Toda lattices with indefinite metric II: 
Topology of the iso-spectral manifolds

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Abstract

We consider the iso-spectral real manifolds of tridiagonal Hessenberg matrices with real eigenvalues. The manifolds are described by the iso-spectral flows of indefinite Toda lattice equations introduced by the authors [Physica, 91D (1996), 321-339]. These Toda lattices consist of $2^{N-1}$ different systems with hamiltonians $H = \frac{1}{2} \sum_{k=1}^{N} y_k^2 + \sum_{k=1}^{N-1} s_k s_{k+1} \exp(x_k - x_{k+1})$, where $s_i = \pm 1$. We compactify the manifolds by adding infinities according to the Toda flows which blow up in finite time except the case with all $s_i s_{i+1} = 1$. The resulting manifolds are shown to be nonorientable for $N > 2$, and the symmetric group is the semi-direct product of $(\mathbb{Z}_2)^{N-1}$ and the permutation group $S_N$. These properties identify themselves with “small covers” introduced by Davis and Januszkiewicz [Duke Mathematical Journal, 62 (1991), 417-451]. As a corollary of our construction, we give a formula on the total numbers of zeroes for a system of exponential polynomials generated as Hankel determinants.

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1 Introduction

The finite non-periodic Toda lattice equation \[14\] describes a hamiltonian system of \( N \) particles on a line interacting pairwise with exponential forces. The hamiltonian \( H_T \) of the system is given by

\[
H_T = \frac{1}{2} \sum_{k=1}^{N} y_k^2 + \sum_{k=1}^{N-1} \exp(x_k - x_{k+1}).
\] (1.1)

Flaschka \[6\] introduced a change of variables

\[
a_k = -\frac{y_k}{2}, \quad k = 1, \ldots, N,
\] (1.2)

and

\[
b_k = \frac{1}{2} \exp \left( \frac{x_k - x_{k+1}}{2} \right), \quad k = 1, \ldots, N - 1,
\] (1.3)

to write the Toda equation in the Lax form:

\[
\frac{d}{dt} L_T = [B_T, L_T],
\] (1.4)

where \( L_T \) is an \( N \times N \) symmetric “tridiagonal” matrix with real entries,

\[
L_T = \begin{pmatrix}
a_1 & b_1 & 0 & \cdots & 0 \\
b_1 & a_2 & b_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & a_{N-1} & b_{N-1} \\
0 & \cdots & \cdots & b_{N-1} & a_N
\end{pmatrix}
\] (1.5)

and \( B_T \) is the skew symmetric matrix defined by

\[
B_T = \prod_a L_T := (L_T)_{>0} - (L_T)_{<0}.
\] (1.6)

Here \((L_T)_{>0} (\leq 0)\) denotes the strictly upper (lower) triangular part of \( L_T \). As an immediate consequence of the Lax form \([14]\), the eigenvalues of \( L_T(t) \) are time-invariant. Moreover, as \( t \to \infty \), \( L_T(t) \) converges to a diagonal matrix with eigenvalues of \( L_T \) in the decreasing order \([14]\), which is referred as “the sorting property”.

Tomei \[17\] used the Toda flow \([14]\) to study the iso-spectral manifolds of symmetric tridiagonal matrices, i.e., in the form of \([1.3]\). He showed several interesting properties of the manifolds: The manifolds are orientable, with Euclidean \( \mathbb{R}^{N-1} \) as the universal covering, and the symmetry group is \((\mathbb{Z}_2)^{N-1} \times S_N\), i.e., the the direct product of \((\mathbb{Z}_2)^{N-1}\)
and the permutation group $S_N$. Davis [3] generalized Tomei’s manifolds using reflection groups acting on some simple polytope.

In [12], the authors considered equation (1.4) replacing $L_T$ with $\tilde{L} = LS$ and $B_T$ with $\tilde{B} = \prod_a L_a$, where $L$ is full symmetric and $S$ is a diagonal matrix, $S = \text{diag}(s_1, \ldots, s_N)$ with $s_i = \pm 1$. An explicit solution formula is given by extending the “orthonormalization” method introduced in [11]. In particular, for the case of tridiagonal $\tilde{L}(= L_T S)$, in analogue to the Toda equation (1.4), the corresponding hamiltonian $H$ is given by

$$H = \frac{1}{2} \sum_{k=1}^{N} y_k^2 + \sum_{k=1}^{N-1} s_k s_{k+1} \exp(x_k - x_{k+1}).$$  \quad (1.7)

In this paper, we consider the integral manifolds of the hamiltonian systems given by (1.7). With different signs of $s_i$’s, there are a total of $2^{N-1}$ different type of Toda lattices. We shall refer them except the one with all positive or negative $s_i$’s as “indefinite Toda lattices”. Note from (1.7) that $H$ includes some attractive forces, thereby is not positive definite. One then expects a “blowing up” in solutions. Indeed, it was shown in [12] that generically there are two types of solutions, having either the sorting property or blowing up to infinity in finite time. The structure of $\tau$-functions describing the solutions was also studied, and they are not positive definite as for the original Toda equation.

Indefinite Toda lattices with the hamiltonian (1.4) arise in the symmetry reduction of the so-called Wess-Zumino-Novikov-Witten model [5], which is one of the most important models in quantum field theory and string theory. It was then shown [5] that the reduced system contains all the indefinite Toda lattices. Even though each of these Toda lattices is singular in the sense of blow-up in the solution, these are expected to be regularized by gluing them together through “infinities” in the full reduced system. The main purpose of the present paper is to give a concrete description of such regularization. Namely we study a compactification of the integral manifolds of these indefinite Toda lattices.

Our method is based on the explicit solution given in [12], which is different from the approach based on the Bruhat decomposition. The Bruhat decomposition of the special linear group $G$ is

$$G = \bigcup_{w \in W} N_- w B_+ ,$$ \quad (1.8)

where $N_-$ is the unipotent subgroup of lower triangular matrices with 1’s on the diagonal, $B_+$ is the Borel subgroup of upper triangular matrices, and $W$ is the Weyl group of $G$. The cell of maximal dimension is $N_- B_+$ (i.e. $w = \text{id}$) which corresponds to the LU factorization. It is shown that the blow-up of the indefinite Toda flow with $\tilde{L} \in G$ at $t_0$ corresponds to the intersection,

$$e^{t_0 \tilde{L}(0)} \in N_- w B_+ , \quad \text{where } w \neq \text{id} .$$ \quad (1.9)
The codimension of $N \cdot wB_+$ (which is given by the length of $w$) then gives the degeneracy of the blow-up. The complex version of the decomposition has been studied by Flaschka and Haine [7, 8] where they use “Painlevé analysis” to study the singularities and to compactify the flag manifold $G/B_+$.

The paper is organized as follows. In Section 2, we give a summary of the results obtained in [12] and reference therein for a basic background of the present paper. In Section 3, we provide a detail study of the solution behaviors. In particular, we give the total number of blow-ups for generic orbits, which extends the well known result of the total positivity of $\tau$-functions studied in [10]. In Section 4, we give an explicit construction of gluing indefinite Toda lattices by adding infinities, i.e., compactification of the iso-spectral manifold. We start with the cases of $N = 2$ and 3 as examples, then proceed to the general case. In Section 5, we give a brief account of CW decomposition of the compactified iso-spectral manifold and show its nonorientability.

## 2 Toda lattice with indefinite metric

For the hamiltonian (1.7), we introduce a change of variables due to Flaschka [6]

$$s_k a_k = -\frac{y_k}{2}, \quad k = 1, \ldots, N,$$

$$b_k = \frac{1}{2} \exp \left( \frac{x_k - x_{k+1}}{2} \right), \quad k = 1, \ldots, N - 1,$$

and $t \to t/2$ to write hamilton’s equations in the following form with $b_0 = b_N = 0$,

$$\frac{da_k}{dt} = s_{k+1}b_k^2 - s_{k-1}b_{k-1}^2,$$  \hspace{1cm} (2.3)

$$\frac{db_k}{dt} = \frac{1}{2} b_k(s_{k+1}a_{k+1} - s_ka_k).$$  \hspace{1cm} (2.4)

This system can be expressed in the Lax form,

$$\frac{d}{dt} L = [B, L],$$  \hspace{1cm} (2.5)

where $L$ is an $N \times N$ tridiagonal matrix with real entries,

$$L = \begin{pmatrix}
    s_1a_1 & s_2b_1 & 0 & \cdots & 0 \\
    s_1b_1 & s_2a_2 & s_3b_2 & \cdots & 0 \\
    \vdots & \ddots & \ddots & \ddots & \vdots \\
    0 & \cdots & \cdots & s_{N-1}a_{N-1} & s_Nb_{N-1} \\
    0 & \cdots & \cdots & s_{N-1}b_{N-1} & s_Na_N
\end{pmatrix}$$  \hspace{1cm} (2.6)
and $B$ is the projection of $L$ given by

$$B = \frac{1}{2} \prod_a L := \frac{1}{2} [(L)_{>0} - (L)_{<0}] .$$

(2.7)

Note from (2.6), $L = L_T S$ where $L_T$ is a symmetric tridiagonal matrix (1.3) and $S$ is a diagonal matrix $S = \text{diag}(s_1, \cdots, s_N)$. In [12], more general situation where $L_T$ is a “full” symmetric matrix is considered. An explicit formula for the solution is given based on the inverse scattering method. Here we give a summary of the results obtained in [12].

The inverse scattering scheme for (2.3) consists of two linear equations,

$$L \Phi = \Phi \Lambda ,$$

(2.8)

$$\frac{d}{dt} \Phi = B \Phi ,$$

(2.9)

where $\Phi$ is the eigenmatrix of $L$, and $\Lambda$ is $\text{diag}(\lambda_1, \cdots, \lambda_N)$. It is shown that $\Phi$ can be normalized to satisfy

$$\Phi S^{-1} \Phi^T = S^{-1}, \quad \Phi^T S \Phi = S. \quad (2.10)$$

In particular, if $S = I$, the identity matrix, then (2.10) implies that $L$ can be diagonalized by an orthogonal matrix $O(N)$, and if $S = \text{diag}(1, \cdots, 1, -1, \cdots, -1)$ then the diagonalization is obtained by a pseudo-orthogonal matrix $O(p, q)$ with $p + q = N$. It should be noted that with the normalization (2.10), the eigenmatrix $\Phi$ becomes complex in general, even in the case that all the eigenvalues are real. This is simply due to a case where the sign of $\sum_{k=1}^N s_k \phi_k^2(\lambda_i)$ in $\Phi^T S \Phi$ differs from that of $s_i$.

The eigenmatrix $\Phi$ consists of the eigenvectors of $L$, $L \phi = \lambda \phi$, with $\phi(\lambda_k) \equiv (\phi_1(\lambda_k), \cdots, \phi_N(\lambda_k))^T$ for $k = 1, 2, \cdots, N$,

$$\Phi \equiv [\phi(\lambda_1), \cdots, \phi(\lambda_N)] = [\phi_i(\lambda_j)]_{1 \leq i,j \leq N} .$$

(2.11)

Then (2.10) give the “orthogonality” relations

$$\sum_{k=1}^N s_k^{-1} \phi_i(\lambda_k) \phi_j(\lambda_k) = \delta_{ij} s_i^{-1} ,$$

(2.12)

$$\sum_{k=1}^N s_k \phi_k(\lambda_i) \phi_k(\lambda_j) = \delta_{ij} s_i .$$

(2.13)

With (2.12), we now define an inner product $\langle \cdot, \cdot \rangle$ for two functions $f$ and $g$ of $\lambda$ as

$$\langle f, g \rangle := \sum_{k=1}^N s_k^{-1} f(\lambda_k) g(\lambda_k),$$

(2.14)
which we write as $<fg>$ in the sequel. The metric in the inner product is given by

$$d\alpha(\lambda) = \sum_{k=1}^{N} s_k^{-1} \delta(\lambda - \lambda_k) d\lambda,$$

(2.15)

which leads to an indefinite metric due to a choice of negative entries $s_k$ in $S$. The entries of $L$ are then expressed by

$$a_{ij} := (L)_{ij} = s_j <\lambda \phi_i \phi_j > .$$

(2.16)

The time evolution of $\Phi(t)$ can be obtained by the orthonormalization procedure of Szegö [16] with respect to the metric (2.14). This also generalizes the orthonormalization method introduced by Kodama and McLaughlin [11]. An explicit form of $\Phi(t)$ is then given by

$$\phi_i(\lambda, t) = \frac{e^{\lambda t}}{\sqrt{D_i(t)D_{i-1}(t)}} \left| \begin{array}{ccc} s_{1c_{11}} & s_{2c_{12}} & \cdots & s_{i-1c_{1,i-1}} & \phi^0_1(\lambda) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{1c_{11}} & s_{2c_{12}} & \cdots & s_{i-1c_{i,i-1}} & \phi^0_i(\lambda) \end{array} \right|,$$

(2.17)

where $\phi^0_i(\lambda) := \phi_i(\lambda, 0)$, $c_{ij}(t) = <\phi^0_i \phi^0_j e^{\lambda t}>$, and $D_k(t)$ is the determinant of the $k \times k$ matrix with entries $s_i c_{ij}(t)$, i.e.,

$$D_k(t) = \left| \left( s_i c_{ij}(t) \right)_{1 \leq i,j \leq k} \right|.$$

(2.18)

The determinants $D_k(t)$ are normalized as $D_k(0) = 1$ for all $k$. With the formula (2.17), we now have the solution (2.16) of the inverse scattering problem (2.8) and (2.9).

It is immediate from the explicit formula (2.17) that

**Proposition 2.1.** Suppose $D_i(t_0) = 0$ for some $t_0$ and some $i$, then $L(t)$ blows up to infinity at $t_0$.

In the tridiagonal case, the determinants $D_i(t)$’s can be written as the so-called $\tau$-functions. Then the solutions $a_i$’s and $b_i(t)$’s are expressed in the form,

$$s_i a_i = \frac{d}{dt} \log \frac{\tau_i}{\tau_{i-1}},$$

(2.19)

$$s_i s_{i+1} b_i^2 = \frac{\tau_{i+1} \tau_{i-1}}{\tau_i^2}.$$  

(2.20)

The indefinite Toda lattice equations (2.3) and (2.4) are written in the bilinear form,

$$\frac{d^2}{dt^2} \log \tau_i = \frac{\tau_{i+1} \tau_{i-1}}{\tau_i^2}.$$  

(2.21)
These $\tau$-functions $\tau_i$ have a simple structure, that is, a Hankel determinant given by

$$
\tau_i = \begin{vmatrix}
\tau_1 & \tau_1' & \tau_1'' & \ldots & \tau_1^{(i-1)} \\
\tau_1' & \tau_1'' & \tau_1^{(3)} & \ldots & \tau_1^{(i)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\tau_1^{(i-1)} & \tau_1^{(i)} & \ldots & \ldots & \tau_1^{(2i-2)}
\end{vmatrix},
$$

(2.22)

where $\tau_1$ is given by $\tau_1 = c_{11} := (\phi_1^0 e^z)^{-1} s_1^{-1} D_1$, and $\tau_1^{(i)} = d^i \tau_1 / dt^i$. The relation between $\tau_i$ and $D_i$ is given by

$$
\tau_i = \frac{1}{s_i^{i-1}} \left[ \prod_{k=1}^{i-1} (s_k s_{k+1} (b_k^0)^2)^{i-k} \right] D_i,
$$

(2.23)

where $b_k^0 = b_k(0)$. (The formula (6.5) in [12] should read as (2.23).)

**Remark 1.** From (2.23), $\tau_i$ becomes 0 if $b_k^0 = 0$ for some $k < i$, thereby (2.24) is no longer valid. This is because the system can be reduced to two or more independent subsytems.

**Remark 2.** The system (2.5) can be also written in the Lax form with a tridiagonal Hessenberg matrix with $\alpha_i := s_i a_i$ and $\beta_i := s_i s_{i+1} b_i^2$

$$
L_H = \begin{pmatrix}
\alpha_1 & 1 & 0 & \cdots & 0 \\
\beta_1 & \alpha_2 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \alpha_{N-1} & 1 \\
0 & \cdots & \cdots & \beta_{N-1} & \alpha_N
\end{pmatrix},
$$

(2.24)

The matrix $L_H$ is similar to $L$ in (2.6), $L_H = H L H^{-1}$, with the diagonal matrix $H = diag(1, s_2 b_1, s_2 s_3 b_2, \ldots, \prod_{i=1}^{N-1} s_i s_{i+1} b_i)$. Then (2.5) becomes

$$
\frac{d}{dt} L_H = [B_H, L_H],
$$

(2.25)

where $B_H = (L_H)_{<0}$, i.e., the lower triangular part of $L_H$. Note from (2.20), if $\tau_{i-1}$ or $\tau_{i+1}$ changes sign, then $b_i^2$ becomes negative. In this paper we use the variables $\alpha_i$ and $\beta_i$, and consider the real form of the iso-spectral manifold of the system of equations (2.3) and (2.4).

**Remark 3.** With the Hessenberg matrix form of (2.23), the Toda equation with the general tridiagonal matrix can be solved in the similar way [13].
We use the $\tau$-functions (2.22) to study the behaviors of solutions for all $t \in \mathbb{R}$. For $t > 0$, a necessary and sufficient condition for solutions being nonsingular is obtained in [9].

First we have the following forms of $\tau$-functions as sums of exponential functions:

**Proposition 3.1** Write $\tau_1 = \sum_{k=1}^{N} \rho_i e^{\lambda_i t}$ where $\rho_i = s_i[\phi_i^0(\lambda_i)]^2 \neq 0$. Then $\tau_i$ for $i = 1, 2, \cdots, N$ in (2.22) can be expressed as

$$
\tau_i(t) = \sum_{J_N=(j_1,\cdots,j_i)} \rho_{j_1}\rho_{j_2}\cdots\rho_{j_i} \begin{vmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ \lambda_{j_1}^{-1} & \cdots & \lambda_{j_i}^{-1} \end{vmatrix}^2 \exp(\sum_{k=1}^{i} \lambda_{j_k} t), \quad (3.1)
$$

where $J_N$ represents all possible combinations for $1 \leq j_1 < \cdots < j_i \leq N$. In particular $\tau_N(t) = \rho_1 \cdots \rho_N \prod_{i<j}(\lambda_i - \lambda_j)^2 \exp(\sum_{k=1}^{N} \lambda_k t)$.

If $\beta_i^0 := s_i s_{i+1}(t_i^0)^2 > 0$ for $i = 1, \cdots, N$, then all $\rho_i > 0$, and from (3.1) we see that $\tau_i(t)$’s are all positive definite. Conversely, if $\tau_i(t)$ are all positive definite, then $\beta_i(t) > 0$ for all $t$ by (2.20). On the other hand, if $\rho_k < 0$ for some $k$, then the existence of zeroes for $\tau_i(t)$’s is guaranteed by Karlin [10]. We summarize these facts as:

**Proposition 3.2** Write $\tau_1 = \sum_{k=1}^{N} \rho_i e^{\lambda_i t}$, $\rho_k \neq 0$ for $k = 1, \cdots, N$. Then $\tau_k(t)$ are all sign definite, i.e. have no zeroes, for $t \in \mathbb{R}$ if and only if either $\rho_k > 0$ or $\rho_k < 0$ for all $k$.

The above proposition implies the Toda flow is necessarily singular for $t \in \mathbb{R}$ if $\beta_k^0 < 0$ for some $k$. Moreover, we can give the precise number of blow-ups for generic orbits based on the construction in the next section.

**Definition** A zero $t_0$ of $\tau_i(t)$’s is said to be nondegenerate if $\tau_k(t_0) = 0$ for some $k$ and $\tau_j(t_0) \neq 0$ for $j \neq k$.

Note that (2.21) can be written as

$$
\tau_i \tau_i'' - (\tau_i')^2 = \tau_{i+1} \tau_{i-1}. \quad (3.2)
$$

If $\tau_i$ has multiple zeroes at $t_0$, then $\tau_{i+1}$ or $\tau_{i-1}$ must be zero at $t_0$. So a zero of multiplicity more than 1 is necessarily degenerate. From the above definition, a zero is nondegenerate if only one $\tau$-function reaches zero. Note that nondegenerate zeroes are generic. Then we have:

**Proposition 3.3** Write $\tau_1 = \sum_{k=1}^{N} \rho_i e^{\lambda_i t}$. Suppose that all zeroes of $\tau$-functions are nondegenerate. Denote by $m$ the number of negative $\rho_i$’s, and by $n_i$ the number of zeroes of $\tau_1(t)$. Then the total number of the zeroes depends only on $m$ and is given by

$$
\sum_{i=1}^{N-1} n_i = m(N - m). \quad (3.3)
$$
Note here that the number of negative $\rho_i$’s takes at most $[N/2]$, the integer part of $N/2$, because of the symmetry of $\alpha_i$ and $\beta_i$ under the change $\tau_1 \rightarrow -\tau_1$. Thus the total number of zeros is a topological quantity of the $\tau$-functions, and this Proposition extends a result of Karlin [10]. The proof will be given in the next section.

Though there are generic orbits that blow up to infinity in finite time, asymptotically, we have:

**Proposition 3.4** Suppose that $\beta^0_i \neq 0$ for $i = 1, \ldots, N-1$, and $\lambda_1 > \cdots > \lambda_N$. Then $L(t) \rightarrow \text{diag}(\lambda_1, \ldots, \lambda_N)$ as $t \rightarrow \infty$, and $L(t) \rightarrow \text{diag}(\lambda_N, \ldots, \lambda_1)$ as $t \rightarrow -\infty$.

**Proof.** Since $\beta^0_i \neq 0$ for all $i$, $\tau_i(t)$ is given by $\sum_{k=1}^{\lambda_{i+1}} \rho_k e^{\lambda_i t}$ with $\rho_i \neq 0$. We calculate directly asymptotic behavior of solution using (2.19) and (2.20). For large $t$, we have

$$
\alpha_i(t) = \frac{d}{dt} \log \frac{\tau_i}{\tau_{i-1}}
= \frac{d}{dt} \log \frac{\rho_1 \cdots \rho_i \prod_{1 < k < 1 < i} (\lambda_k - \lambda_i)^2 e^{\sum_{k=1}^{i-1} \lambda_k t}}{\rho_1 \cdots \rho_{i-1} \prod_{1 < k < i < i-1} (\lambda_k - \lambda_i)^2 e^{\sum_{k=1}^{i-1} \lambda_k t}} = \lambda_i,
$$

$$
\beta_i(t) = \frac{\tau_{i+1} \tau_{i-1} - \tau_i^2}{\tau_i^2}
= \gamma e^{(\lambda_{i+1} - \lambda_i) t} \rightarrow 0,
$$

where $\gamma$ is a nonzero constant. Similarly, one can show $L(t) \rightarrow \text{diag}(\lambda_N, \ldots, \lambda_1)$ as $t \rightarrow -\infty$. ∎

## 4 Topology of indefinite Toda lattices

In this section, we give an explicit construction of a compactification (or regularization) of the integral manifolds of the indefinite Toda lattices. For this purpose, we use $L_H$ given in (2.24) with all eigenvalues being real and distinct as describing the underlying manifolds. We assume that the eigenvalues of $L_H$ are ordered as $\lambda_1 > \cdots > \lambda_N$. We associate $L_H$ with an $S$ matrix through $\text{sgn}(\beta_i) = s_i s_{i+1}$ as in (2.24). With different signs of $\beta_i$, there are a total of $2^{N-1}$ different Toda lattices. As we show here, the compactification is then given by gluing these different Toda lattices.

First we study the simplest case with $N = 2$ which is also the most important case as we shall see later. Then we give a detail discussion on the case $N = 3$, and extend those results to the general case.

Let $L_H$ be a $2 \times 2$ Hessenberg matrix given by

$$
L_H = \begin{pmatrix} \alpha_1 & 1 \\ \beta_1 & \alpha_2 \end{pmatrix} = H L H^{-1}, \quad (4.1)
$$
where $L = L_TS$ is given by (2.6) and $H = \text{diag}(1, s_2 b_1)$ (see Remark 2). We assume that the eigenvalues of $L_H$ are all real and $\lambda_1 > \lambda_2$. The characteristic polynomial for $L_H$ leads to

$$\alpha_1 + \alpha_2 = \lambda_1 + \lambda_2, \quad \alpha_1 \alpha_2 - \beta_1 = \lambda_1 \lambda_2.$$ (4.2)

Eliminating $\alpha_2$ in (4.2), we have

$$\beta_1 = - (\alpha_1 - \lambda_1)(\alpha_1 - \lambda_2).$$ (4.3)

The plot of (4.3) is shown in Fig. 1. There are two critical points ($b_1 = 0$ or $\beta_1 = 0$) corresponding to $L = \text{diag}(\lambda_1, \lambda_2)$ and $\text{diag}(\lambda_2, \lambda_1)$. Let us now give a detail analysis of the orbits of indefinite Toda lattices associated with $L_H$ in (4.1):

(i) The case $S = \text{diag}(1, 1) = I$: Then the initial data $L_0 = L(0)$ is symmetric and diagonalized by an orthogonal matrix $\Phi_0 = \Phi(0)$, i.e. $\Lambda = (\Phi_0)^{-1} L_0 \Phi_0$,

$$\Phi_0 = \begin{pmatrix} \cos \theta_0 & \sin \theta_0 \\ -\sin \theta_0 & \cos \theta_0 \end{pmatrix},$$ (4.4)

where $\theta_0$ is determined by the initial data. Note that the solution of the Toda equation depends on one parameter $\theta_0$ with fixed $\lambda_1$ and $\lambda_2$. From (2.18) and (2.17), we have

$$D_1(t) = \cos^2 \theta_0 \ e^{\lambda_1 t} + \sin^2 \theta_0 \ e^{\lambda_2 t},$$ (4.5)

and

$$\Phi(t) = \frac{1}{\sqrt{D_1(t)}} \begin{pmatrix} \cos \theta_0 \ e^{\frac{\lambda_1 t}{2}} & \sin \theta_0 \ e^{\frac{\lambda_2 t}{2}} \\ -\sin \theta_0 \ e^{\frac{\lambda_1 t}{2}} & \cos \theta_0 \ e^{\frac{\lambda_2 t}{2}} \end{pmatrix}. \quad (4.6)$$

We see from (4.5) that $D_1(t)$ is positive definite, that is, the solution is regular (no blow up) and asymptotically

$$\Phi(t) \to \text{sgn}(\cos \theta_0) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{as } t \to \infty,$$

$$\Phi(t) \to \text{sgn}(\sin \theta_0) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{as } t \to -\infty.$$

Thus $L(t)$ tends to $\text{diag}(\lambda_1, \lambda_2)$ as $t \to \infty$ and $\text{diag}(\lambda_2, \lambda_1)$ as $t \to -\infty$. The orbit corresponds to a closed component with both critical points given as the curve in $\beta_1 \geq 0$.

(ii) The case $S = \text{diag}(1, -1)$: The eigenmatrix $\Phi$ for $L$ is now in $O(1, 1)$. There are two disconnected components in $\beta_1 \leq 0$, and each one connects to either $L =$
$diag(\lambda_1, \lambda_2)$ or $L = diag(\lambda_2, \lambda_1)$. We first consider the component connected to the vertex $L = diag(\lambda_1, \lambda_2)$.

Taking the initial data with $\alpha_1^0 > \alpha_2^0$. The initial eigenmatrix $\Phi^0$ is given by

$$
\Phi^0 = \begin{pmatrix}
\cosh \mu_0 & \sinh \mu_0 \\
\sinh \mu_0 & \cosh \mu_0
\end{pmatrix},
$$

where $\mu_0$ determined by the initial data. From (2.18) and (2.17), we obtain

$$
D_1(t) = \cosh^2 \mu_0 e^{\lambda_1 t} - \sinh^2 \mu_0 e^{\lambda_2 t},
$$

and

$$
\Phi(t) = \frac{1}{\sqrt{D_1(t)}} \begin{pmatrix}
\cosh \mu_0 e^{\frac{\lambda_1}{2} t} & \sinh \mu_0 e^{\frac{\lambda_2}{2} t} \\
\sinh \mu_0 e^{\frac{\lambda_1}{2} t} & \cosh \mu_0 e^{\frac{\lambda_2}{2} t}
\end{pmatrix}.
$$

From (4.8), $D_1(t)$ has a unique zero at

$$
t_0 = \frac{2}{\lambda_1 - \lambda_2} \ln(\tanh \mu_0) < 0.
$$

For $t > t_0$, the eigenmatrix $\Phi(t)$ is real and $\Phi(t) \to I$ as $t \to \infty$, while for $t < t_0$ $\Phi(t)$ becomes pure imaginary and $\Phi(t) \to i \sgn(\sinh \mu_0) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ which implies $L(t) \to diag(\lambda_2, \lambda_1)$. Thus the solutions for $t > t_0$ and $t < t_0$ expresses components connecting to the vertices $L = diag(\lambda_1, \lambda_2)$ and $diag(\lambda_2, \lambda_1)$ respectively. We then connect these orbits at infinity, that is, we compactify the integral manifold of the flow as shown in Fig.1. The resulting manifold is isomorphic to circle $S^1$. This connection of orbits can be viewed as a gluing of two different indefinite Toda systems. To see this more precisely, we first note that the solution $L_H(t)$ is determined by the $\tau$-function $\tau_1(t)$,

$$
\tau_1(t) = s_1[\phi^0_1(\lambda_1)]^2 e^{\lambda_1 t} + s_2[\phi^0_1(\lambda_2)]^2 e^{\lambda_2 t}.
$$

Then the change of $\Phi(t)$ from real to imaginary implies the change of signs in $\rho_1 := [\phi^0_1(\lambda_1)]^2$ and $\rho_2 := [\phi^0_1(\lambda_2)]^2$. This is then equivalent to the change of signs in $s_1$ and $s_2$, or the exchange $s_1 \leftrightarrow s_2$, with fixed signs of $\rho_i$’s. As we will see below, this can be naturally generalized for the $N \times N$ case, that is, in each blow-up corresponding to $\tau_i = 0$ we glue two systems with the $S$-matrices $diag(s_1, \ldots, s_i, s_{i+1}, \ldots, s_N)$ and $diag(s_1, \ldots, s_{i+1}, s_i, \ldots, s_N)$.

Now let us consider the case $N = 3$ which contains main ideas for the general case. For $N = 3$, there are six critical points with $\beta_1 = \beta_2 = 0$ which give the vertices ($0$--$cells$) of the integral manifold. Around each vertex, there are four components assigned by the signs of $\beta_1$ and $\beta_2$, i.e. $++, +-, -+, --$. The component $++$ is shown to be a hexagon. The edges (1--cells) of the hexagon correspond to the Toda flows in the form
Figure 1: The $N=2$ indefinite Toda flow.

Figure 2: The $N=3$ indefinite Toda flow.
of either $\beta_1 = 0$ or $\beta_2 = 0$. Each vertex has four edges, two of which are non-compact corresponding to blow-up solutions. The picture of the underlying manifold is shown in Fig. 2.

We then compactify the manifold as follows: We first glue the edges as in the case $N = 2$. Then for the flows around an outgoing edge, we take a small segment transversal to the flows at some point on the the edge, and determine how the flows transport this segment after the blow-up. In Fig.2, the gluing pattern around one edge, $\beta_1 = 0$, is indicated by arrows. Repeating this process for every outgoing edges, we obtain the compactified manifold. It is immediate from the gluing pattern that the manifold is not orientable. Its Euler characteristic is calculated as $3 - 6 + 1 = -2$ (see below for a detail). Since every compact surfaces are completely classified by orientability and the Euler characteristic, the manifold is topologically isomorphic to a connected sum of two Klein bottles.

We can also describe the gluing patterns from the viewpoint of symmetry groups acting on polytopes [2,3] which provides a useful information for the study of the general case. We first mark the four components with $++$, $+-$, $-+$ and $--$ around the top vertex $V[3,2,1] := \begin{pmatrix} \lambda_3 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$ as $A, B, C$ and $D$, respectively. Then following the flows starting from these components, we mark components around other vertices. For example, the flows starting from $B$ blow up and continue to either the component $--$ of the vertex $V[3,1,2] := \begin{pmatrix} \lambda_3 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_2 \end{pmatrix}$ or $--$ of $V[2,1,3] := \begin{pmatrix} \lambda_2 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_3 \end{pmatrix}$, then they blow up again and end in the component $-+$ of the bottom vertex $V[1,2,3] := \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_2 & 1 \\ 0 & 0 & \lambda_3 \end{pmatrix}$. We then mark all of these components on the pass as $B$. This gives the way of gluing the components around each vertex, and the glued components also form a hexagon. We then have a total of four hexagons. The gluing pattern and the marking for all components are shown in Fig.3. For example, a dotted line segment between a circle and a square is glued with the other segment which has the same vertices and connects with the same edge $\beta_k = 0$. So there are 3 vertices, marked by circle, square and triangle, and 6 edges in the compactified manifold.

As we observed in the case of $N = 2$, the gluing pattern can be regarded as the permutation of vertices resulting from the permutation of the $S$ matrices (i.e. different indefinite Toda lattices). To see this, we associate each component with an $S$ matrix through $sgn(\beta_i) = s_i s_{i+1}$. For instance, $S = diag(1, -1, 1)$ corresponds to $--$. Then the gluing pattern gives for example that the marking of the component associated with $S = diag(s_1, s_2, s_3)$ of the vertex $V[3,2,1]$ is the same as the one with $S = diag(s_2, s_1, s_3)$ of $V[2,3,1]$. In general, let $V_1$ and $V_2$ be two vertices such that $PV_1P^{-1} = V_2$, then $S = diag(s_1, s_2, s_3)$ of $V_1$ is marked the same as $PSP^{-1}$ of $V_2$. In this sense, the
symmetry group of the compactified manifold is the *semi*-direct product of $S_3$ and $(\mathbb{Z}_2)^2$ \[3\]. This manifold is to be compared with the $N = 3$ Tomei’s manifold, i.e., the isospectral manifolds of tridiagonal symmetric matrices $L_T$ in (1.5). There are also four components giving hexagons depending on the signs of $b_1$ and $b_2$. The Toda flow with $L_T$ is complete (no blow-up), and each component is invariant under the flow. So no gluing is necessary, and the marking of the components is just given by the signs of $b_1$ and $b_2$ i.e. no permutation on the signs. Thus the symmetry group for the case of Tomei is the *direct* product of $S_3$ and $(\mathbb{Z}_2)^2$.

With the above marking of the components, the flows in $B, C$ or $D$ generically blow up *twice* in $t \in \mathbb{R}$. This can be verified by finding the zeroes of the $\tau$-functions $\tau_1$ and $\tau_2$: Writing $\tau_1(t) = \sum_{i=1}^3 \rho_i e^{\lambda_i t}$, $\tau_2(t)$ is given by

$$
\tau_2(t) = \rho_1 \rho_2 (\lambda_1 - \lambda_2)^2 e^{(\lambda_1+\lambda_2)t} + \rho_1 \rho_3 (\lambda_1 - \lambda_3)^2 e^{(\lambda_1+\lambda_3)t} + \rho_2 \rho_3 (\lambda_2 - \lambda_3)^2 e^{(\lambda_2+\lambda_3)t}.
$$

(4.12)

Without loss of generality we take $\rho_1 > 0$. Then there are four cases depending on the signs of $\rho_2$ and $\rho_3$:

1. $\rho_2 > 0$, $\rho_3 > 0$. In this case, $\tau_1(t)$ and $\tau_2(t)$ are positive definite. This corresponds to the flows in the component $A$.

2. $\rho_2 > 0$, $\rho_3 < 0$. Since $\tau_1(t) > 0$ for large positive $t$ and $\tau_1(t) < 0$ for large negative $t$, there is at least one zero for $\tau_1$. On the other hand, we find $(\tau_1 e^{-\lambda_3 t})' > 0$, so there is only one simple zero for $\tau_1$. Similarly, one shows that there is only one simple zero for $\tau_2$. Since $\tau_2 = \tau_1 \tau_1'' - (\tau_1)^2$, these zeroes do not coincide.

3. $\rho_2 < 0$, $\rho_3 < 0$. We show similarly as for (2) that both $\tau_1$ and $\tau_2$ have a nondegenerate zero.
(4) \( \rho_2 < 0, \rho_3 > 0 \). This case has the most interesting feature as we will see. From the asymptotic behaviors, it is easy to see both \( \tau_1 \) and \( \tau_2 \) are convex in \( t \) and have either no or two zeroes. To be more precise, we consider \((\tau_1 e^{-\lambda_3 t})' = 0\), and find the root,

\[
t_0 = \frac{1}{\lambda_1 - \lambda_2} \ln \left(\frac{-(\lambda_2 - \lambda_3)\rho_2}{(\lambda_1 - \lambda_3)\rho_1}\right) .
\] (4.13)

Then \( \tau_1(t_0) \) becomes

\[
\tau_1(t_0) = \frac{\lambda_1 - \lambda_2}{\lambda_2 - \lambda_3} \rho_2 e^{\lambda_2 t_0} + \rho_3 e^{\lambda_3 t_0}.
\] (4.14)

Also considering \((\tau_2 e^{-(\lambda_1 + \lambda_2) t})' = 0\), we find that the solution is exactly that given by (4.13), and \( \tau_2(t_0) \) becomes

\[
\tau_2(t_0) = \rho_1 e^{\lambda_1 t}(\lambda_1 - \lambda_2)(\lambda_2 - \lambda_3)\tau_1(t_0) .
\] (4.15)

From (4.14) and (4.15), if \( \tau_1(t_0) \geq 0(\leq 0) \), then \( \tau_2(t_0) \geq 0(\leq 0) \). This implies that (i) if \( \tau_1 \) (or \( \tau_2 \)) has two simple zeroes then \( \tau_2 \) (or \( \tau_1 \)) has no zero, or (ii) both \( \tau_1 \) and \( \tau_2 \) have a doubly degenerated zero at \( t_0 \). Thus the total number of zeroes in \( \tau_1 \) and \( \tau_2 \) functions is generically given by two, i.e. \( 1 + 1 = 0 + 2 = 1 \times (3 - 1) \) (Proposition 3.3).

For the general case, a direct analysis of \( \tau \)-function seems to be very hard for even counting the total number of zeroes in the \( \tau \)-functions. Instead, we study a flow passing through near the edges from the top vertex to the bottom vertex, and counts the number of blow-ups in the flow to determine the number of zeroes in \( \tau \)'s.

The general case with \( N \times N \) matrix \( L_H \) can be obtained by a direct extention of the previous examples. Here we summarize the result with some convenient notations. There are \( N! \) critical points of the flow giving the vertices of the integral manifold of the Toda system, and we denote them as

\[
V[i_1, \ldots, i_N] := \begin{pmatrix} \lambda_{i_1} & 1 & 0 & \cdots & 0 \\ 0 & \lambda_{i_2} & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_{i_{N-1}} & 1 \\ 0 & \cdots & \cdots & 0 & \lambda_{i_N} \end{pmatrix} .
\] (4.16)

We denote the edges connecting the vertices \( V[i_1, \ldots, i_N] \) and \( V[i_1, \ldots, i_{k+1}, i_k, \ldots, i_N] \)
by

\[
E_k^\pm[i_1, \ldots, i_N] := \begin{pmatrix}
\lambda_{i_1} & 1 & 0 & \cdots & 0 \\
0 & \lambda_{i_2} & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & \ldots & \ldots & \lambda_{i_{N-1}} & 1 \\
0 & \ldots & \ldots & 0 & \lambda_{i_N}
\end{pmatrix},
\]

(4.17)

where \( A \) is a \( 2 \times 2 \) matrix defined by

\[
A[i_k, i_{k+1}] = \begin{pmatrix}
\alpha_{i_k} & 1 \\
\beta_{i_k} & \alpha_{i_{k+1}}
\end{pmatrix}
\]

(4.18)

having \( \lambda_{i_k} \) and \( \lambda_{i_{k+1}} \) as the eigenvalues and the superscript \( \pm \) in the edge is assigned by \( \text{sgn}(\beta_{i_k}) \). As we know from the case \( N = 2 \), the flow on \( E_k^\pm[i_1, \ldots, i_N] \) has no singularity, while \( E_k^-[i_1, \ldots, i_N] \) is glued at infinity with \( E_k^+[i_1, \ldots, i_{k+1}, i_k, \ldots, i_N] \) denoted as \( E_k^- \) in Fig.4. We call \( E_k^-[i_1, \ldots, i_N] \) an “outgoing” edge if \( \lambda_{i_k} < \lambda_{i_{k+1}} \), and an “incoming” edge otherwise. Around each vertex, there are \( 2^{N-1} \) different components depending on the signs of \( \beta_i \)’s, that is, these components are separated by the hypersurfaces \( \mathcal{H}_k \) defined by \( \beta_k = 0 \) for \( k = 1, \ldots, N-1 \). We assume that these hypersurfaces intersect transversally in a neighborhood of each vertex. In particular, the edges \( E_k^\pm \) transversally intersect to the surface \( \mathcal{H}_k \), and the superscript \( \pm \) indicates the sides of the component separated by \( \mathcal{H}_k \). We denote these components by \( S[s_{1,2}, \ldots, s_{N-1,N}] \) where \( s_{k,k+1} := s_{k}s_{k+1} \) (note then \( \text{sgn}(\beta_k) = s_{k,k+1} \)), and also define

\[
S_k^\pm[s_{1,2}, \ldots, s_{k-1,k}, s_{k+1,k+2}, \ldots, s_{N-1,N}] := S[s_{1,2}, \ldots, s_{k-1,k}, \pm, s_{k+1,k+2}, \ldots, s_{N-1,N}],
\]

(4.19)

which are the sections of the components \( S \) devided by the surface \( \mathcal{H}_k \), and \( S = S_k^+ \cup S_k^- \). Since the edge \( E_k^+[i_1, \ldots, i_N] \) connects the vertices \( V[i_1, \ldots, i_k, i_{k+1}, \ldots, i_N] \) and \( V[i_1, \ldots, i_k, i_{k+1}, i_k, \ldots, i_N] \) without a singularity, the sections \( S_k^\pm \) of these vertices coincide. However the sections \( S_k^- \) of these vertices are not connected identically along with the signature of \( [s_{1,2}, \ldots, s_{N-1,N}] \), but with permutation on the signs in the \( S \) matrices as we showed in the case of \( 3 \times 3 \) matrix.

We now show that the connection of these sections through blow-up is completely determined by tracing the flows of the Toda lattices with \( L_H \) (i.e. this may be referred as a compactification of the iso-spectral manifold of tridiagonal Hessenberg matrices by the orbit gluing). More precisely, let a flow of \( L_H \) starts in some component of a vertex, then after blow-up, we need to determine which component of other vertex the flow goes in. The key of our approach is to use continuity of \( D_t(t)'s \) (or \( \tau_t(t)'s \)) on the initial data,
which can be easily seen from (2.18) and (2.22). Also from relations (2.19) and (2.20), the flow \( L_H(t) \) is continuous on the initial data except the times of blow-ups given by the zeroes of \( D_i(t) \)'s (or \( \tau_i(t) \)'s). Then we have:

**Lemma 4.1** Suppose \( t_0, \cdots, t_m \) are all zeroes of \( D_i(t) \)'s (\( \tau_i(t) \)'s) with initial data \( L^0_H \). Then for any finite \( \bar{t} \) with \( \bar{t} \neq t_k \) for \( k = 0, \cdots, m \), there exists a neighborhood of \( L^0_H \), such that its time evolution at \( \bar{t} \) is a neighborhood of \( L_H(\bar{t}) \).

We now study the flows around the edges. If the superscript of an edge denoted by (1.17) is “+”, then there is no blow up in the flow. Suppose an initial data \( L^0_H \) is on \( E^-_k[i_1, \cdots, i_N] \). In a small neighborhood of \( L^0_H \), we can take a hypersurface \( G_k \) of \( L^0_H \) transversal to the flow with codimension one. Since the edge \( E^-_k[i_1, \cdots, i_N] \) is shared by all the components \( S_k^-[s_1, \cdots, s_k, \cdots, s_{N+1}] \), \( G_k \) can be divided into these \( 2^{N-1} \) sections. If \( \lambda_{ik} < \lambda_{ik+1} \), \( L_H(t) \) blows up at some \( t_0 \), then it jumps to edge \( E^-_k[i_1, \cdots, i_k, \cdots, i_N] \equiv E^-_k \) as indicated in Fig. 4. By Lemma 4.1, for some \( t > t_0 \), we take \( G_k \) sufficiently small such that \( G_k \) evolves to a hypersurface \( G'_k \) at \( L_H(t) \). We are interested in how sections of \( G_k \) are glued with those of \( G'_k \). We have the following proposition:

**Proposition 4.1** Every point of the section \( S_k^-[s_1, \cdots, s_{k+1}] \) on the hypersurface \( G_k \) is glued by the Toda flow with a unique point in the section marked by \( S_k^-[s_1, \cdots, s_{k+1}] \equiv S_k^- \) in Fig. 4) after the blow-up, that is, the gluing pattern is given by the permutation of the signs \( s_k \leftrightarrow s_{k+1} \).

**Proof.** We calculate directly \( D_i(t) \)'s with the initial data \( L^0_H \) given on the edge \( E_k^- \) as

\[
D_j(t) = \exp\left(\sum_{l=1}^{j} \lambda_{il}t\right), \quad \text{for } j \neq k
\]

\[
D_k(t) = (-\sinh^2 \mu_0 e^{\lambda_{ik}t} + \cosh^2 \mu_0 e^{\lambda_{ik+1}t}) \exp\left(\sum_{l=1}^{k-1} \lambda_{il}t\right).
\]

(4.20)

From (1.20), we see \( D_j(t) \) are positive definite for \( j \neq k \), while \( D_k(t) \) has one zero at some \( t_0 \). By continuity of \( D_i(t) \)'s, \( G_k \) can be chosen such that all the flows starting from \( G_k \) have \( D_k(t) \) reaches zero first and the zero is nondegenerate. So from (2.23), \( \tau_k(t) \) reaches zero first. From (2.20), the sign change of \( \tau_k(t) \) implies the sign changes of \( \beta_{k+1} \) and \( \beta_{k+1} \), that is, \( S_k^-[s_1, \cdots, s_k, \cdots, s_{N+1}] \) connects to \( S_k^-[s_1, \cdots, s_k, \cdots, s_{N+1}] \equiv S_k^- \), as shown in Fig. 4.\]

This proposition gives a clear picture of the gluing of orbits around edges. In some sense, components near an outgoing edge stay close to the edge and connect to appropriate components near the incoming edge. For an orbit whose first zero of \( \tau_i(t) \)'s is nondegenerate, by Lemma 4.1, there exists a neighborhood which will blow up and evolve to the same component as the orbit. So to see the gluing pattern away from
Figure 4: The gluing pattern of the component $S_k$ through the flow.

Proposition 4.2 Suppose $S$ is a component with nonzero outgoing edges $E_1^-, \cdots, E_r^-$. $C_{E_1}, \cdots, C_{E_r}$ as defined above. Then $S = \bigcup_{k=1}^r C_{E_k}$, i.e., the closure of the union of $C_{E_k}$’s.

Proof. We prove by contradiction. Suppose the proposition is not true, that is, there is some region $R$ in $S$ separated from edges, i.e., the boundary of $R$ has a degenerate first zero. By going forward in time, $R$ blows up to some region of some component. This region must also be separated from edges, otherwise, we can apply Lemma 4.1 around the edge back in time, which contradicts $R$ separated from edges. Eventually, $R$ reaches some component of the bottom vertex $V[1, \cdots, N]$, and no more blowup after that. Suppose $\tilde{R}$ is the second last region $R$ visits. Since the boundary of $\tilde{R}$ has degenerate zeroes, it will change to a different component from $\tilde{R}$ does. But by Lemma 4.1, for large $t$, the boundary should get together with $\tilde{R}$, which is a contradiction.

With Propositions 4.1 and 4.2, we have a complete pattern of gluing which provides the compactified manifold of the solution of indefinite Toda lattices. From the gluing pattern, we now prove Proposition 3.3.

Proof of Proposition 3.3. First we determine the signs of $\beta_i(t)$’s for $t \to -\infty$ by studying asymptotic behaviors of $\tau_i(t)$’s. From (2.20) and (3.4), we have for large negative
we start in component discussed in the previous section, the
where $P$ is a component follow edges, all we need to do is to count how many blow-ups are needed through
the edges from $V[N, \cdots, 1]$ to $V[1, \cdots, N]$. We count by induction on $N$. For $N = 2$, the Proposition holds. Suppose it is valid for $N - 1$. If $s_N = 1$, it takes $m$ blow-ups to get to the top, then the system is reduced to $N - 1$. The total number of zeroes is thus $(N - 1 - m)m + m = m(N - m)$. The case $s_N = -1$ can be shown similarly.

5 CW decomposition and nonorientability

In this section, we give a CW decomposition of the compactified manifold $M_N$ obtained in the previous section, and show that $M_N$ is not orientable for $N > 2$. For the iso-spectral manifolds in the case of the tridiagonal symmetric matrices, a CW decomposition was given in [11]. In our case, the $j$-cells are given by the glued components marked by the signs of $\beta_i$’s with $j$ signs and $N - j - 1$ zeroes, say, $\beta_{k_1} = \beta_{k_2} = \cdots = \beta_{k_{N-j-1}} = 0$. Let $S_i^j$ be a component associated with a vertex $V$ of a $j$-cell. Then with the gluing pattern discussed in the previous section, the $j$-cell $S_i^j$ can be written as

$$
\hat{S}_i^j = \bigcup_P (PS_P^jP^{-1})_{PVP^{-1}} \bigcup \{\infty\},
$$

(5.1)

where $P \in S_N$ keeping $\lambda_{k_1}, \cdots, \lambda_{k_{N-j-1}}$ fixed. For example, in the case of $N = 3$, the vertices are 0-cells, the edges are 1-cells, and the hexagons are 2-cells. Then the hexagon marked by $B$ in Fig. 3 is expressed by a union of $S[1, -1]_{V[3,2,1]}$, $S[1, -1]_{V[3,1,2]}$, $S[1, -1]_{V[2,1,3]}$ and $S[1, -1]_{V[1,2,3]}$ together with infinities. In particular, we denote $\hat{S}[s_1, \cdots, s_{N-1,N}]$ as the $(N - 1)$-cell with a component $S[s_1, \cdots, s_{N-1,N}]$ around the top vertex $V[N, \cdots, 1]$. For example, the 2-cell $B$ above is denoted by $S[1, -1]$. Then we have:

**Proposition 5.1** $M_N$ is not orientable, i.e. $H_{N-1}(M_N, Z) = 0$, for $N > 2$.

**Proof.** There are $2^{N-1}$ $(N - 1)$-cells. First we look at them around $V[N, \cdots, 1]$. Since two components differ by one sign have a common $(N - 2)$-face around $V[N, \cdots, 1]$, in order to cancel the boundary, they must have opposite orientations. So starting with $S[1, 1, \cdots, 1]$, the only combination of cells possible to have zero boundary is

$$
\tilde{C} = \sum_{(s_1, \cdots, s_N)} (-1)^{\alpha(s_1, \cdots, s_N)} \hat{S}[s_1, \cdots, s_{N-1,N}],
$$

(5.2)
where \( n(s_1, \ldots, s_N) \) is the number of minus signs in \((s_{1,2}, \ldots, s_{N-1,N})\). However, any component around \( V[N, \cdots, 1] \) is carried by flow to become adjacent to \( S[1, 1, \cdots, 1] \) around some vertices. To see this, note that \((s_1, \cdots, s_n)\) can be permutated to be \((1, \cdots, 1, -1, \cdots, -1)\). Since in (5.2) there are cells having same orientation as \( S[1,1,\cdots,1] \) for \( N > 2 \), it is impossible for \( C \) to have zero boundary. \( \square \)

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