WITTEN’S PERTURBATION ON STRATA WITH GENERAL
ADAPTED METRICS

JESÚS A. ÁLVAREZ LÓPEZ AND MANUEL CALAZA

Abstract. Let \( M \) be a stratum of a compact stratified space \( A \). It is equipped
with a general adapted metric \( g \), which is slightly more general than the
adapted metrics of Nagase and Brasselet-Hector-Saralegi. In particular, \( g \) has a
general type, which is an extension of the type of an adapted metric. A restriction
on this general type is assumed, and then \( g \) is called good. We consider
the maximum/minimum ideal boundary condition, \( d_{\text{max/min}} \), of the compactly
supported de Rham complex on \( M \), in the sense of Brüning-Lesch, defining the
cohomology \( H^*_{\text{max/min}}(M) \), and with corresponding Laplacian \( \Delta_{\text{max/min}} \). The
first main theorem states that \( \Delta_{\text{max/min}} \) has a discrete spectrum satisfying a
weak form of the Weyl’s asymptotic formula. The second main theorem is a
version of Morse inequalities using \( H^*_{\text{max/min}}(M) \) and what we call rel-Morse
functions. The proofs of both theorems involve a version for \( d_{\text{max/min}} \) of the
Witten’s perturbation of the de Rham complex, as well as certain perturbation
of the Dunkl harmonic oscillator previously studied by the authors using clas-
sical perturbation theory. The condition on \( g \) to be good is general enough in
the following sense: using intersection homology when \( A \) is a stratified pseudo-
manifold, for any perversity \( \bar{p} \leq \bar{m} \), there is an associated good adapted metric
on \( M \) satisfying the Nagase isomorphism \( H^r_{\text{max}}(M) \cong I^p H^r_{\bar{p}}(A)^* \ (r \in \mathbb{N}) \). If
\( M \) is oriented and \( \bar{p} \geq \bar{n} \), we also get \( H^r_{\text{min}}(M) \cong I^p H^r_{\bar{p}}(A) \). Thus our version
of the Morse inequalities can be described in terms of \( I^p H^*_{\bar{p}}(A) \).

Contents

1. Introduction 2
  1.1. Ideal boundary conditions of the de Rham complex 2
  1.2. Thom-Mather stratified spaces 3
  1.3. General adapted metrics 4
  1.4. Relatively Morse functions 4
  1.5. Main theorems 6
  1.6. Applications to intersection homology 7
  1.7. Ideas of the proofs 9
  1.8. Some open problems 10

2. Preliminaries 10
  2.1. Products of cones 10
  2.2. General adapted metrics 12
  2.3. Relatively Morse functions 13

1991 Mathematics Subject Classification. 58A14, 32S60.
Key words and phrases. Morse inequalities, ideal boundary condition, stratification, Witten’s
perturbation.

The first author is partially supported by MICINN, Grants MTM2008-02640 and MTM2011-
25656, and by MEC, Grant PR2009-0409.
2.4. Hilbert and elliptic complexes

3. A perturbation of the Dunkl harmonic oscillator

4. Two simple types of elliptic complexes

4.1. An elliptic complex of length one

4.2. An elliptic complex of length two

4.3. The wave operator

5. Witten’s perturbation on a cone

5.1. Witten’s perturbation

5.2. De Rham operators on a cone

5.3. Witten’s perturbation on a cone

6. Splitting of the Witten’s complex on a cone

6.1. Spectral decomposition on the link of the cone

6.2. Subcomplexes of length one

6.3. Subcomplexes of length two

6.4. Splitting into subcomplexes

7. Relatively local model of the Witten’s perturbation

8. Proof of Theorem 1.1

9. Functional calculus

10. The wave operator

11. Proof of Theorem 1.2

References

1. Introduction

1.1. Ideal boundary conditions of the de Rham complex. The following usual notation is used for a densely defined linear operator $T$ in a Hilbert space. Its domain and range are denoted by $D(T)$ and $R(T)$. If $T$ is essentially self-adjoint, its closure is denoted by $\overline{T}$. If $T$ is self-adjoint, its smooth core is $D_\infty(T) := \bigcap_{m=1}^{\infty} D(T^m)$, and its spectrum is denoted by $\sigma(T)$.

A Hilbert complex $(\mathcal{D},d)$ is a differential complex of finite length given by a densely defined closed operator $d$ in a graded separable Hilbert space $\mathcal{H}$ [9]. Then the operator $D = d + d^*$, with $D(\mathcal{D}) = D(d) \cap D(d^*)$, is self-adjoint in $\mathcal{H}$, and therefore the Laplacian $\Delta = D^2 = dd^* + d^*d$ is also self-adjoint. Moreover $D_\infty(\Delta)$ is a subcomplex of $(\mathcal{D},d)$ with the same homology [9] Theorem 2.12; it may be also said that $D_\infty(\Delta)$ is the smooth core of $d$.

The above notion is applied here in the following case. For a Riemannian manifold $M$, let $\Omega_0(M)$ be the space of compactly supported differential forms, $d$ and $\delta$ the de Rham derivative and coderviative acting on $\Omega_0(M)$, and $L^2\Omega(M)$ the graded Hilbert space of square integrable differential forms. Let also $D = d + \delta$ and $\Delta = D^2 = d\delta + \delta d$ (the Laplacian). Each Hilbert complex extension $d$ of $d$ in $L^2\Omega(M)$ is called an ideal boundary conditions (i.b.c.) [9], giving rise to self-adjoint extensions $D$ and $\Delta$ of $D$ and $\Delta$ in $L^2\Omega(M)$. There exists a minimum/maximum i.b.c., $d_{\min} = d$ and $d_{\max} = \delta^*$, inducing self-adjoint extensions $D_{\max}$ and $\Delta_{\max}$ of $D$ and $\Delta$. If $M$ is oriented, then $\Delta_{\max}$ corresponds to $\Delta_{\min}$ by the Hodge star operator. The corresponding cohomologies, $H_{\min/\max}(M)$, and versions of Betti numbers and Euler characteristic, $\beta_{\min/\max} = \beta_{\min/\max}(M)$ and $\chi_{\min/\max} = \chi_{\min/\max}(M)$,
are quasi-isometric invariants of \( M \); for instance, \( H_{\max}(M) \) is the usual \( L^2 \) cohomology \( H_{(2)}(M) \). These concepts can indeed be defined for arbitrary elliptic complexes \([9]\). It is well known that \( d_{\min} = d_{\max} \) if \( M \) is complete. Thus considering an i.b.c. becomes interesting when \( M \) is not complete. For instance, if \( M \) is the interior of a compact Riemannian manifold with \( \partial M \neq \emptyset \), then \( d_{\min/\max} \) is defined by taking relative/absolute boundary conditions. With more generality, we will assume that \( M \) is a stratum of a compact stratified space \([31, 32, 42]\), equipped with a generalization of the adapted metrics considered in \([33, 34, 8]\).

1.2. Thom-Mather stratified spaces. Roughly speaking, a (Thom-Mather) stratified space (or stratification) is a Hausdorff, locally compact and second countable space \( A \) equipped with a partition into \( C^\infty \) manifolds (the strata), satisfying certain conditions \([41, 31]\). In particular, an order relation on the family of strata is defined by declaring \( X \leq Y \) when \( X \subset \overline{Y} \). With respect to this ordering, the maximum length of chains of strata under a stratum \( X \) is called the depth of \( X \), and the supremum of the strata depth is called the depth of \( A \). The precise definition and needed preliminaries were collected in \([3, \text{Section 3}]\), where we have mainly followed \([12]\). Instead of recalling it, let us describe how the strata of \( A \) fit together, describing also morphisms/isomorphisms of stratifications, and, in particular, the group of automorphisms, \( \operatorname{Aut}(A) \). We proceed by induction on its depth. If depth \( A = 0 \), then \( A \) is just a \( C^\infty \) manifold, and \( \operatorname{Aut}(A) \) consists of its diffeomorphisms. Now, given any \( k \in \mathbb{Z}_+ \), assume that any stratified space \( L \) is described if depth \( L < k \), as well as \( \operatorname{Aut}(L) \). If \( L \) is compact, the cone with link \( L \) is \( c(L) = (L \times [0, \infty)) / (L \times \{0\}) \), whose vertex is the point \( * = L \times \{0\} \in c(L) \). Let \( L' \) be another compact stratification of depth \( < k \), and \( \phi : L \to L' \) a morphism, let \( \phi(L) : c(L) \to c(L') \) be the map induced by \( \phi \times \operatorname{id} : L \times [0, \infty) \to L \times [0, \infty) \); in particular, we get the group \( \operatorname{c(Aut}(L)) = \{ \phi(L) \mid \phi \in \operatorname{Aut}(L) \} \). It is also declared that \( \operatorname{c(Aut}(\emptyset)) = \{ * \} \), for the empty stratification, and \( \operatorname{c(Aut}(\emptyset)) = \{ \phi \} \), for the empty map. Then \( c(L) \) is a model stratified space of depth \( k \), whose strata are \( \{ * \} \) and the manifolds \( X \times \mathbb{R}_+ \) for strata \( Y \) of \( L \). The second factor projection \( L \times [0, \infty) \to [0, \infty) \) defines an \( \operatorname{c(Aut}(L))\)-invariant function \( \rho : c(L) \to [0, \infty) \), called the radial function, whose restrictions to the strata are \( C^\infty \). We can also restrict the stratified structure to the \( \operatorname{c(Aut}(L))\)-invariant open subsets \( c_\epsilon(L) \subset c(L) \) determined by the condition \( \rho < \epsilon \) (\( \epsilon > 0 \)), and let \( c_\epsilon(\operatorname{Aut}(L)) \) be the group of homeomorphisms of \( c_\epsilon(L) \) defined by the restriction of maps in \( c(\operatorname{Aut}(L)) \). Now, for any stratification \( A \) of depth \( k \), each stratum \( X \) has an open neighborhood \( T_X \) (a tube representative) that is a fiber bundle over \( X \) with typical fiber \( c_\epsilon(L_X) \), for some \( \epsilon > 0 \) and some compact stratification \( L_X \) of lower depth (the link of \( X \)), whose structural group is \( c_\epsilon(\operatorname{Aut}(L_X)) \), and such that \( X \) corresponds to the vertex section of \( T_X \). The vertex and radial function of \( c(L_X) \) are denoted by \( *_X \) and \( \rho_X \). Note that \( \rho_X \) defines a radial function \( \rho_X \) on \( T_X \). Moreover, for any trivialization \( T_X|_U = U \times c(L_X) \) over some open subset \( U \subset X \), the restriction of the strata of \( A \) to \( T_X|_U \) is given by the product of \( U \) and the strata of \( c_\epsilon(L_X) \). Two such neighborhoods of \( X \) represent the same tube if their structure is equal on some smaller neighborhood of \( X \). Note that \( X \) is open in \( A \) if and only if \( L_X = \emptyset \), and, assuming that \( A \) and \( X \) are connected, this happens if and only if \( X \) is dense in \( A \). Finally, a morphism between two stratifications is a continuous map whose restrictions to the strata are \( C^\infty \), and whose restrictions to small enough tube representatives are fiber-bundle morphisms. Then isomorphisms
and automorphisms of stratifications have the obvious meaning. This completes the description because the depth is locally finite by the local compactness.

The (topological) dimension of a stratification \( A \) equals the supremum of the dimensions of its strata. It may be infinite, but it is locally finite. The codimension of each stratum \( X \) is \( \dim A - \dim X \). Our main results will assume that the stratification is compact, but non-compact stratifications will be also used in the proofs. In any case, we will consider only stratifications of finite dimension. If the above description of \( A \) is modified by requiring that, at each inductive step, only stratifications with no strata of codimension 1 are used, then \( A \) is called a stratified pseudomanifold.

A locally closed subset \( B \subset A \) is called a substratification of \( A \) if the restrictions of the strata and tubes of \( A \) to \( B \) define a stratified structure on \( B \). For instance, \( A \) can be restricted to any open subset, to any locally closed union of strata, and to the closure of any stratum. If moreover there are tube representatives of \( A \) whose restrictions to \( B \) have the same fibers over points of \( B \), then \( B \) is called saturated.

Let \( x \) be a point of a stratum \( X \) of dimension \( m_X \) in a stratification \( A \). A local trivialization of \( TX \) around \( x \) defines a chart \( O \equiv O' \) of \( A \) for some open \( O' \subset \mathbb{R}^{m_X} \times c(L_X) \). This chart is said to be centered at \( x \) if \( x \equiv (0, \ast_X) \in O' \). The corresponding concept of atlas has the obvious meaning. These concepts can be general as follows.

Any finite product of stratifications has a non-canonical stratified structure [3, Section 3.1.2]; in particular, any finite product of cones is isomorphic to a cone [3, Lemma 3.8]. Moreover \( \text{Aut}(P) \times \text{Aut}(Q) \) is canonically injected in \( \text{Aut}(P \times Q) \) for stratifications \( P \) and \( Q \). Thus it makes sense to consider a decomposition \( c(L_X) \cong \prod_{i=1}^{\infty} c(L_{X,i}) \) (\( a_X \in \mathbb{N} \)), for compact stratifications \( L_{X,i} \). The vertex and radial function of each \( c(L_{X,i}) \) are denoted by \( \ast_{X,i} \) and \( \rho_{X,i} \). Then we can also consider general tube representatives \( T_X \) with typical fiber \( \prod_{i=1}^{\infty} c_{X_{i}}(L_{X,i}) \) and structural group \( \prod_{i=1}^{\infty} L_{X,i}(\text{Aut}(L_{X,i})) \). This gives rise to a general chart \( O \equiv O' \) around \( x \) for some open \( O' \subset \mathbb{R}^{m_X} \times \prod_{i=1}^{\infty} c(L_{X,i}) \), which is centered at \( x \) if \( x \equiv (0, \ast_{X,1}, \ldots, \ast_{X,a_X}) \in O' \). Let \( \rho_{X,0} \) denote the norm function on \( \mathbb{R}^{m_X} \). The function \( \rho = (\rho_{X,0}^2 + \cdots + \rho_{X,a_X}^2)^{1/2} \) is called the radial function of \( \mathbb{R}^{m_X} \times \prod_{i=1}^{\infty} c(L_{X,i}) \). A collection of general charts covering \( A \) is called a general atlas.

We can suppose that the strata of \( A \) are connected [3, Remark 1-(v)]. Fix a stratum \( M \) of dimension \( n \) in \( A \). Since the stratified structure of \( A \) can be restricted to \( M \) [3, Section 3.1.1], we can also assume without loss of generality that \( M = A \) (any other stratum is \( < M \)); in particular, depth \( A = \dim M \) and \( \dim A = n \). With the above notation, for a chart \( O \equiv O' \) centered at \( x \), we get \( M \cap O \equiv M' \cap O' \), where \( M' = \mathbb{R}^{m_X} \times N \times \mathbb{R}^+ \) for some dense stratum \( N \) on \( L_X \). In the case of a general chart \( O \equiv O' \) centered at \( x \), we have \( M \cap O \equiv M' \cap O' \) for \( M' = \mathbb{R}^{m_X} \times \prod_{i=1}^{\infty} (N_i \times \mathbb{R}^+) \), where each \( N_i \) is some dense stratum of \( L_{X,i} \).

1.3. General adapted metrics. A general adapted metric \( g \) on \( M \) is defined by induction on the depth of \( M \). It is any (Riemannian) metric if depth \( M = 0 \). Now, assume that depth \( M > 0 \) and general adapted metrics are defined for lower depth. Given any general chart \( O \equiv O' \) above, take any adapted metric \( \hat{g}_i \) on each \( N_i \) (depth \( N_i < \text{depth} M \)), and let \( g_i = \rho_{X,i} \ast_{X,i} \hat{g}_i + (d\rho_{X,i})^2 \) on \( N_i \times \mathbb{R}^+ \) for some \( u_{X,i} > 0 \). Let also \( g_0 \) be the Euclidean metric on \( \mathbb{R}^{m_X} \). Then \( g \) is a general adapted metric if, via any such general chart, \( g|_O \) is quasi-isometric to \( (\sum_{i=0}^{\infty} g_i)|_O' \). In this case, the mapping \( X \mapsto u_X := (u_{X,1}, \ldots, u_{X,a_X}) \in \mathbb{R}^{a_X}_{>0} \) \((X < M)\) is called the
general type of $g$, and such general chart is called compatible with $g$, or with its general type.

Let us point out that a general metric does not completely determine its general type. For instance, suppose $u_{X,i} = u_{X,j} = 1$ for indices $i \neq j$. Write $c(L_{X,i}) \times c(L_{X,j}) \equiv c(L)$, with radial function $\rho$, for some stratification $L$. Then $N_i \times \mathbb{R}_+ \times N_j \times \mathbb{R}_+ \equiv N \times \mathbb{R}_+$ for some dense stratum $N$ of $L$. Moreover there is a general admissible metric $\hat{g}$ on $N$ such that $g_i + g_j$ is quasi-isometric to $\rho^2 \hat{g} + (d\rho)^2$ via the above identity. Therefore we can omit $u_{X,i}$ or $u_{X,j}$ in $u_X$, obtaining a different type of $g$. This cannot be done if $u_{X,i} = u_{X,j} \neq 1$ (Proposition 2.1).

If the above definition of general adapted metric is modified by requiring that, at each inductive step, the general type satisfies

$$u_{X,i} \leq 1, \quad \frac{1}{u_{X,i}} \in 2\mathbb{Z} + k_{X,i} + (0,1] \quad \text{if} \quad \frac{1}{k_{X,i}} \leq u_{X,i} < 1, \quad \left\{ \right.$$  \hspace{1cm} (1)

for all $X < M$ and $i = 1, \ldots, a_X$, where $k_{X,i} = \dim N_i + 1$, then the general adapted metric is called good. On the other hand, if the definition is modified by requiring at each inductive step that $a_X = 1$ and $u_X$ depends only on $k := k_{X,1} = \text{codim} X$ for all $X < M$, then we get the adapted metrics considered in [33, 34, 8]. In this case, the general charts compatible with the general type are indeed charts. Writing $u_k = u_X \equiv u_{X,1} \in \mathbb{R}_+$, the condition (1) becomes

$$u_k \leq 1, \quad \frac{1}{u_k} \in 2\mathbb{Z} + k + (0,1] \quad \text{if} \quad \frac{1}{k} \leq u_k < 1. \quad \left\{ \right.$$  \hspace{1cm} (2)

Thus an adapted metric $g$ is good when (2) is satisfied at each inductive step of its definition. In [33, 34, 8], it is also assumed that $A$ is a stratified pseudomanifold, and then $\hat{a} = (u_2, \ldots, u_n)$ stands for the type of $g$. This $\hat{a}$ is determined by $g$. In particular, if the definition is modified taking $u_k = 1$ for all $k$ at each inductive step, we get the restricted adapted metrics, previously considered in [12, 13, 14]. Be alerted about the three slightly different terms, "(general/restricted) adapted metrics," used for the scope of this paper. The class of (good) general adapted metrics is preserved by products, as well as the class of restricted adapted metrics, but the class of adapted metrics does not have this property. The existence of general adapted metrics with any possible general type can be shown like in the case of adapted metrics [33, Lemma 4.3], [8, Appendix].

Like in [8], the term "relative(ly)" (or simply "rel-") usually means that some condition is required in the intersection of $M$ with small neighborhoods of the points in $\overline{M}$, or some concept can be described using those intersections.

Let $M$ be equipped with a general adapted metric $g$, with a general type $X \mapsto u_X$ as above. The rel-local metric completion $\widehat{M}$ of $M$ consists of the points in the metric completion represented by Cauchy sequences that converge in $\overline{M}$ ($\widehat{M}$ is the metric completion of $M$ if $\overline{M}$ is compact). The limits of Cauchy sequences define a continuous map $\lim : \widehat{M} \to \overline{M}$. The following properties can be proved like in the case of restricted adapted metrics [8, Proposition 3.20(i),(ii)]: $\widehat{M}$ has a unique stratified structure with connected strata such that $\lim : \widehat{M} \to \overline{M}$ is a morphism whose restrictions to the strata are local diffeomorphisms; and $g$ is also a general adapted metric with respect to $\widehat{M}$. 
1.4. **Relatively Morse functions.** A smooth function $f$ on $M$ is called *rel-admissible* when the functions $|df|$ and $|\text{Hess } f|$ are rel-bounded. In this case, $f$ may not have any continuous extension to $\hat{M}$, but it has a continuous extension to $\tilde{M}$. So it makes sense to say that $x \in \tilde{M}$ is a *rel-critical point* of $f$ when there is a sequence $(y_k)$ in $M$ such that $\lim_k y_k = x$ in $\tilde{M}$ and $\lim_k |df(y_k)| = 0$. The set of rel-critical points of $f$ is denoted by $\text{Crit}_{\text{rel}}(f)$. It is said that $f$ is a *rel-Morse function* if it is rel-admissible, and, for each $x \in \text{Crit}_{\text{rel}}(f)$, in some stratum $X$ of $\tilde{M}$, there exists a general chart $O \equiv O'$ of $\hat{M}$, centered at $x$ and compatible with $g$, such that $M \cap O \equiv M' \cap O'$ for $M' = \mathbb{R}^{m_X} \times \prod_{i=1}^{a_X} (N_i \times \mathbb{R}_+)$, and $f|_{M \cap O} \equiv f(x) + \frac{1}{2}(\rho_+^2 - \rho_-^2)|_{M' \cap O'}$, where $\rho_{\pm}$ is the radial function of $\mathbb{R}^{m_{\pm}} \times \prod_{i \in I_{\pm}} c(L_{X,i})$ for some expression $m_X = m_+ + m_- (m_{\pm} \in \mathbb{N})$ and some partition of $\{1, \ldots, a_X\}$ into sets $I_{\pm}$. This local condition is used instead of requiring that Hess $f$ is “rel-non-degenerate” at the rel-critical points because a “rel-Morse lemma” is missing. Moreover, for each $r \in \{0, \ldots, n\}$, let

$$\nu_{r, \text{max/min}}^x = \sum_{(r_1, \ldots, r_{a_X}) \in \mathbb{N}_{a_X}} \prod_{i=1}^{a_X} \beta_{r_i}^X (N_i) ,$$

where $(r_1, \ldots, r_{a_X})$ runs in the subset of $\mathbb{N}_{a_X}$ determined by

$$r = m_- + \sum_{i=1}^{a_X} r_i + |I_-| ,
\begin{align*}
    r_i &< \frac{k_{X,i} - 1}{2} + \frac{1}{2a_{X,i}} & \text{if } i \in I_+ \\
    r_i &\geq \frac{k_{X,i} - 1}{2} + \frac{1}{2a_{X,i}} & \text{if } i \in I_- \\
    r_i &\leq \frac{k_{X,i} - 1}{2} - \frac{1}{2a_{X,i}} & \text{if } i \in I^* \\
    r_i &> \frac{k_{X,i} - 1}{2} - \frac{1}{2a_{X,i}} & \text{if } i \in I^- \\
\end{align*}$$

Finally, let $\nu_{\text{max/min}}^r = \sum_x \nu_{r, \text{max/min}}^x$ with $x$ running in $\text{Crit}_{\text{rel}}(f)$. The notation $\nu_{r, \text{max/min}}^f$ and $\nu_{r, \text{max/min}}^{\max/min}(f)$ may be used if necessary. The existence of rel-Morse functions for general adapted metrics holds like in the case of adapted metrics [4 Proposition 4.9].

1.5. **Main theorems.** The following is our first main theorem, where property (iii) is a weak version of the Weyl’s asymptotic formula.

**Theorem 1.1.** The following properties hold on any stratum of a compact stratification with a good general adapted metric:

(i) $\Delta_{\text{max/min}}$ has a discrete spectrum, $0 \leq \lambda_{\text{max/min},0} \leq \lambda_{\text{max/min},1} \leq \cdots$, where each eigenvalue is repeated according to its multiplicity.

(ii) $\liminf_k \lambda_{\text{max/min},k} k^{-\theta} > 0$ for some $\theta > 0$.

Our second main result is the following version of Morse inequalities for rel-Morse functions.
Theorem 1.2. For any rel-Morse function on a stratum of dimension \( n \) of a compact stratification, equipped with a good general adapted metric, we have

\[
\sum_{r=0}^{k} (-1)^{k-r} \beta^r_{\text{max/min}} \leq \sum_{r=0}^{k} (-1)^{k-r} \nu^r_{\text{max/min}} \quad (0 \leq k < n),
\]

\[
\chi_{\text{max/min}} = \sum_{r=0}^{n} (-1)^r \nu^r_{\text{max/min}}.
\]

In the case of restricted adapted metrics, Theorem 1.2 is essentially due to Cheeger [12, 13] (see also [1, 2, 3]). Theorem 1.1–(ii) was proved by the authors [4, 5], and Theorem 1.2 was proved by the authors [4, 5] and Ludwig [20] (with more restrictive conditions but stronger consequences). Other developments of elliptic theory on strata were made in [10, 16, 17, 18, 10, 2, 11], all of them using restricted adapted metrics. The main novelty of our paper is the extension of the elliptic theory on strata to the wider class of good general adapted metrics, including good adapted metrics.

1.6. Applications to intersection homology. Consider now the case where \( A \) is a stratified pseudomanifold, and therefore \( M \) is its regular stratum. Let \( \hat{P}H_\ast(A) \) denote its intersection homology with perversity \( \hat{p} \) [19, 20], taking real coefficients. Let \( \beta^\hat{p}_r = \beta^{\hat{p}}_r(A) \) and \( \chi^{\hat{p}} = \chi^{\hat{p}}(A) \) denote the versions of Betti numbers and Euler characteristic for \( \hat{P}H_\ast(A) \). Each perversity can be considered as a sequence \( \hat{p} = (p_2, p_3, \ldots) \) in \( \mathbb{N} \) satisfying \( p_2 = 0 \) and \( p_k \leq p_{k+1} \leq p_k + 1 \); for example, the zero perversity is \( \hat{0} = (0, 0, \ldots) \), the top perversity is \( \hat{t} = (0, 1, 2, \ldots) \) \( (t_k = k - 2) \), the lower middle perversity is \( \hat{m} = (0, 0, 1, 1, 2, 2, 3, \ldots) \) \( (m_k = \left\lfloor \frac{k}{2} \right\rfloor - 1) \), and the upper middle perversity is \( \hat{n} = (0, 1, 1, 2, 3, 3, \ldots) \) \( (n_k = \left\lceil \frac{k}{2} \right\rceil - 1) \). Recall also that two perversities \( \hat{p} \) and \( \hat{q} \) are called complementary if \( \hat{p} + \hat{q} = \hat{t} \). Let \( g \) be an adapted metric on \( M \) of type \( \hat{u} = (u_2, \ldots, u_n) \). If \( \hat{u} \) is associated to \( \hat{p} \) in the sense

\[
\begin{align*}
1 - \frac{1}{k-2p_k} & \leq u_k < \frac{1}{k-3-2p_k} & \quad \text{if } & 2p_k \leq k - 3, \\
1 & \leq u_k < \infty & \quad \text{if } & 2p_k = k - 2,
\end{align*}
\]

then \( H^\hat{p}_\ast(M) \cong \hat{P}H_\ast(A)^\ast \) for all perversity \( \hat{p} \leq \hat{m} \) [33, 34, 8]. In particular, \( H^\hat{p}_\ast(M) \cong \hat{P}H_\ast(A)^\ast \) if \( g \) is a restricted adapted metric [13]. Thus the incompatibility of adapted metrics with products is related with the subtleties of the versions of the Künneth theorem for intersection homology [15, 17]; for instance, the isomorphism \( \hat{P}H_\ast(P \times Q) \cong \hat{P}H_\ast(P) \otimes \hat{P}H_\ast(Q) \), for arbitrary pseudomanifolds \( P \) and \( Q \), only holds with some special perversities \( \hat{p} \), including \( \hat{p} = \hat{m} \). The conditions (5) and (2) are satisfied if and only if

\[
\begin{align*}
\frac{1}{k-1-2p_k} & \leq u_k < \frac{1}{k-3-2p_k} & \quad \text{if } & 2p_k \leq k - 3, \\
u_k = 1 & \quad \text{if } & 2p_k = k - 2,
\end{align*}
\]

It follows that there exist good adapted metrics on \( M \) whose type is associated to any given perversity \( \leq \hat{m} \).

Let \( f \) be a rel-Morse function on \( M \), let \( x \in \text{Crit}_{\text{rel}}(f) \), let \( X \) be the stratum of \( \tilde{M} \) containing \( x \), and let \( k = \text{codim} X \). With the above notation for a chart \( O \equiv O' \) of \( \tilde{M} \) centered at \( x \), there is an adapted metric \( \tilde{g} \) on \( N \) so that, via the chart, \( g_{|O} \) is quasi-isometric to the restriction of \( g_0 + \tilde{p}^2 u_k \tilde{g} + (d\rho_X)^2 \) to \( M' \cap O' \). Then the type of \( \tilde{g} \) is also associated to \( \hat{p} \). Moreover there is some expression,
$m_X = m_+ + m_-$ ($m_{\pm} \in \mathbb{N}$), and some decomposition, $c(L_X) \equiv c(L_+) \times c(L_-)$, so that $M' \equiv \mathbb{R}^{m_+} \times N_+ \times \mathbb{R}_+ \times \mathbb{R}^{m_-} \times N_- \times \mathbb{R}_+$ for dense strata $N_{\pm}$ of $L_\pm$, and $f|_O \equiv f(x) + \frac{A}{2}(|D^2 f - \rho_\pm|)|_O$, where $\rho_{\pm}$ is the radial function of $\mathbb{R}^{m_{\pm}} \times c(L_{\pm})$. Let $k_{\pm} = \dim N_{\pm} + 1$; thus $k = k_+ + k_-$. Here, some of the strata $L_{\pm}$ may be empty; in fact, $L_+ \neq \emptyset \neq L_-$ only can happen if $u_k = 1$. From (3), (4) and (6), it follows that the numbers $\nu_{x,\max}$ interact with the choice of $\hat{u}$ associated to $\hat{p}$, and therefore the notation $\nu_{x,r} = \nu_{x,r}^\hat{p}(f)$ will be used. Precisely, they have the following expressions:

- If $L_+ \neq \emptyset \neq L_-$ (only if $2p_k = k - 2$), then
  \[
  \nu_{x,r}^\hat{p} = \sum_{(r_+,r_-)} \beta_{r_+}^\hat{p} (L_+) \beta_{r_-}^\hat{p} (L_-),
  \]
  where $(r_+,r_-)$ runs in the subset of $\mathbb{N}^2$ determined by the conditions
  \[
  r = m_- + r_+ + r_- + 1, \quad r_+ < \frac{k_+}{2}, \quad r_- \geq \frac{k_-}{2}.
  \]

- If $L_X = L_+ \neq \emptyset \neq L_-$, then
  \[
  \nu_{x,r}^\hat{p} = \sum_{r_+} \beta_{r_+}^\hat{p} (L_X),
  \]
  where $r_+$ runs in the subset of $\mathbb{N}$ determined by the conditions
  \[
  r = m_- + r_+, \quad r_+ \leq k - 2 - p_k.
  \]

- If $L_X = L_- \neq \emptyset \neq L_+$, then
  \[
  \nu_{x,r}^\hat{p} = \sum_{r_-} \beta_{r_-}^\hat{p} (L_X),
  \]
  where $r_-$ runs in the subset of $\mathbb{N}$ determined by the conditions
  \[
  r = m_- + r_- + 1, \quad r_- \geq k - 1 - p_k.
  \]

- If $L_X = \emptyset$, then $\nu_{x,r}^\hat{p} = \delta_{x,m_\pm}$.

Finally, let $\nu_{x} = \nu_{x}^\hat{p}(f) = \sum_x \nu_{x,r}^\hat{p}$, which equals $\nu_{x,\max}$.

Suppose now that $A$ is oriented ($M$ is oriented) and compact. On the one hand, we have $\beta_{\min} = \beta_{\max}$ for all $r$ because $\Delta_{\min}$ corresponds to $\Delta_{\max}$ by the Hodge star operator. On the other hand, for any perversity $\hat{q} \geq \hat{n}$, if $\hat{p} \leq \hat{m}$ is complementary of $\hat{q}$, then $t^q H_r(A) \cong t^p H_{\hat{m} - r}(A)^{\hat{q}}$ [14] [18], and therefore $\beta_{\hat{p}}^\hat{q} = \beta_{\hat{m} - r}^\hat{q}$, obtaining $\beta_{\hat{q}}^\hat{q} = \beta_{\min}^\hat{q}$. As before, it follows from (3), (4) and (6) that the numbers $\nu_{x,r}^\hat{p}$ are independent of the choice of $\hat{u}$ associated to $\hat{p}$. Precisely, with the notation $\nu_{x,r}^\hat{q} = \nu_{x,r}^\hat{q}(f) = \nu_{x,r}^{\hat{q},\min}$, they have the following expressions:

- If $L_+ \neq \emptyset \neq L_-$ (only if $2q_k = k - 2$), then
  \[
  \nu_{x,r}^\hat{q} = \sum_{(r_+,r_-)} \beta_{r_+}^\hat{q} (L_+) \beta_{r_-}^\hat{q} (L_-),
  \]
  where $(r_+,r_-)$ runs in the subset of $\mathbb{N}^2$ determined by the conditions
  \[
  r = m_- + r_+ + r_- + 1, \quad r_+ \leq \frac{k_+}{2} - 1, \quad r_- > \frac{k_-}{2} - 1.
  \]

- In the other cases, $\nu_{x,r}^{\hat{q}}$ is defined like $\nu_{x,r}^\hat{p}$ using $\hat{q}$ instead of $\hat{p}$.

Like $\nu_{x,r}^\hat{p}$, we also define $\nu_{x,r}^\hat{q} = \nu_{x,r}^{\hat{q},(f)} = \sum_x \nu_{x,r}^{\hat{q}}$ ($x \in \text{Crit}_{\text{rel}}(f)$), which equals $\nu_{x,r}^{\hat{q},\min}$.

Then Theorem 1.2 has the following direct consequence.

\footnote{Kronecker’s delta symbol is used.}
Corollary 1.3. Let $A$ be a compact pseudomanifold of dimension $n$, let $M$ be its regular stratum, and let $\bar{p}$ be a perversity. If $\bar{p} \leq \bar{m}$, or if $A$ is oriented and $\bar{p} \geq \bar{n}$, then, for any rel-Morse function on $M$ (with respect to any good adapted metric), we have

$$
\sum_{r=0}^{k} (-1)^{k-r} \beta_{r}^{\bar{p}} \leq \sum_{r=0}^{k} (-1)^{k-r} \nu_{r}^{\bar{p}} \quad (0 \leq k < n),
$$

$$
\chi^{\bar{p}} = \sum_{r=0}^{n} (-1)^{r} \nu_{r}^{\bar{p}}.
$$

Stratified Morse theory was introduced by Goresky and MacPherson [21], and has a great wealth of applications. In particular, Goresky and MacPherson have proved Morse inequalities on complex analytic varieties with Whitney stratifications, involving the intersection homology with perversity $\bar{m}$ [21, Chapter 6, Section 6.12]. Ludwig also gave an analytic interpretation of Morse theory in the spirit of Goresky and MacPherson for conformally conic manifolds [26, 27, 28, 29]. Our version of Morse functions, critical points and associated numbers are different from those used in [21], even in the case of perversity $\bar{m}$. To the authors’ knowledge, Corollary 1.3 is the first version of Morse inequalities for intersection homology with perversity $\neq \bar{m}$.

1.7. Ideas of the proofs. In the proofs of Theorems 1.1 and 1.2 several steps are like in the case of restricted adapted metrics [3]. Only brief indications of those steps are given in this paper, whereas the parts with new ideas are explained with detail. We adapt the well known analytic method of Witten [43]; specially, as described in [36] Chapters 9 and 14. Thus, given a rel-Morse function $f$ on $M$, we consider the Witten’s perturbation $d_{s} = e^{-sf} d e^{sf} = d + s df \wedge$ on $\Omega_{0}(M)$ ($s > 0$). Let $d_{s, \max/min}$ denote its maximum/minimum i.b.c., with corresponding Laplacian $\Delta_{s, \max/min}$. Since $\Delta_{s, \max/min} - \Delta_{\max/min}$ is bounded, it is enough to prove the properties of Theorem 1.1 for $\Delta_{s, \max/min}$. Moreover, using a globalization procedure [3, Propositions 14.2 and 14.3] and a version of the Künneth theorem [9, Corollary 2.15], it is enough to consider the case of a stratum $M = N \times \mathbb{R}_{+}$ of a cone $c(L)$ (a non-compact stratification), with a good general adapted metric of the form $g = \rho^{2} \tilde{g} + d \rho^{2}$, and the rel-Morse function $\pm \frac{1}{2} \rho^{2}$, where $\rho$ is the radial function and $L$ (a compact stratification of smaller depth). A tilde is added to the notation of concepts considered for $N$. By induction on the depth, it is assumed that $\tilde{\Delta}_{\max/min}$ satisfies the properties of Theorem 1.1. Then its eigenforms are used like in [3] to split $d_{s, \max/min}$ into a direct sum of Hilbert complexes of length one and two, which can be described as the maximum/minimum i.b.c. of certain elliptic complexes on $\mathbb{R}_{+}$. The elliptic complexes of length one are of the same kind as in [3], so that the Laplacian of their maximum/minimum i.b.c. is induced by the Dunkl harmonic oscillator on $\mathbb{R}$ [4], whose spectrum is well known. However, the Laplacian of the elliptic complexes of length two is a perturbation of the Dunkl harmonic oscillator containing new terms of the form $\rho^{-2n}$ and $\rho^{-n-1}$. A different analytic tool is used here, which was developed by the authors [5]. Precisely, classical perturbation methods were used in [4] to determine self-adjoint operators with discrete spectra defined by this perturbation of the Dunkl harmonic oscillator, giving also upper and lower estimates of its eigenvalues. The application of this analytic tool is what requires $g$ to be good. The information obtained for
2.1. This perturbation is weaker than for the Dunkl harmonic oscillator. For instance, such self-adjoint operators are only known to exist in some cases, and only a core of their square root is known. Thus more work is needed here than in [3] to describe the Laplacians of the maximum/minimum i.h.c. of the simple elliptic complexes of length two, using those self-adjoint operators. The proof of Theorem 1.1 can be completed with such information like in [3]. On the other hand, only eigenvalue estimates of those self-adjoint operators are known, which makes it more difficult to determine the “cohomological contribution” of the rel-critical points. This is the key idea to complete the proof of Theorem 1.2 like in [3].

1.8. Some open problems. We do not know if the condition on \( g \) to be good could be deleted. It depends on whether the result used from [5] holds with weaker hypothesis, which boils down to extending certain estimate of a finite sum.

The applications would increase by extending our version of Morse inequalities to “rel-Morse-Bott functions”, whose rel-critical point set would be a finite union of substratifications.

There should be an extension of the isomorphism \( H^*_A(M) \cong I^F H_*(A)^* \) to the case of general adapted metrics and general perversities [18]. In that direction, an extension of the de Rham theorem with general perversities was proved in [37, 38]; the case with classical perversities was previously considered in [11, 7].

It is also natural to continue with the following program, already achieved on closed manifolds. First, it should be shown that there is a spectral gap of the form \( \sigma(\Delta_{s,\text{max/min}}) \cap (C_1e^{-C_2s}, C_3s) = \emptyset \), for some \( C_1, C_2, C_3 > 0 \), obtaining a finite dimensional complex \( (\mathcal{S}_{s,\text{max/min}}, d_s) \) generated by the eigenforms corresponding to eigenvalues in \( [0, C_1e^{-C_2s}] \) (“small eigenvalues”). Second, it should be proved that \( (\mathcal{S}_{s,\text{max/min}}, d_s) \) “converges” to the “rel-Morse-Thom-Smale complex”, assuming that the function satisfies the “rel-Morse-Smale transversality condition”. It seems that the existence of the above spectral gap would follow easily by adapting the arguments of [3] Propositions 14.2 and 14.3. The comparison of \( (\mathcal{S}_{s,\text{max/min}}, d_s) \) with the “rel-Morse-Thom-Smale complex” would require additional techniques, according to the case of closed manifolds [22, 6, Section 6]. This program was developed by Ludwig in a special case [30].

2. Preliminaries

2.1. Products of cones. Let \( L \) and \( L' \) be compact stratifications, and let \( * \) and \( \rho \), and \( \ast' \) and \( \rho' \) be the vertices and radial functions of \( c(L) \) and \( c(L') \). Any morphism \( \psi : c(L) \to c(L') \) is of the form \( c(\phi) \) around \( * \) for some morphism \( \phi : L \to L' \). In particular, \( \psi(*) = \ast' \), and \( \psi^\ast \rho' = \rho \) around \( * \).

The product of two stratifications, \( A \times A' \), has a stratification structure whose strata are the products of strata of \( A \) and \( A' \). However the tubes in \( A \times A' \) depend on the choice of a function \( h : [0, \infty]^2 \to [0, \infty) \) that is continuous, homogeneous of degree one, smooth on \( \mathbb{R}^2_+ \), with \( h^{-1}(0) = \{(0,0)\} \), and such that, for some \( C > 1 \), we have \( h(r, r') = \max\{r, r'\} \) if \( C \min\{r, r'\} < \max\{r, r'\} \) [3] Section 3.1.2. Thus the stratification structure of \( A \times A' \) is not unique.

In the case of two cones, \( c(L) \times c(L') \) can be described as another cone in the following way [3] Lemma 3.8. The function \( h(\rho \times \rho') : c(L) \times c(L') \to [0, \infty) \) satisfies that \( L'' = (h(\rho \times \rho'))^{-1}(1) \) is a compact saturated substratification of \( c(L) \times c(L') \).
Then the map
\[ \phi : c(L') \to c(L) \times c(L') , \quad \langle [x, r], [x', r'] \rangle, s \mapsto \langle [x, rs], [x', r's] \rangle , \]
is an isomorphism of stratifications. The vertex of \( c(L') \) is \( s'' = \phi^{-1}(s, s') \), and its radial function is \( \rho'' = \phi^*(h(\rho \times \rho')) \). Thus the radial function of \( c(L') \times c(L') \), \((\rho^2 + \rho'')^\sharp \), does not correspond to \( \rho'' \) via \( \phi \) if \( L \neq \emptyset \neq L'' \).

Assume that \( L \neq \emptyset \neq L' \). Let \( N \) and \( N' \) be strata of \( L \) and \( L' \), and let \( M = N \times \mathbb{R}_+ \) and \( M' = N' \times \mathbb{R}_+ \) be the corresponding strata of \( c(L) \) and \( c(L') \). Take general adapted metrics \( \tilde{g} \) and \( \tilde{g}' \) on \( N \) and \( N' \), and fix any \( u > 0 \). We get general adapted metrics \( g = \rho^{2u} \tilde{g} + (dp)^2 \) and \( g' = \rho'^{2u} \tilde{g}' + (dp')^2 \) on \( M \) and \( M' \). On the other hand, with the above notation, we have \( \phi^{-1}(M \times M') = N'' \times \mathbb{R}_+ =: M'' \), where \( N'' = (M \times M') \cap L'' \) (a stratum of \( L'' \)). Let \( \tilde{g}'' \) be any general adapted metric on \( N'' \) so that \( N'' \to M \times M' \) is quasi-isometric; for instance, we may take \( \tilde{g}'' = (g + g')|_{N''} \). We get the general adapted metric \( g'' = \rho''^{2u} \tilde{g}'' + (dp'')^2 \) on \( M'' \).

Equip \( M \times M' \) with \( g + g' \) and \( M'' \) with \( g'' \).

**Proposition 2.1.**

(i) If \( u = 1 \), then \( \phi : M'' \to M \times M' \) is a quasi-isometry.

(ii) If \( u < 1 \), then \( \phi : M'' \cap O \to (M \times M') \cap (\phi(O)) \) is not quasi-isometric for any neighborhood \( O \) of \( s'' \) in \( c(L'') \).

**Proof.** Without lost of generality, we can assume \( \tilde{g}'' = (g + g')|_{N''} \). We have \( M'' = N'' \times \mathbb{R}_+ \subset M \times M' \times \mathbb{R}_+ = N \times \mathbb{R}_+ \times N' \times \mathbb{R}_+ \times \mathbb{R}_+ \). According to this expression, an arbitrary point \( p \in M'' \) can be written as \( p = (x, r, x', r', r'') \), obtaining
\[ \phi(p) = (x, rr'', x', r'r'') \in M \times M' = N \times \mathbb{R}_+ \times N' \times \mathbb{R}_+ . \]
Thus we can canonically consider
\[
\begin{align*}
\mathcal{T}_p N'' &\subset \mathcal{T}_x N \oplus \mathbb{R} \oplus \mathcal{T}_x N' \oplus \mathbb{R} , \\
\mathcal{T}_p M'' &\subset \mathcal{T}_x N \oplus \mathbb{R} \oplus \mathcal{T}_x N' \oplus \mathbb{R} \oplus \mathbb{R} , \\
\mathcal{T}_\phi(p)(M \times M') &\subset \mathcal{T}_x N \oplus \mathbb{R} \oplus \mathcal{T}_x N' \oplus \mathbb{R} ,
\end{align*}
\]
and we easily get
\[
\begin{align*}
\phi_*(\partial_{\rho''}(p)) &= (0, r \partial_r(rr''), 0, r' \partial_{r'}(r'r'')) , \\
\phi_*(X, 0) &= (Y, cr'' \partial_{r''}(rr''), Y', c' r'' \partial_{r''}(r'r'')) ,
\end{align*}
\]
for \( X = (Y, c \partial_r(r), Y', c' \partial_{r'}(r')) \in \mathcal{T}_p N'' \). Hence
\[
\begin{align*}
\|\partial_{\rho''}(p)\|_{g''}^2 &= 1 , \\
\|\phi_*(\partial_{\rho''}(p))\|_{g'' + g'}^2 &= r^2 + r'^2 , \\
\|\phi_*(X, 0)\|_{g'' + g'}^2 &= r''^{2u} \|X\|_{\tilde{g}'' + g'}^2 , \\
&= r''^{2u} \left( \|Y\|_{\tilde{g}''}^2 + c^2 + \|Y'\|_{\tilde{g}''}^2 + c'^2 \right) , \\
\|\phi_*(X, 0)\|_{g'' + g'}^2 &= \left( r''^{2u} \|Y\|_{\tilde{g}''}^2 + c^2 r''^2 + r''^{2u} \|Y'\|_{\tilde{g}''}^2 + c'^2 r'^2 \right) , \\
&= r''^{2u} \left( \|Y\|_{\tilde{g}''}^2 + c^2 r''^2(1-u) + \|Y'\|_{\tilde{g}''}^2 + c'^2 r'^2(1-u) \right) ,
\end{align*}
\]
where each metric is added as subindex of the corresponding norm.
Observe that \( C_0 := \min_{\rho'} (\rho^2 + \rho'^2) > 0 \) and \( C_1 := \max_{\rho'} (\rho^2 + \rho'^2) < \infty \) by the properties of \( h \). So, by (7) and (8),
\[
C_0 \| \partial \rho' (p) \|_{g'}^2 \leq \| \phi_* (\partial \rho' (p)) \|_{g + g'}^2 \leq C_1 \| \partial \rho' (p) \|_{g'}^2 .
\]
Moreover, if \( u = 1 \), then \( \| \phi_* (X, 0) \|_{g + g'}^2 = \| (X, 0) \|_{g'}^2 \) by (9) and (10), obtaining (4).

Now, suppose that \( u < 1 \). With the above notation, by the conditions satisfied by \( h \), we can take \( \bar{p} = (x, r, x', 1) \in N'' \) and \( X = (0, \partial, (r), 0, 0) \in T_p N'' \) for all \( r \) small enough. By (9) and (10), it follows that
\[
\frac{\| \phi_* (X, 0) \|_{g + g'}^2}{\| (X, 0) \|_{g'}^2} = r^{u (1 - u)} \to 0
\]
as \( r'' \to 0 \), giving (11). \( \square \)

Similar observations apply to the product of any finite number of cones.

2.2. General adapted metrics. Consider the notation of Section 1.3.

Remark 1. For each \( m \in \mathbb{Z}_+ \), there is a canonical homeomorphism \( c(S^{m-1}) \approx \mathbb{R}^m \), \( [x, \rho] \mapsto \rho x \), so that the radial function \( \rho \) corresponds to the norm on \( \mathbb{R}^m \). Example 3.6. This is not an isomorphism of stratifications: \( c(S^{m-1}) \) has two strata and \( \mathbb{R}^m \) only one; the stratum \( S^{m-1} \times \mathbb{R}_+ \) of \( c(S^{m-1}) \) corresponds to \( \mathbb{R}^m \setminus \{ 0 \} \). If \( \bar{g} \) denotes the standard metric on \( S^{m-1} \), then \( \rho^2 \bar{g} + (d\rho)^2 \) on \( S^{m-1} \times \mathbb{R}_+ \) corresponds to the Euclidean metric on \( \mathbb{R}^m \setminus \{ 0 \} \). Thus, with the notation of Section 1, the factors \( \mathbb{R}^m \times \) or \( \mathbb{R}^m \) could be also described as cones, or as strata of cones after removing one point.

Remark 2. By taking charts and using induction on the depth, we get the following (cf. [3, Remark 7]):

(i) If two general adapted metrics on \( M \) have the same type with respect to the same general tubes, then they are rel-locally quasi-isometric; in particular, they are quasi-isometric if \( \widehat{M} \) is compact.

(ii) Any point in \( \widehat{M} \) has a countable base \( \{ O_m \mid m \in \mathbb{N} \} \) of open neighborhoods such that, with respect to any general adapted metric, \( \text{vol}(M \cap O_m) \to 0 \) and \( \max \{ \text{diam } P \mid P \in \pi_0 (M \cap O_m) \} \to 0 \) as \( m \to \infty \). Thus, if \( \widehat{M} \) is compact, then \( \text{vol } M < \infty \) and \( \text{diam } P < \infty \) for all \( P \in \pi_0 (M) \).

Remark 3. The argument of [8, Appendix] also shows the following. Let \( \{ O_a \} \) be a locally finite open covering of \( \widehat{M} \), let \( \{ \lambda_a \} \) be a smooth partition of unity of \( M \) subordinated to the open covering \( \{ M \cap O_a \} \), and let \( g_a \) be a general adapted metric on each \( M \cap O_a \). Suppose that the metrics \( g_a \) have the same general type with respect to restrictions to the sets \( O_a \) of the same general tubes. Then the metric \( \sum_a \lambda_a g_a \) is adapted on \( M \) and has the same general type with respect to those general tubes.

When \( M \) is not connected, \( \widehat{M} \) is defined as the disjoint union of the rel-local completion of its connected components of \( M \), using [3, Remark 1-(v)].

Remark 4. (i) By Remark 2-(ii), \( \widehat{M} \) is independent of the choice of the general adapted metric of a given general type. In fact, by Remark 2-(ii), \( \widehat{M} \) is also independent of the general type.

(ii) For any open \( O \subset A \), \( \widehat{M} \cap O \) can be canonically identified to the open subspace \( \lim^{-1} (\widehat{M} \cap O) \subset \widehat{M} \).
Remark 5. The following is a direct consequence of Remark 9 and Proposition 3.20-(iii):

(i) \( \lim_{\hat{M} \to M} \) is surjective with finite fibers.
(ii) \( M \) is rel-locally connected with respect to \( \hat{M} \).
(iii) Let \( M' \) be a connected stratum of another stratification \( A' \) equipped with a general adapted metric. Then, for any morphism \( \phi : A \to A' \) with \( f(M) \subset M' \), the restriction \( \phi : M \to M' \) extends to a morphism \( \hat{\phi} : \hat{M} \to \hat{M}' \). Moreover \( \hat{f} \) is an isomorphism if \( f \) is an isomorphism.

2.3. Relatively Morse functions. Consider the notation of Section 1.4. Besides the observations given in that section, the following holds like in the case of restricted adapted metrics [3, Section 4].

Remark 6. (i) The rel-local boundedness of \( |df| \) is invariant by rel-local quasi-isometries, and therefore it depends only on the general type of \( g \). Similarly, the definition of rel-critical point depends only on the general type of \( g \). But the rel-local boundedness of \( |Hess f| \) depends on the choice of \( g \). However it follows from (iv) and (v) below that the existence of \( g \) so that \( f \) is rel-admissible with respect to \( g \) is a rel-local property.

(ii) If depth \( M = 0 \), then any smooth function is admissible, and its rel-critical points are its critical points.

(iii) With the notation of Section 2.1, for any \( h \in C^\infty(R^+) \) with \( h' \in C^\infty_0(R^+) \), the function \( h(\rho) \) is rel-admissible on the stratum \( M \) of \( c(L) \) with respect to any general adapted metric.

(iv) For any locally finite covering \( \{ O_a \mid a \in A \} \) of \( M \) by open subsets of \( A \), there is a \( C^\infty \) partition of unity \( \{ \lambda_a \} \) on \( M \) subordinated to \( \{ M \cap O_a \} \) such that \( |d\lambda_a| \) is rel-locally bounded for all general adapted metrics on \( M \) of any fixed general type.

(v) Suppose that \( \{ \lambda_a \} \) and \( \{ g_a \} \) satisfy the conditions of Remark 3 and (iv). Let \( f \in C^\infty(M) \) such that each \( f|_{M \cap O_a} \) is rel-admissible with respect to \( g_a \). Then \( f \) is rel-admissible with respect to the general adapted metric \( g = \sum_a \lambda_a g_a \) on \( M \).

(vi) Let \( F \subset C^\infty(M) \) denote the subset of functions with continuous extensions to \( M \) that restrict to rel-Morse functions with respect to all general adapted metrics of all possible types on all strata \( \leq M \). Then \( F \) is dense in \( C^\infty(M) \) with the weak \( C^\infty \) topology.

2.4. Hilbert and elliptic complexes. Consider the notation of Section 1.1.

2.4.1. Hilbert complexes with a discrete positive spectrum. Let \( (D, d) \) be a Hilbert complex in a graded separable Hilbert space \( H \), defining self-adjoint operators \( D \) and \( \Delta \) according to Section 1.1. The direct sum of homogeneous subspaces of even/odd degree are denoted with the subindex “ev/odd”, and the same subindex is used to denote the restriction of homogeneous operators to such subspaces.

Lemma 2.2. The positive spectrum of \( \Delta_{ev} \) is discrete if and only if the positive spectrum of \( \Delta_{odd} \) is discrete. In this case, both operators have the same positive eigenvalues, with the same multiplicity.

\( ^2 \)Recall that a complex number is in the discrete spectrum of a normal operator in a Hilbert space when it is an eigenvalue of finite multiplicity.
Proof. If the positive spectrum of $\Delta_{ev}$ is discrete, it follows from the spectral theorem that
\[ D^\infty(\Delta_{ev/odd}) = \ker \Delta_{ev/odd} \oplus \Delta(D^\infty(\Delta_{ev/odd})) , \]
and
\[ D_{ev} : \Delta(D^\infty(\Delta_{odd})) \to \Delta(D^\infty(\Delta_{odd})) \]
is a linear isomorphism satisfying $D_{ev}\Delta_{ev} = \Delta_{odd}D_{ev}$.

2.4.2. Elliptic complexes with a term that is a direct sum. Let $E = \bigoplus_r E_r$ be a graded Riemannian or Hermitian vector bundle over a Riemannian manifold $M$. The space of its smooth sections will be denoted by $C^\infty(E)$, its subspace of compactly supported smooth sections will be denoted by $C_0^\infty(E)$, and the Hilbert space of square integrable sections of $E$ will be denoted by $L^2(E)$; all of these are graded spaces. Consider differential operators of the same order, $C_{r} : C^\infty(E_r) \to C^\infty(E_{r+1})$, such that $(C^\infty(E), d = \bigoplus_r d_r)$ is an elliptic complex. The simpler notation $(E, d)$ (or even $d$) will be preferred. Elliptic complexes with non-zero terms of negative degrees or homogeneous differential operators of degree $-1$ may be also considered without any essential change. For instance, we have the formal adjoint elliptic complex $(E, \delta)$.

Suppose that there is an orthogonal decomposition $E_{r+1} = E_{r+1,1} \oplus E_{r+1,2}$ for some degree $r + 1$. Thus
\[ C^\infty(E_{r+1}) \equiv C^\infty(E_{r+1,1}) \oplus C^\infty(E_{r+1,2}) , \]
\[ C_0^\infty(E_{r+1}) \equiv C_0^\infty(E_{r+1,1}) \oplus C_0^\infty(E_{r+1,2}) , \]
\[ L^2(E_{r+1}) \equiv L^2(E_{r+1,1}) \oplus L^2(E_{r+1,2}) , \]
and we can write
\[ d_r = \begin{pmatrix} d_{r,1} \\ d_{r,2} \end{pmatrix} , \quad \delta_r = \begin{pmatrix} \delta_{r,1} \\ \delta_{r,2} \end{pmatrix} . \]
The operators $d_{r,i}$ and $\delta_{r,i}$ can be also considered as elliptic complexes of length one, and therefore they have a maximum/minimum i.b.c., $d_{r,i,\max/\min}$ and $\delta_{r,i,\max/\min}$.

Lemma 2.3 (Lemma 8.2). We have:
\[ D(d_{\max,r}) = D(d_{r,1,\max}) \cap D(d_{r,2,\max}) , \quad d_{\max,r} = \begin{pmatrix} d_{r,1,\max} \langle D(d_{\max,r}) \\ d_{r,2,\max} \langle D(d_{\max,r}) \end{pmatrix} . \]

Lemma 2.4. We have:
\[ D(d_{r+1,\max/\min}) \oplus D(d_{r+1,\max/\min}) \subset D(d_{\max,\min,r+1}) . \]

Proof. Take any $(\nu^r) \in D(d_{r+1,\min}) \oplus D(d_{r+1,\min})$, and let $u' = d_{r+1,\min}u$ and $v' = d_{r+1,\min}v$. This means that there are sequences, $u_i \in C_0^\infty(E_{r+1,1})$ and $v_i \in C_0^\infty(E_{r+1,2})$, such that $u_i \to u$ in $L^2(E_{r+1,1})$, $v_i \to v$ in $L^2(E_{r+1,2})$, $d_{r+1,1}u_i \to u'$ and $d_{r+1,2}v_i \to v'$ in $L^2(E_{r+1})$. So $(u_i) \in C_0^\infty(E_{r+1,1}) \oplus C_0^\infty(E_{r+1,2})$, $(u_i) \to (u^r) \in L^2(E_{r+1})$ and $d_{r+1}(u_i) \to u' + v'$ in $L^2(E_{r+1})$, obtaining $(\nu^r) \in D(d_{\min,r+1})$.

Now, take any $(\nu^r) \in D(d_{r+1,\max}) \oplus D(d_{r+1,\max})$, and let $u' = d_{r+1,\max}u$ and $v' = d_{r+1,\max}v$. This means that $\langle u, \delta_{r+1,1}w \rangle = \langle u', w \rangle$ and $\langle v, \delta_{r+1,2}w \rangle = \langle v', w \rangle$ for all $w \in C_0^\infty(E_{r+2})$. Thus $(\nu^r, \delta_{r+1}w) = (u' + v', w)$ for all $w \in C_0^\infty(E_{r+2})$, obtaining that $(\nu^r) \in D(d_{\max,r+1})$.

\[ ^3 \text{Recall that ellipticity means that the sequence of principal symbols of the operators } d_r \text{ is exact over each non-zero cotangent vector.} \]
3. A perturbation of the Dunkl harmonic oscillator

This section is devoted to recall the main analytic tool of the paper: the study of self-adjoint operators on \( \mathbb{R}_+ \) induced by the Dunkl harmonic oscillator on \( \mathbb{R} \) [4], and also by certain perturbation of the Dunkl harmonic oscillator on \( \mathbb{R} \) [5].

Let \( S = \mathcal{S}(\mathbb{R}) \) be the real/complex valued Schwartz space on \( \mathbb{R} \), with its Fréchet topology. It decomposes as direct sum of subspaces of even and odd functions, \( S = S_{\text{ev}} \oplus S_{\text{odd}} \). For \( \sigma > -1/2 \), the sequence of generalized Hermite polynomials, \( p_k = p_{s,\sigma,k}(x) \), consists of the orthogonal polynomials associated with the measure \( e^{-sx^2}|x|^{\sigma/2} \, dx \) on \( \mathbb{R} \) [10, p. 380, Problem 25]; it is assumed that each \( p_k \) is normalized and has positive leading coefficient. They give rise to the general Hermite functions \( \phi_k = \phi_{s,\sigma,k}(x) = p_k e^{-sx^2/2} \in S \). If \( k \) is odd, then \( p_{s,\sigma,k} \) and \( \phi_{s,\tau,k} \) also make sense for \( \tau > -3/2 \).

Now, let \( \rho \) denote the canonical coordinate of \( \mathbb{R}_+ \). Consider the spaces of real/complex valued functions, \( C^\infty = C^\infty(\mathbb{R}) \), \( C^\infty_+ = C^\infty(\mathbb{R}_+) \) and \( C^\infty_{+,0} = C^\infty_0(\mathbb{R}_+) \), where the sub-index 0 is used for compactly supported functions or sections. For each \( a \in \mathbb{R} \), the operator of multiplication by the function \( \rho^a \) on \( C^\infty_+ \) will be also denoted by \( \rho^a \). We have

\[
\left[ \frac{d}{dp} , \rho^a \right] = a \rho^{a-1} , \quad \left[ \frac{d^2}{dp^2} , \rho^a \right] = 2a \rho^{a-1} \frac{d}{dp} + a(a-1) \rho^{a-2} .
\]

For each \( \phi \in C^\infty \), let \( \phi_+ = \phi|_{\mathbb{R}_+} \), and let \( S_{\text{ev/odd},+} = \{ \phi_+ \mid \phi \in S_{\text{ev/odd}} \} \). For \( c,d > -1/2 \), let \( L^2_{c,+} = L^2(\mathbb{R}_+, \rho^c \, d\rho) \) and \( L^2_{c,d,+} = L^2_{c,+} \oplus L^2_{d,+} \), whose scalar products are denoted by \( \langle \cdot , \cdot \rangle_c \) and \( \langle \cdot , \cdot \rangle_{c,d} \), and the corresponding norms by \( || \cdot ||_c \) and \( || \cdot ||_{c,d} \), respectively. The simpler notation \( L^2_{+} \), \( \langle \cdot , \cdot \rangle \) and \( || \cdot || \) is used when \( c = 0 \).

Recall that the harmonic oscillator on \( C^\infty_+ \) is the operator \( H = -\frac{d^2}{dp^2} + s^2 \rho^2 \) \( (s > 0) \). For \( c_1, c_2, d_1, d_2 \in \mathbb{R} \), let

\[
P_0 = H - 2c_1 \rho^{-1} \frac{d}{dp} + c_2 \rho^{-2} , \quad Q_0 = H - 2d_1 \rho^{-1} + d_2 \rho^{-2} ,
\]
whose domains will be determined later. Fix \( a,b \in \mathbb{R} \), and let \( \sigma = a + c_1 \) and \( \tau = b + d_1 \). For the sake of simplicity, define \( c_k = \sigma \) if \( k \) is even, and \( c_k = \tau \) if \( k \) is odd \( (k \in \mathbb{N}) \).

**Proposition 3.1** ([4] Theorem 1.4). If

\[
a^2 + (2c_1 - 1)a - c_2 = 0 , \quad \sigma > -1/2 ,
\]
then the following holds:

(i) \( P_0 \), with \( \mathcal{D}(P_0) = \rho^a S_{\text{ev},+} \), is essentially self-adjoint in \( L^2_{c_1,+} \).

(ii) The spectrum of \( \mathcal{P}_0 := P_0 \) consists of the eigenvalues

\[
\lambda_k = (2k + 1 + 2c_k)s ,
\]

for \( k \in 2\mathbb{N} \), with multiplicity one and corresponding normalized eigenfunctions \( \chi_{k} = \chi_{s,\sigma,a,k} = \sqrt{2} \rho^a \phi_{s,\sigma,a,k,+} \).

(iii) \( \mathcal{D}^{\infty}(P_0) = \rho^a S_{\text{ev},+} \).

**Proposition 3.2** (See [4] Section 5 and [5] Section 7). If

\[
b^2 + (2d_1 + 1)b - d_2 = 0 , \quad \tau > -3/2 ,
\]
then the following holds:
(i) $Q_0$, with $D(Q_0) = \rho^b S_{\text{odd},+}$, is essentially self-adjoint in $L^2_{d_1,+}$.

(ii) The spectrum of $Q_0 := Q_0$ consists of the eigenvalues \[ 16 \] for $k \in 2\mathbb{N}+1$, with multiplicity one and corresponding normalized eigenfunctions $\chi_k = \chi_{s,\tau,b,k} := \sqrt{2} \rho^b \phi_s,\tau,b,k$.

(iii) $D^\infty(Q_0) = \rho^b S_{\text{odd},+}$.

Fix also some $\xi > 0$ and $0 < u < 1$.

**Proposition 3.3** ([5] Corollary 7.1). If \[ 17 \] holds, and

$$\sigma > u - 1/2,$$ \[(19)\]

then there is a positive self-adjoint operator $P$ in $L^2_{c_1,+}$ satisfying the following:

(i) $\rho^a S_{\text{ev},+}$ is a core of $P^{1/2}$ and, for all $\phi, \psi \in \rho^a S_{\text{ev},+}$,

$$\langle P^{1/2} \phi, P^{1/2} \psi \rangle_{c_1} = \langle P_0 \phi, \psi \rangle_{c_1} + \xi \langle \rho^{-u} \phi, \rho^{-u} \psi \rangle_{c_1}.$$

(ii) $P$ has a discrete spectrum. Let $\lambda_0 \leq \lambda_2 \leq \cdots$ be its eigenvalues, repeated according to their multiplicity. There is some $D = D(\sigma, u) > 0$, and, for each $\epsilon > 0$, there is some $C = C(\epsilon, \sigma, u) > 0$ so that, for all $k \in 2\mathbb{N}$,

$$\lambda_k \geq (2k + 1 + 2\xi_k)s + \xi Ds^u(k + 1)^{-u},$$ \[(21)\]

$$\lambda_k \leq (2k + 1 + 2\xi_k)(s + \xi s^u) + \xi Cs^u.$$ \[(22)\]

**Proposition 3.4** ([5] Corollary 7.2). If \[ 18 \] holds, and

$$\tau > u - 3/2,$$ \[(23)\]

then there is a positive self-adjoint operator $Q$ in $L^2_{d_1,+}$ satisfying the following:

(i) $\rho^b S_{\text{odd},+}$ is a core of $Q^{1/2}$ and, for all $\phi, \psi \in \rho^b S_{\text{odd},+}$,

$$\langle Q^{1/2} \phi, Q^{1/2} \psi \rangle_{d_1} = \langle Q_0 \phi, \psi \rangle_{d_1} + d_3 \langle \rho^{-u} \phi, \rho^{-u} \psi \rangle_{d_1}.$$ \[(24)\]

(ii) $Q$ has a discrete spectrum. Let $\lambda_1 \leq \lambda_3 \leq \cdots$ be its eigenvalues, repeated according to their multiplicity. There is some $D = D(\tau, u) > 0$, and, for each $\epsilon > 0$, there is some $C = C(\epsilon, \tau, u) > 0$ so that \[ 21 \] and \[ 22 \] are satisfied for all $k \in 2\mathbb{N} + 1$.

Before stating the next result, let us introduce the following sets:

- $\mathfrak{J}_1$ is the set of points $(\sigma, \tau) \in \mathbb{R}^2$ such that:

$$\frac{1}{4} \leq \tau < \sigma \implies \sigma - 1 < \tau < \frac{\sigma}{2} + \frac{1}{4},$$ \[(25)\]

$$\frac{1}{2}, \sigma \leq \tau \implies \tau < \frac{\sigma}{2} + \frac{1}{4}, \sigma + 1,$$ \[(26)\]

$$\tau < \frac{1}{2}, \sigma \implies \frac{\sigma}{3}, \sigma - 1 < \tau < \frac{\sigma}{2} + \frac{1}{4},$$ \[(27)\]

$$\sigma \leq \tau < \frac{1}{3} \implies -\sigma < \tau < \frac{\sigma}{2} + \frac{1}{4}, \sigma + 1.$$ \[(28)\]
\( J_2 \) is the set of points \((\sigma, \tau) \in \mathbb{R}^2\) such that:

\[
\frac{1}{2} \leq \tau < \sigma - \frac{1}{2} \quad \Rightarrow \quad \sigma - 1 < \tau < \frac{\sigma + 1}{4},
\]
\[
\frac{1}{2}, \sigma - \frac{1}{2} \leq \tau \quad \Rightarrow \quad \tau < \frac{\sigma}{2} + \frac{1}{4}, \sigma,
\]
\[
0 < \tau < \frac{1}{2}, \sigma - \frac{1}{2} \quad \Rightarrow \quad \begin{cases} -\frac{\sigma}{2}, \sigma - 1 < \tau < \frac{\sigma}{2} + \frac{1}{4}, \\ \sigma - 1 < \tau < \frac{\sigma}{2} - \frac{1}{4}. \end{cases}
\]
\[
0 < \tau < \frac{1}{2}, \sigma - \frac{1}{2} \leq \tau \quad \Rightarrow \quad \begin{cases} 1 - \sigma < \tau < \frac{\sigma}{2} + \frac{1}{4}, \sigma, \\ \tau < \frac{\sigma}{2} - \frac{1}{4}, \sigma. \end{cases}
\]

\( R_1 \) is the set of points \((\sigma, \tau, \theta) \in \mathbb{R}^3\) such that:

\[
\theta \leq \sigma - 1, \quad \theta < \tau + 1 \quad \Rightarrow \quad \theta > \frac{\sigma}{3} - \frac{3}{4}, \frac{\sigma + \tau}{4},
\]
\[
\tau + 1 \leq \theta \leq \sigma - 1 \quad \Rightarrow \quad \theta > \frac{\sigma}{3} - \frac{3}{4}, \frac{\sigma - \tau}{4} - 1,
\]
\[
\sigma - 1 < \theta < \tau + 1 \quad \Rightarrow \quad \theta > \frac{\sigma}{3} - \frac{3}{4}, \frac{\sigma - \tau}{4} + 1, \frac{\sigma + \tau}{4},
\]
\[
\sigma - 1 < \theta, \quad \tau + 1 \leq \theta \quad \Rightarrow \quad \theta > \frac{\sigma}{3} - \frac{3}{4}, \frac{\sigma - \tau}{4} - 1, \sigma + \tau > 0.
\]

\( R'_1 \) is the set of points \((\sigma, \tau, \theta) \in \mathbb{R}^3\) such that:

\[
\theta < \sigma, \quad \theta \leq \tau \quad \Rightarrow \quad \theta > \frac{\sigma}{3} - \frac{1}{4}, \frac{\sigma - \tau}{4},
\]
\[
\sigma \leq \theta \leq \tau \quad \Rightarrow \quad \theta > \frac{\sigma}{3} - \frac{1}{4}, \frac{\tau - \sigma}{4},
\]
\[
\tau < \theta < \sigma \quad \Rightarrow \quad \theta > \frac{\sigma}{3} - \frac{1}{4}, \frac{\sigma - \tau}{4} + \frac{\sigma + \tau}{4},
\]
\[
\sigma \leq \theta, \quad \tau < \theta \quad \Rightarrow \quad \theta > \frac{\sigma}{3} - \frac{1}{4}, \frac{\tau - \sigma}{4}, \sigma + \tau > 0.
\]

\( R_2 \) is the set of points \((\sigma, \tau, \theta) \in \mathbb{R}^3\) such that:

\[
\theta \leq \sigma - 1, \quad \theta < \tau + \frac{1}{2} \quad \Rightarrow \quad \theta > \frac{\sigma}{3} - \frac{3}{4}, \frac{\sigma + \tau}{4},
\]
\[
\tau + \frac{1}{2} \leq \theta \leq \sigma - 1 \quad \Rightarrow \quad \theta > \frac{\sigma}{3} - \frac{3}{4}, \frac{\sigma - \tau}{4} - 1,
\]
\[
\sigma - 1 < \theta < \sigma - \frac{1}{2}, \tau + \frac{1}{2} \quad \Rightarrow \quad \begin{cases} \theta > \frac{\sigma}{3} - \frac{3}{4}, \frac{\tau - \sigma}{4} + 1, \frac{\sigma + \tau}{4}, \\ \theta > \frac{\sigma}{3} - \frac{1}{4}, \frac{\sigma - \tau}{4}, \end{cases}
\]
\[
\sigma - 1 < \theta < \sigma - \frac{1}{2}, \tau + \frac{1}{2} \leq \theta \quad \Rightarrow \quad \begin{cases} \theta > \frac{\sigma}{3} - \frac{3}{4}, \frac{\sigma - \tau}{4} - 1, \sigma + \tau > 1, \\ \theta > \frac{\sigma}{3} - \frac{1}{4}, \frac{\sigma - \tau}{4}, \end{cases}
\]
\[
\sigma - \frac{1}{2} = \theta < \tau + \frac{1}{2} \quad \Rightarrow \quad \sigma > \frac{1}{2}, \frac{\tau + 2}{3},
\]
\[
\tau + \frac{1}{2} \leq \theta = \sigma - \frac{1}{2} \quad \Rightarrow \quad \sigma > \frac{1}{2}, -\tau,
\]
\[
\sigma - \frac{1}{2} < \theta < \tau + \frac{1}{2} \quad \Rightarrow \quad \theta > \frac{\sigma}{3} - \frac{1}{4}, \frac{\tau - \sigma + 1}{4}, \frac{\sigma + \tau}{4},
\]
\[
\sigma - \frac{1}{2} < \theta, \tau + \frac{1}{2} \leq \theta \quad \Rightarrow \quad \theta > \frac{\sigma}{3} - \frac{1}{4}, \frac{\tau - \sigma - 1}{4}, \sigma + \tau > 0.
\]
Proposition 3.5. Then there is a positive self-adjoint operator \( J_A \).

Remark (a) ii. If \( \sigma \) is even, and \( \sigma - \frac{1}{2} \leq \theta < \sigma + \frac{1}{2} \), then \( \tau < \sigma - \frac{1}{2} \).

(14) \[ \rho \in L^{1/2} \phi, \mathcal{F}^{1/2} \psi \] \[ \xi (\rho^{-u} \phi, \rho^{-u} \psi) \] \[ \eta (\rho^{-a-b-1} \phi_2, \psi_1) \theta + (\phi_1, \rho^{-a-b-1} \psi_2) \eta \] \[ \lambda_k \leq (2k + 1 + 2c_k)(s + c(s^u + 2|\eta|Es^{\frac{u-1}{2}})) + \xi Cs^u + 2|\eta|Es^{\frac{u-1}{2}} \] .

Remark 7. (i) If \( h \) is a bounded measurable function on \( \mathbb{R}_+ \) with \( h(\rho) \to 1 \) as \( \rho \to 0 \), then \( h(x_0, \chi_0) \to 1 \) as \( s \to \infty \). (Lemma 7.3).

(ii) The existence of \( a \in \mathbb{R} \) satisfying (14) is characterized by the condition \( (2c_1 - 1)^2 + 4c_2 \geq 0 \), which holds if \( c_2 \geq \text{min}\{0, 2c_1\} \). If \( c_2 = 0 \), then (14) means that \( a \in \{0, 1 - 2c_1\} \). If \( c_2 = 2c_1 \), then (14) means that \( a \in \{1, -2c_1\} \).

(iii) The existence of \( b \in \mathbb{R} \) satisfying (17) is characterized by the condition \( (2d_1 + 1)^2 + 4d_2 \geq 0 \), which holds if \( d_2 \geq \text{min}\{0, -2d_1\} \). If \( d_2 = 0 \), then (17) means that \( b \in \{0, -1 - 2d_1\} \). If \( d_2 = -2d_1 \), then (17) means that \( b \in \{1, -2d_1\} \).
(iv) Propositions 3.1 and 3.2 are indeed equivalent, as well as Propositions 3.3 and 3.4, because if $c_1 = d_1 + 1$ and $c_2 = d_2$, then $Q_0 = \rho P_0 \rho^{-1}$ by (12), and $\rho : L^2_{c_1,+} \to L^2_{d_1,+}$ is a unitary isomorphism.

(v) We have $P = P_0 + \xi x^{-2u}$, $Q = Q_0 + \xi x^{-2u}$, and $W = W_0$, where

$$
P = P_0 + \xi x^{-2u}, \quad Q = Q_0 + \xi x^{-2u}, \quad W = \left( \eta x^{2(\theta - d_1) - a - b - 1} \right).
$$

with $D(P) = D^\infty(P)$, $D(Q) = D^\infty(Q)$ and $D(W) = D^\infty(W)$ [5], Remark 1-(ii) and Section 7.

(vi) We have

$$
D(P^{1/2}) = D(P_0^{1/2}), \quad D(Q^{1/2}) = D(Q_0^{1/2}), \quad D(W^{1/2}) = D((P_0 \oplus Q_0)^{1/2}).
$$

Thus the expressions (20), (21) and (22) can be extended to $\phi$ and $\psi$ in $D(P^{1/2})$, $D(Q^{1/2})$ and $D(W^{1/2})$, respectively, using

$$
\langle P_0^{1/2} \phi, Q_0^{1/2} \psi \rangle_{c_1}, \quad \langle Q_0^{1/2} \phi, Q_0^{1/2} \psi \rangle_{d_1},
$$

instead of

$$
\langle P_0 \phi, \psi \rangle_{c_1}, \quad \langle Q_0 \phi, \psi \rangle_{d_1}, \quad \langle (P_0 \oplus Q_0) \phi, \psi \rangle_{c_1,d_1},
$$

respectively [5], Remark 3 and Section 7.

Consider the conditions and notation of Proposition 3.3 and the notation of Proposition 3.1. Take a complete orthonormal system \( \{ \hat{\lambda}_k = \hat{\lambda}_{P,k} \mid k \in 2\mathbb{N} \} \) of $L^2_{c_1,+}$ so that each $\hat{\lambda}_k$ is a $\lambda_k$-eigenfunction of $P$. Let $\hat{\lambda}'_k = \hat{\lambda}'_{P,k}$ and $\hat{\lambda}''_k = \hat{\lambda}''_{P,k}$ denote the orthogonal projections of each $\hat{\lambda}_k$ to the subspaces spanned by $\hat{\lambda}_k$ and \( \{ \hat{\lambda}_i \mid k > i \in 2\mathbb{N} \} \), respectively; in particular, $\hat{\lambda}_0' = 0$. Let also $\hat{\lambda}_k'' = \hat{\lambda}'_{P,k} - \hat{\lambda}_k' - \hat{\lambda}_k''$.

**Lemma 3.6.** $\| \hat{\lambda}'_{P,k} \|_{c_1} \to 1$ as $s \to \infty$ for each $k \in 2\mathbb{N}$.

**Proof.** We proceed by induction on $k$. For $k = 0$, take some $\epsilon > 0$ and $C > 0$ satisfying (22). By Propositions 3.1 [11] and 3.3 [13], and Remark 4 [13],

$$
(1 + 2\sigma)(s + \xi s^n) + \xi C s^n \geq \lambda_0 = \langle P^{1/2}_{0} \hat{\lambda}_0, P^{1/2}_{0} \hat{\lambda}_0 \rangle_{c_1} > \langle P^{1/2}_{0} \hat{\lambda}_0, P^{1/2}_{0} \hat{\lambda}_0 \rangle_{c_1}
$$

$$
= \langle P^{1/2}_{0} \hat{\lambda}_0, P^{1/2}_{0} \hat{\lambda}'_0 \rangle_{c_1} + \langle P^{1/2}_{0} \hat{\lambda}'_0, P^{1/2}_{0} \hat{\lambda}''_0 \rangle_{c_1},
$$

$$
\geq (1 + 2\sigma)s \| \hat{\lambda}'_0 \|_{c_1}^2 + (5 + 2\sigma)s \| \hat{\lambda}''_0 \|_{c_1}^2 = (1 + 2\sigma)s + 4s \| \hat{\lambda}''_0 \|_{c_1}^2,
$$

giving

$$
\| \hat{\lambda}''_0 \|_{c_1}^2 < \frac{((1 + 2\sigma)\epsilon + C)\xi}{4s^{1-n}} \to 0
$$

as $s \to \infty$, and therefore $\| \hat{\lambda}'_0 \|_{c_1}^2 \to 1$.

Now, take any even integer $k > 0$ and suppose that the result holds for all even indices $< k$. This yields $\| \hat{\lambda}''_k \|_{c_1} \to 0$ as $s \to \infty$. Thus, given any $\delta > 0$, we have $\| \hat{\lambda}'_k \|_{c_1} < \delta/k$ for $s$ large enough. Take some $\epsilon > 0$ and $C > 0$ satisfying (22). By
Propositions 3.11 and 3.3, and Remark 7.

\[(2k + 1 + 2\sigma)(s + \xi e^{-u}) + \xi C s^u \geq \lambda_k = \langle P^{1/2} \hat{\chi}_k, P^{1/2} \hat{\chi}_k \rangle_{c_1} \geq \langle P^{1/2} \hat{\chi}_k, P^{-1/2} \hat{\chi}_k \rangle_{c_1} \]

\[= \langle P^{1/2} \hat{\chi}_k, P^{1/2} \hat{\chi}_k, P^{1/2} \hat{\chi}_k, P^{1/2} \hat{\chi}_k \rangle_{c_1} + \langle P^{1/2} \hat{\chi}_k, P^{1/2} \hat{\chi}_k, P^{1/2} \hat{\chi}_k, P^{1/2} \hat{\chi}_k \rangle_{c_1} \]

\[\geq (2k + 1 + 2\sigma)s \| \hat{\chi}_k \|^2_{c_1} + (1 + 2\sigma)s \| \hat{\chi}_k \|^2_{c_1} + (2k + 5 + 2\sigma)s \| \hat{\chi}_k \|^2_{c_1} \]

\[= (1 + 2\sigma)s + 2ks(\| \hat{\chi}_k \|^2_{c_1} + \| \hat{\chi}_k \|^2_{c_1}) + 4s \| \hat{\chi}_k \|^2_{c_1} \]

\[\geq (1 + 2\sigma)s + 2ks(1 - \delta/k) + 4s \| \hat{\chi}_k \|^2_{c_1} \]

giving

\[\| \hat{\chi}''_k \|^2_{c_1} < \frac{(2k + 1 + 2\sigma)e + C\xi}{4s^{u-1}} + \frac{\delta}{2} < \delta \]

for \( s \) large enough. Thus \( \| \hat{\chi}''_k \|^2_{c_1} \to 0 \) as \( s \to \infty \), and the result follows. \( \square \)

Corollary 3.7. If \( h \) is a bounded measurable function on \( \mathbb{R}_+ \) such that \( h(\rho) \to 1 \) as \( \rho \to 0 \), then \( \langle h\hat{\chi}_{P,0}, \hat{\chi}_{P,0} \rangle_{c_1} \to 1 \) as \( s \to \infty \).

Proof. This follows from Lemmas 3.6 and Remark 7. \( \square \)

If we assume the conditions and notation of Proposition 3.3 and the notation of Proposition 3.2 then, defining \( \hat{\chi}_{\mathcal{Q},k} \) and \( \hat{\chi}'_{\mathcal{Q},k} \) for each \( k \in 2N + 1 \) as above, the argument of the proof of Lemma 3.6 gives the following.

Lemma 3.8. \( \| \hat{\chi}'_{\mathcal{Q},k} \|_{d_1} \to 1 \) as \( s \to \infty \) for each \( k \in 2N + 1 \).

Similarly, if we assume the conditions and notation of Proposition 3.5 and the notation of Propositions 3.1 and 3.2 then, defining \( \hat{\chi}_{\mathcal{W},k} \) and \( \hat{\chi}'_{\mathcal{W},k} \) for each \( k \in \mathbb{N} \) as above, the arguments of the proofs of Lemma 3.6 and Corollary 3.7 give the following lemma and corollary.

Lemma 3.9. \( \| \hat{\chi}'_{\mathcal{W},k} \|_{c_1,d_1} \to 1 \) as \( s \to \infty \) for each \( k \in \mathbb{N} \).

Corollary 3.10. If \( h \) is a bounded measurable function on \( \mathbb{R}_+ \) with \( h(\rho) \to 1 \) as \( \rho \to 0 \), then \( \langle h\hat{\chi}_{\mathcal{W},0}, \hat{\chi}_{\mathcal{W},0} \rangle_{c_1,d_1} \to 1 \) as \( s \to \infty \).

4. Two simple types of elliptic complexes

Here, we study two simple elliptic complexes on \( \mathbb{R}_+ \), which will show up in a direct sum splitting of the rel-local model of Witten’s perturbation (Section 3).

4.1. An elliptic complex of length one. Consider the standard metric on \( \mathbb{R}_+ \). Let \( E \) be the graded Riemannian/Hermitian vector bundle over \( \mathbb{R}_+ \) whose non-zero terms are \( E_0 \) and \( E_1 \), which are real/complex trivial line bundles equipped with the standard Riemannian/Hermitian metrics. Thus

\[ C^\infty(E_0) \equiv C^\infty_+ \equiv C^\infty(E_1), \quad L^2(E_0) \equiv L^2_+ \equiv L^2(E_1), \]

where real/complex valued functions are considered in \( C^\infty_+ \) and \( L^2_+ \). For any fixed \( s > 0 \) and \( \kappa \in \mathbb{R} \), let

\[ C^\infty(E_0) \frac{d}{\delta} C^\infty(E_1) \]
be the differential operators defined by
\[ d = \frac{d}{d\rho} - \kappa \rho^{-1} \pm s \rho, \quad \delta = -\frac{d}{d\rho} - \kappa \rho^{-1} \pm s \rho. \]
It is easy to check that \((E, d)\) is an elliptic complex, and let \( \delta = d^\dagger \).

4.1.1. Self-adjoint operators defined by the Laplacian. By (12), the homogeneous components of \( \Delta \) (or \( \Delta^\pm \)) are:
\[
\begin{align*}
\Delta_0 &= H + \kappa(\kappa - 1)\rho^{-2} \mp s(1 + 2\kappa), \\
\Delta_1 &= H + \kappa(\kappa + 1)\rho^{-2} \pm s(1 - 2\kappa),
\end{align*}
\]
where \( H \) is the harmonic oscillator on \( C^\infty \) defined with the constant \( s \). Then \( \Delta_0 \) and \( \Delta_1 \) are like \( P_0 \) and \( Q_0 \) in (13), with \( c_1 = d_1 = 0 \), plus a constant. Then, according to Propositions 3.1 and 3.2, \( \Delta_0 \) and \( \Delta_1 \) define the self-adjoint operators \( A_i \) and \( B_i \) in \( L^2 \) indicated in Table 1. The notation \( A_0^\pm \) and \( B_0^\pm \) may be used as well to specify that these operators are defined by \( \Delta_0^\pm \) and \( \Delta_1^\pm \). In these cases, we have \( c_1 = d_1 = 0 \), and therefore \( \sigma = a \) and \( \tau = b \), which are given by (14) and (17).

|        | \( \sigma \) | \( \tau \) | Condition |
|--------|-------------|-------------|-----------|
| \( \Delta_0 \) | \( A_1 \) | \( \kappa \) | \( \kappa > -\frac{1}{2} \) |
|        | \( A_2 \) | \( 1 - \kappa \) | \( \kappa < \frac{3}{2} \) |
| \( \Delta_1 \) | \( B_1 \) | \( \kappa \) | \( \kappa > -\frac{3}{2} \) |
|        | \( B_2 \) | \( -1 - \kappa \) | \( \kappa < \frac{1}{2} \) |

Table 1. Self-adjoint operators defined by \( \Delta_0 \) and \( \Delta_1 \)

There are the following overlaps in Table 1:

- Both \( A_1 \) and \( A_2 \) are defined if \(-1/2 < \kappa < 3/2\), and they are equal just when \( \kappa = 1/2 \).
- Both \( B_1 \) and \( B_2 \) are defined if \(-3/2 < \kappa < 1/2\), and they are equal just when \( \kappa = -1/2 \).

Propositions 3.1 and 3.2 also describe the spectra of \( A_i \) and \( B_i \):

- The spectrum of \( A_1 \) consists of the eigenvalues
  \[(2k + 4 - (1 \mp 1)(1 + 2\kappa))s \quad (k \in 2\mathbb{N}) \quad (68)\]
  of multiplicity one.
- The spectrum of \( A_2 \) consists of the eigenvalues
  \[(2k + 4 - (1 \pm 1)(1 + 2\kappa))s \quad (k \in 2\mathbb{N}) \quad (69)\]
  of multiplicity one.
- The spectrum of \( B_1 \) consists of the eigenvalues
  \[(2k + 2 + (1 \mp 1)(-1 + 2\kappa))s \quad (k \in 2\mathbb{N} + 1) \quad (70)\]
  of multiplicity one.
- The spectrum of \( B_2 \) consists of the eigenvalues
  \[(2k - 2 - (1 \pm 1)(-1 + 2\kappa))s \quad (k \in 2\mathbb{N} + 1) \quad (71)\]
  of multiplicity one.

\(^4\)The supperindex \( \dagger \) is used to denote the formal adjoint.
These eigenvalues have normalized eigenfunctions $\chi_k$, defined for the corresponding values of $a = \sigma$ and $b = \tau$. For $A_i^+$, (68) becomes $2(k + 1 - 2\kappa)s$. For $A_i^-$, (68) is $2(k + 2)s$. For $A_i^+$, (69) becomes $2(k + 1 - 2\kappa)s$. For $A_i^-$, (69) is $2(k + 2)s$. For $B_i^+$, (70) is $2(k + 1)s$. For $B_i^-$, (70) becomes $2(k + 2)s$. For $B_i^+$, (71) is $2(k - 2\kappa)s$. For $B_i^-$, (71) becomes $2(k - 1)s$. Using this, we get the information about the sign of the eigenvalues of $A_i$ and $B_i$ given in Table 2. In the tables, grey color is used for cases that will be disregarded later (for instance, if there may exist some negative eigenvalue), and a question mark is used for unknown information.

| $A_1^+$ | Sign of eigenvalues |
|---------|---------------------|
| $\kappa > \frac{1}{2}$ | 0 if $k = 0$ + if $k \geq 1$ |
| $\kappa = \frac{1}{2}$ | 0 if $k = 0$ + if $k \geq 1$ |
| $\kappa < \frac{1}{2}$ | + $\forall k$ |
| $A_2^+$ | $\kappa > \frac{1}{2}$ | 0 if $k = 0$ + if $k \geq 1$ |
| $\kappa = \frac{1}{2}$ | 0 if $k = 0$ + if $k \geq 1$ |
| $\kappa < \frac{1}{2}$ | + $\forall k$ |

| $B_1^+$ | Sign of eigenvalues |
|---------|---------------------|
| $\kappa > -\frac{1}{2}$ | + $\forall k$ |
| $\kappa = -\frac{1}{2}$ | 0 if $k = 1$ + if $k > 1$ |
| $\kappa < -\frac{1}{2}$ | 0 if $k = -2\kappa$ + if $k > -2\kappa$ |

| $B_2^+$ | + $\forall k$ |
| $B_2^-$ | 0 if $k = 0$ + if $k \geq 1$ |

Table 2. Sign of the eigenvalues of $A_i$ and $B_i$

4.1.2. Laplacians of the maximum/minimum i.b.c.

**Proposition 4.1** ([3] Proposition 8.4). Table 3 describes $\Delta_{\text{max/min}}$.

| $\Delta_{\text{max,0}}$ | $\Delta_{\text{min,0}}$ | $\Delta_{\text{max,1}}$ | $\Delta_{\text{min,1}}$ |
|---------------------|---------------------|---------------------|---------------------|
| $\kappa \geq \frac{1}{2}$ | $A_1$ | $B_1$ |
| $|\kappa| < \frac{1}{2}$ | $A_1$ | $A_2$ | $B_1$ | $B_2$ |
| $\kappa \leq \frac{1}{2}$ | $A_2$ | $B_2$ |

Table 3. Description of $\Delta_{\text{max/min}}$

**Remark 8.** In [3], the proof of Proposition 4.1 uses the following property [3, Lemma 8.5]. Suppose that either $\theta > 1/2$, or $\theta = 1/2 = \kappa$ (respectively, $\theta = 1/2 = -\kappa$). Then, for each $\xi \in \rho^\theta S_{\text{ev},+}$, considered as subspace of $C^\infty(E_0)$ (respectively, $C^\infty(E_1)$), there is a sequence $(\xi_n)$ in $C^\infty_0(E_0)$ (respectively, $C^\infty_0(E_1)$), independent of $\kappa$, such that $\lim_n \xi_n = \xi$ and $\lim_n d\xi_n = d\xi$ in $L^2(E_0)$ (respectively, $\lim_n d\xi_n = d\xi$ in $L^2(E_1)$). In particular, $\rho^\theta S_{\text{ev},+}$ is contained in $\mathcal{D}(\delta_{\text{min}})$ (respectively, $\mathcal{D}(\delta_{\text{min}})$). Moreover, according to the proof of [3, Lemma 8.5], given $0 < a < b$, we can take $\xi_n = \alpha_n \xi$ for some $\alpha_n \in C^\infty_+$ satisfying $\chi_{[\frac{a}{\alpha_n},\frac{b}{\alpha_n}]} \leq \alpha_n \leq \chi_{[\frac{a}{\alpha_n},\frac{b}{\alpha_n}]}$, where $\chi_S$ denotes the characteristic function of each subset $S \subset \mathbb{R}_+$. 


4.2. An elliptic complex of length two. Consider again the standard metric on \( \mathbb{R}_+ \). Let \( F \) be the graded Riemannian/Hermitean vector bundle over \( \mathbb{R}_+ \) whose non-zero terms are \( F_1 \), \( F_2 \) and \( F_3 \), which are trivial real/complex vector bundles of ranks 1, 2 and 1, respectively, equipped with the standard Riemannian/Hermitean metrics. Thus

\[
C^\infty(F_0) \equiv C^\infty_+ \equiv C^\infty(F_2), \quad C^\infty(F_1) \equiv C^\infty_+ \oplus C^\infty_+,
L^2(F_0) \equiv L^2_+ \equiv L^2(F_2), \quad L^2(F_1) \equiv L^2_+ \oplus L^2_+,
\]

where real/complex valued functions are considered in \( C^\infty_+ \) and \( L^2_+ \). Fix \( s, \mu, u > 0 \) and \( \kappa \in \mathbb{R} \). (Some general observations about this complex hold for all \( u > 0 \), but the main results require \( 0 < u < 1 \).) Let

\[
\begin{align*}
C^\infty(F_0) & \xrightarrow{\delta_0 \equiv (\delta_{0,1} \quad \delta_{0,2})} C^\infty(F_1) & \xrightarrow{\delta_1 \equiv (\delta_{1,1} \quad \delta_{1,2})} C^\infty(F_2),
\end{align*}
\]

be the differential operators defined by

\[
\begin{align*}
d_{0,1} &= \mu \rho^{-u}, & d_{0,2} &= \frac{d}{d\rho} - (\kappa + u)\rho^{-1} \pm s\rho, \\
d_{1,1} &= \frac{d}{d\rho} - \kappa \rho^{-1} \pm s\rho, & d_{1,2} &= -\mu \rho^{-u}, \\
d_{0,1} &= \mu \rho^{-u}, & d_{0,2} &= -\frac{d}{d\rho} - (\kappa + u)\rho^{-1} \pm s\rho, \\
d_{1,1} &= -\frac{d}{d\rho} - \kappa \rho^{-1} \pm s\rho, & d_{1,2} &= -\mu \rho^{-u};
\end{align*}
\]

in particular, \( \delta_0 = d_0^1 \) and \( \delta_1 = d_1^1 \). We may also use the more explicit notation \( d^\pm_r, \delta^\pm_r, \delta^\pm_r \) and \( d^\pm_r \). A direct computation shows that \( d_0 \) and \( d_1 \) define an elliptic complex \( (F, d) \) of length two. Note that, by (12),

\[
d_{1,1} = \rho^{-u} d_{0,2} \rho^u, \quad \delta_{0,2} = \rho^{-u} \delta_{1,1} \rho^u. \tag{72}
\]

4.2.1. Self-adjoint operators defined by the Laplacian. By (12), the homogeneous components of the corresponding Laplacian \( \Delta \) (or \( \Delta^\pm \)) are given by

\[
\begin{align*}
\Delta_0 &= H + (\kappa + u)(\kappa + u - 1)\rho^{-2} + \mu^2 \rho^{-2u} \mp s(1 + 2(\kappa + u)), \\
\Delta_2 &= H + \kappa(\kappa + 1)\rho^{-2} + \mu^2 \rho^{-2u} \mp s(1 - 2\kappa), \\
\Delta_1 &= \begin{pmatrix} \Delta_{1,1} & -2\mu u \rho^{-u-1} \\ -2\mu u \rho^{-u-1} & \Delta_{1,2} \end{pmatrix}, \\
\Delta_{1,1} &= H + \kappa(\kappa + 1)\rho^{-2} + \mu^2 \rho^{-2u} \mp s(1 + 2\kappa), \\
\Delta_{1,2} &= H + (\kappa + u)(\kappa + u + 1)\rho^{-2} + \mu^2 \rho^{-2u} \mp s(1 - 2(\kappa + u)).
\end{align*}
\]

(We may also use (66) and (67) to compute easily some parts of the above components of \( \Delta \).)

Assume that \( u < 1 \). Then \( \Delta_0, \Delta_2, \Delta_{1,1} \) and \( \Delta_{1,2} \) are like \( P \) and \( Q \) in (64), with \( c_1 = 0 = d_1 \), plus a constant term. Write \( \Delta_1 = U \mp sV \), where

\[
V = \begin{pmatrix} 1 + 2\kappa & 0 \\ 0 & -1 + 2(\kappa + u) \end{pmatrix}. \tag{73}
\]

Then, according to Propositions 3.3, 3.4 and 3.5 and Remark 7, \( \Delta_0, \Delta_2 \) and \( \Delta_1 \) define the self-adjoint operators \( P_i \) and \( Q_i \) in \( L^2_{\rho} \), and \( W_{i,j} \) in \( L^2_{\rho} \oplus L^2_{\rho} \), indicated in Table 1. The notation \( P_i^\pm, Q_i^\pm \) and \( W_i^\pm \) may be used as well to specify that these
operators are defined by $\Delta_0^\pm$, $\Delta_2^\pm$ and $\Delta_3^\pm$. Note that $v = u$ for all $W_{i,j}$. The cores of $P_1^{1/2}$, $Q_1^{1/2}$ and $W_{i,j}^{1/2}$, given by Propositions 3.3–3.5 will be denoted by $\mathcal{F}_i^0$, $\mathcal{F}_j^0$ and $\mathcal{F}_{i,j}^1 = \mathcal{F}_i^1 \oplus \mathcal{F}_j^{1,2}$, respectively.

| $\Delta_0$ | $P_1$ | $\kappa + u$ | $\kappa > -\frac{1}{2}$ |
| $P_2$ | $1 - \kappa - u$ | $\kappa < \frac{3}{2} - 2u$ |
| $\Delta_2$ | $Q_1$ | $\kappa$ | $\kappa > u - \frac{1}{2}$ |
| $Q_2$ | $-1 - \kappa$ | $\kappa < \frac{1}{2} - u$ |
| $\Delta_1$ | $W_{1,1}$ | $\kappa$ | $\kappa > u - \frac{1}{2}$ |
| $W_{2,2}$ | $1 - \kappa$ | $-1 - \kappa - u$ | $\kappa < \frac{1}{2} - 2u$ |
| $\not\in W_{1,2}$ | $\kappa$ | $-1 - \kappa - u$ | $\kappa < \frac{1}{2} - 2u$ |
| $W_{2,1}$ | $1 - \kappa$ | $\kappa + u$ | $\frac{1}{2}$ | $\frac{1}{2} < \kappa < \frac{1}{2} - u$, $\kappa = -\frac{1}{2} - u$, $\frac{1}{2}$. |

Table 4. Self-adjoint operators defined by $\Delta_0$, $\Delta_2$ and $\Delta_1$

Let us explain the contents of Table 4. Since $c_1 = d_1 = 0$, we have $\sigma = a$ and $\tau = b$, which are given by (14) and (17). Moreover, $\sigma$, $\tau$ and $u$ determine $\theta$ in Table 4 so that $U$ is of the form (65): $2\theta - \sigma - \tau = -u$. Let us check the conditions written in this table, which are given by the hypothesis of Propositions 3.3–3.5. For $P_i$ and $Q_j$, only (19) and (23) are required. For $W_{i,j}$, we also require (61), and the hypothesis (4)–(6) of Proposition 3.3 obtaining the following:

- For $W_{1,1}$, we have $\sigma = \theta \neq \tau$ and $\tau - \sigma = u \notin -\mathbb{N}$. Thus (23) applies in this case. Note that (19), (23) and (61) mean $\kappa > u - \frac{1}{2}$. Moreover $\sigma - 1 < \tau < \sigma + 1$ means $\kappa - 1 < \kappa + u < \kappa + 1$, which holds because $0 < u < 1$, and $\tau < 2\sigma + \frac{1}{2}$ holds because $\kappa > u - \frac{1}{2}$. So (23) is satisfied.
- For $W_{2,2}$, we have $\sigma \neq \theta = \tau + 1$ and $\sigma - \tau - 1 = 1 + u \notin -\mathbb{N}$. Thus (61) applies in this case. Now, (19), (23) and (61) mean $\kappa < \frac{1}{2} - 2u$. Moreover $-\frac{\kappa}{2} - \frac{1}{2} < \kappa < \frac{1}{2} - 2u$ also means $\kappa < \frac{1}{2} - 2u$, and $\tau < \sigma - \frac{1}{2}$ means $u > -\frac{1}{3}$, which is true. So (23) is satisfied.
- There is no $W_{1,2}$ because $\theta < -\frac{1}{2}$ in that case.
- For $W_{2,1}$, (19), (23) and (61) mean $-\frac{1}{2} < \kappa < \frac{3}{2} - u$, and we have the following possibilities:
  - The case $\sigma = \theta = \tau$ is not possible because $u \neq 0$.
  - The case $\sigma = \theta \neq \tau$ happens when $\kappa = \frac{1}{2}$. Then $\sigma = \frac{1}{2}$ and $\tau = \frac{1}{2} + u$, obtaining $\tau - \sigma = u \notin -\mathbb{N}$. Thus (23) applies in this case. Moreover $\sigma - 1 < \tau < \sigma + 1, 2\sigma + \frac{1}{2}$ means $-\kappa < \kappa + u < \kappa + 1$. So (23) is satisfied.
  - The case $\sigma \neq \theta = \tau$ happens when $\kappa = \frac{1}{2} - u$. Then $\sigma = \frac{1}{2} + u$ and $\tau = \frac{1}{2}$, obtaining $\sigma - \tau = u \notin -\mathbb{N}$. Thus (19) applies in this case. We have $\frac{1}{2} \leq \tau < \sigma$, and moreover $\sigma - 1 < \tau < \sigma/2 + \frac{1}{2}$ means $u - \frac{1}{2} < \frac{1}{2} < \frac{1}{2} + \frac{1}{2}$. So, using (23), we get $(\sigma, \tau) \in J_3$. This shows that (19) is satisfied. Alternatively, it can be shown that $(\sigma, \tau) \in J_2$ using (23) if $u > \frac{1}{2}$, and using (30) if $u \leq \frac{1}{2}$.
The case $\sigma \neq \theta = \tau + 1$ happens when $\kappa = -\frac{1}{2} - u$. Then $\sigma = \frac{3}{2} + u$ and $\tau = -\frac{3}{2}$, obtaining $\sigma - \tau - 1 = 1 + u \notin \mathbb{N}$. Thus (3) applies in this case. Moreover $\tau < \frac{3}{2} - \frac{9}{4}$, $\sigma - \frac{3}{2}$ means $-\frac{1}{2} < u, u - \frac{1}{2}$. Hence (3) is satisfied.

Finally, assume that $\sigma \neq \theta \neq \tau$. The condition $\sigma - \theta, \tau - \theta \notin \mathbb{N}$ means that $\kappa \notin (\frac{1}{2} + \mathbb{N}) \cup (\frac{1}{2} - u - \mathbb{N})$, which in turn means that $\kappa \notin \frac{1}{2}, \frac{1}{2} - u, -\frac{1}{2} - u$ because $-\frac{1}{2} < u < \frac{3}{2} - u$. But $\sigma = \theta$ if $\kappa = \frac{1}{2}$, $\tau = \theta$ if $\kappa = \frac{1}{2} - u$, and $\theta = \tau + 1$ if $\kappa = -\frac{1}{2} - u$, as we have seen in the previous cases. So $\sigma - \theta, \tau - \theta \notin \mathbb{N}$, and (41) applies in this case. On the other hand, 

\[
\begin{align*}
(\sigma, \tau, \theta) \in \mathcal{R}_1 & \iff -1 - \frac{u}{2} < \kappa < -\frac{u}{2}, \\
(\sigma, \tau, \theta) \in \mathcal{R}_1' & \iff -\frac{u}{2} < \kappa < 1 - \frac{u}{2}, \\
(\sigma, \tau, \theta) \in \mathcal{R}_2, \mathcal{R}_2' & \iff -\frac{1+u}{2} < \kappa < \frac{1-u}{2},
\end{align*}
\]

obtaining that 

\[
(\sigma, \tau, \theta) \in (\mathcal{R}_1 \cup \mathcal{R}_2) \cap (\mathcal{R}_1' \cup \mathcal{R}_2') \iff -\frac{1+u}{2} < \kappa < \frac{1-u}{2}.
\]

To check this assertion, Table 5 describes inequalities that are true independently of $\kappa$, which are involved several times in the definitions of these sets. In particular, (37), (11), (13), (48) are satisfied because their right hand sides involve only such true inequalities. On the other hand, (50), (14), (51), (52), (53) and (60) are satisfied because their left hand sides are false independently of $\kappa$, as explained in Table 6. Other conditions explained in Table 7 are also true independently of $\kappa$.

The remaining conditions produce restrictions on $\kappa$ that are explained in Table 8

| $\theta > \frac{2}{3} - \frac{2}{3}$ | $u < 1$ |
| $\sigma + \tau > 1$ | $u > 0$ |
| $\theta > \frac{2}{3} - \frac{1}{2}$ | $\kappa < \frac{2}{3} - u$ |

**Table 5. True inequalities**

Propositions 3.3, 3.4 and 3.5 also give the following spectral estimates:
\[
\begin{array}{|c|c|}
\hline
\sigma - 1 < \theta, \tau + 1 \leq \theta & -\frac{1}{2} < \kappa \leq -\frac{1}{2} - u \\
\sigma \leq \theta, \tau < \theta & \frac{1}{2} \leq \kappa < \frac{1}{2} - u \\
\tau + \frac{1}{2} \leq \theta = \sigma - \frac{1}{2} & u \leq -\kappa = 0 \\
\sigma - \frac{1}{2} < \theta, \tau + \frac{1}{2} \leq \theta & 0 < \kappa \leq -u \\
\sigma - \frac{1}{2} \leq \theta, \tau + \frac{1}{2} < \theta & 0 \leq \kappa = -u \\
\hline
\end{array}
\]

**Table 6. Impossible cases**

\[
\begin{array}{|c|c|}
\hline
\text{Meaning} & \text{Defining} \\
\hline
(47) -\frac{1}{2} - u < \kappa < 0 \Rightarrow (\kappa < -\frac{1}{2} \text{ or } \kappa > -\frac{1}{2}) & \mathcal{A}_1 \\
(48) -\frac{1}{2} < \kappa \leq -u \Rightarrow (\kappa > -\frac{1}{2} + u \text{ or } \kappa > -\frac{1}{2}) & \mathcal{A}_2 \\
\kappa = 0 \Rightarrow 1 > \frac{1}{2}, \frac{u+2}{3} & \mathcal{A}_3 \\
(55) -u < \kappa < 0, \frac{1}{2} - u \Rightarrow (\kappa > -\frac{1}{2} \text{ or } \kappa < \frac{1}{2} - u) & \mathcal{A}_4 \\
(56) 0 \leq \kappa < \frac{1}{2} - u \Rightarrow (\kappa < 1 - \frac{1}{2} - u \text{ or } \kappa < \frac{1}{2} - u) & \mathcal{A}_5 \\
\kappa = -u \Rightarrow 0 > -\frac{1}{2}, \frac{u-1}{3} & \mathcal{A}_6 \\
\hline
\end{array}
\]

**Table 7. Other true conditions**

\[
\begin{array}{|c|c|c|}
\hline
\text{Direct meaning} & \text{Simpler meaning} & \text{Defining} \\
\hline
(38) \kappa \leq -\frac{1}{2} - u \Rightarrow \kappa > -1 - \frac{1}{2} u & \kappa > -\frac{1}{2} - u & \mathcal{A}_1 \\
(59) \kappa > -\frac{1}{2} \Rightarrow \kappa < -\frac{1}{2} & \kappa < -\frac{1}{2} & \mathcal{A}_2 \\
(42) \kappa \geq \frac{1}{2} \Rightarrow \kappa \leq 1 - \frac{1}{2} & \kappa < 1 - \frac{1}{2} & \mathcal{A}_3 \\
(43) \kappa < \frac{1}{2} - u \Rightarrow \kappa > -\frac{1}{2} & \kappa > -\frac{1}{2} & \mathcal{A}_4 \\
(46) \kappa \leq -u, -\frac{1}{2} \Rightarrow \kappa > \frac{1}{2} + u & \kappa > \frac{1}{2} + u & \mathcal{A}_5 \\
\kappa > 0 \Rightarrow \kappa \leq \frac{1}{2} - u & \kappa < \frac{1}{2} - u & \mathcal{A}_6 \\
(51) \kappa > 0, \frac{1}{2} - u \Rightarrow \kappa < \frac{1}{2} - u & \kappa < \frac{1}{2} - u & \mathcal{A}_7 \\
(59) \kappa < -u \Rightarrow \kappa > -1 + \frac{1}{2} & \kappa > -\frac{1}{2} & \mathcal{A}_8 \\
\hline
\end{array}
\]

**Table 8. Conditions that produce restrictions on \( \kappa \)**

- The spectrum of \( P_1 \) consists of eigenvalues \( \lambda_0 \leq \lambda_2 \leq \cdots \), taking multiplicity into account, such that there is some \( D = D(\sigma, u) > 0 \), and, for all \( \epsilon > 0 \), there is some \( C = C(\epsilon, \sigma, u) > 0 \) so that, for all \( k \in 2N \),

\[
\begin{align*}
\lambda_k & \geq (2k + (1 \mp 1)(1 + 2(\kappa + u)))s + \mu^2 Ds^u(k + 1)^{-u}, \\
\lambda_k & \leq (2k + (1 \pm 1)(1 + 2(\kappa + u)))s \\
& + (2k + 1 + 2(\kappa + u))\mu^2 es^u + \mu^2 C s^u.
\end{align*}
\]

The first term of the right hand side of (74) for \( P_1^+ \) and \( P_1^- \) is \( 2ks \) and \( 2(k + 1 + 2(\kappa + u))s \), respectively.
The spectrum of $P_2$ consists of eigenvalues $\lambda_0 \leq \lambda_2 \leq \cdots$, taking multiplicity into account, such that there is some $D = D(\sigma, u) > 0$, and, for all $\epsilon > 0$, there is some $C = C(\epsilon, \sigma, u) > 0$ so that, for all $k \in 2\mathbb{N}$,

$$
\lambda_k \geq (2k + 4 - (1 \pm 1)(1 + 2(k + u)))s + \mu^2Ds^u(k + 1)^{-u},
$$

$$
\lambda_k \leq (2k + 4 - (1 \pm 1)(1 + 2(k + u)))s + (2k + 3 - 2(k + u))\mu^2Cs^u + \mu^2C^u.
$$

The first term of the right hand side of (76) for $P_1^+$ and $P_2^-$ becomes $2(k + 1 - 2(k + u))s$ and $2(k + 2)s$, respectively.

The spectrum of $Q_1$ consists of eigenvalues $\lambda_1 \leq \lambda_3 \leq \cdots$, taking multiplicity into account, such that there is some $D = D(\tau, u) > 0$, and, for all $\epsilon > 0$, there is some $C = C(\epsilon, \tau, u) > 0$ so that, for all $k \in 2\mathbb{N} + 1$,

$$
\lambda_k \geq (2k + 2 - (1 \mp 1)(1 - 2\kappa))s + \mu^2Ds^u(k + 1)^{-u},
$$

$$
\lambda_k \leq (2k + 2 - (1 \mp 1)(1 - 2\kappa))s + (2k + 1 + 2\tau)\mu^2Cs^u + \mu^2C^u.
$$

The first term of the right hand side of (78) for $Q_1^+$ and $Q_1^-$ is $2(k + 1)s$ and $(k + 2)s$, respectively.

The spectrum of $Q_2$ consists of eigenvalues $\lambda_1 \leq \lambda_3 \leq \cdots$, taking multiplicity into account, such that there is some $D = D(\tau, u) > 0$, and, for all $\epsilon > 0$, there is some $C = C(\epsilon, \tau, u) > 0$ so that, for all $k \in 2\mathbb{N} + 1$,

$$
\lambda_k \geq (2k - 2 + (1 \pm 1)(1 - 2\kappa))s + \mu^2Ds^u(k + 1)^{-u},
$$

$$
\lambda_k \leq (2k - 2 + (1 \pm 1)(1 - 2\kappa))s + (2k - 1 + 2\kappa)\mu^2Cs^u + \mu^2C^u.
$$

The first term of the right hand side of (80) for $Q_2^+$ and $Q_2^-$ is $2(k - \kappa)s$ and $(k - 1)s$, respectively.

The spectrum of $W_1,1$ consists of two groups of eigenvalues, $\lambda_0 \leq \lambda_2 \leq \cdots$ and $\lambda_1 \leq \lambda_3 \leq \cdots$, repeated according to multiplicity, such that there is some $D = D(\sigma, \tau, u) > 0$, and, for each $\epsilon > 0$, there is some $C = C(\epsilon, \sigma, \tau, u) > 0$ and $E = E(\epsilon, \sigma, \tau) > 0$ so that, for all $k \in 2\mathbb{N}$,

$$
\lambda_k \geq (2k + 1 \mp 1)(1 + 2\kappa)s + \mu^2Ds^u(k + 1)^{-u},
$$

$$
\lambda_k \leq (2k + 2 - 2u + (1 \mp 1)(2\kappa + u))s + (2k + 1 + 2\kappa)\epsilon s^u + \mu^2Cs^u + 4\mu uEs^{\frac{\kappa + 1}{2}},
$$

and, for all $k \in 2\mathbb{N} + 1$,

$$
\lambda_k \geq (2k + 2 - (1 \mp 1)(1 - 2\kappa))s + \mu^2Ds^u(k + 1)^{-u},
$$

$$
\lambda_k \leq (2k + 2 + (1 \mp 1)(2\kappa + u))s + (2k + 1 + 2\kappa)\epsilon s^u + \mu^2Cs^u + 4\mu uEs^{\frac{\kappa + 1}{2}}.
$$

The spectrum of $W_2,2$ consists of two groups of eigenvalues, $\lambda_0 \leq \lambda_2 \leq \cdots$ and $\lambda_1 \leq \lambda_3 \leq \cdots$, repeated according to multiplicity, such that there is some $D = D(\sigma, \tau, u) > 0$, and, for each $\epsilon > 0$, there is some $C = C(\epsilon, \sigma, \tau, u) > 0$ and $E = E(\epsilon, \sigma, \tau) > 0$ so that, for all $k \in 2\mathbb{N}$,

$$
\lambda_k \geq (2k + 4 - (1 \pm 1)(1 + 2\kappa))s + \mu^2Ds^u(k + 1)^{-u},
$$

$$
\lambda_k \leq (2k + 4 - (1 \pm 1)(2\kappa + u))s + (2k + 3 - 2\kappa)\epsilon s^u + 4\mu uEs^{\frac{\kappa + 1}{2}}.
$$
and, for all \( k \in 2N + 1 \),
\[
\lambda_k \geq (2k - 2 + (1 \pm 1)(1 - 2(\kappa + u)))s + \mu^2Ds^u(k + 1)^{-u}, \tag{88}
\]
\[
\lambda_k \leq (2k - 2u - (1 \pm 1)(2\kappa + u) + (2k - 1 - 2(\kappa + u))\epsilon(\mu^2s^u + 4\muus^{u+1}) + \mu^2Cs^u + 4\muEs^{u+1}. \tag{89}
\]

- The spectrum of \( W_{2,1} \) consists of two groups of eigenvalues, \( \lambda_0 \leq \lambda_2 \leq \cdots \) and \( \lambda_1 \leq \lambda_3 \leq \cdots \), repeated according to multiplicity, such that there is some \( D = D(\sigma, \tau, u) > 0 \), and, for each \( \epsilon > 0 \), there is some \( C = C(\epsilon, \sigma, \tau, u) > 0 \) and \( E = E(\epsilon, \sigma, \tau) > 0 \) so that \( S_6 \) and \( S_7 \) hold for all \( k \in 2N \), and \( S_4 \) and \( S_5 \) hold for all \( k \in 2N + 1 \).

Here, we have used the form version of the min-max principle [35, Theorem XIII.2] to obtain the above estimates as follows:

- The right hand sides of (82) and (86) are the sums of (21) and the eigenvalue \( \mp s(1 + 2\kappa) \) of \( \mp sV \).
- The right hand sides of (84) and (88) are the sums of (21) and the eigenvalue \( \mp s(-1 + 2(\kappa + u)) \) of \( \mp sV \).
- The right hand sides of (83), (85), (87) and (89) are the sums of (63) and the maximum eigenvalue of \( \mp sV \), which is \( s(1 \mp (2\kappa + u) - u) \).

The first term of the right hand side of (82) is \( 2ks \) for \( W_{1,1}^+ \), and \( 2(k + 1 + 2\kappa)s \) for \( W_{1,1}^- \). The first term of the right hand side of (84) is \( 2(k + 1)s \) for \( W_{1,1}^+ \) and \( W_{1,1}^+ \), and \( 2(k + 2(\kappa + u))s \) for \( W_{1,1}^- \) and \( W_{2,2}^- \). The first term of the right hand side of (86) is \( 2(k + 1 - 2\kappa)s \) for \( W_{2,2}^+ \) and \( W_{2,1}^+ \), and \( 2(k + 2)s \) for \( W_{2,2}^- \) and \( W_{2,1}^- \). The first term of the right hand side of (88) is \( 2(k - 2(\kappa + u))s \) for \( W_{2,2}^+ \), and \( 2(k - 1)s \) for \( W_{2,2}^- \). Using this, we get the information about the sign of the eigenvalues of \( P_i, Q_j \) and \( W_{i,j} \) contained in Table 9.

| \( P_1 \) | + \forall k |
| \( P^+_2 \) | \begin{align*}
\kappa > \frac{1}{2} - u & \quad + \quad \text{if } k < 2(\kappa + u) - 1 \\
\kappa \leq \frac{1}{2} - u & \quad + \quad \forall k
\end{align*} |
| \( P^-_2 \) | + \forall k |
| \( Q^+_1 \) | + \forall k |
| \( Q^-_1 \) | \begin{align*}
\kappa \geq -\frac{1}{2} & \quad + \quad \forall k \\
\kappa < -\frac{1}{2} & \quad + \quad \text{if } k < -2\kappa \\
\kappa \geq -2\kappa & \quad + \quad \forall k
\end{align*} |
| \( Q_2 \) | + \forall k |
| \( W_{i,j} \) | + \forall k |

Table 9. Sign of the eigenvalues of \( P_i, Q_i \) and \( W_{i,j} \)
4.2.2. Laplacians of the maximum/minimum i.b.c. We continue taking any \( u > 0 \) for some generalities, and the additional restriction \( u < 1 \) will be assumed when needed.

Remark 9. In contrast with \( \mathcal{E}_i \) in Section 4.1.2, note that the graded subspace \( \mathcal{F}_i^0 \oplus \mathcal{F}_{i,j}^1 \oplus \mathcal{F}_j^2 \) of \( C^\infty(F) \), whenever defined, is not preserved by \( D \). For instance, it is preserved by \( d \) but not by \( \delta \) when \( i = j = 1 \), and it is preserved by \( \delta \) but not by \( d \) when \( i = j = 2 \).

Proposition 4.1 gives the following:

\[
\begin{align*}
\mathcal{D}(d_{0,2,\text{max}}) & \supset \begin{cases} \mathcal{F}_1^0 & \text{if } \kappa > -\frac{1}{2} - u \\ \mathcal{F}_2^0 & \text{if } \kappa \leq -\frac{1}{2} - u \end{cases} , \\
\mathcal{D}(d_{0,2,\text{min}}) & \supset \begin{cases} \mathcal{F}_1^0 & \text{if } \kappa \geq \frac{1}{2} - u \\ \mathcal{F}_2^0 & \text{if } \kappa < \frac{1}{2} - u \end{cases} , \\
\mathcal{D}(\delta_{0,2,\text{max}}) & \supset \begin{cases} \mathcal{F}_{1,2}^1 & \text{if } \kappa \geq \frac{1}{2} - u \\ \mathcal{F}_{1,2}^2 & \text{if } \kappa < \frac{1}{2} - u \end{cases} , \\
\mathcal{D}(\delta_{0,2,\text{min}}) & \supset \begin{cases} \mathcal{F}_{1,2}^1 & \text{if } \kappa > -\frac{1}{2} - u \\ \mathcal{F}_{1,2}^2 & \text{if } \kappa \leq -\frac{1}{2} - u \end{cases} , \\
\mathcal{D}(d_{1,1,\text{max}}) & \supset \begin{cases} \mathcal{F}_{1,1}^1 & \text{if } \kappa > -\frac{1}{2} \\ \mathcal{F}_{1,1}^2 & \text{if } \kappa \leq -\frac{1}{2} \end{cases} , \\
\mathcal{D}(d_{1,1,\text{min}}) & \supset \begin{cases} \mathcal{F}_{1,1}^1 & \text{if } \kappa \geq \frac{1}{2} \\ \mathcal{F}_{1,1}^2 & \text{if } \kappa < \frac{1}{2} \end{cases} , \\
\mathcal{D}(\delta_{1,1,\text{max}}) & \supset \begin{cases} \mathcal{F}_{2}^1 & \text{if } \kappa \geq \frac{1}{2} \\ \mathcal{F}_{2}^2 & \text{if } \kappa < \frac{1}{2} \end{cases} , \\
\mathcal{D}(\delta_{1,1,\text{min}}) & \supset \begin{cases} \mathcal{F}_{2}^1 & \text{if } \kappa > -\frac{1}{2} \\ \mathcal{F}_{2}^2 & \text{if } \kappa \leq -\frac{1}{2} \end{cases} .
\end{align*}
\]

\[
\begin{align*}
d_{0,2,\text{max}} &= d_{0,2,\text{min}} , & \delta_{0,2,\text{max}} &= \delta_{0,2,\text{min}} \text{ if } |\kappa + u| \geq \frac{1}{2} , \\
d_{1,1,\text{max}} &= d_{1,1,\text{min}} , & \delta_{1,1,\text{max}} &= \delta_{1,1,\text{min}} \text{ if } |\kappa| \geq \frac{1}{2} .
\end{align*}
\]

On the other hand, since \( d_{0,1}, \delta_{0,1}, d_{1,2} \) and \( \delta_{1,2} \) are multiplication operators, we have

\[
\begin{align*}
d_{0,1,\text{max}} &= d_{0,1,\text{min}} , & \delta_{0,1,\text{max}} &= \delta_{0,1,\text{min}} , \\
d_{1,2,\text{max}} &= d_{1,2,\text{min}} , & \delta_{1,2,\text{max}} &= \delta_{1,2,\text{min}} ,
\end{align*}
\]
which are the corresponding maximal multiplication operators [24, Examples III-2.2 and V-3.22]. They satisfy the following:

\[
D(d_{0,1,\text{max/min}}) \supset \begin{cases} \mathcal{F}_1^0 & \text{if } \kappa > -\frac{1}{2} \\ \mathcal{F}_2^0 & \text{if } \kappa < \frac{3}{2} - 2u \end{cases}, \quad (98)
\]

\[
D(\delta_{0,1,\text{max/min}}) \supset \begin{cases} \mathcal{F}_1^{1,1} & \text{if } \kappa > u - \frac{1}{2} \\ \mathcal{F}_2^{1,1} & \text{if } \kappa < \frac{1}{2} - u \end{cases}, \quad (99)
\]

\[
D(d_{1,2,\text{max/min}}) \supset \begin{cases} \mathcal{F}_1^{1,2} & \text{if } \kappa > -\frac{3}{2} \\ \mathcal{F}_2^{1,2} & \text{if } \kappa < \frac{1}{2} - 2u \end{cases}, \quad (100)
\]

\[
D(\delta_{1,2,\text{max/min}}) \supset \begin{cases} \mathcal{F}_1^{2} & \text{if } \kappa > u - \frac{3}{2} \\ \mathcal{F}_2^{2} & \text{if } \kappa < \frac{1}{2} - u \end{cases}. \quad (101)
\]

By Remark [8] we also get

\[
D(d_{\text{min,0}}) = D(d_{0,1,\text{min}}) \cap D(d_{0,2,\text{min}}), \quad d_{\text{min,0}} = \left( \frac{d_{0,1,\text{min}} D(d_{\text{min,0}})}{d_{0,2,\text{min}} D(d_{\text{min,0}})} \right), \quad (102)
\]

\[
D(\delta_{\text{min,1}}) = D(\delta_{1,1,\text{min}}) \cap D(\delta_{1,2,\text{min}}), \quad \delta_{\text{min,1}} = \left( \frac{\delta_{1,1,\text{min}} D(\delta_{\text{min,1}})}{\delta_{1,2,\text{min}} D(\delta_{\text{min,1}})} \right), \quad (103)
\]

complementing Lemma [2.3] in this case.

**Proposition 4.2.** If \( u < 1 \), then Tables 10, 11 and 12 describe \( \Delta_{\text{max/min}} \) for the stated values of \( \kappa \).

| \( \kappa \) | \( \Delta_{\text{max,0}} \) | \( \Delta_{\text{min,0}} \) |
|---|---|---|
| \( \kappa > -\frac{1}{2} \) | \( \mathcal{P}_1 \) | \( \kappa > u - \frac{1}{2} \) | \( \mathcal{P}_1 \) |
| \( -\frac{1}{2} - u < \kappa \leq -\frac{1}{2} \) | ? | \( \kappa < \frac{1}{2} - u \) | \( \mathcal{P}_2 \) |
| \( \kappa \leq -\frac{1}{2} - u \) | \( \mathcal{P}_2 \) | |

**Table 10.** Description of \( \Delta_{\text{max/min,0}} \)

| \( \kappa \) | \( \Delta_{\text{max,2}} \) | \( \Delta_{\text{min,2}} \) |
|---|---|---|
| \( \kappa > -\frac{1}{2} \) | \( \mathcal{Q}_1 \) | \( \kappa > \frac{1}{2} \) | \( \mathcal{Q}_1 \) |
| \( \kappa \leq -\frac{1}{2} \) | \( \mathcal{Q}_2 \) | \( \frac{1}{2} - u < \kappa < \frac{1}{2} \) | ? |
| \( \kappa < \frac{1}{2} - u \) | \( \mathcal{Q}_2 \) | |

**Table 11.** Description of \( \Delta_{\text{max/min,2}} \)
Proof. From [93–103], Lemmas 2.3 and 2.4 and [41] Chapter XI-12, p. 338, Eq. (1)), it follows that

\[ D(\Delta_{\text{max,0}}^{1/2}) = D(d_{\text{max,0}}) = D(d_{0,1,\text{max}}) \cap D(d_{0,2,\text{max}}) \supset \begin{cases} \mathcal{F}_1^{0} & \text{if } \kappa > -\frac{1}{2} \\ \mathcal{F}_2^{0} & \text{if } \kappa \leq -\frac{1}{2} - u \end{cases} , \]

\[ D(\Delta_{\text{min,0}}^{1/2}) = D(d_{\text{min,0}}) = D(d_{0,1,\text{min}}) \cap D(d_{0,2,\text{min}}) \supset \begin{cases} \mathcal{F}_1^{0} & \text{if } \kappa \geq \frac{1}{2} - u \\ \mathcal{F}_2^{0} & \text{if } \kappa < \frac{1}{2} - u \end{cases} , \]

\[ D(\Delta_{\text{max,2}}^{1/2}) = D(\delta_{\text{max,1}}) = D(\delta_{1,1,\text{max}}) \cap D(\delta_{1,2,\text{max}}) \supset \begin{cases} \mathcal{F}_1^{2} & \text{if } \kappa > -\frac{1}{2} \\ \mathcal{F}_2^{2} & \text{if } \kappa \leq -\frac{1}{2} \end{cases} , \]

\[ D(\Delta_{\text{min,2}}^{1/2}) = D(\delta_{\text{min,1}}) = D(\delta_{1,1,\text{min}}) \cap D(\delta_{1,2,\text{min}}) \supset \begin{cases} \mathcal{F}_1^{2} & \text{if } \kappa \geq \frac{1}{2} - u \\ \mathcal{F}_2^{2} & \text{if } \kappa < \frac{1}{2} - u \end{cases} , \]

\[ D(\Delta_{\text{max,1}}^{1/2}) = D(\delta_{\text{max,0}} + d_{\text{max,1}}) = D(\delta_{\text{min,0}}) \cap D(d_{\text{max,1}}) \supset (D(\delta_{0,1,\text{min}}) \oplus D(d_{0,2,\text{min}})) \cap (D(d_{1,1,\text{max}}) \oplus D(d_{1,2,\text{max}})) \supset \begin{cases} \mathcal{F}_{1,1}^{1} & \text{if } \kappa > u - \frac{1}{2} \\ \mathcal{F}_{2,1}^{1} & \text{if } -\frac{1}{2} - u < \kappa \leq -\frac{1}{2} \\ \mathcal{F}_{2,2}^{1} & \text{if } \kappa \leq -\frac{1}{2} - u \end{cases} , \]

\[ D(\Delta_{\text{min,1}}^{1/2}) = D(\delta_{\text{max,0}} + d_{\text{min,1}}) = D(\delta_{\text{min,0}}) \cap D(d_{\text{min,1}}) \supset (D(\delta_{0,1,\text{max}}) \oplus D(d_{0,2,\text{min}})) \cap (D(d_{1,1,\text{min}}) \oplus D(d_{1,2,\text{max}})) \supset \begin{cases} \mathcal{F}_{1,1}^{2} & \text{if } \kappa \geq \frac{1}{2} \\ \mathcal{F}_{2,1}^{2} & \text{if } \frac{1}{2} - u \leq \kappa < \frac{1}{2} \\ \mathcal{F}_{2,2}^{2} & \text{if } \kappa < \frac{1}{2} - u \end{cases} . \]

Since \( \mathcal{F}_i^0 \) is a core of \( P_i^{1/2} \), \( \mathcal{F}_j^2 \) is a core of \( Q_j^{1/2} \), and \( \mathcal{F}_{i,j}^0 \) is a core of \( W_i^{1/2} \), and taking into account Table 4 it follows that

\[ \Delta_{\text{max,0}}^{1/2} \supset \begin{cases} P_{1,1}^{1/2} & \text{if } \kappa > -\frac{1}{2} \\ P_{2,1}^{1/2} & \text{if } \kappa \leq -\frac{1}{2} - u \end{cases} , \quad \Delta_{\text{min,0}}^{1/2} \supset \begin{cases} P_{1,1}^{1/2} & \text{if } \kappa \geq \frac{1}{2} - u \\ P_{2,1}^{1/2} & \text{if } \kappa < \frac{1}{2} - u \end{cases} , \]

\[ \Delta_{\text{max,2}}^{1/2} \supset \begin{cases} Q_{1,1}^{1/2} & \text{if } \kappa > -\frac{1}{2} \\ Q_{2,1}^{1/2} & \text{if } \kappa \leq -\frac{1}{2} \end{cases} , \quad \Delta_{\text{min,2}}^{1/2} \supset \begin{cases} Q_{1,1}^{1/2} & \text{if } \kappa \geq \frac{1}{2} \\ Q_{2,1}^{1/2} & \text{if } \kappa < \frac{1}{2} - u \end{cases} . \]
Proof. Property (i) is true because $\ker (\text{Lemma } 2.3, (102) \text{ and } (103))$, since obtaining that $R_{\text{max/min}} \Delta$.

Therefore $L_{\text{dense in }} R$. But these inclusions are equalities because they involve self-adjoint operators. \hfill $\square$

In this case, we have $\Delta_{\text{max/min,ev}} = \Delta_{\text{max/min,0}} \oplus \Delta_{\text{max/min,2}}$, $\Delta_{\text{max/min,odd}} = \Delta_{\text{max/min,1}}$.

Proposition 4.3. (i) $\ker \Delta_{\text{max/min,ev}} = 0$.

(ii) If $\sigma(\Delta_{\text{max/min,ev}})$ is bounded away from 0, then $\ker \Delta_{\text{max/min,1}} = 0$.

Proof. Property (i) is true because $\ker d_{\text{max/min,0}} = 0$ and $\ker \delta_{\text{max/min,1}} = 0$ by Lemma [2.3] [102] and [103], since $d_{0,1,\text{max/min}}$ and $\delta_{1,2,\text{max/min}}$ are maximal multiplication operators in $L_2^2$ by continuous non-vanishing functions $\rho$.

Now, assume that $\sigma(\Delta_{\text{max/min,ev}})$ is bounded away from 0. Then $R(\Delta_{\text{max/min,0}}) = L_2^2 = R(\Delta_{\text{max/min,2}})$ by the spectral theorem. The maximal multiplication operator by $\rho^{\pm u}$ in $L_2^2$ will be also denoted by $\rho^{\pm u}$. Let $\phi \in D(\Delta_{\text{max/min,0}})$ such that $\Delta_{\text{max/min,0}} \phi = D(\rho^u)$. By (72),

$$\psi := \frac{1}{\mu} \rho^u d_{0,2,\text{max/min}} \phi \in D(\delta_{0,2,\text{max/min}} \rho^{-u}) \cap D(\rho^u \delta_{0,2,\text{max/min}} \rho^{-u})$$

$$= D(\rho^{-u} \delta_{1,1,\text{max/min}}) \cap D(\delta_{1,1,\text{max/min}}) .$$

Then $\psi \in D(\delta_{\text{max/min,1}})$ by (103) since $\rho^{-u} \psi \in L_2^2$ and $\delta_{1,2,\text{max/min}}$ is the maximal multiplication operator by $-\mu \rho^{-u}$. In the following, for the sake of simplicity, the notation $d_{0,2, \delta_{1,2,\text{max/min}}}$ and $\Delta_0$ is used for $d_{0,2,\text{max/min}}, \delta_{1,1,\text{max/min}}, \delta_{0,2,\text{max/min}}$ and $\Delta_{\text{max/min,0}}$, respectively. It also follows from (72) that

$$d_{\text{max/min,0}}(\phi) + \delta_{\text{max/min,1}}(\psi) = \begin{pmatrix} \mu \rho^{-u} \phi + \delta_{1,1} \psi \\ d_{0,2} \phi - \mu \rho^{-u} \psi \end{pmatrix} = \begin{pmatrix} \mu \rho^{-u} \phi + \frac{1}{\mu} \rho^u \delta_{0,2} \phi \\ 0 \end{pmatrix} = \begin{pmatrix} \mu \rho^{-u} \Delta_0 \phi \\ 0 \end{pmatrix} .$$

Since $R(\Delta_{\text{max/min,0}}) = L_2^2$, we get

$$R(\rho^u) \oplus 0 \subset R(d_{\text{max/min,0}}) + R(\delta_{\text{max/min,1}}) .$$

With an analogous argument, using Lemma [2.3] instead of (103), we get

$$0 \oplus R(\rho^u) \subset R(d_{\text{max/min,0}}) + R(\delta_{\text{max/min,1}}) .$$

Therefore

$$R(\rho^u) \oplus R(\rho^u) \subset R(d_{\text{max/min,0}}) + R(\delta_{\text{max/min,1}}) ,$$

obtaining that $R(d_{\text{max/min,0}}) + R(\delta_{\text{max/min,1}})$ is dense in $L_2^+ \oplus L_2^+$ because $R(\rho^u)$ is dense in $L_2^+$. Thus $\ker \Delta_{\text{max/min,1}} = 0$ [9] Lemma 2.1]. \hfill $\square$

\footnote{We may also use Table [9] and Proposition [4.2] for some values of $\kappa$ (Tables [10] and [11]).}
Corollary 4.4. \(\Delta_{\text{max/min, ev}}\) has a discrete spectrum if and only if \(\Delta_{\text{max/min, 1}}\) also has a discrete spectrum. In this case, both operators have zero kernels and the same eigenvalues, with the same multiplicity.

**Proof.** This is a direct consequence of Proposition 4.3 and Lemma 2.2. 

Concerning the spectrum, the following corollary fills some gaps in Tables 10–12.

**Corollary 4.5.** If \(u < 1\), Tables 13 and 14 describe the spectra of \(\Delta_{\text{max/min, ev}}\) and \(\Delta_{\text{max/min, 1}}\) in terms of the spectra of \(P_i\), \(Q_j\) and \(W_{i,j}\) for the stated values of \(\kappa\).

| \(\kappa > -\frac{1}{2}\) | \(\kappa \geq \frac{1}{2}\) |
|---------------------------|---------------------------|
| \(\sigma(P_1 \oplus Q_1)\) | \(\sigma(P_1 \oplus Q_1)\) |
| \(-\frac{1}{2} + u < \kappa \leq -\frac{1}{2}\) | \(-\frac{1}{2} < \kappa < \frac{1}{2}\) |
| \(\sigma(W_{2,1})\) | \(\sigma(W_{2,1})\) |
| \(-\frac{1}{2} - u < \kappa \leq -\frac{1}{2}\) | \(\frac{1}{2} - u < \kappa < \frac{1}{2}\) |
| ? | \(\sigma(W_{2,1})\) |
| \(\kappa \leq -\frac{1}{2} - u\) | \(-\frac{1}{2} - u < \kappa < \frac{1}{2}\) |
| \(\sigma(P_2 \oplus Q_2)\) | \(\sigma(P_2 \oplus Q_2)\) |

**Table 13.** Spectrum of \(\Delta_{\text{max/min, ev}}\)

| \(\kappa > u - \frac{1}{2}\) | \(\kappa \geq \frac{1}{2}\) |
|---------------------------|---------------------------|
| \(\sigma(W_{1,1})\) | \(\sigma(W_{1,1})\) |
| \(-\frac{1}{2} < \kappa \leq u - \frac{1}{2}\) | \(-\frac{1}{2} < \kappa < \frac{1}{2}\) |
| \(\sigma(P_1 \oplus Q_1)\) | \(\sigma(W_{2,1})\) |
| \(-\frac{1}{2} + u < \kappa \leq -\frac{1}{2}\) | \(-\frac{1}{2} - 2u < \kappa < \frac{1}{2} - u\) |
| ? | \(\sigma(W_{2,2})\) |
| \(-\frac{1}{2} + u < \kappa \leq -\frac{1}{2}\) | \(\sigma(P_2 \oplus Q_2)\) |
| \(\kappa \leq -\frac{1}{2} - u\) | \(\kappa < \frac{1}{2} - 2u\) |
| \(\sigma(W_{2,2})\) | \(\sigma(W_{2,2})\) |

**Table 14.** Spectrum of \(\Delta_{\text{max/min, 1}}\)

**Proof.** This is a direct consequence of Proposition 4.2 and Corollary 4.3.

4.3. **The wave operator.** For the Hermitian bundle versions of \(E\) and \(F\), consider the wave operator \(\exp(itD_{\text{min/max}})\) \((i = \sqrt{-1})\) on \(L^2(E)\) or \(L^2(F)\), which is bounded.

**Proposition 4.6.** For \(\phi \in L^2(E)\) or \(L^2(F)\), let \(\phi_t = \exp(itD_{\text{min/max}})\phi\). If \(\text{supp } \phi \subset (0,a]\) for some \(a > 0\), then \(\text{supp } \phi_t \subset (0, a + |t|]\) for all \(t \in \mathbb{R}\).

**Proof.** The case of \(E\) is given by [3 Proposition 8.7-(ii)]. Then consider the case of \(F\), where the proof needs a slight change because the needed description of \(D^\infty(\Delta_{\text{max/min}})\) is not available. Since \(\exp(itD_{\text{min/max}})\) is bounded, we can assume that \(\phi \in D^\infty(\Delta_{\text{max/min}})\). Write \(\phi_t = \phi_{t,0} + \phi_{t,1} + \phi_{t,2}\) with \(\phi_{t,r} \in C^\infty(F_r) \equiv C^\infty_{\tau}\) \((r = 0, 2)\), and \(\phi_{t,1} \equiv \left(\frac{\phi_{t,1,1}}{\phi_{t,1,2}}\right) \in C^\infty(F_1) \equiv C^\infty \oplus C^\infty_{\tau}\). Suppose that \(t \geq 0\), the other case being analogous. For any \(c > b\),

\[
\frac{d}{dt} \int_{a+t}^c |\phi_t(\rho)|^2 d\rho = \int_{a+t}^c ((iD\phi_t, \phi_t) + (\phi_t, iD\phi_t))(\rho) d\rho - |\phi_t(a + t)|^2
\]

\[
= i \int_{a+t}^c ((D\phi_t, \phi_t) - (\phi_t, D\phi_t))(\rho) d\rho - |\phi_t(a + t)|^2.
\]
Now, \( d_{0,1} \equiv \delta_{0,1} \) and \( d_{1,2} \equiv \delta_{1,2} \) are multiplication operators by real valued functions. Moreover \( d_{0,2} \) and \( d_{0,1} \) are equal to \( \frac{d}{d\rho} \) and \(-\frac{d}{d\rho}\), respectively, up to the sum of multiplication operators by the same real valued functions, and the same is true for \( d_{1,1} \) and \( \delta_{1,1} \). Thus,

\[
(D_{\phi_t}, \phi_t) - (\phi_t, D_{\phi_t}) = (\delta_{0,1}\phi_{t,1,1} + \delta_{0,2}\phi_{t,1,2}, \phi_t, 0) + (d_{1,1}\phi_{t,1,1} + d_{1,2}\phi_{t,1,2}, \phi_t, 2)
\]

\[
+ (d_{0,1}\phi_{t,1,0} + \delta_{1,1}\phi_{t,2,1,1}) + (d_{0,2}\phi_{t,1,0} + \delta_{1,2}\phi_{t,2,1,2})
\]

\[
- (\phi_{t,0,0}, \delta_{0,1}\phi_{t,1,1} + \delta_{0,2}\phi_{t,1,2}) - (\phi_{t,1,1}, d_{1,1}\phi_{t,1,1} + d_{1,2}\phi_{t,1,2})
\]

\[
- (\phi_{t,1,1}, d_{0,1}\phi_{t,1,0} + \delta_{1,1}\phi_{t,2,1,2}) - (\phi_{t,1,2}, d_{0,2}\phi_{t,1,0} + \delta_{1,2}\phi_{t,2,1,2})
\]

\[
= -\phi_{t,1,1}\phi_{t,0,0} + \phi_{t,1,1}\phi_{t,1,0} - \phi_{t,1,1}\phi_{t,1,0}
\]

\[
\quad + \phi_{t,0,0}\phi_{t,1,1} + \phi_{t,1,1}\phi_{t,1,0} - \phi_{t,1,2}\phi_{t,1,0}
\]

\[
= 2\mathbb{E}(\phi_{t,1,2}\phi_{t,1,0} + \phi_{t,1,1}\phi_{t,1,0} - \phi_{t,1,2}\phi_{t,1,0})
\]

Therefore

\[
\left| \int_{a+t}^c ((D_{\phi_t}, \phi_t) - (\phi_t, D_{\phi_t}))(\rho) \, d\rho \right|
\]

\[
\leq 2|\phi_{t,1,1}\phi_{t,1,0} - \phi_{t,1,1}\phi_{t,1,0} - \phi_{t,1,2}\phi_{t,1,0}|(a + t)|
\]

\[
\leq |\phi_{t,1,1}(c)|^2 + |\phi_{t,1,0}(c)|^2 + |\phi_{t,1,1}(c)|^2 + |\phi_{t,0}(c)|^2
\]

\[
\quad + |\phi_{t,1,0}(a + t)|^2 + |\phi_{t,2}(a + t)|^2 + |\phi_{t,1,2}(a + t)|^2 + |\phi_{t,0}(a + t)|^2
\]

\[
= |\phi_{t,1,1}(c)|^2 + |\phi_{t,1,1}(a + t)|^2
\]

Since \( t \mapsto \phi_t \) defines a differentiable map with values in \( L^2(F) \), it follows that there is a sequence \( b \subset c_t \uparrow \infty \) such that \( \phi_t(c_t) \to 0 \), and

\[
\frac{d}{dt} \int_{a+t}^c |\phi_t(\rho)|^2 \, d\rho = \lim_i \frac{d}{dt} \int_{a+t}^{c_i} |\phi_t(\rho)|^2 \, d\rho \leq \lim_i |\phi_t(c_t)|^2 = 0
\]

So

\[
\int_{a+t}^c |\phi_t(\rho)|^2 \, d\rho \leq \int_a^c |\phi_0(\rho)|^2 \, d\rho = 0
\]

5. Witten’s perturbation on a cone

For the rel-Morse functions, the rel-local analysis of the Witten’s perturbed Laplacian will be reduced to the case of the functions \( \pm \rho^2 \) on a stratum of a cone with a model adapted metric, where \( \rho \) denotes the radial function. This kind of rel-local analysis begins in this section.

5.1. Witten’s perturbation. To begin with, recall the following generalities about Witten’s perturbation. Let \( M \equiv (M, g) \) be a Riemannian \( n \)-manifold. For all \( x \in M \) and \( \alpha \in T_xM^* \), let

\[
\alpha_{\rho} = (-1)^{nr+n+1} \star \alpha \wedge \star \quad \text{on} \quad \bigwedge^r T_xM^* ,
\]

involving the Hodge star operator \( \star \) on \( \bigwedge T_xM^* \) defined by any choice of orientation of \( T_xM \). Writing \( \alpha = g(X, \cdot) \) for \( X \in T_xM \), we have \( \alpha_{\rho} = -i_X \), where \( i_X \) denotes
the inner product by $X$. For any $f \in C^\infty(M)$, E. Witten [13] has introduced the following perturbations of $d$, $\delta$, $D$ and $\Delta$, depending on $s \geq 0$:

\[
d_s = e^{-sF} d e^{sF} = d + s d f \wedge ,
\]
\[
\delta_s = e^{sF} \delta e^{-sF} = \delta - s d f \lrcorner ,
\]
\[
D_s = d_s + \delta_s = D + s R ,
\]
\[
\Delta_s = D_s^2 = d_s \delta_s + \delta_s d_s = \Delta + s (RD + DR) + s^2 R^2 ,
\]

where $R = df \lrcorner - df \lrcorner$. Notice that $\delta_s = d_s^1$; thus $D_s$ and $\Delta_s$ are formally self-adjoint.

By analyzing the terms $RD + DR$ and $R^2$, the expression (106) becomes

\[
\Delta_s = \Delta + s \Hess f + s^2 |df|^2 ,
\]

where $\Hess f$ is an endomorphism defined by $\Hess f$ [36, Lemma 9.17], satisfying $|\Hess f| = |\Hess f| [36, Section 9].$

5.2. De Rham operators on a cone. Let $L$ be a non-empty compact stratification, let $\rho$ be the radial function on $c(L)$, let $N$ be a stratum of $L$ of dimension $\tilde{n}$, let $M = N \times \mathbb{R}_+$ be the corresponding stratum of $c(L)$ with dimension $n = \tilde{n} + 1$, and let $\pi : M \to N$ denote the first factor projection. From $\bigwedge TM^* = \bigwedge T N^* \boxtimes \bigwedge T \mathbb{R}_+$, we get a canonical identity

\[
\bigwedge^r TM^* \equiv \pi^* \bigwedge^r T N^* \oplus \rho \wedge \pi^* \bigwedge^{r-1} T N^* \equiv \pi^* \bigwedge^r T N^* \oplus \pi^* \bigwedge^{r-1} T N^* \]

(108)

for each degree $r$, obtaining

\[
\Omega^r(M) \equiv C^\infty(\mathbb{R}_+, \Omega^r(N)) \oplus \rho \wedge C^\infty(\mathbb{R}_+, \Omega^{r-1}(N)) \equiv C^\infty(\mathbb{R}_+, \Omega^r(N)) \oplus C^\infty(\mathbb{R}_+, \Omega^{r-1}(N)) .
\]

Here, smooth functions $\mathbb{R}_+ \to \Omega(N)$ are defined by considering $\Omega(N)$ as Fréchet space with the weak $C^\infty$ topology. In this section, all matrix expressions of vector bundle homomorphisms on $\bigwedge TM^*$ or differential operators on $\Omega^r(M)$ will be considered with respect to the decompositions (108) and (109).

Let $d$ and $d$ denote the exterior derivatives on $\Omega(M)$ and $\Omega(N)$, respectively. We have [3, Lemma 10.1]

\[
d \equiv \begin{pmatrix} \tilde{d} & 0 \\ \frac{d}{d\rho} & -\tilde{d} \end{pmatrix} .
\]

(111)

Fix a general adapted metric $\tilde{g}$ on $N$. For $u > 0$, the metric $g = \rho^{2u} \tilde{g} + d\rho^2$ is a general adapted metric on $M$. The induced metrics on $\bigwedge TM^*$ and $\bigwedge T N^*$ are also denoted by $g$ and $\tilde{g}$, respectively. Fix some degree $r \in \{0, 1, \ldots, n\}$, and, to simplify the expressions, let

\[
\kappa = (n - 2r - 1) \frac{n}{2} .
\]

(112)

According to (108),

\[
g = \rho^{-2ru} \tilde{g} \oplus \rho^{-2(r-1)u} \tilde{g}
\]

on $\bigwedge^r TM^*$. Given an orientation on an open subset $W \subset N$, and denoting by $\tilde{\omega}$ the corresponding $\tilde{g}$-volume form on $W$, consider the orientation on $W \times \mathbb{R}_+ \subset M$ so that the corresponding $g$-volume form is

\[
\omega = \rho^{(n-1)u} d\rho \wedge \tilde{\omega} .
\]

(114)
The corresponding star operators on $\bigwedge T(W \times \mathbb{R}_+)^*$ and $\bigwedge T W^*$ will be denoted by $\ast$ and $\hat{\ast}$, respectively. Like in [3, Lemma 10.2], from (113) and (114), it follows that
\begin{equation}
\ast \equiv \begin{pmatrix} 0 \\ - (1)^{r} \rho^{2\kappa} \hat{\ast} \\ 0 \end{pmatrix}
\end{equation}
on $\bigwedge T(W \times \mathbb{R}_+)^*$. Let $L^2\Omega^r(M) = L^2\Omega^r(M, g)$ and $L^2\Omega^r(N) = L^2\Omega^r(N, \tilde{g})$. From (113) and (114), we also get that (110) induces the identity of Hilbert spaces
\begin{equation}
L^2\Omega^r(M) \equiv (L^2_{\kappa, +} \otimes L^2\Omega^r(N)) \oplus (L^2_{\kappa+n, +} \otimes L^2\Omega^{r-1}(N)).
\end{equation}
Let $\delta$ and $\tilde{\delta}$ denote the exterior coderivatives on $\Omega(M)$ and $\Omega(N)$, respectively. Like in [3, Lemma 10.2], from (113) and (114), it follows
\begin{equation}
\Delta \equiv \begin{pmatrix} P \\ - 2u \rho^{-2} \delta \\ Q \end{pmatrix}
\end{equation}
on $\Omega^r(M)$, where
\begin{align}
P &= \rho^{-2u} \Delta \frac{d^2}{dp^2} - 2\kappa \rho^{-1} \frac{d}{dp} , \\
Q &= \rho^{-2u} \Delta \frac{d^2}{dp^2} - 2(\kappa + u) \frac{d}{dp} \rho^{-1} .
\end{align}

5.3. Witten’s perturbation on a cone. Let $d_s$, $\delta_s$, $D_s$ and $\Delta_s$ ($s \geq 0$) denote the Witten’s perturbations of $d$, $\delta$, $D$ and $\Delta$ induced by the function $f = \pm \frac{1}{2} \rho^2$ on $M$. The more explicit notation $d_s^\pm$, $\delta_s^\pm$, $D_s^\pm$ and $\Delta_s^\pm$ may be used if needed. In this case, $df = \pm d\rho$. According to (110),
\begin{align}
\rho d\rho \wedge & \equiv \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
- \rho d\rho \wedge & \equiv \begin{pmatrix} 0 \\ \rho \end{pmatrix}.
\end{align}
So, by (111), (117) and (12),
\begin{align}
d_s & \equiv \begin{pmatrix} \hat{\ast} \\ \frac{d}{dp} \pm s \rho \hat{\ast} \\ 0 \end{pmatrix} , \\
\delta_s & \equiv \begin{pmatrix} \rho^{-2u} \tilde{\delta} \\ - \frac{d}{dp} - 2(\kappa + u) \rho^{-1} \pm s \rho \tilde{\delta} \\ 0 \end{pmatrix} ,
\end{align}
on $\Omega^r(M)$. Now,
\begin{align}
R & = \pm \rho(d\rho \wedge - d\rho \wedge) \equiv \pm \begin{pmatrix} 0 \\ \rho \end{pmatrix} ,
\end{align}
and therefore
\begin{equation}
R^2 \equiv \begin{pmatrix} \rho^2 \\ 0 \end{pmatrix} \equiv \rho^2 .
\end{equation}

\footnote{Recall that, for Hilbert spaces $\mathcal{H}'$ and $\mathcal{H}''$, with scalar products $\langle \cdot, \cdot \rangle'$ and $\langle \cdot, \cdot \rangle''$, the notation $\mathcal{H}' \otimes \mathcal{H}''$ is used for the Hilbert space tensor product; i.e., the Hilbert space completion of the algebraic tensor product $\mathcal{H}' \otimes \mathcal{H}''$ with respect to the scalar product defined by $\langle u', v' \otimes v'' \rangle = \langle u', v'' \rangle' \langle u'', v'' \rangle''$.}
Like in [3] Lemma 10.6, we get
\[ RD + DR = V \] (124)
on \( \Omega^r (M) \), where \( V \) is given by (129). As a consequence of (107), (118) and (124), we obtain, on \( \Omega^r (M) \),
\[ \Delta_s \equiv \left( -2u \rho^{-2u-1} \widetilde{\Delta} + 2u \rho^{-1} \partial \right), \] (125)
where
\[ P_s = \rho^{-2u} \widetilde{\Delta} + H - 2 \kappa \rho^{-1} \frac{d}{d\rho} \pm s(1 + 2\kappa), \] (126)
\[ Q_s = \rho^{-2u} \widetilde{\Delta} + H - 2(\kappa + u) \frac{d}{d\rho} \rho^{-1} \pm s(-1 + 2(\kappa + u)). \] (127)

6. Splitting of the Witten’s complex on a cone

6.1. Spectral decomposition on the link of the cone. Theorem 1.1 is proved by induction on the depth. Thus, with the notation of Section 5, suppose that \( \tilde{g} \) is good, and \( \tilde{\Delta}_{\text{max/min}} \) satisfies the statement of Theorem 1.1. Moreover suppose that \( g \) is also good. According to (2), this means that
\[ u \leq 1, \]
\[ \frac{1}{n} \in 2\mathbb{Z} + n + (0, 1] \text{ if } \frac{1}{n} \leq u < 1. \] (128)

Lemma 6.1. \( \kappa \notin \left( -\frac{1}{2} - u, -\frac{1}{2} + u \right] \cup \left[ \frac{1}{2} - u, \frac{1}{2} \right) \) if \( u < 1 \).

Proof. Suppose that \( u < 1 \). Then simple computation shows that the condition of the statement means that

\[ r \notin \left( \frac{n-1}{2u}, \frac{n}{2} - \frac{1}{2u} \right] \cup \left( \frac{n}{2} + \frac{1}{2u}, \frac{n+1}{2} + \frac{1}{2u} \right). \] (129)

When \( n \) is even, (129) holds for all \( 0 \leq r \leq n \) if and only if
\[ \emptyset = \left( \left( -\frac{1}{2u}, -\frac{1}{2} \right] \cup \left( \frac{1}{2u}, \frac{1}{2} \right) \right) \cap \mathbb{Z} \cap \left[ -\frac{n}{2}, \frac{n}{2} \right], \]
which is equivalent to
\[ \emptyset = \left[ \frac{1}{2u}, \frac{1}{2} + \frac{1}{2u} \right) \cap \mathbb{Z} \cap \left[ 0, \frac{n}{2} \right] = \left( \frac{1}{2u} + [0, \frac{1}{2}] \right) \cap \mathbb{Z} \cap \left[ 0, \frac{n}{2} \right]. \]
This means \( \frac{1}{2u} \in \mathbb{Z} + (0, \frac{1}{2}] \) if \( \frac{1}{n} \leq n \), which is equivalent to (128) in this case.

When \( n \) is odd, (129) holds for all \( 0 \leq r \leq n \) if and only if
\[ \emptyset = \left( \left( -\frac{1}{2u}, \frac{1}{2} - \frac{1}{2u} \right] \cup \left[ \frac{1}{2u}, 1 + \frac{1}{2u} \right) \right) \cap \mathbb{Z} \cap \left[ -\frac{n}{2}, \frac{n+1}{2} \right], \]
which is equivalent to
\[ \emptyset = \left( \left( -\frac{1}{2u}, -\frac{1}{2} \right] \cup \left[ -\frac{1}{2}, \frac{1}{2} \right] \right) \cap \mathbb{Z} \cap \left[ -\frac{n+1}{2}, \frac{n+1}{2} \right]. \]
In turn, this is equivalent to
\[ \emptyset = \left[ -\frac{1}{2} + \frac{1}{2u}, \frac{1}{2u} \right) \cap \mathbb{Z} \cap \left[ 0, \frac{n}{2} - 1 \right] = \left( \frac{1}{2u} + [0, \frac{1}{2}] \right) \cap \mathbb{Z} \cap \left[ 0, \frac{n-1}{2} \right]. \]
This means \( \frac{1}{2u} \in \mathbb{Z} + (\frac{1}{2}, 1) \) if \( \frac{1}{n} \leq n \), which is equivalent to (128) in this case. \( \square \)

Remark 10. The above proof in fact shows that the condition of Lemma 6.1, for all \( 0 \leq r \leq n \), characterizes the second condition of (128).
Let \( \tilde{H}_{\text{max/min}} = \text{ker } \tilde{D}_{\text{max/min}} = \text{ker } \tilde{\Delta}_{\text{max/min}} \), which is a graded subspace of \( \Omega(N) \). For each degree \( r \), let \( \tilde{R}_{\text{max/min},r-1}, \tilde{R}_{\text{max/min},r-1}^* \subset L^2 \Omega'(N) \) be the images of \( \tilde{d}_{\text{max/min},r-1} \) and \( \tilde{\delta}_{\text{max/min},r} \), respectively, which are closed subspaces. By restriction, \( \tilde{\Delta}_{\text{max/min}} \) defined self-adjoint operators in \( \tilde{R}_{\text{max/min},r-1} \) and \( \tilde{R}_{\text{max/min},r-1}^* \), with the same eigenvalues \([3 \text{ Section 5.1}]. \) For any eigenvalue \( \tilde{\lambda} \) of the restriction of \( \tilde{\Delta}_{\text{max/min}} \) to \( \tilde{R}_{\text{max/min},r-1} \), let \( \tilde{R}_{\text{max/min},r-1,\tilde{\lambda}} \) and \( \tilde{R}_{\text{max/min},r-1,\tilde{\lambda}}^* \) denote the corresponding \( \tilde{\lambda} \)-eigenspaces. We have\([3 \text{ Proposition 6.3}. \)

\[
L^2 \Omega'(N) = \tilde{H}_{\text{max/min}}^r \oplus \bigoplus_{\tilde{\lambda}} \left( \tilde{R}_{\text{max/min},r-1,\tilde{\lambda}} \oplus \tilde{R}_{\text{max/min},r-1,\tilde{\lambda}}^* \right),
\]

where \( \tilde{\lambda} \) runs in the spectrum of the restrictions of \( \tilde{\Delta}_{\text{max/min}} \) to \( \tilde{R}_{\text{max/min},r-1} \) and \( \tilde{R}_{\text{max/min},r-1}^* \).

6.2. **Subcomplexes of length one.** Given \( 0 \neq \gamma \in \tilde{H}_{\text{max/min}}^r \), consider the canonical identities

\[
C_+^\infty \equiv C_+^\infty \gamma \subset \Omega^r(M), \quad C_+^\infty = C_+^{\infty} \omega \wedge \gamma \subset \Omega^{r+1}(M).
\]

The following result follows from \([12] \text{ and } [12] \).

**Lemma 6.2.** For \( s \geq 0 \), \( d_s \) and \( \delta_s \) define maps

\[
\begin{array}{cccc}
0 & \xleftarrow{d_{s,r-1}} & C_+^\infty & \xrightarrow{\delta_{s,r}} & C_+^\infty \omega \wedge \gamma & \xleftarrow{d_{s,r+1}} & 0.
\end{array}
\]

Moreover, using \([131] \),

\[
d_{s,r} = \frac{d}{d\rho} \pm s\rho, \quad \delta_{s,r} = -\frac{d}{d\rho} - \kappa \rho^{-1} \pm s\rho.
\]

Let \( \mathcal{E}_{r,0} \) denote the dense subcomplex of \( \mathcal{E}_{r,\gamma} \) defined by

\[
\mathcal{E}_{r,0}^\gamma = C_{0,+}^\infty \gamma \equiv C_{0,+}^\infty, \quad \mathcal{E}_{r,0}^{r+1} = C_{0,+}^{\infty} \omega \wedge \gamma \equiv C_{0,+}^\infty.
\]

The closure of \( \mathcal{E}_{r,0} \) in \( L^2 \Omega(M) \) is denoted by \( L^2 \mathcal{E}_{r,0} \). By \([110] \),

\[
L^2 \mathcal{E}_{r,\gamma} = L^2_{\kappa,+} \gamma = L^2_{\kappa,+}, \quad L^2 \mathcal{E}_{r,0}^{r+1} = L^2_{\kappa,+} \omega \wedge \gamma \equiv L^2_{\kappa,+}.
\]

Assume now that \( s > 0 \). With the notation of Section \([11] \), consider the real version of the elliptic complex \( (E,d) \) determined by the constants \( s \) and \( \kappa \) (given by \([112] \)). Using Lemma \([6.2] \) and \([12] \), like in \([3 \text{ Proposition 12.3}] \), we get the following.

**Proposition 6.3.** \( \rho^*: L^2_{\kappa,+} \rightarrow L^2_{\kappa,*} \) defines a unitary isomorphism \( L^2 \mathcal{E}_{r,\gamma} \rightarrow L^2(E) \), which restricts to isomorphisms of complexes, \( (\mathcal{E}_{r,0}, d_s) \rightarrow (C^\infty_0(E), d) \), up to a shift of degree.

---

7For a family of Hilbert spaces, \( \mathcal{H} \), with scalar product \( \langle \cdot, \cdot \rangle_a \), recall that the Hilbert space direct sum, \( \bigoplus_a \mathcal{H} \), is the Hilbert space completion of the algebraic direct sum, \( \bigoplus_a \mathcal{H} \), with respect to the scalar product \( \langle (u^a), (v^a) \rangle = \sum_a \langle u^a, v^a \rangle_a \). Thus \( \bigoplus_a \mathcal{H} = \bigoplus_a \mathcal{H} \) if and only if the family is finite.
By Proposition 6.3, \((E_\gamma, 0, d_s)\) has a maximum/minimum Hilbert complex extension in \(L^2 E_\gamma\). Let \((D_\gamma, d_{s, \gamma})\) denote the maximum/minimum Hilbert complex extension of \((E_\gamma, 0, d_s)\) if \(\gamma \in \tilde{H}_{\text{max/min}}^r\), let \(\Delta_{s, \gamma}\) be the corresponding Laplacian, and let \(H_{s, \gamma} = H_{s, \gamma}^+ \oplus H_{s, \gamma}^{-1} = \ker \Delta_{s, \gamma}\). The more explicit notation \(d_{s, \gamma}^\pm, \Delta_{s, \gamma}^\pm\) and \(H_{s, \gamma}^\pm = H_{s, \gamma}^r \oplus H_{s, \gamma}^{r+1}\) may be also used.

**Corollary 6.4.**

(i) \(\Delta_{s, \gamma}\) has a discrete spectrum.

(ii) The dimensions of \(H_{s, \gamma}^\pm\) and \(H_{s, \gamma}^{r+1}\) are given in Table 15.

(iii) If \(e_s \in H_{s, \gamma}\) with norm one for each \(s\), and \(h\) is a bounded measurable function on \(\mathbb{R}_+\) with \(h(\rho) \to 1\) as \(\rho \to 0\), then \(\langle h e_s, e_s \rangle \to 1\) as \(s \to \infty\).

(iv) All non-zero eigenvalues of \(\Delta_{s, \gamma}\) are in \(O(s)\) as \(s \to \infty\).

| \(\gamma \in H_r^{\text{max}}\) | \(\gamma \in H_r^{\text{min}}\) |
|---|---|
| \(H_{s, \gamma}^{r, \tau}\) | \(H_{s, \gamma}^{r, \tau+1}\) | \(H_{s, \gamma}^{-r, \tau}\) | \(H_{s, \gamma}^{-r, \tau+1}\) |
| \(\kappa \geq \frac{1}{2}\) | 1 | 0 | 0 | 0 |
| \(|\kappa| < \frac{1}{2}\) | 0 | 1 | 0 | 1 |
| \(\kappa \leq -\frac{1}{2}\) | 0 | 0 | 0 | 1 |

Table 15. Dimensions of \(H_{s, \gamma}^{\pm, r}\) and \(H_{s, \gamma}^{\pm, r+1}\)

**Proof.** This follows from Propositions 6.3 and 4.1, Corollary 3.7, Section 4.1.1, and the choice made to define \(d_{s, \gamma}\). \(\square\)

### 6.3. Subomplexes of length two

Let \(\mu = \sqrt{\tilde{\lambda}}\) for an eigenvalue \(\tilde{\lambda}\) of the restriction of \(\tilde{\Delta}_{\text{max/min}}\) to \(\tilde{R}_{\text{max/min}, r-1}\). According to \([3, \text{Section 5.1}]\), there are non-zero differential forms,

\[
\alpha \in \tilde{R}_{\text{max/min}, r-1, \lambda}^r (\mathbb{N}) , \quad \beta \in \tilde{R}_{\text{max/min}, r-1, \lambda}^r (\mathbb{N}) ,
\]

such that \(\tilde{d} \beta = \mu \alpha\) and \(\tilde{d} \alpha = \mu \beta\). Consider the canonical identities

\[
C_+^\infty \equiv C_+^\infty \beta \subset \Omega^{-1}(M) , \quad C_+^\infty \equiv C_+^\infty \delta \beta \subset \Omega^{r+1}(M) , \quad C_+^\infty \equiv C_+^\infty \delta \beta \subset \Omega^r(M) .
\]

The following result follows from \([121]\) and \([122]\).

**Lemma 6.5.** For \(s \geq 0\), \(d_s\) and \(\delta_s\) define maps

\[
0 \xrightarrow{d_{s, r-2}} C_+^\infty \beta \xrightarrow{d_{s, r-1}} C_+^\infty \alpha + C_+^\infty \delta \beta \xrightarrow{d_{s, r}} C_+^\infty d \beta \wedge \alpha \xrightarrow{d_{s, r+1}} 0 .
\]
Moreover, according to (132) and (133),
\[
\begin{align*}
d^+_{s,r-1} &= \left( \frac{d}{dp} s \right)^+,
\delta^+_{s,r-1} &= \left( \mu^2 - \frac{d}{dp} 2(\kappa + u) \rho^{-1} \pm s \rho \right),
d^-_{s,r} &= \left( \frac{d}{dp} + s \rho \right),
\delta^-_{s,r} &= \left( \frac{d}{dp} - 2\kappa \rho^{-1} \pm s \rho \right).
\end{align*}
\]

Let \( \mathcal{F}_{\alpha,\beta,0} = \mathcal{F}_{\alpha,\beta,0}^{-1} \oplus \mathcal{F}_{\alpha,\beta,0}^+ \) denote the subcomplex of length two of \((\Omega(M), d_s)\) defined by
\[
\begin{align*}
\mathcal{F}_{\alpha,\beta,0}^{-1} &= C^0_{\alpha,\beta,0+}, \quad \mathcal{F}_{\alpha,\beta,0}^+ = C^0_{\alpha,\beta,0+} \oplus \alpha \equiv C^0_{\alpha,\beta,0+}, \\
\mathcal{F}_{\alpha,\beta,0}^+ &= C^0_{\alpha,\beta,0} \oplus \alpha \equiv C^0_{\alpha,\beta,0+}. 
\end{align*}
\]
The closure of \( \mathcal{F}_{\alpha,\beta,0} \) in \( L^2 \Omega(M) \) is denoted by \( L^2 \mathcal{F}_{\alpha,\beta} \). By (110),
\[
\begin{align*}
L^2 \mathcal{F}_{\alpha,\beta}^{-1} &= L^2_{\alpha,\beta,0} + \beta \equiv L^2_{\alpha,\beta,0+}, \\
L^2 \mathcal{F}_{\alpha,\beta}^+ &= L^2_{\alpha,\beta,0} + \alpha \equiv L^2_{\alpha,\beta,0+}.
\end{align*}
\]
Assume now that \( s > 0 \). With the notation of Section 4.2 consider the real extension of \((F, d)\) determined by the constants \( s \) and \( \kappa \) (given by 122). Using Lemma 6.9 and 12, we get the following (cf. [3, Proposition 12.9]).

**Proposition 6.6.** If \( u < 1 \), then \( \rho^u : L^2_{\alpha,\beta,0} \to L^2_{\alpha,\beta,0} \) and \( \rho^{\kappa + u} : L^2_{\alpha,\beta,0} \to L^2_{\alpha,\beta,0} \)
define a unitary isomorphism \( L^2 \mathcal{F}_{\alpha,\beta} \to L^2(F) \), which restricts to an isomorphism of complexes, \((\mathcal{F}_{\alpha,\beta,0}, d_s) \to (C^0(F), d)\), up to a shift of degree.

By Proposition 6.6 \((D_{\alpha,\beta}, d_{\alpha,\beta})\) has a maximum/minimum Hilbert complex extension in \( L^2 \mathcal{F}_{\alpha,\beta} \). Let \((D_{\alpha,\beta}, d_{\alpha,\beta})\) denote the maximum/minimum Hilbert complex extension of \((\mathcal{F}_{\alpha,\beta,0}, d_s)\) if \( \alpha \in \max_{\max_{\rho^{\kappa - 1,\beta,0}} \rho^{\kappa - 1,\beta,0}} \) and \( \beta \in \max_{\max_{\rho^{\kappa - 1,\beta,0}} \rho^{\kappa - 1,\beta,0}} \), and let \( \Delta_{\alpha,\beta} \) denote the corresponding Laplacian. The more explicit notation \( d_{\alpha,\beta}^\pm \) and \( \Delta_{\alpha,\beta}^\pm \) may be used.

**Corollary 6.7.** (i) \( \Delta_{\alpha,\beta} \) has a discrete spectrum.
(ii) The eigenvalues of \( \Delta_{\alpha,\beta} \) are positive and in \( O(s) \) as \( s \to \infty \).

**Proof.** In the case \( u < 1 \), this follows from Proposition 6.6, Corollary 4.5 and Lemma 6.1. In the case \( u = 1 \), this is the content of [3, Proposition 12.11]. \( \square \)

**Remark 11.** According to (124)–(126), we have
\[
\begin{align*}
\Delta_s &\equiv H - 2\kappa \rho^{-1} d_d \rho^{-1} \pm s(1 + 2\kappa) \quad \text{on } C^\infty_s \equiv C^\infty_s \gamma, \\
\Delta_s &\equiv H - 2\kappa d_d \rho^{-1} \pm s(-1 + 2\kappa) \quad \text{on } C^\infty_s \equiv C^\infty_s \rho \land \gamma, \\
\Delta_s &\equiv H - 2\kappa \rho^{-1} d_d \rho^{-1} + \mu^2 \rho^{-2u} s(1 + 2\kappa + u) \quad \text{on } C^\infty_s \equiv C^\infty_s \beta, \\
\Delta_s &\equiv H - 2\kappa d_d \rho^{-1} + \mu^2 \rho^{-2u} s(-1 + 2\kappa) \quad \text{on } C^\infty_s \equiv C^\infty_s \rho \land \alpha,
\end{align*}
\]
and
\[
\Delta_s \equiv \begin{pmatrix} P_{\mu, s} & -2\mu s \rho^{-1} \\ -2\mu s \rho^{-1} & Q_{\mu, s} \end{pmatrix}
\]
on $C_+^\infty \oplus C_+^\infty \equiv C_+^\infty \alpha + C_+^\infty \beta$, where

\[ P_{\mu,s} = H - 2\kappa \rho^{-1} \frac{d}{d\rho} + \mu^2 \rho^{-2u} \mp s(1 + 2\kappa), \]

\[ Q_{\mu,s} = H - 2(\kappa + u) \frac{d}{d\rho} \rho^{-1} + \mu^2 \rho^{-2u} \mp s(-1 + 2(\kappa + u)). \]

So the results of Section 3 could be applied to these expressions. We opted for analyzing first the complexes of Section 4 for the sake of simplicity: we have $a = b = 0$, $L^2_{\kappa,+}$ is used instead of $L^2_{\kappa,+}$ or $L^2_{\kappa,+u,+}$, and Remark 3 is directly applied.

6.4. **Splitting into subcomplexes.** Let $C_{\max/min,0}$ denote an orthonormal frame of $H_{\max/min}$ consisting of homogeneous differential forms. For each positive eigenvalue $\mu$ of $D_{\max/min}$, let $C_{\max/min,\mu}$ be an orthonormal frame of $\mu$-eigenspace of $D_{\max/min}$ consisting of differential forms $\alpha + \beta$ like in Section 6.3 Then let

\[ d_{s,\max/min} = \bigoplus_{\gamma} d_{s,\gamma} \oplus \bigoplus_{\mu} d_{s,\mu+\beta}, \]

where $\gamma$ runs in $C_{\max/min,0}$, $\mu$ runs in the positive spectrum of $D_{\max/min}$, and $\alpha + \beta$ runs in $C_{\max/min,\mu}$. The notation $d_{s,\max/min}^\pm$ may be also used when $d_{s,\gamma}^\pm$ and $d_{s,\mu+\beta}^\pm$ are considered.

**Proposition 6.8.** $d_{s,\max/min} = d_{s,\max/min}$. 

**Proof.** This follows like [3, Proposition 12.12], using [3, Lemma 5.2], [9, Lemma 3.6 and (2.38b)], [109] and (1.30). \( \square \)

Let $H_{s,\max/min} = \bigoplus_r H_{s,\max/min}^r = \ker \Delta_{s,\max/min}$. The superindex “±” may be added to this notation to indicate that we are referring to $\Delta_{s,\max/min}^\pm$.

**Corollary 6.9.**

(i) $\Delta_{s,\max/min}$ has a discrete spectrum.

(ii) Table 17 describe the isomorphism class of $H_{s,\max/min}^{\pm,*}$.

(iii) If $e_s \in H_{s,\max/min}$ has norm one for each $s$, and $h$ is a bounded measurable function on $\mathbb{R}_+$ with $h(\rho) \to 1$ as $\rho \to 0$, then $\langle he_s, e_s \rangle \to 1$ as $s \to \infty$.

(iv) Let $0 \leq \lambda_{s,\max/min} \leq \lambda_{s,\max/min,1} \leq \cdots$ be the eigenvalues of $\Delta_{s,\max/min}$, repeated according to their multiplicities. Given $k \in \mathbb{N}$, if $\lambda_{s,\max/min,k} > 0$ for some $s$, then $\lambda_{s,\max/min,k} > 0$ for all $s$, and $\lambda_{s,\max/min,k} \in O(s)$ as $s \to \infty$.

(v) There is some $\theta > 0$ such that $\liminf_k \lambda_{s,\max/min,k}^{-\theta} > 0$.

| $\kappa \geq \frac{1}{2}$ | $H_{\max}^r(N)$ | 0 | $H_{\min}^r(N)$ | 0 |
|---------------------------|-----------------|---|-----------------|---|
| $|\kappa| < \frac{1}{2}$ | 0 | $H_{\max}^r(N)$ | 0 | $H_{\min}^r(N)$ |
| $\kappa \leq -\frac{1}{2}$ | 0 | 0 | 0 |

**Table 16.** Spaces isomorphic to $H_{\max/min}^{\pm,*}$

**Proof.** In the case $u = 1$, this result was already shown in [3, Corollary 12.13]. So we consider only the case $0 < u < 1$. For all $\gamma$, $\mu$ and $\alpha + \beta$ as above, $\Delta_{s,\gamma}$ and $\Delta_{s,\alpha,\beta}$ have a discrete spectrum by Corollaries 6.3, 6.4 and 6.7. Moreover the
union of their spectra has no accumulation points according to Section 6 and since \( \Delta_{\text{max/min}} \) is discrete. Then (i) follows by Proposition 6.8.

Now, properties (ii)–(iv) follow directly from Corollaries 6.4, 6.7 and 6.8.

To prove (v), let \( 0 \leq \lambda_{\text{max/min,0}} \leq \lambda_{\text{max/min,1}} \leq \cdots \) denote the eigenvalues of \( \Delta_{\text{max/min}} \), repeated according to their multiplicities. Since \( N \) satisfies Theorem 1.1, (i) with \( \tilde{g} \), there is some \( C_0, \theta_0 > 0 \) such that

\[
\tilde{\lambda}_{\text{max/min,}\ell} \geq C_0 \ell^{\theta_0}
\]

for all \( \ell \). Consider the counting function

\[
\mathcal{N}^\pm_{s,\text{max/min}}(\lambda) = \# \left\{ k \in \mathbb{N} \mid \lambda^\pm_{s,\text{max/min},k} < \lambda \right\}
\]

for \( \lambda > 0 \). From (68)–(71), (74), (76), (80), (82), (82), (84), (86), (88) and (134), and the choices made to define \( d_\gamma \) and \( d_{\alpha,\beta} \) (Section 6), it follows that there are some \( C_1, C_2, C_3 > 0 \) such that

\[
\mathcal{N}^\pm_{s,\text{max/min}}(\lambda) \leq \# \left\{ (k, \ell) \in \mathbb{N}^2 \mid C_1 k + C_2 \tilde{\lambda}_{\text{max/min,}\ell}(k + 1)^{-u} + C_3 \leq \lambda \right\}
\]

\[
\leq \# \left\{ (k, \ell) \in \mathbb{N}^2 \mid C_1 k + C_2 C_0 \ell^{\theta_0}(k + 1)^{-u} + C_3 \leq \lambda \right\}
\]

\[
\leq \# \left\{ (k, \ell) \in \mathbb{N}^2 \right\}
\]

\[
\leq \left( \int_0^{\lambda-C_3} \left( \frac{\lambda-C_3-C_1 x}{C_2 C_0} \right)^{\frac{1}{2u}} (x + 1)^u \; dx \right)^{\frac{1}{2}}
\]

\[
\leq \left( \int_0^{\lambda-C_3} \left( \frac{\lambda-C_3-C_1 x}{C_2 C_0} \right)^{\frac{1}{2u}} dx \right)^{\frac{1}{2}} \left( \int_0^{\lambda-C_3} (x + 1)^{2u} \; dx \right)^{\frac{1}{2}}
\]

\[
\leq \left( \frac{\theta_0(\lambda-C_3)^{\frac{1}{2u}+1}}{(2+\theta_0)(C_2 C_0)^{\frac{1}{2u}} C_1} \right)^{\frac{1}{2}} \left( \frac{\theta_0(\lambda-C_3+C_1)^{\frac{1}{2u}+1}}{(2+\theta_0)(C_2 C_0)^{\frac{1}{2u}}+1} \right)^{\frac{1}{2}}
\]

\[
= \theta_0(\lambda-C_3)^{\frac{1}{2u}+\frac{1}{2}}(\lambda-C_3+C_1)^{\frac{1}{2u}+\frac{1}{2}}
\]

\[
(2+\theta_0)^{\frac{1}{2}}(2+\theta_0)^{\frac{1}{2}}(C_2 C_0)^{\frac{1}{2u}}+1
\]

So \( \mathcal{N}^\pm_{s,\text{max/min}}(\lambda) \leq C \lambda^{1+\frac{1}{2u}} \) for some \( C > 0 \) and all large enough \( \lambda \), giving (viii) with \( \theta = \frac{1+u}{\theta_0} + 1 \).

Table 17 describes the above conditions on \( \kappa \) in terms of \( r \).

| \( \kappa \geq \frac{1}{2} \) | \( r \leq \frac{u-1}{2} - \frac{1}{2u} \) |
| \( |\kappa| < \frac{1}{2} \) | \( |r - \frac{u-1}{2}| < \frac{1}{2u} \) |
| \( \kappa \leq -\frac{1}{2} \) | \( r \geq \frac{u-1}{2} + \frac{1}{2u} \) |

Table 17. Correspondence between conditions on \( \kappa \) and \( r \).
7. Relatively local model of the Witten’s perturbation

Let \( m \in \mathbb{N} \), let \( L_1, \ldots, L_a \) be compact stratifications, let \( N_i \) be a dense stratum of each \( L_i \), let \( k_i = \dim N_i + 1 \), and let \( \ast_i \) and \( \rho_i \) be the vertex and radial function of every \( c(L_i) \). Then \( M := \mathbb{R}^m \times \prod_{i=1}^a (N_i \times \mathbb{R}_+) \) is a dense stratum of \( A := \mathbb{R}^m \times \prod_{i=1}^a c(L_i) \). For any relatively compact open neighborhood \( O \) of \( x := (0, \ast_1, \ldots, \ast_a) \), all general adapted metrics on \( M \) are quasi-isometric on \( M \cap O \) to a metric of the form \( g = g_0 + \sum_{i=1}^a \rho_i^{2u_i} \hat{g}_i + (d\rho_i)^2 \), where \( g_0 \) is the Euclidean metric on \( \mathbb{R}^m \), each \( \hat{g}_i \) is a general adapted metric on \( N_i \), and \( u_i > 0 \). Suppose that \( g \) is good; i.e., the metrics \( \hat{g}_i \) are good, \( u_i \leq 1 \), and \( \frac{1}{u_i} \in 2\mathbb{Z} + k_i + (0,1] \) if \( \frac{1}{u_i} \leq u_i < 1 \). Note that the fiber of \( \hat{M} \to M \) over \( x \) consists of a unique point, which can be identified to \( x \). According to Section 1.4, the rel-local model of a rel-Morse function around a rel-critical point is of the form \( f := \frac{1}{2} (\rho^2 - \rho^2) \), where \( \rho \) is the radial function of \( \mathbb{R}^m \times \prod_{i \in I_+} c(L_i) \), for some decomposition \( m = m_+ + m_- \ (m_+ \in \mathbb{N}) \), and some partition of \( \{1, \ldots, a\} \) into sets \( I_\pm \). The rel-critical set of \( f \) consists only of \( x \). Let \( d_\ast, \delta_\ast, D_\ast \) and \( \Delta_\ast \) be the Witten’s perturbations of \( d, \delta, D \) and \( \Delta \) on \( \Omega(M) \) induced by \( f \). Let \( \mathcal{H}_{s, \text{max/min}} = \bigoplus_j \mathcal{H}_{s, \text{max/min}}^{\gamma_j} := \ker \Delta_{s, \text{max/min}} \). The following result is a direct consequence of Corollary 6.9 and Example 9.1 and Lemma 5.1, taking also into account Table 17.

**Corollary 7.1.**

(i) \( \Delta_{s, \text{max/min}} \) has a discrete spectrum.

(ii) We have

\[
\mathcal{H}_{s, \text{max/min}}^r \cong \bigoplus_{(r_1, \ldots, r_a) \in I_+} \bigotimes_{i \in I_+} H_{\max/min}^{r_i}(N_i) \otimes \bigotimes_{j \in I_-} H_{\max/min}^{r_j}(N_j),
\]

where \((r_1, \ldots, r_a)\) runs in the subset of \( \mathbb{N}^a \) defined by the conditions

\[
\begin{align*}
    r &= m_- + \sum_{i=1}^a r_i + |I_-|, \\
    r_i &< \frac{k_i - 1}{2} + \frac{1}{2u_i}, \text{ if } i \in I_+ \\
    r_i &\geq \frac{k_i - 1}{2} + \frac{1}{2u_i}, \text{ if } i \in I_- \\
    r_i &\leq \frac{k_i - 1}{2} - \frac{1}{2u_i}, \text{ if } i \in I_+ \\
    r_i &> \frac{k_i - 1}{2} - \frac{1}{2u_i}, \text{ if } i \in I_-
\end{align*}
\]

for \( \mathcal{H}_{s, \text{max}}^r \),

for \( \mathcal{H}_{s, \text{min}}^r \).

(iii) If \( e_s \in \mathcal{H}_{s, \text{max/min}} \) with norm one for each \( s \), and \( h \) is a bounded measurable function on \( \mathbb{R}_+ \) with \( h(\rho) \to 1 \) as \( \rho \to 0 \), then \( \langle h e_s, e_s \rangle \to 1 \) as \( s \to \infty \).

(iv) Let \( 0 \leq \lambda_{s, \text{max/min}, 0} \leq \lambda_{s, \text{max/min}, 1} \leq \cdots \) be the eigenvalues of \( \Delta_{s, \text{max/min}} \), repeated according to their multiplicities. Given \( k \in \mathbb{N} \), if \( \lambda_{s, \text{max/min}, k} > 0 \) for some \( s \), then \( \lambda_{s, \text{max/min}, k} \in O(s) \) as \( s \to \infty \).

(v) There is some \( \theta > 0 \) such that \( \liminf_k \lambda_{s, \text{max/min}, k} k^{-\theta} > 0 \).

For each \( \rho > 0 \), let \( U_{x,\rho} \) be the open ball of center 0 and radius \( \rho \) in \( \mathbb{R}^m \), and let

\[
U_{x,\rho} = B_{\rho} \times \prod_{i=1}^a (N_i \times (0, \rho)) \subset M.
\]

Taking complex coefficients, by Propositions 6.3, 6.6 and 6.8 the following result clearly boils down to the case of Proposition 4.6.
Proposition 7.2. For \( \alpha \in L^2\Omega(M) \), let \( \alpha_t = \exp(itD_{s,\max/min})\alpha \). If \( \text{supp} \alpha \subset \overline{U_{x,a}} \), then \( \text{supp} \alpha_t \subset \overline{U_{x,a+|t|}} \) for all \( t \in \mathbb{R} \).

8. Proof of Theorem 1.1

This theorem follows from Corollary 7.1 [8] with the same arguments as [3] Theorem 1.1, using [3] Propositions 14.2 and 14.3] to globalize the properties of the rel-local model, using the min-max principle (see e.g. [35, Theorem XIII.1]) to show that the properties of the statement are invariant by taking Witten’s perturbation defined by rel-admissible functions, and using Remark 14.2 and 14.3 to produce rel-admissible cut-off functions and partitions of unity with bounded differential; these functions are needed for the Witten’s perturbation and to apply [3] Propositions 14.2 and 14.3.

9. Functional calculus

Let \( M \) be a stratum of a compact stratification, equipped with a good general adapted metric \( g \). Let \( f \) be any rel-admissible function on \( M \), and let \( d_s, \delta_s, D_s \) and \( \Delta_s \) be the corresponding Witten’s perturbations of \( d, \delta, D \) and \( \Delta \). Since \( f \) is rel-admissible, for each \( s, \Delta_s - \Delta \) is a homomorphism with uniformly bounded norm by (107). From (107) and the min-max principle (see e.g. [35, Theorem XIII.1]), it also follows that \( D(\Delta_s,\max/min) = D(\Delta,\max/min) \), \( D^\infty(\Delta_s,\max/min) = D^\infty(\Delta,\max/min) \), and the properties stated in Theorem 1.1 can be extended to the perturbation \( \Delta_s,\max/min \).

For any rapidly decreasing function \( \phi \) on \( \mathbb{R} \), \( \phi(\Delta_s,\max/min) \) is a Hilbert-Schmidt operator on \( L^2\Omega(M) \) by the version of Theorem 1.1 [3] for \( \Delta_s,\max/min \). In fact, \( \phi(\Delta_s,\max/min) \) is a trace class operator because \( \phi \) can be given as the product of two rapidly decreasing functions, \(|\phi|^1/2 \) and \( \text{sign}(\phi)|\phi|^{1/2} \), where \( \text{sign}(\phi)(x) = \text{sign}(\phi(x)) \in \{\pm 1\} \) if \( \phi(x) \neq 0 \).

Like in the case of closed manifolds (see e.g. [36, Chapters 5 and 8]), \( \phi(\Delta_s,\min/max) \) is given by a Schwartz kernel \( K_s \), and \( \text{Tr} \phi(\Delta_s,\min/max) \) equals the integral of the pointwise trace of \( K_s \) on the diagonal. But we do not know whether \( K_s \) is uniformly bounded because a “rel-Sobolev embedding theorem” is missing [3] Section 19]. Theorem 1.1 [3] becomes important in our arguments to make up for this lack.

10. The wave operator

With the notation of Section 9, suppose that \( f \) is a rel-Morse function. Take a general chart \( O \equiv O' \) around each \( x \in \text{Crit}_r(f) \), like in Section 1. Let us add the subindex “\( x \)” to the notation of \( M', N_i, m_\pm \) and \( I_\pm \) in this case. Take a good adapted metric \( g'_x \) on \( M'_x \) of the form used in Section 7. Consider the Witten’s perturbed operators \( d'_{x,s}, \delta'_{x,s}, D'_{x,s} \) and \( \Delta'_{x,s} \) on \( \Omega(M'_x) \) defined by the function \( f' := \frac{1}{2}(\rho^2 - \rho'^2) \) (a prime and the subindex \( x \) is added to their notation). Add also a prime to the notation of the sets \( U_{x,\rho} \) of Section 7 considered in \( M'_x \). Let \( \rho_0 > 0 \) such that \( U_{x,\rho_0} \subset M' \cap O' \). Then, for \( 0 < \rho \leq \rho_0 \), there is some open \( U_{x,\rho} \subset M \) so that \( U_{x,\rho} \equiv U'_{x,\rho} \). Moreover, according to Remark 8 we can assume \( g|_{U_{x,\rho_0}} \equiv g'_{x}|_{U'_{x,\rho_0}} \).

Consider the wave equation
\[
\frac{d\alpha_t}{dt} - iD_s\alpha_t = 0 ,
\]
where \( \alpha_t \in \Omega(M) \) depends smoothly on \( t \). Given any \( \alpha \in D^\infty(\Delta_{s,max/min}) \), its solution with the initial condition \( \alpha_0 = \alpha \) is given by \( \alpha_t = \exp(itD_{s,min/max})\alpha \).

Moreover a usual energy estimate shows the uniqueness of such a solution (see e.g. [36, Proposition 7.4]); in fact, given any \( c > 0 \), such it is also unique for \( |t| \leq c \).

**Proposition 10.1.** Let \( 0 < a < b < \rho_0 \) and \( \alpha \in L^2\Omega(M) \). The following properties hold for \( \alpha_t = \exp(itD_{s,min/max})\alpha \):

(i) If \( \text{supp} \alpha \subset M \setminus U_{x,a} \), then \( \text{supp} \alpha_t \subset M \setminus U_{x,a-|t|} \) for \( 0 < |t| \leq a \).

(ii) If \( \text{supp} \alpha \subset \overline{U}_{x,a} \), then \( \text{supp} \alpha_t \subset U_{x,a+|t|} \) for \( 0 < |t| \leq b - a \).

**Proof.** First, let us prove (ii). We can assume that \( \alpha \in D^\infty(\Delta_{s,min/max}) \) because \( \exp(itD_{s,min/max}) \) is bounded. Since \( \text{supp} \alpha \subset U_{x,a} \), we have \( \alpha|_{U_{x,0}} = \alpha'|_{U_{x,0}} \) for a unique \( \alpha' \in \Omega(M') \) supported in \( U_{x,a} \). We get \( \alpha' \in D^\infty(\Delta_{s,min/max}) \) because \( \alpha \in D^\infty(\Delta_{s,min/max}) \). Let \( \alpha'_t = \exp(i\tau D_{s,min/max})\alpha' \). By Proposition 7.2, we have \( \text{supp} \alpha'_t \subset U_{x,a+|t|} \) for \( 0 < |t| \leq b - a \). Then \( \alpha'_t|_{U_{x,0}} = \beta_t|_{U_{x,0}} \) for a unique \( \beta_t \in \Omega(M) \) supported in \( U_{x,a+|t|} \). Now, \( \beta_t \in D^\infty(\Delta_{s,max/min}) \) because \( \alpha'_t \in D^\infty(\Delta_{s,min/max}) \). Moreover \( \beta_t \) satisfies \( 135 \) for \( |t| \leq b - a \) with initial condition \( \beta_0 = \alpha \). So \( \beta_t = \alpha_t \) by the uniqueness of the solution of \( 135 \), obtaining \( \text{supp} \alpha_t \subset U_{x,a+|t|} \).

Finally, (i) follows from (ii) in the following way. For any \( \beta \in \Omega_0(M) \) with \( \text{supp} \beta \subset U_{x,a-|t|} \), let \( \beta_t = \exp(i\tau D_{s,max/min})\beta \) for \( \tau \in \mathbb{R} \). By (ii), we get \( \text{supp} \beta_{-t} \subset U_{x,a} \), and therefore \( \langle \alpha_t, \beta \rangle = \langle \alpha, \beta_{-t} \rangle = 0 \). This shows that \( \text{supp} \alpha_t \subset M \setminus U_{a-|t|} \).

**Remark 12.** The steps given to achieve Proposition 10.1 are simpler here than in [3]. In fact, it would be difficult to adapt those arguments of [3] since an expression of \( D^\infty(\Delta_{max/min}) \) is missing in Section 4.2.2.

11. **Proof of Theorem 1.2**

This theorem now follows like [3, Theorem 1.2]. Thus the details are omitted.

Consider the notation of Section 10. By [10], the numbers \( \beta_{max/min} \) are also given by the cohomology of \( d_{s,max/min} = d_{max/min} + s\,df \wedge \) on \( D(d_{max/min}) = D(d_{max/min}) \), and we have \( e^{\ell}\,D(d_{max/min}) = D(d_{max/min}) \).

Let \( \phi \) be a smooth rapidly decreasing function on \( \mathbb{R} \) with \( \phi(0) = 1 \). Then \( \phi(\Delta_{s,max/min}) \) is of trace class (Section 11), and let \( \mu_{s,max/min}^r = \text{Tr}(\phi(\Delta_{s,max/min,r})) \).

Then the following result follows formally like [36, Proposition 14.3].

**Proposition 11.1.** We have

\[
\sum_{r=0}^{k} (-1)^{k-r} \beta_{max/min}^r \leq \sum_{r=0}^{k} (-1)^{k-r} \mu_{max/min}^r \quad (0 \leq k < n),
\]

\[
\chi_{max/min} = \sum_{r} (-1)^{r} \mu_{s,max/min}^r.
\]

For \( \rho \leq \rho_0 \), let \( U_{\rho} = \bigcup_x U_{x,\rho} \) with \( x \) running in \( \text{Crit}_{rel}(f) \). Fix some \( \rho_1 > 0 \) such that \( 4\rho_1 < \rho_0 \). Let \( \mathcal{W} \) and \( \mathcal{H} \) be the Hilbert subspaces of \( L^2\Omega(M) \) consisting of forms essentially supported in \( M \setminus U_{\rho_1} \) and \( M \setminus U_{3\rho_1} \), respectively. Since

\[
\Delta_{s,max/min} = \Delta_{max/min} + s\,\text{Hess}f + s^2\,|df|^2
\]
on $D(\Delta_{s,\text{max/min}}) = D(\Delta_{\text{max/min}})$ for all $s \geq 0$ by \eqref{eq:107}, it follows that there is some $C > 0$ so that, if $s$ is large enough,

$$\Delta_{s,\text{max/min}} \geq \Delta_{\text{max/min}} + Cs^2 \quad \text{on} \quad \mathcal{G} \cap D(\Delta_{\text{max/min}}). \quad (136)$$

Let $h$ be a rel-admissible function on $M$ such that $h \leq 0$, $h \equiv 1$ on $U_{\rho_1}$ and $h \equiv 0$ on $M \setminus U_{2\rho_1}$ (Remark \ref{rem:11} \cite{13}). Then $T_{s,\text{max/min}} = \Delta_{s,\text{max/min}} + hCs^2$, with domain $D(\Delta_{\text{max/min}})$, is self-adjoint in $L^2\Omega(M)$ with a discrete spectrum, and moreover

$$T_{s,\text{max/min}} \geq \Delta_{\text{max/min}} + Cs^2 \quad (137)$$

for $s$ is large enough by \eqref{eq:136}.

Take some $\phi \in \mathcal{S}_\nu$ such that $\phi \geq 0$, $\phi(0) = 1$, $\text{supp} \hat{\phi} \subset [-\rho_1, \rho_1]$, and $\phi|_{[0,\infty)}$ is monotone \cite{3} Section 18.2], and write $\phi(x) = \psi(x^2)$ for some $\psi \in \mathcal{S}$. Using Proposition \ref{prop:10.1} \cite{11}, the argument of the first part of the proof of \cite{36} Lemma 14.6 gives the following.

**Lemma 11.2.** $\psi(\Delta_{s,\text{max/min}}) = \psi(T_{s,\text{max/min}})$ on $\mathcal{H}$.

Let $\Pi : L^2\Omega(M) \to \mathcal{H}$ denote the orthogonal projection. According to Section \ref{sec:9} $\psi(\Delta_{s,\text{max/min}})$ is of trace class for all $s \geq 0$. Then the self-adjoint operator $\Pi \psi(\Delta_{s,\text{max/min}})\Pi$ is also of trace class (see e.g. \cite{36} Proposition 8.8]).

**Lemma 11.3.** $\text{Tr}(\Pi \psi(\Delta_{s,\text{max/min}})\Pi) \to 0$ as $s \to \infty$.

**Proof.** This follows like \cite{3} Lemma 18.3], using \eqref{eq:137}, the min-max principle and Lemma \ref{lem:11.2} and expressing the trace as sum of eigenvalues. \hfill $\square$

The following is a direct consequence of Corollary \ref{cor:11} \ref{cor:11}, \ref{cor:11}, \ref{cor:11}, \ref{cor:11}.

**Corollary 11.4.** If $h$ is a bounded measurable function on $\mathbb{R}_+$ such that $h(\rho) \to 1$ as $\rho \to 0$, then $\text{Tr}(h(\rho) \phi(\Delta_{x,s,\text{max/min}})) \to \nu_{x,\text{max/min}}^r$ as $s \to \infty$.

For each $x \in \text{Crit}_{\text{rel}}(f)$, let $\tilde{\mathcal{H}}_x \subset L^2\Omega(M)$ and $\tilde{\mathcal{H}}_x^r \subset L^2\Omega(M_x^r)$ be the Hilbert subspaces of differential forms supported in $U_{x,3\rho_1}$ and $U_{x,3\rho_1}$, respectively. We have $\tilde{\mathcal{H}}_x = \tilde{\mathcal{H}}_x^r$ because $g \equiv g_x^r$ on $U_{x,\rho_0} \equiv U_{x,\rho_0}^r$. Moreover $\Delta_s \equiv \Delta_{x,s}$ on differential forms supported in $U_{x,\rho_0} \equiv U_{x,\rho_0}^r$. By using Proposition \ref{prop:10.1} \cite{11}, the argument of the first part of the proof of \cite{36} Lemma 14.6] can be adapted to show the following.

**Lemma 11.5.** $\phi(\Delta_{s,\text{max/min}}) \equiv \phi(\Delta_{x,s,\text{max/min}})$ on $\tilde{\mathcal{H}}_x \equiv \tilde{\mathcal{H}}_x^r$ for all $x \in \text{Crit}_{\text{rel}}(f)$.

For each $x \in \text{Crit}_{\text{rel}}(f)$, let $\tilde{\Pi}_x : L^2\Omega(M) \to \tilde{\mathcal{H}}_x$ and $\tilde{\Pi}_x^r : L^2\Omega(M_x^r) \to \tilde{\mathcal{H}}_x^r$ denote the orthogonal projections. Since the subspaces $\tilde{\mathcal{H}}_x$ are orthogonal to each other, $\tilde{\Pi} := \sum_x \tilde{\Pi}_x : L^2\Omega(M) \to \tilde{\mathcal{H}} := \sum_x \tilde{\mathcal{H}}_x$ is the orthogonal projection.

**Lemma 11.6.** $\text{Tr}(\tilde{\Pi} \phi(\Delta_{s,\text{max/min}})\tilde{\Pi}) \to \nu_{\text{max/min}}^r$ as $s \to \infty$.

**Proof.** This follows like \cite{3} Lemma 18.3], using Corollary \ref{cor:11} \ref{cor:11} and Lemma \ref{lem:11.5} and, for all $x \in \text{Crit}_{\text{rel}}(f)$, considering $\tilde{\Pi}_x$ as the multiplication operator by the characteristic function of $U_{x,3\rho_1}$.

Since $\Pi + \tilde{\Pi} = 1$, Theorem \ref{thm:1.2} follows from Proposition \ref{prop:10.1} \cite{11} and Lemmas \ref{lem:11.3} and \ref{lem:11.6}.
References

[1] P. Albin, É. Leichtnam, R. Mazzeo, and P. Piazza, *Hodge theory on Cheeger spaces*, J. Reine Angew. Math., to appear, arXiv:1307.5473v2.

[2] ——, *The signature package on Witt spaces*, Ann. Sci. Éc. Norm. Supér. (4) **45** (2012), 241–310. MR 2977620

[3] J.A. Álvarez López and M. Calaza, *Witten’s perturbation on strata*, Asian J. Math., to appear, arXiv:1205.0348.

[4] ——, *Embedding theorems for the Dunkl harmonic oscillator*, SIGMA Symmetry Integrability Geom. Methods Appl. **10** (2014), 004, 16 pages. MR 3210631

[5] ——, *A perturbation of the Dunkl harmonic oscillator on the line*, SIGMA Symmetry Integrability Geom. Methods Appl. **11** (2015), 059, 33 pages. MR 3372950

[6] J.-M. Bismut and W. Zhang, *Milnor and Ray-Singer metrics on the equivariant determinant of a flat vector bundle*, Geom. Funct. Anal. **4** (1994), 136–212. MR 1262703 (96f:58179)

[7] J.P. Brasselet, G. Hector, and M. Saralegi, *Théorème de De Rham pour les variétés stratifiées*, Ann. Global Anal. Geom. **9** (1991), 211–243. MR 1143404 (93g:55009)

[8] ——, *L2-cohomologie des espaces stratifiés*, Manuscripta Math. **76** (1992), 21–32. MR 1171153 (93i:58009)

[9] J. Brüning and M. Lesch, *Kähler-Hodge theory for conformal complex cones*, Geom. Funct. Anal. **3** (1993), 439–473.

[10] ——, *Kähler-Hodge theory for conformal complex cones*, Geom. Funct. Anal. **3** (1993), 439–473.

[11] J.-L. Brylinski, *Equivariant intersection cohomology*, Kazhdan-Lusztig theory and related topics (Chicago, IL, 1989), Contemp. Math., vol. 139, Amer. Math. Soc., Providence, RI, 1992, pp. 5–32. MR 1197827 (94c:55010)

[12] J. Cheeger, *On the Hodge theory of Riemannian pseudomanifolds*, Geometry of the Laplace Operator (Univ. Hawaii, Honolulu, Hawaii, 1979) (Providence, R.I.), Proc. Sympos. Pure Math., vol. XXXVI, Amer. Math. Soc., 1980, pp. 91–146. MR 573430 (83a:58081)

[13] ——, *Spectral geometry of singular Riemannian spaces*, J. Differ. Geom. **18** (1983), 575–657. MR 730920 (85d:58081)

[14] ——, *Intersection homology K"unneth theorems*, Math. Ann. **343** (2009), 371–395. MR 2461258 (2009j:55004)

[15] ——, *An introduction to intersection homology with general perversity functions*, Topology of stratified spaces, Math. Sci. Res. Inst. Publ., vol. 58, Cambridge Univ. Press, Cambridge, 2011, pp. 177–222. MR 2796412 (2012h:55007)

[16] M. Goresky and R. MacPherson, *L2-cohomology and intersection homology of singular varieties*, Seminar on Differential Geometry (Princeton, New Jersey), Ann. Math. Stud., vol. 102, Princeton University Press, 1982, pp. 302–340.

[17] D.C. Cohen, M. Goresky, and L. Ji, *On the K"unneth formula for intersection cohomology*, Trans. Amer. Math. Soc. **333** (1992), 63–69. MR 1052904 (92k:55009)

[18] C. Debord, J.-M. Lescure, and V. Nistor, *Groupoids and an index theorem for conical pseudomodules*, J. Reine Angew. Math. **628** (2009), 1–35.

[19] G. Friedman, *Intersection homology Künneth theorems*, Math. Ann. **343** (2009), 371–395. MR 2461258 (2009j:55004)

[20] ——, *An introduction to intersection homology with general perversity functions*, Topology of stratified spaces, Math. Sci. Res. Inst. Publ., vol. 58, Cambridge Univ. Press, Cambridge, 2011, pp. 177–222. MR 2796412 (2012h:55007)

[21] M. Goresky and R. MacPherson, *Intersection homology theory*, Topology **19** (1980), 135–162.

[22] ——, *Intersection homology II*, Inventiones Math. **71** (1983), 77–129.

[23] ——, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiet, vol. 14, Springer-Verlag, Berlin, Heidelberg, New York, 1988.

[24] B. Helffer and J. Sjöstrand, *Puits multiples en mécanique semi-classique. IV. étude du complexe de Witten*, Comm. Partial Differential Equations **10** (1985), 245–340. MR 780068 (87i:58162)

[25] E. Hunsicker and R. Mazzeo, *Harmonic forms on manifolds with edges*, Int. Math. Res. Not. **2005** (2005), 3229–3272.

[26] T. Kato, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1980 edition. MR 1335452 (96a:47025)

[27] M. Lesch, *Differential operators of Fuchs type, conical singularities, and asymptotic methods*, Teubner-Texte zur Mathematik, vol. 136, B.G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1997.

[28] U. Ludwig, *The geometric complex for algebraic curves with cone-like singularities and admissible Morse functions*, Ann. Inst. Fourier (Grenoble) **60** (2010), 1533–1560.
[27] ______, A proof of stratified Morse inequalities for singular complex algebraic curves using Witten deformation, Ann. Inst. Fourier 61 (2011), 1749–1777.
[28] ______, The Witten complex for singular spaces of dimension 2 with cone-like singularities, Math. Nachrichten 284 (2011), 717–738.
[29] ______, The Witten deformation for even dimensional conformally conic manifolds, Trans. Amer. Math. Soc. 365 (2013), 885–909.
[30] ______, Comparison between two complexes on a singular space, J. Reine Angew. Math., Ahead of Print (2014), DOI 10.1515/crelle-2014-0075.
[31] J.N. Mather, Notes on topological stability, Mimeographed Notes, Harvard University, 1970.
[32] ______, Stratifications and mappings, Dynamical Systems, Academic Press, 1973, pp. 195–232.
[33] N. Nagase, $L^2$-cohomology and intersection cohomology of stratified spaces, Duke Math. J. 50 (1983), 329–368. MR 700144 (84m:58006)
[34] ______, Sheaf theoretic $L^2$-cohomology, Adv. Stud. Pure Math. 8 (1986), 273–279. MR 894298 (88g:58009)
[35] M. Reed and B. Simon, Methods of modern mathematical physics IV: Analysis of operators, Academic Press, New York, 1978.
[36] J. Roe, Elliptic operators, topology and asymptotic methods, second ed., Pitman Research Notes in Mathematics, vol. 395, Addison Wesley Longman Limited, Edinburgh Gate, Harlow, Essex CM20 2JE, England, 1998. MR 1670907 (99m:58182)
[37] M. Saralegi, Homological properties of stratified spaces, Illinois J. Math. 38 (1994), 47–70. MR 1245833 (95a:55011)
[38] ______, de Rham intersection cohomology for general perversities, Illinois J. Math. 49 (2005), 737–758. MR 2210257 (2006k:55013)
[39] B.W. Schulze, The iterative structure of corner operators, arXiv:0905.0977.
[40] G. Szegő, Orthogonal polynomials, fourth ed., Colloquium Publications, vol. 23, Amer. Math. Soc., Providence, RI, 1975. MR 0372517 (51 #8724)
[41] R. Thom, Ensembles et morphismes stratifiés, Bull. Amer. Math. Soc. 75 (1969), 240–284. MR 0239613 (39 #970)
[42] A. Verona, Stratified mappings—structure and triangulability, Lecture Notes in Math., vol. 1102, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1984. MR 771120 (86k:58010)
[43] E. Witten, Supersymmetry and Morse theory, J. Differ. Geom. 17 (1982), 661–692. MR 683171 (84b:58111)
[44] K. Yosida, Functional analysis, sixth ed., Grundlehren der Mathematischen Wissenschaften, vol. 123, Springer-Verlag, Berlin-Heidelberg-New York, 1980. MR 617913 (82j:46002)

Departamento de Xeometría e Topoloxía, Facultade de Matemáticas, Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain
E-mail address: jesus.alvarez@usc.es

Laboratorio de Investigación 2 and Rheumatology Unit, Hospital Clínico Universitario de Santiago, Santiago de Compostela, Spain
E-mail address: manuel.calaza@usc.es