A NEW LOOK AT THE ASHTEKAR-MAGNON ENERGY CONDITION

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In 1975, Ashtekar and Magnon [1] showed that an energy condition selects a unique quantization procedure for certain observers in general, curved spacetimes. We generalize this result in two important ways, by eliminating the need to assume a particular form for the (quantum) Hamiltonian, and by considering the surprisingly nontrivial extension to nonminimal coupling.

I. INTRODUCTION

Ashtekar and Magnon [1] were among the first to consider quantum field theory as seen by observers who were not static or stationary. Remarkably, they were able to give a quantization procedure for the scalar field for any family of hypersurface orthogonal observers in a curved spacetime. Their procedure is based on a single, natural condition: The classical and quantum energies should agree. However, for non-Killing observers, it is not obvious how to define either of these energies. Ashtekar and Magnon choose to use the stress-energy tensor of the scalar field for the classical energy, and to define the quantum energy in terms of a particular choice of quantum Hamiltonian operator.

We extend their work in two important ways. First of all, we show that the construction itself fully determines the Hamiltonian operator, which therefore does not need to be specified in advance. Second, we show that the basic result holds for any choice of the classical energy satisfying certain simple properties. Not surprisingly, for the case of minimal coupling (ξ = 0) considered by Ashtekar and Magnon, if we use the stress-energy tensor to define the classical energy, then we recover not only their complex structure but also their quantum Hamiltonian. However, when ξ ≠ 0, this approach runs into a serious problem: The resulting Hamiltonian and complex structure do not reduce to the known answers for static observers. We show how this problem can be resolved by using the classical Hamiltonian to define the classical energy rather than the stress-energy tensor.

After setting up our formalism in Section II, we summarize the work of Ashtekar and Magnon in Section III and present our generalization in Section IV. Section V shows how to recover Ashtekar and Magnon’s result for ξ = 0, as well as considering the case ξ ≠ 0. Finally, in Section VI we discuss our results.

II. MATHEMATICAL PRELIMINARIES

Let (M, g_{ab}) be a globally hyperbolic spacetime with associated Levi-Civita connection ∇. The action S for a scalar field Φ on M is given by

\[ S = \int_M L \sqrt{-g} \, d^nx \]

\[ L = -\frac{1}{2} (g^{ab} \nabla_a \Phi \nabla_b \Phi + (m^2 + \xi R)\Phi^2) \]

The Klein-Gordon equation, obtained by varying the action S with respect to Φ, is

\[ g^{ab} \nabla_a \nabla_b \Phi - (m^2 + \xi R)\Phi = 0 \]

Let V be the space of smooth, real-valued solutions of (3) which have compact support on any (and hence every) Cauchy surface. Ashtekar and Magnon suggested that, as a real vector space, the one-particle Hilbert space H should be a copy of V.

Introduce coordinates (t = x_0, ..., x_{n-1}) on M so that the hypersurfaces \{t = \text{const}\} are Cauchy surfaces. We assume throughout that the vector field \( t^a \nabla_a = \partial_t \) is hypersurface orthogonal. The standard 3 + 1 formalism leads to a decomposition of the metric g_{ab} in terms of its pullback h_{ij} to Σ and the lapse function N = t^a t_a; the shift is zero. We denote the Levi-Civita connection on (Σ, h_{ij}) by D_i.

On V we have the (nondegenerate) symplectic structure

\[ \Omega(\Phi, \Psi) = \int_\Sigma (\Psi \nabla_a \Phi - \Phi \nabla_a \Psi) \, a^a d\Sigma \]
where \( \Sigma \) is any Cauchy hypersurface, \( n^a \) is the future pointing, unit normal vector field to \( \Sigma \) and \( d\Sigma = \sqrt{h} d^{n-1}x \) is the volume element on \( \Sigma \) induced by the inclusion map. Let \( J \) be any complex structure on \( V \), that is a linear map from \( V \) to itself satisfying

\[ J^2 = -1 \]  

which allows us to view \( V \) as a complex vector space. We will also assume that \( J \) is compatible in the sense that

\[ \Omega(\Phi, J\Psi) \geq 0 \]  

\[ \Omega(J\Phi, J\Psi) = \Omega(\Phi, \Psi) \]  

for any \( \Phi, \Psi \in V \). As discussed in [1], the *-algebra approach leads naturally to the inner product

\[ \langle \Phi, \Psi \rangle = \frac{1}{2} \Omega(\Phi, J\Psi) + i\frac{1}{2} \Omega(\Phi, \Psi) \]  

which is Hermitian under the above assumptions. A candidate for the one-particle Hilbert space \( \mathcal{H} \) is then the Cauchy completion of \((V, J, \langle \cdot, \cdot \rangle)\), so that the problem of identifying the one-particle Hilbert space of states is reduced to that of choosing a suitable complex structure \( J \) on \( V \).

Solutions \( \Phi \in V \) are completely determined by their initial data, so that \( V \) is isomorphic to the vector space \( \hat{V} \) of pairs of smooth, real-valued functions on \( \Sigma \) which have compact support. The isomorphic image of \( \Phi \) is then

\[ \hat{\Phi} = \left( \begin{array}{c} \varphi \\ \pi \end{array} \right) = \left( \begin{array}{c} \Phi|_{\Sigma} \\ \sqrt{h} n^a \nabla_a \Phi|_{\Sigma} \end{array} \right) \]  

We write \( \tau = C^\infty_0(\Sigma, \mathbb{R}) \), with inner product

\[ (f, g) = \int_{\Sigma} fg d^{n-1}x \]  

for \( f, g \in \hat{V} \); note that \( \hat{V} = \tau \oplus \tau \).

We conclude this section with some results about symmetric operators. Any linear operator \( Q \) on \( V \) can be represented as a \( 2 \times 2 \) matrix \( \hat{Q} \) whose elements are linear operators on \( \tau \). In particular, we write

\[ \hat{J} = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \]  

\[ \hat{H} = \left( \begin{array}{cc} W & X \\ Y & Z \end{array} \right) \]  

We define \( Q \) to be symmetric on \( V \) if

\[ \langle \Phi, Q\Psi \rangle = \langle Q\Phi, \Psi \rangle \]  

for all \( \Phi, \Psi \in V \); \( Q \) is antisymmetric if a relative factor of \(-1\) is inserted in (13).

**Lemma 1** Suppose that the linear operator \( Q \) on \( V \) satisfies

\[ \Omega(\Phi, Q\Phi) = 0 \]  

Then \( Q \) is symmetric.

**Proof:** This follows immediately since (14) implies that the expectation value of \( Q \) is always real. \( \Box \)

**Lemma 2** Let \( Q \) be a symmetric operator defined on \( \mathcal{H} \). Then \( [Q, J] = 0 \).

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1 Ashtekar and Magnon omit the factor of \( \sqrt{h} \) from \( \pi \).
Proof: Splitting (13) into its real and imaginary parts, we obtain
\[
\Omega(\Phi, JQ\Psi) = \Omega(Q\Phi, J\Psi) \quad (15)
\]
\[
\Omega(\Phi, Q\Psi) = \Omega(Q\Phi, \Psi) \quad (16)
\]
Using (15) and (16) we obtain:
\[
\Omega(\Phi, (JQ - QJ)\Psi) = \Omega(\Phi, JQ\Psi) - \Omega(\Phi, QJ\Psi) \quad (17)
\]
\[
= \Omega(Q\Phi, J\Psi) - \Omega(Q\Phi, J\Psi) \quad (18)
\]
\[
= 0. \quad (19)
\]
Since \(\Phi\) and \(\Psi\) are arbitrary (and \(\Omega\) is non-degenerate) we conclude that \(JQ - QJ = 0\).

The significance of Lemma 2 comes from the fact that the Hamiltonian operator \(H\) should be self-adjoint and hence symmetric. The total derivative of \(J\) is given by
\[
\frac{\partial}{\partial t} J + J[H, J] = 0 \quad (20)
\]
If \([H, J] = 0\), (20) reduces to \(\frac{\partial}{\partial t} J = 0\). Thus, if \(H\) is self-adjoint, the time derivative of \(J\) measures the amount of particle creation.

Setting \(Q = H\), the conditions (15) and (16) for the symmetry of \(H\) become
\[
X = -X^\dagger \quad (21)
\]
\[
Y = -Y^\dagger \quad (22)
\]
\[
W = Z^\dagger \quad (23)
\]
and
\[
(AX + BZ) = (AX + BZ)^\dagger \quad (24)
\]
\[
(CW + DY) = (CW + DY)^\dagger \quad (25)
\]
\[
(AW + BY) = -(CX + DZ)^\dagger \quad (26)
\]
respectively. But an immediate consequence of Lemma 2 is that (15) and (16) are equivalent. Thus, \(H\) is symmetric if and only if (21)–(23) are satisfied, and this is equivalent to (24)–(26) being satisfied. Furthermore, these latter equations are precisely the condition for \(JH\) to be antisymmetric, so that we have the further result

**Lemma 3** \(Q\) is symmetric if and only if \(JQ\) is antisymmetric.

### III. THE ASHTEKAR-MAGNON ENERGY CONDITION

The essential ingredient in the result of Ashtekar and Magnon [1] is the energy condition. Given a Cauchy hypersurface \(\Sigma\), one can define the classical energy and the quantum energy of a scalar field with respect to those observers orthogonal to \(\Sigma\). The energy condition says that these energies should be equal. Ashtekar and Magnon showed that there is a unique complex structure \(J\) on \(\Sigma\) such that the energy condition is satisfied. Using the results of Section II, they have thus shown that the energy condition selects a unique quantization procedure.

Ashtekar and Magnon define the classical energy of the scalar field with respect to \(\Sigma\) (and the choice of scale implicit in \(t^a\)) to be
\[
CE_T = \int_\Sigma T_{ab} t^a n^b d\Sigma \quad (27)
\]
where
\[
T_{ab} = \nabla_a \Phi \nabla_b \Phi - \frac{1}{2} g_{ab}(\nabla^c \Phi \nabla_c \Phi + m^2 \Phi^2) \quad (28)
\]
is the stress-energy tensor associated with the scalar field.

Ashtekar and Magnon define the quantum energy of the scalar field with respect to \(\Sigma\) (and \(t^a\)) to be the expectation value of the Hamiltonian operator \(H\), i.e.
\[ QE_H = \langle \Phi, H\Phi \rangle \] (29)

But what is the Hamiltonian operator \( H \)?

If the vector field \( t^a \) is Killing, so that the spacetime is stationary, the usual definition for the Hamiltonian operator \( H \) is \( H\Phi = -J(\mathcal{L}_{t^a}\Phi) \), where \( \mathcal{L} \) represents Lie differentiation. But in the present case the vector field \( t^a \) is not necessarily Killing, and so the function \( \mathcal{L}_{t^a}\Phi \) is not necessarily a solution of the Klein-Gordon equation. Therefore \( H \), as defined above, is not necessarily a map into \( \mathcal{H} \).

To overcome this problem, Ashtekar and Magnon used initial data to define \( H \). Let \( \Phi \in V \) be a solution of the Klein-Gordon equation with initial data as in (9). Consider the data to be a function of \( t \) and take its derivative; the result is in \( \hat{V} \) and hence defines a solution \( \dot{\Phi} \in V \). Explicitly, \( \dot{\Phi} \) is the solution with initial data

\[
\left( \frac{\dot{\phi}}{\dot{\pi}} \right) = \partial_t \left( \sqrt{h} n^a \nabla_a \Phi \right) \bigg|_{\Sigma}
\] (30)

It is straightforward but messy to use (27) and (29) to rewrite the time derivatives in terms of spatial derivatives, resulting in

\[
\dot{\phi} = \frac{N}{\sqrt{h}} \pi
\] (31)
\[
\dot{\pi} = \sqrt{h} (Nh^{ij} D_i D_j + h^{ij} D_i N D_j - m^2 N) \phi
\] (32)

Ashtekar and Magnon proceed to define the Hamiltonian operator \( H \) by requiring

\[ H\Phi = -J\dot{\Phi} \] (33)

Using (27) and (29), the energy condition takes the form

\[ \langle \Phi, H\Phi \rangle = \int_{\Sigma} T_{ab} t^a n^b d\Sigma \] (34)

It is again straightforward but messy to verify that

\[ 2\Omega(\Phi, \dot{\Phi}) = \text{Re} \langle \Phi, H\Phi \rangle = CE_T \] (35)

so that the true content of the energy condition is

\[ 2\Omega(\Phi, H\Phi) = \text{Im} \langle \Phi, H\Phi \rangle = 0 \] (36)

We now state without proof Ashtekar and Magnon’s main result.

**Theorem 1** (Ashtekar and Magnon) Let \((M, g_{ab})\) be a globally hyperbolic spacetime with Cauchy surface \( \Sigma \), and let \( V \) be as above. Then there exists a unique compatible complex structure \( J \) on \( V \) such that the energy condition is satisfied. In other words, there is a unique complex structure \( J \) such that

\[ \Omega(\Phi, H\Phi) = 0 \] (37)

for every \( \Phi \in \hat{V} \), where \( H \) is defined in terms of \( J \) via (23).

It is important to note that Ashtekar and Magnon assume a particular form of the Hamiltonian operator \( H \), namely (33).

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2The dot does not refer to a time derivative! (This construction is less intuitive with Ashtekar and Magnon’s choice of data.)
IV. EXTENDING ASHTEKAR AND MAGNON’S RESULT

The main result of this section is Theorem 2 which is a generalization of Theorem 1. There are two main differences. First, we replace the energy condition with a more general condition, which allows some flexibility in defining the classical energy of the scalar field. Second, we eliminate the need for specifying the Hamiltonian operator $H$. Theorem 2 uniquely determines both the complex structure and the operator $\hat{H}$.

**Theorem 2** Let $(M, g_{ab})$ be a globally hyperbolic spacetime with Cauchy surface $\Sigma$. Let $F$ be a real, nonzero smooth function on $\Sigma$, let $K$ be the Cauchy completion of the inner product space $\tau$ with $\langle f, g \rangle_{\tau} = \int_{\Sigma} fgF^{-1} d^{n-1}x$ and let $G$ be a real, semi-bounded, positive-definite symmetric operator on $K$. Suppose we have a linear operator $H$ and a compatible complex structure $J$ defined on $V$ such that

$$\langle \Phi, H\Phi \rangle = \frac{1}{2} \int_{\Sigma} (F\pi^2 + \varphi G\varphi) d^{n-1}x$$

for all $\Phi \in V$ with data $\hat{\Phi} = \begin{pmatrix} \varphi \\ \pi \end{pmatrix} \in \hat{V}$. Then the operators $J$ and $H$ are unique and are given in terms of their action on $\hat{V}$ by

$$\hat{J} = \begin{pmatrix} 0 & \frac{1}{2} (FG)^{-\frac{1}{2}} F \\ -F^{-1}(FG)^{\frac{1}{2}} & 0 \end{pmatrix}$$

and

$$\hat{H} = \begin{pmatrix} \frac{1}{2} (FG)^{\frac{1}{2}} & 0 \\ 0 & F^{-1}(FG)^{\frac{1}{2}} F \end{pmatrix}$$

**Proof 1:** The right-hand-side of (38) can be written as

$$\frac{1}{2} \Omega(\Phi, E\Phi)$$

where

$$\hat{E} = \begin{pmatrix} 0 & F \\ -G & 0 \end{pmatrix}$$

Comparing real and imaginary parts of (38) yields for all $\Phi \in V$:

$$\Omega(\Phi, JH\phi) = \Omega(\Phi, E\Phi)$$

and Lemma 1 now shows that both $H$ and $JH - E$ are symmetric. As discussed previously, $JH$ is antisymmetric if $H$ is symmetric, and (21)–(23) (with appropriate sign changes) show that $E$ is antisymmetric. Thus, $JH - E$ is both symmetric and antisymmetric, and we conclude that

$$JH = E$$

or equivalently

$$H = -JE$$

Using equation (46) we see that equation (44) can be written as

$$\Omega(\Phi, JE\Phi) = 0$$

Careful examination of the proof given by Ashtekar and Magnon shows that it relies only on $J$ being a compatible complex structure and on (47). We can thus use their proof to uniquely determine the complex structure $\hat{J}$ in terms of the operators $F$ and $G$, the only subtlety being the conditions on $G$ which allow square roots to be taken. Finally, writing out the multiplication in (46) and using the identity...
\[ BG = -FC \]  

(which follows naturally from (47) and the symmetry properties) results in the given form for \( \hat{H} \). This completes the proof, full details of which are given in [2]. \( \blacksquare \)

It turns out there is another proof of Theorem 2. We provide this alternate proof here:

**Proof 2:** Writing out the symmetry of \( JH - E \) we obtain

\[
(AW + BY) = -(CX + DZ)\dagger  
(AX + BZ - F) = (AX + BZ - F)\dagger  
((CW + DY + G) = (CW + DY + G)\dagger  
X = -X\dagger  
Y = -Y\dagger  
W = Z\dagger
\]

and the compatibility of \( J \) yields

\[
B = B\dagger  
C = C\dagger  
A = -D\dagger
\]

By using the symmetry and antisymmetry properties of the operators \( A \) through \( Z \), we can rewrite (49)–(51) as

\[
AW + BY = XC + WA  
XD + WB - F = -(AX + BZ - F)  
ZC + YA + G = -(CW + DY + G)
\]

Taking the adjoint of (59) yields

\[
ZD + YB = CX + DZ
\]

Solving for \( F \) in (60) yields

\[
F = \frac{1}{2}(AX + XD + WB + BZ)
\]

Multiplying (63) on the left by \( A \) and on the right by \( D \) and subtracting gives

\[
AF - FD = \frac{1}{2}(A^2X + AWB + ABZ - XD^2 - WBD - BZD)
\]

Multiplying (63) on the left by \( B \) and (59) on the right by \( B \) and solving for \( BZD \) and \( AWB \) yields

\[
BZD = BCX + BDZ - BYB  
AWB = XCB + WAB - BYB
\]

Substituting (63) and (66) into (64) yields

\[
AF - FD = \frac{1}{2}(A^2X + BCX - XD^2 + ABZ - WBD + BDZ - WAB - XCB)
\]

Finally, using (5) we see that the right hand side of (67) is identically zero. Therefore we now know that

\[
D = F^{-1}AF
\]

Using the argument given by Ashtekar and Magnon in proving Theorem 1, we can conclude that

\[
A = 0 = D
\]

It is then straightforward to determine \( B \) and \( C \), thus obtaining the complex structure \( \hat{J} \), and to then solve for \( \hat{H} \). \( \blacksquare \)
V. APPLICATIONS

A. Minimal Coupling \((\xi = 0)\)

We first recover Ashtekar and Magnon’s result. Comparing (31)–(32) and (33) with (38) shows that we should set

\[
G = \frac{\sqrt{h}}{N} \Theta
\]

\[
F = \frac{N}{\sqrt{h}}
\]

(70)

(71)

where the operator \(\Theta\) is defined by

\[
\Theta = - (N^2 h^{ij} D_i D_j + h^{ij} N D_i N D_j - m^2 N^2)
\]

(72)

Theorem 3 selects for us operators \(\hat{J}_0\) and \(\hat{H}_0\) (the zero in the subscripts is being used to emphasize that we are considering the minimally coupled case):

\[
\hat{J}_0 = \begin{pmatrix} 0 & \Theta \frac{\sqrt{N}}{\sqrt{h}} \\ -\frac{\sqrt{N}}{\sqrt{h}} \Theta & 0 \end{pmatrix}
\]

(73)

\[
\hat{H}_0 = \begin{pmatrix} \Theta \frac{\sqrt{N}}{\sqrt{h}} & 0 \\ 0 & -\frac{\sqrt{N}}{\sqrt{h}} \Theta \frac{\sqrt{N}}{\sqrt{h}} \end{pmatrix}
\]

(74)

which agree with [1].

Furthermore, we have

\[
\hat{J}_0 \hat{H}_0 \left( \frac{\varphi}{\pi} \right) = \begin{pmatrix} \frac{\sqrt{N}}{\sqrt{h}} \varphi \\ -\frac{\sqrt{N}}{\sqrt{h}} \Theta \varphi \end{pmatrix}
\]

(75)

and comparing with (31)–(32) shows that

\[
\hat{H}_0 \Phi = -\hat{J}_0 \Phi
\]

(76)

as desired. Therefore the Hamiltonian operator \(H\) and complex structure \(J\) obtained via Theorem 3 satisfy an equation which mimics the Schrödinger equation. It is of course not always true that \(\dot{\Phi} = t^a \nabla_a \Phi\). However, if the vector field \(t^a\) is Killing, then the operators \(H\) and \(J\) determined by Theorem 3 would indeed satisfy the Schrödinger equation

\[
H \Phi = - J \mathcal{L}_t \Phi
\]

(77)

and would therefore correctly reduce to the well-established theory for static spacetimes.

B. Non-Minimal Coupling \((\xi \neq 0)\)

For the second application of Theorem 3 we will allow non-zero values for the coupling constant \(\xi\). As in the previous application we will need to define what is meant by the classical energy of the scalar field. If we stick with the definition for the classical energy which is given by (27) we will find that Theorem 3 selects for us a Hamiltonian operator and complex structure. However, it turns out that these operators do not satisfy (33) and hence, in the static limit, the operators would not satisfy (77). However, by choosing the classical energy of the field to be the surface integral of the classical Hamiltonian, we can still apply Theorem 3 and in this case we obtain a Hamiltonian operator and complex structure which do reduce to the usual Hamiltonian operator and complex structure when considering static spacetimes.

A primary candidate for our definition of the classical energy of the field associated with the Cauchy surface \(\Sigma\) and timelike vector field \(t^a\) is the one given by Ashtekar and Magnon [27], involving the stress-energy tensor \(T_{ab}\). The
stress-energy tensor is obtained by varying the action \( \Pi \) with respect to the metric \( g_{ab} \) (for more details see [3], Chapter 3).

\[
T_{ab} = \frac{2}{\sqrt{-g}} \frac{\delta S}{\delta g^{ab}} = (1 - 2\xi)\nabla_a \Phi \nabla_b \Phi + (2\xi - \frac{1}{2})g_{ab}g^{cd}\nabla_c \Phi \nabla_d \Phi \\
- 2\xi \nabla_a \Phi \nabla_b \Phi + 2\xi g_{ab}\Phi g^{cd}\nabla_c \Phi \nabla_d \Phi \\
+ \frac{1}{2} (g_{ab}m^2 + g_{ab}R \xi - 2\xi R_{ab}) \phi^2
\]  

(78)

In the minimally coupled case, that is when \( \xi = 0 \), the stress-energy tensor reduces to (28). Straightforward computations allow us to put the integral \( \int_{\Sigma} T_{ab}n^a n^b d\Sigma \) in the following form:

\[
\int_{\Sigma} T_{ab}n^a n^b N d\Sigma = \frac{1}{2} \int_{\Sigma} \left( \frac{N}{\sqrt{h}} \nabla^2 + \frac{\sqrt{h}}{N} \phi \Gamma \phi \right) d^{n-1} x
\]

(79)

where

\[
\Gamma = (1 - 4\xi)h^{ij}N D_i N D_j + N^2 (1 - 4\xi)h^{ij} D_i D_j \\
- \xi RN^2 - 2\xi n^a n^b R_{ab} - m^2 N^2
\]

(80)

Since \( \Gamma \) is semi-bounded, symmetric and positive-definite on \( K \) and \( \frac{N}{\sqrt{h}} \) is nonzero, we can apply Theorem 2 and obtain the following operators:

\[
\hat{J}_\xi = \left( \begin{array}{cc}
0 & \frac{N}{\sqrt{h}} \\
-\frac{\sqrt{h}}{N} \Gamma \phi & 0
\end{array} \right)
\]

(82)

\[
\hat{H}_\xi = \left( \begin{array}{cc}
\Gamma \phi & 0 \\
0 & \frac{\sqrt{h}}{N} \Gamma \phi \frac{N}{\sqrt{h}}
\end{array} \right)
\]

(83)

We can proceed as we did in the minimally coupled case and find that the operators (82) and (83) satisfy

\[
\hat{H}_\xi \left( \begin{array}{c}
\phi \\
\pi
\end{array} \right) = -\hat{J}_\xi \left( \begin{array}{c}
\frac{N}{\sqrt{h}} \pi \\
\frac{\sqrt{h}}{N} \Gamma \phi
\end{array} \right)
\]

(84)

As in the minimally coupled case we have that

\[
\frac{N}{\sqrt{h}} \pi = \partial_t \Phi|_{\Sigma}
\]

(85)

However, the function \( -\frac{\sqrt{h}}{N} \Gamma \phi \) does not equal the restricted time derivative of \( \Pi = \sqrt{h} n^a \nabla_a \Phi \):

\[
-\frac{\sqrt{h}}{N} \Gamma \phi = \sqrt{h} (1 - 4\xi)h^{ij} D_i N D_j \phi + N (1 - 4\xi) h^{ij} D_i \phi D_j \phi \\
- (\xi RN + N^{-1} 2\xi n^a n^b R_{ab} + m^2 N) \phi
\]

\[
\partial_t \Pi|_{\Sigma} = \sqrt{h} h^{ij} D_i N D_j \phi + N h^{ij} D_i \phi D_j \phi \\
- (\xi RN + m^2 N) \phi
\]

(86)

(87)

There are several differences between (86) and (87), so that (84) does not mimic the Schrödinger equation. We conclude that using the stress-energy tensor to define the classical energy of the field when \( \xi \neq 0 \) will produce an undesirable choice for the Hamiltonian operator and complex structure.

If \( \xi \neq 0 \), one is either forced to reconsider the definition of the classical energy of the scalar field or abandon the use of Theorem 2. Fortunately there does exist at least one other natural method for defining the classical energy of the field; this alternate definition involves the classical Hamiltonian.

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3There is a sign error in the last term of the third equation of (3.196) on p. 88 in [3].
With $\Phi$ and $\Pi$ as above, the classical Hamiltonian is defined to be

$$H_{cl} = \Pi t^a \nabla_a \Phi - \mathcal{L}$$

$$= \frac{N}{\sqrt{h}} \Pi^2 - \mathcal{L}$$

$$= \frac{1}{2} \left( \frac{N}{\sqrt{h}} \Pi^2 + N \sqrt{h} \partial_i \Phi \partial_j \Phi + N \sqrt{h} (m^2 + \xi R) \Phi^2 \right)$$

The alternate definition for the classical energy of the scalar field associated to the hypersurface $\Sigma$ and the vector field $t^a$ is the surface integral of this Hamiltonian:

$$CE_H = \int_{\Sigma} H_{cl} d^{n-1}x$$

In the case of minimal coupling, this definition for the classical energy of the scalar field coincides with Ashtekar and Magnon’s definition which involves the stress-energy tensor. We now show that by using the surface integral of the classical Hamiltonian to represent the classical energy of the field, we are still able to apply Theorem 2. Moreover, Theorem 2 determines a unique Hamiltonian operator and unique complex structure that reduce to the appropriate operators when considering static spacetimes.

By using the definition of the Hamiltonian and applying integration by parts we obtain

$$CE_H = \frac{1}{2} \int_{\Sigma} \left( \frac{N}{\sqrt{h}} \pi^2 + N \sqrt{h} \partial_i \Phi \partial_j \Phi + N \sqrt{h} (m^2 + \xi R) \Phi^2 \right) d^{n-1}x$$

The operator

$$- \Upsilon = N^2 h^{ij} D_i D_j + h^{ij} N D_i N D_j - N^2 (m^2 + R \xi)$$

is positive-definite, semi-bounded and symmetric on $K$. We can therefore apply Theorem 2 by letting $G = \frac{N}{\sqrt{h}} \Upsilon$ and $F = \frac{N}{\sqrt{h}}$. We obtain for $\hat{J}_\xi$ and $\hat{H}_\xi$ the following operator-valued matrices:

$$\hat{J}_\xi = \left( \begin{array}{cc} 0 & \Upsilon \frac{-N}{\sqrt{h}} \\ -\frac{N}{\sqrt{h}} \frac{\Upsilon}{N} & 0 \end{array} \right)$$

$$\hat{H}_\xi = \left( \begin{array}{cc} \Upsilon \frac{1}{2} & 0 \\ 0 & \frac{N}{\sqrt{h}} \Upsilon \frac{1}{2} \end{array} \right)$$

Proceeding as in the previous cases, we find that the operators (94) and (95) satisfy

$$\hat{H}_\xi \left( \begin{array}{c} \varphi \\ \pi \end{array} \right) = -\hat{J}_\xi \left( \begin{array}{c} \frac{N}{\sqrt{h}} \varphi \\ \frac{N}{\sqrt{h}} \Upsilon \varphi \end{array} \right)$$

However, this time we have

$$\left( \begin{array}{c} \frac{N}{\sqrt{h}} \varphi \\ -\frac{N}{\sqrt{h}} \Upsilon \varphi \end{array} \right) = \partial_t \left( \begin{array}{c} \Phi \\ \sqrt{h} n^a \nabla_a \Phi \end{array} \right) \bigg|_{\Sigma}$$

That is, (96) does indeed reduce to the Schrödinger equation when $t^a$ is a Killing vector field.

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4The usual definition for the classical Hamiltonian assumes that $t^a$ is a Killing vector field [4,5]. Therefore it may be more appropriate to call this a generalized classical Hamiltonian.
VI. DISCUSSION

In Section III we summarized Ashtekar and Magnon’s uniqueness result (Theorem 1). By requiring the quantum energy of the scalar field to be equal to the classical energy of the field at each instant of time, they were able to uniquely specify a complex structure $J$ at each instant of time. However, their result depended on the need to define the Hamiltonian operator.

Our main result (Theorem 2) was discussed in Section IV. We showed that the Ashtekar and Magnon energy condition uniquely determines not only the complex structure, but also the Hamiltonian operator. As shown in Section V A, Ashtekar and Magnon’s result is thus a special case of Theorem 2.

An important consequence of Theorem 2 concerns the case of non-trivial coupling ($\xi \neq 0$). We saw in Section V B that the usual definition for the classical energy produces a complex structure and Hamiltonian operator that do not reduce to the appropriate operators if the spacetime is assumed to be static. However, also in Section V B, we showed that using the classical Hamiltonian to define the classical energy produces via Theorem 2 a complex structure and Hamiltonian operator which do have the correct limits in the static case.

It is somewhat disturbing that the two obvious formulations of the classical energy, namely the stress-energy tensor and the classical Hamiltonian, fail to agree when $\xi \neq 0$. The results of Section V B suggest that the latter is to be preferred.

It would therefore be worthwhile to further examine the properties of the generalized classical energy (90). For instance, under what circumstances is it conserved? The stress-energy tensor (78) is obtained by varying the action (1) with respect to the metric $g_{ab}$. By suitably modifying the action prior to carrying out the variation, is it possible to obtain the same classical energy using the stress-energy tensor that is obtained using the classical Hamiltonian? We have shown that if $\xi \neq 0$ then the complex structure and Hamiltonian operator obtained using the stress-energy tensor are different from those obtained using the classical Hamiltonian. What is the relationship between the two different Fock spaces which are associated with these two different choices for the classical energy?

Finally, we emphasize that both Ashtekar and Magnon’s work and ours consider only hypersurface orthogonal observers. While this description lends itself naturally to globally hyperbolic spacetimes, in which one can always choose such observers, it does not address stationary but non-static observers, let alone more general rotating observers. Some preliminary ideas on how to deal with these cases can be found in [7-8].

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