We study the BPS spectrum of the theory on a D3-brane probe in F theory. The BPS states are realized by multi-string configurations in spacetime. Only certain configurations obeying a selection rule give rise to BPS states in the four-dimensional probe theory. Using these string configurations, we determine the spectrum of N=2 $SU(2)$ Yang-Mills. We also explore the relation between multi-string configurations, M theory membranes and self-dual strings.
1. Introduction

One of the most striking phenomena found in certain supersymmetric theories is the non-analytic behavior of the BPS spectrum as the moduli of the theory are varied. There are generally hypersurfaces of real codimension one in the moduli space where the BPS bound no longer forbids the decay of BPS particles because some of the complex central charges have aligned. When one of these curves of marginal stability (CMS) is crossed, some BPS particles may decay into many particle states, or certain many particle states may bind to give a new single particle BPS state. This phenomena was originally observed in two-dimensional N=2 theories [1], and was seen to be necessary in four-dimensional N=2 gauge theories [2]. The same non-analytic behavior will appear generally in string backgrounds preserving enough supersymmetry.

We will consider the case of N=2 four-dimensional theories. In some of these theories, for example $SU(2)$ Yang-Mills with a massive hypermultiplet, certain decay processes can be studied using semi-classical techniques [3,4]. However, most curves of marginal stability do not extend to the region of weak coupling, so a direct analysis in Yang-Mills is difficult. However, in these cases, consistency of the proposed vacuum solution can often be used to determine the BPS spectrum inside and outside a curve of marginal stability [5]. This technique can be extended to theories whose curves are known, but which have no known Lagrangian description like the d=4 theory associated to an $E_8$ singularity.

There are several ways to realize N=2 d=4 theories in string theory or M theory. Let us consider cases with a one-dimensional Coulomb branch for simplicity. One way is to wrap a type II five-brane on a curve $\Sigma$. This picture can be related by a series of dualities to the type II string on a Calabi-Yau three-fold near a point of enhanced gauge symmetry. The theory at low-energies on the five-brane is an N=2 d=4 theory [6,7]. The BPS states can be constructed in terms of self-dual strings charged under the self-dual two-form $B$ wrapped on a one-cycle of $\Sigma$. The d=4 BPS states then correspond to geodesics on the Seiberg-Witten Riemann surface $\Sigma$. A closely related construction follows from analyzing the strong coupling description of a system of type IIA four-branes and five-branes [8]. At strong coupling, the system is better described in terms of an M theory five-brane wrapping a curve. The BPS states correspond to minimal area membranes with boundaries on the Riemann surface [9,10]. These theories can also be geometrically engineered [11], and the BPS states will correspond to branes wrapping various cycles of the local geometry.

The approach that will primarily concern us in this paper is the probe construction of the N=2 d=4 theory. Our probe is a D3-brane which is placed at a point on the
moduli space of the d=4 theory. The singularities on the moduli space are replaced by
seven-branes, some of which are mutually non-local. This picture can be derived from
F theory \[12\] on K3 which geometrically captures the non-perturbative dynamics of an
orientifold seven-plane \[13\]. The orientifold plane splits into a collection of seven-branes,
some of which we can send to infinity. In this way, the resulting theory is type IIB string
compactified on \(\mathbb{P}^1\) with points deleted and with a varying coupling constant. The coupling
constant is given precisely by the solution of an N=2 d=4 theory. In the case studied by
Sen, the d=4 theory was \(SU(2)\) Yang-Mills with four hypermultiplets. This result was
given a very direct physical interpretation by placing a D3-brane probe at a point on the
base space \[14\]. At low-energies, the theory on the probe is \(SU(2)\) Yang-Mills with four
massive hypermultiplets, and the gauge coupling for this d=4 theory coincides with the
type IIB string coupling. In a similar way, theories with no known Lagrangian descriptions
can be constructed by placing the probe in the vicinity of more exotic singularities such
as an \(E_8\) singularity \[15\].

Sen argued that certain BPS states in the probe theory can be realized by open
strings stretching from the D3-brane to a seven-brane along a geodesic on the moduli
space. This picture was further explored in \[16\]. This geometric realization of the BPS
states is very elegant and compelling physically, yet there was a mystery associated to
this picture. As we shall explain, geodesics involving single strings cannot realize all the
BPS states of the probe theory. Most of the states have to be realized in a slightly more
sophisticated way. The aim of this paper is to explain how these states are constructed, and
to explore the relation between the various geometric realizations of these BPS states in
terms of membranes in M theory, open strings and self-dual strings. Each of these pictures
is perhaps better suited for analyzing specific generalizations of this construction. For
example, the M theory picture can be naturally generalized to the \(SU(N)\) case, while the
probe approach can be extended easily to the case of \(E\)-singularities. Lastly, understanding
BPS states in F theory compactifications with D3-branes has an immediate application to
compactifications of F theory to four dimensions, where one of the ways of obtaining an
anomaly free theory involves placing D3-branes in spacetime \[17\].

While we were in the process of writing up our conclusions, some interesting papers
with overlapping results appeared \[18,19,20\]. Some earlier work on understanding the BPS
spectrum of the probe theory appeared in \[21\].
2. Open String Realizations of BPS States

2.1. Single-string BPS states

We wish to consider F theory on $K3$, which is equivalent to considering the type IIB string with 24 seven-branes, some of which are mutually nonlocal. The positions of these $(p, q)$ seven-branes in the transverse two-dimensional space, parametrized by $u$, are encoded in the choice of elliptically-fibered $K3$. We can describe these spaces by specifying a curve,

$$y^2 = x^3 + f(u)x + g(u),$$

where $f$ and $g$ are polynomials in $u$ of degree 8 and 12, respectively. The zeroes of the discriminant of the curve,

$$\Delta = 4f^3 + 27g^2,$$

describe the location of the seven-branes, while the monodromy of the complex type IIB string coupling around a particular zero encodes the $(p, q)$ charge of the seven-brane located at that point. When some of the seven-branes become coincident, the $K3$ can develop a singularity. By placing a D3-brane probe at a point on the base of the $K3$, we can induce a non-trivial four-dimensional theory on the probe [14].

BPS states in the four-dimensional theory should correspond to string configurations ending on the D3-brane probe [13]. Let us start by considering single string configurations. Only $(p, q)$ strings can end on a $(p, q)$ seven-brane; however, any string can end on the D3-brane. Let us consider a single D7-brane at a point $u_0$. On circling the seven-brane, the string coupling

$$\tau \sim \frac{1}{2\pi i} \ln(u - u_0),$$

undergoes a monodromy. As a result, we need to place branch cuts on the $u$-plane to account for the monodromy around each seven-brane. Let us place the D3-brane at a point $A$ on the $u$-plane. A $(p, q)$ string stretching from the D3-brane along the $u$-plane on a curve $C$ has a total mass given by,

$$\int_C |dw_{p,q}|,$$

where $dw_{p,q}$ is determined by the metric for the F theory background:

$$|dw_{p,q}|^2 = |p + q\tau|^2|dw_{1,0}|^2.$$
The holomorphic form \( dw_{p,q} \) is given by the integral of the holomorphic two-form,

\[
\frac{dx du}{y},
\]

along the \((p, q)\) cycle of the elliptic fiber in the locally flat basis in \( H_1 \) of the fibers. BPS configurations correspond to geodesics in the metric \( |dw_{p,q}|^2 \) and for these strings,

\[
M_{p,q} = \int_C |dw_{p,q}| = \int_C |dw_{p,q}|.
\]

Let us choose a \((p, q)\) seven-brane at some point \( B \). We can first inquire about single-string BPS states. These states correspond to geodesics starting at \( B \) and extending to the D3-brane at \( A \). How many such geodesics exist?

For a given geodesic \( \gamma \), we can define a phase which is constant along the geodesic

\[
\phi = \text{Arg}\left[w_{p,q}(A)\right].
\] (2.5)

Specifying \( \phi \) determines the tangent vector to the curve \( \gamma \) at \( A \). This uniquely determines the geodesic which must terminate with finite length at the location of a \((p, q)\) seven-brane. Therefore, all single-string geodesics are uniquely fixed by the choice of central charge. Now it is clear that we have a problem. For example, the solution for the vacuum structure of \( SU(2) \) \( N=2 \) Yang-Mills only contains two singularities where a monopole or dyon becomes massless. In the F theory picture, these singularities are replaced by appropriate \((p, q)\) seven-branes. From the argument just presented, we only see BPS states that correspond to single strings stretching from the D3-brane to either of the two singularities, where the strings may pass through a branch cut on the \( u \)-plane. As we shall show in the following section, these strings only give a finite number of states. However, the semi-classical spectrum of pure \( SU(2) \) Yang-Mills is easily determined. There are W-bosons with charges \((\pm 2, 0)\) and dyons with charges \((2n, \pm 1)\) where \( n \in \mathbb{Z} \). Higher magnetic charge dyons would correspond to holomorphic \( L^2 \) forms on monopole moduli space, but a non-compact Calabi-Yau manifold has no such forms \[22\]. How are the remainder of these BPS states realized in string theory?
2.2. Various routes to M theory

To realize the full BPS spectrum, we need to consider multi-string configurations \([23]\). Classical string configurations do not generally correspond to quantum mechanical BPS states. In the case of maximal supersymmetry, the existence of such junctions as quantum mechanical bound states was argued in \([24]\) using gauge dynamics and in \([25]\) using the lift to M theory. The dynamics of these junctions has been explored further in \([26,27]\). In the presence of seven-branes, the question of whether a BPS junction exists is more subtle; see \([28]\) for a discussion of non-terminating junctions in the presence of seven-branes. We need to consider the situation where legs of the junction terminate on seven-branes. In this case, we need to determine a selection rule for which junctions are BPS.

We need to consider configurations of string junctions with legs terminating on three-branes and seven-branes. To study when a string junction exists as a BPS configuration, we will lift the configuration to M theory. There are two ways of lifting F theory on \(K3\) with a D3-brane to M theory. For the first route, we consider F theory on \(K3 \times S1\) with the \(S1\) transverse to the D3-brane. Let us recall that M theory on \(T2\) is dual to the type IIB string on a circle. If the Kähler class of the torus is \(A\) then the radius \(R\) of the circle is given by,

\[
R = \frac{1}{M_{pl}^3 A},
\]

where \(M_{pl}\) is the eleven-dimensional Planck scale \([29]\). In this lift, the D3-brane is realized in M theory as an M5-brane wrapping the torus \(T2\). In this way, we connect with the pictures presented in \([3,8]\) of a five-brane wrapped on a curve \(\Sigma\). We are interested in the limit where \(R\) is very large and so the Kähler class of the torus is very small. This limit is essentially the opposite of the limit studied in \([8]\), where the Kähler class was taken to infinity.

A \((p, q)\) stretched on the \(u\)-plane and terminating on the D3-brane lifts to a membrane wrapping the \((p, q)\) cycle of the torus and stretched along the 1-cycle on the \(u\)-plane. We will explore the relation between the boundary of the membrane, which is composed of 1-cycles on \(\Sigma\), and self-dual strings in more detail in the following section. Our starting point is then a membrane wrapped on a curve in the total space of the elliptic fibration. The boundary of the membrane must lie on the curve \(\Sigma\). \(\Sigma\) is the elliptic fiber over the point \(A\) in the \(u\)-plane where our D3-brane was originally placed. The fiber is holomorphic in a distinguished complex structure, so the curve \(\Sigma\) is holomorphic in this particular complex structure. If the configuration is BPS then the world-volume of the membrane
must be a holomorphic surface with a boundary on $\Sigma$. For the world-volume to be a minimal surface, we need to adjust the boundary so that the world-volume intersects $\Sigma$ at right-angles \cite{9,10}. If they do not intersect at right-angles then although the membrane itself is a BPS configuration, the combined M5/M2 system is not BPS \cite {30}. It will turn out that not all string junctions lift to BPS configurations in the total space.

The second route involves taking the D3-brane to be wrapped on the circle $S^1$. In this case, the D3-brane is realized in M theory as an M2-brane transverse to the $T^2$. The end of the membrane representing the $(p, q)$ string now looks like a point-particle in the world-volume of the M2-brane. If we further compactify a circle $S^1$ transverse to this configuration, we can reduce the configuration to a D2-brane intersecting a second D2-brane at a point. The intersection point common to these two D-branes can be deformed to a more general curve, so both branes should be considered part of a single brane. Since the D2-brane taken as the probe is significantly deformed by the intersecting brane, it is no longer clear that there is a reasonable gauge theory interpretation for this configuration.

This second approach corresponds to compactifying the four-dimensional theory to three-dimensions on a circle in the $x_3$ direction with radius $R$. The moduli space of the uncompactified four-dimensional theory is two-dimensional. On compactification, we obtain two new compact scalars giving a four-dimensional moduli space with hyperKähler metric. One scalar $\phi_3$ corresponds to the choice of Wilson line on the compact circle while the other direction is the expectation value of the scalar $\phi_D$ dual to the resulting three-dimensional gauge theory. In the limit where $R \to 0$, the circle corresponding to $\phi_3$ decompactifies, and the moduli space is $\mathbb{R}^3 \times S^1$ equipped with a hyperKähler metric.

2.3. The relation with self-dual strings

Another picture of BPS states in the N=2 theory follows from realizing the theory on an M theory or type IIA five-brane wrapping a Riemann surface $\Sigma$ in a six-dimensional space $C$ of $SU(2)$ holonomy. The five-brane worldvolume theory has $(0, 2)$ superconformal invariance in flat spacetime. The matter content of this theory comprises a tensor multiplet, which contains five scalars describing the transverse coordinates of the brane in eleven dimensions. The theory on the five-brane worldvolume is twisted in the sense that the scalars become sections of the normal bundle $N$ to $\Sigma$ in $C$. The condition for $C$ to have $SU(2)$ holonomy implies that two out of the five scalars become one-forms on $\Sigma$, while the rest are the ordinary functions. Upon reduction to four dimensions the twisted scalars give rise to $2g$ scalars, where $g$ is the genus of the Riemann surface $\Sigma$. After reduction on
Σ, the tensor field $B^+$ gives 2g gauge fields which include pairs of electric-magnetic duals. More precisely, if $A^i, B_i$ is a basis for $H_1(Σ)$ then the two gauge-fields,

$$A_i = \int_{A^i} B^+, \quad \tilde{A}^i = \int_{B_i} B^+, \quad (2.6)$$

are electric-magnetic duals. The three extra scalars which are sections of the trivial part of the normal bundle together with the integral of the two-form $B^+$ over Σ form a neutral hypermultiplet, which decouples from the rest of the fields. In the D3-D7 picture, it describes the relative motion of the D3-brane in the directions within the D7-brane worldvolume. Moreover the periodicity of the scalar coming from the $B^+$ field has to do with the fact that in order to map the D3-D7 picture to M theory, we had to compactify one of the directions along the D7-brane on a circle.

The Riemann surface Σ has a meromorphic one-differential $λ$, which is induced from the ambient space hyperKähler structure: $λ = d^{-1}\omega_c|_Σ$. Here $\omega_c$ is a holomorphic symplectic form on $C$. The five-brane theory contains self-dual strings which are the boundaries of M theory membranes. By wrapping such a string along a cycle $σ ∈ H_1(Σ)$, we get a particle $P_σ$ in the effective four-dimensional theory. The mass of the particle $P_σ$ is given by the area of the membrane which has a curve in the homology class $σ$ as a boundary. Arguments similar to [9] show that such a minimal membrane has to be holomorphic in one of the complex structures of $C$. The area of a holomorphic curve can be computed as an integral of the Kähler form. The hyperKähler manifold $C$ has a two-sphere worth of complex structures. In the complex structure $u ∈ C \cup \{∞\}$, the $(1, 1)$ Kähler form can be written as:

$$ω_r(u) = \frac{1 - |u|^2}{1 + |u|^2} \omega_r + \frac{iu}{1 + |u|^2} \omega_c + \frac{-iu}{1 + |u|^2} \omega_{\bar{c}}, \quad (2.7)$$

$\omega_{\bar{c}} \equiv \bar{ω}_c$. Correspondingly, the $(2, 0)$ form in this complex structure is given by:

$$ω_c(u) = \frac{1}{1 + |u|^2} \omega_c - \frac{2\bar{u}}{1 + |u|^2} ω_r + \frac{\bar{u}^2}{1 + |u|^2} ω_{\bar{c}}. \quad (2.8)$$

Now given a two-surface $Σ_σ$ which is holomorphic in the complex structure $u$, we may write:

$$\text{Area} \ Σ_σ = \int_{Σ_σ} ω_r(u) = \frac{1 - |u|^2}{1 + |u|^2} A_r + \frac{iu}{1 + |u|^2} A_c + \frac{-iu}{1 + |u|^2} A_{\bar{c}},$$

where $A_{r,c,\bar{c}} = \int_{Σ_σ} ω_{r,c,\bar{c}}$. Due to holomorphicity of $Σ_σ$ we have:

$$0 = \int_{Σ_σ} ω_c(u) = \frac{1}{1 + |u|^2} A_c + \frac{2iu}{1 + |u|^2} A_r + \frac{u^2}{1 + |u|^2} A_{\bar{c}},$$

7
hence
\[ \text{Area } \Sigma_\sigma = \sqrt{A_r^2 + |A_c|^2} \]

Now consider varying the surface \( \Sigma_\sigma \) keeping the homology class of the boundary fixed. We get
\[ \delta A_c = \int_{\delta \partial \Sigma_\sigma} \omega_c = 0, \]
since the boundary is bound to lie on the fiber which is holomorphic in the original complex structure corresponding to \( u = 0 \) and moreover Lagrangian with respect to \( \omega_c \). Hence we cannot change \( A_c \) and in order to minimize the area, we must make \( A_r \) as small as possible. The absolute minima would correspond to \( A_r = 0 \), which means that \( \Sigma_\sigma \) is holomorphic in the complex structure \( u \) with \( |u| = 1 \),
\[ u = i \frac{A_c}{|A_c|}. \quad (2.9) \]

We can compare this description to that of self-dual string theory where the BPS states are represented by the geodesics in the metric \( ds^2 = |\lambda|^2 \) in the given homology class:
\[ M_{P_\sigma} \geq |\oint_{\sigma} \lambda| \quad \text{(2.10)} \]
The particle \( P_\sigma \) is charged under the \( U(1) \) gauge symmetry corresponding to \( \sigma \). This follows from the fact that the self-dual string is charged under \( B^+ \). Particles which satisfy the bound (2.10) for a given choice of charges correspond to BPS states. Therefore the BPS counting problem, at least in the cases where the minimal surfaces have \( A_r = 0 \), is equivalent to that in the theory of self-dual strings. Note that the solutions of (2.10) can have moduli. In this case, the moduli space needs to be described and quantum mechanics on this space will determine the degeneracy of BPS states.

The relation between the picture of BPS states as geodesics on the \( u \)-plane and that of geodesics on the Riemann surface \( \Sigma_u \) becomes clear once we lift both pictures to M theory. Then we are really studying minimal holomorphic surfaces representing the configurations of membranes. In the limit where the fiber Kähler class is small, we have a good projection of the membrane onto the \( u \)-plane giving an open string geodesic. By taking the boundary of the membrane on \( \Sigma \), we obtain self-dual strings.
2.4. A selection rule

To derive the selection rule, we need to consider a three-string junction in the neighborhood of a D7-brane. The configuration is displayed in figure 2.1af. We need to know what choices of \( r \) and \((n, m)\) give BPS string configurations, where \( n \) and \( m \) are relatively prime. To determine the selection rules, we perform the lift to M theory as described above with no D3-brane. We will keep \( R \) large so the size of the M theory torus is small and we have a well-defined projection onto the \( u \)-plane. If there were no \((n, m)\) string as in figure 2.1b, then the state certainly exists since a fundamental string can end on a D7-brane. This configuration alone lifts to a holomorphic curve in the total space. To check if the configuration with the \((n, m)\) string lifts to a holomorphic curve, we can compute the intersection number of the curve for the fundamental string with the curve for the junction. If the intersection number is negative then the curve for the junction cannot be holomorphic because the intersection number of two holomorphic surfaces is non-negative.1

\[ \text{Fig. 2.1: Two cycles intersecting.} \]

To compute the intersection number, we deform the cycle corresponding to the fundamental string as in figure 2.1c. This deformation does not change the intersection number. There are two contributions \( i_A \) and \( i_B \) to the intersection number from points \( A \) and \( B \). In general, the intersection number \( i \) of an \((n, m)\) string with a \((q, p)\) string lifted to M theory is given by the product of the intersection number of the strings with the intersection number of the corresponding cycles in the torus. For the case shown in figure 2.2:

1 The self-intersection number of a single curve can be negative even if the curve is holomorphic. For example, a holomorphic two-sphere in \( K3 \) has self-intersection number -2.
Fig. 2.2: Two strings intersecting.

\[ i = pn - qm. \] (2.11)

From point \( A \), we therefore obtain:

\[ i_A = rm. \] (2.12)

From point \( B \), we obtain half the self-intersection number of the cycle for the fundamental string,

\[ i_B = -r^2, \] (2.13)

and we require:

\[ -r^2 + mr \geq 0. \] (2.14)

It follows from (2.9) that the junction and the fundamental string lift to membranes which are holomorphic in the same complex structure; hence the argument of positivity of the intersection index applies. This gives a selection rule for three-string junctions with a leg terminating on a seven-brane.

There is a second way to see this selection rule. If the string junction is deformed as drawn in figure 2.3 using the monodromy,

\[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix},
\]

around a D7-brane then the configuration is no longer BPS unless (2.14) is satisfied. To see this, note that the orientation of the fundamental string in the transformed configuration should be the same as in the original configuration which implies that the sign of \( m + r \) must be opposite to the sign of \( r \). This gives the selection rule (2.14).
2.5. Multi-string junctions and moduli

In the case of string junctions with maximal supersymmetry where there is only a flat metric on the $u$-plane, there are actually moduli for the junction. The simplest configuration is drawn in figure 2.4, where the size of the triangle represents the modulus. When the size of the triangle goes to zero, we recover our usual three-string junction. These moduli would appear, for example, in the string configurations discussed in [31,32] that describe 1/4 BPS states in N=4 Yang-Mills. In these cases, quantum mechanics on the moduli space will give the multiplicity of BPS states, and so the structure of the moduli space needs to be determined.
In general, the moduli space of a given junction consists of all string networks that have the same external legs. This can include gluing in a large, but finite number of triangles. Let us restrict to the subspace of the moduli space where we blow up the vertex of the three-string junction into a single triangle. This is the case displayed in figure 2.4. The external charges are fixed and obey the usual constraint:

$$\sum \vec{p}_i = 0,$$

where we label a \((p, q)\) string by the vector \(\vec{p}\). There are a discrete number of choices for \(\vec{m}\) and these choices parametrize the components of the moduli space. The allowed values of \(\vec{m}\) are restricted in that \(\vec{m}, \vec{m} - \vec{p}_2\) and \(\vec{m} + \vec{p}_1\) must form a closed triangle. This amounts to the statement that these vectors satisfy a linear relation with positive coefficients. This condition is geometrically realized in figure 2.5 by the constraint that the vector \(\vec{m}\) should lie inside the triangle formed by the external vectors \(\vec{p}\):

![Fig. 2.5: Condition on the charge of intermediate leg.](image)

This tells us that the dimension of the complete moduli space, including all possible internal triangles, is given by the number of integral points inside the triangle constructed from the external vectors. Of course, the moduli for junctions with more than three legs are described in an analogous fashion. Similar observations have been made in the study of five-dimensional theories \[33,34\].

2.6. Higher genus curves

We can derive and generalize these results by considering the lift of the string configurations to M theory. Consider the space \(\mathbb{C}^* \times \mathbb{C}^*\) with coordinates \((w, v)\) and let us view it as a torus fibration over the base \(\mathcal{U} = \mathbb{R}^2:\)

\((w, v) \mapsto (x = \log|w|, y = \log|v|).\)
Consider the holomorphic curve \( C \) described by the equation

\[
\sum_{(m,n) \in \Delta C} t_{m,n} w^m v^n = 0, \tag{2.15}
\]

where \( \Delta C \subset \mathbb{Z}^2 \) is just a set of exponents \((m, n)\) for which \( t_{m,n} \neq 0 \). For a given point \( p = (x, y) \in \mathcal{U} \), the intersection of \( C \) with the fiber \( T^2 \) over \( p \) generically consists of a finite set of points.

Let us study the asymptotic behavior of \( C \) along the trajectories of the \( \mathbb{C}^* \) action labelled by a pair of mutually prime integers \((\nu, \lambda)\):

\[
(w, v) = (\mu^\lambda w_0, \mu^\nu v_0), \quad \mu \in \mathbb{C}^*.
\]

In the limit \( \mu \to 0 \), only those exponents \((m, n)\) in (2.15) will be important for which \(- (m\lambda + n\nu)\) is maximal. If the set

\[
L_{\lambda, \nu} = \{(m, n) \mid - (m\lambda + n\nu) \text{ maximal}\}
\]

contains more than one point then \( C \) has an asymptotic component pointing in the \((\lambda, \nu)\) direction in the \( \mathcal{U} \) plane. In fact, we may write down the equation for this component quite explicitly. Let \( a = -(m\lambda + n\nu) \) be that maximal value. We have: \( m = m'a, n = n'a \), for \((m', n')\) mutually prime. The components corresponding to \((\nu, \lambda)\) are labelled by the points \( \xi = w^{\nu}/v^{\lambda} \) in the weighted projective space \( \mathbb{P}^1_{\nu, \lambda} \) which solve the equation:

\[
\sum_{(m', n') \in \frac{1}{a} L_{\lambda, \nu}} t_{m'a, n'a} \xi^{m'a/\nu} = 0 \tag{2.16}
\]

which describes an array of membranes which are wound around a \((p = \lambda, q = \nu)\) cycle of the fiber two-torus. The equation (2.16) implies that \( \xi \) may assume only a finite number of values. For fixed \( \xi \) one gets a cylinder of precisely described form. Its projection to the \( \mathcal{U} \) plane is a line, pointing in the \((p, q)\) direction. All such lines will be parallel to each other, but need not be coincident.

The asymptotic behavior of the curve \( C \) is not changed if we add or remove a point \((m, n)\) in the interior of \( \Delta C \). Therefore the total number of moduli is given by the number of integral points inside the convex hull of \( \Delta C \). This is pictorially illustrated in figure 2.6. Notice that junctions corresponding to non-convex polygons do not exist. A related discussion appeared in [35]. Also note that we have been discussing the case of maximal
supersymmetry. For the situations that we are primarily studying, we have less supersymmetry and these moduli can be lifted. In section three, we will see cases of junctions smoothly connected to single string configurations which cannot therefore have any moduli.

Fig. 2.6: Polyhedron associated to a multi-string junction.

3. BPS states in $SU(2)$ Yang-Mills

We will apply the preceding discussion to the case of $SU(2)$ Yang-Mills without matter. This is the simplest situation where non-trivial strong-coupling dynamics changes the BPS spectrum. The case is easily constructed as a probe theory by placing the probe in the vicinity of a deformed $D_4$ singularity and taking the four mutually local D7-branes to infinity [13,14]. This leaves two $(p, q)$ seven-branes with $(p, q)$ charges corresponding to the two strong coupling singularities in the Seiberg-Witten moduli space. We will normalize our charges so that a $W$-boson has charge $\pm 2$. Then the charges for the two singularities correspond to $(\pm 2, 1)$ and $(0, 1)$ depending on how we approach the singularity.

The moduli space has a curve of marginal stability (CMS) running through both singularities shown in figure 3.1. Let us use the conventions for the branch cuts given in the second paper of reference [5]:

14
The monodromies around $u = -1, 1, \infty$ are given by:

$$
M_{\infty} = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad M'_{-1} = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}.
$$

(3.1)

In general, the monodromy around a $(p, q)$ seven-brane will be given by,

$$
M_{p,q} = \begin{pmatrix} 1 + pq & p^2 \\ -q^2 & 1 - pq \end{pmatrix}.
$$

(3.2)

The only particles that should exist in the strong coupling regime, labelled region II in figure 3.1, are the particles which become light at the two singularities. That consistency of the proposed vacuum structure requires all other states to decay when crossing the CMS is easily seen from the monodromies in (3.1). Otherwise, by taking a semi-classical dyon around one of the two singularities, we could generate a BPS state which does not exist in the semi-classical region [4].

If we place our D3-brane probe at any point $u_0$ on the $u$-plane then we can construct two canonical geodesics going from the probe to either the monopole or dyon singularity. These are simply straight lines in the $w_{p,q}$ plane as we discussed in the previous section. In addition, there can be another single string geodesic which crosses the branch cut shown in figure 3.1 going from the dyon point to infinity. On crossing, the charge of the string is acted on by the appropriate monodromy matrix. If the resulting string has the same charge as either singularity, it can continue on and terminate at the appropriate singularity. Recall that the initial trajectory of a $(p, q)$ string from the point $u_0$ is fixed by the charge of string. A case where there are two single string geodesics is shown in figure 3.2; the existence of this second geodesic can be verified either by numerical integration or analytically.
Fig. 3.2: Two single string BPS states.

If we move $u_0$ in figure 3.2 to the right, we need to drag the string ending on the $(0, 1)$ seven-brane through the other seven-brane. The D3-brane is far from the CMS so the BPS state cannot decay. Rather it splits into a three-string junction drawn in figure 3.3.

Fig. 3.3: A BPS three-string junction.

Under what conditions on $u_c$ is the junction BPS? Let us consider a junction with charge $\vec{p}_1$ on one leg and charge $\vec{p}_2$ on the other leg so that the final leg has charge $\vec{p} = \vec{p}_1 + \vec{p}_2$. Then the mass of the state is given by:

$$m(u_0) = |w_{\vec{p}}(u_0)|$$
$$= |w_{\vec{p}}(u_0) - w_{\vec{p}}(u_c) + w_{\vec{p}_1}(u_c) + w_{\vec{p}_2}(u_c)|$$
$$\leq |w_{\vec{p}}(u_0) - w_{\vec{p}}(u_c)| + |w_{\vec{p}_1}(u_c)| + |w_{\vec{p}_2}(u_c)|. \quad (3.3)$$

To satisfy the inequality, we require that

$$\text{Arg} \left[ w_{\vec{p}}(u_0) - w_{\vec{p}}(u_c) \right] = \text{Arg} \left[ w_{\vec{p}_1}(u_c) \right] = \text{Arg} \left[ w_{\vec{p}_2}(u_c) \right], \quad (3.4)$$
which forces the point \( u_c \) to lie on the CMS. This is rather beautiful and completely in accord with our expectations. If we bring the four local D7-branes that we sent to infinity back to the origin, the metric on the moduli space becomes flat and the 6 seven-branes merge back into an orientifold plane. As we expect, in this limit the string junction reduces to a \((p, q)\) string stretching from the D3-brane to the orientifold plane. If we place our probe inside the CMS, we can have no BPS string configurations except single string configurations. The only geodesics correspond to the two canonical ones and we recover the expected non-analytic behavior of the BPS spectrum.

So far, we have discussed the string junctions that arise by taking a single string configuration around infinity some number of times. This gives us a realization of all dyon states with charge \((2n, \pm 1)\) where \( n \in \mathbb{Z} \) in terms of string configurations. At first sight, it might seem that these junctions have moduli but since they are continuously connected to a single string configuration, it seems unlikely that any moduli actually exist. All these junctions necessarily exist but we can construct more junctions that satisfy (3.3), but do not correspond to BPS states in pure \( SU(2) \) Yang-Mills. The condition of satisfying (3.3) is clearly necessary but not sufficient.

To show that these extra states are actually not BPS, we will apply a generalization of the selection rule from section 2.2. Note that we have all dyons with magnetic charge \( \pm 1 \). Suppose that we have some dyon with higher magnetic charge. We denote the central charge for this dyon by

\[
x(2, 1) + y(0, -1),
\]

where \( x \) and \( y \) have to be of the same sign for the state to be BPS. Without any loss of generality, we may suppose that \( x \) and \( y \) are both positive and coprime. We can then compute the intersection number of the curve for this hypothetical dyon with the curve for some known dyon with central charge

\[
s(2, 1) + t(0, -1)
\]

where \( t = s \pm 1 \).

After deforming the configuration as shown in figure 3.4, we compute the intersection index

\[
i = -(s - t)(x - y) + |sy - tx|.
\]

(3.5)

\footnote{The case shown corresponds to \( \frac{s}{y} < \frac{x}{t} \). In the opposite case we should deform the \( s, t \) cycle in the opposite direction.}
The formal rules for computing the intersection index of two configurations with the same phase for their respective central charges are the following: if they have legs ending on the same seven-brane, with labels $x$ and $s$, then we get a $-xs$ contribution (this is always negative). If two legs intersect, then the contribution is computed as in (2.11) (and this is always positive).

Notice that this intersection index does not change when we change the phase of the central charge, even if we cross the position of the seven-brane and the direction of some leg is changed. This is what we expect since in M theory language, this should correspond to a smooth deformation of membranes. For example, consider moving the junction in figure 3.4 to the right until it crosses the monopole singularity. The labels of the states are changed according to the monodromy transformation. Explicitly, we have

\[
x(2, 1) + y(0, -1) \rightarrow x(2, 1) + (y - 2x)(0, -1)
\]

\[
s(2, 1) + t(0, -1) \rightarrow s(2, 1) + (t - 2s)(0, -1)
\]

(3.6)

Also, after we cross $u = 1$, the intersecting legs will be not $(s, y)$ as before, but $(\tilde{x}, \tilde{t})$. Our rules for the intersection number then give us:

\[-|sx| - |(t - 2s)(y - 2x)| + |2x(t - 2s)| = -sx - ty + 2sy\]

which coincides with what we would have obtained from (3.5). Thus, the intersection pairing may be formally viewed as giving us an invariant of the monodromy transformations.

Suppose that $x = y + b$ where $b > 1$ (the case $b < -1$ can be considered in an analogous way). Then take $s = t + 1$. We can then compute the intersection number:

\[i = -b + |y - tb|,\]

and we can always choose $t = \left\lfloor \frac{y}{b} \right\rfloor$ to get $i < 0$. This proves that given the cycle corresponding to the dyon with higher magnetic charge, we can always find a holomorphic cycle whose intersection with the given cycle is negative. This implies that we cannot realize it as a holomorphic curve.\footnote{If $b = 1$, we would have to take $t = y$ to get $i = -1$, and this would imply that our two cycles are actually the same. That is, we are computing the self-intersection number which does not have to be nonnegative. Thus our argument does not rule out $b = 1$.} At first, we might worry that this argument is faulty because the requirement that the intersection number be non-negative is only true if we demand that both curves are holomorphic in the same complex structure. However, in these cases,
the direction of two legs for both junctions agree and this determines the complex structure in which their respective membranes must be holomorphic. These complex structures coincide so we can apply the intersection rule.

The above argument does not rule out the case \( b = 0 \) because \( i \) given in (3.5) is always non-negative for \( x = y \). This case corresponds to a string junction representing a vector boson. The string leaving the D3-brane has charge \((\pm 2, 0)\) and so represents two fundamental strings which must coincide on the \( u \)-plane by the BPS condition. The lift of this junction has a single complex modulus and there is a single constraint coming from the known projection onto the \( u \)-plane. This leaves a single real modulus in the M theory picture which should be related to the modulus found in \[9,6\].

![Intersection of the cycle in question with an existing cycle.](image)

**Fig. 3.4:** Intersection of the cycle in question with an existing cycle.

The existence of the modulus for the vector boson may also be seen by computing the self-intersection number. Indeed, for the configuration with charge \( x(2, 1) + y(0, -1) \), the self-intersection number is \(-(x - y)^2\); this gives 0 for the vector multiplet \((x = y = 1)\) and \(-1\) for monopoles and dyons \((x = y \pm 1)\). This suggests that hypermultiplets are represented by disks, while the vector multiplet is represented by a cylinder. A membrane with the topology of a cylinder has one real modulus, while the disk does not have any moduli. This picture is very different from what we would expect for the infinite string junction in flat space. According to our description of moduli in section 2.5, we would naively get approximately \( \frac{1}{2}n^2 \) moduli for the dyon with electric charge \( n \). It turns out that a curved metric together with the condition that the membrane end on a given fiber lifts all these moduli.
This method of computing the BPS spectrum of N=2 theories should generalize to the case of multiple D3-brane probes which has been considered in \cite{36,37}. It seems likely that the picture of geodesics on the moduli space of N=2 field theories is actually more general and might well give a universal way of computing the BPS spectrum, even for cases where no probe realization is known.

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