The geometric lattice of embedded subsets

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Abstract

This work proposes an alternative approach to the so-called lattice of embedded subsets, which is included in the product of the subset and partition lattices of a finite set, and whose elements are pairs consisting of a subset and a partition where the former is a block of the latter. The lattice structure proposed in a recent contribution relies on ad-hoc definitions of both the join operator and the bottom element, while also including join-irreducible elements distinct from atoms. Conversely, here embedded subsets obtain through a closure operator defined over the product of the subset and partition lattices, where elements are generic pairs of a subset and a partition. Those such pairs that coincide with their closure are precisely embedded subsets, and since the Steinitz exchange axiom is also satisfied, what results is a geometric (hence atomic) lattice given by a simple matroid (or combinatorial geometry) included in the product of the subset and partition lattices (as the partition lattice itself is the polygon matroid defined on the edges of a complete graph). By focusing on its Möbius function, this geometric lattice of embedded subsets of a \( n \)-set is shown to be isomorphic to the lattice of partitions of a \( n + 1 \)-set.

Keywords: Closure operator, Geometric lattice, Simple matroid, Modular element, Möbius algebra, Lattice isomorphism.

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1 Introduction

For a finite set \( N = \{1, \ldots, n\} \), the framework developed in the sequel obtains from both the lattice \((2^N, \cap, \cup)\) of subsets of \( N \) and the lattice \((\mathcal{P}^N, \wedge, \vee)\) of partitions of \( N \). Generic subsets and partitions are denoted, respectively, by \( A, B \in 2^N \) and \( P, Q \in \mathcal{P}^N \). Recall that a partition \( P = \{A_1, \ldots, A_{|P|}\} \) is a family of (non-empty) pair-wise disjoint subsets \( A_1, \ldots, A_{|P|} \in 2^N \), called blocks, whose union is \( N \). As usual \([2, 18]\), partitions are (partially) ordered by coarsening \( \geq \), meaning that any \( P, Q \in \mathcal{P}^N \) satisfy \( P \geq Q \) if every block \( B \in Q \) is included in a block \( A \in P \), i.e. \( A \supseteq B \). Hence the bottom partition is \( P_\bot = \{\{1\}, \ldots, \{n\}\} \), while the top one is \( P^\top = \{N\} \).

Denote by \( X^N = 2^N \times \mathcal{P}^N \) the product of the subset and partition lattices, whose elements \((A, P), (B, Q) \in X^N\) are pairs consisting of a subset \( A, B \in 2^N \) and a partition \( P, Q \in \mathcal{P}^N \). Lattice \((X^N, \cap, \cup)\) is partially ordered by \( \sqsubseteq \) and naturally obtains through product of the meet, join and order relations for the
subset and partition lattices. That is to say, \((A, P) \cap (B, Q) = (A \cap B, P \cap Q)\) as well as \((A, P) \cup (B, Q) = (A \cup B, P \vee Q)\), while \((A, P) \sqsubseteq (B, Q) \iff A \supseteq B, P \geq Q\), where \(P \cap Q\) is the coarsest partition finer than both \(P, Q\) and \(P \vee Q\) is the finest partition coarser than both \(P, Q\). Thus the bottom element is \(\langle \emptyset, P_\perp \rangle\) while the top one is \((N, P^\top)\).

The existing lattice \(\mathfrak{E}(N)_\perp \sqcap, \sqcup, \sqcap_+\) of embedded subsets has elements given by those pairs \((A, P) \in X^N\) such that \(A \in P\), together with “an artificial bottom element \(\bot\), which is introduced for mathematical convenience and that may be considered as \(\langle \emptyset, P_\perp \rangle\)” [6, p. 481]. It is not hard to see that if \(A, B \in 2^N\) and \(P, Q \in \mathcal{P}^N\) satisfy \(A \in P\) as well as \(B \in Q\), then non-emptiness \(A \cap B \neq \emptyset\) entails \((A \cap B, P \cap Q) \in \mathfrak{E}(N)_\perp\), since \(A \cap B\) is a block of \(P \sqcap Q\). However, in general \(A \cup B\) is not a block of \(P \sqcup Q\), i.e. \((A \cup B, P \vee Q) \notin \mathfrak{E}(N)_\perp\). For this reason, the existing lattice of embedded subsets \(\mathfrak{E}(N)_\perp\) has the same meet \(\sqcap\) and order \(\sqcup\) applying to product lattice \(X^N\), but the join \(\sqcup_+\) is different from \(\sqcup\) as detailed below. The resulting structure [6, Prop. 2, p. 481] has \(n\) atoms of the form \((i, P_\perp), i \in N\) together with additional \(\binom{N}{n}(n-2)\) join-irreducible elements [2 p. 32], and thus the lattice is non-atomic (hence non-geometric) precisely because many elements cannot be decomposed as a join of atoms. In fact, the only embedded subsets \((A, P) \in \mathfrak{E}(N)_\perp\) admitting such a decomposition [6, Prop. 8, pp. 485-486] are those where partition \(P\) is the modular element [2, 15] of the partition lattice whose unique non-singleton block, if any, is precisely \(A\) [2, p. 71] [17, p. 339]. This is denoted by \(P = P^\perp\), with \(P^\perp = \{A, \{i_1\}, \ldots, \{i_{n-|A|}\}\}\) and \(\{i_1, \ldots, i_{n-|A|}\} = N \setminus A = A^c\).

Since \(P^\perp_i = P_\perp\) for all \(i \in N\) and \(P^\perp = P^\top\), there are \(2^n - n\) such modular partitions \(P^\perp\), \(\emptyset \neq A \in 2^N\). They play a key role in the approach proposed here, which relies on a closure operator \(cl : X^N \rightarrow X^N\) over poset \((X^N, \sqsubseteq)\), meaning [2 p. 167] that for all \((A, P), (B, Q) \in X^N\) the following hold:

\[
\begin{align*}
(i) & \quad cl(A, P) \sqsupseteq (A, P), \\
(ii) & \quad (A, P) \sqsubseteq (B, Q) \Rightarrow cl(A, P) \sqsubseteq cl(B, Q), \\
(iii) & \quad cl(cl(A, P)) = cl(A, P).
\end{align*}
\]

This closure is defined by \(cl(\emptyset, P) = (\emptyset, P)\) for all \(P \in \mathcal{P}^N\), while for \(\emptyset \neq A\)

\[
cl(A, P) = (\bar{A}, P \cup P^\perp)
\]

such that \(A \subseteq \bar{A} \in P \vee P^\perp\). In words, \(\bar{A}\) is the block of \(P \vee P^\perp\) satisfying \(\bar{A} \supseteq A\), and \(P^\perp\) is the modular partition where the only non-singleton block, if any, is \(A\). The remainder of this work is concerned with poset \((\mathfrak{E}^N, \sqsubseteq)\), with \(\mathfrak{E}^N = \{(A, P) : cl(A, P) = (A, P)\}\) consisting of those pairs that coincide with their closure. As \(\bot\) corresponds to \((\emptyset, P_\perp)\) (see above), \(\mathfrak{E}(N)_\perp \subset \mathfrak{E}^N \subset X^N\), and in particular \(\mathfrak{E}^N \setminus \mathfrak{E}(N)_\perp = \{(\emptyset, P) : P \neq P_\perp\}\). From the enumerative perspective, \(|\mathfrak{E}(N)_\perp| = 1 + \sum_{1 \leq k \leq n} kS_{n,k}\), where \(S_{n,k}\) is the Stirling number of the second kind, i.e. the number of partitions of a \(n\)-set into \(k\) blocks, while \(|\mathfrak{E}^N| = \sum_{1 \leq k \leq n} (k+1)S_{n,k} = B_{n+1}\), where \(B_{n+1}\) is the Bell number, i.e. the number of partitions of a \(n+1\)-set [2, 3, 12].

The join obtained through closure \(cl(\cdot)\) coincides, over \(\mathfrak{E}(N)_\perp\), with the existing one \(\sqcup_+\) (defined in [6, Prop. 2, p. 481]), i.e. for \((A, P), (B, Q) \in \mathfrak{E}(N)_\perp\),

\[
cl((A, P) \sqcup (B, Q)) = (A, P) \sqcup_+ (B, Q).
\]
However, the novel enlarged lattice $E^N$ of embedded subsets is substantially different from the existing one $C(N)_\perp$, as all its $n + \binom{n}{2} = \binom{n+1}{2}$ join-irreducible elements (apart from $(\emptyset, P_n)$) are atoms. Hence $E^N$ is a geometric lattice with the same atoms as product lattice $X^N$, and in fact the former reproduces within the latter a situation similar to the inclusion of partition lattice $(P^N, \land, \lor)$ within Boolean lattice $(2^{2N_2}, \cap, \cup)$, where $N_2 = \{ \{i, j\} : 1 \leq i < j \leq n \}$ is the $\binom{n}{2}$-set whose elements are all $A \in 2^N$ such that $|A| = 2$. Since $N_2$ is the edge set of the complete graph $K_N = (N, N_2)$ on vertex set $N$, partition lattice $(P^N, \land, \lor)$ obtains through the closure $cl : 2^{2N_2} \to 2^{2N_2}$ such that for every edge set $E \in 2^{2N_2}$ graph $G = (N, cl(E))$ obtains from graph $G = (N, E)$ by adding all edges within each component, i.e. each component of $G'$ is a clique (or maximal complete subgraph) \cite{2} p. 54]. Thus partition lattice $P^N$ corresponds to the family $\{ E : E = cl(E) \} \subset 2^{2N_2}$ of edge sets that coincide with their closure, with the resulting structure known as the polygon matroid defined on the edges of the complete graph on vertex set $N$ \cite{2} pp. 259, 274]. In the sequel, closure $cl : X^N \to X^N$ identifying the novel lattice $E^N$ of embedded subsets is shown to provide a simple matroid (or combinatorial geometry) \cite{2} pp. 52, 256].

The following Section 2 details how the closure defined by expression (1) above satisfies the Steinitz exchange axiom, thereby showing that the resulting lattice of embedded subsets is geometric. Section 3 identifies the modular elements of this lattice for a $n$-set, while also highlighting the isomorphism with the lattice of partitions of a $n + 1$-set. Section 4 is devoted to Möbius inversion, determining the Möbius function of the geometric lattice of embedded subsets of a $n$-set, which equals the Möbius function of the lattice of partitions of a $n + 1$-set. Since functions taking real values on embedded subsets were firstly considered in cooperative game theory (as “games in partition function form”), Section 5 focuses on the (free) vector space of such functions and on the associated Möbius algebra, with emphasis on the vector subspaces identified by those lattice functions whose Möbius inversion lives only on modular elements. Section 6 concludes the paper with some final remarks.

2 Partitions, atoms and closure

The rank $r : P^N \to \mathbb{Z}_+$ of partitions is $r(P) = n - |P|$, and atoms are immediately above the bottom element, with rank 1 \cite{2} p. 52]. Hence they consist of $n - 1$ blocks, out of which $n - 2$ are singletons while the remaining one is a pair. As already outlined, this means that $P^N$ has essentially the same $\binom{n}{2}$ atoms as $2^{2N_2}$. For $1 \leq i < j \leq n$, denote by $[ij]$ the atom of $P^N$ whose non-singleton block is pair $\{i, j\}$. A fundamental difference between Boolean lattice $(2^{2N_2}, \cap, \cup)$ and partition lattice $(P^N, \land, \lor)$ is in terms of join-decompositions \cite{2} Ch. II]. In fact, every $\{\{i,j\}_1, \ldots, \{i,j\}_|E|\} = E \in 2^{2N_2}$ admits a unique decomposition as a join of atoms, namely $E = \{i,j\}_1 \cup \cdots \cup \{i,j\}_|E|$, which is irredundant. Conversely, a generic partition $P \in P^N$ admits several such decompositions $P = [ij]_1 \lor \cdots \lor [ij]_k$ (with $k \geq r(P)$), most of which redundant, while the unique maximal one (in terms of inclusion) clearly is the join of all atoms $[ij]$ such that $[ij] \leq P$. In particular, $r(P)$ is the minimum number of atoms in a join-decomposition of $P$, thus any $P = [ij]_1 \lor \cdots \lor [ij]_r(P)$ is irredundant.

The classical examples of semimodular lattices \cite{15} come from sets endowed with a closure operator satisfying the Steinitz exchange axiom. Specifically, set
$N_2$ endowed with the closure operator $cl : 2^{N_2} \to 2^{N_2}$ introduced in Section 1 yields the semimodular lattice $\mathcal{P}^N$ of partitions. Subsets $E \in 2^{N_2}$ such that

$$cl(E) = E$$

are said to be closed, and the family of closed subsets forms a complete lattice under inclusion with meet given by intersection and join given by the closure of the union. That is, if $E, E'$ are closed, then $E \cap E'$ is closed as well, while $cl(E \cup E')$ is the smallest closed subset containing both $E$ and $E'$.

Furthermore, for all $E \in 2^{N_2}$ and $\{i, j\}, \{i, j\}' \in N_2$, the closure also satisfies the following Steinitz exchange axiom:

$$\text{if } \{i, j\} \not\in cl(E) \text{ and } \{i, j\} \in cl(E \cup \{i, j\}') \text{ then } \{i, j\}' \in cl(E \cup \{i, j\}).$$

Hence the family of closed subsets is a matroid (included in $2^{N_2}$) which, in particular, is said to be simple (or a combinatorial geometry) in that $cl(\emptyset) = \emptyset$ and $cl(\{i, j\}) = \{i, j\}$ for all $\{i, j\} \in N_2$.

As already outlined, a simple matroid also obtains as the novel enlarged lattice $\mathcal{E}^N$ of embedded subsets, whose elements are closed pairs, i.e. pairs $(A, P) \in \mathcal{X}^N$ that coincide with their closure $cl(A, P)$ as defined by expression (1) above. To see this, firstly consider the covering relation between closed pairs, denoted by $\supseteq$, i.e. for $(A, P), (B, Q) \in \mathcal{E}^N$, it holds $(A, P) \supseteq (B, Q)$ whenever $\{(B', Q') : (A, P) \supseteq (B', Q') \supseteq (B, Q), (B', Q') \in \mathcal{E}^N\} = \{(A, P), (B, Q)\}$ (see [2] p. 3). The general situation is when either $A \neq \emptyset \neq B$ or else $A = \emptyset = B$, in which case $(A, P) \supseteq (B, Q)$ attains whenever $P \triangleright Q$, where $\triangleright$ denotes the covering relation between partitions, i.e. $P$ obtains by merging two blocks of $Q$ into a unique block. In addition, it can also be $A \neq \emptyset = B$, and in this case $(A, P) \supseteq (B, Q)$ attains for $Q = \emptyset$. This means that $\mathcal{E}^N$ has $n + \binom{n}{2} = \binom{n+1}{2}$ atoms, in that the bottom pair $(\emptyset, P_{\perp})$ is covered by $n$ embedded subsets $(i, P_{\perp})$ for $i \in N$, and by further $\binom{n}{2}$ embedded subsets $(\emptyset, [ij])$ for $\{i, j\} \in N_2$.

Let $A_1 = \{(i, [i, \perp]) : i \in N\}$ and $A_2 = \{([0, [ij]] : \{i, j\} \in N_2\}$. Hence the set of atoms of $\mathcal{E}^N$ is $A = A_1 \cup A_2$. The closure operator defined by expression (1) above can now be regarded as a mapping $cl : 2^A \to 2^A$, with closed subsets of $A$ being the elements of lattice $\mathcal{E}^N$. In this view, consider a non-closed subset $S \in A$, with $S = S_1 \cup S_2, S_1 \in 2^{A_1}, S_2 \in 2^{A_2}$. Firstly focus on the simple case where $S_1 = \emptyset$, thus $S = S_2 = \{([0, [ij]] \cup \cdots \cup ([0, [ij]]_{S_2})\}$. Here the closure $cl(S)$ of $S$ obtains by adding all atoms $(\emptyset, [ij]) \in A_2$ such that $[ij]$ is finer than $\cup_{i} [ij], \cup \cdots \cup [ij]_{S_2}$. Hence for $S_1 = \emptyset$ the closure $cl : 2^A \to 2^A$ applied to partitions described above. For the general case $\{(i_1, P_{\perp}), \cdots (i_{\mid S_1}, P_{\perp})\} = S_1 \neq \emptyset \neq S_2 = \{([0, [ij]] \cup \cdots \cup ([0, [ij]]_{S_2})\},$ let $\{(i_1, [i_1, \perp]), \cdots [i_{\mid S_1}, \perp]) \} = A$ and $\cup \cdots \cup [ij]_{S_2} = P$, with $(A, P) \in \mathcal{X}^N$. In view of the closure applying to partitions, like in the previous case, if $[ij] \in P$, then $(\emptyset, [ij]) \in cl(S)$. At this point, the closure applying to embedded subsets defined in Section 1 works as follows. If $A \in P$ (entailing $P = P \vee P_{\perp}^\perp$ or, equivalently, $P \geq P_{\perp}^\perp$, of course), then the only atoms to be added are those just described, i.e. $cl(S) = S \cup \{([0, [ij]] : [ij] \leq P\}$, and $(A, P) \in \mathcal{E}^N$. Otherwise, for $k \leq |P|$, let $B_1, \cdots, B_k \in P$ be all the blocks of $P$ whose intersection with $A$ is non-empty. That is, $\{B : B \in P, B \cap A = \emptyset\} = \{B_1, \cdots, B_k\}$. Then, $A = B_1 \cup \cdots \cup B_k$ is a block of partition $P \vee P_{\perp}^\perp$, with proper inclusion $A \supset A$, and therefore $(A, P \vee P_{\perp}^\perp) \in \mathcal{E}^N$ is the minimal (in terms of $\supseteq$) embedded subset $(B, Q)$ satisfying $(B, Q) \supseteq (A, P)$. On the other hand, the smallest closed subset of $A$ containing $S$ is $cl(S) = \{(i, P_{\perp}) : i \in A\} \cup \{([0, [ij]] : [ij] \leq P \vee P_{\perp}^\perp\}$.
Proposition 1 Closure $c: 2^A \to 2^A$ satisfies the Steinitz exchange axiom.

Proof: For $(A, P) \in \mathcal{E}^N$, let atoms $(i, P_\perp), (i', P_\perp) \in A_1, (\emptyset, [ij]), (\emptyset, [ij']) \in A_2$ satisfy $(A, P) \nleq (i, P_\perp), (i', P_\perp), ([ij]), ([ij'])$, i.e., $i, i' \notin A$ and $[ij], [ij'] \notin P$.

In terms of subsets of atoms, let $S_{AP} = \{(i, P_\perp) : i \in A\} \cup \{(\emptyset, [ij]) : [ij] \leq P\}$, hence $S_{AP} = cl(S_{AP})$ and $(i, P_\perp), (i', P_\perp), ([\emptyset, [ij]), ([\emptyset, [ij']]) \notin S_{AP}$. Two cases $(i', P_\perp), ([\emptyset, [ij]) \in cl(S_{AP} \cup (i, P_\perp))$ and $(i', P_\perp), ([\emptyset, [ij']) \in cl(S_{AP} \cup ([\emptyset, [ij]))$ must be distinguished. In the first one, there is a block $A' \in P, A' \neq A$ such that $i \in A'$, and both $i' \in A'$ and $|A \cap \{i,j\}| = 1 = |A' \cap \{i',j\}|$ hold. Therefore, $cl(S_{AP} \cup (i', P_\perp)) = cl(S_{AP} \cup (i, P_\perp)) = cl(S_{AP} \cup ([\emptyset, [ij]))$, entailing that the exchange axiom is satisfied. Analogously, in the second case there is a block $A' \in P, A' \neq A$ such that $|A \cap \{i,j\}| = 1 = |A' \cap \{i',j\}|$, and both $i' \in A'$ and $|A \cap \{i,j\}| = 1 = |A' \cap \{i',j\}|$ hold. Hence again the exchange axiom is satisfied as $cl(S_{AP} \cup (i', P_\perp)) = cl(S_{AP} \cup ([\emptyset, [ij])$.  

Hence, $\mathcal{E}^N$ is a matroid and, in particular, a simple one, in that all $(n+1)_2$ sublists of atoms given by the empty set and the $(n+1)_2$ singletons are closed.

Proposition 2 Poset $(\mathcal{E}^N, \preceq)$ is a geometric lattice where the meet and join of any $(A, P), (B, Q) \in \mathcal{E}^N$ are $(A, P) \cap (B, Q)$ and $cl((A, P) \cup (B, Q))$, respectively.

Proof: Since $|\mathcal{E}^N| < \infty$, it only has to be verified [2, Prop. 2.28, p. 52] that for all $(A, P), (B, Q) \in \mathcal{E}^N$ the following hold:
- if $(A, P) \nleq (B, Q)$, then there is an atom $(i', P_\perp) \in A_1, (B, Q) \nleq (i', P_\perp)$ and/or $(\emptyset, [ij]) \in A_2, (B, Q) \nleq (\emptyset, [ij])$ such that $(A, B) = cl((B, Q) \cup (i', P_\perp))$ and/or $(A, B) = cl((B, Q) \cup ([\emptyset, [ij]))$.
- if $(A, B) = cl((B, Q) \cup (i', P_\perp))$ and/or $(A, B) = cl((B, Q) \cup ([\emptyset, [ij]))$, for some atom $(i', P_\perp) \in A_1, (B, Q) \nleq (i', P_\perp)$ and/or $(\emptyset, [ij]) \in A_1, (B, Q) \nleq ([\emptyset, [ij])$, then $(A, P) \nleq (B, Q)$.

Both conditions are trivially satisfied if $(B, Q)$ is the bottom element, hence let $(B, Q) \neq (\emptyset, P_\perp)$. Then $(A, P) \nleq (B, Q)$ attains for $P \supseteq A \supseteq B$. If $A = B$, then $(A, P) = cl((B, Q) \cup ([\emptyset, [ij]))$ for all atoms $(\emptyset, [ij]) \in A_1, (B, Q) \nleq ([\emptyset, [ij]))$ such that $[i,j] \cap B = \emptyset$ (where of course $Q \nleq [ij]$), while proper inclusion $A \supset B$ entails that for some block $B' \in Q, B' \neq B$ the covering embedded subset $(A, P)$ obtains as $(A, P) = cl((B, Q) \cup ([\emptyset, [ij])) = cl((B, Q) \cup ([\emptyset, [ij]))$ for all atoms $(i', P_\perp) \in A_1, (B, Q) \nleq (i', P_\perp)$, $(\emptyset, [ij]) \in A_2, (B, Q) \nleq ([\emptyset, [ij]))$ such that $i' \in B'$ and $|B \cap \{i,j\}| = 1 = |B' \cap \{i,j\}|$. Accordingly, $A = B \cup B'$.

Turning to the second condition, for all $(i', P_\perp) \in A_1, (B, Q) \nleq (i', P_\perp)$, $(\emptyset, [ij]) \in A_2, (B, Q) \nleq ([\emptyset, [ij]))$ it is easily checked that both $(B, Q) \cup (i', P_\perp))$ and $cl((B, Q) \cup ([\emptyset, [ij]))$ cover $(B, Q)$. In fact, the general case is where there is a block $B' \in Q, B' \neq B$ such that $i' \in B'$ and $|B' \cap \{i,j\}| = 1 = |B \cap \{i,j\}|$, entailing $cl((B, Q) \cup ([\emptyset, [ij])) = cl((B, Q) \cup ([\emptyset, [ij])) = (B \cup B', Q \cup P_{B,B'})$, with $Q \cup P_{B,B'}$ obtained by merging blocks $B$ and $B'$ in $Q$, hence $Q \cup P_{B,B'} \supseteq Q$. Yet, like for the first condition, if $B \cap \{i,j\} = \emptyset$, then there are blocks $B', B'' \in Q$, with $B' \neq B \neq B''$, such that $|B \cap \{i,j\}| = 1 = |B'' \cap \{i,j\}|$. Accordingly, $cl((B, Q) \cup ([\emptyset, [ij])) = (B, Q')$, where $Q'$ obtains by merging blocks $B', B''$ in $Q$, i.e. $Q' \supset Q$.  

Having recognized the lattice structure characterizing $\mathcal{E}^N$, it may now be readily seen that the Jordan-Dedekind chain condition [2, p. 30] is satisfied, and the rank function $r: \mathcal{E}^N \to \mathbb{Z}_+$ is $r(A, P) = r(P) + |\{i \mid |A_i|, 1\}$. In fact, for any $(A, P), (B, Q) \in \mathcal{E}^N$ such that $(A, P) \nleq (B, Q)$ (that is, $(A, P) \nleq (B, Q)$,
From Section 1, all maximal chains \((B', Q')_0, (B', Q')_1, \ldots, (B', Q')_k\) from 
\((B, Q) = (B', Q')_0\) to \((A, P) = (B', Q')_k\) (i.e. \((B', Q')_k+1 \supseteq (B', Q')_k\) for 
\(0 \leq k' < k\) have same length \(k\), in that \((B', Q')_k+1 = cl((B', Q')_k \sqcup (i, P_\perp))\) and/or \((B', Q')_k+1 = cl((B', Q')_k \sqcup (\emptyset, [ij]))\) with atoms \((i, P_\perp), (\emptyset, [ij])\) such 
that \((B', Q')_k+1 \subseteq (i, P_\perp), (\emptyset, [ij]))\) while \((B', Q')_k \nsubseteq (i, P_\perp), (\emptyset, [ij]))\).

\[|A| - |Q^A| + n - |Q| + |Q^A| = n - |Q| + |A| \] as desired.

For the ‘only if’ part, it is not difficult to check that, just like the 
rank of partitions, the rank of embedded subsets is submodular, that is to say 
r\((A, P) + r(B, Q) \geq r(cl((A, P) \sqcup (B, Q))) + r((A, P) \cap (B, Q)), \) with strict 
equality whenever \((A, P), (B, Q)\) is not a modular pair \([2] \text{ p. } 58\) of lattice 
elements.

Let \(N_+ = \{1, \ldots, n, n+1\}\), i.e. \(N = N \cup \{n + 1\}\) and denote by \(P^+ \in \mathcal{P}^N\) 
the lattice of partitions of \(N_+\), with generic element denoted by \(P^+ \in \mathcal{P}^N\). 
From Section 1, \(|\mathcal{E}^N| = \mathcal{B}_{n+1} = \mathcal{P}^N\). In fact, there is a lattice isomorphism 
\(f : \mathcal{E}^N \to \mathcal{P}^N\), meaning that for all \((A, P), (B, Q) \in \mathcal{E}^N\) the following holds:

\[f((A, P) \cap (B, Q)) = f(A, P) \land f(B, Q),\]
\[f(cl((A, P) \sqcup (B, Q))) = f(A, P) \lor f(B, Q).\]

To see this, consider that any partition \(\{A_1, \ldots, A_{|P|}\} = P \in \mathcal{P}^N\) has associated 
\(|P|+1\) partitions \(P^+_1, \ldots, P^+_{|P|}, P^+_{|P|+1} \in \mathcal{P}^N\), all inducing the same partition \(P\)
of \( N \). The first \(|P|\) ones \( P_1^+, \ldots, P_{|P|}^+ \) obtain each by placing additional element \( n + 1 \) into block \( A_k \in P, 1 \leq k \leq |P| \), while the last one \( P_{|P|}^+ \) obtains by placing \( n + 1 \) into a singleton block \( \{n + 1\} \in P_{|P|+1}^+ \). Of course, the former are pair-wise incomparable, i.e. \( P_k^+ \not\preceq P_k^+ \not\preceq P_{k'}^+ \) for \( 1 \leq k < k' \leq |P| \), while the latter is strictly finer than all the others: \( P_k^+ > P_{|P|+1}^+, 1 \leq k \leq |P| \). The same applies to embedded subsets \((A_1, P), \ldots, (A_{|P|}, P), (\emptyset, P) \) \( \in \mathcal{E}_N \), in that \((A_k, P) \not\supseteq (A_{k'}, P) \not\supseteq (A_k, P)\) for \( 1 \leq k < k' \leq |P| \) as well as \((A_k, P) \supseteq (\emptyset, P)\), \( 1 \leq k \leq |P| \). In other terms, letting \( P = \{A, B_1, \ldots, B_{|P|-1}\} \) \( \in \mathcal{P}^+ \) for all \((A, P) \in \mathcal{E}_N\) such that \( A \neq \emptyset \), an isomorphism \( f : \mathcal{E}_N \rightarrow \mathcal{P}^+ \) is defined by
\[
 f(A, P) = P^+ \text{ such that } P^+ = \{A \cup \{n + 1\}, B_1, \ldots, B_{|P|-1}\}.
\]

On the other hand, for every \( P = \{B_1, \ldots, B_{|P|}\} \) \( \in \mathcal{P}^+ \) and \((\emptyset, P) \in \mathcal{E}_N\),
\[
f(\emptyset, P) = P^+ \text{ such that } P^+ = \{\{n + 1\}, B_1, \ldots, B_{|P|}\}.
\]

Evidently, in this way the \( 2n+1 - (n+1) \) modular elements of \( \mathcal{P}^+ \) bijectively correspond to the \( 2^n - n + 2^n - 1 = 2^{n+1} - (n+1) \) modular elements of \( \mathcal{E}_N \).

### 4 Möbius inversion

Möbius inversion applies to locally finite posets \[13\]. In particular, for a finite lattice \((L, \wedge, \vee)\) ordered by \( \geq \), with generic elements \( x, y, z \in L \) and bottom element \( x_\bot \), any lattice function \( f : L \rightarrow \mathbb{R} \) has Möbius inversion (from below)
\[
 \mu^f : L \rightarrow \mathbb{R} \text{ given by } \mu^f(x) = \sum_{y \leq x} \mu_L(y, x)f(y), \text{ where } \mu_L \text{ is the Möbius function of } L, \text{ defined recursively on ordered pairs } (y, x) \in L \times L \text{ by } \mu_L(y, x) = 1 \text{ if } y = x \text{ as well as } \mu_L(y, x) = -\sum_{y \leq z \leq x} \mu_L(y, z) \text{ if } y < x \text{ (i.e. } y \leq x \text{ and } y \neq x), \text{ while } \mu_L(y, x) = 0 \text{ if } y \not\leq x \text{.}
\]

Accordingly, \( f \) and \( \mu^f \) are related by
\[
 f(x) = \sum_y \mu^f(y) \text{ for all } x \in L. \text{ The Möbius function of the subset lattice is } \\
 \mu_\mathcal{P}^+ (B, A) = (-1)^{|A \setminus B|}, \text{ with } B \subseteq A. \text{ Concerning the Möbius function of } \mathcal{P}^+, \text{ given any two partitions } P, Q \in \mathcal{P}^+, \text{ if } Q \prec P = \{A_1, \ldots, A_m\}, \text{ then for every } A \in P \text{ there are } B_1, \ldots, B_{k_A} \in Q \text{ such that } A = B_1 \cup \cdots \cup B_{k_A}, \text{ with } k_A > 1 \text{ for at least one } A \in P. \text{ Segment (or interval) } [Q, P] = \{P' : Q \preceq P' \preceq P\} \text{ is isomorphic to product } \times_{A \in \mathcal{P}^+} A^{k_A}, \text{ where } A^{k_A} \text{ denotes the lattice of partitions of a } k\text{-set. Let } c_{Q}^{P} = \{|A : A \in P, k_A = k|, 1 \leq k \leq n \text{ be the number of blocks of } P \text{ obtained as the union of } k \text{ blocks of } Q, \text{ where } c_{Q}^{P} = (c_{Q}^{P_{1}}, \ldots, c_{Q}^{P_{n}}) \in \mathbb{Z}_{\geq}^n \text{ is the class (or type) of segment } [Q, P]. \text{ Then } [13] \text{ pp. } 359-360, \\
\mu_{Q}^{P_{_{X}}} (Q, P) = (-1)^{-n^2 + \sum_{1 \leq k \leq n} c_{Q}^{P_{_{k}}}} \prod_{1 < k < n} (k!)^{c_{Q}^{P_{_{k+1}}}}, \quad (2)
\]

This result obtains since the Möbius function is multiplicative \[2\] pp. 147-149 \[3\] p. 560 \[9\] Theorem 13.6 \[17\] p. 305, which also entails here that for any two pairs \((A, P), (B, Q) \in X^N \) the Möbius function \( \mu_{X}^{P_{_{X}}} \) of \( X^N \) is given by
\[
 \mu_{X}^{P_{_{X}}} ((B, Q), (A, P)) = \mu_{Q}^{P_{_{X}}} (B, A) \mu_{P_{_{X}}}^{P_{_{X}}} (Q, P).
\]

Another fundamental result in the present setting concerns the link between the Möbius function and a closure operator \[2\] p. 167 \[3\] pp. 561-562 \[8\] p. 179 \[13\] p. 349 \[17\] Ex. 84, p. 425. For closed pairs \((A, P), (B, Q) \in \mathcal{E}_N \), and
denoting generic pairs by \((A', P') \in X^N\), the Möbius function \(\mu_{E^N}\) of \(E^N\) is

\[
\mu_{E^N}((B, Q), (A, P)) = \sum_{\{(A', P') \in X^N : c((A', P') = (A, P)\}} \mu_{X^N}((B, Q), (A', P')).
\]

The Möbius function \(\mu_{E^N}\) of \(E^N\) is now determined through recursion following \cite{13} pp. 359-360. Of course, this leads to re-obtain the isomorphism observed in Section 3. Firstly some further isomorphisms concerning segments \([(B, Q), (A, P)] \subseteq E^N\) have to be highlighted.

Extending the notation introduced above, for \(0 \leq m \leq n\) denote by \(\mathcal{P}^m\) the lattice of partitions of a \(m\)-set. Also let \(E^m\) denote the lattice of embedded subsets of a \(m\)-set, hence \(E^0 = \{\emptyset, \emptyset\}\) consists of only one element. Similarly, recall that the recursions for the Stirling numbers of the second kind and for the Bell numbers \cite{2} pp. 89, 92 rely on the convention that the number of \(\emptyset\) partitions of the empty set is 1, i.e. \(\mathcal{P}^0 = \{\emptyset\}\). Finally, set \(\mu_{\mathcal{P}^m} = \mu_{\mathcal{P}^m}(\perp, \top)\) and \(\mu_{E^m} = \mu_{E^m}(\perp, \top)\), where \(\perp\) and \(\top\) are the bottom and top elements of the corresponding lattice, with \(\mu_{\mathcal{P}^m} = (-1)^{m-1}(m-1)!\) from \cite{13} Prop. 3 p. 359, and \(\mu_{E^m} = 1 = \mu_{E^m}\) (see \cite{17} p. 319); also, \(-1! = -1\) \cite{7}.

Now consider any non-empty segment \([(B, Q), (A, P)] \subseteq E^N\), that is to say \((A, P) \supseteq (B, Q)\). Firstly, if \(B = A\), then \([(B, Q), (A, P)] \cong [Q^A, P^A]\), where \(\cong\) denotes isomorphism and \([Q^A, P^A] \subseteq \mathcal{P}^{|A|}\), hence \(\mu_{E^N}((B, Q), (A, P))\) is given by expression (2), i.e.

\[
\mu_{E^N}((B, Q), (A, P)) = \mu_{\mathcal{P}^{|A|}}([Q^A, P^A]) = (-1)^{|A|+|\Sigma_{1 \leq k \leq |A|} e_q^{Q^A, P^A}} \prod_{1 < k < |A|} (k!)^{e_{Q^A, P^A}},
\]

with \(e_{Q^A, P^A} \in \mathbb{Z}_{|A|}\) being the class of segment \([Q^A, P^A]\).

Secondly, consider the case where \(\emptyset \neq B \subset A\). Letting \(B = B_1\) partition \(Q\) has form \(Q = \{B_1, \ldots, B_{|A|}\} \cup Q^A\), where \(\{B_1, \ldots, B_{|A|}\} = Q^A\). It is readily recognized that for any atom \((i, P) \in A_1\) of \(E^N\) (see Section 2), segment \([(i, P), (N, P^+)] \subseteq E^N\) is isomorphic to \(\mathcal{P}^n\) (see also \cite{6} Prop. 2 (viii), p. 482). Analogously, denoting by \([A]\) the top element of \(\mathcal{P}^{|A|}\), segment \([(B, Q^A), (A, [A])] \subseteq E^{|Q^A|}\) is isomorphic to \(\mathcal{P}^{|A|}\). Therefore,

\[
\mu_{E^N}((B, Q), (A, P)) = \mu_{\mathcal{P}^{|A|}}([Q^A, P^A]) = (-1)^{|Q^A|-1}(|Q^A| - 1)! \mu_{\mathcal{P}^{|A|}}([Q^A, P^A]).
\]

The third and most relevant case is when \(\emptyset = B \subset A\). Like in the previous case, partition \(Q\) has form \(Q^A \cup Q^A\), and thus \([(\emptyset, Q^A), (A, [A])] \cong [Q^A, P^A]\). Accordingly, \([(\emptyset, Q), (A, P)] \cong [Q^A, P^A]\), entailing

\[
\mu_{E^N}((B, Q), (A, P)) = \mu_{\mathcal{P}^{|A|}}([Q^A, P^A]) = \mu_{E^{|Q^A|}}([Q^A, P^A]).
\]

Hence the Möbius function \(\mu_{E^N} : E^N \times E^N \to \mathbb{Z}\) is fully determined once \(\mu_{E^N}((\emptyset, P), (N, P^+)) = \mu_{E^N}\) is known. To this end, focus on the dual atoms of \(E^N\), i.e. closed pairs covered by the top element \((N, P^+).\) There are \(2^n - 2\) such dual atoms with form \((A, \{A, A'\})\) for \(\emptyset \neq A \neq N,\) plus a further one with form \((\emptyset, P^+),\) where \(2^n - 1\) is precisely the number of dual atoms of \(\mathcal{P}^{n+1}\). Also
recall that from Weisner’s theorem [3, Th. 13.9] [13, Cor. (b), p. 351] [17, Cor. 3.9.3, p. 314], for any \((A, P), (B, Q) \in \mathcal{E}^N\) with \((N, P^\top) \subseteq (A, B)\)
\[
\sum_{\substack{(A', P') \in \mathcal{E}^N \\ (A', P') \cap (A, P) = (B, Q)}} \mu_{\mathcal{E}^N}((A', P'), (N, P^\top)) = 0. \tag{4}
\]

**Proposition 4** The M"obius function \(\mu_{\mathcal{E}^n}\) of the lattice of embedded subsets of an n-set is \(\mu_{\mathcal{E}^n} = \mu_{\mathcal{P}^{n+1}} = (-1)^n n!\).

**Proof:** In expression (4), let \((B, Q) = (\emptyset, P_\bot)\) and choose \((A, P)\) to be a dual atom of the form \((N \setminus i, \{N \setminus i, i\})\) for \(i \in N\). Then, \((A', P') \cap (A, P) = (\emptyset, P_\bot)\) entails that \((A', P') \in \mathcal{A}\) is an atom and, in particular, either \((A', P') \in \mathcal{A}_2\) has form \((\emptyset, [ij])\) with \(j \in N \setminus i\), or else \((A', P') \in \mathcal{A}_1\) has form \((i, P_\bot)\). Accordingly,
\[
\mu_{\mathcal{E}^n} = - \sum_{(A', P') \in \mathcal{A}} \mu_{\mathcal{E}^N}((A', P'), (N, P^\top)),
\]
where the sum ranges over the \(n\) atoms just described. From the above isomorphisms, \([i, P_\bot], (N, P^\top) \cong \mathcal{P}^n\) as well as \([j, (ij)], (N, P^\top) \cong \mathcal{E}^{n-1}\). Thus \(\mu_{\mathcal{E}^n} = - (\mu_{\mathcal{P}^n} + (n-1)\mu_{\mathcal{E}^{n-1}})\). Since \(\mu_\emptyset = \mu_{\mathcal{P}^0} = 1\) and \(\mu_{\mathcal{E}^1} = \mu_{\mathcal{P}^1} = -1\), the desired conclusion \(\mu_{\mathcal{E}^n} = - (\mu_{\mathcal{P}^n} + (n-1)\mu_{\mathcal{P}^n}) = -n\mu_{\mathcal{P}^n} = \mu_{\mathcal{P}^{n+1}}\) follows. \(\blacksquare\)

It may be noted that this route aims at following tightly [13 pp. 359-360]. However, if \((A, P)\) is (straightforwardly) chosen to be the dual atom \((\emptyset, P^\top)\) (rather than \((N \setminus i, \{N \setminus i, i\})\)), then the above argument immediately yields \(\mu_{\mathcal{E}^n} = -n\mu_{\mathcal{P}^n}\), as the \(n\) involved atoms are all the elements of \(\mathcal{A}_1\), i.e. \((j, P_\bot), j \in N\).

Proposition 4 confirms that the lattice \(\mathcal{E}^n\) of embedded subsets of an n-set is isomorphic to the lattice \(\mathcal{P}^{n+1}\) of partitions of a \(n+1\)-set. In particular, the characteristic polynomial [2 p. 155], [17 pp. 319-320], [20] is
\[
\chi_{\mathcal{E}^n}(x) = \chi_{\mathcal{P}^{n+1}}(x) = (x-1)(x-2) \cdots (x-n),
\]
with the Whitney numbers of the first kind given by the Stirling numbers of the first kind [1, 2 p. 88], i.e.
\[
\sum_{(A, P) \in \mathcal{E}^N \atop r(A, P) = k} \mu_{\mathcal{E}^N}((\emptyset, P_\bot), (A, P)) = s_{n+1, n+1-k} \text{ for } 0 \leq k \leq n,
\]
and the Whitney numbers of \(\mathcal{E}^n\) of the second kind given by the Stirling numbers of the second kind, i.e.
\[
\left|\{(A, P) : (A, P) \in \mathcal{E}^N, r(A, P) = k\}\right| = S_{n+1, n+1-k} \text{ for } 0 \leq k \leq n.
\]

Furthermore, \(\mathcal{E}^n\) is a supersolvable lattice, since in view of Proposition 3 it admits \(\frac{(n+1)!}{2}\) maximal chains from the bottom element to the top one consisting of modular elements [16 Ex. 2.6, p. 104] [17 Ex. 3.14.4, p. 339].

It seems also worth pointing out (again, see Section 1) that while in the non-atomic lattice \(\mathcal{E}(N)_{\bot}\) of embedded subsets only the \(2^n - 1\) modular elements \((A, P^\top_{\bot}), A \neq \emptyset\) admit a decomposition as a join of the \(n\) available atoms \((i, P_\bot)\)
(see [6] Prop. 8, p. 486), in the geometric lattice $\mathcal{E}^N$ all elements admit a decomposition as a join of the $\binom{n+1}{2}$ atoms. Accordingly [3 Theorem 13.7],

$$
\mu_{\mathcal{C}(N)_+}((\emptyset, P_\perp), (A, P)) = \begin{cases} 
(-1)^{|A|} & \text{if } P = P_\perp^A \\
0 & \text{otherwise}
\end{cases}
$$

while

$$
\mu_{\mathcal{E}^N}((\emptyset, P_\perp), (A, P)) = (-1)^{|A||A|!} \prod_{B \in P^{\perp}} (-1)^{|B|-1}(|B| - 1)!
$$

and $\mu_{\mathcal{E}^N}((\emptyset, P_\perp), (A, P))$ equals the difference between the number of subsets $\{(A', P')_1, \ldots, (A', P')_{|S|}\} = S \subseteq A$ of atoms of $\mathcal{E}^N$ such that $|S|$ is even as well as $cl((A', P')_1 \cup \cdots \cup (A', P')_{|S|}) = (A, P)$ and the number of subsets $\{(A', P')_1, \ldots, (A', P')_{|S'|}\} = S' \subseteq A$ of atoms such that $|S'|$ is odd as well as $cl((A', P')_1 \cup \cdots \cup (A', P')_{|S'|}) = (A, P)$ [9 p. 562].

Finally, it seems worth noticing that

$$
(-1)^{r(N, P^\perp)} \mu_{\mathcal{E}^N}((\emptyset, P_\perp), (N, P^\perp)) = (-1)^n(-1)^n n! = (-1)^{2n} n! > 0
$$

for all $n \geq 0$, which is consistent with [5 Cor. 13.10].

## 5 Möbius algebra, vector subspaces and games

The purpose of this section is to highlight that all cooperative games are in fact elements of subspaces of the free vector space $V(\mathcal{E}^N)$ over $\mathbb{R}$ generated by $\mathcal{E}^N$ [11 p. 143], [2 p. 181]. That is, the elements of $V(\mathcal{E}^N) \subset \mathbb{R}^{2^n+1}$ are real-valued lattice functions $g : \mathcal{E}^N \rightarrow \mathbb{R}$.

Recall that, for $N$ regarded as a player set, cooperative game theory deals with three types of lattice functions:

(a) coalitional games [14] or set functions $v : 2^N \rightarrow \mathbb{R}_+, v(\emptyset) = 0$,

(b) global games [1] or partition functions $h : \mathcal{P}^N \rightarrow \mathbb{R}_+, h(P_\perp) = 0$,

(c) games in partition function form [6] [10] [13], hereafter referred to as PFF games for short, or functions $f : \mathcal{C}(N)_+ \rightarrow \mathbb{R}_+, f(\emptyset, P_\perp) = 0$.

A further fourth type of cooperative games may be defined, which incorporates all previous ones (a), (b) and (c), namely

(d) extended PFF games, or those elements $g \in V(\mathcal{E}^N)$ of the free vector space under concern such that $g : \mathcal{E}^N \rightarrow \mathbb{R}_+, g(\emptyset, P_\perp) = 0$.

Together with the Möbius function, the other element of the incidence algebra (of locally finite posets [2 p. 138] [12 p. 344]) playing a fundamental role in cooperative game theory is the zeta function $\zeta$, as it provides the so-called ‘unanimity games’ [14]. In the incidence algebra of $\mathcal{E}^N$ over $\mathbb{R}$, the zeta function $\zeta_{\mathcal{E}^N} : \mathcal{E}^N \times \mathcal{E}^N \rightarrow \{0, 1\}$ is defined on ordered pairs of embedded subsets by

$$
\zeta_{\mathcal{E}^N}((B, Q), (A, P)) = \begin{cases} 
1 & \text{if } (A, P) \supseteq (B, Q), \\
0 & \text{otherwise}.
\end{cases}
$$
In these terms, unanimity (coalitional) games $u_B, \emptyset \neq B \in 2^N$ are defined by $u_B(A) = \zeta_N(B, A)$ for all subsets (or coalitions) $A \in 2^N$, where $\zeta_N$ is the zeta function in the incidence algebra of $2^N$. They are linearly independent elements of the vector space of coalitional games and thus form a basis. This fact is essential since the very beginning of cooperative game theory.

In the present setting, for $(B, Q) \in \mathcal{E}^N$, let $\zeta_{B, Q}^N(A, P) = \zeta_N((B, Q), (A, P))$ for all $(A, P) \in \mathcal{E}^N$. Then, $\{\zeta_{B, Q}^N : (B, Q) \in \mathcal{E}^N\}$ is a basis of $V(\mathcal{E}^N)$. Also denote by $V(\mathcal{E}^N)_\perp \subset \mathbb{R}_{++}^{2n+1}$ the subspace of extended PFF games. Furthermore, let the Möbius inversion of $g$ be $\mu^g : \mathcal{E}^N \to \mathbb{R}$, i.e.

$$\mu^g(A, P) = \sum_{(B, Q) \subseteq (A, P)} \mu_{\mathcal{E}^N}((B, Q), (A, P))g(B, Q) = g(A, P) - \sum_{(B, Q) \subseteq (A, P)} \mu^g(B, Q),$$

with $\mu^g(\emptyset, P_\perp) = g(\emptyset, P_\perp) (= 0$ if $g \in V(\mathcal{E}^N)_\perp)$, and where

$$\mu_{B, Q}^N(A, P) = \begin{cases} 1 & \text{if } (A, P) = (B, Q) \\ 0 & \text{otherwise} \end{cases}$$

for every element $\zeta_{B, Q}$ of the chosen basis, thereby providing the canonical basis (or Kronecker delta) elements (see [2, Proposition 4.50, p. 181]), while crucially noticing that the alternative definition $\zeta_{B, Q}^N(A, P) = \zeta_N((A, P), (B, Q))$ applies there. Specifically, let $\epsilon_{B, Q} \in V(\mathcal{E}^N)$ be the $\mathbb{B}_{n+1}$-vector with entries indexed by elements $(A, P) \in \mathcal{E}^N$ and such that its unique non-zero entry is the $(B, Q)$ one, which equals 1. Then the Möbius algebra of $\mathcal{E}^N$ over $\mathbb{R}$ (denoted by Möb$(\mathcal{E}^N)$, see [1, p. 143] [2, p. 184]) is defined by the system of orthogonal idempotents $\{\epsilon_{B, Q} : (B, Q) \in \mathcal{E}^N\}$ and linear extension to all of $V(\mathcal{E}^N)$ through multiplication

$$\epsilon_{B, Q} \cdot \epsilon_{A, P} = \begin{cases} 1 & \text{if } (A, P) = (B, Q), \\ 0 & \text{otherwise}. \end{cases}$$

In terms of the above basis $\{\zeta_{B, Q}^N : (B, Q) \in \mathcal{E}^N\}$ of $V(\mathcal{E}^N)$, this multiplication yields $\zeta_{B, Q}^N \cdot \zeta_{A, P}^N = \zeta_N((A, P) \cup (B, Q))$ (see [3, Prop. 4.55, p. 184], crucially noticing again that since the alternative definition of the basis detailed above applies there, then $\zeta_{B, Q}^N \cdot \zeta_{A, P}^N = \zeta_N((A, P) \cup (B, Q))$). It is well known that Möbius inversion $\mu^g$ quantifies precisely the coefficients identifying functions $g \in V(\mathcal{E}^N)$ as linear combinations of the basis elements, that is

$$g(A, P) = \sum_{(B, Q) \in \mathcal{E}^N} \mu^g(B, Q)\zeta_{B, Q}^N(A, P) \text{ for all } (A, P) \in \mathcal{E}^N. \hspace{1cm} (5)$$

If $Y \subset \mathcal{E}^N$ and $\mu^g(A, P) = 0$ for all $(A, P) \in \mathcal{E}^N \setminus Y$, then Möbius inversion $\mu^g$ may be said to live only $Y$. In these terms, the following observations are immediate:

(i) (traditional) PFF games are elements of the subspace $\tilde{V}(\mathcal{E}^N) \subset \mathbb{R}_{++}^{B_{n+1} - B_n}$ of $V(\mathcal{E}^N)$ consisting of those extended PFF games $g \in V(\mathcal{E}^N)$ whose Möbius inversion lives only on embedded subsets $(A, P)$ such that $A \neq \emptyset$, i.e. $A = \emptyset \Rightarrow \mu^g(A, P) = 0$, entailing $g(\emptyset, P) = 0$ for all $P \in \mathcal{P}^N$;
(iii) global games are elements of the subspace $\mathcal{V}(\mathcal{E}^N) \subset \mathbb{R}^{B_n-1}$ consisting of extended PFF games $g \in V(\mathcal{E}^N)$ whose Möbius inversion lives only on embedded subsets $(A, P)$ such that $A = \emptyset$ and $P \neq P_\perp$, entailing $g(A, P) = g(\emptyset, P)$ for all $A \in P$ (thereby somehow strengthening the idea of a global level of satisfaction, common to all players, see [4]);

(iii) coalitional games are elements of the subspace $V(\mathcal{E}^N) \subset \mathbb{R}^{2^n-1}$ consisting of those $g \in V(\mathcal{E}^N)$ whose Möbius inversion lives only on the $2^n - 1$ modular elements $(A, P_A)$ of $\mathcal{E}^N$ defined in Section 3 such that $A \neq \emptyset$, yielding $g(\emptyset, P) = 0$ for all $P \in \mathcal{P}^N$ as well as $g(A, P_A) = g(A, P_A^A)$ for all $P \in \mathcal{P}^N$ such that $A \in P$.

Another subspace of $V(\mathcal{E}^N)$ appearing in cooperative game theory is given by the so-called “additively separable” global games (or partition functions, see [4, 5]), namely those $h : \mathcal{P}^N \to \mathbb{R}_+$ such that $h(P) = h_v(P) := \sum_{A \in P} v(A)$ for some coalitional game (or set function) $v : 2^N \to \mathbb{R}_+$. The Möbius inversion of these partition functions lives only on the $2^n - n$ modular elements $P_A$ of the partition lattice, see Section 1. This is detailed hereafter (see also [4] Prop. 4.4 and Appx. [5, Prop. 3.3] and [11, Prop. 7]).

**Proposition 5** If $h = h_v$ and $\sum_{i \in N} v(i) = h_v(P_\perp) > 0$, then a continuum of set functions $w : 2^N \to \mathbb{R}_+, w \neq v$ satisfies $h = h_w$.

**Proof:** By direct substitution, the Möbius inversion $\mu v : \mathcal{P}^N \to \mathbb{R}$ of partition function $h_v$ satisfies

$$\mu v(P) = \sum_{A \in P} \sum_{B \subseteq A} v(B) \sum_{Q \subseteq P : B \in Q} \mu v(Q, P) \text{ for all } P \in \mathcal{P}^N.$$

If $P \neq P_\perp$, then the recursion for Möbius function $\mu v : \mathcal{P}^N \times \mathcal{P}^N \to \mathbb{R}$ yields

$$\sum_{Q \subseteq P : A \in Q} \mu v(Q, P) = \sum_{P_A \leq Q \subseteq P} \mu v(Q, P) = 0,$$

and the same for proper subsets $B \subset A$. Möbius inversion $\mu v$ thus lives only on modular partitions, where it obtains recursively by $\mu v(P_\perp) = \sum_{i \in N} v(i)$ and $\mu v(P_A) = \mu v(A)$ for $1 < |A| \leq n$, with $P_\perp = P^\perp$. Hence any $w \neq v$ satisfying $\sum_{i \in N} v(i) = \sum_{i \in N} w(i)$ and $\mu v(A) = \mu w(A)$ for all $A \in 2^N$ such that $|A| > 1$ also additively separates $h$, i.e. $h_v = h_w$. ■

If $\sum_{i \in N} v(i) > 0$, then all $w : 2^N \to \mathbb{R}_+$ such that $h_v = h_w$ may be chosen from within a $n - 1$-dimensional simplex $\Delta \subset \mathbb{R}_+^n$. However, if $\sum_{i \in N} v(i) = 0$, then there is no $w : 2^N \to \mathbb{R}_+$ such that $h_v = h_w, v \neq w$. Nevertheless, this technical distinction plays no role as soon as attention is placed on generic partition and set functions $h : \mathcal{P}^N \to \mathbb{R}, w : 2^N \to \mathbb{R}$, i.e. not required to take only non-negative real values. In any case, in view of observation (ii) above, it is evident that additively separable partition functions may be regarded as elements of the subspace of $V(\mathcal{E}^N)$ consisting of those extended PFF games $g$ with Möbius inversion $\mu g$ such that $\mu g(B, Q) \neq 0$ only if $B = \emptyset$ and $Q$ is a modular partition. Developing from additively separable partition functions, now consider those $g \in V(\mathcal{E}^N)$ with Möbius inversion $\mu g$ living only on the $2^{n+1} - (n + 1)$ modular elements of $\mathcal{E}^N$ (see Section 3).
Proposition 6 If \( g \in V(\mathcal{E}^N) \) has Möbius inversion \( \mu^g \) living only on modular elements of \( \mathcal{E}^N \), then there are set functions \( v, w : 2^N \to \mathbb{R}, v(\emptyset) = 0 \) such that
\[
g(A, P) = v(A) + \sum_{B \in P} w(B) \text{ for all } (A, P) \in \mathcal{E}^N.
\]

Proof: Under the above conditions, and given expression (5) above,
\[
g(A, P) = \mu^g(\emptyset, P_{\perp}) + \sum_{B \in P} \sum_{B' \subseteq B} \mu^g(\emptyset, P_{A' \perp}) + \sum_{B' \subseteq A} \mu^g(A', P_{A' \perp})
\]
for all \((A, P) \in \mathcal{E}^N\). In view of Proposition 5,
\[
\mu^g(\emptyset, P_{\perp}) + \sum_{B \in P} \sum_{B' \subseteq B} \mu^g(\emptyset, P_{B'}) = \sum_{B \in P} w(B)
\]
for any set function \( w \) with Möbius inversion \( \mu^w \) such that
\[
\mu^g(\emptyset, P_{\perp}) = \sum_{i \in N} w(\{i\}) \text{ and } \mu^g(\emptyset, P_{B'}) = \mu^w(B') \text{ for } B' \in 2^N, |B'| > 1.
\]
On the other hand, \( \sum_{B' \subseteq A} \mu^g(A', P_{A' \perp}) = v(A) = \sum_{A' \subseteq A} \mu^v(A') \) for any set function \( v \) with Möbius inversion \( \mu^v \) satisfying \( \mu^g(A', P_{A' \perp}) = \mu^v(A') \) for \( A' \in 2^N, |A'| > 0 \) and \( \mu^v(\emptyset) = v(\emptyset) = 0. \]

Like for additively separable partition functions, if the Möbius inversion of extended PFF games lives only on the modular elements of the lattice, then all the values taken on embedded subsets can be recovered from only two set functions, out of which one is unique while the other can be chosen from within a continuum (in view of Proposition 5).

6 Concluding remarks

This paper essentially proposes to look at the so-called lattice of embedded subsets from a wider perspective, such that the novel resulting structure is a combinatorial geometry obtained through a closure operator that satisfies the Steinitz exchange axiom. With respect to the lattice previously proposed in [6], the geometric one presented here additionally includes all elements given by a (non-bottom) partition paired with the empty set, and this evidently yields a most natural lattice embedding, i.e. of \( \mathcal{P}^N \) into \( \mathcal{E}^N \). However, from this perspective what seems mostly important is that the geometric lattice of embedded subsets of a \( n \) set is isomorphic to the lattice of partitions of a \( n + 1 \)-set. In general, this enables to re-obtain a variety of results applying to partitions, ranging from the characteristic polynomial to supersolvability.

Since embedded subsets, or embedded coalitions, were firstly used as modeling tools in cooperative game theory, a meaningful direction for investigation focuses on the free vector space of functions taking real values on lattice elements. In this view, a novel type of games has been defined, namely extended PFF games, which englobe all existing cooperative games as elements of suitable vector subspaces. But perhaps most importantly, the geometric lattice of embedded subsets yields that all existing cooperative games are real-valued functions defined on atomic lattices, and this may have fundamental implications for the solution concept (i.e. how to share the fruits of cooperation) associated with these games. This shall be addressed in future work.
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