 1) \leq 1/r_1 + 1/r_2 \leq s/d \). Moreover, if \( W_1 \in L^{r_1}(\mathbb{R}^d) \) and \( W_2 \in L^{r_2}(\mathbb{R}^d) \), then \( W_1 R_0^{\pm}(z)W_2 \in B_{\infty}(L^2(\mathbb{R}^d)) \) and a map \( z \in I_{\pm} \mapsto W_1 R_0^{\pm}(z)W_2 \) is continuous.

When \( 0 < s < 2d/(d + 1) \), the all results hold if we replace \( L^{r_1}(\mathbb{R}^d) \), \( L^{r_2}(\mathbb{R}^d) \) and \( L^r(\mathbb{R}^d) \) by \( L^{r_1}(\mathbb{R}^d) \cap L^{r_1^\prime}(\mathbb{R}^d) \), \( L^{r_2}(\mathbb{R}^d) \cap L^{r_2^\prime}(\mathbb{R}^d) \) and \( L^r(\mathbb{R}^d) \cap L^{r^\prime}(\mathbb{R}^d) \) respectively with \( r_1, r_2 \in (1, 2(d + 1)] \), \( r \in (1, 2(d + 1) - 4\delta) \) and \( r_1, r_2 \in [2d/s, \infty) \) satisfying \( 2/(d + 1) \leq 1/r_1 + 1/r_2 \) and \( 1/r_1 + 1/r_2 \leq s/d \).

**Proof.** Note that if \( W_1, W_2 \in C_\infty(\mathbb{R}^d) \), then \( W_1(1 - \chi(D))R_0^{\pm}(z)W_2 \) is compact and smooth in \( z \in I_{\pm} \) by \( dR_0(z)/dz = R_0(z) \) and the Rellich-Kondrachov theorem. The other parts of the proof are same as in the proof of Corollary 3.4. \[\square\]

**Appendix A. Some estimates for \( \gamma_{z, \pm} \)**

In this section, we give proofs of the estimates for \( \gamma_{z, \pm} \) which is needed for the proof of Theorem 1.1.

If necessary we take \( \text{supp} \chi \) small, we may assume \( X = \mathbb{R} \). We recall the situation of the proof of Theorem 1.1. Set

\[
\hat{\chi}(\xi', \xi_d, \lambda) = \frac{\chi^2(\xi', \xi_d + h_\lambda(\xi'))}{e(\xi', \xi_d + h_\lambda(\xi'))}, \quad b(\xi', \xi_d, \lambda) = e(\xi', \xi_d + h_\lambda(\xi'))^{-1}.
\]
Note that $b$ is real-valued and $\min(\xi', \xi_d) \in \text{supp} \chi(\cdot, \cdot, \lambda, \kappa) b(\xi', \xi_d, \lambda) > 0$. Recall that
\[
\gamma_{z, \pm}(\xi', x_d) = \int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi} \chi(\xi', \xi_d, \lambda)}{\xi - i(\text{Im} z) b(\xi', \xi_d, \lambda)} d\xi_d, \quad \text{Re } z = \lambda, \pm \text{Im } z \geq 0.
\]
Here if $\pm \text{Im } z = 0$, we interpret $\gamma_{z, \pm}$ as
\[
\gamma_{z, \pm}(\xi', x_d) = \int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi} \chi(\xi', \xi_d + \lambda(\xi'))}{\xi_d + i0} d\xi_d,
\]
where $(\xi_d + i0)^{-1}$ denote the distributions $\lim_{\epsilon > 0, \epsilon \to 0} (\xi_d + i\epsilon)^{-1}$. In order to estimate $\gamma_{z, \pm}$, we need some lemmas.

**Lemma A.1.** Let $\psi, \psi_1 \in C_c^\infty(\mathbb{R})$ and $\mu_1, \mu_2 \in \mathbb{R} \setminus \{0\}$. Then
\[
| \int_{\mathbb{R}} \psi(\mu_1 y_d) p.v. \frac{e^{2\pi i y_d \xi_d}}{y_d} dy_d | \leq \pi \| \hat{\psi} \|_{L^1(\mathbb{R})}, \quad \| \int_{\mathbb{R}} p.v. \frac{e^{2\pi i y_d \xi_d}}{y_d} dy_d \| = \pi.
\]
\[
| \int_{\mathbb{R}} \psi(\mu_1 y_d) \psi_1(\mu_2 y_d) p.v. \frac{e^{2\pi i y_d \xi_d}}{y_d} dy_d | \leq \pi \| \hat{\psi} \|_{L^1(\mathbb{R})} \| \hat{\psi}_1 \|_{L^1(\mathbb{R})},
\]

**Proof.** We learn
\[
| \int_{\mathbb{R}} p.v. \frac{1}{y_d} \psi(y_d) e^{2\pi i y_d \xi_d} dy_d | = \pi \left| \int_{\mathbb{R}} \text{sgn}(\xi_d - \eta_d) \hat{\psi}(-\eta_d) d\eta_d \right|
\leq \pi \| \hat{\psi} \|_{L^1(\mathbb{R})}.
\]
By scaling, we obtain the first inequality. The second equality follows from $\mathcal{F}(p.v. \frac{1}{y_d})(\xi_d) = -i\pi \text{sgn}(\xi_d)$. The third inequality follows from the first inequality and the Young inequality:
\[
\| \hat{\psi} \hat{\psi}_1 \|_{L^1(\mathbb{R})} = \| \hat{\psi} * \hat{\psi}_1 \|_{L^1(\mathbb{R})}
\leq \| \hat{\psi} \|_{L^1(\mathbb{R})} \| \hat{\psi}_1 \|_{L^1(\mathbb{R})}.
\]

**Lemma A.2.** Let $\mu \in \mathbb{R} \setminus \{0\}$ and $\varphi, a, a_1 \in C_c^\infty(\mathbb{R})$ such that $a, a_1$ are real-valued and $a_1 > 0$ on supp $\varphi$.

(i) There exists $C > 0$ independent of $x_d \in \mathbb{R}$, $\varphi, a$ and $\mu \neq 0$ such that
\[
\int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d} \varphi(\mu \xi_d)}{\xi_d - ia(\mu \xi_d)} d\xi_d \leq C(\sup_{\xi_d \in \mathbb{R}} |\varphi(\xi_d)| + \| \varphi \|_{L^1(\mathbb{R})} + \sup_{\xi_d \in \mathbb{R}} |\varphi(\xi_d)a(\xi_d)|).
\]

(ii) Let $l \geq 2$ be an integer. Then there exists $C' > 0$ independent of $x_d \in \mathbb{R}$, $\varphi, a, l$ and $\mu \neq 0$ such that
\[
\int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d} \varphi(\mu \xi_d)}{(\xi_d - ia(\mu \xi_d))^l} d\xi_d \leq C'(\sup_{\xi_d \in \mathbb{R}} |\varphi(\xi_d)| + \| \varphi \|_{L^\infty(\mathbb{R})}).
\]

(iii) Let $l_1, l_2 \geq 1$ be an integer. Then there exists $C'' > 0$ independent of $x_d \in \mathbb{R}$, $\varphi, a, l$ and $\mu \neq 0$ such that
\[
\int_{\mathbb{R}} \frac{e^{2\pi i x_d \xi_d} \varphi(\mu \xi_d)}{(\xi_d - ia(\mu \xi_d))^{l_1}(\xi_d - ia_1(\mu \xi_d))^{l_2}} d\xi_d \leq C''(\sup_{\xi_d \in \mathbb{R}} |\varphi(\xi_d)| + \| \varphi \|_{L^\infty(\mathbb{R})}).
\]
Proof. (i) Take \( \psi \in C^\infty_c(\mathbb{R}, [0, 1]) \) such that \( \psi = 1 \) on \( |t| \leq 1 \) and \( \psi = 0 \) on \( |t| \geq 2 \).

Since \( a \) is real-valued, then
\[
\left| \int_{\mathbb{R}} \frac{e^{2\pi i x \cdot \xi_d} \varphi(\mu_d) \psi(\xi_d)}{\xi_d - ia(\mu_d)} d\xi_d \right| \leq \int_{\mathbb{R}} \frac{|\varphi(\mu_d)\psi(\xi_d)|}{|a(\mu_d)|} d\xi_d \\
\leq \sup_{\xi_d \in \mathbb{R}} \frac{|\varphi(\xi_d)|}{a(\xi_d)} \|\psi\|_{L^1(\mathbb{R})}.
\]

We note that
\[
\int_{\mathbb{R}} \frac{e^{2\pi i x \cdot \xi_d} \varphi(\mu_d)(1 - \psi(\xi_d))}{\xi_d - ia(\mu_d)} d\xi_d = \int_{\mathbb{R}} e^{2\pi i x \cdot \xi_d} \varphi(\mu_d)(1 - \psi(\xi_d)) d\xi_d \\
+ i \int_{\mathbb{R}} \frac{e^{2\pi i x \cdot \xi_d} \varphi(\mu_d)a(\mu_d)(1 - \psi(\xi_d))}{\xi_d - ia(\mu_d)} d\xi_d
\]

By Lemma A.1, we have
\[
|I_1| = \left| \int_{\mathbb{R}} \text{p.v.} \frac{e^{2\pi i x \cdot \xi_d} \varphi(\mu_d)}{\xi_d} d\xi_d - \int_{\mathbb{R}} \text{p.v.} \frac{e^{2\pi i x \cdot \xi_d} \varphi(\mu_d) \psi(\xi_d)}{\xi_d} d\xi_d \right| \\
\leq \pi \|\varphi\|_{L^1(\mathbb{R})}(1 + \|\psi\|_{L^1(\mathbb{R})}).
\]

Moreover, since \( a \) is real-valued, we have
\[
|I_2| \leq \sup_{\xi_d \in \mathbb{R}} |\varphi(\xi_d)a(\xi_d)| \int_{\mathbb{R}} \frac{1 - \psi(\xi_d)}{\xi_d^2} d\xi_d.
\]

Thus we set
\[
C = \max(\|\psi\|_{L^1(\mathbb{R})}, \pi(1 + \|\psi\|_{L^1(\mathbb{R})}), \int_{\mathbb{R}} \frac{1 - \psi(\xi_d)}{\xi_d^2} d\xi_d),
\]

and obtain (34).

(ii) follows from (iii).

(iii) Let \( \psi \) be as above. Then
\[
\left| \int_{\mathbb{R}} \frac{e^{2\pi i x \cdot \xi_d} \varphi(\mu_d)\psi(\xi_d)}{(\xi_d - ia(\mu_d))^{l_1}(\xi_d - ia_1(\mu_d))^{l_2}} d\xi_d \right| \leq \sup_{\xi_d \in \mathbb{R}} \frac{|\varphi(\xi_d)|}{|a(\xi_d)|^{l_1}|a_1(\xi_d)|^{l_2}} \|\psi\|_{L^1(\mathbb{R})}.
\]

Moreover, since \( a, a_1 \) is real-valued and \( l_1 + l_2 \geq 2 \), then
\[
\left| \int_{\mathbb{R}} \frac{e^{2\pi i x \cdot \xi_d} \varphi(\mu_d)(1 - \psi(\xi_d))}{(\xi_d - ia(\mu_d))^{l_1}(\xi_d - ia_1(\mu_d))^{l_2}} d\xi_d \right| \leq \|\varphi\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \frac{1 - \psi(\xi_d)}{|\xi_d|^{l_1 + l_2}} d\xi_d \\
\leq \|\varphi\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \frac{1 - \psi(\xi_d)}{|\xi_d|^2} d\xi_d.
\]

Thus we set \( C'' = \max(\|\psi\|_{L^1(\mathbb{R})}, \int_{\mathbb{R}} \frac{1 - \psi(\xi_d)}{|\xi_d|^2} d\xi_d) \) and obtain (36).

\( \square \)

The main result of this section is the following proposition.

**Proposition A.3.** Fix a signature \( \pm \).

(i) For \( \alpha \in \mathbb{N}^{d-1} \), there exists \( C_\alpha > 0 \) such that
\[
|\partial_\xi^\alpha \gamma_{z, \pm}(\xi', x_d)| \leq C_\alpha
\]
for \( z \in I_\pm, x_d \in \mathbb{R} \) and \( \xi' \in \mathbb{R}^{d-1} \).
(ii) For $\alpha \in \mathbb{N}^{d-1}$, there exists $C'_{\alpha} > 0$ such that

$$|\partial_{\xi}^{\alpha}(\gamma_{z,\pm}(\xi', x_d) - \gamma_{w,\pm}(\xi', x_d))| \leq C'_{\alpha}(1 + |x_d|)|z - w|$$

for $z, w \in I_\pm$ with $|z - w| \leq 1$, $x_d \in \mathbb{R}$ and $\xi' \in \mathbb{R}^{d-1}$.

Remark A.4. Let $0 \leq \delta \leq 1$. Combining (37) with (38), we have

$$|\partial_{\xi}^{\alpha}(\gamma_{z,\pm}(\xi', x_d) - \gamma_{w,\pm}(\xi', x_d))| \leq C'_{\alpha}^{-s}(C'_{\alpha})^s(1 + |x_d|)^{\delta}|z - w|^\delta.$$  

Proof. (i) We follow the argument of the proof of [4, (3.10)]. We may assume $0 \leq \pm \text{Im} z \leq 1$. First, we consider the case of $\pm \text{Im} z = 0$. In this case, the claim follows from the fact that

$$\|e^{2\pi i x_d \xi_d}d\xi_d\|_{L^\infty(\mathbb{R}^d)} < \infty$$

and that $\tilde{\chi}$ is smooth with respect to $(\xi, \xi_d, \lambda) \in \mathbb{R}^d \times I$ and has a compact support with respect to $(\xi', \xi_d)$-variable which is bounded in $\lambda \in I$.

We take $\psi \in C_c^\infty(\mathbb{R}, [0, 1])$ such that $\psi(\xi_d) = 1$ on $|\xi_d| \leq 1$. We learn

$$\gamma_{z,\pm}(\xi', x_d) = \int_{\mathbb{R}} e^{2\pi i (\text{Im} z) x_d \xi_d} \tilde{\chi}(\xi', (\text{Im} z) \xi_d, \lambda) \frac{\xi_d - i b(\xi', (\text{Im} z) \xi_d, \lambda)}{\xi_d - i b(\xi', (\text{Im} z) \xi_d, \lambda)} d\xi_d.$$ 

We note that $\partial_{\xi}^{\alpha}(\gamma_{z,\pm}(\xi', x_d))$ is a linear combination of the form

$$\int_{\mathbb{R}} e^{2\pi i (\text{Im} z) x_d \xi_d} (\partial_{\xi}^{\alpha} \tilde{\chi})(\xi', (\text{Im} z) \xi_d, \lambda) \prod_{j=1}^l (\partial_{\xi}^{\alpha_j} b)(\xi', (\text{Im} z) \xi_d, \lambda) d\xi_d,$$

where $l \geq 1$ is an integer and $\alpha_j \in \mathbb{N}^{d-1}$ for $j = 0, ..., l$. Applying Lemma A.2

(i) if $l = 1$ and (ii) if $l > 1$ with \( \varphi(\xi_d) = (\partial_{\xi}^{\alpha} \tilde{\chi})(\xi', \xi_d, \lambda) \prod_{j=1}^l (\partial_{\xi}^{\alpha_j} b)(\xi', \xi_d, \lambda), \) \( a(\xi_d) = b(\xi', \xi_d, \lambda) \) and $\mu = \text{Im} z$, we obtain (37) with $|\alpha| \geq 1$.

(ii) We take $0 < \varepsilon$ such that

$$\min_{(\xi', \xi_d) \in \text{supp } \chi(\cdot, \cdot, \cdot, \cdot), |z - w| \leq \delta} |b(\xi', \xi_d, \sigma)| > 0.$$ 

Then we may assume $|z - w| < \varepsilon$. In fact, in order to prove (ii), we use (i) if $|z - w| \geq \varepsilon$. Recall that $\lambda = \text{Re} z$ and set $\sigma = \text{Re} w$. Note that

$$\gamma_{z,\pm}(\xi', x_d) - \gamma_{w,\pm}(\xi', x_d) = J_1(x_d) + J_2(x_d) + J_3(x_d),$$

where we set

$$J_1(x_d) = \int_{\mathbb{R}} e^{2\pi i x_d \xi_d} \left( \frac{\tilde{\chi}(\xi', \xi_d, \lambda)}{\xi_d - i (\text{Im} z) b(\xi', \xi_d, \lambda)} - \frac{\tilde{\chi}(\xi', \xi_d, \lambda)}{\xi_d - i (\text{Im} w) b(\xi', \xi_d, \lambda)} \right) d\xi_d,$$

$$J_2(x_d) = \int_{\mathbb{R}} e^{2\pi i x_d \xi_d} \left( \frac{\tilde{\chi}(\xi', \xi_d, \lambda)}{\xi_d - i (\text{Im} z) b(\xi', \xi_d, \lambda)} - \frac{\tilde{\chi}(\xi', \xi_d, \lambda)}{\xi_d - i (\text{Im} w) b(\xi', \xi_d, \lambda)} \right) d\xi_d = \int_{\mathbb{R}} e^{2\pi i (\text{Im} z) x_d \xi_d} \left( \frac{\tilde{\chi}(\xi', (\text{Im} w) \xi_d, \lambda) - \tilde{\chi}(\xi', (\text{Im} w) \xi_d, \lambda)}{\xi_d - i b(\xi', (\text{Im} w) \xi_d, \lambda)} \right) d\xi_d,$$

$$J_3(x_d) = \int_{\mathbb{R}} e^{2\pi i x_d \xi_d} \left( \frac{1}{\xi_d - i (\text{Im} w) b(\xi', \xi_d, \lambda)} - \frac{1}{\xi_d - i (\text{Im} w) b(\xi', \xi_d, \lambda)} \right) d\xi_d = \int_{\mathbb{R}} e^{2\pi i (\text{Im} w) x_d \xi_d} \left( \frac{\tilde{\chi}(\xi', (\text{Im} w) \xi_d, \lambda) - \tilde{\chi}(\xi', (\text{Im} w) \xi_d, \lambda)}{\xi_d - i b(\xi', (\text{Im} w) \xi_d, \lambda)} \right) d\xi_d.$$
First, we estimate $J_2$. Similarly to the proof of (i), $\partial^2_{\zeta} J_2(\zeta')$ is a finite sum of the form
\[
\int_{\mathbb{R}} e^{2\pi i (\text{Im } w) x \cdot \xi_d} (\xi_d, (\text{Im } w) \xi_d, \lambda) - (\partial^2_{\zeta} \tilde{\chi})(\xi_d, (\text{Im } w) \xi_d, \lambda))
\]
\[
\times \prod_{j=1}^l (\partial^2_{\zeta^j} b)(\xi_d, (\text{Im } w) \xi_d, \lambda) d\xi_d,
\]
where $l \geq 1$ is an integer and $\alpha_j \in \mathbb{N}^{d-1}$ for $j = 0, ..., l$. We apply Lemma A.2 (i) if $l = 1$ and (ii) $l \geq 2$ and obtain
\[
(40) \quad |\partial^2_{\zeta} J_2(\zeta')| \leq C_{\alpha}' |z - w|
\]
with $C_{\alpha}' > 0$ independent of $x_d \in \mathbb{R}$, $\xi_d \in \mathbb{R}^{d-1}$ and $z, w \in I_\delta$ with $|z - w| \leq \delta$.

Next, we estimate $J_3$. $\partial^2_{\zeta} J_3$ is a linear combination of the form
\[
\int_{\mathbb{R}} e^{2\pi i (\text{Im } w) x \cdot \xi_d} (\partial^2_{\zeta} \tilde{\chi})(\xi_d, (\text{Im } w) \xi_d, \lambda - b(\xi', (\text{Im } w) \xi_d, \lambda))
\]
\[
\times \prod_{j=2}^{l_1 + l_2 + 1} (\partial^2_{\zeta^j} b)(\xi_d, (\text{Im } w) \xi_d, \lambda) d\xi_d,
\]
where $l_1, l_2 \geq 1$ are integers and $\alpha_j \in \mathbb{N}^{d-1}$ for $j = 0, ..., l_1 + l_2 + 1$. We apply Lemma A.2 (iii) and obtain
\[
(41) \quad |\partial^2_{\zeta} J_3(\zeta')| \leq C_{\alpha}' |z - w|
\]
with $C_{\alpha}' > 0$ independent of $x_d \in \mathbb{R}$, $\xi_d \in \mathbb{R}^{d-1}$ and $z, w \in I_\delta$ with $|z - w| \leq \varepsilon$.

Finally, we estimate $J_1$. Note that $|\partial^2_{\zeta} J_1(x_d)| \leq 2C_0$ by (i). Thus it suffices to prove that $|\partial^2_{\zeta} J_1(x_d)| \leq C_{\alpha}' |\text{Im } z - \text{Im } w|$. Since
\[
\frac{J_1(x_d)}{2\pi i} = \int_{\mathbb{R}} e^{2\pi i x \cdot \xi_d} \frac{\xi_d \tilde{\chi}(\xi_d, \lambda)}{\xi_d - i(\text{Im } w) b(\xi', \lambda)} \frac{d\xi_d}{\xi_d - i(\text{Im } w) b(\xi', \lambda)}
\]
\[
= \int_{\mathbb{R}} e^{2\pi i x \cdot \xi_d} \frac{\xi_d \tilde{\chi}(\xi_d, \lambda) b(\xi', \lambda)}{(\xi_d - i(\text{Im } w) b(\xi', \lambda)) d\xi_d}
\]
\[
= \int_{\mathbb{R}} e^{2\pi i (\text{Im } w) x \cdot \xi_d} \frac{\xi_d - i(\text{Im } z) b(\xi', \lambda)}{(\xi_d - i(\text{Im } w) b(\xi', \lambda)) d\xi_d}
\]
\[
\times \prod_{j=2}^{l_1 + l_2 + 1} (\partial^2_{\zeta^j} b)(\xi_d, \lambda) d\xi_d,
\]
where $l_1, l_2 \geq 1$ are integers, $\alpha_j \in \mathbb{N}^{d-1}$ for $j = 1, ..., l_1 + l_2 + 1$. Applying Lemma A.2 (i) and (ii) with
\[
\varphi(\xi_d) = (\text{Im } z)^{l_1} \frac{\xi_d \tilde{\chi}(\xi_d, \lambda) b(\xi', \lambda)}{(\xi_d - i(\text{Im } z) b(\xi', \lambda)) d\xi_d},
\]
\[
a(\xi_d) = b(\xi', \lambda), \quad l = l_2 \quad \text{and} \quad \mu = \text{Im } w, \quad \text{we have } |\partial^2_{\zeta} J_1(x_d)| \leq C_{\alpha}' |\text{Im } z - \text{Im } w|.
\]
This completes the proof.
APPENDIX B. COMPLEX ANALYSIS

We define \( \log^+ t = \log t \) if \( 1 \leq t \), \( \log^+ t = 0 \) if \( 0 < t \leq 1 \) and \( \log^- t = \log t - \log^+ t \).

**Lemma B.1.** Let \( f : \{ z \in \mathbb{C} \mid |z| \leq 1 \} \to \mathbb{C} \) be a continuous function which is holomorphic on \( \{ |z| < 1 \} \) and has no zero on \( \{ |z| < 1 \} \). Then \( f(e^{i\theta}) \neq 0 \) for almost everywhere \( \theta \in [-\pi, \pi) \).

**Proof.** We follow the argument of [25, Theorem 17.17]. By the mean value properties of the harmonic function, we have

\[
(42) \quad \log |f(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})| d\theta
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta - \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^- |f(re^{i\theta})| d\theta
\]

for \( 0 < r < 1 \). On the other hand, by using \( x \leq e^x \) for \( x \in \mathbb{R} \) and Jensen’s inequality, we have

\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta \leq \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta\right)
\]

\[
\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})| d\theta.
\]

By Fatou’s lemma and (42), we obtain \( \log |f(e^{i\theta})| \in L^1([-\pi, \pi]) \). In particular, \( \log |f(e^{i\theta})| < \infty \) for almost everywhere \( \theta \in [-\pi, \pi) \). Thus \( f(e^{i\theta}) \neq 0 \) for almost everywhere \( \theta \in [-\pi, \pi) \).

**Corollary B.2.** Let \( J = (a, b) \) be an open interval and \( r = (b - a)/2 \). Let \( f : \{ z \in \mathbb{C} \mid |z - (a + b)/2| \leq r, \pm \text{Im} z \geq 0 \} \to \mathbb{C} \) be a continuous function which is holomorphic and has no zero on \( \{ |z - (a + b)/2| < r, \text{Im} z > 0 \} \). Then \( f(\lambda) \neq 0 \) for almost everywhere \( \lambda \in J \).

**Proof.** For simplicity, we assume \( a = -1 \) and \( b = 1 \). Define \( \kappa_1 : D = \{ |z| < 1, \text{Im} z > 0 \} \to \{ \text{Im} z > 0 \} \) and \( \kappa_2 : \{ \text{Im} z > 0 \} \to \{ |z| < 1 \} \) by \( \kappa_1(z) = (1 + z)^2/(1 - z)^2 \) and \( \kappa_2(z) = (z - i)/(z + i) \). Then \( \kappa = \kappa_2 \circ \kappa_1 \) is biholomorphic from \( \{ |z| < 1, \text{Im} z > 0 \} \) to \( \{ |z| < 1 \} \) and homeomorphic from \( \{ |z| \leq 1, \text{Im} z \geq 0 \} \) to \( \{ |z| \leq 1 \} \). Moreover, since

\[
\kappa^{-1}(w) = \frac{\sqrt{1+w} - 1}{\sqrt{1+w} + 1}
\]

where we take a branch such that \( \text{Im} \sqrt{z} > 0 \), then \( \kappa^{-1}|_{|z|=1} : \{ |z| = 1 \} \to \bar{D} \setminus D \) is Hölder continuous. Thus \( \kappa^{-1}|_{|z|=1} \) maps sets of Lebesgue measure zero to sets of Lebesgue measure zero. By Lemma B.1, we obtain the desired result.

Next proposition is a variant of [17, Lemma 4.20]. See also [21, Proposition 4.6].

**Proposition B.3.** Let \( Z \) be a Banach space and fix a signature. For \( J \subset \mathbb{R} \) be an open set, we denote \( J_{\pm} = \{ z \in \mathbb{C} \mid \text{Re} z \in J, \pm \text{Im} z \geq 0 \} \). Let \( K : J_{\pm} \to B_{\infty}(Z) \) be continuous and holomorphic on \( \{ \pm \text{Im} z > 0 \} \). If \( I + K(z) \) has an inverse in \( B(Z) \)
for each \( z \in \{ \pm \text{Im } z > 0 \} \), then \( \Gamma_0 = \{ \lambda \in \mathbb{R} \mid I + K(\lambda) \text{ is not invertible} \} \) is a closed set with Lebesgue measure zero.

**Proof.** Since the set of all invertible operators in \( B(Z) \) is open and since \( K \) is continuous, then \( \Gamma_0 \) is closed. Thus it suffices to prove that the Lebesgue measure of \( \Gamma_0 \) is zero. Note that \( I + K(\lambda) \) is not invertible if and only if \(-1\) is in the spectrum of \( K(\lambda) \) for \( \lambda \in \Gamma_0 \). Fix \( \lambda \in \Gamma_0 \). Since \( K(\lambda) \) is compact, there exists a circle \( C_\lambda \) enclosing \(-1\) such that \( C_\lambda \) is contained in the resolvent set of \( K(\lambda) \). Since \( K \) is continuous, there exists \( r_\lambda > 0 \) such that \( C_\lambda \) is contained in the resolvent set of \( K(z) \) for \( z \in B_{r_\lambda}^\pm(\lambda) \) where \( B_{r_\lambda}^\pm(\lambda) = \{ z \in \mathbb{C} \mid \pm \text{Im } z \geq 0, |z - \lambda| < r_\lambda \} \). We define

\[
P_z = \frac{1}{2\pi i} \int_{C_\lambda} (w - K(z))^{-1}dw,
\]

then \( z \in B_{r_\lambda}^\pm(\lambda) \mapsto P_z \in B(Z) \) is analytic in \( B_{r_\lambda}^\pm(\lambda) \setminus \mathbb{R} \) and continuous in \( B_{r_\lambda}^\pm(\lambda) \). Note that \( n_0 = \dim \text{Ran } P_z < \infty \) is independent of \( z \in B_{r_\lambda}^\pm(\lambda) \). Set \( Z_z = \text{Ran } P_z \) and fix a linear isomorphism \( \Pi_\lambda : \mathbb{C}^{n_0} \rightarrow Z_\lambda \). We choose \( r_\lambda \) smaller such that \( I + P_\lambda(P_z - P_\lambda) \) has an inverse in \( B(Z_\lambda) \). Then \( \Theta_z = P_z|Z_\lambda : Z_\lambda \rightarrow Z_z \) is a linear isomorphism with its inverse

\[
(I + P_\lambda(P_z - P_\lambda))^{-1}P_\lambda : Z_z \rightarrow Z_\lambda.
\]

Now we set

\[
X(z) = \Pi_\lambda^{-1}\Theta_z^{-1}(I + K(z))\Theta_z \Pi_\lambda
\]

for \( z \in B_{r_\lambda}^\pm(\lambda) \). Then \( X \) is continuous on \( B_{r_\lambda}^\pm(\lambda) \) and analytic in \( B_{r_\lambda}^\pm(\lambda) \). Moreover, \( \det X(z) \) is also continuous on \( B_{r_\lambda}^\pm(\lambda) \) and analytic in \( B_{r_\lambda}^\pm(\lambda) \). We note that \( \det X(z) = 0 \) if and only if \(-1\) is in the spectrum of \( K(z) \). By Corollary B.2 and the compactness argument, we conclude that the Lebesgue measure of \( \Gamma_0 \) is zero.

\( \square \)

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