PRIMITIVE IDEALS OF THE RING OF DIFFERENTIAL OPERATORS ON AN AFFINE TORIC VARIETY

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Abstract. Let $A$ be a $d \times n$ integer matrix whose column vectors generate the lattice $\mathbb{Z}^d$, and let $D(R_A)$ be the ring of differential operators on the affine toric variety defined by $A$.

We show that the classification of $A$-hypergeometric systems and that of $\mathbb{Z}^d$-graded simple $D(R_A)$-modules (up to shift) are the same. We then show that the set of $\mathbb{Z}^d$-homogeneous primitive ideals of $D(R_A)$ is finite. Furthermore, we give conditions for the algebra $D(R_A)$ being simple.

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1. Introduction

Let $A$ be a $d \times n$ integer matrix whose column vectors generate the lattice $\mathbb{Z}^d$. Let $R_A$ be the ring of regular functions on the affine toric variety defined by $A$, and $D(R_A)$ its ring of differential operators.

In this paper, we show the following three theorems:

1. The classification of $A$-hypergeometric systems and that of $\mathbb{Z}^d$-graded simple $D(R_A)$-modules (up to shift) are the same (Theorem 5.10).
2. The set of $\mathbb{Z}^d$-homogeneous primitive ideals of $D(R_A)$ is finite (Theorem 7.6).
3. The algebra $D(R_A)$ is simple if and only if $R_A$ is a scored semigroup ring, and $A$ satisfies a certain condition (C2) (Theorem 8.25).

The ring of differential operators was introduced by Grothendieck [6] and Sweedler [13]. As for the ring of differential operators $D(R_A)$ on an affine toric variety, many recent papers such as Jones [8], Musson [11], and Musson and Van den Bergh [12] describe the structure of $D(R_A)$ when $R_A$ is normal. For general $R_A$, we studied the finite generation

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of $D(R_A)$ and its graded ring with respect to the order filtration in [13] and [10]. Moreover, in [15], we showed that the algebra $D(R_A)$ and the symmetry algebra (the algebra of contiguity operators) of $A$-hypergeometric systems are anti-isomorphic to each other. This paper may be considered as a continuation of [15].

The history of $A$-hypergeometric systems (or GKZ hypergeometric systems) goes back to Kalnins, Manocha, and Miller [9], and Hrabowski [7]. After the papers by Gel’fand, Kapranov, and Zelevinskii (e.g. [2]–[5]), researchers in various fields have studied the systems, and established connection with representation theory, algebraic geometry, commutative ring theory, etc. See, for example, the bibliography of [14].

Associated to a parameter vector $\alpha$ and a face $\tau$ of the cone generated by the column vectors of $A$, we defined a finite set $E_\tau(\alpha)$ (see (8) in Section 7) in [13], and proved that the $A$-hypergeometric systems are classified by those finite sets. Hence in order to show that the classification of $A$-hypergeometric systems and that of $\mathbb{Z}^d$-graded simple $D(R_A)$-modules (up to shift) are the same, we only need to show that $\mathbb{Z}^d$-graded simple $D(R_A)$-modules can be classified (up to shift) by the finite sets $E_\tau(\alpha)$ as shown in Theorem 4.4 essentially in [12] and [15]. It is however desirable to show the equivalence intrinsically. We therefore present the second proof of the equivalence by connecting $A$-hypergeometric systems with certain $\mathbb{Z}^d$-graded $D(R_A)$-modules by functors (Corollary 5.9).

Some topics discussed in this paper were treated in [12] under the conditions (A1) and (A2) (see p.4 in [12]). In our case, (A1) is always satisfied, and (A2) requiring that all $\mathbb{Z}^d$-homogeneous components $D(R_A)_\alpha$ are singly generated $D(R_A)_0$-modules is equivalent to requiring that $R_A$ satisfies the Serre’s ($S_2$) condition (see Proposition 5.7). In this paper, we do not assume the Serre’s ($S_2$) condition.

The layout of this paper is as follows: we start with recalling the definitions and some fundamental facts about differential operators in Section 2, and about $A$-hypergeometric systems in Section 3. In Section 4, we recall some results by Musson and Van den Bergh [12] about the category $\mathcal{O}$, analogous to the Bernstein-Gel’fand-Gel’fand’s category $\mathcal{O}$, and the counterpart $R\mathcal{O}$ for right $D(R_A)$-modules. Then we recall realizations $L(\alpha)$ ($R\tilde{L}(\alpha)$) of simple objects in $\mathcal{O}$ ($R\tilde{O}$) from [15], and we show that we have a duality between $\mathcal{O}$ and $R\tilde{O}$ sending $L(\alpha)$ and $R\tilde{L}(\alpha)$ to each other. We also show that any $\mathbb{Z}^d$-graded simple $D(R_A)$-module is isomorphic (up to shift) to $L(\alpha)$. These combined prove that the classifications of $L(\alpha)$, $R\tilde{L}(\alpha)$, their projective covers $M(\alpha)$,
$^R M(\alpha)$, and $\alpha$ with respect to the equivalence relation determined by the finite sets $E_r(\alpha)$ are the same.

In Section 5, we provide two functors between the category of right $D(R_A)$-modules and the category of right $D(R)$-modules supported by the affine toric variety defined by $A$, where $D(R)$ is the $n$-th Weyl algebra. One is the direct image functor, for right $D$-modules, of the closed inclusion of the affine toric variety into $\mathbb{C}^n$, and the other is its right adjoint functor. By using these functors, we prove that $^R M(\alpha) \simeq ^R M(\beta)$ if and only if their corresponding $A$-hypergeometric systems are isomorphic (Theorem 5.10).

After we see a couple of basic facts about primitive ideals in Section 6, we show in Section 7 that if we properly perturb a parameter $\alpha$ then the annihilator ideal $\text{Ann} L(\alpha)$ remains unchanged, and in this way we show that the set $\text{Prim} D(R_A)$ of $\mathbb{Z}^d$-homogeneous primitive ideals of $D(R_A)$ is finite (Theorem 7.6).

In Section 8, the simplicity of $D(R_A)$ is treated. First we consider the conditions: the scoredness and the Serre’s $(S_2)$. We prove that the simplicity of $D(R_A)$ implies the scoredness, and that the conditions $(A2)$ and $(S_2)$ are equivalent. Finally we give a necessary and sufficient condition for the simplicity (Theorem 8.26). In Section 9, we give an example of computation of the set $\text{Prim} D(R_A)$, and an example such that $R_A$ is scored and Cohen-Macaulay, but $D(R_A)$ is not simple.

2. Ring of differential operators on an affine toric variety

In this section, we recall some fundamental facts about the rings of differential operators of semigroup algebras.

Let $A := \{ a_1, a_2, \ldots, a_n \}$ be a finite set of column vectors in $\mathbb{Z}^d$. Sometimes we identify $A$ with the matrix $(a_1, a_2, \ldots, a_n)$. Let $NA$ and $ZA$ denote the monoid and the abelian group generated by $A$, respectively. Throughout this paper, we assume that $ZA = \mathbb{Z}^d$ for simplicity.

Let $R$ denote the polynomial ring $\mathbb{C}[x] := \mathbb{C}[x_1, \ldots, x_n]$. The semigroup algebra $R_A := \mathbb{C}[NA] = \bigoplus_{\alpha \in NA} \mathbb{C} t^\alpha$ is the ring of regular functions on the affine toric variety defined by $A$, where $t^\alpha = t_1^{a_1} t_2^{a_2} \cdots t_d^{a_d}$ for $\alpha = t(a_1, a_2, \ldots, a_d)$. Then we have $R_A \simeq R/I_A(x)$, where $I_A(x)$ is the ideal of $\mathbb{C}[x]$ generated by all $x^u - x^v$ for $u, v \in \mathbb{N}^n$ with $Au = Av$.

Let $M, N$ be $R$-modules. We briefly recall the module $D(M, N)$ of differential operators from $M$ to $N$. For details, see [17]. For $k \in \mathbb{N}$, the subspaces $D^k(M, N)$ of $\text{Hom}_R(M, N)$ are inductively defined by

$$D^0(M, N) = \text{Hom}_R(M, N)$$
and
\[
D^{k+1}(M, N) = \{ P \in \text{Hom}_\mathbb{C}(M, N) : [f, P] \in D^k(M, N) \quad (\forall f \in R) \},
\]
where \([ , ]\) denotes the commutator. Set \(D(M, N) : = \bigcup_{k=0}^\infty D^k(M, N)\), and \(D(M) : = D(M, M)\). Then \(D(M)\) is a \(\mathbb{C}\)-algebra, and \(D(M, N)\) is a \((D(N), D(M))\)-bimodule. Hence \(D(R, R_A)\) is a \((D(R_A), D(R))\)-module.

The ring \(D(R)\) is the \(n\)-th Weyl algebra
\[
D(R) = \mathbb{C}\langle x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \rangle,
\]
where \([\frac{\partial}{\partial x_i}, x_j] = \delta_{ij}\), and the other pairs of generators commute. Here \(\delta_{ij}\) is 1 if \(i = j\) and 0 otherwise.

Let \(\mathbb{C}[t, t^{-1}]\) denote the Laurent polynomial ring \(\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]\). Then its ring of differential operators \(D(\mathbb{C}[t, t^{-1}])\) is the ring
\[
\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]\langle \partial_1, \ldots, \partial_d \rangle,
\]
where \([\partial_i, t_j] = \delta_{ij}\), \([\partial_i, t_j^{-1}] = -\delta_{ij}t_j^{-2}\), and the other pairs of generators commute. The ring of differential operators \(D(R_A)\) can be realized as a subring of the ring \(D(\mathbb{C}[t, t^{-1}])\) by
\[
D(R_A) = \{ P \in \mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]\langle \partial_1, \ldots, \partial_d \rangle : P(R_A) \subseteq R_A \}.
\]

Put \(s_j := t_j \partial_j\) for \(j = 1, 2, \ldots, d\). Then it is easy to see that \(s_j \in D(R_A)\) for all \(j\). We introduce a \(\mathbb{Z}^d\)-grading on the ring \(D(R_A)\); for \(a = (a_1, a_2, \ldots, a_d) \in \mathbb{Z}^d\), set
\[
D(R_A)_a := \{ P \in D(R_A) : [s_j, P] = a_j P \quad \text{for} \quad j = 1, 2, \ldots, d \}.
\]

Then \(D(R_A) = \bigoplus_{a \in \mathbb{Z}^d} D(R_A)_a\).

By regarding \(D(R_A)_a\) as a subset of \(\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]\langle \partial_1, \ldots, \partial_d \rangle\), we see that there exists an ideal \(I \subset \mathbb{C}[s] := \mathbb{C}[s_1, \ldots, s_d]\) such that \(D(R_A)_a = t^a I\). To describe this ideal \(I\) explicitly, we define a subset \(\Omega(a)\) of the semigroup \(\mathbb{N}A\) by
\[
\Omega(a) = \{ b \in \mathbb{N}A : b + a \not\in \mathbb{N}A \} = \mathbb{N}A \setminus (-a + \mathbb{N}A).
\]

Then each \(D(R_A)_a\) is described as follows.

**Theorem 2.1** ([8], Theorem 3.3.1 in [15]).
\[
D(R_A)_a = t^a \mathbb{P}(\Omega(a)) \quad \text{for all} \quad a \in \mathbb{Z}^d,
\]
where
\[
\mathbb{P}(\Omega(a)) := \{ f(s) \in \mathbb{C}[s] : f \ \text{vanishes on} \ \Omega(a) \}.
\]

In particular, \(D(R_A)_a = t^a \mathbb{C}[s] = \mathbb{C}[s]t^a\) for each \(a \in \mathbb{N}A\), since \(\Omega(a) = \emptyset\) in this case.
3. A-hypergeometric systems

Let us briefly recall the definition of an A-hypergeometric system and its classification.

Let $\alpha = t(\alpha_1, \ldots, \alpha_d) \in \mathbb{C}^d$. The A-hypergeometric system with parameter $\alpha$ is the left $D(R)$-module

$$H_A(\alpha) := D(R) / \left( \sum_{i=1}^{d} D(R)(\sum_{j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_j} - \alpha_i) + D(R)I_A(\partial) \right),$$

where $a_j = t(a_{1j}, a_{2j}, \ldots, a_{dj})$, $I_A(\partial)$ is the ideal of $C[\partial_{x_1}, \ldots, \partial_{x_n}]$ generated by all $\prod_{j=1}^{n} \partial_{u_j} - \prod_{j=1}^{n} \partial_{v_j} x_j$ for $u, v \in \mathbb{N}^n$ with $Au = Av$.

Interchanging $x_j$ and $\partial_{x_j}$ for all $j$, we have an anti-automorphism $\iota$ of $D(R)$. Clearly $\iota$ gives rise to a one-to-one correspondence between the left $D(R)$-modules and the right $D(R)$-modules. Thus $\iota$ induces a right $D(R)$-module

$$R^H_A(\alpha) := D(R) / \left( \sum_{i=1}^{d} (\sum_{j=1}^{n} a_{ij} x_j \frac{\partial}{\partial x_j} - \alpha_i)D(R) + I_A(x)D(R) \right).$$

Note that $\iota(x_j \frac{\partial}{\partial x_j}) = \iota(\frac{\partial}{\partial x_j})\iota(x_j) = x_j \frac{\partial}{\partial x_j}$.

In [15, Definition 4.1.1], we have introduced a partial order into the parameter space $\mathbb{C}^d$ (see [10]), which is equivalent by [15, Lemma 4.1.4] to

$$\alpha \preceq \beta \iff I(\Omega(\beta - \alpha)) \not\subseteq m_\alpha,$$

where $m_\alpha$ is the maximal ideal of $\mathbb{C}[s]$ at $\alpha$. Note that, if $\beta - \alpha \notin \mathbb{Z}^d$, then $\Omega(\beta - \alpha) = NA$, and hence $\alpha \not\in \beta$. We write $\alpha \sim \beta$ if $\alpha \preceq \beta$ and $\alpha \succeq \beta$. This equivalence relation was introduced also by Musson and Van den Bergh (see [12, Lemma 3.1.9 (6)]).

This relation classifies A-hypergeometric systems.

**Theorem 3.1** (Theorem 2.1 in [13]). $H_A(\alpha) \simeq H_A(\beta)$ if and only if $\alpha \sim \beta$.

4. $D(R_A)$-modules

In this section, we recall some results by Musson and Van den Bergh [12] about the category $O$, and the counterpart $R^O$ for right $D(R_A)$-modules. Then we recall realizations $L(\alpha)$ ($R^L(\alpha)$) of simple objects in $O$ ($R^O$) from [15], and we show that we have a duality between $O$ and $R^O$ sending $L(\alpha)$ and $R^L(\alpha)$ to each other. We also show that any $\mathbb{Z}^d$-graded simple $D(R_A)$-module is isomorphic (up to shift) to $L(\alpha)$. These combined prove that the classifications of $L(\alpha)$, $R^L(\alpha)$, their
projective covers $M(\alpha)$, $R^*M(\alpha)$, and $\alpha$ with respect to the equivalence relation $\sim$ are the same.

4.1. Left modules. Let us recall the full subcategory $\mathcal{O}$ of the category of left $D(R_A)$-modules introduced in [12], which is an analogue of the Bernstein-Gel’fand-Gel’fand’s category $\mathcal{O}$ for the study of highest weight modules of semisimple Lie algebras. A left $D(R_A)$-module $M$ is an object of $\mathcal{O}$ if $M$ has a weight decomposition $M = \bigoplus_{\lambda \in \mathbb{C}^d} M_\lambda$ with each $M_\lambda$ finite-dimensional, where

$$M_\lambda = \{x \in M : f(s).x = f(\lambda)x \quad (\text{for all } f \in \mathbb{C}[s])\}.$$  

We call $\lambda$ a weight of $M$ if $M_\lambda \neq 0$.

For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{C}^d$, set

$$M(\alpha) := D(R_A)/D(R_A)(s - \alpha),$$

where $D(R_A)(s - \alpha)$ means $\sum_{i=1}^d D(R_A)(s_i - \alpha_i)$. Then $M(\alpha) = \bigoplus_{\lambda \in \alpha + \mathbb{Z}A} M(\alpha)_\lambda$, and $M(\alpha) \in \mathcal{O}$.

Among others, Musson and Van den Bergh proved the following.

**Proposition 4.1** (Proposition 3.1.7 in [12]).

1. $\text{Hom}_{D(R_A)}(M(\alpha), M) = M_\alpha$ for $M \in \mathcal{O}$.
2. $M(\alpha)$ is a projective object in $\mathcal{O}$.
3. $M(\alpha)$ has a unique simple quotient (denoted by $L(\alpha)$).
4. All simple objects in $\mathcal{O}$ are of the form $L(\alpha)$.
5. The natural projection $M(\alpha) \to L(\alpha)$ is the projective cover.
6. $M(\alpha) \simeq M(\beta)$ if and only if $L(\alpha) \simeq L(\beta)$.

**Remark 4.2.** Musson and Van den Bergh assumed the conditions $(A1)$ and $(A2)$ (see p.4 in [12]). In our case, $(A1)$ is always satisfied, and $(A2)$ requiring that all $D(R_A)_\alpha$ are singly generated $\mathbb{C}[s]$-modules is equivalent to requiring that $R_A$ satisfies the Serre’s ($S_2$) condition (see Proposition 8.7). For Proposition 4.1, we do not need the condition $(A2)$.

**Remark 4.3.** Let $M \in \mathcal{O}$, and let $N$ be a left $D(R_A)$-submodule of $M$. Then $N \in \mathcal{O}$. Hence, if $M$ is a simple object in $\mathcal{O}$, then $M$ is a simple left $D(R_A)$-module.

Let $\alpha \in \mathbb{C}^d$. In [15], we studied the composition factors of a $D(R_A)$-module

$$\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]t^\alpha,$$

and we saw that

$$\bigoplus_{\lambda \geq \alpha} \mathbb{C}t^\lambda / \bigoplus_{\lambda \prec \alpha} \mathbb{C}t^\lambda$$
is simple \[15\] Theorem 4.1.6], where \( \lambda \succ \alpha \) means \( \lambda \succeq \alpha \) and \( \lambda \not \sim \alpha \). The \( D(R_A) \)-module \[2\] is a simple quotient of \( M(\alpha) \), and hence a realization of \( L(\alpha) \). In particular, the set of weights of \( L(\alpha) \) is

\[ \{ \lambda \in \mathbb{C}^d : \lambda \sim \alpha \} \]

**Theorem 4.4** (cf. Lemma 3.1.9 (6) in \[12\]). \( L(\alpha) \simeq L(\beta) \) if and only if \( \alpha \sim \beta \).

**Proof.** By the realization \([2]\), \( L(\alpha) = L(\beta) \) if \( \alpha \sim \beta \).

If \( \alpha \not \sim \beta \), then \( L(\alpha) \) and \( L(\beta) \) have different weights. Hence \( L(\alpha) \neq L(\beta) \).

**Proposition 4.5.** Let \( M \) be a \( \mathbb{Z}^d \)-graded simple left \( D(R_A) \)-module. Then \( M \) is isomorphic to \( L(\alpha) \) for some \( \alpha \) as a left \( D(R_A) \)-module.

**Proof.** Recall that \( D(R_A)_0 = \mathbb{C}[s] \). First we show that each nonzero \( M_\lambda \) is a simple \( \mathbb{C}[s] \)-module. Suppose that \( N_\lambda \) is a nontrivial \( \mathbb{C}[s] \)-submodule of \( M_\lambda \). Put \( N := \bigoplus_{a \in \mathbb{Z}^d} N_{\lambda + a} = \bigoplus_{a \in \mathbb{Z}^d} D(R_A)_a N_\lambda \). Then \( N \) is a nontrivial \( \mathbb{Z}^d \)-graded submodule of \( M \), which contradicts the assumption.

Suppose that \( M_\lambda \neq 0 \). By the first paragraph, there exists \( \alpha \in \mathbb{C}^d \) such that \( M_\lambda \simeq \mathbb{C}[s]/m_\alpha \) as \( \mathbb{C}[s] \)-modules. Let \( M[\lambda - \alpha] \) be the \( \mathbb{Z}^d \)-graded \( D(R_A) \)-module shifted by \( \lambda - \alpha \), i.e., \( M[\lambda - \alpha]_\mu = M_{\mu + \lambda - \alpha} \). Then \( M[\lambda - \alpha] = \bigoplus_{\mu \in \mathbb{Z}^d} M[\lambda - \alpha]_\mu \) and \( M[\lambda - \alpha]_{\alpha + a} = D(R_A)_a M_\lambda \). Hence \( M[\lambda - \alpha] \in \mathcal{O} \), and \( M[\lambda - \alpha] \simeq L(\alpha) \in \mathcal{O} \).

4.2. **Right modules.** A right \( D(R_A) \)-module \( M \) is an object of \( R\mathcal{O} \) if \( M \) has a weight decomposition \( M = \bigoplus_{\lambda \in \mathbb{C}^d} M_\lambda \) with each \( M_\lambda \) finite-dimensional, where

\[ M_\lambda = \{ x \in M : x.f(s) = f(-\lambda)x \quad (\forall f \in \mathbb{C}[s]) \} \]

We can make a parallel argument about the categories \( \mathcal{O} \) and \( R\mathcal{O} \). Indeed we shall show that there exists a duality functor between them.

For \( \alpha \in \mathbb{C}^d \), set

\[ R M(\alpha) := D(R_A)/(s - \alpha)D(R_A) \]

Then \( R M(\alpha) = \bigoplus_{\lambda \in \mathbb{C}^d} R M(\alpha)_{\lambda} \), and \( R M(\alpha) \in B\mathcal{O} \).

The following proposition is proved in the same way as Proposition \[13\].

**Proposition 4.6.**

1. \( \text{Hom}_{D(R_A)}(R M(\alpha), M) = M_{-\alpha} \) for \( M \in B\mathcal{O} \).
2. \( R M(\alpha) \) is a projective object in \( B\mathcal{O} \).
3. \( R M(\alpha) \) has a unique simple quotient (denoted by \( R L(\alpha) \)).
4. All simple objects in \( B\mathcal{O} \) are of the form \( R L(\alpha) \).
5. The natural projection \( R M(\alpha) \to R L(\alpha) \) is the projective cover.
(6) $RM(\alpha) \simeq RM(\beta)$ if and only if $RL(\alpha) \simeq RL(\beta)$.

The ring $D(R_A)$ is a subring of $\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}][\partial_1, \ldots, \partial_d]$, where we can take formal adjoint operators, and thus we can consider a right $D(R_A)$-module

$$\mathbb{C}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]t^\alpha dt.$$ 

Here the right action of $P = \sum a^t f_a(s)$ on this module is defined by

$$\left( g(t) \frac{dt}{t} \right)P := P^*(g) \frac{dt}{t},$$ 

where $P^* = \sum a^t f_a(-s)t^a$, and recall that $s_i = t_i\partial_i \ (i = 1, \ldots, d)$.

Then

$$(3) \bigoplus_{\beta \prec \alpha} \mathbb{C}t^{-\beta} \frac{dt}{t} / \bigoplus_{\beta \preceq \alpha} \mathbb{C}t^{-\beta} \frac{dt}{t}.$$ 

is a realization of $RL(\alpha)$.

Let $M \in \mathcal{O}(R\mathcal{O})$. Then $\text{Hom}_\mathbb{C}(M, \mathbb{C})$ is a right (left) $D(R_A)$-module. Define a right (left) $D(R_A)$-submodule $M^*$ of $\text{Hom}_\mathbb{C}(M, \mathbb{C})$ by

$$M^* := \bigoplus_{\lambda} M_{\lambda}^*, \quad M_{\lambda}^* := \text{Hom}_\mathbb{C}(M_{-\lambda}, \mathbb{C}).$$

Then $*: \mathcal{O} \to R\mathcal{O} \quad (*: \mathcal{O} \to \mathcal{O})$ is a duality functor. Hence we have the following proposition.

**Proposition 4.7.**

1. $\text{Hom}_{D(R_A)}(M, R\mathcal{M}(\alpha)^*) = \text{Hom}_\mathbb{C}(M_\alpha, \mathbb{C})$ for $M \in \mathcal{O}$.
2. $R\mathcal{M}(\alpha)^*$ is an injective object in $\mathcal{O}$.
3. $R\mathcal{M}(\alpha)^*$ has a unique simple subobject $RL(\alpha)^*$ in $\mathcal{O}$.
4. $L(\alpha) \simeq RL(\alpha)^*$.
5. The natural inclusion $RL(\alpha)^* \to R\mathcal{M}(\alpha)^*$ is the injective hull.
6. $L(\alpha) \simeq L(\beta)$ if and only if $RL(\alpha) \simeq RL(\beta)$.

**Proof.** (4) follows from the fact that two simple modules $L(\alpha)$ and $RL(\alpha)^*$ have the same weight spaces.

The other statements are clear. \qed

5. **A-hypergeometric systems and Category $\mathcal{O}$**

We have proved that the classification of $A$-hypergeometric systems and that of simple modules $L(\alpha)$ (or $RL(\alpha)$) are the same, by showing that simple modules $L(\alpha)$ are classified according to the equivalence relation $\sim$ in Theorem 4.4. In this section, we make another way to prove
the coincidence of the classifications; we connect \(A\)-hypergeometric systems \(H_A(\alpha)\) and right \(D(R_A)\)-modules \(M(\alpha)\) by functors. This proof is intrinsic, and hence more desirable.

5.1. The bimodule \(D(R, R_A)\). In this subsection, we decompose the \((D(R_A), D(R))\)-bimodule \(D(R, R_A)\) into its \(\mathbb{Z}^d\)-graded parts similarly to Theorem 2.1.

Let \(\mathbb{C}[x, x^{-1}]\) denote the Laurent polynomial ring \(\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]\). By [1, p. 31], we have

\[
D(R, R_A) = \{ P \in D(\mathbb{C}[x, x^{-1}], \mathbb{C}[t, t^{-1}]) : P(R) \subseteq R_A \}.
\]

From [17] 1.3 (e),

\[
D(\mathbb{C}[x, x^{-1}], \mathbb{C}[t, t^{-1}]) = D(\mathbb{C}[x, x^{-1}])/I_A(x)D(\mathbb{C}[x, x^{-1}]),
\]

and hence

\[
D(\mathbb{C}[x, x^{-1}], \mathbb{C}[t, t^{-1}]) = \bigoplus_{a \in \mathbb{Z}^d} t^a \mathbb{C}[\theta_1, \ldots, \theta_n],
\]

where \(\theta_j = x_j \frac{\partial}{\partial x_j} \) (\(j = 1, \ldots, n\)). From [17] 1.3 (e) again, we have

\[
(4) \quad D(R, R_A) = D(R)/I_A(x)D(R).
\]

Note that \(D(R_A) \subseteq D(R, R_A)\) by [11]. Here we identify \(t^a\) and \(s_i\) with \(x_j\) and \(\sum_{j=1}^n a_{ij} \theta_j\) respectively (\(j = 1, \ldots, n, i = 1, \ldots, d\)). In fact, we have

\[
D(R_A) = D(R, R_A) \cap D(\mathbb{C}[t^{\pm 1}])
\]

in \(D(\mathbb{C}[x, x^{-1}], \mathbb{C}[t, t^{-1}])\). The bimodule \(D(R, R_A)\) inherits the \(\mathbb{Z}^d\) grading from \(D(\mathbb{C}[x, x^{-1}], \mathbb{C}[t, t^{-1}])\),

\[
D(R, R_A)_a = D(R, R_A) \cap t^a \mathbb{C}[\theta_1, \ldots, \theta_n].
\]

Proposition 5.1.

\[
D(R, R_A) = \bigoplus_{a \in \mathbb{Z}^d} t^a \mathbb{C}[\tilde{\Omega}_A(a)],
\]

where \(\tilde{\Omega}_A(a) = N^a \cap T_A^{-1}(\Omega_A(a))\), \(T_A\) is the linear map from \(\mathbb{Z}^n\) to \(\mathbb{Z}^d\) defined by \(A\), and \(\mathbb{C}[\tilde{\Omega}_A(a)]\) is the ideal of \(\mathbb{C}[\theta] = \mathbb{C}[\theta_1, \ldots, \theta_n]\) vanishing on \(\tilde{\Omega}_A(a)\).
Proof. We have
\[
\begin{align*}
& t^a p(\theta) \in D(R, R_A)_a \\
& t^a p(\theta)(x^m) \in \mathbb{C}[NA] \quad (\forall m \in \mathbb{N}^n) \\
\Leftrightarrow & \quad p(m)t^{a+Am} \in \mathbb{C}[NA] \quad (\forall m \in \mathbb{N}^n) \\
\Leftrightarrow & \quad a + Am \in NA \text{ or } p(m) = 0 \text{ if } m \in \mathbb{N}^n \\
\Leftrightarrow & \quad p(m) = 0 \quad (\forall m \in \mathbb{N}^n \setminus T_A^{-1}(-a + NA)).
\end{align*}
\]
□

Corollary 5.2. \( D(R, R_A)_a = t^a\mathbb{C}[\theta] \) for all \( a \in NA \).

Proof. In this case \( \Omega(a) = \emptyset \). Hence \( \mathbb{I}(\overline{\Omega_A(a)}) = \mathbb{C}[\theta] \). □

To close this subsection, we describe the weight decomposition of \( RH_A(\alpha) \). Let
\[
RH_A(\alpha)_\lambda := \{ x \in RH_A(\alpha) : x.(\sum_{j=1}^{n} a_{ij}\theta_j + \lambda_i) = 0 \quad (i = 1, \ldots, d) \}
\]
for \( \lambda = t^i(\lambda_1, \ldots, \lambda_d) \). Note that the weight space \( D(R, R_A)_a \) is a \( (\mathbb{C}[s], \mathbb{C}[\theta]) \)-bimodule, since \( D(R_A)_0 = \mathbb{C}[s] \) and \( D(R)_0 = \mathbb{C}[\theta] \). We have
\[
RH_A(\alpha) = \bigoplus_{a \in \mathbb{Z}^d} RH_A(\alpha)_{-\alpha + a},
\]
\[
= \bigoplus_{a \in \mathbb{Z}^d} D(R, R_A)_a/(s - \alpha)D(R, R_A)_a
\]
\[
= \bigoplus_{a \in \mathbb{Z}^d} D(R, R_A)_a/D(R, R_A)_a(A\theta + a - \alpha)
\]
\[
= \bigoplus_{a \in \mathbb{Z}^d} t^a \left( \mathbb{I}(\overline{\Omega}(a))/\mathbb{I}(\overline{\Omega}(a))(A\theta + a - \alpha) \right).
\]

5.2. Functors. Let \( \text{Mod}^R(D(R_A)) \) denote the category of right \( D(R_A) \)-modules, and \( \text{Mod}^R_A(D(R)) \) the category of right \( D(R) \)-modules supported by the affine toric variety \( V(I_A(x)) \) defined by \( A \). A right \( D(R) \)-module \( N \) is said to be supported by \( V(I_A(x)) \) if for every \( x \in N \) there exists \( m \in \mathbb{N} \) such that \( xI_A(x)^m = 0 \).

Let \( \Phi \) denote the functor from \( \text{Mod}^R(D(R_A)) \) to \( \text{Mod}^R_A(D(R)) \) defined by
\[
\Phi(M) := M \otimes_{D(R_A)} D(R, R_A),
\]
and $Ψ$ the functor from $\text{Mod}^R_A(D(R))$ to $\text{Mod}^R(D(R_A))$ defined by

$$Ψ(N) := \text{Hom}_{D(R)}(D(R, R_A), N) = \{x ∈ N : xI_A(x) = 0\}.$$  

Then $Ψ$ is right adjoint to $Φ$,

$$\text{Hom}_{D(R)}(Φ(M), N) \simeq \text{Hom}_{D(R_A)}(M, Ψ(N)).$$

**Remark 5.3.** The functor $Φ$ is the direct image functor, for right $D$-modules, of the closed inclusion

$$V(I_A(x)) = \{x ∈ C^n : f(x) = 0 \quad (∀f ∈ I_A(x))\} → C^n.$$  

For a closed embedding between nonsingular varieties, such a direct image functor gives an equivalence of categories, known as Kashiwara’s equivalence (see e.g. [10, Theorem 4.30]). In our case, the affine toric variety $V(I_A(x))$ is singular whenever $n ≠ d$, and the cone $R_{≥0}A$ generated by $A$ is strongly convex.

We have

$$Φ(D(R_A)) = D(R, R_A),$$

and

$$Ψ(D(R, R_A)) = \text{End}_{D(R)}(D(R)/I_A(x)D(R)) = D(R_A).$$

The following proposition is immediate from the definitions.

**Proposition 5.4.** We have

$$Φ(^RM(\alpha)) = ^RHA(\alpha).$$  

Hence, if $^RM(\alpha) \simeq ^RM(\beta)$, then $^RHA(\alpha) \simeq ^RHA(\beta)$.

To show the inverse of Proposition 5.4, we need the following lemma.

**Lemma 5.5.**

$$\text{End}_{D(R)}(^RHA(\alpha)) = Cid.$$

**Proof.** Let $ψ ∈ \text{End}_{D(R)}(^RHA(\alpha))$. Since $ψ(\overline{1}) ∈ ^RHA(\alpha)−\alpha$, there exists a polynomial $f(θ) ∈ C[θ]$ such that $ψ(\overline{1}) = \overline{f(θ)}$. Here $\overline{f}$ is the element of $^RHA(\alpha)$ represented by $P ∈ D(R)$. Let $u, v ∈ N^n$ satisfy $Au = Av$. Then

$$0 = ψ(x^u − x^v) = ψ(\overline{1})(x^u − x^v) = \overline{f(θ)(x^u − x^v)} = \overline{t^Au(f(θ + u) − f(θ + v))}.$$  

By [5], we have

$$f(θ + u) − f(θ + v) ∈ (Aθ + Au − α)C[θ]$$
for $u,v \in \mathbb{N}^n$ with $Au = Av$. Hence $f(\theta) - f(\theta + l) \in (A\theta - \alpha)\mathbb{C}[\theta]$ for all $l \in L$, where $L = \{ l \in \mathbb{Z}^n : Al = 0 \}$. Letting $A\gamma = \alpha$ ($\gamma \in \mathbb{C}^n$), we have $f(\gamma) - f(\gamma + l) = 0$ for all $l \in L$. Thus $f(\gamma + \theta) \in f(\gamma) + (A\theta)\mathbb{C}[\theta]$, or equivalently

$$f(\theta) \in f(\gamma) + (A\theta - \alpha)\mathbb{C}[\theta].$$

Hence $\psi(\overline{1}) = \overline{f(\theta)} = f(\gamma)$. Therefore $\psi = f(\gamma)id$.

Let $\beta - \alpha \in \mathbb{Z}^d$, and $Q \in D(R_A)_{\beta - \alpha}$. Then

$$\phi_Q : ^R M(\alpha) \ni \overline{F} \mapsto \overline{QP} \in ^R M(\beta)$$

(6)

$$\psi_Q : ^R H_A(\alpha) \ni \overline{P} \mapsto \overline{QP} \in ^R H_A(\beta)$$

(7)

are well-defined morphisms. Clearly $\psi_Q = \Phi(\phi_Q)$.

**Lemma 5.6.** The natural map

$$D(R_A)_{\beta - \alpha} \ni Q \mapsto \phi_Q \in \text{Hom}_{D(R_A)}(^R M(\alpha), ^R M(\beta))$$

is surjective.

**Proof.** Let $\phi \in \text{Hom}_{D(R_A)}(^R M(\alpha), ^R M(\beta))$. Since $\phi(\overline{1}) \in ^R M(\beta)_{-\alpha}$, there exists $Q \in D(R_A)_{\beta - \alpha}$ such that $\phi(\overline{1}) = \overline{Q}$. Then $\phi = \phi_Q$. □

As to $\text{Hom}_{D(R)}(^R H_A(\alpha), ^R H_A(\beta))$, we have the following by Lemma 5.5.

**Corollary 5.7.** Suppose that $^R H_A(\alpha) \simeq ^R H_A(\beta)$. Then

$$\dim_{\mathbb{C}} \text{Hom}_{D(R)}(^R H_A(\alpha), ^R H_A(\beta)) = 1,$$

and the natural map $D(R_A)_{\beta - \alpha} \to \text{Hom}_{D(R)}(^R H_A(\alpha), ^R H_A(\beta))$ is surjective.

**Proof.** The first statement is immediate from Lemma 5.5. The second follows from the fact that in this case the image of the map is not zero by Lemma 5.3. □

Now we are in position of proving the inverse of Proposition 5.4. Note that there exists a natural morphism $M \to \Psi(\Phi(M))$ for $M \in \text{Mod}_R(D(R_A))$.

**Proposition 5.8.** Suppose that $^R H_A(\alpha) \simeq ^R H_A(\beta)$. Then

$$\text{Hom}_{D(R_A)}(^R M(\alpha), ^R M(\beta)) \simeq \text{Hom}_{D(R)}(^R H_A(\alpha), ^R H_A(\beta)).$$

**Proof.** Corollary 5.7 states that there exist $Q, R \in D(R_A)$ such that

$$\psi_Q : ^R H_A(\alpha) \ni \overline{P} \mapsto \overline{QP} \in ^R H_A(\beta)$$

$$\psi_R : ^R H_A(\beta) \ni \overline{P} \mapsto \overline{RP} \in ^R H_A(\alpha)$$

satisfy $\psi_R \circ \psi_Q = id_{^R H_A(\alpha)}$ and $\psi_Q \circ \psi_R = id_{^R H_A(\beta)}$. 
The image of the natural morphism $\mathcal{R} \mathcal{M}(\alpha) \to \Psi(\Phi(\mathcal{R} \mathcal{M}(\alpha))) = \Psi(\mathcal{R} H_A(\alpha))$ is $1_{\alpha} D(R_A)$, where

$$\Psi(\mathcal{R} H_A(\alpha)) = \text{Hom}_{D(R)}(D(R, R_A), \mathcal{R} H_A(\alpha)) \ni 1_{\alpha} : \overline{P} \mapsto \overline{P}.$$ 

The isomorphism $\psi_Q$ induces an isomorphism $\Psi(\psi_Q) : \text{Hom}_{D(R)}(D(R, R_A), \mathcal{R} H_A(\alpha)) \to \text{Hom}_{D(R)}(D(R, R_A), \mathcal{R} H_A(\beta)).$

We show that the restriction of $\Psi(\psi_Q)$ to $1_{\alpha} D(R_A)$ gives an isomorphism $1_{\alpha} D(R_A) \cong 1_{\beta} D(R_A)$. By definition $\Psi(\psi_Q)(1_{\alpha}) = 1_{\beta} Q \in 1_{\beta} D(R_A)$. Hence $\Psi(\psi_Q)(1_{\alpha} D(R_A)) \subseteq 1_{\beta} D(R_A)$. In addition, we have $1_{\beta} = 1_{\beta} Q R = \Psi(\psi_Q)(1_{\alpha} R)$.

Hence $\Psi(\psi_Q)(1_{\alpha} D(R_A)) = 1_{\beta} D(R_A)$. We have thus proved that $\psi_Q$ induces an isomorphism $1_{\alpha} D(R_A) \cong 1_{\beta} D(R_A)$. It is lifted to an isomorphism between their projective covers in $\mathcal{R} \mathcal{O}$, $\mathcal{R} M(\alpha) \cong \mathcal{R} M(\beta)$, which is clearly $\phi_Q$. Thus $\phi_Q$ and $\psi_Q$ correspond to each other.

Combining Propositions 5.4 and 5.8, we have the following.

**Corollary 5.9.** $\mathcal{R} M(\alpha) \cong \mathcal{R} M(\beta)$ if and only if $\mathcal{R} H_A(\alpha) \cong \mathcal{R} H_A(\beta)$.

We summarize the classification.

**Theorem 5.10.** The following are equivalent:

1. $\alpha \sim \beta$.
2. $H_A(\alpha) \cong H_A(\beta)$.
3. $\mathcal{R} H_A(\alpha) \cong \mathcal{R} H_A(\beta)$.
4. $M(\alpha) \cong M(\beta)$.
5. $\mathcal{R} M(\alpha) \cong \mathcal{R} M(\beta)$.
6. $L(\alpha) \cong L(\beta)$.
7. $\mathcal{R} L(\alpha) \cong \mathcal{R} L(\beta)$.

6. **Primitive ideals**

In general, a left primitive ideal is not necessarily a right primitive ideal. In our case, however, we have the following proposition thanks to the duality.

**Proposition 6.1.** For each $\alpha \in \mathbb{C}^d$, the annihilators $\text{Ann}(L(\alpha))$ and $\text{Ann}(\mathcal{R} L(\alpha))$ coincide, i.e.,

$$\text{Ann}(L(\alpha)) = \text{Ann}(\mathcal{R} L(\alpha)).$$

**Proof.** By definition, it is clear that $\text{Ann} M \subseteq \text{Ann} M^*$.
for $M \in \mathcal{O}(R)$. Since $RL(\alpha)^* = L(\alpha)$ and $L(\alpha)^* = RL(\alpha)$, the statement follows.

Hence the set of annihilators of $\mathbb{Z}^d$-graded simple left $D(R_A)$-modules and that of $\mathbb{Z}^d$-graded simple right $D(R_A)$-modules are the same. We denote it by $\text{Prim}(D(R_A))$,

$$\text{Prim}(D(R_A)) = \{ \text{Ann}(L(\alpha)) : \alpha \in \mathbb{C}^d \}.$$ 

Next we describe the graded components of the annihilator ideal $\text{Ann} L(\alpha)$. For $\alpha \in \mathbb{C}^d$ and $a \in \mathbb{Z}^d$, we define a subset $\Lambda_{[\alpha]}(a)$ of $\alpha + \mathbb{Z}^d$ by

$$\Lambda_{[\alpha]}(a) := \{ \beta \in \alpha + \mathbb{Z}^d : \beta \sim \alpha, \beta + a \sim \alpha \}.$$ 

**Proposition 6.2** (Proposition 3.2.2 in [12]). Let

$$(\text{Ann} L(\alpha))_a = \text{Ann} L(\alpha) \cap D(R_A)_a$$

for $a \in \mathbb{Z}^d$. Then

$$(\text{Ann} L(\alpha))_a = t^a(\Omega(a) \cup \Lambda_{[\alpha]}(a))$$

for all $a$.

**Proof.** Let $\beta \sim \alpha$, and let $0 \neq v_{\beta} \in L(\alpha)_{\beta}$. Then

$$\text{Ann}(v_{\beta})_a = \begin{cases} D(R_A)_a & (\beta + a \not\sim \alpha) \\ D(R_A)_a \cap t^a(s - \beta) & (\beta + a \sim \alpha) \end{cases}$$

Hence

$$(\text{Ann} L(\alpha))_a = \bigcap_{\beta \sim \alpha, \beta + a \sim \alpha} D(R_A)_a \cap t^a(s - \beta).$$

We have thus proved the assertion.}

7. **Finiteness**

In this section, we prove that the set $\text{Prim}(D(R_A))$ is finite. If $\alpha - \beta \notin \mathbb{Z}^d$, then $\alpha \not\sim \beta$, and hence there exist infinitely many isomorphism classes of simple objects $L(\alpha)$. We however show that if we properly perturb a parameter $\alpha$ then the annihilator ideal $\text{Ann} L(\alpha)$ remains unchanged.

First we recall the primitive integral support function of a facet (maximal proper face) of the cone $\mathbb{R}_{\geq 0}A$. We denote by $\mathcal{F}$ the set of facets of the cone $\mathbb{R}_{\geq 0}A$. Given $\sigma \in \mathcal{F}$, we denote by $F_{\sigma}$ the primitive integral support function of $\sigma$, i.e., $F_{\sigma}$ is a uniquely determined linear form on $\mathbb{R}^d$ satisfying

1. $F_{\sigma}(\mathbb{R}_{\geq 0}A) \geq 0$, 

linearization of any face of \( R_\sigma \).

Example 7.1. Let

\[
\begin{align*}
F(\alpha) : &= \{ \sigma \in F : F_\sigma(\alpha) \in \mathbb{Z} \}, \\
V(\alpha) : &= \bigcap_{\sigma \in F(\alpha)} (F_\sigma = 0).
\end{align*}
\]

Clearly \( \mu \in V(\alpha) \) implies \( F(\alpha) \subseteq F(\alpha + \mu) \). Note that \( V(\alpha) \) may not be a linearization of a face.

Next we briefly review the finite sets \( E_\tau(\alpha) \) defined in [13]. Associated to a parameter vector \( \alpha \in \mathbb{C}^d \) and a face \( \tau \) of the cone \( \mathbb{R}_{\geq 0}A \), \( E_\tau(\alpha) \) was defined by

\[
E_\tau(\alpha) = \{ \lambda \in \mathbb{C}(A \cap \tau)/\mathbb{Z}(A \cap \tau) : \alpha - \lambda \notin NA + \mathbb{Z}(A \cap \tau) \}.
\]

The set \( E_\tau(\alpha) \) has at most \( \# [\mathbb{Q}(A \cap \tau) \cap \mathbb{Z}^d : \mathbb{Z}(A \cap \tau)] \) elements. Indeed, suppose that \( \lambda \in \mathbb{C}(A \cap \tau) \) satisfies \( \alpha - \lambda \in \mathbb{Z}^d \). Then \( E_\tau(\alpha) \) is a subset of

\[
(\lambda + \mathbb{Q}(A \cap \tau) \cap \mathbb{Z}^d)/\mathbb{Z}(A \cap \tau)
\]

(see [13 Proposition 2.3]). For a facet \( \sigma \), \( E_\sigma(\alpha) = \emptyset \) if and only if \( F_\sigma(\alpha) \in F_\sigma(NA) \), and, for faces \( \tau \preceq \tau' \), \( E_\tau(\alpha) = \emptyset \) implies \( E_{\tau'}(\alpha) = \emptyset \) (see [13 Proposition 2.2]).

We have already introduced a partial ordering \( \preceq \) into the parameter space \( \mathbb{C}^d \) in [11]. The following is its original definition in [15 Definition 4.1.1]: For \( \alpha, \beta \in \mathbb{C}^d \), we write

\[
\alpha \preceq \beta \quad \text{if} \quad E_\tau(\alpha) \subseteq E_\tau(\beta) \quad \text{for all faces} \ \tau.
\]

Hence \( \alpha \sim \beta \) if and only if \( E_\tau(\alpha) = E_\tau(\beta) \) for all faces \( \tau \). Note that \( \alpha + \mathbb{Z}^d \) has only finitely many equivalence classes, since \( E_\tau(\alpha) \) and
Lemma 7.2. Suppose that \( \mu \in V(\alpha) \) and \( F(\alpha + \mu) = F(\alpha) \). Let \( \tau \) be a face of the cone \( \mathbb{R}_{\geq 0}A \). Then \( E_\tau(\alpha + \mu) = \emptyset \) if and only if \( E_\tau(\alpha) = \emptyset \).
Moreover, if \( E_\tau(\alpha) \neq \emptyset \), then
\[
E_\tau(\alpha + \mu) = \mu + E_\tau(\alpha).
\]
Proof. By symmetry, it is sufficient to prove that if \( \lambda \in E_\tau(\alpha) \) then \( \mu \in C(A \cap \tau) \) and \( \lambda + \mu \in E_\tau(\alpha + \mu) \).

Suppose that \( \lambda \in E_\tau(\alpha) \). Then \( F_\sigma(\alpha) = F_\sigma(\alpha - \lambda) \in F_\sigma(N_A) \subseteq N \) for all facets \( \sigma \geq \tau \). Hence \( C(A \cap \tau) \) is the intersection of a subset of \( \mathcal{F}(\alpha) \), and thus \( C(A \cap \tau) \supseteq V(\alpha) \). This proves \( \mu \in C(A \cap \tau) \). Since \( \alpha + \mu - (\lambda + \mu) = \alpha - \lambda \in NA + Z(A \cap \tau) \), we see \( \lambda + \mu \in E_\tau(\alpha + \mu) \).

Corollary 7.3. Suppose that \( \mu \in V(\alpha) \) and \( F(\alpha + \mu) = F(\alpha) \). Then \( \alpha \sim \beta \) if and only if \( \alpha + \mu \sim \beta + \mu \).
Proof. If \( \alpha - \beta \notin ZA \), then \( \alpha \not\sim \beta \) and \( \alpha + \mu \not\sim \beta + \mu \).

Suppose that \( \alpha - \beta \in ZA \). Then \( F(\alpha) = F(\beta) \), and \( F(\alpha + \mu) = F(\beta + \mu) \). The assertion follows from Lemma 7.2.

Proposition 7.4. Suppose that \( \mu \in V(\alpha) \) and \( F(\alpha + \mu) = F(\alpha) \). Then
\[
(\text{Ann}(L(\alpha))) = \text{Ann}(L(\alpha + \mu)).
\]
Proof. Recall Proposition 6.2
\[
(\text{Ann}(L(\alpha)))_\alpha = \mathfrak{a}_\alpha(\Omega(\alpha) \cup \Lambda_{|\alpha|}(\alpha)),
\]
where \( \Lambda_{|\alpha|}(\alpha) = \{ \beta : \beta \sim \beta + a \sim \alpha \} \). We prove that
\[
ZC(\Lambda_{|\alpha|+\mu}(\alpha)) = ZC(\Lambda_{|\alpha|}(\alpha)),
\]
where \( ZC \) stands for Zariski closure. Since \( \Lambda_{|\alpha|+\mu}(\alpha) = \emptyset \) if and only if \( \Lambda_{|\alpha|}(\alpha) = \emptyset \) by Corollary 7.3, we suppose that they are not empty. Then again by Corollary 7.3
\[
\Lambda_{|\alpha|+\mu}(\alpha) = \mu + \Lambda_{|\alpha|}(\alpha).
\]
Hence for the proof it suffices to show that
\[
V(\alpha) + ZC(\Lambda_{|\alpha|}(\alpha)) = ZC(\Lambda_{|\alpha|}(\alpha)).
\]

Let \( \beta \in \Lambda_{|\alpha|}(\alpha) \), and \( \mu' \in V(\alpha) \cap ZA \). Put \( v := \prod_{\mathbb{C}_\tau \not\subseteq V(\alpha)} [ZA \cap \mathbb{Q}_\tau : Z(A \cap \tau)] \). We show that \( \beta + v\mu' \in \Lambda_{|\alpha|}(\alpha) \).

Suppose that \( \mathbb{C}_\tau \not\subseteq V(\alpha) \). Then there exists a facet \( \sigma \geq \tau \) such that \( \mathbb{C}_\sigma \not\supseteq V(\alpha) \). Thus \( F_\sigma(\alpha) \notin Z \). Note that \( v\mu', a, \alpha - \beta \in ZA \), since \( \alpha \sim \beta \). Hence \( F_\sigma(\beta), F_\sigma(\beta + v\mu'), F_\sigma(\beta + a + v\mu') \notin Z \). This
implies \( E_\sigma(\beta) = E_\sigma(\beta + v\mu') = E_\sigma(\beta + a + v\mu') = \emptyset = E_\sigma(\alpha) \), and 
\( E_\tau(\beta) = E_\tau(\beta + v\mu') = E_\tau(\beta + a + v\mu') = \emptyset = E_\tau(\alpha) \) by Proposition \( 2.2 \).

Next suppose that \( C_\tau \supseteq V(\alpha) \). Then 
\( E_\tau(\beta + v\mu') = E_\tau(\beta) \), and 
\( E_\tau(\beta + a + v\mu') = E_\tau(\beta + a) \), since \( v\mu' \in Z(A \cap \tau) \).

Hence \( \beta + v\mu' \in \Lambda(\alpha) \). We have thus proved that

\[
\Lambda(\alpha) + v(\alpha) \cap ZA \subseteq \Lambda(\alpha) \cap ZA.
\]

Taking the Zariski closures, we see that

\[
ZC(\Lambda(\alpha)) + V(\alpha) \subseteq ZC(\Lambda(\alpha))
\]

The other inclusion is trivial. \( \square \)

**Lemma 7.5.** Let \( \mathcal{F}' \) be a subset of \( \mathcal{F} \), and let \( V(\mathcal{F}') \) be the intersection 
\( \bigcap_{\sigma \in \mathcal{F}'}(F_\sigma = 0) \). Then

\[
\{ \alpha : \mathcal{F}(\alpha) \supseteq \mathcal{F}' \}/(ZA + V(\mathcal{F}'))
\]

is finite.

**Proof.** The \( \mathbb{Z} \)-module \( ZA + V(\mathcal{F}')/V(\mathcal{F}') \) is a \( \mathbb{Z} \)-submodule of \( \{ \alpha : \mathcal{F}(\alpha) \supseteq \mathcal{F}' \}/V(\mathcal{F}') \). Both of them are free of rank \( \dim \mathbb{C}^d/V(\mathcal{F}') \). Hence the index is finite. \( \square \)

**Theorem 7.6.** The set \( \text{Prim}(D(R_A)) \) is finite.

**Proof.** It suffices to show that the set

\[
\text{Prim}_{\mathcal{F}'} := \{ \text{Ann} L(\alpha) : \mathcal{F}(\alpha) = \mathcal{F}' \}
\]

is finite for each \( \mathcal{F}' \subseteq \mathcal{F} \).

Let \( \mathcal{F}(\alpha) = \mathcal{F}' \). Let \( \alpha_1, \ldots, \alpha_k \) be a complete set of representatives of \( \{ \alpha : \mathcal{F}(\alpha) \supseteq \mathcal{F}' \}/(ZA + V(\mathcal{F}')) \). We take the representatives so that if a coset \( \alpha_j + ZA + V(\mathcal{F}') \) has an element \( \beta \) with \( \mathcal{F}(\beta) = \mathcal{F}' \) then \( \mathcal{F}(\alpha_j) = \mathcal{F}' \). Then there exist \( j, a \in ZA \), and \( \mu \in V(\mathcal{F}') \) such that

\[
\alpha = \alpha_j + a + \mu.
\]

We have \( \mathcal{F}' = \mathcal{F}(\alpha_j) = \mathcal{F}(\alpha_j + a) \). By Proposition \( 7.4 \) \( \text{Ann} L(\alpha) = \text{Ann} L(\alpha_j + a) \). Since each \( \alpha_j + ZA \) has only finitely many equivalence classes, \( \text{Prim}_{\mathcal{F}'} \) is finite. \( \square \)

We exhibit a computation of \( \text{Prim}(D(R_A)) \) in Example \( 7.1 \).

## 8. Simplicity

In this section, we discuss the simplicity of \( D(R_A) \). In the first subsection, we consider the conditions: the scoredness and the Serre's \( (S_2) \). We prove that the simplicity of \( D(R_A) \) implies the scoredness, and that all \( D(R_A)_\alpha \) are singly generated \( \mathbb{C}[s] \)-modules if and only if
the condition \((S_2)\) is satisfied. In the second subsection, we give a necessary and sufficient condition for the simplicity (Theorem \ref{thm:simplicity}).

We start this section by noting the \(\mathbb{Z}^d\)-graded version of a well-known fact.

**Lemma 8.1.** The ring \(D(R_A)\) is simple if and only if \(\text{Ann} \ L(\alpha) = \{0\}\) for all \(\alpha \in \mathbb{C}^d\).

**Proof.** First note that any two-sided ideal of \(D(R_A)\) is \(\mathbb{Z}^d\)-homogeneous.\(\) An ideal \(I\) is said to be \(\mathbb{Z}^d\)-homogeneous if \(I = \bigoplus_{a \in \mathbb{Z}^d} I \cap D(R_A)a\).

It is enough to show that any maximal ideal of \(D(R_A)\) is the annihilator of a simple \(\mathbb{Z}^d\)-graded module. Let \(I\) be a maximal ideal of \(D(R_A)\). Let \(J\) be a maximal \(\mathbb{Z}^d\)-homogeneous left ideal containing \(I\). Then \(D(R_A)/J\) is a simple \(\mathbb{Z}^d\)-graded \(D(R_A)\)-module, and \(\text{Ann}(D(R_A)/J)\) contains \(I\). Since \(I\) is maximal, we obtain \(I = \text{Ann}(D(R_A)/J)\). \(\square\)

8.1. **Scored semigroups.** We recall the definition of a scored semigroup [13]. The semigroup \(N_A\) is said to be scored if

\[
N_A = \bigcap_{\sigma \in \mathcal{F}} \{ a \in \mathbb{Z}^d : F_\sigma(a) \in F_\sigma(NA) \}.
\]

We know that \(E_\sigma(a) \neq \emptyset\) if and only if \(F_\sigma(a) \in F_\sigma(NA)\) ([13 Proposition 2.2]). Hence a semigroup \(NA\) is scored if and only if

\[
NA = \{ a \in \mathbb{Z}^d : E_\sigma(a) \neq \emptyset \text{ for all } \sigma \in \mathcal{F} \}.
\]

In the following lemma, we characterize the subset \(NA\) of \(\mathbb{Z}^d\) in terms of the finite sets \(E_\tau(a)\).

**Lemma 8.2.**

\[
NA = \{ a \in \mathbb{Z}^d : 0 \in E_\tau(a) \text{ for all faces } \tau \}.
\]

**Proof.** Let \(\tau_0\) be the minimal face of \(\mathbb{R}_{\geq 0}A\). We have

\[
\{ a \in \mathbb{Z}A : 0 \in E_\tau(a) \text{ for all faces } \tau \} = \bigcap_\tau (NA + \mathbb{Z}(A \cap \tau)) = NA + \mathbb{Z}(A \cap \tau_0).
\]

If \(\tau_0 = \{0\}\), or if the cone \(\mathbb{R}_{\geq 0}A\) is strongly convex, then clearly \(NA + \mathbb{Z}(A \cap \tau_0) = NA\). Next suppose \(\tau_0 \neq \{0\}\). Then there exist \(c_j \in \mathbb{Z}_{>0}\) such that \(0 = \sum_{a_j \in A \cap \tau_0} c_j a_j\). Let \(b \in \mathbb{Z}(A \cap \tau_0)\) and \(b = \sum d_j a_j\) with \(d_j \in \mathbb{Z}\). Take \(N \in \mathbb{N}\) so that \(Nc_j + d_j > 0\) for all \(j\). Then \(b = \sum (Nc_j + d_j)a_j \in NA\). \(\square\)
Set

\[(15) \quad S_1 := \{ a \in \mathbb{Z}^d : E_\sigma(a) \neq \emptyset (\forall \sigma \in \mathcal{F}) \},\]

\[(16) \quad S_2 := \{ a \in \mathbb{Z}^d : E_\sigma(a) \ni 0 (\forall \sigma \in \mathcal{F}) \}.\]

Then \(S_2 = \bigcap_{\sigma \in \mathcal{F}} (NA + \mathbb{Z}(A \cap \sigma))\), and

\[NA \subset S_2 \subset S_1\]

by (15), (16), and Lemma 8.2.

Remark 8.3. (1) Serre’s condition \((S_2)\) is the equality \(NA = S_2\).

(2) The semigroup \(NA\) is scored if and only if \(NA = S_1\).

(3) By the proof of Lemma 8.2, we have

\[NA = \{ a \in \mathbb{Z}^d : E_{\tau_0}(a) \ni 0 \},\]

where \(\tau_0\) is the minimal face.

Our first aim in this subsection is to show that the simplicity of \(D(R_A)\) implies the scoredness of \(NA\). We use the following lemma for the proof.

Lemma 8.4.

\[\dim ZC(\Omega(a)) < d \text{ for all } a \in \mathbb{Z}^d.\]

Proof. Take \(M\) so that

\[\{a \in \mathbb{Z}A : F_\sigma(a) \geq M\} \subseteq NA\]

(see e.g. [16, Lemma 3.6]). Then

\[ZC(\Omega(a)) \subseteq \bigcup_{\sigma \in \mathcal{F}, F_\sigma(a) < M} \bigcup_{m=0}^{M-F_\sigma(a)-1} (F_\sigma = m).\]

\[\square\]

Proposition 8.5. If \(D(R_A)\) is simple, then \(NA\) is scored.

Proof. We know

\[S_1 = \bigcap_{\sigma \in \mathcal{F}} \{ a \in \mathbb{Z}A : F_\sigma(a) \in F_\sigma(NA) \}.\]

Take \(M\) as in the proof of Lemma 8.4. Then we have

\[(17) \quad ZC(S_1 \setminus NA) \subseteq \bigcup_{\sigma \in \mathcal{F}} \bigcup_{m=0}^{M-1} (F_\sigma = m).\]

Suppose that \(NA\) is not scored, and \(a \in S_1 \setminus NA\). Since \(S_1 \setminus NA\) is a union of some equivalence classes, we have \(\Lambda_{[a]}(b) \subseteq S_1 \setminus NA\) for all \(b \in \mathbb{Z}A = \mathbb{Z}^d\). Hence by (17) \(\dim ZC(\Lambda_{[a]}(b)) < d\) for all \(b\). By Lemma
we have $\dim ZC(\Omega(\mathbf{b}) \cup \Lambda_{\{\alpha\}}(\mathbf{b})) < d$ for all $\mathbf{b}$. Hence $\text{Ann} L(\alpha) \neq 0$ by Proposition 6.2. \qed

Remark 8.6. By Van den Bergh [20, Theorem 6.2.5], if $D(R_A)$ is simple, then $R_A$ is Cohen-Macaulay. Example 9.2 shows that $NA$ being scored and $R_A$ being Cohen-Macaulay are not enough for the simplicity of $D(R_A)$.

Now we prove the fact announced in Remark 4.2.

Proposition 8.7. The $\mathbb{C}[s]$-modules $D(R_A)_{\mathbf{a}}$ are singly generated for all $\mathbf{a} \in \mathbb{Z}^d$ if and only if the semigroup $NA$ satisfies $(S_2)$.

Proof. First we paraphrase the condition $(S_2)$. In [16, Proposition 3.4], we have shown that there exist $(\mathbf{b}_i, \tau_i) (i = 1, \ldots, l)$, where $\mathbf{b}_i \in \mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d$ and $\tau_i$ is a face of the cone $\mathbb{R}_{\geq 0}A$, such that

$$
(\mathbf{b}_i + \mathbb{Z}(A \cap \tau_i)) \cap \mathbb{R}_{\geq 0}A.
$$

We may assume that this decomposition is irredundant. Then $\{\mathbf{b}_i + \mathbb{Z}(A \cap \tau_i) : i = 1, \ldots, l\}$ is unique. By [16, Lemma 3.6], for $\sigma \in \mathcal{F}$,

$$
NA + \mathbb{Z}(A \cap \sigma) = [\mathbb{R}_{\geq 0}A + \mathbb{R}(A \cap \sigma)] \cap \mathbb{Z}^d \cap \bigcup_{\tau_i = \sigma}(\mathbf{b}_i + \mathbb{Z}(A \cap \tau_i)).
$$

Hence we obtain

$$
\bigcap_{\sigma \in \mathcal{F}}(NA + \mathbb{Z}(A \cap \sigma)) = \mathbb{R}_{\geq 0}A \cap \mathbb{Z}^d \cap \bigcup_{\tau_i \in \mathcal{F}}(\mathbf{b}_i + \mathbb{Z}(A \cap \tau_i)).
$$

This means that $NA$ satisfies $(S_2)$ if and only if each $\tau_i$ appearing in (18) is a facet.

Suppose that $NA$ satisfies $(S_2)$. Then, by the previous paragraph and [16, Proposition 5.1], we have

$$
ZC(\Omega(\mathbf{a})) = \bigcup_{F_{\nu}(\mathbf{a}) < 0} \bigcup_{m < -F_{\nu}(\mathbf{a}), m \in F_{\nu}(NA)} F_{\nu}^{-1}(m) \cup \bigcup_{b_i - \mathbf{a} \in NA + \mathbb{Z}(A \cap \tau_i)} F_{\tau_i}^{-1}(b_i - \mathbf{a}).
$$

Hence $\Omega(\mathbf{a})$ is singly generated.

Next suppose that $NA$ does not satisfy $(S_2)$. Then a face of codimension greater than one appears in the difference (18). Let $\tau_1$ be a face of codimension greater than one, and let $\mathbf{b}_1 + \mathbb{Z}(A \cap \tau_1)$ appear in the difference. Then

$$
ZC(\Omega(\mathbf{b}_1)) = \bigcup_{b_i - \mathbf{b}_1 \in NA + \mathbb{Z}(A \cap \tau_1)} (\mathbf{b}_i - \mathbf{b}_1 + \mathbb{Z}(A \cap \tau_1)).
$$
Lemma 8.8.

Let $\mathcal{C}(A \cap \tau_1)$ be an irreducible component of $\mathcal{Z}(\Omega(b_1))$.

Suppose the contrary. Then there exists $i$ such that

\begin{align}
\mathcal{C}(A \cap \tau_1) \subseteq b_i - b_1 + \mathcal{C}(A \cap \tau_i). \\
\mathcal{C}(A \cap \tau_1) \subseteq \mathbb{N}A + \mathcal{Z}(A \cap \tau_i)
\end{align}

The latter equation (20) means that $b_i - b_1 \in \mathcal{C}(A \cap \tau_i)$ and $\tau_1 \leq \tau_i$. Combining with (19), we have $b_i - b_1 \in \mathcal{Z}(A \cap \tau_i)$. This contradicts the irreducibility of (18). We have thus proved $\mathcal{C}(A \cap \tau_1)$ is an irreducible component of $\mathcal{Z}(\Omega(b_1))$. Hence the ideal $\mathbb{I}(\Omega(b_1))$ is not singly generated.

In the rest of this subsection, we consider the simplicity of $R_A$ and $S_1$. The following lemma is immediate from the definition of $E_\tau(0)$.

Lemma 8.8.

$$E_\tau(0) = \{0\} \quad \text{for all faces } \tau.$$  

Lemma 8.9. Let $a \in \mathbb{Z}^d$. Then there exists $b \in \mathbb{N}A$ such that

$$\mathbb{I}E_\tau(a + b) = [\mathbb{Q}(A \cap \tau) \cap \mathbb{Z}^d : \mathcal{Z}(A \cap \tau)].$$

(In this situation, we write $E_\tau(a + b) = \text{full}.)$

Proof. We may assume that $a \in \mathbb{N}A$. Let $\lambda \in \mathbb{Q}(A \cap \tau) \cap \mathbb{Z}^d / \mathcal{Z}(A \cap \tau)$, take its representative, and denote it by $\lambda$ again. Write $-\lambda = \sum d_k a_k$ with $d_k \in \mathbb{Z}$. Then $-\sum d_k a_k - \lambda = \sum d_k \leq 0 d_k a_k$. Hence $a + (-\sum d_k a_k) - \lambda \in \mathbb{N}A$. Thus $\lambda \in E_\tau(a + (-\sum d_k a_k))$. We repeat this argument for each pair $(\tau, \lambda)$ to prove the assertion.

Proposition 8.10. The semigroup algebra $R_A$ is a simple $\mathbb{Z}^d$-graded $D(R_A)$-module if and only if

$$\mathbb{Q}(A \cap \tau) \cap \mathbb{Z}^d = \mathcal{Z}(A \cap \tau) \quad \text{for all faces } \tau.$$  

Proof. By Lemma 8.8, $R_A = \mathbb{C}[\mathbb{N}A]$ is a simple graded $D(R_A)$-module if and only if $E_\tau(a) = \{0\}$ for all faces $\tau$ and $a \in \mathbb{N}A$. This is equivalent to the condition (21) by Lemma 8.9.

Lemma 8.11. If the semigroup $\mathbb{N}A$ is scored, then it satisfies (21).

Proof. For a facet $\sigma$, take $M_\sigma \in \mathbb{N}$ so that $M_\sigma$ is greater than any number in $\mathbb{N} \setminus F_\sigma(\mathbb{N}A)$.

Let $\tau$ be a face, and let $x \in \mathbb{Q}(A \cap \tau) \cap \mathbb{Z}^d$. For each facet $\sigma \not\supseteq \tau$, there exists $a_\sigma \in A \cap \tau \setminus \sigma$. Take $m_\sigma$ large enough to satisfy $F_\sigma(x) + m_\sigma F_\sigma(a_\sigma) \geq M_\sigma$. Let $y = x + \sum_{\sigma \not\supseteq \tau} m_\sigma a_\sigma$. Then $y \in \mathbb{Q}(A \cap \tau) \cap \mathbb{Z}^d$, and

$$F_\sigma(y) = 0 \quad \text{if } \sigma \supseteq \tau,$$

$$F_\sigma(y) \geq M_\sigma \quad \text{otherwise}.$$  

Since $\mathbb{N}A$ is scored, $y \in \mathbb{N}A$. Hence $y \in \mathbb{N}(A \cap \tau)$, and $x \in \mathcal{Z}(A \cap \tau)$.

□
Corollary 8.12. If the semigroup \( NA \) is scored, then \( R_A \) is a simple \( \mathbb{Z}^d \)-graded \( D(R_A) \)-module.

Proof. This follows from Proposition 8.10 and Lemma 8.11. □

Proposition 8.13. The semigroup \( NA \) is scored if and only if \( C[S_1] \) is a simple \( \mathbb{Z}^d \)-graded \( D(R_A) \)-module.

Proof. Suppose that \( NA \) is scored. Then \( S_1 = NA \). Hence \( C[S_1] \) is a simple \( \mathbb{Z}^d \)-graded \( D(R_A) \)-module by Corollary 8.12.

Suppose that \( C[S_1] \) is a simple \( \mathbb{Z}^d \)-graded \( D(R_A) \)-module. Since \( R_A \) is a nonzero \( \mathbb{Z}^d \)-graded \( D(R_A) \)-submodule of \( C[S_1] \), we have \( R_A = C[S_1] \). Hence \( NA \) is scored. □

8.2. Conditions for simplicity. The aim of this subsection is to give a necessary and sufficient condition for the vanishing of a primitive ideal \( \text{Ann} L(\alpha) \). It leads to a necessary and sufficient condition for the simplicity of \( D(R_A) \).

We start this subsection by introducing some notation. Let \( \alpha \in \mathbb{C}^d \).

Set
\[
  \mathcal{F}_+(\alpha) := \{ \sigma \in \mathcal{F} : F_\sigma(\alpha) \in F_\sigma(NA) \},
\]
\[
  \mathcal{F}_-(\alpha) := \{ \sigma \in \mathcal{F} : F_\sigma(\alpha) \notin F_\sigma(NA) \},
\]
and
\[
  \mathbb{R}_{>0}(\alpha) := \left\{ \gamma \in \mathbb{R}^d : \frac{F_\sigma(\gamma)}{F_\sigma(\alpha)} > 0 \quad (\sigma \in \mathcal{F}_+(\alpha)) \right\}.
\]

Let \( \text{Face}(\alpha) \) denote the set of faces \( \tau \) such that \( \alpha - \lambda \in \mathbb{Z}^d \) for some \( \lambda \in \mathbb{C}(A \cap \tau) \), and that every facet \( \sigma \) containing \( \tau \) belongs to \( \mathcal{F}_+(\alpha) \).

Let \( [\alpha] \) denote the equivalence class that \( \alpha \) belongs to. An equivalence class \( [\alpha] \) is said to be extreme if \( E_\tau(\alpha) \) has \( \mathbb{Q}(A \cap \tau) \cap \mathbb{Z}^d : \mathbb{Z}(A \cap \tau) \)-many elements (i.e., \( E_\tau(\alpha) = \text{full} \)) for every \( \tau \in \text{Face}(\alpha) \), and that \( E_\tau(\alpha) \) is empty for every \( \tau \notin \text{Face}(\alpha) \).

We compare the conditions:

1. An equivalence class \( [\alpha] \) is extreme,
2. \( \mathbb{R}_{>0}(\alpha) \) is not empty,
3. \( \mathbb{Z}\mathbb{C}([\alpha]) = \mathbb{C}^d \),
4. \( \text{Ann} L(\alpha) = 0 \).

Remark 8.14. The conditions (1) and (2) have an advantage over the condition (3), for to check (1) and (2) we do not need the equivalence class \( [\alpha] \), which is not easy to compute.

We need the following technical lemmas.
Lemma 8.15. Let \( \tau \) be a face of \( \mathbb{R}_{\geq 0}A \). Then there exists \( M \in \mathbb{N} \) such that, if \( a \in \mathbb{Z}^d \) satisfies \( F_\sigma(a) \geq M \) for all facets \( \sigma \geq \tau \), then \( E_\tau(a) = \text{full} \).

Proof. Let \( \lambda \in \mathbb{Q}(A \cap \tau) \cap \mathbb{Z}^d \). By [10, Lemma 3.6], there exists \( M \in \mathbb{N} \) such that \( c \in NA + Z(A \cap \tau) \) for all \( c \in \mathbb{Z}^d \) satisfying \( F_\sigma(c) \geq M \) for all facets \( \sigma \geq \tau \). Hence, if \( a \in \mathbb{Z}^d \) satisfies \( F_\sigma(a) \geq M \) for all facets \( \sigma \geq \tau \), then \( a - \lambda \in NA + Z(A \cap \tau) \), or \( \lambda \in E_\tau(a) \).

Lemma 8.16. Let \( \tau \) be a face of \( \mathbb{R}_{\geq 0}A \), and let \( \alpha \in \mathbb{C}^d \). Assume that there exists \( \lambda \in \mathbb{C}(A \cap \tau) \) such that \( \alpha - \lambda \) belongs to \( \mathbb{Z}^d \). Then there exists \( M \in \mathbb{N} \) such that, if \( \gamma \in \alpha + \mathbb{Z}^d \) satisfying \( F_\sigma(\gamma) \in \mathbb{Z}_{\geq M} \) for all facets \( \sigma \geq \tau \), then \( E_\tau(\gamma) = \text{full} \).

Proof. Apply Lemma 8.15 to \( \gamma - \lambda \).

Proposition 8.17. If the Zariski closure of an equivalence class \([\alpha]\) is the whole space \( \mathbb{C}^d \), then \( \mathbb{R}_{>0}(\alpha) \) is not empty.

Proof. Since the set

\[
\{ \gamma \in \alpha + \mathbb{Z}^d : F_\sigma(\gamma) \in F_\sigma(NA) \quad (\sigma \in \mathcal{F}_+(\alpha)) \}
\]

contains \([\alpha]\), its Zariski closure equals \( \mathbb{C}^d \). Take a real number \( \epsilon \) so that \( \epsilon \) is algebraically independent over \( \mathbb{Q}[F_\sigma(\text{Re}(\alpha)), F_\sigma(\text{Im}(\alpha)) : \sigma \in \mathcal{F}] \).

Put \( \alpha_\epsilon := \text{Re}(\alpha) + \epsilon \text{Im}(\alpha) \), where \( \text{Re}(\alpha) \) and \( \text{Im}(\alpha) \) are the vectors in \( \mathbb{R}^d \) with \( \alpha = \text{Re}(\alpha) + \sqrt{-1} \text{Im}(\alpha) \) Then \( \mathcal{F}_+(\alpha) = \mathcal{F}_+(\alpha_\epsilon) \), and \( \mathcal{F}_-(\alpha) = \mathcal{F}_-(\alpha_\epsilon) \), since \( \sigma \in \mathcal{F}_+(\alpha) \) or \( \mathcal{F}_-(\alpha) \) implies \( F_\sigma(\text{Re}(\alpha)) = 0 \).

The set

\[
\{ \gamma \in \alpha_\epsilon + \mathbb{Z}^d : F_\sigma(\gamma) \in F_\sigma(NA) \quad (\sigma \in \mathcal{F}_+(\alpha)) \}
\]

is bijective to the set (22) under the map sending \( \alpha_\epsilon + a \) to \( \alpha + a \), and hence its Zariski closure equals \( \mathbb{C}^d \). The Zariski closure of

\[
\bigcap_{\sigma \in \mathcal{F}_+(\alpha)} F_\sigma = 0 \setminus \bigcup_{\sigma \in \mathcal{F}_-(\alpha)} \bigcup_{m \in \mathbb{N} \setminus F_\sigma(NA)} \{ F_\sigma = m \}
\]

also equals \( \mathbb{C}^d \). Hence \( \mathbb{R}_{>0}(\alpha) \) is not empty.

Proposition 8.18. If the Zariski closure of an equivalence class \([\alpha]\) is the whole space \( \mathbb{C}^d \), then \([\alpha]\) is extreme, i.e.,

\[
[\alpha] = \left\{ \gamma \in \alpha + \mathbb{Z}^d : \begin{array}{ll}
E_\tau(\gamma) = \text{full} & (\tau \in \text{Face}(\alpha)) \\
E_\tau(\gamma) = \emptyset & (\tau \notin \text{Face}(\alpha))
\end{array} \right\}.
\]

Proof. By Lemma 8.16 the equivalence class

\[
[\alpha_1] := \left\{ \gamma \in \alpha + \mathbb{Z}^d : \begin{array}{ll}
E_\tau(\gamma) = \text{full} & (\tau \in \text{Face}(\alpha)) \\
E_\tau(\gamma) = \emptyset & (\tau \notin \text{Face}(\alpha))
\end{array} \right\}
\]
contains
\[
\{ \gamma \in \alpha + \mathbb{Z}^d : \begin{array} {l}
F_\sigma(\gamma) \geq M \\
F_\sigma(\gamma) < 0
\end{array} \ (\sigma \in \mathcal{F}_+(\alpha)) \}
\]
for some $M$ sufficiently large.

Suppose that $[\alpha] \neq [\alpha_1]$. Then $[\alpha]$ does not belong to the set $\{24\}$. Hence we have
\[
[\alpha] \subseteq \bigcup_{\sigma \in \mathcal{F}_+(\alpha)} \bigcup_{m=0}^{M-1} \{ \gamma \in \alpha + \mathbb{Z}^d : F_\sigma(\gamma) = m \}.
\]
This contradicts the assumption that the dimension of $\text{ZC}([\alpha])$ equals $d$. \hfill \Box

**Proposition 8.19** (cf. Proposition 3.3.1 in [12]). \textit{Let} $\alpha \in \mathbb{C}^d$. \textit{Then} $\text{Ann}(L(\alpha)) = 0$ if and only if $\text{ZC}([\alpha]) = \mathbb{C}^d$.

**Proof.** Let $I := \text{Ann}(L(\alpha))$. Recall that we have
\[
I_a = t^a \mathbb{P}((\Omega(a) \cup \Lambda_{[\alpha]}(a))),
\]
where
\[
\Lambda_{[\alpha]}(a) = \{ \gamma : \gamma \sim \alpha, \gamma + a \sim \alpha \}.
\]
Since $I_0 = \mathbb{P}([\alpha])$, the vanishing of $I$ leads to the assertion that $\text{ZC}([\alpha]) = \mathbb{C}^d$.

Next suppose that $\text{ZC}([\alpha]) = \mathbb{C}^d$. As in the proof of Proposition 8.18 there exists $M \in \mathbb{N}$ such that
\[
[\alpha] \supseteq \left\{ \gamma \in \alpha + \mathbb{Z}^d : \begin{array} {l}
F_\sigma(\gamma) \geq M \\
F_\sigma(\gamma) < 0
\end{array} \ (\sigma \in \mathcal{F}_+(\alpha)) \right\}.
\]
Hence
\[
\Lambda_{[\alpha]}(a) \supseteq \left\{ \gamma \in \alpha + \mathbb{Z}^d : \begin{array} {l}
F_\sigma(\gamma) \geq \max\{M, -F_\sigma(a)\} \\
F_\sigma(\gamma) < \min\{0, -F_\sigma(a)\}
\end{array} \ (\sigma \in \mathcal{F}_+(\alpha)), \ (\sigma \in \mathcal{F}_-(\alpha)) \right\}.
\]
Since the right hand side is $d$-dimensional by Proposition 8.17 the Zariski closure $\text{ZC}(\Lambda_{[\alpha]}(a))$ is also $d$-dimensional. Hence $I_a = 0$ for all $a \in \mathbb{Z}^d$. \hfill \Box

**Proposition 8.20.** If $[\alpha]$ is extreme, and $\mathbb{R}_{>0}(\alpha)$ is not empty, then $\text{ZC}([\alpha]) = \mathbb{C}^d$.

**Proof.** As in the proof of Proposition 8.18 $[\alpha]$ contains
\[
\left\{ \gamma \in \alpha + \mathbb{Z}^d : \begin{array} {l}
F_\sigma(\gamma) \geq M \\
F_\sigma(\gamma) < 0
\end{array} \ (\sigma \in \mathcal{F}_+(\alpha)) \right\}.
\]
for some $M$ sufficiently large. By the assumption, the dimension of
\[
\left\{ a \in \mathbb{Z}^d : \begin{array}{ll}
F_{\sigma}(a) > M - F_{\sigma}(\alpha) & (\sigma \in \mathcal{F}_+(\alpha)) \\
F_{\sigma}(a) < -F_{\sigma}(\alpha) & (\sigma \in \mathcal{F}_-(\alpha))
\end{array} \right\}
\]
equals $d$. Hence the proposition follows.

**Theorem 8.21.** Let $\alpha \in \mathbb{C}^d$. Then $\text{Ann}(L(\alpha)) = 0$ if and only if $[\alpha]$ is extreme, and $\mathbb{R}_{>0}(\alpha)$ is not empty.

**Proof.** This follows from Propositions 8.17, 8.18, and 8.20.

**Theorem 8.22.** The algebra $D(R_A)$ is simple if and only if the conditions
\begin{enumerate}
\item[(C1)] Any equivalence class is extreme.
\item[(C2)] For any $\alpha$, $\mathbb{R}_{>0}(\alpha)$ is not empty.
\end{enumerate}
are satisfied.

**Proof.** This follows from Lemma 8.1 and Theorem 8.21.

**Remark 8.23.** To know whether $D(R_A)$ is simple or not, by Theorem 7.6, we need to check (C1) and (C2) only for finitely many $\alpha$.

**Proposition 8.24.** If the semigroup $N_A$ is scored, then it satisfies the condition (C1).

**Proof.** First note that the condition (21) is satisfied in the scored case by Lemma 8.11.

Let $\lambda \in \mathbb{C}(A \cap \tau)$ and $\alpha - \lambda \in \mathbb{Z}^d$. Suppose that $\sigma \in \mathcal{F}_+(\alpha)$ for all facets $\sigma$ containing $\tau$. We need to show $\lambda \in E_\tau(\alpha)$.

If a facet $\sigma$ contains $\tau$, then $\sigma \in \mathcal{F}_+(\alpha)$, or $F_\sigma(\alpha) \in F_\sigma(NA)$. Hence $F_\sigma(\alpha - \lambda) \in F_\sigma(NA)$. If a facet $\sigma$ does not contain $\tau$, then there exists $a_j \in A \cap \tau$ such that $F_\sigma(a_j) > 0$. Hence $F_\sigma(\alpha - \lambda + ma_j) \in F_\sigma(NA)$ for $m \in \mathbb{N}$ sufficiently large. Hence there exists $a \in N(A \cap \tau)$ such that $F_\sigma(\alpha - \lambda + a) \in F_\sigma(NA)$ for all $\sigma \in \mathcal{F}$. Since $NA$ is scored, we obtain $\alpha - \lambda + a \in NA$. This means $\lambda \in E_\tau(\alpha)$.

**Theorem 8.25.** The algebra $D(R_A)$ is simple if and only if the semigroup $NA$ is scored and satisfies the condition (C2).

**Proof.** This immediately follows from Theorem 8.22 and Proposition 8.24.

**Corollary 8.26.** Assume that the cone $\mathbb{R}_{>0}A$ is simplicial. Then the algebra $D(R_A)$ is simple if and only if $NA$ is scored.

**Proof.** In this case the cone $\mathbb{R}_{>0}A$ has exactly $d$ facets. Since the $d$ $F_\sigma$’s are linearly independent, the condition (C2) is satisfied.
Example 9.1. Let
\[
A = \begin{pmatrix} 1 & 1 & 2 & 2 \\ 1 & 2 & 0 & 1 \end{pmatrix}.
\]
Then \( N_A \) is the set of black dots in Figure 1, and \( \mathbb{R}_{\geq 0} A \) has two facets: \( \sigma_2 := \mathbb{R}_{\geq 0} a_2 \) and \( \sigma_3 := \mathbb{R}_{\geq 0} a_3 \). Their primitive integral support functions are \( F_{\sigma_2}(s) = 2s_1 - s_2 \) and \( F_{\sigma_3}(s) = s_2 \) respectively.

\[ \sigma_2 \quad \sigma_3 \]

Figure 1. The semigroup \( N_A \)

The condition (C2) is satisfied, since \( \mathbb{R}_{\geq 0} A \) is simplicial. However \( D(R_A) \) is not simple, since \( N_A \) is not scored. We have
\[
\{ \alpha : F(\alpha) = \emptyset \} / \mathbb{C}^2 = \{ t(\sqrt{2}, \sqrt{3}) \}
\]
\[
\{ \alpha : F_{\sigma_2}(\alpha) \in \mathbb{Z} \} / (\mathbb{Z}^2 + (F_{\sigma_2} = 0)) = \{ t(1/2, \sqrt{2}) \}
\]
\[
\{ \alpha : F_{\sigma_3}(\alpha) \in \mathbb{Z} \} / (\mathbb{Z}^2 + (F_{\sigma_3} = 0)) = \{ t(\sqrt{2}, 0) \}
\]
\[
\{ \alpha : F_{\sigma_2}(\alpha), F_{\sigma_3}(\alpha) \in \mathbb{Z} \} / \mathbb{Z}^2 = \{ t(0, 0), t(1/2, 0) \}.
\]

First we classify \( \mathbb{Z}^2 \). Let \( \alpha \in \mathbb{Z}^2 \). Then we see
- \( E_{\mathbb{R}_{\geq 0}}(\alpha) = \{ 0 \} \),
- \( E_{\sigma_2}(\alpha) = \{ 0 \} \Leftrightarrow 2\alpha_1 - \alpha_2 \geq 0 \),
- \( E_{\sigma_3}(\alpha) = \{ 0, t(1, 0) \} \Leftrightarrow \alpha_2 \geq 1 \),
- \( E_{\sigma_3}(\alpha) = \{ 0 \} \Leftrightarrow \alpha_2 = 0, \alpha_1 \in 2\mathbb{Z} \),
- \( E_{\sigma_3}(\alpha) = \{ t(1, 0) \} \Leftrightarrow \alpha_2 = 0, \alpha_1 \in 2\mathbb{Z} + 1 \),
- \( E_{\{ 0 \}}(\alpha) = \{ 0 \} \Leftrightarrow \alpha \in N_A \).

There are eight classes in \( \mathbb{Z}^2 \):

1. \( \{ \alpha \in \mathbb{Z}^2 : E_{\sigma_2}(\alpha) = \{ 0 \}, E_{\sigma_3}(\alpha) = \{ 0, t(1, 0) \}, E_{\{ 0 \}}(\alpha) = \{ 0 \} \}
   = \{ (\alpha_1, \alpha_2) \in \mathbb{Z}^2 : \alpha_2 \geq 1, 2\alpha_1 - \alpha_2 \geq 0 \} \)
2. \( \{ \alpha \in \mathbb{Z}^2 : E_{\sigma_2}(\alpha) = \emptyset, E_{\sigma_3}(\alpha) = \{ 0, t(1, 0) \}, E_{\{ 0 \}}(\alpha) = \emptyset \}
   = \{ (\alpha_1, \alpha_2) \in \mathbb{Z}^2 : \alpha_2 \geq 1, 2\alpha_1 - \alpha_2 < 0 \} \)
if Ann L see that $\alpha ZC(\Lambda [\alpha]) = 0$,
$E_{\sigma_3}(\alpha) = \emptyset$, $E_{\{0\}}(\alpha) = \emptyset$
$\{t(\alpha_1, \alpha_2) \in \mathbb{Z}^2 : \alpha_2 = 0, \alpha_1 \in 2\mathbb{N}\}$.

(4)
$\{\alpha \in \mathbb{Z}^2 : E_{\sigma_2}(\alpha) = \emptyset, E_{\sigma_3}(\alpha) = \emptyset, E_{\{0\}}(\alpha) = \emptyset\}
= \{t(\alpha_1, \alpha_2) \in \mathbb{Z}^2 : \alpha_2 = 0, \alpha_1 \in 2\mathbb{N} + 1\}.

(5)
$\{\alpha \in \mathbb{Z}^2 : E_{\sigma_2}(\alpha) = \emptyset, E_{\sigma_3}(\alpha) = \emptyset, E_{\{0\}}(\alpha) = \emptyset\}
= \{t(\alpha_1, \alpha_2) \in \mathbb{Z}^2 : \alpha_2 = 0, \alpha_1 \in 2(-\mathbb{N} - 1)\}.

(6)
$\{\alpha \in \mathbb{Z}^2 : E_{\sigma_2}(\alpha) = \emptyset, E_{\sigma_3}(\alpha) = \emptyset, E_{\{0\}}(\alpha) = \emptyset\}
= \{t(\alpha_1, \alpha_2) \in \mathbb{Z}^2 : \alpha_2 = 0, \alpha_1 \in -2\mathbb{N} - 1\}.

(7)
$\{\alpha \in \mathbb{Z}^2 : E_{\sigma_2}(\alpha) = \emptyset, E_{\sigma_3}(\alpha) = \emptyset, E_{\{0\}}(\alpha) = \emptyset\}
= \{t(\alpha_1, \alpha_2) \in \mathbb{Z}^2 : \alpha_2 < 0, 2\alpha_1 - \alpha_2 \geq 0\}.

(8)
$\{\alpha \in \mathbb{Z}^2 : E_{\sigma_2}(\alpha) = \emptyset, E_{\sigma_3}(\alpha) = \emptyset, E_{\{0\}}(\alpha) = \emptyset\}
= \{t(\alpha_1, \alpha_2) \in \mathbb{Z}^2 : \alpha_2 < 0, 2\alpha_1 - \alpha_2 < 0\}.

Let $\alpha$ be not extreme, i.e., let $\alpha$ belong to (3), (4), (5), or (6). Then
$ZC(\Lambda_{\alpha}(a)) = \{\mu : \mu_2 = 0\}$ if $a_2 = 0$ and $a_1 \in 2\mathbb{Z}$, and $\Lambda_{\alpha}(a) = 0$
otherwise.

Let $\alpha$ be extreme, i.e., let $\alpha$ belong to (1), (2), (7), or (8). Then
$ZC(\Lambda_{\alpha}(a)) = \mathbb{C}^2$ for all $a \in \mathbb{Z}^2$.

Similarly $t(\sqrt{2}, \sqrt{3}) + \mathbb{Z}^2$, $t(\frac{1}{2}\sqrt{2}, \sqrt{3}) + \mathbb{Z}^2$, $t(\sqrt{2}, 0) + \mathbb{Z}^2$, and $t(\frac{1}{2}, 0) + \mathbb{Z}^2$
have one, two, four, and eight equivalence classes respectively. We see
that $\alpha = t(\alpha_1, \alpha_2)$ is not extreme if and only if $\alpha_2 = 0$ if and only
if Ann $L(\alpha) = Ann L(0)$.

Hence
$\text{Prim}(D(R_A)) = \{(0), \text{Ann } L(0)\}$.

**Example 9.2.** (cf. [13] Example 4.9,) Let
$$A = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
0 & 2 & 3 & 0 & 2 & 3 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}.$$
Then the cone $\mathbb{R}_{\geq 0}A$ has four facets whose primitive integral support functions are

\[
F_{\sigma_{14}}(s) = s_2, \quad F_{\sigma_{36}}(s) = 3s_1 - s_2, \\
F_{\sigma_{123}}(s) = s_3, \quad F_{\sigma_{456}}(s) = s_1 - s_3.
\]

We have an equality

\[(25) \quad F_{\sigma_{14}} + F_{\sigma_{36}} = 3(F_{\sigma_{123}} + F_{\sigma_{456}}).\]

Then the semigroup

\[S := \mathbb{N}A = \{ a \in \mathbb{R}_{\geq 0}A \cap \mathbb{Z}^3 : F_{\sigma_{14}}(a) \neq 1 \}\]

is scored. Let $\alpha = t(0, 1, 0)$. Then $\text{Face}(\alpha) = \{ F_{\sigma_{123}}, F_{\sigma_{456}} \}$. We see that $\alpha$ is extreme, and $\mathbb{R}_{> 0}(\alpha)$ is empty. Hence $\text{Ann} L(\alpha) \neq 0$.

Let $\lambda := F_{\sigma_{14}} + 2F_{\sigma_{36}} = 6s_1 - s_2$, and let $E_\lambda$ be the blow-up extension of $S$,

\[E_\lambda := \left\{ \left( \begin{array}{c} a \\ p \end{array} \right) \in S \bigoplus \mathbb{N} : p \leq \lambda(a) \right\}.
\]

We can prove that an affine semigroup is scored if and only if its blow-up extension is scored by the same argument as the proof of [19, Lemma 1.1]. Thus $E_\lambda$ is scored. Indeed, the cone $\mathbb{R}_{\geq 0}E_\lambda$ has six facets whose primitive integral support functions are

\[
F_{\sigma_{14}}(s,p) = F_{\sigma_{14}}(s), \quad F_{\sigma_{36}}(s,p) = F_{\sigma_{36}}(s), \\
F_{\sigma_{123}}(s,p) = F_{\sigma_{123}}(s), \quad F_{\sigma_{456}}(s,p) = F_{\sigma_{456}}(s), \\
F_{\sigma_{p}}(s,p) = p, \quad F_{\sigma_{\lambda}}(s,p) = \lambda(s) - p,
\]

and

\[E_\lambda = \mathbb{R}_{\geq 0}E_\lambda \cap \mathbb{Z}^4 \setminus (F_{\sigma_{14}} = 1).\]

In addition, $\mathbb{C}[E_\lambda]$ is Cohen-Macaulay [19, Example 4.9].

However $D(\mathbb{C}[E_\lambda])$ is not simple. To see this, let $\beta = t(0, 1, 0, 0)$. Then

\[\mathcal{F}_+(\beta) = \{ \sigma_{123}, \sigma_{456}, \sigma_{p} \}, \quad \mathcal{F}_-(\beta) = \{ \sigma_{14}, \sigma_{36}, \sigma_{\lambda} \}.
\]

By (25)

\[(F_{\sigma_{123}} > 0) \cap (F_{\sigma_{456}} > 0) \cap (F_{\sigma_{14}} < 0) \cap (F_{\sigma_{36}} < 0)
\]

is empty. Hence $\mathbb{R}_{> 0}(\beta)$ is also empty, and $D(\mathbb{C}[E_\lambda])$ is not simple by Theorem 8.25.
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