Interacting topological mirror excitonic insulator in one dimension

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We introduce the topological mirror excitonic insulator as a new type of interacting topological crystalline phase in one dimension. Its mirror-symmetry-protected topological properties are driven by exciton physics, and it manifests in the quantized bulk polarization and half-charge modes on the boundary. And the bosonization analysis is performed to demonstrate its robustness against strong correlation effects in one dimension. Besides, we also show that Rashba nanowires and Dirac semimetal nanowires could provide ideal experimental platforms to realize this new topological mirror excitonic insulating state. Its experimental consequences, such as quantized tunneling conductance in the tunneling measurement, are also discussed.

Introduction.– The conceptual revolution of topological physics in solids has changed our way of classifying phases of matters, which has made a great impact on the experimental discoveries of the topological insulators (TI) [15–18]. By definition, a time-reversal-invariant TI is featured by the nontrivial Z2 topological invariant, which are protected by the time-reversal symmetry [19–21]. This symmetry protection [13], only upon which the Z2 topological invariant is well-defined, distinguishes TI from the earlier examples such as the integer/fractional quantum Hall systems. And it leads to the concept of the symmetry-protected topological (SPT) state [22]. Beyond the time-reversal symmetry, the framework of symmetry-protected band topology has been successfully extended to the crystalline topological systems protected by space-group symmetries [23–28], magnetic-group symmetries [29–31] and space-time symmetries [32, 33]. Remarkable progresses have been made along this direction, which include the theoretical predictions and experimental discoveries of the crystalline topological phase in SnTe [34], Pb1−xSn2Se [35], KHgSb [36, 37], and recently in MnBi2nTe3n+1 [38–40].

Meanwhile, people have been showing great interests in exploring the fate of SPT phases under strong electron correlations [41–48]. For example, interaction effects could (i) fundamentally change the topology by breaking symmetries spontaneously [49–52], or alter the topological classification of SPT states [53–57], and (ii) enable new topological phases that can never be reached in the free fermion limit [58–70]. However, little is known about how to realize an interacting SPT state in experiments [70–77], especially for those systems with a crystalline-symmetry protection.

In this work, we introduce the topological mirror excitonic insulator (TMEI) as a new type of interacting crystalline topological state in one dimension (1D), and propose two different experimental realizations: the Rashba nanowires [78–80] and the Dirac semimetal nanowires [81–83], to achieve this novel topological state. By definition, the TMEI is featured by a quantized bulk charge polarization and a mirror-protected boundary half-charge mode when interaction-induced excitonic order is driven. To fully incorporate the interaction effects in 1D, we build an effective field theory description to the TMEI in the Rashba nanowire system by imposing of the Abelian bosonization technique, which shows the robustness of TMEI beyond the mean-field level. This allows us to map out the topological phase diagram and clarify the necessary condition for realizing the TMEI phase. Experimentally, we propose the quantized tunneling conductance as the smoking-gun signal for TMEI, which clearly distinguishes the TMEI phase from other states.

Model Hamiltonian.– The minimal system for 1D exciton physics consists of one electron band and one hole band [84]. Such two-channel systems can be realized in: (i) a double-wire setup with one n-type nanowire (electron doping) and a p-type nanowire (hole doping); (ii) a single quantum wire with two conducting channels that have opposite effective masses. In this Letter, we will firstly focus on the TMEI physics in the double-nanowire setup and later briefly discuss its realization in a single Dirac semimetal nanowire.

The double-nanowire system on an insulating substrate is illustrated in Fig. 1. We consider a k · p Hamiltonian [85] to describe the low-energy band structure of the double Rashba nanowires under an in-plane magnetic field, \( H_0 = \epsilon(k_z) + (m_0k_x^2 - \mu)\sigma_0\tau_z + vk_x\sigma_y\tau_0 + h\sigma_z\tau_0, \) (1) in which Pauli matrices \( \tau_z \) and \( \sigma_i \) denote the wire and spin degrees of freedom, respectively; \( \epsilon(k_z) = \delta\mu + \delta mk_x^2; m_0 \) is the inverse of the effective mass; \( \mu \) is the chemical potential; \( v \) characterizes the Rashba spin-orbit coupling; \( h \) represents the Zeeman splitting energy induced by an applied magnetic field \( B \). In the strained nanowires,
\[ \delta m \to 0 \] would be achieved \[ 86 \]. Here we assume \( m_0, v, h \) are all positive, and the existence of a large tunneling barrier between the wires that suppresses possible interwire tunneling events at the single-particle level. Nevertheless, multi-electron processes still exist as a result of inter-wire interactions. When \( B \) is aligned along the wires (\( \hat{x} \) direction), the Hamiltonian in Eq. (1) has a mirror symmetry \( M_x = i \sigma_x \cdot P \), where \( P \) maps \( x \) to \(-x\) \[ 87 \]. In fact, \( M_x \) is a lattice symmetry of the Rashba wires, which holds for both the lattice and continuum limits.

Excitons and Topology. – We first examine the band topology stemming from the exciton physics at the mean-field level. The excitonic orders are driven by the inter-wire interactions \[ 88–92 \]. They could introduce an energy gap to the double-wire system and are thus energetically favorable. Based on their representations of \( M_x \), there are two classes:

\[
\begin{align*}
\text{Mirror-even orders} : & \quad \Delta_{\text{even}} = \sigma_{0,x} \tau_{x,y}, \\
\text{Mirror-odd orders} : & \quad \Delta_{\text{odd}} = \sigma_{y,z} \tau_{x,y}.
\end{align*}
\]

Mirror-odd orders generally spoil \( M_x \) and cannot lead to nontrivial topology in 1D. With mirror-even orders, we can define a quantized bulk electric polarization \( \mathcal{P} \) \[ 89, 90 \] to characterize the bulk topology

\[
\mathcal{P} = \frac{i}{2 \pi} \sum_n \oint dk_x \langle u_n(k_x) | \partial_{k_x} | u_n(k_x) \rangle,
\]

where the \( | u_n(k_x) \rangle \) is the Bloch wave function with an occupied-band index \( n \). Crucially, \( M_x \) enforces \( \mathcal{P} \) to be quantized to half integers (i.e. 0 or \( \frac{1}{2} \) since \( \mathcal{P} \) is well-defined modulo 1). In particular, the TMEI phase is realized when \( \mathcal{P} = \frac{1}{2} \) [see Sec. (1) of Supplementary Material (SM)]. To drive the TMEI phase with \( \mathcal{P} = \frac{1}{2} \) into a trivial phase (or a vacuum state) with \( \mathcal{P} = 0 \), the system must undergo a bulk gap closing process, which manifests itself as a topological phase transition. Moreover, the condition for TMEI phase is \( h > \sqrt{\mu^2 + \Delta_0^2} \), where \( \Delta_0 \) denotes the amplitude of the mirror-even excitonic orders. Given the TMEI condition, we note that the electron and hole bands are “inverted” near the Fermi level to trigger the nontrivial band topology. This is clearly shown in Fig. 1 where we individually plot the bands for both electron and hole nanowires in the \( \Delta_0 \to 0 \) limit. With an open boundary condition, it is easy to check that the system hosts one localized half-charge mode at each end, similar to the case of the Su-Shrieffer-Heeger model.

Bosonization and Phase Diagram. – Next, we establish a Luttinger liquid theory to show that the TMEI physics remains robust when interaction effects are considered, which is beyond the mean-field theory. Since there are one pair of counter-propagating modes in each wire [see Fig. 1], we follow the standard mapping from the fermion fields \( \Psi_{i,s} \) to the chiral boson fields \( \phi_{i,s} \),

\[
\Psi_{i,s} = \frac{n}{\sqrt{4\pi}} e^{i\sqrt{4\pi} \chi_{i,s}},
\]

where \( a \) is the lattice constant and \( \eta_{i,s} \) are Klein factors \[ 96, 97 \]. Here the wire index \( i = \{e, h\} \) labels the \( n/p \)-type nanowire and \( s = \pm \) denotes the right/left-moving fermion modes.

It is convenient to define the dual boson fields as \( \phi_i = \chi_{i,R} + \chi_{i,L} \) and \( \theta_i = \chi_{i,R} - \chi_{i,L} \). Then the fermionic density operators are given by \( \rho_{i,s} = \frac{1}{\sqrt{2}} \phi_{i,s} \).

We first consider the intra-wire density-density interaction \( g_1 \rho_{i,L} \rho_{i,R} \) and the inter-wire density-density interaction \( g_2 (\rho_{e,L} \rho_{h,L} + \rho_{e,R} \rho_{h,R}) \), which are known to renormalize the Fermi velocities and Luttinger parameters. For simplicity, we assume that the bare Fermi velocities, the condition for TMEI phase is

\[
\left( \frac{\Delta_{\text{even}} / \Delta_0}{\mu / h} \right)^2 > 1.
\]

where

\[
\Delta_{\text{even}} = \frac{\mu^2 + \Delta_0^2}{\mu^2},
\]

and

\[
\Delta_0 = \frac{\mu^2 + \Delta_0^2}{\mu^2}.
\]

[83] Also consider that the two-body anharmonic interactions \[ 101 \], which lead

\[
\text{and momentum-conserved scattering processes that arise}
\]

\[
\text{from two-body anharmonic interactions \[ 101 \]}, \text{which lead}
\]

\[
\text{to the Hamiltonian} \quad \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}}, \text{where}
\]

\[
\mathcal{H}_{\text{int}} = \int dx \left[ \alpha_1 \cos(2\sqrt{2\pi} \phi_+) + \alpha_2 \cos(2\sqrt{2\pi} \phi_-) \right],
\]

where the first \( \phi_+ \)-mass term is valid when a pair of “inverted bands” are formed and the chemical potential is around the energy of band crossings; the second \( \phi_- \)-mass term describes the inter-wire pair hopping process. Notably, the \( \phi_- \)-mass term is absent in general

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**FIG. 1.** Sketch of two coupled nanowires on a substrate, one nanowire is n-type and another is p-type. An in-plane magnetic field is applied along the \( \hat{x} \) direction. The Rashba spin-orbit coupling is induced due to the breaking of 3d structural inversion symmetry. Therefore, there are only one left-moving \( (L) \) mode and one right-moving \( (R) \) mode in each nanowire at the chemical potential \( \mu \). To drive the TMEI phase with \( P \) realized when \( P = 1 \) [see Sec. (1) of Supplementary Material (SM)]. Then we include symmetry-allowed \( \phi \) and momentum-conserved scattering processes that arise from two-body anharmonic interactions [101], which lead to the Hamiltonian \( \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{\text{int}} \), where

\[
\mathcal{H}_{\text{int}} = \int dx \left[ \alpha_1 \cos(2\sqrt{2\pi} \phi_+) + \alpha_2 \cos(2\sqrt{2\pi} \phi_-) \right],
\]

where \( \phi_+ \)-mass term is valid when a pair of “inverted bands” are formed and the chemical potential is around the energy of band crossings; the second \( \phi_- \)-mass term describes the inter-wire pair hopping process. Notably, the \( \phi_- \)-mass term is absent in general
nanowires with only electron-like (or hole-like) bands. In particular, the relevance of the cosine terms can be evaluated by the renormalization group (RG) equation \(d\alpha_i/d\ln \lambda = [1 - \Delta(\alpha_i)]\alpha_i\), while the scaling dimensions of the coupling constants are,

\[
\Delta(\alpha_1) = K_+, \quad \Delta(\alpha_2) = \frac{1}{K_-}.
\]

Thus, under RG, we find that \(\alpha_1\) is relevant when \(K_+ < 1\), while \(\alpha_2\) is relevant for \(K_- > 1\). The phase diagram of Eq. (20) is determined by the Luttinger parameters \(K\) (or equivalently \(g_{1,2}\)).

We first notice that the system remains gapless when \(K_+ > 1\). In particular, as for \(K_- > 1\), inter-wire pair-hopping processes are greatly promoted. It gaps out the anti-bonding sector and further leads to number-conserving Majorana physics, which has been intensively discussed in the earlier literature [83, 102, 103]. This phase is denoted as “TSC” in the phase diagram in Fig. 2 (a). On the contrary, when \(K_- < 1\), both anti-bonding and bonding sectors remain gapless. This represents a gapless Luttinger liquid state (“LL\(_a\)” for short in Fig. 2 (a)). Similarly, a Luttinger liquid state “LL\(_b\)” arises for \(K_\pm < 1\) since only the bonding sector is trivially gapped. In this case, the anti-bonding sector could also be trivially gapped when the Umklapp scattering is relevant when \(k_F \to \pi/2\) [see Sec. (2) of SM].

More importantly, we will show that the TMEI is achieved when \(K_+ > 1\) and \(K_- < 1\). Because \(\Delta(\alpha_1) < 1\) and \(\Delta(\alpha_2) < 1\), those cosine terms with \(\alpha_{1,2}\) will flow to the strong coupling limit under RG. As a result, both \(\theta_-\) and \(\phi_+\) will be pinned to the semi-classical values. While the system becomes gapped, we find a set of non-vanishing order parameters,

\[
\Delta^{(1)}_{ex} \sim \langle \Psi^\dagger_{e,R} \Psi_{h,L} \rangle \sim e^{-i\sqrt{2}\pi\theta_+} e^{-i\sqrt{2}\pi\theta_-} \neq 0,
\]

\[
\Delta^{(2)}_{ex} \sim \langle \Psi^\dagger_{e,L} \Psi_{h,R} \rangle \sim e^{i\sqrt{2}\pi\phi_+} e^{-i\sqrt{2}\pi\theta_-} \neq 0,
\]

which imply an excitonic insulating phase. Then, let us analyze the condition for mirror-even excitonic orders. Under \(M_x\), the boson fields are transformed as:

\[
\sqrt{4\pi}\phi_\pm \to -\sqrt{4\pi}\phi_\pm + 2\beta, \quad \sqrt{4\pi}\theta_\pm \to \sqrt{4\pi}\theta_\pm + \pi, \quad \sqrt{4\pi}\phi_\pm \to -\sqrt{4\pi}\phi_\pm - 2\beta, \quad \sqrt{4\pi}\theta_\pm \to \sqrt{4\pi}\theta_\pm - \pi,
\]

where \(\beta\) is an unimportant phase factor. Consequently, the mirror symmetry sends \(\phi_+ \to -\phi_+\) and \(\theta_- \to \theta_- + \sqrt{\pi/2}\). For the excitonic order parameters, we immediately find that \(\Delta^{(1)}_{ex} = -\Delta^{(2)}_{ex}\) respects \(M_x\), while \(\Delta^{(1)}_{ex} = \Delta^{(2)}_{ex}\) does not. The relative phase difference between \(\Delta^{(1)}_{ex}\) and \(\Delta^{(2)}_{ex}\) is determined by the sign of the coupling coefficients \(\alpha_1\) and \(\alpha_2\). For example, when \(\alpha_{1,2} > 0\), the semi-classical limit is given by \(\theta_- = (n_\theta + \frac{1}{2})\sqrt{\pi/2}\) and \(\phi_+ = (n_\phi + \frac{1}{2})\sqrt{\pi/2}\). This leads to \(\Delta^{(1)}_{ex} = -\Delta^{(1)}_{ex} = i\) and thus respects \(M_x\) symmetry. In Fig. 2 (b), we have mapped out the phase diagram with respect to \(\alpha_1\) and \(\alpha_2\). Therefore, the system preserves the mirror symmetry \(M_x\) when \(\alpha_1 > 0\).

To clarify the topological nature of the above mirror-even excitonic insulating physics, it is instructive to map the excitonic orders in the low-energy basis back to the original fermion basis for Eq. (1). We find that the mirror-even case with \(\Delta^{(1)}_{ex} = -\Delta^{(2)}_{ex} = 1\) and \(\Delta^{(1)}_{ex} = -\Delta^{(2)}_{ex} = i\) are respectively equivalent to the mean-field excitonic orders \(\sigma_x\tau_y\) and \(\sigma_x\tau_x\) in the original basis [see Eq. (2)], which are already known for leading to the TMEI phase.

Furthermore, the topological properties of the TMEI could also be understood via the boson theory. With an open boundary condition, let us assume \(x > 0\) to be the vacuum with \(\phi_0 = 0\) and \(x < 0\) to be the TMEI phase with \(\phi_+ = (n_\phi + \frac{1}{2})\pi/\sqrt{2}\). The fractional charge bound to the system end is calculated to be

\[
q = e\sqrt{\frac{2}{\pi}} \int dx \partial_x \phi_+ = (n_\phi + \frac{1}{2})e.
\]

This half-quantized end charge thus confirms the bosonized theory with mirror-even excitonic orders as the TMEI phase.

On the other hand, we can also explore the Luther-Emery physics with \(K_+ = \frac{\pi}{2}\), where the boson system can be renormalized into a noninteracting fermion theory. We focus on the bonding sector and rescale the bonding bosonic fields as \(\phi_\pm/\sqrt{K_\pm} = \tilde{\phi}_\pm\) and \(\sqrt{K_\pm}\theta_\pm = \tilde{\theta}_\pm\). This allows us to introduce a set of new chiral fermion operators as \(\tilde{\psi}_e = \sqrt{4\pi\alpha_1}e^{-i\sqrt{\pi}(\tilde{\phi}_+ + \tilde{\theta}_+)}\) and \(\tilde{\psi}_L = \sqrt{4\pi\alpha_2}e^{i\sqrt{\pi}(\tilde{\phi}_- - \tilde{\theta}_-)}\), which leads to the following
the localized end states, as shown in (b) and (d). Specifically, enforcing a chiral symmetry to the system will lead to a zero-bias peak in (b). However, a general TMEI system that lacks the chiral symmetry would display a peak at finite voltage bias, as shown in (d). Because the evolving slightly from the free-fermion limit with $K_\pm = 1$ to the interacting case with $K_\pm \neq 1$ does not close the energy gap of the system. Therefore, we expect that our transport results in the mean-field limit will hold for an interacting TMEI phase.

**TMEI in a single Dirac Semimetal Nanowire.**—The TMEI phase can also be realized in a single nanowire of rotation-protected Dirac semimetal (e.g., a Cd$_3$As$_2$ nanowire with four-fold rotational symmetry $C_4$). As pointed out in Ref. [33], applying a magnetic field along the wire will naturally drive a 1D band inversion between an electron-like band with angular momentum $J = \frac{1}{2}$ and a hole-like band with $J = \frac{3}{2}$. As a result, any single-particle tunneling from the electron band to the hole band is naturally forbidden by the $C_4$ symmetry. In particular, with both $C_4$ and the spatial inversion $\hat{I}$ symmetry, the Dirac semimetal nanowire also possesses an out-of-plane mirror $M_z = C_2 \hat{I}$ that can protect the exciton-induced band topology in a corresponding nanowire geometry. Thus, without the complexity of aligning two quantum wires and careful band engineering in our double-wire setup, a single Dirac semimetal nanowire naturally fulfill all the symmetry and topological requisites for TMEI physics.

**Conclusions.**—To summarize, we propose a new type of interacting crystalline topological state, the TMEI phase, that can be realized in the Rashba nanowires and

\[
\mathcal{H}_+ = \int dx \; \tilde{\psi}^\dagger (-i\gamma_2 \partial_x + \alpha_1 \gamma_z) \tilde{\psi},
\]  

where $\tilde{\psi} = (\tilde{\psi}_R, \tilde{\psi}_L)^T$ is a spinor and $\gamma_{x,y,z}$ are Pauli matrices in the new chiral fermion basis. Crucially, Eq. (9) describes the low-energy theory of a massive Dirac fermion in 1D. Since the vacuum condition pins $\phi_\pm = 0$ and is equivalent to $\alpha_1 < 0$, the interface between the vacuum and the double wire forms a mass domain wall for the 1D Dirac fermion, which thus hosts a half-charge bound state. Therefore, Eq. (9) is exactly a fermionic model of TMEI phase in the strongly interacting limit, which is consistent with the above bosonic analysis.

**Experimental signature.**—We next propose quantized transport signals in a four-terminal device to distinguish a trivial phase from the TMEI phase. As shown in Fig. [1] the four metallic electrodes (labeled by $i = 1, 2, 3, 4$) are attached to the two-nanowire system. By applying a voltage drop and measuring the corresponding electric current, the conductance $G_{ij}$ between leads $i$ and $j$ can feasibly identified. The intra-wire two-terminal conductance $G_{1,3}$ (or equivalently $G_{2,4}$) measures the electron tunneling probability across the system. Thus, $G_{1,3}$ is expected to show a U-shape dip as a function of voltage bias because of the energy gap.

The non-local inter-wire conductance $G_{1,2}$ (or $G_{3,4}$) is the key to characterize TMEI physics. First, a non-zero $G_{1,2}$ has already implied the strong inter-wire correlation effects, which could further clarify the exciton nature of the energy gap measured in $G_{1,3}$. For a trivial system, we thus expect a similar conductance dip for $G_{1,2}$ due to the exciton gap. For the TMEI phase, however, its half-charge end mode will provide an additional resonant inter-wire conductance contribution to $G_{1,2}$

\[
\Delta G_{1,2} = \frac{e^2}{h} \sqrt{\Gamma^2 + (\omega - E_0)^2}.
\]  

Here, $E_0$ is the energy of the half-charge mode and $\Gamma \sim m_0^2/v_F$ is the transport broadening. Thus, when the half-charge mode is in-gap, $G_{1,2}$ will show a quantized conductance peak of $e^2/h$ as $\omega \to E_0$.

Numerical verifications of the above conductance patterns are performed using Kwant Python package [104]. As shown in Fig. [3] [(a), (c)] and [(b), (d)] show the conductance distributions as a function of voltage bias for the topological trivial and non-trivial phase, respectively. In particular, (a) and (b) has an accidental chiral symmetry in their models, while (c) and (d) do not. Clearly, the conductance patterns behave exactly the same as what we predict above. While every conductance displays a finite gap for the trivial system in (a) and (c), the $G_{1,2}$ and $G_{3,4}$ for the TMEI phase show a quantized conductance peak when the energy of lead electrons matches that of the localized end states, as shown in (b) and (d).
Dirac semimetal nanowires. In particular, we have established a bosonized theory to show the robustness of TMEI phase beyond the mean-field approximation. This idea of exciton-induced crystalline topological states also has interesting higher-dimensional generalizations. For example, let us consider a bilayer two-dimensional systems with the top (bottom) layer contributing a electron (hole) band near the Fermi level. An out-of-plane mirror symmetry $M_y$ in this system can protect a TMEI phase with $|n_M|$ pairs of counter-propagating 1D edge modes, where $n_M \in \mathbb{Z}$ is the mirror Chern number for the system. On the other hand, when the bilayer system possesses in-plane mirror symmetry $M_x$ and $M_y$, it is also possible to realize a higher-order topological insulator with a quantized bulk quadruple moment and corner-localized charges. This is exactly an interacting and excitonic version of the electronic quadruple insulator in Ref. [105, 106]. Detailed discussions on these 2d and excitonic version of the electronic quadruple insulator will be left for future works.

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**Supplementary materials**

This Supplementary materials contain:

1. Mean-field analysis for topology of TMEI
2. Bosonization and Density-wave phase at \( k_F \rightarrow \pi/2 \)

(1) Mean-field analysis for topology of TMEI

Let us briefly discuss the interaction effects and analyze the inter-wire instabilities at the mean-field level. The non-interacting Hamiltonian reads,

\[
\mathcal{H}_0 = \epsilon(k_x) + (m_0 k_x^2 - \mu) \sigma_0 \tau_x + v k_x \sigma_y \tau_0 + h \sigma_x \tau_0, \tag{11}
\]

As discussed in the main text, the excitonic orders can be added into the mean-field Hamiltonian. According to the mirror symmetry \( M_x \), the s-wave inter-wire order parameters \( \Delta = \sigma_{0,x,y,z} \tau_{x,y} \) that couple electrons and holes can be classified into two classes: (1) Mirror-even case: \( \Delta^{\text{even}}_x = \Delta^{\text{even}}_y \); (2) Mirror-odd case: \( \Delta^{\text{odd}}_x = -\Delta^{\text{odd}}_y \). It leads to \( \Delta^{\text{odd}}_x = \sigma_{y,z} \tau_{x,y} \). At \( p_x = 0 \), the eigenstates are given by,

\[
|h \pm \mu⟩ = [1, 1, 0, 0]^T / \sqrt{2} \quad \text{and} \quad m_x = +i,
\]

\[
|h - \mu⟩ = [0, 0, 1, 1]^T / \sqrt{2} \quad \text{and} \quad m_x = +i,
\]

\[
|−h + \mu⟩ = [1, −1, 0, 0]^T / \sqrt{2} \quad \text{and} \quad m_x = −i,
\]

\[
|−h - \mu⟩ = [0, 0, 1, −1]^T / \sqrt{2} \quad \text{and} \quad m_x = −i.
\]

where we assume \( m_0 > 0, v > 0, h > 0 \) and fix \( −h < \mu < h \). And \( m_x \) is the eigenvalue of mirror symmetry \( M_x \). Therefore, the mirror-even order parameters push the state \( |h - \mu⟩ \) to lower energy, at the same time rise the eigen-energy of the state \( |−h + \mu⟩ \). The closing and reopening of the bulk gap indicates there is a topological phase transition. As for the mirror-odd orders, they are trivial since they always increase the bulk gap at \( p_x = 0 \).

Therefore, the key problem is how to realize the mirror-even excitonic order parameters. So we define

\[
\Delta^{(1)}_{\sigma} \sim \langle \Psi_{e,R}^\dagger \Psi_{h,L} \rangle, \quad \Delta^{(2)}_{\sigma} \sim \langle \Psi_{e,L}^\dagger \Psi_{h,R} \rangle. \tag{13}
\]

Under the mirror symmetry \( M_x \), the fermionic modes are transferred as: \( \Psi_{e,L} \rightarrow i \alpha \Psi_{e,R}, \quad \Psi_{h,L} \rightarrow -i \alpha \Psi_{h,R}, \quad \Psi_{e,R} \rightarrow i \alpha \Psi_{e,L}, \quad \text{and} \quad \Psi_{h,R} \rightarrow -i \alpha \Psi_{h,L} \), where we define \( \alpha = (-h + i p_F) / \sqrt{h^2 + p_F^2} \) with \( |\alpha| = 1 \) and \( p_F > 0 \) the Fermi momentum. Thus, the mirror \( M_x \)-even(odd) order parameters correspond with \( \Delta^{(1)}_{\sigma} = -\Delta^{(2)}_{\sigma} \) (\( \Delta^{(1)}_{\sigma} = \Delta^{(2)}_{\sigma} \)). Moreover, the mirror-even case with \( \Delta^{(1)}_{\sigma} = -\Delta^{(2)}_{\sigma} = 1 \) corresponds to the \( \sigma_x \tau_y \) order parameter in the mean-field level. \( \Delta^{(1)}_{\sigma} = -\Delta^{(2)}_{\sigma} = i \) is for \( \sigma_x \tau_y \). Hence the TMEI is established in this system.

In addition, given by \( \delta \mu = \delta \tau = 0 \), there also exists an “accidental” chiral symmetry operation \( C = \sigma_z \tau_y \) (or \( \sigma_x \tau_z \)) that anti-commutes with \( H_0 \) as \( CH_0C^\dagger = -H_0 \).

The topological phase with respect to the mirror-even orders and its topological invariant is captured by the Zak phase, which is calculated and shown in Fig. [3]. The mirror-symmetry protected topological phase is what we discussed in the main text. As a comparison, the chiral symmetry protected topological phase is also considered, similar to the SSH model. However, since the chiral symmetry \( C \) is not often exact in our system, we emphasize that the mirror-enforced boundary half charge is more fundamental and robust than the \( C \)-protected boundary zero mode in our TMEI phase.

(2) Bosonization and Density-wave phase at \( k_F \rightarrow \pi/2 \)

The two-body density-density interactions can renormalize the Fermi velocities and Luttinger parameters. The intra-nanowire density-density interaction is,

\[
H_{\text{int}}^1 = g_1 \sum_{i \in \{e,h\}} \int dx \left( \rho_i \rho_{i,R} + H.c. \right),
\]

\[
= \frac{g_1}{4\pi} \sum_{i \in \{e,h\}} \int dx \left\{ (\partial_x \phi_i)^2 - (\partial_x \theta_i)^2 \right\}. \tag{14}
\]
Here the fermionic density operators are given by $\rho_{i,s} = \frac{1}{\sqrt{\pi}} \partial_x \chi_{i,s}$. The inter-wire density-density interaction is,

$$H^2_{\text{int}} = g_2 \int dx \left( \rho_e, R \rho_{h,L} + \rho_e, L \rho_{h,R} + \text{H.c.} \right),$$

$$= \frac{g_2}{\sqrt{\pi}} \int dx \left\{ (\partial_x \phi_e) (\partial_x \phi_h) - (\partial_x \theta_e) (\partial_x \theta_h) \right\}. \tag{15}$$

In terms of bonding and anti-bonding boson fields $\phi_\pm = (\phi_e \pm \phi_h)/\sqrt{2}$ and $\theta_\pm = (\theta_e \pm \theta_h)/\sqrt{2}$, we again get the renormalized free boson model,

$$\mathcal{H}_0 = \frac{v_\pm}{2} \int dx \left\{ \frac{1}{K_\pm} (\partial_x \phi_\pm)^2 + K_\pm (\partial_x \theta_\pm)^2 \right\}, \tag{16}$$

where the re-normalized Fermi velocities are given by $v_\pm = \sqrt{(v_F + (g_1 \pm g_2)/2\pi)(v_F - (g_1 \pm g_2)/2\pi)}$ and the Luttinger parameters are $K_\pm = \sqrt{(v_F - (g_1 \pm g_2)/2\pi)/(v_F + (g_1 \pm g_2)/2\pi)}$.

Next, the other three scattering processes that arise from two-body anharmonic interactions are,

$$H^5_{\text{int}} = \alpha_1 \int dx \left[ \Psi^\dagger, e, L \Psi^\dagger, e, R \Psi^\dagger, h, L \Psi, h, R + \text{H.c.} \right], \tag{17}$$

$$\sim \alpha_1 \int dx \cos \left( 2\sqrt{2\pi} \phi_+ \right).$$

$$H^3_{\text{int}} = \alpha_2 \int dx \left[ \Psi^\dagger, e, L \Psi, h, R \Psi^\dagger, e, R \Psi, h, L + \text{H.c.} \right], \tag{18}$$

$$\sim \alpha_2 \int dx \cos \left( 2\sqrt{2\pi} \theta_- \right).$$

$$H^5_{\text{int}} = \alpha_3 \int dx \left[ \Psi^\dagger, e, L \Psi^\dagger, e, R \Psi^\dagger, h, R \Psi, h, L + \text{H.c.} \right], \tag{19}$$

$$\sim \alpha_3 \int dx \cos \left( 2\sqrt{2\pi} \phi_- \right).$$

Therefore, the full boson Hamiltonian is,

$$\mathcal{H} = \frac{v_\pm}{2} \int dx \left\{ \frac{1}{K_\pm} (\partial_x \phi_\pm)^2 + K_\pm (\partial_x \theta_\pm)^2 \right\}$$

$$+ \left( \alpha_1 \cos \left( 2\sqrt{2\pi} \phi_+ \right) + \alpha_2 \cos \left( 2\sqrt{2\pi} \theta_- \right) \right. \tag{20}$$

$$\left. + \alpha_3 \cos \left( 2\sqrt{2\pi} \phi_- \right) \right\].$$

The scaling dimensions of the coupling constants $\alpha_1, \alpha_2, \alpha_3$ are,

$$\Delta(\alpha_1) = K_+, \quad \Delta(\alpha_2) = \frac{1}{K_-}, \quad \Delta(\alpha_3) = K_-.$$

from which we find there are two possible phases depending on the relationship between intra-nanowire density-density interaction $g_1$ and inter-wire density-density interaction $g_2$.

When $K_+ < 1$, we find $\Delta(\alpha_1) < 1, \Delta(\alpha_2) > 1$ and $\Delta(\alpha_3) < 1$. Namely, $\alpha_1$ and $\alpha_3$ are relevant under RG and tend to flow to the strong coupling limit. Consider the “semi-classical” limit with $\alpha_1, \alpha_3 \rightarrow +\infty$, $\phi_\pm$ are pinned to classical values with $\phi_\pm = (n_\pm + 1/2)\sqrt{\pi}/2$. Here $n_\pm \in \mathbb{Z}$ are integer-valued operators. As a result, the system develops an energy gap to all of its fermionic excitations. To understand the nature of this gapped phase, we find it possible to define the following density-wave (DW) order parameters $\left| \Psi_e, R \Psi_e, L \right> \sim e^{-\sqrt{2\pi} \phi_e} \neq 0$.

Therefore, we notice that DW orders $\Delta_{\text{dw}}^{(1,2)}$ develop non-zero expectation values directly implies the spontaneous breaking of the translational symmetry. Clearly, the mirror symmetry is spontaneously broken when $\phi_- \neq 0$. In this case, the DW phase is trivial in terms of the mirror-protected topology, which simply because it is defined by a pinned $\phi_-$. 