RATIONAL SINGULARITIES, QUIVER MOMENT MAPS, AND REpresentations of surface Groups

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Abstract. We prove using jet schemes that the zero loci of the moment maps for the quivers with one vertex and at least two loops have rational singularities. This implies that the spaces of representations of the fundamental group of a compact Riemann surface of genus at least two have rational singularities. This has consequences for the numbers of irreducible representations of the special linear groups over the integers and over the $p$-adic integers.

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1. Introduction

Let $\text{Mat}(n, \mathbb{C})$ be the space of $n \times n$ matrices of complex numbers. We prove:

Theorem 1.1. Let $g \geq 2$. The set

$$X = \{(x_1, y_1, \ldots, x_g, y_g) \in \text{Mat}(n, \mathbb{C})^{2g} \mid [x_1, y_1] + \cdots + [x_g, y_g] = 0\},$$

where $[x, y] = xy - yx$, is a variety with rational singularities for all $n \geq 1$.

Throughout this article, a variety over an algebraically closed field is an integral scheme of finite type.

Rational singularities are a natural class of mild singularities. For example, the rational double points of surfaces are classified by the ADE diagrams. While the definition is technical, the importance of rational singularities lies in the abundance of applications they entail. We will see that the theorem has several applications, among which is a bound for the number of irreducible representations of $\text{SL}_n(\mathbb{Z})$ and of $\text{SL}_n(\mathbb{Z}_p)$.

Theorem 1.1 will allow us to prove:

Key words and phrases. Representation; special linear group; surface group; rational singularity; quiver moment map; jet scheme.
**Theorem 1.2.** Let \( C_g \) be a compact Riemann surface of genus \( g \geq 2 \). Then

\[
\text{Hom}(\pi_1(C_g), \text{GL}_n(C)) = \{(x_1, y_1, \ldots, x_g, y_g) \in \text{GL}_n(C)^{\times 2g} \mid \prod_{i=1}^{g} x_i y_i x_i^{-1} y_i^{-1} = I\}
\]
is a variety with rational singularities.

By \( \pi_1(C_g) \) we denote the fundamental group of \( C_g \) based at a fixed point. Simpson [Si94] showed that \( \text{Hom}(\pi_1(C_g), \text{GL}_n(C)) \) with its natural scheme structure is a normal complete intersection variety for \( g \geq 2 \).

The variety \( X \) from Theorem 1.1 is the zero locus of the moment map of the quiver \( Q \) with one vertex and \( g \) loops. In this context, Crawley-Boevey [CB01] also showed that \( X \) with its natural scheme structure is a complete intersection variety for \( g \geq 2 \).

Reducedness in the case \( g = 1 \) of both theorems is an old still open conjecture.

More recently, Aizenbud-Avni [AA16] showed that 0 is a rational singularity of \( X \) for \( g \geq 12 \). The proof of Theorem 1.1 here is different than theirs and works for all \( g \geq 2 \) and for all the points of \( X \).

To show rational singularities for \( X \), we use a criterion due to Mustată [Mu01] saying that the dimension of the scheme of \( m \)-jets passing through the singular locus should be small. To apply this we use induction on strata defined by the representation type of the semisimplifications of the representations in the zero locus of the moment map of the quiver \( Q \). This mirrors the proof of normality for the Mardsen-Weinstein reductions for quiver representations by Crawley-Boevey [CB03]. We use however the full power of Luna’s étale slices, not only the information about the affine quotients. In addition, we are able to control the \( m \)-jets of the zero locus of the moment map of a quiver passing through to the representations fixed by the natural group action. Arguably, this is essentially the only new mathematical ingredient in the whole article, everything else being already known to some sets, perhaps disjoint, of experts.

To conclude the second theorem from the first we use an observation of Kaledin-Lehn-Sorger [KLS06]. Namely, the local structure of \( \text{Hom}(\pi_1(C_g), \text{GL}_n(C)) \) at semisimple representations is modelled by zero loci of moment maps of quivers naturally associated with \( Q \).

The last theorem implies:

**Theorem 1.3.** Let \( C_g \) be a compact Riemann surface of genus \( g \geq 2 \). Then

\[
\text{Hom}(\pi_1(C_g), \text{SL}_n(C)) = \{(x_1, y_1, \ldots, x_g, y_g) \in \text{SL}_n(C)^{\times 2g} \mid \prod_{i=1}^{g} x_i y_i x_i^{-1} y_i^{-1} = I\}
\]
is a variety with rational singularities.

The proof shows also that the natural scheme structure is reduced, a fact which seems to be missing from the literature in full generality, according to A. Sikora. The reduced scheme structure of \( \text{Hom}(\pi_1(C_g), \text{SL}_n(C)) \) was shown to be a complete intersection variety for \( g \geq 2 \) in [RBC96].

For \( g \geq 12 \), Theorem 1.3 is proven in [AA16]. Their proof uses \( p \)-adic group theory to show rational singularities beyond the trivial representation. The proof we give here is entirely geometric and it applies to \( g \geq 2 \).
Since the natural scheme structure on the varieties from Theorems 1.1, 1.2, 1.3 is reduced, and since all these schemes are defined over $\mathbb{Q}$, by general scheme theory it follows that the $\mathbb{Q}$-schemes naturally associated to these sets are $\mathbb{Q}$-varieties with only rational singularities. Here a $\mathbb{Q}$-variety $V$ is a geometrically irreducible, reduced, separated scheme of finite type over $\mathbb{Q}$, and $V$ has rational singularities if for any resolution of singularities $f : W \to V$ over $\mathbb{Q}$, the natural morphism $\mathcal{O}_W \to Rf_*\mathcal{O}_V$ is a quasi-isomorphism. This generalization has the following consequences by Aizenbud-Avni [AA16, AA18], see Section 6.

**Theorem 1.4.** Let $n \geq 3$ be an integer. There exist a positive real number $C$ and a natural number $m_0$ such that for all integers $m \geq m_0$, the number of isomorphism classes of irreducible complex representations of dimension at most $m$ of $\text{SL}_n(\mathbb{Z})$ is less or equal to $Cm^2$.

**Theorem 1.5.** Let $n \geq 1$ be an integer and $p$ a prime number. There exists a positive real number $C$ such that for all integers $m$, the number of isomorphism classes of continuous irreducible $m$-dimensional complex representations of $\text{SL}_n(\mathbb{Z}_p)$ is less than $Cm^2$.

These theorems improve the bound $Cm^{22}$ of [AA16, AA18] to the quadratic bound. The theorems follow from the next two, more general results. See Section 6 for the definition of the abscissa of convergence $\alpha$ of a group.

**Theorem 1.6.** For every non-archimedean local field $F$ containing $\mathbb{Q}$ and every compact open subgroup $\Gamma \subset \text{SL}_n(F)$, the abscissa $\alpha(\Gamma) < 2$.

**Theorem 1.7.** Let $n$ be a positive integer. Then there exists a finite set $S$ of prime numbers such that for every global field $k$ of characteristic not in $S$ and every finite set $T$ of places of $k$ containing all Archimedean ones,

$$\alpha(\text{SL}_n(O_{k,T})) \leq 2,$$

where

$$O_{k,T} = \{ x \in k \mid \forall v \not\in T, ||x||_v \leq 1 \},$$

with the exception of the case $n = 2$ and $k$ equal to $\mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$. In particular,

$$\alpha(\text{SL}_n(\mathbb{Z})) \leq 2 \quad \text{for } n \geq 3.$$

In [AA16, AA18], the abscissae in these theorems have been bounded above by 22, instead of 2.

Another consequence of Theorem 1.3 is:

**Theorem 1.8.** Let $g \geq 2$, $n \geq 1$ be integers. The map of $\mathbb{Q}$-varieties

$$\text{SL}_n^{\times 2g} \to \text{SL}_n, \quad (x_1, y_1, \ldots, x_g, y_g) \mapsto \prod_{i=1}^{g} x_i y_i x_i^{-1} y_i^{-1}$$

is flat and the inverse image of every $z \in \text{SL}_n(\mathbb{C})$ is a variety with rational singularities.
This theorem has been proven in [AA16] for \( g \geq 12 \).

Theorems 1.4-1.8, together with Theorem 1.3 which implies them, were claimed as stated here in [BZo18]. However, [BZo18] contains a gap in an argument involving \( p \)-adic group theory which has not been fixed yet, see [BZo18']. The proof of Theorem 1.3 that we give here serves to fix the gap in [BZo18] in an entirely geometric way, so that all the results of [BZo18, §1] hold, except [BZo18, Proposition 1.5] which remains open.

In connection with the last theorem, we can add:

**Theorem 1.9.** Let \( g \geq 2, n \geq 1 \) be integers.

(a) The map of \( \mathbb{Q} \)-varieties

\[
\text{Mat}(n)^{2g} \to \text{Mat}(n), \quad (x_1, y_1, \ldots, x_g, y_g) \mapsto \sum_{i=1}^{g} [x_i, y_i]
\]

it is flat and the inverse image of every point \( z \in \text{Mat}(n, \mathbb{C}) \) in some open neighborhood of 0 is a variety with rational singularities.

(b) The map of \( \mathbb{Q} \)-varieties

\[
\text{GL}_n^{2g} \to \text{GL}_n, \quad (x_1, y_1, \ldots, x_g, y_g) \mapsto \prod_{i=1}^{g} x_i y_i x_i^{-1} y_i^{-1}
\]

has the following property over a open neighborhood of \( I \in \text{GL}_n \): it is flat and the inverse image of every \( z \in \text{GL}_n(\mathbb{C}) \) is a variety with rational singularities.

The flatness in (a) is a particular case of [CB01, Theorem 1.1] for the quiver with one vertex and \( g \) loops. Hence (a) follows from Theorem 1.1 here and the open nature of rational singularities in flat families [El78, Théorème 4]. It is showed in [AA16, 2.3.1] that (a) implies (b).

Finally, the Isosingularity Principle of Simpson [Si94, Theorem 10.6] applies to transfer the rational singularities property from representations spaces of fundamental groups to the de Rham and Dolbeault analogs, we refer to [Si94] for the precise definitions:

**Theorem 1.10.** Let \( g \geq 2, n \geq 1 \) be integers. Let \( G \) be \( \text{GL}_n \) or \( \text{SL}_n \). Let \( C_g \) be a compact Riemann surface and \( x \in C_g \) a fixed point. Let \( R_{DR}(C_g, G) \) be the fine moduli space of principal \( G \)-bundles with integrable connection and a frame over \( x \). Let \( R_{Dol}(C_g, G) \) be the fine moduli space of principal Higgs bundles for the group \( G \) with a frame over \( x \), which are semistable with vanishing rational Chern classes. Then \( R_{DR}(C_g, G) \) and \( R_{Dol}(C_g, G) \) have rational singularities over \( \mathbb{C} \).

Since GIT quotients of rational singularities are also rational singularities, one obtains as corollaries of our results that for \( g \geq 2 \) and \( G = \text{GL}_n(\mathbb{C}), \text{SL}_n(\mathbb{C}) \) the following schemes have rational singularities: the Mardsen-Weinstein reduction \( X \sslash G \), the moduli of local systems \( M_B(C_g, G) \), the moduli of bundles with integrable connections \( M_{DR}(C_g, G) \), and the moduli of Higgs bundles with vanishing Chern classes \( M_{Dol}(C_g, G) \). This is already known from the fact that all these spaces have symplectic singularities, see [CB03, Corollary 8.4], [BS16], [Ti17], and our results give a different explanation.

In Section 2, we recall the description of the étale slices for zeros of moment maps of quivers due to Crawley-Boevey, and prove some preliminary results on the quiver \( Q \).
In Section 3, the heart of the article, we prove Theorem 1.1. In Section 4 we prove Theorem 1.2 using deformation theory. In Section 5 we prove Theorem 1.3. In Section 6 we address Theorems 1.4-1.8.

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2. Quiver moment maps

We fix an algebraically closed field $k$ of characteristic zero. A quiver $Q$ is a set of directed arrows between a set $I$ of vertices. The representations of $Q$ of dimension vector $\alpha \in \mathbb{N}^I$ are the elements of

$$\text{Rep}(Q, \alpha) = \bigoplus_{a \in Q} \text{Mat}(\alpha_{h(a)} \times \alpha_{t(a)}, k),$$

where $h(a)$ is the head vertex of the arrow $a$, and $t(a)$ is the tail. The group

$$G(\alpha) = \left( \prod_{i \in I} \text{GL}_{\alpha_i}(k) \right) / k^*$$

acts by conjugation.

A morphism between two quiver representations $x \in \text{Rep}(Q, \alpha)$ and $y \in \text{Rep}(Q, \beta)$ is an element

$$f \in \bigoplus_{i \in I} \text{Mat}(\alpha_i \times \beta_i, k)$$

such that $fx = yf$. That is, $f_{h(a)}x_a = y_ao_{t(a)}$ for all $a \in Q$.

The orbits of $G(\alpha)$ on $\text{Rep}(Q, \alpha)$ correspond to isomorphism classes of representations.

The category of representations of a quiver $Q$ is equivalent to the category of left $kQ$-modules, where $kQ$ is the path algebra of $Q$.

For $x \in \text{Rep}(Q, \alpha)$, one defines the groups $\text{Ext}^i(x, x)$ as the extension groups of the associated $kQ$-module.

A representation in $\text{Rep}(Q, \alpha)$ is called simple if the associated $kQ$-module is simple, and semisimple if it is a direct sum of simple representations.

Let $p_Q(\alpha) = 1 - \langle \alpha, \alpha \rangle_Q$, where

$$\langle \alpha, \beta \rangle_Q = \sum_{i \in I} \alpha_i\beta_i - \sum_{a \in Q} \alpha_{t(a)}\beta_{h(a)}.$$

is the pairing on $\mathbb{Z}^I$ associated to $Q$. We also define $(\alpha, \beta)_Q = \langle \alpha, \beta \rangle_Q + \langle \beta, \alpha \rangle_Q$.

The double of $Q$ is the quiver $\overline{Q}$ obtained by adjoining a reverse arrow $a^*$ for each $a \in Q$. There is a $G(\alpha)$-equivariant moment map

$$\mu_{Q, \alpha} : \text{Rep}(\overline{Q}, \alpha) \longrightarrow \text{End}(\alpha) = \bigoplus_{i \in I} \text{Mat}(\alpha_i, k)$$

defined by

$$x \mapsto \sum_{a \in Q} [x_a, x_a^*]$$
where \([x, y] = xy - yx\). We denote the zero locus by
\[
X(Q, \alpha) = \mu_{Q, \alpha}^{-1}(0)
\]
and we consider it as a closed subscheme; it does not depend on the orientation of the arrows of \(Q\) (see [CBH98, Lemma 2.2]). The underlying reduced subscheme of \(X(Q, \alpha)\) is irreducible, but not always a normal variety; see [CB03, §1].

The affine quotient
\[
M(Q, \alpha) = X(Q, \alpha) \sslash G(\alpha)
\]
parametrises isomorphism classes of semisimple representations in \(X(Q, \alpha)\), or, equivalently, the closed orbits of \(G(\alpha)\) on \(X(Q, \alpha)\). Denote by
\[
q : X(Q, \alpha) \to M(Q, \alpha)
\]
the quotient morphism. Every fiber of \(q\) contains a unique closed orbit.

We will use later the following results of Crawley-Boevey from Theorem 1.2, Theorem 1.3, Corollary 1.4, Lemma 6.5, and Proof of Corollary 1.4 on p289 of [CB01]:

**Theorem 2.1.** (Crawley-Boevey) If \(X(Q, \alpha)\) contains a simple representation in \(\text{Rep}(\overline{Q}, \alpha)\), then:

(a) \(X(Q, \alpha)\) is a reduced and irreducible complete intersection of dimension \(\alpha \cdot \alpha - 1 + 2 p_Q(\alpha)\),

(b) the general element of \(X(Q, \alpha)\) is a simple representation,

(c) the dimension of \(M(Q, \alpha)\) is \(2 p_Q(\alpha)\),

(d) \(p_Q(\alpha) > 0\) if and only if \(M(Q, \alpha)\) contains an open dense subset of isomorphism classes of simple representations.

(e) The simple representations in \(X(Q, \alpha)\) are smooth points.

Given a semisimple representation \(x \in X(Q, \alpha)\) of \(\overline{Q}\), we can decompose it into its simple components
\[
x \simeq x_1^{e_1} \oplus \cdots \oplus x_r^{e_r},
\]
where \(x_j\) are non-isomorphic simple representations. If \(\beta^{(j)}\) is the dimension vector of \(x_j\), one says that \(x\) has representation type
\[
\tau = (e_1, \beta^{(1)}; \cdots ; e_r, \beta^{(r)}).
\]
Set
\[
e = (e_1, \ldots, e_r).
\]
Then \(G(e)\) is a conjugate of \(G(\alpha)_x\), the stabilizer of \(x\) under the action of \(G(\alpha)\) [LP90, Theorem 2]. One obtains a stratification of \(M(Q, \alpha)\) according to the representation types of the isomorphism classes of semisimple representations, see [CB01, §11]. This is the same as the stratification according to conjugacy classes of stabilizers of the semisimple representations by [LP90, Theorem 2], [Ma08, Theorem 5.4].

By Crawley-Boevey, the local structure at a semisimple representation of \(X(Q, \alpha)\) reduces to that at the origin with respect to a modified quiver and dimension vector. To state this properly, we need to recall some definitions.
Definition 2.2. Let $G$ be a reductive group acting on an affine variety $X$. Let $H$ be a reductive subgroup of $G$. Let $S$ be a locally closed affine $H$-invariant subvariety of $X$. We define $G \times_H S$ and a $G$-equivariant morphism

$$G \times_H S \to X$$

as follows. Define an action of $H$ on $G \times S$ by $h(g, s) = (gh^{-1}, hs)$. Then the map $G \times S \to X$ sending $(g, s)$ to $gs$ is $H$-equivariant, where $H$ acts trivially on $X$. Then

$$G \times_H S = (G \times S) \sslash H \to X$$

is the associated morphism of affine quotients.

Definition 2.3. Let $G$ be a reductive group acting on affine varieties $X$ and $Y$, and let $f : X \to Y$ be a $G$-equivariant morphism. We say $f$ is strongly étale if

- $f / G : X \sslash G \to Y \sslash G$ is étale, and
- $f, f / G$, and the quotient morphisms induce a $G$-isomorphism $X \simeq Y \times_{Y \sslash G} (X \sslash G)$.

Proposition 2.4. ([Dr04, Proposition 4.15]) If $f : X \to Y$ is a strongly étale $G$-morphism then:

- $f$ is étale and surjective,
- for every $u \in X \sslash G$, $f$ induces an isomorphism from the inverse image of $u$ in $X$ to the inverse image of $f(u)$ in $Y$.
- for every $x \in X$, the restriction of $f$ to $Gx$ is injective, and $Gx$ is closed if and only if $Gf(x)$ is closed.

Lemma 2.5. If $f : X \to Y$ is a strongly étale $G$-morphism and $x \in X$, then $G_x = G_{f(x)}$.

Proof. Since $f$ is $G$-equivariant, $G_x \subset G_{f(x)}$. Conversely, if $g \in G_{f(x)}$, then $f(x) = g f(x) = f(gx)$. By the above proposition, $f$ is injective on orbits. Hence $x = gx$, and $g \in G_x$. \hfill \Box

Definition 2.6. Let $G$ be a reductive group acting on an affine variety $X$. Let $x \in X$ be a point with closed orbit. An étale slice is a $G_x$-invariant locally closed affine subvariety $S$ of $X$ containing $x$ such that the induced $G$-equivariant morphism

$$\psi : G \times_{G_x} S \to X$$

is strongly étale onto a $G$-saturated affine open subset $U$ of $X$.

Recall that a $G$-saturated set is the inverse image of a subset of $X \sslash G$ under the quotient morphism $X \to X \sslash G$.

By the main result of Luna [Lu73] étale slices always exist.

Definition 2.7. Given a semisimple representation $x \in X(Q, \alpha)$ of representation type $\tau = (e_1, \beta^{(1)}; \cdots ; e_r, \beta^{(r)})$, define a new quiver $Q_\tau$ with $r$ vertices and whose double $\overline{Q_\tau}$ has $2pq(\beta^{(i)} + \beta^{(j)})$ loops at vertex $i$, and $-(\beta^{(i)}, \beta^{(j)})_Q$ arrows from $i$ to $j$ if $i \neq j$.

Theorem 2.8. ([CB03, §4]) Let $x \in X(Q, \alpha)$ be a semisimple representation of type $\tau$. There exists a morphism

$$f : S \to X(Q_\tau, e)$$

from an étale slice $S$ for $X(Q, \alpha)$ at $x$, sending $x$ to $0$, such that $f$ is equivariant via the canonical isomorphism $G(\alpha)_x \simeq G(e)$, and the restriction of $f$ is strongly étale from an open $G(\alpha)_x$-saturated neighborhood of $x$ onto an open $G(e)$-saturated neighborhood of $0$. 

Proof. In the proof of [CB03, Lemma 4.4], and with notation from there, the lemma from [Lu73, p95] is applied to $\phi : U \rightarrow V$. The conclusion from this lemma is stronger than as stated by Crawley-Boevey: namely not only $\phi'/G : U' \parallel G \rightarrow V' \parallel G$ étale, but also $\phi' : U' \rightarrow V'$ is an étale, and $U' \rightarrow V' \times_{V'\parallel G} U' \parallel G$ is an isomorphism. Since $U'$ is an open $G$-saturated subset of $G \times_{G, C'} C$ (where $C'$ denote the set $\mu^{-1}_x(0) \cap \nu^{-1}(0)$), by [Lu73, Lemme, p87] one has that $U' = G \times_{G, x} S'$ for some open subset $S'$ of $C'$. Define $S$ to be the image of $S'$ under the translation map $C' \rightarrow \mu^{-1}(0)$ sending $c$ to $x + c$. The remaining properties making $S$ an étale slice are already stated in [CB03, Lemma 4.4].

The morphism $f$ (in our notation) is constructed in [CB03, Lemmas 4.8 and 3.3]. Note that by applying the Fundamental Lemma of [Lu73, p94] to the morphism $\nu^{-1}(0) \rightarrow W$ in the proof of [CB03, Lemma 4.8] (instead of [Lu73, Lemme 3, p93]), one obtains the stronger conclusion that $f$ is strongly étale locally $x$. 

Proposition 2.9. Let $Q$ be a quiver and let $\alpha$ be a dimension vector with $p_Q(\alpha) > 0$. Let $x \in X(Q, \alpha)$ be a semisimple representation of type $\tau$. If $X(Q, \alpha)$ contains a simple representation, then $X(Q_\tau, e)$ also contains a simple representation and $p_{Q_\tau}(e) > 0$.

Proof. The assumptions guarantee that $M(Q, \alpha)$ has an open dense subset of isomorphism classes of simple representations by Theorem 2.1. Let $S$ be an étale slice as in the previous theorem. Let $U$ be an open affine $G(\alpha)$-saturated subset of $X(Q, \alpha)$ such that $\psi : G(\alpha) \times_{G(\alpha)_x} S \rightarrow U \subset X(Q, \alpha)$ is strongly étale, with notation as in Definition 2.6. Then $U \parallel G$, being open $M(Q, \alpha)$, contains an open dense subset of simples. Thus $S$ contains a $G(\alpha)_x$-saturated open dense subset $S^0$ of points $y$ mapping to a simple representation in $U \parallel G$ via the composition $S \rightarrow S \parallel G(\alpha)_x \rightarrow U \parallel G$.

The stabilizer under $G(\alpha)_x$ of any $y \in S$ is trivial if and only if the $G(\alpha)$-stabilizer of the class of $(1, y)$, in $G(\alpha) \times_{G(\alpha)_x} S$ is trivial, by [Dr04, Proposition 4.9-3].

Since $\psi$ is $G(\alpha)$-equivariant strongly étale, $G(\alpha)_{(1, y)} = G(\alpha)_{\psi(1, y)}$ by Lemma 2.5.

Now assume that $y \in S^0$ and let $u$ be its image in $U \parallel G(\alpha)$. Then, the image of $\psi(1, y)$ under the quotient morphism $U \rightarrow U \parallel G(\alpha)$ is also $u$, by the commutativity of the diagram

$$
\begin{array}{ccc}
G(\alpha) \times_{G(\alpha)_x} S & \xrightarrow{\psi} & U \\
\downarrow & & \downarrow \\
S \parallel G(\alpha)_x & \xrightarrow{\psi} & U \parallel G.
\end{array}
$$

Thus, since $u$ is the isomorphism class of a simple representation, $\psi(1, y)$ has trivial stabilizer. Hence $y$ has trivial $G(\alpha)_x$-stabilizer for all $y \in S^0$.

Consider now the morphism $f : S \rightarrow X(Q_\tau, e)$ from the previous theorem, which is locally strongly étale. Since $S^0$ is open dense saturated, it follows from Lemma 2.5 again that there is an open dense saturated subset of $X(Q_\tau, e)$ with trivial stabilizers. Hence $X(Q_\tau, e)$ contains simple representations, and $M(Q_\tau, e)$ contains an open dense subset of isomorphisms classes of simple representations. Equivalently, by Theorem 2.1, $p_{Q_\tau}(e) > 0$. 

\[ \square \]
In principle, for a semisimple representation type $\tau'$ in $X(Q, e)$, it could happen the new variety $X((Q_{\tau'})_{\tau'}, e')$ is not of type $X(Q_{\tau''}, e'')$ for some semisimple representation type $\tau''$ occurring in the initial variety $X(Q, \alpha)$. We show that this is not the case when $Q$ is the quiver with one vertex and $g$ loops.

**Lemma 2.10.** Let $Q$ be the quiver with one vertex and $g \geq 2$ loops. Let $n \geq 1$ be an integer. Let $\tau = (e_i, \beta_i)_{1 \leq i \leq r}$ be a semisimple representation type occurring in $X(Q, n)$. Let $\tau' = (e'_j, \gamma(j))_{1 \leq j \leq r'}$ be a semisimple representation type occurring in $X(Q_{\tau}, e)$. Then $\tau'' = (e'_j, \sum_{i=1}^{r'} \gamma_i^{(j)} \beta_i)_{1 \leq j \leq r'}$ occurs as a semisimple representation type in $X(Q, n)$, and

$$X((Q_{\tau'})_{\tau'}, e') = X(Q_{\tau''}, e'').$$

**Proof.** For an integer $m > 1$, $pq(m) = 1 + (g - 1)m^2 > 0$. So by Theorem 2.1, $X(Q, m)$ has lots of simple representations. By taking direct sums of simple representations of various dimensions, one has that an equivalent condition for $\tau$ to be a representation type occurring in $X(Q, n)$ is that $\sum_{i=1}^{r} e_i \beta_i = n$.

Because $\tau'$ is a type in $X(Q_{\tau}, e)$, one has for all $1 \leq i \leq r$, $e_i = \sum_{j=1}^{r'} e'_j \gamma_i^{(j)}$. Thus

$$\sum_{j=1}^{r'} e'_j \left( \sum_{i=1}^{r} \gamma_i^{(j)} \beta_i \right) = \sum_{i=1}^{r} e_i \beta_i = n.$$ Hence $\tau''$ occurs in $X(Q, n)$.

Consider the zero representation in $X(Q_{\tau}, e)$. It corresponds to a semisimple representation of type $\tau$ in $X(Q, n)$, by Theorem 2.8. Hence it has representation type $(e_i, \epsilon_i)_{1 \leq i \leq r}$ in $X(Q_{\tau}, e)$, where $\epsilon_i$ are the standard coordinate vectors in $\mathbb{Z}^r$. Since applying once more Theorem 2.8 for an étale slice at 0 in $X(Q_{\tau}, e)$ does not change the quiver $Q_{\tau}$, we have

$$pq_{\tau}(e_i) = pq_{\tau}(\beta_i) \quad \text{and} \quad (e_i, \epsilon_j)_{\tau} = (\beta_i, \beta_j)_{\tau} \quad \text{for } i \neq j.$$ Hence there is the following relation between the two symmetric bilinear forms:

$$(\epsilon_i, \epsilon_j)_{\tau} = (\beta_i, \beta_j)_{\tau},$$

for all $1 \leq i, j \leq r$. In particular, letting $b_j = \sum_{i=1}^{r} \gamma_i^{(j)} \beta_i$ for $1 \leq j \leq r'$, we have

$$pq_{\tau}(\gamma(j)) = pq_{\tau}(b_j), \quad \text{and} \quad (\gamma(j), \gamma(j'))_{\tau} = pq_{\tau}(b_j, b_{j'}).$$

Hence $(Q_{\tau})_{\tau'} = Q_{\tau''}$ by Definition 2.6, and the claim follows. \qed

We will use the following estimate on the size of the nilpotent cone:

**Proposition 2.11.** Let $Q$ be the quiver with one vertex and $g \geq 2$ loops. Let $n \geq 1$ be an integer. Let $\tau = (e_i, \beta_i)_{1 \leq i \leq r}$ be a semisimple representation type occurring in $X(Q, n)$ with $\tau \neq (n, 1)$. Let

$$q_{\tau} : X(Q_{\tau}, e) \to M(Q_{\tau}, e)$$

be the affine quotient morphism. Then

$$\dim q^{-1}_{\tau}(0) < 2pq(n) - 2 \sum_{i=1}^{r} pq(\beta_i).$$

**Proof.** In this case, one can work out the right-hand side to be $2(g - 1)(n^2 - \sum \beta_i^2)$. For the left-hand side, we will use [CB03, Lemma 6.2] which we recall now. In $X(Q_{\tau}, e)$,
0 \simeq \oplus_{i=1}^{r} T_{i}^{\oplus \epsilon_{i}} \text{ where } T_{i} \text{ are the simple non-isomorphic zero representations of dimension vectors } \epsilon_{i}. \text{ Then } \dim q_{r}^{-1}(0) \text{ is at most the maximum value among the quantities }

\begin{equation}
(1) \quad e \cdot e - 1 + pQ_{r}(e) + \sum_{s=1}^{h} m_{s} z_{s} - \sum_{s=1}^{h} m_{s}^{2} pQ_{r}(\epsilon_{s})
\end{equation}

for “top-types” \((j_{1}, m_{1}; \ldots; j_{h}, m_{h})\) with \(j_{s} \in \{1, \ldots, r\}\) and \(m_{s}\) positive natural numbers, and

\[ z_{s} = \begin{cases} 
0, & \text{if } pQ_{r}(\epsilon_{j_{s}}) = 0, \text{ or if } \not\exists k < s \text{ with } j_{k} = j_{s}, \\
m_{k}, & \text{for the largest } k < s \text{ with } j_{k} = j_{s}.
\end{cases} \]

A top-type is data coming from an element \(M \in q_{r}^{-1}(0)\) with a filtration \(0 = M_{0} \subset M_{1} \subset \ldots \subset M_{h} = M\) by sub-representations with \(M_{s}/M_{s-1} \simeq T_{j_{s}}^{\oplus m_{s}}\) and such that \(\dim \text{Hom}(M_{s}, T_{j_{s}}) = m_{s}\) for all \(s\). Note that we must have for all \(1 \leq i \leq r\) that

\[ e_{i} = \sum_{s \text{ with } j_{s} = i} m_{s}. \]

Every \(M \in q_{r}^{-1}(0)\) has a top-type for suitable \(j_{s}\) and \(m_{s}\).

In our case, \(pQ_{r}(e) = pQ(n) = 1 + (g-1)n^{2}\) and \(pQ_{r}(\epsilon_{i}) = 1 + (g-1)\beta_{i}^{2}\). We plug this in \((1)\) and we want to check that \((1)\) is strictly smaller than \((2g-1)(n^{2} - \sum_{i} \beta_{i}^{2})\). That is, we want that

\begin{equation}
(2) \quad r \sum_{i=1}^{r} e_{i}^{2} + \sum_{s=1}^{h} m_{s} z_{s} - \sum_{s=1}^{h} m_{s}^{2} (1 + (g-1)\beta_{j_{s}}^{2}) < (2g-1)n^{2} - 2(g-1) \sum_{i=1}^{r} \beta_{i}^{2}.
\end{equation}

We use now the equalities

\[ n = \sum_{i=1}^{r} e_{i} \beta_{i} = \sum_{i=1}^{r} \sum_{s: j_{s} = i} m_{s} \beta_{i} = \sum_{s} m_{s} \beta_{j_{s}} \]

\[ \sum_{i=1}^{r} e_{i}^{2} = \sum_{i=1}^{r} \left( \sum_{s: j_{s} = i} m_{s} \right)^{2} \]

to phrase \((2)\) only in terms of \(j_{s}, m_{s}, z_{s}, \beta_{i},\) and \(g\). We single out \(g\) together with its coefficient, place it on the right-hand side, and move the other summands from one side to the other until they achieve a positive sign. Then \((2)\) becomes equivalent to

\[ \sum_{i} \left( \sum_{s: j_{s} = i} m_{s} \right)^{2} + \sum_{s} m_{s} z_{s} + \sum_{s} m_{s}^{2} \beta_{j_{s}}^{2} + \left( \sum_{s} m_{s} \beta_{j_{s}} \right)^{2} < \]

\[ \sum_{s} m_{s}^{2} + 2 \sum_{i} \beta_{i}^{2} + g \left[ \sum_{s} m_{s}^{2} \beta_{j_{s}}^{2} + \left( \sum_{s} m_{s} \beta_{j_{s}} \right)^{2} - 2 \sum_{i} \beta_{i}^{2} \right]. \]

Since all \(T_{i}\) must appear in the semisimplification of \(M\), the set \(\{ \beta_{j_{s}} \mid s \}\) is the whole set \(\{ \beta_{i} \mid i \}\) with repetitions allowed. Since \(m_{s}, \beta_{i} > 0\), the coefficient of \(g\) must be non-negative. Hence it is enough if we prove the inequality for \(g = 2\). That is, we need to show that

\begin{equation}
(3) \quad \sum_{i} \left( \sum_{s: j_{s} = i} m_{s} \right)^{2} + \sum_{s} m_{s} z_{s} + 2 \sum_{i} \beta_{i}^{2} < \sum_{s} m_{s}^{2} + \sum_{s} m_{s}^{2} \beta_{j_{s}}^{2} + \left( \sum_{s} m_{s} \beta_{j_{s}} \right)^{2}.
\end{equation}
By the definition of \( z_s \), the inequality is equivalent with the one where the top-type is rearranged so that

\[
1 = j_1 = \ldots = j_{s_1}; \ 2 = j_{s_1+1} = \ldots = j_{s_2}; \ \ldots; \ r = j_{s_r-1+1} = \ldots = j_{s_r} = j_h.
\]

Then the sequence \( z_1, \ldots, z_h \) is the sequence

\[
0, m_1, m_2, \ldots, m_{s_1-1}; \ 0, m_{s_1+1}, \ldots, m_{s_2-1}; \ \ldots; \ 0, m_{s_{r-1}+1}, \ldots, m_{s_r-1}.
\]

With this, \((3)\) becomes

\[
[(m_1 + \ldots + m_{s_1})^2 + \ldots + (m_{s_{r-1}+1} + \ldots + m_{s_r})^2] + \\
+[(m_2 m_1 + m_3 m_2 + \ldots + m_{s_1} m_{s_1-1}) + \ldots + (m_{s_{r-1}+2} m_{s_{r-1}+1} + \ldots + m_s m_{s_{r-1}})] + \\
+ 2 \sum_i \beta_i^2 < [m_1^2 + \ldots + m_s^2] + \\
+ [\beta_1^2 (m_1^2 + \ldots + m_{s_1}^2) + \ldots + \beta_r^2 (m_{s_{r-1}+1}^2 + \ldots + m_{s_r}^2)] + \\
+ [\beta_1 (m_1 + \ldots + m_{s_1}) + \ldots + \beta_r (m_{s_{r-1}+1} + \ldots + m_{s_r})]^2.
\]

By bounding the second line of summands via

\[
m_2 m_1 + \ldots + m_{s_1} m_{s_1-1} \leq \frac{1}{2} (m_2^2 + m_1^2 + \ldots + m_{s_1}^2 + m_{s_1-1}^2) = \\
= m_1^2 + \ldots + m_{s_1}^2 - \frac{1}{2} (m_1^2 + m_{s_1}^2),
\]

it is enough to prove that

\[
[(m_1 + \ldots + m_{s_1})^2 + \ldots + (m_{s_{r-1}+1} + \ldots + m_{s_r})^2] + 2 \sum_i \beta_i^2 < \\
1 \left[ m_1^2 + \ldots + m_{s_1}^2 + \ldots + m_{s_{r-1}+1}^2 + m_{s_r}^2 \right] + \\
+ [\beta_1^2 (m_1^2 + \ldots + m_{s_1}^2) + \ldots + \beta_r^2 (m_{s_{r-1}+1}^2 + \ldots + m_{s_r}^2)] + \\
+ [\beta_1 (m_1 + \ldots + m_{s_1}) + \ldots + \beta_r (m_{s_{r-1}+1} + \ldots + m_{s_r})]^2.
\]

Defining

\[
A_i = m_{s_{i-1}+1} + \ldots + m_{s_i},
\]

the inequality is implied by

\[
A_1^2 + \ldots + A_r^2 + \beta_1^2 + \ldots + \beta_r^2 < [\beta_1 A_1 + \ldots + \beta_r A_r]^2
\]

if we can show that this last one is true. Using that

\[
x^2 + y^2 \leq x^2 y^2 + 1 \quad \text{for integers } x, y \geq 1,
\]

the left-hand side of \((4)\) is at most

\[
\beta_1^2 A_1^2 + \ldots + \beta_r^2 A_r^2 + r.
\]

If \( r \geq 2 \),

\[
r \leq \sum_{i \neq j} \beta_i A_i \beta_j A_j.
\]
Therefore in this case
\[ A_1^2 + \ldots + A_r^2 + \beta_1^2 + \ldots + \beta_r^2 \leq \beta_1^2 A_1^2 + \ldots + \beta_r^2 A_r^2 + \sum_{i \neq j} \beta_i A_i \beta_j A_j = \]
\[ = [\beta_1 A_1 + \ldots + \beta_r A_r]^2, \]
which proves (4) for this case. If \( r = 1 \), (4) reduces to \( A_1^2 + \beta_1^2 < \beta_1^2 A_1^2 \). By assumption, \( A_1, \beta_1 > 1 \), so the inequality holds in this case as well. \( \square \)

For a quiver \( Q \) and dimension vector \( \alpha \), we denote the locus in \( X(Q, \alpha) \) fixed by the action of \( G(\alpha) \) by
\[ F(Q, \alpha) = X(Q, \alpha)^{G(\alpha)}. \]

**Lemma 2.12.** For a quiver \( Q \) and dimension vector \( \alpha \), \( F(Q, \alpha) \) is the set of \( x \in \text{Rep}(Q, \alpha) \) such that \( x_a \) and \( x_{a^*} \) are scalar matrices when \( a \in Q \) is a loop at a vertex, and \( x_a \) and \( x_{a^*} \) are 0 otherwise.

**Proof.** If \( x \) is fixed by all \( g \in G(\alpha) \), then \( x_a \) is a square matrix equal to all its conjugates if \( a \) is a loop, and it is a matrix equal to all the matrices similar to it, if \( a \) is not a loop. The same for \( a^* \). Clearly this locus is in the zero locus of the moment map. \( \square \)

**Lemma 2.13.** Let \( Q \) be the quiver with one vertex and \( g \) loops and \( \tau = (e_i, \beta_i)_{i=1}^g \) a representation type occurring in \( X(Q, n) \). Then:
(a) A semisimple point \( x \in X(Q, \tau) \) has representation type \( (e_i, e_i)_{i=1}^g \) if and only if \( x \in F(Q, \tau, e) \).
(b) Let \( \tau' = (e'_j, \gamma_{(j)})_{j=1}^r \) be a representation type occurring in \( X(Q, \tau, e) \). If \( \tau \neq (n, 1) \) and \( \tau' \neq (e_i, e_i)_{i=1}^g \), then
\[ n^2 > e_1^2 + \ldots + e_r^2 > (e'_1)^2 + \ldots + (e'_r)^2. \]

**Proof.** (a) The claim is a particular case of the identification between the representation type stratification of \( M(Q, e) \) and the stratification according to conjugacy classes of stabilizers.
(b) The inequalities follow from \( n = \sum_{i=1}^r e_i \beta_i \) and \( e_i = \sum_{j=1}^{r'} e'_j \gamma_{(j)} \). \( \square \)

We denote the subset of \( X(Q, \alpha) \) consisting of all points whose orbit closure contains a fixed point under the action of \( G(\alpha) \) by \( Z(Q, \alpha) \). So
\[ Z(Q, \alpha) = q^{-1}(g(F(X, \alpha))) \]
where \( q : X(Q, \alpha) \rightarrow M(Q, \alpha) \) is the affine quotient morphism.

**Proposition 2.14.** Let \( Q \) be the quiver with one vertex and \( g \geq 2 \) loops. Let \( n \geq 1 \) be an integer. Let \( \tau = (e_i, \beta_i)_{1 \leq i \leq g} \) be a semisimple representation type occurring in \( X(Q, n) \) with \( \tau \neq (n, 1) \). Let \( q_{\tau} : X(Q, \tau, e) \rightarrow M(Q, \tau, e) \) be the affine quotient morphism. Then
\[ \dim Z(Q, \tau, e) < 2p_{Q, \tau}(e). \]
Proof. It is clear from the proof that Proposition 2.11 also holds for \( q^{-1}_r(q_r(x)) \) for any point \( x \in F(Q_r,e) \), since all points in the fixed locus have the same representation type as the zero representation. By Lemma 2.12, the dimension of \( q_r(F(Q_r,e)) \) is \( \sum_{i=1}^{r} 2p_Q(\beta_i) \), twice the number of loops in \( Q_r \). Hence

\[
\dim Z(Q_r,e) \leq \sum_{i=1}^{r} 2p_Q(\beta_i) + \dim q_r^{-1}(0).
\]

We now use the bound from Proposition 2.11 to obtain that

\[
\dim Z(Q_r,e) < 2p_Q(n) = 2p_Q(e).
\]

\[\square\]

3. Jets along zeros of moment maps

We fix as before an algebraically closed field \( k \) of characteristic zero. For \( X \) a variety over \( k \) and an integer \( m \geq 1 \), we let \( \pi_m : X_m \to X \) denote the projection from the \( m \)-jet scheme. Recall that the \( m \)-jets are the elements of \( X_m(k) = \text{Hom}_{k-\text{sch}}(\text{Spec}(k[t]/t^{m+1}),X) \).

If \( X \subset \mathbb{A}^n \) is given by \( f_1(x) = \ldots = f_r(x) = 0 \) with \( x = (x_1, \ldots, x_n) \) and \( f_i \in k[x] \), then \( X_m \) is given in \( \mathbb{A}^{n(m+1)} \) with coordinates \( x, x', x'', \ldots, x^{(m)} \), where \( x^{(j)} = (x_1^{(j)}, \ldots, x_n^{(j)}) \), by

\[ f_1(x(t)) \equiv \ldots \equiv f_r(x(t)) \equiv 0 \mod t^{m+1} \]

with \( x_i(t) = x_i + x_i't + x_i''t + \ldots + x_i^{(m)}t^m \).

We will use the following relationship between jets schemes and rational singularities due to M. Mustat˘a:

**Theorem 3.1.** ([Mu01, Propositions 1.4 and 1.5]) Let \( X \) be a locally complete intersection variety. The following are equivalent for \( m \geq 1 \):

(i) \( X_m \) is irreducible,

(ii) \( \text{dim } \pi_m^{-1}(X_{\text{sing}}) < (\text{dim } X)(m + 1) \),

(iii) \( X_m \) is a locally complete intersection variety of dimension \( \leq (\text{dim } X)(m + 1) \).

**Theorem 3.2.** ([Mu01, Theorems 0.1 and 3.3]) Let \( X \) be a locally complete intersection variety. The following are also equivalent:

(a) the conditions (i)-(iii) are fulfilled for all \( m \geq 1 \),

(b) \( X \) has rational singularities,

(c) \( X \) has canonical singularities.

**Lemma 3.3.** For a quiver \( Q \) and dimension vector \( \alpha \), let \( \pi_m : X(Q,\alpha)_m \to X(Q,\alpha) \) be the projection from the \( m \)-jet scheme, \( m \geq 1 \). Then

\[
\pi_m^{-1}(F(Q,\alpha)) \simeq \begin{cases} 
F(Q,\alpha) \times X(Q,\alpha)_{m-2} \times \text{Rep}(Q,\alpha) & \text{if } m \geq 2, \\
F(Q,\alpha) \times \text{Rep}(Q,\alpha) & \text{if } m = 1.
\end{cases}
\]
\textbf{Proof.} As before, $F(Q, \alpha)$ denotes the locus in $X(Q, \alpha)$ fixed by the action of $G(\alpha)$. The $m$-jet scheme $X(Q, \alpha)_m$ is the subscheme of 
\[ \bigoplus_{a \in Q} \left( \text{Mat}(\alpha_{h(a)} \times \alpha_{t(a)}, k[t]/t^{m+1}) \oplus \text{Mat}(\alpha_{h(a^*)} \times \alpha_{t(a^*)}, k[t]/t^{m+1}) \right) \]
defined by 
\[ \sum_{a \in Q} [x(t)_a, x(t)_{a^*}] = 0 \in \bigoplus_{i \in I} \text{Mat}(\alpha_i, k[t]/t^{m+1}). \]
Writing $x(t)_a = \sum_{j=0}^m x_{a,j} t^j$ with $x_{a,j}$ a matrix over $k$, this is equivalent with the system of equations 
\[ \sum_{a \in Q} [x_{a,0}, x_{a^*,0}] = 0 \]
\[ \sum_{a \in Q} ([x_{a,0}, x_{a^*,1}] + [x_{a,1}, x_{a^*,0}]) = 0 \]
\[ \sum_{a \in Q} ([x_{a,0}, x_{a^*,2}] + [x_{a,1}, x_{a^*,1}] + [x_{a,2}, x_{a^*,0}]) = 0 \]
\[ \ldots \]
\[ \sum_{a \in Q} ([x_{a,0}, x_{a^*,m}] + [x_{a,1}, x_{a^*,m-1}] + \ldots + [x_{a,m-2}, x_{a^*,1}] + [x_{a,m-1}, x_{a^*,0}]) = 0. \]
If $x(t) = (x(t)_a, x(t)_{a^*})_{a \in Q}$ is an $m$-jet in $X(Q, \alpha)_m$ whose projection is the fixed point 
\[ x_0 = \pi_m(x(t)) = (x_{a,0}, x_{a^*,0})_{a \in Q}, \]
then all the commutator terms involving $x_{a,0}$ or $x_{a^*,0}$ are zero, by Lemma 3.3. The conclusion follows.

As before, let $Z(Q, \alpha)$ denote the subset of points in $X(Q, \alpha)$ with $G(\alpha)$-orbit closure containing a $G(\alpha)$-fixed point.

\textbf{Lemma 3.4.} Let $Q$ be a quiver. Let $\alpha$ be a dimension vector with $p_Q(\alpha) > 0$. Assume that 
\[ \dim Z(Q, \alpha) < 2p_Q(\alpha). \]
Then $X(Q, \alpha)$ has rational singularities if and only if $X(Q, \alpha)$ has rational singularities at all semisimple representations not lying in the fixed point locus $F(Q, \alpha)$.

\textbf{Proof.} For this proof only we will use $X = X(Q, \alpha)$, $F = F(Q, \alpha)$, $Z = Z(Q, \alpha)$, $R = \text{Rep}(\overline{Q}, \alpha)$, $G = G(\alpha)$ to ease the notation.

One implication is clear by the definition of rational singularities.

Conversely, suppose that $X$ has rational singularities at all semisimple representations $x \neq 0$ not lying in $F$. Then $X$ has rational singularities in an open neighborhood $U_x$ of such $x$. Let $y \in X \setminus Z$ be not necessarily semisimple. Then $y$ is a rational singularity if and only if every point of the orbit of $y$ is, since $G$ acts by isomorphisms on $X$. For every such $y$, the closure of its orbit $Gy$ contains a semisimple representation $x \notin F$, and thus $U_x \cap Gy$ is non-empty. Hence $y$ is a rational singularity too. Thus $X$ has rational singularities everywhere except maybe on $Z$. 
We will use Theorem 3.2 to show that $X$ has rational singularities along $Z$. Namely, letting $\pi_m : X_m \to X$ be the projection from the $m$-th jet scheme for $m \geq 1$, we will show that

$$\dim \pi_m^{-1}(X_{\text{sing}}) < (\dim X)(m + 1)$$

for all $m \geq 1$. We already have that

$$\dim \pi_m^{-1}(X_{\text{sing}} \setminus Z) < (\dim X)(m + 1)$$

for all $m \geq 1$ since all points different outside $Z$ are rational singularities.

By upper-semicontinuity of dimensions of $\pi_m^{-1}(x)$, see [Mu02, Proposition 2.3], for every point $x \in F$, and all $y \in Z$ close to $x$,

$$\dim \pi_m^{-1}(y) \leq \dim \pi_m^{-1}(x).$$

Using the group action to map isomorphically a neighborhood of any point in $Z$ to a point in $Z$ arbitrarily close to $F$, the same inequality

$$\dim \pi_m^{-1}(y) \leq \dim \pi_m^{-1}(x).$$

holds for all $y \in Z$ and $x \in F$ in the orbit closure of $y$. By Lemma 3.3, for all $x \in F$

$$\dim \pi_m^{-1}(x) = \dim \pi_m^{-1}(0).$$

Hence

$$\dim \pi_m^{-1}(Z) \leq \dim Z + \dim \pi_m^{-1}(0) - 2pQ(\alpha) + \dim \pi_m^{-1}(0).$$

By Lemma 3.3,

$$\dim \pi_1^{-1}(0) = \dim R = 2(\alpha \cdot \alpha - 1 + pQ(\alpha)).$$

Therefore

$$\dim \pi_1^{-1}(Z) < 2pQ(\alpha) + 2(\alpha \cdot \alpha - 1 + pQ(\alpha)) = 2(\alpha \cdot \alpha - 1 + 2pQ(\alpha)) = 2 \dim X.$$

Together with (6), this implies that (5) holds for $m = 1$.

Let now $m > 1$. By Lemma 3.3,

$$\dim \pi_m^{-1}(0) = \dim X_{m-2} + \dim R.$$

By induction and Theorem 3.1,

$$\dim X_{m-2} \leq (\dim X)(m - 1).$$

Together with (7), this implies that

$$\dim \pi_m^{-1}(Z) < 2pQ(\alpha) + \dim R + (\dim X)(m - 1) = 2 \dim X + (\dim X)(m - 1) = (\dim X)(m + 1).$$

Together with (6), this implies that (5) holds for any $m > 1$ as well. □

Proof of Theorem 1.1. We use Lemma 3.4 inductively by looking at semisimple representations $x$ away from the fixed locus. We can use Theorem 2.8 to model the local structure at $x$ of representation type $\tau$ by the varieties $X(Q_\tau, e)$ because having rational singularities is an étale local property. Picking a semisimple representation away from the fixed points of $X(Q_\tau, e)$ does not introduce any new varieties than those of type $X(Q_{\tau'}, e')$, by Lemma 2.10. At every step corresponding to a representation type
\[ \tau = (e_i, \beta_i), \] occurring in \( X(Q, n) \), the quantity \( \sum_i e_i^2 \) decreases by Lemma 2.13; equivalently, the stabilizer up-to-conjugacy corresponding to \( \tau \) decreases in dimension. So the process will terminate if it runs. It runs because the conditions of Lemma 3.4 are met by Proposition 2.9 and Proposition 2.14. Thus we reduce to simple representations, in which case we know that they are smooth points. \qed

**Remark 3.5.** The proof actually shows that \( X(Q, e) \) has rational singularities for all \( \tau = (e_i, \beta_i) \) with \( n = \sum_i e_i \beta_i \) and \( Q \) the quiver with one vertex and \( g \geq 2 \) loops.

### 4. \( \text{GL}_n \)-Representations of Surface Groups

In this section \( k = \mathbb{C} \). For a compact Riemann surface \( C_g \) of genus \( g \) and a point \( p \in C_g \) we will use the simpler notation
\[
R(g, n) = \text{Hom}(\pi_1(C_g, p), \text{GL}_n(k))
\]
and
\[
M(g, n) = R(g, n) / \text{GL}_n(k)
\]
for the affine quotient. The closed points of \( M(g, n) \) are into 1-1 correspondence with the isomorphism classes of semisimple \( k \)-local systems of rank \( n \) on \( C_g \). For \( g \geq 2 \), \( R(g, n) \) is a normal complete intersection variety by [Si94, Theorem 11.1], while \( M(g, n) \) is a symplectic variety by [BS16, Proposition 8.4], hence it has rational singularities.

We need the following from deformation theory, see for example the case \( k = 0 \) of [BW15, Lemma 7.10]:

**Proposition 4.1.** Let \( g \geq 2 \). Let \( \rho \in R(g, n) \) be a semisimple complex representation and \( L \in M(g, n) \) the associated semisimple complex local system. Let
\[
\mu : \text{Ext}^1(L, L) \longrightarrow \text{Ext}^2(L, L), \quad \mu(e) = e \cup e.
\]
There is an isomorphism of formal germs
\[
\hat{R}(g, n)_\rho \simeq \hat{\mathfrak{h}}_0 \times \mu^{-1}(0)_0,
\]
for some affine space \( \mathfrak{h} \).

**Remark 4.2.** (a) Recall that \( \text{Ext}^*(L, L) = H^*(C_g, L \otimes L^*) \).

(b) For two points \( x \in X \) and \( y \in Y \) on two algebraic varieties, the following are equivalent by Artin approximation: \( x \) and \( y \) have isomorphic complex analytic, or étale, or formal neighborhoods.

(c) The main ingredient of the proof of Proposition 4.1 is the formality of the differential graded Lie algebra controlling the deformation theory of a semisimple representation. This was proved by [Si92] using transcendental techniques, hence the reason for the field \( k \) to be \( \mathbb{C} \).

The map \( \mu \) is the moment map of a double quiver, see [KLS06, 3.1], where one has to replace Serre duality for coherent sheaves with duality for local systems [Di04, Corollary 3.3.12], or see [BS16, Theorem 8.6] for this version with local systems which we state now:
Proposition 4.3. Let
\[ L \simeq L_1^{e_1} \oplus \cdots \oplus L_r^{e_r} \]
be a semisimple complex local system of representation type
\[ \tau = (e_1, \beta_1; \ldots; e_r, \beta_r) \]
on \( C_g \), with \( \text{rank}(L_i) = \beta_i \). Let \( Q \) be the quiver with one vertex and \( g \) loops. Consider the associated quiver \( Q_\tau \), dimension vector \( e \), and group \( G(e) \) as is Definition 2.7. Then:
(a) \( \text{Aut}(L)/k^* \simeq G(e) \).
(b) There is a commutative diagram
\[
\begin{array}{ccc}
\text{Ext}^1(L, L) & \sim & \text{Rep}(Q_\tau, e) \\
\mu \downarrow & & \mu_{Q_\tau, e} \downarrow \\
\text{Ext}^2(L, L) & \sim & \text{End}(e)
\end{array}
\]
where the horizontal maps are \( G(e) \)-equivariant isomorphisms.
(c) There is a \( G(e) \)-equivariant isomorphism
\[ \mu^{-1}(0) \simeq X(Q_\tau, e). \]

Proof. The quiver realizing this identification from [KLS06, BS16] is described as having \( r \) vertices and its double having \( \dim \text{Ext}^1(L_i, L_j) \) arrows from \( i \) to \( j \). The identification with \( Q_\tau \) follows from Definition 2.7 and the equalities
\[ \dim \text{Ext}^1(L_i, L_j) = \begin{cases} 2pQ(\beta_i) = 2(g-1)\beta_i^2 + 2, & \text{if } i = j, \\ -(\beta_i, \beta_j)_Q = 2(g-1)\beta_i \beta_j, & \text{if } i \neq j. \end{cases} \]
We recall how one checks these equalities. One has
\[ \chi(C_g, L_i^\vee \otimes L_j) = \text{rank}(L_i^\vee \otimes L_j)\chi(C_g) = \beta_i \beta_j 2(1 - g) \]
by [Di04, Example 3.3.13]. On the other hand, by duality
\[ h^2(C_g, L_i^\vee \otimes L_j) = h^0(C_g, L_i \otimes L_j^\vee) = \dim \text{Hom}(L_j, L_i). \]
This equals 1 if \( i = j \), and it equals 0 if \( i \neq j \) since the \( L_i \) are simple non-isomorphic local systems. Hence \( h^1(C_g, L_i^\vee \otimes L_j) = \dim \text{Ext}^1(L_i, L_j) \) has the claimed value. \( \square \)

The previous two propositions imply:

Corollary 4.4. Let \( g \geq 2 \). Let \( \rho \in R(g, n) \) be a semisimple complex representation of type \( \tau = (e_1, \beta_1; \ldots; e_r, \beta_r) \). Let \( Q \) be the quiver with one vertex and \( g \) loops. There is an isomorphism of formal germs
\[ \widehat{R(g, n)}_\rho \simeq \widehat{h_0} \times \widehat{X(Q_\tau, e)}_0 \]
for some affine space \( h \).

Remark 4.5. In [KLS06, §3], the analog of this corollary for coherent sheaves on K3 surfaces was proved with some difficulty, see [KLS06, Propositions 3.6 and 3.8], since at the time, formality for polystable sheaves on K3 surfaces was not known. This formality property has been proven recently by Budur-Zhang [BZ18].
Proof of Theorem 1.2. The moduli space $R(g,n)$ contains an open dense subsets of isomorphism classes of simple representations. In particular, by taking direct sums, $R(g,n)$ contains semisimple representations of type $\tau = (e_1, \beta_1; \ldots; e_r, \beta_r)$ for every partition $\sum_i e_i \beta_i = n$ with $r, e_i, \beta_i \in \mathbb{N} \setminus \{0\}$. These are also all the representation types occurring on $X(Q,n)$, where $Q$ is the quiver with one vertex and $g$ loops. The proof of Theorem 1.1 gives that all $X(Q_r, c)$ have rational singularities. By Corollary 4.4, this implies that all semisimple representations in $R(g,n)$ are rational singularities. So every semisimple representation in $R(g,n)$ has an open neighborhood which has rational singularities. Using the group action to map isomorphically a neighborhood of any point to a neighborhood of a point arbitrarily close to a semisimple point, we conclude that $R(g,n)$ has rational singularities everywhere.

5. $SL_n$-representations of surface groups

The sets $\text{Hom}(\pi_1(C_g), \text{GL}_n(C))$ and $\text{Hom}(\pi_1(C_g), \text{SL}_n(C))$ are the sets of complex points of two schemes of finite type over $Q$ which we denote by $R(g, \text{GL}_n)$ and $R(g, \text{SL}_n)$, respectively. The connection between them is given by the following, see [BS16, Lemma 8.17] for the complex points:

Lemma 5.1. Let $g,n \geq 1$. There is an étale morphism of $Q$-schemes

$$R(g, \text{SL}_n) \times_Q (\mathbb{G}_m)^{2g} \to R(g, \text{GL}_n),$$

where $\mathbb{G}_m = \text{Spec } Q[T, T^{-1}]$.

Proof. On the underlying sets of complex points, this just the morphism

$$\text{Hom}(\pi_1(C_g), \text{SL}_n(C)) \times (\mathbb{C}^*)^{2g} \longrightarrow \text{Hom}(\pi_1(C_g), \text{GL}_n(C))$$

$$( (x_i, y_i)_{1 \leq i \leq g}, (\lambda_i, \mu_i)_{1 \leq i \leq g} ) \mapsto ( \lambda_i x_i, \mu_i y_i )_{1 \leq i \leq g}.$$

To prove the claim over $Q$, we need to work with $Q$-algebras though. Let

$$A = Q[s_1^{\pm 1}, t_1^{\pm 1}, \ldots, s_g^{\pm 1}, t_g^{\pm 1}]$$

be ring of regular functions on $(\mathbb{G}_m)^{2g}$. Define the morphism of $Q$-algebras

$$m^\#: A \to A, \quad (s_i, t_i) \mapsto (s_i^n, t_i^n).$$

Let $m : (\mathbb{G}_m)^{2g} \to (\mathbb{G}_m)^{2g}$ be the corresponding morphism on spectra. Then $m$ is an étale morphism sending a closed point $(\lambda_i, \mu_i)_{1 \leq i \leq g}$ to $(\lambda_i^n, \mu_i^n)_{1 \leq i \leq g}$.

Let $B$ be the ring of regular functions on $R(g, \text{GL}_n)$, that is,

$$B = \frac{Q[X_i, Y_i, \det^{-1}(X_i), \det^{-1}(Y_i)]_{1 \leq i \leq g}}{\prod_{i=1}^g [X_i, Y_i] - 1}$$

where $X_i = ((X_{i,j})_{1 \leq j,k \leq n}$, $Y_i = ((Y_{i,j})_{1 \leq j,k \leq n}$ are matrices of indeterminates. Define the $Q$-algebra morphism

$$f^\#: A \to B, \quad (s_i, t_i) \mapsto (\det(X_i), \det(Y_i))$$

Let $f : R(g, \text{GL}_n) \to (\mathbb{G}_m)^{2g}$ be the corresponding morphism of $Q$-schemes. It is given by

$$f : (g_i, h_i)_{1 \leq i \leq g} \mapsto (\det(g_i), \det(h_i))_{1 \leq i \leq g}$$
on closed points. Form the fiber product

\[ R(g, \text{GL}_n) \times (\mathbb{G}_m)^{2g} \to R(g, \text{GL}_n) \]

\[ (\mathbb{G}_m)^{2g} \]

We claim that there is an isomorphism of \((\mathbb{G}_m)^{2g}\)-schemes

\[ R(g, \text{SL}_n) \times \mathbb{Q} (\mathbb{G}_m)^{2g} \sim R(g, \text{GL}_n) \times (\mathbb{G}_m)^{2g} \]

where \(p\) is the second projection. On closed points, \(g\) sends a point \(((g_i, h_i), (\lambda_i, \mu_i))\) of \(R(g, \text{SL}_n) \times \mathbb{Q} (\mathbb{G}_m)^{2g}\) to the point \(((\lambda_i g_i, h_i), (\lambda_i, \mu_i))\) of \(R(g, \text{GL}_n) \times (\mathbb{G}_m)^{2g}\).

To define \(g\) schematically, we define the corresponding \(\mathbb{Q}\)-algebra morphism \(g^\#\). Let

\[ C = \frac{B}{\langle \det(X_i) - 1, \det(Y_i) - 1 \rangle_{1 \leq i \leq g}} \otimes \mathbb{Q} A. \]

Then

\[ R(g, \text{SL}_n) \times \mathbb{Q} (\mathbb{G}_m)^{2g} = \text{Spec}(C). \]

Let \(D = B \otimes_A A\) via the \(A\)-algebra morphisms \(f^\# : A \to B\) and \(m^\# : A \to A\). Then

\[ R(g, \text{GL}_n) \times (\mathbb{G}_m)^{2g} = \text{Spec}(D). \]

Note that there is a natural isomorphism of \(A\)-algebras

\[ D \sim \frac{B \otimes \mathbb{Q} A}{\langle \det(X_i) - s_i^n, \det(Y_i) - t_i^n \rangle} \]

Define first a morphisms of \(A\)-algebras

\[ g^\# : B \otimes \mathbb{Q} A \to B \otimes \mathbb{Q} A \]

by \(X_i \mapsto s_i X_i\) (that is, with \(s_i X_i\) viewed as a matrix), \(Y_i \mapsto t_i Y_i\), \(s_i \mapsto s_i\), \(t_i \mapsto t_i\). This is clearly an isomorphism over \(A\), since \(s_i\) and \(t_i\) are invertible and the inverse is given by \(X_i \mapsto t_i^{-1} X_i\), \(Y_i \mapsto t_i^{-1} Y_i\). Moreover, \(g^\#(\det(X_i) - s_i^n) = s_i^n(\det(X_i) - 1)\), and similarly for \(Y_i\). Hence the ideal generated by the image under \(g^\#\) of the ideal \(\langle \det(X_i) - s_i^n, \det(Y_i) - t_i^n \rangle\) is precisely the ideal \(\langle \det(X_i) - 1, \det(Y_i) - 1 \rangle\), since \(s_i\) and \(t_i\) are invertible. It follows that \(g^\#\) induces a well-defined \(A\)-algebra isomorphism \(g^\# : D \simto C\). Thus \(g\) is an isomorphism over \((\mathbb{G}_m)^{2g}\).

The base change of an étale morphism is étale. Hence \(m^\prime\) is étale. Then the composition

\[ m^\prime \circ g : R(n, \text{SL}_d) \times_k (\mathbb{G}_m)^{2n} \to R(n, \text{GL}_d) \]

is étale since \(g\) is an isomorphism. \(\square\)

**Proof of Theorem 1.3.** By general properties of schemes (see for example the proof of [BZ18, Lemma 3.2]), Theorem 1.2 implies that \(R(g, \text{GL}_n)\) is a \(\mathbb{Q}\)-variety with rational singularities for \(g \geq 2\). By Lemma 5.1, the same is true for \(R(g, \text{SL}_n)\). This
implies in particular Theorem 1.3 and it shows that the natural scheme structure on Hom(π₁(Cₙ), SL_n) is reduced for g ≥ 2.

6. CONSEQUENCES

We recall the following terminology from Theorem 1.6. Let Γ be a topological group. The abscissa of convergence

$$\alpha(\Gamma)$$

is the smallest $s_0 \in \mathbb{R} \cup \{\infty\}$ such that representation zeta function

$$\zeta_\Gamma(s) = \sum_{m \geq 1} r_m(\Gamma)m^{-s},$$

with $r_m(\Gamma)$ being the number of isomorphism classes of continuous irreducible $m$-dimensional complex representations of $\Gamma$, converges for $\text{Re}(s) > s_0$.

In Theorem 1.7, the abscissa $\alpha$ is defined similarly but counting all the irreducible representations without restricting to the continuous ones.

In order for $\zeta_\Gamma(s)$ to be defined, $r_m(\Gamma)$ must be finite. For finitely generated pro-finite groups $\Gamma$, this property is equivalent to having finite abelianization by [JK06]. This is the case for the groups in Theorems 1.5 and 1.6. For arithmetic groups in higher rank semisimple groups, the finiteness of $r_m(\Gamma)$ is a consequence of Margulis superrigidity, see [B+02, p2]. This is the case for the groups in Theorems 1.4 and 1.7.

Proof of Theorem 1.5. Follows from Theorem 1.6 and general facts about abscissae of convergence of Dirichlet generating functions.

Proof of Theorems 1.6 and 1.8. They follow directly from Theorem 1.3 and [AA16, Theorem IV].

Proof of Theorem 1.7. Follows from Theorem 1.3 together with [AA18, Theorem II]. See the comment following Corollary 1.4 in [BZo18].

Proof of Theorem 1.4. Follows from Theorem 1.7 and general facts about abscissae of convergence of Dirichlet generating functions.

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