Martin Schwarzschild’s Contributions to Galaxy Dynamics

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1. Introduction

The astronomical community’s debt to Martin Schwarzschild derives from much more than his published work, as many of us who were his students, collaborators and friends can testify. Nor did Schwarzschild’s contributions to galaxy dynamics constitute more than a small portion of his scientific output. Nevertheless it would be hard to think of another single figure whose work so influenced the development of many of the fields discussed at this meeting.

Those of us who came of scientific age after Schwarzschild’s retirement in 1979 tend to identify his contributions to galaxy dynamics with the remarkable series of papers on elliptical galaxies that began appearing at about the same time. But Schwarzschild’s interest in the structure and dynamics of stellar systems was lifelong; for instance, as early as 1951, he published the first of two papers with L. Spitzer concerning the influence of interstellar clouds on stellar velocities. A number of other papers from this decade dealt with the relation between the chemical composition and kinematics of stars in the Milky Way and other galaxies.

The following review focusses on three areas of galaxy dynamics where Schwarzschild’s contributions were particularly fundamental: the masses of stellar systems; the structure of galactic nuclei; and the dynamics of elliptical galaxies.

2. Masses of Stellar Systems

The study of the distribution of mass in external galaxies was still in its infancy when Schwarzschild published his 1954 paper, “Mass Distribution and Mass-Luminosity Ratio in Galaxies.” Here Schwarzschild re-analyzed the kinematical data in three galaxies – M31, M33 and NGC 3115 – for which earlier workers had found significantly different distributions of light and mass. In each galaxy, he showed that the data were in fact consistent with a constant ratio of mass to light, albeit with rather different values in the three systems. In the case of NGC 3115, for instance, Schwarzschild noted that a high central velocity dispersion recently measured by Minkowski implied a large deviation between circular and rotational velocities near the center of this galaxy, thus allowing \( M/L \) to remain approximately constant in spite of a low central \( v_c \).

But this paper also contained at least three, quite novel approaches to what we would now call the “dark matter problem.” First, Schwarzschild estimated the mass of M32 by assuming that its gravitational pull was responsible for the observed asymmetry in rotation velocity and morphology of its larger companion...
M31. He concluded that the mass-to-light ratio of M32 was of order 200, in approximate agreement with his value for NGC 3115. Second, Schwarzschild presented a new and elegant method for evaluating the virial theorem, the strip-count formula. He showed that the potential energy of a spherical system could be expressed simply in terms of $S(q)$, the observed number of objects in a strip of unit width that passes a distance $q$ from the projected center. He applied his technique to the Coma cluster using Zwicky’s galaxy counts and obtained the “bewilderingly high value” of 800 for its mass-to-light ratio. Finally, this paper contained what was probably the first suggestion that white dwarfs, remnants of an earlier generation of star formation, might constitute a significant fraction of the masses of galaxies.

In “Note on the Mass of M92” (1955), Schwarzschild and S. Bernstein used the strip-count formula to obtain one of the first accurate measurements of the mass-to-light ratio of a globular cluster.²

3. Structure of Galactic Nuclei

Schwarzschild’s pivotal role in the development and deployment of the balloon-borne telescopes Stratoscope I and II is well known.³ After its two initial flights, Stratoscope II, a 36-inch telescope, was reconfigured for high-definition photography and used to obtain images of galactic nuclei unblurred by the atmosphere. In “An Upper Limit to the Angular Diameter of the Nucleus of NGC 4151” (1968, 1973), Schwarzschild, R. Danielson and B. D. Savage reported that the nucleus of NGC 4151 had still not been resolved and accordingly that only an upper limit could be placed on its diameter, which they estimated at 0.08″. They were thus able to show that the non-thermal continuum, which provides most of the nuclear light in this Seyfert galaxy, originated in a region much smaller than that associated with the emission lines.

The eighth, and final, flight of Stratoscope II was used to obtain high-resolution photographs of M31 and M32. The results for M32, while intriguing, were never published; the observations were made shortly before sunrise while the telescope was gradually descending and the resultant temperature differentials caused a substantial degradation in the quality of the images. But the data seemed to show no evidence for a distinct nucleus at a resolution of $\sim 0.5″$, consistent with what we now know about the luminosity profile of this galaxy. Observations taken during the same night of M31 were more successful; in “The Nucleus of M31” (1974), E. S. Light, R. E. Danielson and Schwarzschild presented 0.2″ resolution photographs that clearly resolved the nucleus, showing it

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1 Strip counts had long been used to infer the density profiles of star clusters (e.g. Plummer 1911). Schwarzschild was apparently the first to notice that the potential energy could be computed directly from $S(q)$ without first converting it into a density profile.

2 Those familiar with Schwarzschild’s legendary tact will be struck by the introduction to this paper, which contains a withering (but accurate) critique of a rival formula for evaluating the virial theorem.

3 A wonderfully clear account of the observation of convection cells in the Sun with Stratoscope I was written by Schwarzschild and his wife, Barbara, for Scientific American (1959).
to have a core radius of only 0.48″. More striking was the observed asymmetry of the nucleus, which was revealed to have a low intensity extension on one side of the bright peak. Light et al. raised the possibility that the offset was a result of non-uniform obscuration by dust, and noted that, in the absence of dust, “the observed asymmetry is an intrinsic property of the nucleus which will probably require a dynamic explanation.” The latter picture is now accepted by most astronomers due to the absence of color variations.

4. Elliptical Galaxy Dynamics

Starting in 1976, when he was 64 years old, Schwarzschild wrote or co-authored a remarkable series of 21 papers on the dynamics of elliptical galaxies. The first of these, a collaboration with M. Ruiz, dates from the “early days” of the field when it was still universally assumed that elliptical galaxies and bulges were rotationally-supported, axisymmetric systems. “An Approximate Dynamical Model for Spheroidal Stellar Systems” (1976) presented a novel approach to the problem of elliptical galaxy modelling. Ruiz and Schwarzschild wrote $f(E, L_z) = f_0 e^{-E/\sigma^2} g(L_z)$, and assumed in addition that the density generated by $f$ was constant on spheroids of fixed eccentricity. The two assumptions are mildly inconsistent, as the authors fully realized, but together they permit an extremely elegant derivation of the function $g(L_z)$: one first matches the density profile on the rotation axis, which is independent of $g$, then uses the observed density in the equatorial plane to determine $g(L_z)$. Ruiz (1976) applied the model to the central region of M31, treating the nucleus and bulge as distinct components.

The bulge in Ruiz’s model of M31 was tipped out of the disk plane in order to reproduce the observed twist in the isophotes at about 10′ from the center of this galaxy. Stark (1977) recognized that a coplanar and triaxial bulge could reproduce the twist in M31 equally well. At about the same time, a number of workers began publishing integrated spectra which showed that these objects were rotating much more slowly than expected for centrifugally flattened oblate spheroids. Schwarzschild contributed to the emerging view of early-type galaxies as triaxial ellipsoids in two papers with T. B. Williams, “A Photometric Determination of Twists in Three Early-Type Galaxies,” I & II (1979). These studies revealed significant twists in the inner isophotes of three elliptical galaxies, which the authors cautiously interpreted as evidence that “many elliptical galaxies may have a more complicated basic structure than that of axially symmetric configurations.”

Schwarzschild’s most famous paper from this period is undoubtedly “A Numerical Model for a Triaxial Stellar System in Dynamical Equilibrium” (1979), in which he constructed the first completely self-consistent model of a triaxial galaxy. The approach was at the same time beautifully straightforward and quite novel. Schwarzschild’s insight was to treat individual, time-averaged orbits as building blocks for a galaxy – thus replacing the cumbersome self-consistency equations by a matrix equation that could be solved using standard numerical techniques. In the process, he discovered the four families of regular orbits in triaxial potentials, the boxes and the three types of tubes. His demonstration that most orbits in a non-axisymmetric potential could be regular – i.e. that
they respected three effective integrals of the motion – was quite unexpected at the time.

Schwarzschild went on, in two subsequent studies, to develop a more complete understanding of these major orbit families. “On the Nonexistence of Three-Dimensional Tube Orbits Around the Intermediate Axis in a Triaxial Galaxy Model” (1979), with G. Heiligman, linked the existence of the tube orbits to the stability of the 1:1 resonant orbits in the principal planes. The primary motivation for this work was the apparent absence of intermediate-axis tube orbits in the self-consistent triaxial model. The authors showed that the 1:1 orbit in the $X - Z$ plane (i.e. the plane perpendicular to the intermediate axis) was generally unstable to vertical perturbations, a circumstance which they noted was “quite plausibly destructive for the existence of $Y$-tube orbits.”

A second study with M. Vietri, “Analysis of Box Orbits in a Triaxial Galaxy” (1983) developed the picture of box orbits as perturbations of the stable, long-axis orbit. The key to the analysis was a careful treatment of the second-order terms: these terms were retained in the development of the transverse motion but omitted from the axial motion, thus allowing the equations for the different orders to be solved independently.

A remarkable paper from the following year, “Stellar Orbits in Angle Variables” (1984) with S. J. Ratcliff and K. M. Chang, showed how a complete description of a two-dimensional orbit could be obtained in terms of its action-angle variables. This problem currently goes under the name of “torus construction” but it is actually quite old, with antecedents in work of Einstein and Born on semi-classical quantization. Here again, the approach was beautifully direct. The authors asked simply: How must the Cartesian coordinates depend on the angles if the angles are to increase linearly with time? The result was a set of differential equations for $x$ and $y$ as functions of the angles. These equations are nonlinear, and Ratcliff et al. developed an iterative technique for solving them which worked well whenever the initial guess was sufficiently close to the true solution.

The slow observed rotation of elliptical galaxies was one of the factors that prompted Schwarzschild to construct his first triaxial model. Real elliptical galaxies probably do have rotating figures, and in 1982 Schwarzschild began investigating the effects of slow figure rotation on the triaxial self-consistency problem. “Retrograde Closed Orbits in a Rotating Triaxial Potential” (1982), with J. Heisler and D. Merritt, reported the existence of the “anomalous” orbits, 1:1 resonant orbits that are tipped out of the $Y-Z$ plane by Coriolis forces. The anomalous orbits give rise to two families of $X$-tubes that circulate in opposite directions about the long axis of a rotating triaxial figure. In “A Model for Elliptical Radio Galaxies with Dust Lanes” (1982), T. S. van Albada, C. G. Kotanyi and Schwarzschild suggested that the dust lanes of Centaurus A and M84 consisted of matter moving along these anomalous orbits.  

4Here and below, Schwarzschild’s convention is followed in which the $X$ and $Z$ axes are identified with the long and short axes of the triaxial figure.

5Subsequent observations of Centaurus A revealed that the sense of rotation of the stellar body of this galaxy is probably opposite to that of the van Albada et al. model, implying that the
Schwarzschild made one attempt at achieving self-consistency in a triaxial model with rapid figure rotation; this initial attempt failed, as Schwarzschild reported at one of the Princeton “Tuesday lunches,” and the work was never published. However a subsequent effort, using a more slowly rotating figure, was successful. In “Triaxial Equilibrium Models for Elliptical Galaxies with Slow Figure Rotation” (1982), Schwarzschild chose a value for the rotation period that was long enough (of order $10^9$ years after scaling) that all four of the major orbit families existed out to the truncation radius of the model. He noted that the two branches of $X$-tubes must be equally populated if such a model is to be eight-fold symmetric, which means that a rotating model will lack any streaming around the long axis. This was another example of how the use of orbits as building blocks could lead to insights about a galaxy’s kinematics that would have been difficult to obtain from the Jeans or Boltzmann equations.

In his 1979 self-consistency study, Schwarzschild had found that box orbits alone could not reproduce the mass distribution of his triaxial model, since they tended to place too much mass along the major axis. His solution was to incorporate $X$-tube orbits which avoid the long axis. Schwarzschild noted that solutions incorporating the other major orbit family, the $Z$-tubes, were also likely to exist and that the question of the uniqueness of solutions “is thus left unanswered by the present investigation.” He returned to the uniqueness question in a 1986 paper, “Dynamical Models for Galactic Bars: Truncated Perfect Elliptic Disk.” Schwarzschild considered a strongly truncated, planar mass model that supported only one family of orbits, the boxes, and showed numerically that a self-consistent solution existed and that it was unique. Beyond the truncation radius in this two-dimensional model, tube orbits exist in addition to box orbits, and one might expect to find a certain degree of non-uniqueness in solutions that draw on both orbit families. This was shown to be the case in a study with P. T. de Zeeuw and C. Hunter that appeared the following year, “Nonuniqueness of Self-Consistent Equilibrium Solutions for the Perfect Elliptic Disk” (1987). A further step toward demonstrating non-uniqueness in the three-dimensional problem was taken by Hunter, de Zeeuw, C. Park and Schwarzschild in “Prolate Galaxy Models with Thin-Tube Orbits” (1990). The authors showed that a variety of self-consistent solutions for axisymmetric prolate models could be found by varying the relative occupation numbers of orbits from the two families of thin long-axis tubes.

In 1980, R. H. Miller asked Schwarzschild whether he could test the stability of the nonrotating triaxial model. Schwarzschild agreed, and assigned one of his students the task of re-integrating the orbits to provide initial conditions for the $N$-body code. In the process it was discovered that many of the orbits generated different masses in the grid of cells than they had in the original integrations. The discrepancy was eventually traced to the installation of a new computer at the Princeton Computer Center: the differences in the round-off algorithms of the two machines were sufficient to trigger the exponential instability of those orbits that were stochastic, leading to significantly modified trajectories after many orbital periods. Schwarzschild followed up this hint in the following year in a

outer dust ring has not yet reached a steady state. However a triaxial figure is probably still required to support the inner ring.
study with J. Goodman, “Semistochastic Orbits in a Triaxial Potential” (1981). Goodman and Schwarzschild tested the stability of box orbits by looking for exponential divergence of nearby trajectories. They noted that a large fraction of the box orbits were in fact chaotic, but that the chaos produced only modest changes in the shapes of the orbits over 50 oscillations. They coined the term “semi-stochasticity” to describe this phenomenon. The chaos was tentatively linked to the linear instability of the short- and intermediate-axis orbits.

Schwarzschild’s self-consistent triaxial models from 1979 and 1982 were based on the Hubble density profile, which has a large, constant-density core. It became increasingly clear throughout the 1980’s that the luminosity profiles of many galaxies might increase more steeply at small radii; indeed, Schwarzschild’s own Stratoscope observations of M31 and NGC 4151 had revealed pointlike nuclei in these galaxies. The behavior of box orbits is very sensitive to the central density of a triaxial model, and in 1989 Schwarzschild began to look in detail at the orbits in triaxial models with small or nonexistent cores. His two studies with J. Miralda-Escudé and J. F. Lees – “On the Orbit Structure of the Logarithmic Potential” (1989) and “The Orbital Structure of Galactic Halos” (1992) – revealed that the planar motion in centrally concentrated models is dominated by resonances, which generate families of orbits not seen in models with large cores. Schwarzschild, who was fiercely opposed to opaque terminology, gave these resonant orbits names that evoked their shapes like “banana,” “fish” and “pretzel;” these names have remained in widespread use. He also began to look in these papers at the behavior of orbits in potentials with central point masses representing black holes.

While Miller & Smith’s (1981) $N$-body study did not find any strong evidence for instability in Schwarzschild’s triaxial model, a number of examples of dynamical instabilities in other models of hot stellar systems began to be discussed at about this time. In “Orbital Contributions to the Stability of Triaxial Galaxies” (1989), de Zeeuw and Schwarzschild used an adiabatic deformation technique to evaluate the stability to small perturbations of Statler’s (1987) triaxial models based on the perfect ellipsoid. They found that the response of individual box orbits to barlike perturbations was often destabilizing, in the sense that the response density tended to reinforce the original perturbation; a similar mechanism drives the radial-orbit instability in spherical models. In “The Ring Instability in Radially Cold Oblate Models” (1991), the same authors investigated axisymmetric instabilities in oblate models constructed from thin tube orbits. They found that such models were unstable to radial clumping when sufficiently flat. These stability studies provided yet a further demonstration of the usefulness of an orbit-based approach to galaxy dynamics.

In one of his last papers, “Self-Consistent Models for Galactic Halos,” Schwarzschild revisited the triaxial self-consistency problem, this time using models based on the singular isothermal mass distribution. Such models are scale-free, which allowed Schwarzschild to construct orbit libraries by scaling the orbits computed at a single energy; the increase in efficiency enabled him to compute orbit libraries for six different choices of the model axis ratios. Schwarzschild found that most of the box orbits in these models were significantly stochastic, a rather different situation than he had been led to expect by his earlier work in two dimensions. He showed that the omission of the stochastic
orbits could sometimes preclude a self-consistent solution, implying restrictions on the allowed shapes of isothermal halos. This study demonstrated clearly the importance of chaos in the phase space of realistic triaxial models and opened the door to a wealth of later studies of this fascinating topic.

5. Conclusion

It is sometimes said that a scientist’s career is over by the age of 35. One may safely assume that Martin Schwarzschild would have disagreed with this statement; in any case, all of the work cited here was published after that particular milestone had been passed. Without the contributions which Schwarzschild made in the late stages of his career, the field of galaxy dynamics would be an incomparably less rich and exciting one than it is today.

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