The Devil is in the Details: Spectrum and Eigenvalue Distribution of the Discrete Preisach Memory Model

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Abstract

We consider the adjacency matrix associated with a graph that describes transitions between $2^N$ states of the discrete Preisach memory model. This matrix can also be associated with the “last-in-first-out” inventory management rule. We present an explicit solution for the spectrum by showing that the characteristic polynomial is the product of Chebyshev polynomials. The eigenvalue distribution (density of states) is explicitly calculated and is shown to approach a scaled Devil’s staircase. The eigenvectors of the adjacency matrix are also expressed analytically.

Keywords: Preisach model, Adjacency matrix, Eigenvalue distribution, Chebyshev polynomials, Devil’s staircase

1 Introduction

Hysteresis modeling has been an active area of research for decades with a wide range of physics-based, phenomenological and mathematical models (\cite{11, 28, 33, 34, 44, 45}). In 1935, F. Preisach proposed his well-known input-state-output model for ferromagnetic hysteresis \cite{35}. The evolution of states in this model has been later shown to be universal for many important models of hysteresis with scalar-valued inputs and outputs \cite{28, 33} or, more precisely, for all the models which respect Madelung’s memory update rules (also known as hysteresis with return point memory or the wiping out property \cite{40}).

In its discrete form, the Preisach model describes the state of magnetic domains (moments) in a magnetic medium as illustrated in Figure 1, where the coordinates $\alpha, \beta$ are parameters (called thresholds) associated with the magnetic domains. Here, the center of each dark gray unit box represents a magnetic moment pointing “up” and the center of each light gray unit box corresponds to a magnetic moment pointing “down”. The state of the system is represented by the staircase line $L$ separating the dark and light gray areas. This line of length $N$ consists of horizontal and vertical unit segments and it connects the point $(0, N)$ with a point $(i, i)$ on the diagonal $\alpha = \beta$. Starting at the upper left end $(0, N)$, the line $L$ can be encoded by a unique $N$-tuple of 0’s and 1’s, corresponding to vertical unit segments and horizontal unit segments, respectively.

The input $i \in \{0, 1, \ldots, N\}$ of the discrete Preisach model (which provides the coordinates $(i, i)$ of the lower right end of the staircase line $L$) describes the influence of the external field. The input $i$ can change by $\pm 1$ at each time step, after which the lower right end of $L$ moves accordingly. Further, when $i$ is increased, all the horizontal segments are passed from left to right, and when $i$ is decreased, all the vertical segments are
Figure 1: The state of the Preisach model is associated with the black polyline $L$ separating the dark gray and light gray regions. Here $N = 5$, and $L$ is encoded by a 5-tuple of 0s and 1s. The input changes from the value $i_1 = 3$ on panel (a) to the value $i_2 = i_1 - 1 = 2$ on panel (b) and back to the value $i_3 = i_2 + 1 = 2$ on panel (c). The state of the Preisach model on these panels is $(1,0,1,0)$, $(1,0,1,0,0)$, and $(1,0,1,0,1)$, respectively, where 0’s correspond to vertical and 1’s correspond to horizontal unit segments.

passed in the downward direction. In other words, the rules of updating the state require that the smallest possible number of unit boxes change their color, see Figure 1.

A last-in-first-out (LIFO) model can also be associated with these rules. Consider a one dimensional storage with $N$ spaces. A space is labeled by 0 if empty and by 1 if occupied by a box (element). A stored element takes exactly one space, thus the state of the storage is an $N$-tuple of 0’s and 1’s. Elements can be added to, or removed from, the storage through the entrance/exit at the right end. Suppose the following rules apply: when an element is added to the stock, it is placed to the free space nearest the entrance; similarly, the element nearest the entrance is removed when the “removal” operation is applied. The order in which elements come off the storage according to this protocol can be characterized as last-in-first-out because we always remove the most recently added element. The transition induced by an increase of the input corresponds to adding an element to the storage, while the transition induced by a decrease of the input corresponds to removing an element from the storage.

In the above models, variations of the input induce transitions between the states, which are equivalent to transitions described by an underlying graph $\Gamma$. The graph $\Gamma$ is self-similar with interesting properties. The goal of this paper is to describe and formally prove some of these properties (for example, we show that the adjacency matrix of $\Gamma$ has a self-similar eigenvalue distribution), offering an example of analytical treatment of a self-similar graph to the wider scientific community. Self-similar matrices and graphs appear in distinct areas. Kostadinov [27] expresses the free energy of a one-dimensional Ising model with random couplings in terms of the maximal eigenvalue of a self-similar matrix. He also relates the calculation for the spectra of molecular type of systems in the tight binding approximation to the eigenvalues and eigenvectors of self-similar Hermitian matrices. Stosic et al. [42] find the analytical expression for the residual entropy of the two-dimensional Ising model with nearest-neighbor antiferromagnetic coupling in terms of the fractal Fibonacci matrix. Katsanos and Evangelou [24] study the level-spacing distribution of a fractal Fibonacci matrix to address questions related to Anderson localization and quantum chaos. Hsu et al. introduced Fibonacci cubes as a new class of self-similar graphs [23]. Ferrand [17] considers a self-similar matrix related to the Thue-Morse sequence.

Many real networks can also be categorized as self-similar [41]. Generating realistic networks is a subject of intense study. Barriere et al. [5] investigate the generalized hierarchical product of graphs. Leskovec et al. [31] propose a network generation model based on the Kronecker product. Komjáthy and Simon [25] introduce deterministic scale-free networks derived from a graph directed self-similar fractal. Many books have been written recently on complex networks and graphs, see, for example [?].

\[\text{Note that this protocol is different from a similar protocol known as stack data type in computer science.}\]
The eigenvalues of the adjacency matrix are related to important properties of the graph (for a good introduction to spectral graph theory, see, for example, [12]). The largest eigenvalue of the adjacency matrix of a graph plays a key role in several respects, including in synchronization of oscillators, percolation on directed networks and linear stability of equilibria of coupled systems.

The paper is structured as follows. In Section 2 we introduce the necessary notation and define the graphs studied. Section 3 describes the properties of the adjacency matrix \( A_n \). In Section 4 we look at the eigenvalue distribution of the (non-symmetric) adjacency matrix. Section 5 provides explicit description of the eigenvectors of \( A_n \). Section 6 concludes our work.

2 Preliminaries

Consider the set \( \{0,1\}^N \) of all \( N \)-tuples \( x = (x_0, \ldots, x_{N-1}) \) with \( x_i \in \{0,1\} \) (this is the Hamming space of all \( 2^N \) binary strings of length \( N \)). Let us associate a vertex of a directed graph \( \Gamma \) with every \( N \)-tuple. Define the set of directed edges of \( \Gamma \) according to the rules

\[(H_1) \text{ The vertex } 0 = (0,0,\ldots,0,0,0) \text{ only has one direct successor } (0,0,\ldots,0,0,1);\]

\[(H_2) \text{ The vertex } 1 = (1,1,\ldots,1,1,1) \text{ only has one direct successor } (1,1,\ldots,1,1,0);\]

\[(H_3) \text{ Every other vertex } x \text{ has two direct successors } y \text{ and } z \text{ defined as follows:}\]

\[
x_i = 0; \quad x_{i+1} = \cdots = x_{N-1} = 1 \implies y_i = 1, \quad y_k = x_k \text{ for } k \neq i; \]

\[
x_j = 1; \quad x_{j+1} = \cdots = x_{N-1} = 0 \implies z_j = 0, \quad z_k = x_k \text{ for } k \neq j.\]

In other words, \( N \)-tuple \( y \) is obtained by replacing the rightmost 0 in the \( N \)-tuple \( x = (x_0, \ldots, x_{N-1}) \) with 1 and \( z \) is obtained by replacing the rightmost 1 in the \( N \)-tuple \( x = (x_0, \ldots, x_{N-1}) \) with 0. For example, the vertex \((1,0,1,0,1)\) is connected to the vertices \((1,0,1,0,0)\) and \((1,0,1,1,1)\).

The adjacency matrix \( A_N \) (of order \( 2^N \)) of graph \( \Gamma \) can recursively be defined by the relations

\[
A_0 = (0), \quad A_{k+1} = \begin{pmatrix} A_k & J_k \\ J_k' & A_k \end{pmatrix},
\]

where \( J_k \) and \( J_k' \) are diagonal matrices of order \( 2^k \) defined by

\[
J_k = \text{diag} \{1,0,0,\ldots,0,0\}, \quad J_k' = \text{diag} \{0,0,\ldots,0,0,1\}. \tag{2}
\]

For example,

\[
A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.
\]

Due to its recursive definition, \( A_n \) is a block-hierarchical matrix, i.e. a matrix where hierarchically nested growing blocks are placed along the diagonal, where each (sub-)block is again a block-hierarchical matrix itself [10]. Figure 2 illustrates the graphs corresponding to \( A_1, A_2, A_3, \) and \( A_9 \), while Figure 3 demonstrates the connection structure of the adjacency matrices \( A_6 \) and \( A_7 \) (the 1’s are shown as black dots).

**Remark 1** Matrix \( \frac{1}{2} A_N \) can be viewed as the transition matrix of a discrete time stochastic process naturally associated with the graph \( \Gamma \). This process transits with equal probability \( 1/2 \) from any vertex (state) \( x \neq 0,1 \) to either of its two direct successors along the directed edges. If the process is either in state \( 0 \) or \( 1 \), then it transits to the direct successor of this state with probability \( 1/2 \) or terminates with probability \( 1/2 \). Since the spectral radius \( \rho(\frac{1}{2} A_N) \) is less than 1, this process almost surely terminates in finite time. However, the leading eigenvalue of the matrix \( \frac{1}{2} A_N \) tends to 1 as \( N \to \infty \). Hence, the mean termination time tends to infinity in this limit.

Let us denote by \( \Gamma' \) the graph obtained from \( \Gamma \) by adding self-loops at vertices \( 0 \) and \( 1 \). This is achieved by replacing rules \((H_1), (H_2)\) with
Figure 2: Graphs corresponding to adjacency matrices $A_1$, $A_2$, $A_3$, and $A_9$.

$(H'_1)$ The vertex $0 = (0, 0, \ldots, 0, 0)$ has two direct successors, $(0, 0, \ldots, 0, 1)$ and $0$;

$(H'_2)$ The vertex $1 = (1, 1, \ldots, 1, 1)$ has two direct successors, $(1, 1, \ldots, 1, 0)$ and $1$.

Let $A'_N$ denote the adjacency matrix of the graph $\Gamma'$. Then, $\frac{1}{2}A'_N$ is the transition matrix of the stochastic process associated with the graph $\Gamma'$, which transits from any vertex to either of its two direct successors with equal probability $1/2$. This is a Markov chain because $A'_N$ is a stochastic matrix. The unique stationary probability distribution for this Markov chain has been described in [1], where the Preisach model with random input was considered. Properties of the stochastic output of the Preisach model under various random inputs have been characterized in [15, 26, 30, 36, 37, 39] (in the context of this work, the output is the area (measure) of the dark gray region in Fig. 1).

3 Properties of $A_n$

Spectral graph theory (see, for example, [12]) relates the eigenvalues of the adjacency matrix to other properties of the graph. We are thus interested in the spectrum

$$\text{sp} \left( \frac{1}{2}A_N \right) = \{ \lambda \in \mathbb{C} : \det \left( \frac{1}{2}A_N - \lambda I \right) = 0 \},$$

and eigenvalue distribution

$$F_N(x) = \frac{\# \{ \lambda \in \text{sp} \left( \frac{1}{2}A_N \right) : \lambda < x \}}{2^N},$$

\footnote{One can also consider continuous time Markov chains associated with the graphs $\Gamma$ and $\Gamma'$ with the transition rate matrices $A_N - I_{2N}$ and $A'_N - I_{2N}$, respectively, where $I_n$ denotes the identity matrix of order $n$.}
of the matrix \( \frac{1}{2} A_N \) and the \( N \to \infty \) limit of these objects. Here \( \#\Omega \) denotes the cardinality of the set \( \Omega \). Figure 4 shows the Devil’s staircase eigenvalue distribution for \( A_{12} \).

Xiuqing and Youyan [46] and Fu et al. [18] studied one-dimensional Fibonacci-class quasilattices and showed that for these types of lattices the energy spectrum exhibits a staircase behavior. He et al. [20, 21] studied a family of trees (the interior nodes have degree \( k \) and the boundary nodes have degree 1) and found that the eigenvalue distributions approach a piecewise constant “Cantor function.” Spectral properties of structured matrices have been extensively studied. Banded Toeplitz matrices are the topic of the book by Bottcher and Grudsky [9]. A classical result of Schmidt-Spitzer [38] and Hirschman [22] is that the eigenvalues accumulate on a special curve in the complex plane and the normalized eigenvalue counting measure converges weakly to a measure on this curve as \( N \to \infty \). Duits and Kuijlaars [16] study the limiting eigenvalue distribution of \( N \times N \) banded Toeplitz matrices as \( N \to \infty \), and characterize the limiting measure in terms of an equilibrium problem. Even though the spectrum of the limiting operator need not mimic that of the finite-dimensional operator, pseudospectra and numerical ranges behave nicely [9]. The pseudospectra of \( A_5 \) and \( A_8 \) are shown in Figure 5.

Interestingly, Devil’s staircases have been associated with the one-dimensional Ising model with antiferromagnetic interactions [3]. Exact eigenvalues and eigenvectors of different one-dimensional quantum many-body models starting with the Hamiltonian of the antiferromagnetic Heisenberg model have been obtained using the Bethe ansatz [6, 8].
3.1 Spectrum of the adjacency matrix

Denote by $U_k(\lambda)$ the Chebyshev polynomial of the second kind of degree $k$ \cite{32}. For example, they can be defined by the recursive relations

$$U_0(\lambda) = 1; \quad U_1(\lambda) = 2\lambda; \quad U_{k+1}(\lambda) = 2\lambda U_k(\lambda) - U_{k-1}(\lambda),$$

or by the explicit formula

$$U_k(\cos(\vartheta)) = \frac{\sin((k+1)\vartheta)}{\sin \vartheta}.$$  \hspace{1cm} (5)

The zeros of $U_k(\lambda)$ are given by

$$\lambda_i = \cos \left( \frac{\pi}{k+1} \left( i + \frac{1}{2} \right) \right), \quad i = 0, \ldots, k-1.$$  \hspace{1cm} (6)

The main result of this paper is the following theorem.

\textbf{Theorem 1} The characteristic polynomial of the adjacency matrix $A_N$ of the graph $\Gamma$ defined by $(H_1), (H_2), (H_3)$ equals

$$\chi_N(\lambda) = U_{N+1}(-\lambda/2) \prod_{i=0}^{N-1} (U_i(-\lambda/2))^{2^{N-i-1}}, \quad N \geq 1.$$  \hspace{1cm} (7)

Formulas (6) and (7) explicitly define the spectrum of the matrix $A_N$. 
Theorem 2  The characteristic polynomial of the adjacency matrix $A'_N$ of the graph $\Gamma'$ defined by $(H'_1)$, $(H'_2)$, $(H_3)$ equals

$$
\chi'_N(\lambda) = (2 - \lambda)U_N(-\lambda/2) \prod_{i=0}^{N-1} (U_i(-\lambda/2))^{2N-i-1}, \quad N \geq 1. \quad (8)
$$

Comparing formulas (7) and (8), one can see that adding the two self-loops to the graph $\Gamma$ at vertices 0 and 1 changes exactly $N + 1$ eigenvalues in the spectrum of the adjacency matrix; namely, the roots of the highest degree Chebyshev polynomial $U_{N+1}$ are replaced with the roots of the Chebyshev polynomial $U_N$ and the leading eigenvalue 1.

The proof of Theorem 1 is presented in the next section. The proof of Theorem 2 is similar and is omitted.

The appearance of Chebyshev polynomials in the characteristic function of $A_n$ is not entirely surprising: characteristic functions of tri-diagonal and other structured matrices involve Chebyshev polynomials. Tri-diagonal matrices are naturally associated with a one-dimensional walk on a path graph. Our graph has vertices of out-degree of 2 and thus corresponds to a one-dimensional walk with longer-range jumps.

3.2 Proof of Theorem 1
Consider the matrix $T_k = A_k - \lambda I_2$, where $I_n$ denotes the identity matrix of order $n$. With this notation, the characteristic polynomial $\chi_k = \chi_k(\lambda)$ of $A_k$ equals

$$
\chi_k = \det T_k.
$$

Clearly, matrices $T_k$ satisfy the recursive relationships similar to (1):

$$
T_0 = \{-\lambda\}, \quad T_{k+1} = \left( \begin{array}{c}
T_k \\
J_k
\end{array} \right).
$$

Denote by $Q_k$ the submatrix of $T_k$ formed by deleting the upper row and the right column, and set

$$
\phi_k = \det Q_k.
$$

Denote by $P_k$ the submatrix of $T_k$ formed by deleting the lower row and the right column, and set

$$
\psi_k = \det P_k.
$$

Finally, for any square matrix $B$ of order $n$, denote by $B'$ the matrix obtained by rotating $B$ by $180^\circ$. That is, the elements of $B'$ and $B$ are related by

$$
b'_{i,j} = b_{n+1-i,n+1-j}, \quad i, j = 1, \ldots, n.
$$

3.2.1 Auxiliary lemmas
In this section, we prove a few auxiliary statements.

Lemma 1  Each matrix $T_k$ satisfies $T_k = T'_k$.

Proof. The statement follows from recursive relations (9) by induction in $k$. \qed

Lemma 2  The following recursive relationship holds:

$$
\chi_{k+1} = (\chi_k)^2 - (\phi_k)^2. \quad (10)
$$

Proof. Considering the expression

$$
\det T_{k+1} = \sum_{(i_1 i_2 \ldots i_n) \in S_n} \operatorname{sgn} (i_1 i_2 \ldots i_n) t_{1,i_1} t_{2,i_2} \cdots t_{n,i_n}
$$
for the determinant $\chi_{k+1} = \det T_{k+1}$, where $n = 2^{k+1}$, and using (9), one can see that each term in the product $t_{1,i_1} \cdot t_{2,i_2} \cdots t_{n,i_n}$ either contains elements from the blocks $T_k$ of the matrix $T_{k+1}$ only or both elements 1 from the blocks $J_k = \text{diag}\{1,0,\ldots,0\}$ and $J'_k = \text{diag}\{0,\ldots,0,1\}$. This implies that

$$\det T_{k+1} = (\det T_k)^2 - \det Q_k \det S_k,$$

(11)

where the submatrix $S_k$ of $T_k$ is formed by deleting the lower row and the left column. Lemma 1 implies $S_k = Q'_k$ and since $\det B' = \det B$ for any square matrix $B$, equation (11) yields $\det T_{k+1} = (\det T_k)^2 - (\det Q_k)^2$, which is equivalent to (10).

□

Lemma 3 The following relation holds:

$$\phi_{k+1} = -\phi_k \psi_k.$$  

(12)

Proof. Since the matrix $Q_{k+1}$ is obtained from $T_{k+1}$ by deleting the upper row and the right column, formula (9) implies that $Q_{k+1}$ is an upper block triangular matrix of the form

$$Q_{k+1} = \begin{pmatrix} Q_k & B \\ O & C \end{pmatrix},$$

where $O$ is a zero matrix and the square matrix $C$ is defined by

$$C = \begin{pmatrix} 0 & P_k \\ 1 & \alpha \end{pmatrix},$$

where $0$ is a zero column, and $\alpha$ is a row. Hence, $\det Q_{k+1} = \det Q_k \det C$. Furthermore, using the Laplace expansion along the first column, $\det C = -\det P_k$, hence $\det Q_{k+1} = -\det Q_k \det P_k$, which is equivalent to (12).

□

Lemma 4 The following relation holds:

$$\psi_{k+1} = \chi_k \psi_k.$$  

(13)

Proof. Since the matrix $P_{k+1}$ is obtained from $T_{k+1}$ by deleting the lower row and the right column, formula (9) implies that $P_{k+1}$ is an upper block triangular matrix of the form

$$P_{k+1} = \begin{pmatrix} T_k & B \\ O & P_k \end{pmatrix},$$

hence $\det P_{k+1} = \det T_k \det P_k$, which is equivalent to (13).

□

Lemma 5 Chebyshev polynomials satisfy the relation

$$(U_{k+1})^2 - U_{k+2}U_k = 1.$$  

Proof. Applying the recursive relation (5) gives

$$(U_{k+1})^2 - U_{k+2}U_k = (U_{k+1})^2 - (2\lambda U_{k+1} - U_k)U_k = (2\lambda U_k - U_{k-1})^2 - 2\lambda(2\lambda U_k - U_{k-1})U_k + (U_k)^2 = (U_k)^2 - U_{k+1}U_{k-1}. $$

Hence, the statement follows from $(U_1)^2 - U_2U_0 = 1.$

□

3.2.2 Proof of the theorem

Since $\psi_1 = \chi_0 = -\lambda$, equation (13) implies that

$$\psi_k = \prod_{i=0}^{k-1} \chi_i, \quad k \geq 1.$$
Combining this with (12) and taking into account that \( \phi_1 = 1 \), one obtains

\[
\phi_k = (-1)^{k-1} \prod_{i=0}^{k-2} (x_i)^{k-1-i}, \quad k \geq 2.
\]

Therefore, formula (10) implies the recursive relationship

\[
\chi_{k+1} = (\chi_k)^2 - \prod_{i=0}^{k-2} (x_i)^{2(k-1-i)}, \quad k \geq 2.
\]  \hspace{1cm} (14)

Note that by direct calculation

\[
\chi_0 = -\lambda; \quad \chi_1 = \lambda^2 - 1; \quad \chi_2 = \lambda^4 - 2\lambda^2.
\]  \hspace{1cm} (15)

Since formulas (14), (15) uniquely define the sequence of characteristic polynomials \( \chi_k \), it remains to show that expressions (7) satisfy these formulas. Indeed, for \( N = 1, 2, 3 \), formula (7) gives the expressions

\[
\chi_1(\lambda) = U_2, \quad \chi_2(\lambda) = U_1U_3, \quad \chi_3(\lambda) = U_1^2U_2U_4,
\]

which are compatible with (14), (15); here and henceforth, we omit the argument of Chebyshev polynomials for brevity, i.e. \( U_i \) stands for \( U_i(-\lambda/2) \).

For larger \( N \), rewriting formula (14) equivalently as

\[
(\chi_k)^2 - \chi_{k+1} = \chi_0^{2(k-1)} \prod_{j=1}^{k-2} (x_j)^{2(k-1-j)}
\]

and substituting expressions \( \chi_0 = U_1 \) and (7) into this equation, we obtain

\[
\left( U_{k+1} \prod_{i=0}^{k-1} (U_i)^{2^{k-i-1}} \right)^2 - U_{k+2} \prod_{i=0}^{k} (U_i)^{2^{k-i}} = (U_1)^{2(k-1)} \prod_{j=1}^{k-2} \left( U_{j+1} \prod_{i=0}^{j-1} (U_i)^{2^{j-i-1}} \right)^{2(k-1-j)},
\]

where \( k \geq 3 \). Upon rearrangement of the left hand side, this can be written as

\[
((U_{k+1})^2 - U_{k+2}U_k) \prod_{i=0}^{k-1} (U_i)^{2^{k-i}} = (U_1)^{2(k-1)} \prod_{j=1}^{k-2} \left( U_{j+1} \prod_{i=0}^{j-1} (U_i)^{2^{j-i-1}} \right)^{2(k-1-j)},
\]

which, by Lemma 5, is equivalent to

\[
\prod_{i=0}^{k-1} (U_i)^{2^{k-i}} = (U_1)^{2(k-1)} \prod_{j=1}^{k-2} \left( U_{j+1} \prod_{i=0}^{j-1} (U_i)^{2^{j-i-1}} \right)^{2(k-1-j)}.
\]  \hspace{1cm} (16)

The right hand side of this equation is the product of powers \( (U_i)^{m_i} \) of Chebyshev polynomials with \( i = 1, \ldots, k-1 \) (note that \( U_0 = 1 \)). Counting the number of times each factor \( U_i \) enters the right hand side of equation (16), one obtains

\[
m_i = 2(k-i) + \sum_{j=1}^{k-i-2} 2^{k-i-j-1}j, \quad i = 1, \ldots, k-3; \quad m_{k-2} = 4; \quad m_{k-1} = 2.
\]

It is easy to see by induction that

\[
2n + \sum_{j=1}^{n-2} 2^{n-j-1}j = 2^n
\]

for all \( n \geq 3 \); hence, \( m_i = 2^{k-i} \) for all \( i = 1, \ldots, k-1 \), and therefore, (16) is an identity. This establishes that equations (7) satisfy recursive relations (14), (15) and completes the proof of the theorem.
4 Eigenvalue distribution of the adjacency matrix

In this section, we consider the distribution function (empirical spectral distribution, density of states)

\[ F_N(x) = \# \{ \lambda \in \text{sp}(\frac{1}{2} A_N) : \lambda < x \}, \]  

(17)

of the eigenvalues of the matrix \( \frac{1}{2} A_N \) and its \( N \to \infty \) limit (limiting spectral distribution). By definition, \( F_N \) is an increasing piecewise constant left-continuous function with a finite number of jumps. Since the spectrum of the matrix \( \frac{1}{2} A_N \) belongs to the interval \((-1,1)\), it also follows that \( F_N(x) = 0 \) for \( x \leq -1 \) and \( F_N(x) = 1 \) for \( x \geq 1 \).

The following Devil’s staircase function \([4,7,14]\) has received substantial attention in the literature:

\[ f(x) = \sum_{k=1}^{\infty} \frac{\lfloor kx \rfloor}{2^k}, \quad 0 \leq x < 1, \]  

(18)

where \( \lfloor \cdot \rfloor \) denotes the floor (integer value) function. We consider the extension of \( f \) to the whole axis according to the formulas

\[ f(x) = 0 \quad \text{for} \quad x < 0; \quad f(x) = 1 \quad \text{for} \quad x \geq 1. \]  

(19)

This extension is an increasing right continuous function. At every rational point \( x_i = r/q \in (0,1) \), where \( r,q > 0 \) are coprime integers, the function \( f \) has a jump

\[ \Delta f(x_i) = f(x_i + 0) - f(x_i - 0) = \frac{1}{2q-1}, \quad x_i = \frac{r}{q} \in (0,1), \]

and \( f \) is continuous at every irrational point. As a matter of fact, equation (18) is equivalent to

\[ f(x) = \begin{cases} \sum_{p=1}^{\infty} \frac{1}{2^{\lfloor \frac{p}{x} \rfloor}} & \text{for irrational } x, \\ \sum_{p=1}^{\infty} \frac{1}{2^{\lfloor \frac{p}{x} \rfloor}} + \frac{1}{2q-1} & \text{for rational } x = \frac{r}{q}. \end{cases} \]  

(20)

Further, the total sum of all the jumps of \( f \) satisfies

\[ \sum_i \Delta f(x_i) = \sum_{q=2}^{\infty} \frac{\varphi(q)}{2q-1} = 1, \]

where \( \varphi \) is Euler’s totient function. Hence, this sum equals the total variation \( f(1) - f(0) = 1 \) of \( f \), and therefore \( f \) is a jump function \([2]\), i.e.

\[ f(x) = \sum_{x_i < x} \Delta f(x_i), \quad x \in \mathbb{R}, \]

with the sum over all the rational points \( x_i \in (0, x) \).

**Theorem 3** The distribution function of the spectrum of the matrix \( A_N \) satisfies the limit relationship

\[ \lim_{N \to \infty} F_N(x) = 1 - f \left( \frac{1}{\pi} \arccos x \right), \quad -1 \leq x \leq 1, \]  

(21)

where \( f \) is defined by (18).

This theorem is proved in the next section. From formulas \([7], [8]\), it follows that the distribution function of the spectrum of the matrix \( \frac{1}{2} A_N' \) converges to the same limit as \( N \to \infty \). It should be noted that the algebraic and geometric multiplicities of eigenvalues are different.
4.1 Proof of Theorem 3

Let us order the zeros of the polynomial

\[ \frac{\chi_n(\lambda)}{U_{n+1}(\lambda/2)} = \prod_{q=1}^{n} (U_{q-1}(\lambda/2))^{2^{n-q}} \]  

(cf. (7)) in a \((n-1) \times n\) matrix

\[
D = \begin{pmatrix}
0 & 2^{n-2} & 2^{n-3} & 2^{n-4} & \cdots & 2^0 \\
0 & 0 & 2^{n-3} & 2^{n-4} & \cdots & 2^0 \\
0 & 0 & 0 & 2^{n-4} & \cdots & 2^0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 2^0
\end{pmatrix},
\]

where at position \((p,q)\) with \(1 \leq p \leq n-1, 1 \leq q \leq n\), we put down the entry \(D_{p,q} = 2^{n-q}\) which corresponds to the multiplicity of the factor \(U_q - 1\) in (22). For every \(x\) and \(n\) we define the set of pairs

\[ A_n(x) = \{(p,q) : p/q \leq x \land 1 < q \leq n\} \]

and the function

\[ f_n(x) = \sum_{(p,q) \in A_n(x)} D_{p,q} = \sum_{(p,q) \in A_n(x)} \frac{1}{2^q}. \]

Equation (6) implies that this function and the distribution function (17) are related by

\[ F_n(x) = 1 - f_n\left(\frac{1}{\pi} \arccos x\right). \]  

(23)

Now note that

\[ A_n(x) = \bigcup_{p=1}^{n-1} B_{n,p}(x) \]

with

\[ B_{n,p}(x) = \{(p,q) : p/x \leq q \leq n\} \]

and therefore,

\[ \sum_{(p,q) \in B_{n,p}(x)} \frac{1}{2^q} = \frac{1}{2^{\left\lfloor \frac{p}{x} \right\rfloor}} + \frac{1}{2^{\left\lfloor \frac{p}{x} \right\rfloor + 1}} + \ldots + \frac{1}{2^n} = 2^{1-\left\lfloor \frac{p}{x} \right\rfloor} - 2^{-n}. \]

Summing this over \(p\) gives

\[ f_n(x) = \sum_{p=1}^{n-1} \sum_{(p,q) \in B_{n,p}(x)} \frac{1}{2^q} = \sum_{p=1}^{n-1} \left[ 2^{1-\left\lfloor \frac{p}{x} \right\rfloor} - 2^{-n} \right] = \sum_{p=1}^{n-1} \frac{1}{2^{\left\lfloor \frac{p}{x} \right\rfloor - 1}} \right] - (n-1) 2^{-n}. \]

In the limit \(n \to \infty\) the last term vanishes and we obtain

\[ \lim_{n \to \infty} f_n(x) = \sum_{p=1}^{\infty} \frac{1}{2^{\left\lfloor \frac{p}{x} \right\rfloor - 1}}, \]

(24)

where \([\cdot]\) is the ceiling function. Finally, note that \([\frac{p}{x}] - 1 = \left\lfloor \frac{p}{x} \right\rfloor\) for all \(p\) if \(x\) is irrational, while if \(x = r/q\) is a rational number then \([\frac{p}{x}] - 1 = \left\lfloor \frac{mr}{r} \right\rfloor - 1 = \left\lfloor \frac{p}{x} \right\rfloor - 1\) whenever \(p = mr\) is a multiple of \(r\). We thus see that the limit (24) coincides with the function (20) and hence (23) implies (21), which completes the proof.
5 Eigenvectors

Eigenvectors and eigenspaces of graphs are also important [13, 43]. For example they are used to count walks in a graph and to relate the symmetry of a graph to its spectrum. The principal eigenvector of the adjacency matrix provides vertex centrality information, while the second eigenvector can be used to partition the graph into clusters. The eigenvectors are related to the graph’s automorphic structure.

Let us now study the eigenvectors of $\frac{1}{2}A_N$. In order to do this, let us first make the notation for vertices more efficient. In the following, let us identify a vertex with the corresponding string of 0s and 1s, for example,

$$x = (1, 1, 0, 0, 1) = (11001).$$

Strings are concatenated by simply writing one after the other. We will use the notation $1^j$ to denote a string with $j$ ones and $0^j$ for a string with $j$ zeros. For example, $(11001) = (1^20^21)$. For consistency, it is convenient to agree that $0^0$ and $1^0$ is an empty string. Also for convenience, let us introduce the symmetry operation $S$, which replaces every 0 in a string with a 1 and vice versa, for example, $S(11001) = (00110)$. Obviously, $S$ is an involution.

An arbitrary binary string of length $j$ will be denoted by $q^j$, and the set of all strings of length $j$ will be denoted by $W_j$. In particular, the matrix $\frac{1}{2}A_N$ acts in the $2^N$-dimensional vector space $V$ spanned by the vertices $(q^N) \in W_N$.

Consider an expansion of an eigenvector $v \in V$ of $\frac{1}{2}A_N$ along the basis $W_N$:

$$v = \sum_{(q^N) \in W_N} c(q^N) (q^N), \quad \frac{1}{2}A_N v = \lambda v. \quad (25)$$

According to Theorem 1, $\lambda$ is a root of a Chebyshev polynomial $U_{\ell}$ with $\ell = 1, \ldots, N - 1$ or $\ell = N + 1$. We would like to determine the $2^N$ components $c(q^N)$ of an eigenvector $v$ (up to a scaling factor).

**Theorem 4** Suppose that $\lambda$ is a root of a Chebyshev polynomial $U_{\ell+1}$ with $\ell \leq N - 2$ and is not a root of the Chebyshev polynomials $U_1, \ldots, U_{\ell}$. Then, for every $(q^{N-\ell-2}) \in W_{N-\ell-2}$, the following relations define an eigenvector of the matrix $\frac{1}{2}A_N$:

(i) $c(q^{N-\ell-2,00p^\ell}) = 0$ for all $(p^\ell) \in W_{\ell}$;

(ii) $c(q^{N-\ell-2,10p^\ell}) = 1$;

(iii) For every $p^\ell = 0^{j_k+1}1^{j_k}0^{j_{k-1}1^{j_{k-2}} \cdots 0^{j_2}1^{j_1}}$ with $j_1 + \cdots + j_{k+1} = \ell$ and $j_1, j_{k+1} \geq 0, j_2, \ldots, j_k \geq 1$,

$$c(q^{N-\ell-210p^\ell}) \prod_{i=1}^{k} U_{s_i}(\lambda) = 1, \quad (26)$$

where $s_i = j_1 + \cdots + j_i$;

(iv) $c(q^{N-\ell-211p^\ell}) = 0$ and $c(q^{N-\ell-201p^\ell}) = -U_{\ell}(\lambda)c(q^{N-\ell-210S(p^\ell)}) \quad (27)$

for all $(p^\ell) \in W_{\ell}$.

Similarly, if $\lambda$ is a root of the Chebyshev polynomial $U_{N+1}$ and is not a root of the Chebyshev polynomials $U_1, \ldots, U_{N-1}$, then the following relations define an eigenvector of the matrix $\frac{1}{2}A_N$ with the eigenvalue $\lambda$:

(j) $c(0^N) = 1$;

(jj) For every $q^N = 0^{j_k+1}1^{j_k}0^{j_{k-1}1^{j_{k-2}} \cdots 0^{j_2}1^{j_1}}$ with $j_1 + \cdots + j_{k+1} = N$ and $j_1 \geq 0, j_2, \ldots, j_k, j_{k+1} \geq 1$,

$$c(q^N) \prod_{i=1}^{k} U_{s_i}(\lambda) = 1, \quad (28)$$

where $s_i = j_1 + \cdots + j_i$;
(iii) \( c_{S(q^{N})} = U_N(\lambda)c_{(q^{N})} \) for all \( (q^{N}) \in W_N \).

Note that the identity \( U_k^2 - 1 = U_{k-1}U_{k+1} \) implies \( U_k^2(\lambda) = 1 \) under the conditions of Theorem 4 because \( U_{\ell+1}(\lambda) = 0 \). Since \( (q^{N-\ell-2}) \in W_{N-\ell-2} \) is arbitrary, the geometric multiplicity of such an eigenvalue is \( 2^{N-\ell-2} \).

As an example, if \( \lambda = 1/\sqrt{2} \), then \( U_3(\lambda) = 0 \) and \( U_1(\lambda) = \sqrt{2}.U_2(\lambda) = 1 \). According to Theorem 4, \( 2^{N-4} \) linearly independent eigenvectors corresponding to the eigenvalue \( \lambda = 1/\sqrt{2} \) can be labeled by the strings \( (q^{N-4}) \in W_{N-4} \), and the components of an eigenvector are defined by

\[
\begin{align*}
    c_{(q^{N-4}0000)} &= c_{(q^{N-4}0001)} = c_{(q^{N-4}0010)} = c_{(q^{N-4}0011)} = 0; \\
    c_{(q^{N-4}1111)} &= c_{(q^{N-4}1100)} = c_{(q^{N-4}1010)} = c_{(q^{N-4}1001)} = 0; \\
    c_{(q^{N-4}1000)} &= 1, \quad c_{(q^{N-4}1001)} = \frac{1}{u_1}, \quad c_{(q^{N-4}1010)} = \frac{1}{u_2}, \quad c_{(q^{N-4}1011)} = \frac{1}{u_1u_2}; \\
    c_{(q^{N-4}1000)} &= -u_2, \quad c_{(q^{N-4}1001)} = -\frac{u_2}{u_1}, \quad c_{(q^{N-4}1011)} = -1, \quad c_{(q^{N-4}1010)} = -\frac{1}{u_1},
\end{align*}
\]

where \( u_1 = U_1(\lambda) = \sqrt{2}, u_2 = U_2(\lambda) = 1 \).

As an illustration of formulas \((26), (27)\) for a larger \( m \), if, for example, \( m = 10 \) and \( (p^{m}) = 0100100110 \), then

\[
c_{(q^{N-m-210p^{m}})} = \frac{1}{u_{9}u_{8}u_{6}u_{5}u_{3}u_{1}}, \quad c_{(q^{N-m-201p^{m}})} = -\frac{u_{10}}{u_{9}u_{8}u_{6}u_{5}u_{3}u_{1}},
\]

where \( u_k = U_k(\lambda) \) and, in particular, \( u_{10} = 1 \).

Now, let us consider the matrix \( \frac{1}{2}A_N^\prime \).

Theorem 5 The matrices \( \frac{1}{2}A_N \) and \( \frac{1}{2}A_N^\prime \) have the same eigenvectors (defined by (i) – (iv)) for each eigenvalue \( \lambda \), which is a root of a Chebyshev polynomial \( U_m \) with \( m \leq N-1 \).

If \( \lambda \) is a root of the Chebyshev polynomial \( U_N \) and is not a root of the Chebyshev polynomials \( U_1, \ldots, U_{N-1} \), then the eigenvector of the matrix \( \frac{1}{2}A_N^\prime \) corresponding to the eigenvalue \( \lambda \) is defined by relations (j), (jj) of Theorem 4 and the equality

\[
c_{S(q^{N})} = -U_{N-1}(\lambda)c_{(q^{N})}, \quad (q^{N}) \in W_N.
\]

Finally, the components of the eigenvector corresponding to the eigenvalue 1 are defined by the relations

\[
c_{(q^{N})} \prod_{i=1}^{k} \left( 1 + \sum_{m=1}^{i} j_i \right) = 1, \quad c_{S(q^{N})} = c_{(q^{N})},
\]

for every \( q^{N} = 0^{j_k+1}1^{j_k}0^{j_{k-1}+1}1^{j_{k-2}}\cdots 0^{j_2}1^{j_1} \) with \( j_1 + \cdots + j_{k+1} = N \) and \( j_1 \geq 0, j_2, \ldots, j_k, j_{k+1} \geq 1 \).

Theorem 5 is proved in the next section. The proof of Theorem 5 follows the same line and is omitted.

5.1 Proof of Theorem 4

We again use the notation \( u_j = U_j(\lambda) \). The proof is based on the following lemma.

Lemma 6 For every \( m \leq N-2 \), every \( q^{N-m-2} \in W_{N-m-2} \) and every \( 1 \leq j \leq m \), the components of an eigenvector of \( \frac{1}{2}A_N \) with an eigenvalue \( \lambda \) satisfy

\[
\begin{align*}
    c_{(q^{N-m-201m+1-j}0^{\nu})} + u_{j-1} \sum_{i=j}^{m} c_{(q^{N-m-201m-i}10^{\nu})} &= u_{j}c_{(q^{N-m-201m+1})}, \quad (29) \\
    c_{(q^{N-m-20m+2})} &= u_{m+1}c_{(q^{N-m-201m+1})}, \quad (30) \\
    c_{(q^{N-m-210m+1-j}1^{\nu})} + u_{j-1} \sum_{i=j}^{m} c_{(q^{N-m-210m-i}1^{\nu})} &= u_{j}c_{(q^{N-m-210m+1})}, \quad (31) \\
    c_{(q^{N-m-21m+2})} &= u_{m+1}c_{(q^{N-m-210m+1})} \quad (32)
\end{align*}
\]
provided that \( u_j \neq 0 \) for \( j = 1, \ldots, m-1 \). Further, if \( u_j \neq 0 \) for \( j = 1, \ldots, N-2 \), then

\[
c_{(1N-j)'} + u_{j-1} \sum_{i=j}^{N-1} c_{(1N-i-1)'} = u_j c_{(1N)},
\]

(33)

\[
c_{(0N-j)'} + u_{j-1} \sum_{i=j}^{N-1} c_{(1N-i-1)'} = u_j c_{(0N)},
\]

(34)

for all \( 1 \leq j \leq N-1 \), and

\[
c_{(0N)} = u_N c_{(1N)}, \quad c_{(1N)} = u_N c_{(0N)}.
\]

(35)

Proof. By the definition of \( \frac{1}{A_N} \), for every eigenvector of this matrix with an eigenvalue \( \lambda \), we have

\[
c_{(qN-20^2)} = u_1 c_{(qN-20^2)}, \quad c_{(qN-20^2)} = u_1 c_{(qN-20^2)},
\]

\[
\sum_{j=0}^{m} c_{(qN-m-20^1m-j)0^1'} = u_1 c_{(qN-m-20^1m+1)},
\]

\[
\sum_{j=0}^{m} c_{(qN-m-20^1m-j)1^1'} = u_1 c_{(qN-m-20^1m+1)}
\]

for all \( qN-2 \in W_{N-2}, qN-m-2 \in W_{N-m-2} \) (note that \( u_1 = 2 \lambda \)). These relations coincide with (30), (32) for \( m = 0 \) and with (29), (31) for any \( 1 \leq m \leq N-2, j = 1 \), respectively, and can be used as the basis for the induction in \( m,j \). Due to \( S \)-symmetry, every formula obtained below remains valid if we replace all 0's with 1's and vice versa.

For the induction step, let \( 1 \leq \tilde{m} \leq N-2 \) and assume that (30), (32) hold for \( m = \tilde{m} - 1 \) and (29), (31) hold for \( m = \tilde{m} \) and \( j \leq \tilde{m} - 1 \).

Multiplying equation (29) with \( m = \tilde{m} \) by \( u_j \) and using (30), (32) with \( m = j - 1 \), we obtain

\[
c_{(qN-\tilde{m}-20^1\tilde{m}+1')} + u_{j-1} c_{(qN-\tilde{m}-20^1\tilde{m}-j0^1')} + u_j \sum_{i=j+1}^{\tilde{m}} c_{(qN-\tilde{m}-20^1\tilde{m}-i0^1')} = u_j c_{(qN-\tilde{m}-20^1\tilde{m}+1')}
\]

Since \( u_j^2 - 1 = u_{j-1} u_{j+1} \), this is equivalent to

\[
c_{(qN-\tilde{m}-20^1\tilde{m}-j0^1')} + u_j \sum_{i=j+1}^{\tilde{m}} c_{(qN-\tilde{m}-20^1\tilde{m}-i0^1')} = u_j u_{j+1} c_{(qN-\tilde{m}-20^1\tilde{m}+1)}
\]

provided that \( u_{j-1} \neq 0 \), which is equivalent to (29) with \( m = \tilde{m} \) and \( j \) replaced with \( j + 1 \). By induction, this proves equation (29) (and, similarly, (31)) for all \( j \leq \tilde{m} \).

Further, setting \( j = m = \tilde{m} \) in (29) gives

\[
c_{(qN-\tilde{m}-20^1\tilde{m})} + u_{\tilde{m}-1} c_{(qN-\tilde{m}-20^1\tilde{m})} = u_{\tilde{m}} c_{(qN-\tilde{m}-20^1\tilde{m}+1)}
\]

Multiplying with \( u_{\tilde{m}} \) and using (30), (32) with \( m = \tilde{m} - 1 \), we obtain

\[
c_{(qN-\tilde{m}-20^1\tilde{m}+1')} + u_{\tilde{m}-1} c_{(qN-\tilde{m}-20^1\tilde{m}+2')} = u_{\tilde{m}}^2 c_{(qN-\tilde{m}-20^1\tilde{m}+1)}
\]

which, due to \( u_{\tilde{m}}^2 - 1 = u_{\tilde{m}-1} u_{\tilde{m}+1} \), is equivalent to (29) with \( m = \tilde{m} \), provided that \( u_{\tilde{m}-1} \neq 0 \). The proof of (32) with \( m = \tilde{m} \) is similar. This completes the induction step and the proof of (29) – (32). Formulas (33) – (35) can be obtained in the exact same manner. \( \square \)
The conclusions of the theorem follow easily from Lemma [6]. In particular, if \( u_{\ell} = 0 \) and \( u_k \neq 0 \) for \( k \leq \ell \), then by successively applying formulas (30), (32) with \( m + 1 = s_k, s_k-1, \ldots, s_1 \), we obtain

\[
c_{(q^{N-\ell-210p})} = c_{(q^{N-\ell-210p+1})}, \quad c_{(q^{N-\ell-200p})} = c_{(q^{N-\ell-200p+1})},
\]

(36)

\[
c_{(q^{N-\ell-201p})} = c_{(q^{N-\ell-201p+1})}, \quad c_{(q^{N-\ell-211p})} = c_{(q^{N-\ell-211p+1})}.
\]

(37)

Further, setting \( m = \ell \) in (30), (32) and taking into account that \( u_{\ell+1} = 0 \) gives

\[
c_{(q^{N-\ell-200p+1})} = c_{(q^{N-\ell-210p+2})} = 0.
\]

(38)

On the other hand, formulas (29), (31) with \( j = m = \ell + 1 \) result in the equations

\[
c_{(q^{N-\ell-301p+1})} + u_{\ell} c_{(q^{N-\ell-300p+1})} = 0, \quad c_{(q^{N-\ell-310p+1})} + u_{\ell} c_{(q^{N-\ell-311p+1})} = 0,
\]

which, due to \( u_{\ell}^2 = 1 \), can be combined into one equation

\[
c_{(q^{N-\ell-210p+1})} = -u_{\ell} c_{(q^{N-\ell-301p+1})}.
\]

(39)

Combining relations (36) – (39) with the normalization condition \( c_{(q^{N-\ell-210p})} = 1 \), one obtains statements (i) – (iv) of the theorem. The proof of statements (j) – (jjj) follows from relations (33) – (35) of Lemma [6] in a similar fashion.

6 Conclusions

We considered the graph describing the transitions between the discrete states of the Preisach input-state-output hysteresis model. The graph has a self-similar (block-hierarchical) non-symmetric adjacency matrix. Its eigenvalues, their multiplicities (eigenvalue distribution), and their corresponding eigenvectors were explicitly calculated. These can be used to glean information about the underlying walk, for example, in describing the invariant distribution. These results are expected to prove useful in identifying parameters of the Preisach model (the Preisach measure). Our approach can also be fruitful in studying other systems: in particular, complex networks where the connection structure is self-similar.

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