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THE RELATIVE BOGOMOLOV CONJECTURE FOR FIBERED PRODUCTS OF ELLIPTIC CURVES

LARS KÜHNE

Abstract. We deduce an analogue of the Bogomolov conjecture for non-degenerate subvarieties in fibered products of families of elliptic curves from the author’s recent theorem on equidistribution in families of abelian varieties. This generalizes results of DeMarco and Mavraki and improves certain results of Manin-Mumford type proven by Masser and Zannier to results of Bogomolov type, yielding the first results of this type for subvarieties of relative dimension $> 1$ in families of abelian varieties with trivial trace.

In a previous article [20], the author has established an analogue of the equidistribution conjecture for degenerate subvarieties in families of abelian varieties and deduced uniform versions of the Manin-Mumford and the Bogomolov conjecture for algebraic curves embedded in their Jacobian. In this article, we discuss another application of the same equidistribution result [20 Theorem 1], which has been the original motivation for the author’s work on equidistribution. It should also be remarked that since the preprint [20] appeared, more general equidistribution results have been obtained by Gauthier [16] as well as Yuan and Zhang [34].

Pink has suggested a generalization [29 Conjecture 6.2] of the Manin-Mumford conjecture for families of abelian varieties. The following conjecture is nothing but the Bogomolov-type analogue of this conjecture, which was also proposed as [11 Conjecture 1.2]. We also remark that it overlaps with a conjecture already proposed in Zhang’s 1998 ICM talk [36 Section 4]. Throughout this article, the subfield $K \subset \overline{\mathbb{Q}} \subset \mathbb{C}$ is a number field and $S$ is an irreducible algebraic variety over $K$. The variety $S$ serves as the base of a family $\pi : A \to S$ of abelian varieties. Furthermore, we assume being given an immersion $\iota : A \hookrightarrow \mathbb{P}^N_K$ into projective space and a Weil height $h_{\theta(1)}$ associated with the ample line bundle...
\(O(1)\) on \(\mathbb{P}^N_K\). For each closed point \(x \in A\), we set
\[
\hat{h}_i(x) = \lim_{k \to \infty} \left( h_{O(1)}(\iota \circ [n^k](x)) / n^{2k} \right).
\]
As \([n]\) preserves the proper fibers of \(\pi\), this is just the ordinary Néron-Tate height of \(x\) with respect to (the symmetric part of) the line bundle \(\iota^*O(1)|_{A_\pi(x)}\) on the abelian variety \(A_{\pi(x)}\).

Relative Bogomolov Conjecture (RBC). Let \(X\) be an irreducible subvariety \(X \subset A\) such that \(\pi(X) = S\). Assume that \(X\) is not a subvariety of codimension \(\leq \dim(S)\) in any horizontal torsion coset \(Y \subset A\). Then, there exists some \(\varepsilon(X) > 0\) such that the set
\[
\{\text{closed point } x \in X \mid \hat{h}_i(x) < \varepsilon(X)\}
\]
is not Zariski-dense.

The notion of horizontal torsion coset, which morally is the analogue of an abelian subvariety translated by a torsion point, demands a formal definition: Let \(S' \to S\) be a generically finite map and \(\tau : S' \to A_{S'}\) a torsion section of the base change \(\pi_S : A_{S'} \to S'\). For each subvariety \(X \subset A_{S'}\), we define its translate \(X + \tau\) to be the image of \(X \times_{S'} \tau(S')\) under the (fiberwise) addition \(A_{S'} \times_{S'} A_{S'} \to A_{S'}\). An irreducible variety \(X \subset A\) is called a horizontal torsion coset if there exists a generically finite map \(S' \to S\), an \(S'\)-flat subgroup scheme \(B \subset A_{S'}\), and a torsion section \(\tau : S' \to A_{S'}\) such that the translate \(B + \tau \subset A_{S'}\) projects onto \(X\).

It should be noted that (RBC) is independent of the chosen immersion \(\iota : A \hookrightarrow \mathbb{P}^N_K\), which is not completely trivial as \(\hat{h}_i\) appears in its statement. Let \(\iota, \iota' : A \hookrightarrow \mathbb{P}^N_K\) be two projective immersions and write \(\eta_\iota\) for the generic point of \(S\). Then there exists a positive integer \(k\) such that both \((\iota^*O(1)^{\otimes k} \otimes (\iota')^*O(1)^{\otimes -1}))_{|\eta}\) and \((\iota^*O(1)^{\otimes -1} \otimes (\iota')^*O(1)^{\otimes k}))_{|\eta}\) are ample. There exists thus an open dense subset \(U \subset S\) such that both \((\iota^*O(1)^{\otimes k} \otimes (\iota')^*O(1)^{\otimes -1}))_s\) and \((\iota^*O(1)^{\otimes -1} \otimes (\iota')^*O(1)^{\otimes k}))_s\) are ample for all \(s \in U\). Since \(\hat{h}_i\) and \(\hat{h}_{i'}\) restrict to the usual Néron-Tate heights on fibers, it follows that
\[
k^{-1} \cdot \hat{h}_i(x) \leq \hat{h}_{i'}(x) \leq k \cdot \hat{h}_i(x)
\]
for all closed points \(x \in \pi^{-1}(U)\). As it clearly suffices to prove (RBC) for the restriction \(X|_U \subset A|_U\), this shows the independence of (RBC) from the chosen immersion \(\iota\).

The main result of our article concerns (RBC), and is a generalization of [9, Theorem 1.4]. As usual, (RBC) implies Manin-Mumford type results in the same relative settings. It has been already mentioned that the Manin-Mumford analogue of (RBC) was proposed by Pink [29, Conjecture 6.2]. For \(\dim(X) = 1\), results related to Pink’s conjecture have been obtained by Masser and Zannier

\[\text{Not every family of abelian varieties } \pi : A \to S \text{ admits a projective immersion even if } S \text{ does (compare [30, Chapter XII]), but we can always find a proper closed subset } Z \subset S \text{ such that } A \setminus \pi^{-1}(Z) \text{ is quasi-projective (e.g. by spreading out from the generic point of } S\text{), take an immersion } \iota : A \setminus \pi^{-1}(Z) \hookrightarrow \mathbb{P}^N_K, \text{ and obtain again a canonical height function } \hat{h}_i : (A \setminus \pi^{-1}(Z)) \times \mathcal{O}(1) \to \mathbb{R}^{\geq 0}. \text{ Theorem 4 still makes sense in this setting as (RBC) is invariant under passing to Zariski-dense open subsets of } S\text{. In particular, the Manin-Mumford part of the theorem holds for general families } \pi : A \to S \text{ even without the existence of a projective immersion.}\]
but no result of relative Manin-Mumford type seems to have been known for subvarieties \( X \subset A \) of relative dimension \( > 1 \) up to now. It should be remarked that [9] uses the relative Manin-Mumford conjecture [22] in order to prove (RBC). While this article was in preparation, DeMarco and Mavraki [8] were able to remove this dependence and to generalize their previous work; an essential new ingredient is the separation of holomorphic and anti-holomorphic terms that is also used here (compare our Section 9 below with the first step in [8, Subsection 8.2]) and which has been first introduced by André, Corvaja, and Zannier (see [1, Subsection 5.2]). Our proof, which includes the case considered in [9], also avoids a dependence on the relative Manin-Mumford conjecture so that we obtain it instead as a genuine corollary in all cases under consideration. For general curves \( X \subset A \) defined over \( \mathbb{Q} \), the relative Manin-Mumford conjecture has been proven recently by Masser and Zannier [24, Theorem 1.7].

**Theorem 1.** (RBC) is true if \( A \) is the fibered product \( E_1 \times_S E_2 \times_S \cdots \times_S E_g \) of families of elliptic curves \( E_i \rightarrow S \) \((1 \leq i \leq g)\) over a base variety \( S \).

The reader may note that we include the case of isotrivial families in the theorem. Furthermore, Theorem 1 immediately implies (RBC) in the slightly more general situation that the fiber \( A|_{\eta} \) over the generic point \( \eta \) of \( S \) is isogeneous to a fibered product of families of elliptic curves. The proof of Theorem 1 constitutes the content of Sections 1 to 15. For further details on the structure of the proof, we refer the reader to Section 4.

It is a reasonable first guess that the equidistribution result [20, Theorem 1] implies (RBC) in general just as the classical Bogomolov conjecture can be proven by means of equidistribution. Unfortunately, the analogy with the classical case leads one astray here. The Ullmo-Zhang approach [31, 35] to the Bogomolov conjecture does not transfer well to the relative setting because the Faltings-Zhang map

\[
A^n \rightarrow A^{n-1}, \quad (x_1, x_2, \ldots, x_n) \mapsto (x_1 - x_2, \ldots, x_{n-1} - x_n),
\]

has only an \( S \)-fibered analogue that is too weak for a reproduction of the arguments used in [35, Section 4]. In short, we cannot subtract points in two different fibers as there is no group structure on the total space. One can still subtract points contained in the same fiber, and this gives rise to the uniform results obtained in [12, 13, 20]. Here, we can prove our Theorem 1 by making use of the additional product structure available on \( E_1 \times_S \cdots \times_S E_g \), but our argument definitely breaks down for generically simple families \( A \rightarrow S \). Besides the author’s recent equidistribution result, essential tools for the proof of Theorem 1 are André’s theorem [2] on the normality of the monodromy group in admissible variations of mixed Hodge structures and the Ax-Schanuel conjecture for mixed Shimura varieties proven by Gao [15].

Finally, let us mention that by the argument given in [11], one can easily see that (RBC) implies a uniform version of the Bogomolov conjecture for curves of arbitrary genus \( g \geq 2 \) whose Jacobian is a product of elliptic curves. In other words, one can partially recover the author’s previous result [20, Theorem 2]. Likewise, the result of DeMarco, Krieger, and Ye [7], which answered a question

\[\text{While this article was in revision, Gao and Habegger have announced a general proof of the relative Manin-Mumford conjecture.}\]
of Bogomolov and Tsinkel [1], can be already deduced from the case of (RBC) proven here. However, the results of [20] and their improvement by Yuan [33] present substantially more general cases, in which (RBC) remains widely open.

**Notation and conventions.** Algebraic Geometry (General). Denote by $k$ an arbitrary field. A $k$-variety is a reduced separated scheme of finite type over $k$. By a subvariety of a $k$-variety we mean a reduced closed subscheme. A subvariety is determined by its underlying topological space and we frequently identify both. Furthermore, $X^{sm}$ denotes the smooth locus of $X$.

**Generic sequences.** Let $X$ be an algebraic $k$-variety. If $X$ is irreducible, we say that a sequence $(x_i) \in X^\mathbb{N}$ of closed points is $X$-generic if none of its subsequences is contained in a proper algebraic subvariety of $X$. Note that a sequence is $X$-generic if and only if it converges to the generic point of $X$ in the Zariski topology. If the irreducible variety $X$ can be inferred from context, we simply say generic instead of $X$-generic.

**Continuity and smoothness.** We use $\mathcal{C}^0$ as an abbreviation for continuous. For any topological space $X$, $\mathcal{C}^0(X)$ denotes the real-valued continuous functions on $X$ and $\mathcal{C}^0_c(X)$ the real-valued continuous functions on $X$ having compact support.

**Analytification.** For a number field $K \subset \mathbb{C}$ and a $K$-variety $X$, we write $X(\mathbb{C})$ for the complex analytic space associated with $X_C$.

**Complex spaces.** Let $M$ be a reduced complex (analytic) space (e.g., the analytic space $X(\mathbb{C})$ associated with a $K$-variety $X$). Recall that this means that $M$ is locally biholomorphic to a closed analytic subvariety $V$ in a complex domain $U \subset \mathbb{C}^n$. A $\mathcal{C}^\infty$-form $\omega$ on $M$ is a differential form on the smooth locus $M^{sm}$ of $M$ with the following extension property: $M$ can be covered by local charts $V \subset U \subset \mathbb{C}^n$ as above such that for each chart the differential form $\omega|_{V^{sm}}$ is the restriction of a $\mathcal{C}^\infty$-differential form on $U$. There are also well-defined linear operators $d, \partial, \overline{\partial}$ on the $\mathcal{C}^\infty$-differential forms on $M$. For each local chart $V \subset U \subset \mathbb{C}^n$, these are simply the restrictions of the operators of the same name on $\mathbb{C}^n$.

**Moduli spaces of elliptic curves.** We write $Y(N)$ for the moduli stack over $\mathbb{Q}$ parameterizing elliptic curves with level $N$ structure ([26, Section 13.1]). This is a smooth quasi-projective variety if $N \geq 3$ ([19, Corollary 4.7.2]). For each $N \geq 1$, we write $\xi_N : \mathcal{E}(N) \to Y(N)$ for the universal family of elliptic curves with level $N$ structure.

**Siegel upper half-space.** We write $\mathcal{H}_g$ for the Siegel upper half-space of degree $g$, considered as a complex manifold.

1. **Setting-up the proof of Theorem**

As in the statement of the theorem, let $S$ be a base variety, let $E_j \to S$ ($1 \leq j \leq g$) be families of elliptic curves, and set

$$\pi : A = E_1 \times_S \cdots \times_S E_g \to S.$$  

Furthermore, let $X \subseteq A$ be a subvariety of dimension $d$, for which we want to prove (RBC). We start by making some additional assumptions for the proof of the theorem in Section 2 and introduce coordinates in Section 3. Following this, we give an overview of the main argument in Section 4.
2. Reductions

(i) In our proof, we suppose that the conclusion of (RBC) is false and show that its main assumption cannot hold under this assumption. This means our goal is to show that there exists a horizontal torsion coset \( Y \subseteq A \) such that \( X \) is a subvariety of codimension \( \leq \dim(S) \) in \( Y \). In the sequel, we can hence work with an \( X \)-generic sequence \( (x_i) \in X^N \) such that \( \hat{h}_i(x_i) \to 0 \).

(ii) We can assume that \( g = d + 1 \). In fact, if

\[
\dim(A) - \dim(S) = g < d + 1 = \dim(X) + 1,
\]

then \( \text{codim}_A(X) < \dim(S) + 1 \) so that the assumption of (RBC) is not satisfied. If \( g > d + 1 \), we choose a projection \( \varphi : A \to E_{j_1} \times E_{j_2} \times \cdots \times E_{j_{d+1}}, 1 \leq j_1 < j_2 < \cdots < j_{d+1} \leq g \). It clearly suffices to prove (RBC) for \( \varphi(X) \).

(iii) We can also make the following assumption: For any fibered product

\[
\pi' : A' = E'_{j_1} \times_{S'} E'_{j_2} \times_{S'} \cdots \times_{S'} E'_{j_{d+1}} \to S'
\]

of families of elliptic curves \( E_j \to S \) \((1 \leq j \leq g' \leq g)\) and any commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & A' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
S & \to & S'
\end{array}
\]

with \( \varphi \) a fiberwise homomorphism, we have

\[
\dim(\varphi(X)) \geq \dim(X) - (g - g')
\]

with equality if and only if \( \dim(A) = \dim(A') \).

In fact, the special case \( \dim(A) = \dim(A') \) is trivial as then \( \varphi \) is a fiberwise isogeny and \( g = g' \). Thus, we may suppose that there exist such \( \pi' : A' \to S' \) and \( \varphi : A \to A' \) with \( \dim(A') < \dim(A) \) and

\[
\dim(\varphi(X)) \leq \dim(X) - (g - g').
\]

By an induction on \( \dim(A) \), we can also assume that (RBC) is already proven for the family \( \pi' : A' \to S' \). The sequence \( \varphi(x_i) \) is \( \varphi(X) \)-generic and satisfies \( \hat{h}_{i'}(\varphi(x_i)) \to 0 \) for any immersion \( i' : A' \hookrightarrow \mathbb{P}^{N'} \). Therefore \( \varphi(X) \) has to violate the assumption in (RBC), which means that there exists a horizontal torsion coset \( Y' \subseteq A' \) containing \( \varphi(X) \) and satisfying

\[
\dim(Y') - \dim(\varphi(X)) \leq \dim(S').
\]

Using the inequality \( (3) \), we infer further that

\[
\dim(Y') - \dim(X) + (g - g') \leq \dim(S').
\]

The irreducible components of \( \varphi^{-1}(Y') \) are horizontal torsion cosets. We can pick such an irreducible component \( Y \) containing \( X \) and notice that

\[
\dim(Y) \leq \dim(Y') + \dim(S) - \dim(S') + (g - g').
\]

Combining the last two inequalities, we obtain

\[
\dim(Y) - \dim(X) \leq \dim(S).
\]
This is a violation of the assumption in (RBC) for $X \subseteq A$, and thus there is nothing left to prove.

(iii) Before continuing with our reductions, let us point out some consequences of the previous assumption. For each $1 \leq k \leq g$, we write

$$\text{pr}_k : \prod_{j=1}^{g} E_j \longrightarrow \prod_{j=1, j \neq k}^{g} E_j$$

for the standard projection. The images $\text{pr}_k(X)$, $1 \leq k \leq g$, have then all dimension $\dim(X)$. In fact, we have a commutative square

$$A = \prod_{j=1}^{g} E_j \quad \xrightarrow{\text{pr}_k} \quad \prod_{j=1, j \neq k}^{g} E_j \quad \xrightarrow{\varphi} \quad \prod_{j=1}^{g'} E'_j$$

$$\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow$$

$$S \quad \quad \quad \quad \quad S \quad \quad \quad \quad \quad S'$$

as in (1) above so that (2) implies

$$\dim(\text{pr}_k(X)) > \dim(X) - 1.$$ 

and thus $\dim(\text{pr}_k(X)) = \dim(X)$.

A further consequence of the previous reduction is that $X$ is non-degenerate. In fact, if $X$ were degenerate, then [14, Theorem 1.1 (i)] would produce a new family $\pi' : A' \rightarrow S'$ and a fiberwise homomorphism $\varphi : A \rightarrow A'$ filling a diagram (1) but violating the constraint (2) on the dimension of $\varphi(X)$.

Even more, all the images $\text{pr}_k(X)$, $1 \leq k \leq g$, are non-degenerate as well. Indeed, assume that this is not the case. Then, [14, Theorem 1.1 (i)] supplies a commuting diagram

$$A = \prod_{j=1}^{g} E_j \quad \xrightarrow{\text{pr}_k} \quad \prod_{j=1, j \neq k}^{g} E_j \quad \xrightarrow{\varphi} \quad \prod_{j=1}^{g'} E'_j$$

$$\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow$$

$$S \quad \quad \quad \quad \quad S \quad \quad \quad \quad \quad S'$$

with fiberwise homomorphisms along the upper row and such that

$$\dim(\varphi(\text{pr}_k(X))) < \dim(\text{pr}_k(X)) \) - (g - 1 - g').$$

Using our assumption, we deduce conversely from (2) that

$$\dim(\varphi(\text{pr}_k(X))) > \dim(X) - (g - g').$$

A combination of these two inequalities yields

$$\dim(\text{pr}_k(X)) - (g - 1 - g') \geq \dim(X) - (g - g') + 2,$$

which is equivalent to the absurdity

$$\dim(X) = \dim(\text{pr}_k(X)) \geq \dim(X) + 1.$$ 

This proves our claim about the non-degeneracy of $\text{pr}_k(X)$, $1 \leq k \leq g$.

For the convenience of the reader, let us briefly summarize the assumptions that we can and do tacitly assume in the following:

(i) there exists an $X$-generic sequence $(x_i) \in X^N$ such that $\hat{h}_*(x_i) \rightarrow 0$,

(ii) $\dim(A) - \dim(S) = d + 1 = \dim(X) + 1$,

(iii) $X$ is non-degenerate, and every projection $\text{pr}_k(X)$, $1 \leq k \leq g$, is non-degenerate and has dimension $\dim(X)$. 

We continue with imposing two further restrictions on the family $\pi : A \to S$.

(iv) First, we can assume without loss of generality that it is a subfamily of the $g$-fold self-product of the universal family $\xi : \mathcal{E}(\mathcal{N}) \to Y(\mathcal{N})$, $\mathcal{N} \geq 3$. There exists a classifying map $\nu : S \to Y(1)^g$ (of algebraic stacks) such that
\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & \mathcal{E}(1)^g \\
\downarrow & & \downarrow \\
S & \xrightarrow{\nu} & Y(1)^g
\end{array}
\]
is a cartesian square, and we consider its pullback
\[
\begin{array}{ccc}
A' = A \times_S S' & \xrightarrow{\varphi'} & \mathcal{E}(\mathcal{N})^g \\
\downarrow & & \downarrow \\
S' = S \times_Y Y(1) & \xrightarrow{\nu'} & Y(\mathcal{N})^g
\end{array}
\]
along $Y(\mathcal{N}) \to Y(1)$. As $S' \to S$ is finite, it clearly suffices to prove (RBC) for the subvariety $X' = X \times_S S' \subseteq A'$. Consider the induced subfamily $\pi' : \varphi'(A) \to \varphi'(S)$. Assuming that (RBC) holds for this subfamily, we obtain as above that there exists a horizontal torsion coset $Y'' \subseteq \varphi'(A')$ with the property that
\[
\dim(Y'') - \dim(\varphi'(X')) \leq \dim(\nu'(S')).
\]
The preimage $Y' = (\varphi')^{-1}(Y'')$ is evidently a horizontal torsion coset of dimension $\dim(Y'') + \dim(S') - \dim(\nu'(S'))$ containing $X$, and furthermore
\[
\dim(Y') - \dim(X') \leq \dim(Y'') + \dim(S') - \dim(\varphi'(S')) - \dim(\varphi'(X')) \leq \dim(S')
\]
this is again contradicting the assumption of (RBC) for $X' \subseteq A'$ and hence also for $X \subseteq A$. We can and do hence assume that $A = \mathcal{E}(\mathcal{N})^g|_S$ for some $S \subseteq Y(\mathcal{N})^g$ where $\mathcal{N} \geq 3$ is fixed once and for all in the sequel.

(v) Second, we can use our free choice of the immersion $\iota : A \hookrightarrow \mathbb{P}^N_K$ in the statement of (RBC) to guarantee that
\[
\iota = (\sigma \circ (\iota_0 \times \cdots \times \iota_0))|_S
\]
where $\iota_0 : \mathcal{E}(\mathcal{N}) \hookrightarrow \mathbb{P}^{N_0}_K$ is an arbitrary projective immersion such that $\iota_0^*\mathcal{O}(1)$ is fiberwise symmetric and $\sigma : \mathbb{P}^{N_0}_K \times \cdots \times \mathbb{P}^{N_0}_K \hookrightarrow \mathbb{P}^N_K$ is the Segre embedding.

3. COVERINGS AND COORDINATES

We recall the following commutative diagram, whose horizontal rows are universal coverings:

\[
\begin{array}{ccc}
\Gamma = (\mathbb{Z}^2 \times \Gamma(\mathcal{N}))^g & \xrightarrow{\pi} & \mathcal{E}(\mathcal{N})^g(\mathbb{C}) \\
\downarrow & & \downarrow \\
\Gamma(\mathcal{N})^g & \xrightarrow{\xi} & Y(\mathcal{N})^g(\mathbb{C}).
\end{array}
\]
The covering transformations of $\xi_{\text{pure}}$ (resp. $\xi_{\text{mixed}}$) are given by the group $\Gamma(\mathcal{N})^g$ (resp. $(\mathbb{Z}^2 \times \Gamma(\mathcal{N}))^g$) where
\[
\Gamma(\mathcal{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{\mathcal{N}} \right\}
\]
acts on $\mathbb{Z}^2$ through its standard representation. This identification is such that the group element
\[
\left( \begin{pmatrix} m_1 \\ n_1 \\ \vdots \\ m_g \\ n_g \end{pmatrix}, \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_g & b_g \\ c_g & d_g \end{pmatrix} \right) \in (\mathbb{Z}^2 \rtimes \Gamma(\mathcal{N}))^g
\]
sends $(z_1, \ldots, z_g, \tau_1, \ldots, \tau_g) \in (\mathbb{C} \times \mathcal{H})^g$ to
\[
\left( \frac{z_1 + m_1 + n_1 \tau_1}{c_1 \tau_1 + d_1}, \ldots, \frac{z_g + m_g + n_g \tau_g}{c_g \tau_g + d_g}, \frac{a_1 \tau_1 + b_1}{c_1 \tau_1 + d_1}, \ldots, \frac{a_g \tau_g + b_g}{c_g \tau_g + d_g} \right)
\]
(see e.g. [3, Section 8.8]).

On the complex manifold $(\mathbb{C} \times \mathcal{H}^1)^g$, we have the global holomorphic standard coordinates $z_l, \tau_l$ $(1 \leq l \leq g)$ and their complex-conjugates $\overline{z_l}, \overline{\tau_l}$. In addition, we define $2g$ real-analytic functions $x_l, y_l : (\mathbb{C} \times \mathcal{H})^g \to \mathbb{R}$, $1 \leq l \leq g$, by demanding $z_l = x_l + \tau_l y_l$.

The preimage $u_{\text{pure}}^{-1}(S)$ (resp. $u_{\text{mixed}}^{-1}(X)$) decomposes into irreducible analytic components of dimension $\dim(S)$ (resp. $\dim(X)$), on which the group $\Gamma(\mathcal{N})^g$ (resp. $(\mathbb{Z}^2 \rtimes \Gamma(\mathcal{N}))^g$) acts transitively. Due to the absence of elliptic fixed points on $Y(\mathcal{N})$ (see e.g. [10, Exercise 2.3.7]), the map $u_{\text{pure}}$ (resp. $u_{\text{mixed}}$) is étale and these components coincide with the connected components of $u_{\text{pure}}^{-1}(S)$ (resp. $u_{\text{mixed}}^{-1}(X)$) in the euclidean topology. In the sequel, we keep fixed an irreducible component $\widetilde{X}$ of $X$. Its image $\widetilde{S} = \overline{\pi}(\widetilde{X})$ is then an analytic component of the preimage of $S$.

4. Overview of the proof

For convenience of the reader, we briefly expose the main lines of the argument employed for the proof of Theorem 1 in the following sections. In Section 5 we use the product structure and the equidistribution results of [20] to obtain differential-geometric conditions on $\widetilde{X}$. These conditions appear as real-analytic differential equations (9), which can be written down explicitly in local charts of $\widetilde{X}$ and the local coordinates introduced in Section 3 above.

A natural way to exploit these is to use monodromy, to wit, the fact that the stabilizer $\text{Stab}(\widetilde{X}) \subseteq (\mathbb{Z}^2 \rtimes \Gamma(\mathcal{N}))^g$ is rather large. This largeness follows by Hodge-theoretic techniques, which are exposed in Sections 6 to 8. Besides rather explicit computations, we make use of a theorem of André [2, Theorem 1] on the normality of the algebraic monodromy group. Unfortunately, the equations (9) are invariant under monodromy so that a direct application of monodromy fails; this failure is not really surprising since the Betti form, which gives rise to these equations, is invariant under monodromy.

To the rescue comes an important idea of André, Corvaja, and Zannier [1, Subsection 5.2] that allows us to actually take advantage of the fact that the real-analytic equations (9) contain both holomorphic and anti-holomorphic terms. In short, we replace $\widetilde{X}$ and (9) with $\widetilde{X} \times \overline{\widetilde{X}}$ and a new set of real-analytic differential equations (19). This is the content of Section 9.

Following a computation of the transformation behavior of (19) under monodromy (Section 10) and a final technical preparation in Section 11, we use
explicit elements of the algebraic monodromy group to prove first that—the assumptions of (RBC)—all factors are isogeneous (Sections 12 and 13). With this at our disposal, we then deduce a linear equation (27) on $\tilde{X}$ in the coordinates $z_1, \ldots, z_n$. Gao’s mixed Ax-Schaunel theorem [15] enables us then to conclude the proof of Theorem 1 (Section 13).

5. Equidistribution

Let us start with defining the equilibrium measure. The $(1,1)$-form

$$\frac{i}{\text{Im}(\tau_j)}(dz_j - y_j d\tau_j) \wedge (d\bar{z}_j - y_j d\bar{\tau}_j)$$

on $(\mathbb{C} \times X)^g$ is $(\mathbb{Z}^2 \times \Gamma(\mathcal{N}))^g$-invariant (see [12 Lemma 2.6]) and hence descends to a $(1,1)$-form $\alpha_j$ on $A(\mathbb{C})$. We define the $(1,1)$-form

$$\beta = \sum_{j=1}^{g} \alpha_j$$

and consider its $d$-fold exterior power

$$\beta^\wedge d = d! \cdot \sum_{j=1}^{g} \alpha_j', \ \alpha_j' = \alpha_1 \wedge \cdots \wedge \alpha_{j-1} \wedge \alpha_{j+1} \wedge \cdots \wedge \alpha_g.$$  

Up to multiplication with some (strictly) positive real number, this coincides with the smooth closed $(1,1)$-form on $A(\mathbb{C})$ provided by [13 Lemma 2.6] (compare also the proofs of [20 Lemmas 7 and 11]). By Theorem [20 Theorem 1] (and its proof), there exists some $\mathcal{E}_X > 0$ such that

$$\frac{1}{\#O(x_i)} \sum_{x \in O(x_i)} f(x) \longrightarrow \mathcal{E}_X \int_{X(\mathbb{C})} f \beta^\wedge d, \ i \to \infty,$$

for every continuous function $f \in C(\mathbb{C})(\mathbb{C})$. Note that each $\alpha_j'|_{X(\mathbb{C})}, 1 \leq j \leq g$, is non-zero as $pr_j(X)$ is non-degenerate by the reductions in Section 2 above.

For given integers $n_1, \ldots, n_g > 0$, we consider the homomorphism

$$\varphi : E_1 \times S \cdots \times S E_g \longrightarrow E_1 \times \cdots \times E_g, \ (x_1, \ldots, x_g) \longmapsto ([n_1]x_1, \ldots, [n_g]x_g),$$

and set $Y = \varphi(X)$. We claim that $(\varphi^*\beta|_{Y})^\wedge d$ is real-proportional to $(\beta|_{X})^\wedge d$. As $\varphi$ is étale, there exists a non-empty open subset $U \subseteq X(\mathbb{C})$ such that the restriction $\varphi|_U : U \to \varphi(U)$ is a biholomorphism. Writing $y_i = \varphi(x_i)$, the sequence $(y_i)$ is $Y$-generic and satisfies $\tilde{h}_s(y_i) \to 0$. By the above argument applied to $Y$ instead of $X$, we have hence

$$\frac{1}{\#O(y_i)} \sum_{y \in O(y_i)} g(y) \longrightarrow \mathcal{E}_Y \int_{Y(\mathbb{C})} g \beta^\wedge d, \ i \to \infty,$$

for every $g \in C^0_c(Y(\mathbb{C}))$. For any continuous function $f \in C^0_c(U)$, there exists a (unique) continuous function $g \in C^0_c(\varphi(U))$ such that $f = g \circ \varphi$. As $\varphi(O(x_i)) = O(y_i)$, we have

$$\frac{1}{\#O(x_i)} \sum_{x \in O(x_i)} f(x) = \frac{1}{\#O(y_i)} \sum_{y \in O(y_i)} g(y).$$
in this situation. Hence the limits in (6) and (7) are equal, which means that
\[ \mathcal{E}_X \int_{X(\mathbb{C})} f \beta^{\wedge d} = \mathcal{E}_Y \int_{X(\mathbb{C})} f(\varphi^* \beta)^{\wedge d} \]
for any \( f \in \mathcal{E}^0_c(V) \). Varying the test function \( f \), we infer that \( \mathcal{E}_X(\beta|_V)^{\wedge d} = \mathcal{E}_Y(\varphi^* \beta|_V)^{\wedge d} \). Since \( \beta \) has real-analytic coefficients, this completes the proof of the claim.
As \( \varphi^* \alpha_j = n_j^2 \alpha_j \), we have
\[ \varphi^* \alpha_j' = (\prod_{k \in \{1, \ldots, g\}, k \neq j} n_k^2) \cdot \alpha_j' \]
for every \( j \in \{1, \ldots, g\} \). Thus, we have
\[ (\varphi^* \beta|_{X(\mathbb{C})})^{\wedge d} = d! \cdot \sum_{j=1}^g (\prod_{k \in \{1, \ldots, g\}, k \neq j} n_k^2) \cdot \alpha_j'|_{X(\mathbb{C})} \]
and hence the \((d, d)\)-forms
\[ \sum_{j=1}^g \alpha_j'|_{X(\mathbb{C})} \quad \text{and} \quad \sum_{j=1}^g (\prod_{k \in \{1, \ldots, g\}, k \neq j} n_k^2) \cdot \alpha_j'|_{X(\mathbb{C})} \]
are proportional by a positive real constant, which depends on \( n_1, \ldots, n_g \).

We claim that all \((d, d)\)-forms \( \alpha_j'|_{X(\mathbb{C})} \), \( j \in \{1, \ldots, g\} \), are pairwise proportional up to positive real constants. In fact, choosing for example \( n_1 = n_2 = \cdots = n_{g-1} = 1 \) and \( n_g = 2 \) yields that
\[ \sum_{j=1}^g \alpha_j'|_{X(\mathbb{C})} \quad \text{and} \quad \sum_{j=1}^{g-1} \alpha_j'|_{X(\mathbb{C})} + \frac{1}{4} \cdot \alpha_g'|_{X(\mathbb{C})} \]
are proportional by a real positive constant. Rewriting this proportionality, we obtain that
\[ \sum_{j=1}^{g-1} \alpha_j'|_{X(\mathbb{C})} \quad \text{and} \quad \alpha_g'|_{X(\mathbb{C})} \]
are proportional by a positive real constant. (Note that the positivity of the volume forms \( \alpha_j'|_{X(\mathbb{C})} \), \( 1 \leq j \leq g \), is used here to ensure that these proportionality factors are (strictly) positive.) This implies that also
\[ \sum_{j=1}^g \alpha_j'|_{X(\mathbb{C})} \quad \text{and} \quad \alpha_g'|_{X(\mathbb{C})} \]
are proportional by a positive real constant. We obtain similarly that each \( \alpha_j'|_{X(\mathbb{C})} \), \( j \in \{1, \ldots, g-1\} \), is proportional to \( \sum_{j=1}^g \alpha_j'|_{X(\mathbb{C})} \), whence our claim.
For later reference, let us choose reals \( r_1, \ldots, r_g > 0 \) such that
\[ r_1 \alpha_1'|_{X(\mathbb{C})} = r_2 \alpha_2'|_{X(\mathbb{C})} = \cdots = r_g \alpha_g'|_{X(\mathbb{C})}. \]

This constitutes a differential-geometric restriction on the analytic subset \( X(\mathbb{C}) \) of \( \mathcal{E}(\mathcal{N})(\mathbb{C}) \). Pulling these back along \( \alpha_{\text{mixed}} \), we obtain similar restrictions on \( \tilde{X} \subset (\mathbb{C} \times \mathcal{H})^g \). Let us spell these out in terms of a general local chart
\[ \chi : B_1(0)^d = \{(w_1, \ldots, w_d) \in \mathbb{C}^d \mid \max\{|w_1|, |w_2|, \ldots, |w_d|\} < 1\} \longrightarrow \tilde{X}. \]
For each function $f$ on $\tilde{X}$, we simply write $f$ (resp. $\partial f/\partial w_m$, $\partial f/\partial \bar{w}_m$) instead of $f \circ \chi$ (resp. $\partial(f \circ \chi)/\partial w_m$, $\partial(f \circ \chi)/\partial \bar{w}_m$). With this notation, the $(1, 1)$-form $(\chi \circ \nu_{\text{mixed}}')^* \alpha_j$, $j \in \{1, \ldots, d\}$, on $B_1(0)^d$ equals

$$
\frac{i}{\text{Im}(\tau_j)} \left( \sum_{m=1}^{d} \left[ \frac{\partial z_j}{\partial w_m} - \frac{\text{Im}(z_j)}{\text{Im}(\tau_j)} \frac{\partial \tau_j}{\partial w_m} \right] dw_m \right) \wedge \left( \sum_{m=1}^{d} \left[ \frac{\partial z_j}{\partial \bar{w}_m} - \frac{\text{Im}(z_j)}{\text{Im}(\tau_j)} \frac{\partial \tau_j}{\partial \bar{w}_m} \right] d\bar{w}_m \right).
$$

Consequently, the $(d, d)$-form $(\chi \circ \nu_{\text{mixed}})^* \alpha_{j}', j \in \{1, \ldots, d\}$, on $B_1(0)^d$ equals

$$
\prod_{k \in \{1, \ldots, g\}, k \neq j} \text{Im}(\tau_k) |\det(A_j)|^2 w_1 \wedge \bar{w}_1 \wedge \cdots \wedge w_d \wedge \bar{w}_d
$$

with

$$
A_j = \left( \frac{\partial z_l}{\partial w_m} - \frac{\text{Im}(z_l)}{\text{Im}(\tau_l)} \frac{\partial \tau_l}{\partial w_m} \right)_{l \in \{1, \ldots, g\}, l \neq j, m \in \{1, \ldots, d\}},
$$

Therefore, the equality $r_j \alpha_{j}' \mid_X = r_k \alpha_k' \mid_X$, $j, k \in \{1, \ldots, g\}$, implies

$$
r_j(\tau_j - \tau_l)^3 \det(B_j) \det(C_j) = r_k(\tau_k - \tau_l)^3 \det(B_k) \det(C_k)
$$
on $B_1(0)^d$ where we set

$$
B_j = \left( (\tau_l - \tau_l) \cdot \frac{\partial z_l}{\partial w_m} - (z_l - \tau_l) \cdot \frac{\partial \tau_l}{\partial w_m} \right)_{l \in \{1, \ldots, g\}, l \neq j, m \in \{1, \ldots, d\}},
$$

and

$$
C_j = \left( (\tau_l - \tau_l) \cdot \frac{\partial \tau_l}{\partial w_m} - (z_l - \tau_l) \cdot \frac{\partial \tau_l}{\partial w_m} \right)_{l \in \{1, \ldots, g\}, l \neq j, m \in \{1, \ldots, d\}}
$$

for each $j \in \{1, \ldots, g\}$.

6. A VARIATION OF MIXED HODGE STRUCTURES ON $X(\mathbb{C})$

In this section, we decorate the complex analytic space $X(\mathbb{C})$ with a variation of mixed $\mathbb{Z}$-Hodge structures. We refer to [27, Subsection 14.4.1] for basic definitions concerning (admissible) variations of mixed $\mathbb{Z}$-Hodge structures on complex analytic spaces.

As a starting point, we endow the family $\mathcal{E}(\mathcal{N})(\mathbb{C})$ with a variation $H' = (\nabla, \mathcal{F}, W_\bullet)$ of mixed $\mathbb{Z}$-Hodge structures following Deligne [6, Section 10]: The $\mathbb{Z}$-modules

$$
\nabla_{\mathbb{Z}, x} = \{(l, n) \in \text{Lie}(\mathcal{E}(\mathcal{N})_{\xi_x(x)})(\mathbb{C}) \times \mathbb{Z} \mid \exp_{\xi_x(x)}(l) = [n](x)\}
$$

are naturally the stalks of a local system $\nabla_{\mathbb{Z}}$ on $\mathcal{E}(\mathcal{N})(\mathbb{C})$ having $\mathbb{Z}$-rank 3. Furthermore, the Lie group exponential yields an exact sequence

$$
0 \longrightarrow \xi_x^* (R^1 \xi_x^* \mathbb{Z}_{\mathcal{E}(\mathcal{N})(\mathbb{C})})^\vee \longrightarrow \nabla_{\mathbb{Z}} \longrightarrow \mathbb{Z}_{\mathcal{E}(\mathcal{N})(\mathbb{C})} \longrightarrow 0
$$
of $\mathbb{Z}$-local systems on $\mathcal{E}(\mathcal{N})(\mathbb{C})$. (We write $\mathbb{Z}_{\mathcal{E}(\mathcal{N})(\mathbb{C})}$ for the locally constant sheaf with stalk $\mathbb{Z}$ on $\mathcal{E}(\mathcal{N})(\mathbb{C})$.) We use this to define the weight filtration as

$$
W_0 = \nabla_{\mathbb{Z}}, \ W_{-1} = \xi_x^* (R^1 \xi_x^* \mathbb{Z}_{\mathcal{E}(\mathcal{N})(\mathbb{C})})^\vee, \ W_{-2} = 0.
$$

Writing $\mathcal{V} = \nabla_{\mathbb{Z}} \otimes \mathcal{E}(\mathcal{N})(\mathbb{C})$ for the associated holomorphic vector bundle, we note that the stalk-wise projections $\nabla_{\mathbb{Z}, x} \longrightarrow \text{Lie}(\mathcal{E}(\mathcal{N})_{\xi_x(x)})(\mathbb{C})$ extend to a map

$$
\phi : \mathcal{V} \longrightarrow \xi_x^* \text{Lie}(\mathcal{E}(\mathcal{N}))(\mathbb{C})
$$
of $\mathcal{O}(\mathcal{N})(\mathbb{C})$-sheaves where $\text{Lie}(\mathcal{E}(\mathcal{N}))(\mathbb{C})$ is the sheaf on $Y(\mathcal{N})(\mathbb{C})$ having stalks $\text{Lie}(\mathcal{E}(\mathcal{N}), x) \in Y(\mathcal{N})(\mathbb{C})$. We use this to define the Hodge filtration

$$\mathcal{F}^1 = 0, \quad \mathcal{F}^0 = \ker(\phi), \quad \mathcal{F}^{-1} = \mathcal{V},$$

so that we obtain a mixed $\mathbb{Q}$-Hodge structure of type $\{(0, 0), (-1, 0), (0, -1)\}$.

The induced mixed $\mathbb{Q}$-Hodge structure $H_\mathbb{Q} = (\mathcal{V}_\mathbb{Q}, \mathcal{F}^\bullet, W_\mathbb{Q})$ can also be described in terms of Shimura theory. For this, we note that $\mathcal{E}(\mathcal{N})(\mathbb{C})$ is one of the connected components of the mixed Shimura variety associated to the datum $(G_{a, \mathbb{Q}}^2 \times GL_{2, \mathbb{Q}}, \mathbb{C} \times \mathcal{F}_1)$ where $GL_{2, \mathbb{Q}}$ acts on $G_{a, \mathbb{Q}}^2$ via its standard representation (compare [28, Chapter 10]) and the neat open compact subgroup

$$\mathcal{K} = \left\{ \left( \begin{array}{ccc} m & a & b \\ c & d & e \\ n & f & g \end{array} \right) \in \hat{\mathbb{Z}}^2 \times GL_2(\mathbb{Z}) \mid \left( \begin{array}{ccc} a & b \\ c & d \end{array} \right) \equiv \left( \begin{array}{ccc} 1 & 0 \\ 0 & 1 \end{array} \right) \pmod{\mathcal{N}} \right\};$$

see [28 Section 0.6] for the definition of neatness in adelic groups and the proof that $\mathcal{K}(\mathcal{N}, \mathcal{N} \geq 3)$ is neat. The mixed $\mathbb{Q}$-Hodge structure $H'_\mathbb{Q}$ is then induced in the standard way ([28 Propositions 1.7 and 1.10]) from the representation

$$G_{a, \mathbb{Q}}^2 \times GL_{2, \mathbb{Q}} = \left( \begin{array}{ccc} GL_{2, \mathbb{Q}} & G_{a, \mathbb{Q}}^2 \\ 0 & 1 \end{array} \right) \hookrightarrow GL_{3, \mathbb{Q}}.$$  

Writing $\text{pr}_i : \mathcal{E}(\mathcal{N})^g \to \mathcal{E}(\mathcal{N})$ for the projection to the $i$-th factor, we endow $\mathcal{E}(\mathcal{N})^g(\mathbb{C})$ with the variation $H = \bigoplus_{i=1}^g \text{pr}^*_i H'$ of mixed $\mathbb{Z}$-Hodge structures. The base change $H_\mathbb{Q}$ can again be interpreted in Shimura-theoretic terms as arising from the $g$-fold product of the representation in [10]. This allows to invoke a result of Wildeshaus [32 Theorem II.2.2] implying that $H$ is an admissible variation of mixed $\mathbb{Z}$-Hodge structures. Finally, the restriction $H|_{X(\mathbb{C})}$ is the desired admissible variation of mixed $\mathbb{Z}$-Hodge structures on $X(\mathbb{C})$. (By definition (see e.g. [27, Definition 14.49]), admissibility is trivially preserved by passing to analytic subvarieties.) In the following, we write $H|_X$ instead of $H|_{X(\mathbb{C})}$ to simplify our notation. We also write $H_x$ for the mixed Hodge structure associated with a point $x \in X(\mathbb{C})$.

# 7. The generic Mumford-Tate group of $H|_X$

Write $MT(H|_X)$ for the generic Mumford-Tate group of $H|_X$. It is clear that $MT(H|_X) \subseteq (G_{a, \mathbb{Q}}^2 \times GL_{2, \mathbb{Q}})^g$ – both from the explicit description and the Shimura-theoretic formulation. There exists a countable union $\mathcal{Z} \subseteq X(\mathbb{C})$ of proper analytic subvarieties such that $MT(H|_X) = MT(H_x)$ for all $x \in X(\mathbb{C}) \setminus \mathcal{Z}$ and that, for all points $x \in X(\mathbb{C})$, we have $MT(H_x) \subseteq MT(H|_X)$; we refer the reader to [24 Section 4] or [25 Section 6] for details.

Analogous results are true for the generic Mumford-Tate group $MT(H_{-1}|_X) \subseteq GL_{2, \mathbb{Q}}$ of the variation $H_{-1}|_X = W_{-1}/W_{-2}(H|_X)$ of pure $\mathbb{Z}$-Hodge structures of weight $-1$. There is an evident surjective homomorphism $MT(H|_X) \twoheadrightarrow MT(H_{-1}|_X)$ between the generic Mumford-Tate groups; we hence determine $MT(H_{-1}|_X)$ first. Its structure is mostly related to the presence or absence of generic isogenies between the factors of $A = \prod_{j=1}^g E_j$.

For this reason, we make a further assumption to simplify our notation: Write $\eta$ for the generic point of $S$. In the sequel, we may and do assume that there exist integers

$$i_1 = 1 < i_2 < \cdots < i_{p+1} = g + 1$$
such that the elliptic curves
\[ E_{i_q, q}, E_{i_q + 1, q}, \ldots, E_{i_{q+1} - 1, q} \]
are isogeneous for each \( 1 \leq q \leq p \), and the elliptic curves
\[ E_{i_1, \eta}, E_{i_2, \eta}, \ldots, E_{i_p, \eta} \]
are pairwise non-isogeneous. Set also \( g_q = i_{q+1} - i_q \) for each \( q \in \{1, \ldots, p\} \). For the proof of Theorem \ref{thm:main} we can even assume that
\[ E_{i_q, \eta} = E_{i_{q+1}, \eta} = \cdots = E_{i_{q+1} - 1, \eta} \]
for each \( 1 \leq q \leq p \). We also assume that there exists a \( p' \in \{0, \ldots, p\} \) such that the families
\[ E_{i_1} \to S, \ E_{i_2} \to S, \ldots, \ E_{i_{p'}} \to S \]
are non-isotrivial, and the families
\[ E_{i_{p' + 1}} \to S, \ E_{i_{p' + 2}} \to S, \ldots, \ E_{i_p} \to S \]
are constant. (In particular, all families are constant if \( p' = 0 \) and non-isotrivial if \( p' = p \).) We set \( g' = i_{p' + 1} - 1 \) and \( A_{\text{est}} \times S = E_{g' + 1} \times S \cdots 	imes S E_{g} \). For any sufficiently generic point \( s \in S(\mathbb{C}) \) (i.e., \( s \) is not contained in a countable union of proper analytic subvarieties of \( S(\mathbb{C}) \)), the elliptic curves
\[ E_{i_1, s}, E_{i_2, s}, \ldots, E_{i_p, s} \]
are pairwise non-isogeneous and none of the curves
\[ E_{i_1, s}, E_{i_2, s}, \ldots, E_{i_{p'}, s} \]
has complex multiplication. Using \cite{21} Theorems B.53 and B.72], we obtain for every point \( x \in \pi^{-1}(s) \) that
\begin{equation}
\text{MT}(H_{-1, x}) = \mathbb{G}_m(\Delta_{g_1}(\mathbb{SL}_2, \mathbb{Q})) \times \cdots \times \Delta_{g_{p'}}(\mathbb{SL}_2, \mathbb{Q}) \times Hg(A_{\text{est}}) \subseteq \mathbb{GL}_2(\mathbb{Q})
\end{equation}
where \( \Delta_k : \mathbb{SL}_2, \mathbb{Q} \to \mathbb{SL}_2, \mathbb{Q}, k \in \mathbb{Z}_{>0} \), denotes the diagonal map and \( Hg(A_{\text{est}}) \) is the Hodge group of \( A_{\text{est}} \), which we do not need to determine here. As \( s \) is sufficiently generic, the subgroup in \((11)\) equals the generic Mumford-Tate group \( \text{MT}(H_{-1, x}) \).

For each \( x \in X \), we let \( U_x \) denote the unipotent radical of \( \text{MT}(H_x) \). By \cite{2} Lemma 2.(c)], the sequence
\[ 1 \longrightarrow U_x \longrightarrow \text{MT}(H_x) \longrightarrow \text{MT}(H_{-1, x}) \longrightarrow 1 \]
is exact. We claim that \( \dim(U_x) = 2g = 2\sum_{q=1}^p g_q \) for a sufficiently general \( x \in X(\mathbb{C}) \). This leads immediately to
\begin{equation}
\text{MT}(H_x) = \mathbb{G}_m \prod_{q=1}^{p'} \left( \mathbb{G}_a^{2g_q} \times \mathbb{SL}_2, \mathbb{Q} \right) \times (\mathbb{G}_a^{2(g - g')}) \times Hg(A_{\text{est}}) \subseteq (\mathbb{G}_a^{2}, \mathbb{Q} \times \mathbb{GL}_2, \mathbb{Q})^g
\end{equation}
by comparing dimensions; here each copy of \( \mathbb{SL}_2 \) acts on the respective additive group \( \mathbb{G}_a^{2g_q} = \mathbb{G}_a^{2} \times \cdots \times \mathbb{G}_a^{2} \) diagonally on each factor \( \mathbb{G}_a^{2} \). Similarly, \( Hg(A_{\text{est}}) \) acts on \( \mathbb{G}_a^{2(g - g')} \) but we do not need to specify this action further. Again, it follows that the generic Mumford-Tate group \( \text{MT}(H|_X) \) is the group in \((12)\).
The remaining claim follows by applying [2, Proposition 1] for the mixed Hodge structure \( H_x \) at a sufficiently general point
\begin{equation}
(13) \quad x = (x_1, \ldots, x_g) \in X \subset E_1 \times_S \cdots \times_S E_g = A
\end{equation}
such that the elliptic curves \( \mathcal{E}(\mathcal{N})_{\xi(x_q)} \), \( 1 \leq q \leq p \), are pairwise non-isogeneous. We can freely assume that, as \((13)\) varies, the generic rank of
\begin{equation}
(14) \quad \text{rank}_{R_q}(R_q x_{i_q} + \cdots + R_q x_{i_{q+1-1}}), \quad R_q = \text{End}(\mathcal{E}(\mathcal{N})_{\xi(x_q)}),
\end{equation}
equals \( i_{q+1} - i_q = g_q \), for otherwise \( X \) would be contained in a proper horizontal torsion coset of \( A \), which contradicts the assumption made in the statement of (RBC). This allows us to choose \((13)\) further such that, for all \( q \in \{1, \ldots, p\} \), the rank in \((14)\) is \( g_q \).

We invoke the said proposition from [2] for the 1-motive \([u : \mathbb{Z}^g \to A_{\pi(x)}]\) where
\begin{equation*}
u : (n_1, n_2, \ldots, n_g) \mapsto (n_1 x_1, n_2 x_2, \ldots, n_g x_g).
\end{equation*}
The Zariski closure of \( u(\mathbb{Z}^g) \) is \( A_{\pi(x)} \) by our assumptions \((14)\). Furthermore, we have
\begin{equation*}
\text{End}(A_{\pi(x)}) = R_1^{g_1 \times g_1} \times \cdots \times R_p^{g_p \times g_p},
\end{equation*}
so that
\begin{equation}
(15) \quad \text{Hom}_{\text{End}_Q(A_{\pi(x)})}(\text{End}_Q(A_{\pi(x)}) \cdot u(\mathbb{Z}^g), H_1(A_{\pi(x)}, \mathbb{Q}))
= \prod_{q=1}^p \text{Hom}_{R_q \cdot Q}(R_q \cdot u_q(\mathbb{Z}^{g_q}), H_1(A_{\pi(x)}, \mathbb{Q}))
\end{equation}
where \( u_q : \mathbb{Z}^{g_q} \to E_{i_q} \times E_{i_q+1} \times \cdots \times E_{i_{q+1-1}} = E^{g_q}_{i_q}, \ 1 \leq q \leq p, \) is defined by
\begin{equation*}
u_q(n_{i_q}, n_{i_q+1}, \ldots, n_{i_{q+1}-1}) = (n_{i_q} x_{i_q}, n_{i_q+1} x_{i_q+1}, \ldots, n_{i_{q+1}-1} x_{i_{q+1}-1}).
\end{equation*}
If the elliptic curve \( \mathcal{E}(\mathcal{N})_{\xi(x_q)}, \ q \in \{1, \ldots, p\} \), has no complex multiplication, then \((14)\) implies
\begin{align*}
\text{Hom}_{R_q \cdot Q}(R_q \cdot u_q(\mathbb{Z}^{g_q}), H_1(A_{\pi(x)}, \mathbb{Q})) & \approx \text{Hom}_{\mathbb{Q}^{g_q} \times \mathbb{Q}^{g_q}}(\mathbb{Q}^{g_q} \times \mathbb{Q}^{g_q} \cdot u_q(\mathbb{Z}^{g_q}), (\mathbb{Q}^2)^{g_q}) \\
& \approx \text{Hom}_{\mathbb{Q}}(\mathbb{Q} \cdot u_q(\mathbb{Z}^{g_q}), \mathbb{Q}^2) \\
& \approx \text{Hom}_{\mathbb{Q}}(\mathbb{Q}^{g_q}, \mathbb{Q}^2) \approx \mathbb{Q}^{2g_q}.
\end{align*}
Similarly, if the elliptic curve \( \mathcal{E}(\mathcal{N})_{\xi(x_q)}, \ q \in \{1, \ldots, p\} \), has complex multiplication, then
\begin{align*}
\text{Hom}_{R_q \cdot Q}(R_q \cdot u_q(\mathbb{Z}^{g_q}), H_1(A_{\pi(x)}, \mathbb{Q})) & \approx \text{Hom}_{\mathbb{Q}^{g_q} \times \mathbb{Q}^{g_q}}(R_q^{g_q} \cdot u_q(\mathbb{Z}^{g_q}), R_q^{g_q}) \\
& \approx \text{Hom}_{R_q \cdot Q}(R_q \cdot u_q(\mathbb{Z}^{g_q}), R_q) \\
& \approx \text{Hom}_{R_q \cdot Q}(R_q^{g_q}, R_q) \approx R_q^{g_q} \approx \mathbb{Q}^{2g_q}
\end{align*}
by \((14)\). Thus the \( \mathbb{Q} \)-dimension of \((15)\) is \( 2g \). By André’s proposition, the dimension of \( U_x \) equals the \( \mathbb{Q} \)-dimension of the linear space \((15)\), whence \((12)\).

Finally, let us note that the generic derived Mumford-Tate group is
\begin{equation}
(16) \quad \text{MT}^{\text{der}}(H|_X) = \prod_{q=1}^p (\mathbb{G}_a^{2g_q} \rtimes \text{SL}_2, \mathbb{Q}) \times (\mathbb{G}_a^{2(2g_q')}) \rtimes \text{Hg}(A_{\text{ct}})^{\text{der}}.
\end{equation}
In fact, it is a normal subgroup of $\text{MT}(H|_X)$. Furthermore, its normal subgroup

$$G = \text{MT}^{\text{der}}(H|_X) \cap \left( \prod_{q=1}^{p'} \left( \mathbb{G}_{a,q}^{2g_q} \times \text{SL}_{2,q} \right) \times \{e\} \right)$$

projects surjectively onto

$$\text{MT}^{\text{der}}(H|_{X^1}) = \Delta_{g_1}(\text{SL}_{2,q}) \times \cdots \times \Delta_{g_{p'}}(\text{SL}_{2,q}) = \text{SL}_{2,q}^{p'} \subseteq \text{SL}_{2,q} = (\text{GL}_{2,q})^{\text{der}}.$$  

The following lemma, which is also of use in the next section, yields (16).

**Lemma 2.** Let $G \subseteq \prod_{q=1}^{p'} (\mathbb{G}_{a,q}^{2g_q} \times \text{SL}_{2,q})$ be a normal $\mathbb{Q}$-algebraic subgroup projecting onto $\text{SL}_{2,q}^{p'}$. Then, we have $G = \prod_{q=1}^{p'} (\mathbb{G}_{a,q}^{2g_q} \times \text{SL}_{2,q})$.

**Proof.** We first consider the case $p' = 1$ and write $g$ instead of $g_1$. Note that for every

$$(v_1, \ldots, v_g) \in (\mathbb{Q}^2)^g$$

and every

$$(w_1, \ldots, w_g, \gamma) \in (\mathbb{Q}^2)^g \times \text{SL}_{2}(\mathbb{Q}),$$

the conjugate

$$\begin{pmatrix} v_1, \ldots, v_g, (1 & 0 \\ 0 & 1) \end{pmatrix} \cdot (w_1, \ldots, w_g, \gamma) \cdot \begin{pmatrix} v_1, \ldots, v_g, (1 & 0 \\ 0 & 1) \end{pmatrix}^{-1}$$

equals

$$(v_1 - \gamma(v_1) + w_1, \ldots, v_g - \gamma(v_g) + w_g, \gamma).$$

Choose now a $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_{2}(\overline{\mathbb{Q}})$ such that

$$(1 & 0 \\ 0 & 1) - \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1-a & -b \\ -c & 1-d \end{pmatrix}$$

is invertible; an explicit admissible choice would be $a = d = 0$ and $b = -c = 1$.

By assumption, there exists some

$$(w_1, \ldots, w_g, \gamma) \in G(\overline{\mathbb{Q}}).$$

Moreover, the normality of $G(\overline{\mathbb{Q}})$ implies that

$$(v_1 - \gamma(v_1) + w_1, \ldots, v_g - \gamma(v_g) + w_g, \gamma) \in G(\overline{\mathbb{Q}})$$

for all $(v_1, \ldots, v_g) \in (\mathbb{Q}^2)^g$. The invertibility of (17) implies that the preimage of $\gamma \in \text{SL}_2(\overline{\mathbb{Q}})$ in $G(\overline{\mathbb{Q}}) \subseteq \mathbb{Q}^{2g} \times \text{SL}_2(\overline{\mathbb{Q}})$ is of (algebraic) dimension $2g$ as each of the maps

$$\mathbb{Q}^2 \longrightarrow (\mathbb{Q}^2)^g : v_i \longmapsto v_i - \gamma(v_i), \ i \in \{1, \ldots, g\},$$

is surjective. This means that the quotient map $q : \mathbb{G}_{a,q}^{2g} \times \text{SL}_{2,q} \rightarrow \text{SL}_{2,q}$ restricts to a surjective homomorphism $q|_G : G \rightarrow \text{SL}_{2,q}$ whose kernel is of dimension $2g$. This is only possible if $G = \mathbb{G}_{a,q}^{2g} \times \text{SL}_{2,q}$, whence the assertion of the lemma in case $p = 1$.

The general case $p > 1$ can be proven similarly, working with a lifting of $(\gamma, \ldots, \gamma) \in \text{SL}_2(\overline{\mathbb{Q}})^p$ in $G(\overline{\mathbb{Q}})$ and using the above argument in parallel on each of the $p$ factors. This shows that the kernel of $q|_G : G \rightarrow \text{SL}_{2,q}^p$ is of dimension $\sum_{q=1}^{p} 2(i_{q+1} - i_q) = 2g$, which forces again the asserted equality. \qed
8. The monodromy of $H|_X$

Let $x_0 \in X(\mathbb{C})$ be a sufficiently general point such that $\text{MT}(H_{x_0}) = \text{MT}(H|_X)$ and let $\text{Mon}(H|_X) \subseteq \text{MT}(H|_X)(\mathbb{Q})$ denote the (ordinary) monodromy group at $x_0$ of the local system $V^g_2$ underlying $H|_X$. We write $\text{Mon}^{\text{alg}}(H|_X) \subseteq \text{MT}(H|_X)$ for the connected component of its $\mathbb{Q}$-algebraic closure (i.e., the (connected) algebraic monodromy group of $H|_X$ with base point $x_0$ in [2]). By [2, Theorem 1], the group $\text{Mon}^{\text{alg}}(H|_X)$ (resp. $\text{Mon}^{\text{alg}}(H_{-1}|_X)$) is a $\mathbb{Q}$-normal subgroup of $\text{MT}^{\text{der}}(H|_X)$ (resp. $\text{MT}^{\text{der}}(H_{-1}|_X)$). (Note that the notion of “good” variation of mixed Hodge structures used in [2] agrees with that of an admissible variation of mixed Hodge structures. In fact, the latter notion is even trivially stronger than the former, but a result of Kashiwara [18, Theorem 4.5.2] also allows to prove the converse implication.)

In [17], it is proven that $\text{Mon}^{\text{alg}}(H_{-1}|_X) = \text{SL}_{2,\mathbb{Q}}$ in our situation (see the proof of Equation (10) in loc.cit.). As the natural map $\text{Mon}^{\text{alg}}(H|_X) \to \text{Mon}^{\text{alg}}(H_{-1}|_X)$ is evidently surjective, we infer from another use of Lemma 2 that $\text{Mon}^{\text{alg}}(H|_X)$ projects onto $\prod_{q=1}^s (\mathbb{G}_{a,\mathbb{Q}} \rtimes \text{SL}_{2,\mathbb{Q}})$.

Consider again the connected component $\widetilde{X}$ of $u_{\text{mixed}}^{-1}(X)$ from Section 3 and its stabilizer $\text{Stab}(\widetilde{X})$ under the action of $(\mathbb{Z}^2 \rtimes \Gamma(\mathcal{N}))^g$ on $(\mathbb{C} \times \mathcal{H})^g$. Both $\text{Stab}(\widetilde{X})$ and $\text{Mon}(H|_X)$ are canonically subgroups of $(\mathbb{Z}^2 \rtimes \Gamma(\mathcal{N}))^g$, and in fact they are equal to each other. This seems well-known, but we include the argument here for lack of reference and convenience of the reader. For this purpose, we choose an arbitrary lifting $\widetilde{x}_0 \in \widetilde{X}$ of the point $x_0 \in X(\mathbb{C})$. If $\gamma \in \text{Stab}(\widetilde{X})$, then there exists a path $\tilde{\phi} : [0, 1] \to \widetilde{X}$ with $\tilde{\phi}(0) = \widetilde{x}_0$ and $\tilde{\phi}(1) = \gamma \cdot \widetilde{x}_0$. Through transport along the $\mathbb{Z}$-local system $\tilde{V}^g_\mathbb{Z} = u_{\text{mixed}}^*(V^g_\mathbb{Z})$, the path $\tilde{\phi}$ induces a $\mathbb{Z}$-linear map

$$V^g_{\mathbb{Z},x_0} = \tilde{V}^g_{\mathbb{Z},\widetilde{x}_0} \longrightarrow \tilde{V}^g_{\mathbb{Z},\gamma \cdot \widetilde{x}_0} = V^g_{\mathbb{Z},x_0},$$

which can be seen to equal $\gamma \in \text{MT}(H|_X)$ by unraveling definitions. Thus $\gamma \in \text{Mon}(H|_X)$. If conversely $g \in \text{Mon}(H|_X)$ is induced by a path $\phi : [0, 1] \to X$ with $\phi(0) = \phi(1) = x$, then the lifting $\tilde{\gamma} : [0, 1] \to \widetilde{X}$ with $\tilde{\gamma}(0) = \widetilde{x}_0$ yields a point $\tilde{\phi}(1) = \gamma \cdot \widetilde{x}_0$ for some $\gamma \in (\mathbb{Z}^2 \rtimes \Gamma(\mathcal{N}))^g$. Considering again (18), we infer $\gamma = g$. This means that the intersection $g \widetilde{X} \cap \widetilde{X}$ is non-empty. As both $\widetilde{X}$ and $g \widetilde{X}$ are connected components of $u_{\text{mixed}}^{-1}(X)$, we infer that actually $\widetilde{X} = g \widetilde{X}$, whence $g \in \text{Stab}(\widetilde{X})$.

9. Separating holomorphic and anti-holomorphic terms

The $(\mathbb{Z}^2 \rtimes \Gamma(\mathcal{N}))$-invariance of the $(1, 1)$-form in (1) implies that, for every local chart $\chi : B_1(0)^d \to \widetilde{X}$ and every $\gamma \in \text{Stab}(\widetilde{X})$, the equations (9) associated with the charts $\chi : B_1(0)^d \to \widetilde{X}$ and $\gamma \circ \chi : B_1(0)^d \to \widetilde{X}$ are equivalent. In order to extract non-trivial information from monodromy, we pass to the product $\widetilde{X} \times \widetilde{X} \subseteq (\mathbb{C} \times \mathcal{H})^{2g}$ and exploit the fact that both holomorphic and anti-holomorphic terms appear in (9). The author owes this important idea to [11, Subsection 5.2].
We write $\text{pr}_i : \tilde{X} \times \tilde{X} \to \tilde{X}$, $i \in \{1, 2\}$, for the projection to the $i$-th factor. On $\tilde{X} \times \tilde{X}$, we consider the holomorphic functions

$$z^\sharp_i = z_i \circ \text{pr}_1, \quad \tau^\sharp_i = \tau_i \circ \text{pr}_1, \quad 1 \leq l \leq g,$$

and the antiholomorphic functions

$$\overline{z}^\sharp_i = \overline{z}_i \circ \text{pr}_2, \quad \overline{\tau}^\sharp_i = \overline{\tau}_i \circ \text{pr}_2, \quad 1 \leq l \leq g.$$

We next deduce from the equations (9) on $\tilde{X}$ new equations constraining the analytic subvariety $\tilde{X} \times \tilde{X}$. These equations actually change under the product action of $(\mathbb{Z}^2 \times \Gamma(\mathcal{M}))^{2g}$ on $(\mathbb{C} \times \mathcal{H})^{2g}$, so that we can use the action of $\text{Stab}(\tilde{X}) \times \text{Stab}(\tilde{X})$ to obtain non-trivial information. Let $\chi_1 : B_1(0)^d \to \tilde{X}$ (resp. $\chi_2 : B_1(0)^d \to \tilde{X}$) be a chart with local coordinates $w_1^\sharp, \ldots, w_d^\sharp$ (resp. $w_1^\flat, \ldots, w_d^\flat$) on $B_1(0)^d$. To increase readability, we write here $f$ (resp. $\partial f / \partial w^\sharp_i, \partial f / \partial \overline{w}^\flat_i$) instead of $f \circ (\chi_1, \chi_2)$ (resp. $\partial (f \circ (\chi_1, \chi_2)) / \partial w^\sharp_i, \partial (f \circ (\chi_1, \chi_2)) / \partial \overline{w}^\flat_i$) for functions $f$ on $\tilde{X} \times \tilde{X}$. For each $j \in \{1, \ldots, g\}$, we set

$$B_j' = \frac{(\overline{\tau}^\sharp_j - \overline{\tau}_j) \cdot \partial z^\sharp_i / \partial w^\sharp_i - (\overline{z}^\sharp_i - z^\sharp_i) \cdot \partial \tau^\sharp_i / \partial \overline{w}^\flat_i}{l \in \{1, \ldots, g\}, l \neq j, m \in \{1, \ldots, d\}},$$

and

$$C_j' = \frac{(\tau^\sharp_j - \tau_j) \cdot \partial \overline{z}^\flat_i / \partial \overline{w}^\flat_i - (\overline{z}^\flat_i - \overline{z}_i) \cdot \partial \overline{\tau}^\flat_i / \partial w^\sharp_i}{l \in \{1, \ldots, g\}, l \neq j, m \in \{1, \ldots, d\}}.$$ (Note that $\partial z^\sharp_i / \partial w^\sharp_i$ and $\partial \tau^\sharp_i / \partial w^\flat_i$ (resp. $\overline{\partial z}^\flat_i / \partial \overline{w}^\flat_i = \overline{\partial z}^\flat_i / \partial \overline{w}^\flat_i$ and $\partial \overline{\tau}^\flat_i / \partial w^\flat_i = \partial \overline{\tau}^\flat_i / \partial \overline{w}^\flat_i$) are holomorphic (resp. antiholomorphic) functions on $B_1(0)^d \times B_1(0)^d$.)

We claim that, for every choice of charts $\chi_i : B_1(0)^d \to \tilde{X}$ ($i \in \{1, 2\}$), the product chart

$$(\chi_1, \chi_2) : B_1(0)^d \times B_1(0)^d \to \tilde{X} \times \tilde{X},$$

satisfies the relations

$$r_j(\tau^\sharp_j - \tau_j)^3 \text{det}(B_j') \text{det}(C_j') = r_k(\tau^\flat_k - \tau_k)^3 \text{det}(B_k') \text{det}(C_k'), \quad j, k \in \{1, \ldots, g\},$$

on $B_1(0)^d \times B_1(0)^d$. With this aim in mind, we consider the set

$$\mathcal{R} = \bigcup_{(\chi_1, \chi_2)} (\chi_1(0), \chi_2(0))$$

where

$$(\chi_1 : B_1(0)^d \to \tilde{X}, \chi_2 : B_1(0)^d \to \tilde{X})$$

ranges through all pairs of charts on $\tilde{X}$ such that $\chi_1 \times \chi_2$ satisfies the relation (19) at $(0, 0)$. The following three statements are equivalent:

1. $(\tilde{x}_1, \tilde{x}_2) \in \mathcal{R}$.
2. There exists a pair of charts $(\chi_1 : B_1(0)^d \to \tilde{X}, \chi_2 : B_1(0)^d \to \tilde{X})$ such that $(\tilde{x}_1, \tilde{x}_2) \in \text{im}(\chi_1 \times \chi_2)$ and (19) is satisfied at $(\chi_1, \chi_2)^{-1}(\tilde{x}_1, \tilde{x}_2)$.
3. For every pair of charts $(\chi_1 : B_1(0)^d \to \tilde{X}, \chi_2 : B_1(0)^d \to \tilde{X})$ such that $(\tilde{x}_1, \tilde{x}_2) \in \text{im}(\chi_1 \times \chi_2)$, the equation (19) is satisfied at $(\chi_1, \chi_2)^{-1}(\tilde{x}_1, \tilde{x}_2)$. 

Indeed, the implications (3) ⇒ (2) and (2) ⇒ (1) are trivial and the implication (1) ⇒ (3) follows from the fact that (19) is independent under transformations of the local coordinates $w_1^1, \ldots, w_n^1, w_1^2, \ldots, w_n^2$. From these equivalences, we deduce that $\mathcal{R}$ is locally cut out by a real-analytic equation. We also see that our above claim amounts to $\mathcal{R} = \tilde{X} \times \tilde{X}$.

Since $\tilde{X} \times \tilde{X}$ is irreducible (as a complex-analytic subset of $(\mathbb{C} \times \mathcal{R}_1)^{2g}$), it suffices hence to show that $\mathcal{R}$ contains a non-empty open subset of $\tilde{X} \times \tilde{X}$. For this purpose, we consider an arbitrary point $(\tilde{x}, \tilde{x}) \in \tilde{X} \times \tilde{X}$ on the diagonal. Let $\chi : B_1(0)^d \to \tilde{X}$ be a chart such that $\chi(0) = \tilde{x}$. By the above equivalences, the real-analytic set $(\chi, \chi)^{-1}(\mathcal{R}) \subseteq B_1(0)^d \times B_1(0)^d$ coincides with

$$\mathcal{R}_{(\chi, \chi)} = \left\{ (w_1^1, \ldots, w_d^1, w_1^2, \ldots, w_d^2) \in B_1(0)^d \times B_1(0)^d \mid (\chi \times \chi) \text{ satisfies (19) at } (w_1^1, \ldots, w_d^1, w_1^2, \ldots, w_d^2) \right\}.$$ 

Writing $(\overline{\cdot}) : B_1(0)^d \to B_1(0)^d$ for the component-wise complex conjugation, the set $(\text{id}_{B_1(0)^d} \times (\overline{\cdot}))^{-1}((\mathcal{R}_{(\chi, \chi)})$ is a complex-analytic subset of $B_1(0)^d \times B_1(0)^d$ as can be seen by inspecting (19); this relies on the following elementary fact: If $f(w_1, \ldots, w_d) = \sum_{i_1, \ldots, i_d} a_{i_1, \ldots, i_d} w_1^{i_1} \cdots w_d^{i_d}$ is a holomorphic function on $B_1(0)^d$, then $\overline{f(w_1, \ldots, w_d)} = \sum_{i_1, \ldots, i_d} \overline{a_{i_1, \ldots, i_d}} w_1^{i_1} \cdots w_d^{i_d}$ is holomorphic as well. Furthermore, the fact that the chart $\chi$ satisfies the previous equation (19) implies immediately that

$$\Delta_{\text{skew}} = \left\{ (w_1, \ldots, w_d, \overline{w_1}, \ldots, \overline{w_d}) \mid (w_1, \ldots, w_d) \in B_1(0)^d \right\}$$

in fact, plugging in the local coordinates $(w_1, \ldots, w_d, w_1, \ldots, w_d)$ into the equation (19) for the chart $(\chi \times \chi)$ yields the original equation (9) for the chart $\chi$ back. As the smallest complex-analytic set of $B_1(0)^d \times B_1(0)^d$ containing $\Delta_{\text{skew}}$ is $B_1(0)^d \times B_1(0)^d$, we infer that

$$(\text{id}_{B_1(0)^d} \times (\overline{\cdot}))^{-1}((\mathcal{R}_{(\chi, \chi)}) = B_1(0)^d \times B_1(0)^d$$

and hence $\mathcal{R}_{(\chi, \chi)} = B_1(0)^d \times B_1(0)^d$, whence $\mathcal{R} = \tilde{X} \times \tilde{X}$. In other words, the equations (19) are satisfied on all of $\tilde{X} \times \tilde{X}$.

10. ENTER MONODROMY

Let $\chi_i : B_1(0)^d \to \tilde{X}$, $i \in \{1, 2\}$, two local charts and $\gamma \in \text{Stab}(\tilde{X})$. As $\gamma(\tilde{X}) = \tilde{X}$, the composite $\gamma \circ \chi_i : B_1(0)^d \to \tilde{X}$ is a local chart of $\tilde{X}$ as well. Writing

$$\gamma = \left( \begin{array}{c} m_1 \\ n_1 \\ \vdots \\ m_g \\ n_g \\ c_g \\ d_g \end{array} \right)$$

we note that

$$z_l \circ \gamma = \frac{z_l + m_l + n_l\tau_l}{c_l\tau_l + d_l}$$

and

$$\tau_l \circ \gamma = \frac{a_l\tau_l + b_l}{c_l\tau_l + d_l}, \quad l \in \{1, \ldots, g\},$$

as functions on $\tilde{X}$. We infer that

$$z_l^\# \circ (\gamma, \text{id}) = \frac{z_l^\# + m_l + n_l\tau_l^\#}{c_l\tau_l^\# + d_l}$$

and

$$\tau_l^\# \circ (\gamma, \text{id}) = \frac{a_l\tau_l^\# + b_l}{c_l\tau_l^\# + d_l}, \quad l \in \{1, \ldots, g\},$$
as functions on $\tilde{X} \times \tilde{X}$. We also need the derivatives
\[
\frac{\partial (z_i^\tau \circ (\gamma, \text{id}))}{\partial w_m^{\tau_i}} = \frac{1}{(c_i \tau_i^r + d_i)} \cdot \left( \frac{\partial z_i^\tau}{\partial w_m^{\tau_i}} + n_i \cdot \frac{\partial \tau_i^r}{\partial w_m^{\tau_i}} \right) - \frac{c_i (z_i^\tau + m_i + n_i \tau_i^r)}{(c_i \tau_i^r + d_i)^2} \cdot \frac{\partial \tau_i^r}{\partial w_m^{\tau_i}}
\]
and
\[
\frac{\partial (\tau_i^r \circ (\gamma, \text{id}))}{\partial w_m^{\tau_i}} = \frac{1}{(c_i \tau_i^r + d_i)^2} \cdot \frac{\partial \tau_i^r}{\partial w_m^{\tau_i}}
\]
on $B_1(0)^d \times B_1(0)^d$. With this preparation, we can compute the equations (19) for the chart
\[
(\gamma \circ \chi_1, \chi_2): B_1(0)^d \times B_1(0)^d \longrightarrow \tilde{X} \times \tilde{X};
\]
these are
(20)
\[
r_j \left( \frac{a_j \tau_j^r + b_j}{c_j \tau_j^r + d_j} - \tau_j^r \right)^3 \det(B'_m) \det(C''_m) = r_k \left( \frac{a_k \tau_k^r + b_k}{c_k \tau_k^r + d_k} - \tau_k^r \right)^3 \det(B'_m) \det(C''_m),
\]
with the $(d \times d)$-matrices
\[
(B'_m)_{i \in \{1,...,g\}, i \neq j}^{m \in \{1,...,d\}} \quad \text{and} \quad (C''_m)_{i \in \{1,...,g\}, i \neq j}^{m \in \{1,...,d\}}
\]
where
\[
(B'_m)_{lm} = \left( \frac{a_i \tau_i^r + b_i}{c_i \tau_i^r + d_i} - \tau_i^r \right) \cdot \frac{1}{(c_i \tau_i^r + d_i)} \cdot \left[ \frac{\partial z_i^\tau}{\partial w_m^{\tau_i}} + n_i \cdot \frac{\partial \tau_i^r}{\partial w_m^{\tau_i}} \right] - \frac{c_i (z_i^\tau + m_i + n_i \tau_i^r)}{(c_i \tau_i^r + d_i)^2} \cdot \frac{\partial \tau_i^r}{\partial w_m^{\tau_i}}
\]
and
\[
(C''_m)_{lm} = \left( \frac{a_i \tau_i^r + b_i}{c_i \tau_i^r + d_i} - \tau_i^r \right) \cdot \frac{\partial \tau_i^r}{\partial w_m^{\tau_i}} - \left( \frac{z_i^\tau + m_i + n_i \tau_i^r}{c_i \tau_i^r + d_i} - \tau_i^r \right) \cdot \frac{\partial \tau_i^r}{\partial w_m^{\tau_i}}
\]
In summary, each pair $(\chi_1, \chi_2)$ of charts $\chi_i: B_1(0)^d \rightarrow \tilde{X}$, $i \in \{1, 2\}$, does not only satisfy the equation (19), but also the equations (20) for all $\gamma \in \text{Stab}(\tilde{X})$. Moreover, we can consider each of these equations at each point of $B_1(0)^d \times B_1(0)^d$ as an algebraic equation on $\gamma \in \text{Stab}(\tilde{X})$, giving a $\mathbb{C}$-algebraic hypersurface of $\text{MT}(H|_X)$ containing the $\mathbb{Q}$-rational points $\text{Mon}(H|_X) = \text{Stab}(\tilde{X})$. By [5, Corollary AG.14.6], this hypersurface contains also the algebraic monodromy group $\text{Mon}^\text{alg}(H|_X)$, which is the $\mathbb{Q}$-algebraic closure of these $\mathbb{Q}$-rational points. We infer that each pair $(\chi_1, \chi_2)$ satisfies (20) for all $\gamma \in \text{Mon}^\text{alg}(H|_X)(\mathbb{Q})$.

11. A NON-VANISHING DETERMINANT

We make a final reduction before we start exploiting the equations (20) obtained in the last section. To be precise, we show that we can assume the following: For each chart
\[
\chi_0: B_1(0)^d \longrightarrow \tilde{X}, \quad w = (w_1, \ldots, w_d) \longmapsto (z_i \circ \chi_0(w), \tau_1 \circ \chi_0(w))_{i \leq g},
\]
the determinant
\[
\det \left( \left( \frac{\partial (z_i \circ \chi_0)}{\partial w_m} + \frac{s_i}{t_i} \cdot \frac{\partial (\tau_i \circ \chi_0)}{\partial w_m} \right) \right)_{l \in \{1, \ldots, g-1\} \atop m \in \{1, \ldots, d\}} \left( w \right) = 0
\]
is a non-zero holomorphic function on $B_1(0)^d$. Note that this condition holds for every chart if and only if it holds for a single one.

For each finite map $S' \to S$, note that (RBC) for a subvariety $X$ in a family $\pi : A \to S$ is equivalent to (RBC) for the subvariety $X_{S'} = X \times_S S'$ in the family $\pi_{S'} : A \times_S S' \to S'$. Furthermore, (RBC) for a subvariety $X \subseteq A$ is equivalent to (RBC) for any translate $X + \tau \subseteq A$ by a torsion section $\sigma : S \to A$. By assumption, $S$ is a subvariety of $Y(N')$ and $A = \mathcal{E}(N')|_S$.

Let $\underline{q} = (s_1/t_1, \ldots, s_g/t_g) \in \mathbb{Q}^g$ be given with $\gcd(s_i, t_i) = 1$ for all $i \in \{1, \ldots, g\}$, we set $\mathcal{N}' = \text{lcm}(t_1, \ldots, t_g, N')$, so that all torsion points of order $\text{lcm}(t_1, \ldots, t_g)$ in the generic fiber $\mathcal{E}(N')_{\mathbb{Q}^g(\underline{r})}$ extend to torsion sections $Y(\mathcal{N}') \to \mathcal{E}(\mathcal{N}')$. Writing $\xi_{\mathcal{N}', \mathcal{N}} : \mathcal{E}(\mathcal{N}') \to \mathcal{E}(\mathcal{N})$ for the standard covering and setting $X'_{\underline{q}} = \xi_{\mathcal{N}', \mathcal{N}}^{-1}(X)$, the analytic variety $\tilde{X}$ is also a connected component of $u^{-1}_{\text{mixed}}(X'_{\underline{q}})$. Thus, there exists a torsion section $\sigma : Y(\mathcal{N}') \to \mathcal{E}(\mathcal{N}')$ such that the translate
\[
\tilde{X}_\underline{q} = \left\{ \left( z_i + \frac{s_i}{t_i} \cdot \tau_i, \tau_i \right)_{1 \leq i \leq g} \in (\mathbb{C} \times \mathcal{H}_1)^g \mid (z_i, \tau_i)_{1 \leq i \leq g} \in \tilde{X} \right\}
\]
is a connected component of $u^{-1}_{\text{mixed}}(X'_{\underline{q}})$.

For a fixed local chart
\[
\chi_0 : B_1(0)^d \longrightarrow \tilde{X}, \quad w \longmapsto (z_i \circ \chi_0(w), \tau_i \circ \chi_0(w))_{1 \leq i \leq g},
\]
each of its translates
\[
\chi_{0, \underline{q}} : B_1(0)^d \longrightarrow \tilde{X}_\underline{q}, \quad w \longmapsto \left( z_i \circ \chi_0(w) + \frac{s_i}{t_i} \cdot \tau_i, \tau_i \circ \chi_0(w) \right)_{1 \leq i \leq g},
\]
is a chart of $\tilde{X}_\underline{q}$. As (RBC) for $X'_{\underline{q}}$ is equivalent to (RBC) for $X$ by the above remarks, it suffices to prove that the determinant (21) is a non-zero holomorphic function for a single chart $\chi_{0, \underline{q}} \ (\underline{q} \in \mathbb{Q}^g)$. If this would not be the case, then
\[
\det \left( \left( \frac{\partial (z_i \circ \chi_0)}{\partial w_m} + \frac{s_i}{t_i} \cdot \frac{\partial (\tau_i \circ \chi_0)}{\partial w_m} \right) \right)_{l \in \{1, \ldots, g-1\} \atop m \in \{1, \ldots, d\}} \left( w \right) = 0
\]
for all $(s_1/t_1, \ldots, s_g/t_g) \in \mathbb{Q}^g$ and all $w \in B_1(0)^d$. By continuity, this means
\[
\det \left( \left( \frac{\partial (z_i \circ \chi_0)}{\partial w_m} + u_l \cdot \frac{\partial (\tau_i \circ \chi_0)}{\partial w_m} \right) \right)_{l \in \{1, \ldots, g-1\} \atop m \in \{1, \ldots, d\}} \left( w \right) = 0
\]
for all $(u_1, \ldots, u_g) \in \mathbb{R}^g$ and all $w \in B_1(0)^d$. In particular, we can take
\[
u_l = \frac{-\text{Im}(z_i \circ \chi_0)}{\text{Im}(\tau_i \circ \chi_0)(w)}, \quad 1 \leq l \leq g
\]
so that

\[(22) \quad \det \left( \frac{\partial (z_l \circ w_0)}{\partial w_m} - \frac{\Im(z_l)}{\Im(n_l)} \cdot \frac{\partial (n_l \circ \chi_0)}{\partial w_m} \right)_{l \in \{1, \ldots, g\}} = 0 \]

for all \( w \in B_1(0)^d \). Comparing with (19), we infer that \( \alpha'_1 = 0 \), which is a clear contradiction to the non-degeneracy of \( \text{pr}_1(X) \).

In summary, we can use without loss of generality that (21) is a non-zero holomorphic function for each chart \( \chi : B_1(0)^d \to \tilde{X} \) and a fixed \( j_0 \in \{1, \ldots, g\} \).

12. Proof that \( p' = 0 \) or \( p' = p \)

We claim that either all the families \( \pi : E_i \to S \) are constant (i.e., \( p' = 0 \)) or non-constant (i.e., \( p' = p \)). For this purpose, assume that \( p' \geq 1 \) so that the family \( E_1 = \cdots = E_{i_2-1} \to S \) is non-isotrivial and that the family \( E_{i_p} = \cdots = E_g \to S \) is constant.

From Section 8 we know that \( \text{Mon}^{\text{alg}}(H|_X) \subseteq (\mathbb{G}_{a, \mathbb{Q}}^2 \ltimes \text{SL}_{2, \mathbb{Q}})^g \) projects onto the first \( p' \) factors \( \prod_{i=1}^{p'}(\mathbb{G}_{a, \mathbb{Q}}^2 \ltimes \text{SL}_{2, \mathbb{Q}}) \), which gives rise to a surjective map between their \( \mathbb{Q}_l \)-points. For every integer \( N \), there hence exists an element

\[ \gamma = \left( \frac{m_1}{n_1}, \frac{a_1}{c_1}, \frac{b_1}{d_1}, \ldots, \frac{m_g}{n_g}, \frac{a_g}{c_g}, \frac{b_g}{d_g} \right) \in \text{Mon}^{\text{alg}}(H|_X)(\mathbb{Q}) \]

with

\[ a_l = 1, \quad b_l = N, \quad c_l = 0, \quad d_l = 1, \quad m_l = 0, \quad n_l = 0 \]

for all \( 1 \leq l \leq i_{p'+1} - 1 \). Since the families \( E_l (i_{p'+1} \leq l \leq g) \) are trivial, we have furthermore

\[ a_l = 1, \quad b_l = 0, \quad c_l = 0, \quad d_l = 1 \]

for all \( i_{p'+1} \leq l \leq g \). Let us additionally fix a chart \( \chi_0 : B_1(0)^d \to \tilde{X} \). Specializing to the chart \( \gamma \circ \chi_0, \chi_0 : B_1(0)^d \times B_1(0)^d \to \tilde{X} \times \tilde{X} \), equation (20) for \( j = 1 \) and \( k = g \) becomes

\[(23) \quad r_1(\tau_1^2 + N - \overline{\tau}_1^2)^3 \det(B''_j) \det(C''_j) = r_g(\tau_g^2 - \overline{\tau}_g^2)^3 \det(B''_g) \det(C''_g) \]

with

\[ B''_j = \left( \tau_1^2 + b_l - \overline{\tau}_1^2 \right) \cdot \left( \frac{\partial z_l^2}{\partial w_m} - \left( z_l^2 - \overline{z}_l^2 \right) \cdot \frac{\partial \overline{\tau}_1^2}{\partial w_m} \right)_{l \in \{1, \ldots, g\}, l \neq j} \]

and

\[ C''_j = \left( \tau_1^2 + b_l - \overline{\tau}_1^2 \right) \cdot \left( \frac{\partial \overline{z}_l^2}{\partial w_m} - \left( z_l^2 - \overline{z}_l^2 \right) \cdot \frac{\partial \overline{\tau}_1^2}{\partial w_m} \right)_{l \in \{1, \ldots, g\}, l \neq j} \]

We can consider the difference between the left-hand and the right-hand side of (23) as a polynomial over the ring of (real-analytic) functions on \( B_1(0)^d \times B_1(0)^d \) and indeterminate \( N \). As this polynomial vanishes for each integer, it has to vanish identically. Expanding the left-hand side of the equation for the term of highest order in \( N \), we note that its term of highest degree in \( N \) is

\[ r_1 \prod_{l=i_{p'+1}}^{g} (\tau_l^2 - \overline{\tau}_l^2)^2 \cdot \det \left( \left( \frac{\partial z_l^2}{\partial w_m} \right)_{l \in \{1, \ldots, g\}} \right) \det \left( \left( \frac{\partial \overline{z}_l^2}{\partial w_m} \right)_{l \in \{1, \ldots, g\}} \right) N^{2i_{p'+1} - 1} \]
by our assumption on the non-vanishing of \(p_t^{1}\). However, the leading term on the right-hand side has degree \(\leq 2(t_{p+1} - 1) = 2t_{p+1} - 2\). From this contradiction, we conclude that \(p' = 0\) or \(p' = p\). Note that by our assumptions in Section 2, \(p' = 0\) implies \(\dim(S) = 0\). This means that \(A\) is just an abelian variety, for which (RBC) is proven in [35]. We hence concentrate on the case \(p' = p\) in the following.

13. Existence of generic isogenies

In this section, we prove that the generic fibers \(E_{i, \eta}(1 \leq i \leq g)\) are all isogenous (i.e., \(p_t^{1}\)) and \(\leq p\) for all \(1 \leq i \leq g\), and \(\leq 1\) in the remaining case that all families \(E_i \to S(1 \leq i \leq g)\) are non-isotrivial (i.e., \(p_t^{1}\)). We denote by \(M_{q}, 1 \leq q \leq p\), be arbitrary integers and set

\[N_{1q} = N_{1q+1} = \cdots = N_{q+1-1} = M_{q}\]

for all \(1 \leq q \leq p\). Again by the results from Section 8, we know that \(\text{Mon}_{\text{alg}}(H|_X) = \prod_{q=1}^{p}(\mathbb{G}_{a,q}^{2}\times \text{SL}_{2,q})\) (diagonally embedded in \((\mathbb{G}_{a,q}^{2}\times \text{SL}_{2,q})^p\)). Thus, there exists an element

\[\gamma = \left( \begin{array}{c} m_1 \\ n_1 \\ a_1 \ b_1 \\ \vdots \\ a_g \ b_g \\ m_g \ n_g \\ c_1 \ d_1 \\ \vdots \\ c_g \ d_g \end{array} \right) \in \text{Mon}_{\text{alg}}(H|_X)(\mathbb{Q})\]

with

\[a_l = 1, \ b_l = N_l, \ c_l = 0, \ d_l = 1, \ m_l = 0, \ n_l = 0\]

for all \(1 \leq l \leq g\). Since there is nothing to prove if \(p = 1\), we can assume that there exists \(k \in \{i_2, \ldots, i_3 - 1\}\). Specializing again to the chart \((\gamma \circ \chi_0, \chi_0) : B_1(0)^d \times B_1(0)^d \to X \times X\), the equation (20) for \(j = 1\) and \(k\) as here becomes

\[r_1(\tau_i^+ + N_1 - \tau_1)^3 \det(B_i') \det(C_i') = r_k(\tau_k^+ + N_k - \tau_k)^3 \det(B_k') \det(C_k')\]

with the \((d \times d)\)-matrices

\[B_j'' = \left( (\tau_i^+ + N_1 - \tau_i) \cdot \frac{\partial z_j^2}{\partial w_j^{2}} - (z_j^2 - z_j^2) \cdot \frac{\partial \tau_j^1}{\partial \tau_1} \right)_{l \in \{1, \ldots, g\}, l \neq j, m \in \{1, \ldots, d\}}\]

and

\[C_j'' = \left( (\tau_i^+ + N_1 - \tau_i) \cdot \frac{\partial z_j^2}{\partial w_j^{2}} - (z_j^2 - z_j^2) \cdot \frac{\partial \tau_j^1}{\partial \tau_1} \right)_{l \in \{1, \ldots, g\}, l \neq j, m \in \{1, \ldots, d\}}\]

The difference of both sides in (24) can be considered as a multivariate polynomial in the variables \(M_q = N_{iq} = \cdots = N_{q+1-1}, 1 \leq q \leq p\), of total degree \((2d + 3)\), which has to vanish identically because its evaluations for all \((M_1, \ldots, M_p) \in \mathbb{Z}^p\) vanish. In fact, the sum of its terms of highest degree \(2d + 3\) is

\[r_1 \det \left( \frac{\partial z_j^2}{\partial w_j^{2}} \right)_{l \in \{2, \ldots, g\}, m \in \{1, \ldots, d\}} \cdot \det \left( \frac{\partial \tau_j^1}{\partial \tau_1} \right)_{l \in \{2, \ldots, g\}, m \in \{1, \ldots, d\}} \cdot M_1 \prod_{l=1}^{g} N_l^2\]

\[-r_k \det \left( \frac{\partial z_j^2}{\partial w_j^{2}} \right)_{l \in \{1, \ldots, g\}, l \neq k, m \in \{1, \ldots, d\}} \cdot \det \left( \frac{\partial \tau_j^1}{\partial \tau_1} \right)_{l \in \{1, \ldots, g\}, l \neq k, m \in \{1, \ldots, d\}} \cdot M_2 \prod_{l=1}^{g} N_l^2\]
Since the two terms contain different monomials in \( M_1, \ldots, M_p \), their coefficients must vanish. As in the previous section, the vanishing of the first coefficient yields a contradiction to the non-vanishing of (21), whence \( p = 1 \). Recall that this implies

\[ E_1 = E_2 = \cdots = E_g \]

by the assumptions made in Subsection 7. Our assumptions on \( S \) from Section 2 imply furthermore that \( S \) is the diagonal of \( \mathcal{E}(\mathcal{N})^g \) in this case.

14. Existence of a linear equation on \( z_l|_X \) (1 \( \leq l \leq g \)).

In this section, we deduce a linear equation governing the restrictions of the functions \( z_l, 1 \leq l \leq g \), to \( X \). For each integer \( N \), there exists an element

\[ \gamma = \left( \begin{pmatrix} m_1 \\ n_1 \end{pmatrix}, \left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right), \ldots, \left( \begin{array}{cc} a_g & b_g \\ c_g & d_g \end{array} \right) \right) \in \text{Mon}^g(H|_X)(\mathbb{Q}) \]

with \( a_l = 1, b_l = N, c_l = 0, d_l = 1, m_l = 0, n_l = 0 \) for all \( 1 \leq l \leq g \). Specializing again to the chart

\[ (\gamma \circ \chi_0, \chi_0) : B_1(0)^d \times B_1(0)^d \rightarrow \tilde{X} \times \tilde{X}, \]

the equations (20) become

\[ r_j(\tau_j^2 + N - \tau_j) \det(B_j') \det(C_j') = r_k(\tau_k^2 + N - \tau_k) \det(B_k') \det(C_k'), \quad j, k \in \{1, \ldots, g\}, \]

with the \((d \times d)\)-matrices

\[ B_j' = \left( (\tau_j^2 + N - \tau_j) \cdot \frac{\partial z_j^2}{\partial w_m} - (z_j^2 - \tau_j) \cdot \frac{\partial \tau_j^2}{\partial w_m} \right) \quad (l \in \{1, \ldots, g\}, l \neq j, m \in \{1, \ldots, d\}) \]

and

\[ C_j' = \left( (\tau_j^2 + N - \tau_j) \cdot \frac{\partial z_j^2}{\partial \tau_m} - (z_j^2 - \tau_j) \cdot \frac{\partial \tau_j^2}{\partial \tau_m} \right) \quad (l \in \{1, \ldots, g\}, l \neq j, m \in \{1, \ldots, d\}). \]

Regarding again the difference of both sides of (25) as a polynomial in the indeterminate \( N \) with coefficients in the ring of (real-analytic) functions on \( B_1(0)^d \times B_1(0)^d \) and considering the term of highest degree, we obtain

\[ r_j \det \left( \frac{\partial z_j^2}{\partial w_m} \mid_{l \in \{1, \ldots, g\}, l \neq j} \right) \det \left( \frac{\partial \tau_j^2}{\partial w_m} \mid_{l \in \{1, \ldots, g\}, l \neq j} \right) \]

\[ = r_k \det \left( \frac{\partial z_j^2}{\partial \tau_m} \mid_{l \in \{1, \ldots, g\}, l \neq k} \right) \det \left( \frac{\partial \tau_j^2}{\partial \tau_m} \mid_{l \in \{1, \ldots, g\}, l \neq k} \right) \]

Specializing to the diagonal \( B_1(0)^d \subset B_1(0)^d \times B_1(0)^d \), this yields

\[ r_j \left| \det \left( \frac{\partial z_j}{\partial w_m} \mid_{l \in \{1, \ldots, g\}, l \neq j} \right) \right|^2 = r_k \left| \det \left( \frac{\partial z_j}{\partial \tau_m} \mid_{l \in \{1, \ldots, g\}, l \neq k} \right) \right|^2 \]
for the chart $\chi_0 : B_1(0)^d \to \tilde{X}$. As both determinants are holomorphic functions on $B_1(0)^d$, this implies
\begin{equation}
(26) \quad r_j^{1/2} \det \left( \frac{\partial z_l}{\partial w_m} \right)_{l \in \{1, \ldots, g\}, l \neq j, m \in \{1, \ldots, d\}} = u_{j,k} \cdot r_k^{1/2} \det \left( \frac{\partial z_l}{\partial w_m} \right)_{l \in \{1, \ldots, g\}, l \neq k, m \in \{1, \ldots, d\}}
\end{equation}
for some $u_{j,k} \in S^1 = \{ z \in \mathbb{C} \mid |z| = 1 \}$. Since the determinant (21) does not vanish, we can specialize (26) to $k = g$ and obtain
\begin{equation}
\det \left( \frac{\partial m}{\partial w_m} \right)_{l \in \{1, \ldots, g\}, l \neq j, m \in \{1, \ldots, d\}} = u_{j,g} \cdot r_g^{1/2} \det \left( \frac{\partial m}{\partial w_m} \right)_{l \in \{1, \ldots, g-1\}, l \neq j, m \in \{1, \ldots, d\}}
\end{equation}
for each $j \in \{1, \ldots, g-1\}$. Setting $f_j = (-1)^{(g-l-1)}u_{j,g}r_g^{1/2}/r_j^{1/2} \in \mathbb{C}^\times$ for $j \in \{1, \ldots, g-1\}$, we can use Kramer’s rule to obtain
\begin{equation}
f_1 \begin{pmatrix}
\frac{\partial z_1}{\partial w_1} \\
\cdots \\
\frac{\partial z_1}{\partial w_d}
\end{pmatrix} + f_2 \begin{pmatrix}
\frac{\partial z_2}{\partial w_1} \\
\cdots \\
\frac{\partial z_2}{\partial w_d}
\end{pmatrix} + \cdots + f_{g-1} \begin{pmatrix}
\frac{\partial z_{g-1}}{\partial w_1} \\
\cdots \\
\frac{\partial z_{g-1}}{\partial w_d}
\end{pmatrix} = \begin{pmatrix}
\frac{\partial z_g}{\partial w_1} \\
\cdots \\
\frac{\partial z_g}{\partial w_d}
\end{pmatrix}
\end{equation}
on $B_1(0)^d$. Therefore, we obtain
\begin{equation}
\frac{\partial}{\partial w_m} (f_1z_1 + f_2z_2 + \cdots + f_{g-1}z_{g-1} + z_g) = 0, \quad 1 \leq m \leq d,
\end{equation}
f for all $m \in \{1, \ldots, d\}$. In conclusion, we obtain a non-trivial linear equation
\begin{equation}
f_1z_1 + f_2z_2 + \cdots + f_gz_g = b, \quad b \in \mathbb{C},
\end{equation}
valid on all of $\tilde{X}$ by real-analytic continuation.

15. Completion of the proof of Theorem 11

By Section 13, the variety $S$ is the diagonal in $Y(\mathcal{N})^g$ and hence a special Shimura subvariety. Thus by [13] Section 3.3, the bi-algebraic closure $X^\text{biZar}$ of $X$ as defined in [15] is the minimal horizontal torsion coset containing $X$. It is therefore our goal to prove that $	ext{dim}(X^\text{biZar}) \leq g$ by means of the Ax-Schanuel conjecture for mixed Shimura varieties; in fact, this contradicts the assumption in (RBC). For this purpose, we set
\begin{equation}
Y = \{ (\tilde{x}, x) \in \tilde{X} \times X(\mathbb{C}) \mid \mathfrak{u}_{\text{mixed}}(\tilde{x}) = x \} \subseteq (\mathcal{H}_1 \times \mathbb{C})^g \times \mathcal{E}(\mathcal{N})^g(\mathbb{C}).
\end{equation}
The mixed Ax-Schanuel conjecture in the form of [15] Theorem 1.1] yields
\begin{equation}
\text{dim}(X^\text{biZar}) \leq \text{dim}(Y^\text{Zar}) - \text{dim}(Y)
\end{equation}
where $Y^\text{Zar}$ is the Zariski closure of $Y$ in $(\mathbb{P}^1(\mathbb{C}) \times \mathbb{C})^g \times \mathcal{E}(\mathcal{N})^g(\mathbb{C})$. Writing $H \subset \mathbb{C}^g$ for the linear hypersurface determined by (27) and $\Delta(\mathbb{P}^1(\mathbb{C}))$ for the diagonal in $\mathbb{P}^1(\mathbb{C})$, the analytic variety $Y$ is contained in the algebraic subset
\begin{equation}
(\Delta(\mathcal{H}_1) \times H) \times X \subset (\mathbb{P}^1(\mathbb{C}) \times \mathbb{C})^g \times \mathcal{E}(\mathcal{N})^g(\mathbb{C}).
\end{equation}
It follows that
\begin{equation}
\text{dim}(Y^\text{Zar}) \leq 1 + (g - 1) + \text{dim}(X) = \text{dim}(X) + g.
\end{equation}
As $\dim(Y) = \dim(X)$, we conclude that
\[
\dim(X^{\text{b Zar}}) \leq \dim(Y^{\text{Zar}}) - \dim(Y) \leq g,
\]
which concludes our proof of Theorem \ref{main-thm}.

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**References**

[1] Y. André, P. Corvaja, and U. Zannier. The Betti map associated to a section of an abelian scheme. *Invent. Math.*, 222(1):161–202, 2020.
[2] Yves André. Mumford-Tate groups of mixed Hodge structures and the theorem of the fixed part. *Compositio Math.*, 82(1):1–24, 1992.
[3] Christina Birkenhake and Herbert Lange. *Complex abelian varieties*, volume 302 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 2004.
[4] Fedor Bogomolov, Hang Fu, and Yuri Tschinkel. Torsion of elliptic curves and unlikely intersections. In *Geometry and physics. Vol. I*, pages 19–37. Oxford Univ. Press, Oxford, 2018.
[5] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
[6] Pierre Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.
[7] Laura DeMarco, Holly Krieger, and Hexi Ye. Uniform Manin-Mumford for a family of genus 2 curves. *Ann. of Math. (2)*, 191(3):949–1001, 2020.
[8] Laura DeMarco and Niki Myrto Mavraki. Elliptic surfaces and intersections of adelic $\mathbb{R}$-divisors, to appear in J. Eur. Math. Soc.
[9] Laura DeMarco and Niki Myrto Mavraki. Variation of canonical height and equidistribution. Amer. J. Math., 142(2):443–473, 2020.
[10] Fred Diamond and Jerry Shurman. *A first course in modular forms*, volume 228 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2005.
[11] Vesselin Dimitrov, Ziyang Gao, and Philipp Habegger. A consequence of the relative Bogomolov conjecture. to appear in *J. of Number Theory*.
[12] Vesselin Dimitrov, Ziyang Gao, and Philipp Habegger. Uniform bound for the number of rational points on a pencil of curves. *Int. Math. Res. Not. IMRN*, (2):1138–1159, 2021.
[13] Vesselin Dimitrov, Ziyang Gao, and Philipp Habegger. Uniformity in Mordell-Lang for curves. *Ann. of Math. (2)*, 194(1):237–298, 2021.
[14] Ziyang Gao. Generic rank of Betti map and unlikely intersections. *Compos. Math.*, 156(12):2469–2509, 2020.
[15] Ziyang Gao. Mixed Ax-Schanuel for the universal abelian varieties and some applications. *Compos. Math.*, 156(11):2263–2297, 2020.
[16] Thomas Gauthier. Good height functions on quasi-projective varieties: equidistribution and applications in dynamics. *arXiv e-prints*, page arXiv:2105.02479, May 2021.
[17] Philipp Habegger and Jonathan Pila. Some unlikely intersections beyond André-Oort. *Compos. Math.*, 148(1):1–27, 2012.
[18] Masaki Kashiwara. A study of variation of mixed Hodge structure. *Publ. Res. Inst. Math. Sci.*, 22(5):991–1024, 1986.
[19] Nicholas M. Katz and Barry Mazur. *Arithmetic moduli of elliptic curves*, volume 108 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1985.
[20] Lars Kühne. Equidistribution in Families of Abelian Varieties and Uniformity. *arXiv e-prints*, page arXiv:2101.10272, January 2021.
[21] James D. Lewis. *A survey of the Hodge conjecture*, volume 10 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, second edition, 1999. Appendix B by B. Brent Gordon.

[22] David Masser and Umberto Zannier. Torsion points on families of squares of elliptic curves. *Math. Ann.*, 352(2):453–484, 2012.

[23] David Masser and Umberto Zannier. Torsion points on families of products of elliptic curves. *Adv. Math.*, 259:116–133, 2014.

[24] David Masser and Umberto Zannier. Torsion points, Pell’s equation, and integration in elementary terms. *Acta Math.*, 225(2):227–313, 2020.

[25] J. S. Milne. Shimura varieties and moduli. In *Handbook of moduli. Vol. II*, volume 25 of *Adv. Lect. Math. (ALM)*, pages 467–548. Int. Press, Somerville, MA, 2013.

[26] Martin Olsson. *Algebraic spaces and stacks*, volume 62 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 2016.

[27] Chris A. M. Peters and Joseph H. M. Steenbrink. *Mixed Hodge structures*, volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2008.

[28] Richard Pink. *Arithmetical compactification of mixed Shimura varieties*. Bonner Mathematische Schriften [Bonn Mathematical Publications], 209. Universität Bonn Mathematisches Institut, Bonn, 1990. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 1989.

[29] Richard Pink. A common generalization of the conjectures of Andr´e-Oort, Manin-Mumford, and Mordell-Lang. available from [http://www.math.ethz.ch/~pink](http://www.math.ethz.ch/~pink) 04 2005.

[30] Michel Raynaud. *Faisceaux amples sur les schémas en groupes et les espaces homogènes*. Lecture Notes in Mathematics, Vol. 119. Springer-Verlag, Berlin-New York, 1970.

[31] Emmanuel Ullmo. Positivité et discrétion des points algébriques des courbes. *Ann. of Math. (2)*, 147(1):167–179, 1998.

[32] Jörg Wildeshaus. *Realizations of polylogarithms*, volume 1650 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1997.

[33] Xinyi Yuan. Arithmetic bigness and a uniform Bogomolov-type result, August 2021.

[34] Xinyi Yuan and Shou-Wu Zhang. Adelic line bundles over quasi-projective varieties. *arXiv e-prints*, page arXiv:2105.13587, May 2021.

[35] Shou-Wu Zhang. Equidistribution of small points on abelian varieties. *Ann. of Math. (2)*, 147(1):159–165, 1998.

[36] Shou-Wu Zhang. Small points and Arakelov theory. In *Proceedings of the International Congress of Mathematicians, Vol. II (Berlin, 1998)*, number Extra Vol. II, page 217–225, 1998.

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