Gravitational scattering of two black holes at the fourth post-Newtonian approximation

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(Dated: June 22, 2017)

We compute the (center-of-mass frame) scattering angle $\chi$ of hyperbolic-like encounters of two spinning black holes, at the fourth post-Newtonian approximation level for orbital effects, and at the next-to-next-to-leading order for spin-dependent effects. We find it convenient to compute the gauge-invariant scattering angle (expressed as a function of energy, orbital angular momentum and spins) by using the Effective-One-Body formalism. The contribution to scattering associated with nonlocal, tail effects is computed by generalizing to the case of unbound motions the method of time-localization of the action introduced in the case of (small-eccentricity) bound motions by Damour, Jaranowski and Schäfer [Phys. Rev. D 91, no. 8, 084024 (2015)].

I. INTRODUCTION

The recent observation of gravitational wave signals emitted by the merger of binary black holes\textsuperscript{1,2} makes it very timely to further improve the analytical understanding of the gravitational interaction of compact binary systems in General Relativity (GR). Indeed, the construction\textsuperscript{3,4,5}, within the Effective-One-Body (EOB) formalism\textsuperscript{6–11} of a large bank of (semi-)analytical binary black-hole merger templates has been crucial in the search, significance assessment and parameter estimation of the merger signals.

The motion of comparable-mass binary systems has been tackled by several different approximation methods: i) post-Newtonian (PN) calculations, ii) post-Minkowskian (PM) calculations and iii) numerical relativistic (NR) simulations. In addition, the EOB formalism is a framework within which the results of all the above approximation methods can be usefully combined, thereby extending the realm of validity of the original methods. The traditional way in which the EOB formalism could incorporate both PN and NR results was based on the consideration of bound states (elliptic motions) of compact binaries\textsuperscript{6–9}.

Recently, a novel approach to the EOB description of binary systems was introduced based on the consideration of unbound, scattering states of black hole binaries\textsuperscript{12}. When considering non-spinning systems the key tool for this approach is the functional dependence of the center-of-mass frame scattering angle $\chi$ on the total energy $E$ and the total orbital angular momentum $L$ of the binary system (also considered in the center-of-mass frame). If we consider binary systems of spinning bodies (taken, for simplicity, with parallel spins $S_1$ and $S_2$) the scattering angle $\chi$ will depend on four variables:

\[ \chi = \chi(E, L, S_1, S_2) = \chi_{\text{orb}}(E, L) + \chi_{\text{spin}}(E, L, S_1, S_2). \]  

(1)

In addition, Ref.\textsuperscript{13} showed how to use (in the case of non-spinning bodies) the (orbital) gauge-invariant scattering function $\chi_{\text{orb}}(E, L)$, Eq. (1), to compare analytical results with numerical simulations of hyperbolic encounters of binary black-hole systems.

The (orbital) scattering function $\chi_{\text{orb}}(E, L)$ was analytically computed at the second post-Minkowskian approximation in Refs.\textsuperscript{14} based on the 2PN equations of motion derived in Refs.\textsuperscript{13,15}. Within the PN approximation method the scattering function is (when considering the conservative dynamics) formally expanded according to

\[ \chi(E, L, S_1, S_2) = \chi_{\text{orb}}(E, L) + \chi S_1(E, L)S_1 + \chi S_2(E, L)S_2 + O(\text{spin}^2), \]

(2)

where, as indicated, we will limit our investigation to the linear-in-spin contributions, and where

\[ \chi_{\text{orb}}(E, L) = \chi^N(E', L) + \frac{1}{c^4} \chi^{1PN}(E', L) + \frac{1}{c^6} \chi^{2PN}(E', L) + \frac{1}{c^8} \chi^{3PN}(E', L) + \ldots \]

(3)

where $E' \sim E - M c^2$ is a measure of the nonrelativistic energy content (see below for the definition of the exact energy variable $E' = \mu^2 E$ which we shall use) and where the leading-order, “Newtonian,” term, $\chi^N(E', L)$, is given by

\[ \chi^N(E', L) = 2 \arctan \left( \frac{G m_1 m_2}{L} \sqrt{\frac{\mu}{2E'}} \right). \]

(4)

Our notation here is

\[ M = m_1 + m_2, \quad \mu = \frac{m_1 m_2}{(m_1 + m_2)}, \quad \nu = \frac{\mu}{M}, \]

(5)

where $m_1$ and $m_2$ are the two masses of the components of the binary system. So far only the second post-Newtonian (2PN) approximation to $\chi(E, L)$ ($O(1/c^4)$), Eq. (3), has been explicitly derived\textsuperscript{15}.

The aim of the present paper is: (i) to extend the analytical knowledge of the orbital scattering function up...
to the 4PN $O(1/c^8)$ level; and (ii) to compute the linear-in-spin contribution to $\chi$ to next-to-next-to-leading PN order. Our calculation will be entirely based on the EOB formalism. In particular, we shall make a crucial use of the recent derivation [18, 19] of the 4PN conservative dynamics in Arnowitt-Deser-Misner (ADM) coordinates and on its transcription within the EOB formalism [20]. For earlier partial 4PN results see Refs. [21, 22]. See also [23, 27] for a discussion of the harmonic coordinates counterpart of the 4PN dynamics.

Subtle issues in the computation of the scattering function arise at the 4PN level, because of the nonlocal-in-time character of the two-body action. In the Hamiltonian formalism the conservative two-body dynamics is described by an action of the type

$$S(Q, P) = \int \left( \sum_i p_i Q_i - H_{(\text{tot})}[T; Q(\cdot), P(\cdot)] \right) dT,$$

where $Q$ and $P$ are phase variables describing the relative motion in the center-of-mass frame. It was found in Ref. [18] (and later confirmed in [22]) that the total 4PN-level Hamiltonian $H_{(\text{tot})}$ is the sum of a usual local-in-time Hamiltonian $H_{(\text{loc})}$ and of a nonlocal-in-time contribution $H_{\text{tail}}^{(\text{nonloc})}$, i.e.,

$$H_{(\text{tot})}[T; Q(\cdot), P(\cdot)] = H_{(\text{loc})}[T; Q(T), P(T)] + H_{\text{tail}}^{(\text{nonloc})}[T; Q(\cdot), P(\cdot)].$$

Here the notation $H_{(\text{loc})}(T; Q(T), P(T))$ represents a function of the phase space variables at time $T$, while the notation $H_{\text{tail}}^{(\text{nonloc})}[T; Q(\cdot), P(\cdot)]$ represents, for a given time $T$, a functional of the entire time development of the phase space variables $Q(T')$ and $P(T')$ for $T' \neq T$. More precisely the structure of $H_{\text{tail}}^{(\text{nonloc})}$, which is directly related to tail-transported effects [23], reads (see Eqs. (3.4) and (3.6) of Ref. [21])

$$H_{\text{tail}}^{(\text{nonloc})}[T; Q(\cdot), P(\cdot)] = \frac{G^2 M}{56 \pi^8} I_{ij}^{(3)}(T) \times \frac{\text{Pf}_{2s_{\text{phys}}/c}}{T - T'} I_{ij}^{(3)}(T').$$

Here $I_{ij}^{(3)}(T)$ denotes the third derivative of the quadrupole moment of the system at time $T$ (expressed in terms of the instantaneous values $Q(T)$ and $P(T)$) while $I_{ij}^{(3)}(T')$ denotes the corresponding quantity at a time $T'$ different from $T$ (expressed in terms of $Q(T')$ and $P(T')$). In addition, the logarithmically divergent integral of $T'$ is defined by means of a partie finie operation $\text{Pf}$ using as a regularization time scale the value $T_{\text{scale}} = 2s_{\text{phys}}/c$, where $s_{\text{phys}}$ is an arbitrary length scale. The length scale $s_{\text{phys}}$ enters $H_{(\text{loc})}$ as an infrared regularization scale and it enters $H_{\text{tail}}^{(\text{nonloc})}$ as an ultraviolet regularization scale; the dependence on $s_{\text{phys}}$ cancels between the two contributions.

In view of the dual structure of the action at the 4PN level, Eq. (9), leading to a corresponding dual structure of the scattering function,

$$\chi(E, L) = \chi_{(\text{loc})}(E, L) + \chi_{\text{tail}}(E, L),$$

our computation of the scattering function will be done by combining two separate calculations.

On the one hand, the PN-expansion of the first (local) contribution will be computed by standard Hamiltonian methods within the EOB transcription of the 4PN dynamics [20]

$$\chi_{(\text{loc})}(E, L) = \chi^{(N)}(E, L) + \frac{1}{c^2} \chi^{(1PN)}(E, L)$$

$$+ \frac{1}{c^6} \chi^{(2PN)}(E, L) + \frac{1}{c^8} \chi^{(3PN)}(E, L)$$

$$+ \frac{1}{c^{10}} \chi^{(4PN)}(E, L) + O\left(\frac{1}{c^{12}}\right).$$

On the other hand, the computation of the tail contribution to $\chi(E, L)$ which is entirely at the 4PN level

$$\chi_{\text{tail}}(E, L) = \frac{1}{c^8} \chi_{\text{tail}}^{(4PN)}(E, L),$$

will be performed by two independent calculations. One calculation will be based on a generalization of the “time-localization” method introduced in Ref. [20] and further explained in [21, 20]. Refs. [20, 20] showed, in the case of elliptic-like bound motions, that it was possible to replace the nonlocal-in-time Hamiltonian, Eq. (9a), by a physically equivalent local-in-time Hamiltonian, $H_{\text{tail}}^{(\text{nonloc})}(T; Q(T), P(T))$, obtained in the form of an infinite expansion in eccentricity or equivalently radial momentum. Here, we shall show, for the first time, how to generalize such a time-localization method to the case of hyperbolic-like, unbound motions.

As an independent confirmation of the value of $\chi_{\text{tail}}(E, L)$ obtained from our hyperbolic-type localization method, we will also compute $\chi_{\text{tail}}$ directly from the nonlocal-in-time force $\vec{F}_{\text{tail}}^{(\text{nonloc})}(T)$ associated with the nonlocal Hamiltonian [23] by using the corresponding evolution of the Laplace-Lagrange-Runge-Lenz vector. Let us mention in this respect that an analogous confirmation of the validity of the localization method of Ref. [20] has been obtained, for the case of elliptic-like, bound motions, in several recent works. Indeed, the first analytical, 4PN-level, determination of the periastron advance of small-eccentricity motions was obtained in Ref. [20], see Eqs. (8.1b) to (8.4) there, by using the standard EOB-derived expression [29] yielding periastron precession as function of the two EOB potentials $A(u, \nu)$ and $D(u, \nu)$. The value of the $\nu$-linear contribution (which crucially include the tail contributions) to the potential $\tilde{D}(u, \nu)$ was first confirmed by the analytical self-force results of Refs. [30, 31] (which used the first law of mechanics for eccentric orbits, [32]) as well as by the high-accuracy dynamical numerical self-force calculations of Ref. [33].
further analytical confirmation of the periastron precession result of Ref. [20] was recently obtained by a direct dynamical computation involving the nonlocal tail force \( \mathcal{F}^\text{tail}_{(\text{nonloc})}(T) \) in Ref. [27]. Finally, further confirmations of the reduced, time-localized action of [20] are contained in the recent work of Ref. [34].

II. NOTATION AND BRIEF EOB REMINDERS

Before proceeding with the computation of the scattering angle, let us shortly recall here the EOB description of the orbital motion of a two-body system. The total EOB Hamiltonian of the system is expressed as

\[
H(Q, P) = M c^2 \sqrt{1 + 2 \nu \left( \frac{H_{\text{eff}}}{\mu c^2} - 1 \right)} \equiv M c^2 h, \tag{12}
\]

where we used the notation of Eq. [5].

The “effective Hamiltonian” \( H_{\text{eff}} \) entering Eq. (12) is given by

\[
H_{\text{eff}} = c^2 \left[ A \left( \mu^2 c^2 + P^2 + \left( \frac{1}{B} - 1 \right) P^2_R + Q \right) + \frac{G}{c^2 R^3} (g_s L \cdot S + g_{s*} L \cdot S_*) \right]. \tag{13}
\]

Here \( L \) denotes the orbital angular momentum

\[
L = R \times P, \tag{14}
\]

(with magnitude \( L = P_\phi \)) while

\[
P^2 = P^2_R + \frac{L^2}{R^2}. \tag{15}
\]

The functions \( A(R) \) and \( B(R) \) entering \( H_{\text{eff}} \), Eq. (13), parametrize the effective metric

\[
ds_{\text{eff}}^2 = -A(R)c^2 dt^2_{\text{eff}} + B(R)dR^2 + R^2 d\phi^2, \tag{16}
\]

written in coordinates \((T_{\text{eff}}, R, \phi)\) and specialized here to equatorial motions. \( B \) is often replaced by another EOB function, \( \bar{D} \), defined so that \( A\bar{B}\bar{D} \equiv 1 \). In addition, \( Q \) in Eq. (13) represents a post-geodesic (Finsler-type) contribution which is at least quartic in momenta. Finally, \( S \) and \( S_\ast \) denote the following combinations of the two spin vectors

\[
S = S_1 + S_2; S_\ast = \frac{m_2}{m_1} S_1 + \frac{m_1}{m_2} S_2, \tag{17}
\]

while \( g_s \) and \( g_{s*} \) are some corresponding gyro-gravitomagnetic ratios which will be defined below. In the following, we shall work to linear order in the spins (so that we do not need to discuss the \( O(\text{spin}^2) \) contributions to the EOB Hamiltonian), and we shall only consider parallel (or anti-parallel) spins (i.e. nonprecessing binary systems).

It will be convenient to work with dimensionless rescaled independent variables, namely

\[
t = \frac{c^3 T}{GM}, \quad r = \frac{c^2 R}{GM} = \frac{1}{u}, \quad j = \frac{c P_\phi}{GM\mu}, \quad p_r = \frac{P_r}{\mu c}, \tag{18}
\]

and dimensionless rescaled quantities, e.g.,

\[
\hat{H}_{\text{eff}} = \frac{H_{\text{eff}}}{\mu c^2}, \quad \hat{Q} = \frac{Q}{\mu^2 c^2}. \tag{19}
\]

Note that we use the notation \( j \) (rather than the visually more delicate letter \( \ell \)) for the dimensionless orbital angular momentum. Hopefully, this should not introduce confusion with the usual notation \( J = L + S_1 + S_2 \) for the total conserved angular momentum of the system. [Anyway, in most of the text we shall consider nonspinning bodies, and separately add, to linear order, the effect of the spins.] Moreover, we will often use units such that \( G = 1 = c \), though we will sometimes put back the appropriate factors of \( G \) or \( c \) when it can illuminate the physical meaning of a final result.

Corresponding to the decomposition [7] of the Hamiltonian in local and nonlocal (or tail) parts, the EOB potentials \( A, B \) and \( Q \) admit similar decompositions, which we shall denote by the labels I (for the local piece) and II (for the nonlocal, tail, one), namely

\[
\begin{align*}
A(u; \nu) &= A^I(u; \nu) + A^{II}(u; \nu) , \\
D(u; \nu) &= D^I(u; \nu) + D^{II}(u; \nu) , \\
\bar{Q}(u; \nu) &= \bar{Q}^I(u; \nu) + \bar{Q}^{II}(u; \nu) ,
\end{align*} \tag{20}
\]

Ref. [20] has determined the values of the above potentials. In particular, the values of the local contributions (which are valid for arbitrary motions, including hyperbolic ones) are (see Eqs. (5.2), (5.3), (6.4) in [20])

\[
\begin{align*}
A^I(u; \nu) &= 1 - 2u + \nu a_1(u) + \nu^2 a_2(u) , \\
D^I(u; \nu) &= 1 + \nu \bar{d}_1(u) + \nu^2 \bar{d}_2(u) , \\
\bar{Q}^I(r; p_r; \nu) &= [2(4 - 3\nu)\nu^2 + (20\nu - 83\nu^2 + 10\nu^3)u^3]p_r^4 + \left( -\frac{9}{5} \nu + \frac{27}{5} \nu^2 + 6\nu^3 \right) u^2 p_r^6 ,
\end{align*} \tag{21}
\]

with

\[
\begin{align*}
a_1 &= 2u^3 + \left( \frac{94}{3} - \frac{41}{32} \pi^2 \right) u^4 + (a_{15c}^1 - a_{15n}^1 \ln(u))u^5 , \\
a_2 &= \left( \frac{41}{32} \pi^2 - \frac{221}{6} \right) u^5 , \\
\bar{d}_1 &= 6u^2 + 52u^3 + (\bar{d}_{4c}^1 - \bar{d}_{4n}^1 \ln(u))u^4 , \\
\bar{d}_2 &= -6u^3 + \left( \frac{123}{16} \pi^2 - 260 \right) u^4 .
\end{align*} \tag{22}
\]
and

\[
a_{5c}^{I} = \left( \frac{2275}{512} \pi^2 - \frac{4237}{60} - \frac{128}{5} \ln(s) \right) \nu + \left( \frac{41}{32} \pi^2 - \frac{221}{6} \right) \nu^2
\]

\[
a_{5ln}^{I} = \frac{128}{5} \nu
\]

\[
d_{4c}^{II} = \left( \frac{7243}{45} - \frac{23761}{1536} \pi^2 - \frac{1184}{15} \ln(s) \right) \nu + \frac{123}{16} \pi^2 - 260 \nu^2
\]

\[
d_{4ln}^{II} = \frac{1184}{15} \nu.
\]

Note that the scale \( s \) entering above via its logarithm \( \ln(s) \) is an adimensionalized version of the physical regularization length scale \( s_{\text{phys}} \) mentioned in the Introduction. Namely, \( s \equiv c^2 s_{\text{phys}}/(GM) \).

By contrast, the nonlocal (or tail) contributions to the EOB potentials determined in Eqs. (7.12) of \([20]\), namely

\[
A_{\text{II}}(u, \nu) = (a_{5c}^{I} - a_{5ln}^{I}) \ln(u) u^5
\]

\[
D_{\text{II}}(u, \nu) = (d_{4c}^{II} - d_{4ln}^{II}) \ln(u) u^4
\]

\[
\dot{Q}_{\text{II}}(u, p_r, \nu) = (q_{43c}^{II} - q_{43ln}^{II}) \ln(u) u^3 p_r^2 + q_{62c}^{II} - q_{62ln}^{II} \ln(u) u^2 p_r^2 + O(up_r^6).
\]

where

\[
a_{5c}^{II} = \frac{128}{5} (\gamma + 2 \ln 2 + \ln s) \nu
\]

\[
a_{5ln}^{II} = \frac{192}{5} \nu
\]

\[
d_{4c}^{II} = \left( \frac{845}{5} + \frac{1184}{15} \gamma - \frac{6496}{15} \ln 2 + \frac{2916}{5} \ln 3 + \frac{1184}{15} \ln s \right) \nu
\]

\[
d_{4ln}^{II} = \frac{592}{5} \nu
\]

\[
q_{43c}^{II} = \left( \frac{5608}{15} + \frac{496256}{45} \ln 2 - \frac{33048}{5} \ln 3 \right) \nu
\]

\[
a_{43ln}^{II} = 0
\]

\[
q_{62c}^{II} = \left( \frac{4108}{15} - \frac{2358912}{25} \ln 2 + \frac{1399437}{50} \ln 3 + \frac{390625}{18} \ln 5 \right) \nu
\]

\[
q_{62ln}^{II} = 0
\]

were derived in Ref. \([20]\) by means of an expansion in powers of the eccentricity, and can thereby only be used in the vicinity of circular motions. As we are here interested in motions that are very far from circular ones, we shall not be able to use the latter results, and will have to treat the nonlocal contribution to \( \chi \) by a different expansion (essentially an expansion in inverse powers of the eccentricity).

### III. CONTRIBUTION OF THE LOCAL CONSERVATIVE DYNAMICS TO THE SCATTERING FUNCTION OF NONSPINNING BODIES

As we work linearly in the spins (which only involve local-in-time interactions at the order we consider), and as it is enough to work linearly in the 4PN-level nonlocal tail effects, we can decompose the full scattering function as a sum of separate contributions, namely

\[
\chi(E, L, S, S_*) = \chi_{\text{loc}}(E, L) + \chi_{\text{tail}}(E, L) + \chi_{S}(E, L)S + \chi_{S*}(E, L)S_* + O(\text{spin}^2).
\]

In this section we compute the term \( \chi_{\text{loc}}(E, L) \) in Eq. \((26)\), i.e., the contribution to the scattering function of nonspinning bodies coming from the local-in-time part of the Hamiltonian. This calculation will be done by standard Hamiltonian methods, and will be conveniently performed within the EOB reformulation of the conservative dynamics, restricted here to equatorial motions.

#### A. EOB-derived integral expression for \( \chi_{\text{loc}}(E, L) \)

To compute the scattering angle we use the Hamilton-Jacobi approach. We checked our results by using the alternative method introduced in Ref. \([17]\). We sketch this alternative approach in Appendix \([C]\). The EOB action takes the separated form

\[
S(T_{\text{eff}}, R, \phi; \mathcal{E}_{\text{eff}}, P_{\phi}) = -\mathcal{E}_{\text{eff}} T_{\text{eff}} + P_{\phi} \phi + \int dR P_R(R, \mathcal{E}_{\text{eff}}, P_{\phi}).
\]

Here, \( T_{\text{eff}} \) is the coordinate time of the effective EOB metric, and \( \mathcal{E}_{\text{eff}} \) is the effective EOB energy, whose \( \mu \)-rescaled avatar \( \hat{\mathcal{E}}_{\text{eff}} \equiv \mathcal{E}_{\text{eff}}/(\mu c^2) \) is the energy variable which enters most naturally the EOB formalism. It is related to the total energy \( E \) of the system via

\[
E = Mc^2 \sqrt{1 + 2\nu (\hat{\mathcal{E}}_{\text{eff}} - 1)}
\]

\[
\hat{\mathcal{E}}_{\text{eff}} = \frac{E^2 - m_1^2 c^4 - m_2^2 c^4}{2m_1 m_2 c^4}.
\]

The equation for the orbit is then obtained from

\[
\frac{dS}{dP_{\phi}} = \phi_0 = \text{constant}.
\]

As stated above, it is generally convenient to work with the dimensionless rescaled variables, Eq. \([13]\), namely:
the dimensionless orbital angular momentum \( p_\phi = j \), the dimensionless rescaled radial momentum \( p_r = P_R/\mu c \), and the dimensionless gravitational potential\(^1 \) \( u \equiv 1/r \equiv GM/Rc^2 \). In terms of these, the orbital phase (from which we shall directly deduce the scattering angle \( \chi \)) is given by an integral of the form

\[
\phi - \phi_0 = - \int^u \left( \frac{\partial}{\partial j} p_r(u, \bar{E}, j) \right) \frac{du}{u^2}
\]

\[
eq \int^u U(u, \bar{E}, j) du,
\]

(30)

where

\[
U(u, \bar{E}, j) \equiv - \frac{1}{u^2} \frac{\partial}{\partial j} p_r(u, \bar{E}, j).
\]

(31)

Here, we have introduced a new (dimensionless) energy variable \( \bar{E} \) which is biunivocally related both to the total relativistic energy of the system \( E = Mc^2 + \cdots \), with usual energy dimension, and to the dimensionless relativistic effective energy \( \bar{E}_{\text{eff}} = 1 + O(1/c^2) \). The definition of \( \bar{E} \) is such that, in the non-relativistic limit \( c \to \infty \), it coincides (modulo a factor \( 1/c^2 \)) with the fractional binding energy \( (E - Mc^2)/(Mc^2) \). Namely, we define

\[
\bar{E} \equiv \frac{1}{2} \left( \bar{E}_{\text{eff}}^2 - 1 \right) \equiv \frac{1}{2} \bar{v}_\infty^2.
\]

(32)

In the latter equation we have introduced, in addition to the notation \( \bar{E} \), the related variable \( \bar{v}_\infty^2 = 2 \bar{E} \). [The notation \( \bar{v}_\infty^2 \) has been chosen because, in the case of unbound motions, \( \bar{E} > 0 \), the quantity \( v_\infty = \sqrt{2\bar{E}} \), indeed measures (in units of \( c \)) the relative velocity for infinite separations. In the case of unbound motions only \( v_\infty \) will enter our results.] Note that we have introduced so far, and will indirectly use below, several equivalent energy variables: \( E, \bar{E}_{\text{eff}}, \bar{E} \) and \( v_\infty \).

In order to get the explicit Hamilton-Jacobi integral form (30) of the orbital phase, we need to express \( p_r \) as a function of \( u = 1/r \), for given values of the energy and the angular momentum. This will follow by solving in \( p_r \) the EOB energy conservation law, \( H(r, p_r, j) = 0 \), or, equivalently, \( \bar{E}_{\text{eff}}^2 = \bar{H}_{\text{eff}}^2(r, p_r, j) \). The explicit form of the latter conservation law reads

\[
\bar{E}_{\text{eff}}^2 = A(u; \nu) \left[ 1 + j^2 u^2 + A(u; \nu) \bar{D}(u; \nu) p_r^2 + q_4(u; \nu) p_r^4 + q_6(u; \nu) p_r^6 \right].
\]

(33)

We shall see below how one can, in a PN-expanded sense, iteratively solve Eq. (33) for \( p_r^2 \) to get an explicit form of the function \( p_r(u, \bar{E}, j) \).

Armed with the knowledge of the function \( p_r(u, \bar{E}, j) \), and thereby of the function \( U(u, \bar{E}, j) \), Eq. (31), the scattering angle can then (in keeping with Eq. (5.65) in Ref. [17]) be expressed as the following definite integral

\[
\frac{1}{2} \left( \chi(\bar{E}, j) + \pi \right) = \int^u U(u, \bar{E}, j) du,
\]

(34)

where \( u_{(\text{max})} = u_{(\text{max})}(\bar{E}, j) = 1/r_{(\text{min})} \) corresponds to the distance of closest approach between the two bodies. Note in passing that Eq. (34) is closely similar to the Hamilton-Jacobi integral formula for the dimensionless periastron advance, namely (as in Eq. (5.35) of Ref. [17])

\[
K \equiv 1 + k = \frac{1}{\pi} \int^u U(u, \bar{E}, j) du,
\]

(35)

where \( u_{(\text{min})}(\bar{E}, j) \) and \( u_{(\text{max})}(\bar{E}, j) \) now respectively correspond to apoastron and periastron passages. The exact definitions of the functions \( u_{(\text{min})}(\bar{E}, j) \) and \( u_{(\text{max})}(\bar{E}, j) \) is that they are the two roots closest to zero of the EOB circular relation

\[
\bar{E}_{\text{eff}}^2 = A(u; \nu)(1 + j^2 u^2).
\]

(36)

Note that if we define the function

\[
V(u; j, \bar{E}) = \int^u U(u, \bar{E}, j) du,
\]

(37)

we can write the following compact expressions

\[
\frac{1}{2} \chi(\bar{E}, j) = V(u_{(\text{max})}; j, \bar{E}) - \frac{\pi}{2},
\]

\[
\pi K = V(u_{(\text{max})}; \bar{E}, j) - V(u_{(\text{min})}; \bar{E}, j).
\]

(38)

The first expression holds for \( \bar{E} > 0 \), while the second applies in the case \( \bar{E} < 0 \). Note that, when analytical continuing in \( \bar{E} \) the definitions (31) of the functions \( u_{(\text{min})}(\bar{E}, j) \) and \( u_{(\text{max})}(\bar{E}, j) \), one passes from a configuration where (when \( \bar{E} < 0 \)) \( 0 < u_{(\text{min})} < u_{(\text{max})} \) to a configuration where (when \( \bar{E} > 0 \)) \( u_{(\text{min})} < 0 < u_{(\text{max})} \).

This shows that while \( K \) has the nature of a complete hyperelliptic integral, \( \chi \) has the nature of an incomplete hyperelliptic integral.

B. PN-expanding \( \chi_{\text{loc}}(E, L; 1/c^2) \)

Let us now show how one can explicitly compute (as a PN-expansion) the function \( U(u, \bar{E}, j) \), Eq. (31), whose integral yields \( \chi_{\text{loc}}(E, L) \), Eq. (34). The first step is to compute the function \( p_r(u, \bar{E}, j) \). The latter function is obtained by iteratively solving (in successive powers of \( 1/c^2 \)) Eq. (33) for \( p_r^2 \). This yields

\[
p_r^2 = [p_r^2]^{(0)} + \eta^2 [p_r^2]^{(2)} + \eta^4 [p_r^2]^{(4)} + \eta^6 [p_r^2]^{(6)} + \eta^8 [p_r^2]^{(8)} + O(\eta^{10}).
\]

(39)
Here \( \eta \sim 1/c \) is a PN-order marker (to be taken to one at the end), and

\[
[p_u^2]^{(0)} = 2\bar{E} - j^2 u^2 + 2u
\]
\[
[p_u^2]^{(2)} = 8\bar{E}u + 8u^2 - 2j^2 u^3
\]
\[
[p_u^2]^{(4)} = (24 - 12\nu)\bar{E}u^2 + (24 - 14\nu)u^3 + (6\nu - 4)j^2 u^4
\]
\[
[p_u^2]^{(6)} = (-32 + 24\nu)\nu\bar{E}^2 u^2 + (64 + 60\nu^2 - 224\nu)\bar{E}u^3
+ \left[ (32 - 24\nu)\nu j^2 \bar{E} + 64 + \frac{41}{32}\pi^2 - \frac{694}{3} \right] \nu
+ 36\nu^2 u^4 + (-8 + 98\nu - 30\nu^2)j^2 u^5
+ (-8 + 6\nu)\nu j^4 u^6 ,
\]
with a similar expression for the 4PN term \([p_u^2]^{(8)}\), not displayed here for brevity.

A priori one also needs the PN-expansion\(^3\) of the functions \( u_{\text{min}}(\bar{E}, j; 1/c^2) \) and \( u_{\text{max}}(\bar{E}, j; 1/c^2) \) which define the boundaries of the integral expressions \((34)\) and \((35)\). These expansions have the form

\[
\begin{align*}
 u_{\text{min}}(\bar{E}, j; 1/c^2) &= \frac{1 - \sqrt{1 + 2j^2 \bar{E}}}{j^2} + O\left(\frac{1}{c^2}\right) , \\
 u_{\text{max}}(\bar{E}, j; 1/c^2) &= \frac{1 + \sqrt{1 + 2j^2 \bar{E}}}{j^2} + O\left(\frac{1}{c^2}\right) .
\end{align*}
\]

The original expression \((34)\) for \( \chi \) is given in terms of a convergent integral, having as upper limit the positive root closest to zero, \( u = u_{\text{max}}(\bar{E}, j; 1/c^2) \), of Eq. \((36)\). When PN-expanding the integral expression \((34)\) for \( \chi(\bar{E}, j; 1/c^2) \) one should a priori PN-expand both the integrand \( U(u, \bar{E}, j; 1/c^2) \), and the upper limit \( u_{\text{max}}(\bar{E}, j; 1/c^2) \) (as per Eq. \((35)\)). Such a formal PN expansion generates a sequence of divergent integrals on the interval \([0, u_{\text{max}}^N]\), where

\[
 u_{\text{max}}^N(\bar{E}, j) = 1 + \sqrt{1 + 2j^2 \bar{E}} ,
\]

(42)

with formally infinite contributions coming from the expansion of the upper limit. It was, however, shown in Ref. \[35\] that the correct value of the PN-expanded integral is recovered by: i) using as upper limit of the integral the Newtonian limit \( u_{\text{max}}^N(\bar{E}, j) \), Eq. \((12)\); ii) PN-expanding the integrand; iii) taking the Hadamard partie finie\(^4\) (PF) of the so-generated divergent integrals.

C. Explicit expression of the 4PN level value of \( \frac{1}{2}\chi_{\text{loc}}(\bar{E}, j; 1/c^2) \)

Most of the integrals generated by the PN-expansion technique explained in the previous subsection can be computed by standard techniques, except one of them whose integrand involves \( \ln u \), which will be separately discussed below. To simplify the final expressions it is convenient to introduce the auxiliary variable \( \alpha \), defined as

\[
\alpha \equiv \frac{1}{\sqrt{2j^2 \bar{E}}} = \frac{1}{v_{\text{loc}} j} ,
\]

(43)
as well as the function \( B(\alpha) \), defined as

\[
B(\alpha) \equiv \arctan \alpha + \frac{\pi}{2} .
\]

(44)

Our PN counting is such that \( 2\bar{E} = v_{\text{loc}}^2 = O(\frac{1}{c^2}) \), while \( j = O(c) \). In the Newtonian approximation \( (c \to \infty) \), the dimensionless variable \( \alpha \) has a finite limit that is linked to the (Newtonian-level) eccentricity, say \( e = \sqrt{1 + 2j^2 \bar{E}^2} = 1 + \frac{1}{e} \). As, by contrast, \( j \) grows proportionally to \( c \), the \( n \)-PN contribution to \( \chi \) must scale as \( \chi^{n\text{PN}} = f_n(\alpha)/j^n \).

The result of the calculation of \( \frac{1}{2}\chi_{\text{loc}} \), PN-decomposed as in Eq. \((10)\), is found to be the following

\[
\begin{align*}
\frac{1}{2}\chi^{(N)}(\bar{E}, j) &= \arctan \alpha = B(\alpha) - \frac{\pi}{2} , \\
\frac{1}{2}\chi^{(1\text{PN})}(\bar{E}, j) &= \frac{1}{j^6} \left[ 3B(\alpha) + \frac{(3\alpha^2 + 2)}{\alpha(1 + \alpha^2)} \right] , \\
\frac{1}{2}\chi^{(2\text{PN})}(\bar{E}, j) &= \frac{1}{j^6} \left[ C_B^{(2\text{PN})} B(\alpha) + C_0^{(2\text{PN})} \right] , \\
\frac{1}{2}\chi^{(3\text{PN})}(\bar{E}, j) &= \frac{1}{j^6} \left[ C_B^{(3\text{PN})} B(\alpha) + C_0^{(3\text{PN})} \right] , \\
\frac{1}{2}\chi^{(4\text{PN})}(\bar{E}, j) &= \frac{1}{j^8} \left[ C_B^{(4\text{PN})} B(\alpha) + C_0^{(4\text{PN})} \right] + I_\chi .
\end{align*}
\]

(45)

Here the \( \alpha \)-dependent coefficients entering \( \frac{1}{2}\chi^{(2\text{PN})}, \frac{1}{2}\chi^{(3\text{PN})} \) and \( \frac{1}{2}\chi^{(4\text{PN})} \) are
Finally, the last contribution and whose definition involves the choice of a regularization scale.

\[ C_{(3PN)}^B = \left[ \frac{1155}{4} + \frac{615}{128} \pi^2 - \frac{625}{2} \right] \nu + \frac{105}{8} \nu^2 + \frac{1}{\alpha^2} \left[ \frac{315}{4} + \frac{45}{4} \nu^2 + \left( \frac{123}{128} \pi^2 - 109 \right) \nu \right] + \frac{1}{\alpha^4} \left( - \frac{3}{2} \nu + \frac{9}{8} \nu^2 \right) \]

\[ C_{(3PN)}^0 = \frac{1}{(1 + \alpha^2)^2} \left\{ \alpha^5 \left[ \frac{1155}{4} + \frac{105}{8} \nu^2 + \left( \frac{615}{128} \pi^2 - \frac{625}{2} \right) \nu \right] + \alpha^3 \left[ \frac{3395}{4} + \frac{185}{4} \nu^2 + \left( - \frac{2827}{3} + \frac{1763}{128} \pi^2 \right) \nu \right] \\
+ \alpha \left[ \frac{3381}{4} + 60 \nu^2 + \left( \frac{1681}{128} \pi^2 - \frac{2939}{3} \right) \nu \right] + \frac{1}{\alpha} \left[ \frac{1221}{4} + \frac{135}{4} \nu^2 + \left( - \frac{1153}{3} + \frac{533}{128} \pi^2 \right) \nu \right] \right\} \]

(46)

and

\[ C_{(4PN)}^B = \frac{244}{\alpha^2} + \frac{1190}{3} + \frac{74}{5\alpha^2} \nu \ln s \\
+ \frac{225225}{64} - \frac{315}{16} \nu^3 + \left( \frac{7175}{256} \pi^2 + \frac{132475}{96} \right) \nu^2 + \left( - \frac{1720271}{288} + \frac{2975735}{24576} \pi^2 \right) \nu \]

\[ + \frac{1}{\alpha^2} \left[ \frac{45045}{32} - \frac{525}{16} \nu^3 + \left( \frac{35065}{32} - \frac{615}{32} \pi^2 \right) \nu^2 + \left( \frac{257195}{4096} - \frac{288805}{96} \pi^2 \right) \nu \right] \]

\[ + \frac{1}{\alpha^4} \left( \frac{9}{32} \nu - \frac{15}{16} \nu^3 + \frac{27}{32} \nu^2 \right) \]

\[ C_{(4PN)}^0 = \left( \frac{5758}{15\alpha} + \frac{5606}{45\alpha^2} + \frac{64}{5(\alpha^2 + 1) \alpha} \right) \nu \ln s \\
+ \frac{1}{\alpha} \left[ \frac{197785}{64} - \frac{315}{16} \nu^3 + \left( \frac{43185}{32} - \frac{7011}{256} \pi^2 \right) \nu^2 + \left( - \frac{7890919}{1440} + \frac{2731199}{24576} \pi^2 \right) \nu \right] \]

\[ + \frac{1}{\alpha^3} \left[ \frac{37495}{64} - \frac{105}{4} \nu^3 + \left( \frac{95791}{144} - \frac{8077}{768} \pi^2 \right) \nu^2 + \left( - \frac{3128449}{2160} + \frac{2992919}{73728} \pi^2 \right) \nu \right] \]

\[ + \frac{1}{\alpha^5} \left( \frac{2957}{160} \nu - \frac{113}{16} \nu^3 + \frac{1061}{32} \nu^2 \right) \]

\[ + \frac{2\alpha}{(\alpha^2 + 1)^4} - \frac{\alpha(4\nu - 15)}{(\alpha^2 + 1)^3} \]

\[ + \frac{\alpha}{(1 + \alpha^2)^2} \left[ \nu^2 + \left( \frac{41}{32} \pi^2 - \frac{187}{3} \right) \nu + \frac{155}{2} \right] \]

\[ + \frac{\alpha}{(1 + \alpha^2)^2} \left[ \frac{365}{12} \frac{41}{64} \pi^2 \nu^2 + \left( \frac{10189}{1024} \pi^2 - \frac{59203}{120} \right) \nu + \frac{1715}{4} \right]. \]

(47)

Finally, the last contribution \( I_\chi \) to \( \frac{1}{2} \mathcal{A}_{\text{loc}} \) is defined as the following (Hadamard-regularized) integral

\[ I_\chi = - \frac{16 j \nu}{15} \text{Pf} \int_0^{\mu_{\text{max}}} \frac{u^4 \ln(u) (-74 \bar{E} + 37 j^2 u^2 - 62 u)}{(2 \bar{E} - j^2 u^2 + 2u)^{5/2}} du. \]

(48)
Using a suitable integration by parts, the singular integral $I_\chi$, Eq. (48), can be recast as

$$I_\chi = I_\chi + \frac{(413\alpha^4 - 8\alpha^2 - 37)\nu}{15j^8\alpha^4} B(\alpha) + \frac{\nu(4010\alpha^2 + 6195\alpha^4 - 1461)}{225\alpha^3j^8(\alpha^2 + 1)},$$

(49)

where the first term is now defined in terms of the following convergent integral

$$I_\chi \equiv \frac{16\nu}{15(1 + 2Ej^2)} \int_0^{x_{\max}} \frac{(uj^2 - 1)u^3(-310u - 296E + 222j^2u^2) \ln(u)}{(2E - j^2u^2 + 2u)^{1/2}} \, du.$$  

(50)

D. Computing the large-$j$ expansion of the logarithmic integral $I_\chi$

The integral $I_\chi$, Eq. (50), cannot be expressed in terms of elementary functions. Let us, however, explain how all the terms of the expansion of $I_\chi(\bar{E}, j)$ in powers of $1/j$, at fixed energy, can be explicitly computed in terms of elementary functions.

First, let us replace the integration variable $u$ by a new variable $x$, and introduce a convenient expansion parameter $\epsilon$:

$$u = \sqrt{2\bar{E}j}x; \quad \epsilon \equiv 2\alpha = \frac{2v_\infty}{\bar{E}j}.$$  

(51)

This yields the new form

$$I_\chi = \frac{512\nu}{15\epsilon^4j^8(\epsilon^2 + 4)} \text{Pf} \int_0^{x_{\max}(\epsilon)} \frac{(2x - \epsilon)x^3(-155x\epsilon - 148 + 222x^2) \ln\left(\frac{2x}{\epsilon}\right)}{\sqrt{1 - x^2 + \epsilon}} \, dx,$$  

(52)

where the upper limit is $x_{\max}(\epsilon) = \sqrt{1 + \epsilon^2/4 + \epsilon/2} = 1 + O(\epsilon)$. The large-$j$ expansion, at fixed $v_\infty$, then corresponds (modulo the overall $1/j^8$ factor and the nonlogarithmic integral involving $\ln(1/j^2)$) to a small-$\epsilon$ expansion. Using again the general result of Ref. [33] the expansion in powers of $\epsilon$ of $I_\chi$ is simply obtained by: i) expanding the integrand of Eq. (52) in powers of $\epsilon$ (which generates singular integrals); ii) using as upper limit $x_{\max}(0) = 1$; and iii) taking Hadamard’s partie finie of the singular integrals.

Let us display, for illustration, the first three terms of the $\epsilon$-expansion of the $I_\chi$ integrand

$$\nu \left[ \frac{256}{15\epsilon^4} x^4(-148 + 222x^2) \ln\left(\frac{2x}{\epsilon}\right) + \frac{256}{15\epsilon^3} x^3(155x^4 - 266x^2 + 74) \ln\left(\frac{2x}{\epsilon}\right) \right.$$

$$\left. - \frac{64}{15\epsilon^2} x^2(680x^2 - 703x^4 - 310 + 222x^6) \ln\left(\frac{2x}{\epsilon}\right) + O\left(\frac{1}{\epsilon}\right) \right].$$

(53)

The (finite part of the) integral between 0 and 1 of the latter, expanded, logarithm-dependent integrand can be computed by taking the $a \to 0$ limit of the identity

$$\frac{d}{da} \int_0^1 x^a f(x) \, dx = \int_0^1 x^a \ln(x) f(x) \, dx.$$  

(54)
Actually, it is convenient to consider the following generalized integral
\[
\mathcal{J}_\chi(a,b) = \frac{\nu}{j^8} \int_0^1 \left[ \frac{256}{15\pi^4} x^4 (-148 + 222 x^2) \left( \frac{2x}{\nu x^2} \right)^a (1 - x^2)^{-\frac{b}{2}} + \frac{256}{15\pi^3} x^3 (155x^4 - 266x^2 + 74) \left( \frac{2x}{\nu x^2} \right)^a (1 - x^2)^{-\frac{b}{2}} - \frac{64}{15\pi^2} x^4 (680x^2 - 703x^4 - 310 + 222x^6) \left( \frac{2x}{\nu x^2} \right)^a (1 - x^2)^{-\frac{b}{2} - \frac{5}{2}} + O \left( \frac{1}{j} \right) \right] dx , \tag{55}
\]

in which the powers of the singular denominators have also been shifted by \(-b\). All the integrals entering \(\mathcal{J}_\chi(a,b)\), Eq. (55), can be trivially computed in terms of Euler’s beta function. Finally differentiating the result with respect to \(a\) and taking the limits \(a \to 0\) and \(b \to 0\) we get (restoring \(\bar{E}\) in lieu of \(\epsilon\))
\[
\mathcal{I}_\chi(\bar{E}, j) = \frac{\nu \bar{E}^2}{j^2} \mathcal{I}_4 + \frac{\nu \bar{E}^{3/2}}{j^3} \mathcal{I}_5 + \frac{\nu \bar{E}^1/2}{j^4} \mathcal{I}_6 + \frac{\nu \bar{E}^{1/2}}{j^7} \mathcal{I}_7 + \frac{\nu}{j^8} \mathcal{I}_8 + O \left( \frac{1}{j^9} \right) , \tag{56}
\]

where
\[
\begin{align*}
\mathcal{I}_4 &= -\frac{37\pi}{5} \left[ -3 + \ln \left( \frac{4j^4}{\bar{E}^2} \right) \right] \\
\mathcal{I}_5 &= \frac{448\sqrt{2}}{675} \left[ -218 + 840 \ln(2) - 105 \ln \left( \frac{4j^4}{\bar{E}^2} \right) \right] \\
\mathcal{I}_6 &= -\frac{\pi}{3} \left[ -758 + 183 \ln \left( \frac{4j^4}{\bar{E}^2} \right) \right] \\
\mathcal{I}_7 &= \frac{32\sqrt{2}}{225} \left[ +8760 \ln(2) - 538 - 1095 \ln \left( \frac{4j^4}{\bar{E}^2} \right) \right] \\
\mathcal{I}_8 &= -\frac{\pi}{180} \left[ -57362 + 8925 \ln \left( \frac{4j^4}{\bar{E}^2} \right) \right] . \tag{57}
\end{align*}
\]

Here, for simplicity, only the first terms of the expansion have been displayed, but our method allows one to compute as many terms as one wishes.

The large-\(j\) expansion (56) (at fixed energy) is an expansion in powers of \(\alpha\), and is equivalent to an expansion in inverse powers of the (Newtonian-level) eccentricity \(e = \sqrt{1 + 2\bar{E}^2} = \sqrt{1 + \frac{1}{\nu^2}}\). In addition, one can analytically compute the value of \(\mathcal{I}_\chi(\bar{E}, j)\) at the other boundary (besides \(\alpha = 0\), corresponding to \(e = \infty\)) of the family of unbound, hyperbolic motions, namely \(\alpha = \infty\), corresponding to the marginally bound case \(\bar{E} = 0\), i.e. the parabolic limit \(e = 1\). One finds
\[
\mathcal{I}_\chi(\bar{E} = 0, j) = \frac{\nu \pi}{j^8} \left( 47867 - 35700 \ln(2j^2) \right) . \tag{58}
\]

E. Final result for the large-\(j\) expansion of the local 4PN contribution to \(\chi\)

We have given in Eqs. (45) above the explicit expression of the 4PN-accurate value of the local contribution to the scattering angle. Our result was fully explicit, except for the last, integral, contribution \(\mathcal{I}_\chi(\bar{E}, j)\) in Eq. (45). We have then transformed \(\mathcal{I}_\chi(\bar{E}, j)\), Eq. (58), into the simpler integral \(\mathcal{I}_\chi(\bar{E})\), Eq. (50). In the previous subsection, we have computed the large-\(j\) expansion of \(\mathcal{I}_\chi(\bar{E})\). Let us insert the latter result in our previous results to give the beginning of the large-\(j\) expansion (at fixed energy) of \(\frac{\nu}{2} \chi^{(4\text{PN})}(\bar{E}, j)\).

Both \(\mathcal{I}_\chi\) and \(\mathcal{I}_\chi(\bar{E})\) start at order \(1/j^4\) in their large-\(j\) expansion. Let us first display the large-\(j\) expansion of the (exactly known) part of \(\chi^{(4\text{PN})}/2\), namely the part that does not include \(\mathcal{I}_\chi(\bar{E}, j)\). It reads
\[
\frac{\nu}{2} \chi^{(4\text{PN})}_{\text{loc}} - \mathcal{I}_\chi = \frac{v_6}{j^2} \left[ \frac{27}{64} \pi \nu^2 + \frac{9}{64} \pi \nu - \frac{15}{32} \pi \nu^3 \right] \\
+ \frac{v_5}{j^3} \left[ -8\nu^3 - \frac{91}{5} \nu + 34\nu^2 \right] \\
+ \frac{v_4}{j^4} \left[ -225 \pi \nu^3 + \left( \frac{4827}{64} \pi - \frac{2047}{16384} \pi^3 \right) \nu^2 \right. \\
+ \left( \frac{19373}{192} \pi + \frac{33601}{16384} \pi^3 + \frac{37}{5} \pi \ln(s) \right) \nu \right] \\
+ \frac{3465}{128} \left[ + O \left( \frac{1}{j^5} \right) \right] . \tag{59}
\]

On the other hand, we have computed above the large-\(j\) expansion of \(\mathcal{I}_\chi(\bar{E})\). Its first term reads
\[
\mathcal{I}_\chi = -\frac{37\pi \nu^4}{20j^4} \left[ \ln \left( \frac{4j^4}{\bar{E}^2} \right) - 3 \right] + O \left( \frac{1}{j^5} \right) . \tag{60}
\]

Combining (59) and (60) finally yields the following large-\(j\) expansion of the local 4PN scattering angle
\[
\frac{\nu}{2} \chi^{(4\text{PN})}_{\text{loc}} = \frac{v_6}{j^2} \left[ \frac{27}{64} \pi \nu^2 + \frac{9}{64} \pi \nu - \frac{15}{32} \pi \nu^3 \right] \\
+ \frac{v_5}{j^3} \left[ -8\nu^3 - \frac{91}{5} \nu + 34\nu^2 \right] \\
+ \frac{v_4}{j^4} \left[ -225 \pi \nu^3 + \left( \frac{4827}{64} \pi - \frac{2047}{16384} \pi^3 \right) \nu^2 \right. \\
+ \left( \frac{19373}{192} \pi + \frac{33601}{16384} \pi^3 + \frac{37}{5} \pi \ln(s) \right) \nu \right] \\
+ \frac{3465}{128} \left[ + O \left( \frac{1}{j^5} \right) \right] . \tag{61}
\]
F. Schwarzschild limit

As a check on the above comparable-mass ($\nu = O(1)$) results we have also computed the PN expansion of the extreme-mass-ratio ($\nu \to 0$) scattering angle. This is obtained by considering the scattering angle of a test particle of mass $\mu \to 0$ around a Schwarzschild black hole of mass $M$. It is given in terms of incomplete elliptic integrals \textsuperscript{36} (Eq. (203), page 215 there) and reads

$$\frac{\chi_s}{2} = \frac{\kappa}{\sqrt{e_p u_p}} \left[ K(\kappa) - F \left( \sqrt{\frac{e_p - 1}{2 e_p}} \kappa \right) \right] - \frac{\pi}{2}. \quad (62)$$

where

$$\kappa = 2 \sqrt{\frac{e_p u_p}{1 - 6 u_p + 2 e_p u_p}}. \quad (63)$$

Here $u_p$ and $e_p$ are used to parametrize the specific test-particle energy $E_s = E_{\text{particle}}/\mu$ and angular momentum $j_s = P_{\phi_{\text{particle}}}/\mu$, according to the defining equations

$$E_s = \sqrt{\frac{(1 - 2 u_p)^2 - 4 e_p^2 u_p^2}{1 - 3 u_p - e_p^2 u_p}}$$
$$j_s = \frac{1}{u_p (1 - 3 u_p - e_p^2 u_p)} \quad (64)$$

As the $\nu \to 0$ limits of the EOB quantities $\xi_{\text{eff}}$ and $j$ are equal (by construction of the EOB formalism) to $E_s$ and $j_s$, it is enough to re-express the Schwarzschild scattering angle $\chi_s(E_s, j_s)$ in terms of $j_s$ and of the Schwarzschild analogue of the two-body variable $\alpha$ used above, namely

$$\alpha_s = \frac{1}{j_s \sqrt{E_s^2 - 1}} = \lim_{\nu \to 0} \alpha(E, j). \quad (65)$$

This leads, suppressing for clarity the $s$ subscript on the independent variables and displaying only the first few terms, to

$$\frac{\chi_s(j_s, \alpha_s)}{2} = \arctan(\alpha) + \left[ \frac{3}{2} \pi + 3 \arctan(\alpha) + \frac{(3 \alpha^2 + 2)}{\alpha(\alpha^2 + 1)} \right] \frac{\eta^2}{j^2} + \left[ \frac{15}{4 \alpha^2} + \frac{105}{4} \right] \arctan(\alpha) + \frac{95 \alpha}{8 \alpha^2 \pi} + \frac{81 \alpha^3}{4(\alpha^2 + 1)^2} + \frac{2 \alpha(\alpha^2 + 1)^2 + 105 \pi}{8} + \frac{105 \alpha^3}{4(\alpha^2 + 1)^2} \right] \frac{\eta^4}{j^4} + \ldots. \quad (66)$$

Here we have indicated the PN order by the formal PN-expansion parameter $\eta \sim 1/c$. [As explained above, the rule for doing so is to count $\alpha$ as being independent of $\eta$, while $1/j$ is $O(\eta)$.] Let us also exhibit the beginning of the expansion of $\chi_s(E_s, j_s)$ in the limit where $\alpha_s \to 0$, keeping fixed the value of $E_s$. [This is also the limit $1/j_s \to 0$ with fixed $E_s$.] The result can be displayed in a more compact manner by re-expressing it in terms of both $1/j_s$ and $\alpha = \alpha_s = 1/(j_s \sqrt{E_s^2 - 1})$:

$$\frac{1}{2} \chi_s(E_s, j_s) = \frac{1}{2} \chi_s^{(0)} + \frac{1}{2} \chi_s^{(2)} \frac{\eta^2}{j^2} + \frac{1}{2} \chi_s^{(4)} \frac{\eta^4}{j^4} + \frac{1}{2} \chi_s^{(6)} \frac{\eta^6}{j^6} + \ldots + O \left( \frac{\eta^{10}}{j^{10}} \right), \quad (67)$$

where

$$\frac{1}{2} \chi_s^{(0)} = \alpha - \frac{1}{3} \alpha^3 + \frac{1}{5} \alpha^5 - \frac{1}{7} \alpha^7 + \frac{1}{9} \alpha^9 + O(\alpha^{11})$$
$$\frac{1}{2} \chi_s^{(2)} = \frac{2}{\alpha} + \frac{3}{2} \pi + 4 \alpha - 2 \alpha^3 + \frac{8}{5} \alpha^5 - \frac{10}{7} \alpha^7 + O(\alpha^9)$$
$$\frac{1}{2} \chi_s^{(4)} = \frac{15}{8 \alpha^2 \pi} + \frac{24}{105} \alpha + \frac{105}{8} \pi + 32 \alpha - 16 \alpha^3$$
$$+ \frac{96}{7} \alpha^5 + O(\alpha^7)$$
$$\frac{1}{2} \chi_s^{(6)} = \frac{64}{30} + \frac{315}{8 \alpha^3} + \frac{320}{\alpha} + \frac{115}{8} \pi + 32 \alpha - 448 \alpha^3 + O(\alpha^5)$$
$$\frac{1}{2} \chi_s^{(8)} = \frac{3465}{128 \alpha^5 \pi} + \frac{640}{\alpha^5} + \frac{45045}{64 \alpha^2 \pi} + \frac{4480}{\alpha} + \frac{225225}{128} \pi$$
$$+ \frac{3584 \alpha}{O(\alpha^2)}. \quad (68)$$

Note that, in this expansion, a combination of the type $\alpha^p/j^q$ is of order $1/j^{p+q}$ in the large-$j$ expansion that we are considering.

We have checked that the $\nu \to 0$ limit of our comparable-mass results for $\chi$ given in the previous subsections agree with the Schwarzschild limit.

IV. ON THE TWO APPROACHES TO THE TAIL CONTRIBUTION TO THE CONSERVATIVE DYNAMICS

As recalled in the Introduction, it was shown in Ref. \textsuperscript{18} (see Eq. (5.6) there) that the 4PN Hamiltonian is the sum of a local contribution, $H_{(\text{local})}$, and of a tail contribution originally given as the following nonlocal-in-time Hamiltonian, $H_{(\text{nonloc})}$

$$H_{(\text{nonloc})}(T, Q, P; s) = - \frac{G^2 M}{5} \eta^8 \tilde{I}_{ij}(T) \text{ Pf}_{2s/c} \int_{-\infty}^{\infty} \frac{d\tau}{|\tau|} \tilde{I}_{ij}(T + \tau) \), \quad (69)$$

where, we recall, $I_{ij}$ denotes the Newtonian quadrupole moment of the binary system in the center-of-mass frame,
\[ I_{ij} = \mu \left( Q^{i}Q^{j} - \frac{1}{3}Q^{2}\delta^{ij} \right) \equiv \mu Q^{(i}Q^{j)}. \]  

(70)

Here \( X_{STF}^{\pm} \equiv X^{(i)} \) denotes the symmetric-trace-free part of the 2-tensor \( X^{ij} \); similarly, \( X_{TF}^{\pm} \) denotes its trace-free part.

There are two ways to deal with the tail contribution to the 4PN dynamics. On the one hand, the traditional one would be to compute the nonlocal tail force induced by \( H_{\text{tail(nonloc)}}^{\text{loc}} \), and to estimate how it modifies the dynamical effects linked to the local part of the 4PN-accurate dynamics. On the other hand, a new approach has been introduced in Refs. [20, 26] in the case of ellipticlike motions. This second approach consists in “localizing,” i.e., order-reducing, the nonlocal tail Hamiltonian \( H_{\text{tail}}^{\text{loc}} \). Eq. (69), to a physically equivalent local tail Hamiltonian \( H_{\text{tail}}^{\text{loc}} \).

The basic quantity entering the traditional approach is the nonlocal tail force \( F_{\text{nonloc}} \). Its value is obtained by varying the following nonlocal action

\[
S(Q, P) = \int \left( P_{i}\dot{Q}^{i} - H_{\text{loc}}(Q, P) - H_{\text{tail(nonloc)}}^{\text{loc}}(Q, P, T) \right) dT. 
\]

After some integration by parts, the equations of motion following from this action read (as indicated on pages 5 and 10 of Ref. [18])

\[
\dot{Q}^{i} = \frac{\partial H_{\text{loc}}}{\partial P_{i}}, \quad \dot{P}_{i} = -\frac{\partial H_{\text{loc}}}{\partial Q^{i}} + \frac{4G^{2}M}{5} \mu Q^{(i}(t) P_{f_{2}/c} \int d\tau \frac{I_{ij}^{(6)}(T + \tau)}{\tau} \equiv -\frac{\partial H_{\text{loc}}}{\partial Q^{i}} + F_{\text{nonloc}}^{i}(T), 
\]

(72)

where

\[
F_{\text{nonloc}}^{i}(T) = \frac{4G^{2}M}{5} \mu \delta^{i}Q^{(j}(T) \times \mu P_{f_{2}/c} \int_{-\infty}^{\infty} d\tau \frac{I_{ij}^{(6)}(T + \tau)}{\tau}. 
\]

Note that, as pointed out in [18], the sum of the acausal time-symmetric nonlocal tail force, Eq. (73), and of the acausal time-antisymmetric nonlocal radiation reaction force (written on page 10 of [18]), namely

\[
F_{\text{RR}}^{i}(T) = -\frac{4G^{2}M}{5} \mu \delta^{i}Q^{(j}(T) \times \mu P_{f_{2}/c} \int_{0}^{\infty} d\tau \frac{I_{ij}^{(6)}(T - \tau) - I_{ij}^{(6)}(T + \tau)}{\tau}. 
\]

yields the causal tail-transported nonlocal force first derived in Ref. [35]

\[
F_{\text{tail,even}}^{i}(T) + F_{\text{RR}}^{i}(T) = \frac{8G^{2}M}{5} \mu Q^{(i}(T) P_{f_{2}/c} \int_{0}^{\infty} d\tau \frac{I_{ij}^{(6)}(T - \tau)}{\tau}. 
\]

(74)

Let us now generalize the time-localization procedure of Refs. [20, 26] from the ellipticlike case to the present hyperboliclike one.

Following Refs. [20, 26], \( H_{\text{loc}}^{\text{tail}} \) can be simply obtained by replacing the occurrences of the time-displayed phase-space arguments \( Q(T + \tau) \) and \( P(T + \tau) \) in Eq. (69) by the corresponding solutions of the (Newtonian level) Hamilton equations of motion. [As shown in [20, 26] this is a priori forbidden use of the equations of motion in an action is valid modulo a nonlocal-in-time transformation of the phase space variables.]

Let us introduce a convenient notation to define this time-localization procedure. Given some Hamiltonian flow in phase space (that we can take here to be simply the Newtonian level flow), and given the phase space position, \( Q = Q(T) \) and \( P = P(T) \) at a given time \( T \), we denote by \( [Q]_{Q;T}^{(T+\tau)} \), \( [P]_{Q;T}^{(T+\tau)} \), the solution of Hamilton’s evolution equations at time \( T + \tau \), which takes as initial values at time \( T \) the given data \( Q(T) \) and \( P(T) \). For any phase space function \( F(Q, P) \) we then correspondingly denote the value of \( F \) at \( [Q]_{Q;T}^{(T+\tau)} \), \( [P]_{Q;T}^{(T+\tau)} \) simply as \( [F]_{Q;T}^{(T+\tau)} \). With this notation the time-localized version of the tail action reads

\[
H_{\text{loc}}^{\text{tail}}(T, Q, P; s) = \frac{G^{2}M}{5c^{5}} I_{ij}^{(3)}(Q(T), P(T)) \times \mu P_{f_{2}/c} \int_{-\infty}^{\infty} d\tau \frac{I_{ij}^{(6)}(T + \tau)}{\tau}. 
\]

(76)

Here, the notation \( I_{ij}^{(3)}(Q, P) \) denotes the order-reduced third time derivative of the quadrupole moment, i.e., the function of \( Q \) and \( P \) obtained by using Hamilton’s equations of motion to reduce higher derivatives in \( I_{ij}^{(6)} \).

Refs. [20, 26] showed how to explicitly compute the time-localized Hamiltonian, Eq. (76), in the case of ellipticlike, bound motions. This was done by using both Delaunay variables and the Delaunay averaging technique (to eliminate time-periodic terms). The resulting action was obtained as a power series in the eccentricity \( e \), in a neighborhood of circular motion.

In the present paper we shall show how to generalize the time-localization technique of Refs. [20, 26] to hyperboliclike motions, and to deduce from it the tail contribution to the scattering function. Our result will be expressed as a power series in the reciprocal of the eccentricity \( 1/e \), in a neighborhood of \( e = \infty \), which corresponds to straight line motion. [In addition, we shall show how to compute the value of \( H_{\text{loc}}^{\text{tail}} \) over the full range of hyperbolic motions.] We have explicitly checked that, at leading order in \( 1/e \), the so-deduced result agrees with a direct computation based on the secular evolution of the Newtonian-conserved Laplace-Lagrange-Runge-Lenz, eccentricity vector under the action of the nonlocal, tail 4PN-level force \( F_{\text{nonloc}}^{i} \), Eq. (73). To relieve the tedious, we present this alternative calculation in Appendix B.
V. LINK BETWEEN THE TAIL CONTRIBUTION $\chi^\text{tail}(E,j)$ TO SCATTERING AND THE TIME-INTEGRATED TAIL HAMILTONIAN $\int dt H^\text{tail}$

In this section we show how to explicitly compute the tail contribution $\chi^\text{tail}(E,j)$ to the scattering function from the knowledge of the time-integral (from $-\infty$ to $+\infty$) of the tail Hamiltonian $H^\text{tail}$. We shall assume that we work with the time-localized version, $H^\text{tail}_{\text{loc}}$, of the nonlocal tail Hamiltonian, as defined in the previous section, Eq. (76). [In the next section we shall show how to explicitly compute the time-localized $H^\text{tail}_{\text{loc}}$.]

As $H^\text{tail}_{\text{loc}}$ is of order $O(1/\varepsilon^8)$ one can compute its contribution to $\chi$ simply by considering the Hamiltonian $H^N + H^\text{tail}_{\text{loc}}$, where $H^N$ is the Newtonian-order Hamiltonian, all the other PN contributions having been already incorporated in the local computation of the previous section.

We then start from an ordinary Hamiltonian of the form

$$H(q,p) = H_N(q,p) + H^\text{tail}(q,p). \quad (77)$$

It will be convenient to work with rescaled phase-space variables, notably $r = R/GM$, $p_r = P_R/\mu$, and $p_\phi = P_\phi/\mu$, and the rescaled time $t = T/GM$, while leaving the Hamiltonian unrescaled. [We use here $c = 1$.] Actually, in order to simplify things, we shall consider in this section that we use units such that $M = 1$ (in addition to using $G = c = 1$). In such units, one only needs to keep track of the power of $\mu = \nu$ entering various quantities.

In terms of such units, the lowest-order (here taken as Newtonian-order) Hamiltonian reads

$$H_N = \nu \left[ \frac{1}{2} \left( \frac{p_r^2 + p_\phi^2}{r^2} \right) - \frac{1}{r} \right]. \quad (78)$$

The second, tail contribution, is here assumed to be an ordinary (time-local) Hamiltonian, simply denoted by $H^\text{tail}_{\text{loc}}$. In the following we will denote $p_\phi = j$.

Let us recall the general Hamilton-Jacobi derived formula

$$\chi(E,j) = -\frac{\partial}{\partial j} \int p_r(E,j,r) dr \quad (79)$$

where the function $p_r(E,j,r)$ is defined by solving the energy conservation law

$$\dot{E} = \frac{H(r,p_r,j)}{\nu} = \frac{1}{2} \left( \frac{p_r^2 + p_\phi^2}{r^2} \right) - \frac{1}{r} + \frac{H^\text{tail}(r,p_r,j)}{\nu}. \quad (80)$$

When working to linear order in $H^\text{tail}$ the solution of the equation

$$p_r^2 = 2E - \frac{j^2}{r^2} + \frac{2}{r} - \frac{H^\text{tail}(r,p_r,j)}{\nu} \quad (81)$$

is

$$p_r = p_r^{(0)} - \frac{1}{p_r^{(0)}} \frac{H^\text{tail}(r,p_r^{(0)},j)}{\nu}, \quad (82)$$

where

$$p_r^{(0)} = \pm \sqrt{2E - \frac{j^2}{r^2} + \frac{2}{r}}. \quad (83)$$

Inserting Eq. (82) in Eq. (79) we find

$$\chi(E,j) = \chi_N(E,j) + \chi^\text{tail}(E,j) \quad (84)$$

where

$$\chi_N(E,j) = \arctan \left( \frac{1}{\sqrt{2Ej^2}} \right), \quad (85)$$

and

$$\chi^\text{tail}(E,j) = \frac{1}{\nu} \frac{\partial}{\partial j} \int \frac{dE}{p_r^{(0)}} W^\text{tail}(r,p_r^{(0)},j). \quad (86)$$

We thereby see that the tail correction to the scattering function derives, via a $j$-gradient, from the following “potential” $W^\text{tail}(E,j)$

$$W^\text{tail}(E,j) = \int \frac{dE}{p_r^{(0)}} H^\text{tail}(r,p_r^{(0)},j) \quad (87)$$

In the second expression we have used the property that $dr/p_r^{(0)} = dt$ along the Newtonian Hamiltonian flow, as well as the notation introduced above for signifying that a quantity is computed along a specified flow line of some given zeroth-order Hamiltonian flow. The resulting formula

$$\chi^\text{tail}(E,j) = \frac{1}{\nu} \frac{\partial}{\partial j} W^\text{tail}(E,j), \quad (88)$$

can also be viewed as a hyperbolic analogue of the Delaunay averaging procedure.

The Delaunay approach to ellipticlike motions shows that when one is dealing with a perturbed Hamiltonian of the type

$$H(q,p) = H_0(q,p) + \epsilon H_1(q,p) \quad (89)$$

one can, modulo a canonical transformation of order $\epsilon$, eliminate all oscillatory terms in $H_1(q,p)$, thereby replacing $H_1(q,p)$ by its time average. Similarly here, the result (87) shows that one can eliminate from $H^\text{tail}(q,p)$

---

6 The formula we shall arrive at for the correction to the scattering function induced by an additional contribution to the Hamiltonian is actually very general and applies to the case $H(q,p) = H^0(q,p) + \epsilon H^1(q,p)$. 

any total time derivative contribution (vanishing at infinity separation).

Let us exhibit the explicit form of the potential \( W_{\text{tail}}(E, j) \). To this aim, we start from Eq. (69). To simplify the writing, let us introduce the notation (where \( t' = t + \tau \))

\[
F(t, t') \equiv \frac{1}{5} I_{ij}^{(3)}(t) I_{ij}^{(3)}(t'),
\]

(90)

for the “time-split” gravitational wave flux. [Note that, as said above, we use here units where \( M \) and \( G \) are set to unity. In particular, the quadrupole moment we use corresponds to \( I_{ij}^{(3)}/(GM)^2 \).]

With this notation the potential \( W_{\text{tail}} \), Eq. (87), reads

\[
W_{\text{tail}}(E, j) = - \int dt \, P_f_{2s/c} \int_{-\infty}^{\infty} \frac{dt'}{|t' - t|} F(t, t').
\]

(91)

An exact expression for \( F(t, t') \) along the Newtonian motion (valid both for elliptic and for hyperbolic motions, without any approximation in eccentricity) is

\[
F(t, t') = \frac{4p^2}{15j^{10}} (1 + e \cos \phi')^2 \times
\]

\[
\times \left( 1 + e \cos \phi \right)^2 \left( F_0 + F_1 e + F_2 e^2 \right),
\]

(92)

where

\[
F_0(\phi, \phi') = 24 \cos(2\phi - 2\phi')
\]

\[
F_1(\phi, \phi') = 9 \cos(2\phi - 3\phi') + 15 \cos(-2\phi' + \phi)
\]

\[
+ 9 \cos(3\phi - 2\phi') + 15 \cos(2\phi - \phi')
\]

\[
F_2(\phi, \phi') = -\frac{1}{4} \cos(\phi + \phi') + \frac{45}{8} \cos(3\phi - \phi')
\]

\[
+ \frac{45}{8} \cos(-3\phi' + \phi) + \frac{27}{8} \cos(3\phi - 3\phi')
\]

\[
+ \frac{77}{8} \cos(\phi - \phi').
\]

(93)

Here we have re-expressed the phase space variables \( q \) and \( p \) entering \( I_{ij}^{(3)} \) entirely in terms of the two azimuthal angles

\[
\phi = \phi(t), \quad \phi' = \phi(t'),
\]

(94)

using the Keplerian orbit relations

\[
r = \frac{j^2}{1 + e \cos \phi}, \quad r' = \frac{j^2}{1 + e \cos \phi'}.
\]

(95)

In order to explicitly compute the two time integrals entering the expression (91) for \( W_{\text{tail}} \) one further needs the explicit time dependence of \( \phi \) and \( \phi' \). In the presently considered case of hyperbolic motions, this has to be done via the hyperbolic Kepler equation, namely

\[
\tilde{n} \, t = e \sinh \tilde{u} - \tilde{a},
\]

(96)

where

\[
\tilde{n} = \frac{1}{\tilde{a}^{3/2}}, \quad \tilde{a} = \frac{1}{2E} = -a.
\]

(97)

We recall also that the parameter \( p \) of the considered conic satisfies

\[
p = a(1 - e^2) = \tilde{a}(e^2 - 1) = j^2,
\]

(98)

so that we have also

\[
e^2 = 1 + 2E \cdot j^2 = 1 + \frac{v_c^2}{j^2}.
\]

(99)

Then, given the solution \( \tilde{u}(\tilde{n}t, c) \) of Kepler’s equation (96), one has

\[
\phi(t) = 2 \arctan \left[ \frac{e + 1}{e - 1} \tanh \left( \frac{\tilde{u}}{2} \right) \right].
\]

(100)

A different, but equivalent, expression for \( W_{\text{tail}}(E, j) \) can be obtained by working in the Fourier domain. Our notation for the Fourier-transform \( \tilde{I}_{ij}(\omega) \) of the Newtonian quadrupole moment of the system is

\[
\tilde{I}_{ij}(t) = \int \frac{d\omega}{2\pi} \tilde{I}_{ij}(\omega) e^{-i\omega t},
\]

\[
\tilde{I}_{ij}(\omega) = \int dt I_{ij}(t) e^{i\omega t}.
\]

(101)

Inserting these Fourier representations (together with their “primed” counterparts and the notation \( T_s \equiv 2s/c \)) yields

\[
W_{\text{tail}}(E, j) = \int dt \, H_{\text{tail}} = -\frac{1}{5} \int dt \, P_f T_s \int \frac{dt'}{|t' - t|} I_{ij}^{(3)}(t) I_{ij}^{(3)}(t')
\]

\[
= \frac{1}{5} \int dt P_f T_s \int \frac{dt'}{|t' - t|} \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} (-i\omega)^3 (-i\omega')^3 \tilde{I}_{ij}(\omega) \tilde{I}_{ij}(\omega') e^{-i\omega t} e^{-i\omega' t'}
\]

\[
= -\frac{1}{5} P_f T_s \int \frac{d\tau}{|\tau|} \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi} (-i\omega)^3 (-i\omega')^3 \tilde{I}_{ij}(\omega) \tilde{I}_{ij}(\omega') \int dt e^{-i\omega t} e^{-i\omega'(t+\tau)}
\]

(102)
where we came back to the notation $t' = t + \tau$. Using the fact that integral $\int dt e^{-i(\omega + \omega')t} = 2\pi \delta(\omega + \omega')$ we get

$$W_{\text{tail}}(E, j) = \frac{1}{5} \int \frac{d\omega}{2\pi} \omega^6 |\hat{I}_{ij}(\omega)|^2 \left| \frac{T}{\mu} \right| e^{i\omega \tau}. \quad (103)$$

The partie finie integral entering the latter expression is (see Eq. (5.8) in Ref. [18]; with $\gamma = \gamma_{\text{Euler}}$)

$$\text{Pf}_T \int_{-\infty}^{\infty} \frac{d\tau}{i|\tau|} e^{i\omega \tau} = -2\ln(|\omega| T_s e^\gamma), \quad (104)$$

so that, using the relation $\hat{I}_{ij}^s(-\omega) = \hat{I}_{ij}(\omega)$, we find the final Fourier-domain formula

$$W_{\text{tail}}(E, j) = \frac{2}{5} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega^6 |\hat{I}_{ij}(\omega)|^2 \ln \left( |\omega| E, j \right) e^\gamma. \quad (105)$$

This remarkably compact formula is very simply related to the frequency-domain version of the total gravitational wave energy emitted by the entire hyperbolic motion which reads

$$\Delta E_{\text{GW}} = \frac{1}{5} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \omega^6 |\hat{I}_{ij}(\omega)|^2. \quad (106)$$

More precisely, the only difference consists in inserting the factor $2\ln(|\omega| T_s e^\gamma)$ in the frequency-decomposition of $\Delta E_{\text{GW}}$. As a consequence, the $s$-dependence of $W_{\text{tail}}$ is entirely contained in the term

$$W_{\text{tail}}(E, j) \bigg|_{\ln s} = 2 \Delta E_{\text{GW}}(E, j) \ln s, \quad (107)$$

where we recall the known result for gravitational-wave energy loss along hyperbolic orbits [37] (see also Ref. [38] for its 1PN generalization)

$$\frac{\Delta E_{\text{GW}}}{Mc^2} = \frac{2v^2}{15j} \left( \left( \frac{37e^4 + 292e^2 + 96}{1} \right) \arccos \left( -\frac{1}{e} \right) + \frac{1}{3} \sqrt{e^2 - 1} (673e^2 + 602) \right). \quad (108)$$

Here, we came back provisorily to usual physical units.

To turn the result (108) into an explicit function of $E$ and $j$ we need to insert in it the Fourier-transform of the quadrupole moment of the system computed along a hyperbolic Newtonian orbit. The steps for doing so are the following.

First we note that the $\mu$-rescaled quadrupole moment reads

$$\frac{1}{\mu} I_{ij} = \begin{pmatrix} \frac{2}{3}x^2 - \frac{1}{3}y^2 & xy & 0 \\ xy & \frac{2}{3}y^2 - \frac{1}{3}x^2 & 0 \\ 0 & 0 & -\frac{1}{3}x^2 - \frac{1}{3}y^2 \end{pmatrix}. \quad (109)$$

Second, we rewrite the expressions of the Fourier-transforms of $x^2$, $y^2$ and $xy$, with the notation $(x^2)_\omega = \int dx^2(t)e^{i\omega t}$, etc., as (here we assume $\omega > 0$, and we go back to $GM$-scaled units)

$$(x^2)_\omega = -\frac{2\pi}{u^2 v^2_\infty} \left[ \left( -u e + \frac{u}{e} + i \right) H^{(1)}_1(iu) + u(e^2 - 1) H^{(1)}_{\frac{1}{2} + 1}(iu) \right]$$

$$(y^2)_\omega = -\frac{2\pi}{e^2 v^2_\infty u^2} \left[ \left( \frac{u}{e} + i \right) H^{(1)}_1(iu) - uH^{(1)}_{\frac{1}{2} + 1}(iu) \right]$$

$$(xy)_\omega = -\frac{2\pi}{u^2 v^2_\infty} \left[ \left( -u e + \frac{u}{e} + i \right) H^{(1)}_1(iu) - ieH^{(1)}_{\frac{1}{2} + 1}(iu) \right]. \quad (110)$$

Here we used the notation

$$u = \frac{\omega}{c} > 0, \quad e = \sqrt{1 + 2E j^2} = \sqrt{1 + v^2_\infty j^2} \quad (111)$$

together with the standard notation $H^{(1)}_\nu(z)$ for the Hankel functions of the first type. [The so-introduced notation $u$ for the argument of the Hankel functions should not be confused with the previous notation $u = 1/r$.] See Appendix [A] for the details of the derivation of the expressions (110).

**VI. EXPLICIT COMPUTATION OF $W_{\text{tail}}(E) = \int dt H_{\text{tail}}$ AND $\chi_{\text{tail}}$ IN THE LARGE ECCENTRICITY LIMIT**

In this section we will compute, at leading order (LO) in $1/e$, the potential $W_{\text{tail}}(E) = \int dt H_{\text{tail}}(E)$ (whose $j$-derivative yields the tail contribution $\chi_{\text{tail}}$ to the scattering function, as per Eq. (88)). We will give two independent calculations of $W_{\text{tail}}$. One, in the time domain, i.e. using Eq. (101), and the other, in the Fourier domain, i.e. using Eq. (105). In addition, we shall also check, in Appendix B the correctness of the value of $\chi_{\text{tail}}$ obtained by differentiating $W_{\text{tail}}$, by means of a direct dynamical computation involving the evolution of the Laplace-Lagrange-Runge-Lenz vector.

**A. Time-domain computation of $W_{\text{tail}}(E) = \int dt H_{\text{tail}}$**

In the following, we shall consider hyperbolic motions in the vicinity of an infinite eccentricity (corresponding to
straight-line, uniform motion), and expand the scattering potential \( W^{(\text{tail})} \) in powers of \( 1/e \). A way to set up such a large-\( e \)-expansion is to introduce the eccentricity-rescaled mean motion
\[
\dot{n} \equiv \frac{n}{e} = \frac{(2\varepsilon)^{3/2}}{\sqrt{1 + 2\varepsilon}}.
\] (112)
In terms of \( \dot{n} \), the Kepler equation (110) reads
\[
\dot{n} \tau = \sinh \dot{n} - \frac{1}{e} \dot{n}.
\] (113)
The solution \( \ddot{u}(\dot{n}t) \) of the rescaled Kepler equation (113) can then be expanded in powers of \( 1/e \) (keeping fixed the rescaled time \( \dot{t} \equiv \dot{n}t \)):
\[
\ddot{u}(\dot{n}t, e) = \arcsinh \dot{n}t + \frac{1}{e} \dot{u}_1(\dot{n}t) + \frac{1}{e^2} \dot{u}_2(\dot{n}t) + \cdots
\] (114)
Inserting this expansion in \( \phi(\ddot{u}) \), Eq. (100), then yields the \( 1/e \) expansion of \( \phi(\ddot{n}t) \):
\[
\phi(t) = \phi_0(t) + \frac{1}{e} \phi_1(t) + \frac{1}{e^2} \phi_2(t) + \cdots
\] (115)
The LO terms read
\[
\phi_0(t) = \arctan(\dot{n}t),
\phi_1(t) = \frac{\dot{n}t}{\sqrt{1 + \dot{n}^2 t^2}} + \arcsinh(\dot{n}t) \quad (116)
\]
Note than an alternative way to derive the expansion (115) is to start from Kepler’s area law
\[
r^2 \frac{d\phi}{dt} = j.
\] (118)
Inserting the polar equation
\[
r(t) = \frac{j^2}{1 + e \cos(\phi)}
\] (119)
in the area law (118) yields the following differential equation for \( \phi(t) \):
\[
\frac{d\phi}{dt} = \frac{1}{j^3} (1 + e \cos \phi)^2
\] (120)
which can be integrated term by term in the \( 1/e \) expansion.

Inserting the expansion (115) in the exact expression (122) of the time-split gravitational-wave flux \( F(t, t') \) then yields a large-\( e \)-expansion of \( F(t, t') \) of the form (denoting \( t' \equiv t + \tau \))
\[
F(t, t + \tau) = e^6 f_0(t, \tau) + e^5 f_5(t, \tau) + \cdots
\] (121)
where
\[
f_0(t, \tau) = \frac{4
\nu^2}{15 j^{10}} \cos^2 \phi_0 \cos^2 \phi_0 F_2(\phi_0, \phi_0'),
\] (122)
where \( \phi_0 \equiv \phi_0(t) = \arctan(\dot{n}t), \phi_0' \equiv \phi_0(t + \tau) = \arctan(\dot{n}(t + \tau)) \). Inserting these solutions leads to the compact expression
\[
f_0(t, \tau) = \frac{4 \nu^2}{15 j^{10}} \left[ (1 + (t + \tau)^2)^{5/2}/(1 + t^2)^{5/2} \right].
\] (123)
where we used (on the right-hand-side) the rescaled time variables \( \dot{t} \equiv \dot{n}t, \dot{\tau} \equiv \dot{n}\tau \), and where
\[
f_0(\dot{t}, \dot{\tau}) = 24 + 50\nu^2 + 85\nu^2 \dot{\tau} + 39\nu^2 \dot{\tau}^2 + 6\nu^2 \dot{\tau} + 6\nu^2 \dot{\tau}^2 + 2\nu^2 \dot{\tau}^3 - 12\nu^2 + 50\nu \dot{t} + 28\nu^2 + 2\nu^3.
\] (124)

Let us now show how, using the above large-\( e \) expansion, we can, in principle, compute the large-\( e \)-expansions of both the time-localized value of the tail Hamiltonian \( H^{(\text{tail})}_{\text{(loc)}} \), Eq. (101), and the corresponding scattering potential \( W^{(\text{tail})} = \int dt H^{(\text{tail})}_{\text{(loc)}} \). Here we shall explicitly compute only the LO term in the large-\( e \)-expansions of \( H^{(\text{tail})}_{\text{(loc)}} \) and \( W^{(\text{tail})} \). In principle, our approach gives a way to compute all the terms in the large-\( e \)-expansion, though the higher-order terms involve integrals that seem difficult to compute explicitly. [We had a first look at the problem and found, however, that the large-\( e \) expansion of \( H^{(\text{tail})}_{\text{(loc)}}(t) \) can be evaluated in terms of polylogarithm functions.]

The evaluation of the tail Hamiltonian \( H^{(\text{tail})}_{\text{(loc)}} \), Eq. (101), crucially involves the part of an integral over \( \tau = \dot{n}\tau \). [In view of the rescaling by \( \dot{n} \), the corresponding regularization scale becomes \( \bar{T} \equiv 2\dot{n}s/c \).] Namely,
\[
\frac{1}{e^6} H^{(\text{loc})}_{(\text{tail})} (\dot{t}) = -P_{\dot{T}} \int \frac{d\dot{\tau}}{|\dot{\tau}|} f_0(\dot{t}, \dot{\tau}).
\] (125)
Using the explicit expression of \( f_0(\dot{t}, \dot{\tau}) \), Eq. (123), we found that it was straightforward to compute the regularized integral, with the LO result
\[
\frac{1}{e^6} H^{(\text{loc})}_{(\text{tail})} (\dot{t}) = -8\nu^2 \left[ \frac{2(\dot{t}^2 + 12)(1 + \dot{t}^2)}{15 j^{10}} \ln \left( \frac{2}{T} (1 + \dot{t}^2) \right) \right] + 8\frac{7\dot{t}^2 + 36}{15 j^{10} (1 + \dot{t}^2)^3}.
\] (126)
Finally, we can compute the scattering potential \( W^{(\text{tail})} = (1/\dot{n}) \int d\dot{t} H^{(\text{tail})}_{(\text{loc})} \) (also at the leading order)
\[
W^{(\text{tail})} = \frac{2}{15} \frac{\pi}{\dot{n}} \frac{\nu^2}{j^{10}} \nu^2 \left[ 100 + 37 \ln \left( \frac{T}{8} \right) \right] \approx \frac{2}{15} \pi \nu^2 \nu^4 \frac{\nu^2}{j^3} \left[ 100 + 37 \ln \left( \frac{\nu^2}{4j} \right) \right].
\] (127)
In the last expression, we have used the LO large-\( e \) approximations
\[
e \approx v_{\infty} j, \quad \dot{n} \approx \frac{v_{\infty}^2}{j}.
\] (128)
B. Fourier-domain computation of $W^{\text{tail}} = \int dt H^{\text{tail}}(t)$

It is useful to complement the time-domain computation of $W^{\text{tail}} = \int dt H^{\text{tail}}(t)$ presented in the previous subsection by its frequency-domain counterpart, obtained by considering the large-eccentricity expansion of our general result (105).

Let us start from the Fourier transform of the quadrupole moment, Eqs. (110). [See Appendix A for details of its derivation.] Here, we will limit our discussion to the leading order in the large-eccentricity limit, where the general expressions simplify. We consider the $e \to \infty$ limit of Eqs. (110) keeping fixed the previously defined, Eq. (111), argument of the Hankel functions:

$$u = \frac{\omega}{n} e = \frac{\omega}{\tilde{n}}.$$  \hspace{1cm} (129)

Indeed, our time-domain large-$e$ expansion above has shown that the rescaled time variable adapted to this limit is $\tilde{n}$, so that $\tilde{n}$ yields the characteristic frequency of the large-$e$ hyperbolic motion. One then expects that the Fourier spectrum will be essentially localized around the characteristic frequency $\tilde{n}$, i.e. for values of $u = \omega/\tilde{n}$ of order one. We then see on Eqs. (110) that the limit $e \to \infty$ simplifies the order of the Hankel functions to the values 0 or 1. We note in passing that the latter Hankel functions are related via

$$H_0^{(1)}(q) = -H_1^{(1)}(q).$$  \hspace{1cm} (130)

The LO large-$e$ limit of Eqs. (110) yields

$$\tilde{n} \approx v^3, \quad \tilde{a} \approx \frac{1}{v^2},$$  \hspace{1cm} (131)

and

$$(x^2)_\omega = -\frac{2e^2}{v^3} H_1^{(1)}(q)$$

$$(y^2)_\omega = \frac{2e^2}{q^2 v^3} (q H_1^{(1)}(q) + H_0^{(1)}(q))$$

$$(xy)_\omega = -\frac{2e^2}{q^2 v^3} (q H_0^{(1)}(q) - H_1^{(1)}(q)),$$  \hspace{1cm} (132)

where we denoted $q = iu = i\omega/\tilde{n}$. The latter expressions involve Hankel functions evaluated at purely imaginary arguments. Let us now recall that $H_0^{(1)}(iz)$ is simply related to the modified Bessel function, $K_v$, at argument $z$. (see [39], Eq. (9.6.4)). When the order $\nu$ is zero or one, this relation reads

$$K_0(x) = i\frac{\pi}{2} H_0^{(1)}(ix), \quad K_1(x) = -\frac{\pi}{2} H_1^{(1)}(ix).$$  \hspace{1cm} (133)

This yields

$$(x^2)_\omega = \frac{4e^2}{v^3} K_1(u)$$

$$(y^2)_\omega = -\frac{4e^2}{u^2 v^3} (u K_1(u) + K_0(u))$$

$$(xy)_\omega = \frac{4e^2}{u^2 v^3} (u K_0(u) + K_1(u)).$$  \hspace{1cm} (134)

Inserting these expressions in Eq. (A2) leads to

$$u^0 \bar{I}_{ij}(u)^2 = \frac{32e^4}{v^4} F(u),$$  \hspace{1cm} (135)

where we defined

$$F(u) = \left( \frac{u^2}{3} + u^4 \right) K_0^2(u) + 3u^3 K_0(u) K_1(u) + (u^2 + u^4) K_1^2(u).$$  \hspace{1cm} (136)

Equivalently, this means that the energy flux per frequency interval reads

$$\frac{d\omega}{2\pi} \omega^0 \bar{I}_{ij}(\omega)^2 = \frac{32e^4}{2\pi v^4} F(u) du.$$  \hspace{1cm} (137)

Note that the integrand has to be doubled if one integrates on $\omega$ (or $u$) only from 0 to $+\infty$, instead of $-\infty$ to $+\infty$.

The results (134), (135), (136), (137), are equivalent to the classic results of Ruffini and Wheeler (see Ref. [40], pag. 127, and Ref. [41]) on the “splash gravitational radiation” from a particle in fast hyperbolic motion. To check the equivalence with their results it is useful to introduce the impact parameter (both in physical units and in its rescaled version)

$$b_{\text{phys}} = \frac{L}{\mu v_\infty}; \quad b = \frac{b_{\text{phys}}}{GM} = \frac{j}{v_\infty}.$$  \hspace{1cm} (138)

The argument $u$ of the Bessel functions above is then (consistently with the notation of Refs. [40, 41])

$$u = \frac{\omega}{\tilde{n}} = \frac{\omega b}{v_\infty},$$  \hspace{1cm} (139)

while the gravitational-wave (one-sided; $f = \omega_{\text{phys}}/(2\pi) > 0$) spectrum reads

$$\frac{d\Delta E}{df} = \frac{64G}{5} \left( \frac{G\mu M}{b_{\text{phys}}} \right)^2 F(u).$$  \hspace{1cm} (140)

The integrated gravitational-wave flux, i.e. the total energy emitted by the splash radiation is

$$\Delta E_{\text{splash}} = \frac{64}{5} \frac{v_\infty}{\pi b^3} (\mu M)^2 \int_0^\infty F(u)du.$$  \hspace{1cm} (141)

One finds that

$$\int_0^\infty F(u)du = \frac{37\pi^2}{96},$$  \hspace{1cm} (142)

so that

$$\frac{\Delta E_{\text{splash}}}{M} = \frac{37}{15} \pi^2 \frac{v_\infty^2}{b^3} = \frac{37}{15} \pi^2 \frac{v_\infty^2 v^4}{j^3}.$$  \hspace{1cm} (143)

Note that this result agrees with the large-$e$ limit of Eq. (108) (remembering $e \approx v_\infty j$).
After this check, let us come back to our main purpose, namely the Fourier-domain computation of the scattering potential $W_{\text{tail}}$. It is given by the following logarithmically-modified version of the total emitted gravitational-wave energy (taking into account the factor 2 linked to the one-sidedness of the integral)

$$W_{\text{tail}} = 2 \int_{0}^{\infty} df \frac{d\Delta E}{df} \ln \left( \frac{2s}{c} e^\gamma \right) \tag{144}$$

$$= \frac{64 \pi^2 v_{\infty}^2}{3^2} \int_{0}^{\infty} du F(u) \ln \left( \frac{2s}{c} e^\gamma \right),$$

where

$$\omega = \frac{v_{\infty} u}{b} = \frac{v_{\infty} u}{j} \tag{145}$$

so that the logarithm reads $\ln(\alpha u)$ with $\alpha = 2sv_{\infty}^2 e^\gamma / j$.

Now, we found that

$$\int_{0}^{\infty} F(u) \ln(\alpha u) du = \pi^2 \left[ \frac{25}{24} + \frac{37}{96} \ln \left( \frac{\alpha}{8e^\gamma} \right) \right]$$

$$= \frac{\pi^2}{96} \left[ 100 + 37 \ln \left( \frac{\alpha}{8e^\gamma} \right) \right] \tag{146}$$

Substituting the value of $\alpha = 2sv_{\infty}^2 e^\gamma / j$ then gives

$$\frac{W_{\text{tail}}}{M} = \frac{1}{M} \int dt H_{\text{tail}}$$

$$= \frac{2}{15} \pi v_{\infty}^2 \left[ 100 + 37 \ln \left( \frac{sv_{\infty}^2}{4j} \right) \right], \tag{147}$$

in agreement with the result of our time-domain computation Eq. (127).

C. Computation of the tail contribution to scattering in the large-eccentricity limit

Having confirmed the LO computation of the scattering potential, Eqs. (127), (147), we can now differentiate $W_{\text{tail}}(E, j)$ with respect to $j$ (at fixed $E$, i.e. at fixed $v_{\infty}$) to get the corresponding LO tail contribution to the scattering angle.

Putting back the energy scale $Mc^2$ in Eq. (88), but keeping the integral over the dimensionless time $t = e^3T/GM$, so that

$$\chi_{\text{LO}} = \frac{1}{\nu} \partial_j \left( \frac{1}{Mc^2} \int dt H_{\text{tail}} \right) = \frac{1}{\nu} \partial_j \left( \frac{W_{\text{tail}}}{Mc^2} \right) \tag{148}$$

we finally get

$$\chi_{\text{LO}} = \frac{2}{15} \pi \nu \frac{v_{\infty}^4}{j^3} \left[ 337 + 111 \ln \left( \frac{sv_{\infty}^2}{4j} \right) \right]$$

$$= \frac{2}{5} \pi \nu \frac{v_{\infty}^4}{j^4} \left[ 337 + 37 \ln \left( \frac{sv_{\infty}^2}{4j} \right) \right], \tag{149}$$

where we recall that $s \equiv e^3 a^{\text{phys}} / GM$ is a dimensionless regularization scale defining the nearzone-farzone separation.

As a further check on our result (149), we have shown that it agrees with a direct, dynamical computation based on the secular evolution of the Newtonian-conserved Laplace-Lagrange-Runge-Lenz, eccentricity vector under the action of the nonlocal, tail 4PN-level force $F_{(\text{nonloc})}$, Eq. (73). The reader will find this alternative calculation in Appendix E.

VII. SUMMING THE LOCAL AND NONLOCAL (TAIL) CONTRIBUTIONS TO $\chi^{4\text{PN}}$ IN THE LARGE-$j$ LIMIT

We have separately computed above (using an expansion in powers of $1/(jv_{\infty})$ where necessary) both the local and nonlocal (tail) contributions to the scattering function $\chi(E, j)$, with 4PN accuracy. Our results at the 1PN, 2PN and 3PN levels (see Eqs. (144)) were given as exact, closed-form expressions. By contrast, the 4PN-level value $\chi^{4\text{PN}}$ of the function was obtained as a sum of a closed-form local contribution (modulo the logarithmic term $I_{\chi}$), and of a nonlocal one (given as a large-eccentricity expansion). Let us here combine the two separate (local and nonlocal) 4PN-level contributions to $\chi(E, j)$. Replacing $E$ by $\frac{1}{2}v_{\infty}^2$ at 4PN we find

$$\chi^{4\text{PN}}_{\text{loc}} \left| _{v_{\infty} > \frac{3}{8}} = \frac{v_{\infty}^6}{j^3} \pi \right[ \frac{27}{64} \nu^2 + \frac{9}{64} \nu - \frac{15}{32} \nu^3 \right]$$

$$+ \frac{v_{\infty}^5}{j^3} \pi \left[ -8 \nu^3 - \frac{91}{5} \nu + 34 \nu^2 \right]$$

$$+ \frac{v_{\infty}^4}{j^3} \pi \left[ -225 \nu^3 - \frac{4827}{64} \nu + \frac{369}{512} \nu^2 \right]$$

$$+ \frac{91537}{960} + \frac{33601}{16384} \nu^2 + \frac{37}{5} \ln \left( \frac{v_{\infty}^2}{2j} \right) \nu$$

$$+ \frac{3465}{128}$$

$$+ O \left( \frac{v_{\infty}^3}{j^5} \right) \right.$$}

$$\chi^{4\text{PN}}_{\text{tail}} \left| _{v_{\infty} > \frac{3}{8}} = \frac{1}{5} \pi \frac{v_{\infty}^4}{j^4} \left[ \frac{337}{3} + 37 \ln \left( \frac{sv_{\infty}^2}{4j} \right) \right] \right.$$}

$$+ O \left( \frac{v_{\infty}^3}{j^5} \right). \tag{150}$$

Note that the notion of “n-PN-level contribution” to $\chi$ delicately depends on the choice of the energy variable used as argument for the function $\chi(E, j)$. The use of another energy measure, such as $E - Mc^2$ or $E_{\text{eff}}$ would lead to a complete reshuffling of each individual n-PN-level contribution, while leaving invariant only the sum over all PN levels.
Combining these two pieces only modifies the term $O(\nu)$ in \( v_{\infty}^4/j^4 \) and the final 4PN-level result is the following

\[
\chi_{\text{loc}} \approx 1, \quad \chi_{\text{tail}} \approx 1, \quad 4\text{PN}
\]

\[
\left. \frac{\chi_{\text{loc}} v_{\infty}^6 + \chi_{\text{tail}} v_{\infty}^5}{2} \right|_{4\text{PN}} = \frac{v^6}{j^4} \pi \left[ \frac{27}{64} \nu^2 + \frac{9}{64} \nu - \frac{15}{32} \nu^3 \right] + \frac{v^5}{j^3} \left[ -8 \nu^3 - \frac{91}{5} \nu + 34 \nu^2 \right] + \frac{v^4}{j^2} \pi \left[ -\frac{225}{32} \nu^3 + \left( \frac{4827}{64} - \frac{369}{512} \pi^2 \right) \nu^2 \right] + \frac{3301}{16384} \pi^2 + \frac{37}{5} \ln \left( \frac{2}{v_{\infty}} \right) - \frac{22621}{192} \nu + \frac{345}{128} + O \left( \frac{v_{\infty}^3}{j^5} \right).
\]

It is important to note that the arbitrary scale \( s \) has cancelled between the two contributions (as it should), but that it has left a “large-logarithm” contribution \( \ln \left( \frac{2}{v_{\infty}} \right) \).

**VIII. INCLUDING NNLO LINEAR-IN-SPIN CONTRIBUTIONS TO THE SCATTERING ANGLE**

To complete our 4PN-accurate computation of the orbital (i.e., non-spinning) contribution to the scattering, let us now tackle the additional contributions due to spin effects. For simplicity, we restrict ourselves to the (non-precessing) case of parallel spins, and we work only linearly in the spins.

One of the additional advantages of working within the EOB formalism is that it is rather straightforward to include the linear-in-spin contributions at NNLO to the scattering angle. Indeed, it is enough to complete the local calculation of Section III by including in the PN expansion of the function \( p_r(E, j) \) the linear-in-spin contributions. This is done by including in the EOB energy conservation law both the orbital, and the spin-orbit, terms. In other words, we start with the spin-dependent conserved energy

\[
\hat{\mathcal{E}}_{\text{eff}} = \sqrt{A(1 + j^2 u^2 + ADp_r^2 + \hat{Q})} + \frac{u^3}{M^2 j} [gs S + g_s S_*] .
\]

The two (phase-space-dependent) dimensionless gravitomagnetic ratios \( g_S \) and \( g_{S_*} \) are known (in the PN-expanded sense) at the next-to-next-to-leading-order (NNLO) level [42, 43].

\[
g_S^{\text{PN}}(u, p_r, p_\phi) = 2 + \eta^2 \left[ \frac{5}{8} u - \frac{27}{8} u p_r^2 \right] + \eta^4 \left[ \nu \left( -\frac{51}{4} u^2 - \frac{21}{2} u p_r^2 + \frac{5}{8} p_r^4 \right) \right] + \eta^2 \left[ \nu \left( -\frac{1}{8} u^2 + \frac{23}{8} v p_r^2 + \frac{35}{8} p_r^4 \right) \right] + O(\eta^6).
\]

\[
g_{S*}^{\text{PN}}(u, p_r, p_\phi) = \frac{3}{2} + \eta^2 \left[ -\frac{9}{8} u - \frac{15}{8} p_r^2 + \nu \left( -\frac{3}{4} u - \frac{9}{4} p_r^2 \right) \right] + \eta^4 \left[ \nu \left( -\frac{27}{16} u^2 + \frac{69}{16} u p_r^2 + \frac{35}{16} p_r^4 \right) \right] + \nu \left( -\frac{39}{4} u^2 - \frac{9}{4} u p_r^2 + \frac{5}{2} p_r^4 \right) + \nu^2 \left( -\frac{3}{16} u^2 + \frac{57}{16} u p_r^2 + \frac{45}{16} p_r^4 \right) + O(\eta^6).
\]

The values of \( g_S \) and \( g_{S_*} \) cited above have been expressed in the Damour-Jaranowski-Schaefer (DJS) spin gauge [44,45], which is defined so that these quantities do not actually depend on \( p_\phi \). Let us note in passing that recent gravitational self-force computations have extended the knowledge of \( g_S \) and \( g_{S_*} \) to a high PN level [46, 47]. We will, however, not make use of this (partial) knowledge here.

Passing, as above, to the energy variable \( \tilde{E} = \frac{1}{2} (\hat{\mathcal{E}}_{\text{eff}}^2 - 1) \) and solving (in a PN sense) the energy conservation law (152) for \( p_r = p_r^{(0)} + \eta p_r^{(1)} + \eta^2 p_r^{(2)} + \eta^3 p_r^{(3)} + \ldots \) (where \( \eta = 1/c \)), leads, in addition to the terms involving even powers of \( \eta \) that we used in Section III to the following additional, \( \eta \)-odd contributions:
\[ p_r(1) = 0 \]
\[ p_r(3) = -\frac{j u^3}{2 M^2 p_r^{(0)}(4 S + 3 S_\nu)} \]
\[ p_r(5) = \frac{j u^3}{8 M^2 p_r^{(0)} 3} [A^{(5, S)} S + A^{(5, S_\nu)} S_\nu] \]
\[ p_r(7) = -\frac{u^3 j}{16 M^2 p_r^{(0) 5}} [A^{(7, S)} S + A^{(7, S_\nu)} S_\nu] \]

where

\[ A^{(5, S)} = (-108 E j^2 u^2 + 27 j^4 u^4 + 118 u^2 - 113 j^2 u^3 + 226 u E + 108 E^2) \nu - 64 u^2 + 48 j^2 u^3 - 96 u E + 16 E j^2 u^2 - 32 E^2 \]
\[ A^{(5, S_\nu)} = (-78 j^2 u^3 + 84 u^2 + 72 E^2 + 18 j^4 u^4 - 72 E j^2 u^2 + 156 u E) \nu + 66 u E + 30 u^2 + 15 j^4 u^4 - 48 E j^2 u^2 - 33 j^2 u^3 + 36 E^2 \]
\[ A^{(7, S)} = (-7816 u^2 E^2 - 1680 u^4 E^3 + 7816 u^4 E j^2 + 560 u^6 j E - 1480 u^4 - 1120 E^4 + 606 u^7 j^6 + 2240 E j^2 u^2 - 1954 u^6 j^4 + 7272 u^3 E j^2 - 3636 u^5 j^4 E - 70 u^8 j^8 + 2784 u^4 j^2 u^5 - 5568 u^3 E j^4 \nu^2 + (4632 u^3 E - 606 u^3 E j^2)^2 + 13504 u^2 E j^2 - 12404 u^4 E j^2 - 358 u^7 j^6 + 4992 u^4 - 84 u^4 j^4 E^2 - 10 u^8 j^8 - 328 E^3 j^2 u^2 + 26 u^6 j E + 14136 E j^3 + 1372 E^2 - 660 j^6 u^5 + 2590 u^5 j^4 E + 2826 u^3 j^4 \nu - 512 u^4 + 256 u^3 E j^2 - 128 u^5 j E + 960 u^4 E j^2 - 128 u^7 j E - 960 u^2 E^2 + 16 u^4 j^4 E^2 + 640 j^2 u^5 + 64 E^4 - 96 u^5 j^4 E - 240 u^4 j^4 - 64 E^3 j^2 u^2 \]
\[ A^{(7, S_\nu)} = (-3336 u^3 E^3 + 5004 u^3 E j^2 - 5676 u^2 E^2 + 1440 E^3 j^2 u^2 - 1080 u^4 j^4 E^2 - 1164 u^4 - 720 E^4 + 417 u^7 j^6 + 360 u^6 j^6 E - 1419 u^6 j^2 + 2502 u^5 j^4 E + 576 u^4 E j^2 - 45 u^8 j^8 + 2112 j^2 u^5 - 4224 u^3 E \nu^2 + (368 u E^3 + 2336 u^5 j E + 39724 u^3 E^3 - 744 u^4 j^4 E^2 + 4176 u^2 E^2 + 104 u^7 j^8 + 284 u^6 j^6 E - 352 E^3 - 40 u^8 j^8 + 2608 j^2 u^5 + 848 E^3 j^2 u^2 + 48 u^3 E^2 j^2 - 3432 u^4 E j^2 - 672 u^6 j^4 \nu + 196 u^4 + 1548 u^3 E j^2 + 712 E^3 j^2 u^2 + 100 j^2 u^5 + 199 u^7 j^6 + 250 u^6 j^6 E + 280 u^3 E - 35 u^8 j^8 - 64 u^4 j^4 E^2 - 272 E^4 - 1008 u^3 j^4 E + 936 u^2 E j^2 - 656 u E^3 - 300 u^2 E^2 - 393 u^4 \).

and

\[ p_r^{(0)} = \sqrt{2E - j^2 u^2 + 2u} \]

We compute then

\[ U(E, j, \nu; u) = -\frac{1}{u^2} \partial_j p_r \]

and finally we obtain the (half) of the scattering angle

\[ \frac{1}{2}(\chi + \pi) = \text{Pf} \int_0^{u_{\text{max}}} U(E, j, \nu; u) du \]

with, as usual, \( u_{\text{max}} \) computed at the Newtonian approximation

\[ u_{\text{max}} = \frac{1}{j^2} [1 + \sqrt{1 + 2E j^2}] . \]

The result is the following

\[ \chi^{\text{LO}}_2 = -\frac{2}{j^3} \left[ B(\alpha) + \frac{(1 + 2\alpha^2)}{2\alpha(1 + \alpha^2)} \right] \left( 2S + \frac{3}{2} S_\nu \right) \]
\[ \chi^{\text{NNLO}}_2 = \frac{1}{j^5} \left[ C_{\text{B,NLO}} B(\alpha) + C_{\text{NLO}}^{\text{NNLO}} \right] \]
\[ \chi^{\text{NNLO}}_2 = \frac{1}{j^7} \left[ C_{\text{B,NNLO}} B(\alpha) + C_{\text{NLO}}^{\text{NNLO}} \right] , \]
with
\begin{align*}
C_{0}^{\text{NLO}} &= \frac{1}{8\alpha^{3}(1+\alpha^{2})^{2}}[C_{0}^{\text{NLO},S} S + C_{0}^{\text{NLO},S} S_{*}]
\end{align*}

The part of the background scattering angle \( \lim_{\nu \to 0} \chi \) proportional to \( S \) has been independently checked by studying geodesic motion in the Kerr spacetime, at linear order in the rotational parameter of the black hole.

\section{IX. Determination of the Variation of \( W^{\text{tail}}(e, j) \) (and \( H^{\text{tail}} \)) with the Eccentricity in the Full Hyperbolic Domain \( 1 \leq e \leq +\infty \)}

We have explicitly computed above the LO approximation to the large-eccentricity expansion of the tail scattering potential \( W^{\text{tail}} \), and we have also indicated that, in principle, modulo the tackling of more complicated integrals, our large-\( e \) expansion can be continued to higher orders in \( 1/e \). Here we wish to complete this discussion by briefly considering the global behavior of \( W^{\text{tail}}(e, j) \), considered as a function of \( e \) and \( j \) (rather than \( E = v_{\infty}^{2}/2 \) and \( j \)), when, for a fixed value of \( j \), the eccentricity varies over the complete range of hyperbolic motions, i.e. from \( e = \infty \) down to the parabolic case where \( e = 1 \).

First, we note that our general Fourier-domain result Eq. \( \text{[105]} \) suggests to write the value of \( W^{\text{tail}} = \int dt H^{\text{tail}} \) in the form
\begin{align*}
W^{\text{tail}}(e, j) &= 2 \Delta E_{GW}(e, j) \ln \left( \frac{\omega_{c}(e, j)}{c} \right) \tag{165}
\end{align*}

where the total gravitational-wave energy loss \( \Delta E_{GW}(e, j) \) is given by Eq. \( \text{[105]} \) \cite{105}, and where \( \omega_{c}(e, j) \) is some characteristic frequency of the gravitational-wave spectrum.

Our LO computation above (see \( \text{[127]} \) with \( v_{\infty} \approx c/j \)) has shown that, when \( e \gg 1 \), the characteristic frequency \( \omega_{c}(e, j) \) (or rather its logarithm) is asymptotically equivalent to
\begin{align*}
\ln \omega_{c}(e, j) &\approx \ln \left( \frac{e^{2}}{4j^{3}} \right) + \frac{100}{37} \tag{166}, \quad \text{when } e \to \infty.
\end{align*}

Actually, when considering what is the characteristic orbital frequency of hyperbolic motions as \( e \) varies between \( \infty \) and 1, one can easily see that it will parametrically scale as \( e^{2}/j^{3} \) over the full range of hyperbolic motions. In particular, it must be of order \( 1/j^{3} \) for parabolic motions (\( e = 1 \)), because this is the only frequency scale for such motions. In addition, if we had considered the gravitational-wave spectrum in usual (say CGS) physical units, as the scale \( s \) enters the tail logarithm in the
form of a time scale $s_{\text{phys}}/c$ (as recalled in Eq. (165)), the physical-units version of the characteristic frequency
\[ \omega_c(e,j) \] must indeed have the dimension of $[\text{time}]^{-1}$. Dimensional analysis then shows that, modulo a function of the dimensionless eccentricity $e$, $\omega_c(e,j)$ must scale proportionally to the inverse cube of the angular momentum. In other words, coming back to our scaled units, we conclude that the product $C(e,j)^3$ can only be a function of the (dimensionless) eccentricity $e$. In conclusion, we can parametrize the behavior of $W_{\text{tail}}(e,j)$ over the full range of hyperbolic motions by writing Eq. (165) together with an expression for $\omega_c(e,j)$ of the form
\[ \ln \omega_c(e,j) \equiv \ln \left( \frac{e^2}{4j^3} \right) + C(\frac{1}{e}), \] where the dimensionless contribution $C(\frac{1}{e})$ is only a function of $e$.

The issue of controlling the behavior of $W_{\text{tail}}(e,j)$ over the full range of hyperbolic motions is then reduced to controlling the variation of the (dimensionless) function $C(e)$ as $e \equiv \frac{1}{e}$ increases from 0 to 1. We already know from Eq. (166) that
\[ C(0) = \frac{100}{37}. \] We have studied the variation of $C(e)$ over the full interval $0 \leq e \equiv \frac{1}{e} \leq 1$ by two different approaches: (i) by an analytical study of the limit $e = 1$ (parabolic motion); and (ii) by a numerical study of the Fourier-domain expression of $W_{\text{tail}}(e,j)$ for a sample of intermediate values $0 < e < 1$. Let us here only briefly summarize our results.

In dimensionless units parabolic orbits ($e \equiv 1/e = 1$) can be parametrized as follows [see [48], pag. 75, Ex. 1]
\[ x = \frac{p^2}{2}(1-\eta^2), \quad y = pq, \quad r = \frac{p}{2}(1+\eta^2) \]
\[ t = \frac{p^3/2}{2} \left( 1 + \frac{\eta^2}{3} \right), \] where $p = j^2$ and $\eta \in (-\infty, \infty)$. Comparing with the polar representation of the parabolic orbit,
\[ r = \frac{j^2}{1 + \cos \phi} = \frac{j^2}{2 \cos^2 \frac{\phi}{2}}, \] we find that
\[ \eta = \tan \frac{\phi}{2}, \quad \phi = 2 \arctan \eta. \] The time-split flux function $F(t, t')$, (162), then becomes
\[ F(\eta, \eta') = \frac{512}{15j^{10}(1+\eta^2)^5(1+\eta'^2)^5} P(\eta, \eta'). \] where
\[ P(\eta, \eta') = (\eta^5 + 8\eta^3 - 11\eta)\eta'^5 + (8\eta^7 + 76\eta^5 - 112\eta)\eta'^3 + (-60 + 300\eta^2)\eta'^2 + (-112\eta^3 + 169\eta - 11\eta^5)\eta' + 12 - 60\eta^2. \] We then transform the partie-finie integral over $t'$ we are interested in, namely
\[ \text{Pf}_{2s/c} \int \frac{dt'}{|t-t'|} F(t, t') \] in an integral over $\eta'$, using $dt' = \frac{j^2}{2}(1 + \eta'^2)d\eta'$, $|t-t'| = \frac{1}{j}|\eta - \eta'|((\eta^2 + \eta'^2 + \eta'^2 + 3)$, and also taking care of correspondingly changing the regularization time scale into $T_\eta = 2s/c(\eta dt'/d\eta') = 4s/(c j^3(1+\eta^2))$:
\[ \text{Pf}_{\eta} \int \frac{d\eta'}{|\eta - \eta'|(\eta^2 + \eta'^2 + \eta'^2 + 3)} F(\eta, \eta'). \] We then find
\[ \text{Pf}_{2s} \int \frac{dt'}{|t-t'|} F(t, t') = \frac{1}{j^{10}} \frac{512}{5} \left\{ \frac{1}{3} \left( \frac{\eta^2 + 12}{1 + \eta^2} \right) \ln \left( \frac{3}{T_\eta} \right) (1 + \eta^2) - \frac{\eta^6 + 30\eta^4 + 230\eta^2 + 288}{(1 + \eta^2)^6(\eta^2 + 4)^2} \right. \]
\[ + \frac{2}{\sqrt{3}} \left( \eta^6 + 4\eta^4 - 60\eta^2 + 28 \right) \arctan \left( \frac{\sqrt{3}\eta}{\sqrt{\eta^2 + 4}} \right) \left\} \right. \]
\[ = \frac{1}{j^{10}} F(\eta). \] We can then compute $W_{\text{tail}} = \int dt H_{\text{tail}}$ by integrating over $t$ (using $dt = \frac{j^2}{2}(1 + \eta^2)d\eta$):
\[ W_{\text{tail}}(e = 1,j) = -\int dt \text{Pf}_{2s} \int \frac{dt'}{|t-t'|} F(t, t') \]
\[ = -\int \frac{j^2}{2}(1 + \eta^2) \frac{1}{j^{10}} F(\eta)d\eta \]
\[ = \frac{340}{3j^2} \frac{Mv^2}{c^5} \left[ \frac{416}{85} - \frac{1}{2} \ln 3 + \ln \left( \frac{s}{4j^2} \right) \right]. \]
Factoring

\[ 2\Delta E_{\text{BS}}(e = 1, j) = \frac{340}{3} \pi e^3 M \nu^2 j \]

we then get a result which can be written as in Eq. (105), with \( \ln \omega_\varepsilon \) of the form of Eq. (167) with the dimensionless additional constant having the value

\[ C(1) = \frac{416}{85} \cdot \frac{1}{2} \ln 3 \approx 4.344811502. \] (179)

By comparison, we recall the value \( C(0) = 100/37 \approx 2.702 \), obtained above in Eq. (108).

To complete our analytical determination of the function \( C(\varepsilon) \) at the two extreme values \( \varepsilon = 0 \), Eq. (108), and \( \varepsilon = 1 \), Eq. (179), of hyperbolic motions, we have also computed numerical estimates of \( C(\varepsilon) \) at a sample of intermediate values of the inverse eccentricity: \( 0 < \varepsilon < 1/\epsilon < 1 \). We found useful to compute the function \( W_{\text{tail}}(e, j) \) by rewriting our Fourier-domain formula Eq. (105) in terms of the rescaled frequency variable \( v \), defined as

\[ v \equiv \frac{\omega}{\omega_*}, \quad \text{with} \quad \omega_* = \frac{e^2}{j^2}. \] (180)

Indeed, we have already mentioned that \( \omega_* \) measures, for all values of \( e \geq 1 \), the characteristic frequency of gravitational wave emission. In terms of \( v \) our Fourier-domain integral reads

\[
W_{\text{tail}}(e, j) = \frac{2}{5\pi j^{21}} \times \int_0^\infty dv v^6 |J_3(v; e, j)|^2 \ln \left(2\pi e^2 j^3 v \right). \] (181)

Using (181), we computed the numerical values of the additional constant \( C(\varepsilon) \) in Eq. (167) for a sample of intermediate values of \( \varepsilon \) in the hyperbolic-motion interval

\[ 0 < \varepsilon < 1. \] They are listed in Table I.

| \( \varepsilon = 1/\epsilon \) | \( C(\varepsilon) \) | \( c(\varepsilon) \) |
|---|---|---|
| 0 | 2.702702703 | 0 |
| 10^{-5} | 2.7027272104 | 0.148555134 \times 10^{-4} |
| 1/52345 | 2.702749308 | 0.2838118889 \times 10^{-4} |
| 10^{-4} | 2.702946704 | 0.1485900326 \times 10^{-3} |
| 10^{-3} | 2.705141653 | 0.1485254815 \times 10^{-2} |
| 10^{-1} | 2.935443483 | 0.1417328621 |
| 1/8 | 2.990307450 | 0.1751435393 |
| 1/6.2 | 3.06754516 | 0.2223067151 |
| 1/4.33 | 3.209535882 | 0.3086477457 |
| 1/3.62 | 3.297190578 | 0.3620270931 |
| 1/2.73 | 3.461705299 | 0.4622121241 |
| 1/2.21 | 3.608040700 | 0.5513264392 |
| 1/2.13 | 3.635709539 | 0.5681760165 |
| 1/1.5 | 3.931344289 | 0.7482096112 |
| 1/1.412 | 3.988242935 | 0.7828593530 |
| 1/1.29 | 4.076465716 | 0.8365846489 |
| 1/1.13 | 4.212464303 | 0.9194041229 |
| 1/1.0345 | 4.307440862 | 0.9772422875 |
| 1/1.0243 | 4.318308127 | 0.9838601590 |
| 1 | 4.344811502 | 1 |

We have completed Table I by listing in the last column the values of the normalized version, \( c(\varepsilon) \), of \( C(\varepsilon) \), defined as

\[ c(\varepsilon) = \frac{C(\varepsilon) - C(0)}{C(1) - C(0)}, \]

\[ C(\varepsilon) = C(0) + [C(1) - C(0)] c(\varepsilon). \] (182)

Note that \( c(\varepsilon) \) varies between 0 and 1 as \( \varepsilon \) varies between 0 and 1. [In the following we assume that \( C(0) \) and \( C(1) \) take their exact analytical values (168) and (179).] We have explored several possible simple, analytic fits for \( c(\varepsilon) \). For instance, the cubic polynomial

\[ e_{\text{cubic}}(\varepsilon) = 1.4730\varepsilon - 0.6318\varepsilon^2 + 0.1588\varepsilon^3, \] (183)

agrees with the numerical data of Table 1 within \( 6 \cdot 10^{-4} \) (maximum difference). A more accurate representation (maximum difference \( 5 \cdot 10^{-6} \)) is given by the following Padé approximant

\[ e_{\text{Padé}}(\varepsilon) = \frac{0.018265\varepsilon^2 - 0.715430\varepsilon + 1.485639}{1 - 0.211528\varepsilon^2}. \] (184)

The data points for \( C(\varepsilon) \) given in Table I are plotted in Fig. 1, and compared there with the simple cubic fit (183). By inserting in Eqs. (165), (166), (167), the results on \( C(\varepsilon) \) we have just given, we obtain a global representation of the scattering potential \( W_{\text{tail}}(e, j) = \int dt H_{\text{tail}} \) (and therefore of \( \chi_{\text{tail}} \)) over the full range of hyperbolic motions. [Note that, in order to derive \( \chi_{\text{tail}} \) from \( W_{\text{tail}}(e, j) \),
we have to replace $e$ as a function of $E$ and $j$ (namely $e = \sqrt{1 + 2Ej^2}$ before differentiating with respect to $j$ (at fixed energy).]

Let us go one step further, and deduce from our results a corresponding global description of the non-integrated tail Hamiltonian $H_{\text{tail}}(q,p)$ over the full range of hyperbolic motions. [Here, we denote by $(q,p)$ some canonical phase-space coordinates, say $(r,p_r,\phi,p_\phi)$.] There are several ways of constructing such a (time-localized) tail Hamiltonian. Let us start by remarking that the factor $\Delta E_{\text{GW}}(e,j)$ in Eq. (163) is the time integral of the instantaneous gravitational-wave flux

$$F_E(q,p) = \frac{8\nu^2}{2r^4} \left( 4p^2 - \frac{11}{3}p_r^2 \right).$$

As a consequence, we can define (using one of the global representations of $C(1/e)$ discussed above)

$$H_{\text{tail},C}(q,p) \equiv +2F_E(q,p) \left[ \ln \left( \frac{e^2s}{4c^3p_\phi^3} \right) + C\left(\frac{1}{e}\right) \right]_{e=e(q,p)}.$$  

This local function of $(r,p_r,\phi,p_\phi)$ decreases as $1/r^4$ when the binary separation increases, and its time-integral is equal to $W_{\text{tail}}(e,j)$. Thereby, $H_{\text{tail},C}(q,p)$ provides a time-localized global description of the tail contribution to the Hamiltonian over the full range of hyperbolic motions. It differs from our original time-localized Hamiltonian Eq. (70) by some $O(1/e^8)$ canonical transformation (corresponding to adding a total time derivative).

When considering the vicinity of $1/e = 0$, we have shown that, modulo an additional canonical transformation, one can replace the Hamiltonian (185) by the following more explicit function of positions and momenta

$$\frac{H_{\text{tail}}}{M} = +2F_E(q,p) \ln \left( \frac{js}{cr^2} \right) + \frac{8}{15r^4}s^2 (36p^2 - 29p_r^2) + O\left(\frac{1}{e}\right).$$

The latter, tail Hamiltonian is the large-eccentricity counterpart of the (time-localized) small-eccentricity tail Hamiltonian introduced in Ref. [20] which reads, at order $O(e^0)+O(e^2)$ (after using a canonical transformation with generating function proportional to $p_r/r^3$ to trade the terms $\propto 1/r^3$ by terms $\propto (p^2 - 4p_r^2)/r^4$)

$$\frac{H_{\text{tail}}}{M} = +2F_E(q,p) \ln \left( \frac{s}{cr^{3/2}} \right) + \frac{8}{15r^4}s^2 (\alpha_{\text{ell}} p^2 + \beta_{\text{ell}} p_r^2) + O\left(e^4\right).$$

with

$$\alpha_{\text{ell}} = 24 \ln(4\gamma)$$

$$\beta_{\text{ell}} = -198 - 22\gamma + \frac{2187}{4} \ln 3 - 598 \ln 2.$$  

**X. CONCLUDING REMARKS**

Using the recently computed 4PN-level EOB Hamiltonian [20] (involving both local and nonlocal parts), we have computed here for the first time the corresponding 4PN-accurate (conservative) scattering angle $\chi$ of hyperbolic-like encounters of (comparable-mass) two-body systems. The scattering angle was previously known only up to the 2PN level [17]. While our computation of the local part of $\chi$ makes use of rather well established PN techniques (simplified by the use of the EOB formalism), our computation of the tail part is based on novel techniques. In addition, we have completed our computation of $\chi$ by providing the contributions linear in the spins of the two bodies, at the next-to-next-to-leading PN order.

Both as a check on our computation, and as a way to provide several different viewpoints on the scattering of binary systems, we have implemented several different methods for computing the nonlocal, tail contribution to $\chi$, which is the most subtle part of our calculations.

First, we introduced a generalization of the time-localization technique introduced in Refs. [20, 26]. Indeed, the (Delaunaylike) time-localization technique of Refs. [20, 26] was limited to the case of elliptic-like motions. Here, we have shown how to generalize this idea to the case of hyperbolic-like motions. Our final

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9 A way to deal with the effect of radiation-reaction on $\chi$ has been given in Ref. [17]
result is that the tail part $\chi_{\text{tail}}$ of the scattering angle can be derived via the $j$-gradient of a tail potential: $W^{\text{tail}}(E, j) = \int H^{\text{tail}} dt$. We then gave two different methods for computing the scattering potential $W^{\text{tail}}(E, j)$. One method works directly in the time-domain, while a second method works in the frequency-domain. The final result of the latter method, given in Eq. (105), illuminates the physics of $W^{\text{tail}}(E, j)$ by showing that it is a logarithm-weighted avatar of the frequency-decomposition of the gravitational-wave emission of binary systems.

In addition, we showed (in Appendix B) how to compute $\chi_{\text{tail}}$ by considering the evolution (during the hyperbolic encounter) of the Laplace-Lagrange-Runge-Lenz eccentricity vector $A$. This additional computation provides a nice check of our large-eccentricity generalization of the time-localization technique.

Besides providing general formulas for $\chi_{\text{tail}}$, we have also shown that the functional dependence of the scattering potential $W_{\text{tail}}(e, j)$ on the eccentricity $e$ and the angular momentum $j$ could be described, through Eqs. (105, 107) (where $\Delta_E^{\text{GW}}(e, j)$ is the integrated gravitational wave luminosity, Eq. (108)), by means of a function of the sole eccentricity, namely the function $C(1/e)$ entering Eq. (107). We have analytically computed the value of $C(e)$ (where $e \equiv 1/e$) for $e = 0$ (large eccentricity limit) and for $e = 1$ (parabolic motions). In addition, we have numerically estimated the value of $C(e)$ for a sample of intermediate eccentricity values $0 < e < 1$, and provided some simple analytic fits allowing one to describe $C(e)$ in the full interval $0 \leq e \leq 1$. This allowed us to analytically describe both $W_{\text{tail}}$, and a specific time-localized version of the tail Hamiltonian $H_{\text{tail}}(q, p)$, over the full range of hyperbolic motions (see Eq. (108)). We have also shown how our results for $\chi_{\text{tail}}$ in the large-eccentricity limit are related to the classic results of Ruffini and Wheeler on the “gravitational splash radiation” [40, 41].

In addition to the main results summarized above, we have also provided: (i) a sketch of an alternative method for computing the local part $\chi_{\text{loc}}$ of the scattering angle (see Appendix C); and (ii) the explicit form of a local Hamiltonian yielding the large-eccentricity limit of the scattering.

Summarizing, this work is a contribution to the programme [12, 13, 17] of extending the EOB formalism beyond the inspiral-plunge-merger regime of binary systems to the regime of high-energy hyperbolic encounters of binary systems.

### Appendix A: Computing $\hat{I}_{ij}(\omega)/\mu$ in terms of Hankel’s functions

We give here some details of the derivation of the Fourier transform $\hat{I}_{ij}(\omega)$ of the quadrupole moment along binary hyperbolic motions, as given in Eqs. (119).

The nonvanishing components of $\hat{I}_{ij}(\omega)$ are\(^{10}\)

\[
\hat{I}_{xx}(\omega) = \left( \frac{2}{3} x^2 - \frac{1}{3} y^2 \right) \omega \\
\hat{I}_{yy}(\omega) = \left( \frac{2}{3} y^2 - \frac{1}{3} x^2 \right) \omega \\
\hat{I}_{xy}(\omega) = (xy) \omega \\
\hat{I}_{zz}(\omega) = -\left( \frac{1}{3} x^2 + \frac{1}{3} y^2 \right) \omega \tag{A1}
\]

so that the square of $\hat{I}_{ij}(\omega)$ reads

\[
|\hat{I}_{ij}(\omega)|^2 = \frac{2}{3} \left[ |(x^2)\omega|^2 + |(y^2)\omega|^2 \right] + 2 |(xy)\omega|^2 - \frac{1}{3} \left[ |(x^2)(y^2)^{-\omega} + (x^2)^{-\omega}(y^2)^\omega \right] \tag{A2}
\]

We are interested here in (Newtonian level) hyperbolic motions. This is conveniently parametrized as

\[
x = -a (\cosh \bar{u} - e) \\
y = -a \sqrt{e^2 - 1} \sinh \bar{u} \\
\bar{n}t = e \sinh \bar{u} - \bar{u} \tag{A3}
\]

with $a = (-2E)^{-1}$ and $\bar{n} = (-a)^{-3/2} = v_\infty^2$. [Note that, when discussing hyperbolic motion (with $E > 0$), it is sometimes convenient to replace $a$ by $\bar{a} \equiv -a = (+2E)^{-1} > 0$]

The Fourier transform of the quadrupole tensor is computed by replacing the time integration by an integration over $\bar{u}$ (later denoted as $\xi$):

\[
(x^2)_{\omega} = \int dt e^{i\omega t} x^2(t) = a^2 \int \frac{dt}{du} (\cosh \bar{u} - e)^2 e^{i\hat{\omega} \bar{u}} (e \sinh \bar{u} - \bar{u}) d\bar{u} \\
= \frac{a^2}{\bar{n}} \int d\xi (e \cosh \xi - 1)(e \cosh \xi - e)^2 e^{i\hat{\omega} (e \sinh \xi - \xi)} \\
(y^2)_{\omega} = \int dt e^{i\omega t} y^2(t) = a^2 (e^2 - 1) \int \frac{dt}{du} \sinh^2 \bar{u} e^{i\hat{\omega} (e \sinh \bar{u} - \bar{u})} d\bar{u} \\
= \frac{a^2 (e^2 - 1)}{\bar{n}} \int d\xi (e \cosh \xi - 1)^2 \sinh^2 \xi e^{i\hat{\omega} (e \sinh \xi - \xi)} \\
(xy)_{\omega} = \int dt e^{i\omega t} (x(t)y(t) = a^2 \sqrt{e^2 - 1} \int \frac{dt}{du} (\cosh \bar{u} - e) \sinh \bar{u} e^{i\hat{\omega} (e \sinh \bar{u} - \bar{u})} d\bar{u} \\
= \frac{a^2 \sqrt{e^2 - 1}}{\bar{n}} \int d\xi (e \cosh \xi - 1)(e \cosh \xi - e) \times \sinh \xi e^{i\hat{\omega} (e \sinh \xi - \xi)} \tag{A4}
\]

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\(^{10}\) Here we work, without explicitly indicating it, with the rescaled center-of-mass frame quadrupole moment $I_{ij}/\mu$, and we use $GM$-rescaled space and time units.
from which we see that \((x^2)_{\omega} = [(x^2)_{\omega}]^*\).

The above integrals can be expressed in terms of Hankel’s functions of the first kind \(H^{(1)}\) (see Eq. 9.1.25 in Ref. [39])

\[
\int_{-\infty}^{\infty} e^{q \sinh \xi - \nu \xi} d\xi = i \pi H^{(1)}_p (q) . \tag{A5}
\]

In our case, the argument \(q\) and the order \(p\) are

\[
q = i e^{\omega} n, \quad p = i \omega n = \frac{q}{e} . \tag{A6}
\]

A direct computation (based on decomposing in exponentials of \(\xi\) the factors in front of \(e^{i \pi (\nu \sinh u - u)}\)) gives

\[
(x^2)_{\omega} = \frac{\pi a^2}{2n} \left[ e^{\left(\frac{11}{4} + e^2\right)} [H^{(1)}_{p-1}(q) + H^{(1)}_{p+1}(q)] - \left(1 + 4e^2\right) H^{(1)}_p(q) \right] \tag{A7}
\]

\[
(y^2)_{\omega} = \frac{\pi a^2}{4n} \left[ e^{\left(\frac{e^2 - 1}{4}\right)} \frac{1}{2} [H^{(1)}_{p-1}(q) + H^{(1)}_{p+1}(q)] - [H^{(1)}_{p-2}(q) + H^{(1)}_{p+2}(q)] + \frac{e}{2} [H^{(1)}_{p-1}(q) + H^{(1)}_{p+1}(q)] + 2H^{(1)}_p(q) \right] \tag{A8}
\]

\[
(xy)_{\omega} = \frac{\pi a^2}{4n} \left[ e^{\left(\frac{e^2 - 1}{4}\right)} \frac{1}{2} [H^{(1)}_{p-1}(q) - H^{(1)}_{p+1}(q)] + (\epsilon^2 + 1) [H^{(1)}_{p-2}(q) - H^{(1)}_{p+2}(q)] - \frac{5e}{2} [H^{(1)}_{p-1}(q) - H^{(1)}_{p+1}(q)] \right] . \tag{A9}
\]

These expressions can be simplified by using standard recurrence relations valid for arbitrary Bessel functions \(C_p(q)\). Indeed, using the short-hand notation \(C_{\langle p, n \rangle}(q) \equiv C_{p-n}(q) + C_{p+n}(q)\), \(C_{\langle p, n \rangle}(q) \equiv C_{p+n}(q) - C_{p-n}(q)\), we have

\[
C_{\langle p, 1 \rangle}(q) = \frac{2p}{q} C_p(q)
\]

\[
C_{\langle p, 1 \rangle}(q) = 2C'_p(q)
\]

\[
C_{\langle p, 2 \rangle}(q) = 2 \left(\frac{2p^2 - 1}{q}\right) C_p(q) - \frac{4}{q} C'_p(q)
\]

\[
C_{\langle p, 2 \rangle}(q) = \frac{4p}{q} \left[C'_p(q) - \frac{8}{q} C_p(q)\right]
\]

\[
C_{\langle p, 3 \rangle}(q) = -C_{\langle p, 1 \rangle}(q) + 2p C_{\langle p, 2 \rangle}(q) - \frac{4}{q} C_{\langle p, 1 \rangle}(q)
\]

\[
C_{\langle p, 3 \rangle}(q) = -C_{\langle p, 1 \rangle}(q) + 2p C_{\langle p, 2 \rangle}(q) - \frac{4}{q} C_{\langle p, 1 \rangle}(q) . \tag{A8}
\]

Using these relations, one can simplify the expressions

\[
(x^2)_{\omega} = -\frac{2\pi}{u^2 v^2} \left[ \left(-ue + \frac{u}{e} + i\right) H^{(1)}_{\frac{q}{u}}(iu) + u(e^2 - 1) H^{(1)}_{\frac{q}{ue} + 1}(iu) \right] \tag{A7}
\]

\[
(y^2)_{\omega} = -\frac{2\pi(u^2 - 1)}{e^2 v^2 u^2} \left[ \left(\frac{u}{e} + i\right) H^{(1)}_{\frac{q}{u}}(iu) - u H^{(1)}_{\frac{q}{ue} + 1}(iu) \right] \tag{A8}
\]

\[
(xy)_{\omega} = \frac{2\pi\sqrt{e^2 - 1}}{e^2 v^2} \left[ \left(-ue + \frac{u}{e} + i\right) H^{(1)}_{\frac{q}{u}}(iu) - i e H^{(1)}_{\frac{q}{ue} + 1}(iu) \right] , \tag{A9}
\]

where we introduced the variable \(u \equiv -iq = \frac{\omega}{e}\).

Using the above expressions, one can, in principle, expand \(I_{ij}(\omega)\) in powers of \(\epsilon = 1/e\). In the text, we show the result so obtained at leading order in \(\epsilon\). The higher-order terms in \(\epsilon\) are more complicated and involve derivatives of the Hankel’s functions with respect to the order (in the vicinity of orders 0 or 1).

**Appendix B: Direct dynamical computation of \(\chi_{\text{tail}}\) by using \(\mathcal{F}_{\text{nonloc}}\)**

In this Appendix we consider the traditional approach to computing the tail contribution to the 4PN scattering function; namely, we study the effect of adding to the Hamilton equations of motion derived from the local Hamiltonian \(H_{\text{4PN loc}}\), the additional nonlocal (tail) force \(\mathcal{F}_{\text{nonloc}}\), Eq. (73). As in the main text, this can be done simply by adding the effect of \(\mathcal{F}_{\text{nonloc}}\) on the Newtonian level dynamics. The main conceptual difference with the time-localization technique used in the main text is that, here, we shall time-localize tail effects at the level of the equations of motion, while, in Sec. X, we used a time-localized Hamiltonian tail action to compute \(\chi_{\text{tail}}\).

We then consider the following tail-perturbed Newtonian equations of motion

\[
\frac{dp}{dt} = \hat{\mathcal{F}}_{\text{tot}} - \hat{\mathcal{F}}_{\text{Newton}} + \hat{\mathcal{F}}_{\text{nonloc}} = \frac{n}{r^2} + \mathcal{F}_{\text{nonloc}} , \tag{B1}
\]

where \(n = e \dot{\epsilon}\), and where we used scaled variables \(r, p\) and \(\mu\)-rescaled forces: \(\hat{\mathcal{F}} \equiv F/\mu, \hat{\mathcal{F}}_{\text{nonloc}} \equiv F_{\text{nonloc}}/\mu\).

The tail contribution to \(\chi\) can then be computed by considering the evolution of the Laplace-Lagrange-Runge-Lenz, eccentricity vector (with \(j = r \times p\)),

\[
A = p \times j - n , \tag{B2}
\]

We recall that, in absence of perturbations, \(A\) is conserved: its magnitude is equal to the eccentricity, \(e = \sqrt{1 + 2E_j^2}\), and \(A\) is directed from the origin towards the periastron (i.e., the point of closest approach in the case of hyperbolic motion).
The instantaneous eccentricity vector $\mathbf{A}(t)$ evolves under the effect of the additional force $\hat{\mathbf{F}}_{\text{nonloc}}$:

$$
\frac{d\mathbf{A}}{dt} = \hat{\mathbf{F}}_{\text{nonloc}} \times \mathbf{j} + \mathbf{p} \times (\mathbf{r} \times \hat{\mathbf{F}}_{\text{nonloc}}) 
= \hat{\mathbf{F}}_{\text{nonloc}} \times \mathbf{j} + \mathbf{r}(\mathbf{p} \cdot \hat{\mathbf{F}}_{\text{nonloc}}) - \hat{\mathbf{F}}_{\text{nonloc}}(\mathbf{r} \cdot \mathbf{p}).
$$

(B3)

We specialize Eq. (B3) to equatorial motion, i.e., $\mathbf{j} = je_\perp$. With respect to an adapted Cartesian system with the $x$ direction along the unperturbed apsidal line ($e_\perp$ a unit vector directed from the origin to periastron). We find $A_x = jpy - x/r$, $A_y = -jpx - y/r$ and

$$
\frac{dA_x}{dt} = j\hat{\mathbf{F}}_{\text{nonloc}} y + \left(x\hat{\mathbf{F}}_{\text{nonloc}}^y - y\hat{\mathbf{F}}_{\text{nonloc}}^x\right)py
\frac{dA_y}{dt} = -j\hat{\mathbf{F}}_{\text{nonloc}} x - \left(x\hat{\mathbf{F}}_{\text{nonloc}}^y - y\hat{\mathbf{F}}_{\text{nonloc}}^x\right)px,
$$

(B4)

as well as

$$
\mathbf{n} = \cos \phi e_\perp + \sin \phi e_\|,
\mathbf{p} = \frac{1}{j}[-\sin \phi e_\perp + (\cos \phi + e) e_\|],
p_r = (\mathbf{p} \cdot \mathbf{n}) = \frac{e}{j} \sin \phi.
$$

(B5)

The squared magnitude of the perturbed eccentricity vector $\mathbf{A}(t)$ is given by

$$
A^2(t) = p^2 j^2 + 1 - \frac{2}{r} = 1 + 2E(t)j(t)^2 \equiv e^2(t),
$$

(B6)

where $E(t) = E_\perp^2 + \frac{1}{\mu}$ denotes the perturbed energy ($j(t)$ being the perturbed angular momentum), and where we defined the perturbed eccentricity as $e(t) \equiv A(t) = \sqrt{1 + 2E(t)j(t)^2}$.

As we are considering the conservative 4PN dynamics, both $E(t)$ and $j(t)$ (and therefore $e(t)$) will be globally conserved. I.e., $E_\perp = E(t = \infty) = E(t = -\infty) = E_\perp$, and $j_\perp = j(t = \infty) = j(t = -\infty) = j_\perp$, is easily checked by using suitable integration by parts from the explicit expression of $\hat{\mathbf{F}}_{\text{nonloc}}$. Therefore $e_\perp = e(t = -\infty) = e(t = +\infty) = e_\perp$.

While the magnitude of the perturbed eccentricity vector $\mathbf{A}(t)$ is globally conserved, its direction changes between $t = -\infty$ and $t = +\infty$ and this additional rotation of $\mathbf{A}$ encodes the tail contribution to the scattering function $\chi$.

The simplest way to see this fact, and to extract $\chi_{\text{tail}}$ is, following section V D in Ref. [17], to replace each vector of the equatorial $(x, y)$ plane $\mathbf{V} = V_x e_\perp + V_y e_\|$ by the corresponding complex number $V = V_x + iV_y$. The asymptotic values (at $t = \pm \infty$) of the complex numbers corresponding to $\mathbf{n} = r/r$ and $\mathbf{A}$ are easily found to satisfy

$$
n_\pm = e^{i\phi_\pm},
A_\pm = (-1 + ipj)n_-, \quad A_+ = -(1 + ipj)n_+.
$$

(B7)

where $\phi_\pm$ are the asymptotic values of the polar angle and where we have denoted for brevity as $p = p_{\pm \infty} = \sqrt{2E_\perp}$ and $j = j_\pm$ their asymptotic values. Taking the ratio $A_+/A_-$ then yields

$$
\frac{A_+}{A_-} = \frac{-(1 + ipj)e^{i\phi_+}}{(-1 + ipj)e^{i\phi_-}} = \frac{e^{i\arctan(pj)}e^{i\phi_+}}{e^{-i\arctan(pj)}e^{i\phi_-}}
= e^{i(\phi_+ - \phi_- + \pi/2 + i\arctan(pj))}.
$$

(B8)

This can be rewritten as

$$
\frac{A_+}{A_-} = e^{i(\chi - \chi_{\text{N}}(E, j))}
$$

(B9)

where $\chi \equiv \phi_+ - \phi_- - \pi$ is by definition the total (perturbed) scattering angle and where

$$
\chi_{\text{N}}(E, j) \equiv \arctan \frac{1}{pj} \equiv \arctan \frac{1}{\sqrt{2E_\perp j^2}}.
$$

(B10)

As we see, the total angle of rotation of the $\mathbf{A}$ vector from $t = -\infty$ to $t = +\infty$, say $\alpha$, is equal to

$$
\alpha = \chi_{\text{pert}}(E, j) - \chi_{\text{N}}(E, j) \equiv \chi_{\text{tail}}(E, j),
$$

(B11)

where the last equality is just the usual definition of the tail contribution to the scattering function $\chi(E, j)$ (within our present context where $\chi_{\text{loc}}(E, j) = \chi_{\text{N}}(E, j)$). Neglecting second order effects in $\hat{\mathbf{F}}_{\text{nonloc}}$ in Eqs. (B4) and (B11), we finally get

$$
\chi_{\text{tail}}(E, j) = \int_{-\infty}^{\infty} \frac{dA_y}{dt} dt
= \int_{-\infty}^{\infty} dt \left[ (y(t)p_x(t) - j)\hat{F}_y^{\text{nonloc}}
- x(t)p_x(t)\hat{F}_y^{\text{nonloc}} \right].
$$

(B12)

To first order in $\hat{\mathbf{F}}_{\text{nonloc}}$, we can evaluate the right-hand side of this formula along the unperturbed (Newtonian) motion. Let us see how this allows one to compute the large-eccentricity (or, equivalently, large-$j$) expansion of $\chi_{\text{tail}}(E, j)$.

Recalling the standard polar representation of the Newtonian motion (in rescaled variables)

$$
r(t) = \frac{j^2}{1 + e \cos(\phi)}, \quad \frac{d\phi}{dt} = \frac{1}{j^3}(1 + e \cos(\phi))^2,
$$

(B13)

and the explicit expression of $\hat{\mathbf{F}}_{\text{nonloc}}$ [18]

$$
\hat{F}_{\text{nonloc}} = \frac{-4}{5} r(t)P_{f,2s/c} \int \frac{dr}{|r|} n_i(t)I_{ij}^{(6)}(t + \tau)
$$

(B14)

we find

$$
\chi_{\text{tail}}(E, j) = \int dt P_{f,2s/c} \int \frac{dr}{|r|} \delta(t, \tau)
$$

(B15)
where
\[ \delta(t, \tau) = \alpha^x(t) n^j(t) I_{xj}^{(6)}(t + \tau) + \alpha^\nu(t) n^j(t) I_{yj}^{(6)}(t + \tau), \]
with
\[
\begin{align*}
\alpha^x(t) &= -\frac{4}{5} r(y p_x - j) \\
&= -\frac{4j^3}{5(1 + e \cos \phi)^2}(\cos^2 \phi - e \cos \phi - 2) \\
\alpha^\nu(t) &= \frac{4}{5} r x p_x \\
&= -\frac{4j^3}{5(1 + e \cos \phi)^2} \sin \phi \cos \phi,
\end{align*}
\]
where each orbital function here (apart from \( j \)) depends on \( t \), namely, \( r = r(t), \phi = \phi(t), x = x(t), y = y(t), p_x = p_x(t), p_y = p_y(t) \).

By expanding the above quantities in inverse powers of \( e \) (or, equivalently, remembering \( e = \sqrt{1 + 2E j^2} \), in inverse powers of \( j \), at fixed energy), we can, in principle, compute the nonlocal contribution \( \chi^{\text{tail}}(E, j) \) to any order in \( 1/e \). Here, we will do this calculation at the leading order in \( 1/e \).

For large \( j \) (at fixed \( E \), so that \( e = \sqrt{1 + 2E j^2} \approx \sqrt{2E} j \)), we use the inverse eccentricity expansion of the angular motion \( \phi(t) \):
\[
\phi(t) = \arctan(\hat{n} t) + \frac{1}{e} \left[ \frac{\hat{n} t}{\sqrt{1 + \hat{n}^2 t^2}} + \frac{\arcsinh(\hat{n} t))}{1 + \hat{n}^2 t^2} \right] + O\left( \frac{1}{e^2} \right),
\]
with \( \phi(0) = 0 \) and \( r(0) = r_{(\text{peri})} \) and
\[ \hat{n} \equiv \frac{(2E)^{3/2}}{\sqrt{1 + 2E j^2}} \approx \frac{2E}{j}. \]

Rescaling the temporal variables as \( \hat{t} = \hat{n} t \) and \( \hat{\tau} = \hat{n} \tau \), a straightforward calculation (using Eqs. \[ \text{[B18]} \) in \[ \text{[B15]} \), \[ \text{[B16]} \) and \[ \text{[B17]} \), shows that the crucial integrand \( \delta(\hat{t}, \hat{\tau}) \) giving the value of \( e \chi^{\text{tail}}(E, j) \) has a large-eccentricity expansion that starts as
\[
\delta(\hat{t}, \hat{\tau}) = \nu \left[ e^8 f_8(\hat{t}, \hat{\tau}) + e^7 f_7(\hat{t}, \hat{\tau}) + O(e^6) \right],
\]
where
\[
\begin{align*}
f_8(\hat{t}, \hat{\tau}) &= -\frac{544}{5(1 + (\hat{t} + \hat{\tau})^2)^{11/2}} \left[ \frac{12}{17} + \left( \frac{\hat{t} + \frac{2}{17}}{\hat{t} + \hat{\tau}} \right)^5 \right. \\
&\left. - \frac{271}{68} \left( \frac{\hat{t} - \frac{344}{271}}{\hat{t} + \hat{\tau}} \right)^3 \right] \\
&\left. - \frac{291}{68} \left( \frac{\hat{t} + \frac{187}{97}}{\hat{t} + \hat{\tau}} \right) \right],
\end{align*}
\]
and \( f_7 = f_7^{\text{no-log}} + f_7^{\log} \), where
\[
f_7^{\text{no-log}} = \frac{P_1}{\sqrt{1 + \hat{t}^2(1 + (\hat{t} + \hat{\tau})^2)^{11/2}}} + \frac{P_2}{(1 + (\hat{t} + \hat{\tau})^2)^6},
\]
and
\[
f_7^{\log} = \frac{Q_1}{(1 + (\hat{t} + \hat{\tau})^2)^{13/2}} \ln(\hat{t} + \hat{\tau} + \sqrt{1 + (\hat{t} + \hat{\tau})^2}) + \frac{Q_2}{(1 + (\hat{t} + \hat{\tau})^2)^{11/2}} \ln(\hat{t} + \sqrt{1 + \tilde{\tau}^2}).
\]

The coefficients entering the non-logarithmic and logarithmic parts of \( f_7 \) are
\[ P_1 = \left( -\frac{128}{5} \hat{t}^2 + \frac{64}{5} \hat{t}^6 \right) \hat{t}^6 + \left( -\frac{1248}{5} \hat{t}^3 + \frac{864}{5} \hat{t}^5 \right) \hat{t}^5 + \left( \frac{2752}{5} + \frac{736}{5} \hat{t}^2 - 864 \hat{t}^4 \right) \hat{t}^4 \\
+ \left( \frac{504}{5} \hat{t}^3 + \frac{6088}{5} \hat{t} - 1472 \hat{t}^5 \right) \hat{t}^3 + \left( -1344 \hat{t}^5 - \frac{4488}{5} + \frac{4776}{5} \hat{t}^4 + \frac{6408}{5} \hat{t}^2 \right) \hat{t}^2 \\
+ \left( -3168 \hat{t}^7 - 6816 \hat{t} + 680 \hat{t}^3 + 7048 \hat{t}^5 \right) \hat{t}^2 + \frac{384}{5} - \frac{608}{5} \hat{t}^5 + 568 \hat{t}^6 + \frac{328}{5} \hat{t}^4 - 2736 \hat{t}^2 \\
P_2 = -\frac{192}{5} \hat{t}^7 + \left( -\frac{1344}{5} \hat{t}^2 + \frac{688}{5} \hat{t}^5 \right) \hat{t}^6 + \left( \frac{8272}{5} \hat{t} - 4032 \hat{t}^3 \right) \hat{t}^5 + \left( -\frac{35544}{5} - 8960 \hat{t}^2 - 1344 \hat{t}^4 \right) \hat{t}^4 \\
+ \left( -\frac{55136}{3} \hat{t}^3 - \frac{87816}{5} \hat{t} - 1344 \hat{t}^5 \right) \hat{t}^3 + \left( \frac{44144}{5} \hat{t} - 18608 \hat{t}^4 - 4032 \hat{t}^6 - \frac{50184}{5} \hat{t}^2 \right) \hat{t}^2 \\
+ \left( -\frac{1344}{5} \hat{t}^7 - \frac{4684 \hat{t}^8}{5} + \frac{20904}{5} \hat{t}^3 + \frac{68944}{5} \hat{t}^5 \right) \hat{t} \\
- \frac{28256 \hat{t}^8}{15} + 4960 \hat{t}^2 - \frac{9728}{15} + \frac{18816}{5} \hat{t}^4 - \frac{192}{5} \hat{t}^8 \] 

(B24)

\[ Q_1 = -\frac{608}{5} \left[ -\frac{100}{19} \left( \hat{t} + \frac{1}{10} \hat{t} \right) \left( \hat{t} + \hat{t}^2 \right)^5 \hat{t}^6 + \frac{715}{19} \left( \hat{t} + \hat{t}^4 \right) \left( \hat{t} - \frac{118}{143} \right)^2 \right] \\
+ \frac{470}{19} \left( \hat{t} + \frac{1285}{376} \hat{t} \right) \left( \hat{t} + \hat{t}^2 \right)^2 - \frac{825}{38} \hat{t}^2 - \frac{345}{19} \right] \\
Q_2 = -\frac{96}{5} \left[ \frac{9}{2} + \left( \hat{t} + \hat{t}^2 \right)^4 - \frac{11}{4} \left( \hat{t} + \hat{t}^2 \right)^2 \right] \left( \hat{t} + \hat{t}^2 \right) . 

(B25)

By first evaluating the partie finie in \( \hat{t} \) (with the rescaled regularization scale \( 2s\hat{n}/c \)) we find the intermediate result

\[ \text{Pf}_{2s\hat{n}/c} \int_{-\infty}^{\infty} \frac{d\hat{t}}{\hat{t}} \, f_8(\hat{t}, \hat{\tau}) = \frac{96}{5(1 + \hat{t}^2)^{9/2}} \left[ \left( -\frac{34}{3} \hat{t}^4 + \frac{113}{2} \hat{t}^2 - 8 \right) \ln[2(1 + \hat{t}^2)] - \left( -\frac{68}{3} \hat{t}^4 - 16 + 113 \hat{t}^2 \right) \ln(2s\hat{n}) \right] \\
+ \left( -\frac{1021}{18} \hat{t}^4 - \frac{1159}{6} \hat{t}^2 + \frac{137}{6} \right) . 

(B26)

But then, the explicit evaluation of the remaining \( \hat{t} \) integral yields a vanishing contribution at order \( e^5 \):

\[ \int_{-\infty}^{\infty} \frac{dt}{n} \, \text{Pf}_{2s\hat{n}/c} \int_{-\infty}^{\infty} \frac{d\tau}{|\tau|} \, f_8(\hat{t}, \hat{\tau}) = 0 . \] 

(B27)

The first non-zero contribution to \( e^{\chi_{\text{tail}}} \) comes from the \( e^7 f_7 \) term in \( \delta(\hat{t}, \hat{\tau}) \), Eq. (B29). We managed to compute the double \( \hat{t} \) and \( \hat{\tau} \) integral of \( f_7(\hat{t}, \hat{\tau}) \) by first integrating by parts with respect to \( \hat{t} \) (keeping \( \hat{\tau} \) fixed) the logarithmic terms in \( f_7^\log \), i.e. by rewriting \( f_7^{\log}(\hat{t}, \hat{\tau}) = f_7^{\log -\text{new}}(\hat{t}, \hat{\tau}) + \frac{d}{d\hat{t}} g_7(\hat{t}, \hat{\tau}) \) where

\[ f_7^{\log -\text{new}} = -H_1(\hat{t}, \hat{\tau}) \frac{d}{d\hat{t}} \ln(\hat{t} + \hat{\tau} + \sqrt{1 + (\hat{t} + \hat{\tau})^2}) \\
- H_2(\hat{t}, \hat{\tau}) \frac{d}{d\hat{t}} \ln(\hat{t} + \sqrt{1 + \hat{t}^2}) \] 

(B28)

with \( H_1 = \int h_1 d\hat{t}, \) etc. Because of the final \( \hat{t} \) integration, the term \( g_7(\hat{t}, \hat{\tau}) \) is checked to yield a vanishing boundary contribution. After this transformation, we could perform all the needed integrations, with the final result (using \( e \approx \sqrt{2EJ} = v_{\infty}j \) and \( \hat{\omega} \approx 2EJ = \frac{v_{\infty}^2}{J} \))

\[ \chi_{\text{tail}} = -\frac{2}{5} \frac{v_{\infty}^4}{J^2} \left[ \frac{337}{3} + 37 \ln \left( \frac{\sin^2 \theta}{4j} \right) \right] \\
+ O\left( \frac{v_{\infty}^3}{J^3} \right) . 

(B29)

The agreement of this purely dynamical evaluation of \( \chi_{\text{tail}} \) with our previous result, Eq. (149), obtained via our time-localization of \( H_{\text{tail}} \), is an additional check of the general time-localization technique introduced in [20] for bound motions, and extended in the present work to unbound motions.
Appendix C: An alternative method to compute the local part of the scattering angle

We sketch here the alternative method for computing the PN expansion of the polar equation \( r = r(\phi) \) of the motion introduced in Ref. [17], and we show how we used it to check our computation of the contribution of the local Hamiltonian \( H^{loc} \) to scattering.

By squaring Eq. (B11), we get the following differential equation for the polar equation \( \phi = \phi(u) \) of the orbit (here \( u \equiv 1/r \))

\[
\left( \frac{d\phi}{du} \right)^2 = \frac{j^2}{R''(E, j, u)}, \tag{C1}
\]

where the function \( R''(E, j, u) \) is obtained from the function \( R(E, j, u) \) describing the squared radial momentum,

\[
r_r^2 = R(E, j, u), \tag{C2}
\]

via the definitions

\[
R'(E, j, u) = \frac{1}{u^2} \frac{\partial}{\partial j^2} R(E, j, u)
\]

\[
R''(E, j, u) \equiv \frac{R(E, j, u)}{(R'(E, j, u))^2}. \tag{C3}
\]

In the Newtonian approximation, the function \( R''(E, j, u) \) is a quadratic polynomial in \( u \), namely

\[
R''_{\text{Newton}}(E, j, u) = 2E + 2u - j^2 u^2, \tag{C4}
\]

so that Eq. (C1) can be easily integrated. The basic idea of the alternative approach to scattering discussed in this Appendix (and introduced in Ref. [17]) is to transform the polar equation Eq. (C1) into a Newtonian-like equation by means of an appropriate change of variable \( u \to \bar{u} \).

Under a change of variable \( u = F(\bar{u}) = \bar{u} + O(\bar{u}^2) \), Eq. (C1) gets transformed into

\[
\frac{d\phi^2}{j^2} = \frac{du^2}{R''(u)} = \frac{d\bar{u}^2}{\bar{R}''(\bar{u})}, \tag{C5}
\]

where

\[
\bar{R}''(\bar{u}) = R''(F(\bar{u})) \left( \frac{d\bar{u}}{du} \right)^2. \tag{C6}
\]

The idea is then to define the transformation \( u = F(\bar{u}) \) such that \( \bar{R}'' \) is reduced to a quadratic function of \( \bar{u} \):

\[
\bar{R}''(\bar{u}) = A + 2B\bar{u} + C\bar{u}^2 \tag{C7}
\]

with \( A = 2\bar{E} + O(\eta^2), \ B = 1 + O(\eta^2), \) and \( C = -j^2 + O(\eta^2) \). In view of Eq. (C5), the requirement (C7) yields the following condition on the transformation \( u \to \bar{u} \):

\[
(A + 2B\bar{u} + C\bar{u}^2) \left( \frac{du}{d\bar{u}} \right)^2 = R''|_{u=u(\bar{u})}. \tag{C8}
\]

As emphasized in [17], the three coefficients \( A, B, C \) (all functions of \( \bar{E}, j, j \) and \( \nu \)) entering the reduced quadratic polar equation (C5) contain all the information needed to compute gauge-invariant measures of the relativistic orbit, such as periastron advance (in the bound case) and the scattering angle (in the unbound case). Indeed, the solution of

\[
\left( \frac{d\phi}{d\bar{u}} \right)^2 = \frac{j^2}{(A + 2B\bar{u} + C\bar{u}^2)} \tag{C9}
\]

is the following precessing conic

\[
\bar{u} = \frac{1}{\bar{r}} = u_p \left( 1 + \bar{e} \cos \phi \right), \quad u_p \equiv \frac{1}{\bar{r}}, \tag{C10}
\]

with

\[
A = \frac{u_p^2 j^2 (\bar{e}^2 - 1)}{K^2}, \quad B = \frac{u_p j^2}{K^2}, \quad C = -\frac{j^2}{K^2}. \tag{C11}
\]

The above relations are easily inverted leading to

\[
K^2 = -\frac{j^2}{C}, \quad u_p = -\frac{B}{C}, \quad \bar{e}^2 - 1 = -\frac{AC}{B^2}. \tag{C12}
\]

The periastron-to-periastron precession angle is \( \Phi = \pi K \), while the scattering angle \( \chi \) is given by

\[
\chi + \pi = 2K \arccos \left( \frac{-1}{\bar{e}} \right). \tag{C13}
\]

Note that the function \( K(\bar{E}, j) \) enters both \( \Phi \) and \( \chi \), though one has to analytically continue it from \( \bar{E} < 0 \) to \( \bar{E} > 0 \).

By considering the large-separation limit \( (u \to \infty) \) of the EOB dynamics, one finds that one has simply

\[
A = 2\bar{E}. \tag{C14}
\]

The PN expansion of the transformation \( u = F(\bar{u}) \) is found to have the form...
\[ u = \bar{u} - \bar{u}^2 \eta^2 + \left( \frac{3}{2} \nu + \frac{3}{4} \right) \bar{u}^3 + \left( \frac{5}{2} \nu - \frac{17}{4} \right) \frac{\bar{u}^2}{j^2} \eta^4 \]

\[ + \left\{ \left( 3 \nu + \frac{9}{4} \nu^2 \right) j^2 \bar{u}^5 + \left( \frac{25}{4} \nu^2 + 11 \nu - \frac{1}{2} \right) \bar{u}^4 \right\} \eta^6 \]

\[ + \left[ \left( \frac{23}{4} \nu^2 + 27 \nu - 9 \right) \frac{E}{j^2} + \left( \frac{71}{2} - \frac{35}{8} \right) \nu^2 + \left( - \frac{205}{128} \pi^2 + \frac{245}{3} \right) \nu \right] \frac{1}{j^4} \bar{u}^2 \eta^6 \]

\[ + f(\bar{u}) \sqrt{2 E - j^2 \bar{u} + 2 \bar{u} \eta^8} \]  

(C15)

where we have left unspecified at \( O(\eta^8) \) the 4PN-level transformation \( f(\bar{u}) \) for reasons which will become clear soon.

The corresponding coefficients \( B \) and \( C \), are found to have the form

\[ B = 1 + 4 \bar{E} \eta^2 + \eta^4(17 - 10 \nu) \frac{E}{j^2} + \eta^6 \left[ (23 \nu^2 - 108 \nu + 36) \frac{E^2}{j^4} + \left( \frac{205}{32} \pi^2 - \frac{980}{3} \nu + \frac{35}{2} \nu^2 + 142 \right) \frac{E}{j^4} \right] + B_8 \eta^8 \]

\[ C = -j^2 + 6 \eta^2 + \eta^4 \left[ (15 - 6 \nu) \frac{E}{j^2} + \left( \frac{51}{2} - 15 \nu \right) \frac{1}{j^4} \right] \]

\[ + \eta^6 \left\{ (-12 \nu + 9 \nu^2) \frac{E^2}{j^4} + \left[ 180 + 45 \nu^2 + \left( \frac{123}{32} \pi^2 - 382 \right) \nu \right] \frac{E}{j^4} + \left[ 213 + \frac{105}{4} \nu^2 + \left( -490 + \frac{615}{64} \pi^2 \right) \nu \right] \frac{1}{j^4} \right\} + C_8 \eta^8 \]

(C16)

where we left unspecified the 4PN-level coefficients \( B_8 \) and \( C_8 \).

The formal solution of the condition (C8) for the 4PN-level function \( f(\bar{u}) \) can be cast in the form

\[ f(\bar{u}) = \int_0^\bar{u} \frac{\sigma(u)}{Q(u)^{3/2}} \, du, \quad \sigma(u) = 2 \bar{E} - j^2 \bar{u}^2 + 2 \bar{u} = j^2(u_{(\text{max})} - \bar{u})(\bar{u} - u_{(\text{min})}). \]  

(C17)

Here \( \sigma(u) \) is a 4PN-level “source term” coming from \( R''(u) \) (itself derived from the 4PN-accurate local Hamiltonian). It is such that \( \sigma(0) = 0 \), and it comprises both a log-part, \( \sigma^{\ln}(u) \) (involving \( \ln u \)), and a log-free part, \( \sigma^{\text{no-log}}(u) \). The latter log-free part reads

\[ \sigma^{\ln}(u) = \nu \frac{16}{15} u^4 \left( -62 u + 37 j^2 u^2 - 74 \bar{E} \right) \ln(us). \]  

(C18)

while \( \sigma^{\text{no-log}}(u) \) is an 8th degree polynomial (with coefficients depending on \( \bar{E}, j \) and \( \nu \)), having a simple root at
$u = 0$. Its explicit value reads

$$
\sigma^{\text{no-log}}(u) = \left(\frac{9}{2} \nu + \frac{27}{2} \nu^2 - 15 \nu^3\right) j^6 u^8 + (105 \nu^3 - 27 \nu - 183 \nu^2) j^4 u^7
+ \left\{(-27 \nu + 90 \nu^3 - 81 \nu^2) j^4 \tilde{E} + \left[\frac{5}{4} - 240 \nu^3 + \left(\frac{1453}{2} - \frac{123}{32} \pi^2\right) \nu^2 + \left(\frac{167}{90} + \frac{25729}{3072} \pi^2\right) \nu\right] j^2 \right\} u^6
+ \left[\frac{1581}{2} \nu^2 + 30 \nu - 420 \nu^3\right] j^2 \tilde{E} + \left[\frac{451}{64} \pi^2 - \frac{5153}{6}\right] \nu^2 + \left(\frac{14383}{72} - \frac{60251}{3072} \pi^2\right) \nu\right] j^2 \right\} u^5
+ \left\{(-180 \nu^3 + 162 \nu^2 + 54 \nu) j^2 \tilde{E}^2
+ \left[-\frac{397}{16} + 480 \nu^3 + \left(-\frac{6451}{4} + \frac{123 \pi^2}{16}\right) \nu^2 + \left(-\frac{23761}{1536} \pi^2 + \frac{39547}{180}\right) \nu\right] \tilde{E}
+ \left[-\frac{5889}{32} + \frac{597}{8} \nu^2 + \left(\frac{123 \pi^2}{64} - \frac{661}{8}\right) \nu\right] \frac{1}{j^2} u^4
+ \left\{420 \nu^3 - 867 \nu^2 + 72 \nu\right\} \tilde{E}^2 + \left[\frac{813}{4} + \frac{457}{2} \nu^2 + \left(\frac{369}{32} \pi^2 - 1083\right) \nu\right] \tilde{E}
+ \left[\frac{1425}{2} + 120 \nu^2 + \left(-\frac{4430}{3} + \frac{205}{8} \pi^2\right) \nu\right] \frac{1}{j^2} u^3
+ \left\{(-36 \nu - 108 \nu^2 + 120 \nu^3) \tilde{E}^3 + (-648 \nu + 216 + 138 \nu)^2 \frac{\tilde{E}^2}{j^2}
+ \left[\frac{4275}{4} + 180 \nu^2 + \left(\frac{615}{16} \pi^2 - 2215\right) \nu\right] \tilde{E}^3 - \frac{1}{2} C_S\right\} u^2 - u B_8\right\} \right\}(C19)
$$

Note that the so-far unspecified coefficients $B_8$, and $C_S$ contribute the terms $-B_8 u - \frac{1}{2} C_S u^2$ to $\sigma^{\text{non-log}}(u)$. Note also that the integral (C17) defining the function $f(u)$ entering the 4PN-level transformation $u = F(\bar{u})$ is a priori singular at the two roots of $Q(\nu)$, i.e., at the Newtonian (periastron and apastron) values of $u_{(\text{max/min})}$,

$$
u_{\text{max}} = \frac{1}{j^2} \left(1 + \sqrt{1 + 2 \tilde{E}j^2}\right), \quad \nu_{\text{min}} = \frac{1}{j^2} \left(1 - \sqrt{1 + 2 \tilde{E}j^2}\right).
$$

One finds that one must require the regularity of the transformation $u = F(\bar{u})$ around these roots, and that this regularity condition implies the vanishing of the two following Hadamard-partie-finie integrals

$$
Pf \int_0^{\nu_{(\text{max})}} \frac{\sigma(u)}{Q(u)^{3/2}} du = 0, \quad Pf \int_0^{\nu_{(\text{max})}} \frac{\sigma(u)}{Q(u)^{3/2}} du = 0.
$$

These conditions can be simplified by using the identity

$$
\frac{1}{Q^{3/2}} = \frac{1}{(1 + 2 \tilde{E}j^2)} \frac{d}{du} \left(\frac{uj^2 - 1}{Q^{1/2}}\right) = \frac{d}{du} g(u), \quad g(u) = \frac{1}{(1 + 2 \tilde{E}j^2)} \left(\frac{uj^2 - 1}{Q^{1/2}}\right)
$$

to integrate by parts. This leads to the equivalent conditions

$$
Pf \int_0^{\nu_{(\text{max})}} g(u) \sigma'(u) du = 0, \quad Pf \int_0^{\nu_{(\text{min})}} g(u) \sigma'(u) du = 0.
$$

These two (regularity) conditions completely determine the two parameters $B_8$ and $C_S$. From the practical point of view, while the partie finie of the integrals involving $\sigma^{\text{no-log}}(u)$ can be straightforwardly computed, the partie finie of the integrals involving $\sigma^{\text{ln}}(u)$ can only be computed as an expansion in inverse powers of the eccentricity, as was already the case for the corresponding integrals $I_{\chi}$ and $L_{\chi}$ of sec. III. More precisely, one finds, using the explicit expression,

$$
\frac{d\sigma^{\text{ln}}(u)}{du} = \frac{16}{15} u^3 \left[2 \ln(u)(111 u^2 j^2 - 148 \tilde{E} - 155 u) + 37 u^2 j^2 - 74 \tilde{E} - 62 u\right],
$$

that the log-parts of the two above conditions, namely

$$
Pf \int_0^{\nu_{(\text{max})}} g(u) \frac{d\sigma^{\text{ln}}(u)}{du} du, \quad Pf \int_0^{\nu_{(\text{min})}} g(u) \frac{d\sigma^{\text{ln}}(u)}{du} du,
$$

(C25)
form exactly the combinations $I_\chi$ and $I_\chi$ of sec. III, Eqs. (18) and (50), respectively.

Implementing the above method, we have computed $\chi^{\text{loc}}$, and found that it agreed with the result obtained by the method explained in the main text. For concreteness, let us just indicate the beginnings of the PN expansions of $K$ and $\bar{e}$ (with the notation $\bar{e}_0(\bar{E}, j) \equiv \sqrt{1 + 2 E j^2}$)

\[
K = 1 + \frac{3}{j} \eta^2 + \left( \frac{15}{2} - 3 \nu \right) \frac{\bar{E}}{j^2} + \left( \frac{105}{4} - \frac{15}{2} \nu \right) \frac{1}{j^4} \eta^4
\]

\[
+ \left\{ \left( -6 \nu + \frac{9}{2} \nu^2 \right) \frac{\bar{E}^2}{j^2} + \frac{45}{2} \nu^2 \left( \frac{123}{64} \eta^2 - 218 \nu + \frac{315}{2} \right) \frac{\bar{E}}{j^4} \right\} \eta^6 + O(\eta^8)
\]

\[
\bar{e} = \bar{e}_0 - \frac{2 \bar{E}}{\bar{e}_0} \left( 4 E j^2 - 3 \right) \eta^2
\]

\[
+ \frac{\bar{E}}{\bar{e}_0} \left\{ -512 E^5 j^6 + (176 E - 220 \nu^2) E^4 j^4 + \left( -510 + 540 \nu - \frac{11104}{3} - \frac{533}{8} \eta^2 \right) E^3 j^2
\]

\[
+ \left[ -1532 - 480 \nu^2 + \left( \frac{16612}{3} - \frac{1681}{16} \pi^2 \right) \nu \right] E^2 + \left[ -1061 - 185 \nu^2 + \left( \frac{8536}{3} - \frac{1763}{32} \pi^2 \right) \nu \right] E
\]

\[
+ \left[ -213 - \frac{105}{4} \nu^2 + \left( \frac{490}{64} \eta^2 \right) \nu \right] \frac{1}{j^4} \right\} + O(\eta^8).
\]

(C26)

Acknowledgments

DB thanks ICRANet and the italian INFN for partial support and IHES for warm hospitality at various stages during the development of the present project.

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