A polynomial time infeasible interior-point arc-search algorithm for convex optimization

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Abstract

This paper proposes an infeasible interior-point algorithm for the convex optimization problem using arc-search techniques. The proposed algorithm simultaneously selects the centering parameter and the step size, aiming at optimizing the performance in every iteration. Analytic formulas for the arc-search are provided to make the arc-search method very efficient. The convergence of the algorithm is proved and a polynomial bound of the algorithm is established. The preliminary numerical test results indicate that the algorithm is efficient and effective.

keywords: Infeasible interior-point algorithm; Arc-search; Convex optimization
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1 Introduction

Because of the great success of the interior-point methods for linear programming (LP) problems [25], the methods have been extended to more optimization problems, such as linear complementarity problem [14], convex quadratic optimization problem [22], semidefinite programming problem [2], convex nonlinear optimization problem [36], and non-convex nonlinear programming problem [7] and many references therein.

There are two types of the interior-point methods based on the property of the starting points of the algorithm. The “feasible” interior-point method starts with a feasible initial point and is much easier to analyze the convergence properties but it needs an expensive “phase-I” process to find a feasible starting point. The “infeasible” interior-point method does not need a feasible initial point which is computationally attractive but its convergence analysis is much more difficult and it needs more demanding assumptions in the convergence analysis [25]. For decades, people have realized [19, 20, 31] that infeasible interior-point method is a better strategy than feasible interior-point method for LPs if an initial point is not available.

Another proven strategy of the interior-point methods is to use the central path to guide a series iterates to an optimal solution. Computing the central path of an
optimization problem, however, is very expensive. Most path-following type interior-point algorithms use line segment to approximate the central path and search the optimizer along this line segment. Clearly, this is not a good strategy because the central path is a curve. Therefore, this author proposed an arc-search technique for interior-point method for LPs [28]. The main idea in the arc-search technique is to efficiently and robustly approximate the central path using an arc of part of an ellipse and to search the next iterate along the arc. Since the central path is geometrically a high dimensional curve, the arc can fit the central path better than a line.

Recently, researchers have applied arc-search techniques to different optimization problems. For example, an arc-search interior-point algorithm proposed in [35] shows that it has better polynomial bound and its numerical test is more attractive than a line-search type interior-point algorithm for LPs. In [32], this author showed that an interior-point algorithm using the arc-search technique achieves the best polynomial bound for all interior-point methods, feasible or infeasible, and is numerically competitive to the well-known Mehrotra’s algorithm. Researchers have applied the arc-search technique to the linear complementarity problem [12], convex quadratic programming [29, 38], symmetric programming [27], semidefinite programming [37, 13], and nonlinear programming problem [26]. All these results showed that the arc-search method performs better than the counterpart, the line search method.

In this paper, we extend the arc-search techniques to the convex nonlinear optimization problem for which various line-search interior point algorithms have been developed in [1, 3, 5, 6, 9, 10, 11, 15, 21], because many application problems can be formulated as a convex nonlinear optimization problem [4, 17, 24]. Although a polynomial bound has been proved for a feasible interior-point algorithm for the convex nonlinear optimization problem [15], to our best knowledge, there is no polynomial bound for infeasible interior-point algorithms for the convex nonlinear optimization problem because the latter is much more difficult [25]. We propose an arc-search infeasible interior-point algorithm for the convex nonlinear optimization problem and discuss the convergence property. We show that this algorithm converges under mild conditions in a polynomial bound of \( O(n^{1.5} \log(1/\epsilon)) \).

The remainder of the paper is organized as follows. Section 2 introduces the problem to be discussed. Section 3 describes the proposed arc-search algorithm. Section 4 discusses its convergence properties. Section 5 contains the materials about the Matlab implementation and preliminary numerical test results. Section 6 summarizes the conclusion of the paper. A method about optimal selection of the centering parameter and the step size at the same time is provided in Appendix A.

## 2 Problem description

In the remainder of the paper, we use a superscript T for the transpose of a vector or a matrix, and we use a tuple to denote a stacked vectors, for example, \((x, y)\) stands for \([x^T, y^T]^T\). For a vector \(x \in \mathbb{R}^n\), we denote by \(X \in \mathbb{R}^{n \times n}\) a diagonal matrix whose diagonal elements are \(x\), and by \(\min(x)\) and \(\max(x)\) the minimum and maximum values
of $\mathbf{x}$ respectively. For two vectors $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$, we use $\mathbf{x} \circ \mathbf{y} \in \mathbb{R}^n$ to denote the element-wise product of $\mathbf{x}$ and $\mathbf{y}$. Let $\mathbb{R}_+^n$ (denote the space of nonnegative vectors (positive vectors, respectively), and $\mathbf{e}$ denote a vector of all ones with appropriate dimension. We will use superscript $k$ for the vector iteration count and subscript $k$ for the scalar iteration count, for example, $x^k$ is the value of the vector variable $\mathbf{x}$ at iteration $k$, and $\mu_k$ is the value of the scalar variable $\mu$ at iteration $k$.

We consider the following convex programming problem with linear constraints:

\[
\begin{align*}
\min & \quad f(\mathbf{x}) \\
\text{s.t.} & \quad A_E \mathbf{x} = b_E, \\
& \quad A_I \mathbf{x} \geq b_I, 
\end{align*}
\]

(1)

where $f : \mathbb{R}^n \to \mathbb{R}$ is a nonlinear convex function of $\mathbf{x} \in \mathbb{R}^n$, which is differentiable up to the third order; $A_E \in \mathbb{R}^{m \times n}$, $A_I \in \mathbb{R}^{p \times n}$, $m < n$, $b_E \in \mathbb{R}^m$, and $b_I \in \mathbb{R}^p$ are given constant matrices and vectors; and the decision variable vector is $\mathbf{x}$. We assume that the row of $A_E$ is full rank, which is standard because we can remove dependent rows in finite operations bounded by a polynomial of $m$ and $n$.

Following the treatment of [26], we convert the inequality constraints $A_I \mathbf{x} \geq b_I$ into equality constraints by introducing a slack vector $\mathbf{s} \geq 0$ as follows:

\[
\begin{align*}
\min & \quad f(\mathbf{x}) \\
\text{s.t.} & \quad A_E \mathbf{x} = b_E, \\
& \quad A_I \mathbf{x} - \mathbf{s} = b_I, \quad \mathbf{s} \geq 0.
\end{align*}
\]

(2)

Let Lagrangian multipliers of system (2) be denoted by $y \in \mathbb{R}^m$, $w \in \mathbb{R}^p$, and $z \in \mathbb{R}^p$, and let the tuple $\mathbf{v} = (\mathbf{x}, y, w, s, z) \in \mathbb{R}^{n+m+3p}$ to represent the decision variables and multipliers. Then, the Lagrangian function of (2) is given by

\[L(\mathbf{v}) = f(\mathbf{x}) + y^T(A_E \mathbf{x} - b_E) - w^T(A_I \mathbf{x} - \mathbf{s} - b_I) - z^T \mathbf{s}.
\]

Hence, we have the gradients of Lagrangian with respect to $\mathbf{x}$ and $\mathbf{s}$ given as follows:

\[
\nabla_x L(\mathbf{v}) = \nabla f(\mathbf{x}) + A_E^T y - A_I^T w, \quad \nabla_s L(\mathbf{v}) = w - z.
\]

(3)

Let $\mu$ be the duality measure defined as

\[\mu = \frac{s^T z}{p}.
\]

(4)

The KKT conditions of (2) are

\[g(\mathbf{v}) = 0, \quad (w, s, z) \in \mathbb{R}^{3p},
\]

(5)

where $g : \mathbb{R}^{n+m+3p} \to \mathbb{R}^{n+m+3p}$ is defined by

\[
g(\mathbf{v}) = \begin{bmatrix}
\nabla_x L(\mathbf{v}) \\
A_E \mathbf{x} - b_E \\
A_I \mathbf{x} - \mathbf{s} - b_I \\
w - z \\
Z \mathbf{s}
\end{bmatrix} \approx \begin{bmatrix}
H \mathbf{x} + A_E^T y - A_I^T w \\
A_E \mathbf{x} - b_E \\
A_I \mathbf{x} - \mathbf{s} - b_I \\
w - z \\
Z \mathbf{s}
\end{bmatrix} = \begin{bmatrix}
r_C \\
r_E \\
r_I \\
w - z \\
p \mu \mathbf{e}
\end{bmatrix},
\]

(6)
and

\[ r^k_C = Hx^k + A^T_Ey^k - A^T_Iw^k, \quad (7a) \]
\[ r^k_E = A_Ex^k - b_E, \quad (7b) \]
\[ r^k_I = A_Ix^k - s^k - b_I. \quad (7c) \]

are the approximated residual of the gradient of the Lagrangian function at \( v^k \) as defined in (3), the residual of the equality constraints, and the residual of the inequality constraints, respectively. The last row of (6) requires that the iterate follows the central path as closer as possible.

**Remark 2.1** Please note that \( r^k_C \) is not defined as a strict (but an approximate) residual of the gradient of the Lagrangian function at \( v^k \). This modification is for the purpose to obtain a convergent algorithm.

In view of (3), we have \( \nabla^2_x L(v) = \nabla^2_x f(x) = H_x \) which is a positive definite matrix depending on \( x \) because \( f(x) \) is a nonlinear convex function. To simplify the notation, we will write \( H \) for \( H_x \) but \( H \) needs to be updated in every iteration. The Jacobian of \( g \) is given by

\[
g'(v) = \begin{bmatrix}
\nabla^2_x f(x) & A^T_E & -A^T_I & 0 & 0 \\
A_E & 0 & 0 & 0 & 0 \\
A_I & 0 & 0 & -I & 0 \\
0 & 0 & I & 0 & -I \\
0 & 0 & 0 & Z & S
\end{bmatrix} = \begin{bmatrix}
H & A^T_E & -A^T_I & 0 & 0 \\
A_E & 0 & 0 & 0 & 0 \\
A_I & 0 & 0 & -I & 0 \\
0 & 0 & I & 0 & -I \\
0 & 0 & 0 & Z & S
\end{bmatrix}.
\]

Let \( a_i \) be the \( i \)th row of \( A_I \), \( b_i \) be the \( i \)th element of \( b_I \), \( i \in \{1, \ldots, p\} \), \( a_j \) be the \( j \)th row of \( A_E \), \( b_j \) be the \( j \)th element of \( b_E \), \( j \in \{1, \ldots, m\} \). Let

\[ I(x) = \{ i \in \{1, \ldots, p\} : a_i x = b_i \} \]

be the index set of active inequality constraints at \( x \in \mathbb{R}^n \). It is easy to check that the following properties hold for problem (1).

**Proposition 2.1** Assume that (a) \( A_E \) is full rank, (b) the constraints set of system (1) is not empty, (c) \( f(x) \) is differentiable up to the third order and is locally Lipschitz continuous at optimal solution \( \bar{x} \), then system (1) has the following properties.

(P1) There exists \( v = (\bar{x}, y, w, s, z) \), an optimal solution and its associate multipliers of (2), i.e., KKT conditions (5) has a solution.

(P2) \( g(x) \) are differentiable up to the second order. In addition, \( g(x) \) is locally Lipschitz continuous at \( \bar{x} \).
For the convergence analysis, we make the following assumptions for Problem (1).

**Assumptions:**

(A1) The set \( \{a_j : j = 1, \ldots, m\} \cup \{a_i : i \in I(\bar{x})\} \) is linearly independent.

(A2) For all \( \zeta \in \mathbb{R}^n \setminus \{0\} \), we have \( \zeta^T \nabla_x^2 L(\bar{v}) \zeta > 0 \).

(A3) For each \( i \in \{1, \ldots, p\} \), we have \( \bar{z}_i + \bar{s}_i > 0 \) and \( \bar{z}_i \bar{s}_i = 0 \).

Here (A1) is the linear independence constraint qualification (LICQ); (A2) is the second order sufficient conditions, which is true because Problem (1) is a convex optimization problem; and (A3) is strict complementarity. All these properties are standard and used in convergence analysis in [7, 23]. These properties assure the nonsingularity of the Jacobian matrix at the optimal solution \( \bar{v} \).

**Theorem 2.1** If Conditions (P1), (A1), (A2), and (A3) hold, then the Jacobian matrix \( g'(\bar{v}) \) is nonsingular.

**Proof:** Let \( (\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e}) \) be a constant vector that satisfies

\[
\begin{bmatrix}

\nabla_x^2 L(\bar{v})

\end{bmatrix} +
\begin{bmatrix}

A_E^T

\end{bmatrix} \begin{bmatrix}

\hat{a}

\hat{b}

\hat{c}

\hat{d}

\hat{e}

\end{bmatrix} =
\begin{bmatrix}

-A_f^T

0

0

0

0

\end{bmatrix}
\begin{bmatrix}

0

0

0

0

0

\end{bmatrix}
\begin{bmatrix}

I

0

0

0

0

\end{bmatrix}
\begin{bmatrix}

-\bar{I}

\bar{d}

0

-\bar{I}

S

\end{bmatrix}
\begin{bmatrix}

0

\end{bmatrix}
= 0. \tag{8}
\]

To show the nonsingularity of \( g'(\bar{v}) \), it is enough to show that (8) holds only if \( (\hat{a}, \hat{b}, \hat{c}, \hat{d}, \hat{e}) = 0 \). First, the fourth row indicates that \( \hat{c} = \hat{e} \), therefore, the last row leads to:

\[
\bar{z}_i \hat{d}_i + \bar{s}_i \hat{e}_i = \bar{z} \hat{d}_i + \bar{s} \hat{e}_i = 0 \tag{9}
\]

for each \( i \in \{1, \ldots, p\} \). Therefore, we can derive from (A3) that

\[
\hat{d}^T \hat{c} = 0.
\]

Actually, for each \( i \in \{1, \ldots, p\} \), either \( \bar{z}_i \) or \( \bar{s}_i \) is positive. Thus, if \( \bar{z}_i > 0 \), (A3) implies \( \bar{s}_i = 0 \), therefore we know \( \hat{d}_i = 0 \) due to (9); Similarly, if \( \bar{s}_i > 0 \), (A3) implies \( \bar{z}_i = 0 \), we know \( \hat{c}_i = 0 \) due to (9). From the second and third rows of (8), we have

\[
A_E \hat{a} = 0, \quad A_f \hat{a} - \hat{d} = 0. \tag{10}
\]

Multiplying \( \hat{a}^T \) from the left of the first row of (8) and using (10), we have

\[
\hat{a}^T \nabla_x^2 L(\bar{v}) \hat{a} + \hat{a}^T A_E^T \hat{b} - \hat{a}^T A_f^T \hat{c} = \hat{a}^T \nabla_x^2 L(\bar{v}) \hat{a} - \hat{d}^T \hat{c} = \hat{a}^T \nabla_x^2 L(\bar{v}) \hat{a} = 0.
\]

In view of (A2), we conclude \( \hat{a} = 0 \). Then, it follows from (10) that \( \hat{d} = 0 \), therefore, we know \( \bar{s}_i \hat{c}_i = 0 \) for each \( i \) from (9). If \( \bar{s}_i > 0 \), it holds \( \hat{c}_i = 0 \) for \( i \notin I(\bar{x}) \). On the other hand, if \( \bar{s}_i = 0 \), it holds that \( i \in I(\bar{x}) \), so that the first row of (8) turns to be \( A_f^T \hat{b} - A_f^T \hat{c} = 0 \), since \( \hat{c}_i = 0 \) for \( i \notin I(\bar{x}) \). Consequently, it holds \( \hat{b} = 0 \) and \( \hat{c}_i = 0 \) for \( i \in I(\bar{x}) \) because of (A1). As a result, we obtain \( \hat{c} = 0 \), and we already know \( \hat{c} = \hat{e} \) from the fourth row of (8). This proves the theorem.

**Remark 2.2** For \( v \) not close to the optimal solution, the nonsingularity of the Jacobian of \( g \) is carefully discussed in [26].
3 The interior-point algorithm with arc-search

Let \( v[t] = (x[t], y[t], w[t], s[t], z[t]) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{3p} \) be a function of \( t > 0 \) which is the solution of the modified perturbed KKT conditions \( g(v[t]) = t g(v[1]) \) with nonnegative conditions \((w[t], s[t], z[t]) \in \mathbb{R}^{3p}_+\) given as follows:

\[
\begin{bmatrix}
\nabla_x L(v[t]) \\
(A_E x - b_E)[t] \\
(A_I x - s - b_I)[t] \\
\nabla_s L(v[t]) \\
(Z_s)[t]
\end{bmatrix}
= 
\begin{bmatrix}
tr_C \\
tr_E \\
tr_f \\
t\nabla_s L(v) \\
tZ_s
\end{bmatrix},
\]

(11)

where the current iterate \( v = (x, y, w, s, z) = (x[1], y[1], w[1], s[1], z[1]) \). Clearly, \( v[t] \) defines a curve in \( \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{3p} \) that passes the current iterate point \( v[1] \). We denote the high dimensional curve defined by (11) as

\[
C = \{ v[t] \in \mathbb{R}^{n+m+3p} : t \in (0, 1) \}.
\]

Note that the right-hand-side of (11) goes to zeros when \( t \to 0 \), therefore, \( g(v[t]) \to 0 \) and \( v[t] \) converges to a KKT point given by (5).

Since the calculation of \( v[t] \) is very expensive, the idea of the arc-search is to efficiently approximate the curve \( C \) by using part of an ellipse and searching for optimizer along the ellipse. We denote the ellipse by

\[
\mathcal{E} = \{ v(\alpha) : v(\alpha) = a_\alpha \cos(\alpha) + b_\sin(\alpha) + \bar{c}, \alpha \in [0, 2\pi] \},
\]

(12)

where \( a_\in \mathbb{R}^{n+m+3p} \) and \( b_\in \mathbb{R}^{n+m+3p} \) are the axes of the ellipse, and \( \bar{c} \in \mathbb{R}^{n+m+3p} \) is its center. The calculation of \( a_\), \( b_\), and \( \bar{c} \) can be avoided by using the analytical formulas given in Theorem 3.1 [28]. Denote

\[
v = (\dot{x}, \dot{y}, \dot{w}, \dot{s}, \dot{z}).
\]

Taking the derivative on both sides of (11), we get the linear systems of equations

\[
\begin{bmatrix}
H & A^T_E & -A^T_I & 0 & 0 \\
A_E & 0 & 0 & 0 & 0 \\
A_I & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & Z & S
\end{bmatrix}
\begin{bmatrix}
\dot{x} \\
\dot{y} \\
\dot{w} \\
\dot{s} \\
\dot{z}
\end{bmatrix}
= 
\begin{bmatrix}
rc \\
r_E \\
r_f \\
\nabla_s L(v) \\
Z_s
\end{bmatrix}.
\]

(13)

The first-order derivative of the curve \( v[t] \) at \( t = 1 \) along \( C \) is denoted by \( \dot{v} \). Let \( \sigma \in [0, 1] \) be the centering parameter (see [25]). The second-order derivative \( \ddot{v} = (\ddot{x}, \ddot{y}, \ddot{w}, \ddot{s}, \ddot{z}) \) at \( t = 1 \) along the curve is defined as the solution of the following linear systems of equations:

\[
\begin{bmatrix}
H & A^T_E & -A^T_I & 0 & 0 \\
A_E & 0 & 0 & 0 & 0 \\
A_I & 0 & 0 & 0 & 0 \\
0 & 0 & I & 0 & 0 \\
0 & 0 & 0 & Z & S
\end{bmatrix}
\begin{bmatrix}
\ddot{x} \\
\ddot{y} \\
\ddot{w} \\
\ddot{s} \\
\ddot{z}
\end{bmatrix}
= 
\begin{bmatrix}
-(\nabla^3_x f(x))xx \\
0 \\
0 \\
0 \\
-2Zs
\end{bmatrix}
\approx 
\begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
-2Zs + \sigma \mu e
\end{bmatrix}.
\]

(14)
We add a centering item $\sigma \mu e$ to the last element in right hand side, which is the same strategy used in [33]. This modification assures that a substantial segment of the ellipse satisfies the requirement of $(s, z) > 0$, thereby assures that the step size along the ellipse is greater than zero. Our experience in [26] shows that the computation of $(\nabla^3 f(x)) \ddot{x}$ is very expensive. Therefore, to have an efficient algorithm, we omit this higher order term in the rest discussion. We show that this modification leads to an algorithm that converges in polynomial time. It is worthwhile to mention that, according to (14), $\ddot{v} = (\ddot{x}, \ddot{y}, \ddot{w}, \ddot{z})$ is a function of $\sigma$, i.e., $\ddot{v}$ should be written as $\ddot{v}(\sigma)$. But we use $\ddot{v}$ most time when no confusion is introduced.

Using $\dot{v}$ and $\ddot{v}$, we can approximate $C$ at $t = 1$ by an ellipse (12) that has the explicit form given in Theorem 3.1. We should emphasize that we use $t$ to denote the curve $C$ and $v[t]$ passes $v$ at $t = 1$, while we use the angle $\alpha$ to express an ellipse $E$ and $v(\alpha)$ passes $v$ at $\alpha = 0$, therefore, $v[1] = v(0) = v$.

**Theorem 3.1 ([29])** Assume that an ellipse $E$ of form (12) passes through the current iterate $v$ at $\alpha = 0$, let the first and second order derivatives at $\alpha = 0$ be $\dot{v}$ and $\ddot{v}$ which are defined by (13) and (14), respectively. Then the curve $v(\alpha)$ depends on the selection of $\sigma$ and $v(\alpha, \sigma) = (x(\alpha, \sigma), y(\alpha, \sigma), w(\alpha, \sigma), s(\alpha, \sigma), z(\alpha, \sigma))$ of $E$ is given by

$$v(\alpha, \sigma) = v - \dot{v} \sin(\alpha) + \ddot{v}(\sigma)(1 - \cos(\alpha)), \quad (15)$$

or

$$\begin{bmatrix} x^{k+1} \\ y^{k+1} \\ w^{k+1} \\ s^{k+1} \\ z^{k+1} \end{bmatrix} = \begin{bmatrix} x^k \\ y^k \\ w^k \\ s^k \\ z^k \end{bmatrix} - \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{w} \\ \dot{s} \\ \dot{z} \end{bmatrix} \sin(\alpha^k) + \begin{bmatrix} \ddot{x}(\sigma) \\ \ddot{y}(\sigma) \\ \ddot{w}(\sigma) \\ \ddot{s}(\sigma) \\ \ddot{z}(\sigma) \end{bmatrix}(1 - \cos(\alpha^k)). \quad (16)$$

We would like to emphasize that the second derivatives (therefore the ellipse) are functions of both $\alpha$ and $\sigma$ which we will carefully select simultaneously in every iteration $k$. The following lemma can be used to simplify the computation of (16).

**Lemma 3.1 ([26])** If $v$ satisfies $w = z$, then $w(\alpha) = z(\alpha)$ holds for any $\alpha \in \mathbb{R}$.

**Proof:** The proof is straightforward and therefore is omitted.

For numerical stability, we need that the Jacobian stays away from singularity. Therefore, we make the following assumption.

**Assumption:**

$$(A3') Z^k > 0 \text{ and } S^k > 0$$

are bounded below and away from zeros for all $k$ iterations until the program is terminated.
It will be clear that this assumption is also important in the convergence analysis.

As discussed in [32] and [34], it is a good strategy to simultaneously select the step size \( \alpha \) and the centering parameter \( \sigma \) whenever it is possible. To this end, we should express \( \ddot{v} \) explicitly as a function of \( \sigma \). This can be done by solving two linear systems of equations:

\[
\begin{bmatrix}
    H & A_T^E & -A_I^T & 0 & 0 \\
    A_E & 0 & 0 & 0 & 0 \\
    A_I & 0 & 0 & -I & 0 \\
    0 & 0 & I & 0 & -I \\
    0 & 0 & 0 & Z & S
\end{bmatrix}
\begin{bmatrix}
p_x \\
p_y \\
p_w \\
p_s \\
p_z
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\mu e
\end{bmatrix},
\]  

(17)

and

\[
\begin{bmatrix}
    H & A_T^E & -A_I^T & 0 & 0 \\
    A_E & 0 & 0 & 0 & 0 \\
    A_I & 0 & 0 & -I & 0 \\
    0 & 0 & I & 0 & -I \\
    0 & 0 & 0 & Z & S
\end{bmatrix}
\begin{bmatrix}
q_x \\
q_y \\
q_w \\
q_s \\
q_z
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
-2\dot{Z}\dot{s}
\end{bmatrix},
\]  

(18)

denoting \( p = (p_x, p_y, p_w, p_s, p_z) \) and \( q = (q_x, q_y, q_w, q_s, q_z) \), then we have \( \ddot{v} = \sigma p + q \).

Solving (13), (17), and (18) is equivalent to solve linear systems of equations \( A d_i = b_i \) for \( i = 1, 2, 3 \) with the same \( A \) but different \( b_i \). Therefore we can use the same decomposition of \( A \) three times as indicated in [25], which justifies the strategy of splitting (14) into (17) and (18).

For the convex programming problem (1), it is well-known that a vector \( \bar{v} = (\bar{x}, \bar{y}, \bar{w}, \bar{s}, \bar{z}) \) that meets the KKT conditions is the optimal solution. Therefore, we need to show that the proposed algorithm for problem (1) will generate a sequence \( v^k \) such that it approaches to an point \( \bar{v} \) that meets the approximate KKT conditions:

(C1). \( (\bar{r}_C, \bar{r}_E, \bar{r}_I) \leq \epsilon \).

(C2). \( (w^k, s^k, z^k) > 0 \) before the program terminates at \( (\bar{w}, \bar{s}, \bar{z}) \geq 0 \).

(C3). \( \bar{\mu} \leq \epsilon \) (given \( (\bar{s}, \bar{z}) \geq 0 \), this is equivalent to \( \bar{z}^T\bar{s} \leq \epsilon \).

In addition to the approximate KKT conditions, we will restrict the search in an interior point region given as follows:

\[
\mathcal{F} = \{(s, z) \; : \; (s, z) > 0, \; s^T_i z \geq \theta \mu_k\},
\]  

(19)

where \( \theta \in (0, 1) \) is a constant. Therefore, this imposes one more condition on \( v^k \):  

(C4). \( S^k z^k = Z^k s^k \geq \theta \mu_k e \).
The following proposition shows that searching along the ellipsoidal arc does improve the objective function and the feasibility of the constraints. Moreover, if \( \alpha^k \) is bounded below and away from zero for all \( k \), the above-mentioned Condition (C1) will hold.

**Proposition 3.1** Denote \( \nu_k = \prod_{j=0}^{k-1}(1 - \sin(\alpha^j)) \). We have the following formulas.

\[
\begin{align*}
    r_{C}^{k+1} &= r_{C}^{k}(1 - \sin(\alpha^k)) = \cdots = r_{C}^{0} \prod_{j=0}^{k}(1 - \sin(\alpha^j)) = r_{C}^{0} \nu_k, \\
    r_{E}^{k+1} &= r_{E}^{k}(1 - \sin(\alpha^k)) = \cdots = r_{E}^{0} \prod_{j=0}^{k}(1 - \sin(\alpha^j)) = r_{E}^{0} \nu_k, \\
    r_{I}^{k+1} &= r_{I}^{k}(1 - \sin(\alpha^k)) = \cdots = r_{I}^{0} \prod_{j=0}^{k}(1 - \sin(\alpha^j)) = r_{I}^{0} \nu_k.
\end{align*}
\]

**Proof:** Using (7), (16), and the first lines of (13) and (14), we have

\[
\begin{align*}
    r_{C}^{k+1} - r_{C}^{k} &= H(x^{k+1} - x^k) + A_{E}(y^{k+1} - y^k) - A_I(w^{k+1} - w^k) \\
    &\quad + A_{E}[-\dot{y} \sin(\alpha) + \ddot{y}(1 - \cos(\alpha))] \\
    &\quad - A_I[-\dot{w} \sin(\alpha) + \ddot{w}(1 - \cos(\alpha))] \\
    &\quad - [H \dot{x} + A_{E} \dot{y} - A_I \dot{w}] \sin(\alpha) \\
    &\quad + [H \ddot{x} + A_{E} \ddot{y} - A_I \ddot{w}] (1 - \cos(\alpha)) \\
    &\quad = -r_{C}^{k} \sin(\alpha).
\end{align*}
\]

This shows \( r_{C}^{k+1} = r_{C}^{k}(1 - \sin(\alpha^k)) \). Following a similar argument proves (21b) and (21c).

**Remark 3.1** This proposition indicates that searching along the ellipsoidal curve will improve the objective function and feasibility at the same rate in every iteration. The larger the \( \alpha \) is, the faster the improvement will be.

If \( \nu_k = 0 \), then \( r_E = 0 \) and \( r_I = 0 \), Problem (1) is reduced to a feasible convex programming problem, for which a feasible interior point algorithm such as [15] should be a more appropriate choice. Therefore, we make the following assumption as below.

**Assumption:**

(A4) \( \nu_k > 0 \) for all \( k > 0 \).

To meet the positiveness requirement of Condition (C2), we adopt the strategy described in [32]. Let \( \rho \in (0, 1) \) be a constant, and

\[
\bar{s}_k = \min_i s_i^k, \quad \bar{z}_k = \min_j z_j^k.
\]

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Denote $\phi_k$ and $\psi_k$ such that

$$\phi_k = \min\{\rho s_k, \nu_k\}, \quad \psi_k = \min\{\rho z_k, \nu_k\}.$$  
(24)

It is clear that

$$0 < \phi_k e \leq \rho s_k, \quad 0 < \phi_k e \leq \nu_k e,$$  
(25a)

$$0 < \psi_k e \leq \rho z_k, \quad 0 < \psi_k e \leq \nu_k e.$$  
(25b)

Positivity of $s(\sigma_k, \alpha_k)$ and $z(\sigma_k, \alpha_k)$ is guaranteed if $(s^0, z^0) > 0$ and the following conditions hold.

$$s^{k+1} = s(\sigma_k, \alpha_k) = s^k - \dot{s} \sin(\alpha_k) + \ddot{s}(1 - \cos(\alpha_k))$$
$$= p_s(1 - \cos(\alpha_k)) \sigma_k + [s^k - \dot{s} \sin(\alpha_k) + q_s(1 - \cos(\alpha_k))]$$
$$:= a_s(\alpha_k) \sigma_k + b_s(\alpha_k) \geq \phi_k e.$$  
(26)

$$z^{k+1} = z(\sigma, \alpha_k) = z^k - \dot{z} \sin(\alpha_k) + \ddot{z}(1 - \cos(\alpha_k))$$
$$= p_z(1 - \cos(\alpha_k)) \sigma_k + [z^k - \dot{z} \sin(\alpha_k) + q_z(1 - \cos(\alpha_k))]$$
$$:= a_z(\alpha_k) \sigma_k + b_z(\alpha_k) \geq \psi_k e.$$  
(27)

**Remark 3.2** Conditions of (26) and (27) will be enforced in the algorithm. Given $\phi_k$ and $\psi_k$ as calculated in (24), the corresponding $\alpha_k$ and $\sigma_k$ that meet (26) and (27) will be calculated by the formulas (73)-(99) and a process described in Algorithm A.1.

If $s^{k+1} = s^k - \dot{s} \sin(\alpha_k) + \ddot{s}(1 - \cos(\alpha_k)) \geq \rho s^k$ holds, from (25a), we have $s^{k+1} \geq \phi_k e$. Therefore, inequality (26) will hold if the following inequality holds

$$(1 - \rho)s^k - \dot{s} \sin(\alpha_k) + \ddot{s}(1 - \cos(\alpha_k)) \geq 0.$$  
(28)

In view of (25b), inequality (27) will hold if the following inequality holds

$$(1 - \rho)z^k - \dot{z} \sin(\alpha_k) + \ddot{z}(1 - \cos(\alpha_k)) \geq 0.$$  
(29)

Inequalities (28) and (29) hold for some $\alpha_k > 0$ bounded below and away from zero because $(1 - \rho)s^k > 0$ and $(1 - \rho)z^k > 0$ is bounded below and away from zero due to Assumption (A3').

The following proposition follows immediately from the above discussion.

**Proposition 3.2** There is an $\alpha_k > 0$ bounded below and away from zero such that $(s^{k+1}, z^{k+1}) > 0$ for all iteration $k$.

We will also need the following results in the rest discussions.
Lemma 3.2 Let $\dot{\mathbf{v}}$ and $\ddot{\mathbf{v}}$ be defined as in (13) and (14), and let $\mathbf{p}$ and $\mathbf{q}$ be defined as in (17) and (18). Then the following relations hold.

\begin{align}
\dot{\mathbf{x}}^T \mathbf{A}_I^T \dot{\mathbf{w}} &= \dot{\mathbf{x}}^T \mathbf{A}_I^T \ddot{\mathbf{z}} = \dot{\mathbf{x}}^T \mathbf{H} \ddot{\mathbf{x}} \geq 0, \quad \ddot{\mathbf{z}}^T \mathbf{z} \geq 0. \quad (30a) \\
\mathbf{p}_x^T \mathbf{A}_I^T \mathbf{p}_z &= \mathbf{p}_x^T \mathbf{H} \mathbf{p}_x \geq 0, \quad \mathbf{p}_s^T \mathbf{p}_z \geq 0. \quad (30b) \\
\mathbf{q}_x^T \mathbf{A}_I^T \mathbf{q}_z &= \mathbf{q}_x^T \mathbf{H} \mathbf{q}_x \geq 0, \quad \mathbf{q}_s^T \mathbf{q}_z \geq 0. \quad (30c)
\end{align}

**Proof:** Pre-multiplying $\dot{\mathbf{x}}^T$ in the first line of (14) gives

\[\dot{\mathbf{x}}^T \mathbf{H} \ddot{\mathbf{x}} + \dot{\mathbf{x}}^T \mathbf{A}_E^T \ddot{\mathbf{y}} - \dot{\mathbf{x}}^T \mathbf{A}_I^T \dot{\mathbf{w}} = 0.\]

From the second line of (14), we have $\dot{\mathbf{x}}^T \mathbf{A}_E^T = 0$. Therefore, $\dot{\mathbf{x}}^T \mathbf{H} \ddot{\mathbf{x}} = \dot{\mathbf{x}}^T \mathbf{A}_I^T \dot{\mathbf{w}} \geq 0$ because $\mathbf{H}$ is positive definite.

Pre-multiplying $\ddot{\mathbf{z}}^T$ in the third line of (14) gives

\[\ddot{\mathbf{z}}^T \mathbf{A}_I \ddot{\mathbf{x}} - \ddot{\mathbf{z}}^T \ddot{\mathbf{s}} = 0.\]

Therefore, using $\ddot{\mathbf{z}} = \ddot{\mathbf{w}}$, $\ddot{\mathbf{z}}^T \ddot{\mathbf{s}} = \ddot{\mathbf{z}}^T \mathbf{A}_I \ddot{\mathbf{x}} = \dot{\mathbf{x}}^T \mathbf{H} \ddot{\mathbf{x}} \geq 0$. Similarly, we can prove (30b) and (30c) using (17) and (18).

In addition, we need two simple sinusoidal identities in our proofs.

**Lemma 3.3**

\begin{align}
\sin^2(\alpha) - 2(1 - \cos(\alpha)) &= -(1 - \cos(\alpha))^2, \quad (31a) \\
\sin^2(\alpha) &\geq 1 - \cos(\alpha) \geq \frac{1}{2} \sin^2(\alpha). \quad (31b)
\end{align}

**Proof:** The proof is straightforward, therefore it is omitted.

The above two lemmas, together with (13), (14), (16), (17), and (18), will be used to calculate the value $\mu(\alpha)$.

**Proposition 3.3** Let $\alpha_k$ be the step size at $k$th iteration for $\mathbf{s}(\sigma_k, \alpha_k)$, $\mathbf{z}(\sigma_k, \alpha_k)$ defined in Theorem 3.1. Then, the updated duality measure after the iteration of $k$ can be expressed as

\[\mu_{k+1} := \mu(\sigma_k, \alpha_k) = \frac{1}{p} [a_u(\alpha_k) \sigma_k + b_u(\alpha_k)], \quad (32)\]

where

\[a_u(\alpha_k) = p \mu_k (1 - \cos(\alpha_k)) - (\ddot{\mathbf{z}}^T \mathbf{p}_s + \ddot{\mathbf{s}}^T \mathbf{p}_z) \sin(\alpha_k)(1 - \cos(\alpha_k))\]

and

\[b_u(\alpha_k) = p \mu_k (1 - \sin(\alpha_k)) - [\ddot{\mathbf{z}}^T \ddot{\mathbf{s}}(1 - \cos(\alpha_k))^2 + (\ddot{\mathbf{s}}^T \mathbf{q}_z + \ddot{\mathbf{z}}^T \mathbf{q}_s) \sin(\alpha_k)(1 - \cos(\alpha_k))]\]

are coefficients which are functions of $\alpha_k$. Moreover $a_u(\alpha_k) = O(p \mu_k \sin^2(\alpha))$ and $b_u(\alpha_k) = O(p \mu_k (1 - \sin(\alpha_k)))$. 

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Proof: The proof is similar to the one of [32] and therefore omitted.

Remark 3.3 Proposition 3.3 shows that if a positive $\alpha$ small enough, the duality gap is guaranteed to decrease.

The following proposition assures that Condition (C4) will hold.

Proposition 3.4 There is an $\alpha_k$ bounded below and away from zero for all $k$ such at (20) holds, i.e.,

$$s_{i+1}^{k+1} \geq \theta \mu_k(1 - \sin(\alpha_k)) + \sigma_k \mu_k(1 - \cos(\alpha_k)) - \left( \frac{s_i^{k+1}}{s_i} + \frac{s_i^{k+1}}{s_i} \right) \sin(\alpha_k)(1 - \cos(\alpha_k)) + \left( \frac{s_i^{k+1}}{s_i} - \frac{s_i^{k+1}}{s_i} \right)(1 - \cos(\alpha_k))^2. \tag{33}$$

Proof: The proof is similar to the one of [32], therefore is omitted.

Propositions 3.1, 3.2, 3.3, and 3.4 indicate that an arc-search strategy will generate a sequence of iterates that will eventually meet the approximated KKT conditions, plus Condition (C4) defined by (20). Therefore, we propose the following algorithm.

Algorithm 3.1 (an infeasible arc-search interior-point algorithm)

Parameters: $\theta \in (0, 1)$, and $\epsilon > 0$.

Initial point: $v^0 = (x^0, y^0, w^0, s^0, z^0), (w^0, s^0, z^0) \in \mathbb{R}^{3p}_+, and w^0 = z^0$.

for iteration $k = 0, 1, 2, \ldots$

Step 1: If $\|g(v^k)\| \leq \epsilon$, stop.

Step 2: Calculate $\nabla_z L(v^k), A_E x^k - b_E, A_I x^k - s^k - b_I$, and $H = \nabla^2_z L(v^k)$.

Step 3: Solve (13) to get $\tilde{\nu}^k = (\tilde{x}^k, \tilde{y}^k, \tilde{w}^k, \tilde{s}^k, \tilde{z}^k)$.

Step 4: Calculate $\tilde{Z} s, p$, and $q$.

Step 5: Select $\sigma_k \in [0, 1]$ and $\alpha_k > 0$ such that $\tilde{\nu}^k = p\sigma_k + q, v^{k+1} = v^k(\sigma_k, \alpha_k) = v^k - \tilde{\nu}^k \sin(\alpha_k) + v^k(1 - \cos(\alpha_k)), \mu_{k+1} < \mu_k, (s^{k+1}, w^{k+1}, z^{k+1}) > 0$, and $Z^{k+1} s^{k+1} \geq \theta \mu_{k+1} e$.

Step 6: $k + 1 \rightarrow k$ and go back to Step 1.

end (for)

The next section shows that the proposed algorithm converges in polynomial time.
4 Convergence analysis

In view of Propositions 3.1, 3.2, 3.3, and 3.4, the algorithm will generate a series of $\alpha_k$ which is bounded below and away from zero before the algorithm terminates. Therefore, there exist a constant $\rho \in (0,1)$ satisfying $\rho \geq (1 - \sin(\alpha_k))$ for all $k \geq 0$. Define

$$\beta_k = \frac{\min \{s_k, z_k\}}{\nu_k} \geq 0,$$

and

$$\beta = \inf_k \{\beta_k\} \geq 0.$$  \hspace{1cm} (34)

The next lemma is obtained in [32] and it shows that $\beta$ is bounded below from zero.

**Lemma 4.1 ([32])** Assuming that $\rho \in (0,1)$ is a constant and for all $k \geq 0$, $\rho \geq (1 - \sin(\alpha_k))$. Let $s_0 = \min_i s_i^0$ and $z_0 = \min_i z_i^0$. Then, we have $\beta \geq \min \{s_0, z_0, 1\}$.

Let $D = S^T Z^{-\frac{1}{2}} = \text{diag}(D_{ii})$. The next result can be derived using the same method in [32].

**Lemma 4.2 ([32])** For Algorithm 3.1, there is a constant $C_1$ independent of $n$ and $m$ such that for $\forall i \in \{1, \ldots, n\}$

$$(D_{ii})^{-1} \nu_k \sqrt{\frac{z_i^k}{z_i^k}} \leq C_1 \sqrt{n\mu_k}, \quad D_{ii}^k \nu_k \sqrt{\frac{s_i^k}{z_i^k}} \leq C_1 \sqrt{n\mu_k}.$$  \hspace{1cm} (36)

Note that $\bar{w} = \bar{z}$. Let $(\bar{x}, \bar{y}, \bar{w}, \bar{s}, \bar{z})$ be an optimal solution satisfying

$\begin{align*}
H\bar{x} + A_E^T \bar{y} - A_I^T \bar{z} &= 0, \\
A_E \bar{x} &= b_E, \\
A_I \bar{x} - \bar{s} &= b_I.
\end{align*}$

The following assumption means that the distance between the initial point and the optimal solution is bounded, which is reasonable.

**Assumption:**

(A5) There is a big constant $M$ which is independent to the problem size $n$ and $m$ such that the distance between initial point $(s^0, z^0)$ and $(\bar{s}, \bar{z})$ is smaller than $M$, i.e.,

$$\| (s^0 - \bar{s}, z^0 - \bar{z}) \| < M.$$  \hspace{1cm} (37)

**Lemma 4.3** For an optimal solution $(\bar{x}, \bar{y}, \bar{w}, s, z)$ of (1) that meets (37), the following linear systems of equations

$$\begin{bmatrix}
H & A_E^T & -A_I^T & 0 \\
A_E & 0 & 0 & 0 \\
A_I & 0 & 0 & -I \\
0 & 0 & S & Z
\end{bmatrix}
\begin{bmatrix}
\delta x \\
\delta y \\
\delta z \\
\delta s
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
0 \\
zs - \nu_k s(z^0 - \bar{z}) - \nu_k z(s^0 - \bar{s})
\end{bmatrix}.$$  \hspace{1cm} (38)
has a solution given by

\[
\begin{bmatrix}
\delta x \\
\delta y \\
\delta z \\
\delta s
\end{bmatrix} = \begin{bmatrix}
\dot{x} - \nu_k(x^0 - \bar{x}) \\
\dot{y} - \nu_k(y^0 - \bar{y}) \\
\dot{z} - \nu_k(z^0 - \bar{z}) \\
\dot{s} - \nu_k(s^0 - \bar{s})
\end{bmatrix}
\]

(39)

**Proof:** Using the first row of (13), (21a), (6), and (37a), we have

\[
H\dot{x} + A^T_E\dot{y} - A^T_I\dot{z} = r^k_C = \nu_k r^0_C = \nu_k (Hx^0 + A^T_Ey^0 - A^T_Iz^0 - H\bar{x} - A^T_E\bar{y} + A^T_I\bar{z}).
\]

This gives

\[
H[x - \nu_k(x^0 - \bar{x})] + A^T_E[y - \nu_k(y^0 - \bar{y})] - A^T_I[z - \nu_k(z^0 - \bar{z})] = 0,
\]

(40)

which is the first row of (38). Using the second row of (13), (21b), (6), and (37b), we have

\[
A_E\dot{x} = r^k_E = \nu_k r^0_E = \nu_k (A_Ex^0 - b_E) = \nu_k (A_Ex^0 - A_E\bar{x}).
\]

This gives

\[
A_E[x - \nu_k(x^0 - \bar{x})] = 0,
\]

(41)

which is the second row of (38). Using the third row of (13), (21c), (6), and (37c), we have

\[
A_I\dot{x} - \dot{s} = r^k_I = \nu_k r^0_I = \nu_k (A_Ix^0 - s^0 - b_I) = \nu_k [A_I(x^0 - \bar{x}) - (s^0 - \bar{s})].
\]

This gives

\[
A_I[x - \nu_k(x^0 - \bar{x})] - [\dot{s} - \nu_k(s^0 - \bar{s})] = 0,
\]

(42)

which is the third row of (38). Finally using the last row of (13), we have

\[
S[\dot{z} - \nu_k(z^0 - \bar{z})] + Z[\dot{s} - \nu_k(s^0 - \bar{s})]
= (S\dot{z} + Z\dot{s}) - \nu_k [Z(s^0 - \bar{s}) + S(z^0 - \bar{z})]
= Zs - \nu_k [Z(s^0 - \bar{s})] - \nu_k [S(z^0 - \bar{z})]
\]

(43)

which is the last row of (38). This completes the proof.

Let

\[
\begin{bmatrix}
r^1 \\
r^2 \\
r^3
\end{bmatrix} = \begin{bmatrix}
Zs \\
-\nu_k [Z(s^0 - \bar{s})] \\
-\nu_k [S(z^0 - \bar{z})]
\end{bmatrix},
\]

(44)

and let \((\delta x^i, \delta y^i, \delta z^i, \delta s^i)\), for \(i = 1, 2, 3\), be the solution of

\[
\begin{bmatrix}
H & A^T_E & -A^T_I & 0 \\
A_E & 0 & 0 & 0 \\
A_I & 0 & 0 & -I \\
0 & 0 & S & Z
\end{bmatrix} \begin{bmatrix}
\delta x^i \\
\delta y^i \\
\delta z^i \\
\delta s^i
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
r^i
\end{bmatrix}.
\]

(45)

We will use the following result in our analysis.
Lemma 4.4 The solutions of (38) and (45) meet the following relations.

\[\begin{align*}
\delta x &= \delta x^1 + \delta x^2 + \delta x^3 = \dot{x} - \nu_k(x^0 - \bar{x}), \\
\delta y &= \delta y^1 + \delta y^2 + \delta y^3 = \dot{y} - \nu_k(y^0 - \bar{y}), \\
\delta z &= \delta z^1 + \delta z^2 + \delta z^3 = \dot{z} - \nu_k(z^0 - \bar{z}), \\
\delta s &= \delta s^1 + \delta s^2 + \delta s^3 = \dot{s} - \nu_k(s^0 - \bar{s}).
\end{align*}\]

Moreover, \((D^{-1}\delta z^i)^T(D\delta s^i) \geq 0\) holds, for \(i = 1, 2, 3\).

**Proof:** The first claim follows immediately from the linear systems of equations (38) and (45). The second claim is equivalent to \((\delta z^i)^T(\delta s^i) \geq 0\). Pre-multiplying \((\delta z^i)^T\) in the third row of (45) yields

\[\delta z^T A_1 \delta x^i - (\delta z^i)^T \delta s^i = 0.\] (47)

Rewriting the first row of (45) yields

\[A_1^T \delta z^i = H \delta x^i + A_E^T \delta y^i.\] (48)

Substituting (48) into (47) and using the second row of (45) yields

\[\begin{align*}
(\delta z^i)^T A_1 \delta x^i - (\delta z^i)^T \delta s^i &= \left(\delta x^T H^T + \delta y^T A_E^T\right) \delta x^i - (\delta z^i)^T \delta s^i \\
&= \delta x^T H^T \delta x^i - (\delta z^i)^T \delta s^i = 0.
\end{align*}\] (49)

Since \(H\) is a convex Hessian matrix, therefore, \(H\) is symmetric and positive definite, the last equality indicates that \((\delta z^i)^T \delta s^i = \delta x^T H \delta x^i \geq 0\) for \(i = 1, 2, 3\).

Now, we are ready to provide several estimations that are important to the convergence analysis.

**Lemma 4.5** Let \((x^0, y^0, w^0, s^0, z^0)\) be the initial point of Algorithm 3.1, and \((\bar{x}, \bar{y}, \bar{z}, \bar{w}, \bar{s})\) be an optimal solution of (1). Then

\[\|D\bar{z}\|, \|D^{-1}\dot{s}\| \leq \sqrt{n \mu} + \|D\delta z^2\| + \|D^{-1}\delta s^3\|.\] (50)

**Proof:** Since \((D^{-1}\delta s^i)^T(D\delta z^i) \geq 0\), for \(i = 1, 2, 3\), it follows that

\[\|D^{-1}\delta s^i\|^2, \|D\delta z^i\|^2 \leq \|D^{-1}\delta s^i\|^2 + \|D\delta z^i\|^2 \leq \|D^{-1}\dot{s} + D\delta z^i\|^2.\] (51)

Applying \(S\delta z^i + Z\delta s^i = r^i\) to (51) for \(i = 1, 2, 3\) respectively, we obtain the following relations

\[\|D^{-1}\delta s^1\|, \|D\delta z^1\| \leq \|D^{-1}\delta s^1 + D\delta z^1\| = \|(Sz)^T\| = \sqrt{\bar{s}^Tz} = \sqrt{n \mu},\] (52a)
(52b) \[ \| D^{-1} \delta s^2 \|, \| D \delta z^2 \| \leq \| D^{-1} \delta s^2 + D \delta z^2 \| = \nu_k \| D^{-1}(s^0 - \bar{s}) \|. \]

(52c) \[ \| D^{-1} \delta s^3 \|, \| D \delta z^3 \| \leq \| D^{-1} \delta s^3 + D \delta z^3 \| = \nu_k \| D(z^0 - \bar{z}) \|. \]

Considering the last row of (45) with \( i = 2 \), we have

\[ Z \delta s^2 + S \delta z^2 = r^2 = -\nu_k Z(s^0 - \bar{s}), \]

which is equivalent to

\[ \delta s^2 = -\nu_k(s^0 - \bar{s}) - D^2 \delta z^2. \] (53)

Thus, from (46d), (53), and (52), we have

\[ \| D^{-1} \dot{s} \| = \| D^{-1} \delta s^1 + \delta s^2 + \delta s^3 + \nu_k(s^0 - \bar{s}) \| \]

\[ = \| D^{-1} \delta s^1 - D \delta z^2 + D^{-1} \delta s^3 \| \]

\[ \leq \| D^{-1} \delta s^1 \| + \| D \delta z^2 \| + \| D^{-1} \delta s^3 \|. \] (54)

Considering the last row of (45) with \( i = 3 \), we have

\[ Z \delta s^3 + S \delta z^3 = r^3 = -\nu_k S(z^0 - \bar{z}), \]

which is equivalent to

\[ \delta z^3 = -\nu_k(z^0 - \bar{z}) - D^{-2} \delta s^3. \] (55)

Thus, from (46c), (55), and (52), we have

\[ \| D \dot{z} \| = \| D[\delta z^1 + \delta z^2 + \delta z^3 + \nu_k(z^0 - \bar{z})] \| \]

\[ = \| D \delta z^1 + D \delta z^2 - D^{-1} \delta s^3 \| \]

\[ \leq \| D \delta z^1 \| + \| D \delta z^2 \| + \| D^{-1} \delta s^3 \|. \] (56)

In view of (52a), adjoining (54) and (56) gives (50).

The next lemma provides an estimation which will be used to establish the polynomiality for the proposed algorithm.

**Lemma 4.6** Let \((\dot{x}, \dot{y}, \dot{w}, \dot{s}, \dot{z})\) be defined in (13). Then, there is a constant \( C_2 \) independent of \( n \) and \( m \) such that in every iteration of Algorithm 3.1, the following inequality holds.

\[ \| D \dot{z} \|, \| D^{-1} \dot{s} \| \leq C_2 \sqrt{n \mu_k}. \] (57)

**Proof:** In view of Lemma 4.4, we have

\[ (D \delta z^2)^T(D^{-1} \delta s^2) = (\delta z^2)^T(\delta s^2) \geq 0, \quad (D \delta z^3)^T(D^{-1} \delta s^3) = (\delta z^3)^T(\delta s^3) \geq 0. \]

Using a similar idea of [32], we can derive (57).

Lemma 4.6 can be used to derive several useful inequalities.
Lemma 4.7 Let \((\dot{x}, \dot{y}, \dot{w}, \dot{s}, \dot{z})\) and \((\ddot{x}, \ddot{y}, \ddot{w}, \ddot{s}, \ddot{z})\) be defined in (13) and (14). There is a constant \(C_3 > 0\) independent of \(n\) and \(m\) such that the following inequalities hold.

\[
\|D^{-1}s\|, \|Dz\| \leq C_3n^{0.5} \mu_k^0 \tag{58a}
\]
\[
\|D^{-1}p\|, \|Dp_z\| \leq \sqrt{n/\theta} \mu_k^{0.5}, \tag{58b}
\]
\[
\|D^{-1}q\|, \|Dq_z\| \leq \frac{2C_2^2}{\sqrt{\theta}} n^{0.5} \mu_k^{0.5}. \tag{58c}
\]

Proof: In view of the last row of (18), using the facts that \(q_i^T s_i > 0\) (30c), \(s_i > \theta \mu_k\), and Lemma 4.6, and a similar idea of [32], we can prove (58c). □

The following inequalities follow directly from the results of Lemmas 4.6 and 4.7.

Lemma 4.8 Let \((\dot{x}, \dot{y}, \dot{w}, \dot{s}, \dot{z})\) and \((\ddot{x}, \ddot{y}, \ddot{w}, \ddot{s}, \ddot{z})\) be defined in (13) and (14). The following inequalities hold.

\[
\left| \frac{s^T \dot{z}}{n} \right| \leq C_2^2 \mu_k, \quad \left| \frac{\dot{s}^T \dot{z}}{n} \right| \leq C_2 C_3 \sqrt{n} \mu_k, \quad \left| \frac{\ddot{s}^T \ddot{z}}{n} \right| \leq C_2 C_3 \sqrt{n} \mu_k. \tag{59}
\]

Moreover,

\[
|\dot{s}_i \dot{z}_i| \leq C_2 n \mu_k, \quad |\ddot{s}_i \ddot{z}_i| \leq C_2 C_3 n^{2} \mu_k, \quad |\dot{s}_i \ddot{z}_i| \leq C_2 C_3 n^{2} \mu_k, \quad |\ddot{s}_i \ddot{z}_i| \leq C_2^2 n^{2} \mu_k. \tag{60}
\]

Lemmas 4.6, 4.7, and 4.8 will be used in finding the lower bound of \(\alpha_k\).

Lemma 4.9 There is a positive constant \(C_4\) independent of \(m\) and \(n\), and an \(\bar{\alpha}\) defined by \(\sin(\bar{\alpha}) \geq \frac{C_4}{\sqrt{n}}\) such that for \(\forall k \geq 0\) and \(\sin(\alpha_k) \in (0, \sin(\bar{\alpha})]\),

\[
(s_i^{k+1}, z_i^{k+1}) := (s_i(\sigma_k, \alpha_k), z_i(\sigma_k, \alpha_k)) \geq (\phi_k, \psi_k) > 0 \tag{61}
\]

holds.

Proof: Using (26), (25a), and Lemmas 4.6, 4.8, 3.3, the proof is straightforward and similar to the proof used in [32]. □

Lemma 4.10 There is a positive constant \(C_5\) independent of \(n\) and \(m\), and an \(\bar{\alpha}\) defined by \(\sin(\bar{\alpha}) \geq \frac{C_5}{\sqrt{n}}\) such that for \(\forall k \geq 0\) and \(\sin(\alpha) \in (0, \sin(\bar{\alpha})]\), the following relation

\[
\mu_k(\sigma_k, \alpha_k) \leq \mu_k\left(1 - \frac{\sin(\alpha_k)}{4}\right) \leq \mu_k\left(1 - \frac{C_5}{4n}\right) \tag{62}
\]
holds.

**Proof:** Using Proposition 3.3, Lemmas 3.3 and 4.8 and the similar idea used in [32], we can easily prove the result.

**Lemma 4.11** There is a positive constant \( C_6 \) independent of \( n \) and \( m \), an \( \tilde{\alpha} \) defined by

\[
\sin(\tilde{\alpha}) \geq \frac{C_6}{n^2}
\]

such that if \( s_i^k z_i^k \geq \theta \mu_k \) holds, then for \( \forall k \geq 0, \forall i \in \{1, \ldots, n\} \), and \( \sin(\alpha) \in (0, \sin(\tilde{\alpha})] \), the following relation

\[
s_i^{k+1} z_i^{k+1} \geq \theta \mu_{k+1}
\]

holds.

**Proof:** Using (33), Proposition 3.3, and Lemma 4.8 and the similar idea used in [32], we can easily prove the result.

The following theorem given in [25] has been used to establish the polynomial bound for almost all interior-point algorithms.

**Theorem 4.1 ([25])** Let \( \epsilon \in (0, 1) \) be given. Suppose that an algorithm generates a sequence of iterations \( \{\chi_k\} \) that satisfies

\[
\chi_{k+1} \leq \left(1 - \frac{\delta}{n\omega}\right) \chi_k, \quad k = 0, 1, 2, \ldots,
\]

for some positive constants \( \delta \) and \( \omega \). Then there exists an index \( K \) with

\[
K = O(n^\omega \log(\chi_0/\epsilon))
\]

such that

\[
\chi_k \leq \epsilon \quad \text{for} \quad \forall k \geq K.
\]

The main result of the paper immediately follows from Lemmas 3.1, 4.9, 4.10, 4.11, and Theorem 4.1.

**Theorem 4.2** Algorithms 3.1 is a polynomial algorithm with polynomial complexity bound of \( O(n^{\frac{3}{2}} \max\{\log((s^0)^T z^0/\epsilon), \log(r^0_c/\epsilon), \log(r^0_B/\epsilon), \log(r^0_I/\epsilon)\}) \).

**Proof:** The proof is similar to the one used in [32] and therefore omitted.

## 5 Implementation and numerical test

In this section, we briefly discuss a Matlab implementation of the proposed algorithm and provide some preliminary test results.
5.1 Matlab implementation

Algorithm 3.1 is implemented as a Matlab function:
\[ [x, \text{obj}, \text{kk}, \text{infe}] = \text{arcConvex}(\text{AE}, \text{bE}, \text{AI}, \text{bI}, f, v0, d) \]
where AE and bE are the input matrix and vector for equality constraints, AI and bI are the input matrix and vector for inequality constraints, \( f \) is the objective function of the convex nonlinear optimization problem, which calls a function handle created in a separate file; \( x \) is the output which returns the optimal solution, \( \text{obj} \) is the output which returns the optimal value of the convex nonlinear optimization problem, \( \text{kk} \) is the iteration number which is used to find the optimal solution, and \( \text{infe} \) is the norm of \( \| A_Ex - b_E \| \) which should be small when the program terminates.

In Algorithm 3.1, Step 1 involves the calculation of the gradient of \( g(v^k) \), and Step 2 involves the calculation of the gradient of \( \nabla_x L(v^k) \) and the Hessian \( H = \nabla^2_x L(v^k) \).

To avoid manipulating analytical formulas for every individual problem, which can be tedious and error prone, we adopted a piece of code used in [30] that implements the automatic differentiation method discussed in [23]. The optimal section of \( \sigma \) and \( \alpha \) is implemented exactly as described in the Algorithms A.1 and A.2.

5.2 Preliminary numerical test

The implemented Matlab code is tested for a few problems that were found from different sources. We made no effort to select the initial points for these problems. All problems have a similar linear constraint set but different objective functions.

Example 1 [16]: This problem was posted in Researchgate and a solution was solicited. The objective function is a two dimensional logarithm function.

\[
\begin{align*}
\min & \quad -[(a_1 \log(x_1) - x_1 + b_1) + (a_2 \log(x_2) - x_2 + b_2)] \\
\text{s.t.} & \quad x_1 + x_2 \leq 10 \\
& \quad \ell_1 \leq x_1 \leq u_1 \\
& \quad \ell_2 \leq x_2 \leq u_2, \\
\end{align*}
\]

where we set \( a_1 = 5, b_1 = 7, a_2 = 7, b_2 = 8, \ell_1 = 1, \ell_2 = 1, u_1 = 10, \) and \( u_2 = 10 \).
Starting from initial point \( x^0 = (5, 5) \), \( w^0 = z^0 = (100, 100, 100, 100, 100) \), and \( s^0 = (0.01, 0.01, 0.01, 0.01, 0.01) \), after 68 iterations, we get \( x = (1, 1) \) and the optimal value is \(-13\).

The rest examples are created based on [4, pages 71-73].

Example 2 The objective function of this problem is a two dimensional exponential function.

\[
\begin{align*}
\min & \quad (a_1 e^{x_1} + b_1) + (a_2 e^{x_2} + b_2) \\
\text{s.t.} & \quad x_1 + x_2 \leq 10 \\
& \quad \ell_1 \leq x_1 \leq u_1 \\
\end{align*}
\]

\(^1\)If there is no equality constraint in the problem, then the inputs of AE and bE are \([\ ]\) and \([\ ]\) respectively.
\[ \ell_2 \leq x_2 \leq u_2, \]

where we set \( a_1 = 5, b_1 = 7, a_2 = 7, b_2 = 8, \ell_1 = 2, \ell_2 = 1, u_1 = 10, \) and \( u_2 = 10. \)

Starting from initial point \( x^0 = (5, 5), w^0 = z^0 = (100, 100, 100, 100, 100), \) and \( s^0 = (0.01, 0.01, 0.01, 0.01, 0.01), \) after 66 iterations, we get \( x = (2, 1) \) and the optimal value is 70.9733.

**Example 3** The objective function of this problem is a two dimensional power function of \( x^a \) with the first power greater than 1 and the second power smaller than 0.

\[
\begin{align*}
& \min \ (a_1 x_1^3 + b_1) + (a_2 \frac{1}{x_2} + b_2)] \\
& \text{s.t.} \quad x_1 + x_2 \leq 10 \\
& \quad \ell_1 \leq x_1 \leq u_1 \\
& \quad \ell_2 \leq x_2 \leq u_2, \\
\end{align*}
\]

where we set \( a_1 = 5, b_1 = 7, a_2 = 7, b_2 = 8, \ell_1 = 1, \ell_2 = 2, u_1 = 10, \) and \( u_2 = 10. \)

Starting from initial point \( x^0 = (5, 5), w^0 = z^0 = (100, 100, 100, 100, 100), \) and \( s^0 = (0.01, 0.01, 0.01, 0.01, 0.01), \) after 69 iterations, we get \( x = (1, 2) \) and the optimal value is 23.5000.

**Example 4** The objective function of this problem is a two dimensional negative entropy function.

\[
\begin{align*}
& \min \ (a_1 x_1 \log(x_1) + b_1) + (a_2 x_2 \log(x_2) + b_2) \\
& \text{s.t.} \quad x_1 + x_2 \leq 10 \\
& \quad \ell_1 \leq x_1 \leq u_1 \\
& \quad \ell_2 \leq x_2 \leq u_2, \\
\end{align*}
\]

where we set \( a_1 = 5, b_1 = 7, a_2 = 7, b_2 = 8, \ell_1 = 2, \ell_2 = 2, u_1 = 10, \) and \( u_2 = 10. \)

Starting from initial point \( x^0 = (5, 5), w^0 = z^0 = (100, 100, 100, 100, 100), \) and \( s^0 = (0.01, 0.01, 0.01, 0.01, 0.01), \) after 69 iterations, we get \( x = (2, 2) \) and the optimal value is 31.6355.

**Example 5** The objective function of this problem is a two dimensional quadratic-over-linear function.

\[
\min \ \frac{(a_1 x_1)^2}{a_2 x_2} \\
\text{s.t.} \quad x_1 + x_2 \leq 10 \\
\quad \ell_1 \leq x_1 \leq u_1 \\
\quad \ell_2 \leq x_2 \leq u_2, \\
\]

where we set \( a_1 = 5, a_2 = 7, \ell_1 = 1, \ell_2 = 3, u_1 = 10, \) and \( u_2 = 10. \)

Starting from initial point \( x^0 = (5, 5), w^0 = z^0 = (100, 100, 100, 100, 100), \) and \( s^0 = (0.01, 0.01, 0.01, 0.01, 0.01), \) after 57 iterations, we get \( x = (4.9271, 5.0595) \) and the optimal value is 17.1360.
Example 6 The objective function of this problem is a two dimensional log-sum-exponential function.

\[
\begin{align*}
\min & \quad \log(a_1 e^{x_1} + a_2 e^{x_2}) \\
\text{s.t.} & \quad x_1 + x_2 \leq 10 \\
& \quad \ell_1 \leq x_1 \leq u_1 \\
& \quad \ell_2 \leq x_2 \leq u_2,
\end{align*}
\]

where we set \(a_1 = 5, a_2 = 7, \ell_1 = 3, \ell_2 = 1, u_1 = 10, \) and \(u_2 = 10.\) Starting from initial point \(x^0 = (5, 5), w^0 = z^0 = (100, 100, 100, 100, 100),\) and \(s^0 = (0.01, 0.01, 0.01, 0.01, 0.01),\) after 56 iterations, we get \(x = (4.9924, 4.9924)\) and the optimal value is 7.4773.

Example 7 The objective function of this problem is a two dimensional geometric mean function.

\[
\begin{align*}
\min & \quad (x_1 * x_2)^{1/2} \\
\text{s.t.} & \quad x_1 + x_2 \leq 10 \\
& \quad \ell_1 \leq x_1 \leq u_1 \\
& \quad \ell_2 \leq x_2 \leq u_2,
\end{align*}
\]

where we set \(\ell_1 = 2, \ell_2 = 3, u_1 = 10, \) and \(u_2 = 10.\) Starting from initial point \(x^0 = (5, 5), w^0 = z^0 = (100, 100, 100, 100, 100),\) and \(s^0 = (0.01, 0.01, 0.01, 0.01, 0.01),\) after 59 iterations, we get \(x = (2.0006, 7.9767)\) and the optimal value is 3.9948.

Example 8 The objective function of this problem is a two dimensional log-determinant function for positive definite matrix.

\[
\begin{align*}
\min & \quad -\log \det \left( \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix} \right) \\
\text{s.t.} & \quad x_1 + x_2 \leq 10 \\
& \quad x_2 + x_3 \leq 10 \\
& \quad \ell_1 \leq x_1 \leq u_1 \\
& \quad \ell_2 \leq x_2 \leq u_2, \\
& \quad \ell_3 \leq x_3 \leq u_3,
\end{align*}
\]

where we set \(\ell_1 = 5, \ell_2 = 1, \ell_3 = 5, u_1 = 10, u_2 = 3, \) and \(u_3 = 10.\) Starting from initial point \(x^0 = (6, 2, 6), w^0 = z^0 = (100, 100, 100, 100, 100, 100),\) and \(s^0 = (0.01, 0.01, 0.01, 0.01, 0.01, 0.01),\) after 44 iterations, we get \(x = (5, 3, 5)\) and the optimal value is \(-2.7726.\)

6 Conclusions

In this paper, we proposed an infeasible interior-point arc-search algorithm for convex optimization problem with linear equality and inequality constraints. Many application problems can be formulated as this optimization problem. We showed that this
algorithm is convergent with a nice polynomial iteration bound. To have a good performance, we provided analytic formulas for the arc-search, and developed an efficient algorithm to dynamically select centering parameter and the step size at the same time in Appendix A. In the future, we may consider the general convex programming problem with convex nonlinear inequality constraints.

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A Selection of the centering parameter $\sigma_k$ and step size $\alpha_k$

Although the method of selecting $\alpha_k$ described in Sections 3 and 4 assures that the algorithm converges in polynomial iteration, but this selection is very conservative. A better method is to simultaneously select centering parameter $\sigma_k$ and step size $\alpha_k$ to maximize the step size in every iteration. The merit of this holistic strategy is proved in theory [34], and has been demonstrated in computational experiments [31, 32]. The same strategy is proposed in Step 5 of Algorithm 3.1, but there is no details provided there. In this appendix, we discuss how this strategy is implemented. Although the formulas in this appendix are similar to the ones in [32], they are different. To avoid the confusion and implementation errors, we would like to list them in this appendix. For the sake of completeness, we also provide the proofs even though they follow the same ideas of [32].

Let the current iterate be $\mathbf{v}^k = (\mathbf{x}^k, \mathbf{y}^k, \mathbf{w}^k, \mathbf{s}^k, \mathbf{z}^k)$, $(\tilde{x}, \tilde{y}, \tilde{w}, \tilde{s}, \tilde{z})$, $(\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_w, \mathbf{p}_s, \mathbf{p}_z)$ be computed by solving (13), $(\mathbf{q}_x, \mathbf{q}_y, \mathbf{q}_w, \mathbf{q}_s, \mathbf{q}_z)$ be computed by solving (18), $\phi_k$ and $\psi_k$ be computed by using (23) and (24). An intuition based on Propositions 3.1 and 3.3 is that the step size $\alpha_k$ should be chosen as large as possible provided that Condition (C4), (26) and (27) hold. Given $\mathbf{v}^k = (\mathbf{x}^k, \mathbf{y}^k, \mathbf{w}^k, \mathbf{s}^k, \mathbf{z}^k)$, $(\tilde{x}, \tilde{y}, \tilde{w}, \tilde{s}, \tilde{z})$, $(\mathbf{p}_x, \mathbf{p}_y, \mathbf{p}_w, \mathbf{p}_s, \mathbf{p}_z)$, $(\mathbf{q}_x, \mathbf{q}_y, \mathbf{q}_w, \mathbf{q}_s, \mathbf{q}_z)$, $\phi_k$ and $\psi_k$, similar to the derivation of [31], the largest $\tilde{\alpha}$ that meet conditions (26) and (27) can be expressed as a function of $\sigma_k$. For each $i \in \{1, \ldots, n\}$, given $\sigma$, we can select the largest $\alpha_{s_i}$ such that for any $\alpha \in [0, \alpha_{s_i}]$, the $i$th inequality of (26) holds, and the largest $\alpha_{z_i}$ such that for any $\alpha \in [0, \alpha_{z_i}]$ the $i$th inequality of (27) holds. We then define

$$\alpha^s = \min_{i \in \{1, \ldots, n\}} \{\alpha_{s_i}\},$$

$$\alpha^z = \min_{i \in \{1, \ldots, n\}} \{\alpha_{z_i}\},$$

$$\tilde{\alpha} = \min\{\alpha^s, \alpha^z\},$$

where $\alpha_{s_i}$ and $\alpha_{z_i}$ can be obtained, using a similar argument as in [31], in analytical forms represented by $\phi_k$, $\tilde{s}_i$, $\tilde{s}_i = p_{s_i} \sigma + q_{s_i}$, $\psi_k$, $\tilde{z}_i$, and $\tilde{z}_i = p_{z_i} \sigma + q_{z_i}$. First, from (26), we have

$$s_i + \tilde{s}_i - \phi_k \geq \tilde{s}_i \sin(\alpha) + \tilde{s}_i \cos(\alpha).$$  \hspace{1cm} (76)

Case 1a ($\tilde{s}_i = 0$ and $p_{s_i} \sigma + q_{s_i} \neq 0$):

In this case, if $\tilde{s}_i = -(s_i - \phi_k)$ and $\alpha \in [0, \frac{\pi}{2}]$, then $s_i(\alpha) \geq \phi_k$ follows from (26). If $\tilde{s}_i \leq -(s_i - \phi_k)$ or $s_i + \tilde{s}_i - \phi_k \leq 0$, to meet (76), we must have $\cos(\alpha) \geq \frac{x_i + \tilde{x}_i - \phi_k}{x_i}$, or,
\[ \alpha \leq \cos^{-1}\left(\frac{x_i + \hat{s}_i - \phi_k}{\hat{s}_i}\right). \] Therefore,

\[
\alpha_{s_i}(\sigma) = \begin{cases} 
\frac{\pi}{2} 
\cos^{-1}\left(\frac{s_i - \phi_k + p_{s_i}\sigma + q_{s_i}}{p_{s_i}\sigma + q_{s_i}}\right) & \text{if } s_i + (p_{s_i}\sigma + q_{s_i}) - \phi_k \geq 0 \\
\cos^{-1}\left(\frac{s_i - \phi_k}{p_{s_i}\sigma + q_{s_i}}\right) & \text{if } s_i + (p_{s_i}\sigma + q_{s_i}) - \phi_k \leq 0.
\end{cases} \tag{77}
\]

**Case 2a** \((p_{s_i}\sigma + q_{s_i} = 0 \text{ and } \hat{s}_i \neq 0): \)

In this case, if \(\hat{s}_i \leq s_i - \phi_k \text{ and } \alpha \in \left[0, \frac{\pi}{2}\right]\), then \(s_i(\alpha) \geq \phi_k \) follows from (26). If \(\hat{s}_i \geq s_i - \phi_k\), to meet (76), we must have \(\sin(\alpha) \leq \frac{s_i - \phi_k}{s_i}\), or \(\alpha \leq \sin^{-1}\left(\frac{s_i - \phi_k}{s_i}\right)\). Therefore,

\[
\alpha_{s_i}(\sigma) = \begin{cases} 
\frac{\pi}{2} 
\sin^{-1}\left(\frac{s_i - \phi_k}{s_i}\right) & \text{if } \hat{s}_i \leq s_i - \phi_k \\
\sin^{-1}\left(\frac{s_i - \phi_k}{s_i}\right) - \beta & \text{if } \hat{s}_i \geq s_i - \phi_k
\end{cases} \tag{78}
\]

**Case 3a** \((\hat{s}_i > 0 \text{ and } p_{s_i}\sigma + q_{s_i} > 0): \)

Let \(\hat{s}_i = \sqrt{s_i^2 + \hat{s}_i^2} \cos(\beta)\), and \(\hat{s}_i = \sqrt{s_i^2 + \hat{s}_i^2} \sin(\beta)\), (76) can be rewritten as

\[ s_i + \hat{s}_i - \phi_k \geq \sqrt{s_i^2 + \hat{s}_i^2} \sin(\alpha + \beta), \tag{79} \]

where

\[
\beta = \sin^{-1}\left(\frac{\hat{s}_i}{\sqrt{s_i^2 + \hat{s}_i^2}}\right) = \sin^{-1}\left(\frac{p_{s_i}\sigma + q_{s_i}}{\sqrt{s_i^2 + (p_{s_i}\sigma + q_{s_i})^2}}\right). \tag{80}
\]

If \(\hat{s}_i + s_i - \phi_k \geq \sqrt{s_i^2 + \hat{s}_i^2} \text{ and } \alpha \in \left[0, \frac{\pi}{2}\right]\), then \(s_i(\alpha) \geq \phi_k \) follows from (26). If \(\hat{s}_i + s_i - \phi_k \leq \sqrt{s_i^2 + \hat{s}_i^2}\), to meet (79), we must have \(\sin(\alpha + \beta) \leq \frac{s_i + \hat{s}_i - \phi_k}{\sqrt{s_i^2 + \hat{s}_i^2}}\), or \(\alpha + \beta \leq \sin^{-1}\left(\frac{s_i + \hat{s}_i - \phi_k}{\sqrt{s_i^2 + \hat{s}_i^2}}\right)\). Therefore,

\[
\alpha_{s_i}(\sigma) = \begin{cases} 
\frac{\pi}{2} 
\sin^{-1}\left(\frac{s_i - \phi_k + p_{s_i}\sigma + q_{s_i}}{\sqrt{s_i^2 + (p_{s_i}\sigma + q_{s_i})^2}}\right) - \beta & \text{if } s_i - \phi_k + p_{s_i}\sigma + q_{s_i} \geq \sqrt{s_i^2 + (p_{s_i}\sigma + q_{s_i})^2} \\
\sin^{-1}\left(\frac{s_i - \phi_k}{\sqrt{s_i^2 + \hat{s}_i^2}}\right) - \beta & \text{if } s_i - \phi_k + p_{s_i}\sigma + q_{s_i} \leq \sqrt{s_i^2 + (p_{s_i}\sigma + q_{s_i})^2}
\end{cases} \tag{81}
\]

**Case 4a** \((\hat{s}_i > 0 \text{ and } p_{s_i}\sigma + q_{s_i} < 0): \)

Let \(\hat{s}_i = \sqrt{s_i^2 + \hat{s}_i^2} \cos(\beta)\), and \(\hat{s}_i = -\sqrt{s_i^2 + \hat{s}_i^2} \sin(\beta)\), (76) can be rewritten as

\[ s_i + \hat{s}_i - \phi_k \geq \sqrt{s_i^2 + \hat{s}_i^2} \sin(\alpha - \beta), \tag{82} \]

where

\[
\beta = \sin^{-1}\left(\frac{-\hat{s}_i}{\sqrt{s_i^2 + \hat{s}_i^2}}\right) = \sin^{-1}\left(\frac{- (p_{s_i}\sigma + q_{s_i})}{\sqrt{s_i^2 + (p_{s_i}\sigma + q_{s_i})^2}}\right). \tag{83}
\]

If \(\hat{s}_i + s_i - \phi_k \geq \sqrt{s_i^2 + \hat{s}_i^2} \text{ and } \alpha \in \left[0, \frac{\pi}{2}\right]\), then \(s_i(\alpha) \geq \phi_k \) follows from (26). If \(\hat{s}_i + s_i - \phi_k \leq \sqrt{s_i^2 + \hat{s}_i^2}\), to meet (82), we must have \(\sin(\alpha - \beta) \leq \frac{s_i + \hat{s}_i}{\sqrt{s_i^2 + \hat{s}_i^2}}\), or \(\alpha - \beta \leq \]
\sin^{-1}\left(\frac{s_i + \dot{s}_i}{\sqrt{s_i^2 + \dot{s}_i^2}}\right).\) Therefore,

\[\alpha_{s_i}(\sigma) = \begin{cases} \frac{\pi}{2} & \text{if } s_i - \phi_k + p_{s_i}\sigma + q_{s_i} \geq \sqrt{s_i^2 + (p_{s_i}\sigma + q_{s_i})^2} \\ \pi - \sin^{-1}\left(\frac{-s_i}{\sqrt{s_i^2 + \dot{s}_i^2}}\right) - \beta & \text{if } s_i - \phi_k + p_{s_i}\sigma + q_{s_i} \leq \sqrt{s_i^2 + (p_{s_i}\sigma + q_{s_i})^2} \end{cases}\] (84)

Case 5a \((\dot{s}_i < 0 \text{ and } p_{s_i}\sigma + q_{s_i} < 0)\):

Let \(\dot{s}_i = -\sqrt{s_i^2 + \dot{s}_i^2}\cos(\beta)\), and \(\ddot{s}_i = -\sqrt{s_i^2 + \dot{s}_i^2}\sin(\beta)\), (76) can be rewritten as

\[s_i + \ddot{s}_i - \phi_k \geq -\sqrt{s_i^2 + \dot{s}_i^2}\sin(\alpha + \beta),\] (85)

where

\[\beta = \sin^{-1}\left(\frac{-s_i}{\sqrt{s_i^2 + \dot{s}_i^2}}\right) = \sin^{-1}\left(\frac{-p_{s_i}\sigma + q_{s_i}}{\sqrt{s_i^2 + (p_{s_i}\sigma + q_{s_i})^2}}\right).\] (86)

If \(s_i + \dot{s}_i - \phi_k \geq 0\) and \(\alpha \in [0, \frac{\pi}{2}]\), then \(s_i(\alpha) \geq \phi_k\) follows from (26). If \(s_i + \dot{s}_i - \phi_k \leq 0\), to meet (85), we must have \(\sin(\alpha + \beta) \geq \frac{-s_i + \ddot{s}_i - \phi_k}{\sqrt{s_i^2 + \dot{s}_i^2}},\) or \(\alpha + \beta \leq \pi - \sin^{-1}\left(\frac{s_i + \ddot{s}_i - \phi_k}{\sqrt{s_i^2 + \dot{s}_i^2}}\right).\) Therefore,

\[\alpha_{s_i}(\sigma) = \begin{cases} \frac{\pi}{2} & \text{if } s_i - \phi_k + p_{s_i}\sigma + q_{s_i} \geq 0 \\ \pi - \sin^{-1}\left(\frac{-s_i + \ddot{s}_i - \phi_k}{\sqrt{s_i^2 + \dot{s}_i^2}}\right) - \beta & \text{if } s_i - \phi_k + p_{s_i}\sigma + q_{s_i} \leq 0 \end{cases}\] (87)

Case 6a \((\dot{s}_i < 0 \text{ and } p_{s_i}\sigma + q_{s_i} > 0)\):

\[\alpha_{s_i}(\sigma) = \frac{\pi}{2} \] (88)

Case 7a \((\dot{s}_i = 0 \text{ and } p_{s_i}\sigma + q_{s_i} = 0)\):

\[\alpha_{s_i}(\sigma) = \frac{\pi}{2}.\] (89)

Using the same idea, we can obtain the similar formulas for \(\alpha_{s_i}(\sigma)\).

Case 1b \((\dot{z}_i = 0, p_{z_i}\sigma + q_{z_i} \neq 0)\):

\[\alpha_{z_i}(\sigma) = \begin{cases} \frac{\pi}{2} & \text{if } z_i - \psi_k + p_{z_i}\sigma + q_{z_i} \geq 0 \\ \cos^{-1}\left(\frac{z_i - \psi_k + p_{z_i}\sigma + q_{z_i}}{p_{z_i}\sigma + q_{z_i}}\right) & \text{if } z_i - \psi_k + p_{z_i}\sigma + q_{z_i} \leq 0. \end{cases}\] (90)

Case 2b \((p_{z_i}\sigma + q_{z_i} = 0 \text{ and } \dot{z}_i \neq 0)\):

\[\alpha_{z_i}(\sigma) = \begin{cases} \frac{\pi}{2} & \text{if } \dot{z}_i \leq z_i - \psi_k \\ \sin^{-1}\left(\frac{z_i - \psi_k}{\dot{z}_i}\right) & \text{if } \dot{z}_i \geq z_i - \psi_k \end{cases}\] (91)

Case 3b \((\dot{z}_i > 0 \text{ and } p_{z_i}\sigma + q_{z_i} > 0)\):
Let
\[
\beta = \sin^{-1}\left(\frac{p_z \sigma + q_z}{\sqrt{z_z^2 + (p_z \sigma + q_z)^2}}\right),
\]  
(92)

\[
\alpha_{z_l}(\sigma) = \begin{cases} 
\frac{\pi}{2} & \text{if } z_l - \psi_k + p_z \sigma + q_z \geq \sqrt{z_z^2 + (p_z \sigma + q_z)^2} \\
\sin^{-1}\left(\frac{z_l - \psi_k + p_z \sigma + q_z}{\sqrt{z_z^2 + (p_z \sigma + q_z)^2}}\right) - \beta & \text{if } z_l - \psi_k + p_z \sigma + q_z < \sqrt{z_z^2 + (p_z \sigma + q_z)^2}
\end{cases}
\]  
(93)

Case 4b \((\dot{z}_i > 0 \text{ and } p_{z_i} \sigma + q_{z_i} < 0)\):
Let
\[
\beta = \sin^{-1}\left(\frac{- (p_z \sigma + q_z)}{\sqrt{z_z^2 + (p_z \sigma + q_z)^2}}\right),
\]  
(94)

\[
\alpha_{z_l}(\sigma) = \begin{cases} 
\frac{\pi}{2} & \text{if } z_l - \psi_k + p_z \sigma + q_z \geq \sqrt{z_z^2 + (p_z \sigma + q_z)^2} \\
\sin^{-1}\left(\frac{z_l - \psi_k + p_z \sigma + q_z}{\sqrt{z_z^2 + (p_z \sigma + q_z)^2}}\right) + \beta & \text{if } z_l - \psi_k + p_z \sigma + q_z < \sqrt{z_z^2 + (p_z \sigma + q_z)^2}
\end{cases}
\]  
(95)

Case 5b \((\dot{z}_i < 0 \text{ and } p_{z_i} \sigma + q_{z_i} < 0)\):
Let
\[
\beta = \sin^{-1}\left(\frac{- (p_z \sigma + q_z)}{\sqrt{z_z^2 + (p_z \sigma + q_z)^2}}\right),
\]  
(96)

\[
\alpha_{z_l}(\sigma) = \begin{cases} 
\frac{\pi}{2} & \text{if } z_l - \psi_k + p_z \sigma + q_z \geq 0 \\
\pi - \sin^{-1}\left(\frac{z_l - \psi_k + p_z \sigma + q_z}{\sqrt{z_z^2 + (p_z \sigma + q_z)^2}}\right) - \beta & \text{if } z_l - \psi_k + p_z \sigma + q_z < 0
\end{cases}
\]  
(97)

Case 6b \((\dot{z}_i < 0 \text{ and } p_{z_i} \sigma + q_{z_i} > 0)\):
\[
\alpha_{z_l}(\sigma) = \frac{\pi}{2}
\]  
(98)

Case 7b \((\dot{z}_i = 0 \text{ and } p_{z_i} \sigma + q_{z_i} = 0)\):
\[
\alpha_{z_l}(\sigma) = \frac{\pi}{2}
\]  
(99)

Using this analytic formulas, our strategy to reduce the duality gap is to simultaneously select \(\alpha_k\) and \(\sigma_k\) by an iterative method similar to the idea of [32]. This is implemented as follows: in every iteration \(k\), given fixed \(\phi_k, \psi_k, \hat{s}, \hat{z}, p_\mathbf{s}, p_\mathbf{z}, q_\mathbf{s}\) and \(q_\mathbf{z}\), several different values of \(\sigma\) are tried to find the best \(\sigma_k\) for the maximum of \(\hat{\alpha}\). Therefore, we will find a \(\sigma_k\) which maximizes the step size \(\hat{\alpha}\), i.e.,
\[
\max_{\sigma \in [\sigma_{\min}, \sigma_{\max}]} \min_{i \in \{1, \ldots, n\}} \{\alpha_{s_i}(\sigma), \alpha_{z_i}(\sigma)\},
\]  
(100)
where \(0 \leq \sigma_{\min} < \sigma_{\max} \leq 1\), \(\alpha_{s_i}(\sigma)\) and \(\alpha_{z_i}(\sigma)\) are calculated using (77)-(99) for \(\sigma \in [\sigma_{\min}, \sigma_{\max}]\). Problem (100) has no regularity conditions involving derivatives. Golden section search for variable \(\sigma\) [8] seems to be an appropriate method for solving this problem. Noting the fact from (26) that \(\alpha_{s_i}(\sigma)\) is a monotonic increasing function
of $\sigma$ if $p_{s_i} > 0$ and $\alpha_{s_i}(\sigma)$ is a monotonic decreasing function of $\sigma$ if $p_{s_i} < 0$ (and similar properties hold for $\alpha_{z_i}(\sigma)$), we can use the condition

$$\min\{ \min_{i \in \{i \in p_{s_i} < 0\}} \alpha_{s_i}(\sigma), \min_{i \in \{i \in p_{s_i} > 0\}} \alpha_{z_i}(\sigma) \} > \min\{ \min_{i \in \{i \in p_{s_i} < 0\}} \alpha_{s_i}(\sigma), \min_{i \in \{i \in p_{s_i} > 0\}} \alpha_{z_i}(\sigma) \},$$  

(101)

and the following bisection search for variable $\sigma$ to solve (100).

Algorithm A.1 (bisection search devised for solving (100) )

Data: $(\dot{x}, \dot{s}), (p_x, p_s), (q_x, q_s), (x^k, s^k), \phi_k$, and $\psi_k$.

Parameter: $\epsilon \in (0, 1)$, $\sigma_{lb} = \sigma_{min}$, $\sigma_{ub} = \sigma_{max} \leq 1$.

for iteration $k = 0, 1, 2, \ldots$

Step 0: If $\sigma_{ub} - \sigma_{lb} \leq \epsilon$, set $\alpha = \min_{i \in \{1, \ldots, n\}} \{ \alpha_{x_i}(\sigma), \alpha_{s_i}(\sigma) \}$, stop.

Step 1: Set $\sigma = \sigma_{lb} + 0.5(\sigma_{ub} - \sigma_{lb})$.

Step 2: Calculate $\alpha_{x_i}(\sigma)$ and $\alpha_{s_i}(\sigma)$ using (77)-(99).

Step 3: If (101) holds, set $\sigma_{lb} = \sigma$, otherwise, set $\sigma_{ub} = \sigma$.

Step 4: Set $k + 1 \rightarrow k$. Go back to Step 1.

end (for)

This algorithm reduces interval length by 0.5 in every iteration while golden section method reduces interval length by 0.618. The bisection is more efficient.

In view of Proposition 3.3, if $(\dot{s}^T p_z + \dot{z}^T p_s) < 0$, to minimize $\mu_{k+1}$, we should select $\sigma_k = 0$. Therefore, Problem (100) is reduced to solve a much simpler problem

$$\tilde{\alpha} = \min_\alpha b_u(\alpha).$$  

(102)

This is a one-dimensional unconstrained optimization problem that can be solved by many existing methods, such as golden section method. Given $\tilde{\alpha}$, we still need to find the largest $\alpha_k \in (0, \tilde{\alpha}]$ such that Condition (C4) holds. We summarize the algorithm described above as follows:

Algorithm A.2 (bisection search devised for Step 5 of Algorithm 3.1) )

Data: $(\dot{x}, \dot{s}), (p_x, p_s), (q_x, q_s), (x^k, s^k), \phi_k$, and $\psi_k$.

Parameter: $\epsilon \in (0, 1)$.

Step 1: If $(\dot{s}^T p_z + \dot{z}^T p_s) < 0$, set $\sigma_k = 0$, solve (102) to get $\tilde{\alpha}$.

Step 2: Otherwise, call Algorithm A.1 to get $\tilde{\alpha}$.

Step 3: Find the largest $\alpha_k \in (0, \tilde{\alpha}]$ such that Condition (C4) holds.