STABLE ESTIMATES FOR SOURCE SOLUTION OF CRITICAL FRACTAL BURGERS EQUATION

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Abstract. In this paper, we provide two-sided estimates for the source solution of d-dimensional critical fractal Burgers equation \( u_t - \Delta^{\alpha/2} + b \cdot \nabla (u|u|^q) = 0, \) \( \alpha \in (1, 2), \) by the density function of the isotropic \( \alpha \)-stable process.

1. Introduction

Let \( d \in \mathbb{N} \) and \( \alpha \in (1, 2) \). We consider the following pseudo-differential equation
\[
\begin{cases}
  u_t - \Delta^{\alpha/2} u + b \cdot \nabla (u|u|^q) = 0, & t > 0, \ x \in \mathbb{R}^d, \\
  u(0, x) = M\delta_0(x),
\end{cases}
\]
where \( M > 0 \) is arbitrary constant and \( b \in \mathbb{R}^d \) is a constant vector. In this paper, we focus on the critical case \( q = (\alpha - 1)/d \). Here, \( \Delta^{\alpha/2} \) denotes the fractional Laplacian defined by the Fourier transform
\[
\hat{\Delta}^{\alpha/2} \phi(\xi) = -|\xi|^\alpha \hat{\phi}(\xi), \quad \phi \in C^\infty_0(\mathbb{R}^d).
\]

Equation (1.1) for various values of \( q \) and initial conditions \( u_0 \) was recently intensely studied (\cite{2, 3, 7}). For \( d = 1 \), the case \( q = 2 \) is of particular interest (see e.g. \cite{12, 1, 13, 18}) because it is a natural counterpart of the classical Burgers equation. Another interesting value of \( q \) is \( \frac{\alpha - 1}{d} \). In \cite{4} authors proved that the solution of (1.1), which we denote throughout the paper by \( u_M(t, x) \), exists and is unique and positive. It belongs also to \( L^p(\mathbb{R}^d) \) for every \( p \in [1, \infty] \). The exponent \( q = \frac{\alpha - 1}{d} \) is critical in some sense. It is the only value for which the function \( u_M(t, x) \) is self-similar. Is satisfies the following scaling condition (1.2)
\[
u_M(t, x) = a^d u_M(a^n t, ax), \quad \text{for all } a > 0.
\]

Furthermore, the linear and the nonlinear terms in (1.1) have equivalent influence on the asymptotic behavior of the solution. If \( q > (\alpha - 1)/d \), the operator \( \Delta^{\alpha/2} \) plays the main role. More precisely, for such \( q \) and a function \( u \) satisfying (1.1), with not necessarily the same initial condition, we have
\[
\lim_{t \to \infty} t^{n(1-1/p)/\alpha} \left| u(t, \cdot) - e^{\Delta^{\alpha/2}} u(0, \cdot) \right|_p = 0, \quad \text{for each } p \in [1, \infty].
\]

For \( q < (\alpha - 1)/d \) another asymptotic behavior is expected. In addition, taking \( q = \frac{\alpha - 1}{d} \) for \( d = 1 \) and \( \alpha = 2 \) we obtain the classical case, which makes the equation (1.1) with critical exponent \( q \) one of the natural generalizations of the Burgers equation.

Key words and phrases. fractional Laplacian, critical Burgers equation, source solution
Till the end of the paper we assume that $d \geq 1$, $\alpha \in (1, 2)$ and $q = \frac{\alpha - 1}{d}$. Let $p(t, x)$ be the fundamental solution of

$$v_t = \Delta^{\alpha/2} v.$$  \hfill (1.3)

In [7] the authors proved that for sufficiently small $M$ there is a constant $C = C(d, \alpha, M, b)$ such that

$$u_M(t, x) \leq C p(t, x), \quad t > 0, \quad x \in \mathbb{R}^d.$$  \hfill (1.4)

In this paper we get rid of the smallness assumption of $M$. Furthermore, we also obtain the lower bounds of $u_M$. We propose a new method which allows us to show pointwise estimates of solutions to the nonlinear problem (1.1) without the smallness assumption imposed on $M$. This method has been inspired by the proof of [6, Theorem 1]. Our main result is

**Theorem 1.1.** Let $d \geq 1$ and $\alpha \in (1, 2)$. Let $u_M(t, x)$ be the solution of the equation (1.1) with $q = \frac{\alpha - 1}{d}$. There exists a constant $C = C(d, \alpha, M, b)$ such that

$$C^{-1} p(t, x) \leq u_M(t, x) \leq C p(t, x), \quad t > 0, \quad x \in \mathbb{R}^d.$$  \hfill (1.5)

The fractional Laplacian plays also a very important role in the probability theory as a generator of the so called isotropic stable process. The theory of its linear perturbations has been recently significantly developed, see e.g., [5, 6, 10, 11, 17, 14, 15, 8, 9]. However, since the term $b \cdot \nabla (|u|^q u)$ in (1.1) represents a nonlinear drift, methods used in the linear case often cannot be adapted. In the proofs we mostly use the Duhamel formula and its suitable iteration. The scaling condition (1.2) is also intensively exploit.

The paper is organized as follows. In Preliminaries we collect some basic properties of the function $p(t, x)$ and introduce the Duhamel formula as well. In Section 3 we prove that the solution of (1.1) converges to 0 as $|x| \to \infty$. In section 4 we prove Theorem 1.1.

2. Preliminaries

2.1. Notation. For two positive functions $f, g$ we denote $f \lesssim g$ whenever there exists a constant $c > 1$ such that $f(x) \leq c g(x)$ for every argument $x$. If $f \lesssim g$ and $g \lesssim f$ we write $f \approx g$. If value of a constant in estimates is relevant, we denote it by $C_k$, $k \in \mathbb{N}$, and it does not change throughout the paper.

2.2. Properties of $p(t, x)$. The fundamental solution of (1.3) may be given by the inverse Fourier transform

$$p(t, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-ix \cdot \xi} e^{-t|\xi|^\alpha} d\xi, \quad t > 0, \quad x \in \mathbb{R}^d.$$  \hfill (2.1)

This implies the following scaling property

$$p(t, x) = a^d p(a^\alpha t, ax), \quad \text{for all } a > 0.$$  \hfill (2.2)

Note that $u_M(t, x)$ possesses exactly the same property. Let $p(t, x, y) := p(t, y - x)$. Below, we give two well-known estimates of $p$ and the gradient of $p$ (see [5] for more details).

$$p(t, x, y) \approx \frac{t}{(t^{1/\alpha} + |y - x|)^{d+\alpha}},$$  \hfill (2.3)

$$|\nabla_y p(t, x, y)| \approx \frac{t |y - x|}{(t^{1/\alpha} + |y - x|)^{d+2+\alpha}}.$$  \hfill (2.4)

We will need the following lemma
Lemma 2.1. For \( t, \varepsilon > 0 \) we have

\[
\int_{B(0,\varepsilon)} p(t,0,w)dw \approx \left( \frac{\varepsilon}{t^{1/\alpha} + \varepsilon} \right)^d.
\]

Proof. By formula (2.2),

\[
\int_{B(0,\varepsilon)} p(t,0,w)dw \approx \int_{B(0,\varepsilon)} \frac{t dw}{(t^{1/\alpha} + |w|)^{d+\alpha}}
= c \int_0^\varepsilon \frac{tr^{d-1}dr}{(t^{1/\alpha} + r)^{d+\alpha}}
= c \frac{t}{\varepsilon^\alpha} \int_0^1 \frac{ru^{d-1}dr}{(\frac{1}{\varepsilon} + u)^{d+\alpha}}.
\]

(2.4)

If \( t^{1/\alpha} \geq \varepsilon \), we estimate the denominator in the integral by \( \frac{t^{1/\alpha}}{\varepsilon} \) and we get

\[
\int_{B(0,\varepsilon)} p(t,0,w)dw \approx \left( \frac{\varepsilon}{t^{1/\alpha}} \right)^d, \quad t^{1/\alpha} \geq \varepsilon.
\]

(2.5)

In the case \( t^{1/\alpha} < \varepsilon \) we substitute \( r = \frac{t^{1/\alpha}}{\varepsilon}u \) in (2.4), then

\[
\int_{B(0,\varepsilon)} p(t,0,w)dw \approx \int_0^{t^{1/\alpha}} \frac{w^{d-1}du}{(1 + u)^{d+\alpha}} \approx 1, \quad t^{1/\alpha} < \varepsilon.
\]

(2.6)

Combining (2.5) and (2.6), we obtain the assertion of the lemma. \( \square \)

2.3. Duhamel formula. Our mail tool is the following Duhamel formula,

\[
u_M(t,x) = Mp(t,x) + \int_0^t \int_{\mathbb{R}^d} p(t-s,x,z)b \cdot \nabla_z [u_M(s,z)]^{q+1} dz ds.
\]

(2.7)

In the following, we assume that \( u_M(t,x) = t^{-d/\alpha}u_M(1,xt^{-1/\alpha}) \) is a nonnegative self-similar solution of the equation (2.7) such that \( u_M(1,\cdot) \in L^p(\mathbb{R}^d) \) for each \( p \in [1, \infty] \). As it was mentioned in Introduction, the existence of such a function was shown in [4].

It turns out that the integral in (2.7) is not absolutely convergent, but integrating by parts we obtain a more convenient form

\[
u_M(t,x) = Mp(t,x) - \int_0^t \int_{\mathbb{R}^d} b \cdot \nabla_z p(t-s,x,z)[u_M(s,z)]^{q+1} dz ds,
\]

(2.8)
which is absolutely convergent. Indeed, we have \(|u_M(s, \cdot)|_p \lesssim s^{-(1-1/p)/\alpha}\), \(p \in [1, \infty]\) (see [1]). Hence, by (2.2) and (2.3),
\[
\int_0^t \int_{\mathbb{R}^d} |b \cdot \nabla_z p(t - s, x, z)[u_M(s, z)]^{q+1}| \, dz \, ds \\
\lesssim \int_0^{t/2} \int_{\mathbb{R}^d} |\nabla_z p(t - s, x, z)||u_M(s, z)|^{q+1} \, dz \, ds + \int_{t/2}^t \int_{\mathbb{R}^d} |\nabla_z p(t - s, x, z)||u_M(s, z)|^{q+1} \, dz \, ds \\
\lesssim \int_0^{t/2} \int_{\mathbb{R}^d} (t - s)^{-(d+1)/\alpha}[u_M(s, z)]^{q+1} \, dz \, ds + \int_{t/2}^t \int_{\mathbb{R}^d} (t - s)^{-1/\alpha}p(t - s, x, z)s^{-d(q+1)/\alpha} \, dz \, ds \\
\lesssim t^{-(d+1)/\alpha} \int_0^{t/2} s^{-(\alpha-1)/\alpha} \, ds + t^{-d(q+1)/\alpha} \int_{t/2}^t (t - s)^{-1/\alpha} \, ds \lesssim t^{-d/\alpha}.
\]

Finally, we get
\[
u_M(t, x) = Mp(t, x) - \alpha \int_0^{t/\alpha} \int_{\mathbb{R}^d} \nabla_w p(t - r^\alpha, x, rw)[u_M(1, w)]^{q+1} \, dw \, dr \quad (2.9)
\]

3. BEHAVIOUR OF \(u_M(1, x)\) AT INFINITY

Due to the scaling property [1,2] it suffices to consider \(u_M(t, x)\) only for \(t = 1\).

**Lemma 3.1.** There is a constant \(C_1 > 0\) such that for every \(x \in \mathbb{R}^d\),
\[
u_M(1, x) \leq Mp(1, x) + C_1 \int_0^1 \int_{\mathbb{R}^d} (1 - r^\alpha)^{-1/\alpha}p(1 - r^\alpha, x, rw)[u_M(1, w)]^{q+1} \, dw \, dr. \quad (3.1)
\]

**Proof.** Formulae (2.2) and (2.3) imply that
\[
|\nabla_z p(1 - r, x, z)| \lesssim (1 - r)^{-1/\alpha}p(1 - r, x, z), \quad r \in [0, 1], \quad x, z \in \mathbb{R}^d,
\]
and the assertion follows by (2.9). \(\square\)

Now, we will show that the function \(u_M(1, x)\) vanishes at infinity.

**Proposition 3.2.** We have \(\lim_{|x| \to \infty} u_M(1, x) = 0\).

**Proof.** Let us rewrite (3.1) into the form
\[
u_M(1, x) \lesssim p(1, x) + I_1(x) + I_2(x),
\]

where
\[
I_1(x) = C_1 \int_0^1 \int_{\mathbb{R}^d} (1 - r^\alpha)^{-1/\alpha}p(1 - r^\alpha, x, rw)[u_M(1, w)]^{q+1} \, dw \, dr,
\]
and
\[
I_2(x) = Mp(1, x) - Mp(1, 0).
\]
where

\[
I_1 = \int_0^{1/2} \int_{\mathbb{R}^d} (1 - r^\alpha)^{-1/\alpha} p(1 - r^\alpha, x, rw)[u_M(1, w)]^{1+q} \, dw \, dr,
\]

\[
I_2 = \int_{1/2}^{1} \int_{\mathbb{R}^d} (1 - r^\alpha)^{-1/\alpha} p(1 - r^\alpha, x, rw)[u_M(1, w)]^{1+q} \, dw \, dr.
\]

Note that \(a_R := \int_{B(0,R)} [u_M(1, w)]^{1+q} \, dw \to 0\) as \(R \to \infty\). Using (2.2) and estimating \(1 - r^\alpha \approx 1, r \in (0, 1/2)\), we obtain for \(|x| > R\),

\[
I_1 = \int_0^{1/2} \int_{B(0,R)} \ldots \, dw \, dr + \int_0^{1/2} \int_{B(0,R)^c} \ldots \, dw \, dr
\]

\[
\leq \int_0^{1/2} \int_{B(0,R)} \frac{1}{|x|^{d+\alpha}} u_M(1, w)^{1+q} \, dw \, dr + \int_0^{1/2} \int_{B(0,R)^c} [u_M(1, w)]^{1+q} \, dw \, dr
\]

\[
\leq \frac{|B(0, R)|}{|R|^{d+\alpha}} \|u_M(1, \cdot)\|_{\infty}^{1+q} + a_R,
\]

which is arbitrary small for sufficiently large \(R\). Consider now the integral \(I_2\) and fix \(\varepsilon > 0\). There exist \(R_1, R_2 > 0\) such that

\[
\int_{B(0,R_1)^c} p(1,0,w) dw < \varepsilon,
\]

\[
\int_{B(0,R_2)} u_M(1, w) dw < \varepsilon^{d+1}.
\]

The latter inequality implies that the measure of the set \(\{w \in B(0, R_2)^c : u_M(1, w) > \varepsilon\}\) is less or equal to \(\varepsilon^d\). It gives us for \(r \in (1/2, 1)\) and \(|x| > R_1 + R_2\)

\[
\int_{\mathbb{R}^d} p(1 - r^\alpha, x, rw)[u_M(1, w)]^{1+q} \, dw
\]

\[
= \frac{1}{r^d} \int_{\mathbb{R}^d} p(1 - r^\alpha, x, w)[u_M(1, w/r)]^{1+q} \, dw
\]

\[
\leq \frac{\|u_M(1, \cdot)\|_{\infty}^{1+q}}{r^d} \int_{B(x,R_1)^c} p(1 - r^\alpha, x, w) \, dw + \frac{\varepsilon^{1+q}}{r^d} \int_{B(x,R_1), u_M(1,w/r) \leq \varepsilon} p(1 - r^\alpha, x, w) \, dw
\]

\[
+ \frac{1}{r^d} \|u_M(1, \cdot)\|_{\infty}^{1+q} \int_{B(x,R_1), u_M(1,w/r) > \varepsilon} p(1 - r^\alpha, x, w) \, dw.
\]

Using scaling property (2.1) and substituting \(w = x + (1 - r^\alpha)^{1/\alpha} z\), we get

\[
\int_{B(x,R_1)^c} p(1 - r^\alpha, x, w) \, dw = \int_{B(0,R_1(1-r^\alpha)^{-1/\alpha})^c} p(1,0,z) \, dz \leq \varepsilon.
\]

Moreover,

\[
|\{w \in B(x, R_1) : u_M(1, w/r) > \varepsilon\}| = r^d |\{w \in B(x/r, R_1/r) : u_M(1, w) > \varepsilon\}|
\]

\[
\leq |\{w \in B(0, R_2)^c : u_M(1, w) > \varepsilon\}|
\]

\[
\leq \varepsilon^d \leq |B(0, \varepsilon)|.
\]
Hence, by the fact that $p(s, x, w)$ is a decreasing function of $|x - w|$ and by Lemma 2.1, we obtain

$$I_2(x) \lesssim \varepsilon + \int_{1/2}^{1} \int_{B(x, \varepsilon)} p(1 - r^\alpha, x, w) dw \, dr$$

$$\lesssim \varepsilon + \int_{1/2}^{1} (1 - r^\alpha)^{-1/\alpha} \left( \varepsilon \left( \frac{1}{1 - r^\alpha} + \frac{1}{\varepsilon} \right) \right)^d dr$$

$$\lesssim \varepsilon + \int_0^{1-(1/2)^\alpha} u^{-1/\alpha} \left( \varepsilon \left( \frac{1}{u^{1/\alpha} + \varepsilon} \right) \right)^d du$$

$$= \varepsilon + \int_{\varepsilon/2}^{\varepsilon^{\alpha/2}} du + \int_{\varepsilon/2}^{1-(1/2)^\alpha} \ldots du$$

$$\leq \varepsilon + \int_0^{\varepsilon^{\alpha/2}} u^{-1/\alpha}du + \frac{\varepsilon^{d/2}}{(1 + \sqrt{\varepsilon})^d} \int_{\varepsilon/2}^{1-(1/2)^\alpha} u^{-1/\alpha}du$$

$$\lesssim \varepsilon + \varepsilon^{(\alpha-1)/2} + \varepsilon^{d/2}.$$ 

Therefore, for sufficiently large $|x|$, integrals $I_1$ and $I_2$ are arbitrary small. This ends the proof. 

\[\square\]

4. PROOF OF THEOREM 1.1

In this section, we prove the main theorem of the paper. First, we define some auxiliary functions. Let

$$H(x, w) = \int_0^1 (1 - r^\alpha)^{-1/\alpha} p(1 - r^\alpha, x, rw) dr. \quad (4.1)$$

For $\beta \in (0, 1)$, we define

$$\tilde{H}(x, w) = \int_0^1 r^{-\beta}(1 - r^\alpha)^{-1/\alpha} p(1 - r^\alpha, x, rw) dr. \quad (4.2)$$

Additionally, for $R > 0$, we denote

$$h_R(x) = \int_{B(0, R)} H(x, w) [u_M(1, w)]^{1+q} dw, \quad (4.3)$$

$$\tilde{h}_R(x) = \int_{B(0, R)} \tilde{H}(x, w) [u_M(1, w)]^{1+q} dw, \quad (4.4)$$

$$\tilde{H}_R(x) = \int_{B(0, R) \cap \varepsilon} \tilde{H}(x, w) u_M(1, w) dw. \quad (4.5)$$

Note that $H(x, w) \leq \tilde{H}(x, w)$ and $h_R(x) \leq \tilde{h}_R(x)$.

Lemma 4.1. For $x \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} H(x, w) p(1, w) dw = C_2 p(1, x), \quad (4.6)$$

where

$$C_2 = \frac{\pi}{\alpha \sin (\pi/\alpha)} > 1.$$
Proof. By scaling property and Chapman-Kolmogorov equation for the function \( p(t, x, y) \), we get
\[
\int_{\mathbb{R}^d} p(1 - s^\alpha, x, sw)p(1, w) \, dz = \int_{\mathbb{R}^d} s^{-d}p(s^{-\alpha} - 1, s^{-1}x, w)p(1, w) \, dz
\]
\[
= s^{-d}p(s^{-\alpha}, s^{-1}x, ) = p(1, x).
\] (4.7)

Consequently,
\[
\int_{\mathbb{R}^d} H(x, w)p(1, w) \, dw = \int_0^1 \int_{\mathbb{R}^d} (1 - s^\alpha)^{-1/\alpha}p(1 - s^\alpha, x, sw)p(1, w) \, dw \, ds
\]
\[
= p(1, x) \int_0^1 (1 - s^\alpha)^{-1/\alpha} \, ds = \frac{1}{\alpha} \Gamma \left( 1 - \frac{1}{\alpha} \right) \Gamma \left( \frac{1}{\alpha} \right) p(1, x)
\]
\[
= \frac{\pi}{\alpha \sin (\pi/\alpha)} p(1, x),
\]

where the last equality results from the Euler’s reflection formula. \( \square \)

The next step is to provide a Chapman-Kolmogorov-like inequality involving functions \( H(x, w) \) and \( H_i(x, w) \). At first, we present a technical lemma.

**Lemma 4.2.** Let \( \beta > 0 \) be fixed. For \( v \in (0, 1) \), we have
\[
\int_v^1 r^{-\beta}(1 - r^\alpha)^{-1/\alpha}(r^\alpha - v^\alpha)^{-1/\alpha} \, dr \approx v^{-\beta}(1 - v)^{-2/\alpha}.
\]

**Proof.** Denote the above integral by \( I(v) \). Since \( a^\gamma - b^\gamma \approx (a - b)a^{\gamma - 1} \) for \( a > b > 0 \) and \( \gamma > 0 \) (cf. Lemma 4 in [16]), we have \( 1 - r^\alpha \approx 1 - r \) and \( r^\alpha - v^\alpha \approx (r - v)r^{-\alpha - 1} \). Hence,
\[
I(v) \approx \int_v^1 r^{1/\alpha - 1/\beta}(1 - r)^{-1/\alpha}(r - v)^{-1/\alpha} \, dr.
\]

For \( v \geq 1/4 \), we estimate \( r^{1/\alpha - 1/\beta} \approx 1 \) and substitute \( r = 1 - u(1 - v) \), which gives us
\[
I(v) \approx (1 - v)^{1/2} \int_0^1 u^{-1/\alpha}(1 - u)^{-1/\alpha} \, du \approx (1 - v)^{1/2/\alpha}.
\]

In the case \( v < 1/4 \), we split the integral into \( \int_v^{1/2} + \int_{1/2}^1 \) and obtain
\[
I(v) \approx \int_v^{1/2} r^{1/\alpha - 1/\beta}(r - v)^{-1/\alpha} \, dr + \int_{1/2}^1 (1 - r)^{-1/\alpha} \, dr
\]
\[
= v^{-\beta} \int_1^{1/(2v)} u^{1/\alpha - 1/\beta}(u - 1)^{-1/\alpha} \, du + \frac{\alpha 2^{(\alpha - 1)/\alpha}}{\alpha - 1}
\]
\[
\approx v^{-\beta} + 1 \approx v^{-\beta},
\]

which is equivalent to the required formula. \( \square \)

Since \( \alpha > 1 \), we immediately obtain the following

**Corollary 4.3.** Let \( \beta > 0 \) be fixed. For \( v \in (0, 1) \), we have
\[
\int_v^1 r^{-\beta}(1 - r^\alpha)^{-1/\alpha}(r^\alpha - v^\alpha)^{-1/\alpha} \, dr \lesssim v^{-\beta}(1 - v)^{-1/\alpha}.
\]

This allows us to prove the following lemma.
Lemma 4.4. There exists a constant $C_3 > 0$ such that for $x, z \in \mathbb{R}^d$, we have
\[ \int_{\mathbb{R}^d} H(x, w)\tilde{H}(w, z) dw \leq C_3\tilde{H}(x, z). \] (4.8)

Proof. Scaling property and Chapman-Kolmogorov equation for the function $p(t, x, y)$ give us
\[ \int_{\mathbb{R}^d} H(x, w)\tilde{H}(w, z) dw = \int_0^1 \int_0^1 \int_{\mathbb{R}^d} r^{-\beta}(1 - s^\alpha)^{-1/\alpha} p(1 - s^\alpha, x, sw)(1 - r^\alpha)^{-1/\alpha} p(1 - r^\alpha, w, rz) dw \, ds \, dr \]
\[ = \int_0^1 \int_0^1 \int_{\mathbb{R}^d} r^{-\beta}(1 - s^\alpha)^{-1/\alpha} s^{-d}(s^{-\alpha} - 1, s^{-1} x, w)(1 - r^\alpha)^{-1/\alpha} p(1 - r^\alpha, w, rz) dw \, ds \, dr \]
\[ = \int_0^1 \int_0^1 \int_{\mathbb{R}^d} r^{-\beta}(1 - s^\alpha)^{-1/\alpha}(1 - r^\alpha)^{-1/\alpha} s^{-d}(s^{-\alpha} - s^{-\alpha} - r^\alpha, s^{-1} x, rz) ds \, dr \]
\[ = \int_0^1 \int_0^1 \int_{\mathbb{R}^d} r^{-\beta}(1 - s^\alpha)^{-1/\alpha}(1 - r^\alpha)^{-1/\alpha} p(1 - s^\alpha^{-\alpha}, x, srz) ds \, dr. \]

Substituting $s = v/r$ in the inner integral and then using Fubini-Tonelli theorem, we get
\[ \int_{\mathbb{R}^d} H(x, w)\tilde{H}(w, z) dw = \int_0^1 \int_{\mathbb{R}^d} p(1 - v^\alpha, x, vz) \int_v^1 r^{-\beta}(1 - r^\alpha)^{-1/\alpha}(r^\alpha - v^\alpha)^{-1/\alpha} dr \, dv. \]

By Corollary 4.3, we obtain (4.8). \hfill \Box

Remark 1. Lemma 4.2, which plays an important role in the above-given proof of Lemma 4.4, does not hold for $\beta = 0$. This explains partly the form of the functions $\tilde{H}(x, w)$ and $\tilde{h}_R(x)$.

As a consequence, we get

Corollary 4.5. For $x \in \mathbb{R}^d$, we have
\[ \int_{\mathbb{R}^d} H(x, w)\tilde{h}_R(w) dw \leq C_3\tilde{h}_R(x), \] (4.9)
\[ \int_{\mathbb{R}^d} H(x, w)\tilde{H}(w) dw \leq C_3\tilde{H}(x). \] (4.10)

Now, we pass to the proof of the main result of the paper.

Proof of Theorem 1.1. Let $C_0 = C_1(C_2 \vee C_3)$. By Lemma 3.2, we may choose $\eta \in (0, 1)$ and $R > 0$ such that $|u_M(1, x)| < \left(\frac{\eta}{C_0}\right)^{1/q}$ for $|x| > R$. Thus, by Lemma 3.1, we have
\[ u_M(1, x) \leq Mp(1, x) + C_1\tilde{h}_R(x) + \frac{C_1\eta}{C_0} \int_{B(0, R)^c} H(x, w)u_M(1, w) dw \]
\[ \leq Mp(1, x) + C_1\tilde{h}_R(x) + \frac{C_1\eta}{C_0} \tilde{H}(x). \] (4.11)
We put \(4.12\) to \(4.11\) and, by Lemma \(4.1\) and Corollary \(4.5\) we have
\[
u_M(1, x) \leq Mp(1, x) + C_1 \tilde{h}_R(x)\] (4.13)
\[
\quad + \frac{C_1}{C_0} \int_{B(0, R)^c} H(x, w) \left[ Mp(1, w) + C_1 \tilde{h}_R(w) + \frac{C_1}{C_0} \tilde{h}_R(w) \right] dw
\]
\[
\quad \leq M(1 + \eta)p(1, x) + C_1(1 + \eta)\tilde{h}_R(x) + \eta^2 \tilde{H}_R(x).\] (4.14)

Now, we put \(4.14\) into \(4.11\) and, by Lemma \(4.1\) and Corollary \(4.5\) we obtain
\[
u_M(1, x) \leq M(1 + \eta + \eta^2)p(1, x) + C_1(1 + \eta + \eta^2)\tilde{h}_R(x) + \eta^3 \tilde{H}_R(x).
\]

Hence, by induction,
\[
u_M(1, x) \leq M \sum_{k=0}^{n} \eta^k p(1, x) + C_1 \sum_{k=0}^{n} \eta^k \tilde{h}_R(x) + \eta^{n+1} \tilde{H}_R(x).
\]

Taking \(n \to \infty\), we get
\[
u_M(1, x) \leq \frac{M}{1 - \eta} p(1, x) + \frac{C_1}{1 - \eta} \tilde{h}_R(x).\] (4.15)

Furthermore, since both functions \(p(1, \cdot)\) and \(u_M(1, \cdot)\) are continuous and nonnegative (see [4], proof of Theorem 2.1), they are comparable on every compact set. Hence, we focus only on large values of \(|x|\).

We first prove the upper estimate. Let \(|x| > 2R\). For \(|w| < R\) and \(s \in (0, 1)\), we have \(|x - sw| > |x|/2\), and consequently \(p(s, x, sw) \lesssim |x|^{-d-\alpha} \approx p(1, x)\). Hence,
\[
\tilde{h}_R(x) = \int_{B(0, R)} \int_0^1 r^{-\beta} (1 - r^\alpha)^{-1/\alpha} p(1 - r^\alpha, x, rw)[u_M(1, w)]^{1+q} dr dw
\]
\[
\lesssim p(1, x) \|u_M(1, \cdot)\|_{1+q}^{1+q} \int_{B(0, R)} \int_0^1 r^{-\beta} (1 - r^\alpha)^{-1/\alpha} dr dw
\]
\[
\approx \|u_M(1, \cdot)\|_{1+q}^{1+q} p(1, x),
\]

which, by \(4.15\), gives us
\[
u_M(1, x) \lesssim p(1, x).\] (4.16)

To provide lower estimates we show that the integral in \(2.9\) is much smaller than \(Mp(t, x)\) for sufficiently large \(|x|\). By \(4.16\) and \(2.3\), there are constants \(C_4, C_5 > 0\) such that
\[
\left| \alpha \int_0^1 \int_{\mathbb{R}^d} b \cdot \nabla_w p(1 - s^\alpha, x, sw)[u_M(1, w)]^{q+1} dw ds \right|
\]
\[
\leq C_4 \int_0^1 \int_{\mathbb{R}^d} |\nabla_w p(1 - s^\alpha, x, sw)[p(1, w)]^{q+1}| dw ds
\]
\[
\leq C_5 \int_0^1 \int_{B(0, R)} \frac{1}{(1 - s^\alpha)^{1/\alpha} + |x - sw|} p(1 - s^\alpha, x, sw)p(1, w) dw ds
\]
\[
\quad + C_5 \int_0^1 \int_{B(0, R)^c} (1 - s^\alpha)^{-1/\alpha} p(1 - s^\alpha, x, sw)[p(1, w)]^{q+1} dw ds
\]
for any $\tilde{R} > 0$. We take $\tilde{R}$ such that $p(1, w) < \left(4C_5 \int_0^1 (1 - s^\alpha)^{-1/\alpha} ds/M \right)^{-1/q}$ for $|w| > \tilde{R}$. Then, for $|x| > (2\tilde{R}) \vee (8C_5/M)$, we obtain

$$\left| \alpha \int_0^1 \int_{\mathbb{R}^d} b \cdot \nabla_w p(1 - s^\alpha, x, sw)[u_M(1, w)]^{q+1} dw \right|$$

$$\leq \frac{2C_5}{|x|} \int_0^1 \int_{B(0, \tilde{R})} p(1 - s^\alpha, x, sw)p(1, w) dw ds$$

$$+ \frac{M}{4 \int_0^1 (1 - s^\alpha)^{-1/\alpha} ds} \int_0^1 (1 - s^\alpha)^{-1/\alpha} \int_{B(0, \tilde{R})} p(1 - s^\alpha, x, sw)p(1, w) dw ds.$$

Hence, by (4.7), we get

$$\left| \alpha \int_0^1 \int_{\mathbb{R}^d} b \cdot \nabla_w p(1 - s^\alpha, x, sw)[u_M(1, w)]^{q+1} dw \right| \leq \frac{M}{2} p(1, x),$$

which, together with (2.9), implies

$$u_M(1, x) \geq \frac{M}{2} p(t, x).$$

The proof is complete. \hfill \Box

**Remark 2.** Using the result of Theorem 1.1 as well as the method of proving it one can show that there exists a constant $C = C(d, \alpha, M, b)$ such that

$$|\nabla_x u_M(t, x)| \leq Ct^{-1/\alpha} p(t, x).$$

However, this estimate does not seem to be optimal.

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**References**

[1] N. Alibaud, C. Imbert, and G. Karch. Asymptotic properties of entropy solutions to fractal Burgers equation. *SIAM J. Math. Anal.*, 42(1):354–376, 2010.

[2] P. Biler, T. Funaki, and W. A. Woyczyński. Fractal Burgers equations. *J. Differential Equations*, 148(1):9–46, 1998.

[3] P. Biler, G. Karch, and W. A. Woyczyński. Asymptotics for conservation laws involving Lévy diffusion generators. *Studia Math.*, 148(2):171–192, 2001.

[4] P. Biler, G. Karch, and W. A. Woyczyński. Critical nonlinearity exponent and self-similar asymptotics for Lévy conservation laws. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 18(5):613–637, 2001.

[5] K. Bogdan and T. Jakubowski. Estimates of heat kernel of fractional Laplacian perturbed by gradient operators. *Comm. Math. Phys.*, 271(1):179–198, 2007.

[6] K. Bogdan and T. Jakubowski. Estimates of the Green function for the fractional Laplacian perturbed by gradient. *Potential Anal.*, 36(3):455–481, 2012.

[7] L. Brandolese and G. Karch. Far field asymptotics of solutions to convection equation with anomalous diffusion. *J. Evol. Equ.*, 8(2):307–326, 2008.

[8] Z.-Q. Chen, P. Kim, and R. Song. Dirichlet heat kernel estimates for fractional Laplacian with gradient perturbation. *Ann. Probab.*, 40(6):2483–2538, 2012.

[9] T. Grzywny and M. Ryznar. Estimates of Green functions for some perturbations of fractional Laplacian. *Illinois J. Math.*, 51(4):1409–1438, 2007.

[10] T. Jakubowski and K. Szczykowska. Time-dependent gradient perturbations of fractional Laplacian. *J. Evol. Equ.*, 10(2):319–339, 2010.

[11] T. Jakubowski and K. Szczykowska. Estimates of gradient perturbation series. *J. Math. Anal. Appl.*, 389(1):452–460, 2012.
[12] G. Karch, C. Miao, and X. Xu. On convergence of solutions of fractal Burgers equation toward rarefaction waves. *SIAM J. Math. Anal.*, 39(5):1536–1549, 2008.

[13] A. Kiselev, F. Nazarov, and R. Shterenberg. Blow up and regularity for fractal Burgers equation. *Dyn. Partial Differ. Equ.*, 5(3):211–240, 2008.

[14] Y. Maekawa and H. Miura. On fundamental solutions for non-local parabolic equations with divergence free drift. *Adv. Math.*, 247:123–191, 2013.

[15] Y. Maekawa and H. Miura. Upper bounds for fundamental solutions to non-local diffusion equations with divergence free drift. *J. Funct. Anal.*, 264(10):2245–2268, 2013.

[16] J. Małecki and G. Serafin. Hitting hyperbolic half-space. *Demonstratio Math.*, 45(2):337–360, 2012.

[17] K. Szczypkowski. Gradient perturbations of the sum of two fractional Laplacians. *Probab. Math. Statist.*, 32(1):41–46, 2012.

[18] L. Wang and W. Wang. Large-time behavior of periodic solutions to fractal Burgers equation with large initial data. *Chin. Ann. Math. Ser. B*, 33(3):405–418, 2012.

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