UNIQUE CONTINUATION RESULTS FOR RICCI CURVATURE AND APPLICATIONS

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Abstract. Unique continuation results are proved for metrics with prescribed Ricci curvature in the setting of bounded metrics on compact manifolds with boundary, and in the setting of complete conformally compact metrics on such manifolds. Related to this issue, an isometry extension property is proved: continuous groups of isometries at conformal infinity extend into the bulk of any complete conformally compact Einstein metric. Relations of this property with the invariance of the Gauss-Codazzi constraint equations under deformations are also discussed.

1. Introduction.

In this paper, we study certain issues related to the boundary behavior of metrics with prescribed Ricci curvature. Let $M$ be a compact $(n+1)$-dimensional manifold with compact non-empty boundary $\partial M$. We consider two possible classes of Riemannian metrics $g$ on $M$. First, $g$ may extend smoothly to a Riemannian metric on the closure $\bar{M} = M \cup \partial M$, thus inducing a Riemannian metric $\gamma = g|_{\partial M}$ on $\partial M$. Second, $g$ may be a complete metric on $M$, so that $\partial M$ is “at infinity”. In this case, we assume that $g$ is conformally compact, i.e. there exists a defining function $\rho$ for $\partial M$ in $M$ such that the conformally equivalent metric

\begin{equation}
\tilde{g} = \rho^2 g
\end{equation}

extends at least $C^2$ to $\partial M$. The defining function $\rho$ is unique only up to multiplication by positive functions; hence only the conformal class $[\gamma]$ of the associated boundary metric $\gamma = \tilde{g}|_{\partial M}$ is determined by $(M,g)$.

The issue of boundary regularity of Riemannian metrics $g$ with controlled Ricci curvature has been addressed recently in several papers. Thus, [4] proves boundary regularity for bounded metrics $g$ on $M$ with controlled Ricci curvature, assuming control on the boundary metric $\gamma$ and the mean curvature of $\partial M$ in $M$. In [16], boundary regularity is proved for conformally compact Einstein metrics with smooth conformal infinity; this was previously proved by different methods in dimension 4 in [3], cf. also [5].

One purpose of this paper is to prove a unique continuation property at the boundary $\partial M$ for bounded metrics or for conformally compact metrics. We first state a version of the result for Einstein metrics on bounded domains.

**Theorem 1.1.** Let $(M,g)$ be a $C^{2,\alpha}$ metric on a compact manifold with boundary $M$, with induced metric $\gamma = g|_{\partial M}$, and let $A$ be the 2nd fundamental form of $\partial M$ in $M$. Suppose the Ricci curvature $Ric_g$ satisfies

\begin{equation}
Ric_g = \lambda g,
\end{equation}

where $\lambda$ is a fixed constant.

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Then \((M, g)\) is uniquely determined up to local isometry by the Cauchy data \((\gamma, A)\) on an arbitrary open set \(U\) of \(\partial M\). In particular, the topology of \(M\) and \(\partial M\) are uniquely determined up to covering spaces, and the global Cauchy data \((\gamma, A)\) on \(\partial M\) are determined by their values on \(U\).

Similar results hold for metrics which satisfy other covariant equations involving the metric to 2nd order, for example the Einstein equations coupled to other fields; see Proposition 3.6.

For conformally compact metrics, the 2nd fundamental form \(A\) of the compactified metric \(\bar{g}\) in (1.1) is umbilic, and completely determined by the defining function \(\rho\). In fact, for conformally compact Einstein metrics, the higher order Lie derivatives \(L_N^{(k)} \bar{g}\) at \(\partial M\), where \(N\) is the unit vector in the direction \(\nabla \rho\), are determined by the conformal infinity \([\gamma]\) and \(\rho\) up to order \(k < n\). Supposing \(\rho\) is a geodesic defining function, so that \(\|\nabla \rho\| = 1\), let

\[
g(n) = \frac{1}{n!} L_N^{(n)} \bar{g}.
\]

More precisely, \(g(n)\) is the \(n\)th term in the Fefferman-Graham expansion of the metric \(g\); this is given by (1.3) when \(n\) is odd, and in a similar way when \(n\) is even, cf. [18] and §4 below. The term \(g(n)\) is the natural analogue of \(A\) for conformally compact Einstein metrics.

**Theorem 1.2.** Let \(g\) be a \(C^2\) conformally compact Einstein metric on a compact manifold \(M\) with \(C^\infty\) smooth conformal infinity \([\gamma]\), normalized so that

\[
Ric_g = -ng.
\]

Then the Cauchy data \((\gamma, g(n))\) restricted to any open set \(U\) of \(\partial M\) uniquely determine \((M, g)\) up to local isometry and determine \((\gamma, g(n))\) globally on \(\partial M\).

The recent boundary regularity result of Chruściel et al., [16], implies that \((M, g)\) is \(C^\infty\) polyhomogeneous conformally compact, so that the hypotheses of Theorem 1.2 imply the term \(g(n)\) is well-defined on \(\partial M\). A more general version of Theorem 1.2, (without the smoothness assumption on \([\gamma]\)), is proved in §4, cf. Theorem 4.1. For conformally compact metrics coupled to other fields, see Remark 4.5.

Of course neither Theorem 1.1 or 1.2 hold when just the boundary metric \(\gamma\) on \(U \subset \partial M\) is fixed. For example, in the context of Theorem 1.2, by [20] and [16], given any \(C^\infty\) smooth boundary metric \(\gamma\) sufficiently close to the round metric on \(S^n\), there is a smooth (in the polyhomogeneous sense) conformally compact Einstein metric on the \((n + 1)\)-ball \(B^{n+1}\), close to the Poincaré metric. Hence, the behavior of \(\gamma\) in \(U\) is independent of its behavior on the complement of \(U\) in \(\partial M\).

Theorems 1.1 and 1.2 have been phrased in the context of “global” Einstein metrics, defined on compact manifolds with compact boundary. However, the proofs are local, and these results hold for metrics defined on an open manifold with boundary. From this perspective, the data \((\gamma, A)\) or \((\gamma, g(n))\) on \(U\) determine whether Einstein metric \(g\) has a global extension to an Einstein metric on a compact manifold with boundary, (or conformally compact Einstein metric), and how smooth that extension is at the global boundary.

A second purpose of the paper is to prove the following isometry extension result which is at least conceptually closely related to Theorem 1.2. However, while Theorem 1.2 is valid locally, this result depends crucially on global properties.

**Theorem 1.3.** Let \(g\) be a \(C^2\) conformally compact Einstein metric on a compact manifold \(M\) with \(C^\infty\) boundary metric \((\partial M, \gamma)\), and suppose

\[
\pi_1(M, \partial M) = 0.
\]

Then any connected group of isometries of \((\partial M, \gamma)\) extends to an action by isometries on \((M, g)\).
The condition (1.5) is equivalent to the statement that $\partial M$ is connected and the inclusion map $\iota : \partial M \to M$ induces a surjection $\pi_1(\partial M) \to \pi_1(M) \to 0$.

Rather surprisingly, this result is closely related to the equations at conformal infinity induced by the Gauss-Codazzi equations on hypersurfaces tending to $\partial M$. It turns out that isometry extension from the boundary at least into a thickening of the boundary is equivalent to the requirement that the Gauss-Codazzi equations induced at $\partial M$ are preserved under arbitrary deformations of the boundary metric. This is discussed in detail in §5, see e.g. Proposition 5.4. We note that this result does not hold for complete, asymptotically (locally) flat Einstein metrics, cf. Remark 5.8.

A simple consequence of Theorem 1.3 is the following uniqueness result:

**Corollary 1.4.** A $C^2$ conformally compact Einstein metric with conformal infinity given by the class of the round metric $g_{n+1}$ on the sphere $S^n$ is necessarily isometric to the Poincaré metric on the ball $B^{n+1}$.

Results similar to Theorem 1.3 and Corollary 1.4 have previously been proved in a number of different special cases by several authors, see for example [7], [9], [31], [33]; the proofs in all these cases are very different from the proof given here.

It is well-known that unique continuation does not hold for large classes of elliptic systems of PDE’s, even for general small perturbations of systems which are diagonal at leading order; see for instance [23] and references therein for a discussion related to geometric PDEs. The proofs of Theorems 1.1 and 1.2 rely on unique continuation results of Calderón [13], [14] and Mazzeo [27] respectively, based on Carleman estimates. The main difficulty in reducing the proofs to these results is the diffeomorphism covariance of the Einstein equations and, more importantly, that of the “abstract” Cauchy data $(\gamma, A)$ or $(\gamma, g_{(n)})$ at $\partial M$. The unique continuation theorem of Mazzeo requires a diagonal (i.e. uncoupled) Laplace-type system of equations, at leading (second) order. The unique continuation result of Calderón is more general, but again requires strong restrictions on the structure of the leading order symbol of the operator. For emphasis and clarity, these issues are discussed in more detail in §2. The proofs of Theorems 1.1, 1.2 and 1.3 are then given in §3, §4 and §5 respectively.

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2. Local Coordinates and Cauchy Data

In this section, we discuss in more detail the remarks in the Introduction on classes of local coordinate systems, and their relation with Cauchy data on the boundary $\partial M$.

Thus, consider for example solutions to the system

\[
Ric_g = 0,
\]

defined near the boundary $\partial M$ of an $(n+1)$-dimensional manifold $M$. Since the Ricci curvature involves two derivatives of the metric, Cauchy data at $\partial M$ consist of the boundary metric $\gamma$ and its first derivative, invariantly represented by the 2nd fundamental form $A$ of $\partial M$ in $M$. Thus, we assume $(\gamma, A)$ are prescribed at $\partial M$, (subject to the Gauss and Gauss-Codazzi equations), and call $(\gamma, A)$ abstract Cauchy data. Observe that the abstract Cauchy data are invariant under diffeomorphisms of $M$ equal to the identity at $\partial M$.

The metric $g$ determines the geodesic defining function

\[
t(x) = dist_g(x, \partial M).
\]
The function $t$ depends of course on $g$; however, given any other smooth metric $g'$, there is a diffeomorphism $F$ of a neighborhood of $\partial M$, equal to the identity on $\partial M$, such that $t'(x) = \text{dist}_{F'\gamma'}(x, \partial M)$ satisfies $t' = t$. As noted above, this normalization does not change the abstract Cauchy data $(\gamma, A)$ and preserves the isometry class of the metric.

Let $\{y^\alpha\}$, $0 \leq \alpha \leq n$, be any local coordinates on a domain $\Omega$ in $M$ containing a domain $U$ in $\partial M$. We assume that $\{y^\alpha\}$ for $1 \leq i \leq n$ form local coordinates for $\partial M$ when $y^0 = 0$, so that $\partial/\partial y^0$ is transverse to $\partial M$. Throughout the paper, Greek indices $\alpha, \beta$ run from 0 to $n$, while Latin indices $i, j$ run from 1 to $n$.

If $g_{\alpha\beta}$ are the components of $g$ in these coordinates, then the abstract Cauchy problem associated to (2.1) in the local coordinates $\{y^\alpha\}$ is the system
\[ (\text{Ric}_g)_{\alpha\beta} = 0, \quad \text{with} \quad g_{ij}|_U = \gamma_{ij}, \quad \frac{1}{2}(\mathcal{L}_{\nabla g})_{ij}|_U = a_{ij}, \]
where $\gamma_{ij}$ and $a_{ij}$ are given on $U$, (subject to the constraints of the Gauss and Gauss-Codazzi equations). Here one immediately sees a problem, in that (2.2) on $U \subset \partial M$ involves only the tangential part $g_{ij}$ of the metric (at 0 order), and not the full metric $g_{\alpha\beta}$ at $U$. The normal $g_{00}$ and mixed $g_{0i}$ components of the metric are not prescribed at $U$. As seen below, these components are gauge-dependent; they cannot be prescribed “abstractly”, independent of coordinates, as is the case with $\gamma$ and $A$.

In other words, if (2.1) is expressed in local coordinates $\{y^\alpha\}$ as above, then a well-defined Cauchy or unique continuation problem has the form
\[ (\text{Ric}_g)_{\alpha\beta} = 0, \quad \text{with} \quad g_{\alpha\beta} = \gamma_{\alpha\beta}, \quad \frac{1}{2}\partial t g_{\alpha\beta} = a_{\alpha\beta}, \quad \text{on} \quad U \subset \partial M, \]
where $\Omega$ is an open set in $(\mathbb{R}^{n+1})^+$ with $\partial \Omega = U$ an open set in $\partial (\mathbb{R}^{n+1})^+ = \mathbb{R}^n$. Formally, (2.3) is a determined system, while (2.2) is underdetermined.

Let $g_0$ and $g_1$ be two solutions to (2.1), with the same Cauchy data $(\gamma, A)$, and with geodesic defining functions $t_0, t_1$. Changing the metric $g_1$ by a diffeomorphism if necessary, one may assume that $t_0 = t_1$. One may then write the metrics with respect to a Gaussian or geodesic boundary coordinate system $(t, y^i)$ as
\[ g_k = dt^2 + (g_k)_t, \]
where $(g_k)_t$ is a curve of metrics on $\partial M$ and $k = 0, 1$. Here $y_i$ are coordinates on $\partial M$ which are extended into $M$ to be invariant under the flow of the vector field $\nabla t$. The metric $(g_k)_t$ is the metric induced on $S(t)$ and pulled back to $\partial M$ by the flow of $\nabla t$. One has $(g_k)_0 = \gamma$ and $\frac{1}{2}\partial t (g_k)_t|_{t=0} = A$.

Since $g_{0\alpha} = \delta_{0\alpha}$ in these coordinates, $\nabla t = \partial_t$, and hence the local coordinates are the same for both metrics, (or at least may be chosen to be the same). Thus, geodesic boundary coordinates are natural from the point of view of the Cauchy or unique continuation problem, since in such local coordinates the system (2.2), together with the prescription $g_{0\alpha} = \delta_{0\alpha}$, is equivalent to the system (2.3).

However, the Ricci curvature is not elliptic or diagonal to leading order in these coordinates. The expression of the Ricci curvature in such coordinates does not satisfy the hypotheses of Calderón’s theorem [14], and it appears to be difficult to establish unique continuation of solutions in these coordinates.

Next suppose that $\{x^\alpha\}$ are boundary harmonic coordinates, defined as follows. For $1 \leq i \leq n$, let $\tilde{x}^i$ be local harmonic coordinates on a domain $U$ in $(\partial M, \gamma)$. Extend $\tilde{x}^i$ into $M$ to be harmonic functions in $(\Omega, g)$, $\Omega \subset M$, with Dirichlet boundary data; thus
\[ \Delta_g x^i = 0, \quad x^i|_U = \tilde{x}^i. \]
Let $x^0$ be a harmonic function on $\Omega$ with 0 boundary data, so that
\[ \Delta_g x^0 = 0, \quad x^0|_U = 0. \]
Then the collection \( \{x^\alpha\}, 0 \leq \alpha \leq n \), form a local harmonic coordinate chart on a domain \( \Omega \subset (M,g) \). In such coordinates, one has

\[
(2.7) \quad (\text{Ric}_g)_{\alpha\beta} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} + Q_{\alpha\beta}(g, \partial g),
\]

where \( Q(g, \partial g) \) depends only on \( g \) and its first derivatives. This is an elliptic operator, diagonal at leading order, and satisfies the hypotheses of Calderón’s theorem. However, in general, the local Cauchy problem (2.3) is not well-defined in these coordinates; if \( g_0 \) and \( g_1 \) are two solutions of (2.1), each with corresponding local boundary harmonic coordinates, then the components \( (g_0)_{0\alpha} \) and \( (g_1)_{0\alpha} \) in general will differ at \( U \subset \partial M \). This is of course closely related to the fact that there are many possible choices of harmonic functions \( x^\alpha \) satisfying (2.5) and (2.6), and to the fact that the behavior of harmonic functions depends on global properties of \((\Omega, g)\). In any case, it is not known how to set up a well-defined Cauchy problem in these coordinates for which one can apply standard unique continuation results.

Consider then geodesic-harmonic coordinates “intermediate” between geodesic boundary and boundary harmonic coordinates. Thus, let \( t \) be the geodesic distance to \( \partial M \) as above. Choose local harmonic coordinates \( \tilde{x}^i \) on \( \partial M \) as before and extend them into \( M \) to be harmonic on the level sets \( S(t) \) of \( t \), i.e. locally on \( S(t) \),

\[
(2.8) \quad \Delta_{U(t)} x^i = 0, \quad x^i|_{\partial U(t)} = \tilde{x}^i|_{\partial U(t)};
\]

here the boundary value \( \tilde{x}^i \) is the extension of \( x^i \) on \( U \) into \( M \) which is invariant under the flow \( \phi_t \) of \( \nabla t \), and \( U(t) = \phi_t(U) \subset S(t) \). The functions \( (t, x^i) \) form a coordinate system in a neighborhood \( \Omega \) in \( M \) with \( \Omega \cap \partial M = U \).

It is not difficult to prove that geodesic-harmonic coordinates preserve the Cauchy data, in the sense that the data (2.2) in such coordinates imply the data (2.3). However, the Ricci curvature is not an elliptic operator in the metric in these coordinates, nor is it diagonal at leading order; the main reason is that the mean curvature of the level sets \( S(t) \) is not apriori controlled. So again, it remains an open question whether unique continuation can be proved in these coordinates.

Having listed these attempts which appear to fail, a natural choice of coordinates which do satisfy the necessary requirements are \( H \)-harmonic coordinates \((\tau, x^i)\), whose \( \tau \)-level surfaces \( \Sigma_{\tau} \) are of prescribed mean curvature \( H \) and with \( x^i \) harmonic on \( \Sigma_{\tau} \). These coordinates were introduced by Andersson-Moncrief [8] to prove a well-posedness result for the Cauchy problem for the Einstein equations in general relativity, and, as shown in [8], have a number of advantageous properties. Thus, adapting some of the arguments of [8], we show in §3 that the Einstein equations (1.2) are effectively elliptic in such coordinates, and such coordinates preserve the Cauchy data in the sense above, (i.e. (2.2) implies (2.3)). It will then be shown that unique continuation holds in such coordinates, via application of the Calderón theorem.

3. Proof of Theorem 1.1

Theorem 1.1 follows from a purely local result, which we formulate as follows. Let \( \Omega \) be a domain diffeomorphic to a cylinder \( I \times B^n \subset \mathbb{R}^{n+1} \), with \( U = \partial \Omega \) diffeomorphic to a ball \( B^n \subset \mathbb{R}^n = \{ 0 \} \times \mathbb{R}^n \subset \mathbb{R}^{n+1} \). Let \( g \) be a Riemannian metric on \( \Omega \) which is \( C^{2,\alpha} \) up to \( \partial \Omega \) in a given coordinate system \( \{y^\alpha\} \) with \( y^0 = 0 \) on \( \partial \Omega \). Without loss of generality, we assume that \( \Omega \) is chosen sufficiently small so that \( g \) is close to the Euclidean metric \( \delta \) in the \( C^{2,\alpha} \) topology. For simplicity, rescale \((\Omega, g)\) and the coordinates \( \{y^\alpha\} \) if necessary so that \( (\Omega, g) \) is \( C^{2,\alpha} \) close to the standard cylinder \(((I \times B^n(1), B^n(1))) \subset (\mathbb{R}^{n+1}, \mathbb{R}^n), I = [0, 1] \). The full boundary of \( \Omega \), i.e. \( \Omega \setminus \Omega \) will be denoted by \( \partial \Omega \).

We will prove the following local version of Theorem 1.1.
Theorem 3.1. Let $g_0, g_1$ be two $C^{2,\alpha}$ metrics as above on $\Omega$ satisfying

\begin{equation}
\text{Ric}_{g_k} = \lambda g_k,
\end{equation}

for some fixed constant $\lambda$, $k = 0, 1$. Suppose $g_0$ and $g_1$ have the same abstract Cauchy data on $U$ in the sense of §2, so that $\gamma_0 = \gamma_1$ and $A_0 = A_1$.

Then $(\Omega, g_0)$ is isometric to $(\Omega, g_1)$, by an isometry equal to the identity on $\partial \Omega$. In particular, Theorem 1.1 holds.

The proof of Theorem 3.1 will proceed in several steps, organized around several Lemmas. We first work with a fixed metric $g$ on $\Omega$ as above. Let $N$ be the inward unit normal to $\partial \Omega$ in $\Omega$ and let $A = \nabla N$ be the corresponding 2nd fundamental form, with mean curvature $H = \text{tr}A$ on $U$. By the initial assumptions on $\Omega$ above, one has $A \sim 0$ and $H \sim 0$ in $C^{1,\alpha}$. To begin, we construct the system of $H$-harmonic coordinates discussed at the end of §2.

Lemma 3.2. The domain $\Omega$ has a $C^{2,\alpha}$ foliation by surfaces $\Sigma_s$ with fixed mean curvature $H$, given by the mean curvature of $\partial \Omega$. Thus,

\begin{equation}
H_{\Sigma_s} = H,
\end{equation}

in the coordinates $y^a$, where $\Sigma_s$ is a graph over $U = \partial \Omega$ of the form $\Sigma_s = \{(f_s(y^i), y^i)\}$ such that $y^0(\Sigma_s \cap \partial \Omega) = s$, $s \in [0, 1]$. Hence, in the coordinates $y^i$ for $\Sigma_s$, $H_{\Sigma_s}(y^i) = H(y^i)$.

Proof: For any given $s > 0$, one may solve the prescribed mean curvature equation for graphs over $U = \partial \Omega$ with respect to the coordinates $y^a$ in the metric $g$. The solution $f_s(y^i)$ of (3.2) with $f_s|_{\partial U} = s$ exists and is unique, since the prescribed data $H \in C^{1,\alpha}$ is small and the metric $g$ is $C^{2,\alpha}$ close to the Euclidean metric on the domain $\Omega$. Note here that $U$ is close to the unit ball $B^n \subset \mathbb{R}^n$ with boundary mean curvature close to $n - 1$. Since $g \in C^{2,\alpha}$ and $H \in C^{1,\alpha}$, the solution $f_s \in C^{3,\alpha}(y^i)$. Of course $f_0 = 0$, so that $\Sigma_0 = \partial \Omega$. The existence result above follows from, for instance, [19, Thm. 16.10, Thm. 15.10], cf. also [10], while the uniqueness statement follows from [19, Thm. 10.7(iii)].

The uniqueness implies that the family $\{\Sigma_s\}$ forms a $C^0$ foliation of $\Omega$. The derivative $\frac{df}{ds}$ satisfies a linear elliptic equation, with boundary value 1 and elliptic regularity implies the foliation is $C^{2,\alpha}$ smooth. A basis for the tangent space of $\Sigma_s$ is given by the vectors $\partial_{y^i} + (\partial_{y^i}f)\partial_{y^j}$ and hence the induced metric $g_{ij}$ on each $\Sigma_s$ in the induced $y$-coordinates is $C^{2,\alpha}$. Further, the 2nd fundamental form $A_s$ of $\Sigma_s$ is $C^{1,\alpha}$ in the $y$-coordinates.

Given the foliation $\Sigma_s$ of prescribed mean curvature, we construct a corresponding $H$-harmonic coordinate system essentially as in §2. Thus, define the coordinate function $\tau$ by $\tau^{-1}(s) = \Sigma_s$, so that the level sets of $\tau$ are the leaves $\Sigma_s$, with $\tau = s = y^0$ on $\partial \Sigma_s$. In particular, $\partial \Omega = \tau^{-1}(0)$. For the spatial or tangential coordinates $x^i$, as before, choose harmonic coordinates $x^i$ on $U = \partial \Omega$, $\Delta x^i = 0$, such that in these coordinates $\gamma_{ij}$ is $C^{2,\alpha}$ close to $\delta_{ij}$ on $U$. Extend $x^i$ to coordinates on each $\Sigma_\tau$ by solving the Dirichlet problem

\begin{equation}
\Delta_{\Sigma_\tau} x^i = 0, \quad x^i|_{\partial \Sigma_\tau} = \bar{x}^i|_{\partial \Sigma_\tau};
\end{equation}

here the boundary value $\bar{x}^i$ is the ‘vertical’ extension of $x^i$ on $U$ into $\Omega$, i.e. $\bar{x}^i = \tau = y^0$ on $\partial \Sigma_\tau$. We may assume, without loss of generality, that $\{x^0\} = \{(\tau, x^i)\}$ form a coordinate system for $\Omega$.

As is well-known, the metric $g_{ij}$ induced on each $\Sigma_\tau$ has optimal regularity properties in harmonic coordinates, cf. [11] for instance. Thus, the metric $g_{ij}^y$ on $\Sigma_\tau$ in the $x$-coordinate chart, $g_{ij}^y = g(\partial_{x^i}, \partial_{x^j})$, is as smooth as the induced metric $g_{ij}^y$ on $\Sigma_\tau$ in the $y$-coordinate chart. Since, by the above, $g_{ij}^y \in C^{2,\alpha}$, one has $g_{ij}^y \in C^{2,\alpha}$, and $x^i \in C^{3,\alpha}(\{y^i\})$ on each $\Sigma_\tau$. 

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In the \( \{y^\alpha\} \) coordinates, \( \partial_{y^\rho} H_{\Sigma_\tau} = 0 \) and \( \partial_{y^\rho} H_{\Sigma_\tau} \) is fixed, independent of \( \tau \). In the \( \{x^i\} \) coordinates, one thus has on \( \Sigma_\tau \),
\[
\partial_{x^i} H = \frac{\partial y^j}{\partial x^i} \partial_{y^j} H \quad \text{and} \quad \partial_{\tau} H = \frac{\partial y^j}{\partial \tau} \partial_{y^j} H \in C^\alpha(\{x^k\}).
\]

The metric \( g = g_{\alpha\beta} \) in the \( H \)-harmonic coordinates \( (\tau, x^i) \) has the form
\[
g = u^2 d\tau^2 + g_{ij}(dx_i + \sigma^i d\tau)(dx_j + \sigma^j d\tau),
\]
where \( \sigma = \sigma^i \partial_{x^i} \) is the shift vector and \( u \) is the lapse function, cf. also [8]. One has \( g^{\alpha\beta} = u^{-2} \), and \( g_{00} = u^2 + |\sigma|^2 \), where \( |\sigma|^2 = g_{ij} \sigma^i \sigma^j \). The unit normal \( N \) to the \( \Sigma_\tau \) foliation is given by
\[
N = \nabla \tau/|\nabla \tau|, \quad \nabla \tau = g^{\alpha\beta} \partial_\alpha, \quad \text{so that } |\nabla \tau| = u^{-1} \text{ and}
\]
\[
N = u^{-1}(\partial_\tau + \sigma),
\]
with \( \sigma = u^2 g^{0i} \partial_i \). In particular, \( N(x^i) = u g^{0i} \), or equivalently,
\[
u^2 \nabla \tau(x^i) = \sigma^i.
\]
In these coordinates, the 2nd fundamental form \( A = \frac{1}{2} \mathcal{L}_N g \) of the leaves \( \Sigma_\tau \) has the form
\[
A_{ij} = \frac{1}{2} u^{-1}(\partial_\tau g_{ij} + (\mathcal{L}_\sigma g)_{ij}),
\]
so that
\[
\partial_{\tau} g_{ij} = 2uA_{ij} - (\mathcal{L}_\sigma g)_{ij}.
\]
A standard computation from commuting derivatives gives the Riccati equation
\[
(\mathcal{L}_N A)_{ij} = A_{ij}^2 - u^{-1}(D^2 u)_{ij} - R_N(ij),
\]
where \( R_N(ij) = \langle R(\partial_i, N)N, \partial_j \rangle \) and \( A_{ij}^2 = A_i^k A_j^k \). (The equation (3.10) may also be derived from the 2nd variation formula). Using (3.6) and the fact that \( A \) is tangential, (i.e. \( A(N, \cdot) = 0 \)), this gives
\[
\partial_{\tau} A = -\mathcal{L}_\sigma A - D^2 u + u A^2 - u R_N.
\]
Another straightforward calculation via the Gauss equations shows that \( R_N = Ric_g - Ric_{\Sigma_\tau} + HA - A^2 \), which, via (3.9) and (3.11) gives the system of ‘evolution’ equations for \( g = g_{ij} \) and \( A = A_{ij} \):
\[
\partial_{\tau} g = 2uA - \mathcal{L}_\sigma g,
\]
\[
\partial_{\tau} A = -\mathcal{L}_\sigma A - D^2 u + u[Ric_{\Sigma_\tau} - Ric_g + 2A^2 - HA].
\]
(Up to sign differences, these are the well-known Einstein evolution equations in general relativity, cf. [8], [32]). Substituting (3.12) in (3.13) gives the 2nd order evolution equation for \( g \):
\[
\partial_{\tau}^2 g_{ij} = 2u\partial_{\tau} A_{ij} - (\mathcal{L}_\partial, \mathcal{L}_\sigma g)_{ij} + 2(\partial_\tau u)A_{ij}
\]
\[
= -2u(\mathcal{L}_\sigma A)_{ij} - 2u(D^2 u)_{ij} + 2u^2[Ric_{\Sigma_\tau} - Ric_g + 2A^2 - HA]_{ij} - (\mathcal{L}_\partial, \mathcal{L}_\sigma g)_{ij} + 2(\partial_\tau u)A_{ij}.
\]
Since in \( g \)-harmonic coordinates \( (Ric_{\Sigma_\tau})_{ij} = -\frac{1}{2} \Delta g_{ij} + Q_{ij}(g, \partial g) \), this implies, (via (3.8)),
\[
(\partial_{\tau}^2 + u^2 \Delta) g_{ij} = -2u^2(Ric_g)_{ij} - 2u(D^2 u)_{ij} - 2(\mathcal{L}_\partial \mathcal{L}_\partial g)_{ij} - (\mathcal{L}_\sigma \mathcal{L}_\sigma g)_{ij} + Q_{ij}(g, \partial g),
\]
where \( Q_{ij} \) is a term involving at most the first order derivatives of \( g_{\alpha\beta} \) in all \( x^\alpha \) directions. Thus, for Einstein metrics (1.2),
\[
(\partial_{\tau}^2 + u^2 \Delta + 2\partial_\sigma \partial_{\tau} + 2\partial_\sigma^2) g_{ij} = -2u(D^2 u)_{ij} + Q_{ij}(g, \partial g),
\]
where \( Q_{ij} \) has the same general form as before.
One also has the 'constraint' equations along each leaf \( \Sigma_\tau \), involving the non-tangential part of the Ricci curvature:

\[
\delta(A - H g) = \text{Ric}(N, \cdot) = 0,
\]

\[
|A|^2 - H^2 + R_{\Sigma_\tau} = R_g - 2\text{Ric}_g(N, N) = (n - 1)\lambda.
\]

Next, we derive the equations for the lapse \( u \) and shift \( \sigma \).

**Lemma 3.3.** The lapse \( u \) and shift \( \sigma \) satisfy the following equations:

\[
\Delta u + |A|^2 u + \lambda u = -uN(H) = -(\partial_\tau + \sigma)H.
\]

\[
\Delta \sigma^i = 2u\langle D^2 x^i, A \rangle + u\langle dx^i, dH \rangle + 2\langle dx^i, A(\nabla u) - \frac{1}{2}H du \rangle.
\]

**Proof:** The lapse equation is derived by taking the trace of (3.10), and noting that \( tr\mathcal{L}_N A = N(H) + 2|A|^2 \).

For the shift equation, since the functions \( x^i \) are harmonic on \( \Sigma_\tau \), one has

\[
\Delta((x^i)') + (\Delta')(x^i) = 0,
\]

where \( ' \) denotes the Lie derivative with respect to \( u^i \nabla \tau = uN \) and the Laplacian is taken with respect to the induced metric on the slices \( \Sigma_\tau \). Thus, by (3.7), \( (x^i)' = \sigma^i \).

From standard formulas, cf. [11, Ch. 1K] for example, one has

\[
(\Delta')(x^i) = -2\langle D^2 x^i, \delta^* uN \rangle + 2\langle dx^i, \beta(\delta^* uN) \rangle,
\]

where all the terms on the right are along \( \Sigma_\tau \) and \( \beta \) is the Bianchi operator, \( \beta(k) = \delta k + \frac{1}{2}dtrk \).

Thus, \( \delta^* uN = uA \), and the shift components \( \sigma^i \) satisfy

\[
\Delta \sigma^i = 2u\langle D^2 x^i, A \rangle - 2u\langle dx^i, \delta A + \frac{1}{2}dH \rangle + 2\langle dx^i, A(\nabla u) - \frac{1}{2}H du \rangle.
\]

The relation (3.19) then follows from the constraint equation (3.17).

**Lemma 3.4.** In local \( H \)-harmonic coordinates, the lapse-shift coordinates \( (u, \sigma) \) are uniquely determined by the tangential metric \( g_{ij} \) and \( 2^{nd} \) fundamental form \( A_{ij} \) on each \( \Sigma_\tau \). Moreover, the abstract Cauchy data \( (g_{ij}, A_{ij}) \) on each leaf \( \Sigma_\tau \) determine the local Cauchy data (2.2).

**Proof:** The system (3.18)-(3.19) is a coupled elliptic system in the pair \( (u, \sigma) \) on \( \Sigma_\tau \), with boundary values on \( \partial \Sigma_\tau \) given by

\[
u|_{\partial \Sigma_\tau} = \tau, \quad \sigma|_{\partial \Sigma_\tau} = 0.
\]

Consider first the equations (3.18)-(3.19) in the \( y^i \) coordinates. Elliptic regularity, (cf. [19, Ch. 6] for instance), gives bounds for \( u \) and \( \sigma \) in \( C^{2,\alpha} \) provided the following terms are all bounded in \( C^{\alpha}(y^i) \): \( g^y_{ij}, \partial_y g^y_{ij}, |A|^2, D^2 x^i, \partial_x H, \partial_\tau H, \) and the coefficients of \( \sigma^i \) in \( \sigma(H) \). By Lemma 3.2, \( g^y_{ij} \in C^{2,\alpha} \) and \( A \in C^{1,\alpha} \). Also, \( C^{\alpha} \) bounds on the Hessian \( D^2 x^i \) follow from elliptic regularity for the harmonic functions \( x^i \). The bounds on the derivatives of \( H \) follow from (3.4). Thus, all the coefficients of (3.18)-(3.19) are bounded in \( C^{\alpha} \).

Since the metric \( g_{ij} \) is close to the flat metric, (in the \( C^{2,\alpha} \) topology), it is standard that there is then a unique solution to the elliptic boundary value problem (3.18)-(3.19)-(3.20). The solution \( (u, \sigma) \) is uniquely determined by the coefficients \( (g^y_{ij}, A^y_{ij}) \), and the terms or coefficients containing derivatives of \( H \) and derivatives of \( x^i \) in (3.18)-(3.19). We claim that these are also uniquely determined by \( (g^y_{ij}, A^y_{ij}) \).

First, regarding the term \( N(H) \), write \( N = \omega \partial_{y^0} + \xi \partial_{y^i} \), so that \( (\omega, \xi) \) are the lapse-shift of the foliation \( \Sigma_\tau \) with respect to the \( y^0 \)-coordinates. As preceding (3.4), the derivatives \( \partial_{y^0} H \) are fixed, so that \( N(H) \) depends only on \( (\omega, \xi) \), (to zero order). Hence \( N(H) \) can be expressed uniquely in terms of \( (g^y_{ij}, u, \sigma) \), with \( (u, \sigma) \) inserted into the equation (3.18) on the right side. In addition,
the functions $x^k = x^k(y^\ell)$ are uniquely determined by the metric $g_{ij}^y$, and so, for instance $\partial_x^i H$ is uniquely determined by $g_{ij}^y$ via (3.4).

Combining the facts above, it follows that $(u, \sigma)$ is uniquely determined by $(g_{ij}^y, A_{ij}^y)$. The transformation $\{y^\ell\} \to \{x^k\}$ is invertible and $(g_{ij}^y, A_{ij}^y)$ is thus uniquely determined by $(g^x_{ij}, A^x_{ij}) \equiv (g_{ij}, A_{ij})$. Clearly, $(u, \sigma)$ depends smoothly on $(g_{ij}, A_{ij})$. Of course, by (3.5) the data $(g_{ij}, u, \sigma)$ determine the full metric $g = g_{\alpha\beta}$.

Next we claim that $\partial_r g_{\alpha\beta}$ is also determined by $(g_{ij}, A_{ij})$ along $\Sigma_r$. To see this, one has $\frac{1}{2}(L_N g)_{\alpha\beta} = A(\partial_r, \partial_\gamma) = A(uN - \sigma, \partial_\gamma) = -\phi_{k\ell} A_{k\ell}$, and similarly, $\frac{1}{2}(L_N g)_{\alpha0} = \sigma_j A_{ij}$. Hence $(L_N g)_{0\alpha}$ is determined by $(g_{ij}, A_{ij})$. Expanding this Lie derivative using (3.6), a simple computation gives

\begin{equation}
\partial_r g_{0i} - \partial_r (\log u)(g_{0i} + \sigma^k g_{ik}) = \phi_1,
\end{equation}

\begin{equation}
\partial_r (\log |\sigma|) - \partial_r (\log u) = \phi_2,
\end{equation}

where $\phi_1, \phi_2$ are determined by $(g_{ij}, A_{ij})$. This system of equations is uniquely solvable for $\partial_r g_{\alpha\beta}$, showing that $\partial_r g_{\alpha\beta}$ is determined by $(g_{ij}, A_{ij})$. In particular, it follows that $(\gamma, A)$ determines the full Cauchy data $(g_{\alpha\beta}, \partial_r g_{\alpha\beta})$ at $U = \partial \Omega$.

Summarizing the work above, the Einstein equations in local $H$-harmonic coordinates imply the following system on the data $(g_{ij}, u, \sigma)$:

\begin{equation}
(\partial_r^2 + u^2 \Delta + 2\partial_\sigma \partial_r + 2u \partial_\sigma^2)g_{ij} = -2u(D^2 u)_{ij} + Q(g_{\alpha\beta}, \partial r_g_{\alpha\beta}),
\end{equation}

\begin{equation}
\Delta u + |A|^2 u + \lambda u = -(\partial_r + \sigma) H.
\end{equation}

\begin{equation}
\Delta \sigma^i = 2u(D^2 x^i, A) + u \partial_{x^i} H + 2(dx^i, A(\nabla u) - \frac{1}{2} H du).
\end{equation}

**Remark 3.5.** The system (3.22)-(3.24) is essentially an elliptic system in $(g_{ij}, u, \sigma)$, given that $H$ is prescribed. Thus, assuming $u \sim 1$ and $\sigma \sim 0$, the operator $P = \partial_r^2 + u^2 \Delta + 2\partial_\sigma \partial_r + 2u \partial_\sigma^2$ is elliptic on $\Omega$ and acts diagonally on $(g_{ij})$, as is the Laplace operator on the slices $\Sigma_r$ acting on $(u, \sigma)$. The system (3.22)-(3.24) is of course coupled, but the couplings are all of lower order, i.e. 1st order, except for the term $D^2 u$ in (3.22). However, this term can be controlled or estimated by elliptic regularity applied to the lapse equation (3.23), (as discussed further below).

Given the above, it is not difficult to deduce that local $H$-harmonic coordinates have the optimal regularity property, i.e. if $g$ is in $C^{m,\alpha}(\Omega)$ in some local coordinate system, then $g$ is in $C^{m,\alpha}(\Omega)$ in $H$-harmonic coordinates. Since this will not actually be used here, we omit further details of the proof.

**Proof of Theorem 3.1.**

Lemma 3.4 implies that it suffices to prove the unique continuation property for the tangential metric $g_{ij}$ in $H$-harmonic coordinates, since $g_{ij}$ determines the full metric $g_{\alpha\beta} = (g_{ij}, u, \sigma)$.

Thus suppose $g_0$ and $g_1$ are two Einstein metrics on $\Omega$ with identical $(\gamma, A)$ on $\partial \Omega$. One may construct $H$-harmonic coordinates for each of $g_0$ and $g_1$, and via a diffeomorphism identifying these coordinates, assume that the resulting pair of metrics $g$ and $\bar{g}$ have fixed $H$-harmonic coordinates $(\tau, x^i)$, and both metrics satisfy the system (3.22)-(3.24). Let

\begin{equation} h = h_{ij} = \bar{g}_{ij} - g_{ij}.
\end{equation}

One then takes the difference of (3.22) and freezes the coefficients at $g$ to obtain a linear equation in $h$. Thus, for example, $\Delta g_{ij} - \Delta g g_{ij} = \Delta_g (h_{ij}) - (g^{ab} - \bar{g}^{ab}) \partial_a \partial_b g_{ij}$. The second term here is
of zero order, (rational), in the difference \( h \), with coefficients depending on two derivatives of \( \tilde{g} \). Carrying out the same procedure on the remaining terms in (3.22) gives the equation

\[
(\partial^2_r + u^2 \Delta + 2 \partial_u \partial_r + 2 u \partial^2_u)h_{ij} = -2(\tilde{G}(\tilde{D}^2 \tilde{u})_{ij} - u(D^2 u)_{ij}) + Q_{ij}(h_{\alpha \beta}, \partial_\mu h_{\alpha \beta}),
\]

where \( Q \) depends on two derivatives of the background \( \tilde{g} \), but only on one derivative, (in all directions), of \( h_{\alpha \beta} \). Similarly, \( \tilde{D}^2 \tilde{u} - D^2 u = D^2 v + (\tilde{D}^2 - D^2)\tilde{u} \), where \( v = \tilde{u} - u \) and the second term is of the form \( Q \) above. Hence,

\[
(\partial^2_r + u^2 \Delta + 2 \partial_u \partial_r + 2 u \partial^2_u)h_{ij} = -2u(D^2 v)_{ij} + Q_{ij}(h_{\alpha \beta}, \partial_\mu h_{\alpha \beta}).
\]

Note that since we have linearized, \( Q \) depends linearly on \( h_{\alpha \beta} \) and \( \partial_\mu h_{\alpha \beta} \), with nonlinear coefficients depending on \( \tilde{g} \) and \( g \).

Next we use the lapse and shift equations (3.23)-(3.24) to estimate the differences \( v = \tilde{u} - u \) and \( \chi = \tilde{\sigma} - \sigma \). Thus, as before, \( \Delta \tilde{u} - \Delta_g u = \Delta_g v + D^2_{\tilde{h}}(\tilde{u}) \), where \( D^2_{\tilde{h}} \) is a 2nd order differential operator on \( \tilde{u} \) with coefficients depending on the difference \( h \), to zero order. Similarly, \( |\tilde{A}|^2 \tilde{u} - |A|^2 u = |A|^2 v + D^2_{\tilde{h}}(\tilde{u}) \), the latter depending on \( h \) to first order. As in the proof of Lemma 3.4, the term \( N(H) \) depends only on \( (g_{ij}, u, \sigma) \) to zero order. Taking the difference, it then follows from (3.23) and Lemma 3.4 that

\[
\Delta v + |A|^2 v = Q(h_{ij}, \partial_\mu h_{ij}, v, \chi),
\]

with \( Q \) depending on two derivatives of \( \tilde{g} \) and linear in its arguments. Since \( v = 0 \) on \( \partial \Sigma_\tau \), elliptic regularity then gives

\[
||v||_{L^2_{\Sigma_\tau}} \leq C(\tilde{g}, g)[||h_{ij}||_{L^1_{\Sigma_\tau}} + ||\chi||_{L^2_{\Sigma_\tau}}],
\]

where the \( L^2_{\Sigma_\tau} \) norm on the left is with respect to spatial derivatives \( x \), while the \( L^1_{\Sigma_\tau} \) norm on the right includes also the time derivative \( \tau \); both norms are taken along the leaves \( \Sigma_\tau \). Working in the same way with the shift equation (3.24) gives the analogous estimate

\[
||\chi||_{L^2_{\Sigma_\tau}} \leq C(\tilde{g}, g)[||h_{ij}||_{L^1_{\Sigma_\tau}} + ||v||_{L^2_{\Sigma_\tau}}].
\]

It follows from (3.26) and (3.28) that

\[
||P(h_{ij})||_{L^2_{\Sigma_\tau}} \leq C(\tilde{g}, g)||h_{\alpha \beta}||_{L^1_{\Sigma_\tau}},
\]

where \( P \) is as given following (3.24). We claim that the right side of (3.30) is bounded by \( ||h_{ij}||_{L^1_{\Sigma_\tau}} \). To see this, \( h_{\alpha \beta} = (h_{ij}, v, \chi) \), so that it suffices to show that \( v, \partial_\tau v \) and \( \chi, \partial_\tau \chi \) are determined by \( (h_{ij}, \partial_\tau h_{ij}) \), with corresponding bounds on the \( L^2 \) norms. This follows directly from the proof of Lemma 3.4. Thus, \( u \) and \( \sigma \) are uniquely determined by \( (g_{ij}, A_{ij}) \), and hence by \( (g_{ij}, \partial_\tau g_{ij}) \), which implies that \( v \) and \( \chi \) are determined by linear equations in \( (h_{ij}, \partial_\tau h_{ij}) \); this also follows from (3.28)-(3.29). Similarly, since \( \partial_\tau g_{0\alpha} \) is determined by \( (g_{ij}, A_{ij}) \) as in (3.21), it follows that \( \partial_\tau h_{0\alpha} \) is determined by \( (h_{ij}, \partial_\tau h_{ij}) \). Hence, (3.30) gives

\[
||P(h_{ij})||_{L^2_{\Sigma_\tau}} \leq C(\tilde{g}, g)||h_{ij}||_{L^1_{\Sigma_\tau}},
\]

As noted above, the operator \( P \) is elliptic and diagonal, and by construction, the Cauchy data for \( P \) vanish at \( U = \partial \Omega \), i.e.

\[
h = \partial_\tau h = 0 \text{ at } U,
\]

since the lapse-shift \( (u, \sigma) \) of \( g \) and \( \tilde{g} \) and their \( \tau \)-derivatives agree at \( U \). From now on, \( h = h_{ij} \).

Now we claim that \( P \) satisfies the hypotheses of the Calderon unique continuation theorem [14]. Following [14], decompose the symbol of \( P \) as

\[
A_2(\tau, x, \xi) = (u^2 g^{kl} \xi_k \xi_l + 2 u \sigma^k \sigma^l \xi_k \xi_l) I,
\]
A_1(\tau, x, \xi) = 2\sigma^k \xi_k I,
A_1(\tau, x, \xi) = 2\sigma^k \xi_k I.

Now form the matrix
(3.34)
M = \begin{pmatrix} 0 & -I \\ A_2 & A_1 \end{pmatrix}

The matrices A_1 and A_2 are diagonal, and it is then easy to see that M is diagonalizable, i.e. has a basis of eigenvectors over \( \mathbb{C} \). This implies that M satisfies the hypotheses of [14, Thm. 11(iii)], cf. also [14, Thm. 4]. The bound (3.31) is substituted in the basic Carleman estimate of [14, Thm. 6], cf. also [29, (6.1)], showing that \( h_{ij} \) satisfies the unique continuation property.

It follows from (3.32) and the Calderón unique continuation theorem that
\[
h_{ij} = \tilde{g}_{ij} - g_{ij} = 0,
\]
in an open neighborhood \( \Omega' \subset \Omega \). By Lemma 3.4, this implies
\[
\tilde{g}_{\alpha\beta} = g_{\alpha\beta},
\]
in \( \Omega' \), so that \( g_1 \) is isometric to \( g_0 \) in \( \Omega' \). By construction, the isometry from \( g_0 \) to \( g_1 \) equals the identity on \( \partial \Omega \).

This shows that the metric \( g \) is uniquely determined in \( \Omega' \), up to isometry, by the abstract Cauchy data on \( \partial \Omega \). Since Einstein metrics are real-analytic in the interior in harmonic coordinates, a standard analytic continuation argument, (cf. [25] for instance), then implies that \( g \) is unique up to isometry everywhere in \( \Omega \). In the context of Theorem 1.1, the same analytic continuation argument shows that the Cauchy data of \( g \) on \( U \) uniquely determines the topology of \( M \) and \( \partial M \), up to covering spaces, as well as the Cauchy data \((\gamma, A)\) on \( \partial M \) outside \( U = \partial \Omega \). This completes the proof of Theorems 3.1 and 1.1.

As an illustration, suppose \((M_1, g_1)\) and \((M_2, g_2)\) are a pair of Einstein metrics on compact manifolds-with-boundary and the Cauchy data for \( g_1 \) and \( g_2 \) agree on an open set \( U \) of the boundary. Suppose \( M_i \) are connected and the inclusion map of \( \partial M_i \) into \( M_i \) induces a surjection of fundamental groups, i.e.
(3.35)
\[
\pi_1(\partial M_i) \to \pi_1(M_i) \to 0,
\]
for \( i = 1, 2 \), so that every loop in \( M_i \) is homotopic to a loop in \( \partial M_i \). Then \( M_1 \) is diffeomorphic to \( M_2 \) and \( g_1 \) is isometric to \( g_2 \).

We conclude this section with a discussion of generalizations of Theorem 1.1. First, one might consider the unique continuation problem for
(3.36)
\[
Ric_g = T,
\]
where \( T \) is a fixed symmetric bilinear form on \( M \), at least \( C^{\alpha} \) up to \( \bar{M} \). However, this problem is not natural, in that is not covariant under changes by diffeomorphism. For metrics alone, the Einstein equation (1.2) is the only equation covariant under diffeomorphisms which involves at most the 2nd derivatives of the metric. Nevertheless, the proof of Theorem 1.1 shows that if \( \tilde{g} \) and \( g \) are two solutions of (3.36) which have common \( H \)-harmonic coordinates near (a portion of) \( \partial M \) on which \((\gamma, A) = (\tilde{\gamma}, \tilde{A})\), then \( \tilde{g} \) is isometric to \( g \) near \( \partial M \).

Instead, it is more natural to consider the Einstein equation coupled (covariantly) to other fields \( \chi \) besides the metric; such equations arise naturally in many areas of physics. For example, \( \chi \)}
may be a function on \( M \), i.e. a scalar field, or \( \chi \) may be a connection 1-form (gauge field) on a bundle over \( M \). We assume that the field(s) \( \chi \) arise via a diffeomorphism-invariant Lagrangian \( \mathcal{L} = \mathcal{L}(g, \chi) \), depending on \( \chi \) and its first derivatives in local coordinates, and that \( \chi \) satisfies field equations, i.e. Euler-Lagrange equations, coupled to the metric. For example, for a free massive scalar field, the equation is the eigenfunction equation

\[
\Delta_g \chi = \mu \chi,
\]

while for a connection 1-form, the equations are the Yang-Mills equations, (or Maxwell equations when the bundle is a \( U(1) \) bundle):

\[
dF = d^* F = 0,
\]

where \( F \) is the curvature of the connection \( \chi \). Associated to such fields is the stress-energy tensor \( T = T_{\mu\nu} \); this is a symmetric bilinear form obtained by varying the Lagrangian for \( \chi \) with respect to the metric, cf. [22] for example. For the free massive scalar field \( \chi \) above, one has

\[
T = d\chi \cdot d\chi - \frac{1}{2}(|d\chi|^2 + \mu \chi^2)g,
\]

while for a connection 1-form

\[
T = F \cdot F - \frac{1}{4}|F|^2 g,
\]

where \( (F \cdot F)_{ab} = F_{ac} F_{bd} g^{cd} \).

When the part of the Lagrangian involving the metric to 2nd order only contains the scalar curvature, i.e. the Einstein-Hilbert action, the resulting coupled Euler-Lagrange equations for the system \((g, \chi)\) are

\[
Ric_g - \frac{R}{2} g = T, \quad E_g(\chi) = 0.
\]

By taking the trace, this can be rewritten as

\[
Ric_g = \hat{T} = T - \frac{1}{n-1} tr_g T, \quad E_g(\chi) = 0.
\]

Here we assume \( E_g(\chi) \) is a 2nd order elliptic system for \( \chi \), with coefficients depending on \( g \), as in (3.37) or (3.38), (the latter viewed as an equation for the connection). In case the field(s) \( \chi \) have an internal symmetry group, as in the case of gauge fields, this will require a particular choice of gauge for \( \chi \) in which the Euler-Lagrange equations become an elliptic system in \( \chi \). It is also assumed that solutions \( \chi \) of \( E_g(\chi) = 0 \) satisfy the unique continuation property; for instance \( E_g \) satisfies the hypotheses of the Calderón theorem [14]. Theorem 1.1 now easily extends to cover (3.39) or (3.40).

**Proposition 3.6.** Let \( M \) be a compact manifold with boundary \( \partial M \). Then \( C^{2,\alpha} \) solutions \((g, \chi)\) of (3.39) on \( M \) are uniquely determined, up to local isometry, by the Cauchy data \((\gamma, A)\) of \( g \) and the Cauchy data \((\chi, \partial_t \chi)\) on an open set \( U \subset \partial M \).

**Proof:** The proof is the same as the proof of Theorem 1.1. Briefly, via a suitable diffeomorphism equal to the identity on \( \partial M \), one brings a pair of solutions of (3.39) with common Cauchy data into a fixed system of \( H \)-harmonic coordinates for each metric. As before, one then applies Calderón uniqueness to the resulting system (3.39) in the difference of the metrics and fields. Further details are left to the reader. \( \blacksquare \)
Let \( g \) be a conformally compact metric on a compact \((n+1)\)-manifold \( M \) with boundary which has a \( C^2 \) geodesic compactification

\[
(4.1) \quad \bar{g} = t^2 g,
\]
where \( t(x) = \text{dist}_g(x, \partial M) \). By the Gauss Lemma, one has the splitting

\[
(4.2) \quad \bar{g} = dt^2 + g_t,
\]

near \( \partial M \), where \( g_t \) is a curve of metrics on \( \partial M \) with \( g_0 = \gamma \) the boundary metric. The curve \( g_t \) is obtained by taking the induced metric the level sets \( S(t) \) of \( t \), and pulling back by the flow of \( N = \nabla t \). Note that if \( r = -\log t \), then \( g = dr^2 + t^{-2} g_t \), so the integral curves of \( \nabla r \) with respect to \( g \) are also geodesics. Each choice of boundary metric \( \gamma \in [\gamma] \) determines a unique geodesic defining function \( t \).

Now suppose \( g \) is Einstein, so that (1.4) holds and suppose for the moment that \( g \) is \( C^2 \) conformally compact with \( C^\infty \) smooth boundary metric \( \gamma \). Then the boundary regularity result of [16] implies that \( \bar{g} \) is \( C^\infty \) smooth when \( n \) is odd, and is \( C^\infty \) polyhomogeneous when \( n \) is even. Hence, the curve \( g_t \) has a Taylor-type series in \( t \), called the Fefferman-Graham expansion [18]. The exact form of the expansion depends on whether \( n \) is odd or even. If \( n \) is odd, one has a power series expansion

\[
(4.3) \quad g_t \sim g_{(0)} + t^2 g_{(2)} + \cdots + t^{n-1} g_{(n-1)} + t^n g_{(n)} + \cdots,
\]
while if \( n \) is even, the series is polyhomogeneous,

\[
(4.4) \quad g_t \sim g_{(0)} + t^2 g_{(2)} + \cdots + t^n g_{(n)} + t^n \log t \mathcal{H} + \cdots.
\]

In both cases, this expansion is even in powers of \( t \), up to \( t^n \). It is important to observe that the coefficients \( g_{(2k)} \), \( k \leq [n/2] \), as well as the coefficient \( \mathcal{H} \) when \( n \) is even, are explicitly determined by the boundary metric \( \gamma = g_{(0)} \) and the Einstein condition (1.4), cf. [18], [20]. For \( n \) even, the series (4.4) has terms of the form \( t^{n+k}(\log t)^m \).

For any \( n \), the divergence and trace (with respect to \( g_{(0)} = \gamma \)) of \( g_{(n)} \) are determined by the boundary metric \( \gamma \); in fact there is a symmetric bilinear form \( r_{(n)} \) and scalar function \( a_{(n)} \), both depending only on \( \gamma \) and its derivatives up to order \( n \), such that

\[
(4.5) \quad \delta(\gamma)(g_{(n)} + r_{(n)}) = 0, \quad \text{and} \quad tr(\gamma)(g_{(n)} + r_{(n)}) = a_{(n)}.
\]

For \( n \) odd, \( r_{(n)} = a_{(n)} = 0 \). (The divergence-free tensor \( g_{(n)} + r_{(n)} \) is closely related to the stress-energy of a conformal field theory on \((\partial M, \gamma)\), cf. [17]). The relations (4.5) will be discussed further in §5.

However, beyond the relations (4.5), the term \( g_{(n)} \) is not determined by \( g_{(0)} \); it depends on the “global” structure of the metric \( g \). The higher order coefficients \( g_{(k)} \) of \( t^k \) and coefficients \( h_{(km)} \) of \( t^{n+k}(\log t)^m \), are then determined by \( g_{(0)} \) and \( g_{(n)} \) via the Einstein equations. The equations (4.5) are constraint equations, and arise from the Gauss-Codazzi and Gauss and Riccati equations on the level sets \( S(t) = \{ x : t(x) = t \} \) in the limit \( t \to 0 \); this is also discussed further in §5.

In analogy to the situation in §3, the term \( g_{(n)} \) corresponds to the 2nd fundamental form \( A \) of the boundary, in that, modulo the constraints (4.5), it is freely specifiable as Cauchy data, and is the only such term depending on normal derivatives of the boundary metric.

Suppose now \( g_0 \) and \( g_1 \) are two solutions of

\[
(4.6) \quad \text{Ric}_g + ng = 0,
\]
with the same $C^\infty$ conformal infinity $[\gamma]$. Then there exist geodesic defining functions $t_k$ such that $\bar{g}_k = (t_k)^2g_k$ have a common boundary metric $\gamma \in [\gamma]$, and both metrics are defined for $t_k \leq \varepsilon$, for some $\varepsilon > 0$.

The hypotheses of Theorem 1.2, together with the discussion above concerning (4.3) and (4.4), then imply that

$$
|g_1 - g_0| = o(e^{-\nu r}) = o(t^n),
$$

where the norm is taken with respect to $g_1$, (or $g_0$).

Given this background, we prove the following more general version of Theorem 1.2, analogous to Theorem 3.1. Let $\Omega$ be a domain diffeomorphic to $I \times B^n$, where $B^n$ is a ball in $\mathbb{R}^n$ with boundary $U = \partial \Omega$ diffeomorphic to a ball in $\mathbb{R}^n \simeq \{0\} \times \mathbb{R}^n$.

**Theorem 4.1.** Let $g_0$ and $g_1$ be a pair of conformally compact Einstein metrics on a domain $\Omega$ as above. Suppose $g_0$ and $g_1$ have $C^{2,\alpha}$ geodesic compactifications, and (4.7) holds in $\Omega$.

Then $(\Omega, g_0)$ is isometric to $(\Omega, g_1)$, by an isometry equal to the identity on $\partial \Omega$. Hence, if $(M_0, g_0)$ and $(M_1, g_1)$ are conformally compact Einstein metrics on compact manifolds with boundary, and (4.7) holds on some open domain $\Omega$ in $M_0$ and $M_1$, then the manifolds $M_0$ and $M_1$ are diffeomorphic in some covering space of each and the lifted metrics $\bar{g}_0$ and $\bar{g}_1$ are isometric.

The proof of Theorem 4.1 is very similar to that of Theorem 3.1. For clarity, we first prove the result in case the metrics $g_i$, $i = 0, 1$, have a common $C^\infty$ boundary metric $\gamma$ and then show how the proof can be extended to cover the more general case of metrics with less regularity.

By applying a diffeomorphism if necessary, one may assume that the metrics $g_i$ have a common geodesic defining function $t$ defined near $\partial \Omega$ and common geodesic boundary coordinates. By [16], the geodesically compactified metrics $\bar{g}_i = t^2g_i$ are $C^\infty$ and extend $C^\infty$ to $\partial \Omega$. It follows from the discussion of the Fefferman-Graham expansion following (4.5) that $g_0$ and $g_1$ agree to infinite order at $\partial U$, i.e.

$$
k = g_1 - g_0 = O(t^\nu),
$$

for any $\nu < \infty$. Of course $k_{0\alpha} = 0$.

For the rest of the proof, we work in the setting of the compactified metrics $\bar{g}_i$. As in the proof of Theorem 3.1, we assume that the domain $\Omega$ is sufficiently small so that $(\Omega, \bar{g}_i)$ is smoothly close to the flat metric on the standard cylinder $I \times B^n$, with $A = 0$ on $U = \partial \Omega$. (Note that $g_{(1)} = 0$ in (4.3)-(4.4)). In particular, near $\partial \Omega$, $H = O(t)$. One may construct a foliation $\Sigma_r$ with $H_{\Sigma_r} = 0$ exactly as in Lemma 3.2, together with corresponding $H$-harmonic coordinates $(\tau, x^i)$. All of the analysis carried out in §3 through to Lemma 3.4 carries over to this situation with only a single difference. Namely, for the term $Ric_{\bar{g}}$ in (3.14) or (3.15), one now no longer has $Ric_{\bar{g}} = \lambda g$, but instead the Ricci curvature $Ric$ of the compactified metric $\bar{g}$. Using the facts that $Ric_{\bar{g}} = -ng$ and the compactification $\bar{g}$ is geodesic, standard formulas for the behavior of Ricci curvature under conformal change give

$$
\bar{Ric} = -(n - 1)t^{-1}D^2t - t^{-1}\Delta t\bar{g}.
$$

One has $D^2t = L_{\nabla t}\bar{g} = O(t)$, (recall that $\bar{g}$ is $C^\infty$ up to $\partial \Omega$). If $(t, \bar{y}^i)$ are geodesic boundary coordinates, then $\partial_{\bar{y}^i} = \sum(1 - \varepsilon(\tau))\partial_{\bar{y}^i} + \varepsilon(\tau)\nabla t$, where $\varepsilon(\tau) = O(\tau)$. Similarly, $\tau/t = 1 + \varepsilon(\tau)$. (The specific form of $\varepsilon(\tau)$ of course differs in each occurrence above, but this is insignificant). Since $D^2t$ vanishes on $\nabla t$, it follows from (4.9) that in the $x^a$ coordinates on $\Sigma_r$,(4.10)

$$
\bar{Ric}_{ij} = -(n - 1)(1 - \varepsilon)^2t^{-1}(L_{\nabla t}\bar{g})_{ij} - (1 - \varepsilon)^2t^{-1}(\Delta t)g_{ij} + \varepsilon t^{-1}(\Delta t)q_{ij},
$$

where $q_{ij}$ depends only on $\bar{g}_{0\alpha}$ to zero-order. Next $(L_{\nabla t}\bar{g}) = (1 - \varepsilon)\partial_{\tau}\bar{g} + \varepsilon(\tau)\partial_{x^\alpha}\bar{g}$ and similarly for the Laplace term in (4.10). Substituting (4.10) in (3.15), it follows that the analogue of (3.16)
in this context is the 'evolution equation'
\begin{equation}
\tau^2(\partial^2_t + u^2 \Delta + 2 \partial_t \partial_r + 2 \partial_r^2)g_{ij} = -2\tau^2 u(D^2 u)_{ij} + Q_{ij}(g, \tau \partial g),
\end{equation}
where \( Q_{ij} \) is a term involving \( g_{\alpha\beta} \) with at most first order derivatives of the form \( \tau \partial_u \). Here and below, we drop the bar from the notation.

The lapse \( u \) and shift \( \sigma \) satisfy essentially the same equations as before, namely
\begin{equation}
\Delta u + |A|^2 u - (t^{-1} \Delta t) u = 0,
\end{equation}
\begin{equation}
\Delta \sigma^i = 2 u(D^2 x^i, A) + +2(dx^i, A(\nabla u))
\end{equation}
Comparing with (3.18)-(3.19), one has here \( H = 0 \), with the \( \lambda \) term in (3.17) replaced by \( -t^{-1} \Delta t \).

Lemma 3.4 holds as before, since (4.8), together with elliptic regularity applied to (4.12)-(4.13), as in the proof of Lemma 3.4, one has
\begin{equation}
h_{\alpha\beta} = O(t^\nu),
\end{equation}
for all \( \nu < \infty \). The estimates (3.28)-(3.31) hold as before.

Let \( P(h_{ij}) = \tau^2(\partial^2_t + u^2 \Delta + 2 \partial_t \partial_r + 2 \partial_r^2) \). Then \( P \) is a fully degenerate 2nd order elliptic operator, with smooth coefficients, and one has
\[
||P(h_{ij})||_{L^2_\nu} \leq C||h_{ij}||_{L^1_{\nu+2}},
\]
where the 1st order derivatives on the right are of the form \( \tau \partial \). Further, by (4.14), \( h \) vanishes to infinite order at \( \partial \Omega \). It then follows from a unique continuation theorem of Mazzeo, [27, Thm. 14], that
\[
h_{ij} = 0
\]
in \( \Omega' \subset \Omega \). The vanishing of \( h = h_{\alpha\beta} \) in \( \Omega \) then follows as before in the proof of Theorem 3.1.

Next suppose \( g_0 \) and \( g_1 \) have only a \( C^{2,\alpha} \) geodesic compactification with a common boundary metric \( \gamma \), but that (4.7) holds. All of the arguments above remain valid, except the infinite order vanishing property (4.8), and the corresponding (4.14), which are replaced by the statements \( k = o(t^\nu) \) and \( h = o(t^\nu) \) respectively. The unique continuation result in [27] per se, requires the infinite order decay (4.14). Thus, it suffices to show that (4.14) does in fact hold.

To do this, we first show that \( k = O(t^\nu) \) weakly, for all \( \nu < \infty \). This will imply \( h = O(t^\nu) \) weakly, and the strong or pointwise decay (4.14) then follows from elliptic regularity.

In geodesic boundary coordinates, the geodesic compactification of a conformally compact Einstein metric satisfies the equation
\begin{equation}
t \tilde{g} - (n - 1) \tilde{g} - 2H \tilde{g}^T - 2t Ric_{S(t)} + tH \tilde{g} - t(\tilde{g})^2 = 0,
\end{equation}
where \( \tilde{g} \) is the Lie derivative of \( g \) with respect to \( \nabla t \), cf. [18] or [21]. Thus \( \tilde{g} = 2A \), where \( A \) is the 2nd fundamental form of the level set \( S(t) \) of \( t \), (with respect to the inward normal). Also \( H = tr A \), \( T \) denotes restriction or projection onto \( S(t) \) and \( Ric_{S(t)} \) is the intrinsic Ricci curvature of \( S(t) \). (The equation (4.15) may be derived from (3.13) by setting \( u = 1 \) and \( \sigma = 0 \). We recall, as above, that the bar has been removed from the notation.

As above, the metrics \( g_0 \) and \( g_1 \) are assumed to have a fixed geodesic defining function \( t \) with common boundary metric \( \gamma \) and common geodesic boundary coordinates. Taking the difference of the equation (4.15) evaluated on \( g_1 \) and \( g_0 \) gives the following equation for \( k = g_1 - g_0 \) as in (4.8):
\begin{equation}
t \tilde{k} - (n - 1) \tilde{k} = tr(\tilde{k} \tilde{g}_0^T) + 2t(Ric^1_{S(t)} - Ric^0_{S(t)}) + O(t)k + O(t^2) \tilde{k},
\end{equation}
where \(O(t^k)\) denotes terms of order \(t^k\) with coefficients depending smoothly on \(g_0\). One has \(\text{Ric}_S(t) = D^2_x(g_{ij})\) is a 2nd order operator on \(g_{ij}\), so that (4.16) gives

\[
(4.17) \quad \dot{t}k - (n - 1)\dot{k} = tr(k)g^T_0 + 2tD^2_x(k) + O(t)k + O(t^2)\dot{k},
\]

The (positive) indicial root of the trace-free part of (4.16) or (4.17) is \(n\), in that the formal power series solution of (4.17) has undetermined coefficient at order \(t^n\), as in the Fefferman-Graham expansion (4.3)-(4.4). The hypothesis (4.7) implies that

\[
(4.19) \quad \dot{H} + |A|^2 = -\text{Ric}(T,T).
\]

Via (4.9), this is easily seen to be equivalent to

\[
\dot{t}\dot{H} - H = -t|A|^2.
\]

This holds for each compactified metric \(g_1\) and \(g_0\), and so taking the difference, and computing as in (4.16)-(4.17) gives the equation

\[
(4.20) \quad \frac{d^2}{dt^2}(tk) - \frac{d}{dt}(tk) = O(t)k + O(t^2)\dot{k}.
\]

The positive indicial root of (4.20) is 2, and by (4.7), the \(O(t^2)\) component of the formal expansion of \(tk\) vanishes. Similarly, the trace-free part \(k_0\) of \(k\) satisfies the equation

\[
(4.21) \quad \dot{t}k_0 - (n - 1)\dot{k}_0 = 2t(D^2_x(k))_0 + [O(t)k]_0 + [O(t^2)\dot{k}]_0,
\]

with indicial root \(n\). As in [18], by repeated differentiation of (4.20) and (4.21) it follows from (4.7) that the formal expansion of \(k\) vanishes.

Next we show that (4.8) holds weakly.

**Lemma 4.2.** Suppose \(k = o(t^n)\) weakly, in that, with respect to the compactified metric \((S(t), g), (\phi = g_0)\),

\[
(4.22) \quad \int_{S(t)} \langle k, \phi \rangle = o(t^n), \text{ as } t \to 0,
\]

where \(\phi\) is any symmetric bilinear form, \(C^\infty\) smooth up to \(U = \partial\Omega\) and vanishing to infinite order on \(\partial'\Omega = \Omega \setminus U\). Then

\[
(4.23) \quad k = o(t^\nu), \text{ weakly},
\]

for any \(\nu < \infty\), i.e. (4.22) holds, with \(\nu\) in place of \(n\).

**Proof:** Here smoothness is measured with respect to the given geodesic coordinates \((t, x^i)\) covering \(\Omega\). The proof proceeds by induction, starting at the initial level \(n\). As above, the trace-free and pure trace cases are treated separately, and so we assume in the following first that \(\phi\) is trace-free. Pair \(k\) with \(\phi\) and integrate (4.17) over the level sets \(S(t)\) to obtain

\[
(4.24) \quad t \int_{S(t)} \langle \dot{k}, \phi \rangle - (n - 1) \int_{S(t)} \langle \dot{k}, \phi \rangle = t \int_{S(t)} \langle k, P_2(\phi) \rangle + \int_{S(t)} \langle O(t)k, \phi \rangle + \int_{S(t)} \langle O(t^2)\dot{k}, \phi \rangle.
\]

Here \(P_2(\phi)\) is obtained by integrating the \(D^2\) term on the right in (4.17) by parts over \(S(t)\). Thus \(P_2(\phi)\), and more generally, \(P_k(\phi)\) denote differential operators of order \(k\) on \(\phi\) with coefficients
depending on \( g \) and \( g_1 \) and their derivatives up to order 2 and so at least continuous up to \( \partial \Omega \). We use these expressions generically, so their exact form may change from line-to-line below. Note also there are no boundary terms at \( \partial S(t) \) arising from the integration by parts, by the vanishing hypothesis on \( \partial' \Omega \).

For the terms on the right in (4.24) one then has
\[
\int_{S(t)} \langle O(t)k, \phi \rangle = t \int_{S(t)} \langle k, P_0(\phi) \rangle,
\]
while, since \( A = O(t) \) and \( H = O(t) \),
\[
\int_{S(t)} \langle O(t^2) \delta, \phi \rangle = t^2 \int_{S(t)} \langle \delta, P_0(\phi) \rangle = t^2 \frac{d}{dt} \int_{S(t)} \langle k, P_0(\phi) \rangle - t^2 \int_{S(t)} \langle k, P_1(\phi) \rangle.
\]
Similarly, for the terms on the left in (4.24), one has
\[
\int_{S(t)} \langle \delta, \phi \rangle = \frac{d}{dt} \int_{S(t)} \langle k, \phi \rangle - t \int_{S(t)} \langle k, P_1(\phi) \rangle,
\]
while
\[
\int_{S(t)} \langle \delta, \phi \rangle = \frac{d^2}{dt^2} \int_{S(t)} \langle k, \phi \rangle - 2t \frac{d}{dt} \int_{S(t)} \langle k, P_1(\phi) \rangle + \int_{S(t)} \langle k, P_1(\phi) \rangle + t \int_{S(t)} \langle k, P_2(\phi) \rangle.
\]

Now let
\[
f = f(t) = \int_{S(t)} \langle k, \phi \rangle.
\]
Then the computations above give
\[
t \ddot{f} - (n - 1) \dot{f} = t \int_{S(t)} \langle k, P_2(\phi) \rangle + (1 + t^2) \int_{S(t)} \langle k, P_1(\phi) \rangle
\]
\[
+ \frac{d}{dt} \int_{S(t)} t^2 \langle k, P_0(\phi) \rangle + \frac{d}{dt} \int_{S(t)} t \langle k, P_1(\phi) \rangle.
\]

First observe that
\[
\int_{S(t)} \langle k, \phi \rangle = o(t^n) \Rightarrow \int_{S(t)} \langle k, P_k(\phi) \rangle = o(t^n),
\]
for all \( C^\infty \) forms \( \phi \) vanishing to infinite order at \( \partial' \Omega \). For if the left side of (4.26) holds, then
\[
\int_{S(t)} \langle k, \delta^k \phi \rangle = o(t^n),
\]
since the hypotheses on \( \phi \) are closed under differentiation. The coefficients of \( P_k \) are at least continuous, and it is elementary to verify that if \( \int_{S(t)} \langle k, \delta^k \phi \rangle = o(t^n) \), then
\[
\int_{S(t)} \langle k, \phi \delta^k \phi \rangle = o(t^n),
\]
for any function \( \phi \) continuous on \( \overline{\Omega} \). Note that the same result holds with \( p \) in place of \( n \), for any \( p < \infty \).

It follows from (4.26) and the initial hypothesis (4.22) that the first two terms on the right in (4.25) are \( o(t^n) \) as \( t \to 0 \). Since
\[
t \ddot{f} - (n - 1) \dot{f} = t^n \frac{d}{dt} \left( \frac{\dot{f}}{t^{n-1}} \right),
\]
this gives
\[
\frac{d}{dt} \left( \frac{\dot{f}}{t^{n-1}} \right) = o(1) + t^{-n} \frac{d}{dt} \int_{S(t)} t \langle k, P_1(\phi) \rangle + t^{-n} \frac{d}{dt} \int_{S(t)} t^2 \langle k, P_0(\phi) \rangle.
\]

Integrating from 0 to \( t \) implies
\[
\frac{\dot{f}}{t^{n-1}} = o(t) + t^{-n+1} \int_{S(t)} \langle k, P_1(\phi) \rangle + n \int_{0}^{t} t^{-n} \int_{S(t)} \langle k, P_1(\phi) \rangle + c_1 = o(t) + c_1,
\]
where \( c_1 \) is a constant. A further integration using (4.26) again gives
\[
(4.27) \quad f = o(t^{n+1}) + c_1 t^n + c_2,
\]
where \( c'_1 = \frac{c_1}{n} \). Once more by (4.22), this implies that
\[
f = o(t^{n+1}).
\]
Note the special role played by the indicial root \( n \) here; if instead one had only \( k = O(t^n) \), then the argument above does not give \( k = O(t^{n+1}) \) weakly.

This first estimate holds in fact for any given trace-free \( \phi \) which is \( C^2 \) on \( \bar{\Omega} \), and vanishing to first order on \( \partial' \Omega \). Working in the same way with the trace equation (4.20) shows that the same result holds for pure trace terms. In particular, it follows that
\[
(4.28) \quad k = o(t^{n+1}) \text{ weakly.}
\]

One now just repeats this argument inductively, with the improved estimate (4.28) in place of (4.22), using (4.26) inductively. Note that each inductive step requires higher differentiability of the test function \( \phi \) and its higher order vanishing at \( \partial' \Omega \).

Lemma 4.2 proves that \( k = k_{\alpha \beta} = O(t^\nu) \) weakly, for any \( \nu < \infty \). As discussed in §3, the transition from geodesic boundary coordinates to \( H \)-harmonic coordinates is \( C^{2,\alpha} \) and hence
\[
(4.29) \quad h = h_{\alpha \beta} = O(t^\nu),
\]
weakly, with the level sets \( S(t) \) replaced by \( \Sigma_\tau \). Next, as in Remark 3.5 and the proof of Theorem 3.1, the equations (4.11)-(4.13) satisfy elliptic estimates, and elliptic regularity in weighted Hölder spaces, cf. [26], [20], shows that the weak decay (4.29) implies strong or pointwise decay, i.e. (4.14) holds. The proof of Theorem 4.1 and thus Theorem 1.2 is now completed as before in the \( C^\infty \) smooth case.

Remark 4.3. In [3, Thm. 3.2], a proof of unique continuation of conformally compact Einstein metrics was given in dimension 4, using the fact that the compactified metric \( \tilde{g} \) in (1.1) satisfies the Bach equation, together with the Calderón uniqueness theorem. However, the proof in [3] used harmonic coordinates; as discussed in §2, such coordinates do not preserve the Cauchy data. I am grateful to Robin Graham for pointing this out. Theorem 1.2 thus corrects this error, and generalizes the result to any dimension.

For the work to follow in §5, we note that Theorem 4.1 also holds for linearizations of the Einstein equations, i.e. forms \( k \) satisfying
\[
(4.30) \quad \frac{d}{dt}(\text{Ric}_{g+tk} + n(g + tk))|_{t=0} = 0.
\]
Thus, if \( k \) satisfies (4.30) and the analog of (4.7), i.e. \( |k| = o(t^n) \), then \( k \) is pure gauge in \( \Omega \), in that \( k = \delta^* Z \), where \( Z \) is a vector field on \( \Omega \) with \( Z = 0 \) on \( \partial \Omega \). The proof of this is exactly the same as the proof of Theorem 4.1, replacing the finite difference \( k = g_1 - g_0 \) by an infinitesimal difference.

This has the following consequence:

Corollary 4.4. Let \((M, g)\) be a conformally compact Einstein manifold with metric \( g \) having a \( C^{2,\alpha} \) geodesic compactification. Suppose the topological condition (1.5) holds, i.e. \( \pi_1(M, \partial M) = 0 \).

If \( k \) is an infinitesimal Einstein deformation on \( M \) as in (4.30), in divergence-free gauge, i.e.
\[
(4.31) \quad \delta k = 0,
\]
with \( k = o(t^n) \) on approach to \( \partial M \), then
\[
k = 0 \quad \text{on} \quad M.
\]
Proof: The topological condition (1.5), together with the same analytic continuation argument at the end of the proof of Theorem 3.1, (cf. also (3.35)), implies that $k$ is pure gauge globally on $M$, in that $k = \delta^* Z$ on $M$ with $Z = 0$ on $\partial M$. (Recall that (1.5) implies that $\partial M$ is connected). From (4.31), one then has

$$\delta \delta^* Z = 0,$$

on $M$. Pairing this with $Z$ and integrating over $B(t)$, it follows that

$$\int_{B(t)} |\delta^* Z|^2 = \int_{S(t)} \delta^* Z(Z,N),$$

where $N$ is the unit outward normal. Since $|Z|_g$ is bounded and $|\delta^* Z|\text{vol}(S(t)) = o(1)$, (since $|k| = o(t^n)$), it follows that

$$\int_M |\delta^* Z|^2 = 0,$$

which gives the result.

Of course, analogs of these results also hold for bounded domains, via the proof of Theorem 3.1; the verification is left to the reader.

Remark 4.5. The analogue of Proposition 3.6 most likely also holds in the setting of conformally compact metrics, for fields $\tau$ whose Euler-Lagrange equation is a diagonal system of Laplace-type operators to leading order, as in (3.37) or (3.38). The proof of this is basically the same as that of Proposition 3.6, using the proof of Theorem 1.2 and with the Mazzeo unique continuation result in place of that of Calderón. However, we will not carry out the details of the proof here.

5. ISOMETRY EXTENSION AND THE CONSTRAINT EQUATIONS.

In this section, we prove Theorem 1.3 that continuous groups of isometries at the boundary extend to isometries in the interior of complete conformally compact Einstein metrics and relate this issue in general to the constraint equations induced by the Gauss-Codazzi equations.

We begin with the following elementary consequence of Theorem 4.1.

Proposition 5.1. Let $(\Omega, g)$ be a $C^n$ polyhomogeneous conformally compact Einstein metric on a domain $\Omega \simeq B^{n+1}$ with boundary metric $\gamma$ on $\partial \Omega \simeq B^n$. Suppose $X$ is a Killing field on $(\partial \Omega, \gamma)$ and

$$(5.1) \quad L_X g_{(n)} = 0,$$

where $g_{(n)}$ is the $n^{th}$ term in the Fefferman-Graham expansion (4.3) or (4.4).

Then $X$ extends to a Killing field on $(\Omega, g)$.

Proof: Extend $X$ to a smooth vector field on $\Omega$ by requiring $[X, N] = 0$, where $N = \nabla \log t$ and $t$ is the geodesic defining function determined by $g$ and $\gamma$. Let $\phi_s$ be the corresponding 1-parameter group of diffeomorphisms and set $g_s = \phi_s^* g$. Then $t$ is the geodesic defining function for $g_s$ for any $s$, and the pair $(g, g_s)$ satisfy the hypotheses of Theorem 4.1. Theorem 4.1 then implies that $g_s$ is isometric to $g$, i.e. there exist diffeomorphisms $\psi_s$ of $\Omega$, equal to the identity on $\partial \Omega$, such that $\psi_s^* \phi_s^* g = g$. Thus $\phi_s \circ \psi_s$ is a 1-parameter group of isometries of $g$ defined in $\Omega$, with $Y$ the corresponding Killing field. (In fact, $Y = X$, since any Killing field $Y$ tangent to $\partial \Omega$ preserves the geodesics tangent to $N$, and so $[Y, N] = 0$. This determines $Y$ uniquely in terms of its value at $\partial \Omega$. Since $X$ satisfies the same equation with the same initial value, this gives the claim).
We point out that the same result, and proof, also hold in the case of Einstein metrics on bounded domains, \( \text{via} \) Theorem 3.1. The condition (5.1) is of course replaced by \( \mathcal{L}_X A = 0 \). For some examples and discussion in the bounded domain case, see \([1], [2]\).

Suppose now that \((M, g)\) is a (global) conformally compact Einstein metric and there is a domain \( \Omega \) as in Proposition 5.1 contained in \( M \) on which (5.1) holds. Then by analytic continuation as discussed at the end of the proof of Theorem 3.1, \( X \) extends to a local Killing field on all of \( M \), i.e. \( X \) extends to a Killing field on the universal cover \( \tilde{M} \). In particular, if the condition (3.35) holds, i.e.

\[
\pi_1(\partial \Omega) \to \pi_1(\Omega) \to 0,
\]

then \( X \) extends to a global Killing field on \( M \). Again, the same result holds in the context of bounded domains.

**Remark 5.2.** A natural analogue of Proposition 5.1 holds for conformal Killing fields on \((\partial \Omega, \gamma)\), i.e. vector fields which preserve the conformal class \([\gamma]\) at conformal infinity. Such vector fields satisfy the conformal Killing equation

\[
\mathcal{L}_X \gamma = \mathcal{L}_\gamma \gamma - \frac{\text{tr}(\mathcal{L}_X \gamma)}{n} \gamma = 0.
\]

Namely, since we are working locally, it is well-known - and easy to prove - that any non-vanishing conformal Killing field is Killing with respect to a conformally related metric \( \tilde{\gamma} = \lambda^2 \gamma \), so that

\[
\mathcal{L}_X \tilde{\gamma} = 0.
\]

Hence, if \( \mathcal{L}_X \tilde{g}_{(n)} = 0 \), then Proposition 5.1 implies that \( X \) extends to a Killing field on \( \Omega \).

One may express \( \tilde{g}_{(n)} \) in terms of \( \lambda \) and the lower order terms \( g_{(k)}, k < n \) in the Fefferman-Graham expansion (4.3)-(4.4); however, the expressions become very complicated for \( n \) even and large, cf. \([17]\). Thus, while the equation (5.2) is conformally invariant, the corresponding conformally invariant equation for \( g_{(n)} \) will be complicated in general.

Next we consider the constraint equations (4.5) in detail, i.e.

\[
\delta \tau_{(n)} = 0 \quad \text{and} \quad \text{tr} \, \tau_{(n)} = a_{(n)},
\]

where \( \tau_{(n)} = g_{(n)} + r_{(n)} \); \( r_{(n)} \) and \( a_{(n)} \) are explicitly determined by the boundary metric \( \gamma = g_{(0)} \) and its derivatives up to order \( n \). Both vanish when \( n \) is odd.

As will be seen below, the most important issue is the divergence constraint in (5.3), which arises from the Gauss-Codazzi equations. To see this, in the setting of §4, on \( S(t) \subset (M, g) \), the Gauss-Codazzi equations are

\[
\delta (A - Hg) = -\text{Ric}(N, \cdot),
\]

as 1-forms on \( S(t) \); here \( N = -t \partial_t \) is the unit outward normal. The same equation holds on a geodesic compactification \((M, \tilde{g})\). If \( g \) is Einstein, then \( \text{Ric}(N, \cdot) = \text{Ric}(\tilde{N}, \cdot) = 0 \); the latter equality follows from (4.9). The equation (5.4) holds for all \( t \) small, and differentiating \((n - 1)\) times with respect to \( t \) gives rise to the divergence constraint in (5.3).

The Gauss-Codazzi equations are not used in the derivation and properties of the Fefferman-Graham expansion (4.3)-(4.4) \textit{per se}. The derivation of these equations involves only the tangential \((ij)\) part of the Ricci curvature. The asymptotic behavior of the normal \((00)\) part of the Ricci curvature gives rise to the trace constraint in (5.3), cf. (4.19)-(4.20).

Let \( T \) be the space of pairs \((g_{(0)}, \tau_{(n)})\) satisfying (5.3). If \( \tau^0_{(n)} \) is any fixed solution of (5.3), then any other solution with the same \( g_{(0)} \) is of the form \( \tau_{(n)} = \tau^0_{(n)} + \tau \), where \( \tau \) is transverse-traceless (TT) with respect to \( g_{(0)} \). (Of course if \( n \) is odd, one may take \( \tau^0_{(n)} = 0 \). The space \( T \) is naturally
a bundle over Met(∂M) with fiber at γ an affine space of symmetric tensors and is a subset of the product Met(∂M) × S^2(∂M) ≅ T(Met(∂M)). Let

\[ (5.5) \quad \pi : \mathcal{T} \to \text{Met}(\partial M) \]

be the projection onto the base space Met(∂M), (the first factor projection).

By the discussion in §4, \((g_{(0)}, \tau_{(n)}) \in \mathcal{T}\) if and only if the corresponding pair \((g_{(0)}, g_{(n)})\) determine a formal polyhomogenous solution to the Einstein equations near conformal infinity, i.e. formal series solutions containing log terms, as in (4.3)-(4.4). In fact, if \(g_{(0)}\) and \(g_{(n)}\) are real-analytic on \(\partial M\), a result of Kichenassamy [24] implies that the series (4.3) or (4.4) converges, and gives an Einstein metric \(g\), defined in a neighborhood of \(\partial M\). The metric \(g\) is complete near \(\partial M\) and has a conformal compactification inducing the given data \((g_{(0)}, g_{(n)})\) on \(\partial M\). Here we recall from the discussion in §4 that all coefficients of the expansion (4.3) or (4.4) are determined by \(g_{(0)}\) and \(g_{(n)}\).

In this regard, consider the following:

**Problem.** Is \(\pi : \mathcal{T} \to \text{Met}(\partial M)\) an open map? Thus, given any \((g_{(0)}, \tau_{(n)}) \in \mathcal{T}\) and any boundary metric \(\tilde{g}_{(0)}\) sufficiently close to \(g_{(0)}\), does there exist \(\tilde{\tau}_{(n)}\) close to \(\tau_{(n)}\) such that \((\tilde{g}_{(0)}, \tilde{\tau}_{(n)}) \in \mathcal{T}\).

Although \(\pi\) is obviously globally surjective, the problem above is whether \(\pi\) is locally surjective. For example, a simple fold map \(x \to x^3 - x\) is not locally surjective near \(\pm \sqrt{3}/3\). Observe that the trace condition in (5.3) imposes no constraint on \(g_{(0)}\); given any \(g_{(0)}\), it is easy to find \(g_{(n)}\) such that \(\text{tr}g_{(0)}(g_{(n)} + \tau_{(n)}) = a_{(n)}\); this equation can readily be solved algebraically for many \(g_{(n)}\).

By the inverse function theorem, it suffices, (and is probably also necessary), to examine the problem above at the linearized level. However the linearization of the divergence condition in (5.3) gives a non-trivial constraint on the variation \(h_{(0)}\) of \(g_{(0)}\). Namely, the linearization in this case gives

\[ (5.6) \quad \delta'(\tau_{(n)}) + \delta(\tau_{(n)})' = 0, \]

where \(\delta' = \frac{d}{d\alpha}\delta_{g_{(0)} + \alpha h_{(0)}}\), and similarly for \((\tau_{(n)})'\).

Whether (5.6) is solvable for any \(h_{(0)} \in S^2(\partial M)\) depends on the data \(g_{(0)}\) and \(g_{(n)}\). For example, it is trivially solvable when \(\tau_{(n)} = 0\). For compact \(\partial M\), one has

\[ (5.7) \quad \Omega^1(\partial M) = \text{Im} \delta \oplus \text{Ker} \delta^*, \]

where \(\Omega^1\) is the space of 1-forms, so that solvability in general requires that

\[ (5.8) \quad \delta'(\tau_{(n)}) \in \text{Im} \delta = (\text{Ker} \delta^*)^\perp. \]

Of course \(\text{Ker} \delta^*\) is exactly the space of Killing fields on \((\partial M, \gamma)\), and so this space serves as a potential obstruction space.

Clearly then \(\pi\) is locally surjective when \((\partial M, g_{(0)})\) has no Killing fields. On the other hand, it is easy to construct examples where \((\partial M, \gamma)\) does have Killing fields and \(\pi\) is not locally surjective:

**Example 5.3.** Let \((\partial M, g_{(0)})\) be the flat metric on the \(n\)-torus \(T^n\), \(n \geq 3\), and define \(g_{(n)} = -(n - 2)(d\theta_2)^2 + (d\theta_3)^2 + \cdots + (d\theta^n)^2\). Then \(g_{(n)}\) is transverse-traceless with respect to \(g_{(0)}\). Let \(f = f(\theta^1)\). Then \(\hat{g}_{(n)} = fg_{(n)}\) is still TT with respect to \(g_{(0)}\), so that \((g_{(0)}, \hat{g}_{(n)}) \in \mathcal{T}\), at least for \(n\) odd.

It is then not difficult to see via a direct calculation, or more easily via Proposition 5.4 below, that (5.8) does not hold, so that \(\pi\) is not locally surjective.

Next we relate these two issues, i.e. the general solvability of the divergence constraint (5.8) and the extension of Killing fields on the boundary into the bulk. The following result holds for general \(\phi \in S^2(\partial M)\) with \(\delta \phi = 0\), but will only be used in the case \(\phi = \tau_{(n)}\) on \(\partial M\).
Proposition 5.4. If $X$ is a Killing field on $(\partial M, \gamma)$, with $\partial M$ compact, then

$$
\int_{\partial M} \langle \mathcal{L}_X \tau(n), h(0) \rangle dV = -2 \int_{\partial M} \langle \delta'(\tau(n)), X \rangle dV,
$$

where $\delta' = \frac{d}{d\delta} \delta_{\gamma+sh(0)}$. In particular, (5.1) holds for all Killing fields on $(\partial M, \gamma)$ if and only if the linearized divergence constraint vanishes, i.e. (5.8) holds.

Proof: We will carry out the computation in general, and only use the condition that $X$ is a Killing field at the end of the proof. To simplify the notation, set $h(0) = h$.

The following formulas will be used here and in work to follow:

$$
\mathcal{L}_V \phi = \nabla_V \phi + 2\nabla \circ \phi,
$$

which is standard, and

$$
(\delta^*)' X = \frac{1}{2} \nabla X h + \delta^* X \circ h,
$$

cf. [11] for example. Here $\phi \circ \psi$ is the symmetrized product; in an orthonormal frame, $(\phi \circ \psi)(e_i, e_j) = \frac{1}{2}(\langle \phi(e_i), \psi(e_j) \rangle + \langle \phi(e_j), \psi(e_i) \rangle)$.

To begin $\int_{\partial M} \langle \mathcal{L}_X \tau(n), h \rangle = \int_{\partial M} \langle \nabla_X \tau(n), h \rangle + 2\langle \nabla X \circ \tau(n), h \rangle$. Since $h$ is symmetric, $\langle \nabla X \circ \tau(n), h \rangle = \langle \delta^* X \circ \tau(n), h \rangle$. For the first term, write $\langle \nabla_X \tau(n), h \rangle = \tau(n, h) - \langle \tau(n), \nabla_X h \rangle$. The first term here integrates to $\delta X (\tau(n), h)$, while by (5.11), the second term is $-\langle \tau(n), \nabla_X h \rangle = -2\langle \tau(n), (\delta^*)' X \rangle + 2\langle \tau(n), \delta^* X \circ h \rangle$.

Next, a straightforward computation using the fact that $\delta \tau(n) = 0$ gives

$$
\int_{\partial M} \langle (\delta^*)' X \rangle dV = \int_{\partial M} \langle (\delta')(\tau(n)), X \rangle dV
$$
$$
+ 2\int_{\partial M} \langle \tau(n) \circ \delta^* X, h \rangle - \frac{1}{2} \int_{\partial M} \langle \tau(n), \delta^* X \rangle trh dV.
$$

The last two terms come from variation of the metric and volume form. Combining these computations gives

$$
\int_{\partial M} \langle \mathcal{L}_X \tau(n), h \rangle = -2\int_{\partial M} \langle \delta'(\tau(n)), X \rangle + \int_{\partial M} [\delta X (\tau(n), h) + \frac{1}{2} \langle \tau(n), \delta^* X \rangle trh] dV.
$$

This gives (5.9) when $X$ is Killing, i.e. $\delta^* X = 0$; note that $\mathcal{L}_X \tau(n) = 0$ in this case, since $r(n)$ is determined by the boundary metric.

To prove the last statement, by (5.9), (5.1) holds if and only if $\int_{\partial M} \langle \delta'(\tau(n)), X \rangle = 0$, for all variations $h$. If (5.8) holds, then $\delta'(\tau(n)) = \delta h'(n)$, for some $h'(n)$ and so $\int_{\partial M} \langle \delta'(\tau(n)), X \rangle = \int_{\partial M} \langle h'(n), \delta^* X \rangle = 0$, since $X$ is Killing. The converse of this argument holds equally well.

Proposition 5.4 implies that in general, Killing fields on $\partial M$ do not extend to Killing fields in a neighborhood of $\partial M$, (cf. Example 5.3). (Exactly the same result and proof hold in the bounded domain case, when the term $\tau(n)$ is replaced by $A - Hg$).

Now as noted above, whether isometry extension holds or not depends on the term $\tau(n) = g(n) + r(n)$, or more precisely on the relation of the boundary metric $g(0)$ with $\tau(n)$. For Einstein metrics which are globally conformally compact, the term $\tau(n)$ is determined, up to a finite dimensional moduli space, by the boundary metric $g(0)$; (this is discussed further below). Thus, whether isometry extension holds or not is quite a delicate issue; if so, it must depend crucially on the global structure of $(M, g)$. 

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Before beginning the proof of Theorem 1.3, we first need to discuss some background material from [5]-[6].

Let $E_{AH}$ be the space of conformally compact, or equivalently asymptotically hyperbolic Einstein metrics on $M$ which have a $C^\infty$ polyhomogeneous conformal compactification with respect to a fixed smooth defining function $\rho$, as in (1.1). In [5], it is shown that $E_{AH}$ is a smooth, infinite dimensional manifold. One has a natural smooth boundary map

$$
\Pi : E_{AH} \to \text{Met}(\partial M),
$$

sending $g$ to its boundary metric $\gamma$.

The moduli space $E_{AH}$ is the quotient $E_{AH}/\mathcal{D}_1$, where $\mathcal{D}_1$ is the group of smooth (polyhomogeneous) diffeomorphisms $\phi$ of $M$ equal to the identity on $\partial M$. Thus, $g' \sim g$ if $g' = \phi^* g$, with $\phi \in \mathcal{D}_1$. Changing the defining function $\rho$ in (1.1) changes the boundary metric conformally. Also, if $\phi \in \mathcal{D}_1$ then $\rho \circ \phi$ is another defining function, and all defining functions are of this form near $\partial M$. Hence if $\mathcal{C}$ denotes the space of smooth conformal classes of metrics on $\partial M$, then the boundary map (5.13) descends to a smooth map

$$
\Pi : E_{AH} \to \mathcal{C}
$$

independent of the defining function $\rho$. Either boundary map $\Pi$ in (5.13) or (5.14) is smooth and Fredholm, of Fredholm index 0.

The linearization of the Einstein operator $\text{Ric}_g + ng$ at an Einstein metric $g$ is given by

$$
\hat{\Lambda} = (\text{Ric}_g + ng)' = \frac{1}{2} D^* D - R - \delta^* \beta,
$$

acting on the space of symmetric 2-tensors $S^2(M)$ on $M$, cf. [11]. Here, (as in §3), $\beta$ is the Bianchi operator, $\beta(h) = \delta h + \frac{1}{2} dtr h$, Thus, $h \in T_g E_{AH}$ if and only if

$$
\hat{\Lambda}(h) = 0.
$$

The operator $\hat{\Lambda}$ is not elliptic, due to the $\delta^* \beta$ term. As is well-known, this arises from the diffeomorphism group, and to obtain an elliptic linearization, one needs a gauge choice to break the diffeomorphism invariance of the Einstein equations. We will use a slight modification of the Bianchi gauge introduced in [12].

To describe this, given any fixed $g_0 \in E_{AH}$ with geodesic defining function $t$ and boundary metric $\gamma_0$, let $\gamma$ be a boundary metric near $\gamma_0$ and define the hyperbolic cone metric $g_\gamma$ on $\gamma$ by setting

$$
g_\gamma = t^{-2}(dt^2 + \gamma);
$$

$g_\gamma$ is defined in a neighborhood of $\partial M$. Next, set

$$
g(\gamma) = g_0 + \eta(g_\gamma - g_{\gamma_0}),
$$

where $\eta$ is a non-negative cutoff function supported near $\partial M$ with $\eta = 1$ in a small neighborhood of $\partial M$. Any conformally compact metric $g$ near $g_0$, with boundary metric $\gamma$ then has the form

$$
g = g(\gamma) + h,
$$

where $|h|_{g_0} = O(t^2)$; equivalently $\tilde{h} = t^2 h$ satisfies $\tilde{h}_{ij} = O(t^2)$ in any smooth coordinate chart near $\partial M$. The space of such symmetric bilinear forms $h$ is denoted by $\mathbb{S}_2(M)$ and the space of metrics $g$ of the form (5.17) is denoted by $\text{Met}_{AH}$.

The Bianchi-gauged Einstein operator, (with background metric $g_0$), is defined by

$$
\Phi_{g_0} : \text{Met}_{AH} \to \mathbb{S}_2(M)
$$

$$
\Phi_{g_0}(g) = \Phi(g(\gamma) + h) = \text{Ric}_g + ng + (\delta g)^* \beta_{g(\gamma)}(g),
$$

where $\beta_{g(\gamma)}$ is the Bianchi operator with respect to $g(\gamma)$. By [12, Lemma I.1.4],

$$
Z_{AH} \equiv \Phi^{-1}(0) \cap \{ \text{Ric} < 0 \} \subset E_{AH},
$$

(5.19)
where \( \{ \text{Ric} < 0 \} \) is the open set of metrics with negative Ricci curvature. In fact, if \( g \in E_{AH} \) is close to \( g_0 \), and \( \Phi(g) = 0 \), then \( \beta_{g,0}(g) = 0 \) and moreover

\( \delta_{g,0}(g) = 0, \quad \text{and} \quad tr_{g,0}(g) = 0. \)

The space \( Z_{AH} \) is a local slice for the action of \( D_1 \) on \( E_{AH} \): for any \( g \in E_{AH} \) near \( g_0 \), there exists a diffeomorphism \( \phi \in D_1 \) such that \( \phi^* g \in Z_{AH} \), cf. again [12].

The linearization of \( \Phi \) at \( g_0 \in E_{AH} \) with respect to the 2\(^{\text{nd}}\) variable \( h \) has the simple form

\[
(D_2 \Phi)_{g_0}(h) = \frac{1}{2} D^* D h - R_{g_0}(h),
\]

while the variation of \( \Phi \) at \( g_0 \) with respect to the 1\(^{\text{st}}\) variable \( g(\gamma) \) has the form

\[
(D_1 \Phi)(\gamma)(\tilde{g}(\gamma)) = (D_2 \Phi)(\gamma)(\tilde{g}(\gamma)) - \delta_{g_0} \beta_{g_0}(\tilde{g}(\gamma)) = (\text{Ric}_g + ng)'(\tilde{g}(\gamma)),
\]

as in (5.15). Clearly \( \tilde{g}(\gamma) = \eta t^{-2\gamma} \). The kernel of the elliptic self-adjoint linear operator

\[
L = \frac{1}{2} D^* D - R
\]

acting on the 2\(^{\text{nd}}\) variable \( h \), represents the space of non-trivial infinitesimal Einstein deformations vanishing on \( \partial M \). Let \( K \) denote the \( L^2 \) kernel of \( L \). This is the same as the kernel of \( L \) on \( \mathbb{S}_2(M) \), cf. [12], [26]. An Einstein metric \( g_0 \in E_{AH} \) is called non-degenerate if

\[
K = 0.
\]

For \( g_0 \in E_{AH} \) the kernel \( K = K_{g_0} \) equals the kernel of the linear map \( D_1 \, T_{g_0} \mathcal{E}_{AH} \to T_{\Pi(g_0)} \mathcal{C} \). Hence, \( g_0 \) is non-degenerate if and only if \( g_0 \) is a regular point of the boundary map \( \Pi \) in which case \( \Pi \) is a local diffeomorphism near \( g_0 \). From now on, we denote \( g_0 \) by \( g \).

The components of any \( \kappa \in K \) satisfy the following bounds with respect to a geodesic defining function, cf. [20], [26] or [28, Prop. 5] for example:

\[
\kappa = O(t^n), \quad \kappa(N, Y) = O(t^{n+1}), \quad \kappa(N, N) = O(t^{n+1+\mu}),
\]

where \( N = -t \partial_t \) is the unit outward normal vector to the \( t \)-level set \( S(t) \), \( Y \) is any \( g \)-unit vector tangent to \( S(t) \) and \( \mu > 0 \). Here \( \kappa = O(t^n) \) means \( |\kappa|_g = O(t^n) \). Also by (5.20), any \( \kappa \in K \) is transverse-traceless, i.e.

\[
\delta \kappa = tr \kappa = 0.
\]

Given this background, we are now ready to begin the proof of Theorem 1.3.

**Proof of Theorem 1.3.**

Let \( \bar{g} = t^2 g \) be a geodesic compactification of \( g \) with boundary metric \( \gamma \). By the boundary regularity result of [16], \( \bar{g} \) is \( C^\infty \) polyhomogeneous on \( \bar{M} \). It suffices to prove Theorem 1.3 for arbitrary 1-parameter subgroups of the isometry group of \( (\partial M, \gamma) \). Thus, let \( \phi_s \) be a local 1-parameter group of isometries of \( \gamma \) with \( \phi_0 = id \), so that

\[
\phi_s^* \gamma = \gamma.
\]

The diffeomorphisms \( \phi_s \) of \( \partial M \) may be extended to diffeomorphisms of \( M \), so that the curve

\[
g_s = \phi_s^* g
\]

is a smooth curve in \( E_{AH} \). By construction then, \( \Pi(g_s) = [\gamma] \), so that \( [h] = [d\phi_s/ds] \in Ker D\Pi \), for \( \Pi \) as in (5.14). One may then alter the diffeomorphisms \( \phi_s \) by composition with diffeomorphisms in \( D_1 \) if necessary, so that \( h = d\phi_s/ds \in K_g \), where \( K_g \) is the kernel in (5.24). Denoting \( h = \kappa \), it follows that

\[
\kappa = \delta^* X,
\]

where \( X = d\phi_s/ds \) is smooth up to \( \bar{M} \).
Thus it suffices to prove that $\delta^*X = 0$, since this will imply that $g_s = g$, (when $g_s$ is modified by
the action of $D_1$). If $Kg = 0$, i.e. if $g$ is a regular point of the boundary map $\Pi$, then this is now obvious, (from the above),
and proves the result in this special case; (the proof in this case requires only that $(M, g)$ be $C^{2,\alpha}$ conformally compact).

To prove the result in general, we will prove that any solution of (5.28) necessarily vanishes, so that
$$(5.29) \quad K \cap Im\delta^* = 0.$$

We give two different, (although related), proofs of (5.29), one conceptual and one more computational. The first, conceptual, proof involves an understanding of the cokernel of the map

$$\text{(5.29)}$$

that

Thus, the boundary variations $\dot{\gamma}$ for the $\delta$ it follows that the integral (5.31) reduces to an integral over the boundary, and gives

obvious, (from the above), and proves the result in this special case; (the proof in this case requires

only that $(\mathcal{B}, g)$ be

$$\text{Coker}D$$

such that

$$\text{Ker}L$$

and so one first needs to give an explicit description of this cokernel. To begin, recall

the derivative

$$\text{(5.30)} \quad (D\Phi)g : T_gMet_{AH}(M) \rightarrow T_{\Phi(g)}S_2(M).$$

Via (5.17), one has $T_gMet_{AH} = T_\gamma Met(\partial M) \oplus T_hS_2(M)$ and the derivative with respect to the

second factor is given by (5.21). If $K = 0$, then $D_2\Phi$ is surjective at $g$, (since $D_2\Phi$ has index 0), and

hence so is $DII$. In general, to understand $CokerDII$, we show that $D\Phi$ is always surjective; this follows from the claim that for any non-zero $\kappa \in K$ there is a tangent vector $\dot{\gamma}(\gamma) \in T_\gamma Met(\partial M) \subset T_gMet_{AH}$ such that

$$\text{(5.31)} \quad \int_M \langle (D\Phi)g(\dot{\gamma}(\gamma)), \kappa \rangle dV_g \neq 0.$$ 

Thus, the boundary variations $\dot{\gamma}(\gamma)$ satisfying (5.31) for some $\kappa$ correspond to the cokernel. To

prove (5.31), let $B(t) = \{x \in M : t(x) \geq t\}$ and $S(t) = \partial B(t) = \{x \in M : t(x) = t\}$. Apply the divergence theorem to the integral (5.31) over $B(t)$; twice for the Laplace term in (5.22) and once for the $\delta^*$ term in (5.22). Since

$$\kappa \in Ker L$$

and $\delta\kappa = 0$,

it follows that the integral (5.31) reduces to an integral over the boundary, and gives

$$\text{(5.32)} \quad \int_{B(t)} \langle (D\Phi)g(\dot{\gamma}(\gamma)), \kappa \rangle dV_g = \frac{1}{2} \int_{S(t)} \langle \langle \dot{\gamma}(\gamma), \nabla_N\kappa \rangle - \langle \nabla_N\dot{\gamma}(\gamma), \kappa \rangle - 2\langle \beta(\dot{\gamma}(\gamma)), \kappa N \rangle \rangle dV_{S(t)}.$$ 

Of course $dV_{S(t)} = t^{-n}dV_\gamma + O(t^{-(n-1)})$. By (5.25) the last term in (5.32) is then $O(t)$ and so may be ignored. Let

$$\tilde{\kappa} = t^{-n}\kappa,$$

so that by (5.25), $|\tilde{\kappa}|_g \leq C$. From the definition (5.16), a straightforward computation shows that near $\partial M$,

$$\dot{\gamma}(\gamma) = t^{-2}\dot{\gamma}, \quad \text{and} \quad \nabla_N\dot{\gamma}(\gamma) = 0.$$ 

Note that $|\dot{\gamma}(\gamma)|_g \sim 1$ as $t \rightarrow 0$. Hence,

$$\langle \langle \dot{\gamma}(\gamma), \nabla_N\kappa \rangle - \langle \nabla_N\dot{\gamma}(\gamma), \kappa \rangle \rangle dV_{S(t)} = \langle \nabla_N\kappa, \dot{\gamma} \rangle \gamma dV_{S(t)} + O(t)$$

$$= \langle \nabla_N\tilde{\kappa} - n\tilde{\kappa}, \dot{\gamma} \rangle \gamma dV_\gamma + O(t).$$ 

Thus,

$$\text{(5.34)} \quad \int_{B(t)} \langle (D\Phi)g(\dot{\gamma}(\gamma)), \kappa \rangle dV_g = \frac{1}{2} \int_{S(t)} \langle \nabla_N\tilde{\kappa} - n\tilde{\kappa}, \dot{\gamma} \rangle \gamma dV_\gamma + O(t).$$ 

Now suppose, (contrary to (5.31)),

$$\text{(5.35)} \quad \nabla_N\tilde{\kappa} - n\tilde{\kappa} = O(t).$$
as forms on \((S(t), g)\); note however that \(\nabla\) is taken with respect to \(g\) in (5.35). Since \(\tilde{\kappa}(N) = O(t)\) by (5.25), a simple computation shows that (5.35) implies that \(\frac{1}{2}N(\tilde{\kappa}^2) - n\tilde{\kappa}^2 = O(t)\). Integrating with respect to the induced metric on \((S(t), \bar{g})\) and using the fact that \(\frac{d}{dt}dV_{S(t)} = O(t)\), it follows that

\[
\frac{1}{2}N \int_{S(t)} |\tilde{\kappa}|^2 dV_\gamma - n \int_{S(t)} |\tilde{\kappa}|^2 dV_\gamma = O(t),
\]

as \(t \to 0\). An elementary integration of (5.36) in \(t\) then implies

\[
\int_{S(t)} |\tilde{\kappa}|^2 dV_\gamma = O(t),
\]

and hence, using (5.25) again,

\[
\kappa = o(t^n).
\]

(More precisely (5.37) holds in \(L^2\), but pointwise decay then follows from elliptic regularity in weighted Hölder spaces, as in [26] or [28] for instance). Now \(\kappa\) is an infinitesimal Einstein deformation, divergence-free by (5.26). Then Corollary 4.4, (cf. also [5, Thm. 3.1]), and (5.37), together with the assumption in Theorem 1.3 that \(\pi_1(M, \partial M) = 0\) imply that

\[
\kappa = 0 \quad \text{on} \quad M,
\]

giving a contradiction. This proves the relation (5.31).

The proof above shows that the form

\[
\dot{g}(\gamma) = \lim_{t \to 0} \tilde{\kappa}|S(t)|,
\]

satisfies (5.31). The limit here exists by the regularity results of [16]. Thus, the space

\[
\tilde{K} = \{ \tilde{\kappa} = \lim_{t \to 0} t^{-n}\kappa|_{S(t)} : \kappa \in K \},
\]

is naturally identified with the cokernel of \(\Pi_g\) in \(T_\gamma\text{Met}(\partial M)\). Note that \(\text{dim}\tilde{K} = \text{dim} K\) and also that the estimates (5.25) show that \(\tilde{\kappa} = \tilde{\kappa}^T\) on \(\partial M\).

This means that infinitesimal deformations of the boundary metric \(\gamma\) in the direction \(\tilde{\kappa}, \tilde{\kappa} \in \tilde{K}\), are not realized as \(\frac{d}{ds}\Pi(g_s)|_{s=0}\), where \(g_s\) is a curve in \(E_{AH}\) through \(g\), i.e. a curve of global Einstein metrics on \(M\). However, it is easy to see that \(\tilde{\kappa}\) is realized as the boundary variation of locally defined Einstein metrics. More precisely, choose any sequence \(t_i \to 0\) and consider the metrics

\[
g_{s,i} = g + st_i^{-n}\kappa + O(s^2),
\]

in the region \(A_i = A(\frac{1}{2}, 2t_i) = \{x \in M : \frac{1}{2} \leq t(x) \leq 2t_i\}\), with \(\kappa \in K\). For each \(i\), this is a curve of metrics on \(A_i\), Einstein to 1st order in \(s\) at \(s = 0\). The induced variation of the boundary metric on \(S(t_i)\) is, by construction, \((\tilde{\kappa})^T|_{S(t_i)} \sim \tilde{\kappa}|_{S(t_i)}\). Now note that the linearized divergence constraint (5.6) or (5.8) only involves the behavior at \(\partial M\), or equivalently, the limiting behavior on \((S(t_i), g_{t_i})\), \(g_{t_i} = g|_{S(t_i)}\), as \(t_i \to 0\). This shows that the constraint (5.8) may be solved on the cokernel \(\tilde{K}\), and hence is solvable on all of \(T_\gamma\text{Met}(\partial M)\). This proves the following:

**Corollary 5.5.** Let \(g\) be a conformally compact Einstein metric on a compact manifold \(M\) with \(C^\infty\) boundary metric \(\gamma\). Then the linearized divergence constraint equation (5.8) is always solvable on \((\partial M, \gamma)\), i.e. the map \(\pi\) in (5.5) is locally surjective at \((\gamma, \tau(n))\).  

(A more detailed and computational proof of Corollary 5.5 will also be given below).

Combining Proposition 5.4 and Corollary 5.5, it follows that

\[
\mathcal{L}_X \tau(n) = 0, \quad \text{and hence} \quad \mathcal{L}_X g(n) = 0
\]
on \((\partial M,\gamma)\). (The second statement follows since the term \(r_{(n)}\) is intrinsic to the boundary metric \(\gamma\), so that \(\mathcal{L}_X r_{(n)} = 0\).) Proposition 5.1 and the assumption \(\pi_1(M,\partial M) = 0\) then implies that \(X\) extends to a Killing field \(Y\) on \(M\). This completes the first proof of Theorem 1.3.

Regarding the claim (5.29), since \(\delta^*Y = 0\) and \(Y\) is asymptotic to \(X\) on \(\partial M\), one has \(\kappa = \delta^*(X - Y)\) on \(M\), which implies that \(\kappa = o(t^n)\). By Corollary 4.4, this implies that \(\kappa = 0\), as claimed.

It is useful and of interest to give another, direct computational proof of Theorem 1.3, without using the identification (5.39) as the cokernel of \(\Pi\). The basic idea is to compute as in Proposition 5.4 on \((S(t),g_t)\), with \(A - Hg_t\) in place of \(\tau_{(n)}\), and then pass to the limit on \(\partial M\). Throughout the proof, we assume (5.28) holds.

Before starting the proof proper, we note that the estimates (5.25) and (5.28) imply that \(X\) is tangential, i.e. tangential to \((S(t),g)\), to high order, in that

\[
(5.44) \quad \langle X,N \rangle = O(t^{n+1+\mu}).
\]

To see this, one has \((\delta^*X)(N, N) = \langle \nabla_N X, N \rangle = N\langle X, N \rangle\). Thus (5.42) follows from (5.25) and the claim that \(\langle X, N \rangle = 0\) on \(\partial M\). To prove the latter, consider the compactified metric \(\bar{g} = t^2 g\). One has \(\mathcal{L}_X \bar{g} = \mathcal{L}_X (t^2 g) = 2 \frac{X(t)}{t} \bar{g} + O(t^n)\). Thus for the induced metric \(g\) on \(\partial M\), \(\mathcal{L}_X g = 2 \lambda g\), where \(\lambda = \lim_{t \to 0} \frac{X(t)}{t}\). Since \(X\) is a Killing field on \((\partial M, \gamma)\), this gives \(\lambda = 0\), which is equivalent to the statement that \(\lim_{t \to 0} \langle X, N \rangle g = 0\). Note also that since \(X\) is smooth up to \(\partial M\), \(|X|_g = O(t^{-1})\).

We claim also that

\[
(5.43) \quad [X, N] = O(t^{n+1}),
\]

in norm. First, \(\langle [X,N], N \rangle = \langle \nabla_X N - \nabla_N X, N \rangle = -\langle \delta^*X(N,N) = O(t^{n+1+\mu})\). On the other hand, on tangential \(g\)-unit vectors \(Y\), \(\langle [X,N], Y \rangle = \langle \nabla_X N - \nabla_N X, Y \rangle \sim \langle \nabla_X N, Y \rangle - 2\langle \delta^*X(N,Y) + \langle \nabla_Y N, N \rangle \sim -2\langle \delta^*X(N,Y) = O(t^{n+1})\), as claimed. Here \(\sim\) denotes equality modulo terms of order \(o(t^n)\). We have also used the fact that \(\langle \nabla_X N, Y \rangle + \langle \nabla_Y N, N \rangle \sim X\langle N, N \rangle = 0\).

Now, to begin the proof itself, (assuming (5.28)), as above write

\[g_s = g + s\kappa + O(s^2) = g + s\delta^*X + O(s^2).\]

If \(t_s\) is the geodesic defining function for \(g_s\), (with boundary metric \(\gamma\)), then the Fefferman-Graham expansion gives \(\bar{g}_s = dt_s^2 + (\gamma + t_s^2 g_{(2), s} + \cdots + t^n g_{(n), s}) + O(t^{n+1})\). The estimate (5.42) implies that \(t_s = t + sO(t^{n+2+\alpha}) + O(s^2)\), so that modulo lower order terms, we may view \(t_s \sim t\). Taking the derivative of the FG expansion with respect to \(s\) at \(s = 0\), and using the fact that \(X\) is Killing on \((\partial M, \gamma)\), together with the fact that the lower order terms \(g_{(k)}, k < n\), are determined by \(\gamma\), it follows that, for \(k\) as in (5.39),

\[
(5.44) \quad \bar{\kappa} = \frac{1}{2} \mathcal{L}_X g_{(n)},
\]

at \(\partial M\). Here both \(\bar{\kappa}\) and \(\mathcal{L}_X g_{(n)}\) are viewed as forms on \((\partial M, \gamma)\).

Next, we claim that on \((S(t),g_t)\),

\[
(5.45) \quad \mathcal{L}_X A = -\frac{n}{2} t^{n-2} \mathcal{L}_X g_{(n)} + O(t^{n-1}),
\]

To see this, one has \(A = \frac{1}{2} L_N g = -\frac{1}{2} L_{\partial t} g = -\frac{1}{2} L_{\partial t} (t^{-2} g_t)\). But \(L_{\partial t} (t^{-2} g_t) = \sum L_{\partial t} (t^{-2+k} g_{(k)}) = \sum (k-2) t^{k-2} g_{(k)}\). The same reasoning as before then gives (5.45).

Given these results, we now compute

\[
\int_{S(t)} \langle \mathcal{L}_X(A - Hg_t), \bar{\kappa} \rangle g_t dV_{S(t)};
\]


compare with the left side of (5.9). First, by (5.45),
\[ \int_{S(t)} \langle L_X A, \kappa \rangle_{g_t} dV_{S(t)} = -\frac{2(n-2)}{2} \int_{S(t)} \langle L_X g_{(n), \kappa} \rangle_{g_t} dV_{S(t)} + O(t). \]
Next, one has \( L_X (H g_t) = X(H) g_t + H L_X g_t \). For the first term, \( X(H) = \text{tr} L_X A + O(t^n) = -\frac{n-2}{2} t^{n-2} \text{tr} L_X g_{(n)} + O(t^n) \). Since \( \text{tr} g_{(n)} \) is intrinsic to \( \gamma \) and \( X \) is Killing on \( (\partial M, \gamma) \), it follows that \( X(H) = O(t^{n-1}) \). Also, \( \langle g_t, \kappa \rangle = \text{tr}^T \kappa \), where \( \text{tr}^T \) is the tangential trace. By (5.29) and the fact that \( \kappa \) is trace-free, \( \langle g_t, \kappa \rangle = O(t^{1+\alpha}) \). Hence \( X(H) \langle g_t, \kappa \rangle dV_{S(t)} = O(t^\alpha) \). Similarly, from (5.43) one computes \( L_X g_t = L_X g + O(t^n) = 2t^n \bar{O} + O(t^{n+1}) \). Since \( H \sim n \), using (5.44) this gives
\[ - \int_{S(t)} \langle L_X (H g_t), \kappa \rangle dV_{S(t)} = -n \int_{S(t)} \langle L_X g_{(n), \kappa} \rangle_{g_t} dV_{S(t)} + O(t^n). \]
Combining these computations then gives
\[ \int_{S(t)} \langle L_X (A - H g_t), \kappa \rangle_{g_t} dV_{S(t)} = -(\frac{n-2}{2} + n) \int_{\partial M} \langle L_X g_{(n), \kappa} \rangle_{g_t} dV_{S(t)} + o(1). \]

On the other hand, one may use (5.12), with \( h = \kappa \), to compute the left side of (5.46). Since \( \kappa \) is transverse-traceless, the estimate (5.42) implies that \( \text{tr}^T \kappa = O(t^{n+1}) \) and \( \delta^* X = O(t^{n+1}) \). Hence the last two terms in (5.12) are lower order, (i.e. \( O(t) \)). Also, as in (5.40), the metrics \( g_{(s)} \) are Einstein to \( 1^\text{st} \) order in \( s \) in a neighborhood of \( S(t) \) and hence
\[ \int_{S(t)} \langle L_X (A - H g_t), \kappa \rangle_{g_t} dV_{S(t)} = -2 \int_{S(t)} \langle \delta^* (A - H g_t), X \rangle dV_{g_t} = \]
\[ 2 \int_{S(t)} \langle (A - H g_t)', (\delta^*)^T X \rangle dV_{g_t} + O(t). \]
Now by (5.42), \( (\delta^*)^T X = \delta^* X + O(t^{n+1}) \) while \( A' = A_{g + s\kappa}^\perp = \frac{1}{2} (L_N \kappa + L_{N'} g) = \frac{1}{2} \nabla_N \kappa + O(t) \). It follows that
\[ \int_{S(t)} \langle (A - H g_t)', (\delta^*)^T X \rangle dV_{g_t} = \frac{1}{2} \int_{S(t)} \langle \nabla_N \kappa, \kappa \rangle_{g_t} dV_{g_t} + O(t) = O(t), \]
where the last equality follows by the same reasoning as following (5.36), (since \( N = -t \partial_t \)).

The equations (5.46)-(5.48) imply that
\[ \int_{S(t)} \langle L_X g_{(n), \kappa} \rangle_{g_t} dV_{S(t)} = O(t). \]
Via (5.44) again, this shows that
\[ \bar{\kappa} = 0 \]
on \( \partial M \), and hence via Corollary 4.4 again, \( \kappa = 0 \) on \( M \). This completes the second proof of Theorem 1.3.

**Proof of Corollary 1.4.**

Suppose \( (M, g) \) is a conformally compact Einstein metric with boundary metric given by the round metric \( S^n(1) \) on \( S^n \). Theorem 1.3 implies that the isometry group of \( (M, g) \) contains the isometry group of \( S^n \). This reduces the Einstein equations to a simple system of ODE’s, and it is easily seen that the only solution is given by the Poincaré metric on the ball \( B^{n+1} \).

**Remark 5.6.** By means of Obata’s theorem [30], Theorem 1.3 remains true for continuous groups of conformal isometries at conformal infinity. Thus, the class of the round metric on \( S^n \) is the only conformal class which supports a non-essential conformal Killing field, i.e. a field which is not Killing with respect to some conformally related metric. Corollary 1.4 shows that any \( g \in E_{AH} \)
with boundary metric $S^n(1)$ is necessarily the hyperbolic metric $g_{-1}$ on the ball. For $g_{-1}$, it is well-known that essential conformal Killing fields on $S^n$ extend to Killing fields on $(\mathbb{H}^{n+1}, g_{-1})$.

We expect that a modification of the proof of Theorem 1.3 would give this result directly, without the use of Obata’s theorem. In fact, such would probably give (yet) another proof of Obata’s result.

Corollary 5.5 shows, in the global situation, that the projection $\pi$ of the constraint manifold $T$ to $\text{Met}(\partial M)$ is always locally surjective. Hence there exists a formal solution, and an exact solution in the analytic case, for any nearby boundary metric, which is defined in a neighborhood of the boundary. However, the full boundary map $\Pi$ in (5.13) or (5.14) on global metrics is not locally surjective in general; nor is it always globally surjective.

The simplest example of this behavior is provided by the family of AdS Schwarzschild metrics. These are metrics on $\mathbb{R}^2 \times S^{n-1}$ of the form

$$g_m = V^{-1} dr^2 + V d\theta^2 + r^2 g_{S^{n-1}(1)},$$

where $V = V(r) = 1 + r^2 - \frac{2m}{r^2}$. Here $m > 0$ and $r \in [r_+, \infty]$, where $r_+$ is the largest root of the equation $V(r_+) = 0$. The locus $\{r_+ = 0\}$ is a totally geodesic round $S^{n-1}$ of radius $r_+$. Smoothness of the metric at $\{r_+ = 0\}$ requires that the circular parameter $\theta$ runs over the interval $[0, \beta]$, where

$$\beta = \frac{4\pi r_+}{nr_+^2 + (n-2)}.$$

The metrics $g_m$ are isometrically distinct for distinct values of $m$, and form a curve in $E_{AH}$ with conformal infinity given by the conformal class of the product metric on $S^1(\beta) \times S^{n-1}(1)$. As $m$ ranges over the interval $(0, \infty)$, $\beta$ has a maximum value of

$$\beta \leq \beta_{\text{max}} = 2\pi \sqrt{(n-2)/n}.$$

As $m \to 0$ or $m \to \infty$, $\beta \to 0$.

Hence, the metrics $S^1(L) \times S^{n-1}(1)$ are not in $\Pi(g_m)$ for any $L > \beta_{\text{max}}$. In fact these boundary metrics are not in $\text{Im}(\Pi)$ generally, for any manifold $M^{n+1}$. For Theorem 1.3 implies that any conformally compact Einstein metric with boundary metric $S^1(L) \times S^{n-1}(1)$ has an isometry group containing the isometry group of $S^1(L) \times S^{n-1}(1)$. This again reduces the Einstein equations to a system of ODE’s and it is easy to see, (although we do not give the calculations here), that any such metric is an AdS Schwarzschild metric.

**Remark 5.7.** In the context of Propositions 5.1 and 5.4, it is natural to consider the issue of whether local Killing fields of $\partial M$, (i.e. Killing fields defined on the universal cover), extend to local Killing fields of any global conformally compact Einstein metric. Note that Proposition 5.1 and Proposition 5.4 are both local results, the latter by using variations $h_{(0)}$ which are of compact support. However, the linearized constraint condition (5.8) is not invariant under covering spaces; even the splitting (5.7) is not invariant under coverings, since a Killing field on a covering space need not descend to the base space.

We claim that local Killing fields do not extend even locally into the interior in general. As a specific example, let $N^{n+1}$ be any complete, geometrically finite hyperbolic manifold, with conformal infinity $(\partial N, \gamma)$, and which has at least one parabolic end, i.e. a finite volume cusp end, with cross sections given by flat tori $T^n$. There exist many such manifolds. The metric at conformal infinity is conformally flat, so there are many local Killing fields on $\partial N$. For example, in many examples $N$ itself is a compact hyperbolic manifold. Of course the local (conformal) isometries of $\partial N$ extend here to local isometries of $N$.

However, as shown in [15], the cusp end may be capped off by Dehn filling with a solid torus, to give infinitely many distinct conformally compact Einstein metrics with the same boundary metric $(\partial N, \gamma)$. These Dehn-filled Einstein metrics cannot inherit all the local conformal symmetries of the boundary.
Remark 5.8. We point out that Theorem 1.3 fails for complete Ricci-flat metrics which are ALE (asymptotically locally Euclidean). The simplest counterexamples are the family of Eguchi-Hanson metrics, which have boundary metric at infinity given by the round metric on $S^3/\mathbb{Z}_2$. The symmetry group of these metrics is strictly smaller than the isometry group $\text{Isom}(S^3/\mathbb{Z}_2)$ of the boundary. Similarly, the Gibbons-Hawking family of metrics with boundary metric the round metric on $S^3/\mathbb{Z}_k$ have only an $S^1$ isometry group, much smaller than the group $\text{Isom}(S^3/\mathbb{Z}_k)$.

This indicates that, despite a number of proposals, some important features of holographic renormalization in the AdS context cannot carry over to the asymptotically flat case.

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