METHODS FOR PARAMETRIZING VARIETIES OF LIE ALGEBRAS

Abstract. A real n-dimensional anticommutative nonassociative algebra is represented by an element of $\wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. For each $\mu \in \wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$, there is a unique subset $\Lambda \subseteq \{(i, j, k) : 1 \leq i < j \leq n, 1 \leq k \leq n\}$ so that the structure constant $\mu_{ij}^k$ with $i < j$ is nonzero if and only if $(i, j, k) \in \Lambda$. The set of all $\mu \in \wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$ with subset $\Lambda$ is denoted $S_{\Lambda}(\mathbb{R})$; these sets stratify $\wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$. We describe how to smoothly parametrize the Lie algebras in $S_{\Lambda}(\mathbb{R})$ up to isomorphism, for $\Lambda$ satisfying certain frequently seen hypotheses.

Keywords: varieties of Lie algebras, nilpotent Lie algebra

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1. Introduction

In many areas of mathematics, it is natural to subdivide an object into smaller, simpler pieces and to parametrize each piece in some useful, controlled way ([Yom15]). A semi-algebraic set is a subset of $\mathbb{R}^n$ defined by a finite number of polynomial equations and inequalities, along with the operations of intersection and union. Every closed and bounded semi-algebraic set is semi-algebraically triangulable ([BR90]). A parametrization of semi-algebraic subset $A$ of $\mathbb{R}^n$ is a collection of semi-algebraic subsets $A_j$ that cover $A$, along with surjective charts $\varphi_j : I^n_j \to A_j$, where $I^n_j$ is a cube in $\mathbb{R}^n$, such that each chart $\varphi_j$ is algebraic and is a homeomorphism from the interior of $I^n_j$ to the interior of $A_j$.

Here we are interested in parametrizing varieties of Lie algebras. These sets are not semi-algebraic; rather, they are quotients of semi-algebraic sets by the action of a Lie group. Let $A$ be a semi-algebraic subset of $\mathbb{R}^n$ and let $A/\sim$ be the quotient of $A$ under the action of a group. Let $\{A_j\}$ be a collection of invariant semi-algebraic subsets that...
cover $A$. A parametrization of $A/\sim$ is the collection $\{A_j/\sim\}$ of semi-algebraic quotients, along with charts $\varphi_j : I^{n_j} \to A_j/\sim$, such that each chart $\varphi_j$ is algebraic and is a homeomorphism from the interior of $I^{n_j}$ to the interior of $A_j/\sim$. Our goal is to find explicit parametrizations of specific subsets of the variety $\tilde{N}_n(\mathbb{R})$ of real nilpotent Lie algebras of dimension $n$.

Real nilpotent Lie algebras of dimension 7 and lower have been classified ([Mor58, See93, Gon98]). For $n \leq 6$, $\tilde{N}_n(\mathbb{R})$ is discrete. The space $\tilde{N}_7(\mathbb{R})$ is the union of isolated points and one-parameter families. In dimension eight and higher, nilpotent Lie algebras have not been classified, and there are many components of $\tilde{N}_n(\mathbb{R})$ of higher dimension ([ABGMVC96]). Furthermore, there are large families of “characteristically nilpotent” Lie algebras ([Hak91]; these do not admit nontrivial semisimple derivations. Due to these complications, in dimension eight and higher, instead of analyzing all of $\tilde{N}_n(\mathbb{R})$, it is natural to focus on a tractable subset $\mathcal{R}$ of $\tilde{N}_n(\mathbb{R})$. For example, $\mathcal{R}$ might be

- the set of two-step nilpotent Lie algebras
- the set of filiform or quasi-filiform Lie algebras
- the set of $\mathbb{N}$-graded or naturally graded nilpotent Lie algebras
- a set of nilpotent Lie algebras that admit a special kind of structure (affine, symplectic, contact, almost-complex, Kähler, etc.).

Or, $\mathcal{R}$ might be defined as the intersection of two or more such sets.

The subclass of filiform Lie algebras is not discrete in dimensions seven and higher and has been the setting for many kinds of classification problems. (See, for example, [Ver66, Mil04, Bur06, Arr11].) The classification of complex two-step nilpotent Lie algebras of dimension 9 and lower was completed in [GT99], and dimensions of moduli spaces of two-step nilpotent algebras of types $(p, q)$ were found in [Ebe03].

Another subclass of nilpotent Lie algebras is the set of nilpotent Lie algebras whose “Nikolayevsky derivation” has positive eigenvalues all of multiplicity one. (See [Nik08] for a definition of the Nikolayevsky derivation.) A subset of this subclass in dimensions 7 and 8 was classified in [KP13]. In fact, the motivation for this work is to develop the necessary tools for the completion in [Pay15a] of the classification begun in [KP13]. In [Pay15a], we classify all nilpotent Lie algebras of dimensions 7 and 8 for which the Nikolayevsky derivation is simple with positive eigenvalues. The methods developed here are essential in [Pay15a] for the determination of isomorphism classes of nilpotent
Lie algebras and for the parametrization of continuous families of those isomorphism classes.

Our goal is to find parametrizations of subclasses of the space $\widetilde{N}_n(\mathbb{R})$ of real $n$-dimensional nilpotent Lie algebras, or, more generally, the space $\widetilde{L}_n(K)$ of $n$-dimensional Lie algebras over the field $K$. In attempting to classify some family of Lie algebras up to isomorphism, one frequently shows that there is some efficient basis with respect to which nonzero structure constants for algebras in the class must have specified indices, and then one determines which of the algebras with such structure constants are isomorphic.

Oftentimes a subclass itself may be completely described in terms of indices of nonvanishing structure constants. To be precise, given a Lie algebra with basis $B$, there exists a subset $\Lambda$ of

\[ \Upsilon_n = \{ (i, j, k) : i, j, k \in [n], i < j \} \subseteq [n]^3, \]

where we denote the set $\{1, 2, \ldots, n\}$ by $[n]$, so that the structure constant $\alpha^k_{ij}$ (with $i < j$) is nonzero if and only if $(i, j, k) \in \Lambda$. The subclass $\widetilde{L}_\Lambda(K)$ of $\widetilde{L}_n(K)$ is defined to be the set of all Lie algebras over $K$ that admit a basis $B$ so the structure constants with respect to $B$ are indexed by $\Lambda$. Such sets are quotients of semi-algebraic subsets of $\wedge^2(K^n)^* \otimes K^n$.

Often, subsets $R$ of $\widetilde{L}_n(K)$ may be written as the union of sets $\widetilde{L}_\Lambda(K)$ as $\Lambda$ varies over a subset of $\Upsilon_n$.

For example, let $R = \widetilde{N}_n(K)$, the set of $n$-dimensional nilpotent Lie algebras over $K$. If the Lie algebra $\mathfrak{g}_\alpha$ is nilpotent, then by Engel’s Theorem there is a basis $B = \{x_i\}_{i=1}^n$ for $\mathfrak{g}_\alpha$ with respect to which the operators $\text{ad}_{x_i}$ are simultaneously upper triangularizable. We say that such a basis $B$ is a triangular basis. Relative to $B$, the structure constants for $\mathfrak{g}_\alpha$ (modulo skew-symmetry) are in the set

\[ \Theta_n = \{ (i, j, k) \in [n]^3 : i < j < k \} \subseteq \Upsilon_n. \]

Hence every $n$-dimensional nonabelian nilpotent Lie algebra over $K$ is in the set $\cup_{\Lambda \in \Theta_n} \widetilde{L}_\Lambda(K)$. Conversely, the nilpotency condition holds automatically for every Lie algebra $\mathfrak{g}_\alpha$ with structure constants indexed by a subset of $\Theta_n$ relative to some basis. Many other classes of nilpotent Lie algebras, including filiform and $\mathbb{N}$-graded Lie algebras, may be expressed as the union of sets $\widetilde{L}_\Lambda(K)$.

To parametrize such a set $R = \cup_{\Lambda \in S}\widetilde{L}_\Lambda(K)$, we will find a parametrization of each set $\widetilde{L}_\Lambda(K)$. We assume that the sets $\widetilde{L}_\Lambda(K)$ are disjoint and neglect the issue of when isomorphic Lie algebras occur in different sets $\widetilde{L}_\Lambda(K)$; in practice, the set $S$ may often be chosen so that each Lie algebra in $R$ occurs in exactly one $\widetilde{L}_\Lambda(K)$ with $\Lambda \in S$. 
Although we are primarily motivated by the goal of understanding the class of real nilpotent Lie algebras, we will work in a more general setting. We will drop the constraint imposed by the Jacobi Identity and consider the set $S_\Lambda(K)$ of all anticommutative nonassociative algebras over a field $K$ whose structure constants are indexed by an index set $\Lambda$. We will first consider the question of when products having the same index set $\Lambda$ are isomorphic. We will then consider which products having the index set $\Lambda$ satisfy Jacobi Identity. This will enable us to find a compact semi-algebraic subset $\Sigma$ of $R^{|\Lambda|}$ whose interior is homeomorphic to $\tilde{L}_\Lambda(R)$. Then a parametrization of the set $\Sigma$ yields the desired parametrization for $\tilde{L}_\Lambda(R)$.

The parametrization depends largely on the combinatorics of the index set $\Lambda$. It turns out that many index sets $\Lambda \subseteq \Theta_n$ have the same combinatorial relationships, so the descriptions of the parametrizations of the sets $\tilde{L}_\Lambda(R)$, with $\Lambda \subseteq \Theta_n$, fall into a small number of classes in each dimension.

It should also be mentioned that the methods presented here provide a way to test whether two Lie algebras having the same index set are isomorphic.

The paper is organized as follows. In the next section, we state the main results. In Section 3 necessary background material is covered, and our method of parametrizing classes of Lie algebras is outlined. In Section 4 we focus on defining a subset $\Sigma$ of $S_\Lambda(R)$ so that each Lie algebra in $S_\Lambda(R)$ is isomorphic to exactly one Lie algebra in $\Sigma$. In Section 5 we define some combinatorial objects associated to a set $\Lambda$ and describe some useful properties they have. In Section 6 we address issues related to the Jacobi Identity and find how to parametrize sets of form $\tilde{L}_\Lambda(R)$. Due to the many technicalities involved, and our orientation to applications, we include many concrete examples along the way.

2. Main results

2.1. Necessary definitions. Let $K$ be a field. For $\alpha \in \Lambda^2(K^n)^* \otimes K^n$, we let $g_\alpha$ denote the anticommutative nonassociative algebra whose product is defined by $\alpha$. Let $\mathcal{B} = \{x_i\}_{i=1}^n$ be a basis for $K^n$, and for $\alpha \in \Lambda^2(K^n)^* \otimes K^n$, let $\alpha_{ij}^k$ denote the structure constants for $\alpha$ relative to $\mathcal{B}$.

For a subset $\Lambda$ of $\Upsilon_n$, define the subset $S_\Lambda(K)$ of $\Lambda^2(K^n)^* \otimes K^n$ by

$S_\Lambda(K) = \{ \alpha \in \Lambda^2(K^n)^* \otimes K^n : i < j \text{ and } \alpha_{ij}^k \neq 0 \text{ if and only if } (i, j, k) \in \Lambda \}$. 


Allowing Λ to vary over all subsets of Υ, we obtain a semi-algebraic stratification
\[ \wedge^2 (K^n)^* \otimes K^n = \bigcup_{\Lambda \subseteq \Upsilon_n} S_\Lambda(K) \]
of \( \wedge^2 (K^n)^* \otimes K^n \).

Let \([n] = \{1, 2, \ldots, n\}\). After taking the dictionary ordering on \([n]^{3}\), the set
\[ C = \left\{ (x_i^* \wedge x_j^*) \otimes x_k : (i, j, k) \in \Upsilon_n \right\} \]
is an ordered basis for \( \wedge^2 (K^n)^* \otimes K^n \). Given a vector \( \alpha \) in \( S_\Lambda(K) \subseteq \wedge^2 (K^n)^* \otimes K^n \), the nonzero entries of its coordinate vector with respect to \( C \) are indexed by \( \Lambda \); for this reason we call the sets \( \Lambda \) index sets. The coordinate vectors define a bijection between \( S_\Lambda(K) \) and \( (K \setminus \{0\})^{\left| \Lambda \right|} \).

We sometimes implicitly identify a product in \( S_\Lambda(K) \) with a vector in \( (K \setminus \{0\})^{\left| \Lambda \right|} \).

Let \( L_n(K) \) be the subset of \( \wedge^2 (K^n)^* \otimes K^n \) consisting of those bilinear maps in \( \wedge^2 (K^n)^* \otimes K^n \) that define products that satisfy the Jacobi Identity. The stratification in (2) restricts to a stratification of \( L_n(K) \):

If we let
\[ L_\Lambda(K) = S_\Lambda(K) \cap L_n(K) \]
be the set of Lie brackets whose structure constants are indexed by \( \Lambda \), then
\[ L_n(K) = \bigcup_{\Lambda \subseteq \Upsilon_n} L_\Lambda(K) \]
is a stratification of the set \( L_n(K) \) of all Lie brackets. Although each stratum \( S_\Lambda(K) \) is nonempty, it is possible that a set \( L_\Lambda(K) \) is empty.

A nonabelian Lie algebra is represented by many different maps \( \alpha \) in \( L_n(K) \). The general linear group \( GL_n(K) \) acts on \( \wedge^2 (K^n)^* \otimes K^n \), with
\[ (g \cdot \alpha)(x, y) = g(\alpha(g^{-1}x, g^{-1}y)) \]
for \( g \in GL_n(K) \), \( \alpha \in \wedge^2 (K^n)^* \otimes K^n \) and \( x, y \in K^n \). The space \( \tilde{L}_n(K) \) of all \( n \)-dimensional Lie algebras over \( K \) is the quotient of \( L_n(K) \) under this action. Any choice of initial basis for \( K \) will yield the same quotient space \( \tilde{L}_n(K) \). Therefore, we will often omit mention of a particular basis when referring to the sets \( S_\Lambda(K) \), \( L_\Lambda(K) \), etc.

2.2. **Statements of main results.** We would like to understand how the combinatorics of a set \( \Lambda \subseteq \Theta_n \) relate to properties of the set of nilpotent Lie algebras whose structure constants are indexed by \( \Lambda \). For this purpose, we make some combinatorial definitions in Section 3. We define what it means for a pair of distinct triples \( t_1 \) and \( t_2 \), where
t_1, t_2 \in \Theta_n \subseteq [n]^3$, to be aligned. For an aligned pair of triples $t_1$ and $t_2$ in $\Theta_n \subseteq [n]^3$, we define a quadruple $q(t_1, t_2) \in [n]^4$, and a sign $\text{sign}(t_1, t_2) \in \mathbb{Z}_2$ for that pair of triples. To each $\Lambda \subseteq \Theta_n$, we associate the set $Q \subseteq [n]^4$ of all quadruples arising from all aligned pairs of triples $t_1, t_2 \in \Lambda$.

We would first like to know when $\mathcal{L}_\Lambda(K)$ is guaranteed to be empty: what combinatorial conditions on the set $\Lambda$ imply that no elements of $\mathcal{S}_\Lambda(K)$ satisfy the Jacobi Identity? To this end, we formulate the Jacobi Identity for elements of $\mathcal{S}_\Lambda(K)$ in the language of triples and quadruples.

**Theorem 2.1.** Let $\Lambda \subseteq \Theta_n$ be an index set and let

$$t_1 = (i_1, j_1, k_1), t_2 = (i_2, j_2, k_2), \ldots, t_m = (i_m, j_m, k_m)$$

be an enumeration of the triples in $\Lambda$. Let $Q$ be the set of quadruples for $\Lambda$. The Jacobi Identity for elements of $\mathcal{S}_\Lambda(K)$ is equivalent to the system of equations

$$\sum_{q(t_p, t_r) = (i, j, k, l)} \text{sign}(t_p, t_r) \alpha_{i, j, k}^{p, r} \alpha_{i, j, l}^{r, p} = 0, \quad (i, j, k, l) \in Q.$$  

A corollary to the theorem gives a sufficient condition for $\mathcal{L}_\Lambda(K)$ to be empty.

**Corollary 2.2.** Let $\Lambda \subseteq [n]^3$ be an index set. If $\Lambda$ has a quadruple of multiplicity one, then no elements of $\mathcal{S}_\Lambda(K)$ satisfy the Jacobi Identity.

As another corollary to the previous theorem, if the set of quadruples for an index set $\Lambda$ is empty, then the Jacobi Identity holds automatically for all elements of $\mathcal{S}_\Lambda(K)$.

**Corollary 2.3.** Let $\Lambda \subseteq [n]^3$ be an index set. If $\Lambda$ has no quadruples associated to it, then all elements of $\mathcal{S}_\Lambda(K)$ satisfy the Jacobi Identity.

The first corollary is needed in the computational procedure in [Pay15a]. There we systematically analyze sets $\mathcal{S}_\Lambda(K)$, checking each one for Lie algebras having certain properties. The corollary allows us to disregard all $\mathcal{S}_\Lambda(K)$ for which $\Lambda$ has a quadruple of multiplicity one, thus eliminating irrelevant cases and shortening the computation time.

Given a stratum $\mathcal{S}_\Lambda(K)$ of the stratification (3) we would like to find a subset $\Sigma$ of $\mathcal{S}_\Lambda(K)$ so that each Lie algebra defined by an element of $\mathcal{S}_\Lambda(K)$ is isomorphic to precisely one Lie algebra defined by an element of $\Sigma$. Here, we work over $\mathbb{R}$, and we assume that isomorphism classes of Lie algebras in $\mathcal{S}_\Lambda(\mathbb{R})$ are orbits of the diagonal subgroup $D$ of $GL_n(\mathbb{R})$ under the action in Equation (4). We say that a set which intersects each orbit of an action exactly once is a *simple cross section*. We seek
a simple cross section for the $D$ action so that Lie algebras in that set will parametrize the isomorphism classes of Lie algebras in $S_\Lambda(\mathbb{R})$.

The cross section should be semi-algebraic with compact closure. Other than that, we’d like some flexibility in our definition of cross section. For certain applications we prefer a cross section that contains a prespecified point $\alpha$ as a “center point” at which all parameter values are zero. For example, we might be looking at deformations of a particular Lie algebra of interest. Or, one might prefer to choose the center point so that its structure constants are with respect to a preferred basis. A third reason to vary the center point is that the equations encoding the Jacobi Identity may be simpler in some parametrizations than others.

In Definition 3.4, we define a family of bounded subsets $\Sigma(T, S)$ of $S_\Lambda(\mathbb{R})$ depending on a subset $T$ of $\mathbb{Z}_2^{|\Lambda|}$ and a subset $S$ of $\mathbb{R}_{>0}^{|\Lambda|}$. In Theorem 2.4, we find conditions on a subset $\Sigma(T, S)$ of $S_\Lambda(\mathbb{R})$ that guarantee that it parametrizes algebras in $S_\Lambda(\mathbb{R})$.

**Theorem 2.4.** Let $\Lambda \subseteq [n]^3$ be an index set of cardinality $m > 0$, and let $\dim \text{Null}(Y^T) = d > 0$. Assume that any pair of isomorphic Lie algebras in $S_\Lambda(\mathbb{R})$ lie in the same orbit of the diagonal subgroup under the action in Equation (4).

Let $T \subseteq \mathbb{Z}_2^m$, let $S \subseteq (\mathbb{R}_{>0})^m$, and let $\Sigma(T, S)$ be as in Definition 3.4. For $c \neq 0$, let $F_c$ be as in Definition 4.6.

If $F_c$ maps $S$ onto $\mathbb{R}^d$, then every Lie algebra in $S_\Lambda(\mathbb{R})$ is isomorphic to a Lie algebra in $\Sigma(T, S)$. If in addition $F_c$ is one-to-one, every Lie algebra defined by a Lie bracket in $L_\Lambda(\mathbb{R})$ is isomorphic to precisely one Lie algebra in $\Sigma(T, S) \cap L_\Lambda(\mathbb{R})$.

It is not always easy to verify the hypotheses on $F_c$ in the theorem. Lemma 5.14 gives conditions on $F_c$ that will force bijectivity in some situations. Finding a parametrizing set is less difficult if boundedness and algebraic definitions for charts are not required. (See Remark 4.1.)

In Definition 4.2, we define a subset $\Delta_{a_0}^p$ of $(\mathbb{R}_{>0})^{|\Lambda|}$. We will use sets of the form $\Sigma(T, \Delta_{a_0}^p)$ to parametrize isomorphism classes of products in $S_\Lambda(\mathbb{R})$. The sets $\Sigma(T, \Delta_{a_0}^p)$ are semi-algebraic with compact closure.

**Theorem 2.5.** Let $\Lambda \subseteq \Theta_n$ be an index set of cardinality $m > 0$. Let $T \subseteq \mathbb{Z}_2^n$. Let $p$ be a positive rational number, let $a_0 \in (\mathbb{R}_{>0})^m$ and let $\Delta_{a_0}^p$ be as in Definition 4.2. Let $\Sigma(T, \Delta^p)$ be as in Definition 3.4. Then $\Sigma(T, \Delta_{a_0}^p)$ is a semi-algebraic subset of $\mathbb{R}^m$ with compact closure.

In the computations in [Pay15a], there are hundreds of index sets for which the Jacobi Identity for $S_\Lambda(\mathbb{R})$ is nontrivial, and it is necessary to solve one or more polynomial equations in one or more variables...
for each case. It is initially surprising that almost all of the systems of equations are equivalent to a small number of simple prototypical systems. This happens because the system of equations depends on the combinatorics of the index set, and although there are many index sets, many of those have the same combinatorial relationships. All of the 34 7-dimensional Lie algebras in the classification in [Pay15a] fall into one of the five categories listed in the next theorem. The great majority of the 8-dimensional Lie algebras in the classification in [Pay15a] fall into one of the five categories. Among the remaining examples in dimension 8, most of those remaining Lie algebras fall into three more other categories. There are just a handful of examples that lie in two last categories.

The following theorem from [Pay15b] is proved using the methods developed here. It has a hypothesis involving some more combinatorial definitions. The precise definition of null space spanning is given in Section 5. Among Lie algebras of dimension 7 and 8 arising in [Pay15a], all the Lie algebras of dimension 7 and a vast majority of those of dimension 8 may be represented by a Lie bracket in $S_\Lambda(\mathbb{R})$ where $\Lambda$ is null space spanning. We define what it means for quadruples to have a common triple in Definition 5.4.

**Theorem 2.6** ([Pay15b]). Let $\Lambda \subseteq \Theta_n$ be an index set of cardinality $m$ that is null space spanning. Then

1. If $\Lambda$ has exactly one quadruple of multiplicity two, then $\tilde{L}_\Lambda(\mathbb{R})$ is finite.
2. If $\Lambda$ has exactly two quadruples of multiplicity two, then
   a. If the quadruples do not have a common triple, then $\tilde{L}_\Lambda(\mathbb{R})$ is finite.
   b. If the quadruples have exactly one common triple, then $\tilde{L}_\Lambda(\mathbb{R})$ is one-dimensional.
3. If $\Lambda$ has just one quadruple of multiplicity three, then $\tilde{L}_\Lambda(\mathbb{R})$ is one-dimensional.
4. If $\Lambda$ has one quadruple of multiplicity three, and one quadruple of multiplicity two, and the quadruples have exactly one common triple, then $L_\Lambda(\mathbb{R})$ is one-dimensional.

Furthermore, in [Pay15b] we give formulae depending only on the combinatorics of the set $\Lambda$ for parametrizations of some classes $\tilde{L}_\Lambda(\mathbb{R})$.

3. Preliminaries
3.1. Index sets, root vectors, sign vectors, and root matrices.  
In this section, we give definitions of some algebraic and combinatorial objects that we will use.

Fix an ordering of $\Upsilon_n$. Let $\alpha = [\alpha_{ij}^k]_{(i,j,k) \in \Lambda}$ be in $S_\Lambda(\mathbb{R})$. Define the vector $|\alpha|$ by vector $|\alpha| = [[\alpha_{ij}]_{(i,j,k) \in \Lambda}$ where the entries of $|\alpha|$ are listed in ascending order relative to the fixed ordering of $\Upsilon_n$. Define the sign vector for $\alpha$ to be the vector

$$sgn(\alpha) = [sgn(\alpha_{ij}^k)]_{(i,j,k) \in \Lambda}$$

in $\mathbb{Z}_2^{n^3}$, where $sgn : \mathbb{R}^n \to \mathbb{Z}_2$ is the homomorphism to the additive group $\mathbb{Z}_2$ defined by

$$sgn(x) = \begin{cases} 
1 & \text{if } x < 0 \\
0 & \text{if } x > 0 
\end{cases}$$

for nonzero $x \in \mathbb{R}$. Sometimes it will be more convenient to use the homomorphism $sgn : \mathbb{R}^n \to \{-1,1\}$ with values in the multiplicative group $\mathbb{Z}_2$.

Conversely, a nonempty index set $\Lambda$, a vector $\alpha = [a_{(i,j,k)}]_{(i,j,k) \in \Lambda} \in \mathbb{R}_0^{n^3}$ with positive entries indexed by $\Lambda$, and a vector $s = [s_{(i,j,k)}]_{(i,j,k) \in \Lambda} \in \mathbb{Z}_2^{n^3}$ determine a unique bilinear skew-symmetric map $\alpha \in \Lambda^2(\mathbb{R}^n)^* \otimes \mathbb{R}_0^n$. The map $\alpha$ is given by $\alpha(x_i, x_j) = \sum_{k=1}^n \alpha_{ij}^k x_k$, where

$$\alpha_{ij}^k = sgn(a_{(i,j,k)})|a_{(i,j,k)}| = (-1)^{sgn(a_{(i,j,k)})}|a_{(i,j,k)}|.$$  

(We will repeatedly abuse notation so that when $s \in \mathbb{Z}_2$, we have $(-1)^s \in \mathbb{R}$: i.e., $(-1)^0 = 1 \in \mathbb{R}$, and $(-1)^1 = -1 \in \mathbb{R}$.)

The resulting bilinear map $\alpha$ has $[[\alpha_{(i,j,k)}]_{(i,j,k) \in \Lambda} = \alpha$ and sign vector $s$.

For a nonempty index set $\Lambda \subseteq \Upsilon_n$ of cardinality $m$, we associate to $\Lambda$ a set of $m \times n \times 1$ vectors and an $m \times n$ matrix as follows. To each triple $(i, j, k) \in [n]^3$, we associate the $K$ root vector or root vector $y_{(i,j,k)} = e_i^T + e_j^T - e_k^T$ in the vector space $K^n$, where $\{e_i\}_{i=1}^n$ is the standard basis for $K^n$ (as column vectors). The $K$ root matrix for $\Lambda$ is the $m \times n$ matrix $Y(K)$ whose rows are the root vectors $y_{(i,j,k)}$, $(i, j, k) \in \Lambda$ listed in ascending order relative to the fixed ordering of $\Upsilon_n$. When we write $Y$ omitting mention of the underlying field, we assume that the underlying field is $\mathbb{R}$, and when we write $\hat{Y}$, we assume that the underlying field is $\mathbb{Z}_2$. We will give examples of index sets, sign vectors and root matrices in the coming pages. The reader may also refer to [Pay10] or [Pay12] for more examples and properties of these objects.

3.2. Isomorphism classes of nilpotent Lie algebras. In this section we define the subsets $\Sigma(T, \Delta^p)$ of $S_\Lambda(\mathbb{R})$, and we prove Theorem
2.4. We need to first define the action of the additive group $K^n$ on the vector space $K^m$ determined by an $m \times n$ matrix.

**Definition 3.1.** A $m \times n$ matrix $Y$ with entries in a field $K$ defines an action $\rho_Y : K^n \times K^m \to K^m$ of $K^n$ on $K^m$ with

\[ \rho(d, z) = z + Yd, \]

for $d \in K^n$ and $z \in K^m$.

Orbits of this action are the column space $\text{Col}(Y)$ of $Y$ and translations $z + \text{Col}(Y)$ of the column space, where $z$ varies over $K^m$. If $K = \mathbb{R}$ then each orbit meets $\text{Null}(Y^T)$ exactly once, so $\text{Null}(Y^T)$ is a simple cross section for the action.

Let $(a_1, \ldots, a_m)$ be a point in $(\mathbb{R}_{>0})^m$. Define the coordinate-wise logarithm map $\text{Ln} : (\mathbb{R}_{>0})^m \to \mathbb{R}^m$ by

\[ \text{Ln}((a_1, \ldots, a_m)) = (\ln a_1, \ldots, \ln a_m), \]

and let $E$ denote its inverse, the coordinate-wise exponential map,

\[ E((a_1, \ldots, a_m)) = (e^{a_1}, \ldots, e^{a_m}). \]

The following theorem relates isomorphism classes and orbits of the action defined by a root matrix.

**Theorem 3.2** (Theorem 3.8, [Pay14a]). Let $\Lambda$ be a subset of $\Theta_n$ of cardinality $m$, and let $Y$ and $\hat{Y}$ denote the real and $\mathbb{Z}_2$ root matrices for $\Lambda$ respectively. Let $\alpha$ and $\beta$ be elements of $\mathcal{S}_\Lambda(\mathbb{R}) \subseteq \wedge^2(\mathbb{R}^n)^* \otimes \mathbb{R}^n$.

Let $a^2 = [(\alpha^k_{ij})^2]_{(i,j,k)\in\Lambda}$ and $b^2 = [(\beta^k_{ij})^2]_{(i,j,k)\in\Lambda}$, and let

\[ \text{sgn}(a) = [\text{sgn}(\alpha^k_{ij})]_{(i,j,k)\in\Lambda} \quad \text{and} \quad \text{sgn}(b) = [\text{sgn}(\beta^k_{ij})]_{(i,j,k)\in\Lambda} \]

be the sign vectors for $\alpha$ and $\beta$. Let $g_\alpha$ and $g_\beta$ denote the algebras defined by $\alpha$ and $\beta$ respectively.

Then the algebras $g_\alpha$ and $g_\beta$ are in the same $D$ orbit for the action (1) if and only if

1. the vectors

\[ \text{Ln}(a^2) = [\ln(\alpha^k_{ij})^2]_{(i,j,k)\in\Lambda} \quad \text{and} \quad \text{Ln}(b^2) = [\ln(\beta^k_{ij})^2]_{(i,j,k)\in\Lambda} \]

are in the same orbit for the action $\rho_Y : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ induced by $Y$ as in Definition [3.1], and

2. the sign vectors $\text{sgn}(a)$ and $\text{sgn}(b)$ are in the same orbit for the action $\rho_{\hat{Y}} : \mathbb{Z}_2^n \times \mathbb{Z}_2^m \to \mathbb{Z}_2^m$ as in Definition [3.1].

**Remark 3.3.** One might ask for which $\Lambda \subseteq \Theta_n$ it is true that isomorphism classes in $\mathcal{S}_\Lambda(\mathbb{R})$ are $D$ orbits. This holds if $\mathcal{B}$ is an eigenvector basis for the Nikolayevsky derivation $D^N$ of $g_\alpha$ and all of the eigenvalues of $D^N$ are positive and have multiplicity one. Conveniently, when
\[ \Lambda \subseteq \Theta_n, \text{ all members of a stratum } L_\Lambda(\mathbb{R}) \text{ have the same Nikolayevsky derivation (Theorem 3.1, [Pay14a]). All of the examples of } \Lambda \text{ that we will present in this work have Nikolayevsky derivation with positive eigenvalues each of multiplicity one, with the exception of Example 3.7.} \]

More generally, Proposition 2 and its corollary in [Bra74] present conditions that insure that \( D \)-orbits are isomorphism classes: the basis should be an eigenbasis for a maximal torus in \( \text{Aut}(A) \) and the normalizer of \( T \) in \( \text{Aut}(A) \) should equal its centralizer in \( \text{Aut}(A) \). A careful reading of the proof given there shows that the proof extends to the case that the base field is \( \mathbb{R} \).

Define an action of \( \mathbb{Z}_2^m = \{0, 1\}^m \) on \( \mathbb{R}^m \) by coordinate-wise multiplication:

\[
(s_1, \ldots, s_m) \cdot (a_1, \ldots, a_m) = ((-1)^{s_1}a_1, \ldots, (-1)^{s_m}a_m).
\]

**Definition 3.4.** Let \( \Lambda \) be a subset of \([n]^3\) of cardinality \( m > 0 \). Let \( T \subseteq \mathbb{Z}_2^m \) be a simple cross section for the \( \rho_T \) action, and let \( S \subseteq (\mathbb{R}_{>0})^m \). Define the subset \( \Sigma(T, S) \) of \( S_\Lambda(\mathbb{R}) \) by

\[
\Sigma(T, S) = \{ t \cdot x : t \in T, x \in S \} \subseteq \mathbb{R}^{|\Lambda|},
\]

where \( \cdot \) denotes the action of \( \mathbb{Z}_2^m \) on \( \mathbb{R}^m \) by coordinate-wise multiplication as in (10).

The set \( S_\Lambda(\mathbb{R}) \) is homeomorphic to \((\mathbb{R} \setminus \{0\})^m\), which is homeomorphic to \( \mathbb{Z}_2^m \times \mathbb{R}^m \) via the map \( \psi : (\mathbb{R} \setminus \{0\})^m \to \mathbb{Z}_2^m \times \mathbb{R}^m \) sending \([\alpha(i,j,k)](i,j,k) \in \Lambda \) to the ordered pair

\[
([\text{sgn}(\alpha(i,j,k))](i,j,k) \in \Lambda, [\text{ln}(|\alpha(i,j,k)|)](i,j,k) \in \Lambda) \in \mathbb{Z}_2^m \times (\mathbb{R}_{>0})^m.
\]

The proof of Theorem 3.2 is structured as follows. The action of the diagonal subgroup \( D \cong (\mathbb{R} \setminus \{0\})^n \) of \( GL_n(\mathbb{R}) \) on \( S_\Lambda(\mathbb{R}) \) is topologically conjugate by \( \psi \) to the action \( \rho_D \times \rho_T \) of \( \mathbb{Z}_2^m \times \mathbb{R}^m \) on \( \mathbb{Z}_2^m \times \mathbb{R}^m \). To be precise, let \( g = \text{diag}(c_1, \ldots, c_n) \) be a diagonal matrix in \( GL_n(\mathbb{R}) \), and let \( \text{sgn}(g) \) be the sign vector in \( \mathbb{Z}_2^2 \) and let \( \text{Ln}(|g|) \) be the vector with \( i \)th entry \( \text{ln}(|c_i|) \). Then if \( \alpha = [\alpha(i,j,k)](i,j,k) \in \Lambda \in S_\Lambda(\mathbb{R}) \), the product \( \beta = g \cdot \alpha \) under the action (11) has sign vector

\[
[\text{sgn}(\beta(i,j,k))](i,j,k) \in \Lambda = \rho_Y(\text{sgn}(g), [\text{sgn}(\alpha(i,j,k))](i,j,k) \in \Lambda)
\]

and

\[
[\text{ln}(|\beta(i,j,k)|)](i,j,k) \in \Lambda = \rho_Y(\text{Ln}(|g|), [\text{ln}(|\alpha(i,j,k)|)](i,j,k) \in \Lambda).
\]

Therefore, the map \( \psi \) in (11) induces a one-to-one correspondence between \( D \) orbits in \( S_\Lambda(\mathbb{R}) \) and \( \rho_Y \times \rho_T \) orbits in \( \mathbb{Z}_2^m \times \mathbb{R}^m \). If \( T \) and
Ln(S) are simple cross sections for the $\rho_Y$ and $\rho_Y$ actions respectively, then $T \times Ln(S)$ is a simple cross section for the $\rho_Y \times \rho_Y$ action. But $T \times Ln(S) \subseteq \mathbb{Z}^m_2 \times (\mathbb{R}_{>0})^m$ is homeomorphic to $\psi^{-1}(T, Ln(S)) = \Sigma(T, S) \subseteq (\mathbb{R} \setminus \{0\})^m$. This yields the following corollary to Theorem 2.4.

**Corollary 3.5.** Let $\Lambda$ be a subset of $\Theta_n$ of cardinality $m > 0$, and let $Y$ and $\hat{Y}$ denote the real and $\mathbb{Z}_2$ root matrices for $\Lambda$. Let $T$ be a simple cross section for the action $\rho_Y$ induced by $\hat{Y}$ as in Definition 3.1 and let $S \subseteq (\mathbb{R}_{>0})^m$ be such that $Ln(S)$ is a simple cross section for the action $\rho_Y$ induced by $Y$ as in Definition 3.1.

Then every algebra in $\mathcal{S}_\Lambda(\mathbb{R})$ is represented at least once in the set $\Sigma(T, S)$. If isomorphism classes of algebras (Lie algebras) in $\mathcal{S}_\Lambda(\mathbb{R})$ are $D$ orbits, then every algebra (Lie algebra) in $\mathcal{S}_\Lambda(\mathbb{R})$ is isomorphic to precisely one algebra defined by an element of $\Sigma(T, S)$.

Furthermore, if the natural map from $\mathcal{S}_\Lambda(\mathbb{R})$ to $\Sigma(T, S)$ is continuous, $\Sigma(T, S)$ is homeomorphic to $\tilde{\mathcal{S}}_\Lambda(\mathbb{R})$.

We illustrate the corollary with two examples.

**Example 3.6.** Let $\Lambda = \{(1, 2, 3), (1, 3, 4)\} \subseteq \Theta_4$. The set $\mathcal{S}_\Lambda(\mathbb{R})$ consists of all real anticommutative nonassociative algebras spanned by a fixed basis $\mathcal{B} = \{x_1, x_2, x_3, x_4\}$ with the product determined by the relations

$$[x_1, x_2] = \alpha_{12} x_3, \quad [x_1, x_3] = \alpha_{13} x_4,$$

where $\alpha_{12}$ and $\alpha_{13}$ are nonzero real numbers. By Remark 3.3, isomorphism classes in $\mathcal{S}_\Lambda(\mathbb{R})$ are $D$ orbits.

The $\mathbb{R}$-root matrix $Y$ and the $\mathbb{Z}_2$-root matrix $\hat{Y}$ are given by

$$Y = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \hat{Y} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Because the rank of $Y$ is two, the action $\rho_Y : \mathbb{R}^4 \times \mathbb{R}^2 \to \mathbb{R}^2$ induced by $Y$ as in Definition 3.1 is transitive, and $\text{Null}(Y^T) = \{(0, 0)\} \subseteq \mathbb{R}^2$ is a simple cross section for the action. The rank of $\hat{Y}$ is also two, so the action $\rho_{\hat{Y}} : \mathbb{Z}_2^4 \times \mathbb{Z}_2^2 \to \mathbb{Z}_2^2$ induced by $\hat{Y}$ as in Definition 3.1 is also transitive with simple cross section $T = \{(0, 0)\} \subseteq \mathbb{Z}_2^2$.

If we let $\Delta = \{(1, 1)\}$, then the simple cross section $\{(0, 0)\}$ for the $\rho_Y$ action is equal to $\text{Ln}(\Delta)$. The set $\Sigma(T, \Delta)$ as defined in Definition 3.4 is

$$\Sigma(T, \Delta) = \{(0, 0) \cdot (1, 1)\} = \{(1, 1)\} \subseteq \mathcal{S}_\Lambda(\mathbb{R}).$$

By Corollary 3.5, all Lie algebras in $\mathcal{L}_\Lambda(\mathbb{R})$ are isomorphic to the Lie algebra in $\Sigma(T, \Delta)$. 
Example 3.7. Let $\Lambda = \{(1, 2, 5), (3, 4, 5)\} \subseteq \Theta_5$. Products in $S_\Lambda(\mathbb{R})$ are determined by

$$[x_1, x_2] = \alpha_{12}^5 x_5, \quad [x_3, x_4] = \alpha_{34}^5 x_5,$$

relative to basis $B = \{x_i\}_{i=1}^5$, where $\alpha_{12}^5, \alpha_{34}^5 \neq 0$.

The $\mathbb{R}$-root matrix $Y$ and the $\mathbb{Z}_2$-root matrix $\hat{Y}$ are given by

$$Y = \begin{bmatrix} 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \hat{Y} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

As in the previous example, $\text{Null}(Y^T) = \{(0, 0)\} \subseteq \mathbb{R}^2$ is a simple cross section for the $\rho_Y$ action, and $T = \{(0, 0)\} \subseteq \mathbb{Z}_2^2$ is a simple cross section for the $\rho_{\hat{Y}}$ action. By Corollary 3.5 all algebras in $L_\Lambda(\mathbb{R})$ are isomorphic to the algebra in $\Sigma(T, \Delta) = \{(1, 1)\}$.

Our basic strategy for describing all the Lie algebras in a stratum $S_\Lambda(\mathbb{R})$ is as follows.

1. Find sets $T \subseteq \mathbb{Z}_2^n$ and $S \subseteq (\mathbb{R}_{>0})^m$ so that $T$ and $\text{Ln}(S)$ are cross sections for the $\rho_Y$ and $\rho_{\hat{Y}}$ actions, respectively. Let $\Sigma(T, S)$ be as in Definition 3.4.

2. If $S$ has dimension $d \geq 2$, find a simple parametrization of the set $\text{Ln}(S)$, if possible, in terms of parameters $t_1, t_2, \ldots, t_d$. This will involve using the vectors $w(m_1, m_2, m_3, m_4)$ defined in Definition 5.7. These vectors have the advantage of being linear combinations of just 4 basis vectors, with coefficients of $-1$ and 1.

3. Show that every Lie algebra in $S_\Lambda(\mathbb{R})$ is isomorphic to precisely one Lie algebra in $\Sigma(T, S)$ by using Theorem 2.4 or Corollary 3.5.

4. Use Theorem 2.1 to solve the Jacobi Identity to determine which elements of $\Sigma(T, S)$ are Lie algebras. Sometimes, this may be done in two steps:

   a) Find the values of $|\alpha_{ij}^k|$ by solving a system of polynomial equations in $t_1, t_2, \ldots, t_d$.

   b) Find which possible signs may be assigned to the structure constants so that the Jacobi Identity is satisfied.

For many $\Lambda$'s, the calculations described above are essentially identical. As we proceed, we will repeatedly return to prototypical examples of each type covered by Theorem 2.6:

- Analysis of a stratum $S_\Lambda(\mathbb{R})$ as in Part 1 of Theorem 2.6 is worked out in Examples 4.5 and 6.3.
- Analysis of a set $S_\Lambda(\mathbb{R})$ as in Part 2 of Theorem 2.6 is in Example 6.4.
• Analysis of a set $\mathcal{S}_A(\mathbb{R})$ as in Part 3 of Theorem 2.6 is worked out in Examples 4.7, 5.6, 5.11 and 6.5.
• An example as in Part 4 of Theorem 2.6 is analyzed in Examples 4.9, 5.3, 5.5, 5.8, and 6.6.

4. Cross sections

We would like to generalize the approach used in Example 3.6 to apply to cases in which the simple cross section $\text{Ln}(\Delta)$ is not finite, as it was in Example 3.6. In that example, we found that all algebras in $\mathcal{S}_A(\mathbb{R})$ were isomorphic to the one with structure constants defined by

$$\{(1, 1)\} = \Sigma(\{(0, 0)\}, \{(1, 1)\}) = \Sigma(\text{Null}(\hat{Y}^T), \text{E(Null}(Y^T))).$$

Remark 4.1. There is an obvious generalization. When $Y$ does not have maximal rank, we may use an analogous definition to obtain a set of form $\Sigma(T, \Delta)$ where $T = \text{Null}(\hat{Y}^T)$ and $\Delta = \text{E(Null}(Y^T))$. However, this set is noncompact when $\text{Null}(Y^T)$ is infinite, and requires exponential functions for its parametrization. Yet, this may be a useful simple cross section if boundedness and algebraic charts are not required.

Now we define the type of set that will be our simple cross section for the $\rho_Y$ action in Equation (3.1).

Definition 4.2. Let $Y$ be an $m \times n$ root matrix. Let $a_0$ be a point in $(\mathbb{R}_{>0})^m$ and let

$$\Delta_{a_0} = \{a_0 + w : w \in \text{Null}(Y^T)\} \cap \mathbb{R}_{>0}^m,$$

so

$$\text{Ln}(\Delta_{a_0}) = \{\text{Ln}(a_0 + w) : w \in \text{Null}(Y^T), a_0 + w \in (\mathbb{R}_{>0})^m\}.$$ 

For any $p \neq 0$, let

$$\Delta_{a_0}^p = \{\text{Exp}(pa) : a \in \text{Ln}(\Delta_{a_0})\}$$

$$= \{(a_1^p, a_2^p, \ldots, a_m^p) : (a_1, \ldots, a_m) \in \Delta_{a_0}\}$$

We say that the point $a_0$ is the center point of $\Delta_{a_0}^p$.

Remark 4.3. Because $[\text{ln}(|\alpha_{ij}^k|^p)|_{(i,j,k)\in A} = p[\text{ln}(|\alpha_{ij}^k|)]_{(i,j,k)\in A}$, the condition in Part (1) of the Theorem 3.2 holds if and only if the vectors $[\text{ln}(|\alpha_{ij}^k|^p)|_{(i,j,k)\in A}$ and $[\text{ln}(|\beta_{ij}^k|^p)|_{(i,j,k)\in A}$ are in the same orbit for the action $\rho_Y$ for any nonzero $p$. Therefore, for any $p \neq 0$, $\text{Ln}(\Delta_a)$ is a simple cross section for the $\rho_Y$ action if and only if $\text{Ln}(\Delta)$ is.

Our next goal is to show that under suitable hypotheses, the set $\text{Ln}(\Delta_{a_0}^p)$ is a simple cross section for the $\rho_Y$-action. First we need an elementary lemma.
Lemma 4.4. Let $Y$ be an $m \times n$ matrix over $\mathbb{R}$. Let $\rho_Y : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ be the action defined in Definition 3.1. Suppose that $\text{Null}(Y^T)$ is nontrivial with basis $D = \{w_1, \ldots, w_d\}$. Define the map $\pi_Y : \mathbb{R}^m \to \mathbb{R}^d$ by

$$\pi_Y : v \mapsto (v \cdot w_1, \ldots, v \cdot w_d),$$

where $\cdot$ denotes the standard dot product of two vectors in $\mathbb{R}^m$.

Let $S$ be a nonempty subset of $\mathbb{R}^m$. The set $S$ meets each orbit of $\rho_Y$ at least once if and only if the restriction of $\pi_Y$ to $S$ is surjective, and $S$ is a simple cross section for the action if and only if the restriction of $\pi_Y$ to $S$ is a bijection between $S$ and $\mathbb{R}^m$.

Proof. The orbits of $\rho_Y$ are the sets $z + \text{Col}(Y)$, where $z \in \mathbb{R}^m$. Because the vector space $\mathbb{R}^m$ is the orthogonal direct sum of the null space Null($Y^T$) and the column space Col($Y$), we may always choose a unique $z$ in Null($Y^T$) in each orbit. Let $B = \{v_1, \ldots, v_d\}$ be an orthonormal basis for Null($Y^T$). Let $p : \mathbb{R}^m \to \mathbb{R}^d$ denote the map

$$p(v) = (v \cdot v_1, \ldots, v \cdot v_d),$$

which is orthogonal projection from $\mathbb{R}^m$ to Null($Y^T$), relative to the basis $B$. It follows that the set $S$ meets the orbit $z + \text{Col}(Y)$, with $z \in \text{Null}(Y^T)$, if and only if the coordinate vector $[z]_B$ of $z$ with respect to $B$ is in $p(S)$. Hence $S$ meets each orbit at least once if and only if $p|_S$ is surjective, and $S$ meets each orbit at most once if and only if $p|_S$ is one-to-one.

Since $\{w_1, \ldots, w_d\}$ is a linearly independent set, there exists an invertible linear transformation $T : \mathbb{R}^m \to \mathbb{R}^m$ such that $p \circ T = \pi_Y$. Hence $p|_S$ is bijective if and only if $\pi_Y|_S$ is bijective, and $p|_S$ is onto if and only if $\pi_Y|_S$ is onto. \qed

Now we prove Theorem 2.5, showing that $\Sigma(T, \Delta^p_{a_0})$ is semi-algebraic subset of $\mathbb{R}^m$ with compact closure.

Proof. Let $\{w_1, \ldots, w_d\}$ be a basis for Null($Y^T$). Clearly $\Delta_{a_0}$ is semi-algebraic, as the defining condition

$$x = (x_j) = a(t_1, \ldots, t_d) = a_0 + \sum_{i=1}^d t_i w_i > 0$$

for $\Delta_{a_0}$ is equivalent to requiring that $x_j > 0$ for all $j = 1, \ldots, m$ and that $Y^T(x - a_0) = 0$. Hence, $\Delta_{a_0}$ is a semi-algebraic subset of $\mathbb{R}^m$.

Let $p = r/s$, with $r, s \in \mathbb{Z}_{>0}$. Then $x \in \Delta^p_{a_0}$ if and only if $x_i^r = y^s_i$ for $y \in \Delta_{a_0}$. This means that $\Delta^p_{a_0}$ is the projection of the semi-algebraic set

$$\{(x, y) : y \in \Delta_{a_0}, x_i^r = y_i^s \text{ for all } i\}$$
onto the first factor. The projection of a semi-algebraic set is semi-algebraic, hence \( \Delta_{a_0}^p \) is semi-algebraic.

To see that \( \Delta_{a_0}^p \) has compact closure, by continuity of the map \( x \mapsto x^p \), it suffices to show that \( \Delta_{a_0} \) has compact closure. Since each row of \( Y \) is of the form \( e_i + e_j - e_k \) for \( i < j < k \), \( Y[1]_{k \times 1} = [1]_{m \times 1} \), where \( [1]_{k \times 1} \) denotes the \( k \times 1 \) vector with all entries 1. As \( [1]_{m \times 1} \) is in the column space of \( Y \), \( \text{Null}(Y^T) \) is contained in the hyperplane \([1]_{m \times 1}^T\). Therefore, \( \Delta_{a_0} \) is contained in the intersection of \( a_0 + [1]_{m \times 1}^T \) and \((\mathbb{R}_{>0})^m\), a bounded set. Hence, \( \Delta_{a_0}^p \) is bounded.

Now consider \( \Sigma(T, \Delta^p) \). Let \( T = \{s_1, \ldots, s_k\} \). The set \( \Sigma(T, \Delta^p) \) is the union of the \( k \) disjoint sets \( \{s_i\} \bullet \Delta^p \), where \( i = 1, \ldots, k \). But since coordinate-wise multiplication is a linear map, the sets \( \{s_i\} \bullet \Delta^p \) are all semi-algebraic. The union of the semi-algebraic sets is semi-algebraic, making \( \Sigma(T, \Delta^p) \) algebraic. Clearly the finite union of bounded sets is bounded, so \( \Sigma(T, \Delta^p) \) has compact closure. □

Before we proceed with the proof of Theorem 2.4 we give an example of an application of Lemma 4.3.

**Example 4.5.** Let \( n = 7 \), let

\[
\Lambda = \{(1, 2, 4), (1, 3, 5), (1, 5, 6), (2, 4, 6), (2, 5, 7), (3, 4, 7)\}
\]

and let \( Y \) be the \( 6 \times 7 \) real root matrix for \( \Lambda \). The vector

\[
w_1 = (1, -1, 0, 0, -1, 1)^T
\]

is a basis for \( \text{Null}(Y^T) \). Let \( a_0 = (1, 1, 1, 1, 1, 1)^T \), and for \( s \in \mathbb{R} \) let

\[
a(s) = a_0 + sw_1 = (1, 1, 1, 1, 1, 1)^T + s(1, -1, 0, 0, -1, 1)^T.
\]

The vector \( a(s) \) is in \((\mathbb{R}_{>0})^6\) if and only if \( s \in (-1, 1) \). The set \( \Delta_{a_0} \) as in Definition 4.2 is

\[
\Delta_{a_0} = \{a(s) : -1 < s < 1\}
\]

\[
= \{(1, 1, 1, 1, 1, 1)^T + s(1, -1, 0, 0, -1, 1)^T : -1 < s < 1\}.
\]

For \( s \in (-1, 1) \),

\[
\text{Ln}(a(s)) = (\ln(1 + s), \ln(1 - s), 0, 0, \ln(1 - s), \ln(1 + s)),
\]

and

\[
\text{Ln}(\Delta_{a_0}) = \{\text{Ln}(a(s)) : -1 < s < 1\}.
\]

The map \( \pi_Y : \mathbb{R}^6 \rightarrow \mathbb{R} \) as in Lemma 4.4 is expressed in terms of the parameter \( s \) by

\[
\pi_Y : \text{Ln}(a(s)) \mapsto \text{Ln}(a(s)) : (1, -1, 0, 0, -1, 1) = 2 \ln \left( \frac{1 + s}{1 - s} \right).
\]
The function $s \mapsto \pi_Y(Ln(\mathbf{a}(s)))$ is monotonically increasing on $(-1, 1)$ with $\lim_{s \to -1} \phi(s) = -\infty$ and $\lim_{s \to -1} \phi(s) = \infty$. Hence it is a bijection from $(-1, 1)$ to $\mathbb{R}$. Therefore, by Lemma 4.4, $\text{Ln}(\Delta_{a_0})$ is a simple cross section for the action $\rho_Y$.

Now consider the action $\rho_Y : \mathbb{Z}_2^7 \times \mathbb{Z}_2^6 \to \mathbb{Z}_2^6$ defined by the $\mathbb{Z}_2$ root matrix $\hat{Y}$. It can be shown that the rank of $\hat{Y}$ over $\mathbb{Z}_2$ is 5 and that

$$T = \{(0, 0, 0, 0, 0, 0, 0, 1)^T, (0, 0, 0, 0, 1)^T\}$$

is a simple cross section for the action. The set $\Sigma(T, \Delta_{a_0})$ as in Definition 3.4 is

$$(17) \quad \Sigma(T, \Delta_{a_0}) = \{(1 + s, 1 - s, 1, 1 - s, \pm (1 + s))^T : -1 < s < 1\}.$$ 

Note that this is a semi-algebraic subset of $\mathbb{R}^6$ with compact closure, with two charts, each parametrized linearly by $s$.

Isomorphism classes are $D$ orbits by Remark 3.3. By Corollary 3.5, every Lie algebra in $\mathcal{S}_\Lambda(\mathbb{R})$ is isomorphic to precisely one algebra whose structure constants are encoded by a point in $\Sigma(T, \Delta) \cap \mathcal{L}_7(\mathbb{R})$. In Example 6.3 we shall determine which algebras in $\Sigma(T, \Delta)$ satisfy the Jacobi Identity.

When $S$ is one-dimensional, one may use univariate calculus as in the previous example to show that the map $\pi_Y : S \subseteq \mathbb{R}^m \to \mathbb{R}^d$ as in Equation (16) of Lemma 4.4 is a bijection. It is more difficult to show that a map into a higher-dimensional Euclidean space $\mathbb{R}^d$ is a bijection.

**Definition 4.6.** Let $\Lambda$ be an index set of cardinality $m > 0$ whose root matrix $Y$ has $\dim \text{Null}(Y^T) = d > 0$. Let $S$ be a $d$-dimensional subset of $\mathbb{R}^m$, and let $\pi_Y : \mathbb{R}^m \to \mathbb{R}^d$ be a map of the form in Equation (16) of Lemma 4.4. Let $a : D \to S$ be a parametrization of $S$; that is a bijection sending $(t_1, \ldots, t_d)$ in a subset $D$ of $\mathbb{R}^d$ to $a(t_1, \ldots, t_d)$ in $S$.

Define for $c \in \mathbb{R}$, the map $F_c : D \to \mathbb{R}^d$ by

$$(18) \quad F_c(t_1, \ldots, t_d) = c(\pi_Y \circ \text{Ln} \circ a)(t_1, \ldots, t_d),$$

where Ln is the coordinate-wise logarithm map as defined in (8).

This definition and its utility are illustrated in the next example. For a field $K$, we use $i : K^n \to \mathbb{P}_n(K)$ to denote the map embedding $K^n$ in $n$-dimensional projective space, sending $(x_1, \ldots, x_n)$ in $K^n$ to $[x_1 : \cdots : x_n : 1]$ in $\mathbb{P}_n(K)$. We use $P_n(\mathbb{R})_{\geq 0}$ to indicate the subset

$$P_n(\mathbb{R})_{\geq 0} = \{[y_1 : \cdots : y_{n+1}] : y_1, \ldots, y_{n+1} \geq 0\}$$

of $P_n(\mathbb{R})$, and we define $P_n(\mathbb{R})_{> 0}$ analogously.
Example 4.7. Let \( n = 7 \), and let
\[
\Lambda = \{(1, 2, 4), (1, 3, 5), (1, 4, 6), (1, 6, 7), (2, 3, 6), (2, 5, 7), (3, 4, 7)\}.
\]
The real \( 7 \times 7 \) root matrix \( Y \) for \( \Lambda \) has rank 5 and the vectors
\[
w_1 = (0, 1, 0, -1, -1, 1, 0)^T, \text{ and } w_2 = (1, 0, 0, -1, -1, 0, 1)^T
\]
span \( \text{Null}(Y^T) \). The \( \mathbb{Z}_2 \)-root matrix \( \hat{Y} \) also has rank 5 with
\[
T = \text{span}_{\mathbb{Z}_2} \{(0, 0, 0, 0, 0, 0, 1)^T, (0, 0, 0, 0, 0, 1, 0)^T\}
\]
being a simple cross section for the \( \rho_\hat{Y} \) action on \( \mathbb{Z}_2^7 \).

Let \( a_0 = (1, 2, 1, 1, 1, 2, 1)^T \). We use \( a_0 \) as our center point for \( \Delta_{a_0} \), rather than \( b_0 = (1, 1, 1, 1, 1, 1, 1)^T \), because the algebra defined by \( a_0 \) satisfies the Jacobi Identity, whereas an algebra defined by \( b_0 \) is not a Lie algebra. For \((s, t)\) in \( \mathbb{R}^2 \), let
\[
a(s,t) = a_0 + sw_1 + tw_2 = (1 + t, 2 + s, 1, 1 - s - t, 1 - s - t, 2 + s, 1 + t)^T.
\]
The entries of \( a(s,t) \) are all positive if and only if \((s,t)\) is in the triangle interior
\[
D = \{(s,t) : s > -2, t > -1, \text{ and } s + t < 1\} \subseteq \mathbb{R}^2.
\]
Here, the set \( \Delta_{a_0} \subseteq \mathbb{R}^7 \) as in Definition 4.2 is given by
\[
\Delta_{a_0} = \{a(s,t) : (s,t) \in D\},
\]
and \( \text{Ln}(\Delta_{a_0}) \) is the set of all points of form
\[
(\text{ln}(1 + t), \text{ln}(2 + s), 0, \text{ln}(1 - s - t), \text{ln}(1 - s - t), \text{ln}(2 + s), \text{ln}(1 + t))^T
\]
with \((s,t)\) in \( D \).

In order to apply Lemma 4.4 to show that \( \text{Ln}(\Delta_{a_0}) \) is a simple cross section for the \( \rho_Y \) action we need to show that the map
\[
F_1 = \pi_Y \circ \text{Ln} \circ a : D \to \mathbb{R}^2
\]
given by
\[
F_1(a(s,t)) = (\text{Ln}(a(s,t)) \cdot w_1, \text{Ln}(a(s,t)) \cdot w_2)
\]
\[
= (2 \text{ln}(2 + s) - 2 \text{ln}(1 - s - t), 2 \text{ln}(1 + t) - 2 \text{ln}(1 - s - t))
\]
\[
= \left(2 \ln\left(\frac{2 + s}{1 - s - t}\right), 2 \ln\left(\frac{1 + t}{1 - s - t}\right)\right)
\]
is bijection. Compose this map with coordinate-wise exponentiation to get the map
\[
G_1 = E \circ F_1 : \quad G_1(s,t) = \left(\left(\frac{2 + s}{1 - s - t}\right)^2, \left(\frac{1 + t}{1 - s - t}\right)^2\right).
\]
The map $\pi_Y \circ a$ is a bijection from $D$ to $\mathbb{R}^2$ if and only if $E \circ \pi_Y \circ \text{Ln} \circ a$ has image $(\mathbb{R}_{>0})^2$. This in turn is true if and only if the map 

$$G_{1/2}(s,t) = \left( \frac{2 + s}{1 - s - t}, \frac{1 + t}{1 - s - t} \right),$$

where $F_{1/2}$ is as defined in Definition 4.6, has image $(\mathbb{R}_{>0})^2$.

We compose $G_{1/2}$ with the imbedding $i$ of $\mathbb{R}^2$ into $\mathbb{P}_2(\mathbb{R})$ and obtain 

$$(i \circ G_{1/2})(s,t) = [t + 1 : s + 2 : -s - t + 1].$$

In order to show that $\pi_Y : D \to \mathbb{R}^2$ is bijective it suffices to show that $i \circ G_{1/2}$ is a bijection from $D$ onto $P_2(\mathbb{R})_{>0}$.

Clearly $i \circ G_{1/2}$ is one-to-one as $i \circ G_{1/2}$ extends to the automorphism 

$$(s, t, u) \mapsto [t + u : s + 2u : -s - t + 2u]$$

of $\mathbb{P}_2(\mathbb{R})$. The boundary of $D$ is mapped by $i \circ G_{1/2}$ onto the boundary of $P_2(\mathbb{R})_{>0}$, so by continuity, $P$ sends $D$ onto $P_2(\mathbb{R})_{>0}$. Hence $\pi_Y$ is a bijection and by Lemma 4.4, $\text{Ln}(\Delta_{a_0})$ is a simple cross section for the $\rho_Y$ action.

By Remark 4.3, isomorphism classes in $S_\Lambda(\mathbb{R})$ are $D$ orbits. By Theorem 2.4, every Lie algebra in $S_\Lambda(\mathbb{R})$ is isomorphic to precisely one Lie algebra represented by an element in the intersection parametrizing set 

$$\Sigma(T, \Delta_{a_0}) = \{(1 + t, 2 + s, 1 - s - t, 1 - s - t, \pm(2 + s), \pm(1 + t)) : s, t \in D\}$$

and $\mathcal{L}_7(\mathbb{R})$. By the same theorem, we can use a different simple cross section for the $\rho_Y$ action to define $\Sigma$. By Remark 4.3, the set $\Sigma(T, \Delta_{a_0}^{1/2})$ of points of form 

$$\{(1 + t)^{1/2}, (2 + s)^{1/2}, 1, (1 - s - t)^{1/2}, (1 - s - t)^{1/2}, \pm(2 + s)^{1/2}, \pm(1 + t)^{1/2}\}^T,$$

with $(s,t) \in D$, is also a parametrizing set for $S_\Lambda(\mathbb{R})$. We have not yet considered the issue of the Jacobi Identity; we will return to this in Example 6.5.

Unfortunately, the method for showing bijectivity in the previous example does not generalize broadly. Instead, one may use the following generalization of Hadamard’s Global Inverse Function Theorem to show injectivity.

**Theorem 4.8** ([Gor72]). Let $M_1$ and $M_2$ be connected, oriented $n$-dimensional smooth manifolds of class $C^2$, with $M_2$ simply connected. A $C^1$ map $f : M_1 \to M_2$ is a diffeomorphism if and only if it is proper and the Jacobian $\det(\partial f_i/\partial x_j)$ never vanishes.
Recall that $f : M_1 \to M_2$ is proper if for all compact $K \subseteq M_2$, the preimage $f^{-1}(K)$ in $M_1$ is compact.

In the next example, we show how to use this theorem to show that $\Delta$ is a global cross section.

**Example 4.9.** Let $\Lambda \subseteq \Theta_7$ be the set

$$
\Lambda = \{(1,2,3),(1,3,4),(1,4,5),(1,5,6),(1,6,7),
(2,3,5),(2,4,6),(2,5,7),(3,4,7)\}.
$$

By Remark 3.3, isomorphism classes in $\mathcal{S}_\Lambda(\mathbb{R})$ are $D$ orbits.

The set $T = \text{span}_\mathbb{Z}\{e_7, e_8, e_9\}$ is a simple cross section for the $\rho_Y$ action. The vectors

\[
\begin{align*}
\boldsymbol{w}_1 &= (0, -1, 0, 1, 0, 1, -1, 0, 0)^T \\
\boldsymbol{w}_2 &= (-1, 0, 1, 0, 0, 0, 0, 1, -1)^T \\
\boldsymbol{w}_3 &= (-1, 0, 0, 0, 1, 1, 0, -1)^T
\end{align*}
\]

span $\text{Null}(Y^T)$.

The algebra defined by $a_0 = (1,1,1,1,2,1,2,1,1)^T$ satisfies the Jacobi Identity. For $s,t,u \in \mathbb{R}$, let

\[
\boldsymbol{a}(s,t,u) = a_0 + s\boldsymbol{w}_1 + t\boldsymbol{w}_2 + u\boldsymbol{w}_3,
\]

and let

$$
\Delta_{a_0} = \{\boldsymbol{a}(s,t,u) : (s,t,u) \in \mathbb{R}\} \cap (\mathbb{R}_{>0})^9.
$$

The domain $D$ for the parametrization $a : D \to \mathbb{R}^9$ of $\Delta_{a_0}$ is the bounded convex set defined by the five inequalities

$$
1 - t - u > 0, 1 - s > 0, 1 + t > 0, 1 + s > 0, 1 + u > 0.
$$

For $(s,t,u) \in D$, $F = \pi_Y \circ \text{Ln} \circ \boldsymbol{a}(s,t,u)$ is equal to

$$
\left(\ln\left(\frac{(1+s)(2+s)}{(1-s)(2-s+u)}\right), \ln\left(\frac{(1+t)^2}{(1-t-u)^2}\right), \ln\left(\frac{(1+u)(2-s+u)}{(1-t-u)^2}\right)\right).
$$

Define the map $G : D \to \mathbb{R}^3$ by

\[
\begin{align*}
G(s,t,u) &= (E \circ F)(s,t,u) = \\
&= \left(\frac{(1+s)(2+s)}{(1-s)(2-s+u)}\right), \left(\frac{(1+t)^2}{(1-t-u)^2}\right), \left(\frac{(1+u)(2-s+u)}{(1-t-u)^2}\right).
\end{align*}
\]

If we embed $D$ into $P_3(\mathbb{R})$ using $i : \mathbb{R}^3 \to P_3(\mathbb{R})$, the map $i \circ G : i(D) \to P_3(\mathbb{R})$ may be expressed as

$$
(s,t,u,v) \mapsto [p_1(s,t,u) : p_2(s,t,u) : p_3(s,t,u) : p_4(s,t,u)]
$$
where the polynomials $p_1, p_2, p_3$ and $p_4$ are

\[
\begin{align*}
    p_1(s, t, u) &= (v+s)(2v+s)(v-t-u)^2 \\
    p_2(s, t, u) &= (v+t)^2(v-s)(2v-s+u) \\
    p_3(s, t, u) &= (v+u)(2v-s+u)^2(v-s) \\
    p_4(s, t, u) &= (v-s)(2v-s+u)(v-t-u)^2.
\end{align*}
\]

Note that this map is not defined at $(s, t, u, v) = (1, -1, 2, 1)$, hence can not be extended to the boundary of $i(D)$.

However, the map $G$ is proper. To show this, we need to show that if a sequence of points $x_i$ approaches $\partial D$, the sequence $G(x_i)$ approaches $\partial(\mathbb{R}_{>0})^3$. As $x_i \to \partial D$, some numerator or denominator of a coordinate function in Equation (20) must go to zero. Then that coordinate will go to zero or infinity, unless both the numerator and denominator go to zero as $x_i \to \partial D$. There are three cases to consider, one for each coordinate function. If the numerator and denominator of $G_1$ go to zero simultaneously, then $s \to 1$ and $s \to -1$, a contradiction. (Note that $2-s+u = (1-s) + (1+u)$ can only go to zero if both $1-s$ and $1+u$ go to zero.) If the numerator and denominator of $G_2$ both go to zero, then $u \to 2$, so $1+u$ does not go to zero. This implies that $G_3 \to \infty$. If the numerator and denominator of $G_3$ both go to zero, then $t \to 2$. Then $G_2 \to \infty$. Thus, as $x_i \to \partial D$, $G(x_i) \to \partial(\mathbb{R}_{>0})^3$. Hence, $G$ is proper.

The Jacobian matrix for $F$ at $(s, t, u)$ is

\[
J(s, t, u) = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}.
\]

For all $(s, t, u) \in D$, the entries of $J(s, t, u)$ are positive. For $(s, t, u) \in D$, the matrix $J(s, t, u)$ is strictly diagonally dominant, so by the Lévy-Desplanques Theorem is invertible. As the Jacobian of $E$ is always invertible, the Jacobian of $F$ is everywhere invertible. By Theorem [1.8], $G$ is a diffeomorphism, hence surjective. Since $G$ is a diffeomorphism, $F$ is a diffeomorphism.

Let $\Sigma(T, \Delta_{a_0}) = T \cdot \Delta_{a_0}$. Theorem [2.4] implies that any algebra in $\mathcal{S}_\Lambda(\mathbb{R})$ is isomorphic to precisely one Lie algebra with structure constants given by an element of the set $\mathcal{L}_T(\mathbb{R}) \cap \Sigma(T, \Delta_{a_0})$.

Now we prove Theorem [2.4].

Proof. Let $\alpha \in \mathcal{L}_\Lambda(\mathbb{R})$ correspond to $a = [\alpha_{\{i,j\}}]_{\{i,j,k\} \in \Lambda}$ and let $\beta \in \mathcal{L}_\Lambda(\mathbb{R})$ correspond to $b = [\beta_{\{i,j\}}]_{\{i,j,k\} \in \Lambda}$. Let $s$ denote the sign vector for $\alpha$ and let $t$ denote the sign vector for $\beta$. We may write $a$ and $b$
as $a = s \cdot |a|$ and $b = t \cdot |b|$, where $|a| = [|\alpha_{(i,j)}^k|]_{(i,j,k) \in \Lambda}$ and $|b| = [|\beta_{(i,j,k)}^k|]_{(i,j,k) \in \Lambda}$.

Assume that $F_\varepsilon$ maps $S$ onto $\mathbb{R}^d$. Then $\pi_Y$ maps $L_n(S)$ onto $\mathbb{R}^d$. By Lemma 4.4, the set $L_n(S)$ meets each orbit of $\rho_Y$ at least once, so there exist $a'$ and $b'$ in $S$ so that $L_n(a')$ and $L_n|a|$ are in the same $\rho_Y$ orbit and $L_n b'$ and $L_n|b|$ are in the same $\rho_Y$ orbit. Since $T$ is a simple cross section for the $\rho_Y$ action, there exists a unique $s'$ in $T$ so that $s$ and $s'$ are in the same $\rho_Y$ orbit and there exists a unique $t'$ in $T$ so that $t$ and $t'$ are in the same $\rho_Y$ orbit. Recall that the action of $D$ on $S_\Lambda(\mathbb{R})$ is conjugate to the action $\rho_Y$ on $\mathbb{Z}^m \times (\mathbb{R}_{>0})^m$. Therefore $s \cdot |a|$ and $s' \cdot a'$ are in the same $D$ orbit, and $t \cdot |b|$ and $t' \cdot b'$ are in the same $D$ orbit. Since isomorphism classes in $S_\Lambda(\mathbb{R})$ are $D$ orbits, by Theorem 3.2, the Lie algebras defined by $a = s \cdot |a|$ and $s' \cdot a' \in \Sigma(T, S)$ are isomorphic, and the Lie algebras defined by $b = t \cdot |b|$ and $t' \cdot b' \in \Sigma(T, S)$ are isomorphic. We have shown that every element in $L_\Lambda(\mathbb{R})$ is isomorphic to at least one element of $\Sigma(T, S)$.

If in addition $F_\varepsilon$ is a bijection, then $L_n(S)$ is a simple cross section for the $\rho_Y$ action, and $s' \cdot a'$ and $t' \cdot b'$ are unique. By Theorem 3.2, these points are isomorphic if and only if $s' = t'$ and $a' = b'$. Thus, the Lie brackets $\alpha$ and $\beta$ are isomorphic if and only if $s' \cdot a' = t' \cdot b'$. Thus, every element in $L_\Lambda(\mathbb{R})$ is isomorphic to precisely one element of $\Sigma(T, S)$. \hfill \Box

5. Aligned pairs of triples, quadruples, and $\Lambda$-subspaces

5.1. Triples and quadruples. In this section we define some new kinds of objects: aligned pairs of triples, quadruples of pairs of triples, and the $\Lambda$-subspace for an index set $\Lambda$.

Definition 5.1. Let $t_1 = (i_1, j_1, k_1)$ and $t_2 = (i_2, j_2, k_2)$ be triples in $[n]^3$. Let $y_{(i_1,j_1,k_1)}$ and $y_{(i_2,j_2,k_2)}$ be real root vectors as defined in Section 3.1. We say that the triples $t_1$ and $t_2$ are an aligned pair if the inner product $\langle y_{(i_1,j_1,k_1)}, y_{(i_2,j_2,k_2)} \rangle$ of the corresponding root vectors is $-1$.

It is possible that an index set $\Lambda$ has no triples that form an aligned pair:

Example 5.2. Let $\Lambda \subseteq \Theta_4$ be as in Example 3.6. There are only two distinct triples $(1, 2, 3)$ and $(1, 3, 4)$ whose corresponding root vectors $y_{(1,2,3)} = (1, 1, -1, 0)$ and $y_{(1,3,4)} = (1, 0, 1, -1)$ have the matrix product $y_{(1,2,3)}^T y_{(1,3,4)}$ equal to zero. Therefore $\Lambda$ has no aligned pairs of triples.

It is also possible that an index set $\Lambda$ has many triples that form aligned pairs.
**Example 5.3.** Let $\Lambda \subseteq \Theta_7$ be as in Example 4.9. Denote the triples in $\Lambda$ by $t_1, \ldots, t_s, \ldots, t_9$, where the subscript $s$ ascends concordantly with the dictionary ordering on $\Lambda$:

$$t_1 = (1, 2, 3), t_2 = (1, 3, 4), \ldots, t_9 = (3, 4, 7).$$

There are five pairs of triples root vectors have inner product $-1 : t_4$ and $t_6$; $t_2$ and $t_7$; $t_5$ and $t_7$; $t_3$ and $t_8$; and $t_1$ and $t_9$.

Next we show that every aligned pair of triples $t_1$ and $t_2$ with $t_1, t_2 \in \Theta_n$ determines a unique quadruple in $[n]^4$, and we assign a sign to that quadruple. Let $t_1$ and $t_2$ be an aligned pair of triples in $\Theta_n$. As $t_1$ and $t_2$ are in $\Theta_n$, their root vectors are of form $(\ldots, 1, \ldots, 1, \ldots, -1, \ldots)$ where all entries are zero aside from those indicated. Because $t_1, t_2 \in \Theta_n$, we know that entries of both triples are distinct and in ascending order: $i < j < k$ and either $l < k < m$ or $k < l < m$. The product of $t_1$ and $t_2$ is then

$$-1 = (e_i + e_j - e_k)^T(e_k + e_l - e_m)$$

$$= \delta_{il} + \delta_{jl} - 1 \quad (\text{since } i, j < k < m; l < m; \text{ and } k \neq l).$$

as we have defined the root vectors over $\mathbb{R}$, both $\delta_{il}$ and $\delta_{jl}$ are zero. Therefore $i \neq l$ and $j \neq l$. It follows that the indices $i, j$ and $l$ are pairwise distinct. The index $k$ is characterized by the fact that is it the unique index occurring in both triples; therefore $\{i, j, l, m\}$ is the symmetric difference of the sets $\{i, j, k\}$ and $\{k, l, m\}$.

Suppose that $t_1 = (i_1, j_1, k_1)$ and $t_2 = (i_2, j_2, k_2)$ form an aligned pair of triples in $\Theta_n$. Let $\{q_1, q_2, q_3, q_4\}$ denote the symmetric difference of the sets $\{i_1, j_1, k_1\}$ and $\{i_2, j_2, k_2\}$, where $q_1 < q_2 < q_3 < q_4$. One may verify that there exists $r$ so that $t_1$ and $t_2$ are described by one of the six possibilities listed in Table 1. Note that Cases 4 and 6 cannot actually occur for a pair of triples in $\Theta_n$ because of hypotheses on order relations among the entries of the triples and the quadruple.

| Case | Aligned pair of triples | sign($t_1, t_2$) |
|------|--------------------------|-----------------|
| 1    | $\{t_1, t_2\} = \{(q_1, q_2, r), (r, q_3, q_4)\}$ | 1 |
| 2    | $\{t_1, t_2\} = \{(q_1, q_2, r), (q_3, r, q_4)\}$ | -1 |
| 3    | $\{t_1, t_2\} = \{(q_1, q_3, r), (q_2, r, q_4)\}$ | 1 |
| 4    | $\{t_1, t_2\} = \{(q_1, q_3, r), (r, q_2, q_4)\}$ | -1 |
| 5    | $\{t_1, t_2\} = \{(q_2, q_3, r), (q_1, r, q_4)\}$ | -1 |
| 6    | $\{t_1, t_2\} = \{(q_2, q_3, r), (r, q_1, q_4)\}$ | 1 |

**Table 1.** Possible aligned pairs of triples and their signs
Now we are ready to define the quadruple associated to a pair of triples, and the sign associated to a pair of quadruples.

**Definition 5.4.** Let $t_1 = (i_1, j_1, k_1)$ and $t_2 = (i_2, j_2, k_2)$ be an aligned pair of triples in $\Theta_n$. The quadruple $q(t_1, t_2)$ for $t_1$ and $t_2$ is defined to be

$$q(t_1, t_2) = (q_1, q_2, q_3, q_4) \in [n]^4,$$

where $\{q_1, q_2, q_3, q_4\}$ is the symmetric difference of the sets $\{i_1, j_1, k_1\}$ and $\{i_2, j_2, k_2\}$ and $q_1 < q_2 < q_3 < q_4$.

Define the sign $\text{sign}(t_1, t_2)$ of $t_1$ and $t_2$ by defining $\text{sign}(t_1, t_2)$ to be $-1$ in Cases 2, 4 and 5 of Table 1, and defining $\text{sign}(t_1, t_2)$ to be 1 in Cases 1, 3 and 6.

We say that the triple $s$ is a common triple for quadruples $q_1$ and $q_2$ if there are triples $t_1, t_2$ so that $q_1 = q(s, t_1)$ and $q_2 = q(s, t_2)$.

Let $Q$ be the set of quadruples associated to any aligned pairs of triples in $\Lambda$. We say that a quadruple in $Q$ has multiplicity $m$ if it arises from exactly $m$ distinct pairs of triples.

Note that $q(t_1, t_2) = q(t_2, t_1)$.

**Example 5.5.** Let $\Lambda \subseteq \Theta_7$ be as in Example 4.9 and 5.3. The quadruples associated to the aligned pairs are

$$q(t_4, t_6) = q(t_2, t_7) = (1, 2, 3, 6)$$
$$q(t_5, t_7) = q(t_3, t_8) = q(t_1, t_9) = (1, 2, 4, 7).$$

The set of quadruples for $\Lambda$ is

$$Q = \{(1, 2, 3, 6), (1, 2, 4, 7)\}.$$

The quadruple $(1, 2, 3, 6)$ has multiplicity two and the quadruple $(1, 2, 4, 7)$ has multiplicity three. The signs of the pairs are

$$\text{sign}(t_4, t_6) = \text{sign}(t_5, t_7) = -1, \quad \text{and}$$
$$\text{sign}(t_2, t_7) = \text{sign}(t_3, t_8) = \text{sign}(t_1, t_9) = 1.$$

If there are no aligned pairs of triples in $\Lambda$, then there are no quadruples associated to $\Lambda$. This is the case with Examples 3.6 and 5.2. It is also possible that there is only one quadruple associated to an index set.

**Example 5.6.** Enumerate $\Lambda$ from Example 4.7 in dictionary order. There are three aligned pairs: $t_4$ and $t_5$, $t_2$ and $t_6$, and $t_1$ and $t_7$. For each of these pairs, the associated quadruple is $(1, 2, 3, 7)$. That quadruple has multiplicity three.
5.2. A subspace determined by the index set. If two aligned pairs in an index set \( \Lambda \) have the same quadruple associated to them, they determine a vector \( \mathbf{w} \) defined as follows. In Theorem 5.9, it will be seen that all such vectors are contained in the left null space of the root matrix \( Y \) for \( \Lambda \).

**Definition 5.7.** Fix an index set \( \Lambda \subseteq \Theta_0 \) and enumerate its elements so that \( \Lambda = \{t_1, \ldots, t_m\} \). Suppose that \( \{t_{m_1}, t_{m_2}\} \) and \( \{t_{m_3}, t_{m_4}\} \) are two different aligned pairs from \( \Lambda \) with the same quadruple associated to them, they so that \( \Lambda = \{t_1, \ldots, t_m\} \). Suppose that \( \{t_{m_1}, t_{m_2}\} \) and \( \{t_{m_3}, t_{m_4}\} \) are two different aligned pairs from \( \Lambda \) with the same quadruple

\[
\mathbf{w}(m_1, m_2, m_3, m_4) = e_{m_1} + e_{m_2} - e_{m_3} - e_{m_4}.
\]

If we omit mention of the field we assume that the field is \( \mathbb{R} \).

Define the subspace \( W_\Lambda(K) \) of \( K^m \) to be the span of all vectors \( \mathbf{w}(m_1, m_2, m_3, m_4) \) over \( K \) arising from all aligned pairs \( \{t_{m_1}, t_{m_2}\} \) and \( \{t_{m_3}, t_{m_4}\} \) sharing the same quadruple:

\[
W_\Lambda(K) = \text{span}_K \{ \mathbf{w}(m_1, m_2, m_3, m_4) : q(t_{m_1}, t_{m_2}) = q(t_{m_3}, t_{m_4}) \}.
\]

We call \( W_\Lambda(K) \) the \( \Lambda \)-subspace of \( K^m \).

Note that if \( \{t_{m_1}, t_{m_2}\} \) and \( \{t_{m_3}, t_{m_4}\} \) are aligned pairs with the same quadruple,

\[
\mathbf{w}(m_1, m_2, m_3, m_4) = \mathbf{w}(m_1, m_2, m_3, m_4) = \mathbf{w}(m_2, m_1, m_3, m_4)
\]

and

\[
\mathbf{w}(m_1, m_2, m_3, m_4) = -\mathbf{w}(m_3, m_4, m_1, m_2).
\]

**Example 5.8.** Let \( \Lambda \) be as in Examples 4.9, 5.3 and 5.5. The aligned pairs \( \{t_4, t_6\} \) and \( \{t_2, t_7\} \) have associated quadruple \( (1, 2, 3, 6) \). The vector in \( \mathbb{R}^9 \) associated to these two aligned pairs is

\[
\mathbf{w}(4, 6, 2, 7) = e_4 + e_6 - e_2 - e_7.
\]

We also have (up to sign changes) three other vectors \( \mathbf{w}(m_1, m_2, m_3, m_4) \) arising from the three other aligned pairs of triples, giving a total of four vectors (up to signs)

\[
\begin{align*}
\mathbf{w}_1 &= \mathbf{w}(4, 6, 2, 7) = (0, -1, 0, 1, 0, 1, -1, 0, 0)^T \\
\mathbf{w}_2 &= \mathbf{w}(5, 7, 3, 8) = (0, 0, -1, 0, 1, 0, 1, -1, 0)^T \\
\mathbf{w}_3 &= \mathbf{w}(3, 8, 1, 9) = (-1, 0, 1, 0, 0, 0, 0, 1, -1)^T \\
\mathbf{w}_4 &= \mathbf{w}(1, 9, 5, 7) = (1, 0, 0, 0, -1, 0, -1, 0, 1)^T.
\end{align*}
\]

Observe that \( \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 + \mathbf{w}_4 = 0 \) in \( \mathbb{R}^9 \).
The real Λ-subspace is the three-dimensional subspace
\[ W_\Lambda(\mathbb{R}) = \text{span}_{\mathbb{R}}\{w_1, w_2, w_3\} \subseteq \mathbb{R}^9. \]

The Λ-subspace is always a subspace of the left null space of the root matrix \(Y(K)\) associated to Λ.

**Theorem 5.9.** Let \(\Lambda \subseteq \Theta_n\), and let \(W_\Lambda(K)\) be the Λ-subspace of \(K^m\) as in Definition 5.7. Let \(Y(K)\) be the \(K\) root matrix for \(\Lambda\). Then \(W_\Lambda(K)\) is a subspace of \(\text{Null}(Y(K)^T)\).

**Proof.** Refer to Table 1. If the pair of triples \(\{t_{m_1}, t_{m_2}\}\) has associated quadruple \((q_1, q_2, q_3, q_4)\), then in all six cases
\[ y_{t_{m_1}} + y_{t_{m_2}} = e_{q_1} + e_{q_2} + e_{q_3} - e_{q_4}. \]
Therefore, if \(\{t_{m_1}, t_{m_2}\}\) and \(\{t_{m_3}, t_{m_4}\}\) are two aligned pairs of triples both having the same quadruple \((q_1, q_2, q_3, q_4)\),
\[ y_{t_{m_1}} + y_{t_{m_2}} = y_{t_{m_3}} + y_{t_{m_4}}. \]

Recall that the vector \(y_{m_4}\) is the \(m_4\)th row of the root matrix \(Y(K)\). Hence the dependency
\[ y_{m_1} + y_{m_2} - y_{m_3} - y_{m_4} = 0 \quad (21) \]
of rows may be written as \(wY(K) = 0\), where
\[ w = w(m_1, m_2, m_3, m_4) = e_{m_1} + e_{m_2} - e_{m_3} - e_{m_4}. \]
Therefore \(Y^Tw^T = 0\) and \(w^T\) is in the null space of \(Y\). \(\square\)

**Definition 5.10.** The subset \(\Lambda\) of \(\Theta_n\) is said to be **null space spanning** over \(K\) if the Λ-subspace of \(K^n\) is equal to the full null space: \(W_\Lambda(K) = \text{Null}(Y(K)^T)\).

We revisit Example 4.7.

**Example 5.11.** Let \(\Lambda\) be as in Examples 4.7 and 5.6. We saw that the vectors \(w_1 = (0, 1, 0, -1, -1, -1, 1, 0)\) and \(w_2 = (1, 0, 0, -1, -1, 0, 1)\) spanned \(\text{Null}(Y^T)\). But these vectors are just the vectors determined by the aligned pairs of triples we saw in Example 5.6: \(w_1 = w(2, 6, 4, 5)\) and \(w_2 = w(1, 7, 4, 5)\). Hence \(\Lambda\) is null space spanning.

We leave it to the reader to verify that the index sets \(\Lambda\) in Example 4.5 and Example 4.9 are null space spanning. Not all index sets are null space spanning, as the following example from dimension eight shows.

**Example 5.12.** For the subset \(\Lambda\) of \(\Theta_8\) defined by
\[ \Lambda = \{(1, 2, 4), (1, 3, 5), (1, 4, 6), (1, 5, 7), (1, 7, 8), (2, 3, 6), (2, 4, 7), (2, 6, 8), (3, 5, 8)\}, \]
the only quadruple is \((1, 2, 4, 8)\). It has multiplicity two, arising from the two aligned pairs of triples \(\{t_5, t_7\} = \{(1, 7, 8), (2, 4, 7)\}\) and \(\{t_3, t_8\} = \{(1, 4, 6), (2, 6, 8)\}\). However, the span of \(w_1 = e_5 + e_7 - e_3 - e_8\) is not the full left null space of the associated root matrix \(Y\). Actually, \(\text{Null}(Y^T)\) is spanned by \(w_1\) and \(w_2 = (1, 0, 1, -1, -1, 0, 1)^T\).

**Remark 5.13.** Even in the case that \(\Lambda\) is not null space spanning, the vectors \(w_i\) may still be used as part of a basis for the tangent space. As they have all entries of zero except for two 1’s and two -1’s, a basis including them may be simpler than a full basis found by a computer algebra system.

### 5.3. Criterion for injectivity

Recall that in Example 4.39 the Jacobian matrix of the mapping \(F\) was nonsingular because it was diagonally dominant. For a general index set \(\Lambda\), one can extract a condition on the combinatorics of the set of quadruples that makes the Jacobian matrix of the mapping \(F\) diagonally dominant.

**Lemma 5.14.** Let \(\Lambda \subseteq \Theta_n\) be an index set of cardinality \(m\) with associated root matrix \(Y\). Suppose that \(\Lambda\) is null space spanning and that \(B = \{w_1, w_2, \ldots, w_d\}\) is a basis for \(\text{Null}(Y^T)\), where for \(i = 1, \ldots, d\), the vector \(w_i\) arises from the quadruple \(q_i = (t_1^i, t_2^i, t_3^i, t_4^i) \subseteq [n]^4\):

\[
    w_i = w(t_1^i, t_2^i, t_3^i, t_4^i) = e_{t_1^i} + e_{t_2^i} - e_{t_3^i} - e_{t_4^i}.
\]

Suppose that the set of quadruples for \(\Lambda\) satisfies the following conditions.

1. For each quadruple \(q_i\), there is at least one element from the list \(t_1^i, t_2^i, t_3^i, t_4^i\) that does not occur in any of the other quadruples.
2. For each quadruple \(q_i\), each element \(t_1^i, t_2^i, t_3^i, t_4^i\) of the quadruple occurs at most once in all of the other quadruples.

Let \(a(0) \in (\mathbb{R}_{>0})^m\). For \(s = (s_1, \ldots, s_m) \in \mathbb{R}^m\), let \(a(s) = a(0) + \sum_{i=1}^d s_i w_i\). and let \(\Delta_{a_0} = \{a(s) : s \in \mathbb{R}^m\} \cap (\mathbb{R}_{>0})^m\). Then \(\text{Ln}(\Delta_{a_0})\) is a simple cross section for the \(\rho_Y\) action.

**Proof.** Denote the coordinate functions of \(a(s)\) by \(a_1, \ldots, a_d\). Let \(D\) denote the set of values for \(s\) which parametrize \(\Delta_{a_0}\):

\[
    D = \{s : a_i(s) > 0 \text{ for all } i = 1, \ldots, d\}.
\]

For \(i = 1, \ldots, d\), the function \(a_i\) is given by

\[
    a_i(s) = a_i(0) + \sum_{i \in \{t_1^i, t_2^i\}} s_k - \sum_{i \in \{t_3^i, t_4^i\}} s_k,
\]
so for \( s \in D, \)
\[
\ln(a_i(s)) = \ln \left( a_i(0) + \sum_{t_i t_k} s_k - \sum_{t_i t_k} s_k \right).
\]
The partial derivatives of the functions \( a_i \) with respect to \( s_j \) are
\[
\frac{\partial a_i}{\partial s_j} = \begin{cases} 
0 & i \notin \{t_1, t_2, t_3, t_4\} \\
1 & i \in \{t_1, t_2\} \\
-1 & i \in \{t_3, t_4\}.
\end{cases}
\]
Fix \( i \). The \( i \)th coordinate function of \( F = \pi_Y \circ \ln \circ a \) is
\[
F_i(s) = (\pi_Y \circ \ln \circ a)_i(s) \\
= w_i \cdot (\pi_Y \circ \ln \circ a)(s) \\
= \ln(a_{t_1}(s)) + \ln(a_{t_2}(s)) - \ln(a_{t_3}(s)) - \ln(a_{t_4}(s)).
\]
Then
\[
\frac{\partial F_i}{\partial s_i} = \sum_{k \in \{t_1, t_2, t_3, t_4\}} \frac{\partial F_i}{\partial a_k} \frac{\partial a_k}{\partial s_i} = \sum_{k \in \{t_1, t_2, t_3, t_4\}} \frac{1}{a_k(s)}.
\]
while for \( i \neq j, \)
\[
\frac{\partial F_i}{\partial s_j} = \sum_{k \in \{t_1, t_2, t_3, t_4\}} \frac{\partial F_i}{\partial a_k} \frac{\partial a_k}{\partial s_j} = \sum_{k \in \{t_1, t_2, t_3, t_4\} \cap \{t_1, t_2, t_3, t_4\}} \frac{1}{a_k(s)}.
\]
Therefore the sum of the nondiagonal entries in row \( i \) of the Jacobian matrix for \( F \) is
\[
\sum_{i \neq j} \frac{\partial F_i}{\partial s_j} = \sum_{i \neq j} \sum_{k \in \{t_1, t_2, t_3, t_4\} \cap \{t_1, t_2, t_3, t_4\}} \frac{1}{a_k(s)}.
\]
If it is \( t_3 \) or \( t_4 \) which satisfies hypotheses \( \Box \) of the lemma, we can replace \( w_i = w(t_1, t_2, t_3, t_4) \) by \( -w_i = w(t_3, t_4, t_1, t_2) \) in the parametrization without changing whether or not the Jacobian is nonzero. Then since \( w(t_1, t_2, t_3, t_4) = w(t_3, t_4, t_1, t_2) \) we may assume without loss of generality that \( t_i \notin \{t_1, t_2, t_3, t_4\} \) for all \( j \neq i. \)

By hypothesis \( \Box \),
\[
\sum_{i \neq j} \frac{\partial F_i}{\partial s_j} \leq \sum_{k \in \{t_1, t_2, t_3, t_4\}} \frac{1}{a_k(s)} \leq \frac{\partial F_i}{\partial s_i}.
\]
Thus, the Jacobian is diagonally dominant. Therefore by the Lévy-Desplanques Theorem, it is nonsingular. \( \Box \)
6. The Jacobi Identity

The following theorem from \[\text{Pay10}\] reformulates the Jacobi Identity in terms of structure constants in the case that the basis is triangular. We have rephrased the hypotheses of the theorem using the language of aligned pairs and associated quadruples.

**Theorem 6.1** (Theorem 7, \[\text{Pay10}\]). Let $B = \{x_i\}^n_{i=1}$ be a triangular basis for $\mathbb{R}^n$. Let the vector $[\alpha^k_{ij}]_{(i,j,k) \in \Lambda}$ of nonzero structure constants indexed by $\Lambda \subseteq \Theta_n$, together with the basis $B$, define a skew-symmetric product $\mu : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$.

The product $\mu$ defines a Lie algebra if and only if, whenever there exists an aligned pair of triples $t_p$ and $t_r$ in $\Lambda$ with associated quadruple $(i,j,k,m)$ then the equality

$$\sum_{s<m} \alpha^s_{ij}\alpha^m_{sk} + \alpha^s_{jk}\alpha^m_{si} + \alpha^s_{ki}\alpha^m_{sj} = 0$$

holds. Furthermore, a term of form $\alpha^l_{ij}\alpha^m_{lk}$ in Equation (22) is nonzero if and only if the corresponding triples $(i,j,l)$ or $(j,i,l)$, and $(l,k,m)$ or $(k,l,m)$, are an aligned pair.

Although Theorem 6.1 is proved over $\mathbb{R}$ in \[\text{Pay10}\], it remains true over arbitrary fields. Now we prove Theorem 2.1.

**Proof.** We need to see how to rewrite Equation (22) of Theorem 6.1, in which the subscripts $i, j$ and $k$ are ordered cyclically in each summand, to a convention in which all the terms in the sum are of the form $\alpha^k_{ij}\alpha^m_{kl}$ with $i < j < k$ and $k < l < m$.

Suppose that $(q_1, q_2, q_3, q_4)$ is in the set of quadruples for $\Lambda$. Then $q_1 < q_2 < q_3 < q_4$. Suppose that triples $t_1$ and $t_2$ are an aligned pair with quadruple $(q_1, q_2, q_3, q_4)$. There is an $r \in [\tilde{n}]$ so that one of the triples is in the set $\{(q_1, q_2, r), (q_1, q_3, r), (q_2, q_3, r)\}$.

We need to see whether the corresponding nonvanishing term of Equation (22) changes sign if we re-order the lower subscripts so they are all in ascending order. There are six cases, as listed in Table 2, to consider. In each case, the left side of the equality in the third column expresses term from the Jacobi Identity so that the subscripts come from the triples as in Equation (5), and on the right side, subscripts are in a cyclic form of $(q_1, q_2, q_3)$ as in Equation (22). In each case, the sign change is determined by the value of $\text{sign}(t_1, t_2)$ given in Table 1. By making appropriate substitutions, we get Equation (5).
is equivalent to the equation \[ \sum_{i,j,k} \alpha_{ij}^r \alpha_{jk}^q = \alpha_{ij}^r \alpha_{jk}^q \]

In Example 4.5, we found that the set \( \Sigma(\Delta) \) may solve for elements of \( \Sigma(\Delta) \) for an element of \( \Sigma(T, \Delta) \) into the first equation gives 1 + s = 1 - s. Hence, s = 0. The sign vector must be \((0, 0, 0, 0, 0, 0)\) in order for the second equation to be true. We conclude that every Lie algebra in \( \mathcal{S}_\Lambda(\mathbb{R}) \) is isomorphic to the one with structure constants \( \alpha_{ij}^k = 1 \) for all \((i, j, k) \in \Lambda \).

The next example is of the type described by Part 3 of Theorem 2.6.
Example 6.4. Let
\[ \Lambda = \{(1, 3, 4), (1, 4, 6), (1, 6, 7), (1, 7, 8), (2, 3, 6), (2, 4, 7), (2, 6, 8), (3, 5, 8)\}. \]
There are four aligned pairs. The pairs
\[ \{t_1, t_6\} = \{(1, 3, 4), (2, 4, 7)\} \quad \text{and} \quad \{t_3, t_5\} = \{(1, 6, 7), (2, 3, 6)\} \]
have the quadruple \((1, 2, 3, 7)\) associated to them, while
\[ \{t_2, t_7\} = \{(1, 4, 6), (2, 6, 8)\} \quad \text{and} \quad \{t_4, t_6\} = \{(1, 7, 8), (2, 4, 7)\} \]
have the quadruple \((1, 2, 4, 8)\) associated to them. Accordingly, the vectors
\[ w_1 = (1, 0, -1, 0, -1, 1, 0, 0)^T, \]
\[ w_2 = (0, 1, 0, -1, 0, -1, 1, 0)^T \]
are in \(\text{Null}(Y^T)\) as guaranteed by Proposition 5.9. The null space of \(Y^T\) is two-dimensional so \(\Lambda\) is null space spanning.

For \(s, t \in \mathbb{R}\), let
\[ a(s, t) = (1, 1, 1, 1, 1, 1, 1, 1)^T + sw_1 + tw_2. \]
The simple cross section for the \(\rho_Y\) action of \(\mathbb{Z}_2^n\) on \(\mathbb{Z}_2^m\) has only one element, \(0 \in \mathbb{Z}_2^n\). Methods from Section 4 may be used to show that every Lie algebra in \(\mathcal{S}_\Lambda(\mathbb{R})\) is isomorphic to exactly one Lie algebra in \(\Sigma(T, \Delta) = \{(1+s, 1+t, 1-s, 1-t, 1-s, 1+s-t, 1+t, 1) : (s, t) \in D\}\), where
\[ D = \{(s, t) : |s| < 1, |t| < 1, t < 1+s\}. \]
By Theorem 6.1, the Jacobi Identity for elements of products \(\mu\) in \(\mathcal{S}_\Lambda(\mathbb{R})\) is equivalent to the system of equations
\[ \alpha_{13}^4\alpha_{24}^7 - \alpha_{16}^7\alpha_{23}^6 = 0, \quad \alpha_{14}^6\alpha_{26}^8 - \alpha_{17}^8\alpha_{24}^7 = 0. \]
Equivalently, the absolute values of the structure constants satisfy
\[ |\alpha_{13}^4\alpha_{24}^7| = |\alpha_{16}^7\alpha_{23}^6| \quad \text{and} \quad |\alpha_{14}^6\alpha_{26}^8| = |\alpha_{17}^8\alpha_{24}^7| \]
while simultaneously, their signs satisfy
\[ \text{sgn}(\alpha_{13}^4\alpha_{24}^7) = \text{sgn}(\alpha_{16}^7\alpha_{23}^6), \quad \text{and} \quad \text{sgn}(\alpha_{14}^6\alpha_{26}^8) = \text{sgn}(\alpha_{17}^8\alpha_{24}^7). \]
Since the sign vector \(0 \in T\) has all zero entries, all signs are positive for points in the simple cross section, and the equalities in (25) in hold. Substituting the values of \(\alpha_{ij}^k, (i, j, k) \in \Lambda\), from (23) into Equation (24) yields
\[ (1+s)(1+s-t) = (1-s)^2 \]
\[ (1+t)^2 = (1-t)(1+s-t). \]
It is not hard to show that the only solution yielding positive values for the squares of the structure constants is \( s = t = 0 \). Substituting \( s = t = 0 \) into Equation (23) gives
\[
\Sigma(T, \Delta) \cap L_\Lambda(\mathbb{R}) = \{(1, 1, 1, 1, 1, 1, 1, 1)\}.
\]
Thus, every Lie algebra in \( L_\Lambda(\mathbb{R}) \) is isomorphic to the one with the structure constants
\[
[a_{ijk}^k]_{(i,j,k) \in \Lambda} = (1, 1, 1, 1, 1, 1, 1, 1).
\]

**Example 6.5.** Let \( \Lambda \) be as in Examples 4.7, 5.6, and 5.11. We have established that each Lie algebra in \( S_\Lambda(\mathbb{R}) \) is represented exactly once in the set of points
\[
\Sigma(T, \Delta_{a_0}) = \{\mathbf{a}(s, t) : s, t \in D\},
\]
where the vector \( \mathbf{a}(s, t) = [\alpha_{ijk}^k]_{(i,j,k) \in \Lambda} \) is equal to
\[
(1 + t, 2 + s, 1, 1 - s - t, 1 - s - t, \pm(2 + s), \pm(1 + t))^T
\]
and
\[
D = \{(s, t) : s > -2, t > -1, \text{and } s + t < 1\} \subseteq \mathbb{R}^2.
\]
The set \( \Sigma(T, \Delta_{a_0}^{1/2}) \) of points of form
\[
((1 + t)^{1/2}, (2 + s)^{1/2}, 1, (1 - s - t)^{1/2}, (1 - s - t)^{1/2}, \pm(2 + s)^{1/2}, \pm(1 + t)^{1/2})^T,
\]
with \( (s, t) \in D \), is a second parametrizing set.

By Theorem 2.1, the Jacobi Identity for \( S_\Lambda(\mathbb{R}) \) is equivalent to
\[
\alpha_{13}^5 \alpha_{25}^7 - \alpha_{23}^6 \alpha_{16}^7 - \alpha_{12}^4 \alpha_{34}^7 = 0.
\]
Note that if the signs are given by the sign vector \( e_6 \in \mathbb{Z}_2^7 \), all terms on the left side of the equation are negative, so there are no solutions. Hence the sign vector for a Lie algebra in \( \Sigma(T, \Delta_{a_0}) \) or \( \Sigma(T, \Delta_{a_0}^{1/2}) \) must be in
\[
T_1 = \{(0, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0, 0, 1), (0, 0, 0, 0, 0, 1, 1)\}.
\]

First we consider the Jacobi Identity for products in \( \Sigma(T, \Delta_{a_0}^{1/2}) \):
\[
(26) \quad - \text{sign}(\alpha_{13}^5) \text{sign}(\alpha_{25}^7)(2 + s) + \\
\text{sign}(\alpha_{23}^6) \text{sign}(\alpha_{16}^7)(1 - s - t) + \text{sign}(\alpha_{12}^4) \text{sign}(\alpha_{34}^7)(1 + t) = 0.
\]

There are now three cases to consider, one for each component of \( \Sigma(T_1, \Delta^{1/2}) \).
In sum, each Lie algebra in \( S' \) get the parametrizing set \( \Sigma \) whose structure constants are encoded by a vector in \( \Sigma = T, \Sigma(\Delta') \). When the sign vector is \((0, 0, 0, 0, 0, 1)\), we get

\[
0 = -(2 + s) + (1 - s - t) + (1 + t) = -2s,
\]

which has solutions \( s = 0, -1 < t < 1 \) in \( D \). Therefore, structure constants are in

\[
\Sigma_1 = \{(\sqrt{1 + t}, \sqrt{2}, 1, \sqrt{1 - t}, \sqrt{2}, \sqrt{1 + t}) : -1 < t < 1\}.
\]

If the sign vector is \((0, 0, 0, 0, 0, 1)\), we get

\[
0 = -(2 + s) + (1 - s - t) - (1 + t) = -2 - 2s - 2t,
\]

hence \( s + t = -1 \). and structure constants are

\[
\Sigma_3 = \{((\sqrt{-s}, \sqrt{2 + s}, 1, \sqrt{2}, \sqrt{2 + s}, -\sqrt{-s}) : -2 < s < 0\}
\]

In the last case, when signs are encoded by \((0, 0, 0, 0, 0, 1)\), we have

\[
0 = (2 + s) + (1 - s - t) - (1 + t) = 2 - 2t,
\]

hence \( t = 1 \). The corresponding structure constants are

\[
\Sigma_2 = \{((\sqrt{2}, \sqrt{2 + s}, 1, \sqrt{-s}, \sqrt{2 - s}, -\sqrt{2 + s}) : -2 < s < 0\}.
\]

In sum, each Lie algebra in \( S_\lambda(\mathbb{R}) \) is isomorphic to precisely one Lie algebra whose structure constants are encoded by a vector in \( \Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \).

Now we find an alternate parametrization using \( \Sigma(T, \Delta) \) instead of \( \Sigma(T, \Delta^{1/2}) \). In this case, the Jacobi Identity becomes

\[
\tag{27}
-\text{sign}(\alpha_{12}^5) \text{sign}(\alpha_{25}^7)(2 + s)^2 + \text{sign}(\alpha_{23}^6) \text{sign}(\alpha_{16}^7)(1 - s - t)^2 + \text{sign}(\alpha_{12}^4) \text{sign}(\alpha_{34}^7)(1 + t)^2 = 0.
\]

When the sign vector is \((0, 0, 0, 0, 0, 0)\), the Jacobi Identity becomes

\[
2t^2 + 2st - 6s - 2 = 0,
\]

so

\[
\Sigma'_1 = \{(1 + t, 2 + s(t), 1, 1 - s(t) - t, 1 - s(t) - t, 2 + s(t), 1 + t) : -1 < t < 1\}.
\]

where \( s(t) = \frac{1 - t^2}{t - 3} \). We may solve for \( \Sigma'_2 \) and \( \Sigma'_3 \) in a similar manner to get the parametrizing set \( \Sigma' = \Sigma'_1 \cup \Sigma'_2 \cup \Sigma'_3 \), where

\[
\Sigma'_2 = \{(1 + t_2(s), 2 + s, 1, 1 - s - t_2(s), 1 - s - t_2(s), 2 + s, -1 - t_2(s)) : -2 < s < 0\}
\]

\[
\Sigma'_3 = \{(1 + t_3(s), 2 + s, 1, 1 - s - t_3(s), 1 - s - t_3(s), -2 - s, -1 - t_3(s)) : -2 < s < 0\},
\]

with \( t_2(s) = \frac{3s^2 + 2}{s - 2} \) and \( t_3(s) = \frac{s^2 + s + 2}{2 - s} \).
Example 6.6. Let $\Lambda$ be as in Examples 4.9, 5.3, 5.5, and 5.8. The Jacobi Identity for elements of the stratum $S_\Lambda(\mathbb{R})$ reduces to two equations, one for each of the quadruples $(1, 2, 3, 6)$ and $(1, 2, 4, 7)$:

$$0 = \alpha_6^6\alpha_5^5 - \alpha_4^4\alpha_2^6$$
$$0 = \alpha_7^7\alpha_6^6 - \alpha_5^5\alpha_4^7 + \alpha_3^3\alpha_3^6.$$

There are four sets of sign choices to consider, one for each element of the simple transversal for the $\rho_Y$ action. We need to solve these equations for elements of some set of form $\Sigma(T, \Delta_{a_0})^p$.

For example, if the sign vector is $0$, substituting the values of $a(s, t, u)$ from (19) into the equations above gives

$$0 = 6s - u + su$$
$$0 = -2t^2 - 2ut - s + 5u - su.$$

This set, and the other components of $\Sigma(T, \Delta_{a_0}) \subseteq L_7(\mathbb{R})$ arising from other sign choices, can be parametrized by established methods.

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