Learning game-theoretic models of multiagent trajectories using implicit layers

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Abstract
For prediction of interacting agents’ trajectories, we propose an end-to-end trainable architecture that hybridizes neural nets with game-theoretic reasoning, has interpretable intermediate representations, and transfers to downstream decision making. It uses a net that reveals preferences from the agents’ past joint trajectory, and a differentiable implicit layer that maps these preferences to local Nash equilibria, forming the modes of the predicted future trajectory. Additionally, it learns an equilibrium refinement concept. For tractability, we introduce a new class of continuous potential games and an equilibrium-separating partition of the action space. We provide theoretical results for explicit gradients and soundness. In experiments, we evaluate our approach on two real-world data sets, where we predict highway drivers’ merging trajectories, and on a simple decision-making transfer task.

1 Introduction
Prediction of interacting agents’ trajectories has recently been advanced by flexible, tractable, multi-modal data-driven approaches. But it remains a challenge to use them for safety-critical domains with additional verification and decision-making transfer requirements, like automated driving or mobile robots in interaction with humans. Towards addressing this challenge, the following seem sensible intermediate goals: (1) incorporation of well-understood principles, prior knowledge and reasoning of the multiagent domain, allowing to generalize well and to transfer to robust downstream decision making; (2) interpretability of models’ latent variables, allowing for verification beyond just testing the final output; (3) theoretical analysis of soundness.

In this paper, we take a step towards addressing multiagent trajectory prediction including these intermediate goals, while trying to keep as much as possible of the practical strength of data-driven approaches. For this, we hybridize neural learning with game-theoretic reasoning – because game theory provides well-established explanations of agents’ behavior based on the principle of instrumental rationality, i.e., viewing agents as utility maximizers. Roughly speaking, we “fit a game to the observed trajectory data”. Along this hybrid direction one major obstacle – and a general reason why game theory often remains in abstract settings – lies in classic game-theoretic solution concepts like the Nash equilibrium (NE) notoriously suffering from computational intractability. As one way to overcome this, we build on local NE (Ratliff, Burden, and Sastry 2013, 2016). We combine this with a specific class of games – (continuous) potential games (Monderer and Shapley 1996) – for which local NE usually coincide with local optima of a single objective function, simplifying search. Another challenge lies in combining game theory with neural nets in a way that makes the overall model still efficiently trainable by gradient-based methods. To address this, we build on implicit layers (Amos and Kolter 2017, Bai, Kolter, and Koltun 2019). Implicit layers specify the functional relation between a layer’s input and its output not in closed form, but only implicitly, usually via an equation. Nonetheless they allow to get exact gradients by “differentiating through” the equation, based on the implicit function theorem.

Main contributions and outline. We propose a modular architecture that outputs a multi-modal prediction of interacting agents’ joint trajectory (where modes are interpretable as local Nash equilibria), from their past trajectory as input (Sec. 3). The architecture is depicted in Fig. 1 alongside the motivating example of highway drivers’ merging trajectories. It builds on the following components:

• a tractable, differentiable game solver implicit layer (Sec. 3.1) with explicit gradient formula, mapping game parameters to local Nash equilibria (Thm. 1). It is based on a new class of continuous-action trajectory games that allow to encode prior knowledge on agents’ preferences (Def. 4). We prove that they are potential games (Lem. 1). And it builds on an equilibrium-separating concave partition of the action space that we introduce to ensure tractability (Def. 5).

• Furthermore, the architecture contains a neural net that reveals the agents’ preferences from their past, and a net that learns an equilibrium refinement concept (Sec. 3.2).

This architecture forms a model class where certain latent representations have clear game-theoretic interpretations and certain layers encode game-theoretic principles that help induction (also towards strategically-robust decision-making). At the same time, it has neural net-based capacity for learning, and is end-to-end trainable with analytic gradient formula. Furthermore:
Figure 1: **Bottom:** Our full architecture (Sec. 3.2). **Top Left:** Example (stylized): highway merging scenario, where reliable models of (human) driver interaction are key for safe automated driving. **Top Left:** input $x$: initial trajectories of drivers. **Top Right:** prediction of future trajectory $y$: depicted are two modes $\hat{\gamma}_1, \hat{\gamma}_2$ corresponding to two local Nash eq. $1^\alpha, 2^\alpha$: red going first vs. yellow first.

- In Sec. 4 we give two concrete example scenarios that provably satisfy our approach’s conditions (Prop. 1 etc.).
- In the experiments reported in Sec. 5 we apply our architecture to prediction of real-world highway on-ramp merging driver interaction trajectories, on one established and one new data set we publish alongside this paper. We also apply it to a simple decision-making transfer task.

Keep in mind that proofs are available in Sec. A and that the present paper is an extended version of Geiger and Straehle 2021. In what follows, we first discuss related work and introduce setting and background (Sec. 2).

**Closest related work.** Regarding general multiagent model learning from observational behavioral data with game-theoretic components: closest related is work by Ling, Fang, and Kolter (2018, 2019), who use game solvers as differentiable implicit layers, learning these layers’ input (i.e., agents’ preferences) from covariates. They focus on discrete actions while we address continuous trajectory prediction. And they use different solution concepts, and do not consider equilibrium refinement. There is further work more broadly related in this direction (Kita 1999, Kita, Tanimoto, and Fukuyama 2002, Liu et al. 2007, Kang and Rakha 2017, Tian et al. 2018, Li et al. 2018, Fox et al. 2018, Camara et al. 2018, Ma et al. 2017, Sun, Zhan, and Tomizuka 2018), sometimes also studying driver interaction, but they have no or little data-driven aspects (in particular no implicit layers) and/or use different approximations to rationality than our local NE, such as level-k reasoning, and often are less general than us, often focusing on discrete actions. More broadly related is multiagent inverse reinforcement learning (IRL) (Wang and Klajban 2018, Reddy et al. 2012, Zhang et al. 2019, Etesami and Straehle 2020), often discrete-action.

For multiagent trajectory prediction, there generally is a growing number of papers on the machine learning side, often building on deep learning principles and allowing multimodality – but without game-theoretic components. Without any claim to completeness, there is work using long-term memories (LSTMs) (Alahi et al. 2016, Deo and Trivedi 2018, Salzmann et al. 2020), generative adversarial networks (GANs) (Gupta et al. 2018), and attention-based encoders (Tang and Salakhutdinov 2019). Kuderer et al. (2012) uses a partition (“topological variants”) of the trajectory space related to ours. There is also work related to the principle of “social force” (Helbing and Molnar 1995, Robicquet et al. 2016, Blaiotta 2019), and related rule-based driver modeling approaches (Treiber, Hennecke, and Helbing 2000, Kesting, Treiber, and Helbing 2007).

Regarding additional game-theoretic elements: W.r.t. the class of trajectory potential games we introduce (Det. 4), the closest related work we are aware of is Zazo et al. 2016 who consider a related class, but they do not allow agents’ utilities to have differing additive terms w.r.t. their own actions. Worth mentioning is further related work based on games (different ones than ours though), but towards pure control (not prediction) tasks (Peters et al. 2020, Zhang et al. 2018, Spica et al. 2018, Fisac et al. 2019). Peters et al. (2020) use a latent variable for the equilibrium selection, similar to our equilibrium weighting. For further related work see Sec. 5 in the appendix.

2 General Setting, Goals and Background

We consider scenes, each consisting of:

- a set $I := \{1, \ldots, n\}$ of agents.
- Each agent $i \in I$ at each time $t \in [0, T]$ has an individual state $y^i_t \in \mathbb{R}^{d_y}$. They yield an individual trajectory $y^i = (y^i_t)_{t \in [0, T]}$ (think of 0 as the present time point and $T$ as the horizon up to which we want to predict).
- And $y := ((y^1_t, \ldots, y^n_t))_{t \in [0, T]} \in Y$ denotes the agents’ joint (future) trajectory.
We assume that the past \textit{joint trajectory} \(x \in X\) of the agents until time point 0 is available as side information.

Now, besides the other goals mentioned in Sec.\[1\], we formulate the main (passive) predictive problem as follows:

- \textit{goal:} in a new scene, predict the future joint trajectory \(y\) by a list of pairs \((\hat{y}_i, \hat{q})\), corresponding to \(y\)'s modes, where each \(\hat{q}\) is a point prediction of \(y_i\) and \(\hat{q}\) the associated probability (for more details and metrics etc., see Sec.\[3.2\] [5]):

- \textit{given:} (1) the past trajectory \(x\) of that new scene, as well as (2) a training set consisting of previously sampled scenes, i.e., pairs \((x', y'), (x'', y'')\), \ldots of past and future trajectory (discrete-time subsampled of course). (We assume all scenes sampled from the same underlying distribution.)

We assume that agent \(i\)'s (future) trajectory \(y_i\) is parameterized by a finite-dimensional vector \(a_i \in A_i \subseteq \mathbb{R}^{d_i}\), which we refer to as \(i\)'s action, with \(A_i\) the action space of \(i\). So, in particular, there is a (joint) trajectory parameterization \(r : A \rightarrow Y\), with \(A := A_1 \times \ldots \times A_n\) the joint action space. Keep in mind that \(a = (a_1, \ldots, a_n)\), \(a_i^{-1}\) means \((a_1, \ldots, a_i^{-1}, a_i^{+1}, a_i^n)\) and \((a_i, a_i^{-1})\) reads \(a_i\).

We use games (Shoham and Leyton-Brown\[2008\]; Osborne and Rubinstein\[1994\]) to model our setting. A game specifies the set of agents (also called “players”), their possible actions and their utility functions. The following formal definition is slightly tailored to our setting: utilities are integrals over the trajectories parameterized by the actions.

**Definition 1** (Game). A (trajectory) game consists of: the set \(I\) of agents, and for each agent \(i \in I\) the action space \(A_i \subseteq \mathbb{R}^{d_i}\), and a utility function \(u^i : A \rightarrow \mathbb{R}\). We assume \(u^i_i, i \in I\), to be of the form

\[
u^i(a) = \int_0^T u^i_t(y_t) \, d\mu(t),\]

where \(a \in A, y = r(a); u^i, t \in [0, T]\), are the stage-wise utility functions, and \(\mu\) is a measure on \([0, T]\).

This game formalizes the agents’ “decision-making problem”. Game theory also provides the “Nash equilibrium” as a concept of how rational agents will/should act to “solve” the game. Here we use a “local” version – for tractability:

**Definition 2** (Local Nash equilibrium (NE) (Ratliff, Burden, and Sastry\[2016\];\[2013\]). Given a game, a joint action \(a \in A\) is a (pure) local Nash equilibrium (local NE) if there are open sets \(S_i \subseteq A_i\) such that \(a_i \in S_i\) and for each \(i\),

\[u^i(a', a^{-i}) \geq u^i(a_i', a_i^{-i}),\]

for any \(a' \in S_i\). If \(S_i = A_i\) for all \(i\), then \(a\) is called a (pure, global) NE.

The following type of game can reduce finding local NE to finding local optima of a single objective (“potential function”), allowing for tractable gradient ascent-based search.

**Definition 3** (Potential game (Monderer and Shapley\[1996\]). A game is called an (exact continuous) potential game, if there is a so-called potential function \(\psi\) such that, for all agents \(i\), all actions \(a_i, a_i'\) and remaining actions \(a^{-i}\),

\[u^i(a_i', a_i^{-i}) - u^i(a_i, a_i^{-i}) = \psi(a_i', a_i^{-i}) - \psi(a_i, a_i^{-i}).\]

Let us also give some neural net-related background:

**Remark 1** (Implicit layers (Amos and Kolter\[2017\]; Amos et al.\[2018\]; Bai, Kolter, and Koltun\[2019\]; El Ghaoui et al.\[2019\]). Classically, one specifies a neural net layer by specifying the functional relation between its input \(v\) and output \(w\) explicitly, in closed form, \(w = f(v)\), for some function \(f\) (e.g., a softmax). The idea of implicit layers is to specify the relation implicitly, usually via an equation \(h(v, w) = 0\) (coming from, e.g., a stationarity condition of an optimization or dynamics modeling problem). To ensure that this specification is indeed useful in prediction and training, there are two important requirements: (1) the equation has to determine a unique, tractable function \(f\) that maps \(v\) to \(w\), and (2) \(f\) has to be differentiable, ideally with explicitly given analytic gradients.

### 3 General Approach With Analysis

We now describe our general approach. It consists of (a) a game-theoretic model and differentiable reasoning about how the agents behave (Sec.\[3.1\]), and (b) a neural net architecture that incorporates this game-theoretic model/reasoning as an implicit layer and combines it with learnable modules, with tractable training and decision-making transfer abilities (Sec.\[3.2\]).

#### 3.1 Common-Coupled Games, Equilibrium-Separation and Induced Implicit Layer

For the rest of the paper, let \((\Gamma_\theta)_{\theta \in \Theta} \subseteq \mathbb{R}^{d_{\Theta}}\), be a parametric family of trajectory games (Def.\[1\]). First let us introduce the following type of trajectory game to strike a balance between adequate modeling and tractability:

**Definition 4** (Common-coupled game). We call \(\Gamma_\theta\) a common-coupled(-term trajectory) game, if the stage-wise utility functions (Def.\[7\]) have the following form, for all agents \(i \in I, t \in [0, T]\):

\[u^i_t(y_t) = u^i_{\text{com}, \theta}(y_t) + u^i_{\text{own}, i, \theta}(y_t^i) + u^i_{\text{oth}, i, \theta}(y_t^{-i}),\]

where \(y = r(a)\) (action parameterizes trajectory, Sec.\[2\]),

\[u^i_{\text{com}, \theta}\] is a term that depends on all agents’ trajectories and is common between agents, \(u^i_{\text{own}, i, \theta}\) and \(u^i_{\text{oth}, i, \theta}\) terms that only depend on agent \(i\)’s trajectory, or all other agents’ trajectories, respectively, and may differ between agents.

Common-coupled games adequately approximate many multiagent trajectory settings where agents trade off (a) social norms and/or common interests (say, traffic rules or the common interest to avoid crashes), captured by the common utility term \(u^i_{\text{com}, \theta}\), against (b) individual inclinations related to their own state, captured by the terms \(u^i_{\text{own}, i, \theta}\). It is non-cooperative, i.e., utilities differ, but more on the cooperative than adversarial end of games. For tractability we can state:
Lemma 1. If $\Gamma_\theta$ is a common-coupled game, then it is a potential game with the following potential function, where, as usual, $y = r(a)$:

$$\psi(\theta, a) = \int_0^T u_t^{\text{comp}}(y_t) + \sum_{i \in T} u_t^{\text{comp},i}(y_i^t) d\mu(t).$$

Note that this implies existence of NE, given continuity of the utilities and compactness [Monderer and Shapley 1996].

We now show how the mappings from parameters $\theta$ to local NE of the game $\Gamma_\theta$ can be soundly defined, tractable and differentiable, so that we can use this game-theoretic reasoning as one implicit layer (Rem. 1) in our architecture. For this, a helpful step is to (tractably) partition the action space into subspaces with exactly one equilibrium each – if the game permits this. For the rest of the paper, let $(A_k)_{k \in K}$ be a finite collection of subspaces of $A$, i.e., $A_k \subseteq A$.

Definition 5 (Equilibrium-separating action subspaces). For a common-coupled game $\Gamma_\theta$, we call the action subspace collection $(A_k)_{k \in K}$ equilibrium-separating (partition) if, for all $k \in K$ and $\theta \in \Theta$, the game’s potential function $\psi(\theta, \cdot)$ is strictly concave on $A_k$.

As a simplified example, a first partition towards equilibrium-separation in the highway merging scenario of Fig. 1 would be into two subspaces: (1) those that result in joint trajectories where the red car goes first and (2) those where yellow goes first. More details follow in Scenario 1.

Keep in mind that the equation $\nabla_a \psi(\theta, a)=0$ is a necessary condition for $a$ to be a local NE of $\Gamma_\theta$ (for interior points), since local optima of the potential function correspond to local NE. This equation induces an implicit layer:

Assumption 1. Let $\Gamma_\theta$ be a common-coupled game. Let $(A_k)_{k \in K}$ be equilibrium-separating subspaces for it, and let all $A_k, k \in K$ be compact, given by the intersection of linear inequality constraints. On each subspace $\Theta \times A_k, k \in K$, let $\Gamma_\theta$’s potential function $\psi$ be continuous.

Theorem 1 (Games-induced differentiable implicit layer). Let Assumption 1 hold true.

Then, for each $k \in K$, there is a continuous mapping $g_k : \Theta \to A_k$, such that for any $\theta \in \Theta$, if $g_k(\theta)$ lies in the interior of $A_k$, then

- $g_k(\theta)$ is a local NE of $\Gamma_\theta$.
- $g_k(\theta)$ is given by the unique argmax of $\psi(\theta, \cdot)$ on $A_k$, with $\psi$ the game’s potential function (Lem. 1).
- $g_k$ is continuously differentiable in $\theta$ with gradient $J_\theta g_k(\theta) = -(H_\theta \psi(\theta, a))^{-1} J_\theta \nabla_a \psi(\theta, a)$, whenever $\psi$ is twice continuously differentiable on an open set containing $(\theta, a)$, for $a = g_k(\theta)$, where $\nabla, J$ and $H$ denote gradient, Jacobian and Hessian, respectively.

The specifics of how the $g_k$ of Thm. 1 form an implicit layer will be discussed in Sec. 3.2.

Remark on boundaries. There remain several questions: e.g., whether the action space partition introduces "artificial" local NE at the boundaries of the subspaces; and also regarding what happens to the gradient if $g_k(\theta)$ lies at the boundary of $A$ or $A_k$. Here we state a preliminary answer to the latter:

Lemma 2. Assume Assumption 1 and that $\psi$ is twice continuously differentiable on a neighborhood of $\Theta \times A_k, k \in K$. If $a = g_k(\theta)$ lies exactly on one constraining affine hyperplane of $A_k$, defined by orthogonal vector $v$, with multiplier $\lambda$ and optimum $\lambda^* > 0$ of $\psi(\theta, a)$’s Lagrangian (details see proof), then $J_\theta g_k(\theta)$ is the upper left $n_d A_k \times n_d A_k$-submatrix of

$$- \left( \begin{array}{cc} H_\theta \psi(\theta, a) & v \times T \\ v^T & 0 \end{array} \right)^{-1} \left( \begin{array}{c} J_\theta \nabla_a \psi(\theta, a) \\ 0 \end{array} \right).$$

Remark on identifiability. Another natural question is whether the game’s parameters are identifiable from observations, and, especially, whether the $g_k$ are invertible. While difficult to answer in general, we investigate this for one scenario in Sec. C in the appendix.

3.2 Full Architecture With Further Modules, Tractable Training and Decision Making

Now for the overall problem of mapping past joint trajectories $x$ to predictions of their future continuations $y$, we propose the architecture depicted in Fig. 1 alongside a training procedure. We call it trajectory game learner (TGL). (Its forward pass is explicitly sketched in Alg. 1 in Sec. E in the appendix.) It contains the following modules (here we leave some of the modules fairly abstract because details depend on size of the data set etc.; for one concrete instances see the experimental setup in Sec. C), which are well-defined under Assumption 1.

- $\psi_k(\theta)$ is given by the unique argmax of $\psi(\theta, \cdot)$ on $A_k$.
- $g_k(\theta)$ is continuously differentiable in $\theta$ with gradient $J_\theta g_k(\theta) = -(H_\theta \psi(\theta, a))^{-1} J_\theta \nabla_a \psi(\theta, a)$, whenever $\psi$ twice continuously differentiable on an open set containing $(\theta, a), a = g_k(\theta)$.
- $J_\theta g_k(\theta) = -(H_\theta \psi(\theta, a))^{-1} J_\theta \nabla_a \psi(\theta, a)$.
- $J_\theta g_k(\theta)$ is the upper left $n_d A_k \times n_d A_k$-submatrix of

$$- \left( \begin{array}{cc} H_\theta \psi(\theta, a) & v \times T \\ v^T & 0 \end{array} \right)^{-1} \left( \begin{array}{c} J_\theta \nabla_a \psi(\theta, a) \\ 0 \end{array} \right).$$

- Due to the concavity assumption, we can use established tractable, guaranteeably sound algorithms to calculate this argmax.

- Note that similar results have already been established, in the sense of constrained optimizers as implicit layers [Amos and Kolter 2017], but we give the precise preconditions for our setting. See also Sec. C in the appendix. Moreover, note that under the conditions of this lemma, i.e., when $g_k(\theta)$ lies at the boundary, then the above gradient $J_\theta g_k(\theta)$ in fact often becomes zero, which can be a problem for parameter fitting.
• **Preference revelation net**: It maps the past joint trajectory \( x \in X \) to the inferred game parameters \( \theta \in \Theta \) (encoding agents preferences)\(^\text{8}\). For example, this can be an LSTM.

• **Equilibrium refinement net**: This net maps the past joint trajectory \( x \in X \) to a subset \( \tilde{K} \subseteq K \) (we encode \( \tilde{K} \) e.g. via a multi-hot encoding), with |\( \tilde{K} \)| = \( \tilde{k} \), for \( \tilde{k} \) arbitrary but fixed. This subset \( \tilde{K} \) selects a subcollection \( (\tilde{A}_k)_{k \in \tilde{K}} \) of the full equilibrium-separating action space partition \( (\tilde{A}_k)_{k \in K} \) (introduced in Sec. 3.1, Def. 5). This directly determines a subcollection of local NE of the game \( \Gamma_\theta \), denoted by \( (\tilde{k}a^*)_{k \in \tilde{K}} \), those local NE that lie in one of the subspaces \( \tilde{A}_k, k \in \tilde{K} \). The purpose is to narrow down the set of all local NE to a “refined” set of local NE that form the “most likely” candidates to be selected by the agents.\(^\text{9}\) The reason why we not directly output the refined local NE (instead of the subspaces) is to simplify training (details follow). As a simple example, take a feed forward net with softmax as final layer to get a probability distribution over \( K \), and then take the \( \tilde{k} \) most probable \( k \) to obtain the set \( \tilde{K} \).

• **Game solver implicit layer**\(^\text{10}\) : It maps the revealed game parameters \( \theta \in \Theta \) together with the refined \( \tilde{K} \) to the refined subcollection \( (\tilde{k}a^*)_{k \in \tilde{K}} \) of local NE (described in the equilibrium refinement net above). This is done by performing, for each \( k \in \tilde{K} \), the concave optimization over the subspace \( \tilde{A}_k \):

\[
\tilde{k}a^* = g_k(\theta) = \arg \max_{a \in \tilde{A}_k} \psi(\theta, a),
\]

based on Thm. 1. See also Line 3 to 4 in Alg. 1 in Sec. 3 in the appendix.

• **Equilibrium weighting net**: It outputs probabilities \( (\tilde{k}g)_{k \in \tilde{K}} \) over the refined equilibria, and thus probabilities of the modes of our prediction (introduced in Sec. 2). We think of them as the probabilities of the mixture components in a mixture model, but leave the precise metrics open. As input, in principle the variables \( \theta, (\tilde{k}a^*)_{k \in \tilde{K}} \) are allowed, plus possibly the agents’ utilities attained in the respective equilibria.

\[
\text{In a sense, this net is the inverse of the game solver implicit layer on } x, \text{ but can be more flexible.}
\]

\[
\text{To be exact, in rare cases it can happen that some of these local NE are “artificial” as discussed in Sec. 3.1.}
\]

\[
\text{In game theory, “equilibrium refinement concepts” mean hand-crafted concepts that narrow down the set of equilibria of a game (for various reasons, such as achieving “stable” solutions) (Osborne and Rubinstein 1994). For us, the “locality relaxation” makes the problem of “too many” equilibria particularly severe, since the number of local NE can be even bigger than global NE; it can grow exponentially in the number of agents in our scenarios.}
\]

\[
\text{At first sight, local NE are a poorer approximation to rationality than global NE, and are mainly motivated by tractability. However, we found that in various scenarios, like the highway merging, local NE do seem to correspond to something meaningful, like the intuitive modes of the distribution of joint trajectories. NB: Generally, we do not consider humans as fully (instrumentally) rational, but we see (instrumental) rationality as a useful approximation.}
\]

librium. And one can think of various function classes, for instance a feed forward net with softmax final layer. Its purpose is to (probabilistically) learn agents’ “equilibrium selection” mechanism considered in game theory.\(^\text{14}\)

• **Trajectory parameterization** \( r \): This is the pre-determined parameterization from Sec. 2, it maps each local NE’s joint action \( k a^* \) to the corresponding joint trajectory \( \tilde{k}y \) that results from it, corresponding to mode \( k \) of the prediction, where \( k \in \tilde{K} \) are the indices of the refined equilibria.

Training and tractability. Training the architecture in principle happens as usual by fitting it to past scenes in the training set, sketched in Alg. 2 in Sec. 3 in the appendix. The implicit layer’s gradient for backpropagation is given in Thm. 1. By default, we take the mean absolute error (MAE) averaged over the prediction horizon \([0, T]\) (see also Sec. 5). Note that the full architecture – all modules plugged together – is not differentiable, because the equilibrium refinement net’s output is discrete. However, it is easy to see that (1) the equilibrium refinement net and (2) the rest of the architecture can be trained separately and are both differentiable themselves: in training, for each sample \((x,y)\), we directly know which subspace \( y \) lies in, so we first only train the equilibrium refinement net with this subspace’s index \( k \) as target, and then train the full architecture with the equilibrium refinement net’s weights fixed.\(^\text{14,15}\)

Observe that in training there is an outer (weight fitting) and an inner (game solver, i.e., potential function maximizer, during forward pass) optimization loop, so their speed is crucial. For the game solver, we recommend quasi-Newton methods like L-BFGS, because this is possible due to the subspace-wise concavity of the potential function (Assumption 1). For the outer loop, we recommend recent stochastic quasi-Newton methods (Wang et al. 2017; Li and Liu 2018).

Transferability to decision making. Once the game \( \Gamma_\theta \)’s parameters \( \theta \) are learned (for arbitrary numbers of agents) as described above, it does not just help for prediction – i.e., a model of how an observed set of strategic agents will behave – but also for prescription. This means (among other things) that it tells how a newly introduced agent should decide to maximize its utility, while aware of how the other agents respond to it based on their utilities in \( \Gamma_\theta \) (think of a self-driving car entering a scene with other – human – drivers).\(^\text{16}\) Note: the knowledge of \( \Gamma_\theta \) cannot resolve the remaining equilibrium selection problem (but the equilibrium weighting net may help). For an example see Sec. 5.2.

\[
\text{Equilibrium selection} \quad ^{\text{13}} \quad \text{(Harsanyi, Selten et al. 1988) refers to the problem of which single equilibrium agents will end up choosing if there are multiple – possibly even after a refinement.}
\]

\[
\text{Therefore we loosely refer to the full architecture as “end-to-end trainable”, not “end-to-end differentiable”.}
\]

\[
\text{On a related note, we learn the common term’s parameter } \theta \text{ (see (1)) as shared between all scenes, while the other parameters are predicted from the individual’s sample past trajectory.}
\]

\[
\text{This is the general double nature of game theory – predictive and prescriptive (Shoham and Leyton-Brown 2008).}
\]
4 Concrete Example Scenarios With Analysis

We give two examples of settings alongside games and action space partitions that provably fulfill the conditions for our general approach (Sec. 3) to apply. First we consider a scenario that captures various non-trivial driver interactions like overtaking or merging at on-ramps. Essentially, it consists of a straight road section with multiple (same-direction) lanes, where some lanes can end within the section. Fig. 1 and 2 (left) are examples. This setting will be used in the experiments (Sec. 5).

Scenario 1 (Multi-lane driver interaction). Setting: The set of possible individual states, denote it by \(Y_0\), is of the form \([b, c] \times [d, e] – positions on a road section. There are \(m\) parallel lanes (some of which may end), parallel to the x-axis. Agent \(i\)’s action \(a' \in A'\) is given by the sequence of planar (i.e., 2-D) positions denoted \((v_i^t, w_i^t) \in Y_0, t = 0, \ldots, T\), but not allowing backward moves (and possibly other constraints). Define the states \(y_i^t := ((v_i^t, w_i^t), (v_{i-1}^t, w_{i-1}^t), (v_{i+1}^t, w_{i+1}^t))\). And let \(y^t\) be the linear interpolation. Game: Let, for \(t = 0, \ldots, T\), the stage utilities of agent \(i\) in the game \(\Gamma_\theta\) be the following sum of terms for distance between agents, distance to center of lane, desired velocities, acceleration penalty, and end of lane overshooting penalty, respectively:

\[
\begin{align*}
  u_{t, \theta}^i(y_i^t) & = -\theta_{\text{dist}} \sum \frac{1}{|v_j^t - v_i^t| + \epsilon} - \theta_{\text{cen}}^{i,j}(w_i^t - c_j)^2 \quad (2a) \\
  & - \theta_{\text{vel}}^{i,j}(\delta w_i^t - \delta w_j^t)^2 - \theta_{\text{velw}}^{i,j}(\delta w_i^t)^2 - \theta_{\text{acc}}^{i,j}(\delta^2 v_i^t)^2 \quad (2b) \\
  & - \theta_{\text{end}} \max(0, v_i - c_i^t), \quad (2c)
\end{align*}
\]

where the sum ranges over all \((j, j')\) such that driver \(j\) is right before \(j'\) on the same lane; \(\epsilon > 0\) is a constant, \(c_j^t\) is the respective center of the lane, \(\delta w_i^t\) means velocity along lane, \(\delta w_i^t\) means lateral velocity, \(\delta^2 v_i^t\) means acceleration (vector), \(c_i^t\) is the end of \(i\)’s lane, if it ends, otherwise \(-\infty\); furthermore, \(\mu\) is the counting measure on \([0, \ldots, T]\) (i.e., discrete), and \(\theta = (\theta_{\text{dist}}, \theta_{\text{cen}}^{i,j}, \theta_{\text{vel}}^{i,j}, \theta_{\text{velw}}^{i,j}, \theta_{\text{acc}}^{i,j}, \theta_{\text{end}})_{i, j, t} \in \mathbb{R}^6\).

**Action**

We do this state augmentation so that utilities can also depend on velocity/acceleration (not just position) while still rigorously fitting into Def. 1. When calculating prediction errors for \(\hat{y}\), only the position component is considered.

**Note** that the invariance over time of the utility terms, as we assume it here, is a key element of how rationality principles can give informative priors.

We allow some of the weights to vary with \(t\) to add some flexibility. Consider the following equivalence relation on the trajectory space \(Y\): two joint trajectories \(y, y' \in Y\) are equivalent if at each time point \(t\), (1) each agent \(i\) is on the same lane in \(y\) as in \(y'\), and (2) within each lane, the order of the agents (along the driving direction) is the same in \(y\) as in \(y'\). Now let the subplace collection \(\{\tilde{A}_k\}_{k \in K}\) be obtained by taking the (closures of the) resulting equivalence classes.

**Proposition 1** (Scenario 1) The subspaces \(\tilde{A}_k\)'s are equivalence classes of the resulting partitioning scenario. Hence, \(\Gamma_{\tilde{\theta}}\) satisfies Assumption 1. So, in particular, Thm. 1 is applicable, and the induced implicit layer hold true.

Our general approach (Sec. 3) in principle is also applicable to various other multiagent trajectory settings, such as pedestrian interaction, relevant for mobile robots. We analyze a simplistic such scenario in Sec. 3 in the appendix, for a foretaste.

5 Experiments

We evaluate our approach on (1) an observational prediction task\(^{20}\) on two real-world data sets (Sec. 5.1), as well as (2) a simple decision-making transfer task (Sec. 5.2).

5.1 Prediction Task on Highway Merging Scenarios in Two Real-World Data Set

We consider a highway merging interaction scenario with two cars similar as sketched in Fig. 1. This is considered a challenging scenario for autonomous driving.

Implementation details for our method for these merging scenarios. We use the following generic implementation of our general approach (Sec. 3), with concrete setting, game and action subspaces from Scenario 1 (with \(n = 2\)), referring to it as \(TGL\) (trajectory game learner): We use validation-based early stopping. We combine equilibrium refinement and weighting net into one module, consisting of two nets that predict the weights \((\tilde{\theta})_{k \in \mathbb{R}}\) on the combination of (1) merging order (before/after) probabilities via a cross-entropy loss (2 hidden layers: \(1 \times 16, 1 \times 4\) neurons; dropout 0.6), and (2) Gaussian distribution over merging time point (discretized and truncated, thus the support inducing a refinement; 2 hidden layers: \(1 \times 64, 1 \times 32\) neurons; dropout 0.6), given \(\tilde{\theta}\). For the preference revelation net we use a feed forward net (two hidden layers: \(1 \times 16, 1 \times 24\) neurons).\(^{22}\) As training loss we use mean absolute error (MAE; see also evaluation details below).

\(^{19}\) We use some of the weights to vary with \(t\) to add some flexibility. Consider the following equivalence relation on the trajectory space \(Y\): two joint trajectories \(y, y' \in Y\) are equivalent if at each time point \(t\), (1) each agent \(i\) is on the same lane in \(y\) as in \(y'\), and (2) within each lane, the order of the agents (along the driving direction) is the same in \(y\) as in \(y'\). Now let the subplace collection \(\{\tilde{A}_k\}_{k \in K}\) be obtained by taking the (closures of the) resulting equivalence classes.

\(^{20}\) In the two-driver on-ramp scenario of Fig. 1 and experiments (Sec. 5.1), these subspaces roughly amount to splitting the action space \(A\) w.r.t. (1) time point of merge and (2) which driver goes first. Note that (1) are additional splits beyond the intuitive ones in (2) (see Fig. 1), but they help for concavity and for the analysis.

\(^{21}\) This directly evaluates the method's abilities for the observational/passive prediction task, but it is also a proxy metric/task for decision making.

\(^{22}\) For varying initial trajectory lengths, an LSTM might be more suitable.
Besides this generic instantiation, we also consider a version of it, termed TGL-D: Instead of predicting the desired velocity $v^{\text{desired}}$ itself, the preference revelation net predicts the difference to the past velocity, and then squashes this into a sensible range using a sigmoid. (This can be seen as encoding a bit of additional prior knowledge which may not always be easy to specify and depend on the situation.) For further details, see Sec. D in the appendix. Code is available at: https://github.com/boschresearch/trajectory_games_learning

Table 1: Prediction task: Our method(s) vs. state-of-the-art (CS-LSTM (Deo and Trivedi 2018), MFP (Tang and Salakhutdinov 2019)) for a prediction task on merging scenarios in two real-world highway data sets, averaged over a 7s prediction horizon.

| Data set | Metric | TGL (ours) | TGL-D (ours) | CS-LSTM | MFP |
|----------|--------|------------|--------------|---------|-----|
| highD    | MAE    | 3.6        | 2.9          | 5.0     | 5.2 |
|          | RMSE   | 4.9        | 3.7          | 6.8     | 7.1 |
| HEE (our new data set; Sec. 5.1) | MAE    | 3.7        | 3.2          | 3.6     | 3.7 |
|          | RMSE   | 4.7        | 4.1          | 4.3     | 4.8 |

Figure 3: Decision-making transfer task: Solution trajectory(s) that the (partially learned) game implies for the self-driving car’s decision-making task (each circle/square corresponds to one time step). Left: First local NE: the self-driving car (red) does a full emergency break and the other (blue) merges before it. Right: Second local NE: the other merges after it, both slow down.

Results. The results are in Table 1 (with more details in Sec. D in the appendix). Our generic method TGL outperforms CS-LSTM and MFP on highD. And our slightly more hand-crafted method TGL-D outperforms them on both data sets. Keep in mind that the data sets are small. So, while the results do indicate the practicality of our method in this small-sample regime, their significance is comparably limited.

5.2 Simple Decision-Making Transfer Task in Simulation

As discussed in Sec. 3.2, the game $\Gamma_{\theta}$ – once $\theta$ is given, e.g., by our learned preference revelation net – naturally transfers to decision-making tasks in situations with multiple strategic agents (something which predictive methods like the above CS-LSTM usually cannot do). To test and illustrate its ability for this, we consider a simple scenario: Take the above two-car highway on-ramp situation (Sec. 5.1 Scenario 1), but assume that the car on the highway lane is a self-driving car. Assume it has a technical failure roughly at the height of the on-ramp’s end, and it should do an emergency break (i.e., desired velocity $v^{\text{desired}}$ in (2) is set to 0) while at the same time ensuring that the other car coming from the on-ramp will not crash into it. Which trajectory should it choose? Result. Fed with this situation, our game solver suggests two possible solutions – two local NE, see Fig. 5 (1) the self-driving car completely stops and the on-ramp car will merge in front of it, accepting to touch the on-ramp’s end; (2) the self-driving car moves slowly, but at a non-zero speed, with the other car right behind it (keeping a rational distance). While a toy scenario, we feel that these are sensible solutions. Note that, additionally, this shows that we (similar to IRL) have a strong ability to reasonably generalize out of sample, since a fully braking vehicle is actually not in the data.
6 Conclusion
For modeling of realistic continuous multiagent trajectories, in this work we proposed an end-to-end trainable model class that hybridizes neural nets with game-theoretic reasoning. We accompanied it with theoretical guarantees as well as an empirical demonstration of its practicality, on real-world highway data. We consider this as one step towards machine learning methods for this task that are more interpretable, verifiable and transferable to decision making. This is particularly relevant for safety-critical domains that involve interaction with humans. A major challenge is to make game-theoretic concepts tractable for such settings, and we were only partially able to address this. Specifically, potential for future work lies in relaxing subspace-wise concavity, common-coupled games and related assumptions we made.

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Appendix

A. Proofs and remarks

A.1 Lemma

Let us first restate the result.

**Lemma** If \( y = r(a) \):

\[
\psi(a, \theta) = \int_0^T u_t^{\text{com}, \theta}(y_t) + \sum_{i \in I} u_t^{\text{own}, i, \theta}(y_t^i) d\mu(t).
\]

**Proof of Lemma** Recall that based on the definition of a common-coupled game we have have for the stage-wise utility for all \( i \):

\[
u^{i, \theta}(y_t) = u_t^{\text{com}, \theta}(y_t) + u_t^{\text{own}, i, \theta}(y_t^i) + u_t^{\text{oth}, i, \theta}(y_t^i).
\]

Intuitively, the statement directly follows because taking the “derivative” of \( i \)'s utility above w.r.t. \( y^j \) will just leave the derivative of the common term and \( i \)'s own term, because the other is constant in \( y^j \). And the same happens for \( \psi \).

Formally, observe that for all \( i \) and \( t \),

\[
u^{i, \theta}(y_t) = u_t^{\text{com}, \theta}(y_t) + u_t^{\text{own}, i, \theta}(y_t^i) + u_t^{\text{oth}, i, \theta}(y_t^i).
\]

Integrating w.r.t. \( t \) and using the linearity of the integral completes the proof.

\( \square \)

A.2 Theorem

Let us first restate the result.

**Theorem** (Game-induced differentiable implicit layer). Let Assumption hold true. Then, for each \( k \in K \), there is a continuous mapping \( g_k : \Theta \to \tilde{A}_k \) such that for any \( \theta \in \Theta \), if \( g_k(\theta) \) lies in the interior of \( \tilde{A}_k \), then

• \( g_k(\theta) \) is a local NE of \( \Gamma_\theta \).

• \( g_k(\theta) \) is given by the unique argmax of \( \psi(\theta, \cdot) \) on \( \tilde{A}_k \), with \( \psi \) the game’s potential function (Lem. 1).

• \( g_k \) is continuously differentiable in \( \theta \) with gradient

\[
J_\theta g_k(\theta) = - (H_a \psi(\theta, a))^{-1} J_\theta \nabla_a \psi(\theta, a),
\]

whenever \( \psi \) is twice continuously differentiable on an open set containing \( (\theta, a) \), for \( a = g_k(\theta) \), where \( \nabla, J \) and \( H \) denote gradient, Jacobian and Hessian, respectively.

**Proof of Theorem**

**Gradient etc.** Based on Lemma the potential function \( \psi \) exists. Let \( k \in K \) be arbitrary but fixed. Let \( g_k \) be the function that maps each \( \theta \) to the corresponding unique maximum of \( \psi(\theta, \cdot) \) on \( \tilde{A}_k \) (exists and is unique by the assumption of strict concavity and convexity and compactness of the \( \tilde{A}_k \)). From the definition of the potential function and the local Nash equilibrium (NE), it follows directly that a maximum of the potential function is a local NE of the game.

To apply the implicit function theorem, let us consider the point \( (\theta, g_k(\theta)) \). If the minimum \( g_k(\theta) \) lies in the interior of \( \tilde{A}_k \), and \( \psi \) is continuously differentiable on an open set containing \( (\theta, g_k(\theta)) \), then we have \( \nabla_a \psi(\theta, a)|_{(\theta, a) = (\theta, g_k(\theta))} = 0 \) and furthermore, by assumption of strict concavity, \( J_a \nabla_a \psi(\theta, a)|_{(\theta, a) = (\theta, g_k(\theta))} \) non-singular. Then the implicit function theorem

\( ^{23} \)Note that this theorem in fact holds for potential games in general, not just common-coupled games. Note also that for tractability/analysis reasons we consider the simple deterministic game form of Def. 1 instead of, say, a Markov game.

\( ^{24} \)Due to the concavity assumption, we can use established tractable, guaranteeably sound algorithms to calculate this argmax.
implies that there is an open set $O$ containing $(\theta, g_k(\theta))$, and a unique continuously differentiable function $f : O \to \tilde{A}_k$, such that $\nabla_a \psi'(\theta', f(\theta')) = 0$ for $\theta' \in O$, with gradient

$$J_\theta f(\theta) = -(J_\theta \nabla_a \psi(\theta, a))^{-1} J_\theta \nabla_a \psi(\theta, a) = -(H_\theta \psi(\theta, a))^{-1} J_\theta \nabla_a \psi(\theta, a).$$

(4)

Now on $O$, $f$ and $g_k$ coincide since $f$ is uniquely determined (specifically, based on the implicit function theorem, locally, the graph of $f$ coincides with the solution set of $\nabla_a \psi(\cdot, \cdot) = 0$, and if $f, g_k$ would differ in at least one point $\theta'$, then there would be a solution $(\theta', g_k(\theta'))$ outside the solution set – a contradiction). Therefore $g_k$ is also continuously differentiable on $O$ with gradient

$$J_\theta g_k(\theta) = J_\theta f(\theta).$$

(5)

We can do this for every $\theta$, which completes the proof.

Continuity. Since $\psi$ is continuous, $\tilde{A}_k$ is compact, and the maxima are unique, the maximum theorem implies that the mapping $g_k$ is in fact continuous (hemicontinuity reduces to continuity when the correspondence is in fact a function).

□

A.3 Lemma 2 with remark on zero gradient

Let us first restate the result.

Lemma 2. Let Assumption 7 hold true and additionally assume $\psi$ to be continuously differentiable on (a neighborhood of) $\Theta \times \tilde{A}_k, k \in K$. If $a = g_k(\theta)$ lies on (exactly) one constraining affine hyperplane of $\tilde{A}_k$, defined by orthogonal vector $v$, with multiplier $\lambda$ and optimum $\lambda^* > 0$ of $\psi(\theta, a)$’s Lagrangian (details in the proof), then $J_\theta g_k(\theta) = \left[ - \begin{pmatrix} H_\theta \psi(\theta, a) & v^T \\ \lambda v^T & 0 \end{pmatrix} \right]^{-1} \begin{pmatrix} J_\theta \nabla_a \psi(\theta, a) \\ 0 \end{pmatrix}. \right]_{1: (n \cdot d_A) \times 1: (n \cdot d_A)}$

Remark on zero gradient. Note that under the conditions of this lemma, i.e., when $g_k(\theta)$ lies at the boundary, then the above gradient $J_\theta g_k(\theta)$ in fact often becomes zero, which can be a problem for parameter fitting. So the above result is only meant as a first step.

Proof of Lemma 2. Let $\tilde{A}_k$ be defined by the inequality constraints $v^T_m a \leq b, m = 1, \ldots, M$. Consider the Lagrangian

$$A(\theta, a, \lambda_1, \ldots, \lambda_M) = \psi(\theta, a) + \sum_m \lambda_m (v^T_m a - b).$$

Then the Karush-Kuhn-Tucker optimality conditions (Boyd and Vandenberghe 2004) (note that we assumed differentiability of $\psi$ on a a neighborhood of $\Theta \times \tilde{A}_k, k \in K$) include the following equations:

$$\nabla_a A(\theta, a, \lambda_1, \ldots, \lambda_M) = 0,$nabla_a A(\theta, a, \lambda_1, \ldots, \lambda_M) = 0,$

\begin{align}
\begin{aligned}
\lambda_m (v^T_m a - b) &= 0, \\
& m = 1, \ldots, M.
\end{aligned}
\end{align}

(6a)

(6b)

Now let $a^* = g_k(\theta)$ and let $\lambda_1^*, \ldots, \lambda_M^*$ be the (optimal) duals for $a^*$. And assume that $a^*$ lies on exactly one bounding (affine) hyperplane. W.l.o.g. let this hyperplane correspond to $v_1, \lambda_1^*$. Also recall that $g_k$ is continuous (as in Theorem 1). Therefore, in a neighborhood of $\theta$, the corresponding optimum will not lie within any of the other boundaries. So in this neighborhood of $\theta$, all corresponding optimal duals will be zero (inactive) except for the assumed one.

Therefore, given a $\theta'$ from the mentioned neighborhood of $\theta$, we have that $a, v_1$ satisfy the optimality conditions in (6) (for some remaining duals) iff they satisfy the reduced conditions

$$\nabla_a A(\theta', a, \lambda_1, \lambda_0, \ldots, 0) = 0,$nabla_a A(\theta', a, \lambda_1, \lambda_0, \ldots, 0) = 0,$

$$\begin{align}
\begin{aligned}
\lambda_1 (v_1^T a - b) &= 0.
\end{aligned}
\end{align}

(7a)

(7b)

For succinctness, in what follows we write $\lambda$ instead of $\lambda_1$, i.e., drop the subscript. Let

$$h(\theta', a, \lambda) := (\nabla_a A(\theta', a, \lambda, 0, \ldots, 0), \lambda (v^T a - b))$$

$$= (\nabla_a \psi(\theta', a) + \nabla_a \lambda (v^T - b), \lambda (v^T a - b))$$

$$= (\nabla_a \psi(\theta', a) + \lambda v^T, \lambda (v^T a - b))$$

So the conditions in (7) are

$$h(\theta', a, \lambda) = 0.$$nabla_a A(\theta', a, \lambda_0, \ldots, 0) = 0,$$
Similar as in Theorem 1, around the point $\theta, a^*, \lambda^*, g_k$ satisfies (8) (for some $\lambda$'s). So we can apply the implicit function theorem to get its gradient.

We have

$$J_{(\theta, a, \lambda)} h(\theta, a, \lambda) = \begin{pmatrix} J_{\theta} (\theta, a) & H_{a} (\theta, a) \frac{v^T}{\lambda v^T} & v^T a - b \end{pmatrix}.$$ 

Note that

$$\begin{pmatrix} H_{a} (\theta, a^*) \frac{v^T}{\lambda^* v^T} & v^T a^* - b \end{pmatrix} = \begin{pmatrix} H_{a} (\theta, a^*) & v^T \end{pmatrix}$$

is invertible, since

$$\det \left( \begin{pmatrix} H_{a} (\theta, a^*) & v^T \end{pmatrix} \right) = \det(H_{a} (\theta, a^*)) \det(-\lambda^* v^T (H_{a} (\theta, a^*))^{-1} v)$$

and both factors are non-zero since $H_{a} (\theta, a^*)$ is positive definite and $\lambda^*, v$ are non-zero.

Therefore, the implicit function theorem is applicable to the equation, and we get as gradient

$$J_\theta g_k (\theta) = -\begin{pmatrix} H_{a} (\theta, a^*) & v^T \end{pmatrix}^{-1} \begin{pmatrix} J_{\theta} (\theta, a^*) \end{pmatrix} \begin{pmatrix} 0 \end{pmatrix}.$$ 

\[\Box\]

A.4 Proposition 1

Let us first restate the result.

Proposition 1 (Scenario 1's suitability). Scenario 1 satisfies Assumption 1. So, in particular, Thm. 1's implications on the induced implicit layer hold true.

Proof of Proposition 1

Common-coupled game. It is directly clear from the form of the utilities in Scenario 1 that this forms a common-coupled game.

Strict concavity of the potential function. Observe that, for each $i$, within any one subspace $\tilde{A}_k$, for each $t$,

- $i$ does not change lane, so $c_t, e_t$ are simple constants.
- For each lane, there is a fixed set of agents. Consider the set $S$ of pairs $(j, j')$ of agents that are on this lane and $j$ is right before $j'$. This ordering (and thus $S$) is invariant within $\tilde{A}_k$. Therefore the agent distance term can be rewritten like

$$\theta_{dist} \sum_{j \text{ right before } j' \text{ on same lane}} \frac{1}{|v_t^j - v_t^{j'}| + \zeta}. \quad (9)$$

So all terms are concave (the other terms are obviously concave), therefore the overall potential function, which is just a sum of them, is concave.

Futhermore, note that the sum of all velocity and distance to lane center terms is a sum of functions such that for each component of the vector $a$ there is exactly one function of it, and only of it; and each function is strictly concave. This implies that the overall sum is strictly concave in the whole $a$. So the potential function is a sum of concave and a strictly concave term, meaning it is strictly concave.

NB: On the subspaces, the potential function is also differentiable.


**Algorithm 1:** TGL – forward pass (sketch)

**Input:** past joint trajectory \( x \)

**Output:** modes \( \hat{q}, k \in \tilde{K} \) as prediction for future joint trajectory \( y \)

1. \( \theta \) = preference revelation net \((x)\);
2. \( \hat{K} = \) equilibrium refinement net \((x)\);
3. foreach \( k \in \hat{K} \) do
   4. Solve for local NE \( k^a \) in \( \check{A}_k \):
   5. \( (\hat{q})_{k \in \tilde{K}} = \) trajectory parametrization \( r ((k^a)_{k \in \hat{K}}) \);  
   6. \( (\hat{q})_{k \in \hat{K}} = \) eq. weighting net \((\theta , (k^a)_{k \in \hat{K}}) \); ...

**Algorithm 2:** TGL – training (sketch, in particular how to separate the equilibrium refinement net)

**Input:** set \( S \) of training samples \( (x', y'), (x'', y'') \), ...

1. Train the weights \( w_{er} \) of the equilibrium refinement net on pairs \( (x, k) \), where \( k \in K \) is the index of the subspace the ground truth \( y \) lies in, for \( (x, y) \in S \);  
2. Perform gradient-based training of the full architecture but with the equilibrium refinement net’s weights fixed:

Compactness and linearity of constraints. Besides the constraints that define the compact complete action space \( A \), which are obviously linear, the constraints that define the action subspaces \( \check{A}_k \) are given by the intersection of constraints for each time point \( t \) that are all of the form

- \( w_i^t \geq \text{const.} \) or \( w_i^t \leq \text{const.} \), or
- \( v_i^t \leq v_i^t \),

so they are linear.

**B** Further details on general architecture

Elaborating on Sec. 3.2, Alg. 1 describes TGL’s forward pass, in particular the parallel local NE search over the refined subspaces. Alg. 2 describes the training more formally, in particular the splitting into first training the equilibrium refinement net alone.

**C** Further example scenario: simple pedestrian encounter

As indicated in the main text, let us now elaborate the second – pedestrian scenario – our general method applies to. When considering settings with characteristics such as continuous time, even if they still satisfy the conditions of our framework, to prove so can become arbitrarily complex. Here let us give a simplistic but verified second example with properties somewhat different from the first one: Consider two pedestrians who walk with constant velocity (this velocity is their respective action) along straight paths which are orthogonal and intersect such that they could bump into each other (Fig. 4).

**Scenario 2** (Simple pedestrian encounter). **Setting:** There are \( n = 2 \) agents, the actions parameterize the trajectories via \( y_1^t = (0, ta^1 + z^1) \), \( y_2^t = (ta^2 + z^2, 0) \) (for ease of notation, we put the intersection to the origin but translations are possible of course), the joint action space is \( A = \left[ z_T^1, c \right] \times \left[ z_T^2, c \right] \), for constants \( z^1, z^2 < 0 \) and \( c > 0 \), (the lower bound on the action is

---

\[\text{"A net (B)" reads "output of the A net applied to input B".}\]
to make sure they reach the intersection. **Game:** Let the final stage utility be given by the following sum of a distance penalty term and a desired velocity term

\[
\begin{align*}
\hat{u}_T^{i}\theta(y^i_T) &= -h(\theta^{dist}, a) \max_{t \in [0,T]} \frac{1}{\|y^1_t - y^2_t\|_1} \\
&\quad - \theta^{vel,i}(a^1 - \beta^i) (a^2 - \beta^i)^2 \quad (=-\infty \text{ if division by 0})
\end{align*}
\]  

(11a)

(11b)

for some function \( h, \theta^{vel,i} > 0 \), and let \( \mu \) be the Dirac measure on \( T \). **Action subspaces:** Let the subspaces \((\tilde{A}_k)_{k \in K}\) be given by (1) taking the two subspaces that satisfy \( a_k \geq a_{2-k} + \epsilon \), for \( k = 1, 2 \) respectively, and some small \( \epsilon > 0 \), i.e., split by which agent is faster; and (2) additionally split by which agent first reaches their paths’ intersection (altogether this yields two or three subspaces).

**Proposition 2** (Scenario 2’s suitability and partial identifiability). Assume Scenario 2 with \( h(\theta^{dist}, a) = \frac{1}{a_k} \), with \( k \) the faster agent in \( a \). Then Assumption 1 is satisfied. Furthermore, while the complete game parameter vector \( \theta = (\theta^{vel,1}, \theta^{vel,2}, \theta^{vel,2}) \) is not identifiable in general, if \( \theta^{vel,i} \), \( i = 1, 2 \) is constant, then \((\theta^{vel,1}, \theta^{vel,2})\) is identifiable from \( y \) on the pre-image of the interior of \( \tilde{A}_k \), for any \( k \).

**C.1 Proof of Proposition 2**

**Proof of Proposition 2**

**Remark on utility form.** Note that in Scenario 2 the stage-utility at time \( T \) in its “distance term” uses a form of maximum operator that takes the whole trajectories as input. But according to our Definition 1, actually we allow the stage utility to only depend on respective current state. But this can be resolved by observing that the “distance term” can in fact be rewritten as a function of \((a^1, a^2)\), as we will see in Equation (17) below. And \((a^1, a^2)\) in turn is given by \((y^1_T - z^1, y^2_T - z^2)/T\), i.e., determined by \( y_T \).

**Common-coupled game.** It is directly clear from the form of the utilities in Scenario 2 that this forms a common-coupled game.

**Strict concavity of the potential function.** Consider all subspaces \( \tilde{A}_k \) where agent 1 is faster, i.e., \( a_1 > a_2 \). Note that this can be one or two subspaces; if \( y^1_T > y^2_T \), then it is one subspace, but otherwise it is two (namely, where 1 or 2 reaches the intersection first, respectively).

On any one of these subspaces, the following holds true:

Regarding the distance-based term, observe that

\[
\arg \min_{t \in [0,T]} |t \cdot a^1 + z^1| + |t \cdot a^2 + z^2| = -\frac{z^1}{a^1}.
\]

(12)

To see this, observe that the argmin is given by \( t \) where agent 1 reaches the intersection, i.e, \( t \) where \( t \cdot a^1 + z^1 = 0 \), i.e., \( t = -\frac{z^1}{a^1} \). (And this holds regardless of whether 1 or 2 first reaches the intersection.)

To see this in turn, first consider the case that agent 1 first reaches the intersection. Then, before this \( t \), both terms are bigger (both agents are further away from the origin), while at this \( t \), the first term is 0, and after it the left term grows faster (because the agent is faster) than the right term decreases (until the right term hits 0 as well and then also increases again).

Second, consider the case where agent 2 first reaches the intersection (in case this happens at all – i.e., if 2 starts so much closer to the origin to make this possible). Then, before 2 reaches the intersection: obviously both terms are bigger than when 2 reaches the intersection. Between 2 reaching the intersection and 1 reaching the intersection: in this time span, the right term grows slower than the left term decreases, therefore the minimum (for this time span) happens when 1 reaches the intersection. Now after 1 has reached the intersection obviously both terms just grow.
Therefore,

$$u^{\text{com}, \theta}(y(0,T)) = - \frac{1}{a^1} \max_{t \in [0,T]} \| (0, t \cdot a^1 + z^1) - (t \cdot a^2 + z^2, 0) \|_1$$  \hspace{1cm} (13)

$$= - \frac{1}{a^1} \max_{t \in [0,T]} \| t \cdot a^1 + z^1 + t \cdot a^2 + z^2 \|$$  \hspace{1cm} (14)

$$= - \frac{1}{a^1} \left| \frac{z_2}{a^2} \cdot a^1 + z^1 \right| + \left| \frac{z_2}{a^2} \cdot a^2 + z^2 \right|$$  \hspace{1cm} (15)

$$= - \frac{1}{a^1} \left| \frac{z_2}{a^2} \cdot a^1 + z^1 \right| + \left| \frac{z_2}{a^2} \cdot a^2 + z^2 \right|$$  \hspace{1cm} (16)

$$= - \frac{1}{z^2 a^1 - z^1 a^2}$$  \hspace{1cm} (17)

Keep in mind that for agent \( i \), the time when it reaches the intersection is given by \( t_i = \frac{z_i}{a_i} \). Now, if, for the subspace under consideration, agent 1 first reaches the intersection, i.e., \(-\frac{z_1}{a_1} > -\frac{z_2}{a_2}\), which is obviously concave. Similarly for the case that agent 2 first reaches the intersection.

Regarding the velocity terms, obviously their sum is strictly concave.

So the sum of all terms is strictly concave.

**Linearity and compactness of constraints.** The constraints are all of the form \( a_1 \geq a_2 + \varepsilon \) or \( z^1 a^2 < z^2 a^1 \), i.e., linear.

**Identifiability.** Obviously the full \( \theta \) cannot be identifiable because there are no (local) diffeomorphisms between spaces of differing dimension.

Keep in mind that the parametrization from \( a \) to \( y \) is injective, so we just need to show identifiability from \( a \). That is, we have to show that \( g_k \) is invertible on \( g_k^{-1}(\text{int}(\bigcup_k A_k)) \), where \( \text{int}(\cdot) \) denotes the interior.

Since we fixed \( \theta^{\text{vel}, i}, i = 1, 2 \), consider them as constants, and for what follows, for simplicity let \( \theta \) stand for \( (\theta^{\text{vel}, 1}, \theta^{\text{vel}, 2}) \).

W.l.o.g. (the other cases work similarly) assume \( A_k \) is the subspace where agent 1 is faster and first reaches the intersection, so the potential function becomes

$$\psi(\theta, a) = \frac{1}{z^2 a^1 - z^1 a^2} - \theta^{\text{vel}, 1} (a^1 - \theta^{\text{vel}, 1})^2 - \theta^{\text{vel}, 2} (a^2 - \theta^{\text{vel}, 2})^2.$$  \hspace{1cm} (18)

Now let \( a \) be such that \( a = g_k(\theta) \) for some \( \theta \). We have to show that there can only be one such \( \theta \). To see this, note that \( a \) is a local NE, and thus

$$0 = \nabla_a \psi(\theta, a) = \left( (-1)^{i-1} \frac{z^{3-i}}{(z^2 a^1 - z^1 a^2)^2} + 2a^{\text{vel}, i} a^i - 2a^{\text{vel}, i} \theta^{\text{vel}, i} \right)_{i=1,2}.$$  \hspace{1cm} (19)

But this implies

$$\theta^{\text{vel}, i} = \left( (-1)^{i-1} \frac{z^{3-i}}{(z^2 a^1 - z^1 a^2)^2} - 2a^{\text{vel}, i} a^i \right) \frac{1}{2a^{\text{vel}, i}}.$$  \hspace{1cm} (20)

**D Further details on experiments and implementation**

Keep in mind that in the main text (Sec. 5), we already described two implementations/versions we propose as instantiations of our general method (Sec. 3.2) for the highway merging setting:

- TGL,
- TGL-D.

Here we present one further method:

- TGL-DP (also “TGL-P” for short) – building on TGL-D, we use the interpretable representation of the desired velocity parameter predicted by the preference revelation net, which we first validate, and then encode additional prior-knowledge based constraints (e.g., we clip maximum and minimum desired speed) – see Appendix D.1 for details.

Additionally, in this section we provide further details on the new data set (Appendix D.2) as well as more fine-grained empirical evaluations metrics (individual prediction time points) for all methods.
Figure 5: For the new highway data set, this is roughly the recorded highway section (only the lower lane, incl. exit/entry). Note that the recorded section is in fact slightly more to the right than the picture indicates.

Figure 6: This is a zoom-in on roughly the sub-part of the highway section in Fig. 5 that is relevant for the on-ramp merging trajectories used in the experiment.
Algorithm 3: Part of preference revelation net of TGL-P that outputs the desired velocity game parameters $\theta^{v,2}, \theta^{v,1}$ and differs compared to TGL.

**Input:** old_vx_other, old_vx_merge // velocities along x-axis at the last step of the past trajectory $x$, where 'merger' is the on-ramp car, and 'other' is the highway car

1. desired_vx_other, desired_vx_merge // output $\theta^{v,2}, \theta^{v,1}$ of TGL’s original preference revelation net
2. merger_in_front // most probable subspace given by equilibrium refinement/weighting net, whether merger merges before or after other
3. big_change // parameter -- factor for allowed big change, for the experiment we used $\text{big\_change} = 1.2$
4. small_change // parameter -- factor for allowed small change, for the experiment we used $\text{small\_change} = 1.04$

**Output:** new_vx_other, new_vx_merge // new desired velocity game parameters $\theta^{v,2}, \theta^{v,1}$

5. if $\text{merger\_in\_front} == 0$ then
   - if the highway vehicle is in front
     - new_vx_other = clamp(desired_vx_other, old_vx_other / small_change, old_vx_other*small_change)
     - only allow accelerating merger vehicle, but limit maximum speed change
     - new_vx_merge = clamp(desired_vx_merge, old_vx_merge*big_change)
   - else
     - if the merger vehicle is in front
     - clamp desired merger velocity to make it coherent with driving in front of other
     - new_vx_merge = clamp(desired_vx_merge, min(old_vx_merge*big_change, old_vx_other/big_change), old_vx_merge*big_change)
     - also clamp the desired speed of the other, distinguishing between two cases:
       - if new_vx_merge > old_vx_other then
         - new_vx_other = clamp(desired_vx_other, old_vx_other / small_change, old_vx_other*small_change)
       - else
         - new_vx_other = clamp(desired_vx_other, old_vx_other / big_change, old_vx_other)

D.1 More detailed description of TGL-DP

**NB:** on what follows, we refer to the method TGL-DP also simply by TGL-P:

Let us give further details on our method TGL-P, that was only briefly introduced in Appendix D. This method is the same as TGL-D described in Sec. 5.1 (building on Scenario 1), except that we modified the preference revelation net based on plausible reasoning and using parts of the equilibrium refinement/weighting net. Note that *this is still a special case of our general architecture (Sec. 3.2) in the rigorous sense*, just with a preference revelation net that incorporates a fair amount of additional reasoning and shares some structure with the equilibrium refinement net.

There are two reasons why we introduce TGL-P: First, the preference revelation net has to be trained together with the local game solver implicit layer. This means that training takes comparably long (we have an outer and an inner optimization loop, as described in Sec. 3.2). And one particular problem we experienced and have not fully solved yet, is that often the preference revelation net already starts overfitting while the game parameters that are learned “globally”, i.e., not inferred from the past trajectory by the preference revelation net, are not properly learned yet. Now TGL-P allows to demonstrate what we believe TGL itself can also achieve once the mentioned problems are overcome; in particular, it shows that the game model class $\Gamma_\theta$ has a substantial capacity to resemble the future trajectories. Second, TGL-P shows how the intermediate representation $\theta$ can be inspected and high level knowledge/reasoning about agents’ preferences/utilities and behavior can be incorporated.

Specifically, TGL-P can be described as being the same as TGL, except that the part of preference revelation net of TGL that outputs $\theta^{v,2}, \theta^{v,1}$ is replaced by Alg. 3 (on page 16). The function $\text{clamp}(\cdot)$ is, as usual, defined by

$$\text{clamp}(z, m, M) = \max(m, \min(z, M)),$$

i.e., clipping values to $m / M$ if they are below / above. While the details of the algorithm may look complex, the main idea is simple: we clip the outputs of the preference revelation net if they are too far off compared to what one would expect given initial velocities and output of the equilibrium refinement/weighting net. The algorithm has two parameters $\text{big\_change}$ and $\text{small\_change}$, for which we used the values 1.2 and 1.04, respectively, in the experiment.

Note that, while in general, the equilibrium refinement/weighting net and the preference revelation net serve separate purposes, the reason why in TGL-P we share structure between them is mainly a pragmatic one: the equilibrium refinement/weighting net can be trained separately from the implicit layer and thus much faster – and we made the experience that it learns a reliable signal to predict the future (subspaces / selected equilibria).
D.2 Additional details and link for new highway data set HEE published alongside this paper

One of the data sets we used in the experiment is a new one which we publish along this paper (introduced in Sec. 5.1). We refer to it as HEE (Highway Eagle Eye) data set. Here are the links relevant for the data set:

- Data set itself: [https://github.com/boschresearch/hee_dataset](https://github.com/boschresearch/hee_dataset)
- For example code of how to load/preprocess trajectories from the raw data set, please look at the code repository for this paper: [https://github.com/boschresearch/trajectory_games_learning](https://github.com/boschresearch/trajectory_games_learning). Note that this repository also contains the precise filtered data set of two-player highway merge scenes that we used in the experiments.

Further remarks on HEE:

- The full recorded highway section in HEE is roughly as the lower lane (incl. exit/entry) in Fig. 5 (on page 15) (the recorded section is in fact slightly more to the right than the picture indicates). For the experiments we focused on merging scene trajectories, which roughly take place within the smaller highway section depicted in Fig. 6 (on page 15). Note that the images in Fig. 1 are a stylized (significantly squashed in the x-dimension) version of Fig. 6.
- Note that the recorded highway section in our data set – length ~ 600m – is longer compared to the one section with an on-ramp in the highD data set (Krajewski et al. 2018). But highD also contains other highway sections.
- As stated earlier, keep in mind that this HEE data set can be useful for studies like ours, but some aspects of it may be noisy, so it is only meant for experimental purposes.

D.3 More details on experimental results

In the main text we only gave results averaged over all prediction horizons. Here, Tables 2, 3 (on page 19) give the results – MAE and RMSE for all methods (including CS-LSTM (Deo and Trivedi 2018), MFP (Tang and Salakhutdinov 2019)) – for each prediction horizon individually, from 1s to 7s.

Additionally, in Fig. 7 (on page 18) we show example joint trajectories on highD data set and our new highway data set: the past trajectory $x$, the prediction $\hat{y}$ (most likely mode) of our method TGL-DP, and the ground truth future trajectory $y$. Note that the x-axis is significantly squeezed in these figures.

E Further related work

Regarding work that combines machine learning and game theory, noteworthy is also (Hartford, Wright, and Leyton-Brown 2016) who learn solution concepts, but not equilibrium refinement concepts. Generally, learning agent behavior, but in an active/experimental settings, is also extensively studied, e.g., in multiagent reinforcement learning (Shoham and Leyton-Brown 2008). Recently, also other related setting have been studied, e.g., where a computational assistant interacts with agents to learn their preferences and based on this support their coordination in the game-theoretic sense of equilibrium selection (Geiger et al. 2019).

Regarding imitation learning of agents’ interaction from observational data when there may be relevant unobserved confounders, there is also work based on causal models with a focus on identifiability analysis (Etesami and Geiger 2020; P. Haan 2018; Geiger, Hofmann, and Schölkopf 2016).

Beyond inverse reinforcement learning mentioned already in the introduction, our work is closely related to the general area of imitation learning (Osa et al. 2018).

Regarding probabilistic (multi-)agent trajectory prediction, on the machine learning side there are also recent methods using normalizing flows (Bhattacharyya et al. 2019; 2020).

Regarding implicit layers, the following work is also worth mentioning explicitly: Amos and Kolter (2017) take a (constrained) optimizer as implicit layer and derive the implicit layer’s gradient formula from the (Lagrangian/Karush-Kuhn-Tucker) optimality condition. Note that this is similar to one part of our Thm. 1 which, after the reduction to potential games, also derives the implicit layer’s gradient formula from an optimality condition. But our Thm. 1 contains additional elements in the sense of the precise preconditions to fit with the rest of our setting, the parallelization of solving for several local optima/equilibria, and the continuity implication for the $g_k$, which follows easily but which, so far, we have not seen in other work. Also note Amos et al. (2018) who differentiate through (single-agent) model predictive control (MPC) conditions. Worth mentioning are also Li et al. (2020), who also use implicit differentiation to learn about games.

F Remarks on rationality, local Nash equilibria, equilibrium selection, etc.

Let us make some high level remarks.

- As a remark on the (local) rationality principle in our model: We do not think that humans are perfectly instrumentally rational (i.e., strategic utility maximizers) in general, and in many multiagent trajectory settings bounded rationality and

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\(^{26}\)One thing we found was that there is a slight mismatch between trajectory data compared to a preliminary map of the highway section (available in the data repository as well). This rather seems an error in the map, but could also be a distortion in the trajectory recordings.
Figure 7: Example joint trajectories on highD data set (top) and our new highway data set (bottom). The ground truth future $y$ is green and yellow for on-ramp and highway car, respectively, with the past trajectory segment $x$ (at the very beginning) depicted by 'x' markers. The prediction $\hat{y}$ (most likely mode) of our method TGL-P is in blue and red, respectively. Note that x- and y-axis are in meters; in particular, the x-axis is significantly squeezed.
Table 2: Mean absolute error (MAE) for all prediction horizons for all methods.

| Data set                  | Horizon | TGL (ours) | TGL-D (ours) | TGL-DP (ours) | CS-LSTM | MFP |
|---------------------------|---------|------------|--------------|---------------|---------|-----|
| highD (Krajewski et al. 2018) | 1s      | 0.5        | 0.5          | 0.5           | 1.4     | 1.2 |
|                           | 2s      | 1.2        | 1.1          | 1.0           | 2.4     | 2.3 |
|                           | 3s      | 2.1        | 1.7          | 1.6           | 3.7     | 3.6 |
|                           | 4s      | 3.2        | 2.6          | 2.4           | 4.8     | 5.0 |
|                           | 5s      | 4.5        | 3.6          | 3.2           | 6.0     | 6.6 |
|                           | 6s      | 6.1        | 4.8          | 4.2           | 7.0     | 8.2 |
|                           | 7s      | 7.9        | 6.1          | 5.4           | 10.1    | 9.8 |
| New data set (Sec 5.1)    | 1s      | 0.8        | 0.7          | 0.8           | 0.8     | 0.8 |
|                           | 2s      | 1.5        | 1.4          | 1.5           | 1.6     | 1.6 |
|                           | 3s      | 2.4        | 2.2          | 2.3           | 2.3     | 2.4 |
|                           | 4s      | 3.5        | 3.1          | 3.0           | 3.3     | 3.6 |
|                           | 5s      | 4.6        | 4.0          | 3.9           | 4.4     | 4.6 |
|                           | 6s      | 5.9        | 4.9          | 4.7           | 5.3     | 5.8 |
|                           | 7s      | 7.3        | 5.9          | 5.6           | 7.2     | 7.1 |

Table 3: Root mean square error (RMSE) for all prediction horizons for all methods.

| Data set                  | Horizon | TGL (ours) | TGL-D (ours) | TGL-DP (ours) | CS-LSTM | MFP |
|---------------------------|---------|------------|--------------|---------------|---------|-----|
| highD (Krajewski et al. 2018) | 1s      | 0.6        | 0.5          | 0.5           | 1.8     | 1.6 |
|                           | 2s      | 1.4        | 1.2          | 1.2           | 3.2     | 3.0 |
|                           | 3s      | 2.7        | 2.1          | 2.0           | 5.0     | 4.8 |
|                           | 4s      | 4.3        | 3.2          | 3.0           | 6.3     | 6.8 |
|                           | 5s      | 6.1        | 4.5          | 4.2           | 7.7     | 9.0 |
|                           | 6s      | 8.4        | 6.1          | 5.7           | 9.1     | 11.2 |
|                           | 7s      | 11.0       | 8.0          | 7.6           | 14.5    | 13.6 |
| New data set (Sec 5.1)    | 1s      | 0.8        | 0.7          | 0.8           | 0.9     | 1.0 |
|                           | 2s      | 1.8        | 1.7          | 1.7           | 2.0     | 1.9 |
|                           | 3s      | 3.1        | 2.8          | 2.8           | 2.8     | 3.1 |
|                           | 4s      | 4.4        | 3.9          | 3.9           | 4.0     | 4.5 |
|                           | 5s      | 5.9        | 5.2          | 5.1           | 5.1     | 6.0 |
|                           | 6s      | 7.6        | 6.6          | 6.4           | 6.5     | 7.6 |
|                           | 7s      | 9.4        | 8.1          | 7.7           | 8.7     | 9.3 |

Other phenomena may play an important role. Nonetheless, we believe that instrumental rationality is reasonably good approximation in many settings. And of course, game-theoretic rationality has the advantage that there is an elegant theory for it.

- In fact we use the local Nash equilibrium, which can also be seen as a bounded form of rationality. However, we feel that it can often be a better, more advanced approximation (to reality and/or rationality) than other concepts like level-k game theory (Stahl and Wilson 1995). In particular, we found it interesting that the local Nash equilibria can in fact correspond to intuitive modes of the joint trajectories, like which car goes first in Fig. 1. A reason for this may be that, while there may be one perfect solution (say a global Nash), due to errors and stochasticity of the environment the agents may be perturbed towards ending up at a state where a previously local Nash equilibrium may in fact now be a global Nash equilibrium.
- The problem of equilibrium selection mentioned in the main text may be seen as a form of incompleteness of game theory – it cannot always make a unique prediction (or prescription). Our equilibrium refinement and weighting nets can be seen as a data-driven approach to fill this incompleteness. An interesting question in this context is whether the missing information is actually contained in the preferences of the agents, or if there is additional (hidden) information required to find the unique solution/prediction.
- In some situations, it may be that knowledge of an (local-)equilibrium-separating partition may in fact tractably tell us the global Nash equilibrium (because if we know all local optima then we can also know the global optimum of a function). But this does not always seem to be the case for general common-coupled/potential games and, in particular, also because we do
not require the equilibrium-separation to cover all possible equilibria.

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