Rank-frequency distribution of natural languages: a difference of probabilities approach

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Abstract
The time variation of the rank $k$ of words for six Indo-European languages is obtained using data from Google Books. For low ranks the distinct languages behave differently, maybe due to syntax rules, whereas for $k > 50$ the law of large numbers predominates. The dynamics of $k$ is described stochastically through a master equation governing the time evolution of its probability density, which is approximated by a Fokker-Planck equation that is solved analytically. The difference between the data and the asymptotic solution is identified with the transient, and good agreement is obtained.

Keywords: rank dynamics, languages, master equation, Fokker-Plank equation

1. Introduction
The statistical study of languages has shown an increased interest over the last decades since the pioneering works of Zipf \cite{1} and others \cite{2, 3, 4, 5}. These studies have focused on the rank-frequency distribution of words. Additionally, the rank diversity distribution has recently been proposed as a novel measure to characterize the statistical properties of languages \cite{6}. This distribution can be understood as a measure of how word ranks change in time. This measure has also shown that the size of the language core is similar for most languages. Within this statistical linguistic point of view, in previous work we have introduced a simple Gaussian random walk model for the rank diversity which reproduced some of the observed features of the evolution of this quantity quite well \cite{6}.

Furthermore, in recent years much effort has been given to the study of complex networks associated to physical systems, biological organisms, and social organizations; the structure and dynamics of these networks being a matter of intense research \cite{7}.

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In previous works [9], we have looked into the evolution of complex networks in terms of a master equation (ME) describing birth-death stochastic processes along the lines developed for ecological models [10, 11]. We have shown that under very general conditions in which dynamic conflict (frustration) exists between positive and negative mechanisms, the frequency distribution versus rank is given by the ratio of two power laws. This is also the case for birth and death processes in ecology, or for the excitation-inhibition process for neurons in the central nervous system. In a large variety of systems composed by similar elements and with similar interactions between them, the response of the system is determined by general laws. However, there are always differences in the response of the system in different realizations of the same experiment which can be associated, for instance, to the large numbers law or the central limit theorem, and follow a normal Gaussian distribution. In these cases, the average values are the ones that depend on general laws, whereas the differences among various realizations of the experiments obey a different dynamics, namely, that of the great numbers law.

In this work we use this point of view to study the frequency distribution of words in six languages [6]. In particular, we analyze the difference between the data associated with different realizations of these conflictive dynamics and the adjustments of the real data. We do this in terms of a time dependent probability density distribution, by assuming that the dynamics of the rank distribution may be described by the ME describing an underlying one step, Markovian, birth-death stochastic process [9, 12]. As we have shown in previous work [13], the data describing the frequency of words of several languages can be well adjusted by an asymptotic beta function. However, as it will be shown below, there is always a small difference between the data and this adjustment. Here lies the motivation of this work and one of its main objectives is to analyze and explain the origin of this difference within the context of the proposed stochastic model.

The outline of the present work is as follows. In Sec. 2 we define the stochastic model and construct a ME describing the data obtained for different Indo-European languages. Then in Sec. 3 the initial differential-difference ME is approximated by a (nonlinear) Fokker-Planck equation (FPE) in the continuum limit, where the discrete rank stochastic variable may be treated as a continuous variable. Closed analytic forms for both, the stationary and the time dependent probability density distributions of this equation, are obtained using Padé approximants. In terms of these well defined approximations, we show that the analytic time-dependent solution of the FPE describes well some of the observed features. Finally, in Sec. 4 we summarize our main results and critically discuss the novel features and limitations of our work.

2. Data adjustment for Indo-European languages

The variations of the rank $k$ in time of twenty words for three different $k$-scales for these six languages were obtained for two centuries in [6]; an example for English is given in Fig. 1. From the curves in [6] it can be observed that the behavior of $k(t)$ is similar for all languages. Words with low rank almost do not vary in time and as the value of $k$ increases, its variations depend on the rank itself. Notice that there is a higher variation at all scales before year 1850. As an example, in the case of English and for the $k$-scale between 1-30, the variation of rank with time is very small; in contrast, for the intervals 250-1500 and 4500-15000 its variation is much larger and very irregular. This shows that the variable $k$ exhibits different dynamics in different regions of the $(k,t)$ space.
The fact suggests that the dynamics in the last two intervals may be described by a stochastic model for the random variable $k$.

The normalized word frequencies $f(k)$ associated with the curves in Fig. 1, as a function of the rank $k$, were fitted with different rank distributions $m_i(k)$, $i = 1, 2, 3, 4, 5$, defined by Eqs. (S1) - (S5) in [13]. The models $m_i(k)$ fit better in different regions of $(f,k)$, but for none of them the fit is best for all languages in all regions. However, it was found that the data adjustment is best when the asymptotic beta function

$$m_3(k) = N_3 \left( \frac{N + 1 - k}{N} \right)^b \frac{1}{k^a}$$

is used. Here, $a$ and $b$ are the fitting parameters, $N_3$ is a normalization factor and $N$ is the total number of words. Fig. 2 shows that indeed, there is always a (small) difference between the data and the adjustment.
These plots were obtained for books published in the year 2008. The curves clearly show that none of the distributions captures satisfactorily the entire data behavior. The usual criterion to quantify the quality of an adjustment is to calculate the coefficient of determination (denoted by $R^2$) which is the integral of the squared difference between data and adjustment, or the proportion of the variance in the dependent variable that is predictable from the independent variable; if $R^2$ is near one, the adjustment is considered to be good. However, this quantity does not describe which values of $k$ contribute predominantly to a specific value of $R^2$. One of the objectives of the present work is to take this into account and analyze the origin of this difference. By assuming that the dynamics is originated by the action of multiplicative factors, in the next section we shall describe this difference between data and adjustment in terms of a log-normal distribution.

3. Stochastic model

Given a set of words forming a text, the number of times $N(k, t)$ that a certain word appears with the rank $k$ at time $t$ can be evaluated. If this change in $k$ is modelled by a one-step Markovian stochastic process, and if $b_k \equiv b(k)$ and $c_k \equiv c(k)$ denote arbitrary functions for the transition probabilities per unit time for the rank to increase or to decrease in one unit, the dynamics of the probability density $P_k(t) \equiv P(k, t)$ for the rank to have the value $k$ at time $t$ is given by the nonlinear ME

$$\frac{\partial}{\partial t} P(k, t) = c_{k+1} P_{k+1}(t) + b_{k-1} P_{k-1}(t) - (c_k + b_k) P_k(t).$$

It should be remarked that the ME is always linear in the unknown $P(k, t)$, and that the term nonlinear refers to the generality of the functions $b_k$ and $c_k$. Note that if the range
of values of $k$ is finite, $k = 0, 1, 2, ..., N$, equation (2) is meaningless for $k = 0$, and this value is a boundary of the one step process. However, by assuming $d(0) = b(-1) = 0$, equation (2) is still valid for $k = 0$. It is convenient to rewrite (1) in the more compact form

$$\frac{\partial}{\partial t} P(k, t) = \left[ \left( \hat{E} - 1 \right) d(k) + \left( \hat{E}^{-1} - 1 \right) b(k) \right] P(k, t),$$

where the action of the step operators $\hat{E}^\pm$ over an arbitrary function $f(k)$ is defined by

$$(\hat{E}^\pm f)(k) = f(k \pm 1).$$

4. Fokker-Planck approximation

Since only in rare cases it is possible to solve the ME explicitly, we shall assume that the changes in $k$ are small and that we are only interested in solutions $P(k, t)$ that vary slowly with the discrete variable $k$. In this limit the discrete variable $k$ may be treated as a continuous variable and the operators $\hat{E}^\pm$ may be replaced by a Taylor series expansion in $k$, yielding the following nonlinear FPE approximation for the ME [15]

$$\frac{\partial P(k, t)}{\partial t} = \left\{ -\frac{\partial}{\partial k} g(k) + \frac{1}{2} \frac{\partial^2}{\partial k^2} f(k) \right\} P(k, t) \equiv \hat{L}(k) P(k, t).$$

Here $f(k) \equiv b(k) + c(k)$, $g(k) \equiv b(k) - c(k)$, and $\hat{L}(k)$ defines the Fokker-Planck operator. If the dynamics takes place through multiplicative factors, the system follows a log-normal probability distribution. Assuming this to be the case, we write $P(k, t)$ as $P(x, t)$ with $x \equiv \log k$.

It is well known that the probability density function (PDF) of an additive process depending on multiple, independent stochastic variables, is obtained naturally through the reiterative application of convolution,

$$(P_2 \oplus P_1)(\chi) \equiv N \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} d\xi_1 d\xi_2 P_1(\xi_1) P_2(\xi_2) \delta (\xi_2 + \xi_1 - \chi).$$

As indicated by the $\delta$ function, the integral is performed in the locus of an equal sum of variables. However, if we are interested in modeling a stochastic system in which the dependence of the random variables is not through addition but substraction, $\chi = x_2 - x_1$, it can be shown that the probabilistic outcome of $\chi$ is given by [18]

$$P(\chi) = P_1(\chi_2) \oplus P_1(-\chi_1),$$

where $P_1(\chi_2)$ and $P_1(\chi_1)$ are the PDF’s of $x_2$ and $x_1$, respectively. For most non-symmetrical PDF’s, this result is sufficient to violate the validity of the central limit theorem. However, in Ref. [18] it is also shown that the new product,

$$(P_2 \oplus P_1)(\chi) \equiv N \int_0^1 \int_0^1 d\xi_1 d\xi_2 P_1(\xi_1) P_2(\xi_2) \delta (\xi_2 - \xi_1 - \chi)$$

$$= N \int_\chi^1 d\xi P_1(\xi - \chi) P_2(\xi),$$

(9)
which is the cross-correlation, describes correctly the probabilistic outcome of $\chi$ and that the correlation function between two beta distributions is well described by a beta function. On the other hand, in [6] it is shown that the data of words frequency vs. rank are also well adjusted by a beta distribution. Therefore, these two observations suggest that the dynamics of frequency data as a function of rank might depend on a difference between probability distributions.

Now, according to Fig. 2 there is always a difference between the predicted values and the data adjustments, a fact that suggests the following analysis: If $A$ and $B$ are the probability distributions of two different stochastic variables, $x_1$, $x_2$, and if we define

$$ S \equiv \frac{1}{2} (A + B), \quad D \equiv \frac{1}{2} (A - B), $$

then

$$ A - B = [(P_1 + P_2) \ominus (P_1 + P_2) + (P_1 - P_2) \oplus (P_1 - P_2)]. $$

In previous works, [16], we have shown that for the data associated with the English language, the first term on the right hand side (r.h.s.), i.e. the correlation, can be very well adjusted by a stationary asymptotic distribution function $P_{\text{asym}}$ equal to the $\beta$ distribution. Therefore, the second term on the r.h.s., i.e. the convolution, may be identified with a Gaussian distribution. In this work we show that for the English language data this is indeed the case. Then, as a consequence of (11), $P(k,t)$ can be expressed in the general form

$$ P(x,t) = P_{\text{asym}}(x) + P_1(x,t). $$

Note that for the present model $P_{\text{asym}}(x)$ may be identified with the stationary solution of (6) defined by

$$ \tilde{L}(x)P_{\text{st}}(x) = 0, $$

and that $P_1(x,t)$ satisfies the FPE

$$ \frac{\partial P_1(x,t)}{\partial t} = \tilde{L}(x)P_1(x,t). $$

4.1. Stationary solution

The general form of the stationary $P_{\text{st}}(x)$ solution of (12) is well known [14]

$$ P_{\text{st}}(x) = \frac{N_0}{\mathcal{f}(x)} \exp \left[ 2 \int_0^x \frac{g(x')}{\mathcal{f}(x')} \, dx' \right], $$

where $N_0$ is the integration constant which has to be chosen such that $P_{\text{st}}^k$ is normalized. In [15] we restrict our calculation to the range $1 \ll k \ll N$. However, the fraction $g(x)/\mathcal{f}(x)$ may be expressed in terms of Padé approximants, which are a particular type of rational fraction approximation to the value of a function [19]. The basic idea is to match the Taylor series expansion as far as possible. If we denote the $L$, $M$ Padé approximant to $A(x)$ by

$$ [L/M] = \frac{P_L(x)}{Q_M(x)}, $$

6
where $P_L(x)$ is a polynomial of degree at most $L$ and $Q_M(x)$ is a polynomial of degree at most $M$, the formal power series expansion

$$A(x) = \sum_{j=0}^{\infty} a_j x^j,$$

which is unique if $[L/M]$ exists. The coefficients of $P_L(x)$ and $Q_M(x)$ are determined by the equations

$$A(x) - \frac{P_L(x)}{Q_M(x)} = O(x^{L+M+1}).$$

Since we can obviously multiply the numerator and denominator by any constant and leave $[L/M]$ unchanged, we impose the normalization condition

$$Q_M(0) = 1$$

and write the coefficients of $P_L$ and $Q_M$ as

$$P_L(x) = p_0 + p_1 x + ... + p_L x^L,$$

$$Q_M(x) = 1 + q_1 x + ... + q_M x^M,$$

where $p_0$ is a constant.

In Ref. [9] it is shown that the fraction $g(x)/f(x)$ may be expressed in the form

$$\frac{g_n(x)}{f_n(x)} = A_0 + \sum_{i=1}^{N} \frac{A_i}{x + c_i},$$

where $A_0$ and $A_i$ are well defined constants in terms of the original polynomials $g(x)$ and $f(x)$. [11, 16], and the stationary solution $P_{\text{asym}}(x) \equiv \text{P}_{\text{asym}}(x)$ may be rewritten in the general form

$$P_{\text{asym}}(x) = \mathcal{N} \exp \left[ A_0 x \prod_{i=1}^{N} (x + c_i)^{-A_i} \right],$$

where $\mathcal{N}$ is determined from the normalization condition and the $c_i$ are constants determined by the above procedure.

### 4.2. Time dependent solutions

Since the probability distribution $P_1(x,t)$ satisfies the FPE

$$\frac{\partial P_1(x,t)}{\partial t} = \tilde{L}(x)P_1(x,t),$$

where $\tilde{L}(x)$ is the Fokker-Planck operator [6], by defining $R(x) \equiv -g(x)$, $D(x) \equiv \frac{1}{2}f(x)$ and $U_S(x) \equiv D(x)P_{\text{asym}}(x)$, [13] can be rewritten in the more compact form

$$\frac{R(x)}{D(x)} P_{\text{asym}}(x) - \frac{d}{dx} P_{\text{asym}}(x) = 0.$$
If we introduce the potential $U_1(x) \equiv D(x)P_\text{st}(x)$, then
\[
\frac{R(x)}{D(x)}dx = \frac{dU_1(x)}{U_1(x)}
\]
and
\[
P_\text{st}(x) = \frac{1}{D(x)} \exp \left( \int \frac{R(x')}{D(x')} dx' \right).
\]
As a result Eq. (14) reads
\[
\frac{\partial P_1(x,\tau)}{\partial \tau} = \frac{\partial^2}{\partial x^2} P_1 - \frac{\partial}{\partial x} P_1,
\]
where we have defined $\tau \equiv D(x)t$. This equation can be rewritten as a diffusion equation by introducing the variable $V(x,t)$ through the transformation $V(x,t) \equiv U_1(x)\exp \left( \frac{K}{2} x + \frac{K^2}{4} t \right)$, which yields
\[
\frac{\partial}{\partial t} U_1(x,t) = \frac{\partial^2 U_1}{\partial x^2}.
\]
To find the explicit analytic time dependent solution of this equation we use the method of separation of variables and express $U_1(x,t)$ as
\[
U_1(x,t) = \sum_{n=1}^{\infty} A_n X_n(x) T(t).
\]
This yields the following separation equation for $T(\tau)$
\[
\frac{d}{d\tau} T(\tau) = -\beta T(\tau),
\]
which for a given $T_0 \equiv T(\tau = 0)$ has the solution
\[
T(\tau) = e^{-\beta \tau} T_0.
\]
where $\beta$ is an arbitrary but positive (separation) constant. Similarly, $X(x)$ obeys the ordinary separation equation

$$\frac{d^2}{dx^2}X_n(x) - \beta \frac{d}{dx}X_n(x) + d_nX_n(x) = 0,$$

(36)

where the $d_n$ are separation constants. In terms of the variable $Y(x)$, defined by

$$X(x) \equiv e^{-\beta x}Y(x),$$

(37)

the general solution of (36) reads

$$Y(x) = Ae^{-d_1t} \sin \left(\lambda_0 + \sqrt{d_1}x\right),$$

(38)

where $A$ and $\lambda_0$ are, respectively, an arbitrary amplitude and phase that have to be fixed through the initial and boundary conditions.

We now assume that its possible to replace the infinite sum (33) by an effective term of the form

$$\sum_{n=1}^{\infty} A_nX_n(x)T(t) \to A_{eff}X_{eff}[x, d_{eff}(x)]e^{-d_{eff}(x)t},$$

(39)

where $X_{eff}[x, d_{eff}(x)]$ obeys the equation

$$\frac{d^2}{dx^2}X_{eff} + d_{eff}(x)X_{eff} = 0.$$

(40)

If we parametrize $d_{eff}(x)$ by the linear function $d_{eff}(x) = d_0 + d_1x$, the solution of (40) is

$$X_{eff}[x, d_{eff}(x)] = C_1 \text{AiryA}_i \left(-\frac{d_0 + d_1x}{d_1^{2/3}}\right) + C_2 \text{AiryB}_i \left(-\frac{d_0 + d_1x}{d_1^{2/3}}\right),$$

(41)

where AiryA$_i$ and AiryB$_i$ denote the Airy functions. With these assumptions and taking the Padé approximant [0/0], we may fit the difference between the normalized word frequency $f(k)$ and the asymptotic beta function $m_3(k)$ given by (1). For the different languages this is shown in the plots of Fig. 3.

The curves in Fig. 3 show that for the interval $10^2 \leq k \leq 10^6$ the dynamics of rank variation is very similar for all the languages considered. In contrast, for $k \leq 10^2$ the plots are very different, suggesting that there are other dynamic factors that have to be taken into account.

5. Discussion

In this work we have proposed a stochastic approach to analyze the dynamics of the rank variation ($k$) of words in time for six Indo-European languages: English, French, German, Italian, Russian and Spanish. Based on numerical evidence we here showed that
$k$ may be regarded as a random variable exhibiting complex dynamics in different regions of the $(k, t)$ space. This fact suggests that its dynamics could be adequately described by a stochastic model, and we described it as a Markovian, one-step, stochastic process arising from the conflictive dynamics of appearance and disappearance of words. The time evolution is given by a master equation. For the languages considered here there is always a small difference between the data for $k$ and their adjustment. In this work we have analyzed and proposed an explanation of the origin of this difference within the context of the proposed stochastic model. Actually, in previous works we have introduced a measure of how words ranks change in time and we have called this distribution rank diversity [16].

In this work we have used approximations to obtain stationary and time dependent analytic solutions of the nonlinear Fokker-Planck equation (6) which lead to a good fit of the data. However, there are many open questions and further possibilities regarding a more adequate description of the dynamics of the rank variation. It is likely that a more complex stochastic process is able to describe other regions of $(k, t)$ space, where the dynamics is more complex. Yet, to our knowledge there are no other available descriptions of theoretical linguistics, and the predicted behavior of $k$ should always comply with the analysis based on real linguistic data. However, this remains to be assessed.

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