An analytic proof of the Malgrange–Sibuya theorem on the convergence of formal solutions of an ODE

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Abstract

We propose an analytic proof of the Malgrange–Sibuya theorem concerning a sufficient condition of the convergence of a formal power series satisfying an ordinary differential equation. The proof is based on the majorant method and allows to estimate the radius of convergence of such a series.

§1. Introduction

In the paper we study some properties of formal power series satisfying an ordinary differential equation

$$F(z, u, u', \ldots, u^{(n)}) = 0$$

of order $n$, where $F \not\equiv 0$ is a holomorphic (in some domain) function of $n + 2$ variables.

According to Maillet’s theorem [5], if a formal series $\hat{\phi} = \sum_{j=0}^{\infty} c_j z^j \in \mathbb{C}\langle\langle z \rangle\rangle$ satisfies the equation (1), where $F$ is a polynomial, then there is a real number $s \geq 0$ such that the power series $\sum_{j=0}^{\infty} (c_j / (j!)^s) z^j$ converges in some neighbourhood of zero. In this case one says that the formal series $\hat{\phi}$ has the Gevrey type of order $s$. Furthermore if there is no a real number $s' < s$ such that the series has the Gevrey type of order $s'$, then the order $s$ is called precise.

How small the number $s$ from Maillet’s theorem can be? And when one can guarantee the convergence of the formal solution $\hat{\phi}$ in a neighbourhood of zero, that is, when $s = 0$? Answers on these questions have been obtained in the papers of J.-P. Ramis, B. Malgrange, Y. Sibuya.

Let us first consider the case of a linear differential equation

$$b_n(z)u^{(n)} + b_{n-1}(z)u^{(n-1)} + \ldots + b_0(z)u = 0,$$

whose coefficients $b_i(z)$ are holomorphic functions in a neighbourhood of zero. The point $z = 0$ is said to be a Fuchsian (regular) singular point of this equation, if its coefficients satisfy the conditions

$$\text{ord}_0 b_{n-i}(z) + i \geq \text{ord}_0 b_n(z), \quad i = 1, \ldots, n.$$

The equation (2) can be written in an equivalent form

$$a_n(z)\delta^n u + a_{n-1}(z)\delta^{n-1} u + \ldots + a_0(z)u = 0,$$

where $\delta = z \frac{d}{dz}$ and the coefficients $a_i(z)$ are holomorphic functions in a neighbourhood of zero. With respect to such a form, the point $z = 0$ is a Fuchsian singular point if the inequalities

$$\text{ord}_0 a_i(z) \geq \text{ord}_0 a_n(z), \quad i = 0, \ldots, n - 1,$$

hold.

In the case when $z = 0$ is a Fuchsian singular point of the equation (2), any formal power series that satisfies this equation converges in a neighbourhood of zero. Indeed, using a standard substitution

$$y^1 = u, \quad y^2 = u', \quad \ldots, \quad y^n = u^{(n-1)},$$

the equation can be rewritten as

$$a_{n-1}(z)\delta^n y + a_{n-2}(z)\delta^{n-1} y + \ldots + a_0(z)y = 0.$$
one can pass to a linear differential system with a *regular* singular point \( z = 0 \), which is written in a matrix form:

\[
\frac{dy}{dz} = B(z) y, \quad y = (y^1, \ldots, y^n)^\top.
\]

This system, by means of a gauge transformation \( \tilde{y} = \Gamma(z) y \) with a meromorphic matrix \( \Gamma(z) \) at zero, is equivalent to a (Fuchsian) system

\[
\frac{d\tilde{y}}{dz} = A(z) \tilde{y}, \quad (4)
\]

where \( A \) is a constant \((n \times n)\)-matrix (see, for example [2, Ch. IV, §2]). But any formal Laurent series \( \tilde{y} = \sum_{j \geq -N} c_j z^j, \) \( c_j \in \mathbb{C}^n \), satisfying the system \( (4) \) is in fact a Laurent polynomial, since relations

\[
A c_j = j c_j, \quad j \geq -N,
\]

can hold only for a finite number of non-zero vectors \( c_j \).

In the case when the singular point \( z = 0 \) is *irregular* (i. e., is not a regular singular one), J.-P. Ramis [7] has suggested the following method for estimating the Gevrey order of a formal power series solution. To the linear differential operator

\[
L = a_n(z) \delta^n + a_{n-1}(z) \delta^{n-1} + \ldots + a_0(z),
\]

which corresponds to the equation \( (3) \), one attaches its Newton polygon \( N(L) \), the boundary curve of the smallest convex set containing the union of the sets

\[
X_i = \{(x, y) \in \mathbb{R}^2 | x \leq i, y \geq \text{ord}_0 a_i(z)\}, \quad i = 0, 1, \ldots, n.
\]

Let us note that in the Fuchsian case the Newton polygon consists of two line segments, a horizontal one and a vertical one (see Pic. 1). But if \( z = 0 \) is an irregular singular point of the equation \( (3) \), then the Newton polygon contains line segments with positive slopes. Let \( 0 < r_1 < \ldots < r_m < \infty \) be all of such slopes \((m \leq n, \) see Pic. 2). Then, as Ramis’s theorem asserts, any formal power series satisfying the equation \( (3) \) has the Gevrey type of precise order \( s \in \{0, 1/r_1, \ldots, 1/r_m\} \).

As a generalization of Ramis’s theorem one regards the following theorem for a non-linear differential equation

\[
F(z, u, \delta u, \ldots, \delta^nu) = 0, \quad (5)
\]

where \( F(z, y_0, y_1, \ldots, y_n) \) is a holomorphic function in a neighbourhood of \( 0 \in \mathbb{C}^{n+2} \).

**Theorem 1** (B. Malgrange [6]). Let a formal series \( \tilde{\phi} \in \mathbb{C}[[z]], \tilde{\phi}(0) = 0 \), satisfy the equation \( (5) \) (that is, \( F(z, \Phi) = F(z, \tilde{\phi}, \delta \tilde{\phi}, \ldots, \delta^n \tilde{\phi}) = 0 \)) and \( \frac{\partial F}{\partial y_n}(z, \Phi) \neq 0 \). Then

a) if \( z = 0 \) is a Fuchsian singular point of the operator \( L_{\tilde{\phi}} = \sum_{i=0}^n \frac{\partial F}{\partial y_i}(z, \Phi) \delta^i \), then the series \( \tilde{\phi} \) converges in a neighbourhood of zero;

b) if \( z = 0 \) is an irregular singular point of the operator \( L_{\tilde{\phi}} \), and \( r \) is the smallest of positive slopes of the Newton polygon \( N(L_{\tilde{\phi}}) \), then the formal power series \( \tilde{\phi} \) has the Gevrey type of order \( s = 1/r \).

The part b) of Malgrange’s theorem has been precised by Y. Sibuya [9, App. 2] as follows: the formal power series \( \tilde{\phi} \) has the Gevrey type of the precise order \( s \in \{0, 1/r_1, \ldots, 1/r_m\} \), where \( 0 < r_1 < \ldots < r_m < \infty \) are all of positive slopes of the Newton polygon \( N(L_{\tilde{\phi}}) \).
The Malgrange–Sibuya theorem (Theorem 1a) on a sufficient condition of the convergence of a formal solution of an ordinary differential equation has been proved by these authors in the different ways. First the equation (5) is transformed to some special form, then Malgrange uses the theorem on an implicit mapping for Banach spaces, while Sibuya applies the fundamental Ramis–Sibuya theorem \[8\] on asymptotic expansions. We expose Malgrange’s proof in details in the next section, then in the Section 3 we give an analytic proof which allows to estimate the radius of convergence of the power series \(\hat{\varphi}\) (Malgrange’s and Sibuya’s theorems do not contain estimates for the radius of convergence). An idea of our proof is based on the construction of a majorant equation and was already appeared in the article \[1, Ch. 1, \S 7\].

\section{2. Malgrange’s proof}

For each natural \(k\) the formal power series \(\hat{\varphi}\) may be represented in the form

\[ \hat{\varphi} = \varphi_k + z^k \hat{\psi}, \quad \hat{\psi}(0) = 0. \]

**Lemma 1.** For a sufficiently large \(k\) (under the assumptions of Theorem 1a), the formal power series \(\hat{\psi}\) satisfies the relation

\[ \overline{L}(\delta + k)\hat{\psi} = zM(z, \hat{\psi}, \delta\hat{\psi}, \ldots, \delta^n \hat{\psi}), \]

where \(\overline{L}\) is a polynomial of degree \(n\), \(M\) is a holomorphic function in a neighbourhood of \(0 \in \mathbb{C}^{n+2}\).

**Proof.** Taking into consideration the equality \(\delta(z^k \hat{\psi}) = z^k(\delta + k)\hat{\psi}\) we have the relations

\[ \delta^i(z^k \hat{\psi}) = z^k(\delta + k)^i \hat{\psi}, \quad i = 1, \ldots, n, \]

therefore,

\[ \Phi = (\varphi_k, \delta \varphi_k, \ldots, \delta^n \varphi_k) + z^k (\hat{\psi}, (\delta + k)\hat{\psi}, \ldots, (\delta + k)^n \hat{\psi}) = \Phi_k + z^k \Psi. \]

Further applying Taylor’s formula we obtain

\[ 0 = F(z, \Phi_k + z^k \Psi) = F(z, \Phi_k) + z^k \sum_{i=0}^{n} \frac{\partial F}{\partial y_i}(z, \Phi_k)(\delta + k)^i \hat{\psi} + \]

\[ + z^{2k} \sum_{p,q=0}^{n} H_{pq}(z, \Phi_k, z^k \Psi)(\delta + k)^p \hat{\psi}^p (\delta + k)^q \hat{\psi}, \quad (6) \]
where $H_{pq}$ are holomorphic functions in a neighbourhood of $0 \in \mathbb{C}^{2n+3}$.

Let $l = \text{ord}_0 \frac{\partial F}{\partial y_i}(z, \Phi)$. Then by the assumption of Theorem 1a,

$$\text{ord}_0 \frac{\partial F}{\partial y_i}(z, \Phi) \geq l, \quad i = 0, 1, \ldots, n.$$ 

Let $b_iz^l$ be a summand that the formal power series $\frac{\partial F}{\partial y_i}(z, \Phi)$ begins with:

$$\frac{\partial F}{\partial y_i}(z, \Phi) = b_iz^l + \ldots, \quad i = 0, 1, \ldots, n \quad (b_i \in \mathbb{C}, b_n \neq 0).$$

Define a polynomial

$$L(\xi) = \sum_{i=0}^{n} b_i \xi^i$$  

(7)

of degree $n$ and choose a number $k_0$ such that for any natural $k > k_0$ the inequality $L(k) \neq 0$ holds. Now we show that the number $k$ from the statement of the lemma can be taken as $k = \max(k_0, l + 1)$.

Let us note that

$$\text{ord}_0 \left( \frac{\partial F}{\partial y_i}(z, \Phi) - \frac{\partial F}{\partial y_i}(z, \Phi_k) \right) \geq l + 1, \quad i = 0, 1, \ldots, n$$

(this follows from Taylor’s formula applied to this difference), therefore, $\frac{\partial F}{\partial y_i}(z, \Phi_k) = b_iz^l + o(z^l)$ in a neighbourhood of zero. From the relations (6) and $\hat{\psi}(0) = 0$ it follows that

$$\text{ord}_0 F(z, \Phi_k) \geq k + l + 1.$$ 

Hence the relation (6) can be divided by $z^{k+l}$, and we obtain the equality of the form

$$L(\delta + k)\hat{\psi} - zM(z, \hat{\psi}, \delta\hat{\psi}, \ldots, \delta^n\hat{\psi}) = 0,$$

where the polynomial $L$ is defined by the formula (7), $M$ is a holomorphic function in a neigh- 
bourhood of $0 \in \mathbb{C}^{n+2}$.  \(\square\)

Thus the formal power series $\hat{\psi} = \sum_{j=1}^{\infty} c_jz^j$ is a solution of the ordinary differential equation

$$L(\delta + k)v = zM(z, v, \delta v, \ldots, \delta^n v).$$  

(8)

Let us prove the convergence of this series in some neighbourhood of zero using the theorem on an implicit mapping for Banach spaces. We recall a part of this theorem that we need here (see, for example, [3 Th. 10.2.1]).

Let $E, F, G$ be Banach spaces, $A$ an open subset of the direct product $E \times F$, $f : A \rightarrow G$ a continuously differentiable mapping. Consider a point $(x_0, y_0) \in A$ such that $f(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0)$ is a bijective linear mapping from $F$ to $G$.

Then there is a neighbourhood $U_0 \subset E$ of the point $x_0$ and a unique continuous mapping $u : U_0 \rightarrow F$ such that $u(x_0) = y_0$, $(x, u(x)) \in A$ and $f(x, u(x)) = 0$ for any $x \in U_0$. 4
Note that in view of the condition $\bar{L}(j + k) \neq 0$, $j = 1, 2, \ldots$, the coefficient $c_1$ and all the other coefficients of the power series $\hat{\psi}$ are uniquely determined (every coefficient is expressed via the previous ones). Thus, one has a unique power series that satisfies the equation (8).

For each $m = 0, 1, \ldots, n$ let us define a Banach space

$$H^m = \left\{ \psi = \sum_{j \geq 1} a_j z^j : \sum_{j \geq 1} j^m |a_j| < \infty \right\}$$

with the corresponding norm $\|\psi\|_m = \sum_{j \geq 1} j^m |a_j| = \|\delta^m \psi\|_0$. It is not difficult to see that $H^m \subset H^{m-1}$, $m = 1, 2, \ldots, n$, and all the spaces $H^m$ are contained in the space of functions holomorphic in the open unit disk $D_1$ and contain the space of functions holomorphic in the closed unit disk $\overline{D}_1$:

$$O(D_1) \subset H^n \subset \ldots \subset H^1 \subset H^0 \subset O(D_1).$$

Let the function $M(z, y_0, y_1, \ldots, y_n)$ in the equation (8) be holomorphic in a polydisk $\{|z| < \varepsilon, |y_0| < \varepsilon, \ldots, |y_n| < \varepsilon\}$. In the product $\mathbb{C} \times H^n$ of the Banach spaces we consider an open subset

$$A = \{ (\lambda, \psi) : |\lambda| < \varepsilon, \|\psi\|_n < \varepsilon \}$$

and define a continuously differentiable mapping

$$f : (\lambda, \psi) \mapsto \bar{L}(\delta + k) \psi - \lambda z M(\lambda z, \psi, \delta \psi, \ldots, \delta^n \psi)$$

from $A$ to $H^0$, with $f(0, 0) = 0$. Let us show that the linear operator

$$\frac{\partial f}{\partial \psi}(0, 0) = \bar{L}(\delta + k) : H^n \rightarrow H^0$$

is bijective. Indeed,

$$\bar{L}(\delta + k) a_j z^j = a_j \bar{L}(j + k) z^j = 0 \iff a_j = 0,$$

therefore, $\ker \bar{L}(\delta + k) = \{0\}$. In the same time, if $\sum_{j=1}^{\infty} a_j z^j \in H^0$, then $\sum_{j=1}^{\infty} (a_j/\bar{L}(j+k)) z^j \in H^n$, that is, the image of the operator $\bar{L}(\delta + k)$ coincides with $H^0$.

Thus, the mapping $f$ satisfies the theorem on an implicit mapping, hence there are a real number $\nu > 0$ and a function $\psi_\nu \in H^n \subset O(D_1)$ such that

$$\bar{L}(\delta + k) \psi_\nu(z) - \nu z M(\nu z, \psi_\nu(z), \delta \psi_\nu(z), \ldots, \delta^n \psi_\nu(z)) = 0.$$ 

But then the function $\psi_\nu(z/\nu) \in O(D_\nu)$ is a solution of the equation (8), and the power series $\hat{\psi}$ converges in the disk $D_\nu$ of radius $\nu$. (Here we use the relations $(\delta^p \psi_\nu)(z/\nu) = \delta^p(\psi_\nu(z/\nu))$, $p = 1, 2, \ldots, n$.)

Let us note that in the proof above it is essentially that $\bar{L}$ is a polynomial of degree $n$. If the degree would be less than $n$ (this is the case, when $z = 0$ is an irregular singular point of the operator $L_\varphi$), for example, would be equal to $n - 1$, then the linear operator $\bar{L}(\delta + k) : H^n \rightarrow H^0$ is not surjective, since its image coincides with $H^1$, and we cannot apply the theorem on an implicit mapping.

§3. Proof by the majorant method. An estimate for the radius of convergence

Here we construct a differential equation majorant for the equation (8), in the sense that it will have a unique solution $\psi = \sum_{j=1}^{\infty} C_j z^j$ ($C_j \geq 0$) holomorphic in a neighbourhood of zero,
and its power series will be majorant for the formal power series $\hat{\psi}$. Besides that, we will obtain an estimate for the radius of convergence of the power series $\psi$ and, hence, power series $\hat{\psi}$.

Let us write the function $M(z, y_0, y_1, \ldots, y_n)$ from the right hand side of the equation (5) as a power series converging in some neighbourhood of $0 \in \mathbb{C}^{n+2}$:

$$M(z, y_0, y_1, \ldots, y_n) = \sum_{p=0}^{\infty} \sum_{q=(q_0, q_1, \ldots, q_n) \in \mathbb{Z}_{+1}^{n+1}} \alpha_{p, q} z^p y_0^{q_0} y_1^{q_1} \cdots y_n^{q_n}, \quad \alpha_{p, q} \in \mathbb{C}.$$ 

As we noted before, each coefficient $c_j$ of the formal power series $\hat{\psi} = \sum_{j=1}^{\infty} c_j z^j$ is uniquely determined by the previous ones. Now we find exact expressions for these coefficients using the relation

$$\mathcal{L}(\delta + k) \hat{\psi} = z M(z, \hat{\psi}, \delta \hat{\psi}, \ldots, \delta^n \hat{\psi}).$$

Denoting by $\hat{F}(z)$ the formal power series $M(z, \hat{\psi}, \delta \hat{\psi}, \ldots, \delta^n \hat{\psi})$ one has

$$\mathcal{L}(j + k) c_j = \frac{\hat{F}^{(j-1)}(0)}{(j-1)!}, \quad j = 1, 2, \ldots.$$ 

In order to express $\hat{F}^{(j-1)}(0)$ we use a formula for the derivative of a product:

$$(f_1 \cdots f_q)^{(m)} = \sum_{m_1 + \ldots + m_q = m} \frac{m!}{m_1! \cdots m_q!} f_1^{(m_1)} \cdots f_q^{(m_q)}, \quad f_i \in \mathbb{C}[[z]], \quad m = 1, 2, \ldots.$$ 

For $y_i = \delta^i \hat{\psi} = \sum_{j=1}^{\infty} j^i c_j z^j$ ($i = 0, 1, \ldots, n$) we obtain for each $q \leq m$:

$$(y_i^q)^{(m)}(0) = \sum_{m_1 + \ldots + m_q = m} \frac{m!}{m_1! \cdots m_q!} \frac{y_1^{(m_1)}(0)}{m_1!} \cdots \frac{y_q^{(m_q)}(0)}{m_q!} = \sum_{m_1 + \ldots + m_q = m} m!(m_1 \ldots m_q)^i c_{m_1} \cdots c_{m_q}$$

$$(= 0, \text{ if } q > m), \text{ and also}$$

$$(z^p y_0^{q_0} y_1^{q_1} \cdots y_n^{q_n})^{(j-1)}(0) = \sum_{p+q_0+\ldots+q_n = j-1} \frac{(j-1)!}{j_0! j_1! \cdots j_n!} \frac{(y_0^{(q_0)}(j_0))}{j_0!} \frac{(y_1^{(q_1)}(j_1))}{j_1!} \cdots \frac{(y_n^{(q_n)}(j_n))}{j_n!} \bigg|_{z=0}.$$

Therefore,

$$\hat{F}^{(j-1)}(0) = \sum_{p+q_0+\ldots+q_n = j-1} \frac{\alpha_{p, q}}{p+q_0+\ldots+q_n} \sum_{j_0+\ldots+j_n = j-1-p} \frac{(y_0^{(q_0)}(j_0))}{j_0!} \frac{(y_1^{(q_1)}(j_1))}{j_1!} \cdots \frac{(y_n^{(q_n)}(j_n))}{j_n!} \bigg|_{z=0}, \quad (9)$$

where

$$\frac{(y_i^q)^{(j_i)}(0)}{j_i!} = \sum_{m_1 + \ldots + m_{q_i} = j_i} (m_1 \ldots m_{q_i})^i c_{m_1} \cdots c_{m_{q_i}}, \quad i = 0, 1, \ldots, n. \quad (10)$$

Thus,

$$\mathcal{L}(1+k)c_1 = \alpha_{0,0}, \quad \mathcal{L}(j+k)c_j = P_j(c_1, \ldots, c_{j-1}, \{\alpha_{p,q}\}), \quad j = 2, 3, \ldots, \quad (11)$$

where $P_j$ is a polynomial with positive coefficients that is restored by the formulas (9), (10).
Now we consider an equation majorant, as will be shown further, for the differential equation (8):

$$\sigma \delta^n v = z \tilde{M}(z, \delta^n v), \quad \sigma > 0,$$  

(12)

where

$$\tilde{M}(z, w) = \sum_{p=0}^{\infty} \sum_{q=(q_0, q_1, \ldots, q_n) \in \mathbb{Z}^{n+1}} |\alpha_{p,q}| z^p w^{q_0} w^{q_1} \ldots w^{q_n}$$

is a holomorphic function in a neighbourhood of the point \((0, 0) \in \mathbb{C}^2\), and its power series expansion is produced from the power series expansion of the function \(M(z, y_0, y_1, \ldots, y_n)\) by changing all the coefficients \(\alpha_{p,q}\) to their absolute values and the variables \(y_0, y_1, \ldots, y_n\) to one variable \(w\). The value \(\sigma\) is defined by the formula

$$\sigma = \inf_{j \in \mathbb{N}} \frac{|L(j + k)|}{j^n}.$$

This value is positive, as \(L(j + k) \neq 0\) for \(j \in \mathbb{N}\) and \(\lim_{j \to \infty} |L(j + k)|/j^n = |b_n| > 0\) (recall that \(L(\xi) = \sum_{i=0}^{n} b_i \xi^i\)).

**Lemma 2.** The differential equation (12) has a unique solution of the form \(\psi = \sum_{j=1}^{\infty} C_j z^j\) (i.e., \(\psi(0) = 0\)) holomorphic in a neighbourhood of zero, and its power series is majorant for the power series \(\hat{\psi} = \sum_{j=1}^{\infty} C_j z^j\) satisfying the equation (8).

**Proof.** Existence and uniqueness of a solution \(\psi = \sum_{j=1}^{\infty} C_j z^j\) of the differential equation (12) follows from existence and uniqueness of a holomorphic in a neighbourhood of zero solution \(w = w(z) = \sum_{j=1}^{\infty} a_j z^j\) \((w(0) = 0)\) of the equation \(\sigma w = z \tilde{M}(z, w)\), as the latter satisfies the assumptions of Cauchy’s theorem on an implicit function at the point \((0, 0) \in \mathbb{C}^2\) (then \(C_j = a_j/j^n\)).

Let us show that the power series \(\psi = \sum_{j=1}^{\infty} C_j z^j\) is majorant for the power series \(\hat{\psi} = \sum_{j=1}^{\infty} C_j z^j\). We will find the coefficients \(C_j\) using the relation

$$\sigma \delta^n \psi = z \tilde{M}(z, \delta^n \psi).$$

Denoting by \(\tilde{F}(z)\) the function \(\tilde{M}(z, \delta^n \psi)\) holomorphic in a neighbourhood of zero, one has

$$\sigma j^n C_j = \frac{\tilde{F}(j-1)(0)}{(j-1)!}, \quad j = 1, 2, \ldots.$$

Keeping in mind the difference between \(\tilde{F}(z)\) and \(\hat{F}(z)\), similarly to the formulas (9), (10) for \(\frac{\tilde{F}(j-1)(0)}{(j-1)!}\) we obtain

$$\frac{\tilde{F}(j-1)(0)}{(j-1)!} = \sum_{p+q_0+\ldots+q_n = j-1} |\alpha_{p,q}| \sum_{j_0+\ldots+j_{n-1}+1-p = j} \frac{(w^{q_0})_{j_0}}{j_0!} \frac{(w^{q_1})_{j_1}}{j_1!} \ldots \frac{(w^{q_n})_{j_n}}{j_n!} \bigg|_{z=0},$$  

(13)

where

$$\frac{(w^{q_i})_{j_i}}{j_i!} = \sum_{m_1+\ldots+m_{q_i} = j_i} (m_1 \ldots m_{q_i})^n C_{m_1} \ldots C_{m_{q_i}}, \quad i = 0, 1, \ldots, n.$$

(14)
Hence,
\[ \sigma C_1 = |\alpha_{0,0}| \in \mathbb{R}_+, \quad \sigma j^n C_j = \tilde{P}_j(C_1, \ldots, C_{j-1}, \{ |\alpha_{p,q}| \}) \in \mathbb{R}_+, \quad j = 2, 3, \ldots, \]
where \( \tilde{P}_j \) is a polynomial with positive coefficients that is restored by the formulas \((13), (14)\). Let us note that by the construction one has
\[ \tilde{P}_j(C_1, \ldots, C_{j-1}, \{ |\alpha_{p,q}| \}) \geq P_j(C_1, \ldots, C_{j-1}, \{ |\alpha_{p,q}| \}) \]
(compare the formulas \((10)\) and \((14)\)).

Using the obtained recurrence relations \((11)\) and \((15)\) for the coefficients \( c_j \) and \( C_j \) we see that
\[ |\mathcal{L}(1+k)||c_1| = |\alpha_{0,0}| = \sigma C_1 \implies |c_1| = \frac{\sigma}{|\mathcal{L}(1+k)|} C_1 \leq C_1, \]
and finish the proof by the induction (the second inequality below):
\[ |\mathcal{L}(j+k)||c_j| = |P_j(c_1, \ldots, c_{j-1}, \{ |\alpha_{p,q}| \})| \leq P_j(|c_1|, \ldots, |c_{j-1}|, \{ |\alpha_{p,q}| \}) \leq P_j(C_1, \ldots, C_{j-1}, \{ |\alpha_{p,q}| \}) \leq \tilde{P}_j(C_1, \ldots, C_{j-1}, \{ |\alpha_{p,q}| \}) = \sigma j^n C_j \implies |c_j| \leq \frac{\sigma j^n}{|\mathcal{L}(j+k)|} C_j \leq C_j, \quad j = 2, 3, \ldots. \]
The proof is finished.

**Proposition 1.** Let the function \( M(z, y_0, y_1, \ldots, y_N) \) from the right hand side of the equation \((8)\) be holomorphic in a neighbourhood of a closed polydisk
\[ \Delta = \{ |z| \leq r, |y_0| \leq \rho, \ldots, |y_N| \leq \rho \}, \quad \mu = \max \Delta |M|. \]

Then the power series \( \hat{\psi} = \sum_{j=1}^{\infty} c_j z^j \) satisfying the equation \((8)\) converges in the disk
\[ D_R = \left\{ |z| < r \frac{\rho}{\rho + \mu r / \sigma N} \right\}, \quad N = (n + 1)^{n+1}/(n + 2)^{n+2}. \]

**Proof.** It is enough to check that the power series \( \psi = \sum_{j=1}^{\infty} C_j z^j \), satisfying the equation \((12)\) and majorant for \( \hat{\psi} \), converges in the disk \( D_R \). It will follow from the convergence of the power series \( \delta^n \psi = \sum_{j=1}^{\infty} j^n C_j z^j \), which is a solution of the equation \( \sigma w = z \tilde{M}(z, w) \). Here we apply the majorant method from the proof of the theorem on an implicit function (see, for example, \[4\] Ch. IX, §193).

As
\[ \tilde{M}(z, w) = \sum_{p=0}^{\infty} \sum_{q=(q_0, \ldots, q_n) \in \mathbb{Z}^{n+1}_+} |\alpha_{p,q}| z^p w^{q_0} w^{q_1} \ldots w^{q_n}, \quad |\alpha_{p,q}| \leq \frac{\mu}{r^p \rho^{q_0} \rho^{q_1} \ldots \rho^{q_n}}, \]
the power series \( \delta^n \psi \) satisfying the equation \( \sigma w = z \tilde{M}(z, w) \) converges apriori in a disk where the solution (equal to zero for \( z = 0 \)) of the equation
\[ \sigma w = z \mu \sum_{p=0}^{\infty} \sum_{q=(q_0, \ldots, q_n) \in \mathbb{Z}^{n+1}_+} \left( \frac{z}{r} \right)^p \left( \frac{w}{\rho} \right)^{q_0} \left( \frac{w}{\rho} \right)^{q_1} \ldots \left( \frac{w}{\rho} \right)^{q_n} = \frac{\mu z}{(1-z/r)(1-w/\rho)^{n+1}} \]
is holomorphic. Let us rewrite the last equation in the form

$$f(z, w) = \sigma w (1 - w/\rho)^{n+1} - \frac{\mu z}{1 - z/r} = 0.$$ 

By the theorem on an implicit function, its solution admits a single valued branch that is equal to zero for $z = 0$ and holomorphic in a disk $\{|z| < |z_0|\}$, where $z_0$ is determined from the system of the equations $f(z, w) = 0$, $\frac{\partial f(z, w)}{\partial w} = 0$. A unique solution $w_0 = \rho/(n + 2)$, $z_0 = r \frac{\rho - \mu r/\sigma N}{\rho + \mu r/\sigma N}$ of this system leads to an estimate for the radius of convergence of the power series $\delta^n \psi$, hence of $\psi$ and $\hat{\psi}$ as well. □

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