A topological pinching for the injectivity radius of a compact surface in $S^3$ and in $H^3$

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Abstract

The main result of this article is the following topological injectivity radius pinching theorem for a compact embedded surface $M$ in $S^3$ or $H^3$: If the (extrinsic) injectivity radius $M$ is bigger than or equal to

$$\cot^{-1}\left(\frac{\int_M H d\sigma}{A(M)}\right)$$

in $S^3$, to

$$\coth^{-1}\left(\frac{\int_M H d\sigma}{A(M)}\right)$$

in $H^3$,

where $H$ is the mean curvature and $A(M)$ the area of $M$, then $M$ is homeomorphic to a sphere.

It is given a topological pinching for the injectivity radius of a compact embedded surface either in the sphere or in the hyperbolic space

1 Introduction

Given $k \in \{-1,1\}$ denote by $Q_k$ the complete simply connected 3–dimensional Riemannian manifold of constant sectional curvature $k$; that is $Q_1 = S^3$, the unit sphere, and $Q_{-1} = H^3$, the hyperbolic space of constant sectional curvature $-1$.

Let $M$ be a compact embedded surface in $Q_k$. Consider the unit vector field $\eta$ normal to $M$ such that

$$\int_M H d\sigma \geq 0$$

(1)

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where $H$ is the mean curvature of $M$ with respect to $\eta$. If the integral above is zero $\eta$ may be arbitrarily chosen. Denote by $\Omega_+$ the connected component of $Q_k \setminus M$ which $\eta$ is pointing to. Let $F_{\Omega_+}$ be the set of focal points of $M$ in $\Omega_+$ (see next section). We call

$$\rho_{M^+} = d(F_{\Omega_+}, M) = \inf\{d(p, q) \mid p \in F_{\Omega_+}, \ q \in M\}$$

the (extrinsic) injectivity radius of $M$.

Although there are surfaces of any two given topological type having the same injectivity radius, it is intuitively clear that after some sort of uniformization, the biggest injectivity radius should occur among surfaces homeomorphic to a sphere. Indeed: We prove in this article that if

$$\rho_{M^+} \geq \cot^{-1}\left(\frac{\int_M H d\sigma}{A(M)}\right)$$

if $M$ is in $S^3$ and

$$\rho_{M^+} \geq \coth^{-1}\left(\frac{\int_M H d\sigma}{A(M)}\right)$$

if $M$ is in $H^3$, where $A(M)$ is the area of $M$, then $M$ is homeomorphic to a sphere.

This result is based in a theorem (Theorem 1) that estimates the volume of a tubular neighborhood of the surface, and from which we obtain some other applications, as a weaker version of a disproved conjecture of Blaine Lawson (Corollary 4) and estimates of injectivity radius for surfaces in $S^3$ (Theorem 7).

2 Preliminaries

Let $M$ be a compact, smooth, without boundary, embedded surface of $Q_k$. Consider the unit vector field $\eta$ normal to $M$ such that (1) is true. If the integral in (1) is zero $\eta$ may be arbitrarily chosen. Denote by $\Omega_+$ the connected component of $Q_k \setminus M$ which $\eta$ is pointing to and by $\Omega_-$ the other one.

Denote by $d$ the Riemannian distance in $Q_k$ and by $d_M$ the distance to $M$, that is

$$d_M(p) = \min\{d(p, q) \mid q \in M\}, \ p \in Q_k.$$ 

Recall that $q \in Q_k$ is a focal point of $M$ if $q$ is a critical value of the normal exponential map $\phi : TM^+ \to Q_k, \ \phi(p, \nu) = \exp_p \nu$, where $\exp_p$
is the exponential map of $Q_k$. Denote by $F$ the set of focal points of $M$, $F_{Ω±} = Ω± ∩ F$, set

$$ρ_{M±} = \min\{d(p, q) \mid p ∈ M, q ∈ F_{Ω±}\}$$

and

$$ρ_M = \min\{ρ_{M+}, ρ_{M−}\}.$$

Given $t ∈ [0, ρ_{M±}]$, set

$$Ω^t_{Ω±} = \{p ∈ Ω± \mid d_M(p) < t\}.$$

We use the notations

$$S_k(t) = \begin{cases} \sin t, & \text{if } k = 1 \\ \sinh t, & \text{if } k = -1 \end{cases}$$

and

$$C_k(t) = S'_k(t).$$

We shall make use of a particular version of Reilly’s formula stated as follows. Let $Ω$ be a compact orientable Riemannian manifold with boundary, $\dim Ω = 3$, and let $∇$ be the Riemannian connection of $Ω$. Let $η$ be the unit exterior normal vector of $L := ∂Ω$. Let $H$ be the mean curvature of $L$ with respect to $η$, that is, half of the trace, at each point $p$ of $L$, of the map $A(v) = −∇_v η$, $v ∈ T_p L$. Let $g ∈ C^2(Ω)$ be any given function. The Laplacian and the norm of the Hessian of $g$ are $Δg = \sum (∇_{ei} \text{grad } g, e_i)$ and $|\text{Hes}(g)|^2 = \sum (∇_{e_i} \text{grad } g, ∇_{e_i} \text{grad } g)$ respectively, where $e_i$ is any local orthonormal basis of vector fields of $Ω$. Set $z = g|_{∂Ω}$ and $u = \langle \text{grad } g|_L, η \rangle$ and denote also by $Δ$ and $\text{grad}$ the Laplacian and gradient in $L$. Then (see [4])

\begin{align*}
∫_Ω \left[ (Δg)^2 − |\text{Hes}(g)|^2 − ∫_Ω \text{Ric}(\text{grad } g, \text{grad } g) \right] dΩ \\
= ∫_L [(2Δz - 2Hu) u + \langle A(\text{grad } z), \text{grad } z \rangle] dL
\end{align*}

(2)

where $\text{Ric}(\ , \ )$ is the (non normalized) Ricci tensor of $Ω$. 

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3 Statement and proof of the results

We begin by proving a formula for the volume of a tubular neighborhood of $M$ which is fundamental for the results of the paper and which has interest on its own:

**Theorem 1** Let $M$ be a compact surface embedded either in the sphere $\mathbb{S}^3$ or in the hyperbolic space $\mathbb{H}^3$. Given $t \in [0, \rho_M \pm]$ we have

$$\text{Vol}(\Omega^t_{\pm}) = \pi \chi(M) (t - \sin t \cos t) + \sin t \cos t A(M) \mp \sin^2 t \int_M H\sigma$$  \hspace{1cm} (3)

if $M$ is in $\mathbb{S}^3$ and

$$\text{Vol}(\Omega^t_{\pm}) = \pi \chi(M) (\sinh t \cosh t - t) + \sinh t \cosh t A(M) \mp \sinh^2 t \int_M H\sigma.$$ \hspace{1cm} (4)

where $\chi(M)$ is the Euler characteristic of $M$ (recall that $H$ is the mean curvature of $M$ with respect to the unit normal vector pointing to $\Omega_+$).

**Proof.** The function $d_M$ is smooth on $\Omega^t_\pm$ and it follows from Gauss equation that at a given $p \in \Omega^t_\pm$ we have

$$(\Delta d_M)^2 - |\text{Hes}(d_M)|^2 = 2K_{Q_k}(e_1, e_2) - 2K_{M_{dM}(p)}(e_1, e_2)$$

where $\{e_1, e_2\}$ is an orthonormal basis of $T_pM_{dM(p)}$, $M_{dM(p)}$ being the parallel hypersurface of $M$ such that $d(M, M_{dM(p)}) = d_M(p)$ and $k = K_{Q_k}, K_{M_{dM(p)}}$ the sectional curvatures. Applying Reilly’s formula (2) with $g = d_M$ we then obtain

$$2 \text{Vol}(\Omega^t_{\pm}) = \int_{\Omega^t_{\pm}} K_{M_\pm} \omega + \int_M H\sigma + \int_{M_t} H_t \sigma_t,$$  \hspace{1cm} (5)

where $H_t$ is the mean curvature of $M_t$ with respect to $\eta_t = (\text{grad } d_M)|_{M_t}$. We now compute $H_t$ at a given point $p_t \in M_t$. Assume that $p_t = \varphi_t(p), p \in M$, where $\varphi_t(p) = \exp_p t\eta(p)$. Let $e_1, e_2$ on orthonormal basis of eigenvectors of the second fundamental form $A$ of $M$ at $p$. Thus $A(e_i) = \lambda_i(p)e_i$. Let $e_i(t)$ be the parallel transport of $e_i$ along the geodesic $\gamma(t) = \exp_p t\eta(p)$. It follows that $\{e_1(t), e_2(t)\}$ is an orthonormal basis of $T_{\gamma(t)}M_t$ and then

$$H_t(p_t) = \frac{1}{2} \sum_{i=1}^2 (e_i(t), A_t(e_i(t)))$$  \hspace{1cm} (6)
where $A_t(u) = -\nabla_u \eta_t$ is the second fundamental form of $M_t$, $\eta_t = -(\text{grad } d_M(t))|_{M_t}$.

Let $J_i$ be the Jacobi field along $\gamma$ in $Q_k$ satisfying $J_i(0) = e_i$ and $J'_i(0) = -\lambda_i(p)e_i$. Then $J_i$ is given by

$$J_i(t) = (C_k(t) - \lambda_i(p)S_k(t))e_i(t),$$

from what we have

$$e_i(t) = \frac{J_i(t)}{||J_i(t)||}.$$  \hspace{1cm} (7)

Replacing (7) em (6), and observing that $J_i$ satisfies the equation

$$\langle J_i(t), A_t(J_i(t)) \rangle = -\langle J_i(t), J'_i(t) \rangle$$

along $\gamma$ we obtain

$$H_t(p_t) = \frac{1}{2} \sum_{i=1}^{2} kS_k(t) + \lambda_i(p)C_k(t) \frac{C_k(t) - \lambda_i(p)S_k(t)}{C_k(t) - \lambda_i(p)S_k(t)}.$$ \hspace{1cm} (8)

We may write

$$\int_{S_t} H_t \sigma_t = \int_{M} (H_t \circ \varphi_t) \varphi^*_t(\sigma_t) = \int_{M} (H_t \circ \varphi_t) \delta \sigma.$$ \hspace{1cm} (9)

where $\delta$ is the Jacobian of $\varphi_t$. We have

$$\delta(p) = \varphi^*_t(\sigma_t)(p)\left(e_1, ..., e_{n-1}\right) =$$

$$= \sigma_t(p_t)\left(d(\varphi_t)_p(e_1), ..., d(\varphi_t)_p(e_{n-1})\right) =$$

$$= \sigma_t(p_t)\left(J_1(t), ..., J_{n-1}(t)\right) =$$

$$= \sqrt{\text{det} \langle J_i(t), J_k(t) \rangle} = \prod_{i=1}^{2} \left(\frac{C_k(t) - \lambda_i(p)S_k(t)}{C_k(t) - \lambda_i(p)S_k(t)}\right).$$ \hspace{1cm} (10)

In the spherical case, replacing (10) and (8) in (9) and taking into (5), using the Gauss equation $K = 1 = \lambda_1 \lambda_2$ and Gauss-Bonnet theorem we obtain:

$$2 \text{ Vol } (\Omega^t_{\pm})$$

$$= \int_{\Omega^t_{\pm}} K_{M,t} \omega \mp \int_{M} H \sigma$$

$$\mp \left(\cos^2 t - \sin^2 t\right) \int_{M} H \sigma + 2[(\sin t \cos t)A(M) - \pi \chi(M)]$$

$$= \int_{\Omega^t_{\pm}} K_{M,t} \omega + 2 \sin t \cos t A(M)$$

$$- 2\pi \chi(M) \sin t \cos t \mp 2\sin^2 t \int_{M} H \sigma.$$ \hspace{1cm} (11)
Now, using the coarea formula to integrate $K_M$ on $\Omega^t_\pm$ along the parallel surfaces of $M$, observing that $|\text{grad } d_M| = 1$, we obtain

\[
\int_{\Omega^t_\pm} K_M \omega = \int^t_0 \left[ \int_{M_s} K_{M_s} d\sigma_s \right] \, dt = \int^t_0 2\pi \chi(M_s) \, dt
\]

\[
= \int^t_0 2\pi \chi(M) \, dt = 2\pi t \chi(M).
\]

Taking this into (11) we obtain (3), proving the theorem the spherical case. 

We may use Theorem 1 to obtain the pinching theorems:

**Theorem 2** Let $M$ be a compact surface embedded either in the sphere $\mathbb{Q}_k$. If $k = 1$ and

\[
\cot \rho_M \leq \frac{\int_M H\sigma}{A(M)}
\]

or $k = -1$ and

\[
\coth \rho_M \leq \frac{\int_M H\sigma}{A(M)}
\]

then $M$ is a homeomorphic to a sphere.

**Proof.** We only prove in the spherical case since the hyperbolic case is similar. From (3) we obtain

\[
\chi(M) = \frac{\text{Vol}(\Omega^t_\pm)}{\pi (\rho_M - \sin \rho_M \cos \rho_M)} - \frac{\sin \rho_M \cos \rho_M A(M) + \sin^2 \rho_M \int_M H\sigma}{\pi (\rho_M - \sin \rho_M \cos \rho_M)}
\]

\[
\geq \frac{\sin \rho_M \cos \rho_M A(M) + \sin^2 \rho_M \int_M H\sigma}{\pi (\rho_M - \sin \rho_M \cos \rho_M)}
\]

\[
= \frac{A(M) \sin^2 \rho_M \left( -\cot \rho_M + \frac{\int_M H\sigma}{A(M)} \right)}{\pi (\rho_M - \sin \rho_M \cos \rho_M)} > 0.
\]

which proves the theorem. 

It follows from Theorem 1:

**Theorem 3** Let $M$ be a compact embedded surface in $\mathbb{Q}_k$. Then, the mean curvature of $M$ has zero mean, that is

\[
\int_M H\sigma = 0,
\]

if and only if $M$ divides a $t$–neighborhood about $M$ into two connected components of equal volume, for any $t \in [0, \rho_M]$. 

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Corollary 4 Any compact embedded minimal surface $M$ in $\mathbb{S}^3$ divides any $t$–neighborhood about $M$, with $0 < t \leq \rho_M$, into two connected components of equal volume.

The above results motivate the problem:

**Problem:** Let $M$ be a compact embedded surface in $\mathbb{Q}_k$ such that

$$\int_M H \sigma = 0.$$  \hspace{1cm} (13)

Find the biggest $d \geq \rho_M > 0$ such that $M$ divides

$$\Omega^d = \{ p \in \mathbb{Q}_k \mid d(p, M) \leq d \}$$

into two connected components of equal volume.

In $\mathbb{S}^3$ we could eventually have $\Omega^d = \mathbb{S}^3$; in this case any $M$ satisfying (13) would divide $\mathbb{S}^3$ into two connected components of equal volume. This was conjectured by Blaine Lawson in the case that $M$ is minimal but disproved by Karcher, Pinkall and Sterling (see [3])). Thus, $\Omega^d$ is properly contained in $\mathbb{S}^3$ in general.

Given $t \in [0, \rho_M]$, set

$$\Omega^t = \Omega^t_+ \cup \Omega^t_-$$

We also have from Theorem \[1\]

**Corollary 5** Let $M$ be a compact embedded surface either in $\mathbb{S}^3$ or $\mathbb{H}^3$. Then, for any $t \in [0, \rho_M],$

$$\text{Vol} (\Omega^t) = 2 \pi \chi (M) (\sinh t \cosh t - t) + 2 \sinh t \cosh t A (M)$$

and

$$\text{Vol} (\Omega^t) = 2 \pi \chi (M) (t - \sin t \cos t) + 2 \sin t \cos t A (M).$$  \hspace{1cm} (14)

**Corollary 6** Let $M$ be a compact embedded minimal surface in $\mathbb{S}^3 \,(1)$ with genus $g$. Then

$$\text{Vol} (\Omega^t) \leq 2 \pi \chi (M) [t - 5 \sin t \cos t] + 32 \pi \sin t \cos t$$  \hspace{1cm} (15)

for any $t \in [0, \rho_M]$. 

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Proof. Follows from (14) and Corollary 5 of [2] which asserts that \( A(M) \leq 4\pi (2 - \chi(M)) \). □

We may use Theorem 1 to estimate \( \rho_M \) from above when the ambient space is the sphere. Precisely:

**Theorem 7** Let \( M \) be a compact, embedded surface of genus \( \chi(M) \) in \( S^3(1) \).

If

a) \( \chi(M) = 2 \) then

\[
\rho_M \leq \frac{1}{2} \arcsin \left( \frac{2\pi^2}{A(M)} \right) \tag{16}
\]

b) \( \chi(M) \leq 2 \) then

\[
\rho_M \leq \frac{1}{2} \arcsin \left( \frac{\pi^2(2 - \chi(M))}{A(M) - \pi\chi(M)} \right) \tag{17}
\]

**Proof.** We have from (14)

\[
\text{Vol}(\Omega^{\rho_M}) = 2\pi \chi(M) \rho_M + 2(A(M) - \pi\chi(M)) \sin \rho_M \cos \rho_M.
\]

so that

\[
2\pi \chi(M) \rho_M + 2(A(M) - \pi\chi(M)) \sin \rho_M \cos \rho_M \leq \text{Vol}(S^3) = 2\pi^2,
\]

that is

\[
\pi \chi(M) \rho_M + (A(M) - \pi\chi(M)) \sin \rho_M \cos \rho_M \leq \pi^2. \tag{18}
\]

If \( \chi(M) > 0 \), that is \( \chi(M) = 2 \) then, since

\[
(1/2) \sin(2\rho_M) = \sin \rho_M \cos \rho_M \leq \rho_M
\]

we obtain

\[
2\pi \sin(2\rho_M) + (A(M) - 2\pi) \sin(2\rho_M) \leq 2\pi^2
\]

from what we may conclude that

\[
\rho_M \leq \frac{1}{2} \arcsin \frac{2\pi^2}{A(M)}.
\]

Assume now that \( \chi(M) \leq 0 \). From (18), since \( \rho_M \leq \pi/2 \), we obtain

\[
(A(M) - \pi\chi(M)) \sin \rho_M \cos \rho_M \leq \pi^2 - \frac{\pi^2 \chi(M)}{2}
\]
so that
\[(A(M) - \pi \chi(M)) \frac{\sin(2\rho M)}{2} \leq \pi^2 - \frac{\pi^2 \chi(M)}{2}\]
from what we conclude that
\[\rho_M \leq \frac{1}{2} \arcsin \left( \frac{\pi^2 (2 - \chi(M))}{A(M) - \pi \chi(M)} \right), \quad \chi(M) \leq 2.\]

We remark that the estimate (17) is sharp since the area of the Clifford torus \(T^2\) in \(S^3\) is \(2\pi^2\) and \(\rho_T = \pi/4\).

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