The Bishop-Phelps-Bollobás property for operators on $C(\mathcal{K})$

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Dedicated to Rafael Payá on the occasion of his 60th birthday.

Abstract. We provide a version for operators of the Bishop-Phelps-Bollobás Theorem when the domain space is the complex space $C_0(L)$. In fact we prove that the pair $(C_0(L), Y)$ satisfies the Bishop-Phelps-Bollobás property for operators for every Hausdorff locally compact space $L$ and any $C$-uniformly convex space. As a consequence, this holds for $Y = L_p(\mu)$ ($1 \leq p < \infty$).

1. Introduction

Bishop-Phelps Theorem states the denseness of the subset of norm attaining functionals in the (topological) dual of a Banach space [11]. Since the Bishop-Phelps Theorem was proved in the sixties, some interesting papers provided versions of this result for operators. Related to those results, it is worth to mention the pioneering work by Lindenstrauss [29], the somehow surprising result obtained by Bourgain [14] and also results for concrete classical Banach spaces (see below). In full generality there is no parallel version of Bishop-Phelps Theorem for operators even if the domain space is $c_0$ [29]. Lindenstrauss also provided some results of denseness of the subset of norm attaining operators by assuming some isometric properties either on the domain or on the range space [29]. We mention here two concrete consequences of these results. If the domain space is $\ell_1$ or the range space is $c_0$, every operator can be approximated by norm attaining operators. First Lindenstrauss [29] and later Bourgain [14] proved that certain isomorphic assumptions on the domain space (reflexivity or even Radon-Nikodým property, respectively) implies the denseness of the subset of norm attaining operators in the corresponding space of linear (bounded) operators. For classical Banach spaces, we only mention some papers containing positive results for specific pairs (see for instance [23], [21], [33], [34], [7]) and a few containing counterexamples (see [33], [24], [20], [5] and [11]). The paper [31] also answers an old open problem in the topic.

Recently the paper [2] dealt with “quantitative” versions of the Bishop-Phelps Theorem for operators. The motivating result is known nowadays as Bishop-Phelps-Bollobás Theorem [12, 13] and has been a very useful tool to study numerical ranges of operators (see for instance [13]). This result can be stated as follows.

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Let $X$ be a Banach space and $0 < \varepsilon < 1$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1 - x^*(x)| < \varepsilon^2$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.

Here $X^*$ denotes the (topological) dual of the Banach space $X$ and $S_X$ its unit sphere. We write $B_X$ to denote the closed unit ball of $X$.

Throughout this paper, for two Banach spaces $X$ and $Y$, $L(X, Y)$ is the space of linear bounded operators from $X$ into $Y$. We recall that the pair $(X, Y)$ has the Bishop-Phelps-Bollobás property for operators (BPBp), if for any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that for any $T \in S_{L(X, Y)}$, if $x_0 \in S_X$ is such that $\|Tx_0\| > 1 - \eta(\varepsilon)$, then there exist an element $u_0 \in S_X$ and an operator $S \in S_{L(X, Y)}$ satisfying the following conditions:

$$\|Su_0\| = 1, \quad \|u_0 - x_0\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon.$$ 

Acosta et al proved that for any space $Y$ satisfying the property $\beta$ of Lindenstrauss, the pair $(X, Y)$ has the BPBp for operators for every Banach space $X$ [2] Theorem 2.2]. For the domain space, there is no a reasonably general property implying a positive result. However there are some positive results in concrete cases. For instance, there is a characterization of the spaces $Y$ such that the pair $(\ell_1, Y)$ satisfies the BPBp [2]. As a consequence of this result, it is known that this condition is satisfied by finite-dimensional spaces, uniformly convex spaces, $C(K)$ (K is some compact topological space) and $L_1(\mu)$ (any measure $\mu$). Aron et al showed that the pair $(L_1(\mu), L_{\infty}([0, 1]))$ has also the BPBp for every $\sigma$-finite measure $\mu$ [9]. This result has been extended recently by Choi et al (see [17]). Some related results for operators whose domain is $L_1(\mu)$ can be also found in [16], [4] and [17].

Now we point out results stating that the pair $(X, Y)$ has the BPBp in case that the domain space is $C_0(L)$ (space of continuous functions on a locally compact Hausdorff space $L$ vanishing at infinity). Kim proved that in the real case the pair $(\ell_0, Y)$ has the BPBp for operators whenever $Y$ is uniformly convex [26]. The paper [3] contains also a positive result for the pair $(C(K), C(S))$ in the real case ($K$ and $S$ are compact Hausdorff spaces). Let us point out that in the complex case it is not known yet if the subset of norm attaining operators from $C(K)$ to $C(S)$ is dense in $L(C(K), C(S))$. Very recently Kim, Lee and Lin [28] proved that the pair $(L_{\infty}(\mu), Y)$ has the BPBp whenever $Y$ is a uniformly convex space and $\mu$ is any positive measure. The authors also state the analogous result in complex case for the pairs $(\ell_0, Y)$ and $(L_{\infty}(\mu), Y)$ (\mu is any positive measure) whenever $Y$ is a $\mathbb{C}$-uniformly convex space. It also holds that the pair $(C(K), Y)$ has the BPBp in the real case for any uniformly convex space [27].

In this paper we show that the subspace of weakly compact operators from $C_0(L)$ into $Y$ satisfies the Bishop-Phelps-Bollobás property for operators in the complex case, for every locally compact Hausdorff space $L$ and for any $\mathbb{C}$-uniformly convex (complex) space. Let us notice that this is an extension of the result in [28] for the complex case in two ways. First we consider any space $C(K)$ instead of $L_{\infty}(\mu)$ as the domain space and also we consider a strictly more general property on the range space, namely $\mathbb{C}$-uniform convexity instead of uniform convexity. Our result extends [2] Theorem 5.2] in a satisfactory way and the recent result in [28] for the case that the domain space is $L_{\infty}(\mu)$.
As a consequence, in the complex case the pair \((C(K), L_1(\mu))\) has the BPBp for every compact Hausdorff space \(K\) and any measure \(\mu\).

Let us recall again that it is not trivial at all to obtain the result in the complex case from the real case when the domain space is \(C(K)\). As we already pointed out, it is an open problem whether or not the subset of norm attaining operators between complex spaces \(C(K)\) and \(C(S)\) is dense in \(L(C(K), C(S))\). However a positive result for real \(C(K)\) spaces was proved many years ago [23].

Let us notice that in case that the range space is \(C(K)\) or more generally a uniform algebra, the papers [8] and [15] provides positive results for the BPBp for the class of Asplund operators.

2. The result

Throughout this section it worths to consider only complex normed spaces. For a complex Banach space \(Y\), recall that the \(C\)-modulus of convexity \(\delta\) is defined for every \(\varepsilon > 0\) by

\[
\delta(\varepsilon) = \inf \left\{ \sup \{ \| x + \lambda \varepsilon y \| - 1 : \lambda \in \mathbb{C}, |\lambda| = 1 \} : x, y \in S_Y \right\}.
\]

Recall that the Banach space \(Y\) is \(C\)-uniformly convex if \(\delta(\varepsilon) > 0\) for every \(\varepsilon > 0\) [19]. Every uniformly convex complex space is \(C\)-uniformly convex and the converse is not true. Globevnik proved that the complex space \(L_1(\mu)\) is \(C\)-uniformly convex [19, Theorem 1].

We will denote by \(\overline{D}(0,1)\) the closed unit disc in \(\mathbb{C}\). Let us notice that for \(0 < s < t\) it is satisfied that \(\sup \{ \| x + \lambda sy \| : \lambda \in \overline{D}(0,1) \} \leq \sup \{ \| x + \lambda ty \| : \lambda \in \overline{D}(0,1) \}\). Hence \(\delta\) is an increasing function and \(\delta(t) \leq t\) for every \(t > 0\).

In what follows \(L\) will be a locally compact Hausdorff topological space and \(C_0(L)\) will be the space of continuous complex valued functions on \(L\) vanishing at infinity.

It is convenient to state the next trivial result

**Lemma 2.1.** Assume that \(\lambda, w \in \overline{D}(0,1), t \in ]0,1[\) and \(\text{Re } w\lambda > 1 - t\). Then \(|w - \overline{\lambda}| < \sqrt{2t}\).

As we already mentioned, the subset of norm attaining operators between two Banach spaces is not always dense in the corresponding space of operators in case that the domain space is \(C_0(L)\). Let us notice that there are examples of spaces \(Y\) for which the subspace of finite-dimensional operators from the space \(\ell_2^\infty\) to \(Y\) does not have the Bishop-Phelps-Bollobás property (see [2] Theorem 4.1 and Proposition 3.9 or [10] Corollary 3.3]). For those reasons some restriction is needed on the range space in order to obtain a BPBp result in case that the domain space is \(C_0(L)\).

Schachermayer proved a Bishop-Phelps result in the real case for the subspace of weakly compact operators from any space \(C_0(L)\) into any Banach space [33]. Alaminos et al extended this result to the complex case [6]. The last result is one of the tools essentially used in the proof of the main result. This is our motivation for the next assertion, that might be known, and has interest in itself.
Proposition 2.2. Let \( Y \) be a \( \mathbb{C} \)-uniformly convex Banach space and \( L \) any locally compact Hausdorff space. Then every operator from \( C_0(L) \) into \( Y \) is weakly compact.

Proof. By the proof of James distortion Theorem (see for instance [30, Proposition 2.6.3]) the space \( Y \) cannot contain a copy of \( c_0 \) (the space of complex sequences converging to zero, endowed with the usual norm). Otherwise, by considering a convenient multiple of the norm in \( Y \), \( \| \| \), that it is still \( \mathbb{C} \)-uniformly convex, one can assume that the usual norm of the copy of \( c_0 \) (\( \| \| \)) satisfies

\[
\alpha \| x \| \leq \| x \| \leq \| x \| \quad \forall x \in \mathbb{X},
\]

for some \( \alpha > 0 \). By the proof of [30, Proposition 2.6.3], for any \( \varepsilon > 0 \) there is a sequence \( (y_n) \) in \( Y \) of block basis of the usual basis of \( c_0 \) satisfying

\[
\sum_{k=1}^{\infty} a_n y_n \leq (1 + \varepsilon)^2 \| (a_n) \|_\infty \quad \forall (a_n) \in c_0
\]

and \( \| \sum_{k=1}^{\infty} a_n y_n \| = \| (a_n) \|_\infty \) for every \( (a_n) \in c_0 \). Clearly the above condition contradicts the \( \mathbb{C} \)-uniform convexity of \( Y \).

Now, in view of Bessaga-Pelczynski selection principle, if the underlying real space of a complex space contains a real space isomorphic to \( c_0 \), then it contains the complex space \( c_0 \). So \( Y_\mathbb{R} \) does not contain the real space \( c_0 \), hence for any compact space \( K \), every (real) operator from the space \( C(K) \) (real valued functions) into \( Y \) is weakly compact. As a consequence, every operator from the complex space \( C(K) \) into \( Y \) is also weakly compact. From here it can be easily deduced that every operator from \( C_0(L) \) into \( Y \) is weakly compact, since \( C_0(L) \) is complemented in the space \( C(K) \), being \( K \) the Alexandrov compactification of \( L \). Hence every operator from \( C_0(L) \) into \( Y \) can be extended to an operator from \( C_0(L) \) into \( Y \).

For a locally compact Hausdorff topological space \( L \), we denote by \( \mathcal{B}(L) \) the space of Borel measurable and bounded complex valued functions defined on \( L \), endowed with the sup norm. If \( B \subset L \) is a Borel measurable set, denote by \( P_B \) the projection \( P_B : \mathcal{B}(L) \to \mathcal{B}(L) \) given by \( P_B(f) = f|_B \) for any \( f \in \mathcal{B}(L) \). Of course, in view of Riesz Theorem, the space \( \mathcal{B}(L) \) can be identified in a natural way as a subspace of \( C_0(L)^* \). As a consequence, for an operator \( T \in \mathbb{L}(C_0(L), Y) \) and a Borel set \( B \subset L \), the composition \( T^{**}P_B \) makes sense.

Lemma 2.3. Let \( Y \) be a \( \mathbb{C} \)-uniformly convex space with modulus of \( \mathbb{C} \)-convexity \( \delta \). Let \( L \) be a locally compact Hausdorff topological space and \( A \) a Borel set of \( L \). Assume that for some \( 0 < \varepsilon < 1 \) and \( T \in \mathcal{S}_L(C_0(L), Y) \) it is satisfied \( \| T^{**}P_A \| > 1 - \frac{\delta(\varepsilon)}{1+\delta(\varepsilon)} \). Then \( \| T^{**}(I-P_A) \| \leq \varepsilon \).

Proof. Assume that \( T \) satisfies the assumptions of the result. By Proposition 2.2 \( T \) is a weakly compact operator, so \( T^{**}(C_0(L))^* \subset Y \) and we consider the subspace \( \mathcal{B}(K) \subset C_0(L)^* \). We write \( \eta = \frac{\delta(\varepsilon)}{1+\delta(\varepsilon)} \). By the assumption, there exists \( f \in \mathcal{S}_L \) such that \( f = P_A(f) \) and \( \| T^{**}(f) \| > 1 - \eta > 0 \). For every \( g \in \mathcal{B}(L) \) it is satisfied that \( \| f + (I-P_A)(g) \| \leq 1 \) and so \( \| T^{**}(f + \lambda(I-P_A)g) \| \leq 1 \) for every \( \lambda \in \mathbb{D}(0,1) \). That is,
for any $\lambda \in \overline{D}(0,1)$ we have
\[
\left\| \frac{T^{**}(f)}{\|T^{**}(f)\|} + \lambda \frac{T^{**}(I - P_A)(g)}{\|T^{**}(f)\|} \right\| \leq \frac{1}{\|T^{**}(f)\|} < \frac{1}{1 - \eta} = 1 + \delta(\varepsilon).
\]

As a consequence $\|T^{**}(I - P_A)(g)\| \leq \varepsilon \|T^{**}(f)\| \leq \varepsilon$ and so $\|T^{**}(I - P_A)\| \leq \varepsilon$.

\begin{theorem}
The pair $(C_0(L), Y)$ satisfies the Bishop-Phelps-Bollobás property for operators for any locally compact Hausdorff topological space $L$ and any $C$-uniformly convex space $Y$. Moreover the function $\eta$ appearing in the Definition of BPBp depends only on the modulus of convexity of $Y$.
\end{theorem}

\begin{proof}
Fix $0 < \varepsilon < 1$ and let $\delta(\varepsilon)$ be the modulus of $C$-convexity of $Y$. We denote
\[
\eta = \frac{\varepsilon^2 \delta(\frac{\varepsilon}{2})^2}{10945(1 + \delta(\frac{\varepsilon}{2}))^2}
\]
and $s = \frac{\eta(2 - \varepsilon)\varepsilon^2}{2(\varepsilon^2 + 2\cdot 12^2)}$.

Assume that $T \in S_{L(C_0(L), Y)}$ and $f_0 \in S_{C_0(L)}$ satisfy that
\[
\|T f_0\| > 1 - s.
\]

Our goal is to find an operator $S \in S_{L(C_0(L), Y)}$ and $g \in S_{C_0(L)}$ such that
\[
\|S(g)\| = 1, \quad \|S - T\| < \varepsilon, \quad \text{and} \quad \|g - f_0\| < \varepsilon.
\]

We can choose $y^*_1 \in S_{Y^*}$ such that
\begin{equation}
\label{eq:2.2}
\Re y^*_1(T f_0) = \|T f_0\| > 1 - s.
\end{equation}

We identify $C_0(L)^*$ with the space $M(L)$ of Borel regular complex measures on $L$ in view of Riesz Theorem. We write $\mu_1 = T^*(y^*_1) \in M(L)$. Since $\mu_1$ is absolutely continuous with respect to its variation $|\mu_1|$, by the Radon-Nikodým Theorem there is a Borel measurable function $g_1 \in B(L)$ such that $|g_1| = 1$ and such that
\[
\mu_1(f) = \int_L fg_1 \, d|\mu_1|, \quad \forall f \in C_0(L).
\]

We write $\beta = \frac{\varepsilon^2}{212^2}$ and denote by $A$ the set given by
\[
A = \{ t \in L : \Re f_0(t)g_1(t) > 1 - \beta \}.
\]

By Lemma 2.1 we have that
\begin{equation}
\label{eq:2.3}
\|(f_0 - g_1)\chi_A\|_\infty \leq \sqrt{2\beta} = \frac{\varepsilon}{12}.
\end{equation}

\end{proof}
Clearly $A$ is also Borel measurable and we know that

$$1 - s < \text{Re} y_1^*(Tf_0) = \text{Re} \mu_1(f_0) = \text{Re} \int_L f_0 g_1 \, d|\mu_1|$$

$$= \text{Re} \int_A f_0 g_1 \, d|\mu_1| + \text{Re} \int_{L\setminus A} f_0 g_1 \, d|\mu_1|$$

$$\leq |\mu_1|(A) + (1 - \beta)|\mu_1|(L \setminus A)$$

$$= |\mu_1|(L) - \beta|\mu_1|(L \setminus A)$$

$$\leq 1 - \beta|\mu_1|(L \setminus A).$$

Hence

$$|\mu_1|(L \setminus A) \leq \frac{s}{\beta} = \frac{\eta(2 - \varepsilon)12^2}{\varepsilon^2 + 2 \cdot 12^2}. \tag{2.4}$$

By Lusin’s Theorem (see for instance [32, Theorem 2.23]) and by the inner regularity of $\mu_1$ there is a compact set $B \subset A$ such that the restriction of $g_1$ to $B$ is continuous, and $|\mu_1|(A \setminus B) \leq \frac{s \eta}{2}$ and so

$$|\mu_1|(L \setminus B) \leq |\mu_1|(L \setminus A) + |\mu_1|(A \setminus B) \leq \frac{s}{\beta} + \frac{s \eta}{2}. \tag{2.5}$$

From (2.2) and the previous estimate we obtain

$$|\mu_1|(B) = |\mu_1|(L) - |\mu_1|(L \setminus B) > 1 - s - \frac{s}{\beta} - \frac{s \eta}{2} = 1 - \eta. \tag{2.6}$$

Hence

$$\|T\ast\ast P_B\| \geq |\mu_1|(B)$$

$$> 1 - \eta$$

$$> 1 - \frac{\delta(\frac{\varepsilon}{9})}{1 + \delta(\frac{\varepsilon}{9})}$$

By applying Lemma 2.3 we deduce

$$\|T\ast\ast(I - P_B)\| \leq \frac{\varepsilon}{9}. \tag{2.7}$$

By Proposition 2.2 $T$ is a weakly compact operator and so it is satisfied that $T\ast\ast(C_0(L)\ast\ast) \subset Y$. So we can define the operator $\tilde{S} \in L(C_0(L), Y)$ by

$$\tilde{S}(f) = T\ast\ast(f\chi_B) + \varepsilon_1 y_1^*(T\ast\ast(f\chi_B)) T\ast\ast(\mathcal{F}\chi_B) \quad (f \in C_0(L),)$$

where $\varepsilon_1 = \frac{1}{6} \frac{\delta(\frac{\varepsilon}{9})}{1 + \delta(\frac{\varepsilon}{9})}$. 

By applying Lemma 2.3
Let us notice that \( \tilde{S}^{**} = \tilde{S}^{**} P_B \) and we have that
\[
\| \tilde{S} \| \geq |y_1^*(\tilde{S}^{**}(g_1 \chi_B))|
\]
\[
= |y_1^*(T^{**}(g_1 \chi_B)) + \varepsilon_1(y_1^*(T^{**}(g_1 \chi_B))(y_1^*(T^{**}(g_1 \chi_B)))|
\]
\[
\geq |y_1^*(T^{**}(g_1 \chi_B))| |1 + \varepsilon_1 y_1^*(T^{**}(g_1 \chi_B))|
\]
\[
\geq |\mu_1|(B)(1 + \varepsilon_1 |\mu_1|(B))
\]
\[
> (1 - \eta)(1 + \varepsilon_1(1 - \eta)) \quad \text{(by (2.8))}
\]
As a consequence
\[
1 \leq 1 - \eta + \varepsilon_1(1 - \eta)^2 \leq \| \tilde{S} \| \leq 1 + \varepsilon_1,
\]
and so
\[
|1 - \| \tilde{S} \|| \leq \varepsilon_1.
\]
For every \( h \in C(B) \), we will denote by \( h \chi_B \) the natural extension of \( h \) to \( L \), which is a Borel function on \( L \). Let be \( S_1 \) the operator given by
\[
S_1(h) = \tilde{S}^{**}(h \chi_B) \quad (h \in C(B)),
\]
which is clearly an operator from \( C(B) \) into \( Y \). Since \( \tilde{S}^{**} = \tilde{S}^{**} P_B \), it is clear that \( \| S_1 \| = \| \tilde{S} \| \). We know that \( B \) is a compact set and \( \tilde{S} \) is weakly compact, by [1] Theorem 2] there is an operator \( S_2 \in L(C(B), Y) \) and \( h_1 \in S_{C(B)} \) satisfying that
\[
\| \tilde{S} \| = \| S_2 \| = \| S_2(h_1) \| \quad \text{and} \quad \| S_2 - S_1 \| < \frac{\varepsilon_1}{2}.
\]
We can choose \( y_2^* \in S_2 \) such that
\[
y_2^*(S_2(h_1)) = \| S_2 \|.
\]
By rotating the elements \( h_1 \) and \( y_2^* \) if needed we can also assume that \( y_2^*(T^{**}(h_1 \chi_B)) \in \mathbb{R}_0^+ \). In view of (2.10), the choice of \( y_2^* \) and by using that \( y_1^*(T^{**}(h_1 \chi_B)) \in \mathbb{R}_0^+ \) we have
\[
\| \tilde{S} \| - \frac{\varepsilon_1}{2} \leq \text{Re } y_2^*(S_1(h_1))
\]
\[
= \text{Re } y_2^*(\tilde{S}^{**}(h_1 \chi_B))
\]
\[
= \text{Re } y_2^*(T^{**}(h_1 \chi_B)) + \varepsilon_1 \text{Re } y_1^*(T^{**}(h_1 \chi_B)) y_2^*(T^{**}(g_1 \chi_B))
\]
\[
\leq 1 + \varepsilon_1 \text{Re } y_2^*(T^{**}(g_1 \chi_B)).
\]
Combining this inequality with the estimate (2.8) we deduce that
\[
\text{Re}(y_2^*(T^{**}(g_1 \chi_B)) \geq (1 - \eta)^2 - \frac{\eta(2 + \varepsilon)}{2\varepsilon_1}.
\]
As a consequence we obtain that
\[
\text{Re } y_2^*(\tilde{S}^{**}(g_1 \chi_B)) = \text{Re } y_2^*(T^{**}(g_1 \chi_B)) + \varepsilon_1 \text{Re } y_1^*(T^{**}(g_1 \chi_B)) y_2^*(T^{**}(g_1 \chi_B))
\]
\[
\geq (1 - \eta)^2 - \frac{\eta(2 + \varepsilon)}{2\varepsilon_1} + \varepsilon_1 |\mu_1|(B) \left( (1 - \eta)^2 - \frac{\eta(2 + \varepsilon)}{2\varepsilon_1} \right)
\]
\[
\geq \left( (1 - \eta)^2 - \frac{\eta(2 + \varepsilon)}{2\varepsilon_1} \right) (1 + \varepsilon_1(1 - \eta)) \quad \text{(by (2.6))}.
\]
In view of (2.11) and (2.13) we have that

$$\text{Re} y^*_2(S_2(\overline{g_1}_B)) \geq \text{Re} y^*_2(S_1(\overline{g_1}_B)) - \|S_2 - S_1\|$$

$$\geq \text{Re} y^*_2(\tilde{S}^{**}(\overline{g_1}_B)) - \|S_2 - S_1\|$$

$$\geq \left( (1 - \eta)^2 - \frac{\eta (2 + \varepsilon)}{2\varepsilon_1} \right) \left( 1 + \varepsilon_1 (1 - \eta) \right) - \frac{\eta \varepsilon}{2} \quad \text{by (2.10)}$$

$$\geq \left( (1 - \eta)^2 - \frac{\eta (2 + \varepsilon)}{2\varepsilon_1} \right) \left( 1 + \varepsilon_1 (1 - \eta) \right) - \frac{\eta \varepsilon \|S_2\|}{2} \quad \text{by (2.10) and (2.8)}.$$

Let us write $R_2 = \frac{S_2}{\|S_2\|}$ and $\mu_2 = R_2^*(y^*_2) \in M(B)$. Let $g_2 = \frac{d\mu_2}{d|\mu_2|}$ and we can assume that $|g_2| = 1$. From the previous inequality, in view of (2.10) and (2.8) we have that

$$\text{Re} y^*_2(R_2(\overline{g_1}_B)) \geq \left( (1 - \eta)^2 - \frac{\eta (2 + \varepsilon)}{2\varepsilon_1} \right) \left( 1 + \varepsilon_1 (1 - \eta) \right) - \frac{\eta \varepsilon}{2}$$

$$\geq \left( (1 - \eta)^2 - \frac{\eta (2 + \varepsilon)}{2\varepsilon_1} \right) \left( 1 + \varepsilon_1 (1 - \eta) \right) - \frac{\eta \varepsilon}{2}$$

$$= 1 - \frac{2\eta - 2\eta^2 + \varepsilon_1 (1 - (1 - \eta)^3) + \frac{2\eta + \eta \varepsilon}{2\varepsilon_1} + \frac{\eta \varepsilon}{2} (2 + \varepsilon_1 - \eta)}{1 + \varepsilon_1}$$

$$> 1 - 6\eta - 2\frac{\eta}{\varepsilon_1} - \varepsilon \eta.$$

We consider the measurable set $C$ of $L$ given by

$$C = \{ t \in B : \text{Re} (\overline{g_1}(t) + h_1(t))g_2(t) > 2 - \beta \}.$$

In view of (2.11) and (2.13) we have that

$$2 - 6\eta - 2\frac{\eta}{\varepsilon_1} - \varepsilon \eta < \text{Re} \mu_2(h_1 + \overline{g_1}_B)$$

$$= \int_C \text{Re}(h_1 + \overline{g_1})g_2 \, d|\mu_2| + \int_{B \setminus C} \text{Re}(h_1 + \overline{g_1})g_2 \, d|\mu_2|$$

$$\leq 2|\mu_2|(C) + (2 - \beta)|\mu_2|(B \setminus C)$$

$$= 2|\mu_2|(B) - \beta|\mu_2|(B \setminus C)$$

$$\leq 2 - \beta|\mu_2|(B \setminus C).$$

Hence

$$|\mu_2|(B \setminus C) \leq \frac{6\eta + \frac{2\eta}{\varepsilon_1} + \varepsilon \eta}{\beta}. \quad (2.14)$$

On the other hand, in view of Lemma 2.1 we have that

$$\frac{\|g_1 - g_2\|_\infty}{\sqrt{2\beta}} = \frac{\varepsilon}{12} \quad \text{and} \quad \frac{\|h_1 - \overline{g_2}\|_\infty}{\sqrt{2\beta}} = \frac{\varepsilon}{12}. \quad (2.15)$$

From the previous inequality and (2.3) it follows that

$$\| (h_1 - f_0) \chi_C \|_\infty \leq \| (h_1 - \overline{g_2}) \chi_C \|_\infty + \| (\overline{g_2} - \overline{g_1}) \chi_C \|_\infty + \| (\overline{g_1} - f_0) \chi_C \|_\infty \leq \frac{\varepsilon}{4}. \quad (2.16)$$
By the inner regularity of $\mu_2$ there is a compact set $K_1 \subset C$ such that
\begin{equation}
|\mu_2|(C \setminus K_1) < \frac{\eta \varepsilon}{2}.
\end{equation}

Let us notice that
\begin{equation}
\|R_2^{**} P_{K_1}\| \geq \|y_2^* R_2^{**} P_{K_1}\| = |\mu_2|(K_1)
= |\mu_2|(B) - |\mu_2|(B^\prime \setminus C) - |\mu_2|(C \setminus K_1)
\geq \Re y_2^*(R_2(\overline{y_{1|B}})) - |\mu_2|(B^\prime \setminus C) - |\mu_2|(C \setminus K_1)
\geq \Re y_2^*(R_2(\overline{y_{1|B}})) - |\mu_2|(B^\prime \setminus C) - \frac{\eta \varepsilon}{2}
\end{equation}
(by (2.17))
\begin{equation}
\geq 1 - 6\eta - \frac{\eta}{\varepsilon_1} - \varepsilon - \frac{6\eta + 2\frac{\eta}{\varepsilon_1} + \varepsilon\eta}{\beta} - \frac{\eta \varepsilon}{2}
\end{equation}
(by (2.13) and (2.14))
\begin{equation}
> 1 - \frac{\delta^2}{\varepsilon_1} + \frac{\varepsilon}{\beta} - \frac{\eta \varepsilon}{2}
> 1 - \frac{\delta^2}{\varepsilon_1} + \frac{\varepsilon}{\beta} + \frac{\eta \varepsilon}{2}
> 1 - \frac{\delta^2}{\varepsilon_1} + \frac{\varepsilon}{\beta} + \frac{\eta \varepsilon}{2} > 0.
\end{equation}

Hence $K_1 \neq \emptyset$.

In view of Lemma 2.3 we obtain
\begin{equation}
\|R_2^{**}(P_B - P_{K_1})\| \leq \frac{\varepsilon}{9}.
\end{equation}

We denote by $T_2$ the element in $L(C_0(L), Y)$ defined by
\begin{equation}
T_2(f) = R_2(f_{1|B}) \quad (f \in C_0(L)).
\end{equation}
Clearly it is satisfied that $\|T_2^{**}(I - P_{K_1})\| = \|R_2^{**}(P_B - P_{K_1})\|$ and since $T_2^{**}(P_B - P_{K_1}) = T_2^{**}(I - P_{K_1})P_B$ in view of (2.18) we obtain
\begin{equation}
\|T_2^{**}(P_B - P_{K_1})\| \leq \frac{\varepsilon}{9}.
\end{equation}

We also write $R(f) = T^{**}(f_{1|B})$ for every $f \in C(B)$ and so we have
\begin{equation}
\|(T_2^{**} - T^{**})P_B\| = \|R_2 - R\|.
\end{equation}

By the definition of $S_1$ we know that
\begin{equation}
\|S_1 - R\| \leq \varepsilon_1.
\end{equation}

Since $K_1 \neq \emptyset$, let us fix $t_0 \in K_1$. Since $K_1 \subset C$, we have that $|h_1(t_0)| > 1 - \beta > 1 - \frac{\varepsilon_1}{2}$.
So we can choose and open set $V$ in $B$ such that $t_0 \in V \subset \{t \in B : |h_1(t)| > 1 - \frac{\varepsilon_1}{2}\}$ and a function $v \in C(B)$ satisfying $v(B) \subset [0, 1], v(t_0) = 1$ and $\text{supp } v \subset V$. So there are functions $h_i \in C(B)$ ($i = 2, 3$) such that
\begin{equation}
h_2(t) = h_1(t) + v(t)(1 - |h_1(t)|) \frac{h_1(t)}{|h_1(t)|} \quad (t \in B),
\end{equation}
and
\begin{equation}
h_3(t) = h_1(t) - v(t)(1 - |h_1(t)|) \frac{h_1(t)}{|h_1(t)|} \quad (t \in B).
\end{equation}
It is clear that \( h_i \in B_{C(B)} \) for \( i = 2,3 \) and \( h_1 = \frac{1}{2}(h_2 + h_3) \). By using that the operator \( R_2 \) attains its norm at \( h_1 \) we clearly have that
\[
\|R_2(h_2)\| = 1 \quad \text{and} \quad |h_2(t_0)| = 1. \tag{2.24}
\]
Since \( \text{supp } v \subset V \subset \{ t \in B : |h_1(t)| > 1 - \frac{\varepsilon}{2} \} \) we obtain for \( t \in V \) that
\[
|h_2(t) - h_1(t)| \leq 1 - |h_1(t)| < \frac{\varepsilon}{2}. \tag{2.25}
\]
For \( t \in B \setminus V, \) \( h_2(t) = h_1(t) \) so \( |h_2 - h_1| < \frac{\varepsilon}{2} \). In view of (2.16) we obtain that
\[
\|h_2 - f_0|_C\| \leq \|h_2 - h_1\| + \|h_1 - f_0|_C\|
\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \frac{3\varepsilon}{4}. \tag{2.26}
\]

Since \( B \subset L \) is a compact subset, there is a function \( f_2 \in C_0(L) \) such that it extends the function \( h_2 \) to \( L \) (see for instance [22, Corollary 9.15 and Theorem 12.4] and [25, Theorems 17 and 18]). Since the function \( \Phi : \mathbb{C} \rightarrow \mathbb{C} \) given by \( \Phi(z) = z \) if \( |z| \leq 1 \) and \( \Phi(z) = \frac{z}{|z|} \) if \( |z| > 1 \) is continuous, by using \( \Phi \circ f_2 \) instead of \( f_2 \) if needed, and the fact that \( h_2 \in S_{C(B)} \) we can also assume that \( f_2 \in S_{C_0(L)} \). Since \( f_2 \) is an extension of \( h_2 \), by using (2.26) there is an open set \( G \subset L \) such that \( K_1 \subset G \) and satisfying also that
\[
\|(f_2 - f_0)\chi_G\|_\infty < \frac{7\varepsilon}{8}. \tag{2.27}
\]

By Urysohn’s Lemma there is a function \( u \in C_0(L) \) such that \( u(L) \subset [0,1], \ u|_{K_1} = 1 \) and \( \text{supp } u \subset G \). We define the function \( f_3 \) by
\[
)f_3 = uf_2 + (1 - u)f_0,
\]
that clearly belongs to \( B_{C_0(L)} \).

Notice also that
\[
f_3(t) = f_2(t) = h_2(t) \quad \forall t \in K_1, \quad f_3(t) = f_0(t) \quad \forall t \in L \setminus G
\]
and
\[
|f_3(t) - f_0(t)| = u(t)|f_2(t) - f_0(t)|, \quad \forall t \in G \setminus K_1. \tag{2.29}
\]

In view of (2.27) we obtain that
\[
\|f_3 - f_0\| < \varepsilon. \tag{2.30}
\]

We write \( \lambda_0 = \overline{h_2(t_0)} \) and we know that \( |\lambda_0| = 1 \). Define the operator \( S \in L(C_0(L), Y) \) given by
\[
S(f) = R_2^{**}(f\chi_{K_1})|_B + \lambda_0 f(t_0) R_2^{**}(h_2\chi_{B\setminus K_1}) \quad (f \in C_0(L)).
\]

Since \( R_2 \) is weakly compact, \( S \) is well-defined. For every \( f \in B_{C_0(L)} \) we have that \( |\lambda_0 f(t_0)| \leq 1 \) and so
\[
\|(f\chi_{K_1})|_B + \lambda_0 f(t_0) h_2\chi_{B\setminus K_1}\|_\infty \leq 1.
\]
Since \( \|R_2\| \leq 1 \), then
\[
\|S(f)\| = \|R_2^{**}(f\chi_{K_1})_B + \lambda_0 f(t_0)h_2\chi_{B\setminus K_1}\| \leq 1.
\]
It is also satisfied that
\[
S(f_3) = R_2^{**}(f_3\chi_{K_1})_B + \lambda_0 f_3(t_0)R_2^{**}(h_2\chi_{B\setminus K_1}) = R_2^{**}(h_2) \quad \text{(by (2.28))}
\]
and in view of (2.24) we obtain \( \|S(f_3)\| = \|R_2(h_2)\| = 1 \). Hence \( S \in S_{L(C_0(L),Y)} \) and it attains its norm at \( f_3 \). We also know that \( \|f_3 - f_0\| < \varepsilon \) by inequality (2.30). It suffices to check that \( S \) is close to \( T \). Indeed we obtain the following estimate
\[
\|S - T\| \leq \|S^{**} - T^{**}P_B\| + \|T^{**}(I - P_B)\|
\]
\[
\leq \|T_2^{**}P_{K_1} - T^{**}P_B\| + \|R_2^{**}(P_B - P_{K_1})\| + \frac{\varepsilon}{9} \quad \text{(by (2.27))}
\]
\[
= \|(T_2^{**} - T^{**})P_B\| + \|T_2^{**}(P_B - P_{K_1})\| + \frac{2\varepsilon}{9} \quad \text{(by (2.18))}
\]
\[
\leq \|(T_2^{**} - T^{**})P_B\| + \frac{\varepsilon}{3} \quad \text{(by (2.19))}
\]
\[
= \|R_2 - R\| + \frac{\varepsilon}{3} \quad \text{(by (2.20))}
\]
\[
\leq \|R_2 - S_2\| + \|S_2 - S_1\| + \|S_1 - R\| + \frac{\varepsilon}{3}
\]
\[
\leq \|1 - \|S_2\|\| + \frac{\eta\varepsilon}{2} + \varepsilon_1 + \frac{\varepsilon}{3} \quad \text{(by (2.10) and (2.21))}
\]
\[
\leq 2\varepsilon_1 + \frac{\eta \varepsilon}{2} + \frac{\varepsilon}{3} < \varepsilon \quad \text{(by (2.9) and (2.10)).}
\]

Since the complex spaces \( L_p(\mu) \) (1 \( \leq p < \infty \)) are \( C \)-uniformly convex we obtain the following result:

**Corollary 2.5.** In the complex case the pair \((C_0(L), L_p(\mu))\) does have the Bishop-Phelps-Bollobás property for operators for every Hausdorff locally compact space \( L \), every positive measure \( \mu \) and \( 1 \leq p < \infty \).

As we already mentioned we extended in a non trivial way a result by S.K. Kim, H.J. Lee and P.K. Lin where they consider any (complex) space \( L_\infty(\nu) \) as the domain space \([28]\).

**Open problem:** In the real case it is not known whether or not the pair \((c_0, \ell_1)\) has the BPBp for operators.
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