Spherical symmetry in a dark energy permeated spacetime

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Abstract

The properties of a spherically symmetric static spacetime permeated through dark energy are worked out. Dark energy is viewed as the strain energy of an elastically deformable four-dimensional manifold. The metric is worked out in the vacuum region around a central spherical mass/defect in the linear approximation. We discuss analogies and differences with the analogue in the de Sitter spacetime and how these competing scenarios could be differentiated on an observational ground. The comparison with the tests at the solar system scale puts upper limits to the parameters of the theory, consistent with the values obtained applying the classical cosmological tests.

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1. Introduction

The general relativity (GR) theory has given spacetime a physical status which makes it one of the basic ingredients of the universe, being the other matter/energy; however, usually the nature of spacetime is not really given much attention. Till now, in the unsuccessful attempts to quantize the gravitational field, spacetime is in practice conceived as a field, much like for the other interactions and for matter. Despite the enormous efforts devoted on the quantum gravity front [1–3], both in the string theory and loop quantum gravity approaches, and notwithstanding the undoubtable progress and hindsights obtained with the mathematical machinery of those theories, the main questions still resist answers that can be both globally consistent and unambiguously verifiable.

On the other hand, while quantum gravity tries to solve fundamental problems at the smallest scales and the highest energies, a problem also exists at large scales where classical approaches are in order. Observation [4, 5] has forced people to hypothetically introduce in the universe entities that have scarce or no reference to the matter/energy we know by experiment at intermediate or small scales. We apparently need dark matter and dark energy [6], and, especially for the second, when trying to work out its properties and to build some physical
interpretation of its nature, people are led to results which, to say the least, are far away from our intuition and experience.

Another approach consists in trying to modify the general theory of relativity [7–9], outside and beyond the simplicity criteria that, despite the mathematical complexity, guided its development. Both the dark-something and the modified GR theories are in a sense ad hoc prescriptions. Preserving an internal consistency requirement, the theories of Lagrangians for the universe look apt to yield equations reproducing or mimicking what we observe.

The approach we have already followed in previous works [10] consists in treating spacetime as a classical four-dimensional continuum behaving as three-dimensional material continua [11, 12]. An appropriate name for the theory worked out in this way is strained state theory (SST) since the new features it introduces are contained in the strain tensor expressing the difference between a flat undifferentiated four-dimensional Euclidean manifold and the actual spacetime with its curvature, originated from matter/energy distributions as well as from texture defects in the manifold as such. In a sense, SST is a theory of the dark energy where the latter is a vacuum deformation energy present when the spacetime manifold is curved.

Here, we shall discuss the behavior of such a strained spacetime when some external cause (be it a mass or a defect) induces a spherical symmetry in space. In a sense, we will treat the analogue of the Schwarzschild problem in a dark energy permeated environment.

As it will result, the presence of the strain energy appears at the cosmic scale, without affecting in a sensible way the physics at the scale of the solar system. In any case, the data from the solar system will constrain the value of the parameters of the theory. Since the solution of the problem will be attained by an approximation method, the asymptotic region, where the effect of strain would be dominant, will be excluded from our description.

2. The strained state of spacetime

The essence of the SST is in the idea that spacetime is a four-dimensional manifold endowed with physical properties similar to the ones we know for deformable three-dimensional material continua. In practice, we may think that our spacetime, which we shall call the natural manifold, is obtained from a flat four-dimensional Euclidean manifold, which will be our reference manifold. The deformation, i.e. the curvature, of spacetime is due to the presence of matter fields as in GR or to the presence of texture defects in the manifold; however, here we assume that spacetime resists to deformation more or less as ordinary material continua do. In practice, according to this approach, we introduce in the Lagrangian density of spacetime, besides the traditional Einstein–Hilbert term, an ‘elastic potential term’ built on the strain tensor in the same way as for the classical elasticity theory. The additional term in a sense accounts for the presence of a dark energy or even ‘curvature fluid’ [13]. The bases of SST are described in [10]; here, we review the essential.

The complete action integral of the theory is

$$S = \int \left( R + \frac{1}{2} (\lambda \varepsilon^2 + 2 \mu \varepsilon_{\mu \nu} \varepsilon^{\mu \nu} ) + \mathcal{L}_{\text{matter}} \right) \sqrt{-g} \, d^4x. \quad (1)$$

Of course, $R$ is the scalar curvature of the manifold; the parameters $\lambda$ and $\mu$ are the Lamé coefficients of spacetime, $\varepsilon_{\mu \nu}$ is the strain tensor of the natural manifold and $\varepsilon = \varepsilon^\alpha_\alpha$; $\mathcal{L}_{\text{matter}}$ is the Lagrangian density of matter/energy. The strain tensor is obtained by the comparison of two corresponding line elements, one in the natural frame and the other in the reference frame. By definition, it is

$$\varepsilon_{\mu \nu} = \frac{1}{2} (g_{\mu \nu} - E_{\mu \nu}). \quad (2)$$
where $g_{\mu\nu}$ is the metric tensor of the natural manifold and $E_{\mu\nu}$ is the Euclidean metric tensor of the reference frame.

Action (1) has already been used both in [10] and [14] in order to describe the accelerated expansion of the universe, and has given positive results when tested against four typical cosmological tests [14].

3. Spherical symmetry in space.

Now we focus on a stationary physical system endowed with spherical symmetry in space. Of course, there must be a physical reason for the symmetry to be there, which means that ‘something’ must exist in the central region of the spacetime we are considering. This can be either a time-independent spherical aggregate of mass/energy or a line defect\(^4\). The general form of the line element of a spacetime with the given symmetry is well known:

$$ds^2 = f d\tau^2 - h dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (3)$$

where $f$ and $h$ are the functions of $r$ only, and Schwarzschild coordinates have been used.

The corresponding line element in the flat Euclidean reference frame will be

$$ds_r^2 = d\tau^2 + \left(\frac{dw}{dr}\right)^2 dr^2 + w^2 (d\theta^2 + \sin^2 \theta d\phi^2). \quad (4)$$

In principle, we have four degrees of freedom (together with the flatness condition) in the choice of the coordinates on the reference manifold; however, when we decide to evidence the same symmetry as the one present in the natural frame, the gauge functions in practice reduce to 1. This is the meaning of the $w$ function, only depending on $r$, in equation (4). Figure 1 pictorially clarifies the role of the gauge function.

By the direct inspection of formulae (3) and (4) and using definition (2), we can easily read out the non-zero elements of the strain tensor for this physical configuration:

$$\varepsilon_{00} = \frac{f - 1}{2}, \quad (5)$$

\(^4\) Line defect refers to the full four-dimensional spacetime and the line will be timelike, so that in space the defect will appear to be pointlike.
\[ \varepsilon_{rr} = -\frac{h + w^2}{2}, \quad (6) \]
\[ \varepsilon_{\theta\theta} = -\frac{r^2 + w^2}{2}, \quad (7) \]
\[ \varepsilon_{\phi\phi} = -\frac{r^2 + w^2}{2} \sin^2 \theta. \quad (8) \]

From now on, primes will denote derivatives with respect to \( r \).

Once we have the strain tensor, we are able to write the contribution to the Lagrangian density of spacetime due to the strain present in the natural manifold. The needed ingredients are

\[ \varepsilon = g^{\alpha\beta} \varepsilon_{\alpha\beta} = f - 1 + \frac{h + w^2}{2h} + \frac{r^2 + w^2}{r^2} \quad (9) \]
and

\[ \varepsilon_{\alpha\beta} \varepsilon^{\alpha\beta} = g^{\mu\nu} g_{\alpha\beta} \varepsilon^{\alpha\beta} = \frac{(f - 1)^2}{4f^2} + \frac{(h + w^2)^2}{4h^2} + \frac{(r^2 + w^2)^2}{2r^4}. \quad (10) \]

For completeness, let us recall that it is

\[ R = -\left( \frac{2}{r^2} - \frac{2}{hr^2} - \frac{f''}{f h} + \frac{f'^2}{2f^2 h} + \frac{1}{2fh^2} f' h' - \frac{2}{fh} f + \frac{2h'}{h^2 r} \right) \quad (11) \]
and

\[ \sqrt{-g} = \sqrt{fh r^2} \sin \theta. \quad (12) \]

Going back to equation (1), we are now able to write the full explicit Lagrangian density of our strained spacetime, with the built in Schwarzschild symmetry. We are interested in empty spacetime so that in the region we shall be considering that it will be \( \mathcal{L}_{\text{matter}} = 0 \).

From the Lagrangian density, applying the usual variational procedure, we can obtain the Euler–Lagrange equations for the \( f \), \( h \) and \( w \) functions.

The effective Lagrangian density (modulo a \( \sin \theta \)) is

\[ \mathcal{L} = -\left( \frac{2}{r^2} - \frac{2}{hr^2} + \frac{2h'}{h^2 r} \right) \sqrt{fh r^2} + \lambda \left( \frac{f - 1}{2f^2} + \frac{h + w^2}{2h} + \frac{r^2 + w^2}{r^2} \right)^2 \sqrt{fh r^2} \]
\[ + \mu \left( \frac{(f - 1)^2}{4f^2} + \frac{(h + w^2)^2}{4h^2} + \frac{(r^2 + w^2)^2}{2r^4} \right) \sqrt{fh r^2}. \quad (13) \]

The second derivative appearing in (11) has been eliminated by means of an integration by parts.

The \( w \) function is treated as \( f \) and \( h \), which means that we assume that it has to satisfy Hamilton’s principle just as the others do. The reason for this choice is that we are representing the correspondence between the natural and the reference manifolds as being established by an actual physical deformation process, which is something else from the obvious freedom in the choice of the coordinates. The three explicit final equations are

\[ 0 = h - 1 + \frac{r h'}{h} + \frac{1}{16f^2 h} \lambda r^2 \left( 2f h \frac{w^2}{r^2} + 4fh + 3h + fw^2 \right) \times \left( h - 4fh - 2fh \frac{w^2}{r^2} - fw^2 \right) \]
\[ - \frac{1}{8fh^2} \mu r^2 \left( 2fh^2 + 4f^2 h^2 + 2f^2 h^2 \frac{w^4}{r^4} - 3h^2 + f^2 w^4 + 4f^2 h^2 \frac{w^2}{r^2} + 2f^2 w^2 \right). \quad (14) \]
As is immediately seen, the three equations are highly nonlinear, first-order differential in $f$ and $h$, and second-order differential in $w$. Solving them exactly is apparently a desperate task, but we shall see that it is possible to proceed perturbatively.

4. Approximate solutions

In equations (14) and (15), we see that there are a number of terms multiplying either the $\lambda$ or the $\mu$ parameter, while others do not. From the application of the theory to the cosmic expansion, we know that the values of $\lambda$ and $\mu$ are indeed very small [10, 14]; the dimension of the parameters is the inverse of the square of a length, so we may say that for small distances with respect to some typical radius $\tilde{r}$, the products $\lambda r^2$ and $\mu r^2$ will be much smaller than 1. The typical $\tilde{r}$ is $\sim 10^{26}$ m $\sim 10^4$ Mpc [10, 14].

We are then led to solve the equations by successive approximations. Our first step in the approximation process will be to neglect the terms multiplying $\lambda$ and $\mu$ so that the zero-order equations become

\begin{align*}
0 &= h_0 - 1 - \frac{1}{r} f' - \frac{1}{16h f^2} \lambda r^2 \left( h - 4 f h - 2 f h \frac{w^2}{r^2} + 3 f w^2 \right) \times \left( h - 4 f h - 2 f h \frac{w^2}{r^2} - f w^2 \right) \\
&\quad - \frac{1}{8hf^2} \mu r^2 \left( h^2 + 4 f^2 h^2 + 2 f^2 h^2 \frac{w^4}{r^4} - 2 f h^2 - 3 f^2 w^4 \\
&\quad + 4 f^2 h^2 \frac{w^2}{r^2} - 2 f^2 h w^2 \right) \times \left( h - 4 f h - 2 f h \frac{w^2}{r^2} - f w^2 \right), \quad (15) \\
0 &= \frac{\lambda}{2f h^2} w'' (h r^2 - 3 f r^2 w^2 - 4 f hr^2 - 2 f h w^2) - \frac{\lambda}{h} w w'' - \frac{\lambda}{4h} \left( f' - \frac{3h'}{4h} + 1 \right) w'^3 \\
&\quad + \lambda \frac{w'}{h} \left( - \frac{1}{2} w'' - r^2 - \frac{1}{4f} r^2 \right) f' + \left( r^2 + \frac{1}{2} w^2 - \frac{1}{4f} r^2 \right) \frac{h'}{h} + \frac{1}{r} f - 4r \\
&\quad + \lambda w \left( 4 + \frac{2}{r^2} w^2 - \frac{1}{f} \right) + \mu \frac{r^2}{h^2} w'' (-3w'^2 - h) \\
&\quad - \frac{\mu}{h^2} \left( 2r - \frac{3}{2h} r h' + \frac{1}{2f} r^2 f' \right) w'^3 \\
&\quad + \mu r \frac{f'}{h} \left( \frac{h'}{2h} r - 2 - \frac{f'}{2f} r \right) w' + 2w \mu \left( 1 + \frac{w^2}{r^2} \right) . \quad (16)
\end{align*}

As is immediately seen, the three equations are highly nonlinear, first-order differential in $f$ and $h$, and second-order differential in $w$. Solving them exactly is apparently a desperate task, but we shall see that it is possible to proceed perturbatively.

5 For simplicity, we assume that $\lambda$ and $\mu$ are of the same order of magnitude.
In practice, we can write that the solutions of equations (14), (15) and (16) are of the types
\[
\begin{align*}
    f &= f_0 + \phi \\
    h &= h_0 + \chi \\
    w &= lr(1 + \psi),
\end{align*}
\]
with \( \phi, \chi, \psi \ll 1 \). Up to this moment, we have not said anything about the relative size of \( m/r \) with respect to the \( \lambda r^2 \) or \( \mu r^2 \) terms, inside the fiducial radius \( \tilde{r} \). We know however that, outside any Schwarzschild horizon, it is \( m/r < 1 \), so that any \( m\lambda r \) or \( m\mu r \) term will be smaller than the \( \lambda r^2 \) and \( \mu r^2 \) terms. On these bases, we conclude that at the lowest approximation order, \( \phi, \chi \) and \( \psi \) are the functions of \( \lambda r^2 \) and \( \mu r^2 \).

The adimensional scale factor \( l \) would be arbitrary in a trivial flat spacetime, but this is not the case here.

Introducing the developments (21) into (14) and (15) and keeping the terms up to the first order in \( \lambda r^2 \) and \( \mu r^2 \), we see that only \( w = lr \) plays a role, so that we do not need to worry about the unknown function \( \psi \). In any case, the functional form of \( w \) is determined by requiring that in the absence of elastic deformation the reference metric be Euclidean, which suggests that \( \psi \) in equation (21) must go to zero for \( \lambda = \mu = 0 \). We nevertheless explored the possibility that a different ansatz for \( w \) could bring a new set of solutions; we considered as functional forms for \( w \) either Maclaurin or Taylor expansions in (inverse) powers of \( r \), and we found that higher order terms in the expansion must zero out. The linear \( r \) term considered in equation (21) is then the only relevant one.

Finally, we obtain
\[
\begin{align*}
    \phi &= \Phi r^2, \\
    \chi &= \Psi r^2.
\end{align*}
\]
The explicit expressions of the \( \Phi \) and \( \Psi \) parameters are
\[
\begin{align*}
    \Phi &= \frac{\lambda}{16} (3l^4 + 2l^2 - 1) + \frac{\mu}{8} (l^4 - 1), \\
    \Psi &= \frac{\lambda}{16} (3l^4 + 10l^2 + 7) + \frac{\mu}{8} (l^2 + 1)^2.
\end{align*}
\]
The result does indeed depend on the value of \( l \); different values correspond to different situations. We shall comment on this in a while. In any case, it is \( \Phi \neq \Psi \), unless \( \mu = -2\lambda \).

We could also have started from pure flat spacetime as the zero-order approximation, but at the end we would have found again the same solution, i.e. Schwarzschild plus (22) and (23).

5. The metric tensor

Explicitly writing the results found in the previous section, we see that we have different regions with specific approximate forms for the line element. Cosmological constraints suggest that \( \lambda \sim \mu \sim 10^{-52}\text{m}^{-2} \). Then, for masses as large as those of galaxies or clusters of galaxies, we can distinguish the following three regimes. An internal region, where \( 1 \gg m/r \gg \lambda r^2, \mu r^2 \):
\[
    ds^2 \simeq \left(1 - 2\frac{m}{r} + \Phi r^2\right) d\tau^2 - \left(\frac{1}{1 - \frac{2m}{r}} + \Psi r^2\right) dr^2 - r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2).
\]


An intermediate region, where $1 \gg \frac{m}{r} \sim \lambda r^2, \mu r^2$

\begin{equation}
\text{d}s^2 \simeq \left(1 - 2 \frac{m}{r} + \Phi r^2\right) \text{d}\tau^2 - \left(1 + 2 \frac{m}{r} + \Psi r^2\right) \text{d}r^2 - r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2).
\tag{27}
\end{equation}

An outer region, where $r < \tilde{r}$ but $1 \gg \lambda r^2, \mu r^2 \gg \frac{m}{r}$:

\begin{equation}
\text{d}s^2 \simeq (1 + \Phi r^2) \text{d}\tau^2 - (1 + \Psi r^2) \text{d}r^2 - r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2).
\tag{28}
\end{equation}

Our approximate solutions are unfit to describe the asymptotic region where $\lambda r^2, \mu r^2 \sim \lambda r^2$ or bigger. This is the cosmological domain, and the problem is open for embedding in a given cosmic background spacetime.

The internal metric has vanishing values of $g_{00}$ for

\begin{equation}
r_{00} \simeq \sqrt{\frac{m^2}{\Phi \sigma} + \frac{1}{27 \Phi^3} + \frac{m}{\Phi} - \frac{1}{\sqrt{\Phi \sigma + \frac{1}{27 \Phi^3} + \frac{m}{\Phi}}},
\tag{29}
\end{equation}

whose limit correctly goes to $2m$ when $\Phi \to 0$.

Equation (28) holds also in the case of a defect without mass. In that case, the scalar curvature in the inner region, to the first order in $\lambda r^2, \mu r^2$, is

\begin{equation}
R \simeq 6 (\Psi - \Phi).
\tag{30}
\end{equation}

Explicitly it is

\begin{equation}
R \simeq 3 (1 + I^2) (\lambda + \frac{1}{2} \mu).
\tag{31}
\end{equation}

The curvature is a scalar quantity, independent from the coordinates. As we see, the result depends on $l$ so that we are forced to attach a physical meaning to that parameter. Since we are now treating a mass-free situation, we are led to conclude that some defect is present in the origin and its relevance is quantitatively expressed by the value of $l$. Another remark is that the curvature in the origin, even in the absence of mass, is never zero, if we only allow for real values of $l$: the initial Euclidean reference frame can be brought to locally coincide with a Minkowskian tangent space only for imaginary values of $l$, in which case actually the initial frame would have been Minkowskian.

6. Perihelion precession

Precessions of the perihelia of the solar system planets have provided stringent local tests for competing theories of gravity [15–17]. A metric deviation of the form $\delta g_{00} \simeq \Phi r^2$ from the standard result obtained in GR induces a precession angle after one orbital period of

\begin{equation}
\Delta \phi \simeq 3 \pi \Phi \frac{r_g^3}{r_s} (1 - e^2)^{1/2},
\tag{32}
\end{equation}

where $\Delta \phi$ is in radians; $s$ and $e$ are the semi-major axis and the eccentricity of the unperturbed orbit, respectively, and $r_g = GM/c^2$ is the gravitational radius of the central body.

Data from space flights and modern astrometric methods make it possible to create very accurate planetary ephemerides and to precisely determine orbital elements of the solar system planets [18, 20]. Results are compatible with GR predictions, so that any effect induced by modifications of the gravity law may be to the larger extent of the order of the statistical uncertainty in the measurement of the precession angle. Here, we consider the planetary ephemerides in [20].

The accurate measurement of the Saturn perihelion shift provides the tighter bound on $\Phi$ from solar system tests, $\Phi \lesssim 0.5 \times 10^{-44} \text{ m}^{-2}$, see table 1. Local tests on perihelion precession
Table 1. Limits on $\Phi$ due to extra-precession of the inner planets of the solar system. Extra-precession values $\delta \dot{\omega}$ are from [20].

| Name     | $\delta \dot{\omega}$ (mas century$^{-1}$) | $\Phi$ (m$^{-2}$) |
|-----------|-------------------------------------------|-------------------|
| Mercury   | 0.6                                       | $\lesssim 0.6 \times 10^{-42}$ |
| Venus     | 1.5                                       | $\lesssim 0.6 \times 10^{-42}$ |
| Earth     | 0.9                                       | $\lesssim 0.2 \times 10^{-42}$ |
| Mars      | 0.15                                      | $\lesssim 0.2 \times 10^{-43}$ |
| Jupiter   | 42                                        | $\lesssim 0.8 \times 10^{-42}$ |
| Saturn    | 0.65                                      | $\lesssim 0.5 \times 10^{-44}$ |

put bounds on $\Phi$, whereas cosmological observations constrain a different combination of parameters of the SST theory, the $B\equiv (\mu/4)(2\lambda + \mu)/(\lambda + 2\mu)$ parameter in [14]. Local bounds are anyway seven orders of magnitude less constraining than cosmological tests. Other solar or stellar system tests can probe gravitational theories, but they are usually less constraining than the results from measurements of the precession angle of the planets in the inner solar system [19].

7. Radial acceleration

Another interesting quantity is the radial acceleration of an observer instantaneously at rest. Now we refer to the geodetic equations deducible from line element (26). Being interested to a pure radial fall, we put $d\theta/ds = d\phi/ds = 0$; the remaining pair of equations is

$$\frac{d^2 \tau}{ds^2} + 2 \left( r\Phi + \frac{m}{r} (r - 2m) \right) \frac{d\tau}{ds} \frac{dr}{ds} \simeq 0$$

$$\frac{d^2 r}{ds^2} + \left( r\Phi + \frac{m}{r^3} (r - 2m) \right) \left( \frac{d\tau}{ds} \right)^2 + \left( r\Psi - \frac{m}{r} (r - 2m) \right) \left( \frac{dr}{ds} \right)^2 \simeq 0.$$  (33)

For a momentarily fixed position, it is also $dr/ds = 0$, so that the equations become

$$\frac{d^2 \tau}{ds^2} \simeq 0$$

$$\frac{d^2 r}{ds^2} + \left( r\Phi + \frac{m}{r^3} (r - 2m) \right) \left( \frac{d\tau}{ds} \right)^2 \simeq 0.$$  (34)

Let us evaluate the proper radial acceleration; we see that

$$\frac{d^2 r}{dr^2} \simeq -\frac{m}{r^2} \left( 1 - 2 \frac{m}{r} \right) - r\Phi.$$  (35)

The strained state of spacetime adds a contribution to the Newtonian and post-Newtonian acceleration strengthening (weakening) the force of gravity for a positive (negative) value of $\Phi$.

An additional term in the form of equation (35) causes a change in Kepler’s third law. Because of $\Phi$, the radial motion of a test body around a central mass $M$ is affected by an additional acceleration which perturbs the mean motion. For a radial acceleration in the form of $\Phi r$ perturbing an otherwise Newtonian orbit, the mean motion $n = \sqrt{GM/s^3}$ is changed by [19]

$$\frac{\delta n}{n} = -\Phi \frac{s^3}{rg}.$$  (36)
In principle, the variation of the effective gravitational force felt by the solar system inner planets with respect to the effective forces felt by outer planets could probe new physics. However, observational uncertainties on the mean motion, i.e. on the measured semi-major axis of the solar system planets, are quite large [18]. The tighter constraint comes from the Earth orbit, whose orbital axis is determined with an accuracy of $\delta s = 0.15 \text{ m}$ [18]. This provides an upper bound to $\Phi$ of the order of $\lesssim 0.2 \times 10^{-40} \text{ m}^{-2}$.

8. Matching with the Robertson–Walker (RW) metric

Up to now, we only required the metric to be spherically symmetric. The homogeneous and isotropic spacetime is then a particular case of our local analysis. This highly symmetric case is obtained by considering a manifold without a central mass, i.e. $m = 0$, and with just a central defect that can force the spacetime to be homogeneous too. This condition can fix the size $l$ of the defect. It can be then interesting to compare with the exact solutions obtained with RW coordinates in the cosmological case. With our new result being local, we have to consider the RW metric at the present time. The today’s value of the curvature is

$$R_{\text{RW}} = 12B \left( 1 - \frac{1}{a_0^2} \right),$$

(37)

where $a_0$ is the present value of the scale factor and $B \equiv (\mu/4)(2\lambda + \mu)/(\lambda + 2\mu)$. We can then look for the size $l$ of the central defect such that the resulting spacetime is isotropic and homogeneous at the same time by requiring that the local value of the curvature is equal to the value in the RW metric. We obtain

$$l \simeq \frac{1}{a_0} \sqrt{\frac{2\mu - a_0^2(\lambda + 4\mu)}{\lambda + 2\mu}}.$$

(38)

In [10], cosmological expansion was explained as a consequence of a defect in an elastic medium. The above result describes the today expansion factor in terms of the local size of the defect.

9. Comparison with massive gravity

The SST theory looks very similar to the classical massive gravity theory initially proposed by Fierz and Pauli (FP) [21]. Indeed, at first sight our Lagrangian corresponds to the FP one; if the similarity were an actual coincidence, we would have to face the same kind of inconveniences which are known to plague massive gravity. These are essentially the so-called van Dam–Veltman–Zakharov (vDVZ) discontinuity [22, 23] and the presence of ghosts appearing to various orders. In another work [24], one of us already had considered the problem and the remark had been that the FP theory is based on a first-order perturbative treatment on a flat Minkowskian background; this is not the case of the SST which is ‘exact’ and does not assume that the elements of the strain tensor are small. However, the interest in massive gravity has stimulated a vast effort to formulate a theory valid to all orders and free from the mentioned troubles; a good review of the progress along with the mentioned search can be read in [25] and we will refer to it for further considerations. Again when considering the nonlinear version of massive gravity we find a Lagrangian which apparently corresponds to the one of the SSTs; however, as we shall see in a moment, the two Lagrangians are different. In fact, nonlinear massive gravity can be seen as a four-dimensional bi-metric theory [25]. One metric is dynamical, whereas the second is not coupled to the actual universe and is formally frozen, i.e. it describes a non-dynamical Einstein space background [26]. The non-dynamical
metric is used to raise and lower the indices of the $h_{\mu\nu}$ tensor which is the equivalent of our strain tensor [25] or is combined with the full $g_{\mu\nu}$ to produce the scalars needed for the potential in [26].

In the SST theory, there is just one metric, $g_{\mu\nu}$, which is used for all tasks pertaining to a metric tensor. Our $E_{\mu\nu}$ tensor appearing in equation (2) is indeed described as the metric tensor of the flat reference frame but is not any metric at all for the natural frame. The only existing frame is the natural one; the reference frame belongs to a logically preceding phase in a descriptive paradigm where the present spacetime is obtained as a deformation of some previous undeformed flat state, but the previous stage does not exist or coexist with the natural frame. $E_{\mu\nu}$ is not used to raise or lower any index; rather the full metric $g_{\mu\nu}$ is used to raise and lower all indices including those of $E_{\mu\nu}$, which is a symmetric tensor in the natural manifold. Often we find in the literature also the claim that in massive gravity theories general coordinate transformation invariance is broken by the ‘massive’ term (see for instance [27]) and various devices are needed in order to restore it; this is not the case of SST, since in our theory all objects are true tensors. The $E_{\mu\nu}$ tensor does not even coincide with the metric of the local tangent space, which is Minkowski and position dependent. As a matter of fact, results in the SST theory can equally be well obtained starting from an Euclidean or a Minkowskian reference, which again indicates that the natural metric is the only relevant one.

The difference we have pointed out tells us that there is no obvious affection of the SST by the same difficulties affecting the classical massive gravity theories. By the way, the vDVZ discontinuity is indeed absent in the cosmological application of SST as well as in the case studied here, where the solutions go smoothly to GR when one lets $\lambda$ and $\mu$ go to zero. One further comment about ghosts is in order. The whole discussion of ghosts implies a field theoretical approach to gravity and/or the study of propagating perturbations. As for the former, we know that gravity cannot be described as a spin-2 field on a flat background, furthermore, one cannot even say that the graviton exists, so we continue to use the expression ‘mass of the graviton’ as a sort of abbreviation for something else. Once one analyzes the perturbations, the problem of negative kinetic energy is discussed order by order, but the conclusions that one can draw summing to all orders is not well defined. Various tricks have been devised in order to get rid of ghosts up to a predefined order (e.g. the fourth or the fifth [28]). Here, we do not enter into the discussion and simply stress that: (a) as seen above, we have just one metric, which is a properly defined metric; (b) SST is not based on a peculiar perturbative development. When taken globally, the problems of SST, if any, are shared with the cosmological constant model of spacetime.

Actions in either of the two theories could be formally identified if we lower and raise indices with the full metric rather than the frozen metric in nonlinear massive gravity. Then, in case the full metric and the full determinant can be expanded in powers of the deviation, we can re-organize the terms in the potential and show that the two approaches would carry the same information [25]. However, this analogy has been probed only with this perturbative approach and we have a direct correspondence to first order only. The SST theory is intrinsically nonlinear. Just as an example, the expansion technique cannot be applied in the cosmological case that was exactly analyzed in [10]. We then cannot conclude that the SST theory suffers from the same pathologies as the standard massive gravity.

The comparison of what is known in the spherically symmetric case further shows how known problems affecting massive gravity do not automatically apply to SST. Usual problems in the standard massive gravity have been discussed expanding the equations around the flat solution in terms of small functions. An alternative expansion in the squared mass, which would mimic the expansion technique used in this paper for the SST theory, might hopefully show a smooth limit without discontinuity. Some recent analytic solutions in nonlinear massive
gravity [29] have shown a branch of exact solutions which corresponds to Schwarzschild–de Sitter spacetimes where the curvature scale of de Sitter space is proportional to the squared mass of the graviton. This is similar to the results found in this paper for the SST theory. Even if these arguments are not conclusive they are nevertheless encouraging.

10. Conclusions

We have found the approximate configuration of the spacetime surrounding a spherical mass distribution or a texture defect independent from time, assuming that a dark energy given by the strain of the manifold is present. As expected, we see that the strain of spacetime contributes ‘locally’ extremely tiny corrections to the Schwarzschild solution. These corrections lead to a slight displacement of the horizon in the inner region and to changes of the precession rates of the periapsis of orbiting celestial bodies as well as of the proper radial acceleration.

The comparison of the expected corrections with the data known in the solar system puts upper bounds to the parameters of the theory which are fully consistent with the results found applying the strained state theory (SST) to the universe as a whole. Summing up, the SST, while giving a physical interpretation to the dark energy in vacuo, accounts for the accelerated expansion of the universe and passes other relevant cosmological tests [14]; locally it leads to effects that become visible at the scale of galaxy clusters or bigger.

Our results also show differences between the local predictions of the SST theory versus the standard interpretation of dark energy as a cosmological constant. In particular, we found that in the SST $g_{00} \neq -g_{rr}^{-1}$, which is a main difference with the de Sitter metric and implies that the two competing theories are not degenerate and might be distinguished with very accurate data.

The additional term $\Phi$ to the metric element $g_{00}$ influences the gravitational potential, whereas $\Psi$ contributes to the space curvature perturbation. $\Phi$ directly affects the Poisson equation and determines the modified growth of structure with respect to GR. $\Psi$ together with $\Phi$ influences the null geodesics of light and might be constrained with gravitational lensing measurements.

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