Topological censorship and chronology protection

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Over the past two decades, substantial efforts have been made to understand the way in which physics enforces the ordinary topology and causal structure that we observe, from subnuclear to cosmological scales. We review the status of topological censorship and the topology of event horizons; chronology protection in classical and semiclassical gravity; and related progress in establishing quantum energy inequalities.

This article is dedicated to Rafael Sorkin, whose friendship and tutoring from third grade onward is responsible for one of us (JF) having spent his adult life in physics and whose work has inspired both of us.

I. INTRODUCTION

In addition to the gravitational waves and black holes that reside in our universe, vacuum solutions to the classical Einstein equation generically exhibit white holes and structures with noneuclidean topology, and there is a generically large space of time-nonorientable solutions. There are also vacuum solutions (and positive energy solutions) with closed timelike curves, though how generic these solutions are remains an open question. The absence of white holes is a central mystery, tied to the thermodynamic arrow of time. The absence of macroscopic topological structures and of macroscopic closed timelike curves (CTCs) is a similarly central feature of our experience. Where have they gone? Why is the topology and the causal structure of spacetime “ordinary” on a macroscopic scale, when what we call ordinary is extraordinary in the space of solutions?

Two entirely different answers are consistent with our knowledge. The first is simply that the classical theory has a much broader set of solutions than the correct theory of quantum gravity. It is not implausible that causal structure enters in a fundamental way in quantum gravity and that classical spacetimes with closed timelike curves approximate no quantum states of the spacetime geometry. Less likely to us is the analogous explanation for trivial spacetime topology, that the correct theory of quantum gravity allows only euclidean topology and forbids topology change.

A second possible answer is provided by topological censorship and chronology protection: One supposes that quantum gravity allows microscopic (near Planck-size or near string-size) structures that have nontrivial topology and/or violate causality; and one shows that classical general relativity and the character of macroscopic matter (described by classical or semiclassical fields) forbid exotic structures that are macroscopic in their spatial and temporal size.

This brief review emphasizes a few related areas of recent work and is not intended to be comprehensive in any sense. A monograph by Visser \cite{1} reviews much of the work prior to 1995 and supplies a comprehensive bibliography; informal reviews are given by Thorne \cite{2} and Gott \cite{3} (whose bibliography has brief, useful descriptions of each reference). For a recent review of work on energy inequalities, see, for example, Roman \cite{9}, and an earlier popular article by Ford and Roman \cite{10}. Earlier reviews by one of the present authors are Refs. \cite{11, 12}. For a more detailed (and more technical) review of the Cauchy problem on spacetimes with CTCs and on Lorentzian universes-from-nothing, see Ref. \cite{13}. Because most of chronology protection is concerned with isolated regions of CTCs, and because of space limitations, the extensive work on Gott spacetimes is not covered here and no attempt is made to provide a comprehensive bibliography. Interested readers should balance Gott’s view \cite{3} with articles constraining CTC formation in 3-dimensional spacetimes, beginning for example, with work by Deser, Jackiw and ’t Hooft \cite{4}, Cutler \cite{5}, Carroll, Fahri and Guth \cite{7} (and references therein), and Tiglio \cite{8}.

II. TOPOLOGICAL CENSORSHIP

A. Expectation of nontrivial topology

Beyond the wormholes that science fiction has made a part of popular culture lie an infinite variety of topological structures in three dimensions, a countably infinite set of prime 3-manifolds. Witt \cite{14} showed that all 3-manifolds (prime and composite) occur as spacelike hypersurfaces of vacuum spacetimes – solutions to the vacuum Einstein equations. This is not enough to show that they arise as isolated structures: A flat universe with the topology of a 3-torus satisfies the field equations, but its existence does not imply that one can find a vacuum geometry for which the universe looks everywhere like a 3-sphere except in an isolated region. Isolated systems are ordinarily modeled as asymptotically flat spacetimes; and a recent result by Isenberg...
et al. [15] shows that all 3-manifolds do in fact occur as isolated structures, as spacelike hypersurfaces of asymptotically flat, vacuum spacetimes.

Whether topological structures can arise from a spacetime that is initially topologically trivial is a more difficult question. Given any two 3-manifolds, $S_1$ and $S_2$, one can always find a spacetime (a 4-manifold with Lorentzian metric) that joins them and for which they are each spacelike: There is a spacetime whose spacelike boundary is the disjoint union of $S_1$ and $S_2$ [16,17]. (A strengthened version of this and a review of earlier work on classical topology change is given by Borde [18]). But one pays a price for topology change. A theorem due to Geroch [19] shows that a spacetime whose boundary comprises two disjoint spacelike 3-manifolds always has closed timelike curves. (See [18] for a version appropriate to asymptotically flat spacetimes.)

Topology-changing spacetimes must also have negative energy. Much of the recent work involving restrictions on spacetime topology and on causal structure relies on the null energy condition (NEC): An energy-momentum tensor $T_{\alpha\beta}$ satisfies the null energy condition if, for any null vector $k^\alpha$, $T_{\alpha\beta} k^\alpha k^\beta \geq 0$. In particular, Tipler shows that topology-changing spacetimes violate the null energy condition [20].

Theorems involving the null energy condition use it to infer increasing convergence of light rays. Their proofs require only that the average value of $T_{\alpha\beta} k^\alpha k^\beta$ along null geodesics $\gamma$ be nonnegative. This weaker requirement, the averaged null energy condition (ANEC), has the form

$$\int_\gamma d\lambda T_{\alpha\beta} k^\alpha k^\beta > 0,$$

with $\lambda$ an affine parameter along the null geodesic and $k^\alpha$ the corresponding null tangent vector.

Because classical matter satisfies the null energy condition, it seems unlikely that one can create topological structures with a time evolution that is nearly classical. In path-integral approaches to quantum gravity, however, the amplitude for a transition from one 3-geometry to another is ordinarily written as a sum over all interpolating 4-geometries. Whether one works in a Euclidean or Lorentzian framework, topology change is thus permitted, if only at the Planck scale. One can then ask whether small-scale topological structures can grow to macroscopic size and persist for macroscopically long times.

The meaning of topological censorship is that they cannot: Isolated topological structures with positive energy collapse, and they do so quickly enough that light cannot traverse them.

### B. Gannon’s Singularity Theorem

The first result of this kind was due to Dennis Gannon [21], a similar result obtained independently by Lee [22].

**Theorem 1** Let $M, g$ be an asymptotically flat spacetime, obeying the null energy condition,

$$T_{\alpha\beta} k^\alpha k^\beta \geq 0.$$  

*If* $M, g$ *has a nonsimply connected Cauchy surface, then it is geodesically incomplete.*

Given the recently proven Poincaré conjecture [23,24,25] Gannon’s theorem implies that if the topology of the Cauchy surface $S$ is not trivial (i.e., if adding a point at infinity to $S$ yields any closed 3-manifold other than $S^3$), then the spacetime is geodesically incomplete.

Requiring that the system be isolated is essential to the theorem. Topological structures large enough to expand with the Hubble expansion are not ruled out, and the topology of the universe is unrestricted. Each of the countably many hyperbolic, spherical, and flat 3-manifolds is consistent with the homogeneity and isotropy of the observed universe. In fact, only with the high angular resolution of the recent microwave anisotropy probes has it been possible to show that the apparent size of the visible universe is not an illusion arising from light traversing several times a space whose size is a fraction the Hubble length [26] (for a review, see Levin [27]).  

1 The geometries of the early and present universe are apparently deSitter, and work by Witt and Morrow [28] shows that every 3-manifold occurs as a slice of a deSitter 4-geometry – though not in general a slice consistent with our universe.
C. Topological Censorship

The conjecture can be stated in two equivalent forms. \cite{29, 30}. Recall that to define a black hole – a region from which light cannot escape – one first makes precise the notion of a light ray reaching infinity by attaching to spacetime a boundary, future null infinity ($I^+$) \cite{31}. Light that is not trapped is then light that reaches future null infinity, and a black hole is the region from which no future-directed null geodesic reaches infinity. Past directed null geodesics that are not trapped similarly reach past null infinity. (Unless the spacetime has a white hole, no past-directed light ray will be trapped). An observer that can communicate with the outside world is one whose past and future directed null rays reach null infinity. She is then outside all black (and white) holes, in the \textit{domain of outer communication}. We also need the term \textit{causal curve}, a curve whose tangent is everywhere timelike or null.

\textbf{Theorem 2} Topological Censorship Version A (Friedman, Schleich, Witt). Let $M, g$ be an asymptotically flat, globally hyperbolic spacetime satisfying the averaged null energy condition. Then every causal curve from past null infinity to future null infinity can be deformed to a curve near infinity. (More precisely, each causal curve can be deformed with its endpoints fixed at $I$ to a curve that lies in a simply connected neighborhood of $I$.)

That is, no causal curve can thread the topology. The theorem implies that no observer who remains outside all black holes (and who did not emerge from a white hole) can send a signal that will probe the spacetime topology.

Topological censorship can be regarded as a statement about the topology of the domain of outer communications: The region outside all black (and white) holes is topologically trivial.

\textbf{Theorem 3} Topological Censorship Version B (Galloway). Let $M, g$ be an asymptotically flat spacetime obeying the averaged null energy condition, and suppose the domain of outer communication is globally hyperbolic. Then the domain of outer communications is simply connected.

Again, given the Poincaré conjecture, simply connected is equivalent to topologically trivial. Here this follows from the fact that a globally hyperbolic spacetime has topology $S \times \mathbb{R}$. Topological censorship implies that $S$ is simply connected, and the Poincaré conjecture then implies that $S$ has trivial topology, whence $S \times \mathbb{R}$ has trivial topology.

The proof of Version A that is simplest to outline relies on an argument used by Penrose, Sorkin and Woolgar\cite{32} in their proof of a positive mass theorem. Suppose a causal curve $\gamma$ joining $I^-$ to $I^+$ is not homotopic to an asymptotic curve. Denote by $\lambda^{\pm}$ the generators of $I^\pm$ that contain the endpoints of $\gamma$. Partially order all curves from $\lambda^-$ to $\lambda^+$, writing

$$\gamma_2 \geq \gamma_1, \quad \text{if } \gamma_2 \text{ is faster than } \gamma_1.$$ 

That is, $\gamma_2 \geq \gamma_1$ if $\gamma_2$ leaves $\lambda^-$ later than $\gamma$ (or at the same time) and if $\gamma_2$ reaches $\lambda^+$ earlier (or at the same time). Then a fastest curve $\gamma_\infty$ in the homotopy class of $\gamma$ is a null geodesic without conjugate points.

But the Raychaudhuri equation together with the null energy condition implies that null geodesics have conjugate points in finite affine parameter length, a contradiction.

D. Implications for black holes

Chruściel and Wald showed that, in form B (Theorem 3), topological censorship implies that stationary black holes have spherical topology \cite{33}. The result itself is due initially to Hawking \cite{67}, with a proof that assumes analyticity. The proof based on topological censorship strengthens the theorem: It does not require analyticity (or smoothness) and it uses the weak energy condition. The proof given in Ref. \cite{33} was written before Galloway’s version B appeared, and by reading Galloway first, one can avoid some of the detailed arguments in \cite{33}. The proof uses the fact that the domain of outer communication is simply connected to show that its boundary is homeomorphic to the boundary of a compact three-manifold whose interior is simply connected. This is enough to prove the theorem, because a standard result for 3-manifolds asserts that such a boundary is a disjoint union of spheres.

The result has been amplified in several directions \cite{34, 35, 36, 37, 38, 39}. In particular, one can dispense with stationarity to show spherical topology for slices of the horizon by Cauchy surfaces that lie to the future of a slice of past null infinity \cite{35, 36} or whose topology is unchanging \cite{34}.

At first sight, it would seem that black holes must always have spherical topology, because the event horizon is the boundary of the \textit{simply connected} domain of outer communication. This would match the intuitive picture of their formation and the familiar pair-of-pants coalescence of two black holes. A typical slicing of the horizon begins with two disjoint points that expand to two disjoint spheres; coalescence begins with the intersection of the two spheres in a single point; and the final slices are single spheres.
In fact, however, as Hughes et al. first found numerically [40], one can find examples of collapse in which the intersection of a spacelike hypersurface with the horizon is toroidal. Examples of horizons with slices with higher genus are not difficult to construct, and work by Siino [41] shows that slicings with arbitrary genus can be constructed when the horizon has caustics. The reason is that the past endpoints of the horizon’s null generators form an acausal set; and it is the intersection of wiggly spacelike slices with this acausal set that can have arbitrarily high genus.

It nevertheless appears that there is always an alternative slicing of the horizon in which black holes are spherical. That is, Siino proves the following result: Let \( M, g \) be a strongly causal spacetime with an event horizon that is a smooth \( S^2 \) to the future of some spacelike hypersurface. Then the domain of outer communications of \( M, g \) can be foliated by spacelike slices, each of which intersect the future horizon in a union of disjoint points and of spheres that are either disjoint or that intersect in points. The assumption of a final smooth \( S^2 \) is related to the unproved cosmic censorship conjecture, and one would like to replace that assumption by a positive energy condition.

Underlying the conjecture is the way a toroidal black hole adheres to topological censorship: The torus closes before light can traverse it, and in the examples we know of, that constraint appears to allow a foliation in which black holes are spherical. A simple example, suggested (in one lower dimension) by Greg Galloway, is similar to one given by Shapiro, Teukolsky and Winicour [42] (see also [43]). It is a null surface in Minkowski space that allows an initially toroidal slicing. (One can construct an artificial spacetime for which this null surface is the event horizon, but the construction is unrelated to the nature of the null surface and its slicings.) The surface is generated by the future-directed outward null rays from the spacelike disk \( t = 0, z = 0, x^2 + y^2 \leq 1 \), with \( t, x, y, z \) standard Minkowski coordinates. The surface is rotationally symmetric about the \( z \)-axis. A spacelike surface that cuts through the null rays and then the disk, before dipping below the disk gives a toroidal slice. And spacelike surfaces that lie below the edge of the disk and go upward at the center of the disk give spherical slicings.

### E. Lorentzian universes from nothing

Universes whose spatial slices are compact 3-manifolds are finite in space, but have no boundary. One can similarly construct 4-manifolds with no past boundary that are finite in time – universes from nothing. In a Euclidean framework for quantum gravity, manifolds of this kind arise in the Hartle-Hawking wavefunction of the universe. An example with a Lorentzian metric and with CTCs is given by Gott and Li [44]; but CTCs are not an essential feature of Lorentzian universes from nothing. For a large class of these spacetimes, one can always choose metrics without CTCs; time nonorientability is then their only causal pathology. They are the only examples of topology change in which one has a smooth, nondegenerate Lorentzian metric without closed timelike curves. The double-covering space of these spacetimes is globally hyperbolic, and that fact implies the existence of generalized Cauchy surfaces.

A simple example of such a spacetime is a Möbius strip, whose median circle is taken to be a spacelike hypersurface. Initial data on this median circle consists of a specification of a field and its gradient, and the initial value problem is well defined when the metric has no CTCs. The Möbius strip can be constructed from a cylinder with coordinates \( t, \phi \), by the antipodal identification \( t \rightarrow -t, \phi \rightarrow \phi + \pi \). In four dimensions, antipodally-identified de Sitter space is an example that is locally indistinguishable from ordinary de Sitter. An initial value surface in this case is the antipodally identified 3-sphere at \( t = 0 \), and some initial value surface passes through each point of the spacetime.

More generally, one can construct a Lorentzian universe with no past boundary from any compact 3-manifold \( S \) that admits a diffeomorphism \( I : S \rightarrow S \) by an analogous antipodal identification of \( S \times \mathbb{R} \). If one begins with antipodally symmetric initial data for the Einstein equations on \( S \), then the resulting solution to the field equations on \( S \times \mathbb{R} \) induces a metric on the identified space that is also a solution [45–46].

Because classical physics on Lorentzian universes from nothing has no pathology, it is natural to ask if there is any reason why these spacetimes are forbidden if one incorporates quantum physics. Kay [47] proposes to impose the standard canonical commutation relations in a neighborhood of any point with respect to one time orientation (the “F-locality condition”). This condition rules out time non-orientable spacetime in an obvious manner. Gibbons [48] also rules out conventional quantum field theory based on a complex Hilbert space in some time non-orientable spacetimes, including antipodally identified de Sitter spacetime.

Refs. [45, 46] investigate whether quantum field theory in Lorentzian universes from nothing could be defined in globally hyperbolic neighborhoods with suitable overlap conditions on the intersections. It turns out that there are no difficulties in defining an algebra of fields in this manner. It is also possible to construct states satisfying the positivity conditions in each of the neighborhoods if the union of two neighborhoods is always time orientable. This restriction on the neighborhoods, however, is rather artificial because there will always be sets of points in the spacetime for which no correlation function is defined. If one then allows the union of two neighborhoods on which field theory is defined to be time non-orientable, one can show that there are no physically reasonable states satisfying the positivity condition in each neighborhood. One may be able to construct a consistent field theory in these spacetimes using path-integral quantization, but this possibility has not been explored.
FIG. 1: A simple spacetime with CTCs and a generalized Cauchy surface $S$ is shown in this figure. (The precise definition of a generalized Cauchy surface is given below.) Two parallel segments of equal length are removed from Minkowski space, two disjoint edges are joined to the left and right sides of each slit, and edge points related by the timelike translation $T$ are then identified.

III. CHRONOLOGY PROTECTION

A. Overcoming the grandfather paradox: Existence of solutions for generic data in a class of spacetimes with CTCs

Closed timelike curves were traditionally regarded as unphysical because of the *grandfather paradox*, the fact that a time machine would allow one to go back in time and do away with her grandfather. More precisely, it was thought that most initial data on a spacelike surface would fail to have a consistent evolution: Locally evolving the data would not lead to a global solution, because the local solution would be inconsistent with the data, once the evolution returned to the initial surface.

A simple example of a spacetime for which generic initial data has no solution is the cylinder obtained from the slab of Minkowskian space between $t = -1$ and $t = 1$ by identifying the points $-1, x, y, z$ and $1, x, y, z$. Data on the surface $t = 0$ for, say, a massless scalar field $\psi$ can be locally evolved around the cylinder (to $t = 1$ and then from the identified surface $t = -1$ back to $t = 0$). The locally evolved solution, however, returns to the surface with a value of the field that disagrees with its initial value unless the initial data is chosen to give a solution periodic with period $T = 2$.

One of the surprises that spurred interest in spacetimes with CTCs is the existence of a different class of spacetimes for which solutions exist for free fields with arbitrary initial data, for interacting classical particles (billiard balls), and perhaps for interacting fields as well. Whether physics is consistent with closed timelike curves, however, does not rely on consistency of classical physics. A path-integral assigns probability amplitudes to histories, whether or not there are classical histories. And in the billiard-ball case, where classical solutions exist but are not unique, a path-integral dominated by classical solutions would simply assign large amplitudes to the alternative classical histories.

We review the billiard-ball example and interacting field arguments briefly in the next section, after we have introduced another classical obstacle to closed timelike curves (an instability of the Cauchy horizon) that these examples overcome. In this section we consider free fields.

Two-dimensional spacetimes built from Minkowski space that avoid the grandfather paradox are easily constructed. An example, similar to spaces discussed by Geroch and Horowitz [49] and by Politzer [50], is constructed by removing two parallel timelike slits from Minkowski space and gluing the edges of the slits. Corresponding points on the two inner edges are identified by the translation $T$ shown in Fig. 1 and corresponding points on the two outer edges are similarly identified.

CTCs join identified points of the inner edges, from $Q$ on the left to $T(Q)$ on the right. The hypersurface $S$, lying to the past of the CTCs, is an obvious candidate for a generalized Cauchy surface of $M, g$.

*Definition.* A *generalized Cauchy surface* $S$ is an achronal hypersurface of $M$ for which the initial value problem for the scalar wave equation is well-defined: Any smooth data in $L_2(S)$ with finite energy, for a scalar field $\Phi$, has a unique solution $\Phi$ on $M$. 


FIG. 2: A wormhole is constructed by removing two balls and identifying their spherical boundaries $\Sigma_I$ and $\Sigma_{II}$ after a reflection: Points labeled by the same letter, with subscripts I and II, are identified.

In fact, it is easy to see that initial data in $L_2(S)$ leads to a solution in $L_2(M)$. In the past of the CTCs (in the past of the Cauchy horizon), solutions to the massless wave equation can be written as the sum $f(t-x) + g(t+x)$ of right-moving and left-moving solutions. To obtain a solution in the spacetime $M$, one simply propagates left moving data that encounters the slit in the obvious way. For example, if a left-moving wave enters the left slit at $Q$, it emerges unaltered from the right slit at $T(Q)$. The solution is unique. But it is discontinuous along future-directed null rays that extend from the endpoints of the slits, because the result of the wave propagation is to piece together solutions from disjoint parts of the initial data surface (see Friedman and Morris [51]). Analysis of the initial value problem for a related spacetime with spacelike slits is given by Goldwirth et al. [52].

Although, in our 2-dimensional example, the solution has discontinuities, in four dimensions one can construct spacetimes for which smooth, unique solutions to the scalar wave equation exist for all data on a generalized Cauchy surface. The examples for which these statements are proved are time-independent spacetimes [51, 54], in which CTCs are always present. The spacetimes are asymptotically flat, and one can define future and past null infinity. In Minkowski space past null infinity is a generalized Cauchy surface for massless wave equations, and the theorems show that it is also generalized Cauchy surface for a class of spacetimes with CTCs.

The spacetime considered in Ref. [51, 55] has a wormhole that joins points at one time to points at an earlier time. Recall that a wormhole is constructed from $\mathbb{R}^3$ by removing two balls and identifying their spherical boundaries, $\Sigma_I$ and $\Sigma_{II}$, as shown in Fig. 2. The history of each sphere is a cylinder in spacetime, and one constructs a spacetime with CTCs by removing two solid cylinders from Minkowski space and identifying their boundaries $C_I$ and $C_{II}$ after a time translation, so that the sphere at time $t+T$ is identified with a sphere at an earlier time $t$, as shown in Fig. 3. Thus a particle entering the wormhole mouth at $T(\Sigma_I)$ emerges from the mouth $\Sigma_I$ at an earlier time.

The existence proof relies on a spectral decomposition of the field. Technical difficulties arise from the fact that the boundary conditions are frequency dependent, and solutions with different frequencies are not orthogonal. As a result, one cannot use the spectral theorem, and a separate proof of convergence of the harmonic decomposition is required. Recent work by Bugdayci [53] constructs a solution for the case of a metric that is flat everywhere outside the identified cylinders, as a multiple scattering series.

More recently, Bachelot [54] proved a similar existence theorem and a strong uniqueness theorem for another family of stationary, four-dimensional spacetimes that are flat outside a spatially compact region. These spacetimes have Euclidean topology and their dischronal regions have topology (solid torus) $\times \mathbb{R}$. The metric is axisymmetric, with one free function $\alpha$ that describes the tipping of the light cones in the direction of the rotational Killing vector $\partial_\phi$.

B. Classical chronology protection

Parallel slits do not exhibit a property of almost all other two-dimensional spacetimes whose CTCs lie to one side of a Cauchy horizon: a classical instability of the Cauchy horizon. Once the slits are not parallel (once they are, in effect, walls in relative motion), the instability arises. The paradigm spacetime for this instability is Misner space [31, 56, 57].

Misner space can be obtained from a 1-dimensional room whose walls are moving toward each other at relative speed $v$, by identifying left and right walls at the same proper time read by clocks on each wall. The resulting space is the piece of Minkowski space between two timelike lines (the walls), with the lines identified by the boost that maps one line to the other. In the diagram below, Fig. 4 numbers label readings of a single clock, shown at identified points of the left and right walls. A light ray beginning at the left wall at $t = 0$ is boosted each time it traverses the space, in the same way that light is boosted when reflected by a moving mirror. As it loops around the space, the light ray approaches a closed null geodesic through the clock at
Identified points are spacelike separated for $t < 4$. For $t = 4$ the identified clocks are separated by a null geodesic that marks the boundary of the globally hyperbolic spacetime to its past and the dischronal region above it. Through each point of the dischronal region passes a closed timelike curve; an example is the dashed line joining the clock images at $t = 5$.

The divergence of solutions to the wave equation is clear in the geometrical optics limit. A family of light rays that loop about the space are boosted at each loop, their frequencies increased by the blueshift factor $\left(\frac{1 + v}{1 - v}\right)^{1/2}$. Their energy density, measured in the frame of any inertial observer, diverges as they approach the Cauchy horizon. Each looping light ray is an incomplete null geodesic: It reaches the horizon in finite affine parameter length, because each boost decreases the affine parameter by the factor $\left(\frac{1 + v}{1 - v}\right)^{1/2}$. In discussing Gannon’s singularity theorem for nontrivial topology, we noted that, in the context of gravitational collapse, geodesic incompleteness is thought generically to imply a curvature singularity. Here, however, the spacetime is smooth, and the incomplete geodesic is unpleasant only because it leads to an instability of the Cauchy horizon.
FIG. 5: The wormhole spacetime of Morris et al. is shown in this figure. The two mouths at the same proper time are identified.

This behavior is not unique to Misner space or to two dimensions: A theorem due to Tipler [20] shows that geodesic incompleteness is generic in spacetimes like Misner space in which CTCs are “created” – spacetimes whose dischronal region lies to the future of a spacelike hypersurface. The nature of the geodesic incompleteness is clarified by Hawking [67], at least in the case when the null generators of the Cauchy horizon have the character of a fountain, all springing from a single closed null geodesic (past-directed generators approach the closed null geodesic). The boosted light rays that give rise to the Misner space instability will characterize these more general Cauchy horizons in four dimensional spacetimes. In four dimensions, however, the instability competes with the spreading of the waves. In the geometrical optics limit, the boosting will not lead to a divergent energy density if the area of a beam increases as the beam loops by a factor greater than the increase in energy density due to the boost in frequency. That is, if the ratio $A_{n+1}/A_n$ of beam areas at successive loops is greater than $(\omega_{n+1}/\omega_n)^2$, the boosted frequency will not lead to a divergence of the energy at the Cauchy horizon.

Hawking’s analysis follows an example given by Morris, Thorne and Yurtsever [58, 62] of a wormhole whose mouths move toward each other in a way that initially mimics the Misner-space walls, as shown in Fig. 5.

For the case of a spacetime flat outside the wormhole mouths (the identified spheres), each time a beam of light traverses the wormhole, its frequency is boosted and its area increases by the approximate factor $(d/R)^2$, with $d$ the distance between wormholes, $R$ the radius of the throat. In this example, the horizon generators do have the fountain behavior described above. The fountains assumed by Hawking, however, are probably not generic for the kind of Cauchy horizons he considers (compactly generated, noncompact). (A Cauchy horizon is compactly generated if all its past-directed null generators enter and remain in a compact region.) Chruściel and Isenberg [59] give examples of compactly generated noncompact Cauchy horizons whose generators do not have fountainlike behavior (for which no closed null geodesic is an attractor); they show that fountains are not generic for compact Cauchy horizons and argue that they are not generic for the compactly generated, noncompact case.

Because the evolution of fields on the wormhole spacetime does not lead to an instability of the Cauchy horizon, it appears that asymptotically flat spacelike hypersurfaces to the past of the Cauchy horizon (e.g., a $t =$ constant surface of Minkowski space through $\tau = 0$ in Fig. 5) are generalized Cauchy surfaces. One can formally construct a solution as a multiple scattering series, and, for sufficiently small ratio $R/d$, we expect the series to converge.

For interacting particles, modeled as billiard balls, Echeverria et al. [60] looked at the simpler time-independent wormhole spacetime of Fig. 5 in which billiard balls entering the right mouth of the wormhole at time $t$ exit from the left mouth at time $t - \tau$. Consider a ball that is aimed at the right wormhole along a timelike line that intersects itself. Initial data for the ball has a local solution that, when extended has the exiting billiard ball aimed to strike its earlier self, apparently preventing a solution. In fact, however, there appear always to be glancing blow solutions for arbitrary initial position and velocity of the incoming ball. In these solutions, the incoming ball is hit by its earlier self in just the right way that it enters the wormhole and emerges aimed to strike itself that glancing blow. 2

2 By adding an additional degree of freedom, however, one can apparently obtain systems that do not admit classical solutions [11, 60].
With both the grandfather paradox and the classical Cauchy horizon instability overcome, what prevents the classical formation of spacetimes with CTCs? In the wormhole example, the null energy condition must be violated, if one is to keep the wormholes open long enough for light to traverse them, and hence long enough for closed timelike curves to traverse them. Hawking shows that this is generically true [67].

Classical chronology protection. Let $M, g$ be a spacetime with a compactly generated Cauchy horizon. Assume that the Cauchy horizon is the boundary of the domain of dependence of a noncompact partial Cauchy surface $S$. Then the null energy condition is violated.

The Cauchy horizon is expected to be compactly generated in a nonsingular, asymptotically flat spacetime for which the Cauchy horizon bounds the domain of dependence of an asymptotically flat spacelike hypersurface $S$. The argument is then:

(i) Generators of a future Cauchy horizon are null geodesic segments with no past endpoints.
(ii) Because they enter a compact region, some of the generators must converge to the past.
(iii) Positive convergence for a past-directed generator, together with the Raychaudhuri equation, implies that the null energy condition must be violated on the future horizon. Otherwise the generators would have past endpoints.

If one allows the spacetime to be singular to the future of $S$, as in recent examples by Ori [63], it is not difficult to show that one can find chronology-violating asymptotically flat spacetimes satisfying the null energy condition. In this case the Cauchy horizon is singular and thus not compact (past directed null geodesics run off the manifold in finite affine parameter length). In Ori’s most recent example, a spacetime whose only matter is a compact region of dust has a Cauchy horizon with closed null geodesics that is nonsingular in a neighborhood of these geodesics.

Because quantum fields fail to satisfy the null energy condition, a number of attempts have been made to circumvent the classical chronology protection theorem and topological censorship by finding a way to have negative energy regions that persist for long times; or to have solutions with CTCs or nontrivial topology that require only small amounts or small regions of negative energy. In our opinion, however, the recent work on quantum energy inequalities, discussed in Sect. IV below, has set increasingly stringent constraints on violations of the energy conditions. We think it likely that macroscopic violations of either topological censorship or classical chronology protection are inconsistent with the fundamental properties of semiclassical quantum fields.

C. Quantum chronology protection

Various pathologies have been found in quantum field theory in spacetimes with dischronal regions. A quantum instability in spacetimes with a Cauchy horizon may prevent the formation of CTCs on scales large compared to the Planck scale; or, together with a loss of unitarity for interacting fields on spacetimes with CTCs, it may indicate that chronology violations do not occur in the fundamental theory.

We begin with a brief heuristic summary of the quantum instability and then present a more technical description that includes more recent results.

1. Quantum instability

The classical instability of Misner space has a quantum counterpart that is present in cases where classical fields remain finite. The quantum instability and the loss of unitarity are each related to propagation of fields around closed null or timelike curves. In computing the energy density of a quantum field in the vacuum, one must renormalize the field to produce a finite result, subtracting off a divergent zero-point energy of the vacuum. One can, for example, impose a short-distance cutoff, subtract a spacetime-independent term that would be the zero-point energy of a flat spacetime with a short-distance cutoff, and then take the continuum limit (as the cutoff goes to zero). The subtraction eliminates the divergence in the propagator from nearby points that are separated by a null geodesic. When there are closed null geodesics, however, more than one null geodesic connects nearby points, and additional divergent contributions arise from vacuum fluctuations that propagate around these closed null curves. The result is that, as one approaches a Cauchy horizon, the finite, renormalized energy density can grow without bound.

A divergence of the renormalized energy-momentum tensor at a Cauchy horizon, $\langle T^{\text{ren}}_{\alpha\beta}\rangle$, was examined by various authors for free scalar field theory in several chronology violating spacetimes [64, 65, 66, 67, 68]. These results, together with the Hawking’s energy-condition violation theorem of the previous section (classical chronology protection) led Hawking to propose his Chronology Protection Conjecture: The laws of physics prevent closed timelike curves from appearing. More precisely, Hawking conjectured that the laws of physics prevent local creation of closed timelike curves, as characterized by the existence of a compactly generated Cauchy horizon. In several examples, however, the value of $\langle T^{\text{ren}}_{00}\rangle$ remains finite as the point $x$
approaches the Cauchy horizon \([69, 70, 71, 72]\). Subsequently, Kay, Radzikowski and Wald (KRW) \(73\) showed that the two-point function, from which the energy-momentum tensor is obtained, is singular, in the sense which will be explained below, for a free scalar field at some points on a compactly generated Cauchy horizon. Thus, although \(\langle T^{\text{ren}}_{\alpha \beta} \rangle\) may be bounded as the point approaches the Cauchy horizon, it is not well-defined on the horizon.

The KRW theorem uses the fact that for a Hadamard state the two-point function \(\langle \hat{\varphi}(x)\hat{\varphi}(x') \rangle\) is divergent if and only if the two points \(x\) and \(x'\) can be connected by a null geodesic \(74\). There is now a wide consensus that a physically reasonable state must be a Hadamard state (see, e.g. Ref. \(75\) for the Hadamard condition). This condition roughly states that the light-cone singularity structure must be the same as for the vacuum state in Minkowski spacetime.

In a globally hyperbolic spacetime every point \(x\) has a convex normal neighborhood \(N_x\) \(31\) small enough that no null geodesic leaves and reenters it. This implies for each point \(x\) in a globally hyperbolic spacetime the following property: For a small enough convex normal neighborhood \(N_x\) and for any two points \(y, y' \in N_x\), the two-point function \(\langle \hat{\varphi}(y)\hat{\varphi}(y') \rangle\) is divergent if and only if \(y\) and \(y'\) are connected by a null geodesic inside \(N_x\). Because this property is necessary to define \(T^{\text{ren}}_{\alpha \beta}(x)\), we say that the two-point function is singular at \(x\) if the property is not satisfied.

Now, for a compactly generated Cauchy horizon, every past-directed null geodesic generator \(\lambda\) stays in a compact region. There must then be a point \(x\) such that, given any neighborhood \(N_x\), the geodesic \(\lambda\) passes through \(N_x\) infinitely many times. This implies that for any convex normal neighborhood \(N_x\) of \(x\) there are points \(y\) and \(y'\) in \(N_x\) such that the two-point function \(\langle \hat{\varphi}(y)\hat{\varphi}(y') \rangle\) is divergent. Hence we have the following theorem due to KRW:

**Theorem 4** The two-point function of a free scalar field is singular on certain points on a compactly generated Cauchy horizon.

For explicit verification of the KRW theorem in examples with vanishing \(\langle T^{\text{ren}}_{\alpha \beta} \rangle\), see Ref. \(76\). Although KRW proved this theorem for a scalar field, it would be straightforward to generalize their result to non-interacting fields of any spin.

Because of examples in which the stress tensor does not diverge, and because the theorem does not imply that the strong quantum instability of the earlier examples is generic, one can argue that the quantum singularity is too weak to enforce chronology protection. Kim and Thorne \(66\) had already entertained that argument in their strong-instability example, finally suggesting that the issue can be decided only within a theory of full quantum gravity. Visser \(77, 78\) similarly argues that the above theorem should be interpreted as the statement that quantum field theory in a background spacetime is unreliable on the Cauchy horizon, and that quantum gravity is needed to determine what really happens. The theorem does, however, indicate a drastic difference between the behavior of physical fields and spacetime on observed scales and their behavior on the smallest scales if there is to be chronology violation.

Kay’s F-locality condition mentioned before requires (among other things) that quantum scalar fields in non-globally hyperbolic spacetime commute for any two spacelike separated points in some neighborhood \(N_x\) of any point \(x\). The work of KRW showed that this condition cannot be satisfied in spacetimes with a compactly generated Cauchy horizon. Interestingly, it can be satisfied on the timelike cylinder obtained as the quotient of Minkowski space by a time translation \(47, 79\). However, this was shown not to be the case for a massive scalar field in a two-dimensional spacelike cylinder with a generic metric \(80\).

2. Loss of unitarity

In the classical billiard-ball models mentioned above, it was found that the Cauchy problem is not well defined because there is often more than one solution for a given set of initial data \(58, 60\). In the corresponding quantum theory, however, it was found that the solution of the wave equation is unique. Although this observation gave some hope that the quantum theory of interacting systems with CTCs might be consistent, it turns out that unitarity is lost for interacting quantum fields in spacetimes with CTCs. Ways to recover a consistent quantum theory have been suggested, but all have features that seem undesirable.

Unitarity of the scattering matrix \(S_{fi} = \delta_{fi} - iT_{fi}\) in quantum field theory can be expressed as

\[
2\text{Im }T_{fi} = -\sum_n T_{nf}T_{ni} \quad (3.1)
\]

As for free field theories, it was shown in Ref. \(81\) that, if the Cauchy problem is well defined for classical field equations, then these unitarity relations are satisfied. Thus, for example, the massless scalar field theory in the spacetime with a chronology violating wormhole studied by Friedman and Morris \(51\) is unitary.

The unitarity relations \((3.1)\) were studied for the \(\lambda \varphi^4\) theory in the above-mentioned wormhole spacetime by Friedman, Papastamatiou and Simon \(82\) and in Gott spacetime by Boulware \(83\). They defined the perturbation theory in a path-integral framework and found that the Feynman propagator \(i\Delta_F(x, y)\) has an extra imaginary part \(E(x, y)\):

\[
i\Delta_F(x, y) = \theta(x_0 - y_0)D(x, y) + \theta(y_0 - x_0)\overline{D}(x, y) + E(x, y) \quad (3.2)
\]

At first order in \(\lambda\) the \(T_{fi}\) in \((3.1)\) with the initial and final states both being one-particle states corresponds to a tadpole diagram. The imaginary part of this diagram is essentially given by \(E(x, x)\), which is non-zero. This implies violation of the relation
because the right-hand side starts at order $\lambda^2$. In an unpublished work Klinkhammer and Thorne showed using the WKB approximation that the quantum theory of the billiard system with CTCs is non-unitary. This non-unitarity arises due to the fact that the classical system allows a multitude of solutions for a given set of initial data and that the number of solutions depends on the initial data.

Simple quantum mechanical models which mimic chronology violating interacting field theory were studied by Politzer [50]. He confirmed perturbative non-unitarity in a billiard model. He also studied some exactly solvable models exhibiting non-unitarity [84]. Fewster, Higuchi and Wells [79] studied a generalization of Politzer’s model by solving the differential equation satisfied by the Heisenberg operators. They found that the canonical (anti-)commutation relations are not preserved in time, and, consequently, that the theory is non-unitary. They also compared their method and the path-integral quantization and found that the two methods give different results. This conclusion seems to be related to the observation [85] that the Schrödinger and Heisenberg pictures do not agree in non-unitary quantum mechanics.

Loss of unitarity poses difficulties for the conventional Copenhagen interpretation of quantum mechanics. Suppose, for example, one makes a measurement in a region spacelike separated from the CTCs. One would expect that the result would be unaffected by the chronology violation. However, the measurement could be interpreted to have occurred either before or after the CTCs, and the probability assignment would depend on which interpretation was taken because of non-unitarity [86].

There have been a few proposals for eliminating non-unitarity or for finding a probability interpretation that accepts loss of unitarity: Adopting only the unitary part of the evolution operator [87] and making the non-unitary operator a restriction of a larger norm-preserving operator [85]; but the former would make the evolution of states highly nonlinear, and the latter necessitates the use of negative-norm states. Friedman et al. [82] and Hartle [88] have advocated the sum-over-histories approach to the interpretation of quantum mechanics. This gives a prescription for computing probabilities, but probabilities in a globally hyperbolic past of any CTCs are affected by the existence of CTCs in the future. Hawking [89] has advocated the use of the superscattering operator, i.e. the linear mapping from the initial to final density matrices. However, Cassidy [90] has found that the initial pure state evolves nonlinearly into a mixed state. Also, non-unitarity in the models studied in Ref. [79] is such that evolution cannot even be described by a superscattering matrix.

Let us conclude this subsection by describing the work of Deutsch on the grandfather paradox in the context of quantum information theory [91]. Let the initial state containing the grandfather of the killer be $\psi$ and let $\hat{\rho}$ be the density matrix resulting from the evolution of $|\psi\rangle\langle\psi|$ through the CTC region, which may or may not contain the killer. (Deutsch asserts that a generic pure state would inevitably evolve to a mixed state, violating unitarity, if there was a closed timelike curve.) The state $|\psi\rangle\langle\psi|$ in a Hilbert space $\mathcal{H}_1$ and its future self, $\hat{\rho}$, in a Hilbert space $\mathcal{H}_2$ form a tensor product state $|\psi\rangle\langle\psi| \otimes \hat{\rho}$ in $\mathcal{H}_1 \otimes \mathcal{H}_2$ when they encounter one another. The interaction of the two parts is described by a unitary matrix $U$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$. After the interaction, the Hilbert space $\mathcal{H}_2$ continues to the future and $\mathcal{H}_1$ goes back to the past. The state going back to the past is obtained by tracing out the resulting state over $\mathcal{H}_2$ and must equal $\hat{\rho}$. Thus, the consistency condition is $\text{Tr}_2 \left[U (|\psi\rangle\langle\psi| \otimes \hat{\rho}) U^\dagger \right] = \hat{\rho}$. Deutsch showed that there are solutions to this equation for any initial state $|\psi\rangle$ in simple examples. However, as we have seen, the evolution in chronology violating spacetimes cannot be described in general by a linear superscattering operator as envisaged in these models.

IV. QUANTUM ENERGY INEQUALITIES

A. Introduction

In quantum field theory, the weak energy condition is violated by an energy-density operator whose vacuum expectation value is unbounded from below in Minkowski space, even for a free scalar field. The energy condition is recovered in the classical limit, because the terms responsible for its violation oscillate in time and a classical field measures an average of the fluctuating quantum field. This averaging argument leads in two related ways to energy conditions satisfied by quantum fields: It suggests that the averaged null energy condition, (2.1), may still be valid; the ANEC is sufficient for establishing chronology protection and topological censorship (as well as some singularity theorems [92, 100, 101]). And it suggests that a weighted average of the energy density along a timelike curve may be bounded from below. Indeed, in a Minkowski-space context, Ford and Roman found such bounds [92, 93, 96], commonly called quantum (weak) energy inequalities (QEIs). In the next subsection we briefly discuss some work related to the ANEC in quantum field theory and then review recent developments in QEIs. (See Refs. [9, 97] for more comprehensive reviews of the QEIs.)

First, however, we present a simple example of the violation of the weak energy condition by a free scalar field in Minkowski space. The example relates the violation to the subtraction of an infinite constant from a formally positive energy-density operator $T_{00}$; and it exhibits the oscillation in time that leads to averaged inequalities.

The classical energy density $T_{00}$ for a real massless scalar field $\varphi$ in four dimensions is

$$T_{00} = \frac{1}{2} \left[ \dot{\varphi}^2 + (\nabla \varphi)^2 \right]. \quad (4.1)$$
To define the renormalized energy-density operator, one subtracts the infinite vacuum energy density, \( \langle 0 | \hat{T}_{00} | 0 \rangle \). This renormalized operator is no longer positive-definite, although the energy, its spatial integral, is. One therefore expects states for which the expectation value of the renormalized energy-density operator is negative, and this is indeed the case \([92, 93]\), as seen in the following simple example. Let \( a^\dagger(p) \) be the creation operator for the scalar particle with momentum \( p \) and consider the normalized superposition of the vacuum state and a two-particle state,

\[
|\psi\rangle = \cos \alpha |0\rangle + \frac{\sin \alpha}{\sqrt{2}} \left[ \int \frac{d^3p}{(2\pi)^3 2P} f(p) a^\dagger(p) \right] |0\rangle
\]

with \( P \equiv \|p\| \) and

\[
\int \frac{d^3p}{(2\pi)^3 2P} |f(p)|^2 = 1 ,
\]

where the function \( f(p) \) has been assumed to be real for simplicity. For example, if the function \( f(p) \) is chosen to be peaked about \( p = \overline{p} \) by letting \( f(p) = (12\pi^2 \overline{p}/\delta^3)^{1/2} \) if \( \|p - \overline{p}\| < \delta \) and zero otherwise, then one finds the expectation value of the renormalized energy-momentum tensor at the origin for \( \delta \ll \overline{p} \) as follows:

\[
\langle \psi | \hat{T}_{00}^{\text{ren}}(t,0) |\psi\rangle \approx \frac{\overline{p}^3}{6\pi^2} \sin \alpha \left( - \sqrt{2} \cos \alpha \cos 2pt + 2 \sin \alpha \right) .
\]

This quantity is negative at \( t = 0 \), say, if \( \sin \alpha (\cos \alpha - \sqrt{2} \sin \alpha) > 0 \), and it is unbounded from below as a function of \( \overline{p} \). The NEC is also violated because for any null vector \( t^\alpha \) in the direction perpendicular to \( p \), one finds \( \langle \psi | \hat{T}_{00}^{\text{ren}}(t,0) |\psi\rangle \approx \overline{p}^3 \sqrt{2} \sin \alpha \cos 2pt \). As anticipated, the violation of the weak energy condition is associated with the oscillating term in Eq. (4.4).

Our review reflects the view that overcoming the obstacles to forming CTCs and wormholes has become increasing difficult. The opposite view is taken in a review with extensive references by Lemos et al. \([98]\).

### B. Results and implications

The NEC can be replaced by the ANEC (or a condition similar to it) in proving some singularity theorems \([99, 100, 101]\) as well as chronology protection and topological censorship. It has been shown that the ANEC holds in free quantum scalar field theory in Minkowski space and in two-dimensional curved spacetime under certain assumptions \([102, 103, 107]\). However, Klinkhammer \([103]\) has pointed out that the ANEC does not hold if one compactifies any of the space dimensions in Minkowski space. If, for example, one identifies the space coordinate \( x \) with \( x + L \) in two-dimensional Minkowski space, then the energy-momentum tensor for the massless scalar field in the vacuum state is non-zero due to the Casimir effect and given by \([104, 105, 106]\) \( \langle 0 | \hat{T}_{00}^{\text{ren}} | 0 \rangle = \langle 0 | \hat{T}_{11}^{\text{ren}} | 0 \rangle = -(\pi/6)L^2 \) and \( \langle 0 | \hat{T}_{01}^{\text{ren}} | 0 \rangle = 0 \). Hence for \( k^0 = \pm k^1 = 1 \), one has \( \langle 0 | \hat{T}_{00}^{\text{ren}} | 0 \rangle k^0 k^\beta = -\pi/L^2 = \text{const.} \) and the ANEC is necessarily violated. Wald and Yurtsever also point out that the ANEC is violated in generic spacetimes for the minimally-coupled massless scalar field theory \([107]\). (However, it would be physically wrong to conclude that the Casimir effect with perfectly reflecting mirrors violates the ANEC. See, e.g. \([108]\).)

There have been some attempts to rescue the ANEC. Yurtsever proposed that the integral of \( \langle \psi | \hat{T}_{\alpha\beta}^{\text{ren}} | \psi \rangle k^\alpha k^\beta \) along the null geodesic, to which \( k^\alpha \) is tangent, may be bounded below by a state-independent constant, and he shows that, if true, it can be used to rule out some wormhole spacetimes \([109]\). Flanagan and Wald investigated the ANEC smeared over Planck scale in spacetimes close to Minkowski space, imposing the semi-classical Einstein equations and showed that it holds for pure and mixed states if the curvature scale is much larger than the Planck scale and if incoming gravitational waves do not dominate the spacetime curvature \([110]\).

For a minimally coupled scalar field, the classical energy-density \( \rho = T_{\alpha\beta} t^\alpha t^\beta \), where \( t^\alpha \) is a timelike vector of unit length, is positive definite, and can be given in the following form:

\[
\rho(x) = \sum_j \left[ P^{(j)} \varphi(x) \right]^2 ,
\]

where \( P^{(j)} \) is a differential operator with smooth coefficients. Let us define the energy-density operator, in the corresponding quantum theory, normal-ordered with respect to a reference state \( |\psi_0\rangle \) as follows:

\[
\hat{\rho}(x) : \equiv \sum_j \left\{ \left[ P^{(j)} \varphi(x) \right]^2 - \langle \psi_0 | \left[ P^{(j)} \varphi(x) \right]^2 |\psi_0\rangle \right\} .
\]
Although each of these terms is infinite, the difference is finite and uniquely determined. This operator differs from the renormalized energy-density operator by a smooth c-number function which depends only on spacetime properties. Now, let $x = \gamma(t)$ be a timelike curve and $g(t)$ be a smooth real function satisfying $\int_{-\infty}^{+\infty} g^2(t) \, dt = 1$. A QEI takes the following form in general:

$$\int_{-\infty}^{+\infty} dt \, g^2(t) \langle \psi | \rho(\gamma(t)) | \psi \rangle \geq -C(\gamma, g)$$

(4.7)

for any Hadamard state $|\psi\rangle$, where $C(\gamma, g)$ is a positive number independent of the state $|\psi\rangle$.

One of the first QEIs was for the minimally-coupled massless scalar field in four-dimensional Minkowski space with the Lorentzian sampling function. It has the form $[94, 95, 96]$

$$\int_{-\infty}^{+\infty} dt \langle \psi | \hat{\rho}(t, x) : | \psi \rangle \frac{\tau}{\pi(t^2 + \tau^2)} \geq -\frac{3}{32\pi^2\tau^4}. \quad (4.8)$$

The QEIs appear to place stringent constraints on spacetimes with nontrivial topology, spacetimes with CTCs, as well as on warp-drive [111] spacetimes. To make these constraints rigorous, however, inequalities in general spacetimes are required; and QEIs with compactly-supported sampling functions are more useful in these applications. Flanagan [112] derived QEIs for minimally-coupled massless scalar field in general two-dimensional curved spacetime with general smooth sampling functions. (The bounds given by Flanagan’s QEIs are optimal.) Fewster was able to establish QEIs in general globally hyperbolic spacetime in any dimensions using a general smooth sampling function for minimally-couple scalar field of arbitrary mass [113]. We present his inequality as a representative of the QEIs.

Let us consider, for simplicity, a timelike curve given by $x = 0$ for a given coordinate system in a globally hyperbolic spacetime and write the minimally-coupled scalar field on this curve, $\phi(t, x = 0)$, simply as $\hat{\phi}(t)$. Fewster’s inequality relies on the following lemma, which is an immediate consequence of the work by Radzikowski [74], which gives the Hadamard condition in the language of microlocal analysis:

**Lemma 1** Let $P$ be a differential operator with smooth coefficients. Define the double Fourier transform of the point-separated two-point function for $P\hat{\phi}$ on a Hadamard state $|\psi_0\rangle$ with a smooth compactly-supported sampling function $g(t)$ by

$$\hat{\Delta}^P_g(k_1, k_2) \equiv \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \, e^{i(-k_1 t_1 + k_2 t_2)} g(t_1) g(t_2) \langle \psi_0 | [P\hat{\phi}(t_1)] [P\hat{\phi}(t_2)] | \psi_0 \rangle. \quad (4.9)$$

Then $\hat{\Delta}^P_g(k_1, k_2)$ tends to zero faster than any polynomial as $k_1 \to +\infty$ or $k_2 \to +\infty$.

The physical interpretation of this lemma is that a Hadamard state is annihilated by the “positive-frequency” part $\int_{-\infty}^{+\infty} dt \, e^{ikt} \hat{\phi}(t)$ in the limit $k \to +\infty$. The following result [97], a generalization of the argument used in [114], essentially gives the QEI of Ref. [113]:

**Lemma 2** Define the following normal-ordered product:

$$:[P\hat{\phi}(x)]^2 := [P\hat{\phi}(t)]^2 - \langle \psi_0 | [P\hat{\phi}(t)]^2 | \psi_0 \rangle. \quad (4.10)$$

Then, the expectation value of $[P\hat{\phi}(x)]^2$: averaged over the timelike curve $x = 0$ with the sampling function $g^2(t)$ satisfies

$$\int_{-\infty}^{+\infty} dt \, g^2(t) \langle \psi | [P\hat{\phi}(t)]^2 | \psi \rangle \geq -\frac{1}{\pi} \int_{0}^{\infty} dk \, \hat{\Delta}^P_g(k, k). \quad (4.11)$$

The right-hand side is finite by Lemma 1.

**Proof.** The left-hand side, which we denote by $A$, can be written as

$$A = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} \int_{-\infty}^{+\infty} dt_1 \int_{-\infty}^{+\infty} dt_2 \, g(t_1) g(t_2)$$

$$\times \{ \langle \psi | P\phi(t_1) P\phi(t_2) | \psi \rangle - \langle \psi_0 | P\phi(t_1) P\phi(t_2) | \psi_0 \rangle \} \, e^{-ik(t_1-t_2)}. \quad (4.12)$$

The expression inside the curly brackets is invariant under the interchange $t_1 \leftrightarrow t_2$ because the commutator $[\phi(t_1), \phi(t_2)]$ is state-independent. Hence, one can restrict $k$ to be positive and multiply the integral by two. After this operation, we see that the first term is of the form $\int_{0}^{\infty} dk \langle \psi | \mathcal{O}(k) \mathcal{O}(k) | \psi \rangle \geq 0$ and the second term is exactly the right-hand side of the inequality (4.11).

The Fewster inequality follows immediately from this lemma and Eq. (4.6).
Theorem 5 The expectation value of the normal-ordered energy density given by Eq. (4.6) on a Hadamard state $|\psi\rangle$ satisfies the following inequality:

$$\int_{-\infty}^{\infty} dt \, g^2(t) \langle \psi | \hat{\rho}(t, x = 0) : \psi \rangle \geq - \frac{1}{\pi} \sum_j \int_0^\infty dk \, \Delta_g \rho^{(\psi)}(k, j).$$

Although most of the work on QEIs involves minimally-coupled scalar fields, corresponding inequalities have also been derived for spin-one fields including the electromagnetic field $[115, 116]$ and for spinor fields $[117, 118, 119]$.

The QEIs impose severe restrictions on wormhole spacetimes $[96]$. Since it is difficult to calculate the QEI bounds in curved spacetime (see Ref. $[120, 121]$ for examples of explicit bounds in de Sitter and other spacetimes), these analyses typically use the QEIs in Minkowski spacetime, e.g. Eq. (4.13), with sampling times shorter than the shortest curvature scale involved in the problem to justify their use. QEIs are used in this manner to justify the expectation that the negative energy density, which are necessary for a traversable wormhole to exist as we have seen, cannot be larger than $C^2 \hbar / r_m^4$, where $r_m$ is the smallest scale in the problem and $C$ is a number which is typically taken to be $10^2$ or so to err on the conservative side. Then the argument to put restrictions on wormholes goes very roughly as follows. For a wormhole of size $r_0$ with typical curvature scale $1/r_0^2$ at the throat, the Einstein equations there tell us that $G/r_0^2 \lesssim C \hbar / r_m^4$. Thus, $r_m^2 / r_0 \lesssim C \ell_p$, where $\ell_p = \sqrt{\hbar G} = 1.6 \times 10^{-33} \text{ cm}$ is the Planck length. If the smallest length scale $r_m$ is comparable to the wormhole size $r_0$, then the wormhole cannot be much larger than the Planck length. To circumvent this conclusion, one needs to have $r_m \ll r_0$, typically by concentrating the negative energy in a very narrow region near the throat. Thus, it appears very difficult to construct (theoretically) a traversable wormhole, and that to do so would require fine-tuning of parameters. (See Refs. $[122, 123]$ for recent wormhole models which attempt to evade the argument of Ref. $[96]$. The apparent evasion of the constraints relies on evaluating the energy in one frame, and frame-independent statement of the QEIs reinstates the constraint $[122, 123]$.)

A similar argument was used in Ref. $[126]$ to demonstrate that the warp-drive spacetime as given in Ref. $[111]$ is incompatible with ordinary quantum field theory. This spacetime contains a bubble of nearly flat spacetime which could move faster than light. (Olum shows that superluminal travel of this sort, defined suitably, must involve violation of WEC $[127]$.) A spaceship could sit inside this bubble and thus move superluminally with respect to the metric outside the bubble. However, there is a sphere with a finite thickness where the energy must be negative to satisfy the Einstein equations. It turns out that the smallest scale involved is the thickness $\Delta$ of the sphere of negative energy. The situation is then similar to that of a wormhole with $r_m \sim r_0$. The negative energy must be concentrated on a sphere of thickness at most a few orders of magnitude larger than $\ell_p$, and the energy density there must be only a few orders of magnitude smaller than $\hbar / \ell_p^4$. The total negative energy needed is estimated to be much larger than the (positive) total mass of the visible universe for a bubble size large enough to fit in a spaceship.

C. Future problems

As we have seen, the ANEC is known to be violated even in ordinary quantum field theory in curved spacetime. It would be interesting to see if Yurtsever’s proposal of bounding the ANEC integral from below can be realized in a useful way. It is also interesting to find out whether it continues to be satisfied in spacetimes which are not close to Minkowski space if one enforces the semi-classical Einstein equations. It would also be useful to investigate the validity of the ANEC for interacting field theories (in Minkowski space to start with). Interestingly, Verch has shown that the ANEC holds in two-dimensional Minkowski spacetime for any interacting field theory with a mass gap $[128]$.

QEIs have so far been established primarily for free fields. Recently, Fewster and Hollands $[129]$ have shown that there are QEIs in two-dimensional conformal field theories in Minkowski space. It would be interesting to investigate QEIs in more general interacting field theories, considering the recent work in which a static negative energy region was constructed using a 2+1 dimensional interacting field theory $[130]$. One should also investigate a possible role the QEIs could play in recently proposed models of dark energy with the negative pressure exceeding the energy density $[131, 132, 133]$.

The QEIs place bounds only on the difference of the averaged energy density of the given state and that of the reference state. Strictly speaking, they have nothing to say about the averaged energy density of a single state, and they therefore do not restrict Casimir-type energy. The average energy density of the reference state was implicitly assumed to be at most of order $\hbar / r_m^4$, where $r_m$ was the smallest characteristic scale. This assumption is reasonable, but attempts to rigorously justify it are still in their infancy. It is possible to apply the QEIs to find bounds on Casimir-type energies $[134]$, and “absolute QEIs”, which do not need reference states, have been obtained recently $[135, 136]$.

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