Research Article

Some Results on Strongly Cesáro Ideal Convergent Sequence Spaces

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Some algebraic properties of Cesáro ideal convergent sequence spaces, $C_I$ and $C_{I0}$, are studied in this article and some inclusion relations on these spaces are established.

1. Introduction

Consider the space $\omega = \{x = (x_k): x_k \in \mathbb{R} \text{ or } \mathbb{C}\}$ of all real and complex sequences, where $\mathbb{R}$ and $\mathbb{C}$ are, respectively, the sets of all real and complex numbers.

Suppose that $\ell_\infty$, $c$, and $c_0$ are the linear spaces of bounded, convergent, and null sequences, respectively, normed by

$$\|x\|_\infty = \sup_k |x_k|, \quad \text{where, } k \in \mathbb{N},$$ (1)

$\mathbb{N}$ being the set of all natural numbers.

A sequence space $x = (x_k)$ of complex numbers is said to be $(C, 1)$ summable to $L \in C$ if for $\rho_k = 1/k \sum_{i=1}^{k} x_i$, $\lim_k \rho_k = L$. The sequence $(C, 1)$ is also called Cesáro summable sequence of complex numbers over $C$. Let us denote by $C_1$ the linear space of all $(C, 1)$ summable sequences of complex numbers over $C$, i.e.,

$$C_1 = \left\{x = (x_k) \in \omega: \frac{1}{k} \sum_{i=1}^{k} x_i \in c \right\}.$$ (2)

Hardy and Littlewood [1] initiated the notion of strong Cesáro convergence for real numbers which is defined as follows.

A sequence $(x_k)$ on a normed space $(X, \| \cdot \|)$ is said to be strongly Cesáro convergent to $L$ if

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \|x_k - L\| = 0.$$ (3)

In [2–6], the authors have extended the notion of strong Cesáro convergence in various fields. In 1951, Fast [7] introduced the term statistical convergence, while Steinhaus [8] independently introduced the term “ordinary and asymptotic convergences.”

Later on, Fridy [9, 10] also studied the statistical convergence and he linked it with the summability theory. Kostyrko et al. [11] gave the concept of ideal convergence (I-convergence) which was indeed a generalization of statistical convergence. Salat et al. [12] studied some properties of I-convergence, and further investigations in this field are done by Khan [13], Tripathy and Esi [14], Tripathy and Hazarika [15], and many others.

In this article, further interesting properties of Cesáro Ideal Convergent Sequences are established and a few inclusion relations are also proved.

2. Definitions of the Terms Used

Let us first present some definitions and notions that are required in the sequel.

(i) A family of subsets $I$ of $\mathbb{N}$ is called an ideal set in $\mathbb{N}$.
Remark 1. For every ideal $I$, there is a filter $F(I)$ (associated with $I$) defined as follows:

$$F(I) = \left\{ P \subseteq \mathbb{N} : \frac{\mathbb{N}}{p \notin I} \right\}.$$  
(4)

A sequence $(x_k) \in X$ is said to be $I$-convergent to a number $L$ if, for every $\varepsilon > 0$, the set $\{ x = (x_k) \in X : |x_k - L| \geq \varepsilon \} \in I$. In this case, we write $I$-lim $x_k = L$.

A sequence $x_k \in I$ is said to be $I$-Cauchy if, for every $\varepsilon > 0$, there exists a number $m = m(\varepsilon)$ such that

$$\{ x = x_k \in X : \{|x_n - x_m| \geq \varepsilon \} \in I \}. $$  
(5)

Let $I$ be the class of all finite subsets of $\mathbb{N}$. If $I = \{0\}$, then $I$ is admissible ideal set in $\mathbb{N}$.

A sequence space $X$ is said to be solid (normal) if $(a_k,x_k) \in X$ whenever $(x_k) \in X$ and $(a_k)$ is a sequence of scalars with $|a_k| \leq 1$, for all $k \in \mathbb{N}$.

A sequence space $X$ is a Sequence Algebra if, for every $(x_k), (y_k) \in X$, $(s_k,y_k) \in X$.

Let $K = \{k_1 < k_2 < k_3, \ldots \} \subseteq \mathbb{N}$ and $X$ be a sequence space. A $K$-step space of $X$ is a sequence space $\lambda_K^X = \{ (x_k) \in \omega : (x_k) \in X \}$.

A canonical preimage of a sequence $(x_k) \in \lambda_K^X$ is a sequence $(y_k) \in \omega$ defined by

$$y_k = \begin{cases} x_k, & \text{if } k \in K, \\ 0, & \text{otherwise}, \end{cases} $$  
(6)

A sequence space is monotone if it contains the canonical preimages of its step spaces.

3. Result

A canonical preimage of a step space $\lambda_K^X$ is a set of preimages of all elements in $\lambda_K^X$, i.e., $y$ is in the canonical preimage of $\lambda_K^X$ if and only if $y$ is the canonical preimage of some $x \in \lambda_K^X$.

Let $X$ and $Y$ be two normed linear spaces. An operator $T: X \rightarrow Y$ is known as a compact linear operator if [16].

(b) If, for every bounded subset $D$ of $X$, the image $M(D)$ is relatively compact, i.e., the closure $\overline{T(D)}$ is compact

Lemma 1 (see [12]). Every solid space is monotone.

Lemma 2 (see [12]). Let $K \in F(I)$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \neq I$.

Lemma 3 (see [11]). Let $I \subseteq 2^\mathbb{N}$ and $M \subseteq \mathbb{N}$. If $M \notin I$, then $M \cap K \neq I$.

4. Main Results

Let us first define $C^I$, the space of all Cesáro ideal convergent sequences and $C^I_0$ the space of all Cesáro ideal null sequences which are given as follows:

$$C^I = \left\{ x = (x_k) \in \omega : I - \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \| x_k - L \| = 0, \text{ for some } L \in C \right\},$$
(7)

$$C^I_0 = \left\{ x = (x_k) \in \omega : I - \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \| x_k \| = 0 \right\}.$$  

Theorem 1. The sequence spaces $C^I$ and $C^I_0$ are linear.

Proof. Assume that $x = (x_k), y = (y_k) \in C^I$. Then, one has

$$I - \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \| x_k - L_1 \| = 0, \text{ for some } L_1 \in C,$$  
(8)

$$I - \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \| y_k - L_2 \| = 0, \text{ for some } L_2 \in C.$$  

Let

$$A_1 = \left\{ k \in N : \frac{1}{n} \sum_{k=1}^{n} \| x_k - L_1 \| \right\},$$  
(9)

$$A_2 = \left\{ k \in N : \frac{1}{n} \sum_{k=1}^{n} \| y_k - L_2 \| \right\}. $$  
(10)

Let $\alpha$ and $\beta$ be some scalars.

By using the properties of norm, one can easily see that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \| (\alpha x_k + \beta y_k) - (\alpha L_1 + \beta L_2) \| \leq \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |\alpha| \| x_k - L_1 \| + \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |\beta| \| y_k - L_2 \|.$$  
(11)

Then, from (9) and (10), we have for each $\varepsilon > 0$,}
Let \( x = (x_k) \in \mathcal{C} \) be any sequence. Then, \( \mathcal{C}_0 \subset \mathcal{C} \).

**Proof.** It can be easily observed.

**Theorem 3.** A sequence \( x = (x_k) \in \mathcal{C} \) is \( I \)-convergent if and only if, for every \( \varepsilon > 0 \), there exists \( l = l(\varepsilon) \in \mathbb{N} \) such that

\[
\left\{ k \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \left| x_k - y_k \right| < \varepsilon \right\} \subset F(I).
\]  
(13)

**Proof.** Suppose that \( x = (x_k) \in \mathcal{C} \). Therefore, \( I - \lim_{n \to \infty} 1/n \sum_{k=1}^{n} \| x_k - L \| = 0 \). Then, for all \( \varepsilon > 0 \) the set

\[
C_\varepsilon = \left\{ k \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \| x_k - L \| < \varepsilon \right\} \subset F(I).
\]  
(14)

Fix an \( l(\varepsilon) \in C_\varepsilon \). Then, we have

\[
\frac{1}{n} \sum_{k=1}^{n} \| x_k - x_l \| \leq \frac{1}{n} \sum_{k=1}^{n} \| x_k - L \| + \frac{1}{n} \sum_{k=1}^{n} \| x_l - L \| < \varepsilon \frac{1}{2} + \varepsilon \frac{1}{2} = \varepsilon,
\]  
(15)

which holds for all \( k \in C_\varepsilon \). Hence,

\[
\left\{ k \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \| x_k - x_l \| < \varepsilon \right\} \subset F(I).
\]  
(16)

Conversely, suppose that, for all \( \varepsilon > 0 \), the set

\[
\left\{ k \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \| x_k - x_l \| < \varepsilon \right\} \subset F(I).
\]  
(17)

Then, for every \( \varepsilon > 0 \), we have

\[
B_\varepsilon = \left\{ k \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \| x_k \| \in \left[ \frac{1}{n} \sum_{k=1}^{n} \| x_l \| - \varepsilon, \frac{1}{n} \sum_{k=1}^{n} \| x_l \| + \varepsilon \right] \right\} \subset F(I),
\]  
(18)

Let, \( P_\varepsilon = \left[ \frac{1}{n} \sum_{k=1}^{n} \| x_k \| - \varepsilon, \frac{1}{n} \sum_{k=1}^{n} \| x_l \| + \varepsilon \right] \). For fixed \( \varepsilon > 0 \), one has \( B_\varepsilon \in F(I) \) as well as \( B_\varepsilon \cap B_\varepsilon \in F(I) \). Hence, \( B_\varepsilon \cap B_\varepsilon \not\subset F(I) \).

This implies that \( B_\varepsilon \cap B_\varepsilon \not\subset \phi \), that is,

\[
\left\{ k \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \| x_k \| \not\in P \right\} \subset F(I).
\]  
(19)

That is \( \text{diam } P \leq \text{diam } P_\varepsilon \), where the \( \text{diam } P \) denotes the length of the interval of \( P \).

In this way, by induction, one obtains the sequence of closed intervals:

\[
P_\varepsilon = J_0 \supset J_1 \supset J_2 \supset \cdots,\ V_\varepsilon \supset V_2 \supset V, \cdots, \]
(20)

with the property that \( \text{diam } J_k \leq 1/2 \text{diam } J_{k-1} \) for \( k = 1, 2, 3, \ldots, \), and

\[
\left\{ k \in \mathbb{N} : \frac{1}{n} \sum_{k=1}^{n} \| x_k \| \in J_k \right\} \subset F(I),
\]  
(21)

for \( k = 1, 2, 3, \ldots, \). Then, there exists a \( L \in \bigcap J_k \) such that \( L = I - \lim_{n \to \infty} 1/n \sum_{k=1}^{n} \| x_k \| \) showing that \( x = (x_k) \in \mathcal{C} \) is \( I \)-convergent. Hence, the result holds.

**Theorem 4.** The space \( \mathcal{C}_0 \) is solid and monotone.

**Proof.** Let \( (x_k) \in \mathcal{C}_0 \), be any element. Then, one has

\[
\left\{ k \in \mathbb{N} : I - \lim_{n \to \infty} 1/n \sum_{k=1}^{n} \| x_k \| = 0 \right\}.
\]  
(22)

Let \( (\alpha_k) \) be a sequence of scalars such that \( |\alpha_k| \leq 1 \), for all \( k \in \mathbb{N} \), and hence \( 1/n \sum_{k=1}^{n} |\alpha_k| \leq 1 \).

Then, the result that \( \mathcal{C}_0 \) (is solid) follows from the above equation and inequality:

\[
\frac{1}{n} \sum_{k=1}^{n} \| \alpha_k x_k \| = \frac{1}{n} \sum_{k=1}^{n} |\alpha_k| \frac{1}{n} \sum_{k=1}^{n} \| x_k \| \leq \frac{1}{n} \sum_{k=1}^{n} \| x_k \|,
\]  
(23)

for all \( k \in \mathbb{N} \).

The space \( \mathcal{C}_0 \) is monotone which follows from Lemma 1. Hence, \( \mathcal{C}_0 \) is solid and monotone.

**Theorem 5.** The space \( \mathcal{C}_0 \) is neither solid nor monotone.

**Proof.** For this theorem, we provide a counter example for the proof.

**5. Counter Example**

Let \( I = I_1 \) and consider the \( k \)-step \( x_k \) of \( \chi \) defined as follows.

Let \( (x_k) \in \chi \) and let \( (y_k) \in x_k \) be such that

\[
y_k = \begin{cases} x_k, & \text{if } k \text{ is even,} \\ 0, & \text{otherwise.} \end{cases}
\]  
(24)

Let us consider the sequence \( (x_k) \) defined by \( x_k = 1 \) for all \( k \in \mathbb{N} \). Then, \( (x_k) \in \mathcal{C} \), but its \( K \)-step preimages do not belong to \( \mathcal{C} \). Thus, \( (x_k) \in \mathcal{C} \) is not monotone.

Hence, \( (x_k) \in \mathcal{C} \) is not solid.
Theorem 6. Let \( x = (x_k) \) and \( y = (y_k) \) be two sequences in such a way that \( T(x \cdot y) = T(x)T(y) \). Then, the space \( C^\beta \) and \( C_\delta^\beta \) are sequence algebras.

Proof. Let \( x = (x_k) \) and \( y = (y_k) \) be two elements of \( C^\beta \) with \( T(x \cdot y) = T(x)T(y) \).

For every \( \varepsilon > 0 \) select \( \beta > 0 \) in such a way that \( \varepsilon < \beta \), then

\[
\frac{1}{n} \sum_{k=1}^{n} \| T(x_k, y_k) - L_1 L_2 \| = \frac{1}{n} \sum_{k=1}^{n} \| T(x_k)T(y_k) - L_1 L_2 \| \\
= \frac{1}{n} \sum_{k=1}^{n} \| T(x_k)T(y_k)-L_1 T(y_k)+L_1 T(y_k)-L_1 L_2 \| \\
\leq \frac{1}{n} \sum_{k=1}^{n} \| T(y_k) \| + \frac{1}{n} \sum_{k=1}^{n} \| T(x_k) - L_1 \| + |L_1| \frac{1}{n} \sum_{k=1}^{n} \| T(y_k) - L_2 \| < \frac{\varepsilon^2}{2\beta} + |L_1| \frac{\varepsilon}{2|L_1|} < \varepsilon.
\]

Therefore, the set

\[
\left\{ k \in N: \frac{1}{n} \sum_{k=1}^{n} \| T(x_k, y_k) - L_1 L_2 \| \geq \varepsilon \right\} \in I.
\]  

(28)

Thus, \( (x_k), (y_k) \in \ell^\varepsilon \text{Ces}. \) Hence, \( C^\beta \) is a sequence algebra. On the similar manner, one can prove that space \( C_\delta^\beta \) is also sequence algebra. \( \square \)

Data Availability

The data used to support the findings of the study are obtained from the author upon request.

Conflicts of Interest

The author declares no conflicts of interest.

Authors’ Contributions

The author read and approved the final manuscript.

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References

[1] G. H. Hardy and J. E. Littlewood, “Sur la série de fourier d’une fonction à carré sommable,” *Comptes rendus de l’Académie des Sciences, Paris*, vol. 156, pp. 1307–1309, 1913.
[2] H. Nakano, “Concave modulars,” *Journal of the Mathematical Society of Japan*, vol. 5, no. 1, pp. 29–49, 1953.
[3] H. Şengül, “Some Cesàro-type summability spaces defined by a modulus function of order \( (\alpha, \beta) \),” *Communications Faculty of Sciences University of Ankara Series A1-Mathematics and Statistics*, vol. 66, no. 2, pp. 80–90, 2017.
[4] I. J. Maddox, “Sequence spaces defined by a modulus,” *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 100, no. 1, pp. 161–166, 1986.
[5] M. Et and H. Şengül, “Some cesàro-type summability spaces of order \( \alpha \) and lacunary statistical convergence of order \( \alpha \),” *Filomat*, vol. 28, no. 8, pp. 1593–1602, 2014.
[6] W. H. Ruckle, “FK spaces in which the sequence of coordinate vectors is bounded,” *Canadian Journal of Mathematics*, vol. 25, no. 5, pp. 973–978, 1973.
[7] H. Fast, “Sur la convergence statistique,” *Colloquium Mathematicum*, vol. 2, no. 3–4, pp. 241–244, 1951.
[8] H. Steinhaus, “Sur la convergence ordinaire et la convergence asymptotique,” *Colloquium Mathematicum*, vol. 2, no. 1, pp. 73–74, 1951.
[9] J. A. Fridy, “On statistical convergence,” *Analysis*, vol. 5, pp. 301–313, 1985.
[10] J. A. Fridy, “Statistical limit points,” *Proceedings of the American Mathematical Society*, vol. 118, no. 4, p. 1187, 1993.
[11] P. Kostyrko, T. Šalát, and W. Wilczynski, “\( I \)-convergent,” *Real Analysis Exchange*, vol. 26, no. 2, pp. 669–686, 2000.
[12] T. Salat, B. C. Tripathy, and M. Ziman, “On some properties of \( I \)-convergence,” *Tatra Mountains Mathematical Publications*, vol. 28, pp. 279–286, 2004.
[13] V. A. Khan, “On a sequence space defined by a modulus function,” *Southeast Asian Bulletin of Mathematics*, vol. 29, pp. 747–753, 2005.
[14] B. C. Tripathy and A. Esi, “A new type of sequence spaces,” *International Journal of Environmental Science and Technology*, vol. 1, no. 1, pp. 11–14, 2006.
[15] B. C. Tripathy and B. Hazarika, “Paranormed \( I \)-convergent sequence spaces,” *Mathematica Slovaca*, vol. 59, pp. 485–494, 2009.
[16] E. Kreyszig, *Introductory Functional Analysis with Applications*, Vol. 1, Wiley, New York, NY, USA, 1978.