A Combined Finite-Element-Finite-Volume Method for Thermal Arc Numerical Simulation

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Abstract

Most of the numerical methods used for thermal arc simulation, are often based either on finite volume or finite element approaches but very seldom on both of them. In this paper, we present a new combined finite-element finite-volume method for stationary and unsteady cases. The flow, energy and species equations are solved using a classical finite-volume (FV) method which is conservative and bears a robust physical meaning. The electromagnetic equations (EM) are solved using finite-element (FE) analysis which has the ability to easily deal with their mathematical formulation without transforming the equations or doing any extra manipulation. We propose two separate approaches for the FE-method: one that is only suitable for a steady-state scheme and another that could be applicable for both steady and unsteady cases. Some illustrative numerical examples are also presented in this paper.

Keywords: AC plasma arc, Arc simulation, Finite-Element, Finite-Volume.

1. Introduction

Numerous numerical simulations have been carried out in regard with thermal arc applications: the welding arc [1], arc spraying [2, 3], AC arc furnace and three-phase plasma reactors [4]. In most cases, existing numerical CFD softwares codes are used. They are commonly built for general cases and rely on particular numerical schemes and approaches. As a consequence, the set of equations has to be adjusted accordingly. Given its insightful physical features and its conservative nature, the FV method appears to be very attractive for flow simulation and heat transfer. The EM equations (for the magnetic field B and electric field E) are not expressed as TADR (Transient-advection-diffusion-reaction) equations since they make use of curl-operators instead of div-operators. Transforming the variables is therefore inevitable. The scalar electric potential (φ) and the vector potential (A) formulations are commonly adopted. For transient cases, the A-formulation requires the magneto-quasi-static (MQS) assumption (∂t A = 0 or ∂t B = 0) which may not be valid if the electrodes are modelled. On the other hand, the FE method is mathematically robust and can handle curl-type operators. The drawback of standard FE methods is related to their non-conservative nature. This could lead to nonphysical results or spurious oscillation modes for transport equations with high advection. The variational multiscale approach (VMS), used to model arc dynamics and instabilities in [2, 3] appears to be suitable for handling compressibility effects. The main disadvantage of these methods is the significant computational cost and memory storage required. In this paper, we propose a standard FV method to solve the flow and heat equations and an FE method to solve the EM equations. Two approaches are proposed. The first requires is only suitable for stationary arcs. The second can be applied for a general case. Specific details for the implementation of those methods in 2D-axisymmetric configurations are presented. Some numerical experiments are carried out and the results are compared with previous works.

2. Mathematical model

The physics describing thermal arcs are heavily complex by virtue of the various phenomena that are involved. Flow, energy, species transport and EM equations are to be solved. However, it is common to consider the bulk plasma as

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a single-phase gas in local thermodynamic equilibrium (LTE) with properties depending only on temperature and pressure. Other models have been intensively developed in the literature taking into near-electrode phenomena, the electrode-plasma interaction, species transport and non-local thermodynamic equilibrium [3, 5]. For simplicity, we will list the set of the governing mathematical equations that are used to model a plasma in LTE:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (1)
\]

\[
\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}^T) = \nabla \cdot (-p \mathbf{I} + \tau) + \mathbf{j} \times \mathbf{B} \quad (2)
\]

\[
\frac{\partial \rho h}{\partial t} + \nabla \cdot (\rho h \mathbf{u}) = \nabla \cdot (\kappa C_p \nabla h) + \mathbf{j} \cdot (\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \frac{4}{2} \pi \mathbf{e}_N + \frac{5}{2} \kappa B \nabla T + \frac{D_p}{d} \quad (3)
\]

If a two-temperature model or a multicomponent flow were adopted, energy balance for the electrons and for every species and their transport equations should be separately expressed, see e.g. [3]. A general form of a TADR equation for an arbitrary variable \( \phi \) has the following form:

\[
\frac{\partial \rho \phi}{\partial t} + \nabla \cdot (\rho \mathbf{u} \phi) = \nabla \cdot (\Gamma \nabla \phi) + S \quad (4)
\]

It is obvious that the above-stated equations take already the form of a TADR. Any FV classical approach [6] can be implemented with segregated solvers (SIMPLE or its revised versions) or coupled solvers and staggered or collocated grids. We proceed by stating the EM equations:

\[
\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (5)
\]

\[
\nabla \times \mathbf{B} = \mu_0 \left( \mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \quad (6)
\]

\[
\nabla \cdot \mathbf{B} = 0 \quad (7)
\]

To the Maxwell equations, the charge conservation equation and the Ohm’s law are added:

\[
\nabla \cdot \mathbf{j} = 0 \quad (8)
\]

\[
\mathbf{j} = \mathbf{j}_{\text{diff}} + \mathbf{j}_{\text{diff}} = \sigma (\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \frac{e D_e(T)}{m_e} \nabla \ln T \quad (9)
\]

The coefficient \( D_e(T) \) is a thermal diffusion coefficient. It is a common practice in thermal arcs to assume that the displacement current in equation 6, \( \partial \mathbf{E} / \partial t \) is negligible. The MQS assumption (i.e. omission of the term \( -\partial \mathbf{B} / \partial t \) in equation 5) is also very often made. We believe that it is not valid in particular for AC arcs when electrodes are included in the model. This will be illustrated with some examples later on.

### 3. The standard method for Maxwell’s equations

If a TADR framework is desired, a difficulty arises when dealing with the Maxwell’s equation since curl-operators are involved. To circumvent this problem we use the fact \( \nabla \cdot \mathbf{B} = 0 \), which enables us to write the following:

\[
\mathbf{B} = \nabla \times (A + \nabla \psi) \quad (10)
\]

Replacing this in equation 6 and using the relation \( \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla \cdot \nabla \mathbf{A} \), yields:

\[
-\nabla \cdot \nabla \mathbf{A} = \mu_0 \mathbf{j} \quad (11)
\]
Here for convenience (in order to make the equation for A TADR-conform) the Coulomb gauge is applied A, i.e \( \nabla \cdot A = 0 \). This is possible because A is unique up to a certain gradient, which allows us to gauge fix it. Two important observations can be made here: special BCs must be imposed for (i.e A) and (11) requires somehow the knowledge of j.

3.1. Boundary conditions for A

For the BC of A, null-fluxes, null-values or magnetic insulation (n x A = 0) are proposed in the literature. Unfortunately, no BC guarantees that the solution will be div-free. An inconsistency might arise leading to errors. In order to investigate the appropriate BC, we consider that the magnetic field is normal trace-free on the boundary. In other words, B is tangential at the boundary of the computational domain, i.e B \cdot n = 0. If we substitute for A in this condition we find:

\[
(\nabla \times A) \cdot n = 0 \implies (\nabla \times A) \cdot n = -\nabla \cdot (A \times n) + (\nabla \times n) \cdot A = 0 \implies \nabla \cdot (A \times n) = 0
\]

Here we used the fact that \( \nabla \times n = 0 \). Although its div is null, A \times n is chosen to be equal to zero for practical reasons. This is a Dirichlet-like condition that does not guarantee a divergence-free solution for A. We then resort to the minimization of the energy functional of equation 11, to which we explicitly add the divergence of A:

\[
\mathcal{F}(A) = \frac{1}{2} \int_{\Omega} |\nabla \times A|^2 d\Omega - \int_{\Omega} \mu_0 j \cdot A \, d\Omega + \frac{1}{2} \int_{\Omega} (\nabla \cdot A)^2 d\Omega
\]

(12)

We can easily observe that if A minimizes the above functional, it satisfies both equation 11 and the div-free of A. Using a Taylor series expansion, we get \( \mathcal{F}(A + \varepsilon \delta A) = \mathcal{F}(A) + \varepsilon \delta \mathcal{F}(A, \delta A) + \frac{\varepsilon^2 \delta^2 \mathcal{F}(\delta A)}{2} \). In order for A to minimize \( \mathcal{F}, \delta \mathcal{F}(A, \delta A) = 0 \). We leave the reader to check that \( \varepsilon^2 \delta^2 \mathcal{F}(\delta A) \geq 0 \), the first variation term reads:

\[
\delta \mathcal{F}(A, \delta A) = \left. \frac{\partial \mathcal{F}(A + \varepsilon \delta A)}{\partial \varepsilon} \right|_{\varepsilon=0} = \int_{\Omega} (\nabla \times A) \cdot (\nabla \times \delta A) \, d\Omega - \int_{\Omega} \mu_0 j \cdot \delta A \, d\Omega + \frac{1}{2} \int_{\Omega} (\nabla \cdot A) (\nabla \cdot \delta A) \, d\Omega
\]

Upon using Stokes identities and carrying out some manipulations, we obtain:

\[
\delta \mathcal{F}(A, \delta A) = \int_{\Omega} (\nabla \times (\nabla \times A) - \mu_0 j - (\nabla \cdot A)) \cdot \delta A \, d\Omega + \int_{\Omega} ((\nabla \times A) \times n) \cdot \delta A \, d\Gamma + \int_{\Gamma} (\nabla \cdot A) (\delta A \cdot n) \, d\Gamma
\]

This expression has to identically vanish, i.e. \( \delta \mathcal{F}(A, \delta A) \equiv 0 \). Therefore the first expression in the domain integral must be zero. It is easy to check that the first boundary integral is automatically zero. The last integral must be set to zero and so \( \nabla \cdot A = 0 \) or n \cdot A = 0. To investigate this, we can apply a div-operator on equation 11. Knowing that \( \nabla \cdot j = 0 \) (by the current conservation law), we get: \( \nabla \cdot \nabla^2 A = \nabla^2 (\nabla \cdot A) = 0 \). This equation is a Laplace-type equation for \( \nabla \cdot A \). In order for this latter to be identically zero everywhere, by virtue of the strong maximum principle it must vanish on some parts of the boundary (i.e \( \nabla \cdot A = 0 \)). On the rest of them its flux must be null (i.e \( n \cdot (\nabla \cdot A) = 0 \)). For convenience \( \nabla \cdot A = 0 \) is set on all the boundary. So the complete system to be solved with the appropriate BC is as follows:

\[
\begin{cases}
\nabla \cdot (\nabla \times A) - \mu_0 j - (\nabla \cdot A) = 0 & \text{on } \Omega \\
A \times n = 0 & \text{on } \Gamma \\
\nabla \cdot A = 0 & \text{on } \Gamma
\end{cases}
\]

(13)

So loosely speaking, it is enough that \( \nabla \cdot A = 0 \) be satisfied as a BC to ensure that the solution for A is div-free. It is worthwhile noting that although Biot-Savart law for A is established for a div-free A, the use of it to determine the values of A at the boundary does not guarantee div-free solutions because those are inserted as Dirichlet BC.

3.2. The current density j

The conventional practice suggests to write the current density as follows:

\[ \ldots \]
When steady-state or MQS is considered, using equations (8) and (15), yields:

\[ j = \sigma [\mathbf{E} + \mathbf{u} \times \mathbf{B} - a(T) \nabla \ln T] \]  \hspace{1cm} (14)

\[ \implies j = -\sigma \nabla \varphi_{eff} \]  \hspace{1cm} (15)

Where \( a(T) = \varepsilon D_e(T) / (\sigma m_e) \). In order to express the term inside the brackets in (14) according to (15), each of its components must be expressed either by neglected or irrotational. It can be observed from (5) that \( \nabla \times \mathbf{E} = 0 \) for the steady-state case or when MQS is made. The second term is considered negligible which is valid as the magnitude of the product of the velocity and the magnetic field is small compared to the electric field \( \mathbf{E} \). In high current arcs the contribution of this term could probably be significant. The last term due to thermal diffusion of electrons is irrotational because \( \nabla \times a(T) \nabla \ln T = \nabla a(T) \times \nabla \ln T + a(T) \nabla \times \nabla \ln T = da(T) / dt \nabla T \times \nabla T / T = 0 \). It is assumed here that \( a \) depends only on \( T \). The contribution of the transient term in (5) by inserting the expression (10). This yields: \( \nabla \times \mathbf{E} = -\nabla \times \partial \mathbf{A} / \partial t \). Therefore it is possible to write \( \mathbf{E} = -\nabla \varphi - \partial \mathbf{A} / \partial t \) and then include in (14). However, expression (15) would no longer be valid. Replacing in (8) leads to the emergence of the term \( (\nabla \sigma \cdot \partial \mathbf{A} / \partial t) \). There is tedious to deal with. In the electrodes the term \( \partial \mathbf{A} / \partial t \) is not negligible with respect to \( \mathbf{E} \).

4. Direct resolution in \( B \)

If MQS is assumed and \( j \) is known, then by applying a curl on equation 6, we get after expanding and using \( \nabla \cdot \mathbf{B} = 0 \):

\[ -\nabla \cdot \mathbf{B} = \mu_0 \nabla \times j \]  \hspace{1cm} (16)

This is a TADR equation but we do not recommend it because the computation of \( \nabla \times j \) can lead to severe errors, especially in areas where the change \( j \) is so steep. Another method for solving directly in \( B \) for the general case consists in assuming that \( \mathbf{u} \times \mathbf{B} \) is negligible and rearranging (14) as:

\[ \mathbf{E} = j / \sigma + a(T) \nabla \ln T. \]  \hspace{1cm} Including this in (5) leads to:

\[ \nabla \times \left( \frac{1}{\mu_0 \sigma} \nabla \times \mathbf{B} \right) = -\frac{\partial \mathbf{B}}{\partial t} \]  \hspace{1cm} (17)

The thermal diffusion is cancels out since it is curl-free. The diffusion coefficient \( 1 / \mu_0 \sigma \) is best known as the magnetic diffusivity. We denote it by \( \eta \). Expanding the double curl in 17 we find:

\[ \nabla \times \left( \eta \nabla \times \mathbf{B} \right) = \nabla \eta \times (\nabla \times \mathbf{B}) + \eta \nabla \times (\nabla \times \mathbf{B}) = \nabla \eta \times (\nabla \times \mathbf{B}) - \eta \nabla \cdot \nabla \mathbf{B} \].

Due to the curl-operator in the RHS, the equation for (17) does not have a TADR form. If TADR is desired, then an iterative method must be implemented and \( \nabla \times \mathbf{B} \) must be inserted as a source term. However, convergence is not guaranteed. The use of a variational FE method proves to be very adequate for solving this equation. We propose two FE methods for solving equation (17) directly in \( B \). The first can only be applicable for the steady-state case. The second can be applied for the general case.

4.1. First FE method

When steady-state or MQS is considered, using equations (8) and (15), yields:

\[ \nabla \cdot (\sigma \nabla \varphi_{eff}) = 0 \]  \hspace{1cm} (18)

This is a TADR equation and it can be solved using any FE or FV method. The computation of \( \varphi_{eff} \) enables us to determine \( j \). The system for \( \mathbf{B} \) to be solved is given by:

\[
\begin{align*}
\nabla \times \mathbf{B} &= \mu_0 j \\
\nabla \cdot \mathbf{B} &= 0
\end{align*}
\]  \hspace{1cm} (19)

We introduce first the vector valued Sobolev spaces on the domain \( \Omega \) with boundary \( \Gamma \): \( H^1(\text{curl}; \Omega)^3 = \{ v \in L^2(\Omega)^3; \nabla \times v \in L^2(\Omega)^3 \} \). In the following the subscripts \( t \) and \( n \) designate respectively a tangential trace free \( (n \times \mathbf{v} = 0) \) and normal trace free spaces \( (\mathbf{n} \cdot \mathbf{v} = 0) \). The Hilbert spaces \( L^2(\Omega)^3 \), \( H^1(\Omega)^3 \) and \( H^1(\text{curl}; \Omega)^3 \) are equipped
with their standard norms. We require that \( B \in H^1(\text{div}; \Omega)^3 \) (the space \( H^1(\Omega)^3 \) with \( \nabla \cdot v = 0 \)) so that \( B \) has no normal traces through the boundary \( \Gamma \), i.e. \( n \cdot B = 0 \) and is div-free. By virtue of the generalized Stokes' theorem, a weak variational formulation of the first equation of (19) reads:

\[
(\nabla \times B, v)_{L^2(\Omega)^3} = (\mu_0, v)_{L^2(\Omega)^3}, \quad \forall v \in H^1(\text{curl}; \Omega)^3
\]

\[
(\nabla \times B, v)_{L^2(\Omega)^3} = (B, \nabla \times v)_{L^2(\Omega)^3} + \int_{\Gamma} (v \times n) \cdot B \, ds
\]

Here \((\cdot, \cdot)_{L^2(\Omega)^3}\) denotes the usual inner-product in the Hilbert space \( L^2(\Omega)^3 \). This is to be found in the divergence-free space \( \mathbb{H}^1(\text{curl}; \Omega)^3 \). That is the reason why the div-free constraint in (19) is simply dropped. For the discretization of (20), it is noteworthy to observe that if \( u_h \in V_h \subset V \subset \mathbb{H}^1(\text{curl}; \Omega)^3 \), then \( B_h \in W_h = \nabla \times V_h \subset \mathbb{H}^1(\text{div}0; \Omega)^3 \). The following commuting diagram represents a summary of the relations between the various spaces of interest:

The \( k \)-symbol used in the diagram denotes the order of the finite element polynomials. Suitable high order finite elements in 2D or 3D could be Hermite elements, Bogner-Fox-Schmit rectangles \([7, 8]\), or elements introduced in \([9]\). On structured regular grids, the general idea consists in taking for each component a finite element space composed of ordinary Lagrange polynomials in the direction of that component and of high order Hermite polynomials in the orthogonal directions. The approach is extended to general unstructured grids by means of the \( \mathcal{C}^1 \)–\( P_k \) Morgan-Scott finite element space as per \([10]\) with \( k \geq 5 \). The mapping \( \nabla \times : V_h^k \rightarrow W_h^k \) needs to be bijective. Since the space \( V_h^k \) is too big, its image under the curl-operator is not injective. So only some components are to be accounted for. For instance in 3D, we only take for example the \( x \) and \( y \) components and in 2D axisymmetric, only the \( z \)-components are sufficient. We might also need few degrees of freedom along the remaining component to remove some intrinsic linear dependencies \([8]\). This method is self-consistent as no assumptions on the BC are required.

\[\text{4.2. Second FE method}\]

In the unsteady case, because \( \nabla \times E = -\frac{\partial B}{\partial t} \neq 0 \), \( E \) cannot be expressed as a gradient of a potential field. In this case, equation (17) is to be solved using \( \text{Nédélec} (\text{curl}; \Omega) \)-conforming finite elements introduced in \([11]\). To this end, we write by means of the generalized Stokes’ theorem, the weak formulation of equation (17) as:

\[
(\nabla \times B, v)_{L^2(\Omega)^3} = \int_{\Gamma} (v \times n) \cdot (\eta \nabla \times B) \, ds
\]

The second term cancels not because \( v \) is orthogonal to \( n \) at the boundary as in the first method (since \( v \notin H^1(\text{curl}; \Omega)^3 \)), but because we require that the current density represented by \( \nabla \times B \) has no tangential components. This
boundary conditions bears an important physical meaning since the current either vanishes or exists normally at the boundaries. Replacing this result in the weak formulation, we obtain:

\[
\left( \frac{\partial \mathbf{B}}{\partial t}, \mathbf{v} \right)_{L^2(\Omega)^3} + \left( \eta \nabla \times \mathbf{B}, \nabla \times \mathbf{v} \right)_{L^2(\Omega)^3} = 0 \quad \forall \mathbf{v} \in H^1(\text{curl}; \Omega)^3
\]

(21)

Notice \( \nabla \cdot \mathbf{B} \) does not appear in the weak formulation. A global divergence-free solution is not guaranteed as the normal component across boundaries of the cells are not necessarily continuous. However, in the unsteady case presented above, the global divergence-free solution is bounded, and its bound can be controlled by the mesh size (if any Crank-Nicolson time discretization is used). This can be proven by using similar arguments as in [12]. For low order Nédélec elements on convex domains we can prove \( \| \nabla \cdot \mathbf{B}^h \|_{H^{-1}(\Omega)} \leq C \| \nabla \times \mathbf{B}^0 \|_{L^2(\Omega)} \).

5. Implementation in the axisymmetric configuration

5.1. First method

For structured rectangular meshes, we choose Hermite polynomials of order \( k \geq 3 \) to represent the finite element in the direction of the specified derivative (\( r \)-direction) and Lagrange polynomials of order \( k - 1 \) in the orthogonal direction (\( z \)-direction). In the 2D axisymmetric case, unlike in 3D, Lagrange polynomials could be of any order (not necessarily \( k - 1 \)). We define the reference finite element as the triple \(( \hat{K}, \hat{H}_k - \hat{L}_{k-1}, \hat{\Sigma})\). The \( \hat{H}_k \)-Hermite cubic polynomials \((k = 3)\), to the left, and the \( \hat{L}_{k-1} \)-Lagrange polynomials \((k = 2)\), to the right, are given by:

\[
\begin{align*}
\hat{\phi}_1 &= r^3 - 2r^2 + r \\
\hat{\phi}_2 &= 2r^3 - 3r^2 + 1 \\
\hat{\phi}_3 &= -2r^3 + 3r^2 \\
\hat{\phi}_4 &= -2r^3 + 3r^2 \\
\hat{\phi}_5 &= 2z^2 - 3z + 1 \\
\hat{\phi}_6 &= -4z^2 + 4z \\
\hat{\phi}_7 &= 2z^2 - z \\
\end{align*}
\]

The degrees of freedom (figure 1) represented by the set \( \hat{\Sigma} \) of linear functionals in \( \mathcal{L}(\hat{H}_k - \hat{L}_{k-1}, \mathbb{R}) \) are given by:

\( \hat{\Sigma} = \{ \sigma_i(p_i), i = 1, 2, ..., N \mid \theta_j(p_j), j = 1, 2, ..., M \} \)

The polynomials are constructed by simply taking the tensor product of the polynomials in each direction. \( p_i \) are all the designated nodes, \( N \) is their total number and \( \sigma_i \) is the nodal value at node \( p_i \) (for the polynomials in the above example they correspond to \( \phi_2 \) and \( \phi_3 \)). \( p_j \) are all the nodes on the vertical boundaries of the cell \( K \), including the four vertices (corners), \( M \) is their total number, and \( \theta_j \) designate the nodal value of the derivative in the \( r \)-direction (in the example they are related to \( \hat{\phi}_1 \) and \( \hat{\phi}_5 \)). It is a simple task to show, by means of the barycentric coordinates, that the triple \( (\hat{K}, \hat{H}_k - \hat{L}_{k-1}, \hat{\Sigma}) \) is indeed a finite element. This is done by showing that the elements of \( \hat{\Sigma} \) are linearly independent, i.e. \( \forall \hat{\alpha} \in \hat{H}_k - \hat{L}_{k-1} \mid \forall i, j, \sigma_i(\hat{\alpha}) = 0, \theta_j(\hat{\alpha}) = 0 \implies \hat{\alpha} = 0 \).

On the global domain it is easy to prove, by virtue of continuity and the BC \( \mathbf{n} \times \mathbf{v} = 0 \), that: \( \text{dim}(V^h_k) = [(k - 1)n + 1][(k - 1)n + 1][(k - 1)n + 1][(k - 1)n + 1][(k - 1)n + 1][k - 1]n \). Taking the image under the curl-operator of \( V^h_k \), removing linear dependencies and using the fact that \( \mathbf{w} = 0 \) at the z-axis \( (r = 0) \), we can prove that: \( \text{dim}(W^h_k) = [(k - 1)n + 1][(k - 1)n + 1][(k - 1)n + 1][(k - 1)n + 1][(k - 1)n + 1][k - 1]n \). So \( \text{dim}(W^h_k) = \text{dim}(\nabla \times V^h_k) = \text{dim}(V^h_k) \), which means that \( \nabla \times : V^h_k \to W^h_k \) is bijective. Hence an exact potential space for the div-free magnetic field \( \mathbf{B} \). The image of the nodal derivatives \( \partial r^i(\mathbf{B}) \) gives the nodal values (i.e. the values of the magnetic field \( \mathbf{B} \) in the \( W^h_k \)). The image of the nodal values \( \partial r^i \) correspond to linear functionals that represent the integrals on each edge for the magnetic field and are given by: \( \partial r^i(\alpha) = \frac{1}{\text{meas}(E_i)} \int_{E_i} \mathbf{B} \cdot \mathbf{d}r \).
5.2. Second method

The Nedelec elements correspond to edge elements that are composed of vector polynomials defined in [11] by:

\[ R^k_n = \{ u \in (P^k_{n})^n, \partial_n u = 0 \}. \]

Here, \( k \) and \( n \) represent respectively the order of the polynomials and the dimension of the space.

On rectangular grids of dimension \( n \) and for the order \( k \) the Nedelec elements are defined by:

\[ Q^k_n = \times_{i=1}^{n} Q_{k-i-1,...,k-2,...,k} \]

Where \( Q_{k-i,...,k} \) is the linear space defined as

\[ Q_{k-i,...,k} = \{ q = \sum_{|\alpha_i| \leq k} a_{\alpha} x^{\alpha} \}. \]

\( \alpha \) is a multi-index and the values of \( a_{\alpha} \) are to be determined from the degrees of freedom that are expressed in both triangular and rectangular grids by:

- **edge**: \( \int_E q \cdot t \; ds \quad \forall E, \; p \in P_{k-1} \)
- **face**: \( \int_F (q \times n_F) \cdot p \; d\sigma \quad \forall F, \; p \in P^2_{k-2} \)
- **inner**: \( \int_K q \cdot p \; dx \quad \forall p \in P^3_{k-3} \)

In the 2D case, the set of dofs corresponding to inner dof vanishes. On rectangular grids, the vector-valued polynomials \( p \) for the face dof corresponding must be in \( \times_{i=1}^{n} Q_{k-2,...,k-1,...,k} \). For the inner dofs, they are in \( \times_{i=1}^{n} Q_{k-3,...,k-2,...,k} \).

For proofs that these elements are unisolvent and \( \text{curl} \)-conforming, the reader is referred to [11].

In 2D axisymmetric, only \( B_\theta \) is considered. The projected vector polynomials on the edges in \( rz \) are nothing but ordinary Lagrange polynomials. From a different perspective, if we express 17 in 2D with axisymmetry considerations, we find:

\[
\frac{\partial B_\theta}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} \left( r B_\theta \right) + \frac{\partial^2 B_\theta}{\partial z^2} = 0
\]

(22)

This equation for the \( B_\theta \) component makes use of what is know as the Maxwellian operator \( \partial_t (r^{-1} \partial_r (r B_\theta)) + \partial^2_{zz} B_\theta \).

The discretized weak Galerkin formulation for the 2D problem reads:

\[
\int_{\Omega} \frac{\partial B_\theta}{\partial t} w r \; dr \; dz + \int_{\Omega} \frac{1}{r} \frac{\partial}{\partial r} \left( r B_\theta \right) \frac{\partial}{\partial r} w \; dr \; dz + \int_{\Omega} \frac{\partial B_\theta}{\partial z} \frac{\partial}{\partial z} w \; r \; dz = 0
\]

(23)

In [13] error estimates for Lagrange polynomials finite element space (named bilinear space in [13]) are given for rectangular meshes. Specified values for \( B_\theta \) are given used as BC. They can be obtained using a self-consistent method with the help of the Biot-Savart law that yields volume integration. This is a boundary-element method coupled to the FE method. However and in order to account for the effect the current circulating in the electrodes outside the domain, it is recommended that the Biot-Savart Law under its linear form only along the \( r \)-direction be used (This is only possible if we assumed that the electrodes are long enough so it this formula would be valid). It is expressed by:

\[ B_\theta = \frac{\mu_0 I}{2\pi r}. \]
6. Numerical experiments

In this section, we test our different approaches on the welding arc using the same geometry as in [1] with the tip of the Tungsten electrode tilted by 60° (See figure 3). We also have Argon as working gas at 1 bar with a current of 200 A.

|            | $u_r$ | $u_z$ | $h$  | $p$ | $\phi$ (1st FEM) | $B_0$ (1st FEM): AB Ca | $B_0$ (2nd FEM): AB Ca | $B_0$ (2nd FEM): AB An |
|------------|-------|-------|------|-----|------------------|------------------------|------------------------|------------------------|
| $A'B'$     | -     | -     | -    | -   | $n \cdot \nabla \phi = 0$ | $-\frac{Bt-Sv}{n}$ | $n \cdot \nabla B_0 = 0$ | $n \cdot \nabla B_0 = 0$ |
| $CD$       | 0     | 0     | 1000K| 0   | $n \cdot \nabla \phi = 0$ | $-\frac{Bt-Sv}{n}$ | $n \cdot \nabla B_0 = 0$ | $n \cdot \nabla B_0 = 0$ |
| $ED$       | 0     | 0     | 300K | 1 bar | $n \cdot \nabla \phi = 0$ | $-\frac{Bt-Sv}{n}$ | $n \cdot \nabla B_0 = 0$ | $n \cdot \nabla B_0 = 0$ |

In case of the AC current, a frequency of 50 Hz is used and the electrode radius is increased from 1.6 mm to 2.2 mm. The FE and FV methods are developed on Matlab. For the FE methods, the solver used is based on the conjugate gradient method with an incomplete Cholesky preconditioner. The flow of the current from the electrode to the plasma bulk is modelled exactly as in [1]. And $T_{\infty}$ in the BC presented in table 1 are also taken from [1]. $Bt-Sv$ designates Biot Savart under its linear form. The figures in 4 show good agreement between the FE-FV method and the full FV method with magnetic potential $A$ solved with the appropriate BC presented above. No convergence has been obtained if any other BC are taken when $A$ is considered. All the profiles, including the velocity vectors and the gauge pressure contours (the gauge pressure is difference between the absolute pressure and the reference pressure which in our case corresponds to the atmospheric pressure of 100 kPa) are compared with the results from [1] (where the magnetic field is computed.
For the AC case, we present first a basic elementary case, yet very insightful to show how important the transient term $\partial B / \partial t$ can be if electrodes are taken into account. We consider a current flowing in a metal rod of diameter 3 mm with constant electric conductivity of $10^7 \text{S.m}^{-1}$, which is typical for metals. This problem is well known and analytical solutions for $j_z$ and $B_\theta$ can be found (they make use of complex functions involving Bessel functions). The numerical results from our methods in figure 2 are compared to these solutions and they yield exactly the same plots. We can clearly observe from figure 2 that if the transient term is simply dropped, the magnetic field would vary linearly and the current would be uniform along the cross section sliding up and down as time evolves. The plots are taken at $t = 0.056 \text{ms}$. When the transient term is included, $j_z$ is no longer uniform but there is a delay between the value at the axis and the value at the surface. It is likewise for $B_\theta$. On the welding arc with AC current, this is clearly seen in figures 5 through the much higher values of temperatures when the transient term is not accounted for, especially at the current peak. Finally, to fully appreciate a real AC plasma system, tip-to-tip electrodes have been considered with the same conditions and the same working gas as before. The results at the time steps corresponding to $\omega t = 2k\pi$, are shown in figure 6. We can clearly observe again, that for the case where the transient term is accounted for, the profile of temperature is wider but the peak of temperature is less significant. This again is due to the delayed currents. However, when the transient term is neglected, we notice a sharper temperature profile with a much higher value of the peak (near the cathode tip).

FIG. 5: 200 A(ef f) 50 Hz-AC argon plasma arc (@ 1 bar) at various time steps with (left) and without (right) the transient term using the Maxwell-Ampere law, which is satisfactory for simple geometries). The results are in good agreement.
FIG. 6: 200 A(eff) 50 Hz-AC argon arc (@ 1 bar) at \( \omega t = 2k\pi \) with (left) and without (right) the transient term for a tip-to-tip configuration.

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