A Closed Universe: de Sitter Cosmic Gate

S. Danial Forghani
Faculty of Engineering, Final International University, Kyrenia, North Cyprus via Mersin 10, Turkey

S. Habib Mazharimousavi†
Department of Physics, Faculty of Arts and Sciences, Eastern Mediterranean University, Famagusta, North Cyprus via Mersin 10, Turkey

(Dated: June 20, 2022)

A new cosmological object in analogy with the concept of a wormhole in general relativity is introduced. As wormholes connect two distant points through a tunnel in a spacetime, this new object connects two spacetime through a large mouth which is referred to as a "Cosmic Gate". In this context, two identical copies of the regular part of the de Sitter spacetime are cut through a timelike hyperplane, and are glued at their identical boundaries (the only boundary) to form a complete spacetime. The stability of the cosmic gate against a linear radial perturbation is studied as well. Finally, the radial geodesics of the spacetime for a timelike and a null particle are presented.

I. INTRODUCTION

In his 1985 hard sci-fi novel "Contact", Carl Sagan portrays a spacetime journey via a wormhole in the most scientific way possible for its time. In his endeavor to build up such an accurate presentation of wormholes for a novel, he benefited from Kip Thorne’s consultant who at the time was a physics professor at Caltech. Later in 1987, the communication between the two scientists initiated the idea of two scientific masterworks, published in 1988 by Morris and Thorne [1, 2]. The significance of these articles lies in the fact that, although wormholes and their application for space and time travel were known by 1988, they were not considered to be practically traversable. Therefore, the ever-fainting trend of wormhole physics since the 1950s was considerably revived, which in turn led to many other great papers in the followed by years [3–10]. However, it was already shown by Morris and Thorne [2] that in order to keep the throat of a wormhole open, matter with negative energy density had to be distributed over the spacetime. In 1989, Matt Visser, inspired by thin-shell formalism, constructed a new class of traversable wormholes - known as thin-shell wormholes - which had this exotic matter limited only to the location of the throat [3]. Six years later, Poisson and Visser proposed a method by which the stability of such wormholes could be somehow determined under a linear radial perturbation [11].

In this study, we would like to employ the idea of thin-shell wormholes and apply it to construct a new entity that represents a closed and complete universe. In thin-shell wormhole studies, it is usual to cut two exterior solutions in a way that the existed singularities and their associated event horizons are excised out. Then the two remaining manifolds (that are geodesically incomplete) are identified at the cut hypersurface to form a geodesically complete manifold. Of course, the identification of the two manifolds requires certain conditions to be satisfied which results in determining the surface energy density and pressure of the matter that supports the identification location, known as the throat of the wormhole. In this paper, we instead excise out the exterior solutions and keep the interior ones to construct a closed and complete universe by gluing them. In this regard, we must choose a solution that is free of any singularities and/or horizons at or towards the center at \( r = 0 \). Evidently, the de Sitter solution [12–14] perfectly fits the situation. The de Sitter solution is a maximally symmetric vacuum solution in general relativity that describes a \( 3 + 1 \)-dimensional expanding universe with a positive cosmological constant. The main significance of the de Sitter solution is that our universe is believed to be asymptotically de Sitter, based on cosmological observations [15].

The paper is arranged as follows. In section II, we briefly discuss wormholes in Morris and Thorne’s sense to review some of their important features related to our discussion. Section III goes into the details of our construction. We show that the resultant thin-shell is supported by ordinary matter and is stable under a linear radial perturbation. In the continuation, we show in section IV that our constructed close universe is also geodesically complete. We conclude our paper in section V. All over the paper we have used the conventional units \( c = G = 1 \).

*Electronic address: danial.forghani@final.edu.tr
†Electronic address: habib.mazhari@emu.edu.tr
II. WORMHOLE VS. GATE

The so-called Morris-Thorne wormhole spacetime is described by

\[ ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2d\Omega^2, \]

(1)
in which the red-shift function \( e^{2\Phi(r)} \) admits no root in the domain of \( r \) and \( \lim_{r \to \infty} \Phi \to 0 \). Furthermore, the shape function \( b(r) \) satisfies the so-called \textit{flare-out conditions} [16]. These conditions are obtained through applying the embedding process to (1), which leads to an embedding diagram for the wormhole. The embedding diagram is to visualize a 2-dimensional version of a 3 + 1-dimensional universe by considering an instant of time \( (t = t_0) \) on the equator \( (\theta = \pi/2) \), that reduces the original line element in (1) to

\[ ds^2 = \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2d\phi^2. \]

(2)

Then, we embed this 2-dimensional surface onto a 3-dimensional flat cylindrical space by comparing it to the line element

\[ ds^2 = dr^2 + dz^2 + r^2d\phi^2. \]

(3)

With the cylindrical symmetry configuration, we assume \( z = z(r) \), which takes us to

\[ \frac{1}{1 - \frac{b(r)}{r}} = 1 + z'(r)^2, \]

(4)

where \( z'(r) = \frac{dz(r)}{dr} \). Consequently, we obtain

\[ z(r) = \pm \int_{R_0}^{r} \frac{1}{\sqrt{\frac{b(r)}{r} - 1}}dr, \]

(5)
in which at \( r = R_0 \) we assume \( z(r) = z(R_0) = 0 \). Accordingly, \( z(R_0) \) is called the throat of the wormhole connecting the region \( z \geq 0 \) to the region \( z \leq 0 \) provided that \( r(z) \) is the absolute minimum at \( r(0) = R_0 \). Technically speaking, this condition means that \( b(R_0) = R_0 \) and \( b - rb' \geq 0 \) for \( r \geq R_0 \). In order to keep the signature of the spacetime unaltered for \( r \geq R_0 \), one should also add the condition \( b(r) < r \) to the previous conditions. Therefore, while in the embedding diagram we have \( z \in \mathbb{R} \), the condition \( r \geq R_0 \) always holds. One can mathematically prove that the requirement of these conditions, i.e. the flare-out conditions, implies that the energy-momentum tensor supporting the wormhole spacetime does not satisfy the null energy condition, hence the corresponding matter is exotic [2].

Next, let us assume, there exists a spacetime described by the following line element

\[ ds^2 = -e^{2\Phi(r)}dt^2 + \frac{dr^2}{\frac{b(r)}{r} - 1} + r^2d\Omega^2, \]

(6)
such that \( b(r) \geq r \) for \( r \leq R_0 \) and the equality occurs at \( r = R_0 \). Here, \( r = R_0 \), is not a throat but in the sequel we shall refer to it as a “Gate”. Looking at its embedding diagram, we follow the steps as in the case for the Morris-Thorne wormhole. Hence, we obtain

\[ \frac{1}{\frac{b(r)}{r} - 1} = 1 + z'(r)^2, \]

(7)

and as a result

\[ z'(r) = \pm \sqrt{\frac{2 - \frac{b(r)}{r}}{\frac{b(r)}{r} - 1}}. \]

(8)

Having \( z'(r) \) real, one has to impose the condition

\[ 1 \leq \frac{b}{r} \leq 2. \]

(9)
FIG. 1: Embedding diagram of two identical de-Sitter universes joined at the cosmic gate. The result is a non-singular closed universe that is geodesically complete.

We observe that, at \( r = R_0 \), \( \frac{dr}{dz} = \frac{1}{z} \) is still zero, however, its second derivative is negative, i.e.

\[
\frac{d^2r}{dz^2} < 0. \tag{10}
\]

This implies that \( r(z) \) at \( z = 0 \) becomes an absolute maximum. The function \( z(r) \) is obtained to be

\[
z(r) = \pm \int_{R_0}^{r} \sqrt{\frac{2 - \frac{b(r)}{r}}{\frac{b(r)}{r} - 1}}, \tag{11}
\]

where we again considered \( z(R_0) = 0 \) and \( z \in [-R_0, R_0] \). On this account, unlike the wormhole spacetime, here the gate connects two bounded/closed spacetimes at their maximum surface. The resultant spacetime is complete and consists of two parts connected by what we are calling a "Cosmic Gate" from here onwards, in the hope that the terminology pleases the reader both scientifically and phonaesthetically.

III. DE SITTER THIN-SHELL COSMIC GATE

Let us start by setting the line element of a 3 + 1-dimensional spherically symmetric universe

\[
ds^2 = -f(r) \, dt^2 + \frac{1}{f(r)} \, dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right), \tag{12}
\]

which represents the maximally symmetric de Sitter spacetime with a positive scalar curvature for

\[
f(r) = 1 - \frac{r^2}{\ell^2}, \tag{13}
\]

where \( \ell^2 = \frac{3}{\Lambda} \), with \( \Lambda > 0 \) being the cosmological constant. The spacetime is nonsingular, although it has a cosmological horizon at \( r_c = \ell \). In what follows we construct a closed and geodesically complete universe by virtue of Visser’s cut-and-paste method originally used for constructing thin-shell wormholes [3]. However, what eventually results in our consideration is essentially a thin-shell cosmic gate, as we will show here. Hence, in the bulk metric (12) we make two identical copies of the spacetime inside a timelike hypersurface defined by \( \Sigma : r = R < r_c \), and paste them at this common hypersurface \( \Sigma \) to construct a closed universe. To show the closeness of this universe, we rely on its embedding diagram, which is shown in Fig. 1 for arbitrary values of \( r_c \) and \( R < r_c \). The function regarding the diagram displayed in Fig. 1 is given by

\[
z = \pm \ell \left[ \sqrt{1 - \frac{r^2}{\ell^2}} - \sqrt{1 - \frac{R^2}{\ell^2}} \right], \tag{14}
\]
following the method described in section II. Once again, we emphasize that the flare-out conditions for thin-shell wormholes [17] behave absolutely inversely in the case of thin-shell cosmic gates to satisfy \( \frac{d^2 \gamma}{d \ell^2} = -\frac{c^2}{c^2} < 0 \), in our case.

It is known from Darmois-Israel formalism that the boundary hypersurface \( \Sigma \) should satisfy certain junction conditions [18]. The details of the procedure are explained in many references [19–22], so here we refrain from excessive repetition. In short, the first fundamental form (the induced metric tensor) has to be continuous across the thin-shell whereas the second fundamental form (the induced curvature tensor) is discontinuous due to the presence of matter at the shell. Satisfying these conditions leads to mathematical expressions for the energy density \( \sigma \) and angular pressure \( p \) of the matter present on the shell. The symmetry of the bulk and the hyperplane \( \Sigma \) yields the matter on the shell. Satisfying these conditions leads to mathematical expressions for the energy density \( \sigma \) and angular pressure \( p \) of the matter present on the shell. The symmetry of the bulk and the hyperplane \( \Sigma \) yields the matter on the shell. Satisfying these conditions leads to mathematical expressions for the energy density \( \sigma \) and angular pressure \( p \) of the matter present on the shell. The symmetry of the bulk and the hyperplane \( \Sigma \) yields the matter on the shell. Satisfying these conditions leads to mathematical expressions for the energy density \( \sigma \) and angular pressure \( p \) of the matter present on the shell. The symmetry of the bulk and the hyperplane \( \Sigma \) yields the matter on the shell. Satisfying these conditions leads to mathematical expressions for the energy density \( \sigma \) and angular pressure \( p \) of the matter present on the shell. The symmetry of the bulk and the hyperplane \( \Sigma \) yields the matter on the shell. Satisfying these conditions leads to mathematical expressions for the energy density \( \sigma \) and angular pressure \( p \) of the matter present on the shell. The symmetry of the bulk and the hyperplane \( \Sigma \) yields the matter on the shell.

For \( \beta \) Fig. 2, we are particularly interested in the regions where \( 0 < \beta < 1 \) for stable situations. In Fig. 2, \( \beta^2 \) is graphed versus \( R_0 \) for \( \ell = 1 \) and given values of \( \gamma \).
IV. THE GEODESICS

In this section, we study the geodesics of the constructed closed universe. In this regard, we are interested in investigating the radial timelike and null geodesics for which the angular momentum of the test particles is zero. The Lagrangian of a particle moving on its equatorial plane along its geodesics in a static spherically symmetric space-time is given by [25]

\[ L = \frac{1}{2} \left[ -f(r) \dot{t}^2 + \frac{1}{f(r)} \dot{r}^2 + r^2 \dot{\phi}^2 \right] \]  

(22)

where an overhead dot represents differentiation with respect to an arbitrary affine parameter. Furthermore the 4-velocity satisfies 

\[ g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = \epsilon \]  

or explicitly

\[ -f(r) \dot{t}^2 + \frac{1}{f(r)} \dot{r}^2 + r^2 \dot{\phi}^2 = \epsilon \]  

(23)

with \( \epsilon = -1 \) (\( \epsilon = 0 \)) for a timelike (null) geodesics. In the first observation, since \( L \) is independent of \( t \) and \( \phi \) the energy

\[ E \equiv -\frac{\partial L}{\partial \dot{t}} = f(r) \dot{t} \]  

(24)

and the angular momentum

\[ L \equiv \frac{\partial L}{\partial \dot{\phi}} = r^2 \dot{\phi} \]  

(25)

are conserved. By substituting the energy and angular momentum into Eq. (23) one obtains the differential equation governing the radial coordinate of the geodesics expressed by

\[ \dot{r}^2 = E^2 + f(r) \left( \epsilon - \frac{L^2}{r^2} \right). \]  

(26)

In what follows, we consider both timelike (\( \epsilon = -1 \)) and null (\( \epsilon = 0 \)) cases for a radial motion with no angular momentum (\( L = 0 \)).

A. The Timelike Geodesics

For \( \epsilon = -1 \), once we set \( L = 0 \) in Eq. (26) it reduces to

\[ \frac{dr}{d\tau} = \pm \sqrt{E^2 - 1 + \frac{r^2}{\ell^2}}, \]  

(27)
FIG. 3: Radial timelike-geodesics for a massive particle as seen by a distant observer.

where we considered the affine parameter to be the proper time. The latter equation of motion is exactly solvable with the following solution

\[ r = r_0 \cosh \left( \pm \frac{(\tau - \tau_0)}{r_c} \right), \tag{28} \]

where \( r_0 = \ell \sqrt{1 - E^2} \) \((E^2 < 1)\) is the initial radial position provided that the particle starts from the rest at \( \tau = \tau_0 \). Naturally, we assume that \( r_0 < R_0 \). Solving for \( r = R_0 \) shows that the particle reaches the cosmic gate at \( (\tau_0 = 0) \)

\[ \tau_R = r_c \cosh^{-1} \left( \frac{R_0}{r_0} \right). \tag{29} \]

This, however, is measured by the commoving observer, not by a distant observer. For the distant observer, one needs to solve

\[ \frac{dr}{dt} = \pm \left( 1 - \frac{r^2}{r_c^2} \right) \sqrt{\frac{r^2 - r_0^2}{r_c^2 - r_0^2}}, \tag{30} \]

which directly results from the Lagrangian in Eq. (22) when \( \dot{r}^2/\dot{t}^2 = (dr/dt)^2 \). It can be shown that the solution to this differential equation is

\[ r = \frac{r_0}{\sqrt{1 - \left( 1 - \frac{r_0^2}{r_c^2} \right) \tanh^2 \left( \pm \frac{t}{r_c} \right)}}. \tag{31} \]

where we set \( t_0 = 0 \). Note that as \( t \to \infty \), the radial coordinate approaches \( r_c \), so the distant observer never sees the particle crossing the cosmic horizon in the de Sitter spacetime. However, in the presence of a cosmic gate at \( r = R_0 \), the distant observer witnesses the transition of the particle through the gate at

\[ t_R = r_c \tanh^{-1} \left( \sqrt{\frac{1 - r_0^2/R_0^2}{1 - r_0^2/r_c^2}} \right). \tag{32} \]

In Fig. 3, we plotted the time \( t \) versus radial coordinates \( r \) for a massive particle as observed by a distant observer in the presence (solid line) and absence (dash line) of the cosmic gate. As it is seen, the particle’s geodesics is continuous across the cosmic gate, although it undergoes an impulse in colliding with the matter at the gate.

B. The Null Geodesics

From (24) and (26), with \( L = 0 \) and \( \epsilon = 0 \) for a photon (null particle), one finds

\[ \dot{r}^2 = E^2 = \left( 1 - \frac{r^2}{r_c^2} \right)^2 \left( \frac{dr}{dt} \right)^2. \tag{33} \]
FIG. 4: Radial null-geodesics for a massless particle as seen by a distant observer.

for a radial motion, which admits the integral \((t_0 = 0)\)

\[
r = r_c \left( \frac{r_0 + r_c \tanh \left( \pm \frac{t}{r_c} \right)}{r_c + r_0 \tanh \left( \pm \frac{t}{r_c} \right)} \right) \tag{34}
\]

between the radial coordinate and the time \(t\) measured by a distant observer. It could be easily checked that the distant observer never detects the light reaching the cosmic horizon, although it could pass over the cosmic gate at a finite time given by

\[
t = r_c \tanh^{-1} \left( \frac{r_c (R_0 - r_0)}{r_c^2 - R_0 r_0} \right) \tag{35}
\]

if it existed. The geodesics of the massless particle are plotted in Fig. 4.

V. CONCLUSION

In contrast with the concept of a wormhole that connects two different distant points of the same or different universe(s) through a hyperplane/throat whose surface area is a local minimum, we introduced a cosmic gate that connects two closed universes through a hyperplane/gate whose surface area is a local maximum. In particular, we applied the junction formalism and glued the regular part of two de-Sitter spacetimes at a hyperplane located before the cosmological horizon. The resultant universe in its embedded diagram displayed in Fig. 1 is a closed universe in the sense that the radial coordinate is confined to a finite interval. We investigated the mechanical stability of the newly built cosmological object (we called it Cosmic Gate) and showed that it is stable against a radial linear perturbation. Furthermore, we studied the timelike and null geodesics of the massive and massless particles in this spacetime. Specifically, we focused our attention on radial geodesics to show that the spacetime is geodesically complete. Our results have been presented analytically and as have been depicted in Fig. 3 and 4, the particle reaches the gate in a finite time and travels to the other side. As a final remark, due to the direction of the 4-vector on the gate which is in the opposite direction of its counterpart on a thin-shell wormhole, the matter presented at the gate satisfies the physical energy conditions and therefore is normal matter. This can also be seen from the equal signs of the principal curvature of the gate (see Fig. 1). We recall that at the throat of a wormhole, the signs of the two principal curvatures are opposite of each other indicating the violation of the energy conditions.

[1] M.S. Morris and K.S. Thorne, Am. J. Phys 56 (1988) 395.
[2] M.S. Morris, K.S. Thorne and U. Yurtsever, Phys. Rev. Lett 61 (1988) 1446.
[3] M. Visser, Phys. Rev. D 39 (1989) 3182; Nucl. Phys. B 328 (1989) 203.
[4] I. Klebanov, L. Susskind and T. Banks, Nucl. Phys. B 317 (1989) 665.
[5] S.W. Hawking and D.N. Page, Phys. Rev. D 42 (1990) 2655.
[6] V.P. Frolov and I.D. Novikov, Phys. Rev. D 42 (1990) 1057.
[7] J.G. Cramer et. al., Phys. Rev. D 51 (1995) 3117.
[8] L.H. Ford and T.A. Roman, Phys. Rev. D 53 (1996) 5496.
[9] D. Hochberg and M. Visser, Phys. Rev. D 56 (1997) 4745.
[10] E. Teo, Phys. Rev. D 58 (1998) 024014.
[11] E. Poisson and M. Visser, Phys. Rev. D 52 (1995) 7318.
[12] W. de Sitter, Proc. Kon. Ned. Acad. Wet. 19 (1917) 1217.
[13] W. de Sitter, Proc. Kon. Ned. Acad. Wet. 20 (1917) 229.
[14] T. Levi-Civita, Rendiconti, Reale Accademia dei Lincei, 26 (1917) 519.
[15] I. Steer J. R. Astron. Soc. Can. 105 (2011) 18.
[16] S.W. Kim, J. Korean Phys. Soc 63 (2013) 1887.
[17] S.H. Mazharimousavi and M. Halilsoy, Phys. Rev. D 90 (2014) 087501.
[18] W. Israel, Nuovo Cim. B 44S10 (1966) 1; Erratum ibid. B 48 (1967) 463.
[19] F.S.N. Lobo and P. Crawford, Class. Quantum Grav. 21 (2004) 391.
[20] E.F. Eiroa and G.E. Romero, Gen. Relativ. Gravit 36 (2004) 651.
[21] G.A.S. Dias and J.P.S. Lemos, Phys. Rev. D 82 (2010) 084023.
[22] S.D. Forghani, S.H. Mazharimousavi and M. Halilsoy, Eur. Phys. J. Plus 134 (2019) 342.
[23] S.D. Forghani, S.H. Mazharimousavi and M. Halilsoy, Phys. Lett. B 804 (2020) 135374.
[24] V. Varela, Phys. Rev. D 92 (2015) 044002.
[25] V. Diemer and E. Smolarek, Class. Quantum Grav. 30 (2013) 175014.