The Gravitational Field of String Matter

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Abstract

We study the scattering of a massless and neutral test particle in the gravitational field of a body (the string star) made of a large number of scalar states of the superstring. We consider two cases, the one in which these states are neutral string excitations massive already in ten dimensions and the one in which their masses (and charges) originate in the process of compactification on tori. A perturbative calculation based on superstring amplitudes gives us the deflection angle up to the second order in Newton’s constant. A comparison with field theory explicitly shows which among the various massless fields of the superstring give a contribution to the scattering process. In both cases, the deflection angle is smaller than the one computed in general relativity. The perturbative series can be resummed by finding the exact solution to the classical equations of motion of the corresponding low-energy action. The space-time metric of our two examples of string stars has no horizon.
1 Introduction

A string star is for us a star which is made of string matter, that is, of string states instead of ordinary matter like protons, neutrons and electrons. We believe this to be a useful concept, albeit only a hypothetical one, in studying gravitation in string theory.

To be sure, in the framework of superstring theory, ordinary matter should be made out of the massless string states by compactification and symmetry breaking through some still unknown scenario. In this paper, however, we leave these problems aside because our purpose is to describe string matter in four space-time dimensions after compactification but before symmetry breaking occurs.

We consider two examples of string stars.

The first example is the one in which string matter is made of string excitations that are already massive in ten dimensions and are taken to be neutral with respect to any gauge field, including those arising from the compactification. Such states can be thought of as a peculiar form of matter, not directly related to the one observable at low energies, but dominant at very high energy densities and, in particular, near gravitational singularities [1]. All the results obtained in this case are completely independent of the compactification scheme.

The second example we consider is the one in which string matter is made of massive and charged string states that were originally massless and neutral but acquired a mass and a charge by having a non-zero momentum in one of the compact directions. We take all compact directions to be tori, thus leaving out those more sophisticated compactification scenarios that are perhaps closer to the experimental evidence but for which the computation of string amplitudes becomes harder.

In order for the string star to be an astronomical object it must be made out of a very large number of these massive states. If we assume the string coupling constant to be of the order of $1/10$, the lowest excited state of the superstring has a mass of the order of one hundredth of the Planck mass and about $10^{40}$ of them are necessary to have a star with a mass comparable to our sun ($M_\odot \simeq 2 \times 10^{30}$ kg).

Much of our understanding of general relativity relies on those examples of which we know an exact solution. Among these, the Schwarzschild solution [2] for the spherically symmetric field configuration in vacuum plays a preeminent role because it is a realistic description of the gravitational field of a star like our sun.

In string theory we have a formal equivalence [3] between a piece of the low-
energy action of the superstring on the one hand and the Einstein-Hilbert action of general relativity on the other but an otherwise scarcity of explicit examples on which to train our physical insight.

Recently, interest in exact solutions of string-inspired actions has been revived [4, 5], and similar questions have been addressed, also in comparison with results found in two-dimensional gravity.

Our approach is however different: we start from a perturbative computation based on the standard expression for the superstring scattering amplitude in four space-time dimensions. We show that the amplitude for the scattering of a test particle by the string star is dominated by the expansion into tree diagrams, the legs of which are massless field propagators. Since we can think of the test particle as probing the metric, such an expansion is directly related to the perturbative solution of the low-energy equations for the space-time metric and it can therefore be interpreted as a systematic expansion of the metric in powers of the ratio of the gravitational radius of the star over the impact parameter of the test particle (a computation of that kind was performed for the case of the Schwarzschild metric in ref. [6]). The perturbative series can be resummed by finding an exact solution to the classical equations of motions derived from the low-energy effective action.

Such a computation, based on classical equations, is consistent as long as, like in our case, the gravitational radius of the string star is much larger than the Planck length.

Section 2 of the paper contains a perturbative computation in which the scattering of a massless test particle by the string star is computed to the second order in Newton’s constant. We develop a systematic technique to handle this kind of processes via tree level on-shell superstring amplitudes. In section 3, we compare these results with the corresponding ones in field theory. It is shown that the amplitude can be interpreted in terms of field theory Feynman diagrams where the interaction is carried by massless fields. We can thus identify which among these massless fields take part in the process.

In section 4 we consider directly the classical equations of motion of the relevant massless fields, as derived from the low-energy effective action. We solve these equations exactly; at the lowest and next-to-the-lowest order the solutions reproduce the results obtained by means of the superstring amplitude. The space-time metric is characterized by the absence of any horizon around the string star. This holds in both cases, the one in which we consider neutral massive string excitations and
the one in which we take string matter to be charged through compactification. Otherwise the respective space-time metrics differ very much.

The case of the neutral massive string excitations (also studied in ref. [7]) gives a solution which deviates from the standard Schwarzschild solution in that the string excitations, although neutral with respect to gauge fields, are a source for the dilaton field with a coupling which is fixed by string perturbation theory. Even though the massless test particle interacts only with the graviton—and therefore the deflection is at the leading order the same as in Einstein theory—the metric is strongly affected by the presence of the dilaton field, and the solution belongs to the class of solutions of the Brans-Dicke theory [8] for a special choice of the parameters. As far as we know, this is a result which was not previously discussed in the context of string theory.

The physical relevance of such a scenario stems from several studies [11] which indicate that, at high energy density, the most probable configuration in string theory is the one in which most of the massive states are excited. Thus, one can speculate that an electrically neutral collapsing star of ordinary matter and of sufficiently large mass would eventually evolve into a string star like the one discussed in this paper.

The case of a star made of originally massless string states leads to a solution of the bosonic sector of $N = 8$ supergravity in $D = 4$ space-time dimensions. It belongs to the class of solutions of gravity coupled to a scalar and a Maxwell field arising in the compactification. It corresponds to the extremal case discussed in [4], for which the charge of the star is proportional to its mass. In fact, it represents an example of anti-gravity [9], that is, a theory in which the static gravitational interaction vanishes because the repulsive exchange of spin-1 fields compensates the attraction due to the graviton and the spin-0 fields. This is not manifest in our case because we take the massless test particle to be neutral and therefore insensitive to the vector and scalar field exchange. The signature of the anti-gravity solution is however present in the deflection angle where non-linear effects start only at the third order in Newton’s constant.

Both these solutions have a singularity of the scalar curvature which is not hidden by a horizon. However, a quantum mechanical treatment of the radial motion of a test particle in the presence of the star indicates that the particle would not fall into the singularity. This is similar to the usual problem of the hydrogen atom where the kinetic energy (arising from the indeterminacy principle)
prevents the electron from falling into the nucleus. One can compare this behavior with the black hole case, where instead particles do fall into the horizon. This is discussed in section 5.

In section 6 we put forward a possible argument suggesting why our solution with neutral string states might be of relevance even in the more realistic case in which the dilaton field becomes massive.

2 The Perturbative Computation

In this section we compute the deflection angle in the classical scattering of a massless test particle in the gravitational field of a star made of scalar string states. We consider two possibilities for these states: either they are massive already in ten dimensions or they acquire a mass in the process of compactification. As we will see, the resulting theory of gravity is very different in the two cases.

2.1 Eikonal Multiple Scattering

In quantum field theory, the problem of scattering in a given external field is usually treated by computing Green functions in the presence of an external source. In physical terms, an incoming particle interacts with the background field by a multiple exchange of virtual particles.

In the case of string theory, we have no truly convenient second quantized formulation; string amplitudes are well defined only on the mass shell. A possible procedure to compute the scattering of a particle in a background field by means of on-shell amplitudes is the following.

Consider the scattering of a massless test particle by a distant target (a star of strings or ordinary matter) made of a large collection of $N$ scalar states of mass $M$. In a frame in which all the massive particles are initially at rest, the massless one moves the same way a photon would in the gravitational field of the star.

Since all exchanged momenta are very small compared to the Planck mass, the string amplitudes are dominated by contributions coming from corners of moduli space that correspond to all possible exchanges of massless fields. It is thus possible to analyze the scattering in terms of Feynman-like diagrams in complete analogy with a field theory computation. This point is further discussed in section 3.
To each order in Newton's constant $G_N$, the leading contribution to the amplitude comes from tree diagrams in which the trunk of each massless field tree splits into branches attached to the different scalar string states which make up the star, see figs. 1 and 2. The tree diagrams with the external states removed can be both connected (figs.1a and 1b) and disconnected (fig.1c). The dominance of tree over loop diagrams is true for combinatorial reasons alone. The number of tree-level Feynman diagrams corresponding to a single tree with $m$ branches is

\[
\binom{N}{m} \sim \frac{N^m}{m!} \quad \text{for} \quad N \to \infty,
\]

whereas, for example, a one-loop diagram with $m$ branches can be realized in only

\[
N \binom{N-2}{m-2} \sim \frac{N^{m-1}}{(m-2)!}
\]

ways. We conclude that, in the limit $N \to \infty$, the computation of the scattering of a massless particle by such a composite target is genuinely classical.

Furthermore, if we assume that all exchanged energies are small compared to $M$, we can ignore the back-reaction of the particle on the constituents of the string star and our computation will be equivalent to the background field method in quantum field theory. This means that the massless particle is a test particle in the classical field generated by a stationary star and it is possible to characterize the scattering by a single impact parameter

\[
b = \frac{L}{|p|},
\]
where $L$ is the angular momentum and $p$ the momentum of the incoming test particle. Moreover, the initial and final states of the star, $|i\rangle$ and $|f\rangle$, are approximately equal. They can be expanded on free particle states

$$
|i\rangle = |f\rangle = \int \prod_{j=1}^{N} \left( \frac{d^3p_j}{(2\pi)^3 2E_j} \right) \Psi(p_1, \ldots, p_N) |p_1, \ldots, p_N\rangle,
$$

where the wave function $\Psi(p_1, \ldots, p_N)$ describes an object localized in space inside the surface of the string star and with momenta distributed around the value

$$p_1 = \ldots = p_N = (M, 0)$$

with a spread $|\Delta p| \leq \Lambda$. The cutoff parameter $\Lambda$ gives an upper bound on the allowed energy transfer in a single exchange of a massless field and keeps the scattering within the quasi-elastic regime. In (2.4), $E_j \simeq M$ denotes the energy of the $j$’th particle.

It is convenient to define the matrix element $A$ by factorizing in the $S$-matrix the normalization of the external states of the test particle and the target:

$$(S - 1)_{fi} = 2E \epsilon_{in} \cdot \epsilon_{out} \sqrt{\langle i | i \rangle \langle f | f \rangle} \frac{(2\pi i)}{2} \delta(E_i + E_f) A_{fi}.$$  

In Eq. (2.6), we have considered the case in which the massless test particle in the scattering process is a graviton; the $\epsilon$’s are thus the polarizations. This choice helps in being more specific; however, our results apply also to the case in which the graviton is replaced by any other massless state. $E$ denotes the energy of the test particle.

By definition, the $A$-matrix element is a transition amplitude for free particle states averaged over the wave-function of the target:

$$A_{fi} \equiv \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p'_1}{(2\pi)^3 2E'_1} \cdots \int \frac{d^3p_N}{(2\pi)^3 2E_N} \int \frac{d^3p'_N}{(2\pi)^3 2E'_N} \times \psi^*(p'_1, \ldots, p'_N) \psi(p_1, \ldots, p_N) \times (2\pi)^3 \delta^{(3)}(p_{in} + p_1 + \cdots + p_N + p_{out} + p'_1 + \cdots + p'_N) \times \langle p_{out}; p'_1, \ldots, p'_N | T | p_{in}; p_1, \ldots, p_N \rangle 2E \epsilon_{in} \cdot \epsilon_{out} \sqrt{\langle i | i \rangle \langle f | f \rangle}.$$  

The $T$-matrix appearing in (2.7) is the usual free-particle transition amplitude defined by

$$\langle p_{out}; p'_1, \ldots, p'_N | (S - 1) | p_{in}; p_1, \ldots, p_N \rangle = i(2\pi)^4 \delta^{(4)}(\sum p_i) \langle p_{out}; p'_1, \ldots, p'_N | T | p_{in}; p_1, \ldots, p_N \rangle$$

(2.8)
Figure 2: Parametrization of the tree diagram momenta.

which can be computed by means of Feynman diagrams in field theory and from
the string path integral in string theory.

Let us define

\[ q = p_{\text{in}} + p_{\text{out}} \]  \hspace{1cm} (2.9)

and for each exchange introduce the variables (see fig.2)

\[ q_i = -p_i - p'_1 \]  \hspace{1cm} (2.10)
\[ \Delta_i = \frac{1}{2} (p_i - p'_1) \]  \hspace{1cm} (2.11)

By assumption it is possible to neglect the dependence of the wave function of the
target on the \( q_i \)'s; we can therefore factorize out in (2.7) the normalization factor

\[ \sqrt{\langle i | i \rangle \langle f | f \rangle} = \int \frac{d^3 \Delta_1}{(2\pi)^3 2M} \cdots \int \frac{d^3 \Delta_N}{(2\pi)^3 2M} |\Psi(\Delta_1, \ldots, \Delta_N)|^2 \]  \hspace{1cm} (2.12)

and obtain that

\[ A_{fi} = \int \frac{d^3 q_1}{(2\pi)^3} \cdots \int \frac{d^3 q_N}{(2\pi)^3} \left( \frac{1}{2M} \right)^N (2\pi)^3 \delta^{(3)}(q_1 + \cdots + q_N - q) \times \frac{\langle p_{\text{out}}; p'_1, \ldots, p'_N | T | p_{\text{in}}; p_1, \ldots, p_N \rangle}{2E \epsilon_{\text{in}} \cdot \epsilon_{\text{out}}} \]  \hspace{1cm} (2.13)

\( T \) in eq. (2.13) includes contributions from all massless field exchanges, connected
as well as disconnected.
In general, a connected $m$-branched tree-diagram will have $N - m$ spectator particles that only contribute a normalization factor

$$(2M)^{N-m} \prod_{j=1}^{N-m} \left[ (2\pi)^3 \delta^3(q_j) \right]. \quad (2.14)$$

This fact allows us to write the contribution from such a diagram to the $A$-matrix element in its final form as

$$A^{(m)}_{fi} = \frac{N^m}{m!} \left( \frac{1}{2M} \right)^m \int \frac{d^3q_1}{(2\pi)^3} \cdots \int \frac{d^3q_m}{(2\pi)^3} (2\pi)^3 \delta^3(q_1 + \cdots + q_m - q)$$

$$\times \frac{\langle p_{out}; p'_1, \ldots, p'_m | T^m | p_{in}; p_1, \ldots, p_m \rangle}{2E \epsilon_{in} \cdot \epsilon_{out}}, \quad (2.15)$$

where now $T^m$ refers to the particular connected diagram in which only $m$ scalar massive states partake in the interaction and we have taken into account the combinatorial factor $(2.1)$. Diagrams involving interactions between the constituents of the string star can be consistently ignored in the computation of the $A$-matrix element.

Insofar as we will deal only with processes in which the exchanged momentum is kept fixed and very much smaller than the energy of the incoming particle, it is useful to cast our computation in the framework of the eikonal approximation [10] in which the $A$-matrix for the multiple scattering of the massless test particle on the composite target can be written in the exponential form

$$A_{fi} = \int d^2b e^{i\mathbf{q}_\perp \cdot \mathbf{b}} \left[ e^{i\phi(b)} - 1 \right], \quad (2.16)$$

where $\mathbf{q}_\perp$ is the transverse part of exchanged momentum (see appendix D for a discussion of the kinematics).

At the lowest order in $G_N$, the only contribution to the scattering comes from the process in which the test particle exchanges a single graviton with only one of the $N$ massive scalars (Fig.1a); this can be computed from the four-point Veneziano amplitude [3] in which two gravitons and two massive scalars are taken as external states:

$$\langle p_{out}; p'_1 | T^1 | p_{in}; p_1 \rangle = \epsilon_{in} \cdot \epsilon_{out} \frac{4\kappa^2 M^2 E^2}{q_\perp^2}, \quad (2.17)$$

where $\kappa$ is the gravitational coupling constant.
Eq. (2.17) can be used to define the relationship between $\kappa$ and $G_N$ by imposing that the deflection angle of the massless particle be equal to the one obtained in general relativity.

To calculate the deflection angle $\Delta \vartheta$ we use a stationary phase in (2.16), that is

$$\Delta \vartheta = -\frac{1}{E} \frac{\partial}{\partial b} \delta^m(b).$$

(2.18)

For $T^1$ given by (2.17) we have

$$\delta^m(b) = \int \frac{d^2 q_\perp}{(2\pi)^2} e^{-i q_\perp \cdot b} \frac{\kappa^2 N M E}{q_\perp^2} = -\frac{\kappa^2 N M E}{2\pi} \log b.$$

(2.19)

Eq. (2.18) and (2.19) yield the result

$$\Delta \vartheta = \frac{4M_\star}{b} \frac{\kappa^2}{8\pi},$$

(2.20)

where $M_\star = NM$ is the total rest mass of the star.

The deflection (2.20) is equal to the one in general relativity [2] if we define

$$\kappa^2 = 8\pi G_N,$$

(2.21)

which is the convention we keep throughout this paper.

An important point arises here. Definition (2.21) is based on a process in which only the graviton among the string massless fields is exchanged, the lower spin states being suppressed. Had we defined the string coupling by means of the static potential between two bodies of masses $M_1$ and $M_2$, by taking

$$V \equiv G_N \frac{M_1 M_2}{r},$$

(2.22)

the dilaton (and, for charged string states, other massless fields as well) would have contributed. For instance, neutral, massive and scalar string excitations attract each other by means of two forces, one arising from the graviton and one from the dilaton exchange, the net result being an overall factor two with respect to the usual Newtonian attraction. This can be readily seen by computing the four-point Veneziano amplitude for the external excited string states. Accordingly, the static potential definition would have given us $\kappa^2 = 4\pi G_N$ instead of (2.21).
2.2 The String Amplitude to the Second Order in $G_N$

By counting powers of Newton’s constant, the first non-linear correction to the deflection angle should be extracted from the six-point amplitude. Let us consider first the case in which the string scalar states of the star are massive already in ten dimensions (that is, they are string excitations). We perform the calculation of the superstring amplitude by means of the covariant loop calculus \[11\]. In appendix A we show that the same results can also be obtained by using the four-graviton string amplitude as a building block.

The six-point amplitude for the scattering of a graviton off two massive scalars can be built from the general operator vertex \[11, 12\]

\[
V_6 = \frac{4\pi^3}{\alpha'^2} \left\{ \int \frac{1}{dV_{ABC}} \prod_{k=1}^{6} (dZ_k \, k(q = 0, 0_a)) \right. \\
\left. \times \exp \left[ \frac{1}{2} \sum_{k \neq l} \left( \sqrt{\frac{\alpha'}{2}} p_k + \psi_k DZ_k \right) \left( \sqrt{\frac{\alpha'}{2}} p_l + \psi_l DZ_l \right) \log(Z_k - Z_l) \right] \right\} \wedge \{ \text{c.c.} \}.
\]

(2.23) is written in the Lorentz-covariant world-sheet supersymmetric formulation. Locally on the super Riemann surface we choose holomorphic super coordinates

\[
Z = (z, \theta)
\]

in terms of which the closed string space-time super coordinate can be expanded into a bosonic and a fermionic part and into a left- and a right-moving chirality sector as follows

\[
X(Z, \bar{Z}) = x(z) + \bar{x}(\bar{z}) + i\theta \psi(z) + i\bar{\theta} \bar{\psi}(\bar{z}) ;
\]

(2.25)

these are then decomposed by the oscillator representation into

\[
x(z) = \frac{1}{2} \sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \alpha_n \frac{\alpha'}{2} z^{-n} \\
\psi(z) = \sqrt{\frac{\alpha'}{2}} \sum_{n \in \mathbb{Z}+1/2} \psi_n z^{-n-1/2},
\]

(2.26)

and similarly for the left mover. This decomposition can be performed in the neighborhood of each puncture, thus introducing a set of oscillators for each external state, as in (2.23).
The differences in super space are defined as $Z - Y = z - y - \theta_z \theta_y$. The covariant super derivative is defined as $D_Z \equiv \partial_0 + \theta_0 \partial_z$. Finally, the states $\langle q = 0, 0 \rangle$ are the position zero vacuum states in the appropriate ghost sector.

The super-projective invariant volume $[11]$ is

$$d\nu_{ABC} = \frac{dZ_A dZ_B dZ_C}{[(Z_A - Z_B)(Z_B - Z_C)(Z_C - Z_A)]^{1/2} d\Theta_{ABC}},$$

(2.27)

where

$$\Theta_{ABC} = \frac{\theta_A(Z_B - Z_C) + \theta_B(Z_C - Z_A) + \theta_C(Z_A - Z_B) + \theta_A \theta_B \theta_C}{[(Z_A - Z_B)(Z_B - Z_C)(Z_C - Z_A)]^{1/2}}.$$  

(2.28)

The overall constant appearing in front of (2.23) is the normalization of the string amplitude on the sphere [13].

We can fix the super-projective invariance by choosing the following parametrization for the super Koba-Nielsen variables

$$Z_{in} = (0, \theta_1) \quad Z'_2 = (y, \theta_y) \quad Z'_1 = (x, \theta_x)$$
$$Z_{out} = (z, \theta_2) \quad Z_2 = (1, 0) \quad Z_1 = (\infty, 0).$$

(2.29)

The six-point $T$-matrix element is obtained by acting with the vertex (2.23) on the external states of two gravitons

$$\frac{\kappa}{\pi} \epsilon_{\mu_1 \nu_1}^{\alpha_1} \psi_{\alpha_1 \mu}^\mu [p_i ; 0]_L \otimes \overline{\psi}^{\mu}_{-1/2} \overline{p_i ; 0} R$$

(2.30)

(with $p_i^2 = p_i \cdot e_i = p_i \cdot e_i = 0$), and of four scalars of mass $\alpha' M^2 = 4$

$$\frac{\kappa}{3 \pi} \Xi_{\mu_1 \nu_1}^{\alpha_1} \psi_{\alpha_1 \mu}^\mu [p_i ; 0]_L \otimes \overline{\psi}^{\mu}_{-1/2} \overline{\psi}^{\nu}_{-1/2} \overline{p_i ; 0} R$$

(2.31)

(with $p_i^2 = -M^2$, $\Xi_{\mu_1 \nu_1}^{\alpha_1} = \epsilon_{\mu_1 \nu_1}^{\alpha_1} p_i^\lambda / \sqrt{3\pi}$ $M$, such that $\Xi_{\mu_1 \nu_1}^{\alpha_1} \Xi_{\mu_1 \nu_1}^{\alpha_1} = 1$ and similarly for the barred quantities). The numerical factor in front of both states are fixed by the factorization of the amplitude [13]. No momentum is flowing in the compact directions.

By considering only the part of the complete amplitude that is proportional to $\epsilon_{in} \cdot \epsilon_{out}$, after integrating out the Grassman variables and performing an integration by parts in $z$ to obtain a uniform power of $\alpha'$, we find the following contribution to the $T$-matrix

$$T_6 = \epsilon_{in} \cdot \epsilon_{out} \frac{\kappa^4 (\alpha')^3}{4 \pi^3} \int d^2x d^2y d^2z |1 - y|^{-6 + \alpha' p_2^0} |y|^\alpha p_{in} p_2^0$$
$$\times |1 - x|^{\alpha' p_2^0} |x|^\alpha p_{in} p_2^0 |1 - z|^{\alpha' p_{out} p_2^0} |z|^{-2 + \alpha' p_{in} p_{out}}$$

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Figure 3: The three pinched diagrams contributing to the deflection angle to $O(G_3^2)$. 

\[
\times |z - x|^{\alpha'} |p_{\text{out}} \cdot p'_1| |z - y|^{\alpha'} |p_{\text{out}} \cdot p'_2| |x - y|^{\alpha'} |p'_{2,1}|
\times \left\{ \frac{p_{\text{out}} \cdot p'_1 p'_2 \cdot p'_1}{(y - x)(z - x)} + \frac{p_{\text{in}} \cdot p'_2 p_{\text{out}} \cdot p'_1}{y(z - x)} - \frac{p_{\text{out}} \cdot p'_2 p_{\text{in}} \cdot p'_1}{x(z - y)} \right\} \wedge \{ \text{c.c.} \}.
\]

(2.32)

Because of the three powers of $\alpha'$ in front of $T_6$, in order to get a non-vanishing result in the field theory limit $\alpha' \to 0$ we have to extract three poles in the momenta—each of them bringing down one power of $(\alpha')^{-1}$. Since there are three internal propagators in a six-point $\phi^3$ tree diagram, this is what singles out the corners of moduli space corresponding to $\phi^3$-like diagrams; each pole comes from a pinching limit in which some of the Koba-Nielsen variables come close to each other.

Ignoring all diagrams describing interactions among string star constituents there are three pinching limits we are interested in, as depicted in fig.3. The first one corresponds to taking $z \to 0$ and, successively, $x \to \infty$, $y \to 1$ (fig.3a). In terms of sewing parameters [12] we have $x = 1/A_2$, $y = 1 - A_1$ and $z = A$; these variables are useful in evaluating the amplitude in the pinching limit, since they all go to zero at the same rate. Rewriting (2.32) in terms of the sewing parameters pertaining to the diagram in fig.3a we arrive at:

\[
T_6 = \epsilon_{\text{in}} \cdot \epsilon_{\text{out}} \frac{\kappa^4(\alpha')^3}{4\pi^3} \int \frac{d^2 A d^2 A_1 d^2 A_2}{|A|^2 |A_1|^6 |A_2|^2} |A_2|^{-\alpha'} |p_{\text{in}} + p_{\text{out}} + p'_1 + p'_2| \times |A_1|^{\alpha' p_2 |p'_2|} |1 - A_1|^{\alpha' p_{\text{in}} |p'_2|} |1 - A_2|^{\alpha' p_2 |p'_1|} |1 - A|^{\alpha' p_{\text{out}} |p'_2|} \times |1 - A A_2|^{\alpha' p_{\text{out}} |p'_1|} |1 - A - A_1|^{\alpha' p_{\text{out}} |p'_2|} |1 - A_2 + A_2 A_1|^{\alpha' p'_2 |p'_1|}.
\]

(2.33)
\[
\times \left\{ \frac{A_2 \mathbf{p}_{\text{out}} \cdot \mathbf{p}'_1 \mathbf{p}'_2 \cdot \mathbf{p}'_1}{(1 - AA_2)(1 - A_2 + A_1 A_2)} - \frac{\mathbf{p}_{\text{in}} \cdot \mathbf{p}'_2 \mathbf{p}_{\text{out}} \cdot \mathbf{p}'_1}{(1 - A_1)(1 - AA_2)} + \frac{\mathbf{p}_{\text{out}} \cdot \mathbf{p}'_2 \mathbf{p}_{\text{in}} \cdot \mathbf{p}'_1}{(1 - A - A_1)} \\
+ \frac{\mathbf{p}'_2 \cdot \mathbf{p}_{\text{out}} \cdot \mathbf{p}'_1}{(1 - A_2 + A_1 A_2)(1 - A - A_1)} + \frac{\mathbf{p}_{\text{out}} \cdot \mathbf{p}_2 \mathbf{p}'_2 \cdot \mathbf{p}'_1}{(1 - A)(1 - A_2 + A_1 A_2)} \right\} \wedge \{ \text{c.c.} \}.
\]

By extracting the poles, we obtain (see appendix C)

\[
T^a_6 = \epsilon_{\text{in}} \cdot \epsilon_{\text{out}} \frac{16\kappa^4}{M^4} \frac{[\mathbf{p}_{\text{in}} \cdot \mathbf{p}'_1 \mathbf{p}_{\text{out}} \cdot \mathbf{p}'_2 - \mathbf{p}_{\text{in}} \cdot \mathbf{p}'_2 \mathbf{p}_{\text{out}} \cdot \mathbf{p}'_1 + \mathbf{p}_{\text{out}} \cdot \mathbf{p}_2 \mathbf{p}'_2 \cdot \mathbf{p}'_1 + \mathbf{p}_{\text{out}} \cdot \mathbf{p}_2 \mathbf{p}'_2 \cdot \mathbf{p}'_1]^2}{(p'_1 + p_1)^2 (\mathbf{p}_{\text{in}} + \mathbf{p}_{\text{out}})^2 (p'_2 + p_2)^2}. \tag{2.34}
\]

After substitution of the momenta

\[
\mathbf{p}_{\text{in}} = (E, \mathbf{p}) ; \quad \mathbf{p}_{\text{out}} = (-E', \mathbf{q} - \mathbf{p}) \\
p_1 = p_2 = (M, 0) \\
p'_1 = (-\bar{E}_1, -\mathbf{q}_1) \\
p'_2 = (-\bar{E}_2, -\mathbf{q}_2), \tag{2.35}
\]

where

\[
\bar{E}_i = \sqrt{M^2 + \mathbf{q}_i^2} \simeq M + \frac{\mathbf{q}_i^2}{2M}, \tag{2.36}
\]

and

\[
\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2, \tag{2.37}
\]

eq. (2.34) yields

\[
T^a_6 = -\epsilon_{\text{in}} \cdot \epsilon_{\text{out}} \frac{16\kappa^4 M^4}{q'_1 q'_1 q'_2 q'_2} \left( \mathbf{q}_1 \cdot \mathbf{p} \right) \left( \mathbf{q}_2 \cdot \mathbf{p} \right). \tag{2.38}
\]

We keep in the numerator of (2.38), and subsequent formulas, only terms at most quadratic in the momenta \( \mathbf{q}, \mathbf{q}_1 \) and \( \mathbf{q}_2 \), as higher order terms would not give a contribution to the deflection angle.

The other two pinching limits are the one in which \( x = 1/A_2, z = 1 - A \) and \( y = 1 - AA_1 \) (the ladder, fig.3b) and the one in which \( x = 1/(AA_2), z = 1/A \) and \( y = 1 - A_1 \) (the cross-ladder, fig.3c). Notice that now the Koba-Nielsen variables go to their pinched corners at different rates. These diagrams contain the iteration of the single-graviton exchange but also a contact term where the propagator separating
the two graviton emissions is canceled by momenta factors in the vertices. Hence, we obtain

\[ T^b_6 = \epsilon_{in} \cdot \epsilon_{out} \frac{16\kappa^4}{(p'_1 + p_1)^2 (p_{out} + p_2 + p'_2)^2 (p'_2 + p_2)^2} \left[ p_{out} \cdot p'_2 \cdot p'_1 + p_{out} \cdot p'_1 \cdot p'_2 + p_{out} \cdot p'_1 \cdot p'_2 \right] \]

and

\[ T^c_6 = \epsilon_{in} \cdot \epsilon_{out} \frac{16\kappa^4}{(p'_1 + p_1)^2 (p_{out} + p_1 + p'_1)^2 (p'_2 + p_2)^2} \left[ p_{in} \cdot p'_2 \cdot p_{out} \cdot p'_1 \right] \]

respectively.

Again, by inserting the momenta (2.35) we obtain

\[ T^b_6 + T^c_6 = \epsilon_{in} \cdot \epsilon_{out} \frac{8\kappa^4 E^4 M^4}{q_1^2 q_2^2} \left[ \frac{1}{q_1 \cdot p} + \frac{1}{q_2 \cdot p} \right] + \frac{16\kappa^4 E^2 M^4}{q_1^2 q_2^2} \] (2.41)

Notice that we need not include in our calculation contact terms proportional to \( q_1^2 + q_2^2 \); they vanish after integration over \( q_1 \) and \( q_2 \) (see appendix C).

The first term in the amplitude (2.41) is dominated by momenta \( q_1 \) orthogonal to \( p \) and gives rise to a contribution to the \( A \)-matrix (2.15) that can be written as

\[ \left( A_{f_1}^{(1)} \right)^2 = i \frac{\kappa^4 M^2 E^2}{2} \int \frac{d^2 q_1 d^2 q_2}{(2\pi)^2} \frac{1}{q_1^2 q_2^2} \delta^{(2)}(q_1 + q_2 - q) \] (2.42)

The amplitude (2.42) is a convolution in momentum space of two 4-point tree-level amplitudes. As it has been discussed in [14], such a factorization implies the exponentiation of the lowest order contribution typical of the eikonal approximation [10]. In other words, we witness in our tree-level calculation the same mechanism at work in the loop calculation of [14, 12]: the exponentiation of the amplitude to preserve the unitarity of the theory.

We are not interested here in this part of the amplitude, which is simply the iteration of the four-point amplitude. Instead, we want to study the truly non-linear effects.

2.3 The Deflection Angle, I: Neutral String States

The \( O(G_N^2) \) contribution to the eikonal phase is the sum of two parts. The one from the Y-diagram (fig.3a) given in eq. (2.38) and the second term in eq. (2.41)
that is somewhat hidden as a sub-leading piece of the ladder diagram (figs. 3b and 3c). This latter term was erroneously neglected in refs. [16, 7].

Collecting these two terms together we have that

$$T_6 = \epsilon_{in} \cdot \epsilon_{out} \cdot 16 \kappa^4 M^4 \frac{\mathbf{q}_1^2 \mathbf{q}_2^2}{\mathbf{q}_1^2 \mathbf{q}_2^2} \left[ E^2 \mathbf{q}_1^2 - (\mathbf{q}_1 \cdot \mathbf{p})(\mathbf{q}_2 \cdot \mathbf{p}) \right] ,$$

(2.43)

and therefore

$$A_{fi}^{(2)} = \kappa^4 M^2 \frac{1}{\mathbf{q}_1^2 \mathbf{q}_2^2} \int \frac{d^3 \mathbf{q}_1 d^3 \mathbf{q}_2}{(2\pi)^6} \frac{1}{\mathbf{q}_1^2 \mathbf{q}_2^2} \times \left[ E\mathbf{q}_1^2 - \frac{(\mathbf{q}_1 \cdot \mathbf{p})(\mathbf{q}_2 \cdot \mathbf{p})}{E} \right] (2\pi)^3 \delta^{(3)}(\mathbf{q} - \mathbf{q}_1 - \mathbf{q}_2) .$$

(2.44)

By means of the integrals in appendix C, we readily obtain that

$$\delta^{(m=2)}(b) = 7\pi^2 G_N M^2 E \int \frac{d^2 \mathbf{q}_1}{(2\pi)^2} e^{-i \mathbf{q}_1 \cdot \mathbf{b}} \frac{1}{|\mathbf{q}_1|} = \frac{7\pi G_N^2 M^2 E}{b} .$$

(2.45)

The deflection angle is found by using (2.18) and it is equal to

$$\Delta \vartheta = \frac{4G_N M}{b} + \frac{7\pi}{2} \left( \frac{G_N M}{b} \right)^2 + O(G_N^3) .$$

(2.46)

(2.46) describes the scattering of a test particle in the gravitational field of a string star made of massive and neutral states up to $O(G_N^2)$.

### 2.4 The Deflection Angle, II: Charged String States

The case of massless charged string states can be treated along similar lines. Six massless vertices (the ones in (2.30)) replace the two massless and four massive vertices we used in the previous section. Compactification on tori gives a mass to these scalar states by requiring that they have at least one non-vanishing component of momentum in one of the compact directions. The ten-dimensional momenta are now parametrized as follows:

$$
\begin{align*}
p'_1 &= (-\tilde{E}_1, -\mathbf{q}_1, -\xi_1 M) \\
p_1 &= (M, 0, +\xi_1 M) \\
p'_2 &= (-\tilde{E}_2, -\mathbf{q}_2, -\xi_2 M) \\
p_2 &= (M, 0, +\xi_2 M) ,
\end{align*}
$$

(2.47)
while \( p_{\text{in}} \) and \( p_{\text{out}} \) are the same as before. The last entry in (2.47) gives the non-zero momentum in one of the compact directions, \( \xi_i = \pm 1 \) being the sign and \( M \) is the four-dimensional mass.

The amplitude for this process can be computed and it is identical to (2.32) except for the power of minus six in the first line being replaced by a power of minus two. This is due to the different mass-shell condition (that is, \( \alpha' p^2 = 0 \) instead of \( -4 \)) now valid for the external states. By proceeding in the computation we obtain once again the expressions (2.34), (2.39) and (2.40) for the various pinching limits.

After subtracting the contribution from the iteration of the ladder diagram which gives rise again to (2.42), we can insert the values of the momenta; these are now given by (2.47), this being the real difference with respect to section 2.3 above. At the end we have

\[
T_6 = \epsilon_{\text{in}} \cdot \epsilon_{\text{out}} \frac{32 \kappa^4 M^4}{q_1^2 q_2^2} (1 - \xi_1 \xi_2) \left[ \frac{E^2}{2} q_2^2 - (q_1 \cdot p)(q_2 \cdot p) \right] \tag{2.48}
\]

(recall that \((1 - \xi_1 \xi_2)^2 = 2 (1 - \xi_1 \xi_2)\)).

If the string star contains both positive and negative charges in such a way that its total charge is zero, the term in (2.48) proportional to \( \xi_1 \xi_2 \) averages to zero. In this case the deflection angle can be computed from (2.15), (2.19) and (2.18) and is given by

\[
\Delta \vartheta = \frac{4 G_N M_*}{b} + 3 \pi \left( \frac{G_N M_*}{b} \right)^2 + O(G_N^3). \tag{2.49}
\]

If all the constituents carry charges of the same sign (i.e., \( \xi_1 \xi_2 = 1 \)), the string star has a total charge \( Q_* = \pm \sqrt{2} \kappa M_* \) and the second order correction to the deflection angle vanishes.

### 3 Field Theory Analysis

In the calculation of the previous section, the contribution of the various fields taking part in the interaction are entwined together in the string amplitude. It is therefore useful to reproduce our results by means of a field theory analysis, in which each exchange of field contributes to an independent Feynman diagram, in order to identify exactly which massless fields are excited by the string matter. In section 4 this insight will be used to obtain exact solutions from the low-energy effective action.
3.1 Neutral String States

The massive scalars are neutral and therefore the exchanged fields are only the graviton and the dilaton. This is understood best from the ten-dimensional superstring from which one can compute the amplitude for two massive string states to emit one (slightly off shell) massless state.

Of the many massless states present in the superstring, only those in the $NS \times \overline{NS}$ sector (that is, in the form \(2.30\)) have a non-zero coupling to the massive state \(2.31\), given by

\[
2\kappa \epsilon_\mu \epsilon_\nu p^\mu p^\nu + \left( \text{longitudinal piece} \right). \tag{3.1}
\]

From (3.1) we see that the coupling to a source with purely four-dimensional momentum \(p^\mu\) involves only the graviton and the dilaton in four dimensions. In particular, the anti-symmetric tensor and all states with polarizations pointing in compact directions decouple.

If we parametrize the gravitational field as

\[
\sqrt{-g} g^{\mu \nu} \equiv \eta^{\mu \nu} + 2\kappa h^{\mu \nu}, \tag{3.2}
\]

there exist only two Feynman diagrams (fig.4). The relevant Feynman rules are given in the appendix B.

The computation of the three-graviton vertex is rather tedious; we write here the final result:

\[
T_{\text{graviton}} = \epsilon_{\text{in}} \cdot \epsilon_{\text{out}} \frac{16\kappa^4 M^4}{q_1^2 q_2^2 q_1^2} \left[ E^2 q_1^2 - \frac{1}{2} (q_1 \cdot p)(q_2 \cdot p) \right]. \tag{3.3}
\]
This would be the only contribution in general relativity where only gravitons are exchanged (fig.4a). Notice that this diagram contains a contact term, the term without the pole in $q_2^\perp$, which in the string computation was hidden in the ladder diagrams of fig.3b and 3c.

The corresponding diagram (fig. 4b) in which a dilaton is exchanged between the two massive states gives

$$T_{dilaton} = -\epsilon_{in} \cdot \epsilon_{out} \frac{8\kappa^4 M^4}{q_1^2 q_1^2 q_2^2} (q_1 \cdot p)(q_2 \cdot p),$$  \hspace{1cm} (3.4)

and represents the coupling of gravity to the energy-momentum tensor of the dilaton field.

We see that the string amplitude given by (2.43) originates from these two diagrams. In particular, the dilaton gives the factor two in front of the $(q_1 \cdot p)(q_2 \cdot p)$ piece and does not contribute to the part proportional to $E^2$. The two have opposite sign and therefore the dilaton contribution tends to make the deflection angle smaller.

### 3.2 Charged String States

The theory is more complicated in the case of charged string states acquiring a mass via toroidal compactification. Let us assume for simplicity that the mass is generated by a non-zero value of the momentum in only one (for instance, the fifth) direction. As it can be readily understood from (3.1) such states couple to the graviphoton (whose polarization is $\epsilon_{55}$) and the graviscalar ($\epsilon_{55}$) as well as to the four-dimensional graviton. The Feynman rules and the precise field conventions are given in appendix B. Note that there is no coupling to the ten-dimensional dilaton which couples only via a term proportional to the ten-dimensional mass. Hence, we have to consider the diagrams in fig.5 in addition to the contribution of the graviton, eq. (3.3).

The spin-1 exchange (fig.5a) gives

$$T_{graviphoton} = -\epsilon_{in} \cdot \epsilon_{out} \frac{16\kappa^2 M^2 Q_1 Q_2}{q_1^2 q_1^2 q_2^2} \left[ \frac{E^2}{2} q_2^2 - (q_1 \cdot p)(q_2 \cdot p) \right],$$  \hspace{1cm} (3.5)

where $Q_i = \sqrt{2}\kappa_i M$ are the charges. The spin-0 exchange (fig.5b) gives

$$T_{graviscalar} = -\epsilon_{in} \cdot \epsilon_{out} \frac{24\kappa^4 M^4}{q_2^2 q_1^2 q_2^2} (q_1 \cdot p)(q_2 \cdot p).$$  \hspace{1cm} (3.6)
Together they yield the amplitude
\[
T_6 = \epsilon_{in} \cdot \epsilon_{out} \frac{32k^4M^4}{q_1^2 q_2^2} (1 - \xi_1 \xi_2) \left[ \frac{E^2}{2} q_1^2 - (q_1 \cdot p)(q_2 \cdot p) \right],
\]
which agrees with (2.48).

### 3.3 The Deflection Angle, III: General Relativity

For reference, we report here the computation of the deflection angle in field theory for general relativity. For a different but equivalent computation, see [8]. The amplitude (3.3) can be inserted in (2.13) instead of the string result (2.43) or (2.48). A straightforward computation gives
\[
\Delta \vartheta = \frac{4G_N M_*}{b} + \frac{15\pi}{4} \left( \frac{G_N M_*}{b} \right)^2 + O(G_N^3),
\]
which agrees with the expansion to this order of the exact result [15].

The first order in $G_N$ term agrees with the string result by construction (recall the way (2.21) was defined). The $O(G_N^3)$ terms are different. As a matter of fact, both string results are smaller than the one in general relativity: for a star made out of neutral massive string excitations, the numerical factor in front of the second order term in (2.46) is $14\pi/4$ instead of $15\pi/4$; whereas for the charged string star, in the case where all constituents carry the same charge, this first non-linear correction vanishes (see section 2.4). As we shall see, this decrease of the deflection angle reflects a behavior of the gravitational field of string matter that is less singular at short distances than the one of the Schwarzschild solution.
4 The Exact Solutions

Up to this point we have dealt directly with the string amplitudes and we have analyzed our results in terms of the various fields existing in the string theory. The advantage of this approach is in the string taking automatically into account all the fields that must be included. On the other hand, the perturbative nature of the computation makes it unsuitable for a discussion of strongly non-linear effects, such as the existence or non-existence of horizons.

In this section we leave the string amplitude and move to the effective action of the low-energy field theory, which can be solved exactly.

4.1 Massive String Excitations

The relevant part of the theory is described by the following action (see appendix B for details):

\[ S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} R - \frac{1}{2} \int d^4x \sqrt{-g} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi, \]

(4.1)

where \( \phi \) is the four-dimensional dilaton field. The action (4.1) is equivalent to the Brans-Dicke action \[8\] for the particular value \(-1\) of their parameter \(\omega\).

The equations of motion are

\[ \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \partial_\nu \phi \right) = 0 \]

(4.2)

for the dilaton field, and Einstein’s equations

\[ R_\mu^\nu - \frac{1}{2} \delta_\mu^\nu R = \kappa^2 T_\mu^\nu \]

(4.3)

for the graviton, where

\[ T_\mu^\nu = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \delta_\mu^\nu g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \]

(4.4)

is covariantly conserved.

To find the spherically symmetric solution we proceed in a manner similar to the way the Schwarzschild solution is worked out in, for example, ref. \[17\]. Accordingly, we parametrize the metric tensor in spherical coordinates

\[ ds^2 = -e^{r(r)} dt^2 + e^{\lambda(r)} dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \]

(4.5)

\footnote{The mapping into the Brans-Dicke field variables is given by \( G_N \phi_{BD} = \exp(-2\kappa \phi/\sqrt{2}) \), and \( g^{BD}_{\mu\nu} = g_{\mu\nu}/G_N \phi_{BD} \).}
For this parametrization of the metric, the 11 and 00 component of (4.3) gives us the equations of motion

\[- \frac{1}{r^2} + e^{-\lambda} \left( \frac{\nu}{r} + \frac{1}{r^2} \right) = \frac{\kappa^2}{2} e^{-\lambda} (\phi')^2 \] (4.6)
\[- \frac{1}{r^2} - e^{-\lambda} \left( \frac{\lambda'}{r} - \frac{1}{r^2} \right) = -\frac{\kappa^2}{2} e^{-\lambda} (\phi')^2 \] (4.7)

while the other components yield no further information. From (4.2) we obtain the equation of motion for the dilaton field:

\[\left( e^{-\frac{\nu - \lambda}{2} r^2 \phi'} \right)' = r^2 \phi'' + \left( 2r + r^2 \frac{\nu'}{r} - \lambda' \right) \phi' = 0 \] (4.8)

Everywhere ' denotes differentiation with respect to the coordinate r.

It is now useful to introduce the two functions

\[V(r) \equiv \frac{\nu + \lambda}{2} \] and \[W(r) \equiv \frac{\nu - \lambda}{2} \] (4.9)

and combine (4.7), (4.6) and (4.8) to get

\[\phi'' + \left( \frac{2}{r} + W' \right) \phi' = 0 \] (4.10)
\[\left( re^W \right)' = e^V \] (4.11)
\[V' = \frac{\kappa^2 r}{2} (\phi')^2 . \] (4.12)

Eq. (4.12) can be differentiated to give, by means of (4.10),

\[V'' + \frac{3}{r} V' + 2W'V' = 0 . \] (4.13)

Eq. (4.13) can be integrated to find V' as a function of W:

\[V' = \frac{c_0}{r^3} e^{-2W} . \] (4.14)

Introducing the function

\[y \equiv r e^W , \] (4.15)

and using (4.11), we obtain

\[\frac{y''}{y'} = \frac{c_0}{r} \frac{1}{y^2} \] (4.16)
where $c_0$ is a constant to be determined. Eq. (4.16) is now an ordinary differential equation to be solved. This is possible by noting that (4.16) is obtained from the simpler one,

$$ry' = -\frac{c_0}{y} + y + \Omega_0,$$

by differentiation, $\Omega_0$ being another integration constant. Therefore

$$\left(\frac{r}{r_0}\right) = \left(\frac{y - a}{y_0 - a}\right)^\frac{a}{a+b} \left(\frac{y + b}{y_0 + b}\right)^\frac{b}{a+b},$$

where

$$a \equiv \frac{1}{2} \left(\sqrt{\Omega_0^2 + 4c_0} - \Omega_0\right)$$

$$b \equiv \frac{1}{2} \left(\sqrt{\Omega_0^2 + 4c_0} + \Omega_0\right).$$

Even though it seems that our solution depends on two new constants ($y_0$ and $r_0$) the boundary condition on $W(r)$—that is, the flatness of the metric at infinity which implies that $y \to r$ as $r \to \infty$—tells us that

$$(y_0 - a)^\frac{a}{a+b}(y_0 + b)^\frac{b}{a+b} = r_0.$$

Hence, we have

$$r = f(y) \equiv (y - a)^\frac{a}{a+b}(y + b)^\frac{b}{a+b},$$

which gives $y$ implicitly as a function of $r$.

Finally, from (4.14), (4.16) and (4.21) we have that

$$-g_{00} = \frac{y - a}{y + b}$$

$$g_{11} = \frac{(y - a)(y + b)}{y^2}.$$

We discuss now this solution.

First of all, it is easy to see that the Schwarzschild solution is recovered for $c_0 = 0$ (that is, $a = 0$ and $b = \Omega_0$). In this case, (4.21) reduces to

$$r = y + \Omega_0.$$
A distinctive feature of the Schwarzschild solution is that negative values of $y$ are possible. The horizon is where this change of sign of $y$ takes place. On the contrary, the solution in our example of the string star has no horizon. As it can be understood by studying graphically the solution of eq. (4.21) (fig.6), as $r$ goes to zero, $y$ never crosses over to negative values and reaches the limiting value of $a$ at $r = 0$. Accordingly, $g_{00} < 0$ and $g_{11} > 0$, as it is clear from (4.22), and therefore no component of the metric ever changes sign.

In the opposite limit,

$$y \sim r - \Omega_0 \quad \text{for} \quad r \to \infty. \quad (4.24)$$

This behavior at large $r$ can be used to determine the constant $\Omega_0$ by requiring that the metric goes into the one for Newton theory:

$$- g_{00} \sim 1 - \frac{\Omega_0}{r} \equiv 1 - \frac{2G_N M_*}{r}, \quad (4.25)$$

from which

$$\Omega_0 = \frac{\kappa^2 M_*}{4\pi} = 2G_N M_* . \quad (4.26)$$

By inserting the solution (4.22) for the metric into the equation of motion for the dilaton field (4.12), we find that

$$\phi = \sqrt{\frac{2c_0}{\kappa^2}} \int_{\infty}^{r} \frac{d\varrho}{gy(\varrho)} = \sqrt{\frac{2c_0}{\kappa^2}} \log \left( \frac{y - a}{y + b} \right)^{\frac{1}{1+\kappa}} . \quad (4.27)$$
The solution (4.27) can be used to determine the other unknown constant, \( c_0 \). In fact, at large \( r \)
\[
\phi \sim -\sqrt{\frac{2c_0}{\kappa^2}} \frac{1}{r},
\]
which should correspond to the dilaton field of a collection of \( N \) massive strings at rest (see Feynman rules in appendix B)
\[
-\frac{\kappa M_*}{4\sqrt{2\pi}} \frac{1}{r},
\]
with the result that
\[
c_0 = \left( \frac{\kappa^2 M_*}{8\pi} \right)^2 = \left( G_N M_* \right)^2.
\]
Therefore, \( a = G_N M_* \left( \sqrt{2} - 1 \right) \) and \( b = G_N M_* \left( \sqrt{2} + 1 \right) \).

A closed form of the solution\(^2\) can be found by a change of coordinates in which \( r \) is defined by (4.21). In these coordinates
\[
ds^2 = - \left( \frac{y-a}{y+b} \right) \frac{1}{\sqrt{y-a}} \frac{1}{\sqrt{y+b}} \, dt^2 + \frac{f^2(y)}{(y-a)(y+b)} \, dy^2 + f^2(y) \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right).
\]
The metric (4.31) belongs to the family of solutions related to the Brans-Dicke theory \[^8\]. In our case, the parameters are fixed by matching the superstring perturbative theory.

The value \( r = 0 \) (that is, \( y = a \)) is the position of a singularity of the curvature. In fact,
\[
R = \frac{G^2 N M_*^2}{r^2(y-a)(y+b)}.
\]
Finally, as a consistency check we compare the exact solution to our perturbative computation of section 3 by expanding the solution (4.22) at large distances
\[
-g_{00} = 1 - 2 \frac{G_N M_*}{r} + O(G_N^2) \quad (4.33)
\]
\[
g_{11} = 1 + 2 \frac{G_N M_*}{r} + 3 \left( \frac{G_N M_*}{r} \right)^2 + O(G_N^3) \quad (4.34)
\]
\[^2\] The history of this solution is, as far as we know it, the following. It was written down for the first time in \[^8\] and attributed to a suggestion of C. Misner. In this reference the coordinates are the so-called isotropic ones, which are related to the ours by \( \rho = y/2 + \sqrt{(y-a)(y+b)/2} + (b-a)/4 \).
It was given in \[^18\] (without reference to the previous work) in coordinates equivalent to ours. It has been re-discovered independently in the present context by one of us (R.I.).
and computing the deflection angle for a massless particle by means of the formula [19]

\[ \Delta \vartheta = 2 \left| \vartheta(r) - \vartheta_\infty \right| - \pi, \]  

(4.35)

where

\[ \vartheta(r) - \vartheta_\infty = \int_r^\infty \frac{1}{g_{11}^{1/2}(r)} \left[ \left( \frac{r}{r_0} \right)^2 \left( \frac{g_{00}(r_0)}{g_{00}(r)} \right) - 1 \right]^{-1/2} \frac{dr}{r}, \]  

(4.36)

in which \( r_0 \) is the point of closest approach.

We thus find

\[ \Delta \vartheta = \frac{4 G_N M_*}{r_0} + \left(\frac{7\pi}{2} - 4 \right) \left( \frac{G_N M_*}{r_0} \right)^2 + O(G_N^3) \]

(4.37)

in agreement with (2.46).

4.2 Anti-Gravity

Next, we consider a string star made of states whose charge and mass are given by a non-zero momentum in the compact fifth direction. The exact solution of this theory gives rise to anti-gravity and it has been recently discussed in [9]. It also corresponds to the extremal value of \( Q_*/M_* = \pm \sqrt{2}\kappa \) (see section 2.4) discussed in [4] for a Maxwell field arising in the compactification. We report here only the main results for ease of reference. The relevant part of the effective action involves only the graviton, one scalar (the gravi-scalar \( \phi_{55} \equiv \delta \)) and one vector (the gravi-vector \( A_{\mu}^{5} \equiv A_\mu \)) field and is given in appendix B. Here we re-write the action introducing a scalar field \( \sigma = -\sqrt{3}\log \delta/4 \) in such a way that it can be compared directly to the similar action of ref. [4]. Therefore, we have

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} \mathcal{R} - \frac{1}{4} e^{-2\sqrt{3}\sigma} F_{\mu\nu} F^{\mu\nu} - \frac{1}{\kappa^2} (\partial_\mu \sigma)^2 \right\}, \]  

(4.38)

(4.38) gives rise to the following equations of motion:

\[ \mathcal{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = \kappa^2 T_{\mu\nu} \]  

(4.39)
for the graviton, with

\[ T_{\mu\nu} = e^{-2\sqrt{3}\sigma} \left( F_{\mu\rho}F_{\nu}^{\rho} - \frac{1}{4}g_{\mu\nu}F^{\rho\sigma}F_{\rho\sigma} \right) + \frac{2}{\kappa^2} \left( \partial_{\mu}\sigma \partial_{\nu}\sigma - \frac{1}{2}g_{\mu\nu}(\partial_{\rho}\sigma)^2 \right) \]  

(4.40)

and

\[ \nabla_{\rho} \left( e^{-2\sqrt{3}\sigma} F^{\rho\mu} \right) = 0 \]  

(4.41)

for the gauge field. The scalar field obeys

\[ \nabla_{\rho} \nabla^{\rho} \sigma + \frac{\sqrt{3}}{4} \kappa^2 e^{-2\sqrt{3}\sigma} F_{\mu\nu}F^{\mu\nu} = 0, \]  

(4.42)

where \( \nabla \) is the covariant derivative.

The simplest way of finding a solution to these equations consists in going to the ten dimensional effective field theory where it is just a shock wave \[9\]. In four space-time dimensions it reads as

\[ ds^2 = -\frac{dt^2}{\left( 1 + 4G_N M_* r \right)^{1/2}} + \left( 1 + 4G_N M_* r \right)^{1/2} \left[ dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right] \]

\[ A_{\mu} = \frac{Q_*}{4\pi r} \left( 1 + 4G_N M_* r \right)^{-1} \delta_{\mu}^0 \]

\[ e^{-4\sigma/\sqrt{3}} = 1 + \frac{4G_N M_*}{r}, \]  

(4.43)

where the charge is \( Q_* = \pm \sqrt{2\kappa} M_* \).

The deflection angle for a massless and neutral test particle can be computed according to (4.36) and, at large distances, we obtain

\[ \Delta \vartheta = \frac{4G_N M_*}{b} + O(G_N^3), \]  

(4.44)

in agreement with our perturbative computation in the case of a charged string star.

(4.43) coincides with the extremal solution of \[4\] in isotropic coordinates for the case in which their coupling \( a \) of the scalar field to the vector field is taken to be \( \sqrt{3} \).
5 Quantum Mechanics of the Radial Motion of a Particle near the String Star

In order to discuss the quantum mechanical problem of the motion of a particle in the field of the string star, we consider the equation for the particle’s wave function \( \psi \) in the space-time \( (4.34) \). For a massless particle (a small mass would not alter the discussion), the wave equation follows from the action

\[
S = \frac{1}{2} \int dt d^3x \sqrt{-g} \left( g^{00} |\partial_t \psi|^2 - g^{ij} \partial_i \psi^* \partial_j \psi \right).
\] (5.1)

The stationary and spherically symmetric solution

\[
\psi = e^{iEt} \tilde{\psi}(r)
\] (5.2)

is general enough because a possible centrifugal barrier could only help in keeping the particle from falling into the origin.

The wave equation is obtained by varying \( \tilde{\psi} \) in \( S \). No boundary terms arise if

\[
\sqrt{-g} g^{11} \tilde{\psi}^* \partial_r \tilde{\psi} \to 0
\] (5.3)

at \( r = \infty \) and at \( r = 0 \). Otherwise, a boundary term would appear as a source (or a sink), and we would not have a source-free (or sink-free) wave equation.

We thus have the equation

\[
\frac{1}{\sqrt{-g}} \partial_r \left( \sqrt{-g} g^{11} \partial_r \tilde{\psi} \right) + |g^{00}| E^2 \tilde{\psi} = 0,
\] (5.4)

with the the boundary condition \( (5.3) \) at \( r = 0 \). We assume that at \( \infty \) the condition \( (5.3) \) is always satisfied either because of the exponential decay of \( \tilde{\psi} \) for a bound state, or because we can form suitable wave-packets in the continuum spectrum.

For the metric \( (4.34) \) we have, in the limit \( r \to 0 \)

\[
-g^{00} \sim \left( \frac{r}{a+b} \right)^{-1+b/a}
\] (5.5)

\[
g^{11} \sim \left( \frac{a+b}{a} \right)^2 \left( \frac{r}{a+b} \right)^{1+b/a}
\] (5.6)

\[
\sqrt{-g} \sim r^2 \sin \theta \frac{a+b}{a} \left( \frac{r}{a+b} \right)^{b/a}
\] (5.7)

The wave equation \( (5.4) \) is of the Bessel type

\[
\partial_r^2 \tilde{\psi} + \frac{1}{r} \partial_r \tilde{\psi} + \frac{r^2}{a^2} E^2 \tilde{\psi} = 0,
\] (5.8)
which admits a regular solution:

\[ \tilde{\psi}(r) \sim J_0 \left( \frac{E}{2a} r^2 \right), \tag{5.9} \]

where \( J_0 \) is the zero-order Bessel function which satisfies the boundary condition (the other solution \( Y_0 \left( \frac{E}{2a} r^2 \right) \) would not, and it must be discarded).

The same result holds also in the case of charged scalar states, where the regular solution is proportional to

\[ \frac{1}{\sqrt{r}} J_1 \left( 4E \sqrt{G_N M^* r} \right) \tag{5.10} \]

instead of (5.9).

This way, the situation is essentially similar to the problem of the hydrogen atom in quantum mechanics, where the electron does not fall into the center because the indeterminacy principle prevents it.

Therefore, the quantum mechanical behavior of a particle in the space-time metric of the string star is very mild, contrary to the case of the Schwarzschild solution where the particle falls into the horizon. In fact, in the case of the Schwarzschild metric, a similar analysis leads to the conclusion that

\[ \tilde{\psi} \sim e^{\pm iE r_g \log(r - r_g)} \tag{5.11} \]

where \( r_g = 2G_N M \) is the Schwarzschild radius for the star. Clearly, the boundary condition at \( r = r_g \) can never be satisfied. In terms of the coordinate \( \xi = r_g \log(r - r_g) \), the solution for a particle starting out at \( r = \infty \) represents a plane wave traveling towards and then through the horizon.

6 Speculations

In this final section we would like to go back and discuss some implications of the solution found in section 4.1 for the gravitational field around a string star made from neutral string excitations. As we have seen, the presence of the dilaton field changes drastically the nature of the solution. The strength of the coupling of string matter to the dilaton field \( \phi \) is fixed by string perturbation theory and is given by the parameter \( c_0 \) of eq. (4.30) and thus the parameter \( a \) of eq. (4.17). For \( c_0 = a = 0 \) we recover the standard Schwarzschild solution.
It is interesting that even considering \( c_0 \) as a free parameter and taking it to be very small, the solution remains of the same kind, namely free of horizons and deviating from the Schwarzschild one near the would-be horizon. Therefore, in this sense, the Schwarzschild metric appears to be unstable with respect to the presence of the dilaton field.

In order to understand better this point, it is useful to consider the equation for a static scalar field \( \phi \) in the background of the Schwarzschild metric. For \( x \equiv r - r_g \to 0 \), that is near the horizon at \( r_g \), the equation becomes

\[
\partial^2 \phi + \frac{1}{x} \partial \phi - \frac{m^2 r_g}{x} \phi = 0
\]  

(6.1)

whose solution is \( \phi \approx \log x \) for \( x \to 0 \); this is the only solution if we require that \( \phi \) does not grow exponentially for \( x \to \infty \).

Therefore the energy-momentum tensor of \( \phi \) diverges for \( x \to 0 \) producing a singularity in \( R \). Hence, by computing the back-reaction of the field \( \phi \) on the metric through the Einstein equations, one expects to get a real singularity without horizon. This is indeed what occurs in our solution of section 4.1.

As we have already mentioned, similar solutions of the low-energy action have been recently discussed in the literature \[4\], where there is a dilaton field and still the horizon is present. In these cases however the electric or magnetic charge is also different from zero and the coupled equations for the dilaton and the Maxwell field prevent singularities at the horizon. This is of course not what we find for the case of the neutral string excitation. Similarly, the anti-gravity case we considered corresponds to an extremal solution where there is a Maxwell field but still there is no horizon.

It is also interesting to notice that a mass term for the dilaton field does not alter qualitatively the above picture; we see from eq. (6.1) that for \( x \to 0 \) the behavior is the same as for \( m = 0 \), consistent with the fact that near the horizon any field independent of the external time coordinate \( t \) is propagating almost at the speed of light.

Accordingly, even though we discussed a scenario where compactification has occurred without the dilaton field acquiring a mass, one can speculate that the solution of section 4.1 might be of relevance even in the more realistic case in which the dilaton field becomes massive. For \( r - r_g \gg 1/m_{\text{dilaton}} \) the solution will approach the Schwarzschild one, but for \( r - r_g \ll 1/m_{\text{dilaton}} \) we expect the solution to be indistinguishable from the one discussed in section 4.1. At distances of the order of
the Planck length however interactions of higher orders in $\alpha'$ become important and a classical solution should merge into the full quantum gravity picture, a task beyond the scope of the present work.

A Four-Point Amplitude

In this appendix we show how the six-point string amplitude we have derived within the framework of the covariant loop calculus, can also be obtained by starting out from the four-point graviton-graviton amplitude. This is given in an explicit way in ref. [21] as

$$T_{4}^{\text{tree}} = -\frac{(\alpha')^{3}}{4} \kappa^{2} \frac{\Gamma(-\alpha's/4)\Gamma(-\alpha't/4)\Gamma(-\alpha'u/4)}{\Gamma(1 + \alpha's/4)\Gamma(1 + \alpha't/4)\Gamma(1 + \alpha'u/4)}$$

$$\times \epsilon_{i_{1}j_{1}}^{(1)} \epsilon_{i_{2}j_{2}}^{(2)} \epsilon_{i_{3}j_{3}}^{(3)} \epsilon_{i_{4}j_{4}}^{(4)} K_{i_{1}j_{2}j_{3}j_{4}}$$

$$= \frac{16\kappa^{2}}{stu} \epsilon_{i_{1}j_{1}}^{(1)} \epsilon_{i_{2}j_{2}}^{(2)} \epsilon_{i_{3}j_{3}}^{(3)} \epsilon_{i_{4}j_{4}}^{(4)} \kappa_{i_{1}j_{2}i_{3}j_{4}} K_{j_{1}j_{2}j_{3}j_{4}} + O(\alpha') \quad (A.1)$$

where $s, t, u$ are the usual Mandelstam variables. If we take legs two and three to represent the incoming and outgoing gravitons and legs one and four to represent the virtual gravitons exchanged with the string star, as shown in fig.(7), we have

$$s = -(k_{1} + k_{2})^{2} \simeq 2p \cdot q_{1}$$

$$t = -(k_{2} + k_{3})^{2} \simeq -q_{1}^{2}$$

$$u = -(k_{1} + k_{3})^{2} \simeq -s \quad (A.2)$$

In order to close the two off-shell legs ($n = 1, 4$) and form an on-shell six-point amplitude we make the substitution

$$\epsilon_{i_{n}j_{n}}^{(n)} \rightarrow 2\kappa \frac{\Delta_{\mu}^{(n)} \Delta_{\nu}^{(n)}}{q_{n}^{2}} \quad \text{for } n = 1, 4 \quad (A.3)$$

where

$$\Delta^{(n)}_{\mu} = (M, q_{n}/2) \simeq (M, 0) \quad (A.4)$$

is the average momentum of the string matter inside the star. The normalization of $(A.3)$ is dictated by factorization.

$K$ is given by a rather lengthy expression that contains the kinematical part of the graviton-graviton amplitude, it can be found in full in eq. (2.27) of ref. [21]. Of this, we only need the piece proportional to

$$\delta^{i_{1}i_{3}} \delta^{j_{1}j_{3}} \quad (A.5)$$
which gives rise to the part of the amplitude proportional to $\epsilon_{\text{in}} \cdot \epsilon_{\text{out}}$.

This leaves us with

$$K^{i_1 j_1 i_4 j_4} K^{j_2 j_3} \approx \delta^{i_1 j_1} \delta^{i_2 j_2} \left\{ -\frac{1}{4} u s \delta^{j_1 j_4} + \frac{s}{2} k_3^{j_1} k_2^{j_4} + \frac{u}{2} k_2^{j_1} k_3^{j_4} \right\} \times \left\{ -\frac{1}{4} u s \delta^{j_2 j_3} + \frac{s}{2} k_3^{j_2} k_2^{j_3} + \frac{u}{2} k_2^{j_2} k_3^{j_3} \right\}. \quad (A.6)$$

Inserting this into (A.1) and replacing $\epsilon^{(1)}, \epsilon^{(4)}$ according to (A.3) we find

$$T_6 = \frac{4\kappa^4}{\kappa^4} s^2 t \frac{\epsilon_{\text{in}} \cdot \epsilon_{\text{out}}}{q_1^2 q_2^2} \left( u s \Delta^{(1)} \cdot \Delta^{(2)} + 2 t \Delta^{(1)} \cdot p_{\text{in}} \Delta^{(2)} \cdot p_{\text{in}} \right)^2. \quad (A.7)$$

The square of the first term in the bracket yields

$$- \frac{4\kappa^4 s^2}{t} \frac{\epsilon_{\text{in}} \cdot \epsilon_{\text{out}}}{q_1^2 q_2^2} (\Delta^{(1)} \cdot \Delta^{(2)})^2 = \frac{16\kappa^4 M^4 \epsilon_{\text{in}} \cdot \epsilon_{\text{out}}}{q_1^2 q_2^2} (p \cdot q_1)^2, \quad (A.8)$$

whereas the double product gives the contact term

$$- \frac{16\kappa^4 \epsilon_{\text{in}} \cdot \epsilon_{\text{out}}}{q_1^2 q_2^2} (\Delta^{(1)} \cdot \Delta^{(2)} p_{\text{in}} \cdot \Delta^{(1)} p_{\text{in}} \cdot \Delta^{(2)}) = \epsilon_{\text{in}} \cdot \epsilon_{\text{out}} \frac{16\kappa^4 M^4 E^2}{q_1^2 q_2^2}, \quad (A.9)$$

in agreement with our previous results (2.43).

B Feynman rules

In this appendix we present our definitions for the various field variables which couple to the string matter in $D = 10$ and $d = 4$ dimensions.
In $D = 10$, only the graviton and the dilaton fields are generated (see, eq. (3.1)). The relevant part of the superstring effective action is therefore 

$$\hat{S} = \frac{1}{2\kappa^2} \int d^D \hat{x} \sqrt{-\hat{g}} \ e^{-2\hat{\Phi}} \left( \hat{R} + 4 \partial_{\mu} \hat{\Phi} \partial^{\mu} \hat{\Phi} \right) - \frac{1}{2} \int d^D \hat{x} \sqrt{-\hat{g}} \ e^{-2\hat{\Phi}} \left( \partial_{\mu} \hat{B} \partial^{\mu} \hat{B} + \hat{M}^2 \hat{B}^2 \right), \quad (B.1)$$

where $\hat{\Phi}$ is the 10-dimensional dilaton field and we adopt the convention that all 10-dimensional objects carry a hat. In eq. (B.1) we represented the string matter by a field $\hat{B}$. That this reproduces the correct coupling of string matter to graviton and dilaton is seen by performing the rescalings

$$\hat{g}_{\mu\nu} \rightarrow e^{4\hat{\Phi}/(D-2)} \hat{g}_{\mu\nu} \quad \hat{\Phi} \rightarrow \sqrt{D-2} \frac{\hat{k}}{2} \hat{\Phi} \quad (B.2)$$

in terms of which the effective action (B.1) assumes the canonical form

$$\hat{S} = \int d^D \hat{x} \sqrt{-\hat{g}} \left\{ \frac{1}{2\kappa^2} \hat{R} - \frac{1}{2} \partial_{\mu} \hat{\Phi} \partial^{\mu} \hat{\Phi} - \frac{1}{2} \partial_{\mu} \hat{B} \partial^{\mu} \hat{B} - \frac{1}{2} \hat{M}^2 \ e^{2\hat{\Phi}/\sqrt{D-2}} \hat{B}^2 \right\} \quad (B.3)$$

As we perform a toroidal compactification down to $d = 4$, the internal components of the metric give rise to graviphoton fields $A_\mu^\alpha$ and graviscalar fields $\phi_{\alpha\beta}$, where $\mu, \nu = 1, \ldots, d$ denote $d$-dimensional space-time indices and $\alpha, \beta = 1, \ldots, D - d$ denote the internal ones. We parametrize the 10-dimensional metric as follows

$$\hat{g}_{\mu\nu} = \begin{pmatrix} \delta^\gamma g_{\mu\nu} + 2\kappa^2 A_\mu^\alpha A_\nu^\beta \phi_{\alpha\beta} & -\sqrt{2}\kappa A_\mu^\beta \phi_{\alpha\beta} \\ -\sqrt{2}\kappa A_\nu^\alpha \phi_{\alpha\beta} & \phi_{\alpha\beta} \end{pmatrix} \quad (B.4)$$

where $\delta \equiv \det \phi_{\alpha\beta}$ and $\gamma \equiv -1/(d - 2)$. In terms of these fields the effective action (B.3) becomes

$$S = \int d^d x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{1}{2} \partial_{\mu} \Phi \partial^{\mu} \Phi - \frac{1}{4} \delta^{-\gamma} F^{\mu\nu\alpha} F_{\mu\nu}^\beta \phi_{\alpha\beta} \\ - \frac{1}{8\kappa^2} \phi^{\alpha_1\alpha_2} \phi^{\beta_1\beta_2} \partial_{\mu} \phi_{\alpha_1\beta_1} \partial^{\mu} \phi_{\alpha_2\beta_2} - \frac{1}{8\kappa^2(d - 2)} \partial_{\mu} \log \delta \partial^{\mu} \log \delta \\ - \frac{1}{2} g^{\mu\nu} \left( \partial_{\mu} + i\sqrt{2}\kappa p_\alpha A_\mu^\alpha \right) B \left( \partial_{\nu} - i\sqrt{2}\kappa p_\beta A_\nu^\beta \right) B \\ - \frac{1}{2} \phi_{\alpha\beta} p_\alpha p_\beta B^2 \delta^\gamma - \frac{1}{2} \hat{M}^2 \delta^\gamma e^{2\kappa \Phi/\sqrt{D-2}} \hat{B}^2 \right\} \quad (B.5)$$
where we have introduced compact momenta $p_\alpha$ for the field $B$. We have also introduced the $d$-dimensional gravitational coupling, $\kappa$, and $d$-dimensional canonical fields, $B$ and $\Phi$, by rescaling the $D$-dimensional ones by the appropriate power of the coordinate volume of the compact space.

We can now distinguish between two cases:

1. If the string matter is massless in $D = 10$ (i.e. $\hat{M} = 0$) the 10-dimensional dilaton decouples.

Assuming for simplicity that only one of the compact momenta is non-zero, $p_\beta \equiv M \delta_\beta^\alpha$, we see from (B.3) that only one graviphoton field, $A_\mu^\alpha \equiv A_\mu$, and one graviscalar field, $\phi_{\alpha\alpha} \equiv \delta$, is excited.

The effective action is reduced to (taking $d = 4$):

$$
S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2}R - \frac{1}{4}\delta^{3/2}F_{\mu\nu}F^{\mu\nu} - \frac{3}{16\kappa^2} \partial_\rho \log \delta \partial^\rho \log \delta - \frac{1}{2} \partial_\rho B \partial^\rho B - \frac{1}{2} m^2 \delta^{-3/2} B^2 \right\}
$$

(B.6)

2. If the string matter is massive in $D = 10$ ($\hat{M} \neq 0$) but does not carry any compact momentum, the only relevant fields besides the graviton are the 10-dimensional dilaton $\Phi$ and the determinant of the internal metric, $\delta$. The “transverse” graviscalar fields $\phi^{T}_{\alpha\beta} \equiv \phi_{\alpha\beta}/\delta^{1/(D-d)}$ decouple.

If we define the 4-dimensional dilaton field by

$$
\phi \equiv \sqrt{\frac{d-2}{D-2}} \Phi - \frac{1}{2\kappa} \sqrt{\frac{1}{d-2}} \log \delta,
$$

(B.7)

the orthogonal combination decouples and the effective action is reduced to

$$
S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2}R - \frac{1}{2} \partial_\rho \phi \partial^\rho \phi - \frac{1}{2} \partial_\rho B \partial^\rho B - \frac{1}{2} M^2 e^{2\kappa \phi/\sqrt{d-2}} B^2 \right\}
$$

(B.8)

Notice that the string matter field $B$ does not appear in the Einstein equations (4.3) or (4.39). This is because this field represents the distribution of string matter at the origin, whereas we solve the equations of motion for the massless fields outside this matter distribution. Like in the case of the usual Schwarzschild solution, reference to the matter source is only made in determining the asymptotic behavior at large distance for the massless fields. Accordingly, we only need the linear coupling of the graviton and the dilaton to the string matter field $B$. 

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If we define the graviton and graviscalar perturbations by (3.2) and
$$\phi_{\alpha\beta} \equiv \delta_{\alpha\beta} + 2\kappa h_{\alpha\beta} \tag{B.9}$$
one can derive from (B.3) the set of relevant Feynman rules for the massless fields.

For the **graviton**:

$$\frac{1}{2q^2} (\eta^{\mu_1\mu_2} \eta^{\nu_1\nu_2} + \eta^{\mu_1\nu_2} \eta^{\nu_1\mu_2} - \eta^{\mu_1\nu_1} \eta^{\mu_2\nu_2}) \tag{B.10}$$

$$\frac{\kappa}{2} \text{sym} P_6 \left\{ -4q_2 \cdot q_3 \eta_{\mu_2\nu_3} \eta_{\mu_1\nu_2} \eta_{\nu_1\nu_3} \tag{B.11} \\
+ \frac{4}{d-2} q_2 \cdot q_3 \eta_{\mu_2\nu_2} \eta_{\nu_3\mu_1} \eta_{\nu_3\nu_1} + 2q_1^2 \eta_{\nu_1\mu_1} \eta_{\mu_3\mu_2} \eta_{\nu_2\nu_3} \\
- \frac{2}{d-2} q_1^2 \eta_{\mu_2\nu_2} \eta_{\nu_3\mu_3} + 4q_1^2 \eta_{\mu_3\mu_2} \eta_{\nu_3\nu_1} \right\}$$

where “sym $P_6$” implies: 1) symmetrization in $\mu_1 \leftrightarrow \nu_1$, $\mu_2 \leftrightarrow \nu_2$ and $\mu_3 \leftrightarrow \nu_3$ with unit weight. 2) Adding all permutations of legs 1, 2 and 3.

For the **graviphoton**:

$$\frac{\eta_{\mu\nu} \delta^{\alpha\beta}}{q^2} \tag{B.12}$$

$$-\kappa \left\{ p_1 \cdot p_2 (\eta^{\nu\rho} \eta_{\rho\sigma} - \delta^{\nu}_{\sigma} \delta^{\rho}_{\mu} - \delta^{\rho}_{\sigma} \delta^{\mu}_{\nu} - (p_1^1 p_2^2 + p_1^2 p_2^1) \eta^{\mu\nu} \\
- p_1^1 p_2^2 \eta_{\rho\sigma} + p_1^2 (p_2^1 \delta^{\mu}_{\rho} + p_2^2 \delta^{\mu}_{\sigma}) \\
+ p_2^2 (p_1^1 \delta^{\nu}_{\rho} + p_1^2 \delta^{\nu}_{\sigma}) \right\} \tag{B.13}$$

For the **graviscalar**:

$$\frac{1}{2q^2} (\delta_{\alpha_1\alpha_2} \delta_{\beta_1\beta_2} + \delta_{\alpha_1\beta_2} \delta_{\beta_1\alpha_2} \tag{B.14} - \frac{2}{D-2} \delta_{\alpha_1\beta_1} \delta_{\alpha_2\beta_2})$$
\[
\frac{\kappa}{2} (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) \cdot \left( \delta^{\alpha_1 \alpha_2} \delta^{\beta_1 \beta_2} + \delta^{\alpha_1 \beta_2} \delta^{\beta_1 \alpha_2} \right) 
+ \frac{2}{d-2} \delta^{\alpha_1 \beta_1} \delta^{\alpha_2 \beta_2}
\]
(B.15)

For the \textit{d-dimensional dilaton}:

\[
\frac{1}{q^2}
\]
(B.16)

The coupling to string matter can also be read off from (B.5). The graviton has a universal coupling to energy-momentum

\[
\kappa \left\{ p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - \frac{2}{d-2} M^2 \eta_{\mu \nu} \right\}
\]
(B.17)

where \( M \) is the mass of the string state in \( d \) dimensions.

The graviscalars and graviphotons couple to a string state with compact momentum \( p_\alpha \), mass \( M^2 = \delta^{\alpha \beta} p_\alpha p_\beta \) and charge \( q_\alpha = \sqrt{2\kappa} p_\alpha \) in the following way:

\[
q_\alpha (p_2^\mu - p_1^\mu)
\]
(B.18)

\[
\frac{1}{\kappa} q_\alpha q_\beta + \frac{2\kappa}{d-2} m^2 \delta_\alpha \delta_\beta
\]
(B.19)
The coupling of the $d$-dimensional dilaton to a string state massive already in $D$ dimensions is

$$-\frac{2\kappa}{\sqrt{d-2}}M^2$$  \hspace{1cm} (B.20)

The coupling of the graviton to the dilaton is given by

$$\kappa \left\{ p_\mu^1 p_\nu^2 + p_\nu^1 p_\mu^2 \right\}$$  \hspace{1cm} (B.21)

C \hspace{1cm} Useful Integrals

Integration of Koba-Nielsen variables:

$$\int d^2 A |A|^{-2+\alpha' p_1 \cdot p_2} = \frac{2}{\alpha' p_1 \cdot p_2} \int d^2 \vec{A} \delta_A \left( \vec{A} \alpha' p_1 \cdot p_2 / A^{-1+\alpha' p_1 \cdot p_2} / 2 \right)$$

$$\rightarrow \frac{2\pi}{\alpha' p_1 \cdot p_2} \text{ for } \alpha' \rightarrow 0.$$  \hspace{1cm} (C.1)

Momenta integrals

$$\frac{1}{q^2} \int \frac{d^3 q_1}{(2\pi)^3} \frac{q_1 \cdot q_2}{q_1^2 q_2^2} = \frac{1}{16\sqrt{q^2}}$$  \hspace{1cm} (C.2)

$$\frac{1}{q^2} \int \frac{d^3 q_1}{(2\pi)^3} \frac{q_1^2}{q_1^2 q_2^2} = \frac{1}{8\sqrt{q^2}}$$  \hspace{1cm} (C.3)

$$\frac{1}{q^2} \int \frac{d^3 q_1}{(2\pi)^3} \left( q_1^2 + q_2^2 \right) = 0$$  \hspace{1cm} (C.4)

$$\frac{1}{q^2} \int \frac{d^3 q_1}{(2\pi)^3} \frac{p_i \cdot q_1 p_j \cdot q_2}{q_1^2 q_2^2} = \frac{1}{64} \left( \frac{p_i \cdot q p_j \cdot q}{(q^2)^{3/2}} + \frac{p_i \cdot p_j}{\sqrt{q^2}} \right).$$  \hspace{1cm} (C.5)

\(^{3}\)These integrals, with the exception of (C.3), are ultraviolet divergent and are most conveniently done by dimensional regularization. An ultraviolet cutoff is physically provided by the star wave function, see section 2 and Appendix D.
D-dimensional Fourier Transform:
\[
\int \frac{d^Dq}{(2\pi)^D} F(q) e^{-iq \cdot r} = \frac{1}{(2\pi)^{D/2} r^{(D-2)/2}} \int_0^\infty q^{D/2} F(q) J_{(D-2)/2}(q r) dq, \tag{C.6}
\]
where the last integral can be found by using \[23\]
\[
\int_0^\infty x^\mu J_\nu(ax) dx = 2^\mu a^{-\mu-1} \frac{\Gamma \left( \frac{1}{2} + \frac{\mu+\nu}{2} \right)}{\Gamma \left( \frac{1}{2} + \frac{\nu-\mu}{2} \right)}. \tag{C.7}
\]
For instance, we have
\[
\int \frac{d^2q_\perp}{(2\pi)^2} e^{-i\mathbf{q}_\perp \cdot \mathbf{b}} \frac{1}{q_\perp^2} = -\frac{1}{2\pi} \log b \tag{C.8}
\]
\[
\int \frac{d^2q_\perp}{(2\pi)^2} e^{-i\mathbf{q}_\perp \cdot \mathbf{b}} \frac{1}{|\mathbf{q}_\perp|} = \frac{1}{2\pi b}. \tag{C.9}
\]

\section{D Kinematics}

The kinematics for the amplitude \[2.17\] in which only one graviton is exchanged (fig.1b) is simple. The massless particle comes in with a large energy \(E\). In the Regge regime we are interested in, the square of the exchanged momentum \(q^2\) is fixed and such that also \(q^2/M^2 \ll 1\).

In the laboratory frame in which the \(N\) massive particles are initially at rest, the external momenta (fig.2) are parametrized in terms of

\[
p_{in} = (E, \mathbf{p}) \quad ; \quad p_{out} = (-E', \mathbf{q} - \mathbf{p})
\]
\[
p_1 = (M, 0)
\]
\[
p'_1 = (-\bar{E}, -\mathbf{q}), \tag{D.10}
\]
where \(E = |\mathbf{p}|\),
\[
E' = \sqrt{(\mathbf{q} - \mathbf{p})^2} \simeq E - \frac{\mathbf{p} \cdot \mathbf{q}}{E} \equiv E - q_L \tag{D.11}
\]
and
\[
\bar{E} = \sqrt{M^2 + q^2} \simeq M + \frac{q^2}{2M}. \tag{D.12}
\]

From the conservation of the energy,
\[
q_0 = E - E' = \bar{E} - M, \tag{D.13}
\]

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we obtain that
\[ q_0 = q_L = \frac{q^2}{2M}. \tag{D.14} \]

Eq. (D.14) can be solved to give
\[ q_0 = q_L = \frac{q_1^2}{2M} \left( 1 + O \left( \frac{q_1^2}{M^2} \right) \right), \tag{D.15} \]

where \( q_L \) is the component of \( q \) in the direction of the incoming graviton and \( q_\perp \) is the transverse part. Hence, we have zero energy transfer in the limit \( |q_\perp|/M \to 0 \).

From (D.14) also follows that
\[ q_1^2 = q_\perp^2, \tag{D.16} \]

so that the square of the transferred momentum comes from the transverse part only.

The amplitude (2.32) in which two massive string states partake in the interaction gives a slightly more complicated kinematics (see fig.1.b and fig.2). This time the external momenta are given by (2.33).

Energy conservation yields
\[ q_0 = E - E' = q_L, \tag{D.17} \]

and
\[ q_0 = (\tilde{E}_1 - M) + (\tilde{E}_2 - M) \simeq \frac{q_1^2 + q_2^2}{2M} + \cdots. \tag{D.18} \]

Eq. (D.18) and (D.17) give that
\[ \frac{q_L^2}{M} \simeq \frac{q_1^2 + q_2^2}{2M^2} \ll 1. \tag{D.19} \]

Eq. (D.19) together with the identity
\[ q_1^1 + q_1^2 = q_\perp \tag{D.20} \]

give us two kinematical constraints in the region over which the final momenta of the massive scalars are integrated. In particular we are interested in quasi-elastic scattering where only a relatively small excitation energy is imparted to the scatterer, and in which therefore \( q_0 \leq \Lambda \). This cutoff is taken automatically into account by dimensional regularization which put the infinite part equal to zero.
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