DECAY AND STRICHARTZ ESTIMATES FOR
DISPERSYE EQUATIONS IN AN AHARONOV-BOHM FIELD

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Abstract. We prove decay and Strichartz estimates for the dispersive equations with an Aharonov-Bohm potential which is a singular and scaling-critical potential. The decay estimates generalize the results of [10] and the Strichartz estimates extend the weighted Strichartz estimates for Dirac proved in [7] so that we answer the open problem raised in [7, 8]. In addition, the argument provides a new simple proof of $L^1 \to L^\infty$-decay estimate of Schrödinger equation shown in [17]. The key point of the argument is to develop the distorted Fourier theory based on an observation of the eigenfunction of the Schrödinger operator with Aharonov-Bohm potential.

Key Words: Decay estimates, Strichartz estimates, Aharonov-Bohm potential, Dirac equation

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1. Introduction and main results

In this paper, we study the long time decay behavior of the dispersive equations with an Aharonov-Bohm potential which is a singular homogeneous, scaling-critical magnetic potential. More precisely, in the spirit of [7, 10, 16, 17], we will prove the decay and Strichartz estimates for Schrödinger, wave and Dirac equations in the Aharonov-Bohm magnetic field.

1.1. Introduction. Dispersive behaviors, in particular the decay and Strichartz estimates, of evolution equations play an important role in the study of nonlinear dispersive problems, and this is why they have attracted a great deal of attention in recent years, see [18, 23, 29, 30] and reference therein. The dispersive decay properties of evolution equations are generalized to the variable coefficient case (e.g. the evolution equations on manifolds), however the dispersive property becomes more complicated and is very far from complete even though in the simple perturbation of potentials. There is too much work to cite all here, we refer to [11, 19–22, 26] and the surveys [9, 27] for the results under various assumptions on the potentials.

In the present paper, we are interested in the evolution equations with the perturbation of scaling critical electromagnetic potentials. Let $A(x)$ be a real vector potential, we consider the pure magnetic Schrödinger operator

$$
L_A := (i\nabla + A(x))^2
$$

as the Friedrichs extension of the quadratic form $Q$, defined on $C_c^\infty(\mathbb{R}^n \setminus \{0\})$ via

$$
Q(f) = \left( \int_{\mathbb{R}^n} |(i\nabla + A(x)) f(x)|^2 \, dx \right)^{1/2}.
$$
In [16, 17], Fanelli, Felli, Fontelos, and Primo have studied the dispersive property of Schrödinger equations with more general singular homogeneous electromagnetic potentials
\[ \text{i} u_t - (L_A + r^{-2}V_0(\theta))u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n, \quad r = |x|, \quad \theta = \frac{x}{|x|} \tag{1.2} \]
with \( V_0 \in L^\infty(S^{n-1}) \) and \( |x|A(x) \in C^1(S^{n-1}, \mathbb{R}^n) \) satisfies the transversality condition
\[ A(\theta) \cdot \theta = 0, \quad \forall \; \theta \in S^{n-1}. \tag{1.3} \]
In particular, when \( V_0 = 0 \) and the magnetic potential \( A(x) \) in \( \mathbb{R}^2 \) is given by
\[ A(x) = (A_1(x), A_2(x)) = \alpha \left( -\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right), \quad \alpha \in \mathbb{R}, \tag{1.4} \]
Fanelli et al. [17] proved the dispersive estimate
\[ ||e^{itL_A}||_{L^1(\mathbb{R}^2) \to L^\infty(\mathbb{R}^2)} \lesssim |t|^{-1} \]
which was used to study the scattering theory for nonlinear Schrödinger equation in [31]. It is worth mentioning that the vector potential \( A(x) \) given by (1.4) is associated to the Aharonov-Bohm magnetic field (2-dimensional purely magnetic field), which is given by potentials associated to thin solenoids. More precisely, if the radius of the solenoid tends to zero while the flux through it remains constant, then the particle is subject to a \( \delta \)-type magnetic field. From the physical interesting points, the Aharonov-Bohm effect was initially predicted when electrons propagate in a domain with a zero magnetic field but with a nonzero vector potential \( A \), see [1]. The potential magnetic field is totally confined within a cylindrical tube of infinitesimal radius, we refer to [25] and references.

In this paper, we focus on the dispersive behavior of wave with the Aharonov-Bohm effect. More precisely, we consider
\[
\begin{cases}
\partial_t u + L_A u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\
u(0, x) = f(x), & \partial_t u(0, x) = g(x).
\end{cases}
\tag{1.5}
\]
where the operator \( L_A \) is defined by
\[ L_A f = (\text{i} \nabla + A(x))^2 f = -\Delta f + 2iA(x) \cdot \nabla f + |A(x)|^2 f, \tag{1.6} \]
where \( A(x) \) is the Aharonov-Bohm magnetic potential as in (1.4). Note that the wave equation is perturbed with the first order operator \( 2iA(x) \cdot \nabla \) and the potential \( |A(x)|^2 \). From the physical point of view, this perturbation corresponds to a magnetic potential. Regarding to mathematics interest, the solution of equation (1.5) preserves the scaling \( u \mapsto u_\lambda(t, x) = u(t/\lambda, x/\lambda) \), and for this reason we say \( A(x) \) is a scaling critical potential.

There are a number of work devoted to decay estimates problem for wave with electromagnetic potentials, we refer to [9,10] and the reference therein. However, to our best knowledge, there is little about the scaling critical perturbation on wave equation. In this framework, the Strichartz estimates and local smoothing for the Schrödinger and wave equations with inverse square potential (which is zero order perturbation) are proved in [3,4]. As mentioned above, in [16,17], the \( L^1 \to L^\infty \) time decay is proved for a wide class of Schrödinger flows with critical electromagnetic potentials which includes
the Aharonov-Bohm potential considered here. But the method of \cite{16,17} breaks down for wave equation due to the lack of pseudo conformal transformation which can be used to rewrite the Schrödinger equation in terms of a quantum harmonic oscillator with the singular electromagnetic potential. On the other hand, the Dirac equation is closely linked with the wave equation, and the picture of Dirac is far to complete even though some weak dispersion results are obtained in \cite{2,5-7} and Strichartz estimates are showed in \cite{12,13} with strong decay assumptions on potentials. Our result will be used to conclude the decay estimates of Dirac with scaling critical Aharonov-Bohm potential.

Let $A(x)$ be the Aharonov-Bohm magnetic potential as in (1.4), the Dirac operator in the Aharonov-Bohm magnetic field is given by

$$D_A^m = \sigma_3 m + \sigma_1(i\partial_1 + A_1) + \sigma_2(i\partial_2 + A_2)$$  \hspace{1cm} (1.7)

where $m \geq 0$ and $\sigma_j (j = 1, 2, 3)$ are the standard Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$  \hspace{1cm} (1.8)

which satisfy the following relations of anti-commutations

$$\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_{jk} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$  \hspace{1cm} (1.9)

For the Dirac flow, we will focus on the massless case, i.e. $m = 0$, and we will write the Dirac operator $D_A = D_A^0$. Consider the Cauchy problem for Dirac equation

$$\begin{cases} i\partial_t u = D_A u, & u(t, x) : \mathbb{R}_t \times \mathbb{R}^2_x \to \mathbb{C}^2 \\ u(0, x) = f(x). \end{cases}$$  \hspace{1cm} (1.10)

We emphasize that the Aharonov-Bohm potential $A(x)$ is critical if we rescale the massless Dirac operator. As pointed out in \cite{7}, the study of dispersive estimates for flows with the perturbation of scaling-critical potentials represents a particularly interesting and challenging problem. This is the same model considered in \cite{7}. To study the dynamics of equation (1.10), we should be careful about the self-adjointness of the operator $D_A$ even though one can choose some different self-adjoint extensions depending on the boundary conditions at infinity. For more details, we refer to \cite{8,15,19}.

From (1.7), we can explicitly write the massless Dirac operator as

$$D_A = \begin{pmatrix} 0 & (i\partial_1 + A_1) - i(i\partial_2 + A_2) \\ (i\partial_1 + A_1) + i(i\partial_2 + A_2) & 0 \end{pmatrix}.$$  \hspace{1cm} (1.11)

We will address dispersive problem of (1.10) from the wave equation (1.5) by squaring Dirac operator $D_A$. By squaring the Dirac operator, we have

$$(D_A)^2 = \begin{pmatrix} 0 & (i\partial_1 + A_1)^2 - (i\partial_2 + A_2)^2 \\ (i\partial_1 + A_1)^2 + (i\partial_2 + A_2)^2 & 0 \end{pmatrix}^2 = \mathcal{L}_A \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$  \hspace{1cm} (1.12)
where we have used the facts that $\partial_1 (A_2) = \partial_2 (A_1)$, $\text{div} A = 0$ and
\[(i\partial_t + A_1)^2 + (i\partial_2 + A_2)^2 = -\Delta + i\text{div} A + 2iA \cdot \nabla + |A|^2 = -\Delta + 2iA \cdot \nabla + |A|^2. \tag{1.13}\]
Therefore, by using the squaring trick and (1.6), we apply $i\partial_t + DA$ into the Dirac flow (1.10) to obtain
\[(i\partial_t + DA)(i\partial_t - DA)u = (\partial_t^2 + LA) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u = 0. \tag{1.14}\]
Therefore every component of the solution of (1.10) satisfies the wave equation
\[
\begin{aligned}
\begin{cases}
\partial_t u + LA u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\
u(0, x) = f(x), & \partial_t u(0, x) = g(x).
\end{cases}
\end{aligned}
\tag{1.15}
\]
As a consequence, the free Dirac flow is given by
\[
e^{itDA} f = \cos(t\sqrt{LA}) f - i\frac{\sin(t\sqrt{LA})}{\sqrt{LA}} DA f. \tag{1.16}\]

1.2. The main results. Now we state our first results about the dispersive estimates.

**Theorem 1.1.** Let $LA$ be the Schrödinger operator as in (1.6) and the Aharonov-Bohm potential $A(x)$ be given by (1.4). Then, for the Besov norm $\dot{B}^{3/2}_{s, 1}(\mathbb{R}^2)$ as in (2.14) below, there exists a constant $C$ such that:

- **For Schrödinger flow,**
  \[
  \|e^{itLA} f\|_{L^\infty(\mathbb{R}^2)} \leq C|t|^{-1}\|f\|_{L^1(\mathbb{R}^2)}; \tag{1.17}
  \]

- **For wave flow,**
  \[
  \|e^{it\sqrt{LA}} f\|_{L^\infty(\mathbb{R}^2)} \leq C|t|^{-1/2}\|f\|_{\dot{B}^{3/2}_{s, 1}(\mathbb{R}^2)}; \tag{1.18}
  \]

- **For Dirac flow,**
  \[
  \|e^{itDA} f\|_{L^\infty(\mathbb{R}^2)} \leq C|t|^{-1/2}\|f\|_{\dot{B}^{3/2}_{s, 1}(\mathbb{R}^2)}. \tag{1.19}\]

**Remark 1.1.** The decay estimates (1.18) and (1.19) for wave and Dirac are new. The estimate for Schrödinger (1.17) is not new, which has been proven in [17, Theorem 1.9]. But we provide a new and simple proof.

**Remark 1.2.** The method of [17], in which the Schrödinger equation is rewritten in terms of a quantum harmonic oscillator with the singular electromagnetic potential, does not work for wave equation due to the lack of pseudo conformal transformation.

The second main results are about the Strichartz estimates. Before stating the results, we introduce some notation. We say the pair $(q, r)$ is a **Schrödinger admissible pair**, if
\[
(q, r) \in \Lambda_0^S := \left\{ 2 \leq q, r \leq \infty, \frac{1}{q} + \frac{1}{r} = \frac{1}{2}, (q, r) \neq (2, \infty) \right\}. \tag{1.20}
\]
We say the pair $(q, r)$ is a **wave admissible pair**, if
\[
(q, r) \in \Lambda_s^W := \left\{ 2 \leq q, r \leq \infty, \frac{2}{q} + \frac{1}{r} \leq \frac{1}{2}, s = 2\left(\frac{1}{2} - \frac{1}{r} \right) - \frac{1}{q} \right\}, s \in \mathbb{R}. \tag{1.21}
\]
We state the second results on the Strichartz estimates.
Theorem 1.2. Let $\mathcal{L}_A$ be the Schrödinger operator as in (1.6) and the Aharonov-Bohm potential $A(x)$ be given by (1.4). Then, for the Sobolev norm $\mathcal{H}^s(\mathbb{R}^2)$ defined in (2.15) below, there exists a constant $C$ such that:

- Let $(q, r) \in \Lambda^S_0$, the following Strichartz estimates hold for Schrödinger flow
  \[
  \|e^{it\mathcal{L}_A}f\|_{L^q_t(L^r_x(\mathbb{R}^2))} \leq C\|f\|_{L^2(\mathbb{R}^2)}. \tag{1.22}
  \]

- Let $s \in \mathbb{R}$ and $(q, r) \in \Lambda_s^W$, the Strichartz estimates hold for wave flow
  \[
  \|e^{it\sqrt{\mathcal{L}_A}}f\|_{L^q_t(L^r_x(\mathbb{R}^2))} \leq C\|f\|_{\dot{H}^s(\mathbb{R}^2)}. \tag{1.23}
  \]

- Let $s \in \mathbb{R}$ and $(q, r) \in \Lambda_s^W$, the Strichartz inequalities hold for Dirac flow
  \[
  \|e^{it\mathcal{D}_A}f\|_{L^q_t(L^r_x(\mathbb{R}^2))} \leq C\|f\|_{\dot{H}^s(\mathbb{R}^2)}. \tag{1.24}
  \]

Remark 1.3. To prove the Strichartz estimates, the standard perturbation argument (e.g. see [3, 4, 22, 26]) with local-smoothing estimates does not work in this setting. Indeed (at least for the Dirac equation, and similarly for the wave), due to the long range perturbation of $i\alpha r^{-2}\partial_\theta$, one would need to prove a local smoothing in the form
\[
|||x||^{-1/2-u}\|_{L^2_{t,x}} \leq \|u_0\|_{L^2}
\]
which apparently fails, while one can only reach the weight $|x|^{-1/2-\varepsilon}$, see [7, 8].

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2. The proof of dispersive estimates

In this section, we prove the dispersive estimates. The key point is an observation of the explicit eigenfunction of $\mathcal{L}_A$. By (1.6), we have
\[
\mathcal{L}_A = -\Delta + 2iA(x) \cdot \nabla + |A(x)|^2.
\]
Moreover, under the polar coordinate, this operator can be rewritten as
\[
\mathcal{L}_A = -\partial^2_r - \frac{1}{r}\partial_r + \frac{\alpha^2 - \partial^2_\theta}{r^2} + 2i\alpha \partial_\theta.
\]
To see this, let $r = |x|$, $x_1 = r \cos \theta$, $x_2 = r \sin \theta$, then
\[
\partial_\theta f = \frac{\partial x_1}{\partial \theta} \partial_1 f + \frac{\partial x_2}{\partial \theta} \partial_2 f = -r \sin \theta \partial_1 f + r \cos \theta \partial_2 f = -x_2 \partial_1 f + x_1 \partial_2 f.
\]
We apply (1.4) to get
\[
2iA(x) \cdot \nabla f = 2i\frac{\alpha}{|x|^2}( -x_2 \partial_1 f + x_1 \partial_2 f) = 2i\frac{\alpha}{r^2} \partial_\theta f,
\]
thus we show (2.1). From (2.1), we see that the operator $\mathcal{L}_A$ is the Laplacian perturbated with two terms. The first inverse-square term $\alpha^2 r^{-2}$ is zero oder perturbation which is a short range potential. In [3, 4], the Strichartz estimates are proved for Schrödinger and wave equations only with the inverse-square potential by showing the local smoothing estimates. The second term $i\alpha r^{-2}\partial_\theta$ is first order perturbation which
is in long range case. As mentioned in Remark 1.3, the long range perturbation brings big troubles to the perturbation argument. Instead of the perturbation argument, we directly study the eigenfunction of the operator \( L_A \) and develop a distorted Fourier transform theory associated with \( L_A \).

**Proposition 2.1.** Let \( x = (r \cos \theta, r \sin \theta) \in \mathbb{R}^2, \xi = (\lambda \cos \omega, \lambda \sin \omega) \in \mathbb{R}^2 \), then

\[
L_A(e^{ix \cdot \xi} e^{i\alpha(\theta - \omega)}) = \lambda^2 e^{ix \cdot \xi} e^{i\alpha(\theta - \omega)},
\]

which implies that \( e^{ix \cdot \xi} e^{i\alpha(\theta - \omega)} \) is the distorted plan wave of the operator \( L_A \).

**Remark 2.1.** The above observation on the eigenfunction stems from the Bessel ordinary differential equation and the Jacobi-Anger expansion for plane waves.

**Proof.** We directly verify (2.3). On one hand, we have

\[
-\partial^2_r (e^{ir\lambda \cos(\theta - \omega)} e^{i\alpha(\theta - \omega)}) = \lambda^2 \cos^2(\theta - \omega) (e^{ir\lambda \cos(\theta - \omega)} e^{i\alpha(\theta - \omega)})
\]

and

\[
-\frac{1}{r} \partial_r (e^{ir\lambda \cos(\theta - \omega)} e^{i\alpha(\theta - \omega)}) = -\frac{i\lambda \cos(\theta - \omega)}{r} (e^{ir\lambda \cos(\theta - \omega)} e^{i\alpha(\theta - \omega)}).
\]

On the other hand, we see

\[
\frac{\alpha^2 - \partial^2_\theta}{r^2} (e^{ir\lambda \cos(\theta - \omega)} e^{i\alpha(\theta - \omega)})
\]

\[
= \frac{\alpha^2 - (-ir\lambda \cos(\theta - \omega) - (-ir\lambda \sin(\theta - \omega) + i\alpha)^2)}{r^2} (e^{ir\lambda \cos(\theta - \omega)} e^{i\alpha(\theta - \omega)})
\]

\[
+ \frac{2i\alpha(-ir\lambda \sin(\theta - \omega) + i\alpha)}{r^2} (e^{ir\lambda \cos(\theta - \omega)} e^{i\alpha(\theta - \omega)})
\]

\[
= \frac{ir\lambda \cos(\theta - \omega) + r^2 \lambda^2 \sin^2(\theta - \omega)}{r^2} (e^{ir\lambda \cos(\theta - \omega)} e^{i\alpha(\theta - \omega)}).
\]

Note that \( x \cdot \xi = r\lambda \cos(\theta - \omega) \), from (2.1), we obtain

\[
L_A(e^{ix \cdot \xi} e^{i\alpha(\theta - \omega)}) = |\xi|^2 e^{ix \cdot \xi} e^{i\alpha(\theta - \omega)}.
\]

Using the eigenfunction as in (2.3), we give the definition of the distorted Fourier transform associated with the operator \( L_A \).

**Definition 2.1** (Distorted Fourier transform). For the function \( f, g \in L^2 \cap L^1 \), we define the distorted Fourier transform of \( f \) as follows

\[
\mathcal{F}(f) = \hat{f}(\xi) = \hat{f}(\lambda \cos \omega, \lambda \sin \omega) = \int_0^\infty \int_0^{2\pi} e^{ir\lambda \cos(\theta - \omega)} e^{i\alpha(\theta - \omega)} f(r, \theta) r dr d\theta,
\]

and the inverse distorted Fourier transform of \( g \) is defined by

\[
\mathcal{F}^{-1}(g) = \hat{g}(x) = \hat{g}(r \cos \theta, r \sin \theta) = \int_0^\infty \int_0^{2\pi} e^{-ir\lambda \cos(\theta - \omega)} e^{-i\alpha(\theta - \omega)} g(\lambda, \omega) d\lambda d\omega.
\]

**Lemma 2.1** (The property of the distorted Fourier transform). The distorted Fourier transform satisfies the following properties:

1. The definition of inverse Fourier transform is well-defined: \( \mathcal{F}^{-1} \mathcal{F} = \text{Id}; \)
Lemma 2.2. The Plancherel identity follows the same argument. The definition is well defined. Let $y = (\rho \cos \phi, \rho \sin \phi)$, we have

\[
\mathcal{F}^{-1}(\hat{f})(x) = \int_0^\infty \int_0^{2\pi} e^{-ir\lambda \cos(\theta - \omega)} e^{-i\alpha (\theta - \omega)} \hat{f}(\lambda, \omega) \lambda d\lambda d\omega
\]

\[
= \int_0^\infty \int_0^{2\pi} e^{-ir\lambda \cos(\theta - \omega)} e^{-i\alpha (\theta - \omega)} \int_0^\infty \int_0^{2\pi} e^{ir\rho \cos(\phi - \omega)} e^{i\alpha (\phi - \omega)} f(\rho, \phi) \rho d\rho d\phi \lambda d\omega
\]

\[
= \int_0^\infty \int_0^{2\pi} e^{-i\alpha (\theta - \phi)} f(\rho, \phi) \int_\mathbb{R}^2 e^{-i(x-y)\xi} d\xi \rho d\rho d\phi
\]

\[
= f(x).
\]

The Plancherel identity follows the same argument. □

By using the distorted Fourier transform, we have an explicit formula for the functional calculus

**Lemma 2.2.** Let $F$ be the Borel measure function, and $x = r(\cos \theta, \sin \theta)$ and $y = \rho(\cos \phi, \sin \phi)$. Then, the kernels of the operator $F(\sqrt{\mathcal{L}_A})$ satisfy

\[
F(\sqrt{\mathcal{L}_A})(x, y) = \int_0^\infty \int_0^{2\pi} e^{-ir\lambda \cos(\theta - \omega)} e^{ir\rho \cos(\phi - \omega)} e^{-i\alpha (\theta - \phi)} F(\lambda) \lambda d\lambda d\omega
\]

\[
= e^{-i\alpha (\theta - \phi)} \int_0^\infty \int_0^{2\pi} e^{-ir\lambda \cos(\theta - \omega)} e^{ir\rho \cos(\phi - \omega)} F(\lambda) \lambda d\lambda d\omega
\]

\[
= e^{-i\alpha (\theta - \phi)} \int_{\mathbb{R}^2} e^{-i(x-y)\xi} F(|\xi|) d\xi
\]

\[
= e^{-i\alpha (\theta - \phi)} F(\sqrt{-\Delta})(x, y)
\]

(2.9)

where

\[
F(\sqrt{-\Delta})(x, y) = \int_{\mathbb{R}^2} e^{-i(x-y)\xi} F(|\xi|) d\xi,
\]

(2.10)

**Remark 2.2.** The result implies that the difference between the two kinds of kernel is slight. The difference is harmless to obtain the estimates (e.g. dispersive estimates) which is not sensitive to the angular variables.

Let $\varphi \in C^\infty_c(\mathbb{R} \setminus \{0\})$ take values in $[0, 1]$ and be compactly supported in $[1/2, 2]$ such that

\[
\sum_{j \in \mathbb{Z}} \varphi(2^{-j} \lambda) = 1.
\]

(2.12)

By following (2.10), we define the Littlewood-Paley operator associated with $\mathcal{L}_A$

\[
\varphi_j(\sqrt{\mathcal{L}_A}) f(x) = \int_{\mathbb{R}^2} e^{-i\alpha (\theta - \phi)} \int_{\mathbb{R}^2} e^{-i(x-y)\xi} \varphi(2^{-j} |\xi|) d\xi f(y) dy,
\]

(2.13)

where $x = r(\cos \theta, \sin \theta)$ and $y = \rho(\cos \phi, \sin \phi)$. 
Indeed, by using directly computation and the Plancherel identity, we have
\[
\|f\|_{\dot{B}^s_{p,r}(\mathbb{R}^2)} = \left( \sum_{j \in \mathbb{Z}} 2^{js} \|\hat{\varphi}_j(\sqrt{L_A})f\|_{L^p(\mathbb{R}^2)}^r \right)^{1/r}.
\] (2.14)

In particular, for \( p = r = 2 \), the Besov norm is the same to the Sobolev norm defined by
\[
\|f\|_{\dot{H}^s(\mathbb{R}^2)} = \left( \sum_{j \in \mathbb{Z}} 2^{js} \|\hat{\varphi}_j(\sqrt{L_A})f\|^2 \right)^{1/2} \|\cdot\|_{L^2(\mathbb{R}^2)} = \|f\|_{\dot{H}^s_{2,2}(\mathbb{R}^2)}. \] (2.15)

**Remark 2.3.** The definition is an analogue of the Besov space \( \dot{B}^s_{p,r} \) and Sobolev space \( \dot{H}^s \) defined by the classical Fourier transform. By Lemma 2.2, we have the following equivalence
\[
\|f(r, \theta)\|_{\dot{B}^s_{p,r}(\mathbb{R}^2)} \sim \|e^{i\alpha \theta} f(r, \theta)\|_{\dot{B}^s_{p,r}(\mathbb{R}^2)}. \] (2.16)

**Remark 2.4.** It would be interesting to study the relationship between the spaces defined here and the classical ones. The problem is out of the scope of this paper, however we point out a fact: for \( \alpha \notin \mathbb{Z} \), one has
\[
\|f\|_{\dot{H}^1(\mathbb{R}^2)} \leq C\|f\|_{\dot{H}^1(\mathbb{R}^2)}. \] (2.17)

Recall that
\[
L_A = -\Delta + 2iA(x) \cdot \nabla + |A(x)|^2 = (i\nabla + A(x))^2.
\]

Indeed, by using directly computation and the Plancherel identity, we have
\[
\|f\|_{\dot{H}^1(\mathbb{R}^2)}^2 = \langle |\hat{f}|, |\hat{f}| \rangle = \langle L_A f, f \rangle = \langle (i\nabla + A(x))f, (i\nabla + A(x))f \rangle \] (2.18)

which gives
\[
\|f\|_{\dot{H}^1(\mathbb{R}^2)} = \left( \int_{\mathbb{R}^2} |(i\nabla + A(x))f(x)|^2 \, dx \right)^{1/2}. \] (2.19)

Therefore we have
\[
\|f\|_{\dot{H}^1(\mathbb{R}^2)} = \|\nabla f\|_{L^2} \leq \|(i\nabla + A(x))f\|_{L^2} + \|A(x)f\|_{L^2} \leq C\|f\|_{\dot{H}^1(\mathbb{R}^2)},
\]

where we have used the Hardy inequality for Aharonov-Bohm magnetic Dirichlet forms [24, Lemma 4.3]
\[
\left( \min_{k \in \mathbb{Z}} |k - \alpha|^2 \right) \int_{\mathbb{R}^2} \frac{|f(x)|^2}{|x|^2} \, dx \leq \int_{\mathbb{R}^2} \left| (i\nabla + A(x))f(x) \right|^2 \, dx. \] (2.20)

Now we prove the dispersive estimates in Theorem 1.1.

**The proof of (1.17).** Using Lemma 2.2, we see the kernel of Schrödinger propagator satisfies
\[
e^{itL_A}(x, y) = e^{-i\alpha(\theta - \phi)} e^{-it\Delta}(x, y) = C e^{-i\alpha(\theta - \phi)} |t|^{-1} e^{i|y|/t}. \]

Hence we obtain (1.17). \qed
The proof of (1.18). As above, we write the kernel of half wave propagator
\[ e^{it\sqrt{-\Delta}}(x, y) = e^{-i\alpha(\theta - \phi)}e^{it\sqrt{-\Delta}}(x, y). \] (2.21)
Therefore we show
\[
\|e^{it\sqrt{-\Delta}}f\|_{L^\infty} = \sup_{x \in \mathbb{R}^2} \left| \int_{\mathbb{R}^2} e^{-i\alpha(\theta - \phi)}e^{it\sqrt{-\Delta}}(x, y)f(y)dy \right|
= \sup_{x \in \mathbb{R}^2} \left| \int_{\mathbb{R}^2} e^{it\sqrt{-\Delta}}(x, y)(e^{i\alpha\phi}f(y))dy \right|
\leq C|t|^{-\frac{1}{2}}\|e^{i\alpha\phi}f(y)\|_{B^\frac{1}{2}_{1,1}(\mathbb{R}^2)} \leq C\|f\|_{B^\frac{2}{1,1}(\mathbb{R}^2)},
\] (2.22)
where we use the dispersive estimates for the half wave \( e^{it\sqrt{-\Delta}} \) in the first inequality.

In the following, we provide another detail and direct method based on stationary phase argument. To this end, we first prove the lemma which will also be used to prove the Strichartz estimates.

**Lemma 2.3.** Let \( \varphi \in C_c^\infty(\mathbb{R} \setminus \{0\}) \) take values in \([0, 1]\) and be compactly supported in \([1/2, 2]\) as in (2.12). Then for all \( j \in \mathbb{Z} \), there exists a constant \( C \) independent of \( x, y \) and \( t \) such that
\[
\left| \int_{\mathbb{R}^2} e^{-i(x-y)\cdot\xi}e^{it|\xi|}\varphi(2^{-j}|\xi|)d\xi \right| \leq C2^{\frac{3j}{2}}(2^{-j} + |t|)^{-1/2}.
\] (2.23)

**Proof.** We prove (2.23) by the stationary phase argument. We write
\[
\int_{\mathbb{R}^2} e^{-i(x-y)\cdot\xi}e^{it|\xi|}\varphi(2^{-j}|\xi|)d\xi = \int_{0}^{\infty} e^{it\lambda}\varphi(2^{-j}\lambda) \int_{0}^{2\pi} e^{-i|x-y|\lambda\cos\omega}d\omega d\lambda.
\] (2.24)
From [28, Theorem 1.2.1], we can write
\[
\int_{0}^{2\pi} e^{-i|x-y|\lambda\cos\omega}d\omega = \sum_{\pm} e^{\pm i\lambda|x-y|a_{\pm}(\lambda|x - y|)},
\]
where \( a_{\pm} \) satisfies that
\[
|\partial_{\lambda}^Na_{\pm}(\lambda)| \leq C_N\lambda^{-N}(1 + \lambda)^{-\frac{3}{2}}, \quad \forall N \geq 0.
\] (2.25)
Therefore we obtain
\[
\int_{\mathbb{R}^2} e^{-i(x-y)\cdot\xi}e^{it|\xi|}\varphi(2^{-j}|\xi|)d\xi = \sum_{\pm} \int_{0}^{\infty} e^{i\lambda(t\pm|x-y|)}\varphi(2^{-j}\lambda)a_{\pm}(\lambda|x - y|)d\lambda,
\]
where
\[
|\partial_{\lambda}^Na_{\pm}(\lambda|x - y|)| \leq C_N\lambda^{-N}(1 + \lambda|x - y|)^{-\frac{3}{2}}, \quad \forall N \geq 0.
\] (2.26)
To prove (2.23), it suffices to show
\[
\left| \int_{0}^{\infty} e^{i\lambda(t\pm|x-y|)}\varphi(2^{-j}\lambda)a_{\pm}(\lambda|x - y|)d\lambda \right| \leq C2^{\frac{3j}{2}}(2^{-j} + |t|)^{-1/2}.
\] (2.27)
Indeed, for any $N \geq 0$, we have
\[
\left| \int_0^\infty e^{i\lambda(t|y-x|)}(2^{-j}\lambda)\lambda a_\pm(\lambda|x-y|)d\lambda \right|
\leq C\|t| - |x-y|\|^{-N} \left| \int_0^\infty e^{i\lambda(t|y-x|)} \left( \frac{\partial}{\partial \lambda} \right)^N \left( \varphi(2^{-j}\lambda)\lambda a_\pm(\lambda|x-y|) \right) d\lambda \right|
\leq C\|t| - |x-y|\|^{-N} \left( \int_{2^j}^{2^{j+1}} \lambda^{1-N}(1 + \lambda|x-y|)^{-\frac{3}{2}} d\lambda \right)
\leq C2^j\|t| - |x-y|\|^{-N}(1 + 2^j|x-y|)^{-\frac{1}{2}}.
\] (2.28)

Due to the arbitrary of $N$, therefore we obtain
\[
\left| \int_0^\infty e^{i\lambda(t|y-x|)}(2^{-j}\lambda)\lambda a_\pm(\lambda|x-y|)d\lambda \right|
\leq C2^j\left( 1 + 2^j|t| - |x-y| \right)\|t| - |x-y|\|^{-N}(1 + 2^j|x-y|)^{-\frac{1}{2}}.
\] (2.29)

If $|t| \sim |x-y|$, it is clear to see (2.23). Otherwise, we have $\|t| - |x-y|\| \geq c|t|$ for some small constant $c$, then we use (2.29) with $N = 1$ to prove (2.23).

Choose $\tilde{\varphi} \in C^\infty_c(\mathbb{R} \setminus \{0\})$ such that $\varphi\tilde{\varphi} = 1$. Indeed, we choose $\tilde{\varphi}$ to take values in 1 on the support of $\varphi$ and to be compactly supported in $\left[\frac{1}{4}, 1\right]$. By using (2.23), we prove
\[
\|e^{it\sqrt{\mathcal{L}_A}}f\|_{L^\infty} \leq \sum_{j \in \mathbb{Z}} \left\| e^{it\sqrt{\mathcal{L}_A}} \varphi_j(\sqrt{\mathcal{L}_A})\tilde{\varphi}_j(\sqrt{\mathcal{L}_A}) f \right\|_{L^\infty}
\leq \sum_{j \in \mathbb{Z}} \left\| e^{it\sqrt{\mathcal{L}_A}} \varphi_j(\sqrt{\mathcal{L}_A}) \right\|_{L^\infty} \left\| e^{it\sqrt{\mathcal{L}_A}} \tilde{\varphi}_j(\sqrt{\mathcal{L}_A}) f \right\|_{L^1(\mathbb{R}^2)}
\leq C\|t|^{-\frac{3}{2}} \|f\|_{B^{\frac{3}{2}}_{1,1}(\mathbb{R}^2)}.
\] (2.30)

Hence we prove (1.18).

The proof of (1.19). Now we prove the decay estimate for Dirac equation which is a consequence of (1.18). Indeed, from (1.16), we have seen
\[
e^{it\mathcal{D}_A} f = \cos(t\sqrt{\mathcal{L}_A}) f - i\sin(t\sqrt{\mathcal{L}_A}) \mathcal{D}_A f.
\] (2.31)

To show (1.19), by using (1.18), it suffices to prove that there exists a constant $C$ such that
\[
\|\mathcal{D}_A \mathcal{L}_A^{-1/2} \|_{B^{\frac{3}{2}}_{1,1}(\mathbb{R}^2) \to B^{\frac{3}{2}}_{1,1}(\mathbb{R}^2)} \leq C.
\] (2.32)

Recall the operator $\mathcal{D}_A$ in (1.11)
\[
\mathcal{D}_A = \begin{pmatrix}
0 & (i\partial_1 + A_1) - i(i\partial_2 + A_2) \\
(i\partial_1 + A_1) + i(i\partial_2 + A_2) & 0
\end{pmatrix},
\]
the operator $D_A$ acts on the eigenfunction in (2.3)
\[ D_A \left( e^{ix \xi} e^{i \alpha (\theta - \omega)} \right) = \begin{pmatrix} 0 & -\xi_1 + i \xi_2 \\ -\xi_1 - i \xi_2 & 0 \end{pmatrix} \left( e^{ix \xi} e^{i \alpha (\theta - \omega)} \right). \] (2.33)
Then the symbol of the operator $\mathcal{D}_A \mathcal{L}_A^{-1/2}$ under the distorted Fourier transform is bounded, hence we show (2.32).

\[ \square \]

3. The proof of Strichartz estimates

In this section, we prove the Strichartz estimates in Theorem 1.2.

**The proof of (1.22).** From the abstract method of Keel-Tao [23], the inequality (1.22) is a direct consequence of (1.17) and
\[ \| e^{it \mathcal{L}_A} \|_{L^2 \to L^2} \leq C. \] (3.1)
The $L^2$-estimate is obvious by using the unitary property of the distorted Fourier transform.

**The proof of (1.23).** To this end, we first prove two propositions. The first one is about the Strichartz estimates localized in frequency and the other one is on the Littlewood-Paley theory associated with the operator $\mathcal{L}_A$.

**Proposition 3.1.** Let $U(t) = e^{it \sqrt{\mathcal{L}_A}}$ and $f = \varphi_j(\sqrt{\mathcal{L}_A})f$ as in (2.13) for $j \in \mathbb{Z}$, then
\[ \| U(t)f \|_{L^q_t L^r_x(\mathbb{R} \times \mathbb{R}^2)} \lesssim 2^{js} \| f \|_{L^2(\mathbb{R}^2)}, \] (3.2)
where $s \in \mathbb{R}$ and $(q, r) \in \Lambda^W_s$ defined in (1.21).

**Proof.** Let $\tilde{\varphi} \in C^\infty_0(\mathbb{R} \setminus \{0\})$ take values in $[0, 1]$ such that $\tilde{\varphi} \varphi = \varphi$, hence we can write
\[ U(t)f = U(t)\varphi_j(\sqrt{\mathcal{L}_A})\tilde{\varphi}_j(\sqrt{\mathcal{L}_A})f = \int_{\mathbb{R}^2} e^{-i\alpha(\theta - \phi)} \int_{\mathbb{R}^2} e^{-i(x - y) \xi} e^{it |\xi|} \varphi(2^{-j} |\xi|)d\xi \left( \tilde{\varphi}_j(\sqrt{\mathcal{L}_A})f(y) \right) dy. \]
Define the operator $U_j(t) : L^2 \to L^2$ with the kernel
\[ U_j(t) = e^{-i\alpha(\theta - \phi)} \int_{\mathbb{R}^2} e^{-i(x - y) \xi} e^{it |\xi|} \varphi(2^{-j} |\xi|)d\xi, \] (3.3)
from (2.23) and unitary property of $U_j$, then there exists a constant $C$ such that
\[ \| U_j(t) \|_{L^2 \to L^2} \leq C, \quad t \in \mathbb{R}, \]
\[ \| U_j(t)U_j(s)^* \|_{L^1 \to L^\infty} \leq C 2^{\frac{3j}{2}} \| f \|_{L^q_t L^r_x} \lesssim 2^{2js} \| f \|_{L^q_t L^r_x} \]
(3.4)

Now we prove (3.2). We first consider the estimates on the board line, that is, $(q, r)$ satisfies $\frac{2}{q} + \frac{1}{r} = \frac{1}{2}$. This will be done by following the method of Keel-Tao [23]. Indeed, the Keel-Tao’s argument [23, Sections 3-7] shows (3.2) since we can replace $(|t - s| + 2^{-j})^{-1/2}$ by $|t - s|^{-1/2}$ to satisfy the condition [23, (2)] with $\sigma = 1/2$. Next we only consider $\frac{2}{q} + \frac{1}{r} < \frac{1}{2}$. By the $TT^*$ argument, it suffices to show
\[ \left| \int \int U_j(s)^* f(s), U_j(t)^* g(t) \right| dsdt \lesssim 2^{2js} \| f \|_{L^q_t L^r_x} \| g \|_{L^q_t L^r_x}. \]
Using the bilinear interpolation of (3.4), we have
\[ \langle U_j(s)^*f(s), U_j(t)^*g(t) \rangle \leq C 2^\beta (1 - \frac{2}{p})^j (2^{-j} + |t - s|)^{\frac{1}{p} - \frac{\beta}{2}} \|f\|_{L^p} \|g\|_{L^{p'}}. \]

Therefore, we see by Hölder’s and Young’s inequalities for \( \frac{2}{q} + \frac{1}{r} < \frac{1}{2} \)
\[
\left| \int \int (U_j(s)^*f(s), U_j(t)^*g(t))dsdt \right| \\
\lesssim 2^{\frac{\beta}{2}(1 - \frac{2}{p})^j} \int (2^{-j} + |t - s|)^{\frac{1}{2} - \frac{\beta}{2}} \|f(s)\|_{L^p} \|g(t)\|_{L^{p'}} dtds \\
\lesssim 2^{\frac{\beta}{2}(1 - \frac{2}{p})} 2^{\frac{\beta}{2}(\frac{1}{2} - \frac{1}{q})} \|f\|_{L^{q'}_t L^{p'}_r} \|g\|_{L^{q'}_t L^{p'}_r} = 2^{\beta j \left( \frac{2}{2} - \frac{1}{q} \right)} \|f\|_{L^{q'}_t L^{p'}_r} \|g\|_{L^{q'}_t L^{p'}_r}.
\]
Note \( s = 2 \left( \frac{1}{2} - \frac{1}{p} \right) - \frac{1}{q} \), this proves (3.2). 

The next proposition is the following.

**Proposition 3.2** (Littlewood-Paley square function inequality). Let \( \mathcal{L}_A \) be the Schrödinger operator as in (1.6) and the Aharonov-Bohm potential \( A(x) \) be given by (1.4). Then for \( 1 < p < \infty \), there exist constants \( c_p \) and \( C_p \) depending on \( p \) such that
\[
c_p \|f\|_{L^p(\mathbb{R}^2)} \leq \left\| \left( \sum_{j \in \mathbb{Z}} |\varphi_j(\sqrt{\mathcal{L}_A})f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^2)} \leq C_p \|f\|_{L^p(\mathbb{R}^2)} (3.5)
\]
where the Littlewood-Paley operator \( \varphi_j(\sqrt{\mathcal{L}_A}) \) is defined in (2.13).

**Proof.** From (2.10), we see the relationship between the two kernels
\[
\varphi_j(\sqrt{\mathcal{L}_A})(x, y) = e^{-i \alpha(\theta - \phi)} \varphi_j(\sqrt{-\Delta})(x, y),
\]
where \( x = (r \cos \theta, r \sin \theta), y = (r \cos \phi, r \sin \phi) \). Then we see
\[
|\varphi_j(\sqrt{\mathcal{L}_A})f| = |\varphi_j(\sqrt{-\Delta})g|, \quad g(r, \theta) = e^{i \alpha \theta} f(r, \theta).
\]
By using the Littlewood-Paley square function estimates associated with \( -\Delta \) and the fact \( \|g\|_{L^p} = \|f\|_{L^p} \), we obtain (3.5). 

Now we prove the inequality (1.23). Note that \( q, r \geq 2 \), by using (3.5) and Minkowski’s inequality, we show that
\[
\|e^{it\sqrt{\mathcal{L}_A}} f\|_{L^q(\mathbb{R}; L^r(\mathbb{R}^2))} \lesssim \left( \sum_{j \in \mathbb{Z}} \|e^{it\sqrt{\mathcal{L}_A}} \varphi_j(\sqrt{\mathcal{L}_A})f\|_{L^q(\mathbb{R}; L^r(\mathbb{R}^2))}^2 \right)^{\frac{1}{2}}.
\]

By using (3.2) of Proposition 3.1, we further have
\[
\|e^{it\sqrt{\mathcal{L}_A}} f\|_{L^q(\mathbb{R}; L^r(\mathbb{R}^2))} \lesssim \left( \sum_{j \in \mathbb{Z}} 2^{2sj} \|\varphi_j(\sqrt{\mathcal{L}_A})f\|_{L^2(\mathbb{R}^2)}^2 \right)^{\frac{1}{2}}.
\]

By the definition of Sobolev space (2.15), we prove
\[
\|e^{it\sqrt{\mathcal{L}_A}} f\|_{L^q(\mathbb{R}; L^r(\mathbb{R}^2))} \lesssim \|f\|_{\dot{H}^s(\mathbb{R}^2)}. \]

□
The proof of (1.24). This is direct consequence of (1.23) and
\[ \|D_A L_{A}\|_{H^{s'} \rightarrow H^{s'}} \leq C. \]  
(3.10)
The same argument of (2.32) shows (3.10).

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