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Abstract. Let \( n \geq 2 \) be an integer and \( \alpha_1, \ldots, \alpha_n \) be non-zero algebraic numbers. Let \( b_1, \ldots, b_n \) be integers with \( b_n \neq 0 \), and set \( B = \max\{3, |b_1|, \ldots, |b_n|\} \). For \( j = 1, \ldots, n \), set \( h^*(\alpha_j) = \max\{h(\alpha_j), 1\} \), where \( h \) denotes the (logarithmic) Weil height. Assume that the quantity \( \Lambda = b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n \) is nonzero. A typical lower bound of \( \log |\Lambda| \) given by Baker’s theory of linear forms in logarithms takes the shape
\[
\log |\Lambda| \geq -c(n, D) h^*(\alpha_1) \cdots h^*(\alpha_n) \log B,
\]
where \( c(n, D) \) is positive, effectively computable and depends only on \( n \) and on the degree \( D \) of the field generated by \( \alpha_1, \ldots, \alpha_n \). However, in certain special cases and in particular when \( |b_n| = 1 \), this bound can be improved to
\[
\log |\Lambda| - c(n, D) h^*(\alpha_1) \cdots h^*(\alpha_n) \log \frac{B}{h^*(\alpha_n)}.
\]
The term \( B/h^*(\alpha_n) \) in place of \( B \) originates in works of Feldman and Baker and is a key tool for improving, in an effective way, the upper bound for the irrationality exponent of a real algebraic number of degree at least 3 given by Liouville’s theorem. We survey various applications of this refinement to exponents of approximation evaluated at algebraic numbers, to the \( S \)-part of some integer sequences, and to Diophantine equations. We conclude with some new results on arithmetical properties of convergents to real numbers.

1. Introduction

Baker’s theory of linear forms in the logarithms of algebraic numbers provides us with non-trivial, fully explicit lower bounds for the distance between 1 and a product \( \alpha_1^{b_1} \cdots \alpha_n^{b_n} \) of \( n \) integer powers of algebraic numbers, that is (since \( \log(1 + x) \) is equivalent to \( x \) in a neighborhood of 0), for the absolute value of the linear form
\[
b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n.
\]
It has been developed by Baker [2] and his followers since 1966 and has applications to Diophantine equations and many other topics; the interested reader is directed to the monographs [5, 8, 29, 42, 43, 59, 60, 62] and to the references quoted therein.

In short, Baker’s theory says that if the linear form in logarithms of algebraic numbers
\[
\Lambda := b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n
\]
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is nonzero, then its absolute value is at least equal to some effectively computable positive quantity, expressed in terms of \( n \), the maximum \( B \) of the absolute values of \( b_1, \ldots, b_n \), and the algebraic numbers \( \alpha_1, \ldots, \alpha_n \).

Throughout this paper, \( h(\alpha) \) denotes the (logarithmic) Weil height of the algebraic number \( \alpha \) and we set \( h^*(\alpha) = \max\{h(\alpha), 1\} \). Recall that if \( r/s \) is a nonzero rational number written in its lower form, then its height \( h(r/s) \) is equal to the maximum of \( \log |r| \) and \( \log |s| \). A typical lower bound for \( \log |\Lambda| \), established in [7, 61], takes the form

\[
(1.1) \quad \log |\Lambda| \geq -c h^*(\alpha_1) \cdots h^*(\alpha_n) \log B,
\]

where \( c \) is some effectively computable positive real number depending on \( n \) and on the degree of the number field generated by \( \alpha_1, \ldots, \alpha_n \). But between the birth of the theory and the proof of such a clean result, there have been several steps and many difficulties to overcome.

First, we point out that a Liouville-type estimate yields a bound with \( B \) instead of \( \log B \) in (1.1), while, for most (not all) of the applications, a lower bound with a dependence on \( B \) in \( o(B) \) would be sufficient. The first result of Baker [2] involves a factor \( (\log B)^{n+1+\varepsilon} \), where \( \varepsilon \) is an arbitrary positive real number.

We gather in Theorem 1.1 below immediate consequences of estimates of Waldschmidt [61, 62] and Matveev [52]; see also [29, Theorems 1.1 and 1.2].

**Theorem 1.1.** Let \( n \geq 1 \) be an integer. Let \( \alpha_1, \ldots, \alpha_n \) be non-zero algebraic numbers. Let \( b_1, \ldots, b_n \) be integers with \( b_n \neq 0 \). Let \( D \) be the degree over \( \mathbb{Q} \) of the number field \( \mathbb{Q}(\alpha_1, \ldots, \alpha_n) \). Set

\[
B = \max\{3, |b_1|, \ldots, |b_n|\},
\]

\[
B' = \max\left\{3, \max_{1 \leq j \leq n-1} \left\{ \frac{|b_n|}{h^*(\alpha_j)} + \frac{|b_j|}{h^*(\alpha_n)} \right\} \right\},
\]

and

\[
B'' = \max\left\{3, \max \left\{ \frac{|b_j|}{h^*(\alpha_j)} : 1 \leq j \leq n \right\} \right\}.
\]

Assume that

\[
\Lambda := b_1 \log \alpha_1 + \cdots + b_n \log \alpha_n
\]

is nonzero. Then, there exist effectively computable positive numbers \( c_1, \ldots, c_5 \), depending only on \( D \), such that the following holds. We have

\[
(1.2) \quad \log |\Lambda| \geq -c_1 n h^*(\alpha_1) \cdots h^*(\alpha_n) \log B,
\]

and, furthermore,

\[
(1.3) \quad \log |\Lambda| \geq -c_2 n^3 h^*(\alpha_1) \cdots h^*(\alpha_n) \log B',
\]

and

\[
(1.4) \quad \log |\Lambda| \geq -c_3 n h^*(\alpha_1) \cdots h^*(\alpha_n) \log B''.
\]

In particular, if \( |b_n| = 1 \), then we have

\[
(1.5) \quad \log |\Lambda| \geq -c_4 n^3 h^*(\alpha_1) \cdots h^*(\alpha_n) \log \frac{B}{h^*(\alpha_n)}
\]
and
\[
(1.6) \quad \log |A| \geq -c_0^2 n^3 h^*(\alpha_1) \cdots h^*(\alpha_n) \log \frac{B \max \{h^*(\alpha_1), \ldots, h^*(\alpha_{n-1})\}}{h^*(\alpha_n)}.
\]

The estimate (1.2) takes the shape $|A| \geq B^{-C}$, where $C$ depends only on $n, \alpha_1, \ldots, \alpha_n$. However, it does not include all the refinements. In particular, it is weaker than (1.4) in some important cases.

The quantity $B'$ in (1.3) originates in Feldman’s papers [44, 45]. It is a consequence of the use of the functions $x \mapsto \left(\frac{x}{k}\right)$ instead of $x \mapsto x^k$ in the construction of the auxiliary function. The key point is the presence of the factor $h^*(\alpha_n)$ in the denominator in the definition of $B'$. It is of great interest when $|b_n| = 1$ and $h^*(\alpha_n)$ is large, since it then allows us, roughly speaking, to replace $B$ by the smaller quantity $B/h^*(\alpha_n)$; see (1.5). The difference between (1.3) and (1.4) (and between (1.5) and (1.6)) lies mainly in the dependence on $n$. The improvement in the dependence on $n$ is Matveev’s breakthrough [52]. Observe that $B''$ may be larger than $B'$. This, however, does not cause any trouble in most of the applications (but not in all of them).

The term $B'$ has also its origin in the Sharpenings II of Baker [4], where the following statement is established (with a different notion of height). Throughout this paper, we use $\gg_{a,b,\ldots}$ and $\ll_{a,b,\ldots}$ to indicate that the positive numerical constants implied by $\gg$ and $\ll$ are effectively computable and depend at most on the parameters $a,b,\ldots$.

**Theorem 1.2.** Keep the notation of Theorem 1.1. Set $B_n = \max\{3,|b_n|\}$ and $B_0 = \max\{3,|b_1|,\ldots,|b_{n-1}|\}$. There exists an effectively computable number $C$, depending only on $n, D, \alpha_1, \ldots, \alpha_{n-1}$ such that, for any real number $\delta$ with $0 < \delta < 1/2$, we have

$$|A| \geq (\delta/B_n)^{Ch^*(\alpha_n)} e^{-\delta B_0}.$$  

In particular, if we assume that $n \geq 2$ and $b_n = 1$, and if $\eta$ is a real number with $0 < \eta < 1$, then the inequality

$$|A| < e^{-\eta B},$$

implies the upper bound

$$B = B_0 \ll_{n,D,\alpha_1,\ldots,\alpha_{n-1},\eta} h^*(\alpha_n).$$

The last statement of Theorem 1.2 follows from the first one by taking $\delta = \eta/2$.

Theorem 1.2 improved the earlier results to the extent of the elimination of a factor $\log h^*(\alpha_n)$ from the bound for $B_0$. Note, however, that the dependence on $h^*(\alpha_1), \ldots, h^*(\alpha_{n-1})$ is not specified.

Feldman [45, Theorem 1] obtained a similar result two years before, under some restrictions on $\alpha_n$, which enabled him to get the first effective improvement on Liouville’s bound for the irrationality exponent of an algebraic number of degree at least 3, see Theorem 2.2 below (and [3, p. 118]).

The second assertion of Theorem 1.2 is implied by (1.5). Indeed, under the assumption $\log |A| \leq -\delta B$, we get

$$B \leq \delta^{-1} c_0^2 n^{3n} h^*(\alpha_1) \cdots h^*(\alpha_n) \log \frac{B}{h^*(\alpha_n)}.$$
hence an upper bound for $B$ of the form

$$B \leq_{n,D,\alpha_1,\ldots,\alpha_{n-1},\delta} h^*(\alpha_n).$$

See also [23] for an alternative proof.

Theorem 1.2 is sufficient in most of the situations, but not in all of them. Assume for instance that the upper bound for $|\Lambda|$ takes the form $\log |\Lambda| \leq -\delta Bh^*(\alpha_1)$. Combined with (1.5), this gives an upper bound for $B$ which is linear in $h^*(\alpha_n)$ and independent of $h^*(\alpha_1)$, while Theorem 1.2 yields an upper bound linear in $h^*(\alpha_n)$, but also dependent on $h^*(\alpha_1)$. The same example illustrates the difference between (1.5) and (1.6). As we will see in Subsection 5.3, we encounter such situations when $\alpha_1$ (or one number among $\alpha_1,\ldots,\alpha_{n-1}$) is unknown. Additional explanations are given in [62, p. 361-362].

The term $B'$ has apparently been first introduced in [49], but with a slightly different definition. It is called $M$ in [61] and $b'$ in [48].

There is a non-Archimedean analogue of Baker’s classical theory. Let $p$ be a prime number. For a nonzero rational number $\alpha$, let $v_p(\alpha)$ denote the exponent of $p$ in the decomposition of $\alpha$ as a product of powers of prime numbers. More generally, if $\alpha$ is a nonzero algebraic number in a number field $K$, let $v_p(\alpha)$ denote the exponent of $p$ in the decomposition of the fractional ideal $\alpha O_K$ in a product of prime ideals and set

$$v_p(\alpha) = \frac{v_p(\alpha)}{e_p}.$$

This defines a valuation $v_p$ on $K$ which extends the $p$-adic valuation $v_p$ on $\mathbb{Q}$ normalized in such a way that $v_p(p) = 1$. We reproduce, in a simplified form, estimates obtained by Yu [64]; see also [29, Theorems 2.9 and 2.11].

**Theorem 1.3.** Let $n \geq 2$ be an integer. Let $p$ be a prime number and $\alpha_1,\ldots,\alpha_n$ algebraic numbers in an algebraic number field of degree $D$. Let $b_1,\ldots,b_n$ denote rational integers such that $\alpha_1^{b_1} \cdots \alpha_n^{b_n} = \alpha_1^{b_1} \cdots \alpha_n^{b_n}$ is not equal to 1. Set

$$B = \max\{3, |b_1|, \ldots, |b_n|\}.$$

There exist positive effectively computable real numbers $c_1,\ldots,c_7$, depending only on $D$, such that the following holds. We have

$$v_p(\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1) < c_1^n p^D h^*(\alpha_1) \cdots h^*(\alpha_n) \log B.$$

Let $B_n$ be a real number such that

$$B \geq B_n \geq |b_n|.$$

Assume that

$$v_p(b_n) \leq v_p(b_j), \quad j = 1,\ldots,n.$$

Let $\delta$ be a real number with $0 < \delta \leq \frac{1}{2}$. Then, we have

$$v_p(\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1) < c_2^n \frac{p^D}{(\log p)^2} \max\left\{h^*(\alpha_1) \cdots h^*(\alpha_n)(\log T) \frac{\delta B}{B_n c_3^2}\right\},$$

where

$$T = \frac{B_n}{\delta} c_4^n p^{(n+1)D} h^*(\alpha_1) \cdots h^*(\alpha_{n-1}).$$
In particular, we get either

(1.8) \[ B \leq c_0^n h^*(\alpha_1) \cdots h^*(\alpha_n) B_n \]

or

(1.9) \[ v_p(\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1) < c_0^n p D h^*(\alpha_1) \cdots h^*(\alpha_n) \log \max \left\{ 3, \frac{B}{h^*(\alpha_n)} \right\}. \]

Furthermore, we have

(1.10) \[ v_p(\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1) < c_1^n p D h^*(\alpha_1) \cdots h^*(\alpha_n) \log \frac{B \max\{h^*(\alpha_1), \ldots, h^*(\alpha_{n-1})\}}{h^*(\alpha_n)}. \]

The derivation of ‘(1.8) or (1.9)’ (resp., of (1.10)) from (1.7) is explained on [29, p. 20] (resp., on [42, p. 57]).

Clearly, (1.9) is a \( p \)-adic analogue of (1.5) and (1.6). But, this is not the conclusion of Theorem 1.3: there is an alternative, whose other member is (1.8). In most of the cases, this is sufficient for our applications, but there are examples for which this is not, see Subsection 5.3.

In the case of two logarithms, an analogue of (1.5) has been proved in [37], with, however, a weaker dependence on \( B' \).

**Theorem 1.4.** Let \( p \) be a prime number. Let \( \alpha_1 \) and \( \alpha_2 \) be multiplicatively independent algebraic numbers with \( v_p(\alpha_1) = v_p(\alpha_2) = 0 \). Set \( D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}] \). Let \( b_1 \) and \( b_2 \) be positive integers and set

\[ B' = \max\left\{ 3, \frac{b_1}{h^*(\alpha_2)} + \frac{b_2}{h^*(\alpha_1)} \right\}. \]

Then, there exists an absolute, effectively computable number \( c_1 \) such that

\[ v_p(\alpha_1^{b_1} - \alpha_2^{b_2}) \leq c_1 p D h^*(\alpha_1) h^*(\alpha_2) (\log B')^2. \]

We have slightly simplified the statement of Theorem 1.4. Its interest is more practical than theoretical: indeed, the numerical constant (which is not reproduced here) is much smaller than those in Theorem 1.3, and this is crucial for applications to the resolution of Diophantine equations, even in spite of the appearance of \((\log B')^2\). In principle, the square over the factor \((\log B')\) can be removed. This has been worked out in the Archimedean case by Gouillon [46].

Also, in principle, it should be possible to replace the conclusion ‘we get (1.8) or (1.9)’ by a weaker form of (1.9), with an extra factor \( n^{2n} \) or \( n^{3n} \). This is an open problem, which would have some applications (see Subsection 5.3), even if the dependence on \( n \) is not very good.

**Problem 1.5.** Keep the assumption of Theorem 1.3. Assume that \( |b_n| = 1 \). Establish that there exists a positive effectively computable real number \( c_1 \), depending only on \( D \), such that

\[ v_p(\alpha_1^{b_1} \cdots \alpha_n^{b_n} - 1) < c_1^n n^{3n} p D h^*(\alpha_1) \cdots h^*(\alpha_n) \log \max \left\{ 3, \frac{B}{h^*(\alpha_n)} \right\}. \]
The purpose of this paper is to survey several questions, where the presence of \( B' \) yields stronger, or more general, results than the usual estimate (1.2). We discuss five exponents of Diophantine approximation in Section 2. Sections 3 to 5 are devoted to Diophantine equations. Section 6 contains some new results on arithmetical properties of convergents to real numbers.

This paper can be seen as a complement to the monograph [29], even if several results surveyed below were already discussed in [29].

2. Exponents of Diophantine approximation

In this section, we discuss the values taken by several exponents of Diophantine approximation at algebraic arguments. In each of the five examples considered below, the replacement of \( B \) by \( B' \) is crucial to get a (small) saving on the easy bound derived from a Liouville-type argument. In Sections 2.1 to 2.4 the proofs need some (classical and elementary) algebraic number theory, and we do not reproduce them here. The interested reader is directed to [29] and/or to the original papers.

2.1. Irrationality exponent. The irrationality exponent of an irrational number \( \theta \) measures the quality of approximation to \( \theta \) by rational numbers.

Definition 2.1. Let \( \theta \) be an irrational real number. We denote by \( \mu(\theta) \) (resp., \( \mu_{\text{eff}}(\theta) \)) the infimum of the real numbers \( \mu \) for which there exists a positive real number \( C(\theta) \) (resp., a positive, effectively computable, real number \( C(\theta) \)) such that every rational number \( \frac{p}{q} \) with \( q \geq 1 \) satisfies

\[
|\theta - \frac{p}{q}| > \frac{C(\theta)}{q^\mu}.
\]

Clearly, \( \mu_{\text{eff}}(\theta) \) is always larger than or equal to \( \mu(\theta) \), which is always at least equal to 2, by the theory of continued fractions.

Let \( \xi \) be a real, algebraic, irrational number of degree \( d \). Liouville’s theorem asserts that there exists a positive, effectively computable number \( C(\xi) \) such that

\[
|\xi - \frac{p}{q}| > \frac{C(\xi)}{q^d}, \quad \text{for every } p, q \in \mathbb{Z} \text{ with } q \geq 1.
\]

Thus, we have \( \mu_{\text{eff}}(\xi) \leq d \), while \( \mu(\xi) = 2 \), by Roth’s theorem. For \( d \geq 3 \), Liouville’s bound for \( \mu_{\text{eff}}(\xi) \) can be slightly improved. This was first established by Feldman [45].

Theorem 2.2. Let \( \xi \) be a real, algebraic number of degree \( d \) at least 3. There exists a positive, effectively computable \( \tau = \tau(\xi) \) such that \( \mu_{\text{eff}}(\xi) \leq d - \tau \).

The term \( B' \) is at the source of the proof that \( \mu_{\text{eff}}(\xi) < d \). By using the lower bound (1.2) in place of (1.3), we get an estimate of the form

\[
|\xi - \frac{p}{q}| > \frac{1}{q^{d-c/\log\log q}},
\]

for some effectively computable, positive \( c \) and every \( q \geq 10 \). This effective improvement of Liouville’s inequality (2.1) does not imply anything better than \( \mu_{\text{eff}}(\xi) \leq d \).
In some special cases, for instance for \( \xi = \sqrt{2} \), Theorem 2.2 can be considerably improved; see [1] and subsequent works. Furthermore, Bombieri and Mueller [24] and Bombieri, van der Poorten, and Vaaler [25] proved that, for every \( \varepsilon > 0 \) and every integer \( d \geq 3 \), there exist algebraic real numbers \( \xi \) of degree \( d \) such that \( \mu_{\text{eff}}(\xi) < 2 + \varepsilon \). In [24] the authors consider integral roots of rational numbers close to 1, while in [25] the algebraic numbers are roots of cubic polynomials \( x^3 + px + q \), where \( p > 0 \) and \( q \) are relatively prime integers, with \( p \) sufficiently large in terms of \( |q| \).

### 2.2. On the \( b \)-ary expansion of an algebraic number.

Let \( \theta \) be an irrational real number. We denote by \( v_b(\theta) \) the infimum of the real numbers \( v \) for which the inequality

\[
\|b^n \theta\| > (b^n)^{-v}
\]

holds for every sufficiently large positive integer \( n \). Likewise, \( v_{b,\text{eff}}(\theta) \) denotes the infimum of the real numbers \( v \) for which there exists an effectively computable integer \( n_0(v) \) such that

\[
\|b^n \theta\| > (b^n)^{-v},
\]

for \( n \geq n_0(v) \).

Assume that \( \xi \) is algebraic of degree \( d \geq 2 \). Then Liouville’s inequality yields the trivial upper bound

\[
v_{b,\text{eff}}(\xi) \leq \mu_{\text{eff}}(\xi) - 1 \leq d - 1,
\]

while Ridout’s theorem [54] implies that \( v_b(\xi) = 0 \). Furthermore, if \( d \geq 3 \), then it follows from Theorem 2.2 that

\[
(2.2) \quad v_{b,\text{eff}}(\xi) \leq d - 1 - \tau,
\]

for some effectively computable positive real number \( \tau = \tau(\xi) \). For \( d = 2 \), the theory of \( p \)-adic linear forms in logarithms allows us to improve the trivial estimate \( v_{b,\text{eff}}(\xi) \leq 1 \) as follows.

**Theorem 2.3.** For every integer \( b \geq 2 \) and every quadratic real number \( \xi \), we have

\[
v_{b,\text{eff}}(\xi) \leq 1 - \tau,
\]

where \( \tau = \tau(\xi, S) \) is a positive, effectively computable, constant depending only on \( \xi \) and on the set \( S \) of prime factors of \( b \).

In some specific cases, we can get better upper bounds and even, for every \( \varepsilon > 0 \), construct positive integers \( m \) such that \( v_{b,\text{eff}}(\sqrt{m}) < \varepsilon \). Namely, it has been proved in [15] that for every integers \( b \geq 2 \) and \( k \geq 1 \) we have

\[
v_{b,\text{eff}}(\sqrt{b^{2k}} + 1) \leq \frac{\log 48}{k \log b}.
\]
2.3. **Simultaneous Pell equations.** The definition of (effective) irrationality exponent extends as follows. Let $\theta, \theta'$ be real numbers such that $1, \theta, \theta'$ are linearly independent over the rational numbers. We denote by $\mu(\theta, \theta')$ (resp., $\mu_{\text{eff}}(\theta, \theta')$) the supremum of the real numbers $\mu$ for which there exists a positive real number $c(\theta, \theta')$ (resp., a positive, effectively computable, real number $c(\theta, \theta')$) such that, for every integer triple $(p, q, r)$ with $q \geq 1$, we have

$$\max\left\{ \left| \theta - \frac{p}{q} \right|, \left| \theta' - \frac{r}{q} \right| \right\} > \frac{c(\theta, \theta')}{q^\mu}. $$

Let $\theta, \theta'$ be real numbers such that $1, \theta, \theta'$ are linearly independent over the rational numbers. An easy application of Minkowski’s theorem implies that $\mu(\theta, \theta') \geq \frac{3}{2}$, and a covering lemma shows that equality holds for almost all pairs $(\theta, \theta')$, with respect to the planar Lebesgue measure. Schmidt [56] established that $\mu(\xi, \zeta) = \frac{3}{2}$ if $\xi$ and $\zeta$ are both real and algebraic. His result is ineffective and gives no better information on $\mu_{\text{eff}}(\xi, \zeta)$ than the obvious inequality

$$\mu_{\text{eff}}(\xi, \zeta) \leq \max\{ \mu_{\text{eff}}(\xi), \mu_{\text{eff}}(\zeta) \}. $$

The particular case where $\xi$ and $\zeta$ are quadratic numbers in distinct number fields is of special interest and was considered in [30].

**Theorem 2.4.** Let $\xi, \zeta$ be real quadratic numbers in distinct quadratic fields. Then, we have $\mu(\xi, \zeta) = \frac{3}{2}$ and there exists a positive, effectively computable real number $\tau$, depending only on $\xi$ and $\zeta$, such that

$$\mu_{\text{eff}}(\xi, \zeta) \leq 2 - \tau.$$

Better upper bounds have been obtained in some special cases, in particular by Rickert [53] (see his paper for earlier references), who established among other results that

$$\mu_{\text{eff}}(\sqrt{2}, \sqrt{3}) \leq 1.913,$$

and subsequently by Bennett [12, 13]. Their method applies only to a restricted class of pairs $(\xi, \zeta)$ of quadratic numbers. However, it is strong enough to establish that, for every $\varepsilon > 0$ and every positive integer $N$ sufficiently large in terms of $\varepsilon$, we have

$$\mu_{\text{eff}}(\sqrt{1 - \frac{1}{N}}, \sqrt{1 + \frac{1}{N}}) < \frac{3}{2} + \varepsilon,$$

see [53].

2.4. **Multiplicative $p$-adic approximation.** Let $p$ be a prime number and $\theta$ be an irrational, $p$-adic number. We denote by $\mu^\times(\theta)$ the infimum of the real numbers $\mu$ for which the inequality

$$(2.3) \quad |b\theta - a|_p > |ab|^{-\mu}$$

holds for every nonzero integers $a, b$ with $|ab|$ sufficiently large. Likewise, $\mu_{\text{eff}}^\times(\theta)$ denotes the infimum of the real numbers $\mu$ for which there exists an effectively computable integer $A(\mu)$ such that (2.3) holds for every integers $a, b$ with $|ab| \geq A(\mu)$. 

Assume that $\alpha$ is a $p$-adic algebraic number of degree $d \geq 2$. Then Liouville’s inequality yields the trivial effective lower bound
\[
|b\alpha - a|_p \gg \max\{|a|, |b|\}^{-d}, \quad \text{for } a, b \in \mathbb{Z}_{\neq 0},
\]
thus, since $\max\{|a|, |b|\} \leq |ab|$, we have
\[
\mu_{\text{eff}}(\alpha) \leq \mu_{\text{eff}}(\alpha) \leq d,
\]
while Ridout’s theorem [55] implies that $\mu(\alpha) = 1$. Furthermore, if $d \geq 3$, then the analogue of Theorem 2.2 holds (see [60, Section V.2]), namely we have
\[
(2.4) \quad \mu_{\text{eff}}(\alpha) \leq d - \tau,
\]
for some effectively computable positive real number $\tau = \tau(\alpha)$. For $d = 2$, the theory of Archimedean linear forms in logarithms allows us to improve the trivial estimate $\mu_{\text{eff}}(\alpha) \leq 2$ as follows; see [33].

**Theorem 2.5.** For every prime number $p$ and every $p$-adic quadratic number $\alpha$, there exists a positive, effectively computable real number $\tau = \tau(\alpha)$ such that
\[
\mu_{\text{eff}}(\alpha) \leq 2 - \tau.
\]

We conclude this subsection with an open question.

**Problem 2.6.** For a given prime number $p$ and an arbitrary $\varepsilon > 0$, to construct quadratic $p$-adic numbers $\alpha$ such that $\mu_{\text{eff}}(\alpha) \leq 1 + \varepsilon$.

**2.5. Fractional parts of powers of real algebraic numbers.** For a real number $x$, let
\[
||x|| = \min\{|x - m| : m \in \mathbb{Z}\}
\]
denote the distance to its nearest integer. In 1957 Mahler [51] applied Ridout’s $p$-adic extension [54] of Roth’s theorem to prove the first assertion of the following result. The second assertion was proved in 2004 by Corvaja and Zannier [39], who applied ingeniously the $p$-adic Schmidt Subspace Theorem. Recall that a Pisot number is a real algebraic integer greater than 1 with the property that all of its Galois conjugates (except itself) lie in the open unit disc.

**Theorem 2.7.** Let $r/s$ be a rational number greater than 1 and which is not an integer. Let $\varepsilon$ be a positive real number. Then, there exists an integer $n_0$ such that
\[
\left\| \left( \frac{r}{s} \right)^n \right\| > s^{-\varepsilon n},
\]
for every integer $n$ exceeding $n_0$. More generally, let $\xi$ be a real algebraic number greater than 1. If there are no positive integers $h$ such that $\xi^h$ is a Pisot number, then there exists an integer $n_0$ such that
\[
\|\xi^n\| > \xi^{-\varepsilon n},
\]
for every integer $n$ exceeding $n_0$.

Theorem 2.7 is ineffective in the sense that its proof does not yield an explicit value for the integer $n_0$. To get an effective improvement on the trivial estimate $\|\left(\frac{r}{s}\right)^n\| \geq s^{-n}$, Baker and Coates [6] (see also [26] and [29, Section 6.2]) applied the theory of linear forms in $p$-adic logarithms, with a prime number $p$ which divides $s$. 
Theorem 2.8. Let $r/s$ be a rational number greater than 1 and which is not an integer. Then, there exist effectively computable positive real numbers $\tau = \tau(r/s)$ and $n_0 = n_0(r/s)$ such that

$$\left\| \left( \frac{r}{s} \right)^n \right\| > s^{-(1-\tau)n},$$

for every integer $n$ exceeding $n_0$.

**Proof.** Let $n$ be a positive integer and $A_n$ denote the nearest integer to $(r/s)^n$. Set

$$m_n := r^n - A_n s^n.$$

Let $p$ be a prime divisor of $s$. By applying the first assertion of Theorem 1.3 to estimate $v_p(r^n - m_n)$ we get that

$$n \leq v_p(r^n - m_n) \ll_{r,s} (\log |2m_n|)(\log n),$$

hence

$$n \ll_{r,s} (\log |2m_n|) (\log \log |2m_n|),$$

which is not sufficient for our purpose. Fortunately, the second assertion of Theorem 1.3 shows that

$$n \ll_{r,s} \log |2m_n|$$

or

$$n \leq v_p(r^n - m_n) \ll_{r,s} (\log |2m_n|) \left( \log \frac{n}{\log |2m_n|} \right),$$

giving in both cases that

\begin{equation}
(2.5) \quad n \ll_{r,s} \log |2m_n|.
\end{equation}

Alternatively, Theorem 1.4 implies that

$$n \leq v_p(r^n - m_n) \ll_{r,s} (\log |2m_n|) \left( \log \frac{n}{\log |2m_n|} \right)^2,$$

and we get (2.5) as well. This implies the existence of a positive real number $\delta$, depending only on $r$ and $s$, such that

$$2|m_n| \geq s^{\delta n}.$$

By the definition of $m_n$ we conclude that

$$\left\| \left( \frac{r}{s} \right)^n \right\| = \frac{|m_n|}{s^n} \geq \frac{1}{2s(1-\delta)n}, \quad n \geq 1.$$  

This completes the proof of the theorem. $\Box$

For an arbitrary irrational, real algebraic number $\xi > 1$, Liouville’s theorem also allows us, when $\xi^n$ is not an integer, to bound $\|\xi^n\|$ from below by a positive number raised to the power $-n$. This bound can be improved in an effective way when $\xi^n$ is neither an integer, nor a quadratic Pisot unit.
Theorem 2.9. Let $\xi > 1$ be a real algebraic number of degree $d \geq 1$. Let $a_d$ denote the leading coefficient of its minimal defining polynomial over $\mathbb{Z}$ and $\xi_1, \ldots, \xi_d$ its Galois conjugates, ordered in such a way that $|\xi_1| \leq \ldots \leq |\xi_d|$. Let $j$ be such that $\xi = \xi_j$. Set
\[
C_\xi = a_d \xi^{d-1} \prod_{i>j} \frac{|\xi_i|}{\xi}.
\]
If $\xi$ is not the $d$-th root of an integer, then we have
\[
(2.6) \quad \|\xi^n\| \geq 3^{-(d-1)} C_\xi^{-n}, \quad \text{for } n \geq 1.
\]
Otherwise, $(2.6)$ holds only for the positive integers $n$ such that $\xi^n$ is not an integer.

For a real number $\theta$ which is not an integer define
\[
\nu(\theta) = \limsup_{n \to +\infty, \theta^n \notin \mathbb{Z}} \frac{-\log \|\theta^n\|}{n}
\]
and let $\nu_{\text{eff}}(\theta)$ denote the infimum of the real numbers $\nu$ for which there exists an effectively computable integer $n_0 = n_0(\theta)$ such that $(-\log \|\theta^n\|)/n \leq \nu$ for $n \geq n_0$ and $\theta^n$ is not an integer.

The first two assertions of the next theorem are restatements of Theorems 2.7 and 2.9, while the last one was established in [32].

Theorem 2.10. Let $\xi > 1$ be an algebraic real number which is not an integer. Then, $\nu(\xi) = 0$, unless $\xi$ is an integer root of a Pisot number. Furthermore, $\nu_{\text{eff}}(\xi) \leq \log C_\xi$ and, if $\xi$ is not an integer root of a quadratic Pisot unit, then there exists a positive, effectively computable real number $\tau = \tau(\xi)$ such that $\nu_{\text{eff}}(\xi) \leq (1 - \tau) \log C_\xi$.

Sometimes, the hypergeometric method yields effective improvements of Theorem 2.10. This is the case for the algebraic numbers $\sqrt{2}$ and $3/2$, see Beukers’ seminal papers [20, 21] and the subsequent works [9, 65] where it is shown that
\[
\nu_{\text{eff}}(\sqrt{2}) \leq 0.595, \quad \nu_{\text{eff}}(3/2) < 0.5443,
\]
respectively.

Furthermore, for every $\varepsilon > 0$, Bennett [10] constructed an infinite set $S_\varepsilon$ of rational numbers, dense in $(1, +\infty)$, such that every $p/q$ in $S_\varepsilon$ satisfies $\nu_{\text{eff}}(p/q) < \varepsilon \log q$, that is, $\nu_{\text{eff}}(p/q) < \varepsilon \log C_{p/q}$; see also [11].

2.6. Summary. We have seen that, in each of the five subsections above (with one exception, see Problem 2.6), we have:

(i) an easy, effective bound $M$ (obvious or coming from a Liouville-type inequality);
(ii) the exact result $m$ (but with an ineffective proof);
(iii) a small effective improvement $M - \tau$ on the easy bound (by using the $B'$);
(iv) a more substantial effective improvement in certain specific cases, with

even, for every $\varepsilon > 0$, some explicit examples for which one gets an effective bound smaller than $m + \varepsilon$.  


Moreover, in the cases where the effective bound is very close to the exact value obtained by ineffective methods, the specific numbers constructed with this property are in most cases very close to 1, for the usual or for a $p$-adic absolute value.

3. On the $S$-part of integer sequences

Throughout this section, we let $S = \{\ell_1, \ldots, \ell_s\}$ denote a finite, non-empty set of $s$ distinct prime numbers.

**Definition 3.1.** Let $n$ be a nonzero integer and write $n = A\ell_1^{r_1} \cdots \ell_s^{r_s}$, where $r_1, \ldots, r_s$ are non-negative integers and $A$ is an integer relatively prime to $\ell_1 \cdots \ell_s$. We define the $S$-part $[n]_S$ of $n$ by

$$[n]_S := \ell_1^{r_1} \cdots \ell_s^{r_s}.$$  

We set $[0]_S = 0$.

Let $\mathcal{N}$ denote a sequence of integers, defined in some natural way. We discuss whether one can improve the trivial estimate $[n]_S \leq n$ for integers $n$ in $\mathcal{N}$. When this is the case, then, by taking for $S$ the set composed of the first $s$ prime numbers, this often implies a lower bound for the greatest prime factor of $n$ in $\mathcal{N}$.

We consider several different types of sets $\mathcal{N}$ and survey various recent results obtained in [28, 31, 32, 34–36]. In this section, all the theorems with an arbitrary, positive $\varepsilon$ in their statement are proved by means of the Schmidt Subspace Theorem and are all ineffective, while the corresponding effective statements involve a positive $\tau$ and their proofs depend on the theory of linear forms in logarithms and, in a crucial way, on the term $B'$.

We postpone to Section 6 new results on the $S$-part of sequences of convergents to real numbers.

3.1. Integers with few digits in an integer base. For integers $b \geq 2$ and $k \geq 2$, we denote by $(u_{j}^{(b,k)})_{j \geq 1}$ the sequence, arranged in increasing order, of all positive integers which are not divisible by $b$ and have at most $k$ nonzero digits in their representation in base $b$. Said differently, $(u_{j}^{(b,k)})_{j \geq 1}$ is the ordered sequence composed of the integers $1, 2, \ldots, b - 1$ and those of the form

$$d_k b^{n_k} + \cdots + d_2 b^{n_2} + d_1$$

with

$$n_k \succ \cdots \succ n_2 \succ 0, \quad 0 \leq d_1, \ldots, d_k \leq b - 1, \quad d_1 d_k \neq 0.$$

The following result reproduces [28, Theorem 1.1] and [31, Theorem 1.5]. It extends [28, Theorem 1.2], which deals with the case $k = 3$.

**Theorem 3.2.** Let $b \geq 2, k \geq 2$ be integers and $\varepsilon$ a positive real number. Let $S$ be a finite, non-empty set of prime numbers. Then, we have

$$[u_{j}^{(b,k)}]_S < (u_{j}^{(b,k)})^\varepsilon,$$
for every sufficiently large integer \( j \). In particular, the greatest prime factor of \( u_j^{(b,k)} \) tends to infinity as \( j \) tends to infinity. Furthermore, there exist effectively computable positive numbers \( j_0 \) and \( \tau \), depending only on \( b,k, \) and \( S \), such that

\[
[u_j^{(b,k)}]_S \leq (u_j^{(b,k)})^{1-\tau}, \quad \text{for every } j \geq j_0.
\]

The dependence of \( \tau \) on \( b,k,S \) is made explicit in [31], up to some absolute numerical constants. As a consequence of the main result of [36], we get that, for \( k \geq 3 \), there exists an effectively computable positive integer \( n_0 \), depending only on \( b,k \) and \( \varepsilon \), such that any integer \( n > n_0 \) which is not divisible by \( b \) and has all of its prime factors less than

\[
\left( \frac{1}{k-2} - \varepsilon \right)(\log \log n) \leq (\log \log \log n)
\]

has at least \( k+1 \) nonzero digits in its \( b \)-ary representation. The term \( \frac{1}{k-2} \) can be replaced by \( \frac{1}{k-1} \), see [31].

A version of Theorem 3.2 for the Fibonacci numeration system (Zeckendorf expansions) has been established in [31]. We point out that the proofs in [31] depend only on Theorem 1.1, while Theorems 1.1 and 1.3 are combined in [28, 36].

3.2. Recurrence sequences of integers. Let \( k \) be a positive integer, and let \( a_1, \ldots, a_k \) and \( u_0, \ldots, u_{k-1} \) be integers such that \( a_k \) is non-zero and \( u_0, \ldots, u_{k-1} \) are not all zero. Put

\[
(3.1) \quad u_n = a_1 u_{n-1} + \ldots + a_k u_{n-k}, \quad \text{for } n \geq k.
\]

The sequence \((u_n)_{n \geq 0}\) is a linear recurrence sequence of integers of order \( k \). Its characteristic polynomial

\[
G(X) := X^k - a_1 X^{k-1} - \ldots - a_k
\]

factors as

\[
G(X) = \prod_{i=1}^t (X - \alpha_i)^{h_i},
\]

where \( \alpha_1, \ldots, \alpha_t \) are distinct algebraic numbers with \( |\alpha_1| \geq |\alpha_2| \geq \ldots \geq |\alpha_t| \) and \( h_1, \ldots, h_t \) are positive integers. The recurrence sequence \((u_n)_{n \geq 0}\) is said to be degenerate if there are integers \( i,j \) with \( 1 \leq i < j \leq t \) such that \( \alpha_i/\alpha_j \) is a root of unity. It is said to have a dominant root if \( |\alpha_1| > |\alpha_2| \).

Choose embeddings of \( \mathbb{Q}(\alpha_1, \ldots, \alpha_t) \) in \( \mathbb{C} \) and of \( \mathbb{Q}(\alpha_1, \ldots, \alpha_t) \) in \( \mathbb{Q}_p \), for every prime \( p \). These embeddings define extensions to \( \mathbb{Q}(\alpha_1, \ldots, \alpha_t) \) of the ordinary absolute value \(|\cdot|\) and of the \( p \)-adic absolute value \(|\cdot|_p \) for every prime \( p \), normalized such that \( |p|_p = p^{-1} \).

**Theorem 3.3.** Let \((u_n)_{n \geq 0}\) be a non-degenerate recurrence sequence of integers defined in (3.1). Let \( S := \{\ell_1, \ldots, \ell_s\} \) be a finite, non-empty set of prime numbers, and set

\[
\delta := -\frac{\sum_{i=1}^s \log \max \{ |\alpha_1|_{\ell_i}, \ldots, |\alpha_t|_{\ell_i} \}}{\log \max \{ |\alpha_1|, \ldots, |\alpha_t| \}}.
\]
Let $\varepsilon > 0$. Then, we have
\[ |u_n|^{\delta - \varepsilon} \leq |u_n|_S \leq |u_n|^{\delta + \varepsilon}, \]
for every sufficiently large $n$. In particular, if $\gcd(\ell_1 \cdots \ell_s, a_1, \ldots, a_k) = 1$, then we have
\[ |u_n|_S \leq |u_n|^{\varepsilon}, \]
for every sufficiently large $n$. Furthermore, if $(u_n)_{n \geq 0}$ has a dominant root, then there exist effectively computable positive numbers $n_0$ and $\tau$, depending only on $(u_n)_{n \geq 0}$ and $S$, such that
\[ |u_n|_S \leq |u_n|^{1 - \tau}, \]
for every $n \geq n_0$.

Removing the dominant root assumption in the last statement of Theorem 3.3 seems to be very difficult. However, this can be done for non-degenerate binary recurrence sequences of integers; see [34] for references.

3.3. Polynomials, binary forms, and decomposable forms. The results of this subsection have been established in [35]; see the references therein for earlier works.

Theorem 3.4. Let $f(X)$ be an integer polynomial of degree $d \geq 2$. Let $S$ be a non-empty set of prime numbers. If $f(X)$ has no multiple zeros, then, for every $\varepsilon > 0$ and every integer $n$, we have
\[ [f(n)]_S \ll_{f, S, \varepsilon} |f(n)|^{(1/d) + \varepsilon}. \]
If $f(X)$ has at least two distinct roots, then, for every integer $n$, we have
\[ [f(n)]_S \ll_{f} |f(n)|^{1 - \tau}, \]
where $\tau$ is an effectively computable positive number that depends only on $f(X)$ and $S$.

Bennett, Filaseta, and Trifonov [16, 17] have obtained stronger effective results for the polynomials $X(X + 1)$ and $X^2 + 7$ and special sets $S$.

Note that there are infinitely many primes $p$, and for each of these $p$, there are infinitely many integers $n$, such that $f(n) \neq 0$ and
\[ [f(n)](p) \gg_f |f(n)|^{1/d}. \]

We now formulate an analogue of Theorem 3.4 for binary forms. Denote by $\mathbb{Z}^2_{\text{prim}}$ the set of pairs $(x, y)$ in $\mathbb{Z}^2$ with $\gcd(x, y) = 1$.

Theorem 3.5. Let $F(X, Y)$ be a binary form of degree $d \geq 2$ with integer coefficients. Let $S$ be a non-empty set of primes. If the discriminant of $F(X, Y)$ is nonzero, then, for every $\varepsilon > 0$ and every pair $(x, y)$ in $\mathbb{Z}^2_{\text{prim}}$, we have
\[ [F(x, y)]_S \ll_{F, S, \varepsilon} |F(x, y)|^{(2/d) + \varepsilon}. \]
If $F$ has at least three pairwise non-proportional linear factors over its splitting field, then
\[ [F(x, y)]_S \ll_{F} |F(x, y)|^{1 - \tau}. \]
for every \((x, y)\) in \(\mathbb{Z}_{\text{prim}}^2\), where \(\tau\) is an effectively computable positive number, depending only on \(F\) and \(S\).

Note that there are finite sets of primes \(S\) with the smallest prime in \(S\) being arbitrarily large, and, for each one of these sets \(S\), infinitely many pairs \((x, y)\) in \(\mathbb{Z}_{\text{prim}}^2\), such that \(F(x, y) \neq 0\) and

\[
[F(x, y)]_S \gg_F S |F(x, y)|^{2/d}.
\]

Theorem 3.5 extends to a class of decomposable form equations, see [35].

3.4. **Power sums.** Apparently, arithmetical properties of the sequence of integers of the form \(2^m + 6^n + 1\) were first discussed by Corvaja and Zannier in [40], where they showed, by a clever use of the Schmidt Subspace Theorem, that it contains only finitely many squares. We prove a result related to a special case of [28, Theorem 4.3], namely we bound from above the \(S\)-part of \(2^m + 6^n + 1\) by applying Theorems 1.1 and 1.3.

**Theorem 3.6.** Let \(S\) be a finite, non-empty set of prime numbers. Let \(a, b\) be integers with \(a \geq 2, b \geq 2\). Then, for every \(\varepsilon > 0\), we have

\[
[a^m + b^n + 1]_S \leq (a^m + b^n + 1)^\varepsilon,
\]

if \(m + n\) is sufficiently large. Furthermore, if \(\gcd(a, b) > 1\), then there exist effectively computable positive numbers \(\tau\) and \(n_0\), depending only on \(a, b, \) and \(S\), such that

\[
[a^m + b^n + 1]_S \leq (a^m + b^n + 1)^{1-\tau}, \quad \text{for } m + n \geq n_0.
\]

**Proof.** The first statement easily follows from the \(p\)-adic Schmidt Subspace Theorem. We omit the proof.

We treat briefly the special case \(a = 2, b = 6\). The general case goes along the same lines. Write

\[
2^m + 6^n + 1 = A\ell_1^{r_1} \cdots \ell_s^{r_s},
\]

with \(\gcd(A, \ell_1 \cdots \ell_s) = 1\). Without loss of generality, we can assume that \(A \leq (2^m + 6^n + 1)^{1/2}\). Then, we have

\[
\max\{m, n\} \ll_S \max\{r_1, \ldots, r_s\} \ll_S \max\{m, n\}.
\]

We distinguish three cases.

If \(2^m \geq 6^{2^n}\), then \(A\ell_1^{r_1} \cdots \ell_s^{r_s}2^{-m}\) is very close to 1 and it follows from (1.5) that

\[
m \ll_S \log(2A) \log \frac{m}{\log(2A)}.
\]

If \(6^n \geq 2^{2^m}\), then \(A\ell_1^{r_1} \cdots \ell_s^{r_s}6^{-n}\) is very close to 1 and it follows from (1.5) that

\[
n \ll_S \log(2A) \log \frac{n}{\log(2A)}.
\]

In both cases, we get

\[
\max\{m, n\} \ll_S \log(2A).
\]
It remains for us to treat the case where $2^m \leq 6^{2n} \leq 2^{4m}$. Then, $\min\{m, n\} \gg \max\{m, n\}$ and $A\ell_1^{w_1} \ldots \ell_s^{w_s}$ is 2-adically very close to 1. We derive from (1.10) that

$$\max\{m, n\} \ll_{S} \log(2A) \log \frac{\max\{m, n\}}{\log(2A)}.$$ 

This completes the proof of the theorem. \hfill \Box

A key ingredient in the proof of the second part of Theorem 3.6 is the assumption that $a$ and $b$ are not coprime. It allows us to combine Archimedean and non-Archimedean estimates. We do not see how to get a non-trivial upper bound for the $S$-part of $1 + 2^m + 3^n$. Difficulties occur when the integers $m, n$ are such that $2^m$ is close to $3^n$.

**Problem 3.7.** For a prime number $p \geq 5$, give a non-trivial upper bound for

$$[1 + 2^m + 3^n]_{p}.$$ 

### 3.5. Sums of integral $T$-units.

Let $T$ be a finite set of prime numbers, whose intersection with $S$ is empty. We consider the set $\mathcal{N}$ composed of all the integers whose prime divisors are in $T$.

**Theorem 3.8.** Let $\varepsilon$ be a positive real number. Let $T = \{q_1, \ldots, q_t\}$ be a finite set of prime numbers. Let $x, y$ be coprime integers whose prime divisors are in $T$. If $|x + y|$ is sufficiently large in terms of $\varepsilon$, then

$$[x + y]_S \leq |x + y|^\varepsilon.$$ 

Furthermore, there exist effectively computable positive numbers $\tau$ and $X_0$, depending only on $S$ and $T$, such that

$$[x + y]_S \ll_{S,T} |x + y|^{1-\tau},$$

if $|x + y| > X_0$.

The first assertion is an easy consequence of Ridout’s theorem [54]. It extends to sums of an arbitrary number of $S$-units, by means of the $p$-adic Subspace Theorem. The second assertion does not seem to have been stated previously.

**Proof.** Write $x = q_1^{u_1} \cdots q_t^{u_t}$, $y = q_1^{v_1} \cdots q_t^{v_t}$, and

$$q_1^{u_1} \cdots q_t^{u_t} \pm q_1^{v_1} \cdots q_t^{v_t} = A\ell_1^{w_1} \ldots \ell_s^{w_s},$$

where the integer $A$ is coprime with $\ell_1 \ldots \ell_s$. Assume that $|x + y| \geq 4$ and $|A| \leq |x + y|^{1/2}$. Set

$$B = \max\{u_1, \ldots, u_t, v_1, \ldots, v_t\}, \quad W = \max\{w_1, \ldots, w_s\},$$

and $A^* = \max\{|A|, 2\}$. Throughout the proof, the constants implicit in $\ll$ are effectively computable and depend at most on $S$ and on $T$. Our assumption on $A$ implies that

$$B \gg \ll W.$$ 

Assume that $B = u_1$ and consider

$$v_{q_1}(A\ell_1^{w_1} \ldots \ell_s^{w_s} \pm q_1^{v_1} \cdots q_t^{v_t}).$$
It follows from Theorem 1.3 that

\[ B \ll (\log A^*) \log \frac{B+W}{\log A^*}, \]

thus,

\[ B \ll \log A^*. \]

Setting \( X = \max\{|x|,|y|,2\} \), we get

\[ \log X \ll B \ll \log A^*, \]

thus \( X \ll 1 \) if \(|A| = 1\) and, if \(|A| \geq 2\), then there exist positive effectively computable numbers \( c_1, c_2, c_3 \), depending on \( S \) and \( T \), such that

\[ |A| \gg 2^{c_1 B} \gg 2^{c_2 W} \gg |x+y|^{c_3}. \]

This completes the proof \( \Box \).

3.6. Summary. In all the examples above, there is a big gap between ineffective and effective statements. The term \( B' \) allows us to get a small, effective \( \tau \)-saving on the trivial bound.

4. S-unit equations

Let \( K \) be an algebraic number field. Many Diophantine problems reduce to equations of the form

\[ a_1 x_1 + a_2 x_2 = 1, \]

where \( a_1, a_2 \) are given elements of \( K \) and the unknowns \( x_1, x_2 \) are elements of its unit group \( O_K^* \) or, more generally, of a group of \( S \)-units in \( K \), where \( S \) is a finite set of places on \( K \) containing all the infinite places. Recall that an algebraic number \( x \) in \( K \) is an \( S \)-unit if, by definition, \(|x|_v = 1\) for every place \( v \) not in \( S \).

These equations are called unit equations and \( S \)-unit equations; see the monograph [42] for explicit estimates of the height of their solutions and many bibliographic references.

**Theorem 4.1.** Let \( K \) be an algebraic number field of degree \( d \) and discriminant \( D_K \). Let \( S \) be a finite set of places on \( K \) containing the infinite places. Let \( a_1, a_2 \) be non-zero elements of \( K \). The equation

\[ a_1 x_1 + a_2 x_2 = 1, \quad \text{in } S \text{-units } x_1, x_2 \text{ in } K, \]

has only finitely many solutions, and all of them satisfy

\[ \max\{h(x_1), h(x_2)\} \ll_{d,D_K,S} \max\{h(a_1), h(a_2), 1\}. \]

The fact that the upper bound in (4.1) is linear in \( \max\{h(a_1), h(a_2), 1\} \) is a consequence of term \( B' \) in (1.3). Using (1.2) instead, with \( B \) in place of \( B' \), we would get an extra factor \( \log \max\{h(a_1), h(a_2), 2\} \) in the upper bound.
4.1. **Linear equations in two unknowns from a multiplicative division group.** More generally, we can take an arbitrary finitely generated subgroup $\Gamma$ of positive rank of the multiplicative group $(\mathbb{Q}^*)^2 = \mathbb{Q}^* \times \mathbb{Q}^*$, endowed with coordinatewise multiplication, and consider

\begin{equation}
(4.1) \quad a_1 x_1 + a_2 x_2 = 1, \quad \text{in } (x_1, x_2) \in \Gamma,
\end{equation}

where $a_1, a_2$ are given nonzero complex algebraic numbers. We can compute explicit upper bounds for the heights of $x_1$ and $x_2$. We quote below important results from [19], which deal with extensions of (4.1).

For $x = (x_1, x_2)$ in $(\mathbb{Q}^*)^2$, define $h(x) = h(x_1) + h(x_2)$. The division group of $\Gamma$ is the set

$$\Gamma = \{ x \in (\mathbb{Q}^*)^2 : x^k \in \Gamma \text{ for some } k \in \mathbb{Z}_{>0} \}$$

For $\varepsilon > 0$, the cylinder around $\Gamma$ and the truncated cone around $\Gamma$ are given by

$$\Gamma_{\varepsilon} = \{ x \in (\mathbb{Q}^*)^2 : \exists y, z \text{ with } x = yz, y \in \Gamma, z \in (\mathbb{Q}^*)^2, h(z) < \varepsilon \}$$

and

$$C(\Gamma, \varepsilon) = \{ x \in (\mathbb{Q}^*)^2 : \exists y, z \text{ with } x = yz, y \in \Gamma, z \in (\mathbb{Q}^*)^2, h(z) < \varepsilon(1 + h(y)) \},$$

respectively. We stress that the points in $\Gamma, \Gamma_{\varepsilon}$ or $C(\Gamma, \varepsilon)$ do not have their coordinates in a prescribed number field.

Nevertheless, it is possible to bound the height of a solution $(x_1, x_2)$ in $(\mathbb{Q}^*)^2$ to the equation

$$a_1 x_1 + a_2 x_2 = 1$$

which belongs to $\Gamma$, to $\Gamma_{\varepsilon}$ or to $C(\Gamma, \varepsilon)$, provided that $\varepsilon$ is sufficiently small. Below, $K_0$ denotes the algebraic number field generated by $a_1, a_2$, and the elements of $\Gamma$, and $h_0$ is an upper bound for the heights of the components of a system of generators of $\Gamma$ modulo torsion. Furthermore, $r$ is the rank of $\Gamma$ and $A$ is an effectively computable positive real number, which depends only on $\Gamma$. The reader is referred to [19] for an explicit expression of $A$.

**Theorem 4.2.** Every solution $(x_1, x_2)$ in $\Gamma$ of (4.1) satisfies

$$h(x_1, x_2) < A \max\{ h(a_1), h(a_2), 1 \}.$$

Suppose that $(x_1, x_2)$ is a solution to

$$a_1 x_1 + a_2 x_2 = 1, \quad \text{in } x = (x_1, x_2) \text{ in } \Gamma \text{ or in } \Gamma_{\varepsilon} \text{ with } \varepsilon < 0.0225.$$  

Then we have

$$h(x) \leq Ah(a_1, a_2) + 3rh_0A$$

and

$$[K_0(x_1, x_2) : K_0] \leq 2.$$

Suppose that $(x_1, x_2)$ is a solution to

$$a_1 x_1 + a_2 x_2 = 1, \quad \text{in } x = (x_1, x_2) \text{ in } C(\Gamma, \varepsilon),$$

and that

$$\varepsilon < \frac{0.09}{8Ah(a_1, a_2) + 20rh_0A}.$$
Then we have

\[ h(x) \leq 3Ah(a_1, a_2) + 5rh_0A \]

and

\[ [K_0(x_1, x_2) : K_0] \leq 2. \]

Here also, the fact that the upper bounds for \( h(x) \) are linear in \( h(a_1, a_2) \) is a consequence of term \( B' \) in (1.3). A key auxiliary result in the proof of Theorem 4.2 is a lemma of Beukers and Zagier \([22]\), which gives a lower bound for the sum of the heights of three distinct algebraic points with non-zero coordinates lying on a line \( \lambda x + \mu y + \nu z = 0 \) with \( \lambda \mu \nu = 0 \).

4.2. **Unit equations on quaternions.** Let \( \mathbb{H} = \mathbb{R} \oplus \mathbb{R}i \oplus \mathbb{R}j \oplus \mathbb{R}k \) denote the quaternion algebra \( \mathbb{H} \) over \( \mathbb{R} \), with the standard multiplication law \( i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j \). For an element \( \alpha = a + bi + cj + dk \) in \( \mathbb{H} \), where \( a, b, c, d \) are in \( \mathbb{R} \), define its conjugate to be \( \alpha = a - bi - cj - dk \), its norm to be

\[ N(\alpha) = a\overline{a} = \overline{a}a = a^2 + b^2 + c^2 + d^2, \]

and its trace

\[ \text{tr}(\alpha) = \alpha + \overline{\alpha} = 2a. \]

Write \( |\alpha| = \sqrt{N(\alpha)} \). We say that a quaternion \( \alpha = a + bi + cj + dk \) in \( \mathbb{H} \) is algebraic if all its coordinates \( a, b, c, d \) are algebraic over \( \mathbb{Q} \). This is equivalent to requiring that \( \alpha \) satisfies a polynomial equation with coefficients in \( \mathbb{Q} \), or that \( \mathbb{Q}[\alpha] \) is a finite field extension of \( \mathbb{Q} \). Indeed, \( \alpha \) always satisfies the quadratic equation

\[ X^2 - \text{tr}(\alpha)X + N(\alpha) = 0 \]

and if \( a, b, c, d \) are in \( \mathbb{Q} \), then so are \( \text{tr}(\alpha) \) and \( N(\alpha) \).

Denote by \( \mathbb{H}_a^\times \) the subalgebra of all quaternions that are algebraic. The next statement has been proved by Huang \([47, \text{Theorem 1.2}]\).

**Theorem 4.3.** Let \( \Gamma_1, \Gamma_2 \) be semigroups of \( \mathbb{H}_a^\times \) generated by finitely many elements of norms greater than 1, and fix \( a, a', b, b' \) in \( \mathbb{H}_a^\times \). Then the equation

\[ af' + bgb' = 1 \]

has only finitely many solutions \((f, g)\) in \( \Gamma_1 \times \Gamma_2 \) such that \(|1 - af'| \neq |af'|\).

Moreover, for such pairs \((f, g)\), we have effectively computable upper bounds for \(|f|, |g|\) that depend only on \( a, a', b, b' \) and generators of \( \Gamma_1, \Gamma_2 \).

By inspecting the proof in \([47]\), we see that an application of (1.3) (instead of (1.2) used in \([47]\)) yields an upper bound for \(|f|, |g|\) which is linear in \( \max\{|a|, |a'|, |b|, |b'|\} \).

5. **Almost powers in integer sequences**

As in Section 3, we let \( \mathcal{N} \) be a sequence of positive integers defined in some natural way. Assume that the theory of linear forms in logarithms allows us to compute an upper bound for the prime number \( q \) such that \( y^q \) is in \( \mathcal{N} \) for some integer \( y \geq 2 \). This is for example the case when

\[ \mathcal{N} = \{n(n+1) : n \geq 1\} \quad \text{and} \quad \mathcal{N'} = \{2^n + 3^n : n \geq 1\}. \]
In some situations, the proof can be modified to take advantage of $B'$ in order to show that, more generally, $Ay^q$ cannot be in $\mathcal{N}$ for $y \geq 2$, a prime number $q$ sufficiently large and a rational number $A$ of sufficiently small height in terms of $q$.

We consider three examples below. Other interesting equations, which can be treated in a similar way, include $2^m + 2^n + 1 = Ay^q$ and sums of two integral $T$-units being almost powers, see [14].

5.1. **Almost powers in values of polynomials.** The next result has been established in [18].

**Theorem 5.1.** Let $f(X)$ be an integral polynomial with at least two distinct roots. If there are integers $x, y, q$ and a rational number $A$ such that $|y| \geq 2$ and

$$f(x) = Ay^q,$$

then

$$q \ll_f \max\{1, h(A)\}.$$

By using (1.2) instead of (1.3) in the proof, one gets the weaker estimate

$$q \ll_f \max\{1, h(A)\} \log \max\{2, h(A)\}.$$

5.2. **Almost powers in recurrence sequences.** The next statement deals with recurrence sequences having a dominant root.

**Theorem 5.2.** Let $u := (u_n)_{n \geq 0}$ be a recurrence sequence of integers given by

$$u_n = f_1(n)\alpha_1^n + \ldots + f_t(n)\alpha_t^n, \quad \text{for } n \geq 0,$$

and having a dominant root $\alpha_1$. Assume that $\alpha_1$ is a simple root, that is, the polynomial $f_1(X)$ is equal to a non-zero algebraic number $f_1$. Then, the equation

$$u_n = Ay^q,$$

in rational $A$ and integers $n, y, q$ with $q$ a prime number, $|y| \geq 2$, and $u_n \neq f_1\alpha_1^n$, implies that

$$q \ll_u \max\{1, h(A)\}.$$

**Proof.** We establish a slightly more general result. We consider a sequence of nonzero integers $v := (v_n)_{n \geq 0}$ with the property that there are $\theta$ in $(0, 1)$ and a positive real number $C$ such that

$$|v_n - f\alpha^n| \leq C|\alpha|^{\theta n}, \quad n \geq 0,$$

where $f$ is a non-zero algebraic number and $\alpha$ is an algebraic number with $|\alpha| > 1$.

Let $n, y, q$ with $q$ a prime number be such that $v_n = Ay^q$, $|y| \geq 2$, and $v_n \neq f\alpha^n$. Then, we have

$$|Af^{-1}\alpha^{-n}y^q - 1| \leq \frac{C}{|f|} |\alpha|^{(\theta - 1)n}.$$

There exist integers $k$ and $r$ such that $n = kq + r$ with $|r| \leq \frac{q}{2}$. Rewriting (5.1) as

$$\Lambda := |Af^{-1}\left(\frac{y}{\alpha^k}\right)^q\alpha^{-r} - 1| \leq \frac{C}{|f|} |\alpha|^{(\theta - 1)n},$$
we observe that
\[
\log \Lambda \ll_V (-n).
\]
The height of \( \alpha_k \) is bounded from above by the sum of \( \log |y| \) and \( k \) times the height of \( \alpha \). Since
\[
q \log |y| \ll_V n \leq (k + 1)q \quad \text{and} \quad \frac{kq}{2} \leq n \ll q \log |y|,
\]
we get that \( h(\alpha_k/y) \ll_V \log |y| \). Set \( h^*(A) = \max\{1, h(A)\} \). It then follows from (1.5) that
\[
\log \Lambda \gg_V -h^*(A) (\log |y|) \left( \log \frac{q}{h^*(A)} \right),
\]
which, combined with (5.2) and (5.3), gives
\[
q \log |y| \ll_V n \ll h^*(A) (\log |y|) \left( \log \frac{q}{h^*(A)} \right),
\]
and we get \( q \ll_V h^*(A) \). This proves the theorem. Note that (1.6) does not enable us to conclude. \( \square \)

5.3. Almost powers in power sums. We provide another result on Diophantine properties of the sequence of integers of the form \( 2^m + 6^n + 1 \).

**Theorem 5.3.** All the solutions to the Diophantine equation
\[
2^m + 6^n + 1 = Ay^q,
\]
in positive integers \( m, n, q \) and \( A \) rational, satisfy
\[
q \ll \max\{1, h(A)\}.
\]

**Proof.** As in the proof of Theorem 3.6, we distinguish three cases. If \( 2^m \geq 6^2n \) or if \( 6^n \geq 6^2m \), then (1.5) implies that
\[
q(\log y) \ll \max\{m, n\} \leq h^*(A)(\log y) \log \frac{q}{h^*(A)},
\]
where \( h^*(A) = \max\{1, h(A)\} \). We conclude that \( q \ll \log h^*(A) \).

Now, if \( 2^m \leq 6^2n \leq 2^4m \), then \( m \gg n \) and Theorem 1.4 gives that
\[
q \log y \ll m \ll h^*(A) (\log y)(\log(q/h^*(A)))^2.
\]
In both cases, we get that \( q \ll h^*(A) \), as asserted. \( \square \)

We may try to get one step further. Let \( p \geq 5 \) be a prime number and consider the equation
\[
2^m + 6^n + 1 = Ap^\ell y^q,
\]
Without any restriction we assume that \( \ell \) satisfies \(|\ell| < q\). The cases \( 2^m \geq 6^2n \) and \( 6^n \geq 6^2m \) go as above and yield that
\[
q \ll (\log p)(\log \log p)h^*(A).
\]
Assume now that \( 2^m \leq 6^2n \leq 2^4m \). Theorem 1.3 gives that either
\[
q \ll h^*(A)(\log p)(\log y),
\]
(5.4)
or
\[ q \log y \ll m \ll h^*(A)(\log p)(\log y)(\log(q/h^*(A))). \]

Unfortunately, we cannot deduce anything from (5.4), when \( y \) is assumed to be variable. If one prefers to apply (1.10), then one gets
\[ q \log y \ll m \ll h^*(A)(\log p)(\log y)(\log(q\log y)/h^*(A)), \]
which does not yield a bound for \( q \) independent of \( y \).

6. Arithmetical properties of convergents

In this section, \( \theta \) is an arbitrary irrational, real number and \((p_n(\theta)/q_n(\theta))_{n \geq 1}\) (we will use the shorter notation \( p_n/q_n \) when there is no confusion possible and \( \xi \) instead of \( \theta \) if the number is known to be algebraic) denotes the sequence of its convergents. In particular, we have
\[
\left| \theta - \frac{p_n(\theta)}{q_n(\theta)} \right| < \frac{1}{q_n(\theta)^2}, \quad n \geq 1.
\]
Recall that, by Legendre’s theorem, every rational number \( p/q \) such that \( |\theta - p/q| < 1/(2q^2) \) is a convergent to \( \theta \).

**Theorem 6.1.** Let \( \varepsilon \) be a positive real number. Let \( S \) be a finite, non-empty set of distinct prime numbers. Let \((p_n/q_n)_{n \geq 1}\) denote the sequence of convergents to an irrational real algebraic number \( \xi \). Then, we have
\[
[p_n]_S < p_n^\varepsilon, \quad [q_n]_S < q_n^\varepsilon,
\]
for every sufficiently large integer \( n \). Furthermore, there exist effectively computable positive numbers \( \tau \) and \( n_0 \), depending only on \( \xi \) and \( S \), such that
\[
[p_nq_n]_S \leq (p_nq_n)^{1-\tau}, \quad n \geq n_0.
\]
Moreover, we have
\[
P[p_nq_n] \gg \xi \log \log q_n \frac{\log \log \log q_n}{\log \log \log \log q_n}, \quad n \geq 2.
\]

The first assertion is a direct consequence of Ridout’s theorem; see [57]. The last one has been established in [27], thereby slightly improving a result of [57]. The second assertion is new and follows directly from Theorem 6.3 below. No non-trivial effective upper bounds for \( [p_n]_S \) and \( [q_n]_S \) are known for every algebraic number of degree at least 3.

Erdős and Mahler [41] established that, when \( \theta \) is not a Liouville number (that is, when the irrationality exponent of \( \theta \) is finite), then \( P[q_{n-1}q_nq_{n+1}] \) tends to infinity with \( n \); see also [38]. However, their result is not effective. Using Baker’s theory of linear forms in logarithms, Shorey [58] proved that
\[
P[q_{n-1}q_nq_{n+1}] \gg_\theta \log \log q_n.
\]
We can do slightly better by applying Theorem 1.3.
Theorem 6.2. Let \( \theta \) be a real number, which is not a Liouville number. Let \( S \) be a finite, non-empty set of prime numbers. Then, there exist effectively computable positive numbers \( \tau \) and \( n_0 \), depending only on \( \theta \) and \( S \), such that

\[
[q^{-1}q_1q_n] \leq (q^{-1}q_1q_n)^{1-\tau}, \quad \text{for } n \geq n_0.
\]

Furthermore, we have

\[
P[q^{-1}q_1q_n] \gg \log \log q_n \cdot \frac{\log \log \log q_n}{\log \log \log \log q_n}, \quad n \geq 2.
\]

It follows from the proof of Theorem 6.2 that \( 1/(c\mu(\theta) \log \mu(\theta)) \) is a suitable value for \( \tau \), where \( \mu(\theta) \) is the irrationality exponent of \( \theta \) (see Definition 2.1) and \( c \) a positive, effectively computable, real number depending only on \( S \).

We denote by \( \lfloor x \rfloor \) the integer part of the real number \( x \). Theorem 6.1 is a consequence of the following result.

Theorem 6.3. Let \( S \) be a finite, non-empty set of prime numbers. For every irrational, real algebraic number \( \xi \), there exist effectively computable positive numbers \( q_0 \) and \( \tau \), depending only on \( \xi \) and \( S \), such that

\[
[q|q\xi]|_S \leq (q|q\xi|)^{1-\tau}, \quad \text{for } q \geq q_0.
\]

Furthermore, we have

\[
P[q|q\xi]|_S \gg \log \log q \cdot \frac{\log \log \log q}{\log \log \log \log q}.
\]

Proof. Without any loss of generality, we assume that \( q \) is large enough. Set \( p = \lfloor q\xi \rfloor \) and observe that \( |q\xi - p| < 1 \), thus

\[
0 < q|q^{-1}\xi - 1| < p^{-1}.
\]

Let \( S = \{\ell_1, \ldots, \ell_s\} \) be a finite set of prime numbers with \( \ell_1 < \ldots < \ell_s \). Let \( A_p \) (resp., \( A_q \)) be the greatest divisor of \( p \) (resp., of \( q \)) coprime with \( \ell_1 \ldots \ell_s \). There are integers \( b_1, \ldots, b_s \) such that

\[
(6.1) \quad \Lambda_q := |\xi(A_q/A_p)^{b_1}\ldots\ell^b_s - 1| \leq 1/p.
\]

Since \( \xi \) is irrational, \( \Lambda_q \) is non-zero. It then follows from (1.6) that there exists an effectively computable constant \( c_1 \), depending only on \( \xi \), such that

\[
(6.2) \quad \log \Lambda_q > -c_1 h^*(A) \sum_{i=1}^s (\log \ell_i) \log \frac{B(\log \ell_s)}{h^*(A)},
\]

where \( B = \max\{|b_1|, \ldots, |b_s|, 3\} \) and \( h^*(A) = \max\{\log A_q, \log A_p, 1\} \).

Since \( p \geq 2^B \), we derive from (6.1) and (6.2) that

\[
\frac{B}{h^*(A)} \ll \prod_{i=1}^s (c_1 \log \ell_i) \log \left(\prod_{i=1}^s (\log \ell_i)\right),
\]

thus there exist \( \eta > 0 \) and \( \tau > 0 \), depending only on \( \xi \), such that

\[
h^*(A) \geq e^{\eta B}, \quad [pq]_S \leq \frac{pq}{h^*(A)} \leq (pq)^{1-\tau},
\]

when \( q \) is large enough.
In the special case where $\ell_i$ is the $i$-th prime number and $A_p = A_q = 1$, we derive from the Prime Number Theorem that
\[
\prod_{i=1}^{s} (\log \ell_i) \leq 3^s \log \log s,
\]
thus
\[
\log B \ll \xi \log \log s,
\]
and
\[
\log \log pq \ll \xi \log \log s \ll \xi \ell_s \log \ell_s.
\]
This gives
\[
P[q(q^{\xi})] \gg \xi \log \log q \cdot \frac{\log \log \log \log q}{\log \log q},
\]
as asserted. We see the importance of the dependence on $s$ in (6.2). By using Waldschmidt’s estimate (1.3) in place of Matveev’s one (1.6), we would have to replace the factor $s \log \log s$ by $s \log s$ and we would get the (slightly) weaker statement
\[
P[q(q^{\xi})] \gg_{\text{eff}} \log \log q.
\]
□

Proof of Theorem 6.2. For $n \geq 2$, set $Q_n = q_{n-1}q_nq_{n+1}$ and $A_n$ denote the product $A_{-1,n}A_{0,n}A_{1,n}$, where $A_{j,n}$ is the greatest divisor of $q_{n+j}$ coprime with $\ell_1 \ldots \ell_s$, for $j = -1, 0, 1$. Write also
\[
Q_n = A_n^{b_{1,n}} \ell_1^{b_{s,n}}, \quad B_n = \max\{b_{1,n}, \ldots, b_{s,n}, 3\}
\]
and $h^*(A_n) = \max\{\log A_n, 1\}$. Observe that
\[
q_n \quad \text{divides} \quad \gcd(Q_n, q_{n+1} - q_{n-1}).
\]
For every prime $\ell$ dividing $Q_n$, we deduce from Theorem 1.3 that
\[
B_n \ll \varepsilon h^*(A_n) \quad \text{or} \quad v_\ell(q_{n+1} - q_{n-1}) \ll \varepsilon h^*(A_n) \log \frac{B_n}{h^*(A_n)},
\]
thus
\[
B_n \ll \varepsilon h^*(A_n) \quad \text{or} \quad \log q_n - \log A_{0,n} \ll \varepsilon h^*(A_n) \log \frac{B_n}{h^*(A_n)}.
\]
Note that $Q_n \geq 2^{B_n}$, thus
\[
2^{B_n/3} \leq q_{n+1}.
\]
Since $\theta$ is not a Liouville number, there exists a positive $\mu$ such that $q_{n+1} < q_n^\mu$ and we get
\[
\log q_{n+1} < \mu \log q_n \ll \varepsilon \mu h^*(A_n) \log \frac{\log q_{n+1}}{h^*(A_n)}.
\]
This gives a bound for $n$ such that $h^*(A_n) = 1$ and the existence of $\eta > 0$ and $n_0$, depending only on $\mu$ and $S$, such that
\[
A_n > q_{n+1} > q_n^{\mu/3}, \quad n > n_0.
\]
thus
\[
[Q_n]_S \leq Q_n^{1-\eta/3}, \quad n > n_0.
\]
We derive the last statement of the theorem by taking for $\ell_1, \ldots, \ell_s$ the first $s$ prime numbers and assuming that $A_n = 1$. We omit the details. □

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