On a generalization of the logistic map

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A family of non-conjugate chaotic maps generalizing the well-known logistic function is defined, and some of its basic properties studied. A simple formula for the Lyapunov exponent of all the maps contained in this family is given based on the construction of conjugacies. Moreover, it is shown that, despite the dissimilarity of their polynomial expressions, all the maps possess the same invariant density. Other algebraic properties of the family, which shows some relationship with the set of Tschebysheff polynomials, are also investigated.

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Simple iterated maps, such as the Baker map and the logistic map, are the subject of constant fascination. Part of the interest for these systems is linked to the fact that they provide an easy and pedagogical way to understand how complex and chaotic behavior can arise from simple dynamical models. Even more remarkable, yet, is the fact that studies of low-dimensional maps have proven to be fruitful in understanding the basic mechanisms responsible for the appearance of chaos in a large class of dynamical systems (e.g., differential flows, high-dimensional maps). One paradigm example of such mechanisms is the so-called period-doubling cascades of fixed points, encountered qualitatively in many physical systems of interest. In this paper, we enlarge the set of maps known to be chaotic by presenting a generalization of the logistic map. The generalization, more precisely, enables us to construct an infinite number of one-dimensional maps which are chaotic in the sense that they all have positive Lyapunov exponents, and possess at least one orbit that is not asymptotically periodic.

I. INTRODUCTION

In the theory of nonlinear systems, the logistic map

\[ x_{n+1} = f_r(x_n) = rx_n(1 - x_n), \]

with \( 0 \leq r \leq 4, \ x_n \in [0, 1], \) and \( n = 0, 1, 2, \ldots, \) is well-known to provide one of the simplest example of what is referred to as a chaotic system. That is, a system which, for some range of its parameters, possesses at least one bounded orbit \( \{x_0, x_1, \ldots \} \) such that (i) \( \{x_0, x_1, \ldots \} \) is not asymptotically periodic and (ii) the Lyapunov exponent, defined in the context of Eq.(1) by the usual relation

\[ \lambda(x_0) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=0}^{N-1} \ln |\partial_x f_r(x_i)|, \]

is greater than zero [1,2]. For the logistic map at \( r = 4, \) specifically, \( \lambda(x_0) \) is positive and equals \( \ln 2 \) for almost all \( x_0. \) This value can be easily calculated by noting that the logistic map is conjugate to the tent map or by having recourse to the fact that the dynamical system defined by Eq.(1) is ergodic for this particular value of \( r \). In this latter case, one is justified to calculate analytically the Lyapunov exponent as an ensemble average

\[ \lambda = \int_0^1 \rho(x) \ln |\partial_x f_r(x)| \ dx, \]

the function

\[ \rho(x) = \frac{1}{\pi \sqrt{x(1-x)}}, \quad x \in [0, 1] \]

being the invariant density of the map satisfying \( \rho(A) = \rho(f^{-1}(A)), \) where \( f^{-1}(A) \) is the preimage of an arbitrary subset \( A \) of the unit interval \([0, 1] \).

In this paper, we shall focus on another property of the logistic map at \( r = 4, \) namely the existence of the following closed-form solution

\[ x_n = \sin^2(2^n \pi \theta_0), \]

where \( \theta_0 = \pi^{-1} \arcsin(\sqrt{x_0}). \) The importance of this formula relies evidently on the fact that \( x_n \) can be evaluated directly, for any initial point \( x_0, \) without actually computing the intermediate values \( x_1, x_2, \ldots, x_{n-1}. \) In the remaining of this work, we follow the steps used in the derivation of the above formula to construct similar expressions involving general terms of the form \( \sin^2(N^n \theta). \) In doing so, we shall see that a whole new family of polynomial maps can be defined which, essentially, generalize the logistic map by preserving the density Eq.(3), and whose Lyapunov exponents can be calculated easily. Other properties of these newly defined dynamical maps are also studied. In particular, we shall point out some similarities between the maps and the Tschebysheff polynomials, and prove, finally, that the family of maps as a whole is closed under the composition of functions.
II. DEFINITION OF THE LOGISTIC FAMILY

The explicit solution of the logistic map at the particular value \( r = 4 \) can be derived by substituting in \( f_n(x_n) = 4x_n(1-x_n) \) the change of coordinates \( x_n = \sin^n(\pi \theta_n) \), valid for \( x_n \in [0, 1] \), and by using the trigonometric identity

\[
4 \sin^2 \theta (1 - \sin^2 \theta) = \sin^2 (2 \theta)
\]

in order to obtain \( \sin^2 (\pi \theta_{n+1}) = \sin^2 (2 \pi \theta_n) \). This is equivalent to the map \( \theta_{n+1} = 2 \theta_n \mod 1 \), which has the explicit solution

\[
\theta_n = 2^n \theta_0 \mod 1.
\]

Hence, the complete solution in terms of the coordinate \( x \) must correspond to Eq. (4).

Following these steps, one can imagine to construct a map having the solution \( x_n = \sin(3^n \pi \theta_0) \) by expressing \( \sin^2(3 \theta) \) in terms of \( \sin^2(\theta) \) with the identity

\[
\sin^2(3 \theta) = 16 \sin^6 \theta - 24 \sin^4 \theta + 9 \sin^2 \theta.
\]

In this case, one gets \( g(x) = 16x^3 - 24x^2 + 9x \) as a possible dynamical map on \([0, 1]\) having the required solution. More generally, one can use the recurrence formula

\[
\sin(N \theta) = 2 \cos \theta \sin((N - 1) \theta) - \sin((N - 2) \theta)
\]

to define a whole set of maps which express \( \sin^2(N \theta) \) in terms of \( \sin^2 \theta \). This set, the \textit{sine functions set}, is defined specifically as

\[
S = \{SN(x) : N = 1, 2, 3, \ldots\},
\]

where \( SN(x) = [s_N(\sqrt{x})]^2 \), and

\[
\begin{align*}
s_1(x) &= x \\
s_2(x) &= 2x \sqrt{1 - x^2} \\
& \vdots \\
s_N(x) &= 2 \sqrt{1 - x^2} s_{N-1}(x) - s_{N-2}(x).
\end{align*}
\]

Note that the intermediate functions \( s_N(x) \) are expressed as such in order to verify Eq. (1) with the variable change \( x = \sin \theta \).

Using these definitions, we calculate the first five functions of \( S \) listed in Table 1. Obviously, by construction of \( S \), \( S_2(x) \) is the logistic equation itself with parameter \( r = 4 \). From a more general perspective, it can also be seen that the maps \( SN(x) \) are degree \( N \) polynomials whose leading coefficient, i.e., the coefficient of the highest degree term, is equal to \( 4^{N-1} \) in absolute value. These two results, satisfied by any function \( SN(x) \), is proven more formally in ref. [6]. Note that the latter property allows us to extend the similarity with the logistic map by parameterizing the functions of \( S \) in the following manner

\[
SN_r(x) = r SN(x) / 4^{N-1},
\]

with \( 0 \leq r \leq 4^{N-1} \). We call the set of functions \( \{SN_r(x) : 0 \leq r \leq 4^{N-1}\} \), the \textit{family of SN(x)}, which can be characterized numerically by bifurcation diagrams and Lyapunov spectrums such as the ones shown in Figure 1.

![Figure 1](image-url)

FIG. 1. (a)-(b) Bifurcation diagrams of \( S3(x) \) and \( S4(x) \). (c)-(d) Corresponding Lyapunov spectrums.

III. PARTICULARITIES OF \( S \)

Many of the interesting properties of the logistic map at \( r = 4 \) can be investigated more intuitively by making the fact that Eq. (10) is equivalent to a shift map \( S \) on the binary expression of \( \theta_0 \) [1]. Indeed, if we express \( \theta_0 \) as a binary number

\[
\theta_0 = 0.b_0b_1b_2 \ldots = \sum_{i=0}^{\infty} b_i / 2^{i+1}, \quad b_i \in \{0, 1\},
\]

then applying Eq. (10) to \( \theta_0 \) is equivalent to shifting all the bits of \( \theta_0 \) to the left and dropping the integer part. In
where $S^n = S \circ S^{n-1}$ for $n > 1$, and $S^1 = S$. Not surprisingly, the same is true for the maps $SN(x)$, since Eq. (5) was the guideline in defining the family $S$. However, in the case of $SN(x)$, the shift map to consider takes effect on $\theta_0$ written in base $N$. This follows from the following result which generalizes effectively the solution of Eqs. (3) and (4).

**Theorem 1.** Let $\{x_0, x_1, x_2, \ldots\}$ be the orbit of $x_0$ under $SN(x)$. If we write $x_n = \sin^2(\pi \theta_n)$, then we have that

$$\theta_{n+1} = N^n \theta_0 \mod 1,$$

(15)

where, as usual, $\theta_0 = \pi^{-1} \arcsin(\sqrt{x_0})$.

We omit the proof of this theorem as it follows directly from the next lemma.

**Lemma 1.** Consider $SN(x)$ as defined previously. We have that

$$SN(\sin^2 \theta) = \sin^2(N \theta).$$

(16)

**Proof:** The result is obvious for $N = 1$ and $N = 2$. Suppose Eq. (16) true for $N - 1$ and $N - 2$, that is to say

$$s_{N-1}^2(\sin \theta) = S[N - 1](\sin^2 \theta) = \sin^2(\pi (N - 1) \theta),$$

and $s_{N-2}(\sin \theta) = \sin((N - 2) \theta)$. Then, for $\theta \in \mathbb{R}$,

$$SN(\sin^2 \theta) = s_N^2(\sin \theta) = [2\sqrt{1 - \sin^2 \theta} s_{N-1}(\sin \theta) - s_{N-2}(\sin \theta)]^2 = [2 \cos \theta \sin((N - 1) \theta) - \sin((N - 2) \theta)]^2 = \sin^2(N \theta),$$

where we have used the identity (3). \hfill \Box

Note that we could have proceeded to a similar generalization of the logistic map using cosine functions instead of sine functions, while preserving its shift property. One possible way of achieving this is to define the cosine functions set

$$C = \{CN(x) : N = 1, 2, 3, \ldots\},$$

(17)

where $CN(x) = [c_N(\sqrt{x})]^2$, and

$$c_1(x) = x$$

$$c_2(x) = 2x^2 - 1$$

$$\vdots$$

$$c_N(x) = 2xc_{N-1}(x) - c_{N-2}(x).$$

(18)

Contrary to the $s_N(x)$’s, the functions $c_N(x)$ have the interesting property that they are polynomials of degree $N$. In fact, the set $\{CN(x) : N \in \mathbb{N}\}$ coincides with the set of Tschebysheff polynomials on the unit interval $[0,1]$, the latter set satisfying the exact same recurrence formula as Eq. (18). We thus have that $\{c_N(x)\}$ must constitute a set of orthogonal polynomials, i.e.,

$$\int_0^1 cn(x)c_{n'}(x) \, dx = \delta_{N,N'},$$

(19)

for all integers $N$ and $N'$, where $\delta_{m,n}$ is the delta-Kronecker function. This fact can be further proved using the property $c_N(\cos \theta) = \cos(N \theta)$, well-known to be satisfied by the Tschebysheff functions. Note that $\{s_N(x)\}$ is also a set of orthogonal functions; its members satisfy indeed the relation $s_N(\sin \theta) = \sin(N \theta)$. In the remaining of this work, we shall restrain our study to the set $S$, since the maps $SN(x)$ are directly related to $CN(x)$ by the expression

$$SN(x) = \begin{cases} 
CN(x), & \text{for } N \text{ odd} \\
1 - CN(x), & \text{for } N \text{ even}. 
\end{cases}$$

(20)

Hence, as far as their dynamics are concerned, the functions $SN(x)$ and $CN(x)$ are totally equivalent.

**IV. CONJUGACIES**

The analysis of the chaoticity properties of a map $f$ is greatly simplified by studying conjugate maps of $f$ which are obtained by applying a global change of variables. Recall that two maps $f : I \to I$ and $g : J \to J$ are conjugate if there exists a homeomorphism, i.e., a bijective and continuous map $H : I \to J$ such that $H \circ f = g \circ H$. The function $H$ is called a conjugacy. In the context of $S$, a possible conjugate function of $SN(x)$ can be constructed as follows. Let $TN(x)$ be a piecewise linear function (a generalized tent map) defined on subintervals $[k/N,(k+1)/N]$ of $[0,1]$ by setting

$$TN(x) = \begin{cases} 
N x - k, & \text{for } k \text{ even} \\
-N x + k + 1, & \text{for } k \text{ odd}, 
\end{cases}$$

(21)

with $k = 0,1,\ldots,N-1$.

**Theorem 2.** $SN(x)$ is conjugate to $TN(x)$ with conjugacy $H(x) = \sin^2(\pi x/2)$.

**Proof:** First note that $H(x)$ is both continuous and bijective on the interval $[0,1]$. Now, on the first hand we have that $SN(H(x)) = SN(\sin^2(\pi x/2)) = \sin^2(N \pi x/2)$. On the other hand,

$$H(TN(x)) = \begin{cases} 
\sin^2(N \pi x/2 - k \pi/2), & \text{for } k \text{ even} \\
\sin^2(-N \pi x/2 + (k+1) \pi/2), & \text{for } k \text{ odd} 
\end{cases}$$

$$\begin{align*}
&= (\pm \sin(N \pi x/2))^2 \\
&= \sin^2(N \pi x/2). 
\end{align*}$$

(22)

Thus, we have proved that $SN \circ H = H \circ TN$ for all integers $N$. \hfill \Box
Figure 2 depicts the graphs of $SN(x)$ and the corresponding $TN(x)$ for $N = 3, 4$. ($T2(x)$ is only the tent map since, as we mentioned, $S2(x)$ is the logistic map.) In each case, note that $SN(x)$ is a “smooth” version of $TN(x)$.

Remark. The functions $CN(x)$ are pairwise conjugate to the functions $TN(x)$ with conjugacy map $H(x) = \cos^2(\pi x/2)$. Evidently, since the conjugacy is an equivalence relation, this has the consequence that any $CN(x)$ is conjugate to $SN(x)$.

Now, to evaluate the Lyapunov exponent of $SN(x)$ we may use the fact that $|\partial_x TN(x)| = N$ for almost all $x \in [0, 1]$ to infer that $\lambda(x_0) = \ln N$ almost everywhere in the case of $TN(x)$. Accordingly, since Lyapunov exponents are invariant under smooth and differentiable coordinate transformations \cite{10}, we have the following theorem. (A more extensive proof of this result, which takes care of the pathological points where $\partial_x TN(x)$ is not defined, is contained in ref. \cite{11}.)

Theorem 3. The Lyapunov exponent of $SN(x)$ is $\ln N$ almost everywhere (with respect to the invariant measure $\rho_{SN(x)}$).

The above theorem shows that the members of $S$ are non-conjugate to each other simply because they possess different Lyapunov exponents. It also shows that $S \setminus \{S1\}$, and consequently the set of Tschebyshoeff polynomials, are sets of chaotic maps. Indeed, $\lambda = \ln N > 0$ for $N > 1$, and by using the shift property of $SN(x)$ we can choose $x_0 = \sin^2(\pi \theta_0)$, with $\theta_0$ irrational, to build an orbit that is not asymptotically periodic. Another way to convince ourselves that all the polynomials in $S$ have chaotic orbits is to use the celebrated result “period-3 implies chaos” \cite{14}, and find an initial point $x_0$ of period 3 for each $SN(x)$. For instance, for a $N > 1$ let $x_0 = \sin^2(\pi \theta_0)$ where

$$\theta_0 = \frac{1}{N^3 - 1} = 0.001001 \ldots \quad (\text{in base } N). \quad (26)$$

Again, using the shift map property, we must have

$$x_0 = \sin^2(\pi \cdot 0.001001 \ldots)$$
$$x_1 = \sin^2(\pi \cdot 0.010010 \ldots)$$
$$x_2 = \sin^2(\pi \cdot 0.100100 \ldots)$$
$$x_3 = \sin^2(\pi \cdot 0.001001 \ldots) = x_0. \quad (27)$$

We thus extended the chaoticity properties of the logistic map to an infinite family of polynomials.

VI. ALGEBRAIC PROPERTIES

To complete the study of the properties of $S$, we now deduce that it is an abelian monoid with respect to the composition of functions $(\circ)$. A monoid, precisely, is a non-empty set $M$ together with a binary associative operation, say $*$, such that $x \ast y \in M$ for $x, y \in M$. There must also be an element $e \in M$, called the identity element, for which $x \ast e = e \ast x = x$ for all $x \in M$. Moreover, a monoid is called abelian if the binary operation is commutative \cite{11}. In $S$, the identity element is $S1(x) = x$. Also, the composition of function is clearly associative. Now, to prove that $S$ is indeed abelian monoid, we verify that it is closed under composition and that this composition is commutative, a condition that is not verified in the case of composition of general functions. However, before we do so, we present next a new expression of $SN(x)$ on the unit interval.
Lemma 2. For all \( SN(x) \) and \( x \in [0, 1] \),
\[
SN(x) = \sin^2(N \arcsin \sqrt{x}).
\]

Proof: Let \( x \in [0, 1] \). There exists a \( \theta \in [0, \pi/2] \) such that \( x = \sin^2 \theta \), and thus \( \theta = \arcsin \sqrt{x} \). Now, from Lemma 1 we have \( SN(x) = SN(\sin^2 \theta) = \sin^2(N \theta) = \sin^2(N \arcsin \sqrt{x}) \).

Theorem 4. (Monoid property) Let \( N_1 \) and \( N_2 \) be any positive integers. We have that
\[
SN_1 \circ SN_2 = SN_2 \circ SN_1 = SN_1N_2. \tag{28}
\]

Proof: For \( N_1 \) and \( N_2 \) given, consider \( SN_1(x) \) and \( SN_2(x) \). Then, for any \( x \in [0, 1] \), we obtain from Lemma 2
\[
SN_1(SN_2(x)) = \sin^2(N_1 \arcsin(\sqrt{SN_2(x)})) = \sin^2(N_1 \arcsin(\sin(N_2 \arcsin \sqrt{x}))) = \sin^2(N_1N_2 \arcsin \sqrt{x}) = SN_1N_2(x). \tag{29}
\]

Obviously, \( SN_1N_2(x) = SN_2N_1(x) \), so the composition is commutative.

As a direct consequence of the monoid property, \( k \)-periodic points of a certain polynomial \( SN(x) \) can be looked at as fixed points of the function \( SM(x) \) where \( M = N^k \). Furthermore, a polynomial \( SN(x) \) of very high degree can be computed easily by decomposing its expression using lower degree polynomial of the family \( S \). Explicitly, consider \( SN(x) \in S \). We say that \( SN(x) \) is a prime element of \( S \) if \( N \) is a prime number. Using this definition, we have as a result of Theorem 4 and the Fundamental Theorem of Arithmetic that any polynomial \( SN(x) \) must possess a unique decomposition in prime elements of \( S \).

VII. Final Remarks

To conclude, note that our study of the sine functions, written in the form \( SN(x) = \sin^2(N \arcsin \sqrt{x}) \), have been restricted to positive integers \( N \). In a similar manner, it could be interesting to investigate functions of the type \( S\alpha(x) = \sin^2(\alpha \arcsin \sqrt{x}) \) with \( \alpha \) real. One observation about this extra generalization is that, as for \( S1(x) \), the function \( S\alpha(x) \) does not exhibit chaotic properties for \( 0 \leq \alpha \leq 1 \). The function \( S\frac{1}{2}(x) = \sin^2(\frac{1}{2} \arcsin \sqrt{x}) \), for example, is conjugate to \( g(x) = x/2 \), and has all of its orbits attracted to \( x = 0 \). Yet, this is not surprising since the Lyapunov exponent of this map must be \( \ln(1/2) < 0 \). This brings us to conjecture that \( S\alpha(x) \) must admit chaotic behavior if and only if \( |\alpha| > 1 \), considering that the Lyapunov exponents of \( S\alpha(x) \) should be \( \ln \alpha \). A complete proof of this result, however, cannot be given here using the same symbolic dynamic approach used for \( SN(x) \), for the simple reason that the expression of a point in “base \( N \)” makes sense only if \( N \) is an integer greater than 1.

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