Exact Symmetries of Electron Interactions in the Lowest Landau Level

M. Eliashvili
Department of Theoretical Physics, A. Razmadze Mathematical Institute, Tbilisi 380093 Georgia

Z. F. Ezawa
Department of Physics, Tohoku University, Sendai 980-8578 Japan

G. Tsitsishvili
Department of Theoretical Physics, A. Razmadze Mathematical Institute, Tbilisi 380093 Georgia

Considering the system of interacting electrons in the lowest Landau level we show that the corresponding four-fermion Hamiltonian is invariant with respect to the local area-preserving transformations. Testing a certain class of interaction potentials, we find that this symmetry is universal with respect to a concrete type of potentials.

I. INTRODUCTION

The fractional quantum Hall effects\cite{1,2} emerge due to the strong electron-electron interactions. In the corresponding theoretical constructions these interactions give rise to certain difficulties because of their nonlinear character. Therefore, the usual tool for studying the interaction effects is an approximate analysis or the numerical calculus (see e.g.\cite{3} and references therein).

In the present paper we carry out an exact analysis of electron-electron interactions in the lowest Landau level (LLL) and reveal the exact symmetries of the corresponding four-fermion Hamiltonian.

Symmetries strongly affect the properties of the physical constituents such as the ground states, elementary excitations, etc. Therefore, the matter of symmetries is an essential question, especially in the case of nonlinear theories where the standard methods of studying spectra and eigenstates become inefficient.

The main results of the paper are summarized in the following two points:

(a) The many-body Hamiltonian describing interacting electrons in the LLL is invariant with respect to the local area-preserving transformations for a certain class of interaction potentials. This assertion holds for the systems with a finite number of electrons as well as in the thermodynamic limit.

(b) The algebraic structure of this symmetry is insensitive to a concrete type of potentials within the given class of interaction potentials.

It is to be emphasized that the area-preserving transformations have so far been known only to comprise the symmetries of many-body Hamiltonian of noninteracting electrons in a homogeneous magnetic field\cite{1,3}.

The symmetry stated in point (a) leads to the degeneracy of the corresponding energy levels and endows the eigenstates with specific algebraic structure. For the purpose of verification and in order to outline these features in a relatively transparent way we carry out the exact analytic diagonalization for some few-body quantum states.

The symmetry of electron-electron interactions gives rise to the corresponding quantum number and subsequently, to a novel classification of many-body quantum states in the LLL.

In section II we comment more on our main result together with its physical implications. The detailed verification of the existence of the symmetry is presented in section III. In order to shed more light on the structure of spectrum and eigenstates of the system under consideration, we perform an exact analytic diagonalization in section IV for a finite number of interacting electrons. In section V we present discussions.

II. SYMMETRY OF INTERACTING ELECTRONS

Particles in the LLL carry the vanishing kinetic energy. Hence, the Hamiltonian consists of a pure interaction of the type

\[ H = \int \psi^\ast(r) \psi^\ast(r') V(|r - r'|) \psi(r') \psi(r) \, dr \, dr' \]

(1)

where \( \psi(r) \) is the electron field operator in the LLL, and \( V(r) \) is the interaction potential which is assumed to be rotationally invariant. Below we consider some class of potentials including the Coulomb interaction.

Adopting the circular geometry we present the electron field operator as

\[ \psi(r) = \sum_{n=0}^{\infty} c(n) \psi_n(r) \]

(2)

where

\[ \psi_n(r) = \frac{z^n e^{-\frac{iz}{2}}}{\sqrt{2\pi n!}} \]

(3)

are the one-particle wave functions in the LLL. We employ the magnetic length scale and use the dimensionless quantities \( \sqrt{2} x = x + iy \) and \( \sqrt{2} z = x - iy \). Fermi amplitudes satisfy the standard anticommutation relations \( \{ c(m), c^\ast(n) \} = \delta_{mn} \).
The relations \( H, W, L \) and total angular momentum operators, respectively

\[
\mathcal{L}_{mn} = \int \psi^*(r) \left[ \frac{\sigma}{2} - \frac{\partial}{\partial z} \right]^m \left[ \frac{\sigma}{2} + \frac{\partial}{\partial z} \right]^n \psi(r) \, dr,
\]

where \( m, n \) are nonnegative integers, and \( \psi(r) \) is given by (2) and (3).

Since \( \mathcal{L}_{mn} \) involve the higher powers of derivatives, the corresponding transformations are in general nonlocal. In contrast, the elements \( W_L = \{ L_{00}, L_{11}, L_{01}, L_{10} \} \) form the maximal subalgebra in \( W_\infty \) generating the local transformations. The main purpose of the present paper is to manifest the invariance of the Hamiltonian (1) under these transformations, as expressed by the commutation relations \( [H, W_L] = 0 \).

The elements \( L_{00} \) and \( L_{11} \) represent the particle number and total angular momentum operators, respectively

\[
N \equiv L_{00} = \sum_{n=0}^{\infty} c^*(n) c(n),
\]

\[
M \equiv L_{11} = \sum_{n=0}^{\infty} n c^*(n) c(n).
\]

The relations \( [H, L_{00}] = 0 \) and \( [H, L_{11}] = 0 \) expressing the corresponding conservation laws are obvious. The latter takes place due to the rotational invariance of \( V(r) \). The rest part of \( W_L \) is given by

\[
G^- \equiv L_{10} = \sum_{n=0}^{\infty} \sqrt{n+1} c^*(n) c(n+1),
\]

\[
G^+ \equiv L_{01} = \sum_{n=0}^{\infty} \sqrt{n+1} c^*(n+1) c(n).
\]

In the succeeding section we prove the the relations

\[
[H, G^\pm] = 0
\]

for the class of interaction potentials set by \( V(r) = r^{-2\sigma} \) and \( 0 < \sigma < 3/4 \). In the rest of this section we point out some of their consequences. Let us first remark that the operator \( N \) commutes with \( G^\pm \) and \( H \). Hence, we work within a given \( N \)-particle sector where \( N \) can be treated as an ordinary number.

As a result of this symmetry, the energy levels of the interacting Hamiltonian (1) are infinitely degenerate. From the relations (6) it follows that only the eigenstates belonging to a given energy level can be obtained one from another by repetitive applications of operators \( G^\pm \). The commutation relation \( [G^-, G^+] = N \) indicates that \( G^\pm \) form the ordinary oscillator algebra. Therefore, only one eigenstate of a given energy level can be classified with respect to the amount of \( G \)-quanta. The corresponding number operator is given by \( N_G = N^{-1} G^+ G^- \). Each energy level possesses its own \( G \)-vacuum defined by \( G^- |0\rangle = 0 \). Indeed, the operator \( G^- \) lowers the total angular momentum and its repetitive action on any eigenstate will eventually give zero due to the Fermi statistics. All other states within a given energy level can be obtained from the corresponding \(|0\rangle \) by repetitive action of the operator \( G^+ \) carrying no energy, no particle number and the unit angular momentum: \( [H, G^\pm] = 0, [N, G^\pm] = 0 \) and \( [M, G^\pm] = \mp |G^\pm| \). Some of the details become more pictorial in Section IV where we comment on the system with finite number of interacting electrons.

The complete set of commuting operators employed in applications to quantum Hall systems is usually taken as \( \{ H, N, M \} \). Once the relation \( [H, G^\pm] = 0 \) is found out, any combination of \( G^\pm \) can be taken (instead of \( M \)) for the classification of quantum states. In the preceding paragraph we have discussed the operator \( N_G \). However, it commutes with \( M \), i.e. leads to the equivalent classification scheme.

A novel alternative emerges from the set of commuting operators given by \( \{ H, N, G^- \} \). Namely, the \( N \)-particle eigenstates of \( H \) related with a given energy level can be classified in terms of the coherent states. They are given by \( |w\rangle = \exp(w G^+)|0\rangle \) where \( w \) is a complex number, and \( |0\rangle \) is the \( G \)-vacuum. The corresponding eigenvalue equation reads as \( G^- |w\rangle = w N |w\rangle \). Since the operator \( G^- \) increases the total angular momentum, the coherent states occupy the infinite area. Therefore, the physically interesting case for the application of such a classification scheme will be the thermodynamic limit.

### III. PROOF

In this section we give a proof of the relation \( [H, G^\pm] = 0 \). Due to the identity \( [H, G^+]^* = [-G^-, H] \) it is sufficient to deal with only one of the generators \( G^\pm \). Rewriting \( H \) in terms of \( c^* \) and \( c \) we get

\[
H = \sum_{l,m,t} V_{l,mn} c^*(l) c^*(n) c(t) c(m),
\]

\[
V_{l,mnt} = \int \tilde{\psi}_t(r) \tilde{\psi}_m(r) V \tilde{\psi}_n(r') \psi_t(r') \, dr \, dr'.
\]

Employing \( \{ c(m), c^*(n) \} = \delta_{mn} \) and \( V_{l,mnt} = V_{ntlm} \) we perform some trivial manipulations and arrive at

\[
[H, G^+] = \sum \sqrt{m+1} V_{l,m+1,n,t} c^*(l) c^*(n) c(t) c(m) - \sum \sqrt{m} V_{l-1,n,m,t} c^*(l) c^*(n) c(t) c(m)
\]

Using (3) and (7) we get

\[
\sqrt{m+1} V_{l,m+1,n,t} = \int \! \! dr \tilde{\psi}_t(r) \left( \frac{\sigma}{2} - \frac{\partial}{\partial z} \right) \psi_m(r)
\]

\[
\sqrt{m} V_{l-1,n,m,t} = \int \! \! dr \tilde{\psi}_t(r) \left( \frac{\sigma}{2} - \frac{\partial}{\partial z} \right) \psi_m(r)
\]
\[ \sqrt{\mathcal{L}} V_{-1,m,n,t} = \int \mathcal{D}r \left[ \left( \frac{z}{2} + \frac{\partial}{\partial z} \right) \tilde{\psi}_i(r) \right] \bar{\psi}_m(r) \times \int \mathcal{D}r' V(|r - r'|) \tilde{\psi}_n(r') \psi_t(r'). \]

Plugging these relations in (8) and integrating by parts one gets

\[ [H, G^+] = 2 \sum \{ X_{lmnt} - S_{lmnt} \} c^*(l) c^*(n) c(t) c(m), \]

\[ X_{lmnt} = \int \bar{\psi}_i(r) \psi_m(r) \frac{\partial V}{\partial z} \bar{\psi}_n(r') \psi_t(r') \mathcal{D}r \mathcal{D}r', \]

\[ S_{lmnt} = \int \frac{\partial}{\partial z} \left[ \bar{\psi}_i(r) \psi_m(r) V \right] \bar{\psi}_n(r') \psi_t(r') \mathcal{D}r \mathcal{D}r'. \]

Interchanging the integration variables in \( X_{lmnt} \) we use

\[ \frac{\partial}{\partial z} V(|r - r'|) = - \frac{\partial}{\partial z} V(|r' - r|) \]

and get \( X_{lmnt} = -X_{ntlm} \). This forces the corresponding term in \([H, G^+]\) to vanish and yields

\[ [H, G^+] = -2 \sum S_{lmnt} c^*(l) c^*(n) c(t) c(m). \]

The integrand in \( S_{lmnt} \) represents the total derivative with respect to \( r \). Restricting the corresponding integration to a finite disc of radius \( R \), we subsequently pass to the surface integral and perform the limit \( R \to \infty \). This leads to

\[ \sqrt{2} S_{lmnt} = \int d\theta R e^{i\theta} \bar{\psi}_i(R) \psi_m(R) \times \int V(|R - r'|) \bar{\psi}_n(r') \psi_t(r') \mathcal{D}r' \quad (9) \]

where \( R = \{ R \cos \theta, R \sin \theta \} \) with \( R \to \infty \).

For the sake of convenience we use the Fourier representation of the interaction potential

\[ V(r) = \frac{1}{2\pi} \int V(k) e^{ikr} \mathcal{D}k \quad (10) \]

where \( k = \{ k \cos \gamma, k \sin \gamma \} \). Besides, we assume that \( V(k) \) possesses the necessary behaviour at \( k = 0 \) and \( k = \infty \), so that the Fourier integral (10) is convergent.

Substituting (10) into (9) and performing all the necessary integrations we get

\[ S_{lmnt} = \delta_{l+n,m+t+1} \frac{R^{l+m+1} e^{-R^2/2}}{\sqrt{l! m! 2^{l+m+1}}} \times \int_0^\infty J_{|l-m-1|} (kR) \omega_{nt}(k) V(k) dk \]

Deriving the last expression we have used

\[ \int e^{i\alpha \theta + ikR \cos \theta} d\theta = 2\pi i^{\alpha} J_{|\alpha|}(kR) \]

\[ \int e^{-ikr} \tilde{\psi}_n(r') \psi_t(r') \mathcal{D}r' = (-i)^{|n-t|} e^{-i(n-t)\gamma} \omega_{nt} \quad (11) \]

where \( J_{|\alpha|}(kR) \) is the Bessel function, and the quantity \( \omega_{nt} = \omega_{tn} \) is given by

\[ \omega_{n,n+\alpha} = \frac{\sqrt{n!}}{(n + \alpha)!} \frac{[k^2/2]^{\alpha/2}}{e^{-k^2/2} L_{n}^{\alpha}(k^2/2)} \quad (12) \]

with \( L_{n}^{\alpha} \) denoting the Laguerre polynomials.

We consider \( n \leq t = n + \alpha \) with \( \alpha = 0, 1, \ldots \) separately from \( t < n = t + \alpha \) with \( \alpha = 1, 2, \ldots \) and get

\[ S_{m+\alpha+1,m,n,\alpha+\alpha} = \frac{L_{m+1+\alpha}^{\alpha} e^{-L}}{\sqrt{2^{2m+\alpha+1} \Gamma(m + \alpha + 1)}} I_{\alpha}(L) \quad (13) \]

\[ S_{l,t+\alpha-1,t+\alpha} = \frac{L_{l+\alpha/2}^{\alpha} e^{-L}}{\sqrt{2^{2m+\alpha-1} \Gamma(l + \alpha - 1)}} I_{\alpha}(L) \quad (14) \]

\[ I_{\alpha} = \int_0^\infty V(k) J_{\alpha}(k\sqrt{2L}) k^{\alpha+1} L_{\alpha}^{\alpha}(k^2/2) e^{-k^2/2} dk \]

where \( L \equiv R^2/2 \). In fact, \( L \) sets the order of the maximal one-particle angular momentum which can be accommodated within a finite disk of radius \( R \).

The two cases presented in (13) and (14) can be related with each other as

\[ \frac{S_{l,t+\alpha-1,t+\alpha}}{S_{l,t+\alpha-1,t+\alpha}} = \frac{\sqrt{(l + \alpha)(l + \alpha + 1)}}{L} \quad (15) \]

and therefore, the limit \( L \to \infty \) may be studied for only one of them, say for (13).

If the system contains finite number of electrons, then the quantities \( m, n, \alpha \) in (13) may take the finite values. This corresponds to the finite total angular momentum. In that case the limiting procedure \( L \to \infty \) is trivial and due to the exponential factor \( e^{-L} \) leads to \( S_{lmnt} = 0 \). On the other hand, even if the electron number is finite, the total angular momentum may become infinite, and \( m, n, \alpha \) will take the infinite values. Physically most interesting case where \( m, n, \alpha \) necessarily take the infinite values is the thermodynamic limit, i.e. when the electron number increases together with \( L \) forming the finite density state. Thus, the limiting procedure should be considered in all possible cases where any of \( m, n, \alpha \) in (13) may become infinite together with \( L \). In that case the factor \( e^{-L} \) may be compensated by other ones, and the problem requires the detailed analysis.

The comprehensive analysis for general \( V(r) \) is quite a nontrivial problem and therefore, we concentrate on the particular type of potential given by

\[ V(k) = \frac{\Gamma(1 - \sigma)}{2^{\sigma-1} \Gamma(\sigma)} \frac{1}{k^{2-2\sigma}} \quad (16) \]
where $0 < \sigma < 3/4$. This restriction on $\sigma$ guarantees the integral (10) to be convergent, and gives $V(r) = r^{-2\sigma}$. The Coulomb interaction corresponds to $\sigma = 1/2$.

In this case the integral in the r.h.s. of (13) leads to (see 2.19.12.7 in Ref.[4])

$$S = \frac{2^{-\sigma} \Gamma(n + 1 - \sigma) \Gamma(\alpha + \sigma) L^{m+\sigma+1}e^{-L}}{\Gamma(\sigma) \Gamma(1 - \sigma) \sqrt{m!(m + \alpha + 1)! n!(n + \sigma)!}} \int_{0}^{1} \tau^{\alpha+\sigma-1} (1 - \tau)^{-\sigma} F_{2}(\alpha + \sigma; \sigma - n, \alpha + 1; -L) d\tau$$

where $S$ stands for $S_{m+\sigma+1,m,n,n+\sigma}$ and $2F_{2}$ is the hypergeometric function.

Using the integral representation, we express $2F_{2}$ in terms of $1F_{1}$. Subsequently we perform the Kummer transformation (13.1.27 in Ref.[5]) and arrive at

$$S = \frac{2^{-\sigma} \Gamma(n + 1 - \sigma) L^{m+\sigma+1}e^{-L}}{\Gamma(\sigma) \Gamma(1 - \sigma) \sqrt{m!(m + \alpha + 1)! n!(n + \sigma)!}} \int_{0}^{1} \tau^{\alpha+\sigma-1} (1 - \tau)^{-\sigma} F_{1}(-n; \sigma - n; L\tau) e^{-L\tau} d\tau$$

where $F_{1}(-n; \sigma - n; L\tau)$ is in fact the $n$'th order polynomial with respect to $L\tau$. Using its power expansion we get

$$S = \frac{2^{-\sigma} \Gamma(n + 1 - \sigma) L^{m+\sigma+1}e^{-L}}{\Gamma(\sigma) \Gamma(1 - \sigma) \sqrt{m!(m + \alpha + 1)! n!(n + \sigma)!}} \sum_{k=0}^{n} C_{n}^{k} \Gamma(n - k + 1 - \sigma) L^{k} \int_{0}^{1} \tau^{k+\alpha+\sigma-1} e^{-L\tau} d\tau$$

where $C_{n}^{k}$ is the Newton binomial.

Consider now the integral in the last expression, and remark that the quantity $k + \alpha$ may take infinite as well as finite values. Hence, we need its $(L \to \infty)$-form valid for all (including infinite) values of $k + \alpha$.

The factor $e^{-L\tau}$ provides the exponential suppression of all contributions coming from $\tau \neq 0$. The integrable pole at $\tau = 1$ also fails to contribute because of the same exponential suppression. So, the integral gets the contribution from the vicinity of $\tau = 0$, and we may replace the factor $(1 - \tau)^{\sigma}$ by its value in $\tau = 0$. In this consideration we get

$$\int_{0}^{1} \tau^{k+\alpha+\sigma-1} e^{-L\tau} d\tau \to \int_{0}^{1} \tau^{k+\alpha+\sigma-1} e^{-\tau} d\tau = \frac{1}{L^{k+\alpha+\sigma}} \int_{0}^{L} \tau^{k+\alpha+\sigma-1} e^{-\tau} d\tau.$$

If $k + \alpha + \sigma$ is finite, then for $L \to \infty$ we get

$$\int_{0}^{1} \tau^{k+\alpha+\sigma-1} e^{-\tau} d\tau \to \frac{\Gamma(k + \alpha + \sigma)}{L^{k+\alpha+\sigma}}$$

(17)

This relation is valid also for $k + \alpha + \sigma \to \infty$. Indeed, we have

$$\int_{0}^{L} \tau^{k+\alpha+\sigma-1} e^{-\tau} d\tau = \int_{0}^{1} \tau^{k+\alpha+\sigma-1} e^{-\tau} d\tau + \int_{1}^{L} \tau^{k+\alpha+\sigma-1} e^{-\tau} d\tau$$

Remark, that both of the limits $L$ and $k + \alpha + \sigma$ in the second term bring the divergent effects. Therefore, one may account them in any desired order, say first $L \to \infty$ and then $k + \alpha + \sigma \to \infty$. This leads back to (17), and hence to

$$S = \frac{2^{-\sigma} \Gamma(\sigma) \Gamma(1 - \sigma)}{\sqrt{m!(m + \alpha + 1)! n!(n + \sigma)!}} \sum_{k=0}^{n} C_{n}^{k} \Gamma(n - k + 1 - \sigma) \Gamma(k + \alpha + \sigma).$$

Using

$$\Gamma(a)\Gamma(b) = \Gamma(a + b) \int_{0}^{1} \tau^{a-1} (1 - \tau)^{b-1} d\tau$$

the summation over $k$ can be performed exactly and for $S$ we get

$$S_{m+\alpha+1,m,n,n+\alpha} = \frac{2^{-\sigma} \Gamma(1 - \sigma) L^{m+\sigma+1}e^{-L}}{\Gamma(\sigma) \sqrt{m!(m + \alpha + 1)! n!(n + \alpha)!}} \sqrt{\frac{(n + \alpha)!}{n!} \frac{\Gamma(\alpha + \sigma)}{\Gamma(\alpha + 1)}}$$

(18)

where any kind of limiting procedure is straightforward. As an example we present the case when all of $m, n, \alpha$
increase together with $L$. Introduce the parametrization

$$m = \lambda_m L \quad n = \lambda_n L \quad \alpha = \lambda_\alpha L$$

(19)

where $\lambda_{m,n,\alpha}$ are positive constants (none of them may vanish, otherwise the corresponding momentum will stay finite instead of tending to infinity). Further, the subscripts in $S_{m+n,\alpha+1,m,n,n+\alpha}$ represent the one-particle angular momenta, whose maximal values are of order of $L$, while the matter is confined to the interior (including the edge) of the disk. Hence, $\lambda_{m,n,\alpha}$ must be restricted as

$$0 < \lambda_m + \lambda_n < 1$$

(20)

$$0 < \lambda_n + \lambda_\alpha < 1$$

(21)

$$0 < \lambda_{m,n,\alpha} < 1$$

(22)

Remark, that in contrast with $\lambda_m + \lambda_n$ and $\lambda_n + \lambda_\alpha$, the constants $\lambda_{m,n,\alpha}$ cannot be equal to one, since this would contradict to (20) and (21) due to $\lambda_{m,n,\alpha} > 0$.

We substitute (19) into (18) and employing Stirling’s formula arrive at

$$S_{m+n,\alpha+1,m,n,n+\alpha} = CL^{-1} e^{-LW}$$

where $C$ is an $L$-independent constant, and $W$ looks as

$$W(\lambda_m, \lambda_n, \lambda_\alpha) = 1 - \lambda_m + \frac{1}{2} \lambda_m \ln \lambda_m + \frac{1}{2} \lambda_n \ln \lambda_n +$$

$$+ \frac{1}{2} (\lambda_m + \lambda_n) \ln(\lambda_m + \lambda_n) - \frac{1}{2} (\lambda_n + \lambda_\alpha) \ln(\lambda_n + \lambda_\alpha).$$

The final step is to show that $W > 0$. Remark, that

$$\frac{\partial W}{\partial \lambda_m} = \frac{1}{2} \ln[\lambda_m(\lambda_m + \lambda_\alpha)] < 0,$$

$$\frac{\partial W}{\partial \lambda_n} = \frac{1}{2} \ln \lambda_n \lambda_n + \lambda_\alpha < 0,$$

$$\lambda_m \leq 1 - \lambda_\alpha, \quad \lambda_n \leq 1 - \lambda_\alpha,$$

where the last two relations follow from (20) and (21). Consequently, $W(\lambda_m, \lambda_n, \lambda_\alpha) \geq W(1 - \lambda_\alpha, 1 - \lambda_\alpha, \lambda_\alpha)$ where

$$W(1 - \lambda_\alpha, 1 - \lambda_\alpha, \lambda_\alpha) = \lambda_\alpha + (1 - \lambda_\alpha) \ln(1 - \lambda_\alpha).$$

For $0 < z < 1$ we have $z \ln z > z - 1$. Consequently, $W(1 - \lambda_\alpha, 1 - \lambda_\alpha, \lambda_\alpha > 0$ and $W(\lambda_m, \lambda_n, \lambda_\alpha) > 0$. Hence, $S_{m+n,\alpha+1,m,n,n+\alpha} > 0$, and further, due to (15) $S_{mnt} = 0$.

In the similar way we show that $S_{mnt}$ vanishes in all possible cases when some (or none) of $m$, $n$, $\alpha$ tend to infinity. Thus, for the interacting electron system defined by (1) – (3) and (16) we have demonstrated the relation (6) i.e. the existence of symmetries with respect to the local area-preserving transformation.

IV. DIAGONALIZATION

Due to the symmetry established in the previous sections, the eigenvalue spectrum of the model is highly degenerated, while the eigenstates exhibit a certain kind of structure. In order to outline these features in a relatively transparent way, we present an exact analytic diagonalization procedure.

Since the interaction preserves the number of particles, the procedure can be carried out in a different $N$-particle Fock subspaces independently. The basis is chosen to be $| k_1 \ldots k_N \rangle \equiv c^\alpha(k_1) \cdots c^\alpha(k_N)|0\rangle$ where $k_1 < \ldots < k_N$.

Treating $\{ k_1 \ldots k_N \}$ as a multi-index we consider the quantity $\langle k_1 \ldots k_N | H | l_1 \ldots l_N \rangle$ as an $\infty \times \infty$ square matrix to be diagonalized. Due to the circular symmetry of $V(r)$ we have

$$\langle k_1 \ldots k_N | H | l_1 \ldots l_N \rangle \propto \delta_{k_1 + \ldots + k_N \lambda_1 + \ldots} \delta_{l_1 + \ldots + l_N}$$

reflecting the angular momentum conservation. Hence, the matrix $\langle k_1 \ldots k_N | H | l_1 \ldots l_N \rangle$ has the block-diagonal form, where each separate block is a finite-dimensional square matrix labelled with some fixed value of the total angular momentum $M = k_1 + \ldots + k_N = l_1 + \ldots + l_N$. We regard the blocks to be arranged in the increasing order with respect to $M$, starting from $M_{\text{min}} = N(N - 1)/2$.

The typical structure is shown in Figure 1 exhibiting the lowest momentum blocks in $\langle k_1 k_2 k_3 | H | l_1 l_2 l_3 \rangle$.

Denote the block dimensionality by $d(N, M)$. Proceeding from the Fermi statistics, the quantity $d(N, M)$ represents the number of partitions of the integer $M$ into $N$ distinct nonnegative integers. It is given via the following formal expansion

$$\sum_{M=0}^{\infty} d(N, M) \tau^M = \frac{\tau^{N(N-1)/2}}{(1 - \tau)(1 - \tau^2) \cdots (1 - \tau^N)}$$

(23)
where the factor $M^{-(N-1)/2}$ is related with the minimal value of $M$.

Thus, the general $(N, M)$-block comprises the $d(N, M)$ number of eigenvalues and eigenvectors. Denoting them by $E_{N,M\beta}$ and $\theta_{M\beta}(k_1 \ldots k_N)$ with $\beta = 1, \ldots, d$ we construct the Hamiltonian eigenstates as

$$\|N, M, \beta\rangle = \sum_{k_1 < \cdots < k_N} \theta_{M\beta}(k_1 \ldots k_N) |k_1 \ldots k_N\rangle.$$  

Each block carries some fixed value of the filling factor defined by $\nu = N/K$ where $K$ is the number of available one-particle states in a block. It is given by $K = 1 + l_{\text{max}}$ where $l_{\text{max}}$ is the maximal value of one-particle angular momentum in a block. The later can be calculated from

$$\sum_{1^{st}}^{N-1/2} + \sum_{2^{nd}}^{N-2} + \sum_{(N-1)^{th}}^{N} M = \frac{N}{M + 1 - (N - 1)(N - 2)/2} = \frac{1}{1 + M/N}$$

where $M = M_{\text{min}}$ is the relative momentum.

We move on to the results. Substituting (10) into (7) and using (11) one arrives at

$$V_{\text{mnt}} = \delta_{m+n,l+t} \int_0^\infty V(k) \omega_{ml}(k) \omega_{nt}(k) k dk,$$

where $\omega_{ml}$ is given by (12). We use (16) and replace the Laguerre polynomials in $\omega_{ml}$ and $\omega_{nt}$ by the corresponding power expansions. The integrals become trivial, and $V_{\text{mnt}}$ takes the form of a double finite sum. The later is easily calculable if the one-particle angular momenta $m$, $n$, $l$, $t$ are confined to some reasonable range, say as in the blocks presented in Figure 1. For $V(r)$ given by (16) (with an appropriate normalization) the matrix elements take the rational (exact) values. Hence, the corresponding secular equations can be written in the exact form, as well. In such a way we obtain the exact eigenvalues.

In the Table I we present the eigenvalues of the blocks with $M = 3 \div 8$ shown in Figure 1. The case with $M = 9$ is shown separately. The eigenvalues are labelled with index $\beta$ in the order of their appearance when increasing the quantity $M$.

We observe that the spectrum of the block with total angular momentum $M + 1$ entirely comprises the one corresponding to $M$, and also some additional eigenvalues. The amount of these additional eigenvalues is given by $d(N, M + 1) - d(N, M)$ and increases together with $M$. As an example, passing from $M = 8$ to $M = 9$ in $N = 3$ we get two additional eigenvalues $E[6,7]$ (see Table I and Figure 1) given by

$$10025047 \pm 3\sqrt{107141413705}, \quad V(r) = \frac{A}{r}$$

$$96437129 \pm \sqrt{27563293275793}, \quad V(r) = \frac{B}{r^{2/3}}$$

where the constants $A$ and $B$ are given in the caption of Table I.

Thus, once an eigenvalue appears in any block, it persists in all subsequent ones and gives rise to an infinitely degenerated energy level. The corresponding eigenstates are related with each other by

$$G^\pm \|N, M, \beta\rangle = \lambda^\pm(N, M, \beta)\|N, M \pm 1, \beta\rangle$$

where $\lambda^\pm$ are some constants.

Using these relations we can construct some of eigenstates without solving out them explicitly. Consider the case of $N = 3$ (see Figure 1). The lowest block possesses one single eigenstate $\|3, 3, 1\rangle = |012\rangle$. The next block is also one-dimensional. The corresponding eigenstate can be obtained by applying $G^+$ to $|3, 3, 1\rangle$. Since $G^+$ does not preserve the norm, its action should be accompanied with the subsequent normalization. In such a way we get $|3, 4, 1\rangle = |013\rangle$. The next block ($M = 5$) possesses two eigenstates. One of them can be obtained from $|3, 4, 1\rangle$ by the action of $G^+$. It is given by

$$\|3, 5, 1\rangle = \frac{\sqrt{2}}{\sqrt{3}} |014\rangle + \frac{1}{\sqrt{3}} |023\rangle.$$

The other one is fixed by the orthogonality principle, and appears as

$$\|3, 5, 2\rangle = \frac{1}{\sqrt{3}} |014\rangle - \frac{\sqrt{2}}{\sqrt{3}} |023\rangle.$$

In the similar way we can construct the eigenstates corresponding to $M = 6, 7, 8$.

The starting eigenstate $\|3, 3, 1\rangle \equiv |012\rangle$, the operator $G^+$ and the principle of orthogonality have nothing to do with $V(r)$. Consequently, the eigenstates corresponding to $M = 3 \div 8$ which can be constructed employing only these tools, do not depend on the concrete type of $V(r)$. Passing to $M = 9$ we acquire two additional eigenstates (instead of a single one as for preceding blocks). The principle of orthogonality becomes insufficient for fixing both of them and leaves the one-parameter arbitrariness with respect to rotation in the subspace of those two eigenstates. This degree of freedom is governed by $V(r)$.

As we have remarked, the amount of additional eigenstates given by $d(N, M + 1) - d(N, M)$ increases together with $M$. Hence, the arbitrariness left after the orthogonalization also increases, and the interaction effects become more and more decisive in the formation of quantum states.

The only exception is the two-particle sector ($N = 2$). Indeed, from (23) we get

$$\sum_{M=0}^\infty d(2, M) \tau^M = \frac{\tau}{(1 - \tau)(1 - \tau^2)} = \frac{\tau(1 + \tau)}{(1 - \tau^2)^2} = \tau(1 + \tau)(1 + 2\tau^2 + 3\tau^4 + \cdots)$$

indicating that not more than one additional eigenstate appears when passing from $M$ to $M + 1$. Thus, starting
TABLE I: The values of some $E_{N=3,M,\beta}$. Superscripts in square braces stand for the values of $\beta$. The constants $1/A = 3\Gamma(1/2)$ and $1/B = 2^{1/3}\Gamma(2/3)$ are introduced for the purpose of convenience. $E[^{0,7}]$ are given by (24) and (25) in the text.

| $M$ | $V(r) = \frac{A}{r}$ | $V(r) = \frac{B}{r^{1/3}}$ |
|-----|----------------|------------------|
| 3   | $29[^1]$ 52    | $101[^1]$ 32     |
| 4   | $29[^1]$ 52    | $101[^1]$ 32     |
| 5   | $29[^1]$ 2048  | $101[^1]$ 32     |
| 6   | $29[^1]$ 2048  | $101[^1]$ 32     |
| 7   | $29[^1]$ 2048  | $101[^1]$ 32     |
| 8   | $29[^1]$ 2048  | $101[^1]$ 32     |
| 9   | $29[^1]$ 2048  | $101[^1]$ 32     |

with $\langle 2, 1, 1 \rangle = |01\rangle$, the operator $\hat{G}^+$ and the ortho-

gonization procedure completely fix the set of two-particle eigenstates. So, they do not depend on the concrete type

of $V(r)$, i.e. are universal. Introducing the holomorphic part $P(z_1, z_2)$ as

$$\langle 0 | \psi(r_1) \psi(r_2) | 2, M, \beta \rangle = \frac{P(z_1, z_2)}{2\pi} e^{-(1/2)(|z_1|^2 + |z_2|^2)}$$

we show some of two-particle eigenstates in Table II.

V. DISCUSSIONS

We have shown that the field-theoretical Hamiltonian describing the interacting electrons in the LLL is invariant with respect to the local area-preserving transfor-
mations. We have demonstrated it for the interaction potential given by $V(r) = r^{-\sigma}$ where $0 < \sigma < 3/4$. Mathematically this is expressed by (6), which holds for the systems with a finite electron numbers as well as in the thermodynamic limit.

We wish to stress that the observed symmetry carries the universal character. Namely, it persists and exhibits the unique algebraic properties for the given class of interaction potentials. We tend to conjecture that such an occurrence is supported by the properties of the LLL and therefore, should be the case for a wider class of rotationally invariant potentials. To outline these features we have carried out an exact diagonalization explicitly in a few body interacting electron systems. Results are summarized in Figure 1 and Tables I and II.

Within a given block, only one eigenstate represents the ground state while the rest ones are excitations. This may have interesting developments involving the excita-
tion spectra, spatial structures of charge distribution in a ground and an excited states, etc. However, in that case the analytic studies would be more informative than the ones based on bare numbers. An example, using the

Table I it can be verified that $E[^{1}] > E[^{2}] > E[^{3}] > E[^{4}]$, i.e. as if the ground state of a given block is among the last appeared ones. Actually, this is not always the case: $E[^{3}] > E[^{5}] > E[^{4}]$. Therefore, the knowledge about the global tendency of eigenvalues with respect to $\beta$ would be helpful for searching the ground state in the thermo-
dynamic limit. Remark, that this tendency also seems to be insensitive to the type of $V(r)$: the above inequalities
involving $E^{[2]}$ hold for both types of $V(r)$ presented in Table I. In this connection one may expect the existence of some other peculiarities which are insensitive to a type of $V(r)$ and might shed more light on a common analytic structure of electron-electron interactions in the LLL.

Acknowledgments

One of the authors (G.T.) expresses his sincere gratitude to Nishina Memorial Foundation for supporting his stay at Department of Physics, Tohoku University, where a part of this work was carried out. M.E. and G.T. would like to thank SCOPES for the support under grant No. 7GEPJ62379.

[1] Z. F. Ezawa, Quantum Hall Effects: Field Theoretical Approach and Related Topics (World Scientific, Singapore, 2000).
[2] The Quantum Hall Effect, edited by S. Girvin and R. Prange (Springer-Verlag, New York, 1990).
[3] G. Fano, F. Ortolani and E. Colombo, Phys. Rev. B34 (1986) 2670.
[4] A. Cappelli, C. A. Trugenberger and G. R. Zemba, Nucl. Phys. B396 (1993) 465.
[5] D. Karabali, Nucl. Phys. B419 (1994) 437.
[6] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, Integrals and Series, v.2 Special Functions (Gordon and Breach, New York, 1986).
[7] A. P. Prudnikov, Yu. A. Brychkov and O. I. Marichev, Integrals and Series, v.3 More Special Functions (Gordon and Breach, New York, 1990).
[8] G. E. Andrews, The Theory of Partitions, Encyclopedia of Mathematics and its Applications (Addison-Wesley Publishing Company, 1976).