A Bound Quantum Particle in a Riemann-Cartan space with Topological Defects and Planar Potential

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Starting from a continuum theory of defects, that is the analogous to three-dimensional Einstein-Cartan-Sciama-Kibble gravity, we consider a charged particle with spin \( \frac{1}{2} \) propagating in a uniform magnetic field coincident with a wedge dispiration of finite extent. We assume the particle is bound in the vicinity of the dispiration by long range attractive (harmonic) and short range (inverse square) repulsive potentials. Moreover, we consider the effects of spin-torsion and spin-magnetic field interactions. Exact expressions for the energy eigenfunctions and eigenvalues are determined. The limit, in which the defect region becomes singular, is considered and comparison with the electromagnetic Aharonov-Bohm effect is made.

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I. INTRODUCTION

The investigation of quantum mechanical systems with non-trivial boundary conditions is an active field of research. A fertile ground for such systems is provided by particles moving in a background space with nonvanishing (positive or negative) curvature and/or non-trivial topology. By non-trivial topology we mean multiply connected spaces. Indeed, the prototype representing the case of non-trivial topology is the well known electromagnetic Aharonov-Bohm (AB) effect [1]. In the context of the electromagnetic AB effect, the effect of topology manifests as the phase factor in the wavefunction of an electron moving around a magnetic flux line. The gravitational analogue of this effect has been investigated in [2, 3, 4]. We are going to investigate a space with topological defects characterized by vanishing Riemann-Christoffel curvature and torsion [5] everywhere except on the defects. Such defects arise in gauge theories with spontaneous symmetry breaking and may have played some significant role in the formation of large scale cosmological structure. Some examples in cosmology are domain walls [6], cosmic strings [7, 8] and monopoles [8]. Analogues of such defects in condensed matter physics include vortices in superfluids and superconductors [9], domain walls in magnetic materials, dislocations in solids and disclinations in liquid crystals or two-dimensional graphite [10].

Quantum effects on particles moving in a crystalline media with topological defects have been a subject of investigation since the early 1950’s [11]. In the geometric approach of Katanaev and Volovich [12], the theory of defects in solids is translated into the language of three-dimensional Einstein-Cartan-Sciama-Kibble gravity [13, 14] but it could be extended considering more general forms of torsion [15]. This formalism corresponds to viewing the continuous limit of a crystalline solid in which the defect configuration is characterized by a non-trivial metric describing a static three-space. In this way, elastic deformations introduced in the medium by defects are incorporated into the metric manifold. The boundary conditions required by the presence of such defects are accounted for by the associated non-Euclidean metric. In the continuum limit, that is, for distances much larger than the Bravais lattice spacing, the theory describes a non-Riemannian manifold where curvature and torsion are associated with disclinations and dislocations in the medium, respectively.

In the case of solid crystals, a topological defect can be thought as consisting of a core region characterized by the absence of regular lattice order and an ordered (undistorted) far-field region [10]. Although in the continuum approximation in which we work the core region is usually shrunk to a singularity, we choose to relax this condition by smearing the singularity over a finite region. In this work, we study a charged spin-\( \frac{1}{2} \) particle moving in the space of a defect comprised of a combined screw dislocation and a wedge disclination. Such a combined defect is called a wedge dispiration, a defect possessing nonvanishing local curvature and torsion. A homogeneous, finite magnetic field, concentric with the wedge dispiration
line, is included. As a working hypothesis, we assume the particle is bound in the vicinity of the wedge dispiration by a planar potential given by a long range attractive (harmonic) and short range (inverse square) repulsive terms.

The paper is organized as follows: In Section II, we introduce the topological defect. In Section III, we consider a simple finite defect distribution describing a non-singular dispiration. In Section IV, the corresponding time-independent Schrödinger equation is taken into account. In Section V, exact expressions for the energy eigenfunctions of the particle moving both inside and outside the defect core of the medium are obtained. Boundary conditions on the surface of the defect region are implemented in order to get the wavefunction defined over the whole space. The energy spectrum is determined and we consider the limit in which the defect region is shrunk to a null size. Conclusions are drawn in Section VI.

II. THE MAGNETIC WEDGE DISPIRATION

In the framework of traditional elasticity theory, a Cartesian reference frame $x^i$ is attached to the undistorted medium of an elastic solid with Euclidean metric $\delta_{ab}$. The deformation of the medium is described locally in terms of a continuous displacement function $u(x)$. In this way, after a deformation has occurred, the point $x^i$ will have coordinates $x^i \to y^i(x) = x^i + u^i(x)$. The initial metric $\delta_{ab}$ is transformed into \cite{17,18}

$$g_{ij} := \frac{\partial x^a}{\partial y^i} \frac{\partial x^b}{\partial y^j} \delta_{ab}, \ i, j = 1, 2, 3. \quad (1)$$

We consider a topological defect in three dimensions whose geometry is characterized by the spatial line element \cite{12}

$$dl^2 = g_{ij} dx^i dx^j = d\rho^2 + \kappa^2 \rho^2 d\varphi^2 + (dz + \tau d\varphi)^2 \quad (2)$$

where $(\rho, \varphi, z)$ are cylindrical coordinates, with $\rho \geq 0, 0 \leq \varphi \leq 2\pi, -\infty \leq z \leq \infty, \kappa, \tau \in \mathbb{R}$. The parameter $\kappa$ is related to the Frank vector $\Omega$ \cite{19} of the disclination (describing curvature, i.e. the angular deficit in the manifold) while $\tau$ is related to the Burgers’ vector $\vec{b}$ \cite{17,19} of the dislocation (describing torsion). The three dimensional geometry of the medium is therefore characterized by nonvanishing torsion and curvature. When $\kappa \neq 0$ and $\tau = 0$ we have a wedge disclination: when $\kappa = 0$ and $\tau \neq 0$ we have a screw dislocation. The metric tensor $g_{ij}$ and its inverse $g^{ij} = (g_{ij})^{-1}$ are given by \cite{20}

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \kappa^2 \rho^2 + \tau^2 & \tau \\ 0 & \tau & 1 \end{bmatrix}, \quad g^{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\kappa^2 \rho^2} & -\frac{\tau}{\kappa^2 \rho^2} \\ 0 & -\frac{\tau}{\kappa^2 \rho^2} & 1 + \frac{\tau^2}{\kappa^2 \rho^2} \end{bmatrix}. \quad (3)$$

The line element (2) describes an infinitely long linear wedge dispiration oriented along the $z$-axis. We introduce the dual (1-form) basis vectors $\vartheta^i = e^i_j dx^j$ which describe the background (2), with coframe components \cite{18}

$$\vartheta^1 \equiv d\rho, \ \vartheta^2 \equiv d\varphi = \kappa \rho d\varphi \quad \text{and} \quad \vartheta^3 \equiv dz + \tau d\varphi. \quad (4)$$

The metric $g^{ij}$ and triad components $e^i_a$ are related via $e^i_u e^j_u \delta^{ab} = g^{ij}$, where the triads satisfy $e^a_i e^b_j = \delta^a_b$. In a Riemannian geometry without torsion, the Ricci $R_{ij}$ and Riemann curvature tensor $R^l_{ijk}$ are given by \cite{5}

$$R_{ij} = g^{ab} R_{aibj} = \partial_k \Gamma^i_{jk} - \partial_j \Gamma^i_{ik} + \Gamma^k_{ij} \Gamma^i_{kn} - \Gamma^m_{ik} \Gamma^i_{jm} \quad (5)$$

and

$$R^l_{ijk} = \partial_l \Gamma^d_{jk} - \partial_j \Gamma^d_{lk} + \Gamma^d_{im} \Gamma^m_{jk} - \Gamma^d_{jm} \Gamma^m_{ik}, \quad (6)$$

respectively. The Christoffel symbols $\Gamma_{kij}$ appearing in (5) or (6) are defined by \cite{5}

$$\Gamma_{kij} := \frac{1}{2} (\partial_k g_{ij} + \partial_i g_{jk} - \partial_j g_{ik}) \quad (7)$$

where $\partial_k := \partial/\partial x^k$. The nonvanishing components of the Christoffel symbols in the space with metric (3) are \cite{18}

$$\Gamma^z_{\varphi \rho} = \Gamma^z_{\rho \varphi} = -\frac{\tau}{\rho}, \ \Gamma^\varphi_{\rho \varphi} = -\kappa^2 \rho, \ \Gamma^\rho_{\varphi \varphi} = \frac{\tau}{\rho}, \ \Gamma^\rho_{\rho \varphi} = \frac{1}{\rho} \quad (8)$$
By contrast, a non-Riemannian geometry is one that is characterized by nonvanishing curvature and torsion. The torsion tensor $T_{ab}$ is defined by

$$T_{ij}^k := \partial_i e^k_j - \partial_j e^k_i + \Gamma^k_{ij} e^\nu_j - \Gamma^k_{ji} e^\nu_i.$$  

The Ricci scalar is given by $\[16, 21\]$

$$R_{12} = R_{1}^{1} = R_{2}^{2} = 2\pi \left( \frac{1}{\kappa} \right) \delta^2 (\rho) ,$$  

where $\kappa = 1 + \phi/2\pi$ and $\delta^2 (\rho)$ is a two-dimensional Dirac $\delta$-function. This $\delta$-function reveals the conic singularity in the curvature given in (10). The disclination characterized by (2) can be thought as arising from a "cut and paste" process, known as the Volterra process $[20]$. From the perspective of the Volterra process, the disclination is generated by removing $(\kappa < 1)$ or inserting $(\kappa > 1)$ a wedge of material with deficit angle $\phi = 2\pi (\kappa - 1)$. For $0 < \kappa < 1$ the disclination carries positive curvature while for $1 < \kappa < \infty$ it carries negative curvature. The only non-vanishing component of torsion 2-form $T^k = T^k_{ij} dx^i \wedge dx^j$ is given by $[16, 21]$

$$T^z = 2\pi \tau \delta^2 (\rho) d\rho \wedge d\varphi .$$

It is clear that the space described by metric (3) features two conical singularities at the origin as seen in (10) and (11). The Burger vector can be viewed as a flux of torsion and the Frank vector as a flux of curvature. The Burger vector is calculated by integrating around a closed path $C$ encircling the dislocation $[17, 18]$

$$b^3 = \oint_C \theta^3 = \int_S d\rho \wedge d\varphi T^3_{\varphi \rho} = 2\pi \tau ,$$

implying that $2\pi \tau$ is the flux intensity of the torsion source passing through a closed loop $C$ in the $\hat{z}$-direction. Thus, the parameter $\tau$ is the modulus of the Burger vector. In a similar manner, the Frank vector is determined by $[10, 18]$

$$\Omega^1 = \int_S d\rho \wedge d\varphi R_{12}^{\rho \varphi} = 2\pi \left( \frac{1}{\kappa} \right) ,$$

where $\epsilon_{ijk}$ is the Levi-Civita symbol ($\epsilon_{123} = 1$), $\Omega^1 = -\Omega^2$ and $S$ is a surface perpendicular to the defect line, implying the flux of curvature is the surface density of the Frank vector field.

III. FINITE DEFECT DISTRIBUTION

To avoid the singular nature of (10) and (11), we choose a simple exactly solvable model of finite torsion and curvature. In the former case, we choose a torsion field with a homogeneous flux distribution within the dislocation region. In particular, we choose a torsion field specified by

$$T^z = T^z (\rho, \varphi) = \tau' \Theta (R_c - \rho) d\rho \wedge d\varphi ,$$

where $R_c$ denotes the radius of the defect core, $\tau' = \frac{4\pi \phi}{\pi R_c^2}$ and $s_T = \pm 1$ denotes the handedness of the screw, where $(s_T = 1)$ indicates a left handed screw in which a clockwise rotation of $\varphi = 2\pi \kappa' (\kappa'$ is defined in Eq.(17)) relative to the $(+)$ $x$-axis induces a shift in the $(-)$ $z$-direction; similarly, $(s_T = -1)$ describes a right handed screw in which a counter-clockwise rotation of $\varphi = 2\pi \kappa'$ relative to the $(+)$ $x$-axis induces a shift in the $(+)$ $z$-direction. In (14), $\Theta$ denotes the Heaviside theta function

$$\Theta (R_c - \rho) = \begin{cases} 1 & \text{for } \rho < R_c, \\ 0 & \text{for } \rho > R_c. \end{cases}$$

In the case of non-singular curvature, we consider a disclination characterized by the deficit angle

$$\bar{\phi} = \frac{1}{2} \phi R_c^2 \Theta (R_c - \rho).$$

Under $\phi \rightarrow \bar{\phi}$ transformation, the angular deficit $\kappa = 1 + \phi/2\pi$ transforms into

$$\kappa' = 1 + \frac{1}{2} \phi R_c^2 \Theta (R_c - \rho) = \begin{cases} \kappa_{in} = 1 + \frac{1}{2} \phi R_c^2 & \text{for } \rho < R_c, \\ \kappa_{out} = 1 & \text{for } \rho > R_c. \end{cases}$$
By the change of variables \( \rho \to \pi \), where
\[
\rho \to \pi = \frac{\rho^\prime}{\kappa^\prime} = \begin{cases} \rho_\prec = \frac{\rho}{R_m} & \text{for } \rho < R_c \\ \rho_\succ = \rho & \text{for } \rho > R_c \end{cases}
\] (18)
the metric describing the non-singular wedge dispiration is given by \[16\]
\[
dl^2 = g_{ij}dx^i dx^j = dp^2 + \kappa^2 p^2 d\varphi^2 + (dz + \tau^i d\varphi)^2.
\] (19)

Observe that distributions (14) and (17) are chosen such that the total torsion and curvature flux within the dispiration region are equivalent to initial values \( 2\pi \tau \) and \( 2\pi \left( \frac{\Theta T}{\pi \kappa} \right) \) respectively. Furthermore, we assume the particle is moving in the electromagnetic vector potential \( \vec{A}(\pi) \) \[22\]
\[
\vec{A}(\pi) = \frac{B_0}{2\pi \kappa \pi} \left( \frac{p^2}{R} \Theta (R_c - \rho) + \Theta (\rho - R_c) \right) \hat{\epsilon}_\varphi
\] (20)
where \( B_0 \) is the total magnetic flux. Vector potential (20) gives rise to a uniform magnetic field \( \vec{B} \) within the dispirated region.

**IV. THE SCHröDINGER EQUATION**

We study the non-relativistic quantum dynamics of a charged spin-\( \frac{1}{2} \) particle propagating in a space with finite dispiration defect (19) in presence of a uniform magnetic field \( \vec{B} = \vec{\nabla} \times \vec{A} \) and planar potential
\[
V(\pi) = \frac{1}{2} M \omega^2 \pi^2 \Theta (R_c - \rho) + \frac{\hbar^2}{2M \pi^2}
\] (21)
where \( f \) is a real, dimensionless constant characterizing the medium being probed. Observe that the harmonic potential does not extend beyond the medium \( (R_m) \) being considered, where \( R_m > R_c \) and \( R_m \) is the linear dimension of the medium. Furthermore, following \[23\], the spin-torsion interaction term \( \pi \vec{\sigma} \cdot \vec{T} \) is considered, where \( \vec{\sigma} = \frac{2\hbar}{\pi \kappa} \hat{\epsilon}_S \), \( \hat{\epsilon}_S \) denotes the spin direction, \( \mu = \frac{\hbar}{2M} \) is the Bohr magneton and \( g_e \) is the gyromagnetic ratio of the electron. Finally, the time-independent Schrödinger equation becomes,
\[
\left[ \frac{1}{2M} \Delta + \vec{\sigma} \cdot \left( \vec{B} - \frac{1}{2} \vec{T} \right) + V(\pi) \right] \psi(\pi, \varphi, z) = E \psi(\pi, \varphi, z).
\] (22)

The wavefunction satisfying (22) has periodicity \( \psi(\pi, \varphi, z) = \psi(\pi, \varphi + 2\pi \kappa^\prime, z) \) rather than the usual situation \( \psi(\rho, \varphi, z) = \psi(\rho, \varphi + 2\pi, z) \) in flat space. The Laplace-Beltrami operator \( \Delta \) is defined by
\[
\Delta := \frac{1}{\sqrt{g}} \left( \frac{\hbar}{i\kappa} \partial_i - \frac{q}{e} A_i \right) \left[ g^{ij} \sqrt{g} \left( \frac{\hbar}{i\kappa} \partial_j - \frac{q}{e} A_j \right) \right] = -\hbar^2 \vec{\nabla}^2 + \frac{q^2}{e^2} \vec{A}^2 - \frac{q\hbar}{c\kappa} \left( \vec{\nabla} \cdot \vec{A} + \vec{\nabla} \cdot \vec{\nabla} \right)
\] (23)
where \( i\kappa \) is the imaginary unit, \( c \) is the speed of light, \( q \) is the charge of the electron and \( g = \text{det} |g_{ij}| \) is the determinant of the metric. The Laplacian in the space (2) is given by
\[
\vec{\nabla}^2 = \left[ \frac{1}{p} \frac{\partial}{\partial p} \left( \frac{\partial}{\partial p} \right) + \frac{1}{\kappa^2 p^2} \left( \frac{\partial}{\partial \varphi} - T^z \frac{\partial}{\partial z} \right) \right]^2 + \frac{\partial^2}{\partial z^2}.
\] (24)

The divergence appearing in (23) is computed according to
\[
\vec{\nabla} \cdot \vec{A} = \frac{1}{\kappa p} \left( \frac{\partial}{\partial p} - T^z \frac{\partial}{\partial z} \right) A_p + \frac{1}{\kappa p} \left( \frac{\partial}{\partial \varphi} - T^z \frac{\partial}{\partial z} \right) A_\varphi + \frac{\partial A_z}{\partial z}
\] (25)
and is vanishing since \( \vec{\nabla} \cdot \vec{A} = \frac{1}{\kappa p} \left( \frac{\partial}{\partial p} - T^z \frac{\partial}{\partial z} \right) \frac{\hbar}{2\pi c\kappa p} B_0 = 0 \). With the aid of (24) and (25), operator (23) can be written as
\[
\Delta = -\hbar^2 \vec{\nabla}^2 + \frac{q^2}{e^2} \left( \frac{B_0}{2\pi \kappa \pi} \right)^2 \left[ \frac{p^2}{R^2} \Theta (R_c - \rho) + \Theta (\rho - R_c) \right]^2 + \frac{qB_0}{2\pi c\kappa \pi} \frac{\hbar}{e} \left( \frac{p^2}{R^2} \Theta (R_c - \rho) + \Theta (\rho - R_c) \right) \left( \frac{\partial}{\partial \varphi} - \frac{q\hat{\epsilon}_S}{\pi R^2} \Theta (R_c - \rho) \frac{\partial}{\partial z} \right).
\] (26)
In the space (2), the curl is given by
\[ \vec{\nabla} \times \vec{A} = \left[ \frac{1}{\kappa} \left( \frac{\partial}{\partial \rho} - T^z \frac{\partial}{\partial \rho} A_x - \frac{\partial}{\partial \rho} A_y \right) \hat{\rho} \right] + \frac{1}{\kappa} \left( \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \rho} (\kappa' \rho A_x) - \left( \frac{\partial}{\partial \rho} - T^z \frac{\partial}{\partial \rho} A_y \right) \hat{\rho} \right) \right] \hat{z}. \] (27)

For the vector potential given in (20), the magnetic field is computed with the aid of (27), the result being
\[ \vec{B} = \frac{B_0}{\kappa' \pi R_c^2} \Theta (R_c - \rho) \hat{z} \] (28)
where \( B_0 \) is the total magnetic flux over the dispirated region with radius \( R_c \).

**V. ENERGY EIGENFUNCTION AND EIGENVALUES**

The explicit time-independent Schrödinger equation for the system being considered reads
\[ \left\{ \begin{array}{l}
- \frac{\hbar^2}{2M} \nabla^2 + \frac{1}{2M} \left( \frac{q}{c} \hat{A} \cdot \nabla \right) + \frac{1}{2} \mu g_e \left( s_B \left| \vec{B} \right| - \frac{1}{8} \left| \vec{T} \right| \right) + V(\vec{r}) \right\} \psi = E \psi, (29)
\]
In (29) we have made the replacement
\[ \hat{\sigma} \cdot \left( \vec{B} - \frac{1}{8} \vec{T} \right) = \frac{1}{2} \mu g_e \left( s_B \left| \vec{B} \right| - s_T \frac{1}{8} \left| \vec{T} \right| \right) = \frac{1}{2} \mu g_e \left( s_B \frac{B_0}{\kappa' \pi R_c^2} - s_T \frac{1}{8} \frac{b^2}{\pi R_c^2} \right) \Theta (R_c - \vec{r}) \] (30)

since both \( \vec{B} \) and \( \vec{T} \) are oriented along \( \hat{z} \), where \( s_B = \pm 1 \) describes the orientation of the electron magnetic moment relative to \( \vec{B} \), either (+) parallel or (−) antiparallel. In the region \( \rho < R_c \), \( \Theta (R_c - \rho) = 1 \) and \( \Theta (\rho - R_c) = 0 \) resulting in the Schrödinger equation
\[ \left\{ \begin{array}{l}
- \frac{\hbar^2}{2 \rho c} \left( \rho \frac{\partial}{\partial \rho} \right)^2 + \frac{\hbar^2}{2 \rho c} \left( \frac{\partial}{\partial \rho} \right)^2 + \frac{1}{\kappa' \rho c} \left( \frac{\partial}{\partial \rho} \right)^2 \left( \frac{\partial^2}{\partial \rho^2} \right)^2 \\
+ \frac{1}{2M} \left( \frac{q}{c} \frac{B_0}{2 \pi \kappa' \rho c R_c^2} \right)^2 \rho^2 \left( \frac{\partial}{\partial \rho} \right)^2 + \frac{i q}{c} \frac{B_0}{2 \pi \kappa' \rho c R_c^2} \left( \frac{\partial}{\partial \rho} \right)^2 \left( - \frac{\partial}{\partial \rho} \right)\left( \frac{\partial}{\partial \rho} \right) + \frac{M \omega^2 \rho^2}{2} + \frac{\hbar^2}{2M \rho^2} \\
\end{array} \right\} \psi^{\text{in}} = E \psi^{\text{in}}. (31) \]
In the region \( \rho > R_c \), \( \Theta (R_c - \rho) = 0 \) and \( \Theta (\rho - R_c) = 1 \) yielding
\[ \left\{ \begin{array}{l}
- \frac{\hbar^2}{2 \rho c} \left( \rho \frac{\partial}{\partial \rho} \right)^2 + \frac{\hbar^2}{2 \rho c} \left( \frac{\partial}{\partial \rho} \right)^2 + \frac{1}{\kappa' \rho c} \left( \frac{\partial}{\partial \rho} \right)^2 \\
+ \frac{1}{2M} \left( \frac{q}{c} \frac{B_0}{2 \pi \kappa' \rho c R_c^2} \right)^2 \rho^2 \left( \frac{\partial}{\partial \rho} \right)^2 + \frac{i q}{c} \frac{B_0}{2 \pi \kappa' \rho c R_c^2} \left( \frac{\partial}{\partial \rho} \right)^2 \left( - \frac{\partial}{\partial \rho} \right)\left( \frac{\partial}{\partial \rho} \right) + \frac{M \omega^2 \rho^2}{2} + \frac{\hbar^2}{2M \rho^2} \\
\end{array} \right\} \psi^{\text{out}} = E \psi^{\text{out}}. (32) \]
In order to determine the eigenfunctions, the solutions to (41) and (42) must be matched at the boundary according to
\[ \psi^{\text{in}} (\rho) \big|_{\rho=R_c} = \psi^{\text{out}} (\rho) \big|_{\rho=R_c} \quad \text{and} \quad \frac{\partial}{\partial \rho} \psi^{\text{in}} (\rho) \big|_{\rho=R_c} = \frac{\partial}{\partial \rho} \psi^{\text{out}} (\rho) \big|_{\rho=R_c}. (33) \]
Since we consider a scenario in which the particle is bound in the medium, we require the wavefunctions satisfy the normalization condition
\[ \int_0^{R_c} \int_0^{2\pi} \int_{-\infty}^{\infty} |\psi^{\text{in}}|^2 \rho d\rho d\varphi dz + \int_{R_c}^{R_m} \int_0^{2\pi} \int_{-\infty}^{\infty} |\psi^{\text{out}}|^2 \rho d\rho d\varphi dz = 1. (34) \]
A. The Internal Core Solution

Equation (31) can be rewritten as

\[
\left\{ \left[ \frac{1}{\rho^2} \frac{d}{d\rho^2} \left( \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial z^2} + \frac{1}{\kappa_{\text{in}} \rho^2} \left( \frac{\partial^2}{\partial \rho^2} - \tau' \frac{d}{d\rho} \right) \right] + \frac{M^2 \omega^2}{\hbar^2} \rho^2 + \frac{f}{\rho^2} \right\} \psi^{\text{in}} = 0, \tag{35} \]

where \( Q = \frac{s_B z_{\text{hc}}}{\pi \hbar c} \), \( \Xi = \frac{\hbar c}{M} \left( s_B - \frac{1}{8} R^2 \right) \), \( \tau_0 = \frac{2ME}{\hbar^2} \). Since the space (2) possesses \( \hat{z}_z \)-translational symmetry, we assume a solution of the form \( \psi^{\text{in}}(\rho, \varphi, z) = Z(z) \chi^{\text{in}}(\rho, \varphi) \), with \( Z(z) = e^{ik_z z} \). Moreover, since the cylindrical configuration being considered possesses azimuthal symmetry, we assume \( \chi^{\text{in}}(\rho, \varphi) = \Phi(\varphi) R^{\text{in}}(\rho) \), with \( \Phi(\varphi) = e^{i\hat{\varphi}} \). Thus, (35) reduces to

\[
\left\{ \rho^2 \frac{d^2}{d\rho^2} + \rho \frac{d}{d\rho} + \Lambda_{\text{in}}^2 \rho^2 - \Upsilon^2 \rho^2 - \left[ f + \left( \frac{1 - \tau' k_z}{\kappa_{\text{in}}} \right)^2 \right] \right\} R^{\text{in}}(\rho) = 0, \tag{36} \]

where \( \Upsilon^2 = \left( \frac{Q \Xi}{R^2 \kappa_{\text{in}}} \right)^2 + \frac{M^2 \omega^2}{\kappa_{\text{in}}^2} \), \( \Lambda_{\text{in}}^2 = \frac{(1 - \tau' k_z) Q}{R^2 \Xi} + \epsilon_0 - k_z^2 - g_c \Xi \). The solution to (36) that is regular at \( \rho = 0 \) is given in terms of confluent hypergeometric functions \( F \),

\[
R^{\text{in}}(\rho) = A_l \rho^2 \exp \left( -\frac{\Upsilon^2}{2} \right) F \left( \frac{2 - g_c}{4} + \frac{\lambda}{2} - \frac{\Lambda_{\text{in}}^2}{4\Upsilon}, 1 + \lambda; \Upsilon^2 \right) \tag{37} \]

where \( A_l \) are appropriate expansion coefficients and \( \lambda = \sqrt{\left( \frac{1 - \tau' k_z}{\kappa_{\text{in}}} \right)^2 + f} \).

B. The External Solution

In the region \( R_c < \rho \) we have

\[
\left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \frac{\partial}{\partial \rho} \right) + \frac{\partial^2}{\partial z^2} + \frac{1}{\rho^2} \left( \frac{\partial}{\partial \varphi} - i \zeta Q \right)^2 - \frac{M^2 \omega^2}{\hbar^2} \rho^2 - \frac{f}{\rho^2} + \frac{2ME}{\hbar^2} \right\} \psi^{\text{out}} = 0. \tag{38} \]

Once again using a solution of the form \( \psi^{\text{out}}(\rho, \varphi, z) = Z(z) \Phi(\varphi) R^{\text{out}}(\rho) \), with \( Z(z) = e^{ik_z z} \) and \( \Phi(\varphi) = e^{i\hat{\varphi}} \) we obtain

\[
\left\{ \rho^2 \frac{d^2}{d\rho^2} + \rho \frac{d}{d\rho} + \epsilon_{\text{out}}^2 \rho^2 - \kappa^2 \rho^4 - \left[ (l + Q)^2 + f \right] \right\} R^{\text{out}}(\rho) = 0, \tag{39} \]

with \( \epsilon_{\text{out}}^2 = \frac{2ME}{\hbar^2} - k_z^2, \kappa^2 = \frac{2M^2 \omega^2}{\hbar^2} \). The solution to (39) that is regular at \( \rho = 0 \) is given by

\[
R^{\text{out}}(\rho) = \exp \left( -\frac{\kappa \rho^2}{2} \right) \left[ B_l \rho^{l_0} F \left( \frac{1}{2} + \frac{\nu}{2} - \frac{\epsilon_{\text{out}}^2}{4\kappa^2}, 1 + \nu; \kappa \rho^2 \right) + C_l \rho^{-\nu} F \left( \frac{1}{2} - \frac{\nu}{2} - \frac{\epsilon_{\text{out}}^2}{4\kappa^2}, 1 - \nu; \kappa \rho^2 \right) \right] \tag{40} \]

where \( B_l \) and \( C_l \) are expansion coefficients and \( \nu = \sqrt{(l + Q)^2 + f} \). Coefficients \( A_l, B_l \) and \( C_l \) are connected via the boundary matching conditions (33).

C. Boundary Matching Conditions

From the continuity of the wavefunction \( \psi^{\text{in}}(\rho) |_{\rho = R_c} = \psi^{\text{out}}(\rho) |_{\rho = R_c} \) we obtain

\[
A_l = e^{\Phi} \left( B_l R_{c}^{\nu - \lambda} M_{\nu, \lambda} + C_l R_{c}^{-(\nu + \lambda)} M_{-\nu, \lambda} \right) \tag{41} \]
To implement the second boundary condition in (33) we make use of the formula

\[
M_{\nu, \lambda} := \frac{F\left(\frac{1}{2} + \frac{\nu}{2} - \frac{z^2}{4\nu}, 1 + \nu; \nu R_c^2\right)}{F\left(\frac{z^2}{4} + \frac{1}{2} - \frac{\lambda^2}{4\nu}, 1 + \nu; T R_c^2\right)}.
\]  

(42)

We compute the logarithmic derivative \(\Pi_{\text{in}} := \rho \frac{1}{R_{\text{in}}(\rho)} \frac{d}{d\rho} R_{\text{in}}(\rho)\) |\(\rho=R_c\), the result being

\[
\Pi_{\text{in}} = A_t R_c (\nu R_c^{-1} - \nu R_c + 2 \nu R_c T\lambda),
\]  

(44)

where

\[
T_{\lambda} := \frac{2 - \frac{g_{\nu}}{2} + \lambda}{1 + \lambda} - \frac{\lambda^2}{4\nu} F\left(1 + \frac{2 - \frac{g_{\nu}}{2} + \lambda}{2} - \frac{\lambda^2}{4\nu}, 1 + 1 + \nu; \nu R_c^2\right) \cdot \frac{1 + \frac{2 - \frac{g_{\nu}}{2} + \lambda}{2} - \frac{\lambda^2}{4\nu}}{F\left(\frac{2 - \frac{g_{\nu}}{2} + \lambda}{2} - \frac{\lambda^2}{4\nu}, 1 + \nu; \nu R^2\right)}
\]  

(45)

In computing (44) we made use of the relation

\[
\frac{d}{d\rho} F\left(\frac{2 - \frac{g_{\nu}}{4} + \frac{\lambda}{2} - \frac{\lambda^2}{4\nu}, 1 + \nu; \nu R^2\right) = 2 \nu R_{\text{in}}(\rho) \frac{2 - \frac{g_{\nu}}{4} + \frac{\lambda}{2} - \frac{\lambda^2}{4\nu}}{1 + \lambda} F\left(1 + \frac{2 - \frac{g_{\nu}}{4} + \nu}{2} 1 + 1 + \nu; \nu R^2\right).
\]  

(46)

From the logarithmic derivative of the external radial function, we obtain

\[
\Pi_{\text{out}} = R_c B_t \left(\nu R_{\nu}^{-1} - \nu R_{\nu} + 2 \nu R_{\nu} U_{\nu}\right) + R_c C_t \left(-\nu R_{\nu}^{-1} - \nu R_{\nu} + 2 \nu R_{\nu} U_{-\nu}\right),
\]  

(47)

where

\[
U_{\nu} := \frac{\frac{1}{2} + \frac{\nu}{2} - \frac{\lambda^2}{4\nu}}{1 + \nu} F\left(1 + \frac{\frac{1}{2} + \frac{\nu}{2} - \frac{\lambda^2}{4\nu}}{2} - \frac{\lambda^2}{4\nu}, 1 + 1 + \nu; \nu R_c^2\right) \cdot \frac{1 + \frac{\frac{1}{2} + \frac{\nu}{2} - \frac{\lambda^2}{4\nu}}{2} - \frac{\lambda^2}{4\nu}}{F\left(\frac{\frac{1}{2} + \frac{\nu}{2} - \frac{\lambda^2}{4\nu}}{2} - \frac{\lambda^2}{4\nu}, 1 + \nu; \nu R_c^2\right)}
\]  

(48)

From (44) and (47) we obtain

\[
A_t \left(\lambda - \nu R_c^2 + 2 \nu R_c T\lambda\right) = B_t \left(\nu - \nu R_{\nu}^2 + 2 \nu R_{\nu} U_{\nu}\right) + C_t \left(-\nu - \nu R_{\nu}^2 + 2 \nu R_{\nu} U_{-\nu}\right).
\]  

(49)

Eliminating the coefficient \(A_t\) by substituting (41) in (49), the matching condition reduces to

\[
\nu R_{\nu} \left(\nu R_{-\nu}^2 + 1 + 2 T\lambda\right) = \frac{B_t \left(\nu R_{-\nu}^{-1} + 2 \nu R_{-\nu} U_{\nu}\right) + C_t \left(-\nu - \nu R_{-\nu}^2 + 2 \nu R_{-\nu} U_{-\nu}\right)}{B_t R_{c(\nu - \lambda)} M_{\nu, \lambda} \exp\left[\frac{R_{c(\nu - \lambda)}}{2} (\nu - \nu)\right] + C_t R_{c(\nu - \lambda)} M_{-\nu, \lambda} \exp\left[\frac{R_{c(\nu - \lambda)}}{2} (\nu - \nu)\right]}.
\]  

(50)

Equation (50) can be put into the form

\[
B_t = C_t N^{(1)}_{\nu, \lambda},
\]  

(51)

where

\[
N^{(1)}_{\nu, \lambda} := \frac{-\nu - \nu R_{\nu}^2 + 2 \nu R_{\nu} U_{-\nu} - \exp\left[\frac{R_{c(\nu - \lambda)}}{2} (\nu - \nu)\right] R_{c(\nu + \lambda)} M_{\nu, \lambda} \left(\lambda - \nu R_{\nu}^2 + 2 \nu R_{\nu} T\lambda\right)}{\exp\left[\frac{R_{c(\nu - \lambda)}}{2} (\nu - \nu)\right] R_{c(\nu - \lambda)} M_{\nu, \lambda} \left(\lambda - \nu R_{\nu}^2 + 2 \nu R_{\nu} T\lambda\right) - \left(\nu R_{\nu}^{-1} - \nu R_{\nu}^2 + 2 \nu R_{\nu} U_{\nu}\right)}.
\]  

(52)

Substituting (51) into (41) we obtain

\[
C_t = A_t N^{(2)}_{\nu, \lambda},
\]  

(53)
where
\[ N_{\nu, \lambda}^{(2)} := \frac{\lambda - \nu R_c^2 + 2\nu R_c^2 T \lambda}{N_{\nu, \lambda}^{(1)} (\nu - \nu R_c^2 + 2\nu R_c^2 U_{\nu} + (-\nu - \nu R_c^2 + 2\nu R_c^2 U_{-\nu})}. \] (54)

With the aid of (53), (51) can be rewritten as
\[ B_t = A_t N_{\nu, \lambda}^{(2)} N_{\nu, \lambda}^{(1)}. \] (55)

Conditions (53) and (55) determines the wavefunction up to a normalization constant. In particular, the wavefunction becomes
\[ \psi_{nl} (\rho, \varphi, z) = A_t e^{i c \varphi} e^{i c k_z z} \begin{cases} \rho^\lambda \exp \left( -\frac{\rho^2}{2} \right) F \left( \frac{1 - q}{4}, \frac{1}{2} - \frac{\lambda}{2} - \frac{\nu R_c^2}{4T}, 1 + \lambda; \nu \rho R_c^2 \right) & \text{for } \rho < R_{\text{core}} \text{ (in)} \\ N_{\nu, \lambda}^{(2)} \exp \left( -\frac{\rho^2}{2} \right) \left[ \frac{N_{\nu, \lambda}^{(1)} \rho^\nu F \left( \frac{1}{2} - \frac{\nu R_c^2}{2}, 1, 1 + \nu; \nu \rho^2 \right)}{\rho^\nu} \right] & \text{for } \rho > R_{\text{core}} \text{ (out)}. \end{cases} \] (56)

The asymptotic expansion of the confluent hypergeometric function is given by
\[ F \left( a, c; x \right) \simeq \frac{\Gamma \left( c \right)}{\Gamma \left( a \right)} |x|^{a-c} e^{i c \varphi} e^{i c k_z z} + \frac{\Gamma \left( c \right)}{\Gamma \left( c - a \right)} |x|^{-a} e^{i c (\pi - \varphi)}. \] (57)

If \( x = i c z \) (\( z = |x| \) - real, positive), then \( \varphi = \pi/2 \) and both terms in (57) are approximately equally large and should be taken into account. Provided \( x \) is real, positive (\( \varphi = 0 \)), the first term is considered only. When \( x \) is real, negative (\( \varphi = \pi \)) the second term on the right hand side of (57) is used. The radial function (37) is ensured to be vanishing at the origin \( \rho = 0 \) by expanding (37) according to (57) and choosing
\[ \frac{2 - q_e}{4} + \frac{1}{2} \sqrt{\left( l - \nu' k_z \right)^2 + f - \frac{\Lambda^2_{n_{\text{in}}}}{4T}} = -n \text{ where } n \in \mathbb{Z} \] (58)

which corresponds to the pole in the second term of (57). From (58) the energy eigenvalues in the defect region can be written as
\[ E_{nl}^{\text{in}} = 2h \left( \frac{\omega_B^2}{4} + \omega^2 \right)^{1/2} \left[ n + \sqrt{\left( \frac{l - \nu' k_z}{2\kappa_{n_{\text{in}}}} \right)^2 + f + \frac{2 - q_e}{4}} - \frac{h \omega_B l - \nu' k_z}{2} \frac{1}{\kappa_{n_{\text{in}}}} \right] + \frac{\hbar k^2_{z_0}}{2M}, \] (59)

where \( \omega_B := \frac{q B_0}{\pi \kappa_{n_{\text{in}}} R_c^2 M c} = q \left| \vec{B} / Mc \right| / \kappa_{n_{\text{in}}} R_c^2 \). The Larmor frequency of the electron and we made use of \( \gamma = \sqrt{\left( \frac{M \omega_B}{2h} \right)^2 + \frac{M^2 q_e^2}{\hbar^2}} \). Similarly, demanding (40) be vanishing as \( \rho \to \infty \) requires that we choose
\[ \frac{1}{2} + \frac{1}{2} \sqrt{(l + Q)^2 + f - \frac{\nu^2}{4}} = -n \] (60)

in the expansion of the confluent hypergeometric function whose coefficient is \( B_t \). From (60) the energy eigenvalues external to the defect region can be written as
\[ E_{nl}^{\text{out}} = 2h \omega \left( n + \sqrt{\left( \frac{l + Q}{2} \right)^2 + f + \frac{1}{2}} \right) + \frac{\hbar k^2_{z_0}}{2M}. \] (61)

D. The Limit \( R_c \to 0 \)

In order to facilitate a comparison between our results and the Aharonov-Bohm type scenario, we consider the limit in which the defect region is shrunk to zero, that is \( R_c \to 0 \). In this limit, the dispiration along with the internal
magnetic field is shrunk to an infinitesimal line along the z-axis. In this case the spin-magnetic and spin-torsion interaction vanishes, $T_z$ is given by (11), $R$ by (10) and $V(\rho) = \frac{Me^2 \rho^2}{2} + \frac{\mu B^2}{2M\rho^2}$. The corresponding Schrödinger equation reads

$$
-\left[ \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial z^2} + \frac{1}{\kappa^2 \rho^2} \left( \frac{\partial}{\partial \varphi} - \tau \frac{\partial}{\partial z} \right)^2 \right] - \frac{1}{2} \frac{qB_0}{2\pi \hbar c \kappa \rho^2} \left( \frac{\partial}{\partial \varphi} - \tau \frac{\partial}{\partial z} \right)^2 + \left( \frac{qB_0}{2\pi \hbar c \kappa} \right) + \frac{\mu}{\kappa^2 \rho^2} + \frac{\hbar^2}{\kappa^2 \rho^2}$$

$$
\psi = E\psi. \quad (62)
$$

With a solution of form $\psi (\rho, \varphi, z) = \mathcal{Z}(z) \Phi (\varphi) \mathcal{R} (\rho)$ where $\mathcal{Z}(z) = e^{ik_z z}$ and $\Phi (\varphi) = e^{i\ell \varphi}$, equation (62) is transformed into

$$
\left\{ \rho^2 \frac{d^2}{d\rho^2} + \rho \frac{d}{d\rho} - \left[ \frac{(l + \tau k_z)^2}{\kappa^2} + f \right] + \frac{1}{\kappa^2} \left[ \frac{qB_0}{2\pi \hbar c} (l - \tau k_z) - \left( \frac{qB_0}{2\pi \hbar c} \right)^2 \right] + \kappa \rho^2 + \Lambda^2 \rho^2 \right\} \mathcal{R} (\rho) = 0, \quad (63)
$$

with $\Lambda^2 = \frac{2ME}{\hbar^2} - k$ and $\kappa^2 = \frac{M^2 \omega^2}{\hbar^2}$. The solution to (63) that is regular at $\rho = 0$ is given by

$$
\mathcal{R} (\rho) = \rho^\mu \exp \left( -\frac{\kappa \rho^2}{2} \right) F \left( \frac{1}{2} + \frac{\mu}{2} - \frac{\Lambda^2}{4\kappa}, 1 + \mu; \kappa \rho^2 \right), \quad (64)
$$

where $\mu = \sqrt{\left( \frac{l + Q - \tau k_z}{\kappa} \right)^2 + f}$. In order that (64) be vanishing as $\rho \to \infty$, we require

$$
\frac{1}{2} + \frac{1}{2} \sqrt{\left( \frac{l + Q - \tau k_z}{\kappa} \right)^2 + f} - \frac{\Lambda^2}{4\kappa} = -n \quad (65)
$$

which leads to

$$
E_{nl}^{R_c \to 0} = 2\hbar \omega \left( n + \sqrt{\left( \frac{l + Q - \tau k_z}{2\kappa} \right)^2 + f + \frac{1}{4} + \frac{1}{2}} \right) + \kappa^2 k_z^2 \frac{2M}{\hbar^2} \quad (66)
$$

Finally, the eigenfunction associated to the Aharonov-Bohm type scenario associated with the magnetic wedge dispiration reads

$$
\psi_{nl}^{R_c \to 0} (\rho, \varphi, z) = A_n e^{ik_z z} e^{i\ell \varphi} \rho^\mu \exp \left( -\frac{\kappa \rho^2}{2} \right) F (-n, 1 + \mu; \kappa \rho^2) \quad (67)
$$

where $A_n$ is a normalization constant. We observe the modification of the angular momentum $l \to l' = (l + Q - \tau k_z) / \kappa$, interpreted as an extension of the Aharonov-Bohm effect accounting for the influence of the magnetic flux $Q$ (via the vector potential $\vec{A}$) and the topological defect (via the curvature $\kappa$ and torsion $\tau$ parameters).

**VI. FINAL REMARKS**

In this paper, we obtained exact expressions for the eigenvalues and eigenfunctions of a charged particle with magnetic moment bound in the vicinity of a magnetic wedge dispiration by a short range repulsive and long range attractive potentials. The screw dislocation modified the angular momentum by introducing an additive correction in a similar manner as the magnetic flux. The disclination introduced a multiplicative modification to the angular momentum, the appearance of which is understood as a consequence of the modified periodicity of the wavefunction in the space of the wedge dispiration around the z-axis. Furthermore, due to the finite size of the defect we were able to account for the effects of spin-torsion and spin-magnetic field interactions. The handedness of the screw dislocation simulates the north/south pole of the magnetic field, increasing/decreasing the binding energy of the particle in the core region for left(right) handed screws. In the limit where the defect region is shrunk to zero radius, the angular momentum is modified so as to depend not only on the magnetic flux, but also on the screw dislocation and wedge disclination parameters. Being that the defects characterized by parameters $\kappa$ and $\tau$ are singular at the defect line and vanishing elsewhere in the limit $R_c \to 0$, the appearance of global phenomena represented by the change in angular
momentum is interpreted as a manifestation of the topological features of the space rather than the local geometry induced by the defect.

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