Photon orbits and thermodynamic phase transition in Gauss-Bonnet AdS black holes

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Using the radius of the photon sphere and the minimum impact parameter related to the photon orbits as characteristic quantities, we investigate the thermodynamic phase transition of a five-dimensional Gauss-Bonnet Anti-de Sitter black hole. We find van der Waals like behavior between the thermodynamic pressure of the black hole and these two parameters of the photon orbits around the black hole. As two astronomical observables, this behavior means that we can directly probe black hole phase transition. What’s more, it is shown that the roles of the photon orbit and the black hole horizon are much similar in determining the phase transition of black holes.

I. INTRODUCTION

In the past decades, black hole thermodynamics has become an area of intense investigation [1–3]. It turned out that there indeed exist rich phase structures and many different thermodynamic properties in black hole spacetime. One of the most interesting things is that small black holes are found to be thermodynamically unstable, while large ones are stable in anti-de Sitter(AdS) space. There is a minimum temperature for a black hole in AdS space, no black hole solution can be found below this temperature. Correspondingly, Hawking-Page phase transition was proposed [4]. It is from this seminal start that various research sprung up [5–8].

In the recent past, by regarding cosmological constant as pressure [9–16], a mount of attention has been attracted on the black hole thermodynamics in the extended phase space. Using the standard thermodynamic techniques, Kubiznak and Mann showed that, for a charged AdS black hole, the system indeed exists a first-order small-large black hole phase transition which in many aspects resembles the liquid-gas phase transition occurring in fluids. In fact, it was on this basis that extensive researches were expanded. In the meantime, a lot of novel thermodynamic properties have been discovered, such as the triple point, reentrant phase transition and superfluid black hole phase etc [17–32]. At the same time, there are many attempts to find an observational way to reveal the thermodynamic phase transition of the black hole. Ref. [33] illustrated that, with the value of the horizon radius increasing, the slopes of the quasinormal frequency change drastically different in the small and large black holes. This provides the expectation to find observable signature of black hole phase transitions.

Very recently, Refs. [34, 35] studied the relationship between the photon sphere radius (or minimum impact parameter) and black hole phase transition. It is found that, no matter for d-dimensional charged AdS black holes or for the rotating Kerr-AdS black holes, there exist oscillating behaviors for the pressure below its critical value in terms of the two quantities. Then they present a significant conjecture that thermodynamic phase transition can be unveiled by the sudden change of the photon sphere radius (or minimum impact parameter) in a universal black hole system.

In order to verify this conjecture, we aim to examine the situation for 5 - dimensional Gauss-Bonnet black holes in AdS spacetime [36–44]. Moreover, why can these special quantities of photon orbits betray the information of the black hole phase transition? Up to now, no explanation has been given. Considering the horizon of black holes is a good characteristic parameter to the phase transition of black holes, we speculate that there is a special connection between these two quantities and the horizon of black holes. Based on this speculation, we are going to illustrate the connection and hope to find the answer.

Our paper is organized as follows. In Sec. II, we briefly introduce the $P − V$ criticality in the extended phase space of Gauss-Bonnet black holes in AdS space and give an expression about the photon sphere radius and the minimum impact parameter related to the photon orbits of the black hole. In Sec. III, we study the relationship between these two quantities related to the photon orbits and thermodynamic phase transition for Gauss-Bonnet black holes. By analyzing the connection between these two quantities and the horizon of black holes respectively, we show that these two quantities can indeed be regarded as good characteristic parameters of the black hole phase transition. Based on our findings, the connection in the degenerated case of a charged AdS black hole is investigated in Sec. IV. Finally, the discussion and conclusion are presented in Sec. V.

II. THERMODYNAMICS AND PHOTON ORBITs OF GAUSS-BONNET BLACK HOLES IN ADS SPACETIME

In this section, we will briefly review the $P − V$ criticality in the extended phase space of AdS Gauss-Bonnet black holes [14, 38], and then introduce the photon sphere
radius and another impact parameter related to the photon orbits \[34,36\].

The solution for a Gauss-Bonnet AdS black hole with a negative cosmological constant \( \Lambda = -(d-1)(d-2)/2l^2 \) is

\[
\text{ds}^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2g_{ij}dx^idx^j, \quad (1)
\]

where \( g_{ij}dx^idx^j \) denotes the line element of a \((d-2)\)-dimensional maximally symmetric Einstein space. There are three cases: \( k = -1, k = 0, k = +1 \). \( k = -1 \) corresponds to constant negative curvature on a maximally symmetric space \( \Sigma \), \( k = 0 \) corresponds to no curvature on \( \Sigma \), and \( k = +1 \) corresponds to positive curvature on \( \Sigma \). According to Ref.\[14\], only in the case of \( k \) symmetric space \( \Sigma \), \( s \) corresponds to constant negative curvature on a maximally symmetric space \( \Sigma \), \( k = 0 \) corresponds to no curvature on \( \Sigma \), and \( k = +1 \) corresponds to positive curvature on \( \Sigma \). According to Ref.\[14\], only in the case of \( d = 5 \), \( k = 1 \), the \( P-V \) criticality and the small-large black hole phase transition can appear. Therefore, we will focus on this situation. The blackening factor \( f(r) \) in this case is given as

\[
f(r) = 1 + \frac{r^2}{2\alpha} \left( 1 - \sqrt{1 + \frac{32\alpha M}{3\pi r^4} + \frac{\alpha Q^2}{3\pi r^6} - \frac{16\alpha P}{3}} \right), \quad (2)
\]

where \( \alpha = 2\alpha_{GB} \), \( \alpha_{GB} \) is the Gauss-Bonnet coefficient with dimension \([\text{length}]^2\)\(^2\), \( M \) denotes the black hole mass, \( Q \) corresponds to the charge of the black hole and \( P = \frac{\partial P}{\partial r} \mid_{\text{horizon}} = \frac{\pi Q}{3\pi r^2} \). \( g_{ij}dx^idx^j \) in this case has an explicit form

\[
g_{ij}dx^idx^j = d\theta^2 + \sin^2 \theta d\phi^2 + \cos^2 \theta d\omega^2. \quad (3)
\]

Considering a meaningful vacuum solution with \( M = Q = 0 \), the constraint can be given as

\[
0 \leq \frac{16\pi \alpha P}{3} \leq 1. \quad (4)
\]

In the following discussion, we will replace the variable \( M \) with the horizon \( r_h \) which is the largest real root of the equation \( f(r_h) = 0 \), and it reads

\[
M = \frac{\pi Q^2}{32r_h^2} + \frac{\pi}{8} \left( 3\alpha + 3r_h^2 + 4\pi Pr_h^4 \right). \quad (5)
\]

The black hole temperature is given by

\[
T = \frac{1}{4\pi} f'(r_h) = \frac{4\pi r_h^4 \left( 3 + 8\pi r_h^2 \right) - Q^2}{24\pi^2 r_h^3 (2\alpha + r_h^2)}. \quad (6)
\]

From the Hawking temperature Eq.(6), the equation of state of the black hole can be given

\[
P = \frac{3}{4r_h} \left( 1 + \frac{2\alpha}{r_h^2} \right) T - \frac{3}{8\pi r_h^2} + \frac{Q^2}{32\pi^2 r_h^4}. \quad (7)
\]

The specific volume \( v \) can be identified as \[14\]

\[
v = \frac{4r_h}{3}. \quad (8)
\]

One can see that the specific volume \( v \) depends linearly on the horizon radius \( r_h \). Therefore, we will replace \( v \) with \( r_h \) in this paper. Near the critical point, we have \[28\]

\[
\tilde{v}_g - \tilde{v}_l \propto \tilde{r}_{hd} - \tilde{r}_{ml} \propto |1 - \tilde{T}|^{\frac{1}{2}}, \quad (9)
\]

where subscripts \( g \) and \( l \) stand for the gas phase and liquid phase of the van der Waals liquid-gas system, respectively.

With the thermodynamic pressure and volume defined, the black holes seem to behave like van der Waals behavior. The system does admit a first-order small-large black hole phase transition. According to the equation

\[
\frac{\partial P}{\partial r_h} \mid_{r_h=rc,T=T_c} = \frac{\partial^2 P}{\partial r_h^2} \mid_{r_h=rc,T=T_c} = 0, \quad (10)
\]

the critical point can be derived easily, where the index \( c \) denotes that a quantity takes the value at the critical point of van der Waals liquid-gas system.

Next, let us consider a free photon orbiting around a general static and spherically symmetric black hole in \( d(\geq 4)\)-dimensional spacetime. The line element of the black hole is

\[
ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega_{d-2}^2, \quad (11)
\]

where \( d\Omega_{d-2}^2 \) is the metric element on the unit \((d-2)\)-dimensional sphere, for which the angular coordinates are \( \theta_i \in [0, \pi] \) \((i = 1, ..., d - 3)\) and \( \phi \in [0, 2\pi] \).

For simplicity, we fix the photon in the equatorial plane which means that angular coordinates \( \theta_i = \pi/2 \) for \( i = 1, 2, ..., d - 3 \). Hence, the Lagrangian is

\[
2\mathcal{L} = -f(r)\dot{t}^2 + \frac{1}{f(r)}\dot{r}^2 + r^2\dot{\phi}^2. \quad (12)
\]

The metric has two Killing vectors, so there are two conserved quantities, energy \( E \) and orbital angular momentum \( L \) of the photon, as

\[
\frac{\partial \mathcal{L}}{\partial \dot{t}} = -f(r)\dot{t} = -E = \text{const}, \quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = r^2\dot{\phi} = L = \text{const}. \quad (13)
\]

In addition, we can derive the radial \( r \) motion from Eq. \( (12) \) as

\[
\dot{r} = -V_{\text{eff}}, \quad (14)
\]

where

\[
V_{\text{eff}} = \frac{L^2}{r^2} f(r) - E^2. \quad (15)
\]

When the radial velocity is equal to zero, the photon will round the black hole at that radial distance. For a spherically symmetric static black hole, it corresponds to
a photon sphere. From Eq. (15), one can easily find that there are two key quantities corresponding to the photon sphere, $r_{ps}$ and $u_{ps}$. The circular photon sphere radius $r_{ps}$ and the minimum impact parameter

$$u_{ps} = \frac{L}{E} \bigg|_{r=r_{ps}} = \frac{r}{\sqrt{f(r)}} \bigg|_{r=r_{ps}}$$

(16)

are determined by

$$V_{\text{eff}} = 0, \quad \frac{\partial V_{\text{eff}}}{\partial r} = 0, \quad \frac{\partial^2 V_{\text{eff}}}{\partial r^2} < 0.$$  

(17)

In fact, $u_{ps}$ has a close relation with the deflection angle of the photon. This is just the phenomenon of the black hole lensing. For more details about these two quantities, one can refer to Ref. [34].

Next, we are going to illustrate the relationship between photon orbits and thermodynamic phase transition.

### III. PHOTON ORBITS AND THERMODYNAMIC PHASE TRANSITION

For a 5-dimensional Gauss-Bonnet AdS black hole of $k = 1$, from Eq. (2), we can see that $Q$ determines the lowest power term of $r_h$. So it is necessary to study the difference between the case $Q = 0$ and the case $Q \neq 0$. In order to illustrate the relationship between the physical quantities better, we would like to discuss the thermodynamic quantities in the reduced parameter space, where a reduced quantity is defined as $X = X/X_c$, such as $T/T_c$ and $\tilde{P} = P/P_c$.

#### A. The case of $Q = 0$

In this case, the metric function reads

$$f(r) = 1 + \frac{r^2}{2\alpha} \left(1 - \sqrt{1 + \frac{32\alpha M}{3\pi r^4} - \frac{16\pi \alpha P}{3}}\right).$$  

(18)

Combining the result of Ref. [14]

$$r_{hc} = \sqrt{6\alpha}, \quad P_c = \frac{1}{48\pi \alpha}, \quad T_c = \frac{1}{\pi \sqrt{24\alpha}},$$

(19)

the reduced pressure can be given as

$$\tilde{P} = \frac{\tilde{T} + 3\tilde{r}_h(-1 + \tilde{r}_h\tilde{T})}{\tilde{r}_h^3}.$$  

(20)

Solving Eq. (17), we have

$$\tilde{r}_{ps} = \left(\frac{\tilde{r}_h (3\tilde{r}_h^3 + \tilde{r}_h) (\tilde{T}\tilde{r}_h + 1) (3\tilde{r}_h (\tilde{T}\tilde{r}_h + 1) + \tilde{T})}{7 (3\tilde{r}_h^2 + 1) (3\tilde{r}_h - \tilde{T})}\right)^{\frac{1}{4}},$$

(21)

Since expression of $\tilde{u}_{ps}$ is too long, we only give the function form. It is worth noting that, for fixed reduced pressure, they all depend on $\tilde{r}_h$. Combining Eq.(6) with Eq.(21), we plot the reduced pressure $\tilde{P}$ as a function of the photon sphere radius $\tilde{r}_{ps}$ for fixed temperature $\tilde{T}$, which is shown on the left of Fig.1. It behaves very similar to the isobar of vdW system in the $\tilde{P} - \tilde{v}$ chart which reflects the existence of a first-order phase transition. As we can see, from bottom to top, the oscillations don’t disappear until $\tilde{T} = 1$ at the critical point. Similarly, the behavior of $\tilde{P}$ as a function of $\tilde{u}_{ps}$ is illustrated on the right of Fig.1, which shows that there still exists a non-monotonic behavior when $\tilde{T} < 1$, just like the $P - v$ diagram of vdW fluid.

Next, we would like to verify the exponent of 1/2 near the critical point. Given a certain reduced temperature, such as $\tilde{T} = 0.87$, it corresponds to a phase transition of a reduced pressure denoted by $\tilde{P}^*$. Combining Eq.(20) with Eq.(21), we know that the reduced pressure $\tilde{P}$ must be a function of $\tilde{r}_{ps}$ in this case. Therefore, setting function $\tilde{P}(\tilde{r}_{ps})$ is equal to $\tilde{P}^*$, we can find a minimum solution $\tilde{r}_{ps1}$ and a maximum solution $\tilde{r}_{ps2}$. As the reduced temperature approaches to 1, $\Delta\tilde{r}_{ps}$ can be given by

$$\Delta\tilde{r}_{ps} = \tilde{r}_{ps2} - \tilde{r}_{ps1} = 1.07143\Delta\tilde{r}_h.$$  

(23)

where we have expanded $\tilde{r}_{ps1}$ near $\tilde{r}_h=1$, $i = 1, 2$. Similarly, we have

$$\Delta\tilde{u}_{ps} \sim 0.53347\Delta\tilde{r}_h.$$  

(24)

Combining Eq.(9), one can easily find that they have a same critical exponent of 1/2 near the critical point, just like these order parameters from the thermodynamic view. So the two quantities can be regarded as good characteristic parameters of black hole phase transition. This non-monotonic behavior can be viewed as a signature of the small-large black hole phase transition.

How can these special photon orbits (photon sphere) betray the information about the black hole phase transition? Considering that the horizon is a good characteristic parameter for the phase transition of black holes, we speculate that there is a special connection between these two quantities and the horizon. As shown in Fig.2, the monotonic relation between these two quantities and the horizon $\tilde{r}_h$ means that the oscillatory behavior between the pressure $\tilde{P}$ and the horizon $\tilde{r}_h$ is transferred to the relation between the pressure $\tilde{P}$ and the photon sphere radius $\tilde{r}_{ps}$, as well as the relation between the pressure $\tilde{P}$ and the minimum impact parameter $\tilde{u}_{ps}$.

#### B. The case of $Q \neq 0$

Thinking about

$$r = xQ^\frac{1}{2}, P = pQ^{-1}, \alpha = bQ, M = mQ,$$

(25)
the blackening factor in this case is
\[ f = 1 + \frac{x^2}{2b} \left( 1 - \sqrt{1 - \frac{16bp\pi}{3} - \frac{b}{3\pi x^6} + \frac{32bm}{3\pi x^4}} \right). \tag{26} \]

According to Eq.(5), the mass reads
\[ m = \frac{12\pi bx_h^2 + 16\pi^2 px_h^6 + 12\pi x_h^4 + 1}{32x_h^6}, \tag{27} \]

where \( x_h \) corresponds to horizon radius. According to Eq.(6), the temperature is given as
\[ t = \frac{-1 + 4\pi x_h^4(3 + 8\pi x_h^2)}{24\pi^2 x_h^4(2b + x_h^2)}. \tag{28} \]

Following Eq.(28), we have
\[ p = \frac{1 + 48b\pi^2 t_c x_h^3 - 12\pi x_h^4 + 24\pi^2 t_c x_h^5}{32\pi^2 x_h^6}. \tag{29} \]

Combining Eqs.(10) and (29), one can find that the critical point has to satisfy the following equation
\[ (1 + 24b\pi^2 t_c x_h^3 - 4\pi x_h^4 + 4\pi^2 t_c x_h^5) = 0, \tag{30} \]
\[ (7 + 96b\pi^2 t_c x_h^3 - 12\pi x_h^4 + 8\pi^2 t_c x_h^5) = 0. \]

From Eq.(17), the radius of photon sphere denoted by \( r_{ps} \), has to satisfy the following equation
\[ 12\sqrt{\pi x_{ps}^2 x_h^4} \sqrt{16\pi^2 bpx_{ps}^2 (x_h^4 - x_{ps}^4) + 3\pi x_{ps}^2 (4b (b + x_h^4) + x_{ps}^4) + b \left( \frac{x_{ps}^2}{x_h^4} - 1 \right)} \]
\[ - 2\sqrt{3}x_{ps}^2 (4\pi x_h^2 (3b + 4\pi px_h^4 + 3x_h^4) + 1) + 3\sqrt{3}x_h^2 = 0, \tag{31} \]

and another key quantity, the minimum impact parameter denoted by \( u_{ps} \), can also be derived from Eq.(16).
Therefore, we can notice that $\tilde{p}$, $\tilde{x}_{ps}$ and $\tilde{u}_{ps}$ are all functions of $\tilde{r}_h$ for fixed $b$ and reduced temperature $\tilde{t}$. With the parametric equations, we plot the $\tilde{p} - \tilde{x}_{ps}$ diagram and $\tilde{p} - \tilde{u}_{ps}$ diagram as shown in Fig.3. It is interesting that there is a non-monotonic behavior when $\tilde{t} < 1$. As $\tilde{t}$ approaches to 1, this behavior will disappear.

Using the same method as the previous section, we find that

$$\Delta \tilde{x}_{ps} \sim 1.09904 \Delta \tilde{x}_h,$$

$$\Delta \tilde{u}_{ps} \sim 0.512045 \Delta \tilde{x}_h.$$  \hspace{1cm} (32)  \hspace{1cm} (33)

Obviously, both of them have the same critical exponent $1/2$ at the critical point, just like these order parameters from the thermodynamic view. These results demonstrate that the two quantities can be regarded as good characteristic parameters of black hole phase transition. This non-monotonic behavior can be viewed as a signature of the small-large black hole phase transition.

We analyze the connection between $\tilde{x}_{ps}$ and $\tilde{x}_h$, as well as the connection between $\tilde{u}_{ps}$ and $\tilde{x}_h$ shown in Fig.4. The results are the same as we expected, both of these quantities are monotonically related to $\tilde{x}_h$. It explains why there is a non-monotonic behavior between the reduced pressure $\tilde{p}$ and these two quantities for the temperature below its critical value. The results mean that these two quantities can play the same role as the horizon in determining the phase transition of black hole.

IV. BACK TO THE CASE OF REISSNER-NORDSTROM ADS BLACK HOLES

The photon sphere radius and the minimum impact parameter can be regarded as good characteristic parameters to reflect black hole phase transition, which is consistent with the previous conjecture of Ref. [34] that thermodynamic phase transition is encoded in and can be revealed by the sudden change of the photon sphere radius and minimum impact parameter of the black hole. Considering the horizon of black holes is a good characteristic parameter to reflect black hole phase transition, we speculate that there is a special connection between the photon sphere radius (or the minimum impact parameter) and the horizon of black holes. For Gauss-Bonnet AdS black holes, we have found this connection. Due to the monotonic relationship between these two quantities and the horizon, the oscillating behavior between the pressure $\tilde{P}$ and the horizon $\tilde{r}_h$ is transferred to the pressure $\tilde{P}$ and the photon sphere radius $\tilde{r}_{ps}$ (or the minimum impact parameter $\tilde{u}_{ps}$). However, we don’t know if these two quantities satisfy the equal area rule. As
an example, we examine this back to the simpler case of Reissner-Nordström (RN) black holes.

From Fig.5, we find that there is indeed a non-monotonic behavior as the reduced temperature $\tilde{T}$ is less than 1 which is consistent with Ref.[34]. The connections between these two quantities and the horizon are illustrated in Fig.6. Due to this monotonous relationship, the oscillating behavior between the reduced pressure $\tilde{P}$ and the horizon $\tilde{r}_h$ is transferred to the reduced pressure $\tilde{P}$ and the photon sphere radius $\tilde{r}_{ps}$ (or the minimum impact parameter $\tilde{u}_{ps}$).

As we can see from Fig.7, according to the equation of coexistence curve [43], we derive the difference $\Delta S$ between the two areas enclosed by phase transition pressure line $\tilde{P}^*$ and $\tilde{P} - \tilde{r}_{ps}$ curve. The result is illustrated on the left of Fig.8. It’s worth noting that as the temperature approaches to the critical point, the equal area law can be exactly correct. In other words, the equal area law cannot be applied to photon sphere radius. Similarly, it cannot be applied to the minimum impact parameter $u_{ps}$ too as shown on the right of Fig.8.

FIG. 5. $\tilde{P}$ vs $\tilde{r}_{ps}$ left and $\tilde{P}$ vs $\tilde{u}_{ps}$ right for fixed reduced temperature $\tilde{T} = 0.71, 0.81, 0.91, 1, \text{and} 1.11$ from bottom to top.

FIG. 6. $\tilde{r}_{ps}$ vs $\tilde{r}_h$ left and $\tilde{u}_{ps}$ vs $\tilde{r}_h$ right for fixed reduced temperature $\tilde{T} = 0.71, 0.81, 0.91, 1, \text{and} 1.11$ from bottom to top.

FIG. 7. $\tilde{P}$ vs $\tilde{r}_{ps}$ for fixed reduced temperature $\tilde{T} = 0.81$. The horizontal line corresponds the phase transition pressure $\tilde{P}^*$. We set $\Delta S = |S_{ABC} - S_{CDE}|$ for a fixed reduced temperature $\tilde{T}$.

V. DISCUSSION AND CONCLUSION

The thermodynamic phase transition for 5-dimensional Gauss-Bonnet black holes in anti-de Sitter space has been investigated using the radius of photon sphere and the minimum impact parameter related to the photon orbits. In the reduced parameter space, no matter the black hole
is charged or not, there exist non-monotonic behaviors for the temperature below its critical value in terms of the two quantities, which resemble the behavior of the vdW fluid. At the same time, we find that they have the same critical exponent of 1/2 near the critical point, just like these order parameters from the thermodynamic view. What’s more, we illustrate that the roles of the photon orbit and the black hole horizon are much similar in determining black hole phase transition. As we turn back to the RN-AdS black holes, this connection is still true. However, it’s worth noting that, although these two key quantities can be regarded as good characteristic parameters to reflect black hole phase transition, they cannot observe the equal area rule.

In fact, due to the connection between these two quantities and the horizon of a black hole, such as monotonic increasing, it’s natural to regard these two quantities as good characteristic parameters to reflect black hole phase transition. What really matters is that we can study black hole thermodynamics through astronomical observation. For example, a mutation in the photon sphere radius or the minimum impact parameter means that the black hole must be undergoing a first-order phase transition.

Here, a new astronomical observation method is given to study black hole thermodynamic phase transition. What’s more, it provides a significant insight for establishing the connection between the strong gravitational effect [45–52] and the black hole thermodynamic phase transition. In addition, the relationship between the photon orbits and the maximum hypersurface inside black holes can give some clues about the relation between the photon orbits and the volume inside the black holes [53–58], which can be an interesting topic for the future.

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