High Accuracy Analysis of an Anisotropic Nonconforming Finite Element Method for Two-Dimensional Time Fractional Wave Equation

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Abstract. High-order numerical analysis of a nonconforming finite element method on regular and anisotropic meshes for two dimensional time fractional wave equation is presented. The stability of a fully-discrete approximate scheme based on quasi-Wilson FEM in spatial direction and Crank-Nicolson approximation in temporal direction is proved and spatial global superconvergence and temporal convergence order \(\Theta(h^2 + \tau^{1-\alpha})\) in the broken \(H^1\)-norm is established. For regular and anisotropic meshes, numerical examples are consistent with theoretical results.

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1. Introduction

Fractional differential equations have recently attracted increasing attention in various fields of science and engineering. They play an important role in anomalous transport modeling and in theory of complex systems — cf. [4, 6, 15, 22, 28, 29, 34, 35]. However, analytical solutions of such equations are rarely available and even if they are known, their computation meets essential difficulties because of the presence of special functions. This led to the development of various numerical methods. In particular, for time-fractional
diffusion equations, finite difference methods are considered in [7, 19, 24, 42, 44, 46], Petrov-Galerkin methods in [31], DG methods in [16] and finite element methods in [21, 23, 38, 48–50].

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded convex polygonal region with the boundary \( \partial \Omega \) and let

\[
H^2(H^2(\Omega)) := \{ \varphi(x, t) : \| \varphi(x, t) \|_2 \in H^2[0, T] \}.
\]

In this work, we apply a spatial finite element method and temporal Crank-Nicolson scheme to the following two-dimensional time fractional wave equation (TFWE):

\[
\begin{aligned}
D_t^\alpha u(x, t) - \Delta u(x, t) &= f(x, t), \quad (x, t) \in \Omega \times (0, T], \\
u(x, t) = 0, \quad & (x, t) \in \partial \Omega \times (0, T], \\
u(x, 0) = u_0(x), \quad & u_t(x, 0) = \bar{u}_0(x), \quad x \in \Omega,
\end{aligned}
\]

where \( x = (x, y), u_0(x), \bar{u}_0(x) \) and \( f(x, t) \) are sufficiently smooth functions, \( u(x, t) \in H^2(H^2(\Omega)) \), the operator \( D_t^\alpha \) is the left-sided Caputo fractional derivative of order \( \alpha \) with respect to \( t \) defined by

\[
D_t^\alpha u(x, t) := \frac{1}{\Gamma(2 - \alpha)} \int_0^t \frac{\partial^2 u(x, s)}{\partial s^2} \frac{ds}{(t-s)^{\alpha-1}}, \quad 1 < \alpha < 2,
\]

and \( \Gamma \) is the Gamma function [22].

TFWEs represent a generalisation of classical diffusion and wave equations arising in modeling of diffusion and waves in fluid flow and oil strata. Numerical methods are widely used in their solution. Thus Bhrawy et al. [5], developed an efficient and accurate spectral numerical method for second- and fourth-order TFWEs and TFWEs with damping involving the Jacobi tau spectral procedure and Jacobi operational matrix for fractional integrals. Sun and Wu [40] introduced a fully-discrete scheme for TFWEs and proved that it is solvable, unconditionally stable and converges in \( L_\infty \)-norm. Based on the Crank-Nicolson method combined with the \( L1 \)-approximation, Fairweather et al. [12] developed an alternating direction implicit (ADI) orthogonal spline collocation method for two-dimensional TFWEs and established its optimal accuracy in various norms. Du et al. [9] considered higher order difference methods for TFWEs, proving their unconditional stability and convergence in \( L_\infty \)-norm. Zhang et al. [47] established unconditional stability and convergence of a compact ADI difference scheme and a Crank-Nicolson ADI scheme for two-dimensional TFWE. Ding and Li [8] approximated Riemann-Liouville derivative by second and fourth order difference schemes and constructed two difference schemes for TFDWEs with reaction term. Huang et al. [17] used partial integro-differential equations in finite difference schemes for initial-boundary value TFWEs and proved their convergence with first order accuracy in temporal and second order accuracy in spatial directions. Zeng [44] proposed stable and conditionally stable finite difference schemes for the TFDWE, based on fractional trapezoidal rules and second order generalized Newton-Gregory formulas and central differences. The approximation method developed by Yang et al. [43] is based on the transformation of TFWEs into an integro-differential equation and on Lubich fractional
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multistep method. Piecewise linear Galerkin FEM in space and convolution quadrature in time are used by Jin et al. [21] in fully-discrete schemes for TFWE with nonsmooth data. Li et al. [25] proved the unconditional stability and \( L_2 \)-norm convergence of a numerical method based on Galerkin finite element and Crank-Nicolson ADI methods for spatial and time direction approximations, respectively. Esen et al. [10, 11] employed Galerkin FEM and Petrov-Galerkin FEM to solve fractional diffusion equations and TFWEs. However, to the best of the authors’ knowledge, there is no high accuracy analysis for nonconforming finite element methods.

According to the second Strang lemma, in nonconforming finite elements the error consists of two parts — viz. the consistency error, which arises because of the element’s nonconformity and the interpolation error. For most nonconforming elements, the consistency error order is at most the interpolation error in \( L^2 \)-norm — cf. e.g. Wilson nonconforming element [41], \( P_1 \) nonconforming element [1, 33] and Carey nonconforming element [27]. Besides, it was shown in [36, 37] that if \( u \in H^3(\Omega)/H^4(\Omega) \), the consistency error of quasi-Wilson element in the broken \( H^1 \)-norm on rectangular meshes has order \( \theta(h^2)/\theta(h^3) \). It is one/two orders higher than the interpolation error. This property plays an important role in the study of superconvergence phenomenon. Moreover, the corresponding elements are anisotropic and they can be successfully used in solving boundary value problems for partial differential equations where anisotropic behaviour in parts of the domain is required — cf. [2, 3, 18, 30, 32, 39].

The goal of this work is to construct an unconditionally stable fully-discrete approximation scheme for TFWEs. We first show the unconditional stability of the method under consideration. The convergence and the superclose results in the broken \( H^1 \)-norm are then follows from the properties of quasi-Wilson nonconforming elements. We also use the interpolation postprocessing operator to establish global superconvergence of the method. Another feature of this work is related to the properties of quasi-Wilson elements, which allow the use of anisotropic meshes instead of regular ones. In future, we are also going to study various flexible numerical schemes of nonconforming finite element methods for time fractional partial differential equations on regular and anisotropic meshes.

The remainder of this paper is organised as follows. In Section 2, we introduce nonconforming finite element spaces, \( L^2 \)-approximation and a fully discrete scheme for (1.1). Important auxiliary results are presented in Section 3. Section 4 considers unconditional stability, superclose and superconvergence properties of the fully-discrete scheme. Numerical results of Section 5 are consistent with theoretical analysis and demonstrate a high accuracy of the method. Our conclusions are in Section 6.

2. Quasi-Wilson Elements and a Fully-Discrete Scheme

Let \( \hat{K} = [-1, 1] \times [-1, 1] \) be the reference element on \( \xi-\eta \) plane and \( \hat{A}_1 = (-1, -1), \hat{A}_2 = (1, -1), \hat{A}_3 = (1, 1), \hat{A}_4 = (-1, 1) \) be its vertices. The Quasi-Wilson nonconforming finite elements \( \{\hat{K}, \hat{P}, \hat{S}\} \) are defined by

\[
\hat{P} = \text{span} \{ N_i(\xi, \eta), \ i = 1, 2, 3, 4, \ \hat{\psi}(\xi), \hat{\psi}(\eta) \},
\]

where \( N_i(\xi, \eta) \) are the shape functions.
\[ \hat{\Sigma} = \left\{ \hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{v}_4, \frac{1}{|K|} \int_K \frac{\partial^2 \hat{\varphi}}{\partial \xi^2} d\xi d\eta, \frac{1}{|K|} \int_K \frac{\partial^2 \hat{\varphi}}{\partial \eta^2} d\xi d\eta \right\}, \]

where

\[ N_i(\xi, \eta) = \frac{1}{4} (1 + \xi_i \xi)(1 + \eta_i \eta), \quad \hat{v}_i = \hat{\nu}(\hat{A}_i), \quad i = 1, 2, 3, 4, \]

\[ (\xi_1, \eta_1) = (-1, -1), \quad (\xi_2, \eta_2) = (1, -1), \]

\[ (\xi_3, \eta_3) = (1, 1), \quad (\xi_4, \eta_4) = (-1, 1), \]

\[ \hat{\nu}(s) = \frac{1}{2} (s^2 - 1) - \frac{5}{12} (s^4 - 1). \]

The elements \( \hat{\nu} \in \hat{P} \) can be uniquely represented in the form

\[ \hat{\nu}(\xi, \eta) = \sum_{i=1}^{4} N_i(\xi, \eta) \hat{v}_i - \frac{3}{8} \int_K \frac{\partial^2 \hat{\varphi}}{\partial \xi^2} d\xi d\eta \cdot \hat{\nu}(\xi) - \frac{3}{8} \int_K \frac{\partial^2 \hat{\varphi}}{\partial \eta^2} d\xi d\eta \cdot \hat{\nu}(\eta). \]

Let \( \Gamma_h \) be a decomposition of \( \Omega \) with \( \hat{\Omega} = \bigcup_{K \in \Gamma_h} K \) and for any \( K \in \Gamma_h \), let \( \partial K = (x_K, y_K) \)

be the center of \( K = [x_K-h_{x,K}, x_K+h_{x,K}] \times [y_K-h_{y,K}, y_K+h_{y,K}] \), where \( h_{x,K} \) and \( h_{y,K} \) are, respectively, the distances from \( \partial K \) to edges of \( K \) parallel to the \( x \)- and \( y \)-axis. Moreover, let \( h_K := \max\{h_{x,K}, h_{y,K}\} \) and \( h := \max_K \{h_K\} \).

Considering the affine mapping \( \mathcal{F}_K : \hat{K} \rightarrow K \) defined by

\[ x = x_K + h_{x,K} \xi, \quad y = y_K + h_{y,K} \eta, \]

we introduce the associated finite element space

\[ V^h := \left\{ v_h : v_h |_{K} = \hat{\nu} \circ \mathcal{F}_K^{-1}, \hat{\nu} \in \hat{P}, v_h(a) = 0, \text{ for any nodal point } a \text{ on } \partial \Omega \right\}. \]

Let \((\cdot, \cdot)\) refer to the inner product on the space \( L^2(\Omega) \) and \( \| \cdot \|_0 \) be the corresponding norm. The weak formulation of (1.1) is: Find \( u(x, t) : (0, T) \rightarrow H^1_0(\Omega) \), such that

\[ \left( D_t^n u(x, t), v \right) + (\nabla u(x, t), \nabla v) = (f(x, t), v) \quad \text{for any } v \in H^1_0(\Omega), \]

\[ u(x, 0) = u_0(x), \quad u_t(x, 0) = \tilde{u}_0(x), \quad x \in \Omega. \]

Consider the uniform partition \( 0 = t_0 < t_1 < \ldots < t_N = T \) of the time interval \([0, T]\) — i.e. the partition with the time step \( \tau = T/N \) so that \( t_n = n \tau, n = 0, 1, \ldots, N \). For a smooth function \( \phi(t) \) on \([0, T]\), we set

\[ \phi^n := \phi(t_n), \quad \phi^{n-1/2} := \frac{1}{2} (\phi^n + \phi^{n-1}), \]

\[ \delta_t \phi^n := \frac{\phi^n - \phi^{n-1}}{\tau}, \quad 1 \leq n \leq N, \quad \delta_t \phi^0 = 0. \]

Moreover, we also use the notation

\[ \varphi(x, t) = u_t(x, t), \]

\[ \dot{D}_t^n u^n = \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} \left[ d_0 \delta_t u^n + \sum_{k=1}^{n-1} d_k^n \delta_t u^k - d_{n-1} \tilde{u}_0 \right] \]

\[ = \dot{D}_t^n u^n - \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} d_{n-1} \tilde{u}_0, \]
where
\[ d_{k}^{0} = d_{n-k} - d_{n-k-1}, \quad 1 \leq k \leq n-1, \]
\[ d_{0}^{n} = -d_{n-1}, \quad d_{n}^{n} = 1, \quad d_{k} = (k+1)^{2-a} - k^{2-a}, \quad 0 \leq k \leq N-1. \]

Taking into account the Crank-Nicolson scheme and the Eqs. (2.1), we write the following fully-discrete scheme: Find \( U^{n} \in V^{h} \) such that
\[
\begin{align*}
\left( B^{t} u^{n} - \frac{\tau^{1-a} d_{n-1}}{\Gamma(3-a)} \tilde{U}_{0}, v_{h} \right) + \left( \nabla_{h} u^{n-1/2}, \nabla_{h} v_{h} \right)_{h} &= \left( f^{n-1/2}, v_{h} \right) \quad \text{for all } v_{h} \in V^{h}, \\
U^{0} &= I_{h} u_{0}(x), \quad \tilde{U}_{0} = I_{h} \tilde{u}_{0}(x), \quad x \in \Omega,
\end{align*}
\]

where \( I_{h} \) is the bilinear finite element interpolation operator and
\[(\nabla_{h} \vartheta(x,y), \nabla_{h} \zeta(x,y))_{h} = \sum_{K} \int_{K} \nabla \vartheta(x,y) \cdot \nabla \zeta(x,y) dx dy.\]

### 3. Auxiliary Results

Here, we provide auxiliary results, used in the study of the stability and the accuracy of the method.

**Lemma 3.1** (cf. Refs. [12, 40]). The coefficients \( d_{k}, 0 \leq k \leq N-1 \) have the following properties:
\[
1 = d_{0} > d_{1} > \ldots > d_{N-1} > 0, \quad (3.1)
\]
\[
(2-a)(k+1)^{1-a} \leq d_{k} \leq (2-a)k^{1-a}. \quad (3.2)
\]

Moreover,
\[
d_{k} \geq (2-a)K^{1-a} \tau^{1-a}. \quad (3.3)
\]

**Lemma 3.2** (cf. Refs. [12, 40]). Consider the terms \( R_{t}^{n-1/2} \) and \( R_{t}^{n-1/2} \) defined by
\[
R_{t}^{n-1/2} := \varphi^{n-1/2} - \delta_{t} u^{n},
\]
\[
R_{t}^{n-1/2} := \frac{1}{2} \left( D_{t}^{a} u^{n} + D_{t}^{a} u^{n-1} \right) - \frac{\tau^{1-a}}{\Gamma(3-a)} \times \left[ d_{0} \varphi^{n-1/2} + \sum_{k=1}^{n-1} (d_{n-k} - d_{n-k-1}) \varphi^{k-1/2} - d_{n-1} \tilde{u}_{0} \right].
\]

If \( u_{tt}(x,t) \in L^{2}(\Omega) \) for any \( t \in (0,T) \), then
\[
\| R_{t}^{n-1/2} \|_{0} \leq C \max_{0 \leq t \leq T} \| u_{tt}(x,t) \|_{0} \tau^{3-a}, \quad (3.3)
\]
\[
\| R_{t}^{n-1/2} \|_{0} \leq C \max_{0 \leq t \leq T} \| u_{tt}(x,t) \|_{0} \tau^{2}, \quad (3.4)
\]

where \( C \) is a constant independent of \( h \) and \( \tau \).
Lemma 3.3 (Bu et al. [6]). If \( \{\xi^n\}_{n=0}^N \) is a sequence of functions on \( \Omega \) such that \( \xi^0 = 0 \), then
\[
\left( \xi^n, \sum_{k=1}^{n-1} d_k \xi^k \right) = -\frac{1}{2} \left( \|\xi^n\|^2 - \sum_{k=0}^{n-1} d_k \|\xi^k\|^2 + \sum_{k=0}^{n-1} d_k \|\xi^k - \xi^n\|^2 \right).
\]

Lemma 3.4 (cf. Shi et al. [36,37]). If \( u \in H^4(\Omega) \), then for \( v_n \in V^h \) the inequalities
\[
(\nabla_h (u - I_h u), \nabla_h v_n)_h \leq Ch^2 |u|_{4} \|v_n\|_h, \quad (3.5)
\]
\[
\|v_n\|_0 \leq C \|v_n\|_{h}, \quad (3.6)
\]
where \( \|\cdot\|_h = (\sum_K |\cdot|^2_{1,K})^{1/2} \), hold.

Lemma 3.5 (cf. Shi et al. [36,37]). Let \( n \) be the unit normal vector on \( \partial K \) and \( u \) the solution of the Eq. (2.1). If \( u \in H^4_0(\Omega) \cap H^4(\Omega) \), then for each \( v_n \in V^h \), the inequality
\[
\left| \sum_K \int_{\partial K} \frac{\partial u}{\partial n} \frac{v_n}{h} ds \right| \leq Ch^3 |u|_{4} \|v_n\|_h \leq Ch^2 |u|_{4} \|v_n\|_0
\]
holds.

4. Stability, Convergence and Superconvergence

We start with the stability of the numerical scheme (2.3).

Theorem 4.1. If \( \{U^n\} \) is the solution of (2.3), then
\[
\|U^n\|_0 + \|U^n\|_h \leq \left( C \|\nabla u_0(x)\|^2_0 + C_1 \|\bar{U}_0\|^2_0 + \max_{t \in [0,T]} \|f(t)\|_0^2 \right)^{1/2}.
\]
where
\[
C_1 = \max \left\{ \frac{2T^{2-a}}{\Gamma(3-a)}, \frac{2T^a \Gamma(3-a)}{2-a} \right\}.
\]

Proof. Substituting \( v_n = \delta_i U^n \) in (2.3) yields
\[
\left( \tilde{D}_i^n U^n - \frac{1-a}{\Gamma(3-a)} \bar{U}_0, \delta_i U^n \right) + (\nabla_h U^{n-1/2}, \nabla_h \delta_i U^n)_h = (f^{n-1/2}, \delta_i U^n),
\]
and taking into account the relation (2.2), we write
\[
\frac{\tau^{1-a}}{\Gamma(3-a)} \left( d_0 \delta_i U^n + \sum_{k=1}^{n-1} d_k \delta_i U^k - d_{n-1} \bar{U}_0, \delta_i U^n \right) + (\nabla_h U^{n-1/2}, \nabla_h \delta_i U^n)_h = (f^{n-1/2}, \delta_i U^n).
\]
Recalling that \( d_0 = 1 \) and using Lemma 3.3, we represent the left-hand side of (4.1) in the form
\[
\frac{\tau^{1-\alpha}}{2\Gamma(3-\alpha)} \left( \|\delta_t U^n\|_0^2 + \sum_{k=0}^{n-1} d_k^n \|\delta_t U^k\|_0^2 - \sum_{k=0}^{n-1} d_k^n \|\delta_t U^n - \delta_t U^k\|_0^2 \right)
\]
\[
- \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} d_{n-1} (\bar{U}_0, \delta_t U^n) + \frac{1}{2\tau} \|\nabla_h U^n\|_0^2 - \frac{1}{2\tau} \|\nabla_h U^{n-1}\|_0^2.
\]  
(4.2)

Set \( \lambda := \tau^{1-\alpha}/\Gamma(3-\alpha) \). Substituting (4.2) into (4.1) and multiplying the Eq. (4.2) by \( 2\tau \), we obtain
\[
\lambda \tau \left( \|\delta_t U^n\|_0^2 + \sum_{k=0}^{n-1} d_k^n \|\delta_t U^k\|_0^2 - \sum_{k=0}^{n-1} d_k^n \|\delta_t U^n - \delta_t U^k\|_0^2 \right)
\]
\[
- 2\lambda \tau d_{n-1} (\bar{U}_0, \delta_t U^n) + \|\nabla_h U^n\|_0^2 - \|\nabla_h U^{n-1}\|_0^2 = 2\tau (f^{n-1/2}, \delta_t U^n).
\]  
(4.3)

Since \( \delta_t U^0 = 0 \), then
\[
- \sum_{k=0}^{n-1} d_k^n \|\delta_t U^n - \delta_t U^k\|_0^2 \geq d_{n-1} \|\delta_t U^n\|_0^2
\]
and using (2.2), (3.1), we obtain from the Eq. (4.3) that
\[
\|\nabla_h U^n\|_0^2 + \lambda \tau \sum_{k=1}^n d_{n-k} \|\delta_t U^k\|_0^2 + \lambda \tau d_{n-1} \|\delta_t U^n\|_0^2
\]
\[
\leq \|\nabla_h U^n\|_0^2 + \lambda \tau \sum_{k=1}^n d_{n-k} \|\delta_t U^k\|_0^2 - \lambda \tau \sum_{k=0}^{n-1} d_k^n \|\delta_t U^n - \delta_t U^k\|_0^2
\]
\[
\leq \|\nabla_h U^{n-1}\|_0^2 + \lambda \tau \sum_{k=1}^{n-1} d_{n-k} \|\delta_t U^k\|_0^2 + 2\lambda \tau d_{n-1} (\bar{U}_0, \delta_t U^n) + 2\tau (f^{n-1/2}, \delta_t U^n).
\]  
(4.4)

Introducing the notation
\[
\varrho^n := \|\nabla_h U^n\|_0^2 + \lambda \tau \sum_{k=1}^n d_{n-k} \|\delta_t U^k\|_0^2,
\]
we rewrite the inequality (4.4) in the form
\[
\varrho^n + \lambda \tau d_{n-1} \|\delta_t U^n\|_0^2 \leq \varrho^{n-1} + 2\lambda \tau d_{n-1} (\bar{U}_0, \delta_t U^n) + 2\tau (f^{n-1/2}, \delta_t U^n).
\]  
(4.5)

The right-hand side of (4.5) can be estimated by the Cauchy-Schwarz and Young’s inequalities, so that
\[
|2\lambda \tau d_{n-1} (\bar{U}_0, \delta_t U^n)| \leq 2\lambda \tau d_{n-1} \|\bar{U}_0\|_0 \|\delta_t U^n\|_0 \leq \lambda \tau d_{n-1} \left( 2\|\bar{U}_0\|_0^2 + \frac{1}{2} \|\delta_t U^n\|_0^2 \right),
\]
\[
|2\tau (f^{n-1/2}, \delta_t U^n)| \leq 2\tau (2\lambda^{-1} d_{n-1}^{-1})^{1/2} \|f^{n-1/2}\|_0 \left( \frac{\lambda d_{n-1}^{-1}}{2} \right)^{1/2} \|\delta_t U^n\|_0
\]
\[
\leq 2\tau \lambda^{-1} d_{n-1}^{-1} \|f^{n-1/2}\|_0^2 + \frac{\lambda \tau d_{n-1}^{-1}}{2} \|\delta_t U^n\|_0^2.
\]  
(4.7)
Subsequent substitutions of (4.6) and (4.7) into (4.5) yield
\[
\varepsilon_n \leq \varepsilon^{n-1} + 2\lambda \tau d_{n-1} ||\hat{U}_0||_0^2 + 2\tau \lambda^{-1} d_{n-1} ||f^{n-1/2}||_0^2 \\
\leq \varepsilon^{n-2} + 2\lambda \tau d_{n-1} ||\hat{U}_0||_0^2 + 2\lambda \tau d_{n-2} ||\hat{U}_0||_0^2 \\
+ 2\tau \lambda^{-1} d_{n-1} ||f^{n-1/2}||_0^2 + 2\tau \lambda^{-1} d_{n-2} ||f^{n-1-1/2}||_0^2 \\
\leq \ldots \leq \varepsilon^0 + 2\lambda \tau \sum_{k=1}^n d_{k-1} ||\hat{U}_0||_0^2 + 2\tau \lambda^{-1} \sum_{k=1}^n d_{k-1} ||f^{k-1/2}||_0^2.
\]

The inequality (3.2) implies
\[
\sum_{k=1}^n d_{k-1} \leq \frac{r^a}{2 - \alpha} \tau^{-\alpha},
\]
and since \(\sum_{k=1}^n d_{k-1} = n^{2-\alpha}\), we obtain
\[
2\lambda \tau \sum_{k=1}^n d_{k-1} = 2\lambda \tau n^{2-\alpha} \leq \frac{2\tau^{2-\alpha}}{\Gamma(3 - \alpha)}.
\]

which leads to the estimate
\[
\varepsilon_n \leq \varepsilon^0 + \frac{2T^{2-\alpha}}{\Gamma(3 - \alpha)} ||\hat{U}_0||_0^2 + \frac{2T^a \Gamma(3 - \alpha)}{2 - \alpha} \max_{t \in [0, T]} ||f(t)||_0^2 + \frac{2T^a \Gamma(3 - \alpha)}{2 - \alpha} \max_{t \in [0, T]} ||f(t)||_0^2.
\]

Recalling that \(\varepsilon^0 = ||\nabla U^0||_0^2\) and using the inequality (3.1), we write
\[
||U^n||_h \leq C||\nabla u_0(x)||_0^2 + \frac{2T^{2-\alpha}}{\Gamma(3 - \alpha)} ||\hat{U}_0||_0^2 + \frac{2T^a \Gamma(3 - \alpha)}{2 - \alpha} \max_{t \in [0, T]} ||f(t)||_0^2,
\]
and the application of the estimate (3.6) leads to the result required.

**Theorem 4.2.** Let \(u(t_n)\) and \(U^n\) be, respectively, the solutions of Eqs. (2.1) and Eqs. (2.3) at the point \(t = t_n\). If \(u \in H^4(\Omega)\), \(u_1 \in H^2(\Omega)\), \(u_{11} \in L^2(\Omega)\), \(D^a u \in H^2(\Omega)\), then for any integer \(n \in [1, N]\), we have
\[
||u^n - U^n||_0 = O(h^2 + \tau^{3-\alpha}),
||U^n - I_h u^n||_h = O(h^2 + \tau^{3-\alpha}),
||u^n - U^n||_h = O(h + \tau^{3-\alpha}).
\]

**Proof.** It follows from the Eqs. (1.1) and (2.3) that
\[
\left(\frac{D^a_t u^n - D^a_t U^n, v_h}{h^{1-\alpha}} + \frac{\nabla_h u^{n-1/2} - \nabla_h U^{n-1/2}, \nabla_h v_h}{h} \right)_h
= \frac{\tau^{1-\alpha}}{\Gamma(3 - \alpha)} d_{n-1}(\hat{u}_0 - \hat{U}_0, v_h) - (R^{n-1/2}, v_h) + \sum_{k} \int_{\partial K} \frac{\partial u^{n-1/2}}{\partial n} v_h ds,
\]
(4.9)
where
\[ R^{n-1/2} = R^{n-1/2}_a + \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} R^{n-1/2}_t - \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} \sum_{k=1}^{n-1} (d_{n-k-1} - d_{n-k}) R^{k-1/2}_t. \]

Setting
\[ \rho^n = u^n - I_h u^n, \quad \theta^n = I_h u^n - U^n \]
and choosing \( v_h = \delta_t \theta^n \) in (4.9) yields
\[ (\tilde{D}^a \theta^n, \delta_t \theta^n) + (\nabla_h \theta^{n-1/2}, \nabla_h \delta_t \theta^n)_h = - (\tilde{D}^a \rho^n, \delta_t \theta^n) - (\nabla_h \rho^{n-1/2}, \nabla_h \delta_t \theta^n)_h \]
\[ + \frac{\tau^{1-\alpha}}{\Gamma(3-\alpha)} d_{n-1} (\tilde{u}_0 - \tilde{U}_0, \delta_t \theta^n) - (R^{n-1/2}, \delta_t \theta^n) + \sum_K \int_{\partial K} \frac{\partial u^{n-1/2}}{\partial n} \delta_t \theta^n ds. \]

Expanding the inner products in the left-hand side of the above expression, we write it as
\[ \lambda ||\delta_t \theta^n||_0^2 + \lambda \sum_{k=1}^{n-1} (d_{n-k} - d_{n-k-1}) (\delta_t \theta^k, \delta_t \theta^n) + \frac{1}{2\tau} ||\nabla_h \theta^n||_0^2 - \frac{1}{2\tau} ||\nabla_h \theta^{n-1}||_0^2 \]
\[ = - (\tilde{D}^a \rho^n, \delta_t \theta^n) - (\nabla_h \rho^{n-1/2}, \nabla_h \delta_t \theta^n)_h + \lambda d_{n-1} (\tilde{u}_0 - \tilde{U}_0, \delta_t \theta^n) \]
\[ - (R^{n-1/2}, \delta_t \theta^n) + \sum_K \int_{\partial K} \frac{\partial u^{n-1/2}}{\partial n} \delta_t \theta^n ds. \] (4.10)

Taking into account Lemma 3.3, we simplify the left-hand side of (4.10) and write it in the form
\[ ||\nabla_h \theta^n||_0^2 + \lambda \tau \sum_{k=1}^{n} d_{n-k} ||\delta_t \theta^k||_0^2 - \lambda \tau \sum_{k=0}^{n-1} d_{n-k} ||\delta_t \theta^k - \delta_t \theta^n||_0^2 \]
\[ = ||\nabla_h \theta^{n-1}||_0^2 + \lambda \tau \sum_{k=1}^{n-1} d_{n-k} ||\delta_t \theta^k||_0^2 - 2\tau (\tilde{D}^a \rho^n, \delta_t \theta^n) \]
\[ - 2\tau (\nabla_h \rho^{n-1/2}, \nabla_h \delta_t \theta^n)_h + 2\lambda \tau d_{n-1} (\tilde{u}_0 - \tilde{U}_0, \delta_t \theta^n) \]
\[ - 2\tau (R^{n-1/2}, \delta_t \theta^n) - 2\tau (R^{n-1/2}, \delta_t \theta^n) + 2\tau \sum_K \int_{\partial K} \frac{\partial u^{n-1/2}}{\partial n} \delta_t \theta^n ds. \]

Set
\[ e^n := ||\nabla_h \theta^n||_0^2 + \lambda \tau \sum_{k=1}^{n} d_{n-k} ||\delta_t \theta^k||_0^2 \]
and employ the inequalities (3.1) thus obtaining the estimate
\[ e^n - e^{n-1} + \lambda \tau d_{n-1} ||\delta_t \theta^n||_0^2 \leq -2\tau (\tilde{D}^a \rho^n, \delta_t \theta^n) - 2\tau (\nabla_h \rho^{n-1/2}, \nabla_h \delta_t \theta^n)_h \]
\[ + 2\lambda \tau d_{n-1} (\tilde{u}_0 - \tilde{U}_0, \delta_t \theta^n) - 2\tau (R^{n-1/2}, \delta_t \theta^n) + 2\tau \sum_K \int_{\partial K} \frac{\partial u^{n-1/2}}{\partial n} \delta_t \theta^n ds. \] (4.11)
We now evaluate each term in the right-hand side of (4.11) starting with $-2\tau(\bar{\rho}_t^n, \bar{\theta}_t^n)$. The Cauchy-Schwartz inequality yields

$$
| -2\tau(\bar{\rho}_t^n, \bar{\theta}_t^n) | \leq 2\tau \| \bar{\rho}_t^n \|_0 \| \bar{\theta}_t^n \|_0 \leq 5\tau d_{n-1} \lambda^{-1} \| \bar{\rho}_t^n \|_0^2 + \frac{d_{n-1}}{5} \lambda \tau \| \bar{\theta}_t^n \|_0^2.
$$

(4.12)

In order to estimate the next term, we introduce a new norm — viz.

$$
\| \theta(t) \|_{L^\infty(H^n)} := \max_{0 \leq t \leq T} \| \theta(t) \|_m.
$$

Taking into account the inequality (3.5) and Lemma 3.5, we obtain

$$
| -2\tau \left( \nabla_h \rho_t^{n-1/2}, \nabla_h \bar{\theta}_t^n \right) | \leq Ch^3 \| u_t^{n-1/2} \|_4 \| \bar{\theta}_t^n \|_0
$$

$$
\leq Ch^4 \lambda^{-1} d_{n-1} \| u \|_{L^\infty(H^4)} + \frac{d_{n-1}}{5} \lambda \| \bar{\theta}_t^n \|_0^2,
$$

(4.13)

$$
2\tau \sum_{\kappa} \int_{\partial K} \frac{\partial u_t^{n-1/2}}{\partial n} \bar{\theta}_t^n d\mathbf{s} \leq Ch^4 \lambda^{-1} d_{n-1} \| u_t^{n-1/2} \|_4 + \frac{d_{n-1}}{5} \lambda \| \bar{\theta}_t^n \|_0^2.
$$

(4.14)

The third term in the right-hand side of (4.11) can be again estimated by the Cauchy-Schwartz inequality — viz.

$$
2\lambda \tau d_{n-1} (\bar{u}_t - \bar{U}_t, \bar{\theta}_t^n) \leq 2\lambda \tau d_{n-1} \| \bar{u}_t - \bar{U}_t \|_0 \| \bar{\theta}_t^n \|_0
$$

$$
\leq 5\lambda \tau d_{n-1} \| \bar{u}_t - \bar{U}_t \|_0^2 + \frac{d_{n-1}}{5} \lambda \| \bar{\theta}_t^n \|_0^2
$$

$$
\leq Ch^4 \lambda \tau d_{n-1} \| \bar{u}_t \|_2 + \frac{d_{n-1}}{5} \lambda \| \bar{\theta}_t^n \|_0^2.
$$

(4.15)

Further, it follows from (3.3) and (3.4) that

$$
| -2\tau (R_{t-1/2}, \bar{\theta}_t^n) | \leq 5\lambda \lambda^{-1} d_{n-1} \| R_{t-1/2} \|_0^2 + \frac{d_{n-1}}{5} \lambda \| \bar{\theta}_t^n \|_0^2
$$

$$
\leq C \tau \lambda^{-1} d_{n-1} \max_{0 \leq t \leq T} \| u_{t,t} (x, t) \|_0^2 \tau^{2(3-\alpha)} + \frac{d_{n-1}}{5} \lambda \| \bar{\theta}_t^n \|_0^2.
$$

(4.16)

Using the estimates (4.12)-(4.16) in (4.11), we arrive at the inequality

$$
e^n \leq e^{n-1} + 5\lambda \lambda^{-1} d_{n-1} \| \bar{\rho}_t^n \|_0^2 + Ch^4 \lambda \lambda^{-1} d_{n-1} \| u \|_{L^\infty(H^4)}^2
$$

$$
+ Ch^4 \lambda \tau d_{n-1} \| \bar{u}_t \|_2 + C \tau \lambda^{-1} d_{n-1} \max_{0 \leq t \leq T} \| u_{t,t} (x, t) \|_0^2 \tau^{2(3-\alpha)}
$$

$$
\leq e^0 + 5\lambda \lambda^{-1} \sum_{k=1}^{n} d_{k-1} \| \bar{\rho}_t^k \|_0^2 + Ch^4 \lambda \lambda^{-1} \sum_{k=1}^{n} d_{k-1} \| u \|_{L^\infty(H^4)}^2
$$

$$
+ Ch^4 \sum_{k=1}^{n} \lambda \tau d_{k-1} \| \bar{u}_t \|_2 + C \lambda^{-1} \tau \sum_{k=1}^{n} d_{k-1} \max_{0 \leq t \leq T} \| u_{t,t} (x, t) \|_0^2 \tau^{2(3-\alpha)}.
$$

(4.17)
The estimate (3.2) yields
\[ \sum_{k=1}^{n} \lambda \tau d_{k-1} \leq \frac{T}{1(3-\alpha)}, \quad \sum_{k=1}^{n} d_{k-1} \leq \frac{T^{\alpha}}{2-\alpha} \tau^{-\alpha}, \]
and, consequently,
\[ 5\lambda^{-1} \tau \sum_{k=1}^{n} d_{k-1}^{\alpha} \rho_{k}^{\alpha} \leq C h^{4} \| \tilde{D}_{\alpha} u \|_{2}^{2}, \quad \text{(4.18)} \]
\[ C h^{4} \lambda^{-1} \tau \sum_{k=1}^{n} d_{k-1} \leq C h^{4} \| u \|_{L_{\infty}(\Omega)}^{2}, \quad \text{(4.19)} \]
\[ C \lambda^{-1} \tau \sum_{k=1}^{n} d_{k-1} \frac{\max_{\alpha \leq t \leq T} \| u_{\alpha t t}(x, t) \|_{0}^{2}}{2(3-\alpha)} \leq C \max_{\alpha \leq t \leq T} \| u_{\alpha t t}(x, t) \|_{0}^{2} \tau^{2(3-\alpha)}, \quad \text{(4.20)} \]
Since \( e^{0} = 0 \), the inequalities (4.17)-(4.20) show that
\[ e^{n} = O(h^2 + \tau^{3-\alpha}). \]

Therefore,
\[ \| \theta^{n} \|_{h} = O(h^2 + \tau^{3-\alpha}). \quad \text{(4.21)} \]

Now we can apply (4.21), (3.6) and known results from interpolation theory to obtain estimates (4.8).

Our next goal is to establish global superconvergence results similar to the ones in [26]. For this, we define an interpolation postprocessing operator. Thus let \( I_{2h} \) be the operator defined on \( \overline{K} \) such that \( I_{2h} u |_{\overline{K}} \in Q_{22}(K) \) for all \( u \in C(\overline{K}) \) and
\[ I_{2h} u(A_i) = u(A_i), \quad i = 1, \ldots, 9, \]
where \( \overline{K} \) consists of four neighboring elements which belong to \( \Gamma_{2h} \) and \( A_i \) are vertices of four small elements — cf. Fig. 1.

![Figure 1: Element \( \overline{K} \).](image)

We note that \( I_{2h} \) satisfies the following relations
\[ I_{2h} I_{h} u = I_{2h} u, \quad \| I_{2h} u - u \|_{1} \leq C h^{2} \| u \|_{3} \quad \text{for all } u \in H^{3}(\Omega), \]
\[ ||I_{2h}v_h||_1 \leq C||v_h||_1 \quad \text{for all } v_h \in Q^h_2, \]

where \( Q^h_2 \) is the double quadratic finite element space and \( C(\bar{K}) \) is the space of continuous functions on \( \bar{K} \). These properties of \( I_{2h} \) are used in the proof of the following result.

**Theorem 4.3.** Under assumptions of Theorem 4.2, we have
\[ ||I_{2h}u^n - u^n||_1 = \Theta(h^2 + \varepsilon^{3-a}). \]

5. **Numerical Results**

In this section, we present results of numerical experiments. Here and in what follows, \( m \) and \( n \) denote, respectively, the numbers of elements in the \( x \)- and \( y \)-directions and \( h = \sqrt{1/m^2 + 1/n^2} \). We will study the errors
\[ \text{err}_1 := ||u^n - U^n||_0, \quad \text{err}_2 := ||u^n - U^n||_h, \]
\[ \text{err}_3 := ||I_hu^n - U^n||_h, \quad \text{err}_4 := ||I_{2h}u^n||_1 \]

and the corresponding convergence rates defined by
\[ \text{Rate} \approx \frac{\log(e_{j+1}/e_{j})}{\log(h_j/h_{j+1})}. \]

**Example 5.1.** Consider the equation
\[ D^\alpha_t u(x, y, t) - \Delta u(x, y, t) = f(x, y, t), \quad (x, y, t) \in \Omega \times (0, T], \]
\[ u(x, y, t) = 0, \quad (x, y, t) \in \partial \Omega \times (0, T], \]
\[ u(x, y, 0) = xy(1-x)(1-y), \quad u_t(x, y, 0) = 0, \quad (x, y) \in \Omega, \]
where \( \Omega = (0, 1) \times (0, 1), T = 1 \) and
\[ f(x, y, t) = \frac{2t^{2-a}}{\Gamma(3-a)} xy(1-x)(1-y) + 2(1+t^2)(x-x^2+y-y^2). \]

It has the solution
\[ u(x, y, t) = (1+t^2)xy(1-x)(1-y). \]

**Example 5.2.** Consider the equation
\[ D^\alpha_t u(x, y, t) - \Delta u(x, y, t) = f(x, y, t), \quad (x, y, t) \in \Omega \times (0, T], \]
\[ u(x, y, t) = 0, \quad (x, y, t) \in \partial \Omega \times (0, T], \]
\[ u(x, y, 0) = 0, \quad u_t(x, y, 0) = 0, \quad (x, y) \in \Omega, \]
where \( \Omega = (0, \pi) \times (0, \pi), T = 1 \) and
\[ f(x, y, t) = \left[ \frac{\Gamma(3+\alpha)}{2} t^3 + 2t^{2+\alpha} \right] \sin x \sin y. \]

It has the solution
\[ u(x, y, t) = t^{2+\alpha} \sin x \sin y. \]
We use two types meshes on the domain $\Omega$ — viz. regular rectangular meshes shown in Fig. 2 and anisotropic rectangular meshes shown in Fig. 3.

Tables 1-2 demonstrate the temporal errors and convergence order for regular meshes. The results are consistent with theoretical analysis — e.g. the convergence rates are close to $3-\alpha$. Moreover, for regular meshes the spatial errors and convergence orders are also consistent with the theoretical analysis — e.g. the convergence rate for each of err1, err3 and err4 is close to $\mathcal{O}(h^2)$ and it is close to $\mathcal{O}(h)$ for err2 — cf. Tables 3-5.

In case of an anisotropic mesh we chose $h = \frac{\pi}{\sqrt{1/20^2 + 1/200^2}}$. As it follows from Table 6, temporal errors and convergence order follow theoretical analysis — e.g. the convergence rates for each of err1, err2, err3 and err4 are close to $3-\alpha$. Similar behaviour can be observed in Tables 7-9, where spatial error and convergence orders are presented for an anisotropic mesh. It is easily seen that the convergence rates for err1, err3 and err4 are close to $\mathcal{O}(h^2)$ and close to $\mathcal{O}(h)$ for err2.

We also display error reduction results in Figs. 2-7.

Table 1: Example 5.1. Temporal errors $\|u^n - U^n\|_h$ and temporal convergence order. $\alpha = 1.95, h \approx \tau^{3-\alpha}$ at $t = 1, 0.8$.

| $\tau$ | t/3 | rate | t/6 | rate | t/12 | rate | t/18 | rate |
|-------|-----|------|-----|------|------|------|------|------|
| t = 1 | 1.185e-1 | / | 5.006e-2 | 1.243 | 2.376e-2 | 1.075 | 1.529e-2 | 1.087 |
| t = 0.8 | 1.008e-1 | / | 4.198e-2 | 1.263 | 1.932e-2 | 1.117 | 1.229e-2 | 1.112 |

Table 2: Example 5.1. Temporal errors and temporal convergence order. $\alpha = 1.95, h^2 \approx \tau^{3-\alpha}, t = 1, 0.8$.

| err | t/20 | rate | t/40 | rate | t/80 | rate | t/160 | rate |
|-----|------|------|------|------|------|------|-------|------|
|     | t = 1 |
| err1 | 1.521e-3 | / | 7.485e-4 | 1.021 | 3.837e-4 | 0.964 | 1.888e-4 | 1.021 |
| err3 | 7.445e-3 | / | 3.651e-3 | 1.029 | 1.860e-3 | 0.973 | 9.108e-4 | 1.029 |
| err4 | 4.403e-3 | / | 2.651e-3 | 0.766 | 1.338e-3 | 0.986 | 6.516e-4 | 1.036 |
|     | t = 0.8 |
| err1 | 1.485e-3 | / | 7.443e-4 | 0.997 | 3.835e-4 | 0.957 | 1.882e-4 | 1.027 |
| err3 | 5.007e-3 | / | 2.416e-3 | 1.050 | 1.223e-3 | 0.983 | 6.009e-4 | 1.025 |
| err4 | 2.518e-3 | / | 1.406e-3 | 0.840 | 6.976e-4 | 1.007 | 3.437e-4 | 1.022 |
Table 3: Example 5.1. Numerical results for $u$ at $t = 0.2, 0.4, 0.6, 0.8$; $\alpha = 1.25$, $\tau = 10^{-3}$.

| $m \times n$ | $8 \times 8$ | rate | $16 \times 16$ | rate | $32 \times 32$ | rate | $64 \times 64$ | rate |
|-------------|--------------|--------|----------------|--------|----------------|--------|----------------|--------|
| $t = 0.2$   |              |        |                |        |                |        |                |        |
| err1        | 6.205e-4     | /      | 1.547e-4       | 2.004  | 3.865e-5       | 2.001  | 9.661e-6       | 2.000  |
| err2        | 1.943e-2     | /      | 9.696e-3       | 1.003  | 4.846e-3       | 1.001  | 2.423e-3       | 1.000  |
| err3        | 2.895e-3     | /      | 7.296e-4       | 1.988  | 1.828e-4       | 1.997  | 4.572e-5       | 1.999  |
| err4        | 2.086e-3     | /      | 5.173e-4       | 2.011  | 1.291e-4       | 2.003  | 3.225e-5       | 2.001  |
| $t = 0.4$   |              |        |                |        |                |        |                |        |
| err1        | 6.977e-4     | /      | 1.742e-4       | 2.002  | 4.354e-5       | 2.000  | 1.088e-5       | 2.000  |
| err2        | 2.168e-2     | /      | 1.082e-2       | 1.003  | 5.405e-3       | 1.001  | 2.702e-3       | 1.000  |
| err3        | 3.200e-3     | /      | 8.055e-4       | 1.990  | 2.017e-4       | 1.998  | 5.045e-5       | 1.999  |
| err4        | 2.292e-3     | /      | 5.670e-4       | 2.015  | 1.414e-4       | 2.004  | 3.532e-5       | 2.001  |
| $t = 0.6$   |              |        |                |        |                |        |                |        |
| err1        | 7.935e-4     | /      | 1.981e-4       | 2.002  | 4.949e-5       | 2.001  | 1.237e-5       | 2.000  |
| err2        | 2.541e-2     | /      | 1.268e-2       | 1.003  | 6.337e-3       | 1.001  | 3.168e-3       | 1.000  |
| err3        | 3.876e-3     | /      | 9.756e-4       | 1.990  | 2.443e-4       | 1.998  | 6.110e-5       | 1.999  |
| err4        | 2.843e-3     | /      | 7.031e-4       | 2.015  | 1.753e-4       | 2.004  | 4.380e-5       | 2.001  |
| $t = 0.8$   |              |        |                |        |                |        |                |        |
| err1        | 9.495e-4     | /      | 2.369e-4       | 2.003  | 5.920e-5       | 2.001  | 1.480e-5       | 2.000  |
| err2        | 3.065e-2     | /      | 1.529e-2       | 1.003  | 7.641e-3       | 1.001  | 3.820e-3       | 1.000  |
| err3        | 4.713e-3     | /      | 1.186e-3       | 1.990  | 2.971e-4       | 1.998  | 7.431e-5       | 1.999  |
| err4        | 3.476e-3     | /      | 8.600e-4       | 2.015  | 2.144e-4       | 2.004  | 5.358e-5       | 2.001  |

Table 4: Example 5.1. Numerical results for $u$ at $t = 0.2, 0.4, 0.6, 0.8$; $\alpha = 1.5$, $\tau = 10^{-3}$.

| $m \times n$ | $8 \times 8$ | rate | $16 \times 16$ | rate | $32 \times 32$ | rate | $64 \times 64$ | rate |
|-------------|--------------|--------|----------------|--------|----------------|--------|----------------|--------|
| $t = 0.2$   |              |        |                |        |                |        |                |        |
| err1        | 5.879e-4     | /      | 1.468e-4       | 2.001  | 3.669e-5       | 2.001  | 9.172e-6       | 2.000  |
| err2        | 1.944e-2     | /      | 9.697e-3       | 1.003  | 4.846e-3       | 1.001  | 2.423e-3       | 1.000  |
| err3        | 3.082e-3     | /      | 7.764e-4       | 1.989  | 1.945e-4       | 1.997  | 4.865e-5       | 1.999  |
| err4        | 2.311e-3     | /      | 5.739e-4       | 2.010  | 1.433e-4       | 2.002  | 3.580e-5       | 2.001  |
| $t = 0.4$   |              |        |                |        |                |        |                |        |
| err1        | 7.616e-4     | /      | 1.900e-4       | 2.003  | 4.748e-5       | 2.001  | 1.187e-5       | 2.000  |
| err2        | 2.168e-2     | /      | 1.082e-2       | 1.003  | 5.405e-3       | 1.001  | 2.702e-3       | 1.000  |
| err3        | 2.904e-3     | /      | 7.328e-4       | 1.987  | 1.836e-4       | 1.997  | 4.594e-5       | 1.999  |
| err4        | 1.911e-3     | /      | 4.747e-4       | 2.009  | 1.185e-4       | 2.002  | 2.961e-5       | 2.001  |
| $t = 0.6$   |              |        |                |        |                |        |                |        |
| err1        | 8.211e-4     | /      | 2.052e-4       | 2.000  | 5.130e-5       | 2.000  | 1.282e-5       | 2.000  |
| err2        | 2.541e-2     | /      | 1.268e-2       | 1.003  | 6.337e-3       | 1.001  | 3.168e-3       | 1.000  |
| err3        | 3.737e-3     | /      | 9.398e-4       | 1.991  | 2.353e-4       | 1.998  | 5.884e-5       | 1.999  |
| err4        | 2.670e-3     | /      | 6.591e-4       | 2.018  | 1.643e-4       | 2.005  | 4.103e-5       | 2.001  |
| $t = 0.8$   |              |        |                |        |                |        |                |        |
| err1        | 9.384e-4     | /      | 2.342e-4       | 2.002  | 5.854e-5       | 2.001  | 1.463e-5       | 2.000  |
| err2        | 3.065e-2     | /      | 1.529e-2       | 1.003  | 7.641e-3       | 1.001  | 3.820e-3       | 1.000  |
| err3        | 4.771e-3     | /      | 1.200e-3       | 1.991  | 3.006e-4       | 1.998  | 7.517e-5       | 1.999  |
| err4        | 3.548e-3     | /      | 8.769e-4       | 2.016  | 2.186e-4       | 2.004  | 5.461e-5       | 2.001  |
Table 5: Example 5.1 Numerical results for \( u \) at \( t = 0.2, 0.4, 0.6, 0.8; \) \( \alpha = 1.75, \tau = 10^{-3} \).

| \( m \times n \) | \( 8 \times 8 \) rate | \( 16 \times 16 \) rate | \( 32 \times 32 \) rate | \( 64 \times 64 \) rate |
|------------------|------------------|------------------|------------------|------------------|
| \( t = 0.2 \)    |                  |                  |                  |                  |
| err1             | 5.508e-4 / 1.375e-4 | 2.002 / 3.436e-5 | 2.000 / 8.590e-6 | 2.000 / 10.000 |
| err2             | 1.948e-2 / 9.702e-3 | 1.005 / 4.846e-3 | 1.001 / 2.423e-3 | 1.000 / 1.000 |
| err3             | 3.652e-3 / 9.100e-4 | 2.004 / 2.286e-4 | 1.993 / 5.719e-5 | 1.999 / 1.999 |
| err4             | 2.994e-3 / 7.348e-4 | 2.027 / 1.841e-4 | 1.996 / 4.599e-5 | 2.001 / 2.001 |
| \( t = 0.4 \)    |                  |                  |                  |                  |
| err1             | 7.798e-4 / 1.951e-4 | 1.999 / 4.875e-5 | 2.001 / 1.219e-5 | 2.000 / 2.000 |
| err2             | 2.169e-2 / 1.082e-2 | 1.004 / 5.405e-3 | 1.001 / 2.702e-3 | 1.000 / 1.000 |
| err3             | 2.910e-3 / 7.283e-4 | 1.998 / 1.823e-4 | 1.998 / 4.558e-5 | 2.000 / 2.000 |
| err4             | 1.961e-3 / 4.742e-4 | 2.048 / 1.179e-4 | 2.008 / 2.942e-5 | 2.002 / 2.002 |
| \( t = 0.6 \)    |                  |                  |                  |                  |
| err1             | 9.515e-4 / 2.372e-4 | 2.004 / 5.922e-5 | 2.002 / 1.480e-5 | 2.001 / 2.000 |
| err2             | 2.543e-2 / 1.268e-2 | 1.003 / 6.337e-3 | 1.001 / 3.168e-3 | 1.000 / 1.000 |
| err3             | 3.155e-3 / 7.985e-4 | 1.982 / 2.005e-4 | 1.994 / 5.018e-5 | 1.998 / 1.998 |
| err4             | 1.887e-3 / 4.741e-4 | 1.993 / 1.190e-4 | 1.994 / 2.979e-5 | 1.998 / 1.998 |
| \( t = 0.8 \)    |                  |                  |                  |                  |
| err1             | 9.851e-4 / 2.470e-4 | 1.996 / 6.180e-5 | 1.999 / 1.545e-5 | 2.000 / 2.000 |
| err2             | 3.065e-2 / 1.529e-2 | 1.003 / 7.641e-3 | 1.001 / 3.820e-3 | 1.000 / 1.000 |
| err3             | 4.533e-3 / 1.135e-3 | 1.997 / 2.840e-4 | 1.999 / 7.100e-5 | 2.000 / 2.000 |
| err4             | 3.255e-3 / 7.976e-4 | 2.029 / 1.984e-4 | 2.007 / 4.953e-5 | 2.002 / 2.002 |

Convergence curve for crank–nicolson scheme with \( \alpha = 1.5, \tau = 10^{-3} \) at \( t = 0.2 \)

Convergence curve for crank–nicolson scheme with \( \alpha = 1.5, \tau = 10^{-3} \) at \( t = 0.4 \)

Figure 4: Example 5.1. Error reduction results on regular rectangular meshes.

6. Conclusions

We introduce an unconditionally stable fully-discrete scheme on regular and anisotropic meshes for two-dimensional TFWE based on quasi-Wilson finite elements in spatial direction and the Crank-Nicolson scheme in temporal direction and derive superclose results and global superconvergence in the broken \( H^1 \)-norm. In addition, numerical simulations based on finite element method for anisotropic meshes are presented. The results of numerical experiments are consistent with theoretical analysis.
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Table 6: Example 5.2. Temporal errors and temporal convergence order at $t = 1, 0.1; \alpha = 1.95$.

| $\tau$ | $t/3$ rate | $t/6$ rate | $t/12$ rate | $t/18$ rate |
|--------|------------|------------|-------------|-------------|
| $t = 1$ |
| err1  | 9.960e-2  | 4.525e-2  | 2.107e-2  | 1.103  |
| err2  | 1.729e-1  | 7.891e-2  | 3.738e-2  | 1.077  |
| err3  | 1.731e-1  | 7.902e-2  | 3.712e-2  | 1.150  |
| err4  | 1.734e-1  | 7.910e-2  | 3.713e-2  | 1.091  |
| $t = 0.1$ |
| err1  | 1.436e-5  | 6.034e-6  | 2.719e-6  | 1.151  |
| err2  | 2.496e-5  | 1.051e-5  | 4.805e-6  | 1.129  |
| err3  | 2.499e-5  | 1.053e-5  | 4.782e-6  | 1.139  |
| err4  | 2.503e-5  | 1.054e-5  | 4.784e-6  | 1.139  |

Table 7: Example 5.2. Numerical results for $u$ at $t = 0.3, 0.5, 0.9, 1; \alpha = 1.1, \tau = 10^{-3}$.

| $m \times n$ | $2 \times 20$ rate | $4 \times 40$ rate | $8 \times 80$ rate | $16 \times 160$ rate |
|-------------|------------------|------------------|------------------|------------------|
| $t = 0.3$ |
| err1  | 4.033e-4  | 8.854e-5  | 2.186  | 2.021  |
| err2  | 1.951e-3  | 8.780e-4  | 1.149  | 1.030  |
| err3  | 1.259e-3  | 3.516e-4  | 1.840  | 1.967  |
| err4  | 1.702e-3  | 4.664e-4  | 1.868  | 1.987  |
| $t = 0.5$ |
| err1  | 2.060e-3  | 4.447e-4  | 2.212  | 2.041  |
| err2  | 9.283e-3  | 4.273e-3  | 1.119  | 1.026  |
| err3  | 5.908e-3  | 1.658e-3  | 1.833  | 1.966  |
| err4  | 8.023e-3  | 2.235e-3  | 1.844  | 1.987  |
| $t = 0.9$ |
| err1  | 1.338e-2  | 2.868e-3  | 2.222  | 2.051  |
| err2  | 5.657e-2  | 2.636e-2  | 1.102  | 1.023  |
| err3  | 3.412e-2  | 9.657e-3  | 1.821  | 1.963  |
| err4  | 4.674e-2  | 1.343e-2  | 1.800  | 1.986  |
| $t = 1$ |
| err1  | 1.872e-2  | 4.015e-3  | 2.221  | 2.052  |
| err2  | 7.827e-2  | 3.652e-2  | 1.100  | 1.022  |
| err3  | 4.657e-2  | 1.320e-2  | 1.818  | 1.963  |
| err4  | 6.393e-2  | 1.850e-2  | 1.789  | 1.986  |
Table 8: Example 5.2. Numerical results for $u$ at $t = 0.3, 0.5, 0.9, 1 \alpha = 1.4, \tau = 10^{-3}$.

| $m \times n$ | $2 \times 20$ | $4 \times 40$ | $8 \times 80$ | $16 \times 160$ |
|--------------|---------------|---------------|---------------|---------------|
|              | rate          | rate          | rate          | rate          |
| $t = 0.3$    |               |               |               |               |
| err1         | 2.558e-4      | 5.817e-5      | 2.137         | 1.487e-5      | 1.967         | 3.754e-6      | 1.986         |
| err2         | 1.472e-3      | 6.206e-4      | 1.246         | 3.007e-4      | 1.045         | 1.495e-4      | 1.008         |
| err3         | 9.074e-4      | 2.526e-4      | 1.845         | 6.490e-5      | 1.961         | 1.670e-5      | 1.958         |
| err4         | 1.222e-3      | 3.303e-4      | 1.887         | 8.353e-5      | 1.983         | 2.121e-5      | 1.978         |
| $t = 0.5$    |               |               |               |               |
| err1         | 1.581e-3      | 3.487e-4      | 2.181         | 8.602e-5      | 2.019         | 2.143e-5      | 2.005         |
| err2         | 7.769e-3      | 3.485e-3      | 1.156         | 1.706e-3      | 1.031         | 8.489e-4      | 1.007         |
| err3         | 5.034e-3      | 1.405e-3      | 1.841         | 3.598e-4      | 1.965         | 9.134e-5      | 1.978         |
| err4         | 6.798e-3      | 1.855e-3      | 1.874         | 4.683e-4      | 1.986         | 1.180e-4      | 1.989         |
| $t = 0.9$    |               |               |               |               |
| err1         | 1.249e-2      | 2.683e-3      | 2.219         | 6.496e-4      | 2.046         | 1.610e-4      | 2.013         |
| err2         | 5.540e-2      | 2.559e-2      | 1.114         | 1.258e-2      | 1.025         | 6.262e-3      | 1.006         |
| err3         | 3.521e-2      | 9.891e-3      | 1.831         | 2.534e-3      | 1.965         | 6.391e-4      | 1.987         |
| err4         | 4.785e-2      | 1.336e-2      | 1.840         | 3.372e-3      | 1.986         | 8.463e-4      | 1.994         |
| $t = 1$      |               |               |               |               |
| err1         | 1.807e-2      | 3.871e-3      | 2.222         | 9.354e-4      | 2.049         | 2.317e-4      | 2.013         |
| err2         | 7.899e-2      | 3.659e-2      | 1.110         | 1.799e-2      | 1.024         | 8.960e-3      | 1.006         |
| err3         | 4.968e-2      | 1.398e-2      | 1.829         | 3.582e-3      | 1.964         | 9.031e-4      | 1.988         |
| err4         | 6.763e-2      | 1.900e-2      | 1.831         | 4.795e-3      | 1.986         | 1.203e-3      | 1.995         |

Table 9: Example 5.2. Numerical results for $u$ at $t = 0.3, 0.5, 0.9, 1 \alpha = 1.6, \tau = 10^{-3}$.

| $m \times n$ | $2 \times 20$ | $4 \times 40$ | $8 \times 80$ | $16 \times 160$ |
|--------------|---------------|---------------|---------------|---------------|
|              | rate          | rate          | rate          | rate          |
| $t = 0.3$    |               |               |               |               |
| err1         | 1.863e-4      | 4.259e-5      | 2.129         | 1.148e-5      | 1.891         | 3.087e-6      | 1.895         |
| err2         | 1.276e-3      | 4.994e-4      | 1.354         | 2.369e-4      | 1.076         | 1.176e-4      | 1.010         |
| err3         | 7.233e-4      | 2.018e-4      | 1.840         | 5.282e-5      | 1.934         | 1.460e-4      | 1.855         |
| err4         | 9.715e-4      | 2.619e-4      | 1.891         | 6.695e-5      | 1.968         | 1.775e-5      | 1.915         |
| $t = 0.5$    |               |               |               |               |
| err1         | 1.305e-3      | 2.949e-4      | 2.146         | 7.407e-5      | 1.993         | 1.869e-5      | 1.987         |
| err2         | 7.055e-3      | 3.050e-3      | 1.209         | 1.486e-3      | 1.036         | 7.392e-4      | 1.007         |
| err3         | 4.469e-3      | 1.247e-3      | 1.842         | 3.223e-4      | 1.952         | 8.486e-5      | 1.925         |
| err4         | 6.022e-3      | 1.632e-3      | 1.884         | 4.140e-4      | 1.979         | 1.065e-4      | 1.959         |
| $t = 0.9$    |               |               |               |               |
| err1         | 1.193e-2      | 2.581e-3      | 2.209         | 6.273e-4      | 2.041         | 1.552e-4      | 2.015         |
| err2         | 5.480e-2      | 2.510e-2      | 1.126         | 1.232e-2      | 1.027         | 6.133e-3      | 1.006         |
| err3         | 3.560e-2      | 9.968e-3      | 1.836         | 2.561e-3      | 1.961         | 6.560e-4      | 1.965         |
| err4         | 4.819e-2      | 1.327e-2      | 1.860         | 3.356e-3      | 1.984         | 8.494e-4      | 1.982         |
| $t = 1$      |               |               |               |               |
| err1         | 1.765e-2      | 3.801e-3      | 2.215         | 9.210e-4      | 2.045         | 2.276e-4      | 2.017         |
| err2         | 7.965e-2      | 3.665e-2      | 1.120         | 1.800e-2      | 1.026         | 8.961e-3      | 1.006         |
| err3         | 5.141e-2      | 1.442e-2      | 1.834         | 3.702e-3      | 1.961         | 9.455e-4      | 1.969         |
| err4         | 6.969e-2      | 1.930e-2      | 1.853         | 4.877e-3      | 1.984         | 1.232e-3      | 1.985         |
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