Abstract. Not only the Dirac operator, but also the spinor bundle of a pseudo-Riemannian manifold depends on the underlying metric. This leads to technical difficulties in the study of problems where many metrics are involved, for instance in variational theory. We construct a natural finite dimensional bundle, from which all the metric spinor bundles can be recovered including their extra structure. In the Lorentzian case, we also give some applications to Einstein-Dirac-Maxwell theory as a variational theory and show how to coherently define a maximal Cauchy development for this theory.

Keywords. spin geometry, spinor bundle, jet spaces, natural constructions, Einstein-Dirac-Maxwell equation, Cauchy development

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1. Introduction

Spinor bundles are an important tool in differential geometry as well as in mathematical physics, where they model fermionic particles. As evident from their construction, spinor bundles depend on the underlying metric $g$. This makes it difficult to compare spinor fields $\psi^g$ and $\psi^h$ in spinor bundles formed with respect to different metrics $g$ and $h$. The same problem occurs if one wants to compare the Dirac operators $D^g$ and $D^h$, since their domains of definition are not the same.

A way out of this dilemma is provided by systematically constructing identification isomorphisms between the spinor bundles formed with respect to different metrics. In the Riemannian case, this has been carried out by Bourguignon and Gauduchon in [BG92]. Although using these identification isomorphisms is sufficient for many applications, one might wonder if and how it is possible to define a finite dimensional bundle that is independent of the metric, but nevertheless captures the features of the spinor bundles formed with respect to all the various metrics.

1.1. Statement of the results

In the present article, we propose the following answer to this question: Let $(M, \Theta)$ be a smooth spin manifold of dimension $m$ and $\Theta : \tilde{\text{GL}}^+ M \to \text{GL}^+ M$ be a topological spin structure on $M$. Let $\tau_M : TM \to M$ be the tangent bundle of $M$ and for any $r + s = m$, let $\pi_{r,s} : S_{r,s} M \to M$ be the subbundle of $\tau^*_M \otimes \tau^*_M : T^*M \otimes T^*M \to M$, which over each point $x \in M$ consists of all non-degenerate symmetric bilinear forms $g_x$ on $T_x M$ of signature $(r, s)$. Here, $r$ denotes the dimension of a maximal positive definite subspace, $s$ of a maximal negative definite subspace. We define $S_{r,s}(M) := \Gamma(\pi_{r,s})$ to be the space of all metrics on $M$ of signature $(r, s)$. For any metric $g \in S_{r,s}(M)$, we obtain a metric spinor bundle

$$\pi^g_M : \Sigma^g M := \text{Spin}^g M \times_{\rho} \Sigma_{r,s} M \to M,$$

where $\rho := \rho_{r,s} : \text{Spin}_{r,s} \to \text{GL}(\Sigma_{r,s})$ is a complex spinor representation, $\Theta^g : \text{Spin}^g M \to \text{SO}^g M$ is a metric spin structure and $\text{Spin}_{r,s} \subset \tilde{\text{GL}}^+ m$ is the spin group.

**Main Theorem 1 (universal spinor bundle).** There exists a natural finite dimensional vector bundle $\bar{\pi}^g_{S_M} : \Sigma M \to J^1 \pi^{r,s}$ such that for each metric $g \in S_{r,s}(M)$, there exists $\bar{\iota}_g$ such that

$$
\begin{array}{ccc}
\Sigma^g M & \xrightarrow{\bar{\iota}_g} & \Sigma M \\
\pi^g_M & \downarrow & \pi^g_{S_M} \\
M & \xrightarrow{j^1(\iota)} & J^1\pi^{r,s}
\end{array}
$$

commutes. Here, $J^1\pi^{r,s}$ denotes the first jet bundle of $\pi^{r,s}$. Moreover, $\bar{\pi}^g_{S_M}$ carries a connection, a metric and a Clifford multiplication over $(\pi^g_{0})^*(TM)$ such that $\bar{\iota}_g$ is a morphism of (generalized) Dirac bundles, see Definition 2.28.

The last claim of the assertion means that not only the vector bundle structure of any metric spinor bundle $\pi^g_M$ can be recovered from $\bar{\pi}^g_{S_M}$, but also its spinorial connection, its metric and its Clifford multiplication, see Theorem 2.26 for the precise meaning and a proof.

The rest of this paper is organized as follows: In Section 2.1, we construct a preliminary version $\bar{\pi}^g_{S_M} : \Sigma M \to S_{r,s} M$ of $\bar{\pi}^g_{S_M}$ without using the theory of jet spaces and in Section 2.2, we show that this almost does the job, see Lemma 2.16. In Section 2.3, we introduce some basic notions about jet
spaces, which will then be applied in Section 2.4 to obtain the bundle $\bar{\kappa}_{SM}$ from Main Theorem 1, see Theorem 2.22. We will then show in Section 2.5 that the additional structures constructed in Section 2.2 descend to the jet bundle and give a proof of Theorem 2.26, which directly implies Main Theorem 1. Finally, in Section 2.6 we give the claim that the universal spinor bundle is natural a precise category theoretic meaning and a proof, see Theorem 2.33. We also show that there is no way to define a spinor bundle like (1.1) that is natural under spin diffeomorphisms, see Theorem 2.36.

In Section 3, we indicate how the universal spinor bundle can be used to reformulate the Einstein-Dirac-Maxwell equation as a variational problem for a functional defined on the Fréchet space of sections of a finite dimensional bundle, which makes the equations accessible to techniques like the Palais-Smale condition and Morse theory. Moreover, we show how to define a maximal Cauchy development for this theory, obtaining the following result:

**Main Theorem 2 (Cauchy development for Einstein-Dirac-Maxwell).** There exists a maximal Spin$Mf$-Cauchy development, see Definition 3.17, for the Einstein-Dirac-Maxwell equation.

As the whole construction is natural in the category of spin manifolds with topological spin structures and spin diffeomorphisms, we obtain a spin-topological field theory, i.e. a well-defined action of the group of spin diffeomorphisms on the solution space, whose moduli space could be further examined.

### 1.2. State of the art

The problem how to deal with the technical issues resulting from the fact that the spinor bundle depends on the metric has been approached in various ways. In the Riemannian case, our construction in Definition 2.4 agrees with the classical construction by Bourguignon and Gauduchon in [BG92]; in fact our construction is inspired by this very article.

Its main ingredient is the partially defined vertical connection, see Definition 2.11, which is already enough to construct identification isomorphisms $\beta_{g,h} : \Sigma^g M \to \Sigma^h M$ between the various spinor bundles, see Definition 3.5. In the Riemannian case and in its non-jetted version, the universal spinor bundle $\pi_{SM}$ has been used recently by Ammann, Weiss and Witt to define and study a spinorial energy functional, c.f. [AWW12; AWW14].

One can also construct the identification isomorphisms $\beta_{g,h}$ without the partial connection. The map $b_{g,h}$, which will be constructed in (3.4), induces a map on the frame bundle $GL^+ M$, which can be lifted to the topological spin structure $\tilde{GL}^+ M$, see [Mai97] for that approach. Using this construction, one can even show that the identification isomorphisms $\beta_{g,h}$ themselves depend $C^1$-continuously on the metric, hence providing a topology for the Hilbert bundle

$$L^2(\Sigma M) := \bigsqcup_{g \in R(M)} L^2(\Sigma^g M) \to S_{m,0}(M) \quad \psi \in L^2(\Sigma^g M) \mapsto g,$$

see [Now15, Chapter 4] for a detailed discussion. But since the base and the fiber of that bundle are both infinite dimensional, the space of sections of this bundle does not have a canonical Fréchet space topology, which makes it difficult to do calculus in this space. This becomes much easier in the space of sections of $\pi_{SM}$ from (2.9) respectively $\bar{\pi}_{SM}$ from (2.32).

There is an alternative approach to identify the spinor spaces for two pseudo-Riemannian metrics on $M$, which is conceptually different: For any path $(g_t)_{t \in I}$, one considers the metric $dt^2 + g_t$ on
the generalized cylinder $Z := [0, 1] \times M$. Parallel translation along $t \mapsto (t, x)$, $x \in M$, gives an identification between $g_0$ and $g_1$. Pulling back the Dirac operator using the resulting identification isomorphisms yields a local coordinate formula similar to (3.5), see [BGM05, Thm. 5.1]. This reduces the problem to finding a path between two metrics. Unfortunately, this is not always possible. Moreover, ambiguities appear due to the different choices of connecting paths. For Lorentz metrics, this has been discussed in detail in [BGM05, Sect. 9].

In particular in case of spacetimes, i.e. globally hyperbolic (1, 3) (or (3, 1)) manifolds, one can also use a trivial bundle as a spinor bundle and recover the metric from the operator, see [Fin98]. However, to study the initial value problem of the Einstein-Dirac equation, we need a stronger notion of naturality of the situation. We will discuss this problem in Section 3.

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2. A universal spinor bundle

In this section, we review the classical approach by Bourguignon and Gauduchon in [BG92]. We reformulate and slightly generalize their results to metrics of arbitrary signature and show that the resulting vector bundle already carries a lot of additional structure.

2.1. Construction of the bundle

It turns out that the construction of the universal spinor bundle is easier, if we first consider the case of a real vector space. So, let $V$ be an oriented real $m$-dimensional vector space and let $g$ be a metric on $V$. We will use the following notation from linear algebra:

- $S_{r,s} V := \text{space of inner products of signature } (r, s) \text{ on } V$,
- $\text{GL}^+_m := \{ A \in \mathbb{R}^{m \times m} \mid \det(A) > 0 \}$
- $\widetilde{\text{GL}}^+_m := \text{non-trivial double cover of } \text{GL}^+_m$
- $\text{GL}^+ V := \text{principal } \text{GL}^+_m\text{-bundle of positive bases of } V$
- $\langle \cdot, \cdot \rangle_{r,s} := \text{standard metric of signature } (r, s) \text{ on } \mathbb{R}^m$,
- $I_m := \text{unit matrix in } \mathbb{R}^{m \times m}$
- $I_{r,s} := \text{diagonal matrix with } r \text{ entries of } +1 \text{ followed by } s \text{ entries of } -1$;
- $A^\dagger := I_{r,s} A^T I_{r,s}$, the adjoint with respect to $\langle \cdot, \cdot \rangle_{r,s}$
- $\text{SO}_{r,s} := \{ A \in \text{GL}^+_m \mid AA^\dagger = I_m \}$, $\text{SO}_m := \text{SO}_{m,0}$
- $S_{r,s} := \{ A \in \mathbb{R}^{m \times m} \mid A \text{ is symmetric and has } r \text{ positive and } s \text{ negative eigenvalues} \}$.

We denote the action $a$ of a matrix $A \in \text{GL}^+_m$ on a basis $b \in \text{GL}^+ V$ by $a(b, A) := b.A := b'$, where $b'_j = A^j_i b_i$. We recall that for any metric $g \in S_{r,s} V$, there exists a positive pseudo-orthonormal basis $b$, i.e. a positive basis $b$ such that

\[ \forall 1 \leq i, j \leq m : g(b_i, b_j) = \delta_{ij} \varepsilon_i, \quad \varepsilon_i = (+1, \ldots, +1, -1, \ldots, -1). \]

Conversely, any positive basis $b$ of $V$ determines a metric $g_b$ by declaring $b$ to be a pseudo-orthonormal basis, i.e. by setting $g_b(b_i, b_j) := \delta_{ij} \varepsilon_i$. Assume that $b' = b.A$ for some matrix...
Step 2, which is smooth. Consequently, here,

$$\kappa^r \in \text{GL}^+_m,$$

i.e. \( b'_j = A_j^r b_i, 1 \leq j \leq m \). Clearly, \( b \) and \( b' \) determine the same metric, if and only if for any \( 1 \leq i, j \leq m \)

\[
\delta_{ij} \hat{\varepsilon}_i = g_{\nu}(b_i, b_j) = \sum_{\nu, \mu} g_{\nu}((A^{-1})_i^\nu b'_\mu, (A^{-1})_j^\nu b'_\mu) = \sum_{\nu, \mu} (A^{-1})_i^\nu (A^{-1})_j^\nu \delta_{\mu \nu} \varepsilon_{\nu} = \sum_{\nu, \mu} ((A^{-1})^T)^s_{i \nu} (I_{r,s})_{\nu}^m ((A^{-1})^T)^j_{\nu} = ((A^{-1})^T I_{r,s} (A^{-1}))^i_{j}.
\]

(2.1)

In other words,

\[ g_{\nu} = g_{\nu} \iff I_{r,s} = A I_{r,s} A^T \iff A A^\dagger = I_m \iff A \in \text{SO}_{r,s} . \]

Lemma 2.1. There is a commutative diagram

\[
\begin{array}{ccc}
\text{GL}^+ V & \xrightarrow{\kappa^r} & S_{r,s} V \\
\downarrow{\phi} & & \downarrow{\zeta_{r,s} := s_0 \circ \kappa^r \circ \tau_b^{-1}} \\
\text{GL}^+ V / \text{SO}_{r,s} & \xrightarrow{\phi} & S_{r,s} .
\end{array}
\]

(2.3)

where, \( \kappa^r(b) := g_{\nu} \) and \( q \) is the canonical quotient map. Moreover, \( q \) and \( \kappa^r \) are smooth principal \( \text{SO}_{r,s} \)-bundles and \( \phi \) is a diffeomorphism.

\( \diamond \)

Proof.

Step 1 (\( \kappa^r \) is smooth): We fix a basis \( b \in \text{GL}^+ V \) and consider the commutative diagram

\[
\begin{array}{ccc}
\text{GL}^+ V & \xrightarrow{\tau_b} & \text{GL}^+_m \\
\downarrow{\kappa^r} & & \downarrow{\kappa^r := s_0 \circ \kappa^r \circ \tau_b^{-1}} \\
S_{r,s} V & \xrightarrow{s_b} & S_{r,s} .
\end{array}
\]

(2.4)

Here, \( \tau_b \) and \( s_b \) are the coordinate diffeomorphisms induced by \( b \), i.e.

\[ \forall 1 \leq i, j \leq m : b'_j = \tau_b(b'_j)b_i, \quad \forall 1 \leq i, j \leq m : s_b(g) = g(b_i, b_j). \]

(2.5)

For any \( b' \in \text{GL}^+ V, A := \tau_b(b') \), we obtain from the definitions of \( \tau, s, \kappa^r, A \) and from (2.1)

\[ s_b(\kappa^r(b'))^i_j = \kappa^r(b')(b_i, b_j) = g_{\nu}((A^{-1})_i^\nu b'_\mu, (A^{-1})_j^\nu b'_\mu) = ((A^{-1})^T I_{r,s} (A^{-1}))^i_j. \]

Consequently,

\[ \zeta_{r,s}(A) = \zeta_{r,s}(\tau_b(b')) = (A I_{r,s} A^T)^{-1}, \]

which is smooth.

Step 2 (derivative of \( \zeta_{r,s} \)): Let \( c : I \to \text{GL}^+_m \) be a curve such that \( c(0) =: A \) and consider

\[
d_a \zeta_{r,s}(\hat{c}(0)) = \frac{d}{dt} (\zeta_{r,s} \circ c)(t) |_{t=0} = \frac{d}{dt} (c(t) I_{r,s} c(t)^T)^{-1} |_{t=0} = -(A I_{r,s} A^T)^{-1} \frac{d}{dt} c(t) I_{r,s} c(t)^T |_{t=0} (A I_{r,s} A^T)^{-1}
\]

(2.6)

\[ = -\zeta_{r,s}(A) (A I_{r,s} A^T + A I_{r,s} c(0)^T) \zeta_{r,s}(A). \]

Step 3 (ker \( d\kappa^r \)): Now, if \( b \in \text{GL}^+ V \), then \( \tau_b(b) = I_m \), so if \( k : I \to \text{GL}^+ V \) is a curve with \( k(0) = b \), then \( c := \tau_b \circ k \) is a curve with \( c(0) = I_m \), and we obtain

\[
k(0) \in \ker d\kappa^r \iff d_b k^r(\hat{k}(0)) = 0 \iff 0 = d_b (s_0 \circ \kappa^r)(\hat{k}(0)) = d_b (\zeta_{r,s} \circ \tau_b)(\hat{k}(0))
\]

\[ \iff \hat{c}(0) \in \ker d \zeta_{r,s} \iff \hat{c}(0)^T I_{r,s} + I_{r,s} \hat{c}(0) = 0
\]

\[ \iff \hat{c}(0)^T = -\hat{c}(0) \iff \hat{c}(0) \in T_{I_m} \text{SO}_{r,s}
\]

\[ \iff \hat{k}(0) \in T_b (b, \text{SO}_{r,s}) = T_b q^{-1}(q(b)),
\]
therefore
\[ \ker ds^V = \ker dq. \]  

**Step 4** (q is a principal bundle): Clearly, the action of \( SO_{r,s} \) on \( \text{GL}^+ V \) obtained as restriction of \( a \) is smooth and free. Clearly, the right multiplication of \( SO_{r,s} \) on \( \text{GL}^+_m \) is proper. The map \( \tau_0 \) is \( \text{GL}^+_m \)-equivariant, since for any \( b' \in \text{GL}^+ V, C \in \text{GL}^+_m, 1 \leq j \leq m \), we obtain from (2.5)
\[
(b'C)_j = C_j^u b'_u = C_j^u A_j^u b_j = A_j^u C_j^u b_j = (AC)^j b_j, \quad \implies \tau_0(b'C) = \tau_0(b')C.
\]

Thus, the action of \( SO_{r,s} \) on \( \text{GL}^+ V \) is also proper, for instance by the criterion given in [Lee03, Prop. 9.13]. By the quotient manifold theorem, see for instance [Lee03, 9.16], \( q \) is a smooth submersion and also a principal \( SO_{r,s} \)-bundle by standard results, see e.g. [KN63; Ban06].

**Step 5** (construction of \( \varphi \)): By (2.2), the map \( \kappa^V \) is constant on the fibers of \( q \). Hence, we may pass smoothly to the quotient and obtain existence and smoothness of \( \varphi \) such that (2.3) commutes, see for instance [Lee03, Prop. 7.18]. For any metric \( g \in S_{r,s} V \), there exists a pseudo-orthonormal basis by the pseudo-Gram-Schmidt process, see Theorem A.2, thus \( \kappa^V \) is surjective. Therefore, \( \varphi \) is surjective and by (2.2) it is also injective. Let \( 0 \neq v \in T \text{GL}^+ V/\text{SO}_{r,s} \) be any vector and let \( w \in T \text{GL}^+ V \) such that \( dq(w) = v \), thus \( w \notin \ker dq \) and therefore \( w \notin \ker d\kappa^V \) by (2.7). We calculate
\[
d\varphi(v) = d\varphi dq(w) = d\kappa^V(w) \neq 0,
\]
thus \( d\varphi \) is injective. Thus \( \varphi \) is a diffeomorphism.

All in all, this proves the claim. \( \square \)

**Remark 2.2.** In the Riemannian case, the bundle \( \kappa^V \) admits a global section obtained by fixing any basis \( b \in \text{GL}^+ V \) and setting \( s : S_{m,0} V \to \text{GL}^+ V, g \mapsto GS_V(b,g) \), where \( GS_V \) is the Gram-Schmidt process, see Theorem A.1. Consequently, \( \kappa^V \) is globally trivial then. The same is not possible in the non-Riemannian case. \( \diamond \)

**Remark 2.3.** Let \( \vartheta_{r,s} : \text{Spin}_{r,s} \to \text{SO}_{r,s} \) be the non-trivial \( 2 : 1 \)-cover and \( \theta : \text{GL}^+ V \to \text{GL}^+ V \) be the universal cover of \( \text{GL}^+ V \). Then we can extend (2.3) to
\[
\begin{array}{ccc}
\text{GL}^+ V & \xrightarrow{\theta} & \text{GL}^+ V & \xrightarrow{\kappa^V} & S_{r,s} V \\
\downarrow & & \downarrow & & \downarrow \\
\text{GL}^+ V/\text{Spin}_{r,s} & \xrightarrow{\cong} & \text{GL}^+ V/\text{SO}_{r,s}.
\end{array}
\]

We set \( \tilde{\kappa}^V := \kappa^V \circ \theta \). \( \diamond \)

The diagram (2.3) easily extends to spin manifolds as follows: Let \( \{\rho_{r,s} : \text{Spin}_{r,s} \to \text{GL}(\Sigma_{r,s})\}_{r,s \in \mathbb{N}} \) be a fixed choice of \( \text{Spin}_{r,s} \)-representations. The construction of \( \kappa^V \) and \( \tilde{\kappa}^V \) in Lemma 2.1 respectively Remark 2.3 induces bundle maps
\[
\kappa^M : \text{GL}^+ M \to S_{r,s} M, \quad \tilde{\kappa}^M : \text{GL}^+ M \to S_{r,s} M,
\]
by setting \( \kappa^M|_{\text{GL}^+_M} := \kappa^{T_x M} \) and \( \tilde{\kappa}^M|_{\text{GL}^+_M} := \tilde{\kappa}^{T_x M} \) for any \( x \in M \).

**Definition 2.4 (universal spinor bundle).** The map
\[
\pi\Sigma_{SM} : \Sigma M \to S_{r,s} M \quad [\vec{b}, \sigma] \mapsto \tilde{\kappa}^M(\vec{b}),
\]
where $\Sigma M := \mathring{\text{GL}}^+ M \times_{\rho_{r,s}} \Sigma_{r,s}$ is called **universal spinor (vector) bundle**. We obtain the following diagram

$$
\begin{array}{ccc}
\Sigma M & \xrightarrow{\pi^\Sigma_M} & S_{r,s} M \\
\pi^\Sigma_M & \downarrow & \pi^\Sigma_M \\
& & M,
\end{array}
$$

(2.9)

where $\pi^\Sigma_M := \pi_{r,s} \circ \pi^\Sigma_{SM}$. The map $\pi^\Sigma_M$ is called **universal spinor (fiber) bundle** of $M$. Its sections are called **universal spinor fields**.

Remark 2.5 (universal spinor fields). The relationship between the sections of the metric spinor bundle $\pi^g_M : \Sigma^g M \rightarrow M$ and the universal spinor bundle (2.9) is as follows: Let $\Phi \in \Gamma(\pi^\Sigma_M)$ be a universal spinor field. Then

$$
g := g_\Phi := \pi^\Sigma_M \circ \Phi \in S_{r,s}(M),
$$

$$
\varphi := \varphi_\Phi := \Phi \in \Gamma(\pi^g_M),
$$

defines a metric on $M$ and an associated spinor field in $\Gamma(\pi^g_M)$. This follows from the fact that if $\varphi(x) = [\tilde{b}, v] \in \Sigma_x M, x \in M$, we have that $\Theta(\tilde{b})$ is pseudo-orthonormal with respect to $\kappa^M(\Theta(\tilde{b})) = \kappa^M(\Phi(x)) = g$, thus $\tilde{b} \in \text{Spin}^g M$, thus $\varphi \in \Gamma(\pi^g_M)$. Conversely, any tuple $(g, \varphi)$, $\varphi \in \Gamma(\pi^g_M)$, defines a universal spinor field by setting $\Phi := t_g \circ \varphi$, where $t_g : \Sigma^g M \rightarrow \Sigma M$ denotes the canonical inclusion.

Notation 2.6. For any fiber bundle $\pi_P : P \rightarrow M$, we denote its space of sections by $\Gamma(\pi_P)$ instead of $\Gamma(P)$, because we will deal frequently with doubly fibered bundles as in (2.9) where the projection will be important. We denote by $\tau_P : TP \rightarrow P$ the tangent space of the bundle and by $\tau^\Sigma = T^\Sigma P \rightarrow P$ its vertical subbundle.

2.2. Existence of universal structures on the bundle

We show that the principal $\text{SO}_{r,s}$-fiber bundle $\kappa^V : \text{GL}^+ V \rightarrow S_{r,s} V$ carries much more structure than just being a fiber bundle.

Remark 2.7 (metric). Denote by

$$
\langle \_, \_ \rangle : \mathbb{R}^{m \times m} \times \mathbb{R}^{m \times m} \rightarrow \mathbb{R} \quad (A, B) \mapsto \text{tr}(A^\dagger B),
$$

the metric on $\mathbb{R}^{m \times m}$ and define

$$
\text{Sym}_{r,s} := \{ A \in \mathbb{R}^{m \times m} | A^\dagger = A \},
$$

$$
\text{Asym}_{r,s} := \{ A \in \mathbb{R}^{m \times m} | A^\dagger = -A \}.
$$

An elementary calculation using $\text{tr} \circ (\cdot^\dagger) = \text{tr}$ shows that the natural decomposition

$$
\mathbb{R}^{m \times m} = \text{Asym}_{r,s} \oplus \text{Sym}_{r,s},
$$

(2.10)

is orthogonal with respect to $\langle \_, \_ \rangle$.

Lemma 2.8 (natural metric and connection). Using the coordinate diffeomorphism $\tau_\tilde{b}$ from (2.4), we define

$$
\forall b \in \text{GL}^+ V : \forall X, Y \in T_b \text{GL}^+ V : (X, Y)_b := \langle d\tau_\tilde{b}X, d\tau_\tilde{b}Y \rangle.
$$
Then the \((\_,\_)_b\) assemble to a pseudo-Riemannian metric on \(\text{GL}^+ V\) such that \(\text{SO}_{r,s}\) acts by isometries. In particular, setting
\[
\forall b \in \text{GL}^+ V : T_b^0 \text{GL}^+ V := \ker d_b \kappa V, \quad T^h \text{GL}^+ V := (T_b^0 \text{GL}^+ V)^\perp,
\]
defines an orthogonal decomposition such that \(T^h \text{GL}^+ V\) is a connection on \(\text{GL}^+ V\). This decomposition satisfies
\[
d_b \tau_b (T_b^0 \text{GL}^+ V) = \text{Asym}_{r,s}, \quad d_b \tau_b (T_b^0 \text{GL}^+ V) = \text{Sym}_{r,s},
\]
and thus corresponds to (2.10).

**Proof.** Since \(\tau_b\) is the restriction of the smooth map
\[
\text{GL}^+ V \times \text{GL}^+ V \to \text{GL}^+ m_{\tau_b(b')} \quad (b, b') \mapsto \tau_b(b'),
\]
to the smooth submanifold \(\{b\} \times \text{GL}^+ V\) and thus is smooth itself, we obtain that the \((\_,\_)_b\) constitute a metric on \(\text{GL}^+ V\). For any \(S \in \text{SO}_{r,s}\) and any \(A, B \in T_m \text{GL}^+_m = \mathbb{R}^{m \times m}\), we obtain
\[
\langle AS, BS \rangle = \text{tr}((AS)^\dagger BS) = \text{tr}(S^\dagger A^\dagger BS) = \text{tr}(A^\dagger B SS^\dagger) = \text{tr}(A^\dagger B) = \langle A, B \rangle,
\]
thus the right multiplication action of \(\text{SO}_{r,s}\) on \(\text{GL}^+_m\) is an action by isometries. Since \(\tau_b\) is \(\text{SO}_{r,s}\)-equivariant, this implies that \(\text{SO}_{r,s}\) acts on \(\text{GL}^+ V\) by isometries as well. The last claim follows from the fact that \(T_m \text{SO}_{r,s} = \text{Asym}_{r,s}\) and from (2.10).

**Lemma 2.9.** Let \(I : V \to W\) be an orientation-preserving isomorphism of oriented vector spaces. Then
\[
\begin{array}{ccc}
\text{GL}^+ V & \xrightarrow{\text{GL}^+ I} & \text{GL}^+ W \\
\kappa^V & & \kappa^W \\
S_{r,s} V & \xrightarrow{(I^{-1})^\dagger} & S_{r,s} W
\end{array}
\]
commutes and \(\text{GL}^+ I\) is an \(\text{SO}_{r,s}\)-equivariant isometric diffeomorphism.

**Proof.** For any \(b \in \text{GL}^+ V\) and any \(1 \leq i, j \leq m\),
\[
(I^{-1})^* (\kappa^V)(b)(I(b_i), I(b_j)) = \kappa^V(b)(b_j) = \varepsilon_i \delta_{ij} = \kappa^W(I(b_j))(I(b_i)),
\]
which proves that the two forms agree on the basis \((\text{GL}^+ I)(b) = (I(b_1), \ldots, I(b_m)) \in \text{GL}^+ W\).
This proves the commutativity of the diagram. For any \(b, b' \in \text{GL}^+ V\), we have
\[
b' = b. \tau_b(b') \iff I(b') = I(b). \tau_b(I(b')),
\]
thus \(\tau_b(b') = \tau_b(I(b'))\). Therefore, as \((\_,\_)_b\) is defined via pullback by \(\tau\),
\[
\forall b \in \text{GL}^+ V : \forall X, Y \in T \text{GL}^+_b V : \langle db(\text{GL}^+ I)X, db(\text{GL}^+ I)Y \rangle_{I(b)} = \langle X, Y \rangle_b
\]
and \(\text{GL}^+ I\) is an isometry. For any \(S \in \text{SO}_{r,s}\), \(b \in \text{GL}^+ V\)
\[
(\text{GL}^+ I)(b.S)_j = I(S^\dagger_j b_v) = S^\dagger_j I(b_v) = (\text{GL}^+ I(b).S)_j,
\]
thus \(\text{GL}^+ I\) is \(\text{SO}_{r,s}\)-equivariant.
Definition 2.10. Let $\Theta^V : \tilde{GL}^+ V \to GL^+ V$ be a universal cover. It follows automatically that $\tilde{GL}^+ V$ is a pseudo-Riemannian manifold such that $\Theta^V$ is a local isometry. Therefore, we obtain a connection on $\tilde{GL}^+ V$ by taking the orthogonal complements again. In this manner, $\tilde{\kappa}^V : GL^+ V \to S_{r,s} V$ becomes a principal Spin$_{r,s}$-bundle with a pseudo-Riemannian metric and a connection

$$TGL^+ V = T^v GL^+ V \oplus T^h \tilde{GL}^+ V.$$  \hspace{1cm} (2.12)

The connection $T^h \tilde{GL}^+ V$ is called the \textit{Bourguignon-Gauduchon horizontal distribution}. \hfill \Diamond

These constructions carried out on a vector space can be generalized to a spin manifold as follows.

Definition 2.11 (vertical connection). For any $x \in M$, we define

$$T^v_x GL^+ M := T^v GL^+ (T_x M), \quad T^v_x GL^+ M := T^h GL^+ (T_x M),$$  \hspace{1cm} (2.13)

and analogously for $\tilde{GL}^+ M$. The resulting decomposition $T^v GL^+ M = T^v GL^+ M \oplus T^h GL^+ M$ is called a \textit{vertical distribution} on $\kappa^M : GL^+ M \to S_{r,s} M$ (and analogously on $\tilde{\kappa}^M : \tilde{GL}^+ M \to S_{r,s} M$). We denote by

$$\nabla : \Gamma(\tau^g_{S_{r,s} M}) \times \Gamma(\pi^\Sigma_{SM}) \to \Gamma(\pi^\Sigma_{SM})$$  \hspace{1cm} (2.14)

the induced \textit{vertical connection}, i.e. the connection induced on $\pi^\Sigma_{SM}$ from (2.9) that is only defined for all directions in the vertical space $\tau^g_{S_{r,s} M} : T^v S_{r,s} M \to M$. \hfill \Diamond

In the Riemannian case, this connection agrees with the original construction in [BG92]. We now show that this bundle carries a lot more structure than just being a bundle with a partially defined connection. We denote by

$$\tau^g_{SM} := (\pi^r)^*(TM) \to S_{r,s} M$$

the pullback of the tangent bundle of $M$ to $S_{r,s} M$ by $\pi^r$. \hfill \Diamond

Definition 2.12 (universal metric and Clifford multiplication). We define

$$\eta(\phi, \phi') := g_x (\phi, \phi'), \quad m(X^* \otimes \phi) := X^* \bullet \phi := V \cdot g_x \phi,$$  \hspace{1cm} (2.15, 2.16)

where $\phi, \phi' \in \Sigma M|g_x$, $X^* = (g_x, V) \in (\pi^r)^*(TM)$, $V \in T_x M$. The map $\eta$ is called \textit{universal spinorial metric} and $m$ is called \textit{universal Clifford multiplication} on $\pi^\Sigma_{SM} : \Sigma M \to S_{r,s} M$. \hfill \Diamond

One might wonder to what extent $\nabla$, $\eta$ and $m$ are compatible with each other. To see that, first recall how the analogous structures on the metric spinor bundle $\pi^g_M : \Sigma^g M \to M$ are related to each other.

Remark 2.13 (Dirac structure on the metric spinor bundle). Recall that a metric spinor bundle $\pi^g_M : \Sigma^g M \to M$ carries a canonical structure as a \textit{Dirac bundle}, i.e. there exists an extension of the Levi-Civita connection to the spinor bundle, also denoted by $\nabla^g : \Gamma(\pi^g_M) \to \Gamma(\tau^g_M \otimes \pi^g_M)$, an extension of the metric $g$ to the spinor bundle, also denoted by $g$ and a \textit{Clifford multiplication}, i.e. a morphism of real vector bundles $m^g : TM \otimes \Sigma^g M \to \Sigma^g M$, $V \otimes \psi \mapsto V \cdot g \psi$. \hfill \Diamond
In addition, these structures satisfy the following compatibility relations (see for instance [Bau81, Th. 1.12]):

\[-2g(V, W)\psi = V \cdot g W \cdot g \psi + W \cdot g V \cdot g \psi, \tag{2.17}\]
\[\nabla_{\psi}^{\pi} g(\psi, \psi') = g(\nabla_{\psi}^{\pi} \psi, \psi') + g(\psi, \nabla_{\psi}^{\pi} \psi'), \tag{2.18}\]
\[\nabla_{\psi}^{\pi} (W \cdot g \psi) = \nabla_{\psi}^{\pi} (W \cdot g \psi) + W \cdot g \nabla_{\psi}^{\pi} \psi, \tag{2.19}\]
\[g(V \cdot g \psi, \psi' \cdot g) = (-1)^{s+1} g(\psi, V \cdot g \psi'), \tag{2.20}\]
where \(\psi, \psi' \in \Gamma(\pi^g_M), V, W \in \Gamma(\tau_M).\) \(\Diamond\)

The universal structures \(\nabla, \eta\) and \(m\) satisfy compatibility relations similar to Eqs. (2.18) to (2.20). In order to be able to precisely formulate them, we need the following notion.

**Definition 2.14 (universal metric and vertical Levi-Civita connection).** The metric \(g\) defined by

\[g(X^*, X^*) := g_x(X, X), \tag{2.21}\]

where \(X^* = (g_x, V) \in (\pi^{r,s})^*(TM)\) is called **universal metric**. Let \(\pi^+ : GL^+ M \to M\) the canonical projection. Via the pullback isomorphism

\[
\begin{array}{ccc}
GL^+ M & \xrightarrow{id} & GL^+ M \\
\kappa^M & \downarrow & \downarrow \pi^+ \\
(\pi^{r,s})^*(GL^+ M) & \xrightarrow{\kappa^M} & \pi^{r,s} M \\
\end{array}
\]

we also obtain a vertical connection on \(\kappa^M\) from the vertical connection on \(\kappa^M\) from Definition 2.11. Let \(\gamma : GL^+ M \to GL(\mathbb{R}^m)\) be the standard representation (given by matrix multiplication). Recall that \(GL^+ M \times_\gamma \mathbb{R}^m = TM\), so \((\pi^{r,s})^*(TM) = (\pi^{r,s})^*(GL^+ M) \times_\gamma \mathbb{R}^m\) and therefore, we obtain a vertical connection on \(\tau^M_M\), which is denoted by

\[\nabla : \Gamma(\tau^M_{r,s}) \times \Gamma(\tau^M_{r,s}) \to \Gamma(\tau^M_{r,s}). \tag{2.22}\]

We call \(\nabla\) the **vertical universal Levi-Civita connection**. \(\Diamond\)

**Lemma 2.15 (properties of universal structures).** The universal structures satisfy the compatibility conditions

\[-2g(X^*, Y^*) = X^* \cdot Y^* \cdot \psi + Y^* \cdot X^* \cdot \psi, \tag{2.23}\]
\[\nabla_X(\eta(\phi, \phi')) = \eta(\nabla_X \phi, \phi') + \eta(\phi, \nabla_X \phi'), \tag{2.24}\]
\[\nabla_X(Y^* \cdot \phi) = \nabla_X Y^* \cdot \phi + Y^* \cdot \nabla_X \phi, \tag{2.25}\]
\[\eta(X^* \cdot \phi, \phi') = (-1)^{s+1} \eta(\phi, X^* \cdot \phi). \tag{2.26}\]

where \(X^*, Y^* \in \Gamma(\tau^M_M), X \in \Gamma(\tau^M_{S_{r,s}} M), \phi, \phi' \in \Gamma(\pi^\Sigma_{SM}).\) \(\Diamond\)

**Proof.** (2.23) follows directly from (2.17) and (2.21).

By construction (2.15), the metric \(\eta\) agrees pointwise with the spinorial metric, which is defined via the invariant metric on the spinor space \(\Sigma_{r,s}\) in the spinor representation \(\rho : \text{Spin}_{r,s} \to GL(\Sigma_{r,s}).\)
The representation $\rho$ acts on $\Sigma_{r,s}$ by isometries. Since the vertical connection on $\pi_{SM}^r$ is induced by the vertical connection on $\tilde{\kappa}_M$, the resulting vertical connection is metric. For connections this can be found for instance in [Bau09, Satz 3.13] and the same proof holds for vertical connections. This proves (2.24).

To see (2.25), we just notice that $\nabla \otimes \nabla$ is a connection on $(\pi^r)^*(TM) \otimes \Sigma M$ and $m$ is parallel with respect to this connection. This results from the fact that the ordinary Clifford multiplications $m^g$ are defined via the representation $\rho$.

Finally, (2.26) follows from the definition of $\eta$, $\bullet$ and (2.20).

The relationship between the usual Dirac structure on the metric spinor bundle $\pi^g_M : \Sigma^g M \to M$ from Remark 2.13 and the universal structures on $\pi_{SM}^\Sigma : \Sigma M \to S_{r,s} M$ is as follows.

**Lemma 2.16.** For any metric $g$, there exists a morphism $I_g$ of vector bundles such that

\[
g^* \eta \circ I_g = \eta \circ \iota_g,
\]

where $\psi, \psi' \in \Gamma(\pi^g_M), V \in TM$.

**Proof.** For any $x \in M$, $\psi \in \Sigma^g_x M$, we have $(\pi^g_M \circ \iota_g)(\psi) = g_x = (g \circ \pi^g_M)(\psi)$, thus the desired isomorphism exists by the universal property of the pullback. To see the second claim, we check that

\[
(g^* \eta)(\iota_g(\psi), \iota_g(\psi')) = \eta(\iota_g(\psi), \iota_g(\psi')) = g(\psi, \psi'),
\]

\[
(g^* m)(V \otimes \iota_g(\psi)) = m((g, V) \otimes \iota_g(\psi)) = V \cdot_g \psi = m^g(V \otimes \psi),
\]

Notice that $I_g$ is not compatible with the connections, since we would have to check that $(g^* \nabla)_V(I_g(\psi)) = \nabla_{dgV}(\iota_g(\psi)) = \nabla_{dgV}^g \psi$, which makes no sense, since $dgV$ is not vertical, so $\nabla_{dgV}$ is not defined.

We will discuss this issue in the following.

2.3. Jet spaces

The fact that the connections $\nabla$ and $\nabla$ are defined only on the vertical part of $\pi^{r,s} : S_{r,s} M \to M$ is a bit unpleasant, especially since this prevents us from including them into the compatibility assertion of Lemma 2.16. In this and the next subsection, we show how to obtain a full connection by passing to the first jet space. We consider the first jet bundle, since the spinorial Levi-Civita connection on a spinor bundle depends only on the 1-jet of the metric.

We recall the definition of a jet space, see [Sau89] for a more comprehensive introduction to the topic.
Definition 2.17 (jet bundle). Let \( \pi_P : P \to M \) be any fiber bundle. For any \( x \in M \), denote by \( \Gamma_x(\pi_P) \) the space of sections defined on a local neighborhood near \( x \). Two such sections \( s_1, s_2 \) have the same 1-jet at \( x \) if
\[
s_1(x) = s_2(x) \in P, \quad ds_1|_{T_xM} = ds_2|_{T_xM}.
\]
The equivalence class \( j^1_x(s) \) of a local section \( s \in \Gamma_x(\pi_P) \) is the 1-jet of \( s \) at \( x \). The set
\[
J^1\pi_P := \{ j^1_x(s) \mid x \in M, s \in \Gamma_x(\pi_P) \}
\]
is the first jet space of \( \pi_P \). The space \( J^1\pi_P \) comes along with two canonical projections
\[
\pi_0 : J^1\pi_P \to M \quad \text{and} \quad \pi_1 : J^1\pi_P \to P,
\]
called the source respectively target projection.

It is well known that \( J^1\pi_P \) is a smooth manifold and \( \pi_{1,0} \) is even an affine bundle. One can also define higher jet bundles \( J^k\pi_P \) consisting of equivalence classes of sections that agree up to the \( k \)th-derivative.

Definition 2.18. Let \( \pi_P : P \to M \) be a bundle, \( x \in M, s \in \Gamma_x(\pi_P) \) and \( V \in T_xM \). Then
\[
(j^1_x(s), d_x s(V)) \in \pi_{1,0}^*(TP)
\]
is called the holonomic lift of \( V \) at \( s \).

The following theorem asserts that one can decompose \( \pi_{1,0}^*(TP) \) into vertical vectors and holonomic lifts.

Theorem 2.19 ([Sau89, Thm. 4.3.2]). Let \( \pi_P : P \to M \) be a fiber bundle and \( \pi_{1,0} : J^1\pi_P \to P \) be the target projection. Recall that \( \tau^P_P : T^vP \to P \) denotes the vertical tangent bundle of \( P \). At any point \( j^1_x(s), s \in \Gamma(\pi_P) \), there exists a natural decomposition
\[
\pi_{1,0}^*(TP)|_{j^1_x(s)} = \pi_{1,0}^*(T^vP)|_{j^1_x(s)} \oplus (j^1_x(s), ds(T_xM)).
\]
This decomposition is well-defined, i.e. it does not depend on the choice of \( s \) for a given \( j^1_x(s) \).

Remark 2.20. Notice that the bundles \( T^1\pi_P \) and \( \pi_{1,0}^*(TP) \) are related as follows: By the universal property of the pullback, there exists \( \psi \) such that
\[
\begin{array}{ccc}
TJ^1\pi_P & \xrightarrow{d\pi_{1,0}} & TP \\
\downarrow \psi & & \downarrow \tau_P \\
\pi_{1,0}^*(TP) & \xrightarrow{\pi_{1,0}} & TP \\
\downarrow \tau_{J^1\pi_P} & & \downarrow \pi_{1,0} \\
J^1\pi_P & \xrightarrow{\pi_{1,0}} & P.
\end{array}
\]
This map is explicitly given by \( \psi(\bar{X}) = (\tau_{J^1\pi_P} \bar{X}, d\pi_{1,0} \bar{X}) \). Consequently, for any \( \bar{X} \in T_{j^1_x(s)}J^1\pi_P \), we obtain a decomposition of \( \psi(\bar{X}) \) as in (2.28). Setting \( X := d\pi_{1,0} \bar{X} \), this is can be written explicitly as
\[
X = X^v \oplus X^h \in T^vP \oplus d_x g(T_xM),
\]
i.e. there exists \( X_h \in T_xM \) such that \( X^h = d_x g(X_h) \). Notice that \( X_h = d_{j^1_x(s)} \pi_0 \bar{X} \).
2.4. Construction of the jet version

Before we can state Theorem 2.22, we recall some facts about pullbacks.

Remark 2.21 (description of pull-back connections). Let \( \pi_E : E \to M \) be a vector bundle and \( f : N \to M \) be smooth. Then we denote by \( f^*E \to N \) the pullback, i.e.

\[
\begin{array}{ccc}
f^*E & \xrightarrow{f} & E \\
\downarrow \pi_E & & \downarrow \pi_E \\
N & \xrightarrow{f} & M.
\end{array}
\]

Recall that \( f^*E = \{ (y, e) \in N \times E \mid f(y) = \pi_E(e) \} \). Therefore, a section \( s \in \Gamma(\pi_E) \) determines a section \( \bar{s} = (\bar{s}_1, \bar{s}_2) \in \Gamma(\bar{\pi}_E) \) always satisfies \( \text{id}_N = \bar{\pi}_E \circ \bar{s} = \bar{s}_1 \) and \( (\pi_E \circ f^*)(\bar{s}) = \pi_E \circ \bar{s}_2 = f \) and is therefore already determined by \( \bar{s}_2 \). We call a map \( s : N \to E \) such that \( \pi_E \circ s = f \) a section along \( f \) and identify the space \( \Gamma(\pi_E) \circ f \) of sections along \( f \) with with \( \Gamma(\bar{\pi}_E) \).

In addition, assume that \( E \) carries a connection \( \nabla \). Then we can define

\[
\forall X \in TN : \forall s \in \Gamma(\pi_E) : \bar{D}_X(s \circ f) := \nabla_{f^*X}s.
\]

The result is an \( \mathbb{R} \)-bilinear map \( \bar{D} : \Gamma(\tau_N) \times (\Gamma(\pi_E) \circ f) \to (\Gamma(\pi_E) \circ f) \) such that for all \( \bar{X} \in TN, s \in \Gamma(\pi_E) \),

(i) we have the Leibniz rule

\[
\forall \beta \in C^\infty(M) : \bar{D}_X(\beta s \circ f) = \bar{X}(\beta \circ f)(s \circ f) + (\beta \circ f)\bar{D}_X(s \circ f),
\]

(ii) and the \( C^\infty \)-linearity

\[
\forall \alpha \in C^\infty(N) : \bar{D}_{\alpha X}(s \circ f) = \alpha \bar{D}_X(s \circ f).
\]

It is a standard result in differential geometry that any such map \( \bar{D} \) defines a unique connection \( \bar{\nabla} : \Gamma(\tau_N) \times \Gamma(\bar{\pi}_E) \to \Gamma(\bar{\pi}_E) \) satisfying

\[
\bar{\nabla}_X(\text{id}_N, s \circ f) = (\text{id}_N, \bar{D}_X(s \circ f)). \tag{2.31}
\]

We sometimes write \( s \circ f \) instead of \( (\text{id}_N, s \circ f) \), since it makes no distinction if we use the pullback connection on \( \Gamma(\bar{\pi}_E) \).

With this in mind, we can continue (2.9) (drawn vertically) as follows.

Theorem 2.22 (universal spinor jet bundle). The universal spinor bundle from (2.9) can be extended to a commutative diagram

\[
\begin{array}{ccc}
\bar{\Sigma}M & \xrightarrow{F^\Sigma} & \Sigma M \\
\downarrow \bar{\pi}_M & & \downarrow \pi_M \\
J^1\pi^r,s & \xrightarrow{\pi^r_{\Sigma M}} & S_{r,s}M \\
\downarrow \pi^r_M & & \downarrow \pi_M \\
M & \xrightarrow{\text{id}_M} & M.
\end{array}
\]

\[
\tag{2.32}
\]
Moreover, the vector bundle $\bar{\pi}^\Sigma_{SM}$ carries a connection $\nabla$ satisfying

$$F^\Sigma(\nabla_X \bar{\phi}|_{j^1_2(g)}) = \nabla X \cdot \bar{\phi}|_{g(x)} + \nabla^\partial_x (\phi \circ g)|_x,$$

(2.33)

where $\bar{\phi} := (\text{id}, \pi^r_{1,0} \circ \phi) \in \Gamma(\bar{\pi}^\Sigma_{SM})$, $\phi \in \Gamma(\pi^\Sigma_{SM})$, $\bar{X} \in T\Sigma_{j^1_2(g)}(J^1\pi^r)$, $X := d\pi^r_{1,0} \bar{X}$ and $X^v$ and $X_h$ are a decomposition as in (2.30).

**Proof.** The strategy is to define the connection on $\bar{\Sigma}M$ using Remark 2.21. We define $\bar{\Sigma}M := (\pi^r_{1,0})^* \Sigma M$ and obtain commutativity of (2.32). Using the notation from the assertion and (2.30), we obtain a decomposition

$$X = X^v \oplus X^h \in T^v S_{r,s} M \oplus d_x g(T_x M),$$

(2.34)
i.e. there exists $X_h \in T_x M$ such that $X^h = d_x g(X_h)$. Recall from Remark 2.5 that $\phi \circ g \in \Gamma(\pi^\Sigma_{SM})$ and that the connection $\nabla^\partial$ on the spinor bundle at a point $x \in M$ depends only on $j^1_2(g)$ at $x$, so the right hand side of (2.33) is well-defined. Therefore, we simply define $D_X \phi$ by the right hand side of (2.33). To show that this gives a connection on $\bar{\pi}^\Sigma_{SM}$, it remains only to verify the properties of $\bar{D}$ enlisted in Remark 2.21. It is clear that $\bar{D}$ is $\mathbb{R}$-bilinear in both arguments. To see the Leibniz rule, let $\beta \in C^\infty(S_{r,s} M)$ and calculate

$$\bar{D}_X((\beta \phi) \circ \pi^r_{1,0})|_{j^1_2(g)} = \nabla X^v (\beta \phi)|_{g(x)} + \nabla^\partial_x (\beta \phi \circ g)|_x = \nabla X^v (\beta \phi)|_{g(x)} + \nabla X^v (\phi)|_{g(x)} + (\beta \circ g) \nabla^\partial_x (\phi \circ g)|_x$$

$$= (X^v(\beta) + dg(X_h)(\beta))|_{g(x)} \phi|_{g(x)} + \beta \nabla X^v (\phi)|_{g(x)} + (\beta \circ g) \nabla^\partial_x (\phi \circ g)|_x$$

$$= X(\beta)|_{g(x)} \phi|_{g(x)} + (\beta \circ g)|_x \nabla_\phi (\phi \circ g)|_x$$

$$= \bar{X}(\beta \circ \pi^r_{1,0})|_{j^1_2(g)} \phi|_{g(x)} + (\beta \circ \pi^r_{1,0})|_{j^1_2(g)} \bar{D}_X (\phi \circ \pi^r_{1,0})|_{j^1_2(g)}.$$}

To see the $C^\infty$-linearity, let $\alpha \in C^\infty(J^1\pi^r)$. We obtain $d\pi^r_{1,0} (\alpha \bar{X}) = \alpha d\pi^r_{1,0} \bar{X}$ and clearly $dg(\alpha(j^1_1(g))X_h) = \alpha(j^1_1(g))X^h$, thus

$$\bar{D}_{\alpha(j^1_2(g))}X (\phi \circ \pi^r_{1,0}) = \nabla_{\alpha(j^1_2(g))}X \cdot \phi + \nabla^\partial_{\alpha(j^1_2(g))} \phi|_{g(x)}$$

$$= \alpha(j^1_1(g))(D_X (\phi \circ \pi^r_{1,0})),
$$

which concludes the proof. $\square$

### 2.5. Existence of universal structures on the jet version

We will now show that the structures $g$, $\eta$, $m$ can also be pulled back to the jet bundle $\bar{\pi}^\Sigma_{SM}$ and satisfy compatibility conditions with the connection $\bar{\nabla}$ similar to Lemma 2.15. This also allows us to formulate a version of Lemma 2.16, which includes the connection.

**Definition 2.23 (universal Dirac structure).** Consider the vector bundle $\bar{\pi}^\Sigma_{SM} : \Sigma M \to J^1\pi^r$ from (2.32). We define

$$\mathfrak{g}(\bar{X}^*, \bar{Y}^*) := g_x (V, W),$$

(2.35)

$$\bar{\eta}_{j^1_2(g)}(\bar{\phi}, \bar{\phi}') := \eta_{j^1_1(g)}(\phi, \phi'),$$

(2.36)

$$\bar{m}(X^*, \bar{\phi}) := X^* \cdot \bar{\phi} := V \cdot g_x \phi,$$

(2.37)

where $j^1_1(g) \in J^1\pi^r$, $\bar{X}^* := (j^1_1(g), V)$, $\bar{Y}^* := (j^1_1(g), W) \in (\pi^r_{1,0})^* T\Sigma\rightarrow$, $\bar{\phi}, \bar{\phi}' \in \Gamma(\bar{\pi}^\Sigma_{SM}) \cong F^\Sigma(\bar{\phi})$, $\phi' \cong F^\Sigma(\bar{\phi}')$. $\square$
Of course, one can also pullback the vertical connection $\nabla$ from (2.22) to a vertical connection $(\pi_{1,0}^r)^*\nabla$. This vertical connection can be completed to a full connection in complete analogy to (2.33).

**Definition 2.24 (universal Levi-Civita connection).** For any $\tilde{X} \in T_{j}^{1}(\pi_{j}^r)(T^r M), \tilde{Y} = \tilde{Y}^s = (x^j, V) \in (\pi_{0}^r)^{*} (TM), \tilde{Y} = (g_x, V) \in (\pi_{x}^r)^{*} (TM), V \in T_x M$, we set

$$\nabla_{\tilde{X}} \tilde{Y} := \nabla_{\tilde{X}} \tilde{Y} + \nabla^q_{\tilde{X}} V,$$

where $d\pi_{1,0}^r \tilde{X} := X = X^v \oplus d_x g X_h$ is decomposed as in (2.30). The connection $\nabla$ is called universal Levi-Civita connection.

To see that $\nabla$ is well-defined, we notice that the Levi-Civita connection only depends on the 1-jet of the metric. So one can proceed as in the proof of Theorem 2.22.

In case $r = m$, the metric $\tilde{g}$ is precisely the universal Riemannian metric and $\nabla$ is the universal Levi-Civita connection as considered for instance in [PM08]. The equality between $\nabla$ and the connection considered in [PM08] is evident from [PM08, Thm. 5.1], where the uniqueness of natural metric connections on the bundle is shown, and the fact that the connection constructed here is both metric and natural under the action of diffeomorphisms of $M$, see (2.46).

**Lemma 2.25 (properties of universal Dirac structures).** The structures from Definition 2.23 satisfy the following compatibility relations:

$$-2\tilde{g}(X^v, Y^s) = \tilde{X}^v \tilde{Y} + \tilde{Y}^s \tilde{X}^v + \tilde{Y}^s \tilde{X}^v,$$  \hspace{1cm} (2.38)

$$\nabla_X \tilde{\eta}(\tilde{\phi}, \tilde{\phi}') = \tilde{\eta}(\nabla_X \tilde{\phi}, \tilde{\phi}') + \tilde{\phi}(\nabla_X \tilde{\phi}) = \tilde{\eta}(\tilde{\phi}', \nabla_X \tilde{\phi}),$$ \hspace{1cm} (2.39)

$$\nabla_X (\tilde{X}^v \tilde{\phi}) = \nabla_X \tilde{Y}^s + \tilde{Y}^s \nabla_X \tilde{\phi},$$ \hspace{1cm} (2.40)

$$\tilde{\eta}(\tilde{X}^v \tilde{\phi}, \tilde{\phi}') = (-1)^{s+1} \tilde{\eta}(\tilde{\phi}, \tilde{\phi}'),$$ \hspace{1cm} (2.41)

where $\tilde{X} \in T_{j}^{1} \pi_{j}^r, \tilde{X}^v, \tilde{Y}^s \in (\pi_{0}^r)^{*} (TM), \tilde{\phi}, \tilde{\phi}' \in \Gamma(\pi_{SM}^x)$.

**Proof.** The equation (2.38) follows directly from Eqs. (2.23), (2.35) and (2.37).

To see (2.39), we decompose

$$d\pi_{1,0}^r \tilde{X} =: X = X^v \oplus X^h \in T^v S_{r,s} M \oplus d_x g T_x M, \quad X^h = d_x g X_h, \quad X_h \in T_x M,$$

as in (2.34). We also assume that $\tilde{\phi} = \pi_{1,0}^r \circ \phi$ for some $\phi \in \Gamma(\pi_{SM})$ and analogously for $\tilde{\phi}'$. By definition

$$\nabla_X \tilde{\eta}(\tilde{\phi}, \tilde{\phi}') = \tilde{X}(\tilde{\eta}(\tilde{\phi}, \tilde{\phi}')) = X(\eta(\phi, \phi') \circ \pi_{1,0}^r) = d\pi_{1,0}^r X(\eta(\phi, \phi')) = X^v(\eta(\phi, \phi')) + X^h(\eta(\phi, \phi')).$$

Now, for the vertical part, (2.24) implies

$$X^v(\eta(\phi, \phi')) = \eta(\nabla_X \phi, \phi') + \eta(\phi, \nabla_X \phi').$$

Since $\phi \circ g \in \Gamma(\pi_{SM}^x)$, we obtain for the horizontal part

$$X^h(\eta(\phi, \phi')) = d_x g X_h (\eta(\phi, \phi')) = X_h(g(\phi \circ g, \phi' \circ g)) = g(\nabla_{X_h}^x (\phi \circ g), \phi' \circ g) + g(\phi \circ g, \nabla_{X_h}^x (\phi' \circ g)).$$

Since,

$$\eta(\nabla_X \phi, \phi') + g(\nabla_{X_h}^x (\phi \circ g), \phi' \circ g)) = \eta(id, \nabla_X \phi, \phi') + \eta(\nabla_{X_h}^x (\phi \circ g), \phi' \circ g)) = \eta(\nabla_X \phi, \phi'),$$

$$\eta(\phi, \phi') + g(\nabla_{X_h}^x (\phi \circ g), \phi' \circ g)) = \eta(\phi, \phi') + \eta(\nabla_{X_h}^x (\phi \circ g), \phi' \circ g)) = \eta(\nabla_X \phi, \phi').$$
this implies (2.39).

To see (2.40), we consider \( Y^* = (j_{\ast}^\dagger(g), V) \in (\pi_{r,s}^\ast)^\ast(TM) \), \( V \in T_xM \), \( \bar{Y}^* = (g_{\ast}, V) \in (\pi_{r,s}^\ast)^\ast(T_xM) \).

Using Eqs. (2.19) and (2.25), we obtain

\[
\nabla_X(Y^* \bar{\phi}) = \nabla_X(Y^* \cdot \phi) \circ g + \nabla_{X^g}((V \cdot \phi) \circ g) \\
= \nabla_XY^* \cdot \phi + Y^* \cdot \nabla_X \phi + (\nabla_{X^g}V) \cdot \phi \circ g + V \cdot \nabla_{X^g}(\phi \circ g) \\
= \nabla_X Y^* \bar{\phi} + Y^* \bar{\nabla}_X \phi.
\]

Finally, (2.41) follows directly from the definitions and (2.20).

Now, these universal structures allow us to reformulate Lemma 2.16 as follows.

**Theorem 2.26.** Let \((M, \Theta)\) be a spin manifold with fixed topological spin structure and \(\bar{\pi}_{\Sigma M}^g : \Sigma M \to J^1\pi_{r,s}^\ast\) be the jetted universal spinor bundle from (2.32). For every metric \(g\) on \(M\), there exists a morphism \(\bar{I}_g\) of vector bundles such that

\[
\begin{array}{c}
\Sigma^g M \\
\bar{I}_g \\
j^1(g) \Sigma M
\end{array} \xymatrix{ \ar[d]_{\bar{\pi}_M^g} \ar[r]^{\bar{I}_g} & \Sigma M \\
M \ar[r]_{j^1(g)} & J^1\pi_{r,s}^\ast. }
\]

commutes. In addition, \(\bar{I}_g\) is isometric with respect to the spinorial metric on \(\pi_M^g\) and \(j^1(g)^{\ast} \bar{\eta}\), it is compatible with the Clifford multiplication \(m^g\) and \(j^1(g)^{\ast} \bar{m}\) and it is compatible with the spinorial Levi-Civita connection on \(\pi_M^g\) and \(j^1(g)^{\ast} \bar{\nabla}\).

**Proof.** As in the proof of Lemma 2.16, the commutativity of (2.42) follows again from the universal property of the pullback and we obtain

\[
j^1(g)^{\ast} (\bar{\eta}) (\bar{I}_g(\psi), \bar{I}_g(\psi')) = \bar{\eta}(\bar{I}_g(\psi), \bar{I}_g(\psi')) = \eta(\psi, \psi') = g(\psi, \psi'), \\
j^1(g)^{\ast} (\bar{m})(V \otimes \bar{I}_g(\psi)) = \bar{m}(j^1(g), V) \otimes \bar{I}_g(\psi) = V_\psi \psi = m^g(V \otimes \psi).
\]

Now, since \(d\pi_{r,s}^\ast j^1(g) = d(\pi_{r,s}^\ast) \circ j^1(g)) = V = dgV\), we obtain from (2.33)

\[
(j^1(g)^{\ast} \bar{\nabla})V(\bar{I}_g(\psi)) = \bar{\nabla}_{j^1(g)\psi}(\bar{I}_g(\psi)) = \nabla^g_{\psi} \psi,
\]

which implies the claim.

\[\Box\]

2.6. Naturality questions

To discuss naturality aspects of the spinor bundle, it is helpful to introduce some lightweight category theoretic language and slightly rephrase what we know so far.

**Definition 2.27 (pseudo-Riemannian spin manifolds).** The **category of pseudo-Riemannian spin manifolds**, \(\text{pRiemSpinMf}_{r,s}\), consists of tuples \((M, g, \Theta^g)\) where \(M\) is a smooth manifold, \(g\) is a metric of signature \((r, s)\) and \(\Theta^g\) is a metric spin structure. A **morphism** between two such objects is a tuple \((f, \bar{F})\), where \(f\) is an orientation-preserving isometric diffeomorphism and \(\bar{F}\) is a
lift to the metric spin structures such that

\[
\begin{array}{ccc}
\text{Spin}^q_1 M_1 & \overset{\tilde{F}}{\longrightarrow} & \text{Spin}^q_2 M_2 \\
\downarrow \Theta_1 & & \downarrow \Theta_2 \\
\text{SO}^q_1 M_1 & \overset{df}{\longrightarrow} & \text{SO}^q_2 M_2 \\
\downarrow M_1 & f & \downarrow M_2
\end{array}
\]

(2.43)

commutes.

\[\text{Definition 2.28 (generalized Dirac bundles).} \quad \text{The category of (generalized) Dirac Bundles, } \text{DiracBdl, consists of tuples } (\pi_S : S \to N, h, \nabla^S, \pi_E : E \to N, g, \nabla^E, m^q) , \text{ where } N \text{ is a smooth manifold, } \pi^S \text{ is a complex vector bundle with a metric } h \text{ and a metric connection } \nabla^S, \pi_E \text{ is a real vector bundle with a metric } g \text{ of signature } (r, s) \text{ and a metric connection } \nabla^E, \text{ and } m^q : E \otimes S \to S, X \otimes s \mapsto X \cdot s, \text{ is a Clifford multiplication. In addition these structures satisfy the following compatibility conditions:}
\]

\[-2g(X, Y)\psi = X \cdot Y \cdot \psi + Y \cdot X \cdot \psi,\]
\[\nabla^S_V (X \cdot \psi) = \nabla^S_V X \cdot \psi + X \cdot \nabla^S_V \psi,\]
\[h(X \cdot \psi, \psi') = (-1)^{s+1} h(\psi, X \cdot \psi'),\]

where \(V \in \Gamma(H), X \in \Gamma(\pi_E), Y \in \Gamma(H). \) A morphism between two generalized Dirac bundles is a tuple \((f, F, \tilde{F})\) such that

\[
\begin{array}{ccc}
S_1 & \overset{F}{\longrightarrow} & S_2 \\
\downarrow \pi_{S_1} & & \downarrow \pi_{S_2} \\
N_1 & \overset{f}{\longrightarrow} & N_2
\end{array}
\quad \quad
\begin{array}{ccc}
E_1 & \overset{F}{\longrightarrow} & E_2 \\
\downarrow \pi_{E_1} & & \downarrow \pi_{E_2} \\
N_1 & \overset{f}{\longrightarrow} & N_2
\end{array}
\]

(2.44)

\[\text{commute. In addition, we require } f \text{ to be smooth, } F \text{ and } \tilde{F} \text{ to be isometric, and } F^* \nabla^E_2 = \nabla^E_1, \quad \tilde{F}^* \nabla^S_2 = \nabla^S_1, \quad F \circ m_1 = m_2 \circ (F \otimes \tilde{F}).\]

With this notation, any Dirac bundle \(S \to M\) in the classical sense, see for instance [Roe98], is a generalized Dirac bundle over \(E = TM\). It is well known, see for instance [LMS98], that associating a spinor bundle to a pseudo-Riemannian spin manifold is natural in the following sense.

\[\text{Lemma 2.29.} \quad \text{Let } \Sigma : \text{pRiemSpinMf}_{r,s} \to \text{DiracBdl} \text{ be the map that assigns to each pseudo-Riemannian spin manifold } (M, g, \Theta^q) \text{ its classical spinor bundle } \Sigma^q M := \text{Spin}^q \times_M \Sigma_{r,s} \text{ together with its Dirac structure from Remark 2.13 and its tangent bundle } \tau_M : TM \to M. \text{ Further, we assign to any morphism } (f, \tilde{F}) \text{ of pseudo-Riemannian spin manifolds, the morphism } (f, df, \tilde{F}) \text{ of Dirac bundles, where } \tilde{F} \text{ is defined by setting } \tilde{F} : \Sigma^q_1 M_1 \to \Sigma^q_2 M_2, \quad [b, v] \mapsto [\tilde{f}(b), v]. \text{ Then } \Sigma \text{ is a functor.}\]

\[\text{Remark 2.30.} \quad \text{We can extend } \Sigma \text{ with little modifications to oriented isometric codimension-one immersions instead of isometric diffeomorphisms. Here, } df : \text{SO}^q_1 M_1 \to \text{SO}^q_2 M_2 \text{ is replaced by } (df, \nu) \text{ which completes the pushed-forward basis by the right choice of normal vector } \nu \text{ to an oriented orthonormal basis, for details and applications see [BGM05]} \text{ or [BM08]. If the dimension of the hypersurface is even and we want to interpret } \Sigma \text{ in a contravariant manner, we need two spinor bundles on the hypersurface, which can be identified with the spinor bundle in the ambient space.} \]
space. This is due to the jump in dimension of the spinor module in this case. In the following, by a slight abuse of notation, we will use the symbol $\hat{F}$ also for such a spin lift of an oriented isometric immersion $f$.

**Remark 2.31.** For any morphism $(f, \hat{F})$ between $(M_1, g_1, \Theta_1)$ and $(M_2, g_2, \Theta_2)$, the resulting morphism $(f, df, \hat{F}) = \Sigma(f, \hat{F})$ satisfies $\partial g_2 \circ \hat{F} = \hat{F} \circ \partial g_1$ and the two spin manifolds are Dirac isospectral. So the two spin manifolds are spin isometric in this case.

Now, being spin isometric is a very restrictive condition. We would like to investigate in what sense one can (not) relax this condition without losing the naturality. First, we discuss the universal spinor jet bundle from (2.32).

**Definition 2.32 (spin manifolds).** The category of spin manifolds, $\text{SpinMf}$, consists of tuples $(M, \Theta)$ where $M$ is a smooth manifold and $\Theta$ is a topological spin structure. A morphism between two such objects is a tuple $(f, \hat{F})$, where $f$ is an orientation-preserving diffeomorphism and $\hat{F}$ is a lift to the topological spin structure $\text{GL}^+ M$ such that $\Theta_2 \circ \hat{F} = df \circ \Theta_1$ as in (2.43) (with $\text{SO}^9 M$ replaced by $\text{GL}^+ M$ and Spin$^9 M$ replaced by $\text{GL}^+ M$).

Recall that $\tau_{r,s}^M : (\pi_{r,s}^M)^* (TM) \to S_{r,s} M$ denotes the tangent bundle of $M$ pulled back to $S_{r,s} M$. We also define $\tau_{r,s}^* : (\pi_{0,s}^r)^* (TM) \to J^1 \pi_{r,s}$.

**Theorem 2.33.** The map $\text{usjb} : \text{SpinMf} \to \text{DiracBdl}$, that maps a spin manifold to its universal spinor jet bundle via

$$(M, \Theta) \mapsto (\pi_{S_M}^M : \Sigma M \to J^1 \pi_{r,s}, \hat{\eta}, \nabla, \tau_{F, \pi_{r,s}} : T J^1 \pi_{r,s} \to J^1 \pi_{r,s}, \bar{g}, \nabla, \bar{\pi})$$

is a functor. In particular, for any morphism $(f, \hat{F})$ between spin manifolds $(M_j, \Theta_j)$, $j = 1, 2$, there exist commutative diagrams

$$
\begin{align*}
\Sigma M_1 &\xrightarrow{\pi_{S M_1}^M} \Sigma M_2 \\
J^1 \pi_{r,s}^1 &\xrightarrow{F_{r,s}} J^1 \pi_{r,s}^2,
\end{align*}
$$

and

$$
\begin{align*}
(\pi_{0,s}^{r,1})^* (TM) &\xrightarrow{\hat{F}} (\pi_{0,s}^{r,2})^* (TM) \\
\tau_{s_{M_1}^r} &\xrightarrow{\tau_{s_{M_2}^r}} \tau_{s_{M_1}^r} \\
J^1 \pi_{r,s}^1 &\xrightarrow{F_{r,s}} J^1 \pi_{r,s}^2,
\end{align*}
$$

(2.45)

where $(F_{r,s}, \hat{F})$ and $(F_{r,s}, \hat{F})$ are isomorphisms of vector bundles compatible with the universal structures, i.e. all the relations

$$
\hat{F}^* \bar{\eta}_2 = \bar{\eta}_1, \quad \hat{F}^* \bar{\nabla}^1 = \bar{\nabla}^2, \quad \hat{F}^* \bar{\nabla}^1 = \bar{\nabla}^2, \quad \hat{F} \circ \bar{\pi}_1 = \bar{\pi}_2 \circ (\hat{F} \otimes \hat{F}), \quad \hat{F}^* \bar{g}_2 = \bar{g}_1.
$$

(2.46)

are satisfied.

**Proof.** The pullback of tensor fields via $f^{-1}$ gives a map $F_{r,s} : S_{r,s} M_1 \to S_{r,s} M_2$, which induces the map $\hat{F}_{r,s}$ and also the map $\hat{F}$. The map $\Sigma M_1 \to \Sigma M_2$, $[\bar{b}, v] \mapsto [\hat{F}(\bar{b}), v]$ is well-defined and descends to $\hat{F} : \Sigma M_1 \to \Sigma M_2$. This gives existence and commutativity of (2.45).

One now has to check all the compatibility conditions (2.46) in detail, which is a bit tedious. We only discuss the connection $\bar{\nabla}^1$: Recall that by Lemma 2.9, the map $F_{r,s}$ is an isometry. Since the vertical part of the connection is given via the metric from Lemma 2.8, it is preserved on the level of principal bundles. Since $\Sigma M$ is an associated bundle with induced connection, the claim follows also for $\Sigma M$. The horizontal part follows from the naturality of the spinorial Levi-Civita connection with respect to spin isometries.
Remark 2.34. The modification towards oriented codimension-one immersions \( f \) instead of diffeomorphisms as in Remark 2.30 works here as well, and such a lift will also be denoted by \( \tilde{F} \). We denote by \( \text{SpinMf}^1 \) the category that is identical to \( \text{SpinMf} \), but includes as morphisms oriented codimension-one immersions. The technicality with the jump in the dimension of the spinor module as in Remark 2.30 applies here as well.

Remark 2.35 (non-jetted version). Of course one can get a version of Theorem 2.33 for the non-jetted universal spinor bundle \( \pi^\Sigma : \Sigma \to S_{r,s}M \) and one obtains an analogous functor \( \text{usb} \). In the range category \( \text{DiracBdl} \) one would have to replace the connection by a vertical connection and one has to remove all bars in (2.46). As a result, even the entire diagram (2.32) is natural with respect to spin diffeomorphisms.

For the purpose of variational theory, it would be desirable to have a vector bundle over \( M \) for the spinors that is natural in the category of manifolds and diffeomorphisms and on which only the secondary structure (metric, connection, Clifford multiplication) is induced by the metric of the underlying manifold. This ansatz, however, is doomed to fail as shown in the following theorem:

**Theorem 2.36.** There is no functor \( \Sigma^{\top} \) such that

\[
\begin{array}{ccc}
\text{RiemSpinMf} & \xrightarrow{\Sigma} & \text{DiracBdl} \\
\text{MetrTop} \downarrow & & \downarrow \text{forget} \\
\text{SpinMfds} & \xrightarrow{\Sigma^{\top}} & \text{VB} \\
\downarrow \text{forget} & & \downarrow \text{pr} \\
\text{Mfds} & \xleftarrow{\text{pr}} & \\
\end{array}
\]

is a commutative diagram of categories and functors. Here, \( \text{forget} \) denotes the functor that forgets all additional structure except the manifold and vector bundle structure, \( \text{pr} \) maps a vector bundle \( E \to M \) in the category \( \text{VB} \) of vector bundles to its base \( M \), and \( \text{MetrTop} \) replaces a metric spin structure by its topological one (via the fiber bundle extension corresponding to the inclusion homomorphism \( \text{SO}_m \to \text{GL}^+_m \)).

**Proof.** Assume that (2.47) exists. We let \( \tilde{b} \in \text{GL}^+_m \) be arbitrary and think of \( \vartheta(\tilde{b}) =: b \in \text{GL}^+_m \) as an isomorphism \( \mathbb{R}^m \to \mathbb{R}^m \), \( x = x^i e_i \mapsto x^i b_i \). Let \( \sigma : \Sigma \mathbb{R}^m \to \mathbb{R}^m \) be the spinor bundle with respect to the Euclidean metric, i.e. \( \sigma := \Sigma^{\top} (\text{MetrTop}(\mathbb{R}^m, \text{Spin}_m, \bar{g})) = \Sigma^{\top} (\mathbb{R}^m, \text{GL}^+_m) \). We set \( V := \sigma^{-1}(0) \) and obtain an isomorphism \( \tilde{r}_{\tilde{b}} : \Sigma^{\top} (\tilde{b}, \tilde{b})|_V : V \to V \). All in all, we obtain a finite dimensional representation

\[
\tilde{r} : \text{GL}^+_m \to \text{GL}(V) \\
\tilde{b} \mapsto \tilde{r}_{\tilde{b}}.
\]

By [LM89, Lem. II.5.23], this representation descends to a representation of \( \text{GL}^+_m \), i.e. there exists a representation \( r : \text{GL}^+_m \to \text{GL}(V) \) such that

\[
\begin{array}{ccc}
\text{GL}^+_m & \xrightarrow{r} & \text{GL}(V) \\
\vartheta \downarrow & & \downarrow r \\
\text{GL}^+_m & \xleftarrow{\vartheta_m} & \\
\end{array}
\]

commutes. Now, consider the unit element \( e \in \text{GL}^+_m \) and obtain \( \vartheta^{-1}_m(1) =: \{ \tilde{c}_+, \tilde{c}_- \} \), where \( \tilde{c}_+ \) is
the unit element in $\tilde{GL}_m^+$. Since (2.48) commutes, we obtain that

$$f := \tilde{r}(\tilde{e}_-)(\tilde{r} \circ \vartheta_m)(\tilde{e}_-) = \tilde{r}(\tilde{e}_+) = \text{id}_V.$$  

Notice that $\tilde{e}_- \in \text{Spin}_m$. Therefore, we can consider $\text{forget}(\Sigma(\text{id}_R^m, \tilde{e}_-))|_V = f \in \text{GL}(V)$. By construction $\Sigma^g \mathbb{R}^m = \text{Spin}_m \times_\rho \Sigma_m$, where $\rho$ is the standard spin representation, and the morphism is explicitly given by

$$f : \Sigma \mathbb{R}^m \to \Sigma \mathbb{R}^m$$

$$[\tilde{b}, v] \mapsto [\tilde{b}, \tilde{e}_-, v] = [\tilde{b}, \rho(\tilde{e}_-)^{-1} v].$$

Thus $f \neq \text{id}_V$, since $\rho$ is faithful, which is a contradiction.

\section{The Einstein-Dirac-Maxwell Equation}

Using the universal spinor bundle $\pi^\Sigma_M : \Sigma M \to M$, one can now formulate Einstein-Dirac-Maxwell theory as a variational problem on sections of a finite dimensional fiber bundle in any signature $(r,s)$. The advantage of this approach is that the spaces of sections $W_{k,p}(\pi^\Sigma_M)$ carry canonical topologies as Fréchet manifolds (Banach if $k < \infty$). At the first sight, including Maxwell fields might seem to be an unnecessary complication. However, a good reason for doing so is that, whereas solutions of Einstein-Maxwell theory can be made solutions of Einstein-Dirac-Maxwell theory by including zero spinor fields, the same is not true for Einstein-Dirac solutions: The presence of nonzero spinors entails in general the presence of nonzero Maxwell fields.

\subsection{Notation and basic definitions}

To formulate the Einstein-Dirac-Maxwell equation, we introduce the following notions. Let $g \in S_{r,s}(M)$ be a metric, $\psi \in \Gamma(\pi^\rho_M)$ be a spinor field and $A \in \Omega^1(M)$ (thought of as a connection form on a trivial $U(1)$-bundle over $M$). Each 1-form $A$ induces a connection $\nabla g,A$ on the spinor bundle via $\nabla g,A_X \psi := \nabla g_X \psi + iA(X) \cdot \psi$. The resulting Dirac operator reads $\mathcal{D}^g_A \psi = \mathcal{D}^g \psi - A \cdot \psi$ (using Clifford multiplication on the cotangent bundle).

For $q \in \mathbb{R}$, we define the tensor fields $T^1_{(g,\psi,A)}, T^2_{(g,A)} \in \Gamma(\tau^2_M)$ by

$$T^1_{(g,\psi,A)}(X, Y) := \frac{1}{2} \text{Re}(X \cdot \nabla^g_Y A \psi + Y \cdot \nabla^g_X A \psi),$$

$$T^2_{(g,A)} := \text{tr}_{(1,3)}(F^A \otimes F^A) - \frac{1}{4} g(F^A, F^A) g,$$

for all $X, Y \in \Gamma(\tau_M)$. Here $F^A := dA$, which is the curvature of $A$. The metric $g$ is extended canonically to all tensor powers of the tangent bundle $\tau_M : TM \to M$. We also set

$$T_{(g,\psi,A)} := T^1_{(g,\psi,A)} + T^2_{(g,A)},$$

the \textit{total energy-momentum tensor of $(g, g, \psi, A)$}. The field $j_\psi \in \Omega^1(M)$, defined by

$$j_\psi(X) := \langle X \cdot \psi, \psi \rangle,$$

is called the \textit{Dirac current of $\psi$}. 
Definition 3.1 (Einstein-Dirac-Maxwell equation). Let $(M, \Theta)$ be a spin manifold. For any $\lambda, q \in \mathbb{R}$, the system of equations

\[
\begin{align*}
D^{g,q}A \psi &= \lambda \psi, \\
\text{Ric}^g - \frac{1}{2} \text{scal}^g g &= T(g, \psi, A), \\
\delta^g F^A &= q j \psi,
\end{align*}
\]  

is called $(\lambda, q)$-Einstein-Dirac-Maxwell equation. $q$ is called the charge and $\lambda$ is called the mass. We want to solve this equation for $g, \psi, A$, and denote such a solution by $(M, g, \psi, A)$. ♦

Remark 3.2 (neutral systems). Physically reasonable systems often consist of various particles of different charges and masses. By summation of the Dirac currents in the last equation and of energy-momentum tensors $T^i(g, \psi, A)$ in the second equation, the equations can readily be generalized to a Dirac system comprising various spinor fields, for details cf. [GM14]. This is important as it allows to consider neutral systems, which are systems with $\sum_i q_i = 0$. If, moreover, $\lambda_i = 0$ for all $i$, these systems have a well-posed initial value formulation for small initial values in asymptotically flat spacetimes, cf. [GM14]. ♦

Definition 3.3 (universal Einstein-Dirac-Maxwell operator). The map $\mathcal{D} : \Gamma(\pi^\Sigma_M) \to \Gamma(\pi^\Sigma_M)$, defined by

\[
\Phi = (g, \psi) \mapsto \mathcal{D}(\Phi) := (g, D^g \psi - \lambda \psi)
\]

is called universal Dirac operator. The map $\text{EDM}_{(\lambda,q)} : \Gamma(\pi^\Sigma_M \oplus \tau^*_M) \to \Gamma(\tau^*_M \otimes \tau^*_M \oplus \pi^\Sigma_M \oplus \tau^*_M)$

\[
\Phi = (g, \psi, A) \mapsto (\text{Ric}^g - \frac{1}{2} \text{scal}^g g - T(g, \psi, A), (g, D^{g,q}A \psi - \lambda \psi), \delta^g F^A - q j \psi)
\]  

(3.2)

is called universal Einstein-Dirac-Maxwell operator. ♦

A zero of the universal Einstein-Dirac-Maxwell operator is exactly a solution of the corresponding Einstein-Dirac-Maxwell equation.

Remark 3.4. In view of Remark 3.2, one should say that one can also express the solutions of (3.1) comprising of $k$ spinor fields as zeroes of an analogous functional. In that case, one has to replace $\pi^\Sigma_M$ from the universal spinor bundle in (2.9) by its $k$-fold fiber product. ♦

3.2. Solutions as critical points

Now, want to express solutions of (3.1) as critical points of a functional to be defined in Definition 3.8. This functional acts on the whole space of sections of the universal spinor bundle (or, equivalently, on the space of holonomic sections of its jetted version). If $M$ is compact, then locally, in a convex subset $W$ of this space of sections, we will rewrite it by using trivializations.

An important application of the universal spinor bundle is the construction of identification isomorphisms between the spinor bundles formed with respect to various metrics. This can now be done easily as follows.

Definition 3.5 (identification isomorphisms). Let $g : I \to \mathcal{S}_{r,s}(M)$, $t \mapsto g_t$, be a smooth path of metrics, $g := g_0, h := g_1$. For any $x \in M$ and any $\Phi \in \Sigma M|_{g(x)}$, let $\beta_{g,h}(\Phi) \in \Sigma M|_{h(x)}$ be obtained by $\nabla$-parallel transport of $\Phi$ along $g$. The resulting map $\beta_{g,h} : \Sigma^g M \to \Sigma^h M$ is called an identification isomorphism. ♦
These identification maps satisfy \( \beta_{g,h} \circ \beta_{h,g} = \text{id} \), and one can use them to pull back any Dirac operator \( \mathcal{D}^h \) to \( \Gamma(\pi^g_M) \). This operator can be expressed more explicitly as follows: For any metric \( g \in \mathcal{S}_{r,s}(M) \), we denote by \( \gamma_g : TM \to T^*M \) the musical isomorphism and by \( \bar{\gamma}_g : T^*M \to TM \) its inverse. If \( h \in \mathcal{S}_{r,s}(M) \) is any other metric, the isomorphism \( a_{g,h} := \bar{\gamma}_g \circ \gamma_h \in \text{Iso} \, M \) satisfies
\[
\forall X, Y \in TM : g(a_{g,h}(X), Y) = h(X, Y).
\]
Clearly, if \( g = h \) we obtain \( a_{g,h} = \text{id} \), which is positive definite. Therefore, if \( h \) is in a small neighborhood of \( g \), the map \( a_{g,h} \) is still positive definite. (In case \( g, h \) are Riemannian, \( a_{g,h} \) is always positive definite.) The map \( b_{g,h} := \sqrt{a_{g,h}^{-1}} \) (in the sense of a positive definite square root) satisfies
\[
\forall X, Y \in TM : h(b_{g,h}(X), b_{g,h}(Y)) = g(X, Y),
\]
thus it maps a \( g \)-pseudo-orthonormal basis to an \( h \)-pseudo-orthonormal basis.

Any identification isomorphism \( \beta_{g,h} \) from Definition 3.5 induces a map on sections \( \Gamma(\pi^g_M) \to \Gamma(\pi^h_M) \) also denoted by \( \beta_{g,h} \). The problem is that for any two metrics \( g, h \in \mathcal{S}_{r,s}(M) \) there might be no path in \( \mathcal{S}_{r,s}(M) \) joining them. Even if they can be joined, the path is not unique. To handle this problem, we introduce the following notion.

**Definition 3.6.** Let \( M \) be a (possibly non-compact) manifold and \( g, h \in \mathcal{S}_{r,s}(M) \). We say \( g \) and \( h \) are **joinable**, if the path \( g_t := g + t(h - g), \ t \in [0, 1], \) is contained in \( \mathcal{S}_{r,s}(M) \) and for any \( t \in [0, 1] \), the map \( a_{g,g_t} \) is positive definite.

Obviously, if we are given a compact subset \( C \subset M \), a pseudo-Riemannian metric \( h \) and an auxiliary Riemannian metric \( k \), there is a positive number \( a_C \) such that all pseudo-Riemannian metrics \( g \) coinciding with \( h \) outside of \( C \) and with \( \|g - h\|_k < a_C \) are joinable to \( h \). Even more, there is a positive number \( b_C \) such that all pseudo-Riemannian metrics \( g \) coinciding with \( h \) outside of \( C \) and with \( \|g - h\|_k < b_C \) are joinable to each other. We will call such a neighborhood \( C \)-**convex**. For any two joinable \( g, h \), we obtain a unique identification isomorphism \( \beta_{g,h} \) from Definition 3.5. To compare the induced map on sections on an \( L^2 \)-level, the following is helpful: Since \( g, h \) are of the same signature, there exists a smooth positive function \( f_{g,h} : M \to \mathbb{R}_{>0} \) such that \( \text{d}V^h = f_{g,h}^2 \text{d}V^g \).

Now, whereas \( \beta_{g,h} \) are pointwise isometries and thus \( \beta_{g,h} \circ \) is an isometry \( \Gamma_{C^0}(\pi^g_M) \to \Gamma_{C^0}(\pi^h_M), \ \beta_{g,h} := \frac{1}{f_{g,h}} \beta_{g,h} \) is an isometry \( \beta_{g,h} : \Gamma_{L^2}(\pi^g_M) \to \Gamma_{L^2}(\pi^h_M) \). Of course \( \beta_{g,h} \circ \beta_{h,g} = \text{id} \) as well. Nevertheless, we will work with \( \beta_{g,h} \) in the following.

**Theorem 3.7.** Let \( g, h \in \mathcal{S}_{r,s}(M) \) be joinable. Then the operator
\[
\mathcal{D}^g_h := \beta_{h,g} \circ \mathcal{D}^h \circ \beta_{h,g} : \Gamma(\pi^g_M) \to \Gamma(\pi^g_M)
\]
has the local coordinate representation
\[
\mathcal{D}^g_h \psi = \sum_{i=1}^m e_i \cdot \nabla^g_{b_{g,h}e_i} \psi + \frac{1}{4} \sum_{i,j=1}^m e_i \cdot e_j \cdot \left( b_{h,g}(\nabla^h_{b_{g,h}e_i}(b_{g,h}e_j)) - \nabla^g_{b_{g,h}e_i}e_j \right) \cdot \psi,
\]
where \( e_1, \ldots, e_m \) is a local pseudo-orthonormal frame and \( \psi \in \Gamma(\pi^g_M) \) is any spinor field.

**Proof.** In the case of a compact Riemannian spin manifold, this formula is exactly [BG92, Thm. 20] and holds for any two Riemannian metrics \( g \) and \( h \). The proof goes through in the general case, since we required \( g \) and \( h \) to be joinable.

For the rest of this subsection, we assume the manifold \( M \) to be compact.
In [KF00], using the variation formula for the Dirac operator established by Bourguignon and Gauduchon, Kim and Friedrich show how solutions of the Einstein-Dirac equation in the Riemannian case can be identified with stationary points of some functional w.r.t. special variations. Now we show how to characterize any solution \((g, \psi, A)\) of the Einstein-Dirac-Maxwell equation as a critical point, i.e. in the sense of \(d_{(g, \psi, A)} \text{EDM} = 0\), of a functional EDM defined on the sections of the finite dimensional fiber bundle \(\text{usjb}\) from (2.9).

**Definition 3.8 (Einstein-Dirac-Maxwell functional).** The \((g, \lambda)\)-Einstein-Dirac functional is given by

\[
L_{(g, \lambda)} : \Gamma(\pi_M^\Sigma \oplus \Lambda^1 M) \rightarrow \mathbb{R} \\
(\Phi = (g, \psi), A) \mapsto \int_M L_{(g, \lambda)} \circ j^2(\Phi, A),
\]

where \(L_{(g, \lambda)} \in C^\infty(J^2(\pi_M^\Sigma \oplus \Lambda^1 M), \Lambda^m M)\) is defined by

\[
L_{(g, \lambda)}(j^2(\Phi, A)) := \left( \operatorname{scal}^g + \lambda \langle \psi, \psi \rangle - \langle B^g A, \psi, \psi \rangle - \frac{1}{2} g(F^A, F^A) \right) d\sigma^g.
\]

For simplicity, we set \(L := L_{(g, \lambda, g)}\).

**Remark 3.9 (characterization of critical points).** Let \(\pi : P \rightarrow M\) be a fiber bundle, \(\tau^g_\pi : T^\pi P \rightarrow M\) be the vertical tangent bundle, and \(L : \Gamma(\pi) \rightarrow \mathbb{R}\) be a functional. A section \(s \in \Gamma(\pi)\) is a critical point of \(L\) if and only if \(\frac{d}{dt} s|_{t=0} = 0\) for any 1-parameter family \(s_t \in \Gamma(\pi)\) that is smooth in \(t\). Using the identification between \(\Gamma(s^* \tau^g_\pi)|\) and \(T_s \Gamma(\pi)\), we get that \(s\) is critical if and only if \(\frac{d}{dt} F_{\pi_\pi}(s)|_{t=0} = 0\) for all \(X \in \Gamma(\tau(\pi))\). Here, \(F_{\pi_\pi}\) denotes the flow of \(X\). In case that \(L = L \circ j^2\) as in Definition 3.8, this can be written equivalently as \(\int_M (j^2 X)(L) = 0\).

**Remark 3.10.** Now, we want to examine critical points and we want to justify the product in a trivialization w.r.t. \(\pi_{\Sigma}^M\). Let \(\pi : P \rightarrow M\) be compact, there is an open covering of \(S_{r,s} M\) by \(M\)-convex, that is, convex, subsets. In such a subset \(U\) we trivialize the bundle \(\pi_{\Sigma}^M\) by means of the parallel transport to a fixed metric \(g \in U\): For any \(e \in \Sigma M\), we define \(t_g(e) := (\pi_{\Sigma}^M(e), \beta_{\pi_{\Sigma}^M(e), g} I^\Sigma_{\pi_{\Sigma}^M}(e))\). Consider the open subset set \(W \subseteq \Gamma(\pi_{\Sigma}^M)\) consisting of all the sections \(\Psi\) of \(\pi_{\Sigma}^M\) such that \(\pi_{\Sigma}^M \circ \Psi \in U\). Then \(K_g := t_g \circ -\) is a Fréchet between \(W\) and \(U \times \Gamma(\Sigma^u M)\). If \(g, h \in U\), then \(K_h \circ K_g^{-1}(u, \psi) = (u, \beta_{u, h} \circ \beta_{u, g}(\psi))\) for all \(u \in U, \psi \in \Gamma(\Sigma^u M)\). Consequently, if \(\Psi\) is a critical point of EDM, then \(\Psi\) is a critical point of EDM, so we can show criticality in a trivialization w.r.t. \(\pi_{\Sigma}^M\), as it is done in [KF00].

**Theorem 3.11.** Let \(M\) be compact. A tuple \((\Phi, A) = (g, \psi)\), is a critical point of \(L\) if and only if \((g, \psi, A)\) is a solution of the Einstein-Dirac-Maxwell equation.

**Proof.** Since the inclusion of an additional Maxwell field is trivial, it suffices to discuss the Einstein-Dirac equation (which corresponds to \(A = 0\)).

First of all, note that [KF00, Thm. 2.1] holds for any signature of \(g\), since the proof goes through without modifications. The theorem states that a spinor field \((g, \psi, 0)\) satisfies the Einstein-Dirac equation if and only if the functional \(L(g, \psi, 0)\) is stationary under all variations of type

\[
t \mapsto (g + tk, \psi, 0), \quad t \mapsto (g, \psi + t \varphi, 0),
\]

where the first variation is to be understood using the identifications \(\beta_{g,h}\). Therefore, we can identify these variations with \(t \mapsto \Phi_t := (g + tk, \beta_{g,g+, t \varphi})\) in our setting, see Remark 3.10. The second variation in (3.6) is already a variation of a universal spinor field in the spurious direction.

Now let us show the “⇒” direction: Assume that \((\Phi, 0) = (g, \psi, 0)\) is a critical point of \(L\). In particular, it is critical under the variation (3.6). Therefore, using [KF00, Thm. 2.1] it is a solution of the Einstein-Dirac equation.
Now, for the other direction assume that $\Phi = (g, \psi)$ is a solution of the Einstein-Dirac equation. Then [KF00, Thm. 2.1] implies that it is critical under variations of (3.6). We have to show that those generate all the variations by curves in $\Gamma(\pi^+_X)$. By Remark 3.9 these correspond to $\pi^{-}_X$-vertical vector fields on $\Sigma M$.

We need that $dL(\Phi)X^v = 0$ for all $X^v \in \Phi^* (\tau^u_{\Sigma M})$, where $\tau^u_{\Sigma M} := \tau_{\Sigma M}/(d^{TV})^{-1}(0)$. The Bourguignon-Gauduchon connection gives us a global decomposition $\tau^v_{\Sigma M} = \tau^{vh}_{\Sigma M} \oplus \tau^v_{\Sigma M}$ and corresponding projections $\pi^v$ and $\pi^{vh}$. Now $X^{vv} := \pi^v(\Phi(X^v))$ is the variational vector field of a variation of second type of (3.6) and $X^{vh} := \pi^{vh}(\Phi(X^v))$ is the variational vector field of a variation of first type of (3.6). By linearity of the derivative and by $d_g L(X^{vv}) = 0$ and $d_g L(X^{vh}) = 0$, we obtain $d_g L(X^v) = 0$.

**Remark 3.12.** The description of the solution as critical points of a Lagrangian functional on a finite-dimensional jet bundle now enables us to use all the available tools in this setting. For instance, if one wants to solve extension problems one can ask for Palais-Smale condition or if the functional is a Morse function in some appropriate sense and then use tools like the Mountain Pass Lemma.

**Remark 3.13 (inclusion of boundary values).** Now, assume that $M$ is a compact manifold with boundary. Then we can repeat the proof of Theorem 3.11 and show criticality in the space $\Gamma_Z(\pi^+_M \oplus A^1)$ of fixed boundary values $Z$, where boundary are defined in a natural generalization of the notion of initial values in Definition 3.17.

### 3.3. Existence of a maximal Cauchy development

From now on, we focus on the Lorentzian case solely. The question of a maximal Cauchy development for Maxwell-Einstein theory has been positively answered by reducing the system (3.1) to a symmetric-hyperbolic operator, see for instance [CB09]. Now we want to try the same for Einstein-Dirac-Maxwell Theory. In [FR00], it is indicated how to do so in the framework of two-spinor calculus. As far as we can see, this framework requires the choice of an (unnatural) trivialization of the spinor bundle and is moreover restricted to dimension 4. For the definition of a maximal Cauchy development, we need the stronger naturality properties of Lemma 2.29 to prove geometric uniqueness. Therefore, we choose another approach.

**Definition 3.14 (initial values).** Let $Z = (S, g_0, K, \psi_0, A_0, A_1)$ be a tuple, where $S$ is a $(m - 1)$-dimensional spin manifold, $g_0$ is a Riemannian metric on $S$, $K \in \Gamma(\tau^2_S)$ is a symmetric 2-tensor on $S$, $\psi_0 \in \Gamma(\pi^+_S)$ is one spinor field (for $m$ even) or two (for $m$ odd), $A_0, A_1 \in \Gamma(\tau^1_{S \times \mathbb{R}})^{S \times \{0\}}$ representing initial values for the Maxwell field. Then $Z$ is called an initial value for the $(\lambda, q)$-Einstein-Dirac-Maxwell equation if and only if the initial constraint equations

\[
\begin{align*}
\text{scal}^S g + (tr_g K)^2 - |K|_g = 16\pi T_{(q, g, \psi, A)}(\nu, \nu), \\
tr_{12}(\nabla K) - tr_{23}(\nabla K) = 8\pi T_{(q, g, \psi, A)}(\nu, \cdot)
\end{align*}
\]

are satisfied, where $\nu$ is a future-directed unit normal vector field for $S$.

**Remark 3.15.** It is well-known, see for instance [CB09], that restricting a section that is a solution to the Einstein-Dirac-Maxwell equation to a Cauchy surface yields an initial value (where spinors are re-interpreted as spinors of the submanifold in the sense of Remark 2.30).
There are various notions of a maximal Cauchy development for various field theories (for instance Einstein-Yang-Mills, Klein-Gordon-Einstein, Einstein-Vlasov) in mathematical physics. The following is an attempt to unify these. First, we have to define natural bundles and natural partial differential equations.

**Definition 3.16 (Natural bundles and natural PDEs).** Let $\mathbf{FB}$ be the category of fiber bundles and fiber bundle morphisms and let $\text{Lor}(M) := \mathcal{S}_{-1,1}(M)$ be the Lorentz metrics. Let $\pi : \text{SpinMf}^1 \to \mathbf{FB}$ (where $\text{SpinMf}^1$ has been defined in Remark 2.34) be a contravariant functor admitting a functorial lift to the first jet and with a natural map $W : \pi \to \text{Lor}$. Consequently, there are pullbacks $F^* : \Gamma(\pi(M_2, \Theta_2)) \to \Gamma(\pi(M_1, \Theta_1))$ and $(j^1F)^* : \Gamma(d\pi(M_2, \Theta_2)) \to \Gamma(d\pi(M_1, \Theta_1))$.

Let $Y : \text{SpinMf} \to \text{Sets}$ be a natural PDE of order 2 on $\pi$, i.e. functor such that for each $(M, \Theta)$, $Y(M, \Theta) \in J^2\pi(M, \Theta)$ and for each morphism $(f, \tilde{F})$, we require that $Y(f, \tilde{F}) = j^2(f)$. A solution of $Y(M, \Theta)$ is a section $\gamma$ of $\pi(M, \Theta)$ such that $j^2(\gamma) \in Y(M, \Theta)$. By assumption $\gamma$ is of the form $\gamma = (g, \varphi)$, where $g \in \text{Lor}(M)$.

To make Maxwell-Einstein theory fit into this framework, one would define the natural bundle $\Lambda^1 \boxplus \text{Lor}$ (as a functor) with $W$ being the projection on the second factor. Here, the pull-back of $\text{Lor}(-)$ is defined in Remark 2.34) be a contravariant functor $\pi_{SM} : \mathbf{FB} \to \mathbf{Sets}$ such that for each $(M, \Theta)$, $\pi_{SM}(M, \Theta) \in \mathcal{S}_m(-,1)(M)$ and for each morphism $(f, \tilde{F})$, we require that $\pi_{SM}(f, \tilde{F}) = j^2(f)$. A solution of $\pi_{SM}(M, \Theta)$ is a section $\gamma$ of $\pi_{SM}(M, \Theta)$ such that $j^2(\gamma) \in \pi_{SM}(M, \Theta)$. By assumption $\gamma$ is of the form $\gamma = (g, \varphi)$, where $g \in \text{Lor}(M)$.

**Definition 3.17 (SpinMf-Cauchy development).** Let $\pi$ and $Y$ be as above. Let $Z$ be an $M$-initial value on a spin manifold $S$, i.e. an element of the image of the restriction operator $j^1\pi(f) : J^1\pi(M, \Theta)) \to J^1\Gamma(\pi(M, \Theta))$ to $f(S)$, where $f$ is an embedding onto a Cauchy hypersurface of $(M, g)$ where $\gamma = (g, \varphi)$ is a solution of $Y(M, \Theta)$. Then a SpinMf-Cauchy development of $Z$ is a tuple $(U, \gamma)$ consisting of a topological spin $m$-manifold $U$ and a solution $\gamma = (g, \varphi)$ of $Y_U$ and a spin diffeomorphism $F : S \to F(S) \subset U$ such that

(i) $F(S)$ is a Cauchy surface of $(U, g)$,

(ii) $(j^1F)^*\gamma = Z$.

A development is called maximal, if for every tuple $(U', \gamma', F')$ satisfying items (i), (ii) we have a spin diffeomorphism $\alpha : U' \to \alpha(U') \subset U$ with $\alpha^*(\gamma') = \gamma$ and $j^1(\alpha \circ F') = j^1(F)$. ◊

In fact, the usual definition of 'Cauchy development' would rather work with the corresponding non-spin category, as long as the field theory in question does not include spinors, but we have to keep track of the spin structure, so the definition above is obtained by restricting the more general definition to the subcategory SpinMf.

Note that the following classical definition of Cauchy development for Einstein-Dirac-Maxwell theory is not of this form (simply because the classical spinor bundle is not natural in SpinMf), but tailor-made to be as close to it as possible:

**Definition 3.18 (pRiemSpinMf-Cauchy development).** Let $Z = (S, g_0, K, \psi_0, A_0, A_1)$ be an initial value. Then a pRiemSpinMf-Cauchy development of $Z$ is a tuple $(M, g, \psi, A, f)$ such that $(g, \psi, A)$ is a solution of the Einstein-Dirac-Maxwell equation, $(M, g)$ is a globally hyperbolic
spin $m$-manifold and $f$ is a spin diffeomorphism $S \to f(S) \subset M$ such that

(i) $f(S)$ is a Cauchy surface of $(M, g)$,

(ii) $f^* g = g_0$, $f^* W = K$ and $f^* \psi = \psi_0$, $f^* A = A_0$, $f^* \nabla_0^g A = A_1$.

Here, $W$ denotes the second fundamental form of $S$ (“Weingarten tensor”) and $\nu$ a normal vector field. A development is called maximal, if for every tuple $(M', g', \psi', A', f')$ satisfying items (i) and (ii), we have a spin diffeomorphism $\alpha : M' \to \alpha(M') \subset M$ with $\alpha^*(g) = g'$, $\alpha^*(\psi') = \psi$ (in the sense of (2.43)), $\alpha^* A = A'$ and $d(\alpha \circ f') = df$.

In order to discuss the existence and uniqueness of solutions of (3.1), we require some notions from the theory of symmetric hyperbolic quasilinear systems, see [GM14] for a definition. In particular, we choose a gauge.

**Definition 3.19 (wave-gauged).** For two Lorentzian manifolds $(V, v)$ and $(W, w)$, a map $f : V \to W$ is called a wave map, if $\text{tr}_v (\nabla df) = 0$. Here we consider $df \in \Gamma(\tau^*_V \otimes f^* \tau_W)$, and $\nabla$ is the natural connection on $\tau^*_V \otimes f^* \tau_W$. Let $h$ be a Lorentzian metric on an open subset $U'$ of $W$ with Cauchy hypersurface $S$. A metric $g$ on a neighborhood $U \subset U'$ of $S$ is called wave-gauged to $h$ (on $U$), if the identity on $U$ is a wave map from $(U, g)$ to $(U, h|_U)$.

If $(M, h)$ is an open set of the Minkowski space $\mathbb{R}^{1,m}$, then $g$ is in $h$-wave gauge if and only every coordinate is a $g$-harmonic function. Every metric can be brought into $h$-wave-gauge:

**Lemma 3.20.** For every smooth Lorentzian manifold $(M, g)$ containing as a Cauchy surface the image $f(S)$ of $S$ by an isometric embedding $f$, there is a neighborhood $U$ of $f(S)$ and a diffeomorphism $\alpha$ with $d\alpha|_{\alpha(S)} = \text{id}$ such that $\alpha^* g$ is $h$-wave-gauged. For every globally hyperbolic $U$ the diffeomorphism is unique and spin.

**Proof.** The equation for $g$ to be $h$-wave-gauged is $Q^h(g) := \text{tr}_g (\nabla^g - \nabla^h) = 0$. This is a quasilinear second degree equation for $g$ and thus locally solvable for first order initial values by the usual arguments via the symmetric-hyperbolic first order prolongation of $Q^h$ and then using, e.g. [Tay11, 16.1.2-16.1.7 and 16.2.1] for smooth coefficients, or [GM14, Thm. 4.2] for coefficients of finite regularity. Here, we need the same statement as in the references above, but for a range $G \subset M \times \mathbb{R}^N$ instead of $M \times \mathbb{R}^N$, such that for all $p \in M$, the set $G \cap (\{p\} \times \mathbb{R}^N)$ is convex and reflection around 0. By a well-known procedure [CB09], we can extend the coefficients $A_j$ and $B$ as in [Tay11; GM14] outside $G$ by an appropriate constant map and use stability. $G$ is then chosen in such a way that for any $g$ with $g_p \in G_p$ for all $p$ satisfies that the coefficient $A_0$ as in [GM14] is uniformly positive. The diffeomorphism $\alpha$ is spin, i.e. has a lift to the spin bundle as the identity on $S$ has and because of the homotopy lifting property for the spin bundle, as $U$ is diffeomorphic to $S \times \mathbb{R}$.

**Definition 3.21 (reduced Einstein-Dirac-Maxwell operator).** Let $h$ be a globally hyperbolic metric on a manifold such that $S$ is a Cauchy hypersurface. Let $U$ be a neighborhood of $h$ consisting of metrics joinable to $h$. Then

$$\text{EDM}^h_{(q, \lambda)} : U \oplus \Gamma(\Sigma^h M) \oplus \Omega^1(M) \to \tau^2_M \oplus \Gamma(\Sigma^h M) \oplus \Omega^1(M)$$

defined by

$$(g, \psi^h, A) \mapsto (\text{Ric}^g - \frac{1}{2} \text{scal}^g g - T^q_{(g, \beta_{h,g}(\psi), qA)}, \Box g A - q \psi, \beta_{g,h}(i\nu \cdot g (\bar{D}^{q_A - \lambda} \beta_{h,g} \psi^h)))$$

is called **reduced Einstein-Dirac-Maxwell operator**.
We want to show that EDM\(^{(h)}\) is a symmetric hyperbolic quasilinear operator.

By Lemma 3.20 and by bijectivity of \(\beta_{g,h}\) and the fact that every tuple \((A, \psi)\) can be brought into Lorentz gauge replacing it by \((A + df, e^{if} \psi)\) for some function \(f\), we obtain a one-to-one correspondence between spin diffeomorphism orbits of EDM\(^{-1}\)(0) and EDM\(^{(h)}\)(0).

**Lemma 3.22.** The first-order prolongation of the operator EDM\(^{(h)}\) is symmetric-hyperbolic (if restricted to an appropriate convex set \(G\) of Lorentzian metrics), and therefore there is a pRiemSpinMf-Cauchy development, see Definition 3.18.

**Proof.** As the assertion is well known for Einstein-Maxwell theory (\(\psi = 0\)), we focus on the spinor contributions only. For every precompact open set \(C\) we have a ball \(B\) in \(\Gamma(\Gamma_0^2)\) around \(g_0|_C\) contained in Lor(\(C\)), thus there is an affine path from each metric in \(B\) to \(g_0\). By (3.5) the principal symbol of \((\mathcal{D}_g^h)^2\) can be calculated as

\[
\sigma((\mathcal{D}_g^h)^2)(\omega, \omega) = \sigma(\mathcal{D}_g^h)(\omega) \circ \sigma(\mathcal{D}_g^h)(\omega) = (b_{h,g}(\omega^h), b_{h,g}(\omega^h)) = h(b_{h,g}(\omega^h), b_{h,g}(\omega^h)) = h(\omega^h, a_{h,g}^{-1}(\omega^h)) = g(a_{h,g}^{-1}(\omega^h), a_{h,g}^{-1}(\omega^h)) = g(\omega^h, \omega^g) = g(\omega, \omega)
\]

(where, for any non-degenerate bilinear form \(k\), we define \(\omega^k := \omega^{(k)}\)). Thus \((\mathcal{D}_g^h)^2\) is normally hyperbolic (and consequently \(\mathcal{D}_g^h\) is symmetric-hyperbolic) as long as \(g\) is globally hyperbolic.

The energy momentum tensor \(T_{(g,\psi)} := T_{(1, g, \psi, 0)}\) is first order in \(g\), thus the second order symbol of \(T_{(g,\psi)}\) w.r.t. \(g\) vanishes. To calculate the first order symbol w.r.t. \(\psi\), let \(a\) be a smooth function with \(a(p) = 0\), then \(T_{(g, a\psi)}(X, Y) = 0\). The coupling of one-forms into spinors and the one from spinors into one-forms are of zeroth order, the coupling from metrics to spinors is of first order, thus one order less than the part of the operator acting on the metric. Thus all in all the highest orders vanish in the off-diagonal terms.

Now the usual local existence and uniqueness holds for quasilinear symmetric-hyperbolic operators with range a convex neighborhood of 0, [Tay11, 16.1.2-16.1.7 and 16.2.1] for smooth coefficients, or [GM14, Thm. 4.2] (with the same modifications as above).

The previous Lemma 3.22 shows that, for every initial value set \(Z = (S, g, K, \psi_0, A_0, A_1)\), the class of developments of \(Z\) is nonempty.

**Lemma 3.23 (Geometric Uniqueness, cf. [CB09, Thm. 8.4]).** Let \((M_i, g_i, \psi_i, A_i(f_i))\) be developments of the same initial values \((S, g, K, \psi_0, A_0, A_1)\). Then there are open neighborhoods \(U_i\) of \(f_i(S)\) in \(M_i\) and a spin diffeomorphism \(\tilde{\alpha} : \Sigma U_1 \rightarrow \Sigma U_2\) such that \(\alpha \circ df_1 = df_2 : TS \rightarrow TM_2\), \(\alpha^*g_2 = g_1\), \(\alpha^*A(2) = A(1)\) in Lorentz gauge and \(\tilde{\alpha}\psi_1 = \psi_2\).

**Proof.** This is just an easy consequence of uniqueness of the gauged system as in Lemma 3.22 and of the fact that every metric can be brought into \(h\)-wave gauge as in Lemma 3.20: Take \(U_i\) and \(f_i\) as in Lemma 3.20. By Lemma 2.29 \(f_i^*(g_i, A_i, \psi_i)\) are solutions of EDM\(^{(h)}\) to the same initial values and thus they coincide.

**Definition 3.24 (Cauchy maximal).** A Cauchy development \(J = (M, g, \psi, A, f)\) is called Cauchy maximal, if there is no proper isometric embedding \(f : (M, g) \rightarrow (N, h)\) such that \(f(S)\) is a Cauchy surface of \((N, h)\) if \(S\) is a Cauchy surface of \((M, g)\).

**Theorem 3.25.** For a system of \(k\) spinors, there is a maximal Einstein-Dirac-Maxwell Cauchy development \(J = (M, g, \psi, A, f)\) of the initial values \(Z = (S, g_0, \psi_0^i, A_0, A_1)\). If \(\psi_0^i = 0\) for all \(i = 1, \ldots, k\) or if \(A_0, A_1 = 0\), then \(J\) is Cauchy-maximal. Furthermore, if the system is neutral and
if \( g \) has a conformal extension, there is a weighted Sobolev space \( H \) for \((\psi_0, A_0, A_1)\) and an \( \varepsilon > 0 \) such that if \( \|((\psi_0, A_0, A_1))_H < \varepsilon \), then \( J \) is Cauchy-maximal.

\[ \text{Proof.} \] The proof works via the usual procedure as in Choquet-Bruhat’s and Geroch’s seminal article [CBG69] using Lemmas 3.22 and 3.23. First, we consider the collection \( C \) of all developments of \( Z \).

Now we need a result of Bernal and Sánchez [BS05] to close a gap in the original proof given by Choquet-Bruhat and Geroch, as it was pointed out by Willie Wong [Won13], p.5. The gap is that apriori it is not clear that the collection \( C \) is a set and not a proper class. Wong indicates another possible way to close the gap, nevertheless the only way manifestly without using the axiom of choice is the one via the Bernal-Sánchez theorem, which we will need anyway for the second part of the statement.

The local existence in Lemma 3.22 shows that for every initial value this set of developments is nonempty.

The rest is as in [CBG69], or as in [Sbi13; Won13], where it is shown that the axiom of choice is not needed but only Zermelo-Fraenkel set theory axioms.

Cauchy-maximality of a maximal Cauchy development \((M, g, \psi, 0)\) or \((M, g, 0, A)\) in the sense of Definition 3.18 follows from the fact that, for any fixed globally hyperbolic metric \( g \), the Dirac equation is a linear symmetric hyperbolic system and thus has a global solution. Thus if \( g \) had a Cauchy extension \((\hat{M}, \hat{g})\), then there would be a solution \( \hat{\psi} \) on \((\hat{M}, \hat{g})\) to the same initial values as well, in contradiction to maximality of \((M, g, \psi, 0)\) and correspondingly for \((M, g, 0, A)\). Note that here we need again the Bernal-Sánchez result [BS05] on metric splittings of Lorentzian manifolds by time functions (or its refinements [MS11], [Mül15]) as we do not know beforehand whether the maximal Cauchy development being a globally hyperbolic manifold is regularly sliced (and in general it will not, see [Mül13]). The last statement follows directly from the global existence result in [GM14].

\[ \text{□} \]

Now, one can ask the question of geodesic completeness or at least Lorentzian maximality of the maximal Cauchy development around a solution with vanishing spinor fields, e.g., around the Minkowski solution applying the same machinery as for the Klein-Gordon equation or the Maxwell equation as in [LR10] and [Loi06].

The Definition 3.18 of maximal Cauchy development is tailor-made for Einstein-Dirac-Maxwell theory due to the fact that the spinor bundle is not a \( \text{SpinMf} \)-natural bundle. To obtain a maximal Cauchy development in the sense of Definition 3.17 one replaces the metric \( g \) and the spinor \( \psi \) by a universal spinor field \( \Phi \). Now, from Theorem 3.25, Theorem 2.26, Theorem 2.33 and Remark 2.35 follows immediately:

**Theorem 3.26 (SpinMf-Cauchy development for usb).** For a system of \( k \) spinors, there is a maximal Cauchy development of the universal EMD operator in the sense of Definition 3.17. If the system is neutral and massless and if the metric of the maximal Cauchy development has a conformal extension, then there is a weighted Sobolev ball of initial values around \((\psi, A) = 0\) such that the associated maximal Cauchy developments are Cauchy-maximal.

\[ \text{□} \]

**A. Appendix**

We collect some easy facts from linear algebra for convenient reference.
Theorem A.1 (Gram/Schmidt). Let \( V \) be an \( m \)-dimensional real vector space and \( g \) be a positive or negative definite metric on \( V \). Let \( v = (v_1, \ldots, v_m) \) be a basis of \( V \). Define

\[
    u_1 := v_1, \quad u_k := v_k - \sum_{j=1}^{k-1} \frac{g(v_k, u_j)}{g(u_j, u_j)} u_j, \quad e_j := \frac{u_j}{\|u_j\|}
\]

(A.1)

Then \( e = (e_1, \ldots, e_m) \) is an orthonormal basis. This defines a smooth map

\[
    GS_V : GL_V \times S_{m,0} V \to SO_V
\]

such that \( GS_V(SO^g V \times \{g\}) = \text{id} \).

Theorem A.2 (Pseudo-Gram/Schmidt). Let \( V \) be an \( m \)-dimensional real vector space and \( g \) be a metric on \( V \) of signature \((r,s)\). Let \( v_1, \ldots, v_m \) be a basis of \( V \) such that \( \text{sgn}(g(e_i, e_i)) = \varepsilon_i \). Define \( V^+ := \text{span}(v_1, \ldots, v_r) \). The basis \( e_1, \ldots, e_m \) defined by

\[
    \forall 1 \leq j \leq r : e_j := GS_{V^+}(v_j), \quad \forall r + 1 \leq j \leq m : e_j := GS_{V^+}(\text{pr}_{V^+}(v_j))
\]

is a pseudo-orthonormal basis of \( V \). Here \( \text{pr}_{V^+} : V \to V^+ \) denotes the orthogonal projection. This gives a well-defined continuous map

\[
    GS : GL_{r,s} V \to SO_{r,s} V,
\]

where \( GL_{r,s} V := \{b \in GL_V \mid \text{sgn}(g(b_i, b_i)) = \varepsilon_i\} \).

List of Symbols

\[
    \begin{align*}
        a(b, A) &:= b.A & \text{action of } A \in GL_m \text{ on a basis } b, \text{ page 4} \\
        \text{Asym}_{r,s} &:= (r,s)-\text{anti-symmetric matrices}, \text{ page 7} \\
        \beta_{g,h} &:= \text{identification isomorphism}, \text{ page 22} \\
        A^\dagger &:= \text{adjoint of } A \text{ w.r.t. } \langle_,_\rangle_{r,s}, \text{ page 4} \\
        I_{r,s} &:= \text{diagonal matrix with } r \text{ entries of } -1 \text{ and } s \text{ entries of } +1, \text{ page 4} \\
        \varepsilon_i &:= \text{signature signs}, \text{ page 4} \\
        \eta &:= \text{universal metric on } \Sigma_{SM}, \text{ page 9} \\
        \bar{\eta} &:= \text{universal metric on } \Sigma_{SM}, \text{ page 15} \\
        g_b &:= \text{metric induced by declaring } b \text{ to be pseudo-orthonormal}, \text{ page 4} \\
        \bar{g} &:= \text{universal metric on } (\pi_{0}^{r,s})^*(TM), \text{ page 15} \\
        g &:= \text{universal metric on } (\pi^{r,s})^*(TM), \text{ page 10} \\
        GL^+_m &:= \text{invertible matrices with positive determinant}, \text{ page 4} \\
        GL^+ V &:= \text{positive bases of } V, \text{ page 4} \\
        \tilde{GL}^+_m &:= \text{non-trivial double cover of } GL^+_m, \text{ page 4} \\
        I_m &:= \text{unit matrix}, \text{ page 4} \\
        J^1_{\pi_P} &:= \text{first jet bundle of } \pi_P, \text{ page 12} \\
        j_\psi &:= \text{Dirac current}, \text{ page 20} \\
        \kappa^V &:= GL^+ V \to S_{r,s} V, \text{ page 5} \\
        \bar{\kappa}^V &:= \tilde{GL}^+ V \to S_{r,s} V, \text{ page 6} \\
        m &:= \text{universal Clifford multiplication on } \Sigma_{SM}, \text{ page 9} \\
        M &:= \text{a smooth manifold of dimension } m, \text{ page 2}
    \end{align*}
\]
Clifford multiplication for $\Sigma_{SM}$, page 15

canonical metric on $\mathbb{R}^m$ of signature $(r, s)$, page 4

Clifford multiplication w.r.t. $g$, page 10

vertical universal Levi-Civita connection, page 10

universal Levi-Civita connection, page 15

vertical spinorial connection, page 9

connection on universal spinor jet bundle, page 14

$(\text{spinorial})$ Levi-Civita connection, page 10

jetted universal spinor (fiber) bundle, page 14

jetted universal spinor (vector) bundle, page 14

classical metric spinor bundle, page 2

universal spinor (fiber) bundle, page 7

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fixed choice of spinor representations $\text{Spin}_{r,s} \to \text{GL}^+(\Sigma_{r,s})$, page 6

signature, $r + s = m$, page 2

special orthogonal group of signature $(r, s)$, page 4

a metric spin structure, page 2

space of metrics of signature $(r, s)$, page 2

symmetric matrices with $r$ positive and $s$ negative eigenvalues, page 4

space of the inner products of signature $(r, s)$ on $V$, page 4

$(r, s)$-symmetric matrices, page 7

coordinate matrix of a basis, page 5

$(\pi^r,s_0)^*(TM) \to J^1\pi^r,s$, page 18

tangent bundle, page 2

$(\pi^r,s)^*(TM) \to S_{r,s}M$, page 9

vertical tangent bundle of $P$, page 7

energy momentum tensor of $(g, A)$, page 20

energy momentum tensor of $(g, \psi)$, page 20

total energy momentum tensor of $(g, \psi, A)$, page 20

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$2 : 1$-cover $\text{GL}^+_m \to \text{GL}^+_m$, page 6

$2 : 1$-cover $\text{Spin}_{r,s} \to \text{SO}_{r,s}$, page 6

horizontal distribution on $\text{GL}^+_V$, page 7

horizontal distribution on $\text{GL}^+_V$, page 9

vertical distribution on $\text{GL}^+_V$, page 7

vertical distribution on $\text{GL}^+_V$, page 9

universal spinor bundle functor, page 19

universal spinor jet bundle functor, page 18
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