Model reduction on 2-D incompressible laminar flows: application on the flow over a backward-facing step

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Abstract. The computation of fluid mechanics problems usually leans on a discretization of the Navier-Stokes equations which has to be so fine that the dimensions of the linear systems to be solved are very high. As a direct consequence, the Central Processing Unit time needed to solve complex systems may become extremely large when accuracy is demanded. When coupling numerical modeling schemes to inversion or control problems, the size of linear systems to be solved has to be drastically reduced. Within this context, the identification method consists in identifying the components of a low-order matrix system. The identification process works as an inverse problem of parameter estimation. The test case shows the ability of the proposed method to reduced with accuracy a particular fluid mechanics problem.

1. Introduction
The numerical solution of fluid mechanics problems usually needs a fine spatial discretization of the considered domain. Consequently, the matrix systems may be large and the computational times may be elevated. This common approach gives the so-called detailed model. Though these models may be highly reliable and accurate, the computational cost can be cumbersome to treat applications such as in-time control or inverse problems for instance. The alternative is to build a model of much lower number of degrees of freedom that reproduces well enough the dynamics of the detailed model. Among the large variety of reduction methods, the modal identification method has shown to be efficient in our considered cases in transient heat transfer. The modal identification method was developed earlier on to identify reduced models for both linear [1] and nonlinear systems [2, 3] in heat conduction problems.

The identification method has recently been adapted for some fluid mechanics problems in stationary cases [4]. In [4], the stream function–vorticity (Ψ − ω) formulation was used for simplicity. However, the choice for the application of the boundary conditions is uneasy and impractical. This is the reason why the reduced model formulation has been revisited based on the primal variables that are the velocities and the pressure. The direct Navier-Stokes equations are discretized. The modal form is then derived leading to the structure for the reduced model. The components of the matrix system form the vector of parameters that is to be identified. The identification is based on the solution of an optimization problem that consists in minimizing a cost function. A gradient quasi-Newton method is used. The cost function gradient is computed through the solution of the adjoint state problem [5].
The aim of this paper is to present some results concerning model reduction on laminar flows through the identification method. The laminar steady fluid flow over a backward-facing step is considered to test the developed numerical tools. We present how one chooses the reduced model order and show that the identified reduced models are able to reproduce the behavior of the considered model with accuracy. In the presented test case, the model order is divided by 104.

2. Formulation of the low order model

Let us consider the Navier-Stokes equation:

\[
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \frac{1}{\rho} \nabla p &= 0 \quad \text{in} \quad \Omega \times (0, T), \\
\nabla \cdot \mathbf{u} &= 0 \quad \text{in} \quad \Omega \times (0, T),
\end{aligned}
\]

where \( \mathbf{u}(x,t) \) is the flow velocity, \( p(x,t) \) is the pressure, \( \nu > 0 \) is the kinematic viscosity of the fluid, \( \rho > 0 \) is the density, \( T > 0 \) and \( \Omega \) is an open bounded domain in \( \mathbb{R}^2 \).

In the particular application treated in section 4, the following boundary conditions are applied on some parts of \( \partial \Omega \), boundary of the domain \( \Omega \):

\[
\begin{align}
\mathbf{u} &= \bar{\mathbf{u}} \quad \text{on} \quad \partial \Omega_1 \quad (2a) \\
-\nu \frac{\partial \mathbf{u}}{\partial n} + pm &= 0 \quad \text{on} \quad \partial \Omega_2 \quad (2b)
\end{align}
\]

For sufficiently regular functions \( \mathbf{u} \) and \( p \), the problem (1) becomes [6]:

Find \( \mathbf{u}(x,t), \ p(x,t) \) such that:

\[
\begin{aligned}
\frac{d}{dt} (\mathbf{u}, \mathbf{v})_\Omega + a(\mathbf{u}, \mathbf{v}) + b(\mathbf{u}; \mathbf{u}, \mathbf{v}) + c(\mathbf{v}, p) &= 0 \quad \forall \mathbf{v} \in \mathbf{V} \\
c_0(\mathbf{u}, q) &= 0 \quad \forall q \in \mathbf{Q} \\
+ \text{essential boundary conditions}.
\end{aligned}
\]

In (3) \( \mathbf{V} \) is in a Sobolev space \( H^1_0(\Omega)^2 \), \( \mathbf{Q} \subseteq L^2(\Omega) \), \( (\cdot, \cdot) \) is the inner product of \( L^2(\Omega) \), \( a_0(\mathbf{u}, \mathbf{v}) = \nu (\nabla \mathbf{u}, \nabla \mathbf{v}) \) and \( a_1(\mathbf{u}, \mathbf{v}) = -\nu (\nabla \mathbf{u} \cdot \mathbf{n}, \mathbf{v})_{\partial \Omega} \), the trilinear form \( b(\mathbf{u}; \mathbf{v}, \mathbf{w}) \) is given by \( \sum_{i,j=1}^d \int_\Omega w_i \frac{\partial u_i}{\partial x_j} w_j \) where \( u_i, v_i \) and \( w_i \) are the canonical components of \( \mathbf{u}, \mathbf{v} \) and \( \mathbf{w} \), and \( c = c_0 + c_1 \) where \( c_0(q, q) = -\frac{1}{\rho} (\nabla \cdot \mathbf{v}, q)_{\Omega} \) and \( c_1(q, q) = \frac{1}{\rho} (\mathbf{v} \cdot \mathbf{n}, q)_{\partial \Omega} \).

The space semi-discretized version of (3) yields to finding for all \( t \in (0, T) \): \( (\mathbf{u}, \bar{\mathbf{p}}) \in \mathbb{R}^{\text{dim} (\mathbf{V}_h) + \text{dim} (\mathbf{Q}_h)} \) that satisfies the general model:

\[
\begin{aligned}
M_{i,j} \frac{d \tilde{u}_i}{dt} + A_{i,j} \tilde{u}_j + \tilde{u}_i B_{i,j} \tilde{w}_j + C_{i,j} \bar{p}_j &= 0 \quad \forall i = 1, \ldots, p \\
C_{0,j} \tilde{u}_j &= 0 \quad \forall j = 1, \ldots, q.
\end{aligned}
\]

The second equation in (4) may be eliminated in various ways such as for instance taking the divergence of the first equation of (1) and discretizing as above to get an equation linking the pressure to the velocities, or more directly using a penalty method [6]. In any way, the equation (4) is reduced to the general form (tildes have been removed for clarity)

\[
\frac{d \mathbf{u}}{dt} = A \mathbf{u} + \Psi(\mathbf{u}) + B \bar{\mathbf{u}}
\]

where \( A \mathbf{u} \) represent the linear contributions and \( \Psi(\mathbf{u}) \) represent the quadratic contributions. \( \bar{\mathbf{u}} \) represents the prescribed boundary conditions used as inputs for reduction and \( B \) is thus the input matrix (in some special cases one can get \( B \mathbf{u} + B^+ \)).
A selection of the field is performed through the introduction of the linear relationship:

\[ Y = Cu \]  

where \( C \) is the selection matrix. Relationships (5) and (6) make up the so-called detailed model expressed in a form suitable for reduction.

The formulation of the low order model is deduced from (5) through the truncation of its modal form version (see [1] for instance). Then, the time-independent model structure is:

\[
\begin{aligned}
& x + \omega Z(x) + g\bar{u} = 0 \\
& Y - hx = 0
\end{aligned}
\]  

where \( x \in \mathbb{R}^n \) and \( Y \in \mathbb{R}^q \). Note that while the primal variable \( u \) was in \( \mathbb{R}^N \) where \( N \) is usually large when considering a sufficiently fine mesh (typically \( N \gg 10^6 \)), this number \( n \) is low (typically \( <10 \)). Note also that the quadratic contribution \( \Psi(u) \) in (5) has been rewritten to \( Z(X) \) with components expressed as \( X_i^{m_i}X_j^{m_j} \) with \( i, j = 1, \ldots, N \) and \( m_i, m_j = 0, \ldots, 2 \) and \( m_i + m_j = 2 \). The dimension of this vector is \( \dim(Z)(m) = \frac{m \cdot (m+1)}{2} \) [7].

### 3. The inverse problem solution for model reduction

The identification of vectors and matrices involved in (7) may be computed in the straightforward way from (5) and (6) solving the eigenvalue problem and selecting the most dominant modes. The other way consists in identifying all components involved in (7) through the solution of the optimization problem:

\[
\inf_{\beta \in \mathbb{R}^{\dim(\beta)}} J(\beta)
\]

where \( \beta \) is the collection of \( \omega_{i,j} \), \( g_{i,k} \) and \( h_{l,i} \) with \( i = 1, \ldots, m, \ j = 1, \ldots, \dim(Z)(m), \ k = 1, \ldots, p \) and \( l = 1, \ldots, q \). The cost function \( J \) to be minimized is based on a quadratic norm of errors between the output \( Y \) given by the reduced model and the output \( Y^* \) given by the detailed model. This norm is integrated on all performed comparisons, hence:

\[
J(\beta) = (hx - Y^*, hx - Y^*)_Y
\]

where \((\cdot, \cdot)\) is the inner product in \( Y \) (see [4] for more explanation). The solution of the optimization problem is performed through an iterative quasi-Newton algorithm. At each iteration, the reduced model (7) is integrated, the cost function is computed, and the cost gradient is computed integrating the adjoint problem of (7)-(9). We give here only pieces of information on the adjoint problem solution. The reader may refer to [5] or [8] for more details.

The adjoint equation reads

\[
\chi + \tilde{Z}^t\omega^t\chi + h^t(hx - Y^*) = 0
\]

where \( \chi \) is the adjoint variable and \( \tilde{Z} \) is the jacobian of \( Z \) with respect to the state \( x \). When this adjoint equation is satisfied, then the gradient is given by:

\[
\nabla J = \left( Z_j, \chi_i \right)_{i=1,\ldots,m \atop j=1,\ldots,\dim(Z)(m)} \left( u_j, \chi_i \right)_{i=1,\ldots,m \atop j=1,\ldots,p}^t
\]

both blocks being related to the components of the cost function gradient with respect to \( \omega \) and \( g \) respectively.

Note that the components of the observation matrix \( h \) are not included in (11): we actually get some benefit from the linearity property from \( Y \) and \( x \) to identify the components of \( h \) separately apart using the mean squares method [4]. The general algorithm for the reduced model identification can be found in [5].
4. Numerical results

A schematic diagram of the considered geometry is shown in Figure 1. It consists of a backward-facing step in a duct where the step height is \( h = 1 \) cm. The coordinate system is defined as shown schematically in this figure, where the \( x \)- and \( y \)-coordinate directions denote the streamwise and transverse directions, respectively. The upstream height is also \( h = 1 \) cm, hence the downstream height is \( 2h \).

The length of the upstream region was taken equal to \( 4h \). Doing so, one is sure there is no unwanted direct interaction between the prescribed velocity and the first recirculation bubble. In order to catch the second recirculation, the downstream region length was taken equal to \( 14h \).

The inlet velocity profile is prescribed on \( \partial \Omega_1 \) (i.e. for \( x = 0 \), \( y = [h : 2h] \)) to \( \bar{u}_1(y) = 4U_{max}(y - h)(2h - y) \) and \( \bar{u}_2(y) = 0 \) and the outlet velocity profile is set free, i.e. boundary condition (2b) is applied on \( x = 18h \), \( y = [0 : 2h] \). On all other boundaries a null velocity is prescribed. The chosen fluid assumed to be Newtonian and incompressible is air with dynamic viscosity \( \mu \) and density \( \rho \) respectively equal to \( 1.81 \times 10^{-5} \) kg/m s and \( 1.205 \) kg/m\(^3\). The mean velocity \( U_\infty \) at the inlet is chosen such that the flow is driven at given Reynolds numbers \( \text{Re} = \frac{U_\infty h}{\nu} \) (where \( \nu = \frac{\mu}{\rho} \)) from 100 to 700 by steps of 100.

Seven velocity fields \((K = 7)\) have been computed using the commercial software Fluent®. The number of unknowns needed to ensure the mesh convergence was found to be \( N = 60,751 \). For a prior assessment of the detailed model, the streamline pattern was analysed. For all Reynolds number being tested, downstream of the step, there was a main recirculation region, whose length increased with the Reynolds number. Figure 2 presents the evolution of adimension lengths \( \frac{X_1}{h} \) and \( \frac{X_2}{h} \) with respect to the Reynolds number from 100 up to 700. This figure also compares our results to those from [9], [10] and [11]. The results follow the trend as observed in literature. For a Reynolds number equal to 400, a second recirculation appears attached to the top wall of the channel. Its characteristic lengths also fit with those from literature.

The seven direct detailed model run (from \( \text{Re} = 100 \) to \( \text{Re} = 700 \) by step of 100) were performed to form the output vector \( Y^* \). In there, we considered all nodes satisfying \( \{x \geq 2h\} \cap \Omega \). When considering the mesh with \( N = 60,751 \) nodes, we obtained \( q = 58,190 \) nodes considered as output for the integration of the inverse problem cost function (9) that measures the global distance between the results related to detailed model and to the reduced model. The maximum order for model reduction is \( n = 6 \) so that the identification is performed in an over-determined way.

Table 2 and Figure 3 give the evolution of the cost function \( J \) (defined by (9)), and the mean quadratic errors \( \sigma \) and the maximum error \( \varepsilon \) (defined respectively in (12) and (13)) with respect to the reduced model order. The mean quadratic errors and maximal errors are defined distinctly for velocities \( u_1 \) and \( u_2 \) since the order of magnitude of both velocities are very different (see Table 1).
\[
\sigma_{u_i} = \sqrt{\frac{1}{K \times q} \mathcal{J}} \bigg|_{u_i} 
\]

(12)

\[
(\varepsilon_{u_i})_{\text{max}} = \sup_{j=1,\ldots,K \atop k=1,\ldots,q} |hx - Y^*| \bigg|_{u_i} 
\]

(13)

Both Table 2 and Figure 3 show that the cost function and errors decrease drastically with the reduced model order. One can consider that from the order 4 the errors are acceptable. We however go up to the order six since the errors associated to this model order are the lowest.

To validate the reduction process, we compare the results of both the reduced model of order 6 and the detailed model where other Reynolds numbers are prescribed (through the prescription of other maximal velocities on \(x = 0\)). The considered Reynolds number are taken from 150 to 650 by steps of 100.

Table 3 presents the validation results, i.e. the different errors \(\sigma_{u_i}\) and \(\varepsilon_{u_i,\text{max}}\) taken distinguishably for both velocity components \(u_1\) and \(u_2\). This table shows that errors are very low when compared to the order of magnitude of velocities (compare with velocities presented in Table 1).

Figure 4 presents the values of the stream function lines \(\Psi\) computed with the detailed model and the corresponding one computed with the reduced model of order 6. The vectors of vorticity \(\Omega\) are also compared on Figure 5. It can be seen a very good agreement between these results. Due to the spatial discretization of the velocity field that is obtained through the reduced model, the results are better on the stream function than for the vorticity. This is due to the fact that the stream function is obtained from an integration of the velocity field whereas the vorticity is obtained through a space differentiation of the velocity field. Hence, the stream function field is more continuous than the vorticity one.

Figure 6 shows a good agreement of the \(x\)–velocity profile given by the detailed model and the reduced model of order 6.

5. Conclusion

A model reduction was carried out on a particular laminar and steady incompressible fluid flow by using the identification method just derived from the modal identification method. The reduced-order model formulation is derived from the primal variables, i.e. the velocity and the pressure. The parameter identification process is performed iteratively on the minimization of a cost function. The considered test case results are very satisfying as the identified reduced models can compute with accuracy the velocity fields of different fluid flow regimes. We point out that the computation times for solving the detailed model are of the order of hours and that the solving of the reduced models is of the order of some thousandths seconds.

Future work concerns the coupling of such fluid mechanics problems with heat transfer problems. First results of such convection coupling are given in [12]. Afterwards, since the computation of reduced-order models are very fast, we plan to use such identified models in on-line thermal control applications.

References

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[2] Girault M and Petit P 2005 International Journal of Heat and Mass Transfer 48 105–118
[3] Balima O, Favennec Y and Petit D 2007 Numerical Heat Transfer B 52 107–130
[4] Balima O, Rouiri Y, Favennec Y and Petit D 2007 Proc. Inverse Problems, Design and Optimization Symposium
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Table 1. Maximal velocity and mean velocity for both components $u_1$ and $u_2$ for the case where Re=700

| Components | Maximal Velocity (m/s) | Mean Velocity (m/s) |
|------------|------------------------|---------------------|
| $u_1$      | $7.662 \times 10^{-1}$ | $1.654 \times 10^{-1}$ |
| $u_2$      | $8.276 \times 10^{-2}$ | $8.058 \times 10^{-3}$ |

Table 2. Evolution of the cost function and maximal and mean quadratic errors with respect to the reduced model order. Errors with respect to $u_1$ and $u_2$ are distinguished.

| Model order | $\mathcal{J}$ | $\sigma_{u_1}$ | $\sigma_{u_2}$ | $(\varepsilon_{u_1})_{\text{max}}$ | $(\varepsilon_{u_2})_{\text{max}}$ |
|-------------|---------------|----------------|----------------|-------------------------------|-------------------------------|
| 1           | 276.809       | 3.611×10^{-2}  | 2.978×10^{-4}  | 1.400×10^{-1}                | 3.725×10^{-2}                |
| 2           | 53.3038       | 1.557×10^{-2}  | 1.955×10^{-4}  | 6.135×10^{-2}                | 2.014×10^{-2}                |
| 3           | 7.73241       | 5.813×10^{-3}  | 1.195×10^{-4}  | 2.312×10^{-2}                | 1.052×10^{-2}                |
| 4           | 1.40188       | 2.405×10^{-3}  | 7.683×10^{-5}  | 1.049×10^{-2}                | 4.836×10^{-3}                |
| 5           | 0.21674       | 9.257×10^{-4}  | 4.767×10^{-5}  | 5.010×10^{-3}                | 1.951×10^{-3}                |
| 6           | 0.03931       | 3.926×10^{-4}  | 3.105×10^{-5}  | 1.878×10^{-3}                | 1.195×10^{-3}                |

Table 3. Evolution of errors $\sigma_{u_i}$ and $\varepsilon_{u_{i,\text{max}}}$, $i=1,2$ for the six considered validation test: Re from 150 to 650 by steps of 100.

| Re  | $\sigma_{u_1}$ | $\sigma_{u_2}$ | $(\varepsilon_{u_1})_{\text{max}}$ | $(\varepsilon_{u_2})_{\text{max}}$ |
|-----|----------------|----------------|-------------------------------|-------------------------------|
| 150 | 9.046×10^{-4}  | 4.486×10^{-4}  | 2.885×10^{-3}                | 1.776×10^{-3}                |
| 250 | 4.629×10^{-4}  | 2.625×10^{-4}  | 1.494×10^{-3}                | 9.764×10^{-4}                |
| 350 | 3.528×10^{-4}  | 1.107×10^{-4}  | 1.038×10^{-3}                | 3.972×10^{-4}                |
| 450 | 8.659×10^{-4}  | 2.689×10^{-4}  | 2.322×10^{-3}                | 9.755×10^{-4}                |
| 550 | 3.932×10^{-3}  | 8.663×10^{-4}  | 1.358×10^{-2}                | 3.349×10^{-3}                |
| 650 | 1.385×10^{-2}  | 2.915×10^{-3}  | 5.028×10^{-2}                | 1.014×10^{-2}                |
**Figure 2.** Main figure: size of the main recirculation region $X_1/h$ as a function of the Reynolds number $Re$. Inset: size of the second recirculation $X_2/h$ as a function of the Reynolds number $Re$. See references [9, 10, 11] for comparison. The name Rouizi makes reference to this paper results.

**Figure 3.** Evolution of the cost function and errors $\sigma_{u_i}$ and $\varepsilon_{u_{i\text{max}}}$, $i = 1, 2$ as a function of the reduced model order.

**Figure 4.** Contours of stream line for both the detailed model (middle) and the six-order reduced model (bottom) for $Re=450$. 
Figure 5. Contours of vorticity vector magnitude for both the detailed model (middle) and the six-order reduced model (bottom) for Re=450.

Figure 6. $x$-velocity profile for both the detailed model (top) and the six-order reduced model (bottom) for Re=550.