Minimax Rényi Redundancy

Semih Yagli, Student Member, IEEE, Yücel Altuğ, and Sergio Verdú, Fellow, IEEE

Abstract

The redundancy for universal lossless compression of discrete memoryless sources in Campbell’s setting is characterized as a minimax Rényi divergence, which is shown to be equal to the maximal $\alpha$-mutual information via a generalized redundancy-capacity theorem. Special attention is placed on the analysis of the asymptotics of minimax Rényi divergence, which is determined up to a term vanishing in blocklength.

Keywords: Universal lossless compression, generalized redundancy-capacity theorem, minimax redundancy, minimax regret, Jeffreys’ prior, risk aversion, Rényi divergence, $\alpha$-mutual information.

I. INTRODUCTION

In variable length source coding, expected code length is the usual cost function that one aims to minimize. For discrete memoryless sources, asymptotically, the minimal achievable per-letter expected code length is equal to the entropy. However, if $P_{Y^n|V=\theta}$ is a discrete memoryless source distribution with an unknown parameter $\theta$ and the encoding system assumes a distribution $Q_{Y^n}$, then one needs to pay an extra penalty for the mismatch given by

$$\frac{1}{n} D(P_{Y^n|V=\theta}||Q_{Y^n}) + o(1).$$

(1)

where $D(P||Q)$ stands for the relative entropy between the probability measures $P$ and $Q$. In light of (1), the conventional worst-case measure of redundancy in universal lossless compression is

$$R_n = \inf_{Q_{Y^n}} \sup_\theta D(P_{Y^n|V=\theta}||Q_{Y^n}),$$

(2)

where the infimization is over all distributions on $Y^n$, and the supremum is over all possible values of the unknown parameter. In this zero-sum game, $Q_{Y^n}$ is chosen by the code designer, and $\theta$ is chosen by nature.

Semih Yagli and Sergio Verdú are with the Electrical Engineering Department, Princeton University, Princeton, NJ 08544. Yücel Altuğ is with Natera Inc., San Carlos, CA 94070. E-mail: syagli@princeton.edu, yucelaltug@gmail.com, verdu@princeton.edu. Part of this paper was presented at 2017 IEEE International Symposium on Information Theory [1].

This work has been supported by ARO-MURI contract number W911NF-15-1-0479 and in part by the Center for Science of Information, an NSF Science and Technology Center under Grant CCF-0939370.

1For prefix codes, (1) is well known [2, Theorem 5.4.3]. On the other hand, the loss in rate incurred due to the prefix condition is known to be asymptotically negligible [3].
A relation between \( R_n \) and the maximal mutual information is given by the Redundancy-Capacity Theorem (e.g., [4], and [5]) that states that

\[
R_n = \sup_{P_V} I(P_V, P_{Y^n|V}),
\]

where\(^2\) the supremization is over all possible probability distributions on the parameter space. Through (1), (2) and (3), we see a pleasing relationship between entropy, relative entropy and mutual information in the context of lossless data compression.

Let \( Y^n \sim P_{Y^n|V=\theta} \), and note that

\[
D(P_{Y^n|V=\theta}\|Q_{Y^n}) = \mathbb{E} \left[ \sum_{y^n} I_{P_{Y^n|V=\theta}\|Q_{Y^n}}(y^n) \right],
\]

where the relative information between the discrete probability measures \( P \) and \( Q \) is defined as\(^3\)

\[
I_{P\|Q}(a) = \log \frac{P(a)}{Q(a)}.
\]

A much more stringent performance guarantee than the average of relative information is its pointwise maximum. In particular, if one replaces \( \mathbb{E} \left[ \sum_{y^n} I_{P_{Y^n|V=\theta}\|Q_{Y^n}}(y^n) \right] \) with \( \max_{y^n} \sum_{y^n} I_{P_{Y^n|V=\theta}\|Q_{Y^n}}(y^n) \) in (2), the resulting quantity, i.e.,

\[
r_n = \inf_{Q_{Y^n}} \sup_{\theta} \max_{y^n \in Y^n} I_{P_{Y^n|V=\theta}\|Q_{Y^n}}(y^n),
\]

is called the minimax regret, which has found applications in various settings\(^4\), e.g., [6]–[10]. An analogy to the Redundancy-Capacity Theorem is given by [7]

\[
r_n = \log \sum_{y^n \in Y^n} \sup_{\theta} P_{Y^n|V=\theta}(y^n)
\]

\[
= \sup_{P_V} I_{\infty}(P_V, P_{Y^n|V}),
\]

where \( I_{\infty}(P_X, P_{Y^n|X}) \) denotes the \( \alpha \)-mutual information of infinite order, whose definition is given in (40).

The average and pointwise formulations are two extremes of performance guarantees, which are not quite suitable for certain applications. For this reason, one seeks a compromise between those two. For example, in the economics literature, average and pointwise guarantees are referred as risk-neutral and risk-avoiding, respectively. Since the former is known to be too lenient and the latter is known to be too stringent for typical applications, the notion of risk-aversion has been introduced to provide a more useful compromise between these two extremes [11], [12], which is known to be relevant for diverse applications [13]. In this paper, we introduce the notion of risk-aversion within the universal source coding context and we quantify its effect on the fundamental limit.

\(^2\) \( I(P_X, P_{Y^n|X}) = D(P_{Y^n|X}\|P_{Y|X}) \) is the mutual information between \( X \) and \( Y \) with \( (X, Y) \sim P_X P_{Y|X} \).

\(^3\) Unless otherwise stated, logarithms and exponentials are of arbitrary basis throughout this paper.

\(^4\) For example, in lossless compression with prefix codes, \( I_{P_{Y^n|V=\theta}\|Q_{Y^n}}(y^n) \) is often viewed as a proxy for the mismatch penalty incurred by assuming that \( y^n \) is drawn from \( Q_{Y^n} \) rather than the true distribution \( P_{Y^n|V=\theta} \). Such an approximation can be justified asymptotically.
In the non-universal setting, i.e., when the source distribution is known, a classical result of Campbell [14] introduces such a risk-averse cost function in a discrete memoryless setting. Specifically, [14] proposes to generalize the conventional notion of minimizing the expected code length with the cost function

\[ L_\lambda(Y^n) = \frac{1}{\lambda} \log \mathbb{E} \left[ \exp(\lambda \ell(f(Y^n))) \right], \quad (9) \]

where \( \lambda \in (0, \infty) \), \( f \) denotes the code, and \( \ell(\cdot) \) denotes the length function. In this case, for a discrete memoryless source \( Y^n \), Campbell [14] shows that the minimum per-letter cost asymptotically achievable by prefix codes is given by the \( \text{Rényi entropy} \) \( H_{1+\lambda}(Y^n) \). Notice that \( L_\lambda(Y^n) \) captures the notion of risk-aversion through the parameter \( \lambda \) since

\[ L_\lambda(Y^n) \xrightarrow{\lambda \to 0} \mathbb{E} [\ell(f(Y^n))], \quad (10) \]

\[ L_\lambda(Y^n) \xrightarrow{\lambda \to \infty} \max_{y^n \in Y^n} \ell(f(y^n)). \quad (11) \]

A natural way to introduce risk-aversion in universal source coding is to use Campbell’s formulation and characterize the penalty for the mismatch akin to (1). Indeed, about forty years after Campbell’s work, Sundaresan [15, Theorem 8] showed that if one uses \( L_\lambda(Y^n) \) as the cost function, the penalty paid for universality can be written as

\[ \frac{1}{n} D_{1+\lambda}(\tilde{P}_{Y^n|V=\theta}^{1+\lambda} \parallel \tilde{Q}_{Y^n}^{1+\lambda}) + o(1), \quad (12) \]

where \( D_{1+\lambda}(P||Q) \) denotes the \( \text{Rényi divergence} \) of order \( 1 + \lambda \) (defined in (38)), and \( \tilde{P}_{Y}^{\alpha} \) denotes the \textit{scaled distribution} of \( P_Y \):

\[ \tilde{P}_{Y}^{\alpha}(y) = \frac{P_Y^{\alpha}(y)}{\sum_{b \in Y} P_Y^{\alpha}(b)}. \quad (13) \]

The distance measure

\[ S_\alpha(P||Q) = D_\alpha(\tilde{P}_{Y}^{1/2} \parallel \tilde{Q}_{Y}^{1/2}) \]

is known as the \textit{Sundaresan divergence} of order \( \alpha \) between \( P \) and \( Q \). Following [15], the relevant measure of redundancy for universal lossless compression under Campbell’s performance criterion is

\[ R_{\lambda}(n) = \inf_{Q_{Y^n}} \sup_{\theta} S_{1+\lambda}(P_{Y^n|V=\theta} \parallel Q_{Y^n}). \quad (15) \]

The conventional minimax redundancy in (2) corresponds to \( R_0(n) \) while the minimax regret in (6) corresponds to \( R_{\infty}(n) \). Although, in general, \( S_\alpha(P||Q) \neq D_\alpha(P||Q) \), we are able to establish a pleasing analog to the classical redundancy results such as (2), (3) and (6), (8):

\[ R_{\lambda}(n) = \inf_{Q_{Y^n}} \sup_{\theta} D_{1+\lambda}(P_{Y^n|V=\theta} \parallel Q_{Y^n}) \]

\( ^5 \)Campbell’s and Sundaresan’s results are still valid when \( \lambda \in (-1, 0) \). However, such a formulation corresponds to a \textit{risk-seeking} scheme, which falls outside the philosophy espoused in this paper.
where in (17)
\[
I_{1+\lambda}(P_X, P_Y|X) = \inf_{Q_Y} D_{1+\lambda}(P_Y|X P_X \parallel Q_Y P_X)
\] (18)
is the \(\alpha\)-mutual information of order \(1 + \lambda\) between \(X\) and \(Y\) with \((X, Y) \sim P_X P_Y|X\), see [16], [17]. Note that (16) is analogous to (2) with Rényi divergence replacing the relative entropy. Thus, we refer \(R_{\lambda}(n)\) as the minimax Rényi redundancy. Moreover, (17) generalizes the Redundancy-Capacity Theorem to \(\alpha\)-mutual information thereby finding another operational meaning for the maximal \(\alpha\)-mutual information beyond those that have been shown in the literature on error probability bounds for data transmission (e.g. [16], [18]). Moreover, the \(\alpha\)-mutual information smoothly interpolates between two extremes, namely \(I(P_V, P_{Y|V})\) in (3) and \(I_{\infty}(P_V, P_{Y|V})\) in (8).

Finally, (16) and (17), coupled with Campbell’s result [14], provide a pleasing relationship between Rényi entropy, Rényi divergence and \(\alpha\)-mutual information in the context of universal lossless data compression.

The asymptotic behaviors of the minimax redundancy and minimax regret have also received considerable attention in the literature (e.g., [6], [7], [9], [19]–[24]) since, in addition to compression, they are relevant in applications such as machine learning, finance, prediction, gambling, and so on. In particular, Xie and Barron in their key contributions [19], [6] show that
\[
R_n = R_0(n) = \frac{k - 1}{2} \log \frac{n}{2\pi} + \log \frac{\Gamma^k(1/2)}{\Gamma(k/2)} - \frac{k - 1}{2} \log e + o(1),
\] (19)
\[
r_n = R_\infty(n) = \frac{k - 1}{2} \log \frac{n}{2\pi} + \log \frac{\Gamma^k(1/2)}{\Gamma(k/2)} + o(1),
\] (20)
where \(n\) and \(k\) are the number of observations and the alphabet size, respectively, and \(o(1)\) vanishes as \(n \to \infty\).

While Merhav [25, Theorem 1] gives \(R_{\lambda}(n) = \frac{k - 1}{2} \log n + o(\log n)\), we quantify asymptotically the effect of the risk-aversion parameter \(\lambda\) on the fundamental limit in universal source coding by providing a pleasing interpolation\(^6\) between (19) and (20):
\[
R_{\lambda}(n) = \frac{k - 1}{2} \log \frac{n}{2\pi} + \log \frac{\Gamma^k(1/2)}{\Gamma(k/2)} - \frac{k - 1}{2\lambda} \log (1 + \lambda) + o(1).
\] (21)

In the remainder of the paper, Section II sets the basic notation and definitions. Section III states the main results and gives the outlines of their proofs, which are contained in Section IV. In the Appendices, we prove several lemmas that are used in Section IV.

II. NOTATION AND DEFINITIONS

Let \(\mathcal{Y} = \{1, 2, \ldots, k\}\) and denote the \((k - 1)\)-dimensional simplex of probability mass functions defined on \(\mathcal{Y}\) by
\[
\Delta^{k-1} = \left\{(\theta_1, \ldots, \theta_k) \in \mathbb{R}_+^k: \sum_{i=1}^k \theta_i = 1 \right\}.
\] (22)

\(^6\)In a fundamentally different setup, Hayashi [26, Lemma 3] considers the counterpart of the Clarke and Barron [27, Theorem 2.1] result replacing relative entropy with Rényi divergence.
For each parameter $\theta = (\theta_1, \ldots, \theta_k) \in \Delta^{k-1}$, we define our observation model $P_{Y|V=\theta}: \Delta^{k-1} \rightarrow \mathcal{Y}$ such that

$$P_{Y|V=\theta}(i) = \theta_i,$$  

and the independent identically distributed (i.i.d.) extension of this model $P_{Y^n|V=\theta}: \Delta^{k-1} \rightarrow \mathcal{Y}^n$ such that

$$P_{Y^n|V=\theta}(y^n) = \prod_{i=1}^n P_{Y|V=\theta}(y_i)$$

$$= \theta_1^{t_1} \ldots \theta_k^{t_k},$$

where

$$t_i = \sum_{j=1}^n 1\{y_j = i\},$$

denotes the number of times $i \in \mathcal{Y}$ appears in the vector $y^n$, and therefore

$$\sum_{i=1}^k t_i = n.$$  

It can be verified that the Fisher information matrix (in nats) of $P_{Y|V=\theta}$ for the parameter vector $\theta$ is

$$J(\theta, P_{Y|V}) = \text{diag} \left( \frac{1}{\theta_1}, \frac{1}{\theta_2}, \ldots, \frac{1}{\theta_{k-1}} \right) + \frac{1}{\theta_k} \mathbf{1}_{(k-1) \times (k-1)},$$

where $\mathbf{1}_{l \times l}$ denotes an $l \times l$ matrix all of whose entries are equal to 1. The determinant of the Fisher information matrix in (28) satisfies

$$|J(\theta, P_{Y|V})| = \frac{1}{\prod_{i=1}^k \theta_i}. $$

An important probability measure on $\Delta^{k-1}$ is Jeffreys’ prior [28] defined as

$$P^\star_V(\theta) = \frac{|J(\theta, P_{Y|V})|^{1/2}}{\int_{\Delta^{k-1}} |J(\xi, P_{Y|V})|^{1/2} d\xi}$$

$$= \frac{\theta_1^{-1/2} \cdots \theta_{k-1}^{-1/2}}{D_k(1/2, \ldots, 1/2)},$$

where $D_k(\alpha_1, \ldots, \alpha_k)$ denotes a special form of the Dirichlet integrals of type 1 which can be written in terms of the Gamma function:

$$D_k(\alpha_1, \ldots, \alpha_k) = \int_{\Delta^{k-1}} \xi_1^{\alpha_1-1} \cdots \xi_k^{\alpha_k-1} d\xi$$

$$= \frac{\Gamma(\alpha_1) \cdots \Gamma(\alpha_k)}{\Gamma(\alpha_1 + \cdots + \alpha_k)}.$$  

As a special case, when $k = 2$, we use the short hand notation $P_{Y|V=\theta}$ instead of $P_{Y|V=(\theta, 1-\theta)}$.

Note that the Fisher information matrix is $(k - 1) \times (k - 1)$ since there are $(k - 1)$ free parameters in the model. Nevertheless, it is notationally convenient to denote the parameter vector $\theta$ as if it were $k$-dimensional.
In particular,  
\[ D_k(1/2, \ldots, 1/2) = \frac{\Gamma^k(1/2)}{\Gamma(k/2)} \]
\[ = \begin{cases} 
\pi^{k/2} / (k/2 - 1)! & \text{if } k \text{ is even} \\
\pi^{(k-1)/2} / \prod_{i=1}^{k-1} (i - 1/2) & \text{if } k \text{ is odd}
\end{cases} \] 

(34)

(35)

The source distribution we get by assuming Jeffreys’ prior on the parameter space is referred as Jeffreys’ mixture which we denote by  
\[ Q_Y^*(y^n) = \int_{\Delta_k} P_{Y^n|V=\theta}(y^n) dP_Y^*(\theta) \]
\[ = \frac{D_k(t_1 + 1/2, \ldots, t_k + 1/2)}{D_k(1/2, \ldots, 1/2)}. \] 

(36)

(37)

For discrete probability measures \( P \) and \( Q \) on the set \( Y \) such that \( Q \) dominates \( P \), i.e., \( P \ll Q \), Rényi divergence of order \( \alpha \) between \( P \) and \( Q \) is defined as  
\[ D_\alpha(P\|Q) = \begin{cases} 
D(P\|Q), & \alpha = 1 \\
\frac{1}{\alpha-1} \log \mathbb{E} [\exp((\alpha-1)\bar{\nu}_P(Q(Y)))] , & \alpha \in (1, \infty) \\
\max_{b \in Y} \bar{\nu}_P(b), & \alpha = \infty
\end{cases} \] 

(38)

where \( Y \sim P \). In particular, when \( \alpha \in (1, \infty) \), Rényi divergence of order \( \alpha \) between \( P \) and \( Q \) can be expressed as  
\[ D_\alpha(P\|Q) = \frac{1}{\alpha-1} \log \sum_{b \in Y} P^\alpha(b)Q^{1-\alpha}(b). \] 

(39)

Given \((P_X, P_{Y|X})\), an analogous generalization can be made for mutual information resulting in the \( \alpha \)-mutual information [16]:  
\[ I_\alpha(P_X, P_{Y|X}) = \begin{cases} 
I(P_X, P_{Y|X}), & \alpha = 1 \\
\inf_{Q_Y} D_\alpha(P_{Y|X} P_X\|Q_Y P_X), & \alpha \in (1, \infty) \\
\log \mathbb{E} \left[ \operatorname{ess sup}_X \exp \left( \bar{\nu}_{X,Y}(X; Y) \right) \right], & \alpha = \infty
\end{cases} \] 

(40)

where \( \bar{Y} \sim P_Y \), independent of \( X \sim P_X \), and we have used the conventional notation for information density \( \bar{\nu}_{X,Y}(x; y) = \bar{\nu}_{P_{Y|X=x}}(y) \). As shown in Lemma 1 in Appendix A, the infimum in (40) can be solved explicitly.

\(^9\)Whenever it is advisable to explicitly show the dimensionality of the parameter space in the notation for Jeffreys’ mixture, we do so by replacing \( Q_{Y^n}^* \) with \( Q_{Y^n}^{*(k-1)} \).

\(^{10}\)We are not concerned with Rényi divergences of order \( \alpha \in (0, 1) \). A more general definition can be found in [29].
In parallel with the standard usage for relative entropy, it is common to define the conditional Rényi divergence as

\[ D_\alpha(P_{Y|X}\|Q_{Y|X}|P_X) = D_\alpha(P_X P_{Y|X}\|P_X Q_{Y|X}), \] (41)

therefore, the unconditional Rényi divergence in (40) can be written as \( D_\alpha(P_{Y|X}\|Q_{Y}|P_X) \).

III. STATEMENT OF THE RESULTS

Theorem 1 states that under the minimax operation in (15) the Sundaresan divergence can be replaced by the Rényi divergence. We further show that this minimax operation can be written as the maximization of the \( \alpha \)-mutual information, thus, providing a generalization to the Redundancy-Capacity Theorem in (3). In Theorem 2, we investigate the asymptotic behavior of the minimax Rényi redundancy between \( P_{Y^n|V=\theta} \) and \( Q_{Y^n} \), and we find its precise asymptotic expansion, thereby quantifying the effect of the risk-aversion parameter \( \lambda \).

**Theorem 1** Generalized Redundancy-Capacity Theorem. For any \( \lambda \in (0, \infty) \), and positive integer \( n \)

\[
R_\lambda(n) = \inf_{Q_{Y^n}} \sup_{\theta \in \Delta^{k-1}} D_{1+\lambda}(P_{Y^n|V=\theta}\|Q_{Y^n}) \quad (42)
\]

\[
= \sup_{P_V} I_{1+\lambda}(P_V, P_{Y^n|V}). \quad (43)
\]

As we show in the proof in Section IV, (42) is due to the fact that scaling a distribution is a one-to-one operation that preserves memorylessness while the minimax theorem for Rényi divergence [29, Theorem 34] is the gateway to showing the generalized redundancy-capacity theorem in (43).

Although Theorem 1 holds in great generality, we illustrate its use in the simple example below.

**Example 1** Z-Channel with \( \frac{1}{2} \) crossover probability. Consider the Z-channel with \( \frac{1}{2} \) crossover probability, see, e.g., [2, Problem 7.8]. In this case,

\[
I_{1+\lambda}(P_V, P_{Y|V}) = \frac{1+\lambda}{\lambda} \log \left( \frac{1-P_V(0)}{Q_Y(0)} + (1 + (2^{1+\lambda} - 1)P_V(0)) \right) - \frac{1+\lambda}{\lambda} \log 2, \quad (44)
\]

which is a concave function of \( P_V(0) \) for every value of \( \lambda \in (0, \infty) \), and is maximized when

\[
P_V(0) = \frac{1 - (2^{1+\lambda} - 1)^{-\frac{1}{1+\lambda}}}{1 + (2^{1+\lambda} - 1)^{-\frac{1}{1+\lambda}}}. \quad (45)
\]

After some elementary algebra, plugging (45) into (44) yields

\[
\sup_{P_V} I_{1+\lambda}(P_V, P_{Y|V}) = \log \left( 1 + (2^{1+\lambda} - 1)^{-\frac{1}{1+\lambda}} \right). \quad (46)
\]

Observe that as \( \lambda \to 0 \), the right side of (46) converges to the capacity of the channel, namely, \( \log \frac{5}{4} \). On the other hand, to compute the minimax Rényi redundancy, note that

\[
D_{1+\lambda}(P_{Y|V=0}\|Q_Y) = \log \frac{1}{Q_Y(0)} \quad (47)
\]
\[ D_{1+\lambda}(P_{Y|V=1}||Q_Y) = \frac{1}{\lambda} \log \left( \frac{1}{Q_Y(0)} + \frac{1}{Q_Y(1)} \right) - \frac{1+\lambda}{\lambda} \log 2. \] (48)

Let \( Q_Y^* \) be the distribution such that
\[ D_{1+\lambda}(P_{Y|V=0}||Q_Y^*) = D_{1+\lambda}(P_{Y|V=1}||Q_Y^*). \] (49)

Since
\[ \inf_{Q_Y} \sup_{\theta \in \{0,1\}} D_{1+\lambda}(P_{Y|V=\theta}||Q_Y) = D_{1+\lambda}(P_{Y|V=0}||Q_Y^*) \]
\[ = \log \left( 1 + (2^{1+\lambda} - 1)^{-\frac{1}{\lambda}} \right), \] (51)
through (46) and (51), as enforced by generalized redundancy-capacity theorem, we observe that the maximal \( \alpha \)-mutual information matches the minimax Rényi divergence.

**Theorem 2** Asymptotic Behavior of Minimax Rényi Redundancy. For any \( \lambda \in (0, \infty) \)
\[ \lim_{n \to \infty} \left\{ R_{\lambda}(n) - k \frac{1}{2} \log \frac{n}{2\pi} \right\} = \log \frac{\Gamma(k/2)}{\Gamma(1/2)} - \frac{k-1}{2\lambda} \log(1 + \lambda). \] (52)

We prove Theorem 2 in Section IV by dividing it into two parts: converse and achievability. In both parts, Jeffreys’ prior plays a significant role. However, it is known that Jeffreys’ prior dramatically emphasizes the lower dimensional faces of the simplex. While this is not a problem in proving the converse bound, Jeffreys’ prior achieves a suboptimal minimax value (see Lemma 14 in Appendix J). Similar issues arise in finding the exact asymptotic constant in minimax redundancy [19], and in minimax regret [6]. To overcome this problem, we modify Jeffreys’ prior by placing masses near the faces of the simplex as in [19]. Although this resolves the problem encountered in the minimax redundancy and minimax regret cases, the functional form of Rényi divergence becomes the second obstacle which forces us to show a uniform Laplace approximation thereby making the proof of achievability a much more involved task than that of the converse. For this reason, we start by presenting the achievability proof in the special case of binary alphabets, in which the notation is simplified considerably.

**IV. PROOFS**

**A. Proof of Theorem 1**

To establish (42), for any \( \lambda \in (0, \infty) \), define the bijection \( f_\lambda : \Delta^{k-1} \to \Delta^{k-1} \) as
\[ f_\lambda(\theta_1, \ldots, \theta_k) = \frac{1}{\kappa_\lambda} \left( \theta_1^{1+\lambda}, \ldots, \theta_k^{1+\lambda} \right), \] (53)
where
\[ \kappa_\lambda = \sum_{b \in Y} P_{Y|V=\theta}^b(b). \] (54)
Then, for any \( \theta = (\theta_1, \ldots, \theta_k) \in \Delta^{k-1} \) and \( y^n \in \mathcal{Y}^n \), the scaled version of the conditional distribution (see (13)) satisfies

\[
\tilde{P}_{Y^n | \mathcal{V} = \theta}^{\frac{1}{n}}(y^n) = \prod_{i=1}^{n} \tilde{P}_{Y_i | \mathcal{V} = \theta}^{\frac{1}{n}}(y_i) = \prod_{i=1}^{n} P_{Y_i | \mathcal{V} = f_{\lambda}(\theta)}(y_i) = P_{Y^n | \mathcal{V} = f_{\lambda}(\theta)}(y^n). \tag{55}
\]

Therefore, for any given distribution \( R_{Y^n} \) on \( \mathcal{Y}^n \)

\[
\sup_{\theta \in \Delta^{k-1}} D_{1+\lambda}(\tilde{P}_{Y^n | \mathcal{V} = \theta}^{\frac{1}{n}} \| R_{Y^n}) = \sup_{\theta \in \Delta^{k-1}} D_{1+\lambda}(P_{Y^n | \mathcal{V} = f_{\lambda}(\theta)} \| R_{Y^n}) \tag{58}
\]

\[
= \sup_{\theta \in \Delta^{k-1}} D_{1+\lambda}(P_{Y^n | \mathcal{V} = \theta} \| R_{Y^n}). \tag{59}
\]

As a result of (59),

\[
\inf_{Q_{Y^n}} \sup_{\theta \in \Delta^{k-1}} S_{1+\lambda}(P_{Y^n | \mathcal{V} = \theta} \| Q_{Y^n}) = \inf_{Q_{Y^n}} \sup_{\theta \in \Delta^{k-1}} D_{1+\lambda}(\tilde{P}_{Y^n | \mathcal{V} = \theta}^{\frac{1}{n}} \| \tilde{Q}_{Y^n}^{\frac{1}{n}}) \tag{60}
\]

\[
= \inf_{Q_{Y^n}} \sup_{\theta \in \Delta^{k-1}} D_{1+\lambda}(P_{Y^n | \mathcal{V} = \theta} \| \tilde{Q}_{Y^n}^{\frac{1}{n}}) \tag{61}
\]

\[
= \inf_{Q_{Y^n}} \sup_{\theta \in \Delta^{k-1}} D_{1+\lambda}(P_{Y^n | \mathcal{V} = \theta} \| \tilde{Q}_{Y^n}^{\frac{1}{n}}) \tag{62}
\]

where (62) follows because every probability measure in \( \Delta^{nk-1} \) is a scaled version of another probability measure in \( \Delta^{nk-1} \).

In order to establish (43), note that

\[
\inf_{Q_{Y^n}} \sup_{\theta \in \Delta^{k-1}} D_{1+\lambda} \left( P_{Y^n | \mathcal{V} = \theta} \| Q_{Y^n} \right) = \inf_{Q_{Y^n}} \sup_{P_{V}} \mathbb{E} \left[ D_{1+\lambda} \left( P_{Y^n | \mathcal{V} = \theta} \| Q_{Y^n} \right) \right] \tag{64}
\]

\[
= \sup_{P_{V}} \inf_{Q_{Y^n}} \mathbb{E} \left[ D_{1+\lambda} \left( P_{Y^n | \mathcal{V} = \theta} \| Q_{Y^n} \right) \right] \tag{65}
\]

\[
= \sup_{P_{V}} I_{1+\lambda}(P_{V}, P_{Y^n | \mathcal{V}}), \tag{66}
\]

where the expectation in (64) is with respect to \( V \sim P_{V} \), and (65) follows from [29, Theorem 34], which holds when \( \mathcal{V} \) is finite. The left side of (66) is the maximal \( \alpha \)-mutual information of order\(^{11} 1 + \lambda \) in the sense of Csiszár, see [16] and [18], which is known to equal maximal \( I_{1+\lambda} \) (see [16, Proposition 1], and [18, Theorem 5]) in the

\(^{11}\)When both random variables are discrete, another generalization of mutual information, whose maximum also coincides with (66), is put forward by Arimoto [30]. See [16] for further discussion of the various proposals of \( \alpha \)-mutual information.
discrete parameter case. To see that (66) holds even when the parameter space is continuous, recall the definition of \( \alpha \)-mutual information, (40), which can be written as

\[
I_{1+\lambda}(P_V, P_{Y^n|V}) = \inf_{Q_{Y^n}} \frac{1}{\lambda} \log \mathbb{E} \left[ \exp \left( \lambda t_{P_{Y^n|V}||Q_{Y^n}}(Y^n) \right) \right],
\]

and note that

\[
\sup_{P_V} \inf_{Q_{Y^n}} \mathbb{E} \left[ D_{1+\lambda} \left( P_{Y^n|V}(.|V)||Q_{Y^n} \right) \right] \leq \inf_{Q_{Y^n}} \frac{1}{\lambda} \log \left( \sup_{P_V} \mathbb{E} \left[ \exp \left( \lambda t_{P_{Y^n|V}||Q_{Y^n}}(Y^n) \right) \right] \right),
\]

where (68) follows from Jensen’s inequality, (69) follows from the fact that the maximin value is always less than or equal to the minimax value, and (72) is again due to [29, Theorem 34].

### B. Proof of the Converse of Theorem 2

This section is devoted to the proof of

\[
\liminf_{n \to \infty} \left\{ R_\lambda(n) - \frac{k-1}{2} \log \frac{n}{2\pi} \right\} \geq \log \frac{\Gamma(k/2)}{\Gamma((k-1)/2)} - \frac{k-1}{2\lambda} \log(1+\lambda),
\]

for any \( \lambda \in (0, \infty) \). Define

\[
\mathcal{M}_n = \left\{ \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_+^k : \sum_{i=1}^k a_i = n \right\},
\]

\[
\mathcal{M}_{n,\delta} = \left\{ \mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{Z}_+^k : \sum_{i=1}^k a_i = n \text{ and } \left[ \frac{n\delta}{k} \right] \leq a_i \text{ for each } i \right\},
\]

for any \( \delta \in (0, 1) \). Consider the following

\[
R_\lambda(n) = \sup_{P_V} I_{1+\lambda}(P_V, P_{Y^n|V})
\]

\[
= \sup_{P_V} \frac{1+\lambda}{\lambda} \log \sum_{y^n \in Y^n} \left( \int_{\Delta^{k-1}} P_{Y^n|V=\theta}^{1+\lambda}(y^n) dP_V(\theta) \right)^\frac{1+\lambda}{1+\lambda}
\]

\[
\geq \frac{1+\lambda}{\lambda} \log \sum_{t \in \mathcal{M}_n} \left( \begin{array}{c} n \\ t_1 \cdots t_k \\ \end{array} \right) \left( \int_{\Delta^{k-1}} (\theta_{t_1}^{t_1} \cdots \theta_{t_k}^{t_k})^{1+\lambda} dP_V(\theta) \right)^\frac{1+\lambda}{1+\lambda}
\]

\[
\geq \frac{1+\lambda}{\lambda} \log \sum_{t \in \mathcal{M}_{n,\delta}} \left( \begin{array}{c} n \\ t_1 \cdots t_k \\ \end{array} \right) \left( \int_{\Delta^{k-1}} (\theta_{t_1}^{t_1} \cdots \theta_{t_k}^{t_k})^{1+\lambda} dP_V(\theta) \right)^\frac{1+\lambda}{1+\lambda},
\]
where (76) is due to Theorem 1, (77) follows from a more general result [16, Theorem 1], although, for the sake of completeness, its proof is included in Lemma 1 in Appendix A, (78) is due to the suboptimal choice of Jeffreys’ prior, and (79) follows because \( \mathcal{M}_{n,\delta} \subset \mathcal{M}_n \).

Using Robbins’ sharpening [31] of Stirling’s approximation, one can show that
\[
\frac{e^{nH(\hat{P}_y^n)}}{(2\pi)^{k-1}} \sqrt{\prod_{i=1}^{k} t_i} \prod_{i=1}^{k} e^{\frac{\lambda}{2}} \leq \left( \frac{n}{t_1 \cdots t_k} \right) \leq \frac{e^{nH(\hat{P}_y^n)}}{(2\pi)^{k-1}} \sqrt{\prod_{i=1}^{k} t_i} \prod_{i=1}^{k} e^{\frac{\lambda}{2}},
\]
where the entropy is in nats and \( \hat{P}_y^n \) denotes the empirical distribution of the vector \( y^n \). Since \( t \in \mathcal{M}_{n,\delta} \), (80) particularizes to
\[
\left( \frac{n}{t_1 \cdots t_k} \right) \leq \frac{e^{nH(\hat{P}_y^n)}}{(2\pi)^{k-1}} \sqrt{\prod_{i=1}^{k} t_i} \prod_{i=1}^{k} e^{\frac{\lambda}{2}}.
\]

With the aid of (31) and (33) we can express the integral in the right side of (79) as
\[
\int_{\Delta^{k-1}} (\theta_1^{t_1} \cdots \theta_k^{t_k})^{1+\lambda} dP^*_y(\theta) = \frac{\prod_{i=1}^{k} \Gamma((1+\lambda)t_i + 1/2)}{\Gamma((1+\lambda)n + k/2) D_k(1/2, \ldots, 1/2)}.
\]

The gamma function generalization of Stirling’s approximation (shown to be valid for positive real numbers by Whittaker and Watson [32]) yields
\[
\Gamma(x) = \sqrt{2\pi} x^{-1/2} e^{-x} (1+r), \quad x > 0,
\]
where \(|r| \leq e^{1/(12x)} - 1\). In particular, for \( i = 1, \ldots, k \),
\[
\Gamma((1+\lambda)t_i + 1/2) = \sqrt{2\pi}((1+\lambda)t_i + 1/2)^{(1+\lambda)t_i} e^{-((1+\lambda)t_i - 1/2)(1+r_i)},
\]
\[
\Gamma((1+\lambda)n + k/2) = \sqrt{2\pi}((1+\lambda)n + k/2)^{(1+\lambda)n+k/2} e^{-(1+\lambda)n-k/2(1+r_0)},
\]
where
\[
|r_i| \leq \exp \left( \frac{1}{12(1+\lambda)t_i + 6} \right) - 1,
\]
\[
|r_0| \leq \exp \left( \frac{1}{12(1+\lambda)n + 6k} \right) - 1.
\]

It follows from (85) and (86) that
\[
\frac{\prod_{i=1}^{k} \Gamma((1+\lambda)t_i + 1/2)}{\Gamma((1+\lambda)n + k/2)} = \frac{e^{-n(1+\lambda)H(\hat{P}_y^n)}(2\pi)^{k-1} \prod_{i=1}^{k} \left( 1 + \frac{1}{2(1+\lambda)t_i} \right)^{(1+\lambda)t_i}}{(1+\lambda)^{k-1} n^{k-1} \left( 1 + \frac{k}{2(1+\lambda)n} \right)^{(1+\lambda)n+k/2}(1+r_0)}.
\]
Combining (83) and (89), we can write
\[
\int_{\Delta^{k-1}} \left( \theta_1^{t_1} \cdots \theta_k^{t_k} \right)^{1+\lambda} dP_\lambda^*(\theta)
\]
\[= \frac{(2\pi)^{k-1}}{D_k(1/2, \ldots, 1/2)} \frac{e^{-n(1+\lambda)H(P_\lambda^n)}}{(1+\lambda)^{\frac{k-1}{2}n+\frac{k-1}{2}}} \prod_{i=1}^{k-1} \frac{(1+\frac{1}{2(1+\lambda)t_i})^{(1+\lambda)t_i}}{(1+\frac{k}{2(1+\lambda)n})^{(1+\lambda)n+\frac{k}{2}}} (1+r_i) \] (90)
\[
\geq \frac{(2\pi)^{k-1}}{D_k(1/2, \ldots, 1/2)} \frac{e^{-n(1+\lambda)H(P_\lambda^n)}}{(1+\lambda)^{\frac{k-1}{2}n+\frac{k-1}{2}}} \prod_{i=1}^{k-1} \frac{(1+\frac{1}{2(1+\lambda)n})^{(1+\lambda)n+\frac{k}{2}}}{(1+\frac{k}{2(1+\lambda)n})^{(1+\lambda)n+\frac{k}{2}}} \left(2 - e^{\frac{1}{2(1+\lambda)n+\delta}}\right)^k, \] (91)

where (91) is due to the definition of \(M_{n,\delta}\), (75), the fact that for any positive constant \(c\), \((1+c/x)^x\) is a monotone increasing function of \(x\), and the fact that the error terms (see (87) and (88)) satisfy
\[
\prod_{i=1}^{k-1} \frac{1}{1+r_i} \geq \frac{2 - e^{\frac{1}{2(1+\lambda)n+\delta}}}{e^{\frac{1}{2(1+\lambda)n+\delta}}} \] (92).

Uniting the lower bounds in (79), (82) and (91),
\[
R_\lambda(n) - \frac{k-1}{2} \log \left(\frac{n}{2\pi}\right) \geq \frac{1+\lambda}{\lambda} \log(\beta(n, \delta, k)) - \frac{1}{\lambda} \log(D_k(1/2, \ldots, 1/2))
\]
\[\quad - \frac{k-1}{2\lambda} \log(1+\lambda) + \frac{1+\lambda}{\lambda} \log(\epsilon(n, \delta, k, \lambda)), \] (93)

where
\[
\beta(n, \delta, k) = \sum_{\tau_{\lambda(k-1)}} \frac{1}{n^{k-1}} \prod_{j=1}^{k} \left(\frac{l_j}{n}\right)^{1/2}, \] (94)
\[
\epsilon(n, \delta, k, \lambda) = \frac{e^{\frac{1}{2(1+\lambda)n+\delta}}}{e^{\frac{1}{2(1+\lambda)n}}} \left(1+\frac{k}{2(1+\lambda)n}\right)^{n+\frac{k}{2}} \left(2 - e^{\frac{1}{2(1+\lambda)n+\delta}}\right)^k \] (95).

Notice that
\[
\lim_{n \to \infty} \epsilon(n, \delta, k, \lambda) = 1, \text{ for any } \delta \in (0, 1), \] (96)
\[
\lim_{\delta \to 0} \lim_{n \to \infty} \beta(n, \delta, k) = \int_{\Delta^{k-1}} \tau_{1}^{-1/2} \cdots \tau_{k}^{-1/2} d\tau \] (97)
\[= D_k(1/2, \ldots, 1/2), \] (98)

where (96) follows after noticing that each factor of \(\epsilon(n, \delta, k)\) goes to 1, and (97) follows from the definition of the Riemann integral. Assembling (93), (96) and (98), we obtain the desired bound in (73).

C. Proof of the achievability of Theorem 2 when \(k = 2\)

In this section, we prove \(\leq \) in (52) when \(k = 2\), i.e.,
\[
\limsup_{n \to \infty} R_\lambda(n) - \frac{1}{2} \log \frac{n}{2\pi} \leq \log \frac{\Gamma^2(1/2)}{\Gamma(1)} - \frac{1}{2\lambda} \log(1+\lambda) \] (99)
\[
= \log \pi - \frac{1}{2\lambda} \log(1 + \lambda). \tag{100}
\]

To that end, we modify Jeffreys' prior by placing masses near the vertices of the simplex, i.e., \( \Delta^1 \), which, in turn, enables us to show that when the parameter\(^{12} \theta \) takes values near the vertices of the simplex the value of the minimax Rényi redundancy grows strictly slower than \( \frac{1}{2} \log n + O(1) \). Thus, we focus on values of \( \theta \) that are not close to the vertices of the simplex, thereby enabling us to argue that the minimax Rényi redundancy behaves as in (100).

Inspired by Xie and Barron's [19] modified Jeffreys' prior, for \( \epsilon \in (0, 1) \) and \( c \in (0, 1/(2 \log e)) \), consider the prior

\[
P^\epsilon_V(\theta) = \frac{\epsilon}{2} \left\{ \theta = \frac{c \log n}{n} \right\} + \frac{\epsilon}{2} \left\{ \theta = 1 - \frac{c \log n}{n} \right\} + (1 - \epsilon) P^\epsilon(\theta), \tag{101}
\]

which differs from the one in [19] in the location of the point masses. Because of the modification on Jeffreys' prior, the corresponding \( Y^n \) marginal changes from \( Q^*_{Y^n} \) in (36) to

\[
Q^\epsilon_{Y^n} = \frac{\epsilon}{2} P_{Y^n|V = \frac{c \log n}{n}} + \frac{\epsilon}{2} P_{Y^n|V = 1 - \frac{c \log n}{n}} + (1 - \epsilon) Q^*_{Y^n}. \tag{102}
\]

In view of Theorem 1,

\[
R_\lambda(n) \leq \sup_{\theta \in [0,1]} D_{1+\lambda} (P_{Y^n|V=\theta} \| Q^\epsilon_{Y^n}) \tag{103}
\]

\[
= \max \{ \Xi_1(n, \lambda, \epsilon), \Xi_2(n, \lambda, \epsilon), \Xi_3(n, \lambda, \epsilon) \}, \tag{104}
\]

where

\[
\Xi_1(n, \lambda, \epsilon) = \sup_{\theta \in \left[0, \frac{c \log n}{n}\right]} D_{1+\lambda} (P_{Y^n|V=\theta} \| Q^\epsilon_{Y^n}), \tag{105}
\]

\[
\Xi_2(n, \lambda, \epsilon) = \sup_{\theta \in \left[\frac{c \log n}{n}, 1 - \frac{c \log n}{n}\right]} D_{1+\lambda} (P_{Y^n|V=\theta} \| Q^\epsilon_{Y^n}), \tag{106}
\]

\[
\Xi_3(n, \lambda, \epsilon) = \sup_{\theta \in \left[1 - \frac{c \log n}{n}, 1\right]} D_{1+\lambda} (P_{Y^n|V=\theta} \| Q^\epsilon_{Y^n}). \tag{107}
\]

The following result shows that the first and the third supremizations in the right side of (104) are both dominated by \( \frac{1}{2} \log n + O(1) \).

**Proposition 1.** If \( c \in (0, 1/(2 \log e)) \), then

\[
\max \{ \Xi_1(n, \lambda, \epsilon), \Xi_3(n, \lambda, \epsilon) \} \leq \log \frac{2}{\epsilon} + \frac{c(\log e) \log n}{1 - \frac{c \log n}{n}}. \tag{108}
\]

**Proof:** Assume that \( \theta \in \left[0, \frac{c \log n}{n}\right] \). We have

\[
D_{1+\lambda} (P_{Y^n|V=\theta} \| Q^\epsilon_{Y^n}) \leq \log \frac{2}{\epsilon} + n D_{1+\lambda} (P_{Y|V=\theta} \| P_{Y|V=\frac{c \log n}{n}}) \tag{109}
\]

\(^{12}\)Since \( k = 2 \), we have \( \theta = (\theta, 1 - \theta) \). To simplify the discussion, we prefer the shorthand notation \( \theta \) rather than \( \theta \).
\[ \leq \log \frac{2}{\epsilon} + nD_{1+\lambda} \left( P_{Y|V=0} \| P_{Y|V=c\log n} \right) \]  
(110) 

\[ = \log \frac{2}{\epsilon} - n \log \left( 1 - \frac{c\log n}{n} \right) \]  
(111) 

\[ \leq \log \frac{2}{\epsilon} + \frac{c\log e}{1 - \frac{c\log n}{n}} \log n, \]  
(112) 

where (109) follows from (102), (110) follows because Rényi divergence is monotone decreasing in \( \theta \) (see Lemma 2 in Appendix B) and (112) follows because, for \( x < 1 \), 
\[ \log \left( \frac{1}{1-x} \right) \leq \frac{x}{1-x} \log e. \]  
(113) 

Using a symmetrical argument, one can show that the upper bound in (112) still holds when \( \theta \in [1 - c\log n/n, 1] \). 

It remains to investigate the behavior of the second supremization in the right side of (104). First of all, note that 
\[ \mathbb{E}_2(n, \lambda, \epsilon) \leq \log \frac{1}{1-\epsilon} + \sup_{\theta \in [\epsilon \log n/n, 1 - \epsilon \log n/n]} D_{1+\lambda} \left( P_{Y^n|V=\theta} \| Q_{Y^n}^* \right), \]  
(114) 

which follows from (102). The following proposition bounds the supremum in the right side of (114).

**Proposition 2.** Let \( \epsilon \in (0, 1/(2\log e)) \). For any \( \lambda \in (0, \infty) \), 
\[ \lim_{n \to \infty} \sup \left\{ \sup_{\theta \in [\epsilon \log n/n, 1 - \epsilon \log n/n]} D_{1+\lambda} \left( P_{Y^n|V=\theta} \| Q_{Y^n}^* \right) - \frac{1}{2} \log \frac{n}{2\pi} \right\} \leq \log \frac{\Gamma^2(1/2)}{\Gamma(1)} - \frac{1}{2\lambda} \log(1 + \lambda). \]  
(115) 

**Proof:** Let \( \theta_1 = \theta \) and \( \theta_2 = 1 - \theta \). Without loss of generality, we may assume that \( \theta_1 \leq 1/2 \), otherwise we may interchange the roles of \( \theta_1 \) and \( \theta_2 \) together with the roles of \( t_1 \) and \( t_2 = n - t_1 \) below. Note that 
\[ D_{1+\lambda} \left( P_{Y^n|V=\theta_1} \| Q_{Y^n}^* \right) = \frac{1}{\lambda} \log \left( \mathfrak{B}(\lambda, \theta_1, n) + \mathfrak{B}(\lambda, \theta_1, n) \right), \]  
(116) 

where 
\[ \mathfrak{B}(\lambda, \theta_1, n) = \left( \theta_1^{n(1+\lambda)} + \theta_2^{n(1+\lambda)} \right) \left( \frac{D_2(1/2, 1/2)}{D_2(1/2, n + 1/2)} \right)^{\lambda} \]  
(117) 

\[ \mathfrak{B}(\lambda, \theta_1, n) = \sum_{t_1=1}^{n-1} \binom{n}{t_1} (\theta_1^{t_1} \theta_2^{t_2})^{1+\lambda} \left( \frac{D_2(1/2, 1/2)}{D_2(t_1 + 1/2, t_2 + 1/2)} \right)^{\lambda}. \]  
(118) 

Thanks to Lemma 4 in Appendix D, we know that for all sufficiently large \( n \) satisfying 
\[ \frac{k \ln n}{2n} < 1, \]  
(119) 

we have 
\[ \mathfrak{B}(\lambda, \theta_1, n) \leq 2C_2(2)C_3^{\lambda}(2)n^{-(1+\lambda)c\log e} n^{\frac{1}{\lambda}}, \]  
(120) 

where the explicit expressions for \( C_2(k) \) and \( C_3(k) \) are given in (216) and (226), respectively. Hence, we may now focus attention on \( \mathfrak{B}(\lambda, \theta_1, n) \). Note that 
\[ \binom{n}{t_1} \leq \sqrt{\frac{n}{2\pi t_1 t_2}} e^{n\left(\frac{1}{2} - \frac{1}{e}\right)} n^{t_1}, \]  
(121) 

14
where \( h: [0, 1] \to [0, 1] \) and \( d(\cdot; \cdot): [0, 1] \times [0, 1] \to [0, \infty] \) denote the binary entropy and the binary relative entropy functions in nats, respectively. The bounds in (121) and (124) follow from Stirling’s approximations, (81) and (84), respectively.

By substituting (121), (122), and (124) into the right side of (118), we get

\[
\mathcal{M}(\lambda, \theta_1, n) \leq \left( \frac{n}{2\pi} \right)^{\frac{n}{2}} D_2(1/2, 1/2) \mathcal{G}(\lambda, \theta_1, n),
\]

where

\[
\mathcal{G}(\lambda, \theta_1, n) = \sum_{t_1=1}^{n-1} \sqrt{\frac{n}{2\pi t_1 t_2}} \exp \left( -n(1 + \lambda) d \left( \frac{\lambda}{n} \| \theta_1 \right) \right) K(\lambda, n, t_1),
\]

and

\[
K(\lambda, n, t_1) = e^{\frac{t_1}{2n}} \left( \frac{(1 + \frac{1}{n})^{\frac{n}{2}} e^{\frac{\lambda t_1}{2n} + 1}}{\prod_{i=1}^{\frac{n}{2}} \left( 1 + \frac{1}{2i} \right)^{t_i} \left( 2 - e^{\frac{-t_i}{2}} \right)} \right)^{\lambda}.
\]

Note that we can find an asymptotically suboptimal upper bound on \( \mathcal{G}(\lambda, \theta_1, n) \) that depends only on \( \lambda \) by invoking Lemma 6 in Appendix F, which shows a non-asymptotic uniform upper bound on \( K(\lambda, n, t_1) \), and then by invoking Lemma 5 in Appendix E, which shows a non-asymptotic uniform upper bound on

\[
\mathcal{F}(\lambda, \theta_1, n) = \sum_{t_1=1}^{n-1} \sqrt{\frac{n}{2\pi t_1 t_2}} \exp \left( -n(1 + \lambda) d \left( \frac{\lambda}{n} \| \theta_1 \right) \right).
\]

Finding the optimal upper bound, on the other hand, requires a uniform Laplace approximation on \( \mathcal{G}(\lambda, \theta_1, n) \), which is introduced next. First, given \( \delta \in (0, 1) \), split \( \mathcal{G}(\lambda, \theta_1, n) \) as

\[
\mathcal{G}(\lambda, \theta_1, n) = S_1(\lambda, \theta_1, n, \delta) + S_2(\lambda, \theta_1, n, \delta) + S_3(\lambda, \theta_1, n, \delta),
\]

where

\[
S_1(\lambda, \theta_1, n, \delta) = \sum_{t_1=\lceil n(1-\delta)\theta_1 \rceil}^{\lfloor n(1+\delta)\theta_1 \rfloor} \sqrt{\frac{n}{2\pi t_1 t_2}} \exp \left( -n(1 + \lambda) d \left( \frac{\lambda}{n} \| \theta_1 \right) \right) K(\lambda, n, t_1),
\]

\[
S_2(\lambda, \theta_1, n, \delta) = \sum_{t_1=\lfloor n(1-\delta)\theta_1 \rfloor}^{\lfloor n(1+\delta)\theta_1 \rfloor} \sqrt{\frac{n}{2\pi t_1 t_2}} \exp \left( -n(1 + \lambda) d \left( \frac{\lambda}{n} \| \theta_1 \right) \right) K(\lambda, n, t_1),
\]

\[
S_3(\lambda, \theta_1, n, \delta) = \sum_{t_1=\lfloor n(1+\delta)\theta_1 \rfloor}^{n-1} \sqrt{\frac{n}{2\pi t_1 t_2}} \exp \left( -n(1 + \lambda) d \left( \frac{\lambda}{n} \| \theta_1 \right) \right) K(\lambda, n, t_1).
\]
In Lemmas 8, 9 and 10 in Appendix G, we show each of the following properties:

\[
\lim_{n \to \infty} \sup_{\lambda \in [\log n, \frac{1}{2}]} S_1(\lambda, \theta_1, n, \delta) = 0 \quad \forall \delta \in (0, 1),
\]

(133)

\[
\lim_{\delta \to 0} \lim_{n \to \infty} \sup_{\lambda \in [\log n, \frac{1}{2}]} S_2(\lambda, \theta_1, n, \delta) \leq (1 + \lambda)^{-\frac{1}{2}},
\]

(134)

\[
\lim_{n \to \infty} \sup_{\lambda \in [\log n, \frac{1}{2}]} S_3(\lambda, \theta_1, n, \delta) = 0 \quad \forall \delta \in (0, 1).
\]

(135)

Since the left side of (129) does not depend on \(\delta\), (133)–(135) imply, by letting \(\delta \to 0\), that

\[
\limsup_{n \to \infty} \sup_{\lambda \in [\log n, \frac{1}{2}]} \Xi(\lambda, \theta_1, n) \leq (1 + \lambda)^{-\frac{\beta}{2}}.
\]

(136)

Finally, it follows from (116), (120), (125), and (136) that

\[
\limsup_{n \to \infty} \left\{ \sup_{\theta_1 \in [\log n, \frac{1}{2}]} D_{1+\lambda}(P_{Y^n|V=\theta_1}, Q^*_{Y^n}) - \frac{1}{2} \log \frac{n}{2\pi} \right\} \leq \log \frac{\Gamma^2(1/2)}{\Gamma(1)} - \frac{1}{2\lambda} \log(1 + \lambda).
\]

(137)

Since \(\theta_1 + \theta_2 = 1\), it also follows that

\[
\limsup_{n \to \infty} \left\{ \sup_{\theta_2 \in [\frac{1}{2}, 1-\log n]} D_{1+\lambda}(P_{Y^n|V=\theta_2}, Q^*_{Y^n}) - \frac{1}{2} \log \frac{n}{2\pi} \right\} \leq \log \frac{\Gamma^2(1/2)}{\Gamma(1)} - \frac{1}{2\lambda} \log(1 + \lambda).
\]

(138)

Combining (137) and (138) gives us the promised result of Proposition 2.

Invoking Proposition 1, we see that the functions in (105) and (107) can be bounded by

\[
\Xi_1(n, \lambda, \epsilon) - \frac{1}{2} \log \frac{n}{2\pi} \leq \log \frac{2}{\epsilon} + \frac{1}{2} \log(2\pi) - \left( \frac{1}{2} - \frac{c \log e}{1 - c \log n} \right) \log n,
\]

(139)

\[
\Xi_2(n, \lambda, \epsilon) - \frac{1}{2} \log \frac{n}{2\pi} \leq \log \frac{2}{\epsilon} + \frac{1}{2} \log(2\pi) - \left( \frac{1}{2} - \frac{c \log e}{1 - c \log n} \right) \log n,
\]

(140)

while thanks to (114) and Proposition 2, it follows that

\[
\limsup_{n \to \infty} \left\{ \Xi_2(n, \lambda, \epsilon) - \frac{1}{2} \log \frac{n}{2\pi} \right\} \leq \log \frac{1}{1 - \epsilon} + \log \frac{\Gamma^2(1/2)}{\Gamma(1)} - \frac{1}{2\lambda} \log(1 + \lambda).
\]

(141)

Since \(c \in (0, 1/(2 \log e))\), we see that the right side of (141) asymptotically dominates the right sides of (139) and (140). Due to (104), and (139)–(141), the desired result in (100) follows by choosing an arbitrarily small \(\epsilon\) in (101).

\[\blacksquare\]

**D. Proof of the achievability of Theorem 2 when \(k > 2\)**

In this section, we prove \(\leq\) in (52) when \(k > 2\), i.e.,

\[
\limsup_{n \to \infty} \left\{ R_\lambda(n) - \frac{k - 1}{2} \log \frac{n}{2\pi} \right\} \leq \log \frac{\Gamma^k(1/2)}{\Gamma(k/2)} - \frac{k - 1}{2\lambda} \log(1 + \lambda).
\]

(142)

To do so, we once again modify Jeffreys’ prior as in the previous section by placing masses near the lower dimensional faces of the simplex, i.e., \(\Delta^{k-1}\), which, in turn, enables us to show that when the parameter vector \(\theta\)
takes values near the faces of the simplex, the value of the minimax Rényi redundancy grows strictly slower than $\frac{k-1}{2} \log n + O(1)$. Hence, by focusing on the parameter values that are not close to the faces of the simplex, we show that the minimax Rényi redundancy behaves as in (142).

Following the idea in [19], let $c \in (0, 1/(2 \log e))$ and, for $i = 1, \ldots, k$, define

$$\mathcal{L}_i = \left\{ \theta : \theta_i = \frac{c \log n}{n} \right\} \cap \Delta^{k-1}. \quad (143)$$

Accordingly, we define the probability measure $\mu_i$ with respect to $d_i \xi = d\xi_1 \cdots d\xi_{i-1} d\xi_{i+1} \cdots d\xi_k$, the Lebesgue measure on $\mathbb{R}^{k-2}$, as

$$\mu_i(\theta) = \frac{\theta_{i-1/2} \cdots \theta_{i+1/2} \theta_{k-1/2}}{\int_{\mathcal{L}_i} \xi_{i-1/2} \cdots \xi_{i+1/2} \cdots \xi_k d\xi}. \quad (144)$$

Finally, for $\epsilon \in (0, 1)$, we define the prior distribution $P^\epsilon_V$ on the probability simplex $\Delta^{k-1}$ as

$$P^\epsilon_V = \frac{\epsilon}{k} \sum_{i=1}^{k} \mu_i + (1 - \epsilon) P^*_V, \quad (145)$$

where $P^*_V$ is Jeffreys’ prior. Because of the modification on Jeffreys’ prior in (145), the corresponding $Y^n$ marginal changes from $Q^*_V$ in (36) to

$$Q^\epsilon_V (y^n) = \frac{\epsilon}{k} \sum_{i=1}^{k} M_i (y^n) + (1 - \epsilon) Q^*_V (y^n), \quad (146)$$

where

$$M_i (y^n) = \int_{\mathcal{L}_i} P^{y^n|V=\theta} (y^n) \mu_i (\theta) d\theta \quad (147)$$

$$= \left( \frac{c \log n}{n} \right)^{t_i} \left( 1 - \frac{c \log n}{n} \right)^{n-t_i} \frac{D_{k-1} (t_1 + 1/2, \ldots, t_{i-1} + 1/2, t_{i+1} + 1/2, \ldots, t_k + 1/2)}{D_{k-1} (1/2, \ldots, 1/2)}. \quad (148)$$

Define, for $i = 1, \ldots, k$,

$$\mathcal{R}_i = \left\{ \theta : \theta_i \in \left[ 0, \frac{c \log n}{n} \right] \right\}. \quad (149)$$

$$\mathcal{R}_0 = \Delta^{k-1} - \bigcup_{i=1}^{k} \mathcal{R}_i. \quad (150)$$

Note that $\mathcal{R}_0$ denotes the vectors none of whose coordinates are within close proximity of zero in the sense of (149).

In view of Theorem 1,

$$R_\lambda (n) = \inf_{Q^{y^n}} \sup_{\theta \in \Delta^{k-1}} D_{1+\lambda} (P^{y^n|V=\theta}, Q^{y^n}) \quad (151)$$

$$\leq \sup_{\theta \in \Delta^{k-1}} D_{1+\lambda} (P^{y^n|V=\theta}, Q^{y^n}) \quad (152)$$

$$= \max_{i \in \{0,1,\ldots,k\}} \sup_{\theta \in \mathcal{R}_i} D_{1+\lambda} (P^{y^n|V=\theta}, Q^{y^n}) \quad (153)$$
The following result shows that the supremizations over $R_i$ for $i = 1, \ldots, k$ in (153) are all dominated by $\frac{k-1}{2} \log n + O(1)$.

**Proposition 3.** If $c \in (0, 1/(2 \log e))$, then for each $i \in \{1, \ldots, k\}$

$$
\sup_{\theta \in R_i} D_{1+\lambda} \left( P_{Y^n|V=\theta} \| Q_{Y^n}^{\ast} \right) \leq \log \frac{k}{\epsilon} + \log C_1(k-1) + \left( \frac{k-2}{2} + \frac{c \log e}{1 - \frac{c \log n}{n}} \right) \log n,
$$

(154)

where the explicit value of $C_1(k)$ is given in (201).

**Proof:** We show the result for $i = 1$. Define $f : \Delta^{k-1} \rightarrow \Delta^{k-2}$ as

$$
f(\theta) = \left( \frac{\theta_2}{1-\theta_1}, \ldots, \frac{\theta_k}{1-\theta_1} \right),
$$

(155)

and let $Q_{Y^n}^{\ast(k-2)}$ denote the Jeffreys’ mixture when the underlying parameter space is the $(k-2)$-dimensional simplex. Furthermore, define

$$
\psi(\lambda, n, \theta_1, t_1) = \left( n \right)_{t_1} \frac{[\theta_1^t (1 - \theta_1)^{n-t_1}]^{1+\lambda}}{\left( \frac{c \log n}{n} \right)^{M_1} \left( 1 - \frac{c \log n}{n} \right)^{\lambda(n-t_1)}},
$$

(156)

$$
\zeta(k, \lambda, n, \theta_1) = \exp \left( \lambda D_{1+\lambda} \left( P_{Y^n-t_1|V=\theta} \| Q_{Y^n-t_1}^{\ast(k-2)} \right) \right)
\leq C_1^\lambda (k-1) \exp \left( \lambda \log(n-t_1)^{2+2} \right)
\leq C_1^\lambda (k-1) \exp \left( \lambda \log n^{k+2} \right),
$$

(157)

(158)

(159)

where (158) follows from Lemma 3 in Appendix C. For $\theta \in R_1$,

$$
D_{1+\lambda} \left( P_{Y^n|V=\theta} \| Q_{Y^n}^{\ast} \right) \leq \log \frac{k}{\epsilon} + D_{1+\lambda} \left( P_{Y^n|V=\theta} \| M_1 \right)
$$

(160)

$$
= \log \frac{k}{\epsilon} + \frac{1}{\lambda} \log \sum_{t_1=0}^{n} \psi(\lambda, n, \theta_1, t_1) \zeta(k, \lambda, n, \theta, t_1)
$$

(161)

$$
\leq \log \frac{k}{\epsilon} + \log C_1(k-1) + \frac{k-2}{2} \log n + D_{1+\lambda} \left( P_{Y^n|V=\theta_1} \| P_{Y^n|V=\epsilon \log n} \right)
$$

(162)

$$
\leq \log \frac{k}{\epsilon} + \log C_1(k-1) + \left( \frac{k-2}{2} + \frac{c \log e}{1 - \frac{c \log n}{n}} \right) \log n,
$$

(163)

where (160) follows from (146), (162) follows from (159), and (163) follows because (109)–(112) imply

$$
D_{1+\lambda} \left( P_{Y^n|V=\theta_1} \| P_{Y^n|V=\epsilon \log n} \right) \leq \frac{c \log e}{1 - \frac{c \log n}{n}}.
$$

(164)

It remains to investigate the supremization over $R_0$ in (153). Note that

$$
\sup_{\theta \in R_0} D_{1+\lambda} \left( P_{Y^n|V=\theta} \| Q_{Y^n}^{\ast} \right) \leq \log \frac{1}{1-\epsilon} + \sup_{\theta \in R_0} D_{1+\lambda} \left( P_{Y^n|V=\theta} \| Q_{Y^n}^{\ast} \right),
$$

(165)

which follows from the definition of $Q_{Y^n}^{\ast}$ in (146). Parallel to Proposition 2, the supremum in the right side of (165) behaves as follows.
Proposition 4. For any $\lambda \in (0, \infty)$,

$$
\limsup_{n \to \infty} \left\{ \sup_{\theta \in \mathcal{R}_0} \frac{\ln n}{D_{1+\lambda}(P_{Y^n|V=\theta}||Q_{Y^n}^\ast)} - \frac{k-1}{2} \log \frac{n}{2\pi} \right\} \leq \log \frac{\Gamma^{1/2}(k/2)}{\Gamma(k/2)} - \frac{k-1}{2\lambda} \log(1+\lambda). \tag{166}
$$

Proof: We are only interested in $\theta \in \mathcal{R}_0$. Therefore, for all $i = 1, \ldots, k$,

$$
\theta_i \geq \frac{c \log n}{n}, \tag{167}
$$

where $c \in (0, 1/(2 \log e))$ is a constant. Since there is an index $j$ such that $\theta_j \geq 1/k$, it simplifies notation without loss of generality that $j = k$. Otherwise, the proof remains identical. For a given positive integer $l$, let

$$
\mathcal{I} = \{i_1, \ldots, i_l\} \subset \mathcal{Y} \tag{168}
$$

be a proper subset and note that

$$
D_{1+\lambda}(P_{Y^n|V=\theta}||Q_{Y^n}^\ast) = \frac{1}{\lambda} \log \left( \mathfrak{M}(k, \lambda, \theta, n) + \mathfrak{M}(k, \lambda, \mathfrak{n}, n) \right), \tag{169}
$$

where

$$
\mathfrak{M}(k, \lambda, \theta, n) = \sum_{t_1 \cdots t_k = n} \sum_{l \geq k \lambda} \left( \sum_{t_i \geq 0, \forall i} \left( \sum_{l \geq 0} \frac{n}{t_1 \cdots t_k} \left( T_i^{1+\lambda} \left( \frac{D_k(1/2, \ldots, 1/2)}{D_k(t_1 + 1/2, \ldots, t_k + 1/2)} \right)^{\lambda} \right) \right) \right), \tag{170}
$$

$$
\mathfrak{M}(k, \lambda, \theta, n) = \sum_{t_1 \cdots t_k = n} \sum_{l \geq k \lambda} \left( \sum_{t_i \geq 0, \forall i} \left( \sum_{l \geq 0} \frac{n}{t_1 \cdots t_k} \left( T_i^{1+\lambda} \left( \frac{D_k(1/2, \ldots, 1/2)}{D_k(t_1 + 1/2, \ldots, t_k + 1/2)} \right)^{\lambda} \right) \right) \right). \tag{171}
$$

Thanks to Lemma 4 in Appendix D, we know that for all sufficiently large $n$ satisfying

$$
\frac{k \ln n}{2n} < 1, \tag{172}
$$

it follows that

$$
\mathfrak{M}(k, \lambda, \theta, n) \leq \bar{C}(k, \lambda)n^{-(1+\lambda)c \log e^{\lambda(1+\lambda)}}, \tag{173}
$$

where $\bar{C}(k, \lambda)$ is a constant depending only on $\lambda$ and $k$, which is explicitly given in the proof of Lemma 4, see (229). Hence, we may now focus attention on $\mathfrak{M}(k, \lambda, \mathfrak{n}, n)$. Note that

$$
\left( \sum_{t_1 \cdots t_k} \right) \leq \frac{e^{nH(\tilde{P}_0^n)}}{(2\pi)^{k-1}} \sqrt{k} \prod_{i=1}^k t_i, \tag{174}
$$

$$
\prod_{i=1}^k \theta_i^{t_i} = \exp \left( -n \left[ D(\tilde{P}_0^n||P_{Y^n|V=\theta}) + H(\tilde{P}_0^n) \right] \right), \tag{175}
$$

$$
\frac{1}{D_k(t_1 + 1/2, \ldots, t_k + 1/2)} = \frac{\Gamma(n + k/2)}{\prod_{i=1}^k \Gamma(t_i + 1/2)} \tag{176}
$$

and

$$
\left( \frac{n}{2\pi} \right)^{k-1} \frac{e^{nH(\tilde{P}_0^n)}}{\prod_{i=1}^k \left( 1 + \frac{1}{2\pi} \right) \theta_i^{t_i}} \left( 2 - e^{2\pi \theta t \mathfrak{n}} \right) \leq \left( \frac{n}{2\pi} \right)^{k-1} \frac{e^{nH(\tilde{P}_0^n)}}{\prod_{i=1}^k \left( 1 + \frac{1}{2\pi} \right) \theta_i^{t_i}} \left( 2 - e^{2\pi \theta t \mathfrak{n}} \right), \tag{177}
$$

19
where both the entropy and relative entropy are in nats. The bounds in (174) and (177) follow from Stirling’s approximations, (81) and (84), respectively.

By substituting (174), (175) and (177) into the right side of (171), we get

\[ \mathcal{W}(k, \lambda, \theta, n) \leq \left( \frac{n}{2\pi} \right)^{\frac{k(k-1)}{2}} D_k(1/2, \ldots, 1/2) \mathcal{S}(k, \lambda, \theta, n), \]  

(178)

where

\[ \mathcal{S}(k, \lambda, \theta, n) = \sum_{t_1 \geq 1, \ldots, t_k \geq n} \frac{K(k, \lambda, n, t)}{(2\pi)^{\frac{k}{2}}} \left( \prod_{i=1}^{n} t_i \right)^{\lambda} \exp \left( -n(1 + \lambda) D(\hat{P}_{y^n} || P_{Y|V=\theta}) \right), \]  

(179)

and

\[ K(k, \lambda, n, t) = e^{\frac{1}{2\pi}} \left( \prod_{i=1}^{k} \left( 1 + \frac{1}{2\pi} \right)^{t_i} \left( 2 - e^{2k\theta_i} \right) \right)^{\lambda}. \]  

(180)

Observe once again that we can find an asymptotically suboptimal upper bound on \( \mathcal{S}(k, \lambda, \theta, n) \) that depends only on \( k \) and \( \lambda \) by invoking Lemma 6 in Appendix F, which shows a non-asymptotic uniform upper bound on \( K(\lambda, n, t_1) \), and then by invoking Lemma 5 in Appendix E, which shows a non-asymptotic uniform upper bound on

\[ \mathcal{T}(k, \lambda, \theta, n) \leq \sum_{t_1 \geq 1, \ldots, t_k \geq n} \frac{1}{(2\pi)^{\frac{k}{2}}} \left( \prod_{i=1}^{n} t_i \right)^{\lambda} \exp \left( -n(1 + \lambda) D(\hat{P}_{y^n} || P_{Y|V=\theta}) \right). \]  

(181)

Finding the optimal upper bound, on the other hand, requires a uniform Laplace approximation on \( \mathcal{S}(\lambda, \theta_1, n) \), which is introduced next. First, given \( \delta \in (0, 1/(k-1)) \), define

\[ \mathcal{N}^g_\delta = \left\{ (a_1, \ldots, a_k) \in \mathbb{Z}_+^k : \sum_{i=1}^{k} a_i = n, \text{ and } [n(1-\delta)\theta_i] \leq a_i \leq [n(1+\delta)\theta_i] \text{ for all } 1 \leq i \leq k-1 \right\}. \]  

(182)

and split \( \mathcal{S}(k, \lambda, \theta, n) \) as

\[ \mathcal{S}(k, \lambda, \theta, n) = S_1(k, \lambda, \theta, n, \delta) + S_2(k, \lambda, \theta, n, \delta), \]  

(183)

where

\[ S_1(k, \lambda, \theta, n, \delta) = \sum_{t_1 \geq 1, \ldots, t_k \geq n} \frac{K(k, \lambda, n, t)}{(2\pi)^{\frac{k}{2}}} \left( \prod_{i=1}^{n} t_i \right)^{\lambda} \exp \left( -n(1 + \lambda) D(\hat{P}_{y^n} || P_{Y|V=\theta}) \right), \]  

(184)

\[ S_2(k, \lambda, \theta, n, \delta) = \sum_{t_1 \geq 1, \ldots, t_k \geq n} \frac{K(k, \lambda, n, t)}{(2\pi)^{\frac{k}{2}}} \left( \prod_{i=1}^{n} t_i \right)^{\lambda} \exp \left( -n(1 + \lambda) D(\hat{P}_{y^n} || P_{Y|V=\theta}) \right). \]  

(185)

In Lemmas 12 and 13 in Appendix I, we show that the following properties hold:

\[ \lim_{\delta \to 0} \limsup_{n \to \infty} \sup_{\theta \in \mathbb{R}_0} \sup_{\delta \geq 1/k} S_1(k, \lambda, \theta, n, \delta) \leq (1 + \lambda)^{-\frac{k-1}{k}}, \]  

(186)
\begin{equation}
\lim_{n \to \infty} \sup_{\theta \in \mathbb{R}_0 \atop \theta_k \geq 1/k} S_2(k, \lambda, \theta, n, \delta) = 0 \quad \forall \delta \in (0, 1).
\end{equation}

Since the left side of (183) does not depend on \( \delta \), (186) and (187) imply, by letting \( \delta \to 0 \), that
\begin{equation}
\lim_{n \to \infty} \sup_{\theta \in \mathbb{R}_0 \atop \theta_k \geq 1/k} \mathcal{G}(k, \lambda, \theta, n) \leq (1 + \lambda) - \frac{k}{2} \log n.
\end{equation}

Finally, it follows from (169), (173), (178), and (188), that (166) holds when \( \theta_k \geq 1/k \) as we wanted to show.

Invoking Proposition 3, we see that for each \( i = 1, \ldots, k \)
\begin{equation}
\sup_{\theta \in \mathbb{R}_0} D_{1+\lambda} \left( P_{Y^n|V=\theta} \| Q_{Y^n}^{\epsilon} \right) - \frac{k - 1}{2} \log \frac{n}{2\pi} \leq \log \frac{k}{e} + \log C_1(k - 1) + \frac{k - 1}{2} \log(2\pi) - \left( \frac{1}{2} - \frac{c \log e}{1 - \frac{c \log n}{n}} \right) \log n,
\end{equation}
while thanks to (165) and Proposition 4, it follows that
\begin{equation}
\lim_{n \to \infty} \sup_{\theta \in \mathbb{R}_0} \left\{ \sup_{\theta \in \Theta} D_{1+\lambda} \left( P_{Y^n|V=\theta} \| Q_{Y^n}^{\epsilon} \right) - \frac{k - 1}{2} \log \frac{n}{2\pi} \right\} \leq \log \frac{1}{1 - \epsilon} + \log \frac{\Gamma(k/2)}{\Gamma(k/2)} - \frac{k - 1}{2\lambda} \log(1 + \lambda).
\end{equation}
Since \( c \in (0, 1/(2\log e)) \), we see that, as \( n \to \infty \), the right side of (189) goes to \( -\infty \) whereas the right side of (190) remains constant. In view of (153), (189) and (190), the desired result in (142) follows by choosing an arbitrarily small \( \epsilon \) in (145).

**APPENDIX A**

**EXPLICIT EVALUATION OF \( \alpha \)-MUTUAL INFORMATION**

A more general result that allows non-discrete alphabet \( Y \) can be found in [16].

**Lemma 1.** Let \( \lambda \in (0, \infty) \). Given an arbitrary input distribution \( P_V \) on \( \Theta \) and a random transformation \( P_{Y|V}: \Theta \to Y \) with finite output alphabet \( Y \), the \( \alpha \)-mutual information of order \( 1+\lambda \) induced by \( P_V \) on \( P_{Y|V} \) satisfies
\begin{equation}
I_{1+\lambda}(P_V, P_{Y|V}) = \frac{1 + \lambda}{\lambda} \log \sum_{y \in Y} \left( \int_{\theta \in \Theta} P_{Y^n|V=\theta}^{1+\lambda}(y) dP_V(\theta) \right)^{1+\lambda}.
\end{equation}

**Proof:** Define
\begin{equation}
R_Y(y) = \left( \int_{\theta \in \Theta} P_{Y^n|V=\theta}^{1+\lambda}(y) dP_V(\theta) \right)^{1+\lambda} \frac{1}{\sum_{b \in Y} \left( \int_{\xi \in \Theta} P_{Y^n|V=\xi}^{1+\lambda}(b) dP_V(\xi) \right)^{1+\lambda}},
\end{equation}
and recall that
\begin{equation}
D_{1+\lambda}(R_Y \| Q_Y) \geq 0
\end{equation}
for any distribution \( Q_Y \) on \( Y \). Capitalizing on (193), note that

\[
D_{1+\lambda}(P_{Y|V}P_V||Q_Y P_V) = \frac{1}{\lambda} \log \sum_{y \in Y} \int_{\theta \in \Theta} \frac{P_{Y|V=\theta}^{1+\lambda}(y)}{Q_Y^{\lambda}(y)} dP_V(\theta)
\]

(194)

\[
\geq \frac{1+\lambda}{\lambda} \log \sum_{y \in Y} \left( \int_{\theta \in \Theta} P_{Y|V=\theta}^{1+\lambda}(y) dP_V(\theta) \right)^{\frac{1}{1+\lambda}}
\]

(195)

\[
= D_{1+\lambda}(P_{Y|V}P_V||R_Y P_V).
\]

(196)

By the definition of the \( \alpha \)-mutual information, see (40); (196) implies the result in (191).

\[\blacksquare\]

**APPENDIX B**

**MONOTONICITY OF BINARY RÉNYI DIVERGENCE**

**Lemma 2.** Let \( P_{Y|V=\theta} \) denote a Bernoulli distribution with parameter \( \theta \). For any \( \xi \in (0,1] \) and \( \lambda \in (0,\infty) \),

\[D_{1+\lambda}(P_{Y|V=\theta}||P_{Y|V=\xi}) \text{ is a monotone decreasing function of } \theta \text{ on } [0,\xi].\]

**Proof:** Fix \( \lambda \in (0,\infty) \). Let \( Y \sim P_{Y|V=\theta} \). It suffices to prove that \( E\left[ \left( \frac{P_{Y|V=\theta}(Y)}{P_{Y|V=\xi}(Y)} \right)^{\lambda} \right] \) is a monotone decreasing function of \( \theta \) on \([0,\xi]\). To that end, note that

\[
\frac{d}{d\theta} E\left[ \left( \frac{P_{Y|V=\theta}(Y)}{P_{Y|V=\xi}(Y)} \right)^{\lambda} \right] = (1+\lambda) \left( \frac{\theta^{1-\lambda}}{(1-\theta)^{\lambda}} - \frac{(1-\theta)^{1-\lambda}}{(1-\xi)^{\lambda}} \right)
\]

(197)

\[
\leq 0,
\]

(198)

where (198) follows because \( \theta \in [0,\xi] \) implies

\[
\frac{\theta}{\xi} \leq \frac{1-\theta}{1-\xi}.
\]

(199)

\[\blacksquare\]

**APPENDIX C**

**UNIFORM UPPER BOUND ON D_{1+\lambda} (P_{Y^n|V=\theta}||Q_{Y^n}^*)**

**Lemma 3.** Let \( \theta \in \Delta^{k-1} \) be an element in the \((k-1)\)-dimensional simplex and assume that we are given a discrete i.i.d. model \( P_{Y^n|V=\theta} \). Then, for any \( n \geq 1 \) and \( y^n \in Y^n \), the relative information between the model \( P_{Y^n|V=\theta} \) and Jeffreys’ mixture \( Q_{Y^n}^* \) satisfies the following bound

\[
in_{P_{Y^n|V=\theta}||Q_{Y^n}^*}(y^n) \leq \frac{k-1}{2} \log n + \log C_1(k),
\]

(200)

where

\[
C_1(k) = \frac{e^{\frac{k-1}{2}} D_k \left( \frac{1}{2}, \cdots, \frac{1}{2} \right)}{(2\pi)^{\frac{k-1}{2}} \left( 2-e^{1/6} \right)^{k}} \left( 1 + \frac{k}{2} \right)^{\frac{k-1}{2}}.
\]

(201)
Consequently, for any \( \lambda > 0 \),
\[
D_{1+\lambda} \left( P_{Y^n|V=\theta} \| Q_{Y^n}^{\infty} \right) \leq \frac{k-1}{2} \log n + \log C_1(k),
\]
where \( C_1(k) \) is given in (201).

**Proof:** Immediate consequence of [19, Lemma 4].

### APPENDIX D

#### EDGE CASES OF \( t_i \)

**Lemma 4.** Let \( c \in (0, 1/(2 \log e)) \) and for a given positive integer \( l \), let \( I = \{i_1, \ldots, i_l\} \) be a proper subset of \( \mathcal{Y} \). Then, for any \( n \) satisfying
\[
\frac{k \ln n}{2n} < 1,
\]
and \( \theta \in \mathcal{R}_0 \) (defined in (150))
\[
\mathfrak{N}(k, \lambda, \theta, n) \leq \bar{C}(k, \lambda) n^{-(1+\lambda)c \log e + \lambda(\frac{k}{2} - 1)},
\]
where \( \mathfrak{N}(k, \lambda, \theta, n) \) is defined\(^{13}\) in (170) and \( \bar{C}(k, \lambda) \) is a constant that only depends \( k \) and \( \lambda \).

**Proof:** Denote
\[
\{i_{l+1}, i_{l+2}, \ldots, i_k\} = \{1, \ldots, k\} \setminus \{i_1, i_2, \ldots, i_l\},
\]
and note that
\[
\sum_{t_1 + \cdots + t_k = n \atop \{i_1, \ldots, t_k\} \subseteq I} \left( \frac{\theta_1^{t_1} \cdots \theta_k^{t_k}}{D_k(t_1 + 1/2, \ldots, t_k + 1/2)} \right)^{\lambda} = \sum_{t_1 + \cdots + t_k = n} \left( \frac{\theta_1^{t_1} \cdots \theta_k^{t_k}}{D_k(t_1 + 1/2, \ldots, t_k + 1/2)} \right)^{\lambda}.
\]

Regarding the last term within the summation in the right side of (206),
\[
\frac{D_k(1/2, \ldots, 1/2)}{D_{k-l}(1/2, \ldots, 1/2, t_{i_{l+1}} + 1/2, \ldots, t_k + 1/2)} \leq \frac{\Gamma \left( \frac{k}{2} \right)}{\Gamma \left( \frac{k-1}{2} \right)} \frac{\Gamma \left( n + \frac{k}{2} \right)}{\Gamma \left( n + \frac{k-1}{2} \right)}
\]
\[
\frac{D_{k-l}(t_{i_{l+1}} + 1/2, \ldots, t_k + 1/2)}{D_{k-l}(1/2, \ldots, 1/2, t_{i_{l+1}} + 1/2, \ldots, t_k + 1/2)} \leq \frac{\Gamma \left( \frac{k}{2} \right)}{\Gamma \left( \frac{k-1}{2} \right)} \frac{\Gamma \left( n + \frac{k}{2} \right)}{\Gamma \left( n + \frac{k-1}{2} \right)}.
\]

\(^{13}\)The quantity \( \mathfrak{N}(\lambda, \theta_1, n) \) defined in (117) corresponds to the special case of (170) where \( k = 2, \theta = (\theta_1, 1 - \theta_1) \).
where (207) follows from the definition of the Dirichlet integrals in (33), and (208) follows from the fact that $l \geq 1$. Now, observe that
\[
\frac{\Gamma(n + \frac{k}{2})}{\Gamma(n + \frac{k}{2} + \frac{1}{2})} = \frac{\sqrt{2\pi} (n + \frac{k}{2})^{n + \frac{k-1}{2}} e^{n - \frac{1}{2}} (1 + r_0)}{\sqrt{2\pi} (n + \frac{k-1}{2})^{n + \frac{k-1}{2}} e^{n - \frac{1}{2}} (1 + r_1)} \leq \left( \frac{1 + \frac{k-1}{2}}{2 - e^{1/18}} \right)^{n/2},
\]
where $r_1$ is the remainder in Stirling's approximation of $\Gamma(n + \frac{k-1}{2})$ in (84), and (210) is due to the following simple bounds
\[
\left( 1 + \frac{k-1}{2} \right)^{n + \frac{k-1}{2}} \leq \left( 1 + \frac{k}{2} \right)^{k/2},
\]
\[
1 + r_0 \leq e^{n - \frac{1}{2}} \leq e^{1/24},
\]
\[
1 + r_1 \geq 2 - e^{n - \frac{1}{2}} \geq 2 - e^{1/18}.
\]
It follows that
\[
\frac{D_k(1/2, \ldots, 1/2)}{D_k(1/2, \ldots, 1/2, t_{i+1} + 1/2, \ldots, t_k + 1/2)} \leq \left( \frac{D_{k-1}(1/2, \ldots, 1/2)}{D_{k-1}(t_{i+1} + 1/2, \ldots, t_k + 1/2)} \right) C_2(k) n^{l/2},
\]
where
\[
C_2(k) = \left( \frac{\Gamma(\frac{k-1}{2})}{\Gamma(\frac{k}{2})} \right) \left( \frac{1 + \frac{k-1}{2}}{2 - e^{1/18}} \right)^{n/2}.
\]
Since $\theta \in R_0$,
\[
(1 - (\theta_{i+1} + \ldots + \theta_i))^n \leq \left( 1 - \frac{\lambda c \log n}{n} \right)^n \leq n^{-\lambda c \log e},
\]
where (218) is because $\frac{\lambda c \log n}{n} < \frac{\lambda \ln n}{2n} < 1$ and for any $x < 1$ we have $\log(1 - x) \leq -x \log e$. Let
\[
\bar{\theta} = \left( \frac{\theta_{i+1}}{1 - (\theta_{i+1} + \ldots + \theta_i)} \right) \ldots \left( \frac{\theta_k}{1 - (\theta_{i+1} + \ldots + \theta_i)} \right)
\]
\[
= (\bar{\theta}_{i+1}, \ldots, \bar{\theta}_k).
\]
It follows from (215) and (218) that
\[
\sum_{t_{i+1} + \ldots + t_k = n} \left( \frac{1}{t_{i+1} \ldots t_k} \right)^{1+\lambda} \left( \frac{D_k(1/2, \ldots, 1/2)}{D_k(1/2, \ldots, 1/2, t_{i+1} + 1/2, \ldots, t_k + 1/2)} \right)^{\lambda} \left( \frac{D_{k-1}(1/2, \ldots, 1/2)}{D_{k-1}(t_{i+1} + 1/2, \ldots, t_k + 1/2)} \right)^{\lambda} C_2(k)^{n^{l/2}}
\]
\[
= \sum_{t_{i+1} + \ldots + t_k = n} \left( \frac{1}{t_{i+1} \ldots t_k} \right)^{1+\lambda} \left( \frac{D_k(1/2, \ldots, 1/2)}{D_k(1/2, \ldots, 1/2, t_{i+1} + 1/2, \ldots, t_k + 1/2)} \right)^{\lambda} \left( \frac{D_{k-1}(1/2, \ldots, 1/2)}{D_{k-1}(t_{i+1} + 1/2, \ldots, t_k + 1/2)} \right)^{\lambda} C_2(k)^{n^{l/2}}
\]
\[
\leq \sum_{t_{i+1} + \ldots + t_k = n} \left( \frac{1}{t_{i+1} \ldots t_k} \right)^{1+\lambda} \left( \frac{D_k(1/2, \ldots, 1/2)}{D_k(1/2, \ldots, 1/2, t_{i+1} + 1/2, \ldots, t_k + 1/2)} \right)^{\lambda} \left( \frac{D_{k-1}(1/2, \ldots, 1/2)}{D_{k-1}(t_{i+1} + 1/2, \ldots, t_k + 1/2)} \right)^{\lambda} C_2(k)^{n^{l/2}}
\]

24
Note that
\[
\sum_{t_{i_{i+1}} \cdots t_{i_k} = n} \left( \frac{n}{t_{i_{i+1}} \cdots t_{i_k}} \right)^{1+\lambda} \left( \frac{D_{k-l}(1/2, \ldots, 1/2)}{D_{k-l}(t_{i_{i+1}} + 1/2, \ldots, t_{i_k} + 1/2)} \right)^{\lambda}
\]
(222)
\[= \exp \left( \lambda D_{l+\lambda} \left( P_{Y^n \mid V = \theta} \| Q^*(k-l-1) \right) \right), \]
where \(Q^*(k-l-1)\) denotes the Jeffreys’ mixture when the underlying parameter space is the \((k-l-1)\)-dimensional simplex. Using the uniform upper bound on Rényi divergence in Lemma 3, we get
\[
\exp \left( \lambda D_{l+\lambda} \left( P_{Y^n \mid V = \theta} \| Q^*(k-l-1) \right) \right) \leq C_1 \lambda^{1/k} \left( 2 \pi \right)^{\lambda} \pi^{1/2 \lambda} \exp \left( \lambda + \frac{1}{2} \right),
\]
(223)
where
\[
C_1(k) = \frac{e^{k/2}}{(2\pi)^{k/4 \lambda}} \frac{D_k \left( 1/2, \ldots, 1/2 \right)}{\left( 2 - e^{1/6} \right)^{k}} \left( 1 + \frac{k}{2} \right)^{\frac{k-1}{2}}.
\]
(224)
Since \(l \in \{1, \ldots, k-1\}, D_1(1/2) = 1, \) and \(D_m(1/2, \ldots, 1/2) \leq \pi \) for any integer \(m \geq 2, \) we can upper bound
\[
C_1(k-l) \leq \frac{\pi e^{k/2}}{2 - e^{1/6}} \left( 1 + \frac{k-1}{2} \right)^{\frac{k-2}{2}}
\]
(225)
\[= C_3(k). \]
(226)
As a result,
\[
\mathcal{Q}(k, \lambda, \theta, n) \leq \sum_{l=1}^{k-1} \binom{k}{l} C_2 \left( k \right) C_3 \left( k \right) n^{-(\lambda+1)c \log e + \lambda} \left( \frac{k-1}{2} \right)
\]
(227)
\[= (2^k - 2) C_2 \left( k \right) C_3 \left( k \right) n^{-(\lambda+1)c \log e + \lambda},
\]
(228)
and (204) follows after setting
\[
\widetilde{C}(k, \lambda) = (2^k - 2) C_2 \left( k \right) C_3 \left( k \right).
\]
(229)

**APPENDIX E**

**Uniform Upper Bound on \( \mathcal{Q}(k, \lambda, \theta, n) \)**

The quantity defined\(^{14}\) in (181) satisfies the following upper bound.

**Lemma 5.**

\[
\mathcal{Q}(k, \lambda, \theta, n) \leq \frac{C_1 \left( k \right) (2\pi)^{\frac{\lambda(k-1)}{2}} e^{k/(2^k + 3)}}{D_k \left( 1/2, \ldots, 1/2 \right) \left( 2 - e^{1/6} \right)^{\lambda}}.
\]
(230)

\(^{14}\)The quantity \( \mathcal{Q}(\lambda, \theta_1, n) \) defined in (128) corresponds to the special case of (181) where \( k = 2, \ \theta = (\theta_1, 1 - \theta_1). \)
where \( C_1(k) \) is explicitly given in (201).

\[ \text{Proof:} \]

Define

\[
\tilde{S}(k, \lambda, \theta, n) = \sum_{t_1 \geq 1, \, \forall i} \frac{\widetilde{K}(k, \lambda, n, t)}{(2\pi)^{1/2}} \prod_{i=1}^{n} e^{\frac{-n(1+\lambda)D(\hat{P}_y \parallel P_{Y|V=\theta})}{2}},
\]

and note that

\[
D_{1+\lambda}(P_{Y^n|V=\theta} \parallel Q_{Y^n}) \geq \frac{1}{\lambda} \log \mathcal{N}(k, \lambda, \theta, n) \geq \frac{1}{\lambda} \log \left( \frac{n}{2\pi} \right)^{\frac{\lambda(n-1)}{2}} D_k^\lambda(1/2, \ldots, 1/2) \tilde{S}(k, \lambda, \theta, n),
\]

where (233) follows from (169), and (234) follows from Stirling’s approximations, (80) and (84), as well as the fact that

\[
\prod_{i=1}^{k} \frac{\theta_i^{t_i}}{t_i!} = \exp \left( -n \left[ D(\hat{P}_y \parallel P_{Y|V=\theta}) + H(\hat{P}_y^n) \right] \right).
\]

Regarding \( \widetilde{K}(k, \lambda, n, t) \), one can check that

\[
\widetilde{K}(k, \lambda, n, t) \geq \left( 2 - e^{\frac{k}{2n}} \right)^{\lambda}.
\]

Invoking Lemma 3 in Appendix C to upper bound the left side of (233) and applying the bound in (236) to (234) results in (230).

**APPENDIX F**

**BOUNDS ON \( K(k, \lambda, n, t) \)**

The quantity defined\(^{15}\) in (180) satisfies the following non-asymptotic bound.

**Lemma 6** Uniform Upper Bound on \( K(k, \lambda, n, t) \). Given \( \lambda \in (0, \infty) \),

\[
K(k, \lambda, n, t) \leq e^{\frac{k}{2n}} \left( \frac{e^{\frac{k}{2}}}{\left( \frac{3}{2} \right)^{k}} \right)^{\lambda}.
\]

\[\text{where } k = n - t_1.\]

\(^{15}\) The quantity \( K(\lambda, n, t_1) \) defined in (127) corresponds to the special case of (180) where \( k = 2, t = (t_1, n - t_1) \).
In particular,

\[ K(\lambda, n, t_1) \leq M(2, \lambda) \]

\[ \leq 3^\lambda e^{\frac{1}{12}} \]  \hspace{1cm} (239)

\[ \leq 3^\lambda e^{\frac{1}{12}} \]  \hspace{1cm} (240)

**Proof:** For \( x \geq 1 \),

\[ \left( 1 + \frac{1}{2x} \right)^x \left( 2 - e^{\frac{1}{x+e}} \right) \geq \frac{3}{2} \left( 2 - e^{\frac{1}{x}} \right), \]  \hspace{1cm} (241)

because the function in the left side of (241) is an increasing function. On the other hand,

\[ \left( 1 + \frac{k}{2x} \right)^{\frac{k-1}{2x}} e^{\frac{1}{x+e}} \leq \left( 1 + \frac{k}{2} \right)^{\frac{k-1}{2}} e^{\frac{1}{x+e}}, \]  \hspace{1cm} (242)

because the function of the left side of (242) is a decreasing function. Finally, (237) follows from the fact that \( \lambda \geq 0 \) and \( e^{\frac{1}{x}} \geq \left( 1 + \frac{k}{2x} \right)^n \).

**Lemma 7** Asymptotic Upper Bound on \( K(k, \lambda, n, t) \). Let \( c \in (0, 1/(2 \log e)) \), and \( \delta \in (0, 1/(k-1)) \) be fixed and \( n > 2 \) be an integer. Assume that \( \theta \in \mathcal{R}_0 \) (defined in (150)) satisfies \( \theta_k \geq 1/k \). If

\[ \left\lfloor n(1-\delta)\theta_i \right\rfloor \leq t_i \leq \left\lceil n(1+\delta)\theta_i \right\rceil \]  \hspace{1cm} (243)

for \( i \in \{1, \ldots, k-1\} \),

then

\[ K(k, \lambda, n, t) \leq K(k, \lambda, n, c(1-\delta) \log n, \ldots, c(1-\delta) \log n, (1-(k-1)\delta)n/k) \]  \hspace{1cm} (244)

\[ = M(k, \lambda, n, c, \delta). \]  \hspace{1cm} (245)

Furthermore,

\[ \lim_{n \to \infty} M(k, \lambda, n, c, \delta) = 1. \]  \hspace{1cm} (246)

**Proof:** Note that

\[ t_i \geq c(1-\delta) \log n \]  \hspace{1cm} (247)

\[ t_k \geq \frac{(1-(k-1)\delta)n}{k}. \]  \hspace{1cm} (248)

which, in turn, imply that

\[ \left( 1 + \frac{1}{2t_i} \right)^{t_i} \left( 2 - e^{\frac{1}{t_i+e}} \right) \geq \left( 1 + \frac{1}{2c(1-\delta) \log n} \right)^{c(1-\delta) \log n} \left( 2 - e^{\frac{1}{2c(1-\delta) \log n+e}} \right), \]  \hspace{1cm} (249)

\[ \left( 1 + \frac{1}{2t_k} \right)^{t_k} \left( 2 - e^{\frac{1}{t_k+e}} \right) \geq \left( 1 + \frac{1}{2(1-(k-1)\delta)n/k} \right)^{(1-(k-1)\delta)n/k} \left( 2 - e^{\frac{1}{2(1-(k-1)\delta)n/k+e}} \right). \]  \hspace{1cm} (250)

Hence, inequality (244) follows. It is straightforward to see the limit in (246).

\[ 27 \]
APPENDIX G

LEMmAS FOR THE PROOF IN SECTION IV-C

In the proofs of Lemmas 8, 9 and 10, we use the following bound: for \( \theta \in (0,1/2) \) and \( \delta \in (0,1) \),

\[
|\tau - \theta| \leq \delta \theta \implies d(\tau|\theta) \geq \frac{1}{2} \frac{(1-\delta)(\tau - \theta)^2}{\theta(1-\theta)},
\]

in nats. In particular, when \( 0 < \tau \leq \theta \leq 1/2 \)

\[
d(\tau|\theta) \geq \frac{1}{2} \frac{(\tau - \theta)^2}{\theta(1-\theta)}.
\]

To show (251) and (252), we rely on Taylor’s theorem:

\[
d(\tau|\theta) = \frac{1}{2} \frac{(\tau - \theta)^2}{\theta(1-\theta)} + \frac{2\alpha - 1}{6\alpha^2(1-\alpha)^2}(\tau - \theta)^3,
\]

for some \( \alpha \) in between \( \tau \) and \( \theta \).

**Lemma 8.** Let \( c \in (0, 1/(2 \log e)) \) and fix \( \delta \in (0,1) \).

\[
\lim_{n \to \infty} \sup_{\theta_1 \in [c \log n/n, 1/2]} S_1(\lambda, \theta_1, n, \delta) = 0.
\]

where \( S_1(\lambda, \theta_1, n, \delta) \) is defined in (130).

**Proof:** Assume that \( n \) is a sufficiently large integer, let \( \theta_1 \in [c \log n/n, 1/2] \) be given. Then

\[
S_1(\lambda, \theta_1, n, \delta) \leq \sum_{t_1=1}^{\lfloor n(1-\delta)\theta_1 \rfloor} \sqrt{\frac{n}{2\pi t_1 t_2}} \exp\left(-\frac{1}{2} n(1+\lambda) \left( \frac{t_1}{\theta_1} - \theta_1 \right)^2 \right) 3^\lambda e^{\frac{1}{t_1}}
\]

\[
\leq \sum_{t_1=1}^{\lfloor n(1-\delta)\theta_1 \rfloor} \sqrt{\frac{n}{2\pi t_1 t_2}} \exp\left(-\frac{1}{2} n(1+\lambda) \delta^2 \theta_1 \right) 3^\lambda e^{\frac{1}{t_1}}
\]

\[
\leq n(1-\delta)\theta_1 \sum_{t_1=1}^{\lfloor n(1-\delta)\theta_1 \rfloor} \sqrt{\frac{n}{2\pi (n-1)}} \exp\left(-\frac{1}{2} n(1+\lambda) \delta^2 \theta_1 \right) 3^\lambda e^{\frac{1}{t_1}},
\]

where (255) is due to (252) and the uniform upper bound on \( K(\lambda, n, t_1) \) given in Lemma 6 in Appendix F. (256) follows because \( \left( \frac{t_1}{n} - \theta_1 \right)^2 \geq \delta^2 \theta_1^2 \), (257) follows because for \( 1 \leq t_1 \leq \lfloor n(1-\delta)\theta_1 \rfloor \)

\[
t_1 t_2 \geq n - 1.
\]

Since the supremum in

\[
\sup_{\theta_1 \in [c \log n/n, 1/2]} n(1-\delta)\theta_1 \sqrt{\frac{n}{2\pi (n-1)}} \exp\left(-n(1+\lambda) \frac{1}{2} \delta^2 \theta_1 \right) 3^\lambda e^{\frac{1}{t_1}},
\]

is attained at \( \theta_1 = \frac{c \log n}{n} \), it follows that (254) holds. ■

**Lemma 9.** Let \( c \in (0, 1/(2 \log e)) \).

\[
\lim_{\delta \to 0} \lim_{n \to \infty} \sup_{\theta_1 \in [c \log n/n, 1/2]} S_2(\lambda, \theta_1, n, \delta) \leq (1 + \lambda)^{-\frac{3}{4}},
\]

(260)
where $S_2(\lambda, \theta_1, n, \delta)$ is defined in (131).

**Proof:** Assume that $n$ is a sufficiently large integer, let $\theta_1 \in \left[\frac{c \log n}{n}, \frac{1}{2}\right]$ be given and define

$$\sigma_n = \sqrt{\frac{\theta_1(1 - \theta_1)}{n(1 + \lambda)(1 - \delta)}}. \quad (261)$$

We have

$$S_2(\lambda, \theta_1, n, \delta) \leq \frac{M(2, \lambda, n, c, \delta)}{(1 - \delta)^3(1 + \lambda)} \frac{1 - \theta_1}{1 + \lambda} \sum_{t_1 = [n(1 - \delta)\theta_1]}^{[n(1 + \delta)\theta_1]} \frac{1}{n} \frac{1}{\sqrt{2\pi \sigma_n}} \exp\left(-\frac{(\frac{t_1}{n} - \theta_1)^2}{2\sigma_n^2}\right) \quad (262)$$

$$\leq \frac{M(2, \lambda, n, c, \delta)}{(1 - \delta)^3(1 + \lambda)} \sum_{t_1 = [n(1 - \delta)\theta_1]}^{[n(1 + \delta)\theta_1]} \frac{1}{n} \frac{1}{\sqrt{2\pi \sigma_n}} \exp\left(-\frac{(\frac{t_1}{n} - \theta_1)^2}{2\sigma_n^2}\right), \quad (263)$$

where (262) is due to (251), the bound on $K(\lambda, n, t_1)$ for the given range of $t_1$ (see Lemma 7 in Appendix F), and the fact that for $[n(1 - \delta)\theta_1] \leq t_1 \leq [n(1 + \delta)\theta_1]$,

$$\sqrt{t_1(1 - t_1)} \geq n\sqrt{(1 - \delta)\theta_1(1 - (1 + \delta)\theta_1)}. \quad (264)$$

(263) follows because for $\theta_1 \in \left[\frac{c \log n}{n}, \frac{1}{2}\right]$,

$$\sqrt{\frac{1 - \theta_1}{1 - (1 + \delta)\theta_1}} = \sqrt{\frac{1 + \delta\theta_1}{1 - (1 + \delta)\theta_1}} \leq \frac{1}{\sqrt{1 - \delta}}. \quad (265)$$

In light of Lemma 7 in Appendix F,

$$\lim_{n \to \infty} M(2, \lambda, n, c, \delta) = 1. \quad (267)$$

Moreover, the Riemann sum in (263) can be upper bounded as

$$\limsup_{n \to \infty} \sum_{t_1 = [n(1 - \delta)\theta_1]}^{[n(1 + \delta)\theta_1]} \frac{1}{n} \frac{1}{\sqrt{2\pi \sigma_n}} \exp\left(-\frac{(\frac{t_1}{n} - \theta_1)^2}{2\sigma_n^2}\right) \leq 1. \quad (268)$$

It follows that (260) holds. \hfill \blacksquare

**Lemma 10.** Let $c \in (0, 1/(2 \log e))$ and fix $\delta \in (0, 1)$.

$$\lim_{n \to \infty} \sup_{\theta_1 \in \left[\frac{c \log n}{n}, \frac{1}{2}\right]} S_3(\lambda, \theta_1, n, \delta) = 0, \quad (269)$$

where $S_3(\lambda, \theta_1, n, \delta)$ is defined in (132).

**Proof:** The proof of this lemma is more involved than that of Lemma 8. To proceed, using Pinsker’s inequality (e.g., [33, Ex. 3.18]), namely

$$d(\tau || \theta) \geq 2(\tau - \theta)^2, \quad (270)$$
we first prove that

\[
\lim_{n \to \infty} \sup_{\theta_1 \in \left[ n^{-\frac{\beta}{2}}, \frac{1}{2} \right]} S_3(\lambda, \theta_1, n, \delta) = 0, \tag{271}
\]

where \( \beta \in (0, 1) \) is a fixed constant. Then, we show that

\[
\lim_{n \to \infty} \sup_{\theta_1 \in \left[ \frac{\log n}{n}, n^{-\frac{\beta}{2}} \right]} S_3(\lambda, \theta_1, n, \delta) = 0, \tag{272}
\]

with the help of Lemma 11 in Appendix H. Fix a constant \( \beta \in (0, 1) \), and assume that \( n \) is a sufficiently large integer.

First, let \( \theta_1 \in \left[ n^{-\frac{\beta}{2}}, \frac{1}{2} \right] \) be arbitrary and note that

\[
S_3(\lambda, \theta_1, n, \delta) \leq \sum_{t_1 = \lceil n(1+\delta)\theta_1 \rceil}^{n-1} \sqrt{\frac{n}{2\pi t_1 t_2}} \exp\left( -2n(1+\lambda)\left(\frac{t_1}{n} - \theta_1\right)^2 \right) 3^\lambda e^{\frac{1}{12}} \tag{273}
\]

\[
\leq \sum_{t_1 = \lceil n(1+\delta)\theta_1 \rceil}^{n-1} \sqrt{\frac{n}{2\pi t_1 t_2}} \exp\left( -2n(1+\lambda)\delta^2\theta_1^2 \right) 3^\lambda e^{\frac{1}{12}} \tag{274}
\]

\[
\leq \sqrt{\frac{n^3}{2\pi(n-1)}} \exp\left( -2(1+\lambda)\delta^2 n^{1-\beta} \right) 3^\lambda e^{\frac{1}{12}}, \tag{275}
\]

where (273) follows from Lemma 6 in Appendix F and (270), (274) follows because \( \left(\frac{t_1}{n} - \theta_1\right)^2 \geq \delta^2\theta_1^2 \), (275) follows because \( \theta_1 \geq n^{-\frac{\beta}{2}} \) and for \( \lceil n(1+\delta)\theta_1 \rceil \leq t_1 \leq n - 1 \),

\[
t_1 t_2 \geq (n-1). \tag{276}
\]

Thus, (275) implies that

\[
\sup_{\theta_1 \in \left[ n^{-\frac{\beta}{2}}, \frac{1}{2} \right]} S_3(\lambda, \theta_1, n, \delta) \leq \sqrt{\frac{n^3}{2\pi(n-1)}} \exp\left( -2(1+\lambda)\delta^2 n^{1-\beta} \right) 3^\lambda e^{\frac{1}{12}} \tag{277}
\]

Since \( \beta < 1 \),

\[
\lim_{n \to \infty} \sqrt{\frac{n^3}{2\pi(n-1)}} \exp\left( -2(1+\lambda)\delta^2 n^{1-\beta} \right) 3^\lambda e^{\frac{1}{12}} = 0, \tag{278}
\]

and since

\[
0 \leq \sup_{\theta_1 \in \left[ n^{-\frac{\beta}{2}}, \frac{1}{2} \right]} S_3(\lambda, \theta_1, n, \delta) \tag{279}
\]

\[
\leq \sqrt{\frac{n^3}{2\pi(n-1)}} \exp\left( -2(1+\lambda)\delta^2 n^{1-\beta} \right) 3^\lambda e^{\frac{1}{12}}, \tag{280}
\]

it follows that (271) holds.
Second, let $\theta_1 \in \left[ \frac{c \log n}{n}, n^{-\frac{\beta}{2}} \right]$ be arbitrary and fix some constant $\kappa \in (0, \frac{1}{2})$. Further, separate $S_3(\lambda, \theta_1, n, \delta)$ into two sums as follows

$$S_3(\lambda, \theta_1, n, \delta) = \tilde{S}_3^1(\kappa, \lambda, \theta_1, n, \delta) + \tilde{S}_3^2(\kappa, \lambda, \theta_1, n, \delta),$$  \hspace{1cm} (281)

where

$$\tilde{S}_3^1(\kappa, \lambda, \theta_1, n, \delta) = \sum_{t_1 = [\kappa \mathbf{n}]}^{n-1} \sqrt{\frac{n}{2\pi t_1 t_2}} \exp\left( -n(1+\lambda) d \left( \frac{t_1}{n}, \theta_1 \right) \right) K(\lambda, n, t_1),$$  \hspace{1cm} (282)

$$\tilde{S}_3^2(\kappa, \lambda, \theta_1, n, \delta) = \sum_{t_1 = [n(1+\delta) \theta_1]}^{[n\kappa]} \sqrt{\frac{n}{2\pi t_1 t_2}} \exp\left( -n(1+\lambda) d \left( \frac{t_1}{n}, \theta_1 \right) \right) K(\lambda, n, t_1).$$  \hspace{1cm} (283)

Regarding $\tilde{S}_3^1(\kappa, \lambda, \theta_1, n, \delta)$, we have

$$\tilde{S}_3^1(\kappa, \lambda, \theta_1, n, \delta) \leq \sum_{t_1 = [\kappa \mathbf{n}]}^{n-1} \sqrt{\frac{n}{2\pi t_1 t_2}} \exp\left( -2n(1+\lambda) \left( \frac{t_1}{n} - \theta_1 \right)^2 \right) 3^\lambda e^{\frac{\lambda}{2}}$$  \hspace{1cm} (284)

$$\leq \frac{n}{\sqrt{2\pi \kappa}} \exp\left( -2n(1+\lambda) \left( \kappa - n^{-\frac{\beta}{2}} \right)^2 \right) 3^\lambda e^{\frac{\lambda}{2}},$$  \hspace{1cm} (285)

where (284) follows from Lemma 6 in Appendix F and (270), (285) follows because $\frac{t_1}{n} - \theta_1 \geq \kappa - n^{-\frac{\beta}{2}}$ and $\sqrt{t_1 t_2} \geq \sqrt{\kappa \mathbf{n}}$ for $[n\kappa] \leq t_1 \leq n - 1$ and $c \log n / n \leq \theta_1 \leq n^{-\frac{\beta}{2}}$. Hence,

$$0 \leq \sup_{\theta_1 \in \left[ \frac{c \log n}{n}, n^{-\frac{\beta}{2}} \right]} \tilde{S}_3^1(\kappa, \lambda, \theta_1, n, \delta) \leq \frac{n}{\sqrt{2\pi \kappa}} \exp\left( -2n(1+\lambda) \left( \kappa - n^{-\frac{\beta}{2}} \right)^2 \right) 3^\lambda e^{\frac{\lambda}{2}},$$  \hspace{1cm} (286)

and

$$\lim_{n \to \infty} \sup_{\theta_1 \in \left[ \frac{c \log n}{n}, n^{-\frac{\beta}{2}} \right]} \tilde{S}_3^1(\kappa, \lambda, \theta_1, n, \delta) = 0.$$  \hspace{1cm} (287)

Regarding $\tilde{S}_3^2(\kappa, \lambda, \theta_1, n, \delta)$, we have

$$0 \leq \frac{3^\lambda e^{\frac{\lambda}{2}}}{\sqrt{2\pi \kappa}} \sum_{t_1 = [n(1+\delta) \theta_1]}^{[n\kappa]} \frac{1}{\sqrt{t_1 t_2}} \exp\left( -n(1+\lambda) d \left( \frac{t_1}{n}, \theta_1 \right) \right),$$  \hspace{1cm} (288)

where (289) follows from Lemma 6 in Appendix F. Let $\theta_1^* \in \left[ \frac{c \log n}{n}, n^{-\frac{\beta}{2}} \right]$ be the maximizer of the right side in (289).

Note that

$$\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{t_1 = [n(1+\delta) \theta_1^*]}^{[n\kappa]} \frac{1}{\sqrt{t_1 t_2}} e^{-n(1+\lambda) d \left( \frac{t_1}{n}, \theta_1^* \right)}$$  \hspace{1cm} (289)

$$\leq \limsup_{n \to \infty} \frac{1}{\sqrt{n}} \int_{(1+\delta) \theta_1^*}^{\kappa} \frac{n}{\sqrt{\tau(1-\tau)}} e^{-n(1+\lambda) d(\tau \theta_1^*)} d\tau$$  \hspace{1cm} (290)
\[
\limsup_{n \to \infty} \frac{(1 + \lambda)^{-1} \ln^{-1}(1 + \delta)}{\sqrt{n(1 + \delta)\theta_1^* (1 - (1 + \delta)\theta_1^*)}} \\
\leq \limsup_{n \to \infty} \frac{(1 + \lambda)^{-1} \ln^{-1}(1 + \delta)}{\sqrt{(1 + \delta)c \log n (1 - (1 + \delta)n^{-\frac{2}{3}})}} \\
\leq 0,
\]

where (290) follows after noticing that for any \(\theta \in \left[\frac{c \log n}{n}, n^{-\frac{2}{3}}\right]\), the function
\[
g_\theta(x) = \frac{1}{\sqrt{x(1 - x)}} e^{-n(1 + \lambda)d(x\|\theta)}
\]
is a decreasing function in \(x \in \left((1 + \delta)\theta, \kappa\right)\) and therefore the corresponding Riemann sum in the left side of (290) can be upper bounded by the integral in its right side and (291) follows from Lemma 11 in Appendix H. Hence,
\[
\lim_{n \to \infty} \sup_{\theta_1 \in \left[\frac{c \log n}{n}, n^{-\frac{2}{3}}\right]} \tilde{S}_3^2(\kappa, \lambda, \theta_1, n, \delta) = 0.
\]

As a result of (287) and (295), (272) holds. The desired result follows since we have established (271) and (272).

### Appendix H

**Upper Bound for the Integral in (290)**

**Lemma 11.** Let \(c \in (0, 1/(2 \log e))\) and \(\lambda \in (0, \infty)\). Fix \(\beta \in (0, 1), \delta \in (0, 1)\) and \(\kappa \in (0, 1/2)\). For any \(\theta_1 \in \left[\frac{c \log n}{n}, n^{-\frac{2}{3}}\right]\)
\[
\int_{(1 + \delta)\theta_1}^\kappa \frac{n(1 + \lambda)}{\sqrt{\tau(1 - \tau)}} e^{-n(1 + \lambda)d(\tau\|\theta_1)} d\tau \leq \frac{\ln^{-1}(1 + \delta)}{\sqrt{(1 + \delta)\theta_1^*(1 - (1 + \delta)\theta_1^*)}}.
\]

**Proof:** Abbreviate
\[
a_n = n(1 + \lambda), \\
\varphi(\tau) = \frac{1}{\sqrt{\tau(1 - \tau)}}, \\
\phi(\tau) = d(\tau\|\theta_1).
\]

Applying integration by parts yields
\[
\int a_n \varphi(\tau)e^{-a_n \phi(\tau)} d\tau = -\frac{\varphi(\tau)}{\phi'(\tau)} e^{-a_n \phi(\tau)} + \int e^{-a_n \phi(\tau)} \frac{d}{d\tau} \left(\frac{\varphi(\tau)}{\phi'(\tau)}\right) d\tau.
\]

For \(\tau \in [(1 + \delta)\theta_1, \kappa]\), we have
\[
\frac{d}{d\tau} \left(\frac{\varphi(\tau)}{\phi'(\tau)}\right) \leq 0,
\]

32
because \( \varphi(\tau) \) is a decreasing function and \( \phi(\tau) \) is an increasing convex function for the given range of \( \tau \). Hence, we see that

\[
\int_{(1+\delta)\theta_1}^{\kappa} a_n \varphi(\tau) e^{-a_n \phi(\tau)} d\tau \leq \left. \frac{\varphi(\tau)}{\phi'(\tau)} e^{-a_n \phi(\tau)} \right|_{\tau=(1+\delta)\theta_1}^{\kappa} \leq \left. \frac{\varphi((1+\delta)\theta_1)}{\ln(1+\delta)} \right|_{\tau=(1+\delta)\theta_1} \leq \frac{\varphi((1+\delta)\theta_1)}{\ln(1+\delta)},
\]

(302)

where (303) follows because

\[
\left. \frac{\varphi(\tau)}{\phi'(\tau)} e^{-a_n \phi(\tau)} \right|_{\tau=\kappa} \geq 0,
\]

(305)
since \( \kappa \leq 1/2 \), and finally, (304) follows because

\[
\phi'(\tau)e^{a_n \phi(\tau)} \bigg|_{\tau=(1+\delta)\theta_1} \geq \left( \ln(1+\delta) - \ln \left( 1 - \frac{\delta\theta_i}{1-\theta_1} \right) \right)e^{a_n \phi(\tau)} \geq \ln(1+\delta).
\]

(306)

(307)

\begin{itemize}
  \item \[D(\tau\|\theta) = \sum_{i=1}^{k} \left( \frac{(\tau_i - \theta_i)^2}{2\theta_i} - \frac{(\tau_i - \theta_i)^3}{6\theta_i^2} \right), \quad \text{for some } \alpha = (\alpha_1, \ldots, \alpha_k) \in \Delta^{k-1} \text{ such that } \alpha_i \text{ lies between } \tau_i \text{ and } \theta_i.\]
  \item \textbf{Lemma 12.} The function defined in (184) satisfies
    \[\lim_{\delta \to 0} \limsup_{n \to \infty} \sup_{\theta \in \mathcal{K}_n, \theta_k \geq 1/k} S_1(k, \lambda, \theta, n, \delta) \leq (1 + \lambda)^{-\frac{k-1}{2}}.\]
\end{itemize}
Proof: Assume that \( n \) is a sufficiently large integer, and let \( \theta \in \mathcal{R}_0 \) with \( \theta_k \geq 1/k \) be given. Define

\[
\Sigma_n = \frac{J^{-1}(\theta, P_{V|Y})}{n(1+\lambda)(1-(k-1)\delta)}.
\]

We invoke (308) with

\[
\tau' \leftarrow \left( \frac{t_1}{n}, \ldots, \frac{t_{k-1}}{n} \right).
\]

Hence,

\[
S_1(k, \lambda, \theta, n, \delta) \leq M(k, \lambda, n, c, \delta) \left( \frac{\delta}{\sqrt{1-(k-1)\delta}} \right)^{\frac{k-1}{2}} \prod_{t \in \mathcal{N}_0^\theta} \sum_{i=1}^{k} \frac{e^{-\frac{1}{2}(\tau' - \theta')^T \Sigma_n^{-1}(\tau' - \theta')}}{n^{k-1}(2\pi)^{\frac{k-1}{2}} |\Sigma_n|}\]

(315)

where (315) is due to (308), the bound on \( K(k, \lambda, n, t) \) when \( t \in \mathcal{N}_0^\theta \) (see Lemma 7 in Appendix F), and the fact that for \( t \in \mathcal{N}_0^\theta \),

\[
\prod_{i=1}^{k} t_i^{1/2} \geq n^{k/2} (1-\delta)^{\frac{k-1}{2}} \sqrt{\theta_1 \cdots \theta_{k-1}} \sqrt{(1+\delta)\theta_k - \delta},
\]

(317)

(316) follows because \( \theta_k \geq 1/k \) implies

\[
\sqrt{\frac{\theta_k}{(1+\delta)\theta_k - \delta}} = \sqrt{1 + \frac{\delta(1-\theta_k)}{1-(1+\delta)(1-\theta_k)}} \leq \frac{1}{\sqrt{1-(k-1)\delta}}.
\]

(318)

(319)

In light of Lemma 7 in Appendix F,

\[
\lim_{n \to \infty} M(k, \lambda, n, c, \delta) = 1.
\]

(320)

Since the multi-variable Riemann sum in (316) can be upper bounded as

\[
\limsup_{n \to \infty} \sum_{1 \leq t_i \leq \theta_i \forall i, t \in \mathcal{N}_0^\delta} \frac{e^{-\frac{1}{2}(\tau' - \theta')^T \Sigma_n^{-1}(\tau' - \theta')}}{n^{k-1}(2\pi)^{\frac{k-1}{2}} |\Sigma_n|} \leq 1,
\]

(321)

we can conclude that (312) holds.

Lemma 13. The function defined in (185) satisfies

\[
\lim_{n \to \infty} \sup_{\theta \in \mathcal{R}_0, \theta_k \geq 1/k} S_2(k, \lambda, \theta, n, \delta) = 0.
\]

(322)
Proof: Assume that \( n \) is a sufficiently large integer, and let \( \theta \in \mathcal{R}_0 \) with \( \theta_k \geq 1/k \) be given. Recall the definition of \( \mathcal{N}_q^v \) in (182), and note that if
\[
 t \notin \mathcal{N}_q^v,
\]
then there must exist \( i \in \{1, \ldots, k-1\} \) such that
\[
t_i \notin I_{\delta, \theta_i, n} = \left[ \lfloor n(1-\delta)\theta_i \rfloor, \lceil n(1+\delta)\theta_i \rceil \right].
\]
Moreover, by symmetry, we can write
\[
S_2(k, \lambda, \theta, n, \delta) = \sum_{1 \leq t_i \leq n-(k-1)} \sum_{t_i \geq 1 \forall i} \sum_{t_2, \ldots, t_k} \frac{K(k, \lambda, n, t)}{(2\pi)^{k/2}} \sqrt{\prod_{i=1}^k t_i} e^{-n(1+\lambda)D(\hat{P}_n^v \parallel P_{Y|V=\theta})}
\]
\[
\leq \sum_{1 \leq t_i \leq n-(k-1)} \sum_{t_i \geq 1 \forall i} \sum_{t_2, \ldots, t_k} M(k, \lambda) \sqrt{\prod_{i=1}^k t_i} e^{-n(1+\lambda)D(\hat{P}_n^v \parallel P_{Y|V=\theta})}
\]
\[
= \sum_{1 \leq t_i \leq n-(k-1)} \sqrt{\frac{n}{2\pi t_i(n-t_i)}} e^{-n(1+\lambda)D(\hat{P}_n^v \parallel P_{Y|V=\theta})} M(k, \lambda) \Xi(k-1, \lambda, \theta') n - t_i, \quad (327)
\]
where (326) is due to the uniform upper bound on \( K(k, \lambda, n, t) \) in Lemma 6, in (327), \( \theta' = \left( \frac{\theta_1}{1-\theta_1}, \ldots, \frac{\theta_k}{1-\theta_1} \right) \) and the function denoted by \( \Xi(k, \lambda, \theta, n) \) is defined in (181). By invoking Lemma 5 in Appendix E, we see that \( \Xi(k-1, \lambda, \theta', n - t_i) \) can be upper bounded by a constant depending only on \( \lambda \) and \( k \). On the other hand, the sum without the factor \( \Xi \) vanishes as \( n \to \infty \) (see Lemmas 8 and 10). Therefore, (322) follows.

**Appendix J**

**Jeffreys’ Mixture is not Minimax**

The fact that Jeffreys’ prior is capacity achieving (or least favorable) follows from the converse proof of Theorem 2. Therefore, Jeffreys’ mixture is maximin for Rényi redundancy. Parallel to the results in [19] and [6], Lemma 14 below proves that Jeffreys’ mixture is not minimax.

**Lemma 14.** For any \( l \in \{1, \ldots, k-1\} \),
\[
\liminf_{n \to \infty} \left\{ \sup_{\theta} D_{1+\lambda}(P_{Y|V=\theta} \parallel Q_{Y|V}^n) - \frac{k-1}{2} \log \frac{n}{2\pi} \right\} \geq \log \frac{\Gamma(k/2)}{\Gamma(k/2)} - \frac{k-1}{2\lambda} \log(1+\lambda) + \frac{k-l}{2} \left( \log 2 + \frac{\log(1+\lambda)}{\lambda} \right),
\]
where the supremization is over all \( \theta \in \Delta^{k-1} \) that are on the face of the simplex so that at most \( l \) of its components are known to be non-zero.
Note that the third term in the right side of (328) interpolates the extra constants \( \frac{k-l}{2} \log(2e) \) when \( \lambda = 0 \) and \( \frac{k-l}{2} \log 2 \) when \( \lambda = \infty \), shown in [19] and [6], respectively.

**Proof:** Assuming without loss of generality that the last \( k - l \) entries of \( \theta \) are equal to zero simplifies the notation. Otherwise, the proof remains identical. Define

\[
\tilde{\theta} = (\theta_1, \ldots, \theta_l) \in \Delta^{l-1},
\]

\[
L(k, l, n) = \left( 1 + \frac{k}{2n} \right)^{n+\frac{l}{2}} - 2 e^{-\frac{k}{2n} + \frac{l}{4n}}
\]

where the supremization in the left side of (334) is over all \( \bar{Q} \). Define

\[
\theta_i \text{ denotes the } i\text{-th entry of } \theta.
\]

Note that the third term in the right side of (328) interpolates the extra constants \( \frac{k-l}{2} \log(2e) \) when \( \lambda = 0 \) and \( \frac{k-l}{2} \log 2 \) when \( \lambda = \infty \), shown in [19] and [6], respectively.

\[
D_{1+\lambda}(P_{Y^n|V=\bar{\theta}}\|Q_{Y^n}^{(l-1)}) = D_{1+\lambda}(P_{Y^n|V=\bar{\theta}}\|Q_{Y^n}^{(l-1)}) + \log \frac{\Gamma(l/2) \Gamma(n+k/2)}{\Gamma(k/2) \Gamma(n+l/2)}
\]

\[
\geq D_{1+\lambda}(P_{Y^n|V=\bar{\theta}}\|Q_{Y^n}^{(l-1)}) + \log \frac{\Gamma(l/2)}{\Gamma(k/2)} + \frac{k-l}{2} \log n + \log L(k, l, n),
\]

where \( Q_{Y^n}^{(l-1)} \) denotes the Jeffreys’ mixture when the underlying parameter space is the \( (l-1) \)-dimensional simplex, (331) follows from the fact that

\[
\frac{D_{k}(1/2, \ldots, 1/2)}{D_{k}(t_1 + 1/2, \ldots, t_l + 1/2, 1/2, \ldots, 1/2)} = \frac{D_{k}(1/2, \ldots, 1/2)}{D_{k}(t_1 + 1/2, \ldots, t_l + 1/2, \Gamma(k/2) \Gamma(n+l/2)),}
\]

and (332) follows from Stirling’s approximation which can be seen in (84). Since

\[
\sup_{\theta} D_{1+\lambda}(P_{Y^n|V=\theta}\|Q_{Y^n}^{(l-1)}) \geq \sup_{\theta \in \Delta^{l-1}} D_{1+\lambda}(P_{Y^n|V=\theta}\|Q_{Y^n}^{(l-1)}) + \log \frac{\Gamma(l/2)}{\Gamma(k/2)} + \frac{k-l}{2} \log n + \log L(k, l, n)
\]

\[
\geq \inf_{Q_{Y^n}} \sup_{\theta \in \Delta^{l-1}} D_{1+\lambda}(P_{Y^n|V=\theta}\|Q_{Y^n}) + \log \frac{\Gamma(l/2)}{\Gamma(k/2)} + \frac{k-l}{2} \log n + \log L(k, l, n),
\]

where the supremization in the left side of (334) is over all \( \theta \) whose last \( k - l \) entries are zero, the converse result in Section IV-B with \( k \leftarrow l \), and the fact that

\[
\lim_{n \to \infty} L(k, l, n) = 1,
\]

along with routine algebraic manipulations yield the desired result in (328).

**REFERENCES**

[1] S. Yagli, Y. Altuğ, and S. Verdú, “Minimax Rényi redundancy,” in 2017 IEEE International Symposium on Information Theory (ISIT), June 2017, pp. 2980–2984.

[2] T. M. Cover and J. A. Thomas, *Elements of Information Theory*, 2nd ed. Hoboken, NJ, USA: Wiley, 2006.

[3] I. Kontoyiannis and S. Verdú, “Optimal lossless data compression: non-asymptotics and asymptotics,” *IEEE Transactions on Information Theory*, vol. 60, no. 2, pp. 777–795, Feb. 2014.

[4] R. G. Gallager, “Source coding with side information and universal coding,” September 1976, unpublished manuscript. Available from: http://web.mit.edu/gallager/www/papers/paper5.pdf.
[5] B. Y. Ryabko, “Coding of a source with unknown but ordered probabilities,” *Problems of Information Transmission*, vol. 15, no. 2, pp. 134–138, Oct. 1979.

[6] Q. Xie and A. R. Barron, “Asymptotic minimax regret for data compression, gambling, and prediction,” *IEEE Transactions on Information Theory*, vol. 46, no. 2, pp. 431–445, Mar. 2000.

[7] Y. M. Shtarkov, “Universal sequential coding of single messages,” *Problemy Peredachi Informatsii*, vol. 23, no. 3, pp. 3–17, Jul.–Sep. 1987.

[8] J. Forster and M. K. Warmuth, “Relative expected instantaneous loss bounds,” *Journal of Computer and System Sciences*, vol. 64, no. 1, pp. 76–102, Feb. 2002.

[9] M. Drmota and W. Szpankowski, “Precise minimax redundancy and regret,” *IEEE Transactions on Information Theory*, vol. 50, no. 11, pp. 2686–2707, Nov. 2004.

[10] F. Liang and A. R. Barron, “Exact minimax strategies for predictive density estimation, data compression, and model selection,” *IEEE Transactions on Information Theory*, vol. 50, no. 11, pp. 2708–2726, Nov. 2004.

[11] J. W. Pratt, “Risk aversion in the small and in the large,” *Econometrica: Journal of the Econometric Society*, vol. 32, no. 1–2, pp. 122–136, Jan.–Apr. 1964.

[12] K. J. Arrow, *Aspects of the Theory of Risk-Bearing*. Helsinki, Finland: Yrjö Jahnssonin Säätiö, 1965.

[13] S. A. Ross, “Some stronger measures of risk aversion in the small and the large with applications,” *Econometrica: Journal of the Econometric Society*, vol. 49, no. 3, pp. 621–638, May 1981.

[14] L. L. Campbell, “A coding theorem and Rényi’s entropy,” *Information and Control*, vol. 8, no. 4, pp. 423–429, Aug. 1965.

[15] R. Sundaresan, “Guessing under source uncertainty,” *IEEE Transactions on Information Theory*, vol. 53, no. 1, pp. 269–287, Jan. 2007.

[16] S. Verdú, “c-mutual information,” in *2015 Information Theory and Applications Workshop*, San Diego, Feb. 2015, pp. 1–6.

[17] R. Sibson, “Information radius,” *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, vol. 14, no. 2, pp. 149–160, 1969.

[18] I. Csiszár, “Generalized cutoff rates and Rényi’s information measures,” *IEEE Transactions on Information Theory*, vol. 41, no. 1, pp. 26–34, Jan. 1995.

[19] Q. Xie and A. R. Barron, “Minimax redundancy for the class of memoryless sources,” *IEEE Transactions on Information Theory*, vol. 43, no. 2, pp. 646–657, Mar. 1997.

[20] L. D. Davisson, R. J. McEliece, M. B. Pursley, and M. S. Wallace, “Efficient universal noiseless source codes,” *IEEE Transactions on Information Theory*, vol. 27, no. 3, pp. 269–279, May 1981.

[21] L. Györfi, I. Páli, and E. C. van der Meulen, “There is no universal source code for an infinite source alphabet,” *IEEE Transactions on Information Theory*, vol. 40, no. 1, pp. 267–271, Jan. 1994.

[22] R. E. Krichevsky and V. K. Trofimov, “The performance of universal encoding,” *IEEE Transactions on Information Theory*, vol. 27, no. 2, pp. 199–207, Mar. 1981.

[23] J. Rissanen, “Universal coding, information, prediction, and estimation,” *IEEE Transactions on Information Theory*, vol. 30, no. 4, pp. 629–636, Jul. 1984.

[24] J. Rissanenn, “Stochastic complexity and modeling,” *The Annals of Statistics*, vol. 14, no. 3, pp. 1080–1100, Sep. 1986.

[25] N. Merhav, “On optimum strategies for minimizing the exponential moments of a loss function,” *Communications in Information and Systems*, vol. 11, no. 4, pp. 343–368, 2011.

[26] M. Hayashi, “Universal channel coding for general output alphabet,” 2015, [Online] Available: https://arxiv.org/abs/1502.02218.

[27] B. S. Clarke and A. R. Barron, “Information-theoretic asymptotics of Bayes methods,” *IEEE Transactions on Information Theory*, vol. 36, no. 3, pp. 453–471, May 1990.

[28] H. Jeffreys, “An invariant form for the prior probability in estimation problems,” in *Proceedings of the Royal Society of London Series A: Mathematical, Physical and Engineering Sciences*, vol. 186, no. 1007, Sep. 1946, pp. 453–461.

[29] T. van Erven and P. Harremoës, “Rényi divergence and Kullback-Leibler divergence,” *IEEE Transactions on Information Theory*, vol. 60, no. 7, pp. 3797–3820, Jul. 2014.

[30] S. Arimoto, “Information measures and capacity of order α for discrete memoryless channels,” in *Topics in Information Theory*, Proc. Coll. Math. Soc. János Bolyai. Keszthely, Hungary: Bolyai, 1975, pp. 41–52.
[31] H. Robbins, “A remark on Stirling’s formula,” *The American Mathematical Monthly*, vol. 62, no. 1, pp. 26–29, Jan. 1955.

[32] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th ed. Cambridge, U.K.: Cambridge University Press, 1963.

[33] I. Csiszar and J. Körner, *Information Theory: Coding Theorems for Discrete Memoryless Systems*, 2nd ed. Cambridge, U.K.: Cambridge University Press, 2011.