Weak Saturation of Multipartite Hypergraphs

Denys Bulavka¹ · Martin Tancer¹ · Mykhaylo Tyomkyn¹

Received: 24 September 2021 / Revised: 3 February 2023 / Accepted: 26 April 2023 / Published online: 27 July 2023 © The Author(s) 2023

Abstract

Given $q$-uniform hypergraphs ($q$-graphs) $F$, $G$ and $H$, where $G$ is a spanning subgraph of $F$, $G$ is called weakly $H$-saturated in $F$ if the edges in $E(F) \setminus E(G)$ admit an ordering $e_1, \ldots, e_k$ so that for all $i \in [k]$ the hypergraph $G \cup \{e_1, \ldots, e_i\}$ contains an isomorphic copy of $H$ which in turn contains the edge $e_i$. The weak saturation number of $H$ in $F$ is the smallest size of an $H$-weakly saturated subgraph of $F$. Weak saturation was introduced by Bollobás in 1968, but despite decades of study our understanding of it is still limited. The main difficulty lies in proving lower bounds on weak saturation numbers, which typically withstands combinatorial methods and requires arguments of algebraic or geometrical nature. In our main contribution in this paper we determine exactly the weak saturation number of complete multipartite $q$-graphs in the directed setting, for any choice of parameters. This generalizes a theorem of Alon from 1985. Our proof combines the exterior algebra approach from the works of Kalai with the use of the colorful exterior algebra motivated by the recent work of Bulavka, Goodarzi and Tancer on the colorful fractional Helly theorem. In our second contribution answering a question of Kronenberg, Martins and Morrison, we establish a link between weak saturation numbers of bipartite graphs in the clique versus in a complete bipartite host graph. In a similar fashion we asymptotically determine the weak saturation number of any complete $q$-partite $q$-graph in the clique, generalizing another result of Kronenberg et al.

Denys Bulavka: Supported by Charles University project PRIMUS/21/SCI/014, Grant Schemes at CU, Reg. No. CZ.02.2.69/0.0/0.0/19_073/0016935 and by the Grant SVV–2020–260578. Martin Tancer: Supported by GAČR Grant 22-19073S. Mykhaylo Tyomkyn: Supported by GAČR Grant 22-19073S, ERC Synergy Grant DYNASNET 810115 and the H2020-MSCA-RISE Project CoSP-GA No. 823748.

Denys Bulavka
dbulavka@kam.mff.cuni.cz

Martin Tancer
tancer@kam.mff.cuni.cz

Mykhaylo Tyomkyn
tyomkyn@kam.mff.cuni.cz

¹ Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Prague, Czech Republic
Keywords  Saturation · Hypergraph · Exterior algebra

Mathematics Subject Classification  05C50 · 05C35 · 15A75 · 05C65

1 Introduction

Let $F$ and $H$ be $q$-uniform hypergraphs ($q$-graphs for short); we identify hypergraphs with their edge sets. We say that a subgraph $G \subseteq F$ is weakly $H$-saturated in $F$ if the edges of $F \setminus G$ can be ordered as $e_1, \ldots, e_k$ such that for all $i \in [k]$ the hypergraph $G \cup \{e_1, \ldots, e_i\}$ contains an isomorphic copy of $H$ which in turn contains the edge $e_i$. We call such $e_1, \ldots, e_k$ an $H$-saturating sequence of $G$ in $F$. The weak saturation number of $H$ in $F$, $\text{wsat}(F, H)$ is the minimum number of edges in a weakly $H$-saturated subgraph of $F$. When $F$ is complete of order $n$, we simply write $\text{wsat}(n, H)$.

Weak saturation was introduced by Bollobás [4] in 1968 and is related to (strong) graph saturation: $G$ is $H$-saturated in $F$ if adding any edge of $F \setminus G$ would create a new copy of $H$. However, a number of properties of weak saturation make it a more natural object of study. Firstly, it follows from the definition that any graph $G$ achieving $\text{wsat}(F, H)$ has to be $H$-free (we could otherwise remove an edge from a copy of $H$ in $G$ resulting in a smaller example), while for strong saturation $H$-freeness may or may not be imposed, resulting in two competing notions (see [18] for a discussion). Secondly, a short subadditivity argument originally due to Alon [1] shows that for every 2-uniform $H$, $\lim_{n \to \infty} \text{wsat}(n, H)/n$ exists. Whether the same holds for strong saturation is a longstanding conjecture of Tuza [24]. And thirdly, weak saturation lends itself to be studied via algebraic methods, thus offering insight into algebraic and matroid structures underlying graphs and hypergraphs.

The most natural case when $F$ and $H$ are cliques was the first to be studied. Let $K_{r}^q$ denote the complete $q$-graph of order $r$. Confirming a conjecture of Bollobás, Frankl [10] and Kalai [13, 14] (independently) proved that $\text{wsat}(n, K_q^r) = \binom{n}{q} - \binom{n-r+q}{q}$. Another proof has been given by Alon [1] and in hindsight this conjecture could be also derived from an earlier paper of Lovász [16]. While the upper bound is a construction that is easy to guess (a common feature in weak saturation problems), all of the above lower bound proofs rely on algebraic or geometric methods, and no purely combinatorial proof is known to this date.

In the subsequent years weak saturation has been studied extensively [1, 2, 5–8, 17–20, 22, 23, 25, 26]. Despite this, our understanding of weak saturation numbers is still rather limited. For instance we do not know whether for $q \geq 3$ we have a similar limiting behavior as in the graph case, in that $\lim_{n \to \infty} \text{wsat}(n, H)/n^{q-1}$ always exists; this has been conjectured by Tuza [26].

In this paper we address the case when $H = K_{r_1, \ldots, r_d}^q$ is a complete $d$-partite $q$-graph for arbitrary $d \geq q > 1$. That is, $V(H)$ is a disjoint union of sets $R_1, \ldots, R_d$ with $|R_i| = r_i$ and

$$E(H) = \left\{ e \in \binom{V(H)}{q} : |e \cap R_i| \leq 1 \text{ for all } i \in [d] \right\},$$

in particular, for $q = 2$ we recover the usual complete multipartite graphs. This is perhaps the next most natural class of hypergraphs to consider after the cliques.
For the host graph \( F \), besides the clique it is natural to consider a larger complete \( d \)-partite \( q \)-graph \( K_{n_1, \ldots, n_d}^q \). In the latter case we have a choice between the \textit{undirected} and \textit{directed} versions of the problem. The former follows the definition of weak saturation given at the beginning, while in the latter we additionally impose that the new copies of \( H \) in \( F \) created in every step “point the same way”, i.e. have \( r_i \) vertices in the \( i \)-th partition class for all \( i \in [d] \) (see below for a formal definition).

All three above versions have been studied in the past. For \( q = 2 \), Kalai [14] determined \( \text{wsat}(n, K_{r,r}^q) \) for large enough \( n \). Kronenberg et al. [15] recently extended it to \( \text{wsat}(n, K_{r,r-1}^q) \) and asymptotically to all \( \text{wsat}(n, K_s, q) \). No other values \( \text{wsat}(n, K_{r_1, \ldots, r_d}^q) \) are known except for \( r_1 = \cdots = r_d = 1 \) when \( H \) is a clique and a handful of closely related cases, e.g., when all \( r_i \) but one are 1 [20]. When both \( H \) and \( F \) are complete \( d \)-partite, for \( d = q \) Alon [1] solved the problem in the directed setting. Moshkovitz and Shapira [18], building on Alon’s work, settled the undirected case, determining \( \text{wsat}(K_{n_1, \ldots, n_d}^q, K_{r_1, \ldots, r_d}^q) \). There has been no progress for \( d > q \).

In our main contribution in this paper we settle completely the directed case for all \( q \) and \( d \). To state the problem formally, let \( \mathbf{r} = (r_1, \ldots, r_d) \) and \( \mathbf{n} = (n_1, \ldots, n_d) \) be integer vectors such that \( 1 \leq r_i \leq n_i \). Suppose \( N = N_1 \sqcup \cdots \sqcup N_d \) where \( |N_i| = n_i \) and \( \sqcup \) denotes a disjoint union. Let \( K_{n}^q \) be the complete \( d \)-partite \( q \)-graph on \( N \) whose partition classes are the \( N_i \), and let \( K_{q}^q \) be an unspecified complete \( d \)-partite \( q \)-graph on the same partition classes, with \( r_i \) vertices in each \( N_i \). Given a subgraph \( G \) of \( K_{n}^q \), a sequence of edges \( e_1, \ldots, e_k \) in \( K_{n}^q \) is a \textit{(directed) \( K_{q}^q \)-saturating sequence of} \( G \) in \( K_{n}^q \) if: (i) \( K_{q}^q \setminus G = \{e_1, \ldots, e_k\} \); (ii) for every \( j \in [k] \) there exists \( H_j \subseteq G \cup \{e_1, \ldots, e_j\} \) isomorphic to \( K_{q}^q \) such that \( e_j \in H_j \) and \( V(H_j) \cap N_i = r_i \) for all \( i \in [d] \). The \( q \)-graph \( G \) is said to be \textit{(directed) weakly} \( K_{q}^q \)-saturated in \( K_{n}^q \) if it admits a \( K_{q}^q \)-saturating sequence in the latter. The \textit{(directed) weak saturation number} of \( K_{q}^q \) in \( K_{n}^q \), in notation \( w(K_{q}^q, K_{q}^q) \), is the minimal number of edges in a weakly \( K_{q}^q \)-saturated subgraph of \( K_{n}^q \).

**Theorem 1.1** For all \( d \geq q \geq 2 \), \( \mathbf{n} \) and \( \mathbf{r} \) we have

\[
  w(K_{n}^q, K_{q}^q) = \sum_{\mathbf{i} \in \binom{[d]}{\leq q}} \prod_{i \in \mathbf{i}} n_i - \sum_{\mathbf{i} \in \binom{[d]}{\geq q}} \prod_{i \in \mathbf{i}} (n_i - r_i).
\]

In the above formula \( \binom{[d]}{\leq q} \) stands for the set of all subsets of \( [d] \) of size at most \( q \), and we use the convention that \( \prod_{i \in \emptyset} (n_i - r_i) = 1 \).

As mentioned, the \( d = q \) case of Theorem 1.1 was proved by Alon [1]. Hence our result generalizes Alon’s theorem to arbitrary \( d \geq q \). When \( H \) is balanced, that is when \( r_1 = \cdots = r_d \), there is no difference between the directed and undirected partite settings. Writing \( K^q(r; d) \) for \( K_{1, \ldots, d}^q \) \( (d \times) \), Theorem 1.1 thus determines the weak saturation number of \( K^q(r; d) \) in complete \( d \)-partite \( q \)-graphs.

**Corollary 1.2** For all \( d \geq q \geq 2 \) and \( n_1, \ldots, n_d \geq r \geq 1 \) we have

\[
  \text{wsat}(K_{n_1, \ldots, n_d}^q, K^q(r; d)) = \sum_{\mathbf{i} \in \binom{[d]}{\leq q}} \prod_{i \in \mathbf{i}} n_i - \sum_{\mathbf{i} \in \binom{[d]}{\geq q}} \prod_{i \in \mathbf{i}} (n_i - r).
\]
Our proof of Theorem 1.1 combines exterior algebra techniques in the spirit of [14] with a new ingredient: the use of the colorful exterior algebra inspired by the recent work of Bulavka, Goodarzi and Tancer on the colorful fractional Helly theorem [3].

Kronenberg et al. ([15], Section 5) remarked that while the values wsat(n, Kℓ,m) and wsat(Kℓ,m, Kℓ,i) for ℓ + m = n, which were determined in separate works, are of the same order of magnitude, it is not obvious if there is any direct connection. In our second contribution in this paper we establish such a connection using a tensoring trick. As we have mentioned earlier, 2-graphs H satisfy wsat(n, H) = cHn + o(n), and Alon’s proof of this fact [1] can be straightforwardly adjusted to show that wsat(Kn,n, H) = cH′ · 2n + o(n) when H is bipartite. We show that in fact cH = cH′. A minor adjustment to our proof gives that, for any rational 0 < α < 1, the quantities wsat(n, H) and wsat(Kan,(1−α)n, H), when αn ∈ ℤ, are of the same order of magnitude. Setting H = Kt,i answers the above question of [15].

For q ≥ 3 while we do not have (yet) the same knowledge of limiting constants, a similar method determines asymptotically the weak saturation number of complete d-partite d-graphs in the clique, generalizing Theorem 4 of [15].

**Theorem 1.3** For every bipartite 2-uniform graph H we have
\[
\lim_{n \to \infty} \frac{wsat(n, H)}{n} = \lim_{n \to \infty} \frac{wsat(K_n, H)}{2n}.
\]

Furthermore, for any d ≥ 2 and 1 ≤ r_1 ≤ ⋯ ≤ r_d we have
\[
wsat(n, K_{r_1, \ldots, r_d}^d) = \frac{r_1 - 1}{(d - 1)!} n^{d-1} + O(n^{d-2}).
\]

The rest of the paper is organized as follows. In Sect. 2 we give a construction for the upper bound in Theorem 1.1. In Sect. 3 we review the algebraic tools, setting the stage for the lower bound proof in Sect. 4. In Sect. 5 we discuss weak saturation in the clique and prove Theorem 1.3.

**Notation.** As usual, [n] abbreviates the set {1, 2, ⋯, n}. The symbol □ denotes a disjoint union of sets. For a set M and integer q ≥ 0, \( M_q \) and \( M_{\leq q} \) denote the set of all subsets of M of size exactly q and of size most q, respectively. We use ± to denote an unspecified factor of either +1 or −1.

\( K_n^q \) denotes the complete q-uniform hypergraph (q-graph) of order n. When the vertex set of the said q-graph is [n], we write \( K_n^q \). The complete d-partite q-graph with \( n_i \) vertices in the i-th partition class is denoted by \( K_{n_1, \ldots, n_d}^q \); when \( n_1 = \cdots = n_d = n \) we write simply \( K^q(n; d) \).

Note that in Sects. 2, 3, and 4 we work solely in the directed partite setup (Theorem 1.1), while in Sect. 5 we deal with the undirected partite and the clique setups (Theorem 1.3). In the directed setup our q-graphs are defined on a vertex set N of size n with a fixed d-partition \( N = N_1 \sqcup \cdots \sqcup N_d \), where \( |N_i| = n_i \) for all \( i \in [d] \). Consequently, we use \( K_n^q \) to denote the complete d-partite q-graph on N with respect to this partition. (Up to a graph isomorphism, \( K_n^q \) is uniquely determined by q and n, thus we do not display N in the notation.) For any M ⊆ N the induced subgraph of \( K_n^q \) on M is denoted by \( K_n^q[M] \). The directed weak saturation number defined above
is denoted by \( w(K^q_n, K^q_r) \), as opposed to \( \text{wsat}(K^q_{n_1 \ldots n_d}, K^q_{r_1 \ldots r_d}) \) in the undirected setting, a similar notation was employed in [15].

2 Theorem 1.1: The Upper Bound

In this section we prove the upper bound in Theorem 1.1 by exhibiting a weakly \( K^q_r \)-saturated \( q \)-graph \( G \). Fix a subset \( R \subseteq N \) such that \( |R \cap N_i| = r_i \) for every \( i \in [d] \) and set

\[ \Sigma := \left\{ S \in \left( N \setminus R \right) : |S \cap N_i| \leq 1 \text{ for each } i \in [d] \right\}. \]

We define \( G \) via its complement in \( K^q_n \) as follows. For every \( S \in \Sigma \) choose an edge \( \lambda(S) \) satisfying \( S \subseteq \lambda(S) \). Note that the assignment \( \lambda \) is injective, as \( \lambda(S) \cap (N \setminus R) = S \). Recall that we associate hypergraphs with their edge sets. Define

\[ G := K^q_n \setminus \bigcup_{S \in \Sigma} \lambda(S), \]

so that

\[ |E(G)| = \sum_{I \in \binom{[d]}{q}} \prod_{i \in I} n_i - \sum_{I \in \binom{[d]}{\leq q}} \prod_{i \in I} (n_i - r_i). \]

Notice that the choices of \( \lambda(S) \) are not unique, but as the next lemma shows, each of them yields a weakly \( K^q_r \)-saturated \( q \)-graph. Such non-uniqueness is a common occurrence in weak saturation: for instance, every \( n \)-vertex tree is an extremal example for weak triangle saturation in \( K_n \).

Lemma 2.1 The \( q \)-graph \( G \) defined above is weakly \( K^q_r \)-saturated. Therefore,

\[ w(K^q_n, K^q_r) \leq |E(G)| = \sum_{I \in \binom{[d]}{q}} \prod_{i \in I} n_i - \sum_{I \in \binom{[d]}{\leq q}} \prod_{i \in I} (n_i - r_i). \]

Proof For each \( 0 \leq k \leq q \) let

\[ G_k := G \cup \{ T \in K^q_n : |T \setminus R| \leq k \}, \]

and put \( G_{-1} := G \). We claim that adding any new edge \( L \in K^q_n \) with \( |L \setminus R| = k \) to \( G_{k-1} \) creates a new copy of \( K^q_r \) containing \( L \). This gives rise to a \( K^q_r \)-saturating sequence between \( G_{k-1} \) and \( G_k \) and, by extension, between \( G = G_{-1} \) and \( G_q = K^q_n \).

First, notice that \( G_0 \) is obtained from \( G_{-1} \) by adding the sole missing edge \( \lambda(\emptyset) \). Doing so creates a new copy of \( K^q_r \), namely \( K^q_n[R] \). For an arbitrary \( k \), suppose that \( L \) is a missing edge in \( G_{k-1} \) such that \( S := L \setminus R \) is of size \( k \). Observe that every \( T \in K^q[R \cup S] \) is an edge in \( G_{k-1} \) unless \( T = L \). Indeed, if \( |T \setminus R| < k \) then
this holds by definition of $G_{k-1}$. While otherwise we have $T \setminus R = S$. Hence, by the definition of $G$, we have $L = \lambda(S)$, so that either $T = L$ or $T \in G \subseteq G_{k-1}$. Therefore, adding $L$ to $G_{k-1}$ creates a new copy of $K_q^n[R \cup S]$ containing $L$ and a fortiori also a new copy of $K_q^r$ containing $L$, as desired. □

3 Algebraic Background

In this section we introduce the linear algebra tools needed for the proof of the lower bound in Theorem 1.1. In Sects. 3.1 and 3.2 we largely follow [12, Sec. 2] though we sometimes provide more detail. (For comparison [14] works with a dual generic basis. We believe that the difference is not essential.) In Sect. 3.3 we loosely follow [3].

Before we start explaining the algebraic background, we will try to sketch why algebraic tools can be useful in this context. This sketch should be understood loosely—we do not provide any guarantees for the claims in this sketch. In particular, many important technical details are skipped in the sketch. Understanding this sketch is not required in the following text, thus it can be skipped.

Consider first the somewhat trivial case of providing the lower bound on $\text{wsat}(n, K_3)$, the weak saturation number of the complete graph $K_3$ in $K_n$. Consider a subgraph $G$ of $K_n$ and a saturating sequence $e_1, \ldots, e_k$ of edges in $E(K_n) \setminus E(G)$. Let $G_i := G \cup \{e_1, \ldots, e_i\}$. Because the sequence is saturating, we know that $G_i$ contains a copy of $K_3$ containing $e_i$. This means that the dimension of the cycle space of $G_i$ is strictly larger than the dimension of the cycle space of $G_{i-1}$. Because the final dimension of the cycle space of $K_n$ equals $\binom{n-1}{2}$, we may perform at most $\binom{n-1}{2}$ such steps. In other words $k \leq \binom{n-1}{2}$ and thus $|E(G)| \geq \binom{n}{2} - \binom{n-1}{2}$ as required.

In the language of algebraic topology (which we however do not use in the proofs, no topological background is required), the property that the dimension of the cycle space increases can be phrased so that a new copy of $K_3$ in each step belongs to the kernel of the standard boundary operator. For more complicated (hyper)graphs than $K_3$ it is actually useful to use several independent boundary operators in order to generalize the aforementioned approach. Using such independent operators can be actually efficiently phrased in terms of exterior algebra (without mentioning algebraic topology). They correspond to the left interior product, which we will discuss later on, subject to some suitable independence (genericity) condition. ¹

3.1 Exterior Algebra

Let $N$ be a set of size $n$, ordered with a total order $\lt$. Later on the elements of $N$ will represent vertices of a $q$-graph and we will typically denote them by letters such as $v$ or $w$. Let $V$ be an $n$-dimensional real vector space with a basis $(e_v)_{v \in N}$. The exterior

¹ Perhaps the closest relation between the boundary operators and the left interior product can be seen in Lemma 3.3 interpreting $e_R$ as a simplex with set of vertices $R$, and $f_T$ as an operator removing $t$ times the top-dimensional simplices, yielding a linear combination of simplices $f_S$ with $r-t$ vertices. (However, for this relation, it would be even better to express the right hand side using $e_S$ so that all possible $e_S$ would appear.) Adding a colorful aspect (in our case) then makes it easier to work with multipartite (hyper)graphs rather than complete ones.
algebra of $V$, denoted by $\bigwedge V$, is a $2^n$-dimensional vector space with basis $(e_S)_{S \subseteq N}$ and an associative bilinear product operation, denoted by $\wedge$, that satisfies

(i) $e_{\emptyset}$ is the neutral element, i.e. $e_{\emptyset} \wedge e_S = e_S = e_S \wedge e_{\emptyset}$;
(ii) $e_S = e_{s_1} \wedge \cdots \wedge e_{s_k}$ for $S = \{s_1 < \cdots < s_k\} \subseteq N$;
(iii) $e_v \wedge e_w = -e_w \wedge e_v$ for all $v, w \in N$.

For $0 \leq k \leq n$ we denote by $\bigwedge^k V$ the subspace of $\bigwedge V$ with basis $(e_S)_{S \in \binom{N}{k}}$. Denote by $\langle \cdot, \cdot \rangle$ the standard inner product (dot product) on $V$ as well as on $\bigwedge V$ with respect to the basis $(e_v)_{v \in N}$ and $(e_S)_{S \subseteq N}$ respectively; that is, for every pair of sets $S, T \subseteq N$, the inner product $\langle e_S, e_T \rangle$ is 1 if $S = T$ and 0 otherwise.

If $(f_v)_{v \in N}$ is another basis of $V$, then $(f_S)_{S \subseteq N}$ is a new basis of $\bigwedge V$, where $f_S$ stands for $f_{s_1} \wedge \cdots \wedge f_{s_k}$ for $S = \{s_1 < \cdots < s_k\} \subseteq N$. Similarly, $(f_S)_{S \in \binom{N}{k}}$ is a basis of $\bigwedge^k V$ for $k \in \{0, \ldots, n\}$. The formulas (i), (ii) and (iii) remain valid for the basis $(f_v)_{v \in N}$ due to definition of $f_S$ and bilinearity of $\wedge$. In particular, $\bigwedge V$ and $\bigwedge^k V$ do not depend on the initial choice of the basis. Using (ii) and (iii) iteratively, for $S, T \subseteq N$ we get

$$f_S \wedge f_T = \begin{cases} \text{sgn}(S, T) f_{S \cup T} & \text{if } S \cap T = \emptyset \\ 0 & \text{if } S \cap T \neq \emptyset, \end{cases}$$

where $\text{sgn}(S, T)$ is the sign of the permutation of $S \cup T$ obtained by first placing the elements of $S$ (in our total order $<$) and then the elements of $T$. Equivalently, $\text{sgn}(S, T) = (-1)^{\alpha(S, T)}$ where $\alpha(S, T) = |\{(s, t) \in S \times T : t < s\}|$ is the number of transpositions.

As a consequence we obtain the following useful formula. Let $M_1, \ldots, M_\ell$ be pairwise disjoint subsets of $N$ and $s_1, \ldots, s_\ell$ be integers with $0 \leq s_i \leq |M_i|$. Suppose that for each $i \in [\ell]$ we are given

$$h_i = \sum_{S_i \in \binom{M_i}{s_i}} \lambda_{S_i} f_{S_i}$$

for $\lambda_{S_i} \in \mathbb{R}$ (so that $h_i \in \bigwedge^{s_i} V$). Then by bilinearity of $\wedge$ and (3) we get

$$h_1 \wedge \cdots \wedge h_\ell = \sum_{(S_1, \ldots, S_\ell) \in \binom{M_1}{s_1} \times \cdots \times \binom{M_\ell}{s_\ell}} \left( \prod_{i \in [\ell]} \lambda_{S_i} \right) f_{S_1} \wedge \cdots \wedge f_{S_\ell}$$

$$= \sum_{(S_1, \ldots, S_\ell) \in \binom{M_1}{s_1} \times \cdots \times \binom{M_\ell}{s_\ell}} \pm \left( \prod_{i \in [\ell]} \lambda_{S_i} \right) f_{S_1 \cup \cdots \cup S_\ell}.$$  \hspace{1cm} (4)
Let $A = (a_{vw})_{v,w \in \mathcal{N}}$ be the transition matrix from $(e_v)_{v \in \mathcal{N}}$ to $(f_v)_{v \in \mathcal{N}}$, meaning that $f_v = \sum_{w \in \mathcal{N}} a_{vw} e_w$. Then, for $S \subseteq \mathcal{N}$ of size $k$, $f_S$ can be expressed as

$$f_S = \sum_{T \in \binom{\mathcal{N}}{k}} \det(A_{S|T}) e_T,$$

where $A_{S|T}$ is the submatrix of $A$ formed by rows in $S$ and columns in $T$, i.e. $A_{S|T} = (a_{vw})_{v \in S, w \in T}$.

As noted in [12], it follows from the Cauchy–Binet formula that if the basis $(f_v)_{v \in \mathcal{N}}$ is orthonormal then $(f_S)_{S \subseteq \mathcal{N}}$ is orthonormal as well. For completeness, we provide a short explanation. Let $S, L \subseteq \mathcal{N}$ be a pair of subsets. If $|S| \neq |L|$, then $f_S$ and $f_L$ belong to two orthogonal subspaces of $\bigwedge \mathcal{V}$, namely $\bigwedge |S| \mathcal{V}$ and $\bigwedge |L| \mathcal{V}$, and so $\langle f_S, f_L \rangle = 0$. On the other hand, if $|S| = |L| =: k$, then by writing $f_S$ and $f_L$ in the standard basis $(e_T)_{T \subseteq \mathcal{N}}$ we have that

$$\langle f_S, f_L \rangle = \sum_{T \in \binom{\mathcal{N}}{k}} \det(A_{S|T}) \det(A_{L|T}^t) = \det(A_{S|\mathcal{N}} A_{L|\mathcal{N}}^t),$$

where $B^t$ stands for the transpose matrix of $B$ (and expressions like $A_{L|T}^t$ stand for $(A_{L|T})^t$), and the last equality holds by the Cauchy–Binet formula (see e.g. Section 1.2.4 of [11]). Notice that for any $u \in S$ and $w \in L$ we have $(A_{S|\mathcal{N}} A_{L|\mathcal{N}}^t)_{u,w} = \langle f_u, f_w \rangle$, and since $(f_v)_{v \in \mathcal{N}}$ is orthonormal this is 1 if $u = w$ and 0 otherwise. Therefore, if $S = L$, the product $A_{S|\mathcal{N}} A_{L|\mathcal{N}}^t$ is the identity matrix and consequently the determinant will be 1. On the other hand, if $S \neq L$, the product $A_{S|\mathcal{N}} A_{L|\mathcal{N}}^t$ will have a zero column, and so the determinant will be 0. The above claim follows.

We say that the change of basis from $(e_v)_{v \in \mathcal{N}}$ to $(f_v)_{v \in \mathcal{N}}$ is generic if $\det(A_{S|T}) \neq 0$ for every $S, T \subseteq \mathcal{N}$ of the same size; that is, every square submatrix of $A$ has full rank. It is known (see e.g. [12]) that $(f_v)_{v \in \mathcal{N}}$ can be chosen to be both generic and orthonormal. For a basis $(f_v)_{v \in \mathcal{N}}$ generic with respect to $(e_v)_{v \in \mathcal{N}}$ and a pair of sets $S, T \in \binom{\mathcal{N}}{k}$ we have

$$\langle f_S, e_T \rangle \equiv \left\langle \sum_{T' \in \binom{\mathcal{N}}{k}} \det(A_{S|T'}) e_{T'}, e_T \right\rangle = \sum_{T' \in \binom{\mathcal{N}}{k}} \det(A_{S|T'}) \langle e_{T'}, e_T \rangle = \det(A_{S|T}) \neq 0.$$

(6)

### 3.2 Left Interior Product

The following lemma defines $g \ll f$, the left interior product of $g$ and $f$. We refer to Section 2.2.6 of [21] for a more extensive coverage of the topic.

**Lemma 3.1** For any $f, g \in \bigwedge \mathcal{V}$ there exists a unique element $g \ll f \in \bigwedge \mathcal{V}$ that satisfies

$$\langle h, g \ll f \rangle = \langle h \land g, f \rangle$$

for all $h \in \bigwedge \mathcal{V}$.

(7)
Furthermore, assuming \( f \in \bigwedge^s V \) and \( g \in \bigwedge^t V \), if \( t > s \) then \( g \lhd f = 0 \), while if \( t \leq s \) then \( g \lhd f \in \bigwedge^{s-t} V \).

**Proof** For \( f, g \in \bigwedge V \) we set

\[
g \lhd f := \sum_{S \subseteq N} \langle e_S \land g, f \rangle e_S.
\]

To verify that this satisfies (7) let \( h \in \bigwedge V \) be arbitrary. By bilinearity of \( \langle \cdot, \cdot \rangle \) and \( \land \), and orthonormality of \( (e_S)_{S \subseteq N} \) we have

\[
\langle h, g \lhd f \rangle = \sum_{S \subseteq N} \langle h, e_S \rangle \langle e_S \land g, f \rangle = \sum_{S \subseteq N} \langle h, e_S \rangle \langle e_S \land g, f \rangle = \langle h \land g, f \rangle.
\]

To show uniqueness, suppose that \( z \) is an element in \( \bigwedge V \) that satisfies (7). Then for each \( T \subseteq N \) we have

\[
\langle e_T, z \rangle \overset{(7)}{=} \langle e_T \land g, f \rangle \overset{(7)}{=} \langle e_T, g \lhd f \rangle.
\]

Therefore \( z \) and \( g \lhd f \) are identical, as their inner products with all basis elements coincide.

Now assume that \( f \in \bigwedge^s V \) and \( g \in \bigwedge^t V \), and let \( S \subseteq N \) be arbitrary. By (7) we have

\[
\langle e_S, g \lhd f \rangle = \langle e_S \land g, f \rangle.
\]

Observe that \( e_S \land g \in \bigwedge^{|S|+t} \) while \( f \in \bigwedge^s V \) and these spaces are orthogonal unless \( |S| + t = s \). Hence, \( g \lhd f = 0 \) for \( t > s \) and \( g \lhd f \in \bigwedge^{s-t} V \) otherwise. \( \square \)

It is straightforward to check from the definition that the left interior product is bilinear:

- \( (f + g) \lhd h = (f \lhd h) + (g \lhd h) \),
- \( f \lhd (g + h) = (f \lhd g) + (f \lhd h) \),

and satisfies

\[
h \lhd (g \lhd f) = (h \land g) \lhd f.
\]  

With \( \text{sgn}(\cdot, \cdot) \) as defined in Sect. 3.1 we obtain the following statement.
Lemma 3.2 Let \((f_v)_{v \in N}\) be an orthonormal basis of \(V\). Then, for any \(S, T \subseteq N\) we have
\[
f_{T \setminus S} = \begin{cases} 
\text{sgn}(S \setminus T, T) f_{S \setminus T} & \text{if } T \subseteq S, \\
0 & \text{otherwise.}
\end{cases}
\]

Proof Put \(s := |S|\) and \(t := |T|\). If \(t > s\) then by Lemma 3.1 we have \(f_{T \setminus S} = 0\) and the conclusion follows. So we may assume that \(s \geq t\), and by the same lemma it follows that \(f_{T \setminus S} \in \wedge^{s-t} V\). Since the basis \((f_v)_{v \in N}\) is orthonormal, so is the basis \((f_L)_{L \in \binom{N}{s-t}}\) of \(\wedge^{s-t} V\), as observed in Sect. 3.1. Expressing \(f_{T \setminus S}\) in this basis and using (7), we obtain
\[
f_{T \setminus S} = \sum_{L \in \binom{N}{s-t}} \langle f_L, f_{T \setminus S} \rangle f_L = \sum_{L \in \binom{N}{s-t}} \langle f_L \wedge f_T, f_S \rangle f_L.
\]
Due to (3) and orthonormality of \((f_v)_{v \in N}\) we have \(\langle f_L \wedge f_T, f_S \rangle = 0\) unless \(T \subseteq S\) and \(L = S \setminus T\). Therefore, using (3) again we get
\[
f_{T \setminus S} = \begin{cases} 
\langle f_{S \setminus T} \wedge f_T, f_S \rangle f_{S \setminus T} = \text{sgn}(S \setminus T, T) f_{S \setminus T} & \text{if } T \subseteq S, \\
0 & \text{if } T \nsubseteq S.
\end{cases}
\]

Lemma 3.3 Let \((f_v)_{v \in N}\) be a generic orthonormal basis of \(V\) with respect to \((e_v)_{v \in N}\). For a pair of sets \(T, R \subseteq N\) of sizes \(t\) and \(r\), respectively, such that \(r \geq t\) we have
\[
f_{T \setminus R} = \sum_{S \in \binom{N \setminus T}{r-t}} \lambda_S f_S,
\]
where all the coefficients \(\lambda_S\) are non-zero.

Proof By Lemma 3.1 we have that \(f_{T \setminus R} \in \wedge^{r-t} V\). Since \((f_S)_{S \in \binom{N}{r-t}}\) is an orthonormal basis of \(\wedge^{r-t} V\), we can write
\[
f_{T \setminus R} = \sum_{S \in \binom{N}{r-t}} \langle f_S, f_{T \setminus R} \rangle f_S.
\]
Applying (7) and (3) gives
\[
\langle f_S, f_{T \setminus R} \rangle = \langle f_S \wedge f_T, e_R \rangle
\]
\[
= \begin{cases} 
\pm \langle f_{S \setminus T}, e_R \rangle & \text{if } S \cap T = \emptyset, \text{ equivalently if } S \in \binom{N \setminus T}{r-t}, \\
0 & \text{otherwise.}
\end{cases}
\]
Setting \(\lambda_S = \langle f_S \land f_T, e_R \rangle\) for \(S \in \binom{N \setminus T_{r-t}}{r-t}\), we thus obtain

\[ f_T \land e_R = \sum_{S \in \binom{N \setminus T_{r-t}}{r-t}} \lambda_S f_S, \]

as claimed. In addition, since we assumed that \((f_v)_{v \in N}\) is generic with respect to \((e_v)_{v \in N}\), we have \(\lambda_S = \pm \langle f_S \cup f_T, e_R \rangle \neq 0\) by (6) for all \(S \in \binom{N \setminus T_{r-t}}{r-t}\). \(\square\)

### 3.3 Colorful Exterior Algebra

As we are interested in multipartite hypergraphs it is natural to assume in addition that the set \(N\) is partitioned as a disjoint union \(N = N_1 \sqcup N_2 \sqcup \ldots \sqcup N_d\); consistently with the introduction \(n_i := |N_i|\). Here each \(N_i\) is ordered by a total order \(<_i\). We extend these orders to the whole \(N\) as follows, for \(x \in N_i\) and \(y \in N_j\), we say that \(x < y\) if \(i < j\) or if \(i = j\) and \(x <_i y\).

Given the standard basis \((e_v)_{v \in N}\) of \(V\) we say that a basis \((f_v)_{v \in N}\) is colorful with respect to this partition if \((f_v)_{v \in N_i}\) generates the same subspace of \(V = \mathbb{R}^N\) as \((e_v)_{v \in N_i}\) for every \(i \in [d]\); we denote this subspace \(V_i\). Put differently, the transition matrix \(A\) from \((e_v)_{v \in N_i}\) to \((f_v)_{v \in N_i}\) is a block-diagonal matrix with blocks \(N_i \times N_i\) for \(i \in [d]\).

We also say that \((f_v)_{v \in N}\) is colorful generic (with respect to this partition) if the basis change from \((e_v)_{v \in N_i}\) to \((f_v)_{v \in N_i}\) is generic for every \(i \in [d]\). It is possible to choose a basis which is simultaneously colorful generic with respect to a given partition and orthonormal by choosing each change of basis from \((e_v)_{v \in N_i}\) to \((f_v)_{v \in N_i}\) generic and orthonormal.

By \(\bigwedge V_i\) we denote the subalgebra of \(\bigwedge V\) generated by \(e_{S}\) for \(S \subseteq N_i\) and by \(\bigwedge^k V_i\) the subspace of \(\bigwedge V_i\) with basis \((e_{S})_{S \in \binom{N_i}{k}}\); that is, \(\bigwedge^k V_i = \bigwedge^k V \cap \bigwedge V_i\).

We claim that the left interior product behaves nicely with respect to a colorful partition. To see this, we first need an auxiliary lemma about signs.

**Lemma 3.4** Let \(U\) and \(T\) be disjoint subsets of \(N\) and for all \(i \in [d]\) let \(U_i := U \cap N_i\), \(T_i := T \cap N_i\), \(u_i := |U_i|\) and \(t_i := |T_i|\). Then

\[ \text{sgn}(U, T) = (-1)^c \text{sgn}(U_1, T_1) \cdots \text{sgn}(U_d, T_d), \]

where \(c\) depends only on \(u_1, \ldots, u_d\) and \(t_1, \ldots, t_d\).

**Proof** The value \(\text{sgn}(U, T)\) is \(-1\) to the number of transpositions in the permutation \(\pi\) of \(U \cup T\) where we first place the elements of \(U\) (in our given order on \(N\)) and then the elements of \(T\) (in the same order). Considering that for \(i < j\), \(U_i\) precedes \(U_j\) and \(T_i\) precedes \(T_j\), the order of the blocks \(U_1, \ldots, U_d, T_1, \ldots, T_d\) in \(\pi\) is

\((U_1, \ldots, U_d, T_1, \ldots, T_d)\)
After \( c \) transpositions where \( c \) depends only on \( u_1, \ldots, u_d, t_1, \ldots, t_d \), we get a permutation \( \pi' \) with the following order of blocks

\[
(U_1, T_1, U_2, T_2, \ldots, U_d, T_d).
\]

By the above, the sign of \( \pi' \) equals \((-1)^c \operatorname{sgn}(U, T)\). On the other hand, as \( T_i \) precedes \( U_j \) for \( i < j \) in our order on \( N \), the sign of \( \pi' \) is also equal to the product \( \operatorname{sgn}(U_1, T_1) \cdots \operatorname{sgn}(U_d, T_d) \). Equating these two expressions gives the desired identity.

In the following proposition, the \( f_i \) are not necessarily coming from a colorful generic basis. However, we intend to apply it in this setting. With a slight abuse of notation, we use \( \wedge \) both for the exterior algebra as well as for the wedge product of multiple elements. (This can be easily distinguished from the context.)

**Proposition 3.5** Suppose that \( s_1, \ldots, s_d \) and \( t_1, \ldots, t_d \) are nonnegative integers with \( t_i \leq s_i \leq n_i \) for every \( i \in [d] \). Suppose further that \( f_i \in \bigwedge^{t_i} V_i \) and \( h_i \in \bigwedge^{s_i} V_i \) for all \( i \in [d] \). Then

\[
\left( \bigwedge_{i=1}^d f_i \right) \wedge \left( \bigwedge_{i=1}^d h_i \right) = \pm \bigwedge_{i=1}^d (f_i \wedge h_i).
\]

**Proof** We will show that

\[
\left( \bigwedge_{i=1}^d f_i \right) \wedge \left( \bigwedge_{i=1}^d h_i \right) = (-1)^c \bigwedge_{i=1}^d (f_i \wedge h_i)
\]

where \( c \) comes from Lemma 3.4; in particular, it depends only on \( t_1, \ldots, t_d \) and \( s_1, \ldots, s_d \).

By bilinearity of \( \wedge \) and \( \wedge \) it is sufficient to prove (9) in the case when the \( f_i \) and the \( h_i \) are basis elements of \( \bigwedge^{t_i} V_i \) and \( \bigwedge^{s_i} V_i \) respectively. So, assume for each \( i \in [d] \) that \( f_i = e_{T_i} \) and \( h_i = e_{S_i} \) where \( T_i \in \binom{N_i}{t_i} \) and \( S_i \in \binom{N_i}{s_i} \), and let \( T := T_1 \cup \cdots \cup T_d \) and \( S := S_1 \cup \cdots \cup S_d \). Then \( \bigwedge_{i=1}^d f_i = e_T \) and \( \bigwedge_{i=1}^d h_i = e_S \) by the definition of the exterior product \( \wedge \). If \( T_i \not\subseteq S_i \) for some \( i \in [d] \), then \( T \not\subseteq S \) and both sides of (9) vanish by Lemma 3.2. Therefore, it remains to check the case that \( T_i \subseteq S_i \) for every \( i \in [d] \). Here by Lemma 3.4 (with \( U = S \setminus T \)) and Lemma 3.2 we get

\[
e_{T \cup e_S} = \operatorname{sgn}(S \setminus T, T)e_{S \setminus T}
= (-1)^c \operatorname{sgn}(S_1 \setminus T_1, T_1) \cdots \operatorname{sgn}(S_d \setminus T_d, T_d)e_{S_1 \setminus T_1} \wedge \cdots \wedge e_{S_d \setminus T_d}
= (-1)^c (e_{T_1 \cup e_S} \wedge \cdots \wedge (e_{T_d \cup e_S}),
\]

as required. \[\square\]
4 Theorem 1.1: The Lower Bound

In this section we prove the lower bound in Theorem 1.1. Our proof follows a strategy similar to [2] and [14]. Viewing the edges of \( K^q_n \) as elements of the exterior algebra of \( \mathbb{R}^N \), we will define a linear mapping closely related to the weak saturation process and lower-bound \( w(K^q_n, K^q_r) \) by the rank of the corresponding matrix.

As outlined in Sect. 3, let \( V \) be an \( n \)-dimensional real vector space with a basis \((e_v)_{v \in N}\), equipped with a standard inner product \( \langle \cdot, \cdot \rangle \) with respect to this basis, that is, \((e_v)_{v \in N}\) is orthonormal. Using the exterior product notation of Sect. 3, define

\[
\text{span} \ K^q_n := \text{span} \{e_T : T \in E(K^q_n)\} \subseteq \bigwedge^q V.
\]

For an element \( m \in \bigwedge^k V \) the support of \( m \) is the set

\[
\text{supp}(m) = \left\{ S \in \binom{N}{k} : \langle e_S, m \rangle \neq 0 \right\}.
\]

The following lemma, which converts the problem at hand into a constructive question in linear algebra, is analogous to Lemma 3 in [2].

**Lemma 4.1** Let \( Y \) be a real vector space and \( \Gamma : \text{span} \ K^q_n \rightarrow Y \) a linear map such that for every subset \( R \subseteq N \) with \( |R \cap N_i| = r_i \) for all \( i \in [d] \) there exists an element \( m \in \ker \Gamma \) with \( \text{supp}(m) = E(K^q_n[R]) \). Then

\[
w(K^q_n, K^q_r) \geq \text{rank} \ \Gamma.
\]

**Proof** Suppose the \( q \)-graph \( G_0 \) is weakly \( K^q_r \)-saturated in \( K^q_n \) and \( |E(G_0)| = w(K^q_n, K^q_r) \). Denote by \( \{L_1, \ldots, L_k\} \) a corresponding saturating sequence and by \( H_i \) a new copy of \( K^q_r \) that appears in \( G_i = G_0 \cup \{L_1, \ldots, L_i\} \) with \( L_i \in E(H_i) \). Let \( Y_i = \text{span} \{\Gamma(e_T) : T \in E(G_i)\} \), and note that \( Y_k = \Gamma(\text{span} \ K^q_n) \). By assumption, for each \( i = 1, \ldots, k \) there exist non-zero coefficients \( \{c_T : T \in E(H_i)\} \) such that

\[
\sum_{T \in E(H_i)} c_T \Gamma(e_T) = 0.\]

Therefore,

\[
\Gamma(e_{L_i}) = -\frac{1}{c_{L_i}} \sum_{T \in E(H_i) \setminus L_i} c_T \Gamma(e_T) \in Y_{i-1}.
\]

We conclude that \( Y_i = Y_{i-1} \). By repeating this procedure we obtain

\[
w(K^q_n, K^q_r) = |E(G_0)| \geq \dim Y_0 = \dim Y_k = \text{rank} \ \Gamma.
\]

□

---

2 Put equivalently in the language of [2], we map each edge of \( K^q_n \) to vector in a certain vector space \( \tilde{W} \), so that for each copy of \( K^q_r \) in \( K^q_n \) the underlying vectors are linearly dependent with all coefficients involved being non-zero. This implies \( w(K^q_n, K^q_r) \geq \dim \tilde{W} \).
Our goal now is to define a linear map $\Gamma$ as in Lemma 4.1. For this purpose let us fix an orthonormal colorful generic basis $(f_i)_{i \in N}$ of $V$ with respect to the partition of $N$, as described in Sect. 3.3. Next, for each $i \in [d]$ choose a set $J_i \subseteq N_i$ with $|J_i| = r_i - 1$ and a vertex $w_i \in N_i \setminus J_i$. Put $J := \bigcup_{i \in [d]} J_i$ and $W := \{w_i : i \in [d]\}$. Finally, set $s := d - q$ and

$$g := \sum_{T \in \binom{W}{s}} f_T. \quad (10)$$

We can now state the following auxiliary lemma.

**Lemma 4.2** Let $z$ be an integer with $d \geq z \geq s$ and let $Z \in \binom{N}{z}$. Then

(i) $g \langle f_Z \rangle = 0$ if $|Z \cap W| < s$.

(ii) If $z = s$, then $\langle g, f_Z \rangle = \begin{cases} \pm 1 & \text{if } Z \subseteq W, \\ 0 & \text{if } Z \notin W. \end{cases}$

**Proof** By (10), bilinearity of $\langle \cdot, \cdot \rangle$, and Lemma 3.2 we get

$$g \langle f_Z \rangle = \sum_{W' \in \binom{W}{s}} f_{W' \cup f} = \sum_{W' \in \binom{W\cap Z}{s}} \pm f_{Z \setminus W'}. \quad (11)$$

The last expression is 0 if $|Z \cap W| < s$; this shows (i).

Now, assume that $z = s$. Then

$$\langle g, f_Z \rangle = \langle f_{\emptyset} \cup g, f_Z \rangle = \langle f_{\emptyset}, g \langle f_Z \rangle \rangle = \sum_{W' \in \binom{W\cap Z}{s}} \pm \langle f_{\emptyset}, f_{Z \setminus W'} \rangle. \quad (12)$$

If $Z \notin W$, then $|Z \cap W| < z = s$, so $g \langle f_Z \rangle = 0$ from (i), and thus (12) evaluates to 0. On the other hand, if $Z \subseteq W$, then $\binom{W \cap Z}{s} = \{Z\}$. It follows that

$$\langle g, f_Z \rangle = \pm \langle f_{\emptyset}, f_{\emptyset} \rangle = \pm 1,$$

yielding (ii). \qed

We define the subspace

$$U := \text{span}\{g \langle f_T \rangle : T \in E(K_n^{d \setminus N \setminus J}), |T \cap W| \geq s\}, \quad (13)$$

and observe first that $U \subseteq \text{span } K_n^d$. Indeed, for each $T$ in (13) and $W' \in \binom{W}{s}$, we have by Lemma 3.2 that $f_{W' \cup f} = 0$ if $W' \notin T$ and $f_{W' \cup f} = \pm f_{T \setminus W'}$ if $W' \subseteq T$. In the latter case note that $T \setminus W' \in E(K_n^d)$, and the claim follows by bilinearity of $\langle \cdot, \cdot \rangle$.

Let $Y$ be the orthogonal complement of $U$ in $\text{span } K_n^d$ and let $\Gamma : \text{span } K_n^d \to \text{span } K_n^d$ be the orthogonal projection on $Y$. Our main technical lemma in this paper states that $\Gamma$ satisfies the assumptions of Lemma 4.1.
Lemma 4.3 Suppose that $R \subseteq N$ satisfies $|R \cap N_i| = r_i$ for every $i \in [d]$. Then, there exists $m \in \ker \Gamma$ such that $\text{supp}(m) = E(K^d_n[R])$.

Deferring the proof of Lemma 4.3, let us first compute $\text{rank} \Gamma$ and conclude the proof of Theorem 1.1 assuming Lemma 4.3.

Notice that the sets $T \in K^d_n[N \setminus J]$ with $|T \cap W| \geq s$ are in bijective correspondence with the sets $T \setminus W \in K^d_n[N \setminus (J \cup W)]$ with $p \leq q$. Using this bijection,

$$\dim U \overset{(13)}{\leq} |\{T \in K^d_n[N \setminus J]: |T \cap W| \geq s\}| = \sum_{I \subseteq [d]} \prod_{i \in I} (n_i - r_i).$$

Consequently,

$$\text{rank} \Gamma = \dim(\text{span} K^d_n) - \dim U \geq \sum_{I \in \binom{[d]}{q}} \prod_{i \in I} n_i - \sum_{I \subseteq [d]} \sum_{|I| \leq q} \prod_{i \in I} (n_i - r_i). \quad (14)$$

**Proof of Theorem 1.1** On the one hand, by Lemma 4.3 the map $\Gamma$ satisfies the assumptions of Lemma 4.1. Therefore,

$$w(K^d_n, K^d_R) \geq \text{rank} \Gamma \overset{(14)}{\geq} \sum_{I \in \binom{[d]}{q}} \prod_{i \in I} n_i - \sum_{I \subseteq [d]} \sum_{|I| \leq q} \prod_{i \in I} (n_i - r_i).$$

On the other hand, Lemma 2.1 gives the same upper bound.

**Proof of Lemma 4.3** We claim that

$$m = (g \land f_J)\downarrow e_R$$

is the desired element.\(^3\) Let $R_i := R \cap N_i$ for each $i \in [d]$. First, we verify that $m \in \ker \Gamma = U$. By Proposition 3.5 we have

$$f_J \downarrow e_R = \pm (f_{J_1} \downarrow e_{R_1}) \land \cdots \land (f_{J_d} \downarrow e_{R_d}).$$

By Lemma 3.3 we can write each of these terms as

$$f_{J_i} \downarrow e_{R_i} = \sum_{v \in N_i \setminus J_i} \lambda_v f_v \quad \text{with all } \lambda_v \neq 0. \quad (15)$$

\(^3\) Let us briefly sketch the topological idea hidden behind this choice: As it can be easily deduced from the computations below, $m$ can be also expressed as $\pm g_\cdot((f_{J_1} \downarrow e_{R_1}) \land \cdots \land (f_{J_d} \downarrow e_{R_d}))$. In the terminology of simplicial complexes interpreting loosely (i) $e_R$ as a full simplex on the vertex set $R$, (ii) $\land$ as a join of simplicial complexes and (iii) $\downarrow$ as an operator taking the skeleton of appropriate dimension, we gradually get the following: $f_{J_i} \downarrow e_{R_i}$ corresponds to the 0-skeleton of the simplex on $R_i$, that is, the vertices of $R_i$. Then $(f_{J_1} \downarrow e_{R_1}) \land \cdots \land (f_{J_d} \downarrow e_{R_d})$ corresponds to the join of the sets $R_i$, that is, the complete $d$-partite complex on $R_1, \ldots, R_d$. Finally, applying $g_\cdot$ to this element takes the skeleton again reducing the dimension so that the corresponding hypergraph is the required $K^d_n[R]$. \(\square\)
Combining this with (4) gives
\[ f_{J \cup e_R} = \sum_{Z \in E(K^d_n[N\setminus J])} \pm \left( \prod_{v \in Z} \lambda_v \right) f_Z. \]  
(16)

Therefore, we get
\[ m = (g \land f_J) \cup e_R = g \cup (f_{J \cup e_R} = \sum_{Z \in E(K^d_n[N\setminus J])} \left( \prod_{v \in Z} \lambda_v \right) g \cup f_Z, \]
where the last equality follows by Lemma 4.2(ii) with \( z = d \). Thus \( m \in U \) as wanted.

Next, we show that \( \text{supp}(m) = E(K^d_n[R]) \). As we just have shown, \( m \in U \subseteq \text{span} K^d_n \), i.e. \( \text{supp}(m) \subseteq E(K^d_n) \). Now, for \( T \in E(K^d_n) \) we have
\[ \langle e_T, m \rangle = \pm \langle (g \land f_J), e_R \rangle \]  
(7)
and by Lemma 3.2 we have
\[ \langle e_T, m \rangle = \pm \langle g \land f_J, e_R \rangle \]  
(17)
If \( T \notin E(K^d_n[R]) \), then \( T \not\subseteq R \) and by Lemma 3.2 we have \( (g \land f_J) \subseteq R \), and consequently \( \langle e_T, m \rangle = 0 \). Hence, \( \langle e_T, m \rangle = 0 \).

Now assume that \( T \in E(K^d_n[R]) \), i.e., \( T \subseteq R \). By (17) and Lemma 3.2 we have
\[ \langle e_T, m \rangle = \pm \langle (g \land f_J), e_R \rangle \]  
(18)
Let \( P := \{ i \in [d] : T \cap N_i \neq \emptyset \} \) and \( P' := [d] \setminus P \). Using this notation we can write
\[ e_{R \setminus T} = \pm \left( \bigwedge_{i \in P} e_{R_i \setminus \tau_i} \right) \land \left( \bigwedge_{i \in P'} e_{R_i} \right), \]
where for each \( i \in P \) the set \( \tau_i = T \cap N_i \) contains a single vertex. Applying Proposition 3.5, we deduce
\[ f_{J \cup e_{R \setminus T}} = \pm \left( \bigwedge_{i \in P} f_{J_i \cup e_{R_i \setminus \tau_i}} \right) \land \left( \bigwedge_{i \in P'} f_{J_i \cup e_{R_i}} \right). \]  
(19)
Since \( |J_i| = r_i - 1 = |R_i \setminus \tau_i| \), by Lemma 3.1 for every \( i \in P \) we have \( f_{J_i \cup e_{R_i \setminus \tau_i}} \in \bigwedge^0 V \). Thus
\[ f_{J_i \cup e_{R_i \setminus \tau_i}} = (e_{\emptyset}, f_{J_i \cup e_{R_i \setminus \tau_i}}) e_{\emptyset} = (e_{\emptyset} \land f_{J_i \cup e_{R_i \setminus \tau_i}}) e_{\emptyset} = (f_{J_i}, e_{R_i \setminus \tau_i}) e_{\emptyset}, \]
and notice that \( \langle f_{J_i}, e_{R_i \setminus \tau_i} \rangle \neq 0 \) because \( (f_v)_{v \in N_i} \) is generic with respect to \( (e_v)_{v \in N_i} \).

Plugging it into (19) yields

\[
\begin{align*}
  f_{J_i \cup e_{R_i \setminus T}} &= \pm \left( \bigwedge_{i \in P} \langle f_{J_i}, e_{R_i \setminus \tau_i} \rangle e_{\emptyset} \right) \land \left( \bigwedge_{i \in P'} f_{J_i \cup e_{R_i}} \right) \\
  &= \pm \left( \prod_{i \in P} \langle f_{J_i}, e_{R_i \setminus \tau_i} \rangle \right) \land f_{J_i \cup e_{R_i}}.
\end{align*}
\]

Turning to \( P' \), denote \( N' := \bigcup_{i \in P'} N_i \setminus J_i \). We have

\[
\bigwedge_{i \in P'} f_{J_i \cup e_{R_i}} \overset{(15)}{=} \bigwedge_{i \in P'} \left( \sum_{v \in N_i \setminus J_i} \lambda_v f_v \right) \overset{(4)}{=} \sum_{Z \in E(K^n_4[N'])} \pm \left( \prod_{v \in Z} \lambda_v \right) f_Z.
\]

Therefore,

\[
\langle g, \bigwedge_{i \in P'} f_{J_i \cup e_{R_i}} \rangle = \sum_{Z \in E(K^n_4[N'])} \pm \left( \prod_{v \in Z} \lambda_v \right) \langle g, f_Z \rangle = \pm \prod_{v \in W \cap N'} \lambda_v,
\]

where the second equality is due to Lemma 4.2(ii), using that there is exactly one \( Z \in E(K^n_4[N']) \) with \( Z \subseteq W \), namely \( Z = W \cap N' \). Putting it all together,

\[
\langle e_T, m \rangle \overset{(18)}{=} \pm \langle g, f_{J_i \cup e_{R_i \setminus T}} \rangle \overset{(20)}{=} \pm \left( \prod_{i \in P} \langle f_{J_i}, e_{R_i \setminus \tau_i} \rangle \right) \langle g, \bigwedge_{i \in P'} f_{J_i \cup e_{R_i}} \rangle \overset{(22)}{=} \pm \left( \prod_{i \in P} \langle f_{J_i}, e_{R_i \setminus \tau_i} \rangle \right) \prod_{v \in W \cap N'} \lambda_v \neq 0,
\]

and consequently \( T \in \text{supp}(m) \).

### 5 Weak Saturation in the Clique

In this section we prove Theorem 1.3. Let \( H \) be a \( q \)-graph where \( q \geq 2 \) without isolated vertices. We recall the notion of a link hypergraph of a vertex \( v \in V(H) \): it is the \((q - 1)\)-graph (possibly with isolated vertices) defined via

\[
L_H(v) := \{ e \setminus \{v\} : e \in E(H), v \in e \}.
\]

The co-degree of a set \( W \) of \( q - 1 \) vertices in \( H \) is

\[
d_H(W) := |\{ e \in E(H) : W \subset e \}|.
\]
Define the **minimum positive co-degree** of \( H \), in notation \( \delta^*(H) \), as

\[
\delta^*(H) := \min \left\{ d_H(W) : W \in \binom{V(H)}{q-1}, d_H(W) > 0 \right\}.
\]

Notice that \( \delta^*(H) \leq \delta^*(L_H(v)) \) for all \( v \in V(H) \), and equality holds for some \( v \).

**Lemma 5.1** \( \text{wsat}(n, H) \leq (\delta^*(H) - 1) \binom{n}{q-1} + O_H(n^{q-2}) \).

**Proof** We apply induction on \( q \). For \( q = 2 \) this is a well-known fact ([9], Theorem 4). Suppose now that \( q \geq 3 \) and the statement holds for all smaller values. Let \( H \) be a \( q \)-graph and let \( W = \{v_1, \ldots, v_{q-1}\} \) be a set satisfying \( d_H(W) = \delta^*(H) \). Let \( H_1 = L_H(v_1) \) be the link hypergraph of \( v_1 \), and observe that \( \delta^*(H_1) = \delta^*(H) \). A weakly \( H \)-saturated \( q \)-graph on \( [n] \) is obtained as follows. Take a minimum weakly \( H_1 \)-saturated \( (q-1) \)-graph on \( [n-1] \) and insert \( n \) into each edge; take a union of the resulting \( q \)-graph with a minimum weakly \( H \)-saturated \( q \)-graph on \( [n-1] \). We therefore obtain

\[
\text{wsat}(n, H) \leq \text{wsat}(n - 1, H) + \text{wsat}(n - 1, H_1).
\]

Iterating and applying the induction hypothesis,

\[
\text{wsat}(n, H) \leq \text{wsat}(\lfloor V(H) \rfloor, H) + \sum_{m=\lfloor V(H) \rfloor}^{n-1} \text{wsat}(m, H_1)
\]

\[
\leq (\delta^*(H_1) - 1) \sum_{m=q-2}^{n-1} \binom{m}{q-2} + O_H(n^{q-2})
\]

\[
= (\delta^*(H) - 1) \binom{n}{q-1} + O_H(n^{q-2}).
\]

\[ \square \]

The **tensor product** of two \( q \)-graphs \( G \) and \( J \), \( G \times J \) is defined having the vertex set \( V(G) \times V(J) \) and the edge set

\[
E(G \times J) = \left\{ ((v_1, w_1), \ldots, (v_q, w_q)) : \{v_1, \ldots, v_q\} \in E(G), \{w_1, \ldots, w_q\} \in E(J) \right\}.
\]

(Note that every pair of edges in the original graphs produces \( q! \) edges in the product.)

**Lemma 5.2** Let \( H = K_{d_1}^{k_1} \ldots \times K_{d_r}^{k_r} \) and let \( F_n^d \) be the copy of \( K^d(n; d) \) between the vertex sets \( [n] \times \{1\}, \ldots, [n] \times \{d\} \). Then there exists a \( d \)-graph \( E^d(n, H) \subseteq F_n^d \setminus (K_{[n]}^d \times K_{[d]}^d) \) of size \( O_H(n^{d-2}) \) such that

\[
G(n, H) := (K_{[n]}^d \times K_{[d]}^d) \cup E^d(n, H)
\]

is weakly \( H \)-saturated in \( F_n^d \).
Proof\; It suffices to prove the above statement when \( r_1 = \cdots = r_d =: r \), i.e. when \( H = K^d(r; d) \), as every edge creating a new copy of \( K^d(\max\{r_1, \ldots, r_d\}; d) \) creates in particular a new copy of \( K^d_{r_1, \ldots, r_d} \).

We apply induction on \( d \) and \( n \). For \( d = 2 \) and any \( n \geq |V(H)| \) the graph \( K_{[n]} \times K_{[2]} \) misses only a matching from \( F_n^2 \), making it already \( H \)-saturated in \( F_n^2 \), as can be easily checked. Moreover, for every fixed \( H \) we can assume the statement to hold for all \( n \) less than some large \( C(H) \).

For the induction step, fix \((n, d)\) and suppose that the statement holds for all \((n', d')\) with \( d' < d \) and all \((n'', d)\) with \( n'' < n \). It suffices to show that \( O_H(n^{d-3}) \) edges can be added to \( G(n-1, H) \) to satisfy the assertion; these edges will be as follows.

For each \( i \in [d] \) let the \((d-1)\)-graph \( E'_i \) be an isomorphic copy of \( E^{d-1}(n-1, K^{d-1}(r; d-1)) \) between the sets \([n-1] \times \{j\}\) for \( j \in [d] \setminus \{i\} \), such that \((K_{[n-1]}^{d-1} \times K_{[d]\setminus\{i\}}^{d-1}) \cup E'_i \) is weakly \( K^{d-1}(r; d-1)\)-saturated in the complete \((d-1)\)-partite \((d-1)\)-graph between the sets \([n-1] \times \{j\}\) for \( j \in [d] \setminus \{i\} \). Let

\[
E_i := \{ e \cup \{(n, i)\} : e \in E'_i \}.
\]

By the induction hypothesis \(|E_i| = |E'_i| = O_H(n^{d-3})\).

Similarly, for each \( \{i_1, i_2\} \in \binom{[d]}{2} \) apply Corollary 1.2 to obtain a \((d-2)\)-graph \( E'_{i_1, i_2} \) of size \( O_H(n^{d-3}) \) which is weakly \( K^{d-2}(r; d-2)\)-saturated in the copy of \( K^{d-2}(n-1; d-2) \) between the sets \([n-1] \times \{j\}\) for \( j \in [d] \setminus \{i_1, i_2\} \) for \( d = 3 \) take any \( r - 1 \) vertices in \([n-1] \times [d] \setminus \{i_1, i_2\}\). As above, insert \((n, i_1)\) and \((n, i_2)\) into each edge of \( E'_{i_1, i_2} \), let the resulting edge set be called \( E_{i_1, i_2} \).

Finally, take all edges of \( F_n^{d} \) containing at least three vertices with \( n \) as their first coordinate, and let \( E_0 \) be this edge set; clearly \(|E_0| = O_H(n^{d-3})\) as well. Put

\[
G(n, H) := G(n-1, H) \cup \bigcup_{i \in [d]} E_i \cup \bigcup_{\{i_1, i_2\} \in \binom{[d]}{2}} E_{i_1, i_2} \cup E_0,
\]

and

\[
E^d(n, H) := G(n, H) \setminus (K_{[n]}^d \times K_{[d]}^d).
\]

By the induction hypothesis and the bounds on the \(|E_i|\), the \(|E_{i_1, i_2}|\) and \(|E_0|\), we have \(|E^d(n, H)| = O_H(n^{d-2})\). To see that \( G(n, H) \) is weakly \( H \)-saturated, first note that by induction hypothesis \( G(n-1, H) \) is weakly \( H \)-saturated in \( F_{n-1}^{d} \), hence the \( d \)-graph \( G(n-1, H) \cup (K_{[n]}^d \times K_{[d]}^d) \subseteq G(n, H) \) is weakly \( H \)-saturated in \( J_0 := F_{n-1}^{d} \cup (K_{[n]}^d \times K_{[d]}^d) \). Furthermore, let

\[
K_1 := \{ e \in F_n^d : |e \cap ([n] \times [d])| = 1 \},
\]

and

\[
K_2 := \{ e \in F_n^d : |e \cap ([n] \times [d])| = 2 \}.
\]
Let $J_1 := J_0 \cup K_1$ and $J_2 := J_1 \cup K_2$. By construction, $J_0 \cup \bigcup_{i \in [d]} E_i$ is weakly $H$-saturated in $J_1$, $J_1 \cup \bigcup_{(i_1, i_2) \in \binom{\mathcal{I}}{2}} E_{i_1, i_2}$ is weakly $H$-saturated in $J_2$ and $J_2 \cup E_0 = F_n^d$. Thus, $G(n, H)$ is weakly $H$-saturated in $F_n^d$ as desired. This proves the induction step, and the statement of the lemma follows. \hfill \Box

**Proof of Theorem 1.3** For the first statement, suppose that $G \subseteq K_{n,n}$ is weakly $H$-saturated in $K_{n,n}$. Placing two $|V(H)|$-cliques on the parts of $G$ is easily seen to produce a weakly $H$-saturated graph in $K_{2n}$. Therefore,

$$\text{wsat}(2n, H) \leq \text{wsat}(K_{n,n}, H) + |V(H)|^2. \quad (23)$$

Conversely, suppose that $G = G_0$ is weakly $H$-saturated in $K_{[n]}$ via a saturating sequence $e_1 = \{i_1, j_1\}, \ldots, e_k = \{i_k, j_k\}$. For $1 \leq \ell \leq k$ let $G_\ell = G_0 \cup \{e_1, \ldots, e_\ell\}$, and let $H_\ell$ be a copy of $H$ in $G_\ell$ containing $e_\ell$.

Let $G_{\text{bip}} = G \times K_{[2]}$, i.e., $V(G_{\text{bip}}) = [n] \times \{1, 2\}$ and

$$E(G_{\text{bip}}) = \{(i, 1), (j, 2) \in E(G)\}.$$

We claim that $G_{\text{bip}}$ is weakly $H$-saturated in $K_{[n]} \times K_{[2]}$ via the $H$-saturating sequence $f_1, f_1', \ldots, f_k, f_k'$, where, for each $\ell \in [k]$, $f_\ell = \{(i_\ell, 1), (j_\ell, 2)\}$ and $f_\ell' = \{(i_\ell, 2), (j_\ell, 1)\}$, and that $G_{\ell-1} \cup \{f_\ell, f_\ell'\} = G_{\text{bip}}^\ell$ for all $\ell \in [k]$ (where $G_{\text{bip}}^\ell$ is defined analogously, i.e., $G_{\text{bip}}^\ell = G_\ell \times K_{[2]}$). Indeed, let $(A, B)$ be a bipartition of $V(H_\ell)$ with $i_\ell \in A$ and $j_\ell \in B$, and consider the analogous graph $H_{\ell}^b$ between $A \times \{1\}$ and $B \times \{2\}$, i.e., for every $(i, j) \in A \times B$ we have $\{(i, 1), (j, 2)\} \in E(H_b^\ell)$ if and only if $(i, j) \in E(H_\ell)$. Note that $f_\ell \in E(H_b^\ell)$ is the only edge of $H_b^\ell$ not already present in $G_{\ell-1}^\text{bip}$, therefore we can add it to the latter creating a new copy of $H$, namely $H_b^\ell$. Symmetrically, taking a graph $H_{\ell}^b$ between $A \times \{2\}$ and $B \times \{1\}$ allows to add $f_\ell'$. Since $G_\ell = G_{\ell-1} \cup e_\ell$, we have $G_{\ell-1}^\text{bip} \cup \{f_\ell, f_\ell'\} = G_{\ell}^\text{bip}$. Finally, note that $G_{\text{bip}} \cup \{f_1, \ldots, f_k\} = G_{\text{bip}}^k = K_{[n]}_{\text{bip}}^\text{bip}$.

Note that $K_{[n]}_{\text{bip}}^\text{bip}$ is isomorphic to $K_{n,n}$ minus a perfect matching, and it is a straightforward check that this graph is $H$-saturated in $K_{n,n}$ (we can assume that $|V(H)| \leq n$). We have thus shown

$$\text{wsat}(K_{n,n}, H) \leq 2 \text{wsat}(n, H). \quad (24)$$

Combining (23) and (24) gives

$$\frac{\text{wsat}(2n, H)}{2n} - o(1) \leq \frac{\text{wsat}(K_{n,n}, H)}{2n} \leq \frac{\text{wsat}(n, H)}{n},$$

and taking the limit, (1) follows readily.

For the second statement, denote $H = K_{r_1, \ldots, r_d}^d$ where $1 \leq r_1 \leq \cdots \leq r_d$. Observe that the upper bound in (2) holds by Lemma 5.1, as $\delta^*(H) = r_1$. To prove the lower
bound, suppose $G$ is weakly $H$-saturated in $K^d_{[n]}$, and that $|E(G)| = \text{wsat}(n, H)$. Let $G^{\text{mult}} = G \times K^d_{[d]}$, that is, $V(G^{\text{mult}}) = [n] \times [d]$ and

$$E(G^{\text{mult}}) = \{(i_1, 1), \ldots, (i_d, d) : \{i_1, \ldots, i_d\} \in E(G)\}.$$ 

Essentially the same argument as for $G^{\text{bip}}$ before shows that $G^{\text{mult}}$ is weakly $H$-saturated in $K^d_{[n]} \times K^d_{[d]}$. By Lemma 5.2 adding further $O_H(n^{d-2})$ edges creates a weakly $H$-saturated $d$-graph in $K^d_{(n; d)}$. Hence,

$$\text{wsat}(K^d_{(n; d)}, H) \leq |E(G^{\text{mult}})| + O(n^{d-2}) = d! \text{wsat}(n, H) + O(n^{d-2}). \quad (25)$$

On the other hand, Moshkovitz and Shapira [18] proved that $\text{wsat}(K^d_{(n; d)}, H) = d(r_1 - 1)n^{d-1} + O(n^{d-2})$. Combining this with (25) yields the lower bound in (2).

Acknowledgements We thank the anonymous referees for their careful reading of our article and making many helpful comments.

Funding Open access publishing supported by the National Technical Library in Prague.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

References

1. Alon, N.: An extremal problem for sets with applications to graph theory. J. Comb. Theory A 40(1), 82–89 (1985)
2. Balogh, J., Bollobás, B., Morris, R., Riordan, O.: Linear algebra and bootstrap percolation. J. Comb. Theory A 119(6), 1328–1335 (2012)
3. Bulavka, D., Goodarzi, A., Tancer, M.: Optimal bounds for the colorful fractional Helly theorem. In: Buchin, K., Colin de Verdière, É. (eds.) In: 37th International Symposium on Computational Geometry (SoCG 2021), volume 189 of Leibniz International Proceedings in Informatics (LIPIcs), pp. 19:1–19:14, Dagstuhl, Germany (2021). Schloss Dagstuhl – Leibniz-Zentrum für Informatik
4. Bollobás, B.: Weakly $k$-saturated graphs. In: Beiträge zur Graphentheorie (Kolloquium. Manebach, 1967), pp. 25–31. Teubner, Leipzig (1968)
5. Balogh, J., Pete, G.: Random disease on the square grid. In: Proceedings of the Eighth International Conference “Random Structures and Algorithms” (Poznan, 1997), vol. 13, pp. 409–422 (1998)
6. Borowiecki, M., Sidorowicz, E.: Weakly $P$-saturated graphs. In: Conference on Graph Theory (Elgersburg, 2000), vol. 22, pp. 17–29 (2002)
7. Erdős, P., Füredi, Z., Tuza, Z.: Saturated $r$-uniform hypergraphs. Discret. Math. 98(2), 95–104 (1991)
8. Faudree, R.J., Gould, R.J.: Weak saturation numbers for multiple copies. Discret. Math. 336, 1–6 (2014)
9. Faudree, R.J., Gould, R.J., Jacobson, M.S.: Weak saturation numbers for sparse graphs. Discuss. Math. Graph Theory 33(4), 677–693 (2013)
10. Frankl, P.: An extremal problem for two families of sets. Eur. J. Comb. 3(2), 125–127 (1982)
11. Gantmacher, F.R.: The Theory of Matrices, vol. 1. AMS Chelsea Publishing, Providence, RI (1998). Translated from the Russian by K. A. Hirsch, Reprint of the 1959 translation
12. Kalai, G.: Intersection patterns of convex sets. Israel J. Math. 48(2–3), 161–174 (1984)
13. Kalai, G.: Weakly saturated graphs are rigid. In: Convexity and Graph Theory (Jerusalem, 1981), vol. 87 of North-Holland Math. Stud., pp. 189–190. North-Holland, Amsterdam (1984)
14. Kalai, G.: Hyperconnectivity of graphs. Graphs Comb. 1(1), 65–79 (1985)
15. Kronenberg, G., Martins, T., Morrison, N.: Weak saturation numbers of complete bipartite graphs in the clique. J. Comb. Theory A 178, 105357 (2021)
16. Lovász, L.: Flats in matroids and geometric graphs. In: Combinatorial Surveys (Proc. Sixth British Combinatorial Conf., Royal Holloway Coll., Egham, 1977), pp. 45–86 (1977)
17. Morrison, N., Noel, J.A.: Extremal bounds for bootstrap percolation in the hypercube. J. Comb. Theory A 156, 61–84 (2018)
18. Moshkovitz, G., Shapira, A.: Exact bounds for some hypergraph saturation problems. J. Comb. Theory B 111, 242–248 (2015)
19. Pikhurko, O.: Uniform families and count matroids. Graphs Comb. 17(4), 729–740 (2001)
20. Pikhurko, O.: Weakly saturated hypergraphs and exterior algebra. Comb. Probab. Comput. 10(5), 435–451 (2001)
21. Rosén, A.: Geometric Multivector Analysis: From Grassmann to Dirac. Basler Lehrbücher. Birkhäuser/Springer, Cham, Birkhäuser Advanced Texts (2019)
22. Semanišin, G.: On some variations of extremal graph problems. Discuss. Math. Graph Theory 17(1), 67–76 (1997)
23. Sidorowicz, E.: Size of weakly saturated graphs. Discret. Math. 307(11–12), 1486–1492 (2007)
24. Tuza, Z.: A generalization of saturated graphs for finite languages. In: Proceedings of the 4th International Meeting of Young Computer Scientists, IMYCS ’86 (Smolenice Castle, 1986), vol. 185, pp. 287–293 (1986)
25. Tuza, Z.: Extremal problems on saturated graphs and hypergraphs. In: Eleventh British Combinatorial Conference (London, 1987), vol. 25, pp. 105–113 (1988)
26. Tuza, Z.: Asymptotic growth of sparse saturated structures is locally determined. In: Topological, Algebraical and Combinatorial Structures. Frolík’s memorial volume, vol. 108, pp. 397–402 (1992)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.