EXTENDING THE EXACT SEQUENCE OF NONABELIAN $H^1$, USING NONABELIAN $H^2$
WITH COEFFICIENTS IN CROSSED MODULES

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Abstract. In this note, following Dedecker and Debremaeker, we extend the cohomology exact sequence for nonabelian $H^1$, using nonabelian $H^2$ with coefficients in crossed modules.

Let $\Gamma$ be a fixed group. In this note (not to be published) we consider groups with $\Gamma$-action and crossed modules with $\Gamma$-action. Following Dedecker and Debremaeker, we extend the cohomology exact sequence for $H^1$, using $H^2$ with coefficients in crossed modules. The obtained exact sequence seems to be essentially equivalent to the sequences of Springer [Spr66, Props. 1.27, 1.28, 1.29] and Giraud [Gir71, IV.4.2, Prop. 4.2.8], but looks nicer (more functorial) from our point of view. According to Debremaeker [Deb76], all our results are valid in the more general context of groups and crossed modules in a topos. We claim no originality.

1. Second cohomology with coefficients in a crossed module

Let $(A \xrightarrow{\rho} G)$ be a left crossed module with a $\Gamma$-action (see below). The second nonabelian cohomology with coefficients in a crossed module was considered in [Ded64], [Deb76], [Deb77f], [Br90], [Bor98], [Noo11]. We define $H^2(A \to G)$ in terms of cocycles (note that in [Bor98] this set was denoted by $H^1(\Gamma, A \to G)$, while in [Br90] the corresponding set in a more general setting was denoted by $H^1(A \to G)$). It is important that the set $H^2(A \to G)$ has a distinguished element (the unit element) and a distinguished subset of neutral elements.

Definition 1. A (left) crossed module is a homomorphism of groups $\rho: A \to G$ together with a left action $G \times A \to A$ of $G$ on $A$, denoted $(g, a) \mapsto g^a$, such that

$$aa'a^{-1} = \rho(a)a'$$
$$\rho(g^a) = g \cdot \rho(a) \cdot g^{-1}$$

for all $a, a' \in A, g \in G$.

For examples of crossed modules see e.g. [Bor98, Examples 3.2.2]. Note that for any group $A$ we have crossed modules $A \to \text{Aut} A$ and $A \to \text{Inn} A$.
We say that our fixed group $\Gamma$ acts on a crossed module $(A \to G)$ if $\Gamma$ acts on $A$ and $G$ so that
\[
\rho(\sigma a) = \sigma(\rho(a)), \quad \sigma(ga) = g(\sigma a)
\]
for all $a \in A$, $g \in G$, $\sigma \in \Gamma$.

Let $Z^2(\Gamma, A \to G)$ denote the set of pairs $(u, \psi)$, where $u : \Gamma \times \Gamma \to A$ and $\psi : \Gamma \to G$ are maps satisfying the cocycle conditions of [Bor98, (3.3.2.1-2)]:
\[
\begin{align*}
&u_{\sigma, \tau} \cdot \psi_{\sigma}(\sigma u_{\tau, \nu}) = u_{\sigma, \tau} \cdot u_{\sigma, \nu} \\
&\psi_{\sigma \tau} = \rho(u_{\sigma, \tau}) \cdot \psi_{\sigma} \cdot \sigma \psi_{\tau}
\end{align*}
\]
for all $\sigma, \tau, \nu \in \Gamma$.

**Construction 2.** We define a left action of the group $\text{Maps}(\Gamma, A)$ on $Z^2(\Gamma, A \to G)$ as follows. If $w \in \text{Maps}(\Gamma, A)$, $(u, \psi) \in Z^2(\Gamma, A \to G)$, then we set
\[
w \star (u, \psi) = (u', \psi'),
\]
where
\[
\begin{align*}
u'_{\sigma, \tau} &= w_{\sigma \tau} \cdot u_{\sigma, \tau} \cdot \psi_{\sigma}(\sigma w_{\nu})^{-1} \cdot w_{\sigma}^{-1} \\
\psi'_{\sigma} &= \rho(w_{\sigma}) \cdot \psi_{\sigma}
\end{align*}
\]
for all $\sigma, \tau \in \Gamma$. One checks that $(u', \psi') \in Z^2(\Gamma, A \to G)$.

We define a left action of $G$ on $Z^2(\Gamma, A \to G)$ as follows. If $g \in G$, $(u, \psi) \in Z^2(\Gamma, A \to G)$, then we set
\[
g \star (u, \psi) = (u'', \psi''),
\]
where
\[
\begin{align*}
u''_{\sigma, \tau} &= g_{\sigma, \tau} \\
\psi''_{\sigma} &= g \cdot \psi_{\sigma} \cdot g^{-1}
\end{align*}
\]
for all $\sigma, \tau \in G$. One checks that $(u'', \psi'') \in Z^2(\Gamma, A \to G)$.

The group $G$ acts on the left on the group $\text{Maps}(\Gamma, A)$ by
\[
g \star w = w', \quad \text{where } w'_{\sigma} = g_{\sigma} w_{\sigma}
\]
for $g \in G$, $w \in \text{Maps}(\Gamma, A)$, and $\sigma \in \Gamma$. We consider the semi-direct product
\[
C^1(\Gamma, A \to G) := \text{Maps}(\Gamma, A) \rtimes G.
\]
Then the group $C^1(\Gamma, A \to G)$ acts on the left on the set $Z^2(\Gamma, A \to G)$.

**Definition 3.** The thick cohomology set is $H^2(A \to G) := Z^2(\Gamma, A \to G)/\text{Maps}(\Gamma, A)$.
Definition 4. The thin cohomology set is
\[ H^2(A \to G) := Z^2(\Gamma, A \to G)/C^1(\Gamma, A \to G) = H^2(A \to G)/G. \]

We have a canonical surjective map
\[ \kappa: H^2(A \to G) \to H^2(A \to G). \]

Definition 5. The unit cocycle in \( Z^2(\Gamma, A \to G) \) is the cocycle \((1_A, 1_G)\). The unit classes in \( H^2(A \to G) \) and \( H^2(A \to G) \) are the images of the unit cocycle.

Definition 6. A neutral cocycle in \( Z^2(\Gamma, A \to G) \) is a cocycle of the form \((1_A, \psi)\). The neutral classes in \( H^2(A \to G) \) and \( H^2(A \to G) \) are the images of the neutral cocycles.

Thus the set \( H^2(A \to G) \) contains the distinguished subset \( H^2(A \to G)' \) of neutral elements. This subset \( H^2(A \to G)' \) contains the unit element 1.

2. Second cohomology with coefficients in a group

Let \( A \) be a \( \Gamma \)-group. The \( \Gamma \)-action defines a homomorphism
\[ f_A: \Gamma \to \text{Aut } A, \quad (f_A)_\sigma(a) = \sigma a, \]
and thus it defines a \( \Gamma \)-kernel (\( \Gamma \)-band, \( \Gamma \)-lien)
\[ \kappa_A: \Gamma \xrightarrow{f_A} \text{Aut } A \to \text{Out } A, \]
where \( \text{Out } A := \text{Aut } A/\text{Inn } A \). We write \( H^2(A) \) for \( H^2(\Gamma, A, \kappa_A) \). The second nonabelian cohomology set \( H^2(A) \) was defined by Springer [Spr66] and Giraud [Gir71], see also [Bor93, FSS98, Flo04, and LA15]. By definition [Bor93, Section 1.5], the set of 2-cocycles \( Z^2(\Gamma, A, \kappa_A) \) is the set of pairs \((u, f)\), where \( u \in \text{Maps}(\Gamma \times \Gamma) \to A \) and \( f \in \text{Maps}(\Gamma, \text{Aut } A) \), satisfying the 2-cocycle conditions
\[ f_{\sigma \tau} = \text{inn}(u_{\sigma, \tau} \circ f_{\sigma} \circ f_{\tau}), \]
\[ u_{\sigma, \tau \nu} \cdot f_{\sigma}(u_{\tau, \nu}) = u_{\sigma \tau, \nu} \cdot u_{\sigma, \tau}, \]
\[ f_{\sigma} = \psi_{\sigma} \circ (f_A)_{\sigma} \quad \text{for some } \psi_{\sigma} \in \text{Inn } A \]
for all \( \sigma, \tau, \nu \in \Gamma \). The group \( \text{Maps}(\Gamma, A) \) acts on the left on \( Z^2(\Gamma, A, \kappa_A) \) as follows. If
\[ w \in \text{Maps}(\Gamma, A), \quad (u, f) \in Z^2(\Gamma, A, \kappa_A), \]
then
\[ w \ast (u, f) = (u', f'), \]
where
\[ u'_{\sigma, \tau} = w_{\sigma \tau} \cdot u_{\sigma, \tau} \cdot f_{\sigma}(w_{\tau})^{-1} \cdot w_{\sigma}^{-1}, \]
\[ f'_\sigma = \text{inn}(w_{\sigma}) \circ f_{\sigma} \]
for all \( \sigma, \tau \in \Gamma \).

Definition 7. \( H^2(A) = Z^2(\Gamma, A, \kappa_A)/\text{Maps}(\Gamma, A) \).
By a neutral cocycle in \( Z^2(\Gamma, A, \kappa_A) \) we mean a cocycle of the form \((1_A, f)\), and by the unit cocycle we mean \((1_A, f_A)\). A neutral class in \( H^2(A) \) is the class of a neutral cocycle, and the unit class 1 in \( H^2(A) \) is the class of the unit cocycle. We obtain a distinguished subset \( H^2(A)' \subset H^2(A) \) consisting of the neutral elements, and \( H^2(A)' \) contains the unit element 1.

Note that a 2-cocycle \((u, f) \in Z^2(\Gamma, A, \kappa_A)\) defines a map \( \psi: \Gamma \to \text{Inn} A \) by \( \psi_\sigma = f_\sigma \circ (f_A)_\sigma^{-1} \). One checks immediately that \((u, \psi) \in Z^2(\Gamma, \text{Inn} A)\) and that the bijection

\[
Z^2(\Gamma, A, \kappa_A) \cong Z^2(\Gamma, A \to \text{Inn} A), \quad (u, f) \mapsto (u, \psi),
\]

induces a canonical bijection \( H^2(A) \cong H^2(A \to \text{Inn} A) \). This bijection induces a bijection \( H^2(A)' \cong H^2(A \to \text{Inn} A)' \) on the set of neutral elements and takes the unit element of \( H^2(A) \) to the unit element of \( H^2(A \to \text{Inn} A) \).

We obtain a canonical surjective map

\[
\lambda_A: H^2(A) \cong H^2(A \to \text{Inn} A) \xrightarrow{\kappa} H^2(A \to \text{Inn} A).
\]

**Theorem 8** (Debremaeker [Deb76, Ch. V, Thm. 3, p. 112]). The canonical surjection \((2)\) is a bijection.

**First proof.** Let \( Z_A \) denote the center of \( A \), which is is a \( \Gamma \)-group. Then the group of 2-cocycles \( Z^2(\Gamma, Z_A) \) acts on the left on the set \( Z^2(\Gamma, A, \kappa_A) \) as follows:

if \( z \in Z^2(\Gamma, Z_A) \), \((u, f) \in Z^2(\Gamma, A, \kappa_A)\), then \( z \ast (u, f) = (zu, f) \).

This action induces an action of \( H^2(Z) \) on \( H^2(A) \), which is simply transitive, see [ML63, IV-Thm. 8.8] or [Spr66, Prop. 1.17].

On the other hand, the group \( Z^2(\Gamma, Z_A) \) acts on the left on the set \( Z^2(\Gamma, A \to \text{Inn} A) \) as follows:

if \( z \in Z^2(\Gamma, Z_A) \), \((u, \psi) \in Z^2(\Gamma, A \to \text{Inn} A)\), then \( z \ast (u, \psi) = (zu, \psi) \).

This map induces an action of \( H^2(Z_A) \) on \( H^2(A \to \text{Inn} A) \) and, by the action of \( H^2(Z_A) \) on the unit element \( 1 \in H^2(A \to \text{Inn} A) \), it induces a map

\[
\mu: H^2(Z_A) \to H^2(A \to \text{Inn} A),
\]

which can be factored as

\[
\mu: H^2(Z_A) \cong H^2(Z_A \to 1) \xrightarrow{\iota_*} H^2(A \to \text{Inn} A),
\]

where the map \( \iota_* \) is induced by the embedding of crossed modules

\[
\iota: (Z \to 1) \hookrightarrow (A \to \text{Inn} A).
\]

Since the embedding \( \iota \) is a quasi-isomorphism of crossed modules, the map \( \iota_* \) is bijective (see [Bor92, Thm. 3.3]), hence the map \( \mu \) is bijective and the action of \( H^2(Z_A) \) on \( H^2(A \to \text{Inn} A) \) is simply transitive.

Now, since the map \((2)\) is \( H^2(Z_A) \)-equivariant, we conclude that it is bijective, as required.

**Second proof** (similar to [Spr66, Proof of Prop. 1.19]). We wish to prove that the surjective map

\[
\kappa: H^2(A \to \text{Inn} A) \to H^2(A \to \text{Inn} A)
\]
is bijective. It suffices to show that $\text{Inn} A$ acts on $H^2(A \to \text{Inn} A)$ trivially. 

Let 

$$g \in \text{Inn} A, \ g = \text{inn}(b), \ b \in A.$$ 

One can check that the cocycles $g \star (u, \psi)$ and $(u, \psi)$ give the same class in $H^2(A \to \text{Inn} A)$, namely, that 

$$g \star (u, \psi) = w \star (u, \psi),$$ 

where 

$$w \in \text{Maps}(\Gamma, A), \ w_\sigma = b \cdot \psi_\sigma(\sigma b)^{-1} \quad \text{for} \ \sigma \in \Gamma.$$ 

This completes the second proof. □

3. Cohomology exact sequence

Let

$$1 \to A \overset{i}{\longrightarrow} B \overset{j}{\longrightarrow} C \to 1$$

be a short exact sequence of $\Gamma$-groups. We construct a connecting map 

$$\Delta: H^1(C) \to H^2(A \to \text{Inn} B) \quad \text{(sic!)}. $$

Let $c \in Z^1(\Gamma, C) \subset \text{Maps}(\Gamma, C)$. We lift $c$ to some map $b: \Gamma \to B$ and define 

$$u_{\sigma, \tau} = b_{\sigma \tau} \cdot \sigma b_\tau^{-1} \cdot b_\sigma^{-1} \in A$$

$$\psi_\sigma = \text{inn}(b_\sigma) \in \text{Inn} B$$

for $\sigma, \tau \in \Gamma$. We set $\Delta([c]) = [u, \psi]$, where $[c]$ denotes the class in $H^1(C)$ of $c$, and $[u, \psi]$ denotes the class in $H^2(A \to \text{Inn} B)$ of $(u, \psi) \in Z^2(\Gamma, A \to \text{Inn} B)$. One checks that $(u, \psi) \in Z^2(\Gamma, A \to \text{Inn} B)$ and that the map $\Delta$ is well defined.

Consider the morphisms of crossed modules 

$$(A \to \text{Inn} B) \overset{i_*}{\longrightarrow} (B \to \text{Inn} B) \overset{j_*}{\longrightarrow} (C \to \text{Inn} C)$$

and the sequence

$$(A \to \text{Inn} B) \overset{i_*}{\longrightarrow} H^1(C) \overset{\Delta}{\longrightarrow} H^2(A \to \text{Inn} B) \overset{i_*}{\longrightarrow} H^2(B \to \text{Inn} B) \overset{j_*}{\longrightarrow} H^2(C \to \text{Inn} C)$$

**Theorem 9** (Dedecker [Ded69, Thm. 2.2] and Debremeaker [Deb76, Ch. IV, Thm. 2.1.7, p. 103]). For an exact sequence of $\Gamma$-groups (3), the sequence (4) is exact in the following sense:

(i) an element of $H^1(C)$ is contained in the image of $H^1(B)$ if and only if its image in $H^2(A \to \text{Inn} B)$ is neutral;

(ii) an element of $H^2(A \to \text{Inn} B)$ is contained in the image of $H^1(C)$ if and only if its image in $H^2(B \to \text{Inn} B)$ is the unit element;

(iii) an element of $H^2(B \to \text{Inn} B)$ is contained in the image of $H^2(A \to \text{Inn} B)$ if and only if its image in $H^2(C \to \text{Inn} C)$ is neutral.
Proof. Let \( b \in Z^1(\Gamma, B) \). We show that \( \Delta \circ j_* \) takes \([b]\) to a neutral class. Indeed, by the definition of \( \Delta \), the composite map \( \Delta \circ j_* \) maps \([b]\) to \([u^A, \psi^A]\) where
\[
\sigma_{\sigma, \tau} b = b_{\sigma} \cdot \sigma b^{-1}_{\tau} \cdot b_{\sigma}^{-1} = 1
\]
because \( b \) is a cocycle. Thus \([u^A, \psi^A]\) is a neutral class.

Conversely, let \( c \in Z^1(\Gamma, C) \) and assume that \( \Delta \) takes \([c]\) to a neutral class. Let us lift \( c \) to some map \( b: \Gamma \to B \). Then \( \Delta[c] = [u^A, \psi^A] \), where
\[
\begin{align*}
\sigma_{\sigma, \tau} &= b_{\sigma} \cdot \sigma b^{-1}_{\tau} \cdot b_{\sigma}^{-1}, \\
\psi^A &= \text{inn}(b_{\sigma}).
\end{align*}
\]
By assumption \([u^A, \psi^A]\) is a neutral class in \( H^2(A \to \text{Inn} B) \), i.e., there exists a map \( a: \Gamma \to A \) such that
\[
a \ast (u^A, \psi^A) = (1, \psi^A).
\]
This means that
\[
a_{\sigma, \tau} \cdot b_{\sigma} \cdot \sigma b^{-1}_{\tau} \cdot b_{\sigma}^{-1} \cdot a_{\sigma}^{-1} \cdot b_{\sigma}^{-1} = 1.
\]
that is,
\[
a_{\sigma, \tau} b_{\sigma} \cdot \sigma (a_{\sigma} b_{\sigma})^{-1} \cdot (a_{\sigma} b_{\sigma})^{-1} = 1.
\]
Set \( b'_{\sigma} = a_{\sigma} b_{\sigma} \), then \( b'_{\sigma, \tau} = b'_{\sigma} \cdot \sigma b'_{\tau} \), hence \( b' \) is a cocycle. Clearly \( j_* \) takes \([b']\) to \([c]\), and therefore, \([c] \in \text{im} j_* \), as required.

Let \( c \in Z^1(\Gamma, C) \). We show that \( i_* \circ \Delta \) takes \([c]\) to 1. Indeed, let us lift \( c \) to some map \( b: \Gamma \to B \). Then the composite map \( i_* \circ \Delta \) takes \([c]\) to the class \([u^B, \psi^B]\) where
\[
\begin{align*}
\sigma_{\sigma, \tau} &= b_{\sigma} \cdot \sigma b^{-1}_{\tau} \cdot b_{\sigma}^{-1} \in B, \\
\psi^B &= \text{inn}(b_{\sigma}) \in \text{Inn} B,
\end{align*}
\]
and clearly \((u^B, \psi^B) = b \ast (1, 1)\), hence \([u^B, \psi^B]\) is a neutral class.

Conversely, let \([u^A, \psi^A] \in H^2(A \to \text{Inn} B) \) and assume that \( i_*([u^A, \psi^A]) = [1, 1] \). Clearly \( i_*([u^A, \psi^A]) = [u^A, \psi^A] \), so we obtain that
\[
[u^A, \psi^A] = [1, 1] \in H^2(B \to \text{Inn} B).
\]
By Theorem 8 we have \( H^2(B \to \text{Inn} B) = H^2(B \to \text{Inn} B) \), hence \((u^A, \psi^A) = b \ast (1, 1)\) for some \( b: \Gamma \to B \). We have
\[
\begin{align*}
\sigma_{\sigma, \tau} &= b_{\sigma} \cdot \sigma b^{-1}_{\tau} \cdot b_{\sigma}^{-1}, \\
\psi^A &= \text{inn}(b_{\sigma}).
\end{align*}
\]
Set \( c = j \circ b: \Gamma \to C \). Since \( u^A_{\sigma, \tau} \in A \), we see that
\[
c_{\sigma, \tau} \cdot c^{-1}_{\sigma} \cdot c^{-1}_{\tau} = 1,
\]
hence \( c \) is a cocycle. Clearly, \([u^A, \psi^A] = \Delta([c])\). Thus \([u^A, \psi^A] \in \text{im} \Delta \), as required.

Let \((u^A, \psi^A) \in Z^2(\Gamma, A \to \text{Inn} B) \). We show that \( j_* \circ i_* \) takes \([u^A, \psi^A]\) to a neutral class. Indeed, for any \( \sigma, \tau \in \Gamma \) we have \( u^A_{\sigma, \tau} \in A \). It follows that the image of \([u^A, \psi^A]\) under the composite map \( j_* \circ i_* \) is of the form \([1, \psi^C]\), and hence is neutral.
Conversely, assume that \( j_* \) takes \([u^B, \psi^B]\) to a neutral class \([u^C, \psi^C]\). This means that there exists a map \( c: \Gamma \to C \) such that \( c \ast (u^C, \psi^C) = (1, \psi^C) \). Let us lift \( c \) to a map \( b: \Gamma \to B \) and set \((u^B, \psi^B) = b \ast (u^B, \psi^B)\). Then for any \( \sigma, \tau \in G \) we have \( u^B_{\sigma \tau} \in A \). We see that \([u^B, \psi^B] = [u^B, \psi^B] \) lies in the image of \( i_* \), as required. This completes the proof of the theorem. \( \square \)

4. A VERSION OF THEOREM [9]

We write
\[ G = (\text{Inn } B)|_A := \{ \text{inn}(b)|_A \mid b \in B \}, \]
the group of restrictions to \( A \) of the inner automorphisms of \( B \). Then \( G \subset \text{Aut } A \). We have an epimorphism \( \text{Inn } B \to G \) and a morphism of crossed modules
\[ \pi: (A \to \text{Inn } B) \to (A \to G). \]

**Lemma 10.** For \((u, \psi) \in Z^2(\Gamma, A \to \text{Inn } B)\), its class \([u, \psi] \in H^2(A \to \text{Inn } B)\) is neutral if and only if \( \pi_*([u, \psi]) \in H^2(A \to G) \) is neutral.

**Proof.** Easy. \( \square \)

**Corollary 11.** For an exact sequence of \( \Gamma \)-groups \( \mathfrak{G} \), the sequence
\[
H^1(B) \xrightarrow{j_*} H^1(C) \xrightarrow{\Delta \circ \pi_*} H^2(A \to G)
\]
is exact in the following sense: a cohomology class \( c \in H^1(C) \) comes from \( H^1(B) \) if and only if its image in \( H^2(A \to G) \) is neutral.

5. EXAMPLE

We compute the map \( \Delta \circ \pi_* \): \( \text{H}^1(C) \to \text{H}^2(A \to G) \) in the case when \( A \) is abelian. Here \( G = (\text{Inn } B)|_A \).

Let \((u, \psi) \in Z^2(\Gamma, A \to G)\), then
\[
\begin{align*}
\sigma, \tau \cdot \psi (\sigma, \tau) &= \sigma, \tau \cdot \psi (\sigma, \tau) \\
\psi (\sigma, \tau) &= \text{inn}(u, \psi) \cdot \psi (\sigma, \tau).
\end{align*}
\]
Since \( A \) is abelian, the homomorphism \( A \to G \) is trivial, hence \( \psi \) is a 1-cocycle, \( \psi \in Z^1(\Gamma, G) \). Moreover, \( u \) is a 2-cocycle, \( u \in Z^2(\Gamma, \psi A) \). One checks immediately that the map
\[
Z^2(\Gamma, A \to G) \to Z^1(\Gamma, G), \quad (u, \psi) \mapsto \psi
\]
induces a surjective map
\[
\zeta: H^2(A \to G) \to H^1(G), \quad [u, \psi] \mapsto [\psi].
\]
Moreover, for given \( \psi \in Z^1(\Gamma, G) \), we have a bijection
\[
\lambda_\psi : H^2(\psi A) \xrightarrow{\sim} \zeta^{-1}([\psi]), \quad [u] \mapsto [u, \psi],
\]
and \( \lambda_\psi([u]) \) is neutral in \( H^2(A \to G) \) if and only if \([u] = 0 \in H^2(\psi A) \).

Since \( A \) is abelian, \( C \) acts on \( A \), and we obtain a surjective homomorphism \( p: C \to G \). Let \( c \in Z^1(\Gamma, C) \), \( \psi = p_* (c) \in Z^1(\Gamma, G) \), then we write \( cA \) for \( \psi A \).
Let us lift \( c: \Gamma \to C \) to some map \( b: \Gamma \to B \) and set 
\[
u_{\sigma, \tau} = b_{\sigma \tau} \cdot \sigma b_{\tau}^{-1} \cdot b_{\sigma}^{-1},
\]
then \( u \in Z^2(\Gamma, cA) \). Set 
\[
\Delta_S(c) = [u] \in H^2(\Gamma, cA).
\]
A simple computation shows that the image of \([c]\) in \( H^2(A \to G)\) is 
\[
\lambda_{\psi}(\Delta_S(c)) \in \zeta^{-1}([\psi]) \subset H^2(A \to G),
\]
where \( \psi = p_*(c) \in Z^1(\Gamma, G) \). \( \Delta_S(c) \in H^2(\Gamma, cA) \). This image is a neutral class in \( H^2(A \to G)\) if and only if \( \Delta_S(c) = 0 \).

Applying Corollary 11, we recover a result of Serre [Se94, I. 5.6, Prop. 41]: a cohomology class \([c]\) \( \in H^1(C) \) comes from \( H^1(B) \) if and only if \( \Delta_S(c) = 0 \in H^2(\Gamma, cA) \).

**Remark 12.** This note was inspired by the paper [Dun16] of Alexander Duncan, who in Section 7 constructed "by hand" the cohomology set \( H^2(A \to G) \) with a distinguished subset of neutral elements, where \( A \) is an abelian group, \( G \) is a subgroup of \( \text{Aut} A \), and the homomorphism \( A \to G \) is trivial.

**References**

[Bor92] M. Borovoi, Non-abelian hypercohomology of a group with coefficients in a crossed module, and Galois cohomology. Preprint, 20 pp., 1992, [http://www.math.tau.ac.il/~borovoi/papers/nonab.pdf](http://www.math.tau.ac.il/~borovoi/papers/nonab.pdf).

[Bor93] M. Borovoi, Abelianization of the second nonabelian Galois cohomology, Duke Math. J. 72 (1993), 217-239.

[Bor98] M. Borovoi, Abelian Galois cohomology of reductive groups, Mem. Amer. Math. Soc. 132 (1998), no. 626.

[Br90] L. Breen, Bitorseurs et cohomologie non abélienne, The Grothendieck Festschrift, Vol. I, 401-476, Progr. Math., 86, Birkhäuser Boston, Boston, MA, 1990.

[Deb76] R. Debremaeker, Cohomologie met waarden in een gekruiste groepenschoof op een situs, Thesis, Katholieke Universiteit Leuven, 1976, English translation: [arXiv:1702.02128][math.AG]

[Deb77a] R. Debremaeker, Cohomologie à valeurs dans un faisceau de groupes croisés sur un site, I, II, Acad. Roy. Belg. Bull. Cl. Sci. (5) 63 (1977), 758–764; ibid. (5) 63 (1977), 765–772.

[Ded64] P. Dedecker, Les foncteurs \( \text{Ext}_H, H^2_H \) et \( H^2_G \) non abéliens, C. R. Acad. Sci. Paris 258 (1964) 4891–4894.

[Ded69] P. Dedecker, Three dimensional non-abelian cohomology for groups, 1969, Category Theory, Homology Theory and their Applications, II (Battelle Institute Conference, Seattle, Wash., 1968, Vol, Two) pp. 32–64, Springer, Berlin.

[Dun16] A. Duncan, Twisted forms of toric varieties, Transform. Groups 21 (2016), 763–802.

[FS98] Y. Z. Flicker, C. Scheiderer, and R. Sujatha, Grothendieck's theorem on non-abelian \( H^2 \) and local-global principles, J. Amer. Math. Soc. 11 (1998), 731–750.

[Flo04] M. Florence, Zéro-cycles de degré un sur les espaces homogènes, Int. Math. Res. Not. 2004, no. 54, 2897–2914.

[Gir71] J. Giraud, Cohomologie non abélienne, Die Grundlehren der mathematischen Wissenschaften, Band 179, Springer-Verlag, Berlin-New York, 1971.
[LA15] G. Lucchini Arteche, Extensions of algebraic groups with finite quotient and non-abelian 2-cohomology, J. Algebra 492 (2017), 102–129.

[ML63] S. Mac Lane, Homology, Grundlehren der mathematischen Wissenschaften, Bd. 114, Academic Press, New York; Springer-Verlag, Berlin-Göttingen-Heidelberg, 1963.

[Noo11] B. Noohi, Group cohomology with coefficients in a crossed module, J. Inst. Math. Jussieu 10 (2011), 359–404.

[Se94] J.-P. Serre, Cohomologie galoisienne, LNM 5, 5-ème ed., Springer-Verlag, Berlin, 1994.

[Spr66] T. A. Springer, Nonabelian $H^2$ in Galois cohomology, 1966. Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965) pp. 164–182, Amer. Math. Soc., Providence, R.I.

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