Quantum Statistical Entropy and Minimal Length of 5D Ricci-flat Black String with Generalized Uncertainty Principle

Molin Liu‡, Yuanxing Gui∗ and Hongya Liu‡

School of Physics and Optoelectronic Technology,
Dalian University of Technology, Dalian, 116024, P. R. China

In this paper, we study the quantum statistical entropy in a 5D Ricci-flat black string solution, which contains a 4D Schwarzschild-de Sitter black hole on the brane, by using the improved thin-layer method with the generalized uncertainty principle. The entropy is the linear sum of the areas of the event horizon and the cosmological horizon without any cut-off and any constraint on the bulk’s configuration rather than the usual uncertainty principle. The system’s density of state and free energy are convergent in the neighborhood of horizon. The small-mass approximation is determined by the asymptotic behavior of metric function near horizons. Meanwhile, we obtain the minimal length of the position $\Delta x$ which is restrained by the surface gravities and the thickness of layer near horizons.

PACS numbers: 04.70.Dy; 04.62.+v; 04.70.-s; 04.50.+h
Keywords: black string, entropy, generalized uncertainty principle

I. INTRODUCTION

One decade ago, several attempts on solving the gauge hierarchy problem, based on higher dimensional gravity theories, were proposed (e.g. cite Arkani-Hamed-Dimopoulos-Davli model [1] and Randall-Sundrum model [2] [3]). According to the spirit of these brane world theory, we may live on a (3 + 1) dimensional hypersurface (3-brane) where the standard model fields (such as fermions, gauge bosons, Higgs fields) are confined without accessing along the transversal extra dimensions. The branes are embedded in a higher dimensional space (bulk), in which only gravitons and scalar particles without charges could propagate. If the ordinary matter is confined on the brane, the gravitational collapsing can not be avoided. Hence a black hole can form naturally on the brane and its horizon is extended along the extra dimensions. The collapsed object which is a black hole on the brane is looked actually like a string in the bulk [4], if the total dimensionality is five. On the other hand, the fact that the universe is accelerating expansion is justified by famous astronomical data of type Ia Supernovae (SNe Ia) [5] and the cosmic microwave background (CMB) of the Wilkinson Microwave Anisotropy Probe (WMAP) [6]. Of course, there are many other key observations, including Two-degree-Field Galaxy Redshift Survey (2dFGRS) [7] [8], Sloan Digital Sky Survey (SDSS) [9] and so on, also indicate the acceleration of the universe. These observations in cosmology show that the space-time structure of our universe may be a de Sitter geometry in both the past and the future [10]. In what follows, we consider

∗Electronic address: mlliudl@student.dlut.edu.cn
†Electronic address: guiyx@dlut.edu.cn
‡Electronic address: hyliu@dlut.edu.cn
a black string solution which contains a 4D de Sitter geometry with an effective cosmological constant offering an accelerating universe. This rich class of 5D black brane solutions are found originally by Mashhoon and restudied recently by many people focusing mainly on the induced cosmological constant $\Lambda$, the extra dimensional force, the radiative potential, quasi-normal modes and so on. In a recent work, we used the two-dimensional area, which leads to that we can not consider the contribution of extra dimensional component in total quantum number, to describe the black string’s entropy with the standard uncertainty principle (UP). The small-mass approximation is naturally obtained by the assumption of far-apart two branes. In this paper, we will restudy this problem based on a more general situation with the alternative generalized uncertainty principle (GUP).

In 1970s, Bekenstein and Hawking showed that the black holes should be treated as thermodynamic systems and the entropy of a black hole should be proportional to the area of its horizon. After that, many literatures contributed to the origin of black hole entropy by various approaches. One famous approach is the brick-wall method (BWM) shown by ’t Hooft in 1985. In his work, ’t Hooft studied the quantum statistical property of scalar field in Schwarzschild black hole and used the entropy of exterior field to calculate the interior entropy. In fact, the equivalence of exterior entropy and interior entropy comes from the explanation of entangled entropy, i.e., the exterior field and interior field are in the forms of pure state entanglement. So, the entropy of black hole can be treated as the entropy of the contribution from outside field. Certainly, the standard brick-wall method is used only in the single horizon black hole space in which the external field should be in thermal equilibrium. However, in the aspect of the multi-horizon black hole space, the improved thin-layer method (TLM) shown by Li et al in is more applicable. Meanwhile, the GUP, which is firstly applied to calculate entropy by Li (the related works also can be found in), is an effective method to solve ultraviolet divergences in the vicinity of the horizon. In this paper, we will use the method of GUP to restudy the quantum statistical entropy in a 5D Ricci-flat black string space.

This paper is organized as follows: In Section 2, the 5D Ricci-flat black string metric and the surface gravities near horizons are presented. In section 3, we calculate the proper total number of quantum states including radial and extra dimensional components under GUP. In section 4, the free energy and entropy of this 5D Ricci-flat black string are calculated without configuration assumption. Section 5 is the conclusion. We adopt the signature $(-,+,+,+,+)$ and put $\hbar$, $c$ and $G$ equal to unity. Greek indices $\mu, \nu, \ldots$ will be taken to run over 0, 1, 2, 3 as usual, while capital indices $A, B, C, \ldots$ run over all five coordinates (0, 1, 2, 3, 4).

II. SCALAR FIELD IN THE 5D RICCI-FLAT BLACK STRING COUPLED WITH EFFECTIVE 4D COSMOLOGICAL CONSTANT

Before calculating the entropy, it is necessary to present the background of the space-time. A static, three-dimensional spherically symmetric line element with a 4D effective cosmological constant takes the form

$$ds^2 = \frac{\Lambda\xi^2}{3} \left[ -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2( d\theta^2 + \sin^2\theta d\phi^2) \right] + d\xi^2,$$

where $\xi$ is an open non-compact extra dimensional coordinate. The metric function is $f(r) = 1 - 2M/r - \Lambda r^2/3$ and $M$ is the black hole’s mass on the brane. The part of this metric inside the square bracket is exactly the same line-element as the 4D Schwarzschild-de Sitter (SdS) solution, which is bounded by two horizons — an inner horizon...
(event horizon) and an outer horizon (cosmological horizon). The parameter $\Lambda$ is an induced cosmological constant which is reduced from the $5D$ to $4D$. The metric (1) is Ricci-flat $R_{AB} = 0$ and there is no cosmological constant in $5D$ space. So one can actually treat this $\Lambda$ as a parameter which comes from the fifth dimension.

In this Ricci-flat $5D$ brane world, the binary Randall-Sundrum type branes system can be constructed via a coordinate transformation $\xi = \sqrt{3/\Lambda} \exp(\sqrt{\Lambda/3} y)$ [14]. Hence, the metric (1) takes a conformal form

$$\begin{align*}
ds^2 &= e^{2\sqrt{\Lambda y}} \left[-f(r)dt^2 + \frac{1}{f(r)} dr^2 + r^2 \left(d\theta^2 + \sin^2 \theta d\phi^2\right) + dy^2\right],
\end{align*}$$

where $0 \leq y \leq y_1$ is the extra dimensional coordinate and $y_1$ is essentially a compactification length of extra dimension. It should be notice that the singularity $\xi = 0$ in original metric (1) is removed by this coordinate transformation. Two branes locate at the endpoints of the fifth dimension, which could be very small as in Randall-Sundrum 2-brane model [2] or very large as in Randall-Sundrum 1-brane model [3] by pushing the second brane to the infinity. On the fixed extra dimensional hypersurface this metric describes a SdS black hole. But it actually is a black string intersecting the brane world. The horizon looks like a black string instead of a $4D$ sphere lying along the fifth dimension. So we call the solution (2) black string. The following calculation of entropy will base on this new black brane solution (2).

In order to study the statistic thermodynamics feature in the vicinity of horizons, it is necessary to show its space-time structure near the horizons $r_e$ and $r_c$. The metric function $f(r)$ in Eq. (2) can be recomposed as follows

$$f(r) = \frac{\Lambda}{3r}(r - r_e)(r_c - r)(r - r_o).$$

The singularity of the metric (2) is determined by $f(r) = 0$. The real solutions to this equation are black hole event horizon $r_e$, cosmological horizon $r_c$ and a negative solution $r_o = -(r_e + r_c)$. The last one has no physical significance, and $r_e$ and $r_c$ are given as

$$\begin{align*}
\left\{ \begin{array}{l}
    r_e = \frac{2}{\sqrt{\Lambda}} \cos \chi, \\
    r_c = \frac{2}{\sqrt{\Lambda}} \cos \left(\frac{2\pi}{3} - \chi\right),
\end{array} \right.
\end{align*}$$

where $\chi = \frac{1}{3} \arccos(-3M\sqrt{\Lambda})$ with $\pi/6 \leq \chi \leq \pi/3$. The real physical solutions are accepted only if $\Lambda$ satisfy $\Lambda M^2 \leq \frac{1}{9}$ [30].

The definition of the surface gravity is

$$\kappa_i = \frac{1}{2} \left| \frac{df}{dr} \right|_{r=r_i}.$$ 

These are

$$\begin{align*}
\kappa_e &= \frac{(r_e - r_c)(r_e - r_o)}{6r_e} \Lambda, \\
\kappa_c &= \frac{(r_c - r_e)(r_c - r_o)}{6r_c} \Lambda.
\end{align*}$$

The different surface gravities $\kappa_e$ and $\kappa_c$ give the different temperatures $T_e = \frac{\kappa_e}{2\pi K_B}$ and $T_c = \frac{\kappa_c}{2\pi K_B}$ near inner and outer horizons, where $K_B$ is Boltzmann constant. It indicates that it is impossible to use the original BWM to calculate the free energy in non-equilibrium thermodynamic system. The improved TLM can solve this problem by considering
a thin layer near the event horizon or the cosmological horizon. The non-equilibrium large structure is replaced by the local thermodynamic states where the thickness of layer is a very small quantity. This behavior is similar to the infinitesimal calculus to some degree. Though the whole space is non-equilibrium, the region in the two layers can be generally treated as the equilibrium state. So we can still use equilibrium state’s approach of the statistic physics.

The TLM is an effective method to deal with the multi-horizon space. Many literatures have addressed to the works of Schwarzschild-de Sitter black hole [31], Kerr-de Sitter black hole [32], Vaidya black hole [23] and others [33].

The minimally coupled quantum scalar field $\Phi$ with mass $m$ on the background (2) satisfies

$$
\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^A} \left( \sqrt{g} g^{AB} \frac{\partial}{\partial x^B} \right) \Phi - m^2 \Phi = 0.
$$

The modes of the scalar field can be decomposed as the separable form $\Phi = e^{-i\omega t} \Psi(r, \theta, \phi) L(y)$ and $\omega$ is the particle energy. Then the equations for $L(y)$ and $R_\omega (r)$ read as follows,

$$
e^{-3\sqrt{\Lambda} y} \frac{d}{dy} \left( e^{3\sqrt{\Lambda} y} \frac{d}{dy} \right) L(y) + \left( e^{2\sqrt{\Lambda} y} m^2 + \mu^2 \right) L(y) = 0,
$$

$$
\frac{\partial}{\partial r} \left( r^2 f(r) \frac{\partial}{\partial r} \right) \Psi + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) \Psi 
+ \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \left( \frac{\partial}{\partial \phi} \right) \Psi 
+ \left( \frac{\omega^2}{f(r)} - \mu^2 \right) \Psi = 0,
$$

where the eigenvalue $\mu^2$ is the effective mass squared on the brane. In previous work [17], the effective mass $\mu$ on the brane is associated with the particle mass $m$ via RS Arnowitt-Deser-Misner (ADM) mass relationship i.e. $\mu = m$ on the first brane and $\mu = m e^{\sqrt{\Lambda/3} y}$ is on the second brane. In this paper, we do not consider the RS ADM mass relationship but rather we will study this problem from the perspective of GUP.

III. PROPER TOTAL NUMBER OF QUANTUM STATES WITH ENERGY LESS THAN $\omega$ UNDER GENERALIZED UNCERTAINTY PRINCIPLE

In order to calculate the total number of quantum states, we proceed from two parts, one is the radial component $g_r(\omega)$ and the other is the extra dimensional component $g_y(\omega)$, on the basis of the corresponding Eqs. (9) and (10).

Substituting $\Psi \sim \exp[i S (r, \theta, \phi)]$ into Eq. (10), we get the momentums’s relationship by the effective WKB approximation. Namely, the momentums $p_r = \frac{\partial S}{\partial r}$, $p_\theta = \frac{\partial S}{\partial \theta}$ and $p_\phi = \frac{\partial S}{\partial \phi}$ satisfy

$$
p_r^2 f(r) + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2 \sin^2 \theta} - \frac{\omega^2}{f(r)} + \mu^2 = 0.
$$

Comparing with the formal representations of other usual momentums such as Schwarzschild black hole [24], Reissner-Nordstrom black hole [26], Randall-Sundrum black string [27] and others [25], we find that they are very much alike each other. The differences between this 5D Ricci-flat black string and the others’ are that the metric function $f(r)$ is different and the parameter $\mu$ is the effective mass of $m$. More importantly, there is a novel understand in the view of the higher dimensional space.
The volume of momentum phase space is obtained as follows

\[ V_p = \int dp \cdot dp_\theta \cdot dp_\phi = \frac{4\pi}{3} \sqrt{\frac{1}{f(r)}} \cdot r^2 \cdot r^2 \cdot \sin^2 \theta \cdot p \]

\[ = \frac{4\pi r^2 \sin \theta}{3\sqrt{f(r)}} \left( \frac{\omega^2}{f(r)} - \mu^2 \right)^{3/2}, \tag{12} \]

where \( p \) is the module momentum which is determined by

\[ p^2 = p_i p^i = g_{11} p_r^2 + g_{22} p_\theta^2 + g_{33} p_\phi^2 = \omega^2 f(r)^{-1} - \mu^2. \tag{13} \]

In the standard quantum mechanics, the position \( \hat{x} \) and the momentum \( \hat{p} \) are a pair of conjugate observable parameter which satisfy the uncertainty principle

\[ \Delta x \Delta p \geq \hbar. \tag{14} \]

It means that the uncertainty of position could be arbitrarily small with the increasing uncertainty of momentum. However, in the quantum system under the Planck scale the above UP relation should be corrected to the generalized uncertainty principle\[28\[24\[25\[26\[27\]

\[ \Delta x \Delta p \geq \frac{\hbar}{2} \left[ 1 + \gamma \left( \frac{\Delta p}{\hbar} \right)^2 \right]. \tag{15} \]

This second-order equation gives a minimal length \( \Delta x \geq \sqrt{7} \) by the solution existence condition \( \Delta = \Delta x^2 - \gamma \geq 0 \).

In the coordinate and momentum phase space, the phase space is divided into a lot of ergospheres one by one, i.e., using \( \varepsilon \sim \varepsilon + d\varepsilon \) to describe the quantum state. Furthermore, each ergosphere is also divided into smaller grids and each grid denotes one quantum state. According to the GUP relation (15), the grid or cell gives the length \( 2\pi \hbar(1 + \gamma p^2) \). Hence the number of quantum state in the volume element \( d^3xd^3p \) is

\[ \frac{d^3xd^3p}{(2\pi \hbar)^3(1 + \gamma p^2)^3}. \tag{16} \]

The corresponding quantum state density is

\[ g(\varepsilon) = \frac{1}{(2\pi)^3} \int \frac{dr d\theta d\phi dp_r dp_\theta dp_\phi}{(1 + \gamma p^2)}, \tag{17} \]

where we use the natural unit, i.e., \( \hbar = c = 1 \). Substituting Eqs. (12) and (13) into the density (17), we can get the number of quantum states related to the radial modes as follows,

\[ g_r(\varepsilon) = \frac{1}{(2\pi)^3} \int dr d\theta d\phi \left[ 1 + \gamma \left( \frac{\omega^2}{f(r)} - \mu^2 \right) \right]^{-3} \int dp_r dp_\theta dp_\phi \]

\[ = \frac{2}{3\pi \hbar} \int \frac{r^2 \left[ \omega^2 f(r) - \mu^2 \right]^{3/2}}{f(r)^{1/2} [1 + \gamma \left( \omega^2 / f(r) - \mu^2 \right)]^3}. \tag{18} \]

Here we analyse the asymptotic behavior of integrand near the event horizon and cosmological horizon through the limit \( f(r) \to 0 \). The integrand of Eq. (18) is clearly reduced to

\[ r^2 \gamma^{-2} \omega^{-3} (1 - \frac{2M}{r} - \frac{\Lambda}{3r^2}). \tag{19} \]
Apparently, it is convergent near the two horizons. This is ensured by GUP and the similar behavior can be found in many works [24][25][26][27].

In the previous work [17], two-dimensional area is used to describe this black string’s entropy and the contribution of extra dimensional quantum number was not considered. Then, in this paper we employ the total quantum number, which is composed by the radial part and the extra dimensional part, to calculate the system’s free energy \( F(\beta) \).

The radial part \( g_r(\omega) \) is obtained in the above expression [18] and the extra dimensional \( g_y(\omega) \) can be derived directly from the corresponding Eq. (9). Through the WKB approximation we substitute \( L(y) \sim \exp[iY(y)] \) into Eq. (9) and keep the real part. So the wave number \( k_y = \frac{\partial Y}{\partial y} \) satisfies

\[
k_y^2 = e^{2\sqrt{3}y}m^2 + \mu^2.
\] (20)

The quantum number of the fifth dimensional mode is given by

\[
g_y(\omega) = \frac{1}{\pi} \int_\infty^{\omega/\sqrt{T}} d\mu \int_0^{y_1} dy \frac{\partial k_y(y, \mu)}{\partial \mu}.
\]

\[
= \frac{1}{\pi} \sqrt{3} \int_{y_0}^{\omega/\sqrt{T}} d\mu \left( \text{ArcCoth} \frac{\mu}{\sqrt{1 + \mu^2}} - \text{ArcCoth} \frac{\mu}{\sqrt{e^{2y_1} \sqrt{\Lambda/3} + \mu^2}} \right).
\] (21)

Accordingly, the proper total quantum number \( g_t(\omega) \) with energy less than \( \varepsilon \) can be given formally as follows,

\[
g_t(\omega) = \int dg_t(\omega) = \int dg_r(\omega) dg_y(\omega).\] (22)

IV. CONVERGENT FREE ENERGY AND ENTROPY OF BLACK STRING WITHOUT CUT-OFF

According to the bosons’ ensemble theory, the free energy in terms of the inverse temperature \( \beta \) is written as

\[
F = \frac{1}{\beta} \sum_n \ln \left[ 1 - e^{-\beta \omega_n} \right] \approx \frac{1}{\beta} \int dg_t(\omega) \ln(1 - e^{-\beta \omega})
\]

\[
= - \int_{\mu\sqrt{T}}^{\infty} d\omega \frac{g_t(\omega)}{e^{\beta\omega} - 1}
\]

\[
= - \frac{2}{\sqrt{3} \Lambda \pi^2} \int_{\mu\sqrt{T}}^{\infty} d\omega \frac{1}{e^{\beta\omega} - 1} \int_r f(r) \frac{\mu^2}{\sqrt{f(r)}} \int_{y_0}^{\omega/\sqrt{T}} d\mu \xi(\mu),
\] (23)

where

\[
\xi(\mu) = \frac{(\frac{\omega^2}{f(r)} - \mu^2)^{3/2}}{[1 + \gamma (\omega^2/f(r) - \mu^2)]^3} \left[ \text{ArcCoth} \frac{\mu}{\sqrt{1 + \mu^2}} - \text{ArcCoth} \frac{\mu}{\sqrt{e^{2y_1} \sqrt{\Lambda/3} + \mu^2}} \right].
\] (24)

The summation can be rewritten as an integral form with the continuum limit. The compressed \( \gamma \) term, which is derived directly by GUP, has been taken to ensure the convergence of free energy near horizons. Meanwhile, there is a limit of \( f(r) \rightarrow 0 \) in the vicinity of the horizon. So the term \( \omega^2/f(r) - \mu^2 \) can be transferred to \( \omega^2/f(r) \). Hence, the function \( \xi(\mu) \) can be rewritten as a new expression near both horizons

\[
\xi(\mu) = \frac{\omega^3}{f(r)^{3/2} \left( 1 + \gamma \frac{\omega^2}{f(r)} \right)^3} \left[ \text{ArcCoth} \frac{\mu}{\sqrt{1 + \mu^2}} - \text{ArcCoth} \frac{\mu}{\sqrt{e^{2y_1} \sqrt{\Lambda/3} + \mu^2}} \right].
\] (25)
This approximation simplifies greatly the calculation of the integration about variable $\mu$. The first fraction can be treated as a constant to deal with the $\mu$ aspect. Hence, the last integration of free energy \((23)\) is given as
\[
\left. \int_{m}^{\omega/\sqrt{T}} d\mu \xi(\mu) = \frac{\omega^3 f(r)^{3/2}}{\left(1 + \frac{\omega^2}{f(r)}\right)^3} \left\{ \sqrt{e^{2y_1\sqrt{T}} + \mu^2} - \sqrt{1 + \mu^2} \right. \\
+ \mu \text{ArcCoth} \frac{\mu}{\sqrt{1 + \mu^2}} - \mu \text{ArcCoth} \frac{\mu}{\sqrt{e^{2y_1\sqrt{T}} + \mu^2}} \right|_{m}^{\omega/\sqrt{T}}
\]
\[
= \frac{\omega^3 \alpha}{f(r)^{3/2} \left(1 + \frac{\omega^2}{f(r)}\right)^3},
\]
where
\[
\alpha = \sqrt{1 + m^2 - \sqrt{e^{2y_1\sqrt{T}} + m^2 + m \cdot \text{ArcCoth} \frac{m}{\sqrt{e^{2y_1\sqrt{T}} + m^2}} - m \cdot \text{ArcCoth} \frac{m}{\sqrt{1 + m^2}}}.
\]

It should be noticed that in the last step we use the limit $f(r) \to 0$. Substituting integral result \((26)\) to Eq. \((23)\) and transferring energy range $[\mu \sqrt{T}, +\infty]$ to $[0, +\infty]$, we obtain the free energy of this system
\[
F = -\frac{2\alpha}{\sqrt{3\Lambda^{\pi^2}}} \int_{r_e}^{r_c} dr \frac{r^2}{f^2(r)} \int_0^{+\infty} d\omega \frac{\omega^3}{(e^{\beta\omega} - 1) \left(1 + \frac{\gamma\omega^2}{f(r)}\right)^3}. 
\]

Hence, the scalar field’s entropy in this 5D Ricci-flat black string can be written directly from the derivative of free energy with respect to $\beta$
\[
S = \beta^2 \frac{\partial F}{\partial \beta} = \frac{2\alpha}{\sqrt{3\Lambda^{\pi^2}}} \int_{r_e}^{r_c} dr \frac{r^2}{f^1/2(r)} \int_0^{+\infty} d\zeta \frac{\beta^2 e^{\beta\zeta}}{(e^{a\zeta/2} - e^{-a\zeta/2})^2 \left(1 + \zeta^2\right)^3},
\]
where we adopt the Kim’s coordinate transformation
\[
\zeta = \omega \sqrt{\frac{\gamma}{f}},
\]
\[
a = \beta \sqrt{\frac{f}{\gamma}}.
\]

This transformation will give the deterministic expression about the last integral of variable $\omega$ in the expression of entropy \((29)\). In the improved brick-wall method, the quantum field is restricted in the vicinity of the horizons. The thin-layer BWM boundary conditions are
\[
\Phi(t, r, \theta, \phi, y) = 0 \quad for \quad r_e \leq r \leq r_e + \varepsilon_e,
\]
\[
\Phi(t, r, \theta, \phi, y) = 0 \quad for \quad r_e - \varepsilon_e \leq r \leq r_e,
\]
where the thickness of layers is a small quantity, i.e., $r_e \gg \varepsilon_e$ and $r_e \gg \varepsilon_e$. Hence, according to the expression \((31)\) and the boundary conditions \((32)\), we can get that $a$ goes to zero for $r \to r_e$ or $r \to r_c$. The integral about variable $\zeta$ in Eq. \((29)\) is reduced to
\[
\int_0^{+\infty} d\zeta \frac{a^2 \zeta^4}{(e^{a\zeta/2} - e^{-a\zeta/2})^2 \left(1 + \zeta^2\right)^3} = \int_0^{+\infty} d\zeta \frac{\zeta^2}{(1 + \zeta^2)^3} = \frac{\pi}{16}. 
\]
Finally, from the metric (2) the minimal length is obtained as
\[ \Delta x \geq \sqrt{\frac{3}{4}} \int_{r_e}^{r_e + \varepsilon_e} \frac{dr}{\sqrt{f(r)}} + \int_{r_e - \varepsilon_e}^{r_e} \frac{dr}{\sqrt{f(r)}} = \sqrt{\frac{2\varepsilon_e}{\kappa_e}} + \sqrt{\frac{2\varepsilon_c}{\kappa_c}}, \]
(34)
where \( \kappa_e \) and \( \kappa_c \) are the surface gravities on the horizons \( r_e \) and \( r_c \), which are defined by expressions (6) and (7) respectively. It is clear that the contributions of the minimal length come from the sum of inner horizon and outer horizon.

Substituting Eqs. (33) and (34) into Eq. (29), we obtain the final 5D black string’s entropy
\[ S = \frac{\alpha}{8\sqrt{3}(2\gamma)^{3/2}} \int_{r_e}^{r_e + \varepsilon_e} \int_{r_e - \varepsilon_e}^{r_e} \frac{dr}{\sqrt{f(r)}} + \sqrt{\frac{2\varepsilon_e}{\kappa_e}} + \sqrt{\frac{2\varepsilon_c}{\kappa_c}} \]
\[ = \frac{\alpha}{16\sqrt{6\gamma^{2}}}(\sqrt{\varepsilon_e A_e} + \sqrt{\varepsilon_c A_c}), \]
(35)
where \( A_e = 4\pi r_e^2 \) and \( A_c = 4\pi r_c^2 \) are the areas of black hole horizon and cosmological horizon respectively. Comparing with the previous result [17], this one shows more clearly that the entropy is the linear sum of the area of the event horizon and the cosmological horizon without cut-off factors. It should be noticed that we do not give any constraint on the branes including tension, configuration, ADM mass and so on. This result also justifies that the solution on the brane is indeed a SdS black hole when the 5D Ricci-flat black string with a 4D cosmological constant is reduced to the 4D space-time. The quantum statistical entropy not only varies directly with total areas but also is related to the thickness of the bulk and the field form.

V. CONCLUSION

We have restudied statistical mechanical entropies of a 5D Ricci-flat black string, which contains a 4D Schwarzschild-de Sitter black hole on the brane, by using improved thin-layer BWM under GUP. We obtain the convergent free energy and a minimal length without any cut-off near two horizons which are unsolvable in the classical uncertainty principle scenario.

In usual quantum mechanics, the position-momentum uncertainty relation i.e. Hersenberg uncertainty does not consider the gravity. However, if the gravitational effect is considered, there is an unavoidable minimal length being proportional to Planck scale \( L_p \) in the string theory. As the generalization of uncertainty relation, the GUP tell us some new aspects of quantum systems. Many efforts [34] have been devoted to GUP and its consequences. It is known that GUP can avoid the divergence of quantum number and free energy in the external quantum field outside black hole, so the unnatural cut-off factor in UP is removed in this paper.

In fact, the 5D Ricci-flat black string is different from the usual higher dimensional space. There are two different temperature horizons in this space, one is the inner event horizon and the other is the outer cosmological horizon. It evidently shows that this black string is a nonequilibrium system as a whole. Since the usual BWM is only suitable for
the thermal equilibrium state, it cannot be employed to study the entropy here. However, it is well known that the thin-layer method can overcome this difficulty with GUP. The related domain, which can be looked as an equilibrium state completely, is only a thin layer in Planck scale near horizons.

The last result \( \alpha \) includes a key parameter \( \alpha \) coupled with the magnitude of extra dimension \( y_1 \), the cosmological constant \( \Lambda \) and the mass of particle \( m \). If \( \alpha \) is negative, a singular negative entropy will appear. However, though there is a relationship

\[
\sqrt{1 + m^2} \leq \sqrt{e^{2y_1}\sqrt{\frac{\pi}{\alpha}} + m^2}
\]

in Eq. (27). But we know that the ArcCoth function is a monotone decreasing one in the positive independent variable range and the last two terms satisfy

\[
m \cdot \text{ArcCoth} \left( \frac{m}{\sqrt{e^{2y_1}\sqrt{\frac{\pi}{\alpha}} + m^2}} \right) \geq m \cdot \text{ArcCoth} \left( \frac{m}{\sqrt{1 + m^2}} \right)
\]

(37)

So \( \alpha \) could obtain a positive value theoretically. Because there are four parameters i.e. \( \alpha, \Lambda, m \) and \( \mu \) in the Eq. (27), it is very difficult to solve distinctly the inequality \( \alpha > 0 \) from four free variables. In fact, we can look inequality \( \alpha > 0 \) as a constraint condition of black string.

The GUP parameter \( \gamma \) associated with string theory scale — Planck scale \( l_p \) plays a key role in this model. As we all know, the original GUP is used to explain the fact that string cannot probe distances below the minimum length (its value is \( \sqrt{\gamma} \) in this paper). In the recent years, many people, such as Li [24], Kim [27] and the others [25] [26], use it to calculate black hole’s entropy and find this can obviate the cut-off needed in the usual brick wall model. As we see in this paper, the volume element of phase space changes through the GUP parameter \( \gamma \). After integral in the whole momentum space, referring to equation (18), we find the free energy is convergence in the vicinity of the horizon since there is a suppressing \( \gamma \)-term in the denominator induced from the generalized uncertainty principle. So this convergence free energy ensures the convergence entropy without the artificial cut-off. Else, comparing the early work [17] with this GUP model, the former just is a particular limit that the second brane is push to faraway places. Furthermore, the result (35) contains the string theory scale, which also agree with the strong coupling \( E_8 \times E_8 \) heterotic string theory. Partly, this result supports the statement that when calculating the density of quantum states we should take into account the contribution from the string excitation [35].

Acknowledgments

Project supported by the National Basic Research Program of China (2003CB716300), National Natural Science Foundation of China (10573003) and National Natural Science Foundation of China (10573004).

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