Why Maximum Entropy? A Non-axiomatic Approach

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Abstract. Ill-posed inverse problems of the form \( y = Xp \) where \( y \) is \( J \)-dimensional vector of a data, \( p \) is \( m \)-dimensional probability vector which can not be measured directly and matrix \( X \) of observable variables is a known \( J \times m \) matrix, \( J < m \), are frequently solved by Shannon’s entropy maximization (MaxEnt, ME). Several axiomatizations were proposed (see for instance [1], [2], [3], [4], [5], [6], [7], [8], as well as [9] for a critique of some of them) to justify the MaxEnt method (also) in this context. The main aim of the presented work is two-fold: 1) to view the concept of complementarity of MaxEnt and Maximum Likelihood (ML) tasks introduced at [10] from a geometric perspective, and consequently 2) to provide an intuitive and non-axiomatic answer to the ‘Why MaxEnt?’ question.

INTRODUCTION

The concept of complementarity of maximum entropy and maximum likelihood tasks, proposed at [10], is in this vignette interpreted from a geometrical point of view of vectors. Two key notions are introduced: collinearity and coherence. In addition to shaping the complementarity into an elegant form the collinearity/coherence concepts offer an elementary answer to the persistent ‘Why MaxEnt?’ question.

COLLINEARITY AND COHERENCE: ML AND MAXENT

Definition 1. System of events \( \{A_1, A_2, \ldots, A_m\} \) is equivalently described by its distribution \( p = [p_1, p_2, \ldots, p_m] \), or by its potential \( u = [u_1, u_2, \ldots, u_m] \). The relationship of equivalence is

\[
p = \text{dist}(u) = \frac{1}{\sum_{i=1}^{m} e^{-u_i}} e^{-u_1}, e^{-u_2}, \ldots, e^{-u_m}
\]

Theorem 1. \( \text{dist}(u) = \text{dist}(u + C) \), where \( C \) is any constant.

Note 1. The theorem implies that pmf is determined (induced) uniquely by potential, and potential is determined by pmf up to an additive constant.

Definition 2. Mean value of potential \( u \) weighted by pmf \( p \) is the scalar product of the vectors \( up \).

Definition 3. Two potential vectors \( a, ka \), where \( k \in \mathbb{R} \), are told to be collinear.

Definition 4. Coherence of two pmfs \( p \) and \( q \) on potential \( u \) (or: with respect to potential \( u \), or: relative to potential \( u \)) is defined as

\[
\text{coher}_u(p, q) = up - uq
\]

1 M. Grendar and M. Grendar, Jr., "Why Maximum Entropy? A Non-axiomatic Approach", in Bayesian Inference and Maximum Entropy Methods in Science and Engineering: 21-st International Workshop, edited by R. L. Fry, pp. 375-379, American Institute of Physics, Melville, NY, 2002, vol. CP617
(Zero value means the maximal possible coherence, the greater the (absolute) value, the smaller the coherence.)

**Definition 5.** If coherence of two pmfs is zero, the pmfs are called coherent.

**Theorem 2.** (ML task with simple exponential form) Let $X$ be a random sample with vector of frequencies $r$. Let $u$ be a potential. Then in class of distributions $\text{dist}(\lambda u)$ induced by collinear with $u$ potentials $\lambda u$ the most likely one $\text{dist}(\lambda_0 u)$ is coherent with $r$ relative to potential $u$:

$$\text{coher}_u(r, \text{dist}(\lambda_0 u)) = 0$$

*Proof.* For a proof see [10], proof of the Theorem 1. □

**Definition 6.** The mean value of potential $u$ weighted by its own distribution is called entropy of the potential,

$$\text{ent}(u) = u \text{dist}(u)$$

**Theorem 3.** (MaxEnt task with simple potential) Let $X$ be a random sample with vector of frequencies $r$. Let $u$ be a potential. Then in class of distributions $\text{dist}(v)$ which are coherent with $r$ relative to potential $u$

$$\text{coher}_u(r, \text{dist}(v)) = 0$$

the most entropic one $\text{dist}(v_0)$ is induced by potential $v_0 = \lambda_0 u$ collinear with $u$.

*Proof.* For a proof see [10], proof of the Theorem 1. □

![FIGURE 1. Geometric representation of Theorem 2, 3](image)

From a geometrical standpoint, claims of the Theorem 1 and 2 lead to a search for an intersection of $u$ with a line defined by an orthogonal to $u$ vector $r - q$, as it is depicted on the Figure 1.

The circular relationship of the claims of Theorem 2 and Theorem 3, dubbed in [10] ’complementarity of Maximum Likelihood and MaxEnt tasks’, is visualized by the following diagram:

$$\text{collin} \rightarrow \text{ML} \uparrow \downarrow \text{coher} \leftarrow \text{ME}$$

In an intentionally loose manner the complementarity can bestated as: ‘in the class of collinear potentials the most probable (the most likely) is the coherent one; in the class of coherent distributions the most entropic is the collinear one’.

In light of the MaxProb rationale of MaxEnt (see [11]), the above statement can achieve an even deeper symmetry, in the case of sufficiently large sample. Then the words ‘the most entropic’ can be replaced by ‘the most probable’, and the diagram can be redrawn:
WHY MAXENT? A SIMPLE ANSWER

Theorem 3 offers a simple argument in favor of Shannon’s entropy maximization (MaxEnt) method/criterion in the inverse-problem context, which was mentioned at the abstract.

Consider the following ‘game’ which models the inverse problem: an experiment reveals \( m \) different outcomes. The experiment was repeated sufficiently many times and vector of frequencies of the outcomes in the obtained random sample \( \mathbf{r} = [r_1, r_2, \ldots, r_m] \) is known to us. Also, we are told a potential \( \mathbf{u} \). Given this information \( \{\mathbf{r}, \mathbf{u}\} \) we are asked to pick up a potential \( \mathbf{v}_0 \) of a set of all potentials \( \mathbf{v} \) whose distributions are coherent with \( \mathbf{r} \) on potential \( \mathbf{u} \) (in other words; the set consists of all such potentials \( \mathbf{v} \) that \( \mathbf{r}\mathbf{u} = \text{dist}(\mathbf{v}\mathbf{u}) \)). Common sense dictates to choose \( \mathbf{v}_0 = \mathbf{u} \), but since it is unlikely that this choice will satisfy the coherence condition, the second simplest possible choice is a collinear with \( \mathbf{u} \) potential \( \mathbf{v}_0 = \lambda \mathbf{u} \) where \( \lambda \) should be chosen such that \( \text{dist}(\lambda \mathbf{u}) \) will attain coherence with \( \mathbf{r} \) on the potential \( \mathbf{u} \).

The fact that MaxEnt (and to the best of our knowledge no other method) chooses in the above ‘game’ just the collinear with \( \mathbf{u} \) potential, hence the simplest possible solution, can be used as an answer to ‘Why MaxEnt?’ question.

In order to make relationship of the above mentioned ‘game’ to the ill-posed inverse problem clear, note that the pair of information \( \{\mathbf{r}, \mathbf{u}\} \) forms the value of \( y \), as \( y = \mathbf{r}\mathbf{u} \), and the ill-posed problem becomes \( y = \mathbf{u}\mathbf{p} \). This way obviously extends also to more dimensional \( y \), hence to the case where more potentials are given.

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Note. As compared to the paper which has appeared at the MaxEnt proceedings, here Theorems 2 and 3 were re-stated in terms of random sample, so that the argument made at the last Section should be easier to grasp.

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