Some alternatives for calculating multipole expansions of the electromagnetic radiation field

C. Vrejoiu and R. Zus

University of Bucharest, Department of Physics, Bucharest, Romania

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Abstract

We discuss the multipolar expansion of the electromagnetic field with an emphasis on the radiated field. We investigate if the employment of Jefimenko’s equations brings a new insight into the calculation of the radiation field. We show that the affirmation is valid if one finds an interesting example in which inverting the order between spatial derivatives and integration is not allowed. Further, we consider the generalization of the multipolar expansion of the power radiated by a confined system of charges and currents to a higher arbitrary order.
I. INTRODUCTION

Despite the successful and long history of the electromagnetic field theory, there are several topics open to new theoretical and pedagogical contributions. One of them concerns the formalism of multipole expansion of the field, in general, and of the radiated one, in particular. Another issue is related to the importance of Jefimenko’s equations in the study of such problems. Motivated by recent publications on the topic, we discuss some features of this type of problems.

The multipole expansion of the electromagnetic field in Cartesian coordinates is exposed in electrodynamics textbooks, as the well-known Refs.\textsuperscript{1} and\textsuperscript{2}. Ordinarily, these expansions are calculated only in the first two or three orders, the higher-orders being considered too complicated. As Jackson writes in his textbook, \textit{the labor involved in manipulating terms in the expansion of the vector potential becomes increasingly prohibitive as the expansion is extended beyond the electric quadrupole terms} (see Ref.\textsuperscript{2}, pp 415-416). For this reason and due to the applicability only in the long-wavelength range, another treatment, based on the spherical tensors and the solutions of Helmholtz equation is preferred. This alternative has also a larger domain of applications. Actually, starting from the results obtained employing this calculation technique, the reader can verify what effort is involved when returning to the multipole Cartesian moments which offer a higher physical transparency (see Ref.\textsuperscript{3}). A relatively recent textbook\textsuperscript{4} and a paper\textsuperscript{5}, the last related to the importance of Jefimenko’s equations for expressing the electric and magnetic field when discussing the radiation theory, brought our attention on a very hard formalism employed for the calculation of even the first three or four terms of the expansion series. Though there are some prescriptions in the literature\textsuperscript{6,7,8} for calculating higher-order terms of the multipole series based on a simple algebraic formalism of tensorial analysis, it seems that there is some reticence in using this last technique. For this reason, one of the aims of the present paper is to show how one can hide, as much as possible, the higher-order tensors behind some vectors, reducing the calculation technique to the formalism of an ordinary vectorial algebra or analysis.

Another aim of the paper is to investigate if the use of Jefimenko’s equations brings, as sometimes presented in the literature, a new insight in the calculation of the radiation field. We show that unless one finds an example where the spatial derivative and the integral operations can not be inverted, it is not always a necessary complication. As already
mentioned, we also use the opportunity to explain some advanced features of the multipolar expansions in the field radiation theory.

We start in section II by shortly presenting the notation convention we use and by giving a general formalism for handling multipolar expansions in Cartesian coordinates. In section III we derive the radiated electric and magnetic field without using the retarded potentials, while in section IV we present characteristics of the calculation for the radiation field when employing Jefimenko’s equations. The advantages and disadvantages of different approaches are analyzed. In section V we further discuss some features of the calculation for the radiated power, with an emphasis on the 4th order approximation in \( d/\lambda < 1 \). Finally, in section VI we give the guidelines for the general tensorial calculus of the electric and magnetic moments. The last section is reserved for conclusions.

II. GENERAL FORMALISM

We write Maxwell equations with a notation independent of the unit system ("system free" Maxwell equations):

\[
\begin{align*}
\nabla \times \mathbf{B} &= \frac{\mu_0}{\alpha} \left( \mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right), \quad \nabla \times \mathbf{E} = -\frac{1}{\alpha \varepsilon_0} \frac{\partial \mathbf{B}}{\partial t}, \\
\n\nabla \cdot \mathbf{B} &= 0, \quad \nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0} \rho,
\end{align*}
\]

where \( \varepsilon_0, \mu_0, \alpha \) are proportional factors depending on the system of units and are satisfying the equation

\[
\frac{\alpha^2}{\varepsilon_0 \mu_0} = c^2.
\]

c is the vacuum light speed. Maxwell equations written in SI units are obtained from equations (1) for \( \alpha = 1 \) and the SI values of \( \varepsilon_0, \mu_0 \). For the Gauss system of units, \( \alpha = c, \varepsilon_0 = 1/4\pi, \mu_0 = 4\pi \). With this notation, Jefimenko’s equations can be written as:

\[
\begin{align*}
\mathbf{B}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi \alpha^2} \int_D \nabla \times \left[ \frac{\mathbf{J}}{R} \right] d^3 x', \\
\mathbf{E}(\mathbf{r}, t) &= -\frac{1}{4\pi \varepsilon_0} \int_D \nabla \left[ \frac{\rho}{R} \right] d^3 x' - \frac{\mu_0}{4\pi \alpha^2} \int_D \left[ \frac{\mathbf{J}}{R} \right] d^3 x',
\end{align*}
\]
where \( \mathbf{R} = \mathbf{r} - \mathbf{r}' \), \([f] = f(\mathbf{r}', t - R/c)\) and the origin \( O \) of Cartesian coordinates is in the domain \( \mathcal{D} \). The support of charge and current distribution is supposed included in \( \mathcal{D} \). If in equations (3) and (4) the order of the derivative and the integral is inverted, one obtains the well-known relations between fields and potentials:

\[
\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla \Phi - \frac{1}{\alpha} \frac{\partial \mathbf{A}}{\partial t},
\]

with the retarded potentials

\[
\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi\alpha} \int_{\mathcal{D}} \frac{[\mathbf{J}]}{R} \, d^3x', \quad \Phi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{D}} \frac{[\rho]}{R} \, d^3x'.
\]

In Ref. 5, the authors derive the multipole expansion of the radiation field from equations (3) and (4) claiming to give an original demonstration specific for Jefimenko’s equations, without employing the retarded potentials. We should agree with this claim if at least some calculation of the authors is different from those employing the potential multipole expansions which are generally used in literature. In the following, we search for a difference between the calculation presented in Ref. 5 and the standard one making use of potentials. The goal of the exposition below is to inform on some results regarding multipolar expansion in Cartesian coordinates, too.

Let us derive the multipolar expansion of the field \( \mathbf{B} \) given by equation (3), and written explicitly as:

\[
\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi \alpha} \mathbf{e}_i \varepsilon_{ijk} \int_{\mathcal{D}} \partial_j \left( \frac{[\mathbf{J}]}{R} \right) \, d^3x',
\]

where \( \mathbf{e}_i \) are the unit vectors of the Cartesian axes. Writing the integral from equation (7) as

\[
\int_{\mathcal{D}} \partial_j \left( \frac{[\mathbf{J}]}{R} \right) \, d^3x' = \int_{\mathcal{D}} \partial_j \left( \frac{J_k(\xi, t - R/c)}{R} \right) \, d^3x',
\]

we obtain the multipolar expansion of the magnetic field about \( O \) as function of \( \mathbf{r}' \) using the Taylor series of the integrand:

\[
\partial_j \frac{J_k(\xi, t - R/c)}{R} = \sum_{n \geq 0} \frac{(-1)^n}{n!} x_{i_1}' \ldots x_{i_n}' \partial_{i_1} \ldots \partial_{i_n} \partial_j \left( \frac{1}{r} J_k(\xi, t - \frac{r}{c}) \right).
\]

Equation (7) can now be expressed as

\[
\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi \alpha} \mathbf{e}_i \varepsilon_{ijk} \int_{\mathcal{D}} \sum_{n \geq 0} \frac{(-1)^n}{n!} x_{i_1}' \ldots x_{i_n}' \partial_{i_1} \partial_{i_2} \ldots \partial_{i_n} \left( \frac{1}{r} J_k(\xi, t - \frac{r}{c}) \right) \, d^3x'.
\]
where we employed the notation \([f]_0 = f(r', t - r/c)\).

We assume one is allowed to invert orders of operations in equation (9) and to write:

\[
B(r, t) = \frac{\mu_0}{4\pi} \varepsilon_{ijk} \sum_{n \geq 0} \frac{(-1)^n}{n!} \partial_j \partial_{i_1} \ldots \partial_{i_n} \left( \frac{1}{r} \int_{\mathcal{D}} x'_{i_1} \ldots x'_{i_n} [J_k]_0 \right) \, d^3x'.
\] (10)

Equation (10) represents the curl of the multipolar expansion of the vector potential \(A\). Thus one can perform firstly the multipolar expansion of this potential. It is the usual procedure.

No matter what procedure is employed, a constant in the calculation is the presence of a vector \(a(r, t; \zeta, n)\) defined by the Cartesian components:

\[
a_k(r, t; \zeta, n) = \zeta_{i_1} \ldots \zeta_{i_n} \left( \frac{1}{r} \int_{\mathcal{D}} x'_{i_1} \ldots x'_{i_n} J_k(r', t) \, d^3x' \right).
\] (11)

Here, \(\zeta\) can be either an operator or a number. Generalizing to the dynamic case an algorithm used in\(^9\) for the magnetostatic field, we introduce in equation (11) the consequence of the continuity equation, written for \(t_0 = t - r/c\):

\[
J_k(r', t_0) = \nabla'(x'_k J(r', t_0)) + x'_k \dot{\rho}(r', t_0).
\]

We obtain

\[
a_k(r, t_0; \zeta, n) = \zeta_{i_1} \ldots \zeta_{i_n} \frac{1}{r} \int_{\mathcal{D}} x'_{i_1} \ldots x'_{i_n} \nabla'(x'_k [J]_0) \, d^3x' + \zeta_{i_1} \ldots \zeta_{i_n} \frac{1}{r} \dot{P}_{i_1 \ldots i_n k}(t_0),
\] (12)

where the Cartesian components of the \(n - th\) electric moment of the given charge distribution:

\[
P_{i_1 \ldots i_n}(t) = \int_{\mathcal{D}} x'_{i_1} \ldots x'_{i_n} \rho(r', t) \, d^3x'
\] (13)

are introduced. Let us define, for simplifying the notation, the vector

\[
\mathcal{P}(r, t; \zeta, n) = e_k \zeta_{i_1} \ldots \zeta_{i_{n-1}} \frac{P_{i_1 \ldots i_{n-1} k}(t)}{r}.
\] (14)

Performing partial integration and taking into account that \(J\) vanishes on the surface \(\partial \mathcal{D}\),
equation (12) can be written and processed as follows:

\[ a_k(r, t_0; \zeta, n) - \dot{\mathbf{P}}_k(r, t; \zeta, n + 1) = -\zeta_{i_1} \ldots \zeta_{i_n} \frac{1}{r} \int_{D} x'_{i_1} \ldots x'_{i_{n-1}} x'_{i_n} [J_{i_n}]_0 \, d^3x' \]

or, by a change of the summation index in the second sum,

\[ -n \zeta_{i_1} \ldots \zeta_{i_{n-1}} \frac{1}{r} \int_{D} x'_{i_1} \ldots x'_{i_{n-1}} (x'_{i_n} [J_{i_n}]_0 - x'_{i_n} [J_k]_0) \, d^3x' - n\alpha_k(r, t_0; \zeta, n) \]

Introducing the \( n - \text{th} \) order magnetic moment, as in Ref. 2, by its Cartesian components

\[ \mathbf{M}_{i_1 \ldots i_n}(t) = \frac{n}{(n + 1)\alpha} \int_{D} x'_{i_1} \ldots x'_{i_{n-1}} (r' \times \mathbf{J}(r', t))_{i_n} \, d^3x', \]

equation (15) becomes

\[ a_k(r, t_0; \zeta, n) = -\alpha \varepsilon_{k i n q} \zeta_{i_1} \ldots \zeta_{i_{n-1}} \frac{\mathbf{M}_{i_1 \ldots i_{n-1} q}(t_0)}{r} + \frac{1}{n + 1} \dot{\mathbf{P}}_k(r, t_0; \zeta, n + 1). \]  

Similarly to equation (14), we introduce the vector

\[ \mathbf{M}(r, t; \zeta, n) = e_k \zeta_{i_1} \ldots \zeta_{i_{n-1}} \frac{\mathbf{M}_{i_1 \ldots i_{n-1} k}(t)}{r}, \]

writing finally equation (17) as

\[ \mathbf{a}(r, t_0; \zeta, n) = -\alpha \zeta \times \mathbf{M}(r, t_0; \zeta, n) + \frac{1}{n + 1} \dot{\mathbf{P}}(r, t_0; \zeta, n + 1). \]  

With this result, the magnetic field from equation (11) can be expressed with the help of the vectors \( \mathbf{M} \) and \( \mathbf{P} \):

\[ \mathbf{B}(r, t) = \frac{\mu_0}{4\pi \alpha} e_i \varepsilon_{ijk} \sum_{n \geq 0} \frac{(-1)^n}{n!} \partial_j a_k(r, t_0; \nabla, n) \]

or, by a change of the summation index in the second sum,

\[ \mathbf{B}(r, t) = \nabla \times \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \left( \nabla \times \mathbf{M}(r, t_0; \nabla, n) + \frac{1}{\alpha} \dot{\mathbf{P}}(r, t_0; \nabla, n) \right). \]
From the last expression one has no problem in identifying the multipolar expansion of the vector potential \( A \):

\[
A(r, t) = \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \left( \nabla \times M(r, t_0; \nabla, n) + \frac{1}{\alpha} \dot{P}(r, t_0; \nabla, n) \right).
\] (21)

The calculation for the electric field can be performed in a similar manner. One obtains:

\[
E(r, t) = -\frac{1}{4\pi\varepsilon_0} \sum_{n \geq 0} \frac{(-1)^n}{n!} \nabla (\nabla \cdot P(r, t_0; \nabla, n))
+ \frac{\mu_0}{4\pi\alpha} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \left( \nabla \times \dot{M}(r, t_0; \nabla, n) + \frac{1}{\alpha} \ddot{P}(r, t_0; \nabla, n) \right)
\] (22)

Comparing equations (5) with (21) and (22), we single out the multipole expansion of the potential \( \Phi \):

\[
\Phi(r, t) = \frac{1}{4\pi\varepsilon_0} \sum_{n \geq 0} \frac{(-1)^n}{n!} \nabla \cdot P(r, t_0; \nabla, n).
\] (23)

**III. RADIATION FIELD**

For calculating the radiation field it is sufficient retaining only terms of order \( 1/r \) and \( 1/r^2 \) for \( r \to \infty \). In most textbooks one retains only the terms of order \( 1/r \), the goal being, usually, only the derivation of the radiated energy or of the linear momentum. Actually, when the goal is the complete definition of the radiation field, one must be able to derive all transferring properties, including the angular momentum loss. These are, in fact, minimal conditions for defining a physical system. In the last case, the terms of order \( 1/r^2 \) are also necessary (see Ref.1 - Problem 2 at the end of Section 72, and also Refs.3,10). Although the aim of the present paper is different, we also give the formula for introducing terms of order \( 1/r^2 \) required for the evaluation of the angular momentum loss. The terms of the orders \( 1/r \) and \( 1/r^2 \) are selected making use of formula10:

\[
\partial_{i_1} \ldots \partial_{i_n} \left( \frac{f(t_0)}{r} \right) = \frac{1}{r} \left( -1 \right)^n \frac{1}{e^{n-1}} \nu_{i_1} \ldots \nu_{i_n} \frac{\partial^n f(t_0)}{\partial t^n}
+ \left( -1 \right)^n \frac{1}{e^{n-1}r^2} \left( D_n \nu_{i_1} \ldots \nu_{i_n} - \nu_{\{i_1 \ldots i_{n-2}\}i_{n-1,i_n}} \right) \frac{\partial^{n-1} f(t_0)}{\partial t^{n-1}}.
\] (24)

Again \( t_0 = t - r/c \) and \( \nu_i = x_i/r \). By \( A_{\{i_1 \ldots i_n\}} \) we understand the sum over all the permutations of the symbols \( i_q \) that give distinct terms. The coefficients \( D_n \) are defined by the recurrence relations:

\[
D_n = D_{n-1} + n, \quad D_0 = 0.
\] (25)

7
The formula from equation (24) can be easily proven by recurrence.

In fact, formula (24) represents the sum of the terms corresponding to \( l = 0 \) and 1 of a general formula

\[
\partial_1 \ldots \partial_{i_n} \frac{f(t_0)}{r} = \sum_{l=0}^{n} C_{i_1 \ldots i_n}^{(n,l)}(\nu) \frac{1}{l!+1} \frac{\partial^{l+1} f(t_0)}{\partial t^{n-l}}.
\] (26)

In this equation, \( C_{i_1 \ldots i_n}^{(n,l)}(\nu) \) are symmetric coefficients expressed as linear combinations of products of components \( \nu_i \) and Kronecker symbols \( \delta_{i_q i_p} \), where \( i, i_q, i_p = i_1 \ldots i_n \).

Considering equations (14) and (18), one can see that \( \mathcal{P}(r, t_0; \zeta, n) \) and \( \mathcal{M}(r, t_0; \zeta, n) \) are solutions of the homogeneous wave equation for \( r \neq 0 \). Consequently, one can apply the formula (24) for these quantities. Let us consider the multiple derivative as, for example,

\[
\partial_{j_1} \ldots \partial_{j_m} \mathcal{M}(r, t_0; \nabla, n) = (-1)^{n+m-1} \frac{\partial^{n+m-1}}{\partial t^{n+m-1}} \left( \nabla \cdot \mathcal{M}(r, t_0; \nu, n) \right) - \frac{c}{\alpha} \nu \times \mathcal{P}(r, t_0; \nu, n) + O(1/r^2)
\] (27)

and similarly for \( \mathcal{P}(r, t_0; \nabla, n) \). Using equation (27) and the equivalent relation for \( \mathcal{P}(r, t_0; \nabla, n) \) in equation (20), we obtain the first approximation of the multipolar expansion of radiated magnetic field, which is sufficient for calculating the radiated energy and the linear momentum:

\[
B_{rad}(r, t) = \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{1}{n! \epsilon^{n+1}} \frac{\partial^{n+1}}{\partial t^{n+1}} \left( \nu \times \mathcal{M}(r, t_0; \nu, n) \right) - \frac{c}{\alpha} \nu \times \mathcal{P}(r, t_0; \nu, n)
\] (28)

\[
= \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{1}{n! \epsilon^{n+1}} \frac{\partial^{n+1}}{\partial t^{n+1}} \left( (\nu \cdot \mathcal{M}(r, t_0; \nu, n)) \nu - \mathcal{M}(r, t_0; \nu, n) - \frac{c}{\alpha} \nu \times \mathcal{P}(r, t_0; \nu, n) \right).
\]

For the field \( E_{rad} \), we obtain

\[
E_{rad}(r, t) = \frac{1}{4\pi \epsilon_0} \sum_{n \geq 1} \frac{1}{n! \epsilon^{n+1}} \frac{\partial^{n+1}}{\partial t^{n+1}} (\nu \cdot \mathcal{P}(r, t_0; \nu, n) \nu - \mathcal{P}(r, t_0; \nu, n))
\]

\[
+ \frac{\alpha}{c} \nu \times \mathcal{M}(r, t_0; \nu, n).
\] (29)

We made use of equation (21). From equations (21) and (23), using equation (24), we get easily the expansions of the radiation field potentials:

\[
A_{rad}(r, t) = \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{1}{n! \epsilon^{n}} \frac{\partial^{n}}{\partial t^{n}} \left( \mathcal{M}(r, t_0; \nu, n) \times \nu + \frac{c}{\alpha} \mathcal{P}(r, t_0; \nu, n) \right)
\] (30)

and

\[
\Phi_{rad}(r, t) = \frac{1}{4\pi \epsilon_0} \sum_{n \geq 1} \frac{1}{n! \epsilon^{n}} \frac{\partial^{n}}{\partial t^{n}} \nu \cdot \mathcal{P}(r, t_0; \nu, n).
\] (31)
Given the above expressions, one can verify that the relations between fields and potentials are:

\[ \mathbf{B}_{\text{rad}} = \frac{1}{c}(\mathbf{A}_{\text{rad}} \times \mathbf{\nu}), \quad \mathbf{E}_{\text{rad}} = \frac{1}{\alpha}(\mathbf{A}_{\text{rad}} \times \mathbf{\nu}) \times \mathbf{\nu} = \frac{c}{\alpha} \mathbf{B}_{\text{rad}} \times \mathbf{\nu}. \quad (32) \]

These parts (proportional to \(1/r\)) from the radiated electric and magnetic fields are purely transverse fields, satisfying the properties (see also\(^{10}\)):

\[ \mathbf{\nu} \cdot \mathbf{E}_{\text{rad}} = 0, \quad \mathbf{\nu} \cdot \mathbf{B}_{\text{rad}} = 0, \quad \varepsilon_0 |\mathbf{E}_{\text{rad}}|^2 = \frac{1}{\mu_0} |\mathbf{B}_{\text{rad}}|^2. \quad (33) \]

IV. IS REALLY THE RADIATION FIELD CALCULATION FROM JEFIMENKO’S EQUATIONS A NEW INSIGHT IN THE RADIATION THEORY?

In Ref.\(^1\), calculating the radiation field (in the first approximation), the retarded potentials are approximated by the formulae (66.1) and (66.2) of this reference, written here with the “system free” notation:

\[
\Phi(r, t) \approx \frac{1}{4\pi\varepsilon_0} \int_{D} \rho(r', t_0 + \frac{1}{c} \mathbf{\nu} \cdot r') \, d^3x', \quad \mathbf{A}(r, t) \approx \frac{\mu_0}{4\pi\alpha r} \int_{D} \mathbf{J}(r', t_0 + \frac{1}{c} \mathbf{\nu} \cdot r') \, d^3x'. \quad (34)
\]

As suggested in Ref.\(^1\) (page 184, first footnote), introducing these expressions of the radiated potentials and retaining only the terms of order \(1/r\), one obtains the relations from equation \(^{32}\) (i.e., equation (66.3) from Ref.\(^1\)). In this calculation, the derivative operators must be introduced in the integral and so, the proof is indeed realized directly for the fields \(\mathbf{E}\) and \(\mathbf{B}\). For \(\mathbf{B}\), using the relations

\[
\nabla \times \frac{\mathbf{J}(r', t_0 + \frac{\mathbf{\nu} \cdot r'}{c})}{r} = \left(\nabla \frac{1}{r}\right) \times \mathbf{J} + \frac{1}{r} \nabla(t_0 + \frac{\mathbf{\nu} \cdot r'}{c}) \times \dot{\mathbf{J}}
\]

and

\[
\nabla \frac{1}{r} = \mathcal{O}(\frac{1}{r^2}), \quad \nabla(t_0 + \frac{1}{c} \mathbf{\nu} \cdot r') = -\frac{1}{c} \mathbf{\nu} + \mathcal{O}(\frac{1}{r}), \quad (35)
\]

we can write

\[
\tilde{\mathbf{B}}(r, t) = \frac{\mu_0}{4\pi\alpha} \int_{D} \nabla \times \frac{\mathbf{J}(r', t_0 + \frac{\mathbf{\nu} \cdot r'}{c})}{r} \, d^3x' \]

\[
= - \frac{\mu_0}{4\pi\alpha c} \frac{1}{r} \mathbf{\nu} \times \int_{D} \mathbf{J}(r', t_0 + \frac{1}{c} \mathbf{\nu} \cdot r') \, d^3x' + \mathcal{O}(\frac{1}{r^2}). \quad (36)
\]
From this last equation we can extract the term of order $1/r$ which represents the radiated field (precisely, the first approximation of this field):

$$B_{rad}(r, t) = -\frac{\mu_0}{4\pi \alpha c r} \mathbf{v} \times \int_\mathcal{D} \mathbf{J}(r', t_0 + \frac{1}{c} \mathbf{v} \cdot \mathbf{r}') \, d^3x'. \quad (37)$$

This is, in fact, equation (28) from Ref. 5 and we can consider, by examining Ref. 1, that this result was obtained long time ago from the expression (3) of the field $B$ (see Ref. 11 and Ref. 12). Look also in Ref. 5 to see the usage of Jefimenko’s equation for calculating $B_{rad}$.

Introducing also the approximate expression $\tilde{E}$ starting from equation (4), we obtain

$$\tilde{E}(r, t) = -\frac{\mu_0}{4\pi \alpha^2 r} \int_\mathcal{D} \left( c^2 \nabla \rho(r', t_0 + \frac{1}{c} \mathbf{v} \cdot \mathbf{r}') + \mathbf{J}(r', t_0 + \frac{1}{c} \mathbf{v} \cdot \mathbf{r}') \right) \, d^3x' + O\left(\frac{1}{r^2}\right)$$

$$= \frac{\mu_0}{4\pi \alpha^2 r} \int_\mathcal{D} \left( c \dot{\rho}(r', t_0 + \frac{1}{c} \mathbf{v} \cdot \mathbf{r}') \mathbf{v} - \dot{\mathbf{J}}(r', t_0 + \frac{1}{c} \mathbf{v} \cdot \mathbf{r}') \right) + O\left(\frac{1}{r^2}\right). \quad (38)$$

Using the continuity equation written in the point $r'$ at the retarded time $t - R/c$,

$$[\dot{\rho}] = -[\nabla' \cdot \mathbf{J}(r', t')]_{t'=t-R/c} = -\nabla' \cdot [\mathbf{J}] + [\mathbf{J}] \cdot \nabla'(t - R/c), \quad (39)$$

it results

$$\dot{\rho}(r', t_0 + \frac{1}{c} \mathbf{v} \cdot \mathbf{r}') = -\nabla' \cdot \mathbf{J}(r', t_0 + \frac{1}{c} \mathbf{v} \cdot \mathbf{r}') + \frac{1}{c} \mathbf{v} \cdot \dot{\mathbf{J}}(r', t_0 + \frac{1}{c} \mathbf{v} \cdot \mathbf{r}') + O\left(\frac{1}{r}\right). \quad (40)$$

The first term from the right-hand side of the last equation gives no contribution to the integral from equation (38) and, after a simple algebraic calculation, we obtain the expression of $E_{rad}$ from equation (32) (see also equations (66.3) from Ref. 1 and equation (30) from Ref. 5).

We point out that equation (39) or, generally, the relation between the space derivative of a retarded quantity and the retarded value of the space derivative of the same quantity, should be well-known for each student from a class of electrodynamics since when writing the retarded potentials as solutions of the wave equation, it is necessary to verify the Lorenz condition. The verification can be realized in a direct calculation, a good exercise for a student. Only in this way one can be convinced that the retarded solutions are indeed electromagnetic potentials. For this goal, equation (39) is indispensable since the Lorenz condition appears as a consequence of the continuity equation.
Based on the above results, we can quickly obtain the multipole expansion of the radiation field. Considering the adequate Taylor series for the integrand in equation (36), we write

\[ B_{\text{rad}}(r, t) = -\frac{\mu_0}{4\pi\alpha c} r \times \frac{1}{r} \nabla \cdot (\int_{\mathcal{D}} J(\xi, t_0 + \frac{1}{c} \nu \cdot r') d^3 x') \]

\[ = -\frac{\mu_0}{4\pi\alpha} \nu \times \sum_{n \geq 0} \frac{1}{n!c^{n+1}} \frac{\partial^{n+1}}{\partial t^{n+1}} \left( \nu_{i_1} \cdots \nu_{i_n} \frac{1}{r} \int_{\mathcal{D}} x'_{i_1} \cdots x'_{i_n} J(r', t_0) d^3 x' \right) \]

\[ = -\frac{\mu_0}{4\pi\alpha} \nu \times a(r, t_0; \nu, n). \] (41)

The vector \( a \) was defined in equation (11). After introducing its expression as a function of the vectors associated to the electric and magnetic moment, equation (19), the radiated magnetic field becomes

\[ B_{\text{rad}}(r, t) = \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{1}{n!c^{n+1}} \frac{\partial^{n+1}}{\partial t^{n+1}} \nu \times (\nu \times \mathcal{M}(r, t_0; \nu, n)) \]

\[ - \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{1}{(n+1)!c^{n+1}} \frac{\partial^{n+2}}{\partial t^{n+2}} \nu \times \mathcal{P}(r, t_0; \nu, n+1). \] (42)

The final step is changing \( n \to n - 1 \) in the second sum of equation (42). We obtain again equation (28).

We can say that the result (42), which is the multipole expansion of the radiation field, was determined via Jefimenko’s equations.

Alternatively, after reaching the expression given by equation (9), we can perform the multipolar expansion of the radiation field in such a manner that we can also say that it is obtained via Jefimenko’s equations. We extract from the integrand in equation (9) the terms of the order \( 1/r \) (and \( 1/r^2 \) for a complete definition of this field):

\[ B_{\text{rad}}(r, t) = -\frac{\mu_0}{4\pi\alpha} e_i \varepsilon_{ijk} \nu_j \nu_{i_1} \cdots \nu_{i_n} \frac{1}{r} \int_{\mathcal{D}} \sum_{n \geq 0} \frac{1}{n!c^{n+1}} \frac{\partial^{n+1}}{\partial t^{n+1}} [J_k]_0 d^3 x' \]

\[ = -\frac{\mu_0}{4\pi\alpha} e_i \varepsilon_{ijk} \sum_{n \geq 0} \frac{1}{n!c^{n+1}} \nu_j \frac{\partial^{n+1}}{\partial t^{n+1}} a_k(r', t_0; \nu, n), \] (43)

arriving, as expected, to equation (41).

We remind the reader that for obtaining the results of equations (20) and (22) we admitted the commutation of the derivative with respect to the spatial coordinates, with the series expansion and the integral on the domain \( \mathcal{D} \). In the present section, for the case of the
radiation field, we avoid the inversion of the derivative with the integral operation. Regarding the commutation with the Taylor expansion, we consider that such an operation cannot be avoided as long as we want to emphasize the multipolar moments.

As a short extension in the argumentation of the utility of Jefimenko’s equations, we show how one can avoid the commutation between derivative and integral for an arbitrary point in the exterior of the domain \( D \). Indeed, using equation (26) in equation (9), we can write the magnetic field with the help of time derivatives

\[
B(r, t) = \mu_0 \frac{4\pi}{\varepsilon} \sum_{n \geq 0} \frac{(-1)^n}{n!} \sum_{l=0}^{n+1} \frac{1}{l+1} \frac{\partial^{n+1-l}}{\partial t^{n+1-l}} a_{jk}(r, t_0; \nu, n). \tag{44}
\]

Here,

\[
a_{jk}(r, t_0; \nu, n) = C_{j_1 \ldots i_n}^{(n+1,l)} \int_D x'_i \ldots x'_{i_n} [J_k]_0 d^3 x'.
\]

Similar to the derivation of equation (17) from equation (11), we get:

\[
a_{jk}(r, t_0; \nu, n) = -\alpha \varepsilon_{k i n q} C_{j_1 \ldots i_n}^{(n+1,l)} M_{i_1 \ldots i_{n-1} q}(t_0) + \frac{1}{n+1} C_{j_1 \ldots i_n}^{(n+1,l)} \tilde{P}_{i_1 \ldots i_n k}.
\]

Inserting this equation in equation (44), we obtain

\[
B(r, t) = \frac{\mu_0}{4\pi} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} e_i \varepsilon_{i j k} \sum_{l=0}^{n+1} \varepsilon_{k i n q} C_{j_1 \ldots i_n}^{(n+1,l)} \frac{\partial^{n+1-l}}{\partial t^{n+1-l}} M_{i_1 \ldots i_{n-1} q}
\]

\[
+ \frac{\mu_0}{4\pi\alpha} \sum_{n \geq 0} \frac{(-1)^n}{(n+1)!} e_i \varepsilon_{i j k} \sum_{l=0}^{n+1} \frac{1}{l+1} C_{j_1 \ldots i_n}^{(n+1,l)} \frac{\partial^{n+2-l}}{\partial t^{n+2-l}} P_{i_1 \ldots i_n k} \tag{45}
\]

As anticipated, it represents the multipolar expansion of the magnetic field for any point in the exterior of the domain \( D \) and it is obtained without inverting the spatial derivative with the integral operation. For the approximation of the radiated field (in \( 1/r \)), one can easily verify that writing equation (45) for \( l = 0 \) and taking

\[
C_{j_1 \ldots i_n}^{(n+1,0)} = \frac{(-1)^{n+1}}{e^{n+1}} \nu_1 \nu_2 \ldots \nu_n,
\]

we get equation (28).

In conclusion, employing Jefimenko’s equation in the radiation theory could bring a new insight only if the inversion of the spatial derivative and the integral operation is not allowed. We admit we were unable to find an interesting example where an inversion is not permitted, at least for generalized distributions. However, it might be possible to find such examples, and, in this case, the indispensable character of Jefimenko’s equations would be obvious. Otherwise, for the regular cases, it appears as an unnecessary complication.
V. SOME FEATURES OF THE RADIATED POWER CALCULATION

The Poynting vector is

\[ S = \frac{\alpha}{\mu_0} (\mathbf{E} \times \mathbf{B}) = \frac{c}{\mu_0} |\mathbf{B}_{\text{rad}}|^2 \nu + \mathcal{O}(1/r^3) = \varepsilon_0 c |\mathbf{E}_{\text{rad}}|^2 \nu + \mathcal{O}(1/r^3). \]  

(46)

The total radiated power may be written as the limit of a surface integral on a sphere centered in \( O \), of radius \( r \), for \( r \to \infty \). Let us express the energy current in the radiation approximation corresponding to the sphere of radius \( r \) at the time \( t \):

\[ N(S, \Sigma_r; t) = \oint_{\Sigma_r} r^2 \mathbf{v} \cdot \mathbf{S}(r, t) d\Omega(\nu) = \frac{c}{\mu_0} \oint_{\Sigma_r} r^2 |\mathbf{B}_{\text{rad}}(r, t)|^2 d\Omega(\nu) + \mathcal{O}(\frac{1}{r}). \]  

(47)

The quantity \( N(S, \Sigma_r; t) \) represents, therefore, the energy which crosses the sphere \( \Sigma_r \) in the time interval \((t, t + dt)\) and is determined by the values of the multipolar moments of the source in the interval \((t - r/c, t + dt - r/c)\). For large but finite \( r \), the integrand from equation (47) can be employed for drawing conclusions on the electric charge distribution at the retarded time from observations on the angular distribution of radiation. Since \( dt = dt_0 \), we can say that \( N(S, \Sigma_r; t) dt \) is the part of the energy emitted by source in the given time interval which contributes to the energy intensity corresponding to \( \Sigma_r \) at the time \( t \). As one can see from equation (28), if in equation (47) we put \( t \) instead \( t_0 \), then the quantity \( \lim_{r \to \infty} N(S, \Sigma_r; t) dt \) represents that part of the energy emitted by the source which contributes to the radiated energy or, shortly, radiated by the source. The situation changes when the support of the source depends on time and, in particular, for the radiation of a moving point-like source (see 1, & 73). In conclusion, the energy emitted by the source is characterized by the intensity

\[ I = \lim_{r \to \infty} N(S, \Sigma_r; t) = \frac{cr^2}{\mu_0} \int |\mathbf{B}_{\text{rad}}(r, t)|^2 d\Omega(\nu) = \frac{4\pi c}{\mu_0} \langle r^2 |\mathbf{B}_{\text{rad}}|^2 \rangle, \]  

(48)

where

\[ \langle f \rangle = \frac{1}{4\pi} \int f(\nu) d\Omega(\nu). \]

\( \mathbf{B}_{\text{rad}}(r, t) \) is given by equation (28) substituting the retarded time \( t_0 \) by \( t \). Since in equation (48) the factor \( r^2 \) is simplified by the factors \( 1/r \) included in the definition of \( M \) and \( P \) we should rather introduce the quantities \( \mu \) and \( \pi \):

\[ \mu(t; \nu, n) = r M(r, t; \nu, n), \quad \pi(t; \nu, n) = r P(r, t; \nu, n). \]  

(49)
The expression for the radiation intensity $I$ is

$$I(t) = \frac{\alpha^2}{4\pi \varepsilon_0 c^3} \sum_{n,m \geq 1} \frac{1}{n!m! c^{n+m}} \frac{\partial^{n+1}}{\partial t^{n+1}} \left\{ (\mathbf{\nu} \cdot \mathbf{\mu}(t; \mathbf{\nu}, n)) \mathbf{\nu} - \mathbf{\mu}(t; \mathbf{\nu}, n) - \frac{c}{\alpha} \mathbf{\nu} \times \mathbf{\pi}(t; \mathbf{\nu}, n) \right\}$$

Employing the notation $f_n = \partial^n f(\ldots, t)/\partial t^n$, and specifying only the argument $n$ in $\mathbf{\mu}$ and $\mathbf{\pi}$ when there is not a case of confusion, we can write

$$I(t) = \frac{\alpha^2}{4\pi \varepsilon_0 c^3} \sum_{n,m \geq 1} \frac{1}{n!m! c^{n+m}} \left\{ -\left(\mathbf{\nu} \cdot \mathbf{\mu}_{n+1}(n)\right) \left(\mathbf{\nu} \cdot \mathbf{\mu}_{m+1}(m)\right) \right.$$  

$$+ \mathbf{\mu}_{n+1}(n) \cdot \mathbf{\mu}_{m+1}(m) + \frac{c}{\alpha} \mathbf{\mu}_{n+1}(n) \cdot (\mathbf{\nu} \times \mathbf{\pi}_{m+1}(m))$$

$$+ \frac{c}{\alpha} (\mathbf{\nu} \times \mathbf{\pi}_{n+1}(n)) \cdot \mathbf{\mu}_{m+1}(m) + \frac{\epsilon^2}{\alpha^2} \{\mathbf{\pi}_{n+1}(n) \cdot \mathbf{\pi}_{m+1}(m)$$

$$- (\mathbf{\nu} \cdot \mathbf{\pi}_{n+1}(n)) (\mathbf{\nu} \cdot \mathbf{\pi}_{m+1}(m)) \} \right\}. \quad (51)$$

One can calculate the averaged quantities from the last equation using formula\(^6\):

$$\langle \nu_{i_1} \ldots \nu_{i_n} \rangle = \begin{cases} 0, & n = 2k + 1, \\ \frac{1}{(2k+1)!} \delta_{i_1i_2} \ldots \delta_{i_{n-1}i_n}, & n = 2k, \quad k = 0, 1, \ldots \end{cases} \quad (52)$$

In equation (51) all terms containing an odd number of factors $\nu$ vanish and we can retain only terms with an even number of these factors. We point out that $\mathbf{\mu}(n)$ or $\mathbf{\pi}(n)$ contain $n - 1$ factors $\nu$ (see equations (13), (18) and (49)).

We have to calculate expressions as, for example,

$$\langle (\mathbf{\nu} \cdot \mathbf{\mu}(t; \mathbf{\nu}, n)) (\mathbf{\nu} \cdot \mathbf{\mu}(t; \mathbf{\nu}, m)) \rangle = \langle \nu_{i_1} \nu_{j_1} \ldots \nu_{n-1} \nu_{j_1} \ldots \nu_{j_{m-1}} \rangle \mathbf{M}_{i_1 \ldots i_{n-1} i} \mathbf{M}_{j_1 \ldots j_{m-1} j};$$

This is cumbersome even for the first approximations.

Before discussing and applying the above results, we stress an essential issue for the existence of a precise approximation criteria when a finite number of terms in equation (52) is retained. Let us consider the source from $\mathcal{D}$ being a system of $N$ point-like electric charges $q_1 \ldots q_N$. Therefore,

$$\rho(r, t) = \sum_{i=1}^{N} q_i \delta(r - r^{(i)}(t)), \quad \mathbf{J}(r, t) = \sum_{i=1}^{N} q_i \mathbf{r}^{(i)}(t) \delta(r - r^{(i)}(t)), \quad (53)$$
where \( \mathbf{r}^{(i)}(t) \) represents the position vector of the particle \( i \). In this case,

\[
\mathbf{P}_{i_1 \ldots i_n}(t) = \sum_{i=1}^{N} q_i x_{i_1}^{(i)}(t) \ldots x_{i_n}^{(i)}(t)
\]

(54)

and

\[
\mathbf{M}_{i_1 \ldots i_n}(t) = \sum_{i=1}^{N} q_i x_{i_1}^{(i)}(t) \ldots x_{i_{n-1}}^{(i)}(t) (\mathbf{r}^{(i)} \times \dot{\mathbf{r}}^{(i)})_{i_n}(t).
\]

(55)

Let us suppose the particles oscillating with a pulsation \( \omega = \frac{2\pi c}{\lambda} \), i.e.

\[
\mathbf{r}^{(i)}(t) = \mathbf{r}^{(i)}_0 e^{i\omega t}, \quad \dot{\mathbf{r}}^{(i)}(t) = i\omega \mathbf{r}^{(i)}_0 e^{i\omega t} = \frac{2\pi i c}{\lambda} \mathbf{r}^{(i)}_0 e^{i\omega t}.
\]

(56)

(For the general case of \( N \) wave lengths \( \lambda_i, \ i = 1 \ldots N \), in the following, we will understand by \( \lambda \) the shortest of them.) Denoting by \( d \) the linear dimension of the domain \( D \), we have

\[ |\mathbf{r}^{(i)}_0| \leq d. \]

Considering the radiation in the case of a long wave-length, \( \lambda > d \), we introduce the parameter

\[
\zeta = \frac{d}{\lambda} < 1.
\]

(57)

For the amplitudes of the source, we identify the orders of magnitude:

\[
\frac{|\mathbf{r}^{(i)}|}{\lambda} \lesssim \zeta, \quad |\dot{\mathbf{r}}^{(i)}| \sim \frac{|\mathbf{r}^{(i)}|}{\lambda} \lesssim \zeta.
\]

(58)

Obviously, the same relations can be written for any \( x^{(k)}_i, \dot{x}^{(k)}_i \). For the time derivatives of the tensors \( \mathbf{P}^{(n)} \) and \( \mathbf{M}^{(n)} \) we can conclude that

\[
\frac{\partial^k}{\partial t^k} \mathbf{P}^{(n)} \sim \begin{cases} \zeta^n, & k \geq n \\ \zeta^k, & k \leq n \end{cases}, \quad \frac{\partial^k}{\partial t^k} \mathbf{M}^{(n)} \sim \begin{cases} \zeta^{n+1}, & k \geq n \\ \zeta^{k+1}, & k \leq n \end{cases}.
\]

(59)

With the approximation criteria considered above, we start by calculating the terms from equation (51) up to the 4-th order in the parameter \( \zeta \). For the electric and magnetic dipolar moments we use the usual notation \( p \) and \( m \), respectively. We select from equation (51) all nonvanishing terms for \( (n, m) = (1, 1) \):

\[
\mathcal{I}_{11} = \frac{\alpha^2}{4\pi \varepsilon_0 c^3} \left( -\langle \nu_i \nu_j \rangle \ddot{m}_i \ddot{m}_j + \dddot{m}^2 + \frac{c^2}{\alpha^2} (\ddot{p}^2 - \langle \nu_i \nu_j \rangle \dddot{p}_i \dddot{p}_j) \right).
\]

(60)

Inserting the result for the even combinations \( \langle \nu_i \nu_j \rangle \), we obtain the well-known expression for the radiation in the dipolar approximation:

\[
\mathcal{I}_{11} = \frac{1}{6\pi \varepsilon_0 c^3} \left( \dddot{p}^2 + \frac{\alpha^2}{c^2} \dddot{m}^2 \right).
\]

(61)
Now let us consider in equation (51) the terms with \((n, m) = (1, 2)\) and \((2, 1)\), discarding the vanishing ones:

\[
\mathcal{I}_{12} + \mathcal{I}_{21} = \frac{\alpha}{8\pi \varepsilon_0 c^5} \left\langle \ddot{\mathbf{\mu}}(1) \cdot (\mathbf{\nu} \times \dddot{\mathbf{\pi}}(2)) + (\mathbf{\nu} \times \dddot{\mathbf{\pi}}(1)) \cdot \dddot{\mathbf{\mu}}(2) \right\rangle. \tag{62}
\]

The first term, written explicitly with omission of dots representing the time derivatives of \(\mathbf{n}, \mathbf{\nu}\), is

\[
\left\langle \mathbf{\mu}(1) \cdot (\mathbf{\nu} \times \mathbf{\pi}(2)) \right\rangle = \left\langle m_i (\mathbf{\nu} \times \mathbf{\pi}(2))_i \right\rangle = \varepsilon_{ijk} m_i (\nu_j \nu_k) P_{qk} = \frac{1}{3} \varepsilon_{ijk} \delta_{jq} m_i P_{qk} = 0,
\]

because of the symmetry of the electric quadrupole moment. For the second term,

\[
\left\langle (\mathbf{\nu} \times \mathbf{\pi}(1)) \cdot \mathbf{\mu}(2) \right\rangle = \langle \varepsilon_{ijk} \nu_j \nu_k \nu_l q_i M_{qi} \rangle = \frac{1}{3} \delta_{kl} \varepsilon_{klj} M_{ij},
\]

and finally,

\[
\mathcal{I}_{12} + \mathcal{I}_{21} = -\frac{\alpha}{12\pi \varepsilon_0 c^5} \ddot{\mathbf{p}}_k \varepsilon_{klj} M_{ij}. \tag{63}
\]

Writing equation (51) for \((n, m) = (2, 2)\), we discard the terms of order 6 in \(\zeta\) as, for example, \(\ddot{\mathbf{\mu}}(2) \cdot \dddot{\mathbf{\mu}}(2)\). In this approximation,

\[
\mathcal{I}_{22} = \frac{1}{16\pi \varepsilon_0 c^5} \left\langle \mathbf{\pi}^2(2) \right\rangle + \mathcal{O}(\zeta^6)
\]

\[
= \frac{1}{16\pi \varepsilon_0 c^5} \left( \langle \nu_i \nu_j \rangle \ddot{P}_{ik} \ddot{P}_{jk} - \langle \nu_i \nu_j \nu_k \nu_l \rangle \ddot{P}_{il} \ddot{P}_{qj} \right) + \mathcal{O}(\zeta^6)
\]

\[
= \frac{1}{16\pi \varepsilon_0 c^5} \left( \frac{1}{3} \delta_{ij} \ddot{P}_{ik} \ddot{P}_{jk} - \frac{1}{15} \delta_{ij} \delta_{kl} \ddot{P}_{il} \ddot{P}_{qj} \right) + \mathcal{O}(\zeta^6)
\]

\[
= \frac{1}{16\pi \varepsilon_0 c^5} \left( \frac{1}{3} \ddot{P}_{ij} \ddot{P}_{ij} - \frac{1}{15} \ddot{P}_{il} \ddot{P}_{jl} - \frac{2}{15} \ddot{P}_{ij} \ddot{P}_{ij} \right) + \mathcal{O}(\zeta^6)
\]

\[
= \frac{1}{80\pi \varepsilon_0 c^5} \left( \dddot{P}_{ij} \dddot{P}_{ij} - \frac{1}{3} \dddot{P}_{ii} \dddot{P}_{ii} \right) + \mathcal{O}(\zeta^6). \tag{64}
\]

Terms of order 6 in \(\zeta\) exist also for \((m, n) = (1, 3)\) or \((3, 1)\). Retaining only terms of order 4, we get:

\[
\mathcal{I}_{13} + \mathcal{I}_{31} = \frac{1}{12\pi \varepsilon_0 c^5} \left\langle \dddot{\mathbf{\pi}}(1) \cdot \dddot{\mathbf{\pi}}(3) \right\rangle + \mathcal{O}(\zeta^6)
\]

\[
= \frac{1}{12\pi \varepsilon_0 c^5} \left( \langle \nu_l \nu_q \nu_p \rangle \dddot{P}_{lqi} - \langle \nu_i \nu_j \nu_l \nu_q \rangle \dddot{P}_{il} \dddot{P}_{qj} \right) + \mathcal{O}(\zeta^6)
\]

\[
= \frac{1}{3} \delta_{lq} \dddot{P}_{lqi} - \frac{1}{15} \delta_{ij} \delta_{kl} \dddot{P}_{il} \dddot{P}_{laj} + \mathcal{O}(\zeta^6)
\]

\[
= \frac{1}{90\pi \varepsilon_0 c^5} \dddot{P}_{qqi} + \mathcal{O}(\zeta^6). \tag{65}
\]
Adding up equations (61), (63), (64) and (65), we obtain the 4$-$th order approximation of the total radiated power:

$$I^{(4)} = \frac{1}{4\pi\varepsilon_0 c^3} \left( \frac{2}{3} \ddot{p}^2 + \frac{2}{3} \frac{\alpha^2}{c^2} \ddot{m}^2 - \frac{\alpha}{3c^2} \dddot{p}_k \varepsilon_{kij} M_{ij} + \frac{2}{45c^2} \dddot{p}_i P_{qqi} + \frac{1}{20c^2} \left( \dddot{p}_{ij} P_{ij} - \frac{1}{3} P_{qq} \right) \right).$$  (66)

The output can be partially compared with a well-known result from literature (see Refs. 1 and 2), but for this we have to introduce the *irreducible* electric and magnetic momenta defined as symmetric trace-free (“STF”) Cartesian tensors. Let us consider a $n$$-$th order tensor $T^{(n)}$ and the corresponding projections $S(T^{(n)})$ and $T(S(T^{(n)}))$ on the subspaces of symmetric and STF tensors. For the case of the electric moment $P^{(n)}$, this is a symmetric tensor and one has just to establish their STF projection. Let us consider the simplest case of the quadrupolar electric moment $P^{(2)}$. Writing the components $P_{ij}$ as

$$P_{ij} = \Pi_{ij} + \lambda \delta_{ij},$$

there is a unique value of the parameter $\lambda$ such that $\Pi^{(2)} = T(P^{(2)})$. For $\lambda = P_{qq}/3$, we have

$$\Pi_{ij} = P_{ij} - \frac{1}{3} P_{qq} \delta_{ij} = \int_D (x_i x_j - \frac{1}{3} x^2 \delta_{ij}) \rho \, d^3 x. \quad (67)$$

In equation (66) the octupolar electric moment $P^{(3)}$ is present. The STF projection can be calculated searching the first order tensor $\Lambda^{(1)}$ such that the STF projection $\Pi^{(3)} = T(P^{(3)})$ is given by the components

$$\Pi_{ijk} = P_{ijk} - \delta_{(ij} \Lambda_{k)}.$$

(68)

From the condition of vanishing traces of the tensor $\Pi^{(3)}$, one easily obtains:

$$\Lambda_i = \frac{1}{5} P_{qqi} = \frac{1}{5} \int_D r^2 x_i \rho \, d^3 x. \quad (69)$$

Concerning the magnetic quadrupolar moment $M^{(2)}$, we have a simple procedure for STF projection. Let us write the identity

$$M_{ij} = \frac{1}{2} (M_{ij} + M_{ji}) + \frac{1}{2} (M_{ij} - M_{ji}),$$

where the first bracket represents the symmetric part of this tensor, and the second one, the antisymmetric one. The symmetric part is, for this case ($n = 2$), a STF tensor $\Gamma^{(2)} = T(M^{(2)})$. Therefore,

$$M_{ij} = \Gamma_{ij} + \frac{1}{2} \varepsilon_{ijk} N_k,$$

(70)
\[
N_k = \varepsilon_{kij}M_{ij} = \frac{2}{3\alpha} \int_D (\mathbf{r} \times (\mathbf{r} \times \mathbf{J}))_k \, d^3x = \frac{2}{3\alpha} \int_D ((\mathbf{r} \cdot \mathbf{J}) \mathbf{r} - r^2 \mathbf{J})_k \, d^3x .
\] (71)

Since
\[
P_{ij}P_{ij} = \Pi_{ij}\Pi_{ij} + \frac{1}{3} P_{qq}^2,
\]
we obtain
\[
P_{ij}P_{ij} - \frac{1}{3} P_{qq}^2 = \Pi_{ij}\Pi_{ij} .
\] (72)

Combining the magnetic quadrupolar and the electric octupolar terms from equation (66), with their expressions from equations (67), (68), and (70), we get
\[
-\frac{\alpha}{3c^2} \ddot{\mathbf{t}} k \varepsilon_{kij} \dddot{M}_{ij} + \frac{2}{45c^2} \dddot{\mathbf{p}} k P_{qq} = \frac{4}{3c^2} \dddot{\mathbf{p}} k \left( \frac{\alpha}{4} \dddot{N}_k + \frac{1}{6} \dddot{\Lambda}_k \right) = -\frac{4}{3c^2} \dddot{\mathbf{p}} \cdot \mathbf{t}
\]
a consequence of the traceless character of \(\Pi^{(2)}, \Pi^{(3)}\) and \(\Gamma^{(2)}\). Here, the vector
\[
\mathbf{t} = \frac{\alpha}{4} \mathbf{N} - \frac{1}{6} \dddot{\Lambda} = \frac{1}{10} \int_D ((\mathbf{r} \cdot \mathbf{J}) \mathbf{r} - 2r^2 \mathbf{J}) \, d^3x
\] (73)
is introduced. The last expression is obtained from equations (71), (69) and applying the continuity equation together with an operation of partial integration. This is the so-called electric toroidal dipole moment \(^{16,17}\). We can write the radiation intensity \(I_{(4)}\) in terms of \textbf{STF} projections of electromagnetic momenta:
\[
I_{(4)} = \frac{1}{4\pi\varepsilon_0 c^3} \left( \frac{2}{3} \dddot{\mathbf{p}}^2 + \frac{2\alpha^2}{3c^2} \dddot{\mathbf{m}}^2 + \frac{1}{20c^2} \dddot{\Pi}_{ij}\dddot{\Pi}_{ij} - \frac{4}{3c^2} \dddot{\mathbf{p}} \cdot \dddot{\mathbf{t}} \right) .
\] (74)

Now, we can compare this last result with the one given by equation (71,5) from Ref.\(^1\) where the term corresponding to the quadrupolar electric moment is written in terms of the \textbf{STF} tensor \(D^{(2)}\) defined by the components \(D_{ij} = 3\Pi_{ij}\). We see that the expression given by our equation (74) differs from the equation in Ref.\(^1\) by the term \(-\frac{4}{3c^2} \dddot{\mathbf{p}} \cdot \dddot{\mathbf{t}},\) i.e. the contribution of the toroidal electric moment. In Ref.\(^1\), one calculates the contribution to the radiation of electric and magnetic dipolar and electric quadrupolar momenta. For this calculation, one uses the \textbf{STF} tensor \(D^{(2)}\) based on the invariance of the field to the substitution \(P_{ij} \rightarrow D_{ij}\) or, equivalently, the gauge transformation of potentials for such a substitution. We point out that this invariance is a singular case and, for higher orders
of the multipolar expansion it is not true. The toroidal electric moment does not appear because the orders of magnitude of the different terms in the multipolar expansion are not consequently evaluated. The same omission is done in Ref. In Refs., the toroidal dipole is obtained as a first term from a class of toroidal multipoles. The class is pointed out in a complete multipole analysis, appearing as additional terms beside the STF projections of the primitive moments $P$ and $M$. In Ref., our formula is obtained as a correction to the result of Ref. in order to assure a quantity origin independent, i.e. translation invariant. The toroidal contribution from equation is obtained in Ref., too, by a calculation close to the procedure from the present paper.

Let us consider the contribution of the electric quadrupolar moment which, as seen from equation, is proportional to the third time derivative of the vector

$$ V = \nu \times P = \frac{1}{r} e_i \varepsilon_{ijk} \nu_j P_{qk}. \quad (75) $$

The substitution

$$ P^{(2)} \rightarrow \Pi^{(2)} = T(P^{(2)}) \quad (76) $$

in equation (75) gives

$$ V \rightarrow V - \frac{1}{3r} e_i \varepsilon_{ijk} \nu_j \delta_{qk} P_{ll} = V - \frac{1}{3r} P_{ll}(\nu \times \nu) = V. $$

One can see that $B(r, t)$ and $E(r, t)$ given by equations and are also invariant if one performs the substitution (76). It is an exercise for the reader to prove that the substitution (76) in the expansions of the potentials $A$ and $\Phi$ given by equations and has as effect a gauge transformation of these potentials.

Unfortunately, this type of invariance is not valid for magnetic moments and, for $n \geq 3$, for electric ones. In these cases, the physical results are not invariant with respect to the transformations

$$ P^{(n)} \rightarrow T(P^{(n)}), \quad M^{(n)} \rightarrow T(M^{(n)}). \quad (77) $$

Let us suppose however that, at least for a given pair of numbers $(M, N)$, all the magnetic and electric moments for $m \leq M, n \leq N$ in the expression (51) of the radiation intensity can be substituted by STF tensors $\tilde{M}^{(m)}$, $\tilde{P}^{(n)}$. These new tensors do not necessarily coincide with the corresponding projections $T(M^{(m)})$, $T(P^{(n)})$ as it will be seen in the following. In
this situation, the calculation of the terms from equation (51) obtained by the corresponding substitutions \( \boldsymbol{\mu}(m) \rightarrow \tilde{\boldsymbol{\mu}}(m), \ \pi(n) \rightarrow \tilde{\pi}(n) \) is more easily performed. Indeed, if \( \mathbf{A}^{(n)}, \ \mathbf{B}^{(m)} \) are STF tensors, then:

\[
\langle (\nu_{i_1}, \ldots \nu_{i_k} A_{i_1 \ldots i_n})(\nu_{i_{k+1}}, \ldots \nu_{i_{k+l}} B_{i_{k+1} \ldots i_{m+n}}) \rangle = \frac{k! \delta_{kk'}}{(2k + 1)!!} A_{i_1 \ldots i_n} B_{i_{k+1} \ldots i_{m+n}} \equiv \frac{k! \delta_{kk'}}{(2k + 1)!!} \left( \mathbf{A}^{(n)} \cdot \mathbf{B}^{(m)} \right).
\]

(78)

Here, since \( \mathbf{A}^{(n)} \) and \( \mathbf{B}^{(m)} \) are symmetric tensors, it is of no importance what is the order of factors in the contraction \( \mathbf{A}^{(n)} \cdot \mathbf{B}^{(m)} \) ( noted as \( \mathbf{A}^{(n)}||\mathbf{B}^{(m)} \), too). Obviously, the result of a such contraction is a STF tensor of \(|n - m| - \text{th}\) order. All the terms in equation (51) can be calculated using equation (78) except for those containing vectorial products which vanish as consequence of the equation (52) and of the symmetry properties of the tensors \( \mathbf{P}^{(n)}, \ \mathbf{M}^{(m)} \). Let us apply equation (78) to the term:

\[
\langle \tilde{\mathbf{\mu}}(t; \nu, n) \cdot \tilde{\mathbf{\mu}}(t; \nu, m) \rangle = \langle \nu_{i_1} \ldots \nu_{i_{n-1}} \nu_{j_1} \ldots \nu_{j_{m-1}} \rangle \tilde{M}_{i_1 \ldots i_{n-1}}(t) \tilde{M}_{j_1 \ldots j_{m-1}}(t) = \frac{(n - 1)! \delta_{nm}}{(2n - 1)!!} \left( \tilde{\mathbf{M}}^{(n)}(t) \cdot \tilde{\mathbf{M}}^{(m)}(t) \right),
\]

(79)

and

\[
\langle (\nu \cdot \tilde{\mathbf{\mu}}(t; \nu, n))(\nu \cdot \tilde{\mathbf{\mu}}(t; \nu, m)) \rangle = \frac{n! \delta_{nm}}{(2n + 1)!!} \left( \tilde{\mathbf{M}}^{(n)}(t) \cdot \tilde{\mathbf{M}}^{(m)}(t) \right).
\]

(80)

The expressions for alike terms are similar. The terms including vectorial products as, for example:

\[
\langle \tilde{\mathbf{\mu}}(t; \nu, n) \cdot (\nu \times \tilde{\pi}(t; \nu, m)) \rangle = \langle \tilde{\mu}_i(t; \nu, n) \varepsilon_{ijk} \nu_j \tilde{\pi}_k(t; \nu, m) \rangle = \varepsilon_{ijk} \langle \nu_{i_1} \ldots \nu_{i_{n-1}} \nu_j \nu_{j_1} \ldots \nu_{j_{n-1}} \rangle \tilde{M}_{i_1 \ldots i_{n-1}} \tilde{P}_{j_1 \ldots j_{n-1}} k = 0
\]

(81)

vanish, since each term from the average \( \langle \nu_{i_1} \ldots \nu_{i_{n-1}} \nu_j \nu_{j_1} \ldots \nu_{j_{n-1}} \rangle \) given by equation (52) includes either \( \delta_{ji_q}, q = 1 \ldots n - 1 \) or \( \delta_{ji_q}, q = 1 \ldots m - 1 \) or directly an odd number of \( \nu \).

Using these results, we obtain the final expression of the (total) radiated power:

\[
I(t) = \frac{\alpha^2}{4 \pi \varepsilon_0 c^3} \sum_{n \geq 1} \frac{n + 1}{n \cdot (2n + 1)!!} c^{2n} \left( \langle \tilde{\mathbf{M}}^{(n)}_{n+1}(t) \cdot \tilde{\mathbf{M}}^{(n)}_{n+1}(t) \rangle + \frac{c^2}{\alpha^2} \left( \tilde{\mathbf{P}}^{(n)}_{n+1}(t) \cdot \tilde{\mathbf{P}}^{(n)}_{n+1}(t) \right) \right)
\]

(82)

where we consider the limit for \( M, N \rightarrow \infty \). We point out that considering the infinite sum in equation (82), makes the tensors \( \tilde{\mathbf{M}}^{(n)}, \ \tilde{\mathbf{P}}^{(n)} \) infinite series. This is not so catastrophic since for practically applications only a finite number of terms from equation (82) is necessary,
as it will be seen in the following. Obviously, one can apply equation (59) for the STF-projections of the multipole momenta.

For an answer to the question how one can use equation (82) for calculating the radiation intensity, let us return to the problem of calculating this quantity up to the $4-th$ order in the parameter $\zeta$. Since, together with $P^{(2)}$, the momenta $M^{(2)}$ and $P^{(3)}$ give contributions in this order, we have to consider in equation (82) the STF projections $\Gamma^{(2)}$ and $\Pi^{(3)}$, too. This time the field is not invariant for the substitutions

$$ M^{(2)} \rightarrow \Gamma^{(2)}, \quad P^{(3)} \rightarrow \Pi^{(3)}. \tag{83} $$

Let us calculate the effect of the first substitution from equation (83) in equation (28) for expressing the radiated magnetic field. The terms from the expansion series of $\mathbf{B}$ affected by the substitution are

$$ \frac{1}{2c^3 \partial t^3} \left\{ (\mathbf{\nu} \cdot \mathbf{M}(r, t_0; \mathbf{\nu}, 2)) \mathbf{\nu} - \mathbf{M}(r, t_0; \mathbf{\nu}, 2) \right\} = \frac{1}{2c^3 \partial t^3} \left( \mathbf{e}_i \nu_i \nu_j \nu_l M_{lj} - \mathbf{e}_i \nu_j \mathbf{M}_{ji} \right) $$

$$ \mathbf{M}^{(2)} \rightarrow \mathbf{\Gamma}^{(2)} \rightarrow \frac{1}{2c^3 \partial t^3} \left\{ \mathbf{e}_i \nu_i \nu_j \nu_l M_{lj} - \frac{1}{2} \mathbf{e}_i \nu_j \nu_l \nu_i \epsilon_{ijk} \mathbf{N}_k - \mathbf{e}_i \nu_j \mathbf{M}_{ji} + \frac{1}{2} \mathbf{e}_i \nu_j \epsilon_{ijk} \mathbf{N}_k \right\} $$

$$ \mathbf{\Pi}^{(3)} = \mathbf{\epsilon} \mathbf{N}_i = \mathbf{N}^{(1)}. \quad \text{From equation (28) we can see that the alteration of } \mathbf{B} \text{ by the substitution } \mathbf{M}^{(2)} \rightarrow \Gamma^{(2)} \text{ is compensated by the following transformation of the dipolar electric moment:} $$

$$ \mathbf{p} \rightarrow \mathbf{p} - \frac{\alpha}{4c^2} \dot{\mathbf{N}}. \tag{84} $$

Let us consider the substitution $P^{(3)} \rightarrow \Pi^{(3)}$ in equation (28). The term affected is:

$$ \frac{1}{6\alpha c^3 \partial t^4} (\mathbf{\nu} \times \mathbf{P}(r, t_0; \mathbf{\nu}, 3)) = -\frac{1}{6\alpha c^3 r} \mathbf{e}_i \epsilon_{ijk} \nu_j \nu_l \nu_i \frac{\partial^4}{\partial t^4} \mathbf{P}_{qlk} $$

$$ \mathbf{P}^{(3)} \rightarrow \mathbf{\Pi}^{(3)} \rightarrow -\frac{1}{6\alpha c^3 r} \frac{1}{\partial t^4} \left( \epsilon_{ijk} \nu_j \nu_l \nu_i \mathbf{P}_{qlk} - \epsilon_{ijk} \nu_j \nu_l \nu_i \delta_{ql} \mathbf{\Lambda}_k \right) $$

$$ = -\frac{1}{6\alpha c^3 \partial t^4} (\mathbf{\nu} \times \mathbf{P}(r, t_0; \mathbf{\nu}, 3)) + \frac{1}{6\alpha c^3 r} \left( \mathbf{\nu} \times \mathbf{\Lambda} \right), $$

where $\mathbf{\Lambda} = \mathbf{e}_i \mathbf{\Lambda}_i$. The alteration of $\mathbf{B}$ by the substitution $\mathbf{P}^{(3)} \rightarrow \Pi^{(3)}$ is compensated by the following transformation of the dipolar electric moment:

$$ \mathbf{p} \rightarrow \mathbf{p} + \frac{1}{6c^2} \ddot{\mathbf{\Lambda}}. \tag{85} $$
The total alteration of $B$ by the two substitutions (83) is compensated by the transformation

$$p - \tilde{p} = p - \frac{1}{c^2} \left( \frac{\alpha}{4} \dot{N} - \frac{1}{6} \ddot{\Lambda} \right) = p - \frac{1}{c^2} t,$$

with $t$ defined by equation (83).

Now, in equation (82) we have to consider $\tilde{P}(1) = \tilde{p}$ given by equation (86), $\tilde{M}(1) = m$, $\tilde{P}(2) = \Pi(2)$, $\tilde{M}(2) = \Gamma(2)$, $\tilde{P}(3) = \Pi(3)$ and, for $n \geq 4$, the primitive momenta which contribute only with terms of orders greater than 4 in $\zeta$. We can write:

$$I(t) = \frac{\alpha^2}{4\pi \varepsilon_0 c^3} \left\{ \frac{2}{3c^2} \left( \dot{\bar{m}}^2 + \frac{c^2}{\alpha^2} (\dot{\bar{p}} - \frac{1}{c^2} t)^2 \right) + \frac{1}{20c^2\alpha^2} \Pi^{(2)} \cdot \Pi^{(3)} \right\} + \ldots.$$

The quadrupolar magnetic and octupolar electric terms are not written because these are of order 6 in $\zeta$. Since $t \sim \zeta^2$, from the last expression of $I$ we are left with equation (74).

This calculation offers a suitable example for understanding the general scheme of tensor reduction for electric and magnetic moments, as well as for the usage of formula (82) when describing the radiated power. Replacing an electric moment of order $n$ by its STF-projection induces in the tensors of inferior order, compensating terms of order $\zeta^n$. The general term in the expansion (82) is of order $\zeta^{2n}$. If we are interested in the approximation of $I$ up to terms of order $\zeta^{2k}$, then we have to replace the tensors related to the moments $P(2k-1)$ and $M(2k-2)$ by STF-projections and to take into account all compensating terms for the lower-order tensors.

VI. SOME BASIC FORMULEA FOR GENERALIZING THE PROCEDURE OF REDUCING THE ELECTRIC AND MAGNETIC MOMENTUM TENSORS

In this section we list the required formulae for the generalized tensor reducing procedure. The equations are given without the related proofs since they can be found in the literature.

Let $S^{(n)}$ be a symmetric tensor of rank $n$. Its STF-projection results from the following formula:

$$I(S^{(n)})_{i_1 \ldots i_n} = S_{i_1 \ldots i_n} - \delta_{i_1 i_2} \Lambda(S^{(n)})_{i_3 \ldots i_n}. \quad (87)$$

The operator $\Lambda$ is defined on the account of a formula for the STF-projection of a symmetric tensor given in $^6$ (the book $^2$ is cited as the origin of this formula):

$$\Lambda(S^{(n)})_{i_1 \ldots i_{n-2}} = \sum_{m=0}^{[n/2-1]} \frac{(-1)^m [2n - 1 - 2(m+1)]!!}{(m+1)(2n-1)!!} \delta_{i_1 i_2} \cdots \delta_{i_{2m-1} i_{2m}} S_{i_{2m+1} \ldots i_{n-2}}^{(m+1)}, \quad (88)$$
The proof can be found in Ref.\textsuperscript{22}. \([a]\) stands for the integer part of \(a\) and the notation \(S^{(n,p)}\) indicates \(p\) pairs of contracted indices. In the following, for simplifying the notation, all arguments of the operator \(A\) should be considered as symmetric tensor i.e. \(A(T^{(n)}) = A(S(T^{(n)}))\) for any tensor \(T^{(n)}\). The same applies to the operator \(T\): \(T(T^{(n)}) = T(S(T^{(n)})), S\) being the symmetrization operator.

In the symmetrization process we have to calculate the symmetric projections of some tensors \(L^{(n)}\) of the magnetic moment type: they are symmetric in the first \(n - 1\) indices and the contraction of \(i_n\) with any index \(i_q, q = 1 \ldots n - 1\) gives a null result. For the symmetric projection of such a tensor, we introduce the formula:

\[
(S(L^{(n)}))_{i_1 \ldots i_n} = L_{i_1 \ldots i_n} - \frac{1}{n} \sum_{\lambda=1}^{n-1} \varepsilon_{i_\lambda i_\lambda i_q} \mathcal{N}^{(\lambda)}_{i_1 \ldots i_{n-1} i_q}(L^{(n)}),
\]

(89)

where \(\mathcal{N}^{(\lambda)}_{i_1 \ldots i_{n-1}}\) is the component with the \(i_\lambda\) index suppressed. The operator \(\mathcal{N}\) defines a correspondence between \(L^{(n)}\) and a tensor of rank \((n - 1)\) of the same type from the symmetry point of view. Particularly,

\[
\mathcal{N}^{2k}(M^{(n)}) = \frac{(-1)^k n}{(n + 1)\alpha} \int_D (r^2)^k r^{n-2k} \times J \, d^3 x,
\]

\[
\mathcal{N}^{2k+1}(M^{(n)}) = \frac{(-1)^k n}{(n + 1)\alpha} \int_D (r^2)^k r^{n-2k-1} \times (r \times J) \, d^3 x, \quad k = 0, 1, 2, \ldots
\]

(90)

where \(a^n \times b\) is the tensor defined by the components \((a^n \times b)_{i_1 \ldots i_n} = a_{i_1} \ldots a_{i_{n-1}} (a \times b)_{i_n}\).

Let us consider the process of STF reducing the multipole tensors starting with the rank \(N + 1\) in the electric case, and \(N\) in the magnetic one. In Ref.\textsuperscript{23} one finds a general formula for the resulting tensors \(\tilde{P}^{(n)}\), for \(n = 1, \ldots, N\) and \(\tilde{M}^{(n)}\), for \(n = 1, \ldots, N - 1\). It includes all compensating terms obtained in this process

\[
\tilde{P}^{(n)} = T(P^{(n)}) + \sum_{k=1}^{[(N-n)/2]} \left( (-1)^k c^{2k} \frac{\partial^{2k-1}}{\partial t^{2k-1}} T^{(n)}_k \right),
\]

\[
T^{(n)}_k = (-1)^k c^{2k} T \left( A_k^{(n)} \Lambda^k (P^{(n+2k)}) + \sum_{l=0}^{k-1} B_{k-l, l}^{(n)} \Lambda^l \mathcal{N}^{2k-2l-1}(M^{(n+2k-1)}) \right),
\]

(91)

and

\[
\tilde{M}^{(n)} = T(M^{(n)}) + T \left( \sum_{k=1}^{[(N-n-1)/2]} \frac{\partial^{2k}}{\partial t^{2k}} \sum_{l=0}^{k} C_{kl}^{(n)} \Lambda^l \mathcal{N}^{2k-2l}(M^{(n+2k)}) \right).
\]

(92)
The coefficients are given in a compact algebraic form in Ref. 10:

\[ A^{(n)}_k = \frac{1}{2^k c^{2k}} \frac{n}{n + 2k}, \]
\[ B^{(n)}_{k,l} = \frac{(-1)^{k+l+1} \alpha}{2^l c^{2k+2}} \frac{n(n + 2l)}{(n + 2k + 1)(n + 2k + 1)!}, \]
\[ C^{(n)}_{k,l} = \frac{(-1)^{k-l}}{2^l c^{2k}} \frac{n(n + 2l)}{(n + 2k)(n + 2k)!}, \]

(93)

The reader is encouraged to apply these formulae for the cases \( N = 4, 5, 6 \) and to find convenient intermediary calculation. We point out that in equation (91) the normalisation of the quantities \( T^{(n)} \) is chosen such that the quantities \( T^{(n)}_1 \) coincide with the electric toroidal moments given in literature at least for \( n \) up to 3.

VII. CONCLUSION AND DISCUSSION

After an introduction on the formalism for multipolar expansions of the electric and magnetic field, we dedicated sections III and IV to the main purpose of this article: the analysis of different methods for calculating the radiated field with an emphasis on the novelty the use of Jefimenko’s equations can bring. In the last two sections we presented aspects of the radiated power calculations from the tensorial point of view.

From the analysis of sections III and IV we can draw some conclusions on the utility of Jefimenko’s equations when considering the multipolar expansion problem. As one can notice, when deriving equation (10) from (9), no matter if we work with potentials or directly with fields, the inversion of the spatial derivative with the integral on the domain \( D \) is mandatory. In Section IV it was proven that the multipole expansion of the fields \( E \) and \( B \) can be obtained generally and directly from Jefimenko’s equations. All expanding operations are performed on the integrand, but, still under the assumption that the integration operation is distributive with respect to the Taylor series. It remains only to argue the necessity of the corresponding additional calculation effort for applying this procedure.

As an additional remark, we emphasize that in the same sections we tried to remind the reader that if one wants to completely describe the radiative systems, one has to include besides terms of order \( 1/r \), the \( 1/r^2 \) contributions from the field expansions.

Ref. 5 is part of a paper series trying to emphasize the theoretical and practical importance of Jefimenko’s equations. These equations are considered as extraordinarily powerful and
illuminating as the authors of Ref.\textsuperscript{13} write. We have nothing against the open enthusiasm in these papers. We neither dispute the beauty of the result regarding the calculation of the retarded fields $E$ and $B$ directly from Maxwell’s equations, without having to introduce and handle the potentials. Maybe a series of applications based on these equations are more efficacious and physically more transparent. Although, we are circumspect concerning the axiomatic treatment of the electromagnetic theory starting from these equations (opposite to opinions from e.g. Refs.\textsuperscript{14,15}), but, this subject will be discussed elsewhere.

In the second part of the paper we gave a detailed calculation for the radiated power up to the $4^{th}$ order in $d/\lambda < 1$. The method can be similarly applied to higher-orders. We related our procedure to existing prescriptions in the literature and we underlined what omissions might show when particular terms of the multipolar expansions are neglected. We hope we were able to convince the reader how powerful and not so complicated a consistent vectorial/tensorial computation is.

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\* Electronic address: vrejoiu@fizica.unibuc.ro;roxana.zus@fizica.unibuc.ro

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