What is the Size of the Hilbert Hotel's Computer?

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What is the Size of the Hilbert Hotel's Computer?

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Abstract. The Hilbert's Hotel is a hotel with countably infinitely many rooms. The size of its hypothetical computer was the pretext in order to think about whether it makes sense and what would be \( \log_2(\aleph_0) \). Thus, at the road of this journey, this little paper demonstrates – surprisingly – that there exist countably infinite sets strictly smaller than \( \mathbb{N} \) (the natural numbers), with very elementary mathematics, so shockingly stating the inconsistency of the Zermelo-Fraenkel Set Theory with the Axiom of Choice (ZFC).

Mathematics Subject Classification (2010). Primary 03E50; Secondary 03E35.

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1. Introduction

This paper proves, utilizing the suitable axioms and rules strictly within ZFC, the inconsistency of the proper ZFC. \[^7, \, 16\] The proof relies on the construction of a countably infinite set strictly smaller than \( \mathbb{N} \), which would be impossible, by the Axiom of Countable Choice or Axiom of Denumerable Choice \( (\text{AC}_\omega) \) \[^8\], hence this axiom is unfortunately contradictory with ZFC, which implies that the ZFC is inconsistent, regrettably.

Suppose that the size of a computer is, coarse mode, determined by the size (and quantity) of its internal registers and memory (RAM) \[^14\], mainly when these are huge components. Assume yet that a computer, in order to help controlling the administration of a hotel, must be able to cope efficiently online at least with its number of rooms and guests. Presume also if this [finite] number is \( n \), then, in order to maximize the speed of the processing, the size of its internal registers and memory cells should be at least about \( \lceil \log_2(n) \rceil \), where the size of the computer would be proportional to \( \lceil \log_2(n) \rceil \).\[^14\]

So, with those same [seemingly sensible] assumptions, what would be the [theoretical] size of a computer for the Hilbert's Hotel, \[^15\] in order to help to manage countably infinitely
many rooms and guests ($\aleph_0$)? What about its size being about $\log_2(\aleph_0)$? Would that question make sense in ZFC?\[^1\] What about this problem maybe shed light on the question concerning even to consistency of the proper ZFC?\[^7, 16\]

2. Definition of $\log^i$-panfinite sets ($\log^i(\omega)$, $\log^2(\aleph_0)$ and $\Gamma_4$

**Definition 2.1: $\log^i$-panfinite set ($\log^i(\omega)$ and $\log^2(\aleph_0)$).** Let $X$ and $Y$ be infinite subsets of the natural numbers $\mathbb{N}$\[^19\], and the real numbers $\mathbb{R}$\[^10\], respectively, by the Axiom Schema of Replacement\[^1\] with the functions below:

$$X = \{\llbracket \log_2(n) \rrbracket : n \in \mathbb{N}\setminus\{1\}\} \quad (\llbracket \log_2 \rrbracket : \mathbb{N}\setminus\{1\} \rightarrow \mathbb{N}^+, \text{where } \lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\})$$

$$Y = \{1/(\log_2(n) + 1) : n \in \mathbb{N}^+\} \quad (1/(\log_2 + I) : \mathbb{N}^+ \rightarrow \mathbb{R}, \text{where } \log_2(n) = y \Leftrightarrow 2^y = n)$$

Notice that that set $X$ is very interesting, because even though it is naturally countably infinite, its cardinality or size ($n(X)$) or $|X|$, that we shall call $\log^2(\aleph_0)$ hereafter is strictly less than the cardinality of $\mathbb{N}$ (where $|\mathbb{N}| = \aleph_0$), by the following theorem:

**Theorem 2.1.** $\log_2(\aleph_0) = |X| < |Y| = |\mathbb{N}| = \aleph_0$.

**Proof.** There exists an injective function $f : P(X) \rightarrow Y$. We can see its defining the sets $X_r$ and $Y_r$ below and demonstrating constructively that always $|P(X_r)| \leq |Y_r|$, for every $r$, and then as $r$ approaches $\aleph_0$, this shall necessarily lead to $|P(X)| \leq |Y|$.

**Schema of Definition 2.2: Restricted Sets $X_r$ and $Y_r$.** Let the sets $X_r$ and $Y_r$ be defined as $X$ and $Y$ above, but with a set $\{1, 2, 3, ..., r\}$ replacing $\mathbb{N}^+$, where $r \in \mathbb{N}^+$, or $r = \aleph_0$ (where, in this latter case: $\{1, 2, 3, ..., \aleph_0\} = \mathbb{N}^+$, $X_r = X_{\aleph_0} = X$, and $Y_r = Y_{\aleph_0} = Y$).

$$X_r = \{\llbracket \log_2(n) \rrbracket : n \in \{2, 3, ..., r\}\}$$

$$Y_r = \{1/(\log_2(n) + 1) : n \in \{2, 3, ..., r\}\}$$

So, there exists an injective function $f_r : P(X_r) \rightarrow Y_r$ for every $r$. We can demonstrate it defining that function as $f_r(\emptyset) = 0$, and for each nonempty subset $s = \{k_1, k_2, ..., k_m, ...\}$ of $P(X_r)$ (the power set of $X_r$)\[^1\], $f_r(s) = 1/(\log_2(1 + 2^{k_1} + 2^{k_2} + ... + 2^{k_m} + ...)) + 1$. Note that that $s$ can be an either finite or infinite subset of $P(X_r)$.

Then, we can prove that $f_r$ is really injective by construction, where for every member $p$ of $P(X_r)$ there exists one single member $y$ of $Y_r$, that is if $f_r(p) = y$, and $f_r(q) = y$, then $p = q$. This happens because we need double $r$ in order to generate only one new value to $|\log_2(r)|$, which in its turn will double the sizes of $Y_r$ and of $P(X_r)$, nearly equalizing their sizes ($|Y_r|$ and $|P(X_r)|$), since that if $k_m \in X_r$, then $Y_r$ contains necessarily at least $2^{k_m}$ elements (or members), which implies, as $X_r \subset \mathbb{N}^+$, that $|Y_r| \geq |P(X_r)|$ for all the $r$ varying from 1 up to $\aleph_0$, as shown in the symbolical constructive completed infinite table below:

| $r$ | $1/(\log_2(r)+1)$ | $|Y_r|$ | $|\log_2(r)|$ | $|P(X_r)|$ | $f_r : |P(X_r)| \rightarrow Y_r$ |
|-----|------------------|--------|-------------|-------------|----------------------------------|
| 1   | 1                | $\emptyset$ | 1           | 1           | $f_r(\emptyset) = 1$            |
| 2   | 0.5              | 2      | 1           | 2           | $f_r(\{1\}) = 0.5$              |
| 3   | 0.386...         | 3      | 1           | 2           | $f_r(\{1\}) = 0.5$              |
| 4   | 0.333...         | 4      | 2           | 4           | $f_r(\{1\}) = 0.5$              |
| 5   | 0.301...         | 5      | 2           | 4           | $f_r(\{1\}) = 0.5$              |
| 6   | 0.279...         | 6      | 2           | 4           | $f_r(\{1\}) = 0.5$              |
| 7   | 0.262...         | 7      | 2           | 4           | $f_r(\{1\}) = 0.5$              |
Table 2.1 Symbolical table with the infinite completed construction of all $Y_r$ and $P(X_r)$, varying $r$ from 1 up to $N_0$

| $r$ | $1/(\log(2)+1)$ | $|Y_r|$ | $|\log(2)|$ | $|P(X_r)|$ | $f : \{P(X_r)\} \rightarrow Y_r$ |
|-----|-----------------|-------|--------|----------|------------------|
| 8   | 0.25            | 8     | 3      | 8        | $f(8) = 1$       |
| 9   | 0.293           | 9     | 3      | 8        | $f(9) = 0.5$     |
| 10  | 0.231           | 10    | 3      | 8        | $f(10) = 0.386$  |
| 11  | 0.224           | 11    | 3      | 8        | $f(11, 2) = 0.333$ |
| 12  | 0.218           | 12    | 3      | 8        | $f(3) = 0.301$   |
| 13  | 0.212           | 13    | 3      | 8        | $f(1, 2) = 0.279$ |
| 14  | 0.208           | 14    | 3      | 8        | $f(2, 3) = 0.262$ |
| 15  | 0.203           | 15    | 3      | 8        | $f(1, 2, 3) = 0.25$ |
| 16  | 0.2             | 16    | 4      | 16       | $f(\emptyset) = 1$ |
| ... | ...             | ...   | ...    | ...      | $f(\emptyset) = 1$ |
| $2^k$ | $1/(k+1)$   | $2^k$ | $k$    | $2^k$    | $f(\emptyset) = 1$ |

$N_0$ $1/(\log(2)+1)$ $N_0$ $\log(N_0)$ $N_0$ $f(\emptyset) = 1, f(1) = 0.5, f(2) = 0.386, \ldots, f(1, 2, 3, 4) = 0.2$

Hence, for every finite or infinite subset $\{k_1, k_2, \ldots, k_n, \ldots\}$ of $P(X)$, there exists a definite and distinct value $1/(\log(2)+1)$ of $Y_r$ and then $|Y| = |N| \geq |P(X)|$, thus we can define $\log(N_0) = |X| < |N| = N_0$, because $|X|$ is strictly less than $|P(X)|$, since always $|w| = |P(w)|$ (every set is strictly smaller than its power set) for every [finite or infinite] set $w$, by the Cantor's Theorem $[13]$. 

Verify that the Cantor's diagonal argument $[13]$ is not valid here in order to attempt to prove that $|P(X)| > |N|$, since $\log(N_0) < N_0$, so a supposed anti-diagonal sequence from a countably infinite (supposed exhaustive) $N_0$-enumeration cannot generate another indicator function (or characteristic function) different from all the others ones of this $N_0$-enumeration, since the enumeration is $N_0$-length, but that supposed anti-diagonal is only $\log(N_0)$-length, as shown constructively in the symbolical table below, where all the supposed anti-diagonal sequences can be in that $N_0$-enumeration without being different from any position of their diagonal sequences (otherwise, then it would lead to a contradiction to the exhaustiveness assumption, and then it would prove that $|P(X)| > |N|$, after all, as in that Cantor's argument):

| Enumeration/Indicator Function | 1   | 2   | 3   | ... | n   | ... | $\log(N_0)$ |
|-------------------------------|-----|-----|-----|-----|-----|-----|------------|
| 1                             | 0   | 1   | 1   | ... | 0   | ... | 1          |
| 2                             | 1   | 0   | 1   | ... | 0   | ... | 0          |
| 3                             | 1   | 0   | 0   | ... | 1   | ... | 0          |
| ...                           | ... | ... | ... | ... | ... | ... | ...        |
| $\log(N_0)$                   | 0   | 1   | 0   | ... | 0   | ... | 1          |
| $\log(N_0)+1$                 | 0   | 0   | 0   | ... | 0   | ... | 0          |
| $\log(N_0)+2$                 | 0   | 0   | 0   | ... | 0   | ... | 0          |
| $\log(N_0)+3$                 | 0   | 0   | 0   | ... | 1   | ... | 1          |
| ...                           | ... | ... | ... | ... | ... | ... | ...        |
| $\log(N_0)+i$ (Supposed anti-diagonal above) | 0   | 0   | 0   | ... | 1   | ... | 0          |
| ...                           | ... | ... | ... | ... | ... | ... | ...        |
| $2\log(N_0)$                  | 0   | 0   | 0   | ... | 0   | ... | 0          |
| $2\log(N_0)+1$                | 0   | 0   | 0   | ... | 0   | ... | 0          |
| $2\log(N_0)+2$                | 0   | 0   | 0   | ... | 0   | ... | 0          |
| $2\log(N_0)+3$                | 0   | 0   | 0   | ... | 0   | ... | 0          |
| ...                           | ... | ... | ... | ... | ... | ... | ...        |
| $3\log(N_0)$                  | 0   | 0   | 0   | ... | 1   | ... | 1          |
| ...                           | ... | ... | ... | ... | ... | ... | ...        |
| $N_0$                         | 0   | 0   | 0   | ... | 1   | ... | 0          |

Table 2.2 Symbolical table with the constructive demonstration that Cantor's diagonal argument is not valid here

In order to better understanding of the infinite construction above, let $W$ be a set very similar to $X$, but a finite set instead of an infinite one, for instance, $W = \{\lfloor \log(n) \rfloor : n \in \{2, 3, 4, 5, 6, 7, 8\}\}$. What would be $|W|$ here? $|W| = |\{1, 2, 3\}| = 3$. 

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Notice that $|W| = 3 = \lfloor \log_2(8) \rfloor$, and for every finite or infinite set $\{2, 3, \ldots, m\}$ replacing the $\{2, 3, 4, 5, 6, 7, 8\}$ above, we would have $|W| = \lfloor \log_2(m) \rfloor$, that is this simple mathematical process allows sensibly to define integer logarithm of either finite or infinite sets: Hence, for that $m = \aleph_0$, we can see clearly that $\log_2(\aleph_0) = |W|$.

So, we shall call that set $X$ a log$^{-1}$-panfinite set $(\log^{-1}-\omega)$, where its size $|X|$ is the symbol $\log_2(\aleph_0)$, as defined above, hence $|X| = \log_2(\aleph_0) < \aleph_0$, and $|P(X)| = 2^{\log_2(\aleph_0)} = \aleph_0$.

Therefore, four questions loom about that set $X$, which are readily answered here:

1. “– Is $X$ really a well-defined set within ZFC?”
   
   – Yes, $X$ is very well-defined, since its definition results from ZFC, plainly.

2. “– Aprioristically, $X$ could even be a finite set; so, is $X$ actually infinite?”
   
   – Yes, it is infinite, since for every number $\lfloor \log_2(r) \rfloor$, there is another one greater than it $\lfloor \log_2(r + r) \rfloor = \lfloor \log_2(r) \rfloor + 1$ (see by the way that we “need” $r$ more in the “input” in order to get only 1 more in the “output”, which even assist to explain why that set $X$ “raises” so sluggishly).

3. “– Then, isn’t $X$ in fact a traditional countably infinite set, as $\mathbb{N}$, with cardinality equal to $\aleph_0$ (that is isn’t simply $|X| = \aleph_0$)?”
   
   – No, $X$ cannot be $\aleph_0$-sized, since its cardinality, $\log_2(\aleph_0)$, must be strictly less than $\aleph_0$, as proven within the completed infinite construction shown in the Tab. 2.1 above, unless we conclude otherwise that $|P(X)| = 2^\aleph_0 = \aleph_0$, which would be even very very worse to ZFC. (See within that construction above that $r$ more steps (numbers) are necessary in order to insert only 1 more member to $X$, which even helps to clarify why $X$ “grows” so slowly (logarithmically) on the number $r$ of “steps” or table rows in that construction, and hence it cannot “reach” $\aleph_0$.)

4. “– In truth, isn’t $2^n : \mathbb{N} \rightarrow \mathbb{N}$ an injective [total] function?”
   
   – No, $2^n$, neither every increasing exponential function in $n$, cannot even be just a [total] function from $\mathbb{N}$ to $\mathbb{N}$, since $2^{\aleph_0} > \aleph_0$. On the other hand, every polynomial in $n$ is so, because $k.\aleph_0 \leq \aleph_0$ [18], for every positive finite numbers $i, k$. (But $2^n : \log^{-1}-\omega \rightarrow \mathbb{N}$ is an injective [total] function, as $\log^{-1}-\omega$ is defined herein.)

**Definition 2.3:** $\Gamma_1$ (First Barbosa panfinite number). $\Gamma_1$ is simply another symbol (or name, or label) to represent $\log_2(\aleph_0)$, which leads to $\Gamma_1 = \log_2(\aleph_0)$, and $2^{\Gamma_1} = \aleph_0$.

Remember that $\Gamma_1$ is strictly smaller than $\aleph_0$ ($\Gamma_1 < \aleph_0$), since $|w| < |P(w)|$, by the Cantor’s Theorem [13], although $\Gamma_1$ is greater than every positive finite integer $n$.

**Definition 2.4:** Generalization of Def. 2.1: log$^{-1}$-panfinite sets $(\log^{-1}-\omega)$ and log$^{-2}$-$\omega$ $(\Gamma_1)$. We can now easily generalize the definitions in the Defs. 2.1 and 2.2, considering $\log^{i+1}-\omega = \mathbb{N}^+$, $\Gamma_0 = \aleph_0$, and replacing $\Gamma_{i+1}$ by $\Gamma_i$ over there (where $i \in \mathbb{N}^+$). In more formal terms:

$$Z = \{\lfloor \log_2(n) \rfloor : n \in \log^{i+1}-\omega\{1\}\}$$

The crucial question here is again “– What is the cardinality of $Z$?”

With similar symbolical completed infinite construction used in the constructive proof
of the Theorem 2.1 above, we shall call \( Z \) a \( \log^i \)-panfinite set \((\log^i-\omega)\) and the cardinality of \( Z \) the symbol \( \log_2(\Gamma_i) \), where \(|Z| = \log_2(\Gamma_i) < \Gamma_{i+1}\), and \(|P(Z)| = 2^{\log_2(\Gamma_i)} = \Gamma_{i+1}\).

**Definition 2.5 – Generalization of Def. 2.3: \( \Gamma_i \) (Barbosa panfinite numbers).** \( \Gamma_i \) is simply another symbol (or name, or label) in order to represent \( \log_2(\Gamma_i) \), which leads to \( \Gamma_0 = \mathcal{N}_0 = \mathcal{Z}_0 \), \( 2^{\Gamma_0} = \Gamma_{i+1} \), and \( \Gamma_i < \Gamma_{i+1} \) (for all \( i \in \mathbb{N} \)), whereas \( \mathcal{N}_0 \) leading to \( \Gamma_i = \log_2(\Gamma_0), \Gamma_{i+1} = \log_2(\Gamma_i) \), and so on. (See that all those \( \Gamma_i \) are greater than every positive finite integer \( n \)).

Consequently, initiating with \( \log^i-\omega \), \( \log_2(\mathcal{N}_0) \), and \( \mathcal{Z}_0 \), we can apply recursively the definitions 2.4 and 2.5 in order to define \( \log^i \)-panfinite sets \((\log^i-\omega), \log_2(\Gamma_{i+1}) \) and all the other Barbosa panfinite numbers \((\Gamma_i)\) for every positive finite integer \( i > 1 \).

Note that that concept of Barbosa panfinite numbers encompasses the Beth numbers (infinite cardinal numbers represented by the symbol \( \mathcal{N}_j \), where \( \mathcal{N}_j+1 = 2^{\mathcal{N}_j} \), for all \( j \in \mathbb{N} \))\(^{[2]}\), since \( i \) can be non-positive in the Def. 2.5 above, whereas \( \mathcal{N}_j = \Gamma_j \), for all the integers \( j \geq 0 \), entailing that the Beth numbers are just a proper subset of the Barbosa panfinite numbers.

Notice also that the countably infinite recursive process above generates countably infinite cardinalities \( \Gamma_i \), where all ones are strictly greater than every positive finite number \( n \). See that \( \mathcal{N}_0 = \mathcal{Z}_0 = \mathcal{Z}_0 \), hence there is herein a kind of positive-negative natural symmetry generalizing from Beth numbers to Barbosa panfinite numbers.

### 3. What is the size of the Hilbert Hotel’s Computer?

With the definitions above, we can already easily answer that question “– What is the size of the Hilbert Hotel’s computer?” – It is equal to \( \Gamma_{-1} \). See the construction of this answer in the symbolical infinite table below (note that an \( \eta \)-sized binary register or RAM cell can store one and only one number from exactly \( 2^\eta \) distinct ones: \( 0…2^\eta - 1 \))\(^{[14]}\):

| #Rooms | Range of Numbering | Size of Registers/RAM Cells | Size of Computer |
|--------|-------------------|-----------------------------|-----------------|
| 2      | 0…1              | 1                           | proportional to \((\infty)\) 1 |
| 4      | 0…3              | 2                           | \( \propto 2 \) |
| …      | …                | …                           | …               |
| \( 2^n \) | 0…2\( ^n-1 \) | \( n \)                      | \( \propto n \) |
| …      | …                | …                           | …               |
| \( \mathcal{N}_0 = 2^{\Gamma_{-1}} \) | 0…2\( ^{\Gamma_{-1}}-1 \) | \( \log_2(\mathcal{N}_0) = \Gamma_{-1} \) | \( \Gamma_{-1} \) |

**Table 3.1** Symbolical table representing the theoretical size of a computer in function of its numbers ranging

In fact, we now can answer innumerable theoretical questions of same kind, such as:

1. “– What is the [theoretical] length of a sequence of symbols that represents the cardinality of \( \mathbb{N} \ (\mathcal{N}_0) \) in a base \( b \) [into a numeral system] \(^{[3]}\) strictly greater than \( 1 \)?”

   – It is equal to \( \log_b(\mathcal{N}_0) = \Gamma_{-1} \). (Notice that if that base \( b \) was equal to 1 (unary base), then that length would be equal to \( \mathcal{N}_0 \), instead of \( \Gamma_{-1} \).)

2. “– How many months should we [theoretically] invest our savings at [positive] fixed rate of interest, \(^{[4]}\) in order to get an \( \mathcal{N}_0 \)-moneyed account?”

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– We should do it by $\Gamma_1$ months.

3. “– How many times should you [theoretically] bend an infinitely malleable paper sheet, in order to get an $\aleph_0$-lengthy thread?”

– You should do it $\Gamma_1$ times.

4. “– What is the [theoretical] depth (length) of a perfect binary tree [5] that has $\aleph_0$ leaves?”

– It is equal to $\Gamma_1$.

5. “– What is the [theoretical] maximal size of an NFA that can be converted into an exponentially larger DFA?” [6]

– It is equal to $\Gamma_1$.

6. “– What is the [theoretical] size of RAM memory pointers [14] into a computer with $\Gamma_1$-sized RAM (that is, its primary address space ranging from 0 to $\Gamma_1$)?”

– It is equal to $\Gamma_2 = \log_2(\Gamma_1)$.

7. “– How many terms are there in the infinite sum that is used as a representation of some Zeno’s Paradoxes: $1/2 + 1/4 + \ldots + 1/2^n + \ldots = 1$?” [17]

– If we consider sensibly that all those terms are rational numbers, then $2^n$ is upper bounded by $\aleph_0$, hence there are $\Gamma_1$ terms in that sum.

8. “– Can the Mathematical Induction be used in order to establish a given statement for all $\aleph_0$ natural numbers?” [18]

– No, in general, it cannot; it can do it only for the first $\Gamma_j$ natural numbers, where that statement is proven for all ones only when $j = 0$, since only that $\Gamma_0 = \aleph_0$. The maximum increasing rate (polynomial, exponential, etc.) of the integer formulas that occur within each particular induction shall determine that particular $j$. For instance, the inductive proof that $2^n > n^3$ (for $n \geq 10$) is valid only for the first $\Gamma_1$ natural numbers, not for all $\aleph_0$ ones, as $2^n$ is not integer for $n$ beyond $\Gamma_1$, because naturally $2^{\aleph_0} > \aleph_0$, and $2^{\Gamma_1} = \aleph_0$.

So, like G. Cantor, je le vois, mais je ne le crois pas! [12]: – There exist many countably infinite sets strictly smaller than $\mathbb{N}$.

Thus, as a preliminary result, the cardinalities in this paper can be strictly ordered by magnitude, as simply outlined below:

$$0, 1, \ldots, \aleph_0, \ldots, \Gamma_{\omega+1}, \Gamma_{\omega}, \ldots, \Gamma_1, \Gamma_0 = \aleph_0 = \beth_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_{\omega+1}, \Gamma_\omega, \ldots$$

4. Related Work

The main result of this paper unfortunately asserts that the Axiom of Countable Choice or Axiom of Denumerable Choice (AC$\omega$) [8] (that states that $\aleph_0$ is smaller than every other
transfinite cardinal number) is inconsistent with ZFC (so, the *axiom of choice*, a stronger version of that one)\(^9\), which implies that the ZFC is inconsistent, lamentably.

Therefore, I think we need build a new foundational frame to support and unify the Axiomatic Mathematics, either fixing or replacing the ZFC.

5. Freedom & Mathematics

“– The essence of Mathematics is Freedom.” (Georg Cantor)\(^{11}\)

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