QCD, CHIRAL RANDOM MATRIX THEORY AND INTEGRABILITY

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1. Summary

Random Matrix Theory has been a unifying approach in physics and mathematics. In these lectures we discuss applications of Random Matrix Theory to QCD and emphasize underlying integrable structures. In the first lecture we give an overview of QCD, its low-energy limit and the microscopic limit of the Dirac spectrum which, as we will see in the second lecture, can be described by chiral Random Matrix Theory. The main topic of the third lecture is the recent developments on the relation between the QCD partition function and integrable hierarchies (in our case the Toda lattice hierarchy). This is an efficient way to obtain the QCD Dirac spectrum from the low energy limit of the QCD partition function. Finally, we will discuss the QCD Dirac spectrum at nonzero chemical potential. We will show that the microscopic spectral density is given by the replica limit of the Toda lattice equation. Recent results by Osborn on the Dirac spectrum of full QCD will be discussed.
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References
2. Introduction

Applications of Random Matrix Theories to problems in physics have a long history starting with the idea of Wigner [1] to describe the spacing distribution of nuclear levels by an ensemble of real symmetric matrices. Although this is the first application of Random Matrix Theory (RMT) to strong interactions, applications of RMT to QCD started much later. The first paper that put QCD into the context of RMT was the work of ’t Hooft on two-dimensional QCD in the limit of a large number of colors [2]. It was shown [3] that the combinatorial factors that enter in this large $N_c$ expansion could be obtained from matrix integrals. Even today, as we have seen in the lectures by Di Francesco [4], this work attracts a great deal of attention. It greatly stimulated the analysis of a complicated nonlinear theory such as QCD by means of much simpler matrix models. Because of the success of the application of RMT to the planar expansion of QCD, the hope was that nontrivial results could be derived this way. I will mention three well-known results that have emerged from this line of thought: the Brezin-Gross-Witten model [5, 6], the Eguchi-Kawai [7] reduction, and induced QCD according to Kazakov and Migdal [8].

The Wilson loop in lattice QCD without quarks in 1+1 dimensions can be reduced to the calculation of the unitary matrix integral

$$z(g^2, N_c) = \int_{U \in SU(N_c)} dU e^{g^{-2} Tr(U + U^\dagger)}, \quad (2.1)$$

which is known as the Brezin-Gross-Witten model [9, 5, 6]. This reduction was generalized to an arbitrary Wilson loop amplitude in the large $N_c$ limit of lattice QCD in four dimensions and is known as the Eguchi-Kawai reduction [7]. It was shown that Wilson loop amplitudes do not depend on space-time and can be obtained from a single plaquette integral. However, this reduction is not valid in the weak coupling limit [10]. The idea of induced QCD [8] is to induce the plaquette action by a unitary matrix integral. With a vanishing Wilson line [11] this approach turned out to be unsuccessful as well.

The matrix model (2.1) also appears in the low-energy limit of QCD with quarks. However, in this case the integral over $SU(N_c)$ is not over the color degrees of freedom but rather over the flavor degrees of freedom, and $g^{-2}$ is replaced by $m V \Sigma / 2$ with $\Sigma$ the chiral condensate, $m$ the quark mass and $V$ the volume of space-time. It coincides with the full QCD partition function in a domain where the pion Compton wavelength is much larger than the size of the box [12, 13]. In this limit we have

$$\frac{1}{V N_f} \partial_m z(m \Sigma V / 2, N_f) = \langle \frac{1}{V} \sum_k \frac{1}{i \lambda_k + m} \prod_k (i \lambda_k + m)^{N_f} \rangle, \quad (2.2)$$

where the $\lambda_k$ are the eigenvalues of the Dirac operator. By expanding in powers of the inverse mass, one obtains sum rules for the inverse Dirac eigenvalues.
which put constraints on the Dirac spectrum, but do not determine the average spectral density on the scale of the average level spacing (which is known as the microscopic spectral density) and other spectral correlators.

A Random Matrix Theory that describes the fluctuations of the small eigenvalues of the Dirac operator was introduced in [14, 15]. It was shown that chiral RMT is equivalent to the flavor unitary matrix integral (2.1). The spectral correlation functions were found to be in good agreement with lattice QCD simulations (see [16, 17] for a review of lattice results). One argument to understand this is that level correlations of complex systems on the scale of the average level spacing are universal, i.e. they do not depend on the details of the system. This could be shown rigorously in the context of RMT [18]. However, it was understood later that the generating function for the Dirac spectrum is completely determined by chiral symmetry [19, 20], and its microscopic limit does not change if the Dirac operator is replaced by a random matrix. This is one of the main topics of these lectures.

The question has been raised if the quenched spectral density can be obtained from the limit $N_f \to 0$ of (2.2). This procedure is known as the replica trick. It has been argued that this limit generally gives the wrong result [21]. However, the family of partition functions for different values of $N_f$ are related by the Toda lattice equation [22]. If we take the replica limit of the Toda lattice equation [23], or of the corresponding Painlevé equation [24], we obtain the correct nonperturbative result. This is a second main topic of these lectures.

A third main topic of these lectures is the discussion of QCD at nonzero baryon chemical potential. In that case the Dirac operator is non-Hermitian and its eigenvalues are scattered in the complex plane. We will show that also in this case the spectral density can be obtained from the Toda lattice equation [25, 26]. Recent analytical results by Osborn [27] for the Dirac spectrum of QCD with dynamical quarks at nonzero chemical potential will be discussed.

We start these lectures with an elementary introduction to QCD and its symmetries. The Dirac spectrum is discussed in section 4. The low energy limit of QCD or partially quenched QCD (see section 5) is equivalent to a RMT with the symmetries of QCD that will be introduced in section 6. In section 6 we also calculate the microscopic spectral density by means of orthogonal polynomials and the supersymmetric method. In section 7 we show that this spectral density can be obtained from the replica limit of the Toda lattice equation. In the same section we connect these results with ideas from the theory of exactly solvable systems such as Virasoro constraints, Painlevé equations, Backlund transformations and the Toda lattice. QCD at nonzero chemical potential is discussed in sections 8 and 9. In section 8 we show that the microscopic spectral density can be obtained from the replica limit of a Toda lattice equation. Recent results for full QCD at nonzero chemical potential are discussed in section 9 and concluding remarks are made in section 10.
Table 1. Quark Masses

| Quark | Mass       |
|-------|------------|
| u     | 4 MeV      |
| c     | 1.3 GeV    |
| d     | 8 MeV      |
| b     | 4.4 GeV    |
| s     | 160 MeV    |
| t     | 175 GeV    |

Finally, a note about books and reviews on the subject of these lectures. The classic RMT text is the book by Mehta [28] which emphasizes the orthogonal polynomial method. The third edition of this book appeared recently. In the book by Forrester [29] the emphasis is on the relation between RMT and solvable models and mathematical physics. A comprehensive review of RMT is given in [30]. A collection of recent papers on RMT can be found in [31] which also contains several reviews. Applications to mesoscopic physics are discussed in [32, 33], applications to QCD in [16] and applications to Quantum Gravity in [34]. Among the pedagogical reviews we mention a general review of RMT [35] and an introduction to the supersymmetric method [36, 37]. These lecture notes overlap in part with lecture notes I prepared for the Latin American Summer School in Mexico City [37], where the main emphasis was on the supersymmetric method rather than on applications of RMT to QCD.

3. QCD

3.1 Introduction

QCD (Quantum Chromo Dynamics) is the theory of strong interactions that describes the world of nucleons, pions and the nuclear force. No experimental deviations from this theory have ever been detected. QCD is a theory of quarks which interact via gauge bosons known as gluons. In nature we have 6 different flavors of quarks which each occur in three colors. Each quark is represented by a 4-component Dirac spinor

\[ q_{f,\mu}^i, \quad f = 1, \cdots, N_f = 6, \quad i = 1, \cdots, N_c = 3 \]  

with Dirac index \( \mu \). In total we have 18 quarks (plus an equal number of anti-quarks). The gluon fields are represented by the gauge potentials

\[ A_{ij}^\mu, \quad i, j = 1, \cdots, N_c = 3, \]  

which, as is the case in electrodynamics, are spin 1 vector fields. The gauge fields are Hermitian and traceless; they span the algebra of \( SU(N_c) \). In total we have 8 massless gluons. The 6 quark flavors are known as up, down, strange, charm, bottom and top. The quark masses are given in Table 1. Only the two lightest quarks are important for low-energy nuclear physics.

First principle calculations of QCD can be divided into three different groups, perturbative QCD, lattice QCD and chiral perturbation theory. The main do-
main of applicability of perturbative QCD is for momenta above several GeV. Chiral perturbation theory is an expansion in powers of the momentum and the pion mass and is only reliable below several hundred MeV. Although lattice QCD is an exact reformulation of QCD, in practice both the domain of low momenta and high momenta cannot be accessed, and its main domain of applicability lies somewhere in between the two perturbative schemes.

The reason that perturbative QCD is applicable at high energies is asymptotic freedom: the coupling constant $g \to 0$ for momenta $p \to \infty$. This property was instrumental in gaining broad acceptance for QCD as the theory of strong interactions and its discoverers were awarded this year’s Nobel prize.

A second important property of QCD is confinement, meaning that only color singlets can exist as asymptotic states. This empirically known property has been confirmed by lattice QCD calculations. However, a first principle proof of the existence of a mass gap in QCD is still lacking even in the absence of quarks. Because of confinement the lightest particles of the theory are not quarks or gluons but rather composite mesons.

This brings us to the third important property of QCD: chiral symmetry. At low temperatures and density chiral symmetry is broken spontaneously which, according to Goldstone’s theorem, gives rise to massless Goldstone bosons. Because the chiral symmetry is slightly broken by the light quark masses, the Goldstone bosons are not exactly massless, but the mass of the pions of 135-138 MeV is an order of magnitude less than a typical QCD scale of about 1 GeV. This justifies a systematic expansion in the pion mass and the momenta known as chiral perturbation theory.

A fourth important property of QCD is that a first principle nonperturbative lattice formulation can be simulated numerically. This allows us to compute nonperturbative observables such as for example the nucleon mass and $\rho$-meson mass. Without lattice QCD we would have had only a small number of first principle nonperturbative results and the validity of QCD in this domain would still have been a big question mark.

### 3.2 The QCD partition function

The QCD partition function in a box of volume $V_3 = L^3$ can be expressed in terms of the eigenvalues of the QCD Hamiltonian $E_k$ as

$$Z_{\text{QCD}} = \sum_k e^{-\beta E_k},$$

(3.3)

where $\beta$ is the inverse temperature. At low temperatures, $(\beta \to \infty)$, the partition function is dominated by the lightest states of the theory, namely the vacuum state, with an energy density of $E_0/V_3$ and massless excitations thereof. The partition function $Z_{\text{QCD}}$ can be rewritten as a Euclidean functional inte-
gral over the nonabelian gauge fields $A_\mu$,

$$Z^{\text{QCD}}(M) = \int dA_\mu \prod_{f=1}^{N_f} \det(D + m_f)e^{-S^{\text{YM}}}, \quad (3.4)$$

where $S^{\text{YM}}$ is the Yang-Mills action given by

$$S^{\text{YM}} = \int d^4x \left[ \frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} - i \frac{\theta}{32\pi^2} F^a_{\mu\nu} \tilde{F}^a_{\mu\nu} \right]. \quad (3.5)$$

The field strength and its dual are given by

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu, \quad \tilde{F}_{\mu\nu} = \frac{i}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}. \quad (3.6)$$

The $f^{abc}$ are the structure constants of the gauge group $SU(N_c)$. The gauge fields are denoted by $A_\mu = A_\mu a T^a$, where $T^a$ are the generators of the gauge group. The integral $\nu \equiv \frac{1}{32\pi^2} \int d^4x F^a_{\mu\nu} \tilde{F}^a_{\mu\nu}$ is a topological invariant, i.e. it does not change under continuous transformations of the gauge fields. An important class of field configurations are instantons. These are topological nontrivial field configurations that minimize the classical action. They are classified according to their topological charge $\nu$. The parameter $\theta$ is known as the $\theta$-angle. Experimentally, its value is consistent with zero. In (3.4), the mass matrix is diagonal, $M = \text{diag}(m_1, \ldots, m_{N_f})$, but below we will also consider a general mass matrix. The anti-Hermitian Dirac operator in (3.4) is given by

$$D = \gamma_\mu (\partial_\mu + iA_\mu), \quad (3.7)$$

where the $\gamma_\mu$ are the Euclidean Dirac matrices with anti-commutation relation $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. In the chiral representation the $\gamma$-matrices are given by

$$\gamma_k = \begin{pmatrix} 0 & i \sigma_k \\ -i \sigma_k & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.8)$$

In this representation the Dirac operator has the structure

$$D = \begin{pmatrix} 0 & id \\ id^\dagger & 0 \end{pmatrix}. \quad (3.9)$$

The integration measure is defined by discretizing space-time

$$dA^a_\mu = \prod_x dA^a_\mu(x). \quad (3.10)$$

A particular popular discretization is the lattice discretization where the QCD action is discretized on a hyper-cubic lattice with spacing $a$. The discussion of
lattice QCD would be a lecture by itself. For the interested reader we recommended several excellent textbooks on the subject [38, 39, 40].

A field theory is obtained by taking the continuum limit, i.e. the limit of zero lattice spacing \( a \) for the integration measure discussed above. This limit only exists if we simultaneously adjust the coupling constant, i.e. \( g \to g(a) \). If such limit exists the field theory is called renormalizable. For QCD \( g(a) \) approaches zero in the continuum limit, a property known as asymptotic freedom.

We will be mainly interested in the eigenvalues of the Dirac operator and how they fluctuate for gauge fields \( A_\mu \) distributed according to the QCD action. We will show that below a well-defined scale the fluctuations of the Dirac eigenvalues are given by a RMT with the global symmetries of the QCD.

## 3.3 Symmetries of the QCD Partition Function

It is well-known that the QCD action is greatly constrained by gauge symmetry, Poincaré invariance and renormalizability. These symmetries determine the structure of the Dirac operator and are essential for its infrared spectral properties. In this section we will discuss the global symmetries of the Euclidean Dirac operator. In particular, the chiral symmetry, the flavor symmetry and the anti-unitary symmetry of the continuum Dirac operator are discussed.

### 3.3.1 Axial Symmetry

The axial symmetry, or the \( U_A(1) \) symmetry, can be expressed as the anti-commutation relation

\[
\{ \gamma_5, D \} = 0.
\]  

(3.11)

This implies that all nonzero eigenvalues occur in pairs \( \pm i\lambda_k \) with eigenfunctions given by \( \phi_k \) and \( \gamma_5\phi_k \). If \( \lambda_k = 0 \) the possibility exists that \( \gamma_5\phi_k \sim \phi_k \), so that \( \lambda_k = 0 \) is an unpaired eigenvalue. According to the Atiyah-Singer index theorem, the total number of such zero eigenvalues is a topological invariant, i.e., it does not change under continuous transformations of the gauge field configuration. Indeed, this possibility is realized by the field of an instanton which is a solution of the classical equations of motion. On the other hand, it cannot be excluded that \( \lambda_k = 0 \) while \( \phi_k \) and \( \gamma_5\phi_k \) are linearly independent. However, this imposes additional constraints on the gauge fields that will be violated by infinitesimal deformations. Generically, such situation does not occur.

In a decomposition according to the total number of topological zero modes, the QCD partition function can be written as

\[
Z^{\text{QCD}}(M, \theta) = \sum_{\nu = -\infty}^{\infty} e^{i\nu \theta} Z_{\nu}^{\text{QCD}}(M),
\]  

(3.12)
where
\[ Z^{\text{QCD}}_{\nu}(M) = \langle \prod_f m_f' \prod_k (\lambda_k^2 + m_f^2) \rangle_{\nu}. \]  
(3.13)

Here, \( \langle \cdots \rangle_{\nu} \) denotes the average over gauge-field configurations with topological charge \( \nu \) weighted by the Yang-Mills action. If we introduce right-handed and left-handed masses as complex conjugated masses we find that the \( \theta \) dependence of the QCD partition function is only through the combination \( m e^{i\theta/N_f} \). This property can be used to obtain the \( \theta \)-dependence of the low-energy effective partition function.

### 3.3.2 Flavor Symmetry

A second important global symmetry is the flavor symmetry. This symmetry can be best explained by writing the fermion determinant in the QCD partition function as a functional integral over Grassmann variables,

\[ \prod_f \det(D + m_f) = \int d\psi d\bar{\psi} e^{\int d^4x \sum_{f=1}^{N_f} \bar{\psi}^f (D+m_f) \psi^f}. \]  
(3.14)

In a chiral basis with \( \psi_R = \gamma_5 \psi_R \) and \( \psi_L = -\gamma_5 \psi_L \), the exponent can be rewritten as

\[ \sum_{f=1}^{N_f} \bar{\psi}^f (D+m_f) \psi^f = \bar{\psi}_R^f D \psi_R^f + \bar{\psi}_L^f D \psi_L^f + \bar{\psi}_R^f M_{RL} \psi_L^f + \bar{\psi}_L^f M_{LR} \psi_R^f. \]  
(3.15)

To better illuminate the transformation properties of the partition function we have replaced the diagonal mass matrix by \( M_{RL} \) and \( M_{LR} \).

For \( m_f = 0 \) we have the symmetry

\[ \psi_L \rightarrow U_L \psi_L, \quad \bar{\psi}_L \rightarrow \bar{\psi}_L U_L^{-1}, \quad \psi_R \rightarrow U_R \psi_R, \quad \bar{\psi}_R \rightarrow \bar{\psi}_R U_R^{-1}. \]  
(3.16)

The only condition to be imposed on \( U \) and \( V \) is that their inverse exists. If the number of left-handed modes is equal to the number of right-handed modes we thus have an invariance under \( Gl_R(N_f) \times Gl_L(N_f) \), where \( Gl(N_f) \) is the group of complex \( N_f \times N_f \) matrices with nonzero determinant. However, if the number of left-handed modes is not equal to the number of right-handed modes, the axial-symmetry group is broken to a \( Sl \) subgroup whereas the vector symmetry with \( U_L = U_R \) remains unbroken. For \( m_f = 0 \) the flavor symmetry is thus broken explicitly to \( Gl_V(N_f) \times Sl_A(N_f) \) by instantons or the anomaly. A \( Sl \) subgroup of \( Gl_V(N_f) \) corresponds to baryon number conservation and is usually not considered when flavor symmetries are discussed.
What is much more important, though, is the spontaneous breaking of the axial flavor symmetry. From lattice QCD simulations and phenomenological arguments we know that the expectation value $\langle \bar{\psi}\psi \rangle = \langle \bar{\psi}_R\psi_L \rangle + \langle \bar{\psi}_L\psi_R \rangle \approx -(240 \text{ MeV})^3$ in the vacuum state of QCD instead of the symmetric possibility $\langle \bar{\psi}\psi \rangle = 0$. Phenomenologically, this is known because the pions are much lighter than the $\sigma$ mesons. The spontaneous breaking of the axial symmetry also follows from the absence of parity doublets. For example, the pion mass and the $a_0$ mass are very different ($m_\pi = 135 \text{ MeV}$ and $m_\delta = 980 \text{ MeV}$).

For fermionic quarks there is no need to extend the symmetry group to $Gl_R(N_f) \times Gl_L(N_f)$. In that case, the flavor symmetry is spontaneously broken according to $SU_R(N_f) \times SU_L(N_f) \rightarrow SU_V(N_f)$. For bosonic quarks, which enter in the generating function for the Dirac spectrum, it will be shown in the next section that it is essential to consider the complex extension of $SU(N_f)$. Notice that the complex extension of a symmetry group does not change the number of conserved currents.

On easily verifies that $\langle \bar{\psi}\psi \rangle$ is only invariant for $U_L = U_R$. The vacuum state thus breaks the chiral symmetry down to $Gl_V(N_f)$. In agreement with the Vafa-Witten theorem [41] only the axial symmetries can be broken spontaneously. We also observe that the complete axial group is broken which is known as the maximum breaking [42] of chiral symmetry.

### 3.3.3 Flavor Symmetry for Bosonic Quarks

For bosonic quarks the Goldstone bosons cannot be parameterized by a unitary matrix. The reason is that symmetry transformations have to be consistent with the convergence of the bosonic integrals. Let us consider the case of one bosonic flavor. Then

$$
\det^{-1} \left( \begin{array}{cc} m & id \\ id^\dagger & m \end{array} \right) = \frac{1}{\pi^2} \int d^2 \phi_1 d^2 \phi_2 \exp \left[ - \left( \begin{array}{c} \phi_1^* \\ \phi_2^* \end{array} \right) \left( \begin{array}{cc} m & id \\ id^\dagger & m \end{array} \right) \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right) \right],
$$

(3.17)

so that the integral is convergent for $\text{Re}(m) > 0$. The most general flavor symmetry group of the action in (3.17) is $Gl(2)$ that can be parameterized as

$$U = e^H V \quad \text{with} \quad H^\dagger = H \quad \text{and} \quad VV^\dagger = 1.
$$

(3.18)

For $U$ to be a symmetry transformation for $m = 0$ we require that

$$U^\dagger \left( \begin{array}{cc} 0 & id \\ id^\dagger & 0 \end{array} \right) U = \left( \begin{array}{cc} 0 & id \\ id^\dagger & 0 \end{array} \right),
$$

(3.19)

so that $H$ has to be a multiple of $\sigma_3$, and $V$ has to be a multiple of the identity. The transformations $V$ in (3.18) are not broken by the mass term and therefore represent the vector symmetry. Only the symmetry transformation $\exp(s\sigma_3)$ is
broken by the mass term so that the axial transformations are parameterized by

\[ U = \begin{pmatrix} e^s & 0 \\ 0 & e^{-s} \end{pmatrix} \quad \text{with} \quad s \in (-\infty, \infty). \quad (3.20) \]

For \( N_f \) bosonic flavors the axial transformations are parameterized by

\[ U = \begin{pmatrix} e^H & 0 \\ 0 & e^{-H} \end{pmatrix} \quad \text{with} \quad H^\dagger = H, \quad (3.21) \]

which is the coset \( \text{Gl}(N_f)/U(N_f) \).

### 3.4 Anti-Unitary Symmetries and the Three-fold Way

The QCD partition function with three or more colors in the fundamental representations has no anti-unitary symmetries. As will be discussed below, for two colors with fundamental fermions and for adjoint fermions, the Dirac operator has an anti-unitary symmetry. The classification of the QCD Dirac operator according to anti-unitary symmetries was introduced in [15].

#### 3.4.1 Anti-Unitary symmetries and the Dyson index.

The value of the Dyson index is determined by the anti-unitary symmetries of the system. If there are no anti-unitary symmetries the Hamiltonian is Hermitian and the value of the Dyson index is \( \beta_D = 2 \).

An anti-unitary symmetry operator, which can always be written as \( A = UK \) with \( U \) unitary and \( K \) the complex conjugation operator, commutes with the Hamiltonian of the system

\[ [H, UK] = 0. \quad (3.22) \]

We can distinguish two possibilities,

\[ (UK)^2 = 1 \quad \text{or} \quad (UK)^2 = -1, \quad (3.23) \]

corresponding to \( \beta_D = 1 \) and \( \beta_D = 4 \), respectively. The argument goes as follows. The symmetry operator \( A^2 = (UK)^2 = UU^* \) is unitary, and in an irreducible subspace, it is necessarily a multiple of the identity, \( UU^* = \lambda \). Because of this relation, \( U \) and \( U^* \) commute so that \( \lambda \) is real. By unitarity we have \( |\lambda| = 1 \) which yields \( \lambda = \pm 1 \).

When \( \beta_D = 1 \) it is always possible to find a basis in which the Hamiltonian is real. Starting with basis vector \( \phi_1 \) we can construct \( \psi_1 = \phi_1 + UK\phi_1 \). Then choose \( \phi_2 \) perpendicular to \( \psi_1 \) and define \( \psi_2 = \phi_2 + UK\phi_2 \) with

\[ (\phi_2 + UK\phi_2, \psi_1) = (UK\phi_2, \psi_1) = ((UK)^2\phi_2, UK\psi_1)^* = (\phi_2, \psi_1)^* = 0. \]
The next basis vector is found by choosing $\phi_3$ perpendicular to $\psi_1$ and $\psi_2$, etc. In this basis the Hamiltonian is real

$$H_{kl} = (\psi_k, H \psi_l) = (U K \psi_k, U K H \psi_l)^* = (\psi_k, H U K \psi_l)^* = (\psi_k, H \psi_l)^* = H_{kl}^*.$$  \hspace{1cm} (3.24)

The best known anti-unitary operator in this class is the time-reversal operator for which $U$ is the identity matrix.

In the case $(UK)^2 = -1$ all eigenvalues of the Hamiltonian are doubly degenerate. This can be shown as follows. If $\phi_k$ is and eigenvector with eigenvalue $\lambda_k$, then it follows from (3.22) that also $UK\phi_k$ is an eigenvector of the Hamiltonian with the same eigenvalue. The important thing is that this eigenvector is perpendicular to $\phi_k$ [13],

$$(\phi_k, UK \phi_k) = (UK \phi_k, (UK)^2 \phi_k)^* = -(\phi_k, UK \phi_k). \hspace{1cm} (3.25)$$

In this case it is possible to construct a basis for which the Hamiltonian matrix can be organized into real quaternions [43]. The eigenvalues of a Hermitian quaternion real matrix are quaternion scalars, and the eigenvalues of the original matrix are thus doubly degenerate in agreement with (3.25). The best known example in this class is the Kramers degeneracy for time reversal invariant systems with half-integer spin but no rotational invariance. For example, for spin $\frac{1}{2}$ the time reversal operator is given by $\sigma_2 K$ with $(K \sigma_2)^2 = -1$.

Next we will discuss the anti-unitary symmetries of the QCD Dirac operator.

### 3.4.2 QCD in the Fundamental Representation.

For three or more colors, QCD in the fundamental representation does not have any anti-unitary symmetries and $\beta_D = 2$. QCD with two colors is exceptional. The reason is the pseudo-reality of $SU(2)$:

$$A^*_\mu = (\sum_k A^k_{\mu \frac{T_k}{2}})^* = -\tau_2 A_{\mu \tau_2},$$  \hspace{1cm} (3.26)

where the $\tau_k$ are the Pauli matrices acting in color space. From the explicit representation for the $\gamma$-matrices it follows that

$$\gamma^*_\mu = \gamma_2 \gamma_4 \gamma_\mu \gamma_2 \gamma_4.$$  \hspace{1cm} (3.27)

For the Dirac operator $i D = i \gamma_\mu \partial_\mu + \gamma_\mu A_\mu$ we thus have

$$[KC\gamma_{5 \tau_2}, D] = 0,$$  \hspace{1cm} (3.28)

where $K$ is the complex conjugation operator and $C = \gamma_2 \gamma_4$ is the charge conjugation matrix. Because $(KC\gamma_{5 \tau_2})^2 = 1$ we have that $\beta_D = 1$. Using the argument of Eq. (3.24) a basis can be constructed such that the Dirac matrix is real for any $A_\mu$. 
3.4.3 QCD in the Adjoint Representation. For QCD with gauge fields in the adjoint representation the Dirac operator is given by

$$D = \gamma_\mu \partial_\mu \delta_{bc} + f^{abc} \gamma_\mu A_{a\mu},$$

(3.29)

where the $f^{abc}$ denote the structure constants of the gauge group. Because of the complex conjugation property of the $\gamma$-matrices we have that

$$[\gamma_2 \gamma_4 \gamma_5 K, D] = 0.$$  

(3.30)

One easily verifies that in this case

$$(\gamma_2 \gamma_4 \gamma_5 K)^2 = -1,$$  

(3.31)

so that the eigenvalues of $D$ are doubly degenerate (see section 3.4.1). This corresponds to the case $\beta_D = 4$, so that it is possible to organize the matrix elements of the Dirac operator into real quaternions.

4. The Dirac Spectrum in QCD

In this section we show that the smallest eigenvalues of the QCD Dirac operator are related to the chiral condensate by means of the Banks-Casher relation. This result is used to define the microscopic spectral density.

4.1 Banks-Casher Relation

The order parameter of the chiral phase transition, $\langle \bar{\psi} \psi \rangle$, is nonzero only below a critical temperature. As was shown by Banks and Casher [44], $\langle \bar{\psi} \psi \rangle$ is directly related to the eigenvalue density of the QCD Dirac operator per unit four-volume

$$\Sigma \equiv |\langle \bar{\psi} \psi \rangle| = \lim_{\pi} \frac{\langle \rho(0) \rangle}{V}. \quad (4.1)$$

For eigenvalues $\{\lambda_k\}$ the average spectral density is given by

$$\rho(\lambda) = \langle \sum_k \delta(\lambda - \lambda_k) \rangle. \quad (4.2)$$

To show the Banks-Casher relation we study the resolvent defined by

$$G(z) = \sum_k \frac{1}{z + i\lambda_k}. \quad (4.3)$$

It can be interpreted as the electric field at $z$ of charges at $i\lambda_k$. Using this analogy it is clear that the resolvent changes sign if $z$ crosses the imaginary axis. Let us look at this in more detail. A typical Dirac spectrum is shown in
A typical Dirac spectrum. To derive the Banks-Casher relation we integrate the resolvent over the rectangular contour in this figure. (Figure taken from [45].)

Fig. 1. The average number of eigenvalues in the rectangular contour in this figure is \( \rho(\lambda) l \). If we integrate the resolvent along this contour we find

\[
\oint G(z) = il(G(i\lambda + \epsilon) - G(i\lambda - \epsilon)) = 2\pi i \rho(\lambda) l,
\]

where the second identity follows from Cauchy’s theorem. Using the symmetry of the spectrum we obtain for \( \epsilon \rightarrow 0 \)

\[
\text{Re} G(i\lambda + \epsilon) = \pi \rho(\lambda).
\]

Near the center of the spectrum the imaginary part of the resolvent is negligible. Using that the chiral condensate is related to the resolvent by

\[
\langle \bar{\psi} \psi \rangle = -\lim_{m \rightarrow 0} \lim_{V \rightarrow \infty} \frac{1}{V} G(m),
\]

immediately results in the Banks-Casher relation (4.1). The order of the limits in (4.1) is important. First we take the thermodynamic limit, next the chiral limit (and, finally, the continuum limit).

The resolvent of the QCD Dirac spectrum can be obtained from

\[
G(z, z', m_f) = \frac{\partial}{\partial z} \left|_{z=z'} \log Z_{pq}^{\nu}(z, z', m_f) \right.,
\]

with the so called partially quenched QCD partition function given by

\[
Z_{pq}^{\nu}(z, z', m_f) = \int dA \frac{\det(D + z)}{\det(D + z')} \prod_{f=1}^{N_f} \det(D + m_f) e^{-S_{YM}}.
\]
For \( z = z' \) this partition function coincides with the QCD partition function. In addition to the regular quarks, the partition function (4.8) has additional bosonic and fermionic ghost quarks. Our aim is to find the chiral Lagrangian corresponding to (4.8). If we are successful, we have succeeded in deriving a generating function for the infrared limit of the QCD Dirac spectrum.

### 4.2 Microscopic Spectral Density

An important consequence of the Bank-Casher formula (4.1) is that the eigenvalues near zero virtuality are spaced as

\[
\Delta \lambda = 1/\rho(0) = \pi/\Sigma V. \tag{4.9}
\]

For the average position of the smallest nonzero eigenvalue we obtain the estimate

\[
\lambda_{\text{min}} = \pi/\Sigma V. \tag{4.10}
\]

This should be contrasted with the eigenvalue spectrum of the non-interacting Dirac operator. Then the eigenvalues are those of a free Dirac particle in a box with eigenvalue spacing equal to \( \Delta \lambda \sim 1/V^{1/4} \) for the eigenvalues near \( \lambda = 0 \). Clearly, the presence of gauge fields leads to a strong modification of the spectrum near zero virtuality. Strong interactions result in the coupling of many degrees of freedom leading to extended states and correlated eigenvalues. Because of asymptotic freedom, the spectral density of the Dirac operator for large \( \lambda \) behaves as \( V\lambda^3 \). In Fig. 2 we show a plot of a typical average spectral density of the QCD Dirac operator for \( \lambda \geq 0 \). The spectral density for negative \( \lambda \) is obtained by reflection with respect to the \( y \)-axis. More discussion of this figure will be given in section 5.3.

Because the eigenvalues near zero are spaced as \( \sim 1/\Sigma V \) it is natural to introduce the microscopic spectral density [14]

\[
\rho_s(u) = \lim_{V \to \infty} \frac{1}{V \Sigma} \rho\left( \frac{u}{V \Sigma} \right) \quad \text{with} \quad u = \lambda V \Sigma. \tag{4.11}
\]

We expect that this limit exists and converges to a universal function which is determined by the global symmetries of the QCD Dirac operator. In section 6, we will calculate \( \rho_s(u) \) both for the simplest theory in this universality class, which is chiral Random Matrix Theory (chRMT), and for the partial quenched chiral Lagrangian which describes the low-energy limit of the QCD partition function. We will find that the two results coincide below the Thouless energy.

### 5. Low Energy Limit of QCD

In this section we derive the chiral Lagrangian that provides an exact description of QCD at low energies.
5.1 The chiral Lagrangian

For light quarks the low energy limit of QCD is well understood. Because of confinement, the only light degrees of freedom are the pseudo-scalar mesons, that interact according to a chiral Lagrangian. To lowest order in the quark masses and the momenta, the chiral Lagrangian is completely determined by chiral symmetry and Lorentz invariance. In the case of $N_f$ light quarks with chiral symmetry breaking according to $SU_L(N_f) \times SU_R(N_f) \rightarrow SU_V(N_f)$ the Goldstone fields (pseudo-scalar mesons) are given by $U \in SU(N_f)$. Under an $SU_L(N_f) \times SU_R(N_f)$ transformation of the quark fields given in (3.16), the Goldstone fields $U$ transform in the same way as the chiral condensate

$$U \rightarrow U_R U_L^{-1}. \quad (5.1)$$

The symmetry (3.16) is broken the mass term. However, the full symmetry can be restored if we also transform the mass term as

$$M_{RL} \rightarrow U_R M_{RL} U_L^{-1}, \quad M_{LR} \rightarrow U_L M_{LR} U_R^{-1}. \quad (5.2)$$

The low energy effective theory should have the same invariance properties. To leading order it contains terms of second order in the momenta and of first order in the quark mass matrix. The invariant terms are:

$$\text{Tr}(\partial_\mu U \partial_\mu U^\dagger), \quad \text{Tr}(M_{RL} U^\dagger), \quad \text{Tr}(M_{LR} U). \quad (5.3)$$

Since the QCD partition function is invariant under $M_{RL} \leftrightarrow M_{LR}$, the effective partition function should also have this symmetry. The action of the Goldstone fields is therefore given by the so called Weinberg Lagrangian [47, 48]

$$\mathcal{L}_{\text{eff}}(U) = \frac{F^2}{4} \text{Tr}(\partial_\mu U \partial_\mu U^\dagger) - \frac{\Sigma}{2} \text{Tr}(M_{RL} U^\dagger + M_{LR} U), \quad (5.4)$$
where $F$ is the pion decay constant, and $\Sigma$ is the chiral condensate. The Goldstone fields can be parametrized as $U = \exp(i\sqrt{2}\Pi_{a}t^{a}/F)$, with the generators of $SU(N_{f})$ normalized as $\text{Tr} t^{a}t^{b} = \delta^{ab}$. This chiral Lagrangian has been used extensively for the analysis of pion-pion scattering amplitudes [48].

To lowest order in the pion fields we find for equal quark masses

$$\mathcal{L}_{\text{eff}}(U) = \frac{1}{2} \partial_{\mu}\Pi^{a} \partial^{\mu}\Pi^{a} - N_{f} \Sigma m + \frac{\Sigma m}{F^{2}} \Pi^{a} \Pi^{a}. \quad (5.5)$$

This results in the pion propagator $1/(p^{2} + m_{\pi}^{2})$ with pion mass given by the Gellmann-Oakes-Renner relation

$$m_{\pi}^{2} = \frac{2m\Sigma}{F^{2}}. \quad (5.6)$$

It also illustrates the identification of $\Sigma$ as the chiral condensate.

### 5.2 The Low Energy Limit of $Z_{\nu}^{pq}$

The low-energy limit of the partially quenched QCD partition function can be derived along the same lines as the derivation of the chiral Lagrangian obtained in previous section. In this case, ignoring convergence questions for the moment, the global flavor symmetry of (4.8) is given by the supergroup

$$Gl_{R}(N_{f} + 1|1) \times Gl_{L}(N_{f} + 1|1). \quad (5.7)$$

This reflects that we have $N_{f} + 1$ fermionic quarks and 1 bosonic quark. We already have seen that convergence requirements restrict the axial symmetry for bosonic quarks to $Gl(1)/U(1)$. Although the axial flavor symmetry group of the fermionic quarks is not a priori determined by convergence requirements we will see in this section that supersymmetry necessarily imposes that this symmetry group is compact, i.e. equal to $U(N_{f})$.

Under transformation (5.7) the quarks fields with $N_{f} + 1$ fermionic components and one bosonic component, transform as ($U_{R}, U_{L} \in Gl(N_{f} + 1|1)$)

$$\psi_{R} \rightarrow U_{R}\psi_{R}, \quad \psi_{L} \rightarrow U_{L}\psi_{L}, \quad \bar{\psi}_{R} \rightarrow \bar{\psi}_{R}U_{R}^{-1}, \quad \bar{\psi}_{L} \rightarrow \bar{\psi}_{L}U_{L}^{-1}. \quad (5.8)$$

The subscripts refer to the right-handed (R) or left-handed (L) quarks. For $M = 0$ and $\nu = 0$ this is a symmetry of the QCD action. For $M \neq 0$ this symmetry can be restored if we also transform the mass term according to

$$M_{RL} \rightarrow U_{R}M_{RL}U_{L}^{-1}, \quad M_{LR} \rightarrow U_{L}M_{LR}U_{R}^{-1}. \quad (5.9)$$

In the sector of topological charge $\nu$ the partially quenched partition function transforms as

$$Z_{\nu}^{\text{pq}}(M_{RL}, M_{LR}) \rightarrow \text{Sdet}^{\nu}[U_{R}U_{L}]^{-1}Z_{\nu}^{\text{pq}}(M_{RL}, M_{LR}). \quad (5.10)$$
The Goldstone bosons corresponding to the breaking of the axial subgroup $GL_A(N_f+1|1)$ transform as $Q \rightarrow U_R Q U_L^{-1}$. If we factorize the Goldstone fields into the zero momentum modes $Q_0$ and the nonzero momentum modes $Q(x)$ as
\[ Q = Q_0 Q(x), \] (5.11)
one can easily show that the low energy effective partition function with the above transformation properties is given by
\[ Z^{pq}_\nu(M) = \int_{Q \in GL(N_f+1|1)} dQ S\text{det}^\nu(Q_0) e^{-\int d^4x L^{pq}(Q) }, \] (5.12)
where (see [32, 37] for definitions of the superdeterminant $S\text{det}$ and and the supertrace $\text{Str}$)
\[ L^{pq}(Q) = \frac{F^2}{4} \text{Str} \partial_\mu Q^{-1} \partial_\mu Q - \frac{\Sigma}{2} \text{Str}(M_R L Q^{-1}) - \frac{\Sigma}{2} \text{Str}(M_L R Q). \] (5.13)
We already have seen that the boson-boson block of $GL(N_f+1|1)$ is $GL(1)/U(1)$. If we parameterize the field $Q$ as
\[ Q = e^{\sum_k T_k \pi_k / F}, \] (5.14)
with $T_k$ the generators of $G(N_f+1|1)$, to second order in the Goldstone fields the mass term is given by $\text{Str}(\Sigma M \sum_k T_k^2 \pi_k^2 / F^2)$. Let us take $M$ diagonal positive definite. Because of the supertrace there is a relative minus sign between the boson-boson and fermion-fermion modes. The boson-boson modes are noncompact and require that the overall minus sign of the mass term is negative. In order to avoid tachyonic fermion-fermion Goldstone modes, we have to compensate the minus sign of the supertrace. This can be done by choosing the parameters that multiply the fermion-fermion generators purely imaginary. This corresponds to a compact parametrization of the fermion-fermion Goldstone manifold. This integration manifold is the maximum Riemannian submanifold [49] of $GL(N_f+1|1)$ and will be denoted by $\hat{GL}(N_f+1|1)$.

### 5.3 The Mesoscopic Limit of QCD

In chiral perturbation theory, the different domains of validity were analyzed by Gasser and Leutwyler [12]. A similar analysis applies to partially quenched chiral perturbation theory [51]. The idea is as follows. The $Q$ field can be decomposed as [12]
\[ Q = Q_0 e^{i\psi(x)}, \] (5.15)
where $Q_0$ is a constant (zero-momentum) field. For momenta $p = \pi k/L$ with $k$ integer, the kinetic term of the $\psi$ fields behaves as
\[
\frac{1}{2} \partial_\mu \psi^a(x) \partial_\mu \psi^a(x) \sim L^{-2} \psi^a(x) \psi^a(x).
\]
(5.16)
We observe that the magnitude of the fluctuations of the $\psi^a(x)$ fields are of order $1/L$ which justifies a perturbative expansion of $\exp(i\psi(x))$. The fluctuations of the zero momentum modes are only limited by the mass term
\[
\frac{1}{2} V \Sigma \text{Str} M(Q_0 + Q_0^{-1}).
\]
(5.17)
For quark masses $m \gg 1/V \Sigma$, the field $Q_0$ fluctuates close to the identity and the $Q_0$ field can be expanded around the identity as well. If $m \ll \Lambda_{QCD}$ we are in the domain of chiral perturbation theory. For
\[
\frac{\Sigma m}{F^2} \ll \frac{1}{\sqrt{V}}
\]
(5.18)
the fluctuations of the zero momentum modes dominate the fluctuations of the nonzero momentum modes, and only the contribution from the zero momentum modes has to be taken into account for the calculation of an observable. In this limit the so called finite volume partition function is given by [12, 13]
\[
Z_{N_f}^{\text{eff}}(M, \theta) \sim \int_{U \in SU(N_f)} dU e^{V \Sigma \text{Re} \text{Tr} M U \dagger e^{i\theta/N_f}},
\]
(5.19)
where the $\theta$-dependence follows from the dependence of the QCD partition function on the combination $m e^{i\theta/N_f}$ only (see section 3.3.1). We emphasize that any theory with the same pattern of chiral symmetry breaking as QCD can be reduced to the same extreme infrared limit.

The effective partition function at fixed $\nu$ follows by Fourier inversion [13]
\[
Z_{\nu,N_f}^{\text{eff}}(M) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-i\nu \theta} Z_{\nu,N_f}^{\text{eff}}(M, \theta).
\]
(5.20)
Combining the integral over $SU(N_f)$ and the integral over $U(1)$ we find that
\[
Z_{\nu,N_f}^{\text{eff}}(M) = \int_{U(N_f)} \text{det}^{\nu}(U) e^{V \Sigma \text{Re} \text{Tr} M U \dagger}.
\]
(5.21)
The same arguments apply to the partially quenched chiral Lagrangian. There is an important difference. The mass of the ghost-quarks is an external parameter which can take on any value we wish. Indeed, the mass of the Goldstone modes containing these quarks is given by
\[
M_{zz}^2 = \frac{2\Sigma F^2}{F^2}.
\]
(5.22)
Therefore, independent of the quark masses there is always a domain where
the fluctuations of the zero momentum modes dominate the fluctuations of the
nonzero momentum modes. This domain is given by [51]
\[ z \ll \frac{F^2}{\Sigma L^2} \equiv m_c. \tag{5.23} \]
In this domain, the Compton wavelength of the Goldstone bosons with mass
\( M_{zz} \) is much larger than the size of the box. Because the time scale conjugate
to \( m_c \) is of the order of the diffusion time across the length of the box, this
domain is known as the ergodic domain. For the non-Goldstone modes not to
contribute to the partition function we have to require that \( L \gg 1/\Lambda_{\text{QCD}} \).

In the Dirac spectrum we can thus distinguish three important scales:
\[ \lambda_{\text{min}} \ll m_c \ll \Lambda_{\text{QCD}} \tag{5.24} \]
(see Fig. 2). For \( z \ll m_c \) we are in the zero momentum sector of the theory. If \( z \) is of the order of \( \lambda_{\text{min}} \) or less we have to take into account quantum fluc-
tuations to all orders. For \( \lambda_{\text{min}} \ll z \ll m_c \), the integral over zero modes can
be calculated perturbatively by a loop expansion. For \( m_c \ll z \ll \Lambda_{\text{QCD}} \), chiral
perturbation theory still applies, but the zero momentum modes no longer
dominate the partition function. For \( z \gg \Lambda_{\text{QCD}} \), chiral perturbation theory is
not applicable to the spectrum of the Dirac operator.

In the ergodic domain the QCD partition function in the sector of topological
charge \( \nu \) is given by [19]
\[ Z_{\nu}^{pq}(M) = \int_{Q \in \tilde{G}(N_f+1|1)} dQ \text{Sdet}^\nu Q e^{V \sum \text{Str}(MQ + MQ^{-1})}. \tag{5.25} \]
The number of QCD Dirac eigenvalues that is described by this partition func-
tion is of the order \( m_c/\Delta \lambda \approx F^2 L^2 \). This number increases linearly in \( N_c \) for
\( N_c \rightarrow \infty \) which was recently found in lattice simulations [50].

In section (6.3.2) we will study this partition function in the quenched limit
(\( N_f = 0 \)) and show it coincides with the chRMT result [19, 20].

5.3.1 Comparison to Disordered Systems. In the book by Efetov [32]
it is shown that the diffusion of electrons in a disordered medium can be de-
scribed by the effective action
\[ F(Q) = \int d^d x \left[ \frac{\pi \nu}{8} D \text{Tr} \langle \nabla Q \rangle^2 - \frac{\pi i \nu \omega}{4} \text{Tr} \Lambda Q \right], \tag{5.26} \]
where \( Q \) are the Goldstone fields, \( \nu \) is the density of states, \( D \) is the diffusion
constant and \( \omega \) is the energy difference between the advanced and the retarded
Green’s functions. The matrix \( \Lambda \) is a diagonal matrix with matrix elements \( \pm 1 \).
corresponding to the causal character of the Green’s functions. The Goldstone bosons arise because of the spontaneous breaking of the symmetry between the advanced and retarded Green’s functions.

If we compare this effective action to the chiral Lagrangian (5.13) we can make the identification

\[
\frac{F^2}{4} \leftrightarrow \frac{\pi \nu D}{8}, \quad \frac{\pi \omega \nu}{4} \leftrightarrow \frac{M \Sigma}{2}, \quad \nu \leftrightarrow \frac{\rho(E)}{V},
\]

which can be rewritten as

\[
M \leftrightarrow \frac{\omega}{2}, \quad \Sigma \leftrightarrow \pi \nu, \quad F^2 \leftrightarrow \frac{\pi \nu D}{2}.
\]

The domain where the kinetic term factorizes from the partition function is therefore given by [51, 52]

\[
L^2 \ll \frac{F^2}{M \Sigma} \leftrightarrow \frac{D}{\omega}.
\]

In the theory of disordered mesoscopic systems the corresponding energy scale is known as the Thouless energy. It is defined by [53, 54]:

\[
E_c = \frac{\hbar D}{L^2},
\]

The time conjugate to \(E_c\) is the time scale over which an electron diffuses across the sample. Therefore, the domain where \(\hbar \omega \ll E_c\) is known as the ergodic domain. The time scale in mesoscopic physics corresponding to \(\Lambda_{QCD}\) is the elastic scattering time \(\tau_e\). The domain in between \(E_c\) and \(\hbar/\tau_e\) is known as the diffusive domain. This domain is characterized by diffusive motion of electrons in the disordered sample described by the Lagrangian (5.26).

6. Chiral RMT and the QCD Dirac Spectrum

6.1 The chiral ensembles

The chiral ensembles are defined as the ensembles of \(N \times N\) Hermitian matrices with block structure [14, 15]

\[
D = \begin{pmatrix} 0 & iC \\ iC^\dagger & 0 \end{pmatrix},
\]

and probability distribution given by (for equal quark masses \(m\))

\[
P(C) dC = \mathcal{N} \det^{N_f} (D + m) e^{-\frac{N_f g^2}{4} \text{Tr} C^\dagger C} dC.
\]

The integration measure \(dC\) is the product of differentials of the independent parts of the matrix elements of \(C\), and \(N_f\) is a real parameter (corresponding to
the number of quark flavors in QCD). The matrix $C$ is a rectangular $n \times (n + \nu)$ matrix and $N = 2n + \nu$. The non-zero eigenvalues of the matrix $D$ occur in pairs $\pm \lambda_k$. This can be seen as follows. If

$$D \begin{pmatrix} a \\ b \end{pmatrix} = \lambda \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{then} \quad D \begin{pmatrix} a \\ -b \end{pmatrix} = -\lambda \begin{pmatrix} a \\ -b \end{pmatrix}. \quad (6.3)$$

Generically, the matrix $D$ in (6.1) has exactly $|\nu|$ zero eigenvalues. For this reason, $\nu$ is identified as the topological quantum number. The normalization constant of the probability distribution is denoted by $N$. We can distinguish ensembles with real, complex, or quaternion real matrix elements. They are denoted by $\beta_D = 1$, $\beta_D = 2$, and $\beta_D = 4$, respectively. In addition to the global symmetries of QCD, this partition function has the large unitary invariance

$$C \rightarrow UCV^{-1}, \quad (6.4)$$

where $U$ and $V$ are orthogonal, unitary, or symplectic matrices, respectively. Therefore, the corresponding ensembles are known as the chiral Gaussian Orthogonal Ensemble (chGOE), the chiral Gaussian Unitary Ensemble (chGUE), and the chiral Gaussian Symplectic Ensemble (chGSE), in this order.

Using the invariance (6.4) it is always possible to decompose $C$ as

$$C = U\Lambda V^{-1}, \quad (6.5)$$

where $\Lambda$ is a diagonal matrix with $\lambda_k \geq 0$. The joint probability distribution for the eigenvalues is obtained by transforming to $\Lambda$, $U$ and $V$ as new integration variables. The Jacobian is given by

$$J \sim \prod_k \lambda_k^{\nu\beta_D - 1} \prod_{k<l} |(\lambda_k^2 - \lambda_l^2)|^{\beta_D} \quad (6.6)$$

resulting in the joint eigenvalue distribution

$$P(\{\lambda\})d\{\lambda\} = N|\Delta(\{\lambda^2\})|^{\beta_D} \prod_k \lambda_k^\alpha (\lambda_k^2 + m^2)^{Nf} e^{-N\beta_D \lambda_k^4/4} d\lambda_k, \quad (6.7)$$

where $\alpha = \beta_D - 1 + \beta_D \nu$. We note that the distribution of the eigenvectors factorizes from the distribution of the eigenvalues.

### 6.2 Mathematical Methods

In this section we will discuss the orthogonal polynomial method, the resolvent expansion method, the replica trick and the supersymmetric method. These methods are widely used in Random Matrix Theory.
6.2.1 Resolvent Expansion Methods. These methods are based on expanding the resolvent in a geometric series

\[ G(z) = \langle \text{Tr} \frac{1}{z - H} \rangle = N \frac{1}{z} + \langle \text{Tr} \frac{1}{z} H \frac{1}{z} \rangle + \langle \text{Tr} \frac{1}{z} H \frac{1}{z} \frac{1}{z} \rangle + \cdots. \]  

(6.8)

In the large \( N \) limit the averages are given by a sum of planar diagrams. Let us illustrate this for the GUE. In this case the “propagator” is given by

\[ \langle H_{ij} H_{kl} \rangle = \frac{1}{N} \delta_{il} \delta_{jk}. \]  

(6.9)

For example, as was explained in the course of Di Francesco [4], for \( \text{Tr} H^4 \) term we have two planar diagrams of order \( N^3 \) and one diagram of order \( N^2 \).

6.2.2 The Orthogonal Polynomial Method. The oldest method is the orthogonal polynomial method [28]. In principle, one obtains expressions that are exact for finite size matrices. The drawback of this method is that it requires a probability distribution that is invariant under basis change of the random matrix. In general the probability density can be written as

\[ P(x_1, \ldots, x_n) = \Delta^D(\{x_k\}) \prod_{k=1}^n w(x_k), \]  

(6.10)

where \( w(x) \) is a weight function and the Vandermonde determinant is given by

\[ \Delta(\{x_k\}) = \prod_{k>l} (x_k - x_l). \]  

(6.11)

The method is based on the identity

\[
\Delta(\{x_k\}) = \begin{vmatrix}
1 & \cdots & 1 \\
x_1 & \cdots & x_n \\
\vdots & \vdots & \vdots \\
x_1^{n-1} & \cdots & x_n^{n-1}
\end{vmatrix} = \begin{vmatrix}
P_0(x_1) & \cdots & P_0(x_n) \\
P_1(x_1) & \cdots & P_1(x_n) \\
\vdots & \vdots & \vdots \\
P_{n-1}(x_1) & \cdots & P_{n-1}(x_n)
\end{vmatrix}, \quad (6.12)
\]

where the \( P_k \) are monic orthogonal polynomials defined by

\[
\int dx w(x) P_k(x) P_l(x) = h_k \delta_{kl}. \]  

(6.13)

Because of these relations, integrals over the eigenvalues can be performed by means of orthogonality relations. In the next section we illustrate this method by the calculation of the microscopic spectral density for the chGUE.
6.2.3 The Replica Trick. The replica trick is based on the identity

$$G(z) = \frac{1}{V} \langle \text{Tr} \frac{1}{z+iD} \rangle = \lim_{r\to0} \frac{1}{Vr} \partial_z \langle \det^r(iD + z) \rangle.$$  \hspace{1cm} (6.14)

The recipe is to calculate the partition function for positive or negative integer values of $r$ and then analytically continue to $r = 0$. For positive (negative) integer values of $r$ the average determinant can be calculated by rewriting it as a Grassmann (complex) Gaussian integral. Then $D$ appears linear in the exponent which allows us to perform the average for a Gaussian distribution of $D$. The replica trick works without problems for perturbative calculations [55] but usually fails in nonperturbative calculations [21]. As example consider the following expression for the the modified Bessel function $I_\nu(z)$:

$$I_\nu(z) = \frac{1}{\pi} \int_0^\pi e^{z \cos \theta} \cos \nu \theta d\theta - \frac{\sin \nu \pi}{\pi} \int_0^\infty e^{-z \cosh t} e^{-\nu t} dt.$$  \hspace{1cm} (6.15)

We would have missed the second term if we calculate the Bessel function only for integer values of $\nu$. The replica trick can be made to work if we consider a family of partition functions related by a Toda lattice equation. This will be discussed in detail in the next two lectures.

6.2.4 The Supersymmetric Method. The supersymmetric method [76] is based on the identity

$$G(z) = \frac{1}{V} \partial_z \left\langle \frac{\det(iD + z)}{\det(iD + z')} \right\rangle_{z'=z}.$$  \hspace{1cm} (6.16)

The determinant can be written as a Grassmann integral and the inverse determinant as a complex integral. For $z' = z$ this partition function has an exact supersymmetry. The advantage of this method is that it is mathematically rigorous, but it requires a deep understanding of super mathematics. For example, finite expressions can be obtained from singular terms that do not depend on the Grassmann variables (and are zero upon integration). For a discussion of these so-called Efetov-Wegner terms we refer to the literature [56, 37].

6.3 The Microscopic Spectral Density of the chGUE

In this subsection we calculate the microscopic spectral density by the orthogonal polynomial method [57] and the supersymmetric method [19, 20].

6.3.1 Orthogonal Polynomials. For the chGUE the joint probability distribution only depends on the square of the eigenvalues. In terms of $x_k = \lambda_k^2$ the weight function for $m_f = 0$ is given by

$$w(x) = (xn\sum^2)^a e^{-n\sum^2x},$$  \hspace{1cm} (6.17)
with \( a = N_f + |\nu| \). The monic orthogonal polynomials corresponding to this weight function can be expressed in terms of Laguerre polynomials

\[
P_k(x) = \frac{(-1)^k k!}{(\Sigma^2 n)^k} L_k^a(x \Sigma^2 n)
\]

(6.18)

with normalization constants \( h_k \) given by

\[
h_k = \frac{k!}{(k + a)!} / (n \Sigma^2)^{2k+1}.
\]

The eigenvalue density is given by (with \( c \) a constant)

\[
\rho(x_1) = c \int \prod_{k=2}^n \left[ w(x_k)dx_k \right] \begin{vmatrix}
P_0(x_1) & \cdots & P_0(x_n) \\
\vdots & \ddots & \vdots \\
P_{n-1}(x_1) & \cdots & P_{n-1}(x_n)
\end{vmatrix}^2,
\]

\[
= \sum_{\sigma \pi} \text{sg} (\sigma \pi) \prod_{k=2}^n \left[ w(x_k)dx_k \right] P_{\sigma(0)}(x_1) \cdots P_{\sigma(n-1)}(x_n) \\
\times P_{\pi(0)}(x_1) \cdots P_{\pi(n-1)}(x_n),
\]

\[
= (n - 1)! \prod_{l=0}^{n-1} h_l \sum_{k=0}^{n-1} \frac{1}{h_k} P_k^2(x_1) w(x_1).
\]

(6.19)

The microscopic spectral density is obtained by taking the limit \( n \to \infty \) for fixed \( z = 2n \Sigma \lambda = 2n \Sigma \sqrt{x} \). In this limit the weight function is given by \( w(x) = (x \Sigma^2 n)^a \), and the Laguerre polynomials behave as

\[
L_k^a(x \Sigma^2 n) \to k^a (x \Sigma^2 n k)^{-a/2} J_a(2 \Sigma \sqrt{x nk}).
\]

(6.20)

In the limit \( n \to \infty \), the sum can be replaced by an integral resulting in the microscopic spectral density [57]

\[
\rho(z) \sim z \sum_{k=0}^{n-1} J_a^2 \left( \frac{k}{n} \right) \approx z \int_0^1 t dt J_a^2(zt),
\]

\[
= 2z(J_a^2(z) - J_a(z)J_{a+1}(z)).
\]

(6.21)

### 6.3.2 Supersymmetric Method

In this section we evaluate the resolvent of QCD for the simplest case of \( N_f = 0 \) and \( \nu = 0 \) in the domain \( z \ll F^2 / \Sigma L^2 \). In this domain the partition function is given by (see (5.25))

\[
Z(J) = \int_{Q \in Gl(1|1)} dU \exp \left[ \frac{\Sigma V}{2} \text{Str} \left( \begin{pmatrix} z + J & 0 \\ 0 & z \end{pmatrix} \right)(Q + Q^{-1}) \right],
\]

(6.22)

where the integration is over the maximum super-Riemannian sub-manifold of \( Gl(1|1) \). This manifold is parametrized by

\[
Q = \exp \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \begin{pmatrix} e^{i \phi} & 0 \\ 0 & e^{i \theta} \end{pmatrix}.
\]

(6.23)
The integration measure is the Haar measure which in terms of this parameterization is given by (with $\delta Q \equiv Q^{-1}dQ$)

$$S\det \frac{\delta Q_{kl}}{\delta \phi \delta s \delta \alpha \delta \beta} d\alpha d\beta d\phi ds. \quad (6.24)$$

It is straightforward to calculate the Berezinian going from the variables $\{\delta Q_{11}, \delta Q_{22}, \delta Q_{12}, \delta Q_{21}\}$ to the variables $\{\delta \phi, \delta s, \delta \alpha, \delta \beta\}$. The derivative matrix is given by

$$B = \frac{\delta Q_{kl}}{\delta \phi \delta s \delta \alpha \delta \beta} = \begin{pmatrix} i & 0 & \frac{\beta}{2} & \frac{\alpha}{2} \\ 0 & 1 & \frac{\beta}{2} & \frac{\alpha}{2} \\ 0 & 0 & e^{s-i\phi} & 0 \\ 0 & 0 & 0 & e^{-s+i\phi} \end{pmatrix}. \quad (6.25)$$

Using the definition of the graded determinant one simply finds that $S\det B = i$. Up to a constant, the integration measure is thus given by $d\phi ds d\alpha d\beta$.

We also need

$$\frac{1}{2}(Q + Q^{-1}) = \begin{pmatrix} \cos \phi(1 + \frac{\alpha \beta}{2}) & \alpha(e^s-e^{-i\phi}) \\ \beta(e^{i\phi} - e^{-s}) & \cosh s(1 - \frac{\alpha \beta}{2}) \end{pmatrix}. \quad (6.26)$$

After differentiating with respect to the source term ($G(z) = \partial_J \log Z(J)|_{J=0}$) this results in (with $x = V \Sigma z$)

$$\frac{G(z)}{V \Sigma} = \int \frac{d\phi ds d\alpha d\beta}{4\pi} \cos \phi(1 + \frac{\alpha \beta}{2}) e^x \cos \phi(1 + \frac{\alpha \beta}{2}) - x \cosh s(1 - \frac{\alpha \beta}{2}). \quad (6.27)$$

With the Grassmann integral given by the coefficient of $\alpha \beta$ we obtain

$$\frac{G(z)}{V \Sigma} = \int \frac{ds d\phi}{4\pi} [\cos \phi + x(\cos \phi + \cosh s) \cos \phi] e^x(\cos \phi - \cosh s).$$

All integrals can be expressed in terms of modified Bessel functions. We find

$$\frac{G(z)}{V \Sigma} = I_1(x)K_0(x) + \frac{x}{2}(I_2(x)K_0(x) + I_0(x)K_0(x) + 2I_1(x)K_1(x)), \quad (6.28)$$

which can be further simplified by the recursion relation $I_2(x) = I_0(x) - 2I_1(x)/x$. As final result we obtain [51, 19, 20]

$$\frac{G(z)}{V \Sigma} = x(I_0(x)K_0(x) + I_1(x)K_1(x)). \quad (6.29)$$
This calculation can be generalized to arbitrary $N_f$ and arbitrary $\nu$. The calculation for arbitrary $N_f$ with $m_f = 0$ is much more complicated, but with a natural generalization of the factorized parameterization, and using some known integrals over the unitary group, one arrives at the following expression in terms of modified Bessel functions

$$\frac{G(z)}{\nu} = \frac{\nu}{x} + x(I_{a}(x)K_{a}(x) + I_{a+1}(x)K_{a-1}(x)),$$

(6.30)

where $a = N_f + |\nu|$. This result is in complete agreement with the resolvent obtained [51] from integrating the microscopic spectral density (6.21).

For $a = 0$ this result is plotted in Fig. 3. We observe that, below some scale, lattice QCD data obtained by the Columbia group [58] closely follow this curve. The predictions of chRMT or of the partially quenched chiral Lagrangian have been studied by numerous lattice simulations [16, 17, 59, 50]. In all cases, agreement has been found in the expected domain of applicability.

![Figure 3.](image)

The resolvent of quenched QCD. The points represent lattice data obtained by the Columbia group, and the theoretical prediction (6.29) is given by the solid curve. (Taken from ref. [45].)
7. Integrability and the QCD Partition Function

7.1 Virasoro Constraints

In this section we derive the small mass expansion of the QCD partition function by means of recursion relations for the partition function known as Virasoro constraints. The starting point is the QCD partition function in the ergodic regime given in (5.21). The quantities

\[ G_\nu(t_k) = \det^{\nu}(M)Z_{\nu,N_f}(M) \]  

are invariant under the $U(N_f) \times U(N_f)$ transformations $M \rightarrow V_1MV_2^{-1}$. (Here and below we drop the superscript eff.) Therefore, $G_\nu(t_k)$ only depends on the eigenvalues of $M^\dagger M$ which can be parameterized in terms of the moments

\[ t_k \equiv \frac{1}{k}\text{Tr}\left(\frac{MM^\dagger}{4}\right)^k. \]  

A differential equation for the $G_\nu(t_k)$ is obtained from the unitarity relation

\[ \frac{1}{Z_{\nu,N_f}(M)} \sum_{a=1}^{N_f} \partial^2 Z_{\nu,N_f}(M) = \frac{1}{4} \sum_{a=1}^{N_f} (U_{ab}^\dagger U_{ca}) = \frac{1}{4} \delta_{bc}. \]  

Notice that the factor $\Sigma V$ is included in $M$. This relation can be rewritten as

\[ \left[ \frac{\partial^2}{\partial M_{ba}\partial M_{ac}} + \nu M_{ab}^{-1} \frac{\partial}{\partial M_{bc}} \right] G_\nu(t_k) = \frac{1}{4} G_\nu(t_k)\delta_{bc}. \]  

Using the chain rule and the assumption that the matrix elements of $(MM^\dagger)^{s-1}$ are independent for different values of $s$, we obtain

\[ [\mathcal{L}_s - \delta_{s,1}]G_\nu(t_k) = 0 \quad s \geq 1. \]  

The Virasoro operators $\mathcal{L}_s$ defined by

\[ \mathcal{L}_s = \sum_{k=1}^{s-1} \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_{s-k}} + \sum_{k \geq 1} t_k \frac{\partial}{\partial t_{s+k}} + (N_f + \nu) \frac{\partial}{\partial t_s}, \quad s \geq 1 \]  

satisfy the Virasoro algebra

\[ [\mathcal{L}_r, \mathcal{L}_s] = (r - s)\mathcal{L}_{r+s}. \]  

Therefore, all Virasoro constraints with $s \geq 1$ are satisfied if $G_\nu$ satisfies

\[ [\mathcal{L}_1 - 1]G_\nu = 0 \quad \text{and} \quad \mathcal{L}_2 G_\nu = 0. \]  

This justifies the independence assumption above Eq. (7.5).
7.1.1 Solution of the Virasoro Constraints. We can expand $G_\nu$ as
\[ G_\nu = 1 + a_1 t_1 + a_2 t_2 + a_{11} t_1^2 + \cdots. \] (7.9)
From the first Virasoro constraint we obtain
\[ \mathcal{L}_1 G_\nu = a_2 t_1 + (N_f + \nu)(a_1 + 2a_{11} t_1) + \cdots, \]
\[ = 1 + a_1 t_1 + a_2 t_2 + a_{11} t_1^2 + \cdots. \] (7.10)
By equating the coefficients of the $t_k$ we find
\[ a_1 = \frac{1}{N_f + \nu}, \quad a_2 + 2(N_f + \nu)a_{11} = a_1. \] (7.11)
From the second Virasoro constraint, $\mathcal{L}_2 G_\nu = a_{11} + (N_f + \nu)a_2 = 0$, we obtain\[ a_2 = -a_{11}/(N_f + \nu). \] Continuing this way we can obtain all coefficients in the expansion of the $G(t_k)$. This results in the small mass expansion of the partition function [60]
\[ \frac{Z_\nu^\text{QCD}(M)}{\det^\nu M} = [1 + \frac{\text{Tr} M M^\dagger}{4(N_f + \nu)} + \frac{1}{32} \frac{\text{Tr}(M M^\dagger)^2}{(N_f + \nu)((N_f + \nu)^2 - 1)} + \cdots]. \] (7.12)
An extension of this expansion to all three Dyson classes can be found in [61].

The small mass expansion can be used to obtain sum rules for the inverse eigenvalues of the Dirac operator [13, 60]. The QCD partition function can be expanded as (the prime indicates that $\lambda_k \neq 0$)
\[ Z_\nu^\text{QCD}(M) = m^\nu N_f \langle \prod_k' \lambda_k^{2N_f} (1 + \frac{m^2}{\lambda_k^{2N_f}})^{N_f} \rangle^\nu \]
\[ = m^\nu N_f \langle \prod_k' \lambda_k^{2N_f} \rangle^\nu + m^2 N_f \langle \prod_k' \lambda_k^{2N_f} \sum_k' \frac{1}{\lambda_k^{2N_f}} \rangle + \cdots \] (7.13)
This results in the expansion
\[ \frac{Z_\nu^\text{QCD}(M)}{\lim_{m \to 0} m^{-\nu N_f} Z_\nu^\text{QCD}(M)} = 1 + m^2 N_f \langle \sum_k' \frac{1}{\lambda_k^{2N_f}} \rangle^\nu \text{QCD}. \] (7.14)
By equating this expansion to the expansion (7.12) for equal masses given by
\[ 1 + \frac{N_f(V\Sigma)^2}{4(N_f + \nu)m^2}, \] (7.15)
we obtain a Leutwyler-Smilga sum rule [13] for the inverse Dirac eigenvalues
\[ \frac{1}{V^2} \sum_k' \frac{1}{\lambda_k^2} = \frac{\Sigma^2}{4(N_f + \nu)}. \] (7.16)
7.1.2 Flavor-Topology Duality. If we construct an \((N_f + \nu) \times (N_f + \nu)\) matrix \(\bar{M}\) with \(\bar{M}_{ij} = M_{ij}\) for \(i, j \leq N_f\) and \(\bar{M}_{ij} = 0\) otherwise, we have that \(\text{Tr}(MM^\dagger)^k = \text{Tr}(\bar{M}M^\dagger)^k\). Since the Virasoro constraints only depend on the combination \(N_f + \nu\) we have

\[
\det^{-\nu}M Z_{\nu,N_f}(M) = Z_{\nu=0,N_f+\nu}(\bar{M}).
\]  

(7.17)

This relation is known at the flavor-topology duality [62].

7.2 \(\tau\)-Function

The unitary integral in the QCD partition function can actually be evaluated analytically for an arbitrary number of flavors. We will show that it can be rewritten as a \(\tau\)-function. The unitary matrix integrals can then be evaluated by means of a Harish-Chandra-Itzykson-Zuber type integral and the use of flavor-topology duality.

7.2.1 Itzykson-Zuber Integral. We consider the integral

\[
I = \int dUdV e^{\frac{1}{2}Tr(U^\dagger RV S + SV^\dagger RU)},
\]  

(7.18)

where \(U \in U(N_1)\) and \(V \in U(N_2)/U N_2(1)\) and the integral is over the Haar measure of these groups. The matrices \(R\) and \(S\) are arbitrary rectangular complex matrices. Without loss of generality, they can be taken diagonal with \(R_{kk} = r_k > 0\) and \(S_{kk} = s_k > 0\) and all other matrix element equal to zero. Using the diffusion equation method one can derive the result [63]

\[
I = \prod_k (r_k s_k)^{\nu} \frac{\det I_{\nu}(r_k s_l)}{\Delta^{(s_k^2)} \Delta^{(r_k^2)}}.
\]  

(7.19)

This result first appeared in the Russian literature [64] as a solution of the Laplace equation. It was proved independently in [63].

7.2.2 The QCD Partition Function is a \(\tau\)-Function. In this subsection we show that the finite volume QCD partition function is a \(\tau\)-function. Using the flavor-topology duality (7.17) with \(\bar{M}_{kk} = x_k\) for \(k \leq N_f\) we can write

\[
Z_{\nu,N_f}(M) = \det^{-\nu}M \int_{U \in U(N_f+\nu)} dU e^{\frac{1}{2}Tr(\bar{M}U^\dagger + U^\dagger \bar{M})}.
\]  

(7.20)

This integral is exactly (7.18) with \(R\) equal to the \(N_f \times (N_f + \nu)\) matrix with \(r_k = x_k\). A finite result is obtained if the \(N_f\) diagonal matrix elements of \(S\) are expanded as \(s_k = 1 + \delta s_k\). For \(\delta s_k \to 0\) we obtain

\[
I_{\nu}(x_k s_l) = \sum_{j=1}^{N_f} \frac{x_k^{j-1}}{(j-1)!} I_{\nu}(x_k)(\delta s_l)^{j-1} = \sum_{j=1}^{N_f} A_{kj} B_{jl}.
\]  

(7.21)
with (the upper index between brackets such as \((k)\) denotes the \(k\)’th derivative)

\[ A_{kj} = x_{j}^{(k-1)} I_{\nu}^{(j-1)}(x), \quad B_{jl} = \frac{(\delta s_{k})^{j-1}}{(j-1)!}. \]  

(7.22)

Up to a constant, the determinant of \(B\) is given by

\[ \det B \sim \Delta(\{(1 + \delta s_{k})^{2}\}), \]  

(7.23)

and cancels against the denominator in (7.19). We finally obtain the result

\[ Z_{\nu,Nf}(M) = \frac{\det[x_{k}^{(j-1)} I_{\nu}^{(j-1)}(x)]}{\Delta(\{x_{k}^{2}\})}. \]  

(7.24)

This result was first obtained in [65] and independently for equal masses in [9, 66]. Using identities such as

\[ x^2 \partial_{x}^2 = (x \partial_{x})^2 - x \partial_{x}, \quad x^3 \partial_{x}^3 = (x \partial_{x})^3 - 3(\partial_{x})^2 + 2x \partial_{x}, \]  

(7.25)

we can rewrite this partition function in terms of derivatives \(\delta_{k} \equiv x_{k} \partial_{x_{k}}\) as

\[ Z_{\nu,Nf}(M) = \frac{\det[\delta_{k}^{j-1} I_{\nu}(x)]}{\Delta(\{x_{k}^{2}\})}. \]  

(7.26)

This form of the partition function is also known as a \(\tau\)-function [22, 67].

**7.2.3 QCD Partition Function for Equal Masses.** The limit of equal masses in the partition function (7.26) can be obtained by writing

\[ x_{k} = x(1 + \delta x_{k}), \]  

(7.27)

and taking the limit \(\delta x_{k} \to 0\). Because all columns are the same for \(\delta x_{k} = 0\), we have to expand the matrix elements to order \((\delta x_{k})^{N_{f}-1}\). The expansion to this order can be combined into

\[ \det[\delta_{k}^{j-1} I_{\nu}(x)] = \det[x^{l-1}(\delta_{k}^{j-1} I_{\nu}(x))^{(l-1)}] \det\left[\frac{(\delta x_{k})^{l-1}}{(l-1)!}\right]. \]  

(7.28)

Using that

\[ \det[(\delta x_{k})^{l-1}] = \Delta(\{\delta x_{k}\}), \quad \Delta(\{x_{k}^{2}\}) = x^{N_{f}(N_{f}-1)} \Delta(\{\delta x_{k}\}). \]  

(7.29)

we obtain the partition function

\[ Z_{\nu,Nf}(x) = cx^{-N_{f}(N_{f}-1)} \det[x^{l-1}(x \partial_{x})^{j-1} I_{\nu}(x)]^{(l-1)}]. \]  

(7.30)

With the help of the identities (7.25) the derivatives can be combined into derivatives \((x \partial_{x})^{p}\) resulting in

\[ Z_{\nu,Nf}(x) = cx^{-N_{f}(N_{f}-1)} \det[(x \partial_{x})^{l+j-2} I_{\nu}(x)]_{1 \leq l \leq N_{f}, 1 \leq j \leq N_{f}}. \]  

(7.31)
7.3 Toda Lattice Equation

In this section we show that the QCD partition function for equal masses satisfies a Toda lattice equation. This result and generalizations thereof were first obtained in [68]. The Toda lattice was originally introduced as a one dimensional lattice in which neighboring atoms interact via a potential that depends exponentially on the distance. The Hamiltonian equations of motion of this system can be written in the form of the Toda lattice equation discussed below. Because of the existence of a Lax pair, they have infinitely many constants of motion. For a more elaborate discussion of the Toda lattice equation and the relation to integrable systems, we refer to [29, 69, 70]. Several subsections below are based on the paper by Forrester and Witte [71].

7.3.1 The Sylvester Identity. We all know how to expand a determinant matrix with respect to its co-factors given by

\[ C_{ij} = \frac{\partial}{\partial A_{ij}} \det A. \]  

(7.32)

What is less known is that there exists a remarkable identity that relates co-factors to the double co-factors defined by

\[ C_{ij;pq} = \frac{\partial^2}{\partial A_{ij} \partial A_{pq}} \det A. \]  

(7.33)

This identity, which is known as the Sylvester identity [72], is given by

\[ C_{ij} C_{pq} - C_{iq} C_{pj} = \det AC_{ij;pq}. \]  

(7.34)

For example, it holds for a \( 2 \times 2 \) matrix with \( i = j = 1 \) and \( p = q = 2 \).

7.3.2 Toda Lattice Equation. We apply the Sylvester identity to the determinant that appears in the partition function (7.31). For \( i = j = N_f - 1 \) and \( p = q = N_f \) we obtain

\[ C_{N_f-1,N_f-1} C_{N_f,N_f} - C_{N_f-1,N_f} C_{N_f,N_f-1} = \det AC_{N_f-1,N_f-1;N_f,N_f}, \]  

(7.35)

with matrix \( A \) given by

\[ A_{jl} \equiv (x\partial_x)^{l+j-2} I_\nu(x). \]  

(7.36)

The derivative of a determinant is equal to the sum of determinants with one of the rows replaced by its derivatives, or is equal to sum of the determinants with one of the columns replaced by its derivative (in both cases we have in total \( N_f \) terms). For the matrix \( A \) in (7.36) only differentiating the last row or column
gives a nonzero result. This allows us to rewrite the co-factors as derivatives of \( \det A \). In particular, we find

\[
C_{N_f-1,N_f} = -x \partial_x \det A_{N_f-1}, \quad C_{N_f,N_f-1} = -x \partial_x \det A_{N_f-1},
\]

\[
C_{N_f,N_f} = \det A_{N_f-1}, \quad C_{N_f-1,N_f-1} = (x \partial_x)^2 \det A_{N_f-1}.
\]

(7.37)

To obtain the second last identity we first differentiate the columns and then the rows. Inserting this in (7.35) we find

\[
(x \partial_x)^2 \log \det A_{N_f-1} = \frac{\det A_{N_f} \det A_{N_f-2}}{\det^2 A_{N_f-1}}.
\]

(7.38)

Next we substitute the relation between \( \det A_{N_f} \) and the partition function

\[
\det A_{N_f} = \frac{1}{c} x^{N_f(N_f-1)} Z_{\nu,N_f}^N(x).
\]

(7.39)

The prefactor contributes a factor \( x^2 \) to the r.h.s. of (7.38) but does not contribute to its l.h.s.. After raising \( N_f \) by 1 we obtain the celebrated Toda lattice equation [22, 67]

\[
(x \partial_x)^2 \log Z_{\nu,N_f}(x) = cx^2 Z_{\nu,N_f+1}(x) Z_{\nu,N_f-1}(x) \left[ Z_{\nu,N_f}(x) \right]^2.
\]

(7.40)

The large-\( x \) limit of the partition function for equal masses,

\[
Z_{\nu,N_f}(x) = \int_{U \in U(N_f)} dU \det^n U e^{\frac{x}{2} \text{Tr}(U + U^{-1})},
\]

(7.41)

is obtained by performing the \( U \)-integral by a saddle point approximation including the Gaussian fluctuations. The asymptotic behavior is given by

\[
Z_{\nu,N_f}(x) \sim e^{N_f x / x^{N_f/2}}.
\]

(7.42)

Using this result to normalize the partition function we find that \( c = N_f \).

### 7.4 Painlevé System

In this subsection we derive the Toda lattice from the Backlund transformation of a Painlevé equation [71].

The partition function (7.41) can be obtained from the “double” scaling limit of the random matrix partition function

\[
Z_{\nu,N_f}^{\text{RMT}}(x) = \int \prod_{k=1}^{N_f} d\lambda_k \lambda_k^{2\nu+1} (\lambda_k^2 + m^2)^{N_f} m^{\nu N_f} |\Delta(\{\lambda_k^2\})|^2 e^{-\frac{N_f x}{2} \sum_i \lambda_i^2},
\]

(7.43)
where \( x = mN\Sigma \) is kept fixed for \( N \to \infty \). Using \( x_k^2 = \lambda_k^2 \Sigma^2 N^2 + x^2 \) as new integration variables, we obtain in this limit

\[
Z_{\nu,N_f}^{\text{RMT}}(x) = \prod_k \left[ \int_{x^2}^{\infty} dx_k^2 (x_k^2 - x^2)^{2N_f} x_k^{\nu N_f} e^{x^2/4} e^{-\sum_k x_k^2/2N} \right]^{x^\nu N_f} e^{x^2/4} E_N([0,x^2],N_f,\nu). \tag{7.44}
\]

\( E_N([0,s = x^2],N_f,\nu) \) can be interpreted as the probability that there are no eigenvalues in the interval \([0,x^2]\) for the joint probability distribution given by the integrand of (7.44). For \( \nu = 0 \) this is the probability of the partition function with topological charge \( N_f \) and no flavors. If we introduce \( \sigma(t) \) by

\[
E_N([0,s = x^2],N_f,\nu) = e^{-\int_0^s \frac{4t}{N_f} \sigma(t) + \frac{1}{2} \nu (N_f + \nu)), \tag{7.45}\]

then the function \( \sigma(t) \) satisfies the Painlevé equation [71],

\[
(t \sigma'' - (N_f^2 - \nu^2)(\sigma')^2 + \sigma'(4\sigma' - 1)(\sigma - t \sigma') - \frac{\nu^2}{16} = 0. \tag{7.46}\]

The boundary conditions for this differential equation follow from the asymptotic behavior of the partition function (7.41). Using (7.42) we find for the large \( x \) behavior of \( E_N([0,s = x^2],N_f,\nu) \),

\[
E_N([0,s = x^2],N_f,\nu) \sim x^{-N_f^2/2 - \nu N_f} e^{N_f x - x^2/4}, \tag{7.47}
\]

so that the large-\( s \) behavior of \( \sigma(s) \) is given by

\[
\sigma(s) \sim \frac{N_f}{2} \sqrt{s} + \frac{s}{4} - \frac{\nu^2}{2} + \frac{N_f^2}{4}. \tag{7.48}
\]

The Painlevé equation (7.46) can be derived from the Hamiltonian [71]

\[
\dot{t}H = q^2 p^2 - (q^2 + (N_f + \nu)q - t)p + N_f q \tag{7.49}
\]

with the identification

\[
\sigma(t) = - \left. (tH) \right|_{t \to \frac{t}{4}} - \frac{1}{2} (N_f + \nu) \nu + \frac{t}{4}. \tag{7.50}
\]

For example, we have the Hamiltonian equations of motion

\[
(tH)' = p, \quad (tH)'' = p' = -\frac{\partial H}{\partial q}, \quad tq' = \frac{\partial tH}{\partial p}. \tag{7.51}
\]

Such Hamiltonians play an important role in the theory of exactly solvable models. Hamiltonians with different values of \( \nu \) and \( N_f \) are connected by
a Backlund transformation. This is a canonical transformation together with 
\((\nu, N_f) \rightarrow (\bar{\nu}, \bar{N}_f)\) such that, in the new variables, the same Painlevé equation 
is satisfied. In our case, we have the Backlund transformation (at fixed \(\nu\))

\[
T : \quad \begin{align*}
N_f & \rightarrow N_f + 1, \\
H_{N_f} & \rightarrow H_{N_f+1} = H_{N_f} + q_{N_f} - q_{N_f} p_{N_f}, \\
q_{N_f} & \rightarrow q_{N_f+1}, \\
p_{N_f} & \rightarrow p_{N_f+1}.
\end{align*}
\] (7.52)

Below we do not need the explicit transformation rules of \(q_{N_f}\) and \(p_{N_f}\), but of 
the inverse transformation of \(q_{N_f}(p_{N_f} - 1)\) which is given by

\[
T^{-1} q_{N_f}(p_{N_f} - 1) \rightarrow -q_{N_f}(p_{N_f} - 1) + N_f + \nu - \frac{N_f}{p_{N_f}}. \quad (7.53)
\]

If we define the \(\tau\)-function by

\[
\tau_{N_f} = e^{\int_0^t H_{N_f} dt},
\] (7.54)

one can easily derive the equalities

\[
\begin{align*}
t \partial_t \log \frac{\tau_{N_f-1} \tau_{N_f+1}}{\tau_{N_f}^2} &= tH_{N_f-1} + tH_{N_f+1} - tH_{N_f}, \\
&= -T^{-1} (q_{N_f}(1 - p_{N_f}) + (q_{N_f}(1 - p_{N_f})), \\
&= t \partial_t \log \partial_t [tH_{N_f}], \\
&= t \partial_t \log \partial_t [t \partial_t \log \tau_{N_f}].
\end{align*}
\] (7.55)

To derive the second last equality we have used the inverse Backlund transformation and the Hamilton equations (7.51). Integrating this equation once and putting the integration constant equal to zero we find the Toda lattice equation

\[
(t \partial_t)^2 \log \tau_{N_f} = t \frac{\tau_{N_f+1} \tau_{N_f-1}}{\tau_{N_f}^2}. \quad (7.56)
\]

### 7.4.1 Solutions of the Painlevé equation

The probability \(E([0, s], N_f, \nu)\) is related to the partition function (7.41) by

\[
E([0, s], N_f, \nu) = s^{-N_f(\nu/2)} e^{-s/4} Z_{\nu, N_f}(\sqrt{s}). \quad (7.57)
\]

For \(N_f = 0\) the partition function is normalized to 1 so that

\[
\sigma_{N_f=0}(s) = s - \frac{\nu^2}{2}. \quad (7.58)
\]
Indeed this is a solution of the PIII Painlevé equation (7.46).

For \( N_f = 1 \) we have that

\[
E([0, s], N_f = 1, \nu) = s^{-\nu/2}e^{-s/4}I_\nu(\sqrt{s}) = e^{-\int_0^s \frac{d}{dt}(\sigma(t) + \frac{1}{2} \nu(1+\nu))}, \tag{7.59}
\]

resulting in another solution of (7.46):

\[
\sigma_{N_f=1}(s) = \frac{s}{4} - \frac{\nu^2}{2} - s \frac{d}{ds} \log I_\nu(\sqrt{s}). \tag{7.60}
\]

For \( x \to 0 \) the modified Bessel function behaves as \( I_\nu(x) \sim x^\nu \) so that the \( t \)-integral in (7.59) is well-behaved for \( t \to 0 \). Only recursion relations for Bessel functions are required to show that (7.59) is a solution of the Painlevé equation. Since \((-1)^\nu K_\nu(x)\) satisfies the same recursion relations as \( I_\nu(x) \) this provides us with another solution of the Painlevé equation. This solution corresponds to the partition function with \( N_f = -1 \) where

\[
E([0, s], N_f = -1, \nu) = s^{\nu/2}e^{-s/4}K_\nu(\sqrt{s}) = e^{-\int_0^s \frac{d}{dt}(\sigma(t) + \frac{1}{2} \nu(-1+\nu))}, \tag{7.61}
\]

and satisfies the boundary condition for \( N_f = -1 \).

7.4.2 The Bosonic Partition Function. The natural interpretation of \( N_f = -1 \) is as a bosonic flavor. In this section we will derive the low-energy limit of the QCD partition function for \( N_f \) bosonic flavors with equal masses. We already have seen in section 3.3.2 that the Goldstone manifold for \( n \) bosonic quarks is given by \( \text{Gl}(n)/U(n) \). Using the same invariance arguments as before one obtains the low-energy effective partition function

\[
Z_{\nu,-n} = \int_{Q \in \text{Gl}(n)/U(n)} \text{det}^{\nu}(Q) e^{\frac{1}{2} V \text{Tr} M(Q+Q^{-1})}. \tag{7.62}
\]

In this case \( Q \) can be diagonalized as \( Q = U \text{diag}(e^{s_k})U^{-1} \), so that an eigenvalue representation of this partition function is given by [73]

\[
\int \prod_k ds_k \prod_k e^{\nu s_k} \prod_{k<l} (e^{s_k} - e^{s_l})(e^{-s_k} - e^{-s_l}) e^x \sum_k \cosh s_k. \tag{7.63}
\]

The Vandermonde determinant can be written as

\[
\prod_{k<l} (e^{s_k} - e^{s_l}) = \text{det} [e^{p s_k}]_{0 \leq p \leq n-1, 1 \leq q \leq n} \tag{7.64}
\]

and a similar expression for \( s_k \to -s_k \). By expanding the two determinants the integrals can be written as modified Bessel functions which can be combined into a determinant as follows [73]

\[
Z_{\nu,-n}(x) = c_{-n} \text{det} [K_{\nu+k+i}(x)]_{0 \leq k, i \leq n-1}. \tag{7.65}
\]
From the observation that \((-1)^\nu K_\nu(x)\) and \(I_\nu(x)\) satisfy the same recursion relations, and that the factor \((-1)^\nu\) does not affect the determinant, (7.65) can be rewritten as the \(\tau\)-function

\[
Z_{\nu,-n}(x) = \frac{c-n}{n^{n(n-1)}} \det[(x\partial_x)^{k+1}Z_{\nu,-1}(x)]_{0\leq k,l \leq n-1}. \tag{7.66}
\]

with \(Z_{\nu,-1}(x) = K_\nu(x)\). The bosonic partition function can also be analyzed along the same lines as the fermionic partition function. On the other hand this derivation can be simply modified to obtain the partition function for \(N_f\) fermionic flavors with equal mass.

The bosonic partition functions thus satisfy the same Toda lattice equations as the fermionic partition functions. The semi-infinite hierarchies are connected by

\[
\lim_{n \to 0} \frac{1}{n} (x\partial_x)^2 \log Z_{\nu,n}(x), \tag{7.67}
\]

which is related to a derivative of the resolvent.

### 7.5 Replica Limit of the Toda Lattice Equation

The resolvent can be obtained from the replica limit

\[
G(z) = \lim_{n \to 0} \frac{1}{n} \log Z_{\nu,n}(z). \tag{7.68}
\]

If we take the replica limit of the fermionic \((n > 0)\) or bosonic \((n < 0)\) partition functions directly, we will obtain a result that differs from the supersymmetric calculation. These problems can be avoided if the take the replica limit of the Toda lattice equation (7.40). With the normalization \(Z_{\nu,0}(x) = 1\) we obtain the relation

\[
x\partial_x xG(x) = 2x^2Z_{\nu,1}(x)Z_{\nu,-1}(x). \tag{7.69}
\]

Inserting the expressions for \(Z_{\nu,1}\) and \(Z_{\nu,-1}\) we find [23]

\[
G(x) = \frac{\nu}{x} + x(K_\nu(x)I_\nu(x) + K_{\nu-1}(x)I_{\nu+1}(x)), \tag{7.70}
\]

which agrees with the result obtained from the supersymmetric method (6.30). This result has also been derived from the solution of the Painlevé equation (7.46) for \(n \to 0\) [24].

The validity of the replica limit of the Toda lattice equation can be proved by extending the Toda lattice hierarchy to include an additional spectator boson with mass \(y\) and using the identity [74]

\[
\lim_{n \to 0} \frac{1}{n} (x\partial_x)^2 \log Z_{\nu,n}(x) = \lim_{y \to x} x\partial_x(x\partial_x + y\partial_y) \log Z_{\nu,1,-1}(x,y). \tag{7.71}
\]
7.6 Replica Limit for the GUE Two-Point Function

We have two possibilities for the generating function of the two-point function of the Gaussian Unitary Ensemble: a fermionic generating function or a bosonic generating function. The fermionic (bosonic) generating function for the two-point function is defined by

\[ Z_n(x, y) = \int dHP(H) \det^n(x + i\epsilon + H) \det^n(y - i\epsilon + H), \quad (7.72) \]

with \( n > 0 \) (\( n < 0 \)). We will consider the microscopic limit where \( \pi(x - y)N\rho(x) \equiv r \) is kept fixed for \( N \to \infty \). In that case the two-point function in the center of the spectrum only depends on \( r \) and is given by

\[ R_2(r) = -\lim_{n \to 0} \frac{1}{n^2} \partial_r^2 Z_n(r), \quad (7.73) \]

both in the fermionic and the bosonic case. In an eigenvalue representation of the Goldstone fields the microscopic limit of the generating function \( Z_n(r) \) can be written as \([21, 37]\)

\[ Z_n(r) = \int_{-1}^{1} \prod_k du_k \prod_{k<l} (u_k - u_l)^2 e^{ir \sum_k u_k}. \quad (7.74) \]

This partition (7.74) can be written as a \( \tau \)-function. The first step is to expand the Vandermonde determinant

\[ Z_n(r) = \int_{-1}^{1} \prod_k du_k \sum_{\sigma \pi} s g(\sigma \pi) u_1^{\sigma(1) + \pi(1)} \cdots u_n^{\sigma(n) + \pi(n)} e^{ir \sum_k u_k}. \quad (7.75) \]

Next we use that

\[ \int_{-1}^{1} du_k u_k^a e^{ir u_k} = (\partial_{ir})^a Z_1(r), \quad (7.76) \]

which results in [75]

\[ Z_n(r) = n! [\det(\partial_{ir})^{i+j} Z_1(x)]_{0 \leq i, j \leq n-1}. \quad (7.77) \]

The partition function \( Z_1(x) \) is given by

\[ Z_1(r) = \int_{-1}^{1} du e^{ir u}. \quad (7.78) \]

The microscopic limit of bosonic partition function can be rewritten similarly. The main difference is the convergence requirements of the bosonic
integrals which are essential for the structure of the Goldstone manifold. In an
eigenvalue representation of the Goldstone fields we find \[21\]

\[
Z_{-n}(r) = \int_1^\infty \prod_k du_k \prod_{k<l} (u_k - u_l)^2 e^{ir \sum_k u_k}.
\] (7.79)

This partition function can also be written as a \(\tau\)-function. By expanding the
Vandermonde determinant we can express this generating function as a determinant of derivatives

\[
Z_{-n}(r) = n! [\det (\partial_{ir})^{i+j} Z_{-1}]_{0\leq i,j \leq n-1},
\] (7.80)

with \(Z_{-1}(r)\) given by

\[
Z_{-1}(r) = \int_1^\infty d\xi e^{ir \xi}.
\] (7.81)

Because of the derivative structure of the partition function, we can again
use the Sylvester identity to derive a Toda lattice equation. In this case we find

\[
\partial_r^2 \log Z_n(x) = n^2 \frac{Z_{n+1}(r)Z_{n-1}(r)}{[Z_n(r)]^2},
\] (7.82)

where the factor \(n^2\) follows from the choice of the normalization constants.
We have made this choice because the left hand side is proportional to \(n^2\). The
two-point correlation is given by the replica limit of (7.82)

\[
R_2(r) = - \lim_{n \to 0} \frac{1}{n^2 \partial_r^2} \log Z_n(r) = Z_1(r)Z_{-1}(r)
\]

\[
= \int_{-1}^1 d\xi e^{i\xi x} \int_1^\infty e^{-ix} = 2i \frac{\sin x}{x} e^{ix},
\] (7.83)

which is the correct analytical result for the two-point function. This derivation
explains the factorization of the two-point function into a compact and a
non-compact integral which characterizes the result obtained by a supersym-
metric calculation \[76\]. The fermionic partition functions, the bosonic partition
functions and the super-symmetric partition function form a single integrable
hierarchy which are related by the Toda lattice equation \[23\]. A closely related
way to derive the two-point function of the GUE is to take the replica limit
of the corresponding Painlevé equation. For a discussion of this approach we
refer to \[24\] which preceded our work \[23\] on the Toda lattice.

### 8. QCD at Finite Baryon Density

In this Chapter we study the quenched microscopic spectrum of the QCD
Dirac operator at nonzero chemical potential when the Dirac operator is non-
Hermitian with eigenvalues scattered in the complex plane. Using the replica
limit of the Toda lattice equation we obtain the exact analytical result for the
microscopic spectral density \[25\].
8.1 General Remarks

The average spectral density of a non-Hermitian operator is given by
\[ \rho(\lambda) = \langle \sum_k \delta^2(\lambda - \lambda_k) \rangle, \quad (8.1) \]
and the average resolvent is defined by
\[ G(z) = \left\langle \sum_k \frac{1}{i\lambda_k + z} \right\rangle. \quad (8.2) \]
Using that \( \partial_z (1/z) = \pi \delta^2(z) \) we easily derive
\[ \partial_z G(z) |_{z=\lambda} = \pi \rho(\lambda). \quad (8.3) \]
The resolvent can be interpreted as the electric field in the plane at point \( z \) from charges located at the position of the eigenvalues. For example, Gauss law is given by
\[ \oint_C G(z) dz = 2\pi i Q, \quad (8.4) \]
where \( Q \) is the number of eigenvalues enclosed by \( C \).

8.2 The Ginibre Ensemble

To obtain a better understanding of the resolvent for a non-Hermitian random matrix ensemble, we first consider the Ginibre ensemble \([77]\) defined by the probability distribution
\[ \rho(C) = e^{-N \text{Tr} CC^\dagger}, \quad (8.5) \]
with \( C \) a complex \( N \times N \) matrix. The eigenvalues of \( C \) are given by the solutions of the secular equation \( \det(C - \lambda_k) = 0 \). If all eigenvalues are different, the matrix \( C \) can be decomposed as
\[ C = V \Lambda V^{-1}, \quad (8.6) \]
where \( V \) is a similarity transformation and \( \Lambda = \text{diag}(\lambda_1, \cdots, \lambda_N) \). The joint eigenvalue distribution is obtained by using the decomposition
\[ C = U T U^{-1}, \quad (8.7) \]
with \( U \) a unitary matrix and \( T \) an triangular matrix with the eigenvalues of \( C \) on the diagonal. After integrating out the upper triangular matrix elements we obtain (see the lectures by Zabrodin \([78]\) for a derivation),
\[ \rho(\{\lambda_k\}) = |\Delta(\{\lambda_k\})|^2 e^{-N \sum_k |\lambda_k|^2}. \quad (8.8) \]
This distribution can be interpreted as repulsive charges in the plane balanced by an external force $N|z|$. The resolvent is equal to the electric field of the eigenvalues. For an equilibrium distribution we have $G(z) = N|z|$. Using that the eigenvalue density is spherically symmetric, we find from Gauss law

$$2\pi r|G| = 2\pi \int_0^r \rho(r')dr', \quad (8.9)$$

so that $\rho(r) = N/\pi$. Because the total number of eigenvalues is equal to $N$, they are located inside the circle $|z| = 1$. The resolvent is thus given by

$$G(z) = Nz^* \theta(1 - |z|) + \frac{N}{z} \theta(|z| - 1). \quad (8.10)$$

### 8.3 QCD at Nonzero Chemical Potential

The QCD partition function at nonzero chemical potential $\mu$ is given by

$$Z_{\text{QCD}} = \sum_k e^{-\beta(E_k - \mu N_k)}, \quad (8.11)$$

where $E_k$ is the energy of the state, and $N_k$ is the quark number of the state. At zero temperature ($\beta \to \infty$) the partition function does not depend on $\mu$ for $\mu < m_N/N_N$, where $N$ is the particle with the smallest value of $m_N/N_N$. For QCD $N$ is the nucleon with quark number $N_N = 3$. This implies that the chiral condensate does not depend on $\mu$ for $\mu < m_N/N_N$.

The QCD partition function can be written as a Euclidean path integral with the fermionic part of the Lagrangian density given by

$$\mathcal{L} = \bar{\psi}D\psi + m\bar{\psi}\psi + \mu\bar{\psi}\gamma_0\psi. \quad (8.12)$$

with $D$ the anti-Hermitian Dirac operator. Since $\mu\gamma_0$ is Hermitian, the Dirac operator as a whole is non-Hermitian. As a consequence, the eigenvalues are scattered in the complex plane [79]. The fermion determinant is in general complex. This means that it is not possible to study the QCD partition function by means stochastic methods which severely limits our knowledge of QCD at nonzero chemical potential.

The question we wish to address is if there is a domain where the fluctuations of the Dirac eigenvalues are universal and can be obtained from a random matrix partition function with the global symmetries QCD, or equivalently from a chiral Lagrangian. In this domain we will calculate the resolvent and the spectral density from the replica limit of the Toda lattice equation [25, 26].

#### 8.3.1 Generating Function for the Quenched Spectral Density.

The quenched spectral density is given by the replica limit [80, 81, 82]

$$\rho^{\text{quen}}(z, z^*, \mu) = \lim_{n \to 0} \frac{1}{\pi n} \partial_z \partial_{z^*} \log Z_{\nu,n}(z, z^*, \mu), \quad (8.13)$$
with generating function given by
\[
Z_{\nu,n}(z, z^*, \mu) = \langle \det^n(D + \mu \gamma_0 + z) \det^n(-D + \mu \gamma_0 + z^*) \rangle. \tag{8.14}
\]
The product of the determinants in (8.14) can be written as the determinant of [83, 84]
\[
\begin{pmatrix}
id + \mu & 0 & z & 0 \\
0 & id - \mu & 0 & z^* \\
z & 0 & id^1 + \mu & 0 \\
z^* & 0 & id^1 - \mu & \\
\end{pmatrix}
\equiv \begin{pmatrix}
id + \mu_1 & M_{RL} \\
M_{LR} & id^1 + \mu_2 \\
\end{pmatrix}.
\tag{8.15}
\]
Here we have used the decomposition of the Dirac operator given in (3.9). We observe that the \(U(2n) \times U(2n)\) flavor symmetry is broken by the chemical potential term and the mass term. Invariance is recovered by transforming the mass term as in the case of zero chemical potential (see (5.2)) and the chemical potential term by a local gauge transformation [85]. For the chemical potential matrices the latter transformation is simply given by
\[
\mu_1 \to U_R \mu_1 U_R^{-1}, \quad \mu_2 \to U_L \mu_2 U_L^{-1}. \tag{8.16}
\]
The low-energy limit of quenched QCD should have the same transformation properties. In the domain \(\mu \ll 1/L\) and \(z \Sigma \ll F^2/L^2\), we only have to consider the zero momentum modes. Using that the Goldstone fields transform as \(U \to U_R U U_L^{-1}\) we can write down the following invariants to first order in the quark mass and to second order in the chemical potential
\[
\text{Tr} \mu_1^2, \quad \text{Tr} U^{-1} \mu_1 U \mu_2, \quad \text{Tr} M_{RL} U^{-1}, \quad \text{Tr} M_{LR} U. \tag{8.17}
\]
The low energy effective partition function is therefore given by
\[
Z_{\nu,n}(z, z^*, \mu) = \int_{U(2n)} dU \det^n U e^{-\frac{F^2 \text{Tr} B U^{-1} B + \Sigma V \text{Tr} M (U + U^{-1})}{2}}, \tag{8.18}
\]
where
\[
B = \begin{pmatrix}
1_n & 0 \\
0 & -1_n
\end{pmatrix}, \quad M = \begin{pmatrix}
z 1_n & 0 \\
0 & z^* 1_n
\end{pmatrix}. \tag{8.19}
\]

8.3.2 Random Matrix Model. The partition function (8.18) can be obtained from the large \(N\) limit of a random matrix model with the global symmetries of the QCD partition function. For \(\nu = 0\), the model is defined by an integral over \(N/2 \times N/2\) complex matrices [14, 86, 82],
\[
Z_{\nu=0}(m_f, \mu) = \int dW \prod_{f=1}^{N_f} \det(D(\mu) + m_f) e^{-\frac{N}{2} \Sigma^2 \text{Tr} W W^T}. \tag{8.20}
\]
The Dirac matrix has the structure

\[ D(\mu) = \begin{pmatrix} 0 & iW + \mu \\ iW^\dagger + \mu & 0 \end{pmatrix}. \tag{8.21} \]

For QCD with three or more colors in the fundamental representation, the matrix \( W \) is complex (\( \beta_D = 2 \)). One can also introduce random matrix ensembles with \( \beta_D = 1 \) or \( \beta_D = 4 \) by choosing the matrix elements of \( W \) real or quaternion real, respectively [87].

An alternative random matrix model [27] is obtained by replacing the identity matrix that multiplies \( \mu \) by a complex matrix with the same distribution as \( W \). This random matrix model is in the same universality class but turns out to be mathematically simpler. In particular, the joint eigenvalue distribution has been derived [27] which makes it possible to calculate correlation functions by means of the orthogonal polynomial method.

### 8.3.3 Mean Field Analysis.

The macroscopic spectral density of the partition function (8.18) can be easily obtained by means of a saddle point approximation [84]. Using an Ansatz that is diagonal in replica space,

\[ U = \begin{pmatrix} \cos \theta & e^{i\phi} \sin \theta \\ -e^{-i\phi} \sin \theta & \cos \theta \end{pmatrix}, \tag{8.22} \]

the partition function is given by

\[ Z_n(z, z^*, \mu) = e^{nV[2\mu^2F^2\sin^2 \theta + \Sigma(z+z^*) \cos \theta]} \tag{8.23} \]

The extrema are at

\[ \cos \theta = 1, \quad \text{or} \quad \cos \theta = \frac{\Sigma(z+z^*)}{4F^2\mu^2}. \tag{8.24} \]

The critical value of \( \mu \) is at the point where the two saddle points coincide,

\[ \mu_c^2 = \frac{\Sigma|z+z^*|}{4F^2}. \tag{8.25} \]

The partition function at the saddle point is given by

\begin{align*}
\mu < \mu_c : \quad Z_n(z, z^*, \mu) &= e^{nV\Sigma(z+z^*)}, \\
\mu > \mu_c : \quad Z_n(z, z^*, \mu) &= e^{nV(2\mu^2F^2+\Sigma^2(z+z^*)^2/(8F^2\mu^2))}. \tag{8.26}
\end{align*}

For the resolvent and the spectral density we thus find

\begin{align*}
\mu < \mu_c : \quad G_{\text{quen}}(z, z^*, \mu) &= V\Sigma, \quad \rho_{\text{quen}}(z, z^*, \mu) = 0, \tag{8.27} \\
\mu > \mu_c : \quad G_{\text{quen}}(z, z^*, \mu) &= \frac{V\Sigma^2(z+z^*)}{4\mu^2F^2}, \quad \rho_{\text{quen}}(z, z^*, \mu) = \frac{\Sigma^2V}{4\mu^2F^2}. \tag{8.28}
\end{align*}

The eigenvalues are located inside a strip of width \( 4F^2\mu_c^2/\Sigma \).
8.4 The Microscopic Spectral Density

The microscopic spectral density cannot be obtained from a mean field analysis. The assumption in the derivation of the previous section is that saddle point is proportional to the identity in replica space, so that the replica limit can be obtained from the calculation with one replica. The generating function for the microscopic spectral density depends in a nontrivial way on the number of replicas which, as we have seen before, can be obtained from the replica limit of a Toda lattice equation. In this subsection we closely follow [25]. The starting point is a remarkable integration formula to be discussed next.

8.4.1 Integration Formula. By decomposing a $U(2n)$ matrix as
\[
U = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \left( \begin{array}{cc} \sqrt{1-b^2} & b \\ b & -\sqrt{1-b^2} \end{array} \right) \begin{pmatrix} v_1^\dagger \\ v_2^\dagger \end{pmatrix},
\]
with $u_1, u_2, v_1 \in U(n), v_2 \in U(n)/U^n(1)$ and $b$ a diagonal matrix, the following integration formula can be proved [25]
\[
\int_{U(2n)} dU \det \nu \ e^{\frac{1}{2} \text{Tr}[M(U+U^{-1})+\sum_p a_p \text{Tr}((UBU^{-1}B)^p)]} \det[(x\partial_x)^k(y\partial_y)^l Z_{\nu,1}(x,y)]_{0 \leq k,l \leq n-1},
\]
where $c_n$ is an $n$-dependent constant and
\[
Z_{\nu,1}(x,y) = \int_0^1 \lambda d\lambda I_{\nu}(\lambda x) I_{\nu}(-\lambda y) e^{2 \sum_p a_p \cos(2p \cos^{-1} \lambda)}.
\]

8.4.2 Toda Lattice Equation at Nonzero Chemical Potential. Using the integration formula (8.29) for $p = 1$ we find that the zero momentum partition function $Z_{\nu,n}(z, z^*, \mu)$ (see Eq. (8.18)) can be written as
\[
Z_{\nu,n}(z, z^*, \mu) = \frac{c_n}{(z z^*)^{n(n-1)}} \det[(z\partial_z)^k(z^*\partial_{z^*})^l Z_{\nu,1}(z, z^*)]_{0 \leq k,l \leq n-1},
\]
where
\[
Z_{\nu,1}(z, z^*, \mu) = \int_0^1 \lambda d\lambda e^{-2V \Sigma} |I_{\nu}(\lambda z \Sigma)|^2.
\]
By applying the Sylvester identity to the determinant in (8.31) for $i = j = n - 1$ and $p = q = n$ and expressing the cofactors as derivatives, we find a recursion relation that can be written in the form of the Toda lattice equation
\[
z\partial_z z^* \partial_{z^*} \log Z_{\nu,n}(z, z^*, \mu) = \frac{\pi n}{2} (z z^*)^2 \frac{Z_{\nu,n+1}(z, z^*, \mu) Z_{\nu,n-1}(z, z^*, \mu)}{[Z_{\nu,n}(z, z^*, \mu)]^2},
\]
(8.33)
For the spectral density we find the simple expression \( (Z_{\nu,0}(z, z^*, \mu) = 1) \)

\[
\rho_{\text{quen}}(z, z^*, \mu) = \lim_{n \to 0} \frac{1}{\pi n} \partial_z \partial_{z^*} \log Z_{\nu,n}(z, z^*) = \frac{zz^*}{2} Z_{\nu,1}(z, z^*) Z_{\nu,-1}(z, z^*). \tag{8.34}
\]

What remains to be done is to calculate the bosonic partition function for \( n = -1 \) which will be completed in the next subsections.

### 8.4.3 The Bosonic Partition Function

In this subsection we evaluate the low-energy limit of the QCD partition function at nonzero chemical potential for one bosonic quark and one conjugate bosonic quark and \( \nu = 0 \). We closely follow [25]. Because of convergence requirements, the inverse determinants of nonhermitian operators have to be regulated. This is achieved by expressing them as the determinant of a larger Hermitian operator [83]

\[
\begin{align*}
\det^{-1} & \left( \begin{array}{cc} z & id+\mu \\ id^\dagger+\mu & z^* \end{array} \right) \det^{-1} \left( \begin{array}{cc} z^* & -id+\mu \\ -id^\dagger+\mu & z^* \end{array} \right) \\
= & \lim_{\epsilon \to 0} \det^{-1} \left( \begin{array}{cccc} \epsilon & 0 & z & id+\mu \\ 0 & \epsilon & id^\dagger+\mu & z \\ -id^\dagger+\mu & \epsilon & 0 & z^* \\ -id^\dagger+\mu & \epsilon & 0 & z^* \end{array} \right) \tag{8.35} \\
= & \int \exp \left[ i \sum_{j=1}^{N/2} \phi_k^{j*} \left( \begin{array}{cccc} \epsilon & z & 0 \\ z^* & \epsilon & 0 \\ 0 & -id^\dagger+\mu & -z^* \end{array} \right) \phi_k^j \right].
\end{align*}
\]

The mass matrices are given by

\[
\zeta_1 = \left( \begin{array}{cc} \epsilon & z \\ z^* & \epsilon \end{array} \right) \quad \text{and} \quad \zeta_2 = \left( \begin{array}{cc} \epsilon & -z^* \\ -z & \epsilon \end{array} \right) = -I \zeta_1 I. \tag{8.36}
\]

with \( I \equiv i\sigma_2 \). For the random matrix model (8.21) we have that \( d = W \). The Gaussian integral over \( W \) results in the 4-boson term \( \exp[-2\text{Tr}(\overline{Q}_1 \overline{Q}_2)/N] \)

with

\[
\overline{Q}_1 = \left( \begin{array}{cc} \phi_1^* \cdot \phi_1 & \phi_2^* \cdot \phi_2 \\ \phi_2^* \cdot \phi_1 & \phi_1^* \cdot \phi_2 \end{array} \right), \quad \overline{Q}_2 = \left( \begin{array}{cc} \phi_3^* \cdot \phi_3 & \phi_4^* \cdot \phi_4 \\ \phi_4^* \cdot \phi_3 & \phi_3^* \cdot \phi_4 \end{array} \right), \tag{8.37}
\]

and we have used the notation \( \phi_k^i \cdot \phi_l = \sum_{i=1}^{N/2} \phi_k^{i*} \phi_l^i \). Instead of the usual Hubbard-Stratonovitch transformation, we linearize the 4-boson interaction term by the Hermitian matrix \( \delta \) function

\[
\delta(Q_i - \overline{Q}_i) = \frac{1}{(2\pi)^4} \int dF e^{-i\text{Tr}F(Q_i - \overline{Q}_i)}, \tag{8.38}
\]
where the integral is over Hermitian matrices $F$. We thus find the identity

$$
\exp \left[ -\frac{2}{N} \text{Tr} Q_1 Q_2 \right] \sim \int dQ_1 dQ_2 \int dF dG e^{\text{Tr}[-iF(Q_1 - Q_1) + iG(Q_2 - Q_2) - \frac{N}{2}Q_1 Q_2]}.
$$

(8.39)

The integral over the $\phi_k$ is uniformly convergent in $F$ and $G$ which justifies the interchange of the order of the integrals. This results in the partition function

$$
Z_{-1} = \int dQ_1 dQ_2 \int dF dG e^{\text{Tr}[-i\frac{N}{2}Q_1 iQ_2 + i\frac{N}{2} \zeta Q_1 (Q_1 - IQ_2 I) - \frac{N}{2}Q_1 Q_2]}
$$

\times \det^{-\frac{N}{2}} \left( \begin{array}{cc} \epsilon + F & \mu \sigma_3 \\ \mu \sigma_3 & \epsilon + G \end{array} \right),

(8.40)

where we have used a block notation and the mass matrices (8.36). We have also simplified this integral by changing integration variables according to $F \rightarrow F - \zeta^T_1$ and $G \rightarrow G + I\zeta^T_1 I$ and $Q_i \rightarrow NQ_i/2$, $i = 1, 2$. For reasons of convergence we have kept the infinitesimal increments inside the determinant.

In the weak nonhermiticity limit, where $\mu^2 N$ is kept fixed for $N \rightarrow \infty$, the determinant can be approximated by

$$
\det^{-\frac{N}{2}} \left( \begin{array}{cc} \epsilon + F & \mu \sigma_3 \\ \mu \sigma_3 & \epsilon + G \end{array} \right) = \det^{-\frac{N}{2}} (\epsilon + F) \det^{-\frac{N}{2}} (\epsilon + G)
$$

\times \exp \left[ \frac{N \mu^2}{2} \det \left( \begin{array}{cc} 1 & \frac{1}{\epsilon + F} \sigma_3 \\ \frac{1}{\epsilon + G} \sigma_3 & 1 \end{array} \right) \right] (1 + O \left( \frac{1}{N} \right)).

(8.41)

The $F$ and $G$ variables in the $\mu^2 N$ term can be replaced the saddle point values of $F$ and $G$ at $\mu = 0$ given by $(\epsilon + F)Q_1 = i$ and $(\epsilon + G)Q_2 = i$. The remaining integrals over $F$ and $G$ are Ingham-Siegel integrals given by [88]

$$
\int dF \det^{-n} (\epsilon + F) e^{i \text{Tr} QF} \sim \theta(Q) \det^{-n-p} (Q) e^{-i \epsilon \text{Tr} Q},
$$

(8.42)

where the integral is over $p \times p$ Hermitian matrices, $\text{Im} \epsilon < 0$, and $\theta(Q)$ denotes that $Q$ is positive definite. These manipulations result in

$$
Z_{-1}(z, z^*, \mu) = \int dQ_1 dQ_2 \theta(Q_1) \theta(Q_2) \det^{-\frac{N}{2}} (Q_1 Q_2)
$$

\times e^{\text{Tr} \left[ i \frac{N}{2} \zeta^T_1 (Q_1 - IQ_2 I) - \frac{N}{2} \zeta Q_1 Q_2 - \frac{N}{2} \mu^2 Q_1 \sigma_3 Q_2 \sigma_3 \right]}.

(8.43)

In the limit $N \rightarrow \infty$ the integrals over the massive modes can be performed by a saddle point approximation. The saddle point equations are given by

$$
Q_1^{-1} - Q_2 = 0, \quad Q_2^{-1} - Q_1 = 0.
$$

(8.44)
Both equations can be rewritten as
\[ Q_1 = Q_2^{-1}, \tag{8.45} \]
and therefore only four of the modes, which we choose to be \( Q_2 \), can be integrated out by a saddle-point approximation. The quadratic fluctuations give rise to a factor \( \pi^2/\det^2 Q_1 \). The integral over the remaining modes has to be performed exactly. We thus arrive at the partition function \[ Z_{-1}(z, z^*, \mu) = \int \frac{dQ_1}{\det^2 Q_1} e^{\text{Tr} \left[ \frac{\pi}{2} Q_1^T (Q_1 - i Q_1^{-1} i) - \frac{\mu^2}{2} Q_1 \sigma_3 Q_1^{-1} \sigma_3 \right]} \tag{8.46} \]
Before evaluating this integral, we rederive this partition function based on the symmetries of the QCD partition function.

### 8.4.4 Symmetries of \( Z_{-1}(\mu) \)

For \( \mu = 0 \) and \( \zeta_1 = \zeta_2 = 0 \) the symmetry of the partition function (8.35) are the \( \text{Gl}(2) \times \text{Gl}(2) \) transformations,

\[
\begin{align*}
(\phi_1 & \quad \phi_2) \to U_1 (\phi_1 & \quad \phi_2), \quad (\phi_1^* & \quad \phi_2^*) \to (\phi_1^* & \quad \phi_2^*) U_2^{-1}, \\
(\phi_3 & \quad \phi_4) \to U_2 (\phi_3 & \quad \phi_4), \quad (\phi_3^* & \quad \phi_4^*) \to (\phi_3^* & \quad \phi_4^*) U_1^{-1}, \tag{8.47}
\end{align*}
\]
where we have disregarded convergence. This symmetry can be extended to nonzero mass or chemical potential if we adopt the transformation rules

\[
\begin{align*}
\zeta_1 & \to U_2 \zeta_1 U_1^{-1}, \quad \zeta_2 \to U_1 \zeta_2 U_2^{-1}, \\
\mu_1 & \to U_2 \mu_1 U_1^{-1}, \quad \mu_2 \to U_1 \mu_2 U_1^{-1}, \tag{8.48}
\end{align*}
\]
where \( \mu_1 \) is the chemical potential matrix that is added \( id \) and \( \mu_2 \) is the chemical potential matrix that is added to \( -id^t \). These matrices are introduced for the sake of discussing the transformation properties of the partition function (8.35) and will ultimately be replaced by their original values \( \mu_1 = \mu_2 = \mu \sigma_3 \). The chiral symmetry is broken spontaneously to \( \text{Gl}(2) \) by the chiral condensate. Because the bosonic integral has to converge, the Goldstone manifold is not \( \text{Gl}(2) \) but rather \( \text{Gl}(2)/U(2) \), i.e. the coset of positive definite matrices as in the case of zero chemical potential. Under a \( \text{Gl}(2) \times \text{Gl}(2) \) transformation the Goldstone fields transform as

\[ Q \to U_1 Q U_2^{-1}. \tag{8.49} \]

The low energy effective partition function should have the same transformation properties as the microscopic partition function (8.35). To second order in \( \mu \) and first order in the mass matrix we can write down the following invariants

\[
\begin{align*}
\text{Tr} \zeta_1 Q, \quad \text{Tr} \zeta_2 Q^{-1}, \quad \text{Tr} Q \mu_1 Q^{-1} \mu_2, \quad \text{Tr} \mu_1^2, \quad \text{Tr} \mu_2^2. \tag{8.50}
\end{align*}
\]
We also have the discrete symmetry that the partition function is invariant under the interchange of $\zeta_1$ and $\zeta_2$. This symmetry implies that the coefficients of the two mass terms in the effective partition function are the same. Using that the integration measure on positive definite Hermitian matrices is given by $dQ/\det^2 Q$, we finally arrive at the effective partition function

$$Z_{\nu,-1}(z,z^*) = \int_{Q \in GL(2)/U(2)} \frac{\det^\nu(Q)}{\det^2(Q)} e^{-\frac{\mu^2}{4} V \text{Tr}[Q,B][Q^{-1},B] + \frac{i \Sigma V}{2} \text{Tr}(\zeta_1 Q + \zeta_2 Q^{-1})},$$

(8.51)

The partition function (8.46) for $\nu = 0$ is recovered after making the identification $V \rightarrow N, \Sigma \rightarrow 1$ and $F^2 \rightarrow 1$ and $Q \rightarrow Q^T$.

### 8.4.5 Calculation of the Integral over $Q$. To evaluate the integral

(8.51) we use the parameterization

$$Q = e^{t} \begin{pmatrix} e^r \cosh s & e^{i\theta} \sinh s \\ e^{-i\theta} \sinh s & e^{-r} \cosh s \end{pmatrix},$$

(8.52)

where

$$r \in (-\infty, \infty), \quad s \in (-\infty, \infty), \quad t \in (-\infty, \infty), \quad \theta \in (0, \pi).$$

(8.53)

The Jacobian relating the measures $dQ/\det^2 Q$ and $drdsdt\theta$ is given by

$$J = 4e^{4t} \cosh s \sinh s.$$

(8.54)

We first perform the integral over $r$, which gives a factor $2K_0(2N e \cosh s \cosh t)$ with leading singularity given by $\sim -\log \epsilon$. This factor is absorbed in the normalization of the partition function. Then the integral over $\theta$ gives a Bessel function. Introducing $u = \sinh s$ as new integration variable we find [25]

$$Z_{\nu,-1}(z,z^*,\mu) = C_{-1} \int_0^\infty dt \int_0^\infty du e^{2ut} J_0(2Vu(x^2 \cosh^2 t + y^2 \sinh^2 t)^{1/2})$$

$$\times e^{-\mu^2 F^2 V(1+2u^2)}.$$  

(8.55)

To do the integral over $u$ we use the known integral

$$\int_0^\infty dx x^{a+1} e^{-ax^2} J_0(\beta x) = \frac{\beta^a}{(2\alpha)^{a+1}} e^{-\beta^2/4\alpha}.$$  

(8.56)

This results in

$$Z_{\nu,-1}(z,z^*,\mu) = \frac{C_{-1}}{4\mu^2 F^2 V} \int_{-\infty}^\infty dt e^{2ut} e^{-\frac{V(x^2 \cosh^2 t + y^2 \sinh^2 t)}{2\mu^2 F^2}}.$$  

(8.57)
Using that \( \cosh^2 t = \frac{1}{2} + \frac{1}{2} \cosh 2t \) and \( \sinh^2 t = -\frac{1}{2} + \frac{1}{2} \cosh 2t \), the integral over \( t \) can be rewritten as a modified Bessel function resulting in [25]

\[
Z_{\nu,-1}(z = x + iy, z^*) = C_{-1} e^{-\frac{V\Sigma^2(\nu^2 - \nu^2)}{4\mu^2 F^2}} K_{\nu}\left(\frac{V\Sigma^2(x^2 + y^2)}{4\mu^2 F^2}\right).
\] (8.58)

### 8.4.6 The Dirac Spectrum at Nonzero Chemical Potential

The final result for the quenched spectral density is obtained by substituting the partition functions \( Z_{\nu,1}(z, z^*, \mu) \) and \( Z_{\nu,-1}(z = x + iy, z^*, \mu) \) in expression (8.34) obtained from the replica limit of the Toda lattice equation. We find,

\[
\rho^{\text{quen}}(x, y, \mu) = \frac{V^3 \Sigma^4}{2\pi F^2 \mu^2} \left(\frac{x^2 + y^2}{\mu^2 F^2}\right) e^{-\frac{V\Sigma^2}{4\mu^2 F^2}} K_{\nu}\left(\frac{V\Sigma^2(x^2 + y^2)}{4\mu^2 F^2}\right) \times \int_0^1 d\lambda \lambda^\nu e^{-\frac{2VF^2\mu^2 \lambda^2}{\mu^2 F^2}} |I_{\nu}(\lambda^2 V\Sigma)|^2.
\] (8.59)

The normalization constant has been chosen such that the \( \mu \to 0 \) limit of \( \rho^{\text{quen}}(x, y, \mu) \) for large \( y \) is given by \( \Sigma V/\pi \) (see below).

In the limit \( \text{Re}(z) \Sigma/\mu^2 F^2 \ll 1 \) the upper limit of the integral in (8.59) can be extended to infinity. Using the known integral

\[
\int_0^\infty \lambda d\lambda e^{-2VF^2\mu^2 \lambda^2} |I_{\nu}(\lambda^2 V\Sigma)|^2 = e^{-\frac{(z^2 + z^*2 \lambda^2 V)}{4\mu^2 F^2}} I_{\nu}\left(\frac{zz^* V\Sigma^2}{4\mu^2 F^2}\right),
\] (8.60)

the spectral density can be expressed as

\[
\rho^{\text{quen}}(x, y, \mu) = \frac{2}{\pi} u^2 zz^* K_{\nu}(zz^* u) I_{\nu}(zz^* u) \quad \text{with} \quad u = \frac{V\Sigma^2}{4\mu^2 F^2}.
\] (8.61)

Therefore, the spectral density becomes a universal function that only depends on a single parameter \( u \). This parameter can be rewritten in a more physical way as \( u = \pi \rho^{\text{asym}}(x, y, \mu) \). For the dimensionless ratio we obtain

\[
\frac{\rho^{\text{quen}}(x, y, \mu)}{\rho^{\text{asym}}(x, y, \mu)} = 2 u zz^* K_{\nu}(zz^* u) I_{\nu}(zz^* u),
\] (8.62)

which is universal combination that depends only on a single universal combination \( zz^* u \). (This result was obtained in collaboration with Tilo Wettig).

In the thermodynamic limit the Bessel functions can be approximated by their asymptotic limit. This results in

\[
\rho^{\text{quen}}(x, y, \mu) = \frac{V^2 \Sigma^2}{2\pi F \mu \sqrt{2\pi V}} \int_0^1 d\lambda e^{-2VF^2\mu^2 (\lambda - \frac{|z|\Sigma}{2F^2\mu^2})^2}.
\] (8.63)
For $V \to \infty$ the integral over $\lambda$ can be performed by a saddle point approximation. If the saddle point is outside the range $[0, 1]$ the integral vanishes for $V \to \infty$. We thus find for the spectral density
\[
\rho_{\text{quen}}(x,y,\mu) = \frac{V \Sigma^2}{4\pi \mu^2 F^2} \mathrm{for} \quad |x| < \frac{2F^2 \mu^2}{\Sigma} \tag{8.64}
\]
and $\rho_{\text{quen}}(x,y,\mu) = 0$ outside this strip. This result is in agreement with the mean field analysis [84] of the effective partition function given in section 8.3.3. For the integrated eigenvalue density we find
\[
\int_{-\infty}^{\infty} dx \rho_{\text{quen}}(x,y,\mu) = \frac{\Sigma V}{\pi} \tag{8.65}
\]
in agreement with the eigenvalue density at $\mu = 0$.

9. Full QCD at Nonzero Chemical Potential

In quenched QCD the chiral condensate $G(m,\mu) \sim m/\mu^2$ in the region where the eigenvalues are located. In section 8.3 we have argued that that chiral condensate in full QCD does not depend on $\mu$ for $\mu < mN_N/N_N$. The conclusion is that the presence of the fermion determinant completely alters the vacuum structure of the theory. The question we wish to address is how we can understand this based on the spectrum of the QCD Dirac operator. For simplicity we only consider the case of $N_f = 1$ and $\nu = 0$.

The average spectral density in full QCD is defined by
\[
\rho^{\text{full}}(x,y,\mu) = \langle \sum_k \delta^2(x + iy - \lambda_k) \det(D + m + \mu \gamma_0) \rangle. \tag{9.1}
\]
The low-energy limit of the generating function for the spectral density can again be written as a $\tau$-function [26]. The spectral density is then obtained from the replica limit of the corresponding Toda lattice equation. The result is
\[
\rho^{\text{full}}(x,y,\mu) = \frac{V^3(x^2 + y^2)\Sigma^2}{2\pi \mu^2 F^2} e^{\frac{V(x^2 + y^2)\Sigma^2}{4\mu^2 F^2}} K_0\left(\frac{V(x^2 + y^2)\Sigma^2}{4\mu^2 F^2}\right) \times \int_0^1 t dt e^{-2V\mu^2 F^2 t} (I_0(z\Sigma V t) - \frac{I_0(z\Sigma V) I_0(mV \Sigma t)}{I_0(mV \Sigma)}) \tag{9.2}
\]
It was first obtained from the random matrix model [27] using the method of complex orthogonal polynomials developed in [89, 90]. To appreciate this result, we consider its asymptotic expansion for $V \to \infty$. For $m = 0$ and $x > 0$ we can derive the asymptotic result for the difference
\[
\rho^{\text{quen}}(x,y,\mu) - \rho^{\text{full}}(x,y,\mu) \sim \frac{\sqrt{2\Sigma}}{2\mu^2 F^3} e^{\frac{\sqrt{2}\Sigma^2}{2\mu^2 F^2}} e^{\frac{\sqrt{2}\Sigma^2}{2\mu^2 F^2}} e^{\Sigma V}. \tag{9.3}
\]
The behavior near the extremum at \((x, y) = (4\mu^2 F^2 / 3\Sigma, 0)\) is given by
\[
\rho^{\text{quen}}(x, y, \mu) - \rho^{\text{full}}(x, y, \mu) \sim e^{\frac{2}{3} \mu^2 F^2 V} e^{\frac{4}{3} \mu \Sigma V}.
\] (9.4)

We have oscillations on the scale \(1/\Sigma V\) with an amplitude that diverges exponentially with the volume [26]. In the thermodynamic limit, these oscillations are visible in a domain where the real part of the exponent in (9.3) is positive. This region is given by intersection of the inside of the ellipses
\[
3\left(x \pm \frac{4}{3} \mu^2 F^2 \Sigma\right)^2 + y^2 = \frac{16}{3} \mu^4 F^2 \Sigma^2
\] (9.5)

and the strip \(|x| < 2F^2 \mu^2 / \Sigma\). At the mean field level this can be reinterpreted as a region where Kaon condensation takes place [94, 95].

10. Conclusions

The existence of two formulations of QCD at low energy, first as a microscopic theory of quarks and gluons and second as an effective theory of weakly interacting Goldstone bosons, imposes powerful constraints on either of the theories. The effective theory is completely determined by the symmetries of the microscopic theory, and the mass dependence of the effective theory imposes sum rules on the inverse Dirac eigenvalues. In particular this means that any theory with the same symmetry breaking pattern and a mass gap will be subject to the same constraints. The simplest microscopic theory is chiral Random Matrix Theory.

However, more can be done than constraining the inverse Dirac eigenvalues by sum rules. The key observation is that the generating function for the resolvent amounts to a partition function with additional flavors with a mass \(z\) equal to the value for which the resolvent is calculated. Again we have a microscopic theory and an effective theory with the same low energy limit. Because \(z\) is a free parameter, it can always be chosen such that the Compton wavelength of the corresponding Goldstone boson is much larger than the size of the box. In this region the \(z\) dependence of the partition function is determined by the mass term of the chiral Lagrangian which is a simple matrix integral.

To obtain the Dirac spectrum we have to quench the determinant corresponding to \(z\). This can be done in two ways: by the replica trick or by the supersymmetric method. Although, the supersymmetric method is straightforward, the naive replica trick is technically somewhat simpler. The problem is that the naive replica trick does not give correct nonperturbative results. One way out is if the dependence on the number of replicas \(n\) is known as an analytical function of \(n\) around \(n = 0\). It was shown by Kanzieper that the \(n\)-dependence can be obtained from the solution of the Painlevé equation. The other way out, which has been advocated in these lectures, is if the partition
function for \( n = 0 \) is related by a recursion relation to partition functions with a nonzero integer number of flavors. The replica limit of this Toda lattice equation gives us nonperturbative correlation functions. This is an efficient formulation of the problem. The structure of the final answer already has the factorized structure of the Toda lattice formulation. We could also say that the supersymmetric partition function connects two semi-infinite hierarchies.

New results with the Toda lattice method were obtained for QCD at nonzero chemical potential. In this case the low-energy effective partition functions are also related by the Toda lattice equation. This made it possible to express the microscopic spectral density as the product of the partition function with one fermionic flavor and the partition function with one bosonic flavor. This result has later been reproduced by RMT with the method of orthogonal polynomials.

More surprisingly the Toda lattice method also gives the correct result for QCD at nonzero chemical potential with dynamical fermions. Because of the phase of the fermion determinant, a breakdown of this method for this case would not have been a surprise. However, the concept of integrability that also reigns this case, is so powerful that replica limit can be taken in exactly the same way as in the quenched case. The result for the spectral density shows oscillations on the scale of \( 1/V \) and an amplitude that diverges exponentially with \( V \). This structure is necessary to obtain a nonzero chiral condensate in the chiral limit.

The Toda lattice method has been applied to quite a few cases in the symmetry class \( \beta = 2 \) (see [24, 23, 92, 25, 26, 93]). Our conjecture is that all microscopic correlation functions in this class can be obtained from the replica limit of a Toda lattice equation. A much tougher problem is analysis of the replica limit for the other Dyson classes. We are not aware of any progress on this problem and encourage the reader to confront this challenge.

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