Research Memorandum
Faculty of Economics and Business Administration

Tilburg University
Palm theory of random time changes

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FEW 754

Communicated by Prof.dr. B.B. van der Genugten
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November 1997
Abstract

Palm distributions are basic tools when studying point processes, queueing systems, fluid queues or random measures. The framework usually varies with the random phenomenon of interest, but commonly a one-dimensional group of measure-preserving shifts is present. In this research we study random time changes (RTC's, right-continuous and non-decreasing functionals which pass the zero-level at 0), appropriate to characterize all above systems. We assume the existence of a two-dimensional family of shifts which behaves consistently like a group along the extended graph of the RTC. In canonical settings this assumption is trivially satisfied. From this family we derive two one-dimensional groups of shifts and assume for one of them that the elements preserve the underlying distribution $P$. As a consequence, the elements of the second group preserve the detailed Palm distribution. This DPD has a very natural interpretation and satisfies a duality criterion: the DPD of the DPD gives $P$ in return.

For this framework of time changes, we also consider the version of the "classical" Palm distribution. The relationship with the DPD is studied. We prove that Palm theory for random measures is indeed included in Palm theory for RTC's. In a theoretical application we consider the important non-simple (marked) point process case and define another distribution of Palm type. It can discriminate between (the marks of) simultaneous occurrences and has nice stationarity properties.
1 Introduction

Palm theory is especially known because of its merits for the study of stationary queueing systems. See, e.g., Franken et al. (1982), Brandt et al. (1990), Baccelli & Brémaud (1994), and Sigman (1995). In the presence of a group of measure-preserving, one-dimensional time-shifts the theory considers the relationship between two distributions, a time-stationary distribution and a Palm distribution (PD). Both describe the stochastic behavior of the system. The first as it is seen from a randomly chosen time and the second from a randomly chosen arrival (point) or arrival epoch. In classical queueing theory there are two basic frameworks to describe the process of arriving customers, each with its own merits. In Franken et al. (1982) the realizations are characterized as integer-valued measures counting the numbers of arrivals. For simple queueing systems (i.e., only single arrivals) a natural group of point-shifts exists which is stationary under the PD. An advantage of this framework is that the setting of counting measures naturally generalizes to a setting of measures to cover the more general framework needed for Palm theory for modern fluid queues and random measures. See, e.g., Schmidt & Serfozo (1994) and Miyazawa (1994). However, a disadvantage is that this framework and the PD are less appropriate in the case of non-simple queueing systems. The PD loses its nice stationarity properties and it cannot discriminate between two simultaneous arrivals within a batch of arrivals. (Cf. Section 1.3.7 in Franken et al. (1982) and page 87 in König & Schmidt (1992).) In Brandt et al. (1990) the realizations of a system of arriving customers are characterized as sequences of non-decreasing times which correspond to arrival epochs of customers. The distribution P defined on page 82 of this reference is of Palm type and can discriminate between (the marks of) two simultaneous arrivals. However, this framework can only be used to describe point process systems and cannot be generalized to a more general framework for fluid queues.

The general framework of measures mentioned above can equivalently be characterized by the set of right-continuous and non-decreasing functions g on R with g(0) = 0. The PD of a random functional with realizations in this set follows immediately from (1.4) of Geman & Horowitz (1973) and is equivalent to the PD of the corresponding random measure. So, this framework of functions has the same advantages and disadvantages as the framework of measures. As in this last reference, we point out that in general the relationship between the time-stationary distribution and its PD is not dual.

In the present research we will consider a framework and a distribution of Palm type which overcome all disadvantages mentioned above. This DPD, detailed Palm distribution, restores duality and stationarity. In a slightly modified form it was first mentioned in Miyazawa (1994), on a smaller σ-field from a more applied point of view.
We will study the properties of this (general) DPD. It behaves like "the mother of all PD's" in the sense that all well-known distributions of Palm type follow immediately from it, in a natural and very intuitive way. To compare these distributions of Palm type, we will write them all as integrals along the vertical axis. This turns out to be more natural. Essentially, we generalize Geman & Horowitz' set of realizations to a set $G$ by replacing $g(0) = 0$ for the right-continuous and non-decreasing functions $g$ by $g(0-) \leq 0 \leq g(0)$. We will consider random time changes (RTC's) $\Lambda$, random functionals on $(\Omega, \mathcal{F}, P)$ with values in $G$, and study Palm theory for such random phenomena. In contrast with the frameworks mentioned above, a two-dimensional family $\Theta$ of transformations (shifts) on $\Omega$ will be considered. It will be assumed that this family of shifts behaves like the natural two-dimensional family in the canonical case, the case that $\Lambda$ is the identity on $\Omega = G$. That is, $\Theta$ behaves like a group along the extended graph (i.e., the graph extended with the vertical jump-parts) of $\Lambda(\cdot, \omega)$, in a consistent way. Assuming time-stationarity (as usual, one-dimensional along the horizontal axis) with respect to $P$, we define the detailed Palm distribution (DPD) $P_\Lambda$ of $P$ with respect to $\Lambda$. With $\Lambda'(\cdot, \omega)$ the right-continuous inverse of $\Lambda(\cdot, \omega)$, it is proved that the pairs $(\Lambda, P)$ and $(\Lambda', P_\Lambda)$ are dual: taking the DPD of the DPD gives $P$ in return. Both $P$ and $P_\Lambda$ describe the stochastic behavior of the system. The first as it is seen on the graph of $\Lambda$ from a position with the first coordinate chosen at random, the second as it is seen on the extended graph of $\Lambda$ from a position with the second coordinate chosen at random. Stating it otherwise, $P_\Lambda$ describes the stochastic behavior of the system as it is seen on the graph of $\Lambda'$ from a position with the first coordinate chosen at random. From this heuristic description of the relationship between $P$ and its DPD it immediately becomes clear how -for the present framework- the "ordinary" PD's of special systems should be defined. For instance in the case of a non-simple queueing system, a PD which can discriminate between two simultaneous arrivals and which has nice stationarity properties follows immediately from the DPD.

In Section 2 we first introduce the framework and give the definition of a random time change $\Lambda$. It is assumed that a two-dimensional family $\Theta$ of transformations exists which behaves as a group along the extended graph of $\Lambda$, in a consistent way. In a canonical setting this assumption is naturally satisfied. The family $\Theta$ induces one dimensional groups $\theta$ and $\eta$ of transformations (shifts) on $\Omega$, the first on the horizontal axis and the second on the vertical axis. The relationship between $\theta, \eta, \Lambda$, and its generalized, right-continuous inverse $\Lambda'$ is considered in a few lemma's. In Section 3 we formulate the further assumptions, that $\theta$ is stationary with respect to $P$ and that the (possibly random) long-run average $\lim_{t \to \infty} \Lambda(t)/t$ is positive and finite. The detailed Palm dis-
tribution $P_\Lambda$ is defined and it is proved that the group $\eta$ is stationary w.r.t. it. The intuitive interpretation of $P_\Lambda$ mentioned before is formalized. While $\Lambda(0)$ is zero a.s. under $P$, it turns out that under $P_\Lambda$ it is -given the jumps size $\Lambda(0) - \Lambda(0^-)$ at zero- uniform $[0, \Lambda(0) - \Lambda(0^-)]$ distributed. We also consider the Palm distribution $P^0$ which in the present framework is most similar to the PD's of the textbooks and papers mentioned above. Just like $P_\Lambda$, this PD is also defined in terms of an integral along the vertical axis. In the canonical case it simply arises from the DPD by shifting the origin (on the extended graph) to the position $(0, \Lambda(0))$ on the graph of $\Lambda$. Things are generalized further by letting the random time change $\Lambda$ be accompanied by a stochastic process $S$ defined on its extended graph while the consistency property is generalized. The pair $(\Lambda, S)$ is called a marked time change. It turns out that on the set $\{(t, \Lambda(t)) : t \in \mathbb{R}\}$ the process $S$ is stationary under $P$ w.r.t. $\theta$, while on the set $\{(\Lambda'(x), x) : x \in \mathbb{R}\}$ it is stationary under $P_\Lambda$ w.r.t. $\eta$. Section 4 is about duality. It is proved that reflecting $\Theta_{(t,x)}$ into $\Theta_{(x,t)}$ leads to a family $\Theta'$ of transformations such that the assumptions of Sections 2 and 3 for the triple $(\Theta, \Lambda, P)$, also hold for the triple $(\Theta', \Lambda', P_\Lambda)$. So, we can define the DPD of the DPD. And this distribution turns out to be equal to $P$ again. This duality principle can be used to derive results for $P_\Lambda$ from similar results for $P$ (and vice versa). The duality between $P$ and $P_\Lambda$, and the simple relationship between the two Palm distributions $P_\Lambda$ and $P^0$ is used to obtain a general inversion formula to express $P$ in terms of $P^0$.

In Section 5 we show that Palm theory for random measures is indeed included in Palm theory for RTC's. From a random measure satisfying the usual assumptions (see, e.g., Schmidt & Serfozo (1994) and Miyazawa (1994)), we constructively create an RTC which satisfies the assumptions of the present research, without additional assumptions. The (ordinary) PD of this RTC corresponds to the PD of the random measure we started with. Section 6 can be seen as a theoretical application. It shows that the present framework is appropriate in cases where the counting measure framework is less suitable. For not necessarily simple point processes $\Lambda = \Phi$, it is shown that the DPD $P_\Phi$ naturally leads to a distribution $\overline{P}_\Phi$ of Palm type which can discriminate between simultaneous occurrences, having nice stationarity properties. Under this distribution we consider a stationary family of shifts, and show that the sequence of interval-lengths and - in the case of a marked point process (an example of a marked time change) - the sequence of marks are both stationary if suitably indexed. This distribution is just the equivalent for the PD in Brandt et al. (1990) when the framework of time changes is used. The relationship between the three Palm distributions $P^0, P_\Phi$ and $\overline{P}_\Phi$ is considered. Some of the proofs are given in the Appendix.
2 Framework

Let $\tilde{G}$ denote the set of functions $g : \mathbb{R} \to \mathbb{R}$ such that $g$ is non-decreasing, continuous from the right, and $\lim_{t \to -\infty} g(t) = \pm \infty$. Set $G := \{ g \in \tilde{G} : g(0-) \leq 0 \leq g(0) \}$. Endow $\tilde{G}$ with the smallest $\sigma$-field making all the projection mappings $t \to g(t), g \in \tilde{G}$, measurable; denote this by $\tilde{G}$ and set $G := \tilde{G} \cap G$. We view $\mathbb{R}$ as the time line, and call $g \in G$ a time change. For $g \in \tilde{G}$, the set

$$ \Gamma(g) := \{(t, x) \in \mathbb{R}^2 : g(t-) \leq x \leq g(t) \} $$

is called the extended graph of $g$, and the function $g'$ with

$$ g'(x) := \sup \{ s \in \mathbb{R} : g(s) \leq x \}, \quad x \in \mathbb{R}, $$

the (generalized) inverse of $g$. By identifying $g \in \tilde{G}$ with its extended graph $\Gamma(g)$, we obtain measurable spaces $(\Gamma(\tilde{G}), \Gamma(\tilde{G}))$ and $(\Gamma(G), \Gamma(G))$. For a proof of the following lemma we refer to the Appendix.

Lemma 2.1. For all $g \in G$ we have:
(a) $g' \in G$, (b) $(g')' = g$, (c) $(t, x) \in \Gamma(g)$ iff $(x, t) \in \Gamma(g')$, (d) $(g'(x), x) \in \Gamma(g)$ for all $x \in \mathbb{R}$.

Let $(\mathcal{S}, \mathcal{F})$ be a measurable space. A random time change (shortly, RTC) $\Lambda$ is a measurable mapping $\mathcal{S} \to G$. For $w \in \mathcal{S}$ we will write $\Lambda(\cdot, w)$ for the corresponding function in $G$ and $\Lambda(t, w)$ for its value in $t \in \mathbb{R}$. The generalized inverse of $\Lambda(\cdot, w)$ is denoted by $\Lambda'(\cdot, w)$. So, $\Lambda'$ is another random time change. The extended graphs of $\Lambda(\cdot, w)$ and $\Lambda'(\cdot, w)$ are shortly denoted by $\Gamma(w)$ and $\Gamma'(w)$, respectively. In this context we will usually use $s$ and $t$ to denote elements of the horizontal axis of $\Gamma(w)$, and $x$ and $y$ for elements of the vertical axis.

Let $(\hat{\Omega}, \hat{\mathcal{F}})$ be a measurable space. A random time change (shortly, RTC) $\Delta$ is a measurable mapping $\hat{\Omega} \to G$. For $\omega \in \hat{\Omega}$ we will write $\Delta(\cdot, \omega)$ for the corresponding function in $G$ and $\Delta(t, \omega)$ for its value in $t \in \mathbb{R}$. The generalized inverse of $\Delta(\cdot, \omega)$ is denoted by $\Delta'(\cdot, \omega)$. So, $\Delta'$ is another random time change. The extended graphs of $\Delta(\cdot, \omega)$ and $\Delta'(\cdot, \omega)$ are shortly denoted by $\Gamma(\omega)$ and $\Gamma'(\omega)$, respectively. In this context we will usually use $s$ and $t$ to denote elements of the horizontal axis of $\Gamma(\omega)$, and $x$ and $y$ for elements of the vertical axis.

Let $(\hat{\Omega}, \hat{\mathcal{F}})$ be a measurable space such that $\hat{\Omega} \supset \Omega$ and $\hat{\mathcal{F}} \cap \Omega = \mathcal{F}$. We call $(\hat{\Omega}, \hat{\mathcal{F}})$ an extension of $(\Omega, \mathcal{F})$. Let $\Theta = \{ \Theta_{(t,x)} : (t, x) \in \mathbb{R}^2 \}$ be a family of transformations on $\hat{\Omega}$, not necessarily a group. I.e., $\Theta_{(t,x)}(\omega) \quad ((t, x, \omega) \in \mathbb{R} \times \mathbb{R} \times \hat{\Omega})$ is a measurable mapping from $(\mathbb{R}^2 \times \hat{\Omega}, B(\mathbb{R}^2) \times \hat{\mathcal{F}})$ to $(\hat{\Omega}, \hat{\mathcal{F}})$. The assumption below expresses that the (random) extended graph $\Gamma$ of $\Delta$ is consistent with $\Theta$, and that the family $\Theta$ behaves itself on $\Gamma$ as a group. Assume:

(i) For all $\omega \in \Omega, (t, x) \in \Gamma(\omega)$ and $(s, y) \in \Gamma(\Theta_{(t,x)}(\omega))$ we have:
(a) $\Lambda(\cdot, \Theta(t,x)\omega) = \Lambda(t + \cdot, \omega) - x$,

(b) $\Theta(s,y)(\Theta(t,x)\omega) = \Theta(s+t,x+y)\omega$.

(Note that with $(t, x) \in \Gamma(\omega)$ and $(s, y) \in \Gamma(\Theta(t,x)\omega)$, indeed $(s + t, x + y) \in \Gamma(\omega)$.)

Assumption (i) is motivated by canonical settings (useful in applications) as in the following example.

**Example 2.1.** In the canonical case, we take $(\tilde{\Omega}, \tilde{\mathcal{F}}) = (\tilde{G}, \tilde{\mathcal{G}})$ and $(\Omega, \mathcal{F}) = (G, \mathcal{G})$. The RTC $\Lambda$ is the identity mapping on $G$. In this case a natural family $\Theta$ is defined by $\Theta(t,x)g := g(t + \cdot) - x$, $(t, x) \in \mathbb{R}^2$ and $g \in \tilde{G}$. Assumption (i) is trivially satisfied.

A more general canonical case (see also the marked time change in Section 3) arises as follows. Let $\Omega$ be the set of pairs $(g, \rho)$ with $g \in \tilde{G}$ and $\rho$ a measurable function on $\Gamma(g)$. Let $\Omega$ be the restriction of $\tilde{\Omega}$ to $g \in G$. $\sigma$-Fields $\tilde{\mathcal{F}}$ and $\mathcal{F}$ are constituted by the sets $\{(g, \rho) \in \tilde{\Omega} : g \in B\}$ with $B \in \tilde{\mathcal{G}}$ and $B \in \mathcal{G}$, respectively. A natural family $\Theta$ is defined by

$$\Theta(t,x)(g, \rho) := (g(t + \cdot) - x, \rho(t + \cdot, x + \cdot)), \ (t, x) \in \mathbb{R}^2 \text{ and } (g, \rho) \in \tilde{\Omega},$$

and an RTC $\Lambda$ by

$$\Lambda(\cdot, (g, \rho)) := g(\cdot), \ (g, \rho) \in \Omega.$$ 

It is an easy exercise to prove that the consistency in (a) and the group-structure in (b) are indeed satisfied.

Define, for $\omega \in \Omega$, $t \in \mathbb{R}$, and $x \in \mathbb{R}$,

$$\theta_t \omega := \Theta(t,\Lambda(t,\omega)) \omega \text{ and } \eta_x \omega := \Theta(\Lambda'(x,\omega), x)\omega,$$

and put $\theta := \{\theta_t : t \in \mathbb{R}\}$ and $\eta := \{\eta_x : x \in \mathbb{R}\}$ for the corresponding families of transformations (shifts) on $\Omega$. The results in the following lemma can be proved easily.

**Lemma 2.2.** Under Assumption (i), $\theta$ and $\eta$ are groups. For all $s, t, x, y \in \mathbb{R}$ and $\omega \in \Omega$ we have:
\begin{align*}
\Lambda(t, \theta_s \omega) &= \Lambda(t + s, \omega) - \Lambda(s, \omega), \\
\Lambda'(y, \eta_x \omega) &= \Lambda'(y + x, \omega) - \Lambda'(x, \omega), \\
\Lambda'(x, \theta_t \omega) &= \Lambda'(x + \Lambda(t, \omega), \omega) - t, \\
\Lambda(t, \eta_x \omega) &= \Lambda(t + \Lambda'(x, \omega), \omega) - x, \\
\eta_x(\theta_t \omega) = \eta_x + \Lambda(t, \omega) \omega \quad \text{and} \quad \theta_t(\eta_x \omega) = \theta_t + \Lambda'(x, \omega) \omega.
\end{align*}

Note that $\theta_0 \omega$ and $\eta_0 \omega$ are not necessarily equal to $\omega$. In the canonical setting of Example 2.1, $\theta_t$ is the shift operator which moves the origin to the position (on the graph) belonging to $t$ on the horizontal axis, while $\eta_x$ moves the origin to the position (on the extended graph) which belongs to $x$ on the vertical axis. Note also that $\Lambda(0, \theta_t \omega)$ is always zero, while $\Lambda(0, \eta_x \omega)$ need not.

We next introduce shift-invariant sets. Define

$$
\mathcal{I}^{(\theta)} := \{ A \in \mathcal{F} : \theta_t^{-1} A = A \text{ for all } t \in \mathbb{R} \}, \quad \mathcal{I}^{(\eta)} := \{ A \in \mathcal{F} : \eta_x^{-1} A = A \text{ for all } x \in \mathbb{R} \}.
$$

The next lemma is an extension of Lemma 2 of Nieuwenhuis (1994). See the Appendix for a proof.

\textbf{Lemma 2.3.} Under Assumption (i), the above invariant $\sigma$-fields coincide.

In view of this lemma, we denote $\mathcal{I}^{(\theta)}$ and $\mathcal{I}^{(\eta)}$ by a single notation $\mathcal{I}$. Note that, as an immediate consequence of the lemma,

$$
f \circ \theta_t = f \quad \text{and} \quad f \circ \eta_x = f \quad (2.1)
$$

for all $\mathcal{I}$-measurable functions $f : \Omega \to \mathbb{R}$ and all $t, x \in \mathbb{R}$.

In the next sections we will occasionally use the left-continuous inverse $g^{-1}$ of $g \in \mathcal{G}$, defined by $g^{-1}(x) = \inf\{ s \in \mathbb{R} : g(s) \geq x \}, x \in \mathbb{R}$. Let $g^*$ be the measure generated by $g$. I.e.,

$$
g^*((s, t]) := g(t) - g(s), \quad s < t. \quad (2.2)
$$

The following lemma enables us to transform integrals with respect to $g^*$, on the horizontal axis, into Lebesgue-integrals on the vertical axis. It will be proved in the Appendix.
Lemma 2.4. Let \( g \in \hat{G} \) and let \( f : \mathbb{R} \to \mathbb{R} \) be \( g^* \)-integrable. Then we have, for all \( a, b \in \mathbb{R} \) with \( a < b \),

\[
\int_{g(a)}^{g(b)} f(g'(x)) dx = \int_{g(a)}^{g(b)} f(g^{-1}(x)) dx = \int_{(a,b)} f(s) g^*(ds).
\]

3 Two distributions of Palm type

An RTC is a right-continuous and non-decreasing random functional, not necessarily zero at 0 but passing the zero-level at 0. For such a functional we define two Palm distributions (PD's). The detailed PD has a nice stationarity property, the "ordinary" PD is very similar to the well-known PD for random measures. Heuristically, both describe the stochastic behavior of the functional as it is seen from a randomly chosen position on the extended graph; however, the randomness procedures differ. The relationship between the two distributions is studied. Things are generalized by considering marked time changes: RTC's accompanied by a stochastic process on their extended graphs.

We first introduce a probability measure \( P \) on \((\Omega, \mathcal{F})\). Apart from Assumption (i), we will also assume that the family \( \theta \) is stationary with respect to \( P \), i.e.:

\[ (ii) \quad P(\theta_t^{-1}A) = P(A) \text{ for all } t \in \mathbb{R} \text{ and } A \in \mathcal{F}, \]

and that the (possibly non-degenerate) limit \( \bar{\Lambda} := \lim_{t \to \infty} \Lambda(t)/t = E(\Lambda(1)/t) \) satisfies

\[ (iii) \quad P(0 < \bar{\Lambda} < \infty) = 1. \]

Assumptions (i) and (ii) imply that the RTC \( \Lambda \) has stationary increments.

**Definition 3.1.** Under Assumptions (i)-(iii), the probability measure \( P_\Lambda \) on \((\Omega, \mathcal{F})\), the detailed Palm distribution (DPD) of \( P \) with respect to \( \Lambda \), is defined by

\[
P_\Lambda(A) := E \left( \frac{1}{\Lambda} \int_{\Lambda(0)}^{\Lambda(1)} 1_A \circ \eta_x dx \right), \quad A \in \mathcal{F}.
\]

(3.1)

In Miyazawa (1994) a slightly modified version of (3.1) is presented. It is defined from a more applied point of view, on a smaller \( \sigma \)-field.
Theorem 3.1. Assume (i)-(iii). Then $P = P_\Lambda$ on $\mathcal{I}$, and the group $\eta$ of transformations on $\Omega$ is stationary with respect to $P_\Lambda$:

$$P_\Lambda(\eta_y^{-1}A) = P_\Lambda(A) \text{ for all } y \in \mathbb{R} \text{ and } A \in \mathcal{F}. \quad (3.2)$$

Proof. By (2.1) it is obvious that $P|_\mathcal{I} = P_\Lambda|_\mathcal{I}$. Let $y \in \mathbb{R}$ and $A \in \mathcal{F}$. Then

$$P_\Lambda(\eta_y^{-1}A) = E \left( \frac{1}{\Lambda} \int_{\Lambda(1)}^{\Lambda(1)+y} 1_A \circ \eta_x dx \right)$$

$$= P_\Lambda(A) + E \left( \frac{1}{\Lambda} \int_{\Lambda(1)}^{\Lambda(1)+y} 1_A \circ \eta_x dx \right) - E \left( \frac{1}{\Lambda} \int_{0}^{y} 1_A \circ \eta_x dx \right),$$

which equals $P_\Lambda(A)$ by Lemma 2.2 and stationarity of $\theta$. \Box

Expectations under $P_\Lambda$ are denoted by $E_\Lambda$. With $\Lambda \in G$, we also have $\Lambda' \in G$. As an immediate consequence of Theorem 3.1 it follows:

$$\Lambda' = \lim_{x \to \infty} \frac{1}{x} \Lambda'(x) = E_\Lambda(\Lambda'(1)|\mathcal{I}) \quad P_\Lambda- \text{ and } P\text{-a.s.} \quad (3.3)$$

(Note that $\Lambda'(0) = 0$ $P_\Lambda$-a.s.) By part (d) of Lemma 2.1 we obtain that for all $\omega \in \Omega$ and $\varepsilon > 0$,

$$\frac{\Lambda'(x)}{\Lambda(\Lambda(x))} \leq \frac{\Lambda'(x)}{x} \leq \frac{\Lambda'(x)}{\Lambda(\Lambda(x)) - \varepsilon}$$

if $x$ is sufficiently large. Hence, by Assumption (iii) and the first part of Theorem 3.1, we have:

$$\Lambda' = \frac{1}{\Lambda} \quad P_\Lambda- \text{ and } P\text{-a.s.} \quad (3.4)$$

The following theorem gives (at least in the canonical case) the intuitive meaning for the detailed Palm distribution $P_\Lambda$ as defined in Definition 3.1. Consider $\Lambda$ by way of its extended graph $\Gamma$. Starting with $P$, the DPD $P_\Lambda$ arises intuitively by choosing at random an $x$ on the positive half-line of the vertical axis and shifting the origin to the corresponding position $(\Lambda'(x), x)$ on the extended graph of $\Lambda$. 

Theorem 3.2. Assume (i)-(iii). Then, for $A \in \mathcal{F}$,
\[
\frac{1}{\Lambda(t)} \int_0^{A(t)} 1_A \circ \eta_x dx \to P_A(A|\mathcal{I}) = \frac{1}{\Lambda} E \left( \int_{\Lambda(0)}^{A(1)} 1_A \circ \eta_x dx | \mathcal{I} \right) \quad P\text{- and } P_\Lambda\text{-a.s.,}
\]
\[
\frac{1}{y} \int_0^y P(\eta_x^{-1} A) dx \to P_A(A).
\]

Proof. Set $\psi(t) := \int_0^{\Lambda(t)} 1_A \circ \eta_y dy$, $t \in \mathbb{R}$. By Lemma 2.2 it follows that $\psi(t) \circ \theta_s = \psi(t+s) - \psi(s)$ for all $s, t \in \mathbb{R}$. Note that the limits
\[
\lim_{y \to \infty} \frac{1}{y} \int_0^y 1_A \circ \eta_x dx \quad \text{and} \quad \lim_{t \to \infty} \frac{1}{\Lambda(t)} \int_0^{\Lambda(t)} 1_A \circ \eta_x dx
\]
exist and are equal (for all $\omega \in \Omega$). Under $P_\Lambda$ the left-hand limit equals $P_\Lambda(A|\mathcal{I})$ a.s., while under $P$ the right-hand limit equals
\[
\lim_{n \to \infty} \frac{1}{\Lambda(n)} \sum_{i=1}^n \psi(1) \circ \theta_i = \frac{1}{\Lambda} E \left( \int_{\Lambda(0)}^{A(1)} 1_A \circ \eta_x dx | \mathcal{I} \right) \quad \text{a.s.}
\]

Since $P = P_\Lambda$ on $\mathcal{I}$, the first part of the theorem follows immediately. The second part follows by taking $P$-expectation in the left-hand part of (3.5) and by noting that $E(P_\Lambda(A|\mathcal{I})) = P_\Lambda(A)$.

Set $\Lambda\{t\} := \Lambda(t) - \Lambda(-t)$, $t \in \mathbb{R}$. The following corollary concludes that under $P_\Lambda$ the conditional distribution of $\Lambda(0)$ given $\Lambda\{0\}$ is the uniform $[0, \Lambda\{0\}]$ distribution. A proof is given in the Appendix.

Corollary 3.1. Assume (i)-(iii). Then $P_\Lambda(\Lambda(0) \in B|\Lambda\{0\})$ is $P_\Lambda$-a.s. equal to
\[
\frac{1}{\Lambda(0)} \int_0^{\Lambda(0)} 1_B(s) ds 1_{\Lambda(0) > 0} + 1_B(0) 1_{\Lambda(0) = 0}, \quad B \in \mathcal{B}([0, \infty)).
\]

In Section 5 we will include Palm theory for random measures in Palm theory for RTC's. In advance, note that two RTC's $\Lambda_1$ and $\Lambda_2$ on $(\Omega, \mathcal{F}, P)$ which generate the same random measure $\Lambda^*$, i.e.
\[
\Lambda^*((s, t]) = \Lambda_1(t) - \Lambda_1(s) = \Lambda_2(t) - \Lambda_2(s)
\]
for all \( s \leq t \), may have different DPD’s. It can, however, easily be proved that the DPD’s coincide on the sub-\( \sigma \)-field generated by \( \Lambda^* \), and that they coincide on the whole of \( \mathcal{F} \) if the respective families \( \Theta^{(1)} \) and \( \Theta^{(2)} \) satisfy

\[
\Theta^{(2)}_{(t, x)} \omega = \Theta^{(1)}_{(t, x - X(\omega))} \omega \quad \text{for all } \omega \in \Omega \text{ and } (t, x) \in \Gamma_2(\omega).
\]

(3.6)

Here \( X = \Lambda_2(0) - \Lambda_1(0) \).

We next introduce a distribution of Palm type which looks more familiar than the DPD mentioned above. A random time change \( \Lambda \) generates a random measure \( \Lambda^* \). By Assumptions (i) and (ii), \( \Lambda^* \) is stationary under \( P \). In accordance with Palm theory for random measures we define the (ordinary) Palm distribution (PD) of \( P \) with respect to \( \Lambda \) as the well-known PD of \( P \) w.r.t. \( \Lambda^* \), and call it \( P^0 \). I.e.,

\[
P^0(A) := E \left( \frac{1}{\Lambda} \int_{[0,1]} 1_A \circ \theta_t \lambda^*(dt) \right), \quad A \in \mathcal{F}.
\]

(3.7)

This definition corresponds to (2) in Schmidt and Serfozo (1994), modified along the lines of Sigman (1995) and Nieuwenhuis (1997). (As discussed in the last two references, it is more natural to use the random intensity \( \lambda^* \) (instead of its \( P \)-expectation) in the definition of PD.) Note that two RTC’s which generate the same random measure \( \Lambda^* \) have the same PD if their respective \( \theta \)-groups are equal. This hypothesis is satisfied if (3.6) holds. In order to compare PD and DPD, it seems natural to obtain an expression for \( P^0(A) \) in terms of an integral on the other (vertical) axis. The following result is an immediate consequence of Lemma 2.4.

\[
P^0(A) = E \left( \frac{1}{\Lambda} \int_{0}^{\Lambda(1)} 1_A \circ \theta_{\lambda^*(x)}(dx) \right), \quad A \in \mathcal{F}.
\]

(3.8)

We will write \( E^0 \) for expectations under \( P^0 \). In the next theorem the relationship between PD and DPD is studied. It is proved in the Appendix.

**Theorem 3.3.** Let \( \Lambda \) be an RTC on \( (\Omega, \mathcal{F}, P) \) which satisfies (i)-(iii). Then the relationship between \( P^0 \) and \( P_{\Lambda} \) is as follows:

(a) \( P^0 = P_{\Lambda} \theta_0^{-1} \),

(b) \( P_{\Lambda}(A) = E^0 \left( \frac{1}{\Lambda\{0\}} \int_{-\Lambda\{0\}}^{0} 1_A \circ \eta_\tau dx \right), \quad A \in \mathcal{F} \).
The averaged integral under (b) is interpreted as $1_A(\omega)$ if $\Lambda(\{0\}, \omega) = 0$.

**Remark 3.1.** As an immediate consequence of (a), Theorem 3.2, and the last equation in Lemma 2.2, we have:

$$
\frac{1}{y} \int_0^y P \left( \theta_{\Lambda(x)}^{-1} A \right) \, dx \to P^0(A), \quad A \in \mathcal{F}.
$$

(3.9)

In the canonical case $P^0$ arises from $P$ by randomly choosing an $x$ on the positive vertical axis, and shifting the origin to the position $(\Lambda'(x), \Lambda(\Lambda'(x)))$ on the graph of $\Lambda$. With this intuitive result in mind, part (b) of Theorem 3.3 is very obvious. If $\Lambda$ is a pure jump process with jump-times $T_i$ and jump-sizes $X_i$ (under the convention that $\cdots < T_{-1} < T_0 < 0 < T_1 < \cdots$), relation (3.9) becomes

$$
\frac{1}{n} \sum_{i=1}^n E(X_i 1_A \circ \theta_{T_i}) \to P^0(A), \quad A \in \mathcal{F}. \quad \square
$$

(3.10)

At the end of this section, we consider the more general situation that the RTC is accompanied by a stochastic process on its extended graph. A *marked time change* is a pair $(\Lambda, S)$ consisting of a random time change $\Lambda$ and a stochastic process $S$, on a common probability space $(\Omega, \mathcal{F}, P)$, such that $S((\cdot, \cdot), \omega)$ is a measurable function on $\Gamma(\omega)$ for all $\omega \in \Omega$. It is assumed that a family $\Theta$ of transformations exists such that $\Lambda$ satisfies Assumptions (i)-(iii). Furthermore, we assume that for all $\omega \in \Omega$ and $(t, x) \in \Gamma(\omega),$

(iv) $S((s, y), \Theta_{(t,x)}\omega) = S((s + t, y + x), \omega)$ for all $(s, y) \in \Gamma(\Theta_{(t,x)}\omega)$.

(See Example 2.1 for a canonical version.) Set $S_1(t) := S((t, \Lambda(t)))$ and $S_2(x) := S((\Lambda'(x), x)); \quad t, x \in \mathbb{R}$. It is an easy exercise to prove that the stochastic processes $S_1$ and $S_2$ satisfy

$$
S_1(t) \circ \theta_s = S_1(t + s) \quad \text{and} \quad S_2(x) \circ \eta_y = S_2(x + y),
$$

$$
S_1(t) \circ \eta_y = S_1(t + \Lambda'(y)) \quad \text{and} \quad S_2(x) \circ \theta_t = S_2(x + \Lambda(t)),
$$

(3.11)

for all $s, t, x, y \in \mathbb{R}$. Consequently, $S_1$ is stationary under $P$ w.r.t. $\theta$, while $S_2$ is stationary under $P_{\Lambda}$ w.r.t. $\eta$. 
4 Inversion by duality

Starting with Assumptions (i)-(iii) for the pair $(\Lambda, P)$ we defined $P_\Lambda$, the DPD w.r.t. $\Lambda$. A similar approach for the pair $(\Lambda', P_\Lambda)$ leads to a duality criterion. This criterion is used to derive an inversion formula for the (ordinary) PD.

Assume (i)-(iii). We next consider $\Lambda'$ instead of $\Lambda$; we will give corresponding quantities a prime. Define the family $\Theta'$ of transformations $\Theta'_{(x,t)}$ by $\Theta'_{(x,t)}(\omega) := \Theta_{(t,x)}(\omega)$, $\omega \in \hat{\Omega}$ and $(x,t) \in \mathbb{R}^2$. By Lemma 2.1 it is an easy exercise to prove that $\Theta'$ satisfies Assumption (i)' which arises from (i) by replacing $\Lambda$ by $\Lambda'$ and $\Gamma(\omega)$ by $\Gamma'(\omega)$. From $\Theta'$ we define $\theta'_x$ and $\eta'_t$; $t, x \in \mathbb{R}$. Part (b) of Lemma 2.1 ensures that $\theta' = \eta$ and $\eta' = \theta$. So, we have

(ii)' $\theta'$ is stationary w.r.t. $P_\Lambda$,

(iii)' $P_\Lambda(0 < \Lambda' < \infty) = 1$.

(The last assertion is a consequence of (3.4).) Consequently, the DPD of $P_\Lambda$ with respect to $\Lambda'$, notation $(P_\Lambda)_{\Lambda'}$, is well-defined:

$$(P_\Lambda)_{\Lambda'}(A) = E_\Lambda \left( \frac{1}{\Lambda'} \int_{\Lambda'(0)}^{\Lambda'(1)} 1_A \circ \theta_s ds \right).$$

**Theorem 4.1.** The detailed Palm distribution of $P_\Lambda$ with respect to $\Lambda'$ is equal to $P$. Especially, for $A \in \mathcal{F}$,

$$P(A) = E_\Lambda \left( \frac{1}{\Lambda'} \int_{\Lambda'(0)}^{\Lambda'(1)} 1_A \circ \theta_s ds \right),$$

$$\frac{1}{t} \int_0^t P_\Lambda(\theta_s^{-1} A) ds \to P(A).$$

**Proof.** Since (i)'-(iii)' are satisfied, we can apply Theorem 3.2 replacing $\Lambda$ by $\Lambda'$, $P$ by $P_\Lambda$, and $P_\Lambda$ by $(P_\Lambda)_{\Lambda'}$. This yields, for an equivalent version of the first part of Theorem 3.2,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t 1_A \circ \theta_s ds = \frac{1}{\Lambda'} E_\Lambda \left( \int_{\Lambda'(0)}^{\Lambda'(1)} 1_A \circ \theta_s ds \right) \ P_\Lambda{\text{-a.s.}}$$

Since $P = P_\Lambda$ on $\mathcal{I}$, we obtain:

$$P(A) = \lim_{t \to \infty} \frac{1}{t} \int_0^t P(\theta_s^{-1} A) ds = (P_\Lambda)_{\Lambda'}(A),$$
which gives the first assertion of the present theorem. The second is an immediate consequence.

\[ \square \]

**Remark 4.1.** By the above approach it follows that duality holds between \( P \) and its DPD w.r.t. \( \Lambda \), a property which in general does not hold for classical PD's. See also Geman and Horowitz (1973). Properties for \( P \) can immediately be translated into dual properties for \( P_{\Lambda} \), and vice versa. For instance, from Theorem 3.2 and Corollary 3.1 we immediately obtain the following dual assertions:

\[
P(A|\mathcal{I}) = \frac{1}{\Lambda'} E_{\Lambda} \left( \int_{\Lambda'(0)}^{\Lambda'(1)} 1_A \circ \theta ds | \mathcal{I} \right) \quad P_{\Lambda^-} \text{ and } P\text{-a.s.},
\]

Under \( P \), the conditional distribution of \( \Lambda'(0) \) given \( \Lambda' \{0\} \) is the uniform \([0, \Lambda' \{0\}]\) distribution.

The last result is well-known in the case that \( \Lambda \) characterizes a simple point process (see Section 6), and obviously holds more generally, for instance for a pure jump process: with \( \Lambda'(0) =: T_1 \) the first jump-time on \((0, \infty)\) and \( \Lambda'(0-) =: T_0 \) the last jump-time on \((-\infty, 0]\), the conditional distribution of \( T_1 \) given \( T_1 - T_0 \) is the uniform \([0, T_1 - T_0]\) distribution. Note also that the convergence result of Theorem 4.1 means that intuitively \( P \) arises from \( P_{\Lambda} \) by choosing at random an \( s \) on the positive half-line of the horizontal axis and shifting the origin to the corresponding position \((s, \Lambda(s))\) on the graph of \( \Lambda \). \( \square \)

Relations (3.7) and (3.8) express how \( P \) can be transformed into \( P^0 \). An expression which works the other way round, is historically called an inversion formula. See also Schmidt and Serfozo (1994), Corollary 1 in Section 2. We use inversion of \( P_{\Lambda} \) to \( P \), managed in Theorem 4.1 by using the duality approach, to accomplish inversion of \( P^0 \) to \( P \). The proof of the following theorem is included in the Appendix. Recall that \( \overline{N} = E_{\Lambda}(\Lambda'(1)|\mathcal{I}) \).

**Theorem 4.2.** Let \( P^0 \) be the PD of \( P \) with respect to \( \Lambda \). Then

\[
P(A) = E^0 \left( \frac{1}{\overline{N}} \int_0^{\Lambda'(1)} \left( 1 \wedge \frac{1 - \Lambda(t)}{\Lambda \{0\}} \right) 1_A \circ \theta dt \right), \quad A \in \mathcal{F}.
\]
Here the minimum in the integrand is interpreted as 1 if $\Lambda (\{0\}, \omega) = 0$.

5 Stationary random measures and PD's

In this section we include Palm theory for random measures in Palm theory for RTC's. Starting with a random measure and the well-known PD in a common stationary setting, we construct an RTC which generates the random measure and which satisfies (i)-(iii). No additional assumptions are needed. In a sense the PD of this RTC is equal to the PD of the random measure we started with. The DPD of the random measure is defined as the DPD of this RTC.

Let $M$ be the set of all measures $\mu$ on $\mathcal{B}(\mathbb{R})$ for which $\mu(B) < \infty$ for all bounded $B \in \mathcal{B}(\mathbb{R})$. $M$ is endowed with the $\sigma$-field $\mathcal{M}$ generated by the sets $\{\mu \in M : \mu(B) = k\}$, $k \in \mathbb{N}$, and $B \in \mathcal{B}(\mathbb{R})$. A random measure on $\mathbb{R}$ is a measurable mapping $\Lambda_0^*$ from a measurable space $(\Omega_0, \mathcal{F}_0)$ to $(M, \mathcal{M})$. Let $Q$ be a probability measure on $(\Omega_0, \mathcal{F}_0)$. We write $E$ for expectations under $Q$. We assume that a group $\tau := \{\tau_t : t \in \mathbb{R}\}$ of transformations on $\Omega_0$ exists such that $\Lambda_0$ is consistent with $\tau$, and $\tau$ is stationary with respect to $Q$; i.e.

(i-a) $\Lambda_0^*(B) \circ \tau_t = \Lambda_0^*(B + t)$ for all $B \in \mathcal{B}(\mathbb{R})$ and $t \in \mathbb{R}$,

(ii-a) $Q \tau_t^{-1} = Q$ for all $t \in \mathbb{R}$.

Hence, $\Lambda_0^*$ is stationary under $Q$. It can be characterized by the random time change $\Lambda_0$ defined by

$$\Lambda_0(t) := \begin{cases} 
\Lambda_0^*((0, t]) & \text{if } t \geq 0 \\
-\Lambda_0^*((t, 0]) & \text{if } t < 0.
\end{cases}$$

Note that $\Lambda_0(0) = 0$ and that $\Lambda_0$ generates $\Lambda_0^*$; see (2.2). In case $\Lambda_0^*$ is a point process, i.e. an integer-valued random measure, the RTC $\Lambda_0$ is also integer-valued and can never satisfy part (a) of Assumption (i), notwithstanding the choice of the family $\Theta$. So, we must choose the RTC generating $\Lambda_0^*$, in a more clever way.

Furthermore, we assume that

(iii-a) $Q(0 < \Lambda_0 < \infty) = 1$. 
Here $\overline{\Lambda}_0$ is the long-run average $E(\Lambda_0(1)|\mathcal{I}_0) = \lim_{t \to \infty} \Lambda_0(t)/t$ with $\mathcal{I}_0$ the invariant $\sigma$-field of $\tau$. Similar to Schmidt & Serfozo (1994), we define the Palm distribution $Q^0$ of $Q$ with respect to $\Lambda_0^*$ by

$$
Q^0(A) := E \left( \frac{1}{\overline{\Lambda}_0} \int_{[0,1]} 1_A \circ \tau_1 \Lambda_0^*(dt) \right), \quad A \in \mathcal{F}_0.
$$

As in (3.7), we use the random intensity; see Sigman (1995) and Nieuwenhuis (1997). For a fixed $\Lambda_0^*$, this PD does not depend on the choice made for the RTC which generates $\Lambda_0^*$. By Lemma 2.4 we can also consider $Q^0(A)$ along the vertical axis:

$$
Q^0(A) = E \left( \frac{1}{\overline{\Lambda}_0} \int_0^{\Lambda_0(1)} 1_A \circ \tau_{\Lambda_0^*(x)} dx \right), \quad A \in \mathcal{F}_0.
$$

Since, for fixed $\omega \in \Omega_0$, $\Lambda_0^*(x)$ and $\Lambda_0^*(x^-)$ can be unequal for at most countably many $x \in \mathbb{R}$, we may equivalently use $\Lambda_0^{-1}(x) := \Lambda_0^*(x^-)$ in (5.3) instead of $\Lambda_0^*(x)$. I.e., we may also use the left-continuous version $\Lambda^{-1}$ of $\Lambda$.

As mentioned above, a family $\Theta$ of transformations need not satisfy Assumption (i), not even if (i-a) holds. We have to make the measurable space $(\Omega_0, \mathcal{F}_0)$ richer. Assume that (i-a), (ii-a) and (iii-a) are satisfied, and define

$$
\tilde{\Omega} := \Omega_0 \times \mathbb{R} \quad \text{and} \quad \tilde{\mathcal{F}} := \mathcal{F}_0 \times \mathcal{B}(\mathbb{R}),
$$

$$
\Omega := \{(\omega_0, z) \in \tilde{\Omega} : 0 \leq z \leq \Lambda_0^*(\{0\}, \omega_0)\},
$$

$$
\mathcal{F} := \tilde{\mathcal{F}} \cap \mathcal{F}.
$$

Let $\omega = (\omega_0, z)$ be an element of $\tilde{\Omega}$. For $s, t, x \in \mathbb{R}$ we define:

$$
\Theta(t, x)\omega := (\tau_t \omega_0, \Lambda_0(t, \omega_0) + z - x) \in \tilde{\Omega},
$$

$$
\Lambda(t, \omega) := \Lambda_0(t, \omega_0) + z,
$$

$$
\Lambda^*((s, t], \omega) := \Lambda(t, \omega) - \Lambda(s, \omega) \quad \text{for} \ s \leq t.
$$

Next, we identify $\Omega_0$ and $\Omega_0 \times \{0\}$. With this identification, $\Lambda$ and $\Lambda^*$ are extensions of $\Lambda_0$ and $\Lambda_0^*$. Note, however, that $\Omega = \Omega_0$ if $\Lambda_0(\cdot, \omega)$ is continuous on $\mathbb{R}$ for all $\omega \in \Omega_0$. Note also that the last definition above implies a measure $\Lambda^*(\cdot, \omega)$ on $\mathcal{B}(\mathbb{R})$ with $\Lambda^*(B, \omega) = \Lambda_0^*(B, \omega_0)$ for all $B \in \mathcal{B}(\mathbb{R})$, and that the random function $\Lambda$, defined on $(\Omega, \mathcal{F})$, is indeed a random time change since $\Lambda(\cdot, \omega) \in G$ for all $\omega \in \Omega$. The family $\Theta$ of transformations on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ satisfies part (b) of Assumption (i), even for all $\omega = (\omega_0, z)$ in $\tilde{\Omega}$ and for all $t, x, s, y \in \mathbb{R}$:
\[ \Theta_{(s,y)}(\Theta_{(t,z)}\omega) = \Theta_{(s,y)}(\tau_{t}\omega_0, \Lambda_0(t,\omega_0) + z - x) \\
= (\tau_{s}(\tau_{t}\omega_0), \Lambda_0(s, \tau_{t}\omega_0) + \Lambda_0(t,\omega_0) + z - x - y) \\
= (\tau_{s+t}\omega_0, \Lambda_0(s + t,\omega_0) + z - (x + y)), \]

which equals \( \Theta_{(s+t,x+y)}\omega \). (In the last equality, we used (i-a) and the group property of the family \( \tau \) on \((\Omega_0, \mathcal{F}_0)\).) Again with (i-a), it is an easy exercise to prove that part (a) of (i) also holds. Hence, we can define groups \( \theta \) and \( \eta \) of transformations on \( \Omega \), as in Section 2. Note that, with the identification \( \omega_0 = (\omega_0, 0) \), we have for \( \omega = (\omega_0, z) \):

\[ \theta_t \omega = (\tau_t \omega_0, 0) = \tau_t \omega_0 \in \Omega_0. \quad (5.5) \]

Especially, \( \tau_t \) is just the restriction of \( \theta_t \) to \( \Omega_0 \) (as it should be). We can extend \((\Omega_0, \mathcal{F}_0, Q)\) to \((\Omega, \mathcal{F}, P)\) by the definition:

\[ P(A) := Q(A \cap \Omega_0), \ A \in \mathcal{F}. \quad (5.6) \]

The pair \((\theta, P)\) also satisfies (ii). So, \( \theta \) is stationary with respect to \( P \). Concerning the invariant \( \sigma \)-fields \( \mathcal{I}_0 \) and \( \mathcal{I} \) of \( \tau \) and \( \theta \), respectively, we note that: \( A \cap \Omega_0 \in \mathcal{I}_0 \) if \( A \in \mathcal{I} \). Hence, Assumption (iii-a), with \( E \) denoting expectation under \( Q \), implies Assumption (iii), with \( E \) denoting expectation under \( P \).

We conclude that a random measure \( \Lambda^*_0 \) satisfying (i-a), (ii-a), and (iii-a), can (in a natural way) be extended to a random measure \( \Lambda^* \) and a corresponding random time change \( \Lambda \) which satisfies Assumptions (i)-(iii); without additional assumptions. Reverse-ly, a random time change \( \Lambda \) satisfying (i)-(iii) implies a random time change \( \Lambda_0 := \Lambda \circ \theta_0 \) which satisfies (i-a), (ii-a), and (iii-a).

Having extended \((\Omega_0, \mathcal{F}_0, Q, \tau, \Lambda^*_0, \Lambda_0)\) to \((\Omega, \mathcal{F}, P, \theta, \Lambda^*, \Lambda)\), the definition of \( Q^0 \) in (5.2) transforms into the definition of \( P^0 \) -the PD of \( P \) w.r.t. \( \Lambda \) in (3.7). Note that \( P^0(A) = Q^0(A \cap \Omega_0) \). We will interpret \( P^0 \) as the PD of \( Q \) w.r.t. the random measure \( \Lambda^*_0 \). Similarly, we will call \( P_{\Lambda} \) the DPD of \( Q \) w.r.t. \( \Lambda^*_0 \). The relationship between these two distributions of Palm type is described in Theorem 3.3.

6 PD's in the point process case

When the random time change is a stepfunction with integer-valued stepsizes, a modified version of the DPD is of interest. This distribution of Palm type, defined for the setting
of time changes, can discriminate between (the marks of) simultaneous occurrences. This is illustrated in the marked point process case. It is equivalent to the PD in Brandt et al. (1990) for the setting of sequences of occurrences. The several distributions of Palm type are compared.

Let $\Phi$ be a (random) point process (PP) on $\mathbb{R}$, defined on a probability space $(\Omega, \mathcal{F}, P)$. That is, $\Phi$ is an RTC with $\Phi(t) - \Phi(s) \in \mathbb{Z}$ for all $\omega \in \Omega$ and $s, t \in \mathbb{R}$. Note that $\Phi(0)$ can be unequal to zero and $\Phi(t)$ need not be integer-valued. We suppose that Assumptions (i)-(iii) are satisfied.

Repeat that $P_{\Phi}$ and $P^0$, the DPD and the (ordinary) PD of $P$ w.r.t $\Phi$, are defined by

$$P_{\Phi}(A) = E \left( \frac{1}{\Phi_0} \int_0^{\Phi(1)} 1_A \circ \eta_x dx \right), \quad P^0(A) = E \left( \frac{1}{\Phi_0} \int_0^{\Phi(1)} 1_A \circ \theta_{\eta_x} dx \right), \quad A \in \mathcal{F} \quad (6.1)$$

(cf. (3.1) and (3.8)), and that

$$P_{\Phi}(A) = \lim_{y \to -\infty} \frac{1}{y} \int_0^y P \left( \eta_x^{-1} A \right) dx, \quad P^0(A) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^n P \left( \theta_{T_i}^{-1} A \right), \quad A \in \mathcal{F} \quad (6.2)$$

(cf. Theorem 3.2 and (3.10)). Here $\Phi = E(\Phi(1)|\mathcal{I})$ and the $T_i$ are the times of occurrences (arrivals) defined by $T_i = \Phi'(i - 1 + \Phi(0)) = \Phi'(i - 1) \circ \theta_0$. So,

$$... \leq T_{-2} \leq T_{-1} \leq T_0 \leq 0 < T_1 \leq T_2 \leq ... \quad (6.3)$$

Recall, for the canonical settings, the intuitive interpretations of $P_{\Phi}$ and $P^0$ following (3.4) and (3.9), respectively. Obviously, $P^0$ cannot discriminate between two simultaneous occurrences within one batch. A modified version of the DPD seems to overcome this disadvantage. Define the distribution $\overline{P}_{\Phi}$ by

$$\overline{P}_{\Phi}(A) = E \left( \frac{1}{\Phi_0} \int_0^{\Phi(1)} 1_A \circ \eta_x m(dx) \right), \quad A \in \mathcal{F} \quad (6.4)$$

Here $m$ is the lattice-measure concentrated on the set $\mathbb{Z}$ of integers, and $\Phi_0 := E(\Phi(1)|\mathcal{I}_{00})$ with $\mathcal{I}_{00}$ the invariant $\sigma$-field of the group $\{\eta_i : i \in \mathbb{Z}\}$ of transformations on $\Omega$. Note that $\mathcal{I} \subset \mathcal{I}_{00}$ by Lemma 2.3, that $\mathcal{I} \neq \mathcal{I}_{00}$ since the $\omega$-set ($\Phi(0) \in \mathbb{N}$) does not belong to $\mathcal{I}$, and that

$$\overline{P}_{\Phi}(\eta_i^{-1} A) = \overline{P}_{\Phi}(A), \quad A \in \mathcal{F} \text{ and } i \in \mathbb{Z} \quad (6.5)$$
So, \( \{\eta_i\} \) is stationary w.r.t. \( \overline{P}_\Phi \), and

\[
\frac{1}{n} \sum_{i=1}^{n} 1_A \circ \eta_i \rightarrow \overline{P}_\Phi(A|\mathcal{I}_0) \quad \overline{P}_\Phi- \text{and } P-\text{a.s.}
\]

since \( P = \overline{P}_\Phi \) on \( \mathcal{I}_{00} \). Hence,

\[
\frac{1}{n} \sum_{i=1}^{n} P(\eta_i^{-1} A) \rightarrow \overline{P}_\Phi(A), \quad A \in \mathcal{F}.
\] (6.6)

In the canonical setting, we can interpret \( \overline{P}_\Phi \) as arising from \( P \) by randomly choosing a positive integer \( i \) on the vertical axis and shifting the origin to \((\Phi'(i), i) = (T_{i+1}, i)\) on the extended graph of \( \Phi \). In case of a non-simple PP, relation (6.6) makes clear that \( \overline{P}_\Phi \) gives the opportunity to discriminate between the arrivals within a batch and that it is equivalent to the distribution \( P \) on page 82 of Brandt et. al. (1990).

Let \( \beta_i := T_{i+1} - T_i, i \in \mathbb{Z}, \) be the sequence of interval lengths (interarrivals) of the PP. It can easily be proved that

\[
\beta_j \circ \theta_t = \beta_j + \Phi'(t) - \Phi(0), \quad j \in \mathbb{Z} \text{ and } t \in \mathbb{R},
\] (6.7)

and that in general \( \beta_j \circ \eta_1 \) is not equal to \( \beta_{j+1} \). However, by renumbering the interarrivals by making use of the special character of our framework we can regain this property. Set

\[
\alpha := \max \{ \Phi(0) - i : i \in \mathbb{N}_0 \text{ and } \Phi(0) - i \leq 0 \}.
\] (6.8)

In a canonical setting, \( \Theta_{(0,\alpha(w))} \omega \) moves the origin \((0,0)\) of \( \Gamma(\omega) \) downwards to the first position on this extended graph which is integer-distanced from \((0,\Phi(0,\omega))\). (If \( \Phi(0,\omega) = 0 \), nothing happens.) With \( \Phi_\alpha := \Phi - \alpha \), we define

\[
\tilde{T}_j := \Phi'(j + \alpha) = T_{j+1} - \Phi_\alpha(0) \quad \text{and} \quad \tilde{\beta}_j := \tilde{T}_{j+1} - \tilde{T}_j, \quad j \in \mathbb{Z}.
\] (6.9)

It is obvious that \( \alpha \circ \eta_1 = \alpha \) since (6.8) does not change by adding an integer to the \( \Phi(0) - i \). Consequently,

\[
\tilde{\beta}_j \circ \eta_1 = \tilde{\beta}_{j+1}, \quad j \in \mathbb{Z}.
\] (6.10)
Hence, \((\tilde{\beta}_i)\) is stationary under \(\tilde{P}_\Phi\).

We next compare the distributions \(P, P^0, P_\Phi\) and \(\bar{P}_\Phi\). At first, note that

\[
P(\Phi(0) = 0) = 1 \quad \text{and} \quad P^0(\Phi(0) = 0) = 1,
\]
\[
P_\Phi(\Phi(0) \in \mathbb{N}_0) = 0 \quad \text{and} \quad \bar{P}_\Phi(\Phi(0) \in \mathbb{N}) = 1, 0
\]
\[
P(\Phi(0-) = 0) = 1 \quad \text{and} \quad P^0(\Phi(0-) < 0) = 1,
\]
\[
P_\Phi(\Phi(0-) = 0) = 0 \quad \text{and} \quad \bar{P}_\Phi(\Phi(0-) = 0) > 0.
\]

(Here \(\mathbb{N}\) does not contain 0, but \(\mathbb{N}_0\) does.) In the following theorem we write \(E^0, E_\Phi\) and \(\bar{E}_\Phi\) for expectations under \(P^0, P_\Phi\) and \(\bar{P}_\Phi\), respectively.

**Theorem 6.1.** Let \(\Phi\) be a PP on \((\Omega, \mathcal{F}, P)\) which satisfies Assumptions (i)-(iii). Then, for \(A \in \mathcal{F}\),

(a) \(P_\Phi(A) = \bar{E}_\Phi\left(\int_0^1 1_A \circ \eta_x \, dx\right)\),
(b) \(\bar{P}_\Phi(A) = P_\Phi(\eta^{-1}_\sigma A)\),
(c) \(P^0(A) = P_\Phi(\theta_0^{-1} A) = \bar{P}_\Phi(\theta_0^{-1} A)\),
(d) \(P_\Phi(A) = E^0 \left(\frac{1}{\Phi(0)} \int_{-\Phi(0)}^0 1_A \circ \eta_x \, dx\right)\),
(e) \(\bar{P}_\Phi(A) = E^0 \left(\frac{1}{\Phi(0)} \sum_{i=-\Phi(0)}^{n-1} 1_A \circ \eta_i\right)\).

**Proof.** Note that, for \(n \in \mathbb{N}\) and \(A \in \mathcal{F}\),

\[
\frac{1}{n} \int_0^n 1_A \circ \eta_x \, dy = \frac{1}{n} \sum_{i=1}^{n} \int_0^1 1_A \circ \eta_x \circ \eta_{i-1} \, dx.
\]

As \(n \to \infty\), the LHS tends to \(P_\Phi(A|\mathcal{I})\), both \(P_\Phi\)-a.s. and \(P\)-a.s. The RHS of (6.12) tends to \(\bar{E}_\Phi\left(\int_0^1 1_A \circ \eta_x \, dx|\mathcal{I}_{00}\right)\), both \(\bar{P}_\Phi\)-a.s. and \(P\)-a.s. Since \(P = P_\Phi\) on \(\mathcal{I}\) and \(P = \bar{P}_\Phi\) on \(\mathcal{I}_{00}\), we obtain both sides of (a) as limits of \(\frac{1}{n} \int_0^n P(\eta_x^{-1} A) \, dy\) as \(n \to \infty\). So, the two sides have to be equal. For part (b), note that under \(\bar{P}_\Phi\) we have by (6.11) that \(\Phi'(x) = 0\) for all \(x \in (0, 1)\). Hence, \(\bar{P}_\Phi\)-a.s., the composition \(\alpha \circ \eta_x\) equals \(\alpha - x\) for all \(x \in (0, 1)\). With this result, (b) follows from (a). Part (d) and the first equality in (c) follow from Theorem 3.3. The second equality in (c) is a consequence of (a) and the RHS of (6.11).
Part (e) follows from (b) and (d).

In order to show how $\bar{P}_\Phi$ can discriminate between simultaneous arrivals within a batch, we consider a generalisation. Let $K$ be a metric space, assumed to be complete and separable. $B(K)$ denotes the Borel-$\sigma$-field on $K$. A marked point process (MPP) on $\mathbb{R}$ with mark space $K$ is a random pair $(\Phi, (m_i)_{i \in \mathbb{Z}})$ where $\Phi$ is a point process and $(m_i)_{i \in \mathbb{Z}}$ is a random sequence in $K$. The two elements of the pair are defined on a common probability space $(\Omega, \mathcal{F}, P)$. We interpret $m_i$ as the mark of $T_i, i \in \mathbb{Z}$, and assume that $\Phi$ satisfies Assumptions (i)-(iii). Furthermore, we assume that

$$(iv - a) \quad m_i(\Theta_{(t,x)}(\omega)) = m_{i+\Phi(t,\omega)-\Phi(0,\omega)}(\omega), \quad i \in \mathbb{Z}, \quad \omega \in \Omega, \quad (t, x) \in \Gamma(\omega). \quad (6.13)$$

An MPP is indeed a marked time change (cf. Section 3) since the stochastic process $S$ with

$$S((s,y),\omega) := \begin{cases} m_i(\omega) & \text{if } y = \Phi(0,\omega) + i - 1 \\ 0 & \text{otherwise,} \end{cases}$$

$\omega \in \Omega$ and $(s,y) \in \Gamma(\omega)$, is defined on $\Gamma$ and satisfies Assumption (iv) by (6.13). Note that $S$ is constant on horizontal parts of $\Gamma$ and that $m_i$ is just the value of $S$ at the position $(T_i, \Phi(0) + i - 1)$ on $\Gamma$. As in (3.11), we could create a stochastic process $S_2$ which is stationary under $P_{\Phi}$. In view of (6.9), a renumbering of the sequence $(m_i)_{i \in \mathbb{Z}}$ seems to be of more importance. Set

$$\tilde{m}_j := m_{j+\Phi_{\alpha}(0)} = S(\tilde{T}_j, j + \alpha), \quad j \in \mathbb{Z}.$$

Hence, $\tilde{m}_j$ is the mark of $\tilde{T}_j$. Since $\alpha \circ \eta_1 = \alpha$, it is an easy exercise to prove that

$$\tilde{m}_j \circ \eta_1 = \tilde{m}_{j+1}, \quad j \in \mathbb{Z}.$$

So, the sequence $(\tilde{m}_j)_{j \in \mathbb{Z}}$ is stationary under $P_{\Phi}$. In view of (6.6) this result is intuitively clear (and can also be proved from it), at least in the canonical setting. Under $P_{\Phi}$ we can, for instance, consider the probability that the mark of the first customer in order has some property, even in the case that customers arrive in batches.
Appendix 1

Proof of Lemma 2.1. Let \( g \in G \).

(a) Only the fact that \( g'(0-) \leq 0 \leq g'(0) \) needs an argument. For \( y < 0 \) we have: \( g(0) \geq 0 > y \) and hence \( g'(y) \leq 0 \). By letting \( y \) tend to 0 from below, we obtain that \( g'(0-) \leq 0 \). For \( s < 0 \) we have: \( g(s) \leq g(0-) \leq 0 \). So, \( g'(0) \geq 0 \).

(b) Let \( t \in \mathbb{R} \) and \( \varepsilon > 0 \). Then:

\[
g'(g(t + \varepsilon)) = \sup\{s \in \mathbb{R} : g(s) \leq g(t + \varepsilon)\} \geq t + \varepsilon > t.
\]

So, \( g(t + \varepsilon) \not\in \{y \in \mathbb{R} : g'(y) \leq t\} \), and

\[
g(t + \varepsilon) \geq \sup\{y \in \mathbb{R} : g'(y) \leq t\} = (g')'(t).
\]

By letting \( \varepsilon \) tend to 0, we obtain \( g(t) \geq (g')'(t) \). Suppose that \( g(t) \) is strictly larger than \( (g')'(t) \). Then \( y \in \mathbb{R} \) would exist such that \( y > (g')'(t) \) and \( y < g(t) \). On one hand, \( g'(y) \) would be larger than \( t \) because of (A.1). On the other hand, we could choose a positive \( \varepsilon \) such that \( y < g(t) - \varepsilon \), and hence \( g'(y) \leq g'(g(t) - \varepsilon) \leq t \). We conclude that \( g(t) = (g')'(t) \) for all \( t \in \mathbb{R} \).

(c) Suppose that \( (t, x) \in \Gamma(g) \), i.e. \( g(t-) \leq x \leq g(t) \). Then

\[
g'(x) \geq g'(g(t-)) = \sup\{s \in \mathbb{R} : g(s) \leq g(t-)\} \geq t.
\]

For \( \varepsilon > 0 \) we have: \( x - \varepsilon < g(t) - \frac{1}{2} \varepsilon \) and \( g'(x - \varepsilon) \leq g'(g(t) - \frac{1}{2} \varepsilon) \leq t \). Hence, \( g'(x-) \leq t \leq g'(x) \) and \((x, t) \in \Gamma(g')\). The reversed implication follows from (b).

(d) Follows from (c). \( \square \)

Proof of Lemma 2.3. We prove that \( \mathcal{T}^{(n)} \subset \mathcal{T}^{(\theta)} \) for a family \( \Theta = \{\Theta_{(t, x)} : (t, x) \in \mathbb{R}^2\} \) of transformations (on \( \Omega \)) which satisfy Assumption (i). The reversed implication follows by duality arguments as in the first part of Section 4.

Let \( A \in \mathcal{T}^{(n)} \), i.e.

\[
\text{for all } \omega' \in \Omega \text{ and } x \in \mathbb{R} : \omega' \in A \iff \eta_x\omega' \in A. \tag{A.2}
\]

We prove that \( \omega \in A \iff \theta_{\omega} A \subset \mathbb{R} \), for all \( \omega \in \Omega \) and \( s \in \mathbb{R} \). Let \( \omega \in A \) and \( s \in \mathbb{R} \). For \( x := -\Lambda(s, \omega) \) we obtain by Lemma 2.2 that
\[ \eta_x(\theta_s \omega) = \eta_{x + \Lambda(x, \omega)} \omega = \eta_0 \omega, \]

which belongs to \( A \) by (A.2). Again by (A.2), with \( \omega' = \theta_s \omega \), we conclude that \( \theta_s \omega \in A \). Let \( \omega \in \Omega \) be such that \( \theta_\omega \in A \). Note that \( \theta_0 \omega = \theta_{-\Lambda(0, \omega)} \) belongs to \( A \) because of the above arguments. By (A.2), with \( x = -\Lambda(0, \omega) \), we obtain that \( \eta_0 \omega = \eta_x(\theta_0 \omega) \) belongs to \( A \). Again by (A.2), with \( \omega' = \omega \) and \( x = 0 \), we conclude that \( \omega \in A \). \( \square \)

**Proof of Lemma 2.4.** It is an easy exercise to prove that, for all \( s, t \in \mathbb{R} \) with \( s < t \),

\[ g(s) \geq x \iff s \geq g^{-1}(x). \tag{A.3} \]

Hence, the integral in the middle is equal to

\[ \int_{-\infty}^{+\infty} 1_{(a,b)}(g^{-1}(x)) f(g^{-1}(x)) \, dx. \tag{A.4} \]

Note that \( g^{-1} \) induces on \( \mathbb{R} \) the measure \( \mu \) defined by

\[ \mu((s, t]) = \text{Leb}\{x \in \mathbb{R} : s \leq g^{-1}(x) \leq t\}, \quad s < t. \]

Again by (A.3) it follows that \( \mu = g^* \). This proves the right-hand equality. The left-hand equality follows immediately since \( g^{-1} \) and \( g' \) can only differ in countably many points. \( \square \)

**Proof of Corollary 3.1.** Let \( T_i, \quad i \geq 1, \) be the subsequent times (if any) in \((0, \infty)\) where \( \Lambda \) is discontinuous. Set \( D_i := \Lambda(T_i -) \) and \( S_i := \Lambda(T_i), \quad i \geq 1, \) and note that for \( y \in [D_i, S_i) \) we have: \( \Lambda\{0\} \circ \eta_y = S_i - D_i \) and \( \Lambda(0) \circ \eta_y = S_i - y \). For \( y \in [D_i, \infty) \) but \( y \) outside the intervals \([D_i, S_i)\) we have: \( \Lambda\{0\} \circ \eta_y = 0 \).

Let \( B, C \in \mathcal{B}([0, \infty)) \). By Theorem 3.1 we obtain on one hand that

\[
\begin{align*}
P_\Lambda(\Lambda\{0\} \in (C \cap (0, \infty)) \text{ and } \Lambda(0) \in B) &= \lim_{n \to \infty} E \left( \frac{1}{S^n} \sum_{i=1}^{n} \int_{D_i}^{S_i} (1_{\Lambda(0) \in C} \circ \eta_y \cdot 1_{\Lambda(0) \in B} \circ \eta_y) \, dy \right) \\
&= \lim_{n \to \infty} E \left( \frac{1}{S^n} \sum_{i=1}^{n} 1_{(S_i - D_i, \infty) \cap \Lambda(0) \in C} \cdot \int_{0}^{S_i - D_i} 1_{B}(s) \, ds \right),
\end{align*}
\]
while on the other hand
\[
E_{A} \left( \frac{1}{\Lambda(0)} \int_{0}^{\Lambda(0)} 1_{B}(s) ds \right) = \lim_{n \to \infty} E \left( \frac{1}{S_n} \sum_{i=1}^{n} 1_{(S_i - D_i, C)} \int_{D_i}^{S_i} 1_{B}(s) ds dy \right) = \lim_{n \to \infty} E \left( \frac{1}{S_n} \sum_{i=1}^{n} 1_{(S_i - D_i, C)} \int_{0}^{S_i - D_i} 1_{B}(s) ds \right).
\]

The corollary follows immediately.

\[\Box\]

**Proof of Theorem 3.3.** By (3.1) and the last equality in Lemma 2.2, part (a) follows immediately from (3.8). Since \( P_{A} \Theta_{(0,0)}^{-1} = P_{\Lambda} \), we obtain with (a) that
\[
P_{0}(A \cap (\Lambda \{0\} = 0)) = P_{\Lambda}((\Theta_{(0,0)}^{-1} A) \cap (\Lambda \{0\} = 0)) = P_{\Lambda}(A \cap (\Lambda \{0\} = 0)).
\]

By part (a) and again by Lemma 2.2, we have
\[
E_{0} \left( \frac{1}{\Lambda \{0\}} \int_{-\Lambda \{0\}}^{0} 1_{A \circ \eta_{x}} dx \cdot 1_{(\Lambda \{0\} > 0)} \right) = E_{\Lambda} \left( \frac{1}{\int_{\Lambda(0) \setminus \Lambda(0)}^{\Lambda(0)}} 1_{A \circ \eta_{y}} \cdot 1_{(\Lambda(0) > 0)} dy \right)
\]
since \( \Lambda'(y) = 0 \) for all \( y \in (\Lambda(0-), \Lambda(0)) \). By Fubini's theorem and Theorem 3.1 this last expression is equal to
\[
\int_{-\infty}^{+\infty} E_{\Lambda} \left( \frac{1}{\Lambda \{0\}} 1_{A \circ \eta_{y}} \cdot \frac{1}{\Lambda \{0\}} 1_{(\Lambda \{0\} > 0)} dy \right)
\]
\[
= E_{\Lambda} \left( \frac{1}{\Lambda \{0\}} 1_{A \circ \eta_{y}} \cdot \text{Leb} \{ y \in \mathbb{R} : \Lambda(\Lambda'(y)) - y < \Lambda(0) \}< \Lambda(\Lambda'(y)) \} \right)
\]
\[
= P_{\Lambda}(A \cap (\Lambda \{0\} > 0)).
\]
(Here Leb represents Lebesgue measure.) Part (b) follows.

\[\Box\]

**Proof of Theorem 4.2.** Starting with \( P(A) \), we use inversion of \( P_{\Lambda} \) into \( P \) as expressed in Theorem 4.1, and then we apply Theorem 3.3(b). Splitting the resulting
$P^0$-expectation into two parts according to whether $\Lambda\{0\} = 0$ or $\Lambda\{0\} > 0$, only the second part, i.e.

$$E^0\left(\frac{1}{\Lambda\{0\}} \frac{1}{A} \int_{-\Lambda\{0\}}^{0-} \int_0^{\Lambda'(1+x)} 1_A \circ \theta_t dt dx \cdot 1_{(\Lambda\{0\} > 0)}\right),$$

needs some arguments. Since $\Lambda'(1 + x) = 0$ for all $x$ with $-\Lambda\{0\} < x < -1$ (if any), we can restrict the outer integral. Concerning the inner integral, note that there are at most countably many $t$ where $\Lambda(\cdot, \omega)$ is discontinuous; we omit them. Note also that the remaining $t$ satisfy $\Lambda'(1 + x) \geq t$ iff $\Lambda(t) \leq 1 + x$. Applying Fubini's theorem to the resulting expression, we obtain:

$$E^0\left(\frac{1}{\Lambda\{0\}} \frac{1}{A} \int_0^{\Lambda'(1-)} 1_A \circ \theta_t \int_{-(\Lambda\{0\} \Lambda 1)}^{t} 1_{[\Lambda(t) - 1, \infty)}(x) dx dt \cdot 1_{(\Lambda\{0\} > 0)}\right),$$

$$= E^0\left(\frac{1}{\Lambda\{0\}} \frac{1}{A} \int_0^{\Lambda'(1-)} (\Lambda\{0\} \Lambda (1 - \Lambda(t))) \cdot 1_A \circ \theta_t dt \cdot 1_{(\Lambda\{0\} > 0)}\right).$$

The theorem follows by noting that for all $t \in (\Lambda'(1-, \omega), \Lambda'(1, \omega))$, we have: $1 - \Lambda(t, \omega) = 0$. \qed
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