Ternary Codes Associated with $O^{-}(2n,q)$ and Power Moments of Kloosterman Sums with Square Arguments

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Abstract. In this paper, we construct three ternary linear codes associated with the orthogonal group $O^{-}(2,q)$ and the special orthogonal groups $SO^{-}(2,q)$ and $SO^{-}(4,q)$. Here $q$ is a power of three. Then we obtain recursive formulas for the power moments of Kloosterman sums with square arguments and for the even power moments of those in terms of the frequencies of weights in the codes. This is done via Pless power moment identity and by utilizing the explicit expressions of “Gauss sums” for the orthogonal and special orthogonal groups $O^{-}(2n,q)$ and $SO^{-}(2n,q)$.

Key words—ternary linear code, power moment, Kloosterman sum, square argument, Pless power moment identity, Gauss sum, orthogonal group, weight distribution.

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1. Introduction

Let $\psi$ be a nontrivial additive character of the finite field $\mathbb{F}_q$ with $q = p^r$ elements ($p$ a prime). Then the Kloosterman sum $K(\psi; a)$ ([11]) is defined by

$$K(\psi; a) = \sum_{\alpha \in \mathbb{F}_q^*} \psi(\alpha + a\alpha^{-1})(a \in \mathbb{F}_q^*).$$

For this, we have the Weil bound

$$|K(\psi; a)| \leq 2\sqrt{q}. \quad (1.1)$$

The Kloosterman sum was introduced in 1926 ([10]) to give an estimate for the Fourier coefficients of modular forms.

For each nonnegative integer $h$, by $MK(\psi)^h$ we will denote the $h$-th moment of the Kloosterman sum $K(\psi; a)$. Namely, it is given by

$$MK(\psi)^h = \sum_{a \in \mathbb{F}_q^*} K(\psi; a)^h.$$ 

If $\psi = \lambda$ is the canonical additive character of $\mathbb{F}_q$, then $MK(\lambda)^h$ will be simply denoted by $MK^h$.

Also, we introduce an incomplete power moments of Kloosterman sums. Namely, for every nonnegative integer $h$, and $\psi$ as before, we define

$$SK(\psi)^h = \sum_{a \in \mathbb{F}_q^*, \text{ a square}} K(\psi; a)^h, \quad (1.2)$$

which is called the $h$-th moment of Kloosterman sums with “square arguments”. If $\psi = \lambda$ is the canonical additive character of $\mathbb{F}_q$, then $SK(\lambda)^h$ will be denoted by $SK^h$, for brevity.

Explicit computations on power moments of Kloosterman sums were begun with the paper [16] of Salié in 1931, where he showed, for any odd prime $q$,

$$MK^h = q^2 M_{h-1} - (q - 1)^{h-1} + 2(-1)^{h-1} (h \geq 1).$$

Here $M_0 = 0$, and for $h \in \mathbb{Z}_{>0}$,

$$M_h = |\{(\alpha_1, \cdots, \alpha_h) \in (\mathbb{F}_q^*)^h | \sum_{j=1}^h \alpha_j = 1 = \sum_{j=1}^h \alpha_j^{-1}\}|.$$

For $q = p$ odd prime, Salié obtained $MK^1, MK^2, MK^3, MK^4$ in [16] by determining $M_1, M_2, M_3$. On the other hand, $MK^5$ can be expressed in terms of the $p$-th eigenvalue for a weight 3 newform on $\Gamma_0(15)$ (cf. [12], [15]). $MK^6$ can be expressed in terms of the $p$-th eigenvalue for a weight 4 newform on $\Gamma_0(6)$ (cf. [3]). Also, based on numerical evidence, in [1] Evans was led to propose a conjecture which expresses $MK^7$ in terms of Hecke eigenvalues for a weight 3 newform on $\Gamma_0(525)$ with quartic nebentypus of conductor 105.

Assume now that $q = 3^r$. Recently, Moisio was able to find explicit expressions of $MK^h$, for $h \leq 10$ (cf. [14]). This was done, via Pless power moment identity, by connecting moments of Kloosterman sums and the frequencies of weights in the ternary Melas code of length $q - 1$, which were known by the work of Geer, Schoof and Vlugt in [2]. In this paper, we will be able to produce three recursive formulas generating power moments of Kloosterman sums with square arguments over finite fields of characteristic three. To do that, we will construct three ternary linear codes $C(SO^- (2, q))$, $C(O^- (2, q))$, and $C(SO^- (4, q))$, respectively associated with $SO^- (2, q)$, $O^- (2, q)$, and $SO^- (4, q)$, and express those power moments in terms of the frequencies of weights in each code. Then, thanks to our previous results on the explicit expressions of “Gauss sums” for the orthogonal and special orthogonal groups $O^- (2n, q)$ and $SO^- (2n, q)$, we can express the weight of each codeword in the duals of the codes in terms of Kloosterman sums with square arguments. Then our formulas will follow immediately from the Pless power moment identity (cf. [5, 11]). Similar results of this paper were obtained in [9] for the case of finite symplectic groups over finite fields of characteristic three. Also, in the same case infinite families of recursive formulas were derived in [8] by using explicit expressions of exponential sums associated with certain double cosets.

Theorem 1.1 in the following (cf. [1, 5], [1, 6], [1, 8], [1, 12]) is the main result of this paper. Henceforth, we agree that, for nonnegative integers $a, b, c$,

\begin{equation}
\binom{c}{a, b} = \frac{c!}{a! b! (c - a - b)!}, \text{ if } a + b \leq c,
\end{equation}

and

\begin{equation}
\binom{c}{a, b} = 0, \text{ if } a + b > c.
\end{equation}
Theorem 1.1. Let \( q = 3^r \). Then we have the following.

(1) For \( h = 1, 2, \ldots \),

\[
SK^h = -\sum_{j=0}^{h-1} \binom{h}{j} (q+1)^{h-j} SK^j
\]

\[
+ q \sum_{j=0}^{\min\{N_1, h\}} (-1)^j C_{1,j} \sum_{t=j}^{h} t! S(h, t) 3^{h-t} 2^{t-h-j-1} (N_1 - j) \allowbreak \left(\frac{N_1 - t}{N_1 - j}\right),
\]

where \( N_1 = |SO^-(2, q)| = q + 1 \), and \( \{C_{1,j}\}_{j=0}^{N_1} \) is the weight distribution of the ternary linear code \( C(SO^-(2, q)) \) given by

\[
C_{1,j} = \sum \left( \frac{1}{\nu_1, \mu_1} \right) \left( \frac{1}{\nu_{-1}, \mu_{-1}} \right) \prod_{\beta = 1}^{nonsquare} \left( \frac{2}{\nu_{\beta}, \mu_{\beta}} \right) \text{ for } j = 0, \ldots, N_1.
\]

Here the sum is over all the sets of nonnegative integers \( \{\nu_\beta\}_{\beta = 1}^{nonsquare} \cup \{\nu_{\pm 1}\} \) and \( \{\mu_\beta\}_{\beta = 1}^{nonsquare} \cup \{\mu_{\pm 1}\} \) satisfying

\[
\nu_1 + \nu_{-1} + \sum_{\beta = 1}^{nonsquare} \nu_\beta + \mu_1 + \mu_{-1} + \sum_{\beta = 1}^{nonsquare} \mu_\beta = j
\]

and

\[
\nu_1 + \nu_{-1} (-1) + \sum_{\beta = 1}^{nonsquare} \nu_\beta \beta = \mu_1 + \mu_{-1} (-1) + \sum_{\beta = 1}^{nonsquare} \mu_\beta \beta.
\]

In addition, \( S(h, t) \) is the Stirling number of the second kind defined by

\[
S(h, t) = \frac{1}{h!} \sum_{j=0}^{t} (-1)^{t-j} \binom{t}{j} j^h.
\]

(2) For \( h = 1, 2, \ldots \),

\[
SK^h = -\sum_{j=0}^{h-1} \binom{h}{j} (q+1)^{h-j} SK^j
\]

\[
+ q \sum_{j=0}^{\min\{N_2, h\}} (-1)^j C_{2,j} \sum_{t=j}^{h} t! S(h, t) 3^{h-t} 2^{t-h-j-1} (N_2 - j) \allowbreak \left(\frac{N_2 - t}{N_2 - j}\right),
\]

where \( N_2 = |O^-(2, q)| = 2(q+1) \), and \( \{C_{2,j}\}_{j=0}^{N_2} \) is the weight distribution of the ternary linear code \( C(O^-(2, q)) \) given by:

(a) For \( r \) even,

\[
C_{2,j} = \sum \left( \frac{1}{\nu_0, \mu_0} \right) \left( \frac{1}{\nu_1, \mu_1} \right) \prod_{\beta = 1}^{nonsquare} \left( \frac{2}{\nu_{\beta}, \mu_{\beta}} \right) \text{ for } j = 0, \ldots, N_2.
\]
Here the sum is over all the sets of nonnegative integers \( \{ \nu_\beta \} \) \( \beta^{2-1} \) nonsquare \( \cup \{ \nu_{\pm 1} \} \) \( \cup \{ \nu_0 \} \) and \( \{ \mu_\beta \} \) \( \beta^{2-1} \) nonsquare \( \cup \{ \mu_{\pm 1} \} \) \( \cup \{ \mu_0 \} \) satisfying

\[
\nu_0 + \nu_1 + \nu_{-1} + \sum_{\beta^{2-1} \text{ nonsquare}} \nu_\beta \\
+ \mu_0 + \mu_1 + \mu_{-1} + \sum_{\beta^{2-1} \text{ nonsquare}} \mu_\beta = j
\]

and

\[
\nu_1 + \nu_{-1}(-1) + \sum_{\beta^{2-1} \text{ nonsquare}} \nu_\beta \beta \\
= \mu_1 + \mu_{-1}(-1) + \sum_{\beta^{2-1} \text{ nonsquare}} \mu_\beta \beta.
\]

(b) For \( r \) odd,

\[
C_{2,j} = \sum \left( \frac{1}{\nu_1, \mu_1} \right) \left( \frac{1}{\nu_{-1}, \mu_{-1}} \right) \prod_{\beta^{2-1} \neq -1 \text{ nonsquare}} \left( \frac{2}{\nu_\beta, \mu_\beta} \right) \left( q + 3 \right) \left( q_0, \mu_0 \right)
\]

\[(j = 0, \cdots, N_2).\]

Here the sum is over all the sets of nonnegative integers \( \{ \nu_\beta \} \) \( \beta^{2-1} \neq -1 \) nonsquare \( \cup \{ \nu_{\pm 1} \} \) \( \cup \{ \nu_0 \} \) and \( \{ \mu_\beta \} \) \( \beta^{2-1} \neq -1 \) nonsquare \( \cup \{ \mu_{\pm 1} \} \) \( \cup \{ \mu_0 \} \) satisfying

\[
\nu_0 + \nu_1 + \nu_{-1} + \sum_{\beta^{2-1} \neq -1 \text{ nonsquare}} \nu_\beta \\
+ \mu_0 + \mu_1 + \mu_{-1} + \sum_{\beta^{2-1} \neq -1 \text{ nonsquare}} \mu_\beta = j
\]

and

\[
\nu_1 + \nu_{-1}(-1) + \sum_{\beta^{2-1} \neq -1 \text{ nonsquare}} \nu_\beta \beta \\
= \mu_1 + \mu_{-1}(-1) + \sum_{\beta^{2-1} \neq -1 \text{ nonsquare}} \mu_\beta \beta.
\]

(3) For \( h = 1, 2, \cdots \),

\[
SK^{2h} = -h^{-1} \sum_{j=0}^{h-1} \binom{h}{j} (q^4 + q^3 - q - 1)^{h-j} SK^{2j}
\]

\[(1.11)
+ q^{1-2h} \sum_{j=0}^{\min(N_3, h)} (-1)^j C_{3,j} \sum_{t=j}^{h} \prod_{t=0}^{h} S(h, t) 3^{h - t} 2^{h - j - 1} \left( N_3 - j \right) \left( N_3 - t \right),
\]

where \( N_3 = |SO^{-}(4, q)| = q^2(q^3 - 1) \), and \( \{ C_{3,j} \}^{N_3}_{j=0} \) is the weight distribution of the ternary linear code \( C(SO^{-}(4, q)) \) given by
\[ C_{3,j} = \sum_{\nu_0, \mu_0} \left( -q^2 \delta(2, q; 0) + q^4 + 2q^3 - 3q^2 \right) \]

\[ \times \prod_{\beta \in \mathbb{F}_q^*} \left( -q^2 \delta(2, q; \beta) + q^5 + q^4 + q^3 - 3q^2 \right) \]

\[ (j = 0, \ldots, N_3). \]

Here the sum runs over all the sets of nonnegative integers \( \{\nu_\beta\}_{\beta \in \mathbb{F}_q} \) and \( \{\mu_\beta\}_{\beta \in \mathbb{F}_q} \) satisfying \( \sum_{\beta \in \mathbb{F}_q} \nu_\beta + \sum_{\beta \in \mathbb{F}_q} \mu_\beta = j \) and \( \sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = \sum_{\beta \in \mathbb{F}_q} \mu_\beta \beta \), and, for every \( \beta \in \mathbb{F}_q \), \( \delta(2, q; \beta) = |\{(\alpha_1, \alpha_2) \in (\mathbb{F}_q^*)^2 | \alpha_1 + \alpha_1^{-1} + \alpha_2 + \alpha_2^{-1} = \beta\}|. \)

2. \( O^{-}(2n, q) \)

For more details about this section, one is referred to the paper [4] and [5].

Throughout this paper, the following notations will be used:

\[ q = 3^r \ (r \in \mathbb{Z}_{>0}), \]

\[ \mathbb{F}_q = \text{the finite field with } q \text{ elements}, \]

\[ TrA = \text{the trace of } A \text{ for a square matrix } A, \]

\[ ^tB = \text{the transpose of } B \text{ for any matrix } B. \]

The orthogonal group \( O^{-}(2n, q) \) over the field \( \mathbb{F}_q \) is defined as:

\[ O^{-}(2n, q) = \{ w \in GL(2n, q) | ^twwJw = J \}, \]

where

\[ J = \begin{bmatrix}
0 & 1_{n-1} & 0 & 0 \\
1_{n-1} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -\epsilon
\end{bmatrix}, \]

and \( \epsilon \) is a fixed element in \( \mathbb{F}_q^* \setminus \mathbb{F}_q^{*2} \), here and throughout this paper.

For convenience, we put

\[ \delta_\epsilon = \begin{bmatrix}
1 & 0 \\
0 & -\epsilon
\end{bmatrix}. \]

Then \( O^{-}(2n, q) \) consists of all matrices

\[
\begin{bmatrix}
A & B & e \\
C & D & f \\
g & h & i
\end{bmatrix}
(A, B, C, D \ (n - 1) \times (n - 1), e, f \ (n - 1) \times 2, g, h \ 2 \times (n - 1), i \ 2 \times 2)
\]
\( t^t AC + t^t CA + g^t \delta g = 0, \)
\( t^t BD + t^t DB + t^t h \delta h = 0, \)
\( t^t ef + t^t fe + i^t \delta i = \delta, \)
\( t^t AD + t^t CB + g^t \delta h = 1_{n-1}, \)
\( t^t Af + t^t Ce + g^t \delta i = 0, \)
\( t^t Bf + t^t De + h^t \delta i = 0. \)

The special orthogonal group \( SO^{-}(2n, q) \) over the field \( \mathbb{F}_q \) is defined as:
\[
SO^{-}(2n, q) = \{ w \in O^{-}(2n, q) | \det w = 1 \},
\]
which is a subgroup of index 2 in \( O^{-}(2n, q) \).

In particular, we have
\[
O^{-}(2, q) = \{ i \in GL(2, q) | i^t \delta i = \delta \}
\]
(2.2) \( = SO^{-}(2, q) \Pi \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} SO^{-}(2, q), \)

with
\[
SO^{-}(2, q) = \left\{ \begin{bmatrix} a & bc \\ b & a \end{bmatrix} | a, b \in \mathbb{F}_q, \ a^2 - b^2 = 1 \right\} = \left\{ \begin{bmatrix} a & bc \\ b & a \end{bmatrix} | a + b \in \mathbb{F}_q(\epsilon) \text{ with } N_{\mathbb{F}_q(\epsilon)/\mathbb{F}_q}(a + b) = 1 \right\}.
\]

Let \( P(2n, q) \) be the maximal parabolic subgroup of \( O^{-}(2n, q) \) given by
\[
P = P(2n, q)
\]
\[
= \left\{ \begin{bmatrix} A & 0 & 0 \\ \text{t} A^{-1} & 0 & 0 \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} 1_{n-1} & B & -h \delta \epsilon \\ 0 & 1_{n-1} & 0 \\ h & 1_{2} \end{bmatrix} | A \in GL(n-1, q), \ i \in O^{-}(2n, q), \ t^t B + B + h \delta \epsilon h = 0 \right\},
\]

and let \( Q(2n, q) \) be the subgroup of \( P(2n, q) \) of index 2 defined by
\[
Q = Q(2n, q)
\]
\[
= \left\{ \begin{bmatrix} A & 0 & 0 \\ \text{t} A^{-1} & 0 & 0 \\ 0 & 0 & i \end{bmatrix} \begin{bmatrix} 1_{n-1} & B & -h \delta \epsilon \\ 0 & 1_{n-1} & 0 \\ h & 1_{2} \end{bmatrix} | A \in GL(n-1, q), \ i \in SO^{-}(2n, q), \ t^t B + B + h \delta \epsilon h = 0 \right\}.
\]

From (2.2), we see that
\[
P = Q \Pi \rho Q,
\]
with
\[
\rho = \begin{bmatrix} 1_{n-1} & 0 & 0 & 0 \\ 0 & 1_{n-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.
\]
The Bruhat decomposition of $O^{-}(2n, q)$ with respect to $P = P(2n, q)$ is given by

$$O^{-}(2n, q) = \prod_{r=0}^{n-1} P\sigma_r P = \prod_{r=0}^{n-1} P\sigma_r Q,$$

which can be modified to give

(2.4) $$O^{-}(2n, q) = \prod_{r=0}^{n-1} P\sigma_r (B_r \setminus Q),$$

with

$$B_r = B_r(q) = \{ w \in Q(2n, q) | \sigma_r w\sigma_{r-1}^{-1} \in P \}.$$  

Here $\sigma_r$ denotes the following matrix in $O^{-}(2n, q)$

$$\sigma_r = \begin{bmatrix} 0 & 0 & 1_r & 0 & 0 & 0 \\ 0 & 1_{n-1-r} & 0 & 0 & 0 & 0 \\ 1_r & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-1-r} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix} (0 \leq r \leq n-1).$$

One can also show that

$$|B_r \setminus Q| = \left[ \begin{array}{c} n-1 \\ r \end{array} \right]_q q^{r(r+3)/2} (0 \leq r \leq n-1) \quad (c.f. [4], (3.12), (3.21)),$$

where, for integers $n, r$ with $0 \leq r \leq n$, the $q$-binomial coefficients are defined as:

$$\left[ \begin{array}{c} n \\ r \end{array} \right]_q = \prod_{j=0}^{r-1} (q^{n-j} - 1)/(q^{r-j} - 1).$$

Taking the decomposition in (2.3) into consideration, we see that (2.4) can further be modified as

(2.5) $$O^{-}(2n, q) = \prod_{r=0}^{n-1} Q\sigma_r (B_r \setminus Q) \sqcup \prod_{r=0}^{n-1} (\rho Q)\sigma_r (B_r \setminus Q),$$

and

(2.6) $$SO^{-}(2n, q) = \prod_{0 \leq r \leq n-1 \atop r, even} Q\sigma_r (B_r \setminus Q) \sqcup \prod_{0 \leq r \leq n-1 \atop r, odd} (\rho Q)\sigma_r (B_r \setminus Q).$$

As is well-known or mentioned in [1], we have

(2.7) $$|O^{-}(2n, q)| = 2q^{n^2-n}(q^n + 1) \prod_{j=1}^{n-1} (q^{2j} - 1),$$

(2.8) $$|SO^{-}(2n, q)| = q^{n^2-n}(q^n + 1) \prod_{j=1}^{n-1} (q^{2j} - 1).$$

In particular, from (2.7) and (2.8), we have

(2.9) $$[O^{-}(2n, q) : SO^{-}(2n, q)] = 2, \quad |SO^{-}(2, q)| = q + 1.$$
3. Gauss sums for $O^-(2n, q)$

The following notations will be employed throughout this paper.

- $tr(x) = x + x^3 + \cdots + x^{3r-1}$ the trace function $\mathbb{F}_q \to \mathbb{F}_3$,
- $\lambda_0(x) = e^{2\pi ix/3}$ the canonical additive character of $\mathbb{F}_3$,
- $\lambda(x) = e^{2\pi itr(x)/3}$ the canonical additive character of $\mathbb{F}_q$.

Then any nontrivial additive character $\psi$ of $\mathbb{F}_q$ is given by $\psi(x) = \lambda(ax)$, for a unique $a \in \mathbb{F}_q^*$.

For any nontrivial additive character $\psi$ of $\mathbb{F}_q$ and $a \in \mathbb{F}_q^*$, the Kloosterman sum $K_{GL(t,q)}(\psi; a)$ for $GL(t, q)$ is defined as

$$K_{GL(t,q)}(\psi; a) = \sum_{w \in GL(t,q)} \psi(Trw + aTrw^{-1}).$$

Notice that, for $t = 1$, $K_{GL(1,q)}(\psi; a)$ denotes the Kloosterman sum $K(\psi; a)$.

In [6], it is shown that $K_{GL(t,q)}(\psi; a)$ satisfies the following recursive relation:

for integers $t \geq 2$, $a \in \mathbb{F}_q^*$,

$$K_{GL(t,q)}(\psi; a) = q^{t-1}K_{GL(t-1,q)}(\psi; a)K(\psi; a) + q^{2t-2}(q^{t-1} - 1)K_{GL(t-2,q)}(\psi; a),$$

where we understand that $K_{GL(0,q)}(\psi, a) = 1$.

Proposition 3.1. ([1]) Let $\psi$ be a nontrivial additive character of $\mathbb{F}_q$. For each positive integer $r$, let $\Omega_r$ be the set of all $r \times r$ nonsingular symmetric matrices over $\mathbb{F}_q$. Then, with $\delta_\epsilon$ as in (2.1), we have

$$b_r(\psi) = \sum_{B \in \Omega_r} \sum_{h \in \mathbb{F}_q^{r \times 2}} \psi(Tr\delta_\epsilon hBh)$$

$$= \begin{cases} q^{r(r+6)/4} \prod_{j=1}^{r/2} (q^{2j-1} - 1), & \text{for } r \text{ even,} \\ -q^{r^2+4r-1)/4} \prod_{j=1}^{(r+1)/2} (q^{2j-1} - 1), & \text{for } r \text{ odd.}
\end{cases}$$

Proposition 3.2. ([5]) Let $\psi$ be a nontrivial additive character of $\mathbb{F}_q$. Then

(1) $\sum_{w \in SO^-(2,q)} \psi(Trw) = -K(\psi; 1),$

(2) $\sum_{w \in SO^-(2,q)} \psi(Tr\delta_1 w) = q + 1,$

(3) $\sum_{w \in SO^-(2,q)} \psi(Trw) = -K(\lambda; 1) + q + 1$ (c.f. (2.2)),

where

$$\delta_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. $$
In Section 5 of [4], it is shown that the Gauss sum for $SO^{-}(2n, q)$ is given by:

$$
\sum_{w \in SO^{-}(2n, q)} \psi(Trw)
\begin{aligned}
= & \sum_{0 \leq r \leq n-1, \; r \text{ even}} |B_r \setminus Q| \sum_{w \in Q} \psi(Trw \sigma_r) + \sum_{0 \leq r \leq n-1, \; r \text{ odd}} |B_r \setminus Q| \sum_{w \in Q} \psi(Trpw \sigma_r) \quad \text{(cf. (2.6))}
\end{aligned}
$$

$$
= q^{(n-1)(n+2)/2} \left\{ \sum_{i \in SO^{-}(2, q)} \psi(Tr1) \sum_{0 \leq r \leq n-1, \; r \text{ even}} |B_r \setminus Q| q^{r(n-r)-3} b_r(\psi) K_{GL(n-1-r,q)}(\psi; 1)
\begin{aligned}
&+ \sum_{i \in SO^{-}(2, q)} \psi(Tr1) \sum_{0 \leq r \leq n-1, \; r \text{ odd}} |B_r \setminus Q| q^{r(n-r)-3} b_r(\psi) K_{GL(n-1-r,q)}(\psi; 1)
\end{aligned}
$$

$$
= -q^{(n-1)(n+2)/2} \left\{ K(\psi; 1) \sum_{0 \leq r \leq n-1, \; r \text{ even}} [n-1]_r q^{r(n-r)-3} \prod_{j=1}^{r/2} (q^{2j-1} - 1) K_{GL(n-1-r,q)}(\psi; 1)
\begin{aligned}
&+ (q + 1) \sum_{0 \leq r \leq n-1, \; r \text{ odd}} [n-1]_r q^{r(n-r)-3} \prod_{j=1}^{(r+1)/2} (q^{2j-1} - 1) K_{GL(n-1-r,q)}(\psi; 1)
\end{aligned}
$$

(3.2)

Also, from Section 6 of [4] the Gauss sum for $O^{-}(2n, q)$ is given by:

$$
\sum_{w \in O^{-}(2n, q)} \psi(Trw)
\begin{aligned}
= & \sum_{0 \leq r \leq n-1} |B_r \setminus Q| \sum_{w \in Q} \psi(Trw \sigma_r) + \sum_{0 \leq r \leq n-1} |B_r \setminus Q| \sum_{w \in Q} \psi(Trpw \sigma_r) \quad \text{(cf. (2.5))}
\end{aligned}
$$

$$
= q^{(n-1)(n+2)/2} (-K(\psi; 1) + q + 1) \sum_{0 \leq r \leq n-1} |B_r \setminus Q| q^{r(n-r)-3} b_r(\psi) K_{GL(n-1-r,q)}(\psi; 1)
$$

(3.3)

$$
= q^{(n-1)(n+2)/2} (-K(\psi; 1) + q + 1)
\begin{aligned}
&\times \left\{ \sum_{0 \leq r \leq n-1, \; r \text{ even}} [n-1]_r q^{r(n-r)-3} \prod_{j=1}^{r/2} (q^{2j-1} - 1) K_{GL(n-1-r,q)}(\psi; 1)
\begin{aligned}
&- \sum_{0 \leq r \leq n-1, \; r \text{ odd}} [n-1]_r q^{r(n-r)-3} \prod_{j=1}^{(r+1)/2} (q^{2j-1} - 1) K_{GL(n-1-r,q)}(\psi; 1)
\end{aligned}
\end{aligned}
$$

Note here that the results in Proposition 3.2 are incorporated in (3.2) and (3.3).

For our purposes, we only need the following three expressions of the Gauss sums for $SO^{-}(2, q)$, $O^{-}(2, q)$, and $SO^{-}(4, q)$. So we state them separately as a theorem which follows from (3.1) - (3.3) by a simple change of variables. Also, for the ease of notations, we introduce

$$
G_1(q) = SO^{-}(2, q), \quad G_2(q) = O^{-}(2, q), \quad G_3(q) = SO^{-}(4, q).
$$
Theorem 3.3. Let $\lambda$ be the canonical additive character of $\mathbb{F}_q$, and let $a \in \mathbb{F}_q^*$. Then we have

\[ \sum_{w \in G_1(q)} \lambda(aTrw) = -K(\lambda; a^2), \]

\[ \sum_{w \in G_2(q)} \lambda(aTrw) = -K(\lambda; a^2) + q + 1, \]

\[ \sum_{w \in G_3(q)} \lambda(aTrw) = -q^2(K(\lambda; a^2)^2 + q^3 - q). \]

Proposition 3.4. \cite[(5.3 - 5)]{7} Let $\lambda$ be the canonical additive character of $\mathbb{F}_q$, $m \in \mathbb{Z}_{\geq 0}$, $\beta \in \mathbb{F}_q$. Then

\[ \sum_{a \in \mathbb{F}_q^*} \lambda(-a\beta)K(\lambda; a^2)^m = q\delta(m, q; \beta) - (q - 1)^m, \]

where, for $m \geq 1$,

\[ \delta(m, q; \beta) = \left| \{(\alpha_1, \cdots, \alpha_m) \in (\mathbb{F}_q^*)^m | \alpha_1 + \alpha_1^{-1} + \cdots + \alpha_m + \alpha_m^{-1} = \beta \} \right|, \]

and

\[ \delta(0, q; \beta) = \begin{cases} 1, & \beta = 0, \\ 0, & \text{otherwise}. \end{cases} \]

Remark 3.5. Here one notes that

\[ \delta(1, q; \beta) = |\{x \in \mathbb{F}_q | x^2 - \beta x + 1 = 0\}| \]

\[ = \begin{cases} 2, & \text{if } \beta^2 - 1 \neq 0 \text{ is a square}, \\ 1, & \text{if } \beta^2 - 1 = 0, \\ 0, & \text{if } \beta^2 - 1 \text{ is a nonsquare}. \end{cases} \]

Let $G(q)$ be one of finite classical groups over $\mathbb{F}_q$. Then we put, for each $\beta \in \mathbb{F}_q$,

\[ N_{G(q)}(\beta) = |\{w \in G(q) | \text{Tr}(w) = \beta\}|. \]

Then it is easy to see that

\[ qN_{G(q)}(\beta) = |G(q)| + \sum_{a \in \mathbb{F}_q^*} \lambda(-a\beta) \sum_{w \in G(q)} \lambda(aTrw). \]

For brevity, we write

\[ n_1(\beta) = N_{G_1(q)}(\beta), \quad n_2(\beta) = N_{G_2(q)}(\beta), \quad n_3(\beta) = N_{G_3(q)}(\beta). \]

Using \eqref{3.11}, \eqref{4.1}, \eqref{3.4}, \eqref{3.6}, one derives the following.

Proposition 3.6. Let $n_1(\beta)$, $n_2(\beta)$, $n_3(\beta)$ be as in \eqref{3.12}. Then we have the following.

(1)

\[ n_1(\beta) = 2 - \delta(1, q; \beta) = \begin{cases} 0, & \text{if } \beta^2 - 1 \neq 0 \text{ is a square}, \\ 1, & \text{if } \beta^2 - 1 = 0, \\ 2, & \text{if } \beta^2 - 1 \text{ is a nonsquare}. \end{cases} \]
Here $\delta$ groups $G_m$ from (3.5), (3.7), (3.11) and (4.1), we have

Proof. Here we only provide the proof for (2). The others can be shown in a similar manner. From (3.17), (3.1), (3.11) and (4.1), we have

$$q n_2(\beta) = -q \delta (1, q; \beta) + 3q + 1 + (q + 1) \sum_{\lambda \in \mathbb{F}_g} \lambda (-a \beta),$$

and hence

$$n_2(\beta) = \begin{cases} -\delta (1, q; \beta) + 2, & \text{if } \beta \neq 0, \\ -\delta (1, q; 0) + q + 3, & \text{if } \beta = 0. \end{cases}$$

Noting now that, for $\beta = 0$, $\beta^2 - 1 = -1$ is a square in $\mathbb{F}_q = \mathbb{F}_3$, if and only if $r$ is even, the results now follow from Remark 3.5. \hfill \Box

4. Construction of codes

Let

$$N_1 = |G_1(q)| = q + 1, \quad N_2 = |G_2(q)| = 2(q + 1), \quad N_3 = |G_3(q)| = q^2(q^4 - 1).$$

Here we will construct three ternary linear codes $C(G_1(q))$ of length $N_1$, $C(G_2(q))$ of length $N_2$, and $C(G_3(q))$ of length $N_3$, respectively associated with the orthogonal groups $G_1(q)$, $G_2(q)$, and $G_3(q)$.

By abuse of notations, for $i = 1, 2, 3$, let $g_1, g_2, \cdots, g_N$, be a fixed ordering of the elements in the group $G_i(q)$.

Also, for $i = 1, 2, 3$, we put

$$v_i = (\text{Tr}g_1, \text{Tr}g_2, \cdots, \text{Tr}g_N) \in \mathbb{F}_q^{N_i}.$$

Then, for $i = 1, 2, 3$, the ternary linear code $C(G_i(q))$ is defined as

$$C(G_i(q)) = \{ u \in \mathbb{F}_2^{N_i} | u \cdot v_i = 0 \},$$

where the dot denotes the usual inner product in $\mathbb{F}_q^{N_i}$.

The following Delsarte's theorem is well-known.
Theorem 4.1. ([13]) Let $B$ be a linear code over $\mathbb{F}_q$. Then
\[(B|_{\mathbb{F}_3})^\perp = \text{tr}(B^\perp).\]

In view of this theorem, the dual $C(G_i(q))^\perp$ $(i = 1, 2, 3)$ is given by
\[(4.3) \quad C(G_i(q))^\perp = \{c_i(a) = (\text{tr}(a T r g_1), \cdots, \text{tr}(a T r g_N)) | a \in \mathbb{F}_q\}.

Lemma 4.2. Let $\delta(1, q; \beta)$ be as in (3.8) and (3.10), and let $a \in \mathbb{F}_q^*$. Then we have
\[(4.4) \quad \sum_{\beta \in \mathbb{F}_q} \delta(1, q; \beta) \lambda(a \beta) = K(\lambda; a^2).

Proof. The LHS of (4.4) is equal to
\[
\sum_{\beta \in \mathbb{F}_q} (q^{-1} \sum_{x \in \mathbb{F}_q} \sum_{\alpha \in \mathbb{F}_q} \lambda(\alpha(x^2 - \beta x + 1)) \lambda(a \beta))
= q^{-1} \sum_{x \in \mathbb{F}_q} \sum_{\alpha \in \mathbb{F}_q} \lambda(\alpha(x^2 + 1)) \sum_{\beta \in \mathbb{F}_q} \lambda(\beta(a - \alpha x))
= \sum_{\alpha \in \mathbb{F}_q^*} \lambda(\alpha(a^{-2}a^2 + 1))
= K(\lambda; a^2).
\]

Proposition 4.3. The map $\mathbb{F}_q \to C(G_i(q))^\perp$ $(a \mapsto c_i(a))$ is an $\mathbb{F}_3$-linear isomorphism, for every $q = 3^r$, and $i = 1, 2, 3$.

Proof. As the proofs for $i = 1$ and $i = 2$ are similar, we will treat only the cases $i = 2$ and $i = 3$. Let $i = 2$. The map is clearly $\mathbb{F}_3$-linear and surjective. Let $a$ be in the kernel of the map. Then, $\text{tr}(a T r g) = 0$, for all $g \in G_2(q)$. Assume that $a \neq 0$. Then
\[2(q + 1) = |G_2(q)| = \sum_{g \in G_2(q)} e^{2\pi i \text{tr}(a T r g)/3}
= \sum_{\beta \in \mathbb{F}_q} n_2(\beta) \lambda(a \beta)
= q + 1 + \sum_{\beta \in \mathbb{F}_q} (-\delta(1, q; \beta) + 2) \lambda(a \beta) \quad (\text{from } (3.17))
= q + 1 - \sum_{\beta \in \mathbb{F}_q} \delta(1, q; \beta) \lambda(a \beta)
= q + 1 - K(\lambda; a^2). \quad (\text{by } (4.4))
\]
So $q + 1 = -K(\lambda; a^2)$. This yields from Weil bound \[11\] that $q + 1 \leq 2 \sqrt{q}$, equivalently $q = 1$. As $q = 3^r \geq 3$, we must have $a = 0$. Now, let $i = 3$. Then again the map is $\mathbb{F}_3$-linear and surjective. Let $a$ be in the kernel of the map. Then $\text{tr}(a T r g) = 0$, for all $g \in G_3(q)$. Since $n_3(\beta) > 0$, for all $\beta \in \mathbb{F}_q^*$ (cf. (3.16)), this in turn yields that $\text{tr}(a \beta) = 0$. As the trace function $\mathbb{F}_q \to \mathbb{F}_3$ is surjective, we must have $a = 0$. \[\square\]
5. Recursive formulas for power moments of Kloosterman sums with square arguments

In this section, we will be able to find, via Pless power moment identity, recursive formulas for the power moments of Kloosterman sums with square arguments and even power moments of those with square arguments in terms of the frequencies of weights in \( C(G_i(q)) \), for each \( i = 1, 2, 3 \).

**Theorem 5.1.** (Pless power moment identity, [13]) Let \( B \) be an \( q \)-ary \([n, k]\) code, and let \( B_i \) (resp. \( B_i^\perp \)) denote the number of codewords of weight \( i \) in \( B \) (resp. in \( B_i^\perp \)). Then, for \( h = 0, 1, 2, \cdots \),

\[
(5.1) \quad \sum_{j=0}^{n} j^h B_j = \sum_{j=0}^{\min\{n, h\}} (-1)^j B_j^\perp \sum_{t=0}^{h} t! S(h, t) q^{k-t}(q-1)^{t-j} \binom{n-j}{n-t},
\]

where \( S(h, t) \) is the Stirling number of the second kind defined in [1.7].

**Lemma 5.2.** Let \( c_i(a) = (\text{tr}(aTr_{g_1}), \cdots, \text{tr}(aTr_{g_{N_i}})) \in C(G_i(q))^\perp \), for \( a \in \mathbb{F}_q^3 \), and \( i = 1, 2, 3 \). Then the Hamming weight \( w(c_i(a)) \) can be expressed as follows:

\[
(1) \quad w(c_i(a)) = \frac{2}{3}(q+1 + K(\lambda; a^2)), \quad \text{for } i = 1, 2,
\]

\[
(2) \quad w(c_3(a)) = \frac{2}{3}q^2(K(\lambda; a^2)^2 + q^4 + q^3 - q - 1).
\]

**Proof.** For \( i = 1, 2, 3 \),

\[
\begin{align*}
\quad w(c_i(a)) &= \sum_{j=1}^{N_i} (1 - \frac{1}{3} \sum_{\alpha \in \mathbb{F}_3} \lambda_0(\alpha \text{tr}(aTr_{g_j}))) \\
&= N_i - \frac{1}{3} \sum_{\alpha \in \mathbb{F}_3} \sum_{w \in G_i(q)} \lambda(\alpha \text{tr}w) \\
&= \frac{2}{3}N_i - \frac{1}{3} \sum_{\alpha \in \mathbb{F}_3} \sum_{w \in G_i(q)} \lambda(\alpha \text{tr}w).
\end{align*}
\]

Our results now follow from (4.11) and (3.14)- (8.10). \( \square \)

Fix \( i(i = 1, 2, 3) \), and let \( u = (u_1, \cdots, u_{N_i}) \in \mathbb{F}_3^{N_i} \), with \( \nu_\beta \) 1’s and \( \mu_\beta \) 2’s in the coordinate places where \( \text{Tr}(g_j) = \beta \), for each \( \beta \in \mathbb{F}_q \). Then we see from the definition of the code \( C(G_i(q)) \) (cf. (1.2)) that \( u \) is a codeword with weight \( j \) if and only if \( \sum_{\beta \in \mathbb{F}_q} \nu_\beta + \sum_{\beta \in \mathbb{F}_q} \mu_\beta = j \) and \( \sum_{\beta \in \mathbb{F}_q} \nu_\beta \beta = \sum_{\beta \in \mathbb{F}_q} \mu_\beta \beta \) (an identity in \( \mathbb{F}_q \)). Note that there are \( \prod_{\beta \in \mathbb{F}_q} \binom{n_i(\beta)}{\nu_\beta, \mu_\beta} \) (cf. (1.3), (1.4)) many such codewords with weight \( j \). Now, we get the following formulas in (5.4)-(5.7), by using the explicit values of \( n_i(\beta) \) in (3.13), (3.16).

**Theorem 5.3.** Let \( \{C_{i,j}\}_{j=0}^{N_i} \) be the weight distribution of \( C(G_i(q)) \), for \( i = 1, 2, 3 \). Then we have the following.
\[ C_{1,j} = \sum_{\nu, \mu} \left( \frac{1}{1} \right) \left( \frac{1}{\nu_1, \mu_1} \right) \prod_{\beta=1}^{\text{nonsquare}} \left( \frac{2}{\nu_\beta, \mu_\beta} \right) (j = 0, \ldots, N_1), \]

where the sum is over all the sets of nonnegative integers \( \{\nu\} \cup \{\nu\} \cup \{\mu\} \) and \( \{\mu\} \cup \{\mu\} \) satisfying
\[
\nu_1 + \nu - 1 + \sum_{\beta=1}^{\text{nonsquare}} \nu_\beta + \mu_1 + \mu - 1 + \sum_{\beta=1}^{\text{nonsquare}} \mu_\beta = j
\]

and
\[
\nu_1 + \nu_1(-1) + \sum_{\beta=1}^{\text{nonsquare}} \nu_\beta = \mu_1 + \mu - 1 + \sum_{\beta=1}^{\text{nonsquare}} \mu_\beta.
\]

For \( r \) even,
\[ C_{2,j} = \sum_{\nu, \mu} \left( \frac{q+1}{\nu_0, \mu_0} \right) \left( \frac{1}{\nu_1, \mu_1} \right) \left( \frac{1}{\nu_1, \mu_1} \right) \prod_{\beta=1}^{\text{nonsquare}} \left( \frac{2}{\nu_\beta, \mu_\beta} \right) (j = 0, \ldots, N_1), \]

where the sum is over all the sets of nonnegative integers \( \{\nu\} \cup \{\nu\} \cup \{\mu\} \) and \( \{\mu\} \cup \{\mu\} \) satisfying
\[
\nu_0 + \nu_1 + \nu - 1 + \sum_{\beta=1}^{\text{nonsquare}} \nu_\beta
\]
\[ + \mu_0 + \mu_1 + \mu - 1 + \sum_{\beta=1}^{\text{nonsquare}} \mu_\beta = j
\]

and
\[
\nu_1 + \nu_1(-1) + \sum_{\beta=1}^{\text{nonsquare}} \nu_\beta = \mu_1 + \mu - 1 + \sum_{\beta=1}^{\text{nonsquare}} \mu_\beta.
\]

For \( r \) odd,
\[ C_{2,j} = \sum_{\nu, \mu} \left( \frac{1}{\nu_1, \mu_1} \right) \left( \frac{1}{\nu_1, \mu_1} \right) \prod_{\beta=1}^{\text{nonsquare}} \left( \frac{2}{\nu_\beta, \mu_\beta} \right) (j = 0, \ldots, N_2), \]

where the sum is over all the sets of nonnegative integers \( \{\nu\} \cup \{\nu\} \cup \{\mu\} \) and \( \{\mu\} \cup \{\mu\} \) satisfying
\[
\nu_0 + \nu_1 + \nu - 1 + \sum_{\beta=1}^{\text{nonsquare}} \nu_\beta
\]
\[ + \mu_0 + \mu_1 + \mu - 1 + \sum_{\beta=1}^{\text{nonsquare}} \mu_\beta = j
\]

and
\[
\nu_1 + \nu_1(-1) + \sum_{\beta=1}^{\text{nonsquare}} \nu_\beta = \mu_1 + \mu - 1 + \sum_{\beta=1}^{\text{nonsquare}} \mu_\beta.
\]
\[ C_{3,j} = \sum_{\nu_0, \mu_0} \left( -q^2 \delta(2, q; 0) + q^4 + 2q^3 - 3q^2 \right) \times \prod_{\beta \in \mathbb{F}_q^*} \left( -q^2 \delta(2, q; \beta) + q^5 + q^4 + q^3 - 3q^2 \right) \quad (j = 0, \ldots, N_3), \]

where the sum runs over all the sets of nonnegative integers \( \{\nu_\beta\} \in \mathbb{F}_q^* \) and \( \{\mu_\beta\} \in \mathbb{F}_q^* \) satisfying

\[ \sum_{\beta \in \mathbb{F}_q^*} \nu_\beta + \sum_{\beta \in \mathbb{F}_q^*} \mu_\beta = j \]

and, for every \( \beta \in \mathbb{F}_q^* \),

\[ \delta(2, q; \beta) = \left| \{ (\alpha_1, \alpha_2) \in (\mathbb{F}_q^*)^2 | \alpha_1 + \alpha_1^{-1} + \alpha_2 + \alpha_2^{-1} = \beta \} \right|. \]

We now apply the Pless power moment identity in (5.1) to each \( C(G_i(q)) \) for \( i = 1, 2, 3 \), in order to obtain the results in Theorem 1.1 about recursive formulas.

Then the left hand side of the identity in (5.1) is equal to

\[ (5.8) \sum_{a \in \mathbb{F}_q^*} w(c_i(a))^h, \]

with the \( w(c_i(a)) \) in each case given by (5.2) and (5.3).

For \( i = 1, 2 \), (5.8) is

\[ \left( \frac{2}{3} \right)^h \sum_{a \in \mathbb{F}_q^*} (q + 1 + K(\lambda; a^2))^h \]

\[ = \left( \frac{2}{3} \right)^h \sum_{a \in \mathbb{F}_q^*} \sum_{j=0}^{h} \binom{h}{j} (q + 1)^{h-j} K(\lambda; a^2)^j \]

\[ = 2 \left( \frac{2}{3} \right)^h \sum_{j=0}^{h} \binom{h}{j} (q + 1)^{h-j} SK^j. \]

Similarly, for \( i = 3 \), (5.8) equals

\[ (5.10) 2 \left( \frac{2}{3} \right)^h \sum_{j=0}^{h} \binom{h}{j} (q^4 + q^3 - q - 1)^{h-j} SK^j. \]

Here one has to separate the term corresponding to \( j = h \) in (5.9) and (5.10), and note \( \dim_{\mathbb{F}_q} C(G_i(q)) = r \).

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