The Gaussian approximation for multi-color generalized Friedman’s urn model

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Abstract

The Friedman’s urn model is a popular urn model which is widely used in many disciplines. In particular, it is extensively used in treatment allocation schemes in clinical trials. In this paper, we prove that both the urn composition process and the allocation proportion process can be approximated by a multi-dimensional Gaussian process almost surely for a multi-color generalized Friedman’s urn model with non-homogeneous generating matrices. The Gaussian process is a solution of a stochastic differential equation. This Gaussian approximation together with the properties of the Gaussian process is important for the understanding of the behavior of the urn process and is also useful for statistical inferences. As an application, we obtain the asymptotic properties including the asymptotic normality and the law of the iterated logarithm for a multi-color generalized Friedman’s urn model as well as the randomized-play-the-winner rule as a special case.

Abbreviated Title: Approximation for multi-color urn models

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1 Introduction.

Urn models have long been recognized as valuable mathematical apparatus in many areas including physical sciences, biological sciences and engineering (Johnson and Kotz, 1977; Kotz and Balakrishnan, 1997). Urn models are also extensively applied in clinical studies. The applications are mostly found in the area of adaptive design which is utilized to provide a response-adaptive allocation scheme. In clinical trials, suppose patients accrue sequentially and assume the availability of several treatments. Adaptive designs are inclining to assign more patients to the better treatments, while seeking to maintain randomness as a basis for statistical inference. Thus the cumulative information of the response of treatments on previous patients will be used to adjust treatment assignment to coming patients. For this purpose, various urn models have been proposed and used extensively in adaptive designs (Wei and Durham (1978), Wei (1979), Flournoy and Rosenberger (1995), Rosenberger (1996), Bai and Hu (1999,2005)). One large family of randomized adaptive designs is based on the Generalized Friedman’s Urn (GFU) model (also named as Generalized Pólya Urn (GPU) in literature). For more detailed reference, the reader is referred to Flournoy and Rosenberger (1995), Rosenberger (1996), Hu and Rosenberger (2006).

A general description of the GPU model is as follows. Consider an urn containing particles of $d$ types, respectively representing $d$ 'treatments' in a clinical trial. At the beginning, the urn contains $Y_0 = (Y_{01}, \ldots, Y_{0d})$ particles, where $Y_{0k} > 0$ denotes the number of particles of type $k$, $k = 1, \ldots, d$. At the stage $m$, $m = 1, 2, \ldots$, a particle is drawn from the urn and replaced. If the particle is of type $k$, then the treatment $k$ is assigned to the $m$th patient, $k = 1, \ldots, d$. We then wait for observing a random variable $\xi(m)$, the response of the treatment at the patient $m$. Here $\xi(m)$ may be a random vector. After that, an additional $D_{k,q}(m)$ particles of type $q$, $q = 1, \ldots, d$, are added to the urn, where $D_{k,q}(m)$ is a function of $\xi(m)$ and also may be a function of urn compositions, assignments and responses of previous stages. This procedure is repeated throughout $n$ stages. After $n$ draws and generations, the urn composition is denoted by the row vector $Y_n = (Y_{n1}, \ldots, Y_{nd})$, where $Y_{nk}$ stands for the number of particles of type $k$ in the urn after the $n$th draw. This relation can be written as the following recursive formula:

$$Y_m = Y_{m-1} + X_mD_m,$$  \hspace{1cm} (1.1)
where $D_m = (D_{k,q}(m))_{k,q=1}^d$, and $X_m$ is the result of the $n$th draw, distributed according to the urn composition at the previous stage, i.e., if the $m$th draw is type $k$ particle, then the $k$th component of $X_m$ is 1 and other components are 0. The matrices $D_m$'s are named as the addition rules. Furthermore, write $N_n = (N_{n1}, \ldots, N_{nd})$, where $N_{nk}$ is the number of times a type $k$ particle drawn in the first $n$ stages. In clinical trials, $N_{nk}$ represents the number of patients assigned to the treatment $k$ in the first $n$ trials. Obviously,

$$N_n = \sum_{k=1}^n X_n. \quad (1.2)$$

In clinical applications, $Y_{n-1}/\sum_{k=1}^d Y_{n-1,k}$ are the probabilities of the patient $n$ being allocated to treatments, and $N_n/n$ are sample allocation proportions. The asymptotic behavior of $Y_n$ and $N_n$ is of immense importance (Hu and Rosenberger, 2003, 2006). Obviously, the asymptotic behavior of $Y_n$ and $N_n$ will depend on the addition rules $D_m$, especially the conditional expectations $H_m = (E[D_{k,q}(m)|\mathcal{F}_{m-1}])_{k,q=1}^d$ for given the history sigma field $\mathcal{F}_{m-1}$ generated by the urn compositions $Y_1, \ldots, Y_{m-1}$, the assignments $X_1, \ldots, X_{m-1}$ and the responses $\xi(1), \ldots, \xi(m-1)$ of all previous stages, $m = 1, 2, \ldots$. The conditional expectations $H_m$'s are named as the generating matrices. In some usual cases, the addition rules are assumed to be independent of the previous process. Thus, we may define $H_m$ is the expectation of the rule matrix $D_m$. For more generality, in the sequel of this paper, we define $H_m$ to be the conditional expectation of $D_m$ when the history sigma field $\mathcal{F}_{m-1}$ is given, also we assume that at the stage $m$, the adding rule $D_m$ is independent of the assignment $X_m$ when the history sigma field is given.

When $D_m$, $m = 1, 2, \ldots$, are independent and identical distributed, the GFU model is usually said to be homogeneous. In such case $H_m = H$ are identical and nonrandom, and usually the addition rule $D_m$ is merely function of the $m$th patient's observed outcome. In the general non-homogeneous cases, both $D_m$ and $H_m$ depend on the entire history of all previous trials which provides more information of the efficacy of the treatments. Interesting examples of non-homogeneous urn models and their applications can be found in Andersen, Faries and Tamura (1994) and Bai, Hu and Shen (2002).

Athreya and Karlin (1967, 1968) first considered the asymptotic properties of the GFU model with homogeneous generating matrix and conjecture that $N_n$ is asymptotically normal. This conjecture has not been solved for almost three decades until Janson (2004).
and Bai and Hu (2005) solved it independently. Janson (2004) established functional limit theorems of $Y_n$ and $N_n$ for the homogenous case by using the theory of continuous-time branching processes. Bai and Hu (2005) established the consistency and the asymptotic normality of the non-homogeneous GFU model by applying the central limit of martingales and the matrix theory. However, the asymptotic variances of $Y_n$ and $N_n$ are complicated and not easy to be understood.

In the two-arm clinical trial, Bai, Hu and Zhang (2002) showed that the urn process \{$Y_n$\} with nonhomogeneous generating matrices $H_m$’s can be approximated by a Gaussian process almost surely under some suitable conditions, where $Y_n = Y_{n1}$ represents the number of type 1 balls in the urn after the $n$th draw. As an application, the weak invariance principle and the law of the iterated logarithm for \{$Y_n$\} are established. However, the results for the allocation proportion $N_{n1}/n$ is not obtained. In this paper, we consider the general multi-color case. The strong approximation of the process $(Y_n, N_n)$ are established. In particular, the asymptotic normality and the law of the iterated logarithm for the multi-dimensional process $(Y_n, N_n)$ are obtained. We will prove that under some mild conditions, the process $(Y_n', N_n')$ can be approximated by a multi-dimensional Gaussian process which is a solution of a simple multi-dimensional stochastic differential equation. This differential equation and the behavior of the Gaussian process make us to understand the complex asymptotic variances and the asymptotic behavior of $Y_n$ and $N_n$ more easily.

The approximation theorems will be presented in Section 2 whose technical proofs are stated in the last section. Some important properties of the limit processes are given in Section 3. By combining the approximation theorems and the properties of the limit processes, important asymptotic properties including the asymptotic normality and the law of the iterated logarithm of $Y_n$ and $N_n$ are derived in Section 4. Throughout this paper, $C, C_\epsilon$, etc. denote positive constants whose values can differ in different places, $\log x = \ln(e \vee x)$. For a vector $x$, $\|x\|$ denote its Euclidean norm, and $\|M\| = \sup\{\|xM\| : \|x\| = 1\}$ for a matrix $M$. Also we denote $a_n = Y_n1'$ to be the total number of balls in the urn after stage $n$. 
2 Strong approximation.

In this section, we will give our main results on the Gaussian approximation of both the
urn composition $Y_n$ and the allocation numbers $N_n$. We first need two assumptions on
the addition rules $D_m$. We let $\mathcal{F}_m = \sigma(Y_1, \ldots, Y_m, X_1, \ldots, X_m, \xi(1), \ldots, \xi(m))$ be history sigma field. and $H_m = \mathbb{E}[D_m | \mathcal{F}_{m-1}]$ be the generating matrix.

**Assumption 2.1** Suppose there is a $\tau \geq 0$ such that the generating matrices $H_m$ satisfy
\[
\sum_{m=1}^{n} \|H_m - H\| = o(n^{1/2-\tau}) \text{ a.s., } \tag{2.1}
\]
with $H$ being a deterministic matrix and
\[
H_{qk} \geq 0 \text{ for } k \neq q \text{ and } \sum_{k=1}^{d} H_{qk} = s \text{ for all } q = 1, \ldots, d,
\]
where $H_{qk}$ is the $(q,k)$-entry of the matrix $H$ and $s$ is a positive constant. Without loss of
generality, we assume $s = 1$ throughout this paper. For otherwise, we may consider $Y_m/s$, $H_m/s$ instead.

**Assumption 2.2** Let
\[
V_{qkl}(n) =: \text{Cov}[(D_{qk}(n), D_{ql}(n)) | \mathcal{F}_{n-1}], \quad q, k, l = 1, \ldots, d
\]
and denote by $V_n = (V_{qkl}(n))_{k,l=1}^{d}$. Suppose for some $0 < \epsilon < 1/2$,
\[
\mathbb{E}\|D_n\|^{2+\epsilon} \leq C < \infty \text{ for all } n = 1, 2, \ldots, \tag{2.2}
\]
\[
\sum_{m=1}^{n} V_{mq} = nV_q + o(n^{1-\epsilon}) \text{ a.s., for all } q = 1, \ldots, d, \tag{2.3}
\]
where $V_q = (V_{qkl})_{k,l=1}^{d}, q = 1, \ldots, d$, are $d \times d$ non-negative definite matrices.

By Assumption 2.1, $H$ has a maximal eigenvalue 1 and a corresponding right eigenvector $\mathbf{1} = (1, \ldots, 1)$. Let $\lambda_2, \ldots, \lambda_d$ be other $d-1$ eigenvalues of $H$. Then $H$ has the following Jordan form decomposition
\[
T^{-1}HT = \text{diag} (1, J) \quad \text{and} \quad J = \text{diag}(J_2, \ldots, J_s) \tag{2.4}
\]
with

\[
J_t = \begin{pmatrix}
\lambda_t & 1 & 0 & \ldots & 0 \\
0 & \lambda_t & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & \lambda_t & 1 \\
0 & 0 & 0 & \ldots & \lambda_t
\end{pmatrix},
\]

(2.5)

where \( T = (t'_1, \ldots, t'_d) \) and \( t_1 = 1 \). Denote by \( \rho = \max\{Re(\lambda_2), \ldots, Re(\lambda_2)\} \), where \( Re(\lambda_k) \) is the real part of the complex number \( \lambda_k \). And denote the order of \( J_t \) by \( \nu_t \) and \( \nu = \max\{\nu_t : Re(\lambda_t) = \rho\} \). Let \( v \) be the left eigenvector of \( H \) associated with the positive maximal eigenvalue 1 and satisfy \( v1' = 1 \). We denote \( \tilde{H} = H - 1'v, \Sigma_1 = diag(v) - v'v, \Sigma_2 = \sum_{q=1}^d v_q V_q \) and \( \Sigma = H'\Sigma_1 H + \Sigma_2 \). For a \( d \)-dimensional Brownian motion \( \{W_t; t \geq 0\} \) with a co-variance \( \Lambda \), we denote the solution of the equation:

\[
S_t = W_t + \int_0^t \frac{S_s \tilde{H}}{s} ds, \quad t \geq 0, \quad S_0 = 0 \quad \text{Equ1}
\]

by \( \{S_t = \text{Solut}(\text{Equ1}, W_t); t \geq 0\} \). In the next section, we will show that \( S_t \) is well defined if \( \rho < 1/2 \), and

\[
S_t = \int_0^t W_s \frac{\tilde{H}}{s} \left( t \right) \tilde{H} ds + W_t = \int_0^t (dW_s) \left( t \right) \tilde{H}.
\]

where, for any \( t > 0 \) and any matrix \( M \), \( t^M \) is defined to be

\[
\exp\{M \ln t\} := \sum_{k=0}^\infty \frac{M^k (\ln t)^k}{k!}.
\]

Also we denote the solution of the equation:

\[
\tilde{S}_t = W_t - W_1 + \int_1^t \frac{\tilde{S}_s \tilde{H}}{s} ds, \quad t > 0, \quad \tilde{S}_1 = 0 \quad \text{Equ2}
\]

by \( \{\tilde{S}_t = \text{Solut}(\text{Equ2}, W_t); t > 0\} \). And we will show that \( \tilde{S}_t \) is well defined and

\[
\tilde{S}_t = \int_1^t W_s \frac{\tilde{H}}{s} \left( t \right) \tilde{H} ds + W_t - W_1 t^\tilde{H} = \int_1^t (dW_s) \left( t \right) \tilde{H}.
\]

Now let \( B_{t1} \) and \( B_{t2} \) be two independent \( d \)-dimensional standard Brownian motions. Define \( G_{t_1} = \text{Solut}(\text{Equ1}, B_{t1} \Sigma_{t_1}^{1/2}) \) and \( \tilde{G}_{t_1} = \text{Solut}(\text{Equ2}, B_{t1} \Sigma_{t_1}^{1/2}), i = 1, 2 \). Let \( G_t = G_{t_1} H + G_{t_2} \) and \( \tilde{G}_t = \tilde{G}_{t_1} H + \tilde{G}_{t_2} \). Then \( G_t = \text{Solut}(\text{Equ1}, B_{t1} \Sigma_{t_1}^{1/2} H + B_{t2} \Sigma_{t_2}^{1/2}) \) and \( \tilde{G}_t = \text{Solut}(\text{Equ2}, B_{t1} \Sigma_{t_1}^{1/2} H + B_{t2} \Sigma_{t_2}^{1/2}) \).

The next two theorems are on the strong approximation.
Theorem 2.1 Suppose \( \rho < 1/2 \). Under Assumptions 2.1 and 2.2, there are two independent \( d \)-dimensional standard Brownian motions \( B_{t1} \) and \( B_{t2} \) (possibly in an enlarged probability space with the process \((Y_n, N_n)\) being redefined without changing the distribution) such that for some \( \gamma > 0 \),

\[
Y_n - n\nu = G_{n1}H + G_{n2} + o(n^{1/2-\tau\wedge\gamma}) \text{ a.s.}, \tag{2.6}
\]

\[
N_n - n\nu = G_{n1} + \int_0^n \frac{G_{2x}}{x} \, dx (I - 1'\nu) + o(n^{1/2-\tau\wedge\gamma}) \text{ a.s.}, \tag{2.7}
\]

where \( \tau \wedge \gamma = \min\{\tau, \gamma\} \).

Theorem 2.2 Suppose \( \rho = 1/2 \) and Assumptions 2.1 and 2.2 are satisfied. Further, assume that

\[
\sum_{m=1}^{\infty} \frac{\|H_m - H\|}{m^{1/2}} < \infty. \tag{2.8}
\]

Then there are two independent \( d \)-dimensional standard Brownian motions \( B_{t1} \) and \( B_{t2} \) (possibly in an enlarged probability space with the process \((Y_n, N_n)\) being redefined without changing the distribution) such that

\[
Y_n - n\nu = \tilde{G}_{n1}H + \tilde{G}_{n2} + O(n^{1/2}\log^{\nu-1} n) \text{ a.s.} \tag{2.9}
\]

\[
N_n - n\nu = \tilde{G}_{n1} + \int_1^n \frac{\tilde{G}_{2x}}{x} \, dx (I - 1'\nu) + O(n^{1/2}\log^{\nu-1} n) \text{ a.s.} \tag{2.10}
\]

Also,

\[
(\tilde{G}_{t1} + \int_1^t \frac{\tilde{G}_{2x}}{x} \, dx (I - 1'\nu))H = \tilde{G}_{t1}H + \tilde{G}_{t2} - B_{t2}\Sigma_{1/2}^2. \tag{2.11}
\]

Remark 2.1 The condition (2.8) is used by Bai and Hu (2005) to obtain the asymptotic normality. It is easily seen that it implies the condition (2.1) with \( \tau = 0 \). Bai and Hu (2005) also assumed that \( H_m1' = 1' \) and \( H_m \to H \).

Remark 2.2 By (2.9)-(2.11), under the assumptions in Theorem 2.2,

\[
Y_n - n\nu = (N_n - n\nu)H + O(n^{1/2}\log^{\nu-1} n) \text{ a.s.} \quad \text{if} \quad \nu > 1.
\]

The proof of Theorems 2.1 and 2.2 will be given in the last section. Before that, we give some properties of the limit processes and several application of these approximations.
3 Properties of the limit processes.

This section will give several properties of the solutions of equations (Equ1) and (Equ2) and Gaussian processes $G_t, S_t$, etc. By combining these properties with the approximation theorems in the above section we can obtain important properties of the urn models which will be given in the next section. The properties of the Gaussian processes will be also used in the proofs of the approximation theorems.

We first need two lemmas.

**Lemma 3.1** Under Assumption 2.1, there exists a constant $C$ such that for any $a \geq 1$,

$$\|a \hat{H}\| \leq Ca^{\lambda} \log^{\nu} a.$$  \hspace{1cm} (3.1)

**Proof** It is obvious that $\hat{H} = T \text{diag}(0, J) T^{-1} := T J T^{-1}$. It follows that

$$a \hat{H} = T a J T^{-1}.$$

So it is enough to show that for any $a > 1$, $\|a J_t\| \leq C a^{\lambda} \log^{\nu} a$. Denote $J_t = \lambda_t I + \tilde{I}_t$ where

$$\tilde{I}_t = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \hspace{1cm} (3.2)$$

Then $\|a J_t\| = \|a^{\lambda_t} a \tilde{I}_t\| \leq C a^{\lambda} \log^{\nu} a$. Obviously,

$$\tilde{I}_t^2 = \begin{pmatrix} 0 & 0 & 1 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \ldots, \tilde{I}_t^{\nu-1} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

and $\tilde{I}_t^{\nu} = 0$. It follows that

$$a \tilde{I}_t = \sum_{k=0}^{\infty} \frac{\tilde{I}_t^k (\ln a)^k}{k!} = \sum_{k=0}^{\nu-1} \frac{\tilde{I}_t^k (\ln a)^k}{k!}.$$
Then
\[ \| a T_t \| \leq C \sum_{k=0}^{\nu a - 1} \frac{(\ln a)^k}{k!} \leq C (\log a)^{\nu a - 1}. \]
Hence (3.1) is proved.

**Lemma 3.2** For any \( a \geq 0 \), the equation
\[ Z_t = \int_t^s \frac{Z_s}{s} \tilde{H} ds, \quad Z_a = 0 \]
or equivalently
\[ dZ_t = \frac{Z_t}{t} \tilde{H} dt, \quad Z_a = 0 \]
has an unique solution \( Z_t \equiv 0 \).

**Proof** It is obvious that \( \tilde{H} = H - 1'v \) has the Jordan form decomposition
\[ T^{-1} \tilde{H} T = \text{diag}(0, J_2, \ldots, J_s) \]
and (3.3) is equivalent to
\[ Z_tT = \frac{Z_t}{t} \text{diag}(0, J_2, \ldots, J_s) dt, \quad Z_aT = 0. \]
On the other hand, for each \( s \),
\[ d\tilde{Z}_t^{(s)} = \frac{\tilde{Z}_t^{(s)}}{t} J_s dt, \quad \tilde{Z}_a^{(s)} = 0 \]
has an unique solution \( \tilde{Z}_t^{(s)} \equiv 0 \). The proof is completed.

From this Lemma, it follows that the solutions of (Equ1) and (Equ2) are unique. The following two propositions tells us that the solutions exist.

**Proposition 3.1** Let \( \{W_t; t \geq 0\} \) be a \( d \)-dimensional Brownian motion with a co-variance matrix \( \Lambda \). Suppose that Assumption 2.1 is satisfied and \( \rho < 1/2 \). Then the unique solution \( S_t = \text{Solut}(\text{Equ1}, W_t) \) of the equation (Equ1) is
\[ S_t = \int_0^t W_x \frac{\tilde{H}}{x} (t) \tilde{H} dx + W_t. \]
Also
\[ S_t = \left( \int_0^t (dW_x)x^{-\tilde{H}} \right) t \tilde{H} \quad \text{a.s.} \]
Furthermore, with probability one \( S_t \) is continuous on \([0, \infty)\).
Proof Fist, since \( \|x^{-\tilde{H}}\| \leq Cx^{-\rho}(\log x^{-1})^{\nu-1} \) for \( 0 < x < 1 \) by Lemma 3.1, and \( W_x \overset{a.s.}{\longrightarrow} O(\sqrt{x \log \log x^{-1}}) \) as \( x \to 0 \), we have

\[
W_x x^{-\tilde{H}} = O(1) x^{1/2-\rho}(\log x^{-1})^\nu - 0 \quad a.s.,
\]

\[
W_x \frac{x^{-\tilde{H}}(t)}{x^{-\tilde{H}}(t)} = O(1) x^{-1/2-\rho}(\log x^{-1})^\nu \quad a.s.
\]
as \( x \to 0 \). Since \( -1/2 - \rho > -1 \) and \( W_x \frac{x^{-\tilde{H}}(t)}{x^{-\tilde{H}}(t)} \) is continuous on \((0, \infty)\), it follows that the integral \( \int_0^t W_x \frac{x^{-\tilde{H}}(t)}{x^{-\tilde{H}}(t)} H \, dx \) exists, and then \( S_t \) in (3.3) is well defined and

\[
\int_0^t W_x \frac{x^{-\tilde{H}}(t)}{x^{-\tilde{H}}(t)} H \, dx = \left( \int_0^t W_x d(x^{-\tilde{H}}(t)) \right) t^{-\tilde{H}}(t) - W_t + \left( \int_0^t W_x d(x^{-\tilde{H}}(t)) \right) t^{-\tilde{H}}(t).
\]

It follows that (3.5) is true. Now we show that \( S_t \) is the solution of equation (Equ1). Note that

\[
S_t = O(1) \int_0^t \left( x \log x^{-1} \right)^{1/2} (t/x)^{\rho}(\log(t/x))^{\nu-1} \, dx = O(1) t^{1/2-\rho}(\log t^{-1})^\nu
\]
as \( t \to \infty \). It follows that \( S_0 = 0 \), the integral \( \int_0^t \frac{S_s}{s} \, ds \) exists and

\[
\int_0^t \frac{S_s}{s} \, ds = \int_0^t ds \int_0^s W_x \frac{x^{-\tilde{H}}(s)}{x^{-\tilde{H}}(s)} H \, dx + \int_0^t W_s \, ds
\]

\[
= \int_0^t W_x \frac{x^{-\tilde{H}}(t)}{x^{-\tilde{H}}(t)} H \, dx \int_0^s x^{-\tilde{H}}(s) \, ds + \int_0^t W_s \, ds
\]

\[
= \int_0^t W_x \frac{x^{-\tilde{H}}(t)}{x^{-\tilde{H}}(t)} H \, dx \int_0^t W_s \, ds
\]

\[
= \int_0^t W_x \frac{x^{-\tilde{H}}(s)}{x^{-\tilde{H}}(s)} H \, dx - \int_0^t \frac{W_x}{x} \, dx + \int_0^t \frac{W_s}{s} \, dx = \int_0^t W_x \frac{1}{x} \left( \frac{t}{x} \right) H \, dx.
\]

Then

\[
\int_0^t \frac{S_s}{s} \, ds = \int_0^t W_x \frac{H(t)}{x} \, H \, dx = S_t - W_t.
\]

So, \( S_t \) is the solution of equation (Equ1). Finally, the continuity of \( S_t \) follows from the continuity of the Brownian motion \( W_t \).

**Proposition 3.2** Let \( \{W_t; t \geq 0\} \) be a \( d \)-dimensional Brownian motion with some covariance matrix. Suppose Assumption 2.1 is satisfied. Then the unique solution \( \hat{S}_t = \text{Solut(Equ2, W)} \) of the equation (Equ2) is

\[
\hat{S}_t = \int_0^t W_x \frac{\tilde{H}(s)}{x} H \, dx + W_t - W_t \tilde{H}.
\]
Also

\[ \tilde{S}_t = \left( \int_1^t (dW_x)x^{-\tilde{H}} \right) t^{\tilde{H}} \text{ a.s.} \] (3.7)

Furthermore, with probability one \( \tilde{S}_t \) is continuous on \((0, \infty)\).

**Proof** The proof is similar to Proposition 3.1 and so omitted.

**Proposition 3.3** Let \( \{W_t; t \geq 0\} \) be a \( d \)-dimensional Brownian motion with some covariance matrix. Suppose Assumption 2.1 is satisfied. If \( \rho < 1/2 \), then for \( S_t = \text{Solut(Equ1, } W_t) \) we have

\[ \int_0^n S_t \frac{dt}{t} = \sum_{m=1}^{n-1} S_m \frac{m}{m} + O(1) \text{ a.s.} \] (3.8)

If \( \rho < 1 \), then for \( \hat{S}_t = \text{Solut(Equ2, } W_t) \) we have

\[ \int_1^n \hat{S}_t \frac{dt}{t} = \sum_{m=1}^{n-1} \hat{S}_m \frac{m}{m} + O(1) \text{ a.s.} \] (3.9)

**Proof** We only give a proof of (3.9) since the proof of (3.8) is similar. First, from (3.7) it follows that for all \( t > 1 \),

\[
\| \text{Var}(\hat{S}_t) \| = \left\| \int_1^t \left( \frac{t}{x} \right)^{-\tilde{H}} \text{Var}(W_1)(\frac{t}{x})^{\tilde{H}} dx \right\| \leq C \int_1^t (t/x)^{2\rho} \log^{2\nu - 2}(t/x) dx
\]

\[
\leq \begin{cases} 
C t \log^{2\nu - 1} t, & \text{if } \rho = 1/2 \\
C t, & \text{if } \rho < 1/2 \\
C t^{2\rho} \log^{2\nu - 2} \log t, & \text{if } \rho > 1/2
\end{cases}
\]

So, \( E\|\hat{S}_t\| \leq C t^{\rho \nu - 1/2} \log^{\nu - 1/2} t \). According to equation (Equ2),

\[ \hat{S}_t - \hat{S}_s = W_t - W_s + \int_s^t \frac{\hat{S}_x H}{x} dx, \text{ } t \geq s \geq 1. \]

It follows that

\[
\sum_{m=1}^{\infty} \int_m^{m+1} \left( \frac{\hat{S}_t}{m} - \frac{\hat{S}_m}{m} \right) dt \\
= \sum_{m=1}^{\infty} \int_m^{m+1} \hat{S}_t \left( \frac{1}{t} - \frac{1}{m} \right) dt + \sum_{m=1}^{\infty} \int_m^{m+1} \frac{\hat{S}_t - \hat{S}_m}{m} dt \\
= \sum_{m=1}^{\infty} \int_m^{m+1} \hat{S}_t \left( \frac{1}{t} - \frac{1}{m} \right) dt + \sum_{m=1}^{\infty} \int_m^{m+1} \frac{W_t - W_m}{m} dt \\
+ \sum_{m=1}^{\infty} \frac{1}{m} \int_m^{m+1} \frac{\hat{S}_x H}{x} dx dt.
\]
The first and the third term above are a.s. convergent because
\[
\sum_{m=1}^{\infty} \int_{m}^{m+1} E\|\tilde{S}_t\| \frac{1}{t} - \frac{1}{m} dt \leq C \sum_{m=1}^{\infty} \frac{(m+1)^{\rho\frac{1}{2}} \log^{n-1/2}(m+1)}{m^2} < \infty,
\]
and
\[
\sum_{m=1}^{\infty} \frac{1}{m} \int_{m}^{m+1} \int_{m}^{t} E\|\tilde{S}_x\tilde{H}\| dx dt \leq C \sum_{m=1}^{\infty} \frac{(m+1)^{\rho\frac{1}{2}} \log^{n-1/2}(m+1)}{m^2} < \infty.
\]
The second term is a.s. convergent because it is an infinite series of independent normal random variables with
\[
\sum_{m=1}^{\infty} \left\| \text{Var} \left\{ \int_{m}^{m+1} \frac{W_i - W_m}{m} dt \right\} \right\| \leq C \sum_{m=1}^{\infty} \frac{1}{m^2} < \infty.
\]
It follows that
\[
\sum_{m=1}^{n-1} \int_{m}^{m+1} \left( \frac{\tilde{S}_t}{t} - \frac{\tilde{S}_m}{m} \right) dt = O(1) \text{ a.s.}
\]
The proof of (3.9) is completed.

Propositions 3.1 and 3.2 give the solutions of equations (Equ1) and (Equ2). To give further properties of the \( G_t \) and \( \tilde{G}_t \), we need the analytic representation of the solutions. Recall \( T = (t_1', \ldots, t_d') \), where \( t_1 = 1 \). Let \( \{W_t; t \geq 0\} \) be a \( d \)-dimensional Brownian motion with some co-variance matrix \( \Lambda \). First we consider (Equ1). Let \( \{S_t = \text{Solut}(\text{Equ1}, W_t); t \geq 0\} \) be the solution of (Equ1) and \( U_t = S_tT \). Then \( U_t \) is the unique solution of the equation
\[
U_t = W_tT + \int_0^t \frac{U_s}{s} J_s ds \quad t \geq 0, \quad U_0 = 0
\]
(3.10)
Note that \( \tilde{J} = \text{diag}(0, J) = \text{diag}(0, J_2, \ldots, J_s) \), where \( J_i \)'s are defined as in (2.5). Write \( U_t = (U_{i1}, U^{(2)}_t, \ldots, U^{(s)}_t) \), where \( U^{(i)}_t = (U^{(i)}_{t1}, \ldots, U^{(i)}_{ts}) \) is the vector which contains \( \nu_i \) coordinate variables corresponding to \( J_i \). Also write \( T = (1', T^{(2)}, \ldots, T^{(s)}) \), where \( T^{(i)} = (t_1^{(i)}, \ldots, t_{\nu_i}^{(i)}) \) is the \( \nu_i \times d \) matrix which contains \( \nu_i \) columns of \( T \) corresponding to \( J_i \). Obviously, \( U^{(i)}_{tj} = U_{t1+\nu_2+\ldots+\nu_{i-1}+j} \) and \( t_{ij} = t_{1+\nu_2+\ldots+\nu_{i-1}+j} \). It is easily seen that (3.10) is equivalent to
\[
U_{i1} = W_{i1}1'
\]
\[
d U^{(i)}_{t1} = d(W_{i1}t_{1i1}' + \lambda_i \frac{U^{(i)}_{t1}}{t} dt), \quad U^{(i)}_{01} = 0,
\]
\[
d U^{(i)}_{tj} = d(W_{ij}t_{ji}' + \frac{U^{(i)}_{tc}}{t} + \lambda_{ij} \frac{U^{(i)}_{tj}}{t} dt), \quad U^{(i)}_{0j} = 0,
\]
\( j = 2, \ldots, \nu_i; \quad i = 2, \ldots, s \).
On can show that the solution of equation (3.11) is

\[ U_{t1} = W_t 1' \]
\[ U_{t1}^{(i)} = t^{\lambda_i} \int_0^t \frac{dW_s t_{ij}}{x^\gamma_s}, \]
\[ U_{tj}^{(i)} = t^{\lambda_i} \int_0^t \frac{dW_s t_{ij}}{x^\gamma_s} + t^{\lambda_i} \int_0^t \frac{U_{t1}^{(i)}}{x^\gamma_s} dx, \]

\[ j = 2, \ldots, \nu_t; \quad i = 2, \ldots, s. \]  

(3.12)

Putting all the \( U \)'s to \( S_t = U_t T^{-1} \), we obtain the solution of (Equ1).

Similarly, we have \( \hat{S}_t = \hat{U}_t T^{-1} \), where

\[ U_t = W_t T - W_1 T + \int_1^t \hat{U}_s T ds \quad t > 0, \quad \hat{U}_1 = 0 \]  

(3.13)

and, \( \hat{U}_t = (\hat{U}_{t1}, \hat{U}_{t2}^{(2)}, \ldots, \hat{U}_{ts}^{(s)}), \hat{U}_t^{(i)} = (\hat{U}_{t1}^{(i)}, \ldots, \hat{U}_{t\nu_t}^{(i)}), \)

\[ \hat{U}_{t1} = (W_t - W_1) 1', \]
\[ \hat{U}_{t1}^{(i)} = t^{\lambda_i} \int_1^t \frac{dW_s t_{ij}}{x^\gamma_s}, \]
\[ \hat{U}_{tj}^{(i)} = t^{\lambda_i} \int_1^t \frac{dW_s t_{ij}}{x^\gamma_s} + t^{\lambda_i} \int_1^t \frac{U_{t1}^{(i)}}{x^\gamma_s} dx, \]

\[ j = 2, \ldots, \nu_t; \quad i = 2, \ldots, s. \]  

(3.14)

**Proposition 3.4** Under Assumption 27 and \( \rho < 1/2, \)

\[ \text{Var}\{ (G_{t1} H + G_{t2}, G_{t1} + \int_0^t \frac{G_{x2}}{x} dx (I - 1'v)) \} = t \Gamma \]  

(3.15)

with

\[ \Gamma = \text{Var}\{ (G_{11} H + G_{12}, G_{11} + \int_0^1 \frac{G_{x2}}{x} dx (I - 1'v)) \} \]

\[ =: \begin{pmatrix} \Gamma^{(11)} & \Gamma^{(12)} \\ \Gamma^{(21)} & \Gamma^{(22)} \end{pmatrix} \]  

(3.16)

and

\[ \Gamma^{(11)} = \int_0^1 \frac{1}{x} \overline{H}' \left( H' \Sigma_1 H + \Sigma_2 \right) \left( \frac{1}{x} \right) \overline{H} dx, \]

\[ \Gamma^{(22)} = \int_0^1 \frac{1}{x} \overline{H}' \Sigma_1 \left( \frac{1}{x} \right) \overline{H} dx \]

\[ + (I - 1'v)' \int_0^1 \left[ \int x \left( \frac{1}{x} \overline{H} dy \right) \right] ' \Sigma_2 \left[ \int x \left( \frac{1}{x} \overline{H} dy \right) \right] dx (I - 1'v), \]

\[ \Gamma^{(12)} = \Gamma^{(21)} = H' \int_0^1 \frac{1}{x} \overline{H}' \Sigma_1 \left( \frac{1}{x} \right) \overline{H} dx \]

\[ + (I - \overline{H}')^{-1} \int_0^1 \frac{1}{x} \overline{H}' \Sigma_2 \left( \frac{1}{x} \right) \overline{H} dx (I - 1'v). \]
Proof Let \( \{W_t; t \geq 0\} \) be a \( d \)-dimensional Brownian motion with some co-variance matrix \( \Lambda \), \( \{S_t = \text{Solut}(\text{Equ}1, W_t); t \geq 0\} \) be the solution of (Equ1). Notice \( \{T^{-1/2}W_{Tt}, t \geq 0\} \) and \( \{W_t, t \geq 0\} \) are identical distributed. So \( \{T^{-1/2}S_{Tt}, t \geq 0\} \) and \( \{S_t, t \geq 0\} \) are identical distributed. Hence (3.15) is true. By (3.5),

\[
\int_0^t \frac{S_{2u}}{x} dy = \int_0^t \left[ \frac{1}{y} \int_0^y dW_x \left( \frac{y}{x} \right)^H \right] dy = \int_0^t dW_x \left[ \int_x^y \frac{1}{y} \left( \frac{y}{x} \right)^H dy \right].
\]

It follows that

\[
\text{Var}\{S_1\} = \int_0^1 x^{-H'} \Lambda x^{-H} dx,
\]

\[
\text{Cov}\{S_t, S_s\} = \int_0^1 \left( \begin{array}{c} t \cr x \end{array} \right)^H \Lambda \left( \begin{array}{c} s \cr x \end{array} \right)^H dx = s \left( \begin{array}{c} t \cr s \end{array} \right)^H \text{Var}\{S_1\}, \quad t \geq s,
\]

\[
\text{Var}\left\{ \int_0^1 \frac{S_u}{x} dy \right\} = \int_0^1 \left[ \int_x^1 \frac{1}{y} \left( \frac{y}{x} \right)^H dy \right]' \Lambda \left[ \int_x^1 \frac{1}{y} \left( \frac{y}{x} \right)^H dy \right] dx,
\]

\[
\text{Cov}\left\{ S_1, \int_0^1 \frac{S_y}{x} dy \right\} = \int_0^1 \frac{1}{y} \text{Cov}\{S_1, S_y\} dy = \int_0^1 \left( \frac{1}{y} \right)^H \text{Var}\{S_1\} dy \text{Var}\{S_1\} = (I - \tilde{H}')^{-1} \text{Var}\{S_1\}.
\]

The proof is now completed by noticing the independence of \( G_{t1} \) and \( G_{t2} \).

**Proposition 3.5** Under Assumption 2.1 and \( \rho = 1/2 \), the limit

\[
\tilde{\Gamma} = \lim_{t \to \infty} t^{-1}(\log t)^{1-2\nu} \text{Var}\{ \hat{G}_{t1}H + \hat{G}_{t2}, \hat{G}_{t1} + \int_1^t \hat{G}_{x;2} dx (I - 1'v) \} \tag{3.17}
\]

exists, and

\[
\tilde{\Gamma} = \left( \begin{array}{cc} \tilde{\Gamma}^{(11)} & \tilde{\Gamma}^{(12)} \\ \tilde{\Gamma}^{(21)} & \tilde{\Gamma}^{(22)} \end{array} \right), \tag{3.18}
\]

where

\[
(T^* \tilde{\Gamma}^{(11)} T)_{ij} = \frac{1}{(\nu - 1)^2} \frac{1}{2 \nu - 1} (|\lambda_t|^2 t_{1i} \Sigma_1 t_{j1}^t + \lambda_t \Sigma_2 t_{j1}^t),
\]

\[
(T^* \tilde{\Gamma}^{(22)} T)_{ij} = \frac{1}{(\nu - 1)^2} \frac{1}{2 \nu - 1} (t_{1i} \Sigma_1 t_{j1}^t + |\lambda_t|^2 t_{1i} \Sigma_2 t_{j1}^t), \tag{3.19}
\]

\[
(T^* \tilde{\Gamma}^{(12)} T)_{ij} = (T^* \tilde{\Gamma}^{(21)} T)_{ij} = \frac{1}{(\nu - 1)^2} \frac{1}{2 \nu - 1} (t_{1i} \Sigma_1 t_{j1}^t + \lambda_t t_{1i} \Sigma_2 t_{j1}^t)
\]

whenever \( i = j = 1 + \nu_2 + \ldots + \nu_1 \) and \( \text{Re}(\lambda_t) = 1/2, \nu_1 = \nu \), and \( (T^* \tilde{\Gamma}^{(uv)} T)_{ij} = 0 \) for otherwise \( u, v = 1, 2 \). Here \( \overline{a} \) is the conjugate vector of a complex vector \( a \).
Proof Let $W_t$ be a $d$-dimensional Brownian motion with a co-variance $\lambda$, $\hat{S}_t$ a solution of (Equ2) and $\hat{U}_t = \hat{S}_t T$. Then by (3.13) and Proposition 3.2,

$$\hat{U}_{11} = (W_t - W_1)^T L^2 o(t^{1/2} \log^{v-1/2} t)$$

and

$$\hat{U}^{(i)}_t = W_t T^{(i)}_t - W_1 T^{(i)}_1 + \int_1^t W_x T^{(i)}_t \frac{1}{x} (\frac{t}{x})^J_i J_i \, dx$$

$$\overset{L^2}{=} o(t^{1/2} \log^{v-1/2} t) + \sum_{k=0}^{\nu_i-1} \frac{1}{k!} \int_1^t W_x T^{(i)}_t \frac{1}{x} \lambda_i \log^{k} \frac{t}{x} T^k_i J_i \, dx$$

$$= o(t^{1/2} \log^{v-1/2} t) + \sum_{k=0}^{\nu_i-1} \frac{1}{k!} \lambda_i \int_1^t W_x T^{(i)}_t \frac{1}{x} \lambda_i \log^{k} \frac{t}{x} T^k_i \, dx,$$

where $T_i$ is defined as in (3.2). It is easily seen that

$$\int_1^t W_x T^{(i)}_t \frac{1}{x} \lambda_i \log^{k} \frac{t}{x} \overset{L^2}{=} O(1) \int_1^t x^{1/2} \frac{1}{x} \log^{k} \frac{t}{x}$$

$$= \begin{cases} O(t^{1/2}), & \text{if } Re(\lambda_i) < 1/2, \\ O(t^{1/2} \log^{k+1} t), & \text{if } Re(\lambda_i) = 1/2. \end{cases}$$

So

$$\hat{U}^{(i)}_t \overset{L^2}{=} \frac{\lambda_i}{(\nu_i - 1)!} \int_1^t W_x T^{(i)}_t \frac{1}{x} \lambda_i \left( \log^{\nu_i-1} \frac{t}{x} J_i - T^{\nu_i-1}_i \right) \frac{dx}{x} + o(t^{1/2} \log^{v-1/2} t).$$

It follows that

$$\hat{U}^{(i)}_{1j} \overset{L^2}{=} \begin{cases} \frac{\lambda_i}{(\nu_i - 1)!} \int_1^t W_x T^{(i)}_t \frac{1}{x} \lambda_i \log^{\nu_i-1} \frac{t}{x} J_i \frac{dx}{x}, & \text{if } Re(\lambda_i) = 1/2, j = \nu_i = \nu, \\ 0, & \text{otherwise}. \end{cases}$$

Similarly,

$$\int_1^t \frac{x \hat{U}^{(i)}_{1j}}{x} \, dx \overset{L^2}{=} \begin{cases} \frac{1}{(\nu_i - 1)!} \int_1^t W_x t^{(i)}_1 \frac{1}{x} \lambda_i \log^{\nu_i-1} \frac{t}{x} \frac{dx}{x}, & \text{if } Re(\lambda_i) = 1/2, j = \nu_i = \nu, \\ 0, & \text{otherwise}. \end{cases}$$

On the other hand, if $Re(\lambda_i) = Re(\lambda_j) = 1/2$ and $\nu_i = \nu_j = \nu$, then

$$\text{Cov}\left\{ \int_1^t W_x t^{(i)}_1 \frac{1}{x} \lambda_i \log^{\nu_i-1} \frac{t}{x} \, dx, \int_1^t W_x t^{(j)}_1 \frac{1}{x} \lambda_j \log^{\nu_j-1} \frac{t}{x} \, dx \right\}$$

$$= \begin{cases} (2\nu - 1)^{-1} |\lambda_i|^{-2} \bar{e}_{1i} \bar{\Lambda}_j^{(1)} (1 + o(1)) \log^{2\nu_i-1} t, & \text{if } \lambda_i = \lambda_j, \\ (2\nu_i - 1/\lambda_i + 1) \bar{\lambda}_i \bar{\Lambda}_j^{(1)} (1 + o(1)) \log^{2\nu_i-2} t, & \text{if } \lambda_i \neq \lambda_j. \end{cases}$$

15
It follows that
\[
\lim_{t \to \infty} t^{-1}(\log t)^{-2} \text{Var} \left\{ (\hat{G}_{11} H + \hat{G}_{12}) T, (\hat{G}_{21} + t \frac{\hat{G}_{x1}}{x} d(x - 1')v) T \right\} = \left( \begin{array}{c}
(\hat{\Xi}^{(11)}) \\
(\hat{\Xi}^{(12)}) \\
(\hat{\Xi}^{(21)}) \\
(\hat{\Xi}^{(22)}) 
\end{array} \right)
\]
exists. Also if \( i = j = 1 + \nu_2 + \ldots + \nu_\eta \) and \( \Re(\lambda_i) = 1/2, \nu_\eta = \nu \), then
\[
(\hat{\Xi}^{(11)})_{ij} = \frac{1}{(\nu_0 - 1)!} \frac{1}{2^{\nu_0 - 1}} (|\lambda_1^2| \bar{W}_{11}^1 \Sigma_{1j}^t + \bar{W}_{11}^2 \Sigma_{2j}^t),
\]
\[
(\hat{\Xi}^{(22)})_{ij} = \frac{1}{(\nu_0 - 1)!} \frac{1}{2^{\nu_0 - 1}} (|\lambda_2^2| \bar{W}_{11}^1 \Sigma_{1j}^t + |\lambda_2|^{-2} \bar{W}_{11}^2 \Sigma_{2j}^t),
\]

(3.20)
\[
(\hat{\Xi}^{(12)})_{ij} = \frac{1}{(\nu_0 - 1)!} \frac{1}{2^{\nu_0 - 1}} (\bar{W}_{11}^1 \Sigma_{1j}^t \chi_{1j}^1 + \lambda_1^{-1} \bar{W}_{11}^2 \Sigma_{2j}^t),
\]
\[
(\hat{\Xi}^{(1u)})_{ij} = 0 \text{ for other cases, } u, v = 1, 2. \text{ The proof is completed.}
\]

**Proposition 3.6** Suppose Assumption [2.1] is satisfied. If \( \rho < 1/2 \), then for \( i = 1, 2 \),
\[
G_{ii} = O(t \log t)^{1/2} \quad \text{and} \quad \int_0^t \frac{G_{x1}}{x} dx = O(t \log t)^{1/2} \text{ a.s. } t \to \infty.
\]

(3.21)
\[
\text{If } \rho = 1/2, \text{ then for } i = 1, 2,
\]
\[
G_{ii} = O((t \log t)^{1/2} \log^{-1/2} t) \text{ a.s. } t \to \infty,
\]

(3.22)
\[
\int_0^t \frac{G_{x1}}{x} dx = O((t \log t)^{1/2} \log^{-1/2} t) \text{ a.s. } t \to \infty.
\]

**Proof** Let \( W_t \) be a \( d \)-dimensional Brownian motion, \( S_t \) a solution of (Equ1). If \( \rho < 1/2 \), then by (3.4) and Lemma 3.1
\[
\|S_t\| = O((t \log t)^{1/2}) + \int_0^t \frac{O(x \log \log x)^{1/2}}{x} (\frac{t}{x})^\rho \log^{-1} (\frac{t}{x})
\]
\[
= O((t \log t)^{1/2}) \text{ a.s.}
\]
which implies (3.21).

When \( \rho = 1/2 \), let \( \hat{S}_t \) be a solution of (Equ1). Then \( \hat{U}_t = \hat{S}_t T \) a solution of (Equ1). It is easily seen that (cf. Bai, Hu and Zhang 2002)
\[
t^{\lambda_i} \int_0^t \frac{d(W_t \Sigma_{ij}^t)}{x^\lambda_i} \text{ a.s.} =
\begin{cases}
O((t \log t)^{1/2}) \text{ a.s.,} & \text{if } \Re(\lambda_i) < 1/2 \\
O((t \log t)^{1/2} \log^{1/2} t) \text{ a.s.,} & \text{if } \Re(\lambda_i) = 1/2
\end{cases}
\]
as \( t \to \infty \). From (3.14) it follows that, if \( \Re(\lambda_i) < 1/2 \),
\[
\hat{U}_{11}^{(i)} = O((t \log t)^{1/2}) \text{ a.s.}
\]
\[
\hat{U}_{1j}^{(i)} = O((t \log t)^{1/2}) + t^{\Re(\lambda_i)} \int_0^t \frac{O(x \log \log x)^{1/2}}{x^{1+\Re(\lambda_i)}} = O((t \log t)^{1/2}) \text{ a.s.}
\]
\[
j = 2, \ldots, v; i = 1, \ldots, s,
\]

16
and if $\text{Re}(\lambda_i) = 1/2$,

\[
\hat{U}_t^{(i)} = O\left((t \log \log \log t)^{1/2} \log \log t \right) \quad \text{a.s.}
\]

\[
\hat{U}_{tj}^{(i)} = O\left((t \log \log \log t)^{1/2} \log \log t \right) + t^{1/2} \int_1^t O\left((x \log \log \log x)^{1/2} \log (t^{1/2} - x) \right) dx \quad \text{a.s.}
\]

\[j = 2, \ldots, v; i = 1, \ldots, s.\]

It follows that $\hat{S}_t = \hat{U}_t T^{-1} = O\left((t \log \log \log t)^{1/2} \log \log (v^{1/2} \log t) \right) \quad \text{a.s.}$ Also,

\[
\int_1^t \frac{\hat{S}_x}{x} dx = \int_1^t \frac{O\left((x \log \log \log x)^{1/2} \log \log (v^{1/2} x) \right)}{x} dx = O\left((t \log \log \log t)^{1/2} \log \log (v^{1/2} x) \right) \quad \text{a.s.}
\]

So (3.22) is proved.

4 Applications.

In this section, we give several applications of the approximation theorems. First, by combining Theorem 2.1 with Proposition 3.4 and Theorem 2.2 with Proposition 3.5 respectively, we have the following asymptotic normalities for $(Y_n, N_n)$.

**Theorem 4.1** Under Assumptions 2.1 and 2.2, and $\rho < 1/2$,

\[n^{-1/2} (Y_n - n \nu, N_n - n \nu) \overset{D}{\to} N(0, \Gamma),\]

where $\Gamma$ is defined in Proposition 3.4.

**Theorem 4.2** Suppose Assumptions 2.1 and 2.2 are satisfied. Further, assume (2.8) is satisfied and $\rho = 1/2$. Then

\[n^{-1/2} (\log n)^{1/2-\nu} (Y_n - n \nu, N_n - n \nu) \overset{D}{\to} N(0, \tilde{\Gamma}),\]

where $\tilde{\Gamma}$ is defined in Proposition 3.5.

Also, by combining Proposition 3.6 with Theorem 2.1 and Theorem 2.2 respectively, we have the following laws of the iterated logarithm.
Theorem 4.3 Suppose Assumptions 2.1 and 2.2 are satisfied. If $\rho < 1/2$, then

$$Y_n - n \nu = O((n \log \log n)^{1/2}) \text{ a.s.},$$

$$N_n - n \nu = O((n \log \log n)^{1/2}) \text{ a.s.}$$

If (2.5) is satisfied and $\rho = 1/2$, then

$$Y_n - n \nu = O((n \log \log n)^{1/2} \log^{\nu-1/2} n) \text{ a.s.},$$

$$N_n - n \nu = O((n \log \log n)^{1/2} \log^{\nu-1/2} n) \text{ a.s.}$$

Next, we consider a two-treatment case in which the addition rule matrices are denoted by

$$D_m = \begin{pmatrix} d_1(\xi_{m1}), & 1 - d_1(\xi_{m1}) \\ 1 - d_2(\xi_{m2}), & d_2(\xi_{m2}) \end{pmatrix},$$

where $(\xi_{11}, \xi_{12}), \ldots, (\xi_{m1}, \xi_{m2})$ are assumed to be i.i.d. random variables with $0 \leq d_k(\xi_{mk}) \leq 1$ for $k = 1, 2$. This is a generalized randomized play-the-winner (RPW) rule (Bai and Hu, 1999). When $\xi_{m1}$ and $\xi_{m2}$ are dichotomous and $d_k(x) = x$, then generalized RPW model is the well-known RPW model proposed by Wei and Durham (1978). In using the generalized RPW rule, at the stage $m$, if the patient $m$ is allocated to treatment 1 and the response $\xi_{m1}$ is observed, then $d_1(\xi_{m1})$ balls of type 1 and $1 - d_1(\xi_{m1})$ balls of type 2 are added to the urn. And, if the patient $m$ is allocated to treatment 2 and the response $\xi_{m2}$ is observed, then $d_2(\xi_{m2})$ balls of type 2 and $1 - d_2(\xi_{m2})$ balls of type 1 are added to the urn. It is obvious that the generating matrix is

$$H_m = H = E[D_m | \mathcal{F}_{m-1}] = E[D_m] = \begin{pmatrix} p_1, & q_1 \\ q_2, & p_2 \end{pmatrix},$$

where $p_k = E[d_k(\xi_{mk})]$ and $q_k = 1 - p_k$ for $k = 1, 2$. It is easily checked that Assumptions 2.1 and 2.2 are satisfied, and $v_1 = q_2/(q_1 + q_2)$, $v_2 = q_1/(q_1 + q_2)$, $1 = 1$, $\rho = \lambda_2 = p_1 - q_2$. Denote $\sigma_1^2 = v_1 v_2 = \frac{a_1 q_2}{(q_1 + q_2)^2}$ and $\sigma_2^2 = \frac{a_1 q_2 + a_2 q_1}{q_1 + q_2}$, where $a_k = \text{Var}(d_k(\xi_1))$ for $k = 1, 2$. Then

$$\Sigma_1 = \sigma_1^2 \begin{pmatrix} 1, & -1 \\ -1, & 1 \end{pmatrix} = \sigma_1^2 (1, -1)'(1, -1), \quad \Sigma_2 = \sigma_2^2 \begin{pmatrix} 1, & -1 \\ -1, & 1 \end{pmatrix} \sigma_2^2 (1, -1)'(1, -1),$$

$$\bar{H} = \rho(v_2, -v_1)'(1, -1).$$
Further, it is trivial that, if $W_t$ is a Brownian motion with a variance-covariance matrix $\sigma^2(1,-1)(1,-1)$, then $W_t = \sigma(B_t - B_t)$ where $B_t$ is a standard Brownian motion. When $\rho < 1/2$, multiplying $1'$ in both side of the equation (Equ1) yields $S_t' = 0$, which implies $S_t = (S_t, -S_t)$ and $S_t$ is a solution of

$$S_t = \sigma B_t + \rho \int_0^t \frac{S_x}{x} dx, \quad S_0 = 0.$$ 

It is easily check that

$$S_t = \sigma t^\rho \int_0^t x^{-\rho} dB_x = \sigma B_t + \sigma t^\rho \int_0^t B_x x^{-\rho-1} dx$$

and

$$\int_0^t \frac{S_x}{x} dx = \sigma t^\rho \int_0^t B_t x^{-\rho-1} dx.$$

Also,

$$\text{Var}(S_t) = \sigma^2 t^{2\rho} \int_0^t x^{-2\rho} dx = \frac{\sigma^2}{1-2\rho} t,$$

$$\text{Var}\left( \int_0^t \frac{S_x}{x} dx \right) = \sigma^2 t^{2\rho} \int_0^t \int_0^t (x \wedge y)x^{-\rho-1}y^{-\rho-1} dxdy = \frac{2\sigma^2}{(1-2\rho)(1-\rho)} t,$$

$$\text{Cov}(S_t, S_s) = \sigma^2 (\frac{t}{s})^{\rho} \text{Var}(S_s) = \frac{\sigma^2}{1-2\rho} (\frac{t}{s})^{\rho} s, \quad t \geq s,$$

$$\text{Cov}\left( S_t, \int_0^t \frac{S_x}{x} dx \right) = \int_0^t \text{Cov}(S_t, S_x) dx = \frac{\sigma^2}{1-2\rho} (1-\rho) t.$$

Hence by applying Theorem 2.1 we conclude the following theorem.

**Theorem 4.4** For the generalized RPW rule, if $\rho = p_1 - q_2 < 1/2$, then there are two independent standard Brownian motion $B_{t1}$ and $B_{t2}$ such that for some $\gamma > 0$,

$$Y_{n1} - n\gamma_1 = n\rho \int_0^n x^{-\rho} d(\rho \sigma_1 B_{x1} + \sigma_2 B_{x2}) + o(n^{1/2-\gamma}) \ a.s.,$$

$$N_{n1} - n\gamma_1 = \sigma_1 n\rho \int_0^n x^{-\rho} dB_{x1} + \sigma_2 n\rho \int_0^n B_{x2} x^{-\rho-1} dx + o(n^{1/2-\gamma}) \ a.s.$$  

and

$$\rho(N_{n1} - n\gamma_1) = n\rho \int_0^n x^{-\rho} d(\rho \sigma_1 B_{x1} + \sigma_2 B_{x2}) - \sigma_2 B_{n2} + o(n^{1/2-\gamma}) \ a.s.$$  

In particular,

$$n^{1/2} \left( \frac{Y_{n1}}{n} - \frac{q_2}{q_1 + q_2}, \frac{N_{n1}}{n} - \frac{q_2}{q_1 + q_2} \right) \xrightarrow{p} N(0, \Sigma),$$

19
where \( \Sigma = (\sigma_{ij})_{i,j=1}^4 \) and

\[
\begin{align*}
\sigma_{11} &= \frac{(p_1 - q_2)^2 q_1 q_2 + (q_1 + q_2)(a_1 q_2 + a_2 q_1)}{(1 - 2(p_1 - q_2))(q_1 + q_2)^2}, \\
\sigma_{22} &= \frac{q_1 q_2 + 2(a_1 q_2 + a_2 q_1)}{(1 - 2(p_1 - q_2))(q_1 + q_2)^2}, \\
\sigma_{12} &= \sigma_{21} = \frac{(p_1 - q_2)q_1 q_2 + (a_1 q_2 + a_2 q_1)}{(1 - 2(p_1 - q_2))(q_1 + q_2)^2}.
\end{align*}
\]

When \( \rho = p_1 - q_2 = 1/2 \), by considering the equation \( \text{Equ2} \) and applying Theorem 2.2 instead, we can define two independent standard Brownian motion \( B_{11} \) and \( B_{12} \) such that

\[
Y_{n1} - n v_1 = n^{1/2} \int_1^n x^{-1/2} d\left(\frac{1}{2} \sigma_1 B_{x1} + \sigma_2 B_{x2}\right) + O(\sqrt{n}) \text{ a.s.,}
\]

\[
N_{n1} - n v_1 = \sigma_1 n^{1/2} \int_1^n x^{-1/2} dB_{x1} + \sigma_2 n^{1/2} \int_1^n B_{x2} x^{-3/2} dx + O(\sqrt{n}) \text{ a.s.}
\]

and

\[
\frac{1}{2}(N_{n1} - n v_1) = n^{1/2} \int_1^n x^{-1/2} d\left(\frac{1}{2} \sigma_1 B_{x1} + \sigma_2 B_{x2}\right) - (B_{n2} - n^{1/2} B_{12}) + O(\sqrt{n}) \text{ a.s.}
\]

If we denote

\[
\tilde{\sigma}^2 = \frac{1}{4} \sigma_1^2 + \sigma_2^2 = \frac{q_1 q_2}{4(q_1 + q_2)^2} + \frac{a_1 q_2 + a_2 q_1}{q_1 + q_2} = q_1 q_2 + 2(a_1 q_2 + a_2 q_1),
\]

and

\[
B(t) = \frac{1}{\sigma} \int_1^t x^{-1/2} d\left(\frac{1}{2} \sigma_1 B_{x1} + \sigma_2 B_{x2}\right),
\]

it is easily to check that \( B(t) \) is a standard Brownian motion. Hence we obtain the following theorem for the case of \( \rho = 1/2 \).

**Theorem 4.5** For the generalized RPW rule, if \( \rho = p_1 - q_2 = 1/2 \), then there a standard Brownian motion \( B(t) \) such that

\[
Y_{n1} - n \frac{q_2}{q_1 + q_2} = \tilde{\sigma} n^{1/2} B(\log n) + O(\sqrt{n}) \text{ a.s.,}
\]

\[
N_{n1} - n \frac{q_2}{q_1 + q_2} = 2 \tilde{\sigma} n^{1/2} B(\log n) + \begin{cases} O(\sqrt{n}) \quad \text{in probability,} \\ O(n \log \log n)^{1/2} \quad \text{a.s.} \end{cases}
\]

where \( \tilde{\sigma}^2 = q_1 q_2 + 2(q_1 q_2 + a_2 q_1) \). In particular,

\[
\limsup_{n \to \infty} \frac{Y_{n1} - n q_2/(q_1 + q_2)}{\sqrt{2n(\log n)(\log \log \log n)}} = \tilde{\sigma} \text{ a.s.,}
\]

20
Recall that 

Define

\[ \tilde{\Sigma} = (\tilde{\sigma}_{ij})_{i,j=1}^{4} \]

where \( \tilde{\sigma}_{11} = \tilde{\sigma}^2, \tilde{\sigma}_{12} = \tilde{\sigma}_{21} = 2\tilde{\sigma}^2, \tilde{\sigma}_{22} = 4\tilde{\sigma}^2. \]

5 Proof of the approximation theorems.

Define

\[
M_{n1} = \sum_{k=1}^{n} \{ X_k - \mathbb{E}[X_k | \mathcal{F}_{k-1}] \} =: \sum_{k=1}^{n} \Delta M_{k1},
\]

\[
M_{n2} = \sum_{m=1}^{n} X_m \{ D_m - \mathbb{E}[D_m | \mathcal{F}_{m-1}] \} =: \sum_{m=1}^{n} \Delta M_{m2}.
\]

(5.1)

Recall that \( a_n = Y_n 1' \) is the total number of balls in the urn after stage \( n \). By (1.1) we have

\[
Y_n = Y_0 + \sum_{k=1}^{n} X_k D_k
\]

\[
= Y_0 + \sum_{m=1}^{n} \left\{ X_m \{ D_m - \mathbb{E}[D_m | \mathcal{F}_{m-1}] \} \right\} + (X_m - \mathbb{E}[X_m | \mathcal{F}_{m-1}] + \frac{Y_{m-1}}{a_{m-1}}) H + X_m (H_m - H)
\]

\[
= Y_0 + M_{n2} + M_{n1} H + \sum_{m=0}^{n-1} \left( \frac{Y_m}{a_m} H + \sum_{m=1}^{n} X_m (H_m - H) \right) (\text{since} Y_m 1' = a_m, \tilde{H} = H - 1' v, \ v \tilde{H} = 0)
\]

\[
= n v + Y_0 + M_{n2} + M_{n1} H + \sum_{m=1}^{n-1} \left( \frac{Y_m}{m} v - v \tilde{H} \right)
\]

\[
+ (\frac{Y_0}{a_0} - v) \tilde{H} + \sum_{m=1}^{n-1} \frac{m - a_m}{m} \left( \frac{Y_m}{a_m} v - v \tilde{H} \right) + \sum_{m=1}^{n} X_m (H_m - H)
\]

\[
=: n v + M_{n2} + M_{n1} H + \sum_{m=1}^{n-1} \frac{Y_m - mv}{m} \tilde{H} + R_{n1} + Y_0,
\]

(5.2)

where

\[
R_{n1} = \left( \frac{Y_0}{a_0} - v \right) \tilde{H} + \sum_{m=1}^{n-1} \frac{m - a_m}{m} \left( \frac{Y_m}{a_m} - v \right) \tilde{H} + \sum_{m=1}^{n} X_m (H_m - H).
\]

(5.3)
Also by (1.2),

\[
N_n = \sum_{m=1}^{n} (X_m - E[X_m | \mathcal{F}_{m-1}]) + \sum_{m=1}^{n} E[X_m | \mathcal{F}_{m-1}] = M_{n1} + \sum_{m=0}^{n-1} \frac{Y_m}{a_m}
\]

\[
= n\nu + M_{n1} + \sum_{m=0}^{n-1} \left( \frac{Y_m}{a_m} - \nu \right) (I - 1')
\]

\[
+(Y_0/a_0 - \nu) + \sum_{m=1}^{n-1} m - a_m \left( \frac{Y_m}{a_m} - \nu \right) (I - 1')
\]

\[
= n\nu + M_{n1} + \sum_{m=1}^{n-1} \frac{Y_m - m\nu}{m} (I - 1') + R_{n2},
\]

(5.4)

where

\[
R_{n2} = \left( Y_0/a_0 - \nu \right) + \sum_{m=1}^{n-1} \frac{m - a_m}{m} \left( \frac{Y_m}{a_m} - \nu \right) (I - 1').
\]

(5.5)

The expansions given in (5.2) and (5.4) are the key component in asymptotic analysis of \( Y_n \) and \( N_n \). Actually, if we neglect the remainder \( R_{n1} \) and replace \( M_{n1} \tilde{H} + M_{n2} \) by a Brownian motion \( W_n \), then

\[
Y_n - n\nu \approx W_n + \sum_{m=1}^{n-1} \frac{Y_m - m\nu}{m} \tilde{H},
\]

which is very similar to the equations (Equ1) or (Equ2). We will show \( Y_n - n\nu, N_n - n\nu \) can be approximated by a 2d-dimensional Gaussian process by approximating the martingale \( (M_{n1}, M_{n2}) \) to a 2d-dimensional Brownian motion. First show that the remainders \( R_{n1} \) and \( R_{n2} \) can be neglected.

**Proposition 5.1** Under Assumptions (2.1) and (2.2), we have for any \( \delta > 0 \),

\[
R_{n1} = o(n^\delta) + \sum_{m=1}^{n} X_m (H_m - H) = o(n^{1/2-\tau}) \text{ a.s.},
\]

\[
R_{n1} = o(n^\delta) \text{ a.s.}
\]

To proving this proposition, we need two lemmas, the first one can be found in Hu and Zhang (2004).

**Lemma 5.1** (Hu and Zhang (2004)) If \( \Delta Q_n = \Delta P_n + Q_{n-1} \tilde{H} / (n-1), n \geq 2 \), then

\[
\|Q_n\| = O(\|P_n\|) + \sum_{m=1}^{n} \frac{O(\|P_m\|)}{m}(n/m)^{\rho} \log^{\nu-1}(n/m).
\]
Lemma 5.2 Suppose \( \sup_m E\|D_m\|^2 < \infty \). Under Assumptions [2.4]

\[
M_n^{L_2} = O(n^{1/2}) \quad \text{and} \quad M_{n1}^{L_2} = O(n^{1/2}),
\]

\[
M_n \overset{a.s.}{=} O(n^{1/2+\delta}) \quad \forall \delta > 0 \quad \text{and} \quad M_{n1} \overset{a.s.}{=} O((\log \log n)^{1/2}),
\]

\[
a_n - n \overset{a.s.}{=} O(n^{1/2+\delta}) \quad \forall \delta > 0.
\]

Furthermore, under Assumption [2.2]

\[
M_n \overset{a.s.}{=} O((\log \log n)^{1/2}) \quad \text{and} \quad a_n - n \overset{a.s.}{=} O((\log \log n)^{1/2}).
\]

Proof Note that \( \|\Delta M_{n1}\| \leq \|X_n\| + E[\|X_n\| | \mathcal{F}_{n-1}] \leq 2 \), \( \|\Delta M_n\| \leq \|D_n\| + E[\|D_n\| | \mathcal{F}_{n-1}] \)

and \( a_n = n + Y_0' + M_n' + \sum_{m=1}^n X_m(H_m - H)' \). By the properties of martingale, the results follow easily.

Lemma 5.3 Suppose \( \rho \leq 1/2 \) and \( \sup_m E\|D_m\|^2 < \infty \). Under Assumptions [2.7]

\[
\frac{Y_n}{a_n} - v = o(n^{-1/2+\delta}) \quad \text{a.s. for any} \quad \delta > 0.
\]

(5.6)

Proof By (5.3) and Lemma 5.2, it is obvious that

\[
\|R_{n1}\| \leq C \sum_{m=1}^{n-1} \frac{|m - a_m|}{m} + \sum_{m=1}^n \|H_m - H\| = o(n^{1/2+\delta}) \quad \text{a.s. for any} \quad \delta > 0.
\]

From (5.2) and Lemma 5.2 it follows that

\[
Y_n - nv = \sum_{m=1}^{n-1} \frac{Y_m - m\nu H}{m} + o(n^{1/2+\delta}) \quad \text{a.s.}
\]

By Lemma 5.1 it follows that

\[
Y_n - nv = o(n^{1/2+\delta}) + \sum_{m=1}^n \frac{m^{(1/2+\delta)}}{m} (n/m)^{\rho} \log^{\rho-1} (n/m) = o(n^{1/2+\delta}) \quad \text{a.s.}
\]

Hence

\[
\frac{Y_n}{a_n} - v = \frac{Y_n - nv}{n} + \frac{(Y_n - nv)' Y_n}{n a_n} = o\left(\frac{n^{1/2+\delta}}{n}\right) = o(n^{-1/2+\delta}) \quad \text{a.s.}
\]

(5.6) is proved.

Now, we tend to

Proof of Proposition 5.1. Notice \( \frac{n-a_n}{m} (Y_m - v) = o(n^{-1+\delta}) \quad \text{a.s. by Lemma 5.3} \)

The proof is completed by noticing (5.3) and (5.5).

The next result is about the conditional variance-covariance matrix of the 2d-dimensional martingale \( (M_{n1}, M_{n2}) \).
Proposition 5.2 We have

$$E[(\Delta M_{m1})'(\Delta M_{m2})|\mathcal{F}_{m-1}] = 0$$  \hspace{1cm} (5.7)

and under Assumptions 2.1 and 2.2,

$$\sum_{m=1}^{n} E(\Delta M_{mi})'(\Delta M_{mi})|\mathcal{F}_{m-1}] = n\Sigma_i + o(n^{1-\epsilon}) \text{ a.s. } i = 1, 2. \hspace{1cm} (5.8)$$

Proof (5.7) is trivial. For (5.8), we have

$$E[\Delta M_n'(\Delta M_n)|\mathcal{F}_{n-1}] = E[(D_n - H_n)'\text{diag}(X_n)(D_n - H_n)|\mathcal{F}_{n-1}]$$

$$= E[(D_n - H_n)'\text{diag}(Y_{n-1}/a_{n-1})(D_n - H_n)|\mathcal{F}_{n-1}]$$

$$= E[(D_n - H_n)'\text{diag}(v)(D_n - H_n)|\mathcal{F}_{n-1}]$$

$$+ E[(D_n - H_n)'(\text{diag}(Y_{n-1}/a_{n-1}) - \text{diag}(v))(D_n - H_n)|\mathcal{F}_{n-1}]$$

$$= \sum_{q=1}^{d} v_q V_{nq} + \sum_{q=1}^{d} (Y_{n-1} - v_q) V_{nq}.$$

Under Assumptions 2.1 2.2 by Lemma 5.3 we have

$$\sum_{m=1}^{n} E[\Delta M_{m2}'\Delta M_{m2}|\mathcal{F}_{m-1}] = n\Sigma_2 + o(n^{1-\epsilon}) \text{ a.s.}$$

Also

$$E[\Delta M_1'(\Delta M_1)|\mathcal{F}_{n-1}]$$

$$= E[X_n'X_n|\mathcal{F}_{n-1}] - (E[X_n|\mathcal{F}_{n-1}])'E[X_n|\mathcal{F}_{n-1}]$$

$$= E[\text{diag}(X_n)|\mathcal{F}_{n-1}] - \frac{Y_{n-1}Y_{n-1}}{a_{n-1}a_{n-1}} = \text{diag}(\frac{Y_{n-1}}{a_{n-1}}) - \frac{Y_{n-1}Y_{n-1}}{a_{n-1}a_{n-1}}$$

$$= \text{diag}(v) - v'v + o(n^{-1/2+\delta}) \text{ a.s.}$$

(5.8) is proved.

Proof of Theorem 2.1 Suppose that Assumption 2.2 is satisfied. According to (5.8),

$$\sum_{m=1}^{n} E[(\Delta M_{m1}, \Delta M_{m2})'(\Delta M_{m1}, \Delta M_{m2})|\mathcal{F}_{m-1}] = n\text{ diag(}\Sigma_1, \Sigma_2\text{)} + o(n^{1-\epsilon}) \text{ a.s.}$$

It follows from Theorem 1.3 of Zhang (2004) that, there exist two standard $d$-dimensional Brownian motions $B_{t1}$ and $B_{t2}$ for which

$$(M_{n1}, M_{n2}) - (B_{n1}\Sigma_1^{1/2}, B_{n2}\Sigma_2^{1/2}) = o(n^{1/2-\gamma}) \text{ a.s.} \hspace{1cm} (5.9)$$
Here \( \gamma > 0 \) depends only on \( d \) and \( \epsilon \). Without loss of generality, we assume \( \gamma < \epsilon / 3 \).

Now let \( G_{ti} = \text{Solut}(\text{Equ}1, B_{ti}\Sigma_i^{1/2}) \) \((i = 1, 2)\). Then by Proposition 3.3,

\[
\int_0^n \frac{G_{xi}}{x} \, dx = \sum_{m=1}^{n-1} \frac{G_{mi}}{m} + O(1) \text{ a.s.} \quad i = 1, 2.
\]

Write \( G_t = G_{t2}\Sigma_2^{1/2} + G_{t1}\Sigma_1^{1/2}H \). Combining the above equality with \( 5.2 \), \( 5.9 \) and Proposition 5.1 yields

\[
Y_n - n\upsilon - G_n = \sum_{m=1}^{n-1} \frac{Y_m - m\upsilon - G_m}{m} \tilde{H} + o(n^{1/2 - \tau \wedge \gamma}) \quad \text{a.s.}
\]

By Proposition 5.1,

\[
Y_n - n\upsilon - G_n = o(n^{1/2 - \tau \wedge \gamma}) + \sum_{m=1}^{n} \frac{o(m^{1/2 - \tau \wedge \gamma})}{m} (n/m)^\rho \log^{v-1}(n/m) = o(n^{1/2 - \tau \wedge \gamma}) \quad \text{a.s.}
\]

Finally, combining the above equality with \( 5.4 \), \( 5.9 \) and Proposition 5.1 yields

\[
N_n - n\upsilon = M_{n1} + \sum_{m=0}^{n-1} \frac{Y_m - \text{E}Y_m (I - 1'\upsilon)}{m} + o(n^\delta)
\]

\[
= B_{n1}\Sigma_1^{1/2} + \sum_{m=1}^{n} \frac{G_m}{m} (I - 1'\upsilon) + o(n^{1/2 - \tau \wedge \gamma})
\]

\[
= B_{n1}\Sigma_1^{1/2} + \int_0^n \frac{G_x}{x} \, dx (I - 1'\upsilon) + o(n^{1/2 - \tau \wedge \gamma})
\]

\[
= G_{n1} + \int_0^n \frac{G_{x2}}{x} \, dx (I - 1'\upsilon) + o(n^{1/2 - \tau \wedge \gamma}) \quad \text{a.s.}
\]

The proof is now completed.

**Proof of Theorem 2.2.** \( 5.9 \) remains true. Let \( \tilde{G}_{ti} = \text{Solut}(\text{Equ}2, B_{ti}\Sigma_i^{1/2}) \) \((i = 1, 2)\). Then by Proposition 3.3

\[
\int_1^n \frac{\tilde{G}_{xi}}{x} \, dx = \sum_{m=1}^{n-1} \frac{\tilde{G}_{mi}}{m} + O(1) \text{ a.s.} \quad i = 1, 2.
\]

Write \( \tilde{G}_t = \tilde{G}_{t2}\Sigma_2^{1/2} + \tilde{G}_{t1}\Sigma_1^{1/2}H \). Combining the above equality with \( 5.2 \), \( 5.9 \) and Proposition 5.1 yields

\[
Y_n - n\upsilon - \tilde{G}_n = \sum_{m=1}^{n-1} \frac{Y_m - m\upsilon - \tilde{G}_m}{m} \tilde{H} + o(n^{1/2 - \gamma}) + \sum_{m=1}^{n} X_m(X_m - H) \quad \text{a.s.}
\]
By Proposition 5.1,

\[ Y_n - n\mathbf{v} - \hat{G}_n = o(n^{1/2-\gamma}) + \sum_{m=1}^{n} \frac{o(m^{1/2-\gamma})}{m} (n/m)^{1/2} \log^v(n/m) \]

\[ + \, O(\sum_{m=1}^{n} \|H_m - H\|) + \sum_{m=1}^{n} \frac{O(\sum_{j=1}^{m} \|H_m - H\|)}{m} (n/m)^{1/2} \log^v(n/m) \]

\[ = o(n^{1/2}) + O(\sum_{j=1}^{n} \|H_j - H\|) n^{1/2} \log^v(n) \]

Finally, combining the above equality with (5.4), (5.9) and Proposition 5.1 yields

\[ N_n - n\mathbf{v} = M_n + \sum_{m=0}^{n-1} \frac{Y_m - EY_m}{m} (I - 1'\mathbf{v}) + o(n^{\delta}) \]

\[ = B_n \Sigma_1^{1/2} + \sum_{m=1}^{n} \frac{\hat{G}_m}{m} (I - 1'\mathbf{v}) + O(n^{1/2} \log^v n) \]

\[ = B_n \Sigma_1^{1/2} + \int_{1}^{n} \frac{\hat{G}_x}{x} dx (I - 1'\mathbf{v}) + O(n^{1/2} \log^v n) \]

\[ = \hat{G}_n + \int_{1}^{n} \frac{\hat{G}_x^2}{x} dx (I - 1'\mathbf{v}) + O(n^{1/2} \log^v n) \text{ a.s.} \]

The proof is now completed.
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