Realization of Global Symmetries in the Wilsonian Renormalization Group

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Abstract

We present a method to solve the master equation for the Wilsonian action in the antifield formalism. This is based on a representation theory for cutoff dependent global symmetries along the Wilsonian renormalization group (RG) flow. For the chiral symmetry, the master equation for the free theory yields a continuum version of the Ginsparg-Wilson relation. We construct chiral invariant operators describing fermionic self-interactions. The use of canonically transformed variables is shown to simplify the underlying algebraic structure of the symmetry. We also give another non-trivial example, a realization of SU(2) vector symmetry. Our formalism may be used for a non-perturbative truncation of the Wilsonian action preserving global symmetries.

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1 Introduction

The recent discovery of chiral symmetry on the lattice [1] has important implications not restricted to lattice theories. We strongly believe that it is the prototype of new realization of symmetry which is not compatible, in the ordinary sense, with a given regularization. The symmetry is present, though it undergoes deformation due to the regularization.

In the Wilsonian renormalization group (RG) [2], a promising non-perturbative approach to continuum theories, a regulator with the IR cutoff \( k \) is introduced to yield the Wilsonian effective action for lower momentum modes. The regularization is, often and sometimes inevitably, in conflict with the standard form of a symmetry. In such a case, the above realization may be the only possible way of preserving the symmetry.

In previous papers [3, 4, 5], we gave a general formalism for the realization of symmetries, called renormalized symmetries, along the RG flow. It applies to global as well as local symmetries. What plays a crucial role in describing renormalized symmetries is the master equation (ME) for the Wilsonian action. The ME defines an invariant hypersurface in the theory space, ie, in the space spanned with couplings. An important observation is that once a solution of the ME is found, it stays on the hypersurface during its RG evolution. The main task in this formulation is therefore to solve the ME at some scale of \( k \). In ref. [6], we gave a general perturbative method for solving the ME in gauge theories.

It is difficult but highly desirable to construct non-perturbative solutions to the ME. If such solutions are obtained, they can be used to make symmetry-preserving truncations of the effective action: the exact renormalized symmetry is realized in truncated effective action at every scale along the RG flow.

The purpose of this paper is to discuss, as a first step towards a consistent truncation, non-perturbative solutions to the ME for global symmetries. We consider the chiral symmetry and the SU(2) (vector) symmetry. Although there are regularizations which manifestly preserve these symmetries, we take here regularization schemes which are incompatible with the standard form of symmetries. By doing this, we can see how the renormalized symmetries are realized and how their algebraic structures are related to the ME.

In one of our previous papers [3], we showed that, for the chiral symmetry, the associated ME and the symmetry transformations are precisely the continuum counterparts of the Ginsparg-Wilson (GW) [7] relation and the Lüscher’s chiral transformations [1]. In this paper, we use the antifield formalism of Batalin-Vilkovisky [8] to construct fermionic interactions which are invariant under the renormalized chiral transformations. A Wilsonian action consisting of such invariants solves the ME and gives a symmetry-preserving truncation.

We start our construction of invariant interactions from the average action for a free theory, which is obtained via the “block-spin transformations” from the UV (microscopic)

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1For recent progress in this subject, see, for example, ref. [9].
fields to the IR (macroscopic) fields. After an appropriate canonical transformation in the field-antifield space, the average action generates a continuum analog of Lüscher’s chiral transformation. It is also shown that the corresponding algebra is related to the standard (cutoff independent) chiral algebra via a unitary transformation. We may use the algebra to define chiral projections and chiral charges in a consistent manner. Based on this representation theory of the cutoff dependent algebra, we easily obtain invariant interaction terms.

In addition to the chiral symmetry, we also give another non-trivial example, renormalized non-abelian SU(2) symmetry. Following the same procedure used for the chiral symmetry, we find that the ME for the free theory leads to a simple algebraic GW-like relation for the Dirac operator. New generators of the algebra, which depend on the IR cutoff, can be obtained from the standard generators via a similarity transformation. Using this fact, we may obtain a representation of the renormalized SU(2) symmetry and construct fermionic invariants.

This paper is organized as follows. In the next section, we consider the chiral symmetry using the antifield formalism. The section 3 describes the application of our formalism to SU(2) symmetry. The final section is devoted to discussion.

2 Chiral symmetry

Let us consider a generic UV action of the Dirac fields, \( S[\bar{\psi}, \psi] \), in \( d=4 \) Euclidean momentum space. The UV action given at a UV scale \( \Lambda \) is assumed to be invariant under the standard chiral transformations:

\[
\delta \psi(p) = i \, c \, \gamma_5 \psi(p),
\]

\[
\delta \bar{\psi}(p) = i \, c \, \bar{\psi}(p) \, \gamma_5,
\]

where \( c \) is a constant ghost with Grassmann parity \( \epsilon(c) = 1 \). Let \( \{ \psi^*(p), \bar{\psi}^*(p) \} \) be the antifields of \( \{ \psi(p), \bar{\psi}(p) \} \). These fields with \( \epsilon(\psi^*) = \epsilon(\bar{\psi}^*) = 0 \) play the role of source terms for \( \{ \delta \psi, \delta \bar{\psi} \} \).

According to the general formalism given in ref. [5], we make the block-spin transformation from the UV fields to the IR fields \( \{ \Psi(p), \bar{\Psi}(p) \} \). This is achieved by using a gaussian term

\[
\int \frac{d^4 p}{(2\pi)^4} \left( \bar{\Psi}(-p) - f_k(p^2) \bar{\psi}(-p) \right) \alpha^k(p^2) \left( \Psi(p) - f_k(p^2) \psi(p) \right)
\]

\[
\equiv \int_p \left( \Psi - f_k \bar{\psi} \right)(-p) \alpha^k \left( \Psi - f_k \psi \right)(p),
\]

where \( f_k(p^2) \) is a function for a “coarse graining”. For a given IR cutoff \( k \), \( f_k(p^2) \approx 0 \) for \( p^2 < k^2 \), and \( f_k(p^2) \approx 1 \) for \( p^2 > k^2 \). The cutoff function \( \alpha^k(p^2) \) is introduced to relate

\footnote{These two kinds of fields were introduced by Ginsparg and Wilson \[8\] in lattice theories, and by Wetterich \[9\] in continuum theories.}
the UV fields to the IR fields. It behaves as \( f_k \alpha^k \approx 1 \) for \( p^2 < k^2 \), and \( f_k \alpha^k \rightarrow \infty \) for \( p^2 > k^2 \). At the UV scale \( k = \Lambda \), \( f_\Lambda \approx 1 \), and \( \alpha^\Lambda \rightarrow \infty \).

Adding the antifield contributions and the gaussian term to \( S[\bar{\psi}, \psi] \), we have a cutoff dependent action

\[
S_k[\psi, \bar{\psi}, \Psi, \bar{\Psi}, \psi^*, \bar{\psi}^*] = S_0[\psi, \bar{\psi}]
+ \int_p \left[ \psi^*(-p)\delta \psi(p) + \delta \bar{\psi}(-p)\psi^*(p) + (\bar{\Psi} - f_k \bar{\psi})(-p)\alpha^k(\Psi - f_k \psi)(p) \right].
\]

Using this in the path integral over the UV fields, we may define a Wilsonian effective action called the average action. In order to formulate it, we introduce the antibracket

\[
\{ F, G \} \equiv \int_p \left[ \frac{\partial^r F}{\partial \Psi(-p)} \frac{\partial^r G}{\partial \Psi^*(p)} - \frac{\partial^r F}{\partial \Psi^*(-p)} \frac{\partial^r G}{\partial \Psi(p)} \right]
+ \frac{\partial^r F}{\partial \bar{\Psi}(-p)} \frac{\partial^r G}{\partial \bar{\Psi}^*(p)} - \frac{\partial^r F}{\partial \bar{\Psi}^*(-p)} \frac{\partial^r G}{\partial \bar{\Psi}(p)}
\]

for any functionals \( F \) and \( G \) of \( \{ \Psi, \bar{\Psi}, \Psi^*, \bar{\Psi}^* \} \). Then, the renormalized symmetry transformations are given by

\[
\begin{align*}
\delta \Psi &= (\Psi, W_k), \\
\delta \bar{\Psi} &= (\bar{\Psi}, W_k).
\end{align*}
\]

Invariance of the action under these transformations is expressed by the classical ME

\[
(W_k, W_k) = 0.
\]

For the free theory, it is easy to solve the ME. Note first that in the path integral (2.3), the effective source terms for the UV fields \( \psi \) and \( \bar{\psi} \) are proportional to \( (\bar{\Psi} - \Psi^* i \gamma_5 c(\alpha^k)) \) and \( (\Psi + i \gamma_5 c(\alpha^k)^{-1} \bar{\Psi}^*) \), respectively. Therefore, except the bilinear term \( \bar{\Psi} \alpha^k \Psi \) arising from the block-spin transformation (2.3), the average action is a functional of the block-spin variables only through these combinations. For instance, its free action takes the form

\[
W_k^{(0)} = \int_p \left[ (\bar{\Psi} - \Psi^* i \gamma_5 c(\alpha^k)^{-1})(-p)(D - \alpha^k)(\Psi + i \gamma_5 c(\alpha^k)^{-1} \bar{\Psi}^*)(p) \right]
+ \bar{\Psi}(-p)\alpha^k \Psi(p),
\]

(2.9)
where $D$ is the Dirac operator for the IR fields. Eqs. (2.7) and (2.8) written for the free action are,

$$
\delta \Psi(p) = i c \gamma_5 \left(1 - ((\alpha^k)^{-1}D(p))\right) \Psi(p), \\
\delta \bar{\Psi}(-p) = i c \bar{\Psi}(-p) \left(1 - ((\alpha^k)^{-1}D(p))\right) \gamma_5,
$$

(2.10)

and

$$
D(p)\gamma_5 + \gamma_5 D(p) = 2((\alpha^k)^{-1}D(p))\gamma_5 D(p).
$$

(2.11)

These are continuum analogs of the Lüscher’s chiral transformations [1] and the GW relation [7] for fermion fields on the lattice.

A simple solution to the GW relation is given by

$$
D(p) = \frac{\alpha^k(p)}{2} \left[ 1 + \frac{\bar{\Psi}(-p) \Psi(p) + \Psi^*(-p)\bar{\Psi}^*(p) + \Psi^*(-p)}{\sqrt{p^2 + (\alpha^k)^2(p)}} \right].
$$

(2.12)

This is the continuum version of the lattice solution given by Neuberger [10], and satisfies the boundary condition, $D \rightarrow i/\bar{\gamma}^k/2$ in the limit of $k \rightarrow 0$ and $k \rightarrow \Lambda$.

We now consider fermionic self-interaction terms. As we explained already, they must be functions of the block-spin variables, $(\bar{\Psi} - \Psi^*\gamma_5c(\alpha^k)^{-1})$ and $(\Psi + i\gamma_5c(\alpha^k)^{-1}\bar{\Psi})$. Obviously the dependence on the antifields makes it difficult to solve the ME. This suggests that the original block-spin variables may not suit for the description of interactions. Therefore, we look for a canonical transformation [11], [12] from the set of variables $\{\Psi, \bar{\Psi}, \Psi^*, \bar{\Psi}^*\}$ to a new set of variables $\{\Psi', \bar{\Psi}', \Psi'^*, \bar{\Psi}'^*\}$ in such a way that the interaction terms can be described only with the new fields $\{\Psi', \bar{\Psi}'\}$. The canonical transformation would give us simpler algebraic relations of renormalized symmetry. For the free theory under consideration, the generator of such transformation is found to be

$$
G[\Psi, \bar{\Psi}, \Psi'^*, \bar{\Psi}'^*] = \int_p \left[ \Psi'^*(-p)\Psi(p) + \bar{\Psi}(p)\bar{\Psi}'^*(p) + \Psi'^*(-p)c\gamma_5(\alpha^k)^{-1}(p)\bar{\Psi}'^*(p) \right].
$$

(2.13)

This leads to the relations

$$
\Psi'^*(-p) = \frac{\partial G}{\partial \Psi(p)} = \Psi'^*(-p), \\
\bar{\Psi}'^*(p) = \frac{\partial G}{\partial \bar{\Psi}(-p)} = \bar{\Psi}'^*(p), \\
\Psi'(p) = \frac{\partial G}{\partial \Psi'^*(p)} = \Psi(p) + ci\gamma_5\alpha_k^{-1}(p)\bar{\Psi}'^*(p), \\
\bar{\Psi}'(-p) = \frac{\partial G}{\partial \bar{\Psi}'(p)} = \bar{\Psi}(-p) + \Psi'^*(-p)c\gamma_5\alpha_k^{-1}(p).
$$

(2.14)

Replacing the old variables by new ones in (2.9), we obtain

$$
W^{(0)}_k = \int_p \left[ \Psi'^*(-p)D(p)\Psi'(p) + \Psi'^*(-p)c\gamma_5(p)\Psi' - \Psi'(p)c\gamma_5\Psi'^*(p) \right],
$$

(2.15)

Since the Jacobian factor from the canonical transformation (2.14) is trivial, there is no additional contribution to the free action.
\[ \hat{\gamma}_5(p) \equiv \gamma_5 \left( 1 - 2\hat{D}(p) \right), \]
\[ \hat{D}(p) \equiv (\alpha^k)^{-1}(p)D(p). \]  
(2.16)

From now on, we denote the new variables simply as \{\Psi, \bar{\Psi}\}, discarding primes. The renormalized chiral transformations are given by

\[ \delta \Psi(p) = i c \hat{\gamma}_5(p)\Psi(p), \]
\[ \delta \bar{\Psi}(-p) = i c \bar{\Psi}(-p)\gamma_5. \]  
(2.17)

The asymmetric transformations on \Psi and \bar{\Psi} appeared in (2.17) have been known in the lattice chiral theory. The ME \( (W_k^{(0)}, W_k^{(0)}) = 0 \) yields

\[ \hat{D}(p)\hat{\gamma}_5(p) + \gamma_5\hat{D}(p) = 0. \]  
(2.18)

This is another form of the GW relation and extensively used below. We find that

\[ \hat{\gamma}_5(p)^2 = 1, \quad \hat{\gamma}_5^\dagger(p) = \hat{\gamma}_5(p), \]  
(2.19)

where \( D^\dagger = \gamma_5D\gamma_5 \). Using the above relations (2.18) and (2.19), we can define the Lüsher’s chiral projection

\[ \hat{\Psi}_R = \frac{1 + \hat{\gamma}_5\Psi}{2}, \]
\[ \hat{\Psi}_L = \frac{1 - \hat{\gamma}_5\Psi}{2}, \]
\[ \bar{\Psi}_R = \bar{\Psi}\frac{1 + \gamma_5}{2}, \]
\[ \bar{\Psi}_L = \bar{\Psi}\frac{1 - \gamma_5}{2}. \]  
(2.20)

The projected fields obey

\[ \delta \hat{\Psi}_R = i c \hat{\Psi}_R, \quad \delta \hat{\Psi}_L = -i c \hat{\Psi}_L, \]
\[ \delta \bar{\Psi}_R = i c \bar{\Psi}_R, \quad \delta \bar{\Psi}_L = -i c \bar{\Psi}_L. \]  
(2.21)

Therefore, we may assign chiral charges (+, +, −, −) to \( (\hat{\Psi}_R, \hat{\Psi}_R, \hat{\Psi}_L, \hat{\Psi}_L) \).

It should be remarked that the \( \hat{\gamma}_5(p) \) is related to \( \gamma_5 \) via a unitary transformation

\[ \hat{\gamma}_5(p) = \gamma_5 \left( 1 - 2\hat{D}(p) \right) = V^\dagger(p)\gamma_5V(p), \]
\[ V(p) \equiv \frac{1}{\sqrt{1 - 2\hat{D}(p)}}, \]  
(2.22)

where \( V(p) \) is defined as a power series expansion wrt \( \hat{D}(p) \). The unitarity of \( V(p) \) follows from the GW relation (2.11). Using \( \hat{\gamma}_\mu(p) \equiv V^\dagger(p)\gamma_\mu V(p) \), we define operators

\[ O_-(p) = \left\{ 1, \hat{\gamma}_5(p), \frac{i}{2}[\hat{\gamma}_\mu(p), \hat{\gamma}_5(p)] \right\} \times (p - \text{dep. factor}), \]
\[ O_+(p) = \left\{ \hat{\gamma}_\mu(p), \hat{\gamma}_5(p)\hat{\gamma}_\mu(p) \right\} \times (p - \text{dep. factor}). \]  
(2.23)

\(^4\)Note that \( \Psi \) and \( \bar{\Psi} \) are independent and the mass term is \( \bar{\Psi}_R\Psi_R + \bar{\Psi}_L\Psi_L \) in our notation.
which satisfy
\[ O_-(p)\hat{\gamma}_5(p) - \hat{\gamma}_5(p)O_- = 0, \quad O_+(p)\hat{\gamma}_5(p) + \hat{\gamma}_5(p)O_+ = 0. \] (2.24)

It is then easy to list operators which are invariant under the parity and the chiral transformations (2.21). Using the shorthand notations,
\[ \bar{\Psi}_L O_- \hat{\Psi}_R \equiv \int_p \bar{\Psi}_L(-p)O_-\hat{\Psi}_R(p), \]
\[ \left( \bar{\Psi}_L O_- \hat{\Psi}_L \right) \left( \bar{\Psi}_R O_- \hat{\Psi}_R \right) \equiv \prod_{i=1}^{4} \int_{p_i} \delta (\sum_{i=1}^{4} p_i) \left( \bar{\Psi}_L(p_1)O_-\hat{\Psi}_L(p_2) \right) \times \left( \bar{\Psi}_R(p_3)O_-\hat{\Psi}_R(p_4) \right), \]

we may construct invariants as

1) bilinear operators: \( \bar{\Psi}_L O_\pm \hat{\Psi}_R + \bar{\Psi}_R O_\pm \hat{\Psi}_L \),
2) 4-fermi operators: \( \left( \bar{\Psi}_L O_\pm \hat{\Psi}_L \right) \left( \bar{\Psi}_R O_\pm \hat{\Psi}_R \right), \quad \left( \bar{\Psi}_L O_\pm \hat{\Psi}_R \right) \left( \bar{\Psi}_R O_\pm \hat{\Psi}_L \right), \quad \left( \bar{\Psi}_L O_\pm \hat{\Psi}_L \right)^2 \quad \left( \bar{\Psi}_R O_\pm \hat{\Psi}_R \right)^2 \).

Let us write down some typical invariants. As \( O_\pm(p) \), we take those with no additional momentum dependent factors (cf. (2.23)). The first examples are
\[ \left( \bar{\Psi}_L \hat{\Psi}_L \right) \left( \bar{\Psi}_R \hat{\Psi}_R \right), \quad \left( \bar{\Psi}_L \hat{\gamma}_\mu \hat{\Psi}_R \right)^2 \quad \left( \bar{\Psi}_R \hat{\gamma}_\mu \hat{\Psi}_L \right)^2. \] (2.25)

These are obtained by appropriately replacing the projection operators in the well-known invariants of the standard chiral symmetry
\[ (\bar{\Psi}_L \Psi_L)(\bar{\Psi}_R \Psi_R) = \frac{1}{4} \left[ (\bar{\Psi}\Psi)^2 - (\bar{\Psi}\hat{\gamma}_5\Psi)^2 \right], \]
\[ (\bar{\Psi}_R \hat{\gamma}_\mu \Psi_L)^2 + (\bar{\Psi}_L \hat{\gamma}_\mu \Psi_R)^2 = \frac{1}{4} \left[ (\bar{\Psi}\hat{\gamma}_\mu\Psi)^2 + (\bar{\Psi}\hat{\gamma}_5\hat{\gamma}_\mu\Psi)^2 \right]. \] (2.26)

The first invariant in (2.25) reads
\[ \left( \bar{\Psi}_L \hat{\Psi}_L \right) \left( \bar{\Psi}_R \hat{\Psi}_R \right) = \frac{1}{4} \left[ (\bar{\Psi}\Psi)^2 - (\bar{\Psi}\hat{\gamma}_5\Psi)^2 - 2 \left( \bar{\Psi}\hat{D}\Psi \right) (\bar{\Psi}\Psi) \right. \]
\[ \left. + 2 \left( \bar{\Psi}\hat{\gamma}_5\hat{D}\Psi \right) (\bar{\Psi}\hat{\gamma}_5\Psi) + \left( \bar{\Psi}\hat{D}\Psi \right)^2 - \left( \bar{\Psi}\hat{\gamma}_5\hat{D}\Psi \right)^2 \right]. \] (2.27)

A lattice counterpart of this operator (2.27) expressed by auxiliary fields was given in ref. [13]. In our general construction, we also have the second invariant in (2.23) which cannot be expressed as a polynomial in \( \hat{D} \).

The above invariants (2.23) reduce to the conventional ones (2.26) in the limit of \( \alpha \to \infty \) (\( \hat{D} \to 0 \)). In addition to such invariants, we have another type of invariants which vanish in this limit. For example,
\[ \left( \bar{\Psi}_L \hat{\Psi}_R \right)^2 + \left( \bar{\Psi}_R \hat{\Psi}_L \right)^2 = \frac{1}{2} \left[ (\bar{\Psi}\hat{D}\Psi)^2 + (\bar{\Psi}\hat{\gamma}_5\hat{D}\Psi)^2 \right]. \] (2.28)
The existence of this kind of invariants is characteristic to the renormalized chiral symmetry.

Let $W_k^{(1)}[\Psi, \bar{\Psi}]$ be an action that consists of invariant operators constructed above. Since the action $W_k^{(1)}$ contains no antifields, the total action $W_k = W_k^{(0)} + W_k^{(1)}$ is a non-perturbative solution of the ME (2.8), and gives a symmetry-preserving truncation of the average action.

3 Global SU(2) symmetry

In order to show that our formalism may be applicable to a non-abelian symmetry, we consider an SU(2) vector symmetry in this section. The antifield formalism requires to include constant ghosts $C^a$ for generators $T_a = \sigma_a/2 \ (a = 1, 2, 3)$. The SU(2) transformations on the UV fields read

$$
\delta \psi(p) = i C^a T_a \psi(p),
\delta \bar{\psi}(p) = -i C^a \bar{\psi}(p) T_a,
\delta C^a = -\frac{1}{2} \varepsilon_{abc} C^b C^c = -\frac{1}{2} (C \times C)^a.
$$

(3.1)

The free field action of the IR fields is given by

$$
W_k^{(0)} = \int_p \left[ (\bar{\Psi} - \Psi^* i C^a T_a (\alpha^k)^{-1})(-p)(D - \alpha^k)(p)(\Psi - i (\alpha^k)^{-1} C^a T_a \bar{\Psi}^*)(p) + \bar{\Psi}(-p) \alpha^k \Psi(p) \right] - \frac{1}{2} C^a (C \times C)^a,
$$

(3.2)

with the matrix $\alpha^k$ chosen as

$$
\alpha^k(p) = \alpha_0^k(p) + \alpha_3^k(p) \sigma_3.
$$

(3.3)

The regularization using $\alpha^k$ clearly violates the standard SU(2) symmetry.

The action (3.2) contains bilinear terms of the antifields. We make a canonical transformation on the IR fields so that the action becomes linear in the antifields when expressed in terms of new variables. We take the generator

$$
G[\Psi, \bar{\Psi}, \Psi^*, \bar{\Psi}^*, C, C^*] = \int_p \left[ \Psi^* (-p) \Psi(p) + \bar{\Psi}(-p) \bar{\Psi}^*(p) 
- i \Psi^* (-p)(\alpha^k)^{-1}(p) C^a T_a \bar{\Psi}^*(p) \right] + C^a C^* ,
$$

(3.4)

and find

$$
\Psi'(p) = \Psi(p) - i (\alpha^k)^{-1}(p) C^a T_a \bar{\Psi}^*(p),
\bar{\Psi}'(-p) = \bar{\Psi}(-p) + i \Psi^* (-p)(\alpha^k)^{-1}(p) C^a T_a,
C^*_a = C^*_a + i \int_p \Psi^* (-p)(\alpha^k)^{-1}(p) C^a T_a \bar{\Psi}^*(p),
\Psi^* (-p) = \Psi^* (-p), \quad \bar{\Psi}^* (p) = \bar{\Psi}^* (p).
$$

(3.5)
In terms of new variables, we obtain

\[
W^{(0)}_k = \int_p \left[ \bar{\Psi}(-p) D(p) \Psi'(p) + \Psi'^*(-p)iC^a \hat{T}_a(p) \Psi' + \bar{\Psi}'*(-p)i C^a \hat{T}_a \bar{\Psi}'(p) \right] \\
- \frac{1}{2} C^a_* (C^a \times C^a)^r ,
\]

where the renormalized generators are given by

\[
\hat{T}_a(p) \equiv T_a + [(\alpha^k)^{-1}, T_a] D(p).
\]

Then the ME \( W^{(0)}_k , W^{(0)}_k = 0 \) yields a GW-like algebraic relation

\[
D(p) \hat{T}_a(p) - T_a D(p) = 0 .
\]

Thanks to this relation, the new generators are shown to satisfy

\[
[\hat{T}_a(p), \hat{T}_b(p)] = i \varepsilon_{abc} \hat{T}_c(p) .
\]

Note that the choice (3.3) for \( \alpha^k \) made the diagonal generator remains invariant, \( \hat{T}_3(p) = T_3 \). Then, since \( [T_3, D(p)] = 0 \), the Dirac operator takes the form,

\[
D(p) = D_0(p) + D_3(p) \sigma_3 .
\]

The GW-like relation (3.8) leads to

\[
D_3(p) = \frac{\alpha^k_3(p)}{[\alpha^k_0(p)]^2 - [\alpha^k_3(p)]^2} \left( D_0^2(p) - D_3^2(p) \right) .
\]

The Dirac operator that solve this equation is given by

\[
D(p) \equiv \frac{1}{2} \left( \alpha^k_0(p) + \alpha^k_3(p) \sigma_3 \right) \left[ 1 + \frac{\bar{\Psi}' - \alpha^k_0(p)}{\sqrt{p^2 + (\alpha^k_0)^2(p)}} \right] ,
\]

where we impose the boundary condition, \( \alpha^k_3/\alpha^k_0 \to 0 \) in the limits of \( k \to 0 \) and \( k \to \Lambda \).

Using the relation (3.11), one can find the similarity transformation which relates the standard generators \( \{T_a\} \) to the renormalized ones \( \{\hat{T}_a(p)\} \),

\[
\hat{T}_a(p) = \exp (-\theta(p)T_3) T_a \exp (\theta(p)T_3) ,
\]

where

\[
\tanh \theta(p) = \frac{2 \alpha^k_3(p) D_0(p)}{[\alpha^k_0(p)]^2 - [\alpha^k_3(p)]^2 + 2 \alpha^k_3(p) D_3(p)} .
\]

Obviously, the SU(2) algebra in (3.9) for \( \hat{T}_a(p) \) is obtained from the standard one by using the similarity transformation.
The renormalized transformations on the new variables \( \{ \Psi, \bar{\Psi}, C \} \) are given by

\[
\begin{align*}
\delta \Psi(p) &= i C^a \hat{T}_a(p) \Psi(p), \\
\delta \bar{\Psi}(p) &= -i C^a \bar{\Psi}(p) T_a, \\
\delta C^a &= -\frac{1}{2} (C \times C)^a.
\end{align*}
\] (3.15)

The presence of the similarity transformations implies that bilinear operators

\[
\bar{\Psi}(p)D(p)\Psi(p), \quad \bar{\Psi}(-p) \left[ \exp \left( \theta(p)T_3 \right) \right] \Psi(p),
\] (3.16)

are invariant under (3.15). Typical 4-fermi invariant operators are given by

\[
\prod_{i=1}^4 \int_{p_i} \delta \left( \sum_{j=1}^4 p_j \right) \mathcal{O},
\] (3.17)

where \( \mathcal{O} = \mathcal{O}_1 \) or \( \mathcal{O}_2 \),

\[
\begin{align*}
\mathcal{O}_1 &= \left[ \bar{\Psi}(p_1) \exp \left( \theta(p_2)T_3 \right) \Psi(p_2) \right] \left[ \bar{\Psi}(p_3) \exp \left( \theta(p_4)T_3 \right) \Psi(p_4) \right], \\
\mathcal{O}_2 &= \left[ \bar{\Psi}(p_1) \exp \left( \theta(p_2)T_3 \right) \hat{T}_a(p_2) \Psi(p_2) \right] \left[ \bar{\Psi}(p_3) \exp \left( \theta(p_4)T_3 \right) \hat{T}_a(p_4) \Psi(p_4) \right].
\end{align*}
\] (3.18)

The average action with such invariants solves the ME and gives a symmetry-preserving truncation.

4 Discussion

Our method for formulating the renormalized symmetries uses the notion of the block-spin transformations. The standard form of a symmetry is assumed to be realized in the UV action. The block-spin variables are useful in deriving the ME, which ensures the presence of the renormalized symmetry. In solving the ME, however, the variables are found not to be convenient due to the following reasons. (1) The antifields appear in the interaction terms and therefore the symmetry transformations will depend on the interaction terms. (2) As seen in the case of SU(2) symmetry, the average action generally becomes nonlinear in the antifields even for a free theory.

The above problems are closely related to the choices of the dynamical variables in the average action. We have shown that the use of new variables obtained by the canonical transformations makes the average action linear in antifields, and simplifies representations of the underlying algebra of the renormalized symmetries. For the SU(2) symmetry, we have obtained the closed algebra. The GW relations in both symmetries obtained from the ME for the free actions played an important role in constructing invariant operators.

Our discussion for the fermionic systems may be extended to SU(N) \((N > 2)\) symmetry in such a way that the Cartan’s subalgebra remains unchanged. In our specific examples

\[5\] From now on, the primes will be discarded.
including these cases, the renormalized symmetries are realized in an asymmetric way: the transformations on $\Psi$ undergo deformation, while those on $\bar{\Psi}$ remain intact.

The solutions we have constructed in this paper are those to the classical ME rather to the quantum ME. Let $W^{(q)}_k$ be a solution of the quantum ME: $(W^{(q)}_k, W^{(q)}_k)/2 - h\Delta W^{(q)}_k = 0$. The $\Delta$-derivative term may arise from the jacobian factor \[12\] of the functional measure associated with the renormalized symmetry transformations. For an anomaly-free theory, all terms arising from the jacobian factor must be the coboundary terms which can be removed by introducing a suitable counter action $\tilde{W}$, ie, $(W^{(q)}_k, \tilde{W}) - h\Delta W^{(q)}_k = 0$. Furthermore, if the counter action $\tilde{W}$ has no antifield dependence, we find that $(W^{(q)}_k - \tilde{W}, W^{(q)}_k - \tilde{W}) = 0$. Therefore, the action $W^{(c)}_k \equiv W^{(q)}_k - \tilde{W}$ obeys the classical ME. Note that the concrete form of the counter action $\tilde{W}$ depends on the UV regularization scheme which is needed to make the $\Delta$-derivative well-defined. Up to this action, the $W^{(q)}_k$ is equivalent to $W^{(c)}_k$. Thus, we may conclude that our solutions of the classical ME provide a reasonable truncation for the average action.

It is worth studying the evolution of couplings for the invariant operators. This may be achieved by solving the exact flow equations. We expect that these considerations provide us with some important clues for the realization of gauge (BRS) symmetries along the RG flow.

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