SEMICSORED PRODUCTS AND REFLEXIVITY

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Abstract. Given a w*-closed unital algebra $A$ acting on $H_0$ and a contractive w*-continuous endomorphism $\beta$ of $A$, there is a w*-closed (non-selfadjoint) unital algebra $Z_+\times_\beta A$ acting on $H_0 \otimes \ell^2(\mathbb{Z}_+)$, called the w*-semicrossed product of $A$ with $\beta$. We prove that $Z_+\times_\beta A$ is a reflexive operator algebra provided $A$ is reflexive and $\beta$ is unitarily implemented, and that $Z_+\times_\beta A$ has the bicommutant property if and only if so does $A$. Also, we show that the w*-semicrossed product generated by a commutative C*-algebra and a *-endomorphism is reflexive.

Introduction

As is well known, to construct the C*-crossed product of a unital C*-algebra $C$ by a *-isomorphism $\alpha : C \to C$, we begin with the Banach space $\ell^1(\mathbb{Z}, C, \alpha)$ which is the closed linear span of the monomials $\delta_n \otimes x$, $n \in \mathbb{Z}$, $x \in C$, under the norm $|\sum_{n=-k}^k \delta_n \otimes x_n|_1 = \sum_{n=-k}^k \|x_n\|_C$, equipped with the (isometric) involution $(\delta_n \otimes x)^* = \delta_{-n} \otimes a^{-n}(x^*)$. Now, there are two “natural” ways to define multiplication in $\ell^1(\mathbb{Z}, C, \alpha)$: either the left multiplication $(\delta_n \otimes x) *_l (\delta_m \otimes y) = \delta_{n+m} \otimes a^m(x)y$, or the right one $(\delta_n \otimes x) *_r (\delta_m \otimes y) = \delta_{n+m} \otimes xa^m(y)$. Then the corresponding algebras are isometrically *-isomorphic via the map $\Psi(\delta_n \otimes x) = \delta_{-n} \otimes a^{-n}(x)$. We can see that $(\ell^1(\mathbb{Z}, C, \alpha)_l)^{opp} = \ell^1(\mathbb{Z}, C^{opp}, \alpha)_r$, where for an algebra $B$, $B^{opp}$ is the space $B$ along with the multiplication $x \otimes y := yx$; hence, in case $C$ is commutative, each algebra is the opposite of the other. The left and right crossed product are the completion of the corresponding involutive Banach algebras under a universal norm induced by the $| \cdot |_1$-contractive *-representations (hence, they are C*-algebras characterized by a universal property) and the map $\Psi$ extends to a C*-isomorphism. Moreover, it can be proved that the crossed product is *-isomorphic to the reduced crossed product $C^\pi_l(C)$, i.e. the norm closure of the range of the left regular representation, and thus we end up with just one object to which we refer as the

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crossed product of the dynamical system \((\mathcal{C}, \alpha)\). The key fact is that there is a bijection between the \(|·|_1\)-contractive \(*\)-representations of each of these \(\ell^1\)-algebras and the (left or right) covariant unitary pairs (see section 1).

If we wish to construct a non-selfadjoint analogue, we can see that there are more possibilities. For example, Peters defined the semicrossed product as the completion of the Banach algebra \(\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l\) under the universal norm that arises from the left covariant isometric pairs and examined the case when \(\alpha\) is an injective \(*\)-endomorphism of \(\mathcal{C}\). He proved that this semicrossed product embeds isometrically in a crossed product (see [11]) and, for the commutative case, that this crossed product is the \(C^*\)-envelope of the semicrossed product (see [12]).

In section 1 we use an alternative definition using “sufficiently many” homomorphisms of the Banach algebra \(\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l\) (see also [3]). The advantage is that there is a bijection between the left covariant contractive pairs and the homomorphisms of the Banach algebra \(\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l\). Moreover, there is a duality between the left covariant contractive pairs and the right covariant contractive pairs, which induce the homomorphisms of the Banach algebra \(\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r\); hence, we get similar results for the right version. Also, using a dilation theorem of [9], we can see that this definition gives the one in [11]. If we consider the maximal operator space structure, then the semicrossed products are operator algebras with a universal property that characterizes them up to completely isometric isomorphism. In theorem 1.4 we prove that the semicrossed product is independent of the way \(\mathcal{C}\) is (faithfully) represented and in theorem 1.5 we prove that in case \(\alpha\) is a \(*\)-isomorphism, its \(C^*\)-envelope is exactly the crossed product. So, in order to define a \(w^*\)-analogue of the semicrossed product that arises by a \(w^*\)-continuous contractive endomorphism \(\beta\) of a \(w^*\)-closed subalgebra \(\mathcal{A}\) of some \(\mathcal{B}(H_0)\) (for example, a von Neumann algebra), either we take the \(w^*\)-closed linear span of a non-selfadjoint left regular representation or the \(w^*\)-closed linear span of the analytic polynomials of the von Neumann crossed product, depending on the properties of \(\beta\).

In section 2 we analyze the properties of the \(w^*\)-semicrossed product, in case \(\beta\) is unitarily implemented. First of all, we study the connection between the semicrossed product and the \(w^*\)-tensor product \(\mathcal{A} \overline{\otimes} \mathcal{T}\), where \(\mathcal{T}\) is the algebra of the analytic Toeplitz operators, and give an example when these two algebras are incomparable. A main result of this section is the reflexivity of the \(w^*\)-semicrossed product, when \(\mathcal{A}\) is reflexive. Recall that a subspace \(\mathcal{S} \subseteq \mathcal{B}(H)\) is reflexive if it coincides with its reflexive cover, namely \(\text{Ref}(\mathcal{S}) = \{T \in \mathcal{B}(H) : T\xi \in \mathcal{S} \text{ for all } \xi \}\).
semicrossed products and reflexivity

Sξ, for all ξ ∈ H} (see [7]); unlike [7], we will call S hereditarily reflexive if every w*-closed subspace of S is reflexive. As a consequence we have that, when a unitary implementation condition holds, the w*-closed image of ṯπ induced by a representation (H₀, π) of C is reflexive. Also, we get several known results as applications. As another main result, we prove that the w*-semicrossed product is the commutant of a w*-semicrossed product and is its own bicommutant if and only if the same holds for A.

In the last section we consider the semicrossed product of a commutative C*-algebra C(K) with a continuous map φ : K → K. As observed in theorem 1.4, the representations induced by a character of C(K), say evt, t ∈ K, suffice to obtain the norm of the semicrossed product and play a significant role for its study. First, we show that the w*-closure of such representations is always reflexive; in fact, it has the form (T₀P₀) ⊕ (T₀P₀) ⊕ · · · ⊕ (T₀P₀), where T is the algebra of lower triangular operators in B(ℓ²(Z⁺)), T is the algebra of analytic Toeplitz operators and P₀, P₀, . . . P₀ some projections determined by the orbit of the point t ∈ K.

In what follows we use standard notation, as in [4] for example. Z⁺ = {0, 1, 2, . . .} and all infinite sums are considered in the strong-convergent sense. Throughout, we use the symbol v for the unilateral shift on B(ℓ²(Z⁺)), given by v(en) = en+1. A useful tool for the proofs in sections 2 and 3 is a Féjer-type Lemma; consider the unitary action of T on H = H₀ ⊗ ℓ²(Z⁺) induced by the operators Uₙ, s ∈ R, given by Uₙ(ξ ⊗ eₙ) = eᵢₙsξ ⊗ eₙ. For every T ∈ B(H) and every m ∈ Z we define the “m-Fourier coefficient”

\[ G_m(T) = \int_0^{2\pi} U_s T U_s^* e^{-imt} ds \]

the integral taken as the w*-limit of Riemann sums. If we set σₗ(T)(t) = \(\frac{1}{2\pi} \sum_{n=0}^{t} \sum_{m=-n}^{n} G_m(T) \exp(amt)\), then σₗ(T)(0) \(\xrightarrow{w^*}\) T. Note that Gₘ(·) is w*-continuous for every m ∈ Z.

Now, for every κ, λ ∈ Z⁺, and T ∈ B(H₀) let the “matrix elements” Tₖ,λ ∈ B(H₀) be defined by ⟨Tₖ,λξ, η⟩ = ⟨T(ξ ⊗ eₙ), η ⊗ eₙ⟩, ξ, η ∈ H₀; then we can write the Fourier coefficients explicitly by the formula

\[ G_m(T) = \begin{cases} 
V_m(\sum_{n≥0} T_{m,n,n} ⊗ p_n) & , \text{when } m ≥ 0, \\
(\sum_{n≥0} T_{n,-m+n} ⊗ p_n)(V^*)^{-m} & , \text{when } m < 0,
\end{cases} \]
where $V = 1_{H_0} \otimes v$. For simplicity, we define the diagonal matrices

$$T_{(m)} = \begin{cases} \sum_{n \geq 0} T_{m+n,n} \otimes p_n & \text{, when } m \geq 0, \\ \sum_{n \geq 0} T_{n,-m+n} \otimes p_n & \text{, when } m < 0, \end{cases}$$

Note that the sums converge in the $w^*$-topology as well, since the partial sums are uniformly bounded by $\|T\|$. Hence, $G_m(T)$ is the $m$-diagonal of $T$, when we view $H$ as the $\ell^2$-sum of copies of $H_0$.

1. Semicrossed products of $C^*$-algebras

Let $C$ be a unital $C^*$-algebra and $\alpha : C \to C$ a $*$-morphism; define $\ell^1(\mathbb{Z}_+, C, \alpha)$ to be the closed linear span of the monomials $\delta_n \otimes x$, $n \in \mathbb{Z}_+$, $x \in C$, under the norm

$$\left\| \sum_{n=0}^k \delta_n \otimes x_n \right\|_1 = \sum_{n=0}^k \|x_n\|_C.$$

We endow $\ell^1(\mathbb{Z}_+, C, \alpha)$ either with the left multiplication $(\delta_n \otimes x) \ast_l (\delta_m \otimes y) = \delta_{n+m} \otimes a^m(x)y$, or with the right one $(\delta_n \otimes x) \ast_r (\delta_m \otimes y) = \delta_{n+m} \otimes xa^m(y)$, and denote the corresponding Banach algebras by $\ell^1(\mathbb{Z}_+, C, \alpha)_l$ and $\ell^1(\mathbb{Z}_+, C, \alpha)_r$, respectively. One can see that $(\ell^1(\mathbb{Z}_+, C, \alpha)_l)^{opp}$ is exactly $\ell^1(\mathbb{Z}_+, C^{opp}, \alpha)_r$, where, if $B$ is an algebra, $B^{opp}$ is the space $B$ with the multiplication $x \otimes y := yx$. Thus, in case $C$ is commutative, each algebra is the opposite of the other.

Let $(H, \pi)$ be a $*$-representation of $C$ and $T$ a contraction in $B(H)$. The pair $(\pi, T)$ is called a left covariant contractive (l-cov.con.) pair, if the left covariance relation is satisfied, i.e. $\pi(x)T = T\pi(\alpha(x))$, $x \in C$. If, in particular, $T$ is an isometry, pure isometry, co-isometry or unitary, then we will call such a pair a left covariant isometric, purely isometric, co-isometric or unitary pair. We can see that every l-cov.con. pair induces a contractive representation $(H, T \times \pi)$ of $\ell^1(\mathbb{Z}_+, C, \alpha)_l$, given by

$$(T \times \pi) \left( \sum_{n=0}^k \delta_n \otimes x_n \right) = \sum_{n=0}^k T^n \pi(x_n).$$

Conversely, if $\rho : \ell^1(\mathbb{Z}_+, C, \alpha)_l \to B(H)$ is a contractive representation, then $(H, \rho)$ restricts to a contractive representation $(H, \pi)$ of the $C^*$-algebra $C$, thus a $*$-representation. If we set $\rho(\delta_1 \otimes e) = T$, then $\|T^n\| \leq 1$, for every $n \in \mathbb{Z}_+$. It is easy to check that the pair $(\pi, T)$ satisfies the left covariance relation.

Analogously, there is a bijection between right covariant contractive (r-cov.con.) pairs $(\pi, T)$ (i.e. satisfying the right covariance condition
$T\pi(x) = \pi(\alpha(x))T$, $x \in \mathcal{C}$) and contractive representations $\pi \times T$ of the algebra $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r$. Note that if $(\pi, T)$ is a l-cov.con. pair then $(\pi, T^*)$ is a r-cov.con. pair. Thus $TT^*$ commutes with $\pi(\mathcal{C})$.

**Example 1.1.** Let $(H_0, \pi)$ be a faithful *-representation of $\mathcal{C}$ and define on $H_0 \otimes \ell^2(\mathbb{Z}_+)$ the representation $\tilde{\pi}(x) = \text{diag}\{\pi(\alpha^n(x)) : n \in \mathbb{Z}_+\}$ and $V = 1_{H_0} \otimes v$, where $v$ is the unilateral shift. Then $(\tilde{\pi}, V)$ is a l-cov.is. pair. For simplicity we will denote the corresponding representation $V \times \tilde{\pi}$, by $lt_{\pi}$. As mentioned before, the pair $(\tilde{\pi}, V^*)$ is a r-cov.con. pair which induces the representation $rt_{\pi} := \tilde{\pi} \times V^*$. One can check that $lt_{\pi}$ and $rt_{\pi}$ are faithful.

**Definition 1.2.** The (left) semicrossed product $\mathbb{Z}_+ \times_{\alpha} \mathcal{C}$ is the completion of $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l$ under the norm

$$
\|F\|_l = \sup\{\|((T \times \pi)(F))\| : (\pi, T) \text{ is a l-cov.con. pair}\}.
$$

Analogously, the (right) semicrossed product $\mathcal{C} \times_{\alpha} \mathbb{Z}_+$ is the completion of $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r$ under the norm

$$
\|F\|_r = \sup\{\|((\pi \times T)(F))\| : (\pi, T) \text{ is a r-cov.con. pair}\}.
$$

The left semicrossed product is endowed with an operator space structure (the maximal one, see [1, 1.2.22]) induced by the matrix norms

$$
\|[F_{i,j}]\|_l = \sup\{\|((T \times \pi)(F_{i,j}))\| : (\pi, T) \text{ l-con.cov. pair}\}.
$$

We note that there is a bijective correspondence between the l-cov.con. pairs $(\pi, T)$ and the unital completely contractive representations of $\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l$. Thus, the left semicrossed product has the following universal property (up to completely isometric isomorphisms): for any unital operator algebra $\mathcal{B}$ and for any unital completely contractive morphism $\rho : \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l \rightarrow \mathcal{B}$, there exists a unique unital completely contractive morphism $\tilde{\rho} : \mathbb{Z}_+ \times_{\alpha} \mathcal{C} \rightarrow \mathcal{B}$ that extends $\rho$.

In theorem 1.4 we prove that the semicrossed product, as an operator algebra, is independent of the way $\mathcal{C}$ is (faithfully) represented. In order to do so, we use some dilations theorems of [9] and [11] and arguments similar to the ones in [6, theorem 6.2].

First of all, every l-cov.con. pair $(\pi, T)$ on a Hilbert space $H$ dilates to a l-cov.is. pair $(\eta, W)$ on a Hilbert space $H_1 \supseteq H$, such that $\eta(x)H \subseteq H$ and $\eta(x)|_H = \pi(x)$, for every $x \in \mathcal{C}$, and $T^n = P_HW^n|_H$, for every $n \in \mathbb{Z}_+$, where $W$ is an isometry (see [9]). Hence, by [11, II.5] we see that the norm $\|\cdot\|$ is the supremum over all left covariant purely isometric pairs. By [11, proposition I.4], for such a pair $(\eta, W)$ on a
Hilbert space $H_1$ there is a representation $(H_2, \pi')$ of $\mathcal{C}$ such that $W \times \eta$ is unitarily equivalent to $l_t \pi'$. Thus, eventually we have that, for $F \in \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_t$, $\|F\|_t = \text{sup}\{\|lt_\pi(F)\| : (H, \pi) \text{ a } *\text{-representation of } \mathcal{C}\}$. Moreover, $\|\{F_{i,j}\}_t\|_t = \text{sup}\{\|lt_\pi(F_{i,j})\| : (H, \pi) \text{ a } *\text{-representation of } \mathcal{C}\}$.

**Proposition 1.3.** If $F_{i,j} \in \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_t$, then $\|\{F_{i,j}\}_t\|_t = \|lt_{\pi_u}(F_{i,j})\|$, where $(H_u, \pi_u)$ is the universal representation of $\mathcal{C}$. Analogously, for every $F_{i,j} \in \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_r$, $\|\{F_{i,j}\}_r\|_r = \|lt_{\pi_u}(F_{i,j})\|$.

*Proof.* Let $(H, \pi)$ be a $*$-representation of $\mathcal{C}$. By definition of the universal representation we have that $\pi_u|_H = \pi$ and $\pi_u(x)H \subseteq H$. Let $H_0 = H \otimes \ell^2(\mathbb{Z}_+)$. We denote by $P_{H_0}$ the projection onto $H \otimes \ell^2(\mathbb{Z}_+) \subseteq H_u \otimes \ell^2(\mathbb{Z}_+)$ and observe that $P_{H_0}(1_{H_u} \otimes v)^n|_{H_0} = (1_{H_0} \otimes v)^n$, for every $n \in \mathbb{Z}_+$. Thus, for every $\nu \in \mathbb{Z}_+$ and for every $F_{i,j} \in \mathcal{M}_\nu(\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha))$, we have that $\|lt_\pi(F_{i,j})\| = (P_{H_0} \otimes I_\nu)[lt_{\pi_u}(F_{i,j})]|_{(H_0)^{\nu}}$, and so $\|lt_\pi(F_{i,j})\| \leq \|lt_{\pi_u}(F_{i,j})\|$.

If $(H, \pi)$ is a faithful $*$-representation of $\mathcal{C}$, we denote by $C^*(\pi, V)$ the $C^*$-algebra generated by the representation $lt_\pi$ in $\mathcal{B}(H \otimes \ell^2(\mathbb{Z}_+))$. The covariance relation shows that $C^*(\pi, V)$ is the norm-closed linear span of the monomials $V^m \pi(x)(V^*)^\lambda$, $m, \lambda \in \mathbb{Z}_+$. Since, $C^*(\pi, V)$ is a direct summand of $C^*(\pi_u, V_u)$, the compression $\Phi : \mathcal{B}(H_u \otimes \ell^2(\mathbb{Z}_+)) \to \mathcal{B}(H \otimes \ell^2(\mathbb{Z}_+))$ is a $*$-epimorphism when restricted on $C^*(\pi_u, V_u)$. We will prove that it is also faithful, hence completely isometric.

To this end, for every $s \in [0, 2\pi]$, we define $u_s : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ by $u_s(e_m) = e^{2\pi i s} e_m$. Let $\widetilde{U}_s = 1_{H_u} \otimes u_s$ and $U_s = 1_H \otimes u_s$. The map $\tilde{\gamma}_s = ad_{\widetilde{U}_s}$ is a $*$-automorphism of $C^*(\pi_u, V_u)$, since $\tilde{\gamma}_s(\pi_u(x)) = \pi_u(x)$ and $\tilde{\gamma}_s(V_u)^n = e^{2\pi i n s} V_u^n$. Similarly, $\gamma_s = ad_{U_s}$ is a $*$-automorphism of $C^*(\pi, V)$. It is clear that $\Phi \circ \tilde{\gamma}_s = \gamma_s \circ \Phi$, because $\Phi(\tilde{U}_s) = U_s$. We denote by $C^*(\pi_u, V_u)^\gamma$ the fixed point algebra of $\tilde{\gamma}$ and define the contractive, faithful projection $\widetilde{E} : C^*(\pi_u, V_u) \to C^*(\pi_u, V_u)^\gamma$ by

$$\widetilde{E}(X) := \int_0^{2\pi} \tilde{\gamma}_s(X) \frac{ds}{2\pi},$$

(as a Riemann integral of a norm-continuous function). Let $\mathcal{B}_k := \{\sum_{n=0}^{k} V_u^n \pi_u(x_n)(V_u^*)^n : x_n \in \mathcal{C}\}$; then we can check that $C^*(\pi_u, V_u)^\gamma$ is the norm-closure of $\cup_{k \in \mathbb{Z}_+} \mathcal{B}_k$. Let $X_k$ be an element of $\mathcal{B}_k$. Since, $V_u^n \pi_u(x)(V_u^*)^n = \text{diag}\{0, \ldots, 0, \pi_u(x), \pi_u(\alpha(x)), \ldots\}$, we see that $X_k$ is a diagonal matrix whose $(m, m)$-entry is the element $(X_{k})_{m,m} = \pi_u\left(\sum_{j=0}^{\min\{m,k\}} \alpha^{m-j}(x_{m-j})\right)$. So, if $(H, \pi)$ is a faithful $*$-representation
of $C$,

$$
\|(X_k)_{m,m}\| = \left\| \pi_u \left( \sum_{j=0}^{\min\{m,k\}} \alpha^{m-j}(x_{m-j}) \right) \right\|
$$

$$
= \left\| \sum_{j=0}^{\min\{m,k\}} \alpha^{m-j}(x_{m-j}) \right\|_C = \left\| \pi \left( \sum_{j=0}^{\min\{m,k\}} \alpha^{m-j}(x_{m-j}) \right) \right\|
$$

$$
= \left\| (\Phi(X_k))_{m,m} \right\|.
$$

So $\|X_k\| = \sup_m \{\|(X_k)_{m,m}\|\} = \sup_m \{\|(\Phi(X_k))_{m,m}\|\} = \|\Phi(X_k)\|$; hence $\Phi : C^*(\pi_u, V_u) \to C^*(\pi, V)$ is isometric on each $B_k$. Thus, $\Phi$ is injective when restricted to the fixed point algebra $C^*(\pi_u, V_u)^\gamma$.

**Theorem 1.4.** The left semicrossed product $\mathbb{Z}_+ \times_\alpha C$ is completely isometrically isomorphic to the norm-closed linear span of $\sum_{n=0}^{k} V^n \tilde{\pi}(x_n), x_n \in C$, where $(H, \pi)$ is any faithful $*$-representation of $C$. Respectively, the right semicrossed product $C \times_\alpha \mathbb{Z}_+$ is completely isometrically isomorphic to the norm-closed linear span of $\sum_{n=0}^{k} \tilde{\pi}(x_n)(V^*)^n, x_n \in C$, where $(H, \pi)$ is any faithful $*$-representation of $C$.

**Proof.** It suffices to prove that the natural $*$-epimorphism $\Phi$ is faithful, hence a (completely) $*$-isometric isomorphism. Let $X \in \ker \Phi$, then $X^*X \in \ker \Phi$. Hence,

$$
\Phi(\tilde{E}(X^*X)) = \Phi \left( \int_0^{2\pi} \tilde{\gamma}_s(X^*X) \frac{ds}{2\pi} \right)
$$

$$
= \int_0^{2\pi} \Phi(\tilde{\gamma}_s(X^*X)) \frac{ds}{2\pi} = \int_0^{2\pi} \gamma_s(\Phi(X^*X)) \frac{ds}{2\pi} = 0.
$$

Now $\tilde{E}(X^*X)$ is in $C^*(\pi_u, V_u)^\gamma$ and $\Phi$ is faithful there; hence $\tilde{E}(X^*X) = 0$ and so $X^*X = 0$.

For the right semicrossed product, note that $C^*(\pi, V^*) = C^*(\pi, V)$. 

If, in particular, $\alpha$ is a $*$-isomorphism, then there is a natural way to identify the left semicrossed product as a closed subalgebra of the (reduced) crossed product, i.e. $C^*_l(C)$. In this case, we refer to this closed subalgebra as the left reduced semicrossed product. In a dual way, we can define the right reduced semicrossed product. The following is proved in [12], when $C$ is abelian.

**Theorem 1.5.** If $\alpha$ is a $*$-isomorphism, then the $C^*$-envelope of the semicrossed product is the (reduced) crossed product.
Proof. Since \( \alpha \) is a \(*\)-isomorphism, we can view \( \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha) \) as a \( \cdot \mid_\cdot \)-closed subalgebra of \( \ell^1(\mathbb{Z}, \mathcal{C}, \alpha)_l \). First we prove that the inclusion map \( \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha) \hookrightarrow \ell^1(\mathbb{Z}, \mathcal{C}, \alpha) \) is completely isometric. The key is to prove that

\[
\|F\|_l = \sup\{\|(U \times \pi)(F)\| : (\pi, U) \text{ l-cov.un. pair of } \ell^1(\mathbb{Z}, \mathcal{C}, \alpha)_l\},
\]

for every \( F \in \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l \), since the right hand side is exactly the norm of the (left) crossed product. For simplicity, we denote this norm by \( \|\cdot\|_l \).

It is obvious that \( \|F\| \leq \|F\|_l \), since every l-cov.un pair of \( \ell^1(\mathbb{Z}, \mathcal{C}, \alpha)_l \) restricts to a l-cov.un. pair of the subalgebra \( \ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l \). Also, if \((H_0, \pi)\) is a faithful \(*\)-representation of \( \mathcal{C} \), then \( t_\pi \) is the compression of the left regular representation of \( \ell^1(\mathbb{Z}, \mathcal{C}, \alpha)_l \) induced by \( \pi \), denoted simply by \( lt \). So, \( \|lt_\pi(F)\| \leq \|lt(F)\|_l \), thus \( \|F\| \leq \|F\|_l \) by theorem 1.4.

Arguing in the same way, we get that \( \|[F_{i,j}]\| \leq \|[F_{i,j}]\|_l \) and \( \|[lt_\pi(F_{i,j})]\| \leq \|[lt(F_{i,j})]\|_l \), for every \([F_{i,j}] \in \mathcal{M}_\psi(\ell^1(\mathbb{Z}_+, \mathcal{C}, \alpha)_l) \). But \( lt \) is a \(*\)-morphism of the crossed product, hence completely contractive.

Thus, \( \|[F_{i,j}]\|_l \leq \|[F_{i,j}]\| \) and equality holds.

Hence, if \( \tilde{\pi}(x) = \text{diag}\{\pi(a^m(x))\}, m \in \mathbb{Z} \) and \( U = 1_{H_0} \otimes u \), where \( u \) is the bilateral shift, then the map \( x \mapsto U^n\tilde{\pi}(x) \) extends to a complete isometry \( \iota : \mathbb{Z}_+ \times_\alpha \mathcal{C} \to \mathcal{C}_1^*(\mathcal{C}) \), whose image generates \( \mathcal{C}_1^*(\mathcal{C}) \) as a \( \mathcal{C}^* \)-algebra.

Let \( \mathcal{B} \) be the \( \mathcal{C}^* \)-envelope of \( \mathbb{Z}_+ \times_\alpha \mathcal{C} \). Then, by the universal property of \( \mathcal{C}^* \)-envelopes, there is a surjective \( \mathcal{C}^* \)-homomorphism \( \Psi : \mathcal{C}_1^*(\mathcal{C}) \to \mathcal{B} \), which restricts to a completely isometry on \( \iota(\mathbb{Z}_+ \times_\alpha \mathcal{C}) \).

Let \( G \in \ker \Phi \) be of unit norm, and choose \( F = \sum_{n=0}^k U^n\tilde{\pi}(x_n) \) with \( \|G - F\| < \frac{1}{2} \). Thus \( UG \in \ker \Psi, \iota^{-1}(UG) \in \mathbb{Z}_+ \times_\alpha \mathcal{C} \), \( \|\iota^{-1}(UG)\| = \|UG\| = \|F\| > \frac{1}{2} \) and \( \|U^kG - U^kF\| = \|G - F\| < \frac{1}{2} \). Then \( \frac{1}{2} < \|\iota^{-1}(UG)\| = \|\Psi(UG)\| = \|\Psi(U^kG - U^kF)\| \leq \|U^kG - U^kF\| < \frac{1}{2} \), which is a contradiction. \( \square \)

## 2. \( w^* \)-Semicrossed Products

Let \( \mathcal{A} \subseteq B(H_0) \) be a unital subalgebra, closed in the \( w^* \)-operator topology, and \( \beta : \mathcal{A} \to \mathcal{A} \), a contractive \( w^* \)-continuous endomorphism of \( \mathcal{A} \). From now on we fix \( H = H_0 \otimes \ell^2(\mathbb{Z}_+) \) and \( \pi := \text{id}_{\mathcal{A}} \), as in example 1.1. Then \( \pi \) is a faithful representation of \( \mathcal{A} \) on \( H \), and we can write \( \pi(b) = \sum_{n \geq 0} \beta^n(b) \otimes p_n \), where \( p_n \in \mathcal{B}(\ell^2(\mathbb{Z}_+)) \) is the projection onto \( [e_n] \). Note that the sum converges in the \( w^* \)-topology as well. Hence, \( \pi(b) \) belongs to the \( w^* \)-tensor product algebra \( \mathcal{A} \otimes \mathcal{B}(\ell^2(\mathbb{Z}_+)) \). This is, by definition, the \( w^* \)-closed linear span in \( \mathcal{B}(H) \) of the operators \( b \otimes a \), with \( b \in \mathcal{A} \) and \( a \in \mathcal{B}(\ell^2(\mathbb{Z}_+)) \). We also represent \( \mathbb{Z}_+ \) on \( H \) by the isometries \( V^n = 1_{H_0} \otimes v^n \), where \( v \) is the unilateral shift on \( \ell^2(\mathbb{Z}_+) \).

Thus, \( V^n \in \mathcal{A} \otimes \mathcal{B}(\ell^2(\mathbb{Z}_+)) \).
Definition 2.1. The \( w^*-\)semicrossed product \( \mathbb{Z}_+ \times_{\beta} \mathcal{A} \) is the \( w^* \)-closure of the linear space of the ‘analytic polynomials’ \( \sum_{n=0}^{k} V^n \pi(b_n), b_n \in \mathcal{A}, k \geq 0 \).

It is easy to check that the left covariance relation \( \pi(b)V = V \pi(\beta(b)) \) holds. Hence, \( (\pi, V) \) is a left covariant isometric pair. Thus, the \( w^* \)-semicrossed product is a unital (non-selfadjoint) subalgebra of \( \mathcal{B}(H) \) and by definition, \( \mathbb{Z}_+ \times_{\beta} \mathcal{A} \subseteq \mathcal{A} \overline{\otimes} \mathcal{B}(l^2(\mathbb{Z}_+)) \).

Proposition 2.2. An operator \( T \in \mathcal{B}(H) \) is in the \( w^* \)-semicrossed product if and only if \( T_{\kappa, \lambda} \in \mathcal{A} \) and \( G_m(T) = V^m \pi(T_{m,0}) \) when \( m \in \mathbb{Z}_+ \) while \( G_m(T) = 0 \) for \( m < 0 \). Equivalently, when \( T_{\kappa, \lambda} \in \mathcal{A} \) and \( \beta(T_{m+\lambda, \lambda}) = T_{m+\lambda+1, \lambda+1} \) for every \( m, \lambda \in \mathbb{Z}_+ \), while \( T_{\kappa, \lambda} = 0 \) when \( \kappa < \lambda \).

Proof. If \( T = \sum_{\kappa=0}^{n} V^\kappa \pi(b_\kappa) \) with \( b_\kappa \in \mathcal{A} \), then \( G_m(T) = V^m \pi(b_m) \) when \( m \in \{0, 1, \ldots, n\} \) and \( G_m(T) = 0 \) otherwise. Let \( T \in \mathbb{Z}_+ \times_{\beta} \mathcal{A} \) and a net \( A_\iota = \sum_{\kappa=0}^{n} V^\kappa \pi(b_{\iota, \kappa}) \) of analytic polynomials converging to \( T \) in the \( w^* \)-topology. Since \( G_m \) is \( w^* \)-continuous, we have that \( G_m(T) = w^* \lim G_m(A_\iota) \) for every \( m \in \mathbb{Z} \). Thus \( G_m(T) = 0 \) when \( m < 0 \). If \( m \geq 0 \), then \( T_{(m)} = (V^*)^m G_m(T) = w^* \lim_i (V^*)^m G_m(A_i) = w^* \lim_i \pi(b_{i,m}) \). Let \( \phi \in \mathcal{B}(H_0)_* \) and \( k \in \mathbb{Z}_+ \), then \( \phi \otimes \omega_{e_k, e_k} \in \mathcal{B}(H)_* \); hence we get \( \phi(T_{m+k, \kappa}) = (\phi \otimes \omega_{e_k, e_k})(T_{(m)}) \) is \( w^* \lim_i \pi(b_{i,m}) \). Thus \( T_{m+k, \kappa} = w^* \lim_i \beta^k(b_{i,m}) \), for every \( \kappa \in \mathbb{Z}_+ \), so \( T_{m+k, \kappa} \in \mathcal{A} \). Also, since \( \beta \) is \( w^* \)-continuous, we get that \( \beta^k(T_{m,0}) = w^* \lim_i \beta^k(b_{i,m}) = T_{m+k, \kappa} \), for every \( \kappa \in \mathbb{Z}_+ \). Hence, we get that \( G_m(T) = V^m \pi(T_{m,0}) \), for every \( m \geq 0 \).

For the opposite direction, if \( T \in \mathcal{B}(H) \) satisfies the conditions, we can see that \( G_m(T) \in \mathbb{Z}_+ \times_{\beta} \mathcal{A} \), and so by the Féjer Lemma, \( T \in \mathbb{Z}_+ \times_{\beta} \mathcal{A} \) as well. The last equivalence is trivial. \( \square \)

Remark 2.3. Note that each \( ad_w \) leaves \( \mathbb{Z}_+ \times_{\beta} \mathcal{A} \) invariant, and hence, being unitarily implemented, also leaves its reflexive cover invariant. Thus, so does \( G_m(\cdot) \).

Suppose now that the endomorphism \( \beta \) is implemented by a unitary \( w \) acting on \( H_0 \), so that \( \beta(b) = wbw^* \), for all \( b \in \mathcal{A} \). Let \( \rho(b) = b \otimes 1_{\mathcal{F}(\mathbb{Z}_+)} \), for \( b \in \mathcal{A} \) and \( W = w^* \otimes v \). Then \( (\rho, W) \) is a left covariant isometric pair and we denote by \( \mathbb{Z}_+ \times_w \mathcal{A} \) the \( w^* \)-closure of the linear space of the ‘analytic polynomials’ \( \sum_{n=0}^{k} W^n \rho(b_n), b_n \in \mathcal{A}, k \geq 0 \).

It is easy to check that \( \mathbb{Z}_+ \times_w \mathcal{A} \) is unitarily equivalent to \( \mathbb{Z}_+ \times_{\beta} \mathcal{A} \), via \( Q = \sum_{n \geq 0} w^{-n} \otimes p_n \). Thus we refer to \( \mathbb{Z}_+ \times_w \mathcal{A} \) as the \( w^* \)-semicrossed product, as well. Using the unitary operator \( Q \) and proposition 2.2 we get the following characterization.
Proposition 2.4. An operator \( T \in \mathcal{B}(H) \) is in \( \mathbb{Z}_+ \otimes w \mathcal{A} \) if and only if \( G_m(T) = W^m \rho(b_m) \), for some \( b_m \in \mathcal{A} \), when \( m \in \mathbb{Z}_+ \) and \( G_m(T) = 0 \) for \( m < 0 \). Equivalently, when \( T_{m+\lambda,\lambda} = (w^*)^m b_m \), for every \( m, \lambda \in \mathbb{Z}_+ \) and \( T_{\kappa,\lambda} = 0 \), when \( \kappa < \lambda \). \( \square \)

The relation between the \( w^* \)-semicrossed product \( \mathcal{A} \otimes T \) and \( \mathbb{Z}_+ \otimes w \mathcal{A} \) depends on some properties of \( w \). Specifically,

- \( \mathcal{A} \otimes T = \mathbb{Z}_+ \otimes w \mathcal{A} \) if and only if \( w^* \in \mathcal{A} \).
- \( \mathcal{A} \otimes T \subset \mathbb{Z}_+ \otimes w \mathcal{A} \) if and only if \( w^* \notin \mathcal{A} \), \( w \notin \mathcal{A} \).
- \( \mathbb{Z}_+ \otimes w \mathcal{A} \subset \mathcal{A} \otimes T \) if and only if \( w \notin \mathcal{A} \), \( w^* \in \mathcal{A} \).
- \( (\mathbb{Z}_+ \otimes w \mathcal{A}) \cap (\mathcal{A} \otimes T) = \rho(\mathcal{A}) \), if and only if \( (w^* \mathcal{A}) \cap \mathcal{A} = \{0\} \), \( \forall n \in \mathbb{Z}_+ \).

It is easy to verify that, when \( (w^* \mathcal{A}) \cap \mathcal{A} = \{0\} \) for every \( n \in \mathbb{Z}_+ \), then \( w, w^* \notin \mathcal{A} \), but the converse is not always true.

Example 2.5. Take \( \mathcal{A} = L^\infty(\mathbb{T}) \) acting on \( L^2(\mathbb{T}) \) and \( \beta(f)(z) = f(\lambda z) \), where \( \lambda \) is a \( q \)-th root of unity. Then \( \beta \) is unitarily implemented by \( w \in \mathcal{B}(L^2(\mathbb{T})) \), with \( (w(g))(z) = g(\lambda z) \). Then \( w^{mq} = I_{H_0} \), for every \( m \in \mathbb{Z}_+ \), hence \( w^{mq} \mathcal{A} \cap \mathcal{A} = \mathcal{A} \). In this case, \( (\mathbb{Z}_+ \otimes w \mathcal{A}) \cap (\mathcal{A} \otimes T) \) contains the \( w^* \)-closed algebra generated by \( \sum_{n=0}^k W^{mq}_n \rho(b_n), b_n \in \mathcal{A} \), which properly contains \( \rho(\mathcal{A}) \).

The following lemma will be superseded below (theorem 2.9).

Lemma 2.6. The \( w^* \)-semicrossed product \( \mathbb{Z}_+ \otimes w \mathcal{B}(H_0) \) is reflexive, for every unitary \( w \in \mathcal{B}(H_0) \).

Proof. Let \( T \in \text{Ref}(\mathbb{Z}_+ \otimes w \mathcal{B}(H_0)) \). By remark 2.3 each \( G_m(T) \) belongs to the reflexive cover of the \( w^* \)-semicrossed product. For \( \kappa < \lambda \) and \( \xi, \eta \in H_0 \) there is a sequence \( A_n \in \mathbb{Z}_+ \otimes w \mathcal{B}(H_0) \) such that \( \langle T(\xi \otimes e_\lambda), \xi \otimes e_\kappa \rangle = \lim_n \langle A_n(\xi \otimes e_\lambda), \eta \otimes e_\kappa \rangle \). Hence, \( \langle T_{\kappa,\lambda}^* \xi, \eta \rangle = \lim_n \langle (A_n)^{\kappa,\lambda}_\kappa \xi, \eta \rangle = 0 \), since \( (A_n)^{\kappa,\lambda}_\kappa = 0 \), for \( \kappa < \lambda \). Hence \( G_m(T) = 0 \) for every \( m < 0 \). Now, fix \( m \in \mathbb{Z}_+ \) and consider \( \xi \in H_0 \), \( g_r = \sum r^m e_n, 0 \leq r < 1 \). We can check that the subspace \( \mathcal{F} = \{ (b\xi) \otimes g_r : b \in \mathcal{B}(H_0) \} \) is \( (\mathbb{Z}_+ \otimes w \mathcal{B}(H_0))^* \)-invariant, hence \( G_m(T)^* \)-invariant. Since \( \xi \otimes g_r \in \mathcal{F} \), there is a sequence \( (b_n) \) in \( \mathcal{B}(H_0) \) such that \( G_m(T)^*(\xi \otimes g_r) = \lim_n (b_n \xi) \otimes g_r \). Thus, \( \sum \gamma^{m+n+k} T_{m+k,\kappa}^\kappa \xi \otimes e_\kappa = \lim_n (b_n \xi) \otimes g_r \). Taking scalar product with \( \eta \otimes e_\kappa \), where \( \eta \in H_0 \) and \( \kappa \geq 0 \), we have that \( \gamma^{m+k} \langle T_{m+k,\kappa}^\kappa \xi, \eta \rangle = \lim_n \gamma^\kappa \langle b_n \xi, \eta \rangle \). Hence, \( r^\kappa \langle T_{m+k,\kappa}^\kappa \xi, \eta \rangle = \lim_n (b_n \xi, \eta) = r^\kappa \langle T_{m,0}^\kappa \xi, \eta \rangle \), for every \( \eta \). Thus, \( T_{m+k,\kappa}^\kappa \xi = T_{m,0}^\kappa \xi \), for arbitrary \( \xi \in H_0 \), so \( T_{m+k,\kappa} = T_{m,0} \) for every \( \kappa \in \mathbb{Z}_+ \). Hence, \( G_m(T) \in \mathcal{B}(H_0) \otimes T \), which coincides with \( \mathbb{Z}_+ \otimes w \mathcal{B}(H_0) \) since \( w \in \mathcal{B}(H_0) \). \( \square \)
Let $S$ be a $w^*$-closed subspace of $\mathcal{B}(H)$. We say that $S$ is $G$-invariant if $G_m(S) \subseteq S$ for every $m \in \mathbb{Z}$. If, in particular, $S$ is a $w^*$-closed subspace of $Z_+ \overline{\times}_w \mathcal{B}(H_0)$, then $G_m(S) = 0$, for every $m < 0$.

In the next proposition we prove that we can associate a sequence $(S_m)_{m \geq 0}$ of $w^*$-closed subspaces of $\mathcal{B}(H_0)$ to such an $S$, and vice versa.

**Proposition 2.7.** A $w^*$-closed subspace $S$ of $\mathcal{B}(H)$ is a $G$-invariant subspace of $Z_+ \overline{\times}_w \mathcal{B}(H_0)$ if and only if it is the $w^*$-closure of the linear space of the analytic polynomials $\sum_{n=0}^{k} W^n \rho(x_n)$, $x_n \in S_n$, $k \in \mathbb{Z}_+$, where $S_n$ are $w^*$-closed subspaces of $\mathcal{B}(H_0)$.

**Proof.** Let $S$ be a $G$-invariant $w^*$-closed subspace of $Z_+ \overline{\times}_w \mathcal{B}(H_0)$ and let $S_m = \{w^m T_{m,0} : T \in S\}$, for every $m \geq 0$. Then $S_m$ is a $w^*$-closed subspace of $\mathcal{B}(H_0)$. Indeed, let $x = w^*\lim_m w^m(T_{i,m,0})$, for $T_i \in S$. Then $\rho((w^*)^{m}x) = w^*\lim_m \rho(T_{i,m,0})$, so $W^m \rho(x) = w^*\lim_m V^m \rho(T_{i,m,0}) = w^*\lim_m G_m(T_i)$, since $T_i \in Z_+ \overline{\times}_w \mathcal{B}(H_0)$. But $S$ is $G$-invariant, hence $W^m \rho(x) \in S$, thus $x = (w^m(V^m \rho(x)))_{m,0} \in S_m$. A use of the Féjer Lemma and proposition 2.4 completes the forward implication.

For the converse, let $S$ be a $w^*$-closed subspace as in the statement and $A \in S$; so $A = w^*\lim_i A_i$, where $A_i = \sum_{\kappa=0}^{n_i} W^\kappa \rho(x_{i,\kappa})$, with $x_{i,\kappa} \in S_{\kappa}$. Then $A \in Z_+ \overline{\times}_w \mathcal{B}(H_0)$ and $G_m(A) = w^*\lim_i G_m(A_i) = w^*\lim_i W^m \rho(x_{i,\kappa})$. Thus, $w^m A_{m,0} = w^*\lim_i x_{i,m} \in S_m$. Hence, we have that $G_m(A) = W^m \rho(w^m A_{m,0}) \in S$. \(\square\)

**Theorem 2.8.** Let $(S_m)_{m \geq 0}$ be the sequence associated to a $G$-invariant $w^*$-closed subspace $S$ of $Z_+ \overline{\times}_w \mathcal{B}(H_0)$. If every $S_m$ is reflexive then $S$ is reflexive.

**Proof.** By lemma 2.6 $\text{Ref}(S) \subseteq \text{Ref}(Z_+ \overline{\times}_w \mathcal{B}(H_0)) = Z_+ \overline{\times}_w \mathcal{B}(H_0)$. So, for every $T$ in the reflexive cover of $S$ and every $m, \lambda \in \mathbb{Z}_+$, we have that $T_{m,\lambda,\lambda} = (w^*)^m b_m$, where $b_m \in \mathcal{B}(H_0)$. Thus, it suffices to prove that $b_m \in S_m$. Since $T \in \text{Ref}(S)$, for every $\xi, \eta \in H_0$, there is a sequence $(A_n)$ in $S$ such that $\langle T(\xi \otimes e_\lambda), (w^*)^m \eta \otimes e_{m+\lambda} \rangle = \lim_n \langle A_n(\xi \otimes e_\lambda), (w^*)^m \eta \otimes e_{m+\lambda} \rangle$. So, $\langle b_m \xi, \eta \rangle = \langle T_{m,\lambda,\lambda} \xi, (w^*)^m \eta \rangle = \lim_n \langle (A_n)_{m,\lambda,\lambda} \xi, (w^*)^m \eta \rangle$. Since each $A_n \in S$, then $(A_n)_{m,\lambda,\lambda} = (w^*)^m b_{n,m}$ for some $b_{n,m} \in S_m$. Thus $\langle b_m \xi, \eta \rangle = \lim_n \langle b_{n,m} \xi, \eta \rangle$, which means that $b_m \in \text{Ref}(S_m) = S_m$. \(\square\)

**Theorem 2.9.** If $A$ is a reflexive algebra, then $Z_+ \overline{\times}_w A$ is reflexive. In addition, if $A$ is hereditarily reflexive, then every $G$-invariant $w^*$-closed subspace of $Z_+ \overline{\times}_w A$ is reflexive.

**Proof.** The algebra $Z_+ \overline{\times}_w A$ is associated to the sequence $(A)_{m \geq 0}$; hence it is reflexive by the previous theorem. \(\square\)
Applications 2.10. A. (Sarason’s result, [13] theorem 3]) Consider the case of a reflexive subalgebra $\mathcal{A}$ of $M_n(\mathbb{C})$ and a unitary $w \in M_n(\mathbb{C})$ such that $w \mathcal{A} w^* \subseteq \mathcal{A}$. Then $Z_+ \mathcal{A}$ is reflexive. Note that $Z_+ \mathcal{A} = \mathcal{T}$ when $n = 1$ and $w = I_{H_0}$.

B. (Ptak’s result, [13] theorem 2]) More generally, $\mathcal{A} \otimes \mathcal{T}$ coincides with $Z_+ \mathcal{A}$. So $\mathcal{A} \otimes \mathcal{T}$ is reflexive, when $\mathcal{A}$ is reflexive.

C. If $\mathcal{M}$ is a maximal abelian selfadjoint algebra and let $\beta$ be a *-automorphism, then $Z_+ \beta \mathcal{M}$ is reflexive, since every *-automorphism of a m.a.s.a. is unitarily implemented. For example let $\mathcal{M} = L^\infty(\mathbb{T})$ acting on $L^2(\mathbb{T})$ and $\beta$ the rotation by $\theta \in \mathbb{R}$. Also $Z_+ \beta \mathcal{A}$ is reflexive whenever $\mathcal{A}$ is a $\beta$-invariant $w^*$-closed subalgebra of $\mathcal{M}$, since $\mathcal{M}$ is hereditarily reflexive (see [7]).

D. Consider $\mathcal{T}$ acting on $H^2(\mathbb{T})$ and $\beta$ as in the previous example. Then, $\mathcal{T}$ is reflexive and so $Z_+ \beta \mathcal{T}$ is a reflexive subalgebra of $\mathcal{B}(H^2(\mathbb{T})) \otimes \mathcal{B}(\ell^2(\mathbb{Z}_+))$.

E. If $\mathcal{A}$ is a nest algebra and $\beta$ is an isometric automorphism, then it is unitarily implemented (see [2]). Thus, $Z_+ \beta \mathcal{A}$ is reflexive.

F. Consider a $C^*$-algebra $\mathcal{C}$ and a *-morphism $\alpha : \mathcal{C} \rightarrow \mathcal{C}$. Let $(H_0, \sigma)$ be a faithful *-representation of $\mathcal{C}$ such that the induced *-morphism

$$\beta : \sigma(\mathcal{C}) \rightarrow \sigma(\mathcal{C}) : \sigma(x) \mapsto \beta(\sigma(x)) = \sigma(\alpha(x))$$

is implemented by a unitary $w \in \mathcal{B}(H_0)$. Then the induced representation $lt_\alpha$ is faithful on $Z_+ \times_\alpha \mathcal{C}$. Thus, $lt_\alpha(Z_+ \times_\alpha \mathcal{C})$ is the $w^*$-closed linear span of the analytic polynomials $\sum_{n=0}^{k} V^n \pi(\sigma(x))$, and it is unitarily equivalent to the algebra $\mathcal{E} := \text{span}\{\rho(\sigma(x)), W^n : x \in \mathcal{C}, n \in \mathbb{Z}_+\}$, via $Q = \sum_{n \geq 0} w^{-n} \otimes p_n$. But $\mathcal{E}$ is exactly the $w^*$-semicrossed product $Z_+ \times_\alpha \sigma(\mathcal{C})^{w^*}$. Thus, $lt_\sigma(Z_+ \times_\alpha \mathcal{C})^{w^*}$ is reflexive.

In particular, let $K$ be a compact, Hausdorff space, $\mu$ a positive, regular Borel measure on $K$ and $\sigma : C(K) \rightarrow \mathcal{B}(L^2(K, \mu)) : f \mapsto M_f$. Consider a homeomorphism $\phi$ of $K$, such that $\phi$ and $\phi^{-1}$ preserve the $\mu$-null sets and let $\alpha(f) = f \circ \phi$. Then the map $M_f \rightarrow M_{f \circ \phi}$ extends to a *-automorphism of $L^\infty(K, \mu)$, hence it is unitarily implemented. Thus, $lt_\sigma(Z_+ \times_\alpha C(K))^{w^*}$ is reflexive.

H. Let $(\mathcal{M}, \tau)$ be a von Neumann algebra with a faithful, normal, tracial state $\tau$ and let $L^2(\mathcal{M}, \tau)$ be the Hilbert space associated to $(\mathcal{M}, \tau)$. Let $\beta : \mathcal{M} \rightarrow \mathcal{M}$ be a trace-preserving *-automorphism and consider $\mathcal{M}$ acting on $L^2(\mathcal{M}, \tau)$ by left multiplication. Then $\beta$ is
unitarily implemented and it can be verified that the $w^*$-semicrossed product $Z_+\overline{\times}_w\mathcal{A}$ coincides with the adjoint of the analytic semicrossed product defined in [8] and [10]. Hence, we obtain [10, proposition 4.5] for $p = 2$.

We conclude the analysis of the $w^*$-semicrossed product $Z_+\overline{\times}_w\mathcal{A}$ by finding its commutant. We know that $U_s(Z_+\overline{\times}_w\mathcal{A})U_s^* = Z_+\overline{\times}_w\mathcal{A}$, for all $s \in [0, 2\pi]$, hence, $U_s(Z_+\overline{\times}_w\mathcal{A})U_s^* = (Z_+\overline{\times}_w\mathcal{A})'$. Thus, $T \in (Z_+\overline{\times}_w\mathcal{A})'$ if and only if $G_m(T) \in (Z_+\overline{\times}_w\mathcal{A})'$, for every $m \in \mathbb{Z}$. Now, recall that $w\mathcal{A}w^* \subseteq \mathcal{A}$, hence $w^*\mathcal{A}'w \subseteq \mathcal{A}'$. So, we can define the $w^*$-semicrossed product $Z_+\overline{\times}_w\mathcal{A}'$, where $\gamma \equiv ad_{w^*} : \mathcal{A}' \to \mathcal{A}'$.

**Theorem 2.11.** If $\gamma \equiv ad_{w^*}$, then $(Z_+\overline{\times}_w\mathcal{A})' = Z_+\overline{\times}_\gamma\mathcal{A}'$. □

**Proof.** Obviously $T \in (Z_+\overline{\times}_w\mathcal{A})'$ if and only if $T \in \{b \otimes 1, w^* \otimes v : b \in \mathcal{A}\}'$; note also that $V \in (Z_+\overline{\times}_w\mathcal{A})'$. Let $T \in (Z_+\overline{\times}_w\mathcal{A})'$, then for $m \geq 0$ and $b \in \mathcal{A}$, $n \geq 0$,

$$G_m(T)(b \otimes 1) = (b \otimes 1)G_m(T) \quad \text{and} \quad G_m(T)(w^* \otimes v) = (w^* \otimes v)G_m(T),$$

hence, $T_{m+n,n}b = bT_{m,n+n}$ and $T_{m+n+1,n+1}(w^*)_n = (w^*)_nT_{m+n,n}$,

so, $T_{m+n,n} \in \mathcal{A}'$ and $T_{m+n,n} = \gamma^n(T_{0,-m})$.

Thus, if we set $\pi'(T_{m,0}) = \sum_{n \geq 0} \gamma^n(T_{m,0}) \otimes p_n$, we get that $G_m(T) = V^m \pi'(T_{m,0})$, for $m \geq 0$.

Now, let $m < 0$, hence $G_m(T) = T_{(m)}(V^*)^{-m}$. Since, $V \in (Z_+\overline{\times}_w\mathcal{A})'$, we have that $T_{(m)} = G_m(T)V^{-m} \in (Z_+\overline{\times}_w\mathcal{A})'$. Thus, $G_0(T_{(m)}) = T_{(m)} \in (Z_+\overline{\times}_w\mathcal{A})'$ and so, by what we have proved, $T_{n,-m+n} = \gamma^n(T_{0,-m})$.

Since $G_m(T)(\xi \otimes e_0) = 0$, then

$$G_m(T)(w^* \otimes v)^{-m}(\xi \otimes e_0) = (w^* \otimes v)^{-m}G_m(T)(\xi \otimes e_0) = 0,$$

so $(T_{0,-m}(w^*)^{-m}\xi) \otimes e_0 = 0$. Thus, $T_{0,-m} = 0$ and therefore $T_{n,-m+n} = \gamma^n(T_{0,-m}) = 0$, for every $n \geq 0$; hence $G_m(T) = 0$, for every $m < 0$.

Hence, by proposition 2.2, we get that $T \in Z_+\overline{\times}_\gamma\mathcal{A}'$.

For the converse, let $T \in Z_+\overline{\times}_\gamma\mathcal{A}'$, then $G_m(T) \in Z_+\overline{\times}_\gamma\mathcal{A}'$ for every $m \in \mathbb{Z}$, and we can see that $G_m(T) \in \{b \otimes 1, w^* \otimes v : b \in \mathcal{A}\}'$. Hence, $G_m(T) \in (Z_+\overline{\times}_w\mathcal{A})'$ for every $m \in \mathbb{Z}$, so $T \in (Z_+\overline{\times}_w\mathcal{A})'$. □

**Theorem 2.12.** The double commutant of $Z_+\overline{\times}_w\mathcal{A}$ is $Z_+\overline{\times}_w\mathcal{A}''$. Thus, the $w^*$-semicrossed product is its own bicommutant if and only if $\mathcal{A} = \mathcal{A}''$.

**Proof.** We recall that $Q(Z_+\overline{\times}_\beta\mathcal{A})Q^* = Z_+\overline{\times}_w\mathcal{A}$, where $Q = \sum_{n} w^{-n} \otimes p_n$; hence $Q'\overline{(Z_+\overline{\times}_\beta\mathcal{A})}' = Z_+\overline{\times}_w\mathcal{A}'$. Thus, $(Z_+\overline{\times}_w\mathcal{A})' = (Z_+\overline{\times}_w\mathcal{A})' = (Q(Z_+\overline{\times}_w\mathcal{A})Q)^* = Q(Z_+\overline{\times}_w\mathcal{A})' Q^* = Q(Z_+\overline{\times}_w\mathcal{A})' Q^* = Z_+\overline{\times}_w\mathcal{A}''$. 
We end this section with a note on the reduced $w^*$-semicrossed products (see the definition below). Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $H_0$, $\beta$ a $*$-automorphism of $\mathcal{M}$ and consider $\mathbb{Z} \rtimes_{\beta} \mathcal{M}$ to be the usual $w^*$-crossed product, a von Neumann subalgebra of $\mathcal{M}\overline{\otimes}\mathcal{B}(\ell^2(\mathbb{Z}))$. This is by definition the von Neumann algebra $\{\hat{\pi}(b), U : b \in \mathcal{M}\}''$, where $\hat{\pi}(b) = \sum_{n \in \mathbb{Z}} \beta^n(b) \otimes p_n$ and $U = 1_{H_0} \otimes u$, the ampliation of the bilateral shift $u \in \mathcal{B}(\ell^2(\mathbb{Z}))$.

**Definition 2.13.** The reduced $w^*$-semicrossed product $\mathbb{Z}^+ \rtimes_{\beta} \mathcal{M}$ is the $w^*$-closure of the linear space of ‘analytic polynomials’ $\sum_{k \geq 0} U^k \pi(b_n)$, $b_n \in \mathcal{M}$.

Since $(\hat{\pi}, U)$ is a 1-cov.un. pair, the reduced $w^*$-semicrossed product is a $w^*$-closed subalgebra of the $w^*$-crossed product. In fact, note that $\mathbb{Z}^+ \rtimes_{\beta} \mathcal{M}$ is the intersection of $\mathbb{Z} \rtimes_{\beta} \mathcal{M}$ with the ‘lower triangular’ matrices. Hence, we have the following proposition.

**Proposition 2.14.** The reduced $w^*$-semicrossed product of a von Neumann algebra is reflexive. □

Now, take $\mathcal{A}$ to be a $w^*$-closed subalgebra of $\mathcal{M}$ which is invariant under $\beta$. We define $\mathbb{Z}^+ \rtimes_{\beta} \mathcal{A}$ to be the $w^*$-closure of the linear space of ‘analytic polynomials’ $\sum_{k \geq 0} U^k \pi(b_n)$, $b_n \in \mathcal{A}$, $k \geq 0$. Using the technique of theorem 2.9 one can show the following.

**Corollary 2.15.** If $\mathcal{A}$ is reflexive subalgebra of $\mathcal{M}$ which is invariant under $\beta$, then $\mathbb{Z}^+ \rtimes_{\beta} \mathcal{A}$ is reflexive. □

3. **The Commutative Case**

Now, we examine the case where $\mathcal{C}$ is a commutative, unital C*-algebra, $\mathcal{C} = C(K)$, and the $*$-endomorphism $\alpha$ is induced by a continuous map $\phi : K \to K$. Let $ev_t$ be the evaluation at $t \in K$, i.e. $ev_t(f) = f(t)$; then $(\ell^2(K), \oplus_t ev_t)$ is a faithful $*$-representation of $C(K)$. If some $t \in K$ has dense orbit, we obtain a faithful representation of $C(K)$ on $\ell^2(\mathbb{Z}^+)$. As observed in theorem 1.1, such representations play a fundamental role for the semicrossed product $\mathbb{Z}^+ \times_{\alpha} C(K)$, since they are “enough” to obtain the norm. Let $\pi_t := ev_t$, as in example 1.1. So, $\pi_t : C(K) \to \mathcal{B}(\ell^2(\mathbb{Z}^+))$ is given by $\pi_t(f) := \sum_{n \geq 0} f(\phi^n(t)) p_n$, where $p_n$ is the one-dimensional projection on $[e_n]$. Then $(\pi_t, v)$ is a left covariant isometric pair. We define the one point $w^*$-semicrossed product to be $\mathcal{C}_t = \ell^2(\mathbb{Z}^+) \rtimes_{\alpha} \mathcal{C}(K)$, i.e. the $w^*$-closed linear span in $\mathcal{B}(\ell^2(\mathbb{Z}^+))$.
of the ‘analytic polynomials’ $\sum_{n=0}^{k} v^n \pi_t(f_n)$, $f_n \in C(K)$.

Let $t' = \phi^{n_0}(t)$ be the first periodic element of the orbit of $t$ with period $p$, as in the following diagram:

$$t \xrightarrow{\phi(t)} \phi^{n_0-1}(t) \xrightarrow{\phi^{n_0}(t) = t'} \phi(t')$$

Then $\text{orb}(t) = \{t, \ldots, \phi^{n_0-1}(t), t', \ldots, \phi^{p-1}(t')\}$ induces a family of projections $\{P_{n_0}, P_0, \ldots, P_{p-1}\}$ such that $I = P_{n_0} \oplus P_0 \oplus \cdots \oplus P_{p-1}$. Indeed, let $P_{n_0}$ be the projection on $[e_0, \ldots, e_{n_0-1}]$ and $P_i$ be the projection on $[e_{n_0+i} : j \in \mathbb{Z}_+]$ for $i = 0, \ldots, p-1$. Note that if $f \in C(K)$, then $\pi_t(f)(e_{n_0+i+j}) = f(\phi^{n_0+i+j}(t))e_{n_0+i+j}$, for $j \in \mathbb{Z}_+$. Hence, $\pi_t(f)P_i = f(\phi^i(t))P_i$, for every $i = 0, \ldots, p-1$.

**Proposition 3.1.** The algebra $C_t$ is the linear sum $(\Sigma P_{n_0}) \oplus (TP_0) \oplus \cdots \oplus (TP_{p-1})$, where $\Sigma$ is the algebra of lower triangular operators in $B(\ell^2(\mathbb{Z}_+))$, $T$ is the algebra of analytic Toeplitz operators and $P_{n_0}, P_0, \ldots, P_{p-1}$ are the projections induced by the orbit of $t$.

**Proof.** For any $n \in \mathbb{Z}_+$ and $f \in C(K)$, we have

$$v^n \pi_t(f) = v^n \pi_t(f) P_{n_0} \oplus f(t') v^n P_0 \oplus f(\phi^{p-1}(t')) v^n P_{p-1}.$$ 

Thus, $C_t \subseteq (\Sigma P_{n_0}) \oplus (TP_0) \oplus \cdots \oplus (TP_{p-1})$.

For the converse, first let $TP_{n_0} \in \Sigma P_{n_0}$ and note that $(TP_{n_0})_{\kappa, \lambda} = 0$ when $\kappa < \lambda$ or $n_0 - 1 < \lambda$. So, $G_m(TP_{n_0}) = 0$, when $m < 0$, and $G_m(TP_{n_0}) = v^m(\sum_{n=0}^{n_0-1}(TP_{n_0})_{m+n,n} P_n)$, when $m \geq 0$. Note that $(TP_{n_0})_{\kappa, \lambda} \in \mathbb{C}$ for every $\kappa, \lambda \in \mathbb{Z}_+$. Fix $m \geq 0$ and let $n \in \{0, \ldots, n_0 - 1\}$.

Then by Urysohn’s Lemma there is a sequence $(f_{n,j})_j$ of continuous functions on $K$, such that $\lim_j f_{n,j}(\phi^n(s)) = (TP_{n_0})_{m+n,n}$ and $f_{n,j}(s) = 0$ for $s \in \text{orb}(t) \setminus \{\phi^n(t)\}$. Hence, $(TP_{n_0})_{m+n,n} P_n = w^*-\lim_j \pi_t(f_{n,j}) \subseteq C_t$ and so $v^m(TP_{n_0})_{m+n,n} P_n \subseteq C_t$. Thus $G_m(TP_{n_0}) \in C_t$, and, by the Féjer Lemma, $TP_{n_0} \subseteq C_t$.

Also, for fixed $i \in \{0, \ldots, p-1\}$ and $m \in \mathbb{Z}_+$, consider $v^m P_i \in TP_i$. Again by Urysohn’s Lemma, there is a sequence $(f_{i,j})_j$ of continuous functions on $K$, such that $\lim_j f_{i,j}(\phi^i(s')) = 1$ and $f_{i,j}(s) = 0$ for $s \in \text{orb}(t) \setminus \{\phi^i(t')\}$. Then $w^*-\lim_j \pi_t(f_{i,j}) = P_i$, so $v^m P_i \subseteq C_t$. Hence, $TP_i \subseteq C_t$, for every $i \in \{0, \ldots, p-1\}$. Thus, $(\Sigma P_{n_0}) \oplus (TP_0) \oplus \cdots \oplus (TP_{p-1}) \subseteq C_t$. □
Note that if \( orb(t) \) has no periodic points, then \( C_t = \mathcal{F} \), since \( P_{n_0} = 1_{\ell^2(\mathbb{Z}_+)} \). Also, if \( orb(t) \) has exactly one periodic point \( t' \), then \( \phi^n(t) = t' \) for every \( n \geq n_0 \) (i.e. \( t' \) is a fixed point); thus \( C_t = \mathcal{F} P_{n_0} \oplus T P_{n_0}^+ \). If \( t \) is itself a fixed point, then \( C_t = \mathcal{T} \).

**Remark 3.2.** Let \( \mathcal{D} \) be the algebra of diagonal operators in \( B(\ell^2(\mathbb{Z}_+)) \) and \( \mathcal{D}_\phi = \{ T \in \mathcal{D} : T_{\kappa,\kappa} = T_{n,n} \text{ when } \phi^n(t) = \phi^n(t) \} \) which is a \( w^* \)-closed subalgebra of \( \mathcal{D} \). Hence, \( T \in \mathcal{D}_\phi \) if and only if \( T \) is of the form

\[
T = \text{diag}\{y_0, \ldots, y_{n_0-1}, y_{n_0}, \ldots, y_p-1, y_{n_0}, \ldots, y_{p-1}, \ldots\}.
\]

It is immediate from the previous proposition that \( C_t \) is generated by the unilateral shift in \( B(\ell^2(\mathbb{Z}_+)) \) and the diagonal matrices \( \text{id} \) \( \mathcal{D}_\phi \). Thus, an operator \( T \in B(\ell^2(\mathbb{Z}_+)) \) is in \( C_t \) if and only if for every \( m < 0 \), \( G_m(T) = 0 \), and for every \( m \geq 0 \), \( G_m(T) = v^m \sum_n T_{m+n,n}p_n \) where \( T_{m+n,n} = T_{m+n,n} \) whenever \( \phi^n(t) = \phi^n(t) \).

**Theorem 3.3.** The algebra \( C_t \) is reflexive.

**Proof.** If \( T \in \text{Ref}(C_t) \), then \( G_m(T) \in \text{Ref}(C_t) \); thus \( G_m(T) = 0 \), for \( m < 0 \). Let \( g_r = \sum_{n=0}^r r^n e_n \), with \( 0 \leq r < 1 \), and \( \mathcal{F} = [\pi_t(f)g_r : f \in C(K)] \). Then \( \mathcal{F} \) is \( (C_t)^* \)-invariant; thus \( G_m(T)^* \)-invariant, for \( m \in \mathbb{Z}_+ \). So, there is a sequence of \( f_j \in C(K) \) such that \( G_m(T)^* g_r = \lim_j \pi_t(f_j)g_r \). Hence \( r^m \sum_{n=0}^r \pi_t(f_j)g_r \), for every \( n \in \mathbb{Z}_+ \). Thus, \( T_{m+n,n} = T_{m+n,n} \) if \( \phi^n(t) = \phi^n(t) \). So, by remark 3.2, \( T \in C_t \). \( \square \)

**Remark 3.4.** In order to construct \( C_t \), it is sufficient to take coefficients from any uniform algebra \( \mathfrak{A} \) on \( K \).

Indeed, let \( \mathfrak{A} \) be a norm closed subalgebra of \( C(K) \) containing the constant functions which separates the points of \( K \) and form the polynomials \( \sum_{k=0}^k v^n \pi_t(f_n) \), \( f_n \in \mathfrak{A} \). By remark 3.2, it suffices to prove that \( \pi_t(\text{ball}(\mathfrak{A})) \) is \( w^* \)-dense in \( \text{ball}(\mathcal{D}_\phi) \). Fix \( z \in \mathbb{T} \) and \( n_0 \in \mathbb{Z}_+ \), and take \( T \in \mathcal{D}_\phi \), such that \( T_{n_0,n_0} = z \) and \( T_{n,n} = 1 \), if \( \phi^n(t) \neq \phi^n(t) \). Using the argument of the claim of \( \mathfrak{A} \) theorem 2.9) we can find a sequence of \( (f_j)_j \) in \( \text{ball}(\mathfrak{A}) \) such that \( w^*\lim_j \pi_t(f_j) = T \). To complete the proof, observe that products of elements of this form approximate the unitaries in \( \mathcal{D}_\phi \) in the \( w^* \)-topology and that the strong closure of \( \pi_t(\text{ball}(\mathfrak{A})) \) is closed under multiplication.

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