Graphical Quantum Error-Correcting Codes

Sixia Yu\textsuperscript{1,2}, Qing Chen\textsuperscript{1}, and Choo Hiap Oh\textsuperscript{2}

\textsuperscript{1}Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei 230026, P.R. China

\textsuperscript{2}Department of Physics, National University of Singapore, 2 Science Drive 3, Singapore 117542

(Dated: February 5, 2008)

We introduce a purely graph-theoretical object, namely the coding clique, to construct quantum error-correcting codes. Almost all quantum codes constructed so far are stabilizer (additive) codes and the construction of nonadditive codes, which are potentially more efficient, is not as well understood as that of stabilizer codes. Our graphical approach provides a unified and classical way to construct both stabilizer and nonadditive codes. In particular we have explicitly constructed the optimal ((10,24,3)) code and a family of 1-error detecting nonadditive codes with the highest encoding rate so far. In the case of stabilizer codes a thorough search becomes tangible and we have classified all the extremal stabilizer codes up to 8 qubits.

Introduction

Quantum error correcting code (QECC) \textsuperscript{[123]} has become an indispensable element in many quantum informational tasks such as the fault-tolerant quantum computation \textsuperscript{[5]}, the quantum key distributions \textsuperscript{[6]}, and so on, to fight the noises. Roughly speaking, a QECC is a subspace of the Hilbert space of a system of many qubits (physical qubits) with the property that the quantum data (logical qubits) encoded in this subspace can be recovered faithfully, even though some physical qubits may suffer arbitrary errors, by a measurement followed by suitable unitary transformations.

Almost all QECCs constructed so far are stabilizer or additive codes \textsuperscript{[8910]} whose code space is specified by the joint +1 eigenspace of a stabilizer, an Abelian group of tensor products of Pauli operators. Special examples include the CSS codes \textsuperscript{[11]}, the topological codes \textsuperscript{[12]}, color codes \textsuperscript{[13]}, and the recently introduced entanglement-assisted codes \textsuperscript{[14]}. Because of its structure a stabilizer code has always a dimension of a power of 2 and is usually denoted as \([\bigotimes_{i=1}^n|0\rangle\] \(n\)-qubit Hilbert space. Here \(d\) denotes the distance of a quantum code meaning that \([\frac{d-1}{2}]\)-qubit errors can be corrected and all the errors that acts non-trivially on less than \(d\) qubits either can be detected or stabilize the code subspace. Apart from a classification of the \([n,0,d]\) codes up to 12 qubits \textsuperscript{[15]}, the classification of stabilizer codes encoding a nonzero number of logical qubits has not been achieved yet.

The code without a stabilizer structure is called a nonadditive code. Usually we denote by \((n,K,d)\) a nonadditive QECC of distance \(d\) that is a \(K\)-dimensional subspace of an \(n\)-qubit Hilbert space correcting \([\frac{d+1}{2}]\)-qubit. Since less structured than the additive codes, the nonadditive codes may be more efficient in the sense of a larger code subspace on one hand, harder to be constructed on the other hand. The first nonadditive code \textsuperscript{[16]} that outperforms the stabilizer codes is the 1-error detecting code \((5,6,2)\). Its generalization \textsuperscript{[17]} is outperformed by another family of codes of distance 2 \textsuperscript{[18]}. There are also some efforts to construct nonadditive codes systematically \textsuperscript{[192021]}. Only recently the first error-correcting code, namely a \((9,12,3)\) code, that outperforms the optimal stabilizer code is explicitly constructed in \textsuperscript{[22]}. Later on some other nonadditive codes have also been found \textsuperscript{[23]}.

In this article we present a graphical approach to construct both additive and nonadditive codes in a unified and systematic manner. Graph is related intimately to the construction of classical error-correcting codes (e.g. Tanner graph \textsuperscript{[24]}) and also finds applications in the quantum stabilizer code via graph states \textsuperscript{[25262728]}. With the help of graph states, we relate a pure graph-theoretic object — the coding cliques — to the construction of quantum codes, especially of nonadditive codes. An algorithm to search systematically for quantum codes via coding cliques is outlined.

Graph and graph-state basis

A graph \(G = (V,\Gamma)\) is composed of a set \(V\) of \(n\) vertices and a set of edges specified by the adjacency matrix \(\Gamma\), which is an \(n\times n\) symmetric matrix with vanishing diagonal entries and \(\Gamma_{ab} = 1\) if vertices \(a, b\) are connected and \(\Gamma_{ab} = 0\) otherwise. Here we consider only undirected simple graphs. The neighborhood of a vertex \(a\) is denoted by \(N_a = \{v \in V | \Gamma_{av} = 1\}\), i.e., the set of all the vertices that are connected to \(a\).

The graph states \textsuperscript{[2526]} are useful multipartite entangled states that are essential resources for the one-way computing \textsuperscript{[29]} and can be experimentally demonstrated \textsuperscript{[30]}. Also the graph state plays the key role in our graphical construction of QECC. Consider a system of \(n\) qubits that are labeled by those \(n\) vertices in \(V\) and denote by \(X_a, Y_a, Z_a, I_a\) three Pauli operators and identity matrix acting on the qubit \(a \in V\). The \(n\)-qubit graph state associated with \(G\) reads \textsuperscript{[2526]}

\[
|\Gamma\rangle = \prod_{\Gamma_{ab}=1}^{\Gamma} |U_{ab}|+1_{\chi_{y}} |\frac{1}{\sqrt{2^n}} \sum_{\vec{\mu}=0}^{\Gamma} (-1)^{\vec{\mu} \cdot \vec{\Gamma}} |\vec{\mu}\rangle_\chi, \tag{1}
\]
A coding clique $C^k_d$ of a graph $G$ is a collection of vertex subsets $\{C_1, C_2, \ldots, C_k\}$ that satisfies

**Condition 0.** $\emptyset \in C^k_d$.

**Condition 1.** $|C_i \cap S| \text{ is even for all } 1 \leq i \leq K \text{ and } S \in \mathbb{S}_d$.

**Condition 2.** $C_i \Delta C_j \in \mathbb{D}_d$ for all $1 \leq i \neq j \leq K$.

A $K$-clique of a given graph refers to a subset of $K$ vertices that are pairwise connected. Our coding clique $C^k_d$ of a graph $G$ is exactly a $K$-clique of the super graph $\mathcal{G}$ defined as follows. The vertices of the super graph $\mathcal{G}$ include the empty set and all the nonempty vertex subsets that belong to $\mathbb{D}_d$ and satisfy Condition 1. Two different vertices $C, C'$ of the super graph $\mathcal{G}$ are connected if $C \Delta C' \in \mathbb{D}_d$. Thus the coding clique $C^k_d$ is a pure graph-theoretic object that is formulated in a constructive way. The following theorem relates the coding cliques to the construction of QECCs. (See Appendix for proofs.)

**Theorem 1** Given a coding clique $C^k_d$ of a graph $G$ on $n$ vertices, the subspace spanned by the graph-state basis $|\Gamma_C\rangle |C \in C^k_d\rangle$, denoted as $(G, K, d)$, is an $(n, (K, d))$ code.

As an application, a systematic search for the quantum codes can be done according to the following algorithm: i) To input a graph $G = (V, \Gamma)$ on $n$ vertices that may be connected or disconnected; ii) To choose a distance $d$ and compute the $d$-purity set $\mathbb{S}_d$ and the $d$-uncoverable set $\mathbb{D}_d$ so that a super graph $\mathcal{G}$ can be built; iii) To find all the $K$-clique $C^k_d$ of the super graph $\mathcal{G}$; iv) For every coding clique we obtain a $(G, K, d)$ code, i.e., an $(n, (K, d))$ code that is spanned by the graph-state basis $|\Gamma_C\rangle = |Z_C\rangle |C \in C^k_d\rangle$.

It is not difficult to see that if the $d$-purity set $\mathbb{S}_d$ is empty then the $(G, K, d)$ code is pure, i.e., every error acting nontrivially on less than $d$ qubits can be detected.

If a coding clique form a group with respect to the symmetric difference $\Delta$, then this coding clique is referred to as a coding group of the graph. Because the coding group is an Abelian group with self inverse, the number $K$ of its elements must be a power of 2, i.e., $K = 2^k$ for some integer $k$, which is referred to as the dimension of the coding group. A $k$-dimensional coding group has $k$ independent generators. If we find a coding group then we obtain a stabilizer code and all the stabilizer codes can be found this way because of the following theorem.

**Theorem 2** Every coding group $C^k_d(G)$ provides a stabilizer code $[(n, k, d)]$, denoted as a $[G, k, d]$ code. Every stabilizer code $[(n, k, d)]$ is equivalent to a $[G, k, d]$ code for some graph $G$ on $n$ vertices.

From the same graph we may obtain inequivalent $(G, K, d)$ codes and different graphs may provide equivalent codes. To reduce the number of the graphs to be searched we have also investigated how the coding clique changes under the local complements of the graph. For convenience we denote by $G_v$ the graph obtained from $G$ by making an LC on vertex $v$. For a given vertex $v$ and a subset $C \subseteq V$ we denote $C_v = C$ if $v \notin C$ and $C_v = C \Delta N_v$ if $v \in C$.
LC rule for coding cliques. If \( C^K_d \) is a coding clique (group) of the graph \( G \), then \( C^K_d = \{C_i | C_i \in C^K_d \} \) is the coding clique (group) of the graph \( G \). Two codes specified by coding cliques (groups) \( C^K_d \) and \( C^K_d' \) of \( G \) and \( G' \) are equivalent under LCTs.

Two quantum codes are regarded to be equivalent if they are related to each other by LCTs plus permutations of physical qubits. Therefore we need only to take those inequivalent classes of graphs under LCs and graph isomorphisms as inputs to our algorithm. In what follows we shall document some results obtained via the algorithm outlined above [32]. A QECC will be specified by a graph together with a coding clique. For a stabilizer code we will only specify the generators of the coding group and for a nonadditive code we will list all the members of the coding clique.

Nonadditive codes

At first let us reproduce some known nonadditive codes via our graphical construction. The first example is the first nonadditive 1-error correcting nonadditive code ((9, 12, 3)) that outperforms the optimal stabilizer code [5, 2, 2]. The graph provides the code is the loop graph \( L_5 \) on 5 vertices as shown in Fig. 1(a) and the coding clique is \( \{C_i \}_{i=1}^6 \) where

\[
\begin{align*}
C_1 &= \emptyset, \\
C_2 &= \{235\}, \\
C_3 &= \{341\}, \\
C_4 &= \{452\}, \\
C_5 &= \{513\}, \\
C_6 &= \{124\}.
\end{align*}
\]

The second example comes from the first 1-error correcting nonadditive code ((9, 12, 3)) that outperforms the optimal stabilizer code [9, 3, 3]. The code is specified by the loop graph \( L_5 \) on 9 vertices and the coding clique contains those 12 subsets \( \{V_i\}_{i=1}^{12} \) defined as in [22]. Our search results show that the code \( (L_5, 6, 2) \) and the code \( (L_5, 12, 3) \) are unique. In addition there is no \( (G, 13, 3) \) code for any graph \( G \) on 9 vertices.

Another example is Rain’s \((2m + 3, 6, 4m - 1, 2)\) code [17], which can be specified by the graph made up of \( L_5 \) and \( 2(m-1) \) pairwise connected vertices as shown in Fig. 1(b). The coding clique contains all the subsets of form \( \{C_i \triangle U_j \}_{i=1}^{6} \), where \( C_i \)’s are given in Eq. (5) and \( U \) can be any one of \( 2^{2m-2} \) subsets of vertices generated by \( \{2, a_j\} \) and \( \{5, b_l\} \) for \( j, l = 1, 2, \ldots, m - 1 \) with respect to the symmetric difference \( \Delta \).

Now let us construct some new codes. The first result is a family of 1-error detecting codes \((4n + 1, 1, M_n + 1, 2)\) where \( M_n = 2^{4n-1} - \frac{1}{3}C_{2n}^{n} \). A family of nonadditive codes of distance 2 [18] is constructed which encodes an \( M_n \)-dimensional subspace if there are 4n + 1 physical qubits. The code can be specified by the star graph on \( 4n + 1 \) vertices \( V_{4n+1} \) centered on vertex \( o \) as shown in Fig. 1 (c) (indicated by red edges). The coding clique reads

\[
\left\{ C \subseteq V_{4n+1} \middle| o \notin C; |C| = 2l, \text{ or } |C| = 2n + 2l + 1, 0 \leq l \leq n - 1 \right\}.
\]

For the star graph the dimension \( M_n \) of the code space is optimal [18]. We consider instead the graph as shown in Fig. 1(c) that is built on a star graph with red edges and some additional black edges. The coding clique, in addition to Eq. (6), contains one more subset of all \( n + 1 \) blue-colored vertices \( \{o, 4l - 2, 4l - 1 \}_{l=1}^{n} \) as indicated in Fig. 1(c). Therefore we obtain a family of nonadditive code of distance 2 with an encoding rate \( M_n + 1 \).

The second result is the optimal \((10, 24, 3)\) code. The linear programming bound [10] indicates that the largest code subspace of a 1-error correcting code on 10 qubits is of dimension 24. Among 3132 inequivalent connected graphs and all the disconnected graphs on 10 vertices there is a unique \((G_{10}, 24, 3)\) code as depicted in Fig. 1(d) in which blue-colored vertex sets indicate all the nonempty elements of the coding clique.

Stabilizer codes

Because of Theorem 2 the search on all the inequivalent graphs on \( n \) vertices for all the coding groups according to our algorithm provided above will exhaust all the stabilizer codes of \( n \) qubits. Therefore a classification of all stabilizer codes is tangible. We consider in the following the classification of those extremal codes, i.e., the code with a maximal \( k \) or a maximal \( d \) given \( n \), up to 8 qubits, as listed in the table in [10].

Up to six qubits, there are three extremal codes \([4, 2, 2]\), \([5, 1, 3]\), and \([6, 1, 3]\), which are all unique and corresponding graphs and coding groups are shown in Fig. 2. For \([4, 2, 2]\) we have shown 4 different graphs and coding groups, all resulting an equivalent code. There is also a trivial
Among 101 inequivalent connected graphs on 8 vertices there are only 6 LU-inequivalent graphs that admit 3-dimensional coding groups as shown in Fig.2. In each graph there may be many different coding groups and only one coding group is shown for each graph. Disconnected graphs on 8 vertices provide no further [[8, 3, 3]] codes. From the weight distributions of corresponding codes it can be proved that the code [[8, 3, 3]] is unique (Appendix). In Fig.2 we also show three different graphs and the coding groups that provide codes that are equivalent to Shor’s [[9, 1, 3]] code \[^{11}\] and a LC-equivalent graph for the impure [[10, 1, 4]] code constructed in \[^{26}\]. In comparison we have found a pure [[10, 2, 4]] code with a weight distribution (906, 1356, 3010).

**Discussion**

We provide a classical recipe to cook quantum codes, which is exhaustive for stabilizer codes and systematic for nonadditive codes. A classification of stabilizer codes has been done up to 8 qubits. Since all the known good nonadditive codes can be reproduced and an optimal code ((10, 24, 3)) is constructed via our graphical approach, we speculate that our algorithm is also exhaustive for nonadditive codes.

With an improved computational power a classification of the stabilizer codes on more qubits and the discovery of more good codes either stabilizer or nonadditive codes are expected. It should admitted that the clique finding problem is intrinsically a NP-complete problem. Therefore analytical constructions of the coding cliques, at least for some special family of graphs such as loop graphs and hyper cubic graphs, deserve exploring. A direct construction of a series nonadditive 1-error detecting codes with highest encoding rate so far described here may provide a clue. In addition it is not difficult to generalize the idea of coding cliques to the construction of nonbinary codes via graph states for systems with more than 2 levels.

Our precious quantum data can be protected from decoherences either in a dynamical manner as in the decoherence-free subspace approach, or in a geometric manner as in geometric computations, or in a topological manner as in the topological quantum computations based on the topological codes, a special stabilizer codes. Our results make a bridge between the exciting classical field of graph theory and the quantum error correction. Equipped with the one-way computation model based on graph states and the graphical QECCs, we may envision a graphical quantum computation based directly on the graphical objects.

SXY acknowledges financial support from NNSF of China (Grant No. 90303023 and Grant No. 10675107), CAS, and WBS (Project Account No): R-144-000-189-305, Quantum information and Storage (QIS).
Appendix

**Proof of Theorem 1** It is enough to prove that for any error $E_n$ acting nontrivially on less than $d$ qubits we have $(\Gamma C)|E_n|\Gamma C) = f(E_n)|\Gamma C)\equiv 0$ for all $C, C' \in \mathbb{C}_d^2$. Without loss of generality we assume that $E_n = X_n Z_n$ for some pair of subsets $\delta, \omega$ with $|\omega \cup \delta| < d$, which represents that there are $X, Y$, and $Z$ errors on the qubits in $\omega - \delta \cap \omega, \omega \cap \delta$, and $\delta - \omega \cap \delta$ respectively. When acting on the graph state the error $E_n \otimes \mathcal{G}_d \mathcal{Z}_\Omega$ can be replaced by phase flip errors $\mathcal{Z}_\Omega$ on $\Omega := \delta \cup N_\omega$. If $\Omega$ is empty then $\delta = N_\omega$ and the error is proportional to $\mathcal{G}_d$. In this case we have $\mathcal{G}_d|\Gamma C) = |\Gamma C)\times C$ for all $C \in \mathbb{C}_d^2$ because $|\omega \cap C|$ is even which stems from the fact that $|\omega \cup N_\omega| < d$, i.e., $\omega \in \mathbb{S}_F$ and Condition 1. Thus the behavior errors like a constant operator on the coding subspace and can be neglected. If $\Omega$ is not empty then $\Omega \not\subseteq \mathbb{D}_d \cup \emptyset$ because it is covered by $(\delta, \omega)$ and $|\delta \cup \omega| < d$. As a result $(\Gamma C)|E_n|\Gamma C) = (\Gamma)|ZC_{\delta}C\mathcal{Z}_\Omega|\Gamma = 0$ for all $C, C' \in \mathbb{C}_d^2 \cup \emptyset$ because condition 2 ensures that $C \cap C' \not\subseteq \emptyset$.

**Proof of Theorem 2** We prove the second part first, i.e. to construct a graph and a coding clique from a given stabilizer code. We do not need to introduce those input vertices as in Ref. [26].

A stabilizer is a set of commuting observables that are tensor products of Pauli operators $\{X, Y, Z\}$ and the identity operator $I$ on each qubit. A stabilizer code $[n, k, d]$, etc. is the simultaneous $+1$ eigenspace of $n-k$ generators $\{S_1, S_2, \ldots, S_{n-k}\}$ of the stabilizer. Stabilizer codes can be related to each other by a local Clifford transformation (LCTs) and permutations of physical qubits are equivalent.

With the help of the binary representation $(I \to 00, X \to 10, Z \to 01, Y \to 11)$ of the local operators we can describe the stabilizer a check matrix $[G_d|G_r]$ where $(n-k) \times n$ matrix $G_d|G_r$ is formed by all the first (second) digits of the local operators of the generators of the stabilizer. By relabeling or a different choice of the generators of the stabilizer, permuting the qubits, and making suitable LCTs, one can always bring the check matrix into its standard form 

$$[I_r, A_{rsk}][D_{rsk} + AE, E_{rsk}] \quad (r = n-k)$$

where $I_r$ denotes the identity matrix and the submatrix $D$ is symmetric as required by the commutativity among the generators, i.e., $G_d|G_r$ being symmetric. Here all the additions are additions modulo 2.

A basis of the code subspace can be constructed by adding to the stabilizer additional $k$ independent generators. The most general choice, up to an LCT and a choice of different generators, is to add the $k$ generators specified by the check matrix

$$[0, I_k][E^T + FA^T, F_{rsk}]$$

where $F$ is an arbitrary symmetric matrix. By adding the check matrix Eq. (8) multiplied by $A$ from left to the standard form Eq. (1) we obtain the check matrix of $n$ generators as

$$[I_r, 0][D + AFT + E + AF, E^T + FA^T + F] = [I_n]|\Gamma)$$

If one of the diagonal element $\Gamma$, say $\Gamma_{mm}$, is nonzero then we can make it vanish by performing a LCT ($Y \to X \to -Y; Z \to Z$) to the $m$-th qubit. Therefore we obtain a standard adjacency matrix $\Gamma$ of a graph: $n \times n$ symmetric matrix with vanishing diagonal entries. In terms of the vertex stabilizers $[G_d]$ of this graph, the stabilizer is generated by

$$\mathcal{G}_d \mathcal{Z}_\Omega^k$$

Denote by $C_m$ the set of vertices on which the entries of the $m$-th row $(m = 1, \ldots, k)$ of the $k \times n$ matrix $[A^T, I_k]$ do not vanish and by $C$ the subset of the state generated by $[C_m]_{m=1}^k$ with respect to the symmetric difference $\triangle$. Then the graph-state basis $[\mathcal{Z}_C|\Gamma)(C \in \mathbb{C}_d^2]$ spans the $[n, k, d]$ code. By definition $C$ is a group with respect to the symmetric difference $\triangle$ and $0 \in C$. Secondly for every $S \in \mathbb{S}_F$ the graph stabilizer $\mathcal{G}_S$, which can be a possible error, must stabilize the code subspace, i.e., $[S \cap \Gamma]$ must be even for all $C \in \mathbb{C}_d^2$.

Thirdly, let us suppose $\mathcal{G}_S$ is nonzero. Then the graph-state basis $[\mathcal{Z}_C|\Gamma)(C \in \mathbb{C}_d^2]$ spans $[n, k, d]$ code. By definition $C$ is a group with respect to the symmetric difference $\triangle$ and $0 \in C$. Secondly for every $S \in \mathbb{S}_F$ the graph stabilizer $\mathcal{G}_S$, which can be a possible error, must stabilize the code subspace, i.e., $[S \cap \Gamma]$ must be even for all $C \in \mathbb{C}_d^2$. Finally, let us suppose $\mathcal{G}_S$ is nonzero. Then the graph-state basis $[\mathcal{Z}_C|\Gamma)(C \in \mathbb{C}_d^2]$ spans $[n, k, d]$ code. By definition $C$ is a group with respect to the symmetric difference $\triangle$ and $0 \in C$. Secondly for every $S \in \mathbb{S}_F$ the graph stabilizer $\mathcal{G}_S$, which can be a possible error, must stabilize the code subspace, i.e., $[S \cap \Gamma]$ must be even for all $C \in \mathbb{C}_d^2$.
for $C \Delta C'$ in $G$, i.e. $C \Delta C' = \delta' \Delta N_{\omega'}$ where $\delta' = \delta (\omega \cap N_{\omega'})$ and $\omega' = \omega \Delta m$ where

$$m = |\omega \cap N_{\omega}| + |\nu \cap (C \Delta C')|$$  \hspace{1cm} (13)

Since $m + |\nu \cap \delta| = p$ is even we obtain $|\nu' \cup \omega'| = p$. Thus $C \Delta C'$ has a $p < d$ cover in graph $G$ which is in contradiction with Condition 2 of a coding clique. Therefore, $C_1 \Delta C_1'$ cannot be covered by less than $d$ vertices and belongs to the $d$-uncoverable set of $G_r$, i.e., $C_r \Delta C_r' \in \bar{\mathcal{D}}_d(G_r)$ for every $C_r, C_r' \in \bar{\mathcal{C}}_d^K(G)$. Condition 2 is also satisfied. In addition if $C_d^K$ is a coding group of $G$ then $\bar{\mathcal{C}}_d^K$ is a coding group of $G_r$ since $C_r \Delta C_r' = (C \Delta C')_r$.

Let us denote by $P$ and $\bar{P}$ two projectors of the code subspaces specified by $C_d^K$ of $G = (V, \Gamma)$ and $\bar{C}_d^K$ of $G_r = (V, \bar{\Gamma})$ i.e.,

$$P = \sum_{C \in \bar{\mathcal{C}}_d^K} Z_C |\Gamma \rangle \langle Z_C|, \quad \bar{P} = \sum_{C \in \bar{\mathcal{C}}_d^K} Z_C |\bar{\Gamma} \rangle \langle Z_C|. \hspace{1cm} (14)$$

By denoting an LCT as $\mathcal{U} = \sqrt{-1} X_C \prod_{\omega \in N_{\omega}} \sqrt{1 - Z_{\omega}}$ we have $\mathcal{U}^\dagger |\Gamma \rangle \langle \bar{\Gamma}| \mathcal{U} = |\bar{\Gamma} \rangle \langle \bar{\Gamma}|$ and $\mathcal{U} Z_C |\bar{\Gamma} \rangle \langle Z_C| \mathcal{U}^\dagger = Z_{\bar{C}}$ if $\nu \notin C$ and $\mathcal{U} Z_C |\bar{\Gamma} \rangle \langle Z_C| \mathcal{U}^\dagger = i Z_{\bar{C}} C_G$ if $\nu \in C$. As a result $\mathcal{U}^\dagger |\bar{\Gamma} \rangle \langle \bar{\Gamma}| \mathcal{U} = \bar{P}$, i.e., two codes are related to each other by the LCT $\mathcal{U}$.

**Weight distribution and Classification of all the [[7, 1, 3]] codes**

Given a $K$ dimensional subspace with projector denoted by $P$ of $n$ qubit Hilbert space and an arbitrary set $\omega$ of qubits, one can build an invariant [35] under local unitary transformations as

$$A_\omega = \frac{1}{K^2} \sum_{\text{supp}(E) = \omega} |\text{Tr}(E P)|^2 \hspace{1cm} (15)$$

where the summation is taken over all Hermitian Pauli errors that act nontrivially on the qubits in $\omega$. If $\omega = \emptyset$ we have $A_\emptyset = 1$. If we sum over all possible subsets containing $d$ qubits then we obtain the weight distribution [33] $(A_0, A_1, A_2, \ldots, A_n)$ of a code where

$$A_d = \sum_{\omega \subseteq d} A_\omega, \quad d = 0, 1, \ldots, n \hspace{1cm} (16)$$

are invariant under LUTs and permutations of qubits. Obviously $\sum_{d=0}^{n} A_d = 2^n / K$ and we neglect $A_0$ and those zero entries sometimes in the weight distributions. For nonadditive codes $A_d$ may be a fractional. For example the weight distribution of the nonadditive ($G_{10}, 24, 3$) code is $(\{20, 16, 3, 8\})$ meaning that $A_0 = 1$, $A_6 = 20 / 3$, $A_8 = 35$, and $A_4 = 0$ otherwise.

For stabilizer codes $A_d$ is an integer which equals to the number of the stabilizers with the same weight $d$. The weight of a Pauli operator is the number of qubits on which it acts nontrivially. For all the stabilizer codes [[7, 1, 3]] obtained by searching the coding groups can have only 10 different weight distributions as documented in the following table

| $W_0$ | $W_1$ | $W_2$ | $W_3$ | $W_4$ | $W_5$ | $W_6$ | $W_7$ | $W_8$ | $W_9$ |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $A_0$ | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     | 0     |
| $A_1$ | 5     | 3     | 2     | 1     | 1     | 0     | 0     | 0     | 0     |
| $A_2$ | 0     | 0     | 0     | 0     | 2     | 0     | 0     | 0     | 0     |
| $A_3$ | 11    | 15    | 17    | 9     | 7     | 19    | 11    | 9     | 21    |
| $A_4$ | 0     | 0     | 24    | 24    | 24    | 24    | 0     | 24    | 0     |
| $A_5$ | 47    | 45    | 44    | 20    | 23    | 43    | 19    | 22    | 42    |
| $A_6$ | 0     | 0     | 0     | 8     | 6     | 0     | 8     | 6     | 0     |

By using the weight distributions we can only identify 10 different classes of [[7, 1, 3]] codes. Further classification is done by the frequency analysis described as follows. Given $d$ and a subset $S$ of qubits we define

$$F_d(S) = \sum_{\omega \subseteq S, |\omega| = d} A_\omega, \quad d = 1, 2, \ldots, n \hspace{1cm} (17)$$

to be the frequency of $S$, which is obviously an LU-invariant quantity. For every $d$ we order all the frequencies $F_d(S)$ of $S$ containing the same number of qubits by their magnitudes. The resulting ordered series of frequencies is invariant under permutations of qubits. Any difference between the corresponding series of two codes will witness their inequivalency. As a result of frequency analysis we obtain 16 different inequivalent codes and within each equivalent class all codes can be related to each other via explicit local Clifford transformations.

**Proof of the uniqueness of the code [[8, 3, 3]]**

All the stabilizer codes specified by the coding groups of graphs on 8 vertices have the same weight distribution $(28_8, 3_8)$ and 3 stabilizers with full support form a group. Therefore the code must be pure $(A_1, A_2 = 0)$ and it is possible to bring two weight 8 generators of the stabilizer to $X_V$ and $Z_V$ by LCTs. ($V$ denotes 8 qubits here.) As a result one can always bring the local operator acting on the first qubit of the remaining three generators to the identity operator by choosing different generators. Denote $[G_x, G_z]$ as the check matrix of the remaining 3 generators. The $3 \times 8$ matrix $G_x$ or $G_z$ must have distinct columns because all single-qubit $Z$ or $X$ errors can be corrected and two matrices $G_x$ and $G_z$ cannot have 2 or more identical columns on the same qubit because all the single-qubit $Y$ errors can be corrected and the code is pure. Thus the columns of $G_x$ form a map of the the columns of $G_x$ with only one fix point and this map is unique up to LCTs and permutations [10]. As a result the code [[8, 3, 3]] is unique.

---

[1] P.W. Shor, Phys. Rev. A 52, R2493 (1995).
[2] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wooters, Phys. Rev. A 54, 3824 (1996).
[3] E. Knill and R. Laflamme, Phys. Rev. A 55, 900 (1997).
[4] A. Steane, Phys. Rev. Lett. 77, 793 (1996).
[5] E. Knill, R. Laflamme, W. H. Zurek, Science 279, 342 (1998); D. Gottesman, Phys. Rev. A 57, 127 (1998).
[6] E. Knill, G. Brassard, Proceedings of IEEE International Conference on Computers, Systems, and Signal Processing, 175 (1984); A. K. Ekert, Phys. Rev. Lett. 67, 661 (1991).
