VOLterra type integral operator and analytic function spaces

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Abstract. We obtain the radius of convexity of the Volterra type integral operator
\[ T_g f(z) = \int_0^z f(s) g'(s) ds \quad (|z| < 1), \]
when \( f \) and \( g \) belong to the some certain subclass of analytic functions.

1. Volterra type operator

Let \( \Delta := \{ z \in \mathbb{C} : |z| < 1 \} \). In 1977 Pommerenke \[17\] Lemma 1 introduced an integral operator, called Volterra type operator as follows:
\[ J_g f(z) := \int_0^z f'(s) g(s) ds \quad (z \in \Delta). \]
Pommerenke proved that \( J_g f(z) \) is a bounded operator on the Hardy space \( H^2 \) if and only if \( g \) belongs to the class \( BMOA \). Also Aleman and Siskakis \[1\] proved that this characterization (boundedness) is valid on each \( H^p \) for \( 1 \leq p < \infty \) and that \( J_g \) is compact on \( H^p \) if and only if \( g \in VMOA \). An another natural integral operator is defined as follows:
\[ T_g f(z) := \int_0^z f(s) g'(s) ds \quad (z \in \Delta). \]
It is necessary to refer to the this fact that
\[ J_g f(z) + T_g f(z) = M_g f - f(0)g(0), \]
where \( M_g \) is the multiplication operator and is defined by
\[ (M_g f)(z) = g(z)f(z) \quad (f \in H(\Delta), z \in \Delta) \]
and \( H(\Delta) \) denotes the class of all analytic functions on \( \Delta \). Indeed, if \( f \) and \( g \) are two normalized analytic functions, then
\[ J_g f(z) + T_g f(z) = g(z)f(z). \]
We note that the integral operators \( J_g f(z) \) and \( T_g f(z) \) contain the well-known integral operators in the analytic function theory and geometric function theory, such as the generalized Bernardi–Libera–Livingston linear integral operator (see \[2\] \[9\] \[10\]), Srivastava–Owa fractional derivative operators \[12\] \[14\] and the Cesáro integral operator, \[18\] \[19\].

Recently many researchers have been studied the integral operators \( J_g f(z) \) and \( T_g f(z) \). For example, Li and Stević \[7\] studied the boundedness and the compactness of \( J_g f(z) \) and \( T_g f(z) \) on the Zygmund space and the little Zygmund space. Also, many authors have been studied the essential norm of the integral operators. Laitila et al. \[6\] studied the essential norm of the operator \( T_g \) on the Hardy space. Or Liu et al. (see \[8\]) studied the essential norm of the operator \( T_g \) on the Bloch

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space and some other spaces. Zhuo and Ye [20] studied the essential norm of the
operator $T_g$ from Morrey spaces to the Bloch space. Finally, Zhou and Zhu studied
the essential norm of the operator $T_g$ from Hardy spaces to the BMOA space, Besov
spaces, Bergman spaces and Bloch-type spaces, see [21].

The basis of this article is to study of the integral operator

$$T_g f(z) = \int_0^z f(s)g'(s)ds = \int_0^1 f(tz)g'(tz)dt = \int_0^z f dg.$$  

2. Some Subclass of Analytic Functions

In this section we recall some certain subclass of analytic functions. Further,
we denote by $\mathcal{A}$ the class of all analytic and normalized functions in the open unit
disk $\Delta$. The subclass of $\mathcal{A}$ that the function $f \in A$ holds throughout $\Delta$, then

$\text{Re} \left\{ z f'(z) \right\} > \alpha \quad (z \in \Delta).$

We denote by $S^*(\alpha)$ the class of starlike functions of order $\alpha$. Also, $f \in \mathcal{U}$ is convex
of order $0 \leq \alpha < 1$ if $z f'(z) \in S^*(\alpha)$. The class of convex functions of order alpha
is denoted by $\mathcal{K}(\alpha)$. Analytically, $f \in \mathcal{K}(\alpha)$ iff

$$\text{Re} \left\{ 1 + z f''(z) \right\} > \alpha \quad (z \in \Delta).$$

Let $A$ and $B$ be two complex numbers such that $|A| > 1$ and $|B| \leq 1$. We say
that the function $f \in \mathcal{A}$ belongs to the class $S^*(A, B)$ if it satisfies the following
subordination relation:

$$zf'(z) - \frac{A + Az}{1 + Bz} (z \in \Delta).$$

We note that if $-1 \leq B < A \leq 1$ and are real, then $S^*(A, B)$ becomes the family of
Janowski starlike functions. Also, we say that $f \in \mathcal{K}(A, B)$ if and only if $z f'(z) \in S^*(A, B)$. Indeed, if $f$ belongs to the class $\mathcal{K}(A, B)$, then it satisfies

$$1 + z f''(z) < \frac{1 + A}{1 + B} (z \in \Delta).$$

We remark that $\mathcal{K}(2, 1)$ and $\mathcal{K}(2, -1)$ become to the Ozaki conditions. Moreover,
by the Lindelöf subordination principle (this principle states that if $f(z) \prec g(z)$,
then $|f'(0)| \leq |g'(0)|$ and $f(\Delta) \subset g(\Delta)$), if $f \in \mathcal{K}(2, 1)$, then we have

$$(2.1) \quad \text{Re} \left\{ 1 + z f''(z) \right\} < \frac{3}{2} (z \in \Delta).$$

Also, if $f \in \mathcal{K}(2, -1)$, then

$$(2.2) \quad \text{Re} \left\{ 1 + z f''(z) \right\} > -\frac{1}{2} (z \in \Delta).$$

Ozaki proved that if $f \in \mathcal{A}$ with $f(z)f(z)/z \neq 0$, there, and if either (2.1) or (2.2)
holds throughout $\Delta$, then $f$ is univalent and convex in at least one direction in $\Delta$, see [15].

Let $\mathcal{LU}$ denote the family of normalized locally univalent functions in $\Delta$. For
$\beta \in \mathbb{R}$, we consider the class $\mathcal{G}(\beta)$ consisting of all functions $f \in \mathcal{LU}$ which satisfy the condition

$$(2.3) \quad \text{Re} \left\{ 1 + z f''(z) \right\} < 1 + \frac{\beta}{2} (z \in \Delta).$$
We note that $G(1) \subset U$ and $G(1) \equiv K(2,1)$. Also the functions in the class $G(1)$ are starlike of order zero in $\Delta$. The functions class $G(\beta)$ was studied extensively by Kargar et al. [5] (see also [11]). For more details about the class $G(\beta)$ see [5] and its references.

Let $Aut(\Delta)$ be the class of holomorphic automorphisms in $\Delta$. Any $\phi \in Aut(\Delta)$ has the following representation:

$$\phi(z) = e^{i\theta} \frac{z + a}{1 + az} \quad (\theta, a \in \mathbb{R}, z \in \Delta).$$

The family $F$ of $A$ is called a linear–invariant family (L.I.F.), if $F \subset LU$ and for all $f \in F$ and $\phi \in Aut(\Delta)$

$$(2.4) \quad F_\phi(f)(z) := f(\phi(z)) - f(\phi(0))$$

and the universal linear invariant family of order $\gamma \geq 1$ as

$$UL_\gamma := \{ f \in F : \text{ord } f \leq \gamma \}.$$  

We remark that $UL_1 \equiv K(0)$ and $U \subset UL_2$. For more details about the L.I.F. see [4, Chapter 5].

3. Some Key Lemmas

In this section we recall some lemmas which help us in order to prove of main results.

**Lemma 3.1.** (see [3]) If $f \in UL_\gamma$ and $\gamma \geq 1$, then

$$\max \left\{ \left| \frac{zf''(z)}{f'(z)} \right| - \frac{2|z|^2}{1 - |z|^2} : f \in F \right\} \leq \frac{2\gamma|z|}{1 - |z|^2} \quad (z \in \Delta).$$

The next lemma is due to Pommerenke [16], see also [4, Lemma 5.1.3].

**Lemma 3.2.** Let $F$ be a linear–invariant family and $\delta = \text{ord } F$. Then

$$(3.1) \quad \delta = \sup \sup_{f \in F, |z| < 1} \left| -z + \frac{1}{2} \frac{f''(z)}{f'(z)} \right| (z \in \Delta).$$

The following lemma gives a basic estimate which leads to the distortion theorem for univalent functions.

**Lemma 3.3.** (see [4] p. 15) If $f \in U$, then

$$\max \left\{ \left| \frac{zf''(z)}{f'(z)} - \frac{2r^2}{1 - r^2} \right| : |z| = r < 1 \right\} \leq \frac{4r}{1 - r^2}.$$

The estimate is sharp for rotation of Koebe function.

**Lemma 3.4.** (see [11]) Let $\beta \in (0,1]$ be fixed. If $f \in G(\beta)$, then

$$\max \left\{ \left| \frac{f''(z)}{f'(z)} \right| : z \in \Delta \right\} \leq \frac{\beta}{1 - z}.$$

The result is sharp for the function $f'(z) = (1 - z)^\beta$.

In the present paper, we obtain the radius of convexity of the Volterra-type integral operator $T_\beta f(z)$ given by (1.2) when the functions $f$ and $g$ belonging to the some certain subclasses of analytic functions which are defined in the Section 2.
4. Main Results

We begin this section with the following result.

**Theorem 4.1.** Let $A$ and $B$ be two complex numbers, $|A| > 1$ and $|B| ≤ 1$. Also, let $0 ≤ \alpha < 1$ be real number. If $f \in S^*(A, B)$ and $g \in \mathcal{K}(A, B)$, then the Volterra-type integral operator $T_g f(z)$ given by (1.2) is convex of order $\alpha$ in $|z| ≤ r_c(A, B, \alpha)$ where

$$r_c(A, B, \alpha) = \begin{cases} \frac{|B - A| - (\alpha - 1)|B - A|}{\alpha |B|^2 - 2 \text{Re}(AB)} & B = 0 \\ \frac{|B - A| - (\alpha - 1)|B - A|}{\alpha |B|^2 - 2 \text{Re}(AB)} & B \neq 0. \end{cases}$$

*Proof.* From now on, for convenience we put $T(z) := T_g f(z)$. By the analytic definition of convexity of order $0 ≤ \alpha < 1$ it is enough to find out the largest number $0 < r < 1$ for which

$$\min_{|z| = r} \text{Re} \left\{ 1 + z \frac{T''(z)}{T'(z)} \right\} - \alpha ≥ 0.$$ 

Further, from (1.2) we have

$$1 + z \frac{T''(z)}{T'(z)} - \alpha = z \frac{f''(z)}{f(z)} - \alpha + z \frac{g''(z)}{g'(z)} + 1.$$ 

On the other hand, since $f \in S^*(A, B)$ we get

$$z \frac{f''(z)}{f(z)} = \frac{1 + A z}{1 + B z} \quad (z \in \Delta).$$

Thus by using the Lindelöf subordination principle, we get

$$\left| z \frac{f''(z)}{f(z)} - \frac{1 - AB r^2}{1 - |B|^2 r^2} \right| ≤ \frac{|B - A|^r}{1 - |B|^2 r^2} \quad (|r| = r < 1).$$

From the above inequality (4.5), we obtain

$$\text{Re} \left\{ z \frac{f''(z)}{f(z)} \right\} - \alpha ≥ \text{Re} \left\{ 1 - AB r^2 \right\} - \alpha \quad \geq \frac{1 - |B - A| r - (\text{Re} \{AB\} - \alpha |B|^2) r^2}{1 - |B|^2 r^2}.$$ 

Also, because $g \in \mathcal{K}(A, B)$, by the same proof we obtain

$$\text{Re} \left\{ 1 + z \frac{g''(z)}{g'(z)} \right\} ≥ \frac{1 - |B - A| r - \text{Re} \{AB\} r^2}{1 - |B|^2 r^2}.$$ 

Now from (4.3), (4.6) and (4.7) we get

$$\text{Re} \left\{ 1 + z \frac{T''(z)}{T'(z)} \right\} - \alpha ≥ \frac{2 - \alpha - 2|B - A| r - (2 \text{Re} \{AB\} - \alpha |B|^2) r^2}{1 - |B|^2 r^2} > 0$$

provided $\phi(r) := 2 - \alpha - 2|B - A| r - (2 \text{Re} \{AB\} - \alpha |B|^2) r^2 > 0$. A simple calculation gives that the roots of $\phi(r)$ are

$$\frac{|B - A| ± |(\alpha - 1)B - A|}{\alpha |B|^2 - 2 \text{Re}(AB)}.$$ 

Also, it is clear that if $B = 0$, then $\phi(r)$ yields that $\phi(r) = 2 - \alpha - 2|A| r$ and it will be positive if $r < (2 - \alpha)/2|A|$. This completes the proof. □

If we take $A = 2$ and $B = -1$ in the Theorem 1.1 then we have the following.

**Corollary 4.1.** Let $\text{Re}\{zf'(z)/f(z)\} > -1/2$ where $z \in \Delta$ and $g$ satisfies the condition (2.2). Then the Volterra-type integral operator $T_g f(z)$ given by (1.2) is convex of order $\alpha$ ($0 ≤ \alpha < 1$) in the disk $|z| ≤ (2 - \alpha)/(4 + \alpha)$.
Putting $A = 2$ and $B = 1$ in the Theorem 4.1, we get.

**Corollary 4.2.** Let $\text{Re}\{zf'(z)/f(z)\} < 3/2$ where $z \in \Delta$ and $g$ satisfies the condition (4.11). Then the radius of convexity of order $\alpha$ of the Volterra-type integral operator $T_g f(z)$ given by (1.2) is

$$
(4.11) \quad \text{-Re} \left\{ z \frac{g''(z)}{g'(z)} \right\} \geq \frac{2r^2}{1 - r^2} - \frac{2\gamma r}{1 - r^2}.
$$

Now by definition of starlikeness of order alpha and using the above inequality (4.11), and applying the relation (4.10) we obtain

$$
\Re \left\{ 1 + z \frac{T''(z)}{T'(z)} \right\} = \Re \left\{ z f'(z) + z \frac{g''(z)}{g'(z)} + 1 \right\}
$$

$$
> \frac{1 + \alpha - 2\gamma r + (2 - \alpha - 1)r^2}{1 - r^2} > 0,
$$

provided $\varphi(r) = 1 + \alpha - 2\gamma r + (2 - \alpha - 1)r^2 > 0$. It is easy to see that the roots of $\varphi(r)$ are

$$
\gamma \pm \sqrt{\alpha^2 + \gamma^2 - 1}.
$$

Also we see that if $0 \leq \alpha < 1$ and $\gamma \geq 1$, then $0 < r_c^{-}(\alpha, \gamma) < 1$ where $r_c(\alpha, \gamma)$ defined in (4.10). This is the end of proof. \text{□}

Putting $\alpha = 0$ and $\gamma = 1$ in the Theorem 4.2, we get.

**Corollary 4.3.** Let $f$ and $g$ be starlike and convex univalent functions in the open unit disk $\Delta$, respectively. Then the Volterra-type integral operator $T_g f(z)$ is convex univalent function in $\Delta$, too.

**Theorem 4.3.** Let $0 \leq \alpha < 1$. If $f$ is starlike of order $\alpha$ and $g \in \mathcal{F}$ with $\text{ord} \mathcal{F} = 1$, then the Volterra type integral operator $T_g f(z)$ given by (1.2) is convex univalent in the disk $\Delta$.

**Proof.** Since $f$ is starlike of order $0 \leq \alpha < 1$ we have

$$
(4.12) \quad \text{Re} \left\{ z \frac{f'(z)}{f(z)} \right\} > \alpha \quad (z \in \Delta).
$$

Also since $g \in \mathcal{F}$, by using the relation (5.1) and with a little calculation we get

$$
(4.13) \quad \Re \left\{ 1 + z \frac{g''(z)}{g'(z)} \right\} \geq \frac{1 - 2\delta r + r^2}{1 - r^2} \quad (|z| = r < 1).
$$

Now, from (4.10), (4.12) and (4.13) and since $\text{ord} \mathcal{F} = 1$, we get

$$
\Re \left\{ 1 + z \frac{T''(z)}{T'(z)} \right\} \geq \frac{1 + \alpha - 2\gamma r + (1 - \alpha)r^2}{1 - r^2}.
$$
and for $z \in \Delta$ one deduces that

$$\Re \left\{ 1 + z \frac{T''(z)}{T'(z)} \right\} > 0 \quad (z \in \Delta).$$

Thus by definition we conclude that $T(z)$ is convex univalent in the unit disk $\Delta$. This is the end of proof.

The next theorem is the following.

**Theorem 4.4.** Let $f$ be starlike function of order $\alpha$ and $g$ be univalent function. Then the radius of convexity of the Volterra-type integral operator $T_g f(z)$ is

$$r_c^\alpha (\alpha) = \frac{2 - \sqrt{3 + \alpha^2}}{1 - \alpha} \quad (0 \leq \alpha < 1).$$

**Proof.** Let $f \in S^*(\alpha)$ where $0 \leq \alpha < 1$ and $g \in \mathcal{U}$. From Lemma 3.3 and by (4.10), we get

$$\Re \left\{ 1 + z \frac{T''(z)}{T'(z)} \right\} = \Re \left\{ z \frac{f'(z)}{f(z)} + z \frac{g''(z)}{g'(z)} + 1 \right\} > \frac{1 + \alpha - 4r + (1 - \alpha)r^2}{1 - r^2} > 0,$$

when $|z| \leq (2 - \sqrt{3 + \alpha^2})/(1 - \alpha)$ and concluding the proof.

**Remark 4.1.** Taking $\alpha = 0$ in the Theorem 4.4, we see that if $f$ is starlike univalent function and $g \in \mathcal{U}$, then the radius of convexity of the Volterra type integral operator is $2 - \sqrt{3}$. Indeed, in this case the radius of convexity the Volterra type integral operator is equal to familiar radius of convexity for the class $\mathcal{U}$ (see [4, Theorem 2.2.22, p. 51]).

Finally we have.

**Theorem 4.5.** Let $f$ be starlike function of order $0 \leq \alpha < 1$ and $g$ be locally univalent function which satisfies in (2.3) where $0 < \beta \leq 1$. Then the radius of convexity of the Volterra-type integral operator $T_g f(z)$ is

$$r_c(\alpha, \beta) = \frac{1 + \alpha}{1 + \alpha + \beta}.$$

**Proof.** Assume that $f \in S^*(\alpha)$ and $g \in \mathcal{G}(\beta)$, where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. By Lemma 3.3 and from (4.10) we have

$$\Re \left\{ 1 + z \frac{T''(z)}{T'(z)} \right\} > 1 + \alpha - \frac{\beta |z|}{1 - |z|} \geq 0,$$

where $|z| \leq (1 + \alpha)/(1 + \alpha + \beta)$. This completes the proof.

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