1. Introduction. The concept of differential privacy (DP) was introduced in Dwork et al. (2006), which offered a framework for the construction of private mechanisms and a rigorous notion of what it means to limit privacy loss when performing statistical releases on sensitive data. DP requires that the randomized algorithm \( M \) performing the release has the property that for any two datasets \( X \) and \( X' \) which differ in one individual’s data (adjacent datasets), the distributions of \( M(X) \) and \( M(X') \) are “close.” Since this seminal paper, many variants of differential privacy have been proposed; the variants primarily differ in how they formulate the notion of closeness. For example, pure and approximate DP are phrased in terms of bounding the probabilities of sets of outputs, according to \( M(X) \) versus \( M(X') \) (Dwork et al., 2014), whereas concentrated (Bun and Steinke, 2016) and Renyi (Mironov, 2017) DP are based on bounding a divergence between \( M(X) \) and \( M(X') \).

Wasserman and Zhou (2010) and Kairouz, Oh and Viswanath (2017) showed that pure and approximate DP can be expressed as imposing constraints on the type I and type II errors of hypothesis tests which seek to discriminate between two adjacent databases. Recently Dong et al. (2022) expanded this view, defining \( f \)-DP which allows for an arbitrary bound to be placed on the receiver-operator curve (ROC) or tradeoff function when testing between two adjacent databases. It is shown in Dong et al. (2022) that \( f \)-DP retains many of the useful properties of DP such as post-processing, composition, and subsampling and allows for lossless calculation of the privacy cost of each of these operations. Furthermore, as special cases, \( f \)-DP contains both pure and approximate DP, and contains relatives of divergence-based...
notions of DP as well (e.g., Gaussian DP (GDP) is slightly stronger than zero-concentrated DP).

In this paper, we study two basic and fundamental privacy questions in the framework of \( f\)-DP. The first is based on optimizing the basic mechanism of adding independent noise to a real-valued statistic, and the second is about constructing hypothesis tests under the constraint of DP. We show that in fact, the two problems are intricately related, where the “canonical additive noise distribution” enables private \( p\)-values “for free,” and gives a closed form construction of certain optimal hypothesis tests.

One of the most basic and fundamental types of privacy mechanisms is noise addition, where independent noise is added to a real-valued statistic. Additive mechanisms are not only widely used by themselves, but are also often a key ingredient to more complex mechanisms such as functional mechanism (Zhang et al., 2012), objective perturbation (Chaudhuri, Monteleoni and Sarwate, 2011), stochastic gradient descent (Abadi et al., 2016), and the sparse vector technique (Dwork et al., 2009), to name a few. The oldest and most widely used additive mechanisms are the Laplace and Gaussian mechanisms, but there have since been many proposed distributions which satisfy different definitions of DP. A natural question is what noise distributions are “optimal” or “canonical” for a given definition of privacy. The geometric mechanism/discrete Laplace mechanism is optimal for \( \epsilon\)-DP counts, in terms of maximizing Bayesian utility (Ghosh, Roughgarden and Sundararajan, 2012), the staircase mechanism is optimal for \( \epsilon\)-DP in terms of \( \ell_1\) or \( \ell_2\)-error (Geng and Viswanath, 2015), and the truncated-uniform-Laplace (Tulap) distribution generalizes both the discrete Laplace and staircase mechanisms and is optimal for \( (\epsilon, \delta)\)-DP in terms of generating uniformly most powerful (UMP) hypothesis tests and uniformly most accurate (UMA) confidence intervals for Bernoulli data (Awan and Slavković, 2018; Awan and Slavković, 2020). With divergence-based definitions of privacy, Gaussian noise is argued to be canonical for (zero) concentrated DP (Bun and Steinke, 2016), and the sinh-normal distribution is argued to be canonical for truncated concentrated DP (Bun et al., 2018).

In this paper, we give the first formal definition of a canonical noise distribution (CND) which captures the notion of whether a distribution tightly matches a privacy guarantee \( f\)-DP. We show that the Gaussian distribution is canonical for Gaussian differential privacy (GDP), and the Tulap distribution is canonical for \( (\epsilon, \delta)\)-DP. We prove that a CND always exists for any nontrivial symmetric tradeoff function \( f\), and give a general construction to generate a CND given any tradeoff function \( f\). This construction results in the first general mechanism for an arbitrary \( f\)-DP guarantee. In the special case of \( (\epsilon, \delta)\)-DP, our construction results in the Tulap distribution.

Another basic privacy question is on the nature of DP hypothesis tests. Awan and Slavković (2018) showed that for independent Bernoulli data, there exists uniformly most powerful (UMP) \( (\epsilon, \delta)\)-DP tests which are based on the Tulap distribution, enabling “free” private \( p\)-values, at no additional cost to privacy.

We show that for an arbitrary tradeoff function \( f\) and any \( f\)-DP test, a free private \( p\)-value can always be generated in terms of a CND for \( f\). We also extend the main results of Awan and Slavković (2018) from \( (\epsilon, \delta)\)-DP to \( f\)-DP as well as from i.i.d. Bernoulli variables to exchangeable binary data. This extension shows that the CND is the proper generalization of the Tulap distribution, and gives an explicit construction of the most powerful \( f\)-DP test for binary data, the first DP hypothesis test for a general \( f\)-DP guarantee.

We end with an extensive application to private difference-of-proportions testing. Testing two population proportions is a very basic and common hypothesis testing setting that arises when there are two groups with binary responses, such as A/B testing, clinical trials, and observational studies. As such, the techniques for testing these hypotheses are standardized and included in most introductory statistics textbooks. However, there currently lacks a theoretically based private test with accurate sensitivity and specificity. Karwa and Vadhan (2018)
was the first attempt at tackling the private difference-of-proportions testing problem, and recently Awan and Cai (2020) used a novel asymptotic method to calibrate the type I errors of a related DP test in large sample sizes. Our application builds on these prior works, with a much improved analysis and strong theoretical basis in the $f$-DP framework.

We show that in general, there does not exist a UMP unbiased $f$-DP test for this problem, but using our earlier results on most powerful $f$-DP tests for binary data, we show that there does exist a UMP unbiased “semi-private” test, which satisfies a weakened version of $f$-DP. While this test does not satisfy $f$-DP, it does provide an upper bound on the power of any $f$-DP test, and gives intuition on the structure of a good $f$-DP test for this problem. We then design a novel $f$-DP test for the testing problem, based on using CNDs and an expression of the sampling distribution in terms of characteristic functions, enabling efficient computation via Gil-Pelaez inversion. Using theory of the parametric bootstrap, we argue that the test is asymptotically unbiased and has asymptotically accurate type I errors. Empirically, we show that the test has more accurate type I errors and $p$-values than the popularly used normal approximation test, and that the power of our proposed test is nearly as powerful as the semi-private UMP unbiased test. In the case of $\epsilon$-DP, we demonstrate through simulations that our test has higher power than any $(\epsilon/\sqrt{2})$-DP test, indicating that it is near optimal. Furthermore, our test has the benefit of allowing for optimal hypothesis tests and confidence intervals for each of the population proportions, using the techniques of Awan and Slavković (2020), as the proposed test is based on the same DP summary statistics.

**Organization** In Section 2 we review background on hypothesis tests and differential privacy. In Section 3, we introduce the concept of a canonical noise distribution, give some basic properties of CNDs, and provide a general construction of a CND for any $f$-DP privacy notion. In Section 4, we show that any $f$-DP hypothesis test must satisfy constraints based on the function $f$, we give a general result for “free” DP $p$-values given an $f$-DP test function, and develop most powerful $f$-DP tests for exchangeable binary data. In Section 5, we consider the problem of privately testing the difference of population proportions. Specifically in Section 5.1, we develop a uniformly most powerful unbiased “semi-private” test, which gives an upper bound on the power of any $f$-DP test, in Section 5.2 we propose an $f$-DP test based on the inversion of characteristic functions, and in Section 5.3 we evaluate the type I error and power of our two sample test in simulations. Proofs and technical details are deferred to the supplementary materials.

**Related work** Private hypothesis testing was first tackled by Vu and Slavković (2009), developing DP tests for population proportions as well as independence tests for $2 \times 2$ tables. These tests use additive Laplace noise, and use a normal approximation to the sampling distribution to calibrate the type I errors. Solea (2014) develop tests for normally distributed data using similar techniques. Wang, Lee and Kifer (2015) and Gaboardi et al. (2016) expanded on Vu and Slavković (2009), developing additional tests for multinomials. Wang, Lee and Kifer (2015) developed asymptotic sampling distributions for their tests, verifying the type I errors via simulations, whereas Gaboardi et al. (2016) use Monte Carlo methods to estimate and control the type I error. Uhler, Slavković and Fienberg (2013) develop DP $p$-values for chi-squared tests of GWAS data, and derive the exact sampling distribution of the noisy statistic. Kifer and Rogers (2016) develop private $\chi^2$ tests for goodness-of-fit and identity problems which are designed to have the same asymptotic properties as the non-private tests.

Under “local differential privacy,” a notion of DP where even the data curator does not have access to the original dataset, Gaboardi and Rogers (2018) develop multinomial tests based on asymptotic distributions.

The first uniformly most powerful hypothesis tests under DP for the testing of i.i.d. Bernoulli data were developed by Awan and Slavković (2018). Their tests were based on the Tulap distribution, an extension of the discrete Laplace and Staircase mechanisms. Awan
and Slavković (2020) expanded on these results to offer UMP unbiased two-sided DP tests as well as optimal DP confidence intervals and confidence distributions for Bernoulli data.

Given a DP output, Sheffet (2017) and Barrientos et al. (2019) develop significance tests for regression coefficients. Wang et al. (2018) develop general approximating distributions for DP statistics, which can be used to construct hypothesis tests and confidence intervals, but which are only applicable to limited models. Awan and Cai (2020) also provide asymptotic techniques that can be used to conduct approximate hypothesis tests, given DP summary statistics, but which have limited accuracy in finite samples.

Rather than the classical regime of fixing the type I error, and minimizing the type II error, there are several works on DP testing, where the goal is to optimize the sample complexity required to generate a test which places both the type I and type II errors below a certain threshold. Canonne et al. (2019) show that for simple hypothesis tests, a noisy clamped likelihood ratio test achieves optimal sample complexity. Cai, Daskalakis and Kamath (2017) and Kakizaki, Fukuchi and Sakuma (2017) both study the problem of \( \epsilon \)-DP discrete identity testing from the sampling complexity perspective. Aliakbarpour, Diakonikolas and Rubinfeld (2018) also studies \( \epsilon \)-DP identity testing as well as DP equivalence testing. Acharya, Sun and Zhang (2018) study identity and closeness testing of discrete distributions in the \((\epsilon, \delta)\)-DP framework. Bun et al. (2019) derive sample complexity bounds for differentially privacy hypothesis selection, where the goal is to choose among a set of potential data generating distributions, which one has the smallest total variation distance to the true distribution. Suresh (2021) develop an alternative to the Neyman-Pearson Lemma for simple hypotheses, which is robust to misspecification of the hypotheses; due to the connection between robustness and differential privacy (Dwork and Lei, 2009), this could be a promising tool for developing private tests.

Outside the hypothesis testing setting, there is some additional work on optimal population inference under DP. Duchi, Jordan and Wainwright (2018) give general techniques to derive minimax rates under local DP, and in particular give minimax optimal point estimates for the mean, median, generalized linear models, and nonparametric density estimation. Karwa and Vadhan (2017) develop nearly optimal confidence intervals for normally distributed data with finite sample guarantees, which could potentially be inverted to give approximately UMP unbiased tests.

Notable works that develop optimal DP mechanisms for general loss functions are Geng and Viswanath (2015) and Ghosh, Roughgarden and Sundararajan (2012), which give mechanisms that optimize symmetric convex loss functions, centered at a real-valued statistic. Similarly, Awan and Slavković (2021) derive optimal mechanisms among the class of \( K \)-Norm Mechanisms for a fixed statistic and sample size.

2. Background. In this section, we review some basic notation as well as background on differential privacy. Notation and terminology regarding hypothesis testing is deferred to Appendix A.

we say that a real-valued function \( f(x) \) is increasing (decreasing) if \( a \leq b \) implies \( f(a) \leq f(b) \) (resp. \( f(a) \geq f(b) \)). We say that \( f \) is strictly increasing (strictly decreasing) if \( a < b \) implies \( f(a) < f(b) \) (resp. \( f(a) > f(b) \)). Given an increasing function \( f \), we define its inverse to be \( f^{-1}(y) = \inf \{ x \in \mathbb{R} \mid y \leq f(x) \} \). For a decreasing function \( f \), the inverse is defined to be \( f^{-1}(y) = \inf \{ x \in \mathbb{R} \mid y \geq f(x) \} \).

For a real-valued random variable \( X \), its cumulative distribution function (cdf) is defined as \( F_X(t) = P(X \leq t) \), and its quantile function is \( F_X^{-1} \). A real-valued random variable is continuous if its cdf \( F_X(t) \) is continuous in \( t \), and \( X \) is symmetric about zero if \( F_X(t) = 1 - F_X(-t) \). For a random variable \( X \sim P \), with cdf \( F \), we use \( P \) and \( F(\cdot) \) interchangeably to denote the distribution of \( X \).
2.1. Differential Privacy. In this section, we review the definition of $f$-DP which is formulated in terms of constraints on hypothesis tests and relate it to other notions of DP in the literature. A mechanism $M$ is a randomized algorithm that takes as input a database $D$, and outputs a (randomized) statistic $M(D)$ in an abstract space $Y$. Given two databases $X$ and $X'$ which differ in one person’s contribution, a mechanism $M$ satisfies differential privacy if given the output of $M$ it is difficult to determine whether the original database was $X$ or $X'$.

The notion “differing in one person’s contribution” is often formalized in terms of a metric. In this paper, we use the Hamming metric, which is defined as follows: For any set $X$, we write $X^n = \{(x_1, x_2, \ldots, x_n) | x_i \in X \text{ for all } 1 \leq i \leq n\}$. The Hamming metric on $X^n$ is defined by $H(X, X') = \#\{i | X_i \neq X'_i\}$. If $H(X, X') \leq 1$, we call $X$ and $X'$ adjacent databases. Note that by using the Hamming metric, we assume that the sample size $n$ is a public value and does not require privacy protection.

All of the major variants of DP state that given a randomized algorithm $M$, for any two adjacent databases $X, X'$, the distributions of $M(X)$ and $M(X')$ should be “similar.” While many DP variants measure similarity in terms of divergences, recently Dong et al. (2022) proposed $f$-DP, which formalizes similarity in terms of constraints on hypothesis tests.

For two probability distributions $P$ and $Q$, the tradeoff function $T(P, Q) : [0, 1] \rightarrow [0, 1]$ is defined as $T(P, Q)(\alpha) = \inf \{1 - \mathbb{E}_Q \phi | \mathbb{E}_P(\phi) \leq \alpha\}$, where the infimum is over all measurable tests $\phi$. The tradeoff function can be interpreted as follows: If $T(P, Q)(\alpha) = \beta$, then the most powerful test $\phi$ which is trying to distinguish between $H_0 = \{P\}$ and $H_1 = \{Q\}$ at type I error $\leq \alpha$ has type II error $\beta$. A larger tradeoff function means that it is harder to distinguish between $P$ and $Q$. Note that the tradeoff function is closely related to the receiver-operator curve (ROC), and captures the difficulty of distinguishing between $P$ and $Q$. A function $f : [0, 1] \rightarrow [0, 1]$ is a tradeoff function if and only if $f$ is convex, continuous, decreasing, and $f(x) \leq 1 - x$ for all $x \in [0, 1]$ (Dong et al., 2022, Proposition 2.2). We say that a tradeoff function $f$ is nontrivial if $f(\alpha) < 1 - \alpha$ for some $\alpha \in (0, 1)$; that is if $f$ is not identically equal to $1 - \alpha$.

DEFINITION 2.1 ($f$-DP: Dong et al., 2022). Let $f$ be a tradeoff function. A mechanism $M$ satisfies $f$-DP if

$$T(M(D), M(D')) \geq f$$
for all \(D, D' \in \mathcal{X}^n \) such that \(H(D, D') \leq 1\).

See Figure 1 for examples of tradeoff functions which do and do not satisfy \(f\)-DP for a particular \(f\). In the above definition, the inequality \(T(M(D), M(D')) \geq f\) is shorthand for \(T(M(D), M(D'))(\alpha) \geq f(\alpha)\) for all \(\alpha \in [0, 1]\). Without loss of generality we can assume that \(f\) is symmetric: \(f(\alpha) = f^{-1}(\alpha)\), where \(f^{-1}(\alpha) = \inf\{t \in [0, 1] \mid f(t) \leq \alpha\}\). This is due to the fact that adjacency of databases is a symmetric relation (Dong et al., 2022, Proposition 2.4). For the remainder of the paper, we assume that \(f\)-DP also requires this symmetry.

Wasserman and Zhou (2010) and Kairouz, Oh and Viswanath (2017) both showed that \((\epsilon, \delta)\)-DP can be expressed in terms of hypothesis testing, and in fact Dong et al. (2022) showed that \((\epsilon, \delta)\)-DP can be expressed as a special case of \(f\)-DP.

**Definition 2.2** \((\epsilon, \delta)\)-DP: Dwork et al., 2006. Let \(\epsilon > 0\) and \(\delta \geq 0\), and define \(f_{\epsilon, \delta}(\alpha) = \max\{0, 1 - \delta - \exp(\epsilon)\alpha, \exp(-\epsilon)(1 - \delta - \alpha)\}\). Then we say that a mechanism \(M\) satisfies \((\epsilon, \delta)\)-DP if it satisfies \(f_{\epsilon, \delta}\)-DP.

Another notable special case of \(f\)-DP is Gaussian DP (\(\mu\)-GDP). Dong et al. (2022) showed that \(\mu\)-GDP is perhaps the most natural single parameter privacy definition, due to the central limit theorem for composition. Gaussian DP is closely related to zero-concentrated differential privacy (zCDP) (Bun and Steinke, 2016), a very popular relaxation of DP. GDP is slightly stronger than zCDP in that a mechanism satisfying GDP satisfies zCDP (Dong et al., 2022, Corollary B.6), but the converse is not true (Dong et al., 2022, Proposition B.7).

**Definition 2.3** (Gaussian differential privacy: Dong et al., 2022). Let \(\mu > 0\) and define \(G_\mu(\alpha) = T(N(0, 1), N(\mu, 1))(\alpha) = \Phi(\Phi^{-1}(1 - \alpha) - \mu)\), where \(\Phi\) is the cdf of \(N(0, 1)\). We say that a mechanism \(M\) satisfies \(\mu\)-Gaussian differential privacy (\(\mu\)-GDP) if it is \(G_\mu\)-DP.

3. **Canonical noise distributions.** One of the most basic techniques of designing a privacy mechanism is through adding data-independent noise. The earliest DP mechanisms add either Laplace or Gaussian noise, and there have since been several works developing optimal additive mechanisms including the geometric (discrete Laplace) (Ghosh, Roughgarden and Sundararajan, 2012), truncated-uniform-Laplace (Tulap) (Awan and Slavković, 2018; Awan and Slavković, 2020), and staircase mechanisms (Geng and Viswanath, 2015). There have also been several works exploring multivariate and infinite-dimensional additive mechanisms such as \(K\)-norm (Hardt and Talwar, 2010; Awan and Slavković, 2021), elliptical perturbations (Reimherr and Awan, 2019), and Gaussian processes (Hall, Rinaldo and Wasserman, 2013; Mirshahi, Reimherr and Slavković, 2019).

While there are many choices of additive mechanisms to achieve \(f\)-DP, we are interested in adding the least noise necessary in order to maximize the utility of the output. Rather than measuring the amount of noise by its variance or entropy, we focus on whether the privacy guarantee is tight.

In this section, we introduce the concept *canonical noise distribution* (CND), which captures whether a real-valued distribution is perfectly tailored to satisfy \(f\)-DP. We formalize this in Definition 3.1. We then show that for any symmetric \(f\), we can always construct a CND, where the construction is given in Definition 3.7 and proved to be a CND in Theorem 3.9. We will see in Section 4 that CNDs are fundamental for understanding the nature of \(f\)-DP hypothesis tests, for constructing “free” \(p\)-values, and for the design of uniformly most powerful \(f\)-DP tests for binary data. We also see in Section 5 that CNDs are central to our application of difference-of-proportions tests as well.
Before we define canonical noise distribution, we must introduce the sensitivity of a statistic, a central concept of DP (Dwork et al., 2006). A statistic $T : \mathcal{X}^n \rightarrow \mathbb{R}$ has sensitivity $\Delta > 0$ if $|T(X) - T(X')| \leq \Delta$ for all $H(X, X') \leq 1$. As the sensitivity measures how much a statistic can change when one person’s data is modified, additive noise must be scaled proportionally to the sensitivity in order to protect privacy.

**Definition 3.1.** Let $f$ be a symmetric nontrivial tradeoff function. A continuous distribution function $F$ is a canonical noise distribution (CND) for $f$ if

1. for every statistic $S : \mathcal{X}^n \rightarrow \mathbb{R}$ with sensitivity $\Delta > 0$, and $N \sim F(\cdot)$, the mechanism $S(X) + \Delta N$ satisfies $f$-DP. Equivalently, for every $m \in [0, 1], T(F(\cdot), F(\cdot - m)) \geq f,$
2. $f(\alpha) = T(F(\cdot), F(\cdot - 1))((\alpha))$ for all $\alpha \in (0, 1)$,
3. $T(F(\cdot), F(\cdot - 1))(\alpha) = F(F^{-1}(1 - \alpha) - 1)$ for all $\alpha \in (0, 1),$
4. $F(x) = 1 - F(-x)$ for all $x \in \mathbb{R}$; that is, $F$ is the cdf of a random variable which is symmetric about zero.

The most important conditions of Definition 3.1 are 1 and 2, which state that the distribution can be used to satisfy $f$-DP and that the privacy bound is tight. For property 1, the value $m$ can be interpreted as the quantity $|S(X) - S(X')|/\Delta$; then by the symmetry of $F$, it can be seen that $T(S(X) + \Delta N, S(X') + \Delta N) = T(F(\cdot), F(\cdot - m))$. Condition 3 of Definition 3.1 gives a closed form for the tradeoff function, and is equivalent to requiring that the optimal rejection set for discerning between $F(\cdot)$ and $F(\cdot - 1)$ is of the form $(x, \infty)$ for some $x \in \mathbb{R}$. The last condition of Definition 3.1 enforces symmetry of the distribution, which makes CNDs much easier to work with.

Finally note that conditions 1 and 2 are not equivalent. Adding excessive noise would satisfy 1, but not 2, whereas a mechanism which fails $T(F(\cdot), F(\cdot - m)) \geq T(F(\cdot), F(\cdot - 1))$ for some $m \in (0, 1)$ would not satisfy property 1. The following example illustrates both cases.

**Example 3.2.** Consider the discrete Laplace mechanism, which has cdf $F(t) = \frac{1 - b}{t + b}$ for $t \in \mathbb{Z}$ and $b \in (0, 1)$. Then it can be verified that the discrete Laplace distribution with $b = \exp(-\epsilon)$ satisfies $T(F(\cdot), F(\cdot - 1)) = f_{\epsilon, 0},$ but not part 1 of Definition 3.1. For example, if $S(X) = 0$ and $S(X') = .1$, adding discrete Laplace noise $N \sim F$ results in distributions with disjoint support, since $S(X) + N$ takes values in $\mathbb{Z}$, whereas $S(X') + N$ takes values in $\mathbb{Z} + .1$. As the supports of the distributions are disjoint, we can have zero type I and type II error when testing between $X$ and $X'$, violating the $f_{\epsilon, 0}$ bound.

It is well known that the continuous Laplace mechanism with scale parameter $\Delta/\epsilon$ satisfies $\epsilon$-DP, when added to a $\Delta$-sensitivity statistic, and so satisfies property 1 of Definition 3.1 for $f_{\epsilon, 0}$. However, as Dong et al. (2022) noted, it can be verified that the Laplace distribution does not satisfy property 2 of Definition 3.1, as there exists $\alpha \in (0, 1)$ such that the tradeoff function is strictly greater than $f_{\epsilon, 0}$ at $\alpha$.

**Remark 3.3.** Note that property 2 of Definition 3.1 captures the intuition that a privacy mechanism should match the tradeoff function in the privacy guarantee to avoid introducing excessive noise. While this is indeed an intuitive idea, this has never previously been formalized into a precise criterion for a privacy mechanism, as we do in Definition 3.1. Furthermore, no prior work has attempted to build a mechanism that matches the tradeoff function for an arbitrary $f$-DP guarantee. In Theorem 3.9, we not only prove that a CND exists, but give a construction to build a CND for every $f$. 
**Example 3.4 (CND for GDP).** The distribution \( N(0, 1/\mu) \), which has cdf \( \Phi(1/\mu) \) (\( \Phi \) is the cdf of a standard normal) is a CND for \( G_\mu \), defined in Definition 2.3. Property 1 is proved in (Dong et al., 2022), properties 2 and 3 are easily verified, and the distribution is obviously symmetric. Dong et al. (2022) state that “GDP precisely characterizes the Gaussian mechanism.” From the opposite perspective, we argue that this is because the normal distribution is a CND for \( G_\mu \).

**Proposition 3.5.** Let \( f \) be a symmetric nontrivial tradeoff function. Let \( F \) be a CND for \( f \), and \( G \) be another cdf such that \( T(G(\cdot), G(\cdot - 1)) \geq f \). Let \( N \sim F \) and \( M \sim G \). Then there exists a randomized function \( \operatorname{Proc} : \mathbb{R} \to \mathbb{R} \) which satisfies \( \operatorname{Proc}(N) \overset{d}{=} M \) and \( \operatorname{Proc}(N + 1) \overset{d}{=} M + 1 \), where "\( \overset{d}{=} \)" means equal in distribution.

Proposition 3.5 follows from property 2 in Definition 3.1 along with Dong et al. (2022, Theorem 2.10), which is based on Blackwell’s Theorem (Blackwell, 1950). Proposition 3.5 shows that if we add noise from a CND to a statistic \( S(X) \) versus \( S(X) + 1 \), we can post-process the result to obtain the same result as if we added noise from another distribution that achieves \( f \)-DP. This shows in some sense that a CND adds the least noise necessary to achieve \( f \)-DP. Note that Proposition 3.5 does not imply that a CND is optimal in every sense: for example, Geng and Viswanath (2015) derived the minimum variance additive \( (\epsilon, 0) \)-DP mechanism, which is not a CND for \( f_{\epsilon, 0} \). We will see in Section 4 that the properties of Definition 3.1 do lead to optimal properties of DP hypothesis tests.

In the remainder of this section, we show that given any tradeoff function \( f \), we can always construct a canonical noise distribution (CND), but that a CND need not be unique.

**Lemma 3.6.** Let \( f \) be a symmetric nontrivial tradeoff function and let \( F \) be a CND for \( f \). Then \( F(x) = 1 - f(F(x - 1)) \) when \( F(x - 1) > 0 \) and \( F(x) = f(1 - F(x + 1)) \) when \( F(x + 1) < 1 \).

**Proof Sketch.** The result follows from properties 2, 3, and 4 of Definition 3.1 along with some algebra of cdfs.

In the Lemma 3.6, we see that a CND satisfies an interesting recurrence relation. If we know the value \( F(x) = c \) for some \( x \in \mathbb{R} \) and \( c \in (0, 1) \), then we know the value of \( F(y) \) for all \( y \in \mathbb{Z} + x \). This means that if we specify \( F \) on an interval of length 1, such as \([-1/2, 1/2] \), then \( F \) is completely determined by the recurrence relation. While there are many choices to specify \( F \) on \([-1/2, 1/2] \), each of which may or may not lead to a CND. We show that using a particular linear function in \([-1/2, 1/2] \) does indeed give a CND. The remainder of this section is devoted to this construction of a CND and the proof that it has the properties of Definition 3.1.

**Definition 3.7.** Let \( f \) be a symmetric nontrivial tradeoff function, and let \( c \in [0, 1/2) \) be the unique fixed point of \( f \): \( f(c) = c \). We define \( \mathcal{F}_f : \mathbb{R} \to \mathbb{R} \) as

\[
\mathcal{F}_f(x) = \begin{cases} 
 1 - f(F_f(x + 1)) & x < -1/2 \\
 1/2 - x + (1 - c)(x + 1/2) & -1/2 \leq x \leq 1/2 \\
 1 - f(F_f(x - 1)) & x > 1/2.
\end{cases}
\]

In Definition 3.7, the fact that there is a unique fixed point follows from the fact that \( f \) is convex and decreasing, and so intersects the line \( y = \alpha \) at a unique value. In Lemma F.4
we establish that the fixed point \( c \) lies in the interval \([0, 1/2)\). Note that in Definition 3.7, the cdf corresponds to a uniform random variable on the interval \([-1/2, 1/2]\), but due to the recursive nature of \( F_f \) and the fact that \( f \) is in general non-linear, the CND of Definition 3.7 need not be uniformly distributed on any other intervals. See Figure 2 for a plot of the pdf and cdf of the CND of Definition 3.7 corresponding to the tradeoff function \( G_1 \).

The following proposition verifies that \( F_f \) is a distribution function, as well as some other properties, such as continuity, symmetry, and concavity/convexity.

**Proposition 3.8.** Let \( f \) be a symmetric nontrivial tradeoff function, and let \( F := F_f \). Then

1. \( F(x) \) is a cdf for a symmetric, continuous, real-valued random variable,
2. \( F(x) \) satisfies \( F(x) = 1 - f(F(x - 1)) \) whenever \( F(x - 1) > 0 \) and \( F(x) = f(1 - F(x + 1)) \) whenever \( F(x + 1) < 1 \).
3. \( F'(x) \) is decreasing on \((-1/2, \infty)\) and increasing on \((-\infty, 1/2)\).
4. \( F(x) \) is strictly increasing on \( \{x \mid 0 < F(x) < 1\} \).

**Proof Sketch.** Most of the properties are proved by induction, checking that the properties hold on intervals of the type \([x - 1/2, x + 1/2]\) for \( x \in \mathbb{Z} \) as well as at the break points at half-integer values. The full proof is found in Appendix F.

Theorem 3.9 below states that for any nontrivial tradeoff function, this construction yields a canonical noise distribution, which can be constructed as in Definition 3.7. This CND can be used to add perfectly calibrated noise to a statistic to achieve \( f\text{-DP} \). As we will see later, the existence (and construction) of a CND will enable us to prove that any \( f\text{-DP} \) test can be post-processed from a test statistic, and this implies that we can always obtain hypothesis testing \( p \)-values at no additional privacy cost, a generalization of the result of Awan and Slavković (2018) which previously only held for \((\epsilon, \delta)\)-DP and for Bernoulli data.

**Theorem 3.9.** Let \( f \) be a symmetric nontrivial tradeoff function and let \( F_f \) be as in Definition 3.7. Then \( F_f \) is a canonical noise distribution for \( f \).

**Proof Sketch.** \( F_f \) was already shown to be symmetric in Proposition 3.8. The two equalities, \( f(\alpha) = T(F(\cdot), F(\cdot - 1))(\alpha) = F(F^{-1}(1 - \alpha) - 1) \) can also be easily verified using the properties of Proposition 3.8. The main challenge is to show that \( T(F(\cdot), F(\cdot - m)) \geq T(F(\cdot), F(\cdot - 1)) \) for \( m \in (0, 1) \). Lemma F.5 in the appendix gives an alternative technical condition which makes it easier to verify property 1 of Definition 3.1.

It turns out that the requirements of Definition 3.1 do not uniquely determine a distribution. For instance, \( \Phi \) the cdf of a standard normal is a CND for \( 1 \text{-GDP} \), but \( \Phi \) is different from the construction in Definition 3.7. See Figure 2 for the cdf and pdf of these two CNDs. Note that the CND of Definition 3.7 is uniform in \([-1/2, 1/2]\) and has “kinks” at each half-integer value. On the other hand, the standard normal is smooth. This example shows that for certain tradeoff functions there may be a more natural CND than the one constructed in Definition 3.7.

While there may be more natural CNDs in some settings, we emphasize the generality of the construction in Definition 3.7. In Proposition F.6, we present an exact method to sample from the CND of Definition 3.7 based on inverse transform sampling, allowing for straightforward implementation and application of our CND results.
3.1. **Canonical noise for \((\epsilon, \delta)\)-DP.** So far, we have developed a constructive and general method of generating canonical noise distributions for \(f\)-DP. In the special case of \((\epsilon, \delta)\)-DP, the CND \(F_f\) is equal to the cdf of the Tulap distribution, proposed in Awan and Slavkovič (2018), which is an extension of the Staircase mechanism (Geng and Viswanath, 2015) from \((\epsilon, 0)\)-DP to \((\epsilon, \delta)\)-DP.

**COROLLARY 3.10.** The distribution \(\text{Tulap}(0, b, q)\), where \(b = \exp(-\epsilon)\) and \(q = \frac{2b^2}{1-b+2b^2}\), is a CND for \(f_{\epsilon, \delta}\)-DP, which agrees with the construction of Definition 3.7.

**PROOF SKETCH.** The cdf of \(\text{Tulap}(0, b, q)\) is defined in the full proof. From the definition, it is easy to verify that the cdf of a Tulap random variable agrees with \(F_f\) on \([-1/2, 1/2]\). By Awan and Slavkovič (2020, Lemma 2.8), the Tulap cdf also satisfies the recurrence relation of Definition 3.7.

It was claimed in both Awan and Slavkovič (2018) and Awan and Slavkovič (2020) that adding Tulap noise satisfied \((\epsilon, \delta)\)-DP, but their proof is actually incorrect and only holds for integer valued statistics. The above Corollary along with Theorem 3.9 offers a complete and correct argument for Awan and Slavkovič (2020, Theorem 2.11).

In Awan and Slavkovič (2018) and Awan and Slavkovič (2020), it was shown that the Tulap distribution could be used to design optimal hypothesis tests and confidence intervals for Bernoulli data. Our notion of a canonical noise distribution, and the fact that Tulap is a CND for \((\epsilon, \delta)\)-DP sheds some light on why it had such optimality properties (even further explored in Section 4). The Tulap distribution is also closely related to discrete Laplace and the Staircase distributions, which were shown by Ghosh, Roughgarden and Sundararajan (2012) and Geng and Viswanath (2015) respectively to be optimal in terms of maximizing various definitions of utility in \((\epsilon, 0)\)-DP.

While continuous Laplace noise is commonly used in \((\epsilon, 0)\)-DP, Dong et al. (2022) pointed out that the tradeoff function for Laplace noise does not agree with \(f_{\epsilon, \delta}\) for any values of \(\epsilon\).
and δ. From this observation, we conclude from Definition 3.1 that Laplace is not a CND for \((\epsilon, \delta)\)-DP. From the perspective of CNDs, Tulap noise is preferable over the Laplace mechanism.

4. The nature of \(f\)-DP tests. Recall that a test is a function \(\phi: \mathcal{X}^n \to [0, 1]\), where \(\phi(x)\) represents the probability of rejecting the null hypothesis given that we observed \(x\). However, the mechanism corresponding to this test releases a random value drawn as \(\text{Bern}(\phi(x))\), where 1 represents “Reject” and 0 represents “Accept.” we say that the test \(\phi\) satisfies \(f\)-DP if the corresponding mechanism \(\text{Bern}(\phi(x))\) satisfies \(f\)-DP. Intuitively, Lemma 4.1 shows that a test satisfies \(f\)-DP if for adjacent databases \(x\) and \(x'\), the values \(\phi(x)\) and \(\phi(x')\) are close in terms of an inequality based on \(f\).

**Lemma 4.1.** Let \(f\) be a symmetric tradeoff function. A test \(\phi: \mathcal{X}^n \to [0, 1]\) satisfies \(f\)-DP if and only if \(\phi(x) \leq 1 - f(\phi(x'))\) for all \(x, x' \in \mathcal{X}^n\) such that \(H(x, x') \leq 1\).

**Proof sketch.** If we take the rejection region to be the set \(\{1\}\) then \(\phi(x)\) is the type I error and \(1 - \phi(x')\) is the type II error. The \(f\)-DP guarantee requires that \(f(\phi(x)) \leq 1 - \phi(x')\), or equivalently, \(\phi(x') \leq 1 - f(\phi(x))\). Using the rejection region \(\{0\}\) and some algebra, we get \(\phi(x) \leq 1 - f(\phi(x'))\). The full proof argues more precisely using the Neyman Pearson Lemma, considering also randomized tests.

Lemma 4.1 greatly simplifies the search for \(f\)-DP hypothesis tests and generalizes the bounds on private tests established in Awan and Slavković (2018).

**Example 4.2 ((\(\epsilon, \delta)\)-DP tests).** When we apply Lemma 4.1 to the setting of \((\epsilon, \delta)\)-DP, we have the two inequalities: \((1 - \phi(x)) \geq 1 - \delta - \exp(\epsilon)\phi(x')\) and \((1 - \phi(x)) \geq \exp(-\epsilon)(1 - \delta - \phi(x'))\). Some algebra gives

\[
\phi(x) \leq \begin{cases} 
\delta + \exp(\epsilon)\phi(x') \\
1 - \exp(-\epsilon)(1 - \delta - \phi(x'))
\end{cases},
\]

which agrees with the constraints derived in Awan and Slavković (2018).

The result of Lemma 4.1 can also be expressed in terms of canonical noise distributions in Corollary 4.3, giving the elegant relation that \(F^{-1}(\phi(x))\) and \(F^{-1}(\phi(x'))\) differ by at most 1 when \(x\) and \(x'\) are adjacent.

**Corollary 4.3 (Canonical Noise Distributions).** Let \(f\) be a symmetric nontrivial tradeoff function and \(F\) be a canonical noise distribution for \(f\). Then a test \(\phi\) satisfies \(f\)-DP if and only if \(F^{-1}(\phi(x)) \leq F^{-1}(\phi(x')) + 1\) for all \(x, x' \in \mathcal{X}^n\) such that \(H(x, x') \leq 1\).

**Proof sketch.** The result follows from the fact that \(f(\alpha) = F(F^{-1}(1 - \alpha) - 1),\) the symmetry of \(F\), and some algebra of cdfs.

Corollary 4.3 is also important for the construction of “free” DP \(p\)-values in Section 4.1.

4.1. Free \(f\)-DP \(p\)-values. In Awan and Slavković (2018), it was shown that for Bernoulli data, the uniformly most powerful DP test could also be expressed as the post-processing of a privatized test statistic, offering \(p\)-values at no additional privacy cost. We generalize this result using the concept of canonical noise distributions and show that any \(f\)-DP test can be
expressed as a post-processing threshold test based on a privatized test statistic, and that the test statistic can also be used to give private p-values.

Typically in statistics, it is preferred to report a p-value rather than an accept/reject decision at a single type I error. A p-value provides a continuous summary of how much evidence there is for the alternative hypothesis and allows for the reader to determine whether there is enough evidence to reject at the reader’s personal type I error. Lower p-values give more evidence for the alternative hypothesis.

However, with privacy, one may wonder whether releasing a p-value rather than just the accept/reject decision would result in an increased privacy cost, or conversely whether a p-value at the same privacy level would have lower power. In fact, this question is related to fundamental concepts in differential privacy such as post-processing, privacy amplification, and composition. In Lemma 4.4, we recall the post-processing property of DP, which states that after a DP result is released, no post-processing can compromise the DP guarantee.

**Lemma 4.4 (Post-processing: Dong et al., 2022).** Let $M$ be an $f$-DP mechanism taking values in $Y$. Let $Proc$ be a mechanism from $Y$ to $Z$. Then $Proc \circ M$ satisfies $f$-DP.

Theorem 4.5 is the main result of this section, demonstrating that given an arbitrary $f$-DP hypothesis test, we can construct a summary statistic and p-values, all with no additional privacy cost, using a CND.

**Theorem 4.5.** Let $\phi : X^n \to [0, 1]$ be an $f$-DP test. Let $F$ be a CND for $f$, and draw $N \sim F$. Then

1. releasing $T = F^{-1}(\phi(x)) + N$ satisfies $f$-DP,
2. the variable $Z = I(T \geq 0)$, a post-processing of $T$, is distributed as $Z \mid X = x \sim \text{Bern}(\phi(x))$,
3. the value $p = \sup_{\theta_0 \in H_0} E_{X \sim \theta_0} (F^{-1}(\phi(X)) - T)$ is also a post-processing of $T$ and is a p-value for $H_0$,
4. if $H_0$ is a simple hypothesis and $\mathbb{E}_{H_0} \phi = \alpha$, then at type I error $\alpha$, the p-value from part 3 is as powerful as $\phi$ at every alternative.

**Proof Sketch.** Property 1 follows from Corollary 4.3, the observation that $F^{-1}(\phi(x))$ has sensitivity 1, and property 1 of Definition 3.1. Property 2 can be verified using algebra of cdfs. Property 3 is a standard construction of a p-value (Casella and Berger, 2002, Theorem 8.3.27). Property 4 is a special case of Lemma F.8, a general lemma about p-values.

We see from Theorem 4.5 that given an $f$-DP test $\phi$, we can report both a summary statistic (namely, $T$) as well as a p-value (a post-processing of $T$) which contain strictly more information than only sampling $\text{Bern}(\phi(x))$. This shows that for simple null hypotheses, there is no general privacy amplification when post-processing a p-value or test statistic to a binary accept/reject decision.

While in part 3 of Theorem 4.5 there are no assumptions on $H_0$, for some composite null hypotheses, the resulting p-value may have very low power. Part 4 states that if the null hypothesis is a singleton, then the power is perfectly preserved.

We also remark that while the proof of Theorem 4.5 is not technical, it heavily relies on the properties of the CND, showing that the notion of CND has exactly the right properties for Theorem 4.5 to hold.

Note that Theorem 4.5 starts with an $f$-DP test, and shows how to get a private summary statistic and p-values. However, constructing a private test $\phi$ is another matter. In Section 4.2, we show that for exchangeable binary data, we can construct a most powerful $f$-DP test in terms of a CND.
Remark 4.6. While recently there has been controversy around the use of p-values in scientific research (Colquhoun, 2017; Wasserstein and Lazar, 2016), this is mostly due to the misuse or misinterpretation of a p-value. Many of the criticisms of p-values can be addressed by including additional statistical measures such as the effect size, confidence intervals, likelihood ratios, or Bayes factors. We view p-values as a valuable tool that is a component of a complete statistical analysis. Since the p-values of Theorem 4.5 are a post-processing of a private summary statistic, that statistic can also be potentially used for other statistical inference tasks, such as in Awan and Slavković (2020).

4.2. Most powerful tests for exchangeable binary data. In this section, we extend the main result of Awan and Slavković (2018), that of constructing most powerful DP tests, to general f-DP as well as exchangeable distributions on \( \{0, 1\}^n \). In contrast, the hypothesis tests of Awan and Slavković (2018) were limited to \((\epsilon, \delta)\)-DP and i.i.d. Bernoulli data. A distribution \( P \) on a set \( \mathcal{X}^n \) is exchangeable if given \( X \sim P \) and a permutation \( \pi, X \overset{d}{=} \pi(X) \). Note that i.i.d. data are always exchangeable, but there are exchangeable distributions that are not i.i.d. For example, sampling without replacement results in exchangeable but non-i.i.d. data.

In the next result, we extend Theorem 3.2 of Awan and Slavković (2018) from \((\epsilon, \delta)\)-DP to the setting of general f-DP. The argument is essentially identical. We include the proof for completeness.

Lemma 4.7 (Theorem 3.2 of Awan and Slavković (2018)). Let \( \mathcal{P} \) be a set of exchangeable distributions on \( \mathcal{X}^n \). Let \( f : \mathcal{X}^n \rightarrow [0, 1] \) be a test satisfying f-DP. Then there exists a test \( f' : \mathcal{X}^n \rightarrow [0, 1] \) such that for all \( x \in \mathcal{X}^n \), \( f'(x) \) only depends on the empirical distribution of \( x \), and \( \int f'(x) \ dP = \int f(x) \ dP \) for all \( P \in \mathcal{P} \).

Proof. Define \( f'(x) = \frac{1}{n!} \sum_{\pi \in \sigma(n)} f(\pi(x)) \), where \( \sigma(n) \) is the symmetric group on \( n \) letters. Note that for any \( \pi \in \sigma(n) \), \( f(\pi(x)) \) satisfies f-DP (just rearranging the sample space). Furthermore, \( \int f(\pi(x)) \ dP = \int f(x) \ dP \) by exchangeability. Finally, by the convexity of \( f \), the set of tests \( \phi \) which satisfy \( \phi(x) \leq 1 - f(\phi(x')) \) is a convex set, and so is closed under convex combinations. So, \( f' \) defined above satisfies f-DP, and by the linearity of integrals, preserves the expectations. \( \square \)

We work with the sample space \( \mathcal{X} = \{0, 1\} \). Note that by Lemma 4.7, because we are dealing with exchangeable distributions, the test need only depend on \( X = \sum_{i=1}^{n} X_i \), so we define \( \phi(x) \) for \( x = 0, 1, 2, \ldots, n \). Since changing one \( X_i \) only changes \( X \) by \( \pm 1 \), we need only relate \( \phi(x) \) and \( \phi(x - 1) \).

The main result of this section, Theorem 4.8 constructs not only the first private hypothesis test in the general f-DP framework, but derives a most powerful f-DP test as well as a corresponding p-value in terms of the canonical noise distribution. The proof of Theorem 4.8 is similar to the proof of Awan and Slavković (2018, Theorem 4.5), further demonstrating that the canonical noise distribution is the appropriate concept needed to extend their result from \((\epsilon, \delta)\)-DP to arbitrary f-DP. Just like in Awan and Slavković (2018), we have the surprising result that the UMP DP test in this case only depends on the summary statistic \( x + N \), where \( N \) is a CND. The extension from Bernoulli distributions to arbitrary exchangeable binary variables is simply an observation that the argument only depends on the likelihood ratio. However, the extension to exchangeable distributions will allow us to apply Theorem 4.8 to the difference-of-proportions problem in Section 5.
**THEOREM 4.8.** Let $f$ be a symmetric nontrivial tradeoff function and let $F$ be a CND of $f$. Let $X = \{0, 1\}$. Let $P$ and $Q$ be two exchangeable distributions on $X^n$ with pmfs $p$ and $q$ such that $q/p$ is an increasing function of $x = \sum_{i=1}^n x_i$. Let $\alpha \in (0, 1)$. Then a most powerful $f$-DP test $\phi$ with level $\alpha$ for $H_0 : X \sim P$ versus $H_1 : X \sim Q$ can be expressed in any of the following forms:

1. There exists $y \in \{0, 1, 2, \ldots, n\}$ and $c \in (0, 1)$ such that for all $x \in \{0, 1, 2, \ldots, n\}$, 
   \[ \phi(x) = \begin{cases} 
   0 & x < y, \\
   c & x = y, \\
   1 - f(\phi(x - 1)) & x > y, 
   \end{cases} \]
   where if $y > 0$ then $c$ satisfies $c \leq 1 - f(0)$, and $c$ and $y$ are chosen such that $\mathbb{E}_P \phi(x) = \alpha$.
   If $f(0) = 1$, then $y = 0$.
2. $\phi(x) = F(x - m)$, where $m \in \mathbb{R}$ is chosen such that $\mathbb{E}_P \phi(x) = \alpha$.
3. Let $N \sim F$. The variable $T = X + N$ satisfies $f$-DP. Then $p = \mathbb{E}_{X \sim P} F(X - T)$ is a $p$-value and $I(p \leq \alpha) | X = I(T \geq m) | X \sim \text{Bern}(\phi(X))$, where $\phi(x)$ agrees with 1 and 2 above.

**PROOF SKETCH.** Similar to the proof of Awan and Slavković (2018, Theorem 4.5), we begin by establishing the equivalence of forms 1 and 2, and arguing that there exists a test of the form 2 by the Intermediate Value Theorem. Using Awan and Slavković (2018, Lemma 4.4), a variation of the Neyman Pearson Lemma, we argue that the proposed $\phi$ is most powerful. Statement 3 uses the expressions from Theorem 4.5 as well as some distributional algebra of CNDs to get the more explicit formula.

While Theorem 4.5 took an $f$-DP test and produced “free” private $p$-values, Theorem 4.8 constructs an optimal test from scratch beginning only with a CND.

**EXAMPLE 4.9.** Let us consider what distributions fit within the framework of Theorem 4.8. If the variables $X_i$ are i.i.d., then they are distributed as Bernoulli. However, it is possible for the variables to be exchangeable and not independent. For example, the sum $X = \sum_{i=1}^n X_i$ could be distributed as a hypergeometric or Fisher’s noncentral hypergeometric, which arises in two sample tests of proportions, see Section 5. For other exchangeable binary distributions, see Dang, Keeton and Peng (2009).

**REMARK 4.10.** Theorem 4.8 and Corollary 4.3 show that the results of Awan and Slavković (2020) extend to arbitrary $f$-DP. By simply modifying the Tulap distribution to a CND, all of the other results of Awan and Slavković (2020) carry over as well. In particular, for Bernoulli data, there exists a UMP one-sided test, a UMP unbiased two-sided test, UMA one sided confidence interval and UMA unbiased two-sided confidence interval. All of these quantities are a post-processing of the summary value $X + N$, where the noise $N$ is drawn from a CND $F$ of $f$.

**5. Extension to semi-private difference-of-proportions tests.** Testing two population proportions is a very common hypothesis testing problem, which arises in clinical trials with control and test groups, A/B testing, and observation studies comparing two groups (such as men and women, students from two universities, or aspects of two different countries). As such, the techniques for testing such hypotheses are very standardized and taught in many introductory statistics textbooks. However, there are limited techniques to test these hypotheses under $f$-DP.
In Appendix D we show that subject to differential privacy, there does not exist a UMP (unbiased) \( f \)-DP test. Nevertheless, we use the techniques developed earlier in this paper to derive a “semi-private” UMP unbiased test, which gives an upper bound on the power of any \( f \)-DP UMP unbiased test. The novel concept of “semi-privacy” enforces some of the DP constraints but not others, and this framework may be of independent interest when analyzing a combination of private and non-private releases (see Remark 5.4 for more details). We then construct an \( f \)-DP test which allows for optimal inference for the two population parameters, and which we show through simulations to have comparable power to the semi-private UMP unbiased test. In the case of \( \epsilon \)-DP, we show through simulations that the proposed DP test is similar to the semi-private UMP unbiased test with privacy parameter \((\epsilon/\sqrt{2})\). We also demonstrate that the proposed test has more accurate \( p \)-values and type I error than commonly used Normal approximation tests.

5.1. Semi-private UMP unbiased test. In this section, we simplify the search for an \( f \)-DP test for the difference of proportions, establishing a condition for the test to be unbiased. However, as demonstrated through an example in Appendix D, there does not in general exist a UMP unbiased (UMPU) \( f \)-DP test. By weakening the privacy guarantee, we develop a “semi-private” UMPU test which can be efficiently implemented. While the “semi-private” test does not satisfy \( f \)-DP, it gives an upper bound on the power of any other unbiased \( f \)-DP test, and serves as a useful baseline in Section 5.3.

We observe independent \( X_i \sim \text{Bern}(\theta_X) \) for \( i = 1, \ldots, n \) and \( Y_j \sim \text{Bern}(\theta_Y) \) for \( j = 1, \ldots, m \). For privacy, we consider two datasets adjacent if either one of the \( X_i \) is changed or one of the \( Y_j \) is changed (but only one total value). We consider \( m \) and \( n \) to be publicly known values. We wish to test \( H_0 : \theta_X \geq \theta_Y \) versus \( H_1 : \theta_X < \theta_Y \), subject to the constraint of differential privacy. Such one-sided tests can also be converted to two-sided tests using a Bonferroni correction, as discussed in Remark 5.9, at the end of Section 5.2.

By a similar argument as in Lemma 4.7, it is sufficient to consider tests which are functions of the empirical distributions of \( X \) and \( Y \). Equivalently, we may restrict to tests which are functions of \( X = \sum_{i=1}^{n} X_i \) and \( Y = \sum_{j=1}^{m} Y_j \). We consider two databases adjacent if either \( X \) changes by 1 or if \( Y \) changes by 1 (but not both). By Lemma 4.1, a test \( \phi(x,y) \) satisfies \( f \)-DP if the following set of inequalities hold

\[
\begin{align*}
\phi(x,y) &\leq 1 - f(\phi(x+1,y)) \\
\phi(x,y) &\leq 1 - f(\phi(x-1,y)) \\
\phi(x,y) &\leq 1 - f(\phi(x,y+1)) \\
\phi(x,y) &\leq 1 - f(\phi(x,y-1)),
\end{align*}
\]

for all pairs of \((x,y)\).

Classically, it is known that even without privacy there is no uniformly most powerful test for this problem. Traditionally, attention is restricted to unbiased tests. Recall that a test is unbiased if for all \( \theta_1 \in \Theta_1 \) and \( \theta_0 \in \Theta_0 \), the power at \( \theta_1 \) is higher than at \( \theta_0 \) (here, \( \theta \) represents the pair \((\theta_X, \theta_Y)\)). Because the variables \((X,Y)\) have distribution in the exponential family, the search for a UMP unbiased test can be restricted to tests which satisfy \( \mathbb{E}_{\theta_X=\theta_Y} (\phi(X,Y) \mid X + Y = z) = \alpha \) (Schervish, 2012, Proof of Theorem 4.124), since \( X + Y \) is a complete sufficient statistic under \( H_0 \). When \( \theta_X = \theta_Y = \theta_0 \), \( X + Y \sim \text{Binom}(m+n, \theta_0) \), and \( Y \mid (X+Y = z) \sim \text{Hyper}(m, n, z) \), where \( \text{Hyper}(m, n, z) \) is the hypergeometric distribution, where we draw \( m \) balls out of a total of \( m+n \) balls, and where \( z \) balls are white, and the random variable counts the number of drawn white balls. This is equivalent to a permutation test where we shuffle the labels of the observations. Lemma 5.1 summarizes these observations.
**Lemma 5.1.** Let \( X \sim \text{Binom}(n, \theta_X) \) and \( Y \sim \text{Binom}(m, \theta_Y) \) be independent. Consider the test \( H_0 : \theta_X \geq \theta_Y \) and \( H_1 : \theta_X < \theta_Y \). Let \( \Phi \) be a set of tests. If there exists a UMP test \( \phi \in \Phi \) among those which satisfy
\[
\mathbb{E}_{H \sim \text{Hyper}(m,n,z)} \phi(z - H, H) = \alpha,
\]
for all \( \alpha \), then \( \phi \) is UMP unbiased size \( \alpha \) among \( \Phi \).

**Proof.** It is easy to verify that the power function is continuous, and that \( X + Y \) is a boundedly complete sufficient statistic under \( H_0 \). By Schervish (2012, Proposition 4.92) and Schervish (2012, Lemma 4.122), the set of unbiased tests for this problem is a subset of the boundedly complete sufficient statistic under \( H \). For all \( \phi \) satisfying Equation (2) it is also clear that Equation (2) implies that the test is size \( \alpha \). It follows that if a test is UMP among the tests in \( \Phi \) satisfying Equation (2) then it is UMP unbiased size \( \alpha \) among \( \Phi \).

However, as demonstrated by an example given later in Appendix D, in general there is no UMP test for the hypothesis \( H_0 : \theta_X \geq \theta_Y \) versus \( H_1 : \theta_X < \theta_Y \) among the set
\[
\Phi_f = \{ \phi(x,y) \mid \phi \text{ satisfies inequalities (1) and Equation (2)} \}.
\]
The reason for this is that Lemma 5.1 suggests that a UMP unbiased test relies on being able to construct a UMP test, given \( X + Y = z \). However, the inequalities (1) put constraints, relating \( \phi(x,y) \) for different values of \( z \).

Instead of requiring that all of the inequalities (1) hold, we weaken the requirement of differential privacy, to only include the constraints relating \( (x,y) \) with the same sum \( x + y = z \). We call the following the set of “semi-private” tests:
\[
\Phi_{\text{semi}}^f = \left\{ \phi(x,y) \mid \phi \text{ satisfies constraints of (1), and for each } x \in \{0, 1, \ldots, m+n\}, \text{ there exists } \psi \in \Phi_f, \text{ s.t. } \phi(x,y) = \psi(x,y) \text{ for all } x + y = z \right\}.
\]

Intuitively, \( \Phi_{\text{semi}}^f \) is the set of tests, which satisfy the set of implied constraints of (1), which only relate \( (x,y) \) and \( (x+1, y-1) \). So, the summary \( z = X + Y \) is not protected at all, but for any \( X + Y = z \), \( (X,Y) \) must satisfy \( f\)-DP. While these semi-private tests are not necessarily intended for the purpose of privacy protection, by weakening the privacy requirement, they offer an upper bound on the performance of any DP test, as stated in Corollary 5.3.

**Theorem 5.2 (Semi-Private UMPU).** Let \( f \) be a symmetric nontrivial tradeoff function and let \( F \) be a CND for \( f \). Let \( X \sim \text{Binom}(n, \theta_X) \) and \( Y \sim \text{Binom}(m, \theta_Y) \) be independent. Let \( \alpha \in (0,1) \) be given. For the hypothesis \( H_0 : \theta_X \geq \theta_Y \) versus \( H_1 : \theta_X < \theta_Y \),
1. \( \phi^*(x,y) = F(y - x - c(x+y)) \) is the UMPU test of size \( \alpha \) among \( \Phi_{\text{semi}}^f \), where \( c(x+y) \) is chosen such that \( \mathbb{E}_{H \sim \text{Hyper}(m,n,x+y)} \phi^*(x+y) - H, H = \alpha \).
2. Set \( T = Y - X + N \), where \( N \sim F \), and set \( Z = X + Y \). Then
\[
p = \mathbb{E}_{H \sim \text{Hyper}(m,n,z)} F(2H - Z - T)
\]
is the exact \( p \)-value corresponding to \( \phi^* \).

**Proof Sketch.** Lemma 5.1 reduced the problem to determining whether the test is UMP among those which satisfy Equation (2). The technical lemmas F.10 and F.12, given in Appendix F, quantify the privacy of the semi-private tests when viewed as a function of \( y \) (where \( z \) is fixed), and determine the CND of the derived tradeoff function. Conditional on \( z \), the distribution of \( Y \) is a Fisher noncentral hypergeometric distribution (Harkness, 1965; Fog, 2008). By Theorem 4.8 we can construct the most powerful DP test based on the CND. Finally, we verify a monotone likelihood ratio property of the noncentral hypergeometrics to argue that the test is in fact uniformly most powerful. \( \square \)
Corollary 5.3 shows that while the semiprivate UMPU test does not satisfy $f$-DP, we can use it as a benchmark to compare other tests, as it gives an upper bound on the highest possible power of any unbiased $f$-DP level $\alpha$ test.

**Corollary 5.3.** Let $\phi^*(x, y)$ be the UMPU size $\alpha$ test among $\Phi_{\text{semi}}^f$, and let $\phi(x, y)$ be any unbiased, level $\alpha$ test in $\Phi_f$. Then

$$E_{X \sim \theta_0, Y \sim \theta_Y} \phi^*(X, Y) \geq E_{X \sim \theta_0, Y \sim \theta_Y} \phi(X, Y),$$

for any values of $\theta_X \leq \theta_Y$.

**Remark 5.4.** The semi-private framework could potentially be of independent interest, as it is an example of a setting where some statistics are preserved exactly, whereas others are protected with privacy noise. For example, this is similar to the framework used for the 2020 Decennial Census, where certain counts are preserved without any privacy noise, and the other counts are sanitized by an additive noise mechanism. While they phrase their privacy guarantee in terms of post-processing, one could also view it as a “semi-private” procedure, where their privacy guarantee only holds for the databases which agree with the preserved counts. This is an alternative perspective to subspace differential privacy (Gao, Gong and Yu, 2022), which restricts the output of a mechanism rather than the input database.

5.2. **Designing an $f$-DP test for difference-of-proportions.** Based on the negative result of Appendix D, we consider a different approach to building a well-performing DP test. A very common non-private test used to test $H_0 : \theta_X \geq \theta_Y$ versus $H_1 : \theta_X < \theta_Y$ for $X \sim \text{Binom}(n, \theta_X)$ and $Y \sim \text{Binom}(m, \theta_Y)$ is based on the test statistic

$$Y/m - X/n,$

which is intuitive as this quantity captures the sample evidence for the difference between $\theta_X$ and $\theta_Y$. In fact this statistic has the important property that its expectation under the null does not depend on the parameter $\theta_X = \theta_Y$. If this were not the case, then tests based on this statistic would have limited power (Robins, van der Vaart and Ventura, 2000). However, the sampling distribution of this quantity depends on the parameter $\theta = \theta_X = \theta_Y$ under the null (e.g., for $\theta = 1/2$, the variance of $Y/m - X/n$ is higher than when $\theta$ is larger or smaller). Typically, the central limit theorem is used to justify that

$$\frac{Y/m - X/n}{\sqrt{(1/m + 1/n)\hat{\theta}_0(1 - \hat{\theta}_0)}} \approx N(0, 1),$$

where $\hat{\theta}_0 = \frac{\bar{X} + \bar{Y}}{m + n}$ is the maximum likelihood estimator for $\theta$ under the null. The central limit approximation works well in large samples, but for small samples this approximation can be inadequate as demonstrated in the simulations of Section 5.3.

5.2.1. **Inversion-based parametric bootstrap $f$-DP test.** In this section, we consider tests based on the following privatized summary quantities $X + N_1$ and $Y + N_2$, where $N_1, N_2 \sim F$ where $F$ is a CND of $f$. The vector $(X + N_1, Y + N_2)$ satisfies $f$-DP, since only one of $X$ and $Y$ changes by at most 1, between adjacent databases.

**Remark 5.5.** Basing our test on these two noisy statistics has a few important benefits. As noted in Remark 4.10, given $X + N_1$ and $Y + N_2$ we can perform optimal hypothesis tests and confidence intervals for $\theta_X$ and $\theta_Y$ combining Theorem 4.8, Corollary 4.3 and the other results of Awan and Slavković (2020). In general this is not the case for an arbitrary
$f$-DP test of $H_0 : \theta_X \geq \theta_Y$ versus $H_1 : \theta_X < \theta_Y$. While 4.5 says that we can always get a summary statistic and $p$-value out of an arbitrary $f$-DP test, these values may not contain enough information to do inference (let alone optimal inference) for $\theta_X$ and $\theta_Y$ separately.

Then we consider the quantity $T = m^{-1}(Y + N_2) - n^{-1}(X + N_1)$. Asymptotics tells us that under the null hypothesis, $T / \sqrt{(1/m + 1/n) \theta (1 - \theta)} \overset{d}{\to} N(0, 1)$, which is the same sampling distribution as without privacy. However, as many other researchers have noted, while these approximations are serviceable in classical settings, the approximations are too poor when privacy noise is introduced (Wang et al., 2018). One reason for this is that the noise introduced to achieve privacy, such as Laplace or Tulap, often has heavier tails than the limit distribution, which is often Gaussian.

We notice that $T$ is a linear combination of independent random variables. So, we can use characteristic functions to derive the sampling distribution of $T$ under a specific null parameter $\theta$.

We use $\psi_X(\cdot)$ to denote the characteristic function of a random variable $X$: $\psi_X(t) := \mathbb{E}[X e^{itX}]$. Recall that for independent random variables $X_1, \ldots, X_n$ and real values $a_1, \ldots, a_n$, if $X = \sum_{i=1}^{n} a_iX_i$, then $\psi_X(t) = \prod_{i=1}^{n} \psi_{X_i}(a_i t)$.

Then the characteristic function of our test statistic $T$ is given by

$$\psi_{T \sim \theta}(t) = \psi_{Y \sim \theta}(t/m) \psi_{N_2}(t/m) \psi_{X \sim \theta}(-t/n) \psi_{N_1}(-t/n).$$

We know the characteristic function for a binomial random variable, and for many common DP distributions $N$, we have formulas for $\psi_N$ as well.

We can use the following inversion formula to evaluate the cdf of $T$.

**Lemma 5.6** (Inversion Formula: Gil-Pelaez). Let $X$ be a real-valued continuous random variable, with characteristic function $\psi_X(t)$. Then the cdf of $X$ can be evaluated as

$$F_X(x) = \int_0^{\infty} \frac{\text{Im}(e^{-itx} \psi_X(t))}{t} \, dt,$$

where $\text{Im}(\cdot)$ returns the imaginary component of a complex number: $\text{Im}(z) = (z - z^*)/(2i)$, where $z^*$ is the complex conjugate of $z$.

Lemma 5.6 gives a computationally tractable method of evaluating the exact sampling distribution of $T$ at a given null parameter. Since larger values of $T$ give more evidence of the alternative hypothesis, $p(T) = 1 - F_{T \sim \theta_0}(T)$ is a $p$-value for the null hypothesis $H_0 : \theta_X = \theta_Y = \theta_0$ (Casella and Berger, 2002, Theorem 8.3.27). However, this $p$-value depends on the null parameter $\theta_0$, which we likely do not know. A solution is to substitute an estimator for $\theta_0$ under the null hypothesis that $\theta_X = \theta_Y$, based on the privatized statistics $X + N_1$ and $Y + N_2$. A natural estimator is $\hat{\theta}_0 = \min\{\max\{X + N_1, Y + N_2\}, 0\}, 1\}$. Plugging this estimate in for $\theta_0$ gives the approximate $p$-value:

$$\hat{p}(T, \hat{\theta}_0) = 1 - F_{T \sim \hat{\theta}_0}(T).$$

This approximate $p$-value is our recommended $f$-DP test for the difference-of-proportions testing problem, and the procedure is summarized in Algorithm 1 for the cases of $(\epsilon, 0)$-DP and $\mu$-GDP. While $p$-value is not exact, and is thus not guaranteed to have the intended type I error, the results of Robins, van der Vaart and Ventura (2000) imply that this $p$-value is asymptotically uniform under the null, implying that the test is asymptotically unbiased, with asymptotically accurate type I errors. Furthermore, as we demonstrate in Section 5.3, for even sample sizes as small as $n, m \geq 30$, the approximation is incredibly accurate, offering
Algorithm 1: $\epsilon$-DP or $\mu$-GDP approximate $p$-value, based on inversion.

1. Let $X$, $Y$, $m$, and $n$ be given. Let either $\epsilon$ or $\mu$ be given.
2. if $\epsilon$-DP then
   3. Draw $N_1, N_2 \sim \text{Tulap}(0, \exp(-\epsilon), 0)$;
   4. Set $\psi_N(t) = \frac{[1-\exp(-t)]\exp(-it/2)\exp(it/2)}{it[1-\exp(it)][1-\exp(-it-\epsilon)]}$;
5. end
6. if $\mu$-GDP then
   7. Draw $N_1, N_2 \sim N(0, 1/\mu^2)$;
   8. Set $\psi_N(t) = \exp(-t^2/(2\mu^2))$;
9. end
10. Set $\psi_{Y \sim \theta}(t) = ((1-\theta) + \theta \exp(it))^m$ and $\psi_{X \sim \theta}(t) = ((1-\theta) + \theta \exp(it))^n$;
11. Set $X = X + N_1$ and $Y = Y + N_2$;
12. Set $T = \bar{Y}/m - \bar{X}/n$;
13. Set $\hat{\theta} = \min \left\{ \max \left\{ \frac{X+Y}{m+n}, 0 \right\}, 1 \right\}$;
14. Set $\psi_{T \sim \hat{\theta}}(t) = \psi_{Y \sim \theta}(t/m)\psi_{X \sim \theta}(-t/n)\psi_N(t/m)\psi_N(-t/n)$;
15. Output $p$-value and summary values: $p = 1 - \int_0^\infty \frac{\text{Im}(\exp(itT)\psi_{T \sim \hat{\theta}}(t))}{t} dt$, $X + N_1$, $Y + N_2$

accuracy even higher than the classic normal approximation test, which is widely used and accepted. We also show in Section 5.3 that the power of the test is comparable to the semi-private test of Section 5.1 indicating that it is near optimal.

REMARK 5.7. While the $p$-value generated from Algorithm 1 may seem complex, it is relatively easy to implement. For instance in R, the command `integrate` can perform an accurate numerical integral. Another strength of Algorithm 1 is that the running time does not depend on the sample size $m$ or $n$, whereas the semi-private test runs in $O(m)$ time.

REMARK 5.8. Algorithm 1 can be viewed as an exact evaluation of a parametric bootstrap, where we by-pass the need for sampling by numerically computing the cdf. As such, we avoid the additional error and running time produced by the Monte Carlo sampling.

REMARK 5.9. While we focus on the one-sided hypothesis $H_0 : \theta_X \geq \theta_Y$ versus $H_1 : \theta_X < \theta_Y$, the test of Algorithm 1 can be easily modified to produce a “two-sided” test for $H_0 : \theta_X = \theta_Y$ versus $H_1 : \theta_X \neq \theta_Y$. Call $p$ the one-sided $p$-value from Algorithm 1. Then $p_2 = 2 \min\{p, 1-p\}$ is a $p$-value for the two-sided test. This method of combining multiple tests called a Bonferroni correction or an intersection-union test (Casella and Berger, 2002, Section 8.2.3).

5.3. Simulations. In this section, we perform several simulations to compare the performance of our proposed DP test to other competing DP tests, the semi-private UMPU test, as well as popularly used non-private tests. While our results can be applied to arbitrary $f$-DP, we only run our simulations for $(\epsilon, 0)$-DP as this privacy definition is commonly used and introduces noise that is difficult to incorporate.

In Section 5.3.3, we consider the empirical power of the tests, and show that the inversion DP test out-performs other DP tests, and by comparing against the semi-private test with privacy budget $\epsilon/\sqrt{2}$, show that it is observed to be more powerful than any $(\epsilon/\sqrt{2})$-DP test (see Remark 5.10 for the intuition behind the factor of $1/\sqrt{2}$). In Section 5.3.1, we consider the type I error of the various tests, and show that the observed type I error of the inversion
test is more accurate than the commonly used non-private normal approximation test. We also show that naive DP normal approximation tests have unacceptably inaccurate empirical type I errors. In Section 5.3.2, we plot the empirical cumulative distribution functions (cdf) of the p-values from the various tests demonstrating from another perspective that the proposed test has accurate type I error.

5.3.1. Type I Error. The first simulation that we will consider, and one of the most important, demonstrates the reliability of the type I error guarantees of our proposed test against alternative tests. Recall that in the best practices of scientific research, many approximate statistical tests are widely used and accepted. For instance, most hypothesis testing tools are based on asymptotic theory which approximates the sampling distribution, such as the central limit theorem. As such, many widely used tests do not have exact type I error guarantees, but the error of these tests has been determined to be small enough for practical purposes. In Section 5.2, our proposed inversion-based test also involves an approximation to the sampling distribution. We demonstrate in the following simulation that the type I errors of this proposed test are more accurate than the widely accepted normal-approximation test.

For the simulation, we measure the empirical Type I error as the null $\theta_0$ takes values in \{.05, .1, ..., .95\} and sample sizes are set to $m = n = 30$, based on 20,000 replicates for each $\theta_0$ value. We consider two values for the nominal type I error: in the left plot of Figure 3 we set $\alpha = .01$ and in the right plot of Figure 3 we set $\alpha = .05$. The dotted horizontal lines represent a 95% Monte Carlo confidence interval assuming that the true type I error is equal to the nominal level. As there are 19 unique theta values, if a curve crosses these thresholds more than once, this is evidence that the type I error is not appropriately calibrated. For this simulation, we only consider approximate tests as the non-private UMPU test and the semi-private test have perfectly calibrated type I errors.

In red is the classic normal approximation test, described in Section 5.2. Such approximations are often considered accurate enough when the sample sizes $n$ and $m$ are greater than 30. Some rules of thumb for this problem require that there are at least 8 successes and failures in each group for the approximation to be accurate enough (Akritas, 2015, p. 321). We see in the left plot of Figure 3 that while this test has reasonable empirical type I error for moderate values of $\theta_0$, the test is overly conservative for extreme values of $\theta_0$. In the right plot of Figure 3, we see that the normal approximation test is much less reliable in this setting, with seven of the nineteen values outside of the 95% confidence region. We see that at extreme values of $\theta_0$, the actual type I error rates are much higher than the nominal level, resulting in excessive false positives. It is interesting that the type I errors are over-conservative when $\alpha = .01$ and inflated when $\alpha = .05$. In general, it is hard to predict whether in a particular setting the type I errors will be too high or too low.

In green is an $\epsilon$-DP normal approximation test, proposed by Karwa and Vadhan (2018) which is analogous to the one-sample test of Vu and Slavković (2009). See Appendix E for a description of the method. While the empirical type I errors of this test are acceptable when $\alpha = .05$, we see that for $\alpha = .01$, the empirical type I error is approximately .016 and is entirely outside the confidence region. We conclude that the type I errors for this normal approximation test are unreliable for these settings.

In light blue is an $\epsilon$-DP, which splits the budget between privatizing $T = Y - X$ and $Z = X + Y$, and plugs in the results into the semi-private test of Theorem 5.2. The test is described in Algorithm 3, which appears in Appendix E. The empirical type I errors for the plugin test are slightly higher than expected, crossing the confidence band three times in the left plot and once in the right plot, but are much more reliable than either of the normal approximation tests discussed above.

Finally, in magenta is the inversion-based test of Algorithm 1. The empirical type I errors of the inversion-based test lie entirely within the confidence bands for both settings of $\alpha$. This
Fig 3: Empirical Type I error as $\theta_0$ varies in $\{.05,.1,\ldots,.95\}$. The nominal $\alpha$ level is .01 (left) and .05 (right). $m = n = 30$, $\epsilon = .1$, and results are over 20,000 replicates for each $\theta_0$ value.

indicates that for the settings of these simulations, the type I errors of the inversion test are indistinguishable from the nominal level, and are much more accurate than the classic normal approximation test or a DP normal approximation test, such as in Vu and Slavković (2009).

5.3.2. P-values. In this section, we consider the empirical cumulative distribution function (cdf) of the $p$-values, while holding $\theta_0$ fixed. This can be interpreted as varying the nominal $\alpha$ value on the $x$-axis, with the empirical type I error on the $y$-axis. This differs from the previous simulation, where we varied the null value of $\theta$ along the $x$-axis, but left the nominal value of $\alpha$ fixed. Combined with the previous results, this simulation gives a more complete picture of how accurate the type I errors are, for a spectrum of nominal $\alpha$ values.

For the simulation, we set $\theta_0 = .95$, $n = 30$, $m = 40$, and $\epsilon = .1$. We chose to investigate $\theta_0 = .95$ since the type I errors in Section 5.3.1 were found to be more inaccurate for extreme values of $\theta_0$. The results are based on 100,000 replicates with these settings. The simulation includes the same tests as in Section 5.3.1, marked with the same color scheme, as well as a test based on the simulation-based method of Awan and Cai (2020). Included is a dotted black line of intercept 0 and slope 1, which represents perfectly calibrated type I error rates.

We see that for these simulation settings, the non-private normal approximation test has inflated type I errors for nominal $\alpha$ values between .02 and .2. The DP normal approximation test has inflated type I error rates for nominal alpha values below .05, and deflated type I error rates for larger values of $\alpha$. The plugin test also has inflated type I errors in this setting, while not as extreme as the normal approximate test. Finally, the curve for the inversion test is visually indistinguishable from the dotted black line, indicating that this tests has well-calibrated type I errors for this simulation setting, much improved over the other approximate tests considered here.

Awan and Cai (2020) tackled the same DP testing problem, and also based their test on adding Tulap noise to both $X$ and $Y$. They implement their test using the OASIS algorithm, which they argue gives asymptotically accurate type I errors. We include their test in this section for comparison, and while Awan and Cai (2020) advocated this approach in large samples, we see in the left plot of Figure 4 that it has greatly inflated type I errors for the smaller sample sizes considered in this simulation.
5.3.3. **Power.** Finally, we compare the power of our candidate tests. We use the semi-private UMPU test as a baseline for comparison: recall from Theorem 5.2 that the semi-private test has perfectly calibrated type I errors, and is uniformly more powerful than any DP unbiased test. As such, it serves as an upper bound on the power of the other candidate tests. We will see that the inversion test (with $\epsilon = .1$) has power similar to the semi-private UMPU with $\epsilon = (1/\sqrt{2})$, indicating that its power cannot be beaten by the most powerful $(\epsilon/\sqrt{2})$-DP unbiased test.

For the simulation, we vary the sample size $n = m$ along the $x$-axis and measure the empirical power on the $y$-axis, at a nominal $\alpha$ level of .05. The privacy parameter is set to $\epsilon = .1$ and the results are based on 1000 replicates for each sample size. In black is the non-private UMPU test, described Appendix C, which is guaranteed to be more powerful than any of the private tests considered in this paper. The dotted dark blue curve is the semi-private UMPU test of Section 5.1. Since the semi-private UMPU has a weaker privacy guarantee than DP, this test should also give an upper bound on the power of any DP test. We also include the semi-private test implemented with $\epsilon = .1/\sqrt{2}$ and $\epsilon = .1/2$, with the same color and line scheme. We see that the plugin test, appearing in light blue, has similar power as the semi-private test with $\epsilon = .1/\sqrt{2}$, indicating that this test is more powerful than any $\epsilon/2$ test. In magenta, we have the inversion-based test, which we see has similar power as the semi-private test with $\epsilon = .1/\sqrt{2}$, indicating that it is more powerful than any $\epsilon/\sqrt{2}$ test.

**Remark 5.10.** That the inversion test has comparable power to the semi-private test with $\epsilon/\sqrt{2}$ can be understood as follows: the semi-private test is based on the test statistic $S = Y - X + N$, where $N$ is a TuIap random variable. On the other hand, the inversion test is based on $\tilde{X} = X + N_1$ and $\tilde{Y} = Y + N_2$. If we tried to approximate the test statistic $S$ using $\tilde{X}$ and $\tilde{Y}$, we end up with $\tilde{S} = Y - X + (N_1 - N_2)$. If the same privacy parameters are used for $N$ and $N_1$, $N_2$, then $\text{Var}(N_1 - N_2) = 2 \text{Var}(N)$. By decreasing the privacy parameter of $N$ to $\epsilon/\sqrt{2}$, we obtain equality of the variances.
6. Discussion. In this paper we proposed the new concept canonical noise distribution, which expanded upon previous notions of an optimal noise adding mechanism for privacy. We showed that a CND is a fundamental concept in $f$-DP, connecting it to optimality properties of private hypothesis testing. Using CNDs and the theoretical results on $f$-DP hypothesis tests, we also developed a novel DP test for the difference-of-proportions, which was shown to have accurate type I errors and near optimal power. The introduction of CNDs also raises several questions:

It was noted in Section 3 that the CND is in general not unique for a given tradeoff function. While the construction in Definition 3.7 always results in a CND, and has a simple sampling procedure, it may not be the most natural CND. For example, when applied to the tradeoff function $G_1$, we see in Figure 2 that the CND constructed by Definition 3.7 has a non-differentiable pdf. On the other hand, $N(0, 1)$ is also a CND for $G_1$ which has a smooth pdf. One may wonder if there a more natural construction of a CND which recovers $N(0, 1)$ in the case of $G_1$, and similarly, if there is a CND for $f_{\epsilon,\delta}$ which has a continuous or smooth pdf. A recent paper that builds upon the present work, Awan and Dong (2022), partially answers these questions, showing that in some cases it is possible to construct a log-concave CND, which recovers $N(0, 1)$ in the case of $G_1$; surprisingly, Awan and Dong (2022) also show that the Tulap distribution is the unique CND for $f_{\epsilon,0}$, ruling out the possibility of a smooth CND for $f_{\epsilon,0}$.

Another question is whether there is a natural and meaningful extension of CNDs to vector-valued statistics. The follow-up paper, Awan and Dong (2022), partially answers this question, giving a definition of a multivariate CND and general constructions under various assumptions. While they show that there exists multivariate CNDs for many general classes of tradeoff functions, including GDP, Laplace-DP, and $(\epsilon, \delta)$-DP, they also prove that there is no multivariate CND for $f_{\epsilon,0}$.

While this paper focused on the connection between CNDs and private hypothesis tests, it is an open question whether there are other fundamental optimality properties of CNDs. It was also noted in the introduction that additive noise mechanisms often appear as a component of more complex DP mechanisms, and it is worth investigating whether CNDs can be used to optimize these other mechanisms for a particular $f$-DP guarantee.

The applications to DP hypothesis tests also raise many interesting questions. In general, there always exists a most powerful DP test for any composite null and simple alternative, as shown in Proposition B.1, which can be expressed as the solution to a convex optimization problem. However, solving the optimization problem is computationally burdensome for all but the simplest of problems. In Theorem 4.8, we were able to derive closed-form expressions for the most powerful DP tests. Do there exist closed-form expressions for other UMP DP tests to avoid computational optimization?

We also introduced the semi-private framework which allowed us to derive an upper bound on the power of any unbiased $f$-DP test. Can this framework be applied to other DP testing problems to derive similar bounds? We also remarked that the semi-private framework may be useful to better understand the privacy guarantee of mechanisms where certain statistics are privatized, whereas others are reported exactly, such as by in the 2020 Decennial US Census – it remains to be seen whether the semi-private framework can give new results or new understanding in these settings.

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APPENDIX A: BACKGROUND ON HYPOTHESIS TESTING

In this section, we review the definitions of randomized hypothesis tests and $p$-values.

DEFINITION A.1 (Hypothesis Test). Let $X \in \mathcal{X}$ be distributed $X \sim P_\theta$, where $\theta \in \Theta$. Let $\Theta_0, \Theta_1$ be a partition of $\Theta$. A (randomized) test of $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$ is a measurable function $\phi : \mathcal{X} \to [0, 1]$. We call $H_0 : \theta \in \Theta_0$ the null hypothesis and $H_1 : \theta \in \Theta_1$ the alternative hypothesis. We interpret the test $\phi(x)$ as the probability of rejecting the null hypothesis after observing $x \in \mathcal{X}$. We say a test $\phi$ is at level $\alpha$ if $\sup_{\theta \in \Theta_0} E_{P_\theta} \phi \leq \alpha$, and at size $\alpha$ if $\sup_{\theta \in \Theta_0} E_{P_\theta} \phi = \alpha$. The size is also called the type I error and represents the probability of mistakenly rejecting the null hypothesis. The power of $\phi$ at $\theta$ is denoted $\beta_\phi(\theta) = E_{P_\theta} \phi$, which is the probability of rejecting when the true parameter is $\theta$. A test $\phi$ is unbiased if $\beta_\phi(\theta_0) \leq \beta_\phi(\theta_1)$ for all $\theta_0 \in \Theta_0$ and $\theta_1 \in \Theta_1$; that is, the power is always higher at any alternative than at any null value.

Let $\Phi$ be a set of tests for $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$. We say that $\phi^* \in \Phi$ is the uniformly most powerful (UMP) test among $\Phi$ at level $\alpha$ if it is level $\alpha$ and for any other level $\alpha$ test $\phi \in \Phi$, we have $\beta_{\phi^*}(\theta) \geq \beta_\phi(\theta)$, for all $\theta \in \Theta_1$. If $\Theta_1$ has cardinality one, we simply say that $\phi$ is the most powerful test.

Classically, randomized tests appear in the Neyman-Pearson Lemma and their role in that setting is to allow a test to achieve a specified size. However, for privacy, we require that all of our tests are randomized, and use the randomness to achieve differential privacy.

Usually, rather than a binary accept/reject decision from a randomized test, it is preferable to report a $p$-value, which gives a continuous measure of how much evidence there is for the alternative hypothesis over the null. Smaller values of $p$ give more evidence for the alternative.

DEFINITION A.2 ($p$-Value). Let $X \in \mathcal{X}$ be distributed $X \sim P_\theta$, where $\theta \in \Theta$. Let $\Theta_0, \Theta_1$ be a partition of $\Theta$. Let $p$ be a random variable, taking values in $[0, 1]$. Define $p(X) := p|X$ to be the random variable $p$ conditioned on $X$. We say that $p$ is a $p$-value for the test $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta \in \Theta_1$ if

$$\sup_{\theta_0 \in \Theta_0} P_{\theta_0}(p(X) \leq \alpha) \leq \alpha,$$

where the probability is over both $p$ and $X$. In other words, for every $\theta \in \Theta_0$, the distribution of $p(X)$ stochastically dominates $U(0, 1)$.

A $p$-value represents the probability of observing data as extreme or more extreme as the present sample, when the null hypothesis is true. Often the measure of “extreme” is based on a specific test statistic. A small $p$-value offers evidence that the present sample is unlikely to have been generated by the null model.

Given a $p$-value $p(X)$, $\phi(X) = P(p(X) < \alpha \mid X)$ is a test for the same hypothesis, at level $\alpha$. For each $\alpha$, let $\phi_\alpha : \mathcal{X} \to [0, 1]$ be a test at level $\alpha$. Let $U \sim U[0, 1]$. Then $p(X) = \inf\{\alpha \mid \phi_\alpha \geq U\}$ is a $p$-value for the same test. See Geyer and Meeden (2005) for a deeper understanding of randomized tests, $p$-values, and confidence sets in terms of fuzzy set theory.
APPENDIX B: MOST POWERFUL $f$-DP TEST AS CONVEX OPTIMIZATION

In this section, we show that for an arbitrary null hypothesis, and a simple alternative hypothesis, there exists a most powerful $\alpha$-level $f$-DP test, which can be expressed as the solution to a convex optimization problem. This result is an extension of Awan and Slavković (2018, Remark 3.1), which showed that in the case of $(\epsilon, \delta)$-DP the most powerful test is the solution to a linear program.

**PROPOSITION B.1.** Let $\Theta$ be a set of parameters, and $\{P_{\theta} : \theta \in \Theta\}$ be a set of distributions on $X^n$. Let $\Theta_0 \subset \Theta$ and $\theta_1 \in \Theta \setminus \Theta_0$. Then a most powerful $\alpha$-level $f$-DP for $H_0 : \theta \in \Theta_0$ versus $H_1 : \theta = \theta_1$ is the solution to a convex optimization problem.

**PROOF.** First, note that the $f$-DP constraint on tests: $0 \geq \phi(x') - 1 + f(\phi(x))$ is a convex constraint, since $f$ is convex. Furthermore, the type I error constraints $\mathbb{E}_{P_{\theta_0}} \phi(x) \leq \alpha$ are linear and hence convex. The intersection of the privacy constraints and the type I error constraints is thus a convex set. This set is non-empty as the constant test $\phi(x) = c$ lies inside the set for all $c \in [0, \alpha]$. Finally, the power $\mathbb{E}_{P_{\theta_1}} \phi(x)$ is a linear objective.

APPENDIX C: DIFFERENCE-OF-PROPORtIONS NON-PRIVATE UMPU

Suppose we observe $X_i \sim \text{Bern}(\theta_X)$ for $i = 1, \ldots, n$ and $Y_j \sim \text{Bern}(\theta_Y)$ for $j = 1, \ldots, m$, and we wish to test $H_0 : \theta_X \leq \theta_Y$ versus $H_1 : \theta_X < \theta_Y$. Denote $X = \sum_{i=1}^n X_i$ and $Y = \sum_{j=1}^m Y_j$. The joint distribution of $(X_i, Y_j)_{i,j}$ is

$$f_{\theta_X, \theta_Y}(x, y) = \prod_{i=1}^n \theta_X^{x_i} (1 - \theta_X)^{1-x_i} \prod_{j=1}^m \theta_Y^{y_j} (1 - \theta_Y)^{1-y_j},$$

$$= \theta_X^{\sum_{i=1}^n x_i} (1 - \theta_X)^{n - \sum_{i=1}^n x_i} \theta_Y^{\sum_{j=1}^m y_j} (1 - \theta_Y)^{m - \sum_{j=1}^m y_j},$$

$$= (1 - \theta_X)^n (1 - \theta_Y)^m \exp \left( \sum_{i=1}^n x_i \log \left( \frac{\theta_X}{1 - \theta_X} \right) + \sum_{j=1}^m y_j \log \left( \frac{\theta_Y}{1 - \theta_Y} \right) \right).$$

By relabeling $\eta_0 = \log \left( \frac{\theta_X}{\theta_Y} \right)$ and $\eta_1 = \log \left( \frac{\theta_Y}{1 - \theta_Y} \right) - \eta_0$, and setting $x = \sum_{i=1}^n x_i$ and $y = \sum_{j=1}^m y_j$, we can write

$$f_{\theta_X, \theta_Y}(x, y) = (1 - \theta_X)^n (1 - \theta_Y)^m \exp \left( (x + y) \eta_0 + y \eta_1 \right),$$

and from this expression we see that $\eta_0$ and $\eta_1$ are natural exponential family parameters for the sufficient statistics $\{(X+Y), Y\}$. We can also re-express our test $H_0 : \theta_X \geq \theta_Y$ versus $H_1 : \theta_X < \theta_Y$ as $H_0 : \eta_1 \leq 0$ versus $H_1 : \eta_1 > 0$. This now fits the assumptions of Schervish (2012, Theorem 4.124). Since $Y \mid X + Y = z$ has a monotone likelihood ratio in $\eta_1$, we know that the UMP unbiased test for the above hypothesis is of the form

$$\phi(X,Y) = \begin{cases} 
0 & Y < c \\
\alpha & Y = c \\
1 & Y > c 
\end{cases}$$

where $c$ and $\alpha$ depend on the value of $X + Y = z$, and are chosen such that $\mathbb{E}_{\theta_X = \theta_Y}(\phi \mid X + Y = z) = \alpha$.

In fact there is a more convenient formulation of this test, which gives exact $p$-values. First note that $\phi(X,Y)$ can be written in the form $\phi(X,Y) = F_V(Y - \delta')$, where $F_V(\cdot)$ is the cdf...
of \( U \sim \text{Unif}(-1/2,1/2) \), and \( c' \) is a real number, which depends on \( z = X + Y \). Then we can write

\[
\phi(X,Y) = F_U(Y - c') = P(U \leq Y - c'(z) \mid X,Y) = P(c' \leq Y + U \mid X,Y)
\]

From the last equality, we see that the UMPU test depends on the (random) test statistic \( T = Y + U \), and on the value \( z = X + Y \). The \( p \)-value corresponding to \( T \) is

\[
p = P(Y + U \geq T \mid T, X + Y = z) = P(Y - T \geq U \mid T, X + Y = z) = \mathbb{E}[F_U(Y - T) \mid T, X + Y = z]
\]

This \( p \)-value can be computed fairly efficiently, since the expected value is over the \( m \) hyper-geometric values of \( Y \) given \( z \). Lastly, to check that this \( p \)-value agrees with the UMPU, we want to show that \( P(p(T, z) \leq \alpha \mid T, X + Y = z) = \phi(X,Y) \). To this end,

\[
P(p(T, z) \leq \alpha \mid T, X + Y = z) = P(1 - F_{Y + U|z}(T) \leq \alpha \mid T, z)
\]

\[
= P(1 - \alpha \leq F_{T|z}(T) \mid T, z)
\]

\[
= P(F_{T|z}^{-1}(1 - \alpha) \leq T \mid T, z)
\]

\[
= P(F_{T|z}^{-1}(1 - \alpha) \leq Y + U \mid Y, z)
\]

\[
= P(c \leq Y + U \mid Y, z)
\]

\[
= \phi(X,Y)
\]

where \( c = F_{T|z}^{-1}(1 - \alpha) \) is a constant, which only depends on \( z \).

**APPENDIX D: NON-EXISTENCE OF UMPU IN DIFFERENCE-OF-PROPORTIONS**

In this section, we give a simple example demonstrating that there is no UMP unbiased \( f \)-DP test for the problem of Section 5. In particular, we work with \((\epsilon, 0)\)-DP.

Suppose that \( m = 1 \) and \( n = 2 \). Then \( Y \sim \text{Binom}(1, \theta_Y) \) and \( X \sim \text{Binom}(2, \theta_X) \). we consider unbiased tests which satisfy \((1,0)\)-DP, at level .05. Equation (2) imposes the following constraints on a test \( \phi(x,y) \):

\[
\phi(2,1) = .05
\]

\[
\phi(0,0) = .05
\]

\[
(1/3)\phi(0,1) + (2/3)\phi(1,0) = .05
\]

\[
(2/3)\phi(1,1) + (1/3)\phi(2,0) = .05
\]

1) Suppose that \( \theta_X = 0 \) and \( \theta_Y = 1 \). The following test maximizes the power in this case

\[
\begin{pmatrix}
0 & .05 & .05 e^{-1} & .05 \\
1 & .15 & .1 e^{-1} & .05 & .05 \\
x : & 0 & 1 & 2
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & .05 & .05 e^{-1} & .05 \\
1 & .15 & .1 e^{-1} & .05 & .05 \\
x : & 0 & 1 & 2
\end{pmatrix}
\]

We can see that maximizing the power is equivalent to maximizing the value of \( \phi(0,1) \), as \( P(X = 0, Y = 1) = 1 \). Increasing \( \phi(0,1) \) any further, would require decreasing \( \phi(1,0) \). But for privacy we require \( \phi(1,0) \geq \exp(-1)\phi(0,0) \), which is tight. The other privacy constraints
can be easily verified. So, the above test is the most powerful unbiased test for \( \theta_X = 0 \), \( \theta_Y = 1 \).

2) Suppose that \( \theta_X = 1/2 \) and \( \theta_Y = 1 \), and consider the following test:

\[
\begin{pmatrix}
    y = 0 & 0.05 & 0.075e^{-1} - 0.025e^{-2}e^{-1} \cdot 0.05 \\
    y = 1 & 0.15(1 - e^{-1}) + 0.05e^{-2} & 0.075 - 0.025e^{-1} \\
    x: & 0 & 1 & 2
\end{pmatrix}
\approx
\begin{pmatrix}
    y = 0 & 0.05 & 0.0242 & 0.184 \\
    y = 1 & 0.1016 & 0.0658 & 0.05 \\
    x: & 0 & 1 & 2
\end{pmatrix}
\]

It can be verified that this test satisfies the constraints of Equation (4) as well as the \( \epsilon \)-DP constraints. Note that the power formula for \( \theta_X = 1/2 \) and \( \theta_Y = 1 \) is

\[
0.5^2[\phi(0, 1) + 2\phi(1, 1) + \phi(2, 1)],
\]

and we see that the test above has higher power compared to the test from part 1). Since the test in part 1) was most powerful unbiased 1-DP test for \( \theta_X = 0 \) and \( \theta_Y = 1 \), it is not most powerful unbiased 1-DP test for \( \theta_X = 1/2 \) and \( \theta_Y = 1 \), we conclude that there is no uniformly most powerful unbiased 1-DP test in this setting.

APPENDIX E: ALTERNATIVE DP TESTS

In this section, we describe the other DP tests that appear in the simulations of Section 5.3.

In Algorithm 2, we describe an \( \epsilon \)-DP normal approximation test, proposed by Karwa and Vadhan (2018). This test is analogous to the test of a single population proportion described in Vu and Slavković (2009), where a normal approximation with inflated variance is used to approximate the sampling distribution. This test adds independent Laplace noise to \( X \) and \( Y \), bases the test statistic on the difference of the estimated proportions, estimates the variance of the test statistic using a plug-in estimate, and then approximates the sampling distribution of the test statistic as normal. In Algorithm 2, \( \Phi \) denotes the cdf of \( N(0, 1) \).

**Algorithm 2: \( \epsilon \)-DP Normal approximation p-value.**

**Data:** Let \( X, Y, m, \) and \( n \) be given. Let \( \epsilon > 0 \) be given.

1. Draw \( L_1, L_2 \overset{iid}{\sim} \text{Laplace}(0, 1/\epsilon) \);
2. Set \( \tilde{X} = X + L_1 \) and \( \tilde{Y} = Y + L_2 \);
3. Set \( \tilde{\theta} = \min\{\max\{\frac{\tilde{X} + \tilde{Y}}{m+n}, 0\}, 1\} \);
4. Set \( T = \frac{\tilde{Y}}{m} - \frac{\tilde{X}}{n} \);
5. Set \( \text{var} = \tilde{\theta}(1 - \tilde{\theta}) + \frac{2}{(m+n)^2} + \frac{2}{(m+n)^2} \);

**Result:** p-value: \( p = 1 - \Phi(T/\sqrt{\text{var}}) \), \( \tilde{X}, \tilde{Y} \).

Another DP test would be to take the semiprivate UMPU test of Section 5.1, and using composition, produce both a privatized test statistic as well as a private estimate of the value \( Z = X + Y \). Then plugging in the estimate of \( Z \) gives a fully \( \epsilon \)-DP version of the semiprivate...
Note that $F$ represents the cdf of the variables Tulap(0, $\exp(-\epsilon/2)$, 0).

Algorithm 3: $\epsilon$-DP plug-in $p$-value.

**Data:** Let $X$, $Y$, $m$, and $n$ be given. Let $\epsilon > 0$ be given.
1. Draw $L_1, L_2 \sim \text{Tulap}(0, \exp(-\epsilon/2), 0)$;
2. Set $\tilde{T} = (Y - X) + L_1$ and $\tilde{Z} = (X + Y) + L_2$;
3. Set $T = \tilde{Y}/m - \tilde{X}/n$;

**Result:** $p$-value: $p = \mathbb{E}_{H \sim \text{Hyper}(m, n, \tilde{Z})} F(2H - \tilde{Z})$

### APPENDIX F: PROOFS

The Galois inequalities are a well-known property of cdfs and their quantile functions. We include a short proof for completeness.

**Lemma F.1** (Galois Inequalities). Let $F$ be a cdf and $F^{-1}(p) := \inf \{x \mid p \leq F(x)\}$ be its quantile function. Then $F^{-1}(p) \leq x$ if and only if $p \leq F(x)$.

**Proof.** Suppose that $p \leq F(x)$. This holds if and only if $x \in \{t \mid p \leq F(t)\}$. Since $F$ is monotone increasing, $\{t \mid p \leq F(t)\}$ is of the form $[F^{-1}(p), \infty)$. Therefore, $x \in \{t \mid p \leq F(t)\}$ holds if and only if $x \geq \inf \{t \mid p \leq F(t)\} = F^{-1}(p)$.

Let $F$ be a cdf. We say that $F$ is invertible at $t \in \mathbb{R}$ if $F^{-1} \circ F(t) = t$. Similarly, we say that a symmetric tradeoff function $f$ is invertible at $\alpha \in [0, 1]$ if $f^{-1} \circ f(\alpha) = \alpha$ (or equivalently, $f \circ f(\alpha) = \alpha$).

**Lemma F.2.** Let $f$ be a symmetric tradeoff function and let $F$ be a cdf. Then

1. $f$ is invertible for all $\alpha \in [0, f(0)]$ and $f \circ f(\alpha) \leq \alpha$ otherwise.
2. If $F$ is continuous, then $F \circ F^{-1}(p) = p$ for all $p \in [0, 1]$.
3. $F^{-1} \circ F(t) \leq t$.

**Proof.** 1. Since, $f^{-1}(\alpha) = \inf \{t \mid f(t) \leq \alpha\}$, we have that $f^{-1}(f(\alpha)) = \inf \{t \mid f(t) \leq f(\alpha)\}$. We notice that $\alpha \in \{t \mid f(t) \leq f(\alpha)\}$, and so we have that $f^{-1}(f(\alpha)) = \inf \{t \mid f(t) \leq f(\alpha)\} \leq \alpha$.

   Next, notice that because $f$ is a tradeoff function, it is convex, decreasing, and $f(1) = 0$. This implies that the only possibility for $f(a) = f(b)$ is either $a = b$ or $f(a) = f(b) = 0$. Hence, $f$ is invertible on $[0, f^{-1}(0)] = [0, f(0)]$.

2. Since $F$ is continuous, by the Intermediate Value Theorem, for any $p \in (0, 1)$, there exists $x \in \mathbb{R}$ such that $p = F(x)$. So, we can write $F^{-1}(p) = \inf \{t \mid p = F(t)\}$. Since $F$ is continuous, $\{t \mid p = F(t)\}$ is a closed set, and so we have that $F^{-1}(p) \in \{t \mid p = F(t)\}$. The result follows: $F(F^{-1}(p)) = p$. If $F^{-1}(0) = -\infty$ or $F^{-1}(1) = \infty$, we can allow $F$ to take as input $-\infty$ and $+\infty$, with $F(-\infty) = 0$ and $F(\infty) = 1$. Then we get $F(F^{-1}(p)) = p$ when $p \in \{0, 1\}$ as well.

3. Note that $F^{-1} \circ F(t) = \inf \{x \mid F(t) \leq F(x)\}$. We see that $t \in \{x \mid F(t) \leq F(x)\}$, so $F^{-1} \circ F(t) = \inf \{x \mid F(t) \leq F(x)\} \leq t$.

Lemma F.3 is a technical lemma establishing that the patterns of invertibility/non-invertibility of a CND $F$ satisfy a recurrence.
**Lemma F.3.** Let $F$ be a CND for some tradeoff function $f$, and call $(-U,U) = [F^{-1}(0), -F^{-1}(0)]$ (which may be $(-\infty, \infty)$). Suppose for contradiction that there exists $-U < a < b < U$ such that $F(a) = F(b)$. Then $F(a + 1) = F(b + 1)$ and $F(a - 1) = F(b - 1)$.

**Proof.** We use a different parametrization for the proof. Suppose that there exists $-U < a - 1 < b - 1 < U$ such that $F(a - 1) = F(b - 1)$ but that $F(a) < F(b)$. Then there exists $[c, d] \subset [a, b]$ such that $F$ is invertible for all $t \in [c, d]$, since $F$ is a continuous cdf. To see this, let $A = \{ t \mid \exists x \neq t \text{ s.t. } F(x) = F(t) \}$, which we identify as the union of disjoint closed sets. Then $\mathbb{R} \setminus A$ is an open set. Because $F$ is continuous, $\mathbb{R} \setminus A$ is non-empty, and so there exists the desired interval $[c, d] \subset \mathbb{R} \setminus A$.

Call $\alpha_c = 1 - F(c)$ and $\alpha_d = 1 - F(d)$. Then $\alpha_c > \alpha_d$. Note that

$$F^{-1}(1 - \alpha_c) = F^{-1}(F(c)) = c,$$

$$F^{-1}(1 - \alpha_d) = F^{-1}(F(d)) = d,$$

since $F$ is invertible on $[c, d]$. Then $f(\alpha_c) = F(F^{-1}(1 - \alpha_c) - 1) = F(c - 1)$ and $f(\alpha_d) = F(F^{-1}(1 - \alpha_d) - 1) = F(d - 1)$ because $F$ is a CND for $f$, and using the above identities. However, since $F(a - 1) = F(b - 1)$, and $F$ is monotone, we have that $F(c - 1) = F(d - 1)$, which implies that $f(\alpha_c) = f(\alpha_d)$. Earlier we noted that $\alpha_c > \alpha_d$; since tradeoff functions are decreasing and convex, the only possibility for $f(\alpha_c) = f(\alpha_d)$ is for $f(\alpha_c) = f(\alpha_d) = 0$. Now let $t \in [c, d]$. Just as above, we denote $\alpha_t = 1 - F(t)$, which satisfies $t = F^{-1}(1 - \alpha_t)$ and $f(\alpha_t) = 0$ just like above. Consider the following inequalities, where each line is implied by the line above:

1. $F(F^{-1}(1 - \alpha_t) - 1) = f(\alpha_t) = 0$
2. $F^{-1}(1 - \alpha_t) - 1 \leq -U$
3. $F^{-1}(1 - \alpha_t) \leq 1 - U$
4. $(1 - \alpha_t) \leq F(1 - U)$
5. $1 - F(1 - U) \leq \alpha_t = 1 - F(t)$
6. $F(t) \leq F(1 - U)$
7. $t \leq F^{-1}(F(1 - U)) \leq 1 - U,$

where (5) used our observation that $f(\alpha_c) = f(\alpha_d) = 0$, the monotonicity of $f$, and the fact that $F$ is a CND for $f$, (6) used the fact that $-U = F^{-1}(0)$ and part 3 of Lemma F.2, Lemma F.1 gives (8), we used the fact that $\alpha_t = 1 - F(t)$ for (9), and part 3 of Lemma F.2 for (11). We then have that $t - 1 \leq -U$, which implies that $a - 1 \leq c - 1 \leq t - 1 \leq -U$, which contradicts the assumption that $-U < a - 1$. We conclude that $F(a) = F(b)$. Using the parametrization in the Lemma statement, we have that if there exists $-U < a < b < U$ such that $F(a) = F(b)$, then $F(a + 1) = F(b + 1)$.

Now suppose that $-U < a < b < U$ such that $F(a) = F(b)$, and we will show that $F(a - 1) = F(b - 1)$. By symmetry, we have that $F(-b) = F(-a)$ and $-U < -b < -a < U$. By the above work, we have that $F(-b - 1) = F(-a + 1)$. Applying symmetry again, we have $F(b - 1) = F(a - 1)$, which establishes the result.

**Lemma 3.6.** Let $f$ be a symmetric nontrivial tradeoff function and let $F$ be a CND for $f$. Then $F(x) = 1 - f(F(x - 1))$ when $F(x - 1) > 0$ and $F(x) = f(1 - F(x + 1))$ when $F(x + 1) < 1$. 


PROOF. We prove the first recurrence in detail and remark that the second recurrence is obtained by a similar argument. We know that \( f(\alpha) = F(F^{-1}(1 - \alpha) - 1) \) for all \( \alpha \in (0, 1) \). Assume that \( F(x - 1) \in (0, 1) \). If \( F \) is invertible at \( 1 - x \), then plugging in \( \alpha = F(x - 1) \) gives

\[
\begin{align*}
f(F(x - 1)) &= F(F^{-1}(1 - F(x - 1)) - 1) \\
&= F(F^{-1}(F(1 - x)) - 1) \\
&= F(-x) \\
&= 1 - F(x),
\end{align*}
\]

where we used the fact that \( F \) is symmetric and that \( F \) is invertible at \( 1 - x \).

If \( 1 - x \) is not an invertible point of \( F \), then it lies in the interval \([a, b] := [\inf\{t \mid F(1 - x) = F(t)\}, \sup\{t \mid F(1 - x) = F(t)\}] \). Note that \( F \) is constant on this interval, and \( F \) is invertible at \( a \). By Lemma F.3, we have that \( F \) is also constant on the interval \([a - 1, b - 1] \), which contains \(-x\). Then

\[
\begin{align*}
f(F(x - 1)) &= f(1 - F(1 - x)) \\
&= f(1 - F(a)) \\
&= F(F^{-1}(F(a)) - 1) \\
&= F(a - 1) \\
&= F(-x) \\
&= 1 - F(x),
\end{align*}
\]

where for (13) we use the fact that \( F(1 - x) = F(a) \), for (14) we use the fact that \( F \) is a CND for \( f \), for (15) we use the invertibility of \( F \) at \( a \), and for (16) we use the facts that \(-x \in [a - 1, b - 1]\) and \( F \) is constant on \([a - 1, b - 1] \).

Finally, for the case that \( F(x - 1) = 1 \), we have that \( F(x) = 1 \) since \( F \) is increasing, and \( 1 - f(F(x - 1)) = 1 - f(1) = 1 - 0 = 1 \) and we see that the recurrence holds for this case as well. \( \square \)

**Lemma F.4.** Let \( f \) be a nontrivial symmetric tradeoff function, and let \( c \) be the fixed point of \( f \). Then

1. \( c \in [0, 1/2] \),
2. iteratively applying \( 1 - f(\cdot) \) to any point in \((c, 1]\) approaches 1 in the limit.

**Proof.** Since \( f(\alpha) < 1 - \alpha \) for some \( \alpha \), it follows that \( c < 1/2 \): Suppose to the contrary that \( f(1/2) = 1/2 \); by symmetry and convexity, \( f(\alpha) = 1 - \alpha \) for all \( \alpha \).

Then the function \( 1 - f(\cdot) \) is concave, increasing, and has slope < 1 on the set \((c, 1]\). By the mean value inequality (Shifrin, 2005, Proposition 1.3, Chapter 6.1), we have that \( 1 - f(\cdot) \) is a contraction map on \((c, 1]\). By the contraction mapping theorem (Shifrin, 2005, Theorem 1.2, Chapter 6.1), \( 1 - f(\cdot) \) has a unique fixed point on \((c, 1]\). By definition of \( f \) as a tradeoff function, we know that \( 1 - f(1) = 1 \), so the value 1 must be the unique fixed point. The contraction mapping theorem also tells us that iteratively applying \( 1 - f(\cdot) \) to any point in \((c, 1]\) approaches the fixed point 1 in the limit. \( \square \)

**Proposition 3.8.** Let \( f \) be a symmetric nontrivial tradeoff function, and let \( F := F_f \). Then
1. $F(x)$ is a cdf for a symmetric, continuous, real-valued random variable.
2. $F(x)$ satisfies $F(x) = 1 - f(F(x-1))$ whenever $F(x-1) > 0$ and $F(x) = f(1 - F(x+1))$ whenever $F(x+1) < 1$.
3. $F'(x)$ is decreasing on $(-1/2, \infty)$ and increasing on $(-\infty, 1/2)$.
4. $F(x)$ is strictly increasing on $\{ x \mid 0 < F(x) < 1 \}$.

**Proof.** 1. First we will show that $F$ is a continuous cdf, which represents a symmetric random variable. We need to verify the following properties:

a) $F(x)$ takes values in $[0,1]$:

First note that $c \in [0,1]$, since $f : [0,1] \to [0,1]$. Then $F(x) \in [0,1]$ for $x \in [-1/2, 1/2]$. Finally, as $f$ takes values in $[0,1]$, by the recurrence relation of Definition 3.7, we have that $F(x) \in [0,1]$ for all $x \in \mathbb{R}$.

b) $F(x) = 1 - F(-x)$:

For values $x \in [-1/2, 1/2]$, it is easy to verify that $F(x) = 1 - F(-x)$. Now assume that the relation $F(x) = 1 - F(-x)$ holds on an interval $[-a,a]$ for some $a \geq 1/2$. Let $x \in [a,a+1]$. Then $F(x) = 1 - f(F(x-1)) = 1 - f(1 - F(-x+1)) = 1 - F(-x)$. By a symmetric argument, we have that the relation now holds on $[-a-1, a+1]$. By induction, we conclude that $F(x) = 1 - F(-x)$ on $\mathbb{R}$.

c) $F(x)$ is continuous:

First note that $F(x)$ is continuous on $(-1/2, 1/2)$. Next, as $f$ is convex, it is continuous on $(0,1)$. So, we have that $F(x)$ is continuous everywhere except potentially at half-integer values. We can verify that $F$ is continuous at 1/2: $\lim_{x \to 1/2} F(x) = F(1/2) = 1 - c$, where $\lim_{x \to 1/2} F(x) = \lim_{x \to 1/2} 1 - f(F(x-1)) = 1 - f(c) = 1 - c$, where we used the fact that $f$ is continuous on $(0,1)$. Now, assume that $F$ is continuous at a half integer value $x \geq 1/2$. Then $\lim_{y \to x+1} F(y) = \lim_{y \to x+1} 1 - f(F(y-1)) = 1 - f(F(x)) = F(x+1)$. By induction, continuity holds on $[0, \infty)$. By symmetry (b), we have continuity on $\mathbb{R}$.

d) $F'(x)$ is defined almost everywhere, and $F(x)$ is increasing:

Note that $F(x)$ is differentiable and $F'(x) > 0$ on $(-1/2, 1/2)$. As $f$ is convex it is differentiable almost everywhere on $(0,1)$. Applying the recurrence relation, we have that $F(x)$ is differentiable a.e. on $(n - 1/2, n + 1/2)$ for all $n \in \mathbb{Z}$. We conclude that $F(x)$ is differentiable a.e., as a countable union of measure zero sets has measure zero. As $F(x)$ is continuous, it suffices to verify that $F'(x) \geq 0$ almost everywhere. Let $x \geq 1/2$ such that $x \in \mathbb{R} \backslash (\mathbb{Z} + 1/2)$ and both $F'(x-1)$ and $F'(x-1)$ are defined. Then $\frac{d}{dx} F(x) = \frac{d}{dx} (1 - f(F(x-1))) = -f'(F(x-1)) F'(x-1)$. For induction, we assume that $F'(x-1) \geq 0$. As $f'(y) \leq 0$ for all $y \in (0,1)$ where $f'$ is defined, we have that $F'(x) \geq 0$. By symmetry and induction, we have that $F'(x) \geq 0$ almost everywhere. Thus, $F$ is increasing.

e) $\lim_{x \to \infty} F(x) = 1$ and $\lim_{x \to -\infty} F(x) = 0$:

By symmetry, it suffices to show that $\lim_{x \to \infty} F(x) = 1$. By Lemma F.4, we have that $[1-c,1] \subset (c,1]$, and that iteratively applying $1 - f(\cdot)$ to any point in $[1-c,1]$ approaches the fixed point 1 in the limit.

Consider the sequence $Y_n$, where $Y_1 = 1 - c$, and $Y_n = 1 - f(Y_{n-1})$. Note that $Y_n = F(n-1/2)$ for $n \in \mathbb{Z}^+$. Then as $F$ is bounded and increasing, $\lim_{n \to \infty} F(x) = \lim_{n \to \infty} Y_n = 1$, where in the last equality, we used the fact that $Y_n$ is constructed by iteratively applying $1 - f(\cdot)$ and applying Lemma F.4.

2. First we will check that $1 - f(F(-1/2)) = 1 - f(c) = 1 - c = F(1/2)$. Let $x \in \mathbb{R}$ such that $F(x-1) > 0$. If $x \geq 1/2$ then we have $F(x) = 1 - f(F(x-1))$ by construction. If $x < 1/2$, then $x - 1 < -1/2$. So, $F(x-1) = f(1 - F(x))$ by construction. We will apply $f$ to both sides of this last equation. By part 1 of Lemma F.2, to justify that $f \circ f(1-
\( F(x) = 1 - F(x) \), we need to show that \( 1 - F(x) \leq f(0) \). However, if \( 1 - F(x) > f(0) \), then \( F(x - 1) = f(1 - F(x)) = 0 \), which contradicts our earlier assumption. So, applying \( f \), we obtain \( f(F(x - 1)) = 1 - F(x) \). The other recurrence holds by a similar argument due to the symmetries of \( F \) and \( f \).

3. By symmetry, it suffices to check only that \( F'(x) \) is decreasing on \((-1/2, \infty)\). Note that it holds trivially on \((-1/2, 1/2)\), since \( F \) is linear on this interval. For \( x \geq 1/2 \) a half integer, \( F' \) is decreasing on \((x, x + 1)\), since \( 1 - f(\cdot) \) is a concave function.

At \( 1/2 \), we check the two limits \( \lim_{y \downarrow 1/2} F'(y) = -c + 1 - c = 1 - 2c \) and \( \lim_{y \uparrow 1/2} F'(y) = \lim_{y \downarrow 1/2} F'(y - 1) \lim_{y \downarrow 1/2} F' (y - 1) = \lim_{z \downarrow c} -f'(z)(1 - 2c) \leq 1 - 2c \), where we use the fact that \( -f'(y) \leq 1 \) for all \( y \geq c \), since \( c \) is the point of symmetry.

Now for induction, suppose that for some half integer \( x \geq 1/2 \), we have \( \lim_{y \downarrow x} F'(y) \leq \lim_{y \uparrow x} F'(y) \). Then

\[
\lim_{y \downarrow x+1} F'(y) = \lim_{y \uparrow x} F'(y + 1) \\
= \lim_{y \uparrow x} -f'(F(y)) F'(y) \\
= \lim_{y \downarrow x} -f'(F(y)) \lim_{y \uparrow x} F'(y) \\
\geq \lim_{y \downarrow x} -f'(F(y)) F'(y) \\
= \lim_{y \downarrow x} -f'(F(y)) F'(y) \\
= \lim_{y \downarrow x} F'(y) \\
= \lim_{y \downarrow x+1} F'(y),
\]

where the inequality used the inductive hypothesis as well as the fact that \( -f'(F(y)) \) is positive and decreasing in \( y \).

4. We will establish that \( F \) is strictly increasing within its support \( \{x \mid F(x) \in (0, 1)\} \). It is non-decreasing and continuous by property 1. Suppose that \( F \) is constant on an interval \((a, b) \subset \mathbb{R}\). Then \( F'(x) = 0 \) on \((a, b)\). By construction, we know that \( F \) is strictly increasing on \((-1/2, 1/2)\). By symmetry of \( F \), we may assume that \((a, b) \subset (1/2, \infty)\).

However, property 3 states that \( F'(x) \) is weakly decreasing on \((1/2, \infty)\). But then \( F \) must be constant on \((a, \infty)\). This implies that \((a, b) \not\subset \{x \mid F(x) \in (0, 1)\} \).

\[ \square \]

Lemma F.5 is a technical lemma that is important for the proof of Theorem 3.9.

**Lemma F.5.** Let \( X \sim F \) be a real-valued continuous random variable which is symmetric about zero. Let \( m > 0 \) be given. Let \( f \) be an arbitrary symmetric tradeoff function. Then

1. \( T(F(\cdot), F(\cdot - m)) = T(F(\cdot - m), F(\cdot)) \) or equivalently \( T(X, X + m) = T(X + m, X) \).
2. To verify \( f \leq T(F(\cdot), F(\cdot - m)) \), it suffices to check that \( f(\mathbb{E}_F \phi(X)) \leq 1 - \mathbb{E}_F (F(x - m)) \phi(X) \), where \( \phi(x) \) is either of the form \( I(F(x - m)/F(x) > k) \) for \( k \geq 1 \) or \( I(F(x - m)/F(x) \geq k) \) for \( k \geq 1 \).

**Proof.** 1. Note that the mapping \( g(t) = -t + m \) is a bijection, hence applying it to both entries preserves the tradeoff function: \( T(X, X + m) = T(-X + m, -X) = T(X + m, X) \), where the last equality uses the fact that \( -X \overset{d}{=} X \), as \( X \) is symmetric.
2. Since $F$ is a continuous cdf, the derivative $F'$ is defined almost everywhere, and $F'$ is a pdf for $F$. We will denote $F_m(x) = F(x - m)$ and $F_0(x) = F(x)$. Then, the Neyman-Pearson Lemma tells us that the optimal test is $\phi^*(x) = (1 - \alpha)\phi_{>k}(x) + \alpha\phi_{\geq k}(x)$, for some $\alpha \in [0, 1]$ and $k \in \mathbb{R}^\geq 0$, where $\phi_{>k}(x) = I\left(\frac{F_m(x)}{F_0(x)} > k\right)$ and $\phi_{\geq k}(x) = I\left(\frac{F_m(x)}{F_0(x)} \geq k\right)$.

The points of the tradeoff function consist of the type I and type II error of the test $\phi$. Since both type I and type II error of $\phi^*$ are linear in $\alpha$, we have that the tradeoff function is linear on the interval $[\mathbb{E}_F\phi_{>k}, \mathbb{E}_F\phi_{\geq k}]$. Since the tradeoff function $f$ is a convex function, it is upper bounded by secant lines; so, it suffices to check that $f(\mathbb{E}_F\phi(X)) \leq 1 - \mathbb{E}_F(-\phi)(X)$ for $\phi = \phi_{>k}$ or $\phi = \phi_{\geq k}$.

Next, we argue that we need only consider $\phi_{>k}$ for $k \geq 1$ and $\phi_{\geq k}$ for $k > 1$. We know from part 1 that $T(F(x), F(x - m))$ is symmetric. So, we need to verify that these tests fully specify the tradeoff function up to the point of symmetry, or equivalently up until the fixed point of $T(F(x), F(x - m))$. Call

$$p = P_{F_0}\left(\frac{F_m'(x)}{F_0'(x)} = 1\right)$$

$$= \int I\left(\frac{F_m'(x)}{F_0'(x)} = 1\right) F_0'(x) \, dx$$

$$= \int I\left(\frac{F_m'(x)}{F_0'(x)} = 1\right) F_m'(x) \, dx$$

$$= P_{F_m}\left(\frac{F_m'(x)}{F_0'(x)} = 1\right).$$

If $p > 0$, then it suffices to verify that between the points corresponding to the tests $\phi_{>1}$ and $\phi_{\geq 1}$, the tradeoff function has slope -1. This is sufficient since the point of symmetry of a symmetric tradeoff function has -1 as a sub-derivative, and by concavity the derivative is increasing. Then

$$\mathbb{E}_{F_0}\phi_{\geq 1} = \mathbb{E}_{F_0}\phi_{>1} + P_{F_0}\left(\frac{F_m'(x)}{F_0'(x)} = 1\right) = \mathbb{E}_{F_0}\phi_{>1} + p,$$

$$1 - \mathbb{E}_{F_m}\phi_{\geq 1} = 1 - \mathbb{E}_{F_m}\phi_{>1} - P_{F_m}\left(\frac{F_m'(x)}{F_0'(x)} = 1\right) = 1 - \mathbb{E}_{F_m}\phi_{>1} - p.$$

We see that the slope is $-p/p = -1$.

If $p = 0$, then $\mathbb{E}_{F_0}\phi_{>1} = \mathbb{E}_{F_0}\phi_{>k}$ and $\mathbb{E}_{F_m}\phi_{>1} = \mathbb{E}_{F_m}\phi_{\geq 1}$. We will show that the type I and type II errors are equal for the test $\phi_{>1}$, and hence $\mathbb{E}_{F}\phi_{>1}$ is the fixed point of $T(F_0, F_m)$. For an indeterminate $x$, call $y = m - x$. Then,

$$\mathbb{E}_{F_0}\phi_{>1} = \int I\left(\frac{F'(x - m)}{F'(x)} \geq 1\right) F'(x) \, dx$$

$$= \int I\left(\frac{F'(-y)}{F'(m - y)} \geq 1\right) F'(m - y) \, dx$$

$$= \int I\left(\frac{F'(y)}{F'(y - m)} \geq 1\right) F'(y - m) \, dx$$

$$= 1 - \int I\left(\frac{F'(y - m)}{F'(y)} < 1\right)$$

$$= 1 - \mathbb{E}_{F_m}\phi_{>1}$$

$$= 1 - \mathbb{E}_{F_m}\phi_{>1}. $$
We see that $\mathbb{E}_{F_0} \phi_{\geq 1}$ is the fixed point of $T(F_0, F_m)$.

\[ \square \]

**Theorem 3.9.** Let $f$ be a symmetric nontrivial tradeoff function and let $F_f$ be as in Definition 3.7. Then $F_f$ is a canonical noise distribution for $f$.

**Proof.** For simplicity of notation, we denote $F := F_f$. We verify the four points of Definition 3.1. It is easiest to prove the points in reverse order.

4. Symmetry was already shown in Proposition 3.8.
3. First, we will show that $\frac{d}{dx} F(x - 1) = \frac{d}{dx} F(x)$ is increasing in $x$. Let $x \in \mathbb{R}$ be a point of differentiability of $F(x)$ and $F(x - 1)$ and be such that $0 < F(x) < 1$. Then $F'(x) > 0$ by property 4 of Proposition 3.8. So,

$$
\frac{d}{dx} F(x - 1) = \frac{d}{dx} f(1 - F(x)) = \frac{d}{dx} F(x)
$$

$$
= f'(1 - F(x)) \frac{F''(x)}{F'(x)}
$$

$$
= -f'(1 - F(x)),
$$

where in the end, we note that $1 - F(x)$ is decreasing, and $f'(\alpha)$ is increasing. We see that this quantity is increasing. By property 4 of Proposition 3.5, $F'(x) = 0$ only if $F(x) = 0$ or $F(x) = 1$. When either $X \sim F$ or $X \sim F(\cdot - 1)$, the probability that either $F(X) = F(X - 1) = 0$ or $F(X) = F(X - 1) = 1$ is zero, so we can disregard the case that $F'(x - 1)/F'(x)$ has the form $0/0$. If $x$ satisfies $F(x - 1) = 0$ and $F(x) \in (0, 1)$, then the ratio $F'(x - 1)/F'(x) = 0$, whereas if $F(x) = 1$ and $F(x - 1) \in (0, 1)$, then the ratio $F'(x - 1)/F'(x) = +\infty$. These special cases preserve the increasing nature of the ratio.

Thus, we know that the optimal rejection set is of the form $(a, \infty)$. The type I error of this test is $\alpha = 1 - F(a)$, whereas the type II is $F(a - 1)$. Then we have that the tradeoff function is $T(F(\cdot), F(\cdot - 1))(\alpha) = F^{-1}(1 - \alpha) - 1$.

2. Let $\alpha \in (0, 1)$. Then

$$
T(F(\cdot), F(\cdot - 1))(\alpha) = F(F^{-1}(1 - \alpha) - 1)
$$

$$
= f(1 - F(F^{-1}(1 - \alpha)))
$$

$$
= f(1 - (1 - \alpha))
$$

$$
= f(\alpha),
$$

where we used the identity $F(x - 1) = f(1 - F(x))$, provided that $F(x) > 0$ (property 2 of Proposition 3.8), as well as property 2 of Lemma F.2.

1. We need to show that $T(F(\cdot), F(\cdot - m)) \geq T(F(\cdot), F(\cdot - 1))$ for all $m \in [0, 1]$. We will denote $F_0 = F$, $F_m = F(\cdot - m)$ and $F_1 = F(\cdot - 1)$. By Lemma F.5, when testing $F_0$ versus $F_m$, it suffices to check rejection regions of the form $S_k = \{x \mid F_m(x) / F(x) > k\}$ for $k \geq 1$ or $S'_k = \{x \mid F_m(x) / F(x) \geq k\}$ for $k > 1$. First we will check $S_k$ for $k \geq 1$. Since $F'$ is decreasing on $(-1/2, \infty)$ and increasing on $(-\infty, 1/2)$, we have that $S_k \subseteq S_1 = \{x \mid F'(x - m) > F'(x)\} \subset (1/2, \infty)$. Furthermore, on $(1/2, \infty)$, we have that $F'_1(x) / F'_m(x) \geq 1$, since $F'$ is decreasing.

Then $\mathbb{E}_{X \sim F_m} I(X \in S_k) = \int_{S_k} F_m'(x) \, dx \leq \int_{S_k} F_1'(x) F_m'(x) \, dx = \int_{S_k} F_1'(x) \, dx = \mathbb{E}_{X \sim F_1} I(X \in S_k)$, so that the power under $F_1$ is greater than the power under $F_m$. We
can repeat the argument for rejection sets of the form $S'_k$ for $k > 1$. By part 2 of Lemma F.5, we have that $T(F(\cdot), F(\cdot - m)) \geq T(F(\cdot), F(\cdot - 1))$, establishing part 1 of Definition 3.1.

In Definition 3.7 we defined the cdf of the constructed CND. While this expression is very useful for deriving properties of this distribution, the quantile function is important for sampling. In Proposition F.6, we give a recursive expression for the quantile function of the CND constructed in Definition 3.7 and show that it can be evaluated in a finite number of steps.

**Proposition F.6.** Let $f$ be a symmetric nontrivial tradeoff function and let $F_f$ be as in Definition 3.7. Then the quantile function $F_f^{-1} : (0, 1) \to \mathbb{R}$ for $F_f$ can be expressed as

$$F_f^{-1}(u) = \begin{cases} F_f^{-1}(1 - f(u)) - 1 & u < c \\ \frac{u}{1 - 2c} & c \leq u \leq 1 - c \\ F_f^{-1}(f(1 - u)) + 1 & u > 1 - c, \end{cases}$$

where $c$ is the unique fixed point of $f$. Furthermore, for any $u \in (0, 1)$, the expression $Q_f(u)$ takes a finite number of recursive steps to evaluate. Thus, if $U \sim U(0,1)$, then $F_f^{-1}(U) \sim F_f$.

**Proof.** For ease of notation, we will drop the subscripts of $F_f$ and $F_f^{-1}$. By Proposition 3.8, we have established that $F$ is strictly increasing on $\{x \mid F(x) \in (0,1)\}$. So, $F$ is invertible on $\{x \mid F(x) \in (0,1)\}$. In the case that $u \in [c, 1 - c]$, it is easy to verify the expression of $F^{-1}$ by the construction of $F$ in Definition 3.7.

Suppose that $u \in (1 - c, 1)$. Then $F^{-1}(u) > 1/2$, so by Definition 3.7, we know that $F^{-1}(u)$ satisfies $u = 1 - f(F(F^{-1}(u) - 1)).$ Since $u \in (0,1)$, this implies that $F(F^{-1}(u) - 1) = f(1 - u) < f(0)$, by parts 2 and 3 of Theorem 3.9, so the previous equation implies that $f(1 - u) = F(F^{-1}(u) - 1).$ Next we will apply $F^{-1}$ to both sides, but we need to verify that $F$ is invertible at $F^{-1}(u) - 1$. Call $M = \inf \{x \mid F(x) > 0\}$; by the construction of Definition 3.7, note that $M \leq -1/2.$ It suffices to show that $F^{-1}(u) - 1 \geq M$, since $F$ is invertible whenever $F \in (0,1)$. Note that

$$u > 1 - c = F(1/2) \geq F(1 + M),$$

where the inequality uses the fact that $M \leq -1/2$. So, we have that $u \geq F(1 + M)$, which implies that $F^{-1}(u) - 1 \geq M$, by Lemma F.1. Now that we know $F$ is invertible at $F^{-1}(u) - 1$, we have $F^{-1}(f(1 - u)) = F^{-1}(u) - 1$, which is equivalent to $F^{-1}(f(1 - u)) + 1 = F^{-1}(u)$, as claimed in the prescription of $F^{-1}$.

For the case where $u < c$, note that since $F$ corresponds to a symmetric random variable, we have that $F^{-1}(u) = -F^{-1}(1 - u)$. Applying the recursive formula for $u > 1 - c$ gives the result

$$F^{-1}(u) = -F^{-1}(1 - u) = -[F^{-1}(f(u)) + 1] = -F^{-1}(f(u)) - 1 = F^{-1}(1 - f(u)) - 1,$$

for $u < c$.

To see that the recursion only requires a finite number of iterations, note that if $u \in (1 - c, 1)$, then by Lemma F.4 there exists $k \in \mathbb{Z}^+$ such that $u \in ((1 - f)^{\circ k}(c), (1 - f)^{\circ (k+1)}(c))$ and necessarily $(1 - f)^{\circ k}(c) < 1$, where the notation means $(1 - f)^{\circ k}(c) := (1 - f) \circ (1 - f)^{\circ (k-1)}(c)$.
\( f \circ \cdots \circ (1 - f)(c) \), where \( 1 - f \) is composed \( k \) times. Then,
\begin{align*}
(18) \quad f(1 - u) &\in \left( f(1 - (1 - f)^{\circ k}(c)), f(1 - (1 - f)^{\circ (k+1)}(c)) \right) \\
(19) \quad &= \left( f \circ f \circ (1 - f)^{\circ (k-1)}(c), f \circ f \circ (1 - f)^{\circ k}(c) \right) \\
(20) \quad &= \left( (1 - f)^{\circ (k-1)}(c), f \circ f \circ (1 - f)^{\circ k}(c) \right) \\
(21) \quad &\subset \left( (1 - f)^{\circ (k-1)}(c), (1 - f)^{\circ k}(c) \right),
\end{align*}

where for (20), we use the equation \( f \circ f \circ (1 - f)^{\circ (k-1)}(c) = (1 - f)^{\circ (k-1)}(c) \), which is justified as follows: notice that \( (1 - f)^{\circ (k-1)}(c) \leq f(0) \), as if \( (1 - f)^{\circ (k-1)}(c) > f(0) \) then \( f \circ (1 - f)^{\circ (k-1)}(c) = 1 \), contradicting our assumption about \( (1 - f)^{\circ k}(c) \). For (21), we use the fact that \( f \circ f(\alpha) \leq \alpha \). We see that after \( k \) iterations of the recursive formula, the evaluation reduces to \( F^{-1}(u^*) \) for some \( u^* \in [c, 1 - c] \). By symmetry, we have that when \( u \in (0, c) \) the recursion finishes in a finite number of steps as well.

By inverse transform sampling, when \( U \sim U(0, 1) \) we have that \( F^{-1}(U) \sim F \). \( \square \)

**Corollary 3.10.** The distribution Tulap(0, b, q), where \( b = \exp(-c) \) and \( q = \frac{2b}{1 - b + 2b} \) is a CND for \( f_c, \delta \)-DP, which agrees with the construction of Definition 3.7.

**Proof.** Recall that the cdf of Tulap(0, b, 0), defined in Awan and Slavković (2018), is
\[
F_N(x) = \begin{cases} 
\frac{b - \lfloor x \rfloor}{1 + b} \left(b + \{x - \lfloor x \rfloor + 1/2\}(1 - b)\right) & x \leq 0 \\
1 - \frac{b - \lfloor x \rfloor}{1 + b} \left(b + \{x - \lfloor x \rfloor + 1/2\}(1 - b)\right) & x > 0,
\end{cases}
\]
where \( \lfloor x \rfloor \) is the nearest integer function. The cdf of Tulap(0, b, q) is
\[
F_N(x) = \begin{cases} 
0 & F_N(x) < q/2 \\
\frac{F_N(x) - q/2}{1 - q} & q/2 \leq F_N(x) \leq 1 - q/2 \\
1 & F_N(x) > 1 - q/2.
\end{cases}
\]
By inspection, the fixed point of \( f_c, \delta \) is \( c = \frac{1 - \delta}{1 + \epsilon} \). It is easy to verify that \( F_N(x) = c(1/2 - x) + (1 - c)(x + 1/2) \) for \( x \in (-1/2, 1/2) \). By Awan and Slavković (2020, Lemma 2.8), we have that \( F_N \) satisfies the recurrence relation in Definition 3.7. We conclude that \( F_N = F_f \). \( \square \)

**Lemma 4.1.** Let \( f \) be a symmetric tradeoff function. A test \( \phi : X^n \to [0, 1] \) satisfies \( f \)-DP if and only if \( \phi(x) \leq 1 - f(\phi(x')) \) for all \( x, x' \in X^n \) such that \( H(x, x') \leq 1 \).

**Proof.** The proof is based on applying the Neyman Pearson Lemma to the testing of two Bernoulli random variables. Let \( x, x' \in X^n \) such that \( H(x, x') \leq 1 \) be given. We need to show that \( T(\text{Bern}(\phi(x))), \text{Bern}(\phi(x')) ) \geq f \). Call \( p := \phi(x) \) and \( q := \phi(x') \).

Note that if \( p = q \), the result is trivial since the tradeoff function \( T(\text{Bern}(p), \text{Bern}(q)) = \text{Id} \geq f \), and both \( q = p \leq 1 - f(p) = 1 - f(q), \) since \( 1 - f(p) \geq p \). Next we will assume that \( p < q \).

By the Neyman Pearson Lemma, recall that the most powerful test \( \psi \) to distinguish \( H_0 : \text{Bern}(p) \) versus \( H_1 : \text{Bern}(q) \) is of one of the following forms:
\[
\psi_1(x) = \begin{cases} 
1 & x = 1 \\
\frac{1}{c} & x = 0
\end{cases}, \quad \psi_2(x) = \begin{cases} 
c & x = 1 \\
0 & x = 0,
\end{cases}
\]
where the case and the value $c$ are chosen such that the size is $\alpha$.

In the case where $p > \alpha$, setting $c = \alpha/p$ allows for $E_{\text{Bern}(p)} \psi_2(x) = cp = \alpha$. The type II error in this case is then $1 - E_{\text{Bern}(q)} \psi_2(x) = 1 - cq = 1 - \frac{\alpha}{p} q$.

On the other hand if $p \leq \alpha$, then for $c = \frac{\alpha - p}{1 - p}$ we have that $E_{\text{Bern}(p)} \psi_1(x) = 1p + c(1 - p) = \alpha$. Then the type II error is $1 - E_{\text{Bern}(q)} \psi_2(x) = 1 - \left(1q + \frac{\alpha - p}{1 - p} (1 - q)\right) = \frac{\alpha - p}{1 - p} (1 - q)$.

Combining these two cases, we see that the tradeoff function is

$$ T(\text{Bern}(p), \text{Bern}(q)) = \begin{cases} \frac{1 - \alpha}{1 - p} (1 - q) & p \leq \alpha \\ 1 - \frac{q}{p} (q) & p > \alpha, \end{cases} $$

which is a piece-wise linear function with break points $(0, 1)$, $(p, 1 - q)$ and $(1, 0)$. Because $f$ is a convex function which satisfies $f(0) \leq 1$ and $f(1) = 0$ (implied by $f(x) \leq 1 - x$), we have that $T(\text{Bern}(p), \text{Bern}(q)) \geq f$ if and only if $1 - q \geq f(p)$ or equivalently $q \leq 1 - f(p)$.

Now suppose that $p > q$. Note that by symmetry, establishing $T(\text{Bern}(p), \text{Bern}(q)) \geq f$ is equivalent to establishing $T(\text{Bern}(q), \text{Bern}(p)) \geq f^{-1} = f$. By our earlier work, swapping the roles of $p$ and $q$, we have that this inequality holds if and only if $p \leq 1 - f(q)$.

**Corollary 4.3 (Canonical Noise Distributions).** Let $f$ be a symmetric nontrivial tradeoff function and let $F$ be a canonical noise distribution for $f$. Then a test $\phi$ satisfies $f$-DP if and only if $F^{-1}(\phi(x)) \leq F^{-1}(\phi(x')) + 1$ for all $x, x' \in \mathcal{X}^n$ such that $H(x, x') \leq 1$.

**Proof.** Let $x, x' \in \mathcal{X}^n$ such that $H(x, x') \leq 1$. For the reverse direction of the statement, suppose that $F^{-1}(\phi(x)) \leq F^{-1}(\phi(x')) + 1$. Applying $F$ preserves this inequality since $F$ is increasing, and $F \circ F^{-1}(\phi(x)) = \phi(x)$ by Lemma 5.6. So,

$$ \phi(x) \leq F(F^{-1}(\phi(x')) + 1) $$
$$ = 1 - F(-F^{-1}(\phi(x')) - 1) $$
$$ = 1 - F(F^{-1}(1 - \phi(x')) - 1) $$
$$ = 1 - \phi(x'), $$

where we used the symmetry of $F$, and the fact that $F$ is a CND for $f$. By Lemma 4.1 we conclude that $\phi$ satisfies $f$-DP.

For the forward direction, suppose that $\phi(x) \leq 1 - f(\phi(x'))$, or equivalently, $F(F^{-1}(1 - \phi(x')) - 1) \leq 1 - \phi(x)$. Then each of the following inequalities follows from the one above:

(22)  \( F(F^{-1}(1 - \phi(x')) \leq 1 - \phi(x) \)
(23)  \( 1 - F(F^{-1}(1 - \phi(x')) - 1) \geq \phi(x) \)
(24)  \( F(1 - F^{-1}(1 - \phi(x'))) \geq \phi(x) \)
(25)  \( 1 - F^{-1}(1 - \phi(x')) \geq F^{-1}(\phi(x)) \)
(26)  \( F^{-1}(1 - \phi(x')) - 1 \leq -F^{-1}(\phi(x)) \)
(27)  \( F^{-1}(1 - \phi(x')) - 1 \leq F^{-1}(1 - \phi(x)) \)
(28)  \( -F^{-1}(1 - \phi(x)) \leq -F^{-1}(1 - \phi(x')) + 1 \)
(29)  \( F^{-1}(\phi(x)) \leq F^{-1}(\phi(x')) + 1, \)

where (25) used the Galois inequalities of Lemma F.1, and the other steps used the symmetry of $F$ and basic algebraic manipulations.

\[\square\]
LEMMA F.7 (Theorem 8.3.27 of Casella and Berger, 2002). Let \( H_0 : \theta = \theta_0 \) versus \( H_1 : \theta \in \Theta_1 \) be a hypothesis test with a simple null hypothesis. Let \( T \sim \theta \) be a real-valued (continuous) test statistic. Assuming that large values of \( T \) give evidence for \( H_1 \), a \( p \)-value for the hypothesis is
\[
p(T) = 1 - F_{T \sim \theta_0}(T) = P_{\theta_0}(T_0 > T).
\]

LEMMA F.8. Let \( I \) be an arbitrary index set. Let \( X \) be a random variable, and consider the following simple hypothesis test \( H_0 : X \sim P \) versus \( H_1 : X \sim Q_i \) for some \( i \in I \). Let \( T(X) \) be a continuous real-valued statistic, and consider the threshold test which rejects for large values of \( T \). Let \( t \in \mathbb{R} \) and call \( \alpha = P_{X \sim P}(T \geq t) \) and \( \beta_i = P_{X \sim Q_i}(T \geq t) \). Then the \( p \)-value \( p(T) = P_{X \sim P}(T(X) \geq T) \) satisfies \( P_{\phi}(p(T) \leq \alpha) = \alpha \) and \( P_{Q_i}(p(T) \leq \alpha) = \beta_i \) for all \( i \in I \).

PROOF. Call \( F_P \) and \( F_{Q_i} \), the cdf of the random variable \( T(X) \), when \( X \sim P \) and \( X \sim Q_i \), respectively. Then \( p(T) = 1 - F_P(T) \). We have that \( P_{\phi}(p(T) \leq \alpha) = P_P(1 - F_P(T) \leq \alpha) = P_P(F_P^{-1}(1 - \alpha) \leq T) = 1 - F_P(F_P^{-1}(1 - \alpha)) = \alpha \), where we use the fact that \( F_P^{-1} \circ F_P(T) = T \) with probability one, and that \( F_P \circ F_P^{-1}(\alpha) = \alpha \) by part 2 of Lemma 5.6. Next consider
\[
\begin{align*}
P_{Q_i}(p(T) \leq \alpha) &= P_{Q_i}(1 - F_P(T) \leq \alpha) \\
&= P_{Q_i}(F_P^{-1}(1 - \alpha) \leq T) \\
&= 1 - F_{Q_i}(t) \\
&= P_{Q_i}(T \geq t) \\
&= \beta_i,
\end{align*}
\]
where (31) used the Galois inequalities (Lemma F.1), and we used the fact that \( F_P^{-1}(1 - \alpha) = t \) or equivalently, that \( \alpha = 1 - F_P(t) = P_P(T \geq t) \). \( \square \)

THEOREM 4.5. Let \( \phi : X^n \to [0,1] \) be an \( f \)-DP test. Let \( F \) be a CND for \( f \), and draw \( N \sim F \). Then
1. releasing \( T = F^{-1}(\phi(x)) + N \) satisfies \( f \)-DP;
2. the variable \( Z = I(T \geq 0) \), a post-processing of \( T \), is distributed as \( Z | X = x \sim \text{Bern}(\phi(x)) \);
3. the value \( p = \sup_{\theta_0 \in H_0} \mathbb{E}_{X \sim \theta_0} F(F^{-1}(\phi(X)) - T) \) is also a post-processing of \( T \) and is a \( p \)-value for \( H_0 \);
4. if \( \mathcal{H}_0 \) is a simple hypothesis and \( \mathbb{E}_{\mathcal{H}_0} \phi = \alpha \), then at type I error \( \alpha \), the \( p \)-value from part 3 is as powerful as \( \phi \) at every alternative.

PROOF. 1. In Corollary 4.3, we saw that for any \( x \) and \( x' \) adjacent, \( F^{-1}(\phi(x)) \leq F^{-1}(\phi(x')) + 1 \). This implies that \( F^{-1}(\phi(x)) \) has sensitivity 1. The result follows from Definition 3.1.
2. It suffices to show that \( P(T \geq 0) = \phi(x) \):
\[
\begin{align*}
P(T \geq 0) &= P(F^{-1}(\phi(x)) + N \geq 0) \\
&= P(N \geq F^{-1}(\phi(x))) \\
&= F(F^{-1}(\phi(x))) \\
&= \phi(x),
\end{align*}
\]
where \( F \circ F^{-1} = \text{Id} \) since \( F \) is continuous, and we used the symmetry of \( N \).
3. We can express

\[ p = \sup_{\theta_0 \in H_0} \mathbb{E}_{X \sim \theta_0} F(F^{-1}(\phi(X)) - T) = \sup_{\theta_0 \in H_0} P_{X \sim \theta_0, N}(N \leq F^{-1}(\phi(X)) - T) = \sup_{\theta_0 \in H_0} P_{X \sim \theta_0, N}(T \leq F^{-1}(\phi(X)) + N), \]

where we used the fact that \( N \overset{d}{=} -N \). By Casella and Berger (2002, Theorem 8.3.27), this is a valid \( p \)-value.

4. The result follows from Lemma F.8. \( \square \)

The following lemma is one of several techniques to prove the Neyman Pearson Lemma, and appears as Lemma 4.4 in Awan and Slavković (2018).

**Lemma F.9.** Let \((\mathcal{X}, \mathcal{F}, \mu)\) be a measure space and let \( f \) and \( g \) be two densities on \( X \) with respect to \( \mu \). Suppose that \( \phi_1, \phi_2 : \mathcal{X} \to [0, 1] \) are such that \( \int \phi_1 \, d\mu \geq \int \phi_2 \, d\mu \), and there exists \( k \geq 0 \) such that \( \phi_1 \geq \phi_2 \) when \( g \geq kf \) and \( \phi_1 \leq \phi_2 \) when \( g < kf \). Then \( \int \phi_1 \, d\mu \geq \int \phi_2 \, d\mu \).

**Proof.** Note that \( (\phi_1 - \phi_2)(g - kf) \geq 0 \) for all \( x \in \mathcal{X} \). This implies that \( \int (\phi_1 - \phi_2)(g - kf) \, d\mu \geq 0 \). Hence, \( \int \phi_1 \, d\mu - \int \phi_2 \, d\mu \geq k \left( \int \phi_1 \, d\mu - \int \phi_2 \, d\mu \right) \geq 0 \). \( \square \)

**Theorem 4.8.** Let \( f \) be a symmetric nontrivial tradeoff function and let \( F \) be a CND of \( f \). Let \( \mathcal{X} = \{0, 1\} \). Let \( P \) and \( Q \) be two exchangeable distributions on \( \mathcal{X}^n \) with pmfs \( p \) and \( q \) such that \( \frac{q}{p} \) is an increasing function of \( x = \sum_{i=1}^n x_i \). Let \( \alpha \in (0, 1) \). Then a most powerful \( f \)-DP test \( \phi \) with level \( \alpha \) for \( H_0 : X \sim P \) versus \( H_1 : X \sim Q \) can be expressed in any of the following forms:

1. There exists \( y \in \{0, 1, 2, \ldots, n\} \) and \( c \in (0, 1) \) such that for all \( x \in \{0, 1, 2, \ldots, n\} \),

\[
\phi(x) = \begin{cases} 
0 & x < y, \\
c & x = y, \\
1 - f(\phi(x - 1)) & x > y,
\end{cases}
\]

where if \( y > 0 \) then \( c \) satisfies \( c \leq 1 - f(0) \), and \( c \) and \( y \) are chosen such that \( \mathbb{E}_P \phi(x) = \alpha \). If \( f(0) = 1 \), then \( y = 0 \).

2. \( \phi(x) = F(x - m) \), where \( m \in \mathbb{R} \) is chosen such that \( \mathbb{E}_P \phi(x) = \alpha \).

3. Let \( N \sim F \). The variable \( T = X + N \) satisfies \( f \)-DP. Then \( p = \mathbb{E}_{X \sim P} F(X - T) \) is a \( p \)-value and \( I(\phi(x) \leq \alpha) \mid X = I(T \geq m) \mid X \sim \text{Bern}(\phi(X)) \), where \( \phi(x) \) agrees with 1 and 2 above.

**Proof.** First we will establish the equivalence of 1 and 2. Given a test \( \phi \) of the form 2, we know by Lemma 3.6 that \( \phi \) satisfies the recurrence in 1. Set \( y \) to be the smallest \( x \in \{0, 1, 2, \ldots, n\} \) such that \( \phi(x) > 0 \). Then set \( c = \phi(y) \). We have that \( \phi \) fits the form of 1.

Now let \( \phi \) be of the form 1. Solve \( c = F(y - m) \) for \( m \), which has a solution by the Intermediate Value Theorem as \( \lim_{m \to -\infty} F(y - m) = 0 \) and \( \lim_{m \to \infty} F(y - m) = 1 \). By Lemma 3.6, \( F(x - m) = \phi(x) \) for \( x \in \{0, 1, 2, \ldots, n\} \). We conclude that 1 and 2 are equivalent.

Next we argue that the prescribed \( \phi \) satisfies \( f \)-DP, using form 1. Note that we have \( \phi(x) \leq 1 - f(\phi(x - 1)) \) for all \( x = 1, \ldots, n \). We also need to show that \( \phi(x - 1) \leq 1 - f(\phi(x)) \). To
this end, we first observe that \( \phi(x - 1) \leq \phi(x) \). This follows from the fact that \( f(t) \leq 1 - t \) or equivalently that \( t \leq 1 - f(t) \). So, we have \( \phi(x - 1) \leq \phi(x) \leq 1 - f(\phi(x)) \).

Next given \( \alpha \in (0, 1) \), we need to argue that there exists a test \( \phi \) of the prescribed form which has \( \mathbb{E}_P \phi(x) = \alpha \). We use form 2 for this part, so we need to show that there exists \( m \in \mathbb{R} \) such that \( \alpha = \mathbb{E}_X \phi F(X - m) \). Note that \( \mathbb{E}_X \phi F(X - m) \) is a continuous function in \( m \), where the limit as \( m \to -\infty \) is zero and the limit as \( m \to \infty \) is 1. By the Intermediate Value Theorem there exists \( m \) such that \( \alpha = \mathbb{E}_X \phi F(X - m) \).

Let \( \phi_0(x) \) be a test of form 1 which has \( \mathbb{E}_P \phi_0(x) = \alpha \), and let \( \psi \) be another level \( \alpha \) \( f \)-DP test. We will show that \( \phi_0 \) is more powerful than \( \psi \). First we claim that there exists a value \( z \) such that \( \psi(z) \leq \phi_0(z) \). If this were not the case, then \( \psi(x) > \phi_0(x) \) for all \( x \), which implies that \( \mathbb{E}_P \psi(x) > \mathbb{E}_P \phi_0(x) = \alpha \), contradicting the level of \( \psi \).

Now, let \( z_m \) be the smallest value such that \( \psi(z_m) \leq \phi_0(z_m) \). Then by assumption, for all \( x < z_m \), \( \psi(x) > \phi_0(x) \). Next, note that \( \psi(z_m + 1) \leq 1 - f(\psi(z_m)) \leq 1 - f(\phi_0(z_m)) \leq \phi_0(z_m + 1) \). By induction, we have that for all \( x \geq z_m \), \( \psi(x) \leq \phi_0(x) \).

We conclude that \( \phi_0(x) \geq \psi(x) \) for all \( x \geq z_m \) and \( \phi_0(x) \leq \psi(x) \) for all \( x < z_m \). In other words, there exists a threshold \( t^* \) such that \( \phi_0(x) \geq \psi(x) \) when \( \frac{g(x)}{p(x)} \geq t^* \) and \( \phi_0(x) \leq \psi(x) \) when \( \frac{g(x)}{p(x)} \leq t^* \). By Lemma F.9, we have that \( \mathbb{E}_Q \phi_0(x) \geq \mathbb{E}_Q \psi(x) \).

Last, we verify the claim of form 3. The variable \( T = X + N \) satisfies \( f \)-DP by Definition 3.1, since \( X \) has sensitivity 1. Call \( F_{T \sim P}(t) = P_{X \sim P, N \sim F}(X + N \leq t) \) the cdf of \( T = X + N \) when \( X \sim P \). The variable \( p \) can be expressed as

\[
 p = \mathbb{E}_{X \sim P} F(X - T) \\
 = P_{X \sim P, N \sim F}(N \leq X - T) \\
 = P_{X \sim P, N \sim F}(T \leq X + N) \\
 = 1 - P_{X \sim P, N \sim F}(X + N \leq T) \\
 = 1 - F_{T \sim P}(T),
\]

where we used the fact that \( N \overset{d}{=} -N \). We see that this is a \( p \)-value by Lemma F.7.

It is easy to verify that \( P(I(T \geq m) = 1 \mid X = x) = P_N(x + N \geq m) = P_N(N \leq x - m) = F(x - m) = \phi(x) \). We then check

\[
 P(p \leq \alpha \mid X = x) = P_N(1 - F_{T \sim P}(x + N) \leq \alpha) \\
 = P_N(1 - \alpha \leq F_{T \sim P}(x + N)) \\
 = P_N(F_{T \sim P}^{-1}(1 - \alpha) \leq x + N) \\
 = P_N(m \leq x + N) \\
 = F_N(x - m) \\
 = \phi(x)
\]

where we used the Galois inequalities (Lemma F.1), and that \( \alpha = \mathbb{E}_X \phi F(X - m) = P_{X,N}(N \leq X - m) = 1 - F_{T \sim P}(m) \), which implies that \( F_{T \sim P}^{-1}(1 - \alpha) = m \). \( \square \)

F.1. Proof of Theorem 5.2. This section is devoted to the proof of Theorem 5.2. First we need to establish notation and a few lemmas.

Given \( f \) be a symmetric nontrivial tradeoff function, define \( \Psi_f \) to be the set of \( f \)-DP tests on \( \{0, 1, 2, \ldots, m\} \):

\[
 \Psi_f = \{ \psi : \{0, 1, 2, \ldots, m\} \to [0, 1] \mid \psi(y) \leq 1 - f(\psi(y')) \text{, for all } |y - y'| = 1 \}.
\]
Given a function $f : \mathbb{R} \to \mathbb{R}$ we define $f^{\circ k} = f \circ f \circ \cdots \circ f$, where there are $k$ appearances of $f$. For example, we write $(1 - f)^{\circ 2}(x) = (1 - f) \circ (1 - f)(x) = 1 - f(1 - f(x))$.

**Lemma F.10.** Given $\phi \in \Phi_f$ defined in Equation (3), and given $z \in \{0, 1, \ldots, m + n\}$ define $\psi_z : \{0, 1, \ldots, m\} \to [0, 1]$ by

$$
\psi_z(y) = \begin{cases} 
\phi(z - L, L) & \text{if } y \leq L \\
\phi(z - y, y) & \text{if } L \leq y \leq U \\
\phi(z - U, U) & \text{if } y \geq U,
\end{cases}
$$

where $L = \max\{0, n - z\}$ and $U = \min\{m, z\}$. Then $\psi_z \in \Psi_g$ where $g = 1 - (1 - f)^{\circ 2}$.

**Proof.** Let $y, y' \in \{0, \ldots, m\}$ such that $|y - y'| = 1$. Define $x = z - y$ and $x' = z - y'$. Then $(x, y)$ and $(x', y')$ both lie in $\{0, 1, \ldots, n\} \times \{0, 1, \ldots, m\}$. Then $(x', y')$ is adjacent to $(x, y')$ and $(x, y)$ is adjacent to $(x, y)$.

If $y, y' \geq U$ or $y, y' \leq L$, then $\psi_z(y') = \psi_z(y) \leq (1 - g) \circ \psi_z(y)$. If $y, y' \in \{L, L + 1, \ldots, U - 1, U\}$, then

$$
\psi_z(y') = \phi(x', y') \leq (1 - f) \circ \phi(x, y') \leq (1 - f)^{\circ 2} \circ \phi(x, y) = (1 - g) \circ \phi(x, y) = (1 - g) \circ \psi_z(y)
$$

where $g = 1 - (1 - f)^{\circ 2}$.

**Lemma F.11 (Lemma A.5, Dong et al., 2022).** Let $P, Q$, and $R$ be distributions, and let $f$ and $g$ be tradeoff functions. If $T(P, Q) \geq f$ and $T(Q, R) \geq g$ then $T(P, R) \geq g \circ (1 - f)$.

**Lemma F.12.** Let $f$ be a symmetric nontrivial tradeoff function, and let $F_f$ be a CND for $f$. Then $F(2) : F_f$ is a CND for $g = 1 - (1 - f)^{\circ 2} = f(1 - f)$.

**Proof.** Recall from Dong et al. (2022, Section 2.5) that $g(\alpha) = 1 - (1 - f)^{\circ 2}(\alpha)$ is also symmetric, and for all $\alpha$, $g(\alpha) \leq f(\alpha)$, so $g$ is also nontrivial. By Theorem 3.9 there exists a CND for $g$.

We drop the subscript and write $F = F_f$. We write $F_2(\cdot) = F(2 \cdot)$, and $F_2^{-1}(\cdot) = \frac{1}{2} F^{-1}(\cdot)$, where $F_2^{-1}$ is the quantile function for $F_2$. $F_2$ clearly satisfies property 4 of Definition 3.1, since $F$ is a CND. For $\alpha \in (0, 1)$, note the following connection between $F$ and $F_2$:

$$
F(F^{-1}(1 - \alpha) - 2) = F\left(\frac{1}{2}\left[F^{-1}(1 - \alpha) - 1\right]\right) = F_2(F_2^{-1}(1 - \alpha) - 1).
$$

Let $M := \inf\{t \mid 0 < F(t)\}$ by symmetry of $F$, we know that $M \leq 0$. If $M \geq -1/2$, then we have that $g(\alpha) \leq f(\alpha) = F(F^{-1}(1 - \alpha) - 1) = 0$ for all $\alpha \in (0, 1)$; we also have $F_2(F_2^{-1}(1 - \alpha) - 1) = F(F^{-1}(1 - \alpha) - 2) = 0 = g(\alpha)$ justifying property 3 of Definition 3.1. Furthermore, $T(F_2(\cdot), F_2(\cdot - 1)) = T(F(\cdot), F(\cdot - 2)) = 0 = g$, since $F(\cdot)$ and $F(\cdot - 2)$ have disjoint support. Finally, note that $T(F_2(\cdot), F_2(\cdot - m)) \geq T(F_2(\cdot), F_2(\cdot - 1)) = 0$, since 0 is a trivial lower bound for any tradeoff function. We conclude that when $M \geq -1/2$, $F_2$ is a CND for $g$. 

Now suppose that $M < -1/2$ and let $\alpha \in (0, 1)$. If $\alpha \geq f(0)$, then $g(\alpha) = f(1 - f(\alpha)) = f(1 - 0) = 0$ and $F(F^{-1}(1 - \alpha) - 2) \leq F(F^{-1}(1 - \alpha) - 1) = f(\alpha) = 0$ because $F$ is increasing. We see that property 3 of Definition 3.1 holds in this case. Now assume that $\alpha < f(0)$, or equivalently $1 - \alpha > 1 - f(0)$. For the following calculations, we will need to justify that $F$ is invertible at $F^{-1}(1 - \alpha) - 1$. To see this, note that $F$ is invertible at $F^{-1}(1 - \alpha)$, and by Lemma F.3 $F$ is also invertible at $F^{-1}(1 - \alpha) - 1$ unless $F^{-1}(1 - \alpha) - 1 < M$. So, we need to show that $F^{-1}(1 - \alpha) \geq M + 1$:

\begin{equation}
M + 1 = \inf \{ t + 1 \mid 0 < F(t) \} \label{eq:46}
\end{equation}

\begin{equation}
= \inf \{ t \mid 0 < F(t - 1) \} \label{eq:47}
\end{equation}

\begin{equation}
= \inf \{ t \mid f(0) > f(F(t - 1)) \& 0 < F(t - 1) \} \label{eq:48}
\end{equation}

\begin{equation}
= \inf \{ t \mid 1 - f(0) < 1 - f(F(t - 1)) \& 0 < F(t - 1) \} \label{eq:49}
\end{equation}

where \eqref{eq:48} uses the fact that $f$ is strictly decreasing at 0; \eqref{eq:49} uses the fact that $0 < F(t - 1)$ to apply the recursion of Lemma 3.6. Now, suppose that $F(t - 1) = 0$: then $t - 1 \leq M$ and because $M < -1/2$, $F(t) < 1$. So, $0 = F(t - 1)$ implies that $0 = f(1 - F(t))$. But this in turn implies that $1 - F(t) \geq f(0)$ or equivalently $1 - f(0) \geq F(t)$. We see that $1 - f(0) < F(t)$ implies that $0 < F(t - 1)$. So,

\begin{equation}
M + 1 = \inf \{ t \mid 1 - f(0) < F(t) \} \label{eq:41}
\end{equation}

\begin{equation}
\leq \inf \{ t \mid 1 - \alpha \leq F(t) \} \label{eq:42}
\end{equation}

\begin{equation}
= F^{-1}(1 - \alpha), \label{eq:43}
\end{equation}

where \eqref{eq:42} uses the fact that $1 - \alpha > 1 - f(0)$. We are now ready to verify that $F_2$ satisfies property 3 of Definition 3.1 for $g$ when $M < -1/2$ and $\alpha \in (0, f(0))$:

\begin{equation}
g(\alpha) = f(1 - f(\alpha)) \label{eq:44}
\end{equation}

\begin{equation}
= f(1 - F[F^{-1}(1 - \alpha) - 1]) \label{eq:45}
\end{equation}

\begin{equation}
= F\{F^{-1}[1 - (1 - F[F^{-1}(1 - \alpha) - 1])] - 1\} \label{eq:46}
\end{equation}

\begin{equation}
= F(F^{-1}(1 - \alpha) - 2) \label{eq:47}
\end{equation}

\begin{equation}
= F\left(2 \left[\left(1/2\right)F^{-1}(1 - \alpha) - 1\right]\right) \label{eq:48}
\end{equation}

\begin{equation}
= F_2(F_2^{-1}(1 - \alpha) - 1), \label{eq:49}
\end{equation}

where \eqref{eq:47} used the fact that $F$ is invertible at $F^{-1}(1 - \alpha) - 1$.

For property 2 of Definition 3.1, we need to show that $\frac{d}{dx} F(2x) = \frac{d}{dx} F(2x - 1)$ is increasing in $x$. Let $x$ be such that $0 < F(2x) < 1$ and a point where $\frac{d}{dx} F(2x)$ and $\frac{d}{dx} F(2(x - 1))$ are well defined. Then $\frac{d}{dx} F(2x) > 0$. Setting $y = 2x$, we have

\begin{equation}
\frac{d}{dx} F(2(x - 1)) = \frac{d}{dy} F(y) = \frac{d}{dy} F(2(y - 1)) \frac{d}{dy} F(y) \label{eq:50}
\end{equation}

\begin{equation}
= \frac{d}{dy} f(1 - F(y - 1)) \frac{F'(y)}{F'(y)} \label{eq:51}
\end{equation}
\[
\frac{d}{dy} f(1 - [f(1 - F(y))]) = \frac{d}{dy} g(1 - F(y)) = -g'(1 - F(y)),
\]
which is increasing because \( g \) is convex and \( 1 - F(x) \) is decreasing. If \( F(2x) = 0 \), then \( F(2(x - 1)) = 0 \) as well, and the ratio of the derivatives is 0/0. If \( F(2x) = 1 \) and \( F(2(x - 1)) \in \{0, 1\} \), then we also get the undefined ratio of 0/0. In the case that \( F(2x) = 1 \), but \( F(2(x - 1)) < 1 \), the ratio is \( \frac{d}{dy} F(2(x-1)) = +\infty \). In each case, we have that the ratio is increasing, except when it has the form 0/0 (which has probability zero under either \( F(2(\cdot)) \) or \( F(2(\cdot - 1)) \)).

It remains to verify property 1 of Definition 3.1. Let \( S_0 \) and \( S_1 \) be two real values such that \( |S_0 - S_1| \leq \Delta \). Let \( N \sim F \) and \( N_2 \sim F_2 \). Note that \( N_2 \frac{d}{dy} F = \frac{1}{2} N \). Call \( S_2 = (1/2)(S_0 + S_1) \), and observe that \( |S_2 - S_0| \leq \Delta /2 \) and \( |S_2 - S_1| \leq \Delta /2 \). Then since \( N \) is drawn from a CND for \( f \),
\[
T(S_0 + \Delta N_2, S_2 + \Delta N_2) = T \left( S_0 + \frac{\Delta}{2} N, S_2 + \frac{\Delta}{2} N \right) \geq f
\]
\[
T(S_2 + \Delta N_2, S_1 + \Delta N_2) = T \left( S_2 + \frac{\Delta}{2} N, S_1 + \frac{\Delta}{2} N \right) \geq f,
\]
since \( F \) is a CND for \( f \). Then by Lemma F.11, we have \( T(S_0 + \Delta N_2, S_1 + \Delta N_2) \geq f(1 - f) = g. \)

Before we finally prove Theorem 5.2, we recall the definition of Neyman structure, and its connection to unbiased tests.

**DEFINITION F.13** (Definition 4.120 of Schervish). Let \( G \subset \Theta \). If \( T \) is a sufficient statistic for \( G \), then a test \( \phi \) has **Neyman structure relative to \( G \) and \( T \)** if \( \mathbb{E}_\theta [\phi(X) \mid T = t] \) is constant in \( t \) for all \( \theta \in G \).

**THEOREM F.14** (Theorem 4.123 of Schervish). Let \( G = \overline{\Theta}_0 \cap \overline{\Theta}_1 \). Let \( T \) be a boundedly complete sufficient statistic for \( G \). Assume that the power function is continuous. If there is a UMP unbiased level \( \alpha \) test \( \phi \) among those which have Neyman structure relative to \( G \) and \( T \), then \( \phi \) is UMP unbiased level \( \alpha \).

**THEOREM 5.2** (Semi-Private UMPU). Let \( f \) be a symmetric nontrivial tradeoff function and let \( F \) be a CND for \( f \). Let \( X \sim \text{Binom}(n, \theta_X) \) and \( Y \sim \text{Binom}(m, \theta_Y) \) be independent. Let \( \alpha \in (0, 1) \) be given. For the hypothesis \( H_0: \theta_X \geq \theta_Y \) versus \( H_1: \theta_X < \theta_Y \),
\begin{enumerate}
\item \( \phi^*(x, y) = F(y - x - c(x + y)) \) is the UMPU test of size \( \alpha \) among \( \Phi_f^{semi} \), where \( c(x + y) \) is chosen such that \( \mathbb{E}_{H \sim \text{Hyper}(m, n, x + y)} \phi^*((x + y) - H, H) = \alpha \).
\item Set \( T = Y - X + N \), where \( N \sim F \), and set \( Z = X + Y \). Then
\[
p = \mathbb{E}_{H \sim \text{Hyper}(m, n, Z)} F(2H - Z - T)
\]
is the exact \( p \)-value corresponding to \( \phi^* \).
\end{enumerate}
PROOF. By Theorem F.14, it suffices to consider tests which have Neyman structure relative to \( \{(\theta_x, \theta_y) \mid \theta_x = \theta_y \} \) and \( Z = X + Y \). So, we need only consider tests \( \phi(x, y) \) that satisfy \( \mathbb{E}_{\theta_x, \theta_y}[\phi(x, y) \mid x + y = Z] = \alpha \) for all \( Z \in \{0, 1, 2, \ldots, m + n \} \). By Theorem F.14, it suffices to show that \( \phi^* \) is UMP among the tests in \( \Phi^\text{semi} \) which also satisfy \( \mathbb{E}_{\theta_x, \theta_y}[\phi(x, y) \mid x + y = Z] = \alpha \).

Recall that if \( X \sim \text{Binom}(n, \theta_X) \) and \( Y \sim \text{Binom}(m, \theta_Y) \), then \( (X, Y) \mid X + Y = z \) is equal in distribution to \( (z - H, H) \), where \( H \sim \text{Hyper}(m, n, z, \omega) \), with \( \omega = \theta_Y/(1 - \theta_Y) \) and where \( \text{Hyper}(m, n, z, \omega) \) is the Fisher noncentral hypergeometric distribution, which has pmf

\[
P_{\omega}(H = x) = \frac{\binom{n}{z}(\binom{m}{x})^{\omega x}}{\sum_{x=L}^{U} \binom{n}{z}(\binom{m}{x})^{\omega x}},
\]

with support \( \{L = \max\{0, z - n\}, L + 1, \ldots, U - 1, U = \min\{z, m\} \} \). Then unbiased testing \( H_0 : \theta_X \leq \theta_Y \) versus \( H_1 : \theta_Y \geq \theta_X \) in the original model is equivalent to testing \( H_0 : \omega \leq 1 \) versus \( H_1 : \omega > 1 \) in the hypergeometric model.

Next, note that \( \text{Hyper}(m, n, z, \omega) \) has an increasing likelihood ratio in \( \omega \), meaning that for \( \omega_1 \leq \omega_2 \), we have \( \frac{P_{\omega_1}(H = x)}{P_{\omega_2}(H = x)} \) is an increasing function of \( x \). Then a test on hypergeometrics for \( H_0 : \omega \leq 1 \) versus \( H_1 : \omega_1 > 1 \) has size \( \alpha \) if and only if \( \mathbb{E}_{H \sim \text{Hyper}(m, n, z, 1)} \phi(H) = \alpha \). For the tradeoff function \( g = 1 - (1 - f)^{\alpha^2} \), by Theorem 4.8, there exists a most powerful \( \Psi_g \) test for \( H_0 : \omega = 1 \) versus \( H_1 : \omega = \omega_1 \) where \( \omega_1 > 1 \), which is of the form \( \psi^*_g(x) = F_{g}(x - m(z)) \), where \( m(z) \) is chosen such that \( \mathbb{E}_{H \sim \text{Hyper}(m, n, z, 1)} F_{g}(H - m) = \alpha \). Since this test does not depend on the specific alternative, it is UMP for \( H_0 : \omega \leq 1 \) versus \( H_1 : \omega > 1 \).

Now, given \( \psi^*_g \) for \( z \in \{0, 1, 2, \ldots, m + n\} \), the statement of Lemma F.14 follows. Let \( z \in \{0, 1, 2, \ldots, m + n\} \). Then for \( x + y = z \), we have

\[
\phi^*(x, y) = F_f(y - x - c(z)) = F_f(2y - z - c(z)) = F_f(2y - 2m(z)) = F_f(y - m(z)) = \psi^*_g(y),
\]

where \( m(z) = (1/2)(z + c(z)) \), and where we used Lemma F.12 to justify that \( F_f(2\cdot) = F_f(\cdot) \).

Now, suppose that there is another \( \psi^\text{semi} \) test which satisfies \( \mathbb{E}_{H \sim \text{Hyper}(m, n, z, 1)} \phi(z - H, H) = \alpha \), and which has higher power than \( \psi^*_g \) for some \( \theta_X < \theta_Y \) (call \( \omega_1 = \theta_Y/(1 - \theta_Y) \)). Because power can be expressed as \( \mathbb{E}_Z \mathbb{E}_{H \sim \text{Hyper}(m, n, z, \omega_1)} \phi(Z - H, H) \), where the first expectation is over the marginal distribution of \( Z \), this implies that there exists \( z \in \{0, 1, 2, \ldots, m + n\} \) such that \( \mathbb{E}_{Z \sim \text{Hyper}(m, n, z, \omega_1)} \phi(z - H, H) > \mathbb{E}_{Z \sim \text{Hyper}(m, n, z, \omega_1)} \phi^*(z - H, H) \). However, applying the transformation in Lemma F.10 gives test in \( \Psi_g \) with size \( \alpha \) for testing \( H_0 : \omega \leq 1 \) versus \( H_1 : \omega_1 > 1 \) in the family \( \text{Hyper}(m, n, z, \omega) \), with power at \( \omega_1 \) higher than \( \psi^*_g \). This contradicts that \( \psi^*_g \) is UMP size \( \alpha \) in \( \Psi_g \). We conclude that \( \phi^* \) is UMP unbiased size \( \alpha \) among \( \Psi^\text{semi} \) for the hypothesis \( H_0 : \theta_X \leq \theta_Y \) versus \( H_1 : \theta_X < \theta_Y \).

Line 2 of the theorem statement follows from the monotone likelihood ratio property of the Fisher noncentral hypergeometric distribution along with parts 3 and 4 of Theorem 4.5.