A Class of Representations of Hecke Algebras II

Dean Alvis

Abstract

Let $W$ be a Coxeter group whose proper parabolic subgroups are finite. According to Theorem 1.12 of [1], if the module of a finite $W$-digraph $\Gamma$ is isomorphic to the module of a $W$-graph over $\mathbb{Q}$, then $\Gamma$ is acyclic. We extend this result to Coxeter groups with finite dihedral parabolic subgroups and $W$-graphs over arbitrary fields $F \subseteq \mathbb{C}$. Also, an example is provided showing the converse of this theorem is false. That is, there is an example of a finite, acyclic $W$-digraph whose module does not afford a $W$-graph.

1 An extension of Theorem 1.12 of [1]

Let $(W, S)$ be a Coxeter system with presentation

$$W = \left\langle s \in S \mid (rs)^{n(r, s)} = e \text{ for } r, s \in S \text{ whenever } n(r, s) < \infty \right\rangle,$$

where $n(s, s) = 1$ and $1 < n(r, s) = n(s, r) \leq \infty$ for $r, s \in S$, $r \neq s$. Let $\ell$ be the length function of $(W, S)$. Let $u$ be an indeterminate over $\mathbb{C}$, and let $H$ be the Hecke algebra of $(W, S)$ over $\mathbb{Q}(u)$.

See [1] for the definition of $W$-digraph. For $\Gamma$ a $W$-digraph and $\beta \in V(\Gamma)$, define $\text{In}(\beta)$ to be the set of all $s \in S$ such that $\Gamma$ has an edge of the form $\alpha \overset{s}{\rightarrow} \beta$ or $\alpha \overset{s}{\leftarrow} \beta$ for some $\alpha \in V(\Gamma)$. Observe $\beta$ is a source (sink) in $\Gamma$ if and only if $\text{In}(\beta) = \emptyset$ ($\text{In}(\beta) = S$, respectively). For $J \subseteq S$, put

$$N_{\Gamma}(J) = |\{\beta \in V(\Gamma) \mid \text{In}(\beta) = J\}|.$$

This definition will only be applied when $\Gamma$ is finite, i.e. when $V(\Gamma)$ and $\mathcal{E}(\Gamma)$ are finite.

Let $\Psi$ be a $W$-graph over the subfield $F$ of $\mathbb{C}$, in the sense of [2], Definition 2.1, with $u$ playing the role of $q^{1/2}$ in [2]. Thus $I$ is the vertex-labeling function $x \mapsto I_x \subseteq S$, $x \in X$, and $\mu$ is the edge-labeling function $\mu : X \times X \setminus \Delta \rightarrow F$. For $J \subseteq S$, put

$$N_{\Psi}(J) = |\{x \in V(\Psi) \mid I_x = J\}|.$$

This definition will only be applied when $\Psi$ is finite, i.e. when $V(\Psi)$ is finite.

The goal of this section is to prove the following.

Theorem 1.1. If $n(s, t) < \infty$ for $s, t \in S$, $\Gamma$ is a finite $W$-digraph, $\Psi$ is a $W$-graph over a subfield $F$ of $\mathbb{C}$, and $M(\Gamma)^F = F(u) \otimes_{\mathbb{Q}(u)} M(\Gamma)$ is isomorphic to $M(\Psi)$ as $H^F$-modules, then the following hold:

(i) $N_{\Gamma}(J) = N_{\Psi}(J)$ for all $J \subseteq S$.

(ii) $\Gamma$ is acyclic.
We require the following results. For $F$ a subfield of $\mathbb{C}$, $\lambda$ a linear character of $H^F$, and $V$ an $H^F$-module, define

$$V_\lambda = \{ v \in V \mid hv = \lambda(h)v \text{ for all } h \in H^F \}$$

and

$$\langle V, \lambda \rangle_{H^F} = \dim_{F(u)} V_\lambda.$$

Also, let $\text{ind}^F$ and $\text{sgn}^F$ be the linear characters of $H^F$ determined by $\text{ind}^F(T_w) = u_w = u^{2\ell(w)}$ and $\text{sgn}^F(T_w) = \varepsilon_w = (-1)^{\ell(w)}$ for $w \in W$.

**Lemma 1.2.** Let $F$ a subfield of $\mathbb{C}$. Let $\Gamma$ be a $W$-digraph, and put $V = M(\Gamma)^F$. Suppose $s \in S$ and $v = \sum_{\gamma \in X} \lambda_\gamma \gamma \in V$. Then the following hold:

1. $T_sv = u^2v$ if and only if $\lambda_\beta = \lambda_\alpha$ whenever $\alpha \xrightarrow{\gamma} \beta$ or $\alpha \xrightarrow{\gamma} \beta$ is an edge of $\Gamma$.

2. $T_sv = -v$ if and only if

$$\lambda_\beta = \begin{cases} -u^{-2}\lambda_\alpha & \text{whenever } \alpha \xrightarrow{\gamma} \beta \in \mathcal{E}(\Gamma), \\ -(u+1)(u^2-u)^{-1}\lambda_\alpha & \text{whenever } \alpha \xrightarrow{\gamma} \beta \in \mathcal{E}_0(\Gamma). \end{cases}$$

**Proof.** The argument given for Lemma 2.4 in [1] applies, with $F$, $H^F$, and $V = M(\Gamma)^F$ replacing $\mathbb{Q}$, $H$, and $M = M(\Gamma)$, respectively.

**Lemma 1.3.** If $\Gamma$ is a $W$-digraph, $\mathcal{V}(\Gamma)$ is finite, and $F$ is a subfield of $\mathbb{C}$, then the following hold:

1. The number of connected components of $\Gamma$ is equal to $\langle M(\Gamma)^F, \text{ind}^F \rangle_{H^F}$.

2. If $n(s,t) < \infty$ for all $s,t \in S$, then the number of acyclic connected components of $\Gamma$ is equal to $\langle M(\Gamma)^F, \text{sgn}^F \rangle_{H^F}$.

**Proof.** The proof for Theorem 1.7 in [1] applies here if $\mathbb{Q}$, $H$, and $M(\Gamma)$ are replaced by $F$, $H^F$, $M(\Gamma)^F$, respectively, using Lemma 1.2 in place of [1], Lemma 2.4.

**Lemma 1.4.** If $F$ is a subfield of $\mathbb{C}$, $\Psi$ is a $W$-graph over $F$, $x \mapsto I_x \subseteq S$ is the vertex-labeling function for $\Psi$, and $M(\Psi)_{\text{ind}^F} \neq \{0\}$, then there is some $x_0 \in \mathcal{V}(\Psi)$ such that $I_{x_0} = \emptyset$.

**Proof.** Let $X = \mathcal{V}(\Psi)$, and let $\mu$ be the edge-labeling function of $\Psi$. Suppose $v = \sum_{x \in X} \gamma_x x \in M(\Psi)_{\text{ind}^F}$ and $v \neq 0$, where all but finitely many coefficients $\gamma_x$ are zero. Replacing $v$ by a scalar multiple if necessary, we can assume $\gamma_x \in F[u]$ for all $x \in X$ and $\gcd \{ \gamma_x \mid x \in X \} = 1$. Choose $x_0 \in X$ such that $\gamma_{x_0} \not\in uF[u]$. Suppose $I_{x_0} \neq \emptyset$, so that $s \in I_{x_0}$ for some $s \in S$. Since

$$u^2v = T_sv = -\sum_{x \in X, s \in I_x} \gamma_x x + \sum_{y \in X, s \notin I_y} \gamma_y \left( u^2y + u \sum_{z \in X, s \in I_z} \mu(z, x)z \right),$$

comparing coefficients of $x_0$ shows $\gamma_{x_0} \in uF[u]$, and so a contradiction is reached. Therefore $I_{x_0} = \emptyset$. 

\[2\]
Proof of Theorem 1.1. For the remainder of this proof it is assumed that \( n(s, t) < \infty \) for \( s, t \in S \), \( \Gamma \) is a finite \( W \)-digraph, \( \Psi \) is a \( W \)-graph over the subfield \( F \) of \( \mathbb{C} \), and \( M(\Gamma)^F \) is isomorphic to \( M(\Psi) \) as \( H^F \)-modules.

Any connected component of \( \Gamma \) that contains a sink is acyclic by \([1]\), Theorem 1.5(ii). On the other hand, any acyclic connected component \( C \) of \( \Gamma \) contains some sink \( \sigma \) because \( \Gamma \) is finite, and \( \sigma \) is the unique sink in \( C \) by \([1]\), Theorem 1.5(i). Thus the number of sinks in \( \Gamma \), that is, \( N_\Gamma(S) \), is equal to the number of acyclic connected components of \( \Gamma \). Thus by Lemma 1.3(ii), \( N_\Gamma(S) \) is equal to \( \langle M(\Gamma)^F, sgn^F \rangle_{H^F} \). Also, \( \langle M(\Psi), sgn^F \rangle_{H^F} = N_\Psi(S) \) because \( M(\Psi)_{sgn^F} \) has basis \( \{ x \in \Psi | I_x = S \} \) over \( F(u) \). Hence
\[
N_\Gamma(S) = \langle M(\Gamma)^F, sgn^F \rangle_{H^F} = \langle M(\Psi), sgn^F \rangle_{H^F} = N_\Psi(S).
\]

Now suppose \( J \subseteq S \). Let \( \Psi_J \) be the \( W_J \)-graph obtained from \( \Psi \) by replacing \( I_x \) by \( I_x \cap J \) for \( x \in \Psi(\Psi) \). Also, let \( \Gamma_J \) be the \( W_J \)-digraph obtained from \( \Gamma \) by removing all edges with labels in \( S \setminus J \). Then \( M(\Gamma_J)^F \equiv M(\Gamma)^F |_{H^F_J} \equiv M(\Psi)|_{H^F_J} \equiv M(\Psi_J) \) as \( H^F_J \)-modules, and so the reasoning above gives \( N_{\Gamma_J}(J) = N_{\Psi_J}(J) \). Therefore
\[
\sum_{J \subseteq K \subseteq S} N_{\Gamma}(K) = N_{\Gamma_J}(J) = N_{\Psi_J}(J) = \sum_{J \subseteq K \subseteq S} N_{\Psi}(K).
\]
(Note \( S \) itself is finite because \( \mathcal{E}(\Gamma) \) is finite, so the sums above are finite.) Thus part (i) of the theorem holds by induction on \( |S \setminus J| \).

Let \( \widehat{M} \) be the \( H^F \)-submodule of \( M(\Psi) \) with basis \( \{ x \in \Psi(\Psi) | I_x \neq \emptyset \} \). By Lemma 1.3
\[
M(\Psi)_{\text{ind}^F} \cap \widehat{M} = \widehat{M}_{\text{ind}^F} = \{ 0 \}.
\]
Thus
\[
\langle M(\Gamma)^F, \text{ind}^F \rangle_{H^F} = \langle M(\Psi), \text{ind}^F \rangle_{H^F} = \dim_{F(u)} M(\Psi)_{\text{ind}^F} \\
\leq \dim_{F(u)} \left( M(\Psi)/\widehat{M} \right) = |\{ x \in \Psi(\Psi) | I_x = \emptyset \}| = N_\Psi(\emptyset) = N_\Gamma(\emptyset),
\]
with the last equality holding by part (i) of the theorem. Now, \( \langle M(\Gamma)^F, \text{ind}^F \rangle_{H^F} \) is equal to the number of connected components of \( \Gamma \) by Lemma 1.3(i), while \( N_\Gamma(\emptyset) \) is equal to the number of sources of \( \Gamma \). Therefore \( \Gamma \) has at least as many sources as connected components. Because each connected component contains at most one source by \([1]\), Theorem 1.5(i), it follows that every connected component of \( \Gamma \) contains a (unique) source. Hence every connected component of \( \Gamma \) is acyclic by \([1]\), Theorem 1.5(ii). Therefore \( \Gamma \) itself is acyclic, so part (ii) of the theorem holds and the proof of the theorem is complete. \( \square \)

2 An example

In Figure 1, a \( W \)-digraph \( \Gamma \) is given for the affine group \( W = W(\tilde{A}_2) \), with generators \( S = \{ r, s, t \} \) satisfying \( n(r, s) = n(s, t) = n(r, t) = 3 \). (The digraph \( \Gamma \) is in fact a \( W \)-digraph by the classification given in Theorem 1.3 of \([1]\)). Let \( F \) be a subfield of \( \mathbb{C} \). We show \( M(\Gamma)^F \) does not afford a \( W \)-graph over \( F \), arguing by contradiction.
Suppose to the contrary that $\Psi = (\mathcal{V}(\Psi), I, \mu)$ is a $W$-graph over $F \leq \mathbb{C}$ such that $M(\Gamma)^F \cong M(\Psi)$. Note that $\Gamma$ satisfies $N_1(J) = 1$ for all $J \subseteq S$. Thus $N_\Psi(J) = 1$ for $J \subseteq S$ by Theorem (1.1(i)). We can order $\mathcal{V}(\Psi) = \{x_0, x_1, x_2, \ldots, x_7\}$ so that, with $I_j = I_{x_j}$, we have

$\begin{align*}
I_0 &= \emptyset, I_1 = \{r\}, I_2 = \{s\}, I_3 = \{t\}, \\
I_4 &= \{r,s\}, I_5 = \{r,t\}, I_6 = \{s,t\}, I_7 = \{r,s,t\}.
\end{align*}$

Define $M_1 = \bigoplus_{1 \leq i \leq 7} F(u)x_i$, $M_2 = F(u)x_7$. It is clear that $M(\Psi) \geq M_1 \geq M_2$ as $H^F$-modules. Moreover, $M(\Psi)/M_1 \cong M(\Psi)_{\text{ind}^F}$ and $M_2 = M(\Psi)_{\text{sgn}^F}$ are 1-dimensional. Put $\overline{M_1} = M_1/M_2$, an $H^F$-module with basis $\mathcal{V}_1 \setminus \{x_7\}$.

Then $\Psi_1 = (\mathcal{V}_1, I_1, \mu_1)$ is a $W$-graph over $F$ with module $M(\Psi_1) = \overline{M_1}$.

Put $m_{ij} = \mu(x_i, x_j)$ for $1 \leq i, j \leq 6, i \neq j$. The matrices $B_r$, $B_s$, $B_t$ of $T_r$, $T_s$, $T_t$ acting on $M(\Psi_1)$ with respect to the basis $\mathcal{V}_1$ are

$$
\begin{pmatrix}
-1 & m_{12}u & m_{13}u & 0 & 0 & m_{16}u \\
0 & u^2 & 0 & 0 & 0 & 0 \\
0 & 0 & u^2 & 0 & 0 & 0 \\
0 & m_{42}u & m_{43}u & -1 & 0 & m_{46}u \\
0 & m_{52}u & m_{53}u & 0 & -1 & m_{56}u \\
0 & 0 & 0 & 0 & u^2 & 0
\end{pmatrix},
\begin{pmatrix}
u^2 & 0 & 0 & 0 & 0 & 0 \\
m_{21}u & -1 & m_{23}u & 0 & m_{25}u & 0 \\
0 & 0 & u^2 & 0 & 0 & 0 \\
m_{41}u & 0 & m_{43}u & -1 & m_{45}u & 0 \\
0 & 0 & 0 & u^2 & 0 & 0 \\
m_{61}u & 0 & m_{63}u & 0 & m_{65}u & -1
\end{pmatrix},
\begin{pmatrix}
u^2 & 0 & 0 & 0 & 0 & 0 \\
m_{31}u & m_{32}u & -1 & m_{34}u & 0 & 0 \\
0 & 0 & 0 & u^2 & 0 & 0 \\
m_{51}u & m_{52}u & 0 & m_{54}u & -1 & 0 \\
m_{61}u & m_{62}u & 0 & m_{64}u & 0 & -1
\end{pmatrix},
$$

respectively. For any finite sequence $(s_1, s_2, \ldots, s_\ell)$ in $S = \{r, s, t\}$, define

$$T_{(s_1, s_2, \ldots, s_\ell)} = T_{s_1}T_{s_2} \cdots T_{s_\ell} \quad \text{and} \quad B_{(s_1, s_2, \ldots, s_\ell)} = B_{s_1}B_{s_2} \cdots B_{s_\ell}.$$
Let \( \chi_\Gamma : H^F \rightarrow F(u) \) be the character afforded by \( M(\Gamma)^F \), and let \( \chi_\Psi \) and \( \chi_{\Psi,1} \) denote the characters of \( M(\Psi) \) and \( M(\Psi_1) \), respectively. Then

\[
\chi_\Gamma = \chi_\Psi = \chi_{\Psi,1} + \text{ind}^F + \text{sgn}^F.
\]

Thus

\[
\chi_\Gamma(T(s_1, s_2, \ldots, s_\ell)) - \chi_\Gamma(T(B(s_1, s_2, \ldots, s_\ell)) - u^{2\ell} - (-1)^\ell = 0
\]

for any finite sequence \((s_1, s_2, \ldots, s_\ell)\) in \( S \).

Direct calculations show

\[
\chi_\Gamma(T(T(s, t, r, s, t, r, t)) - \chi_\Gamma(T(B(s, t, r, s, t, r, t))) - u^2 - 1
\]

\[
= (1 - m_{56}m_{65})u^2 + \cdots + (1 - m_{12}m_{21})u^{10},
\]

where the omitted terms have degrees in \( u \) between 3 and 9, inclusively. This proves the first pair of equations in (2.1).

(2.1) \( m_{12}m_{21} = m_{56}m_{65} = 1, \quad m_{13}m_{31} = m_{46}m_{64} = 1, \quad m_{23}m_{32} = m_{45}m_{54} = 1 \)

Similar calculations, using the sequences \((s, t, r, s, r, t)\) and \((r, s, t, r, t, s)\), establish the remaining equations of (2.1). Calculations also show

\[
\chi_\Gamma(T(T(s, r, s, r, t, s, r, t)) - \chi_\Gamma(T(B(s, r, s, r, t, s, r, t))) - u^4 - 1
\]

\[
= (1 - m_{46}m_{54}m_{65})u^3 + \cdots + (1 - m_{12}m_{23}m_{31})u^{13},
\]

so

(2.2) \( m_{12}m_{23}m_{31} = m_{46}m_{54}m_{65} = 1. \)

Next, the coefficient of \( u^2 \) in

\[
\chi_\Gamma(T(T(s, r, t, r, s)) - \chi_\Gamma(T(B(s, r, t, r, s))) - u^{10} - 1
\]

is

\[
-2 + m_{16}m_{61} + m_{46}m_{64} + m_{56}m_{65},
\]

which is equal to \( m_{16}m_{61} \) by (2.1), so the first equation of (2.3) holds.

(2.3) \( m_{16}m_{61} = 0, \quad m_{25}m_{52} = 0, \quad m_{34}m_{43} = 0 \)

The remaining equations of (2.3) are verified by similar calculations using the sequences \((t, r, s, t, r)\) and \((r, s, t, r, s)\).

Now,

\[
\chi_\Gamma(T(T(r, s, t, r, s)) - \chi_\Gamma(T(B(r, s, t, r, s))) - u^6 + 1 = \alpha u^2 - \beta u^3 + \gamma u^4,
\]

where

\[
\begin{align*}
\alpha &= -3 + m_{34}m_{43} + m_{25}m_{52} + m_{45}m_{54} + m_{16}m_{61} + m_{46}m_{64} + m_{56}m_{65}, \\
\beta &= m_{12}m_{23}m_{31} + m_{23}m_{34}m_{42} + m_{12}m_{25}m_{51} + m_{25}m_{42}m_{54} \\
&\quad + m_{16}m_{31}m_{63} + m_{34}m_{46}m_{63} + m_{16}m_{51}m_{65} + m_{46}m_{54}m_{65}, \\
\gamma &= 3 - m_{12}m_{21} - m_{13}m_{31} - m_{23}m_{32} - m_{34}m_{43} - m_{25}m_{52} - m_{16}m_{61}.
\end{align*}
\]

Applying the relations (2.2), it follows that

\[
\beta = 2 + m_{23}m_{34}m_{42} + m_{12}m_{25}m_{51} + m_{25}m_{42}m_{54} \\
+ m_{16}m_{31}m_{63} + m_{34}m_{46}m_{63} + m_{16}m_{51}m_{65}.
\]
We consider eight cases of the form \((a, b, c) = (0, 0, 0)\), where \(a \in \{m_{16}, m_{61}\}\), \(b \in \{m_{25}, m_{52}\}\), \(c \in \{m_{34}, m_{43}\}\). These cases are exhaustive by \((2.3)\). In each case it is shown that \(\beta = 2 \neq 0\), giving a contradiction.

Case 1. \((m_{16}, m_{25}, m_{34}) = (0, 0, 0)\). In this case it is clear that \(\beta = 2\).

Cases 2–7. \((a, b, c) = (0, 0, 0)\), where \((a, b, c) \neq (m_{16}, m_{25}, m_{34})\), \((a, b, c) \neq (m_{61}, m_{52}, m_{43})\).

In these six cases, we use the dihedral relations

\[
\begin{align*}
B_{(r,s,r)} - B_{(s,r,s)} &= 0 \\
B_{(r,t,r)} - B_{(t,r,t)} &= 0 \\
B_{(s,t,s)} - B_{(t,s,t)} &= 0
\end{align*}
\]  

(2.4)

to show

\[
\mathcal{M} = \{0\},
\]

where \(\mathcal{M} = \{m_{16}, m_{61}, m_{25}, m_{52}, m_{34}, m_{43}\}\). In each case \((2.5)\) can be established by considering certain monomial entries in the matrices on the left sides of \((2.4)\).

For example, consider the case \((m_{16}, m_{25}, m_{43}) = (0, 0, 0)\). In this case the \((6, 1)\)-entry of \(B_{(r,t,r)} - B_{(t,r,t)}\) is

\[-m_{61}(-1 + m_{13}m_{31} + m_{46}m_{64})u^3,
\]

which is equal to \(-m_{61}u^3\) in view of \((2.1)\). Therefore \(m_{61} = 0\). Also, again using \((2.1)\), the \((3, 4)\)- and \((5, 2)\)-entries of \(B_{(s,t,s)} - B_{(t,s,t)}\) are equal to \(-m_{34}u^3\) and \(-m_{52}u^3\), respectively, so \(m_{34} = m_{52} = 0\) and \((2.5)\) holds.

As a second example, consider the case \((m_{61}, m_{25}, m_{43}) = (0, 0, 0)\). Using \((2.1)\), the \((1, 6)\)- and \((5, 2)\)-entries of \(B_{(r,s,r)} - B_{(s,r,s)}\) are equal to \(m_{16}u^3\) and \(m_{52}u^3\), respectively, so \(m_{16} = m_{52} = 0\). Also, the \((3, 4)\)-entry of \(B_{(s,t,s)} - B_{(t,s,t)}\) is equal to \(-m_{34}u^3\), and thus \(m_{34} = 0\), so again \((2.5)\) holds.

Similary calculations establish \((2.5)\) in the remaining cases in this group. From \((2.5)\), it follows that \(\beta = 2\). (In any of Cases 2–7, to show \(m_{ij} = 0\), where \(m_{ij} \in \mathcal{M} \setminus \{a, b, c\}\), it suffices to look at the \((i, j)\)-entry of one matrix on the left side of \((2.4)\) and apply \((2.1)\). The author has no explanation for this pattern.)

Case 8. \((m_{61}, m_{52}, m_{43}) = (0, 0, 0)\). The \((1, 5)\)-entry of \(B_{(r,s,r)} - B_{(s,r,s)}\) is

\[-(m_{12}m_{25} + m_{16}m_{65})(u^2 + u^4),
\]

and thus \(m_{12}m_{25} + m_{16}m_{65} = 0\). Also, the \((3, 6)\)-entry of \(B_{(r,t,r)} - B_{(t,r,t)}\) is

\[(m_{16}m_{31} + m_{34}m_{46})(u^2 + u^4),
\]

so \(m_{16}m_{31} + m_{34}m_{46} = 0\). Finally, the \((2, 4)\)-entry of \(B_{(s,t,s)} - B_{(t,s,t)}\) is

\[-(m_{23}m_{34} + m_{25}m_{54})(u^2 + u^4),
\]

and thus \(m_{23}m_{34} + m_{25}m_{54} = 0\). Hence

\[
\beta = 2 + (m_{23}m_{34} + m_{25}m_{54})m_{42} + (m_{16}m_{31} + m_{34}m_{46})m_{63} + (m_{12}m_{25} + m_{16}m_{65})m_{51} = 2.
\]

Since \(\beta = 2\) in all cases, we have arrived at a contradiction Therefore \(M(\Gamma)^F\) is not isomorphic to \(M(\Psi)\).
References

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