SUB-CRITICAL AND CRITICAL STOCHASTIC QUASI-GEOSTROPHIC EQUATIONS WITH INFINITE DELAY

TONGTONG LIANG AND YEJUAN WANG *

School of Mathematics and Statistics
Gansu Key Laboratory of Applied Mathematics and Complex Systems
Lanzhou University
Lanzhou 730000, China

(Communicated by Tomasz Dlotko)

Abstract. In this paper, we investigate a stochastic fractionally dissipative quasi-geostrophic equation driven by a multiplicative white noise, whose external forces contain hereditary characteristics. The existence and uniqueness of both local martingale and local pathwise solutions are established in $H^s$ with \( s \geq 2 - 2\alpha \), where \( \alpha \in \left( \frac{1}{2}, 1 \right) \). For the critical case \( \alpha = \frac{1}{2} \), we obtain the similar results in $H^s$ with \( s > 1 \).

1. Introduction. We consider the following 2D stochastic quasi-geostrophic equation with infinite delay in the periodic domain $\mathbb{T}^2 = \mathbb{R}^2/(2\pi \mathbb{Z})^2$:

\[
\begin{aligned}
    d\theta + (u \cdot \nabla \theta + \kappa (-\Delta)^{\alpha} \theta) dt &= f dt + g(\theta_t) dt + h(\theta_t) dW(t), \quad \text{in } (0, \infty) \times \mathbb{T}^2, \\
    \theta(t, x) &= \varphi(t, x), \quad t \in (-\infty, 0], \ x \in \mathbb{T}^2,
\end{aligned}
\]

(1.1)

where $\alpha \in \left[ \frac{1}{2}, 1 \right)$ (the cases $\alpha = \frac{1}{2}$ and $\alpha < \frac{1}{2}$ are called critical and sub-critical, respectively), $\kappa \geq 0$ is a diffusivity coefficient, $\theta$ represents the potential temperature, $f$ is a non-delayed external force independent of time, $g$ and $h$ are external forces containing some hereditary characteristics (memory, unbounded variable or distributed delay, etc), $W(t)$ is a Wiener process on a suitable probability space to be described below, and $u = (u_1, u_2)$ is the velocity field determined by $\theta$ via the following relations:

\[
    u = (u_1, u_2) = (-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1}) \quad \text{and} \quad (-\Delta)^{\frac{1}{2}} \psi = -\theta.
\]

(1.2)

The fractionally dissipative 2D quasi-geostrophic equation describes a kind of dynamics of large-scale phenomena in the atmosphere and ocean, see [23] for more details. In the deterministic case, 2D quasi-geostrophic equations have been investigated extensively, see, e.g. [7, 8, 12, 13, 14, 32] and the references therein, due to mathematical importance and potential applications in meteorology. This equation

2020 Mathematics Subject Classification. Primary: 35Q86, 60H15, 76B03.

Key words and phrases. Stochastic quasi-geostrophic equation, fractional Laplacian, infinite delay, martingale solution, pathwise solution.

This work was supported by the National Natural Science Foundation of China under grant 41875084.

* Corresponding author.
is an important model in geophysical fluid dynamics, in the case of $\alpha = \frac{1}{2}$, it exhibits strikingly similar features (singularities) as 3D Navier-Stokes equations. The existence of global strong solutions for 2D primitive equations of the ocean driven by multiplicative noise has been established in [10] for the case when $u$ and its vertical gradient are initially in $L^2$. The existence of global strong solutions for stochastic Navier-Stokes equations with multiplicative noise has been studied in $H^1$, see, e.g. [2, 20] for the 2D case and [11] for the 3D case.

In control problems, or when some memory is present in the real phenomenon, we need to consider some delay terms in the models, see, e.g. the functional Navier-Stokes equation [3, 4, 16, 18, 29] and the quasi-geostrophic equation with delay [19, 21, 31]. For the atmosphere, the main external forcing coming from the solar radiation is stochastic, and the external forcing of the ocean comes mainly from the atmosphere. Therefore, it is very interesting to study the stochastic quasi-geostrophic equation with delay. The existence and uniqueness of martingale solutions and strong solutions have been proved in [26, 27] for 2D stochastic quasi-geostrophic equations in $L^2$. For the long time behavior of solutions to the 2D stochastic quasi-geostrophic equation, we refer the reader to [17, 33]. Very recently, the existence of martingale solutions to a stochastic fractionally quasi-geostrophic equation on $\mathbb{R}^d$ has been proved in [1]. It is worth mentioning that, all of these problems were considered in $L^2$. As far as we know, there has been little literature on quasi-geostrophic equations with delay even in the deterministic case. For stochastic Navier-Stokes equations with delay, the literature is also scarce even in the bounded delay case.

The purpose of this paper is to investigate the existence and uniqueness of both local martingale and local pathwise solutions to problem (1.1) in $H^s$. The essential difficulty is the lack of cancellation property for the quadratic type of nonlinear term in $H^s$. On the other hand, thanks to weak dissipation, moment estimates become highly non-trivial, especially for critical case. In order to overcome these difficulties, we consider a modified version:

$$
\begin{cases}
\frac{d\theta}{dt} + (\chi(||\theta - \theta^\ast||_{H^s}))u \cdot \nabla \theta + \kappa(-\Delta)\alpha \theta = f dt + g(\theta_t) dt + h(\theta_t) dW(t), \\
\theta(t,x) = \varphi(t,x), \quad t \in (-\infty,0], \quad x \in \mathbb{T}^2,
\end{cases}
$$

(1.3)

where

$$
\begin{cases}
\frac{d}{dt}\theta^\ast + \kappa(-\Delta)\alpha \theta^\ast = 0, \\
\theta^\ast(t,x) = \varphi(t,x), \quad t \in (-\infty,0], \quad x \in \mathbb{T}^2,
\end{cases}
$$

(1.4)

and $\chi : \mathbb{R} \rightarrow [0,1]$ is a cutoff function given in Section 3.

We first investigate global martingale solutions of the modified system (1.3) by the classical Faedo-Galerkin approximation, the compactness method, the Skorohod Theorem and the Martingale Representation Theorem used in [5, Chapter 8]. To associate the modified system (1.3) with the original system (1.1), the continuity in time of martingale solutions is also established. The construction of the pathwise unique global solution is based on pathwise uniqueness of solutions and the Yamada-Watanabe Theorem. Then the existence and uniqueness of both local martingale and local pathwise solutions to the original system (1.1) follow immediately. In order to overcome the above mentioned difficulties, we use the decomposition technique proposed in [6], and give some new and complicated moment estimates on solutions by using appropriate stopping time and commutator estimates. For the infinite
delay term, in addition to the difficulty on estimates, we show a different convergence analysis with non-delay case, and present a regularity property in time of solutions.

The article is organized as follows. In Section 2, we introduce some notations, and briefly recall some useful estimates and relevant mathematical preliminaries from probability theory and functional analysis. The existence of local martingale solutions are considered in Section 3 for problem (1.1) in the sub-critical and critical cases. In Section 4, the existence and uniqueness of local pathwise solutions to problem (1.1) are established in the sub-critical and critical cases.

2. Preliminaries.

2.1. Abstract spaces. Throughout this manuscript, we consider mean-zero (zero average) solutions of (1.1), that is

$$\int_{\mathbb{T}^2} \theta(t,x)dx = 0, \text{ for any } t \in \mathbb{R}.$$  

We denote the square root of Laplacian $$(-\Delta)^{\frac{1}{2}}$$ by $$\Lambda$$. The fractional Laplacian $$\Lambda^s$$, with $$s \in \mathbb{R}$$ can be defined by

$$\hat{\Lambda^s f}(k) = |k|^s \hat{f}(k),$$

where $$\hat{f}$$ denotes the Fourier transform of $$f$$, i.e.

$$\hat{f}(k) = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} f(x)e^{-ikx}dx.$$  

For $$1 \leq p \leq \infty$$, $$L^p$$ denotes the Banach space of $$p$$th-power integrable functions if $$p < \infty$$ and the essentially bounded functions when $$p = \infty$$. The following standard notations are used:

$$\|f\|_{L^p} = \int_{\mathbb{T}^2} |f|^p dx, \quad \|f\|_{L^\infty} = \text{ess sup}_{x \in \mathbb{T}^2} |f(x)|.$$  

For any tempered distribution $$f$$ on $$\mathbb{T}^2$$ and $$s \in \mathbb{R}$$, we define

$$\|f\|_{H^s}^2 = \|\Lambda^s f\|_{L^2}^2 = \sum_{k \in \mathbb{Z}^2} |k|^{2s} |\hat{f}(k)|^2,$$

and $$H^s$$ denotes the Sobolev space of all $$f$$ for which $$\|f\|_{H^s}$$ is finite. For $$1 \leq p \leq \infty$$ and $$s \in \mathbb{R}$$, the space $$H^{s,p}$$ is a subspace of $$L^p$$, consisting of all $$f$$ which can be written in the form $$f = \Lambda^{-s} g$$, $$g \in L^p$$, and the $$H^{s,p}$$ norm of $$f$$ is defined by

$$\|f\|_{H^{s,p}} = \|\Lambda f\|_{L^p}.$$  

We denote by $$\langle \cdot, \cdot \rangle$$ the inner product of $$L^2$$. Given a Banach space $$X$$ and its dual $$X'$$, we also denote the dual pairing between $$X$$ and $$X'$$ by $$\langle \cdot, \cdot \rangle$$, unless noted otherwise.

The equality relating $$u$$ to $$\theta$$ in (1.2) can be rewritten in terms of periodic Riesz transforms as:

$$u = (\partial_{x_2} \Lambda^{-1} \theta, -\partial_{x_1} \Lambda^{-1} \theta) = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta) \in \mathcal{R}^\perp \theta,$$

where $$\mathcal{R}_j, j = 1, 2$$ denote the Riesz transforms defined by

$$\mathcal{R}_j \hat{f}(k) = -i \frac{k_j}{|k|} \hat{f}(k), \quad k \in \mathbb{Z}^2 \setminus \{0\}.$$
The following result can be obtained by the fact that the Riesz transforms commute with \((-\Delta)^l\) and the boundedness of the Riesz transforms in \(L^p\), see [28, Chapter III] for more details.

**Lemma 1.** Let \(1 < p < \infty\) and \(l \geq 0\). Then there exists a constant \(C(l,p)\) such that

\[
\|(-\Delta)^l u\|_{L^p} \leq C(l,p)\|(-\Delta)^l \theta\|_{L^p}. \tag{2.1}
\]

If \(p = 2\), the inequality (2.1) can be strengthened to

\[
\|(-\Delta)^l u\|_{L^2} = \|(-\Delta)^l \theta\|_{L^2}. \tag{2.2}
\]

We recall the following important commutator and product estimates:

**Lemma 2.** Suppose that \(s > 0\) and \(p \in (1, \infty)\). If \(f, g \in \mathcal{S}\), the Schwartz class, then

\[
\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \leq C_0(\|\nabla f\|_{L^{p_1}} \|g\|_{H^{-s} p_2} + \|f\|_{H^{-s} p_3} \|g\|_{L^p})
\]

and

\[
\|\Lambda^s(fg)\|_{L^p} \leq C_0(\|f\|_{L^{p_1}} \|g\|_{H^{-s} p_2} + \|f\|_{H^{-s} p_3} \|g\|_{L^p}),
\]

with \(p_2, p_3 \in (1, \infty)\) such that

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}.
\]

Throughout the paper, we denote by \(C\) a real positive constant which can vary from a line to another and even in the same line. If the constant \(C\) depends on some variable \(x\), we denote it by \(C_x\).

### 2.2. The stochastic framework and notations.

Given a filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P, \{W_k\}_{k \geq 1})\), where \(\{W_k\}_{k \geq 1}\) is a sequence of mutually independent standard one dimensional Brownian motions adapted to a complete and right continuous filtration \(\{\mathcal{F}_t\}_{t \geq 0}\). Let \(\mathcal{U}\) be a second separable Hilbert space with the inner product \(\langle \cdot, \cdot \rangle\), the norm \(\|\cdot\|\), and the associated orthonormal basis \(\{e_k\}_{k \geq 1}\). Set

\[
W(t) = \sum_{k=1}^{\infty} W_k e_k, \quad t \geq 0.
\]

The above \(\mathcal{U}\)-valued stochastic process \(W(t)\) is called a cylindrical Wiener process.

Let \(L_2(\mathcal{U}, X)\) be the collection of Hilbert-Schmidt operators from \(\mathcal{U}\) to \(X\), endowed with the norm

\[
\|\phi\|_{L_2(\mathcal{U}, X)}^2 = \sum_{k=1}^{\infty} \|\phi e_k\|_X^2.
\]

Given real numbers \(a < b\), we will denote by \(C([a, b]; X)\) the Banach space of all continuous functions from \([a, b]\) into \(X\) equipped with sup norm. In general, we will write \(L^2(\Omega; C([a, b]; X))\) instead of \(L^2(\Omega, \mathcal{F}, dP; C([a, b]; X))\). Let \(T > 0\) be a fixed real number. If a function \(x \in C((-\infty, T]; X)\), for each \(t \in [0, T]\), we denote by \(x_t \in C((-\infty, 0]; X)\) the function defined by \(x_t(r) = x(t + r), \forall r \in (-\infty, 0]\).

Consider the phase space \(\mathcal{E}(X)\) by

\[
\mathcal{E}(X) = \{\psi \in C((-\infty, 0]; X); \lim_{r \to -\infty} \psi(r) \text{ exists in } X\},
\]

which is a Banach space equipped with the norm

\[
\|\psi\|_{\mathcal{E}(X)} = \sup_{r \in (-\infty, 0]} \|\psi(r)\|_X.
\]
Given an $X$-valued progressively measurable process $G \in L^2(\Omega \times [0,T]; L_2(\mathcal{U}, X))$, we can define the Itô stochastic integral

$$M_t := \int_0^t G dW = \sum_k \int_0^t G_k dW_k, \quad \text{where } G_k = G\varepsilon, t \in [0,T].$$

Clearly $\{M_t\}_{t \geq 0}$ is an $X$-valued square integrable martingale (see [24, Section 2.2, 2.3]).

Let $X$ be a separable Hilbert space with its associated norm denoted by $\| \cdot \|_X$. For fixed $p > 1$ and $0 \leq \alpha < 1$, let $W^{\alpha,p}(0,T;X)$ be the Sobolev space of all $u \in L^p(0,T;X)$ such that

$$\int_0^T \int_0^T \frac{\|u(t_1) - u(t_2)\|_X^p}{|t_1 - t_2|^{1+\alpha p}} dt_1 dt_2 < \infty,$$

endowed with the norm

$$\|u\|_{W^{\alpha,p}(0,T;X)}^p = \int_0^T \|u(r)\|_X^p dr + \int_0^T \int_0^T \frac{\|u(t_1) - u(t_2)\|_X^p}{|t_1 - t_2|^{1+\alpha p}} dt_1 dt_2.$$

For the case when $\alpha = 1$, we take

$$W^{1,p}(0,T;X) := \left\{ u \in L^p(0,T;X); \frac{du}{dt} \in L^p(0,T;X) \right\},$$

to be the classical Sobolev space with its usual norm

$$\|u\|_{W^{1,p}(0,T;X)}^p = \int_0^T \|u(r)\|_X^p + \left\| \frac{du}{dt}(r) \right\|_X^p dr.$$

Note that for $\alpha \in (0,1)$, $W^{1,p}(0,T;X) \subset W^{\alpha,p}(0,T;X)$.

Now we recall the compact embeddings needed below; see [9] for more details.

Lemma 3. (i) Suppose that $X_1 \subset X_0 \subset X_2$ are Banach spaces with $X_1$ and $X_2$ reflexive, and the embedding of $X_1$ into $X_0$ compact. Then for any $1 < p < \infty$ and $0 < \alpha < 1$, the embedding:

$$L^p(0,T;X_1) \cap W^{\alpha,p}(0,T;X_2) \subset \subset L^p(0,T;X_0)$$

is compact.

(ii) Suppose that $Y \subset Y_0$ are Banach spaces with $Y$ compactly embedded in $Y_0$. Let $\alpha \in (0,1]$ and $p \in (1,\infty)$ be such that $\alpha p > 1$ then

$$W^{\alpha,p}(0,T;Y) \subset \subset C([0,T];Y_0)$$

and the embedding is compact.

Most notably for the analysis here, the Burkholder-Davis-Gundy inequality holds which in the present context takes the form

$$\mathbb{E} \left( \sup_{t \in [0,T]} \left\| \int_0^t G dW \right\|_X^p \right) \leq C \mathbb{E} \left( \int_0^T \|G\|_{L_2(\mathcal{U},X)}^2 dt \right)^{\frac{p}{2}},$$

where $r \geq 1$ and $C$ is an absolute constant depending only on $r$.

We shall also need a variation of this inequality, established in [9] which applies to fractional derivatives of $M_t$. 


Lemma 4. Let $p \geq 2$ and $\alpha \in [0, \frac{1}{2})$ be given. Then for any progressively measurable process $G \in L^p(\Omega \times [0, T]; L_2(\mathcal{U}, X))$, there exists a constant $C_{p, \alpha} > 0$ independent of $G$ such that
\[
\mathbb{E}\left(\left\| \int_0^t G dW \right\|_{W^{p,\infty}(0,T;X)}^p \right) \leq C_{p,\alpha} \mathbb{E}\left( \int_0^T \| G \|_{L_2(\mathcal{U}, X)}^p dt \right).
\]

2.3. Definitions of the solution and assumptions. First we introduce the following notion of martingale solutions to Eq. (1.1).

**Definition 5 (Local and Global Martingale Solutions).** We say that there exists a martingale solution to Eq. (1.1) if given every $\varphi \in \mathcal{C}(H^s)$, there exist a stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P, W)$, a stopping time $\tau$ relative to $\mathcal{F}_t$, and a progressively measurable process $\theta(\cdot) = \theta(\cdot \wedge \tau) : \Omega \times [0, \infty) \to H^s$ such that

(i) $\theta(t) = \varphi(t), \ t \in (-\infty, 0]$;

(ii) for almost all $\omega \in \Omega$,

\[
\theta(\cdot \wedge \tau, \omega) \in C([0, \infty); H^s) \quad \text{and} \quad \theta(\cdot \wedge \tau, \omega) \in L^2_{\text{loc}}(0, \infty; H^{s+\alpha});
\]

(iii) for every $t \in [0, \infty)$,

\[
\theta(t \wedge \tau) = \theta(0) - \kappa \int_0^{t \wedge \tau} (-\Delta)^{\alpha} \theta(r) dr - \int_0^{t \wedge \tau} u(r) \cdot \nabla \theta(r) dr \\
+ \int_0^{t \wedge \tau} (f + g(\theta_r)) dr + \int_0^{t \wedge \tau} h(\theta_r) dW(r), \ \text{P-a.s.}
\]

We say that the martingale solution $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P, W, \theta, \tau)$ is global if $\tau = \infty$, P-a.s.

**Definition 6 (Pathwise Uniqueness).** Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P, W)$ be a fixed stochastic basis and suppose that $\varphi \in \mathcal{C}(H^s)$. We say that the pathwise uniqueness up to a stopping time $\tau > 0$ holds for Eq. (1.1) if whenever we are given two martingale solutions $\theta^1$ and $\theta^2$ of Eq. (1.1) defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P, W)$,

\[
P(\omega : \theta^1(t \wedge \tau, \omega) = \theta^2(t \wedge \tau, \omega); \forall t \geq 0) = 1.
\]

**Definition 7 (Local Pathwise Solutions).** Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P, W)$ be a fixed stochastic basis and suppose that $\varphi \in \mathcal{C}(H^s)$. We say that problem (1.1) has a local strong (pathwise) solution if there exist a strictly positive stopping time $\tau$ relative to $\mathcal{F}_t$, and a progressively measurable process $\theta(\cdot) = \theta(\cdot \wedge \tau) : \Omega \times [0, \infty) \to H^s$ such that $\theta(t) = \varphi(t), \ \forall t \in (-\infty, 0]$ and (2.4)-(2.5) hold.

To describe the conditions imposed on the delay terms $g$ and $h$, we first introduce some notations. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Banach spaces. We denote by $Bndu(\mathcal{X}, \mathcal{Y})$, the collection of all continuous mappings $G : \mathcal{X} \to \mathcal{Y}$ so that

\[
\| G(x) \|_{\mathcal{Y}} \leq L(1 + \| x \|_{\mathcal{X}}), \quad x \in \mathcal{X},
\]

where $L > 0$ is a constant. If, in addition,

\[
\| G(x) - G(y) \|_{\mathcal{Y}} \leq L_0 \| x - y \|_{\mathcal{X}}, \quad x, y \in \mathcal{X},
\]

we say that $G$ is in $Lipu(\mathcal{X}, \mathcal{Y})$.

For $s > 0$ and $g : \mathcal{C}(H^s) \to H^s$, we assume that $g \in Bndu(\mathcal{C}(H^s), H^s)$ for martingale solutions and $g \in Lipu(\mathcal{C}(H^s), H^s)$ for pathwise solutions. Similar conditions are also imposed on $h : \mathcal{C}(H^s) \to L_2(\mathcal{U}, H^s)$. For the case of martingale solutions, we assume that $h \in Bndu(\mathcal{C}(H^s), L_2(\mathcal{U}, H^s))$. For another case of pathwise solutions, we assume that $h \in Lipu(\mathcal{C}(H^s), L_2(\mathcal{U}, H^s))$. 
3. Local existence of martingale solutions. In this section, we establish the local existence of martingale solutions for (1.1) in $H^s$ with $s \geq 2-2\alpha$ and $\alpha \in \left(\frac{1}{2},1\right)$. We first show the global existence of a martingale solution of the modified system

$$
\begin{aligned}
\dot{\theta} + (\chi(\theta - \theta_s)_{H^s}) u \cdot \nabla \theta + \kappa (-\Delta)^\alpha \theta dt &= f dt + g(\theta_t)dt + h(\theta_t)dW(t), \\
\theta(t, x) &= \varphi(t, x), \quad t \in (-\infty, 0], \ x \in \mathbb{T}^2,
\end{aligned}
$$

(3.1)

where

$$
\begin{aligned}
\frac{d}{dt} \theta_s + \kappa (-\Delta)^\alpha \theta_s &= 0, \\
\theta_s(t, x) &= \varphi(t, x), \quad t \in (-\infty, 0], \ x \in \mathbb{T}^2,
\end{aligned}
$$

(3.2)

and $\chi : \mathbb{R} \rightarrow [0,1]$ is a cutoff function which is $C^\infty$ and such that

$$
\chi(x) = \begin{cases} 1, & \text{if } |x| < \varsigma, \\ 0, & \text{if } |x| > 2\varsigma. \end{cases}
$$

(3.3)

Here we choose $\varsigma$ to be any positive constant such that

$$
\varsigma \leq \frac{\kappa}{64C'_0C'_1 \max\{C(\frac{4}{2}, p_1), C(\frac{4}{2}, p_2)\}},
$$

(3.4)

where $C'_0$ and $\max\{C(\frac{4}{2}, p_1), C(\frac{4}{2}, p_2)\}$ are the constants appearing in Lemma 2 and Lemma 1 respectively, for other related constants, see (3.12).

3.1. The approximation scheme. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis of $L^2$ comprised of $L^2$-normalized eigenfunctions $e_i$ of $-\Delta$, i.e.

$$
-\Delta e_i = \lambda_i e_i, \quad \int_{\mathbb{T}^2} e_i^2 dx = 1,
$$

with $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \rightarrow \infty$. Denote by $P_n$ the projection in $L^2$ onto the linear span $L^2_n$ of eigenfunctions $\{e_1, \ldots, e_n\}$, i.e.

$$
P_n f = \sum_{i=1}^n f_ie_i \quad \text{for} \quad f = \sum_{i=1}^\infty f_ie_i.
$$

Note that $P_n$ commutes with $\Lambda^s$ on $H^s$ for any $s \geq 0$.

We take $\theta_{n}^{s}$ to be the unique solution of

$$
\begin{aligned}
\frac{d\theta_{n}^{s}}{dt} + \kappa (-\Delta)^\alpha \theta_{n}^{s} &= 0, \\
\theta_{n}^{s}(t, x) &= P_n\varphi(t, x), \quad t \in (-\infty, 0], \ x \in \mathbb{T}^2.
\end{aligned}
$$

(3.5)

Then we define the $n$-th Galerkin approximation of (3.1) as the following ODE system with infinite delay:

$$
\begin{aligned}
\dot{\theta}_{n}^{s} + (\chi(\theta_{n}^{s} - \theta_{n}^{s})_{H^s}) P_n(u^n \cdot \nabla \theta_{n}^{s}) + \kappa (-\Delta)^\alpha \theta_{n}^{s} dt &= f dt + P_n(g(\dot{\theta}_{n}^{s}))dt + h(\theta_{n}^{s})dW(t), \\
\theta_{n}^{s}(t, x) &= P_n\varphi(t, x), \quad t \in (-\infty, 0], \ x \in \mathbb{T}^2,
\end{aligned}
$$

(3.6)

with $u^n = \mathcal{R}^{\perp} \theta_{n}^{s}$ satisfying $\nabla \cdot u^n = 0$.

**Lemma 8.** Fix $\alpha \in \left(\frac{1}{2},1\right)$ and $s \geq 2 - 2\alpha$. For any $p \geq 2$, $T > 0$ and $n \in \mathbb{N}$,

$$
\sup_{0 \leq t \leq T} \|\theta_{n}^{s}(t)\|_{H^s}^{p} + \kappa p \int_0^T \|\theta_{n}^{s}(t)\|_{H^s}^{p-2} \|\theta_{n}^{s}(t)\|_{H^{s+\alpha}}^{2} dt \leq \|\theta_{n}^{s}(0)\|_{H^s}^{p}.
$$

(3.7)
Proof. Applying \( \Lambda^n \) to the system (3.5) and taking the inner product in \( L^2 \) with \( \Lambda^n \theta^n \), we obtain
\[
\int_{T^2} \frac{d\Lambda^n \theta^n}{dt} \Lambda^n \theta^n \, dx + \kappa \int_{T^2} \Lambda^{2\alpha+x} \theta^n \Lambda^n \theta^n \, dx = 0.
\]
Note that
\[
\frac{d}{dt} \left( \int_{T^2} |\Lambda^n \theta^n|^2 \, dx \right)^\frac{p}{2} = p \|\theta^n\|_{L^p}^{-2} \int_{T^2} \frac{d\Lambda^n \theta^n}{dt} \Lambda^n \theta^n \, dx.
\]
Therefore,
\[
\frac{d}{dt} \|\theta^n\|_{L^p}^p + \kappa p \|\theta^n\|_{L^p}^{-2} \|\theta^n\|_{H^{\alpha+\alpha}}^2 = 0.
\] (3.8)
Integrating the equality (3.8) on \([0, t]\) and taking the supremum over \([0, T]\), then the conclusion (3.7) follows immediately.

Lemma 9. Fix \( \alpha \in (\frac{1}{2}, 1) \) and \( s \geq 2 - 2\alpha \). Assume that \( f \in H^{s+\alpha} \), \( g \in B(\mathcal{E}(H^s), H^s) \), \( h \in B(\mathcal{E}(H^s), L_2(U, H^s)) \) and the constant \( c \) appearing in the cutoff function \( \chi \) satisfies (3.4). Let \( p \geq 2 \) and \( \varphi \in \mathcal{C}(H^s) \) be an initial value. Then for any \( T > 0 \), there exists a positive constant \( C_{p,T} \) such that for any \( n \in \mathbb{N} \),
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \|\theta^n(t)\|_{L^p} + \int_0^T \|\theta^n(t)\|_{L^p}^{-2} \|\theta^n(t)\|_{H^{\alpha+\alpha}}^2 \, dt \right) \\
\leq C_{p,T} \|f\|_{L^p}^p + C_{p,T} (1 + \|\varphi\|_{\mathcal{C}(H^s)})^{\max\{p,4\}} + C_{p,T} \|\varphi(0)\|_{H^{2p}}^2.
\]
Proof. We define \( \Theta^n = \theta^n - \theta_0^n \), then it follows from (3.5) and (3.6) that \( \Theta^n \) satisfies
\[
\begin{cases}
\begin{aligned}
d\Theta^n + (\chi(\|\Theta^n\|_{H^s})P_n((U^n + u^n_0) \cdot \nabla (\Theta^n + \theta^n_0)) + \kappa (-\Delta)^\alpha (\Theta^n) dt \\
= P_n f dt + P_n g((\Theta^n + \theta^n_0)_t) dt + P_n h((\Theta^n + \theta^n_0)_t) dW(t),
\end{aligned}
\end{cases}
\] (3.9)
\[
\Theta^n(t, x) = 0, \quad t \in (-\infty, 0], \quad x \in T^2,
\]
where \( U^n = \mathcal{R}^n \Theta^n \) and \( u^n_0 = \mathcal{R}^n \theta^n_0 \). Applying \( \Lambda^n \) to system (3.9) and using the Itô formula to the function \( \|\Lambda^n \Theta^n\|_{L^2}^2 \) with \( p \geq 2 \), we have that
\[
d\|\Theta^n\|_{L^p}^p + \kappa p \|\Theta^n\|_{L^p}^{-2} \|\Theta^n\|_{H^{\alpha+\alpha}}^2 dt \\
\leq -p \chi(\|\Theta^n\|_{H^s}) \|\Theta^n\|_{L^p}^{-2} (\Lambda^n P_n ((U^n + u^n_0) \cdot \nabla (\Theta^n + \theta^n_0))) \cdot \Lambda^n \Theta^n dt \\
+ p \|\Theta^n\|_{L^p}^{-2} (\Lambda^n P_n f, \Lambda^n \Theta^n) dt + p \|\Theta^n\|_{L^p}^{-2} (\Lambda^n P_n g((\Theta^n + \theta^n_0)_t), \Lambda^n \Theta^n) dt \\
+ \frac{p(p-1)}{2} \|\Theta^n\|_{L^p}^{-2} \|P_n h((\Theta^n + \theta^n_0)_t)\|_{L_2(U, H^s)} dt \\
+ p \|\Theta^n\|_{L^p}^{-2} (\Lambda^n P_n h((\Theta^n + \theta^n_0)_t), \Lambda^n \Theta^n) dW(t) \\
:=(J_1^p + J_2^p + J_3^p + J_4^p + J_5^p) dt + J_6^p dW(t).
\] (3.10)
Now we estimate the terms of the right-hand side of (3.10) in several steps.

Step 1. Note that \( P_n \) commutes with \( \Lambda^n \) on \( H^s \) and \( P_n \) is self-adjoint in \( L^2 \), in view of the bilinearity of the quadratic nonlinear term, we obtain
\[
\langle \Lambda^n P_n ((U^n + u^n_0) \cdot \nabla (\Theta^n + \theta^n_0)), \Lambda^n \Theta^n \rangle \\
= \langle P_n \Lambda^n ((U^n + u^n_0) \cdot \nabla (\Theta^n + \theta^n_0)), \Lambda^n \Theta^n \rangle \\
= \langle \Lambda^n (U^n \cdot \nabla \Theta^n), \Lambda^n \Theta^n \rangle + \langle \Lambda^n (u^n_0 \cdot \nabla \Theta^n), \Lambda^n \Theta^n \rangle \\
+ \langle \Lambda^n (U^n \cdot \nabla \theta^n_0), \Lambda^n \Theta^n \rangle + \langle \Lambda^n (u^n_0 \cdot \nabla \theta^n_0), \Lambda^n \Theta^n \rangle.
\]
Hence,

$$|J^p_1| \leq p\chi(|\Theta^n||H^s)|\Theta^n||p^{-2}\Lambda^s(U^n \cdot \nabla \Theta^n), \Lambda^s \Theta^n)|$$

$$+ p\chi(|\Theta^n||H^s)|\Theta^n||p^{-2}\Lambda^s(u^n \cdot \nabla \Theta^n), \Lambda^s \Theta^n)|$$

$$+ p\chi(|\Theta^n||H^s)|\Theta^n||p^{-2}\Lambda^s(U^n \cdot \nabla \theta^n), \Lambda^s \Theta^n)|$$

$$+ p\chi(|\Theta^n||H^s)|\Theta^n||p^{-2}\Lambda^s(u^n \cdot \nabla \theta^n), \Lambda^s \Theta^n)|$$

$$:= J^p_{1,1} + J^p_{1,2} + J^p_{1,3} + J^p_{1,4}. \quad (3.11)$$

We estimate each of these terms. Note that $\nabla \cdot U^n = 0$ ensures

$$\langle U^n \cdot \nabla (\Lambda^s \Theta^n), \Lambda^s \Theta^n \rangle = 0.$$ 

Since $\nabla$ and $\Lambda^s$ are commutative ([7, Remark 5.3] or [12]), using Lemmas 1 and 2, we deduce that

$$|\langle \Lambda^s(U^n \cdot \nabla \Theta^n), \Lambda^s \Theta^n \rangle|$$

$$= |\langle \Lambda^s(U^n \cdot \nabla \Theta^n) - U^n \cdot \nabla (\Lambda^s \Theta^n), \Lambda^s \Theta^n \rangle|$$

$$= |\Lambda^s(U^n \cdot \nabla \Theta^n) - U^n \cdot \nabla (\Lambda^s \Theta^n), \Lambda^s \Theta^n \rangle|$$

$$\leq \|\Lambda^s(U^n \cdot \nabla \Theta^n) - U^n \cdot \nabla (\nabla) \Theta^n)\|_{L^2} \|\Lambda^s \Theta^n\|_{L^2}$$

$$\leq C_0(\|\nabla U^n\|_{L^p} \|\nabla \Theta^n\|_{H^{s-1,p}} + \|U^n\|_{H^{s,p}} \|\nabla \Theta^n\|_{L^p}) \|\Theta^n\|_{H^s}$$

$$\leq C_0 C^p(\frac{1}{2}, p_1) \|\nabla \Theta^n\|_{L^p} \|\Theta^n\|_{H^{s,p}} \|\Theta^n\|_{H^s}$$

$$+ C_0 C^p(\frac{s}{2}, p_2) \|\Theta^n\|_{H^{s,p}} \|\nabla \Theta^n\|_{L^p} \|\Theta^n\|_{H^s}$$

$$\leq 2C_1 C_0^p \max(C(\frac{1}{2}, p_1), C(\frac{s}{2}, p_2)) \|\Theta^n\|_{H^{s,p}} \|\Theta^n\|_{H^s}, \quad (3.12)$$

where we have used the Sobolev embeddings

$$\|\nabla \Theta^n\|_{L^p} \leq \sqrt{C_1^p} \|\Theta^n\|_{H^{s,p}} \quad \text{and} \quad \|\Theta^n\|_{H^{s,p}} \leq \sqrt{C_1^p} \|\Theta^n\|_{H^{s,p}}$$

for $p_1 = \frac{2}{n}$ and $p_2 = \frac{2}{n-1}$. It follows from (3.12) that

$$J^p_{1,1} \leq 2pC_1 C_0^p \max(C(\frac{1}{2}, p_1), C(\frac{s}{2}, p_2)) \chi(|\Theta^n||H^s)|\Theta^n||p^{-1}\|\Theta^n||^2_{H^{s,p}}$$

$$\leq 4pC_1 C_0^p \max(C(\frac{1}{2}, p_1), C(\frac{s}{2}, p_2)) \|\Theta^n||p^{-2}\|\Theta^n||^2_{H^{s,p}}$$

$$\leq \frac{K p}{16} \|\Theta^n||p^{-2}\|\Theta^n||^2_{H^{s,p}}, \quad (3.13)$$

where we have used (3.4) in the last inequality.

For $J^p_{1,2}$, in a similar way as above, applying the Young inequality gives

$$J^p_{1,2} \leq CP\chi(|\Theta^n||H^s)|\Theta^n||p^{-1}\|\theta^n\|_{H^{s+p}} \|\Theta^n||_{H^{s+p}}$$

$$\leq \frac{K p}{16} \|\Theta^n||p^{-2}\|\Theta^n||^2_{H^{s+p}} + C_p\chi(|\Theta^n||H^s)|\Theta^n||p^{-2}\|\theta^n\|_{H^{s+p}}$$

$$\leq \frac{K p}{16} \|\Theta^n||p^{-2}\|\Theta^n||^2_{H^{s+p}} + C_p \|\theta^n||^2_{H^{s+p}}. \quad (3.14)$$

For $J^p_{1,3}$, similar to (3.12), we have

$$|\langle \Lambda^s(U^n \cdot \nabla \theta^n), \Lambda^s \Theta^n \rangle|$$

$$\leq |\langle \Lambda^s(U^n \cdot \nabla \theta^n) - U^n \cdot \nabla (\Lambda^s \theta^n), \Lambda^s \Theta^n \rangle| + |\langle U^n \cdot \nabla (\Lambda^s \theta^n), \Lambda^s \Theta^n \rangle|$$

$$\leq C \|\Theta^n||H^{s+p}|\|\theta^n||^2_{H^{s+p}} \|\Theta^n||H^s + |\langle U^n \cdot \nabla (\Lambda^s \theta^n), \Lambda^s \Theta^n \rangle|. \quad (3.15)$$
Using the fact that $\nabla \cdot U^n = 0$ and Lemmas 1 and 2, we find that
\[
\begin{align*}
|\langle U^n \cdot \nabla (\Lambda^s \theta^n), \Lambda^s \Theta^n \rangle | \\
&= |\langle \nabla \cdot (U^n \Lambda^s \theta^n), \Lambda^s \Theta^n \rangle | \\
&\leq |\langle \Lambda^{1-\alpha} (U^n \Lambda^s \theta^n), \Lambda^{s+\alpha} \Theta^n \rangle | \\
&\leq C(\|U^n\|_{L^{p_3}} \|\theta^n\|_{H^{s+1-\alpha,p_4}} + \|U^n\|_{H^{1-\alpha,p_1}} \|\theta^n\|_{H^{s,p_2}}) \|\Theta^n\|_{H^{s+\alpha}} \\
&\leq C\|\Theta^n\|_{H^s} \|\theta^n\|_{H^{s+\alpha}} \|\Theta^n\|_{H^{s+\alpha}},
\end{align*}
\] (3.16)
where $p_3 = \frac{2}{2-\alpha}$, $p_4 = \frac{1}{1-\alpha}$, $p_1$ and $p_2$ are given in (3.12), and we have used the Sobolev embeddings $H^s \subset L^{p_3}$, $H^{s+\alpha} \subset H^{s+1-\alpha,p_4}$, $H^s \subset H^{1-\alpha,p_1}$, $H^{s+\alpha} \subset H^{s,p_2}$ in the last inequality. Substituting (3.16) into (3.15) results in
\[
|\langle \Lambda^s (U^n \cdot \nabla \theta^n), \Lambda^s \Theta^n \rangle | \leq C\|\Theta^n\|_{H^s} \|\theta^n\|_{H^{s+\alpha}} \|\Theta^n\|_{H^{s+\alpha}}.
\] (3.17)

Then, arguing as in (3.14), the term $J_{1,3}^p$ can be estimated by
\[
J_{1,3}^p \leq C_p \chi(\|\Theta^n\|_{H^s}) \|\Theta^n\|_{H^s} \|\theta^n\|_{H^{s+\alpha}}^2 \|\Theta^n\|_{H^{s+\alpha}} \\
\leq \frac{K_p}{16} \|\Theta^n\|_{H^s} \|\Theta^n\|_{H^{s+\alpha}}^2 \|\theta^n\|_{H^{s+\alpha}}^2 + C_p \|\theta^n\|_{H^{s+\alpha}}^2 \|\Theta^n\|_{H^{s+\alpha}}.
\] (3.18)

In a similar way as in (3.15)-(3.18), we infer that
\[
J_{1,4}^p \leq C_p \chi(\|\Theta^n\|_{H^s}) \|\Theta^n\|_{H^s} \|\theta^n\|_{H^{s+\alpha}}^2 \|\Theta^n\|_{H^{s+\alpha}} + \|\theta^n\|_{H^s} \|\theta^n\|_{H^{s+\alpha}} \|\Theta^n\|_{H^{s+\alpha}} \\
\leq \frac{K_p}{16} \|\Theta^n\|_{H^s} \|\Theta^n\|_{H^{s+\alpha}}^2 \|\theta^n\|_{H^{s+\alpha}}^2 + C_p \|\theta^n\|_{H^{s+\alpha}}^2 \|\Theta^n\|_{H^{s+\alpha}}^2 + C_p \|\theta^n\|_{H^s} \|\theta^n\|_{H^{s+\alpha}}^2 \|\Theta^n\|_{H^{s+\alpha}}.
\] (3.19)

Substituting (3.13)-(3.14) and (3.18)-(3.19) into (3.11), we obtain
\[
|J_1^p| \leq \frac{K_p}{4} \|\Theta^n\|_{H^s} \|\Theta^n\|_{H^{s+\alpha}} \|\theta^n\|_{H^{s+\alpha}}^2 + C_p (\varsigma^p + \varsigma^{p-1}) \|\theta^n\|_{H^{s+\alpha}}^2 \\
+ C_p \|\theta^n\|_{H^s} \|\theta^n\|_{H^{s+\alpha}}^2.
\] (3.20)

Step 2. For $J_2^p$, since $P_n$ commutes with $\Lambda^{s-\alpha}$ on $H^{s-\alpha}$, in view of $f \in H^{s-\alpha}$ and the Young inequality, we find that
\[
|J_2^p| = p \|\Theta^n\|_{H^s} \|\theta^n\|_{H^{s+\alpha}} \|\Theta^n\|_{H^{s+\alpha}} \\
= p \|\Theta^n\|_{H^s} \|\theta^n\|_{H^{s+\alpha}} \|\Theta^n\|_{H^{s+\alpha}} \\
\leq p \|\Theta^n\|_{H^s} \|\theta^n\|_{H^{s+\alpha}} \|f\|_{H^{s-\alpha}} \\
\leq \frac{K_p}{4} \|\Theta^n\|_{H^s} \|\Theta^n\|_{H^{s+\alpha}} \|f\|_{H^{s-\alpha}} + C_p \|\Theta^n\|_{H^s} \|f\|_{H^{s-\alpha}}^2 \\
\leq \frac{K_p}{4} \|\Theta^n\|_{H^s} \|\Theta^n\|_{H^{s+\alpha}} + C_p \|\Theta^n\|_{H^s} \|f\|_{H^{s-\alpha}}^2.
\] (3.21)

Note that $g \in B_{p\omega}(C(H^s), H^s)$, arguing as in (3.21), the term $J_3^p$ is bounded by
\[
|J_3^p| = p \|\Theta^n\|_{H^s} \|\theta^n\|_{H^{s+\alpha}} \|\Theta^n\|_{H^{s+\alpha}} \\
\leq p \|\Theta^n\|_{H^s} \|\theta^n\|_{H^{s+\alpha}} \|\Theta^n\|_{H^{s+\alpha}} \\
\leq C_p \|\Theta^n\|_{H^s} + C_p \|\theta^n\|_{H^{s+\alpha}} \|\Theta^n\|_{H^{s+\alpha}} \\
\leq C_p \left(1 + \|\varphi\|_{C(H^s)} + \sup_{0 \leq \tau \leq t} \|\Theta^n(\tau)\|_{H^s} + \sup_{0 \leq \tau \leq t} \|\theta^n(\tau)\|_{H^s}\right).
\] (3.22)
For $J^p_1$, using $h \in Bu(P(H^s), L^2_2(U, H^s))$ and the Young inequality, we have

$$\begin{align*}
|J^p_1| \leq & \frac{p^2(p-1)}{2} \|\Theta^n\|_{L^2_2(U, H^s)}^2 \\
\leq & C_p \|\Theta^n\|_{H^s}^2 + C_p h((\Theta^n + \Theta^p_\sigma))_t^2 \\
\leq & C_p \left(1 + \|\varphi\|_{\mathcal{E}(H^s)} + \sup_{0 \leq \tau \leq t} \|\Theta^n(\tau)\|_{H^s}^2 + \sup_{0 \leq \tau \leq t} \|\Theta^p_\sigma(\tau)\|_{H^s}^2 \right). \quad (3.23)
\end{align*}$$

**Step 3.** Finally, we estimate the stochastic term $J^p_2$. Taking into account that $h \in Bu(P(H^s), L^2_2(U, H^s))$ and $P_n$ commutes with $\Lambda^s$ on $H^s$. Thanks to the Burkholder-Davis-Gundy inequality, we deduce that for any $T > 0$,

$$\begin{align*}
\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t J^p_2 dW(\sigma) \right| \\
\leq & C_p \mathbb{E} \left( \int_0^T \|\Theta^n(\sigma)\|_{H^s}^p \|H((\Theta^n + \Theta^p_\sigma))_t^2 \right) \\
\leq & C_p \mathbb{E} \left( \sup_{0 \leq \sigma \leq T} \|\Theta^n(\sigma)\|_{H^s}^p \int_0^T \|\Theta^n(\sigma)\|_{H^s}^p \|H((\Theta^n + \Theta^p_\sigma))_t^2 \right) \\
\leq & \frac{1}{2} \mathbb{E} \left( \sup_{0 \leq \sigma \leq T} \|\Theta^n(\sigma)\|_{H^s}^p \right) + C_p \mathbb{E} \int_0^T \|\Theta^n(\sigma)\|_{H^s}^p \left(1 + \|\varphi\|_{\mathcal{E}(H^s)} \right) \\
& + \sup_{0 \leq \tau \leq s} \|\Theta^n(\tau)\|_{H^s} + \sup_{0 \leq \tau \leq s} \|\Theta^p_\sigma(\tau)\|_{H^s} \\
\leq & \frac{1}{2} \mathbb{E} \left( \sup_{0 \leq \sigma \leq T} \|\Theta^n(\sigma)\|_{H^s}^p \right) + C_p \mathbb{E} \int_0^T \left(1 + \|\varphi\|_{\mathcal{E}(H^s)} \right) + \sup_{0 \leq \tau \leq s} \|\Theta^n(\tau)\|_{H^s} \, d\sigma,
\end{align*}$$

where we have used the Young inequality in the last two inequalities.

Combining the estimates (3.20)-(3.24) together, it follows from (3.10) that

$$\begin{align*}
\mathbb{E} \left( \sup_{0 \leq t \leq T} \|\Theta^n(t)\|_{H^s}^p + \int_0^T \|\Theta^n(t)\|_{H^s}^p \|\Theta^n(t)\|_{H^{s+\alpha}}^2 \right) \\
\leq & C_p \int_0^T \|\Theta^n(t)\|_{H^{s+\alpha}}^2 + \|\Theta^n(t)\|_{H^s}^2 \|\Theta^n(t)\|_{H^{s+\alpha}}^2 + \|f\|_{H^{s+\alpha}}^p \, dt \\
& + C_p \int_0^T \left(1 + \|\varphi\|_{\mathcal{E}(H^s)} \right) + \sup_{0 \leq \tau \leq t} \|\Theta^p_\sigma(\tau)\|_{H^s} \, d\sigma.
\end{align*}$$

Applying the Gronwall inequality, we conclude from Lemma 8 that

$$\begin{align*}
\mathbb{E} \left( \sup_{0 \leq t \leq T} \|\Theta^n(t)\|_{H^s}^p + \int_0^T \|\Theta^n(t)\|_{H^s}^p \|\Theta^n(t)\|_{H^{s+\alpha}}^2 \right) \\
\leq & C_{p,T} \int_0^T \|\Theta^n(t)\|_{H^{s+\alpha}}^2 + \|\Theta^n(t)\|_{H^s}^2 \|\Theta^n(t)\|_{H^s}^2 + \|f\|_{H^{s+\alpha}}^p \, dt \\
& + \int_0^T \left(1 + \|\varphi\|_{\mathcal{E}(H^s)} \right) + \sup_{0 \leq \tau \leq t} \|\Theta^p_\sigma(\tau)\|_{H^s} \, dt \\
\leq & C_{p,T} \|f\|_{H^{s+\alpha}}^p + C_{p,T} \left(1 + \|\varphi\|_{\mathcal{E}(H^s)}^\max\{0,4\} \right). \quad (3.25)
\end{align*}$$
In order to complete the proof of this lemma, recall that $\Theta^n = \theta^n - \theta^n_*$, using the Young inequality, we obtain that

$$
\mathbb{E} \left( \sup_{0 \leq t \leq T} \|\theta^n(t)\|_{H^{\alpha}}^2 + \int_0^T \|\theta^n(t)\|_{H^{\alpha+\alpha}}^{p-2} \|\theta^n(t)\|_{H^{\alpha+\alpha}}^2 dt \right) 
\leq C_p \mathbb{E} \left( \sup_{0 \leq t \leq T} \|\theta^n(t)\|_{H^{\alpha}}^2 + \left( \int_0^T \|\theta^n(t)\|_{H^{\alpha+\alpha}}^2 dt \right)^{\frac{p}{2}} \right) 
\leq C_p \mathbb{E} \left( \sup_{0 \leq t \leq T} \|\theta^n(t)\|_{H^{\alpha}}^2 + \left( \int_0^T \|\theta^n(t)\|_{H^{\alpha+\alpha}}^2 dt \right)^{\frac{p}{2}} \right) 
+ C_p \mathbb{E} \left( \sup_{0 \leq t \leq T} \|\Theta^n(t)\|_{H^{\alpha}}^2 + \left( \int_0^T \|\Theta^n(t)\|_{H^{\alpha+\alpha}}^2 dt \right)^{\frac{p}{2}} \right) 
\leq C_p T \|f\|_{H^{\alpha-\alpha}} + C_p T \left( 1 + \|\varphi\|_{C^1(H^{\alpha})} \right)^{\max\{p,4\}} + C_p \mathbb{E} \left( \int_0^T \|\Theta^n(t)\|_{H^{\alpha+\alpha}}^2 dt \right)^{\frac{p}{2}},
$$

(3.26)

where we have used (3.7) and (3.25) in the last inequality.

Now it only remains to estimate the last term in (3.26). Returning to (3.10) for the case $p = 2$, we have

$$
\mathbb{E} \left( \int_0^T \|\Theta^n(t)\|_{H^{\alpha+\alpha}}^2 dt \right)^{\frac{2}{p}} 
\leq (2\kappa)^{-\frac{p}{2}} \mathbb{E} \left( \int_0^T |J_1^n| + |J_2^n| + |J_3^n| + |J_4^n| dt + \sup_{0 \leq t \leq T} \left| \int_0^t J_5^n dW \right| \right)^{\frac{p}{2}}.
$$

(3.27)

Arguing as in (3.13)-(3.14) and (3.18)-(3.23), we infer that

$$
|J_1^n| + |J_2^n| + |J_3^n| + |J_4^n| 
\leq 8 \kappa C_1 C_0 \max\{C_{\frac{1}{2},1}, C_{\frac{1}{2},2}\} \|\Theta^n(t)\|_{H^{\alpha+\alpha}}^2 + 2^{-2(1+\frac{\alpha}{2})} \kappa \|\Theta^n(t)\|_{H^{\alpha+\alpha}}^2 
+ C \|\theta^n(t)\|_{H^{\alpha+\alpha}}^2 + C \|\theta^n(t)\|_{H^{\alpha+\alpha}}^2 + C \|\theta^n(t)\|_{H^{\alpha+\alpha}}^2 
+ C \left( 1 + \|\varphi\|_{C^1(H^{\alpha})}^2 \right) \sup_{0 \leq \tau \leq t} \|\theta^n(\tau)\|_{H^{\alpha+\alpha}}^2 + \sup_{0 \leq \tau \leq t} \|\theta^n(\tau)\|_{H^{\alpha+\alpha}}^2 
\leq 2^{-1+\frac{\alpha}{2}} \kappa \|\Theta^n(t)\|_{H^{\alpha+\alpha}}^2 + C \|\theta^n(t)\|_{H^{\alpha+\alpha}}^2 
+ C \|\theta^n(t)\|_{H^{\alpha+\alpha}}^2 + C \|f\|_{H^{\alpha+\alpha}}^2 
+ C \left( 1 + \|\varphi\|_{C^1(H^{\alpha})}^2 \right) \sup_{0 \leq \tau \leq t} \|\theta^n(\tau)\|_{H^{\alpha+\alpha}}^2 + \sup_{0 \leq \tau \leq t} \|\theta^n(\tau)\|_{H^{\alpha+\alpha}}^2.
$$

(3.28)

For the stochastic integral term in (3.27), we apply the Burkholder-Davis-Gundy inequality, in view of $h \in Bndu(C(H^\alpha), L_2(U,H^\alpha))$ and the Young inequality, we deduce that

$$
\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t J_5^n dW \right|^{\frac{2}{p}} 
\leq C \mathbb{E} \left( \int_0^T \|\Theta^n(t)\|_{H^{\alpha+\alpha}}^2 \|h(\Theta^n + \theta^n_*)\|_{L_2(U,H^\alpha)}^2 dt \right)^{\frac{p}{2}}.
$$
3.2. Existence of martingale solutions. Let \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P, W) \) be a fixed stochastic basis. Consider the sequence of Galerkin approximation \( \{\theta^n\}_{n \in \mathbb{N}} \) solving (3.6) relative to this stochastic basis, where \( \{\theta^n\}_{n \in \mathbb{N}} \) satisfy (3.5) with the initial value \( \varphi \in \mathcal{C}(H^s) \).

**Lemma 10.** Suppose that \( g \in \text{Bd} u(\mathcal{C}(H^s), H^s) \), \( h \in \text{Bd} u(\mathcal{C}(H^s), L_2(U, H^s)) \), \( f \in H^{s-\alpha} \) and the initial value \( \varphi \in \mathcal{C}(H^s) \), where \( \alpha \in (\frac{1}{2}, 1) \), \( s \geq 2 - 2\alpha \). Let \( \mu_n \) be the law of \( \theta^n \) on \( L^2(0, T; H^s) \cap C([0, T]; H^\beta) \) with \( \beta < s - \alpha \). Then the family of probability measure \( \{\mu_n\}_{n \in \mathbb{N}} \) is tight on \( L^2(0, T; H^s) \cap C([0, T]; H^\beta) \).

*Proof.* Integrating (3.6) on \((0, t)\), we find that

\[
\theta^n(t) = P_n\varphi(0) - \int_0^t \chi(||\theta^n(\sigma) - \theta^n(\sigma)||_{H^s}) P_n(u^n(\sigma) \cdot \nabla \theta^n(\sigma)) d\sigma
\]
\[- \kappa \int_0^t (\Delta) \eta^n (\sigma) d\sigma + \int_0^t P_n f d\sigma + \int_0^t P_n g(\eta^n) d\sigma + \int_0^t P_n h(\eta^n) dW(\sigma) := P_n \varphi(0) + I_1 + I_2 + I_3 + I_4 + I_5. \tag{3.32} \]

Note that \( \nabla \cdot u^n = 0 \). Thanks to Lemmas 1 and 2, we obtain that for \( v \in D(\Lambda^{\alpha-s}) \),
\[
|\langle u \cdot \nabla \theta, v \rangle| \leq \| \Lambda^{\alpha+1-\alpha} (u \theta) \|_{L^2} \| v \|_{H^{\alpha-s}}
\leq C (\| u \|_{L^p} \| \theta \|_{H^{\alpha+1-\alpha},p_4} + \| u \|_{H^{\alpha+1-\alpha},p_4} \| \theta \|_{L^p}) \| v \|_{H^{\alpha-s}}
\leq C \| \theta \|_{L^p} \| \theta \|_{H^{\alpha+1-\alpha},p_4} \| v \|_{H^{\alpha-s}}
\leq C \| \theta \|_{H^s} \| \theta \|_{H^{\alpha+1-\alpha}} \| v \|_{H^{\alpha-s}},
\]
where \( p_3 = \frac{2}{\alpha-1}, p_4 = \frac{1}{\alpha} \) and we have used the Sobolev embeddings \( H^s \subset L^{p_3} \) and \( H^{s+\alpha} \subset H^{s+1-\alpha,p_4} \) in the last inequality. Therefore,
\[
\| P_n (u^n \cdot \nabla \theta^n) \|^2_{L^2(0,T;H^{s-\alpha})} \leq C \int_0^T \| \theta^n(t) \|^2_{H^s} \| \theta^n(t) \|^2_{H^{s+\alpha}} dt,
\]
which implies that
\[
E \| I_1 \|^2_{W^{1,2}(0,T;H^{s-\alpha})} \leq C_T E \int_0^T \| \theta^n(t) \|^2_{H^s} \| \theta^n(t) \|^2_{H^{s+\alpha}} dt \leq C_T, \tag{3.33}
\]
where we have used (3.31) for the case \( p = 4 \) in the last inequality. Using (3.31) again for the case \( p = 2 \), we obtain that
\[
E \| I_2 \|^2_{W^{1,2}(0,T;H^{s-\alpha})} \leq C_T E \int_0^T \| \theta^n(t) \|^2_{H^{s+\alpha}} dt \leq C_T. \tag{3.34}
\]
For \( I_3 \), since \( f \in H^{s-\alpha} \), we have
\[
\| I_3 \|^2_{W^{1,2}(0,T;H^{s-\alpha})} \leq C_T \| f \|^2_{H^{s-\alpha}}. \tag{3.35}
\]
For the nonlinear term with delay, in view of \( g \in Bndu(C(H^s),H^s) \), it follows from (3.31) that
\[
E \| I_4 \|^2_{W^{1,2}(0,T;H^s)} \leq C_T E \int_0^T \| P_n g(\eta^n(t)) \|^2_{H^s} dt \leq C_T E \int_0^T \| \theta^n(t) \|^2_{H^s} dt \leq C_T. \tag{3.36}
\]
Thanks to Lemma 4, using \( h \in Bndu(C(H^s),L^2(\mathcal{U},H^s)) \) and (3.31), the last term can be estimated by
\[
E \| I_5 \|^2_{W^{1,2}(0,T;H^s)} \leq C_\gamma E \int_0^T \| P_n h(\eta^n(t)) \|^2_{L^2(\mathcal{U},H^s)} dt \leq C_\gamma E \int_0^T \| \theta^n(t) \|^2_{H^s} dt \leq C_\gamma T, \tag{3.37}
\]
where \( \gamma \in (0,\frac{1}{2}) \). Note that \( W^{1,2}(0,T;H^{s-\alpha}) \subset W^{\gamma,2}(0,T;H^{s-\alpha}) \), hence we conclude from (3.32)-(3.37) that
\[
E \| \theta^n \|^2_{W^{\gamma,2}(0,T;H^{s-\alpha})} \leq C_\gamma T.
\]
Recalling (3.31) for the case \( p = 2 \), we find that the laws \{\mu_n\}_{n \in \mathbb{N}} are bounded in probability in
\[ L^2(0,T;H^{s+\alpha}) \cap W^{\gamma,2}(0,T;H^{s-\alpha}), \]
and thus the family \{\mu_n\}_{n \in \mathbb{N}} is tight in \( L^2(0,T;H^s) \) due to Lemma 3.
Arguing as in (3.37), the term $I_5$ is also bounded by

$$
\mathbb{E}\|I_5\|_{W^{q,1}(0,T;H^r)}^q \leq C_{\gamma,q} \mathbb{E} \int_0^T \|P_n h(\theta^n_t)\|_{L_2(U,H^r)}^q dt
\leq C_{\gamma,q} \mathbb{E} \int_0^T 1 + \|\varphi\|_{C(H^r)}^q + \sup_{0 < \tau \leq t} \|\theta^n(\tau)\|_{H^r}^q dt \leq C_{\gamma,q,T},
$$

(3.38)

where $\gamma_1 \in (0, \frac{1}{2})$, $q \geq 2$, $\gamma q > 1$ and we have used (3.31) for the case $p = q$ in the last inequality. Combining (3.33)-(3.36) and (3.38), we deduce from Lemma 3 that the set $\gamma_1$ and by (3.31) we infer that for all $n \theta$ Note that $\tilde{\alpha}$ \theta, we obtain that there exist a probability space $(\Omega, \tilde{\mathcal{F}}, \tilde{P})$, and all $\theta \in \mathcal{C}([0,T];H^\beta)$ with $\beta < s - \alpha$. The proof of this lemma is completed.

We now prove the existence of a martingale solution to (1.1).

**Theorem 11.** Fix $\alpha \in (\frac{1}{2}, 1)$ and $s \geq 2 - 2\alpha$. Suppose that $g \in \text{Bndu}(\mathcal{C}(H^s), H^*) \cap \text{Bndu}(\mathcal{C}(H^\beta), H^*)$ and $h \in \text{Bndu}(\mathcal{C}(H^s), L_2(U, H^*)) \cap \text{Bndu}(\mathcal{C}(H^\beta), L_2(U, H^*))$, with $\beta < s - \alpha$. If $f \in H^{s-\alpha}$ and the initial value $\varphi \in \mathcal{C}(H^s)$, there exists a local martingale solution for (1.1) in the sense of Definition 5.

**Proof.** Let $\beta < s - \alpha$ be given, Lemma 10 ensures that the family $\{\mu_n\}_{n \in \mathbb{N}}$ is tight in $L^2(0,T;H^s) \cap \mathcal{C}([0,T];H^\beta)$. By Prokhorov’s theorem [5, Theorem 2.3], we can find a subsequence, still denoted later by $\{\mu_{n_k}\}_{n_k \in \mathbb{N}}$, such that $\mu_{n_k}$ converges weakly to $\mu$ in $L^2(0,T;H^s) \cap \mathcal{C}([0,T];H^\beta)$. Applying Skorohod’s theorem [5, Theorem 2.4], we obtain that there exist a probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$, and $\mathcal{L}^2(0,T;H^s) \cap \mathcal{C}([0,T];H^\beta)$-valued random variables $\{\bar{\theta}^n\}_{n \in \mathbb{N}}$ and $\bar{\theta}$ such that

(i) $\bar{\theta}^n$ has the same law as $\theta^n$ on $L^2(0,T;H^s) \cap \mathcal{C}([0,T];H^\beta)$ for each $n \in \mathbb{N}$;

(ii) $\bar{\theta}^n \rightarrow \bar{\theta}$ in $L^2(0,T;H^s) \cap \mathcal{C}([0,T];H^\beta)$, $\bar{P}$-a.s., and $\bar{\theta}$ has law $\mu$.

Note that $\theta^n \in \mathcal{C}([0,T];P_n H^s)$, $\bar{P}$-a.s. Since $\bar{\theta}^n$ and $\theta^n$ have the same laws, and the set $\mathcal{C}([0,T];P_n H^s)$ is a Borel subset of $L^2(0,T;H^s) \cap \mathcal{C}([0,T];H^\beta)$, then

$$
\bar{P}(\bar{\theta}^n \in \mathcal{C}([0,T];P_n H^s)) = 1, \quad n \in \mathbb{N},
$$

and by (3.31) we infer that for all $n \in \mathbb{N}$ and all $p \geq 2$,

$$
\mathbb{E} \left( \sup_{0 \leq t \leq T} \|\bar{\theta}^n(t)\|_{H^s}^p \right) \leq C_{p,T},
$$

(3.39)

$$
\mathbb{E} \int_0^T \|\bar{\theta}^n(t)\|_{H^s}^2 dt \leq C_T.
$$

(3.40)

We follow from (3.40) that there exists a subsequence (relabelled the same) $\{\bar{\theta}^n\}_{n \in \mathbb{N}}$ such that $\bar{\theta}^n$ converges weakly in $L^2(\Omega \times [0,T];H^{s+\alpha})$. On the other hand, recall that

$$
\bar{\theta}^n \rightarrow \bar{\theta} \text{ in } L^2(0,T;H^s) \cap \mathcal{C}([0,T];H^\beta), \quad \bar{P} \text{-a.s.},
$$

(3.41)

thus we have $\bar{\theta} \in L^2(\Omega \times [0,T];H^{s+\alpha})$, i.e.

$$
\mathbb{E} \int_0^T \|\bar{\theta}(t)\|_{H^{s+\alpha}}^2 dt < \infty.
$$

(3.42)
In a similar way, by (3.39) for the case $p = 2$, we can choose a subsequence $\{\tilde{\theta}^n\}_{n \in \mathbb{N}}$ (after relabelling) such that $\theta^n$ converges weakly star in $L^2(\Omega; L^\infty(0, T; H^s))$, and using (3.41), we obtain that
\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} \|\tilde{\theta}(t)\|_{H^s}^2 \right) < \infty. \tag{3.43}
\]
Consider a process $\tilde{M}_n$ with trajectories in $C([0, T]; H^s)$ defined by
\[
\tilde{M}_n(t) = \tilde{\theta}^n(t) - \tilde{\theta}^n(0) + \int_0^t \chi(||\tilde{\theta}^n(\sigma) - \theta^*_n(\sigma)||_{H^s}) P_n(\tilde{u}^n(\sigma) \cdot \nabla \tilde{\theta}^n(\sigma)) d\sigma
+ \int_0^t (-\Delta)^{\alpha/2} \tilde{\theta}^n(\sigma) d\sigma - \int_0^t P_n f d\sigma - \int_0^t P_n g(\tilde{\theta}^n_\sigma) d\sigma, \quad n \in \mathbb{N}, \quad t \in [0, T].
\]
It is a continuous, square integrable martingale with respect to the filtration $\mathcal{F}_{n,t} = \sigma(\tilde{\theta}^n(r), r \leq t)$, with quadratic variation
\[
\langle \langle \tilde{M}_n(t) \rangle \rangle_t = \int_0^t P_n h(\tilde{\theta}^n_\sigma) h(\tilde{\theta}^n_\sigma)^* P_n d\sigma, \quad t \in [0, T]. \tag{3.44}
\]
Since $\tilde{\theta}^n$ and $\theta^n$ have the same laws, arguing as in [5, Section 8.4], we obtain that for all $t_1, t_2 \in [0, T]$ with $t_1 \leq t_2$, all bounded continuous functions $\phi$ on $L^2(0, T; H^s)$ or $C([0, T]; H^s)$, and for all $v, z \in C^\infty(T^2)$,
\[
\mathbb{E}(\langle \langle \tilde{M}_n(t_2) - \tilde{M}_n(t_1), v \rangle \rangle_{[0, t_1]}) = 0, \tag{3.45}
\]
and
\[
\mathbb{E} \left( \langle \langle \tilde{M}_n(t_2), v \rangle \langle \tilde{M}_n(t_2), z \rangle - \langle \langle \tilde{M}_n(t_1), v \rangle \langle \tilde{M}_n(t_1), z \rangle \right.
- \int_{t_1}^{t_2} \langle h(\tilde{\theta}^n_\sigma)^* P_n v, h(\tilde{\theta}^n_\sigma)^* P_n z \rangle d\sigma \bigg) \phi(\tilde{\theta}^n_{[0, t_1]}) \bigg) = 0. \tag{3.46}
\]
Observe that $\theta^*_n$ satisfies Eq. (3.5), by (3.7) for the case $p = 2$ and the compact embedding $W^{1,2}(0, T; H^{s-\alpha}) \cap L^2(0, T; H^{s+\alpha}) \subset L^2(0, T; H^s)$ (see Lemma 3), we have
\[
\theta^*_n \rightarrow \theta_* \quad \text{in} \quad L^2(0, T; H^s). \tag{3.47}
\]
Let $\tilde{M}$ be a $H^\beta$-valued process defined by
\[
\tilde{M}(t) = \tilde{\theta}(t) - \tilde{\theta}(0) + \int_0^t \chi(||\tilde{\theta}(\sigma) - \theta_*(\sigma)||_{H^s}) (\tilde{u}(\sigma) \cdot \nabla \tilde{\theta}(\sigma)) d\sigma
+ \kappa \int_0^t (-\Delta)^{\alpha/2} \tilde{\theta}(\sigma) d\sigma - \int_0^t f d\sigma - \int_0^t g(\tilde{\theta}_\sigma) d\sigma, \quad t \in [0, T]. \tag{3.48}
\]
Now, we will take the limits in (3.45)-(3.46), and prove the existence of martingale solutions. We divide the proof into several steps.

**Step 1.** We shall prove that for all $t_1, t_2 \in [0, T]$ with $t_1 \leq t_2$ and all $v \in C^\infty(T^2)$,
\begin{align*}
(a) & \lim_{n \to \infty} \langle \tilde{\theta}^n(t_2), v \rangle = \langle \tilde{\theta}(t_2), v \rangle, \quad \tilde{P}\text{-a.s.,} \\
(b) & \lim_{n \to \infty} \int_{t_1}^{t_2} \chi(||\tilde{\theta}^n(\sigma) - \theta^*_n(\sigma)||_{H^s}) \langle P_n(\tilde{u}^n(\sigma) \cdot \nabla \tilde{\theta}^n(\sigma)), v \rangle d\sigma \\
& = \int_{t_1}^{t_2} \chi(||\tilde{\theta}(\sigma) - \theta_*(\sigma)||_{H^s}) \langle \tilde{u}(\sigma) \cdot \nabla \tilde{\theta}(\sigma), v \rangle d\sigma, \quad \tilde{P}\text{-a.s.,} \\
(c) & \lim_{n \to \infty} \int_{t_1}^{t_2} \langle (-\Delta)^{\alpha/2} \tilde{\theta}^n(\sigma), v \rangle d\sigma = \int_{t_1}^{t_2} \langle (-\Delta)^{\alpha/2} \tilde{\theta}(\sigma), v \rangle d\sigma, \quad \tilde{P}\text{-a.s.},
\end{align*}
\[(d) \lim_{n \to \infty} \int_{t_1}^{t_2} \langle P_n f, v \rangle d\sigma = \int_{t_1}^{t_2} \langle f, v \rangle d\sigma, \ P\text{-a.s.,}\]
\[(e) \lim_{n \to \infty} \int_{t_1}^{t_2} \langle P_n g(\tilde{\theta}_n^\ast), v \rangle d\sigma = \int_{t_1}^{t_2} \langle g(\tilde{\theta}), v \rangle d\sigma, \ P\text{-a.s.,}\]
where the sequence \(n\) can be thinned if it is necessary.

Let us fix \(t_1, t_2 \in [0, T]\), \(t_1 \leq t_2\) and \(v \in C_0^\infty(\mathbb{T}^2)\). By (3.41) we have \(\tilde{\theta}^n \to \tilde{\theta}\) in \(C([0, T]; H^3)\), \(P\)-a.s. This implies that (a) holds true.

For (b), we estimate
\[
\int_{t_1}^{t_2} \chi(\|\tilde{\theta}_n^\ast(\sigma) - \tilde{\theta}_n^\ast(\sigma)\|_{H^4}) P_n(\tilde{u}_n(\sigma) \cdot \nabla \tilde{\theta}(\sigma)) - \chi(\|\tilde{\theta}(\sigma) - \theta_\ast(\sigma)\|_{H^4}) \tilde{u}(\sigma) \cdot \nabla \tilde{\theta}(\sigma), v d\sigma
\]
\[
\leq \left| \int_{t_1}^{t_2} \chi(\|\tilde{\theta}_n(\sigma) - \theta_\ast(\sigma)\|_{H^4}) \langle \tilde{u}_n(\sigma) \cdot \nabla \tilde{\theta}(\sigma) - \tilde{u}_n(\sigma) \cdot \nabla \tilde{\theta}(\sigma), P_n(v) \rangle d\sigma \right|
\]
\[
+ \int_{t_1}^{t_2} \left( \chi(\|\tilde{\theta}_n(\sigma) - \theta_\ast(\sigma)\|_{H^4}) - \chi(\|\tilde{\theta}(\sigma) - \theta_\ast(\sigma)\|_{H^4}) \right) \langle \tilde{u}_n(\sigma) \cdot \nabla \tilde{\theta}(\sigma), P_n(v) \rangle d\sigma
\]
\[
+ \int_{t_1}^{t_2} \chi(\|\tilde{\theta}(\sigma) - \theta_\ast(\sigma)\|_{H^4}) \langle \tilde{u}(\sigma) \cdot \nabla \tilde{\theta}(\sigma), (I - P_n) v \rangle d\sigma
\]
\[
:= J_1^n + J_2^n + J_3^n, \tag{3.49}
\]
where we have used the self-adjoint property of \(P_n\) in \(L^2\). We address the first term on the right hand side of (3.49). By (3.40)-(3.42), in view of \(\nabla \cdot \tilde{u}_n = \nabla \cdot \tilde{u} = 0\), we obtain
\[
J_1^n \leq \left| \int_{t_1}^{t_2} \langle \tilde{u}_n(\sigma) - \tilde{u}(\sigma) \rangle \cdot \nabla \tilde{\theta}_n(\sigma), P_n(v) \rangle d\sigma \right|
\]
\[
+ \left| \int_{t_1}^{t_2} \langle \tilde{u}(\sigma) \cdot \nabla (\tilde{\theta}_n(\sigma) - \tilde{\theta}(\sigma)), P_n(v) \rangle d\sigma \right|
\]
\[
\leq \|\nabla v\|_{L^4} \|\tilde{u}_n - \tilde{u}\|_{L^2} \|\tilde{\theta}_n\|_{L^4} d\sigma
\]
\[
+ \|\nabla v\|_{L^4} \|\tilde{u}(\sigma)\|_{L^4} \|\tilde{\theta}_n(\sigma) - \tilde{\theta}(\sigma)\|_{L^2} d\sigma
\]
\[
\leq C\|v\|_{H^2} \|\tilde{u}_n\|_{L^2(0, T; H^{\ast + \alpha})} \|\tilde{\theta}_n - \tilde{\theta}\|_{L^2(0, T; H^\ast)}
\]
\[
+ C\|v\|_{H^2} \|\tilde{\theta}\|_{L^2(0, T; H^{\ast + \alpha})} \|\tilde{\theta}_n - \tilde{\theta}\|_{L^2(0, T; H^\ast)} \to 0, \ P\text{-a.s.}, \tag{3.50}
\]
where we have used Lemma 1 and the Sobolev embeddings \(H^\ast \subset L^2, H^{\ast + \alpha} \subset L^4\) and \(H^2 \subset H^{1,4}\) in the last inequality. For \(J_2^n\), thanks to (3.41) and (3.47), we deduce that there exist a subsequence \(\{\tilde{\theta}_n^\ast\}_{n \in \mathbb{N}}\) (after relabelling) and a corresponding subsequence \(\{\theta_\ast^\ast\}_{n \in \mathbb{N}}\) such that for a.e. \((t, \omega) \in [0, T] \times \Omega,\)
\[
\|\tilde{\theta}_n^\ast - \tilde{\theta}\|_{H^\ast} \to 0 \quad \text{and} \quad \|\theta_\ast^\ast - \theta_\ast\|_{H^\ast} \to 0,
\]
therefore we infer that
\[
\chi(\|\tilde{\theta}_n^\ast - \theta_\ast^\ast\|_{H^\ast}) - \chi(\|\tilde{\theta} - \theta_\ast\|_{H^\ast}) \to 0 \quad \text{almost everywhere on } [0, T] \times \Omega.
\]
Arguing as in (3.50), it follows from (3.42) that for almost every \(\omega \in \Omega,\)
\[
J_3^n \leq 2 \left| \int_{t_1}^{t_2} \langle \tilde{u}(\sigma) \cdot \nabla \tilde{\theta}(\sigma), P_n(v) \rangle d\sigma \right| \leq 2\|\nabla v\|_{L^2} \int_{t_1}^{t_2} \|\tilde{u}(\sigma)\|_{L^4} \|\tilde{\theta}(\sigma)\|_{L^4} d\sigma
\]
Then we can use the Lebesgue dominated convergence theorem to obtain
\[ J^n_2 \to 0, \quad \bar{P}\text{-a.s.} \tag{3.52} \]

For \( J^n_1 \), in a similar way as in (3.51), we find that
\[ J^n_1 \leq C \|v\|_{H^1} \int_{t_1}^{t_2} \|\bar{\theta}(\sigma)\|_{H^{s+\alpha}}^2 d\sigma < \infty. \tag{3.51} \]

Combining (3.49)-(3.50) and (3.52)-(3.53) we deduce that (b) holds.

For (c), due to (3.41) and the Sobolev embedding \( H^s \subset L^2 \), we have
\[
\left| \int_{t_1}^{t_2} \left\langle \nabla (\bar{\theta}^n(\sigma) - \bar{\theta}(\sigma)), v \right\rangle d\sigma \right| \\
\leq C \|v\|_{H^{2\alpha}} \int_{t_1}^{t_2} \|\bar{\theta}^n(\sigma) - \bar{\theta}(\sigma)\|_{H^\alpha}^2 d\sigma \\
\leq C_T \|v\|_{H^2} \|\bar{\theta}^n - \bar{\theta}\|_{L^2(0,T;H^s)} \to 0, \quad \bar{P}\text{-a.s.}
\]

Hence assertion (c) follows immediately.

For (d), since \( f \in H^{s-\alpha} \), we obtain
\[
\left| \int_{t_1}^{t_2} \left\langle P_n f - f, v \right\rangle d\sigma \right| \leq C_T \|(I - P_n)f\|_{H^{s-\alpha}} \cdot \|v\|_{H^{s-\alpha}} \to 0, \quad \bar{P}\text{-a.s.},
\]

which implies assertion (d) holds true.

For (e), we estimate
\[
\left| \int_{t_1}^{t_2} \left\langle P_n g(\bar{\theta}^n_\sigma) - g(\bar{\theta}_\sigma), v \right\rangle d\sigma \right| \\
\leq \left| \int_{t_1}^{t_2} \left\langle g(\bar{\theta}^n_\sigma) - g(\bar{\theta}_\sigma), P_n v \right\rangle d\sigma \right| + \left| \int_{t_1}^{t_2} \left\langle g(\bar{\theta}_\sigma), (I - P_n)v \right\rangle d\sigma \right| \\
:= J^n_4 + J^n_5, \tag{3.54}
\]

where we have used the self-adjoint property of \( P_n \) in \( L^2 \). Note that \( \varphi \in \mathcal{C}(H^s) \), by the definition of \( \mathcal{C}(H^s) \), we deduce that
\[
\|P_n \varphi - \varphi\|_{\mathcal{C}(H^s)} \to 0.
\]

Due to (3.41), in view of the Sobolev embedding \( H^s \subset H^\beta \) with \( \beta < s - \alpha \), we conclude that for every \( \sigma \in [t_1, t_2] \),
\[
\|\bar{\theta}^n_\sigma - \bar{\theta}_\sigma\|_{\mathcal{C}(H^s)} = \sup_{-\infty < \tau \leq 0} \|\bar{\theta}^n(\sigma + \tau) - \bar{\theta}(\sigma + \tau)\|_{H^s} \\
= \max \left\{ \sup_{-\infty < \tau \leq 0} \|P_n \varphi(\tau) - \varphi(\tau)\|_{H^s}, \sup_{0 \leq \tau \leq \sigma} \|\bar{\theta}^n(\tau) - \bar{\theta}(\tau)\|_{H^s} \right\} \\
\leq \max \left\{ \|P_n \varphi - \varphi\|_{\mathcal{C}(H^s)}, \sup_{0 \leq \tau \leq \sigma} \|\bar{\theta}^n(\tau) - \bar{\theta}(\tau)\|_{H^s} \right\} \to 0, \quad \bar{P}\text{-a.s.} \tag{3.55}
\]
This together with the continuity assumption for $g$ implies that for every $\sigma \in [t_1, t_2]$,
\[
\|g(\tilde{\theta}_n) - g(\tilde{\theta}_s)\|_{H^s} \to 0, \ P\text{-a.s.}
\]
On the other hand, taking into account $g \in Bndu(\mathcal{C}(H^\beta), H^\beta)$ and $\varphi \in \mathcal{C}(H^s)$, it follows from (3.39) and (3.43) that for almost every $\omega \in \Omega$,
\[
\left| \int_{t_1}^{t_2} (g(\tilde{\theta}_n) - g(\tilde{\theta}_s), P_n v) d\sigma \right|
\leq \int_{t_1}^{t_2} \|g(\tilde{\theta}_n) - g(\tilde{\theta}_s)\|_{H^s} \|v\|_{H^{-\beta}} d\sigma
\leq C\|v\|_{H^{-\beta}} \int_{t_1}^{t_2} 1 + \|\tilde{\theta}_n\|_{\mathcal{C}(H^\beta)} + \|\tilde{\theta}_s\|_{\mathcal{C}(H^\beta)} d\sigma
\leq C_T \|v\|_{H^{-\beta}} \left(1 + \|\varphi\|_{\mathcal{C}(H^s)} + \sup_{0 \leq \tau \leq T} \|\tilde{\theta}_n(\tau)\|_{H^s} + \sup_{0 \leq \tau \leq T} \|\tilde{\theta}(\tau)\|_{H^s} \right) < \infty,
\]
(3.56)
where we have used the Sobolev embedding $H^s \subset H^\beta$ in the last inequality. Then the Lebesgue dominated convergence theorem ensures that
\[
J^p_5 \to 0, \ \tilde{P}\text{-a.s.}
\]
(3.57)
For $J^p_5$, arguing as in (3.56), we obtain that
\[
J^p_5 \leq \int_{t_1}^{t_2} \|g(\tilde{\theta}_n)\|_{H^s} \|(I - P_n) v\|_{H^{-\beta}} d\sigma
\leq C\|(I - P_n) v\|_{H^{-\beta}} \int_{t_1}^{t_2} 1 + \|\tilde{\theta}_n\|_{\mathcal{C}(H^\beta)} d\sigma
\leq C_T \|(I - P_n) v\|_{H^{-\beta}} \left(1 + \|\varphi\|_{\mathcal{C}(H^s)} + \sup_{0 \leq \tau \leq T} \|\tilde{\theta}_n(\tau)\|_{H^s} \right) \to 0, \ \tilde{P}\text{-a.s.}
\]
(3.58)
Therefore assertion (e) follows from (3.54), (3.57) and (3.58).

**Step 2.** We will show that for all $t_1, t_2 \in [0, T]$ with $t_1 \leq t_2$, all bounded continuous functions $\phi$ on $L^2(0, T; H^s)$ or $C([0, T]; H^\beta)$, and all $v \in C^\infty(T^2)$,
\[
\lim_{n \to \infty} \mathbb{E} \left( (\tilde{M}_n(t_2) - \tilde{M}_n(t_1), v) \phi(\tilde{\theta}^n_{[0, t_1]}) \right) = \mathbb{E} \left( (\tilde{M}(t_2) - \tilde{M}(t_1), v) \phi(\tilde{\theta}_{[0, t_1]}) \right),
\]
(3.59)
where the sequence $n$ can be thinned if it is necessary.

By assertions (a)-(e) in Step 1, we have
\[
\lim_{n \to \infty} (\tilde{M}_n(t_2) - \tilde{M}_n(t_1), v) = (\tilde{M}(t_2) - \tilde{M}(t_1), v), \ \tilde{P}\text{-a.s.}
\]
(3.60)
Recall that $\phi$ is a bounded continuous function on $L^2(0, T; H^s)$ or $C([0, T]; H^\beta)$, in view of (3.41), we deduce that
\[
\lim_{n \to \infty} \phi(\tilde{\theta}^n_{[0, t_1]}) \to \phi(\tilde{\theta}_{[0, t_1]}) \quad \text{and} \quad \sup_{n \in \mathbb{N}} \|\phi(\tilde{\theta}^n_{[0, t_1]})\|_{L^\infty} < \infty, \ \tilde{P}\text{-a.e.}
\]
(3.61)
Then it follows from (3.60) and (3.61) that
\[
\lim_{n \to \infty} (\tilde{M}_n(t_2) - \tilde{M}_n(t_1), v) \phi(\tilde{\theta}^n_{[0, t_1]}) = (\tilde{M}(t_2) - \tilde{M}(t_1), v) \phi(\tilde{\theta}_{[0, t_1]}), \ \tilde{P}\text{-a.e.}
\]
(3.62)
Let us denote 
\[ f_n := \langle \tilde{M}_n(t_2) - \tilde{M}_n(t_1), v \rangle \phi(\tilde{\theta}^n|_{[0,t_1]}), \quad \omega \in \tilde{\Omega}. \]

In order to apply the Vitali convergence theorem, we claim that 
\[
\sup_{n \geq 1} \mathbb{E}(|f_n|^2) < \infty. \tag{3.63}
\]

Indeed, using the Sobolev embedding \( H^s \subset L^2 \) and the Schwarz inequality, we obtain that for each \( n \in \mathbb{N}, \)
\[
\mathbb{E}(|f_n|^2) \leq C\|\phi\|_{L^\infty}^2 \|v\|_{L^2}^2 \mathbb{E} \left( \|\tilde{M}_n(t_2)\|_{L^2}^2 + \|\tilde{M}_n(t_1)\|_{L^2}^2 \right) \\
\leq C\|\phi\|_{L^\infty}^2 \|v\|_{L^2}^2 \mathbb{E} \left( \|\tilde{M}_n(t_2)\|_{H^s}^2 + \|\tilde{M}_n(t_1)\|_{H^s}^2 \right). \tag{3.64}
\]
Since \( \tilde{M}_n \) is a continuous martingale with quadratic variation defined in (3.44), by the Burkholder-Davis-Gundy inequality, in view of \( h \in \mathcal{B}u(\mathcal{C}(H^s), L_2(\mathcal{U}, H^s)), \)
\( \varphi \in \mathcal{C}(H^s) \) and (3.39) for the case \( p = 2, \) we have
\[
\mathbb{E} \left( \sup_{t \in [0,T]} \|\tilde{M}_n(t)\|_{H^s}^2 \right) \leq C\mathbb{E} \int_0^T \|P_n h(\tilde{\theta}^n)\|_{L_2(\mathcal{U}, H^s)}^2 d\sigma \\
\leq C\mathbb{E} \int_0^T 1 + \|P_n \mathcal{F}_{H^s}\|_{\mathcal{F}_{H^s}}^2 + \sup_{0 \leq \tau \leq \sigma} \|\tilde{\theta}^n(\tau)\|_{H^s}^2 d\sigma < \infty. \tag{3.65}
\]
This together with (3.64) implies that (3.63) holds true, and consequently \( \{f_n\}_{n \in \mathbb{N}} \)
are uniformly integrable. Then (3.59) follows from (3.62) and the Vitali convergence theorem.

**Step 3.** We will prove that for all \( t_1, t_2 \in [0,T] \) with \( t_1 \leq t_2, \) all bounded continuous functions \( \phi \) on \( L^2(0,T; H^s) \) or \( C([0,T]; H^\beta), \) and all \( v, z \in C^\infty(\mathbb{T}^2), \)
\[
\lim_{n \to \infty} \mathbb{E} \left( \left( \langle \tilde{M}_n(t_2), v \rangle \langle \tilde{M}_n(t_2), z \rangle - \langle \tilde{M}_n(t_1), v \rangle \langle \tilde{M}_n(t_1), z \rangle \right) \phi(\tilde{\theta}^n|_{[0,t_1]}) \right) \\
= \mathbb{E} \left( \left( \langle \tilde{M}(t_2), v \rangle \langle \tilde{M}(t_2), z \rangle - \langle \tilde{M}(t_1), v \rangle \langle \tilde{M}(t_1), z \rangle \right) \phi(\tilde{\theta}|_{[0,t_1]}) \right). \tag{3.66}
\]
where the sequence \( n \) can be thinned if it is necessary.
Let us denote
\[
f'_n(\omega) = \left( \langle \tilde{M}_n(t_2), v \rangle \langle \tilde{M}_n(t_2), z \rangle - \langle \tilde{M}_n(t_1), v \rangle \langle \tilde{M}_n(t_1), z \rangle \right) \phi(\tilde{\theta}^n|_{[0,t_1]}), \\
f'(\omega) = \left( \langle \tilde{M}(t_2), v \rangle \langle \tilde{M}(t_2), z \rangle - \langle \tilde{M}(t_1), v \rangle \langle \tilde{M}(t_1), z \rangle \right) \phi(\tilde{\theta}|_{[0,t_1]}), \quad \omega \in \tilde{\Omega}.
\]
By assertions (a)-(e) in Step 1 and (3.61), we infer that
\[
\lim_{n \to \infty} f'_n(\omega) = f'(\omega), \quad \bar{P}\text{-a.s.} \tag{3.67}
\]
Note that for any \( r > 1, \) similar to the arguments of (3.64), we find that for each \( n \in \mathbb{N}, \)
\[
\mathbb{E}(|f'_n|^r) \leq C\|\phi\|_{L^\infty}^r \|v\|_{L^r}^r \mathbb{E} \left( \|\tilde{M}_n(t_2)\|_{H^r}^r + \|\tilde{M}_n(t_1)\|_{H^r}^r \right). \tag{3.68}
\]
On the other hand, arguing as in (3.65), using (3.39) for the case \( p = 2r \) and the Burkholder-Davis-Gundy inequality, we have
\[
\mathbb{E} \left( \sup_{t \in [0,T]} \|\tilde{M}_n(t)\|_{H^r}^r \right)
\]
uniformly integrable and \( \lim \) in the last inequality. Hence, (3.70) follows from (3.67), and thanks to the Vitali convergence theorem, the conclusion (3.66) follows from (3.67) and the uniform integrability.

**Step 4.** We will show that for all \( t_1, t_2 \in [0, T] \) with \( t_1 \leq t_2 \), all bounded continuous functions \( \phi \) on \( L^2(0, T; H^s) \) or \( C([0, T]; H^s) \), and all \( v \in C^\infty(T^2) \),

\[
\lim_{n \to \infty} \mathbb{E}\left( \int_{t_1}^{t_2} \langle h(\tilde{\theta}_n^\sigma)^* P_n v, h(\tilde{\theta}_n^\sigma)^* P_n z \rangle_{\mathcal{U}} d\sigma \phi(\tilde{\theta}_n^\sigma|_{[0,t_1]}) \right) = \mathbb{E}\left( \int_{t_1}^{t_2} \langle h(\tilde{\theta}_\sigma)^* v, h(\tilde{\theta}_\sigma)^* z \rangle_{\mathcal{U}} d\sigma \phi(\tilde{\theta}|_{[0,t_1]}) \right),
\]

(3.69)

where the sequence \( n \) can be thinned if it is necessary.

Let us denote

\[
\tilde{f}_n(\omega) := \int_{t_1}^{t_2} \langle h(\tilde{\theta}_n^\sigma)^* P_n v, h(\tilde{\theta}_n^\sigma)^* P_n z \rangle_{\mathcal{U}} d\sigma \phi(\tilde{\theta}_n^\sigma|_{[0,t_1]})
\]

\[
f(\omega) := \int_{t_1}^{t_2} \langle h(\tilde{\theta}_\sigma)^* v, h(\tilde{\theta}_\sigma)^* z \rangle_{\mathcal{U}} d\sigma \phi(\tilde{\theta}|_{[0,t_1]}) \quad \omega \in \tilde{\Omega}.
\]

In order to show (3.69) holds true, we need to prove that the functions \( \{\tilde{f}_n\}_{n \in \mathbb{N}} \) are uniformly integrable and \( \lim_{n \to \infty} \tilde{f}_n(\omega) = f(\omega), \ P\text{-}a.s. \)

For the uniform integrability of \( \{\tilde{f}_n\}_{n \in \mathbb{N}} \), it suffices to prove that for some \( r > 1 \),

\[
\sup_{n \geq 1} \mathbb{E}(|\tilde{f}_n|^r) < \infty.
\]

(3.70)

Indeed, since \( h \in Bndu(\mathcal{E}(H^s), L_2(\mathcal{U}, H^s)) \), we obtain that for any \( r > 1 \),

\[
|\tilde{f}_n|^r = \left| \int_{t_1}^{t_2} \langle h(\tilde{\theta}_n^\sigma)^* P_n v, h(\tilde{\theta}_n^\sigma)^* P_n z \rangle_{\mathcal{U}} d\sigma \phi(\tilde{\theta}_n^\sigma|_{[0,t_1]}) \right|^r
\]

\[
\leq \|\phi\|_{L^\infty}^r \left( \int_{t_1}^{t_2} \|h(\tilde{\theta}_n^\sigma)^*\|^2_{L_2(\mathcal{U}, H^s)} \|P_n v\|_{H^s} \|P_n z\|_{H^s} d\sigma \right)^r
\]

\[
\leq C\|\phi\|_{L^\infty} \|v\|_{H^s} \|z\|_{H^s} \left( \int_{t_1}^{t_2} \left( 1 + \|\phi\|_{\mathcal{E}(H^s)} + \sup_{0 \leq \tau \leq \sigma} \|\tilde{\theta}_n^\sigma(\tau)\|_{H^s}^2 \right)^2 d\sigma \right)^r
\]

\[
\leq C_T \left( 1 + \|\phi\|_{\mathcal{E}(H^s)}^2 r + \sup_{0 \leq \tau \leq T} \|\tilde{\theta}_n^\sigma(\tau)\|_{H^s}^{2r} \right),
\]

(3.71)

where we have used the bounded property of \( \phi \) in the last inequality. Hence, (3.70) follows from (3.39) for the case \( p = 2r \).
Now we will prove that \( \lim_{n \to \infty} \tilde{f}_n(\omega) = \tilde{f}(\omega) \), \( \tilde{P} \)-a.s. In a similar way as in (3.71), we deduce that for almost every \( \omega \in \Omega \),
\[
\begin{align*}
\left| \int_{t_1}^{t_2} \langle h(\tilde{\sigma}_n)^* P_n v, h(\tilde{\sigma}_n)^* P_n z \rangle |\mathcal{U}| d\sigma - \int_{t_1}^{t_2} \langle h(\tilde{\sigma})^* v, h(\tilde{\sigma})^* z \rangle |\mathcal{U}| d\sigma \right| & \\
\leq \int_{t_1}^{t_2} \| h(\tilde{\sigma}_n)^* P_n v - h(\tilde{\sigma})^* v \|_{|\mathcal{U}|} \| h(\tilde{\sigma}_n)^* P_n z \|_{|\mathcal{U}|} d\sigma \\
& + \int_{t_1}^{t_2} \| h(\tilde{\sigma}_n)^* P_n z - h(\tilde{\sigma})^* z \|_{|\mathcal{U}|} \| h(\tilde{\sigma}_n)^* v \|_{|\mathcal{U}|} d\sigma \\
\leq C \| z \|_{H^r} \left( 1 + \| \varphi \|_{\mathcal{C}(H^r)} + \sup_{0 \leq \tau \leq T} \| \tilde{\sigma}_n(\tau) \|_{H^r} \right) \int_{t_1}^{t_2} \| h(\tilde{\sigma}_n)^* P_n v - h(\tilde{\sigma})^* v \|_{|\mathcal{U}|} d\sigma \\
& + C \| v \|_{H^r} \left( 1 + \| \varphi \|_{\mathcal{C}(H^r)} + \sup_{0 \leq \tau \leq T} \| \tilde{\sigma}(\tau) \|_{H^r} \right) \int_{t_1}^{t_2} \| h(\tilde{\sigma}_n)^* P_n z - h(\tilde{\sigma})^* z \|_{|\mathcal{U}|} d\sigma \\
\leq C_T \int_{t_1}^{t_2} \| h(\tilde{\sigma}_n)^* P_n v - h(\tilde{\sigma})^* v \|_{|\mathcal{U}|} d\sigma + C_T \int_{t_1}^{t_2} \| h(\tilde{\sigma}_n)^* P_n z - h(\tilde{\sigma})^* z \|_{|\mathcal{U}|} d\sigma,
\end{align*}
\]
(3.72)
where we have used (3.39) and (3.43) in the last inequality. Therefore if we show that both terms in the last line of (3.72) tend to zero, combining (3.61), we will obtain the pointwise convergence of \( \{ \tilde{f}_n \}_{n \in \mathbb{N}} \). We are ready to only prove that for all \( v \in C^\infty(\mathbb{T}^2) \),
\[
\int_{t_1}^{t_2} \| h(\tilde{\sigma}_n)^* P_n v - h(\tilde{\sigma})^* v \|_{|\mathcal{U}|} d\sigma \to 0, \quad \tilde{P} \text{-a.s.},
\]
(3.73)
since another term can be addressed analogously. Note that
\[
\int_{t_1}^{t_2} \| h(\tilde{\sigma}_n)^* P_n v - h(\tilde{\sigma})^* v \|_{|\mathcal{U}|} d\sigma \\
\leq \int_{t_1}^{t_2} \| h(\tilde{\sigma}_n)^* (P_n v - v) \|_{|\mathcal{U}|} d\sigma + \int_{t_1}^{t_2} \| h(\tilde{\sigma}_n)^* v - h(\tilde{\sigma})^* v \|_{|\mathcal{U}|} d\sigma \\
:= J_{0n}^v + J_{1n}^v.
\]
For \( J_{0n}^v \), recall that \( h \in Bndu(\mathcal{C}(H^r), L_2(\mathcal{U}, H^r)) \), we infer from (3.39) that
\[
J_{0n}^v \leq \| P_n v - v \|_{H^r} \int_{t_1}^{t_2} \| h(\tilde{\sigma}_n)^* \|_{L_2(\mathcal{U}, H^r)} d\sigma \\
\leq C \| P_n v - v \|_{H^r} \int_{t_1}^{t_2} 1 + \| P_n \varphi \|_{\mathcal{C}(H^r)} + \sup_{0 \leq \tau \leq T} \| \tilde{\sigma}_n(\tau) \|_{H^r} d\sigma \\
\leq C_T \left( 1 + \| \varphi \|_{\mathcal{C}(H^r)} + \sup_{0 \leq \tau \leq T} \| \tilde{\sigma}_n(\tau) \|_{H^r} \right) \| (I - P_n) v \|_{H^r} \to 0, \quad \tilde{P} \text{-a.s.}
\]
(3.74)
Let us move to the term \( J_{1n}^v \). By (3.55) and the continuity assumption for \( h \), we have that for every \( \sigma \in [t_1, t_2] \),
\[
\| h(\tilde{\sigma}_n)^* v - h(\tilde{\sigma})^* v \|_{|\mathcal{U}|} \to 0, \quad \tilde{P} \text{-a.s.}
\]
Modifying slightly the arguments in (3.71), we conclude from (3.39) and (3.43) that for almost every \( \omega \in \Omega \),
\[
\{ \| h(\tilde{\sigma}_n)^* v - h(\tilde{\sigma})^* v \|_{|\mathcal{U}|} \}_{n \in \mathbb{N}}
\]
are bounded. Then applying the Lebesgue dominated convergence theorem gives
\[ J^2_\tau \to 0, \quad \tilde{P}\text{-a.s.}, \]
which together with (3.74) implies that (3.73) holds true.

Thanks to the uniform integrability and the pointwise convergence of \( \{ \tilde{f}_n \}_{n \in \mathbb{N}} \), the Vitali convergence theorem allows us to obtain the conclusion (3.69).

**Step 5.** We show that there exists a martingale solution of problem (3.1).

By Steps 2-4, we can pass to the limits in (3.45) and (3.46), thus we obtain that for all \( t_1, t_2 \in [0, T] \) with \( t_1 \leq t_2 \), all bounded continuous functions \( \phi \) on \( L^2(0, T; H^s) \) or \( C([0, T]; H^\beta) \), and all \( v, z \in C^\infty(\mathbb{T}^2) \),
\[ \mathbb{E} \left( \langle \tilde{M}(t_2) - \tilde{M}(t_1), v \rangle \phi(\tilde{\theta}|[0,t_1]) \right) = 0, \tag{3.75} \]
and
\[ \mathbb{E} \left( \left( \langle \tilde{M}(t_2), v \rangle \langle \tilde{M}(t_2), z \rangle - \langle \tilde{M}(t_1), v \rangle \langle \tilde{M}(t_1), z \rangle \right) - \int_{t_1}^{t_2} \langle \tilde{h}(\tilde{\theta})^* v, \tilde{h}(\tilde{\theta})^* z \rangle \tilde{d} \sigma \right) \phi(\tilde{\theta}|[0,t_1]) \right) = 0, \tag{3.76} \]
where \( \tilde{M} \) is a \( H^\beta \)-valued process defined by (3.48). From (3.75) and (3.76), we see that \( \Lambda^{\beta-s} \tilde{M}(t), t \in [0, T] \) is a continuous square integrable martingale in \( H^s \) with respect to the filtration \( \tilde{\mathcal{F}}_t = \sigma(\tilde{\theta}(r), r \leq t) \), with quadratic variation
\[ \langle \Lambda^{\beta-s} \tilde{M} \rangle_t = \int_0^t [\Lambda^{\beta-s} \tilde{h}(\tilde{\theta}_\sigma)][\Lambda^{\beta-s} \tilde{h}(\tilde{\theta}_\sigma)]^* d\sigma. \]

By the Martingale Representation Theorem, see [5, Theorem 8.2], there exist a probability space \( (\Omega, \tilde{\mathcal{F}}, \{ \tilde{\mathcal{F}}_t \}_{t \geq 0}, \tilde{P}) \), a cylindrical Wiener process \( \tilde{W} \) defined on this space, and a progressively measurable process \( \tilde{\theta}(t) \) such that
\[
\Lambda^{\beta-s} \tilde{\theta}(t) - \Lambda^{\beta-s} \tilde{\theta}(0) + \Lambda^{\beta-s} \int_0^t \chi(\| \tilde{\theta}(\sigma) - \theta_\sigma \|_{H^s}) \tilde{u}(\sigma) \cdot \nabla \tilde{\theta}(\sigma) d\sigma \\
+ \kappa \Lambda^{\beta-s} \int_0^t (-\Delta)^{\alpha} \tilde{\theta}(\sigma) d\sigma - \Lambda^{\beta-s} \int_0^t f d\sigma - \Lambda^{\beta-s} \int_0^t g(\tilde{\theta}_\sigma) d\sigma \\
= \int_0^t \Lambda^{\beta-s} \tilde{h}(\tilde{\theta}_\sigma)d\tilde{W}(\sigma).
\]

Note that
\[ \int_0^t \Lambda^{\beta-s} \tilde{h}(\tilde{\theta}_\sigma)d\tilde{W}(\sigma) = \Lambda^{\beta-s} \int_0^t h(\tilde{\theta}_\sigma)d\tilde{W}(\sigma). \]

Hence we obtain that
\[ \tilde{\theta}(t) - \tilde{\theta}(0) + \int_0^t \chi(\| \tilde{\theta}(\sigma) - \theta_\sigma \|_{H^s}) \tilde{u}(\sigma) \cdot \nabla \tilde{\theta}(\sigma) d\sigma + \kappa \int_0^t (-\Delta)^{\alpha} \tilde{\theta}(\sigma) d\sigma \\
- \int_0^t f d\sigma - \int_0^t g(\tilde{\theta}_\sigma) d\sigma \\
= \int_0^t h(\tilde{\theta}_\sigma)d\tilde{W}(\sigma), \quad t \in [0, T], \quad \tilde{P}\text{-a.s.,} \tag{3.77} \]
where \( \tilde{u} = R^\perp \tilde{\theta} \).

**Step 6.** Finally, we will establish better regularity property for \( \tilde{\theta} \).
Consider
\[ dz + \kappa(-\Delta)^\alpha zdt = h(\hat{\theta}_t)d\hat{W}(t), \quad z(0) = 0. \tag{3.78} \]
It follows from (3.77) that $$\hat{\theta}$$ satisfies (3.1), arguing as in Lemma 9, we obtain that
\[ \hat{\theta} \in L^2(\hat{\Omega} \times [0, T]; H^{s+\alpha}) \cap L^2(\hat{\Omega}; L^\infty(0, T; H^s)). \tag{3.79} \]
Since $$\varphi \in \mathcal{C}(H^s)$$, using (3.79) and the assumption $$h \in Bndu(\mathcal{C}(H^s), L_2(\mathcal{U}, H^s))$$, we find that $$h(\hat{\theta}_t) \in L^2(\hat{\Omega} \times [0, T]; L_2(\mathcal{U}, H^s))$$. Then by Theorem 3.1 in [22], we infer that Eq. (3.78) has a unique progressively measurable solution $$z$$ such that $$\hat{P}$$-a.s. $$z \in C([0, T]; H^s)$$ and
\[ \mathbb{E} \left( \sup_{t \in [0, T]} \|z(t)\|^2_{H^s} + \int_0^T \|z(t)\|^2_{H^{s+\alpha}} dt \right) < \infty. \tag{3.80} \]
Let $$\bar{\theta} := \hat{\theta} - z$$. Subtracting (3.78) from (3.1), we have
\[ \begin{cases} \frac{d}{dt}\bar{\theta} + \kappa(-\Delta)^\alpha \bar{\theta} + \chi(||\bar{\theta} + z - \theta_*||_{H^s})(\mathcal{R}^\perp(\bar{\theta} + z) \cdot \nabla(\bar{\theta} + z)) = f + g((\bar{\theta} + z)_t), \\ \bar{\theta}(t, x) = \varphi(t, x), \quad t \in (-\infty, 0], \quad x \in \mathbb{T}^2. \end{cases} \tag{3.81} \]
Due to (3.79) and (3.80), we find that $$\bar{\theta} \in L^2(\hat{\Omega} \times [0, T]; H^{s+\alpha}) \cap L^2(\hat{\Omega}; L^\infty(0, T; H^s))$$, therefore, $$(-\Delta)^\alpha \bar{\theta} \in L^2(\hat{\Omega} \times [0, T]; H^{s-\alpha})$$, and by the assumption on $$g$$, i.e., $$g \in Bndu(\mathcal{C}(H^s), H^\alpha)$$, we have $$g((\bar{\theta} + z)_t) \in L^2(\hat{\Omega} \times [0, T]; H^\alpha)$$. On the other hand, using Lemmas 1 and 2, we find that
\[ \|\Lambda^{s-\alpha}(\mathcal{R}^\perp(\bar{\theta} + z) \cdot \nabla(\bar{\theta} + z))\|_{L^2} \leq \|\Lambda^{s+1-\alpha}(\mathcal{R}^\perp(\bar{\theta} + z) \cdot (\bar{\theta} + z))\|_{L^2} \]
\[ \leq C\|\bar{\theta} + z\|_{L^p} \|\bar{\theta} + z\|_{H^{s+1-\alpha}, p} \]
\[ \leq C\|\bar{\theta} + z\|^2_{H^{s+\alpha}}, \tag{3.82} \]
where $$p_3 = \frac{2}{2\alpha - 1}$$, $$p_4 = \frac{1}{1-\alpha}$$, and we have used the Sobolev embeddings $$H^{s+\alpha} \subset L^{p_3}$$ and $$H^{s-\alpha} \subset H^{s+1-\alpha, p_4}$$. Thus,
\[ \chi(||\bar{\theta} + z - \theta_*||_{H^s})(\mathcal{R}^\perp(\bar{\theta} + z) \cdot \nabla(\bar{\theta} + z)) \in L^2(\hat{\Omega} \times [0, T]; H^{s-\alpha}). \]
Then it follows from (3.81) that
\[ \frac{d}{dt}\Lambda^s\bar{\theta} \in L^2(\hat{\Omega} \times [0, T]; H^{-\alpha}), \quad \Lambda^s\bar{\theta} \in L^2(\hat{\Omega} \times [0, T]; H^\alpha). \]
Applying [30, Chapter 3, Lemma 1.2], we have $$\Lambda^s\bar{\theta} \in C([0, T]; H^s)$$, $$\hat{P}$$-a.s. Recall that $$z \in C([0, T]; H^s)$$, $$\hat{P}$$-a.s., then
\[ \bar{\theta} \in C([0, T]; H^s), \quad \hat{P}$$-a.s. \tag{3.83} \]
By (3.77), (3.79) and (3.83), we conclude that $$(\hat{\Omega}, \hat{\mathbb{F}}, \{\hat{\mathcal{F}}_t\}_{t \geq 0}, \hat{P}, \hat{\mathbb{W}}, \hat{\theta})$$ is a global martingale solution of problem (3.1). Furthermore, define the stopping time
\[ \tau := \inf_{t \geq 0} \{||\hat{\theta}(t) - \theta_*(t)||_{H^\ast} \geq \varsigma \}, \]
where $$\varsigma$$ is the constant appearing in the definition of $$\chi$$, see (3.3). We observe that
\[ \int_0^{t \land \tau} \chi(||\hat{\theta} - \theta_*||_{H^\ast})(\hat{\theta} \cdot \nabla \hat{\theta}) d\sigma = \int_0^{t \land \tau} \hat{u} \cdot \nabla \hat{u} d\sigma, \quad \text{for any } t \geq 0. \]
Therefore, $$(\hat{\Omega}, \hat{\mathbb{F}}, \{\hat{\mathcal{F}}_t\}_{t \geq 0}, \hat{P}, \hat{\mathbb{W}}, \hat{\theta}, \tau)$$ is a local martingale solution of problem (1.1), and consequently the proof of Theorem 11 is completed.
Remark 12. It follows from Theorem 11 that problem (1.1) has a local martingale solution in $H^s$ with $s \geq 2 - 2\alpha$ for the sub-critical case $\alpha \in (\frac{1}{2}, 1)$. In fact, for the critical case $\alpha = \frac{1}{2}$, the existence of a local martingale solution still holds in $H^s$ with $s > 1$ since the Sobolev embedding $H^s \subset L^\infty$ used in (3.16) and (3.82) holds for $s > 1$.

4. Local existence and uniqueness of pathwise solutions. In this section, we shall prove the local existence and uniqueness of pathwise solutions for (1.1) in $H^s$ with $s \geq 2 - 2\alpha$ and $\alpha \in (\frac{1}{2}, 1)$.

We first need a Gronwall lemma for stochastic processes. The reader is referred to [11, Lemma 5.3] for the similar result of stochastic processes.

**Lemma 13.** Fix $T > 0$. Assume that $X, Y, Z, R : [0, T] \times \Omega \to \mathbb{R}$ are real-valued non-negative stochastic processes. Let $\tau < T$ be a stopping time such that

$$\mathbb{E} \int_0^\tau (RX + Z)dr < \infty.$$  

Assume, moreover, that for some fixed constant $k$,

$$\int_0^\tau Rdr < k, \text{ P-a.s.}$$  

Suppose that for all stopping times $0 \leq \tau_a \leq \tau_b \leq \tau$,

$$\mathbb{E} \left( \sup_{t \in [\tau_a, \tau_b]} X + \int_{\tau_a}^{\tau_b} Ydr \right) \leq C_0 \mathbb{E} \left( \sup_{t \in [0, \tau_a]} X + \int_{\tau_a}^{\tau_b} R \sup_{t \in [\tau_a, \tau_b]} X + Zdr \right),$$

where $C_0$ is a constant independent of the choice of $\tau_a, \tau_b$. Then

$$\mathbb{E} \left( \sup_{t \in [0, \tau]} X + \int_0^\tau Ydr \right) \leq C \mathbb{E} \left( X(0) + \int_0^\tau Zdr \right),$$

where $C = C(C_0, T, k)$.

**Proof.** Choose a sequence of stopping times $0 = \tau_0 < \tau_1 < \ldots < \tau_n < \tau_{n+1} = \tau$ such that

$$\int_{\tau_{k-1}}^{\tau_k} Rdr < \frac{1}{2C_0}, \text{ P-a.s.}$$

Taking $\tau_a = \tau_{k-1}$ and $\tau_b = \tau_k$ in (4.1), in view of (4.2), we obtain that for each pair $\tau_{k-1}, \tau_k$,

$$\mathbb{E} \left( \sup_{t \in [\tau_{k-1}, \tau_k]} X + \int_{\tau_{k-1}}^{\tau_k} Ydr \right) \leq C \mathbb{E} \left( \sup_{t \in [0, \tau_{k-1}]} X + \int_{\tau_{k-1}}^{\tau_k} Zdr \right).$$

Assuming, by induction on $j$, that

$$\mathbb{E} \left( \sup_{t \in [0, \tau_j]} X + \int_0^{\tau_j} Ydr \right) \leq C \mathbb{E} \left( X(0) + \int_0^{\tau_j} Zdr \right),$$

then by (4.3) and (4.4),

$$\mathbb{E} \left( \sup_{t \in [0, \tau_{j+1}]} X + \int_0^{\tau_{j+1}} Ydr \right) \leq C \mathbb{E} \left( \sup_{t \in [0, \tau_j]} X + \int_0^{\tau_j} Ydr \right) + C \mathbb{E} \left( \sup_{t \in [\tau_j, \tau_{j+1}]} X + \int_{\tau_j}^{\tau_{j+1}} Ydr \right)$$

and we are done.
\[
\begin{align*}
&\leq C \mathbb{E} \left( X(0) + \int_0^{\tau_j} Z \, dr \right) + C \mathbb{E} \left( \sup_{t \in [0, \tau_j]} X + \int_{\tau_j}^{\tau_{j+1}} Z \, dr \right) \\
&\leq C \mathbb{E} \left( X(0) + \int_0^{\tau_{j+1}} Z \, dr \right).
\end{align*}
\]

This completes the proof. \(\square\)

We now show that the solutions of the modified system (3.1) are pathwise unique.

**Theorem 14.** Assume that, in addition to the conditions imposed in Theorem 11, \(g \in \text{Lip}(\mathcal{C}(H^s), H^s)\) and \(h \in \text{Lip}(\mathcal{C}(H^s), L_2(\mathcal{U}, H^s))\), where \(s \geq 2 - 2\alpha\) and \(\alpha \in (\frac{1}{2}, 1)\). Let \(\theta^{(1)}\) and \(\theta^{(2)}\) be global martingale solutions of (3.1) defined on the same stochastic basis \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P, W)\) and starting from the same initial value \(\varphi\). Then

\[
P \left( \omega : \theta^{(1)}(t, \omega) = \theta^{(2)}(t, \omega), \forall t \geq 0 \right) = 1.
\]

**Proof.** Let \(\varrho := \theta^{(1)} - \theta^{(2)}\). Then \(\varrho\) satisfies the following equation

\[
\begin{cases}
\frac{d\varrho}{dt} + \kappa(-\Delta)^{\alpha} \varrho dt + \chi(||\theta^{(1)} - \theta^{(2)}||_{H^s}) u^{(1)} \cdot \nabla \theta^{(1)} dt \\
- \chi(||\theta^{(2)} - \theta^{(1)}||_{H^s}) u^{(2)} \cdot \nabla \theta^{(2)} dt \\
= (g(\theta^{(1)}_t) - g(\theta^{(2)}_t)) dt + (h(\theta^{(1)}_t) - h(\theta^{(2)}_t)) dW(t),
\end{cases}
\]

where \(t \in (-\infty, 0]\), \(x \in \mathbb{T}^2\).

By the definition of martingale solutions, we have

\[
\varrho \in C([0, \infty); H^s) \cap L^2_{\text{loc}}(0, \infty; H^{s+\alpha}), \quad P\text{-a.s.}
\]

Let us define the stopping time

\[
\tau_N := \inf_{t \geq 0} \left\{ \int_0^t ||\theta^{(1)}||_{H^s}^2 + ||\theta^{(1)}||_{H^{s+\alpha}}^2 + ||\theta^{(2)}||_{H^{s+\alpha}}^2 + 1 d\sigma \geq N \right\},
\]

where \(N \in \mathbb{N}\). It is clear that \(\{\tau_N\}_{N \in \mathbb{N}}\) is an increasing sequence. Furthermore, since \(\theta^{(1)}\) and \(\theta^{(2)}\) are global martingale solutions, we deduce that \(\lim_{n \to \infty} \tau_N = \infty\), \(P\text{-a.s.}\). Hence, the desired result can be obtained if we prove that for any \(N, T > 0\),

\[
\mathbb{E} \left( \sup_{t \in [0, \tau_N \wedge T]} ||\varrho(t)||_{H^s}^2 \right) = 0. \quad (4.5)
\]

Applying the Itô formula to the function \(||\Lambda^s \varrho||_{L^2}^2\), we find that

\[
\begin{align*}
&d||\varrho||_{H^s}^2 + 2\kappa||\varrho||_{H^{s+\alpha}}^2 dt \\
&\leq -2(\Lambda^s(\chi(||\theta^{(1)} - \theta^{(2)}||_{H^s}) u^{(1)} \cdot \nabla \theta^{(1)} \\
- \chi(||\theta^{(2)} - \theta^{(1)}||_{H^s}) u^{(2)} \cdot \nabla \theta^{(2)})) \Lambda^s \varrho dt + 2(\Lambda^s(g(\theta^{(1)}_t) - g(\theta^{(2)}_t)) \Lambda^s \varrho dt \\
+ ||h(\theta^{(1)}_t) - h(\theta^{(2)}_t)||_{L^2(\mathcal{U}, H^s)}^2 dt + 2(\Lambda^s(h(\theta^{(1)}_t) - h(\theta^{(2)}_t)) dW(t), \Lambda^s \varrho dt.
\end{align*}
\]

Fix stopping times \(\tau_a, \tau_b\) such that \(0 \leq \tau_a \leq \tau_b \leq \tau_N \wedge T\). Let \(N\) and \(T\) be given arbitrarily. Then it follows from (4.6) that

\[
\begin{align*}
&\sup_{t \in [\tau_a, \tau_b]} ||\varrho(t)||_{H^s}^2 + 2\kappa \int_{\tau_a}^{\tau_b} ||\varrho(\sigma)||_{H^{s+\alpha}}^2 d\sigma \\
&\leq ||\varrho(\tau_a)||_{H^s}^2 - 2 \int_{\tau_a}^{\tau_b} (\Lambda^s(\chi(||\theta^{(1)}(\sigma) - \theta^{(2)}(\sigma)||_{H^s}) u^{(1)}(\sigma) \cdot \nabla \theta^{(1)}(\sigma)) \Lambda^s \varrho(\sigma) dt \\
- \chi(||\theta^{(2)}(\sigma) - \theta^{(1)}(\sigma)||_{H^s}) u^{(2)}(\sigma) \cdot \nabla \theta^{(2)}(\sigma)) \Lambda^s \varrho(\sigma) d\sigma.
\end{align*}
\]
Inserting (4.9)-(4.10) into (4.8) yields
\[
J_1 \leq C \int_{\tau_a}^{\tau_b} \|\theta^{(1)}(\sigma) - \theta^{(2)}(\sigma)\|_{H^s} d\sigma
\]
where we have used \(\theta^{(1)}(\sigma) - \theta^{(2)}(\sigma) = 0\) for every \(\sigma \geq 0\), since \(\theta^{(1)}_{\sigma}\) and \(\theta^{(2)}_{\sigma}\) satisfy the linear equation (3.2) with the same initial value \(\varphi\). Similar to (3.12) and (3.16), we estimate
\[
J_{1,1} \leq C \int_{\tau_a}^{\tau_b} \|\theta^{(1)}(\sigma)\|_{H^s}^2 \|\theta^{(1)}(\sigma)\|_{H^{s+\alpha}}^2 d\sigma \tag{4.9}
\]
and
\[
J_{1,2} \leq C \int_{\tau_a}^{\tau_b} \|\theta^{(1)}(\sigma)\|_{H^s}^2 \|\theta^{(1)}(\sigma)\|_{H^{s+\alpha}}^2 \|\theta^{(1)}(\sigma)\|_{H^{s+\alpha}} \|\varphi(\sigma)\|_{H^s}^2 d\sigma \tag{4.10}
\]
Inserting (4.9)-(4.10) into (4.8) yields
\[
J_1 \leq \frac{K}{2} \int_{\tau_a}^{\tau_b} \|\varphi(\sigma)\|_{H^{s+\alpha}}^2 d\sigma + C \int_{\tau_a}^{\tau_b} \|\theta^{(1)}(\sigma)\|_{H^s}^2 \|\theta^{(1)}(\sigma)\|_{H^{s+\alpha}}^2 \|\varphi(\sigma)\|_{H^s}^2 d\sigma \tag{4.11}
\]
For \(J_2\), arguing as in (3.12) and (3.15)-(3.17), by the bilinearity of \(u \cdot \nabla \theta\), we have
\[
J_2 \leq 2 \int_{\tau_a}^{\tau_b} \langle \Lambda^s(u^{(1)}(\sigma) \cdot \nabla \varphi(\sigma)), \Lambda^s \varphi(\sigma) \rangle d\sigma
\]
\[
+ 2 \int_{\tau_a}^{\tau_b} \langle \Lambda^s((u^{(1)}(\sigma) - u^{(2)}(\sigma)) \cdot \nabla \theta^{(2)}(\sigma)), \Lambda^s \varphi(\sigma) \rangle d\sigma
\]
\[
\leq C \int_{\tau_a}^{\tau_b} \|\theta^{(1)}(\sigma)\|_{H^{s+\alpha}} \|\varphi(\sigma)\|_{H^{s+\alpha}} \|\theta^{(2)}(\sigma)\|_{H^{s+\alpha}} \|\varphi(\sigma)\|_{H^s} d\sigma
\]
\[
+ C \int_{\tau_a}^{\tau_b} \|\varphi(\sigma)\|_{H^{s+\alpha}} \|\theta^{(2)}(\sigma)\|_{H^{s+\alpha}} \|\varphi(\sigma)\|_{H^s} d\sigma
\]
\[
\frac{\kappa}{2} \int_{\tau_n}^{\tau_B} \|\varrho(\sigma)\|_{H^{s+\alpha}}^2 \, d\sigma + C \int_{\tau_n}^{\tau_B} \left( \|\theta^{(1)}(\sigma)\|_{H^{s+\alpha}}^2 + \|\theta^{(2)}(\sigma)\|_{H^{s+\alpha}}^2 \right) \|\varrho(\sigma)\|_{H^s}^2 \, d\sigma.
\] (4.12)

Using \( g \in \text{Lipu}(\mathcal{C}(H^s), H^s) \), \( h \in \text{Lipu}(\mathcal{C}(H^s), L_2(\mathcal{U}, H^s)) \), and the Young inequality, the terms \( J_3 \) and \( J_4 \) are bounded by

\[
J_3 \leq 2 \int_{\tau_n}^{\tau_B} \|g(\theta^{(1)}_{\sigma}) - g(\theta^{(2)}_{\sigma})\|_{H^s} \|\varrho(\sigma)\|_{H^s} \, d\sigma
\leq C \int_{\tau_n}^{\tau_B} \|\theta^{(1)}_{\sigma} - \theta^{(2)}_{\sigma}\|_{\mathcal{C}(H^s)} \|\varrho(\sigma)\|_{H^s} \, d\sigma
\leq C \sup_{t \in [0, \tau_B]} \|\varrho(t)\|_{H^s} \int_{\tau_n}^{\tau_B} \|\varrho(\sigma)\|_{H^s} \, d\sigma
\leq \frac{1}{4} \left( \sup_{t \in [0, \tau_n]} \|\varrho(t)\|_{H^s}^2 + \sup_{t \in [\tau_n, \tau_B]} \|\varrho(t)\|_{H^s}^2 \right) + C \int_{\tau_n}^{\tau_B} \|\varrho(\sigma)\|_{H^s}^2 \, d\sigma, \tag{4.13}\]

and

\[
J_4 \leq C \int_{\tau_n}^{\tau_B} \|\theta^{(1)}_{\sigma} - \theta^{(2)}_{\sigma}\|_{\mathcal{C}(H^s)}^2 \, d\sigma \leq C \int_{\tau_n}^{\tau_B} \sup_{t \in [0, \sigma]} \|\varrho(t)\|_{H^s}^2 \, d\sigma
\leq C \sup_{t \in [0, \tau_n]} \|\varrho(t)\|_{H^s}^2 + C \int_{\tau_n}^{\tau_B} \sup_{t \in [\tau_n, \sigma]} \|\varrho(t)\|_{H^s}^2 \, d\sigma. \tag{4.14}\]

Finally, taking into account that \( h \in \text{Lipu}(\mathcal{C}(H^s), L_2(\mathcal{U}, H^s)) \), by the Burkholder-Davis-Gundy inequality, we deduce that

\[
2\mathbb{E} \sup_{t \in [\tau_n, \tau_B]} \left| \int_{\tau_n}^{t} \langle \Lambda^s(h(\theta^{(1)}_{\sigma}) - h(\theta^{(2)}_{\sigma})), dW(\sigma), \Lambda^s \varrho(\sigma) \rangle \right|
\leq \mathbb{E} \left( \int_{\tau_n}^{\tau_B} \|h(\theta^{(1)}_{\sigma}) - h(\theta^{(2)}_{\sigma})\|_{L_2(\mathcal{U}, H^s)}^2 \|\varrho(\sigma)\|_{H^s}^2 \, d\sigma \right)^{\frac{1}{2}}
\leq \mathbb{E} \left( \left( \sup_{t \in [0, \tau_n]} \|\varrho(t)\|_{H^s}^2 \right)^{\frac{1}{2}} \left( \int_{\tau_n}^{\tau_B} \|\varrho(\sigma)\|_{H^s}^2 \, d\sigma \right)^{\frac{1}{2}} \right)
\leq \frac{1}{4} \left( \sup_{t \in [0, \tau_n]} \|\varrho(t)\|_{H^s}^2 + \sup_{t \in [\tau_n, \tau_B]} \|\varrho(t)\|_{H^s}^2 \right) + C \int_{\tau_n}^{\tau_B} \|\varrho(\sigma)\|_{H^s}^2 \, d\sigma, \tag{4.15}\]

where we have used the Young inequality in the last inequality. Applying the estimates (4.11)-(4.15) to (4.7), we obtain that

\[
\mathbb{E} \sup_{t \in [\tau_n, \tau_B]} \|\varrho(t)\|_{H^s}^2 + 2\kappa \mathbb{E} \int_{\tau_n}^{\tau_B} \|\varrho(\sigma)\|_{H^{s+\alpha}}^2 \, d\sigma
\leq \mathbb{E} \sup_{t \in [0, \tau_n]} \|\varrho(t)\|_{H^s}^2 + C \mathbb{E} \int_{\tau_n}^{\tau_B} \left( \|\theta^{(1)}(\sigma)\|_{H^s}^2 + \|\theta^{(2)}(\sigma)\|_{H^s}^2 \right) \, d\sigma
\leq \frac{1}{4} \left( \sup_{t \in [0, \tau_n]} \|\varrho(t)\|_{H^s}^2 + \sup_{t \in [\tau_n, \sigma]} \|\varrho(t)\|_{H^s}^2 + 1 \right) \sup_{t \in [\tau_n, \sigma]} \|\varrho(t)\|_{H^s}^2 \, d\sigma. \tag{4.16}\]

Then (4.5) follows from (4.16) and Lemma 13, and thus the proof is complete.

The following theorem shows the global existence of pathwise solutions for the modified system (3.1).
Theorem 15. In addition to the hypotheses imposed in Theorem 11, suppose that $g \in \text{Lip}_{u}(C(H^s, H^s))$ and $h \in \text{Lip}_{u}(C(H^s, L_2(U, H^s)))$. Then there exists a path-wise unique global strong solution of problem (3.1).

Proof. By the proof of Step 6 in Theorem 11, we see that there exists a global martingale solution of problem (3.1). Furthermore, Theorem 14 ensures the path-wise uniqueness of martingale solutions. Hence the conclusion follows from the Yamada-Watanabe Theorem (see, [15, 25]).

As a direct result of Theorem 15, we have

Corollary 16. Under the assumptions of Theorem 15, there exists a pathwise unique local strong solution of problem (1.1).

Remark 17. For the critical case $\alpha = \frac{1}{2}$ of problem (1.1), note that the Sobolev embedding $H^s \subset L^\infty$ holds for $s > 1$, hence problem (1.1) has a pathwise unique local strong solution in $H^s$ with $s > 1$ when $\alpha = \frac{1}{2}$.

REFERENCES

[1] Z. Brzeźniak and E. Motyl, Fractionally dissipative stochastic quasi-geostrophic type equations on $\mathbb{R}^d$, SIAM J. Math. Anal., 51 (2019), 2306–2358.

[2] Z. Brzeźniak and S. Peszat, Strong local and global solutions for stochastic Navier-Stokes equations, infinite dimensional stochastic analysis, Verh. Afd. Natuurkd. 1. Reeks. K. Ned. Akad. Wet., 52 (2000), 85–98.

[3] T. Caraballo and J. Real, Attractors for 2D-Navier-Stokes models with delays, J. Differential Equations, 205 (2004), 271–297.

[4] T. Caraballo and X. Han, A survey on Navier-Stokes models with delays, Discrete Contin. Dyn. Syst. Ser. S, 8 (2015), 1079–1101.

[5] G. Da Prato and J. Zabczyk, Stochastic Equations in Infinite Dimensions, Encyclopedia Math. Appl., vol. 152, Cambridge University Press, 2014.

[6] A. Debussche, N. Glatt-Holtz and R. Temam, Local martingale and pathwise solutions for an abstract fluids model, Phys. D, 240 (2011), 1123–1144.

[7] T. Dlotko, T. Liang and Y. Wang, Critical and super-critical abstract parabolic equations, Discrete Contin. Dyn. Syst. Ser. B, 25 (2020), 1517–1541.

[8] R. Farwig and C. Qian, Asymptotic behavior for the quasi-geostrophic equations with fractional dissipation in $\mathbb{R}^2$, J. Differential Equations, 266 (2019), 6525–6579.

[9] F. Flandoli and D. Gatarek, Martingale and stationary solutions for stochastic Navier-Stokes equations, Probab. Theory Related Fields, 102 (1995), 367–391.

[10] N. Glatt-Holtz and M. Ziane, The stochastic primitive equations in two space dimensions with multiplicative noise, Discrete Contin. Dyn. Syst. Ser. B, 10 (2008), 801–822.

[11] N. Glatt-Holtz and M. Ziane, Strong pathwise solutions of the stochastic Navier-Stokes system, Adv. Differential Equations, 14 (2009), 567–600.

[12] N. Ju, Global solutions to the two dimensional quasi-geostrophic equation with critical or super-critical dissipation, Math. Ann., 334 (2006), 627–642.

[13] N. Ju, Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space, Comm. Math. Phys., 251 (2004), 365–376.

[14] N. Ju, The maximum principle and the global attractor for the dissipative 2D quasi-geostrophic equations, Comm. Math. Phys., 255 (2005), 161–181.

[15] T. G. Kurtz, The Yamada-Watanabe-Engelbert theorem for general stochastic equations and inequalities, Electr. J. Probab., 12 (2007), 951–965.

[16] L. Liu and T. Caraballo, Analysis of a stochastic 2D-Navier-Stokes model with infinite delay, J. Dynam. Differential Equations, 31 (2019), 2249–2274.

[17] H. Lu, S. Lü, J. Xin and D. Huang, A random attractor for the stochastic quasi-geostrophic dynamical system on unbounded domains, Nonlinear Anal., 90 (2013), 96–112.

[18] P. Marín-Rubio, A. M. Márquez-Durán and J. Real, Three dimensional system of globally modified Navier-Stokes equations with infinite delays, Discrete Contin. Dyn. Syst. Ser. B, 14 (2010), 655–673.
[19] T. T. Medjo, Attractors for the multilayer quasi-geostrophic equations of the ocean with delays, Appl. Anal., 87 (2008), 325–347.
[20] R. Mikulevicius and B. L. Rozovskii, Stochastic Navier-Stokes equations for turbulent flows, SIAM J. Math. Anal., 35 (2004), 1250–1310.
[21] C. J. Niche and G. Planas, Existence and decay of solutions to the dissipative quasi-geostrophic equation with delays, Nonlinear Anal., 75 (2012), 3936–3950.
[22] E. Pardoux, Stochastic partial differential equations and filtering of diffusion processes, Stochastics, 3 (1979), 127–167.
[23] J. Pedlosky, Geophysical Fluid Dynamics, Springer-Verlag, New York, 1987.
[24] C. Prévôt and M. Röckner, A Concise Course on Stochastic Partial Differential Equations, in: Lecture Notes in Mathematics, vol. 1905, Springer, Berlin, 2007.
[25] M. Röckner, B. Schmuland and X. Zhang, Yamada-Watanabe theorem for stochastic evolution equations in infinite dimensions, Condensed Matter Phys., 2 (2008), 247–259.
[26] M. Röckner, R. Zhu and X. Zhu, Sub and supercritical stochastic quasi-geostrophic equation, Ann. Probab., 43 (2015), 1202–1273.
[27] M. Röckner, R. Zhu and X. Zhu, Stochastic quasi-geostrophic equation, Infin. Dimens. Anal. Quantum Probab. Relat. Top., 15 (2012), 1–6.
[28] E. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1970.
[29] T. Taniguchi, The existence and asymptotic behaviour of energy solutions to stochastic 2D functional Navier-Stokes equations driven by Levy processes, J. Math. Anal. Appl., 385 (2012), 634–654.
[30] R. Temam, Navier-Stokes Equations. Theory and Numerical Analysis, North-Holland, Amsterdam-New York-Oxford, 1977.
[31] L. Wan and J. Duan, Exponential stability of the multi-layer quasi-geostrophic ocean model with delays, Nonlinear Anal., 71 (2009), 799–811.
[32] J. Wu, Dissipative quasi-geostrophic equations with $L^p$ data, Electron. J. Differential Equations, 56 (2001), 1–13.
[33] R. Zhu and X. Zhu, Random attractor associated with the quasi-geostrophic equation, J. Dynam. Differential Equations, 29 (2017), 289–322.

Received June 2020; revised September 2020.

E-mail address: liangtt18@lzu.edu.cn
E-mail address: wangyj@lzu.edu.cn