Flow of A Visco-Elastic Rivlin-Ericksen (1955) Fluid Through Porous Medium in a Long Uniform Rectangular Duct Due to an Impulsive Pressure Gradient Acting at the Central Part of a Section

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Abstract

The aim of the present paper is to study the unsteady flow of a visco-elastic Rivlin-Ericksen (1955) type fluid through porous medium in a long uniform rectangular duct under the influence of an impulsive pressure gradient acting at the central part of a section. The analytical expressions for velocity profile and flux have been determined by the application of transform technique. Some particular cases have been discussed in detail. The results for the flow of ordinary viscous fluid have also been deduced by taking parameter $\mu_1 \to 0$.

Key words: visco-elastic, Rivlin-Ericksen (1955), fluid, unsteady flow

Introduction

Ghosh$^6$ discussed the viscous fluid flow through a long rectangular channel under the influence of an impulsive pressure gradient acting at the central part of a section. The problem of viscous fluid flow subjected to uniform and periodic body force for a finite time has been studied by Dutta$^2$. Roy, Sen and Lahiri$^{13}$ investigated the problem of unsteady flow of Rivlin-Ericksen visco-elastic fluid through a rectangular duct due to an impulsive pressure gradient acting at the central part of a section. Many researchers have paid their attention towards the application of visco-elastic fluid flow through porous medium such as Ahmad and Taqui$^1$; Thingarajan, Ramamurthi and Rahim$^{19}$; Pundir and Pundir$^{12}$; Sharma and Sharma$^{14}$; Prakash and Aggarwal$^{11}$; Kumar, Sharma and Singh$^8$; Singh, Mishra and Sharma$^{17}$; Singh, Kumar and Sharma$^{16}$; Singh$^{15}$ and Kumar, Singh and Sharma$^9$; Kumar, Gupta and Jain$^7$; Panda and Mohapatra$^{10}$ have studied the flow problems of visco-elastic fluid through porous medium under the influence of magnetic field applied perpendicularly to the flow of fluid. Alle et. al.$^2$ have studied flow of a dusty visco-elastic fluid through a inclined channel. Banayal$^8$ studied a characterization of Rivlin-Ericksen visco-elastic fluid in the presence of a magnetic field and rotation. Arya, Kumar and Singh$^3$ have discussed the unsteady flow of visco-elastic Rivlin-Ericksen fluid through porous medium in a long uniform rectangular channel. Venkateswarlu and Narayana$^{20}$ studied MHD visco-elastic fluid over a continuously moving vertical surface with chemical reaction. Tripathi$^9$ studied flow of a visco-elastic Rivlin-Ericksen type fluid through a long
uniform rectangular duct when an impulsive pressure gradient acting at the central part of a section.

In the present paper, the flow of visco-elastic Rivlin-Ericksen type fluid through porous medium in a rectangular duct due to an impulsive pressure gradient acting at the central part of a section has been studied. The analytical expressions for the velocity profiles and the flux have been determined by using Laplace integral transform. Some particular cases and deduction are also considered.

**Basic Theory:**

For slow motion, the rheological equations for Rivlin-Ericksen (1955) visco-elastic fluid are:

\[
\tau_{ij} = -p\delta_{ij} + \tau'_{ij} \\
\tau'_{ij} = 2\mu_1 \left(1 + \mu_1 \frac{\partial}{\partial t}\right) e_{ij} \\
e_{ij} = \frac{1}{2} \left(u_{i,j} + u_{j,i}\right)
\]

where \(\tau_{ij}\) is the stress tensor, \(\tau'_{ij}\) the deviatoric stress tensor, \(e_{ij}\) the rate of strain tensor, \(\mu_1\) the coefficient of viscoelasticity, \(p\) the fluid presser, \(u_i\) the velocity components of the fluid and \(\delta_{ij}\) is the kronecker delta.

The Navier-Stokes equation of motion for Rivlin-Ericksen fluid is

\[
\frac{d\vec{q}}{dt} = -\frac{1}{\rho} \vec{v} p + \nu \left(1 + \mu_1 \frac{\partial}{\partial t}\right) \nabla^2 \vec{q} + \vec{F}
\]

where \(\vec{q}\) is the velocity, \(\vec{F}\) the body force, \(\nu = \mu_1 / \rho\) the coefficient of viscosity and \(\rho\) is the density of fluid.

**Formulation of the problem:**

Consider the rectangular Cartesian coordinates system \((x,y,z)\); z-axis is taken towards the direction of motion of the fluid. The wall of the rectangular duct are taken to be the planes \(x=0, x=a\) and \(y=0, y=b\). Let \(u, v, w\) be the components of the velocity of the fluid along positive direction of x-axis, y-axis and z-axis respectively. Also consider rectangular duct is long and the fluid velocity \(w\) in the direction of z-axis only, therefore

\[u = 0, v = 0, w = W(x,y,t)\]

In view of the above assumptions the equation of visco-elastic Rivlin-Ericksen fluid through porous medium is given by

\[
\frac{\partial w}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial z} \frac{\partial}{\partial z} \left(1 + \mu_1 \frac{\partial}{\partial t}\right) \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right) - \frac{v}{K} W
\]

where \(W\) is the velocity of the fluid in z-direction, \(t\) the time, \(K\) the permeability of porous media.

Introducing the following non-dimensional quantities:

\[x^* = \frac{x}{a}, y^* = \frac{y}{a}, z^* = \frac{z}{a}, t^* = \frac{v}{a^2} t, p^* = \frac{a^2}{\rho v^2} p,\]

\[\mu_1^* = \frac{v}{a^2} \mu_1, K^* = \frac{1}{a^2} K\]

in equation (5), there is found (after dropping stars)

\[
\frac{\partial w}{\partial t} = -\frac{\partial p}{\partial z} + \left(1 + \mu_1 \frac{\partial}{\partial t}\right) \left(\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2}\right) - \frac{1}{K} W
\]
Let \( -\frac{\partial p}{\partial z} = f(x, y)\delta t \), \( \delta t \) is Dirac delta function, be an impulsive pressure gradient acting at the central part of any section such that:

Boundary conditions are

\[
f(x, y) = p \quad \text{for} \quad \begin{cases} \frac{1}{2} - l_1 \leq x \leq \frac{1}{2} + l_1 \\ \frac{c}{2} - l_2 \leq y \leq \frac{c}{2} + l_2 \end{cases}
\]

\[
= 0 \quad \text{for} \quad \begin{cases} 0 \leq x < \frac{1}{2} - l_1 \\
 \frac{c}{2} + l_2 < y \leq c \end{cases}
\]  

\[
= 0 \quad \text{for} \quad t \leq 0
\]

where \( c = \frac{b}{a} < 1 \), a non-dimensional quantity.

To find the solution of equation (6), it is defined as

\[
f(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{1} \sin \frac{n\pi y}{c}
\]  

(8)

and Integrating, it is found

\[
B_{mn} \int_{0}^{1} \int_{0}^{1} \sin^2 \frac{m\pi x}{1} \sin^2 \frac{n\pi y}{c} \, dx \\ dy = p \int_{c^{-1} + \frac{1}{2} + l_1}^{1} \int_{\frac{c}{2} + l_2}^{c} \sin \frac{m\pi x}{1} \sin \frac{n\pi y}{c} \, dx \\ dy
\]

\[
\Rightarrow B_{mn} = \frac{16p}{m\pi n^2} \sin \frac{ml_1}{2} \sin \frac{nl_2}{2c}
\]  

(9)

Now let it be assumed that the velocity of fluid be

\[
W = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}(t) \sin \frac{m\pi x}{1} \sin \frac{n\pi y}{c}
\]  

(10)

Substituting equations (8) and (10) in equation (6), it is found

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\partial}{\partial t} \left( A_{mn}(t) \sin \frac{m\pi x}{1} \sin \frac{n\pi y}{c} \right) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin \frac{m\pi x}{1} \sin \frac{n\pi y}{c} +
\]

\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn}(t) \left( \left( -\frac{m^2\pi^2}{1} \right) + \left( -\frac{n^2\pi^2}{c^2} \right) \right) \sin \frac{m\pi x}{1} \sin \frac{n\pi y}{c} +
\]
Taking Laplace transform of equation (11), it is found

\[ s \bar{A}_{mn}(s) = \frac{B_{mn}}{s} - \bar{A}_{mn}(s)C_{mn} - \mu_1 s \bar{A}_{mn}(s)C_{mn} - \frac{1}{K} \bar{A}_{mn} \]

or \( \left\{ (1 + \mu_1 C_{mn})s + \left( \frac{1}{K} + C_{mn} \right) \right\} \bar{A}_{mn} = \frac{B_{mn}}{s} \)

or \( \bar{A}_{mn}(s) = \frac{B_{mn}}{s \left\{ (1 + \mu_1 C_{mn})s + \left( \frac{1}{K} + C_{mn} \right) \right\}} \)

or \( \bar{A}_{mn}(s) = \frac{B_{mn}}{(1 + \mu_1 C_{mn})s + \left( \frac{1}{K} + C_{mn} \right)} \)

or \( \bar{A}_{mn}(s) = \frac{B_{mn}}{(1 + \mu_1 C_{mn})} \left[ \frac{1}{s} - \frac{1}{s + \left( \frac{1}{K} + C_{mn} \right)} \right] \)

where \( \bar{A}_{mn}(s) = \int_0^\infty e^{-st} A_{mn}(t) \, dt \) (Laplace transform of function \( A_{mn}(t) \))

and \( C_{mn} = \frac{m^2 \pi^2}{1} + \frac{n^2 \pi^2}{c^2} = \left( \frac{m^2}{c^2} + \frac{n^2}{c^2} \right) \pi^2 \)

Now applying the inverse Laplace transform of (12), we get

\[ L^{-1}\{ \bar{A}_{mn}(s) \} = A_{mn}(t) \]

\[ = \frac{B_{mn}}{(1 + \frac{1}{c^2} + \frac{C_{mn}}{1 + \mu_1 C_{mn}})} \left[ L^{-1}\left( \frac{1}{s} \right) - L^{-1}\left( \frac{1}{s + \left( \frac{1}{K} + C_{mn} \right)} \right) \right] \]
From equations (9), (10) and (13), the velocity of visco-elastic Rivlin-Ericksen fluid is obtained as

\[ W = \frac{16P}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(\frac{1}{K} + C_{mn})} \sin \left( \frac{m\pi l_1}{2} \right) \sin \left( \frac{n\pi l_2}{2} \right) \sin \left( \frac{m\pi x}{1} \right) \sin \left( \frac{n\pi y}{c} \right) \]  

(14)

The flux \( Q \) across any section at any instant is given by

\[ Q = \int \int w \, dx \, dy \]

c

\[ = \frac{64cP}{\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2n^2(\frac{1}{K} + C_{mn})} \sin \left( \frac{m\pi l_1}{2} \right) \sin \left( \frac{n\pi l_2}{2} \right) \sin \left( \frac{m\pi x}{1} \right) \sin \left( \frac{n\pi y}{c} \right) \]  

(15)

when \( m \) and \( n \) both are odd.

**Particular cases:**

**Case I:** When \( \mu_1 \to 0 \) \( i.e. \) the coefficient of visco-elasticity becomes zero, the velocity of purely viscous fluid through porous medium is given from equation (14) is

\[ W = \frac{16P}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{mn(\frac{1}{K} + C_{mn})} \sin \left( \frac{m\pi l_1}{2} \right) \sin \left( \frac{n\pi l_2}{2} \right) \sin \left( \frac{m\pi x}{1} \right) \sin \left( \frac{n\pi y}{c} \right) \]  

(16)

and flux from equation (15) is

\[ Q = \frac{64cP}{\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{m^2n^2(\frac{1}{K} + C_{mn})} \sin \left( \frac{m\pi l_1}{2} \right) \sin \left( \frac{n\pi l_2}{2} \right) \]  

(17)

**Case II:** When \( K \to \infty \) \( i.e. \) porous medium is withdrawn, the velocity of visco-elastic Rivlin-Ericksen fluid is given from equation (14) is
and flux from (15) is

\[ Q = \frac{64cP}{\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ 1 - e^{-\frac{C_{mn}}{K}} \right] \frac{m\pi l_1}{2} \sin \frac{n\pi l_2}{2c} \sin \frac{m\pi x}{2c} \sin \frac{m\pi y}{c} \]  

which are the same result obtained by Roy, Sen and Lahiri\textsuperscript{13} with slight change of notations.

**Case III:** When \( \mu_1 \to 0, K \to \infty \) then velocity of purely incompressible viscous fluid is given from equation (14) is

\[ W = \frac{16P}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ 1 - e^{-\frac{C_{mn}}{\alpha}} \right] \frac{m\pi l_1}{2} \sin \frac{n\pi l_2}{2c} \sin \frac{m\pi x}{2c} \sin \frac{m\pi y}{c} \]  

and flux from (15) is

\[ Q = \frac{64cP}{\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[ 1 - e^{-\frac{C_{mn}}{\alpha}} \right] \frac{m\pi l_1}{2} \sin \frac{n\pi l_2}{2c} \]  

which are the same result obtained by Gosh\textsuperscript{6} with slight change of notations.

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