ON REVERSES OF THE GOLDEN-THOMPSON TYPE INEQUALITIES

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Abstract. In this paper we present some reverses of the Golden-Thompson type inequalities: Let $H$ and $K$ be Hermitian matrices such that $e^s e^H \preceq_{ols} e^t e^H$ for some scalars $s \leq t$, and $\alpha \in [0,1]$. Then for all $p > 0$ and $k = 1, 2, \ldots, n$

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq \left(\max\{S(e^{sp}), S(e^{tp})\}\right)^{\frac{1}{p}} \lambda_k(e^{pH_{\alpha}e^pK})^\frac{1}{p},$$

where $A_{\alpha} B = A^\frac{1}{2} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^\frac{1}{2}$ is $\alpha$-geometric mean, $S(t)$ is the so-called Specht’s ratio and $\preceq_{ols}$ is the so-called Olson order. The same inequalities are also provided with other constants. The obtained inequalities improve some known results.

1. Introduction

In what follows, capital letters $A, B, H$ and $K$ stand for $n \times n$ matrices or bounded linear operators on an $n$-dimensional complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. For a pair $A, B$ of Hermitian matrices, we say $A \preceq B$ if $B - A \succeq 0$. Let $A$ and $B$ be two positive definite matrices. For each $\alpha \in [0,1]$, the weighted geometric mean $A_{\alpha} B$ of $A$ and $B$ in the sense of Kubo-Ando is defined by

$$A_{\alpha} B = A^\frac{1}{2} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\alpha A^\frac{1}{2}.$$ 

Also for positive definite matrices $A$ and $B$, the weak log-majorization $A \prec_{wlog} B$ means that

$$\prod_{j=1}^{k} \lambda_j(A) \leq \prod_{j=1}^{k} \lambda_j(B), \quad k = 1, 2, \ldots, n,$$

where $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ are the eigenvalues of $A$ listed in decreasing order. If equality holds when $k = n$, we have the log-majorization $A \prec_{log} B$. It is known that the weak log-majorization $A \prec_{wlog} B$ implies $\|A\|_u \leq \|B\|_u$ for any unitarily invariant norm $\|\cdot\|_u$, i.e. $\|UAV\|_u = \|A\|_u$ for all $A$ and all unitaries $U,V$. See [2] for theory of majorization.

In [15], Specht obtained an inequality for the arithmetic and geometric means of positive numbers: Let $x_1 \geq \cdots \geq x_n > 0$ and set $t = x_1/x_n$. Then

$$\frac{x_1 + \cdots + x_n}{n} \leq S(t)(x_1 \cdots x_n)^\frac{1}{n},$$

2010 Mathematics Subject Classification. Primary 15A42; Secondary 15A60, 47A63.

Key words and phrases. Ando-Hiai inequality, Golden-Thompson inequality, Eigenvalue inequality, Geometric mean, Olson order, Specht ratio, Generalized Kantorovich constant, Unitarily invariant norm.
where

\[ S(t) = \frac{(t-1)t^{1/(t-1)}}{e \log t} \quad (t \neq 1) \quad \text{and} \quad S(1) = 1 \quad (1.1) \]

is called the Specht ratio at \( t \). Note that \( \lim_{p \to 0} S(tp)^{1/p} = 1 \), \( S(t^{-1}) = S(t) > 1 \) for \( t \neq 1, t > 0 \). Specht’s inequality is a ratio type reverse inequality of the classical arithmetic-geometric mean inequality. Using this nice ratio we can state our main result in Section 2.

The Golden-Thompson trace inequality, which is of importance in statistical mechanics and in the theory of random matrices, states that \( \text{Tr} e^{H + K} \leq \text{Tr} e^H e^K \) for arbitrary Hermitian matrices \( H \) and \( K \). This inequality has been complemented in several ways [1, 9]. Ando and Hiai in [1] proved that for every unitarily invariant norm \( \| \cdot \|_u \) and \( p > 0 \)

\[ \| (e^{pH} e^{pK})^{1/p} \|_u \leq \| e^{(1-\alpha)H + \alpha K} \|_u. \quad (1.2) \]

Seo in [13] found some upper bounds on \( \| e^{(1-\alpha)H + \alpha K} \|_u \) in terms of scalar multiples of \( \| (e^{pH} e^{pK})^{1/p} \|_u \), which show reverse of the Golden-Thompson type inequality (1.2). In this paper we establish another reverse of this inequality, which improve and refine Seo’s results. In fact the general sandwich conditions \( sA \leq B \leq tA \) for positive definite matrices, is the key for our statements. Also, the so called Olson order \( \preceq_{\text{ols}} \) is used. For positive operators, \( A \preceq_{\text{ols}} B \) if and only if \( A^r \leq B^r \) for every \( r \geq 1 \) [12]. Our results are parallel to eigenvalue inequalities obtained in [3] and [8].

2. REVERSE INEQUALITIES VIA SPECHT RATIO

To study the Golden-Thompson inequality, Ando-Hiai in [1] developed the following log-majorizations:

\[ A^r \preceq_{\text{log}} (A^s B)^r, \quad r \geq 1, \]

or equivalently

\[ (A^p \preceq_{\text{log}} (A^q B^p)^{1/p}, \quad 0 < q \leq p. \]

There are some literatures [14] on the converse of these inequalities in terms of unitarily invariant norm \( \| \cdot \|_u \). By the following lemmas, we obtain a new reverse of these inequalities in terms of eigenvalue inequalities.

**Lemma 2.1.** Let \( A \) and \( B \) be positive definite matrices such that \( sA \leq B \leq tA \) for some scalars \( 0 < s \leq t \), and \( \alpha \in [0, 1] \). Then

\[ A^r \preceq_{\text{log}} (A^s B)^r, \quad 0 < r \leq 1, \]

where \( S(t) \) is the Specht ratio defined as (4.3).

**Proof.** Let \( f \) be an operator monotone function on \( [0, \infty) \). Then according to the proof of Theorem 1 in [7], we have

\[ f(A) \preceq (f(M(A^\alpha B))), \]

where

\[ S(t) = \frac{(t-1)t^{1/(t-1)}}{e \log t} \quad (t \neq 1) \quad \text{and} \quad S(1) = 1 \quad (1.1) \]
where \( M = \max\{S(s), S(t)\} \). Putting \( f(t) = t^r \) for \( 0 < r \leq 1 \), we reach inequality (2.1).

**Lemma 2.2.** Let \( A \) and \( B \) be positive definite matrices such that \( sA \preceq_{ols} B \preceq_{ols} tA \) for some scalars \( 0 < s \leq t \), and \( \alpha \in [0, 1] \). Then

\[
\lambda_k(A^{\frac{1}{\alpha}}_{\#\alpha} B^r) \leq \max\{S(s^r), S(t^r)\} \lambda_k(A^{\frac{r}{\alpha}}_{\#\alpha} B^r), \quad r \geq 1, \tag{2.2}
\]

and hence,

\[
\lambda_k(A^{q\frac{1}{\alpha}}_{\#\alpha} B^r)^{\frac{1}{q}} \leq \left( \max\{S(s^p), S(t^p)\} \right)^{\frac{1}{q}} \lambda_k(A^{p\frac{r}{\alpha}}_{\#\alpha} B^p)^{\frac{1}{q}}, \quad 0 < q \leq p, \tag{2.3}
\]

where \( S(t) \) is the Specht’s ratio defined as (4.3) and \( k = 1, 2, \ldots, n \).

**Proof.** First note that the condition \( sA \preceq_{ols} B \preceq_{ols} tA \), is equivalent to the condition \( s^\nu A^\nu \preceq B^\nu \preceq t^\nu A^\nu \) for every \( \nu \geq 1 \). In particular, we have \( sA \leq B \leq tA \) for \( \nu = 1 \). Also, for \( r \geq 1 \) we have \( 0 < \frac{1}{r} \leq 1 \) and by (2.1)

\[
A^{\frac{1}{\alpha}}_{\#\alpha} B^r \leq \left( \max\{S(s), S(t)\} \right)^{\frac{1}{r}} (A^{\frac{r}{\alpha}}_{\#\alpha} B^\frac{1}{r}). \tag{2.4}
\]

On the other hand, from the condition \( s^\nu A^\nu \preceq B^\nu \preceq t^\nu A^\nu \) for every \( \nu \geq 1 \) and letting \( \nu = r \), we have

\[
s^r A^r \preceq B^r \preceq t^r A^r.
\]

Now if we let \( X = A^r \), \( Y = B^r \), \( w = s^r \) and \( z = t^r \), then

\[
wX \leq Y \leq zX. \tag{2.5}
\]

Using (2.4) under the condition (2.5), we have

\[
X^{\frac{1}{\alpha}}_{\#\alpha} Y^{\frac{1}{\alpha}} \leq \left( \max\{S(w), S(z)\} \right)^{\frac{1}{\alpha}} (X^{\frac{1}{\alpha}}_{\#\alpha} Y^{\frac{1}{\alpha}}),
\]

and this is the same as

\[
A^{\frac{1}{\alpha}}_{\#\alpha} B \leq \left( \max\{S(s^r), S(t^r)\} \right)^{\frac{1}{r}} (A^{\frac{r}{\alpha}}_{\#\alpha} B^r)^{\frac{1}{r}}.
\]

Hence

\[
\lambda_k(A^{\frac{1}{\alpha}}_{\#\alpha} B) \leq \left( \max\{S(s^r), S(t^r)\} \right)^{\frac{1}{r}} \lambda_k(A^{\frac{r}{\alpha}}_{\#\alpha} B^r)^{\frac{1}{r}}.
\]

By taking \( r \)-th power on both sides and using the Spectral Mapping Theorem, we get the desired inequality (2.2). Note that from the minimax characterization of eigenvalues of a Hermitian matrix [2] it follows immediately that \( A \leq B \) implies \( \lambda_k(A) \leq \lambda_k(B) \) for each \( k \). Similarly since \( p/q \geq 1 \), from inequality (2.2)

\[
\lambda_k(A^{\frac{1}{\alpha}}_{\#\alpha} B) \leq \max\{S(s^\frac{p}{q}), S(t^\frac{p}{q})\} \lambda_k(A^{\frac{p}{q}\frac{r}{\alpha}}_{\#\alpha} B^\frac{p}{q}). \tag{2.6}
\]

Replacing \( A \) and \( B \) by \( A^q \) and \( B^q \) in (2.6), and using the sandwich condition \( s^q A^q \preceq B^q \preceq t^q A^q \), we have

\[
\lambda_k(A^{q\frac{1}{\alpha}}_{\#\alpha} B^q)^{\frac{1}{q}} \leq \max\{S(s^p), S(t^p)\} \lambda_k(A^{p\frac{r}{\alpha}}_{\#\alpha} B^p)^{\frac{1}{p}}.
\]

This completes the proof. \( \square \)

Note that eigenvalue inequalities immediately imply log-majorization and unitarily invariant norm inequalities.
Corollary 2.3. Let $A$ and $B$ be positive definite matrices such that $mI \leq A, B \leq MI$ for some scalars $0 < m \leq M$ with $h = M/m$, and let $\alpha \in [0, 1]$. Then

$$A^\sharp_{\alpha} B^r \leq S(h^r) (A^\sharp_{\alpha} B)^r, \quad 0 < r \leq 1,$$

and hence

$$\lambda_k(A^\sharp_{\alpha} B)^r \leq S(h^r) \lambda_k(A^\sharp_{\alpha} B)^r, \quad r \geq 1,$$

$$\lambda_k(A^\sharp_{\alpha} B^q) ^\frac{1}{q} \leq S(h^p) ^\frac{1}{p} \lambda_k(A^\sharp_{\alpha} B^p) ^\frac{1}{p}, \quad 0 < q \leq p.$$  \hspace{1cm} (2.8)

where $S(t)$ is the Specht's ratio defined as (4.3) and $k = 1, 2, \ldots, n$.

Proof. Since $mI \leq A, B \leq MI$ implies $\frac{m}{M} A \leq B \leq \frac{M}{m} A$, the inequality (2.7) is obtained by letting $s = m/M$, $t = M/m$ in Lemma 2.1. Also from $mI \leq A, B \leq MI$, we have $m^\nu I \leq A^\nu, B^\nu \leq M^\nu I$ for every $\nu \geq 1$, and so

$$\left(\frac{m}{M}\right)^\nu A^\nu \leq B^\nu \leq \left(\frac{M}{m}\right)^\nu A^\nu.$$  \hspace{1cm} (2.10)

Using Lemma 2.2 under the condition (2.10), we reach inequalities (2.8) and (2.9). Note that $S(h) = S(\frac{1}{h})$. \hfill $\square$

Remark 2.4. We remark that the matrix inequality (2.7) is more stronger than corresponding norm inequality obtained by Seo in [13, Corollary 3.2]. Also inequality (2.9) is presented in [13, Lemma 3.1].

In the sequel we show a reverse of the Golden-Thompson type inequality (1.2), which is our main result.

Theorem 2.5. Let $H$ and $K$ be Hermitian matrices such that $e^s e^H \preceq_{ols} e^K \preceq_{ols} e^t e^H$ for some scalars $s \leq t$, and let $\alpha \in [0, 1]$. Then for all $p > 0$,

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq (\max\{S(e^{sp}), S(e^{tp})\})^\frac{1}{p} \lambda_k(e^{pH_{\#_{\alpha}} e^{pK})^\frac{1}{p},$$

where $S(t)$ is the so called Specht’s ratio defined as (4.3) and $k = 1, 2, \ldots, n$.

Proof. Replacing $A$ and $B$ by $e^H$ and $e^K$ in the inequality (2.3) of Lemma 2.2, we can write

$$\lambda_k(e^{qH_{\#_{\alpha}} e^{qK})^\frac{1}{q} \leq (\max\{S(e^{sp}), S(e^{tp})\})^\frac{1}{p} \lambda_k(e^{pH_{\#_{\alpha}} e^{pK})^\frac{1}{p}, \quad 0 < q \leq p.$$  \hspace{1cm} (2.8)

By [9, Lemma 3.3], we have

$$e^{(1-\alpha)H+\alpha K} = \lim_{q \to 0} (e^{qH_{\#_{\alpha}} e^{qK})^\frac{1}{q},$$

and hence it follows that for each $p > 0$,

$$\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq (\max\{S(e^{sp}), S(e^{tp})\})^\frac{1}{p} \lambda_k(e^{pH_{\#_{\alpha}} e^{pK})^\frac{1}{p}. \hfill \square$$

Corollary 2.6. Let $H$ and $K$ be Hermitian matrices such that $e^s e^H \preceq_{ols} e^K \preceq_{ols} e^t e^H$ for some scalars $s \leq t$, and let $\alpha \in [0, 1]$. Then for every unitarily invariant norm $\| \cdot \|_u$ and all $p > 0$,

$$\|e^{(1-\alpha)H+\alpha K}\|_u \leq (\max\{S(e^{sp}), S(e^{tp})\})^\frac{1}{p} \|(e^{pH_{\#_{\alpha}} e^{pK})^\frac{1}{p}\|_u, \quad (2.11)$$
and the right-hand side of (2.11) converges to the left-hand side as \( p \downarrow 0 \). In particular,

\[
\| e^{H+K} \|_u \leq \max \{ S(e^{2s}), S(e^{2t}) \} \| (e^{2H} + e^{2K}) \|_u.
\]

**Corollary 2.7.** [13, Theorem 3.3-Theorem 3.4] Let \( H \) and \( K \) be Hermitian matrices such that \( mI \leq H, K \leq MI \) for some scalars \( m \leq M \), and let \( \alpha \in [0,1] \). Then for all \( p > 0 \),

\[
\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq S(e^{(M-m)p})^{\frac{1}{p}} \lambda_k(e^{pH})^{\frac{1}{p}} e^{\alpha K} \leq m, \quad k = 1, 2, \ldots, n.
\]

So, for every unitarily invariant norm \( \| \cdot \|_u \)

\[
\| e^{(1-\alpha)H+\alpha K} \|_u \leq S(e^{(M-m)p})^{\frac{1}{p}} \| (e^{pH} + e^{\alpha K}) \|_u,
\]

and the right-hand side of these inequalities converges to the left-hand side as \( p \downarrow 0 \).

**Proof.** From \( mI \leq H, K \leq MI \), we have \( e^{\nu m} \leq e^{\nu H}, e^{\nu K} \leq e^{\nu M} \) for every \( \nu \geq 1 \) and so we can derive \( e^{m-M} e^{H} \leq_{ols} e^{K} \leq_{ols} e^{M-m} e^{H} \). Now the assertion is obtained by applying Theorem 2.5 and the fact that for every \( t > 0 \), \( S(t) = S(\frac{1}{t}) \).

\( \square \)

3. REVERSE INEQUALITIES VIA KANTOROVICH CONSTANT

A well-known matrix version of the Kantorovich inequality [11] asserts that if \( A \) and \( U \) are two matrices such that \( 0 < mI \leq A \leq MI \) and \( UU^* = I \), then

\[
UA^{-1}U^* \leq \frac{(m + M)^2}{4mM} (UAU^*)^{-1}.
\]

Let \( w > 0 \). The generalized Kantorovich constant \( K(w, \alpha) \) is defined by

\[
K(w, \alpha) := \frac{w^\alpha - w}{(\alpha - 1)(w - 1)} \left( \frac{\alpha - 1}{\alpha} \right) \frac{w^\alpha - 1}{w^\alpha - w},
\]

for any real number \( \alpha \in \mathbb{R} \) [6]. In fact, \( K\left(\frac{M}{m}, -1\right) = K\left(\frac{M}{m}, 2\right) \) is the constant occurring in (3.1).

Now as a result of following statement, we have another reverse Golden-Thompson type inequality which refines corresponding inequality in [13].

**Proposition 3.1.** [8, Theorem 3] Let \( H \) and \( K \) be Hermitian matrices such that \( e^s e^H \leq_{ols} e^K \leq_{ols} e^t e^H \) for some scalars \( s \leq t \), and let \( \alpha \in [0,1] \). Then

\[
\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq K(e^{p(t-s)}, \alpha)^{-\frac{1}{p}} \lambda_k(e^{pH} + e^{pK})^{\frac{1}{p}}, \quad p > 0,
\]

where \( K(w, \alpha) \) is the generalized Kantorovich constant defined as (3.2).

**Theorem 3.2.** Let \( H \) and \( K \) be Hermitian matrices such that \( mI \leq H, K \leq MI \) for some scalars \( m \leq M \) and let \( \alpha \in [0,1] \). Then for every \( p > 0 \)

\[
\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq K(e^{2p(M-m)}, \alpha)^{-\frac{1}{p}} \lambda_k(e^{pH} + e^{pK})^{\frac{1}{p}}, \quad k = 1, 2, \ldots, n,
\]
and the right-hand side of this inequality converges to the left-hand side as $p \downarrow 0$. In particular,

$$
\lambda_k(e^{H+K}) \leq \frac{e^{2M} + e^{2m}}{2e^Me^m} \lambda_k(e^{2H\#e^{2K}}), \quad k = 1, 2, \ldots, n.
$$

**Proof.** Since $MI \leq K, H \leq MI$ implies $e^{m-M}e^H \preceq_{ols} e^K \preceq_{ols} e^{M-m}e^H$, desired inequalities are obtained by letting $s = m - M$ and $t = M - m$ in Proposition 3.1.

For the convergence, we know that $\frac{2w^p}{w^p + 1} \leq K(w, \alpha) \leq 1$, for every $\alpha \in [0, 1]$. So, for every $p > 0$

$$
1 \leq K(w^p, \alpha)^{-\frac{1}{p}} \leq (\frac{2w^p}{w^p + 1})^{-\frac{1}{p}}.
$$

A simple calculation shows that

$$
\lim_{p \to 0} \frac{1}{p} \ln \left( \frac{2w^p}{w^p + 1} \right) = \lim_{p \to 0} \frac{\ln (w^p)(w^p - 1)}{4w^p + 1} = 0,
$$

and hence $\lim_{p \to 0} (\frac{2w^p}{w^p + 1})^{-\frac{1}{p}} = 1$. Now by using the sandwich condition and letting $w = e^{2(M-m)}$, we have $\lim_{p \to 0} K(e^{2p(M-m)}, \alpha)^{-\frac{1}{p}} = 1$. \hfill \Box

**Remark 3.3.** Under the assumptions of Theorem 3.2, Seo in [13, Theorem 4.2] proved that

$$
\|e^{(1-\alpha)H+\alpha K}\|_u \leq K(e^{(M-m)}, p)^{-\frac{1}{p}} K(e^{2p(M-m)}, \alpha)^{-\frac{1}{p}} \|(e^{pH\#e^{pK}})^{\frac{1}{p}}\|_u,
$$

and

$$
\|e^{(1-\alpha)H+\alpha K}\|_u \leq K(e^{2p(M-m)}, \alpha)^{-\frac{1}{p}} \|(e^{pH\#e^{pK}})^{\frac{1}{p}}\|_u, \quad p \geq 1.
$$

But the inequality (3.3) shows that the sharper constant for all $p > 0$ is $K(e^{2p(M-m)}, \alpha)^{-\frac{1}{p}}$. Since for $0 < p \leq 1$, $K(e^{(M-m)}, p)^{-\frac{1}{p}} \geq 1$ and hence

$$
K(e^{2p(M-m)}, \alpha)^{-\frac{1}{p}} \leq K(e^{(M-m)}, p)^{-\frac{1}{p}} K(e^{2p(M-m)}, \alpha)^{-\frac{1}{p}}.
$$

4. SOME RELATED RESULTS

It has been shown [7] that if $f : [0, \infty) \rightarrow [0, \infty)$ is operator monotone function and $0 < mI \leq A \leq B \leq MI \leq I$ with $h = \frac{M}{m}$, then for all $\alpha \in [0, 1]$

$$
f(A)^{\#_\alpha}f(B) \leq \exp (\alpha(1-\alpha)(1-\frac{1}{h})) f(A^{\#_\alpha}B), \quad (4.1)
$$

This new ratio has been introduced by Furuichi and Minculete in [4], which is different from Specht ratio and Kantorovich constant. By applying (4.4) for $f(t) = t^r$, $0 < r \leq 1$ we have the following results similar to Lemma 2.1 and Lemma 2.2.

**Lemma 4.1.** Let $A$ and $B$ be positive definite matrices such that $0 < mI \leq A \leq B \leq MI \leq I$ with $h = M/m$, and let $\alpha \in [0, 1]$. Then

$$
A^{\#_\alpha}B^r \leq \exp (r\alpha(1-\alpha)(1-\frac{1}{h})^2) (A^{\#_\alpha}B)^r, \quad 0 < r \leq 1,
$$
Lemma 4.2. Let $A$ and $B$ be positive definite matrices such that $0 < mI \preceq_{ols} A \preceq_{ols} B \preceq_{ols} MI \preceq_{ols} I$ with $h = M/m$, and let $\alpha \in [0,1]$. Then for all $k = 1,2,\ldots,n$,
\[
\lambda_k(A^+_\alpha B)^r \leq \exp \left( \frac{1}{p} \alpha (1 - \alpha) \left( 1 - \frac{1}{h^p} \right)^2 \right) \lambda_k(A^+^\# \alpha B^r), \quad r \geq 1,
\]
\[
\lambda_k(A^\#_\alpha B^p) \leq \exp \left( \frac{1}{p} \alpha (1 - \alpha) \left( 1 - \frac{1}{e} \right)^2 \right) \lambda_k(A^{\#^\#}_\alpha B^p), \quad 0 < q \leq p. \quad (4.2)
\]

Theorem 4.3. Let $H$ and $K$ be Hermitian matrices such that $e^m I \preceq_{ols} e^H \preceq_{ols} e^K \preceq_{ols} e^M I \preceq_{ols} I$ for some scalars $m \leq M$, and let $\alpha \in [0,1]$. Then for all $p > 0$ and $k = 1,2,\ldots,n$
\[
\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq \exp \left( \frac{1}{p} \alpha (1 - \alpha) \left( 1 - \frac{1}{e^p(M-m)} \right)^2 \right) \lambda_k(e^{pH} \#_\alpha e^{pK})^{\frac{1}{p}},
\]
and so, for every unitarily invariant norm $\| \cdot \|_u$
\[
\|e^{(1-\alpha)H+\alpha K}\|_u \leq \exp \left( \frac{1}{p} \alpha (1 - \alpha) \left( 1 - \frac{1}{e^p(M-m)} \right)^2 \right) \|e^{pH} \#_\alpha e^{pK}\|^{\frac{1}{p}}_u.
\]

Proof. The proof is similar to that of Theorem 2.5, by replacing $A$ and $B$ with $e^H$ and $e^K$, and $h = e^{M-m}$ in the inequality $(4.2)$. \hfill \square

Remark 4.4. Under the different conditions, the different coefficients are not comparable. But it is known that if we have a certain statement under the sandwich condition $0 < sA \leq B \leq tA$, then the same statement is also true under the condition $0 < mI \leq A, B \leq MI$ and $0 < mI \leq A \leq B \leq MI \leq I$. Hence, we can compare the following special cases:

1. Comparison of the constants in Theorem 4.3 and in Theorem 3.2:
   Let $e^m I \preceq_{ols} e^H \preceq_{ols} e^K \preceq_{ols} e^M I \preceq_{ols} I$. Operator monotony of $\log(t)$ leads to $mI \leq H \leq K \leq MI \leq I$, and so $mI \leq H, K \leq MI$. Now by applying Theorem 3.2 we have
\[
\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq K(e^{2p(M-m)}, \alpha)^{-\frac{1}{p}} \lambda_k(e^{pH} \#_\alpha e^{pK})^{\frac{1}{p}}, \quad p > 0.
\]
Also, by Theorem 4.3
\[
\lambda_k(e^{(1-\alpha)H+\alpha K}) \leq \exp \left( \frac{1}{p} \alpha (1 - \alpha) \left( 1 - \frac{1}{e^p(M-m)} \right)^2 \right) \lambda_k(e^{pH} \#_\alpha e^{pK})^{\frac{1}{p}}, \quad p > 0.
\]

Letting $h = e^{M-m} \geq 1$, the following numerical examples show that there is no ordering between these inequalities.

(i) Take $\alpha = \frac{1}{2}$, $p = \frac{1}{2}$ and $h = 2$, then we have
\[
K(h^{2p}, \alpha)^{-\frac{1}{p}} - \exp \left( \frac{1}{p} \alpha (1 - \alpha) \left( 1 - \frac{1}{h^p} \right)^2 \right) \approx -0.0134963.
\]

(ii) Take $\alpha = \frac{1}{2}$, $p = \frac{1}{2}$ and $h = 8$, then we have
\[
K(h^{2p}, \alpha)^{-\frac{1}{p}} - \exp \left( \frac{1}{p} \alpha (1 - \alpha) \left( 1 - \frac{1}{h^p} \right)^2 \right) \approx 0.0631159.
\]
(2) Comparison of the constants in Lemma 4.1 and in Lemma 2.1:
Let $0 < mI \leq A \leq B \leq MI \leq I$. Then the following sandwich condition is obtained

$$m \leq \frac{m}{M} \leq 1 \leq A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \leq \frac{M}{m} \leq \frac{1}{m}.$$ 

Now by letting $s = 1$ and $t = \frac{M}{m} = h$ in Lemma 2.1, we get

$$A^{\frac{\alpha}{n}}B^r \leq S(h)^r(A^{\frac{\alpha}{n}}B)^r, \quad 0 < r \leq 1. \quad (4.3)$$

Also, by Lemma 4.1

$$A^{\frac{\alpha}{n}}B^r \leq \exp \left( r(1 - \alpha)(1 - \frac{1}{h})^2 \right)(A^{\frac{\alpha}{n}}B)^r, \quad 0 < r \leq 1. \quad (4.4)$$

It is shown in [4, Remark 2.4] that there is no ordering between coefficients of (4.3) and (4.4). Therefore, we may conclude evaluation of Lemma 4.1 and Lemma 2.1 are different.

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