Hamilton-Jacobi Equations for Controlled Magnetic Hamiltonian Systems

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Abstract. In this paper, we first define a kind of controlled magnetic Hamiltonian (CMH) system, and give a good expression of the dynamical vector field of the CMH system, such that we can describe the magnetic vanishing condition and the CMH-equivalence, and derive precisely the geometric constraint conditions of the magnetic symplectic form for the dynamical vector field of the CMH system, which are called the Type I and Type II of Hamilton-Jacobi equation. Next, we prove that the CMH-equivalence for the CMH systems leaves the solutions of the associated Hamilton-Jacobi equations invariant, if the associated magnetic Hamiltonian systems are equivalent. These research works reveal the deeply internal relationships of the magnetic symplectic forms, the dynamical vector fields and controls of the CMH systems.

Keywords: regular controlled Hamiltonian system, controlled magnetic Hamiltonian system, geometric constraint condition, Hamilton-Jacobi equation, CMH-equivalence.

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1 Introduction

It is well-known that Hamilton-Jacobi theory is an important research subject in mathematics and analytical mechanics, see Abraham and Marsden [1], Arnold [2] and Marsden and Ratiu [10], and the Hamilton-Jacobi equation is also fundamental in the study of the quantum-classical relationship in quantization, and it also plays an important role in the study of stochastic dynamical systems, see Woodhouse [22], Ge and Marsden [3], and Lázaro-Camí and Ortega [4]. For these reasons it is described as a useful tool in the study of Hamiltonian system theory, and has been extensively developed in past many years and become one of the most active subjects in the study of modern
Just as we have known that Hamilton-Jacobi theory from the variational point of view is originally developed by Jacobi in 1866, which state that the integral of Lagrangian of a mechanical system along the solution of its Euler-Lagrange equation satisfies the Hamilton-Jacobi equation. The classical description of this problem from the generating function and the geometrical point of view is given by Abraham and Marsden in [1] as follows: Let $Q$ be a smooth manifold and $TQ$ the tangent bundle, $T^*Q$ the cotangent bundle with a canonical symplectic form $\omega$ and the projection $\pi_Q : T^*Q \to Q$ induces the map $T\pi_Q : TT^*Q \to TQ$.

**Theorem 1.1** Assume that the triple $(T^*Q, \omega, H)$ is a Hamiltonian system with Hamiltonian vector field $X_H$, and $W : Q \to \mathbb{R}$ is a given generating function. Then the following two assertions are equivalent:

(i) For every curve $\sigma : \mathbb{R} \to Q$ satisfying $\dot{\sigma}(t) = T\pi_Q(X_H(dW(\sigma(t))))$, $\forall t \in \mathbb{R}$, then $dW \cdot \sigma$ is an integral curve of the Hamiltonian vector field $X_H$.

(ii) $W$ satisfies the Hamilton-Jacobi equation $H(q^i, \frac{\partial W}{\partial q^i}) = E$, where $E$ is a constant.

From the proof of the above theorem given in Abraham and Marsden [1], we know that the assertion (i) with equivalent to Hamilton-Jacobi equation by the generating function (ii), gives a geometric constraint condition of the canonical symplectic form on the cotangent bundle $T^*Q$ for Hamiltonian vector field of the system. Thus, the Hamilton-Jacobi equation reveals the deeply internal relationships of the generating function, the canonical symplectic form and the dynamical vector field of a Hamiltonian system.

On the other hand, the authors in Marsden et al. [11] define a regular controlled Hamiltonian (RCH) system, which is a Hamiltonian system with external force and control. In general, an RCH system, under the actions of external force and control, is not Hamiltonian, however, it is a dynamical system closely related to a Hamiltonian system, and it can be explored and studied by extending the methods for external force and control in the study of Hamiltonian systems. Thus, one can emphasize explicitly the impact of external force and control in the study for the RCH systems. However, since an RCH system defined on the cotangent bundle $T^*Q$, may not be a Hamiltonian system, and it may have no generating function, we cannot give the Hamilton-Jacobi theorem for the RCH system just like same as the above Theorem 1.1. We have to look for a new way. It is worthy of noting that, in Wang [18] the author derives precisely the geometric constraint conditions of canonical symplectic form for the dynamical vector field of an RCH system. These conditions are called the Type I and Type II of Hamilton-Jacobi equation, which are the development of the Type I and Type II of Hamilton-Jacobi equation for a Hamiltonian system given in Wang [17]. Moreover, the author proves that the RCH-equivalence for the RCH systems leaves the solutions of corresponding Hamilton-Jacobi equations invariant.

In order to describe the impact of different structures of geometry for the RCH system and Hamilton-Jacobi equations, we consider the magnetic symplectic form $\omega^B = \omega - \pi_Q^* B$, where $\omega$ is the usual canonical symplectic form on $T^*Q$, and $B$ is the closed two-form on $Q$. A magnetic Hamiltonian system is a Hamiltonian system defined by the magnetic symplectic form, which is a canonical Hamiltonian system coupling the action of a magnetic field $B$. A controlled magnetic Hamiltonian (CMH) system on $T^*Q$ is a magnetic Hamiltonian system $(T^*Q, \omega^B, H)$ with external force $F$ and control $W$, where $F : T^*Q \to T^*Q$ is the fiber-preserving map, and $W \subset T^*Q$ is a fiber submanifold of $T^*Q$. see Wang [16]. Thus, it is a natural problem how to derive precisely the geometric constraint conditions of the magnetic symplectic form for the dynamical vector field of a CMH system, and how to describe explicitly the relationship between the CMH-equivalence and
the solutions of corresponding Hamilton-Jacobi equations. These research are our goal in this paper.

A brief of outline of this paper is as follows. In the second section, we first define a kind of controlled magnetic Hamiltonian (CMH) system by using magnetic symplectic form, and then give a good expression of the dynamical vector field of the CMH system, such that we can also describe the magnetic vanishing condition. In the third section, we first prove a key lemma, which is an important tool for the proofs of two types of Hamilton-Jacobi theorems of the CMH system. Then we derive precisely the geometric constraint conditions of the magnetic symplectic form for the dynamical vector field of a CMH system on the cotangent bundle of a configuration manifold, that is, the Type I and Type II of Hamilton-Jacobi equation for a CMH system. Moreover, in the fourth section, we describe the CMH-equivalence for the CMH system, and prove that the CMH-equivalence for the CMH systems leaves the solutions of the associated Hamilton-Jacobi equations invariant, if the associated magnetic Hamiltonian systems are equivalent. These research works reveal the deeply internal relationships of the geometrical structures of phase spaces, the dynamical vector fields and controls of the CMH systems, and make us have much deeper understanding and recognition for the structures of Hamiltonian system, RCH system and CMH system.

2 Controlled Magnetic Hamiltonian System

In this section, we first define a kind of controlled magnetic Hamiltonian (CMH) system by using magnetic symplectic form, and then give a good expression of the dynamical vector field of the CMH system, by using the vertical lift map of a vector along a fiber. We shall follow some of the notations and conventions introduced in Abraham and Marsden [1], Arnold [2], Marsden et al. [8, 11], Marsden and Ratiu [10], Ortega and Ratiu [13], Wang [16] and Wang [17]. For convenience, in this paper, we assume that all manifolds are real, smooth and finite dimensional and all actions are smooth left actions. and all controls appearing in this paper are the admissible controls.

In the reduction theory and application of Hamiltonian systems, the Marsden-Weinstein reduction for a Hamiltonian system with symmetry and momentum map is very important and foundational, see Marsden and Weinstein [12], and Libermann and Marle [6], Marsden [7]. But, from the classification of symplectic reduced space of the cotangent bundle $T^*Q$, see Marsden et al. [8] and Marsden and Perlmutter [9], we know that the set of Hamiltonian systems with symmetries and momentum maps on the cotangent bundle $T^*Q$ is not complete under the Marsden-Weinstein reduction, that is, the symplectic reduced system of a Hamiltonian system with symmetry and momentum map defined on the cotangent bundle $T^*Q$ may not be a Hamiltonian system on a cotangent bundle. In consequence, if we define directly a controlled Hamiltonian system with symmetry on the cotangent bundle $T^*Q$, then the symplectic reduced controlled Hamiltonian system may not have definition.

In order to describe uniformly RCH systems defined on a cotangent bundle and on the regular reduced spaces, in this subsection we first define an RCH system on a symplectic fiber bundle, see Marsden et al. [11]. Then we can obtain the RCH system and the CMH system on the cotangent bundle of a configuration manifold as the special cases, and give a good expression of the dynamical vector field of the CMH system, such that we can describe the magnetic vanishing condition, and discuss CMH-equivalence in fourth section. In consequence, we can regard the associated Hamiltonian system and the magnetic Hamiltonian system on the cotangent bundle as the spacial cases of the RCH system and the CMH system without external forces and controls, such that we can study the RCH system and the CMH system by extending the methods for external force and control in the study of (magnetic) Hamiltonian system.
Let \((E, M, N, \pi, G)\) be a fiber bundle and \((E, \omega_E)\) be a symplectic fiber bundle. If for any function \(H : E \to \mathbb{R}\), we have a Hamiltonian vector field \(X_H\), which satisfies the Hamilton’s equation, that is, \(i_{X_H} \omega_E = dH\), then \((E, \omega_E, H)\) is a Hamiltonian system. Moreover, if considering the external force and control, we can define a kind of regular controlled Hamiltonian (RCH) system on the symplectic fiber bundle \(E\) as follows.

**Definition 2.1 (RCH System)** A regular controlled Hamiltonian (RCH) system on \(E\) is a 5-tuple \((E, \omega_E, H, F, W)\), where \((E, \omega_E, H)\) is a Hamiltonian system, and the function \(H : E \to \mathbb{R}\) is called the Hamiltonian, a fiber-preserving map \(F : E \to E\) is called the (external) force map, and a fiber submanifold \(W \subset E\) is called the control subset.

Sometimes, \(W\) is also denoted the set of fiber-preserving maps from \(E\) to \(W\). When a feedback control law \(u : E \to W\) is chosen, the 5-tuple \((E, \omega_E, H, F, u)\) is a regular closed-loop dynamic system. In particular, when \(Q\) is a smooth manifold, and \(T^*Q\) its cotangent bundle with a symplectic form \(\omega\) (not necessarily canonical symplectic form), then \((T^*Q, \omega)\) is a symplectic vector bundle. If we take that \(E = T^*Q\), from above definition we can obtain an RCH system on the cotangent bundle \(T^*Q\), that is, 5-tuple \((T^*Q, \omega, H, F, W)\). Where the fiber-preserving map \(F : T^*Q \to T^*Q\) is the (external) force map, which is the reason that the fiber-preserving map \(F : E \to E\) is called an (external) force map in above definition.

In order to describe the impact of different structures of geometry for the RCH systems, we shall consider the magnetic symplectic form on \(T^*Q\) as follows: Assume that \(T^*Q\) with the canonical symplectic form \(\omega\), and \(B\) is a closed two-form on \(Q\), then \(\omega^B = \omega - \pi^*_Q B\) is a symplectic form on \(T^*Q\), where \(\pi^*_Q : T^*Q \to T^*T^*Q\). The \(\omega^B\) is called a magnetic symplectic form, and \(\pi^*_Q B\) is called a magnetic term on \(T^*Q\), see Marsden et al. [8].

A magnetic Hamiltonian system is a 3-tuple \((T^*Q, \omega^B, H)\), which is a Hamiltonian system defined by the magnetic symplectic form \(\omega^B\), that is, a canonical Hamiltonian system coupling the action of a magnetic field \(B\). For a given Hamiltonian \(H\), the dynamical vector field \(X^B_H\), which is called the magnetic Hamiltonian vector field, satisfies the magnetic Hamilton’s equation, that is, \(i_{X^B_H} \omega^B = dH\). In canonical cotangent bundle coordinates, for any \(q \in Q\), \((q, p) \in T^*Q\), we have that

\[
\omega = \sum_{i=1}^{n} dq^i \wedge dp_i, \quad B = \sum_{i,j=1}^{n} B_{ij} dq^i \wedge dq^j, \quad dB = 0,
\]

\[
\omega^B = \omega - \pi^*_Q B = \sum_{i=1}^{n} dq^i \wedge dp_i - \sum_{i,j=1}^{n} B_{ij} dq^i \wedge dq^j,
\]

and the magnetic Hamiltonian vector field \(X^B_H\) with respect to the magnetic symplectic form \(\omega^B\) can be expressed that

\[
X^B_H = \sum_{i=1}^{n} \left( \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i} \frac{\partial}{\partial p_i} \right) - \sum_{i,j=1}^{n} B_{ij} \frac{\partial H}{\partial p_j} \frac{\partial}{\partial p_i}.
\]

See Marsden et al. [8].

Moreover, if considering the external force and control, we can define a kind of controlled magnetic Hamiltonian (CMH) system on \(T^*Q\) as follows.

**Definition 2.2 (CMH System)** A controlled magnetic Hamiltonian (CMH) system on \(T^*Q\) is a 5-tuple \((T^*Q, \omega^B, H, F, W)\), which is a magnetic Hamiltonian system \((T^*Q, \omega^B, H)\) with external force \(F\) and control \(W\), where \(F : T^*Q \to T^*Q\) is the fiber-preserving map, and \(W \subset T^*Q\) is a fiber submanifold, which is called the control subset.
From the above Definition 2.1 and Definition 2.2 we know that a CMH system on $T^*Q$ is also an RCH system on $T^*Q$, but its symplectic structure is given by a magnetic symplectic form, and the set of the CMH systems on $T^*Q$ is a subset of the set of the RCH systems on $T^*Q$. When a feedback control law $u : T^*Q \to W$ is chosen, the 5-tuple $(T^*Q, \omega^B, H, F, u)$ is a regular closed-loop dynamic system.

In order to describe the dynamics of the CMH system $(T^*Q, \omega^B, H, F, W)$ with a control law $u$, we need to give a good expression of the dynamical vector field of the CMH system. First, we introduce a notations of vertical lift maps of a vector along a fiber, also see Marsden et al. [11]. For a smooth manifold $E$, its tangent bundle $TE$ is a vector bundle, and for the fiber bundle $\pi : E \to M$, we consider the tangent mapping $T\pi : TE \to TM$ and its kernel $ker(T\pi) = \{ \rho \in TE | T\pi(\rho) = 0 \}$, which is a vector subbundle of $TE$. Denote by $VE := ker(T\pi)$, which is called a vertical bundle of $E$. Assume that there is a metric on $E$, and we take a Levi-Civita connection $A$ on $TE$, and denote by $HE := ker(A)$, which is called a horizontal bundle of $E$, such that $TE = HE \oplus VE$. For any $x \in M$, $a_x, b_x \in E_x$, any tangent vector $\rho(b_x) \in T_{b_x}E$ can be split into horizontal and vertical parts, that is, $\rho(b_x) = \rho^h(b_x) \oplus \rho^v(b_x)$, where $\rho^h(b_x) \in H_{b_x}E$ and $\rho^v(b_x) \in V_{b_x}E$. Let $\gamma$ be a geodesic in $E_x$ connecting $a_x$ and $b_x$, and denote by $\rho^\gamma_x(a_x)$ a tangent vector at $a_x$, which is a parallel displacement of the vertical vector $\rho^v(b_x)$ along the geodesic $\gamma$ from $b_x$ to $a_x$. Since the angle between two vectors is invariant under a parallel displacement along a geodesic, then $T\pi(\rho^\gamma_x(a_x)) = 0$, and hence $\rho^\gamma_x(a_x) \in V_{a_x}E$. Now, for $a_x, b_x \in E_x$ and tangent vector $\rho(b_x) \in T_{b_x}E$, we can define the vertical lift map of a vector along a fiber given by

$$vlift : TE_x \times E_x \to TE_x; \quad vlift(\rho(b_x), a_x) = \rho^\gamma_x(a_x).$$

It is easy to check from the basic fact in differential geometry that this map does not depend on the choice of $\gamma$. If $F : E \to E$ is a fiber-preserving map, for any $x \in M$, we have that $F_x : E_x \to E_x$ and $TF_x : TE_x \to TE_x$, then for any $a_x \in E_x$ and $\rho \in TE_x$, the vertical lift of $\rho$ under the action of $F$ along a fiber is defined by

$$(vlift(F_x)\rho)(a_x) = vlift((TF_x\rho)(F_x(a_x)), a_x) = (TF_x\rho)^\gamma_x(a_x),$$

where $\gamma$ is a geodesic in $E_x$ connecting $F_x(a_x)$ and $a_x$.

In particular, when $\pi : E \to M$ is a vector bundle, for any $x \in M$, the fiber $E_x = \pi^{-1}(x)$ is a vector space. In this case, we can choose the geodesic $\gamma$ to be a straight line, and the vertical vector is invariant under a parallel displacement along a straight line, that is, $\rho^\gamma_x(a_x) = \rho^v(b_x)$. Moreover, when $E = T^*Q$, $M = Q$, by using the local trivialization of $TT^*Q$, we have that $TT^*Q \cong TQ \times T^*Q$. Because of $\pi : T^*Q \to Q$, and $T\pi : TT^*Q \to TQ$, then in this case, for any $a_x, b_x \in T_y^*Q, x \in Q$, we know that $(0, b_x) \in V_{b_x}T^*Q$, and hence we can get that

$$vlift((0, b_x)(\beta_x), a_x) = (0, \beta_x)(a_x) = \frac{d}{ds} \bigg|_{s=0} (\alpha_x + s\beta_x),$$

which is consistent with the definition of vertical lift map along fiber in Marsden and Ratiu [10].

For a given CMH system $(T^*Q, \omega^B, H, F, W)$, the dynamical vector field of the associated magnetic Hamiltonian system $(T^*Q, \omega^B, H)$ is $X_H^B$, which satisfies the equation $i_{X_H^B} \omega^B = dH$. If considering the external force $F : T^*Q \to T^*Q$, by using the above notation of vertical lift map of a vector along a fiber, the change of $X_H^B$ under the action of $F$ is that

$$vlift(F)X_H^B(a_x) = vlift((TFX_H^B)(F(a_x)), a_x) = (TFX_H^B)^\gamma_x(a_x),$$

where $\gamma$ is a geodesic in $E_x$ connecting $F_x(a_x)$ and $a_x$.
where $\alpha_x \in T_x^*Q$, $x \in Q$ and $\gamma$ is a straight line in $T_x^*Q$ connecting $F_x(\alpha_x)$ and $\alpha_x$. In the same way, when a feedback control law $u : T^*Q \to W$ is chosen, the change of $X^B_H$ under the action of $u$ is that

$$\text{vlift}(u)X^B_H(\alpha_x) = \text{vlift}((TuX^B_H)(u(\alpha_x)), \alpha_x) = (TuX^B_H)^*\gamma(\alpha_x).$$

In consequence, we can give an expression of the dynamical vector field of the CMH system as follows.

**Theorem 2.3** The dynamical vector field of a CMH system $(T^*Q, \omega^B, H, F, W)$ with a control law $u$ is the synthetic of magnetic Hamiltonian vector field $X^B_H$ and its changes under the actions of the external force $F$ and control law $u$, that is,

$$X_{(T^*Q, \omega^B, H, F, W)}(\alpha_x) = X^B_H(\alpha_x) + \text{vlift}(F)X^B_H(\alpha_x) + \text{vlift}(u)X^B_H(\alpha_x), \quad (2.1)$$

for any $\alpha_x \in T_x^*Q$, $x \in Q$. For convenience, it is simply written as

$$X_{(T^*Q, \omega^B, H, F, u)} = X^B_H + \text{vlift}(F)^B + \text{vlift}(u)^B. \quad (2.2)$$

Where $\text{vlift}(F)^B = \text{vlift}(F)X^B_H$, and $\text{vlift}(u)^B = \text{vlift}(u)X^B_H$, are the changes of $X^B_H$ under the actions of $F$ and $u$. We also denote that $\text{vlift}(W)^B = \bigcup \{\text{vlift}(u)X^B_H | u \in W\}$. It is worthy of noting that, in order to deduce and calculate easily, we always use the simple expression of dynamical vector field $X_{(T^*Q, \omega^B, H, F, u)}$.

From the expression (2.2) of the dynamical vector field of a CMH system, we know that under the actions of the external force $F$ and control law $u$, in general, the dynamical vector field may not be magnetic Hamiltonian, and hence the CMH system may not be yet a magnetic Hamiltonian system. However, it is a dynamical system closed relative to a magnetic Hamiltonian system, and it can be explored and studied by extending the methods for external force and control in the study of magnetic Hamiltonian system.

For the magnetic Hamiltonian system $(T^*Q, \omega^B, H)$, its magnetic Hamiltonian vector field $X^B_H$ satisfies the equation $i_{X^B_H}\omega^B = dH$, and for the associated canonical Hamiltonian system $(T^*Q, \omega, H)$, its canonical Hamiltonian vector field $X_H$ satisfies the equation $i_{X_H}\omega = dH$. Denote by the vector field $X^0 = X^B_H - X_H$, and from the magnetic symplectic form $\omega^B = \omega - \pi_Q^*B$, we have that

$$i_{X^0}\omega = i_{(X^B_H - X_H)}\omega = i_{X^B_H}\omega - i_{X_H}\omega = i_{X^B_H}(\omega^B + \pi_Q^*B) - i_{X_H}\omega = i_{X^B_H}(\pi_Q^*B).$$

Thus, $X^0$ is called the magnetic vector field and $i_{X^0}\omega = i_{X^B_H}(\pi_Q^*B)$ is called the magnetic equation, which is determined by the magnetic term $\pi_Q^*B$ on $T^*Q$. When $B = 0$, then $X^0 = 0$, the magnetic equation holds trivially. For the CMH system $(T^*Q, \omega^B, H, F, W)$, from the expression (2.2) of its dynamical vector field, we have that

$$X_{(T^*Q, \omega^B, H, F, u)} = X_H + X^0 + \text{vlift}(F)^B + \text{vlift}(u)^B. \quad (2.3)$$

If we choose the external force $F$ and control law $u$, such that

$$X^0 + \text{vlift}(F)^B + \text{vlift}(u)^B = 0, \quad (2.4)$$

then from (2.3) we have that $X_{(T^*Q, \omega^B, H, F, u)} = X_H$, that is, in this case the dynamical vector field of the CMH system is just the canonical Hamiltonian vector field, and the motion of the CMH system is just the same like the motion of canonical Hamiltonian system without the actions of magnetic, external force and control. Thus, (2.4) is called the magnetic vanishing condition for the CMH system $(T^*Q, \omega^B, H, F, W)$.

To sum up the above discussion, we have the following theorem.
Theorem 2.4 If the external force $F$ and the control law $u$ for a CMH system $(T^*Q, \omega^B, H, F, u)$ satisfy the magnetic vanishing condition (2.4), then its dynamical vector field $X_{(T^*Q, \omega^B, H, F, u)}$ is just the canonical Hamiltonian vector field $X_H$ for the associated canonical Hamiltonian system $(T^*Q, \omega, H)$.

3 Two Types of Hamilton-Jacobi Equation for a CMH System

In this section, we shall derive precisely the geometric constraint conditions of the magnetic symplectic form for the dynamical vector field of a CMH system, that is, Type I and Type II of Hamilton-Jacobi equation for the CMH system. In order to do this, in the following we first give an important notion and prove a key lemma, which is an important tool for the proofs of two types of Hamilton-Jacobi theorem for the CMH system.

Denote by $\Omega^i(Q)$ the set of all $i$-forms on $Q$, $i = 1, 2$. For any $\gamma \in \Omega^1(Q)$, $q \in Q$, then $\gamma(q) \in T^*_qQ$, and we can define a map $\gamma : Q \rightarrow T^*_Q, q \rightarrow (q, \gamma(q))$. Hence we say often that the map $\gamma : Q \rightarrow T^*Q$ is an one-form on $Q$. If the one-form $\gamma$ is closed, then $d\gamma(x, y) = 0, \forall x, y \in TQ$. In the following we give a weaker notion.

Definition 3.1 The one-form $\gamma$ is called to be closed with respect to $T\pi_Q : TT^*Q \rightarrow TQ$, if for any $v, w \in TT^*Q$, we have $d\gamma(T\pi_Q(v), T\pi_Q(w)) = 0$.

From the above definition we know that, if $\gamma$ is a closed one-form, then it must be closed with respect to $T\pi_Q : TT^*Q \rightarrow TQ$. Conversely, if $\gamma$ is closed with respect to $T\pi_Q : TT^*Q \rightarrow TQ$, then it may not be closed. We can prove a general result as follows, its proof given in Wang [17].

Proposition 3.2 Assume that $\gamma : Q \rightarrow T^*Q$ is an one-form on $Q$ and it is not closed. we define the set $N$, which is a subset of $TQ$, such that the one-form $\gamma$ on $N$ satisfies the condition that for any $x, y \in N$, $d\gamma(x, y) \neq 0$. Denote by $\text{Ker}(T\pi_Q) = \{u \in TT^*Q \mid T\pi_Q(u) = 0\}$, and $T\gamma : TQ \rightarrow TT^*Q$ is the tangent map of $\gamma : Q \rightarrow T^*Q$. If $T\gamma(N) \subset \text{Ker}(T\pi_Q)$, then $\gamma$ is closed with respect to $T\pi_Q : TT^*Q \rightarrow TQ$.

For the one-form $\gamma : Q \rightarrow T^*Q$, $d\gamma$ is a two-form on $Q$. Assume that $B$ is a closed two-form on $Q$, we say that the $\gamma$ satisfies condition $d\gamma = -B$, if for any $x, y \in TQ$, we have that $(d\gamma + B)(x, y) = 0$. In the following we give a new notion.

Definition 3.3 Assume that $\gamma : Q \rightarrow T^*Q$ is an one-form on $Q$, we say that the $\gamma$ satisfies condition $d\gamma = -B$ with respect to $T\pi_Q : TT^*Q \rightarrow TQ$, if for any $v, w \in TT^*Q$, we have $(d\gamma + B)(T\pi_Q(v), T\pi_Q(w)) = 0$.

From the above Definition 3.1 and Definition 3.3, we know that, when $B = 0$, the condition that, $\gamma$ satisfies condition $d\gamma = -B$ with respect to $T\pi_Q : TT^*Q \rightarrow TQ$, become that $\gamma$ is closed with respect to $T\pi_Q : TT^*Q \rightarrow TQ$. Now, we prove the following lemma, which is a generalization of a corresponding to lemma given by Wang [17], and the lemma is a very important tool for our research.

Lemma 3.4 Assume that $\gamma : Q \rightarrow T^*Q$ is an one-form on $Q$, and $\lambda = \gamma \cdot \pi_Q : T^*Q \rightarrow T^*Q$. For the magnetic symplectic form $\omega^B = \omega - \pi^*_QB$ on $T^*Q$, where $\omega$ is the canonical symplectic form on $T^*Q$, then we have that the following two assertions hold.
(i) For any $v, w \in TT^*Q$, $\Lambda^\omega^B(v, w) = -(d\gamma + B)(T\pi_Q(v), T\pi_Q(w))$;
(ii) For any $v, w \in TT^*Q$, $\omega^B(T\lambda \cdot v, w) = \omega^B(v, w - T\lambda \cdot w) - (d\gamma + B)(T\pi_Q(v), T\pi_Q(w))$. 

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Proof: We first prove the assertion (i). Since $\omega$ is the canonical symplectic form on $T^*Q$, we know that there is an unique canonical one-form $\theta$, such that $\omega = -d\theta$. From the Proposition 3.2.11 in Abraham and Marsden [1], we have that for the one-form $\gamma : Q \to T^*Q$, $\gamma^* \theta = \gamma$. Then we can obtain that for any $x, y \in TQ$,

$$\gamma^*(x, y) = \gamma(-d\theta)(x, y) = -d(\gamma^*\theta)(x, y) = -d\gamma(x, y).$$

Note that $\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q$, and $\lambda^* = \pi_Q^* \cdot \gamma^* : T^*T^*Q \to T^*T^*Q$, then we have that for any $v, w \in TT^*Q$,

$$\lambda^*(v, w) = \lambda^*(-d\theta)(v, w) = -d(\lambda^*\theta)(v, w) = -d(\pi_Q^* \cdot \gamma^*)(v, w) = -d(\pi_Q^* \cdot \gamma)(v, w) = -d\gamma(T\pi_Q(v), T\pi_Q(w)).$$

Hence, we have that

$$\lambda^*\omega^B(v, w) = \lambda^*\omega(v, w) - \lambda^* \cdot \pi_Q^* B(v, w) = -d\gamma(T\pi_Q(v), T\pi_Q(w)) - (\pi_Q \cdot \gamma \cdot \pi_Q^*) B(v, w) = -d\gamma(T\pi_Q(v), T\pi_Q(w)) - \pi_Q^* B(v, w) = -(d\gamma + B)(T\pi_Q(v), T\pi_Q(w)),$$

where we have used the relation $\pi_Q \cdot \gamma \cdot \pi_Q = \pi_Q$. It follows that the assertion (i) holds.

Next, we prove the assertion (ii). For any $v, w \in TT^*Q$, note that $v - T(\gamma \cdot \pi_Q) \cdot v$ is vertical, because

$$T\pi_Q(v - T(\gamma \cdot \pi_Q) \cdot v) = T\pi_Q(v) - T(\pi_Q \cdot \gamma \cdot \pi_Q) \cdot v = T\pi_Q(v) - T\pi_Q(v) = 0,$$

Thus, $\omega(v - T(\gamma \cdot \pi_Q) \cdot v, w - T(\gamma \cdot \pi_Q) \cdot w) = 0$, and hence,

$$\omega(T(\gamma \cdot \pi_Q) \cdot v, w) = \omega(v, w - T(\gamma \cdot \pi_Q) \cdot w) + \omega(T(\gamma \cdot \pi_Q) \cdot v, T(\gamma \cdot \pi_Q) \cdot w).$$

However, the second term on the right-hand side is given by

$$\omega(T(\gamma \cdot \pi_Q) \cdot v, T(\gamma \cdot \pi_Q) \cdot w) = \gamma^* \omega(T\pi_Q(v), T\pi_Q(w)) = -d\gamma(T\pi_Q(v), T\pi_Q(w)), $$

It follows that

$$\omega(T\lambda \cdot v, w) = \omega(T(\gamma \cdot \pi_Q) \cdot v, w) = \omega(v, w - T(\gamma \cdot \pi_Q) \cdot w) - d\gamma(T\pi_Q(v), T\pi_Q(w)) = \omega(v, w - T\lambda \cdot w) - d\gamma(T\pi_Q(v), T\pi_Q(w)).$$

Hence, we have that

$$\omega^B(T\lambda \cdot v, w) = \omega(T\lambda \cdot v, w) - \pi_Q^* B(T\lambda \cdot v, w)$$

$$= \omega(v, w - T\lambda \cdot w) - d\gamma(T\pi_Q(v), T\pi_Q(w)) - B(T\pi_Q \cdot T\lambda \cdot v, T\pi_Q(w))$$

$$= \omega^B(v, w - T\lambda \cdot w) + \pi_Q^* B(v, w - T\lambda \cdot w) - d\gamma(T\pi_Q(v), T\pi_Q(w)) - B(T\pi_Q \cdot \lambda \cdot v, T\pi_Q(w))$$

$$= \omega^B(v, w - T\lambda \cdot w) + \pi_Q^* B(v, w - B(T\pi_Q(v), T\pi_Q(w))$$

$$= \omega^B(v, w - T\lambda \cdot w) + \pi_Q^* B(v, w - B(T\pi_Q(w), T\pi_Q(w))$$

$$= \omega^B(v, w - T\lambda \cdot w) - (d\gamma + B)(T\pi_Q(v), T\pi_Q(w)).$$

$$= \omega^B(v, w - T\lambda \cdot w) - (d\gamma + B)(T\pi_Q(v), T\pi_Q(w)).$$
Thus, the assertion (ii) holds.

For a given CMH system \((T^*Q, \omega^B, H, F, W)\) on \(T^*Q\), by using the above Lemma 3.4, we can derive precisely the geometric constraint conditions of the magnetic symplectic form \(\omega^B\) for the dynamical vector field \(X_{(T^*Q, \omega^B, H, F, W)}\) of the CMH system with a control law \(u\), that is, Type I and Type II of Hamilton-Jacobi equation for the CMH system.

**Theorem 3.5 (Type I of Hamilton-Jacobi Theorem for a CMH System)** For the CMH system \((T^*Q, \omega^B, H, F, W)\) with the magnetic symplectic form \(\omega^B = \omega - \pi_B^*B\) on \(T^*Q\), where \(\omega\) is the canonical symplectic form on \(T^*Q\) and \(B\) is a closed two-form on \(Q\), assume that \(\gamma : Q \to T^*Q\) is an one-form on \(Q\), and \(\hat{X}_\gamma = T\pi_Q \cdot \hat{X} \cdot \gamma\), where \(\hat{X} = X_{(T^*Q, \omega^B, H, F, W)}\) is the dynamical vector field of the CMH system \((T^*Q, \omega^B, H, F, W)\) with a control law \(u\). If the one-form \(\gamma : Q \to T^*Q\) satisfies the condition \(d\gamma = -B\) with respect to \(T\pi_Q : TT^*Q \to TQ\), then \(\gamma\) is a solution of the equation \(T\gamma \cdot \hat{X}_\gamma = X^B_H \cdot \gamma\), where \(X^B_H\) is the magnetic Hamiltonian vector field of the associated magnetic Hamiltonian system \((T^*Q, \omega^B, H, F, W)\), and the equation is called the Type I of Hamilton-Jacobi equation for the CMH system \((T^*Q, \omega^B, H, F, W)\) with a control law \(u\). Here the maps involved in the theorem are shown in the following Diagram-1.

![Diagram-1](image)

**Proof:** Since \(\hat{X} = \hat{X}_{(T^*Q, \omega^B, H, F, W)} = X^B_H + vlift(F)^B + vlift(u)^B\), and \(T\pi_Q \cdot vlift(F)^B = T\pi_Q \cdot vlift(u)^B = 0\), then we have that \(T\pi_Q \cdot \hat{X} \cdot \gamma = T\pi_Q \cdot X^B_H \cdot \gamma\). If we take that \(v = X^B_H \cdot \gamma \in TT^*Q\), and for any \(w \in TT^*Q\), \(T\pi_Q(w) \neq 0\), from Lemma 3.4(ii) and \(d\gamma = -B\) with respect to \(T\pi_Q : TT^*Q \to TQ\), that is, \((d\gamma + B)(T\pi_Q \cdot X^B_H \cdot \gamma, T\pi_Q \cdot w) = 0\), we have that

\[
\omega^B(T\gamma \cdot \hat{X}_\gamma, w) = \omega^B(T\gamma \cdot T\pi_Q \cdot \hat{X} \cdot \gamma, w)
= \omega^B(T\gamma \cdot T\pi_Q \cdot X^B_H \cdot \gamma, w) = \omega^B(T(\gamma \cdot T\pi_Q) \cdot X^B_H \cdot \gamma, w)
= \omega^B(X^B_H \cdot \gamma, w - T(\gamma \cdot T\pi_Q) \cdot w) - (d\gamma + B)(T\pi_Q \cdot X^B_H \cdot \gamma, T\pi_Q \cdot w)
= \omega^B(X^B_H \cdot \gamma, w) - \omega^B(X^B_H \cdot \gamma, T\lambda \cdot w).
\]

Hence, we have that

\[
\omega^B(T\gamma \cdot \hat{X}_\gamma, w) - \omega^B(X^B_H \cdot \gamma, w) = -\omega^B(X^B_H \cdot \gamma, T\lambda \cdot w). \tag{3.1}
\]

If \(\gamma\) satisfies the equation \(T\gamma \cdot \hat{X}_\gamma = X^B_H \cdot \gamma\), from Lemma 3.4(i) we can obtain that

\[
\omega^B(X^B_H \cdot \gamma, T\lambda \cdot w) = \omega^B(T\gamma \cdot \hat{X}_\gamma, T\lambda \cdot w)
= \omega^B(T\gamma \cdot T\pi_Q \cdot \hat{X} \cdot \gamma, T\lambda \cdot w)
= \omega^B(T\gamma \cdot T\pi_Q \cdot X^B_H \cdot \gamma, T\lambda \cdot w)
= \omega^B(T\lambda \cdot X^B_H \cdot \gamma, T\lambda \cdot w)
= \lambda^*\omega^B(X^B_H \cdot \gamma, w)
= -(d\gamma + B)(T\pi_Q \cdot X^B_H \cdot \gamma, T\pi_Q \cdot w) = 0,
\]

since \(\gamma : Q \to T^*Q\) satisfies the condition \(d\gamma = -B\) with respect to \(T\pi_Q : TT^*Q \to TQ\). But, because the magnetic symplectic form \(\omega^B\) is non-degenerate, the left side of (3.1) equals zero, only
when $\gamma$ satisfies the equation $T\gamma \cdot \dot{X} \gamma = X_B^B \cdot \gamma$. Thus, if the one-form $\gamma : Q \to T^*Q$ satisfies the condition $d\gamma = -B$ with respect to $T\pi_Q : TT^*Q \to TQ$, then $\gamma$ must be a solution of the Type I of Hamilton-Jacobi equation $T\gamma \cdot \dot{X} \gamma = X_B^B \cdot \gamma$, for the CMH system $(T^*Q, \omega^B, H, F, W)$ with a control law $u$. ■

When $B = 0$, in this case the magnetic symplectic form $\omega^B$ is just the canonical symplectic form $\omega$ on $T^*Q$, and the CMH system $(T^*Q, \omega^B, H, F, W)$ is just a canonical RCH system $(T^*Q, \omega, H, F, W)$ and the condition that, the one-form $\gamma : Q \to T^*Q$ satisfies the condition $d\gamma = -B$ with respect to $T\pi_Q : TT^*Q \to TQ$, becomes that $\gamma$ is closed with respect to $T\pi_Q : TT^*Q \to TQ$. Thus, from above Theorem 3.5, we can obtain Theorem 2.6 in Wang [18]. On the other hand, it is a natural problem what and how we could do, if an one-form $\gamma : Q \to T^*Q$ is not closed on $Q$ with respect to $T\pi_Q : TT^*Q \to TQ$ in Theorem 2.6 in Wang [18], and hence $\gamma$ is not a solution of the Type I of Hamilton-Jacobi equation for the canonical RCH system. In this case, our idea is that we hope to look for a new RCH system, such that $\gamma$ is a solution of the Type I of Hamilton-Jacobi equation for the new RCH system. Note that, if $\gamma : Q \to T^*Q$ is closed on $Q$ with respect to $T\pi_Q : TT^*Q \to TQ$, that is, there exist $v, w \in TT^*Q$, such that $d\gamma(T\pi_Q(v), T\pi_Q(w)) \neq 0$, and hence $\gamma$ is not yet closed on $Q$. In this case, we note that $d \cdot d\gamma = d^2\gamma = 0$, and hence the $d\gamma$ is a closed two-form on $Q$. Thus, we can construct a magnetic symplectic form on $T^*Q$, that is, $\omega^B = \omega - \pi_Q^B B = \omega + \pi_Q^B d\gamma$, where $B = -d\gamma$, and $\omega$ is the canonical symplectic form on $T^*Q$, and $\pi_Q^B : T^*Q \to T^*T^*Q$. In this case, for any $x, y \in T^*Q$, we have that $(d\gamma + B)(x, y) = 0$, and hence for any $v, w \in TT^*Q$, we have $(d\gamma + B)(T\pi_Q(v), T\pi_Q(w)) = 0$, that is, the one-form $\gamma : Q \to T^*Q$ satisfies the condition $d\gamma = -B$ with respect to $T\pi_Q : TT^*Q \to TQ$. Thus, we can construct a CMH system $(T^*Q, \omega^B, H, F, W)$, its dynamical vector field with a control law $u$ is given by $X_{(T^*Q, \omega_B, H, F, u)} = X_H^B + \text{vlift}(F)^B + \text{vlift}(u)^B$, where $X_H^B$ satisfies the magnetic Hamiltonian equation, that is, $\iota_{X_H^B} \omega^B = dH$, and $\text{vlift}(F)^B = \text{vlift}(F)X_H^B$, $\text{vlift}(u)^B = \text{vlift}(u)X_H^B$. In this case, by using Lemma 3.4 and the dynamical vector field $X_{(T^*Q, \omega_B, H, F, u)}$, from Theorem 3.5 we can obtain the following theorem.

Theorem 3.6 For a given RCH system $(T^*Q, \omega, H, F, W)$ with the canonical symplectic form $\omega$ on $T^*Q$, assume that the one-form $\gamma : Q \to T^*Q$ is not closed with respect to $T\pi_Q : TT^*Q \to TQ$. Construct a magnetic symplectic form on $T^*Q$, $\omega^B = \omega - \pi_Q^B B$, where $B = -d\gamma$, and a CMH system $(T^*Q, \omega^B, H, F, W)$. Denote $\tilde{X} \gamma = T\pi_Q \cdot \tilde{X} \cdot \gamma$, where $\tilde{X} = X_{(T^*Q, \omega_B, H, F, u)}$ is the dynamical vector field of the CMH system $(T^*Q, \omega^B, H, F, W)$ with a control law $u$. Then the one-form $\gamma$ is just a solution of the Type I of Hamilton-Jacobi equation $T\gamma \cdot \tilde{X} \gamma = X_H^B \cdot \gamma$, for the CMH system with a control law $u$.

Next, for any symplectic map $\varepsilon : T^*Q \to T^*Q$ with respect to the magnetic symplectic form $\omega^B$, we can prove the following Type II of geometric Hamilton-Jacobi theorem for the CMH system $(T^*Q, \omega^B, H, F, W)$. For convenience, the maps involved in the following theorem and its proof are shown in Diagram-2.

\[ T^*Q \xrightarrow{\varepsilon} T^*Q \xrightarrow{\pi_Q} Q \xrightarrow{\gamma} T^*Q \]
\[ X_H^B \iota_{\tilde{X} \varepsilon} \]
\[ T(T^*Q) \xrightarrow{T\gamma} TQ \xrightarrow{T\pi_Q} T(T^*Q) \]

Diagram-2

Theorem 3.7 (Type II of Hamilton-Jacobi Theorem for a CMH System) For the CMH system $(T^*Q, \omega^B, H, F, W)$ with the magnetic symplectic form $\omega^B = \omega - \pi_Q^B B$ on $T^*Q$, where $\omega$ is the
canonical symplectic form on $T^*Q$ and $B$ is a closed two-form on $Q$, assume that $\gamma : Q \to T^*Q$ is an one-form on $Q$, and $\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q$, and for any symplectic map $\varepsilon : T^*Q \to T^*Q$ with respect to $\omega^B$, denote by $\tilde{\varepsilon} = T\pi_Q \cdot \tilde{X} \cdot \varepsilon$, where $\tilde{X} = X_{(T\pi_Q,\omega^B,H,F,u)}$ is the dynamical vector field of the CMH system $(T^*Q, \omega^B, H, F, W)$ with a control law $u$. Then $\varepsilon$ is a solution of the equation $T\varepsilon \cdot X^B_{H,\varepsilon} = T\lambda \cdot \tilde{X} \cdot \varepsilon$, if and only if it is a solution of the equation $T\gamma \cdot \tilde{\varepsilon} = X^B_{H,\varepsilon} \cdot \varepsilon$, where $X^B_{H,\varepsilon}$ and $X^B_{H,\varepsilon} \in T^*Q$ are the magnetic Hamiltonian vector fields of the functions $H$ and $H \cdot \varepsilon : T^*Q \to \mathbb{R}$, respectively. The equation $T\gamma \cdot \tilde{\varepsilon} = X^B_{H,\varepsilon} \cdot \varepsilon$, is called the Type II of Hamilton-Jacobi equation for the CMH system $(T^*Q, \omega^B, H, F, W)$ with a control law $u$.

Proof: Since $\tilde{X} = X_{(T\pi_Q,\omega^B,H,F,u)} = X^B_{H} + \text{vlift}(F)^B + \text{vlift}(u)^B$, and $T\pi_Q \cdot \text{vlift}(F)^B = T\pi_Q \cdot \text{vlift}(u)^B = 0$, then we have that $T\pi_Q \cdot \tilde{X} \cdot \varepsilon = T\pi_Q \cdot X^B_{H} \cdot \varepsilon$. If we take that $\varepsilon = X^B_{H} \cdot \varepsilon \in T^*Q$, and for any $w \in T^*Q$, $T\lambda(w) \neq 0$, from Lemma 3.4(ii) we have that

$$
\omega^B(T\gamma \cdot \tilde{\varepsilon}, w) = \omega^B(T\gamma \cdot T\pi_Q \cdot \tilde{X} \cdot \varepsilon, w) = \omega^B(T\gamma \cdot T\pi_Q \cdot X^B_{H} \cdot \varepsilon, w)
$$

$$
= \omega^B(T(\gamma \cdot \pi_Q) \cdot X^B_{H} \cdot \varepsilon, w) = \omega^B(X^B_{H} \cdot \varepsilon, w - T(\gamma \cdot \pi_Q) \cdot w) - (d\gamma + B)(T\pi_Q(X^B_{H} \cdot \varepsilon), T\pi_Q(w))
$$

$$
= \omega^B(X^B_{H} \cdot \varepsilon, w) - \omega^B(X^B_{H} \cdot \varepsilon, T\lambda \cdot w) + \lambda \cdot \omega^B(X^B_{H} \cdot \varepsilon, w)
$$

$$
= \omega^B(X^B_{H} \cdot \varepsilon, w) - \omega^B(X^B_{H} \cdot \varepsilon, T\lambda \cdot w) + \omega^B(T\lambda \cdot X^B_{H} \cdot \varepsilon, T\lambda \cdot w).
$$

Because $\varepsilon : T^*Q \to T^*Q$ is symplectic with respect to $\omega^B$, and hence $X^B_{H} \cdot \varepsilon = T\varepsilon \cdot X^B_{H}, \varepsilon$, along $\varepsilon$. Note that $T\lambda \cdot X^B_{H} \cdot \varepsilon = T\gamma \cdot T\pi_Q \cdot X^B_{H} \cdot \varepsilon = T\gamma \cdot T\pi_Q \cdot \tilde{X} \cdot \varepsilon = T\lambda \cdot \tilde{X} \cdot \varepsilon$. From the above arguments, we can obtain that

$$
\omega^B(T\gamma \cdot \tilde{\varepsilon}, w) - \omega^B(X^B_{H} \cdot \varepsilon, w)
$$

$$
= -\omega^B(X^B_{H} \cdot \varepsilon, T\lambda \cdot w) + \omega^B(T\lambda \cdot X^B_{H} \cdot \varepsilon, T\lambda \cdot w)
$$

$$
= -\omega^B(T\varepsilon \cdot X^B_{H,\varepsilon}, T\lambda \cdot w) + \omega^B(T\lambda \cdot \tilde{X} \cdot \varepsilon, T\lambda \cdot w)
$$

$$
= \omega^B(T\lambda \cdot \tilde{X} \cdot \varepsilon - T\varepsilon \cdot X^B_{H,\varepsilon}, T\lambda \cdot w).
$$

Because the magnetic symplectic form $\omega^B$ is non-degenerate, it follows that $T\gamma \cdot \tilde{\varepsilon} = X^B_{H} \cdot \varepsilon$, is equivalent to $T\varepsilon \cdot X^B_{H,\varepsilon} = T\lambda \cdot \tilde{X} \cdot \varepsilon$. Thus, $\varepsilon$ is a solution of the equation $T\varepsilon \cdot X^B_{H,\varepsilon} = T\lambda \cdot \tilde{X} \cdot \varepsilon$, if and only if it is a solution of the Type II of Hamilton-Jacobi equation $T\gamma \cdot \tilde{\varepsilon} = X^B_{H} \cdot \varepsilon$.

It is worthy of noting that, if the external force and control of a CMH system $(T^*Q, \omega^B, H, F, u)$ are both zeros, in this case the system is just a magnetic Hamiltonian system $(T^*Q, \omega^B, H)$, and from the proofs of the above Theorem 3.5 and Theorem 3.7, we can obtain two types of Hamilton-Jacobi equation for the associated magnetic Hamiltonian system as follows.

**Theorem 3.8** For the magnetic Hamiltonian system $(T^*Q, \omega^B, H)$ with the magnetic symplectic form $\omega^B = \omega - \pi^*_Q B$ on $T^*Q$, assume that $\gamma : Q \to T^*Q$ is an one-form on $Q$, and $\lambda = \gamma \cdot \pi_Q : T^*Q \to T^*Q$, and for any symplectic map $\varepsilon : T^*Q \to T^*Q$ with respect to $\omega^B$, denote by $X^\gamma = T\pi_Q \cdot X^B_{H} \cdot \gamma$, and $X^\varepsilon = T\pi_Q \cdot X^B_{H} \cdot \varepsilon$, where $X^B_{H}$ is the magnetic Hamiltonian vector field.

Then the following two assertions hold:

(i) If the one-form $\gamma : Q \to T^*Q$ satisfies the condition $d\gamma = -B$ with respect to $T\pi_Q : TT^*Q \to TQ$, then $\gamma$ is a solution of the Type I of Hamilton-Jacobi equation $T\gamma \cdot X^\gamma = X^B_{H} \cdot \gamma$.

(ii) The $\varepsilon$ is a solution of the Type II of Hamilton-Jacobi equation $T\gamma \cdot X^\varepsilon = X^B_{H} \cdot \varepsilon$, if and only if it is a solution of the equation $T\varepsilon \cdot X^B_{H,\varepsilon} = T\lambda \cdot X^B_{H} \cdot \varepsilon$, where $X^B_{H,\varepsilon} \in TT^*Q$ is the magnetic Hamiltonian vector field of the function $H \cdot \varepsilon : T^*Q \to \mathbb{R}$. Here the maps involved in the theorem are shown in the following Diagram-3.
Thus, Theorem 3.5 and Theorem 3.7 can be regarded as an extension of two types of Hamilton-Jacobi equation for a magnetic Hamiltonian system to that for the system with external force and control.

Remark 3.9 When \( B = 0 \), in this case the magnetic symplectic form \( \omega^B \) is just the canonical symplectic form \( \omega \) on \( T^*Q \), and the condition that the one-form \( \gamma : Q \to T^*Q \) satisfies the condition \( d\gamma = -B \) with respect to \( T\pi_Q : TT^*Q \to TQ \), becomes that \( \gamma \) is closed with respect to \( T\pi_Q : TT^*Q \to TQ \). Thus, from above Theorem 3.8, we can obtain Theorem 2.5 and Theorem 2.6 in Wang [17]. Thus, Theorem 3.8 can be regarded as an extension of two types of Hamilton-Jacobi equation for a canonical Hamiltonian system to that for the magnetic Hamiltonian system.

4 CMH-equivalence and the Solutions of Hamilton-Jacobi Equations

In the following we first give the definition of CMH-equivalence for the CMH systems, then prove that the solutions of the associated Hamilton-Jacobi equations leave invariant under the conditions of CMH-equivalence, if the associated magnetic Hamiltonian systems are equivalent. This result describes the relationship between the CMH-equivalence for the CMH systems and the solutions of the associated Hamilton-Jacobi equations.

For two given Hamiltonian systems \((T^*Q_i, \omega_i, H_i)\), \(i = 1, 2\), we say them to be equivalent, if there exists a diffeomorphism \( \varphi : Q_1 \to Q_2 \), such that their Hamiltonian vector fields \( X_{H_i}, i = 1, 2 \) satisfy the condition \( X_{H_1} \cdot \varphi^* = T(\varphi^*)X_{H_2} \), where the map \( \varphi^* = T^*\varphi \) : \( T^*Q_2 \to T^*Q_1 \) is the cotangent lifted map of \( \varphi \), and the map \( T(\varphi^*) : TT^*Q_2 \to TT^*Q_1 \) is the tangent map of \( \varphi^* \). From Marsden and Ratiu [10], we know that the condition \( X_{H_1} \cdot \varphi^* = T(\varphi^*)X_{H_2} \) is equivalent the fact that the map \( \varphi^* : T^*Q_2 \to T^*Q_1 \) is symplectic with respect to the canonical symplectic forms on \( T^*Q_i, i = 1, 2 \). In the same way, for two given magnetic Hamiltonian systems \((T^*Q_i, \omega^B_i, H_i)\), \(i = 1, 2\), we say them to be equivalent, if there exists a diffeomorphism \( \varphi : Q_1 \to Q_2 \), which is symplectic with respect to their magnetic symplectic forms, such that their magnetic Hamiltonian vector fields \( X^B_{H_i}, i = 1, 2 \) satisfy the condition \( X^B_{H_1} \cdot \varphi^* = T(\varphi^*)X^B_{H_2} \).

For two given CMH systems \((T^*Q_i, \omega^B_i, H_i, F_i, W_i)\), \(i = 1, 2\), we also want to define their equivalence, that is, to look for a diffeomorphism \( \varphi : Q_1 \to Q_2 \), such that \( X_{(T^*Q_1, \omega^B_1, H_1, F_1, W_1)} \cdot \varphi^* = T(\varphi^*)X_{(T^*Q_2, \omega^B_2, H_2, F_2, W_2)} \). But, it is worthy of noting that, when a CMH system is given, the force map \( F \) is determined, but the feedback control law \( u : T^*Q \to W \) could be chosen. In order to describe the feedback control law to modify the structure of the CMH system, the controlled magnetic Hamiltonian matching conditions and CMH-equivalence are induced as follows.

**Definition 4.1** (CMH-equivalence) Suppose that we have two CMH systems \((T^*Q_i, \omega^B_i, H_i, F_i, W_i)\), \(i = 1, 2\), we say them to be CMH-equivalent, or simply, \((T^*Q_1, \omega^B_1, H_1, F_1, W_1) \underset{CMH}{\sim} (T^*Q_2, \omega^B_2, H_2, F_2, W_2)\), if there exists a diffeomorphism \( \varphi : Q_1 \to Q_2 \), such that the following controlled magnetic Hamiltonian matching conditions hold:
CMH-1: The control subsets \( W_i, \ i = 1, 2 \) satisfy the condition \( W_1 = \varphi^*(W_2) \), where the map \( \varphi^* = T^*\varphi : T^*Q_2 \to T^*Q_1 \) is cotangent lifted map of \( \varphi \).

CMH-2: For each control law \( u_1 : T^*Q_1 \to W_1 \), there exists the control law \( u_2 : T^*Q_2 \to W_2 \), such that the two closed-loop dynamical systems produce the same dynamical vector fields, that is, 
\[
X(T^*Q_1, \omega^B_1, H_1, F_1, u_1) \cdot \varphi^* = T(\varphi^*)X(T^*Q_2, \omega^B_2, H_2, F_2, u_2),
\]
where \( T(\varphi^*) : T^*Q_2 \to T^*Q_1 \) is the tangent map of \( \varphi^* \).

From the expression (2.1) of the dynamical vector field of the CMH system and the condition 
\[
X(T^*Q_1, \omega^B_1, H_1, F_1, u_1) \cdot \varphi^* = T(\varphi^*)X(T^*Q_2, \omega^B_2, H_2, F_2, u_2),
\]
we have that 
\[
(X^B_{H_1} + \text{vlift}(F_1))X^B_{H_2} + \text{vlift}(u_1)X^B_{H_2} \cdot \varphi^* = T(\varphi^*)[X^B_{H_2} + \text{vlift}(F_2)X^B_{H_2} + \text{vlift}(u_2)X^B_{H_2}].
\]
By using the notation of vertical lift map of a vector along a fiber, for \( \alpha_x \in T^*_xQ_2, x \in Q_2 \), we have that 
\[
T(\varphi^*)\text{vlift}(F_2)X^B_{H_2}(\alpha_x) = T(\varphi^*)\text{vlift}((TF_2X^B_{H_2})(F_2(\alpha_x)), \alpha_x)
\]
\[
= \text{vlift}(T(\varphi^*) \cdot TF_2 \cdot T(\varphi^*)X^B_{H_2}(\varphi^*F_2\varphi^*(\varphi^*\alpha_i)), \varphi^*\alpha_i)
\]
\[
= \text{vlift}(T(\varphi^*F_2\varphi^*)X^B_{H_2}(\varphi^*F_2\varphi^*(\varphi^*\alpha_i)), \varphi^*\alpha_i)
\]
\[
= \text{vlift}(\varphi^*F_2\varphi^*)X^B_{H_2}(\varphi^*\alpha_i),
\]
where the map \( \varphi^* = (\varphi^{-1})^* : T^*Q_1 \to T^*Q_2 \). In the same way, we have that 
\[
T(\varphi^*)\text{vlift}(u_2)X^B_{H_2} = \text{vlift}(\varphi^*u_2\varphi^*)X^B_{H_2} \cdot \varphi^*.
\]
Note that \( \text{vlift}(F)^B = \text{vlift}(F)X^B_{H_2} \), and \( \text{vlift}(u)^B = \text{vlift}(u)X^B_{H_2} \), and hence we have that the explicit relation between the two control laws \( u_i \in W_i, i = 1, 2 \) in RCH-2 is given by
\[
(vlilt(u_1)^B - \text{vlift}(\varphi^*u_2\varphi^*)) \cdot \varphi^* = -X^B_{H_1} \cdot \varphi^* + T(\varphi^*)X^B_{H_2} + (-\text{vlift}(F_1)^B + \text{vlift}(\varphi^*F_2\varphi^*)) \cdot \varphi^*.
\]
From the above relation (4.1) we know that, when two CMH systems \( T^*Q_i, \omega^B_i, H_i, F_i, W_i \), \( i = 1, 2 \), are CMH-equivalent with respect to \( \varphi^* \), the associated magnetic Hamiltonian systems \( (T^*Q_i, \omega^B_i, H_i) \), \( i = 1, 2 \), may not be equivalent with respect to \( \varphi^* \).

On the other hand, note that the magnetic vector field \( X^0 = X^B_H - X_H \), from (2.3) and (4.1) we have that
\[
(vlilt(u_1)^B - \text{vlift}(\varphi^*u_2\varphi^*)) \cdot \varphi^* = -(X^B_{H_1} + X^0) \cdot \varphi^* + T(\varphi^*)X^B_{H_2} + (-\text{vlift}(F_1)^B + \text{vlift}(\varphi^*F_2\varphi^*)) \cdot \varphi^*.
\]
and hence we have that
\[
(X^B_{H_1} + \text{vlift}(F_1)^B + \text{vlift}(u_1)^B) \cdot \varphi^* = -X^B_{H_1} \cdot \varphi^* + T(\varphi^*)X^B_{H_2} + T(\varphi^*)X^0 + \text{vlift}(F_2)^B + \text{vlift}(u_2)^B.
\]
If the associated canonical Hamiltonian systems \( (T^*Q_i, \omega_i, H_i) \), \( i = 1, 2 \), are also equivalent with respect to \( \varphi^* \), that is, \( T(\varphi^*) \cdot X_{H_2} = X_{H_1} \cdot \varphi^* \). In this case, from (2.4) and (4.2), we have the following theorem.

**Theorem 4.2** Suppose that two CMH systems \( (T^*Q_i, \omega^B_i, H_i, F_i, W_i) \), \( i = 1, 2 \), are CMH-equivalent with respect to \( \varphi^* \), and the associated canonical Hamiltonian systems \( (T^*Q_i, \omega_i, H_i) \), \( i = 1, 2 \), are also equivalent with respect to \( \varphi^* \). Then we have the following fact that, if one system satisfies the magnetic vanishing condition, then another CMH-equivalent system must satisfy the associated magnetic vanishing condition.
Moreover, if considering the CMH-equivalence of the CMH systems, we can prove the following
Theorem 4.3, which states that the solutions of two types of Hamilton-Jacobi equations for the
CMH systems leave invariant under the conditions of CMH-equivalence, if the associated magnetic
Hamiltonian systems are equivalent.

**Theorem 4.3** Suppose that two CMH systems \((T^*Q_i, \omega_i^B, H_i, F_i, W_i), i = 1, 2,\) are CMH-equivalent
with an equivalent map \(\varphi : Q_1 \rightarrow Q_2,\) and the associated magnetic Hamiltonian systems \((T^*Q_i, \omega_i^B, H_i), i = 1, 2,\) are also equivalent with respect to \(\varphi^*\), under the hypotheses and notations of Theorem 3.5, 
Theorem 3.7, we have that

(i) If the one-form \(\gamma_2 : Q_2 \rightarrow T^*Q_2\) satisfies the condition that \(d\gamma_2 = -B_2\) with respect to
\(T\pi_{Q_2} : TT^*Q_2 \rightarrow TQ_2,\) then \(\gamma_1 = \varphi^* \cdot \gamma_2 \cdot \varphi : Q_1 \rightarrow T^*Q_1\) satisfies also the condition that
\(d\gamma_1 = -B_1\) with respect to \(T\pi_{Q_1} : TT^*Q_1 \rightarrow TQ_1,\) and hence it is a solution of the Type I of
Hamilton-Jacobi equation for the CMH system \((T^*Q_1, \omega_1^B, H_1, F_1, W_1).\) Vice versa;

(ii) If the symplectic map \(\varepsilon_2 : T^*Q_2 \rightarrow T^*Q_2\) with respect to \(\omega_2^B\) is a solution of the Type II of
Hamilton-Jacobi equation for the CMH system \((T^*Q_2, \omega_2^B, H_2, F_2, W_2),\) then \(\varepsilon_1 = \varphi^* \cdot \varepsilon_2 \cdot \varphi^* : T^*Q_1 \rightarrow T^*Q_1\) is a symplectic map with respect to \(\omega_1^B,\) and it is a solution of the Type II of
Hamilton-Jacobi equation for the CMH system \((T^*Q_1, \omega_1^B, H_1, F_1, W_1).\) Vice versa.

**Proof:** We first prove the assertion (i). If two given CMH systems \((T^*Q_i, \omega_i^B, H_i, F_i, W_i), i = 1, 2,\)
are CMH-equivalent with an equivalent map \(\varphi : Q_1 \rightarrow Q_2,\) from the definition of CMH-equivalence,
we know that for each control law \(u_i : T^*Q_1 \rightarrow W_1,\) there exists the control law \(u_2 : T^*Q_2 \rightarrow W_2,\)
such that the two closed-loop dynamical systems produce the same dynamical vector fields, that
is, \(\vec{X}_1 \cdot \varphi^* = T(\varphi^*) \cdot \vec{X}_2,\) where \(\vec{X}_i = X(T^*Q_i, \omega_i^B, H_i, F_i, u_i), i = 1, 2.\) From the following commutative
Diagram-4:

\[
\begin{array}{ccc}
Q_1 & \xrightarrow{\gamma_1} & T^*Q_1 \\
\varphi & \downarrow & \varphi^* \\
Q_2 & \xrightarrow{\gamma_2} & T^*Q_2 \\
\end{array}
\]

Diagram-4

we have that \(\gamma_1 = \varphi^* \cdot \gamma_2 \cdot \varphi, d\gamma_1 = \varphi^* \cdot d\gamma_2 \cdot \varphi, B_1 = \varphi^* \cdot B_2 \cdot \varphi,\) and \(T\varphi \cdot T\pi_{Q_1} \cdot T\varphi^* = T\pi_{Q_2}.\) For
\(x \in Q_1,\) and \(v, w \in TT^*Q_1,\) then \(\varphi(x) \in Q_2\) and \(T\varphi_*(v), T\varphi_*(w) \in TT^*Q_2.\) Since the one-form
\(\gamma_2 : Q_2 \rightarrow T^*Q_2\) satisfies the condition that \(d\gamma_2 = -B_2\) with respect to \(T\pi_{Q_2} : TT^*Q_2 \rightarrow TQ_2,\) then

\[
(d\gamma_2 + B_2)(T\pi_{Q_2} \cdot T\varphi_*(v), T\pi_{Q_2} \cdot T\varphi_*(w))(\varphi(x)) = 0.
\]

Thus,

\[
(d\gamma_1 + B_1)(T\pi_{Q_1}(v), T\pi_{Q_1}(w))(x)
\]

\[
= \varphi^* \cdot (d\gamma_2 + B_2) \cdot \varphi(T\pi_{Q_1}(v), T\pi_{Q_1}(w))(x)
\]

\[
= (d\gamma_2 + B_2)(T\varphi \cdot T\pi_{Q_1}(v), T\varphi \cdot T\pi_{Q_1}(w))(\varphi(x))
\]

\[
= (d\gamma_2 + B_2)(T\varphi \cdot T\pi_{Q_1} \cdot T\varphi^* \cdot T(\varphi^{-1})^*(v), T\varphi \cdot T\pi_{Q_1} \cdot T\varphi^* \cdot T(\varphi^{-1})^*(w))(\varphi(x))
\]

\[
= (d\gamma_2 + B_2)(T\pi_{Q_2} \cdot T\varphi_*(v), T\pi_{Q_2} \cdot T\varphi_*(w))(\varphi(x)) = 0,
\]

that is, the one-form \(\gamma_1 = \varphi^* \cdot \gamma_2 \cdot \varphi : Q_1 \rightarrow T^*Q_1\) satisfies the condition that \(d\gamma_1 = -B_1\) with respect to \(T\pi_{Q_1} : TT^*Q_1 \rightarrow TQ_1.\) Moreover, from Theorem 3.5 we know that, the one-form \(\gamma_2\) is
a solution of the Type I of Hamilton-Jacobi equation for the CMH system \((T^*Q_2, \omega_2^B, H_2, F_2, W_2),\)
that is, \( T\gamma_2 \cdot \dot{X}^{\gamma_2} = X^{B}_{H_2} \cdot \gamma_2 \), where \( \dot{X}^{\gamma_i} = T\pi_{Q_i} \cdot \dot{X}_i \cdot \gamma_i, \ i = 1, 2 \). Hence,

\[
T\gamma_1 \cdot \dot{X}^{\gamma_1} = T(\varphi^* \cdot \gamma_2 \cdot \varphi) \cdot T\pi_{Q_1} \cdot \dot{X}_1 \cdot \gamma_1
\]

\[
= T(\varphi^*) \cdot T\gamma_2 \cdot T\varphi \cdot T\pi_{Q_1} \cdot \dot{X}_1 \cdot (\varphi^* \cdot \gamma_2 \cdot \varphi)
\]

\[
= T(\varphi^*) \cdot T\gamma_2 \cdot T\varphi \cdot T\pi_{Q_1} \cdot (T(\varphi^*) \cdot \dot{X}_2) \cdot \gamma_2 \cdot \varphi
\]

\[
= T(\varphi^*) \cdot T\gamma_2 \cdot (T\pi_{Q_2} \cdot \dot{X}_2 \cdot \gamma_2) \cdot \varphi = T(\varphi^*) \cdot T\gamma_2 \cdot \dot{X}^{\varphi}\cdot \varphi
\]

\[
= T(\varphi^*) \cdot X^{B}_{H_2} \cdot \gamma_2 \cdot \varphi = X^{B}_{H_1} \cdot \gamma_1,
\]

where we have used that \( T(\varphi^*) \cdot X^{B}_{H_2} = X^{B}_{H_1} \cdot \varphi^* \), because the associated magnetic Hamiltonian systems \( (T^*Q_i, \omega^B_i, H_i), \ i = 1, 2 \), are equivalent with respect to \( \varphi^* \). Thus, the one-form \( \gamma_1 = \varphi^* \cdot \gamma_2 \cdot \varphi \) is a solution of the Type I of Hamilton-Jacobi equation for the CMH system \( (T^*Q_1, \omega^B_1, H_1, F_1, W_1) \). Note that the map \( \varphi : Q_1 \to Q_2 \) is a diffeomorphism, and \( \varphi^* : T^*Q_2 \to T^*Q_1 \) is a symplectic isomorphisms, vice versa. It follows that the assertion (i) of Theorem 4.3 holds.

Next, we prove the assertion (ii). From the following commutative Diagram-5:

\[
\begin{array}{ccc}
Q_1 & \xrightarrow{\gamma_1} & T^*Q_1 & \xrightarrow{\varepsilon_1} & T^*Q_1 & \xrightarrow{\dot{X}_1} & TT^*Q_1 & \xrightarrow{T\pi_{Q_1}} & TQ_1 \\
\varphi \downarrow & & \varphi \downarrow & & \varphi \uparrow & & T\varphi & \leftarrow & T\varphi \\
Q_2 & \xrightarrow{\gamma_2} & T^*Q_2 & \xrightarrow{\varepsilon_2} & T^*Q_2 & \xrightarrow{\dot{X}_2} & TT^*Q_2 & \xrightarrow{T\pi_{Q_2}} & TQ_2
\end{array}
\]

Diagram-5

we have that \( \varepsilon_1 = \varphi^* \cdot \varepsilon_2 \cdot \varphi_* : T^*Q_1 \to T^*Q_1 \). Since \( \varepsilon_2 : T^*Q_2 \to T^*Q_2 \) is symplectic with respect to \( \omega^B_i \), then for \( x \in Q_1, v, w \in TT^*_x Q_1 \), and \( \varphi(x) \in Q_2, T\varphi_*(v), T\varphi_*(w) \in TT^*_{\varphi(x)} Q_2 \), we have that \( \varepsilon_2^* \cdot \omega^B_2 \cdot (T\varphi_*(v), T\varphi_*(w))(\varphi(x)) = \omega^B_2 \cdot (T\varphi_*(v), T\varphi_*(w))(\varphi(x)). \) Note that the associated magnetic Hamiltonian systems \( (T^*Q_i, \omega^B_i, H_i), \ i = 1, 2 \), are equivalent with respect to \( \varphi^* \), and hence \( \varphi^* : T^*Q_2 \to T^*Q_1 \) is symplectic with respect to their magnetic symplectic forms, that is, \( (\varphi^*)^* \cdot \omega^B_2 \cdot (v, w)(x) = \omega^B_2 \cdot (T\varphi_*(v), T\varphi_*(w))(\varphi(x)) \), then we have that

\[
\varepsilon_1^* \cdot \omega^B_1 \cdot (v, w)(x) = (\varphi^* \cdot \varepsilon_2 \cdot \varphi_*)^* \cdot \omega^B_1 \cdot \varphi^*(v, w)(x)
\]

\[
= (\varphi^*)^* \cdot \varepsilon_2^* \cdot \omega^B_2 \cdot (T\varphi_*(v), T\varphi_*(w))(\varphi(x)) = ((\varphi^{-1})^*)^* \cdot \omega^B_2 \cdot (T\varphi_*(v), T\varphi_*(w))(\varphi(x))
\]

\[
= \omega^B_1 \cdot (T(\varphi^{-1})_*) \cdot T\varphi_*(v), T(\varphi^{-1})_* \cdot T\varphi_*(w))(\varphi^{-1} \cdot \varphi(x)) = \omega^B_1 \cdot (v, w)(x),
\]

that is, the map \( \varepsilon_1 = \varphi^* \cdot \varepsilon_2 \cdot \varphi_* : T^*Q_1 \to T^*Q_1 \) is symplectic map with respect to \( \omega^B_1 \). Moreover, because the symplectic map \( \varepsilon_2 : T^*Q_2 \to T^*Q_2 \) is a solution of the Type II of Hamilton-Jacobi equation for the CMH system \( (T^*Q_2, \omega^B_2, H_2, F_2, W_2) \), that is, \( T\gamma_2 \cdot \dot{X}_2^{\varepsilon_2} = X^{B}_{H_2} \cdot \varepsilon_2 \), where \( \dot{X}_2^{\varepsilon_i} = T\pi_{Q_i} \cdot \dot{X}_i \cdot \varepsilon_i, \ i = 1, 2 \). Hence, we have that

\[
T\gamma_1 \cdot \dot{X}^{\varepsilon_1} = T(\varphi^* \cdot \gamma_2 \cdot \varphi) \cdot T\pi_{Q_1} \cdot \dot{X}_1 \cdot \varepsilon_1
\]

\[
= T(\varphi^*) \cdot T\gamma_2 \cdot T\varphi \cdot T\pi_{Q_1} \cdot \dot{X}_1 \cdot (\varphi^* \cdot \varepsilon_2 \cdot \varphi_*)
\]

\[
= T(\varphi^*) \cdot T\gamma_2 \cdot T\varphi \cdot T\pi_{Q_1} \cdot (T(\varphi^*) \cdot \dot{X}_2) \cdot \varepsilon_2 \cdot \varphi_*
\]

\[
= T(\varphi^*) \cdot T\gamma_2 \cdot (T\pi_{Q_2} \cdot \dot{X}_2 \cdot \varepsilon_2) \cdot \varphi_* = T(\varphi^*) \cdot T\gamma_2 \cdot \dot{X}^{\varepsilon_2} \cdot \varphi_*
\]

\[
= T(\varphi^*) \cdot X^{B}_{H_2} \cdot \varepsilon_2 \cdot \varphi_* = X^{B}_{H_1} \cdot \varepsilon_2 \cdot \varphi_* = X^{B}_{H_1} \cdot \varepsilon_1,
\]

that is, the symplectic map \( \varepsilon_1 = \varphi^* \cdot \varepsilon_2 \cdot \varphi_* \) is a solution of the Type II of Hamilton-Jacobi equation for the CMH system \( (T^*Q_1, \omega^B_1, H_1, F_1, W_1) \). In the same way, because the map \( \varphi : Q_1 \to Q_2 \) is a diffeomorphism, and \( \varphi^* : T^*Q_2 \to T^*Q_1 \) is a symplectic isomorphisms, vice versa. Hence we prove
the assertion (ii) of Theorem 4.3. ■

The theory of controlled mechanical system is a very important subject, its research gathers together some separate areas of research such as mechanics, differential geometry and nonlinear control theory, etc., and the emphasis of this research on geometry is motivated by the aim of understanding the structure of equations of motion of the system in a way that helps both analysis and design. Thus, it is natural to study the controlled mechanical systems by combining with the analysis of dynamical systems and the geometric reduction theory of Hamiltonian and Lagrangian systems. Following the theoretical development of geometric mechanics, a lot of important problems about this subject are being explored and studied, see León and Wang [5], Marsden et al. [11], Ratiu and Wang [14], Wang [15–20], and Wang and Zhang [21]. These research works reveal from the geometrical point of view the internal relationships of the geometrical structures of phase spaces, symmetric reductions, constraints, dynamical vector fields and controls of a mechanical system and its regular reduced systems. In particular, it is the key thought of the researches of geometrical mechanics of the professor Jerrold E. Marsden to explore and reveal the deeply internal relationship between the geometrical structure of phase space and the dynamical vector field of a mechanical system. It is also our goal of pursuing and inheriting.

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