On the von Neumann equation with time-dependent Hamiltonian. Part II: Applications

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Abstract

This second part deals with applications of a general method to describe the quantum time evolution determined by a Schrödinger equation with time-dependent Hamiltonian. A new aspect of our approach is that we find all solutions starting from one special solution. The two main applications are reviewed, namely the Bloch equations and the harmonic oscillator with time-dependent frequency. Even in these well-known examples some new results are obtained.

1 Introduction

The study of quantum models with a time-dependent Hamiltonian is relevant for many branches of physics. Instead of solving the Schrödinger equation for a quantum system together with its environment, it is often possible to replace the action of the environment by a time-dependent term in the Hamiltonian of the system. A slightly more general problem is that of solving the von Neumann equation

\[ \frac{d}{dt}\rho(t) = i[\rho(t), H(t)], \quad (1) \]

which describes the time evolution of the density matrix \( \rho \). Its formal solution is

\[ \rho(t) = U(t)\rho(0)U(t)^*, \quad (2) \]

The unitary operators \( U(t) \) satisfy the equation of motion

\[ i\left( \frac{d}{dt}U(t) \right) U^*(t) = H(t). \quad (3) \]
A considerable body of knowledge exists about solving these equations in the case that both the Hamiltonian and the density matrix (minus an operator commuting with the Hamiltonian) are linear combinations of the generators $S_1, S_2, \ldots, S_n$ of a finite Lie algebra. This knowledge has been reviewed in the first part of the present work [31], hereafter called Part I. In addition, a method was presented to obtain expressions for $U(t)$ in a systematic manner. The aim of what follows is to show that the method is capable of reproducing known results and of obtaining new results.

Sections 2 and 3 of the paper deal with the SU(2) symmetry. Section 2 discusses the Bloch equations and a generalisation involving Jacobi’s elliptic functions. Section 3 discusses the case when phase modulation is included. Section 4 deals with SU(1,1) symmetry, more specifically, the harmonic oscillator with time-dependent frequency. Section 5 considers more general oscillators. Finally, Section 6 contains a short discussion of the obtained results and of the possibilities for further work.

## 2 SU(2)

The generators of SU(2) satisfy the commutation relations

$$[S_1, S_2] = iS_3 \quad \text{and cyclic permutations.} \quad (4)$$

A well-known application of this Lie algebra concerns the Bloch equations — see [4]. It is treated in the present Section. The modification obtained by considering phase modulation is treated in the next Section. The combination of amplitude and of phase modulation was considered in [5] but will not be considered here.

Note that the incorporation of the Bloch equations into the Maxwell-Bloch equations have been studied by many authors, including [10] [26].

More general applications of SU(2) symmetry have been considered in the literature as well. Campolieti and Sanctuary [6] applied the Wei-Norman technique to field modulation in NMR. Zhou and Ye [12] study the case where all coefficients are time-dependent. They introduce Euler angles with the same purpose as in the present work. Finally, Dasgupta [17] studies the Jaynes-Cummings model with time-dependent coupling between the spin and the photon field.

### 2.1 The Bloch equations

A magnetic spin in a magnetic field is usually described by a Hamiltonian of the form

$$H = \frac{1}{2} \epsilon \sigma_3 - \frac{1}{2} \xi \sigma_1. \quad (5)$$

The Pauli matrices $\sigma_1, \sigma_2, \sigma_3$ are related to the generators of the Lie algebra by $\sigma_\alpha = 2S_\alpha$. One has $h = (-\xi, 0, \epsilon)^T$. The equations of motion $\dot{a} = h \times a$ are known as the Bloch equations — see [4]. Written explicitly, they are

$$\begin{align*}
\dot{a}_1 &= -\epsilon a_2, \\
\dot{a}_2 &= \epsilon a_1 + \xi a_3 \\
\dot{a}_3 &= -\xi a_2.
\end{align*} \quad (6)$$
2.2 Special solutions

Assume that the constant $\epsilon$ does not depend on time. If $\xi(t)$ does not depend on time then any solution $x(t)$ of the harmonic oscillator equation $\ddot{x} + (\epsilon^2 + \xi^2)x = 0$ determines a solution of (6), given by

$$
\begin{align*}
a_1 &= \frac{\epsilon \dot{x}}{\epsilon^2 + \xi^2} + \xi \dot{C}, \\
a_2 &= \dot{x}, \quad \dot{a}_2 = \frac{\epsilon^2}{\epsilon^2 + \xi^2}, \\
a_3 &= \frac{\epsilon \dot{x}}{\epsilon^2 + \xi^2} - \epsilon \dot{C}.
\end{align*}
$$

with integration constant $C$. When $\xi(t)$ is time-dependent then a solution is known only in very specific cases. One such case involves Jacobi’s elliptic functions sn, cn, and dn, with elliptic modulus $k$. Let

$$
\xi(t) = 2\omega k \operatorname{cn}(\omega t; k).
$$

Then a solution of (6) exists of the form

$$
\begin{align*}
a_1(t) &= \epsilon \operatorname{cn}(\omega t; k), \\
a_2(t) &= \omega \operatorname{sn}(\omega t; k) \operatorname{dn}(\omega t; k), \\
a_3(t) &= -\omega k \operatorname{sn}^2(\omega t; k) + \gamma,
\end{align*}
$$

with

$$
\gamma = -\frac{\epsilon^2 - \omega^2}{2\omega k}.
$$

In the limit $k = 1$ these expressions lead to the well-known result (see Eq. 4.21 of [1])

$$
\begin{align*}
\xi(t) &= 2\omega \operatorname{sech}(\omega t), \\
a_1(t) &= \epsilon \operatorname{sech}(\omega t), \\
a_2(t) &= \omega \operatorname{tanh}(\omega t) \operatorname{sech}(\omega t), \\
a_3(t) &= -\omega \operatorname{tanh}^2(\omega t) + \gamma.
\end{align*}
$$

When $\omega = \epsilon$ then the limit $k = 0$ can be taken. The rather trivial result is

$$
\begin{align*}
\xi(t) &= 0, \\
a_1(t) &= \omega \cos(\omega t), \\
a_2(t) &= \omega \sin(\omega t), \\
a_3(t) &= 0.
\end{align*}
$$

2.3 Automorphisms

Consider the special solution (9). Following the general method of Section 4.4 of Part I the transformation $V(t)$ is determined by two angles $\phi(t), \theta(t)$. The special solution $a(t)$ at time $t = 0$ reads

$$
a(0) = (\epsilon, 0, \gamma)^\top.
$$
It is rotated into the fixed vector $\lambda(0, 1, 0)^T$. One has

\[
\begin{align*}
z(t) &= \sqrt{a_1^2(t) + a_2^2(t)} = \sqrt{\epsilon^2 \cos^2(\omega t; k) + \omega^2 \sin^2(\omega t; k) \sin^2(\omega t; k)}, \\
z(0) &= e, \\
\lambda &= \sqrt{z^2(t) + a_3^2(t)} = \sqrt{\epsilon^2 + \gamma^2}.
\end{align*}
\] (14)

The angles $\phi(t)$ and $\theta(t)$ are determined by (62) of Part I. In particular, at $t = 0$ is

\[
\begin{align*}
sin(\phi(0)) &= \frac{a_1(0)}{z(0)} = 1 \quad \text{and} \quad cos(\phi(0)) = \frac{a_2(0)}{z(0)} = 0, \\
sin(\theta(0)) &= -\frac{a_3(0)}{\lambda} = -\frac{\gamma}{\lambda} \quad \text{and} \quad cos(\theta(0)) = \frac{z(0)}{\lambda} = \frac{\epsilon}{\lambda}.
\end{align*}
\] (15)

One can understand these values as follows. By a rotation of $-\phi(0) = -\pi/2$ around the third axis the initial vector $a(0)$ becomes $(0, \epsilon, \gamma)^T$. Then by a rotation with angle $-\theta(0)$ around the first axis it becomes $(0, \lambda, 0)^T$. Next, the rotation $R_1(\theta(t))$, followed by the rotation $R_3(\phi(t))$ maps this fixed vector onto the time-dependent $a(t)$.

The Hamiltonian $K(t)$ equals (see (63) of Part I)

\[
K(t) = \frac{a_2 \dot{a}_3 S_1 - a_1 \dot{a}_3 S_2 + a_1 \dot{a}_2 - a_2 \dot{a}_1}{z^2} S_3
= \frac{\xi(t)}{z^2(t)} [-a_2^2(t) S_1 + a_1(t) a_2(t) S_2 + a_1(t) a_3(t) S_3] + \epsilon S_3.
\] (16)

The difference between this $K(t)$ and the Hamiltonian $H(t)$ as given by (5) makes an extra rotation necessary. It involves the function $\alpha(t)$, given by (64) of Part I. It evaluates to

\[
\alpha(t) = -\frac{a_1(t)}{z^2(t)} \xi(t) = -\frac{2\epsilon \omega k \cos(\omega t; k)}{\epsilon^2 \cos^2(\omega t; k) + \omega^2 \sin^2(\omega t; k) \sin^2(\omega t; k)}.
\] (17)

The final result then becomes

\[
U(t) = e^{i\phi(t) S_1} e^{i(\theta(t) - \theta(0)) S_1} e^{-i(\pi/2) S_3} e^{-i(\int_0^t ds \alpha(s)) (\epsilon S_1 + \gamma S_3)}.
\] (18)

Note that this expression can be simplified to

\[
U(t) = e^{i(\phi(t) - \pi/2) S_3} e^{-i(\theta(t) - \theta(0)) S_2} e^{i\lambda \tau X}.
\] (19)

with $\tau \equiv \tau(t) = -\int_0^t ds \alpha(s)$ and $X = \frac{\epsilon S_1 + \gamma S_3}{\lambda}$. For further use note that

\[
e^{i\lambda \tau X} S_j e^{-i\lambda \tau X} = S_j + i \sin(\lambda \tau)[X, S_j] + (\cos(\lambda \tau) - 1)[X, [X, S_j]].
\] (20)

### 2.4 General solution of the Bloch equations

The general solution of the Bloch equations, given arbitrary initial conditions and with time-dependent Hamiltonian determined by (5), can be derived using (19). Note that (omitting time dependences and denoting $\theta_0 \equiv \theta(0)$)

\[
\epsilon \cos(\theta - \theta_0) - \gamma \sin(\theta - \theta_0) = \frac{\epsilon^2 - \gamma^2}{\lambda} \cos(\theta) - 2\frac{\epsilon \gamma}{\lambda} \sin(\theta)
\]
\[
\gamma \cos(\theta - \theta_0) + \epsilon \sin(\theta - \theta_0) = 2 \frac{\epsilon \gamma}{\lambda} \cos(\theta) + \frac{\epsilon^2 - \gamma^2}{\lambda} \sin(\theta)
\]
\[
\gamma \cos(\theta - \theta_0) - \epsilon \sin(\theta - \theta_0) = -\lambda \sin(\theta)
\]
\[
\epsilon \cos(\theta - \theta_0) + \gamma \sin(\theta - \theta_0) = \lambda \cos(\theta).
\]

These relations can be used to calculate
\[
U(t)(\epsilon S_1 + \gamma S_2)U(t)^* = -\lambda \sin(\theta)S_3 + \lambda \cos(\theta)S_+,
\]
\[
\lambda U(t)(\epsilon S_1 - \gamma S_2)U(t)^* = \left[ (\epsilon^2 - \gamma^2) \cos(\theta) - 2\epsilon \gamma \sin(\theta) \cos(\lambda \tau) \right] S_+
\]
\[+ 2\epsilon \gamma \sin(\lambda \tau) S_-.
\]
\[
U(t)S_2U(t)^* = -\sin(\lambda \tau) \cos(\theta)S_3 - \cos(\lambda \tau)S_+ - \sin(\lambda \tau) S_-,
\]

(22)

with
\[
S_+ = \sin(\phi)S_1 + \cos(\phi)S_2 \quad \text{and} \quad S_- = \cos(\phi)S_1 - \sin(\phi)S_2.
\]

(23)

Note further that
\[
\dot{\phi} = -\epsilon + \xi \sin(\phi) \tan(\theta)
\]
\[
\dot{\theta} = \xi \cos(\phi)
\]
\[
\dot{\gamma} = \frac{\xi \sin(\phi)}{\lambda \cos(\theta)}.
\]

(24)

Using these equations one can verify explicitly that the time-dependent operators \([22]\) satisfy indeed the von Neumann equation of motion.

From \([22]\) one can obtain the three independent solutions \(a^{(1)}, a^{(2)}, a^{(3)}\) of the Bloch equations.

\[
a^{(1)} = \frac{\epsilon}{\lambda} \left( \begin{array}{cc} \sin(\phi) \cos(\theta) \\ \cos(\phi) \cos(\theta) \\ -\sin(\theta) \end{array} \right) - \frac{\gamma}{\lambda} \cos(\lambda \tau) \left( \begin{array}{cc} \sin(\phi) \sin(\theta) \\ \cos(\phi) \sin(\theta) \\ \cos(\theta) \end{array} \right) + \frac{\gamma}{\lambda} \sin(\lambda \tau) \left( \begin{array}{cc} \cos(\phi) \\ -\sin(\phi) \\ 0 \end{array} \right)
\]
\[
a^{(2)} = -\sin(\theta) \sin(\lambda \tau) \left( \begin{array}{cc} \sin(\phi) \\ \cos(\phi) \\ 0 \end{array} \right) - \cos(\lambda \tau) \left( \begin{array}{cc} \cos(\phi) \\ -\sin(\phi) \\ 0 \end{array} \right) - \cos(\theta) \sin(\lambda \tau) \left( \begin{array}{cc} 0 \\ 0 \\ 1 \end{array} \right)
\]
\[
a^{(3)} = \frac{\gamma}{\lambda} \left( \begin{array}{cc} \sin(\phi) \cos(\theta) \\ \cos(\phi) \cos(\theta) \\ -\sin(\theta) \end{array} \right) + \frac{\epsilon}{\lambda} \cos(\lambda \tau) \left( \begin{array}{cc} \sin(\phi) \sin(\theta) \\ \cos(\phi) \sin(\theta) \\ \cos(\theta) \end{array} \right) - \frac{\epsilon}{\lambda} \sin(\lambda \tau) \left( \begin{array}{cc} \cos(\phi) \\ -\sin(\phi) \\ 0 \end{array} \right)
\]

(25)

The special solution \([9]\) satisfies \(a = \epsilon a^{(1)} + \gamma a^{(3)}\). The general solution of the Bloch equations \(\dot{x} = \hbar \times x\) is given by expression (67) of Part I and agrees with the results given above.

### 2.5 The \(k = 1\)-limit

We now focus on the case \(k = 1\) because then the formulas simplify. The relevant expressions become

\[
\sin(\phi) = \frac{a_1}{z} = \frac{\epsilon}{\sqrt{\epsilon^2 + \omega^2 \tanh^2(\omega t)}},
\]
\[
\cos(\phi) = \frac{a_2}{z} = \frac{\omega \tanh(\omega t)}{\sqrt{\epsilon^2 + \omega^2 \tanh^2(\omega t)}},
\]
\[
\sin(\theta) = -\frac{a_3}{\lambda} = \frac{\omega \tanh^2(\omega t) - \gamma}{\sqrt{\epsilon^2 + \gamma^2}},
\]
\[
\cos(\theta) = \frac{z}{\lambda} = \text{sech}(\omega t) \sqrt{\frac{\epsilon^2 + \omega^2 \tanh^2(\omega t)}{\epsilon^2 + \gamma^2}},
\]
\[
z = \sqrt{a_1^2 + a_2^2} = \text{sech}(\omega t) \sqrt{\frac{\epsilon^2 + \omega^2 \tanh(\omega t)}{\epsilon^2 + \gamma^2}},
\]
\[
\lambda = \sqrt{z^2 + a_3^2} = \sqrt{\epsilon^2 + \gamma^2},
\]
\[
\alpha = -\frac{a_1}{z^2 \xi} = -\frac{2\epsilon \omega}{\epsilon^2 + \omega^2 \tanh^2(\omega t)}. \tag{26}
\]

with \(\gamma = (\omega^2 - \epsilon^2)/2\omega\). The integral of \(\alpha(t)\) can be done analytically. It yields
\[
\tau(t) = -\int_0^t ds \alpha(s) = \frac{2\omega}{\epsilon^2 + \omega^2} \left\{ ct + \arctan \left( \frac{\omega}{\epsilon} \tanh(\omega t) \right) \right\}. \tag{27}
\]

### 2.6 The Bloch equations at resonance

The resonant condition is \(\epsilon = \omega\). In that case, \(\gamma\), as given by (10), vanishes. Hence, the normalisations \(z(t)\) and \(\lambda\), given by (14) and appearing in the expressions for the angles \(\phi\) and \(\theta\) (see (62) of Part I), become
\[
z(t) = \omega \sqrt{\frac{\epsilon^2}{\epsilon^2 + \omega^2}} \text{cn}^2(\omega t; k) + \text{sn}^2(\omega t; k) \text{dn}^2(\omega t; k), \tag{28}
\]
\[
\lambda = \omega. \tag{29}
\]

There follows
\[
\sin(2\phi) = \frac{2a_1}{z^2} \text{sn}(2\omega t; k), \tag{30}
\]
\[
\cos(2\phi) = \frac{a_2^2 - a_1^2}{z^2} = -\text{cn}(2\omega t; k), \tag{31}
\]
\[
\sin(\theta) = -\frac{a_3}{\lambda} = k \text{sn}^2(\omega t; k), \tag{32}
\]
\[
\cos(\theta) = \frac{z}{\lambda} = \sqrt{\frac{\epsilon^2}{\epsilon^2 + \omega^2}} \text{cn}^2(\omega t; k) + \text{sn}^2(\omega t; k) \text{dn}^2(\omega t; k). \tag{33}
\]

In particular is \(\theta(0) = 0\). The equations (22) become
\[
U(t)S_1U(t)^* = -\sin(\theta)S_3 + \cos(\theta)S_+,
\]
\[
U(t)S_3U(t)^* = \sin(\theta) \cos(\omega \tau)S_+ - \sin(\omega \tau)S_- + \cos(\omega \tau) \cos(\theta)S_3,
\]
\[
U(t)S_2U(t)^* = -\sin(\omega \tau) \cos(\theta)S_3 - \cos(\omega \tau)S_- - \sin(\theta) \sin(\omega \tau)S_+. \tag{34}
\]

They can be used to obtain the general solution of the Bloch equations (6) at resonance
\[
a_1^{\text{gen}}(t) = a_1^{\text{gen}}(0) \cos(\theta) \sin(\phi)
\]

\(a_1^{\text{gen}}(0)\) is...
With some effort, (35) can now be written as

\[ a_2^{\text{gen}}(t) = a_1^{\text{gen}}(0) \cos(\theta) \cos(\phi) + a_2^{\text{gen}}(0) \cos(\omega t) - a_2^{\text{gen}}(0) \sin(\omega t) \]

\[ a_2^{\text{gen}}(t) = a_1^{\text{gen}}(0) \cos(\theta) \cos(\phi) + a_2^{\text{gen}}(0) \cos(\omega t) + a_3^{\text{gen}}(0) \sin(\omega t) \]

\[ a_3^{\text{gen}}(t) = -a_1^{\text{gen}}(0) \sin(\theta) + [a_3^{\text{gen}}(0) \cos(\omega t) - a_2^{\text{gen}}(0) \sin(\omega t)] \cos(\theta). \]  

(35)

The correction angle \( \alpha \), given by (17), simplifies to

\[ \alpha(t) = -\frac{a_1}{z^2} \xi = -k \omega (1 + cn(2\omega t; k)). \]  

(36)

This expression can be integrated analytically. The result is

\[ \tau(t) = -\int_0^t ds \alpha(s) = \frac{1}{\omega} \text{arctan} \left( \frac{\text{dn}(\omega t; k)}{k \text{sn}(\omega t; k) \text{cn}(\omega t; k)} \right). \]  

(37)

With some effort, (35) can now be written as

\[ a_1^{\text{gen}}(t) = a_1^{\text{gen}}(0) \text{cn}(\omega t; k) + a_2^{\text{gen}}(0) \text{sn}(\omega t; k) \cos(k\omega t) + a_3^{\text{gen}}(0) \text{sn}(\omega t; k) \sin(k\omega t), \]

\[ a_2^{\text{gen}}(t) = a_1^{\text{gen}}(0) \text{sn}(\omega t; k) \text{dn}(\omega t; k) + a_2^{\text{gen}}(0) (\text{cn}(\omega t; k) \text{dn}(\omega t; k) \cos(k\omega t) - k \text{sn}(\omega t; k) \sin(k\omega t)) + a_3^{\text{gen}}(0) (\text{cn}(\omega t; k) \text{dn}(\omega t; k) \sin(k\omega t) + k \text{sn}(\omega t; k) \cos(k\omega t)), \]

\[ a_3^{\text{gen}}(t) = -a_1^{\text{gen}}(0) k \text{sn}^2(\omega t; k) + a_2^{\text{gen}}(0) (k \text{cn}(\omega t; k) \text{sn}(\omega t; k) \cos(k\omega t) + \text{dn}(\omega t; k) \sin(k\omega t)) + a_3^{\text{gen}}(0) (k \text{cn}(\omega t; k) \text{sn}(\omega t; k) \sin(k\omega t) - \text{dn}(\omega t; k) \cos(k\omega t)). \]  

(38)

The three independent solutions are therefore

\[ a^{(1)} = \begin{pmatrix} \text{cn}(\omega t; k) \\ \text{sn}(\omega t; k) \text{dn}(\omega t; k) \\ -k \text{sn}^2(\omega t; k) \end{pmatrix}, \]

\[ a^{(2)} = \begin{pmatrix} \text{sn}(\omega t; k) \cos(k\omega t) \\ \text{cn}(\omega t; k) \text{dn}(\omega t; k) \cos(k\omega t) - k \text{sn}(\omega t; k) \sin(k\omega t) \\ k \text{cn}(\omega t; k) \text{sn}(\omega t; k) \cos(k\omega t) + \text{dn}(\omega t; k) \sin(k\omega t) \end{pmatrix}, \]

\[ a^{(3)} = \begin{pmatrix} \text{sn}(\omega t; k) \sin(k\omega t) \\ \text{cn}(\omega t; k) \text{dn}(\omega t; k) \sin(k\omega t) + k \text{sn}(\omega t; k) \cos(k\omega t) \\ k \text{cn}(\omega t; k) \text{sn}(\omega t; k) \sin(k\omega t) - \text{dn}(\omega t; k) \cos(k\omega t) \end{pmatrix}. \]  

(39)

One clearly has \( a = \omega a^{(1)} \), the special solution we started with. It is easy to verify that also \( a^{(2)} \) and \( a^{(3)} \) are solutions of \( \dot{a} = a \times h \).
3 SU(2) continued

3.1 Including phase modulation

A slightly different solution is obtained when the parameter $\epsilon$ in (6) is made time dependent in the following way

$$\epsilon = \epsilon_0 \tanh(\omega t).$$

(40)

In [4], the resulting equations are called the Bloch equations including phase modulation. Also in this case a special solution is known. The generalisation to Jacobi’s elliptic functions, as given below, can be done in two different ways.

3.2 Special solutions

Assume that

$$\xi(t) = \xi_0 \text{cn}(\omega t; k),$$
$$\epsilon(t) = \epsilon_0 \text{sn}(\omega t; k).$$

(41)

or

$$\xi(t) = \xi_0 \text{dn}(\omega t; k),$$
$$\epsilon(t) = \epsilon_0 \text{sn}(\omega t; k).$$

(42)

Both assumptions reduce to $\xi(t) = \xi_0 \text{sech}(\omega t)$ and (40) in the limit $k = 1$.

A solution of the equations (6) is given by

$$a_1(t) = \epsilon_0 \text{cn}(\omega t; k),$$
$$a_2(t) = \omega \text{dn}(\omega t; k),$$
$$a_3(t) = -\xi_0 \text{sn}(\omega t; k).$$

(43)

or

$$a_1(t) = \omega \text{cn}(\omega t; k),$$
$$a_2(t) = \epsilon_0 \text{dn}(\omega t; k),$$
$$a_3(t) = -\xi_0 \text{sn}(\omega t; k).$$

(44)

provided that $\xi_0^2 = \epsilon_0^2 + \omega^2 k^2$, respectively $\xi_0^2 = \epsilon_0^2 k^2 + \omega^2$, is satisfied.

Only the first of the cases is treated below. The other case is completely analogous.

Note that in the limit $k = 1$ the solution (43) reduces to the well-known solution

$$a_1(t) = \epsilon_0 \text{sech}(\omega t),$$
$$a_2(t) = \omega \text{sech}(\omega t),$$
$$a_3(t) = -\xi_0 \tanh(\omega t).$$

(45)

In the limit $k = 0$ it reduces to a harmonic precession

$$\xi(t) = \xi_0 \cos(\omega t),$$
$$\epsilon(t) = \epsilon_0 \sin(\omega t),$$
$$a_1(t) = \epsilon_0 \cos(\omega t),$$
$$a_2(t) = \omega,$$
$$a_3(t) = -\xi_0 \sin(\omega t).$$

(46)
3.3 Automorphisms

Consider the special solution (43). The angles $\phi(t)$ and $\theta(t)$ are determined by (62) of Part I. In particular, at $t = 0$ is

$$\sin(\phi(0)) = \frac{\epsilon_0}{\lambda}, \quad \cos(\phi(0)) = \frac{\omega}{\lambda},$$

(47)

and $\theta(0) = 0$, with $\lambda = \sqrt{\epsilon_0^2 + \omega^2}$. This means that by a rotation $R_3(-\phi(0))$ around the third axis the initial vector $a(0)$ is rotated into the fixed vector $\lambda(0, 1, 0)^T$. Next, the rotation $R_1(\theta(t))$, followed by the rotation $R_3(\phi(t))$ maps this fixed vector onto the time-dependent $a(t)$.

The Hamiltonian $K(t)$ equals (see (63) of Part I)

$$K(t) = \frac{a_2 \dot{a}_3}{z^2} S_1 - \frac{a_1 \dot{a}_3}{z^2} S_2 + \frac{a_1 \dot{a}_2 - a_2 \dot{a}_1}{z^2} S_3$$

$$= \frac{\xi(t)}{z^2(t)} \left[ -a_2^2(t) S_1 + a_1(t) a_2(t) S_2 + a_1(t) a_3(t) S_3 \right] + \epsilon S_3.$$

(48)

This is the same expression as (16). The difference between this $K(t)$ and the Hamiltonian $H(t)$ as given by (5) makes an extra rotation necessary. It involves the function $\alpha(t)$, given by (64) of Part I. It evaluates to

$$\alpha(t) = -\frac{a_1(t)}{z^2(t)} \xi(t) = -\frac{\epsilon_0 \zeta_0 \operatorname{cn}(\omega t; k) \operatorname{dn}(\omega t; k)}{\epsilon_0^2 \operatorname{cn}^2(\omega t; k) + \omega^2 \operatorname{dn}^2(\omega t; k)}.$$

(49)

The final result then becomes

$$U(t) = e^{i\phi(t) S_3} e^{i\theta(t) S_1} e^{-i\phi(0) S_3} e^{i\lambda \tau(t) X},$$

(50)

with $\tau(t) = -\int_0^t ds \alpha(s)$ and $X = \frac{\epsilon_0 S_1 + \omega S_2}{\lambda}$. Note that (49) can be integrated analytically.

The result is

$$\tau(t) = \frac{\epsilon_0 \zeta_0}{2\omega \lambda \mu} \ln \frac{\lambda + \mu \operatorname{sn}(\omega t; k)}{\lambda - \mu \operatorname{sn}(\omega t; k)}$$

(51)

with $\mu = \sqrt{\epsilon_0^2 + k^2 \omega^2}$.

3.4 General solution including phase modulation

It is now possible to obtain the general solution of the Bloch equations including phase modulation, with driving fields of the form (41). One calculates (again omitting time dependencies and denoting $\phi_0 \equiv \phi(0)$)

$$U(t)(\epsilon_0 S_1 + \omega S_2) U(t)^* = \lambda \cos(\theta) S_+ - \lambda \sin(\theta) S_3$$

$$\lambda U(t)(\epsilon_0 S_1 - \omega S_2) U(t)^* = \left[ (\epsilon_0^2 - \omega^2) \cos(\theta) + 2\epsilon_0 \omega \sin(\theta) \sin(\lambda \tau) \right] S_+$$

$$+ 2\epsilon_0 \omega \cos(\lambda \tau) S_-$$

$$+ \left[ 2\epsilon_0 \omega \cos(\theta) \sin(\lambda \tau) - (\epsilon_0^2 - \omega^2) \sin(\theta) \right] S_3$$

9
with
\[ S_+ = \sin(\phi)S_1 + \cos(\phi)S_2 \quad \text{and} \quad S_- = \cos(\phi)S_1 - \sin(\phi)S_2. \]

The three independent solutions of the Bloch equations including phase modulation are therefore
\[
\begin{align*}
a^{(1)}(t) &= \frac{\epsilon_0}{\lambda} \left( \begin{array}{c}
\sin(\phi) \cos(\theta) \\
\cos(\phi) \cos(\theta) \\
- \sin(\theta)
\end{array} \right) \\
&\quad + \frac{\omega}{\lambda} \sin(\lambda \tau) \left( \begin{array}{c}
\sin(\phi) \sin(\theta) \\
\cos(\phi) \sin(\theta) \\
\cos(\theta)
\end{array} \right) \\
&\quad + \frac{\omega}{\lambda} \cos(\lambda \tau) \left( \begin{array}{c}
\cos(\phi) \\
- \sin(\phi) \\
0
\end{array} \right), \\
a^{(2)}(t) &= \frac{\omega}{\lambda} \left( \begin{array}{c}
\sin(\phi) \cos(\theta) \\
\cos(\phi) \cos(\theta) \\
- \sin(\theta)
\end{array} \right) \\
&\quad - \frac{\epsilon_0}{\lambda} \sin(\lambda \tau) \left( \begin{array}{c}
\sin(\phi) \sin(\theta) \\
\cos(\phi) \sin(\theta) \\
\cos(\theta)
\end{array} \right) \\
&\quad - \frac{\epsilon_0}{\lambda} \cos(\lambda \tau) \left( \begin{array}{c}
\cos(\phi) \\
- \sin(\phi) \\
0
\end{array} \right), \\
a^{(3)}(t) &= \cos(\lambda \tau) \left( \begin{array}{c}
\sin(\phi) \sin(\theta) \\
\cos(\phi) \sin(\theta) \\
\cos(\theta)
\end{array} \right) \\
&\quad - \sin(\lambda \tau) \left( \begin{array}{c}
\cos(\phi) \\
- \sin(\phi) \\
0
\end{array} \right).
\end{align*}
\]

The special solution \([43]\) satisfies \(a = \epsilon_0 a^{(1)} + \omega a^{(2)}\). The general solution of the equations \(\dot{x} = h \times x\) has been given in Part I of the paper and can be rederived from the knowledge of \(a^{(1)}, a^{(2)}, a^{(3)}\).

3.5 The \(k = 1\)-limit

In this limit the existence of the special solution requires that \(\xi_0 = \lambda\). The angle \(\phi\) and the phase \(\alpha\) become constants
\[
\sin(\phi) = \frac{\epsilon_0}{\lambda} \quad \text{and} \quad \cos(\phi) = \frac{\omega}{\lambda}
\]
and \(\alpha = -\epsilon_0 \xi_0 / \lambda^2\). Hence one has \(\lambda \tau(t) = \epsilon_0 t\). The angle \(\theta\) satisfies
\[
\sin(\theta) = -\tanh(\omega t) \quad \text{and} \quad \cos(\theta) = \sech(\omega t).
\]

The three independent solutions are
\[
\begin{align*}
a^{(1)}(t) &= \frac{\epsilon_0}{\lambda^2} \left( \begin{array}{c}
\epsilon_0 \sech(\omega t) \\
\omega \sech(\omega t) \\
\lambda \tanh(\omega t)
\end{array} \right) + \frac{\omega}{\lambda^2} \sin(\epsilon_0 t) \left( \begin{array}{c}
-\epsilon_0 \tanh(\omega t) \\
-\omega \tanh(\omega t) \\
\lambda \sech(\omega t)
\end{array} \right) \\
&\quad + \frac{\omega}{\lambda^2} \cos(\epsilon_0 t) \left( \begin{array}{c}
\omega \\
-\epsilon_0 \\
0
\end{array} \right), \\
a^{(2)}(t) &= \frac{\omega}{\lambda^2} \left( \begin{array}{c}
\epsilon_0 \sech(\omega t) \\
\omega \sech(\omega t) \\
\lambda \tanh(\omega t)
\end{array} \right) - \frac{\epsilon_0}{\lambda^2} \sin(\epsilon_0 t) \left( \begin{array}{c}
-\epsilon_0 \tanh(\omega t) \\
-\omega \tanh(\omega t) \\
\lambda \sech(\omega t)
\end{array} \right) \\
&\quad - \frac{\epsilon_0}{\lambda^2} \cos(\epsilon_0 t) \left( \begin{array}{c}
\omega \\
-\epsilon_0 \\
0
\end{array} \right).
\end{align*}
\]
\[ a^{(3)}(t) = \frac{1}{\lambda} \cos(\epsilon_0 t) \begin{pmatrix} -\epsilon_0 \tanh(\omega t) \\ -\omega \tanh(\omega t) \end{pmatrix} - \frac{1}{\lambda} \sin(\epsilon_0 t) \begin{pmatrix} \omega \\ -\epsilon_0 \end{pmatrix}. \] (57)

4 SU(1,1)

The generators of SU(1,1) satisfy the commutation relations

\[
\begin{align*}
[S_1, S_2] & = iS_3, \\
[S_2, S_3] & = -iS_1, \\
[S_3, S_1] & = -iS_2.
\end{align*}
\] (58)

Consider creation and annihilation operators \(b^\dagger\) and \(b\) satisfying the canonical commutation relations \([b, b^\dagger] = I\). Then the operators

\[
\begin{align*}
S_1 & = \frac{1}{4}((b^\dagger)^2 + b^2), \\
S_2 & = \frac{i}{4}((b^\dagger)^2 - b^2), \\
S_3 & = \frac{1}{4}(bb^\dagger + b^\dagger b)
\end{align*}
\] (59)

satisfy (58). Hamiltonians which can be written as a linear combination of these generators are the generalised harmonic oscillators.

4.1 Time-dependent frequency

Consider the harmonic oscillator with arbitrary time-dependent frequency \(\omega(t)\)

\[ H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2(t)Q^2. \] (60)

Here, \(Q\) is the position operator and \(P\) the momentum operator. They satisfy \([Q, P] = i\).

This problem was studied by Lewis and Riesenfeld [2]. See also [1, 9, 13, 18, 19, 23, 24, 27, 30].

Introduce an annihilation operator defined by

\[ b = \frac{1}{r\sqrt{2}}Q + i\frac{r}{\sqrt{2}}P, \] (61)

with \(r = \frac{1}{\sqrt{m\omega_0}}\) and \(\omega_0\) some constant frequency. Then the generators equal

\[
\begin{align*}
S_1 & = \frac{1}{4r^2}Q^2 - \frac{r^2}{4}P^2, \\
S_2 & = \frac{1}{4}(QP + PQ), \\
S_3 & = \frac{1}{4r^2}Q^2 + \frac{r^2}{4}P^2.
\end{align*}
\] (62-64)
Introduce the function $\gamma(t)$, modulating the frequency $\omega_0$, defined by $\omega(t) = \gamma(t)\omega_0$. The Hamiltonian becomes
\begin{equation}
H(t) = -\frac{1}{4}\omega_0(b - b^\dagger)^2 + \frac{1}{4}\omega_0\gamma^2(t)(b + b^\dagger)^2
= \omega_0(\gamma^2(t) - 1)S_1 + \omega_0(\gamma^2(t) + 1)S_3.
\tag{65}
\end{equation}
Hence one has $h = \omega_0(\gamma^2(t) - 1, 0, \gamma^2(t) + 1)^T$.

4.2 Special solution

The time evolution equation reads (see the definition of the Lie bracket in Part I, equation (69))
\begin{align*}
\dot{a} &= h \times a \\
&= (h_3a_2, h_1a_3 - h_3a_1, h_1a_2)^T \\
&= \omega_0(\gamma^2(t) + 1)(a_2, -a_1, 0)^T + \omega_0(\gamma^2(t) - 1)(0, a_3, a_2)^T. 
\tag{66}
\end{align*}

**Proposition 1** Let $x(t)$ be a solution of the classical oscillator equation
\begin{equation}
\ddot{x} + \omega^2(t)x = 0. 
\tag{67}
\end{equation}
Then $a$, defined by
\begin{equation}
a = \frac{1}{2}(\dot{x})^2(1, 0, 1)^T + \frac{1}{2}\omega_0^2x^2(-1, 0, 1)^T - \omega_0\dot{x}(0, 1, 0)^T, 
\tag{68}
\end{equation}
is a solution of (66).

The proof is done by explicit calculation. One concludes that, to find a special solution of the von Neumann equation, it suffices to find a special solution of the classical equation (67). Note that the latter problem is equivalent with solving Riccati’s equation
\begin{equation}
\dot{g} - g^2 = \omega^2(t). 
\tag{69}
\end{equation}
The corresponding solution of (67) is
\begin{equation}
x = C \exp \left( -\int dt \, g(t) \right), 
\tag{70}
\end{equation}
with integration constant $C$. In the case that $\omega(t)$ is constant one finds
\begin{equation}
g(t) = \omega \tan(\omega t) 
\tag{71}
\end{equation}
so that $x(t) = C \cos(\omega t)$. With $C = 1$ and $\omega_0 = \omega$ one obtains the special solution
\begin{equation}
a = \frac{1}{2}\omega^2(-\cos(2\omega t), \sin(2\omega t), 1)^T. 
\tag{72}
\end{equation}
If $\omega(t)$ is of the form $\omega(t) = \omega_0(1 + \epsilon \cos(2\lambda t))$ then the equation (67) is related to Mathieu’s equation. The solution $x = \text{cn}(\omega_0, t; k)$, involving Jacobi’s elliptic function with $0 < k < 1/\sqrt{2}$, is obtained when
\begin{equation}
\omega(t) = \omega_0 \sqrt{2 \text{dn}^2(\omega_0 t; k)} - 1. 
\tag{73}
\end{equation}
4.3 Automorphisms

Note that the special solution (68) satisfies \( \langle a|a \rangle = 0 \) (using the metric with signature \(-, -, +\)). Hence, we have to apply the exceptional case discussed in Part I, at the end of Section 5.

Write \( a(t) \) into the form

\[
a(t) = R_3(\phi)P_1(\chi)R_3(-\phi(0))a(0). \tag{74}
\]

From (77) of Part I follows

\[
\begin{align*}
\sin(\phi) &= \frac{a_1}{a_3} = \frac{(\dot{x})^2 - \omega_0^2 x^2}{(\dot{x})^2 + \omega_0^2 x^2}, \\
\cos(\phi) &= \frac{a_2}{a_3} = \frac{2 \omega_0 x \dot{x}}{(\dot{x})^2 + \omega_0^2 x^2}.
\end{align*} \tag{75}
\]

From (83) of Part I follows

\[
\chi = \ln \frac{a_3(t)}{a_3(0)} = \ln \frac{(\dot{x})^2 + \omega_0^2 x^2}{[(\dot{x})^2 + \omega_0^2 x^2]_{t=0}}. \tag{76}
\]

Note that \( \phi(0) \) simplifies if either \( x(0) = 0 \) or \( \dot{x}(0) = 0 \).

From the general theory now follows that (see (76) of Part I)

\[
V(t) = e^{-i \phi(t)S_3} e^{-i \chi(t) S_1} e^{i \phi(0) S_3}. \tag{77}
\]

The corresponding Hamiltonian is

\[
K = \dot{\chi} \{ \cos(\phi) S_1 - \sin(\phi) S_2 \} + \dot{\phi} S_3. \tag{78}
\]

Note that

\[
\dot{\chi} = \frac{\dot{a}_3}{a_3} = h_1 \frac{a_2}{a_3} \quad \text{and} \quad \dot{\phi} = \frac{a_2 \dot{a}_1 - a_1 \dot{a}_2}{a_3^2} = h_3 - h_1 \frac{a_1}{a_3}. \tag{79}
\]

Hence \( K \) can be written as

\[
K = \begin{pmatrix} \cos(\phi) \dot{\chi}, & -\sin(\phi) \dot{\chi}, & \dot{\phi} \end{pmatrix}^T
= \frac{1}{a_3^2} \begin{pmatrix} h_1 a_2^2, & -h_1 a_1 a_2, & h_3 a_3^2 - h_1 a_1 a_3 \end{pmatrix}^T. \tag{80}
\]

One obtains \( h - k = \alpha a \), with

\[
\alpha = h_1 \frac{a_1}{a_3^2}. \tag{81}
\]

The final result is

\[
U(t) = V(t) e^{i \int_{a_3(0)}^{a_3(t)} -C} = e^{-i \phi(t) S_3} e^{-i \chi(t) S_1} e^{i \phi(0) S_3} e^{i \sum_{j} a_j(0) S_j} \int_0^t ds \alpha(s). \tag{82}
\]
More general oscillators

More general time-dependent harmonic oscillators have been considered in the literature [3, 7, 8, 15, 20, 22, 27]. Even damped oscillators have been studied — see for instance [11, 14, 21]. Some of them can be treated by the present method. An example not yet considered in the literature is

\[ H = \sum_{j=1}^{3} h_j(t) S_j \]

with constants \( \omega, a, c \), and \( 0 \leq k \leq 1 \). A special solution of the von Neumann equation is given by (see [25])

\[ \rho_s(t) = \sum_{j=1}^{3} a_j(t) S_j \]

with

\[ a_1(t) = a \, \text{cn}(\omega t; k) \]  
\[ a_2(t) = -a \, \text{sn}(\omega t; k) \]  
\[ a_3(t) = c. \]  

Note that \( \rho_s(t) \) is not a density operator. But this does not harm our method.

Introduce unitary operators \( V(t) \) by (see (76) of Part I)

\[ V(t) = e^{-i\phi(t)S_3} e^{-i\chi(t)S_1} e^{i\chi(0)S_1} e^{i\phi(0)S_3} \]

with angles \( \phi(t) \) and \( \chi(t) \) satisfying (see (77) of Part I)

\[ \sin(\phi) = \text{cn}(\omega t; k), \quad \cos(\phi) = \text{sn}(\omega t; k) \]
\[ \sinh(\chi) = \frac{c}{\sqrt{a^2 - c^2}}, \quad \cosh(\chi) = \frac{a}{\sqrt{a^2 - c^2}}. \]

Note that we assume that \( a^2 > c^2 \). Also, note that \( \phi(0) = \pi/2 \). Because \( \chi(t) \) turns out to be independent of time \( t \), (89) simplifies to

\[ V(t) = e^{-i(\phi(t) - \phi(0))S_3}. \]

Next calculate

\[ \alpha(t) = \frac{a_1 h_1 + a_2 h_2}{a^2} = 1. \]

Hence, the time evolution is described by the unitary operators

\[ U(t) = V(t) e^{-i\rho_s(0)t} = e^{-i(\phi(t) - \phi(0))S_3} e^{-i(a S_1 + c S_3)}. \]

It is now possible to calculate the time evolution of the generators in the Heisenberg picture. For simplicity take \( c = 0 \). Then one obtains

\[ S_1(t) = U(t)^* S_1 U(t) \]
\[ = \cos(\phi(t) - \phi(0)) S_1 + \sin(\phi(t) - \phi(0)) S_2. \]
\[ S_1 - \cos(\phi(t))S_2 \]
\[ = \sin(\phi(t))S_1 - \cos(\phi(t))S_2 \]
\[ = \text{cn}(\omega t; k)S_1 - \text{sn}(\omega t; k)S_2. \quad (95) \]

Similarly is
\[ S_2(t) = U(t)^*S_2U(t) \]
\[ = \cosh(at)[\cos(\phi(t) - \phi(0))S_2 - \sin(\phi(t) - \phi(0))S_1] - \sinh(at)S_3 \]
\[ = \cosh(at)[\sin(\phi(t))S_2 + \cos(\phi(t))S_1] - \sinh(at)S_3 \]
\[ = \cosh(at)[\text{cn}(\omega t; k)S_2 + \text{sn}(\omega t; k)S_1] - \sinh(at)S_3, \quad (96) \]

and
\[ S_3(t) = U(t)^*S_3U(t) \]
\[ = \cosh(at)S_3 - \sinh(at)[\cos(\phi(t) - \phi(0))S_2 - \sin(\phi(t) - \phi(0))S_1] \]
\[ = \cosh(at)S_3 - \sinh(at)[\sin(\phi(t))S_2 + \cos(\phi(t))S_1] \]
\[ = \cosh(at)S_3 - \sinh(at)[\text{cn}(\omega t; k)S_2 + \text{sn}(\omega t; k)S_1]. \quad (97) \]

Note that \( S_3 \) is the energy of the unperturbed harmonic oscillator. Clearly, this quantity explodes for large times. Hence, the time-dependent harmonic oscillator described by (85) is at resonance.

6 Discussion

In Part I of this work a method was developed to solve time-dependent Hamiltonians with the assumption that they equal a time-dependent linear combination of generators of a finite Lie algebra. The method aims at finding all solutions, given one special solution. This Part II demonstrates how the method of Part I can be applied in two well-known cases, one corresponding with SU(2) symmetry, the other with SU(1,1). In this way, a number of known results are treated in a unified way. But in both cases the method is shown to produce new results as well.

We did not try to reproduce the most general results found in the literature. We are confident that we could do so, at the expense of writing a more technical and less pedagogical paper. Of more interest is the application of our method to other Lie algebras.

In [15], the Lie algebra that we used in Section 4 is extended to contain 6 elements \( \frac{1}{2}P^2, \frac{1}{2}Q^2, \frac{1}{2}(PQ + QP), P, Q, \mathbb{I} \). With this extension it becomes possible to calculate the time evolution in the Heisenberg picture of physically interesting quantities such as the position \( Q \) and the momentum \( P \). One then can calculate the classical phase portrait by studying the orbit \( t \rightarrow \langle \psi|Q(t)|\psi\rangle, \langle \psi|P(t)|\psi\rangle \) for an arbitrary wavefunction \( \psi \). This will be done in a future work.

Weigert [16] has considered the general case of SU(N) symmetry. Lopez and Suslov [28] used the Heisenberg-Weyl group N(3) to describe a forced harmonic oscillator. Finally, Cariñena et al [29], among others, are interested in developing a superposition principle for nonlinear equations by mapping the solutions onto the solutions of linear equations with time-dependent coefficients.
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