Pairing in Asymmetric Many-Fermion Systems: 
Functional Renormalisation Group Approach

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(Dated: July 22, 2014)

Functional renormalisation group approach is applied to a imbalanced many-fermion system with a short-range attractive force. We introduce a composite boson field to describe pairing effects, and assume a simple ansatz for the effective action. A set of approximate flow equations for the effective coupling including boson and fermionic fluctuations is derived and solved. We identify the critical values of particle number density mismatch when the system undergoes to a normal state. We determine the phase diagram both at unitarity and around. The obtained phase diagram is in a reasonable agreement with the experimental data.

The mechanism of pairing in imbalanced many-fermion systems is nowadays a subject of the intensive theoretical and experimental studies (see ref. [1] for review). This phenomena occurs in many physical systems from molecular physics to quark matter at finite density. Being different in details, the underlying dynamical mechanisms share a common feature related to Cooper instability leading to a rearrangement of the ground state and associated spontaneous symmetry breaking.

In this paper we focus on the asymmetric ultracold atomic Fermi mixture of two fermion flavours, which realizes a highly tunable system of strongly interacting fermions. This tunability is provided by a Feshbach resonance, which allows to control the interaction strength between two different species of fermions and explore the BEC-BCS crossover in a wide range of physical parameters. Another tunable parameter (in asymmetric systems) is the population imbalance which can be used to probe how stable the superfluid phase is. The problem was studied long time ago by Clogston and Chandrasekhar [2] who found that in the BCS limit the system with the chemical potential mismatch $\delta \mu$ undergoes first order phase transition to a normal phase at $\delta \mu = 0.71 \Delta_0$ where $\Delta_0$ is the gap at zero temperature for balanced system. Recently, the issue has been looked at again but now in the
case of strongly interacting fermions with infinite scattering length (unitary limit) [1]. Most theoretical studies have been performed in the framework of the mean-field (MF) type of approaches which are of limited use for the imbalanced many-fermion systems and may not be reliable in providing quantitative answers. In many cases the effects of quantum fluctuations turn out to be important.

The aim of the present paper is to set up a framework to study pairing phenomena in imbalanced many-fermion systems using the formalism of Functional Renormalisation Group [3] (FRG) where the effects of quantum fluctuations are included in a consistent and reliable way. The FRG approach makes use of the Legendre transformed effective action: \( \Gamma[\phi_c] = W[J] - J \cdot \phi_c \), where \( W \) is the usual partition function in the presence of an external source \( J \). The action functional \( \Gamma \) generates the 1PI Green’s functions and it reduces to the effective potential for homogeneous systems. In the FRG one introduces an artificial renormalisation group flow, generated by a momentum scale \( k \) and we define the effective action by integrating over components of the fields with \( q \gtrsim k \). The RG trajectory then interpolates between the classical action of the underlying field theory (at large \( k \)), and the full effective action (at \( k = 0 \)). This method has been successfully applied to a range of problems, from condensed matter physics [4] to particle physics [5].

The evolution equation for \( \Gamma \) in the ERG has a one-loop structure and can be written as

\[
\partial_k \Gamma = -\frac{i}{2} \text{Tr} \left[ (\Gamma^{(2)}_{BB} - R_B)^{-1} \partial_k R_B \right] + \frac{i}{2} \text{Tr} \left[ (\Gamma^{(2)}_{FF} - R_F)^{-1} \partial_k R_F \right].
\] (1)

Here \( \Gamma^{(2)}_{FF(BB)} \) is the matrix containing second functional derivatives of the effective action with respect to the fermion (boson) fields and \( R_{B(F)} \) is a matrix containing the corresponding boson (fermion) regulators which must vanish when the running scale approaches zero. A 2 \( \times \) 2 matrix structure arises for the bosons because we treat \( \phi \) and \( \phi^\dagger \) as independent fields in order to include the number-violating condensate. A similar structure also appears for the fermions. By inserting the ansatz for \( \Gamma \) into this equation one can turn it into a set of coupled equations for the various couplings.

Here we study a system of fermions with population imbalances interacting through an attractive two-body point-like potential and consider pairing between the fermions with different flavours assuming that the interaction between the identical ones is negligible. We take as our starting point an EFT that describes the s-wave scattering of two nonidentical fermions with a \( T \)-matrix determined by the scattering length \( a \). A positive scattering length
corresponds to a system with a two-body bound state (and hence repulsive phase-shifts for low-energy scattering) whereas a negative scattering length to one without a bound state. The binding energy gets deeper as \( a \) gets smaller, while the limit \( a \to \pm \infty \) corresponds to a zero-energy bound state.

Since we are interested in the appearance of a gap in the fermion spectrum, we need to parametrise our effective action in a way that can describe the qualitative change in the physics when this occurs. A natural way to do this is to introduce a boson field whose vacuum expectation value (VEV) describes the gap and so acts as the corresponding order parameter. At the start of the RG flow, the boson field is not dynamical and is introduced through a Hubbard-Stratonovich transformation of the four-fermion pointlike interaction. As we integrate out more and more of the fermion degrees of freedom by running \( k \) to lower values, we generate dynamical terms in the bosonic effective action.

We take the following ansatz for \( \Gamma \)

\[
\Gamma[\psi, \psi^\dagger, \phi, \phi^\dagger, \mu, k] = \int d^4x \left[ \phi^\dagger(x) \left( Z_\phi i \partial_t + \frac{Z_m}{2m} \nabla^2 \right) \phi(x) - U(\phi, \phi^\dagger) \right. \\
+ \sum_{i=1}^{i=2} \psi^\dagger \left( Z_\psi (i \partial_t + \mu_i) + \frac{Z_{M_i}}{2M_i} \nabla^2 \right) \psi \\
- g \left( i \frac{1}{2} \psi^T \psi \phi^\dagger - i \frac{1}{2} \psi^\dagger \psi^T \phi \right),
\]

Here \( M_i \) and \( m \) are masses of fermions and composite boson. All renormalisation factors, couplings and chemical potentials run with the scale \( k \). The term containing the boson chemical potential is quadratic in \( \phi \) so it can be absorbed into effective potential \( U \) and the Yukawa coupling is assumed to describe the decay (creation) of a pair of nonidentical fermions. Due to \( U(1) \) symmetry the effective potential depends on the combination \( \phi^\dagger \phi \). We expand the potential \( U(\rho) \) near its minima

\[
U(\phi, \phi^\dagger) = u_0 + u_1 (\rho - \rho_0) + \frac{1}{2} u_2 (\rho - \rho_0)^2,
\]

where \( \rho = \phi^\dagger \phi \). We truncate the expansion to quartic order and assume \( Z_{\psi i} = Z_{M_i} = 1 \) and neglect running of Yukawa coupling. One notes that the expansion near minimum of the effective potential (either trivial or nontrivial), being quite reliable in the case of second order phase transition, may not be sufficient to quantitatively describe the first order one. It is worth emphasizing that the CC limit related transition from the superfluid phase to a normal one is of the first order so that the proper FRG treatment aiming at simultaneous
description of the phase boundary and normal phase would probably require solving the flow equation with unexpanded potential, which is outside of the scope of the present paper. However, as we will discuss below, even a simple ansatz for the effective action including the expansion of the effective potential near the minimum up to quartic order is able to give a reasonable description of the corresponding phase diagram.

At the starting scale the system is in a symmetric regime with a trivial minimum so that $u_1(k)$ is positive. At some lower scale $k = k_{\text{crit}}$ the coupling $u_1(k)$ becomes zero and the system undergoes a transition to the broken phase with a nontrivial minimum and develops the energy gap.

In our RG evolution we have chosen the trajectory when chemical potentials run in the broken phase and the corresponding particle densities $n_i$ remain fixed so that we define "running Fermi-momenta" for two fermionic species as $p_i = \sqrt{2M_im_i}$. It is convenient to work with the total chemical potential and their difference so we define

$$\mu = \frac{\mu_1 + \mu_2}{2}; \quad \delta = \frac{\mu_1 - \mu_2}{2}$$

and assume that $\mu_1$ is always larger then $\mu_2$. Calculating corresponding functional derivatives, taking the trace and performing a contour integration results in the following flow equation for the effective potential

$$\partial_k U = -\frac{1}{2Z_\psi} \int \frac{d^3q}{(2\pi)^3} \frac{E_{1F} + E_{2F}}{\sqrt{(E_{1F} + E_{2F})^2 + 4\Delta^2}} (\partial_k R_{1F} + R_{2F})$$

$$+ \frac{1}{2Z_\phi} \int \frac{d^3q}{(2\pi)^3} \frac{E_{BR}}{\sqrt{E_{BR}^2 - V_B^2}} \partial_k R_B,$$

where

$$E_{BR}(q) = \frac{Z_m}{2m} q^2 + u_1 + u_2(2\rho - \rho_0) + R_B(q, k), \quad V_B = u_2\rho,$$

and

$$E_{iF}(q, k) = \epsilon_i(q) - \mu_i + R_{iF}(q, p_i, k), \quad \epsilon_i(q) = q^2/2M_i$$

One notes that the position of the pole in the fermion loop integral which defines the corresponding dispersion relation is given by

$$q_0 = \frac{E_{2F} - E_{1F} \pm \sqrt{(E_{2F} + E_{1F})^2 + 4\Delta^2}}{2},$$

which (in the physical limit of vanishing scale) indicates a possibility of the gapless exitation in asymmetric many-fermion systems (much discussed Sarma phase [7]). The
gappless exitation occurs (in the physical limit at vanishing running scale) at $\Delta \delta < 1$. As we will show below, this condition is never fulfilled so that Sarma phase does not occur. We note, however, that this conclusion is valid at zero temperature case and can be altered at finite temperature where the possibility for the Sarma phase is known to still exists[1]. The corresponding bosonic excitations are just gapless "Goldstone" bosons as it should be.

In order to follow the evolution at constant density and running chemical potential we define the total derivative

$$d_k = \partial_k + (d_k \mu) \frac{\partial}{\partial \mu}, \quad (9)$$

where $d_k \mu = d\mu/dk$. Applying this to effective potential, demanding that $n$ is constant ($d_k n = 0$) and neglecting higher order contributions gives the set of the flow equations

$$2Z_\phi d_k \rho = \left. \frac{\partial}{\partial \mu} \left( \partial_k U \right) \right|_{\mu=\mu_0}, \quad (10)$$
$$d_k u_0 + n d_k \mu = \left. \partial_k U \right|_{\rho=\rho_0}, \quad (11)$$
$$d_k u_1 - u_2 d_k \rho + 2Z_\phi d_k \mu = \left. \frac{\partial}{\partial \rho} \left( \partial_k U \right) \right|_{\rho=\rho_0}, \quad (12)$$
$$d_k u_2 = \left. \frac{\partial^2}{\partial \rho^2} \left( \partial_k U \right) \right|_{\rho=\rho_0}, \quad (13)$$
$$d_k Z_\phi = - \frac{1}{2} \left. \frac{\partial^2}{\partial \mu \partial \rho} \left( \partial_k U \right) \right|_{\rho=\rho_0}, \quad (14)$$

The driving terms in these evolution equations are given by appropriate derivatives of Eq. (5). In the symmetric phase we evaluate these expressions at $\rho = 0$. The driving term for the chemical potential evolution vanishes in this case, and hence $\mu$ remains constant. In the broken phase we keep $\rho$ non-zero and set $u_1 = 0$. The details of the derivation can be found in ref. [6].

Neglecting the effect of bosonic fluctuations leads to the mean-field expression for the effective potential

$$U = -\frac{M_r}{2\pi a} \rho + \frac{1}{2} \int \frac{d^3 q}{(2\pi)^3} \left[ \epsilon_1 + \epsilon_2 - \mu_1 - \mu_2 + \rho \frac{\epsilon_1 + \epsilon_2}{2\epsilon_1 \epsilon_2} + \sqrt{(E_1 F + E_2 F)^2 + 4\Delta^2} \right]. \quad (15)$$

Here $a$ is the fermion-fermion scattering length and $M_r$ is the reduced mass. Imposing the condition $\partial_\mu U|_{\rho=\rho_0} = 0$ we recover the BCS-like gap equation.
FIG. 1: Phase diagram as a function of $-1/p_Fa$ with $p_F$ corresponding to the fermions with a larger density. The upper curve (red online) is the result of the calculations and lower curve (blue online) corresponds to experimental data from [8].

\[- \frac{M_r}{2\pi a} + \int \frac{d^3 q}{(2\pi)^3} \left[ \frac{1}{E_{1F} + E_{2F}} - \frac{1}{\sqrt{(E_{1F} + E_{2F})^2 + 4\Delta^2}} \right] = 0, \quad (16)\]

Our approach can be applied to any type of many-fermion system but for a concreteness we use a parameter set relevant to nuclear matter: $M_1 = M_2 = 4.76$ fm$^{-1}$, $p_{1(2)} \simeq 1$ fm$^{-1}$ and large fermion-fermion scattering length ($a > 1$) fm.

We use the regulators in the form suggested in [9] for both bosons and fermions

\[ R_{Fi} = \frac{1}{2M_i} \left[ (k^2 \text{sgn}(q - p_{\mu_i}) - (q^2 - p_{\mu_i}^2)) \right] \theta(k^2 - |q^2 - p_{\mu_i}^2|), \quad (17)\]

where $p_{\mu_i} = (2M_i\mu_i)^{1/2}$, and

\[ R_B = \frac{1}{2m} (k^2 - q^2)\theta(k - q). \quad (18)\]

The initial conditions for $u$'s and $Z$ can be obtained by differentiating the expression for the effective potential at the starting scale $k = k_{st}$ and setting the parameter $\rho$ to zero.

Now we turn to the results. First note that the system undergoes the transition to the broken phase at critical scale $k_{cr} \simeq \frac{p_1 + p_2}{2}$. Its value slowly decreases when the asymmetry is increased while keeping the total chemical potential fixed. We found that the value of the $k_{kr}$ is practically insensitive to the starting scale provided the scale is chosen to be larger than 10 fm.
In Fig.1 we show the results for the critical line, separating the gapped and normal phases as a function of the dimensionless parameter $1/p_{1}a$, where $p_{1}$ corresponds to the state with larger density and particle density asymmetry $\alpha = \frac{n_{1}-n_{2}}{n_{1}+n_{2}}$. The experimental data are from [8]. The curve going through the experimental points is the exponential fit from [8]. Our theoretical curve approaches the experimental fit with decreasing $p_{1}|a|$ although always lies above the experimental data thus indicating the room for a further improvement of our ansatz.

We show on Table 1 the results of the calculations for the superfluid gap in the limit of small density imbalance $\alpha = 0.03$ in comparison with the experimental data from [10]. As in the case of the phase diagram the theoretical points are not far from the experimental data but still lie above them indicating that higher order terms should be included in our truncation for the effective action to achieve better agreement with the data.

| $1/p_{F}a$ | $\Delta$(exp) | $\Delta$(calc) |
|------------|---------------|----------------|
| 0          | 0.44          | 0.69           |
| -0.25      | 0.22          | 0.35           |

We have also calculated the critical value of the chemical potential mismatch $\delta\mu_{c}$ with parameters typical for neutron matter (scattering length $a_{nn} \simeq -18.6$) fm. At large enough $\delta\mu > \delta\mu_{c}$ the pairing is disrupted and the system undergoes to a normal phase. The value of $\delta\mu_{c}$ can be important for the phenomenology of neutron stars because the transport properties of the normal and superconducting phase are very different [11]. Our calculations gives the value $\delta\mu_{c} = 0.41\mu_{c}$ to be compared with the QMC based results $\delta\mu_{c} = 0.27\mu_{c}$ [12].

In general, one can conclude that, in spite of a relative simplicity of the assumed ansatz for the effective action FRG provides a good starting point for a reasonable description of the phase diagram of asymmetric many-fermion systems. However, in order to achieve a better quantitative agreement with the data our truncation must be modified by adding some higher order terms.

One of the most obvious improvements of our approximation is to use a complete effective potential instead of expanding it near a possible scale dependent minimum. Given the fact
that we deal with first order phase transition using unexpanded effective potential will clearly be important in identifying the correct CC limit in the strongly interacting regime.

Another potentially important improvement of the formalism would be an inclusion of the fermion-fermion interaction in the particle-hole (ph) channel leading to the Gorkov-Melik-Barkhudarov (GMB) corrections [13]. The FRG based studies of the GMB corrections have been performed in [14] for the case of the balanced many-fermion systems. A generalisation of the approach developed in [14] to the imbalanced systems is highly nontrivial and requires a serious technical and computational efforts, Although the full size FRG calculations including the ph channel are beyond the scope of this paper some preliminary results indicate that the inclusion of the particle-hole interactions brings the theoretical results closer to the experimental data [15].

I. ACKNOWLEDGEMENT

The author is grateful to M. Birse and N. Walet for valuable discussions.

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