Analysis of 3D poroelastodynamics using BEM based on modified time-step scheme

I A Igumnov 1, A N Petrov 2, I V Vorobtsov 1

1 Research Institute for Mechanics, Lobachevsky State University of Nizhni Novgorod, 23 Prospekt Gagarina, bld. 6, 603950, Nizhni Novgorod, Russia
2 Research and Education Center “Materials”, Don State Technical University, 1 Gagarin Sq., 344010, Rostov-on-Don, Russia

E-mail: aigumnov@mech.unn.ru

Abstract. The development of 3d boundary elements modeling of dynamic partially saturated poroelastic media using a stepping scheme is presented in this paper. Boundary Element Method (BEM) in Laplace domain and the time-stepping scheme for numerical inversion of the Laplace transform are used to solve the boundary value problem. The modified stepping scheme with a varied integration step for quadrature coefficients calculation using the symmetry of the integrand function and integral formulas of Strongly Oscillating Functions was applied. The problem with force acting on a poroelastic prismatic console end was solved using the developed method. A comparison of the results obtained by the traditional stepping scheme with the solutions obtained by this modified scheme shows that the computational efficiency is better with usage of combined formulas.

Introduction
There are two major approaches to dynamic processes modeling by means of BEM: solving directly in time domain using stepping scheme and in Laplace or Fourier domain followed by the respective transform inversion [1]. The first approach with traditional stepping schemes has some disadvantages, e.g. requires fundamental solutions in time, significant computing costs and a low stability rate. The New Convolution Quadrature Method (CQM), which is widely applied in construction of time-step boundary element schemes on the basis of fundamental solutions in Laplace domain, was introduced in [2, 3]. It helps to get rid of fundamental solutions in time requirement and shows a better stability [4]. A similar boundary element scheme based on the stepping method for Laplace transform numerical inversion is considered in [5]. To solve a 3d poroelastodynamic problem in the scope of the boundary element method, the modification of the scheme with a varied integration step relying on highly oscillatory quadrature principles and symmetry of the integrand function is employed in this paper.

Problem formulation
A set of fully coupled governing differential equations of a porous medium saturated by two compressible fluids (water and air) subjected to dynamic loadings is considered. In this formulation, solid skeleton displacements $u_i$, water pressure $p^w$ and air pressure $p^a$ are presumed to be independent variables [6]. The final differential equations in Laplace domain yield:
\[
G \bar{u}_{i,j} + (K + \frac{G}{3}) \bar{u}_{i,j,j} - (\rho - \beta S_w \rho_w - \gamma S_a \rho_a) s^2 \bar{u}_i - (\alpha - \beta) S_w \bar{p}^w_j + (\alpha - \gamma) S_a \bar{p}^a_j = -F_i,
\]
(1)

\[
- (\alpha - \beta) S_w \bar{u}_{i,j} - (\zeta - S_{na} S_w + S_a) s \bar{p}^w - \frac{\beta S_w}{\rho_w s} \bar{p}^w_{ij} - (\zeta S_{ww} S_w + \frac{\phi}{K_w} S_w - S_a) s \bar{p}^w = -\bar{T}^w,
\]
(2)

\[
- (\alpha - \gamma) S_a \bar{u}_{i,j} - (\zeta S_{aw} S_a + S_a) s \bar{p}^a + \frac{\gamma S_a}{\rho_a s} \bar{p}^a_{ij} - (\zeta S_{aa} + \frac{\phi}{K_a} S_a - S_a) s \bar{p}^a = -\bar{T}^a, \quad x \in \Omega,
\]
(3)

where \( K_w \) and \( K_a \) are bulk moduli of the fluid. \( \phi \) is porosity, \( \bar{F}_i, \bar{T}^w, \bar{T}^a \) are bulk body forces. The bulk density is denoted by \( \rho = (1 - \phi) \rho_s + \phi \rho_w + \phi \rho_a \), where \( \rho_s \) is the density of the solid, \( \rho_w \) is wetting fluid density, \( \rho_a \) is non-wetting fluid density. The saturation degrees are defined as the ratios of the volume occupied by the fluid \( V_w \) or \( V_a \) to the void volume, i.e. it holds true for

\[
S_w = \frac{V_w}{V_{void}}, \quad S_a = \frac{V_a}{V_{void}}, \quad S_w + S_a = 1.
\]
(4)

The abbreviations

\[
\zeta = \frac{\alpha - \phi}{K}, \quad S_{ww} = S_w - \vartheta (S_w - S_{rw}), \quad S_{na} = S_a + \vartheta (S_w - S_{rw}).
\]
(5)

\[
S_a = -\frac{\vartheta (S_{ra} - S_{rw})}{p^d} \left( \frac{S_w - S_{ra}}{S_{wa}} \right)^{\vartheta + 1} - \frac{\vartheta (S_{ra} - S_{rw})}{p^d} \left( \frac{S_a - S_{ra}}{S_{ra} - S_{rw}} \right)^{\vartheta + 1}.
\]
(6)

are introduced, where \( S_{rw} \) is the residual wetting fluid saturation and \( S_{ra} \) is the non-wetting fluid entry saturation. Symbol \( p^d \) is non-wetting fluid entry pressure, \( \vartheta \) is the pore distribution index while the value of \( \vartheta \) normally lies between 0.2 and 3. Symbols \( \beta \) and \( \gamma \) are Laplace parameter dependent variables and are expressed as:

\[
\beta = \frac{\kappa_w \phi \rho_w s}{\phi \rho_w + \kappa_w \rho_w s}, \quad \gamma = \frac{\kappa_a \phi \rho_a s}{\phi \rho_a + \kappa_a \rho_a s},
\]
(7)

where \( \kappa_w \) and \( \kappa_a \) - the phase permeability of the wetting and the non-wetting fluid given by \( \kappa_w = K_{rw} k / \eta_w \) and \( \kappa_a = K_{ra} k / \eta_a \) respectively. \( K_{rw} \) and \( K_{ra} \) denote the relative fluid phase permeability, \( k \) denotes the intrinsic fluid permeability, \( \eta_w \) and \( \eta_a \) are viscosity of the fluid. To
evaluate the relative phase permeability, the following equations are used: \( K_{rw} = S_e^{(2+3\delta)/\theta} \) and \( K_{ra} = (1 - S_e) \left( 1 - S_e^{(2+3\delta)/\theta} \right) \). \( S_e \) denotes the effective wetting fluid saturation degree given by:

\[
S_e = \begin{cases} 
0, & S_w \leq S_{rw} \\
\frac{S_w - S_{rw}}{S_{ra} - S_{rw}}, & S_{rw} < S_w < S_{ra} \\
1, & S_w \geq S_{ra} 
\end{cases}
\]

(8)

For extreme case \( S_w = 0 \) the equations turn out to be a simple elasticity problem; for extreme case \( S_w = 1 \), they are a saturated poroelasticity problem.

**BEM application**

The boundary-value problem can be reduced to the following boundary integral equation [4, 7]:

\[
\begin{align*}
\alpha_i \tilde{U}_i(x,s) + \int_\Gamma \tilde{T}_{ik}(x,y,s)\tilde{u}_k(y,s) - \tilde{T}_{ik}(x,y,s)\tilde{u}_k(y,s) - \tilde{U}_{ik}(x,y,s)c_i(y,s)dy & = 0, \\
\end{align*}
\]

(9)

where \( \tilde{U}(x,s) \), \( \tilde{T}(x,s) \) are fundamental and singular solutions for boundary integral equation (BIE), \( \tilde{T}^0(x,s) \) contains the singularities isolation, \( x \in \Gamma \), \( \vec{r} = (\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{q}^e, \vec{q}^w)^T \), \( \tilde{u} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{p}^e, \tilde{p}^w)^T \) for partially saturated poroelasticity.

To approximate the boundary of a piecewise homogeneous body, a set of quadrangular and triangular elements is used, where triangular elements are considered singular quadrangular. Quadrangular elements help to obtain a higher precision as compared to fully triangular approximation. Every boundary element is mapped to a reference one (canonical square \([0, 1]^2\) or triangle \([0, 1, 0]^2\) with the help of the formula:

\[
y_i(\xi) = \sum_{l=1}^{8} N_l'(\xi)\psi_{(k,l)}(\xi), \quad i = 1, 2, 3,
\]

(10)

where \( \psi_{(k,l)} \) is the global node number, \( l \) – local node number in element \( k \), \( N_l'(\xi) \) – shape functions. Boundary functions are interpolated by a subset of geometrical grid nodes. Local approximation of generalized boundary displacements and tractions follows the Goldshleyn’s displacement-stress matched model, meaning that generalized displacements are approximated by a bilinear element, and tractions – by a constant one. The integrals are calculated using Gaussian quadrature based on the elements containing no singularities. Once an element contains singularity, singularity decreasing or an eliminating algorithm is used. An adaptive integration algorithm is used: the order of Gaussian quadrature in an element is chosen by the satisfaction of the prescribed precision if possible. Otherwise, the element is recursively subdivided into smaller elements for integration.

**Laplace transform inversion**
The time-step method for Laplace transform numerical inversion is similar to CQM, but whereas CQM is based on the convolution theorem, it is based on the integration theorem, hence dedicated to the calculation of the original function integral. In order to get the original function \( f(t) \) such as \( f(0) = 0 \) from \( \tilde{f}(s) \) as a result, one needs to apply the theorem to its derivative:

\[
f(t) = \int_0^t f'(\tau) d\tau = \frac{1}{2\pi i} \lim_{R \to \infty} \int_{c-iR}^{c+iR} \tilde{f}(s) s e^{st} d\tau ds = ...
\]

(11)

Following the theorem, the integral can be found as follows:

\[
f(0) = 0, \quad f(n\Delta t) = \sum_{k=0}^{n} \omega_n (\Delta t), \quad n = 1,...N,
\]

(12)

\[
\omega_n (\Delta t) = \frac{R^n}{L} \sum_{k=0}^{L-1} \tilde{f}(s_k) s e^{-i\phi_k}, \quad s = \frac{\gamma(z)}{\Delta t}, \quad z = Re^{i\phi}, \quad \phi = 2\pi \frac{k}{L}.
\]

(13)

Several modifications of the method can be derived from different ways to calculate \( \omega_n (\Delta t) \). For the varied step by \( \phi \) and linear approximation of the integrand function, one obtains:

\[
\omega_n (\Delta t) = \frac{R^n}{2\pi} \sum_{k=0}^{L-1} \left[ \tilde{f}(s_k) s e^{-i\phi_k} + \tilde{f}(s_{k+1}) s_{k+1} e^{-i\phi_{k+1}} \right] \left( \frac{\phi_{k+1} - \phi_k}{2} \right), \quad s_k = \frac{\gamma(Re^{i\phi_k})}{\Delta t}
\]

(14)

or, basing on the highly oscillatory quadrature [5]:

\[
\omega_n (\Delta t) = \frac{R^n}{2\pi} \sum_{k=0}^{L-1} \frac{\phi_{k+1} - \phi_k}{2} e^{-i\phi_k} \left[ D_1(w) \tilde{f}(s_k) s_k + D_2(w) \tilde{f}(s_{k+1}) s_{k+1} \right], \quad s_k = \frac{\gamma(Re^{i\phi_k})}{\Delta t}
\]

\[w = -n \frac{\phi_{k+1} - \phi_k}{2}, \quad D_{1,2}(w) = \begin{cases} \sin w + w \cos w - \sin w / i & \text{where } |w| > w_*, \\
\frac{w^2}{w^2 + w^2} e^{\pm w / i} & \text{where } |w| \leq w_* \end{cases}
\]

(15)

where \( w_* \) is a specific key value, and the overall scheme using this formula is informally called ‘scheme with a key’. \( R \) denotes the radius of the analyticity region for \( \tilde{f}(\gamma(z) / \Delta t) \), and \( \gamma(z) \) is the characteristic function for the linear multistep method applied to the Cauchy problem arising within the integral evaluation. Backward differentiation (BDF) based methods of order \( \leq 6 \) are applicable in the scope of this solution scheme. For BDF-2, one has \( \gamma(z) = 3/2 - 2z + z^2 / 2 \).

**Numerical example**
A problem of frontal thrust $F(t) = 1 N/m^2$ to 3d poroelastic column of 3m x 1m x 1m with another fixed front is considered (Fig. 1), traction $t_z$ on the clamped end and displacement $u_z$ on another end are calculated. Pore size distribution index $\vartheta$ is set to 1.5, residual water saturation $S_{rw}$ is set to 0, and the air entry saturation $S_{ra}$ is set to 1, $S_w = 0.9$. The column is made of Massilon sandstone with the following properties:

- $K = 1.02 \cdot 10^9 N/m^2$, $G = 1.44 \cdot 10^9 N/m^2$, $\phi = 0.23$, $\rho_w = 2650 kg/m^3$, $\rho_a = 1.10 kg/m^3$, $K_\zeta = 3.55 \cdot 10^{10} N/m^2$, $K_w = 2.25 \cdot 10^9 N/m^2$, $K_a = 1.10 \cdot 10^5 N/m^2$, $\kappa = 2.5 \cdot 10^{-12} m^2$, $\eta_w = 1.0 \cdot 10^{-3} Ns/m^2$, $\eta_a = 1.8 \cdot 10^{-5} Ns/m^2$. A boundary-element grid constructed accounting to the two symmetry planes was used in the computations – one quarter of the grid comprised 56 elements.

Figure 1. Geometry and boundary conditions of a partially saturated poroelastic column

Figure 2. Displacement $u_z$ versus time at the top center of the column.
The following values of stepping scheme parameters are used for calculations: $R = 0.997$, $N = 800$, $\Delta t = 0.000437$. Considering the symmetry of the integrand function, the number of nodes for integration with respect to argument $\varphi$ for the traditional scheme on interval $[0; \pi]$ is 16000, for scheme with varied integration step and scheme with a key – 645 on interval $[0; 0.3]$, 55 on interval $[0.3; 0.45]$, 75 on interval $[0.45; 1]$ and 25 on interval $[1; \pi]$. Parameter $w$ is 0.5. The difference in displacements $u_z$ (Fig. 2) and tractions $t_z$ (Fig. 3) values calculated with the scheme using the varied step and with the traditional stepping scheme is not more than 6%. However, it requires the number, which is 20 times less than that of the integration nodes. Unwanted oscillations may be avoided if the scheme with a key is applied on the same set of integration points.

Conclusions
The mathematical model of the partially saturated porous media with five basic function is described, based on Biot’s model. The mathematical model in Laplace transform for 3d dynamic poroelastic boundary value problem is presented. The time-stepping scheme for numerical inversion of the Laplace transform is also presented. The description of the BEM modeling is given. The numerical example demonstrates the effectiveness of the joint use of the BEM, of the stepping scheme with the varied integration step and linear approximation of the integrand function in comparison with the traditional time-stepping scheme for dynamic poroelastic problems.

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