An asymptotic formula of the divergent bilateral basic hypergeometric series

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Abstract

We show an asymptotic formula of the divergent bilateral basic hypergeometric series \( \psi_0(a; -; q, \cdot) \) with using the \( q \)-Borel-Laplace method. We also give the limit \( q \to 1 - 0 \) of our asymptotic formula.

1 Introduction

In this paper, we show an asymptotic formula of the divergent bilateral basic hypergeometric series

\[
\psi_0(a; -; q; x) := \sum_{n \in \mathbb{Z}} (a; q)_n \left\{ (-1)^n q^{n(n-1)/2} \right\}^{-1} x^n.
\]

Here, \((a; q)_n\) is the \(q\)-shifted factorial;

\[
(a; q)_n := \begin{cases} 1, & n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & n \geq 1, \\ [(1-aq^{-1})(1-aq^{-2})\cdots(1-aq^n)]^{-1}, & n \leq -1 \end{cases}
\]

moreover, \((a; q)_\infty := \lim_{n \to \infty} (a; q)_n\) and

\[
(a_1, a_2, \ldots, a_m; q)_\infty := (a_1; q)_\infty(a_2; q)_\infty \cdots (a_m; q)_\infty.
\]

The \(q\)-shifted factorial \((a; q)_n\) is a \(q\)-analogue of the shifted factorial

\[
(a)_n = a\{a + 1\} \cdots \{a + (n - 1)\}.
\]

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The series $\psi_0(a; -; q, \cdot)$ is related to the bilateral basic hypergeometric series $\psi_1(a; b; q, \cdot)$. At first, we review the series

$$\psi_1(a; b; q, z) := \sum_{n \in \mathbb{Z}} \frac{(a; q)_n}{(b; q)_n} z^n.$$ 

This series has Ramanujan’s summation formula [3]

$$\psi_1(a; b; q, z) = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \quad |b/a| < |z| < 1, \quad (1)$$

which was first given by Ramanujan [4]. This formula is considered as an extension of the $q$-binomial theorem [3, 1]:

$$\sum_{n \geq 0} \frac{(a; q)_n}{(q; q)_n} z^n = \frac{(az; q)_\infty}{(z; q)_\infty}, \quad |z| < 1.$$ 

We also regard the summation (1) as a $q$-analogue of the bilateral binomial theorem [6] discovered by M. E. Horn. If $\alpha$ and $\beta$ are complex numbers $\Re(\beta - \alpha) > 1$ and $z$ is a complex number with $|z| = 1$, then

$$\sum_{n \in \mathbb{Z}} \frac{(\alpha)_n}{(\beta)_n} z^n = \frac{(1 - z)^{\beta - \alpha - 1} \Gamma(1 - \alpha) \Gamma(\beta)}{(-z)^{\beta - 1} \Gamma(\beta - \alpha)}. \quad (2)$$

We remark that the limit $q \to 1 - 0$ of (1) with suitable condition gives the bilateral binomial theorem (2).

The series $\psi_1(a; b; q, z)$ satisfies the following $q$-difference equation

$$\left(\frac{b}{q} - az\right) u(qz) + (z - 1)u(z) = 0.$$

In this paper, we study the degeneration of this equation as follows;

$$\left(\frac{1}{q} - ax\right) \tilde{u}(qx) + x\tilde{u}(x) = 0.$$

This equation has a formal solution

$$\tilde{u}(x) = \psi_0(a; -; q, x). \quad (3)$$
But $\tilde{u}(x)$ is a divergent around the origin \[3\] and its properties are not clear. In section three, we show an asymptotic formula of the series \[3\] with using the $q$-Borel-Laplace transformations. The $q$-Borel-Laplace transformations are introduced in the study of connection problems on linear $q$-difference equations between the origin and the infinity.

Connection problems on $q$-difference equations with regular singular points are studied by G. D. Birkhoff \[2\]. The first example of the connection formula is given by G. N. Watson \[13\] in 1910 as follows;

$$2\varphi_1(a, b; c; q, x) = \frac{(b, c/a; q)\infty (ax, q/ax; q)\infty}{(c, b/a; q)\infty (x, q/x; q)\infty} 2\varphi_1 (a, aq/c; aq/b; q, cq/abx)$$

$$+ \frac{(a, c/b; q)\infty (bx, q/bx; q)\infty}{(c, a/b; q)\infty (x, q/x; q)\infty} 2\varphi_1 (b, bq/c; bq/a; q, cq/abx). \quad (4)$$

The function $2\varphi_1(a, b; c; q, x)$ is the basic hypergeometric series

$$2\varphi_1(a, b; c; q, x) = \sum_{n \geq 0} \frac{(a, b; q)_n}{(c; q)_n (q; q)_n} x^n,$$

which is introduced by E. Heine \[3\]. This series satisfies the second order linear $q$-difference equation

$$(c - abqx)u(q^2x) - \{c + q - (a + b)qx\} u(qx) + q(1 - x)u(x) = 0. \quad (5)$$

around the origin. On the other hand, the equation \[3\] has a fundamental system of solutions

$$v_1(x) = \frac{\theta(-ax)}{\theta(-x)} 2\varphi_1 \left( a, \frac{aq}{c}; b; q, \frac{cq}{abx} \right), \quad v_2(x) = \frac{\theta(-bx)}{\theta(-x)} 2\varphi_1 \left( b, \frac{bq}{c}; a; q, \frac{cq}{abx} \right)$$

around the infinity \[14\]. Here, $\theta(\cdot)$ is the theta function of Jacobi (see section two). In the connection formula \[4\], we remark that the connection coefficients are given by $q$-elliptic functions.

Other connection formulae, especially connection formulae of irregular singular type $q$-special functions has not known for a long time. Recently, C. Zhang gives some connection formulae by two different types of $q$-Borel transformations and $q$-Laplace transformations \[14, 15, 16\], which are introduced by J. Sauloy \[11\]. We assume that $f(x)$ is a formal power series

$$f(x) = \sum_{n \geq 0} a_n x^n, \quad a_0 = 1.$$
Definition 1. For any power series \( f(x) \), the \( q \)-Borel transformation of the first kind \( \mathcal{B}_q^+ \) is

\[
(\mathcal{B}_q^+ f)(\xi) := \sum_{n \geq 0} a_n q^{\frac{n(n-1)}{2}} \frac{\xi^n}{n!} =: \varphi(\xi).
\]

For an entire function \( \varphi \), the \( q \)-Laplace transformation of the first kind \( \mathcal{L}_{q,\lambda}^+ \) is

\[
(\mathcal{L}_{q,\lambda}^+ \varphi)(x) := \sum_{n \in \mathbb{Z}} \frac{\varphi(\lambda q^n)}{\theta(\lambda q^n/x)}, \quad \lambda \notin q \mathbb{Z}.
\]

Similarly, the \( q \)-Borel transformation of the second kind \( \mathcal{B}_q^- \) is

\[
(\mathcal{B}_q^- f)(\xi) := \sum_{n \geq 0} a_n q^{-\frac{n(n-1)}{2}} \xi^n =: g(\xi)
\]

and the \( q \)-Laplace transformation of the second kind \( \mathcal{L}_q^- \) is

\[
(\mathcal{L}_q^- g)(x) := \frac{1}{2\pi i} \int_{|\xi|=r} g(\xi) \theta\left(\frac{x}{\xi}\right) \frac{d\xi}{\xi}.
\]

We remark that each \( q \)-Borel transformation is a formal inverse of each \( q \)-Laplace transformation;

\[\mathcal{L}_{q,\lambda}^+ \circ \mathcal{B}_q^+ f = f, \quad \mathcal{L}_q^- \circ \mathcal{B}_q^- f = f.\]

We can find applications of the \( q \)-Borel-Laplace method of the first kind in [14, 15, 10]. This summation method is powerful to deal with divergent type basic hypergeometric series and to study the \( q \)-Stokes phenomenon. Applications of the method of the second kind can be found in [16, 8, 9, 10]. But other examples, for example, applications to bilateral basic hypergeometric series, has not known.

In this paper, we study an asymptotic formula of the divergent bilateral basic hypergeometric series \( _1\psi_0(a; -; q, \cdot) \) from viewpoint of connection problems on linear \( q \)-difference equations. We show the following theorem.

**Theorem.** For any \( x \in \mathbb{C}^* \setminus [-\lambda; q] \), we have

\[
_1\psi_0(a; -; \lambda; q, x) = \frac{(q; q)_\infty}{(q/a; q)_\infty} \frac{\theta(aq\lambda) \theta(ax/\lambda)}{\theta(q\lambda) \theta(x/\lambda)} \frac{1}{(1/ax; q)_\infty},
\]

where \( 1 < |ax| \).
Here, the function $\hat{\psi}_0(a; -; \lambda; q, x)$ is the $q$-Borel-Laplace transform of the series $\psi_0(a; -; q, x)$ and the set $[-\lambda; q]$ is the $q$-spiral which is given by $[\lambda; q] := \{\lambda q^k : k \in \mathbb{Z}\}$. We also show the limit $q \to 1 - 0$ of our asymptotic formula in section four. If we take a suitable limit $q \to 1 - 0$ of the theorem above, we formally obtain the asymptotic formula of the bilateral hypergeometric series $\hat{1}H_0$.

2 Basic notations

In this section, we fix our notations. We assume that $0 < |q| < 1$ and $\sigma_q$ is the $q$-difference operator such that $\sigma_q f(x) = f(qx)$. The basic hypergeometric series is

$$r \varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, x) := \sum_{n \geq 0} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{1+s-r} x^n. \quad (6)$$

The radius of convergence $\rho$ of the basic hypergeometric series is given by [7]

$$\rho = \begin{cases} \infty, & \text{if } r < s + 1, \\ 1, & \text{if } r = s + 1, \\ 0, & \text{if } r > s + 1. \end{cases}$$

The bilateral basic hypergeometric series is

$$r \psi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, x) := \sum_{n \in \mathbb{Z}} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n} \left\{ (-1)^n q^{\frac{n(n-1)}{2}} \right\}^{s-r} x^n. \quad (7)$$

If $s < r$, the series (7) diverges for $x \neq 0$ and if $r = s$, the series (7) converges $|b_1 \ldots b_s/a_1 \ldots a_r| < |x| < 1$ (see [3] for more detail). The series (6) is a $q$-analogue of the generalized hypergeometric function

$$r F_s(a_1, \ldots, a_r; \beta_1, \ldots, \beta_s; x) := \sum_{n \geq 0} \frac{(a_1, \ldots, a_r)_n}{(\beta_1, \ldots, \beta_s)_n n!} x^n$$

and the series (7) is a $q$-analogue of the bilateral hypergeometric function

$$r H_s(a_1, \ldots, a_r; \beta_1, \ldots, \beta_s; x) := \sum_{n \in \mathbb{Z}} \frac{(a_1, \ldots, a_r)_n}{(\beta_1, \ldots, \beta_s)_n} x^n.$$

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By D’Alembert’s ratio test, it can be checked that $H_r$ converges only for $|x| = 1$ \cite{12}, provided that $\Re(\beta_1 + \cdots + \beta_r - \alpha_1 - \cdots - \alpha_r) > 1$.

The $q$-exponential function $e_q(x)$ is

$$e_q(x) := _1\varphi_0(0; -q, x) = \sum_{n \geq 0} \frac{x^n}{(q; q)_n}$$

The function $e_q(x)$ has the infinite product representation as follows;

$$e_q(x) = \frac{1}{(x; q)_{\infty}}, \quad |x| < 1.$$  

The limit $q \to 1 - 0$ of $e_q(x)$ is the exponential function \cite{3};

$$\lim_{q \to 1 - 0} e_q(x(1 - q)) = e^x. \quad (8)$$

The $q$-gamma function $\Gamma_q(x)$ is

$$\Gamma_q(x) := \frac{(q; q)_{\infty}}{(q^x; q)_{\infty}}(1 - q)^{1-x}, \quad 0 < q < 1.$$  

The limit $q \to 1 - 0$ of $\Gamma_q(x)$ gives the gamma function \cite{3};

$$\lim_{q \to 1 - 0} \Gamma_q(x) = \Gamma(x). \quad (9)$$

The theta function of Jacobi is given by

$$\theta(x) := \sum_{n \in \mathbb{Z}} q^{n(n-1)/2} x^n, \quad \forall x \in \mathbb{C}^*.$$  

Jacobi’s triple product identity is

$$\theta(x) = (q, -x, -q/x; q)_{\infty}$$

and the theta function satisfies the following $q$-difference equation

$$\theta(q^k x) = x^{-k} q^{-\frac{k(k-1)}{2}} \theta(x), \quad \forall k \in \mathbb{Z}.$$  

The inversion formula is

$$\theta(x) = x \theta(1/x). \quad (10)$$

We remark that $\theta(\lambda q^k / x) = 0$ if and only if $x \in [-\lambda; q]$. In our study, the following proposition \cite{13} is useful to consider the limit $q \to 1 - 0$ of our formula.
Proposition 1. For any \( x \in \mathbb{C}^*(-\pi < \arg x < \pi) \), we have

\[
\lim_{q \to 1^{-0}} \frac{\theta(q^\alpha x)}{\theta(q^\beta x)} = x^{\beta - \alpha}
\]

and

\[
\lim_{q \to 1^{-0}} \frac{\theta \left( \frac{q^\alpha x}{1-q} \right)}{\theta \left( \frac{q^\beta x}{1-q} \right)} (1-q)^{\beta - \alpha} = \left( \frac{1}{x} \right)^{\alpha - \beta}.
\]

We also use the limit \( [3] \) as follows;

\[
\lim_{q \to 1^{-0}} \frac{(xq^\alpha; q)_\infty}{(x; q)_\infty} = (1-x)^{-\alpha}, \quad |x| < 1.
\]

3 An asymptotic formula of the divergent series \( 1\psi_0(a; -; q, x) \)

In this section, we show an asymptotic formula of the divergent series \( 1\psi_0(a; -; q, x) \). At first, we review Ramanujan’s sum for \( 1\psi_1(a; b; q, z) \) and its property. Ramanujan gives the following sum \( [4] \)

\[
1\psi_1(a; b; q, z) = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \quad |b/a| < |z| < 1.
\]

We can regard this sum as a \( q \)-analogue of the bilateral binomial theorem \( [6] \) as follows;

**Theorem.** (Horn \([8]\)) If \( z \) is a complex number such that \( |z| = 1 \), we have

\[
1H_1(\alpha; \beta; z) = \frac{\Gamma(\beta)\Gamma(1-\alpha)}{\Gamma(\beta-\alpha)} \frac{(1-z)^{\beta-\alpha-1}}{(-z)^{\beta-1}},
\]

provided that \( \Re(\beta - \alpha) > 1 \).

Let \( z \) is a complex number such that \( -\pi < \arg z < \pi \). The summation \( [14] \) is rewritten by

\[
\frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty} = \frac{(q, b/a; q)_\infty}{(b, q/a; q)_\infty} \frac{\theta(-az)}{\theta(-aqz/b)} \frac{(aqz/b; q)_\infty}{(z; q)_\infty}.
\]
In (16), we put \( a = q^\alpha \) and \( b = q^\beta \) (with \( \Re(\beta - \alpha) > 1 \)), we obtain

\[
1\psi_1(q^\alpha; q^\beta; q, z) = \frac{\Gamma_q(\beta)\Gamma_q(1-\alpha)}{\Gamma_q(\beta - \alpha)} \frac{\theta(q^\alpha(-z))}{\theta(q^{\alpha+1-\beta}(-z))} \frac{(q^{\alpha+1-\beta}z; q)_\infty}{(z; q)_\infty}, \tag{17}
\]

where \(|q^{\beta-\alpha}| < |z| < 1\). Combining (9), (11) and (13), we can check out that the limit \( q \to 1-0 \) of (17) gives the bilateral binomial theorem (15), provided that \(-\pi < \arg z < \pi\).

The bilateral basic hypergeometric series \( 1\psi_1(a; b; q, z) \) satisfies the \( q \)-difference equation

\[
\left( \frac{b}{q} - az \right) u(qz) + (z-1)u(z) = 0. \tag{18}
\]

We consider the degeneration of the equation (18) in the next section.

### 3.1 Local solutions of the degenerated equation

In the equation (18), we put \( z = bx \) and take the limit \( b \to \infty \), we obtain the equation

\[
\left\{ \left( \frac{1}{q} - ax \right) \sigma_q + x \right\} \tilde{u}(x) = 0. \tag{19}
\]

The formal solution of (19) is

\[
\tilde{u}(x) = 1\psi_1(a; -; q, x), \tag{20}
\]

which is divergent around the origin. We consider “the basic hypergeometric type” solution around the infinity. We assume that the solution around the infinity is given by

\[
\tilde{u}_\infty(x) := \frac{\theta(ax)}{\theta(x)} v_\infty(x) = \frac{\theta(ax)}{\theta(x)} \sum_{n \geq 0} v_n x^{-n}, \quad v_0 = 1.
\]

By the \( q \)-difference equation of the theta function, we obtain the equation

\[
\left\{ \left( \frac{1}{aq} - x \right) \sigma_q + x \right\} v_\infty(x) = 0.
\]

We remark that the function \( \theta(ax)/\theta(x) \) satisfies the following \( q \)-difference equation

\[
u(qx) = q^{-\alpha} u(x)
\]
which is also satisfied by the function \( u(x) = x^{-\alpha} \) with \( \log_q a = \alpha \). We can check out that the series \( v_\infty(x) \) is

\[
v_\infty(x) = e_q \left( \frac{1}{ax} \right) = \frac{1}{(1/ax; q)_\infty}, \quad \left| \frac{1}{ax} \right| < 1.
\]

Therefore, one of the solution of (19) around the infinity is

\[
\tilde{u}_\infty(x) = \frac{\theta(ax)}{\theta(x)} e_q \left( \frac{1}{ax} \right).
\]

(21)

In the following section, we study the relation between these solutions (20) and (21) with using the \( q \)-Borel-Laplace method of the first kind.

**Remark 1.** We remark that “the bilateral basic hypergeometric type solution” of (19) around the infinity is given by

\[
w_\infty(x) = 1\psi_0(0; \frac{q}{a}; q) \frac{1}{(1/ax; q)_\infty} = \sum_{n \in \mathbb{Z}} \frac{1}{(q/a; q)_n} \left( \frac{1}{ax} \right)^n.
\]

But this solution is not suitable for our argument. We choose and deal with the solution (21).

### 3.2 An asymptotic formula

In this section, we show an asymptotic formula of (21) with using the \( q \)-Borel-Laplace transformations of the first kind. We show the following theorem.

**Theorem 1.** For any \( x \in \mathbb{C}^* \setminus [-\lambda; q] \), we have

\[
\hat{\psi}_0(a; -\lambda; q, x) = \frac{(q; q)_\infty}{(q/a; q)_\infty} \frac{\theta(aq\lambda) \theta(ax/\lambda)}{\theta(q\lambda) \theta(x/\lambda)} \frac{1}{(1/ax; q)_\infty},
\]

where \( 1 < |ax| \).

**Proof.** We apply the \( q \)-Borel transformation \( B_q^+ \) to the series \( \hat{\psi}_0(a; -q, x) \). Then,

\[
\psi(\xi) := (B_q^+ \hat{\psi}_0)(\xi) = \hat{\psi}_0(a; 0; q, -\xi).
\]

By Ramamujan’s sum (14), we obtain the infinite product representation of \( \psi(\xi) \) as follows;

\[
\psi(\xi) = \frac{(q; q)_\infty}{(q/a; q)_\infty} \frac{\theta(a\xi)}{\theta(\xi)} (-q/\xi; q)_\infty.
\]
We apply the $q$-Laplace transformation $\mathcal{L}_{q,\lambda}^+$ to $\psi(\xi)$.

\[
(\mathcal{L}_{q,\lambda}^+ \psi)(x) = \sum_{n \in \mathbb{Z}} \frac{\psi(\lambda q^n)}{\theta(\frac{\lambda q^n}{x})} = (q; q)_\infty \frac{1}{(q/a; q)_\infty} \sum_{n \in \mathbb{Z}} \frac{\theta(a \lambda q^n)}{\theta(\lambda q^n)} \left( -\frac{q}{\lambda q^n}; q \right)_\infty
\]

We remark that

\[
1 \psi_1(-\lambda; 0; q, 1/ax) = \frac{\theta(\lambda/ax)}{(-q/\lambda; q)_\infty(1/ax; q)_\infty}.
\]

Combining (22) and (10), we obtain the conclusion.

We define the function $C(x; q)$ as follows;

\[
C(x; q) := \frac{(q; q)_\infty}{(q/a; q)_\infty} \frac{1}{\theta(\lambda/\lambda) \theta(\lambda/\lambda)} \left( -q/\lambda; q \right)_\infty 1 \psi_1(-\lambda; 0; q, 1/ax).
\]

Then, $C(x; q)$ is single valued as a function of $x$.

**Corollary 1.** The function $C(x; q)$ is the $q$-elliptic function, i.e., $C(x; q)$ satisfies

\[
C(qx; q) = C(x; q), \quad C(e^{2\pi i}x; q) = C(x; q).
\]

### 4 The limit $q \to 1 - 0$ of the asymptotic formula

In this section, we give the limit $q \to 1 - 0$ of our asymptotic formula. We assume that $x \in \mathbb{C}^* \setminus [-\lambda; q] (-\pi < \arg x < \pi)$. We put $a = q^n$ and $x \mapsto x/(1-q)$ in theorem 1 to consider the limit. We remark that the limit of the left-hand side in theorem 1 formally gives the bilateral hypergeometric series $\psi_{10}(\alpha; -; -x)$. We show the following theorem.
Theorem 2. For any $x \in \mathbb{C} \setminus [-\lambda; q]$ ($-\pi < \arg x < \pi$), we have

$$
\lim_{q \to 1-0} \frac{(q; q)_\infty}{(q^{1-\alpha}; q)_\infty} \frac{\theta(q^{\alpha+1}\lambda)}{\theta(q\lambda)} \frac{\theta \left( \frac{q^\alpha}{(1-q)\lambda} \right)}{\theta \left( \frac{x}{(1-q)\lambda} \right)} \frac{1}{(1-q)^{\frac{\alpha-\alpha}{x}; q)_\infty} = \frac{\Gamma(1 - \alpha)}{x^\alpha} e^{\frac{\lambda}{2}},
$$

where $1 < |x|$.

We give the proof of theorem 2.

Proof. The right-hand side of theorem 1 is rewritten by

$$
\frac{(q; q)_\infty}{(q^{1-\alpha}; q)_\infty} \frac{\theta(q^{\alpha+1}\lambda)}{\theta(q\lambda)} \frac{\theta \left( \frac{q^\alpha}{(1-q)\lambda} \right)}{\theta \left( \frac{x}{(1-q)\lambda} \right)} \frac{1}{(1-q)^{\frac{\alpha-\alpha}{x}; q)_\infty} = \Gamma_q(1 - \alpha) \frac{\theta(q^{\alpha+1}\lambda)}{\theta(q\lambda)} \left\{ \frac{\theta \left( \frac{q^\alpha}{(1-q)\lambda} \right)}{\theta \left( \frac{q^\alpha}{(1-q)\lambda} \right)} (1-q)^{-\alpha} \right\} e_q \left( (1-q)^{\frac{1}{q^\alpha x}} \right).
$$

Combining (9), (11), (12) and (8), we obtain the conclusion. 

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