SCREENING OPERATORS FOR $\mathcal{W}$-ALGEBRAS

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Abstract. Let $\mathfrak{g}$ be a simple finite-dimensional Lie superalgebra with a non-degenerate supersymmetric even invariant bilinear form, $f$ a nilpotent element in the even part of $\mathfrak{g}$, $\Gamma$ a good grading of $\mathfrak{g}$ for $f$ and $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$ the (affine) $\mathcal{W}$-algebra associated with $\mathfrak{g}, f, k, \Gamma$ defined by the generalized Drinfeld-Sokolov reduction. In this paper, we present each $\mathcal{W}$-algebra as the intersection of kernels of the screening operators, acting on the tensor vertex superalgebra of an affine vertex superalgebra and a neutral free superfermion vertex superalgebra. As applications, we prove that the $\mathcal{W}$-algebra associated with a regular nilpotent element in $\mathfrak{osp}(1, 2n)$ is isomorphic to the $\mathcal{W}B_n$-algebra introduced by Fateev and Lukyanov, and that the $\mathcal{W}$-algebra associated with a subregular nilpotent element in $\mathfrak{sl}_n$ is isomorphic to the $\mathcal{W}^{(2)}_n$-algebra introduced by Feigin and Semikhatov.

1. Introduction

Let $\mathfrak{g}$ be a simple finite-dimensional Lie superalgebra with a non-degenerate supersymmetric even invariant bilinear form $(\cdot | \cdot)$, $f$ be a nilpotent element in the even part of $\mathfrak{g}$, $\Gamma$ be a good grading of $\mathfrak{g}$ for $f$, denoted by

$$\Gamma : \mathfrak{g} = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} \mathfrak{g}_j$$

and $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$ be the (affine) $\mathcal{W}$-algebra defined as the BRST cohomology associated with $\mathfrak{g}, f, k$ and $\Gamma$ (FF1, KRW). The $\mathcal{W}$-algebra is a $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded vertex superalgebra and it is conformal if its level is not critical i.e. $k \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$. For fixed $\mathfrak{g}, f$ and $k$, the vertex superalgebra structure of the $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{g}, f; \Gamma)$ does not depend on the choice of the good grading $\Gamma$, although the conformal grading does; see AKM, BG.

In this paper, we present an arbitrary $\mathcal{W}$-algebra as the intersection of kernels of the screening operators. In the special case that the good grading $\Gamma$ can be chosen such that $\mathfrak{g}_0$ is a Cartan subalgebra, our presentation gives a free field realization in terms of the screening operators, which recovers a result of Feigin and Frenkel [FF2] in the case that $\mathfrak{g}$ is a simple Lie algebra and $f$ is a regular nilpotent element. As applications, we prove that the $\mathcal{W}$-algebra associated with a regular nilpotent element in $\mathfrak{osp}(1, 2n)$ is isomorphic to the $\mathcal{W}B_n$-algebra introduced by Fateev and Lukyanov, and that the $\mathcal{W}$-algebra associated with a subregular nilpotent element in $\mathfrak{sl}_n$ is isomorphic to the $\mathcal{W}^{(2)}_n$-algebra introduced by Feigin and Semikhatov. These results were conjectured in [IMP, FOR, Watts] and [ACGHR], respectively.

To clarify our results, let $\Delta$ be a set of roots of $\mathfrak{g}$ and $\Pi$ be a set of simple roots compatible with $\Gamma$. For $j \in \frac{1}{2}\mathbb{Z}$, we denote by $\Delta_j$ the set of roots whose root vector lies in $\mathfrak{g}_j$ and set $\Pi_j = \Pi \cap \Delta_j$. Then, there is a decomposition $\Pi = \Pi_0 \cup \Pi_{\frac{1}{2}} \cup \Pi_1$;
see e.g. [EK] [H]. Denote by $\Delta^\Gamma = \Delta \backslash \Delta_0$ the restricted root system associated with $\Gamma$ ([BG]) and by

$$\Pi^\Gamma = \{ \alpha \in \Delta_{>0} \mid \alpha \text{ is indecomposable in } \Delta_{>0} \}$$

the base of $\Delta^\Gamma$. Then, $\Pi^\Gamma = \Pi^\Gamma_+ \sqcup \Pi^\Gamma_1$, where $\Pi^\Gamma_j = \Pi^\Gamma \cap \Delta_j$. Let $Q_0$ be a root lattice of $g_0$. The equivalence relation on $\Pi^\Gamma$ defined by

$$\alpha \sim \beta \iff \alpha - \beta \in Q_0$$

for $\alpha, \beta \in \Pi^\Gamma$ gives the quotient set $[\Pi^\Gamma]$ of $\Pi^\Gamma$. Let $[\beta]$ be the equivalent class of $\beta \in \Pi^\Gamma$ in $[\Pi^\Gamma]$. Since the equivalence relation on $\Pi^\Gamma$ is homogeneous with respect to the grading, the decomposition of $\Pi^\Gamma$ induces that of $[\Pi^\Gamma]$:

$$[\Pi^\Gamma] = [\Pi^\Gamma_+] \sqcup [\Pi^\Gamma_1].$$

Let $V_{\tau_k}(g_0)$ be an affine vertex superalgebra associated with the subalgebra $g_0$ of $g$ with degree 0 and its invariant bilinear form $\tau_k$ (see (2.2) for the definition), and $F(g_{\frac{1}{2}})$ be a neutral free superfermion vertex superalgebra associated with the subspace $g_{\frac{1}{2}}$ of $g$ with degree $\frac{1}{2}$ ([KRW]).

**Theorem A** (Theorem 3.8). For generic $k$, the $\mathcal{W}$-algebra $\mathcal{W}_k(g, f; \Gamma)$ is isomorphic to the vertex subalgebra of $V_{\tau_k}(g_0) \otimes F(g_{\frac{1}{2}})$, which is the intersection of kernels of the screening operators $Q_{[\beta]}$ for all $[\beta] \in [\Pi^\Gamma]$:

$$\mathcal{W}_k(g, f; \Gamma) \simeq \bigcap_{[\beta] \in [\Pi^\Gamma]} \text{Ker } Q_{[\beta]}.$$  

(1.1)

See [5.7] and [5.8] for the definitions of the screening operators $Q_{[\beta]}$.

We note that Theorem A describes the image of the Miura map defined in [KW2], see Section 5.

In the special case that $g_0$ is a Cartan subalgebra $\mathfrak{h}$, the vertex superalgebra $V_{\tau_k}(g_0)$ is isomorphic to the Heisenberg vertex algebra $\mathcal{H}$ associated with $\mathfrak{h}$. Moreover, since the equivalence relation on $\Pi^\Gamma$ is trivial and $\Delta_0$ is empty, the set $[\Pi^\Gamma]$ is equal to $\Pi$. Let $\chi$ be a linear function on $g$ defined by $\chi(u) = (f|u)$ for all $u \in g$, $\alpha(z)$ be the vertex operator on $\mathcal{H}$ associated with $\alpha \in \Pi$, where we identify the dual space $\mathfrak{h}^*$ with $\mathfrak{h}$ via the fixed bilinear form, and $\Phi_{\beta}(z)$ be the vertex operator on $F(g_{\frac{1}{2}})$ associated with $\beta \in \Pi_{\frac{1}{2}}$. Then we obtain a more explicit presentation of the $\mathcal{W}$-algebra.

**Theorem B** (Theorem 3.9). Assume that $\Gamma$ can be chosen so that $g_0$ is a Cartan subalgebra $\mathfrak{h}$ of $g$. Then, for generic $k$, the $\mathcal{W}$-algebra $\mathcal{W}_k(g, f; \Gamma)$ is isomorphic to the vertex subalgebra of $\mathcal{H} \otimes F(g_{\frac{1}{2}})$, which is the intersection of kernels of the screening operators:

$$\mathcal{W}_k(g, f; \Gamma) \simeq \bigcap_{\alpha \in \Pi^1} \text{Ker } \int \frac{e^{-\frac{\tau}{2}} f \alpha(z)}{\chi(\alpha)} \, dz \cap \bigcap_{\alpha \in \Pi_{\frac{1}{2}}} \text{Ker } \int : e^{-\frac{\nu}{2}} f \alpha(z) \Phi_{\alpha}(z) : \, dz,$$

where $\nu = \sqrt{k + h^2}$. 
Theorem B for a simple Lie algebra and a regular nilpotent element was known by Feigin and Frenkel \([\text{FF2}]\), and for a simple Lie superalgebra and a superprincipal \(f\) nilpotent element by Heluani and Dias \([\text{HD}]\) (see Remark 3.4 of \([\text{HD}]\)).

As applications of Theorem A and B, we have the following results that were conjectured in \([\text{IMP}, \text{FOR}, \text{Watts}]\) and \([\text{ACGHR}]\).

**Theorem C** (Theorem 6.4 and Theorem 6.9).

1. The \(W\)-algebra associated with a regular nilpotent even element \(f_{\text{reg}}\) in \(\mathfrak{osp}(1, 2n)\) at level \(k \neq -n - \frac{1}{2}\) is isomorphic to the Fateev-Lukyanov \(WB_n\)-algebra (\([\text{FL}]\)):
   \[W^k(\mathfrak{osp}(1, 2n), f_{\text{reg}}; \Gamma) \cong WB_n.\]

2. The \(W\)-algebra associated with a subregular nilpotent element \(f_{\text{sub}}\) in \(\mathfrak{sl}_n\) at level \(k \neq -n\) is isomorphic to the Feigin-Semikhatov \(W_{n}^{(2)}\)-algebra (\([\text{FS}]\)):
   \[W^k(\mathfrak{sl}_n, f_{\text{sub}}; \Gamma) \cong W_{n}^{(2)}.\]

For \(n = 1\), Theorem C (1) was shown in \([\text{KRW}]\) as the \(WB_1\)-algebra is the Neveu-Schwarz vertex superalgebra (or the Super Virasoro vertex superalgebra) (\([\text{FL}]\)). We apply Theorem B to \(\mathfrak{osp}(1, 2n)\) and \(f_{\text{reg}}\) to prove Theorem C (1) for generic \(k\), and use a result related to the Miura map to extend the proof to non-critical level \(k\), see the proof of Theorem 6.4 for the details.

For \(n = 3\), Theorem C (2) follows from results of \([\text{KRW}, \text{FS}]\), for the \(W_{3}^{(2)}\)-algebra is the Bershadsky-Polyakov vertex algebra (\([\text{Ber}, \text{P}]\)). We apply Theorem A to \(\mathfrak{g} = \mathfrak{sl}_n\) and \(f = f_{\text{sub}}\), and prove Theorem C (2) using the Wakimoto construction of the affine vertex algebra of \(\mathfrak{sl}_2\) for generic \(k\) and a result related to the Miura map for non-critical level \(k\), see the proof of Theorem 6.9.

The paper is organized as follows. In Section 2.1 we review the definition of the \(W\)-algebra \(W^k(\mathfrak{g}, f; \Gamma)\). In Section 2.2 we recall the definition of the complex \(C_k\) that is used to compute \(W^k(\mathfrak{g}, f; \Gamma)\). In Section 2.3 we consider the classical limit of \(C_k\), which is used to prove Theorem A. In Section 3.1 we set up some notations to state our main results. In particular, we define the screening operators, which consist of the vertex operators on \(F(\mathfrak{g}_2)\) and certain operators \(S_{\alpha}(z)\). In Section 3.2 we deduce Theorem B from Theorem A. Section 4 is devoted to the proof of Theorem A. We define the weight filtration on \(C_k\) in Section 4.1 and study the first spectral sequence \(E_1\) in Section 1.2 by using the classical limit. The key fact that we use to analyze \(E_1\) is Lemma 4.2 which is proved in Section 4.3. In Section 4.4 we give another construction of \(S_{\alpha}(z)\) and derive some results from it. In Section 4.5, we prove Theorem A. Section 5 is devoted to the study of the Miura map. In Section 6 we prove Theorem C.

Acknowledgments This paper is the master thesis of the author and he wishes to express his gratitude to his supervisor Tomoyuki Arakawa for suggesting the problems and lots of advice to improve this paper. He thanks Hiroshi Yamauchi and Kazuya Kawasetsu for useful comments and discussions. He is deeply grateful to Toshiro Kuwabara for explaining him to the geometric aspects of the \(W_{n}^{(2)}\)-algebra and giving him the crucial insight to prove Theorem C (2).

\(^{1}\)There exists an odd nilpotent element \(F \in \mathfrak{g}_{-\frac{1}{2}}\) with \([F, F] = f\) (\(f\) being regular nilpotent) and these two vectors form part of \(\mathfrak{osp}(1, 2) \subset \mathfrak{g}\).
2. Affine $\mathcal{W}$-algebras

2.1. Definitions. In this section we review the definition of the (affine) $\mathcal{W}$-algebras via the generalized Drinfeld-Sokolov reduction given by Kac, Roan and Wakimoto [KRW]. Let $g$ be a simple Lie superalgebra with a non-degenerate even supersymmetric invariant bilinear form $(\cdot | \cdot)$, $f$ be a nilpotent element of the even part of $g$, $k$ be a complex number and $\Gamma$ be a good grading of $g$ with respect to $f$, where a $\frac{1}{2}\mathbb{Z}$-grading

$$\Gamma : g = \bigoplus_{j \in \frac{1}{2}\mathbb{Z}} g_j$$

is called good if $f \in g_{-1}$ and $\text{ad} f : g_j \to g_{j-1}$ is injective for $j \geq \frac{1}{2}$ and surjective for $j \leq \frac{1}{2}$. Let $x$ be a semisimple element in $g$ such that $g_j = \{ u \in g \mid [x, u] = j u \}$. Choose a Cartan subalgebra $h$ containing $x$, a basis $\{ e_i \}_i \in I$ of $h$, where $I = \{ 1, \ldots, \text{rank } g \}$, the root system $\Delta$ of $(g, h)$ and a set of simple roots $\Pi$ such that $e_\alpha \in g_{\geq 0}$ for all positive roots $\alpha$, where $e_\alpha$ is a non-zero root vector. We normalize the invariant bilinear form $(\cdot | \cdot)$ on $g$ by $(\theta | \theta) = 2$, where $\theta$ is a highest root of the even part of $g$. Let $\Delta_j = \{ \alpha \in \Delta \mid e_\alpha \in g_j \}$ and set $\Pi_j = \Pi \cap \Delta_j$ for all $j \in \frac{1}{2}\mathbb{Z}$. Let $x$ be a linear function on $g$ defined by $\chi(u) = (f | u)$ for $u \in g$, $p(\alpha) \in \mathbb{Z}/2\mathbb{Z}$ be the parity of $e_\alpha$ for all $\alpha \in I \cup \Delta$ and $c_{\alpha, \beta} \chi \in \mathbb{C}$ be a structure constant for all $\alpha, \beta, \gamma \in I \cup \Delta$ such that $[e_\alpha, e_\beta] = \sum_{\gamma \in I \cup \Delta} c_{\alpha, \beta} e_\gamma$. Denote by $\kappa_g$ the Killing form of $g$ and by $\text{str}_g$ the supertrace on $g$.

Given a vertex superalgebra $V$ (see [FBZ, Kac3] for the definitions), we denote by $Y(A, z) = A(z) = \sum_{n \in \mathbb{Z}} A(n) z^{-n-1}$ the vertex operator for $A \in V$ and by $[A, B] = \sum_{n=0}^\infty (A(n) B) \frac{z^n}{n!}$ the $\lambda$-bracket of $A, B \in V$.

Consider the vertex superalgebra

$$C^k(g, f; \Gamma) = V^k(g) \otimes F^{\text{ch}}(g_+) \otimes F(g_0^-),$$

where $V^k(g)$ is the universal affine vertex superalgebra associated with $g$, $F^{\text{ch}}(g_+)$ is the charged free superfermion vertex superalgebra associated with $g_+ \oplus g^*_-$ with reversed parities and $F(g_0^-)$ is the neutral free superfermion vertex superalgebra associated with $g_-$ (see [KRW] for definitions), where

$$g_+ = \bigoplus_{j > 0} g_j.$$

Denote by $u(z)$ the vertex operator on $V^k(g)$ associated with $u \in g$, by $\varphi_\alpha(z), \varphi^\alpha(z)$ the vertex operators on $F^{\text{ch}}(g_+)$ associated with $e_\alpha, e^*_\alpha$ for $\alpha \in \Delta_{>0}$, by $\Phi_\alpha(z)$ the vertex operator on $F(g_0^-)$ associated with $\alpha \in \Delta_{>0}$. Note that

$$[u_\lambda v] = [u, v] + k(u | v) \lambda, \quad [\varphi_\alpha \varphi^\alpha] = \delta_{\alpha, \alpha'},$$

for all $u, v \in g$, $\alpha, \alpha' \in \Delta_{>0}$ and $\beta, \beta' \in \Delta_{<0}$. Using the charges in $F^{\text{ch}}(g_+)$ defined by $\text{charge}(\varphi_\alpha) = -1$ and $\text{charge}(\varphi^\alpha) = 1$, one has the induced charge decomposition of $C^k(g, f; \Gamma)$. Let

$$d(z) = d_{\text{st}}(z) + d_{\text{nec}}(z) + d_{\chi}(z).$$
where
\[ d_{st}(z) = \sum_{\alpha \in \Delta_{>0}} (-1)^{p(\alpha)} :e_{\alpha}(z)\varphi^\alpha(z) : - \frac{1}{2} \sum_{\alpha, \beta, \gamma \in \Delta_{>0}} (-1)^{p(\alpha)p(\gamma)} \epsilon_{\alpha, \beta, \gamma} :\varphi_{\gamma}(z)\varphi^\alpha(z)\varphi^\beta(z) :, \]
\[ d_{ne}(z) = \sum_{\alpha \in \Delta_{\frac{1}{2}}} :\varphi^\alpha(z)\Phi_\alpha(z) :, \quad d_{\lambda}(z) = \sum_{\alpha \in \Delta_{>0}} \chi(\epsilon_\alpha)\varphi^\alpha(z). \]

Then \( d^2_{(0)} = 0 \) and \( d_{(0)} \) is a differential compatible with the charge decomposition of \( C^k(\mathfrak{g}, f; \Gamma) \). Thus, \( (C^k(\mathfrak{g}, f; \Gamma), d_{(0)}) \) is a cohomology complex. We denote the cohomology of this complex by
\begin{equation}
(2.1) \quad W^k(\mathfrak{g}, f; \Gamma) = H^*(C^k(\mathfrak{g}, f; \Gamma), d_{(0)}),
\end{equation}
which inherits a vertex superalgebra structure from \( C^k(\mathfrak{g}, f; \Gamma) \). Since the \( j \)-th cohomology of \( (2.1) \) is zero if \( j \neq 0 \) \cite{KW2, KW3}, \( W^k(\mathfrak{g}, f; \Gamma) \) is the 0-th cohomology of \( (2.1) \). This vertex superalgebra is called the \( \mathcal{W} \)-algebra associated with \( \mathfrak{g}, f, k, \) and \( \Gamma \). We note that the vertex superalgebra structure of the \( \mathcal{W} \)-algebra does not depend on the choice of the good grading \( \Gamma \) \cite{DJ}, \cite{AKM} (but the conformal grading of the \( \mathcal{W} \)-algebra depends on \( \Gamma \)).

### 2.2. Decomposition of complex.

We have \cite{KW2, KW3} the decomposition of the complex \( C^k(\mathfrak{g}, f; \Gamma) = C^- \otimes C_k \) such that \( \mathcal{H}(\mathcal{C}^-) = C \) and \( C_k \) is a vertex superalgebra generated by \( J^u(z) \) for all \( u \in \mathfrak{g}_{\leq 0}, \varphi^\alpha(z) \) for all \( \alpha \in \Delta_{>0} \) and \( \Phi_\alpha(z) \) for all \( \alpha \in \Delta_{\frac{1}{2}}, \) where
\[ J^u(z) = u(z) + \sum_{\alpha, \beta \in \Delta_{>0}} (-1)^{p(\alpha)} c_{\alpha, \beta}^\alpha :\varphi_\alpha(z)\varphi^\beta(z) :, \]
for \( u \in \mathfrak{g}_{\leq 0} \). Note that
\[ [J^u, J^v] = J^{[u, v]}, \quad \tau_k(u|v) = \sum_{\beta \in \Delta_{>0}} \epsilon_{\alpha, \beta}^\alpha \varphi^\beta, \quad \Phi_\lambda J^u = 0 \]
for all \( u, v \in \mathfrak{g}_{\leq 0}, \alpha \in \Delta_{>0}, \gamma \in \Delta_{\frac{1}{2}}, \) where
\begin{equation}
(2.2) \quad \tau_k(u|v) = k(u|v) + \frac{1}{2}\kappa_{\alpha}(u|v) - \frac{1}{2}\kappa_{\alpha}(u|v).
\end{equation}
The vertex superalgebra \( C_k \) is a subcomplex of \( C^k(\mathfrak{g}, f; \Gamma) \) and we have
\[ [d_{st}\lambda J^u] = - \sum_{\beta \in \Delta_{>0}} \sum_{\alpha \in \Delta_{\leq 0}} (-1)^{p(\alpha)} c_{\alpha, \beta}^\alpha \varphi^\alpha, \quad [d_{ne}\lambda J^u] = \sum_{\beta \in \Delta_{>0}} \chi([u, e_\beta])\varphi^\beta, \]
\[ [d_{ne}\varphi^\alpha] = \frac{1}{2} \sum_{\beta, \gamma \in \Delta_{\frac{1}{2}}} (-1)^{p(\beta)p(\gamma)} c_{\beta, \gamma}^\beta \varphi^\beta \varphi^\gamma, \quad [d_{st}\lambda \Phi_\alpha] = \sum_{\beta \in \Delta_{\frac{1}{2}}} \chi([e_\beta, e_\alpha])\varphi^\beta, \]
where
\[ a_k(v|w) = \text{str}_g((ad v)p_+(ad w)) + k(v|w), \]
\[ b_k(v|w) = \text{str}_g(p_+(ad v)(ad w)) + k(v|w) \]
for $v, w \in g$ and $p_+ : g \to g_+$ is a projection map. Since $H(C) = \mathbb{C}, H(C_k(g, f; \Gamma)) = H(C_k)$. Moreover, $H^n(C_k) = 0$ for all $n \neq 0$, see [KW2, KW3]. Therefore

$$W_k(g, f; \Gamma) = H(C_k) = H^0(C_k).$$

Let $C'_k$ be a vertex subalgebra of $C_k$ generated by $J^n(z)$ for all $u \in g_{>0}$ and $\varphi^\alpha(z)$ for all $\alpha \in \Delta_{>0}$. Since $d_{st}^2(0) = 0$ and by the above formulas, the vertex superalgebra $C'_k$ is a complex with the differential $d_{st}(0)$. We have

$$(2.3) \quad C_k = C'_k \otimes F(g_+).$$

Let $V_{>0}(g_{>0})$ be a vertex subalgebra of $C'_k$ generated by $J^n(z)$ for all $u \in g_{>0}$ and $V_{>0}(g_{>0})$ be a vertex subalgebra of $V_{>0}(g_{>0})$ generated by $J^n(z)$ for all $u \in g_{>0}$. Then

$$(2.4) \quad C^{(0)}_k = V_{>0}(g_{>0}) \otimes F(g_+),$$

$$(2.5) \quad d_{st}(0)(V_{>0}(g_{>0}) \otimes F(g_+)) = 0.$$

2.3. Classical Limit. Provided that $k + h^\vee \neq 0$, define

$$\epsilon = \frac{1}{k + h^\vee}, \quad \tilde{J}^u(z) = \epsilon J^u(z), \quad \tilde{\Phi}_\alpha(z) = \epsilon \Phi_\alpha(z)$$

for $u \in g_{>0}$ and $\alpha \in \Delta_{>0}$.

Lemma 2.1. Suppose $k + h^\vee \neq 0$ and replace $f$ by $f^{-1}$. Then

$$[J^u, J^v] = \epsilon([J^u, v] + \tilde{\tau}_u(v) \lambda), \quad [\varphi^\alpha J^u] = \epsilon \sum_{\beta \in \Delta_{>0}} c_{u, \beta}^\alpha \varphi^\beta;$$

$$[\tilde{\Phi}_\alpha \tilde{\Phi}_\beta] = \epsilon \cdot \chi([\epsilon \alpha, \epsilon \beta]), \quad [J^u \tilde{\Phi}_\alpha] = [\varphi^\alpha \tilde{\Phi}_\beta] = [\varphi^\alpha \tilde{\Phi}_\beta] = 0;$$

$$[d_{st}(0)(J^u)] = -\sum_{\beta \in \Delta_{>0}} \sum_{\alpha \in \{1, \Delta_{>0}\}} (-1)^{p(\alpha)} c_{u, \beta}^\alpha J^u \varphi^\beta \cdot + \sum_{\beta \in \Delta_{>0}} (\tilde{a}_c(u|e_{\beta}) \varphi^\beta + (\tilde{b}_c(u|e_{\beta}) \lambda) \varphi^\beta;$$

$$[d_{ne}(J^u)] = \sum_{\beta \in \Delta_{>0}} c_{u, \beta} \tilde{\Phi}_\alpha \varphi^\beta; \quad [d_{\chi}(J^u)] = \sum_{\beta \in \Delta_{>0}} \chi([u, e_{\beta}]) \varphi^\beta;$$

$$[d_{ne}(\varphi^\alpha)] = -\frac{1}{2} \sum_{\beta, \gamma \in \Delta_{>0}} (-1)^{p(\alpha) p(\beta) c_{\beta, \gamma}^\alpha \varphi^\beta \varphi^\gamma; \quad [d_{ne}(\tilde{\Phi}_\alpha)] = \sum_{\beta \in \Delta_{>0}} \chi([e_{\beta}, e_{\alpha}]) \varphi^\beta;$$

$$[d_{ne}(\varphi^\alpha)] = [d_{\chi}(\tilde{\Phi}_\alpha)] = [d_{\chi}(\tilde{\Phi}_\alpha)] = 0;$$

where

$$\tilde{\tau}_u(v) = (u|v) + \frac{\epsilon}{2} (\kappa_g(u|v) - \kappa_{g_0}(u|v) - 2h^\vee(u|v));$$

$$\tilde{a}_c(u|v) = (u|v) + \epsilon \{\text{str}_b((\text{ad} u)p_+ (\text{ad} v)) - h^\vee(u|v)\};$$

$$\tilde{b}_c(u|v) = (u|v) + \epsilon \{\text{str}_b(p_+ (\text{ad} u)(\text{ad} v)) - h^\vee(u|v)\}.$$
C[y] is a cochain complex over C[y] with the differential \( d_{(0)} \) by Lemma 2.1. Define a vertex superalgebra \( \hat{C}_\epsilon \) by

\[ \hat{C}_\epsilon = \hat{C}_y \otimes \mathbb{C}_\epsilon \]

for \( \epsilon \in \mathbb{C} \), where \( \mathbb{C}_\epsilon \) is a one-dimensional \( \mathbb{C}[y] \)-module defined by \( y = \epsilon \) in \( \mathbb{C}_\epsilon \). The vertex superalgebra \( \hat{C}_\epsilon \) is also a complex with the differential \( d_{(0)} \). By construction,

\[ \hat{C}_\epsilon \simeq C_k \]

if \( \epsilon = (k + h\lambda)_{-1} \).

Let \( \hat{C}'_y \) be a vertex subalgebra over \( \mathbb{C}[y] \) in \( \hat{C}_y \) generated by \( \bar{J}(z) \) for all \( u \in \mathfrak{g}_{\leq 0} \) and \( \varphi^\alpha(z) \) for all \( \alpha \in \Delta_{>0} \). The vertex superalgebra \( \hat{C}'_y \) over \( \mathbb{C}[y] \) is a chain complex over \( \mathbb{C}[y] \) with the differential \( d_{st(0)} \) by Lemma 2.1 and the formula \( d_{st(0)}^2 = 0 \).

Denote \( \hat{C}'_\epsilon = \hat{C}'_y \otimes \mathbb{C}[y] \mathbb{C}_\epsilon \). Thus, \( \hat{C}'_\epsilon \otimes F(\mathfrak{g}_{k,2}) = \hat{C}_\epsilon \) and \( \hat{C}'_\epsilon \) is also a complex with the differential \( d_{st(0)} \). Let

\[ C_\infty = \hat{C}_y |_{y=0}, \quad C'_{\infty} = \hat{C}'_y |_{y=0}. \]

Then \( C_\infty \) is a complex with the differential \( d_{(0)} \). It has a supercommutative vertex superalgebra structure by Lemma 2.1 in the case of \( \epsilon = 0 \) and analogously for \( C'_{\infty} \) and \( d_{st(0)} \). The vertex superalgebra \( C_\infty \) is a Poisson vertex superalgebra through the Poisson \( \lambda \)-bracket

\[ \{A_\lambda B\} = \frac{1}{y}[A_\lambda B]|_{y=0}, \]

for \( A, B \in C_\infty \), where the \( \lambda \)-bracket is that of \( \hat{C}_y \). Then \( C_\infty \) is called the classical limit of \( C_k \) and \( H(C_\infty, d_{(0)}) \) is called the classical \( \mathcal{W} \)-algebra associated with \( \mathfrak{g}, f, k \) and \( \Gamma \). The cohomology \( H(C_\infty, d_{(0)}) \) inherits the Poisson vertex superalgebra structure from \( C_\infty \).

3. Main Results

3.1. Restricted root system and screening operators. In this section we fix some notations that will be used in the sequel. Let

\[ \Delta^{\Gamma} = \Delta \setminus \Delta_0, \]

which is a restricted root system ([15],

\[ \Pi^{\Gamma} = \{ \alpha \in \Delta_{>0} \mid \alpha \text{ is indecomposable in } \Delta_{>0} \} \]

be the base of \( \Delta^{\Gamma} \) such that the positive part of \( \Delta^{\Gamma} \) coincides with a set of positive roots in \( \Delta^{\Gamma} \) and \( Q_0 \) be the root lattice of \( \mathfrak{g}_0 \) i.e. \( Q_0 = \bigoplus_{\alpha \in \Pi_0} \mathbb{Z} \alpha \). Set \( \Pi_j^{\Gamma} = \Pi^{\Gamma} \cap \Delta_j \).

We define the equivalence relation in \( \Pi^{\Gamma} \) by

\[ \alpha \sim \beta \iff \alpha - \beta \in Q_0 \]

for \( \alpha, \beta \in \Pi^{\Gamma} \). Denote by \( [\Pi^{\Gamma}] \) the quotient set of \( \Pi^{\Gamma} \) and by \( [\alpha] \) the equivalent class of \( \alpha \in \Pi^{\Gamma} \) in \( [\Pi^{\Gamma}] \).

Lemma 3.1. \( \Pi^{\Gamma} = \Pi_j^{\Gamma} \sqcup \Pi_{\frac{3}{2}}^{\Gamma} \).
Proof. If there exists \( \alpha \in \Pi^+_j \) for \( j > 1 \), the positive root \( \alpha \) is not simple because
\[
\Pi = \Pi_0 \sqcup \Pi_1 \sqcup \Pi_1
\]
as is shown in [EK] [Ho]. Thus, \( \alpha \) is decomposed to two positive roots in \( \Delta_{>0} \), which contradicts that \( \alpha \in \Pi^+_1 \).

Lemma 3.1 induces the decomposition
\[
[\Pi^1] = [\Pi^1_1] \sqcup [\Pi^1_2].
\]

Lemma 3.2. Let
\[
\mathbb{C}^{[\beta]} = \bigoplus_{\alpha \in [\beta]} \mathbb{C} x_\alpha
\]
be the \( g_0 \)-module defined by
\[
u \cdot x_\alpha = \sum_{\gamma \in [\beta]} c^\alpha_{\gamma,u} x_\gamma
\]
for \( u \in g_0 \), where the complex number \( c^\alpha_{\gamma,u} \) is defined by \( [e_\gamma, u] = \sum_{\alpha \in [\beta]} c^\alpha_{\gamma,u} e_\alpha \). Then \( \mathbb{C}^{[\beta]} \) is well-defined.

Proof. We show that
\[
u \cdot v \cdot x_\alpha - (-1)^{p(u)p(v)} v \cdot u \cdot x_\alpha = [\nu, v] \cdot x_\alpha
\]
for all \( u, v \in g_0 \) and \( \alpha \in [\beta] \). Thus, it is enough to show that
\[
\sum_{\rho \in [\beta]} (c^\rho_{\gamma,u} c^\alpha_{\rho,u} - (-1)^{p(u)p(v)} c^\rho_{\gamma,u} c^\alpha_{\rho,u}) = c^\alpha_{\gamma,[u,v]}
\]
for all \( \gamma \in [\beta] \). By Jacobi identity,
\[
[e_\gamma, [u, v]] = [[e_\gamma, u], v] + (-1)^{p(u)p(v)}[[e_\gamma, v], u].
\]
This implies (3.3). Therefore the assertion follows.

Remark 3.3. Let \( H(g_+, \mathbb{C}) \) be the Lie superalgebra cohomology with the coefficients in the trivial \( g_+ \)-module \( \mathbb{C} \). Then \( H^1(g_+, \mathbb{C}) \) is naturally a \( g_0 \)-module induced by the adjoint action of \( g_0 \) on \( g_+ \), and
\[
H^1(g_+, \mathbb{C}) \simeq \bigoplus_{[\beta] \in [\Pi^1]} \mathbb{C}^{[\beta]}
\]
is the irreducible \( g_0 \)-module decomposition described in [Ko].

Denote by \( \hat{g}_0 = g_0[t, t^{-1}] \oplus \mathbb{C} K \) the central extension of the loop superalgebra \( g_0[t, t^{-1}] \) via the even supersymmetric invariant bilinear form \( \tau_K \) defined by (2.2). We extend the \( g_0 \)-module \( \mathbb{C}^{[\beta]} \) to a \( g_0[t] \oplus \mathbb{C} K \)-module by \( g_0[t]t = 0 \) and \( K = 1 \) on \( \mathbb{C}^{[\beta]} \). Define a \( \hat{g}_0 \)-module \( M_{[\beta]} \) by
\[
M_{[\beta]} = \text{Ind}^{\hat{g}_0}_{g_0[t] \oplus \mathbb{C} K} \mathbb{C}^{[\beta]} = U(\hat{g}_0) \otimes_{U(g_0[t] \oplus \mathbb{C} K)} \mathbb{C}^{[\beta]}
\]
\[
\simeq V^{\tau_K}(g_0) \otimes \bigoplus_{\alpha \in [\beta]} \mathbb{C} x_\alpha.
\]
Then \( M_{[\beta]} \) is naturally a \( V^{\tau_K}(g_0) \)-module.
Lemma 3.4. Suppose that \( k + h^\vee \neq 0 \). Let \( L(z) \) be the field on \( V^{\tau_0}(\mathfrak{g}_0) \) defined by
\[
L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} = \frac{1}{2(k + h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}_0} :J^{u_i}(z)J^{u_i}(z):,
\]
where \( \{u_i\}_{i=1}^{\dim \mathfrak{g}_0} \) is a basis of \( \mathfrak{g}_0 \) and \( \{u^i\}_{i=1}^{\dim \mathfrak{g}_0} \) is the dual basis of \( \mathfrak{g}_0 \) such that \( (u^i|u_j) = \delta_{i,j} \). Then \( L(z) \) is a Virasoro field on \( V^{\tau_0}(\mathfrak{g}_0) \).

Proof. Consider the decomposition
\[
\mathfrak{g}_0 = \bigoplus_{s=0}^{n} \mathfrak{g}'_s,
\]
where \( \mathfrak{g}'_0 \) is the center of \( \mathfrak{g}_0 \) and \( \mathfrak{g}'_s \) is a simple Lie superalgebra for \( s > 0 \). Let \( r_s \) for \( s = 1, \ldots, n \) be the complex number defined by
\[
\kappa_{\mathfrak{g}_0}(u|v) = 2r_s(u|v)
\]
for \( u, v \in \mathfrak{g}'_s \). Set \( r_0 = 0 \). Then, by definition,
\[
\tau_k(u|v) = (k + h^\vee - r_s)(u|v)
\]
for \( u, v \in \mathfrak{g}'_s \) and \( s = 0, \ldots, n \). Let \( \{u_{s,i}\}_{i=1}^{\dim \mathfrak{g}'_s} \) be a basis of \( \mathfrak{g}'_s \) and \( \{u^i\}_{i=1}^{\dim \mathfrak{g}'_s} \) be the dual basis of \( \mathfrak{g}'_s \) so that \( (u^i|u_{s,j}) = \delta_{i,j} \). By the Sugawara construction, we have a Virasoro field on \( V^{\tau_0}(\mathfrak{g}_0) \) defined by
\[
L(z) = \frac{1}{2(k + h^\vee)} \sum_{s=0}^{n} \sum_{i=1}^{\dim \mathfrak{g}'_s} :J^{u^i_s}(z)J^{u_{s,i}}(z):.
\]
Therefore the assertion follows.

Remark 3.5. There exists the canonical isomorphism
\[
V^{\tau_0}(\mathfrak{g}_0) = \bigotimes_{s=0}^{n} V^{k + h^\vee - r_s}(\mathfrak{g}'_s),
\]
where we follow the notation in the proof of Lemma 3.4.

Lemma 3.6.
\[
L_{-1}x_\alpha = -\frac{(k + h^\vee)^{-1}}{\epsilon(\alpha | \epsilon - \alpha)} \sum_{\beta \in \Delta_0} (-1)^{p(\beta)p(\gamma)} c_{\beta, \gamma} J^{\tau^\gamma_{-1}}x_\beta.
\]

Proof. Direct calculations.

We introduce a formal power series
\[
S^\alpha(z) = \sum_{n \in \mathbb{Z}} S_n^\alpha z^{-n}
\]
for \( \alpha \in \Pi^F \), where its coefficients are the operators
\[
S_n^\alpha : V^{\tau_0}(\mathfrak{g}_0) \rightarrow M_{[\alpha]}
\]
defined by
\[
S_n^\alpha(z)A := (-1)^{p(\alpha)p(A) + p(A)} e^{zL_{-1}}Y(A, -z) x_\alpha
\]
for all \( A \in V^{\tau_0}(\mathfrak{g}_0) \), where \( Y(A, z) \) is the vertex operator of \( A \) on \( M_{[\alpha]} \). The formal power series \( S^\alpha(z) \) satisfies the following formulas for generic \( k \) (see Definition 4.3).
Proposition 3.7. If \( k \) is generic,
\[
S_0^\alpha |0\rangle = x_\alpha, \quad S_n^\alpha |0\rangle = 0 \quad \text{for all } n \geq 1,
\]
\[
[J^u(m), S_n^\alpha] = \sum_{\beta \in [\alpha]} c_{\beta,u}^\alpha S_{m+n}^\beta,
\]
\[
\partial S^\alpha(z) = \frac{(k + h^\vee - 1)}{\left(e_\alpha | e_{-\alpha}\right)} \sum_{\beta \in [\alpha]} (-1)^{p(\beta)p(\gamma)} c_{\beta,-\alpha}^\gamma J^{\gamma}(z) S^\beta(z):
\]
for all \( u \in g_0 \) and \( \alpha \in \Pi^\Gamma \).

Proposition 3.7 will be proved in Section 4.3.

We define the screening operators
\[
Q_{[\beta]} : V^{\tau_k}(g_0) \otimes F(g_\frac{1}{2}) \to M_{[\beta]} \otimes F(g_\frac{1}{2})
\]
for \([\beta] \in [\Pi^\Gamma]\) by
\[
Q_{[\beta]} = \sum_{\alpha \in [\beta]} \int S^\alpha(z) \Phi_\alpha(z) : dz \quad \text{if } [\beta] \in [\Pi^\Gamma_1],
\]
\[
Q_{[\beta]} = \sum_{\alpha \in [\beta]} \chi(e_\alpha) \int S^\alpha(z) dz \quad \text{if } [\beta] \in [\Pi^\Gamma_2],
\]
where we have set
\[
\int a(z) \, dz = a(0)
\]
for a formal power series \( a(z) = \sum_{n \in \mathbb{Z}} a(n) z^{-n-1} \).

3.2. Main Theorems.

Theorem 3.8. If \( k \) is generic, \( W^k(g, f; \Gamma) \) may be described as a vertex subalgebra in \( V^{\tau_k}(g_0) \otimes F(g_\frac{1}{2}) \), which is the intersection of kernels of the screening operators:
\[
W^k(g, f; \Gamma) \simeq \bigcap_{[\beta] \in [\Pi^\Gamma]} \text{Ker } Q_{[\beta]}.
\]

Here the screening operators
\[
Q_{[\beta]} : V^{\tau_k}(g_0) \otimes F(g_\frac{1}{2}) \to M_{[\beta]} \otimes F(g_\frac{1}{2})
\]
for \([\beta] \in [\Pi^\Gamma]\) are defined by (3.7) and (3.8).

The proof of Theorem 3.8 is given in Section 4.

Suppose that \( g_0 = h \). Let \( \mathcal{H} \) be the Heisenberg vertex algebra associated with \( h^* \) and its non-degenerate bilinear form \( (\cdot | \cdot) \) induced by \( h \), \( \mathcal{H}_\beta \) be the \( \mathcal{H} \)-module with the highest weight \( \beta \in h^* \) and \( e^{f(\beta(z))} \) be the formal power series whose coefficients are operators from \( \mathcal{H} \) to \( \mathcal{H}_\beta \) defined by
\[
e^{f(\beta(z))} = s_\beta z^{\beta(0)} \exp(-\sum_{n<0} \frac{\beta(n)}{n} z^{-n}) \exp(-\sum_{n>0} \frac{\beta(n)}{n} z^{-n}),
\]
where \( s_\beta \) is the shift operator determined by the following formulas
\[
s_\beta |0\rangle = |\beta\rangle, \quad [s_\beta, e^\alpha] = 0 \quad \text{for all } n \neq 0
\]
for \( \alpha \in h^* \), where \( |\beta\rangle \) is the highest weight vector of \( \mathcal{H}_\beta \).
Theorem 3.9. If $k$ is generic and $\mathfrak{g}_0 = \mathfrak{h}$, then $\mathcal{W}^k(\mathfrak{g}, f)$ may be described as a vertex subalgebra in $\mathcal{H} \otimes F(\mathfrak{g}_0)$ as follows:

$$
\mathcal{W}^k(\mathfrak{g}, f) \simeq \bigcap_{\alpha \in \Pi_2} \operatorname{Ker} \int :e^{-\sqrt{k}} f_\alpha(z)\Phi_\alpha(z) :dz \cap \bigcap_{\alpha \in \Pi_1} \operatorname{Ker} \int e^{-\sqrt{k}} f_\alpha(z) dz,
$$

where $\nu = \sqrt{k + h^\vee}$.

Proof. If $k$ is generic and $\mathfrak{g}_0 = \mathfrak{h}$, then $\Delta_{> 0} = \Delta_+$ and $\Pi_\Gamma = \Pi$. Since $Q_0$ is empty, the equivalence relation in $\Pi_\Gamma$ defined by (3.2) is trivial. Therefore

$$
\mathcal{W}^k(\mathfrak{g}, f) \simeq \bigcap_{\alpha \in \Pi_2} \operatorname{Ker} \int :S^\alpha(z)\Phi_\alpha(z) :dz \cap \bigcap_{\alpha \in \Pi_1} \operatorname{Ker} \chi(e_\alpha) \int S^\alpha(z) dz
$$

$$
\simeq \bigcap_{\alpha \in \Pi_2} \operatorname{Ker} \int :S^\alpha(z)\Phi_\alpha(z) :dz \cap \bigcap_{\alpha \in \Pi_1} \operatorname{Ker} \int S^\alpha(z) dz
$$

by Theorem 3.8. The vertex algebra $V^{\tau_\alpha}(\mathfrak{g}_0) = V^{\tau_\alpha}(\mathfrak{h})$ is generated by $J^u(z)$ for all $u \in \mathfrak{h}$ satisfying

$$
[J^u, J^v] = (k + h^\vee)(u|v)\lambda
$$

for $u, v \in \mathfrak{h}$ since

$$
\tau_k(u|v) = k(u|v) + \frac{1}{2} \kappa_\mathfrak{g}(u|v) - \frac{1}{2} \kappa_{\mathfrak{g}_0}(u|v)
$$

$$
= k(u|v) + \frac{1}{2} \kappa_\mathfrak{g}(u|v)
$$

$$
= (k + h^\vee)(u|v).
$$

We identify $\mathfrak{h}$ with $\mathfrak{h}^*$ via the non-degenerate bilinear form $(\cdot | \cdot)$, that is, $\mathfrak{h}^* \ni \beta \mapsto t_\beta \in \mathfrak{h}$ such that $(t_\beta | h) = \beta(h)$ for all $h \in \mathfrak{h}$. Denote

$$
\beta(z) = \frac{1}{\nu} f^\beta (z)
$$

for $\beta \in \mathfrak{h}^*$, where $\nu = \sqrt{k + h^\vee}$. Then

$$
[\beta | \gamma] = (\beta | \gamma) \lambda
$$

for $\beta, \gamma \in \mathfrak{h}^*$. Hence $V^{\tau_\alpha}(\mathfrak{h})$ is the Heisenberg vertex algebra $\mathcal{H}$ associated with $\mathfrak{h}^*$.

We show

$$
(3.10) \quad S^\alpha(z) = e^{-\sqrt{k} f_\alpha(z)}
$$

for $\alpha \in \Pi$. We have

$$
[e_\alpha, e_{-\alpha}] = (e_\alpha | e_{-\alpha}) t_\alpha \quad \text{for all } \alpha \in \Delta_+.
$$

by the invariant properties of $(\cdot | \cdot)$. Using this formula and Proposition 3.7, we obtain the following relations:

$$
S^\alpha_0 |0\rangle = x_\alpha \text{ and } S^\alpha_n |0\rangle = 0 \text{ for all } n \geq 1,
$$

$$
[\beta_{(m)}, S^\alpha_n] = \frac{1}{\nu} (\beta | \alpha) S^\alpha_{m+n},
$$

$$
\partial S^\alpha(z) = -\frac{1}{\nu} :\alpha(z) S^\alpha(z) :.
$$
for $\alpha \in \Pi$ and $\beta \in \mathfrak{h}^*$. This implies (3.10). Therefore the assertion follows. \hfill \Box

**Remark 3.10.** For $\mathfrak{g} = \mathfrak{sl}_n$, there is a one-to-one correspondence between nilpotent orbits in $\mathfrak{sl}_n$ and partitions of $n$. If $f$ is a nilpotent element corresponding to the partition $(s, t)$ with $s + t = n, s > t$, there exists a good grading for $f$ such that $(\mathfrak{sl}_n)_0 = \mathfrak{h}$ (see [EK]).

4. **Proof of Theorem 3.8**

4.1. **Weight filtration.** We define a degree map $\text{deg} : C_k \to \mathbb{Z}$ by
\[
\text{deg } J^u = -2j \quad \text{for all } u \in \mathfrak{g}_j \ (j \leq 0),
\]
\[
\text{deg } \varphi^\alpha = 2j \quad \text{for all } \alpha \in \Delta_j \ (j > 0),
\]
\[
\text{deg } \Phi_\alpha = 0 \quad \text{for all } \alpha \in \Delta^\beta,
\]
\[
\text{deg } (0) = 0, \quad \text{deg } \partial A = \text{deg } A \quad \text{for all } A \in C_k,
\]
\[
\text{deg } : AB := \text{deg } A + \text{deg } B \quad \text{for all } A, B \in C_k.
\]

Let
\[
F_pC_k = \{ A \in C_k \mid \text{deg } A \geq p \}
\]
and $F_*C_k = \{ F_pC_k \}_{p \geq 0}$. Since
\[
d_{(0)}(F_pC_k) \subset F_pC_k
\]
for all $p \geq 0$, $F_*C_k$ is a filtration of a complex $C_k$. Thus, the filtration $F_*C_k$ gives a spectral sequence $\{ E_n \}_{n \geq 0}$. We call $F_*C_k$ a **weight filtration** on $C_k$. Define the conformal weight of $C_k$ inherited from $C^G(\mathfrak{g}, f; \Gamma)$, that is,
\[
\Delta(J^u) = 1 - j \quad \text{for all } u \in \mathfrak{g}_j \ (j \leq 0),
\]
\[
\Delta(\varphi^\alpha) = j \quad \text{for all } \alpha \in \Delta_j \ (j > 0),
\]
\[
\Delta(\Phi_\alpha) = \frac{1}{2} \quad \text{for all } \alpha \in \Delta^\beta,
\]
\[
\Delta(0) = 0, \quad \Delta(\partial A) = \Delta(A) + 1 \quad \text{for all } A \in C_k,
\]
\[
\Delta(AB) = \Delta(A) + \Delta(B) \quad \text{for all } A, B \in C_k,
\]
where we denote by $\Delta(A) \in \frac{1}{2}\mathbb{Z}$ the conformal weight of $A \in C_k$. Let $C_k(n)$ be the homogeneous conformal weight space with the conformal weight $n$ in $C_k$. We have
\[
(4.1) \quad C_k = \bigoplus_{n \geq 0} C_k(n).
\]
Since $d_{(0)}$ preserves each homogeneous conformal weight spaces, $C_k(n)$ is a subcomplex of $C_k$. Hence the filtration $F_*C_k$ induces a filtration and a spectral sequence on the subcomplex $C_k(n)$. Since $\dim C_k(n) < \infty$, the spectral sequence on $C_k(n)$ converges for all $n \geq 0$ so that the total spectral sequence $\{ E_n \}_{n \geq 0}$ also converges. Denote by $d_r$ the differential of $E_r$ and
\[
d^{(i)}_r : E^{(i)}_{r-1} \to E^{(i+1)}_{r-1} \quad \text{for } i \geq 0.
\]
By construction,
\[
d_0 = d_{\text{st}(0)}, \quad d_1 = d_{\text{st}(0)}, \quad d_2 = d_{\chi(0)}
\]
and $d_r = 0$ for $r \geq 3$.

**Lemma 4.1.** $E_1 = H(C_k', d_{\text{st}(0)}) \otimes F(\mathfrak{h}^*)$. 

Proof. The assertion follows from the fact that the differential $d_{st(0)}$ acts only on the subcomplex $C'_h$ and $\{\}$.

4.2. The first spectral sequence. Suppose that $V$ is a super space. Then we denote by $S(V)$ the supersymmetric algebra of $V$, which is the tensor algebra of $V$ quotient by the two-sided ideal generated by $ab-(-1)^{|a||b|}ba$ for all $a,b \in V$.

Let $S(g_0[t^{-1}]t^{-1})$ be the supersymmetric algebra of $g_0[t^{-1}]t^{-1}$ and $H(g_+,-)$ be the Lie superalgebra cohomology with the coefficients in the trivial $g_+$-module $\mathbb{C}$. The Chevalley complex of $H(g_+,-)$ is the superexterior algebra $\Lambda(g_+)$. We denote by $J^u(-n)$ the element of $S(g_0[t^{-1}]t^{-1})$ corresponding to $ut^{-n} \in g_0[t^{-1}]t^{-1}$ and by $\varphi^\alpha(0)$ the element of $\Lambda(g_+)$ corresponding to $e_\alpha \in g_+$ with reversed parity for $\alpha \in \Delta_{>0}$. We can regard $S(g_0[t^{-1}]t^{-1})$ and $\Lambda(g_+)$ as subalgebras of the supercommutative algebra $C'_{\infty}$ by

$$S(g_0[t^{-1}]t^{-1}) \ni J^{u_1}(-n_1) \cdots J^{u_s}(-n_s) \mapsto \prod_{i=1}^s J^{u_i}(-n_i) |0\rangle \in C'_{\infty},$$

$$\Lambda(g_+) \ni \varphi_{\beta_1}(0) \cdots \varphi_{\beta_s}(0) \mapsto \prod_{i=1}^s \varphi_{\beta_i} |0\rangle \in C'_{\infty}$$

for $u_i \in g_0$ and $\beta_i \in \Delta_{>0}$.

Lemma 4.2. $H(C'_{\infty},d_{st(0)}) \simeq S(g_0[t^{-1}]t^{-1}) \otimes H(g_+,-)$.

The proof of Lemma 4.2 will be given in Section 4.5.

Corollary 4.3. $H(\tilde{C}'_y,d_{st(0)}) \simeq H(C'_{\infty},d_{st(0)}) \otimes \mathbb{C}[y]$.

Proof. Let

$$F_p\tilde{C}'_y = y^p\tilde{C}'_y$$

for all $p \geq 0$. Since

$$d_{st(0)}(F_p\tilde{C}'_y) \subset F_p\tilde{C}'_y$$

for all $p \geq 0$, $\{F_p\tilde{C}'_y\}_{p \geq 0}$ is a filtration of the complex $\tilde{C}'_y$ over $\mathbb{C}[y]$ and gives a spectral sequence $\{E_{y,n}\}_{n \geq 0}$. We extend the conformal weight on $C'_{\infty}$ to $\tilde{C}'_y$ by $\Delta(y) = 0$ and get the homogeneous conformal weight decomposition of $\tilde{C}'_y$. Since each of the homogeneous spaces is a free $\mathbb{C}[y]$-module of finite rank, the induced spectral sequence on it converges so that the total spectral sequence $\{E_{y,n}\}_{n \geq 0}$ also converges. Denote by $d_{y,r}$ the differential of $E_{y,r}$ for $r \geq 0$. It is straightforward that

$$E_{y,1} = H(E_{y,0},d_{y,0}) = H(C'_{\infty}) \otimes \mathbb{C}[y].$$

By Lemma 4.2,

$$H(C'_{\infty}) \simeq S(g_0[t^{-1}]t^{-1}) \otimes H(g_+,-)$$

and this implies

$$E_{y,1} = V^{\tau_\gamma}(g_0) \otimes_{\mathbb{C}[y]} H(g_+,-),$$

where $V^{\tau_\gamma}(g_0)$ is a vertex subalgebra over $\mathbb{C}[y]$ in $\tilde{C}'_y$ generated by $J^u(z)$ for all $u \in g_0$ and $H(g_+,-)$ is the Lie superalgebra cohomology with the coefficients in the trivial $g_+$-module $\mathbb{C}[y]$ whose complex is identified with a subspace in $\tilde{C}'_y$ spanned by all the elements

$$\varphi_0^{\beta_1} \cdots \varphi_0^{\beta_s} |0\rangle$$
with $\beta_i \in \Delta_{>0}$ and $s \geq 0$ over $\mathbb{C}[y]$. Since
\[ d_{st(0)}V^{\tau y}(g_0) = d_{st(0)}H(g_+, \mathbb{C}[y]) = 0, \]
d_{y,r} = 0 for all $r \geq 2$ and then the spectral sequence collapses at $E_{y,1} = E_{y,\infty}$. Therefore the assertion follows from (4.2). \hfill \Box

**Lemma 4.4.** The set
\[ S = \{ k \in \mathbb{C} \mid H(C'_k, d_{st(0)}) \simeq H(C'_\infty, d_{st(0)}) \} \]
is Zariski dense in $\mathbb{C}$.

**Proof.** By Corollary 4.3
\[ H(\hat{C}'_y, d_{st(0)}) \otimes \mathbb{C}_\epsilon \simeq H(C'_\infty, d_{st(0)}). \]
Suppose that $k \in \mathbb{C}\setminus\{-h^\vee\}$. If
\[ H(\hat{C}'_y \otimes \mathbb{C}_\epsilon, d_{st(0)}) = H(\hat{C}'_y, d_{st(0)}) \otimes \mathbb{C}_\epsilon \]
holds for $\epsilon = (k + h^\vee)^{-1}$, it follows that $k \in S$ because
\[ C'_k = \hat{C}'_\epsilon = \hat{C}'_y \otimes \mathbb{C}_\epsilon. \]
Furthermore (4.3) holds for $\epsilon = 0$ because
\[ C'_\infty = \hat{C}'_y \otimes \mathbb{C}_0. \]
Therefore
\[ H(\hat{C}'_y(n) \otimes \mathbb{C}_\epsilon, d_{st(0)}) = H(\hat{C}'_y(n), d_{st(0)}) \otimes \mathbb{C}_\epsilon \]
holds for $\epsilon = 0$ and all $n \in \frac{1}{2}\mathbb{Z}_{\geq 0}$, where $\hat{C}'_y(n)$ is the homogeneous conformal weight space of conformal weight $n$ in $\hat{C}'_y$. Since $\hat{C}'_y(n)$ is a free $\mathbb{C}[y]$-module of finite rank, the upper semi-continuous theorem (cf. [Ha]) in a neighborhood of $y = 0$ implies that (4.3) holds for some neighborhood $U(\epsilon)$ of $\epsilon = 0$ in $\mathbb{C}$. The complement $\mathbb{C}\setminus U(\epsilon)$ is a finite set because $U(\epsilon)$ is a non-empty Zariski open set in $\mathbb{C}$. Hence
\[ \mathbb{C} \setminus \bigcap_{n \geq 0} U(n) = \bigcup_{n \geq 0} (\mathbb{C}\setminus U(n)) \]
is a countable set. Therefore a set
\[ S' = \{ k \in \mathbb{C}\setminus\{-h^\vee\} \mid (k + h^\vee)^{-1} \in \bigcap_{n \geq 0} U(n) \} \]
is dense in $\mathbb{C}$. The assertion follows from the fact $S' \subset S$ because (4.3) holds for $k \in S'$.

**Definition 4.5.** $k \in \mathbb{C}$ is called generic if $H(C'_k, d_{st(0)}) \simeq H(C'_\infty, d_{st(0)})$.

If $k$ is generic, Lemma 4.2 implies that
\[ H(C'_k, d_{st(0)}) \simeq V^{\tau_k}(g_0) \otimes H(g_+, \mathbb{C}), \]
where the complex $\Lambda(g_+) \otimes H(g_+, \mathbb{C})$ is identified with the subspace of $C_k^I$ spanned by

$$\varphi^{\beta_1}_{0} \cdots \varphi^{\beta_s}_{0} |0\rangle$$

for all $\beta_i \in \Delta_{>,0}$ and $s \geq 0$.

**Proposition 4.6.** If $k$ is generic,

$$E_1 \cong V^{\tau_k}(g_0) \otimes H(g_+, \mathbb{C}) \otimes F(g_{12})$$

**Proof.** By Lemma 4.1,

$$E_1 = H(C^I_k, d_{st}(0)) \otimes F(g_{12}).$$

The assertion immediately follows from (4.5). □

Recall that $\Pi^\Gamma$ is the set of all indecomposable roots in $\Delta_{>,0}$ defined in (3.1).

**Lemma 4.7.** Let $\alpha \in \Pi^\Gamma$. Then

$$d_{st}(0) J^{\tau_\alpha} = - \sum_{\substack{\beta \in [\alpha] \\
\gamma \in I \cup \Delta_{\leq 0}}} (-1)^{p(\gamma)} c^{\gamma}_{\alpha,\beta} \cdot J^{\tau_\gamma} \varphi^\beta : + (e_{-\alpha}|e_\alpha)(k + h^\vee) \partial \varphi^\alpha$$

**Proof.** Let $\alpha \in \Pi^\Gamma$. We have

$$d_{st}(0) J^{\tau_\alpha} = - \sum_{\substack{\beta \in \Delta_{>,0} \\
\gamma \in I \cup \Delta_{\leq 0}}} (-1)^{p(\gamma)} c^{\gamma}_{-\alpha,\beta} \cdot J^{\tau_\gamma} \varphi^\beta : + a_k(e_{-\alpha}|e_\alpha) \partial \varphi^\alpha,$$

where

$$a_k(e_{-\alpha}|e_\alpha) = \text{str}_g((\text{ad } e_{-\alpha}) p_+ (\text{ad } e_\alpha)) + k(e_{-\alpha}|e_\alpha).$$

The indecomposability of $\alpha$ implies that $\beta \in [\alpha]$ if $c^{\gamma}_{-\alpha,\beta} \neq 0$ for $\beta \in \Delta_{>,0}$ and $\gamma \in I \cup \Delta_{\leq 0}$. It is enough to show that

(4.6) $$a_k(e_{-\alpha}|e_\alpha) = (k + h^\vee)(e_{-\alpha}|e_\alpha).$$

First,

$$\kappa_g(e_{-\alpha}|e_\alpha) = \text{str}_g((\text{ad } e_{-\alpha}) p_+ (\text{ad } e_\alpha)) + \text{str}_g((\text{ad } e_{-\alpha}) p_{\leq 0} (\text{ad } e_\alpha)),$$

where $p_{\leq 0}: g \rightarrow g_{\leq 0}$ is the projection map. Next, we have

$$\text{str}_g((\text{ad } e_{-\alpha}) p_+ (\text{ad } e_\alpha)) = \text{str}_g((\text{ad } e_{-\alpha}) p_{\leq 0} (\text{ad } e_\alpha))$$

by the indecomposability of $\alpha$ and the invariant property of $(\cdot | \cdot)$. Thus,

$$\text{str}_g((\text{ad } e_{-\alpha}) p_+ (\text{ad } e_\alpha)) = \frac{1}{2} \kappa_g(e_{-\alpha}|e_\alpha) = h^\vee (e_{-\alpha}|e_\alpha).$$

Therefore the formula (4.6) holds. This completes the proof. □
4.3. The vertex superalgebra structure of $E_1$. Notice that the cohomology 
$H(C_k', d_{st(0)})$ inherits a vertex superalgebra structure from $C_k'$. Thus,

$$E_1 = H(C_k', d_{st(0)}) \otimes F(g_+^\bullet)$$

is a vertex superalgebra. Assume that $k$ is generic. By (4.5),

$$H(C_k', d_{st(0)}) \simeq V^{\tau_k}(g_0) \otimes H(g_+, \mathbb{C}).$$

Recall that we denote by $\varphi^\alpha(0)$ the element of the complex $\Lambda(g_+)$ of $H(g_+, \mathbb{C})$ corresponding to $e_\alpha \in g_0$ with reversed parity for $\alpha \in \Delta_{>0}$.

**Lemma 4.8.**

$$H^0(g_+, \mathbb{C}) = \mathbb{C}, \quad H^1(g_+, \mathbb{C}) = \bigoplus_{\alpha \in \Pi^F} \mathbb{C} \psi_\alpha,$$

where $\psi_\alpha$ is the non-zero cohomology class of $\varphi^\alpha(0)$ in $H^1(g_+, \mathbb{C})$.

**Proof.** Denote by $d_+$ be the differential of the complex $\Lambda(g_+)$ of $H(g_+, \mathbb{C})$. Since

$$\Lambda^0(g_+) = \mathbb{C}, \quad d_+ \cdot 1 = 0,$$

we obtain the first formula. Moreover

$$\Lambda^1(g_+) = \bigoplus_{\alpha \in \Delta_{>0}} \mathbb{C} \varphi^\alpha(0)$$

and

$$d_+ \cdot \varphi^\alpha(0) = -\frac{1}{2} \sum_{\beta, \gamma \in \Delta_{>0}} (-1)^{p(\alpha)p(\beta)} c_{\beta, \gamma}^{\alpha \beta} \varphi^\beta(0) \varphi^\gamma(0)$$

$$= - \sum_{\beta < \gamma \in \Delta_{>0}, \alpha = \beta + \gamma} (-1)^{p(\alpha)p(\beta)} c_{\beta, \gamma}^{\alpha \beta} \varphi^\beta(0) \varphi^\gamma(0) - \frac{1}{2} \sum_{\beta \in \Delta_{>0}, \alpha = \beta} c_{\beta, \gamma}^{\alpha \beta} \varphi^\beta(0)^2,$$

where $\prec$ is any total order in $\Delta_{>0}$. Therefore $d_+ \cdot \varphi^\alpha(0) = 0$ if and only if $\alpha$ is indecomposable in $\Delta_{>0}$. This implies the second formula. \qed

Here,

$$H^0(C_k', d_{st(0)}) \simeq V^{\tau_k}(g_0),$$

$$H^1(C_k', d_{st(0)}) \simeq V^{\tau_k}(g_0) \otimes \bigoplus_{\alpha \in \Pi^F} \mathbb{C} \psi_\alpha$$

by Lemma 4.8. The first formula is also a vertex superalgebra isomorphism. Let

$$\tilde{S}^\alpha(z) = \sum_{n \in \mathbb{Z}} \tilde{S}_n^\alpha z^{-n} = Y(\psi_\alpha, z)$$

be the vertex operator of $\psi_\alpha$ on $H(C_k', d_{st(0)})$ for $\alpha \in \Pi^F$.

**Lemma 4.9.** The following formulas hold:

(4.7) $\tilde{S}_0^\alpha |0\rangle = \psi_\alpha, \quad \tilde{S}_n^\alpha |0\rangle = 0$ for $n \geq 1$,

(4.8) $[J^\nu, \tilde{S}^\alpha] = \sum_{\beta \in [\alpha]} c_{\beta, \alpha}^{\alpha} \tilde{S}^\beta$,

(4.9) $\partial \tilde{S}^\alpha(z) = -\frac{(k+\hbar^\gamma)^{-1}}{e_\alpha e_{-\alpha}} \sum_{\beta \in [\alpha], \gamma \in \Pi \cup \Delta_0} (-1)^{p(\beta)p(\gamma)} c_{\beta, -\alpha}^{\gamma} J^\epsilon_\gamma(z) \tilde{S}^\beta(z)$.
for \( \alpha \in \Pi^\Gamma \) and \( u \in g_0 \).

**Proof.** The following formulas hold in \( C'_k \):

\[
\varphi_\alpha^0 |0\rangle = \varphi_\alpha(0), \quad \varphi_\alpha^n |0\rangle = 0 \quad \text{for } n \geq 1, \\
[J^u_\alpha \varphi_\alpha] = \sum_{\beta \in [\alpha]} c^\alpha_{\beta, u} \varphi_\beta
\]

for \( \alpha \in \Pi^\Gamma \) and \( u \in g_0 \). These formulas imply (4.7) and (4.8). Moreover

\[
d_{st(0)} J^{e - \alpha} \equiv 0
\]

for all \( \alpha \in \Pi^\Gamma \) in \( H(C'_k, d_{st(0)}) \). Since

\[
d_{st(0)} J^{e - \alpha} = - \sum_{\beta \in [\alpha]} (-1)^{p(\gamma)} c^\gamma_{-\alpha, \beta} : J^{e - \gamma} \varphi_\beta \equiv + (e_{-\alpha} | e_\alpha)(k + h^\gamma) \partial \varphi_\alpha
\]

for all \( \alpha \in \Pi^\Gamma \) by Lemma 4.9, the formula \( d_{st(0)} J^{e - \alpha} \equiv 0 \) holds if and only if

\[
\partial \varphi_\alpha(z) \equiv - \frac{(k + h^\gamma)^{-1}}{(e_\alpha | e_{-\alpha})} \sum_{\beta \in [\alpha]} (-1)^{p(\beta)p(\gamma)} c^\gamma_{\beta, -\alpha} : J^{e - \gamma}(z) \varphi_\beta(z) :
\]

This implies (4.3) and completes the proof. \( \square \)

By Lemma 4.10, \( H^1(C'_k, d_{st(0)}) \) is a \( V^{\tau_\kappa}(g_0) \)-module and the operator \( \tilde{S}^\alpha_n \) restricts to the maps

\[
\tilde{S}^\alpha_n : V^{\tau_\kappa}(g_0) \rightarrow V^{\tau_\kappa}(g_0) \otimes \bigoplus_{\beta \in [\alpha]} \mathbb{C} \psi_\beta
\]

for all \( \alpha \in \Pi^\Gamma \) and \( n \in \mathbb{Z} \). Recall that \( M_{[\alpha]} \) is a \( V^{\tau_\kappa}(g_0) \)-module defined by (3.4).

**Lemma 4.10.** Let

\[
H^1(C'_k, d_{st(0)}) \simeq \bigoplus_{[\alpha] \in [\Pi^\Gamma]} M_{[\alpha]}
\]

be the vector space isomorphism defined by \( A \psi_{\alpha} \mapsto Ax_{\alpha} \) for all \( A \in V^{\tau_\kappa}(g_0) \) and \( \alpha \in \Pi^\Gamma \). Then this is also a \( V^{\tau_\kappa}(g_0) \)-module isomorphism.

**Proof.** By (4.8),

\[
[J^u_\alpha, \tilde{S}^\alpha_m] = \sum_{\beta \in [\alpha]} c^\alpha_{\beta, u} \tilde{S}^\beta_{n+m}
\]

for \( u \in g_0, \alpha \in \Pi^\Gamma, m, n \in \mathbb{Z} \). Therefore

\[
J^u_{(0)} \psi_{\alpha} = \sum_{\beta \in [\alpha]} c^\alpha_{\beta, u} \psi_{\beta}, \quad J^u_{(n)} \psi_{\alpha} = 0 \quad \text{for } n \geq 1
\]

by (4.7). This implies that (4.10) is also a \( V^{\tau_\kappa}(g_0) \)-module isomorphism. \( \square \)

By Lemma 4.10 we can regard the operators \( \tilde{S}^\alpha_n \) as the maps

\[
\tilde{S}^\alpha_n : V^{\tau_\kappa}(g_0) \rightarrow M_{[\alpha]}
\]

Recall that \( S^\alpha(z) \) is the formal power series whose coefficients are the operators from \( V^{\tau_\kappa}(g_0) \) to \( M_{[\alpha]} \) defined in (3.3).
Proposition 4.11.

\[ S^α(z) = \tilde{S}^α(z) \]

as the operator from \( V^γ(\mathfrak{g}_0) \) to \( M_{[α]}((z)) \) for all \( α \in Π^Γ \).

Proof. By the skew-symmetry in \( H(C'_k, d_{st}(0)) \),

\[ \tilde{S}^α(z)A = (-1)^{p(α)p(A) + p(A)} e^z Y(A, -z)ψ_α \]

for all \( A \in V^γ(\mathfrak{g}_0) \), where \( Y(A, z) \) is the vertex operator on \( H(C'_k, d_{st}(0)) \) and \( T \) is the translation operator. All we need to show is that the actions of \( L_{-1} \) and \( T \) on \( M_{[α]} \) are the same via the isomorphism (4.10). However,

(4.11) \[ L_{-1}(Ax_α) = (L_{-1}A)x_α + A(L_{-1}x_α), \]

(4.12) \[ T(Aψ_α) = (TA)ψ_α + A(Tψ_α) \]

for all \( A \in V^γ(\mathfrak{g}_0) \) and

\[ L_{-1}A = TA \]

because \( L(z) \) is a Virasoro field on \( V^γ(\mathfrak{g}_0) \) by Lemma 3.4. Therefore the first terms of (4.11) and (4.12) are the same. Thus, it is enough to show that

\[ L_{-1}x_α = Tψ_α \]

via the isomorphism (4.10). By (4.9),

\[ Tψ_α = -\frac{(k + h^γ)}{2} \sum_{β \in Π^Γ \setminus Δ_α} \frac{(-1)^{p(β)p(γ)} c^γ_β}{e^{β|e_β}} J^{γ}_β ψ_β. \]

This coincides with \( L_{-1}x_α \) by Lemma 3.6. So the proof is completed. ∎

Proof of Proposition 3.7. By Proposition 4.11

\[ S^α(z) = \tilde{S}^α(z). \]

Hence the assertion follows from Lemma 4.9. ∎

4.4. Proof of Theorem 3.8. Assume that \( k \) is generic. By Proposition 4.6,

\[ E_1 = H(C'_k, d_{st}(0)) \otimes F(\mathfrak{g}_+^2) \simeq V^γ(\mathfrak{g}_0) \otimes H(\mathfrak{g}_+^2) \otimes F(\mathfrak{g}_+^2). \]

Since \( d_1 \) is the vertex operator on \( E_1 \) induced by \( d_{ne(0)} \),

\[ d_1 = \sum_{α ∈ Π^Γ \setminus \setminus} \int : \tilde{S}^α(z) Φ_α(z) : dz. \]

Let \( \tilde{d}_2 \) be a vertex operator on \( E_1 \) defined by

\[ \tilde{d}_2 = \sum_{α ∈ Π^Γ \setminus \setminus} \chi(e_α) \int \tilde{S}^α(z) dz, \]

which is the vertex operator induced by \( d_χ(0) \).
Lemma 4.12. Assume that $k$ is generic. Let $Q$ be a vertex operator on $E_1$ defined by

$$Q = \sum_{\alpha \in \Pi_1^G} \int : S^\alpha(\zeta) \Phi_\alpha(z) : dz + \sum_{\alpha \in \Pi_1^G} \chi(\epsilon_\alpha) \int S^\alpha(\zeta) \ dz.$$ 

Then $Q^2 = 0$ and $(E_1, Q)$ is a complex with respect to the charge degrees. Moreover, $H(C_k, d_{(0)}) \simeq H(E_1, Q)$.

Proof. Recall that $Q = d_1 + \tilde{d}_2$. Since $[d_{\text{ne}(0)}, d_{\chi(0)}] = [d_{\chi(0)}, d_{\chi(0)}] = 0$,

$$Q^2 = \frac{1}{2}[Q, Q] = \frac{1}{2}[d_1 + \tilde{d}_2, d_1 + \tilde{d}_2] = \frac{1}{2}[d_1, d_1] = d_1^2 = 0.$$ 

Therefore $(E_1, Q)$ is a complex with respect to the charge degrees. The filtration on $E_1$ induced by the weight filtration on $C_k$ gives a convergent spectral sequence $\{\tilde{E}_n\}$ such that

$$E_n \simeq \tilde{E}_n$$

for all $n \geq 1$. Therefore

$$H(C_k, d_{(0)}) \simeq E_\infty \simeq \tilde{E}_\infty \simeq H(E_1, Q).$$

This completes the proof. \qed

Proof of Theorem 3.8. Recall

$$W^k(g, f; \Gamma) = H(C_k, d_{(0)}) = H^0(C_k, d_{(0)}).$$

By Lemma 4.12

(4.13) $$W^k(g, f; \Gamma) \simeq H^0(E_1, Q) = \text{Ker } Q.$$ 

Notice that

$$Q = \sum_{[\beta] \in \Pi^\Gamma} Q_{[\beta]}$$

by definition in (3.7) and (3.8), and that

$$Q_{[\beta]} : (V^\infty(g_0) \otimes F(g_0)) \subset M_{[\beta]} \otimes F(g_0)$$

for $[\beta] \in \Pi^\Gamma$. This implies the decomposition of the kernel of the operator $Q$:

$$\text{Ker } Q = \bigcap_{[\beta] \in \Pi^\Gamma} \text{Ker } Q_{[\beta]}.$$ 

The isomorphism (4.13) is the vertex superalgebra isomorphism (Lemma 5.1 below). Therefore this completes the proof. \qed
4.5. **Proof of Lemma 4.2.** Define the \( \mathfrak{g}_+[t] \) action on the affine superspace \( \text{Conn} = \partial + \mathfrak{g}_{\geq 0}[t] \) by
\[
a(t) \cdot (\partial + b(t)) = \partial + [a(t), b(t)] - \partial a(t)
\]
for \( a(t) \in \mathfrak{g}_+[t] \) and \( b(t) \in \mathfrak{g}_{\geq 0}[t] \). This action is induced by the derivation of the gauge action of the loop supergroup associated with \( \mathfrak{g}_+[t] \). This \( \mathfrak{g}_+[t] \) action on \( \text{Conn} \) gives rise to the \( \mathfrak{g}_+[t] \)-module structure on the space of the functions of \( \text{Conn} \). Identify the space of the functions of \( \text{Conn} \) with the non-degenerate bilinear form \( \langle \cdot | \cdot \rangle_{\text{res}} : \text{Conn} \times \mathfrak{g}_{\leq 0} \to \mathbb{C} \) defined by the following formula
\[
\langle \partial + a \otimes f(t) | b \otimes g(t) \rangle_{\text{res}} = (a|b) \text{ Res}_{t=0} f(t)g(t) \, dt
\]
for \( a \in \mathfrak{g}_{\geq 0}, b \in \mathfrak{g}_{\leq 0}, f(t) \in \mathbb{C}[t] \) and \( g(t) \in \mathbb{C}[t^{-1}] \). Then \( S(\mathfrak{g}_{\leq 0}[t^{-1}]) \) is a \( \mathfrak{g}_+[t] \)-module. Set
\[
u \otimes t^n = J^n(u) \in S(\mathfrak{g}_{\leq 0}[t^{-1}])
\]
for \( u \in \mathfrak{g}_{\leq 0}, n \in \mathbb{Z}_{<0} \) and
\[
a \otimes t^m = a(m) \in \mathfrak{g}_+[t]
\]
for \( a \in \mathfrak{g}_+, m \in \mathbb{Z}_{\geq 0} \). Here, \( \mathfrak{g}_+[t] \) acts on \( S(\mathfrak{g}_{\leq 0}[t^{-1}]) \) by
\[
e_{\alpha(m)} \cdot J^n(u) = -\sum_{\beta \in \Lambda^0 \Delta_{\leq 0}} (\pm)^{p(u)p(\alpha)}e^{\alpha,\beta}J^n\beta(-n + m) + n(e_{\alpha}|u)\delta_{m,n}
\]
for \( \alpha \in \Delta_{>0}, u \in \mathfrak{g}_{\leq 0}, m \geq 0 \) and \( n > 0 \), and \( e_{\alpha(m)} \) acts on \( S(\mathfrak{g}_{\leq 0}[t^{-1}]) \) as a super derivation with a parity \( p(\alpha) \). Consider the semi-infinite cohomology (cf. [Fei] [Fu]) of the \( \mathfrak{g}_+[t] \)-module \( S(\mathfrak{g}_{\leq 0}[t^{-1}]) \), denoted by
\[
H(\mathfrak{g}_+[t], S(\mathfrak{g}_{\leq 0}[t^{-1}])).
\]
The complex of \( \{4.14\} \) is
\[
S(\mathfrak{g}_{\leq 0}[t^{-1}]) \otimes \Lambda((\mathfrak{g}_+[t])^*),
\]
where \( \Lambda((\mathfrak{g}_+[t])^*) \) is the superexterior algebra of \( (\mathfrak{g}_+[t])^* \). Let \( \varphi^\alpha(-n) \) be a generator of \( \Lambda((\mathfrak{g}_+[t])^*) \) associated with the element \( (e_{\alpha} \otimes t^n)^* = (\mathfrak{g}_+[t])^* \) for \( \alpha \in \Delta_{>0} \) and \( n \in \mathbb{Z}_{\geq 0} \).

**Lemma 4.13.** \( H(\mathfrak{g}_+, S(\mathfrak{g}_{\leq 0}[t^{-1}])) = H(C_{\infty}', d_{\text{str}}(0)) \).

**Proof.** The map \( C_{\infty}' \to S(\mathfrak{g}_{\leq 0}[t^{-1}]) \otimes \Lambda((\mathfrak{g}_+[t])^*) \) defined by
\[
|0\rangle \mapsto 1, \quad J^n_{(-n)} \mapsto J^n(-n), \quad \varphi^\alpha_{-m} \mapsto \varphi^\alpha(-m)
\]
for \( u \in \mathfrak{g}_{\leq 0}, \alpha \in \Delta_{>0}, n \geq 1, m \geq 0 \) is a superalgebra isomorphism by construction. The derivation on \( C_{\infty}' \) coincides with the derivation on \( S(\mathfrak{g}_{\leq 0}[t^{-1}]) \otimes \Lambda((\mathfrak{g}_+[t])^*) \) via this map by direct calculations. Therefore the assertion follows.

**Proof of Lemma 4.2.** Denote by
\[
\mathcal{C} = S(\mathfrak{g}_{\leq 0}[t^{-1}]) \otimes \Lambda((\mathfrak{g}_+[t])^*)
\]
the complex of \( \{4.14\} \) and by \( d_{\mathcal{C}} \) the derivation of \( \mathcal{C} \). Let \( F_\mathcal{C} \) be a subspace of \( \mathcal{C} \) spanned by all the elements
\[
J^{n_1}(-n_1) \cdots J^{n_r}(-n_r) \otimes \varphi^{\beta_1}(-m_1) \cdots \varphi^{\beta_s}(-m_s)
\]
with $u_i \in \mathfrak{g}_{<0}$, $\beta_i \in \Delta_{>0}$, $n_i > 0$, $m_i \geq 0$ and $r + s \geq p$. Since

$$d_c F_p C \subset F_p C,$$

$F_p C$ is a filtration of the complex $C$ and gives the spectral sequence $\{E_{c,n}\}_{n \geq 0}$ on $C$. The conformal weight on $C$ is defined in the same way as $C'_k$. Since each of homogeneous conformal weight spaces is finite dimensional, the spectral sequence restricted to each of homogeneous conformal weight spaces converges so that $\{E_{c,n}\}_{n \geq 0}$ also converges. Set

$$E_{c,n+1} = H(E_{c,n}, d_{c,n})$$

for $n \geq 0$. We have

(4.15) $$d_{c,0} \cdot J^c_{-\alpha}(-n) = n(e_{-\alpha}|e_\alpha)\varphi^\alpha(-n),$$

(4.16) $$d_{c,0} \cdot J^c_{-\alpha}(-n) = d_{c,0}\varphi^\alpha(-m) = 0$$

for $\alpha \in \Delta_{>0}$, $u \in \mathfrak{g}_0$, $n > 0$ and $m \geq 0$. For $\alpha \in \Delta_{>0}$ and $n \geq 1$, denote by $C_{\alpha,n}$ a subalgebra of $C$ generated by $J^c_{-\alpha}(-n)$ and $\varphi^\alpha(-n)$. Let $C_0$ be a subalgebra generated by $\varphi^\alpha(0)$ for all $\alpha \in \Delta_{>0}$. Then

$$C = S(\mathfrak{g}_0[t^{-1}]t^{-1}) \otimes C_0 \otimes \bigotimes_{\alpha \in \Delta_{>0}} C_{\alpha,n}$$

is a decomposition of the complex $C$ by (4.15) (4.16). Therefore

$$H(C, d_{c,0}) = S(\mathfrak{g}_0[t^{-1}]t^{-1}) \otimes C_0 \otimes \bigotimes_{\alpha \in \Delta_{>0}} H(C_{\alpha,n}, d_{c,0})$$

since $d_{c,0} \cdot S(\mathfrak{g}_0[t^{-1}]t^{-1}) = d_{c,0} \cdot C_0 = 0$. We show that it is sufficient to prove that

(4.17) $$H(C_{\alpha,n}, d_{c,0}) = C$$

for all $\alpha \in \Delta_{>0}$ and $n \geq 1$. Then

$$E_{c,1} = H(C, d_{c,0}) = S(\mathfrak{g}_0[t^{-1}]t^{-1}) \otimes C_0.$$

Thus,

$$E_{c,2} = H(E_{c,1}, d_{c,1}) = S(\mathfrak{g}_0[t^{-1}]t^{-1}) \otimes H(C_0, d_{c,1})$$

since $d_{c,1} \cdot S(\mathfrak{g}_0[t^{-1}]t^{-1}) = 0$. We have

$$d_{c,1} \cdot \varphi^\alpha(0) = -\frac{1}{2} \sum_{\beta, \gamma \in \Delta_{>0}} (\alpha)_{\beta\gamma} c_{\alpha,\beta\gamma} \varphi^\beta(0)\varphi^\gamma(0)$$

for $\alpha \in \Delta_{>0}$. This implies

$$H(C_0, d_{c,1}) = H(\mathfrak{g}_+, \mathbb{C}).$$

Since $d_{c,r} = 0$ for $r \geq 2$, the assertion follows. Therefore we only need to show (4.17) for all $\alpha \in \Delta_{>0}$ and $n \geq 1$. If $p(\alpha)$ is even, $J^c_{-\alpha}(-n)$ is even and $\varphi^\alpha(-n)$ is odd. Therefore $C_{\alpha,n}$ is spanned by all the elements

$$(J^c_{-\alpha}(-n))^r, \quad (J^c_{-\alpha}(-n))^r \varphi^\alpha(-n)$$

with $r \geq 0$. We have

$$d_{c,0} \cdot (J^c_{-\alpha}(-n))^r = r(n_{-\alpha}|e_\alpha)(J^c_{-\alpha}(-n))^r \varphi^\alpha(-n),$$

$$d_{c,0} \cdot (J^c_{-\alpha}(-n))^r \varphi^\alpha(-n) = 0$$
by (4.15) and (4.16). Hence (4.17) follows. If \( p(\alpha) \) is odd, \( J^{e-\alpha}(-n) \) is odd and \( \varphi^\alpha(-n) \) is even. Therefore \( C_{\alpha,n} \) is spanned by all the elements
\[
(\varphi^\alpha(-n))^r, \quad J^{e-\alpha}(-n)(\varphi^\alpha(-n))^r
\]
with \( r \geq 0 \). We have
\[
d_{e,0} \cdot J^{e-\alpha}(-n)(\varphi^\alpha(-n))^r = n(e_{-\alpha}(e_\alpha)(\varphi^\alpha(-n))^r + 1,
\]
d_{e,0} \cdot (\varphi^\alpha(-n))^r = 0
by (4.15) and (4.16). Hence (4.17) follows. Therefore this completes the proof. \( \square \)

5. Non-generic Level

In this section we consider the \( \mathcal{W} \)-algebra \( \mathcal{W}^k(g, f; \Gamma) \) at non-generic level. Recall that, following [KW3], there exists an ascending vertex superalgebra filtration on \( C_k \) such that the induced filtration on \( H(C_k, d_{(0)}) \) gives an isomorphism
\[
gr H(C_k, d_{(0)}) \simeq V^\tau_k(g^f),
\]
where \( g^f \subset g_{\leq 0} \) is the centralizer of \( f \) in \( g \). Hence if
\[
dim g^f = n
\]
and \( \{u_1, \ldots, u_n\} \) is a basis of \( g^f \) such that
\[
u_i \in g_{-j_i},
\]
for all \( i \), where all \( j_i \)'s are some non-negative half-integers, then the \( \mathcal{W} \)-algebra \( \mathcal{W}^k(g, f; \Gamma) \) is strongly generated by
\[
W^{u_i}(z) = J^{u_i}(z) + (\text{lower terms})
\]
for all \( i \) with conformal weights
\[
\Delta(W^{u_i}) = j_i + 1
\]
since the differential \( d_{(0)} \) preserves the conformal weights on \( C_k \). Thus, \( \mathcal{W}^k(g, f; \Gamma) \) is realized as a vertex subalgebra of \( V^\tau_k(g_{\leq 0}) \otimes F(g_{(\frac{1}{2})}) \) generated by \( W^{u_i}(z) \) for all \( i \). Let
\[
\tilde{\mu} : V^\tau_k(g_{\leq 0}) \otimes F(g_{(\frac{1}{2})}) \to V^\tau_k(g_0) \otimes F(g_{(\frac{1}{2})})
\]
be the vertex superalgebra homomorphism defined by \( \tilde{\mu}(J_u^\Gamma) = 0 \) for all \( u \in g_{< 0} \) and \( n < 0 \), and \( \tilde{\mu} = \text{Id} \) on \( V^\tau_k(g_0) \otimes F(g_{(\frac{1}{2})}) \). This map \( \tilde{\mu} \) induces the vertex superalgebra homomorphism
\[
\mu : \mathcal{W}^k(g, f; \Gamma) \to V^\tau_k(g_0) \otimes F(g_{(\frac{1}{2})})
\]
by restriction, where we regard the \( \mathcal{W} \)-algebra \( \mathcal{W}^k(g, f; \Gamma) \) as a vertex subalgebra of \( V^\tau_k(g_{\leq 0}) \otimes F(g_{(\frac{1}{2})}) \). This map \( \mu \) is called the Miura map.

**Lemma 5.1.** The image of the isomorphism (3.9) in Theorem 3.8 coincides with that of the Miura map \( \mu \).

**Proof.** Assume that \( k \) is generic. By Lemma 4.12
\[
(5.1) \quad \mathcal{W}^k(g, f; \Gamma) = H^0(C_k, d_{(0)}) \simeq H^0(E_1, Q),
\]
which is the isomorphism (5.9). This is the vector isomorphism induced by the weight filtration on $C_k$. By Proposition 4.6

$$E_1^{(0)} = V_{\tau k} (g_0) \otimes F(g_1).$$

This implies that the leading terms of the cohomology classes in $C_k^{(0)}$ concentrate on the homogeneous space with degree 0. Therefore the isomorphism (5.1) is the projection of the homogeneous terms with degree 0 in the cohomology classes in $C_k^{(0)}$. Hence the image of (5.1) coincides with that of the Miura map by definition. This completes the proof.

Corollary 5.2. If $k$ is generic, the Miura map $\mu$ is injective and there exist strongly generating fields $X_1(z), \ldots, X_n(z)$ of the vertex subalgebra $\mu(W_k(g, f; \Gamma)) \cong W_k(g, f; \Gamma)$ of $V_{\tau k} (g_0) \otimes F(g_1)$ such that

$$\mu(W_{u_i}(z)) = X_i(z), \quad \Delta(X_i) = j_i + 1$$

for all $i$.

Though the following assertion is proved in [A3] for regular nilpotent elements $f$, the same proof applies for arbitrary $f$ (see Section 5.9 of [A3] for the details).

Lemma 5.3. The Miura map $\mu$ is injective for all $k$ if $g$ is a simple Lie algebra.

We note that the analogue of Lemma 5.3 for the classical $W$-algebras is proved in [DKV] and this can be used to give an yet another proof of Lemma 5.3.

Recall that $\hat{C}_y$ is a vertex superalgebra over $C[y]$ generated by $\overline{J}_u(z), \overline{\Phi}_\alpha(z), \overline{\Phi}_\beta(z)$ for all $u \in g_{\leq 0}, \alpha \in \Delta_{>0}, \beta \in \Delta_{\frac{1}{2}}$ and also a chain complex over $C[y]$ with the differential $d_{(0)}$ (see Section 2.3 for definitions). Let

$$\hat{C}_y, \quad V_{\tau k} (g_{\leq 0}), \quad V_{\tau k} (g_0)$$

be the vertex subalgebras of $\hat{C}_y$ over $C[y]$ generated by

$$\overline{\Phi}_\alpha(z), \quad \overline{J}_u(z), \quad \overline{J}_v(z).$$

for all $\alpha \in \Delta_{\frac{1}{2}}, u \in g_{\leq 0}, v \in g_0$. By Kac and Wakimoto arguments in the case of $C_k$, one finds that the cohomology $H(\hat{C}_y, d_{(0)})$ is a free $C[y]$-module vertex superalgebra generated by $W_{u_i}(z) = J_{u_i}(z) + (\text{lower terms})$ for all $i$ in $V_{\tau k} (g_{\leq 0}) \otimes F(g_1)_y$ and we call it the $W$-algebra over $C[y]$ denoted by $W^y(g, f; \Gamma) = H(\hat{C}_y, d_{(0)})$. By construction, if $k + h^\vee \neq 0$,

$$W^y(g, f; \Gamma) = \mathcal{W}^y(g, f; \Gamma) \otimes_{C[y]} C_\epsilon,$$

where $C_\epsilon$ is a one-dimensional $C[y]$-module defined by $y = \epsilon$ and $\epsilon = (k + h^\vee)^{-1}$. We can also define the Miura map

$$\mu_y : W^y(g, f) \rightarrow V_{\tau k} (g_0) \otimes F(g_1)_y$$

where
in the same way and we have
\begin{equation}
\mu = \mu_y \otimes \text{ev}_{y=\epsilon}
\end{equation}
for all \( k \neq -h^\vee \), where \( \text{ev}_{y=\epsilon} \) is the evaluation map defined by \( y \mapsto \epsilon \).

**Lemma 5.4.** The Miura map \( \mu_y \) is injective.

**Proof.** We extend the weight filtration on \( C_k \) to \( \widehat{C}_y \) by \( \deg_y = 0 \). This filtration gives the convergent spectral sequence \( \widehat{E}_1 = H(\widehat{C}_y, d_{\text{st}(0)}) \otimes F(y) \).

By Corollary 4.3
\begin{equation}
H(\widehat{C}_y, d_{\text{st}(0)}) = V(\tau_y(g_0)) \otimes H(g_1, C[y]).
\end{equation}
Therefore, in the same way as Theorem 3.8, one finds that the \( W \)-algebra \( W_y(g, f; \Gamma) \) over \( C[y] \) is isomorphic to the vertex subalgebra over \( C[y] \) of \( V(\tau_y(g_0)) \otimes F(g_1, y) \), which is the intersection of kernels of the screening operators:
\begin{equation}
W_y(g, f; \Gamma) \simeq \bigcap_{[\beta] \in \Pi^\Gamma} \text{Ker}Q_{[\beta], y},
\end{equation}
where
\begin{align*}
Q_{[\beta], y} &= \sum_{\alpha \in [\beta]} \int S^\alpha_y(z) \Phi_\alpha(z) :dz: \text{ if } [\beta] \in [\Pi^\Gamma], \\
Q_{[\beta], y} &= \sum_{\alpha \in [\beta]} \chi(e_\alpha) \int S^\alpha_y(z) dz \text{ if } [\beta] \in [\Pi^1].
\end{align*}
and \( S^\alpha_y(z) \) is defined in Proposition 3.7 by replacing \( \epsilon = (k + h^\vee)^{-1} \) by \( y \). This implies that the Miura map \( \mu_y \) is injective. \( \square \)

The following lemma is useful in applications.

**Lemma 5.5.** Let
\( X^i_y = \mu_y(W^u_y) \)
for all \( i \). Then
\( X^i_y|_{y=\epsilon} = \epsilon \cdot \mu(W^u) \)
for all \( k \neq -h^\vee \).

**Proof.** Compare the leading term of \( W^u_y(z) \) with that of \( W^u(z) \). The assertion follows from the definition \( J^u_z = \epsilon J^u_y(z) \). \( \square \)

6. **Applications**

6.1. \( WB_n \)-algebras. Let \( n \in \mathbb{Z}_{\geq 1} \) and \( \mathfrak{h} \) be a Cartan subalgebra of a simple Lie algebra of type \( B_n \) with the non-degenerate symmetric bilinear form \( (\cdot, \cdot) \) such that \( (\theta, \theta) = 2 \). Denote by \( \{ \alpha_i \}_{i=1}^n \) a set of simple roots of type \( B_n \) such that the simple root \( \alpha_i \) is short.

Let \( \mathcal{H} \) be the Heisenberg vertex algebra associated with \( \mathfrak{h}^* \) i.e. the vertex algebra \( \mathcal{H} \) is generated by the even fields \( \alpha_i(z) \) for all \( i = 1, \ldots, n \) satisfying
\( [\alpha_i, \alpha_j] = (\alpha_i, \alpha_j) \lambda \),
and \( \mathcal{F} \) be the vertex superalgebra generated by the odd field \( \Psi(z) \) satisfying
\[
[\Psi, \Psi] = 1.
\]
Choose \( \gamma \in \mathbb{C} \). Let \( G(z) \) be an odd field on \( \mathcal{H} \otimes \mathcal{F} \) defined by
\[
G(z) = : (\gamma \partial + b_1(z))(\gamma \partial + b_2(z)) \cdots (\gamma \partial + b_n(z))\Psi(z) :,
\]
where
\[
b_i(z) = \sum_{j=1}^{n} \alpha_j(z)
\]
for \( i = 1, \ldots, n \), and \( W_i(z) \) is an even field on \( \mathcal{H} \otimes \mathcal{F} \) for \( i = 0, \ldots, 2n - 2 \) defined by
\[
[G_{\lambda}G] = W_0 + \sum_{i=1}^{n-1} \gamma_i (W_{2i-1} \lambda^{2i-1} (2i-1)! + W_{2i} \lambda^{2i} (2i)! + \gamma_n \lambda^{2n} (2n)!),
\]
where
\[
\gamma_i = \prod_{j=1}^{i} (1 - 2j(2j - 1) \gamma^2).
\]

**Lemma 6.1.** Let \( C_2(\mathcal{H} \otimes \mathcal{F}) \) be the subspace of \( \mathcal{H} \otimes \mathcal{F} \) spanned by \( a(-2) b \) for all \( a, b \in \mathcal{H} \otimes \mathcal{F} \). Then
\[
[G_{\lambda}G] \equiv W_0 + \sum_{i=1}^{n-1} \gamma_i W_{2i} \lambda^{2i} (2i)! + \gamma_n \lambda^{2n} (2n)! \pmod{C_2(\mathcal{H} \otimes \mathcal{F})}
\]
and
\[
W_{2i} \equiv \sum_{1 \leq j_1 < \cdots < j_{n-i} \leq n} : b_{j_1}^2 \cdots b_{j_{n-i}}^2 : \pmod{C_2(\mathcal{H} \otimes \mathcal{F})}
\]
for all \( i = 0, \ldots, n-1 \).

**Proof.** Notice that
\[
[b_i b_j] = \delta_{i,j} \lambda
\]
for all \( i, j \). If \( n = 1 \),
\[
[G_{\lambda}G] = : b_1^2 + \gamma \partial b_1 + : (\partial \Psi) \Psi : + (1 - 2 \gamma^2) \frac{\lambda^2}{2}
\]
\[
\equiv : b_1^2 : + (1 - 2 \gamma^2) \frac{\lambda^2}{2} \pmod{C_2(\mathcal{H} \otimes \mathcal{F})}.
\]
Let \( n \geq 2 \) and
\[
G'(z) = : (\gamma \partial + b_2(z)) \cdots (\gamma \partial + b_n(z))\Psi(z) :.
\]
Then
\[
G(z) = : (\gamma \partial + b_1(z))G'(z) :.
\]
By inductions on \( n \), we can assume that \( G'(z) \) satisfies our assertions. Hence
\[
[G'_{\lambda}G'] \equiv W_0' + \sum_{i=1}^{n-2} \gamma_i W_{2i} \lambda^{2i} (2i)! + \gamma_{n-1} \lambda^{2n-2} (2n-2)! \pmod{C_2(\mathcal{H} \otimes \mathcal{F})}
\]
and
\[ W_{2i}' = \sum_{2 \leq j_2 < \cdots < j_{n-1} \leq n} b^2_{j_2} \cdots b^2_{j_{n-1}} : (\text{mod } C_2(H \otimes F)). \]

Therefore
\[ [G_\lambda G] \equiv -\gamma^2 \lambda^2 [G'_\lambda G'_\lambda] - \gamma \lambda [G'_\lambda : b_1 G'_\lambda :] + \gamma \lambda [G'_\lambda : b_1 G'_\lambda :] + [ : b_1 G'_\lambda : \lambda : b_1 G'_\lambda :] \]
\[ \equiv : b^2_i W_0' : + \sum_{i=1}^{n-2} \gamma_i \left( \frac{b^2_i W_{2i}' : + W_{2i-2}' : \lambda^{2i}}{(2i)!} \right) + \gamma_{n-1} \left( \frac{b^2_i : + W_{2n-4}' : \lambda^{2n-2}}{(2n-2)!} \right) + \gamma_n \left( \frac{\lambda^{2n}}{(2n)!} \right) \]

Furthermore
\[ W_0' \equiv : b^2_i W_0' : \equiv : b^2_i \cdots b^2_n : \quad (\text{mod } C_2(H \otimes F)), \]
\[ W_{2n-2}' \equiv : b^2_i : + W_{2n-4}' \equiv : b^2_i : + \cdots + b^2_n : \quad (\text{mod } C_2(H \otimes F)), \]
\[ W_{2i}' \equiv : b^2_i W_{2i}' : + W_{2i-2}' \equiv \sum_{1 \leq j_1 < \cdots < j_{n-1} \leq n} b^2_{j_1} \cdots b^2_{j_{n-1}} : \quad (\text{mod } C_2(H \otimes F)) \]

for \( i = 1, \ldots, n-2 \) by our assumptions. This completes the proof. \( \square \)

The vertex subalgebra of \( H \otimes F \) generated by the odd field \( G(z) \) and the even fields \( W_{2i}(z) \) for \( i = 0, \ldots, n-1 \) is the \( \mathcal{WB}_n \)-algebra for \( \gamma \in \mathbb{C} \) introduced by Fateev and Lukyanov [FL]. Let \( \gamma_+ \in \mathbb{C} \) such that \( \gamma_+ + \gamma_- = \gamma \) and \( \gamma_+ + \gamma_- = -1 \). Define the operators
\[ Q_i : H \otimes F \to H_{\gamma_+ \alpha_i} \otimes F \]
for \( i = 1, \ldots, n \) by
\[ Q_i = \int e^{\gamma_+ f(\alpha_i(z))} \, dz \quad \text{for} \quad i \neq n, \]
\[ Q_n = \int e^{\gamma_+ f(\alpha_n(z))} \Psi(z) \, dz, \]
where \( e^{f(\beta(z))} \) for \( \beta \in h^\ast \) is defined in Section 3.2. According to [FL],
\[ \mathcal{WB}_n = \bigcap_{i=1}^{n} \ker Q_i \]
for generic \( \gamma \).

**Lemma 6.2.** The odd vector \( G \) and even vectors \( W_{2i} \) for \( i = 0, \ldots, n-1 \) belong to the subspace \( \bigcap_{i=1}^{n} \ker Q_i \) of \( H \otimes F \).

**Proof.** It is enough to show that \( Q_i \cdot G = 0 \) for all \( i \). Let
\[ S^{\alpha_i}(z) = e^{\gamma_+ f(\alpha_i(z))}, \]
\[ G_i(z) = : (\gamma \partial + b_i(z)) \cdots (\gamma \partial + b_n(z)) \Psi(z) : \]
for \( i = 1, \ldots, n \) and \( G_{n+1}(z) = \Psi(z) \). Then, for \( i = 1, \ldots, n-1 \),
\[ [S^{\alpha_i} G]_{\lambda=0} = : (\gamma \partial + b_i) \cdots (\gamma_+ + b_i-1) [S^{\alpha_i} \lambda : (\gamma \partial + b_i)(\gamma \partial + b_i+1) G_{i+2} :]_{\lambda=0} : \]
and
\[ [S^\alpha, \lambda : (\gamma \partial + b_i)(\gamma \partial + b_{i+1})G_{i+2}] :\lambda = 0 \]
\[ = \gamma \partial + S^\alpha_i G_{i+2} : - \gamma_i : S^\alpha_i b_{i+1} G_{i+2} : + \gamma_i : b_i S^\alpha_i G_{i+2} : \]
\[ = \gamma_i : (\partial S^\alpha_i - \gamma_i \partial S^\alpha_i - b_{i+1} S^\alpha_i + b_i S^\alpha_i)G_{i+2} : \]
\[ = : (-\partial S^\alpha_i + \gamma_i \alpha_i S^\alpha)G_{i+2} : \]
\[ = 0. \]

Hence
\[ Q_i \cdot G = 0 \quad \text{for } i = 1, \ldots, n - 1. \]

Moreover
\[ [\alpha^\alpha \Psi : \lambda \in \mathbb{C}] = : (\gamma \partial + b_1) \cdots (\gamma + b_{n-1} : S^\alpha \Psi : \lambda : (\gamma \partial + b_n) : \Psi : \lambda = 0 : \]

and
\[ [: S^\alpha \Psi : \lambda : (\gamma \partial + b_n) : \Psi : : = 0 = \gamma \partial S^\alpha_n - \gamma_+ \partial S^\alpha_n + : b_n S^\alpha_n : = 0. \]

Therefore
\[ Q_n \cdot G = 0. \]

This completes the proof. \(\square\)

**Lemma 6.3.** If \( k \) is generic,
\[ \mathcal{W}^k(\mathfrak{osp}(1,2n), f_{\text{reg}}; \Gamma) \simeq \bigcap_{i=1}^n \text{Ker} Q_i, \]

where \( \gamma_+ = -1/\sqrt{2k + 2n + 1} \), \( f_{\text{reg}} \) is a regular nilpotent element of the even part of \( \mathfrak{osp}(1,2n) \) and \( \Gamma \) is the Dynkin grading of \( f_{\text{reg}} \).

**Proof.** Let \( \mathfrak{g} = \mathfrak{osp}(1,2n) \) and \( \{ \beta_i \}_{i=1}^n \) be a set of simple roots of \( \mathfrak{g} \) such that \( \beta_n \) is an odd root and \( \{ \beta_1, \ldots, \beta_{n-2}, 2\beta_n \} \) is a set of simple roots of the even part of \( \mathfrak{g} \). Note that the even part of \( \mathfrak{g} \) is the symplectic Lie algebra \( \mathfrak{sp}(2n) \) of rank \( n \). Let \( f_{\text{reg}} \) be a regular nilpotent element of the even part of \( \mathfrak{g} \) defined by
\[ f_{\text{reg}} = e_{-\beta_1} + \cdots + e_{-\beta_{n-1}} + e_{-2\beta_n} \]
and \( \{ e, h, f_{\text{reg}} \} \) be a \( \mathfrak{sl}_2 \)-triple. We set \( x = \frac{1}{2} h \) and denote by \( \langle \cdot | \cdot \rangle \) the normalized even supersymmetric non-degenerate invariant bilinear form on \( \mathfrak{g} \). Notice that the dual Coxeter number \( h^\vee \) of \( \mathfrak{g} \) is equal to \( n + \frac{1}{2} \). A semisimple element \( x \) defines the Dynkin grading \( \Gamma \) on \( \mathfrak{g} \) with respect to \( f_{\text{reg}} \) denoted by
\[ \Gamma : \mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j, \]
where \( \mathfrak{g}_j \) is the eigenspace of \( \text{ad} x \) with an eigenvalue \( j \). Then
\[ \mathfrak{g}_0 = \mathfrak{h}', \quad \Pi_{\frac{1}{2}} = \{ \beta_n \}, \quad \Pi_1 = \{ \beta_1, \ldots, \beta_{n-1} \}, \]
where \( \mathfrak{h}' \) is a Cartan subalgebra of \( \mathfrak{g} \) containing \( x \). The subspace \( \mathfrak{g}_0 \) of \( \mathfrak{g} \) is the one-dimensional odd space spanned by \( e_{\beta_n} \). The neutral free superfermion vertex superalgebra associated with \( \mathfrak{g}_0 \) is isomorphic to \( \mathcal{F} \) since \( \chi(\langle e_{\beta_n}, e_{\beta_n} \rangle) \neq 0 \). Let \( \mathcal{H}' \) be the Heisenberg vertex algebra associated with \( \mathfrak{h}' \). If \( k \) is generic, then by Theorem
the vertex superalgebra $W^k(g, f_{\text{reg}}; \Gamma)$ is isomorphic to the vertex subalgebra of $\mathcal{H}' \otimes F$, which is the intersection of kernels of the screening operators:

$$W^k(g, f_{\text{reg}}; \Gamma) \simeq \bigcap_{i=1}^{n-1} \text{Ker} \int e^{-\beta_i(z)} dz \cap \text{Ker} \int :e^{-\bar{\phi}} f_{\beta_i(z)} \bar{\Psi}(z) : dz,$$

where $\nu' = \sqrt{k + n + \frac{1}{2}}$. Let $\alpha_i = \sqrt{2} \beta_i \in h^*$ for $i = 1, \ldots, n$.

Then $(\alpha_i|\alpha_j)_{g} = (\alpha_i|\alpha_j)$ for all $i, j$ and the vertex algebra $\mathcal{H}'$ is isomorphic to $\mathcal{H}$. Moreover

$$e^{-\nu'} f_{\beta_i(z)} = e^{\gamma_+} f_{\alpha_i(z)}$$

for all $i$, where $\gamma_+ = -1/\sqrt{2\nu'}$. Therefore our assertion follows. \qed

**Theorem 6.4.** If $k + n + \frac{1}{2} \neq 0$,

$$W^k(osp(1, 2n), f_{\text{reg}}; \Gamma) \simeq WB_n,$$

where $\gamma = \nu - \frac{1}{2}$, $\nu = \sqrt{2k + 2n + 1}$, $f_{\text{reg}}$ is a regular nilpotent element of the even part of $osp(1, 2n)$ and $\Gamma$ is the Dynkin grading of $f_{\text{reg}}$. Moreover the fields

$$G(z), \ W_{2i}(z) (i = 0, \ldots, n - 1)$$

are strongly generating fields of $W^k(osp(1, 2n), f_{\text{reg}}; \Gamma)$ with conformal weights $\Delta(G) = n + \frac{1}{2}$ and $\Delta(W_{2i}) = 2n - 2i$.

**Proof.** Let $f = f_{\text{reg}}, \ g = osp(1, 2n)$.

If $k$ is generic, by Lemma 6.3

(6.1) $$W^k(g, f; \Gamma) \simeq \bigcap_{i=1}^{n} \text{Ker} Q_i,$$

where $\gamma_+ = -1/\sqrt{2k + 2n + 1}$, which describes the image of the Miura map $\mu$ (see Section 5 for the definition). The conformal weights of $\mathcal{H} \otimes F$ induced by this isomorphism is determined by

$$\Delta(\alpha_i) = 1, \quad \Delta(\Psi) = \frac{1}{2}.$$

By Lemma 6.2 the odd vector $G$ and $W_{2i}$ for $i = 0, \ldots, n - 1$ belong to $\bigcap_{i=1}^{n} \text{Ker} Q_i$ and

$$\Delta(G) = n + \frac{1}{2}, \quad \Delta(W_{2i}) = 2n - 2i.$$

By definition, the $WB_n$-algebra is generated by $G(z)$ and $W_{2i}(z)$ for all $i$ as a vertex subalgebra in $\mathcal{H} \otimes F$. Since $W_{2i}(z)$ has a leading term

$$W_{2i}(z) = \sum_{1 \leq j_1 < \cdots < j_{n-i} \leq n} :b_{j_1}^2(z) \cdots b_{j_{n-i}}^2(z) : + \cdots$$

for $i = 0, \ldots, n - 1$ by Lemma 6.1 there exists a set of strongly generating fields of the $WB_n$-algebras such that $G(z)$ and $W_{2i}(z)$ are included in this set. Let $g_{ev}$ be
the even part of $g$ and $g_{od}$ be the odd part of $g$. Denote by $g^f$ the centralizer of $f$ in $g$. Then

$$g^f = g_{ev}^f \oplus g_{od}^f.$$  

Since $g_{ev} = \mathfrak{sp}(2n)$ and $f$ is a regular nilpotent element of $g_{ev}$,

$$\dim g_{ev}^f = n$$

and there exists a basis $\{u_i\}_{i=1}^n$ of $g_{ev}^f$ such that

$$u_i \in g_{-j_i}, \quad j_i = 2i - 1$$

for $i = 1, \ldots, n$, where $j_i$ is called the $i$-th exponent of $\mathfrak{sp}(2n)$. A subspace $g_{od}^f$ of $g^f$ is a one-dimensional odd space spanned by

$$e^{-\beta_1} \cdots e^{-\beta_{n-1}} e^{-\beta_n} \in g_{-n+\frac{1}{2}},$$

where we follow the notation in the proof of Lemma 6.3. Therefore, by Corollary 5.2, the vertex superalgebra $W^k(g, f; \Gamma)$ is generated by an odd field $G'(z)$ and even fields $X^i(z)$ with conformal weights

$$\Delta(G') = n + \frac{1}{2}, \quad \Delta(X^i) = j_i + 1 = 2i$$

for $i = 1, \ldots, n$. Therefore, for generic $k$, we can choose $X^i(z)$ and $G'(z)$ such that

$$(6.2) \quad \mu(G') = G, \quad \mu(X^i) = W_{2n-2i}.$$  

Hence the isomorphism (6.1) implies that the Miura map

$$(6.3) \quad \mu : W^k(g, f; \Gamma) \rightarrow WB_n$$

is an isomorphism for generic $k$. Since the equations (6.2) hold not only for generic $k$ but also for $k \neq -n - \frac{1}{2}$ by Lemma 5.5, the Miura map (6.3) is also an isomorphism for $k \neq -n - \frac{1}{2}$. This completes the proof. \(\square\)

**Remark 6.5.** For $k \neq -h'(= -n - \frac{1}{2})$, the $W$-algebra $W^k(\mathfrak{osp}(1, 2n), f_{reg}; \Gamma)$ is isomorphic to the vertex subalgebra of $\mathcal{H} \otimes F$ generated by $G(z)$. Therefore the Feigin-Frenkel duality

$$W^k(\mathfrak{osp}(1, 2n), f_{reg}; \Gamma) \simeq W^{k'}(\mathfrak{osp}(1, 2n), f_{reg}; \Gamma)$$

holds for this case, where $4(k + h')(k' + h') = 1$.

6.2. $W^{(2)}_n$-algebras. Let $n \in \mathbb{Z}_{\geq 2}$, $k \in \mathbb{C}$ such that $k + n \neq 0$ and $V$ be a vector space spanned by

$$a_{n-1}, a_{n-2}, \ldots, a_1, \psi, \xi$$
with non-degenerate \( \mathbb{C} \)-bilinear form \( (\cdot | \cdot)_V : V \times V \to \mathbb{C} \) defined by the following Gram matrix:

\[
\begin{pmatrix}
 a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_1 & \psi & \xi \\
 2(k+n) & -k-n & 0 & \cdots & 0 & 0 & 0 \\
 -k-n & 2(k+n) & -k-n & \cdots & 0 & 0 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
 a_1 & 0 & 0 & 0 & \cdots & 2(k+n) & -k-n & 0 \\
 \psi & 0 & 0 & 0 & \cdots & -k-n & 1 & 1 \\
 \xi & 0 & 0 & 0 & \cdots & 0 & 1 & 0
\end{pmatrix}
\]

Let \( H \) be the Heisenberg vertex algebra associated with \( V \) and \( (\cdot | \cdot)_V \), and \( H_{m\xi} \) be the \( H \)-module with the highest weight \( m\xi \in V \) for \( m \in \mathbb{Z} \). Denote by \( e^{m\xi} \) the highest weight vector of \( H_{m\xi} \). Then

\[
V_\xi = \bigoplus_{m \in \mathbb{Z}} H_{m\xi}
\]

has a vertex algebra structure such that the vertex operator of \( e^{m\xi} \) is defined by

\[
e^{m\xi}(z) = Y(e^{m\xi}, z) := e^{m\int \xi(z)}
\]

for all \( m \in \mathbb{Z} \). Since \( (\xi | \xi)_V = 0 \), the parity of \( e^{m\xi}(z) \) is even for all \( m \in \mathbb{Z} \) and the following formula holds:

\[
[e^{m\xi}, e^{m'\xi}] = 0
\]

for all \( m, m' \in \mathbb{Z} \). The \( \mathcal{W}_n^{(2)} \)-algebra is defined as a vertex subalgebra of \( V_\xi \) generated by

\[
\mathcal{E}(z) = e^{\xi}(z), \quad \mathcal{F}(z) =: P(z)e^{-\xi}(z) ;,
\]

where

\[
P = -((k + n - 1)\partial + \psi + \sum_{i=1}^{n-1} a_i) \cdots ((k + n - 1)\partial + \psi + \sum_{i=1}^{n-2} a_i) \cdots ((k + n - 1)\partial + \psi + a_1)\psi.
\]

We remark that we should substitute \( P \) for \( F \) after expanding \( P \) formally. For example, if \( n = 2 \),

\[
P = -(k+1)\partial\psi - (\psi + a_1)\psi,
\]

\[
\mathcal{F}(z) = -(k+1) : (\partial \psi)(z)e^{-\xi}(z) : - : (\psi(z) + a_1(z))\psi(z)e^{-\xi}(z) :.
\]

Let

\[
A_i(z) = e^{\int a_i(z)} \text{ for } i = 1, \ldots, n-1, \quad Q(z) = e^{\int \psi(z)}.
\]

By a result in \([4S]\),

\[
\mathcal{W}_n^{(2)} = \bigcap_{i=1}^{n-1} \ker \int A_i(z) \, dz \cap \ker \int Q(z) \, dz
\]

for generic \( k \).

For \( n = 2 \), the vertex algebra \( \mathcal{W}_2^{(2)} \) is isomorphic to the universal affine vertex algebra associated with \( \mathfrak{sl}_2 \) at level \( k \) and for \( n = 3 \), the vertex algebra \( \mathcal{W}_3^{(2)} \) is isomorphic to the Bershadsky-Polyakov algebra \(([3]) \).
Remark 6.6. Feigin and Semikhatov gave equivalent $n+1$ definitions of the $W_n^{(2)}$-algebra, denoted by $\mathcal{W}_n^{(2)}$ for $m \in \mathbb{Z}$ with $0 \leq m \leq n$. In this paper, we use only the definition of $W_n^{(2)}$.

Lemma 6.7.

$$F(z) = - ((k + n - 1)(\partial + \xi(z)) + \psi(z) + \sum_{i=1}^{n-1} a_i(z))$$

$$\cdots ((k + n - 1)(\partial + \xi(z)) + \psi(z) + a_1(z)) \psi(z) e^{-\xi(z)} : .$$

Proof. If $n = 2$,

$$F(z) = -(k + 1)(\partial \psi)(z)e^{-\xi(z)} + (\psi(z) + a_1(z)) \psi(z)e^{-\xi(z)} :$$

$$= -(k + 1)(\partial : (\psi(z))e^{-\xi(z)} : - (\psi(z) + a_1(z)) \psi(z)e^{-\xi(z)} :$$

$$= - ((k + 1)(\partial + \xi(z)) + \psi(z) + a_1(z)) \psi(z)e^{-\xi(z)} : .$$

For $n \geq 3$, the above formula is proved similarly using the induction. \hfill \Box

Assume $n \geq 3$. Let $\{\epsilon_{i,j}\}_{1 \leq i, j \leq n}$ be the standard basis of $\mathfrak{gl}_n$ and

$$h_i = \epsilon_{i,i} - \epsilon_{i+1,i+1}$$

for $i = 1, \ldots, n-1$. Denote by

$$\mathfrak{g} = \mathfrak{sl}_n \subset \mathfrak{gl}_n$$

the special linear Lie algebra of rank $n$. Then $\{h_i\}_{i=1}^{n-1}$ is a basis of the Cartan subalgebra $\mathfrak{h}$. Let $\epsilon_i$ be the dual element of $e_{i,i}$ for $i = 1, \ldots, n$ and $\alpha_j = \epsilon_j - \epsilon_{j+1}$ for $j = 1, \ldots, n-1$. Then $\Delta = \{\epsilon_i - \epsilon_j\}_{1 \leq i \neq j \leq n}$ is a root system of $\mathfrak{g}$ and

$$\Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$$

is a set of simple roots of $\Delta$. Choose a root vector $\epsilon_{i,j} = e_{i,j}$. Define a subregular nilpotent element $f = f_{\text{sub}}$ of $\mathfrak{g}$ by

$$f_{\text{sub}} := e_{-\alpha_2} + \cdots + e_{-\alpha_{n-1}}$$

and a semisimple element $x$ of $\mathfrak{g}$ by

$$x = \frac{1}{2n} \sum_{i=1}^{n-1} (n - i)(in - 2)h_i.$$
where
\[ [\alpha_2] = \{ \alpha_1 + \alpha_2, \alpha_2 \}, \quad [\alpha_i] = \{ \alpha_i \} \quad \text{for } i > 2. \]

Let \((\cdot|\cdot)\) be the non-degenerate symmetric invariant bilinear form on \(g\) defined by \((u|v) = \text{tr}(uv)\) for \(u, v \in g\). By Theorem 3.8, the vertex algebra \(W_k(\mathfrak{sl}_n, f_{\text{sub}}; \Gamma)\) is isomorphic to the vertex subalgebra of \(V_{\tau_{k}}(g_0)\), which is the intersection of kernels of the screening operators:
\[
W_k(\mathfrak{sl}_n, f_{\text{sub}}; \Gamma) \simeq \bigcap_{i=2}^{n-1} \ker \int S^{\alpha_i}(z) \, dz
\]
for generic \(k\). Let \(E(z)\) and \(F(z)\) be the fields on \(V_{\tau_{k}}(g_0)\) defined by
\[
E(z) = J^{e_{-\alpha_1}}(z),
\]
\[
F(z) = :((k + n - 1)\partial + \sum_{i=1}^{n-1} J^{h_i}(z)) \cdot ((k + n - 1)\partial + \sum_{i=1}^{n-2} J^{h_i}(z)) \cdot \cdots (J^{h_n}(z) + J^{h_{n-1}}(z)) J^{e_{-\alpha_1}}(z) :.
\]

**Lemma 6.8.**
\[
E, F \in \bigcap_{i=2}^{n-1} \ker \int S^{\alpha_i}(z) \, dz.
\]

**Proof.** It is enough to show that
\[
[S^{\alpha_i}_{\lambda} E]_{\lambda=0} = [S^{\alpha_i}_{\lambda} F]_{\lambda=0} = 0
\]
for all \(i = 2, \ldots, n - 1\). By Proposition 3.7,
\[
[S^\alpha_m, J^h_{(n)}] = \alpha(h) S^\alpha_{m+n} \quad \text{for all } \alpha \in \Pi', \ h \in \mathfrak{h},
\]
\[
[S^\alpha_1, S^\alpha_2] = S^{\alpha_1+\alpha_2}_{m+n}, \quad [S^\alpha_m, J^{e_{-\alpha_1}}_{(n)}] = 0 \quad \text{for all } \alpha \neq \alpha_1 + \alpha_2,
\]
\[
[S^\alpha_1, J^{e_{-\alpha_2}}_{(n)}] = S^{\alpha_1+\alpha_2}_{m+n}, \quad [S^\alpha_m, J^{e_{-\alpha_2}}_{(n)}] = 0 \quad \text{for all } \alpha \neq \alpha_2
\]
and
\[
\partial S^{\alpha_1+\alpha_2}(z) = -\frac{1}{k+n} : (J^{h_1+h_2}(z) S^{\alpha_1+\alpha_2}(z) + J^{e_{-\alpha_1}}(z) S^{\alpha_2}(z)) :,
\]
\[
\partial S^{\alpha_2}(z) = -\frac{1}{k+n} : (J^{h_2}(z) S^{\alpha_2}(z) + J^{e_{-\alpha_1}}(z) S^{\alpha_1+\alpha_2}(z)) :,
\]
\[
\partial S^{\alpha_i}(z) = -\frac{1}{k+n} : J^{h_i}(z) S^{\alpha_i}(z) : \quad \text{for } i = 3, \ldots, n - 1.
\]

Therefore
\[
[S^{\alpha_i}_{\lambda} E]_{\lambda=0} = 0
\]
for all \(i\). Next, we show \([S^{\alpha_i}_{\lambda} F]_{\lambda=0} = 0\) for all \(i\). Let \(F_1(z) = J^{e_{-\alpha_1}}(z)\) and
\[
F_i(z) = :((k + n - 1)\partial + J^{h_i}(z)) \cdots ((k + n - 1)\partial + J^{h_2}(z)) J^{e_{-\alpha_1}}(z) :.
\]
for \(i \geq 2\), where
\[
b_i = \sum_{i=1}^{j} h_i
for all $j$. First,
\[ S^{α_i}F_i \|_{λ=0} = ((k + n - 1)\partial + J^{h_{i-1}}(z)) \cdots ((k + n - 1)\partial + J^{h_i}(z))[S^{α_i}F_i]_{λ=0} : . \]
Therefore it is enough to show that
\[ [S^{α_i}F_i]_{λ=0} = 0 \]
for all $i$. For $i = 2$, we have
\[ [S^{α_2}F_2]_{λ=0} = (k + n - 1)\partial S^{α_1+α_2} + : J^{b_2}S^{α_1+α_2} : + : S^{α_2}J^{e-α_2} : = (k + n)\partial S^{α_1+α_2} + : (J^{b_2}S^{α_1+α_2} + J^{e-α_2}S^{α_2}) : = 0. \]
For $i \geq 3$, since $[S^{α_i}F_{i-2}]_{λ=0} = 0$, we obtain that
\[ [S^{α_i}F_i]_{λ=0} = [S^{α_i} : ((k + n - 1)\partial + J^{h_i})(k + n - 1)\partial + J^{h_{i-1}}F_{i-2} :]_{λ=0} = - ((k + n - 1)\partial S^{α_i}F_{i-2} : - : J^{b_i}S^{α_i}F_{i-2} : + : S^{α_i}J^{b_{i-1}}F_{i-2} : = - : ((k + n)\partial S^{α_i} + J^{b_i}S^{α_i})F_{i-2} : = 0. \]
This completes the proof.

**Theorem 6.9.** If $k+n \neq 0$,
\[ \mathcal{W}^k(\mathfrak{sl}_n, f_{\text{sub}}; \Gamma) \simeq \mathcal{W}^{(2)}_n, \]
where $f_{\text{sub}}$ is a subregular nilpotent element of $\mathfrak{sl}_n$ and $\Gamma$ is the good grading such that the corresponding weighted Dynkin diagram is $[0,4]$.

**Proof.** Define a vertex algebra homomorphism
\[ π : V^{τ_k}(\mathfrak{g}_0) \rightarrow V_ξ \]
by
\[ J^{h_1}(z) \mapsto (k + n - 2)ξ(z) + 2ψ(z) + a_1(z), \]
\[ J^{h_2}(z) \mapsto ξ(z) - ψ(z) + a_2(z), \]
\[ J^{h_i}(z) \mapsto a_i(z) \quad (i = 3, \ldots, n - 1), \]
\[ J^{ε_{α_i}}(z) \mapsto e^{f}ξ(z), \]
\[ J^{ε_{-α_i}}(z) \mapsto - : (k + n - 1)(∂ + ξ(z)) + ψ(z) + a_1(z)ψ(z)e^{-f}ξ(z) : . \]
Then
\[ \text{Im}(π) \subset \text{Ker} \int A_1(z) \, dz \cap \text{Ker} \int Q(z) \, dz. \]
Consider the Lie algebra decomposition
\[ \mathfrak{g}_0 = \mathfrak{sl}_2 \oplus \mathfrak{z}, \]
where $\mathfrak{sl}_2$ is the subalgebra of $\mathfrak{g}_0$ spanned by $e_{α_1}, h_1, e_{-α_1}$ and $\mathfrak{z}$ is the center of $\mathfrak{g}_0$. If we restrict $π$ to the vertex subalgebra $V^{k+n-2}(\mathfrak{sl}_2)$ of $V^{τ_k}(\mathfrak{g}_0)$, the map $π$ coincides with the vertex algebra homomorphism of the Wakimoto construction of $V^{k+n-2}(\mathfrak{sl}_2)$, which is injective ([2]). Since
\[ V^{τ_k}(\mathfrak{g}_0) = V^{k+n-2}(\mathfrak{sl}_2) \otimes V^{k+n}(\mathfrak{z}) \]
(see Remark 3.3 and $V^{k+n}(z)$ is the Heisenberg vertex algebra associated with $z$, $\pi$ is injective and we can regard $V^\tau\mathfrak{(g_0)}$ as a vertex subalgebra of $V_\xi$ via $\pi$. Then $S^{\alpha_i}(z)$ acts on $V^\tau\mathfrak{(g_0)}$ as
\begin{align*}
S^{\alpha_1+\alpha_2}(z) &= -:Q(z)e^{-\frac{1}{k+n}\int a_2(z)-f(z)}:, \\
S^{\alpha_i} &= e^{-\frac{1}{k+n}\int a_i(z)} \quad (i = 2, \ldots, n - 1).
\end{align*}
Therefore, for generic $k$,
\begin{align*}
\pi(\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{sub}}; \Gamma)) &\subset \bigcap_{i=2}^{n-1} \text{Ker} \int S^{\alpha_i}(z) \, dz \cap \text{Ker} \int A_1(z) \, dz \cap \text{Ker} \int Q(z) \, dz \\
&= \bigcap_{i=2}^{n-1} \text{Ker} \int A_i(z) \, dz \cap \text{Ker} \int A_1(z) \, dz \cap \text{Ker} \int Q(z) \, dz \\
&= \mathcal{W}_n^{(2)}
\end{align*}
because
\[ \text{Ker} \int S^{\alpha_i}(z) \, dz = \text{Ker} \int A_i(z) \, dz \]
for all $i = 2, \ldots, n - 1$. Moreover
\[ \pi(E) = \mathcal{E}, \quad \pi(F) = \mathcal{F} \]
and these are generating fields of $\mathcal{W}_n^{(2)}$ for all $k+n \neq 0$. Hence $\pi|_{\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{sub}})}$ induces a vertex algebra isomorphism
\[ \mathcal{W}^k(\mathfrak{sl}_n, f_{\text{sub}}) \simeq \mathcal{W}_n^{(2)} \]
for generic $k$. By Lemma 5.3 this isomorphism holds for all $k+n \neq 0$. □

**Remark 6.10.** For a subregular nilpotent element $f_{\text{sub}}$ in $\mathfrak{g} = \mathfrak{sl}_n$, there exists a good grading $\Gamma$ such that $\mathfrak{g}_0 = \mathfrak{h}$. Thus, Theorem 3.9 gives a free field realization of the $\mathcal{W}_n^{(2)}$-algebra.

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