A GLOBAL EXISTENCE OF CLASSICAL SOLUTIONS TO THE TWO-DIMENSIONAL KINETIC-FLUID MODEL FOR FLOCKING WITH LARGE INITIAL DATA

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ABSTRACT. We present a two-dimensional coupled system for flocking particle-compressible fluid interactions, and study its global solvability for the proposed coupled system. For particle and fluid dynamics, we employ the kinetic Cucker-Smale-Fokker-Planck (CS-FP) model for flocking particle part, and the isentropic compressible Navier-Stokes (N-S) equations for the fluid part, respectively, and these separate systems are coupled through the drag force. For the global solvability of the coupled system, we present a sufficient framework for the global existence of classical solutions with large initial data which can contain vacuum using the weighted energy method. We extend an earlier global solvability result [20] in the one-dimensional setting to the two-dimensional setting.

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1. Introduction. The purpose of this paper is to provide a global existence theory of classical solutions for the coupled Cucker-Smale-Navier-Stokes system describing the interaction of Cucker-Smale flocking particles and the isentropic viscous compressible fluid. This kind of coupled system arises in many industrial applications such as the sedimentation phenomenon analysis and the modeling of aerosols and sprays. When the flocking particles are surrounded by the viscous fluids, the force per unit mass exerted on a flocking particle by the surrounding fluid will come from several effects, e.g., skin friction, separation drag, gravity and body forces, rotation of the particle with respect to the gas, pressure gradient in the gas, etc (see [37] for details). In this paper, we consider a coupled kinetic-fluid model for the interactions between Cucker-Smale particles and compressible viscous fluid via a friction force in a random environment, which can be modeled by the coupled system of kinetic CS-FP type equation with a degenerate diffusion coefficient and compressible isentropic Navier-Stokes equations. The nonlinear convection term is difficult to be controlled due to the highly nonlinearity of the 3-D Navier-Stokes equations. Therefore, we will consider 2-D case in this paper. To make the coupled kinetic equation be consistent with the fluid equation, we also consider the kinetic equation in 2-D case, which can be rigorously derived by the 2-D microscopic dynamical model as in [22]. More precisely, let \( f = f(x, v, t) \) be the one-particle distribution function of a Cucker-Smale (C-S) ensemble with velocity \( v = (v_1, v_2) \in \mathbb{R}^2 \) at position \( x = (x_1, x_2) \in \mathbb{R}^2 \) at time \( t > 0 \) for particle side, and let \( \rho(x, t) \) be the density and \( u = u(x, t) = (u_1, u_2)(x, t) \), be the bulk velocity of the compressible fluid. Then the coupled dynamics of \([f, \rho, u]\) is governed by the following kinetic-fluid system:

\[
\begin{align*}
\partial_t f + v \cdot \nabla_x f + \kappa_1 \nabla_v \cdot (fL[f]) + \kappa_2 \nabla_v \cdot ((u - v) f) &= \sigma \Delta_v (|v - v_c|^2 f), \\
\partial_t \rho + \nabla_x \cdot (\rho u) &= 0, \\
\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u) + \nabla_x P(\rho) &= \mu \Delta_x u + \nabla_x ((\mu + \lambda(\rho)) \text{div} u) + \kappa_2 \int_{\mathbb{R}^2} (v - u) f dv, \\
v_c := \frac{\int_{\mathbb{R}^2} v f dv dx}{\int_{\mathbb{R}^2} f dv dx}, & \quad L[f](x, v, t) := \int_{\mathbb{R}^2} \psi(x - y)(v_* - v) f(v_*, y, t) dv_* dy,
\end{align*}
\]

subject to initial conditions:

\[
(f(x, v, 0), \rho(x, 0), u(x, 0)) = (f_0(x, v), \rho_0(x), u_0(x)), \quad (x, v) \in \mathbb{R}^2 \times \mathbb{R}^2,
\]

where \( P(\rho) \) is the pressure given by

\[
P(\rho) = \rho^\gamma, \quad \gamma > 1,
\]

\( \kappa_i, i = 1, 2 \) are positive constants and \( \psi(|x - y|) \) in \( L[f] \) is a communication weight representing a degree of communication between particles located at \( x \) and \( y \). For definiteness, we assumed that \( \psi \) is uniformly bounded and away from zero, and is sufficiently regular: there exist positive constants \( \psi_m \) and \( \psi_M \) such that the communication function \( \psi \) satisfies the following properties:

\[
0 < \psi(s) \leq \psi_M < +\infty, \quad s \geq 0, \quad \sum_{1 \leq i \leq 3} \left\| \frac{d^i}{ds^i} \psi \right\|_{L^\infty(\mathbb{R}^+)} < \infty.
\]

Nonnegative constants \( \mu \) and \( \sigma \) are proportional to the viscosity of the fluid and squared strength of the noise in a random part of the communication weight, respectively. In the sequel, the constants \( \kappa_1, \kappa_2 \) and \( \sigma \) will be normalized to the unity for simplicity. There are three mechanisms for flocking particles: the first one is the
alignment force which drives particles to flocking state; the second one is from random noise, which is expressed as a degenerate diffusion in PDE level; the last one is the fluid coupling strength i.e., the compressible fluid will interact with the flocking particles through the drag force. We assume some regularity of \( \psi \) to guarantee the existence of the coupled system. The positive lower bound of \( \psi \) can balance the diffusion and thus the emergence of flocking can be expected.

For notational simplicity, we denote the right hand side of (1) by

\[
\Gamma_{\rho} u := \mu \Delta_x u + \nabla_x ((\mu + \lambda(\rho)) \text{div} u),
\]

where the shear and bulk viscosity coefficients \( \mu \) and \( \lambda \) are assumed to satisfy

\[
\mu = \text{const.} > 0, \quad \lambda(\rho) = \rho^\beta, \quad \beta > 1,
\]

such that \( \Gamma_{\rho} \) is strictly elliptic. Moreover, \( \rho_f \) and \( u_f \) denote the local mass density, and average local velocity of particle ensemble, respectively:

\[
\rho_f := \int_{\mathbb{R}^2} f dv \quad \text{and} \quad u_f := \begin{cases} 
\frac{\int_{\mathbb{R}^2} x f dv}{\int_{\mathbb{R}^2} f dv} & \text{if } \rho_f \neq 0, \\
0 & \text{if } \rho_f = 0.
\end{cases}
\]

The modeling of collective dynamics via a coupled kinetic-fluid system is one of the hottest topics in the field of nonlinear partial differential equations in recent years. When the number of flocking particles is sufficiently large, it is almost impossible to track the motion of each particle. Therefore, we use the corresponding CS-FP type mean-field kinetic equation to describe the motion of flocking particles under random communication [1, 20]. On the other hand, the fluid dynamics can be described by various types of hydrodynamic models such as compressible Navier-Stokes (N-S) equation, please refer to [26, 27, 38] for more details. Accordingly, we will study a coupled system consisting of the compressible N-S equations and the CS-FP equation in the present paper.

Next, we briefly review some earlier results. As mentioned before, coupled systems for the fluid-particles interaction have been used in many contexts, for example, biosprays in medicine [4], sedimentation of solid grain in physics [5, 8], fuel-droplets in combustion theory [37] etc. The CS-FP type equation can arise from the kinetic description of the Cucker-Smale flocking ensemble under random communication [1, 20], whereas the compressible Navier-Stokes equations have been studied in many references e.g. [23, 26, 38]. We mention that the global existence of classical solutions to 2-D N-S equations has been established using Caffarelli-Kohn-Nirenberg inequality in [27], respectively. There are also extensive mathematical analysis studies on coupled systems. To understand the coupled system between solid particles and incompressible flow, the authors in [19] applied Vlasov-Stokes equations to model the motion of particles with white noise interacting with incompressible viscous stokes flow. In this work, a global-in-time weak solution was constructed in both two and three dimensions. In [24], the authors further considered the nonlinear effect of the fluid and used Navier-Stokes-Vlasov-Fokker-Planck equations to model the particle-fluid coupled system. Under this setting, they showed the asymptotic stability of the equilibrium solutions under small perturbation. Later on, in [10], the global existence of classical solutions to the Euler-Vlasov-Fokker-Planck equations was studied and convergence rate of solutions around the equilibrium state was established as well. Due to loss of diffusion effect in inviscid Euler flow, authors there had to refine the energy estimates and yielded the large time behavior. Moreover, in [36, 39], the authors constructed global weak solutions to the Navier-Stokes-Vlasov system without the white noise of the particle. On the other hand, for the coupled
system involving compressible fluid in a bounded domain of $\mathbb{R}^3$, the coupled system of the Vlasov-Fokker-Planck equation with the compressible Navier-Stokes equations was studied in [31, 32], in which global weak solutions were constructed and the asymptotic analysis has been done for strong interaction and diffusion effect. Later on, in [16], the authors proved the global existence of classical solutions to the coupled system of Vlasov-Fokker-Planck equation and compressible Euler equations for small initial data. We also refer to related works in [2, 3, 7].

The main goal of the present paper is to obtain global classical solutions to the coupled system (1) with large initial data. According to the brief review of previous works, for initial data away from equilibrium, the results on global existence of classical solutions are far from being completed. As in [27], in order to extend the classical solutions globally in time, the key point is to obtain the upper bound for fluid density $\rho$. However, we cannot directly apply the methods employed in [21, 27] to derive the upper bound of $\rho$ or high-order energy estimates due to the additional nonlinear coupling terms in (1):

$$\kappa_2 \nabla v \cdot ((u - v)f), \quad -\kappa_2 (u - u_f)\rho_f.$$

Due to the complicated and delicate treatment of these coupling terms in the a priori high-order estimates, there are three key ingredients to overcome the difficulties and obtain the upper bound of fluid density: First, we show that the $L^\infty(0, T; L^1(\mathbb{R}^2))$-norm of the local momentum $m_k f(x, t)$ can be controlled by the initial data and the $L^1(0, T; L^{k+2}(\mathbb{R}^2))$-norm of fluid velocity $u$ (Lemma 3.2). This is different from the case for the coupled system with the incompressible fluids [20], where the corresponding norm can be controlled by the initial data uniformly. The momentum estimate plays a very important role through the paper since it connects the integrability of the coupling terms with the integrability of fluid velocity $u$, which is necessary for the energy estimates of the fluid equation. Second, we obtain a modified elementary weighted energy estimate by combining the basic energy estimate, momentum estimate and the Caffarelli-Kohn-Nirenberg inequality, see the estimate of $I_{22}$ in Lemma 3.3 for details. This fundamental energy estimate is crucial because it yields the $L^\infty(0, T; L^p(\mathbb{R}^2))$-integrability of both $\rho$ and $m_k f(x, t)$ with $p \geq 1$, see Lemma 3.6 and Corollary 1 for details. With these a priori estimates in hand, we can further obtain two different kinds of estimates for $\|\rho u\|_{L^p}$ ($p \geq 2\gamma$), which are essential to the estimate of the commutator $\vartheta = [u_i, R_i R_j] (\rho u_{ij})$ and $\|\psi\|_{L^\infty}$ in the derivation of upper bound of $\rho$. Third, with proper observation, we use the uniform boundedness of the particle kinetic energy functional $\int_{\mathbb{R}^4} |v|^2 f dv dx$ and obtain a modified nonlinear functional $Z^2(t)$ including the functional $\int_{\mathbb{R}^4} (u - v)^2 f dv dx$, which is compatible with the coupled system (1), see Lemma 3.7 for details. The nonlinear functional $Z^2(t)$ implies that $\log(1 + \|\nabla_x u\|_{L^2})$ can be controlled by $\|\rho\|_{L^\infty}^{1+\beta\varepsilon}$. Finally, combining aforementioned ingredients and estimates, we can apply Brezis-Wainger inequality in Lemma A.6 and the estimates of commutator in Lemma A.7 to obtain the desired upper bound of density $\rho$ (see Lemma 3.9).

The rest of paper is organized as follows. In Section 2, we briefly discuss a framework and present our main results. In Section 3, we provide several lemmas to be used later. In Section 4, we derive a priori estimates in the whole space. In Section 5, we provide a proof of main result. Finally, Section 6 is devoted to a summary of our main results.
Notation. Throughout the paper, \( C \) denotes a generic positive constant which may change line by line. The small constants to be chosen are denoted by \( \varepsilon, \delta \). For function spaces, \( W^{k,p}(\mathbb{R}^2) \) and \( W^{k,p}(\mathbb{R}^3) \) denote the standard Sobolev spaces with standard norm \( \| \cdot \|_{W^{k,p}} \) and \( H^k := W^{k,2} \). \( \| \cdot \|_p := (\int_{\mathbb{R}^2} |p| dx)^{1/2} \) or \( (\int_{\mathbb{R}^3} |p| dx)^{1/2} \) with \( 1 \leq p \leq +\infty \). For notational simplicity, we denote
\[
\frac{\partial^{\alpha_s} f}{\partial x_1^{\alpha_1} \cdots \partial x_L^{\alpha_L}} := \frac{\partial^{\alpha_j} f}{\partial x_1^{\alpha_j}}, \quad \alpha_s = [\alpha_s^1, \alpha_s^2], \quad \beta_s = [\beta_s^1, \beta_s^2],
\]
and
\[
\| f \|_{W^{k,p}} := \sum_{|\alpha_s| + |\beta_s| \leq k} \| \langle v \rangle^k \partial^{\alpha_2} \partial^{\beta_2} f \|_{L^p(\mathbb{R}^2)} , \quad k \geq 0.
\]

Homogeneous Sobolev space \( D^{\ell,p}(\mathbb{R}^2) \) is defined by
\[
D^{\ell,p}(\mathbb{R}^2) = \{ u \in L^1_{loc}(\mathbb{R}^2) : \| \nabla^j u \|_p < +\infty \} \text{ with } \| u \|_{D^{\ell,p}} := \| \nabla^\ell u \|_p. \]
Here \( \nabla^\ell \) means the vector with components \( \partial_{x_1}^\ell \partial_{x_2}^\ell \), where \( \alpha_1 + \alpha_2 = \ell \). For the special case \( p = 2 \), we denote \( D^\ell \) as \( D^{\ell,2} \).

2. Discussion of a framework and main result. In this section, we briefly recall the CS–FP equation, list several elementary inequalities which will be used later, and derive the weighted basic energy estimates of the compressible Navier-Stokes equations and the upper bound of the fluid density \( \rho \). Meanwhile, the estimate of \( \| \nabla_x u \|^2_2 \) will be obtained.

Next, we briefly discuss how the kinetic CS–FP equation (1) can be derived from the corresponding stochastic C–S model via the mean-field limit \((N \to \infty)\). Let \( x^i \) and \( v^i \) be the position and velocity of the \( i \)-th particle in \( \mathbb{R}^3 \), respectively. Then, the C–S flocking model reads as follows:
\[
\frac{dx^i}{dt} = v^i, \quad t > 0, \quad 1 \leq i \leq N, \quad (3)
\]
\[
\frac{dv^i}{dt} = \frac{\kappa_1}{N} \sum_{j=1}^N \psi(|x^j - x^i|)(v^j - v^i),
\]
where \( \psi(|x^j - x^i|) \) denotes the communication weight between the \( i \)-th and \( j \)-th particles.

For a recent survey on the C–S model, we refer to [13]. We now assume that the communication weight \( \psi \) contains Gaussian white noise in the form of:
\[
\tilde{\psi}(|x^j - x^i|) = \psi(|x^j - x^i|) + \frac{\sqrt{2\pi}}{\kappa_1} \eta^i_j, \quad (4)
\]
where \( \eta^i_j \) is \( d \)-dimensional Gaussian white noise. Thus, the C–S model in (3) incorporated with the random communication ansatz in (4) can be rewritten as the stochastic C–S model with a multiplicative noise [1]:
\[
\frac{dx^i}{dt} = v^i dt, \quad t > 0, \quad 1 \leq i \leq N, \quad (5)
\]
\[
\frac{dv^i}{dt} = \frac{1}{N} \sum_{j=1}^N \psi(|x^j - x^i|)(v^j - v^i) dt + \sqrt{2\sigma} (v^i - v_c^i) dB_t^i,
\]
where \( v_c^i := \frac{1}{N} \sum_{j=1}^N v^j \). In a mean-field limit \((N \to \infty)\) [21], the system (5) can be approximated by the following mean-field equation:
\[
\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (\kappa_1 L[f] f) = \sigma \Delta_v (|v - v_c|^2 f).
\]
Throughout the paper, for notational simplicity, we suppress the $t$-dependence in $f$:

$$f(x,v) := f(x,v,t), \quad (x,v) \in \mathbb{R}^2 \times \mathbb{R}^2.$$ 

Next, we present our main results whose proof will be given in Section 4.

**Theorem 2.1.** Suppose that the following conditions hold.

1. The parameters $\alpha, \beta$ and $p$ satisfy

$$1 < \alpha < 2\sqrt{2} - 1, \quad \beta > \frac{4}{3}, \quad p > 4.$$ 

2. Initial data $[f_0, \rho_0, u_0]$ satisfy regularities and integrability:

$$x^2 f_0 \in L^1(\mathbb{R}^4), \quad f_0 \in W^{3,p}_k \text{ for sufficiently large } k, \quad \rho_0 > 0, \quad (\rho_0, P(\rho_0)) \in W^{2,p}(\mathbb{R}^2) \times W^{2,p}(\mathbb{R}^2), \quad u_0 \in D^1 \cap D^2(\mathbb{R}^2), \quad (\rho_0, \rho_0) \in L^1(\mathbb{R}^2),$$

$$P(\rho_0)(1 + |x|^\alpha) \in L^1(\mathbb{R}^2), \quad \sqrt{\rho_0} u_0(1 + |x|^\frac{\alpha}{2}) \in L^2(\mathbb{R}^2), \quad \nabla_x u_0 |x|^\frac{\alpha}{2} \in L^2(\mathbb{R}^2).\tag{6}$$

3. Initial data $[f_0, \rho_0, u_0]$ satisfy the following compatibility condition:

$$\Gamma_{\rho_0} u_0 - \nabla_x P(\rho_0) - (u_0 - u_{f_0}) \rho_{f_0} = \sqrt{\rho_0} g(x) \quad \text{where } g(1 + |x|^\frac{\alpha}{2}) \in L^2(\mathbb{R}^2). \tag{7}$$

Then, the Cauchy problem (1)-(2) admits a unique global classical solution $[f, \rho, u]$ to the Navier-Stokes-Fokker-Planck equations satisfying the regularity and integrability: for $T \in (0, \infty)$ and $\forall (x,t) \in \mathbb{R}^2 \times [0, T],$

1. $0 \leq \rho(x,t) \leq C(T), \quad (\rho, P(\rho)) \in C([0,T]; W^{2,p}(\mathbb{R}^2))$,
2. $P(\rho)(1 + |x|^\alpha) \in C([0,T]; L^1(\mathbb{R}^2))$,
3. $u \in C([0,T]; L^\alpha \cap D^2(\mathbb{R}^2)) \cap L^2(0,T; L^{\frac{\alpha}{2}} \cap D^3(\mathbb{R}^2))$,
4. $\sqrt{\rho} u \in L^\infty(0,T; D^3(\mathbb{R}^2)), \quad tu \in L^\infty(0,T; D^3(\mathbb{R}^2))$,
5. $ut \in L^2(0,T; D^1(\mathbb{R}^2)), \quad tu_t \in L^\infty(0,T; D^1(\mathbb{R}^2))$,
6. $\sqrt{\rho} u_{tt} \in L^2(0,T; D^2(\mathbb{R}^2)), \quad tu_{tt} \in L^2(0,T; D^2(\mathbb{R}^2))$,
7. $\sqrt{\rho} u_{ttt} \in L^2(0,T; D^3(\mathbb{R}^2)), \quad tu_{ttt} \in L^\infty(0,T; L^2(\mathbb{R}^2)), \quad \nabla_x u_{tt} \in L^2(0,T; L^2(\mathbb{R}^2)), \quad C \rho \leq L^\infty(0,T; W^{3,p}_k(\mathbb{R}^4))$, \quad $|x|^2 f \in L^\infty(0,T; L^1(\mathbb{R}^4)). \tag{8}$

**Remark 1.**

1. From the regularity (8) of the solutions $[f, \rho, u]$, we can see that $[f, \rho, u]$ are classical solutions to system (1) by well-known Aubin-Lions Lemma (cf. [34])
2. Compatibility condition (7) was proposed by Cho and Kim in [12], when they considered the local existence of classical solutions to isentropic fluids. Roughly speaking, $g(x)$ is equivalent to $u_t$ at $t = 0$.
3. For simplicity, we require that $k$ is suitably large. Since the weight is used many times throughout the paper, it is really not easy to give an exact expression of $k$.

So we do not track the optimal $k$.

3. **A priori lower-order estimates.** In this section, we present lower-order energy estimates for the coupled system (1), and derive several momentum estimates for $f$. 

3.1. Propagation of velocity moments. First, we state elementary energy estimates for the coupled system without proofs.

Lemma 3.1.\textsuperscript{[20, 26]} Suppose that initial data \([f_0, \rho_0, u_0]\) satisfy the conditions (6). For a positive constant \(T \in (0, \infty]\), let \([f, \rho, u]\) be a smooth solution to system (1)-(2) in \([0, T]\). Then, we have

\begin{itemize}
    \item[(i)] \(\sup_{0 \leq t \leq T} \int_{\mathbb{R}^4} f v dv dx \leq C(T), \quad \sup_{0 \leq t \leq T} \int_{\mathbb{R}^4} |x|^2 f dv dx \leq C(T), \quad \sup_{0 \leq t \leq T} |v_c| \leq C(T)\).
    \item[(ii)] \(\left(\int_{\mathbb{R}^4} |v|^2 f dv + \int_{\mathbb{R}^2} \rho u^2 dx + \int_{\mathbb{R}^2} (\rho^\gamma + \rho) dx\right)(t) + \int_{[0,t] \times \mathbb{R}^2} (\mu |\nabla_x u|^2 + (\mu + \lambda(\rho))(\text{div} u)^2) dx d\tau \leq C(T)\).
    \item[(iii)] \(\|f\|_{L^\infty(0, T; L^p(\mathbb{R}^4))} + \|v - v_c\|_{L^2(0, T; L^2(\mathbb{R}^4))} \leq C(T), \quad 1 \leq p < \infty\).
    \item[(iv)] \(\|f\|_{L^{\infty}(0, T; L^{\infty}(\mathbb{R}^4))} \leq C(T), \quad \inf_{0 \leq t \leq T} \int_{B_N(t)(0)} \rho dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \rho dx\),
\end{itemize}

for some constant \(N(t) > 0\) depending on \(t\) and \(B_N(t)(0) := \{x \in \mathbb{R}^2 | |x| < N(t)\}\).

Next, we show some momentum (velocity) estimates for the kinetic part \(f\). For this, we set
\[m_k f(x, t) := \int_{\mathbb{R}^3} |v|^k f(x, v, t) dv.\]

Lemma 3.2. Under the same setting in Lemma 3.1, we have

\begin{itemize}
    \item[(i)] \(m_{k_1} f(x, t) \leq C(1 + \|f\|_{L^\infty_{x,v,t}})(m_{k_2} f(x, t))^{\frac{k_1 + 2}{k_2 + 2}}, \quad \forall k_2 > k_1 \geq 0,\)
    \item[(ii)] \(\sup_{0 \leq t \leq T} \int_{\mathbb{R}^4} (1 + |v|^{k_3}) f dv dx \leq C(T) \left(\int_{\mathbb{R}^4} (1 + |v|^{k_3}) f_0 dv dx + \left(\int_0^T \|u\|_{k_3 + 2} dt\right)^{k_3 + 2}\right),\)
\end{itemize}

where \(k_3 \geq 2\) is a positive constant.

Proof. (i) Note that for \(R > 0\),
\[\int_{\mathbb{R}^2} |v|^{k_1} f dv = \int_{|v| \leq R} |v|^{k_1} f dv + \int_{|v| > R} |v|^{k_1} f dv \leq \|f\|_{L^\infty_{x,v,t}} R^{k_1 + 2} + \frac{1}{R^{k_2 - k_1}} \int_{\mathbb{R}^2} |v|^{k_2} f dv.\]

We now choose \(R = (\int_{\mathbb{R}^2} |v|^{k_2} f dv)^{\frac{1}{k_2 + 2}}\) in the above relation to obtain
\[\int_{\mathbb{R}^2} |v|^{k_1} f dv \leq \left(\|f\|_{L^\infty_{x,v,t}} + 1\right) \left(\int_{\mathbb{R}^2} |v|^{k_2} f dv\right)^{\frac{k_1 + 2}{k_2 + 2}}.\]
(ii) We multiply (1) by \((1 + |v|^{k_3})\) to have

\[
\frac{d}{dt} \int_{\mathbb{R}^4} (1 + |v|^{k_3}) f dv dx = - \int_{\mathbb{R}^4} |v|^{k_3} \nabla v \cdot [L(f) f + (u - v)f + \nabla v (|v|^{2} f)] d\nu dx
\]

\[
= \int_{\mathbb{R}^4} v \cdot (v - u) \psi(|x - y|) k_3 |v|^{k_3-2} f(y, v_*, t) f(x, v, t) dv_* dy d\nu dx
\]

\[
+ \int_{\mathbb{R}^4} k_3 |v|^{k_3-2} v \cdot (u - v) f dv dx - \int_{\mathbb{R}^4} k_3 |v|^{k_3-2} \nabla v (|v - v_c|^{2} f) dv dx
\]

\[
:= I_{11} + I_{12} + I_{13}.
\]

Below, we estimate the terms \(I_{1i}(1 \leq i \leq 3)\), separately.

- **Case A.1** (estimate of \(I_{11}\)): We use the Hölder inequality and Lemma 3.1 to obtain

\[
I_{11} \leq C \int_{\mathbb{R}^4} |v|^{k_3-1} f dv dx \int_{\mathbb{R}^4} |v_*| f dv_* dy
\]

\[
\leq C \left( \int_{\mathbb{R}^4} |v|^{k_3} f dv dx \right)^{\frac{k_3-1}{k_3}} \left( \int_{\mathbb{R}^4} f dv dx \right)^{\frac{1}{k_3}} \int_{\mathbb{R}^4} (v_*^2 + 1) f dv_* dy
\]

\[
\leq C(T) \left( \int_{\mathbb{R}^4} (1 + |v|^{k_3}) f dv dx \right)^{\frac{k_3+1}{k_3+2}}.
\]

- **Case A.2** (estimate of \(I_{12}\)): Again we apply the Hölder inequality and the result (i) to obtain

\[
I_{12} \leq - \int_{\mathbb{R}^4} k_3 |v|^{k_3} f dv dx
\]

\[
+ C \left( \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} |v|^{k_3-1} f dv dx \right)^{\frac{k_3+2}{k_3+1}} \right) \left( \int_{\mathbb{R}^2} |u|^{k_3+2} dx \right)^{\frac{1}{k_3+2}}
\]

\[
\leq C \left( \int_{\mathbb{R}^4} (1 + |v|^{k_3}) f dv dx \right)^{\frac{k_3+1}{k_3+2}} \left( \int_{\mathbb{R}^2} |u|^{k_3+2} dx \right)^{\frac{1}{k_3+2}}.
\]

- **Case A.3** (estimate of \(I_{13}\)): We use integration by parts and the estimate \(|v_c| \leq C(T)\) to get

\[
I_{13} = \int_{\mathbb{R}^4} k_3^2 |v|^{k_3-2}(v^2 - 2v \cdot v_c + v_c^2) f dv dx \lesssim C(T) \int_{\mathbb{R}^4} (1 + |v|^{k_3}) f dv dx.
\]

In (9), we collect all estimates in Case A.1 - Case A.3 to find

\[
\frac{d}{dt} \left( \int_{\mathbb{R}^4} (1 + |v|^{k_3}) f dv dx \right)^{\frac{1}{k_3+2}} \lesssim C(T) \left( \int_{\mathbb{R}^4} (1 + |v|^{k_3}) f dv dx \right)^{\frac{1}{k_3+2}} + C(T) ||u||_{k_3+2}.
\]

Finally, we integrate the above inequality over \([0, t]\) and use the Gronwall lemma to derive the desired estimate.

3.2. **Weighted energy estimates of the fluid variables.** In this part, we show the weighted basic energy estimates of the fluid part. This will be achieved by combing the estimates of \(||\rho||_p\) \((1 \leq p < +\infty)\).
Lemma 3.3. Suppose that initial data set \([f_0, \rho_0, u_0]\) satisfy the conditions (6), and for a positive constant \(T \in (0, \infty)\), let \([f, \rho, u]\) be a smooth solution to system (1)-(2) in \([0, T)\), and the parameters \(\alpha, \gamma\) and \(m\) satisfy
\[
1 < \alpha < 2\sqrt{2} - 1, \quad \gamma > 1, \quad m \gg 1.
\]
Then, we have
\[
\int_{\mathbb{R}^2} |x|^\alpha (\rho u^2 + \rho^\gamma) dx + \int_0^t \int_{\mathbb{R}^4} |x|^\alpha f|u|^2 dx dv dt
+ \int_0^t \left[ \|x|^\frac{\alpha}{2} \nabla_u u\|^2_2 + \|x|^\frac{\alpha}{2} \text{div} u\|^2_2 + \|x|^\frac{\alpha}{2} \sqrt{\lambda(x)} \text{div} u\|^2_2 \right] dt
\leq C(T) \left[ 1 + \int_0^t \|\rho\|_{2m\beta}^2 \|\nabla u\|^2_2 d\tau \right].
\]
Proof. We multiply \((1)_2\) by \(\frac{\gamma}{\gamma - 1}\) and \((1)_3\) \(|x|^\alpha u\) to obtain
\[
\frac{d}{dt} \int_{\mathbb{R}^2} |x|^\alpha \left( \frac{1}{2} \rho u^2 + \frac{\rho^\gamma}{\gamma - 1} \right) dx + \kappa_2 \int_{\mathbb{R}^4} |x|^\alpha f|u|^2 dx dv
+ \mu \|x|^\frac{\alpha}{2} \nabla_u u\|^2_2 + \mu \|x|^\frac{\alpha}{2} \text{div} u\|^2_2 + \|x|^\frac{\alpha}{2} \sqrt{\lambda(x)} \text{div} u\|^2_2
= \int_{\mathbb{R}^2} \nabla_x (|x|^\alpha) \cdot \left[ u \left( \frac{1}{2} \rho u^2 + \frac{\rho^\gamma}{\gamma - 1} \right) dx - \left( \mu \nabla_x \frac{u^2}{2} + (\mu + \lambda(x)) (\text{div} u) \right) \right] dx
+ \kappa_2 \int_{\mathbb{R}^4} |x|^\alpha v \cdot uf dx dv
=: I_{21} + I_{22}.
\]
Next, we provide estimates for \(I_{2i}\), \(i = 1, 2\).
- Case B.1 (Estimate of \(I_{21}\)): As in Lemma 3.2 [27], we have
\[
|I_{21}| \leq \varepsilon \left( \|x|^\frac{\alpha}{2} \sqrt{\lambda(x)} \text{div} u\|^2_2 + \|x|^\frac{\alpha}{2} \nabla_u u\|^2_2 \right) + \frac{\mu \alpha^2}{2} \|x|^\frac{\alpha}{2} \nabla_u u\|^2_2
+ \frac{\mu \alpha^2}{2} \|x|^\frac{\alpha}{2} \nabla_u u\|^2_2 + C(T) \left( 1 + \|\rho\|_{2m\beta}^2 \|\nabla u\|^2_2 \right),
\]
where \(\varepsilon > 0\) is a constant which can be arbitrarily small.
- Case B.2 (Estimate of \(I_{22}\)): We use (2) in Lemma A.4, Lemma 3.1 and 3.2 to have
\[
|I_{22}| \leq \left( \int_{\mathbb{R}^4} \|u \cdot v\|^\frac{2\alpha}{\alpha + 1} f dv dx \right) \frac{2^\alpha}{\alpha + 1} \left( \int_{\mathbb{R}^4} |x|^2 f dv dx \right)^\frac{2}{\alpha + 1}
\leq C(T) \|u\|^\frac{2}{\alpha}\left( \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^4} |v|^\frac{2\alpha}{\alpha + 1} f dv \right) \frac{4\alpha^3}{\alpha^3 - 3\alpha} \right)^\frac{4\alpha^3}{\alpha^3 - 3\alpha}
\leq C(T) \|x|^\frac{\alpha}{2} \nabla_x u\|_2 \left( \int_{\mathbb{R}^4} |v|^{4\alpha^2 + 3\alpha} f dv dx \right)^\frac{4\alpha^3}{\alpha^3 - 3\alpha}
\leq C(T) \|x|^\frac{\alpha}{2} \nabla_x u\|_2 \left( \int_0^t \|u\|^2_2 + \frac{4\alpha^2}{\alpha^3 - 3\alpha} \left( 2 + \frac{\alpha^2}{\alpha^3 - 3\alpha} \right) \right)
\leq C(T) \|x|^\frac{\alpha}{2} \nabla_x u\|_2 \frac{4\alpha^3}{\alpha^3 - 3\alpha} \left( 2 + \frac{\alpha^2}{\alpha^3 - 3\alpha} \right)\left( \int_0^t \|x|^\frac{\alpha}{2} \nabla_x u\|_2 \frac{4\alpha^2}{\alpha^3 - 3\alpha} \left( 2 + \frac{\alpha^2}{\alpha^3 - 3\alpha} \right) \right)^\frac{4\alpha^3}{\alpha^3 - 3\alpha} \left( 2 + \frac{\alpha^2}{\alpha^3 - 3\alpha} \right)
\[\leq C(T) \||x|^{\frac{2}{3}} \nabla_x u\|_2 \left( \int_0^t \||x|^{\frac{2}{3}} \nabla_x u\|_2^2 \, dt \right)^{\frac{4 - 3\alpha}{2}},\]

where we have used the following relation:

\[4 - 3\alpha > 0 \text{ for } 1 < \alpha < 2\sqrt{2} - 1.\]

Note that there exists a positive constant \(C(\alpha)\) such that

\[
\mu \left(1 - \frac{\alpha^2}{2}\right) \||x|^{\frac{2}{3}} \nabla_x u\|_2^2 - \frac{\mu \alpha^2}{2} \||x|^{\frac{2}{3}} \nabla_x u\|_2 \||x|^{\frac{2}{3}} \text{div} u\|_2 - \mu \||x|^{\frac{2}{3}} \text{div} u\|_2^2 \\
\geq C(\alpha) \left(\||x|^{\frac{2}{3}} \nabla_x u\|_2^2 + \||x|^{\frac{2}{3}} \text{div} u\|_2^2\right).
\]

We combine all estimates for \(\mathcal{L}_2(i = 1, 2)\) in (11) and choose \(\varepsilon\) sufficiently small, and then integrate (11) with respect to \(t\) over \([0, T]\) to derive (10). \(\square\)

We apply the operator \(\text{div}\) to the momentum equation (1), to have

\[\text{div}(\rho u) + \text{div}(\text{div}(\rho u \otimes u) + (u - u_f) \rho_f) = \Delta_x F,\]

where the effective viscous flux \(F\) is defined by

\[F := (2\mu + \lambda(\rho))\text{div}u - P(\rho).
\]

On the other hand, consider the following three elliptic problems on the whole space \(\mathbb{R}^2:\)

\[- \Delta_x \psi = \text{div}(\rho u), \quad \lim_{|x| \to +\infty} \psi(x) = 0,\]

\[- \Delta_x \eta_1 = \text{div}(\rho u \otimes u), \quad \lim_{|x| \to +\infty} \eta_1(x) = 0,\]

\[- \Delta_x \eta_2 = \text{div}((u - u_f) \rho_f), \quad \lim_{|x| \to +\infty} \eta_2(x) = 0.\]

For (13), we can derive the following elliptic estimates in the following lemma.

**Lemma 3.4.** [26, 38] Let \([\psi, \eta_1, \eta_2]\) be solutions to the elliptic problems in (13). Then, we have

(i) \(\|\nabla_x \psi\|_{2m} \lesssim m\|\rho\|_{\frac{2m}{2m + k}} \|u\|_{2mk}, \quad \forall k > 1, m \geq 1;\)

\(\|\nabla_x \psi\|_{2 - r} \lesssim \|\sqrt{\rho} u\|_2 \|\rho\|_{\frac{3}{2 - r}}, \quad 0 < r < 1.\)

(ii) \(\|\eta_1\|_{2m} \lesssim m\|\rho\|_{\frac{2m}{2m + k}} \|u\|_{2mk}, \quad \forall k > 1, m \geq 1;\)

\(\|\nabla_x \eta_2\|_{2m} \lesssim m\|(u - u_f) \rho_f\|_{2m}, \quad \forall m \geq 1.\)

With Lemma 3.4, we can further derive estimates for \(u, \psi, \eta_i\) as follows.
Lemma 3.5. The following estimates hold.

(i) \( \|\psi\|_{2m} \leq m^{\frac{1}{2}} \|\nabla_x \psi\|_{\frac{2m}{m+1}} \leq C(T)m^{\frac{1}{2}} \|\rho\|_{\frac{1}{m}}, \quad \forall \ m \geq 1, \)

(ii) \( \|u\|_{m} \leq m^{\frac{1}{2}} \|\nabla_x u\|_{2} \frac{1}{\alpha} \|\rho\|_{\frac{2m}{2m+1}} \|\nabla_x u\|_{\frac{2m}{2}}, \quad \forall \ m \geq \frac{4}{\alpha}, \)

(iii) \( \|\nabla_x \psi\|_{2m} \leq m^{\frac{1}{2}} k \|\rho\|_{\frac{2m}{2m+1}} \|\nabla_x u\|_{2} \frac{1}{\alpha} \|\nabla_x u\|_{\frac{2m}{2}}, \quad \forall \ k > 1, \ mk \geq \frac{2}{\alpha}, \)

(iv) \( \|\eta_1\|_{2m} \leq m^{\frac{1}{2}} k \|\rho\|_{\frac{2m}{2m+1}} \|\nabla_x u\|_{2} \frac{2+\frac{4}{\alpha}}{2m} \|\nabla_x u\|_{\frac{2m}{2}}, \quad \forall \ k > 1, \ m \geq 1, \)

(v) \( \|\eta_2\|_{2m} \leq C(T)m \left( \int_{\mathbb{R}^4} (u - v)^2 f dv dx \right)^{\frac{1}{2}} \times \int_{0}^{T} \|\nabla_x u\|_{2} \frac{1}{\alpha} \|\nabla_x u\|_{\frac{2m}{2}}, \quad \forall \ t \in [0, T], \forall \ m \geq \frac{2}{\alpha}. \)

Proof. For (i), we use Lemma 3.2 and (ii) in Lemma 3.4 to see

\( \|\psi\|_{2m} \leq Cm^{\frac{1}{2}} \|\nabla_x \psi\|_{\frac{2m}{m+1}} \leq Cm^{\frac{1}{2}} \sqrt{\rho} \|u\|_{m}. \)

To prove (ii), we write \( m = 2^n a, (1 \leq a \leq 2), \) where \( n \) is a positive integer, and without loss of generality, we may assume \( n \geq 2. \) Then we use (1) in Lemma A.2 with \( \frac{2m'}{2m'} = 2^n a, \) i.e. \( m' = \frac{2^n a}{2n+1}, \) to have

\( \|u\|_{2^n a} \leq C \left( \frac{2^{n-1} a + 1}{2} \right)^{\frac{1}{2}} \|\nabla_x u\|_{\frac{2^n a + 1}{2}}. \)

Thus, we have

\[ \|u\|_{2^n a} = \|u^2\|_{\frac{1}{2} - 1, a} \leq C \left( \frac{2^{n-1} a + 1}{2} \right)^{\frac{1}{2}} \|\nabla_x (u^2)\|_{\frac{1}{2} - 1, a+1} \]

\[ \leq C \left( \frac{2^{n-1} a + 1}{2} \right)^{\frac{1}{2}} (\|\nabla_x u\|_{2})^{\frac{1}{2}} \|u\|_{2^n a - a}, \]

and for \( n \geq 3, \)

\[ \|u\|_{\frac{1}{2} - 1, a} = \|u^2\|_{\frac{1}{2} - 2, a} \leq C \left( \frac{2^{n-2} a + 1}{2} \right)^{\frac{1}{2}} \|\nabla_x (u^2)\|_{\frac{1}{2} - 2, a+1} \]

\[ \leq C \left( \frac{2^{n-2} a + 1}{2} \right)^{\frac{1}{2}} \|\nabla_x u\|_{2}^{\frac{1}{2}} \|u\|_{2^n a - a}, \]

Therefore, by mathematical induction, we have

\[ \|u\|_{m} = \|u\|_{2^n a} \leq C \left( \sum_{i=2}^{n-2} \frac{n-1}{2^{i-1}} \right) \left( \frac{2^{n-1} a + 1}{2} \right)^{\frac{1}{2}} (\|\nabla_x u\|_{2})^{\frac{1}{2}} \|u\|_{2^n a - a} \]

\[ \leq C \left( 1 - 2^{-n} \right) \left( \frac{2^{n-1} a + 1}{2} \right)^{\frac{1}{2}} \|\nabla_x u\|_{2}^{\frac{1}{2}} \|u\|_{2^n a - a} \]

\[ \times \|\nabla_x u\|_{2}^{\frac{1}{2}} \|\nabla_x u\|_{\frac{1}{2} - 1, a} \]

\[ \leq C \|\nabla_x u\|_{2} \frac{1}{\alpha} \|u\|_{2^n a - a} \]

where we have used the relation \( m = 2^n a \) and the following interpolation inequality

\[ \|u\|_{4a} \leq C \|\nabla_x u\|_{2}^{\frac{1}{2}} \|u\|_{\frac{1}{2}} \leq C \|\nabla_x u\|_{2}^{\frac{1}{2}} \|u\|_{\frac{1}{2}} \]
by (3). Similarly, (iii) and (iv) can be derived.

Now we prove the last estimate (v). From (i), we have

$$
\|\eta_2\|_{2m} \lesssim m^{\frac{1}{2}} \|\nabla_x \eta_2\|_{\frac{2m}{m+1}} \lesssim m^{\frac{1}{2}} \|(u - u_f)\rho_f\|_{\frac{2m}{m+1}}.
$$

(14)

On the other hand, we use (ii) in Lemma 3.1, Lemma 3.2 and (ii) in this lemma to have

$$
\|(u - u_f)\rho_f\|_{\frac{2m}{m+1}} 
\leq C \left( \int_{\mathbb{R}^4} (u - v)^2 f \, dv \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^4} |u|^{2m-2} f \, dv \right) \, dx \right)^{\frac{1}{2m}}
$$

$$
\leq C \left( \int_{\mathbb{R}^4} (u - v)^2 f \, dv \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} |u|^{2m-2} f \, dv \right) \, dx
$$

$$
\leq C(T) \left( \int_{\mathbb{R}^4} (u - v)^2 f \, dv \, dx \right)^{\frac{1}{2}} \int_0^T \|u\|_{2m} \, dt
$$

$$
\leq C(T) m^{\frac{1}{2}} \left( \int_{\mathbb{R}^4} (u - v)^2 f \, dv \, dx \right)^{\frac{1}{2}} \int_0^T \|\nabla_x u\|_{2m}^{1-\frac{1}{2m}} \|x\|^{\frac{1}{2}} \nabla_x u\|_{2m}^{\frac{1}{2}} \, dt.
$$

We combine (14) and (15) to derive a desired estimate (v).

It follows from (13) and (12) that

$$
\Delta \psi (\psi + \eta_1 + \eta_2 + F) = 0, \quad \int_{\mathbb{R}^2} (\psi + \eta_1 + \eta_2 + F) \, dx = 0,
$$

which yields

$$
\psi + \eta_1 + \eta_2 + F = 0.
$$

We define

$$
\Lambda(\rho) := \int_1^\rho \frac{2\mu + \lambda(s)}{s} \, ds = 2\mu \log \rho + \frac{1}{\beta} (\rho^\beta - 1).
$$

It follows from the definition of the effective viscous flux $F$ and (1) that

$$
(\Lambda(\rho) - \psi)_t + u \cdot \nabla_x (\Lambda(\rho) - \psi) + P(\rho) - \eta_1 - \eta_2 + u \cdot \nabla_x \psi = 0.
$$

(16)

Next, we derive the $L_t^\infty L_x^p$ estimate of the density $\rho(x, t)$ by using (16).

**Lemma 3.6.** Let $\beta > 1$, and assume that the same conditions in Lemma 3.3 hold. Then, we have

$$
\sup_{0 \leq t \leq T} \|\rho\|_{p}(t) \leq C(T, p, \beta), \quad p \geq 1,
$$

$$
\int_{\mathbb{R}^2} |x|^\alpha (\rho u^2 + \rho^\gamma) \, dx + \int_0^t \int_{\mathbb{R}^4} |x|^\alpha f |u|^2 \, dv \, dx \, dt \quad + \quad \int_0^t \|\|x\|^{\frac{1}{2}} \nabla_x u\|_2^2 \, dt \quad - \quad \int_0^t \|\|x\|^{\frac{1}{2}} \nabla_x u\|_2^2 \, dt \leq C(T).
$$

(17)

**Proof.** We set $(h)_+$ to be the positive part of a function $h$, and we multiply (16) by $\rho((\Lambda(\rho) - \psi)_t)^{2m-1}$ with $m \gg 1$ and integrate the resulting relation over $\mathbb{R}^2$ to
obtain

\[
\begin{aligned}
\frac{1}{2m} \frac{d}{dt} \int_{\mathbb{R}^2} \rho([\Lambda(\rho) - \psi]_+)^{2m} dx + \int_{\mathbb{R}^2} \rho P(\rho)[(\Lambda(\rho) - \psi)]^{2m-1} dx \\
= \int_{\mathbb{R}^2} \rho \eta_1([\Lambda(\rho) - \psi]_+)^{2m-1} dx + \int_{\mathbb{R}^2} \rho \eta_2([\Lambda(\rho) - \psi])^{2m-1} dx \\
- \int_{\mathbb{R}^2} \rho u \cdot \nabla_x \psi((\Lambda(\rho) - \psi))^{2m-1} dx \\
= : \sum_{i=1}^3 \mathcal{I}_{3i}.
\end{aligned}
\]

For notational simplicity, we define

\[
g(t) := \left( \int_{\mathbb{R}^2} \rho([\Lambda(\rho) - \psi]_+)^{2m} dx \right)^{\frac{1}{2m}},
\]

and estimate \( \mathcal{I}_{3i} \) one by one as follows.

- **Case C.1 (estimate of \( \mathcal{I}_{31} \))** We use \((iv)\) in Lemma 3.5 to have

\[
|\mathcal{I}_{31}| \leq \int_{\mathbb{R}^2} \rho^{\frac{1}{2m}} \eta_1 \| \rho(\Lambda(\rho) - \psi)_+ \|^{2m-1} dx \\
\leq \| \rho \|^{\frac{1}{2m + \frac{1}{2}}} \eta_1 \| \rho(\Lambda(\rho) - \psi)_+ \|^2 \| \nabla_x u \|^2 \\
\leq C(m, k) \| \rho \|^{\frac{1}{2m + \frac{1}{2}}} \| \nabla_x u \|^2 \| \nabla_x u \|^2 g^{2m-1}(t),
\]

where we have chosen \( k = \frac{\beta}{\beta - 2} \) in the last inequality.

- **Case C.2 (Estimate of \( \mathcal{I}_{32} \))** We use \((v)\) in Lemma 3.5 to have

\[
|\mathcal{I}_{32}| \leq \int_{\mathbb{R}^2} \rho^{\frac{1}{2m}} \| \rho(\Lambda(\rho) - \psi)_+ \|^{2m-1} dx \\
\leq C \| \rho \|^{\frac{1}{2m + \frac{1}{2}}} \eta_2 \| \rho(\Lambda(\rho) - \psi)_+ \|^2 \| \nabla_x u \|^2 g^{2m-1}(t) \\
\leq C(T, m) \| \rho \|^{\frac{1}{2m + \frac{1}{2}}} \eta_2 \| \nabla_x u \|^2 g^{2m-1}(t) \left( \int_{\mathbb{R}^4} (u - v)^2 f d\nu dx \right)^{\frac{1}{2}} \\
\times \int_0^t \| \nabla_x u \|^2 \| |x| \frac{1}{2} \nabla_x u \|^{\frac{2}{m}} \| \right)^{\frac{1}{2}} d\tau,
\]

where \( m' = m + \frac{1}{2} \).

- **Case C.3 (Estimate of \( \mathcal{I}_{33} \))** We choose

\[
p' = q' = \frac{2m\beta + 1}{m\beta}, \quad k = \frac{\beta}{\beta - 2}.
\]
and use Lemma 3.5 to have

\[
|I_{33}| \leq \int_{\mathbb{R}^2} \rho^{\frac{1}{2m \beta + 1}} |u||\nabla_x \psi||\rho(\Lambda(\rho) - \psi)^{2 \frac{1}{2}}|^{\frac{2m-1}{2m}} \mathrm{d}x \\
\leq \|ho\|_{2m \beta + 1}^{\frac{1}{2m \beta + 1}} \|
abla_x \psi\|_{2m \beta + 1}^{\frac{1}{2m \beta + 1}} \|ho(\Lambda(\rho) - \psi)^{2 \frac{1}{2}}\|_{2m \beta + 1}^{\frac{2m-1}{2m}} \\
\leq C(m, \beta) \rho \|
abla_x u\|_{2}^{1+ \frac{1}{2m \beta + 1}} \|
abla_x \psi\|_{2m \beta + 1}^{\frac{1}{2m \beta + 1}} \|\nabla_x u\|_{2m \beta + 1}^{\frac{1}{2m \beta + 1}} \|\nabla_x u\|_{2m \beta + 1}^{\frac{1}{2m \beta + 1}} \\
\leq C(m, \beta) \rho \|
abla_x u\|_{2}^{1+ \frac{1}{2m \beta + 1}} \|\nabla_x \psi\|_{2m \beta + 1}^{\frac{1}{2m \beta + 1}} \|\nabla_x u\|_{2m \beta + 1}^{\frac{1}{2m \beta + 1}} g^{2m-1}(t) \\
\leq C(m, \beta) \rho \|
abla_x u\|_{2}^{1+ \frac{1}{2m \beta + 1}} \|\nabla_x \psi\|_{2m \beta + 1}^{\frac{1}{2m \beta + 1}} \|\nabla_x u\|_{2m \beta + 1}^{\frac{1}{2m \beta + 1}} g^{2m-1}(t).
\]

In (18), we collect all estimates to find

\[
g(t) \leq C(T, m, \beta) \left( \int_{0}^{t} \|ho\|_{2m \beta + 1}^{\frac{1}{2m \beta + 1}} \|\nabla_x u\|_{2}^{1+ \frac{1}{2m \beta + 1}} \|\nabla_x \psi\|_{2m \beta + 1}^{\frac{1}{2m \beta + 1}} \|\nabla_x u\|_{2m \beta + 1}^{\frac{1}{2m \beta + 1}} + \int_{0}^{t} \|\nabla_x u\|_{2}^{2- \frac{1}{2m \beta + 1}} \|\nabla_x u\|_{2m \beta + 1}^{\frac{2m-1}{2m \beta + 1}} \|\nabla_x u\|_{2m \beta + 1}^{\frac{1}{2m \beta + 1}} \mathrm{d}r \right) + \int_{t}^{T} \|\nabla_x u\|_{2}^{2- \frac{1}{2m \beta + 1}} \|\nabla_x u\|_{2m \beta + 1}^{\frac{2m-1}{2m \beta + 1}} \|\nabla_x u\|_{2m \beta + 1}^{\frac{1}{2m \beta + 1}} \mathrm{d}r \right). (19)
\]

We set

\[\Omega_1(t) := \{x \in \mathbb{R}^2 : \rho(x, t) > 2\} \quad \text{and} \quad \Omega_2(t) := \{x \in \Omega_1(t) : \Lambda(\rho)(x, t) - \psi(x, t) > 2\}.
\]

Note that, \(\Lambda(\rho)(x, t) > 0\) in \(\Omega_1(t)\). Therefore, we have

\[|\Lambda(\rho)(x, t) - \psi(x, t)| \leq 2 + |\psi|, \quad x \in \Omega_1(t) \setminus \Omega_2(t).
\]

Then, we have

\[\|\rho\|_{2m \beta + 1}(t) \leq \left( \int_{\Omega_1(t)} \rho^{2m \beta + 1} \mathrm{d}x + \int_{\mathbb{R}^2 \setminus \Omega_1(t)} \rho^{2m \beta + 1} \mathrm{d}x \right)^{\frac{1}{2m \beta + 1}} \leq 2 \left( \int_{\Omega_1(t)} \rho^{2m \beta + 1} \mathrm{d}x \right)^{\frac{1}{2m \beta + 1}} + 2 \left( \int_{\mathbb{R}^2 \setminus \Omega_1(t)} 2m \beta \rho \mathrm{d}x \right)^{\frac{1}{2m \beta + 1}} \leq C \left( \int_{\Omega_1(t)} \rho^{2m \beta + 1} \mathrm{d}x \right)^{\frac{1}{2m \beta + 1}} + C \leq C \left( \int_{\Omega_1(t)} \rho \Lambda(\rho) \mathrm{d}x + \int_{\Omega_1(t)} \rho \psi \|\rho\|_{2m \beta + 1} \mathrm{d}x \right) \leq C \left( \int_{\Omega_1(t)} \rho \Lambda(\rho) \mathrm{d}x + \int_{\Omega_1(t)} \rho \psi \|\rho\|_{2m \beta + 1} \mathrm{d}x \right) \leq C \left( g^{2m}(t) + \int_{\mathbb{R}^2} \rho \psi \|\rho\|_{2m \beta + 1} \mathrm{d}x \right)^{\frac{1}{2m \beta + 1}} + C \leq C \left[ g(t) + \left( \int_{\mathbb{R}^2} \rho \psi \|\rho\|_{2m \beta + 1} \mathrm{d}x \right)^{\frac{1}{2m \beta + 1}} + 1 \right].
\]
Corollary 1. Under the conditions in Lemma 3.3, we have

On the other hand, by (i) in Lemma 3.5 and Young’s inequality, we have

Note that

We use (ii) in Lemma 3.1, (10) and (19) to have

Then (17) follows from (10) and (17).

We use (ii) in Lemma 3.1, (10) and (19) to have

We now apply the Gronwall Lemma for (20) using the estimate (ii) in Lemma 3.1 to find (17)1. Then (17)2 follows from (10) and (17)1.

**Corollary 1.** Under the conditions in Lemma 3.3, we have

\[ (i) \sup_{0 \leq t \leq T} \int_{\mathbb{R}^4} (1 + |v|^{k_3}) f \, dx \leq C(T), \quad 2 \leq k_3 < \infty. \]

\[ (ii) \sup_{0 \leq t \leq T} \|\rho_f\|_p^p \leq C(T), \quad p \geq 2, \quad \sup_{0 \leq t \leq T} \|\rho_f \|_p^p \leq C(T), \quad p \geq \frac{4}{3}. \]  

\[ (iii) \sup_{0 \leq t \leq T} \|m_2 f\|_p^p \leq C(T), \quad p \geq 1. \]
Proof. The estimates in (21) are easy consequences of Lemma 3.2 and Lemma 3.6. For brevity, we only show the estimate (iii). In fact, we use (i) in Lemma 3.2 for $k_1 = 2$ and $k_2 = 4p - 2$ to have

$$\|m_2 f\|_p^p \leq C(T) \int_{\mathbb{R}^4} |v|^{4p-2} f dv dx.$$  

Then we further use (ii) in Lemma 3.2 for $k = 4p - 2$, (3) in Lemma A.4, (ii) in Lemma 3.5 and Lemma 3.6 to have

$$\|m_2 f\|_p^p \leq C(T) \left[ 1 + \left( \int_0^T \|u\|_{4p} dt \right)^{4p} \right] \leq C(T) \left[ 1 + \left( \int_0^T \|x\|_{2}^2 \|\nabla x u\|_{4p}^{1-\frac{2}{p}} dt \right)^{4p} \right] \leq C(T).$$

$\square$

3.3. Estimates on the fluid density. In this subsection, we provide an upper bound of the fluid density by the method of characteristics. With the help of the Brezis-Wainger inequality, we combine the estimate of $\|\nabla x u\|_2^2$ and $\|\rho u\|_p$ ($p \geq 2\gamma$) to derive the upper bound estimate of the fluid density. We set the material derivative of the fluid velocity by $\dot{u}$:

$$\dot{u} = \partial_t u + u \cdot \nabla x u,$$

and introduce nonlinear functionals:

$$Z^2(t) := \int_{\mathbb{R}^2} \left( \mu \omega^2 + \frac{F^2}{2\mu + \lambda(\rho)} \right) dx + \int_{\mathbb{R}^4} (u - v)^2 f dv dx,$$

$$\chi^2(t) := \int_{\mathbb{R}^2} \rho |\dot{u}|^2 dx, \quad \Phi_T := \|\rho\|_{L^\infty_{x,t}} + 1.$$

Lemma 3.7. Let $T$ be a positive constant and suppose that the conditions in Lemma 3.3 hold. Then, for $t \in [0, T]$ we have

$$\log \left\{ e + Z^2(t) \right\} + \int_0^t \frac{\chi^2}{e + Z^2} d\tau \leq C(T) \Phi_T^{1+\beta_t},$$

where $\varepsilon > 0$ is a constant which can be arbitrarily small.

Proof. We denote the perpendicular gradient by $\nabla^\perp := (\partial_{x_2}, -\partial_{x_1})$. Then, the momentum equation can be rewritten as follows:

$$\rho \dot{u} = \nabla x F + \mu \nabla^\perp \omega - (u - u_f) \rho_f.$$

We multiply the above identity by $\dot{u}$ to obtain

$$\int_{\mathbb{R}^2} \rho |\dot{u}|^2 dx = - \int_{\mathbb{R}^2} F \text{div} \dot{u} dx - \mu \int_{\mathbb{R}^2} \omega \nabla^\perp \dot{u} dx - \int_{\mathbb{R}^2} \dot{u} \cdot (u - u_f) \rho_f dx.$$  

On the other hand, we use the relations:

$$\text{div} \dot{u} = (\text{div} u)_t + u \cdot \nabla x \text{div} u - 2\nabla x u_1 \cdot \nabla^\perp u_2 + (\text{div} u)^2,$$

$$\nabla^\perp \cdot \dot{u} = (\omega)_t + u \cdot \nabla x \omega + \omega \text{div} u,$$

to find

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \left( \mu \omega^2 + \frac{F^2}{2\mu + \lambda(\rho)} \right) dx + \frac{d}{dt} \int_{\mathbb{R}^4} \left( \frac{1}{2} u^2 f - u f x + f dx + \int_{\mathbb{R}^2} \rho |\dot{u}|^2 dx.$$
Then, the above estimates in (25) yield
\[ \begin{align*}
&= -\frac{\mu}{2} \int_{\mathbb{R}^2} \omega^2 \text{div} u dx + 2 \int_{\mathbb{R}^2} F \nabla_x u_1 \cdot \nabla^\perp u_2 dx \\
&\quad + \frac{1}{2} \int_{\mathbb{R}^2} F^2 \text{div} u \left[ \rho \left( \frac{1}{2\mu + \lambda(\rho)} \right) - \frac{1}{2\mu + \lambda(\rho)} \right] dx \\
&\quad + \int_{\mathbb{R}^2} F \text{div} u \left[ \rho \left( \frac{P(\rho)}{2\mu + \lambda(\rho)} \right) - \frac{P(\rho)}{2\mu + \lambda(\rho)} \right] dx - \int_{\mathbb{R}^2} u \cdot \nabla_x u(u - u_f) \rho_f dx \\
&\quad + \int_{\mathbb{R}^2} \left( \frac{1}{2} u^2 f_t - u f_x \right) dv dx \\
&=: \sum_{i=1}^6 \mathcal{I}_{4i}.
\end{align*} \]

Now, we estimate the terms \( \mathcal{I}_{4i}, i = 1, \ldots, 6 \), separately.

- Case D.1 (Estimate of \( \mathcal{I}_{41} \)): We use the relations
  \[ \Delta_x F = \text{div}(\rho \dot{u} + (u - u_f) \rho_f), \quad \mu \Delta_x \omega = \nabla^\perp \cdot (\rho \dot{u} + (u - u_f) \rho_f), \]
  the elliptic estimates, Sobolev inequality and Corollary 1 to get
  \[ \begin{align*}
  \| \nabla_x F \|_2 + \| \nabla_x \omega \|_2 &\leq C(\| \rho \dot{u} \|_2 + \| (u - u_f) \rho_f \|_2) \\
  &\leq C(\Phi_T^\frac{1}{2} \chi(t) + \| \sqrt{\rho_f} (u - u_f) \|_2 + 1), \\
  \| \omega \|_4 &\leq C(\| \omega \|_2 \| \nabla_x \omega \|_2) \leq C(T) Z^\frac{1}{2}(t)(\Phi_T^\frac{1}{2} \chi(t) + \| \sqrt{\rho_f} (u - u_f) \|_2 + 1)^\frac{1}{2}.
  \end{align*} \]
  Then, the above estimates in (25) yield
  \[ \begin{align*}
  |\mathcal{I}_{41}| &\leq C \| \omega \|_2^2 \| \text{div} u \|_2 \leq C(T) Z(t)(\Phi_T^\frac{1}{2} \chi(t) + \| \sqrt{\rho_f} (u - u_f) \|_2 + 1) \| \nabla_x u \|_2 \\
  &\leq \frac{1}{8} \chi^2(t) + C(T)(Z^2(t) + 1)(\| \sqrt{\rho_f} (u - u_f) \|_2 + 1) \Phi_T.
  \end{align*} \]

- Case D.2 (Estimate of \( \mathcal{I}_{42} \)): By the duality between Hardy \( \mathcal{H}^1 \) and \( \text{BMO} \) spaces, we have
  \[ \begin{align*}
  |\mathcal{I}_{42}| &\leq C \| F \|_{\text{BMO}} \| \nabla_x u_1 \cdot \nabla^\perp u_2 \|_{\mathcal{H}^1} \leq C \| \nabla_x F \|_2 \| \nabla_x u_1 \|_2 \| \nabla^\perp u_2 \|_2 \\
  &\leq C(T)(\Phi_T^\frac{1}{2} \chi(t) + \| \sqrt{\rho_f} (u - u_f) \|_2 + 1) \| \nabla_x u \|_2^2 \\
  &\leq \frac{1}{8} \chi^2(t) + C(T)(Z^2(t) + 1)(\| \sqrt{\rho_f} (u - u_f) \|_2 + 1) \Phi_T.
  \end{align*} \]

- Case D.3 (Estimate of \( \mathcal{I}_{43} \) and \( \mathcal{I}_{44} \)): We use \( \| F \|_2 \leq \Phi_T^\frac{2}{T} Z(t) \) and (25) to get
  \[ \begin{align*}
  \| \frac{F^2}{2\mu + \lambda(\rho)} \|_2 &\leq C \| \frac{F}{\sqrt{2\mu + \lambda(\rho)}} \|_2 \| F \|^{1-\varepsilon}_2 \| F \|^{1+\varepsilon}_2 \| \nabla_x F \|_2 \leq C(T)(\Phi_T^\frac{8}{T} \chi(t) + \| \sqrt{\rho_f} (u - u_f) \|_2 + 1),
  \end{align*} \]
for arbitrarily small constant $\varepsilon > 0$. Then we have

$$|I_{43} + I_{44}| \leq C \int_{\mathbb{R}^d} \left( |F|^2 |\text{div}u| \frac{1}{2\mu + \lambda(\rho)} + |F| |\text{div}u| \frac{P(\rho)}{2\mu + \lambda(\rho)} \right) \, dx$$

$$\leq C \|\nabla x u\|_2 \left( \left\| \frac{F^2}{2\mu + \lambda(\rho)} \right\|_2 + \|F\|_{2(\alpha+2)} \|P(\rho)\|_{2+} \right)$$

$$\leq C \|\nabla x u\|_2 \left( \left\| \frac{F^2}{2\mu + \lambda(\rho)} \right\|_2 + \|F\|_{2+} \|\nabla F\|_{2+} \right)$$

$$\leq C(T) \|\nabla x u\|_2 \left( Z(t) \Phi_T^{1+\beta \varepsilon} (\Phi_T^{3}(t) + \|\sqrt{p_f}(u - u_f)\|_2 + 1) \right)$$

$$\leq \frac{1}{8} \chi^2(t) + C(T) (Z^2(t) + 1)(\|\nabla x u\|_2^2 + \|\nabla z u\|_2^2).$$

• Case D.4 (Estimate of $I_{45}$) From (ii) in Lemma 3.5 and Corollary 1, we have

$$|I_{45}| \leq C(\|\rho_f\|_4 \|\nabla x u\|_2 \|u\|_8^2 + \|u\|_8 \|\nabla x u\|_2 \|u_f \rho_f\|_2)$$

$$\leq C(T) (Z^2(t) + 1)(\|\nabla x u\|_2^2 + \|\nabla z u\|_2^2).$$

• Case D.5 (Estimate of $I_{46}$) We have from (1) that

$$\int_{\mathbb{R}^2} f_t \, dv = - \int_{\mathbb{R}^2} v \cdot \nabla_x f \, dv,$$

$$\int_{\mathbb{R}^2} v f_t \, dv = - \int_{\mathbb{R}^2} v \cdot \nabla x f \, dv + \int_{\mathbb{R}^2} (L(f) + u - v) f \, dv.$$ (26)

Applying the integration by parts on space variables, we use Corollary 1 to have

$$|I_{46}| = \left| \int_{\mathbb{R}^4} u \cdot \nabla x u \cdot v f \, dv dx - \int_{\mathbb{R}^4} \nabla x u \cdot v \cdot f \, dv dx - \int_{\mathbb{R}^4} u \cdot (L(f) + u - v) f \, dv dx \right|$$

$$\leq C \left( \|u\|_8 \|\nabla x u\|_2 \|u_f \rho_f\|_2 + \|\nabla x u\|_2 \|u_f \|_2 + \int_{\mathbb{R}^4} f(|u - v|^2 + |v|^2) \, dv dx \right)$$

$$\leq C(T) (Z^2(t) + 1)(\|\nabla x u\|_2^2 + \|\nabla z u\|_2^2).$$

Collecting the estimates of $I_{4i}(1 \leq i \leq 6)$ in (23), we have

$$\frac{d}{dt} \left[ Z^2(t) - \int_{\mathbb{R}^4} \left[ (|u - v|^2 - |v|^2) f \right] \, dv dx \right] + \chi^2(t)$$

$$\leq C(T) (Z^2(t) + 1)(\|\nabla x u\|_2^2 + \|\nabla z u\|_2^2 + \|\sqrt{p_f}(u - u_f)\|_2^2 + 1) \Phi_T^{1+\beta \varepsilon}.$$ (27)

On the other hand, we apply the integration by parts, and use Lemma 3.1, (3), (21) to have

$$\frac{1}{2} \int_{\mathbb{R}^4} v^2 f \, dv dx = \int_{\mathbb{R}^4} [v \cdot (L(f) + u - v) f] \, dv dx + 2 \int_{\mathbb{R}^4} |v - v_c|^2 f \, dv dx$$

$$\leq C(\psi_M + 1) \left( 1 + \int_{\mathbb{R}^4} v^2 f \, dv dx \right) + \|u\|_{\frac{4}{3}} \|\rho_f u_f\|_{\frac{4}{3}} + C |v_c|^2$$

$$\leq C(T) (1 + \|\nabla x u\|_2).$$

This together with (ii) in Lemma 3.1 implies

$$\frac{d}{dt} Z^2(t) + \chi^2(t) \leq C(T) (Z^2(t) + 1)$$

$$\times (\|\nabla x u\|_2^2 + \|\nabla z u\|_2^2 + \|\sqrt{p_f}(u - u_f)\|_2^2 + 1) \Phi_T^{1+\beta \varepsilon}.$$ (28)

Then we further use Lemma 3.6 to derive (22).
Now we estimate with (α)

\[ \alpha = (2 + \alpha') \]

Lemma 3.8. Assume the conditions in Lemma 3.3 hold. Then it holds that

\[
\begin{align*}
(i) \quad & \| \rho u \|_p \leq C(T) \Phi_T^{1+\frac{\alpha}{2}} \left( \| \nabla_x u \|_2 + \| |x|^\frac{\alpha}{2} \nabla_x u \|_2 \right)^{1 - \frac{\alpha}{2}}, \\
(ii) \quad & \| \rho u \|_{2\gamma} \leq C(T) \Phi_T^{\frac{\alpha}{2}} (1 + \| \nabla_x u \|_2).
\end{align*}
\]

Proof. Before the proof of (i) in (27), we first derive the estimate of \( \int_{\mathbb{R}^2} \rho |u|^{2+\alpha'} \, dx \)

with \( \alpha' = \frac{\alpha}{2(\gamma+1)} \Phi_T^{-\frac{\alpha}{2}} \). To the end, we multiply the momentum equation by \( (2 + \alpha') |u|^{\alpha'} \) to have

\[
\rho |u|^{2+\alpha'} \frac{d}{dt} \rho u + \rho u \cdot \nabla_x |u|^{2+\alpha'} = (2 + \alpha') |u|^{\alpha'} \left( -\nabla_x P(\rho) + \mu \Delta_x u + \nabla_x ((\mu + \lambda(\rho)) \text{div} u) - \int_{\mathbb{R}^2} (u - v) f \, dv \right).
\]

We integrate the above equation over \( \mathbb{R}^2 \), and use integration by parts to obtain

\[
\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}^2} \rho |u|^{2+\alpha'} \, dx & + \mu (2 + \alpha') \int_{\mathbb{R}^2} |u|^{\alpha'} |\nabla_x u|^{2} \, dx + \mu (2 + \alpha') \int_{\mathbb{R}^2} \nabla_x |\nabla_x u|^{2} \, dx \\
& + (2 + \alpha') \int_{\mathbb{R}^2} (\mu + \lambda(\rho)) (\text{div} u)^2 |u|^{\alpha'} \, dx + (2 + \alpha') \int_{\mathbb{R}^2} (u - v)^2 f |u|^{\alpha'} \, dv \, dx \\
& = (2 + \alpha') \int_{\mathbb{R}^2} P(\rho) \text{div}(u |u|^{\alpha'}) \, dx - (2 + \alpha') \int_{\mathbb{R}^2} (\mu + \lambda(\rho)) \text{div} u \cdot \nabla_x |u|^{\alpha'} \, dx \\
& - (2 + \alpha') \int_{\mathbb{R}^2} (u - v) \cdot v f |u|^{\alpha'} \, dv \, dx \\
& =: \sum_{i=1}^{3} I_{5i}.
\end{align*}
\]

Now we estimate \( I_{5i} (1 \leq i \leq 3) \) one by one.

- Case E.1 (Estimate of \( I_{51} \) and \( I_{52} \)): Similar to the computation in [29, 30], we use (ii) in Lemma 3.5 to have

\[
\begin{align*}
|I_{51}| & \leq \| \nabla_x u \|_2^2 + C \int_{\mathbb{R}^2} (P(\rho))^2 |u|^{2\alpha'} \, dx \\
& \leq \| \nabla_x u \|_2^2 + C \| P(\rho) \|_2^{2+\frac{\alpha}{4-\alpha}} \| u \|_8^{\alpha'} \\
& \leq C(T)(\| \nabla_x u \|_2^2 + \| |x|^\frac{\alpha}{2} \nabla_x u \|_3^2),
\end{align*}
\]

\[
|I_{52}| \leq \frac{(2 + \alpha')}{2} \int_{\mathbb{R}^2} (\mu + \lambda(\rho)) (\text{div} u)^2 |u|^{\alpha'} \, dx + \frac{\mu(2 + \alpha')}{4} \int_{\mathbb{R}^2} |u|^{\alpha'} |\nabla_x u|^{2} \, dx,
\]

- Case E.2 (Estimate of \( I_{53} \)): We use (ii) in Lemma 3.5 and Corollary 1 to have

\[
\begin{align*}
|I_{53}| & \leq C \int_{\mathbb{R}^2} \left( (u^2 + 1)|u|f |\rho_f| + (|u| + 1)m_2 f \right) \, dx \\
& \leq C(1 + \| u_f \rho_f \|_4 \| u \|_8 + \| u_f \rho_f \|_4 \| u \|_4 + \| m_2 f \|_2 \| u \|_8 + \| m_2 f \|_1) \\
& \leq C(T)(\| \nabla_x u \|_2^2 + \| |x|^\frac{\alpha}{2} \nabla_x u \|_3^2 + 1).
\end{align*}
\]

Therefore, we combine the above three estimates and (28) to have

\[
\frac{d}{dt} \int_{\mathbb{R}^2} \rho |u|^{2+\alpha'} \, dx \leq C(T)(\| \nabla_x u \|_2^2 + \| |x|^\frac{\alpha}{2} \nabla_x u \|_3^2 + 1),
\]
which implies
\[
\sup_{0 \leq t \leq T} \int_{\mathbb{R}^2} \rho \lvert u \rvert^{2+\alpha'} \, dx \leq C(T).
\]
Thus, for \( q = (1 + \frac{2}{\alpha'}) (p - 2) \leq C \Phi_\frac{3}{T} \), we use interpolation inequality to obtain the first result:
\[
\begin{align*}
\|\rho u\|_p &\leq \|\rho u\|_{p_{2+\alpha'}}^{\frac{3}{2}} \|\rho u\|_q^{1-\frac{3}{2}} \\
&\leq C(T) \Phi_\frac{3}{T}^{1-\frac{3}{2(p-2)}} \bigl(\|\Phi_\frac{3}{T}\|_q\bigr) \bigl(\|\nabla_x u\|_2 + \|\rho^\frac{1}{2} \nabla_x u\|_2\bigr)^{1-\frac{3}{2}} \\
&\leq C(T) \Phi_\frac{3}{T}^{1-\frac{3}{2}} \bigl(\|\nabla_x u\|_2 + \|\rho^\frac{1}{2} \nabla_x u\|_2\bigr)^{1-\frac{3}{2}}.
\end{align*}
\]
(ii) By the Hölder inequality, we have
\[
\|\rho u\|_{2+\gamma} \leq \|\rho(x)^\frac{\gamma}{\alpha'}\|_{3\gamma} \|u(x)^{-\frac{\gamma}{\alpha'}}\|_{6\gamma}.
\] (29)
We use (ii) in Lemma 3.1 to get
\[
\|\rho(x)^\frac{\gamma}{\alpha'}\|_{3\gamma} \leq C \Phi_\frac{3}{T} \left( \int_{\mathbb{R}^2} \rho^\gamma (1 + |x|)^{\alpha} \, dx \right)^\frac{1}{\gamma} \leq C(T) \Phi_\frac{3}{T}.
\] (30)
Note that
\[
\langle x \rangle^{-2\alpha} \leq C \langle x \rangle^{-2} (\log \langle x \rangle)^{-4\gamma}, \quad \gamma \in (1, \infty), \quad 1 < \alpha < 2 \sqrt{2} - 1.
\]
Then, we use Lemma A.3 to have
\[
\|u(x)^{-\frac{\gamma}{\alpha'}}\|_{6\gamma} \leq \left( \int_{\mathbb{R}^2} \frac{|u|^6_{\gamma}(\log \langle x \rangle)^{-4\gamma} \, dx}{{\langle x \rangle}^{\gamma}} \right)^\frac{1}{\gamma} \lesssim \|u\|_{L^2(B_1)} + \|\nabla_x u\|_2.
\]
On the other hand, we use the Poincaré type inequality in Lemma 3.2 of [18] and get from (ii) and (iv) in Lemma 3.1 that
\[
\|u\|_{L^2(B_1)} \leq \|u\|_{L^2(B_{N(t)})} \leq C(\|\sqrt{\rho} u\|_{L^2(B_{N(t)})} + \|\nabla_x u\|_{L^2(B_{N(t)})}) \leq C(T) (1 + \|\nabla_x u\|_2).
\]
The above two estimates imply
\[
\|u(x)^{-\frac{\gamma}{\alpha'}}\|_{6\gamma} \leq C(T) (1 + \|\nabla_x u\|_2).
\] (31)
Therefore, we substitute the estimates (30) and (31) in (29) to obtain (ii) in (27). \( \square \)

Now we are in position to construct the upper bound of \( \rho(x, t) \).

Lemma 3.9. Suppose that the conditions in Lemma 3.3 hold. Then, we have
\[
0 \leq \rho(x, t) \leq C(T).
\] (32)

Proof. It follows from definition of \( \psi \) and \( \eta_1 \) that we have
\[
0 \leq u \cdot \nabla_x \psi - \eta_1 = [u, R_i R_j] (\rho u_j).
\]
It follows from (16) that
\[
(\Lambda(\rho) - \psi)_t + u \cdot \nabla_x (\Lambda(\rho) - \psi) + P(\rho) + [u, R_i R_j] (\rho u_j) - \eta_2 = 0.
\]
Consider the trajectory defined by
\[
\begin{align*}
\frac{d\zeta}{d\tau}(\tau; x, t) &= u(\tau, \zeta(\tau; x, t)) \\
\zeta(\tau; x, t)|_{\tau=t} &= x,
\end{align*}
\]
Then, we have
\[
\frac{d}{dt}(\Lambda(\rho) - \psi)(\tau; \varsigma(\tau; x, t)) + P(\rho)(\tau; \varsigma(\tau; x, t))
= -[u_i, R_i R_j](\rho u_j)(\tau; \varsigma(\tau; x, t)) + \eta_2(\tau; \varsigma(\tau; x, t)).
\]
We integrate the above equation over \([0, t]\) to obtain
\[
2\mu \log \frac{\rho(x, t)}{\rho_0(s_0)} + \frac{1}{\beta}(\rho^\beta(x, t) - \rho^\beta_0(s_0))
\leq \psi(x, t) - \psi_0(s_0) - \int_0^t [u_i, R_i R_j](\rho u_j) d\tau + \int_0^t \eta_2 d\tau. \tag{33}
\]
Now, we estimate the terms on the RHS of (33). For \(\psi\), we use Lemma A.6 to have
\[
\|\psi\|_\infty \leq C\|\nabla_x \psi\|_{2\log^\frac{1}{2}(e + \|\nabla_x \psi\|_{L^2})} + C(\|\psi\|_2 + 1) \tag{34}
\]
From (ii) in Lemma 3.1 and (i) in Lemma 3.5, we have
\[
(\|\psi\|_2 + \|\psi\|_{L^2}) \leq C(T)(\|\rho\|_1 + \|\rho\|_\infty) \leq C(T).
\]
From the elliptic equation (13)\textsubscript{1}, we use (ii) in Lemma 3.1 and (ii) in (27), Lemma 3.8 to get
\[
\|\nabla_x \psi\|_2 \leq C\|\rho u\|_2 \leq C(T)\Phi^\frac{3}{4}_{T},
\]
\[
\|\nabla_x \psi\|_{L^2} \leq C\|\rho u\|_{L^2} \leq C(T)\Phi^\frac{3}{4}_{T}(1 + \|\nabla_x u\|_2).
\]
Collecting the estimates (33) and (34), we use (22) to have
\[
\|\psi\|_\infty \leq C(T)\Phi^{1+\beta_e}_{T}.\tag{35}
\]
Now we turn to the estimate of the third term in the RHS of (33). To the end, we first use (i) in Lemma A.5 and (25) to have
\[
\|\nabla_x u\|_4 \leq C(\|\text{div} u\|_4 + \|\omega\|_4) \leq C \left( \frac{\chi^2}{2\mu + \lambda(\rho)} \right)^\frac{1}{4} + \|\omega\|_4 + \|P(\rho)\|_4)
\leq CZ^{\frac{3}{4}} \Phi^{\frac{3}{2}}_{T} (\Phi^\frac{3}{4} \chi^\frac{1}{2} + \|\nabla_x u\|_2^2) + 1
\leq C\Phi^{\frac{3}{2}}_{T} Z^\frac{1}{4} \chi^\frac{1}{2} + C\Phi^{\frac{3}{4}}_{T} (Z + 1)
\leq C\Phi^{\frac{3}{2}}_{T} Z^\frac{1}{4} \chi^\frac{1}{2} + C\Phi^{\frac{3}{4}}_{T} (Z + 1)
\leq C\Phi^{\frac{3}{2}}_{T} Z^\frac{1}{4} \chi^\frac{1}{2} \left( e + \|\nabla_x u\|_2 + \int_{\mathbb{R}^4} (u - v)^2 f dv dx \right)^\frac{1}{4} \left( \frac{\chi^2}{e + Z^2} \right)^\frac{1}{2}
+ C\Phi^{\frac{3}{4}}_{T} (Z + 1)
\]
where in the last inequality one has used
\[
Z^2 \leq C\Phi^{\frac{3}{2}}_{T} \|\nabla_x u\|_2^2 + \int_{\mathbb{R}^4} (u - v)^2 f dv dx.
\]
Denoting the commutator \(\vartheta = [u_i, R_i R_j](\rho u_j)\), we use Lemma A.7 and (i) in (27) to have
\[
\|\vartheta\|_\infty \leq C\|u\|_p^{1-\frac{2}{p}} \|\nabla_x \vartheta\|^{\frac{2}{p}}_{\text{BMO}} \leq C(\|u\|_{\text{BMO}} \|\rho u\|_p)^{1-\frac{2}{p}} (\|\nabla_x u\|_4 \|\rho u\|_p)^{\frac{2}{p}}
\leq C\Phi^{1+\frac{2}{p}}_{T} (\|\nabla_x u\|_2 + \|x\|^{\frac{2}{2}} \nabla_x u\|_2)\|\nabla_x u\|_4^\frac{1}{4}.
\]
Then we choose $p > 4$ sufficiently large and use (35) to have

$$\int_{0}^{t} \| \vartheta \|_{\infty} \, d\tau \leq C \Phi_{T}^{1 + \frac{4}{3} + \beta \varepsilon}.$$  

Finally, we treat the forth term on the RHS of (33). From the elliptic equation (13), we use $(i)$ in Lemma 3.4 and $(v)$ in Lemma 3.5 for suitably large $m$ to have

$$\int_{0}^{t} \| \eta_{2} \|_{\infty} \, d\tau \leq C \int_{0}^{t} (\| \eta_{2} \|_{2m} + \| \nabla_{x} \eta_{2} \|_{2m}) \, d\tau \leq C \int_{0}^{t} (\| \nabla_{x} u \|_{2}^{2} + \| x |^{2} \nabla_{x} u \|_{2}^{2} + \int_{\mathbb{R}^{4}} (u - v)^{2} f \, dv \, dx) \, d\tau \leq C(T).$$

Finally, we substitute all above estimates into (33) to derive

$$\Phi_{T}^{\beta} \leq C(T) \Phi_{T}^{1 + \frac{4}{3} + \beta \varepsilon}.$$  

When $\beta > \frac{4}{3}$, we take positive constant $\varepsilon$ sufficiently small to have

$$\sup_{0 \leq t \leq T} \| \rho \|_{\infty}(t) \leq C(T).$$

With Lemma 3.7, 3.9 in hand, we immediately have

**Corollary 2.** Assume the conditions in Lemma 3.3 hold. Then it holds that

$$\left( \| \nabla_{x} u \|_{2} + \int_{\mathbb{R}^{4}} (u - v)^{2} f \, dv \, dx \right)(t) + \int_{0}^{t} (\| \sqrt{\rho} \dot{u} \|_{2}^{2} + \| x |^{2} \nabla_{x} \dot{u} \|_{2}^{2}) \, d\tau \leq C(T).$$  

(36)

4. **A priori high-order estimates.** In this section, we present higher-order estimates. Let $[f, \rho, u]$ be a classical solution to the coupled system on $[0, T] \times \mathbb{R}^{4}$, we derive some a priori estimates for the system (1)-(2) with $(x, v) \in \mathbb{R}^{4}$.

4.1. **$W^{1, p}$-estimates with $p > 4$.** In this subsection, we derive the $W^{1, p}$ estimates of the classical solution $[f, \rho, u]$ to the system (1)-(2): $\| (\rho, u) \|_{W^{1, p}}$ and $\| (u)^{2} \nabla_{x} f \|_{p}$ for $4 < p < \infty$.

**Lemma 4.1.** Suppose that the conditions in Lemma 3.3 hold. Then, we have

$$\left( (1 + |x|^{2}) \sqrt{\rho} \cdot (\dot{u}_{j}) \right)(t) + \int_{0}^{t} ((1 + |x|^{2}) \nabla_{x} \dot{u}_{j})_{2} + ((1 + |x|^{2}) \sqrt{\rho} \cdot (\dot{u}_{j})_{2}) \, d\tau \leq C(T).$$  

(37)

**Proof.** Note that

$$\dot{u}_{j} [\partial_{i}(\rho \dot{u}_{j}) + \text{div}(u \rho \dot{u}_{j})] = \frac{1}{2} \rho \partial_{i}(\dot{u}_{j})^{2} + \frac{1}{2} \rho \cdot \nabla_{x}(\dot{u}_{j})^{2}.$$
Then, we apply the operator $\dot{u}_j[\partial_t + \text{div}(u \cdot \nabla)]$ to $(1)_{2,j}$ and use $(26)$ to have

$$
\frac{d}{dt} \int_{\mathbb{R}^2} \rho|\dot{u}|^2dx + 2 \int_{\mathbb{R}^2} \rho_f|\dot{u}|^2dx = -2 \int_{\mathbb{R}^2} \dot{u}_j[\partial_t P(\rho) + \text{div}(u\partial_j P(\rho))]dx \\
+ 2\mu \int_{\mathbb{R}^2} \dot{u}_j[\partial_t \Delta_x u_j + \text{div}(u \Delta_x u_j)]dx \\
+ 2 \int_{\mathbb{R}^2} \dot{u}_j[\partial_j((\mu + \lambda(\rho))\text{div}u) + \text{div}(u\partial_j((\mu + \lambda(\rho))\text{div}u))]dx \\
+ 2 \int_{\mathbb{R}^2} \dot{u}_j[u \cdot \nabla_x u_j \rho_f + u_j \text{div}(u_f \rho_f) - \text{div}(u(u_j \rho_f))]dx \\
+ 2 \int_{\mathbb{R}^2} \dot{u}_j((L_j(f) + u_j - v_j)f - v_j(v \cdot \nabla_x F)) \text{div}(u(u_f \rho_f))dx
$$

$$
= : \sum_{i=1}^{3} I_{6i},
$$

(38)

• Case F.1 (Estimate of $\sum_{i=1}^{3} I_{6i}$): For this term, we have

$$
\sum_{i=1}^{3} I_{6i} \leq \frac{-3\mu}{2} ||\nabla_x u_\ast ||^2_2 - \frac{3\mu}{2} ||\partial_t \text{div}u + u \cdot \nabla_x \text{div}u||^2_2 + C(T)(1 + ||\nabla_x u||^2_4).
$$

• Case F.2 (Estimate of $I_{64}$ and $I_{65}$): We apply the integration by parts and use (ii) in Lemma 3.5, Corollary 1 and (36) to have

$$
I_{64} \leq C \left( ||\dot{u}_s||^2_2 ||u||^2_2 ||\nabla_x u||^2_2 ||\rho_f||^4_4 + ||\nabla_x \dot{u}_s||^2_2 ||u||^2_s ||u_f \rho_f||^2_2 \right) \\
+ ||\dot{u}_s||^2_2 ||\nabla_x u||^2_2 ||u_f \rho_f||^2_2 + ||\nabla_x \dot{u}_s||^2_2 ||u||^2_4 \rho_f||^4_4 \\
\leq C(T) ||\nabla_x u||^2_2 \left( \frac{1}{2} |||x||^2_2 \nabla \dot{u}_s||^2_2 \left( |||x||^2_2 \nabla_x u||^2_2 \right)^{\frac{1}{2}} + 1 \right) + C ||\nabla_x \dot{u}_s||^2_2 ||x||^2_2 ||\nabla_x u||^2_2^{\frac{1}{2}} \\
\leq \varepsilon \left( (1 + ||x||^2_2) \nabla_x \dot{u}_s||^2_2 + C(T)(1 + ||x||^2_2 \nabla_x u||^2_2) \right),
$$

where $\varepsilon > 0$ is a small positive constant. Similarly, we have

$$
I_{65} \leq C \left( ||\nabla_x \dot{u}_s||^2_2 ||u||^2_s ||u_f \rho_f||^2_2 + C ||\nabla_x \dot{u}_s||^2_2 ||m_2 ||^2_2 \\
+ C ||\dot{u}_s||^2_2 ||u||^2_s ||\rho_f||^4_4 + C ||\dot{u}_s||^2_2 ||u_f \rho_f||^4_4 \\
\leq \varepsilon \left( (1 + ||x||^2_2) \nabla_x \dot{u}_s||^2_2 + C(T)(1 + ||x||^2_2 \nabla_x u||^2_2) \right).
$$

We collect all estimates of $I_{6i}$ in (38) to obtain

$$
\frac{d}{dt} ||\sqrt{\rho} \dot{u}||^2_2 + ||\nabla_x \dot{u}||^2_2 + ||\sqrt{\rho_f} \dot{u}||^2_2 \\
\leq \varepsilon \left( (1 + ||x||^2_2) \nabla_x \dot{u}||^2_2 + C(T)(1 + ||\nabla_x u||^2_4 + ||x||^2_2 \nabla_x u||^2_2) \right).
$$

Note that

$$
||\nabla_x u||^4_4 \leq C(||\omega||^4_4 + ||\text{div}u||^4_4) \leq C(T)(||F||^4_4 + ||\omega||^4_4 + ||P(\rho)||^4_4) \\
\leq C(T)(||\nabla_x F||^2_2 + ||\nabla_x \omega||^2_2 + 1) \\
\leq C(||\rho \dot{u}||^2_2 + ||u||^2_2 \rho_f||^2_2 + ||u_f \rho_f||^2_2 + 1) \\
\leq C(||\sqrt{\rho} \dot{u}||^2_2 + C ||x||^2_2 \nabla_x u||^2_2 + C(T),
$$
by (24) and (36). We can apply the Gronwall inequality and use (17) to further obtain

\[
\|\sqrt{\rho} \dot{u}\|^2_2(t) + \int_0^t \left( \|\nabla_x \dot{u}\|^2_2 + \|\sqrt{\rho_f} \dot{u}\|^2_2 \right) dt \leq 2\varepsilon \int_0^t \|x|^{\frac{3}{2}} \nabla_x \dot{u}\|^2_2 dt + C(T). \tag{39}
\]

In order to estimate the first term on the RHS of (39), we apply the operator \(|x|^\alpha \dot{u}_j[\partial_t + \text{div}(\cdot)]\) to (1)2,j, and use the derivation similar to (38) to obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^2} \rho |\dot{u}|^2 |x|^\alpha dx + 2 \int_{\mathbb{R}^2} \rho_f |\dot{u}|^2 |x|^\alpha dx
\]

\[
= -2 \int_{\mathbb{R}^2} \dot{u}_j[\partial_t P(\rho) + \text{div}(u \partial_j P(\rho))] |x|^\alpha dx
\]

\[
+ 2\mu \int_{\mathbb{R}^2} \dot{u}_j[\partial_t \Delta_x u_j + \text{div}(u \Delta_x u_j)] |x|^\alpha dx + 2 \int_{\mathbb{R}^2} \dot{u}_j[\partial_t ((\mu + \lambda) \text{div} u) + \text{div}(u \partial_j ((\mu + \lambda) \text{div} u))] |x|^\alpha dx
\]

\[
+ 2 \int_{\mathbb{R}^2} \dot{u}_j [u \cdot \nabla_x u_j \rho_f + u_j \text{div}(u_f \rho_f) - \text{div}(u(u_j \rho_f))] |x|^\alpha dx
\]

\[
+ 2 \int_{\mathbb{R}^2} \dot{u}_j [\int_{\mathbb{R}^2} ((L_f(f) + u_j - v_j) f - v_j (v \cdot \nabla_x F)) dv + \text{div}(u(u_f \rho_f))_j] |x|^\alpha dx
\]

\[
+ \int_{\mathbb{R}^2} |\dot{u}|^2 \alpha |x|^{\alpha - 2} \rho u \cdot x dx
\]

\[
= \sum_{i=1}^{6} \mathcal{I}_{7i}. \tag{40}
\]

For terms \(\mathcal{I}_{7i}, i = 1, 2, 3,\) we have

\[
\mathcal{I}_{71} \leq \varepsilon \|x|^{\frac{3}{2}} \nabla_x \dot{u}\|^2_2 + C(T) \|x|^{\frac{3}{2}} \nabla_x \dot{u}\|^2_2,
\]

\[
\mathcal{I}_{72} \leq -2\mu \left(1 - \frac{\alpha^2}{2}\right) \|x|^{\frac{3}{2}} \nabla_x \dot{u}\|^2_2 + \varepsilon \|x|^{\frac{3}{2}} \nabla_x \dot{u}\|^2_2
\]

\[
+ C(T)(1 + \|1 + |x|^{\frac{3}{2}}\| \sqrt{\rho} \dot{u}\|^2_2 + \|x|^{\frac{3}{2}} \nabla_x \dot{u}\|^2_2),
\]

\[
\mathcal{I}_{73} \leq -2\mu |f|^{\frac{3}{2}} \text{div} \dot{u}\|^2_2 + \|\lambda(\rho)|x|^{\frac{3}{2}} \text{div} \dot{u}\|^2_2 + \mu \alpha^2 |x|^{\frac{3}{2}} \text{div} \dot{u}\|^2_2 + \|x|^{\frac{3}{2}} \nabla_x \dot{u}\|^2
\]

\[
+ \varepsilon \|x|^{\frac{3}{2}} \nabla_x \dot{u}\|^2_2 + \|\lambda(\rho)|x|^{\frac{3}{2}} \text{div} \dot{u}\|^2_2
\]

\[
+ C(T)(1 + \|1 + |x|^{\frac{3}{2}}\| \sqrt{\rho} \dot{u}\|^2_2 + \|x|^{\frac{3}{2}} \nabla_x \dot{u}\|^2_2).
\]

Note that we have

\[
\int_{\mathbb{R}^2} |\dot{u} u \nabla_x u \rho_f| |x|^\alpha dx
\]

\[
\leq C \|\nabla_x u\|_2 \|x|^{\frac{3}{2} - \frac{\alpha}{2}} \dot{u}\|_{12} \|x|^{\frac{3}{2} - \frac{\alpha}{2}} \|\dot{u}\|_{12} \|x|^{\frac{3}{2}} \rho_f\|_3
\]

\[
\leq C |x|^{\frac{3}{2}} \|\nabla_x u\|_2 \|x|^{\frac{3}{2}} \|\dot{u}\|_2 \left( \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} f |x|^2 dv \right)^{\frac{3}{2}} \rho_f^{\frac{3}{2}} dx \right)^{\frac{1}{3}}
\]

\[
\leq C |x|^{\frac{3}{2}} \|\nabla_x u\|_2 \|x|^{\frac{3}{2}} \|\dot{u}\|_2 \|x\|^{\frac{3}{2}} \rho_f \|\rho_f\|^{\frac{3}{2}}_3
\]

\[
\leq C(T) |x|^{\frac{3}{2}} \|\nabla_x u\|_2 \|x|^{\frac{3}{2}} \|\dot{u}\|_2.
\]
and
\[ \int_{\mathbb{R}^2} |\dot{u} \nabla_x uu_f \rho_f||x|^\alpha dx \]
\[ \leq C||x|^{\frac{3}{2}} \nabla_x u||_2||x|^{\frac{3}{2} - \frac{1}{2}} \dot{u}||_6||x|^{\frac{1}{2}} uu_f \rho_f||_3 \]
\[ \leq C||x|^{\frac{3}{2}} \nabla_x u||_2||x|^{\frac{3}{2}} \dot{u}||_2 (\int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^2} f||x|^2 dv \right)^{\frac{1}{2}} \rho_f^{\frac{1}{2}} m_2 f dx)^{\frac{1}{2}} \]
\[ \leq C||x|^{\frac{3}{2}} \nabla_x u||_2||x|^{\frac{3}{2}} \dot{u}||_2 ||x|^{\frac{1}{2}} \rho_f \rho_f||_2^{\frac{1}{2}} ||m_2 f||_3^{\frac{1}{2}} \]
\[ \leq C(T)||x|^{\frac{3}{2}} \nabla_x u||_2||x|^{\frac{3}{2}} \dot{u}||_2, \]
by the Caffarelli-Kohn-Nirenberg inequality (3) in Lemma A.4, (iii) in Lemma 3.1 and Corollary 1. Then we have
\[ I_{74} \leq C \int_{\mathbb{R}^2} \left[ (|\dot{u} u \nabla_x u \rho_f| + |\nabla_x \dot{u} uu_f \rho_f| + |\dot{u} \nabla_x uu_f \rho_f|) ||x|^\alpha \right. \]
\[ + \left( (|\dot{u} uu_f \rho_f| + |\dot{u} uu_f \rho_f|) ||x|^{\alpha - 1} \right) dx \]
\[ \leq \epsilon ||x|^{\frac{3}{2}} \nabla_x \dot{u}||_2^2 + C(T)||x|^{\frac{3}{2}} \nabla_x u||_2^2. \]
Similarly, we obtain
\[ I_{75} \leq C \int_{\mathbb{R}^2} \left[ (|\dot{u} \rho_f| + |\dot{u} uu_f \rho_f| + |\nabla_x \dot{u} m_2 f| + |\nabla_x \ddot{u} uu_f \rho_f|) ||x|^\alpha \right. \]
\[ + \left( (|\ddot{u} m_2 f| + |\ddot{u} uu_f \rho_f|) ||x|^{\alpha - 1} \right) dx \]
\[ \leq \epsilon ||x|^{\frac{3}{2}} \nabla_x \dot{u}||_2^2 + C(T)||x|^{\frac{3}{2}} \nabla_x u||_2^2. \]
Finally, for \( I_{76}, \) we have
\[ I_{76} \leq C||\sqrt{\rho} \ddot{u}||_2 ||x||_s||u||_s ||x||^{\alpha - 1}_s \rho \leq \epsilon ||x|^{\frac{3}{2}} \nabla_x \ddot{u}||_2^2 + ||\nabla_x \dot{u}||_2^2 \]
\[ + C(T)||\sqrt{\rho} \ddot{u}||_2^2 ||x|^{\frac{3}{2}} \nabla_x u||_2^2. \]
Note that for \( 1 < \alpha < 2 \sqrt{2} - 1, \) there exists a positive constant \( C(\alpha) \) such that
\[ 2\mu \left( 1 - \frac{\alpha^2}{2} \right) |||x|^{\frac{3}{2}} \nabla_x \ddot{u}||_2^2 - \mu \alpha^2 ||x|^{\frac{3}{2}} \nabla_x \dot{u}||_2 ||x|^{\frac{3}{2}} \nabla_x \dot{u}||_2 + 2\mu ||x|^{\frac{3}{2}} \nabla_x \dot{u}||_2^2 \]
\[ \geq C(\alpha)(||x|^{\frac{3}{2}} \nabla_x \ddot{u}||_2^2 + ||x|^{\frac{3}{2}} \nabla_x \dot{u}||_2^2). \]
Finally, we collect all estimates of \( I_{71} \) in (40), combine the resulting inequality and (39), and choose suitably small \( \epsilon \) to have
\[ ||(1 + |x|^{\frac{3}{2}}) \sqrt{\rho} \ddot{u}||_2^2(t) + \int_0^t \left[ ||(1 + |x|^{\frac{3}{2}}) \nabla_x \dot{u}||_2^2 + ||(1 + |x|^{\frac{3}{2}}) \sqrt{\rho} \ddot{u}||_2^2 \right] d\tau \leq C(T). \]

\[ \Box \]

**Lemma 4.2.** Suppose that the conditions in Lemma 3.3 hold. Then, we have
\[ ||(\nabla_x p, \nabla_x P(p))||_p(t) + \int_0^t ||\nabla_x u||_p^2 d\tau \leq C(T), \quad 4 < p < \infty. \]  

**Proof.** We apply the operator \( \nabla \) to the continuity equation (1) \(_2, \) multiply the resulted equation by \( p|\nabla_x \rho|^{p-2} \nabla_x \rho, \) and integrate over \( \mathbb{R}^2 \) to get
\[ \frac{d}{dt} ||\nabla_x \rho||_p \leq C(T)(||\nabla_x u||_\infty ||\nabla_x \rho||_p + ||\nabla_x^2 u||_p). \]  

(43)
Case G.1 (Estimate of $\|\nabla_x^2 u\|_p$): By the interpolation inequality, we use (3) and (37) to have

$$
\|\nabla^2 u\|_p \leq C(\|\nabla_x x u\|_p + \|\nabla_x \omega\|_p)
\leq C(\|\nabla_x((2u + \lambda(\rho)) x u)\|_p + \|u x \omega\|_\infty \|\nabla_x \lambda(\rho)\|_p + \|\nabla_x \omega\|_p)
\leq C(T)(\|\nabla_x F\|_p + \|\nabla_x \rho\|_p + \|u x \omega\|_\infty \|\nabla_x \rho\|_p)
\leq C(T)(\|\rho x u\|_p + \|u x \|_p \|\rho\|_p + \|u f x \rho\|_p + \|\omega x u\|_\infty + 1)(1 + \|\nabla_x \rho\|_p)
\leq C(T)(\|\nabla_x \|_\infty + \|u x \|_p \|\rho\|_p + \|\omega x u\|_\infty + 1)(1 + \|\nabla_x \rho\|_p).
$$

On the other hand, we use Lemma 3.1, (17)_1 and Corollary 2 to have

$$
\|\nabla_x u\|_\infty \leq C(T)(\|\nabla_x F\|_p + \|\nabla_x \omega\|_\infty + 1) \leq C(T)(\|\nabla_x F\|_p^\frac{2}{3} + \|\nabla_x \omega\|_p^\frac{2}{3} + 1)
\leq C(T)(\|\rho x u\|_p^\frac{2}{3} + \|u x \|_p^\frac{2}{3} \|\rho\|_p^\frac{2}{3} + \|u f x \rho\|_p^\frac{2}{3} + 1)
\leq C(T)(\|\nabla_x u\|_p^\frac{2}{3} + \|u x \|_p^\frac{2}{3} \|\rho\|_p^\frac{2}{3} + \|u f x \rho\|_p^\frac{2}{3} + 1).
$$

Then we can further estimate $\|\nabla_x^2 u\|_p$ as

$$
\|\nabla_x^2 u\|_p \leq C(T)(\|\nabla_x u\|_\infty + 1)(1 + \|u x \|_p)\|\nabla_x \omega\|_2 + \|u x \|_p \|\nabla_x u\|_2 + \|u x \|_p \|\nabla_x \rho\|_p + \|\omega x u\|_\infty + 1)(1 + \|\nabla_x \rho\|_p).
$$

Case G.2 (Estimate of $\|\nabla_x u\|_\infty$): By Lemma A.8, we use the estimate of $\|\nabla_x^2 u\|_p$ to have

$$
\|\nabla_x u\|_\infty \leq C(\|\nabla_x \|_\infty + \|\omega\|_\infty)\log(\|\nabla_x^2 u\|_p + \|\nabla_x u\|_2 + C)
\leq C(T)(\|\nabla_x^2 u\|_p + \|\nabla_x u\|_2 + \|\nabla_x \rho\|_p + \|\omega x u\|_\infty + 1)(\|\nabla_x \|_\infty + \|\omega x u\|_\infty + 1)(1 + \|\nabla_x \rho\|_p).
$$

Collecting the estimates of $\|\nabla_x^2 u\|_p$ and $\|\nabla_x u\|_\infty$ in (43), we have

$$
\frac{d}{dt} \|\nabla_x \rho\|_p \leq C(T)(\|\nabla_x^2 u\|_p + \|\nabla_x \|_\infty + \|\omega x u\|_\infty + 1)(\|\nabla_x \rho\|_p + \|\omega x u\|_\infty + 1)(\|\nabla_x \|_\infty + \|\omega x u\|_\infty + 1)(1 + \|\nabla_x \rho\|_p + 1).
$$

We apply the Gronwall lemma, and use (17)_2 and Lemma 4.1 to have

$$
\sup_{0 \leq t \leq T} \|\nabla_x \rho\|_p \leq C(T).
$$

This inequality together with (44) implies

$$
\int_0^T \|\nabla_x u\|_\infty^2 \, dt \leq C(T).
$$
Lemma 4.3. Suppose that the conditions in Lemma 3.3 hold. Then, for $4 < p < \infty$ we have

$$\sup_{0 \leq t \leq T} \left( \| (v)^k \nabla_x f \|_p + \| (v)^k \nabla_v f \|^p_p \right) + \int_0^t \left( \| v - v_c \| (v)^k \nabla_v (|\nabla_x f|^2) \right) \frac{\partial}{\partial t} \left( \| v - v_c \| (v)^k \nabla_v (|\nabla_x f|^2) \right) \leq C(T, p).$$

(45)

Proof. We apply the operator $\partial_{x_i}$ to (1.3) to obtain

$$\partial_t \partial_{x_i} f + v \cdot \nabla \partial_{x_i} f + \nabla_v \cdot \left( \int_{\mathbb{R}^4} \partial_{x_i}(\psi(x-y))(v_* - v)f(v_*, y, t)dv_*dyf \right) \nabla_v \cdot \left( \int_{\mathbb{R}^4} (\psi(x-y))(v_* - v)f(v_*, y, t)dv_*dy\partial_{x_i} f \right) + \kappa_2 \nabla_v \cdot ((u - v) \partial_{x_i} f) + \nabla_v \cdot (\partial_{x_i} uf) = \partial_{x_i} \Delta_v (|v - v_c|^2 f).$$

We multiply the above equation by $(v)^{kp} \partial_{x_i} f |p-2 \partial_{x_i} f$, and integrate the resulted equations with respect to $x, v$ over $\mathbb{R}^4$ to give

$$\frac{d}{dt} \| (v)^k \partial_{x_i} f \|^p_p$$

$$= - \int_{\mathbb{R}^4} \langle v \rangle^k \partial_{x_i} f |p-2 \partial_{x_i} f \nabla_v \cdot \left( \int_{\mathbb{R}^4} \partial_{x_i}(\psi(x-y))(v_* - v)f(v_*, y, t)dv_*dyf \right)dvdx$$

$$- \int_{\mathbb{R}^4} \langle v \rangle^k \partial_{x_i} f |p-2 \partial_{x_i} f \nabla_v \cdot \left( \int_{\mathbb{R}^4} (\psi(x-y))(v_* - v)f(v_*, y, t)dv_*dy\partial_{x_i} f \right)dvdx$$

$$- \int_{\mathbb{R}^4} \langle v \rangle^k \partial_{x_i} f |p-2 \partial_{x_i} f \kappa_2 \nabla_v \cdot ((u - v) \partial_{x_i} f)dvdx$$

$$- \int_{\mathbb{R}^4} \langle v \rangle^k \partial_{x_i} f |p-2 \partial_{x_i} f \nabla_v \cdot (\partial_{x_i} uf)dvdx$$

$$+ \int_{\mathbb{R}^4} \langle v \rangle^k \partial_{x_i} f |p-2 \partial_{x_i} f \partial_{x_i} \Delta_v (|v - v_c|^2 f)dvdx$$

$$=: \sum_{i=1}^5 \mathcal{I}_{8i}.$$ 

Now we deal with $\mathcal{I}_{8i}, i = 1, \ldots, 5$, as below.

- Case H.1 (Estimate of $\mathcal{I}_{81}$): By direct calculation, we have

$$\mathcal{I}_{81}$$

$$= - \int_{\mathbb{R}^4} \langle v \rangle^k \partial_{x_i} f |p-2 \partial_{x_i} f \nabla_v \cdot \left( \int_{\mathbb{R}^4} \partial_{x_i}(\psi(x-y))(v_* - v_c)f(v_*, y, t)dv_*dyf \right)dvdx$$

$$- \int_{\mathbb{R}^4} \langle v \rangle^k \partial_{x_i} f |p-2 \partial_{x_i} f \nabla_v \cdot \left( \int_{\mathbb{R}^4} \partial_{x_i}(\psi(x-y))(v_* - v)f(v_*, y, t)dv_*dyf \right)dvdx$$

$$:= \mathcal{I}_{811} + \mathcal{I}_{812}.$$
We further apply the Gronwall inequality and use (42) to derive (45).

\[ |I_{S11}| \leq \sup_{x \in \mathbb{R}^2} |\nabla_x \psi| \int_{\mathbb{R}^4} \langle v \rangle^{kp} |\partial_\xi f|^{p-1} |\partial_\nu f| \left( \int_{\mathbb{R}^4} |v_{j+} - v_{c,j}| f(v_+, y, t) dv_x dv_y \right) dx \]

\[ \leq C(T)(\langle \langle v \rangle \rangle^k |\partial_\xi f|^{\|p}\rangle + \langle \langle v \rangle \rangle^k |\nabla_x f|^{\|p}\rangle), \]

\[ |I_{S12}| \leq p(p-1) \int_{\mathbb{R}^4} \langle v \rangle^{kp} |\partial_\xi f|^{p-2} |\partial_\xi \partial_\nu f| |v_j - v_{c,j}| dx \]

\[ + kp^2 \int_{\mathbb{R}^4} \langle v \rangle^{kp-1} |\partial_\xi f|^{p-1} dx \]

\[ \leq \varepsilon \|v - v_c\| \langle \langle v \rangle \rangle^k |\nabla_x f|^{\|p\}} + C(p)(\langle \langle v \rangle \rangle^k |\partial_\xi f|^{\|p\} + \langle \langle v \rangle \rangle^k |\nabla_x f|^{\|p\}), \]

where \( \varepsilon > 0 \) is a small constant. We further estimate \( I_{S1} \) as

\[ |I_{S1}| \leq \frac{p-1}{p} \|v - v_c\| \langle \langle v \rangle \rangle^k |\nabla_x f|^{\|p\}} + C(T, p)(\langle \langle v \rangle \rangle^k |\partial_\xi f|^{\|p\} + \langle \langle v \rangle \rangle^k |\nabla_x f|^{\|p\}). \]

- Case H.2 (Estimate of \( I_{S1} (i = 2, 3, 4) \): We use integration by parts to obtain

\[ |I_{S2}| + |I_{S3}| \leq C(\langle \langle v \rangle \rangle^k |\partial_\xi f|^{\|p\} \]

\[ |I_{S4}| \leq \|\nabla_x u\|_{\infty} \int_{\mathbb{R}^4} \langle v \rangle^{kp} |\partial_\xi f|^{p-1} |\nabla_x f| dx \]

\[ \leq C(\|\nabla_x u\|_{\infty}(\langle \langle v \rangle \rangle^k |\partial_\xi f|^{\|p\} + \langle \langle v \rangle \rangle^k |\nabla_x f|^{\|p\})). \]

- Case H.3 (Estimate of \( I_{S5} \): Again, we use the integration by parts to have

\[ I_{S5} = \int_{\mathbb{R}^4} p(v)^{kp} |\partial_\xi f|^{p-2} |\partial_\xi f| \partial_\xi (4f + 4(v - v_c) \cdot \nabla_v f + |v - v_c|^2 |\nabla_v f|) dv_x dv_y \]

\[ = - \int_{\mathbb{R}^4} \left[ 4p(v)^{kp} |\partial_\xi f|^p + kp v(v - v_c)^{k^{p-2}} |\partial_\xi f|^p \right] dv_x dv_y \]

\[ + 2(v)^{kp} |v - v_c|^2 \nabla_v (|\partial_\xi f|^p) + kp(v)^{kp-2} |v - v_c|^2 v \cdot \nabla_v (|\partial_\xi f|^p) ] dx \text{ and dx} \]

\[ \frac{4(p-1)}{p} \|v - v_c\| \langle \langle v \rangle \rangle^k |\nabla_x f|^{\|p\}} \]

\[ \leq - \frac{2p-1}{p} \|v - v_c\| \langle \langle v \rangle \rangle^k |\nabla_x f|^{\|p\}}, \]

We collect all estimates of \( I_{S1} \) in (46) to find

\[ \frac{d}{dt} \|\langle \langle v \rangle \rangle^k |\nabla_x f|^{\|p\}} + \frac{p-1}{p} \|v - v_c\| \langle \langle v \rangle \rangle^k |\nabla_x f|^{\|p\}} \]

\[ \leq C(T, p)(1 + \|\nabla_x u\|_{\infty})(\langle \langle v \rangle \rangle^k |\nabla_x f|^{\|p\} + \langle \langle v \rangle \rangle^k |\nabla_v f|^{\|p\}) \]

Similarly, we can obtain

\[ \frac{d}{dt} \|\langle \langle v \rangle \rangle^k |\nabla_v f|^{\|p\}} + \frac{p-1}{p} \|v - v_c\| \langle \langle v \rangle \rangle^k |\nabla_v f|^{\|p\}} \]

\[ \leq C(T, p)(\|\langle \langle v \rangle \rangle^k |\nabla_x f|^{\|p\} + \langle \langle v \rangle \rangle^k |\nabla_v f|^{\|p\}). \]

Then we combine the above two estimates to have

\[ \frac{d}{dt} \|\langle \langle v \rangle \rangle^k |\nabla_x f|^{\|p\} + \|\langle \langle v \rangle \rangle^k |\nabla_v f|^{\|p\}} \leq C(T)(1 + \|\nabla_x u\|_{\infty})(\langle \langle v \rangle \rangle^k |\nabla_x f|^{\|p\} + \langle \langle v \rangle \rangle^k |\nabla_v f|^{\|p\}). \]

We further apply the Gronwall inequality and use (42) to derive (45).
Lemma 4.4. Suppose that the conditions in Lemma 3.3 hold. Then, we have
\[ (\|u\|^2 + \|x\| \sqrt{\|\nabla u\|^2} ) (t) + \frac{\mu}{2} \sqrt{\|\nabla \dot{u}\|^2} \leq C(T), \quad q \geq \frac{4}{\alpha}. \]

Proof. First, we rewrite the momentum equation (1) as
\[ \rho \dot{u} + \nabla P = \mu \Delta u + \nabla ((\mu + \lambda(\rho)) \text{div} u) - \int \rho \dot{u} dx. \]
We multiply the above equation by \(|x|^\alpha \dot{u}\), and integrate the resulting relation with respect to \(x\) over \(\mathbb{R}^2\) to get
\[
\frac{d}{dt} \int_{\mathbb{R}^2} \left( \frac{\mu}{2} |\nabla u|^2 + (\mu + \lambda(\rho)) \frac{\text{div} u}{2} - P(\rho) \text{div} u \right) |x|^\alpha dx + \int_{\mathbb{R}^2} \rho |\dot{u}|^2 |x|^\alpha dx \]
\[ = \int_{\mathbb{R}^2} (P(\rho) (\dot{u} - u(\text{div} u)) + \frac{\mu}{2} u |\nabla u|^2 - \mu \nabla u \cdot \dot{u}) \]
\[ + (\mu + \lambda(\rho)) u (\frac{\text{div} u}{2}) - (\mu + \lambda(\rho)) \text{div} u \dot{u}\cdot \nabla (|x|^\alpha) dx \]
\[ + \int_{\mathbb{R}^2} \left( \mu \partial_{x_i} u_j \partial_{x_i} u_k \partial_{x_k} u_j + \frac{\mu}{2} |\nabla u|^2 \text{div} u - (\mu + \lambda(\rho)) \text{div} u \partial_{x_i} u_j \partial_{x_j} u_i \right) \]
\[ - \rho \lambda(\rho) (\frac{\text{div} u}{2})^3 |x|^\alpha dx \]
\[ = \sum_{i=1}^{4} I_{9i}. \]
Now, we estimate the terms \(I_{9i}(1 \leq i \leq 4)\), separately.
- Case I.1 (Estimate of \(I_{9i}(1 \leq i \leq 3)\)): As the corresponding estimates in Lemma 4.2 [27], we have
  \[ |I_{91}| \leq C(T) [\|\nabla u\|_{\infty} + 1] |x|^{\frac{3}{2}} |\nabla u|^2 + |x|^{\frac{3}{2}} |\nabla \dot{u}|^2 + 1, \]
  \[ |I_{92}| \leq C(T) \int_{\mathbb{R}^2} (1 + \lambda(\rho)) |\nabla u|^3 |x|^\alpha dx \leq C(T) \|\nabla u\|_{\infty} |x|^{\frac{3}{2}} |\nabla u|^2, \]
  \[ |I_{93}| \leq C(T) \int_{\mathbb{R}^2} P(\rho) |\nabla u|^2 |x|^\alpha dx \leq C(T) |x|^{\frac{3}{2}} |\nabla u|^2. \]
- Case I.2 (Estimate of \(I_{94}\)): We use (i) in Lemma 3.1 and (i) in Corollary 1 to have
  \[ |I_{94}| \leq \int_{\mathbb{R}^4} f |\dot{u}|^2 |x|^\alpha dx + \int_{\mathbb{R}^4} f |u - v|^2 |x|^\alpha dx \]
  \[ \leq \int_{\mathbb{R}^4} f (|\dot{u}|^2 + 2|u|^2) |x|^\alpha dx + 2 \int_{\mathbb{R}^4} f |v|^2 |x|^\alpha dx \]
  \[ \leq \int_{\mathbb{R}^4} f (|\dot{u}|^2 + 2|u|^2) |x|^\alpha dx + 2 \left( \int_{\mathbb{R}^4} f |x|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^4} f |x|^2 dx \right)^{\frac{2}{2}} \]
  \[ \leq C(T) + \int_{\mathbb{R}^4} f (|\dot{u}|^2 + 2|u|^2) |x|^\alpha dx. \]
Note that
\[
\int_{\mathbb{R}^2} \left( \frac{\mu}{2} |\nabla u|^2 + (\mu + \lambda(\rho)) \frac{(\text{div} u)^2}{2} - P(\rho) \text{div} u \right) |x|^\alpha \, dx \\
\geq \int_{\mathbb{R}^2} \left( \frac{\mu}{2} |\nabla u|^2 + (\mu + \lambda(\rho)) \frac{(\text{div} u)^2}{2} - \mu \frac{(\text{div} u)^2}{4} - CP(\rho)^2 \right) |x|^\alpha \, dx \\
\geq \int_{\mathbb{R}^2} \left( \frac{\mu}{2} |\nabla u|^2 + (\mu + \lambda(\rho)) \frac{(\text{div} u)^2}{4} \right) |x|^\alpha \, dx - C(T),
\]
by (17) in Lemma 3.6 and (32). Collecting the above estimates in (47), integrating the resulted equation with respect to $t$ over $[0, T]$, we apply the Gronwall inequality and use Lemma 3.6, 4.1, 4.2 to have
\[
\|x^{\frac{q}{2}} \nabla_x u\|_2^2(t) + \int_0^t \|x^{\frac{q}{2}} \sqrt{\rho u_t}\|_2^2 \, d\tau \leq C(T), \quad \forall t \in [0, T].
\]
For $q \geq \frac{4}{\alpha}$, we use (3) in Lemma A.4 to have
\[
\|u\|_q \leq C(1 + |x|^{\frac{q}{2}}) \nabla_x u\|_2 \leq C(T).
\]

\begin{lemma}
Suppose that the conditions in Lemma 3.3 hold. Then, we have
\[
\left( \| (1 + |x|^{\frac{q}{2}}) \sqrt{\rho u_t} \|_2^2 + \| (\rho_t, P(\rho), \lambda(\rho)) \|_H^2 + \| \nabla_x u \|_2^2 \right)(t) \\
+ \int_0^t \left( \|u_t\|_q^2 + \| \nabla_x u_t \|_2^2 + \| \nabla_x u \|_2^2 + \| (\rho_t, P(\rho), \lambda(\rho)) \|_2^2 \right) \, d\tau \leq C(T),
\]
where $q \geq \frac{4}{\alpha}$.
\end{lemma}

\begin{proof}
By the standard $L^2$-estimates for (1), we use Corollary 2, Lemma 4.1 and Lemma 4.2 to have
\[
\| \nabla_x u \|_2 \leq C(\| \rho u_t \|_2 + \| \nabla_x P(\rho) \|_2 + \| (u - u_f) \rho_f \|_2) \leq C(T),
\]
By the Sobolev inequalities, we use (3) and the above estimate to have
\[
\sup_{0 \leq t \leq T} \|u\|_{\infty} \leq C \sup_{0 \leq t \leq T} \|u\|_{\frac{2+\frac{q}{2}}{2}} \| \nabla_x u \|_{2} \leq C(T),
\]
\[
\sup_{0 \leq t \leq T} \| \nabla_x u \|_p \leq C \sup_{0 \leq t \leq T} \| \nabla_x u \|_{H^{1}} \leq C(T), \quad 2 \leq p < \infty.
\]
Then we can further use Corollary 2, Lemma 4.1, 4.2, 4.4 to obtain
\[
\left( \| (1 + |x|^{\frac{q}{2}}) \sqrt{\rho u_t} \|_2^2 \right) \leq 2 \|(1 + |x|^{\frac{q}{2}}) \sqrt{\rho u} \|_2^2 + \| \nabla_x \cdot (\rho_u \cdot \nabla_x u) \|_2^2 \\
\leq 2 \|(1 + |x|^{\frac{q}{2}}) \sqrt{\rho u} \|_2^2 + C(T) \| (1 + |x|^{\frac{q}{2}}) \nabla_x u \|_2^2, \\
\| \nabla_x u_t \|_2^2 \leq \| \nabla_x u \|_2^2 + \|u_t\|_{\infty}^2 \| \nabla_x u \|_2^2 + \| \nabla_x u \|_2^2, \\
\|u_t\|_q^2 \leq C(\| (1 + |x|^{\frac{q}{2}}) \nabla_x u \|_2^2 + \|u_t\|_{\infty}^2 \| \nabla_x u \|_2^2 \\
\leq C(T) + C(1 + |x|^{\frac{q}{2}}) \nabla_x u \|_2^2, \quad q \geq \frac{4}{\alpha},
\]
and
\[
\left( \| (1 + |x|^{\frac{q}{2}}) \sqrt{\rho u_t} \|_2^2(t) \right) + \int_0^t (\| \nabla_x u_t \|_2^2 + \|u_t\|_q^2) \, d\tau \leq C(T).
\]
On the other hand, we apply the operator $\nabla^2$ to the continuity equation (1) and obtain
\[
\frac{d}{dt}\|\nabla^2 \rho\|_2^2 \leq C(T)(\|\nabla \rho\|_\infty + 1)\|\nabla^2 \rho\|_2^2 + \|\nabla \rho\|_2^2 + 1).
\] (48)

Similarly, we have
\[
\frac{d}{dt}\|\nabla^2 P(\rho)\|_2^2 \leq C(T)(\|\nabla \rho\|_\infty + 1)\|\nabla^2 P(\rho)\|_2^2 + \|\nabla^2 \rho\|_2^2 + 1).
\] (49)

We first note that for $1 < p_1 < \infty$, $b_{p_1 - \frac{1}{4}} > 2$,
\[
\left\|\int_{\mathbb{R}^2} |v|^i \nabla_x f dv\right\|_{p_1} \leq C\left(\int_{\mathbb{R}^2} (v)^{-b_{p_1 - \frac{1}{4}} + 1} dv\right)^{\frac{p_1 - 1}{p_1}}\int_{\mathbb{R}^d} (v)^{(b_{p_1} - 1)p_1} |\nabla_x f|^{p_1} dv dx \leq C(T),
\] (50)

by Lemma 4.3 for $i = 0, 1, 2, 3$ and suitably large $k$. We set $m_k \nabla_x f := \int_{\mathbb{R}^2} |v|^k |\nabla_x f| dv$. Then, by standard elliptic estimates, we use (32), (50) and Lemma 4.2 to have
\[
\|\nabla^3 u\|_2 \leq C\|\nabla^2 \text{div} u\|_2 + \|\nabla^2 \omega\|_2
\]
\[
\leq C(T)(\|\nabla^2 F\|_2 + \|\nabla^2 \omega\|_2 + \|\nabla^2 P(\rho)\|_2 + \|\text{div} \nabla^2 \omega\|_2 + \|\text{div} \nabla^2 u\|_2)
\]
\[
\leq C(T)(\|\nabla \rho\|_2 + \|u\|_\infty m_0 \nabla x f|_2 + \|\nabla u\|_2 \rho|_2 + \|\Delta^2 \rho\|_2)
\]
\[
+ \|\nabla \nabla^2 \rho\|_2 + \|\Delta^2 \rho\|_2 + \|\Delta^2 \rho\|_2 + 1)
\]
\[
\leq C(T)(\|\nabla \rho\|_2 \nabla x \rho\|_2 + \|\nabla^2 \rho\|_2 + \|\nabla^2 \rho\|_2 + \|\nabla^2 \rho\|_2 + 1).
\]

We combine (48), (49) and (51) to obtain
\[
\frac{d}{dt}\|\nabla^2 \rho, \nabla^2 P(\rho)\|_2^2 \leq C([\|\nabla \rho\|_2^2 + 1]([\|\nabla \rho\|_2^2 + 1] + \|\nabla^2 \rho\|_2 + \|\nabla^2 \rho\|_2 + 1].
\]

We apply the Gronwall inequality and use Lemma 4.1, 4.2, (63) to have
\[
\sup_{0 \leq t \leq T} \|\nabla^2 \rho, \nabla^2 P(\rho)\|_2^2 + \int_0^T \|\nabla^2 \rho\|_2^2 d\tau \leq C(T).
\]

By the continuity equation (1) and, we have
\[
P(\rho) = -u \cdot \nabla_x P(\rho) - \gamma P(\rho) \text{div} u, \quad \lambda(\rho) = -u \cdot \nabla_x \lambda(\rho) - \beta \lambda(\rho) \text{div} u.
\] (52)

Then we use (32) and Lemma 4.2 to have
\[
\|\rho_0, P(\rho)\|_{H^1} \leq C\|\rho_0\|_\infty \|u\|_\infty \|\nabla \rho\|_2 + \|\nabla u\|_2\)
\]
\[
+ C(\|\rho_0\|_\infty \|u\|_\infty \|\nabla \rho\|_2 \|\nabla u\|_2 + \|\nabla^2 \rho\|_2)
\]
\[
\leq C(T).
\]

Similarly, we apply the operator $\partial_t$ to (52) to obtain
\[
\int_0^t \|\rho_t, P(\rho)\|_{H^1} d\tau \leq C(T)\int_0^t \|\rho\|_2^2 + \|u\|_2^2 \|\nabla \rho\|_2^2 + \|\nabla u\|_2^2 d\tau
\]
\[
+ \|\rho_t\|_2^2 \|\nabla \rho\|_2^2 + \|\nabla\rho_t\|_2^2 d\tau
\]
\[
\leq C(T)\int_0^t \|\nabla \rho\|_2^2 + \|\rho\|_2^2 + 1) d\tau \leq C(T).
\]
\[
\square
4.2. Second order estimates. In this subsection, we derive the second order estimates of the classical solution \([f, \rho, u]\) to the system (1)-(2): \(\|\nabla^2(\rho, u)\|_p\) and \(\|u^k\nabla_x^k f\|_p\) for \(4 < p < \infty\).

**Lemma 4.6.** Suppose that the conditions in Lemma 3.3 hold. Then, for \(4 < p < \infty\) we have

\[
(t\|\nabla u_t, \nabla u\|^2_2 + t\|\sqrt{\rho}\rho u_t\|^2_2 + \|\rho, P(\rho)\|_{W^{2,p}})(t)
+ \int_0^t [\tau(\|\sqrt{\rho}\rho u_t\|^2_2 + \|\sqrt{\rho}\rho u_t\|^2_2 + \|\nabla^2 u_t\|^2_2) + \|\nabla u\|_{W^{2,p}}]d\tau
\leq C(T).
\]

**Proof.** We apply the operator \(\partial_t\) to (1) to obtain

\[
\rho u_{tt} + \rho u \cdot \nabla u_t + \nabla_x P(\rho)_t
= \mu \Delta_x u_t + \nabla_x ((\mu + \lambda(\rho))\text{div} u_t) - \rho_t u_t - \rho_t u \cdot \nabla_x u - \rho u_t \cdot \nabla_x u
+ \nabla_x (\lambda(\rho)\text{div} u_t) - u_t \rho_f - u(\rho f)_t + (u f \rho f)_t.
\]

We multiply the above equation by \(u_{tt}\), and use the integration by parts to obtain

\[
\|\sqrt{\rho}\rho u_{tt}\|^2_2 + \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} (\mu \nabla u_t)^2 + (\mu + \lambda(\rho))|\text{div} u_t|^2 + 2\lambda(\rho) |\text{div} u_t \text{div} u_t| \, dx
+ \frac{d}{dt} \int_{\mathbb{R}^2} \left(\frac{3}{2} \lambda(\rho)|\text{div} u_t|^2 + \lambda(\rho)|\text{div} u_t \text{div} u_t| \right) dx
= \int_{\mathbb{R}^2} (\nabla_x P(\rho)_t + \rho_t u_t + \rho_t u \cdot \nabla_x u + \rho u \cdot \nabla_x u)
+ \rho u_t \cdot \nabla_x u) \cdot u_{tt} \, dx + \int_{\mathbb{R}^2} \left(\frac{3}{2} (u_t)^2 |\rho f|_t + u \cdot u(\rho f)_t - u_t \cdot (u f \rho f)_t \right) dx.
\]

As the corresponding estimates in Lemma 3.12 [29], the first term and the second term in the RHS of (53) can be estimated as follows:

\[
\int_{\mathbb{R}^2} \left(\frac{3}{2} \lambda(\rho)|\text{div} u_t|^2 + \lambda(\rho)|\text{div} u_t \text{div} u_t| \right) dx
\leq \frac{1}{8} \|\sqrt{\rho}\rho u_{tt}\|^2_2 + C(T)(\|\nabla u_t\|^2_\infty + 1)\|\nabla u_t\|^2_2
+ C(T)(\|\lambda(\rho)\|^2_2 + \|u_t\|^2_4 + \|\nabla u\|^2_\infty + \|\nabla^3 u\|^2_2 + 1)
- \int_{\mathbb{R}^2} (\nabla_x P(\rho)_t + \rho_t u_t + \rho_t u \cdot \nabla_x u + \rho u \cdot \nabla_x u_t + \rho u_t \cdot \nabla_x u) \cdot u_{tt} \, dx
\leq \frac{d}{dt} \int_{\mathbb{R}^2} (P(\rho)_t \text{div} u_t - \rho_t \frac{|u_t|^2}{2} - \rho_t u \cdot \nabla_x u \cdot u_t) \, dx
+ \frac{1}{8} \|\sqrt{\rho}\rho u_{tt}\|^2_2 + C(T)(\|\rho_t\|^2_2 + \|P(\rho)\|^2_2 + \|u_t\|^4_4
+ \|\nabla_x u_t\|^2_2 + \|\nabla_x u\|^2_\infty + \|\nabla^3 u\|^2_2 + 1).
\]
Now, we turn to treat the third term on the RHS of (53). Rewrite this term as
\[
\int_{\mathbb{R}^2} \frac{3}{2} (u_t)^2 (\rho f)_t + u \cdot u_t (\rho f)_t - (u \cdot (u f)_{tt}) dx
\]
\[
= \int_{\mathbb{R}^2} \frac{3}{2} |u|^2 (\rho f)_t dx + \int_{\mathbb{R}^2} u \cdot u_t (\rho f)_t dx + \int_{\mathbb{R}^2} -u_t \cdot (u f_{tt}) dx
\]
\[
= \sum_{i=1}^{3} I_{10i}.
\]
Before the estimation of $I_{10i}$, we first note that
\[
(\rho f)_t = - \int_{\mathbb{R}^2} v \cdot \nabla_x f dv,
\]
\[
(\rho f)_{tt} = \int_{\mathbb{R}^2} (v_i v_j \partial_{x_i x_j}^2 f - \partial_{x_i}((L_i f) + u_i - v_i) f) dv,
\]
\[
(u f f)_{tt} = \int_{\mathbb{R}^2} (-v v \cdot \nabla_x f + (L(f) + u - v) f) dv,
\]
\[
(u f f)_{tt} = \int_{\mathbb{R}^2} v v_i [v_j \partial_{x_i x_j}^2 f + \partial_{x_i x_j}^2 ((L_j f) + u_j - v_j) f]
\]
\[
- \partial_{x_i}(\Delta_v(|v - v_c|^2 f)) - (L(f) + u - v)(v \cdot \nabla_x f)
\]
\[
+ \nabla_v \cdot (L(f) f + (u - v) f) + L(f) f) dv + u_t \rho f,
\]
and
\[
\mu \Delta_x u_t + \nabla_x ((\mu + \lambda(\rho)) \text{div} u_t)
\]
\[
= (\rho u)_t + \nabla_x P(\rho)_t - \nabla_x (\lambda(\rho) \text{div} u_t) + [(u - u_f) \rho]_t,
\]
\[
||\nabla^2 u||_2 \leq C( ||\sqrt{\rho}||_\infty ||\sqrt{\rho} u_t||_2 + ||\rho_t||_4 ||u||_4
\]
\[
+ ||\rho||_\infty ||u_t||_4 ||\nabla x u||_4 + ||\rho u||_\infty ||\nabla x u_t||_2 + ||\rho_t||_4 ||u||_\infty ||\nabla x u_t||_4
\]
\[
+ ||\nabla x P(\rho)\|_2 + ||\nabla x \lambda(\rho)\|_2 ||\text{div} u_t||_\infty + ||\lambda(\rho)||_4 ||\nabla^2 u||_4
\]
\[
+ ||u_t||_4 ||\rho_t||_4 + ||u f f||_2 + ||(u f f)_t|| + 11)
\]
\[
\times (||\rho f||_2 + ||m_1 \nabla x f||_2) + ||m_2 \nabla x f||_2
\]
\[
\leq C(T)( ||\nabla^4 u||_2 + ||u_t||_4 + ||\nabla x u_t||_2 + ||\nabla^2 u||_2 + 1),
\]
by the estimates in Lemma 4.5.

- Case 1.1 (Estimate of $I_{101}$): By the integration by parts, we use (ii) in Corollary 1, (56) to have
\[
I_{101} = \int_{\mathbb{R}^2} 3 u_t \cdot \nabla_x u_t \cdot u f f dx
\]
\[
\leq C ||u_t||_4 ||\nabla x u_t||_2 ||u f f||_4 \leq ||u||_4^2 + C(T)||\nabla x u_t||_2^2.
\]
- Case 1.2 (Estimate of $I_{102}$): We apply the integration by parts and use (ii) in Corollary 1, (56) and (57) to get
\[
I_{102} = \int_{\mathbb{R}^2} u \cdot u_t (v_i v_j \partial_{x_i x_j}^2 f - \partial_{x_i}((L_i f) + u_i - v_i) f) dv dx
\]
\[
\leq (||\nabla^2 u||_4 ||u_t||_4 + ||\nabla x u||_\infty ||\nabla x u_t||_2 + ||u||_\infty ||\nabla^2 u_t||_2 ||m_2 f||_2
\]
\[
+ C(T)(||\nabla x u||_2 ||u_t||_4 + ||\nabla x u_t||_2 ||u_t||_4) ||u f f||_4 + ||\rho f||_4
\]
\[
+ C(T)||u||_4^2 ||\nabla x u_t||_2 ||\rho f||_2 + C ||\nabla x u||_\infty ||u||_\infty ||u_t||_4 ||\rho f||_4
\]
We apply the integration by parts to use Corollary 1, (56) and (57) to obtain
\[
\leq \frac{1}{16} \| \sqrt{\rho_0 t} u_t \|^2 + C(T) [\| u_t \|^2 + \| \nabla_x u \|^2 + (\| \nabla_x u \|^2 + 1)(\| \nabla_x u_t \|^2 + 1)].
\]

Case J.3 (Estimate of $I_{103}$): We rewrite $I_{103}$ as
\[
I_{103} = - \int_{\mathbb{R}^t} u_t [v_v v_j \partial_{x, x_j} v_j + v_v \partial_{x, x_j}(L_j(f) + u_j - v_j) f] \\
- v_v \partial_{x_t}(\Delta_v (\| v - v_c \|^2)) dv dx \\
- \int_{\mathbb{R}^t} u_t [-(L(f) + u - v)(v \cdot \nabla_x f + \nabla_v \cdot (L(f) f + (u - v) f)) \\
+ L(f) f + u_t f] dv dx
\]
:= $I_{1031} + I_{1032}$.

We apply the integration by parts to use Corollary 1, (56) and (57) to obtain
\[
I_{1031} \leq C(T) [\| \nabla^2 u_t \|_2 \| m_3 f \|_2 \\
+ C(T) [\| v_{x} u_t \|_2 \| \rho f \|_2 + \| u_f \rho f \|_2 + \| u_l \|_4 \| u_f \rho f \|_4 + \| m_2 f \|_2]
\]
\[
\leq \frac{1}{16} \| \sqrt{\rho_0 t} u_t \|^2 + C(T) [\| u_t \|^2 + \| \nabla^2 u_t \|^2 + (\| \nabla_x u \|^2 + 1)(\| \nabla_x u_t \|^2 + 1)]
\]
\[
I_{1032} \leq - \frac{1}{2} \| \sqrt{\rho^f} u_t \|^2 + C(T) [\| v_{x} u_t \|_2 \| u_f \rho f \|_2 + \| u_l \|_4 \| u_f \rho f \|_4 + \| m_2 f \|_2] \\
+ C(T) [\| u_t \|_4 (\| \nabla_x u \|_2 \| u_f \rho f \|_4 + \| u_l \|_4 \| \rho f \|_4 + \| u_f \rho f \|_4 + \| m_2 f \|_4)
\]
\[
\leq C(T) [\| \nabla_x u_t \|^2_2 \| u_t \|^2_4].
\]

We collect all estimates of $I_{831}, I_{832}$ in (59) to obtain
\[
I_{103} \leq \frac{1}{16} \| \sqrt{\rho_0 t} u_t \|^2 + C(T) [\| u_t \|^2 + \| \nabla^2 u_t \|^2 + (\| \nabla_x u \|^2 + 1)(\| \nabla_x u_t \|^2 + 1)].
\]

We combine the estimates of $I_{8i}(1 \leq i \leq 3)$ and (55) to have
\[
\int_{\mathbb{R}^2} \frac{3}{2} (u_t)^2 (\rho f)_t + u \cdot u_t (\rho f)_t - u_t \cdot (u_f \rho f)_t dx \\
\leq \frac{1}{8} \| \sqrt{\rho_0 t} u_t \|^2 + C(T) [\| u_t \|^2 + \| \nabla^2 u_t \|^2 + (\| \nabla_x u \|^2 + 1)(\| \nabla_x u_t \|^2 + 1)].
\]

With the estimates (54) and (60) in hands, we can further estimate (53) as
\[
\frac{1}{2} \| \sqrt{\rho_0 t} u_t \|^2 + \frac{1}{2} \| \sqrt{\rho^f} u_t \|^2 + \frac{d}{dt} \Pi(t) \\
\leq C(T) [\| (\rho_0 t, P(\rho_t, \lambda(\rho_t)) \|^2_2 + \| u_t \|^2 + \| \nabla^2 u_t \|^2_2 \\
+ (\| \nabla_x u \|^2 + 1)(\| \nabla_x u_t \|^2 + 1)],
\]

where
\[
\Pi(t) = \int_{\mathbb{R}^2} \left( \mu [\nabla_x u_t]^2 + (\mu + \lambda(\rho)) [\div u_t]^2 + \lambda(\rho) \div u_t \div \div u_t \\
- P(\rho_t) \div u_t + \rho_t \left( \frac{(u_t)^2}{2} + \rho_t u \cdot \nabla x u \cdot u_t \right) dx \\
+ \int_{\mathbb{R}^2} \left( \frac{(u_t)^2}{2} \rho f + u \cdot u_t (\rho f)_t - u_t \cdot (u_f \rho f)_t dx.\right.
\]

Note that
\[
\left| \int_{\mathbb{R}^2} (\lambda(\rho) \div u_t - P(\rho_t) \div u_t) dx \right|
\]
We can further obtain
\[ \frac{1}{2} \|
abla u_t \|_2^2 + \frac{1}{2} \|
abla P u_t \|_2^2 + \frac{d}{dt} \Pi(t) \leq C(T)(\|\rho u_t\|_2^2 + \|
abla u_t\|_2^2 + \Pi(t) + \|
abla u_t\|_2^2 + \|\rho u_t\|_2^2 + \|u_t\|_2^2 + \|
abla u_t\|_2^2 + \Pi(t) + 1) \]

We multiply the above inequality by \( t \), integrate the resulting inequality with respect to \( t \) over the interval \([\tau, t_1]\) with \( \tau, t_1 \in [0, T] \), and we use the similar arguments as in Lemma 3.12 [29] to obtain
\[ t\|
abla u\|_2^2 \leq t\|
abla u_t\|_2^2 + t\|u\|_2^2 \|
abla^2 u\|_2^2 + t\|
abla u\|_2^2 \leq C(T), \quad \text{and} \]
\[ (t\|
abla u_t\|_2^2 + \|\nabla P u_t\|_2^2) + \int_0^t \tau(\|
abla u_t\|_2^2 + \|\nabla P u_t\|_2^2 + \|\nabla^2 u\|_2^2) d\tau \leq C(T), \quad t \in [0, T]. \]

On the other hand, we apply operator \( \nabla^2 \) to the continuity equation (1) and obtain
\[ \frac{d}{dt}\|\nabla^2 \rho\|_p \leq C(T)(\|\nabla u\|_\infty \|\nabla^2 \rho\|_p + \|\nabla^2 u\|_{W^{1,p}}). \quad (61) \]

For \( P(\rho) \), we also have
\[ \frac{d}{dt}\|\nabla^2 P(\rho)\|_p \leq C(T)(\|\nabla u\|_\infty \|\nabla^2 P(\rho)\|_p + \|\nabla^2 u\|_{W^{1,p}}). \quad (62) \]

To estimate \( \|\nabla^2 u\|_{W^{1,p}} \) on the RHS of (61) and (62), we apply \( \partial_x \) to (1) and obtain
\[ \mu \Delta(\partial_x, u) + \nabla_x((\mu + \lambda(\rho))\text{div}(\partial_x, u)) = \nabla_x(\partial_x, \rho(\rho)\text{div}u) + u_t \partial_x, \rho + \rho \partial_x, u_t + \rho \nabla_x u \cdot \partial_x, u + \rho \partial_x, u \cdot \nabla_x u + \nabla_x \rho \cdot \partial_x, u + \nabla_x \partial_x, P(\rho) + \rho_f \partial_x, u + u \partial_x, (\rho_f) - \partial_x, (u f \rho_f). \]
Then, we use Lemma 4.4 and (50) to have
\[
|\nabla_x u|_{W^{2,p}} \leq C(T)(|\nabla_x u|_\infty + 1)||\nabla_x^2 p, \nabla_x^2 P(\rho)||_p + |\nabla_x u|_{W^{1,p}} + |\nabla_x u_t|_p \\
+ |\partial_x \rho f|_p + |\partial_x(u f \rho_f)|_p + 1 \\
\leq C(T)(|\nabla_x u|_\infty + 1)||\nabla_x^2 p, \nabla_x^2 P(\rho)||_p + |\nabla^3 u|_2 + |\nabla_x u_t|_p + 1.
\]

We combine this estimate and (61), (62) to have
\[
\frac{d}{dt} ||\nabla_x^2 p, \nabla_x^2 P(\rho)||_p \\
\leq C(T)(|\nabla_x u|_\infty + 1)||\nabla_x^2 p, \nabla_x^2 P(\rho)||_p + |\nabla^3 u|_2 + |\nabla_x u_t|_p + 1.
\]

Note that for any \( t \in [0,T], \)
\[
\int_0^t |\nabla_x u_t|_p d\tau \leq C \int_0^t \tau^{-\frac{\alpha - 2}{2}} |\nabla_x^2 u_t|_2 \tau^{\frac{\alpha - 2}{2}} |\nabla_x u_t|_2^{1-\frac{\alpha}{2}} d\tau \\
\leq C \sup_{0 \leq \tau \leq t} (\tau |\nabla_x u_t|_2(\tau))^{\frac{\alpha - 2}{2}} \left( \int_0^t \tau^{-\frac{\alpha - 2}{2}} d\tau \right)^{\frac{\alpha - 1}{2}} \left( \int_0^t \tau^2 |\nabla_x u_t|_2 d\tau \right)^{\frac{\alpha - 1}{2}} \\
\leq C(T).
\]

Then, by Grönwall’s lemma, we have
\[
||\nabla_x^2 p, \nabla_x^2 P(\rho)||_p(t) \leq C(T), \quad \int_0^t |\nabla_x u|_{W^{2,p}} d\tau \leq C(T), \quad t \in [0,T]. \tag{63}
\]

With the bound of \(|\nabla_x u|_{L^2(0,T;W^{2,p})}, q > 2\), one can get the second order estimates of \( f(x,v,t) \) as follows.

**Lemma 4.7.** Suppose that the conditions in Lemma 3.3 hold. Then, we have
\[
\sum_{|\alpha| + |\beta| = 2} \| (v)_k^\alpha \partial_x^\beta f \|_p(t) \leq C(T), \quad 4 < p < \infty. \tag{64}
\]

**Proof.** We apply \( \partial_x^\alpha (|\alpha| = 2) \) to the (1)_3, and multiply the above equation by
\[
(v)_k^p \partial_x^\alpha f \|^p - 2 \partial_x^\alpha f,
\]
and integrate the resulted equations with respect to \( x, v \) over \( \mathbb{R}^4 \) to obtain
\[
\frac{d}{dt} \| (v)_k^p \partial_x^\alpha f \|^p = - \int_{\mathbb{R}^4} (v)_k^p \partial_x^\alpha f \|^{p-2} \partial_x^\alpha f \\
\times \nabla_v \cdot \left( \int_{\mathbb{R}^4} \partial_x^\alpha (\psi(x-v))(v_4-v) f(v_4, y, t) dv_4, dy f \right) dx \\
- 2 \sum_{|\alpha'| = 1} \int_{\mathbb{R}^4} (v)_k^p \partial_x^\alpha f \|^{p-2} \partial_x^\alpha f \\
\times \nabla_v \cdot \left( \int_{\mathbb{R}^4} \partial_x^\alpha (\psi(x-v))(v_4-v) f(v_4, y, t) dv_4, dy \partial_x^\alpha f \right) dx \\
- \int_{\mathbb{R}^4} (v)_k^p \partial_x^\alpha f \|^{p-2} \partial_x^\alpha f \\
\times \nabla_v \cdot \left( \int_{\mathbb{R}^4} (\psi(x-v))(v_4-v) f(v_4, y, t) dv_4, dy \partial_x^\alpha f \right) dx \\
- \int_{\mathbb{R}^4} (v)_k^p \partial_x^\alpha f \|^{p-2} \partial_x^\alpha f \nabla_v \cdot ((v_4-v) \partial_x^\alpha f) dx \tag{65}
\]
Similarly, we use integration by parts to have

\begin{align*}
-2 \sum_{|\alpha|=1}^{\infty} \int_{\mathbb{R}^d} \langle v \rangle^{kp} p |\partial^{\alpha} f|^{p-2} |\partial^{\alpha} f| \nabla_v \cdot (\partial^\alpha u \partial^{\alpha} f) \, dv dx \\
- \int_{\mathbb{R}^d} \langle v \rangle^{kp} p |\partial^{\alpha} f|^{p-2} |\partial^{\alpha} f| \nabla_v \cdot (\partial^\alpha u f) \, dv dx \\
+ \int_{\mathbb{R}^d} \langle v \rangle^{kp} p |\partial^{\alpha} f|^{p-2} |\partial^{\alpha} f| \Delta_v (|v| - v_c)^2 f \, dv dx \\
= : \sum_{i=1}^{7} I_{11i}.
\end{align*}

In the sequel, we estimate the terms $I_{11i}$ separately.

- Case K.1 (Estimate of $I_{111}$): We rewrite $I_{111}$ as follows.

\begin{align*}
I_{111} &= - \int_{\mathbb{R}^d} \langle v \rangle^{kp} p |\partial^{\alpha} f|^{p-2} |\partial^{\alpha} f| \\
&\times \nabla_v \cdot \left( \int_{\mathbb{R}^d} \partial^{\alpha} (\psi(x-y))(v_c - v_c) f(v_c, y) \, dv_x dy f \right) \, dv dx \\
&- \int_{\mathbb{R}^d} \langle v \rangle^{kp} p |\partial^{\alpha} f|^{p-2} |\partial^{\alpha} f| \\
&\times \nabla_v \cdot \left( \int_{\mathbb{R}^d} \partial^{\alpha} (\psi(x-y))(v_c - v) f(v_c, y) \, dv_x dy f \right) \, dv dx \\
&=: I_{1111} + I_{1112}.
\end{align*}

By direct computation, we have

\begin{align*}
|I_{1111}| &\leq \sup_{x \in \mathbb{R}^d} |\nabla^2 \psi| \int_{\mathbb{R}^d} \langle v \rangle^{kp} p |\partial^{\alpha} f|^{p-1} |\nabla_v f| \, dv dx \int_{\mathbb{R}^d} |v_c - v_c| f(v_c, y) \, dv_x dy f \\
&\leq C(T)(\|\langle v \rangle^{kp} p |\partial^{\alpha} f|\|_p + \|\langle v \rangle^{kp} |\nabla_v f|\|_p),
\end{align*}

\begin{align*}
|I_{1112}| &\leq p(p - 1) \int_{\mathbb{R}^d} \langle v \rangle^{kp} p |\partial^{\alpha} f|^{p-2} |\nabla_v |\partial^{\alpha} f| |v| - v_c| \, dv dx \\
&\quad + kp^2 \int_{\mathbb{R}^d} \langle v \rangle^{kp-1} f |\partial_x^{\alpha} f|^{p-1} \, dv dx \\
&\leq \frac{p-1}{p} \|v| - v_c\|\langle v \rangle^{kp} \nabla_v (|\nabla_v f|^2)^{\frac{p}{2}} + C(T)(\|\langle v \rangle^{kp} |\partial^{\alpha} f|\|_p + \|\langle v \rangle^{kp} |\nabla_v f|\|_p).
\end{align*}

Then, we can further estimate $I_{111}$ as

\begin{align*}
|I_{111}| &\leq \frac{p-1}{p} \|v| - v_c\|\langle v \rangle^{kp} \nabla_v (|\nabla^2_v f|^2)^{\frac{p}{2}} + C(T)(\|\langle v \rangle^{kp} |\partial^{\alpha} f|\|_p + \|\langle v \rangle^{kp} |\nabla_v f|\|_p).
\end{align*}

Similarly, we use integration by parts to have

\begin{align*}
|I_{112}| + |I_{113}| + |I_{114}| &\leq C(\|\langle v \rangle^{kp} |\partial^{\alpha} f|\|_p + \|\langle v \rangle^{kp} |\partial^{\alpha} f|\|_p + \|v| - v_c\|\langle v \rangle^{kp} \nabla_v (|\nabla_v f|^2)^{\frac{p}{2}},
\end{align*}

\begin{align*}
|I_{115}| + |I_{116}| &\leq C(T)(\|\Delta_v u\|_\infty + \|\nabla^2_v u\|_\infty) \|\langle v \rangle^{kp} |\partial^{\alpha} f|\|_p \times (\|\langle v \rangle^{kp} |\nabla_v f|\|_p + \|\langle v \rangle^{kp} |\nabla^2_v f|\|_p) \\
&\leq C(T)(\|\Delta_v u\|_{W^{2,p}}(\|\langle v \rangle^{kp} |\partial^{\alpha} f|\|_p + \|\langle v \rangle^{kp} |\nabla_v f|\|_p + \|\langle v \rangle^{kp} |\nabla^2_v f|\|_p).
Lemma 4.8. We combine (66) and (67), use Lemma 4.3 to have
\[ I_{117} \leq -\frac{2(p-1)}{p} \|v - v_c\| (v) \|\nabla_v (|\nabla_v f|^2)\|_2^2 + C(p)\|\langle v \rangle^k \partial^\alpha f\|_p. \]

We collect the estimates of $I_{117}$ in (65) to have
\[
\frac{d}{dt} \|\langle v \rangle^k \partial^\alpha f\|_p^p + \frac{p-1}{p} \|v - v_c\| (v) \|\nabla_v (|\nabla_v f|^2)\|_2^2 \\
\leq C(T,p)(1 + \|\nabla_v u\|_2^2)\|\langle v \rangle^k \partial^\alpha f\|_{W^{2,p}(\mathbb{R}^3)}.
\]

Similarly, we can obtain
\[
\frac{d}{dt} \|\langle v \rangle^k \nabla_x \nabla_v f\|_p^p + \frac{p-1}{p} \|v - v_c\| (v) \|\nabla_v^2 (|\nabla_v f|^2)\|_2^2 \\
\leq C(T,p)\|\langle v \rangle^k \partial^\alpha f\|_{W^{2,p}(\mathbb{R}^3)}.
\]

We combine (66) and (67), use Lemma 4.3 to have
\[
\frac{d}{dt} \left( \sum_{|\alpha_s| + |\beta_s| = 2} \|\langle v \rangle^k \partial^\alpha_{\beta_s} f\|_p \right) \leq C(T)(1 + \|\nabla_x u\|_{W^{2,p}}^2) \left( 1 + \sum_{|\alpha_s| + |\beta_s| = 2} \|\langle v \rangle^k \partial^\alpha_{\beta_s} f\|_p \right).
\]

We further apply the Gronwall inequality and use (63) to have (64). \qed

4.3. Third-order estimates. In this subsection, we derive the third order estimates of the classical solution $[f, \rho, u]$ to the system (1)-(2): $\|\nabla_v^3 u\|_p$ and $\|\langle v \rangle^k \nabla_{x,v}^3 f\|_p$ for $4 < p < \infty$. The third order estimates for $f(x, v, t)$ can be obtained as below.

Lemma 4.8. Suppose that the conditions in Lemma 3.3 hold. Then, we have
\[
\sum_{|\alpha_s| + |\beta_s| = 3} \|\langle v \rangle^k \partial^\alpha_{\beta_s} f\|_p(t) \leq C(T), \quad 4 < p < \infty.
\]

Proof. We apply $\partial^\alpha_{\beta_s}$ to (1), $|\alpha_s| + |\beta_s| = 3$, multiply the equation by
\[ (v)^{kp} \rho(\partial^\alpha_{\beta_s} f)^{p-2} \partial^\alpha_{\beta_s} f, \]
summing up $\alpha_s, \beta_s$, and using 4.7 to obtain
\[
\frac{d}{dt} \left( \sum_{|\alpha_s| + |\beta_s| = 3} \|\langle v \rangle^k \partial^\alpha_{\beta_s} f\|_p \right) \leq C(T, p)(1 + \|\nabla_x u\|_{W^{2,p}}^2) \left( 1 + \sum_{|\alpha_s| + |\beta_s| = 3} \|\langle v \rangle^k \partial^\alpha_{\beta_s} f\|_p \right).
\]

By Gronwall’s lemma, we use (63) to have (68). \qed

In order to obtain the estimate $t^2 \|\nabla_v^3 u(t)\|_p$, $(t > 0)$, we need some preparation. To the end, we first show the following estimate.

Lemma 4.9. Suppose that the conditions in Lemma 3.3 hold. Then, we have
\[
t^2 \|\sqrt{\rho} u_{tt}\|_2^2(t) + \int_0^t \tau^2 (\|\nabla_x u_{tt}\|_2^2 + \|\sqrt{\rho} u_{tt}\|_2^2)\,d\tau \leq C(T).
\]
Proof. We apply the operator \(\partial_t\) to (1.2) to obtain
\[
\rho u_{tt} + \rho \cdot \nabla_x u_t - \mu \Delta_x u_t - \nabla_x ((\mu + \lambda(\rho)) \div u_{tt})
\]
\[
= -\nabla_x P(\rho)_{tt} - \rho_t u_t + u_t \cdot \nabla_x u
-2 \rho_t (u_{tt} + u_t \cdot \nabla_x u + u \cdot \nabla_x u_t) - 2 \rho u_t \cdot \nabla_x u_t
- \rho u_{tt} \cdot \nabla_x u + 2 \nabla_x (\lambda(\rho)_t \div u_t) + \nabla_x (\lambda(\rho)_{tt} \div u_t)
- u_{tt} \rho f - 2 u_t (\rho f)_t - u (\rho f)_tt.
\]
We multiply the above equation by \(t^2 u_{tt}\) and integrate the resulted equation with respect to \(x\) over \(\mathbb{R}^2\) to obtain
\[
\frac{1}{2} \frac{d}{dt} \left( t^2 \int_{\mathbb{R}^2} \rho |u_{tt}|^2 dx \right) - t \int_{\mathbb{R}^2} \rho |u_{tt}|^2 dx + t^2 \int_{\mathbb{R}^2} (\mu |\nabla_x u_{tt}|^2
+ (\mu + \lambda(\rho)) (\div u_{tt})^2 ) dx + t^2 \int_{\mathbb{R}^2} \rho |u_{tt}|^2 dx
= t^2 \int_{\mathbb{R}^2} P(\rho)_{tt} \div u_{tt} dx - \int_{\mathbb{R}^2} \rho_t (u_t + u \cdot \nabla_x u) \cdot u_{tt} dx
- 2 \int_{\mathbb{R}^2} \rho u_t \cdot \nabla_x u \cdot u_{tt} dx
- 2 \int_{\mathbb{R}^2} \rho u_{tt} \cdot \nabla_x u \cdot u_{tt} dx + t^2 \int_{\mathbb{R}^2} (\lambda(\rho)_{tt} \div u_{tt}) dx
- \int_{\mathbb{R}^2} u \cdot u_{tt} (\rho f)_t dx - 2 \int_{\mathbb{R}^2} u_t \cdot u_{tt} (\rho f)_t dx + 2 \int_{\mathbb{R}^2} u_{tt} \cdot (u f)_t dx
=: \sum_{i=1}^{10} \mathcal{I}_{12i}.
\]
In the sequel, we will estimate the terms \(\mathcal{I}_{12i}\), separately.

- **Case L1** (Estimates of \(\mathcal{I}_{12i}, 1 \leq i \leq 7\)): Similar to Lemma 4.6 [26], we have
\[
\sum_{i=1}^{7} |\mathcal{I}_{12i}| \leq \frac{1}{8} t^2 \left( \|\nabla_x u_{tt}\|^2 + \|\nabla_x u_{tt}\|^2 \right) + C(T) \left[ t^2 \sqrt{\rho_{tt}} \|\nabla_x u_{tt}\|^2 \right.
\]
\[
\times \left( \|\nabla_x u\|^2 + \|\nabla_x u\|^2 \right) + t^2 \left( \|\nabla_x u_{tt}\|^2 \right) + \|\nabla_x u\|^2 \right] + \|P(\rho)_{tt}\|^2 + \|\lambda(\rho)_{tt}\|^2 \|\nabla_x u\|^2 \right) + \left( (1 + \|x\|)^2 \|\nabla_x u\|^2 \right)
\]
- **Case L2** (Estimates of \(\mathcal{I}_{128}\)): We apply the integration by parts and the Sobolev inequalities to use (56) and (50) to get
\[
|\mathcal{I}_{128}| \leq C t^2 \int_{\mathbb{R}^2} \left( \|\nabla_x^2 u_{tt}\|_{L^2} + \|\nabla_x u\| \|\nabla_x u_{tt}\|_{H^2} + \|u\| \|\nabla_x u_{tt}\|_{L^2} \right) dx
\]
\[
\leq C t^2 \|\nabla_x u\| \|\nabla_x u_{tt}\|_{L^2} + \|u\| \|\nabla_x u_{tt}\|_{L^2} \|\nabla_x u_{tt}\|_{L^2}
+ \|u\| \|\nabla_x u_{tt}\|_{L^2} \|\rho f\|_{L^2} + \|u\| \|\nabla_x u_{tt}\|_{L^2} \|u f\|_{L^2} + \|u\| \|\nabla_x u_{tt}\|_{L^2} \|\rho f\|_{L^2}
+ C t^2 \sqrt{\rho f} \|\nabla_x u_{tt}\|_{L^2} \left( \|\nabla_x^2 u_{tt}\|_{L^2} \right) dx
+ \|\nabla_x u\| \|\rho f\|_{L^2} \right) dx
\[
\leq \frac{\mu}{8} t^2 \|\nabla x u_{tt}\|_2^2 + \frac{1}{4} t^2 \|\sqrt{\rho} u_{tt}\|_2^2 + C(T)(\|\nabla^3 u\|_2^2 + 1).
\]

Case L3 (Estimates of \(I_{129}\) and \(I_{1210}\)): Similar to the estimate of \(I_{128}\), we have
\[
|I_{129}| \leq C t^2 \int_{R^2} (\|\nabla x u_t\|_2 + |u_t| |\nabla x u_{tt}|) u_f \rho f \|dx \leq C t^2 \|u_t\|_\infty \|\nabla x u_{tt}\|_2 \|u_f \rho f\|_2 + C t^2 \|\sqrt{\rho} u_{tt}\|_2 \|\nabla x u_t\|_4 \int_{R^4} v^2 f^2 dv dx \leq \frac{\mu}{8} t^2 \|\nabla x u_{tt}\|_2^2 + \frac{1}{4} t^2 \|\sqrt{\rho} u_{tt}\|_2^2 + C(T)(\|\nabla x u_{tt}\|_{H^1}^2 + \|u_{tt}\|_2^2 + 1),
\]
\[
|I_{1210}| \leq C t^2 \int_{R^2} (|\nabla x u_{tt}|(|m_3 \nabla x f| + |u| |u_f \rho f| + |m_2 f| + |u_f \rho_f| + |\rho_f|) + |u_{tt}|(|u_f \rho f| + |\rho_f|) + |u| |u_f \rho f| + |\nabla x u| |u_f \rho f| + |u_t| |\rho_f|) dx \leq \frac{\mu}{8} t^2 \|\nabla x u_{tt}\|_2^2 + \frac{1}{4} t^2 \|\sqrt{\rho} u_{tt}\|_2^2 + C(T)(t^2 \|\sqrt{\rho} u_{tt}\|_2^2 + \|\nabla x u_{tt}\|_{H^1}^2 + 1).
\]

We collect the estimates of \(I_{12}\), in (69) to obtain
\[
\frac{d}{dt} \left( t^2 \int_{R^2} \rho |u_{tt}|^2 dx \right) + t^2 \int_{R^2} (|\nabla x u_{tt}|^2 + \|\sqrt{\rho} u_{tt}\|^2) dx \leq C(T) t^2 \|\sqrt{\rho} u_{tt}\|_2^2 (\|\nabla x u\|_2^2 + \|\nabla x u\|_{H^1}^2 + 1) + \|\nabla x u_t\|^2_{H^2} + \|P(\rho)_{tt}\|^2_2 + \|\lambda(\rho)_{tt}\|^2_2 \|\nabla x u\|^2_{H^2} + \|(1 + |x|)^2 \|\nabla x u\|^2_2 + C(T).
\]

We integrate the above inequality with respect \(t\) over \([\tau, t]\), use Lemma 4.1, 4.2 and 4.5 to have
\[
t^2 \|\sqrt{\rho} u_{tt}\|_2^2 (t) + \int_{\tau}^{t} t^2 (\|\nabla x u_{tt}\|_2^2 + \|\sqrt{\rho} u_{tt}\|_2^2) dt \leq C(T) (1 + t^2 \|\sqrt{\rho} u_{tt}\|_2^2 (\tau)).
\]

Since \(t \|\sqrt{\rho} u_{tt}\|_2^2 (t) \in L^1 (0, T)\), there exists a subsequence \(\tau_k\) such that
\[
\tau_k \rightarrow 0, \quad \tau_k^2 \|\sqrt{\rho} u_{tt}\|_2^2 (\tau_k) \rightarrow 0, \quad \text{as} \quad k \rightarrow +\infty.
\]

Taking \(\tau = \tau_k\) and \(k \rightarrow +\infty\), we have
\[
t^2 \|\sqrt{\rho} u_{tt}\|_2^2 (t) + \int_{0}^{\tau} t^2 (\|\nabla x u_{tt}\|_2^2 + \|\sqrt{\rho} u_{tt}\|_2^2) dt \leq C(T), \quad t \in [0, T].
\]

Lemma 4.10. Suppose that the conditions in Lemma 3.3 hold. Then, for \(4 < p < \infty\), we have
\[
\sup_{0 \leq t \leq T} \|t \|x\|_4 \|\nabla u\|_2^2 + \|t \|\sqrt{\rho} x\|_4 \|\dot{u}\|_2^2 + \|t \|u_t\|_2^2 + \|t \|\nabla^2 u\|_2 + \|t \|\nabla^2 u_{tt}\|_2 + \|t \|\nabla^2 u\|_p^2 (t) + \int_{0}^{T} t \|\sqrt{\rho} u_t \|_4 \|\dot{u}\|_2^2 dt \leq C(T).
\]

Proof. We apply the operator \(\partial_t + div(\dot{u})\) to the momentum equation (1) to obtain
\[
\rho \dot{u}_{tt} + pu \cdot \nabla x u_t = \mu \Delta_x u_t + \mu \text{div}(u \Delta_x u_t) + \partial^2_\rho ((\mu + \lambda(\rho)) \text{div} u)
\]
\[
+ \text{div}(u \partial_x ((\mu + \lambda(\rho)) \text{div} u)) - \partial^2_{x_i} P(\rho) - \text{div}(u \partial_x (P(\rho))) - \partial_t (u \rho_f) - \text{div}(u u_i \rho_f) + \partial_t ((u f)_i \rho_f) + \text{div}(u (u f)_i \rho_f).
\]
We multiply the above equation by $|x|^{\alpha} \dot{u}_t$ and summing over $i = 1, 2$ to have

\[
\frac{d}{dt} \int_{\mathbb{R}^2} \left[ \frac{\mu}{2} \nabla_x \dot{u}^2 + \frac{\mu + \lambda(\rho)}{2} (\text{div}\dot{u})^2 + \frac{|\dot{u}|^2}{2} \rho_f \right] |x|^{\alpha} dx + \int_{\mathbb{R}^2} \rho |\dot{u}_t|^2 |x|^{\alpha} dx
\]

\[
= \int_{\mathbb{R}^2} \left( \lambda(\rho) \frac{(\text{div}\dot{u})^2}{2} - \rho u \cdot \nabla_x \dot{u} \cdot \dot{u}_t \right) |x|^{\alpha} dx
\]

\[- \int_{\mathbb{R}^2} \left[ \mu \dot{u}_t \cdot \dot{u}_t \cdot \nabla_x \dot{u}_t \cdot \nabla_x (|x|^{\alpha}) \right] dx + \int_{\mathbb{R}^2} \mu \partial_{x_j} u \cdot \nabla_2 u \cdot (\dot{u} |x|^{\alpha})_{x_j} - \mu (\text{div}\mu) \partial_{x_j} u \cdot (\dot{u} |x|^{\alpha})_{x_j} + \mu \lambda(\rho) \partial_{x_j} u \cdot \mu u_{x_j} \cdot \nabla_x (|x|^{\alpha})_{x_j}
\]

\[- \int_{\mathbb{R}^2} \left[ P(\rho)u_{x_j} \cdot \nabla_x (\dot{u} |x|^{\alpha})_{x_j} + (\gamma - 1) P(\rho) (\text{div}\mu) \dot{u} |x|^{\alpha} \right] dx
\]

\[
+ \int_{\mathbb{R}^2} \dot{u}_j u \cdot \nabla_2 u_{x_j} |x|^{\alpha} dx + \frac{1}{2} \int_{\mathbb{R}^2} |\dot{u}|^2 |x|^{\alpha} (\rho f) dx
\]

\[
+ \int_{\mathbb{R}^2} \left[ \partial_{x_j} |u_{x_j} \rho f| + \text{div}(uu_j \rho f) - u_j (\rho f) - \text{div}(uu_j \rho f) \right] |\dot{u}_i|^2 |x|^{\alpha} dx
\]

\[
:= \sum_{i=1}^{7} I_{13i}.
\]

Now we estimate the terms $I_{13i}$, $i = 1 \cdots 7$, separately.

- Case M.1 (Estimates of $I_{13i}$, $1 \leq i \leq 4$): Similar to Lemma 4.8 [27], we have

\[
\sum_{i=1}^{4} I_{13i} \leq - \frac{d}{dt} \bar{Y}(t) + \frac{1}{2} \| \sqrt{\mu} \dot{u}_t |x|^{\alpha} \|_2^2 + C(T) \| \nabla_x \dot{u} (1 + |x|^{\alpha}) \|_2^2
\]

\[
+ (1 + \| \nabla_x u \|_\infty^2) (1 + \| \nabla_x u_t \|_2^2) + (1 + \| \text{div} u \|_\infty) \| \nabla_x \dot{u} |x|^{\alpha} \|_2^2
\]

\[
+ C \| \nabla_x \dot{u} |x|^{\alpha} \|_2 \| \nabla_x u_t \|_2 + \| \nabla_x u \|_\infty (\| \nabla_x u_t \|_2 + \| \nabla_x \dot{u} \|_2)
\]

\[
+ \| \nabla_x u |x|^{\alpha} \|_2 \| \nabla_x u_t \|_2 + \| \nabla_x \dot{u} \|_2,
\]

where

\[
\bar{Y}(t) = \int_{\mathbb{R}^2} \left( \mu \dot{u} \cdot \dot{u}_x \cdot \nabla_x ((|x|^{\alpha}) \right) + (\mu + \lambda(\rho))(\text{div}\mu) \dot{u} \cdot \nabla_x (|x|^{\alpha}) \right) dx
\]

\[
+ \int_{\mathbb{R}^2} \left[ \mu (\text{div}\mu) \partial_{x_j} u \cdot (\dot{u} |x|^{\alpha})_{x_j} - \mu \partial_{x_j} u \cdot \nabla_x u \cdot (\dot{u} |x|^{\alpha})_{x_j} - \mu u_{x_j} \cdot \nabla_x (\dot{u} |x|^{\alpha})_{x_j}
\]

\[- \int_{\mathbb{R}^2} \left( P(\rho)u_{x_j} \cdot \nabla_x (\dot{u} |x|^{\alpha})_{x_j} + (\gamma - 1) P(\rho) (\text{div}\mu) \dot{u} |x|^{\alpha} \right) dx
\]

\[
+ \int_{\mathbb{R}^2} \left[ \partial_{x_j} |u_{x_j} \rho f| + \text{div}(uu_j \rho f) - u_j (\rho f) - \text{div}(uu_j \rho f) \right] |\dot{u}_i|^2 |x|^{\alpha} dx
\]

\[
+ (\gamma - 1) P(\rho) (\text{div}\mu) \dot{u} |x|^{\alpha} dx.
\]
• Case M.2 (Estimates of $I_{135}$): By direct computation, we have

$$|I_{135}| = \frac{d}{dt} \int_{\mathbb{R}^2} \hat{u}_j \cdot \nabla_x u_j \rho_f \, |x|^{\frac{3}{2}} \, dx$$

$$- \int_{\mathbb{R}^2} \hat{u}_j \left( u_j \cdot \nabla_x u_j + u \cdot \nabla_x u_j \right) \rho_f + u \cdot \nabla_x u_j) \partial_t \rho_f |x|^{\frac{3}{2}} \, dx$$

$$\leq \frac{d}{dt} \int_{\mathbb{R}^2} \hat{u}_j \cdot \nabla_x u_j \rho_f \, |x|^{\frac{3}{2}} \, dx$$

$$+ \left( \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} \left( u_j \cdot \nabla_x u_j \right) \rho_f \, |x|^{\frac{3}{2}} \, dx$$

$$\leq C(T) \left( \left( |t| \frac{1}{2} \sqrt{\rho_f} \hat{u} \right)^{\frac{3}{4}} \| \nabla_x u \|_{L^1} + \| u \|_{L^1} \right)$$

$$\leq C(T) \left( \left( |t| \frac{1}{2} \sqrt{\rho_f} \hat{u} \right)^{\frac{3}{4}} \| \nabla_x u \|_{L^1} + \| u \|_{L^1} \right).$$

• Case M.3 (Estimates of $I_{136}$): We apply the integration by parts, use (3), Corollary 1 and (56) to have

$$|I_{136}| \leq \int_{\mathbb{R}^2} \left( |\nabla_x \hat{u} | |x|^{\frac{3}{2}} |u_f \rho_f| + |\hat{u}| |x|^{\frac{3}{2} - 1} |u_f \rho_f| \right) \, dx$$

$$\leq C \left( |\nabla_x \hat{u} | |x|^{\frac{3}{2}} \right) \left( \| u_f \|_{L^1} \| \nabla_x u \|_{L^1} + \| u \|_{L^1} \right)^{\frac{3}{2} - 1} \| u_f \rho_f \|_{L^1}$$

$$\leq C(T) \left( \| \nabla_x \hat{u} | |x|^{\frac{3}{2}} \right)^{\frac{3}{2}} + 1.$$

• Case M.3 (Estimates of $I_{137}$): By the integration by parts, we use Corollary 1, (50), Lemma 4.5 and (56) to have

$$I_{137} = \frac{d}{dt} \int_{\mathbb{R}^2} \left( v_j (u_f \rho_f) \right)_j + \text{div}(uu_f \rho_f) + u_j (\rho_f)_t - \text{div}(uu_j \rho_f) \right) \hat{u}_j |x|^{\frac{3}{2}} \, dx$$

$$\leq \frac{d}{dt} \int_{\mathbb{R}^2} \left[ \frac{d}{dt} \left( u_f \rho_f \right) \right]_j + \text{div}(uu_f \rho_f) + u_j (\rho_f)_t - \text{div}(uu_j \rho_f) \right) \hat{u}_j |x|^{\frac{3}{2}} \, dx$$

$$\leq C(T) \int_{\mathbb{R}^2} \left( |t| \frac{1}{2} \sqrt{\rho_f} \hat{u} \right)^{\frac{3}{4}} \left( \| u_f \|_{L^1} + \| \rho_f \|_{L^1} + \| u_f \rho_f \|_{L^1} + \| \nabla_x u \|_{L^1} \left( \| u_f \|_{L^1} \right) \right)$$

$$\leq C(T) \left( \| u_f \|_{L^1} \| \nabla_x \hat{u} | |x|^{\frac{3}{2}} \right)^{\frac{3}{2}} + 1.$$
where

\[ Y(t) = \int_{\mathbb{R}^2} \left[ \frac{\mu}{2} |\nabla u|^2 + \frac{\mu + \lambda(p)}{2} (\text{div } u)^2 + \frac{|\dot{u}|^2}{2} \rho_f \right] |x|^2 \, dx + \tilde{Y}(t), \]

\[ \dot{Y}(t) = \int_{\mathbb{R}^2} [-\dot{u} \cdot \nabla u + \partial_t (u_f \rho_f) + \dot{\partial}(u_f \rho_f)_f + \text{div}(u u_f \rho_f)_f - \text{div}(u u_f \rho_f)] \dot{u}_j |x|^2 \, dx. \]

As in Lemma 4.8 [27], we have for arbitrarily small constant \( \sigma > 0 \) that

\[ |\tilde{Y}(t)| \leq \frac{\mu \alpha^2}{4} \| \text{div}_x |x|^2 \| + \| \nabla_x \dot{u} \|_2^2 + \sigma \sqrt{\lambda(p)} \| \text{div}_x \dot{u} \|_2^2 \]

\[ + \| |x|^2 \nabla_x \dot{u} \|_2^2 + C(T)(1 + \| \nabla_x \dot{u} \|_2^2). \]

Based on the fact that \( f(x, v, t) \) can absorb some weight of spatial variables, namely \( \int_{\mathbb{R}^2} |x|^2 f \, dv \, dx \leq C \), similar to (41), we have

\[ |\tilde{Y}| \leq C \int_{\mathbb{R}^2} |\nabla_x (\tilde{u}|x|^2)|\| u \|_2 |u_f \rho_f| + u^2 |\rho_f| + |m_2 f| + |\dot{u}| |x|^2 (|u| |\rho_f| \]

\[ + |u_f \rho_f| + |\rho_f| + |\nabla_x u| |u_f \rho_f|) \, dx \leq \sigma \| |x|^2 \nabla_x \dot{u} \|_2^2 + C(T). \]

Then we choose \( \varepsilon \) suitably small to have

\[ Y(t) \geq \frac{\mu}{2} \left[ \left( 1 - \frac{\alpha^2}{4} \right) \| |x|^2 \nabla_x \dot{u} \|_2^2 + \| |x|^2 \text{div}_x \dot{u} \|_2^2 \right] \]

\[ + \frac{1}{2} \| \sqrt{\rho_f} |x|^2 \|_2^2 + \left( \frac{1}{2} - \varepsilon \right) \| |x|^2 \lambda(p) \text{div}_x \dot{u} \|_2^2 \]

\[ - 2\varepsilon \| \nabla_x \dot{u} \|_2^2 - \sigma \| |x|^2 \nabla_x \dot{u} \|_2^2 + C(T)(1 + \| \nabla_x \dot{u} \|_2^2) \]

\[ \geq \kappa \left( \| |x|^2 \nabla_x \dot{u} \|_2^2 + \| |x|^2 \text{div}_x \dot{u} \|_2^2 + \| |x|^2 \lambda(p) \text{div}_x \dot{u} \|^2 \right) \]

\[ + \| \sqrt{\rho_f} |x|^2 \|_2^2 - C(T)(1 + \| \nabla_x \dot{u} \|_2^2), \]

where \( 1 < \alpha^2 < 4(\sqrt{2} - 1) \), \( \kappa > 0 \) is a small constant.

We multiply the inequality (71) by \( t \) and integrate the resulting inequality with respect to \( t \) over \([\tau, t_1]\) to derive

\[ t_1 Y(t_1) + \frac{1}{2} \int_{\tau}^{t_1} t \| \sqrt{\rho_f} |x|^2 \|_2^2 \, dt \]

\[ \leq \tau Y(\tau) + C \int_{\tau}^{t_1} \left[ t(1 + \| \nabla_x u \|_2^2) \right] \left[ 1 + \| \nabla_x u \|_2^2 + \| \nabla_x \dot{u} \|_2^2 \right] \]

\[ + \| \nabla_x \dot{u} \|_2^2 + \| \nabla_x u_t \|_2^2 + t^2 \| \nabla_x u_t \|_2^2 + t \| u_t \|_2^2 + Y(t) \] \, dt \]

\[ \leq C \tau Y(\tau) + C \int_{\tau}^{t_1} \left[ 1 + \| \nabla_x u \|_2^2 + \| \nabla_x \dot{u} \|_2^2 + \| \nabla_x u_t \|_2^2 + \| u_t \|_2^2 + Y(t) \right] \, dt \]

Since that

\[ \int_{0}^{T} Y(t) \, dt \leq \int_{0}^{T} \left( \| \nabla_x u \|_2^2 + \| \nabla_x \dot{u} \|_2^2 + \| \nabla_x u_t \|_2^2 + \| (1 + |x|^2) \| \sqrt{\rho_f} \|_2^2 \right) \, dt \]

\[ \leq \int_{0}^{T} C(T), \]

there exists a subsequence \( \tau_k \) such that

\[ \tau_k \rightarrow 0, \quad \tau_k Y(\tau_k) \rightarrow 0, \quad \text{as} \quad k \rightarrow +\infty. \]
Taking \( \tau = \tau_k \) and \( k \to +\infty \), it gets that
\[
\sup_{0 \leq t \leq T} t Y(t) + \int_0^T \| \sqrt{\rho} \hat{u}_t \| \| \bar{\varphi} \|_{L^2} dt \leq C(T),
\]
which together with Lemma 4.6 yields that
\[
\sup_{0 \leq t \leq T} [|t|\|\bar{\varphi}| \| \bar{\varphi} \|_{L^2} + t \| (u_t, \hat{u}_t) \|_{L^2} + \int_0^T \| \sqrt{\rho} \hat{u}_t \| \| \bar{\varphi} \|_{L^2} dt \leq C(T).
\]
By (63), it holds that
\[
\| \nabla_x^2 u \|_2 \leq C(T)(1 + \| \nabla_x^2 u \|_2^2 + \| \nabla_x \hat{u} \|_2 + \| \hat{u} \|_{\mathcal{A}}),
\]
which implies that
\[
t \| \nabla_x^2 u \|_2^2 \leq C(T)(1 + t \| \nabla_x \hat{u} \|_2^2 + t \| u_t \|_{\mathcal{A}}) \leq C(T).
\]
Thus, from (57), (63) and Lemma 4.9, it holds that
\[
t^2 \| \nabla_x^2 u_t \|_2^2 \leq C(T)(1 + t^2 \| \nabla u_t \|_2^2 + t^2 \| u_t \|_{\mathcal{A}} + t^2 \sqrt{\rho} u_t \|_2^2 + t^2 \| \nabla_x^2 u \|_2^2) \leq C(T),
\]
and
\[
t^2 \| \nabla_x u \|_{W^{2,p}} \leq C(T)(1 + t^2 \| \nabla u_t \|_{H^1}^2 + t^2 \| \nabla_x u \|_{H^2}) \leq C(T).
\]

5. A proof of Theorem 2.1. In this section, we provide the proof of Theorem 2.1. Using linearization, Schauder fixed point theorem and borrowing a priori estimates in Section 4, we can obtain the following local existence of classical solutions to the coupled systems as in [12]. We omit the details for simplicity.

Lemma 5.1. Under the assumptions of Theorem 2.1, there exists a small \( T_* > 0 \) and a unique classical solution \( [f, \rho, u] \) to coupled systems satisfying the regularity properties (8) with \( T \) replaced by \( T_* \).

Firstly, we show that \( [f, \rho, u] \) is a classical solution to (1) if \( [f, \rho, u] \) satisfying (6).

Notice that \( u \in L^2(0, T; L^{\frac{3}{2}} \cap D^1(\mathbb{R}^2)) \) and \( u_t \in L^2(0, T; L^{\frac{3}{2}} \cap D^1(\mathbb{R}^2)) \), so we can get
\[
u \in C(0, T; L^{\frac{3}{2}} \cap D^1(\mathbb{R}^2)) \to C(0, T \times \mathbb{R}^2).
\]
From \( (\rho, P(\rho)) \in L^\infty(0, T; W^{2,q}(\mathbb{R}^2)) \) and \( (\rho, P(\rho)) \in L^\infty(0, T; H^1(\mathbb{R}^2)) \), \( q > 2 \), it holds that
\[(\rho, P(\rho)) \in C(0, T; W^{1,q}(\mathbb{R}^2)) \cap C(0, T; W^{2,q}(\mathbb{R}^2) - weak).
\]
Combining with Lemma 4.8, it implies that
\[(\rho, P(\rho)) \in C(0, T; W^{2,q}(\mathbb{R}^2)).
\]
It follows from \( f \in L^\infty(0, T; W^{3,p}(\mathbb{R}^4)) \) with \( p > 4 \) and \( \partial_t f \in L^\infty(0, T; H^1(\mathbb{R}^4)) \) that
\[f \in C(0, T; W^{3,p}(\mathbb{R}^4)) \cap C(0, T; W^{3,p}(\mathbb{R}^4) - weak).
\]
Together with Lemma 4.8, it implies that
\[f \in C(0, T; W^{3,p}(\mathbb{R}^4)) \to C(0, T; C^{3-\frac{1}{p}}(\mathbb{R}^4)).
\]
Note from Lemma 4.10 that for any \( \tau \in (0, T) \),
\[\nabla_x u, \nabla_x^2 u \in L^\infty(\tau, T; W^{1,q}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)), \quad (\nabla_x u_t, \nabla_x^2 u_t) \in L^\infty(\tau, T; L^2(\mathbb{R}^2)).
\]
So, 
\[(\nabla_x u, \nabla_x^2 u) \in C([\tau, T] \times \mathbb{R}^2).\]

By (1)\(_3\), it holds that
\[
\partial_t f = -v \cdot \nabla_x f - \nabla_v \cdot (L(f)f + (u - v)f) + \Delta_v(|v - v_r|^2 f) \in C([0, T] \times \mathbb{R}^4).
\]

From
\[
(\nabla_x \rho, \nabla_x P(\rho)) \in C(0, T; \mathcal{W}^{1, q}(\mathbb{R}^2)) \hookrightarrow C([0, T] \times \mathbb{R}^2),
\]
continuity equation (1)\(_1\) and momentum equation (1)\(_2\), we have
\[
\rho_t = -(u \cdot \nabla_x \rho + \rho \text{div} u) \in C([\tau, T] \times \mathbb{R}^2),
\]
\[
(\rho u)_t = \mu \Delta_x u + ((\mu + \lambda(\rho)) \nabla_x \text{div} u + \text{div} u \nabla_x \lambda(\rho) + \rho u \cdot \nabla_x u + \rho \text{div} u
\]
\[+ (u \cdot \nabla_x \rho) u - \nabla_x P(\rho) - \int_{\mathbb{R}^2} (u - v) f dv \in C([\tau, T] \times \mathbb{R}^4).
\]

Then Theorem 2.1 can be proved combining the local existence of classical solution and a priori estimates in Section 4.

6. Conclusion. In this paper, we provided a global existence of the coupled system of a kinetic CS-FP equation and compressible N-S equations in the whole space \(\mathbb{R}^2\). Note that the initial data can be arbitrarily large to contain vacuum states. This is motivated by [26, 27, 29, 21]. Compared with the previous results in [27], we need to deal with the drag term in momentum equations (1)\(_2\) additionally. In order to control the drag term, there are three main ingredients in our strategy. First, the global momentum of \(f\) can be bounded by \(\int_0^t ||u||_p d\tau\) with some \(p > 1\) as in Lemma 4.2. Second, by using Caffarelli-Kohn-Nirenberg inequality with best constant, we need to obtain the space weighted estimate of velocity \(u\). Third, the velocity weighted space \(W^k_{3,p}\) with some suitably large \(k\) is introduced in our proof. Through the weighted energy estimates and elaborate analysis, the global classical solution to the coupled system is established eventually.

Appendix A. Elementary inequalities. In this section, we recall several elementary inequalities without the proofs to be used in this paper. These inequalities play an important role in our proof. First, we state the Garliardo-Nirenberg inequality with the best constant.

**Lemma A.1 ([33]).** (1) For any \(h \in (W^{1,m} \cap L^r)(\mathbb{R}^2)\), we have
\[
||h||_q \leq C||\nabla_x h||_m^\theta ||h||_p^{1-\theta},
\]
where \(\theta = (\frac{1}{r} - \frac{1}{q})(\frac{1}{r} - \frac{1}{m} + \frac{1}{2})^{-1}\), and if \(m < 2\), then \(q\) is between \(r\) and \(\frac{2m}{2-m}\), if \(m = 2\) then \(q\) is in \([r, +\infty)\), if \(m > 2\), then \(q\) is in \([r, +\infty]\).

(2) For any \(h \in H^1(\mathbb{R}^2)\), we have
\[
||h||_p \leq C||\nabla_x h||_2^{\frac{p-2}{2}} ||h||_2^{\frac{2}{2}}, \quad p \in [2, +\infty).
\]

We also need the Sobolev-Poincare inequality:

**Lemma A.2 ([25]).** For any \(h \in W^{1,m}(\mathbb{R}^2)\) with \(m \in [1, 2]\), we have
\[
||h||_{\frac{2m}{2-m}} \leq C(2 - m)^{-\frac{1}{2}} ||\nabla_x h||_m,
\]
where the positive constant \(C\) is independent of \(m\).
We denote the Hilbert space by $\tilde{D}^{1,2}(\mathbb{R}^2) := \{H^1_{\text{loc}}(\mathbb{R}^2)\nabla u \in L^2(\mathbb{R}^2)\}$. Then, we have the following weighted $L^p$ bounds for elements of $\tilde{D}^{1,2}(\mathbb{R}^2)$ (cf. Theorem B.1 in [28]).

Lemma A.3. For $m \in [2, \infty)$, $\theta \in (1 + m/2, \infty)$ and $h \in \tilde{D}^{1,2}(\mathbb{R}^2)$, we have
\[
\left(\int_{\mathbb{R}^2} \frac{|h|^m}{(\log \|x\|)^\theta} \, dx \right)^{\frac{1}{m}} \lesssim \|h\|_{L^2(B_1)} + \|\nabla h\|_2,
\]
where $B_1 := \{x \in \mathbb{R}^2 | |x| < 1\}$.

The following Caffarelli-Kohn-Nirenberg weighted inequalities are crucial to the weighted estimates in $\mathbb{R}^2$.

Lemma A.4 ([9, 11]). The following inequalities hold.

1. For any $h \in C_0^\infty(\mathbb{R}^2)$, we have
\[
\| |x|^k h \|_r \leq C \| |x|^a \nabla x h \|_p \| |x|^\beta h \|_q^{1-\theta}, \tag{2}
\]
where $1 \leq p, q < +\infty$, $0 < r < +\infty$, $0 \leq \theta \leq 1$, $\frac{1}{r} + \frac{1}{q} > 0$, $\frac{1}{q} + \frac{\beta}{2} > 0$, $\frac{1}{r} + \frac{k}{2} > 0$ and satisfying
\[
\frac{1}{r} + \frac{k}{2} = \theta \left(\frac{1}{p} + \frac{a - 1}{2}\right) + (1 - \theta) \left(\frac{1}{q} + \frac{\beta}{2}\right), \quad k = \theta \sigma + (1 - \theta)\beta,
\]
with $0 \leq \alpha - \sigma$ if $\theta > 0$, $0 \leq \alpha - \sigma \leq 1$ if $\theta > 0$ and $\frac{1}{r} + \frac{a-1}{2} = \frac{1}{r} + \frac{k}{2}$.

2. (Best constant for Caffarelli-Kohn-Nirenberg inequality): For any $h \in C_0^\infty(\mathbb{R}^2)$, we have
\[
\| |x|^b h \|_p \leq C_{a,b} \| |x|^a \nabla_x h \|_2, \tag{3}
\]
where $a > 0$, $a - 1 \leq b \leq a$ and $p = \frac{2}{a-\frac{b}{\alpha}}$. If $b = a - 1$, then $p = 2$ and the best constant in the above inequality is $C_{a,b} = C_{a,a-1} = a$.

Now we state the weighted-$L^p$-estimates which can be proved through the $A_p$-weighted theory.

Lemma A.5 ([35]). (1) For any $1 < p < +\infty$ and $u \in C_0^\infty(\mathbb{R}^2)$, we have
\[
\|\nabla_x u\|_p \leq C(\|\text{div} u\|_p + \|\omega\|_p).
\]

2. (2) For any $1 < p < +\infty$, $-2 < \alpha < 2(p-1)$ and $u \in C_0^\infty(\mathbb{R}^2)$, we have
\[
\left\| |x|^\gamma \nabla_x u \right\|_p \leq C \left(\left\| |x|^\gamma \text{div} u \right\|_p + \left\| |x|^\gamma \omega \right\|_p\right).
\]

The following Brezis-Wainger inequalities and properties of the commutator $[b, R_i R_j](f)$ will be used to derive the upper bound of the density $\rho$, with $R_i := (-\Delta)^{\frac{1}{2}} \partial x_i$.

Lemma A.6 ([6, 17]). For $m > 2$, there exists a positive constant $C$ such that every function $h \in W^{1,m}(\mathbb{R}^2)$ satisfies
\[
\|h\|_\infty \leq C\|\nabla_x h\|_2 \log^\frac{1}{2} (e + \|\nabla_x h\|_m) + C(\|h\|_2 + 1).
\]
Lemma A.7 ([14, 15]). Let \( b, f \in C_0^\infty(\mathbb{R}^2) \). Then for \( p \in (1, +\infty) \), there exists a constant \( C(p) \) such that
\[
\| [b, R_i R_j](f) \|_p \leq C(p) \| b \|_{B^0_{\infty \infty}} \| f \|_p.
\]
Moreover, for \( q_k \in (1, +\infty) (k = 1, 2 , 3) \) with \( \frac{1}{q_1} = \frac{1}{q_2} + \frac{1}{q_3} \), there exists \( C \) depending on \( q_k \) such that
\[
\| \nabla [b, R_i R_j](f) \|_{q_i} \leq C \| \nabla b \|_{q_2} \| f \|_{q_3},
\]
where \([\cdot, \cdot]\) and \( R_i \) are standard Lie bracket and Riesz transform respectively, that is,
\[
[b, R_i R_j](f) := b R_i \circ R_j(f) - R_i \circ R_j(bf), \quad i, j = 1, 2.
\]

The following Beale-Kato-Majda type inequality will be crucial to derive the \( L^\infty \)-norm of \( \nabla x u \).

Lemma A.8 (6). For \( 2 < q < +\infty \), there exists a positive constant \( C \) may depend on \( q \) such that the following estimate holds for all \( \nabla x u \in W^{1,q}(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \),
\[
\| \nabla x u \|_{\infty} \leq C(\| \text{div} u \|_{\infty} + \| \omega \|_{\infty}) \log(e + \| \nabla^2 u \|_{q}) + C \| \nabla x u \|_2 + C.
\]

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