MÖBIUS DISJOINTNESS CONJECTURE ON THE PRODUCT OF A CIRCLE AND THE HEISENBERG NILMANIFOLD

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Abstract. Let $\mathbb{T}$ be the unit circle and $\Gamma \backslash G$ the 3-dimensional Heisenberg nilmanifold. We prove that the Möbius function is linearly disjoint from a class of distal skew products on $\mathbb{T} \times \Gamma \backslash G$. These results generalize a recent work of Huang-Liu-Wang.

1. Introduction

Let $(X, T)$ be a flow, where $X$ is a compact metric space and $T : X \to X$ a continuous map. Let $\mu(n)$ be the Möbius function, that is $\mu(n) = 0$ if $n$ is not square-free, and $\mu(n) = (-1)^q$ if $n$ is a product of $q$ distinct primes. $\mu$ is said to be linearly disjoint from $(X, T)$ if
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n \leq N} \mu(n) f(T^n x) = 0
\]
for any $f \in C(X)$ and any $x \in X$. The Möbius Disjointness Conjecture of Sarnak [16, 17] states that the function $\mu$ is linearly disjoint from every $(X, T)$ whose entropy is 0. This conjecture has been established for many cases, and we refer to the survey paper [4] for recent progresses. An incomplete list for works related to the present paper is: Bourgain [1], Bourgain-Sarnak-Ziegler [2], Green-Tao [7], Liu-Sarnak [12, 13], Wang [18], Peckner [15], Huang-Wang-Ye [10], Litman-Wang [11], and Huang-Liu-Wang [9].

Distal flows are typical examples of zero-entropy flows, see Parry [14]. A flow $(X, T)$ with a compatible metric $d$ is called distal if
\[
\inf_{n \geq 0} d(T^n x, T^n y) > 0
\]
whenever $x \neq y$. According to Furstenberg’s structure theorem of minimal distal flows [6], skew products are building blocks of distal flows.

An example of distal flow is the skew product $T$ on the 2-torus $\mathbb{T}^2 = (\mathbb{R}/\mathbb{Z})^2$ given by
\[
T : (x, y) \mapsto (x + \alpha, y + h(x)),
\]
where $\alpha \in [0, 1)$ and $h : \mathbb{T} \to \mathbb{R}$ is a continuous function. For dynamical properties of this skew product, see for example Furstenberg [5]. The Möbius disjointness for the skew product (1.1) was first studied by Liu and Sarnak in [12, 13]. A result in [12] states that, if $h$ is analytic with an additional assumption on its Fourier coefficients, then the Möbius Disjointness Conjecture is true for the skew product $(\mathbb{T}^2, T)$. This result holds for all $\alpha$, as is not common in the KAM theory. The aforementioned additional assumption was removed in Wang [18]. It has been further generalized by Huang-Wang-Ye [10] to the case that $h(x)$ is $C^\infty$-smooth.

Date: May 10, 2022.

Key words and phrases. Möbius Disjointness Conjecture, distal flow, skew product, Heisenberg nilmanifold, measure complexity.
Another example of distal flow is nilsystem. Let $G$ be a nilpotent Lie group with a discrete cocompact subgroup $\Gamma$. The group $G$ acts in a natural way on the homogeneous space $\Gamma \backslash G$. Fix $h \in G$. Then the transformation $T$ given by $T(\Gamma g) = \Gamma gh$ makes $(\Gamma \backslash G, T)$ a nilsystem. The Möbius Disjointness Conjecture for these nilsystems was proved by Green and Tao in [7].

Now let $G$ be the 3-dimensional Heisenberg group with the cocompact discrete subgroup $\Gamma$, namely

$$G = \begin{pmatrix} 1 & \mathbb{R} & \mathbb{R} \\ 0 & 1 & \mathbb{R} \\ 0 & 0 & 1 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & \mathbb{Z} & \mathbb{Z} \\ 0 & 1 & \mathbb{Z} \\ 0 & 0 & 1 \end{pmatrix}. $$

Then $\Gamma \backslash G$ is the 3-dimensional Heisenberg nilmanifold. Let $T$ be the unit circle. Huang-Liu-Wang [9] define a skew product $S$ on $T \times \Gamma \backslash G$

$$(1.2) \quad S : (t, \Gamma g) \mapsto \left( t + \alpha, \Gamma g \begin{pmatrix} 1 & \varphi_2(t) & \psi(t) \\ 0 & 1 & \varphi_1(t) \\ 0 & 0 & 1 \end{pmatrix} \right),$$

where $\varphi_1, \varphi_2$ and $\psi$ are $C^\infty$-smooth periodic functions with period 1. They proved that the flow $(T \times \Gamma \backslash G, S)$ is distal, and in the case that $\varphi_1 = \varphi_2 = \phi$, the skew product

$$S_0 : (t, \Gamma g) \mapsto \left( t + \alpha, \Gamma g \begin{pmatrix} 1 & \phi(t) & \psi(t) \\ 0 & 1 & \phi(t) \\ 0 & 0 & 1 \end{pmatrix} \right)$$

is linearly disjoint with the Möbius function $\mu$.

The entropy of $(T \times \Gamma \backslash G, S)$ is a zero, so the Möbius Disjointness Conjecture is expected to hold on $(T \times \Gamma \backslash G, S)$. In this manuscript, we generalize the result of Huang-Liu-Wang [9] by showing that the Möbius Disjointness Conjecture holds on $(T \times \Gamma \backslash G, S)$ if $\alpha$ is rational. Let the skew product $T$ on $T \times \Gamma \backslash G$ be given by

$$(1.3) \quad T : (t, \Gamma g) \mapsto \left( t + \alpha, \Gamma g \begin{pmatrix} 1 & k\phi(t) & \psi(t) \\ 0 & 1 & \phi(t) \\ 0 & 0 & 1 \end{pmatrix} \right),$$

where $\alpha \in [0, 1)$, $k \in \mathbb{R}$ and $\phi, \psi$ be $C^\infty$-smooth periodic functions from $\mathbb{R}$ to $\mathbb{R}$ with period 1. We also prove that the Möbius Disjointness Conjecture holds on $(T \times \Gamma \backslash G, T)$.

In particular, the main results of this manuscript are as follows.

**Theorem 1.1.** Let $T$ be the unit circle and $\Gamma \backslash G$ the 3-dimensional Heisenberg nilmanifold. Let $\alpha \in \mathbb{Q} \cap [0, 1)$, $\varphi_1, \varphi_2, \psi$ be $C^\infty$-smooth periodic functions from $\mathbb{R}$ to $\mathbb{R}$ with period 1. Let $S$ be the skew product on $T \times \Gamma \backslash G$ defined in (1.2). Then the Möbius function $\mu$ is linearly disjoint from $(T \times \Gamma \backslash G, S)$.

**Theorem 1.2.** Let $T$ be the unit circle and $\Gamma \backslash G$ the 3-dimensional Heisenberg nilmanifold. Let $\alpha \in [0, 1)$, $k \in \mathbb{R}$ and $\phi, \psi$ be $C^\infty$-smooth periodic functions from $\mathbb{R}$ to $\mathbb{R}$ with period 1 such that

$$(1.4) \quad \int_0^1 \phi(t) dt = 0.$$ 

Let the skew product $T$ on $T \times \Gamma \backslash G$ be given by (1.3). Then the Möbius function $\mu$ is linearly disjoint from $(T \times \Gamma \backslash G, T)$. 

Notations. We list some notations that we use in this paper. We write \(e(x)\) for \(e^{2\pi i x}\), and write \(\|x\|\) for the distance between \(x\) and the nearest integer, that is
\[
\|x\| = \min_{n \in \mathbb{Z}} |x - n|.
\]
For positive \(A\), the notations \(B = O(A)\) or \(B \ll A\) mean that there exists a positive constant \(c\) such that \(|B| \leq cA\). If the constant \(c\) depends on a parameter \(b\), we write \(B = O_b(A)\) or \(B \ll_b A\). The notation \(A \asymp B\) means that \(A \ll B\) and \(B \ll A\). For a topological space \(X\), we use \(C(X)\) to denote the set of all continuous complex-valued functions on \(X\). If \(X\) is a smooth manifold and \(r \geq 1\) is an integer, then we use \(C^r(X)\) to denote the set of all \(f \in C(X)\) that have continuous \(r\)-th derivatives.

2. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 where \(\alpha\) is a rational number.

The Proposition 2.3 in [9] shows that the \(\mathbb{C}\)-linear subspace spanned by two classes of functions \(A\) and \(B\) is dense in \(C(\mathbb{T} \times \Gamma \backslash G)\).

For integers \(m, j\) with \(0 \leq j \leq m - 1\), the functions \(\psi_{mj}\) and \(\psi^*_{mj}\) on \(G\) are defined by
\[
\psi_{mj} \left( \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \right) = e(mz + jx) \sum_{b \in \mathbb{Z}} e^{-\pi (y + b + \frac{1}{m} j)^2} e(mb), \\
\psi^*_{mj} \left( \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \right) = ie(mz + jx) \sum_{b \in \mathbb{Z}} e^{-\pi (y + b + \frac{1}{m} j)^2} e \left( \frac{1}{2} (y + b + \frac{1}{m}) + (mb) \right).
\]
Then \(\psi_{mj}\) and \(\psi^*_{mj}\) are \(\Gamma\)-invariant so that they can be regarded as functions on \(\Gamma \backslash G\).

Recall that there is a unique Borel probability measure on \(\Gamma \backslash G\) that is invariant under the right translations, and therefore \(L^2(\Gamma \backslash G)\) can be defined. Let \(V_0\) be the subspace of \(L^2(\Gamma \backslash G)\) consisting of all functions \(f \in L^2(\Gamma \backslash G)\) satisfying
\[
f \left( \Gamma g \begin{pmatrix} 1 & 0 & z \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right) = f(\Gamma g)
\]
for any \(g \in G\) and \(z \in \mathbb{R}\). Set \(C_0 = V_0 \cap C(\Gamma \backslash G)\).

Lemma 2.1. ([9] Proposition 2.3) Let \(A\) be the subset of \(f \in C(\mathbb{T} \times \Gamma \backslash G)\) satisfying
\[
f : \begin{pmatrix} t, \Gamma \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \end{pmatrix} \mapsto e(\xi_1 t + \xi_2 x + \xi_3 y) \psi \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix},
\]
where \(\xi_1, \xi_2, \xi_3 \in \mathbb{Z}\) and \(\psi = \psi_{mj}, \psi^*_{mj}, \psi_{mj}^*\) or \(\psi^*_{mj}\) for some \(0 \leq j \leq m - 1\). Here \(\psi_{mj}\) and \(\psi^*_{mj}\) stand for the complex conjugates of \(\psi_{mj}\) and \(\psi^*_{mj}\), respectively. Let \(B\) be subset of \(f \in C(\mathbb{T} \times \Gamma \backslash G)\) satisfying \(f((t, \Gamma g)) = f_1(t) f_2(\Gamma g)\) with \(f_1 \in C(\mathbb{T})\) and \(f_2 \in C_0(\Gamma \backslash G)\). Then the \(\mathbb{C}\)-linear subspace spanned by \(A \cup B\) is dense in \(C(\mathbb{T} \times \Gamma \backslash G)\).

Following [9], we should separately consider the two cases, namely \(f \in A\) and \(f \in B\). The case \(f \in A\) will be handled by Fourier analysis and a classical result of Hua [8], which is a generalization of Davenport [3].
Lemma 2.2. ([8]) Let $f(x) = \alpha_d x^d + \alpha_{d-1} x^{d-1} + \cdots + \alpha_1 x + \alpha_0 \in \mathbb{R}[x]$ and $0 \leq a < q$. Then, for arbitrary $A > 0$,

$$
\sum_{n \leq N, n \equiv a \mod q} \mu(n) e(f(n)) \ll_A \frac{N}{\log^A N},
$$

where the implied constant may depend on $A$, $q$ and $d$, but is independent of $\alpha_d, \ldots, \alpha_0$.

For the case that $f \in \mathcal{A}$ we have the following proposition.

Proposition 2.3. Let $(\mathbb{T} \times \Gamma \setminus \Gamma, S)$ be as in Theorem [1,4] with $\alpha \in \mathbb{Q} \cap [0, 1)$. Let $\mathcal{A}$ be as above. Then, for any $(t_0, \Gamma g_0) \in \mathbb{T} \times \Gamma \setminus \Gamma$, any $f \in \mathcal{A}$ and any $A > 0$,

$$
\sum_{n \leq N} \mu(n) f(S^n(t_0, \Gamma g_0)) \ll_A \frac{N}{\log^A N},
$$

where the implied constant depends on $A$ and $\alpha$ only.

Proof. For simplicity, we only consider a typical $f \in \mathcal{A}$ defined by

$$
(2.1) \quad f\left(t, \Gamma \left( \begin{array}{ccc} 1 & y & z \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{array} \right) \right) = e(t + x + y + z) \sum_{l \in \mathbb{Z}} e^{-\pi(y+l)^2} e(lx).
$$

A general $f$ can be treated in the same way.

To compute $f(S^n(t_0, \Gamma g_0))$ via (2.1), we define

$$
S_1(n; t) = \sum_{l=0}^{n-1} \varphi_1(\alpha l + t), \quad S_2(n; t) = \sum_{l=0}^{n-1} \varphi_2(\alpha l + t),
$$

$$
S_3(n; t) = \sum_{l=0}^{n-1} \psi(\alpha l + t), \quad S_4(n; t) = \sum_{l=0}^{n-2} \varphi_2(\alpha l + t) \sum_{j=l+1}^{n-1} \varphi_1(\alpha j + t)
$$

for $n \geq 1$, $t \in \mathbb{T}$ and set $S_1(0; t) = S_2(0; t) = S_3(0; t) = S_4(0; t) = S_4(1; t) = 0$ for simplicity. A straightforward calculation gives

$$
(2.2) \quad S^n : (t_0, \Gamma g_0) \mapsto (t_0 + n\alpha, \Gamma g_n),
$$

where

$$
(2.3) \quad g_n := g_0 \begin{pmatrix} 1 & S_2(n; t_0) & S_3(n; t_0) + S_4(n; t_0) \\ 0 & 1 & S_1(n; t_0) \\ 0 & 0 & 1 \end{pmatrix}.
$$

Now write

$$
g_0 = \begin{pmatrix} 1 & y_0 & z_0 \\ 0 & 1 & x_0 \\ 0 & 0 & 1 \end{pmatrix}, \quad g_n = \begin{pmatrix} 1 & y_n & z_n \\ 0 & 1 & x_n \\ 0 & 0 & 1 \end{pmatrix},
$$

where we may assume $x_0, y_0, z_0 \in [0, 1)$ without loss of generality, so that (2.3) gives

$$
x_n = x_0 + S_1(n; t_0),
$$

$$
y_n = y_0 + S_2(n; t_0),
$$

$$
z_n = z_0 + y_0 S_1(n; t_0) + S_3(n; t_0) + S_4(n; t_0).$$
Substituting (2.3) into (2.2) and combining with (2.1), we obtain that

\[
f(S^n(t_0, \Gamma g_0)) = f \left( t_0 + n \alpha, \Gamma \begin{pmatrix} \frac{1}{y_n} & z_n \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \right) \\
= e(t_0 + n \alpha + x_n + y_n + z_n) \sum_{l \in \mathbb{Z}} e^{-\pi(y_n+l)^2} e(lx_n).
\]

To analyze (2.4), we define \( \omega : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) by

\[
\omega(u,v) = \sum_{l \in \mathbb{Z}} e^{-\pi(u+y_0+l)^2} e(l(u + x_0)).
\]

Then \( \omega \) is an double periodic analytic function which is bounded on \( \mathbb{R} \times \mathbb{R} \). With this function \( \omega \) we can rewrite (2.4) as

\[
f(S^n(t_0, \Gamma g_0)) = \rho \omega \left( S_1(n; t_0), S_2(n; t_0) \right) \\
\times e \left( n \alpha + (y_0 + 1)S_1(n; t_0) + S_2(n; t_0) + S_3(n; t_0) + S_4(n; t_0) \right),
\]

where \( \rho := e(t_0 + x_0 + y_0 + z_0) \).

Recall that \( \alpha \in \mathbb{Q} \cap [0, 1) \) in the present situation, so that we can write \( \alpha = a/q \) with \( 0 \leq a < q \) and \( (a, q) = 1 \). Thus for any periodic function \( h \) with period 1, we have \( h(l_1 \alpha + t_0) = h(l_2 \alpha + t_0) \) whenever \( l_1 \equiv l_2 \) mod \( q \). For any \( 0 \leq b < q \) and any periodic function \( h \) with period 1, define

\[
\gamma(h, b) = \sum_{l=0}^{b-1} h(l \alpha + t_0)
\]

and set \( \gamma(h) = \gamma(h, q)/q \). For any \( n \equiv b \) mod \( q \), write \( n - 1 \equiv b' \) mod \( q \), \( 0 \leq b, b' < q \). Then

\[
S_1(n; t_0) = (n - b) \gamma(\varphi_1) + \gamma(\varphi_1, b), \\
S_2(n; t_0) = (n - b) \gamma(\varphi_2) + \gamma(\varphi_2, b), \\
S_3(n; t_0) = (n - b) \gamma(\psi) + \gamma(\psi, b)
\]

and

\[
S_4(n; t_0) = \sum_{l=0}^{n-2} \varphi_2(\alpha l + t_0) \sum_{j=l+1}^{n-1} \varphi_1(\alpha j + t_0) \\
= \left( \frac{n - 1 - q - b'}{q} + \cdots + 1 \right) S_1(q; t_0) S_2(q; t_0) + \sum_{l=0}^{b'-1} \varphi_2(\alpha l + t_0) \sum_{j=l+1}^{b'} \varphi_1(\alpha j + t_0) \\
+ (n - 1 - b')q^{-1} \sum_{l=0}^{q-1} \varphi_2(\alpha l + t_0) \sum_{j=l+1}^{q+b'} \varphi_1(\alpha j + t_0)
\]

are all real-valued polynomials in \( n \) of degree less than 3 with coefficients depending on \( \alpha, b, b' \) and \( m \). It follows from this and (2.5) that

\[
f(S^n(t_0, \Gamma g_0)) = \rho \omega \left( S_1(n; t_0), S_2(n; t_0) \right) e(P_{b,b'}(n)),
\]
where $P_{b,b'}(n)$ is a real-valued polynomial in $n$ of degree less than 3 with coefficients depending on $\alpha, b, b'$ and $m$, and then

$$\sum_{n \leq N} \mu(n) f(S^n(t_0, \Gamma g_0)) = \rho \sum_{b=0}^{q-1} \sum_{n \equiv b \mod q} \mu(n) \tilde{\omega}(n) e(P(n, b)), \tag{2.6}$$

where $\tilde{\omega}(n) = \omega(S_1(n; t_0), S_2(n; t_0))$ is a bounded smooth weight. Hence, by Lemma 2.2 we have

$$\sum_{n \equiv b \mod q} \mu(n) e(P(n, b)) \ll A \frac{N}{\log^4 N} \tag{2.7}$$

for arbitrary $A > 0$, where the implied constant depending on $q$ (hence on $\alpha$) and $A$ only. Substituting this to (2.6) we obtain the desired estimate. □

The other case $f \in B$ can be reduced to the case of skew products on $T^3$ which is already known as [9, Corollary 3.2].

**Lemma 2.4.** ([9, Corollary 3.2]) Let $\alpha \in \mathbb{R}$, and let $h_1, h_2 : T \to \mathbb{R}$ be $C^\infty$-smooth functions, $T : T^3 \to T^3$ be given by $T(x, y, z) = (x + \alpha, y + h_1(x), z + h_2(x))$. Then the Möbius Disjointness Conjecture holds for $(T^3, T)$.

For the case that $f \in B$ we have the following proposition.

**Proposition 2.5.** Let $B \subset C(T \times \Gamma \setminus G)$ as above. Let $S$ be as in Theorem 1.1 and let $f \in B$. Then, for any $(t_0, \Gamma g_0) \in T \times \Gamma \setminus G$,

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} \mu(n) f(S^n(t_0, \Gamma g_0)) = 0.$$

**Proof.** Let $\tilde{S} : T^3 \to T^3$ be given by

$$\tilde{S} : (t, x, y) \mapsto (t + \alpha, x + \varphi_1(t), y + \varphi_2(t)),$$

and $\pi$ be the projection of $T \times \Gamma \setminus G$ onto $T^3$ given by

$$\pi : \left( t, \Gamma \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \right) \mapsto (t, x, y).$$

Then we have $\pi \circ S = \tilde{S} \circ \pi$, and hence $(T^3, \tilde{S})$ is a topological factor of $(T \times \Gamma \setminus G, S)$.

Since $f \in B$, there are some $f_1 \in C(T)$ and $f_2 \in V_0 \cap C(\Gamma \setminus G)$ such that $f(t, \Gamma g) = f_1(t) f_2(\Gamma g)$. Then for any $z' \in \mathbb{R}$,

$$f_2 \left( \Gamma \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \right) = f_2 \left( \Gamma \begin{pmatrix} 1 & y & z + z' \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \right)$$

which shows that $f_2$ is independent of the $z$-component and induces a well-defined continuous function $\tilde{f}_2 \in C(T^2)$ given by

$$\tilde{f}_2(x, y) = f_2 \left( \Gamma \begin{pmatrix} 1 & y & z \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix} \right)$$
for any \( z \in \mathbb{R} \). Define \( \tilde{f}(t, x, y) \in C(\mathbb{T}^3) \) by
\[
\tilde{f}(t, x, y) = f_1(t) \tilde{f}_2(x, y).
\]
Then \( f(t, \Gamma g) = \tilde{f} \circ \pi(t, \Gamma g) \) for any \( (t, \Gamma g) \in \mathbb{T} \times \Gamma \backslash G \). Hence
\[
f(S^n(t_0, \Gamma g_0)) = \tilde{f} \circ \pi \circ S^n(t_0, \Gamma g_0) = \tilde{f} \circ \tilde{S}^n \circ (t_0, \Gamma g_0)
\]
for any \( n \geq 1 \), and the sequence \( \{f(S^n(t_0, \Gamma g_0))\}_{n \geq 1} \) is observed in \((\mathbb{T}^3, \tilde{S})\). The desired result follows from Lemma 2.4. \( \square \)

**Proof of Theorem 1.1** Combining Propositions 2.3 and 2.5 we get that Theorem 1.1 holds. \( \square \)

### 3. Measure complexity

To prove Theorem 1.2 for irrational \( \alpha \), we will use the criterion given by Huang-Wang-Ye in [10]. To do this, we need the concept of measure complexity. In this section, we will collect some concepts and facts from [10] without proof.

Let \((X, \mathcal{T})\) be a flow, and \(M(X, \mathcal{T})\) the set of all \(\mathcal{T}\)-invariant Borel probability measures on \(X\). A metric \(d\) on \(X\) is said to be compatible if the topology induced by \(d\) is the same as the given topology on \(X\). For a compatible metric \(d\) and an \(n \in \mathbb{N}\), define
\[
\bar{d}_n(x, y) = \frac{1}{n} \sum_{j=0}^{n-1} d(T^jx, T^jy)
\]
for \(x, y \in X\). Then, for \(\varepsilon > 0\), let
\[
B_{\bar{d}_n}(x, \varepsilon) = \{y \in X : \bar{d}_n(x, y) < \varepsilon\},
\]
with which we can further define, for \(\rho \in M(X, \mathcal{T})\),
\[
s_n(X, \mathcal{T}, d, \rho, \varepsilon) = \min\{m \in \mathbb{N} : \exists x_1, \ldots, x_m \in X, \text{ s. t. } \rho(\cup_{j=1}^{m} B_{\bar{d}_n}(x_j, \varepsilon)) > 1 - \varepsilon\}.
\]

Let \((X, d, \mathcal{T}, \rho)\) be as above, and let \(\{u(n)\}_{n \geq 1}\) be an increasing sequence satisfying \(1 \leq u(n) \to \infty\) as \(n \to \infty\). We say that the measure complexity of \((X, d, \mathcal{T}, \rho)\) is weaker than \(u(n)\) if
\[
\liminf_{n \to \infty} \frac{s_n(X, \mathcal{T}, d, \rho, \varepsilon)}{u(n)} = 0
\]
for any \(\varepsilon > 0\). In view of Lemma 3.2 this property is independent of the choice of compatible metrics. Hence we can say instead that the measure complexity of \((X, \mathcal{T}, \rho)\) is weaker than \(u(n)\). We say the measure complexity of \((X, \mathcal{T}, \rho)\) is sub-polynomial if the measure complexity of \((X, \mathcal{T}, \rho)\) is weaker than \(n^\tau\) for any \(\tau > 0\).

We will use the equivalent condition provided by Huang-Wang-Ye [10] to derive that the Möbius Disjointness Conjecture holds for our flow \((\mathbb{T} \times \Gamma \backslash G, T)\) as in Theorem 1.2 for \(\alpha\) irrational. The following lemma is the main theorem of [10] which gives a criterion of the Möbius Disjointness Conjecture.

**Lemma 3.1.** ([10, Theorem 1.1]) If the measure complexity of \((X, \mathcal{T}, \rho)\) is sub-polynomial for any \(\rho \in M(X; T)\), then the Möbius Disjointness Conjecture holds for \((X, T)\).
We say \((X,\theta)\) preserving map \(Y\) and id\(^d\) and the metric \(d\) with \((3.2)\) and \((3.4)\) \(d\) can be proved that \(d\) from which we see that \(d\) finally take \(d\) to be the canonical Euclidean metric on \(\mathbb{T}\), and \(d = d_{\mathbb{T} \times \Gamma \setminus G}\) the \(l^\infty\)-product metric of \(d\) and \(d_{\Gamma \setminus G}\) given by
\[
d((t_1, \Gamma g_1), (t_2, \Gamma g_2)) = \max (d_T(t_1, t_2), d_{\Gamma \setminus G}(\Gamma g_1, \Gamma g_2)).
\]
In view of Lemma 3.2 the choice of compatible metrics does not affect the measure complexity. Thus the above choice of \(d\) is admissible.
4. Proof of Theorem 1.2

In this section, we assume that \( \alpha \) is irrational and prove Theorem 1.2 for irrational \( \alpha \). Then combine with Theorem 1.1 we get Theorem 1.2.

Let \( \alpha = [0; a_1, a_2, \ldots, a_i, \ldots] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} \) be the continued fraction expansion of \( \alpha \). Then the expansion is infinite since \( \alpha \) is irrational. Let \( \frac{l_i}{q_i} = [0; a_1, a_2, \ldots, a_i] \) be the \( i \)-th convergent of \( \alpha \). Let \( Q = \{ q_i : i \geq 1 \} \).

Furthermore, we define \( M_1(B) = \bigcup_{q_i \in Q(B)} \{ m \in \mathbb{Z} : q_i \leq |m| < q_{i+1}, q_i|m \} \) and define \( M_2(B) = \mathbb{Z} \setminus M_1(B) \). Now expand \( \phi \) into Fourier series

\[
\phi(t) = \sum_{m \in \mathbb{Z}} \hat{\phi}(m)e(mt),
\]

and further decompose \( \phi \) as

\[
\phi(t) = \phi_1(t) + \phi_2(t) = \sum_{m \in M_1(B)} \hat{\phi}(m)e(mt) + \sum_{m \in M_2(B)} \hat{\phi}(m)e(mt).
\]

We call \( \phi_1 \) and \( \phi_2 \) the resonant and non-resonant part of \( \phi \), respectively. Let \( \eta(t) = \phi_2^2(t) \).

We can do the same decompositions for \( \eta \) and \( \psi \), getting

\[
\eta(t) = \eta_1(t) + \eta_2(t), \quad \psi(t) = \psi_1(t) + \psi_2(t).
\]

Note that the above decompositions of \( \phi \), \( \eta \) and \( \psi \) depend on the parameter \( B \), though we do not make it explicit.

In our proof of Theorem 1.2 for irrational \( \alpha \) we still need a lemma from [9].

**Lemma 4.1.** ([9 Lemma 4.2]) Let \( B > 2 \) and let \( \{ a(m) \}_{m \in \mathbb{Z}} \) be a sequence such that \( |a(m)| \ll m^{-2B} \). Then the series

\[
\sum_{m \in M_2(B)} \frac{a(m)}{e(m\alpha) - 1}
\]

is absolutely convergent.

Define

\[
g_{\phi}(t) = \sum_{m \in M_2(B)} \hat{\phi}(m)\frac{e(mt)}{e(m\alpha) - 1}.
\]

Since \( \phi \) is assumed to be \( C^\infty \)-smooth, we have \( \hat{\phi}(m) \ll |m|^{-2B} \) for any \( B > 0 \). Therefore, by Lemma 1.1, \( g_{\phi} \) is a continuous periodic function with period 1 and satisfies

\[
g_{\phi}(t + \alpha) - g_{\phi}(t) = \phi_2(t).
\]
Similarly, there exist continuous periodic functions $g_\eta$ and $g_\psi$ with period 1 such that
\begin{equation}
\eta_2(t) = g_\eta(t + \alpha) - g_\eta(t), \quad \psi_2(t) = g_\psi(t + \alpha) - g_\psi(t).
\end{equation}

To investigate the resonant part, define
\begin{equation}
\Phi_n(t) = \sum_{l=0}^{n-1} \phi_1(l\alpha + t), \quad H_n(t) = \sum_{l=0}^{n-1} \eta_1(l\alpha + t), \quad \Psi_n(t) = \sum_{l=0}^{n-1} \psi_1(l\alpha + t)
\end{equation}
for $n \in \mathbb{N}$ and $t \in \mathbb{T}$. For $n = q_i$, Huang-Liu-Wang [9] proved the following proposition.

**Lemma 4.2.** ([9] Lemma 4.3) Let $B > 2$. Then there exists a positive constant $C_1 = C_1(B)$ depending on $B$ only, such that the three inequalities
\[|\Phi_q(t) - q_i\hat{\phi}(0)| \leq C_1q_i^{-B+1}, \quad |H_q(t) - q_i\hat{\eta}(0)| \leq C_1q_i^{-B+1}, \quad |\Psi_q(t) - q_i\hat{\psi}(0)| \leq C_1q_i^{-B+1}\]
hold simultaneously for all $t \in \mathbb{T}$ and all $q_i \in \mathbb{Q}(B)$.

Now we are ready to prove Theorem 1.2 for irrational $\alpha$ and the skew product $T$ defined in (1.3). We will prove the following proposition.

**Proposition 4.3.** Let $(\mathbb{T} \times \mathbb{R} \backslash G, T)$ be as in Theorem 1.2 with $\alpha$ irrational. Then the measure complexity of $(\mathbb{T} \times \mathbb{R} \backslash G, T)$ is sub-polynomial for any $\rho \in M(\mathbb{T} \times \mathbb{R} \backslash G, T)$.

**Proof.** Fix $\tau > 0$. We want to show that, for any $\varepsilon > 0$,
\[\lim_{n \to \infty} \inf \frac{\mathcal{S}_n(\mathbb{T} \times \mathbb{R} \backslash G, T, d, \rho, \varepsilon)}{n^\tau} = 0.\]
Without loss of generality, we assume that both $\tau, \varepsilon < 10^{-2}$, and $\tau^{-1}, \varepsilon^{-1}$ are integers. Set $B = 8\tau^{-1} + 1$.

Firstly, assume that $\mathbb{Q}(B)$ is infinite. Construct a transformation $R : \mathbb{T} \times \mathbb{R} \backslash G \to \mathbb{T} \times \mathbb{R} \backslash G$ as
\begin{equation}
R : (t, \Gamma) \mapsto \begin{pmatrix} t, \Gamma g \begin{pmatrix} 1 & k\phi(t) \frac{1}{2}k^2\phi(t) - \frac{1}{2}k\phi(t) + g_\phi(t) \\ 0 & 1 \end{pmatrix} \end{pmatrix} \end{equation}
Write $T_1 = R^{-1} \circ T \circ R$. Then a straightforward calculation gives
\begin{align*}
T_1(t, \Gamma) &= R^{-1} \circ T \circ R(t, \Gamma) \\
&= R^{-1} \left( t + \alpha, \Gamma g \begin{pmatrix} 1 & k\phi(t) + k\phi(t) \frac{1}{2}k^2\phi(t) - \frac{1}{2}k\phi(t) + g_\phi(t) \\ 0 & 1 \end{pmatrix} \right) \\
&= R^{-1} \begin{pmatrix} t + \alpha, \Gamma g \begin{pmatrix} 1 & k\phi(t) + k\phi(t) \omega - g_\phi(t) + g_\phi(t) \\ 0 & 1 \end{pmatrix} \end{pmatrix} \\
&= \begin{pmatrix} t + \alpha, \Gamma g \begin{pmatrix} 1 & k\phi(t) + k\phi(t) - k\phi(t + \alpha) \omega - g_\phi(t) + g_\phi(t) \\ 0 & 1 \end{pmatrix} \end{pmatrix},
\end{align*}
where
\[\omega = \frac{1}{2}k^2\phi(t) - \frac{1}{2}k\phi(t)g_\phi(t) + g_\phi(t) + \frac{1}{2}kg_\phi(t + \alpha) - g_\psi(t + \alpha) - k\phi(t)g_\phi(t + \alpha) + \psi(t) + k\phi(t)g_\phi(t) - \frac{1}{2}kg_\psi(t) + g_\psi(t).\]
So $\omega$ can be simplified as
\[
\omega = \frac{1}{2}k(g_\phi(t + \alpha) - g_\phi(t))^2 + \frac{1}{2}k(g_\eta(t + \alpha) - g_\eta(t)) - k\phi(t)(g_\phi(t + \alpha) - g_\phi(t)) + \psi_1(t)
\]
\[
\phi
\]
\[
(4.7)
\]
\[
= \frac{1}{2}k\psi_2'(t) + \frac{1}{2}k\eta_2(t) - k\phi(t)\phi_2(t) + \psi_1(t)
\]
\[
= \frac{1}{2}k(\phi(t) - \phi_2(t))^2 - \frac{1}{2}k\phi_2'(t) + \frac{1}{2}k\eta_2(t) + \psi_1(t)
\]
\[
= \frac{1}{2}k\psi_2'(t) - \frac{1}{2}k\eta_1(t) + \psi_1(t),
\]
where we have used (4.2), (4.4) and (4.3). It follows that
\[
(4.8)
\]
\[
T_1 : (t, \Gamma) \mapsto \left( t + \alpha, \Gamma g \left( \begin{array}{c} k\phi_1(t) \\ 0 \\ 0 \\ 0 \\ 0 \\ k\psi_1(t) \end{array} \right) \right)
\]
and then
\[
T_1^n : (t, \Gamma) \mapsto \left( t + n\alpha, \Gamma g \left( \begin{array}{c} k\phi_1(t) \\ 0 \\ 0 \\ 0 \\ 0 \\ k\psi_1(t) \end{array} \right) \right).
\]
Clearly, $R$ is a homeomorphism on $\mathbb{T} \times \Gamma \setminus G$. Hence by Lemma 3.2, we need only to show that the measure complexity of $(\mathbb{T} \times \Gamma \setminus G, T_1, v)$ is weak than $n^r$, where $v = \rho \circ R$.

Let $C_1 = C_1(B) > 0$ be the constant in Lemma 4.2. The functions $\phi_1(t)$, $\eta_1(t)$ and $\psi_1(t)$ are Lipschitz continuous, and therefore there exists $L > 0$ such that
\[
|\phi_1(t_1) - \phi_1(t_2)| \leq L\|t_1 - t_2\|,
\]
\[
|\eta_1(t_1) - \eta_1(t_2)| \leq L\|t_1 - t_2\|,
\]
\[
|\psi_1(t_1) - \psi_1(t_2)| \leq L\|t_1 - t_2\|
\]
for any $t_1, t_2 \in \mathbb{T}$. We also assume that $L$ is large enough such that $L > \varepsilon^{-1}$. Moreover, since $\phi_1(t)$, $\eta_1(t)$ and $\psi_1(t)$ are continuous, there exists a constant $C_2 > 0$ such that
\[
|\phi_1(t)| \leq C_2, \quad |\eta_1(t)| \leq C_2, \quad |\psi_1(t)| \leq C_2
\]
for all $t \in \mathbb{T}$. Since $q_i \to \infty$ as $i \to \infty$, there exists a constant $I_0 > 0$ such that $(C_1 + C_2)/q_i < \varepsilon$ for all $i \geq I_0$. For $i \geq I_0$, define
\[
F_1(i) = \left\{ t = \frac{j\varepsilon}{Lq_i} \in \mathbb{T} : j = 0, 1, \ldots, \frac{Lq_i}{\varepsilon} - 1 \right\}
\]
and
\[
F_2(i) = \left\{ \Gamma g = \left( \begin{array}{cccc} 1 & j_2(q_i^2L)^{-1} & j_3(q_i^2L)^{-1} \\ 0 & 1 & j_1(q_i^2L)^{-1} \\ 0 & 0 & 1 \end{array} \right) \in \Gamma \setminus G : j_1, j_2, j_3 = 0, 1, \ldots, q_i^2L - 1 \right\}.
\]
Let
\[
F(i) = \{(t, \Gamma g) \in \mathbb{T} \times \Gamma \setminus G : t \in F_1(i), \ \Gamma g \in F_2(i) \}.
\]
Then $\#F(i) = \varepsilon^{-1}L^2q_i^2$.

Now assume that $q_i \in \mathcal{Q}^2(B)$ with $i \geq I_0$ and set
\[
in = q_i^2 - 1.
\]
Then any positive integer $m \leq n_i$ can be uniquely written as
\[
m = a_m q_i + b_m
\]
with $0 \leq b_m < q_i$ and $a_m \leq q_i^{-2}$. By the definition of $F(i)$, for any
\[
(t, \Gamma g) = \begin{pmatrix} t & 1 & y & z \\ 0 & 1 & x & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in T \times \Gamma \setminus G
\]
with $x, y, z \in [0, 1)$, there exists
\[
(t^*, \Gamma g^*) = \begin{pmatrix} t^* & 1 & y^* & z^* \\ 0 & 1 & x^* & 0 \end{pmatrix} \in F(i)
\]
such that $||t - t^*|| \leq \varepsilon/(L q_i)$ and
\[
\max \{|x - x^*|, |y - y^*|, |z - z^*|\} \leq \frac{1}{q_i L}.
\]
We want to show that $d(T_1^m(t, \Gamma g), T_1^m(t^*, \Gamma g^*))$ is small for any $m \leq n_i$, where $n_i$ is as in (4.10). Let
\[
Y(m) = \begin{pmatrix} 1 & k \Phi m(t) & \frac{1}{2} k \Phi^2 m(t) - \frac{1}{2} k H m(t) + \Psi m(t) \\ 0 & 1 & 0 \end{pmatrix}
\]
and
\[
Y^*(m) = \begin{pmatrix} 1 & k \Phi m(t^*) & \frac{1}{2} k \Phi^2 m(t^*) - \frac{1}{2} k H m(t^*) + \Psi m(t^*) \\ 0 & 1 & 0 \end{pmatrix}
\]
Then we have
\[
T_1^m(t, \Gamma g) = (t + \alpha m, \Gamma g Y(m)), \quad T_1^m(t^*, \Gamma g^*) = (t^* + \alpha m, \Gamma g^* Y^*(m)).
\]
Therefore, by our choice of the metric on $T \times \Gamma \setminus G$ in (3.5), we have
\[
d(T_1^m(t, \Gamma g), T_1^m(t^*, \Gamma g^*)) \leq \max \left( ||t - t^*||, d_{\Gamma \setminus G}(\Gamma g Y(m), \Gamma g^* Y^*(m)) \right).
\]
Since the term $||t - t^*||$ can be arbitrarily small as $q_i \to \infty$, it remains only to bound the last term in (4.13). By the triangle inequality and (3.4) we get
\[
d_{\Gamma \setminus G}(\Gamma g Y(m), \Gamma g^* Y^*(m)) \leq d_{\Gamma \setminus G}(\Gamma g^* Y(m), \Gamma g Y(m)) + d_{\Gamma \setminus G}(\Gamma g^* Y^*(m), \Gamma g^* Y(m))
\]
\[
(4.14)
\]
\[
\leq d_G(g^* Y(m), g Y(m)) + d_G(g^* Y^*(m), g Y(m)) = d_G(g^* Y(m), g Y(m)) + d_G(Y^*(m), Y(m)),
\]
where the last equality follows from the left invariance of $d_G$. Furthermore, from the definition of $d_G$ we get
\[
d_G(g^* Y(m), g Y(m)) \leq |\kappa(Y(m)^{-1} g^{-1} g^* Y(m))|, \quad d_G(Y^*(m), Y(m)) \leq |\kappa(Y(m)^{-1} Y^*(m))|,
\]
where $\kappa$ is the Mal’cev coordinate map (3.2), and $|\cdot|$ is the $l^\infty$-norm on $\mathbb{R}^3$. 

A straightforward calculation gives

\[
Y(m)^{-1} g^{-1} g^* Y(m) = \begin{pmatrix}
1 & -k\Phi_m(t) & \frac{1}{2}k\Phi_m^2(t) + \frac{1}{2}kH_m(t) - \Psi_m(t) \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
y^* - y \\
z^* - z + xy - x^*y \\
x^* - x
\end{pmatrix}
= \begin{pmatrix}
y^* - y \\
z^* - z + y(x - x^*) + \Phi_m(t)(kx - ky^* - y + y^*) \\
x^* - x
\end{pmatrix}
\]

Since \(x, y \in [0, 1]\), from (3.3) we get

\[
|\kappa(Y(m)^{-1} g^{-1} g^* Y(m))| \leq \left( \max(|k|, 1)|\Phi_m(t)| + 2 \right)(|x - x^*| + |y - y^*| + |z - z^*|).
\]

By Lemma 4.2 and noticing that \(\tilde{\phi}(0) = 0\) as in (1.4), we get

\[
|\Phi_{q_i}| \leq C_1 q_i^{-B+1}.
\]

Hence by the definition of \(\Phi_n(t)\) and (4.11), we obtain

\[
|\Phi_m(t)| \leq \sum_{r=0}^{a_m-1} |\Phi_{q_i}(t + rq_i\alpha)| + \sum_{l=0}^{b_m} |\phi_1(t + (a_m q_i + l)\alpha)| \leq \frac{C_1 a_m}{q_i^B} + C_2 q_i \leq \frac{C_1}{q_i} + C_2 q_i.
\]

Thus by (4.12), (4.14) and (4.15), we obtain

\[
d_G(g^* Y(m), g Y(m)) \leq \left( \max(|k|, 1)|\Phi_m(t)| + 2 \right)(|x - x^*| + |y - y^*| + |z - z^*|)
\[
\leq \left( \max(|k|, 1)(1 + C_2 q_i) + 2 \right) \frac{3}{q_i^2 L} \leq 6(\max(|k|, 1) + 1)\varepsilon.
\]

The treatment of \(d_G(Y^*(m), Y(m))\) is similar. We calculate that

\[
Y^{-1}(m)Y^*(m) = \begin{pmatrix}
1 & -k\Phi_m(t) & \frac{1}{2}k\Phi_m^2(t) + \frac{1}{2}kH_m(t) - \Psi_m(t) \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
y \\
z - y \\
x
\end{pmatrix}
= \begin{pmatrix}
y \\
z - y + \Phi_m(t)(kx - ky^* - y + y^*) \\
x
\end{pmatrix}
\]

where

\[
\tilde{\omega} = \frac{1}{2}k(\Phi_m(t^*) - \Phi_m(t))^2 + \frac{1}{2}k(H_m(t) - H_m(t^*)) + (\Psi_m(t^*) - \Psi_m(t)).
\]
By Lemma (4.2), (4.9) and (4.11), we get

\[
|\Phi_m(t^*) - \Phi_m(t)| \leq \sum_{r=0}^{a_m-1} \left( |\Phi_q(t^* + r\alpha)| + |\Phi_q(t + r\alpha)| \right) + \sum_{l=0}^{b_m} |\phi_1(t^* + (a_mq_l + l)\alpha) - \phi_1(t + (a_mq_l + l)\alpha)|
\]

\[
\leq \frac{C_1}{q_i} + K L \|t^* - t\| < 2\varepsilon.
\]

Similarly, we can also obtain \( |H_m(t^*) - H_m(t)| \leq 2\varepsilon \) and \( |\Psi_m(t^*) - \Psi_m(t)| \leq 2\varepsilon \). Therefore, by the definition of \( d_G \) and (3.3), we get

\[
d_G(Y^*(m), Y(m)) \leq |\kappa Y(m)^{-1} Y^*(m)| < 8 \max(|k|, 1) \varepsilon.
\]

From (4.13), (4.14), (4.16) and (4.17), we conclude that

\[
d(T_1^m(t, \Gamma G), T_1^m(t^*, \Gamma G^*)) \leq (14 \max(|k|, 1) + 6) \varepsilon
\]

for all \( m \leq n_i \). Here, and in what follows, \( n_i \) as in (4.10). So, by the definition of \( \tilde{d}_n \) in (3.1), we have

\[
\tilde{d}_n_i((t, \Gamma G), (t^*, \Gamma G^*)) = \frac{1}{n_i} \sum_{m=0}^{n_i-1} d(T_1^m(t, \Gamma G), T_1^m(t^*, \Gamma G^*)) \leq (14 \max(|k|, 1) + 6) \varepsilon.
\]

This means that \( \mathbb{T} \times \Gamma \backslash G \) can be covered by \( \sharp F(k) = \varepsilon^{-1} L^4 q_i^7 \) balls of radius \( \delta := (14 \max(|k|, 1) + 6) \varepsilon \) under the metric \( \tilde{d}_n_i \), since \( (t, \Gamma G) \) can be chosen arbitrarily. It follows that

\[
s_n_i(\mathbb{T} \times \Gamma \backslash G, T_1, d, v, \delta) \leq \varepsilon^{-1} L^4 q_i^7.
\]

Since \( Q^2(B) \) is infinite, we can let \( q_i \) tend to infinity along \( Q^2(B) \), getting

\[
\liminf_{n_i \to \infty} \frac{s_n_i(\mathbb{T} \times \Gamma \backslash G, T_1, d, v, \delta)}{n_i^7} \leq \liminf_{q_i \in Q^2(B), i \geq i_0} \frac{s_n_i(\mathbb{T} \times \Gamma \backslash G, T_1, d, v, \delta)}{q_i^7} \leq \liminf_{q_i \in Q^2(B), i \geq i_0} \frac{\varepsilon^{-1} L^4 q_i^7}{q_i^7} = 0.
\]

Since \( \varepsilon \) can be arbitrary small, this means that the measure complexity of \( (\mathbb{T} \times \Gamma \backslash G, T, \rho) \) is weaker than \( n_i^7 \) when \( Q^2(B) \) is infinite.

Finally, we deal with the case that \( Q^2(B) \) is finite. Now \( M_1(B) \) is also finite. Hence the conclusion of Lemma (4.11) still holds if we replace \( M_2(B) \) by \( \mathbb{Z} \backslash \{0\} \). Hence the functions \( \hat{g}_\phi(t), \hat{g}_\eta(t), \) and \( \hat{g}_\psi(t) \) defined by

\[
\hat{g}_\phi(t) = \sum_{m \neq 0} \frac{\hat{\phi}(m)e(mt)}{e(m\alpha) - 1}, \quad \hat{g}_\eta(t) = \sum_{m \neq 0} \frac{\hat{\eta}(m)e(mt)}{e(m\alpha) - 1}, \quad \hat{g}_\psi(t) = \sum_{m \neq 0} \frac{\hat{\psi}(m)e(mt)}{e(m\alpha) - 1}
\]
are continuous and periodic with period 1. Thus we can write
\[ \phi(t) = \tilde{g}_\phi(t + \alpha) - \tilde{g}_\phi(t), \]
\[ \eta(t) = \tilde{h}(0) + \tilde{g}_\eta(t + \alpha) - \tilde{g}_\eta(t), \]
\[ \psi(t) = \tilde{\psi}(0) + \tilde{g}_\psi(t + \alpha) - \tilde{g}_\psi(t). \]
Notice that \( \tilde{\phi}(0) = 0 \), and so there is no constant term in the first equation. Similarly to (4.6), we define \( \tilde{R} : T \times \Gamma \backslash G \to T \times \Gamma \backslash G \) by
\[ \tilde{R}(t, \Gamma \tilde{g}) \mapsto \begin{pmatrix} t \Gamma \tilde{g} & \frac{1}{1} k \tilde{g}_\phi(t) \frac{1}{1} k \tilde{g}_\psi(t) \\ 0 & 1 \end{pmatrix}, \]
Then \( T_1 = \tilde{R}^{-1} \circ T \circ \tilde{R} \) is given by
\[ T_1 : (t, \Gamma \tilde{g}) \mapsto \begin{pmatrix} t + \alpha \Gamma \tilde{g} & 0 \frac{1}{1} k \tilde{h}(0) + \tilde{\psi}(0) \\ 0 & 1 \end{pmatrix}. \]
As in (4.8). Again by Lemma 3.2, the measure complexity of \((T \times \Gamma \backslash G, T, \rho)\) is weaker than \( n^\tau \) if and only if the measure complexity of \((T \times \Gamma \backslash G, T_1, v)\) is weaker than \( n^\tau \), where \( v = \rho \circ \tilde{R} \). However, \( d \) is invariant under \( T_1 \). So we have for any \( n \geq 1 \) and \( \varepsilon > 0 \) that
\[ s_n(T \times \Gamma \backslash G, T_1, d, v, \varepsilon) = s_1(T \times \Gamma \backslash G, T_1, d, v, \varepsilon). \]
Since \( T \times \Gamma \backslash G \) is compact, we have \( s_1(T \times \Gamma \backslash G, T_1, d, v, \varepsilon) < \infty \) and consequently
\[ \lim_{n \to \infty} \frac{s_n(T \times \Gamma \backslash G, T_1, d, v, \varepsilon)}{n^\tau} = \lim_{n \to \infty} \frac{s_1(T \times \Gamma \backslash G, T_1, d, v, \varepsilon)}{n^\tau} = 0. \]
Hence the measure complexity of \((T \times \Gamma \backslash G, T, \rho)\) is also weaker than \( n^\tau \) if \( Q^t(B) \) is finite. The proof is complete. \( \square \)

**Proof of Theorem 1.2** Note that the the skew product \( T \) in Theorem 1.2 is a special case of the skew product \( S \) in Theorem 1.1. Theorem 1.2 follows from Theorem 1.1, Proposition 4.3 and Lemma 3.1. \( \square \)

**Acknowledgments** This work is supported by the National Natural Science Foundation of China (Grant No. 11771252).

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