Analysis of a splitting-differentiation population model leading to cross-diffusion *

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Abstract

Starting from the dynamical system model capturing the splitting-differentiation process of populations, we extend this notion to show how the speciation mechanism from a single species leads to the consideration of several well known evolution cross-diffusion partial differential equations.

Among the different alternatives for the diffusion terms, we study the model introduced by Busenberg and Travis, for which we prove the existence of solutions in the one-dimensional spatial case. Using a direct parabolic regularization technique, we show that the problem is well posed in the space of bounded variation functions, and demonstrate with a simple example that this is the best regularity expected for solutions.

We numerically compare our approach to other alternative regularizations previously introduced in the literature, for the particular case of the contact inhibition problem. Simulation experiments indicate that the numerical scheme arising from the approximation introduced in this article outperforms those of the existent models from the stability point of view.

Keywords: Cross-diffusion system, population dynamics, contact inhibition, splitting and differentiation, numerical simulations, existence of solutions.

1 Mathematical model and main result

1.1 The splitting-differentiation model in terms of ODEs

In [27], Sánchez-Palencia analyzes the situation in which a species, with population density $U$, splits, due to a number of factors, into two different species with population densities $U_1$ and $U_2$, but still keeping the original ecological behavior.

Thus, we assume that $U$ followed a logistics law until time $t^*$, i.e.

$$U'(t) = U(t)(\alpha - \beta U(t)), \quad \text{for } t \in (0, t^*), \quad U(0) = U_0 > 0.$$  \hspace{1cm} (1)

Then, after splitting, $(U_1, U_2)$ is assumed to satisfy the Lotka-Volterra system

$$U'_i(t) = U_i(t)(\alpha - \beta (U_1(t) + U_2(t))), \quad \text{for } t \in (t^*, T), \quad U_i(t^*) = U_{i0} > 0,$$  \hspace{1cm} (2)

for $i = 1, 2$, with $U_10 + U_20 = U(t^*)$. Note that, under this splitting without differentiation, $U_1 + U_2$ still satisfies (1) for $t \geq t^*$.

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Although problem (1) has a unique non-trivial equilibrium, \( U_0 = \alpha/\beta \), problem (2) has a continuum set of non-trivial equilibria given by all the combinations of \( U_{10} \geq 0 \) and \( U_{20} \geq 0 \) such that \( U_{10} + U_{20} = \alpha/\beta \).

In [27], the author analyzes how the differentiation of populations \( U_1, U_2 \) after splitting, understood as a perturbation in the Lotka-Volterra coefficients, affects the equilibrium of the system. By splitting \textit{with differentiation} we mean that \( (U_1, U_2) \) is a solution of the following problem:

\[
U_i'(t) = U_i(t)\left(\alpha_i - \left(\beta_i U_1(t) + \beta_2 U_2(t)\right)\right), \quad \text{for } t \in (t^*, T), \quad U_i(t^*) = U_{i0} > 0, \quad \text{for } i = 1, 2,
\]

with \( U_{10} + U_{20} = U(t^*) \). Observe that, under this splitting, \( U_1 + U_2 \) does not satisfy, in general, problem (1) for \( t \geq t^* \).

The main conclusion of [27] is that the differentiation mechanism selects, in general, a unique solution for the equilibrium system, having therefore a stabilizing effect.

### 1.2 The splitting model in terms of PDEs: cross-diffusion

In this article, we extend the previous dynamical system models to the case of space dependent population densities. We start considering the dynamics of one single species population satisfying

\[
\begin{aligned}
\begin{cases}
\partial_t u - \text{div} J(u) = f(u) & \text{in } Q_{(0,t^*)}, \\
J(u) \cdot \nu = 0 & \text{on } \Gamma_{(0,t^*)}, \\
u(0, \cdot) = u_0 & \geq 0 \text{ on } \Omega,
\end{cases}
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded domain with Lipschitz continuous boundary, \( \partial \Omega, Q_{(0,t^*)} = (0, t^*) \times \Omega \), and \( \Gamma_{(0,t^*)} = (0, t^*) \times \partial \Omega \) is the parabolic boundary of \( Q_{(0,t^*)} \). The vector \( \nu \) is the outwards canonical normal to \( \partial \Omega \). The growth-competition term is assumed to have the logistic form \( f(u) = u(\alpha - \beta u) \), and the flow to be given by \( J(u) = u \nabla u + u q \).

As it is well known, the term \( u \nabla u \) captures the individuals aversion to overcrowding, while \( q \) is usually determined by an environmental potential, \( q = -\nabla \varphi \), whose minima represent attracting points for the populations. For biological background and origins of the model see, for instance, [25].

After splitting, the new two populations, \( u_1 \) and \( u_2 \), satisfy, for \( i = 1, 2 \),

\[
\begin{aligned}
\begin{cases}
\partial_t u_i - \text{div} J_i(u_1, u_2) = f_i(u_1, u_2) & \text{in } Q_{(t^*, T)}, \\
J_i(u_1, u_2) \cdot \nu = 0 & \text{on } \Gamma_{(t^*, T)}, \\
u_i(t^*, \cdot) = u_{i0} & \text{on } \Omega,
\end{cases}
\end{aligned}
\]

with \( u_{i0} \) such that \( u_{10} + u_{20} = u(t^*, \cdot) \), and with \( J_i \) and \( f_i \) to be defined.

Assuming, like in the dynamical system model, that the possible differentiation process only takes place through the growth and the inter- and intra-competitive behavior of the new species implies that the split flows must satisfy \( J_1(u_1, u_2) + J_2(u_1, u_2) = J(u_1 + u_2) \), that is

\[
J_1(u_1, u_2) + J_2(u_1, u_2) = (u_1 + u_2) \nabla (u_1 + u_2) + (u_1 + u_2) q.
\]

Being clear the way of defining the linear transport term of \( J_i \), the nonlinear diffusive term admits several reasonable decompositions. For instance, in [12], the following splitting was considered

\[
J_i(u_1, u_2) = u_i \nabla u_i + b_i \nabla (u_1 u_2) + u_i q_i, \quad \text{(6)}
\]
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with $b_i \geq 0$, and $b_1 + b_2 = 1$. Under this splitting, problem (5) takes the form of the cross-diffusion model introduced by Shigesada et al. [28], for which a thorough mathematical analysis does exist, see for instance [13, 19, 1, 2, 26, 22] for numerical approaches, [8, 18, 20, 21, 9] for analytical and qualitative results, or [14, 11, 23] for applications.

In this paper, we consider the alternative splitting

$$J_i(u_1, u_2) = u_i \nabla (u_1 + u_2) + u_i q,$$

which brings problem (5) to the form of the Busenberg and Travis model [6]. Although apparently simpler than (6), no general proof of existence of solutions does exist for problem (5) with flows given by (7). However, some partial results related to the cell-growth contact-inhibition problem may be found in [3, 4, 5, 17], as well as in [15, 16] for other specific situations.

1.3 Differentiation after splitting

According to the species behavior after splitting, we consider two problems arising from two different sets of Lotka-Volterra terms:

$$f_i(u_1, u_2) = u_i(\alpha - \beta(u_1 + u_2)), \quad \text{(non-differentiation)} \quad (8)$$

$$f_i(u_1, u_2) = u_i(\alpha_i - (\beta_{i1}u_1 + \beta_{i2}u_2)), \quad \text{(differentiation)} \quad (9)$$

for $i = 1, 2$. We shall refer to problem (5) with flows given by (7) and with $f_i$ given by (8) and (9) as to problems (ND) and (D), respectively. Observe that these are the PDE versions that generalize the non-differentiation and differentiation ODE problems (2) and (3), introduced in [27].

The existence of solutions of problem (ND) was proven in [15] in the multi-dimensional case. The proof is based on the construction of solutions, $(u_1^\delta, u_2^\delta)$, to a nonlinear parabolic regularization of problem (ND) such that, although $u_j^\delta$ only converges weakly to the solution of the limit problem, $u_i$, the global density, $u_1^\delta + u_2^\delta$, converges strongly to $u_1 + u_2$. Thus, weak and strong convergences are compensated so that the limit problem (ND) is shown to have a solution.

However, since no a.e. convergence of $u_i^\delta$ to $u_i$ was proven in [15], the case of differentiated Lotka-Volterra terms, i.e. problem (D), may not be handled directly with this technique.

Previously to [15], in the one-dimensional setting and for $q = 0$, Bertsch et al. [4] proved the existence of a solution of problem (D) with the form

$$u_1(t, x) = r(t, x)u(t, x), \quad u_2(t, x) = (1 - r(t, x))u(t, x),$$

where $u = u_1 + u_2$ and $0 \leq r \leq 1$ solve certain parabolic-hyperbolic auxiliary problem, see problem (P)$_B$ in Section 2.

Their proof is based on the parabolic regularization of the auxiliary problem together with the use of the Lagrangian flows (characteristics) associated to $\partial_x u$. In particular, and important for the results proven in this article, they obtained strong convergence of the sequence of approximated solutions to a limit, which is identified as a solution of problem (D).

The first aim of this paper is to show the existence of solutions of problem (D) with an alternative proof to that given in [4]. Our proof is based on the direct parabolic regularization used for problem (ND) in [15], and takes advantage of the techniques employed in [4] to obtain the strong convergence of each component of the regularized problem.
In fact, like in [4], the space regularity of solutions of problem (D) is shown to be of bounded variation, $BV(\Omega)$, which seems to be the optimal expected regularity, see Theorem 2 in Section 3, for an example.

More explicitly, we prove the following result in Section 4. Here, we retake the usual notation $[0, T]$ for the time domain, replacing $[t^*, T]$.

**Theorem 1** Let $\Omega \subset \mathbb{R}$ be a bounded interval and $T > 0$ be arbitrarily fixed. Let $q(t, \cdot) = 0$ on $\partial \Omega$ for all $t \in [0, T]$, and $f_{i} : \mathbb{R}^{2} \to \mathbb{R}$ be given by (9). Assume that $u_{10}, u_{20} \in BV(\Omega)$ are non-negative, with $u_{0} := u_{10} + u_{20} > 0$ in $\Omega$, $u_{0} \in C^{0,1}(\bar{\Omega})$ and $\partial_{x} u_{0} = 0$ on $\partial \Omega$. Then, there exists a weak solution of problem (D), $(u_{1}, u_{2})$, with, for $i = 1, 2$,

1. $u_{i} \geq 0$ a.e. in $Q_{T}$.
2. $u_{i} \in L^{\infty}(0, T; BV(\Omega)) \cap BV(0, T; L^{1}(\Omega))$.
3. $u := u_{1} + u_{2} \in C^{0,1}(\bar{Q}_{T}) \cap L^{2}(0, T; H^{1}(\Omega))$.
4. For all $\varphi \in C^{1}(0, T; H^{1}(\Omega))$ with $\varphi(T, \cdot) = 0$ in $\Omega$,

$$
\int_{Q_{T}} u_{i} \partial_{t} \varphi = \int_{Q_{T}} (u_{i} \partial_{x} u + u_{i} q) \partial_{x} \varphi - \int_{Q_{T}} f_{i}(u_{1}, u_{2}) \varphi - \int_{\Omega} u_{i0} \varphi(0, \cdot).
$$

The second aim of this article is to numerically investigate and compare the resulting Finite Element schemes of each regularizing approach, see problems $(P)_{\delta}$ and $(P)_{B}$ in Section 2. We focus our attention into two model problems: (i) the invasion problem, arising in Tumor theory, as suggested by Bertsch et al. [4], in which an initial small perturbation (tumor) inside a healthy tissue evolves in time keeping a sharp interface; and (ii) a model problem with a Barenblatt-based explicit solution, for which a detailed comparison to the approximated solutions is given, see Section 3.

After the proofs of our results in Section 4, we conclude the article with some conclusions, in Section 5.

## 2 Approximated problems and discretization

In this section we compare the approximated solutions to problem (D) constructed via the regularization scheme employed in the proof of Theorem 1, and those corresponding to the approximated problem introduced in [4]. The formulation of these problems is the following, respectively.

$$(P)_{\delta} \begin{cases}
\partial_{t} u_{i} - \partial_{x} (u_{i} \partial_{x} (u_{1} + u_{2}) + u_{i} q) - \frac{1}{2} \partial_{xx} (u_{i} (u_{1} + u_{2})) = f_{i}(u_{1}, u_{2}) & \text{in } Q_{T}, \\
\partial_{x} u_{i} = 0 & \text{on } \Gamma_{T}, \\
u_{i}(0, \cdot) = u_{i0}^{\delta} & \text{in } \Omega,
\end{cases}$$

for $\delta > 0$, and some non-negative $u_{i0}^{\delta} \in C^{1}(\bar{\Omega})$ such that $u_{i0}^{\delta} \to u_{i0}$ strongly in $BV(\Omega)$, for $i = 1, 2$.

$$(P)_{B} \begin{cases}
\partial_{t} u - \partial_{x} (u (\partial_{x} u + q)) = F_{1}(u, r) & \text{in } Q_{T}, \\
\partial_{t} r - (\partial_{x} u + q) \partial_{x} r - \delta_{B} \partial_{xx} r = F_{2}(u, r) & \text{in } Q_{T}, \\
\partial_{x} u = \partial_{x} r = 0 & \text{on } \Gamma_{T}, \\
u(0, \cdot) = u_{0}, \quad r(0, \cdot) = r_{0}^{\delta_{B}} & \text{in } \Omega,
\end{cases}$$

for $\delta_{B} > 0$.
for \( \delta_B > 0 \), and for some \( r_0^{\delta_B} \in C^1(\bar{\Omega}) \) such that \( r_0^{\delta_B} \to r_0 \) strongly in \( BV(\Omega) \), and with

\[
F_1(u, r) = f_1(ru, (1 - r)u) + f_2(ru, (1 - r)u),
\]

\[
F_2(u, r) = r(1 - r)\left( \frac{f_1(ru, (1 - r)u)}{ru} - \frac{f_2(ru, (1 - r)u)}{(1 - r)u} \right).
\]

Recall that, according to [4], for any \( \delta_B > 0 \) there exists a regular solution to problem \((P)_B \), \((u^{\delta_B}, r^{\delta_B}) \), and that this sequence converges strongly in \( L^1(Q_T) \), as \( \delta_B \to 0 \), to some \((u, r)\) such that \( u_1 = ru \) and \( u_2 = (1 - r)u \) are a solution of problem \((D)\).

For the numerical discretization of problems \((P)_\delta \) and \((P)_B \), we consider a fully discrete approximation using finite elements in space and backward finite differences in time. The proof of the convergence of the numerical scheme for problem \((P)_\delta \) may be found in [15].

We consider a quasi-uniform mesh on the interval \( \Omega \), \( \{T_h\}_h \), with \( h \) representing step size. We introduce the finite element space of continuous \( P_1 \)-piecewise elements:

\[
S^h = \{ \chi \in C(\bar{\Omega}); \chi|_\kappa \in P_1 \text{ for all } \kappa \in T_h \}.
\]

The Lagrange interpolation operator is denoted by \( \pi^h : C(\bar{\Omega}) \to S^h \). We also introduce the discrete semi-inner product on \( C(\bar{\Omega}) \) and its induced discrete seminorm:

\[
(\eta_1, \eta_2)^h = \int_{\Omega} \pi^h(\eta_1\eta_2), \quad |\eta|^h = \sqrt{(\eta, \eta)^h}.
\]

For each \( \varepsilon \in (0, 1) \) we consider the function

\[
\lambda_\varepsilon(s) = \begin{cases} 
\varepsilon & \text{if } s \leq \varepsilon, \\
\varepsilon^{-1} & \text{if } s \geq \varepsilon^{-1},
\end{cases}
\]

and the linear operator \( \Lambda_\varepsilon : S^h \to L^\infty(\Omega) \) given by, for \( x_{mp} = (x_j + x_{j+1})/2 \)

\[
\Lambda_\varepsilon(z^h) = \lambda_\varepsilon(z^h(x_{mp})), \quad \text{in } (x_j, x_{j+1}).
\]

For the time discretization, we take in the experiments a uniform partition of \([0, T]\) of time step \( \tau \). For \( t = t_0 = 0 \), set \( u_{i,\varepsilon}^0 = u_{i,\varepsilon}^1 \), for \( i = 1, 2 \). Then, for \( n \geq 1 \) the full discretization of problem \((P)_\delta \) reads: Find \( u_{i,\varepsilon}^n \in S^h \) such that

\[
\frac{1}{\tau}(u_{i,\varepsilon}^n - u_{i,\varepsilon}^{n-1}, \chi)^h + \left( \alpha_1 \frac{U_{i,\varepsilon}^n}{\varepsilon} \partial_x u_{i,\varepsilon}^n, \partial_x \chi \right) + \beta_1 \left( \frac{U_{i,\varepsilon}^n}{\varepsilon} \partial_x u_{i,\varepsilon}^n, \partial_x \chi \right) + \frac{\varepsilon}{2} \left( \alpha_2 \frac{U_{i,\varepsilon}^n}{\varepsilon} \partial_x u_{i,\varepsilon}^n, \partial_x \chi \right)
\]

\[
+ \frac{\varepsilon}{2} \left( \alpha_2 \frac{U_{i,\varepsilon}^n}{\varepsilon} \partial_x u_{i,\varepsilon}^n, \partial_x \chi \right) + \left( \alpha_3 \frac{U_{i,\varepsilon}^n}{\varepsilon} \partial_x u_{i,\varepsilon}^n, \partial_x \chi \right) = \left( f_{1,\varepsilon}(u_{i,\varepsilon}^n, r_{i,\varepsilon}^n), \chi \right)^h,
\]

for every \( \chi \in S^h \), where we introduced the notation \( U_{i,\varepsilon}^n = \Lambda_\varepsilon(u_{i,\varepsilon}^n) \).

Similarly, the full discretization of problem \((P)_B \) reads: Set \((u_{i,\varepsilon}^0, r_{i,\varepsilon}^0) = (u_0, r_0)\). Then, for \( n \geq 1 \), find \((u_{i,\varepsilon}^n, r_{i,\varepsilon}^n) \in S^h \times S^h \) such that

\[
\frac{1}{\tau}(u_{i,\varepsilon}^n - u_{i,\varepsilon}^{n-1}, \chi)^h + \left( U_{i,\varepsilon}^n \partial_x u_{i,\varepsilon}^n, \partial_x \chi \right) + \left( \pi^h(q)U_{i,\varepsilon}^n, \partial_x \chi \right) = \left( F_{1,\varepsilon}(u_{i,\varepsilon}^n, r_{i,\varepsilon}^n), \chi \right)^h,
\]

\[
\frac{1}{\tau}(r_{i,\varepsilon}^n - r_{i,\varepsilon}^{n-1}, \chi)^h + \beta_B \left( \partial_x r_{i,\varepsilon}^n, \partial_x \chi \right) = \left( \frac{(\partial_x u_{i,\varepsilon}^n + \pi^h(q)) \partial_x r_{i,\varepsilon}^n, \partial_x \chi}{} \right),
\]

\[
\left( F_{2,\varepsilon}(u_{i,\varepsilon}^n, r_{i,\varepsilon}^n), \chi \right)^h.
\]
for every \( \chi \in S^h \), where \( U^n = \Lambda_\varepsilon(u^n) \), and \( F_\varepsilon(s, \sigma) = F_\varepsilon(\lambda_\varepsilon(s), \lambda_\varepsilon(\sigma)) \).

Since the above systems are nonlinear algebraic problems, we use a fixed point argument to approximate their solution at each time slice \( t = t_n \), from the previous approximation at \( t = t_{n-1} \). Thus, for problem \((P)_\delta\), let \( u_{\varepsilon,i}^0 = u_{\varepsilon,i}^{n-1} \). Then, for \( k \geq 1 \) the problem is to find \( u_{\varepsilon,i}^{n,k} \in S^h \) such that for \( i = 1, 2 \), and for all \( \chi \in S^h \)

\[
\frac{1}{h}(u_{\varepsilon,i}^{n,k} - u_{\varepsilon,i}^{n-1}, \chi) + (1 + \frac{\delta}{2})(U_{\varepsilon,i}^{n,k-1} \frac{\partial u_{\varepsilon,i}^{n,k}}{\partial t} + u_{\varepsilon,i}^{n,k}, \partial_x \chi) + \frac{\delta}{2}((U_{\varepsilon,i}^{n,k-1} + U_{\varepsilon,i}^{n,k}) \frac{\partial u_{\varepsilon,i}^{n,k}}{\partial t} + u_{\varepsilon,i}^{n,k}, \partial_x \chi) + (\pi^h(q)U_{\varepsilon,i}^{n,k-1} + \partial_x \chi) = (\alpha_i u_{\varepsilon,i}^{n,k} - \lambda_\varepsilon(u_{\varepsilon,i}^{n,k-1})(\beta_1 \lambda_\varepsilon(u_{\varepsilon,i}^{n-1}) + \beta_2 \lambda_\varepsilon(u_{\varepsilon,i}^{n-1})), \chi) + \delta\chi.
\]

We then use the stopping criterion \( \max_{i=1,2} \|u_{\varepsilon,i}^{n,k} - u_{\varepsilon,i}^{n,k-1}\|_\infty < \text{tol} \), for values of \( \text{tol} \) chosen empirically, and set \( u^n = u_{\varepsilon,i}^{n,k} \).

Similarly, for problem \((P)_B\) we use the scheme

\[
\frac{1}{h}(u_{\varepsilon,i}^{n,k} - u_{\varepsilon,i}^{n-1}, \chi) + (U_{\varepsilon,i}^{n,k-1} \frac{\partial u_{\varepsilon,i}^{n,k}}{\partial t} + u_{\varepsilon,i}^{n,k}, \partial_x \chi) + (\pi^h(q)U_{\varepsilon,i}^{n,k-1}, \partial_x \chi) = (F_{\varepsilon,i}(u_{\varepsilon,i}^{n,k-1}, r_{\varepsilon,i}^{n,k-1}), \chi) + \delta_B((\partial_x u_{\varepsilon,i}^{n,k-1} + \pi^h(q)) \partial_x r_{\varepsilon,i}^{n,k-1}, \partial_x \chi) = (F_{\varepsilon,i}(u_{\varepsilon,i}^{n,k-1}, r_{\varepsilon,i}^{n,k-1}), \chi)
\]

with an analogous stopping criterion than above.

### 3 Numerical simulations

In this section, we present numerical simulations for two sets of initial and Lotka-Volterra data aiming to clarify the advantages of approximating the solutions of problem \((D)\) either by the scheme derived from the regularized parabolic-hyperbolic formulation \((P)_B\) of [4], or by the direct viscosity approximation introduced in this article, \((P)_\delta\).

The general conclusion is that approximating the solution, \((u_1, u_2)\), of problem \((D)\) by the sequence \((u_{\delta}, u_{2\delta})\) of solutions of problem \((P)_\delta\) is more robust against instabilities produced in discontinuity points than that given by the sequence \((u_{\delta}, r_{\delta})\) corresponding to \((P)_B\). Although we only show the results for two standard examples, we confirmed this conclusion for a variety of data problem.

Parameter data was fixed as follows. In all the examples, we consider the spatial domain \( \Omega = (-2, 2) \) and investigate the mesh sizes \( h = 0.04, 0.013 \) and 0.008, whereas the time step is empirically chosen as \( \tau = 1.e - 3 \) (in the first experiment) and \( 1.e - 4 \) (in the second). The regularization parameters are taken as \( \delta = h^2, \delta_B = 2h^2 \). To avoid negative values of the discrete solutions, we use the functions \( \lambda_\varepsilon \) and \( \Lambda_\varepsilon \), see (10) and (11), respectively, setting \( \varepsilon = 1.e - 10 \). Finally, the tolerance for the fixed point stopping criterion is set to \( \text{tol} = 1.e - 8 \); in all the examples, we observed good convergence properties of the fixed point algorithm, reaching the prescribed tolerance in less than ten iterations.

In all the cases, the solutions corresponding to problem \((P)_B\) produce more oscillations than those of problem \((P)_\delta\). To measure these oscillations, we compute an approximation to the zero-crossings of \( \partial_x u(t, \cdot) \) in \( \Omega \) by

\[
\text{osc}(u)(t) = h \sum |\Delta(\text{sign}(\Delta u(t, \cdot)))|,
\]

where \( \Delta(\text{sign}(\Delta u(t, \cdot))) \) is the absolute value of the sign function of the second derivative of \( u(t, \cdot) \).
where $\Delta$ stands for the difference between two consecutive node values, and the sum runs over the spatial nodes. In fact, we may observe that while solutions of $({P})_\delta$ attenuate the oscillations when $t$ increases, those corresponding to solutions of $({P})_B$ are always above some threshold value. In addition, in all the experiments we observe that the size of oscillations diminish according to the mesh size, for both approximations.

### 3.1 Experiment 1: Invasion

In this example we investigate the question (Q2) stated in [4], concerning the invasion of a population (which simulates mutated abnormal cells) over an initially dominant population (which represents the normal cell). More precisely, we take the initial data

$$u_{10}(x) = 0.22 \exp(-(x - 0.25)^2/0.001), \quad u_{20}(x) = 0.45 - u_{10}(x) \quad \text{for} \ x \in \Omega.$$  

Besides, we deal with the Lotka-Volterra coefficients

$$\alpha_i = 1, \quad \beta_{ij} = i \quad \text{for} \ i, j = 1, 2.$$  

We conducted the experiments for several choices of the mesh size, $h$. Only the experiments corresponding to $h = 0.04$ (101 nodes), $h = 0.013$ (301 nodes), and $h = 0.008$ (501 nodes) are shown in Figure 1.

We may check that for the finer mesh (last two rows), both approximations give similar results. However, instabilities are already present in the approximation obtained from problem $({P})_B$ for the medium size mesh (row four), which appear amplified for the coarsest mesh (row two). For the printed resolution, instabilities are not visible in the approximation obtained from problem $({P})_\delta$ (rows one and three). This graphical evidences are confirmed through the oscillation measure $\text{osc}(u)(t)$, plotted in Figure 2.

### 3.2 Experiment 2: A Barenblatt-based explicit solution

The contact inhibition problem is a particular case of problem (D) arising in tumor dynamics theory, see [7] for the modeling background. In this problem, the initial density distributions are segregated, satisfying

$$\text{supp } u_{10} \cup \text{supp } u_{20} = \Omega, \quad \text{supp } u_{10} \cap \text{supp } u_{20} = \omega_0,$$  

where $\omega_0 \subset \Omega$ is the contact hypersurface, one or several points in our one-dimensional simulations. The theory predicts that the initial segregation is kept for later times, that is, no mixing between species is possible.

Under our modeling point of view, the initial data segregation can be interpreted as if the process of species splitting (tumor cells and normal cells) took place in coincidence with that of niches segregation: tumor tissue on one side of $\omega_0$, and normal tissue on the other side.

For the one-dimensional model, the results of Bertsch et al. [4], or the results proven in the present article include this special case of initial data.

For the multi-dimensional model, existence of solutions of problem (D) with initial data satisfying (12) is proven in [5, 17]. These multi-dimensional results are based on the use of suitable Lagrangian formulations of the problem but, as suggested by its parabolic-hyperbolic nature
evidenced in the auxiliary formulation introduced in [4], see Problem \((P)_B\), this construction lacks of uniqueness, as proven in [17].

Thus, the viscosity approximation used to prove the existence of solutions in the one-dimensional case [15] could be a method to select one of the infinitely many solutions obtained by the Lagrangian approach. This, however, is just a conjecture.

Finally, notice that the case of general initial data in the multi-dimensional framework is still an open problem.

In this example, we consider a particular situation of the contact-inhibition problem where a space discontinuous explicit solution of problem (ND) may be computed in terms of the Barenblatt explicit solution of the porous medium equation, the Heaviside function and the trajectory of the contact-inhibition point. To be precise, we construct a solution to the problem,

\[
\begin{align*}
\partial_t u_i - \partial_x(\rho_i \partial_x (u_1 + u_2)) &= 0 & \text{in } Q_T, \\
\rho_i \partial_x (u_1 + u_2) &= 0 & \text{on } \Gamma_T,
\end{align*}
\]

with, for \(x, x_0 \in \Omega = (-L, L)\),

\[
u_{10}(x) = H(x - x_0)B(0, x), \quad u_{20}(x) = H(x_0 - x)B(0, x).
\]

Here, \(H\) is the Heaviside function and \(B\) is the Barenblatt solution of the porous medium equation corresponding to the initial datum \(B(-t^*, \cdot) = \delta_0\) (Dirac delta function), i.e.

\[
B(t, x) = 2(t + t^*)^{-1/3}[1 - \frac{1}{12}x^2(t + t^*)^{-2/3}]_+,
\]

with \(t^* > 0\). For simplicity, we consider problem \((13)-(15)\) for \(T > 0\) such that \(\rho(T) < L^2\), with \(\rho(t) = \sqrt{12(t + t^*)^{1/3}}\), so that \(B(t, \pm L) = 0\) for all \(t \in [0, T]\). The point \(x_0\) is the initial contact inhibition point, for which we assume \(|x_0| < \rho(0)\), i.e. it belongs to the interior of the support of \(B(0, \cdot)\), implying that the initial mass of both populations is positive.

**Theorem 2** The \(L^\infty(0, T; BV(\Omega))\) functions

\[
u_1(t, x) = H(x - \eta(t))B(t, x), \quad u_2(t, x) = H(\eta(t) - x)B(t, x),
\]

with \(\eta(t) = x_0 (1 + \frac{t}{t^*})^{1/3}\), are a weak solution of problem \((13)-(15)\) in the following sense: For any \(\varphi \in C^1(0, T; H^1(\Omega))\) with \(\varphi(T, \cdot) = 0\) in \(\Omega\),

\[
\int_{Q_T} u_i \partial_t \varphi = \int_{Q_T} (u_i \partial_x u) \partial_x \varphi - \int_{-L}^L u_{i0} \varphi(0, \cdot).
\]

With this experiment, we check whether the condition \(u_1 + u_2 > 0\) in \(Q_T\) needed for proving the convergence of the solutions of problems \((P)_\delta\) and \((P)_B\) to a solution of problem (D) is necessary or not. In particular, we consider \(L = 2\), \(x_0 = -0, 25\) and \(t^* = 0.01\), so that \(u_1(t, \cdot) + u_2(t, \cdot) = 0\) in some regions of \(\Omega = (-2, 2)\) for \(t < T = 0.15\), see Figure 3. In fact, we already have initial conditions such that \(u_{10} + u_{20} = 0\) in some regions of \(\Omega\). According to this, and on the contrary of problem \((P)_\delta\), the initialization of the data for problem \((P)_B\) requires special care since by construction we have \(r_0 = u_{10}/(u_{10} + u_{20})\) in \(\Omega\). In this sense, among several choices to define \(r_0\) the following gave us the best results: Considering the perturbation, for \(\gamma > 0\),

\[
u_{10} = \begin{cases} 
\gamma + u_{10} & \text{if } x \leq x_0, \\
0 & \text{if } x > x_0,
\end{cases} \quad u_{20} = \begin{cases} 
0 & \text{if } x < x_0, \\
\gamma + u_{20} & \text{if } x \geq x_0,
\end{cases}
\]
we get \( u_{10}^\gamma + u_{20}^\gamma = \gamma + u_{10} + u_{20} > 0 \). Then, \( r_0^\gamma(x) = 1 \) if \( x < x_0 \) and \( r_0^\gamma(x) = 0 \) if \( x > x_0 \), and thus, taking \( \gamma \to 0 \) we obtain the same definition for \( r_0 \).

In Figures 3 and 4 we plot the discrete solutions obtained from both approximation schemes together with the explicit solution, and the corresponding oscillation measure. Just as happened in the previous example, in all the cases the solutions corresponding to problem \( (P)_B \) produce more oscillations than those of problem \( (P)_\delta \), and solutions of \( (P)_\delta \) attenuate the oscillations when \( t \) increases whereas those corresponding to solutions of \( (P)_B \) are always above some threshold value. Moreover, in this example the oscillations come up around the contact inhibition point.

It is also interesting to notice that, since this example has an explicitly known solution, we can check the convergence of the numerical solutions to the exact one. Indeed, the relative \( L^2 \) errors of both solutions are similar, and decreasing with respect to the mesh size, see Figure 4, last two rows.

## 4 Proofs of the theorems

### 4.1 Proof of Theorem 1

We divide the proof in several steps.

**Step 1. An approximation problem.** We consider problem \( (P)_\delta \) for some non-negative \( u^\delta_{i0} \in C^1(\bar{\Omega}) \) with \( u^\delta_{i0} \to u_{i0} \) strongly in \( BV(\Omega) \), \( \min(u_0) \leq u^\delta_{10} + u^\delta_{20} \leq \max(u_0) \) in \( \Omega \), and

\[
\sqrt{\delta} \| \partial_x u^\delta_{i0} \|_{L^2(\Omega)} \text{ uniformly bounded. (17)}
\]

This nonlinear viscosity regularization allows us to use the results in [13, 8] to deduce the existence of a solution \((u^\delta_1, u^\delta_2)\), with non-negative \( u^\delta_i \in L^2(0, T; H^1(\Omega)) \cap W^{1,4/3}(0, T; (W^{1,4}(\Omega))^\prime) \), satisfying \( (P)_\delta \) in the weak sense:

\[
\int_0^T < \partial_t u^\delta_i, \varphi > + \int_{Q_T} (u^\delta_i \partial_x u^\delta + u^\delta_i q + \frac{\delta}{2} \partial_x (u^\delta_i u^\delta)) \partial_x \varphi = \int_{Q_T} f_i(u^\delta_1, u^\delta_2) \varphi, \tag{18}
\]

for all \( \varphi \in L^4(0, T; W^{1,4}(\Omega)) \cap L^\infty(Q_T) \). Here, \( < \cdot, \cdot > \) denotes the duality product \((W^{1,4}(\Omega))^\prime \times W^{1,4}(\Omega)\).

Adding the corresponding equations of problem \( (P)_\delta \), for \( i = 1, 2 \), we find that \( u^\delta = u^\delta_1 + u^\delta_2 \) satisfies, in a weak sense, the following problem \( (PS)_\delta \)

\[
\begin{align*}
\partial_t u^\delta - (1 + \delta) \partial_x (u^\delta \partial_x u^\delta) + \partial_x (qu^\delta) &= F(u^\delta_1, u^\delta_2) \quad \text{in } Q_T, \\
\partial_x u^\delta &= 0 \quad \text{on } \Gamma_T, \\
u^\delta(0, \cdot) &= u^\delta_{10} + u^\delta_{20} \quad \text{in } \Omega,
\end{align*}
\]

where \( F(s_1, s_2) = f_1(s_1, s_2) + f_2(s_1, s_2) \). Using Theorem 2.2 (step 5) of [10] and the pointwise bounds for \( u^\delta_{i0} \), we get

\[
\min(u_0)e^{-\lambda t} \leq u^\delta(t, \cdot) \leq \max(u_0)e^{\lambda t} \quad \text{in } \Omega, \quad \text{for a.e. } t \in (0, T), \tag{19}
\]
for some \( \lambda > 0 \) independent of \( \delta \). Thus, since \( u^\delta_i \geq 0 \) in \( \Omega \), we also get
\[
0 \leq u^\delta_i(t, \cdot) \leq \max(u_0)e^{\lambda t} \quad \text{in} \quad \Omega, \quad \text{for a.e.} \quad t \in (0, T).
\] (20)

The uniform estimate (19) together with the regularity of \( u_0 \) and \( q \), and the compatibility conditions satisfied by these functions, allow us to apply the general theory for evolution quasilinear equations to deduce that the unique solution of (PS)\( _\delta \) is Lipschitz continuous in \( Q_T \), and that, for any \( 1 \leq p \leq \infty \),
\[
\|\partial_t u^\delta_i\|_{L^p(Q_T)}, \quad \|\partial_{xx} u^\delta_i\|_{L^p(Q_T)} \text{ are uniformly bounded},
\] (21)
see [24, Theorem 7.20].

Returning to problem (P)\( _\delta \), we may see that the additional regularity \( u^\delta_i \in L^\infty(Q_T) \) given by (20) allows us to expand the sense of weak solution to the regularity
\[
u^\delta_i \in L^2(0, T; H^1(\Omega)) \cap H^1(0, T; (H^1(\Omega)') \cap L^\infty(Q_T),
\]
and for test functions \( \varphi \in L^2(0, T; H^1(\Omega)) \). Using the arguments of the proof of [15, Lemma 3], we easily get
\[
\sqrt{\delta}\|\partial_x u^\delta_i\|_{L^2(Q_T)}), \quad \|\partial_t u^\delta_i\|_{L^2(0, T; (H^1(\Omega)')} \text{ are uniformly bounded}.
\] (22)

Moreover, the above mentioned regularity \( \partial_{xx} u^\delta_i \in L^\infty(Q_T) \) allows us to express (18) as, for all \( \varphi \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q_T),
\[
\int_0^T < \partial_t u^\delta_i, \varphi > + \frac{\delta}{2} \int_{Q_T} u^\delta_i \partial_x u^\delta_i \partial_x \varphi = \int_{Q_T} \psi \varphi,
\]
with
\[
\psi = (1 + \frac{\delta}{2})(\partial_x u^\delta_i \partial_x u^\delta + u^\delta_i \partial_{xx} u^\delta) + u^\delta_i \partial_x q + \partial_x u^\delta q + f_i(u^\delta_1, u^\delta_2).
\]
Since \( \psi \in L^2(Q_T) \), we have that \( u^\delta_i \) satisfies
\[
\partial_t u^\delta_i - \frac{\delta}{2} \partial_x (u^\delta \partial_x u^\delta_i) = \psi \in L^2(Q_T),
\]
with \( u^\delta \in C^{0,1}(\bar{Q}_T) \) bounded away from zero, see (19). Thus, we have \( \partial_t u^\delta_i, \partial_{xx} u^\delta_i \in L^2(Q_T) \), implying \( \partial_x u^\delta_i \in L^4(Q_T) \). We then deduce that, in fact, \( \psi \in L^4(Q_T) \), improving in this way the regularity of \( u^\delta_i \). A boot-strap argument allows us to deduce
\[
\partial_t u^\delta_i, \partial_{xx} u^\delta_i \in L^\infty(Q_T), \text{(23)}
\]
implying the Lipschitz continuity of \( u^\delta_i \) (with norm depending on \( \delta \)).

**Step 2. Uniform estimates for** \( r^\delta = u^\delta_i/u^\delta \). For clarity, we omit the super-index \( \delta \) in this Step. Due to the regularity (23), the derivatives \( \partial_tr, \partial_xr, \) and \( \partial_{xx}r \) are well defined as \( L^\infty(Q_T) \) functions. After some computations, we obtain that \( r \) satisfies
\[
\partial_t r - \frac{\delta}{2} \partial_{xx} r - (q + (1 + 2\delta)\partial_xu)\partial_x r = G(u, r) \quad \text{in} \quad Q_T,
\]
(24)

\[
\partial_x r = 0 \quad \text{on} \quad \Gamma_T,
\]
\[
r(0, \cdot) = u_{10}/u_0 \quad \text{in} \quad \Omega,
\]
with,

\[ G(u, r) = \frac{1 - r}{u} f_1(u, r, u, 1 - r)) - \frac{r}{u} f_2(u, 1 - r). \]

Since \( \partial_u G \) and \( \partial_r G \) are bounded, and \( u \) and \( r \) are Lipschitz continuous, we deduce \( \partial_r G(u, r) \in L^\infty(Q_T) \). Therefore, the solution \( r \) of (24) is regular enough to allow us to differentiate equation (24) with respect to \( x \), for a.e. \((t, x) \in Q_T \). We then multiply the differentiated equation by \( \xi = \partial_x r/\sqrt{\delta + (\partial_x r)^2} \) and integrate by parts to get, term by term,

\[
\begin{align*}
\int_{Q_T} \partial_x \partial_t r \xi &= \int_\Omega \left( \frac{\sqrt{\delta + (\partial_x r)^2}}{\partial_t r} \right)_0^T, \\
- \delta \int_{Q_T} \partial_x (u \partial_{xx} r) \xi &= \frac{\delta^2}{2} \int_{Q_T} u (\partial_{xx} r)^2, \\
- \int_{Q_T} \partial_x ((q + (1 + 2\delta) \partial_x u) \partial_x r) \xi &= \delta \int_{Q_T} \frac{\partial_x (q + (1 + 2\delta) \partial_x u)}{\sqrt{\delta + (\partial_x r)^2}}, \\
\int_{Q_T} \partial_x G(u, r) \xi &= \int_{Q_T} (\partial_u G(u, r) \partial_x u + \partial_r G(u, r) \partial_x r) \xi.
\end{align*}
\]

Therefore, thanks to the regularity \( \partial_x q \in L^1(Q_T) \) and to the uniform bounds for \( \partial_x u \) and \( \partial_{xx} u \) in \( L^1(Q_T) \) we obtain

\[
\int_\Omega \frac{\sqrt{\delta + (\partial_x r)^2}}{\partial_t r} \Big|_0^T \leq c_1 + c_2 \int_{Q_T} |\partial_x r|,
\]

with \( c_1, c_2 \) independent of \( \delta \). We then deduce from Gronwall’s lemma that

\[
\sup_{[0, T]} \int_\Omega |\partial_x r| \text{ is uniformly bounded with respect to } \delta. \tag{25}
\]

We finish this step by showing a uniform bound for \( \partial_t r \). From equation (24) we get

\[
\int_{Q_T} |\partial_t r| \leq \frac{\delta}{2} \|u\|_{L^\infty(Q_T)} \int_{Q_T} |\partial_{xx} r| + c_3 \int_{Q_T} |\partial_x r| + \int_{Q_T} |G(u, r)|, \tag{26}
\]

with \( c_3 = \|q\|_{L^\infty(Q_T)} + (1 + 2\delta) \|\partial_x u\|_{L^\infty(Q_T)} \). We have already shown that the two last terms of the right hand side of (26) are uniformly bounded. For estimating the first term, we multiply equation (24) by \( \delta \partial_{xx} r \) and integrate by parts. We obtain

\[
\begin{align*}
\min(u) \frac{\delta^2}{2} \int_{Q_T} |\partial_{xx} r|^2 \leq & \frac{\delta}{2} \int_\Omega |\partial_x r(0, \cdot)|^2 + \frac{\delta(1 + 2\delta)}{2} \int_{Q_T} (|\partial_{xx} u| + |\partial_x q|) |\partial_x r|^2 \\
& + \delta \int_{Q_T} |G(u, r)| |\partial_{xx} r|.
\end{align*}
\]

Using property (17), we find that the first term of the right hand side is uniformly bounded. The third term may be handled by Hölder’s and Young’s inequality to get

\[
\begin{align*}
\min(u) \frac{\delta^2}{4} \int_{Q_T} |\partial_{xx} r|^2 \leq & c_4 + \frac{\delta(1 + 2\delta)}{2} \int_{Q_T} (|\partial_{xx} u| + |\partial_x q|) |\partial_x r|^2,
\end{align*}
\]
with $c_4$ independent of $\delta$. Finally, using the uniform lower bound for $u$ in $Q_T$, the uniform bound for $\partial_x u$ in $L^2(Q_T)$, the regularity $\partial_x q \in L^2(Q_T)$ and applying the last argument of point (v) of the proof of Lemma 3.1 of [4], we deduce that $\delta \|\partial_x r\|_{L^1(Q_T)}$ is uniformly bounded, and therefore, from (26) we also deduce
\[
\|\partial_t r\|_{L^1(Q_T)} \text{ is uniformly bounded with respect to } \delta.
\] (27)

Step 3. Passing to the limit $\delta \to 0$. From the uniform estimates (20) and (22), we deduce the existence of $u_i \in L^\infty(Q_T) \cap L^2(0,T; (H^1(\Omega))')$ such that (for a subsequence, not relabeled)
\[
u_i^\delta \to u_i \text{ weakly * in } L^\infty(Q_T),
\]
\[
\partial_t u_i^\delta \to \partial_t u_i \text{ weakly in } L^2(0,T; (H^1(\Omega))'),
\]
\[
\delta \partial_x u_i^\delta \to 0 \text{ strongly in } L^2(Q_T),
\]
and from (21), we also deduce the existence of $u \in C^{0,1}(\bar{Q}_T) \cap L^2(0,T; H^2(\Omega))$ such that
\[
u^\delta \to u \text{ uniformly in } C^{0,1}(\bar{Q}_T),
\]
\[
\partial_x u^\delta \to \partial_x u \text{ strongly in } L^2(Q_T).
\]

In particular, the weak and strong convergences of $u_i^\delta$ and $u^\delta$ imply $u = u_1 + u_2$. Using the uniform estimates (25) and (27) we deduce the existence of $r \in L^\infty(0,T; BV(\Omega)) \cap BV(0,T; L^1(\Omega))$ such that, for $1 \leq p < \infty$,
\[
r^\delta \to r \text{ strongly in } L^p(Q_T).
\]

Since $r^\delta = u_1^\delta / u^\delta$ the weak and strong convergences of $u_i^\delta$ and $u^\delta$ in $L^2(Q_T)$, respectively, imply $r = u_1 / u$. Then, the strong convergences of $r^\delta$ and $u^\delta$ in $L^p(Q_T)$ and $L^\infty(Q_T)$ imply
\[
u_1^\delta = r^\delta u^\delta \to ru = u_1 \text{ strongly in } L^p(Q_T).
\]

Similarly, we obtain
\[
u_2^\delta = (1 - r^\delta)u^\delta \to (1 - r)u = u_2 \text{ strongly in } L^p(Q_T).
\]

Finally, using again the uniform estimates (25) and (27), we also obtain
\[
\|\partial_x u_1^\delta\|_{L^\infty(0,T; L^1(\Omega))}, \|\partial_t u_1^\delta\|_{L^1(Q_T)} \text{ uniformly bounded},
\]
from where we, additionally, deduce
\[
u_i \in L^\infty(0,T; BV(\Omega)) \cap BV(0,T; L^1(\Omega)).
\]

Thus, the passing to the limit $\delta \to 0$ in (18) is justified, and so we identify the limit $(u_1, u_2)$ as a weak solution of problem (D). □
4.2 Proof of Theorem 2

Proof. Let $H_\epsilon$ be the regularization of the Heaviside function taking the values \{0, 1/2(1 + x/\epsilon), 1\} in the intervals $(-L, -\epsilon)$, $(-\epsilon, \epsilon)$ and $(\epsilon, L)$, respectively, for $\epsilon > 0$ small.

Define the functions $u_i^\epsilon$ as in (16), with $H$ replaced by $H_\epsilon$, and let $\varphi \in H^1(Q_T)$, with $\varphi(T, \cdot) = 0$ in $\Omega$. Using $\varphi_i(x,t) = \varphi(t,x)H_i(x - \eta(t))$ as a test function in the weak formulation of (13)-(14) corresponding to the initial datum $B(0, \cdot)$, that is, the problem satisfied by the Barenblatt solution in $Q_T$, we obtain

\[
\int_{Q_T} H_\epsilon(x - \eta(t))B(t,x)(\partial_t \varphi(t,x) - \partial_x B(t,x)\partial_x \varphi(t,x))dxdt \tag{28}
\]

\[+ \int_{-L}^L H_\epsilon(x - x_0)B(0,x)\varphi(0,x)dx = I_1^\epsilon,
\]

with

\[I_1^\epsilon = \int_{Q_T} \varphi(t,x)B(t,x)H_\epsilon'(x - \eta(t))(\eta'(t) + \partial_x B(t,x))dxdt.
\]

Since $|x_0| < \rho(0)$, we have $|\eta(t)| < L^2 - \epsilon$, for $\epsilon$ small enough and $t \in (0,T)$. Using the explicit expression of $\partial_x B$ and $\eta'$ we deduce

\[I_1^\epsilon = -\frac{1}{6\epsilon} \int_0^T \int_{-\epsilon}^\epsilon y\varphi(t,y + \eta(t))B(t,y + \eta(t))dydt.
\]

Since $\varphi$ and $B$ are uniformly bounded in $L^\infty(Q_T)$, we obtain

\[|I_1^\epsilon| \leq C\epsilon, \tag{29}
\]

with $C > 0$ independent of $\epsilon$. The computation using $\varphi(t,x)H_\epsilon(\eta(t) - x)$ as test function gives similar results for some $I_2^\epsilon$ satisfying the same estimate (29) than $I_1^\epsilon$. Since, by definition, $u_1^\epsilon + u_2^\epsilon = B$, we obtain

\[\left| \int_{Q_T} u_i^\epsilon(\partial_t \varphi - \partial_x (u_1^\epsilon + u_2^\epsilon)\partial_x \varphi) + \int_{-L}^L (u_i^\epsilon \varphi)(0,\cdot) \right| \leq C\epsilon
\]

for all $\varphi \in H^1(Q_T)$. Thus, using that $u_i^\epsilon \to u_i$ uniformly in $L^\infty(Q_T)$, and performing the limit $\epsilon \to 0$ in (28), we deduce that $(u_1, u_2)$ is a solution of problem (13)-(15). □

Remark 1 It is not difficult to extend the above construction to problem (ND) for non-vanishing $q$ and $f_i$. Indeed, reconsider the solution $u$ of problem (4) and the corresponding approximations $u_i^\epsilon$ given in the proof of the previous theorem. Then, to handle the integrals $I_1^\epsilon$, we first observe that for $\epsilon \to 0$ we get

\[I_1^\epsilon \to - \int_0^T \varphi(t,\eta(t))u(t,\eta(t))(\eta'(t) - b(t,\eta(t)))dt,
\]

with $b = -(\partial_x u + q)$. Therefore, if the ODE problem

\[
\begin{cases}
\eta'(t) = b(t,\eta(t)) \quad \text{for } t \in (0,T), \\
\eta(0) = x_0,
\end{cases}
\]

(30)
is solvable, a solution for problem (ND) may be constructed as before. Typical conditions on $b$ for (30) to be solvable (in the one-dimensional case) are given in terms of spatial Lipschitz continuity. That is, the solution of problem (4) must satisfy $\partial_{xx}u \in L^\infty(Q_T)$, which is true for smooth data.

5 Conclusions

The generalization of the dynamical system model for the splitting - differentiation process of populations [27] to the space dependent situation leads to a family of cross-diffusion PDE problems including several well known segregation models.

In a previous work [12], we analyzed the case in which the cross-diffusion is of the type introduced by Shigesada et al. [28], a model well studied in the literature, see [15] and its references. In this article, we turned to another important case, the model of Busenberg and Travis [6], in which the segregation effects are stronger, as motivated in [15].

The model of Busenberg and Travis was studied by Bertsch et al. in a series of papers [3, 4, 5] focusing in a special case of initial data arising from Tumor modeling which gives rise to the so-called contact-inhibition problem.

From the analytical side, we have produced a proof of existence of solutions conceptually simpler than previous proofs [4, 17], since ours derives from a direct parabolic regularization of the problem meanwhile previous involved a change of unknowns rendering the problem to a parabolic-hyperbolic formulation.

Although difficult to ensure, we believe that our approach also gives a way to select a unique natural solution of the problem as a limit of vanishing viscosity solutions of the regularized parabolic problems, tackling thus the non-uniqueness issue of the parabolic-hyperbolic formulation [17].

From the numerical side, we have introduced a Finite Element discretization of both formulations and compared the results in a series of experiments (only two of them showed in the article). The general observation is that our approach is always more stable in the tricky regions where the solutions exhibit discontinuities.

Finally, there are evidences in our numerical experiments showing that the main data restriction of the existence theorems, the positivity of the initial total mass, is not a necessary condition. Thus, future research will focus in the replacement of this condition, as well as in the generalization to the multi-dimensional case.

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Figure 1: Experiment 1. Odd rows correspond to the approximated solution of problem \((P)_δ\) for several time slices, \(t\), while even rows correspond to the solution of problem \((P)_B\). Rows correspond to different mesh sizes, captured by parameter \(h\). Mind the different vertical scales among time slices.
Figure 2: Experiment 1. Each row corresponds to the oscillation measure $\text{osc}(u)(t)$ of the solutions approximated by $(P)_\delta$ and $(P)_B$, respectively. Columns correspond to different mesh sizes, captured by parameter $h$. 
Figure 3: Experiment 2. Exact solution (dots) against approximated solutions (solid lines). Odd rows correspond to the approximated solution of problem (P) for several time slices, \(t\), while even rows correspond to the solution of problem (P)$_B$. Rows correspond to different mesh sizes, captured by parameter \(h\). Mind the different vertical scales among time slices.
Figure 4: Experiment 2. Two first rows correspond to the oscillation measure \( \text{osc}(u)(t) \) of the solutions approximated by \((P)_\delta\) and \((P)_B\), respectively. Two last rows correspond to the relative error between the exact solution and the solution approximated by \((P)_\delta\) and \((P)_B\), respectively. Columns correspond to different mesh sizes, captured by parameter \( h \).