Compositeness of weakly S-wave bound states from next-to-leading Weinberg’s relations

M. Albaladejo and J. Nieves
Instituto de Física Corpuscular (centro mixto CSIC-UV), Institutos de Investigación de Paterna, C/Catedrático José Beltrán 2, E-46980 Paterna, Valencia, Spain
(Dated: March 10, 2022)

We discuss a model-independent estimator of the likelihood of the compositeness of a shallow S-wave bound or virtual state. The approach is based on an extension of Weinberg’s relations in Phys. Rev. 137, B672 (1965) [1] and it relies on the proximity of the energy of the state to the two-hadron threshold to which it significantly couples. As explicit applications, we analyse the case of the deuteron, the \(^1S_0\) nucleon-nucleon virtual state and the exotic \(D_{s0}^+(2317)^+\), and find strong support to the molecular interpretation in all cases. Results are less conclusive for the \(D_{s0}^+(2317)^+\), since this narrow resonance is not a loosely bound state, and the approach employed here is in at the limit of its applicability.

Introduction.—Quantum Chromodynamics (QCD), the theory of strong interactions, generates a rich spectrum of hadrons, most of which can be classified according to simple constituent quark models [2–6] as \(q\bar{q}\) (mesons) and \(qqq\) (baryons) states. Despite this fact, the last two decades have witnessed the discovery of many states that defies this simple classification [7]. Different worldwide experiments (BaBar, Belle, BES, LHCb, ...) have reported the observation of a plethora of unstable states and peaks in mass-distributions located surprisingly close to different two bottomed/charmed-hadron thresholds, such as the \(XYZ\) mesons, the \(P_c\) pentaquarks [8–16], or the doubly charmed \(T_{cc}^+\) state [17, 18]. There exist also clear examples of exotic candidates in the open charm and bottom sectors, e.g. \(D_{s0}^+(2317), D_s^+(2300), \Lambda_c(2595)\) [19–21], \(B_1(5721), B_2^+(5747), \Xi_b(6227)\) [22–26], etc. These states are often interpreted as hadron molecules [27–76], or compact tetraquarks/pentaquarks [77–100], and when allowed by their quantum numbers and flavour content, there are also attempts to describe these states as predominant \(q\bar{q}\) or \(qqq\) structures [101–113]. Other possibilities (hybrids, virtual poles, etc.) for the nature of these exotics [114–120] or the role of kinematic (non-dynamical) effects (chfiely triangle singularity) in the interpretation of the observed peaks have also been stressed [55, 120–130]. Further discussions and references can be found in Refs. [131–136].

In addition to knowing how many of these states exist and their masses and widths, it is clearly a fundamental task in hadron physics to study the dynamical details of their structure. Such analysis will be invaluable in improving our understanding of strong interactions.

It is a direct consequence of unitarity that the inverse single-channel two-particle scattering amplitude \(f\) is given in terms of the phase shift \(\delta\) by \(f(E)^{-1} = k \cot(\delta(k) - i\kappa)\), with \(k = \sqrt{2\mu E}\), for non-relativistic kinematics, \(\mu\) the reduced mass of the scattering particles, and \(E\) the energy of the system relative to the threshold. The real part of the inverse scattering amplitude is a polynomial in even powers of \(k\) and, in the case of S-wave, it leads to the effective range expansion (ERE):

\[
k\cot(\delta(k)) = \frac{1}{a} + \frac{1}{2}r k^2 + O(k^4),
\]

where the parameters \(a\) and \(r\) are called the scattering length and effective range, respectively. Weinberg’s compositeness rules [1] connect these low energy observables with the probability \((X = 1 - Z)\) that an S-wave shallow bound state is a two-particle molecule,

\[
a = a_{LO}(Z) + O(\beta^{-1}), \quad a_{LO}(Z) = -\frac{2}{\gamma_b} \left( \frac{1 - Z}{2 - Z} \right),
\]

\[
r = r_{LO}(Z) + O(\beta^{-1}), \quad r_{LO}(Z) = -\frac{1}{\gamma_b} \left( \frac{Z}{1 - Z} \right),
\]

where \(\gamma_b = \sqrt{2\mu|E_b|}\) (with \(E_b < 0\), the binding energy) and \(\beta\) denotes the next momentum scale that is not treated explicitly in the ERE, with \(1/\beta\) providing an estimate for the interaction range corrections. The work of Ref. [1] showed that the experimental values for \(a\) and \(r\) from \(pn\) scattering give strong model-independent evidence that the deuteron is composite, i.e., the probability \(Z\) of ending the deuteron in a bare elementary particle state is very small. However, as commonly acknowledged (see e.g. Refs. [137–140]), this does not follow from the naive evaluation of \(X(a, r) = 1/\sqrt{1 + 2r/a}\), easily derived from Eqs. (2), which gives the meaningless result of \(X = 1.68 > 1\) for a probability. Indeed, the above formula and this numerical value for \(X\) are not given in Ref. [1]. As explicitly pointed out by Weinberg, the key token for the deuteron compositeness is the fact that \(r\) is small and positive of the order of the range \(~ m_n^2\) of the \(pn\) interaction, rather than large and negative. This discussion clearly shows the ambiguities affecting any conclusion about the nature of an exotic state based uniquely on a blind numerical computation of \(X\), neglecting \(O(\beta^{-1})\) corrections. Different applications, re-derivations, re-interpretation and extensions of Weinberg’s compositeness relations have been proposed [137–155], discussing or trying to improve upon various aspects of the derivation of the criterion in Ref. [1], but so far there is not a well established tool to decide whether a particle is composite or elementary.
In this work, we will present an estimator of the compositeness of a state relying only on the proximity of its mass to a two-hadron threshold to which it significantly couples. Our conclusions will apply only to weakly bound or virtual states since model-independent statements are possible only if \( \gamma_b \ll \beta \). The internal structure of a non-shallow bound state cannot be studied without having a knowledge of its wave-function more detailed than what can be inferred from the experimental value of just a few scattering parameters. Hence, one cannot extract model-independent conclusions on the compositeness of a particle located far from the relevant threshold, since one would necessarily have to rely on certain interaction, re-summation and renormalization models. In this respect, the calculation of hadron form factors, which can be accessed by LQCD simulations, can play an important role in determining the internal structure of hadrons [37, 156, 157].

For simplicity, we will ignore coupled-channels dynamics, though we will comment in the concluding remarks on this issue. We will also assume that the particle is stable, otherwise \( Z \) is complex. However, it might be an adequate approximation to ignore the decay modes of a very narrow resonance.

**Likelihood of the compositeness of a weakly bound state.**—
The parameters of the ERE [cf. Eq. (1)] depend in turn on \( \gamma_b \), which is determined by the binding energy of the state. The large values for both \( a \) and \( r \) when \( Z \) is not zero appear because of the \( 1/\gamma_b \) contributions in Eq. (2). They may suggest that the next-to-leading order (NLO) approximation to the ERE, consisting in neglecting \( O(k^4) \) terms, may itself break down when the particle is elementary. This, however, does not happen since only the first two terms in the ERE become of order \( 1/\gamma_b \) for \( Z \neq 0 \) and \( k \approx 1/\gamma_b \). The third and higher terms are smaller by powers of \( \gamma_b/\beta \) [1]. As a consequence, the approximate relation

\[
\gamma_b \approx -\frac{1}{a} + \frac{1}{2} r \gamma_b^2
\]

(3)

is expected to be fulfilled with great accuracy for weakly bound states. This is indeed the case for the deuteron. The above relation does not tell anything about the elementarity of the particle, since it follows from the requirement \( \cot \delta(\gamma_b) = i \). In fact, it is exactly satisfied by \( a_{LO} \) and \( r_{LO} \) for all \( Z \) [cf. Eqs. (2)]. However, the deviations of the actual scattering length and effective range from their \( \gamma_b \)–expansion LO values \( a_{LO} \) and \( r_{LO} \) encode some valuable information on the compositeness of the state. Moreover, from the discussion above, the third and higher terms in the ERE of Eq. (1) could provide corrections of order \( O(Z \gamma_b^2/\beta) \), at most, to the difference \( \gamma_b - (1/a + r \gamma_b^2/2) \). With these ideas in mind, we introduce a phenomenological term \( \delta r \) to estimate the NLO contribution to the effective range \( r \), within its expansion in powers of the binding momentum \( \gamma_b \),

\[
r = r_{NLO} + O \left( \frac{\gamma_b}{\beta^2} \right), \quad r_{NLO} = r_{LO} + \delta r,
\]

(4a)

This correction, \( \delta r \), is expected to be of the order of the range of the interaction \([O(1/\beta)]\). In addition, we fix the analogous NLO contribution to the scattering length such that the difference \( \gamma_b - (1/a_{NLO} + r_{NLO} \gamma_b^2/2) \) [cf. Eq. (3)] deviates from zero in terms of the order \( O \left( \gamma_b^3/\beta^2 \right) \), obtaining:

\[
a = a_{NLO} + O \left( \frac{\gamma_b}{\beta^2} \right), \quad a_{NLO} = a_{LO} - \frac{1}{2} (1 - Z)^2 \left( 1 - \frac{Z}{Z'} \right) \frac{1}{2} a_{LO}^2 - \frac{1}{2} \left( \frac{1 - Z}{1 - Z'} \right) \frac{1}{2} a_{LO}^2.
\]

The relations given in Eqs. (4) are a key result of this work. They provide a model-independent scheme to correlate the NLO corrections to \( a \) and \( r \), which turns out to be consistent as long as \( Z \approx 0 \), of the order of \( O \left( \gamma_b/\beta \ll 1 \right) \).

Given the experimental values for the mass, or equivalently the binding momentum \( \gamma_b^{exp} \), the associated scattering length \( a_{exp} \), and the effective range \( r_{exp} \) parameters of a two-particle weakly bound state, we propose to study the following two dimensional (2D) distribution:

\[
\mathcal{L}(Z, \delta r) = \frac{1}{3} \left( \frac{a_{exp} - a_{NLO}}{\Delta a_{exp}} \right)^2 + \frac{\left( r_{exp} - r_{NLO} \right)}{\Delta r_{exp}}^2 + \frac{\left( \gamma_b^{exp} - \gamma_b^{NLO} \right)}{\Delta \gamma_b}^2
\]

(5)

to estimate the likelihood of the compositeness of the state. Above, \( \gamma_b^{NLO} \) is given by

\[
\gamma_b^{NLO} = 1 - \sqrt{1 + 2 \gamma_b^{exp}/\gamma_b^{NLO}},
\]

which exactly satisfies Eq. (3). In addition, \( \Delta \gamma_b^{exp}, \Delta a_{exp}, \) and \( \Delta r_{exp} \) should be fixed taking into account not only the uncertainties on the determinations of these observables, but also the expected accuracy (relative error of the order of \( \gamma_b^{2}/\beta^2 \)) of the NLO approximation to them. Consistent with the order we are working, we evaluate \( a_{NLO} \) and \( r_{NLO} \) using \( \gamma_b^{exp} \).

**The deuteron.**—We first apply the compositeness distribution of Eq. (5) to the paradigmatic case of the deuteron, whose properties are known very precisely: \( E_p^{exp} = -2.224575(9) \) MeV [or equivalently \( \gamma_p^{exp} = 0.2316068(5) \text{fm}^{-1} \)] [158], and \( a_{exp} = -5.42(1) \) fm and \( r_{exp} = 1.75(1) \) fm from the Granada-group analysis of the \( pn \) isoscalar \( ^3S_1 \) wave [159] (see also Ref. [160]). The analysis of the latter work includes statistical errors, stemming from the data uncertainties for a fixed form of the potential, and systematic errors arising from the different most-likely forms of the potentials. Assuming they are independent, the total uncertainty corresponds to adding both errors in quadrature. Despite including systematic uncertainties, the errors of \( a \) and \( r \) turn out to be much smaller than the accuracy that can be expected from the NLO approximation, \( (\gamma_b^2/m_p^2) 

\approx 10\% \), taking \( \beta \approx m_p \). Therefore, we fix \( \Delta \gamma_b^{exp}, \Delta a_{exp}, \) and \( \Delta r_{exp} \) in Eq. (5) assuming a relative error of 10\%. With all these inputs, we show in Fig. 1 the 2D distribution \( \mathcal{L}(Z, \delta r) \) for the case of the deuteron. It strongly supports molecular probabilities \( (1 - Z) \) quite close to one, in conjunction with values of the NLO \( \delta r \) contribution of the order of...
commenced masses are used for the kaon and $D$ mesons. Due to the large uncertainties affecting both $a_{\text{exp}}$ and $r_{\text{exp}}$, it is not necessary to take into account the subtleties associated with the $D^0K^*-D^+K^0$ isospin breaking effects. The scale $\beta$ is in this case of the order of 300 MeV, estimated from the expected effects induced by the nearest $D_s\eta$ threshold [137] and/or by the two-pion exchange interaction, none of which are explicitly treated in the ERE. Hence, we expect the accuracy of the NLO $\gamma_b$-expansion to be of the order of $(\gamma_b^\text{exp}/\beta^2) \sim 40\%$, which we adopt for $\Delta\gamma_b^\text{exp}$, while for $\Delta a_{\text{exp}}$ and $\Delta r_{\text{exp}}$, we use the errors quoted above.

The compositeness 2D distribution of Eq. (5) for the $D_{s0}^*(2317)^+$ is shown in Fig. 2. The results favor $DK$ molecular probabilities of at least 50%, which is agreement with previous model-dependent predictions [50, 73, 137, 140, 166, 168, 169]. We cannot be as predictive in this case as for the deuteron, not only because of the bigger uncertainties of the input, but also because of the larger size of the power-counting parameter $\gamma_b/\beta \sim 0.6$. The $D_{s0}^*(2317)^+$ is not a loosely bound state, and the NLO approach employed here is in at the limit of its applicability. To be quantitatively more precise, it would be necessary to know more details about the dynamics at short distances of this exotic state than those encoded in the first two parameters of the ERE.

The $1^S_0$ nucleon-nucleon virtual state.—The $pn$ scattering length and effective range determined in Ref. [160] for this isovector partial wave are $a_{\text{exp}} = 23.735(16)$ fm and $r_{\text{exp}} = 2.68(3)$ fm, respectively, with the total uncertainties obtained by adding statistical and systematic errors in quadrature. This partial wave has a shallow virtual state (pole on the real energy-axis below the threshold on the unphysi-
ich sheet), which position is determined by the condition 
\( \cot(\delta - i \gamma_v) = +1 \), with \( \gamma_v = \sqrt{2 \mu |E_v|} \) and \( E_v < 0 \), the binding energy of the virtual state. From the ERE, it follows

\[
\gamma_v \approx \frac{1}{a} - \frac{1}{2} \gamma_v^2 + O(\gamma_v^3). \quad (7)
\]

Neglecting \( O(\gamma_v^4) \) corrections and using the experimental \( a_{\exp} \) and \( r_{\exp} \) values, the above equation leads to

\[
\gamma^\exp_v \approx \frac{-1 + \sqrt{1 + 2r_{\exp}/a_{\exp}}}{r_{\exp}} = 0.03999(5) \text{ fm}^{-1}, \quad (8)
\]

where the error comes from the uncertainties of the experimental ERE parameters. This error turns out to be around a factor of ten greater than the corrections induced by the \( O(\gamma_v^4) \) term in Eq. (8). This virtual binding momentum corresponds to \( E_v = -0.0663(4) \text{ MeV} \).

It is not trivial to extend the notion of compositeness to states other than bound states, since wave functions derived from poles on the unphysical sheet are not normalizable and the probabilistic interpretation is lost. They are not QCD asymptotic states, and thus it seems difficult to argue about wave-function components. However, one could think of some variation of QCD parameters, e.g. quark masses, such that these virtual states could become physical, bound states. From this perspective, it would make sense to generalize the notion of compositeness. On the other hand, formally relying on the definition of the field renormalization \( Z \) in the nonrelativistic theory [137], relations between \( a, r \) and \( Z \) can be derived also for a virtual state with a pole at \( k = -i \gamma_v \), and they are similar to those of a bound state. One should simply replace \( \gamma_b \) with \( -\gamma_v \) in \( a_{\text{NLO}} \) and \( r_{\text{NLO}} \) given in Eqs. (2) and (4). In addition, \( \gamma_v^{\text{NLO}} \) should be evaluated using Eq. (8), but with the NLO ERE parameters. Thus, we can use the definition of Eq. (5) for the compositeness distribution of a virtual state. The accuracy of the NLO \( \gamma \)-expansion \( (\gamma_v^2/m^2) \sim 0.3\% \) is larger than the experimental errors on \( a_{\exp} \) and \( r_{\exp} \), and we take it to set \( \Delta \gamma^\exp_v \) and \( \Delta r_{\exp} \), while we use the experimental error to fix \( \Delta r_{\exp} \). Due to the high precision of the input parameters, very small variations of \( Z \) and \( \delta r \) produce quite large changes of \( \mathcal{L}(Z, \delta r) \). In Fig. 3, we show the neperian logarithm of the compositeness distribution for this case. We see that for \( Z > 0.2 \), the 2D function \( \mathcal{L}(Z, \delta r) \) takes values larger than 2, while its minimum values are found for \( Z \sim (0.05 - 0.1) \). This is consistent with the general understanding that virtual states are the result of two-particle interactions, and in this sense can be considered of molecular type [137].

**Summary.**—We have discussed a model-independent estimator of the likelihood of the compositeness of a shallow S-wave bound or virtual state. It relies only on the proximity of the energy of the state to the two-hadron threshold to which it significantly couples. The approach is based on next-to-leading Weinberg’s relations. We have analysed the case of the deuteron, the \(^1S_0\) nucleon-nucleon virtual state and the exotic \( D_{slb}^*(2317)^+ \) resonance, and found strong support to the molecular interpretation in all cases. Nevertheless, results are less conclusive for the \( D_{slb}^*(2317)^+ \) due to the large size of the power-counting parameter \( \gamma_b/\beta \sim 0.6 \). This is because the \( D_{slb}^*(2317)^+ \) is not really a loosely bound state, and the NLO approach employed here is at the limit of its applicability. The approach should be modified in the presence of close coupled channels that play an important role on the long-distance dynamics of the state, and as a consequence, they significantly modify the effective range parameter [137]. In Ref. [170], it is shown for the \( \chi_{c1}(3872) \) and \( T_{cc} \) exotic states that the appearance of a large and negative effective range, which in the one-channel case would indicate the dominance of a compact component [1], can be naturally generated by the coupled-channel dynamics. Hence, in presence of coupled channels, and before analyzing the estimator of the compositeness proposed in this work, it would be necessary to correct the effective range by a term [137, 170] that stems from coupled-channel effects, and which clearly needs to be attributed to the molecular component of the state.

**ACKNOWLEDGEMENTS**

We warmly thank E. Ruiz-Arriola for useful discussions. This research has been supported by the Spanish Ministerio de Ciencia e Innovación (MICINN) and the European Regional Development Fund (ERDF) under contract PID2020-112777GB-I00, the EU STRONG-2020 project under the program H2020-INFRAIA-2018-1, grant agreement no. 824093 and by Generalitat Valenciana under contract PROMETEO/2020/023. M. A. is supported by Generalitat Valenciana under Grant No. CIDEGENT/2020/002.

[1] S. Weinberg, Phys. Rev. 137, B672 (1965).
[2] M. Gell-Mann, Phys. Lett. 8, 214 (1964).
[58] M.-L. Du, V. Baru, F.-K. Guo, C. Hanhart, U.-G. Meißner, J. A. Ollier, and Q. Wang, Phys. Rev. Lett. 124, 072001 (2020), arXiv:1910.11846 [hep-ph].

[59] M.-Z. Liu, T.-W. Wu, M. Pavon Valderrama, J.-J. Xie, and L.-S. Geng, Phys. Rev. D 99, 094018 (2019), arXiv:1902.0344 [hep-ph].

[60] J.-X. Lu, Y. Zhou, H.-X. Chen, J.-J. Xie, and L.-S. Geng, Phys. Rev. D 92, 014036 (2015), arXiv:1409.3133 [hep-ph].

[61] X.-K. Dong, F.-K. Guo, and B.-S. Zou, Commun. Theor. Phys. 73, 125201 (2021), arXiv:2108.02673 [hep-ph].

[62] M.-L. Du, V. Baru, F.-K. Guo, C. Hanhart, U.-G. Meißner, J. A. Ollier, and Q. Wang, JHEP 08, 157 (2021), arXiv:2102.07159 [hep-ph].

[63] M.-L. Du, V. Baru, X.-K. Dong, A. Filin, F.-K. Guo, C. Hanhart, A. Nefediev, J. Nieves, and Q. Wang, Phys. Rev. D 105, 014024 (2022), arXiv:2110.13765 [hep-ph].

[64] X.-K. Dong, F.-K. Guo, and B.-S. Zou, Progr. Phys. 41, 65 (2021), arXiv:2101.01021 [hep-ph].

[65] F.-K. Guo, C. Hanhart, U.-G. Meißner, Q. Wang, Q. Zhao, and B.-S. Zou, Rev. Mod. Phys. 90, 015004 (2018), arXiv:1705.00141 [hep-ph].

[66] Z.-H. Guo and J. A. Ollier, Phys. Rev. D 103, 054021 (2021), arXiv:2012.11904 [hep-ph].

[67] A. Feijoo, W. H. Liang, and E. Oset, Phys. Rev. D 104, 114015 (2021), arXiv:2108.02730 [hep-ph].

[68] M. Albaldadejo, (2021), arXiv:2110.02944 [hep-ph].

[69] A. Esposito, A. Pilloni, and A. D. Polosa, Phys. Rept. 668, 1 (2017), arXiv:1611.07920 [hep-ph].

[70] L. Maiani, F. Piccinini, A. D. Polosa, and V. Riquer, Phys. Rev. D 71, 014028 (2005), arXiv:hep-ph/0412098.

[71] L. Maiani, F. Piccinini, A. D. Polosa, and V. Riquer, Phys. Rev. D 89, 114010 (2014), arXiv:1405.1551 [hep-ph].

[72] E. Braaten, Phys. Rev. Lett. 111, 162003 (2013), arXiv:1305.6905 [hep-ph].

[73] J. M. Dias, F. S. Navarra, M. Nielsen, and C. M. Zanetti, Phys. Rev. D 88, 016004 (2013), arXiv:1304.6433 [hep-ph].

[74] C.-F. Qiao and L. Tang, Eur. Phys. J. C 74, 3122 (2014), arXiv:1307.6654 [hep-ph].

[75] C. Deng, J. Ping, and F. Wang, Phys. Rev. D 90, 054009 (2014), arXiv:1402.0777 [hep-ph].

[76] A. Ali, C. Hambrock, and W. Wang, Phys. Rev. D 85, 054011 (2012), arXiv:1110.1333 [hep-ph].

[77] H.-Y. Cheng and W.-S. Hou, Phys. Lett. B 566, 193 (2003), arXiv:hep-ph/0305308.

[78] K. Terasaki, Phys. Rev. D 68, 011501 (2003), arXiv:hep-ph/0305213.

[79] V. Dmitrasinovic, Phys. Rev. Lett. 94, 162002 (2005).

[80] M. E. Bracco, A. Lozea, R. D. Matheus, F. S. Navarra, and M. Nielsen, Phys. Lett. B 624, 217 (2005), arXiv:hep-ph/0503137.

[81] Z.-G. Wang and S.-L. Wan, Nucl. Phys. A 778, 22 (2006), arXiv:hep-ph/0602080.

[82] Y.-Q. Chen and X.-Q. Li, Phys. Rev. Lett. 93, 232001 (2004), arXiv:hep-ph/0407062.

[83] Y. Kim, M. Oka, and K. Suzuki, (2022), arXiv:2202.06520 [hep-ph].

[84] L. Maiani, A. D. Polosa, and V. Riquer, Phys. Lett. B 749, 289 (2015), arXiv:1507.04980 [hep-ph].
