All minimal $[9, 4]_2$-codes are hyperbolic quadrics

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Abstract

Minimal codes are being intensively studied in last years. $[n, k]_q$-minimal linear codes are in bijection with strong blocking sets of size $n$ in $PG(k-1, q)$ and a lower bound for the size of strong blocking sets is given by $(k-1)(q+1) \leq n$. In this note we show that all strong blocking sets of length 9 in $PG(3, 2)$ are the hyperbolic quadrics $Q^+(3, 2)$.

1 Introduction

1.1 First definitions

In this section we fix the notation that we will use in the paper and some preliminary results on minimal codes.

Definition 1.1. • An $[n, k]_q$-linear code $C$ is a subspace of $\mathbb{F}_q^n$ of dimension $k$.

• All the elements of $C$ are said codewords.

• The (Hamming) support of a vector $(v_1, v_2, \ldots, v_n) = v \in \mathbb{F}_q^n$ is $\sigma(v) = \{i|v_i \neq 0\} \subseteq \{1, 2, \ldots, n\}$.

• The (Hamming) weight of $v$ is $w(v) = |\sigma(v)|$.

• A Generator matrix $G$ for $C$ is an $k \times n$ matrix over $\mathbb{F}_q$ such that $C = \text{rowspace}(G)$.

In [11], J. L. Massey used minimal codewords to determine the access structure in its codebased secrete sharing scheme.

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Definition 1.2. Let \( C \) be an \( [n, k]_q \) code. A nonzero codeword \( c \in C \) is minimal if for every codeword \( c' \in C \) such that \( \sigma(c') \subseteq \sigma(c) \), \( c' = \lambda c \), for some \( \lambda \) in \( \mathbb{F}_q^* \). We say that \( C \) is minimal if all its codewords are minimal.

As it was noticed in [2], minimal codes are in bijection with strong blocking sets in projective spaces. A strong blocking set in \( \text{PG}(k-1, q) \) is a set of points \( M \) such that, for each hyperplane \( \sigma \), \( \langle \sigma \cap M \rangle = \sigma \). Strong blocking sets were introduced by A. A. Davydov, M. Giulietti, S. Marcugini and F. Pambianco in [6]. Later, the same concept was also studied by Sz. Fancsali and P. Sziklai in [7], under the name of generator sets, by M. Bonini and M. Borello in [4], under the name of cutting blocking sets, and by T. Héger, B. Patkós and M. Takáts in [8], under the name of hyperplane generating set.

Theorem 1.3. [2, Theorem 3.4] Let \( C \) be a non-degenerate \( [n, k]_q \)-code with generator matrix \( G = (G_1, \ldots, G_n) \). Let \( S = \{G_1, \ldots, G_n\} \) be the corresponding point set of \( \text{PG}(k-1, q) \). Then, \( C \) is a minimal code if and only if \( S \) is a strong blocking set.

From the previous theorem it follows the correspondence between \( [n, k]_q \)-minimal code and strong blocking sets of \( \text{PG}(k-1, q) \) of size \( n \). We refer the reader to [2] for more details on minimal codes, while we refer to [8] for a more geometrical approach.

1.2 Minimal length of a minimal code

A general problem is to determine the minimal length of a \( [n, k]_q \)-minimal code can have, fixing \( k \) and \( q \). Equivalently, the problem is to determine the minimum size \( n \) of a strong blocking set in \( \text{PG}(k-1, q) \). In [3], A. Beutelspacher found a lower bound for \( n \), under the hypothesis \( k-1 \leq q \). Later, G. N. Alfarano, M. Borello, A. Neri and A. Ravagnani extended the result for all values of \( k \) and \( q \).

Theorem 1.4. [1, Theorem 2.14] Let \( C \) be a \( [n, k]_q \)-minimal code. Then \( n \geq (k-1)(q+1) \).

Corollary 1.5. Let \( q = 2 \). If \( n \) is the size of a strong blocking set in \( \text{PG}(k-1, 2) \), then \( n \geq 3(k-1) \).

2 Minimal strong blocking sets in \( \text{PG}(3, 2) \)

Consider now the projective space \( \text{PG}(3, 2) \). From Corollary 1.5 we know that the minimum possible size for a strong blocking set is 9. In [2] Example 5.10],
is pointed out that a computer search showed that there is only one minimal code of length 9 up to equivalence. Here we prove in terms of minimal strong blocking sets of $PG(3, 2)$. We call minimal strong blocking set a strong blocking set meeting the lower bound given by Theorem 1.4.

It is easy to see that hyperbolic quadrics $Q^\pm (3, 2)$ are minimal strong blocking sets, i.e. all planar intersections generate the whole plane. The proof is contained also in [2], and belongs to a more general result in [5]. In fact, it is well known that plane section of a quadric is a non-degenerate conic, or a degenerate conic consisting on two concurrent lines, and in both cases the whole plane is generated by the intersection.

We now focus on a geometric characterization of minimal strong blocking sets of $PG(3, 2)$

**Lemma 2.1.** Consider a minimal strong blocking set $S$ of $PG(3, 2)$.

- Any projective plane $\pi$ in $PG(3, 2)$ contains at most 5 points of $S$.
- If $\pi$ contains a line of $S$, then it contains 5 points of $S$, forming two lines meeting in a point.

**Proof.**

- Fix a plane $\pi$. Since $|S| = 9$, we want to prove that $|S \setminus \pi| > 3$. Assume by contradiction that $|S \setminus \pi| \leq 3$, then consider the plane $\sigma$ through the points in $S \setminus \pi$. The intersection of $\sigma$ and $\pi$ is a line $\ell$, and through a $\ell$ there pass exactly three planes: $\sigma$, $\pi$ and $\delta$. But now $\delta \cap S \subseteq \ell$, which does not generate the whole plane $\delta$.

- Consider a line $\ell \subset S$. Through $\ell$ there pass the 3 planes $\sigma$, $\pi$ and $\delta$. Since by the previous a plane may not contain more than 5 points of $S$, and since $|S| = 9$, we have $|S \cap \pi| = |S \cap \sigma| = |S \cap \delta| = 5$.

**Lemma 2.2.** Let $S$ be a minimal strong blocking set of $PG(3, 2)$. Through each point $P \in S$ there pass exactly 2 lines entirely contained in $S$.

**Proof.** Firstly we prove that through $P$ may not pass more than 2 lines of $S$. Otherwise, say $\ell_1, \ell_2, \ell_3$ are 3 lines through $P$ contained in $S$. Now, $|\ell_1 \cup \ell_2 \cup \ell_3| = 7$, and the other 2 points of the minimal strong blocking set $S$ are on two more lines $\ell_4$ and $\ell_5$ through $P$. Since $P$ lies on 7 lines, take $\ell_6$ and $\ell_7$ meeting $S$ only in $P$. Now the plane $\langle \ell_6, \ell_7 \rangle$ is such that $S \cap \langle \ell_6, \ell_7 \rangle \in \{S \cap \ell_1, S \cap \ell_2, S \cap \ell_3, S \cap \ell_4, S \cap \ell_5\}$, and in all these cases the intersection does not generate $\langle \ell_6, \ell_7 \rangle$. The second step is to prove that through each $P \in S$ there pass at least two lines of $S$. Since $P$ lies on 7 lines and $|S \setminus \{P\}| = 8$, by counting we see that at least
one of the lines through $P$ is entirely contained in $S$ (recall that a projective line in $PG(3, 2)$ has 3 points). Now fix a line $\ell \subset S$, and the 3 planes $\pi_1$, $\pi_2$ and $\pi_3$ through $\ell$. Since by Lemma 2.1 $|S \cap \pi_1| = |S \cap \pi_2| = |S \cap \pi_3| = 5$, each of the planes through $\ell$ contains an other line in $S$. But we just proved that a point may not lie on 3 lines in $S$, so taking the 3 points $P_1, P_2, P_3 \in \ell$, we see that, up to a possible permutation on points, through $P_i$ there pass exactly an other line in $S \cap \pi_i$, $i \in \{1, 2, 3\}$.

We are now ready to prove the main theorem.

**Theorem 2.3.** The minimal strong blocking sets of $PG(3, 2)$ are exactly the hyperbolic quadrics $Q^+(3, 2)$.

**Proof.** The thesis arises from the characterization given in Lemma 2.1 and Lemma 2.2, together with the known classification of quadrics in [10].

**Remark 2.4.** An other proof of the theorem, which does not involve the classification of polar spaces, arises by checking all projectively non-equivalent configurations of points in $PG(3, 2)$. Since $|PG(3, 2)| = 15$, the set $X$ of all possible configurations of 9 points in $PG(3, 2)$ has size $|X| = \binom{15}{9} = 5005$. In $PG(3, 2)$ there are exactly 5 projectively non-equivalent configurations of 9 points, i.e. $PGL(4, 2)$ has exactly 5 orbits on $X$:

1. 280 hyperbolic quadrics $Q^+(3, 2)$;
2. 105 configurations that consist on a point $P$ and all the other points out of a fixed plane $\pi \ni P$;
3. 420 configurations that consist on a plane $\pi$ and two points $P, Q \notin \pi$;
4. 1680 configurations that consist on a punctured plane $\pi \setminus \{P\}$ and other 3 points $Q, R, S \notin \pi$ such that $\langle Q, R, S \rangle \cap \pi = \ell \ni P$;
5. 2520 configurations that consist on a punctured plane $\pi \setminus \{P\}$ and other 3 points $Q, R, S \notin \pi$ such that $\langle Q, R, S \rangle \cap \pi = \ell \ni P$.

By the Orbit-Stabilizer Theorem we have $\frac{|PGL(4, 2)|}{|PGL^+(4, 2)|} = \frac{20160}{72} = 280$ hyperbolic quadrics. Now we consider the second configuration. The intersection with the plane $\pi$ consists of the singleton $\{P\}$, while the intersections with the other planes have size 4 or 5, so we have no equivalence with the hyperbolic quadrics. The number of point in $|PG(3, 2)|$ is 15 and through each point there pass 7 planes, so the second set has size $7 \cdot 15 = 105$. The third configuration considers

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1 We thank the reviewer for the shortcut in the proof of the main theorem.
the 15 planes and all possible pairs of points in the remaining \(15 - 7 = |\text{PG}(3, 2) \setminus \pi|\), so we get \(15 \cdot \binom{8}{3} = 420\). Now we consider all the possibilities made of 6 points on a plane \(\pi\) and three points \(Q, R, S \notin \pi\). Since the intersection with \(\pi\) is 6, the configuration is non-equivalent to all the other considered. We have 15 planes, and each time we take out \(P\), one of the 7 points on the plane. Moreover, we should consider all the possible triple of the 8 points out of \(\pi\), so we get \((8^3) = 5880\). Here we count twice the 1680 configurations as in 4, and we have 2520 configurations as in 5. But now we have to take care on two different cases: if \(P \in \ell = \langle Q, R, S \rangle \cap \pi\) the intersection with \(\langle Q, R, S \rangle\) has size 5, while if \(P \notin \ell = \langle Q, R, S \rangle \cap \pi\), the intersection with \(\langle Q, R, S \rangle\) consists on \(\ell \cup \{Q, R, S\}\) and has size 6, so we have counted twice this configuration in the 5880 possibilities. An other way to consider the latter, is to fix the line \(\ell\) and fix two of the three planes through \(\ell\), taking out one point from each plane. Since we have 35 lines in \(\text{PG}(3, 2)\) and we consider all the \(\binom{3}{2} = 3\) pairs of planes \(\pi, \pi'\) through \(\ell\), excluding one of the 4 points of \(\pi \setminus \ell\) and one of the 4 points of \(\pi' \setminus \ell\), the total number is \(35 \cdot 3 \cdot 4 \cdot 4 = 1680\). The last configuration, when \(\langle Q, R, S \rangle \cap \pi = \ell \ni P\), contains \(5880 - (2 \cdot 1680) = 2520\) sets. We are actually considering all the possibilities since \(5005 = 280 + 105 + 420 + 1680 + 2520\).

3 Conclusion

In this paper we provided examples of strong blocking set of the minimal possible size allowed by Theorem 1.4 in \(\text{PG}(3, 2)\). From Corollary 1.5 we know that the size of a minimal strong blocking set in a projective \((k - 1)\)-dimensional space over \(F_2\) is \(3(k - 1)\). In the Fano plane we find the trivial 7 configurations of 6 points given by all the point of the plane except one. The following step should rely on \(\text{PG}(4, 2)\). In this case \(n \geq 12\), while, for example, a parabolic quadric \(Q(4, 2)\) is a strong blocking set of size 15. Computer-aided search does not show any configuration of 12 points such that the intersection with all hyperplanes generate the whole 3-space, while it is possible to show minimal strong blocking set of size 15 in \(\text{PG}(5, 2)\).

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