On the number of light rings in curved spacetimes of ultra-compact objects

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In a very interesting paper, Cunha, Berti, and Herdeiro have recently claimed that ultra-compact objects, self-gravitating horizonless solutions of the Einstein field equations which have a light ring, must possess at least two (and, in general, an even number of) light rings, of which the inner one is stable. In the present compact paper we explicitly prove that, while this intriguing theorem is generally true, there is an important exception in the presence of degenerate light rings which, in the spherically symmetric static case, are characterized by the simple dimensionless relation $8\pi r^2(\rho + p_T) = 1$ [here $r$, is the radius of the light ring and $\{\rho, p_T\}$ are respectively the energy density and tangential pressure of the matter fields]. Ultra-compact objects which belong to this unique family can have an odd number of light rings. As a concrete example, we show that spherically symmetric constant density stars with dimensionless compactness $M/R = 1/3$ possess only one light ring which, interestingly, is shown to be unstable.

I. INTRODUCTION

One of the most physically interesting predictions of general relativity is the existence of closed light rings in curved spacetimes of compact astrophysical objects. These null circular geodesics are usually associated with black-hole spacetimes (see [1–5] and references therein), but horizonless compact objects like boson stars may also possess light rings [6–8].

Intriguingly, Cunha, Berti, and Herdeiro [6] (see also [7]) have recently asserted that horizonless matter configurations that have a light ring (the term ultra-compact objects is usually used in the literature to describe these self-gravitating field configurations) must have pairs (that is, an even number) of light rings. In particular, the interesting claim has been made [6, 7] that, for these regular ultra-compact horizonless objects, one of the closed light rings is stable [9].

Combining this claimed property of ultra-compact objects with the intriguing suggestion made in [11] (see also [3, 4]) that, due to the fact that massless perturbation fields tend to pile up on stable null geodesics, curved spacetimes with stable light rings may develop non-linear instabilities, it has been argued [4, 6, 7] that horizonless ultra-compact objects are non-linearly unstable and thus cannot provide physically acceptable alternatives to the canonical black-hole solutions of the Einstein field equations [12].

The main goal of the present paper is to prove that, while the physically intriguing theorem presented in [6] is generally true, it may be violated by ultra-compact objects with degenerate null circular geodesics which, in the spherically symmetric case, are characterized by the simple dimensionless relation $8\pi r^2(\rho + p_T) = 1$ [13]. In particular, below we shall explicitly demonstrate that, in principle, one can have special horizonless ultra-compact objects which possess only one light ring which, interestingly, is shown to be unstable.

II. DESCRIPTION OF THE SYSTEM

We consider spherically symmetric nonlinear matter configurations which are characterized by the static line element [1–5, 14, 15]

$$ds^2 = -e^{-2\delta}\mu dt^2 + \mu^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) ,$$

where $\delta = \delta(r)$ and $\mu = \mu(r)$. Regular spacetimes are characterized by the near-origin behavior [4]

$$\mu(r \to 0) = 1 + O(r^2) \quad \text{and} \quad \delta(0) < \infty .$$

In addition, asymptotically flat matter configurations are characterized by the functional relations [4]

$$\mu(r \to \infty) \to 1 \quad \text{and} \quad \delta(r \to \infty) \to 0 .$$

For spherically symmetric static spacetimes, the composed Einstein-matter field equations, $G^\mu_\nu = 8\pi T^\mu_\nu$, are given by [4, 16]

$$\mu' = -8\pi r\rho + \frac{1-\mu}{r}$$

(4)
and

\[ \delta' = -\frac{4\pi r(\rho + p)}{\mu}, \]  

where \((\rho, p, p_T) \equiv (-T^t_t, T^r_r, T^\theta_\theta = T^\phi_\phi)\) are respectively the energy density, the radial pressure, and the tangential pressure of the horizonless matter configuration \[17\]. We shall assume that the matter fields satisfy the dominant energy condition \[18\]

\[ \rho \geq |p|, |p_T| \geq 0. \]  

The gravitational mass \(m(r)\) of the field configuration contained within a sphere of areal radius \(r\) is given by the integral relation \[4, 19\]

\[ m(r) = 4\pi \int_0^r x^2 \rho(x) dx. \]  

The relations (6) and (7) imply that regular matter configurations with finite ADM masses (as measured by asymptotic observers) are characterized by the asymptotic functional behavior

\[ r^3 p(r) \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty. \]  

Defining the dimensionless radial function

\[ R(r) \equiv 3\mu - 1 - 8\pi r^2 p, \]  

one finds that the conservation equation

\[ T^\mu_{;\mu} = 0 \]  

together with the Einstein differential equations \[4\] and \[5\] yield, for spherically symmetric static spacetimes, the radial pressure gradient

\[ p'(r) = \frac{1}{2\mu r} \left[ R(\rho + p) + 2\mu T - 8\mu p \right], \]  

where

\[ T = -\rho + p + 2p_T \]

is the trace of the energy-momentum tensor which characterizes the self-gravitating matter fields.

### III. THE NUMBER OF LIGHT RINGS OF SPHERICALLY SYMMETRIC ULTRA-COMPACT OBJECTS

In the present section we shall explicitly prove that, while the intriguing Cunha-Berti-Herdeiro theorem \[6\] is generally true, it may be violated by a special family of horizonless ultra-compact objects that have a light ring which is characterized by the dimensionless functional relation \(8\pi r^2(\rho + p_T) = 1\).

To this end, we shall first derive, following the analysis of \[1, 3, 4\], the characteristic functional relation of null circular geodesics (light rings) in the spherically symmetric static spacetime \[1\]. Taking cognizance of the fact that the curved line element \[1\] is independent of the time and angular coordinates \(\{t, \phi\}\), one deduces that the geodesic trajectories are characterized by two conserved physical quantities: the energy \(E\) and the angular momentum \(L\) \[1, 3, 4\]. In particular, the null geodesics of the spherically symmetric static spacetime \[1\] are governed by an effective radial potential \(V_r\) which satisfies the characteristic equation \[1, 3, 4, 20\]

\[ E^2 - V_r \equiv r^2 = \mu \left( \frac{E^2}{e^{-2\delta} \mu} - \frac{L^2}{r^2} \right). \]  

The characteristic circular geodesics of the horizonless curved spacetime \[1\] are determined by the effective radial potential \[13\] through the relations \[1, 3, 4, 21\]

\[ V_r = E^2 \quad \text{and} \quad V'_r = 0. \]  

The gravitational mass \(m(r)\) of the field configuration contained within a sphere of areal radius \(r\) is given by the integral relation \[4, 19\]

\[ m(r) = 4\pi \int_0^r x^2 \rho(x) dx. \]  

The relations (6) and (7) imply that regular matter configurations with finite ADM masses (as measured by asymptotic observers) are characterized by the asymptotic functional behavior

\[ r^3 p(r) \rightarrow 0 \quad \text{as} \quad r \rightarrow \infty. \]  

Defining the dimensionless radial function

\[ R(r) \equiv 3\mu - 1 - 8\pi r^2 p, \]  

one finds that the conservation equation

\[ T^\mu_{;\mu} = 0 \]  

together with the Einstein differential equations \[4\] and \[5\] yield, for spherically symmetric static spacetimes, the radial pressure gradient

\[ p'(r) = \frac{1}{2\mu r} \left[ R(\rho + p) + 2\mu T - 8\mu p \right], \]  

where

\[ T = -\rho + p + 2p_T \]

is the trace of the energy-momentum tensor which characterizes the self-gravitating matter fields.
Taking cognizance of Eqs. (14), (5), and (13), one can express (14) in the simple dimensionless form

\[ \mathcal{R}(r = r_\gamma) = 0 \]  

(15)

The remarkably compact functional relation (15) determines the characteristic null circular geodesics of the spherically symmetric static spacetime (11). For later purposes we note that one learns from Eqs. (2), (3), and (8) that the dimensionless radial function \( \mathcal{R}(r) \) [see Eq. (3)] is characterized by the simple relations

\[ \mathcal{R}(r = 0) = 2 \quad \text{and} \quad \mathcal{R}(r \to \infty) \to 2 \]  

(16)

As explicitly shown in (12), stable circular geodesics of the spherically symmetric static spacetime (11) are characterized by a locally convex effective radial potential with the property \( V''_r(r = r_\gamma) > 0 \), whereas unstable circular geodesics are characterized by a locally concave effective radial potential with the opposite property \( V''_r(r = r_\gamma) < 0 \). Using Eqs. (4), (5), and (11), one finds the compact functional relation

\[ V''_r(r = r_\gamma) = -\frac{E^2c^2\delta}{\mu r_\gamma} \times \mathcal{R}'(r = r_\gamma) \]  

(17)

for the second spatial derivative of the effective radial potential (13), where [see Eqs. (1), (9), and (11)]

\[ \mathcal{R}'(r = r_\gamma) = \frac{2}{r_\gamma} \left[ 1 - 8\pi r_\gamma^3 (\rho + p_T) \right] \]  

(18)

Taking cognizance of Eqs. (15) and (16), one deduces that horizonless ultra-compact objects are generally (but, as will be discussed below, not always) characterized by a discrete set \( \{ r_\gamma^1, r_\gamma^2, \ldots, r_\gamma^{2n-1}, r_\gamma^{2n} \} \) of even number of light rings with the property \( \mathcal{R}(r = r_\gamma^i) = 0 \). In particular, the subset \( \{ r_\gamma^1, r_\gamma^3, \ldots, r_\gamma^{2n-1} \} \) of odd light rings is generally characterized by the property \( \mathcal{R}'(r = r_\gamma^i) < 0 \), whereas the subset \( \{ r_\gamma^2, r_\gamma^4, \ldots, r_\gamma^{2n} \} \) of even light rings is generally characterized by the property \( \mathcal{R}'(r = r_\gamma^i) > 0 \). Thus, the first subset corresponds to stable light rings with the property \( V''_r(r = r_\gamma^i) > 0 \) [see Eq. (17)], whereas the second subset corresponds to unstable light rings with the property \( V''_r(r = r_\gamma^i) < 0 \). It is important to stress the fact that this conclusion agrees with the recently published interesting theorem of Cunha, Berti, and Herdeiro (6).

However, it is also physically important to stress the fact that there are special cases in which one (or more) of the light rings, \( r = r_\gamma^n \), which characterize the horizonless ultra-compact objects is characterized by the functional relation

\[ \mathcal{R}'(r = r_\gamma^n) = 0 \]  

(19)

In this case the horizonless ultra-compact objects may be characterized by an odd number of light rings.

In particular, as a special case that will be demonstrated explicitly in the next section, we note that there are ultra-compact objects which possess only one degenerate light ring. In particular, this unique light ring is characterized by the simple properties \( \mathcal{R}(r = r_\gamma^n) = \mathcal{R}'(r = r_\gamma^n) = 0 \) and \( \mathcal{R}''(r = r_\gamma^n) > 0 \). Interestingly, taking cognizance of the characteristic properties \( V_r(r \to 0) \to \infty \) and \( V_r(r \to \infty) \to 0 \) of the effective radial potential (13) [see Eqs. (1) and (3)] which determines the null geodesics of the spacetime (11), one deduces that, in this special case, the effective radial potential \( V_r(r) \) is a monotonically decreasing function with the simple property \( V_r(r = r_\gamma^n - \epsilon) > V_r(r = r_\gamma^n) > V_r(r = r_\gamma^n + \epsilon) \) (23). Thus, the unique null circular geodesic \( r = r_\gamma^n \) is unstable to outward radial perturbations. One therefore concludes that there are special cases of horizonless ultra-compact objects which are characterized by the relations (15) and (19) and which possess no stable light rings.

IV. ULTRA-COMPACT OBJECTS WITH ONE LIGHT RING: CONSTANT DENSITY STARS

In the present section we shall explicitly demonstrate that there are ultra-compact horizonless objects which are characterized by an odd number of light rings. In particular, we shall show that constant density stars may possess only one light ring.

The Einstein-matter differential equations (4), (5), and (11) can be solved analytically in the case of constant density configurations, yielding the simple expression (2, 7, 25)

\[ p(r) = \frac{M}{2\pi R^3} \cdot \frac{\sqrt{3 - \frac{6M}{R}} - \sqrt{3 - \frac{6Mr^2}{R^2}}}{\sqrt{3 - \frac{6Mr^2}{R^2}} - 3\sqrt{3 - \frac{6Mr^2}{R^2}}} ; \quad 0 \leq 0 \leq R \]  

(20)
for the radial pressure, where $M$ and $R$ are respectively the total mass and surface radius of the compact star [26]. In addition, the exterior (vacuum) spacetime $r \geq R$ is described by the Schwarzschild line element with total mass $M$, whereas the interior spacetime of the compact star is characterized by the metric component [2, 7]

$$
\mu(r) = 1 - \frac{2Mr^2}{R^3}; \quad 0 \leq 0 \leq R.
$$

(21)

Compact density stars with $M/R \geq 1/3$ possess a light ring in the exterior vacuum ($p = 0$) region which, like the familiar Schwarzschild black-hole spacetime, is characterized by the areal radius

$$
r_\gamma = 3M.
$$

(22)

In addition, substituting Eqs. (20) and (21) into the characteristic equation (15), which determines the null circular geodesics of the spherically symmetric static spacetime (1), one finds after some algebra that these compact objects also possess a second (inner) light ring which is characterized by the remarkably compact dimensionless relation

$$
\mu(r_\gamma) \cdot \left(1 - \frac{2M}{r_\gamma}\right) = \frac{1}{9}.
$$

(23)

Interestingly, from Eqs. (21), (22), and (23) one deduces that constant density stars with $M/R = 1/3$ are characterized by only one light ring with $r_\gamma = R = 3M$ [27, 30]. These horizonless matter configurations therefore violate the physically interesting theorem presented in [6, 31].

V. SUMMARY

Horizonless ultra-compact objects with closed null circular geodesics (light rings) have recently attracted much attention from physicists and mathematicians (see [6, 7] and references therein) as possible exotic alternatives to black-hole spacetimes [9].

In a physically interesting paper, Cunha, Berti, and Herdeiro [6] have recently claimed that self-gravitating horizonless solutions of the Einstein field equations which have a light ring, must have at least two (and, in general, an even number of) light rings, of which one of them is stable. Combined with the non-linear instability to massless perturbation fields which is expected to characterize curved spacetimes that possess stable light rings [3, 4, 11], it has been claimed [6, 7] that ultra-compact horizonless objects (self-gravitating field configurations with light rings) cannot provide physically acceptable alternatives to classical black-hole spacetimes [10, 12].

In the present ultra-compact paper we have proved that, while the physically important theorem presented in [6] is generally true, there is an intriguing exception for horizonless spacetimes that possess degenerate null circular geodesics which, in the spherically symmetric static case, are characterized by the simple dimensionless functional relation [see Eqs. (17), (18), and (19)] [32]

$$
8\pi r_\gamma^2 (\rho + p_T) = 1.
$$

(24)

We have stressed the fact that horizonless ultra-compact matter configurations with the property (24) may have an odd number of null circular geodesics. In particular, it has been explicitly demonstrated that there are special cases of ultra-compact objects which possess no stable light rings.

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We shall use natural units in which $G = 1$. It is important to stress the fact that, according to the interesting theorem presented in [6], one of the non-degenerate rate

It is important to note that this physically interesting conclusion also depends on the characteristic timescale of the expected non-linear instabilities [6, 7].

Here $\{r_\gamma, \rho, \rho_T\}$ are respectively the radius of the light ring, the energy density, and the tangential pressure of the matter fields.

We shall use natural units in which $G = c = 1$.

Here $(t, r, \theta, \phi)$ are the familiar Schwarzschild spacetime coordinates.

Here a prime $'$ denotes a spatial derivative with respect to the radial coordinate $r$.

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Taking cognizance of Eqs. (1), (4) and (7), one deduces the simple functional relation $\rho(r) \equiv 1 - 2\rho(r)/r$.

Here a dot $'$ denotes a derivative with respect to an affine parameter.

Note that these two characteristic conditions correspond to $r^2 = (t^2)' = 0$.

Note that the relation (19) corresponds to a special light ring with the property $V''(r = r_\gamma) = 0$ [see Eq. (17)].

Here we assume that $\epsilon > 0$.

It is worth emphasizing again that this special null circular geodesic is characterized by the properties $V'(r = r_\gamma) = V''(r = r_\gamma) = 0$.

Note that the expression $3M/4\pi R^3$ in (20) is simply the constant matter density of the star.

Note that $p(r = R) = 0$.

It is worth noting that, for constant density stars, the dominant energy condition $\rho \geq |p|, |\rho_T| \geq 0$ [18] is respected in the physical regime $M/R \leq 3/8$ [see Eq. (21)]. Thus, our constant density stars with $M/R = 1/3$ respect this energy condition.

Note that, for constant density stars, $R'(r)$ is not well defined at the surface $r = R$. This, however, does not harm our observation that horizonless constant density stars with $M/R = 1/3$ possess only one closed light ring.

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It is worth emphasizing that the results of the present paper reveal the intriguing fact that in some special (degenerate) cases, one can have horizonless ultra-compact objects which possess no stable null circular geodesics. In particular, as opposed to generic ultra-compact objects which possess stable light rings, the unstable degenerate light ring of these special ultra-compact matter configurations is not expected to induce non-linear instabilities in the corresponding spatially regular horizonless curved spacetimes. Thus, one cannot rule out a priori the existence of stable horizonless ultra-compact objects which possess degenerate light rings. It would therefore be physically interesting to examine the possibility of producing dynamically these ultra-compact objects with degenerate null circular geodesics.

We believe that it would be physically interesting to generalize the relation (24), which characterizes ultra-compact objects that can violate the important theorem presented in [6], to the non-spherically symmetric regime of self-gravitating horizonless matter configurations.