Scalar Casimir densities induced by a cylindrical shell in de Sitter spacetime

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Abstract
We evaluate the positive-frequency Wightman function, the vacuum expectation values (VEVs) of the field squared, and the energy–momentum tensor for a massive scalar field with general curvature coupling for a cylindrical shell in the background of de Sitter (dS) spacetime. The field is prepared in the Bunch–Davies vacuum state and on the shell, and the corresponding operator obeys the Robin boundary condition (BC). In the region inside the shell and for non-Neumann BC, the Bunch–Davies vacuum is a physically realizable state for all values of the mass and curvature coupling parameter. For both interior and exterior regions, the VEVs are decomposed into boundary-free dS and shell-induced parts. We show that the shell-induced part of the vacuum energy–momentum tensor has a nonzero off-diagonal component corresponding to the energy flux along the radial direction. Unlike in the case of a shell in Minkowski bulk, for the dS background, the axial stresses are not equal to the energy density. In dependence of the mass and the coefficient in the BC, the vacuum energy density and the energy flux can be either positive or negative. The influence of the background gravitational field on the boundary-induced effects is crucial at distances from the shell larger than the dS curvature scale. In particular, the decay of the VEVs with distance is power-law (monotonic or oscillatory with dependence of the mass) for both massless and massive fields. For the Neumann BC, the decay is faster than that for non-Neumann conditions.

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(Some figures may appear in colour only in the online journal)
1. Introduction

The study of the Casimir effect (for reviews, see [1–5]) for geometries involving cylindrical boundaries have attracted considerable theoretical and experimental interest. In addition to traditional problems of quantum electrodynamics under the presence of material boundaries, the Casimir effect for cylindrical geometries can also be important to the flux tube models of confinement in quantum chromodynamics [6, 7] and for determining the structure of the vacuum state in interacting field theories [8]. A number of widely used nanostructures, such as single- and multi-walled carbon nanotubes, have cylindrical shapes. From the point of view of the experimental studies, the geometries with cylindrical boundaries are among the most optimal candidates for the precision measurements of the Casimir force. Compared to the case of spherical boundaries, in these geometries, the effective area of interaction is larger [9–11]. Considering this, the cylindrically symmetric boundary geometries are becoming increasingly important in the investigations of the Casimir effect. First, the Casimir energy of an infinite perfectly conducting cylindrical shell has been evaluated in [12] based on a Green’s function technique with an ultraviolet regulator. Later, the corresponding result was rederived using the zeta function [13, 14] and the mode-by-mode summation techniques [15] (for the Casimir energy and self-stresses in a more general problem of a dielectric-diamagnetic cylinder, see [16] and references therein). The vacuum expectation value (VEV) of the energy–momentum tensor for the electromagnetic field in the interior and exterior regions of a conducting cylindrical shell are investigated in [17]. The geometry of two coaxial cylindrical shells is considered in [18, 19]. The scalar Casimir densities and the vacuum energy for a single and two coaxial cylindrical shells with Robin BCs are studied in [20, 21]. The zero-point energy of an arbitrary number of perfectly conducting coaxial cylindrical shells is calculated in [22] with the help of the mode summation technique. A less symmetric configuration of two eccentric cylinders is considered in [23] by using the mode summation and functional determinant methods. The Casimir self-energies for an elliptic cylinder are studied in [24]. The Casimir forces acting on two parallel plates inside a conducting cylindrical shell are investigated in [25]. The combined geometry of a wedge and coaxial cylindrical boundary is considered in [26]. The Casimir interaction energy in the configurations involving cylinders, plates, and spheres has been discussed in [4, 5, 27].

In most studies of the Casimir effect with cylindrical boundaries, the geometry of the background spacetime is Minkowskian. Combined effects of a cylindrical boundary and nontrivial topology induced by a cosmic string are discussed in [28]. For an idealized infinite straight cosmic string, the spacetime is locally flat, except on the top of the string where it has a delta-shaped curvature tensor. To see the effects of the curvature on the Casimir densities induced by a cylindrical boundary, in the present paper, we consider the background geometry described by de Sitter (dS) spacetime. The corresponding features for planar and spherically symmetric boundaries are discussed in [29, 30]. The importance of dS background in gravitational physics is motivated by several factors. First, dS spacetime is maximally symmetric and a better understanding of physical effects on its backgrounds could serve as a way to deal with more general geometries. The investigation of physical effects in dS spacetime is important for understanding both the early Universe and its future. In most inflationary scenarios, the dS spacetime is employed to solve a number of problems in standard cosmology related to initial conditions in the early Universe. During an inflationary epoch, the quantum fluctuations generate seeds for the formation of large scale structures in the Universe. More recently, cosmological observations have indicated that the expansion of the Universe at the present epoch is accelerating and the corresponding dynamics are well
approximated by the model, with a positive cosmological constant as a dominant source. For this source, the standard cosmology would lead to an asymptotic dS universe in the future.

We have organized the paper as follows. In the next section, we evaluate the positive-frequency Wightman function for a scalar field with general curvature coupling inside and outside of a cylindrical shell, on which the field obeys the Robin BC. We assume that the field is prepared in the Bunch–Davies vacuum state. The VEV of the field squared is investigated in section 3. The asymptotics are studied in detail at distances larger than the dS curvature scale. Section 4 is devoted to the investigation of the VEV of the energy–momentum tensor for both interior and exterior regions. We show that, in addition to the diagonal components, the vacuum energy–momentum tensor has an off-diagonal component that describes an energy flux along the radial direction. The main results are summarized and discussed in section 5.

2. Wightman function

We consider a quantum scalar field $\phi(x)$ in the background of a $(D + 1)$-dimensional dS spacetime, with the Robin BC

$$\left( A + Bn^l \nabla_l \right) \phi(x) = 0, \tag{2.1}$$

imposed on a cylindrical shell having the radius $a$. Here, $n^l$ is the normal to the shell, $\nabla_l$ is the covariant derivative operator, and $A$ and $B$ are constants. Special cases of equation (2.1) correspond to Dirichlet ($B = 0$) and Neumann ($A = 0$) BCs. In accordance with the problem symmetry, we will write the dS line element in cylindrical coordinates $(r, \phi, z)$:

$$ds^2 = dr^2 - e^{2\alpha} \left[ d\phi^2 + r^2 d\phi^2 + (dz)^2 \right], \tag{2.2}$$

where $z = (z^1, \ldots, z^D)$. The Ricci scalar $R$ and the corresponding cosmological constant $\Lambda$ are expressed in terms of the parameter $\alpha$ as

$$R = D(D + 1)\alpha^{-2}, \quad \Lambda = D(D - 1)\alpha^{-2}/2. \tag{2.3}$$

In addition to the synchronous time coordinate $t$, it is convenient to introduce the conformal time in accordance with

$$\tau = -ae^{-\alpha/\alpha}, \quad -\infty < \tau < 0. \tag{2.4}$$

The corresponding metric tensor is written in a conformally flat form, $g_{ik} = \Omega^2 \eta_{ik}$, with the Minkowskian metric tensor $\eta_{ik}$ and with the conformal factor $\Omega^2 = (a/\tau)^2$.

For a free scalar field with a curvature coupling parameter $\xi$, the field equation is in the form

$$\left( \nabla_l \nabla^l + m^2 + \xi R \right) \phi(x) = 0. \tag{2.5}$$

For the special cases of minimally and conformally coupled fields, one has the values of the curvature coupling $\xi = 0$ and $\xi = \xi_D = (D - 1)/(4D)$, respectively. The imposition of BC (2.1) on the field leads to modifications in the vacuum fluctuations spectrum and, as a result, to the change in the expectation values of the physical characteristics of the vacuum state $|0\rangle$.

For a free field under consideration, all properties of the vacuum state are encoded in two-point functions. Here, we will investigate the positive-frequency Wightman function, defined as the VEV $W(x, x') = \langle 0 | \phi(x) \phi(x') | 0 \rangle$, assuming that the state $|0\rangle$ corresponds to the Bunch–Davies vacuum. Among the set of maximally symmetric quantum states in dS
spacetime, the Bunch–Davies vacuum is the only one for which the ultraviolet behavior of the two-point functions is the same as in Minkowski spacetime.

The Wightman function can be presented in the form of the sum over a complete set of mode functions \( \{ q_{\sigma}(x), q_{\sigma}^*(x') \} \), obeying the field equation (2.5) and the BC (2.1). The collective index \( \sigma \) will be specified next. The Wightman function is given by the expression

\[
W(x, x') = \sum_{\sigma} q_{\sigma}(x)q_{\sigma}^*(x').
\] (2.6)

Having this function, we can evaluate the VEVs of the field squared and of the energy–momentum tensor. In addition, the Wightman function determines the transition rate of an Unruh–DeWitt particle detector in a given state of motion (see [31]).

2.1. Interior region

First, we consider the region inside the cylindrical shell, \( r < a \). In the cylindrical spatial coordinates, for the corresponding mode functions, realizing the Bunch–Davies vacuum state, one has

\[
q_{\sigma}(x) = C_\sigma \eta^{D/2}H_\nu^{(2)}(\sqrt{D}r)J_{\nu}(\lambda r)e^{i(\eta k + k z)},
\] (2.7)

where \( \eta = 1, 0 \), \( H_\nu^{(2)}(x) \) and \( J_{\nu}(x) \) are the Hankel and Bessel functions, \( k = (k_1, ..., k_D) \), \( k = |k| \), and we have defined

\[
\nu = \left[ D^2/4 - D(D + 1)\xi^2 - a^2m^2 \right]^{1/2},
\]

\[
\gamma = \sqrt{\lambda^2 + k^2}.
\] (2.8)

With the choice (2.7), the collective index \( \sigma \) is specified to \( (n, \lambda, k) \), where \( n = 0, \pm 1, \pm 2, ..., \) and \( -\infty < k_i < +\infty, i = 3, ..., D \). Note that the parameter \( \nu \) can be either real or purely imaginary. For a conformally coupled massless field, one has \( \nu = 1/2 \) and the Hankel function in equation (2.7) is expressed in terms of the exponential function. In this case, the mode functions in dS spacetime are related to the corresponding functions for the shell in Minkowski spacetime by a conformal transformation.

The eigenvalues of the quantum number \( \lambda \) are determined from the BC (2.1). Substituting the modes (2.7), we see that these eigenvalues are solutions to the equation

\[
AJ_n(\lambda a) + B\lambda J'_n(\lambda a) = 0,
\] (2.9)

where the prime means the derivative with respect to the argument of the function. For real \( A \) and \( B \), the roots of equation (2.9) are simple and real. We will denote the corresponding positive zeros by \( \lambda a = \lambda a_i, l = 1, 2, ..., \) assuming that they are arranged in ascending order: \( \lambda a_{i+1} < \lambda a_i \). Now, for the set of quantum numbers specifying the modes, one has \( \sigma = (n, l, k) \). Note that for Neumann BC (\( A = 0 \)), the zero mode is present corresponding to \( n = 0, \lambda = 0 \).

The normalization coefficient \( C_\sigma \) in equation (2.7) is determined from the standard condition

\[
\int d^Dx \sqrt{|g^{\mu\nu}(x)|} \bar{q}_{\sigma}^*(x) \gamma_{\sigma}^*(x) = 0.
\] (2.10)

where the integration over the radial coordinate goes over the region inside the cylinder. In equation (2.10), \( \delta_{\sigma\rho} \) stands for the Kronecker delta in the case of discrete components of \( \sigma \) (quantum numbers \( n \) and \( l \)) and for the Dirac delta function for continuous ones (\( k \)). The normalization condition leads to the result
\[
C_\alpha = \frac{\Lambda T_n(\lambda a) e^{i(x-y)v/2}}{4(2\pi)^{D-2}\alpha^{D-1}a},
\]
with the notation
\[
T_n(z) = \frac{z}{(z^2 - n^2)f^2_n(z) + z^2g^2_n(z)}.
\]

In the case of Neumann BC, the normalization coefficient for the zero mode is obtained from equation (2.11), putting \( n = 0 \) and taking the limit \( \lambda \to 0 \).

Substituting the eigenfunctions (2.7) into the mode sum formula (2.6) and by considering that \( H_0^{(2)}(\pi r) = (2i/\pi)e^{i\pi/2}K_0(\eta e^{-\pi r/2}) \), with \( K_\nu(x) \) being the Macdonald function, for the Wightman function, one finds the expression
\[
W(x, x') = \frac{4(\eta \eta')^{D/2}}{(2\pi)^D a^{D-1}} \sum_{n=-\infty}^{\infty} \frac{e^{i\Delta\phi} \int dk e^{ik\Delta x} \sum_{\nu=1}^{\infty} \lambda T_n(\lambda a)}{2^{\nu} \nu!} J_{\nu}(\lambda r) J_{\nu}(\lambda' r') \left[ e^{-\nu \pi i/2} K_{\nu}(e^{-\pi r/2}) K_{\nu}(e^{-\pi r'/2}) \right] I_{\nu}(\lambda r) I_{\nu}(\lambda' r'),
\]
where \( \Delta\phi = \phi - \phi' \), \( \Delta z = z - z' \), and \( \gamma \) is defined in equation (2.8). For the case of Neumann BC, the contribution of the zero mode should be added to the right-hand side of equation (2.13). To separate explicitly the contribution induced by the cylindrical shell, we apply to the series over \( l \) the generalized Abel–Plana summation formula [32, 33]
\[
2 \sum_{l=1}^{\infty} T_n(\lambda, l) \bar{f}(\lambda, l) = \int_0^\infty dx f(x) + \frac{\pi}{2} \text{Res}_{z=\alpha f(z)} \frac{\bar{Y}_n(z)}{J_n(z)},
\]
where \( f(z) \) is an analytic function on the right half-plane, \( Y_n(z) \) is the Neumann function and, for a given function \( F(z) \), we use the notation
\[
F(z) = AF(z) + (B/a)z^2F'(z).
\]
As the function \( f(z) \) in equation (2.14), we take
\[
f(z) = zK_\nu\left( e^{-\pi r/2} \sqrt{x^2 + y^2} + k^2 \right) K_\nu\left( e^{\pi r'/2} \sqrt{x^2 + y^2} + k^2 \right) I_{\nu}(\lambda r/a) I_{\nu}(\lambda' r'/a).
\]
First, let us consider the part in the Wightman function corresponding to the first term in the right-hand side of equation (2.14). We will denote it by \( W_0(x, x') \):
\[
W_0(x, x') = \frac{2(\eta \eta')^{D/2}}{(2\pi)^D a^{D-1}} \sum_{n=-\infty}^{\infty} \frac{e^{i\Delta\phi} \int dk e^{ik\Delta x} \int_0^\infty dk \lambda T_n(\lambda a)}{2^{\nu} \nu!} J_{\nu}(\lambda r) J_{\nu}(\lambda' r') \left[ e^{-\nu \pi i/2} K_{\nu}(e^{-\pi r/2}) K_{\nu}(e^{-\pi r'/2}) \right].
\]

By using the relations
\[
\sum_{n=-\infty}^{\infty} e^{i\Delta\phi} J_{\nu}(\lambda r) J_{\nu}(\lambda' r') = J_0\left( \lambda \sqrt{r^2 + r'^2 - 2rr' \cos \Delta\phi} \right),
\]

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\[ \int_0^\infty du \, u F(u) I_0\left(u\sqrt{z_1^2 + z_2^2}\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk_1 \int_{-\infty}^{+\infty} dk_2 \, e^{ik_1z_1 + ik_2z_2} F\left(\sqrt{k_1^2 + k_2^2}\right). \] (2.19)

one can see that

\[ W_0(x, x') = \frac{2(\eta\eta')^{D/2}}{(2\pi)^{D+1}a^{D-1}} \int dK \, e^{iK\Delta x} K_\nu\left(e^{-\eta/2\eta'} |K|\right) K_\nu\left(e^{\eta/2\eta'} |K|\right), \] (2.20)

where \( K = (k_1, k_2, k_3, \ldots, k_D) \), \( x = (z_1, z_2, z) \), \( \Delta x = x - x' \), \( x = (\tau, x) \), \( x' = (\tau, x') \). After the evaluation of the integral (see [34]), we obtain the expression

\[ W_0(x, x') = \frac{I(D/2 + \nu)I(D/2 - \nu)}{(4\pi)^{D+1}/2^D \Gamma((D + 1)/2)\alpha^{D-1}} F\left(\frac{D}{2} + \nu, \frac{D}{2} - \nu, \frac{D + 1}{2}; w\right), \] (2.21)

where

\[ w = 1 + \frac{(\Delta\eta)^2 - \Delta x^2}{4\eta\eta'}, \] (2.22)

and \( F(a, b; c; w) \) is the hypergeometric function. The function (2.21) is the Wightman function for the boundary-free dS spacetime.

Note that for \( \text{Re} \, \nu \gtrsim D/2 \) the integral in (2.20) contains infrared divergences arising from long-wavelength modes. For these values of \( \nu \) the Bunch–Davies vacuum state is not a physically realizable state in the boundary-free dS spacetime. The BCs imposed on the field may exclude the modes leading to divergences. This is the case for the region inside the cylindrical shell with non-Neumann BCs. For these conditions, in equation (2.13), one has \( \gamma \gtrsim \lambda_{n,1}/a \) and the infrared divergences are absent regardless of \( \nu \). Consequently, the Bunch–Davies vacuum is a realizable state for all values of \( \nu \).

Now, after the application of equation (2.14) to the series over \( l \) in equation (2.13), we obtain the representation

\[ W(x, x') = W_0(x, x') + W_0(x, x'), \] (2.23)

where the contribution induced by the cylindrical shell is given by the expression

\[ W_0(x, x') = -\frac{4(\eta\eta')^{D/2}}{(2\pi)^{D+1}a^{D-1}} \sum_{n=0}^\infty \cos(n\Delta\phi) \int dK \, e^{iK\Delta x}
\times \int_0^\infty du \, u \frac{I_n(au)}{I_n(ay)} I_n(yr) I_n(uy) \bigg|_{y = \sqrt{u^2 + k^2}}. \] (2.24)

With the function

\[ I_n(x, y) = I_{n-1}(x) K_n(y) + K_n(x) I_n(y). \] (2.25)

In equation (2.24), the prime on the sign of the summation means that the term \( n = 0 \) is taken with the coefficient 1/2. The formula (2.24) is valid for \( \text{Re} \, \nu < 1 \) and, in what follows, we assume the values of \( \nu \) in this range. Note that the contribution of the zero mode for the Neumann BC is canceled by the second term in the right-hand side of equation (2.14). The representation (2.23) has two important advantages compared to equation (2.13). First, the explicit knowledge of the roots \( \lambda_{n,1} \) is not required. Second, the effects induced by the shell are explicitly separated and, for points away from the shell, the boundary-induced contribution is finite in the coincidence limit of the arguments. In this way, the renormalization of the VEVs of the field squared and the energy–momentum tensor is...
reduced to the one for the boundary-free dS spacetime. In addition, the integrand in equation (2.24) is an exponentially decreasing function at the upper limit of the integration instead of strongly oscillating function in equation (2.13).

2.2. Exterior region

In the exterior region, \( r > a \), the radial part of the mode functions is a linear combination of the Bessel and Neumann functions. The relative coefficient in this combination is determined from the BC (2.1) imposed on the shell. The mode functions realizing the Bunch–Davies vacuum state are written as

\[
\phi_{\eta \gamma \lambda \lambda}(x) = C_{\eta \gamma} J_{\lambda}^{(2)}(\lambda r) \eta \gamma \lambda \lambda, (2.26)
\]

where \( 0 \leq \lambda < \infty \) and

\[
g_{\eta \gamma}(\lambda a, \lambda r) = \tilde{Y}_{\eta \gamma}(\lambda a)J_{\lambda}(\lambda r) - \tilde{J}_{\eta \gamma}(\lambda a)Y_{\lambda}(\lambda r), (2.27)
\]

with the notation defined by equation (2.15). Now, in equation (2.10), the integration over the radial coordinate goes over the region \( a \leq r < \infty \) and, in the right-hand side for the part corresponding to the quantum number \( \lambda \), one has \( \delta(\lambda - \lambda') \). For the normalization coefficient, we find

\[
C_{\eta \gamma}^2 = \frac{\lambda \alpha^{(1)\lambda/2}}{8(2\pi)^{D-2}a^{D-1}} \left[ J_{\lambda}^2(\lambda a) + \tilde{Y}_{\eta \gamma}^2(\lambda a) \right]^{-1}. (2.28)
\]

Substituting the functions (2.26) into the mode sum (2.6), and introducing the Macdonald function instead of the Hankel function, we obtain the following expression for the exterior Wightman function:

\[
W(x, x') = \frac{2(\eta \gamma)^{D/2}}{(2\pi)^{D/2}a^{D-1}} \sum_{n=-\infty}^{\infty} e^{i\Delta \phi} \int dk \ e^{ik \Delta x} \int_0^\infty d\lambda \ \lambda \frac{g_{\eta \gamma}(\lambda a, \lambda r)g_{\eta \gamma}(\lambda a, \lambda r')}{J_{\lambda}^2(\lambda a) + \tilde{Y}_{\eta \gamma}^2(\lambda a)} K_i(\lambda \alpha^{(1)\lambda/2} \eta \gamma) K_i(\lambda \alpha^{(1)\lambda/2} \eta \gamma). (2.29)
\]

By using the identity

\[
\frac{g_{\eta \gamma}(\lambda a, \lambda r)g_{\eta \gamma}(\lambda a, \lambda r')}{J_{\lambda}^2(\lambda a) + \tilde{Y}_{\eta \gamma}^2(\lambda a)} = \frac{J_{\lambda}(\lambda r)J_{\lambda}(\lambda r') - \frac{1}{2} \sum_{j=1,2} \tilde{H}_{\lambda}^{(j)0}(\lambda a)H_{\lambda}^{(j)}(\lambda r) - H_{\lambda}^{(j)}(\lambda r)}{J_{\lambda}^2(\lambda a) + \tilde{Y}_{\eta \gamma}^2(\lambda a)}, (2.30)
\]

the Wightman function is presented in the decomposed form (2.23) with the shell-induced part

\[
W_b(x, x') = \frac{2(\eta \gamma)^{D/2}}{(2\pi)^{D/2}a^{D-1}} \sum_{n=-\infty}^{\infty} e^{i\Delta \phi} \int dk \ e^{ik \Delta x} \int_0^\infty d\lambda \ \lambda \frac{J_{\lambda}(\lambda a)H_{\lambda}^{(j)}(\lambda a)}{\tilde{H}_{\lambda}^{(j)0}(\lambda a)} K_i(\lambda \alpha^{(1)\lambda/2} \eta \gamma) K_i(\lambda \alpha^{(1)\lambda/2} \eta \gamma). (2.31)
\]

Assuming that the functions \( \tilde{H}_{\lambda}^{(j)}(\zeta) \) and \( H_{\lambda}^{(j)}(\zeta) \) have no zeros for \( 0 < \arg \zeta < \pi/2 \) and \( -\pi/2 \leq \arg \zeta < 0 \), respectively, we rotate the contour of integration over \( \lambda \) by the angle \( \pi/2 \) for the term with \( j = 1 \) and by the angle \( -\pi/2 \) for \( j = 2 \). The expression (2.31) takes the form
Comparing with equation (2.24), we see that the boundary-induced part of the Wightman function in the exterior region is obtained from the corresponding expression in the interior region by the interchange $I_n \leftrightarrow K_n$.

The expressions of the Wightman functions inside and outside a cylindrical shell in Minkowski spacetime are obtained from equations (2.24) and (2.32) in the limit $\alpha \to \infty$. To show this, we note that, for large values of $\alpha$, one has $\nu \approx im \alpha$ and $\eta \approx \alpha - t$. By using the uniform asymptotic expansions of the functions $I_{\pm}(u\nu z)$ and $K_{\nu}(u\nu z)$ (see [35]) for purely imaginary values of the order with a large modulus, it can be seen that the main contribution to the integrals comes from the range $u > m$, in which one has [30]

$$I_n(u\nu, u\nu') \approx \cosh \left( \Delta t\sqrt{u^2 - m^2} \right) \frac{\cosh \left( \Delta t\sqrt{u^2 - m^2} \right)}{\alpha\sqrt{u^2 - m^2}},$$

with $\Delta t = t' - t$. Substituting into equation (2.24), for the Wightman function in the interior region, we find

$$W_{b}(x, x') = -\frac{4}{(2\pi)^D} \sum_{n=0}^{\infty} \cos \left( n\Delta \phi \right) \int \frac{d^Dk}{(2\pi)^D} \left[ I_n(u\nu)K_n(u\nu') - K_n(u\nu)I_n(u\nu') \right] \bigg|_{y'=y^++k^+}.$$  (2.34)

The expression in the exterior region is obtained by the interchange $I_n \leftrightarrow K_n$. The corresponding VEVs for the both interior and exterior regions are investigated in [20].

### 3. VEV of the field squared

The VEVs of the field squared and of the energy–momentum tensor are among the most important characteristics of the vacuum state. The VEV of the field squared is obtained from the Wightman function by taking the coincidence limit of the arguments. Similar to the Wightman function, it is presented in the decomposed form

$$\langle \phi^2 \rangle = \langle \phi^2 \rangle_0 + \langle \phi^2 \rangle_b,$$

where $\langle \phi^2 \rangle_0$ is the VEV in the boundary-free dS spacetime and the part $\langle \phi^2 \rangle_b$ is induced by the cylindrical shell. The boundary-free part is widely investigated in the literature. Because of the maximal symmetry of the dS spacetime and of the Bunch–Davies vacuum state, the renormalized boundary-free VEV does not depend on the spacetime point. In what follows, we will be concerned with the boundary-induced effects.

#### 3.1. Interior region

In the region inside the shell, for the boundary-induced contribution from equation (2.24), one has
\[
\langle \psi^2 \rangle_b = -\frac{A_D}{\alpha^{D-1}} \sum_{n=0}^{\infty} \int_0^{\infty} du \, u^{D-1-\frac{\eta}{\alpha}} \frac{I_n(au/\eta)}{I_n(au)} \tilde{K}_n(au) \sum_{\nu=0}^{\infty} \int_0^{\infty} du \, u^{D-2-\frac{\nu}{\alpha}} \frac{I_n(au)}{I_n(au)} h_\nu(u),
\]
\[
A_D = \frac{\pi^{D/2-1}}{2^{D-3} \Gamma(D/2 - 1)}. \tag{3.3}
\]

In equation (3.2), we have defined the function
\[
h_\nu(u) = \int_0^1 \frac{ds}{s} \left( 1 - s^2 \right)^{D/2-2} f_\nu (us), \tag{3.4}
\]
with
\[
f_\nu (y) = \left[ I_\nu (y) + I_\nu (y) \right] K_\nu (y). \tag{3.5}
\]

In deriving (3.2), we first integrated over the angular part of \( k \) in equation (2.24) and then introduced polar coordinates into the \((k,u)-plane\). The integral in equation (3.4) is obtained from the integral over the polar angle. For points outside the shell, \( r < a \), the boundary-induced part is finite and the renormalization is needed for the boundary-free part only.

The boundary-induced contribution to the VEV depends on \( \eta, \alpha, \) and \( r \) in the form of the ratios \( a/\eta \) and \( r/\eta \). This property is a consequence of the maximal symmetry of dS spacetime. By considering that \( aa/\eta \) is the proper radius of the cylinder and \( ar/\eta \) is the proper distance from the cylinder axis, we see that \( a/\eta \) and \( r/\eta \) are the proper radius and the proper distance, measured in units of the dS curvature scale \( \alpha \). The function \( f_\nu (y) \) is positive for \( \nu \geq 0 \). In this case, the part in the VEV of the field squared induced by the cylindrical shell is negative for the Dirichlet BC and positive for the Neumann BC.

For points near the shell, the dominant contribution to equation (3.2) comes from large values of \( u \) and \( n \). By considering that for large \( y \), one has \( \approx \nu y^{1/2} \), we conclude that the leading term in the asymptotic expansion over the distance

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from the boundary coincides with that for a conformally coupled massless field. By considering the relation (3.8) and using the corresponding asymptotic for the shell in the Minkowski spacetime, we get

\[
\langle \phi^2 \rangle_b \approx -\frac{\Gamma((D - 1)/2)(2\delta_{AB} - 1)}{(4\pi)^{(D+1)/2}|\alpha(a - r)/\eta|^{D-1}}. \tag{3.10}
\]

In deriving equation (3.10) for \( B \neq 0 \), we have assumed that \( a - r \ll |B| \). Note that \( \alpha(a - r)/\eta \) is the proper distance from the shell. As it is seen, near the shell, the boundary-induced part in the VEV of the field squared is negative for the Dirichlet BC \( (B = 0) \) and positive for the non-Dirichlet BC.

On the axis of the shell, only the contribution of the term with \( n = 0 \) survives and we get

\[
\int_0^\infty \phi_\alpha \eta = -\nu \mathbb{A}_{D,0} \mathbb{K}_0(x) \mathbb{I}_0(x), \tag{3.11}
\]

This expression is further simplified for large values of \( a/\eta \) corresponding to large values of the shell proper radius compared to the dS curvature radius. By using the asymptotic expressions for the modified Bessel functions with small values of the argument, to the leading order, for \( u \ll 1 \), one gets

\[
h_u(a) \approx \frac{\Gamma(D/2 - 1)}{4\sigma_u} \frac{(2/a)^2\Gamma(\nu)}{\Gamma(D/2 - \nu)}, \tag{3.12}
\]

where \( \sigma_u = 1 \) for positive \( \nu \) and \( \sigma_u = 2 \) for purely imaginary \( \nu \). Substituting into equation (3.11), for positive \( \nu \), we find

\[
\langle \phi^2 \rangle_{b, r=0} \approx -\frac{\pi^{-D/2-1}A(\nu)}{a^{D-1}(2a/\eta)^{D-2}}, \tag{3.13}
\]

where

\[
A(\nu) = \frac{\Gamma(\nu)}{\Gamma(D/2 - \nu)} \int_0^{\infty} dx \ x^{D-2-\nu} \mathbb{K}_0(x) \mathbb{I}_0(x). \tag{3.14}
\]

In this case, \( \langle \phi^2 \rangle_{b, r=0} \) is a monotonic function of \( a/\eta \).

For purely imaginary \( \nu \) and for \( a/\eta \gg 1 \), the decay of the leading term is oscillatory:

\[
\langle \phi^2 \rangle_{b, r=0} \approx -\frac{2\pi^{-D/2-1}M^{(i)}}{a^{D-1}(2a/\eta)^D} \cos \left[ 2\nu \ln (2a/\eta) + \phi^{(i)} \right], \tag{3.15}
\]

where \( M^{(i)} \) and \( \phi^{(i)} \) are defined by the relation

\[
A^{(i)}(\nu) = M^{(i)} e^{i\phi^{(i)}}, \tag{3.16}
\]

with \( M^{(i)} = |A(\nu)| \). For a given value of \( a \), the expressions (3.13) and (3.15) describe the behavior of the VEV at late times of the expansion, \( t \gg \alpha \). In the case of positive \( \nu \), the shell-induced VEV on the axis decays as \( e^{-D/2(\nu/\alpha)} \), whereas for purely imaginary \( \nu \), the decay is like \( e^{-D/2(\nu/\alpha)} \cos (\omega t + \phi^{(i)}) \) with \( \omega = 2\nu/\alpha \ln (2a/\eta) \).

### 3.2. Exterior region

In the region outside the cylindrical shell, taking the coincidence limit of the arguments in equation (2.32), for the boundary-induced part in the VEV of the field squared, we obtain
\[ \langle \phi^2 \rangle_b = -\frac{A_D}{\alpha^{D-1}} \sum_{n=0}^{\infty} \int_0^\infty du \ u^{D-1} \frac{\tilde{I}_n(ua/\eta)}{K_n(ua/\eta)} K^2_n(u/\eta) h_\eta(n, u), \quad (3.17) \]

with the function \( h_\eta(n, u) \) defined in equation (3.4). For \( \nu \geq 0 \), the latter is positive and, similar to the case of the interior region, the shell-induced VEV is negative for the Dirichlet BC and positive for the Neumann BC. The expression in the right-hand side of equation (3.17) diverges on the shell. The leading term in the asymptotic expansion over the distance from the shell is given by expression (3.10), with \( a - r \) replaced by \( r - a \). In this region, the effects of the curvature are small and the leading term coincides with that for the shell in Minkowski spacetime with the distance \( r - a \) replaced by the proper distance \( \alpha(r - a)/\eta \).

At large proper distances from the shell compared with the dS curvature radius, we have \( r/\eta \gg 1 \) for a fixed value \( a/\eta \). In this limit, the dominant contribution to the integral in equation (3.17) comes from the region near the lower limit of the integration, \( u \lesssim \eta/r \). For positive values of considering \( \nu \) and for \( A \neq 0 \), the dominant contribution to equation (3.17) comes from the term \( n = 0 \). By considering that, for small values of \( z \), one has \( I_0(z)/K_0(z) \approx -1/\ln z \), to the leading order, we find

\[ \langle \phi^2 \rangle_b \approx -\frac{\pi^{-(D+1)/2}a^{1-D}A^{(e)}(\nu)}{4(2r/\eta)^{D-2} \ln (r/a)}, \quad (3.18) \]

where

\[ A^{(e)}(\nu) = \frac{\Gamma(\nu)\Gamma^2(D/2 - \nu)}{\Gamma(D/2 - \nu + 1/2)}. \quad (3.19) \]

Here, for \( B \neq 0 \), we have assumed that \( r \gg |B| \). With this condition, the leading term does not depend on the values of the coefficients in the BC and is negative. In the case of the Neumann BC \( (A = 0) \) and for positive \( \nu \), the leading contribution comes from the terms \( n = 0 \) and \( n = 1 \) with the asymptotic

\[ \langle \phi^2 \rangle_b \approx -\frac{a^{1-D}A^{(e)}_N(\nu)(a/\eta)^2}{2\pi^{(D+1)/2}(2r/\eta)^{D-2} \ln (r/a)}, \quad (3.20) \]

where

\[ A^{(e)}_N(\nu) = (3D/2 - 3\nu + 2)\frac{\Gamma(\nu)\Gamma^2(D/2 - \nu + 1)}{\Gamma(D/2 - \nu + 3/2)}. \quad (3.21) \]

For this case, the decay of the boundary-induced part at large distances from the shell is faster and this part is positive. Combining with the asymptotic analysis for the region near the shell, we conclude that for the Robin BC with \( A, B \neq 0 \), the shell-induced contribution in the VEV of the field squared is positive for points near the shell and negative at large distances. Hence, for some intermediate value of \( r \), it becomes zero. Note that, at large distances, the decay of the shell-induced VEV is power-law for both massless and massive fields. For a cylindrical shell in the Minkowski bulk and for a massive field, the VEV of the field squared decays exponentially with the distance from the shell.

For purely imaginary values of the parameter \( \nu \) and for \( A \neq 0 \), the leading asymptotic term is in the form

\[ \langle \phi^2 \rangle_b \approx \frac{\pi^{-(D+1)/2}a^{1-D}M^{(e)}}{2(2r/\eta)^D \ln (a/r)} \cos \left[ 2 |\nu| \ln (2r/\eta) + \phi^{(e)} \right], \quad (3.22) \]
where the constants $M^{(e)}$ and $\phi^{(e)}$ are defined by the relation
\[ M^{(e)} e^{\phi^{(e)}} = A^{(e)}(\nu). \tag{3.23} \]
For the Neumann BC, the asymptotic has the form
\[ \langle \phi^2 \rangle_b \approx \frac{\alpha^{1-D}(a/\eta)^2 M_N^{(e)}}{\pi^{D+1/2}(2r/\eta)^{D+2}} \cos \left[ 2 |\nu| \ln (2r/\eta) + \phi_N^{(e)} \right]. \tag{3.24} \]
with $M_N^{(e)}$ and $\phi_N^{(e)}$ defined as
\[ M_N^{(e)} e^{\phi_N^{(e)}} = A_N^{(e)}(\nu). \tag{3.25} \]
As we see, for imaginary $\nu$, the damping of the boundary-induced part with the distance from the shell is oscillatory.

4. Energy–momentum tensor

Now we turn to the investigation of the VEV for the energy–momentum tensor. In addition to describing the physical structure of a quantum field at a given point, it acts as the source of gravity in the quasiclassical Einstein equations and plays an important role in modeling self-consistent dynamics involving the gravitational field. Similar to the mean field squared, the VEV is decomposed as
\[ \langle T_{ab} \rangle = \langle T_{ab} \rangle_0 + \langle T_{ab} \rangle_b, \tag{4.1} \]
where $\langle T_{ab} \rangle_0$ is the part corresponding to the boundary-free dS spacetime and $\langle T_{ab} \rangle_b$ is the boundary-induced part. From the maximal symmetry of dS spacetime and of the Bunch–Davies vacuum state, it follows that the renormalized boundary-free part has the form $\langle T_{ab} \rangle_0 = \text{const} \cdot \delta_{ab}$. The boundary-induced contribution is obtained from the corresponding parts in the Wightman function and in the VEV of the field squared by using the formula
\[ \langle T_{ab} \rangle_b = \lim_{x' \rightarrow x} \partial_a \partial_b W_b(x, x') + \left[ (\xi - 1/4) g_{ab} V_i V^i - \xi V_i V^i V^i - \xi R_{ab} \right] \left\langle \phi^2 \right\rangle_b, \tag{4.2} \]
with $R_{ab} = D g_{ab}/\alpha^2$ being the Ricci tensor for dS spacetime. In the right-hand side of equation (4.2) we have used the expression for the energy–momentum tensor of a scalar field that differs from the standard one (given in [31]) by a term that vanishes on the solutions of the field equation (2.5) (see [36]).

4.1. Interior region

First, we consider the region inside the cylindrical shell. After lengthy but straightforward calculations, the VEVs for the diagonal components are presented in the form (no summation over $i$)
\[ \langle T_{ii} \rangle_b = -\frac{A_D}{\alpha^{D+1}} \sum_{n=0}^{\infty} \int_0^\infty du \, u^{D+1} \frac{K_n(u\alpha/\eta)}{T_n(u\alpha/\eta)} \]
\[ \times \left\{ G_i \left[ I_n(\eta s) \right] h_i(u) + L_n(\eta s) \int_0^1 ds \, s^3 \left( 1 - s^2 \right)^{D/2-2} F_i(su) \right\}. \tag{4.3} \]
In this formula, we have defined the functions

\[
G_0[f(z)] = \left( \frac{1}{2} - 2\xi \right) \left[ f'^2(z) + \left( 1 + \frac{n^2}{z^2} \right) f^2(z) \right],
\]

\[
G_1[f(z)] = -\frac{f'^2(z)}{2} - \frac{2\xi}{z} f(z) f' + \frac{1}{2} \left( 1 + \frac{n^2}{z^2} \right) f^2(z),
\]

\[
G_2[f(z)] = G_0[f(z)] + \frac{2\xi}{z} f(z) f' - \frac{n^2}{z^2} f^2(z),
\]

\[
G_l[f(z)] = G_0[f(z)] - \frac{1}{D-2} f^2(z),
\]

with \( l = 3, \ldots, D \), and

\[
F_0(z) = \left[ \frac{1}{4} \partial_z^2 + \left( \frac{D+1}{4} - D\xi \right) \frac{1}{z} \partial_z + \frac{m^2\alpha^2}{z^2} - 1 \right] \delta f(z),
\]

\[
F_1(z) = F_2(z) = \left( \xi - \frac{1}{4} \right) \partial_z^2 \delta f(z) + \left[ \xi (D + 2) - \frac{D+1}{4} \right] \frac{1}{z} \partial_z \delta f(z),
\]

\[
F_l(z) = \frac{1}{D-2} \delta f(z).
\]

Note that, unlike to the case of a shell in Minkowski bulk, here the stresses along the axial directions are not equal to the energy density.

In addition to the diagonal components, the shell-induced VEV of the energy–momentum tensor has also a nonzero off-diagonal component

\[
\langle T^b_0 \rangle = -\frac{A_D}{2^{D+1}} \sum_{n=0}^{\infty} \int_0^\infty du \, u^D \frac{K_n(ua/\eta)}{I_n(ua/\eta)} I_n'(ur/\eta) \times I_n'(ur/\eta) \left( (1 - 4\xi) u^2 a + D - 4(D + 1) \xi \right) b_s(u).
\]

which corresponds to the energy flux along the radial direction.

The components of the energy–momentum tensor (4.3) and (4.6) are given in the coordinates \((r, \theta, \phi, t)\). For the VEVs in the coordinates \((t, r, \theta, \phi)\) with the synchronous time \(t\), denoted here as \( \langle T^b_0 \rangle \), one has the relations (no summation over \(i\)) \( \langle T^b_0 \rangle = \langle T^b_l \rangle \) and \( \langle T^b_{(i)} \rangle = \langle T^b_{(i)} \rangle \). For a conformally coupled massless field, by using relations (3.6) and (3.7), we can see that the off-diagonal component vanishes and the diagonal components of the vacuum energy–momentum tensor are related to the corresponding quantities inside a cylindrical shell in the Minkowski bulk, given in [20], by the conformal relation (no summation over \(i\)) \( \langle T^b_0 \rangle \approx (\eta/\alpha)^{D+1} \langle T^b_0 \rangle \).

As an additional check of the calculations, it can be seen that the shell-induced VEVs obey the covariant continuity equation, \( \nabla_b \langle T^b_i \rangle = 0 \), and the trace relation

\[
\langle T^i_i \rangle = \left[ D \left( \xi - \xi_D \right) V^i V^i + m^2 \right] \langle \phi^2 \rangle.
\]

In particular, the shell-induced contribution is traceless for a conformally coupled massless field. The trace anomaly is contained in the boundary-free part of the VEV. The continuity equation is reduced to two relations between the components of the shell-induced part:
\[
\begin{align*}
\left( \partial_r - \frac{D + 1}{r} \right) \langle T^0_0 \rangle_b + \frac{1}{r} \langle T^k_k \rangle_b + \left( \partial_r + \frac{1}{r} \right) \langle T^0_1 \rangle_b &= 0, \\
\left( \partial_r - \frac{D + 1}{r} \right) \langle T^0_1 \rangle_b + \left( \partial_r + \frac{1}{r} \right) \langle T^1_1 \rangle_b - \frac{1}{r} \langle T^2_2 \rangle_b &= 0.
\end{align*}
\]
(4.8)

Note that \(\langle T^1_0 \rangle_b = -\langle T^0_1 \rangle_b\).

The shell-induced part of the vacuum energy in the region \(r \leq r_0 < a\), per unit coordinate lengths along the directions \(z^1, \ldots, z^D\), is given by
\[
E_r \approx 2\pi (\alpha/\eta)^D \int_0^{r_0} dr' \langle T^0_0 \rangle_b.
\]
(4.9)

For the corresponding time derivative from the first relation in equation (4.8), one gets
\[
\partial_t E_r \approx \frac{2\pi}{\alpha} \left( \frac{\alpha}{\eta} \right)^D \int_0^{r_0} dr' \sum_{i=1}^D \langle T^i_i \rangle_b - 2\pi \left( \frac{\alpha}{\eta} \right)^{D-1} \left( \langle T^0_1 \rangle_b \right)_{r=r_0}. \quad (4.10)
\]

From here, it is seen that \(\langle T^0_1 \rangle_b\) is the energy flux per unit proper surface area. Note that \(\langle T^i_i \rangle_b\) is the boundary-induced part of the vacuum pressure along the \(i\)th direction. Equation (4.10) shows that the change of the energy is caused by two factors: by the work done by the surrounding (first term in the right-hand side of equation (4.10)) and by the energy flux through the boundary of the selected volume (second term).

Now let us discuss the asymptotics of the vacuum energy–momentum tensor. Near the cylindrical surface, the dominant contribution to the boundary-induced VEVs comes from large values of \(n\) and \(u\). By using the uniform asymptotic expansions for the modified Bessel functions (see [37]), it can be seen that the leading terms in the diagonal components for a scalar field with non-conformal coupling (\(\xi \neq \xi_D\)) are related to the corresponding terms for a cylindrical boundary in Minkowski spacetime by (no summation over \(i\))
\[
\langle T^i_i \rangle_b \approx \left( \frac{\eta}{a} \right)^{D+1} \langle T^i_i \rangle_M. \quad (4.11)
\]

These leading terms are given by the expression (no summation over \(i\))
\[
\langle T^i_i \rangle_b \approx \frac{D\Gamma((D + 1)/2)(\xi - \xi_D)}{2^{D+1}a(D+1)/2[\alpha(a - r)/\eta]^{D+1}(2\delta_{B0} - 1)},
\]
(4.11)

for the components with \(i=0, 2, \ldots, D\). For the radial stress and the energy flux, to the leading order, one has
\[
\langle T^1_1 \rangle_b \approx \frac{1 - r/a}{D} \langle T^0_0 \rangle_b, \quad \langle T^0_1 \rangle_b \approx \frac{a - r}{\eta} \langle T^0_0 \rangle_b. \quad (4.12)
\]
From equation (4.11), it may seem that near the shell, the VEV of the energy–momentum tensor, as a function of \(B\), is discontinuous at \(B = 0\). However, this is not the case. The reason is that, in deriving the asymptotic expression for \(B \neq 0\), we have assumed that \((a - r)/\eta \ll |B/(\alpha A)|\) (in this case, in equation (4.3) to the leading order one has \(K_n(x)/I_n(x) \sim K_n(x)/I'_n(x)\) with \(x = ua/\eta\) and, hence, equation (4.11) holds only for the values of \(B\) not too close to 0. For \((a - r)/\eta \sim |B/(\alpha A)|\), the asymptotic expression is more complicated and is given in an integral form. As is seen from equation (4.11), the leading terms have opposite signs for the Dirichlet BC and the non-Dirichlet BC with \(|B/(\alpha A)| \gg (a - r)/\eta\). In particular, for a minimally coupled field the energy density and the energy flux are negative for the Dirichlet BC and positive for the non-Dirichlet BC. Near the shell, the VEVs are dominated by the boundary-induced parts and the same is the case for the total energy density. In general, for the VEV of the energy–momentum tensor, an asymptotic expansion can be given in terms of the distance from the shell. For the case of
$D = 3$, static backgrounds and for a general boundary geometry this type of asymptotic expansion is discussed in [38] for Dirichlet and Neumann BCs and in [39] for a scalar field with a Robin BC at finite temperatures. The coefficients of the next-to-leading terms involve the extrinsic curvature tensor of the boundary. For $D = 3$, the leading term given by equation (4.11) coincides with the corresponding result in [38, 39] taking $\epsilon = a(a - r) / \eta$ (proper distance from the shell).

On the axis of the shell, $r = 0$, the only nonzero contribution to the diagonal components of the boundary-induced VEV comes from the terms in equation (4.3) with $n = 0, 1$. The energy flux vanishes on the axis as $\eta r$. Simple expressions on the axis are obtained for large values of the shell’s proper radius compared with the dS curvature scale, $a / \eta \gg 1$. For positive values of $\nu$, to the leading order, we have (no summation over $i$)

$$
\langle T_{i}^{i} \rangle_{b, r=0} \approx - \frac{\pi^{-D/2}}{\alpha^{D+1}(2a/\eta)^{D-2+2/\nu}} A_{i}^{j},
$$

(4.13)

where $A^{(j)}(\nu)$ is defined by equation (3.14) and

$$
A_{0}^{0} = D \left[ \xi (2\nu - D - 1) + \frac{D - 2\nu}{4} \right],
$$

$$
A_{i}^{j} = \frac{2\nu}{D} A_{0}^{0}, \quad l = 1, \ldots, D.
$$

(4.14)

For a conformally coupled field, one has $A_{0}^{0} = (1 - 2\nu) / 4$ and the leading term (4.13) vanishes in the massless case. For minimally and conformally coupled massive fields, $A_{0}^{0} > 0$. Now, by considering that $A^{(0)}(\nu) > 0$ for the Dirichlet BC and $A^{(0)}(\nu) < 0$ for the Neumann BC, we conclude that in these cases, $\langle T_{i}^{i} \rangle_{b, r=0} < 0$ for the Dirichlet BC and $\langle T_{i}^{i} \rangle_{b, r=0} < 0$ for the Neumann BC.

For the energy flux in the limit $r \to 0$ and for $a / \eta \gg 1$, to the leading order, one has

$$
\langle T_{0}^{0} \rangle \approx - \frac{4\pi^{-D/2 - 1} A_{10}^{(0)}(\nu) r / \eta}{\alpha^{D+1}(2a/\eta)^{D-2+2/\nu}},
$$

(4.15)

with the function

$$
A_{10}^{(0)}(\nu) = \frac{\Gamma(\nu) A_{0}^{0}}{D \Gamma(D/2 - \nu)} \int_{0}^{\infty} dx \ x^{D-2+2/\nu} \left[ \frac{\tilde{K}_{0}(x)}{I_{0}(x)} + \frac{\tilde{K}_{1}(x)}{I_{1}(x)} \right].
$$

(4.16)

As before, for a conformally coupled massless field, the leading term vanishes. For minimally and conformally coupled massive fields, the energy flux corresponding to equation (4.15) is negative the for Dirichlet BC and positive for the Neumann BC.

For purely imaginary $\nu$ and for $a / \eta \gg 1$, the behavior of the diagonal components on the axis is described by (no summation over $i$)

$$
\langle T_{i}^{i} \rangle_{b, r=0} \approx - \frac{2\pi^{-D/2 - 1} M_{i}^{(i)}(\nu)}{\alpha^{D+1}(2a/\eta)^{D}} \cos \left[ 2 | \nu | \ln (2a / \eta) + \phi_{i}^{(i)} \right]\),
$$

(4.17)

with $M_{i}^{(i)} = |A_{i}^{(i)}(\nu) A_{i}^{(i)}|$ and the phase $\phi_{i}^{(i)}$ defined by the relation

$$
A_{i}^{(i)}(\nu) A_{i}^{(i)} = M_{i}^{(i)} e^{i \phi_{i}^{(i)}}.
$$

(4.18)
For the energy flux near the axis, in the case of imaginary $\nu$, one has the leading term
\[
\left< T^0_b \right>_b \approx -\frac{8\pi^{D/2-1}M^{(i)}_{10}r/\eta}{2^{D+1}(2a/\eta)^{D+2}} \cos \left[ \ln \left( \frac{2a/\eta}{\phi_{10}^{(i)}} \right) + \phi_{10}^{(i)} \right].
\] (4.19)

where
\[
A^{(i)}_{10}(\nu) = M^{(i)}_{10} e^{\phi_{10}^{(i)}}.
\] (4.20)

For a given $a$, the expressions (4.13), (4.15), (4.17), and (4.19) describe the asymptotic behavior of the shell-induced VEVs at late stages of the expansion, $t \gg a$.

4.2. Exterior region

Now we turn to the investigation of the VEV for the energy–momentum tensor outside the cylindrical shell. The VEV is presented in the decomposed form (4.1) with the diagonal components of the boundary-induced part (no summation over $i$)
\[
\left< T^i_b \right>_b = -\frac{A_D}{2^{D+1}} \sum_{n=0}^{\infty} \int_0^{\infty} du^{D+1} \frac{\bar{I}_n(ua/\eta)}{K_n(ua/\eta)}
\times \left\{ G_i\left[ K_n(ua/\eta)h_v(u) + K^2_n(ua/\eta)\int_0^1 ds \left( 1 - s^2 \right)^{D/2-2} s^2 F_i(su) \right] \right\},
\] (4.21)

where the functions $G_i[f(z)]$ and $F_i(z)$ are defined by equations (4.4) and (4.5). The off-diagonal component, corresponding to the energy flux along the radial direction, has the form
\[
\left< T^0_b \right>_b = -\frac{A_D}{2^{D+1}} \sum_{n=0}^{\infty} \int_0^{\infty} du^{D+1} \frac{\bar{I}_n(ua/\eta)}{K_n(ua/\eta)}
\times K'_n(ua/\eta)\left[ (1 - 4\xi)ua + D - 4(D + 1)\xi \right] h_v(u).
\] (4.22)

Similar to the case of the interior region, for the components of the vacuum energy–momentum tensor, we have the relations (4.7) and (4.8).

On the boundary, the VEVs diverge. The leading terms in the asymptotic expansion over the distance from the shell for the components $\langle T^i_b \rangle$ with $i = 0, 2, \ldots, D$, are obtained from equation (4.11) by the replacement $(a - r) \rightarrow (r - a)$. Hence, these components have the same sign on both sides of the shell for points near the boundary. The leading terms for the radial stress $\langle T^1_b \rangle_b$ and the energy flux $\langle T^2_b \rangle_b$ are related to the energy density by equation (4.12). In particular, near the shell, the radial stress and the energy flux for non-conformally coupled fields have opposite signs in the exterior and interior regions. For a minimally coupled field, the energy density near the shell is negative for the Dirichlet BC and positive for the non-Dirichlet BC ($B \neq 0$). Near the boundary, for both interior and exterior regions, the corresponding energy flux is directed from the boundary for the Dirichlet BC and to the boundary for the non-Dirichlet BC.

Now we consider the asymptotic behavior of VEV for the energy–momentum tensor at large distances from the cylindrical shell, $r \gg a$. For the diagonal components, the dominant contribution comes from the second term in the figure braces of equation (4.21). For the functions $F_i(z)$ in this term, for small values of the argument, one has (no summation over $i$)
\[
F_i(z) \approx \sigma_i \Re \left[ \frac{2^{2i-1}T_i(\nu)}{T(1-\nu)} A_i^i z^{2-2r} \right].
\] (4.23)
with $A_i^0$ defined by equation (4.14). By considering that at large distances from the shell the main contribution to the integrals in (4.21) and (4.22) comes from the region near the lower limit of the integral, for positive values of $\nu$ and for $A \neq 0$, to the leading order we get

$$\langle T_i^k \rangle_b \approx -\frac{\pi^{-D+1/2} a^{-D-1} A_i^k A^{(e)}(\nu)}{2^1 + \delta_i^1 (2r/\eta)^{D-2+1-\delta_i^1} \ln (r/a)},$$  \hspace{1cm} (4.24)$$

where $A^{(e)}(\nu)$ is given by equation (3.19) and

$$A_i^0 = (2\nu/D - 1) A_i^0.$$  \hspace{1cm} (4.25)$$

The leading term (4.24) does not depend on the specific value of the ratio $B/A$. As is seen, at large distances, to the leading order, the shell-induced stresses are isotropic. For a conformally coupled massless field, $A_i^0 = 0$ and the leading term given by equation (4.24) vanishes. In this case, the diagonal components decay as $1/r^{D+2}$. For minimally and conformally coupled massive fields, $A_i^0 > 0$ and the boundary-induced part in the energy density is negative at large distances. In accordance with equation (4.25), $\langle T_i^0 \rangle_b > 0$ and the energy flux is directed from the shell. For a cylindrical shell in Minkowski spacetime and for a massless field, at large distances, the diagonal components decay as $-r/a \ln (r/a)$ and the leading terms vanish for a conformally coupled field. For massive fields, the decay of the Minkowskian VEVs with the distance from the shell is exponential.

For the Neumann BC ($A = 0$) and for positive values of the parameter $\nu$, at large distances, to the leading order, one has

$$\langle T_i^k \rangle_b \approx \frac{\pi^{-D+1/2} A_i^0 A_i^0 (\nu) (a/\eta)^2}{2^1 + \delta_i^1 (2r/\eta)^{D-2+1-\delta_i^1} \ln (r/a)},$$  \hspace{1cm} (4.26)$$

where $A_i^0 = A_i^0$ (no summation over $i$) and

$$A_i^0 = -2(D - 2\nu + 2) A_i^0 / D.$$  \hspace{1cm} (4.27)$$

As before, for a conformally coupled massless field, the leading term vanishes. For minimally and conformally coupled massive fields, the boundary-induced part in the energy density is positive. In these cases, $\langle T_i^0 \rangle_b < 0$ and the energy flux is directed toward the shell.

Combining with the results of the asymptotic analysis for the region near the shell, we conclude that for positive values of $\nu$ and for minimally and conformally coupled massive fields, the shell-induced contribution in the VEV of the energy density is negative/positive near the shell and at large distances for the Dirichlet/Neumann BC. For the Robin BC with $A, B \neq 0$, this contribution is positive near the shell and negative at large distances. The energy flux is positive/negative for Dirichlet/Neumann BC near the shell and at large distances. For the Robin BC with $A, B \neq 0$, the energy flux is negative near the shell and positive at large distances. At some intermediate value, the energy flux vanishes.

Asymptotic behavior of the vacuum energy–momentum tensor at large distances from the shell is qualitatively different for imaginary values of $\nu$. In this case and for the non-Neumann BC ($A \neq 0$), the leading term has the form

$$\langle T_i^k \rangle_b \approx \frac{\pi^{-D+1/2} a^{-D-1} M_{(e)k}^i}{2^1 + \delta_i^1 (2r/\eta)^{D+1-\delta_i^1} \ln (r/a)} \cos \left[ 2 \nu \ln (2r/\eta) + \phi_{(e)k}^i \right],$$  \hspace{1cm} (4.28)$$

where the coefficient $M_{(e)k}^i$ and the phase $\phi_{(e)k}^i$ are defined by the relation

$$M_{(e)k}^i e^{\phi_{(e)k}^i} = A_i^k A^{(e)}(\nu).$$  \hspace{1cm} (4.29)$$
The decay of the VEVs is oscillatory. For the Neumann BC and in the case of imaginary $\nu$, for the leading term in the asymptotic expansion, one has

$$\alpha_\eta \pi_\eta \nu_\eta \phi \approx + \delta_+ - \delta_0 \left[ \ln (2r/\eta) \right].$$

(4.30)

where

$$M(\eta) = \eta^\nu \phi(\eta).$$

(4.31)

As is seen, for the Neumann BC, the suppression of the VEVs at large distances is faster. For a fixed value of $r$, the asymptotic expressions given previously describe the behavior of the shell-induced contributions to the VEV of the energy–momentum tensor at late stages of the expansion corresponding to $t \gg \alpha$.

In figure 1, for a $D = 3$ conformally coupled field, we display the boundary-induced part in the energy density as a function of the proper distance from the cylindrical shell axis (measured in units of the dS curvature scale $\alpha$). The graphs are plotted for the radius of the shell corresponding to $a/\eta = 2$ and the numbers near the curves are the values of the parameter $m\alpha$. The left/right panel corresponds to Dirichlet/Neumann BC, respectively. For $m\alpha = 2.5$, the parameter $\nu$ is purely imaginary and the oscillatory behavior at large distances from the shell is seen. Similar to the minimal coupling, for a conformally coupled field, the energy density near the shell is negative for the Dirichlet BC and positive for the Neumann BC.

The same graphs for the energy flux are plotted in figure 2. Inside the shell, the energy flux is negative for the Dirichlet BC and positive for the Neumann BC. This means that the flux is directed from the shell for the first case and toward the shell in the second case. On the shell axis, the energy flux vanishes as $r/\eta$.

Figure 3 shows the dependence of the boundary-induced parts in the VEVs of the energy density (left panel) and the energy flux (right panel) on the mass for a conformally coupled field in $D = 3$. The graphs are plotted for $a/\eta = 2$ and for fixed values of $r/\eta = 0.5$ and $r/\eta = 3$ (numbers near the curves). The full/dashed curves correspond to Dirichlet/Neumann
BCs, respectively. As seen from the presented examples, the VEVs for massive fields can be essentially larger than those in the massless case.

5. Conclusion

In the present paper, for a free scalar field with general curvature coupling, we have investigated the change in the properties of the vacuum state induced by a cylindrical shell in the background of dS spacetime. The Robin BC is imposed on the shell, which includes Dirichlet and Neumann BCs as special cases. We have assumed that the field is prepared in the Bunch–Davies vacuum state. In the region inside the shell and for non-Neumann BCs, the zero mode is absent and, in this region, the Bunch–Davies vacuum is a physically realizable state for all values of the mass. All properties of the vacuum are encoded in two-point functions. As such a function, we have taken the positive-frequency Wightman function. Our method for the
evaluation of this function employs the mode summation and, for the interior region, is based on a variant of the generalized Abel–Plana formula. This enabled us to extract explicitly the boundary-free dS part and to present the contribution induced by the shell in terms of strongly convergent integrals. The latter is given by the expression (2.24) in the interior region and by equation (2.32) for the exterior region. In the limit \( \alpha \to \infty \), by using the uniform asymptotic expansions for the modified Bessel functions, the Wightman function is obtained for a shell in Minkowski spacetime.

Having the Wightman function, we have evaluated the VEVs of the field squared and of the energy–momentum tensor. These VEVs are decomposed into boundary-free dS and shell-induced parts. The boundary-free parts are widely discussed in the literature and we were concerned with the boundary-induced effects. For points outside the shell, the shell-induced parts of the VEVs are finite and the renormalization is reduced to that for the boundary-free geometry. The shell-induced contributions depend on the variables \( \eta, a, r \) through the ratios \( a/\eta \) and \( r/\eta \), which are the proper radius and the proper distance from the shell axis, measured in units of the dS curvature scale. This property is a consequence of the maximal symmetry of dS spacetime and of the Bunch–Davies vacuum state.

The shell-induced parts in the VEV of the field squared is given by expressions (3.2) and (3.17) for the interior and exterior regions, respectively. For \( \nu \geq 0 \), the VEV is negative for the Dirichlet BC and positive for the Neumann BC in both regions. The boundary-induced part diverges on the shell with the leading term given by equation (3.10) for the interior region (in the exterior region \( a – r \) should be replaced by \( r – a \)). For points near the shell, the effects of the curvature are subdominant and the leading term coincides with that for the shell in Minkowski bulk. On the axis of the shell, only the term \( n=0 \) contributes and one obtains formula (3.11). Simple expressions are obtained for large values of the shell proper radius compared to the dS curvature scale, \( a/\eta \gg 1 \). For positive \( \nu \), on the axis, the shell-induced VEV of the field squared behaves like \( (a/\eta)^{2\nu-D} \), whereas for purely imaginary \( \nu \), the corresponding behavior, as a function of \( a/\eta \), is damping oscillatory (see equation (3.15)). In the exterior region, at proper distances from the shell larger than the curvature radius of the background spacetime, the influence of the gravitational field on the boundary-induced VEVs is crucial. For positive values of \( \nu \) and for non-Neumann BC, the shell-induced part in the VEV of the field squared decays as \( (r/\eta)^{2\nu-D}/\ln (r/a) \). For the Neumann BC, the decay is stronger, like \( (r/\eta)^{2\nu-D-2} \). In the exterior region and for the Robin BC with \( A, B \neq 0 \), the shell contribution in the mean field squared is positive for points near the shell and negative at large distances. For purely imaginary values of \( \nu \), the behavior of the boundary-induced VEV at large distances is damping oscillatory and the leading term is given by equation (3.18) for the non-Neumann BC and by equation (3.24) for the Neumann BC. As before, the damping for Neumann BC is faster.

The diagonal components of the shell-induced contribution in the VEV of the energy–momentum tensor are given by expression (4.3) inside the shell and by equation (4.21) outside the shell. In addition to the diagonal components, the vacuum energy–momentum tensor has a nonzero off-diagonal component given by expressions (4.6) and (4.22) for the interior and exterior regions, respectively. This component describes the energy flux along the radial direction and, in dependence of the parameters, it can be either positive or negative. Note that unlike the case of a shell in Minkowski bulk, for dS background, the axial stresses are not equal to the energy density. Near the shell, the leading term in the expansion of the energy density and of the stresses parallel to the shell is given by equation (4.11). For nonconformally coupled fields, these VEVs diverge as the inverse \( (D + 1) \)th power of the proper distance from the boundary and near the shell, they have the same sign in the interior
and exterior regions. The leading terms for the normal stress and energy flux are given by equation (4.12) and the corresponding divergences are weaker. Near the shell, these components have opposite signs for the interior and exterior regions. For a minimally coupled field, the energy flux near the shell is directed from the shell for the Dirichlet BC and toward the shell for the Neumann BC.

On the shell axis and for large values of the shell radius compared to the dS curvature scale, the diagonal components decay as \((a/\eta)^{2-D}\) for positive values of \(\nu\) and exhibit a damping oscillatory behavior, given by equation (4.17), for imaginary \(\nu\). The energy flux vanishes on the axis as \(r/\eta\) for \(r \to 0\) and the corresponding asymptotic expressions are given by equations (4.15) and (4.19) for positive and imaginary \(\nu\), respectively. At distances from the shell larger than the dS curvature scale, to the leading order, the vacuum stresses are isotropic. For non-Neumann BCs, the leading terms in the asymptotic expansion of the VEV of the energy–momentum tensor are given by equations (4.24) and (4.28) for positive and imaginary values of \(\nu\). In the first case, the diagonal components decay as \((r/\eta)^{2-\nu}\) for positive values of \(\nu\) and exhibit a damping oscillatory behavior. For imaginary \(\nu\), the decay of the VEVs is oscillatory. This type of behavior is a gravitationally induced effect and is absent in Minkowski bulk. For non-Neumann BCs, the leading terms in the asymptotic expansions at large distances do not depend on the specific values of the coefficients in the Robin BC. For minimally and conformally coupled massive fields and for positive \(\nu\), the shell-induced contribution in the energy density is negative at large distances and the energy flux is directed from the shell. For Neumann BC, the leading terms in the asymptotic expansion at large distances are given by expressions (4.26) and (4.30) for positive and imaginary \(\nu\), respectively. In this case, the decay of the VEVs at large distances is faster (by an additional factor \((\eta/r)^2\)) than that for non-Neumann BCs. For minimally and conformally coupled massive fields with the Neumann BC and for positive values of \(\nu\), the shell-induced contribution in the energy density is positive and the energy flux is directed toward the shell. For the Robin BC with \(A, B \neq 0\), the shell-induced energy density in the exterior region is positive near the shell and negative at large distances, whereas the energy flux is negative near the shell and positive at large distances. At some intermediate value, of the radial coordinate, these quantities vanish.

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