DIMENSION TOWERS OF SICS. I.

ALIGNED SICS AND EMBEDDED TIGHT FRAMES

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Abstract:

Algebraic number theory relates SIC-POVMs in dimension $d > 3$ to those in dimension $d(d - 2)$. We define a SIC in dimension $d(d - 2)$ to be aligned to a SIC in dimension $d$ if and only if the squares of the overlap phases in dimension $d$ appear as a subset of the overlap phases in dimension $d(d - 2)$ in a specified way. We give 19 (mostly numerical) examples of aligned SICs. We conjecture that given any SIC in dimension $d$ there exists an aligned SIC in dimension $d(d - 2)$. In all our examples the aligned SIC has lower dimensional equiangular tight frames embedded in it. If $d$ is odd so that a natural tensor product structure exists, we prove that the individual vectors in the aligned SIC have a very special entanglement structure, and the existence of the embedded tight frames follows as a theorem. If $d - 2$ is an odd prime number we prove that a complete set of mutually unbiased bases can be obtained by reducing an aligned SIC to this dimension.
1. Introduction

It sometimes happens that an apparently simple question leads into very deep waters. We are concerned with just such a question here \[1\,2\]. To begin at the beginning, a SIC (also known as a SIC-POVM, or as a maximal complex equiangular tight frame) is a collection of \(d^2\) unit vectors in \(\mathbb{C}^d\) such that they resolve the identity,

\[
\sum_{I=1}^{d^2} |\psi_I\rangle\langle \psi_I| = d \mathbb{1},
\]

and such that the absolute values of the overlaps \(\langle \psi_I|\psi_J\rangle\) are equal (to \(1/\sqrt{d+1}\) in fact) whenever \(I \neq J\). The acronym stands for Symmetric Informationally Complete, and betrays the quantum state tomographical origin of the concept. In ‘Bloch space’—the affine space of Hermitian operators with unit trace equipped with the Hilbert-Schmidt inner product—a SIC is a maximal regular simplex, inscribed in the set of pure states. An obvious question is: Do SICs exist in all dimensions?

At the outset the SIC existence problem shows almost no structure. However, the known solutions make it clear that SICs are deeply implicated in a major open question in algebraic number theory. In every dimension that has been studied so far \[3\,6\] there are SICs which are orbits under the discrete Weyl–Heisenberg group, a group with many applications in quantum mechanics \[7\], in radar and communication \[8\], and in some approaches to Hilbert’s 12th problem \[9\]. Remarkably, in every known example, in the preferred basis singled out by the Weyl–Heisenberg group the components of the SIC vectors belong to abelian extensions of a real quadratic number field \[10\]. (We assume throughout that \(d > 3\) and the SIC is Weyl–Heisenberg covariant.) Which real quadratic field that comes into play depends, contingent on a conjecture \[11\], in a known way on the dimension \(d\). After a highly non-trivial but well understood extension of the quadratic field one arrives at a ray class field with conductor \(d\) (or \(2d\) if \(d\) is even), and it appears that this always suffices to construct a SIC in dimension \(d\) \[11\]. See ref. \[12\] for an account that assumes little or no background in number theory. Ray class fields are important because every abelian extension is contained in some ray class field. In many (presumably most) dimensions several unitarily inequivalent SICs exist, and further extensions of the ray class field are needed to construct them all.

This particular connection between number theory and a simple geometric question was unexpected. It may be worthwhile to recall the connection between the geometry of regular polygons and the roots of unity. In number theoretic language the roots of unity generate extensions of the rational numbers, called cyclotomic fields. They are abelian extensions because the Galois group of the extension is abelian \[13\]. Moreover the cyclotomic field generated by an \(n\)th root of unity is a ray class field over the rational number field \(\mathbb{Q}\), with conductor \(n\) \[14\]. The importance of the conductor is that one cyclotomic field is a subfield of another if the conductor of the one divides the conductor of the second. Every abelian extension of the rational numbers is a subfield of one of these ray class fields.

A more pertinent example may be that of mutually unbiased bases (MUB) in dimensions \(d\) such that \(d\) is a prime number. Complete sets of such bases can be constructed using the Weyl–Heisenberg group, and in the preferred basis singled out by the group the components of all the MUB vectors can be constructed using \(d\)th roots of unity only (with a slight complication for \(d = 2\) \[15\]). Thus, to construct MUB in \(d\) dimensions one needs cyclotomic fields with conductor \(d\). Keep in mind that the roots of unity look extremely complex if one expresses them in terms of nested radicals, but they appear simple once it is realized that they can be obtained by evaluating the transcendental function \(e^{2\pi i z}\) at rational points. (See Appendix\[A\]) SICs are two orders of magnitude more difficult, because the relevant number fields are not yet fully understood. In particular, a description making use of special values of transcendental functions is conspicuously missing. Finding such a description forms an important part of the
unsolved 12th problem on Hilbert’s famous list. We say ‘two orders of magnitude’ because there is a completed theory of abelian extensions of imaginary quadratic fields, one order of magnitude more difficult than the theory of the cyclotomic fields, and relying on the geometry of elliptic curves. Hilbert is reported as saying that this theory “is not only the most beautiful part of mathematics but also of all science” [16]. But he wanted more, and understanding abelian extensions of the real quadratic fields seems a natural next step.

We have reached the deep waters. To see how the dimension towers arise out of them, we need to add some details. The real quadratic field \(\mathbb{Q}(\sqrt{D})\) conjectured to be relevant to SICs in dimension \(d\) consists of the set of all numbers of the form \(x + \sqrt{D}y\), where \(x, y\) are rational numbers \[D = \text{square-free part of } (d + 1)(d - 3)\].

Starting from this real quadratic number field one may perform further extensions to reach the ray class fields with conductor \(d\) (or \(2d\) if \(d\) is even).

The next question is what dimensions \(d\) correspond to what square-free integers \(D\). To see this one fixes a square free integer \(D > 1\) and solves the Diophantine equation \((d + 1)(d - 3) = m^2D\) \[\iff (d - 1)^2 - m^2D = 4\] for the integers \(m\) and \(d\). The solution consists of infinite sequences in each case [11, 12]. The beginnings of the sequences corresponding to the first three values of \(D\) are

\[
\begin{align*}
d &= 7, 35, 199, 1155, 6727, 39203, 228487 \ldots & \text{corresponding to } & D &= 2 \quad (4) \\
d &= 5, 15, 53, 195, 725, 2703, 10085 \ldots & \text{corresponding to } & D &= 3 \quad (5) \\
d &= 4, 8, 19, 48, 124, 323, 844, \ldots & \text{corresponding to } & D &= 5 \quad . \quad (6)
\end{align*}
\]

The last of these sequences is noteworthy for the fact that it contains no less than seven dimensions less than 1000, and is the subject of an important recent study by Grassl and Scott [17].

As with the cyclotomic fields, one field is a subfield of another if the conductor of the first divides the conductor of the other. Consequently, the divisibility properties of the dimensions give rise to an intricate partially ordered set ordered by field inclusions [11, 12]. See Figure 1. Its structure is the same for each \(D\). For instance, the first dimension in every sequence divides the second but not the third. In this paper we will be concerned with subsequences of the form \(d_1, d_2, \ldots\) with the property \(d_{j+1} = d_j(d_j - 2)\) for all \(j\). It is easily seen that the elements of such subsequences correspond to the same value of \(D\). In fact, if \(N = d(d - 2)\) then

\[
(N + 1)(N - 3) = (d^2 - 2d + 1)(d^2 - 2d - 3) = (d - 1)^2(d + 1)(d - 3). \quad (7)
\]

The square-free part is \((d + 1)(d - 3)\). Since \(d\) divides \(N\) the ray class field with conductor \(d\) is a subfield of that with conductor \(N\). The replacement \(d \to d(d - 2)\) thus generates an infinite ‘tower’ (or ‘ladder’) of ray class fields over the same real quadratic field, each one contained in the next. Examples of towers of this form include

\[
\begin{align*}
7 & \to 35 & \to 1155 & \to \ldots & \text{corresponding to } & D &= 2 \quad (8) \\
5 & \to 15 & \to 195 & \to \ldots & \text{corresponding to } & D &= 3 \quad (9) \\
4 & \to 8 & \to 48 & \to \ldots & \text{corresponding to } & D &= 5 \quad . \quad (10)
\end{align*}
\]

As a glance at Figure 1 makes clear, there are other towers (such as \(4 \to 124 \to 15128 \to \ldots\)) not considered here.
Figure 1. Ray class field inclusions for $D = 5$ and $D = 3$. A field at the upper end of a line contains the field at the lower end. When $d$ is even the conductor equals $2d$, but this does not affect the links. The intricate structure of the partially ordered set does not come through because only the ten lowest dimensions are shown. In this paper we will be concerned with the vertical connections only.

When translated into Hilbert space, this means that the number field from which one constructs $d$-dimensional SICs embeds into that used to construct $d(d-2)$-dimensional SICs. We are then led to ask how this number theoretic embedding manifests itself in terms of the geometry of Hilbert space. This question was first addressed by Gary McConnell, who studied the scalar products among SIC vectors and found that some of the overlap phases in dimension $d(d-2)$ actually belong to the smaller field. The pattern is subtle and has many facets. Here we focus on one of them: in every known example, we find that some of the overlap phases in dimension $d(d-2)$ are squares of overlap phases from dimension $d$, or the negative thereof. The precise relationship is described in Observations 1 and 2 in Section 3. This facet has significant geometrical consequences which we explore in Sections 4–8.

This relationship between the phases leads to our definition of aligned SICs, and we conjecture that corresponding to every SIC in dimension $d$ there is an aligned SIC in dimension $d(d-2)$. We observe that lower dimensional equiangular tight frames (ETFs) can be found embedded in all our examples of aligned SICs, as described in Section 4.

We then specialize to the case of odd dimensions. We study the entanglement properties of an aligned SIC in $(d)$ dimension $d(d-2)$, and prove two theorems regarding the spectrum of their reduced density operators in Section 5. We show that starting with an aligned SIC in dimension $p(p+2)$, for $p$ an odd prime, we can obtain a full set of MUB in dimension $p$ via an affine map; this is shown in Theorem 3 in Section 6. We then show in Theorem 4 in Section 7 that an aligned SIC in odd dimension $d(d-2)$ necessarily contains two ETFs of the kind whose existence was observed in Section 4. Finally, we show in Theorem 5 in Section 8 that such a SIC necessarily has the $F_b$ symmetry whose existence was noted empirically by Scott and Grassl [3, 4].

Proving the even dimensional analogs of the results proven in Sections 5–8 involves some significant complications, arising because in even dimensions $d$ and $d-2$ are not relatively prime. This case will be discussed in a subsequent publication.

Our conclusions are given in Section 9, where we also comment on the very recent and important results of Grassl and Scott [17].
2. Preliminaries

A Weyl–Heisenberg SIC in dimension $d$ is defined by a fiducial vector $|\psi_{0,0}\rangle$, from which the remaining SIC vectors $|\psi_{i,j}\rangle$ are obtained by acting with the $d^2$ displacement operators $D_{i,j}$. The labels are pairs of non-negative integers $0 \leq i, j < d$. For convenience these operators are often indexed by a two-component ‘vector’ $p$, and the SIC vectors are then written as $|\psi_p\rangle = D_p|\psi_0\rangle$. We use both notations interchangeably, guided by convenience rather than principle. Readers unfamiliar with these matters are referred to Appendix B, and readers who need to be convinced of the preferred role of the Weyl–Heisenberg group are referred to the literature [18]. In dimension 8 there exists a sporadic SIC covariant under a related Heisenberg group. See ref. [19] for a recent discussion. It will be completely ignored here.

The SIC overlap phases in dimension $d$ are defined by

$$e^{i\theta_p} = \begin{cases} 1 & \text{if } p = 0 \\ \sqrt{d+1}\langle\psi_0|D_p|\psi_0\rangle & \text{if } p \neq 0 \end{cases}.$$  \hfill (11)

It turns out, in every case where an exact fiducial is known, that the overlap phases are algebraic integers, and in fact algebraic units, in the number fields they give rise to [11,12]. In this respect they are similar to the roots of unity, which are algebraic units in the cyclotomic fields.

The importance of the Weyl–Heisenberg group derives largely from the fact that it is a unitary operator basis [20], which means that every operator $A$ acting on $\mathbb{C}^d$ admits a unique expansion

$$A = \sum_p a_p D_{-p}, \quad a_p = \frac{1}{d} \text{Tr} D_p A.$$  \hfill (12)

In particular, for a one-dimensional projector this specializes to

$$|\psi\rangle\langle\psi| = \frac{1}{d} \sum_p D_{-p} |\psi|D_p|\psi\rangle.$$  \hfill (13)

This formula will enter most of our arguments. In particular it means that the vectors in a SIC can be reconstructed from their overlap phases.

A technicality needs to be mentioned here, because it plays a large role in the intermediate stages of our argument. The choice of the fiducial vector—among the vectors in a given SIC—seems at first sight to be arbitrary, so that we might just as well consider the overlap phases

$$\langle\psi_q|D_p|\psi_q\rangle = \langle\psi_0|D_{-q}D_pD_q|\psi_0\rangle = \omega^{\langle p,q \rangle} \langle\psi_0|D_p|\psi_0\rangle,$$  \hfill (14)

where $\omega$ is a $d$th root of unity, $\langle p, q \rangle$ is an integer modulo $d$, and we used properties of the displacement operators that are explained in Appendix B. But then the number theoretical properties of the overlap phases can get ‘polluted’ by roots of unity. A good choice of the fiducial vector can be made by observing that the Clifford group (the unitary automorphism group of the Weyl–Heisenberg group) contains the symplectic group as a factor group. A definite copy of this group is represented by unitary operators $U_F$, where $F$ is a symplectic two-by-two matrix, with entries that are integers modulo $d$ (or $2d$ if $d$ is even) [21]. It turns out, in every case where an exact or numerical fiducial is known, that there always exist special choices of $F$ and of the vectors such that $|\psi_0\rangle$ is an eigenvector of $U_F$. Such SIC vectors are called centred. The SIC vector $|\psi_q\rangle$ is left invariant by $D_q U_F D_{-q}$, and is said to be displaced. Centred SIC vectors are our preferred fiducial vectors, because the overlaps then...
lie in a smaller field, and the action of the Galois group simplifies. In dimensions divisible by 3 there is a further complication, because then there are displacement operators commuting with the relevant $U_F$. As a result, centred SIC vectors come in triplets. It turns out, in every case where an exact fiducial is known, that one of them is singled out by the number theoretical properties of its overlap phases, and is said to be strongly centred \[11,12\].

We will need to distinguish SIC overlap phases in dimensions $d$ from those in dimension $N = d(d-2)$. The latter are defined, using a strongly centred SIC fiducial $|\Psi_0\rangle$ in dimension $N$, by

$$e^{i\Theta_p} = \sqrt{N+1}\langle\Psi_0|D^{(N)}_p|\Psi_0\rangle = (d-1)\langle\Psi_0|D^{(N)}_p|\Psi_0\rangle.$$ (15)

Again we set $e^{i\Theta_{0,0}} = 1$ by convention. We label the operators with a superscript to signify the dimension, whenever this is demanded for clarity. The other convention established here is that capital letters $\Theta$ and $\Psi$ are associated to the larger dimension $N$, whereas lower case $\theta$ and $\psi$ refer to overlap phases and fiducials in the smaller dimension $d$.

Given that we know $e^{i\Theta_p}$ in dimension $d$, what can we say about $e^{i\Theta_p}$ in dimension $d(d-2)$? If there is a pattern, what are the geometrical consequences? We will present some theorems concerning the second question, but for a technical reason we will restrict ourselves to the case of odd dimensions $d$. The reason is that the integers $d$ and $d-2$ are relatively prime if the dimension $d$ is odd, and then the Weyl–Heisenberg group, and indeed the whole Clifford group, splits as a direct product. The Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^{d-2}$, with $d$ odd, is thus displayed as a tensor product in a preferred way. The (known) details revolve around the Chinese remainder theorem from elementary number theory. They are spelled out in Appendix D. The tensor product structure makes it much easier to describe the geometrical consequences that we have found. In particular we can then use the language of entanglement theory, and it is irresistible to make use of this when we can. We will prove that the entanglement properties of a SIC in $d(d-2)$ dimensions are very special if it is aligned to one in dimension $d$. Moreover, when $d-2$ is an odd prime number we can include mutually unbiased bases (MUB) in the picture, and we do so in Section 6.

3. Squared phases in dimensional towers

The observations that will lead to our definition of aligned SICs are summarized in Tables 1 and 2. Every SIC in the tables is aligned to the one immediately below it (if any), in a sense to be explained. Our calculations are numerical, and the precision limited. For $d \leq 15$ we used the numerical fiducials given by Scott and Grassl [3].

Before presenting the tables, we make an important clarifying remark. It must be understood that none of the phenomena we describe in this section has been proved to be a necessary consequence of the definition of a SIC. Each property of SICs that we discuss in this section as being universal (i.e. holding for all SICs, assuming further yet unknown ones exist) should be read with the caveat, ‘in every known case’. Still, the claims are based on a large number of examples. At the end of this section we will frame a definition motivated by some of them.

First we should explain the labeling system used for SICs [3]. SICs in a given dimension fall into orbits of the extended Clifford group (see Appendix A, which includes both unitary and anti-unitary transformations. The number of such orbits varies with the dimension, in ways that are not yet understood. Every SIC is labeled by the dimension and a letter labeling the extended Clifford orbit to which it belongs.

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1In five cases exact calculations have been made by Gary McConnell.
Table 1. SIC ladders with three known rungs. Exactly known SICs are in boldface, and they are underlined if they are ray class SICs. The pair 15ac are surrounded by brackets because they are constructed from the same field. The order of the symmetry group is given below the label, with an asterisk if anti-unitary symmetries are included, a subscript $a$ if the Zauner symmetry is of the unusual kind (see Eq. (69) for definitions), and a subscript $s$ if the fiducial sits in the smallest of the three Zauner subspaces, as explained further in the main text.

| $48g$ | $48f$ | 195d | 195b | 195a | 195c |
|-------|-------|------|------|------|------|
| 24$^*_a$ | 6 | 12 | 6 | 6 | 6 |
| 8b | 8a | 15d | 15b | (15a | 15c) |
| 12$^*_a$ | 3 | 6 | 3 | 3 | 3 |
| 4a | 5a | |
| 6$^*$ | 3 | |

Table 2. SIC ladders with only two known rungs, with the same conventions as in the previous table.

| 24c | 35j | 35i | 63b | 63c | 80i | 99b | 99c | 99d | 120c | 120b | 143a | 143b | 168a |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|------|------|------|------|------|
| 6 | 12$^*_a$ | 6 | 6 | 6 | 6 | 6 | 6 | 6 | 12a | 6 | 6 | 6 | 6 |
| 6a | 7b | 7a | (9a | 9b) | 10a | 11c | (11a | 11b) | 12b | 12a | (13a | 13b) | 14b |
| 3 | 6$^*$ | 3 | 3 | 3 | 3 | 3 | 3 | 6$^*_a$ | 3 | 3 | 3 | 3 |

Every SIC vector is left invariant by a subgroup of the extended Clifford group, that also transforms the SIC into itself. For centred fiducials this symmetry group is a subgroup of the extended symplectic group. As suggested by a conjecture of Zauner’s [1], and confirmed in all the examples, the symmetry group always contains a cyclic subgroup of order 3. It is generated by a unitary operator called the Zauner operator.

For $d \leq 50$ the order of the symmetry group may increase with the labeling letter’s position in the alphabet [3]. For higher dimensions no such system has been adopted. Then the lexicographical order reflects the order in which the various orbits were found [4]. Thus 4a is on a unique orbit in dimension 4, 48g has the highest symmetry of all SICs in dimension 48, and 63p is the last orbit that was discovered in dimension 63. If the labeling system reminds the reader of the labeling system used for spectral classes of stars (in logical order, OBABFGKM), then so be it.

A striking fact is that the order of the symmetry group doubles for each rung of the ladder in the tables. The tables contain some extra information that can be ignored for the time being: In dimensions $d = 3$ or 6 modulo 9 the symplectic group contains two different conjugacy classes of order 3 elements, represented by the matrices $F_z$ and $F_a$. See Eq. (69). SICs invariant under $U_{F_z}$ exist in all dimensions, but if $d = 3$ modulo 9 SICs invariant under $U_{F_a}$ exist too. Being of order 3, the Zauner operators split the Hilbert space into three Zauner subspaces. SIC vectors are always to be found in the largest of these, but in dimensions $d = 8$ modulo 9 the smallest subspace also contains SIC fiducials. There holds

$$d = 3 \text{ or } 8 \text{ mod } 9 \iff d(d - 2) = 3 \text{ mod } 9$$

(16)
Thus the first exceptional property is ‘inherited’ by the next rung, the second is not.

Each dimension contains a SIC known as a ray class SIC, constructed using a ray class field over the real quadratic field \( \mathbb{Q}(\sqrt{D}) \), where \( D \) is the square free part of the integer \( (d + 1)(d - 3) \). Other SICs in the same dimension are constructed from extensions of the ray class field. More precisely, there is a unique Galois multiplet (i.e. an orbit under the joint action of the Galois group and the extended Clifford group) of SICs belonging to the same ray class field; examples where the multiplet has more than one member include 9ab and 13ab [5]. Field inclusions give rise to a partial ordering among the fields, given in Figure 2 in the two cases where we have exact solutions available for more than one aligned SIC in the higher dimension. This pattern is not clear to us.

\[
d = 1 \text{ or } 4 \text{ or } 7 \mod 9 \quad \Leftrightarrow \quad d(d - 2) = 8 \mod 9.
\] (17)

Figure 2. Field inclusions in three of the towers. A field at an upper end of a line contains the field at the lower end. We walk up the ladders by stepping rightwards.

Our special concern in this paper is the phenomenology of squared SIC overlap phases. This can be summarized in two observations, relating some of the overlap phases in dimension \( N = d(d - 2) \) to those in dimension \( d \):

**First observation.** For SICs in dimension \( d \) there exists a SIC in dimension \( N = d(d - 2) \), and a choice of fiducials, such that for \( p = (d_i, d_j) \) we have

\[
e^{i\Theta_{d_i,d_j}} = \begin{cases} 
+1 & \text{if } d \text{ is odd} \\
-(1)^{(i+1)(j+1)} & \text{if } d \text{ is even.}
\end{cases}
\] (18)

**Second observation.** For SICs in odd dimensions \( d \) there exists a SIC in dimension \( N = d(d - 2), \) and a choice of fiducials, such that \( e^{i\Theta_{(d-2),(d-2)}} \) is the negative of a square of an overlap phase from dimension \( d \) if \( d \) is odd. The relation between the phases is given by

\[
e^{i\Theta_{(d-2),(d-2)}} = \begin{cases} 
-e^{2i\alpha + \beta \gamma + \delta} & \text{if } d \text{ is odd} \\
-(1)^{(i+1)(j+1)} e^{2i\gamma + \beta \gamma + \delta} & \text{if } d \text{ is even.}
\end{cases}
\] (19)

where \( \alpha, \beta, \gamma, \delta \) are integers modulo \( d \) such that \( \alpha \delta - \beta \gamma = \pm 1 \).

The fiducial 14a (which is in the same field as 14b [3]) does not appear in the tables because its higher dimensional cousin is not available at the moment.\(^2\) With this exception the observations have been made starting from every SIC in dimension \( 4 \leq d \leq 15 \).

\(^2\) Andrew Scott kindly produced the fiducials 120c and 195bcd when we asked for them.
The integers occurring in the second observation can be collected into a matrix $M$,

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad \det M = \pm 1 \mod d. \quad (20)$$

(The arithmetic is modulo $d$ also if $d$ is even.) In general this is an ESL matrix belonging to some coset of the symmetry group of the SIC. One can change the coset by choosing different SICs belonging to the same Clifford orbit.

The observations hold as stated only if the SIC fiducials are centred. If a displaced fiducial is used to calculate the overlaps then $(d - 2)$th roots of unity appear in $e^{i \Theta_{d_{i,d_j}}}$, and $d$th roots of unity in $e^{i \Theta_{(d-2)i, (d-2)j}}$. If the dimension $N$ is divisible by 3, as will always be the case from the third rung of the ladders and upwards, there are three SIC vectors in the same Zauner subspace. Unless one chooses the right one, roots of unity will again complicate the observations. It is natural to expect that the ‘right ones’ can be taken to be strongly centred, but in those cases where an exact solution is missing we are unable to check this. Instead we refer to ‘suitably chosen’ SIC vectors.

With this understanding the observations hold for every adjacent pair of SICs in the columns of Tables 1 and 2. They motivate a formal definition:

**Definition.** Pairs of SICs for which fiducial vectors can be chosen so that the two observations hold are aligned. The higher dimensional member of an aligned pair is called an aligned SIC.

There may well be logical dependencies among the two observations. Indeed, as we proceed we will find some evidence that this is so. Hence a more economical statement of the definition should be possible.

Based on the fact that the two observations hold in every case we have looked at, we make the following conjecture.

**Conjecture.** Every $d$-dimensional Weyl–Heisenberg SIC has a corresponding aligned SIC in dimension $d(d-2)$.

It is worth noting that this conjecture is both stronger and weaker than the simple conjecture that SICs exist in every dimension. It posits significantly more structure on the problem, and is in that sense stronger. But it allows for the possibility that some dimensions might not contain SICs, or be otherwise sporadic, while still positing the existence of infinite families. It also suggests a natural line of attack using inductive reasoning, though our own efforts in this direction have not yet been successful. But note also that the theorems in Sections 5–8 do not depend on the conjecture. They only depend on the (non-empty) definition.

### 4. Equiangular tight frames

The previous section clearly draws attention to two special subsets of vectors in an $N = d(d-2)$ dimensional SIC, namely

$$\{ |\Psi_{(d-2)i, (d-2)j} \rangle \}_{i,j=0}^{d-1} \quad \text{and} \quad \{ |\Psi_{d_{i,d_j}} \rangle \}_{i,j=0}^{d-3}. \quad (21)$$

The mutual overlaps within these subsets are very special numbers. What geometrical properties do these sets of vectors have?
A symmetric rank 1 POVM, also known as an equiangular tight frame (ETF), is a collection of $n$ unit vectors in $\mathbb{C}^m$ such that they resolve the identity,

$$\sum_{I=1}^{n} |\psi_I\rangle\langle\psi_I| = \frac{n}{m} \mathbb{1} ,$$

and such that the absolute values $|\langle\psi_I|\psi_J\rangle|$ are equal whenever $I \neq J$. (We denote the dimension by $m$ since we cannot use $d$, for a reason that will soon be evident.) It is easy to show that $n$ cannot be smaller than $m$, and it cannot be larger than $m^2$. A minimal ETF is an orthonormal basis and a maximal ETF is a SIC, but there are many interesting intermediate cases. Because the overlaps $\langle\psi_I|\psi_J\rangle$ have constant absolute values it is easy to show—by squaring and taking the trace—that we must have

$$|\text{overlap}|^2 = \frac{n-m}{m(n-1)} .$$

Now let us fix an arbitrary integer $d > 3$, and ask for solutions of the Diophantine equation

$$\frac{n-m}{m(n-1)} = \frac{1}{d(d-2)+1} = \frac{1}{(d-1)^2} .$$

There are typically many solutions. We are interested in four of them, namely

$$(m, n) = \begin{cases} (d(d-2), d^2(d-2)^2) & \text{SIC} \\ \left(\frac{d(d-1)}{2}, d^2\right) & \text{ETF}_1 \\ \left(\frac{(d-1)(d-2)}{2}, (d-2)^2\right) & \text{ETF}_2 \\ (d-1, d) & \text{ETF}_3 \end{cases} .$$

The first is that of a SIC in dimension $N = d(d-2)$. The fourth is a regular simplex in dimension $d$. The second and third solutions have just the right number of vectors to be identified with the equiangular subsets of the $N$-dimensional SIC that we identified above.

The point here is that we have checked numerically, with a precision of 120 digits, that in each of the 19 aligned SICs listed in Section 4 the $d^2$ vectors in the first subset identified in (21) are linearly dependent and belong to a subspace of dimension $d(d-1)/2$. Similarly, the $(d-2)^2$ vectors in the second subset of (21) are linearly dependent and belong to a subspace of dimension $(d-1)(d-2)/2$. Hence they form smaller equiangular tight frames embedded in the aligned SIC. In the sequel, we will prove that this must happen in all aligned SICs (although the case of even $d$ is postponed to a later publication). We will also identify special aligned SICs which contain embedded $(d-1)$-dimensional simplices.

5. Entanglement properties of SIC vectors

We now restrict the dimension of Hilbert space to be odd, for the pragmatic reason that then the Weyl–Heisenberg group defines a preferred tensor product decomposition $\mathbb{C}^{d(d-2)} = \mathbb{C}^d \otimes \mathbb{C}^{d-2}$. As a result every vector in $\mathbb{C}^{d(d-2)}$ can be described in the language of entanglement theory. In particular we will find the Schmidt decomposition very useful. Although this
Suppose that $\mathbb{C}^N = \mathbb{C}^{n_1} \otimes \mathbb{C}^{n_2}$ where $n_1 \geq n_2$. There will be local operators affecting only one of the factors of the Hilbert space. Given a pure state vector $|\Psi\rangle$ in the large Hilbert space, we define a reduced state $\rho_1$, which is a density matrix acting on $\mathbb{C}^{n_1}$, by the requirement that for all operators of the form $A_1 \otimes 1$ there holds

$$\text{Tr}|\Psi\rangle\langle\Psi|(A_1 \otimes 1) = \text{Tr}_1 \rho_1 A_1,$$

where $\text{Tr}_1$ denotes the trace over matrices acting on $\mathbb{C}^{n_1}$. This is enough to define $\rho_1$. One can explicitly write

$$\rho_1 = \text{Tr}_2|\Psi\rangle\langle\Psi|,$$

where $\text{Tr}_2$ denotes the partial trace over the second factor. The reduced state $\rho_2$ is defined similarly, using a partial trace over the first factor. Although the state we start out from is pure (defines a one-dimensional projector), the reduced state $\rho_1$ is typically a convex mixture of more than one pure state acting on $\mathbb{C}^{n_1}$. Generically it will have $n_2$ non-vanishing eigenvalues.

A comfortable theorem says that the spectra of $\rho_1$ and $\rho_2$ are identical, except for additional zero eigenvalues in the larger dimension. The eigenvalues $\lambda_k$ of the reduced density matrices are called Schmidt coefficients, and they completely determine the entanglement properties of a pure state $|\Psi\rangle$ in dimension $N = n_1 n_2$. Indeed, given any such pure state $|\Psi\rangle$ one can always adapt the orthonormal bases $\{|e_k\rangle\}_{k=0}^{n_1-1}$ and $\{|f_k\rangle\}_{k=0}^{n_2-1}$ in the factors, such that $|\Psi\rangle$ is given by the single sum

$$|\Psi\rangle = \sum_{k=0}^{n_2-1} \sqrt{\lambda_k} |e_k\rangle |f_k\rangle.$$

This is called the Schmidt decomposition of the state, and the coefficients in this expansion are the positive square roots of the Schmidt coefficients. Practical computation of the Schmidt decomposition follows by noting that the singular value decomposition of the $n_1 \times n_2$ matrix whose entries are the components of $|\Psi\rangle$ gives the same information.

We can now ask: what are the entanglement properties of a SIC vector in dimension $N = d(d-2)$? For generic pure states one expects $d-2$ different, and non-vanishing, Schmidt coefficients, but we will prove that the vectors in an aligned SIC are highly non-generic in this regard.

At the outset we consider dimension $N = n_1 n_2$, where $n_1$ and $n_2$ are relatively prime and odd. We use the fact that the Weyl–Heisenberg group is a unitary operator basis, and then the group isomorphism provided by the Chinese remainder theorem, to conclude for any vector $|\Psi\rangle \in \mathbb{C}^N$ that

$$|\Psi\rangle\langle\Psi| = \frac{1}{N} \sum_{i,j=0}^{N-1} D_{-i,-j}^{(N)} \langle \Psi | D_{i,j}^{(N)} |\Psi\rangle =$$

$$= \frac{1}{n_1 n_2} \sum_{i_1,j_1=0}^{n_1} \sum_{i_2,j_2=0}^{n_2} D_{-i_1,-n_2^{-1}j_1}^{(n_1)} \otimes D_{-i_2,-n_1^{-1}j_2}^{(n_2)} \langle \Psi | D_{i,j}^{(N)} |\Psi\rangle,$$

where applying the Chinese remainder theorem (see Appendix D) allows us to express

$$\langle \Psi | D_{i,j}^{(N)} |\Psi\rangle = \langle \Psi | D_{i_1 n_2 n_2^{-1}+i_2 n_1 n_1^{-1}, j_1 n_2 n_2^{-1}+j_2 n_1 n_1^{-1}}^{(N)} |\Psi\rangle.$$
If we now take the partial trace over, say, the first factor only the terms with \( i_1 = j_1 = 0 \) contribute. In this way we obtain the reduced density matrix

\[
\rho^{(n_2)} = \text{Tr}_{n_1} |\Psi\rangle \langle \Psi| = \frac{1}{n_2} \sum_{i_2,j_2=0}^{n_2-1} D^{(n_2)}_{-i_2,-j_2} (|\Psi D^{(N)}_{i_2n_1n_1^{-1}, j_2n_1}| \langle \Psi|).
\]

(31)

One summation index was shifted, which is allowed.

Now we specialize to the case of interest, namely

\[
\begin{align*}
n_1 &= d, \\
n_2 &= d - 2, \\
n_1^{-1} &= n_2^{-1} = \frac{d - 1}{2} \equiv \kappa,
\end{align*}
\]

(32)

and to the case that \( |\Psi\rangle \) is a vector in an aligned SIC. We drop the subscripts on the indices—which are no longer needed since they are summation indices only—and conclude from the above that

\[
\rho^{(d-2)} = \frac{1}{d-2} \sum_{i,j=0}^{d-3} D^{(d-2)}_{-i,-j} |\Psi D^{(N)}_{id\kappa,jd}| \langle \Psi|).
\]

(33)

We are now ready to prove our first theorem. The parity operator that occurs in its statement is defined in Appendix C.

**Theorem 1.** If \( d \) is odd and if \( |\Psi_0\rangle \) is a suitably chosen SIC vector in an aligned SIC in dimension \( d(d-2) \), the density matrix reduced to dimension \( d-2 \) is

\[
\rho_0^{(d-2)} = \text{Tr}_d |\Psi_0\rangle \langle \Psi_0| = \frac{1}{d-1} (\mathbb{1}_{d-2} + P^{(d-2)}) ,
\]

(34)

where \( P^{(d-2)} \) is the parity operator in dimension \( d-2 \). Hence \( \rho_0^{(d-2)} \) is proportional to a projector from \( \mathbb{C}^{d-2} \) onto a subspace of dimension \( (d-1)/2 \).

**Proof:** Recalling that we defined \( e^{i\Theta_{\delta,0,0}} = 1 \) we rewrite Eq. (33) as

\[
\text{Tr}_d |\Psi_0\rangle \langle \Psi_0| = \frac{1}{d - 2} \left( 1 - \frac{1}{d - 1} \right) \mathbb{1} + \frac{1}{d - 1} \sum_{i,j=0}^{d-3} D^{(d-2)}_{-i,-j} e^{i\Theta_{d\kappa,i,j}} \right) .
\]

(35)

The definition of an aligned SIC implies that we can choose the fiducial so that

\[
e^{i\Theta_{d\kappa,i,j}} = 1.
\]

(36)

Equation (33) then becomes

\[
\rho_0^{(d-2)} = \frac{1}{d - 1} \left( \mathbb{1} + \frac{1}{d - 2} \sum_{i,j=0}^{d-3} D^{(d-2)}_{-i,-j} \right) = \frac{1}{d - 1} (\mathbb{1}_{d-2} + P^{(d-2)}) ,
\]

(37)

where Eq. (72) for the parity operator was used in the last step. In dimension \( d-2 \) the operator \((\mathbb{1} + P)/2\) is a projection operator of rank \((d-1)/2\), which gives the final part of the statement. □

Thus we find only \((d-1)/2\) non-vanishing Schmidt coefficients, and they are all equal. Indeed the entanglement properties of a vector belonging to an aligned SIC are very special.
The theorem applies only to aligned SICs, such as 15d and 195abcd. A calculation shows that the non-aligned fiducials 15abc have non-degenerate Schmidt coefficients, as expected for generic vectors. (Compare Table 1.) On the other hand the restriction to special choices of SIC vectors can be removed, except that one then encounters displaced parity operators on the right hand side. The proof simplifies considerably if we choose the fiducials suitably.

The next task is to find the state reduced to dimension \( d \). From entanglement theory we know that the spectra of \( \text{Tr}_{d-2}|\Psi_0\rangle \langle \Psi_0| \) and \( \text{Tr}_{d-2}|\Psi_0\rangle \langle \Psi_0| \) coincide. However, the precise mechanism that allows this to happen is worth studying because it depends on the details of our definition of aligned SICs. This will show that the two observations we made are in fact related.

The preliminary steps are the same as before. In Eq. \[ \text{(31)} \], set \((n_1,n_2) = (d - 2, d)\), and rewrite

\[
\text{Tr}_{d-2}|\Psi_0\rangle \langle \Psi_0| = \frac{1}{d} \left( 1 + \frac{1}{d-1} \sum_{i,j \neq (0,0)}^{d-1} D_{-i,-j}^{(d)} e^{i\Theta_{(d-2)i,(d-2)j}} \right)
\]

(38)

We are now ready to bring in the squared overlap phases in dimension \( d \) by applying the full definition of an aligned SIC.

**Theorem 2.** If \( d \) is odd and if \(|\Psi_0\rangle\) is a suitable SIC vector in an aligned SIC in dimension \( d(d-2)/2 \), the density matrix reduced to dimension \( d \) is

\[
\rho_0^{(d)} \equiv \text{Tr}_{d-2}|\Psi_0\rangle \langle \Psi_0| = \frac{1}{d-1} \left( \mathbb{1}_d - P_{\theta}^{(d)} \right),
\]

(39)

where \( P_{\theta}^{(d)} \) is a generalized parity operator in dimension \( d \). Hence \( \rho_0^{(d)} \) is proportional to a projector from \( \mathbb{C}^d \) onto a subspace of dimension \( (d - 1)/2 \).

**Proof:** Applying the definition of an aligned SIC to Eq. \[ \text{(38)} \] we obtain

\[
\rho_0^{(d)} = \frac{1}{d} \left( 1 + \frac{1}{d-1} \sum_{i,j = 0}^{d-1} D_{2i,-j}^{(d)} e^{2i\Theta_{2i,-j}} \right)
\]

(40)

We relabeled the summation index and introduced the multiplicative inverse of 2 modulo \( d \). Making use of Eq. \[ \text{(20)} \]

\[
\rho_0^{(d)} = \frac{1}{d-1} \left( \mathbb{1}_d - \frac{1}{d} \sum_{p} D_{-p}^{(d)} e^{2i\Theta_{-p}} \right),
\]

(41)
where the $GL(2,\mathbb{Z}_d)$ matrix $M'$ obeys $\det M'^{-1} = \pm 2$. We now appeal to a result from ref. [25], which says that, under the conditions stated, the generalized parity operator

$$P_\theta = \frac{1}{d} \sum_{p} D_p e^{2i \theta M'_p}$$

obeys $P_\theta^2 = 1$ and has $(d + 1)/2$ eigenvalues equal to $+1$ and $(d - 1)/2$ eigenvalues equal to $-1$. □

Concerning the result from ref. [25] we observe that it is a consequence of a key property of SICs, that they form projective 2-designs. This goes some way towards explaining why squared overlap phases play a role. See ref. [26] for a review of projective $t$-designs.

Again the restriction to special choices of fiducials can be dropped at the expense of complicating the statement of the theorem a little, and significantly complicating the direct proof. In Section 7 we will formulate a geometrical theorem where this restriction is dropped.

### 6. Mutually unbiased bases

The appearance of the parity operator $P$ in the preceding section allows us to give a resolution of the long-standing question of how to relate SICs to mutually unbiased bases (MUB) in prime dimensions. By definition a complete set of MUB in dimension $p$ is a collection of $p + 1$ orthonormal bases such that every overlap between vectors in different bases has absolute value squared equal to $1/p$ [15]. This definition, like the definition of a SIC, has its origin in quantum state tomography, and MUB have found a number of interesting applications over the years. Complete sets of MUB do exist in all dimensions equal to a power of a prime number [27], and if the dimension $p$ is a prime number they arise as eigenbases of the $p + 1$ cyclic subgroups of the Weyl–Heisenberg group. (If the dimension is equal to a higher power of a prime number a multipartite Heisenberg group appears. In non-prime power dimensions complete sets of MUB may well not exist, and if they do they are unrelated to the Heisenberg groups [11,28].) Given this group theoretical connection one expects to find a tight geometrical connection between MUB and SICs in prime dimensional Hilbert spaces. This is indeed so in the very special case of $d = 3$, which was cleared up in 1844 [29]. When $d > 3$ it has to be kept in mind that MUB are based on cyclotomic fields, while SICs are two steps beyond that since ray class fields over real quadratic fields come in. Although a loose connection between SICs and MUB in prime dimensions exists [30], the details have remained elusive.

We can now offer an answer to this question, because our Theorem 1 provides us with the means to construct a complete set of MUB in dimension $p = d - 2$ (assumed to be a prime number) from an aligned SIC in dimension $N = d(d - 2)$ [15]. In fact, given Wootters’ elegant construction of complete sets of MUB in prime dimensions [31], this result follows trivially from the above, but the details are worth spelling out. The starting point is the observation that in prime dimension the vectors labeling the displacement operators form a true vector space. This is so because the set of integers modulo a prime number form a finite field. This vector space can be regarded as a finite affine plane consisting of $p^2$ points and $p(p + 1)$ lines containing $p$ points each. The lines are given by the equation

$$j = zi + a,$$

where $i, j, a$ are integers modulo $p$ while $z$ can also take the formal value $\infty$, corresponding to a set of ‘vertical’ lines [30]. Thus a line is given by fixing the pair $(z, a)$. Next, consider the
displaced parity operators

\[ P_{i,j} = D_{i,j} P D_{-i,-j}. \]  \hspace{1cm} (44)

They are renamed as phase point operators, and associated with the \( p^2 \) points of the affine plane. We also need operators associated with the \( p(p+1) \) lines of the affine plane. A key fact proved by Wootters is that the operators

\[ W^{(z,a)} = \frac{1}{p} \sum_{\text{line}} P_{i,j} \]  \hspace{1cm} (45)

are one-dimensional projectors projecting to the vectors in a complete set of MUB. The sum goes over all \( i,j \) consistent with Eq. (43) for some given \( z,a \). The construction needs the combinatorics of the affine plane to work, which is certainly available when \( p \) is prime.

We now have:

**Theorem 3.** If \( p = d - 2 \) is an odd prime then a complete set of MUB in dimension \( p \) can be obtained by taking affine combinations of projectors to the vectors in an aligned SIC in dimension \( d(d-2) \), and then performing a partial trace.

**Proof:** By Theorem 1 and the properties of the partial trace

\[ \text{Tr}_d \left( \mathbb{1}_d \otimes D_{i,j}^{(d-2)} \right) \langle \Psi_0 | \langle \Psi_0 | \left( \mathbb{1}_d \otimes D_{-i,-j}^{(d-2)} \right) = \frac{1}{d-1} (\mathbb{1}_{d-2} + P_{i,j}) \right), \]  \hspace{1cm} (46)

where we used definition (44) for the displaced parity operators in dimension \( d - 2 \). The construction uses the \( p^2 = (d-2)^2 \) SIC vectors

\[ |\Psi_{di,dj}\rangle = \mathbb{1}_d \otimes D_{di,j}^{(d-2)} |\Psi_0\rangle. \]  \hspace{1cm} (47)

Using Wootters’ formula (45), and the linearity of the trace, we immediately obtain

\[ W^{(z,a)} = \text{Tr}_d \left[ \frac{d-1}{d-2} \sum_{\text{line}} |\Psi_{di,dj}\rangle \langle \Psi_{di,dj}| - \frac{1}{d} \mathbb{1}_N \right]. \]  \hspace{1cm} (48)

By construction the \( (p+1)p \) operators \( W^{(z,a)} \) project to the vectors in a complete set of MUB.

\( \Box \)

Hence we have a firm relation between MUB in dimension \( p \) and SICs in dimension \( (p+2)p \). Unfortunately we do not have a way to go from SICs in dimension \( d \) to SICs in dimension \( d(d-2) \), nor are we close to having this, but if we had we would have a firm relation between MUB in dimension \( p \) and SICs in dimension \( p+2 \).

7. The embedding of the equiangular tight frames

We are now ready to prove (for odd \( d \)) that the equiangular tight frames observed in Section 4 have to appear in every aligned SIC. Because the Weyl-Heisenberg group is an operator basis
Schur’s lemma implies, for any operator $A$, that

$$\frac{1}{N} \sum_p D_p A D_p^\dagger = 1_N \text{Tr} A \ . \quad (49)$$

Now suppose the dimension is composite, $N = n_1 n_2$, and assume that the factors are relatively prime and odd. Then Chinese remaindering can be applied, and one can show that

$$\frac{1}{n_1} \sum_{p_1} (D_{p_1}^{(n_1)} \otimes 1_{n_2}) A (D_{-p_1}^{(n_1)} \otimes 1_{n_2}) = 1_{n_1} \otimes \text{Tr}_{n_1} A \ . \quad (50)$$

We have ‘isotropized’ one factor of the tensor product, and a partial trace appears on the other. A similar equation, with the role of the factors interchanged, will also be used below.

We now specialize to the case $n_1 = d$, $n_2 = d - 2$, and assume $A = |\Psi_0\rangle\langle\Psi_0|$, where $|\Psi_0\rangle$ is a suitably chosen SIC vector aligned with a SIC vector in dimension $d$. Then Theorems 1 and 2 give us information about the partial trace that appears on the right hand side. On the other hand, the left hand side has an interesting interpretation. Indeed, we can consider the two operators

$$\Pi_1 \equiv \frac{d-1}{2d} \sum_{i,j=0}^{d-1} |\Psi_{(d-2)i,(d-2)j}\rangle \langle \Psi_{(d-2)i,(d-2)j}| =$$

$$= \frac{d-1}{2d} \sum_{p_1} (D_{p_1}^{(d)} \otimes 1_{d-2}) |\Psi_0\rangle \langle \Psi_0| (D_{-p_1}^{(d)} \otimes 1_{d-2}) \quad (51)$$

$$\Pi_2 \equiv \frac{d-1}{2(d-2)} \sum_{i,j=0}^{d-3} |\Psi_{d_1,dj}\rangle \langle \Psi_{d_1,dj}| =$$

$$= \frac{d-1}{2(d-2)} \sum_{p_2} (1_d \otimes D_{p_2}^{(d-2)}) |\Psi_0\rangle \langle \Psi_0| (1_d \otimes D_{-p_2}^{(d-2)}) \ . \quad (52)$$

The idea behind the next theorem is that these operators are projectors, and can be substituted for the unit operator in the POVM condition (22) provided we restrict ourselves to the subspaces of $\mathbb{C}^N$ to which these operators project.

**Theorem 4.** If $d$ is odd, then every aligned SIC in dimension $d(d-2)$ contains two multiplets of smaller equiangular tight frames embedded in it. Each individual SIC vector in an aligned SIC belongs to an equiangular tight frame of $d^2$ vectors spanning a subspace of dimension $d(d-1)/2$, and another consisting of $(d-2)^2$ vectors spanning a subspace of dimension $(d-1)(d-2)/2$.

**Proof:** Combining the definitions (51) and (52), Eq. (50), and Theorems 1 and 2 gives immediately that

$$\Pi_1 = 1_d \otimes \frac{1}{2} (1_{d-2} + P^{(d-2)}) \quad (53)$$

$$\Pi_2 = \frac{1}{2} (1_d - P_0^{(d)}) \otimes 1_{d-2} \ . \quad (54)$$
It follows that $\Pi_1$ and $\Pi_2$ are projectors, to subspaces of dimension $d(d-1)/2$ and $(d-1)(d-2)/2$, respectively. To see that the support of $\Pi_1$ contains $d^2$ equiangular SIC vectors one performs the calculation

$$\langle \Psi_{(d-2)i,(d-2)j} | \Pi_1 | \Psi_{(d-2)i,(d-2)j} \rangle = \text{Tr} \Pi_1 | \Psi_{(d-2)i,(d-2)j} \rangle \langle \Psi_{(d-2)i,(d-2)j} | = 1 ,$$

and similarly for $\Pi_2$. The fiducial $|\Psi_0\rangle$ belongs to both subspaces. Conjugating with the Weyl–Heisenberg group one finds that the subspace defined by the projector $\Pi_1$ belongs to an orbit of $(d-2)^2$ subspaces each containing an ETF of type $(d(d-1)/2, d^2)$, and similarly for $\Pi_2$. □

The projectors $\Pi_1$ and $\Pi_2$, and the Gram matrices of the resulting ETFs, are constructed entirely out of numbers present in the $d$-dimensional SIC and of suitable roots of unity. Waldron [32] and Goyeneche have already noted that given a SIC in dimension $d$ one can always construct the Gram matrices corresponding to equiangular tight frames of the types we have here found to be embedded in the aligned $d(d-2)$-dimensional SICs. This result is valid regardless of whether $d$ is odd and even. A version of Theorem 4 that holds for arbitrary $d$ is in fact known, but we postpone its presentation to a companion paper.

In Eq. (25) we also raised the possibility that a regular $(d-1)$-dimensional simplex can be embedded in a $d(d-2)$-dimensional SIC. This happens in three of our examples, namely 8b, 35j, and 120c, and is connected (via our definition of aligned SICs) to the fact that $d-1$ real overlap phases $e^{i\theta_{i,j}}$ occur in the relevant $d$-dimensional SICs 4a, 7b, and 12b, all of which have an extra anti-unitary symmetry beyond the Zauner symmetry. This is not a property that is inherited on higher rungs of the ladder though; 8b has only 3 real phases, and 35j only 30 real phases.

The embedding of lower dimensional ETFs in the SIC means that non-trivial linear dependencies are present among the vectors of the latter. The general question under what conditions sets of vectors in Weyl–Heisenberg orbits can be linearly dependent has been studied [33, 34], and it is known that linear dependencies do occur, in such orbits, whenever the order of their symmetry group fails to be coprime with the dimension. Some of the linear dependencies that we report here are not covered by these results.

8. Symmetries

A striking feature of Tables 1 and 2 is that the order of the intrinsic symmetry group of the SICs increases with a factor of two for each rung of the ladder. In fact several of the numerical fiducials in these high dimensions were found because Scott and Grassl [3, 4] conjectured the presence of an extra symmetry of order 2 (beyond the order 3 Zauner symmetry), given by the symplectic matrix

$$F_b = \begin{pmatrix} 1-d & 0 \\ 0 & 1-d \end{pmatrix} \in SL(2,\mathbb{Z}_N) .$$

In the standard representation that we use [21] an easy calculation gives, after Chinese remaindering according to Eq. (30), that the corresponding unitary operator is

$$U_b = 1 \otimes P ,$$
where $P$ is the parity operator in dimension $d - 2$. It is easy to prove that this symmetry has to be there.

**Theorem 5.** An aligned SIC in an odd dimension is invariant under $U_b$.

**Proof:** Let $|\Psi_0\rangle$ be a strongly centred SIC fiducial. Then Theorem 1 states that the reduced density matrix is

$$\rho_2 = \text{Tr}_1 |\Psi_0\rangle\langle\Psi_0| = \frac{1}{d - 1}(1 + P). \quad (58)$$

The Schmidt decomposition [24] of such a state is

$$|\Psi_0\rangle = \sqrt{\frac{2}{d - 1}} \sum_{k=1}^{d-1} |e_k\rangle|f_k\rangle. \quad (59)$$

Moreover $\rho_2$ and $U_b = 1 \otimes P$ are diagonal in the Schmidt basis, and $U_b$ manifestly leaves $|\Psi_0\rangle$ invariant. Being a member of the Clifford group it will permute the remaining SIC vectors among themselves. □

A similar argument fails on the left hand factor. The generalized parity operator can be used to construct an operator that leaves $|\Psi_0\rangle$ invariant, but since it is not a member of the Clifford group the last line in the proof fails. This is also the reason why, in Section 6, we were able to connect aligned SICs to mutually unbiased bases in dimension $d - 2$, but not to MUB in dimension $d$.

There is more to say about symmetries and dimension towers, and we hope to come back to these issues in a later publication.

### 9. Conclusions

The number theoretical connections between SICs in dimension $d$ and dimension $d(d - 2)$ manifest themselves very explicitly in the case of aligned SICs. The number field needed to construct the former is a subfield of that needed to construct the latter [11, 12]. Gary McConnell has noted that it can happen that some of the overlap phases in dimension $d(d - 2)$ actually belong to the subfield. We have explored a part of this pattern, and it enables us to make significant statements about the Hilbert space geometry of the relevant $d(d - 2)$ dimensional SICs. Moreover we have collected evidence, in the form of 19 mostly numerical examples, suggesting that every SIC in dimension $d$ gives rise to a SIC in dimension $d(d - 2)$ where this pattern occurs. The higher dimensional member of such a pair is said to be an aligned SIC, and we offered a precise definition of aligned SICs.

In this paper we concentrated on the case of odd dimensions, in which case there is a canonical tensor product structure. Then the alignment manifests itself as very special entanglement properties (Theorems 1 and 2). If $d - 2 = p$ is an odd prime number a complete set of mutually unbiased bases in dimension $p$ can be derived from a higher dimensional SIC (Theorem 3). We also proved that there are non-trivial equiangular tight frames embedded in the $d(d - 2)$ dimensional aligned SICs (Theorem 4). This property generalizes to even dimensions, as we will prove in a companion paper. Finally we proved that a conjectured extra symmetry is indeed always present in the aligned SICs (Theorem 5).

We stress that we have only scratched the surface of an intricate pattern. There is more to the story than just squared phases. Then, as we discussed in the introduction, there are
other dimension towers to consider. The field inclusions organize the dimension towers into partially ordered sets with a very intricate structure. Moreover, very recently Grassl and Scott [17] published the results of an investigation of the full sequence $\mathcal{D}$, corresponding to $D = 5$. They conjecture that the ray class SICs in these dimensions have a special symmetry that grows with $d$, and verify this conjecture by calculating an exact solution for $d = 124 (!)$ as well as numerical solutions in dimensions 323 and 844 (!). Their approach is in a way complementary to ours, since we have not focussed on the ray class SICs exclusively. In fact, as our Figure 2 may make clear, the full picture is likely to be even richer than what Figure 1 begins to suggest.

There is a hope that one can find a way to construct higher dimensional SICs starting from lower dimensional ones, and this hope has served as one of our motivations. There is also an over-riding question: What is the ‘mechanism’ forcing certain algebraic number fields of great independent interest to manifest themselves in Hilbert space in the precise way they do? We are far from an answer, but we hope our results represent a small step forwards.

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Appendix A. Roots of unity

When it was first calculated the SIC in dimension 6 seemed to cement the idea that SICs are significantly more complex than mutually unbiased bases [35]. However, on further reflection it will be seen that we were not really comparing apples to apples. The exact solutions for the known SICs are written in radicals. If the number $e^{\frac{2\pi i}{d}}$ is written out in radicals the expression which results is also very complicated (except in special cases). Thus, using the techniques developed by Lagrange, Vandermonde, and Gauss [13], one finds that the primitive eleventh root of unity is

$$\omega_{11} = \frac{1}{10} + \frac{1}{10}(-1 + b_1) + \frac{1}{20}(1 + b_1)b_2 + \frac{1}{30}(-1 + 5b_1) + \frac{1}{20}(-5 - b_1)b_2 + \frac{1}{120}\left(\frac{1}{4}(-2 - 2b_1)b_2\right)b_3^3 + \frac{1}{605}\left(\frac{1}{4}(-2 - 2b_1)b_2\right)b_3^3 + \frac{1}{605}\left(\frac{1}{6}(-2 - 2b_1)b_2\right)b_3^3 + \frac{1}{605}\left(\frac{1}{8}(-2 - 2b_1)b_2\right)b_3^3 + \frac{1}{605}\left(\frac{1}{10}(-2 - 2b_1)b_2\right)b_3^3$$

where

$$b_1 = \sqrt{5}, \quad b_2 = \frac{1}{4}\sqrt{10 - 2b_1}, \quad b_3 = \left(\frac{1}{4}(561671 + 29975b_1) + (-24365 + 37620b_1)b_2\right)^{1/10}.$$

If this formula was used to calculate MUB in dimension 11 the complexity of the resulting expressions would be similar to the complexity of the expressions for the $d = 11$ SICs given by Scott and Grassl [3]. On the other hand, using the transcendental function $f(z) = e^{2\pi iz}$, we find

$$\omega_{11} = f\left(\frac{1}{11}\right).$$

Hilbert’s 12th problem asks for a representation of the numbers needed to construct SICs analogous to the second description of the 11th root of unity. The suggestion is that SICs, if they could be seen through the right number theoretical glasses, are as simple as MUB in prime dimensions are.

Appendix B. The Weyl–Heisenberg and Clifford groups

We define the Weyl–Heisenberg group $H(d)$ in dimension $d$ to contain central elements represented by the phase factors [21]

$$\tau = -e^{\frac{2\pi i}{d}}, \quad \omega = \tau^2 = e^{\frac{4\pi i}{d}}.$$
Multiplication with the unit matrix is left understood whenever this cannot cause confusion.

If the dimension $d$ is odd, as we assume, then $(d + 1)/2$ is an integer and there holds

$$\omega^{\frac{d+1}{2}} = \left( e^{\pi i} \right)^{\frac{d+1}{2}} = \tau . \quad (64)$$

Both $\tau$ and $\omega$ are $d$th roots of unity in this case. If $d$ is even some complications arise, and we postpone this case to a separate paper. Here we only wish to note the fact, evident from the introduction, that odd and even $d$ show some differences also at the level of algebraic number theory.

The remaining group elements are given by $d^2$ displacement operators which we write interchangeably as $D_{i,j}$ and $D_p$, with the understanding that $p$ is a two-component ‘vector’ with components $i,j$ that are integers modulo $d$. The displacement operators obey $D_p D_q = \omega^{\langle p,q \rangle} D_q D_p$ and

$$D_p D_q = \tau^{\langle p,q \rangle} D_{p+q} = \omega^{\langle p,q \rangle} D_q D_p , \quad (65)$$

where the exponent is given in terms of the components of the ‘vectors’,

$$p = \begin{pmatrix} i \\ j \end{pmatrix}, \quad q = \begin{pmatrix} k \\ l \end{pmatrix} \Rightarrow \langle p,q \rangle = kj - li . \quad (66)$$

Thus $\langle , \rangle$ is a symplectic form. An explicit matrix representation is

$$(D_{i,j})_{r,s} = \tau^{ij+2js} \delta_{r,s+i} . \quad (67)$$

This representation is essentially unique, once $D_{0,j}$ is chosen to be diagonal.

Frequently we will have displacement operators for dimensions $d$ and $d(d-2)$ occurring in the same formula. When necessary to avoid confusion operators are supplied with superscripts denoting the dimension in which they act, eg. $D_p^{(d)}$, $D_p^{(d-2)}$, $D_p^{(N)}$. In this appendix no superscripts are necessary because the dimension is always an arbitrary integer $d$. Occasionally we use subscripts for the same purpose, thus $\omega_d$ is the $d$th root of unity whenever this is not obvious.

If $F$ is a $GL(2, \mathbb{Z}_d)$ matrix, that is to say a $2 \times 2$ matrix with entries that are integers modulo $d$, then we find when we calculate in modulo $d$ arithmetic that

$$\langle Fp, Fq \rangle = \langle p, q \rangle \det F . \quad (68)$$

The condition $\det F = 1$ defines the symplectic subgroup $SL(2, \mathbb{Z}_d)$. This group is part of the unitary automorphism group of the Weyl-Heisenberg group, also known as the Clifford group. Every matrix $F \in SL(2, \mathbb{Z}_d)$ is represented by a unitary matrix $U_F$. By definition a Zauner operator is associated to a matrix of order three and trace equal to $-1$. The matrices $F_z$ and $F_a$, corresponding respectively to the ‘universal’ Zauner operator and to the ‘unusual’ Zauner operator in dimensions of the form $d = 9k + 3$, are

$$F_z = \begin{pmatrix} 0 & d-1 \\ 1 & -1 \end{pmatrix} , \quad F_a = \begin{pmatrix} 1 & 3 \\ 3k & d-2 \end{pmatrix} . \quad (69)$$

See refs. [3,5] for more. Matrices with $\det F = -1$ are represented as anti-unitary operators, and as such belong to the extended Clifford group [21].
Appendix C. Parity operators

The symplectic group contains a special involution of order 2, whose unitary representative is known as the parity operator,

\[
F = \begin{pmatrix}
-1 & 0 \\
0 & -1
\end{pmatrix}
\Rightarrow
U_F \equiv P.
\tag{70}
\]

If \(d\) is odd this is a unitary Hermitian operator with spectrum \(\left((d+1)/2, (d-1)/2\right)\). When \(d\) is odd the integer 2 has a multiplicative inverse \(2^{-1}\) in arithmetic modulo \(d\), and we can calculate that

\[
\text{Tr} D_p P = \text{Tr} D_{-1} P = \text{Tr} P = 1.
\tag{71}
\]

Hence the parity operator can be expanded as

\[
P = \frac{1}{d} \sum_{p} D_{-p}.
\tag{72}
\]

Conjugating with the Weyl-Heisenberg group we obtain \(d^2\) parity operators belonging to the Clifford group. They are the displaced parity operators used in Section 6, and were called phase point operators by Wootters [31].

It is a property of SIC overlap phases that the generalized parity operator \(P_{\theta}\) occurring in Eq. (42) is isospectral with the parity operator \(P\) [25], but \(P_{\theta}\) does not belong to the Clifford group.

Appendix D. The Chinese remainder theorem

We are interested in dimensions of the form \(N = d(d-2)\). When \(N\) is odd \(d\) and \(d-2\) are relatively prime integers. A theorem from elementary number theory then comes into play: the Chinese remainder theorem states that if \(n_1\) and \(n_2\) are relatively prime then any integer \(r\) modulo \(N = n_1 n_2\) can be uniquely expressed in terms of a pair of integers \(r_i = r \mod n_i\) as

\[
r = r_1 n_2 n_1^{-1} + r_2 n_1 n_2^{-1}.
\tag{73}
\]

Throughout, \(n_2^{-1} (n_1^{-1})\) denotes the inverse of the integer \(n_2\) \((n_1)\) in arithmetic modulo \(n_1\) \((n_2)\). The formula expresses a ring isomorphism between \(\mathbb{Z}_N\), the ring of integers modulo \(N\), and the ring \(\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}\). This was appreciated in ancient China because it allows arithmetic modulo a large integer \(N\) to be carried out modulo the smaller integers \(n_1\) and \(n_2\), and the end result reconverted to an integer modulo \(N\). The application to Weyl-Heisenberg groups as an approach to the SIC problem was pioneered by David Gross [36].

The Chinese remainder theorem can be used to express the isomorphism between the corresponding cyclic groups, and also the isomorphism \(H(N) = H(n_1) \times H(n_2)\). We use \(\omega = e^{2\pi i}/N\) to represent \(H(N)\). There holds

\[
\omega = \omega_{n_1} n_2^{-1} \omega_{n_2} n_1^{-1}.
\tag{74}
\]

Namely

\[
e^{2\pi i}/N = e^{2\pi i}/n_2 n_1^{-1} = e^{2\pi i}/n_2 n_1^{-1} (n_2 n_1^{-1} + n_2 n_1^{-1}) = e^{2\pi i}/n_1 e^{2\pi i}/n_2 n_1^{-1}.
\tag{75}
\]
Given that $\omega_1$ is a primitive root of unity, so is $\omega_1^{n_2^{-1}}$, so it would be possible to use this to represent $H(n_1)$. However, we choose not to. We then find that

$$D_{i,j} = D^{(n_1)}_{i_1,n_2^{-1}j_1} \otimes D^{(n_2)}_{i_2,n_1^{-1}j_2},$$

(76)

where the matrix representation is, say,

$$D^{(n_1)}_{i_1,n_2^{-1}j_1} = \omega_1^{(2n_2^{-1})i_1j_1+n_2^{-1}j_1s_1} \delta_{r_1,s_1+i_1},$$

(77)

The subscripts on the indices are superfluous, since the arithmetic used for the indices is automatically modulo $n_1$. Using vector notation we write

$$D_P = D^{(n_1)}_{H_1P} \otimes D^{(n_2)}_{H_2P},$$

(78)

where

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ n_1 & 0 & n_2^{-1} \\ \end{pmatrix}, \quad H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ n_2^{-1} & 0 & 1 \\ \end{pmatrix}.$$  

(79)

The Clifford group also splits into a direct product. One finds

$$U_F = U^{(n_1)}_{F_1} \otimes U^{(n_2)}_{F_2} = U^{(n_1)}_{H_1F} \otimes U^{(n_2)}_{H_2F}.$$  

(80)

Now we specialize to $n_1 = d$, $n_2 = d - 2$. Then

$$n_2^{-1} \mod n_1 = n_1^{-1} \mod n_2 = \frac{d - 1}{2} \equiv \kappa,$$

(81)

where the integer $\kappa$ was defined in the last step. (Proof: Calculating modulo $d - 2$ we find $d(d - 1)/2 = 2(d - 1)/2 = d - 1 = 1$. The point is that $(d - 1)/2$ is an ordinary integer. Mutatis mutandis when calculating modulo $d$.) Thus

$$H \equiv H_1 = H_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \kappa \\ \end{pmatrix}.$$  

(82)

For the symplectic matrices one finds

$$F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \\ \end{pmatrix} \quad \Rightarrow \quad HFH^{-1} = \begin{pmatrix} \alpha & \kappa \beta \\ \kappa \gamma & \delta \\ \end{pmatrix},$$

(83)

where we decide on the modulus in the last step.

In conclusion, in dimensions $N = d(d - 2)$ with $d$ odd the Weyl–Heisenberg group allows us to express the Hilbert space as $\mathbb{C}^N = \mathbb{C}^d \otimes \mathbb{C}^{d-2}$ in a preferred way.

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