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On simulating a medium with special reflecting properties
by Lobachevsky geometry
(One exactly solvable electromagnetic problem)

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Lobachevsky geometry simulates a medium with special constitutive relations, $D^i = \epsilon_0 \epsilon^{ik} E^k$, $B^i = \mu_0 \mu^{ik} H^k$, where two matrices coincide: $\epsilon^{ik}(x) = \mu^{ik}(x)$. The situation is specified in quasi-cartesian coordinates $(x, y, z)$. Exact solutions of the Maxwell equations in complex 3-vector $E + iB$ form, extended to curved space models within the tetrad formalism, have been found in Lobachevsky space. The problem reduces to a second order differential equation which can be associated with an 1-dimensional Schrödinger problem for a particle in external potential field $U(z) = U_0 e^{2z}$. In quantum mechanics, curved geometry acts as an effective potential barrier with reflection coefficient $R = 1$; in electrodynamic context results similar to quantum-mechanical ones arise: the Lobachevsky geometry simulates a medium that effectively acts as an ideal mirror. Penetration of the electromagnetic field into the effective medium, depends on the parameters of an electromagnetic wave, frequency $\omega$, $k_1^2 + k_2^2$, and the curvature radius $\rho$.

1 Introduction

An aim of the present paper is to obtain exact solutions of the Maxwell equations in 3-dimensional Lobachevsky space $H_3$. A coordinate system used is one from the list given by Olevsky [1], which generalizes Cartesian coordinate in flat Euclidean space.

To treat Maxwell equations we make use of complex representation of these according to the known approach by Riemann–Silberstein–Oppenheimer–Majorana [2, 3, 4, 5] (see also in [6 – 30]), extended to curved space-time models in the frames of tetrad formalism of Tetrode–Weyl–Fock–Ivanenko [31, 32, 33]; see also in [34]). On the base of this technique, new exact solutions of the type of extended plane wave in Lobachevsky space have been constructed explicitly. These may be interesting in the cosmological sense; besides, they may be interesting in the context of geometric simulating electromagnetic field in a special medium [35], [34].

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2 Cartezian coordinates in Lobachevsky space

In Olevsky paper [1], under the number 2 the following coordinate system in Lobachevsky space \(H_3\) is specified

\[ x^a = (t, x, y, z) , \quad dS^2 = dt^2 - e^{-2z}(dx^2 + dy^2) - dz^2 , \quad (1) \]

the element of volume is given by

\[ dV = \sqrt{-g} \, dx \, dy \, dz = e^{-2z} \, dx \, dy \, dz , \quad x, y, z \in (-\infty, +\infty) ; \]

the magnitude and sign of the \(z\) are substantial, in particular, when dealing with localization, for example, the energy of the field

\[ dW = \frac{1}{2} (E^2 + B^2) dV = \frac{1}{2} (E^2 + B^2) \, e^{-2z} \, dx \, dy \, dz . \quad (2) \]

It is helpful to have at hand some detail of the parametrization of the model \(H_3\) by \(x, y, z\).

It is known that this model can be identified with a branch of hyperboloid in 4-dimension flat space

\[ u_0^2 - u_1^2 - u_2^2 - u_3^2 = \rho^2 , \quad u_0 = +\sqrt{\rho^2 + u_1^2} . \quad (3) \]

Coordinate in use, \(x, y, z\), are referred to \(u_a\) by relations

\[ u_1 = xe^{-z} , \quad u_2 = ye^{-z} , \]

\[ u_3 = \frac{1}{2} [(e^z - e^{-z}) + (x^2 + y^2)e^{-z}] , \]

\[ u_0 = \frac{1}{2} [(e^z + e^{-z}) + (x^2 + y^2)e^{-z}] . \quad (3) \]

It is convenient to employ 3-dimensional Poincaré realization for Lobachevsky space as inside part of 3-sphere

\[ q_i = \frac{u_i}{u_0} = \frac{u_i}{\sqrt{\rho^2 + u_1^2 + u_2^2 + u_3^2}} , \quad q_i q_i < +1 . \quad (4) \]

Quasi-Cartesian coordinates \((x, y, z)\) are referred to \(q_i\) as follows

\[ q_1 = \frac{2x}{x^2 + y^2 + e^{2z} + 1} , \]

\[ q_2 = \frac{2y}{x^2 + y^2 + e^{2z} + 1} , \]

\[ q_3 = \frac{x^2 + y^2 + e^{2z} - 1}{x^2 + y^2 + e^{2z} + 1} ; \quad (5) \]

Inverses to (5) relations are

\[ x = \frac{q_1}{1 - q_3} , \quad y = \frac{q_2}{1 - q_3} , \quad e^z = \frac{\sqrt{1 - q_2^2}}{1 - q_3} . \quad (6) \]
In particular, note that on the axis \( q_1 = 0, q_2 = 0, q \in (-1, +1) \) relations (6) assume the form

\[
x = 0, \quad y = 0, \quad e^z = \sqrt{\frac{1 + q_3}{1 - q_3}}.
\]

that is

\[
q_3 \rightarrow +1, \quad e^z \rightarrow +\infty, \quad z \rightarrow +\infty;
\]

\[
q_3 \rightarrow -1, \quad e^z \rightarrow +0, \quad z \rightarrow -\infty.
\]

Solutions of the Maxwell equation, constructed below, can be of interest in the context of description of electromagnetic waves in special media, because the Lobachevsky geometry simulates effectively a definite special medium [36], inhomogeneous along the axis \( z \). Effective electric permittivity tensor \( \epsilon^{ik}(x) \) is given by

\[
\epsilon^{ik}(x) = -\sqrt{-g} g^{00}(x) g^{ik}(x) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-2z} \end{vmatrix},
\]

whereas the corresponding effective magnetic permittivity tensor is

\[
(\mu^{-1})^{ik}(x) = \sqrt{-g} \begin{vmatrix} g^{22} g^{33} & 0 & 0 \\ 0 & g^{33} g^{11} & 0 \\ 0 & 0 & g^{11} g^{22} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{2z} \end{vmatrix}.
\]

In explicit form, effective constitutive relations (the system SI is used) are

\[
D^i = \epsilon_0 e^{ik} E_k, \quad B_i = \mu_0 \mu^{ik} H_k,
\]

note that two matrices coincide: \( e^{ik}(x) = \mu^{ik}(x) \).

3 Tetrads and Maxwell equations in complex form

In the coordinate (11), let us introduce a tetrad

\[
\epsilon_{(a)}^\beta = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & e^2 & 0 & 0 \\ 0 & 0 & e^2 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad \epsilon_{(a)}^{(\beta)} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -e^{-z} & 0 & 0 \\ 0 & 0 & -e^{-z} & 0 \\ 0 & 0 & 0 & -1 \end{vmatrix}.
\]

One should find Christoffel symbols; some of them evidently vanish: \( \Gamma^0_{\beta\sigma} = 0, \Gamma^i_{00} = 0, \Gamma^i_{0j} = 0 \), remaining ones are determined by relations

\[
\Gamma^x_{jk} = \begin{vmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{vmatrix}, \quad \Gamma^y_{jk} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{vmatrix}, \quad \Gamma^z_{jk} = \begin{vmatrix} e^{-2z} & 0 & 0 \\ 0 & e^{-2z} & 0 \\ 0 & 0 & 0 \end{vmatrix}.
\]
Ricci rotation coefficients are (only not vanishing ones are written down)

$$\gamma_{311} = -1, \quad \gamma_{232} = 1.$$  

Using the notation [34]

$$e^\rho_{(0)} \partial_\rho = \partial_0 = \partial_t, \quad e^\rho_{(1)} \partial_\rho = \partial_1 = e^x \partial_x,$$

$$e^\rho_{(2)} \partial_\rho = \partial_2 = e^y \partial_y, \quad e^\rho_{(3)} \partial_\rho = \partial_3 = e_z \partial_z,$$

$$v_0 = (\gamma_{010}, \gamma_{020}, \gamma_{030}) \equiv 0, \quad v_1 = (\gamma_{011}, \gamma_{021}, \gamma_{031}) \equiv 0,$$

$$v_2 = (\gamma_{012}, \gamma_{022}, \gamma_{032}) \equiv 0, \quad v_3 = (\gamma_{013}, \gamma_{023}, \gamma_{033}) \equiv 0,$$

$$p_0 = (\gamma_{230}, \gamma_{310}, \gamma_{120}) = 0, \quad p_1 = (\gamma_{231}, \gamma_{311}, \gamma_{121}) = (0, -1, 0),$$

$$p_2 = (\gamma_{232}, \gamma_{312}, \gamma_{122}) = (1, 0, 0), \quad p_3 = (\gamma_{233}, \gamma_{313}, \gamma_{123}) = 0;$$

the Maxwell equations in the complex matrix form [34] read

$$\begin{bmatrix} \alpha^k \partial_k + sv_0 + \alpha^k sp_k - i(\partial_0 + sp_0 - \alpha^k sv_k) \end{bmatrix} \begin{bmatrix} 0 \\ E + iB \end{bmatrix} = 0; \quad (12)$$

in the used retraed it assumes the form

$$\begin{bmatrix} -i\partial_t + \alpha^1 e^x \partial_x + \alpha^2 e^y \partial_y + \alpha^3 \partial_z - \alpha^1 s_2 + \alpha^2 s_1 \end{bmatrix} \begin{bmatrix} 0 \\ E + iB \end{bmatrix} = 0. \quad (13)$$

Matrices involved in [13] are

$$\alpha^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \alpha^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix},$$

$$\alpha^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, \quad s^1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad s^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$  

4 Separation of the variables

Let us use the substitution

$$\begin{bmatrix} 0 \\ E + iB \end{bmatrix} = e^{-i\omega t} e^{ik_1 x} e^{ik_2 y} \begin{bmatrix} 0 \\ f(z) \end{bmatrix}. \quad (14)$$

correspondingly eq. [14] gives

$$\begin{bmatrix} -\omega + \alpha^1 e^x ik_1 + \alpha^2 e^y ik_2 + \alpha^3 \frac{d}{dz} - \alpha^1 s_2 + \alpha^2 s_1 \end{bmatrix} \begin{bmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \end{bmatrix} = 0. \quad (15)$$
After simple calculation, we derive a first order system for \( f_i \):

\[
ik_1 e^z f_1 + ik_2 e^z f_2 + \left( \frac{d}{dz} - 2 \right) f_3 = 0 , \\
-\omega f_1 - \left( \frac{d}{dz} - 1 \right) f_2 + ik_2 e^z f_3 = 0 , \\
-\omega f_2 + \left( \frac{d}{dz} - 1 \right) f_1 - ik_1 e^z f_3 = 0 , \\
-\omega f_3 - e^z ik_2 f_1 + ik_1 e^z f_2 = 0 .
\] (16)

Allowing three last equations in the first one, we get an identity 0 = 0. So, there exist only three independent equations (below the notation \( k_1 = a, k_2 = b \) is used):

\[
\omega f_3 = -ib e^z f_1 + ia e^z f_2 , \\
\omega f_1 = -(\frac{d}{dz} - 1) f_2 + ib e^z f_3 , \\
\omega f_2 = +(\frac{d}{dz} - 1) f_1 - ia e^z f_3 , 
\] (17)

With substitutions \( f_1 = e^z F_1(z) \), \( f_2 = e^z F_2(z) \), eqs. (17) give

\[
\omega F_3 = -ib e^z F_1 + ia e^z F_2 , \\
\omega F_1 = -\frac{d}{dz} F_2 + ib f_3 , \\
\omega F_2 = +\frac{d}{dz} F_1 - ia f_3 . 
\] (18)

There exist a particular case readily treatable, when \( a = 0, b = 0, f_3 = 0 \):

\[
\omega F_1 = -\frac{d}{dz} F_2 , \quad \omega F_2 = +\frac{d}{dz} F_1 \implies \\
F_1(z) = e^{\pm i\omega z} , \quad F_2 = \pm i e^{\pm i\omega z} , 
\] (19)

which gives

\[
\Phi^\pm = \begin{vmatrix} 0 & 0 \\ E + iB & e^{\pm i\omega z} \end{vmatrix} = e^{\pm i\omega z} \\
E - iB & \pm i e^{\pm i\omega z} 
\] (20)

or (let it be \( \varphi(\pm) = \omega t \mp \omega z \))

\[
E_1^{(\pm)} + i B_1^{(\pm)} = \cos(\omega t \mp \omega z) - i \sin(\omega t \mp \omega z) , \\
E_2^{(\pm)} + i B_2^{(\pm)} = \pm \sin(\omega t \mp \omega z) \pm i \cos(\omega t \mp \omega z) . 
\] (21)

It is easily checked the known presupposed property \( E^{(\pm)} \times B^{(\pm)} = \pm e_z \).

Let us turn back to the generale case (18), from the first equation it follows

\[
f_3 = \frac{-ib}{\omega} e^{2z} F_1 + \frac{ia}{\omega} e^{2z} F_2 ,
\] (22)
and further we get a system for $F_1$ and $F_2$

\[
\begin{align*}
\left( \frac{d}{dz} + \frac{ab e^{2z}}{\omega} \right) F_2 &= \frac{b^2 e^{2z} - \omega^2}{\omega} F_1, \\
\left( \frac{d}{dz} - \frac{ab e^{2z}}{\omega} \right) F_1 &= \frac{\omega^2 - a^2 e^{2z}}{\omega} F_2.
\end{align*}
\] (23)

With the help of a new variable $e^z = \sqrt{\omega} Z$, two last are written as

\[
\begin{align*}
Z \left( \frac{d}{dZ} + ab Z \right) F_2 &= +\left( b^2 Z^2 - \omega \right) F_1, \\
Z \left( \frac{d}{dZ} - ab Z \right) F_1 &= -\left( a^2 Z^2 - \omega \right) F_2.
\end{align*}
\] (24)

This system can be solved straightforwardly in terms if Heun confluent functions. Indeed, from (24) it follows a second order differential equation for $F_1$

\[
\frac{d^2 F_1}{dZ^2} - \frac{a^2 Z^2 + \omega}{Z(a^2 Z^2 - \omega)} \frac{dF_1}{dZ} + \left[ \frac{\omega^2}{Z^2} + \frac{2ab \omega}{a^2 Z^2 - \omega} - (a^2 + b^2) \omega \right] F_1 = 0,
\] (25)

here we note additional singular point at $Z = \pm \sqrt{\omega}/a$. With the new variable, we get

\[
\begin{align*}
y &= \frac{a^2 Z^2}{\omega}, \quad \frac{d^2 F_1}{dy^2} + \left[ \frac{1}{y} - \frac{1}{y-1} \right] \frac{dF_1}{dy} \\
&+ \left[ \frac{\omega^2}{4 y^2} - \frac{2ab \omega + (a^2 + b^2) \omega^2}{4 a^2 y} + \frac{b \omega}{2a(y-1)} \right] F_1 = 0.
\end{align*}
\] (26)

from whence or with the substitution $F_1(y) = y^c g_1(y)$ we arrive at

\[
\begin{align*}
\frac{d^2 g_1}{dy^2} + \left[ \frac{2c + 1}{y} - \frac{1}{y-1} \right] \frac{dg_1}{dy} + \left[ \frac{\omega^2/4 + c^2}{y^2} + \frac{2c - \omega^2/2 - b \omega/a - b^2 \omega^2/(2a^2)}{2y} + \frac{-2c + b \omega/a}{2(y-1)} \right] g_1 &= 0.
\end{align*}
\] (27)

When $c = \pm i \omega/2$, eq. (27) is simplified

\[
\begin{align*}
\frac{d^2 g_1}{dy^2} + \left[ \frac{2c + 1}{y} - \frac{1}{y-1} \right] \frac{dg_1}{dy} \\
+ \left[ \frac{2c - \omega^2/2 - b \omega/a - b^2 \omega^2/(2a^2)}{2y} + \frac{-2c + b \omega/a}{2(y-1)} \right] g_1 &= 0
\end{align*}
\]

which can be identified with confluent Heun function

\[
H(\alpha, \beta, \gamma, \delta, \eta, z), \quad \frac{d^2 H}{dz^2} + \left[ \alpha + \frac{1 + \beta}{z} + \frac{1 + \gamma}{z-1} \right] \frac{dH}{dz} \\
+ \left[ \frac{1}{2} \frac{\alpha + \alpha \beta - \beta \gamma - \beta - \gamma - 2 \eta}{z} + \frac{1}{2} \frac{\alpha \gamma + \beta + \alpha + 2 \eta + 2 \delta + \beta \gamma + \gamma}{z-1} \right] H = 0
\] (28)
with parameters
\[ \alpha = 0, \quad \beta = 2c, \quad \gamma = -2, \quad \delta = -\frac{1}{4} \frac{(a^2 + b^2) \omega^2}{a^2}, \]
\[ \eta = \frac{1}{4} \frac{2a b \omega + (a^2 + b^2) \omega^2 + 4a^2}{a^2}, \quad F_1 = y^{\pm \omega/2} H(\alpha, \beta, \gamma, \delta, \eta, y). \] (29)

Below we will develop a method that makes possible to construct solutions of the system (23) in more simple functions, solution of the Bessel equation.

5 Additional studying of the system

Let us perform a special transformation in (23) (suppose \( \alpha n - \beta m = 1 \))
\[ F_1 = \alpha G_1 + \beta G_2, \quad F_2 = m G_1 + n G_2; \]
\[ G_1 = n F_1 - \beta F_2, \quad G_2 = -m F_1 + \alpha F_2. \] (30)

Combining equations from (23), we get
\[ n Z \left( \frac{d}{dZ} - ab Z \right) F_1 - \beta Z \left( \frac{d}{dZ} + ab Z \right) F_2 = -n (a^2 Z^2 - \omega) F_2 - \beta (b^2 Z^2 - \omega) F_1, \]
\[ -m Z \left( \frac{d}{dZ} - ab Z \right) F_1 + \alpha Z \left( \frac{d}{dZ} + ab Z \right) F_2 = m (a^2 Z^2 - \omega) F_2 + \alpha (b^2 Z^2 - \omega) F_1, \]
from whence it follows
\[ Z \frac{d}{dZ} G_1 - Z^2 ab (n F_1 + \beta F_2) = -Z^2 (n a^2 F_2 + \beta b^2 F_1) + \omega (n F_2 + \beta F_1), \]
\[ Z \frac{d}{dZ} G_2 + Z^2 ab (m F_1 + \alpha F_2) = Z^2 (m a^2 F_2 + \alpha b^2 F_1) - \omega (m F_2 + \alpha F_1). \] (31)

Taking into account (30), eqs. (31) reduce to
\[ \left[ Z \frac{d}{dZ} - Z^2 ab (n \alpha + \beta m) + Z^2 (a^2 mn + b^2 \alpha \beta) - \omega (nm + \alpha \beta) \right] G_1 \]
\[ = \left[ -Z^2 (an - b\beta)^2 + \omega (n^2 + \beta^2) \right] G_2, \]
\[ \left[ Z \frac{d}{dZ} + Z^2 ab (n \beta + m \alpha) - Z^2 (a^2 mn + b^2 \alpha \beta) + \omega (nm + \alpha \beta) \right] G_2 \]
\[ = \left[ Z^2 (am - b\alpha)^2 - \omega (m^2 + \alpha^2) \right] G_1. \] (32)

Let us impose additional restriction (there exist two possibilities): \( an - b\beta = 0 \quad \implies \quad \beta = \frac{a}{n}, \)
\[ \left[ Z \frac{d}{dZ} - Z^2 ab (n \alpha + \beta m) + Z^2 (a^2 mn + b^2 \alpha \beta) - \omega (nm + \alpha \beta) \right] G_1 \]
\[ = +\omega (n^2 + \beta^2) G_2, \]
\[ \left[ Z \frac{d}{dZ} + Z^2 ab (m \beta + n \alpha) - Z^2 (a^2 mn + b^2 \alpha \beta) + \omega (nm + \alpha \beta) \right] G_2 \]
\[ = \left[ Z^2 (am - b\alpha)^2 - \omega (m^2 + \alpha^2) \right] G_1. \] (33)
or

\[ am - b; \alpha = 0 \implies \frac{\alpha}{m} = \frac{a}{b}, \]

\[
\begin{align*}
Z \frac{d}{dZ} - Z^2 ab(n\alpha + \beta m) + Z^2 (a^2 mn + b^2 \alpha \beta) - \omega (nm + \alpha \beta) \big] G_1 \\
&= \big[ -Z^2 (an - b\beta)^2 + \omega (n^2 + \beta^2) \big] G_2, \\
\end{align*}
\]

\[
\begin{align*}
Z \frac{d}{dZ} + Z^2 ab(m\beta + n\alpha) - Z^2 (a^2 mn + b^2 \alpha \beta) + \omega (nm + \alpha \beta) \big] G_2 \\
&= -\omega (m^2 + \alpha^2) G_1. 
\end{align*}
\]

The two variant are equivalent each other, for definiteness we will use the variant (33). It can be presented in more symmetrical form

\[
F_1 = \alpha G_1 + \beta G_2 = + \frac{b}{\sqrt{a^2 + b^2}} G_1 + \frac{a}{\sqrt{a^2 + b^2}} G_2,
\]

\[
F_2 = m G_1 + n G_2 = - \frac{a}{\sqrt{a^2 + b^2}} G_1 + \frac{b}{\sqrt{a^2 + b^2}} G_2;
\]

at this eqs. (18) assume the form

\[
\begin{align*}
Z \frac{d}{dZ} - Z^2 ab b^2 - a^2 + Z^2 ab b^2 - a^2 - \omega (\frac{ab}{a^2 + b^2} + \frac{ab}{a^2 + b^2}) \big] G_1 \\
&= +\omega (\frac{b^2}{a^2 + b^2} + \frac{a^2}{a^2 + b^2}) G_2, \\
\end{align*}
\]

\[
\begin{align*}
Z \frac{d}{dZ} + Z^2 ab b^2 - a^2 - Z^2 ab b^2 - a^2 + \omega (\frac{ab}{a^2 + b^2} + \frac{ab}{a^2 + b^2}) \big] G_2 \\
&= \big[ Z^2 (\frac{a^2}{\sqrt{a^2 + b^2}} - \frac{b^2}{\sqrt{a^2 + b^2}})^2 - \omega (\frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2}) \big] G_1,
\end{align*}
\]

that is

\[
Z \frac{d}{dZ} G_1 = \omega G_2, \quad Z \frac{d}{dZ} G_2 = \big[ Z^2 (a^2 + b^2) - \omega \big] G_1. \tag{36}
\]

From (36) we derive a second order equation for \(G_1\)

\[
\left( Z^2 \frac{d^2}{dZ^2} + Z \frac{d}{dZ} + \omega^2 - \omega (a^2 + b^2) Z^2 \right) G_1 = 0. \tag{37}
\]

To understand better the physical meaning of the equation (37), it is convenient to translate the equation to variable \(z\), then it reads

\[
e^z = \sqrt{\omega} Z, \quad \left( \frac{d^2}{dz^2} + \omega^2 - (a^2 + b^2) e^{2z} \right) G_1 = 0. \tag{38}
\]
It can be associated with the Schrödinger equation

\[
\left( \frac{d^2}{dz^2} + \epsilon - U(z) \right) \varphi(z) = 0 \quad (39)
\]

with potential function \( U(z) = (a^2 + b^2)e^{2z} \), and an effective force acting on the left \( F_z = -2(a^2 + b^2)e^{2z} \). Note that when \( a = k_1 = 0, b = k_2 = 0 \), the effective force vanishes. The corresponding quantum-mechanical system can be illustrated by Fig.1.

![Figure 1: Effective potential curve](image)

Therefore, we should expect properties of the electromagnetic solutions similar to those existing in the associated quantum-mechanical problem.

Let us turn back to eq. (37) – in the variable

\[
x = i \sqrt{\omega(a^2 + b^2)} \quad Z = i \sqrt{a^2 + b^2} e^z
\]

it assumes the form of the Bessel equation

\[
\left( \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 + \frac{\omega^2}{x^2} \right) G_1 = 0 . \quad (40)
\]

The first order system \((36)\) in variable \( x \) takes the form

\[
x \frac{d}{dx} G_1 = \omega G_2, \quad x \frac{d}{dx} G_2 = -\frac{\omega^2 + x^2}{\omega} G_1 . \quad (41)
\]

A second order equation for \( G_2 \) reads

\[
\left[ \frac{d^2}{dx^2} + \left( \frac{1}{x} - \frac{2x}{\omega^2 + x^2} \right) \frac{d}{dx} + \frac{x^2 + \omega^2}{x^2} \right] G_2 = 0 . \quad (42)
\]

Note that substituting \((35)\) into \((22)\), we get

\[
F_1 = \frac{b}{\sqrt{a^2 + b^2}} G_1 + \frac{a}{\sqrt{a^2 + b^2}} G_2, \quad F_2 = -\frac{a}{\sqrt{a^2 + b^2}} G_1 + \frac{b}{\sqrt{a^2 + b^2}} G_2
\]

into \((22)\), we get

\[
f_3 = \frac{e^{2z}}{\omega} (-ib \ F_1 + ia \ F_2) = \frac{\sqrt{a^2 + b^2}}{i \ \omega} G_1 . \quad (43)
\]
6 Asymptotic behavior of solutions

Mostly used for Bessel equation [37] are solutions

\[ G^I_1(x) = J_{+i\omega}(x) \quad \text{and} \quad G^{II}_1(x) = J_{-i\omega}(x) ; \quad (44) \]

\[ G^I_1(x) = H^{(1)}_{i\omega}(x) \quad \text{and} \quad G^{II}_1(x) = H^{(2)}_{i\omega}(x) , \]

\[ G^I_1(x) = H^{(1)}_{-i\omega}(x) \quad \text{and} \quad G^{II}_1(x) = H^{(2)}_{-i\omega}(x) ; \quad (45) \]

\[ G^I_1(x) = N_{+i\omega}(x) , \quad G^{II}_1(x) = N_{-i\omega}(x) . \quad (46) \]

For shortness, below the notation \( +\sqrt{a^2 + b^2} = 2\sigma \) is used. First, let us consider solutions in Bessel’s functions [37] when

\[ z \to -\infty, \quad x = i\sigma e^z \to i0 , \]

\[ G^I_1(x) = J_{+i\omega}(x) = \frac{1}{\Gamma(1 + i\omega)} (\frac{x}{2})^{i\omega} = \frac{(i\sigma)^{i\omega}}{\Gamma(1 + i\omega)} e^{i\omega z} , \]

\[ G^{II}_1(x) = J_{-i\omega}(x) = \frac{1}{\Gamma(1 - i\omega)} (\frac{x}{2})^{-i\omega} = \frac{(i\sigma)^{-i\omega}}{\Gamma(1 - i\omega)} e^{-i\omega z} . \quad (47) \]

In the region \( z \to +\infty, \quad (x = i\sigma e^z = iX \to i\infty) \), using the knows asymptotic formula [37]

\[ J_{i\omega}(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left( x - (i\omega + \frac{1}{2})\frac{\pi}{2} \right) , \]

we get

\[ G^I_1(z \to \infty) = J_{+i\omega}(z \to \infty) \sim e^{i\pi/4} \sqrt{\frac{1}{2\pi i X}} e^{-\omega\pi/2} e^{+X} , \]

\[ G^{II}_1(z \to \infty) = J_{-i\omega}(z \to \infty) \sim e^{i\pi/4} \sqrt{\frac{1}{2\pi i X}} e^{+\omega\pi/2} e^{+X} . \quad (48) \]

Let us consider solutions in Hankel’s functions [37], determined in terms of \( J_{\pm i\omega}(x) \) as follows

\[ H^{(1)}_{i\omega}(x) = +\frac{i}{\sin(i\omega\pi)} \left( e^{i\omega x} J_{+i\omega}(x) - J_{-i\omega}(x) \right) , \]

\[ H^{(2)}_{i\omega}(x) = -\frac{i}{\sin(i\omega\pi)} \left( e^{-i\omega x} J_{+i\omega}(x) - J_{-i\omega}(x) \right) . \quad (49) \]
so that \( z \to -\infty, \ x \to i0 \),

\[
G_I^I(x) = H_{i\omega}^{(1)}(x) = \frac{i}{\sin(i\omega \pi)} \left( e^{+\omega \pi} \frac{(i \sigma)^{i\omega}}{\Gamma(1 + i\omega)} e^{+i\omega z} - \frac{(i \sigma)^{-i\omega}}{\Gamma(1 - i\omega)} e^{-i\omega z} \right),
\]

\[
G_{II}^I(x) = H_{i\omega}^{(2)}(x) = -\frac{i}{\sin(i\omega \pi)} \left( e^{-\omega \pi} \frac{(i \sigma)^{i\omega}}{\Gamma(1 + i\omega)} e^{+i\omega z} - \frac{(i \sigma)^{-i\omega}}{\Gamma(1 - i\omega)} e^{-i\omega z} \right).
\]

(50)

Behavior of them when \( z \to +\infty \) is governed the known relation [37]

\[
H_{i\omega}^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}} \exp \left[ +i \left( x - \frac{\pi}{2} (i\omega + \frac{1}{2}) \right) \right],
\]

\[
H_{i\omega}^{(2)}(x) \sim \sqrt{\frac{2}{\pi x}} \exp \left[ -i \left( x - \frac{\pi}{2} (i\omega + \frac{1}{2}) \right) \right];
\]

from whence it follows

\[
z \to +\infty, \ x = iX \to i\infty,
\]

\[
G_I^I(x) = H_{i\omega}^{(1)}(x) \sim e^{-i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{+\omega \pi/2} e^{-X},
\]

\[
G_{II}^I(x) = H_{i\omega}^{(2)}(x) \sim e^{+i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{-\omega \pi/2} e^{+X}.
\]

(51)

Let us consider interpretation of the first type solution: this wave goes from the left, then it is partly reflected and partly goes forward through an effective potential barrier but gradually damping as \( z \) rises. The corresponding reflection coefficient is determined as follows

\[
G(z) \sim M_+ e^{+i\omega z} + M_- e^{-i\omega z}, \quad R = \frac{|M_+|^2}{|M_+|^2}. \quad (52)
\]

Taking into account identities

\[
(i\sigma)^{+i\omega} = (e^{i\pi/2} e^{\ln\sigma})^{+i\omega} = e^{-\omega \pi/2} e^{+i\omega \ln\sigma},
\]

\[
(i\sigma)^{-i\omega} = (e^{i\pi/2} e^{\ln\sigma})^{-i\omega} = e^{+\omega \pi/2} e^{-i\omega \ln\sigma}; \quad (53)
\]

we derive

\[
|M_+|^2 = \frac{1}{\sin(i\omega \pi) \sin(-i\omega \pi)} \frac{e^{+\omega \pi}}{\Gamma(1 - i\omega) \Gamma(1 + i\omega)}, \quad \frac{e^{+\omega \pi}}{\Gamma(1 - i\omega) \Gamma(1 + i\omega)},
\]

\[
|M_-|^2 = \frac{1}{\sin(i\omega \pi) \sin(-i\omega \pi)} \frac{e^{+\omega \pi}}{\Gamma(1 - i\omega) \Gamma(1 + i\omega)}. \quad (54)
\]

This means that for all solutions of that type the reflection coefficient always equals to 1:

\[
R = 1. \quad (55)
\]

11
Solutions of the second type, rising to infinity as \( z \to +\infty \), are characterized by
\[
M^I_\omega e^{i\omega z} + M^I_{-\omega} e^{-i\omega z}, \quad R = \frac{|M^I_\omega|^2}{|M^I_{-\omega}|^2} = e^{4\omega \pi} > 1.
\]

Finally, let us specify asymptotic behavior of solutions in terms of Neyman functions. They functions are defined by \[37\]
\[
N_{i\omega}(x) = \cos(i\omega \pi) J_{i\omega}(x) - J_{-i\omega}(x), \quad N_{-i\omega}(x) = J_{i\omega}(x) - \cos(i\omega \pi) J_{-i\omega}(x).
\]

In the region \( z \to +\infty \), \((x = iX \to i\infty)\), with the use of the known relation \[37\]
\[
N_{i\omega}(x) \sim \sqrt{\frac{2}{i\pi X}} \sin \left( iX - (i\omega + \frac{1}{2}) \frac{\pi}{2} \right),
\]
we get
\[
G^I_1(x) = N_{i\omega}(x) \sim ie^{i\pi/4} \sqrt{\frac{1}{2i\pi X}} e^{-\omega \pi/2} e^{X},
\]
\[
G^I_2(x) = N_{-i\omega}(x) \sim +ie^{i\pi/4} \sqrt{\frac{1}{2i\pi X}} e^{\omega \pi/2} e^{X}.
\]

In the region \( z \to -\infty \) their behavior is given by
\[
G^I(z) = \frac{\cos(i\omega \pi)}{\sin(i\omega \pi)} \frac{(i\sigma)^{i\omega}}{\Gamma(1 + i\omega)} e^{+i\omega z} - \frac{1}{\sin(i\omega \pi)} \frac{(i\sigma)^{-i\omega}}{\Gamma(1 - i\omega)} e^{-i\omega z},
\]
\[
G^{II}(z) = \frac{1}{\sin(i\omega \pi)} \frac{(i\sigma)^{i\omega}}{\Gamma(1 + i\omega)} e^{+i\omega z} - \frac{\cos(i\omega \pi)}{\sin(i\omega \pi)} \frac{(i\sigma)^{-i\omega}}{\Gamma(1 - i\omega)} e^{-i\omega z}.
\]

For these solutions we have respectively
\[
R^I = \frac{e^{2\omega \pi}}{(e^{2\omega \pi} + e^{-2\omega \pi})/4} = \frac{4}{1 + e^{-4\omega \pi}},
\]
\[
R^{II} = e^{2\omega \pi} \frac{(e^{2\omega \pi} + e^{-2\omega \pi})/4}{4} = \frac{1 + e^{4\omega \pi}}{4}.
\]

7 On explicit form of the function \( G_2 \)

The function \( G_1(x) \) satisfies the Bessel equation
\[
\left( \frac{d^2}{dx^2} + \frac{1}{x} \frac{d}{dx} + 1 + \frac{\omega^2}{x^2} \right) G_1 = 0;
\]
the second function \( G_2(x) \) is determined by
\[
G_2 = \frac{x}{\omega} \frac{d}{dx} G_1.
\]
Solutions of the Bessel equation obey the following recurrent formulas [37]

\[ x \frac{d}{dx} F_{i\omega} = i\omega F_{i\omega} - x F_{i\omega+1}, \]
\[ x \frac{d}{dx} F_{-i\omega} = +i\omega F_{-i\omega}(x) + x F_{-i\omega-1}, \]

(63)

where \( F_{\pm\nu} \) stands for \( J_{\pm\nu}(x) \), \( H^{(1)}_{\pm\nu}(x) \), \( H^{(2)}_{\pm\nu}(x) \), \( N_{\pm\nu}(x) \).

Therefore, with the help of (63), one can express \( G_2 \) in terms of the known \( G_1 \). For instance,

\[ G^I_1(x) = H^{(1)}_{+i\omega}(x), \quad G^I_2(x) = i H^{(1)}_{+i\omega}(x) - \frac{x}{\omega} H^{(1)}_{i\omega+1}(x), \]
\[ G^{II}_1(x) = H^{(2)}_{+i\omega}(x), \quad G^{II}_2(x) = i H^{(2)}_{+i\omega}(x) - \frac{x}{\omega} H^{(2)}_{i\omega+1}(x). \]

(64)

Remember that

\[ F^I_1 = \frac{b}{\sqrt{a^2 + b^2}} G_1 + \frac{a}{\sqrt{a^2 + b^2}} G_2, \]
\[ F^I_2 = -\frac{a}{\sqrt{a^2 + b^2}} G_1 + \frac{b}{\sqrt{a^2 + b^2}} G_2, \]
\[ f^I_3 = \frac{e^{2z}}{\omega} (-ib F^I_1 + ia F^I_2) = \frac{\sqrt{a^2 + b^2}}{i \omega} G_1. \]

(65)

Let us examine asymptotic behavior of \( G_2 \). Starting with

\[ H^{(1)}_{i\omega}(x) = +\frac{i}{\sin(i\omega \pi)} (e^{i\omega\pi} J_{+i\omega}(x) - J_{-i\omega}(x)), \]
\[ H^{(2)}_{i\omega}(x) = -\frac{i}{\sin(i\omega \pi)} (e^{-i\omega\pi} J_{+i\omega}(x) - J_{-i\omega}(x)), \]
\[ H^{(1)}_{i\omega+1}(x) = +\frac{i}{\sin(i\omega + 1)\pi} (e^{-i(\omega+1)\pi} J_{+i\omega+1}(x) - J_{-(\omega+1)}(x)), \]
\[ H^{(2)}_{i\omega+1}(x) = -\frac{i}{\sin(i\omega + 1)\pi} (e^{i(\omega+1)\pi} J_{+i\omega+1}(x) - J_{-(\omega+1)}(x)), \]

(66)

with the help of relations

\[ z \to -\infty, \ x \to i0, \]

\[ J_{+i\omega}(x) \sim \frac{(i \sigma)^{i\omega}}{\Gamma(1 + i\omega)} e^{+i\omega z}, \quad J_{-i\omega}(x) \sim \frac{(i \sigma)^{-i\omega}}{\Gamma(1 - i\omega)} e^{-i\omega z}. \]

we get

\[ z \to -\infty, \ x \to i0, \]

\[ H^{(1)}_{i\omega} \sim +\frac{i}{\sin(i\omega \pi)} \left( e^{+\omega\pi} \frac{(i \sigma)^{i\omega}}{\Gamma(1 + i\omega)} e^{+i\omega z} - \frac{(i \sigma)^{-i\omega}}{\Gamma(1 - i\omega)} e^{-i\omega z} \right), \]
\[ H^{(2)}_{i\omega} \sim -\frac{i}{\sin(i\omega \pi)} \left( e^{-\omega\pi} \frac{(i \sigma)^{i\omega}}{\Gamma(1 + i\omega)} e^{+i\omega z} - \frac{(i \sigma)^{-i\omega}}{\Gamma(1 - i\omega)} e^{-i\omega z} \right), \]
\[ H^{(1)}_{\omega+1} \sim \frac{i}{\sin(\omega + 1)\pi} \left( e^{-i(\omega+1)\pi} \frac{(i\sigma)^{\omega+1}}{\Gamma(2 + i\omega)} e^{i\omega z} e^z - \frac{(i\sigma)^{-\omega-1}}{\Gamma(-i\omega)} e^{-i\omega z} e^{-z} \right) \]
\[ \sim \frac{i}{\sin(\omega + 1)\pi} \frac{(i\sigma)^{-\omega-1}}{\Gamma(-i\omega)} e^{-i\omega z} e^{-z}, \]

\[ H^{(2)}_{\omega+1}(x) \sim \frac{-i}{\sin(\omega + 1)\pi} \left( e^{i(\omega+1)\pi} \frac{(i\sigma)^{\omega+1}}{\Gamma(2 + i\omega)} e^{i\omega z} e^z - \frac{(i\sigma)^{-\omega-1}}{\Gamma(-i\omega)} e^{-i\omega z} e^{-z} \right) \]
\[ \sim \frac{i}{\sin(\omega + 1)\pi} \frac{(i\sigma)^{-\omega-1}}{\Gamma(-i\omega)} e^{-i\omega z} e^{-z}. \]

So we get
\[ G_I^2(x) = -\frac{1}{\sin(i\omega\pi)} \left( e^{+\omega\pi} \frac{(i\sigma)^{\omega}}{\Gamma(1 + i\omega)} e^{+i\omega z} e^z - \frac{(i\sigma)^{-\omega}}{\Gamma(1 - i\omega)} e^{-i\omega z} \right) \]
\[ -\frac{2\sigma}{\omega} \frac{1}{\sin(i\omega\pi)} \frac{(i\sigma)^{-\omega-1}}{\Gamma(-i\omega)} e^{-i\omega z}. \] (67)

Taking into consideration an identity
\[ = \frac{2\sigma}{\omega} \frac{1}{\sin(i\omega\pi)} \frac{(i\sigma)^{-\omega}(-i\omega)}{\Gamma(1 - i\omega)} e^{-i\omega z} = -2 \frac{1}{\sin(i\omega\pi)} \frac{(i\sigma)^{-\omega}}{\Gamma(1 - i\omega)} e^{-i\omega z} \] (68)

one reduces the above relation to the form
\[ G_I^2(x) = -\frac{1}{\sin(i\omega\pi)} \left( e^{+\omega\pi} \frac{(i\sigma)^{\omega}}{\Gamma(1 + i\omega)} e^{+i\omega z} e^z + \frac{(i\sigma)^{-\omega}}{\Gamma(1 - i\omega)} e^{-i\omega z} \right). \] (70)

In similar manner one can treat the case
\[ G_{II}^1 = \frac{1}{\sin(i\omega\pi)} \left( e^{-\omega\pi} \frac{(i\sigma)^{\omega}}{\Gamma(1 + i\omega)} e^{+i\omega z} e^z - \frac{(i\sigma)^{-\omega}}{\Gamma(1 - i\omega)} e^{-i\omega z} \right) \]
\[ + \frac{2\sigma}{\omega} \frac{1}{\sin(i\omega\pi)} \frac{(i\sigma)^{-\omega-1}}{\Gamma(-i\omega)} e^{-i\omega z} \]
\[ = \frac{1}{\sin(i\omega\pi)} \left( e^{-\omega\pi} \frac{(i\sigma)^{\omega}}{\Gamma(1 + i\omega)} e^{+i\omega z} e^z + \frac{(i\sigma)^{-\omega}}{\Gamma(1 - i\omega)} e^{-i\omega z} \right). \] (71)

Behavior of these solutions when \( z \to +\infty \) is governed the relation
\[ H^{(1)}_{\omega}(x) \sim \sqrt{\frac{2}{\pi x}} \exp \left[ +i \left( x - \frac{\pi}{2}(i\omega + \frac{1}{2}) \right) \right], \]
\[ H^{(2)}_{\omega}(x) \sim \sqrt{\frac{2}{\pi x}} \exp \left[ -i \left( x - \frac{\pi}{2}(i\omega + \frac{1}{2}) \right) \right]; \]
from whence it follows

\[ H_{i\omega}^{(1)}(x) \sim e^{-i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{+\omega\pi/2} e^{-X}, \]

\[ H_{i\omega}^{(2)}(x) \sim e^{+i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{-\omega\pi/2} e^{+X}, \]

\[ H_{i\omega+1}^{(1)}(x) \sim \sqrt{\frac{2}{i\pi X}} \exp \left[ +i \left( iX - \frac{\pi}{2} (i\omega + 1 + \frac{1}{2}) \right) \right] \]

\[ \sim -i e^{-i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{+\omega\pi/2} e^{-X}, \]

\[ H_{i\omega+1}^{(2)}(x) \sim \sqrt{\frac{2}{i\pi X}} \exp \left[ -i \left( iX - \frac{\pi}{2} (i\omega + 1 + \frac{1}{2}) \right) \right] \]

\[ \sim i e^{+i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{-\omega\pi/2} e^{+X}. \] (72)

Therefore, we arrive at the formulas

\[ G_{I}^{(1)}(x) = i \frac{1}{\omega} \frac{d}{dz} h_{i\omega}(x) - \frac{x}{\omega} H_{i\omega+1}^{(1)}(x) \]

\[ \sim ie^{-i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{+\omega\pi/2} e^{-X} - \frac{X}{\omega} e^{-i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{+\omega\pi/2} e^{-X}, \]

\[ G_{I}^{(2)} = i \frac{1}{\omega} \frac{d}{dz} h_{i\omega}(x) - \frac{x}{\omega} H_{i\omega+1}^{(2)}(x) \]

\[ \sim ie^{+i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{-\omega\pi/2} e^{+X} + \frac{X}{\omega} e^{+i\pi/4} \sqrt{\frac{2}{i\pi X}} e^{-\omega\pi/2} e^{+X}. \] (73)

Evidently, to find asymptotic for \( G_{2} \), it is sufficient to make use of the known asymptotic for \( G_{1} \). For instance,

\[ G_{2} \sim \frac{1}{i\omega} \frac{d}{dz} \left( \frac{i}{\sin(i\omega\pi)} \left( e^{+\omega\pi} \frac{(i\sigma)^{i\omega}}{\Gamma(1+i\omega)} e^{+i\omega z} - \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{-i\omega z} \right) \right) \]

\[ = -\frac{1}{\sin(i\omega\pi)} \left( e^{+\omega\pi} \frac{(i\sigma)^{i\omega}}{\Gamma(1+i\omega)} e^{+i\omega z} + \frac{(i\sigma)^{-i\omega}}{\Gamma(1-i\omega)} e^{-i\omega z} \right); \] (74)

which coincides with (70). It is a superposition of two plane waves with reflection coefficient \( R = 1 \).

8 Concluding remarks

In accordance with (39), an equation below

\[ \omega^2 = U(z) \quad \omega^2 = (a^2 + b^2) e^{2z_0} \] (75)

determines a critical point \( z_0 \) in which behavior of the function \( G_{1}(x) \) must change dramatically. To such a point \( z_0 \) there corresponds

\[ x_0 = i\sqrt{a^2 + b^2} e^{z_0} = i\omega. \] (76)
In order to examine behavior of solutions in vicinity of \( x_0 \), it is convenient to introduce a new coordinate

\[
x = x_0 + i\omega \ u = i\omega (1 + u) , \quad \frac{d}{dx} = \frac{1}{i\omega} \frac{d}{du} ;
\]

(77)
eq. (40) for \( G_1(x) \) assumes the form

\[
\left( \frac{d^2}{du^2} + \frac{1}{1 + u} \frac{d}{du} - \frac{1}{(1 + u)^2} \right) G_1 = 0 .
\]

(78)

Close to \( u = 0 \), we have

\[
\left( \frac{d^2}{du^2} + \frac{d}{du} \right) G_1 = 0 .
\]

(79)

that is

\[
G_1 = e^{Bu}, \quad B^2 + B = 0, \quad B = 0, -1 ;
\]

physically interesting is the choice \( B = -1 \).

To such a critical value \( x_0 = i\omega \), there correspond

\[
\omega = \sqrt{k_1^2 + k_2^2} e^{z_0} \implies z_0 = \ln \frac{\omega}{\sqrt{k_1^2 + k_2^2}} ;
\]

(80)
in usual units, this relation reads

\[
z_0 = \rho \ln \frac{\omega}{c \sqrt{k_1^2 + k_2^2}} ,
\]

(81)

where \( \rho \) is a curvature radius of the Lobachevsky space.

Let us summarize results.

Lobachevsky geometry simulates a medium with special constitutive relations. The situation is specified in quasi-cartesian coordinates \((x, y, z)\). Exact solutions of the Maxwell equations in complex 3-vector \( E + iB \) form, extended to curved space models within the tetrad formalism, have been found in Lobachevsky space. The problem reduces to a second order differential equation which can be associated with an 1-dimensional Schrödinger problem for a particle in external potential field \( U(z) = U_0 e^{2z} \).

In quantum mechanics, curved geometry acts as an effective potential barrier with reflection coefficient \( R = 1 \); in electrodynamic context results similar to quantum-mechanical ones arise: the Lobachevsky geometry simulates a medium that effectively acts as an ideal mirror. Penetration of the electromagnetic field into the effective medium, depends on the parameters of an electromagnetic wave, frequency \( \omega \), \( k_1^2 + k_2^2 \), and the curvature radius \( \rho \) – see (81). See illustrations in Fig. 2,3.
Figure 2: $\text{Im} H^{(1)}_{+i\omega}, \omega = 10$

Figure 3: $\text{Im} H^{(1)}_{+i\omega}, \omega = 20$

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