Manifolds of positive Ricci curvature, quadratically asymptotically nonnegative curvature, and infinite Betti numbers

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Abstract In a previous paper (Jiang and Yang (2021)), we constructed complete manifolds of positive Ricci curvature with quadratically asymptotically nonnegative curvature and infinite topological type but dimensions greater than or equal to 6. The purpose of the present paper is to use a different technique to exhibit a family of complete $I$-dimensional ($I \geq 5$) Riemannian manifolds of positive Ricci curvature, quadratically asymptotically nonnegative sectional curvature, and certain infinite Betti numbers $b_j$ ($2 \leq j \leq I - 2$).

Keywords Riemannian manifold, positive Ricci curvature, quadratically asymptotically nonnegative curvature

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1 Introduction

An interesting topic in Riemannian geometry is to give some topological constraints of complete manifolds under certain prescribed curvature assumptions. For example, a remarkable theorem due to Gromov [5] says that the total Betti number of complete manifolds $M^n$ (either compact or noncompact) with nonnegative sectional curvature is bounded by a constant only depending on $n$. A natural question is: Can one bound the Betti numbers of open manifolds with nonnegative Ricci curvature?

For the first Betti number, Anderson [3] proved that $b_1(M^n) \leq n$ for a complete manifold with nonnegative Ricci curvature and $b_1(M^n) \leq n - 3$ if the manifold has positive Ricci curvature. For the codimension one Betti number of open manifolds with nonnegative Ricci curvature, Shen and Sormani [16] showed that either $M^n$ is a flat normal bundle over a compact totally geodesic submanifold or $M^n$ has a trivial codimension one integer homology. So $b_{n-1}(M^n) = \dim H_{n-1}(M^n; \mathbb{Z}) \leq 1$.

For other Betti numbers $b_k(M^n)$ ($2 \leq k \leq n - 2$), it is difficult to find out some similar control, even with some additional assumptions. In fact, Sha and Yang [15] (see also [14,17]) constructed $n$-dimensional ($n \geq 4$) open manifolds with positive Ricci curvature and certain infinite Betti numbers $b_k$.

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asymptotically nonnegative curvature, and infinite Betti numbers $b_k$ ($2 \leq k \leq n - 2$). Later, using a totally different method, Menguy [9, 10] also constructed 4-dimensional open manifolds with positive Ricci curvature, maximal volume growth or minimal volume growth, and the infinite second Betti number; moreover, he showed that similar things can be done so that one can get higher-dimensional examples with infinite even-dimensional Betti numbers.

So, in order to get certain control of Betti numbers, it seems to need some additional assumptions. There is a natural and interesting sectional curvature condition, quadratically asymptotically nonnegative curvature, which can be defined as follows. Let

$$K_{p_0}(t) = \inf_{M^n \setminus B(p_0, t)} K,$$

where $K$ denotes the sectional curvature of $M$, and the infimum is taken over all the 2-planes at the points in $M \setminus B(p_0, t)$. We say that $M$ is of quadratically asymptotically nonnegative curvature if for all $t \geq 0$,

$$K_{p_0}(t) \geq -\frac{K_0}{1 + t^2}$$

for some positive constant $K_0$. We also say that $M$ is of finite topological type if it is homeomorphic to the interior of a compact manifold with boundary, otherwise, infinite topological type. In particular, manifolds of finite topological type must have finite Betti numbers. A little additional computation shows that all the examples mentioned above are not of quadratically asymptotically nonnegative curvature.

Actually, Sha and Shen [13] conjectured that a complete manifold $M^n$ of nonnegative Ricci curvature and quadratically asymptotically nonnegative curvature should be of finite topological type. In [6], Jiang and Yang constructed a counterexample with the infinite second Betti number in the case of dimensions not less than 6, based on works of Perelman [11] and Menguy [8–10].

The purpose of the present paper is to use a topologically and metrically different technique to construct a family of complete 1-dimensional ($I \geq 5$) Riemannian manifolds of positive Ricci curvature, quadratically asymptotically nonnegative sectional curvature, and certain infinite Betti numbers $b_j$ ($2 \leq j \leq I - 2$). In particular, this again gives a negative answer to Sha and Shen’s conjecture in dimension 5. More precisely, we have the following theorem.

**Theorem 1.1.** For all integers $m$ and $n$ satisfying $n \geq m \geq 2$ and $n \geq 3$, there exists an $(m+n)$-dimensional complete Riemannian manifold $M$, which is of positive Ricci curvature, quadratically asymptotically nonnegative curvature, and infinite Betti numbers $b_m$ and $b_n$. In particular, it is of infinite topological type.

**Remark 1.2.** (1) Abresch [1, 2] showed that if a complete $n$-dimensional ($n \geq 3$) manifold satisfies that the integral

$$\int_0^\infty rK_{p_0}^-(r)dr = b_0 < \infty,$$

where $K_{p_0}^-(r) = \max\{-K_{p_0}(r), 0\}$ (a certain kind of decay on the lower sectional curvature bound which is a little stronger than quadratically asymptotically nonnegative curvature), and then the manifold is of finite topological type; moreover, the total Betti number of the manifold is uniformly bounded.

(2) For the 3-dimensional case, complete noncompact Riemannian manifolds with nonnegative Ricci curvature were completely classified by Liu [7] based on the technique earlier developed by Schoen and Yau [12], which must be of finite topological type.

(3) Combine our examples in dimensions greater than or equal to 5 with Liu’s result in dimension 3, and then the remaining is the case of dimension 4. But, technically, it is difficult to use the present method to construct counterexamples of dimension 4. On the other hand, this seems to be the most interesting case in some sense.

Topologically, the construction of the manifolds is similar to that of Sha and Yang [15], i.e., removing some balls $D^{m+1}_i (1 \leq i < \infty)$ from $\mathbb{R}^{m+1}$, and then gluing $(\mathbb{R}^{m+1} \setminus D^{m+1}) \times S^{n-1}$ with $P_i = S^m \times D^n$ along the boundary, which is different from our previous one [6]. Then, the resulting $(m+n)$-dimensional manifold has infinite Betti numbers $b_m$ and $b_n$. 
The construction for the metrics however is totally different from that of Sha and Yang [15]; in their construction, a C¹ metric on the whole manifold is directly given. Instead, we first give a C⁰ metric on the whole manifold; more precisely, we give a C¹ metric on each part (i.e., \([\mathbb{R}^{m+1} \setminus D^{m+1}_i] \times S^{n-1}\) and \(P_i = S^m \times D^n\)) but with isometric boundaries and then glue them together along the boundaries, so that the metric is only C⁰ on the glued part, but C¹ on the remaining part. On the other hand, the above C¹ metrics actually have some additional restrictions on the normal curvature of the corresponding boundaries, so that we next can use a C¹ metric to replace the C⁰ metric near the glued part; moreover, the corresponding Ricci and sectional curvature properties are preserved. Thus, we can get a required complete C¹ metric on the whole manifold. Essentially, the existence of such a metric is guaranteed by the following gluing criterion due to Perelman [11] (see also [4] and Remark 4.1 in Section 4), but here for the sake of clarity, we give an explicit C¹ construction.

**Gluing criterion.** Let \(M_1\) and \(M_2\) be two compact smooth manifolds of positive Ricci curvature with isometric boundaries \(\partial M_1 \cong \partial M_2 = X\). Suppose that the normal curvature of \(\partial M_1\) is larger than the negative of the normal curvature of \(\partial M_2\). Then the glued manifold \(M_1 \cup_X M_2\) along the boundary \(X\) can be smoothened near \(X\) to produce a manifold of positive Ricci curvature.

Based on the above idea, the construction can be divided into the following three steps. First, we construct a C¹ metric \(ds^2\) on

\[ Q = \mathbb{R}^{m+1} \times S^{n-1} = [t_0, +\infty) \times_{u(t)} S^m \times_{g(t)} S^{n-1} \]

and a C¹ metric \(ds^2\) on \(P_i = S^m \times D^n\) (\(1 \leq i < \infty\)). These are done in Section 2.

Then we compute the curvature tensors and show that both metrics have positive Ricci curvature and quadratically asymptotically nonnegative sectional curvature by choosing some appropriate constants. These will be done in Section 3.

Finally, we remove the geodesic balls \(B_{\epsilon r_i}(\epsilon)\) from \([t_0, +\infty) \times_{u(t)} S^m\) (for the precise meanings of \(r_i\) and \(\epsilon\), see Subsection 2.1) and glue \(Q \setminus \bigcup_{i=1}^{+\infty} (B_{\epsilon r_i}(\epsilon) \times g_i, S^{n-1})\) with \(P_i\) along the boundary \(\partial B_{\epsilon r_i}(\epsilon) \times g_i, S^{n-1} = S^m \times S^{n-1}\),

this gives a C⁰ metric \(h_0\) on the whole manifold. Then we use a C¹ metric \(h_1\) to replace the above \(h_0\) near the glued boundary and verify that it still has positive Ricci curvature and quadratically asymptotically nonnegative curvature. These will be done in Section 4.

So our manifold is actually

\[ M^{m+n} = \left( Q \setminus \bigsqcup_{i=1}^{+\infty} (B_{\epsilon r_i}(\epsilon) \times g_i, S^{n-1}) \right) \cup_{\text{Id}} \bigsqcup_{i=1}^{+\infty} P_i \]

\[ = \left( \left( \mathbb{R}^{m+1} \setminus \bigsqcup_{i=1}^{+\infty} D^{m+1}_i \right) \times S^{n-1} \right) \cup_{\text{Id}} \bigsqcup_{i=1}^{+\infty} (S^m \times D^n)_i, \]

where by ‘\(\text{Id}\)’ we mean gluing along the corresponding boundaries through the identity map.

2 The construction of the metrics

As seen in Section 1, our topological manifold is

\[ M^{m+n} = \left( Q \setminus \bigsqcup_{i=1}^{+\infty} (B_{\epsilon r_i}(\epsilon) \times g_i, S^{n-1}) \right) \cup_{\text{Id}} \bigsqcup_{i=1}^{+\infty} P_i. \]

In the following, we construct the required metrics on \(Q\) and \(P_i\) \((i = 1, 2, \ldots)\), respectively.
2.1 The construction of $Q$

We equip $Q = [t_0, +\infty) \times u(t) S^m \times g(t) S^{n-1}$ (for some $t_0 = t_1 - \psi_1$, where $t_1$ and $\psi_1$ will be given later) with the following metric:
\[
ds^2 = dt^2 + u^2(t)d\theta^2_{S^m} + g^2(t)ds^2_{S^{n-1}},
\]
where $d\theta^2_{S^m}$ and $ds^2_{S^{n-1}}$ are the standard metrics on $S^m$ and $S^{n-1}$, respectively.

Give some constants $c, \gamma, \alpha$ and $t_1$ satisfying
\[
0 < c < \frac{1}{3}, \quad 0 < \gamma < \frac{1}{4}, \quad \alpha > 1, \quad t_1 > 1.
\]

Set
\[
K = K(c) = 1 - \frac{c^2}{c^2},
\]
and define $\psi = \psi(c)$ by
\[
\sin(\sqrt{K}\psi) = \sqrt{1 - c^2}.
\]

Take $r > 0$ satisfying
\[
r \leq r(c) = \frac{\pi}{4\sqrt{K}} - \frac{\psi}{2}.
\]

Note that $c$ first is fixed while $r, \gamma, \alpha$ and $t_1$ will be determined later. We define for $i = 1, 2, \ldots$,
\[
t_i = t_1 \alpha^{i-1},
\]
\[
r_i = rt_i,
\]
\[
\psi_i = \psi t_i,
\]
\[
K_i = \frac{K}{t_i^2},
\]
and
\[
\Delta = \sqrt{K_i}(2r_i + \psi_i) = \sqrt{K}(2r + \psi).
\]

In the following, we give the constructions of $u$ and $g$, respectively.

2.1.1 The construction of $u(t)$

For $t_0 \leq t \leq t_1$, we set
\[
u(t) = \frac{1}{\sqrt{K_1}} \sin(\sqrt{K_1}(t - t_1 + \psi_1)).
\]

Here, $t_0 = t_1 - \psi_1 = (1 - \psi)t_1$.

For $t_i < t < t_i + 2r_i$ ($i = 1, 2, \ldots$), we set
\[
u(t) = \frac{1}{\sqrt{K_i}} \sin(\sqrt{K_i}(t - t_i + \psi_i));
\]

for $t_i + 2r_i < t < t_{i+1}$, we set
\[
u(t) = t_iw\left(\frac{t}{t_i}\right),
\]
where $w$ is a $C^2$ function on $[1 + 2r, \alpha]$, which is independent of $i$.

Actually, we can define $w$ as follows. First, set
\[
w(t) = \min\left\{\sin\frac{\Delta}{\sqrt{K}} + (t - 1 - 2r)\cos\Delta + \frac{c + 1}{2\log\frac{t}{1 + 2r}}\left(t \log\frac{t}{1 + 2r} - t + 1 + 2r\right), ct\right\},
\]
and then smoothen $w(t)$ to be a $C^2$ function. Note that at $t = 1 + 2r$,
\[
\sin\frac{\Delta}{\sqrt{K}} < c(1 + 2r),
\]
while at $t = \alpha$,
\[
\sin \frac{\Delta}{\sqrt{K}} + (\alpha - 1 - 2r) \cos \Delta + \frac{c + 1}{2 \log \frac{}\alpha + 1 \alpha \gamma} \left( \alpha \log \frac{\alpha}{1 + 2r} - \alpha + 1 + 2r \right) > ca,
\]
provided that $\cos \Delta \geq \frac{c + 1}{2 \log \frac{\alpha}{1 + 2r}}$, which is possible for $\alpha \geq \alpha_0(c, r)$.

Thus, $u(t)$ is $C^1$ at the endpoints $t_i + 2r_i = (1 + 2r)t_i$ and $t_{i+1} = \alpha t_i$. On the other hand, if $w(t) = ct$, then
\[
\begin{cases}
w_t \equiv c, \\
w_{tt} \equiv 0;
\end{cases}
\]
if $w(t) = \frac{\sin \Delta}{\sqrt{K}} + (t - 1 - 2r) \cos \Delta + \frac{c + 1}{2 \log \frac{\alpha}{1 + 2r}} (t \log \frac{1}{1 + 2r} - t + 1 + 2r)$, then
\[
\begin{cases}
\cos \Delta \leq w_t \leq \frac{1 + 3c}{2} < 1, \\
-3 \frac{w_{tt}}{w} + \frac{\gamma(1 - 2\gamma)}{t^2} > 0,
\end{cases}
\]
when $\cos \Delta \geq \frac{c + 1}{2 \log \frac{\alpha}{1 + 2r}}$ and $\alpha \geq \alpha_1(c, r, \gamma)$.

**Conclusion.** When $\alpha \geq \alpha_2(c, r, \gamma)$, we have a $u(t)$ satisfying for $t > t_1$,
\[
\begin{cases}
\cos \Delta \leq u_t \leq c, -\frac{u_{tt}}{u} = K \frac{1}{t^2}, t_i < t < t_i + 2r_i, \\
\cos \Delta \leq u_t \leq \frac{1 + 3c}{2}, -3 \frac{u_{tt}}{u} + \frac{\gamma(1 - 2\gamma)}{t^2} > 0, t_i + 2r_i < t < t_{i+1},
\end{cases}
\]

2.1.2 *The construction of $g(t)$*

For $t_0 \leq t \leq t_1 + \frac{r}{6}$, we set
\[
g(t) \equiv g_1 = g \left( t_1 + \frac{r_1}{6} \right) = \left( t_1 + \frac{r_1}{6} \right) ^\gamma = \left( 1 + \frac{r}{6} \right) ^\gamma t_1 ^\gamma.
\]

For $t_1 + \frac{r}{6} < t < t_{i+1} + \frac{r_{i+1}}{6} = \alpha(t_i + \frac{r}{6})$ $(i = 1, 2, \ldots)$, we set
\[
g_i(t) = \begin{cases}
0, & t_i + \frac{r_i}{6} < t < t_i + \frac{11r_i}{6}, \\
\frac{6\gamma\beta}{(1 + 2r)r t_i ^2} \left( t - \left( t_i + \frac{11r_i}{6} \right) \right), & t_i + \frac{11r_i}{6} < t < t_i + 2r_i, \\
\frac{\gamma\beta}{r}, & t_i + 2r_i < t < t_{i+1}, \\
\frac{6\gamma\beta}{\alpha^2 r t_i ^2} \left( \left( t_{i+1} + \frac{r_{i+1}}{6} \right) - t \right), & t_{i+1} < t < t_{i+1} + \frac{r_{i+1}}{6},
\end{cases}
\]
where
\[
\beta = \frac{\log \alpha}{\log \alpha - \log(1 + 2r) + \frac{r(1 + r)}{r(1 + 2r)}}, \quad \beta(r, \alpha),
\]
so that
\[
g(t) = g \left( t_i + \frac{r}{6} \right) = \left( t_i + \frac{r}{6} \right) ^\gamma, \quad \forall i \geq 1.
\]

Thus, for $t_i + \frac{r}{6} < t < t_{i+1} + \frac{r_{i+1}}{6} = \alpha(t_i + \frac{r}{6})$, we have
\[
g(t) \leq g \left( t_{i+1} + \frac{r_{i+1}}{6} \right) = \alpha ^\gamma \left( t_i + \frac{r}{6} \right) ^\gamma \leq \alpha ^\gamma t_i ^\gamma.
\]
and
\[ g(t) \geq g\left(t_i + \frac{r_i}{6}\right) = \left(t_i + \frac{r_i}{6}\right)^\gamma \geq \alpha^{-\gamma}t^\gamma. \]

Note that for \( \alpha \geq \alpha_3(r) \), we can have \( \frac{1}{2} \leq \beta \leq 2 \). Then one has
\[ \left| \frac{g_t}{g} \right| \leq \frac{2(1 + \frac{r_i}{6})^\gamma}{t} \]
and
\[
\begin{cases}
\left| \frac{g_u}{g} \right| \leq \frac{12\gamma(1 + \frac{r_i}{6})}{rt_i^2}, & t_i < t < t_i + 2r_i, \\
- \frac{g_t}{g} \geq \gamma(1 - 2\gamma)\frac{1}{2t^2}, & t_i + 2r_i < t < t_{i+1}.
\end{cases}
\]

**Conclusion.** When \( \alpha \geq \alpha_3(r) \), we have a \( C^1 \) function \( g(t) \) satisfying for \( t > t_1 \),
\[ \alpha^{-\gamma}t^\gamma \leq g(t) \leq \alpha^{\gamma}t^\gamma, \]
\[ \left| \frac{g_t}{g} \right| \leq \frac{2(1 + \frac{r_i}{6})^\gamma}{t} \]
and
\[
\begin{cases}
\left| \frac{g_u}{g} \right| \leq \frac{12\gamma(1 + \frac{r_i}{6})}{rt_i^2}, & t_i < t < t_i + 2r_i, \\
- \frac{g_t}{g} \geq \gamma(1 - 2\gamma)\frac{1}{2t^2}, & t_i + 2r_i < t < t_{i+1}.
\end{cases}
\]

**Remark 2.1.** We remark that at some discrete points, \( u(t) \) and \( g(t) \) are only \( C^1 \). However, if the manifold constructed in this manner has positive Ricci curvature on the complement of those \( C^1 \) parts, the manifold can then be smoothened to be a \( C^2 \) manifold of positive Ricci curvature; for this, one can refer to [11] (see also [4,10]).

**2.2 The construction of \( P_i \) \( (1 \leq i < +\infty) \)**

\( P_i \) topologically is \( S^n \times D^n \) but with a metric
\[ ds^2 = dt^2 + \bar{u}^2(t) d\bar{\theta}_{S^m}^2 + \bar{g}^2(t) d\sigma_{S^{n-1}}, \]
where \( d\bar{\theta}_{S^m}^2 \) and \( d\sigma_{S^{n-1}}^2 \) are the standard metrics on \( S^m \) and \( S^{n-1} \), respectively, and \( 0 \leq \bar{u} \leq R_i \) (and \( R_i \) will be fixed in the following).

First, some constants \( C_1, C_2, C_3, N, R_i, b_i, l_i \) and \( C_4 \) are given as follows:
\[ C_1 = \left( \sin \frac{4}{5}\sqrt{K}r \right)^{-1}, \]
\[ C_2 = \cos \frac{3}{5}\sqrt{K}r > \cos \frac{4}{5}\sqrt{K}r, \]
\[ 0 < C_3 < 1 - C_2^2, \]
\[ N \geq 5n, \]
\[ R_i > \frac{C_4}{C_1C_2(1 - C_3)} t_i, \]
\[ b_i = R_i - \frac{C_4}{C_1C_2(1 - C_3)} t_i, \]
\[ l_i = \frac{C_i(NC_2)^{1/C_3}}{t_i} \]
and \( C_4 \) is given by \( C_2(1 - C_4)^{\frac{C_2}{C_3}} = \frac{4}{5}, \) provided that \( N > \frac{1}{C_2^2} \); clearly, \( 0 < C_4 < 1 \). Note that \( C_1 \) and \( C_2 \) are given by \( c \) and \( r \); \( C_3, N \) and \( R_i \) will be fixed later; \( b_i, l_i \) and \( C_4 \) are given by \( C_1, C_2, C_3, N \) and \( R_i \).
2.2.1 The construction of \( \bar{u}(\ell) \)

\( \bar{u}(\ell) \) is given by

\[
\bar{u}_{\ell} = \begin{cases} 
\frac{1}{(1-C_3)\ell + \frac{N}{t_i} - (1-C_3)b_i}, & b_i \leq \ell \leq R_i, \\
\frac{l_i}{N l_i}, & 0 \leq \ell < b_i,
\end{cases}
\]

and \( \bar{u}(R_i) = \frac{l_i}{N} \) and \( \bar{u}(\ell) = C_2 \). Thus,

\[
\bar{u}_{\ell} = \begin{cases} 
C_3 \frac{\bar{u}_{\ell}^2}{\bar{u}} = \frac{C_3}{(1-C_3)\ell + \frac{N}{t_i} - (1-C_3)b_i}^2, & b_i \leq \ell \leq R_i, \\
\frac{l_i}{N l_i} + \left( \frac{l_i}{N l_i} \right)^2, & 0 \leq \ell < b_i.
\end{cases}
\]

Then for \( b_i \leq \ell \leq R_i \),

\[
\bar{u}(\ell) = C_2 \left( \frac{t_i}{C_1 C_2} \right)^{-\frac{c_4}{C_1}} \left( (1-C_3)\ell + \frac{N}{t_i} - (1-C_3)b_i \right)^{-\frac{1}{C_1}},
\]

\( \bar{u}(b_i) = \frac{1}{l_i} \) and \( \bar{u}(b_i) = \frac{1}{N} \), while for \( 0 \leq \ell < b_i \),

\[
0 < \frac{\bar{u}_{\ell}}{\bar{u}} \leq \frac{l_i}{N l_i} + \left( \frac{l_i^2}{N^2} \right)^2,
\]

and \( \bar{u}(0) = \frac{1}{l_i} \exp(-\frac{b_i}{2N}) > 0 \).

2.2.2 The construction of \( \bar{g}(\ell) \)

Set

\[
A = \frac{(1 + \frac{r}{6})^\gamma}{c_1(N C_2)^{1+c_3}} + \frac{C_3}{c_4(C_2(1-C_3))^\gamma} = A(r, c, \gamma, N, C_3) = O(1),
\]

and then \( d_i = d_i(A, t_i) \) is given by the equation \( d_i^{-\gamma} \cos(l_i d_i) = \gamma A \). We remark that when \( d_i = 0 \), the left-hand side of the equation is 0, which is smaller than the right-hand side \( \gamma A \); when \( d_i = O(t_i) \), the left-hand side is \( O(t_i^{-\gamma}) \) (since \( l_i = O(\frac{1}{t_i}) \)), which is much larger than the right-hand side. Thus, there exists such a positive \( d_i \) satisfying the above equation; moreover, \( d_i = O(1) \).

Set

\[
B = \frac{\sin(l_i d_i)}{l_i} - A d_i = A d_i \left( \gamma \tan(l_i d_i) - 1 \right),
\]

which is negative when \( t_i \) is sufficiently large (i.e., \( i \) is sufficiently large). Moreover, \( B = O(1) \).

Finally, \( R_i \) is given by the following equation:

\[
AR_i = B = \left( 1 + \frac{r}{6} \right)^\gamma t_i^\gamma,
\]

i.e.,

\[
\left( b_i + \frac{C_4}{C_1 C_2(1-C_3)} t_i \right)^\gamma = R_i = \frac{(1 + \frac{r}{6})^\gamma t_i^\gamma - B}{A}.
\]

Since \( B < 0 \) and \( B = O(1) \), we have

\[
\frac{(1 + \frac{r}{6})^\gamma t_i^\gamma - B}{A} \leq \frac{(1 + \frac{r}{6})^\gamma t_i^\gamma}{A} \leq \frac{(1 + \frac{r}{6})^\gamma t_i^\gamma}{A} \left( \frac{2}{C_1(N C_2)^{1+c_3}} + \frac{C_4}{C_4(C_2(1-C_3))^\gamma} \right),
\]
as \( t_i > t'(c, r, \gamma, C_3, N) \), i.e., \( t_i \) is sufficiently large.

Then, we get

\[
R_i > \frac{C_4}{C_1C_2(1-C_3)} t_i
\]

and

\[
\frac{1}{l_i} \leq b_i \leq \frac{2}{l_i}.
\]

Moreover, \( R_i = O(t_i) \) and \( b_i = O(t_i) > d_i \), since \( l_i = O\left(\frac{1}{t_i}\right) \).

Now, \( \bar{g}(\bar{t}) \) can be defined as follows:

\[
\bar{g}(\bar{t}) = \begin{cases} 
A\bar{r}^2 + B, & d_i \leq \bar{t} \leq R_i, \\
\sin(l_i\bar{t})/l_i, & 0 \leq \bar{t} \leq d_i,
\end{cases}
\]

which satisfies

\[
\frac{\bar{g}}{g} = \begin{cases} 
\gamma A\bar{r}^{\gamma-1}, & d_i \leq \bar{t} \leq R_i, \\
A\bar{r}^{\gamma} + B/l_i, & 0 \leq \bar{t} \leq d_i
\end{cases}
\]

and

\[
\frac{\bar{g}^{\bar{t}t}}{g} = \begin{cases} 
-\gamma(1-\gamma)A\bar{r}^{-2}/l_i^{\gamma-1}, & d_i \leq \bar{t} \leq R_i, \\
-l_i^{\gamma-2}, & 0 \leq \bar{t} \leq d_i
\end{cases}
\]

3 Quadratically asymptotic nonnegativeness of curvature and positiveness of Ricci curvature

In this section, we calculate the curvature tensors of \( Q \) and \( P_t \) with the given metrics in Section 2, and show that they have both positive Ricci curvature and quadratically asymptotically nonnegative sectional curvature by taking some appropriate constants, i.e., \( \eta, r, \gamma \) and \( C_3 \) are sufficiently small and \( \alpha, t_1 \) and \( N \) are sufficiently large.

3.1 Curvature of \( Q \)

For \( Q \), the metric is

\[
ds^2 = dt^2 + u^2(t)d\theta_{S^m}^2 + g^2(t)d\sigma_{S^{n-1}}^2,
\]

where \( m \geq 2, n \geq 3 \) and \( n \geq m \), and thus \( 2(n-1) \geq m \geq 2 \). Let \( T, \{\Theta_k\} \) and \( \{\Sigma_l\} \) be orthonormal bases of the tangent spaces corresponding to the directions \( dt, d\theta_{S^m} \) and \( d\sigma_{S^{n-1}} \), respectively. Then for \( t \geq t_1 \), the sectional curvature can be computed as follows:

\[
K(T, \Theta_k, \Theta_k, T) = -\frac{u_t}{u} \geq -\frac{\gamma(1-2\gamma)}{3l^2},
\]

\[
K(T, \Sigma_l, \Sigma_l, T) = -\frac{g_t}{g} \geq -\frac{12\gamma(1 + \frac{r^2}{l^2})(1 + 2r)}{rt^2}\frac{1}{2} > 0,
\]

\[
K(\Theta_k, \Theta_p, \Theta_p, \Theta_k) = \frac{1}{u^2} - \frac{u_t^2}{u^2} \geq \frac{1 - (1+3c)^2}{\cos^2 \Delta} \frac{1}{l^2} > 0,
\]

\[
K(\Sigma_l, \Sigma_q, \Sigma_q, \Sigma_l) = \frac{1}{g^2} - \frac{g_t^2}{g^2} \geq \frac{1}{l^2} - \frac{4(1 + \frac{r^2}{l^2})^2\gamma^2}{l^2} > 0,
\]

\[
K(\Theta_k, \Sigma_l, \Sigma_l, \Theta_k) = -\frac{u_t}{u} \geq -\frac{(1+3c)(1+\frac{r}{l})\gamma}{\cos \Delta} \frac{1}{l^2},
\]

the other terms of curvature tensors are zero. Thus, we have

\[
K_{P_1}(t) \geq -\frac{K_2^2(c, r, \gamma)}{l^2},
\]
i.e., $Q$ has quadratically asymptotically nonnegative sectional curvature.

The nonzero Ricci curvature of $Q$ is as follows:

$$
\text{Ric}(T, T) = -m \frac{u_t}{u} - (n - 1) \frac{g_t}{g}
$$

for $i = 1, 2, \ldots$, provided that $\gamma < \gamma_0(n, c, r);

$$
\text{Ric}(\Theta_i, \Theta_i) = (m - 1) \left( \frac{1}{u_t^2} - \frac{u_t^2}{u^2} \right) - \frac{u_t g_t}{u g}
$$

for $i = 1, 2, \ldots$, provided that $\gamma < \gamma_1(m, n, c, r);

$$
\text{Ric}(\Sigma_l, \Sigma_l) = (n - 2) \left( \frac{1}{g^2} - \frac{g_t^2}{g^2} \right) - \frac{g_t}{g} - m \frac{u_t g_t}{u g}
$$

provided that $t_1 > t_1(c, r, \gamma)$. Moreover, all the off-diagonal terms of the Ricci tensor vanish.

On the other hand, when $t_0 \leq t \leq t_1$, the metric is actually

$$
\text{ds}^2 = dt^2 + \left( \frac{1}{\sqrt{K_1}} \sin(\sqrt{K_1}(t - t_1 + \psi_1)) \right)^2 d\psi_{m}^2 + \tilde{g}_t^2 d\sigma_{\gamma-1}^2,
$$

which can be considered as part of $S^{m+1}(\frac{1}{\sqrt{K_1}}) \times (t_1 + \frac{\psi}{\gamma}) S^{\gamma-1}$, which obviously is of positive Ricci curvature.

Thus, the Ricci curvature of $Q$ is positive.

### 3.2 Curvature of $P_i$

For $P_i$, the metric is

$$
\text{ds}^2 = dt^2 + \tilde{u}^2(t) d\psi_{m}^2 + \tilde{g}_t^2(t) d\sigma_{\gamma-1}^2,
$$
where \( m \geq 2, n \geq 3 \) and \( n \geq m \) so that \( 2(n-1) \geq m \geq 2 \), and \( 0 \leq \ell \leq R_i \). Similarly, let \( T, \{\Theta_k\} \) and \( \{\Sigma_l\} \) be orthonormal bases of the tangent spaces corresponding to the directions \( dl, d\theta_{\Sigma m}^l \) and \( d\sigma_{\Sigma n-1}^l \), respectively. Then the sectional curvature can be computed as follows:

\[
K(T, \Theta_k, \Theta_k, T) = -\frac{\ddot{u}^2}{\dot{u}}, \\
K(\Theta_k, \Theta_p, \Theta_p, \Theta_k) = \frac{1}{\dot{u}^2} - \frac{\ddot{u}^2}{\dot{u}^2}, \\
K(T, \Sigma_l, \Sigma_l, T) = -\frac{\ddot{g}_{\Sigma l}}{\dot{g}}, \\
K(\Sigma_l, \Sigma_q, \Sigma_q, \Sigma_l) = \frac{1}{\ddot{g}} - \frac{\ddot{g}_{\Sigma l}}{\ddot{g}}, \\
K(\Theta_k, \Sigma_l, \Sigma_l, \Theta_k) = -\frac{\ddot{u} \ddot{g}_{\Sigma l}}{\dot{u} \dot{g}}.
\]

The other terms of curvature tensors are zero.

The nonzero Ricci curvature of \( P_i \) is as follows:

\[
\text{Ric}(T, T) = -m \frac{\ddot{u}}{\dot{u}} - (n-1) \frac{\ddot{g}}{\dot{g}}, \\
\text{Ric}(\Theta_k, \Theta_k) = (m-1) \left( \frac{1}{\dot{u}^2} - \frac{\ddot{u}^2}{\dot{u}^2} \right) - \frac{\ddot{u}}{\dot{u}} - (n-1) \frac{\ddot{g}_{\Sigma l}(b_i)}{\dot{u} \dot{g}}, \\
\text{Ric}(\Sigma_l, \Sigma_l) = (n-2) \left( \frac{1}{\ddot{g}} - \frac{\ddot{g}_{\Sigma l}}{\ddot{g}} \right) - \frac{\ddot{g}_{\Sigma l}}{\ddot{g}} - m \frac{\ddot{u} \ddot{g}_{\Sigma l}}{\dot{u} \dot{g}}.
\]

In the following, we discuss the sectional and Ricci curvature as \( \ell \) is in different intervals, i.e., \([0, d_1], [d_1, b_i] \) and \([b_i, R_i] \), respectively.

3.2.1 In the interval \([0, d_1] \)

For \( 0 \leq \ell \leq d_1 \),

\[
K(T, \Theta_k, \Theta_k, T) = -\frac{\ddot{u}}{\dot{u}} \geq \left( \frac{1}{N} + \frac{1}{N^2} \right) \ell^2 = O \left( \frac{1}{\ell^2} \right), \\
K(\Theta_k, \Theta_p, \Theta_p, \Theta_k) = \frac{1}{\dot{u}^2} - \frac{\ddot{u}^2}{\dot{u}^2} \geq \left( 1 - \frac{1}{N^2} \right) \frac{1}{\dot{u}^2} \geq \left( 1 - \frac{1}{N^2} \right) \ell^2 > 0, \\
K(T, \Sigma_l, \Sigma_l, T) = -\frac{\ddot{g}}{\dot{g}} = \ell^2 > 0, \\
K(\Sigma_l, \Sigma_q, \Sigma_q, \Sigma_l) = \frac{1}{\ddot{g}} - \frac{\ddot{g}_{\Sigma l}}{\ddot{g}} = \ell^2 > 0, \\
K(\Theta_k, \Sigma_l, \Sigma_l, \Theta_k) = -\frac{\ddot{u} \ddot{g}_{\Sigma l}}{\dot{u} \dot{g}} = -\frac{l_i}{N b_i \tan(l_i)} \geq -\frac{l_i}{N b_i} \geq -\frac{\ell^2}{5m} = O \left( \frac{1}{\ell^2} \right).
\]

For the last estimate, note that it comes from \( N \geq 5m \) and \( b_i \geq \frac{\ell}{5} \). Thus, the condition of quadratically asymptotically nonnegative sectional curvature holds in this interval.

For Ricci curvature, we have

\[
\text{Ric}(T, T) = -m \frac{\ddot{u}}{\dot{u}} - (n-1) \frac{\ddot{g}}{\dot{g}} \geq (n-1) \ell^2 - m \left( \frac{1}{N} + \frac{1}{N^2} \right) \ell^2 \geq \frac{m}{2} \ell^2 - m \left( \frac{1}{5} + \frac{1}{25} \right) \ell^2 > 0,
\]

For in the interval provided that asymptotically nonnegative sectional curvature also holds in this interval.

For Ricci curvature,

\[ \text{Ric}(\Theta, \Theta) = \left( m - 1 \right) \left( \frac{1}{u^2} - \frac{\dot{u}^2}{u^2} \right) - \frac{\ddot{u} \dot{u}}{u} - (n - 1) \frac{\ddot{u}}{u} \frac{\dot{u}}{g} \]

\[ \geq \left( m - 1 \right) \left( 1 - \frac{1}{N^2} \right) \frac{t^2_i}{g} - \frac{1}{N} \frac{1}{N^2} \frac{t^2_i}{g} - \frac{n - 1}{5n} \frac{t^2_i}{g} \]

\[ \geq \left( 1 - \frac{1}{N^2} - \frac{1}{N} - \frac{1}{N^2} - \frac{1}{5} \right) \frac{t^2_i}{g} > 0, \]

\[ \text{Ric}(\Sigma, \Sigma) = \left( n - 2 \right) \left( \frac{1}{g^2} - \frac{\dot{g}^2}{g^2} \right) - \frac{\ddot{g} \dot{g}}{g} - m \frac{\ddot{g}}{g} \frac{\dot{g}}{g} \]

\[ \geq \left( n - 2 \right) \frac{l^2_i}{g} + \frac{l^2_i}{2} - \frac{m l^2_i}{5n} \]

\[ \geq \left( n - 1 \right) \frac{1}{5} \frac{l^2_i}{g} > 0. \]

3.2.2 In the interval \([d_i, b_i]\)

For \(d_i \leq t \leq b_i\),

\[ K(T, \Theta, \Theta, T) = -\frac{\ddot{u}}{u} \geq - \frac{1}{N} + \frac{1}{N^2} \frac{l^2_i}{g} = O \left( \frac{1}{l^2_i} \right), \]

\[ K(\Theta, \Theta, \Theta, \Theta) = \frac{1}{u^2} - \frac{\dot{u}^2}{u^2} \geq \frac{1 - \ddot{u} \dot{u}}{u^2} = \left( 1 - \frac{1}{N^2} \right) \frac{1}{u^2} \geq \left( 1 - \frac{1}{2} \right) \frac{1}{u^2} > 0, \]

\[ K(T, \Sigma, \Sigma, T) = -\frac{\ddot{g}}{g} \geq -\frac{\gamma (1 - \gamma)A}{t^2 - \gamma (A t^2 + B)} > 0, \]

\[ K(\Sigma, \Sigma, \Sigma, \Sigma) = \frac{1}{g^2} - \frac{\dot{g}^2}{g^2} + \frac{1}{g^2} - \frac{\gamma A}{t^2 - \gamma (A t^2 + B)} \geq \frac{1 - \frac{\gamma A}{t^2 - \gamma (A t^2 + B)}}{g^2} = \frac{\sin^2(l_i d_i)}{g^2} > 0, \]

\[ K(\Theta, \Sigma, \Sigma, \Theta) = -\frac{\ddot{u} \dot{u}}{u} \geq - \frac{l_i}{N b_i} \frac{\gamma A}{t^2 - \gamma (A t^2 + B)} = \frac{l_i}{N b_i} \frac{\gamma A}{t^2 - \gamma (A t^2 + B)} \]

\[ \geq - \frac{l_i}{N b_i} \frac{\gamma A}{t^2 - \gamma (A t^2 + B)} + \frac{l_i}{N b_i} \frac{\gamma A}{t^2 - \gamma (A t^2 + B)} \geq - \frac{l_i}{N b_i} \geq - \frac{l^2_i}{5n}. \]

For the last estimate, note that it comes from \(N \geq 5n\) and \(b_i \geq \frac{1}{2} \frac{1}{l_i}\). Thus, the condition of quadratically asymptotically nonnegative sectional curvature also holds in this interval.

For Ricci curvature,

\[ \text{Ric}(T, T) = -m \frac{\ddot{u}}{u} - (n - 1) \frac{\ddot{u}}{g} \]

\[ \geq -m \left( \frac{1}{N} + \frac{1}{N^2} \right) \frac{l^2_i}{g} + \frac{m}{2} \frac{\gamma A}{t^2 - \gamma (A t^2 + B)} \]

\[ \geq -m \left( \frac{1}{N} + \frac{1}{N^2} \right) \frac{l^2_i}{g} + \frac{m}{2} \frac{\gamma A}{t^2 - \gamma (A t^2 + B)} \]

\[ \geq -m \left( \frac{1}{N} + \frac{1}{N^2} \right) \frac{l^2_i}{g} + \frac{m}{2} \frac{\gamma A}{t^2 - \gamma (A t^2 + B)} \]

\[ \geq -m \left( \frac{1}{N} + \frac{1}{N^2} \right) \frac{l^2_i}{g} + m \frac{(1 - \gamma)A}{8} \frac{l^2_i}{g} \]

\[ > 0, \]

provided that \(N \geq N(\gamma)\). It holds that

\[ \text{Ric}(\Theta, \Theta) = \left( m - 1 \right) \left( \frac{1}{u^2} - \frac{\dot{u}^2}{u^2} \right) - \frac{\ddot{u} \dot{u}}{u} - (n - 1) \frac{\ddot{u}}{u} \frac{\dot{u}}{g} \]
\[
\begin{align*}
&\geq (m-1) \left( 1 - \frac{1}{N^2} \right) t_i^2 - \left( \frac{1}{N} + \frac{1}{N^2} \right) t_i^2 - \frac{n-1}{5n} t_i^2 \\
&\geq \left( 1 - \frac{1}{N^2} - \frac{1}{N} - \frac{1}{5} \right) t_i^2 \\
&> 0
\end{align*}
\]

and

\[
\begin{align*}
\text{Ric}(\Sigma_t, \Sigma_t) &= (n-2) \left( \frac{1}{y^2} - \frac{\tilde{g}^2}{y^2} \right) - \frac{\tilde{g}_{tt}}{y} - m \frac{\bar{u}_t \tilde{g}_t}{u} \\
&\geq (n-2) \frac{\sin^2(l_i d_i)}{\tilde{g}^2} + \frac{\gamma(1-\gamma)A t^2}{t^2(t^2 + B)} - m \frac{l_i}{N b_i} \frac{\gamma A t^2}{A t^2 + B} \\
&> \frac{\gamma A t^2}{A t^2 + B} \left( \frac{1}{t^2} - m \frac{l_i}{N b_i} \right) \\
&\geq \frac{\gamma A t^2}{A t^2 + B} \left( \frac{1}{b_i^2} - \frac{2m}{N b_i^2} \right) \\
&> 0.
\end{align*}
\]

3.2.3 *In the interval* \([b_i, R_i]\)

For \(b_i \leq t \leq R_i\), by the previous construction, we have

\[
0 < \bar{u}_t = \frac{1}{(1 - C_3) t + \frac{N}{t_i} - (1 - C_3)b_i} \\
\leq \frac{1}{(1 - C_3) t + \frac{N}{t_i} - \frac{2(1 - C_3)}{t_i}} \\
\leq \frac{1}{(1 - C_3) t}.
\]

Thus,

\[
\begin{align*}
K(T, \Theta_k, \Theta_k, T) &= -\frac{\tilde{g}_{tt}}{u} - C_3 \frac{\tilde{g}_t^2}{u^2} \geq - \frac{C_3}{(1 - C_3)^2 t_i^2} \geq - \frac{C_3}{(1 - C_3)^2 t_i^2} = O \left( \frac{1}{t_i^2} \right) , \\
K(\Theta_k, \Theta_p, \Theta_p, \Theta_k) &= \frac{1}{u^2} - \frac{\tilde{g}_t^2}{u^2} \geq \frac{1 - C_3^2}{u^2} > 0, \\
K(T, \Sigma_t, \Sigma_t, T) &= -\frac{\tilde{g}_{tt}}{y} = \frac{\gamma(1-\gamma)A}{t^2 - \gamma(A t^2 + B)} \geq \frac{\gamma(1-\gamma)}{t^2} \geq \frac{\gamma(1-\gamma)}{t^2} \geq \frac{2}{(1 - C_3)^2 t_i^2} > 0,
\end{align*}
\]

provided that \(C_3 \leq C_3(\gamma)\).

\[
K(\Sigma_t, \Sigma_q, \Sigma_q, \Sigma_t) = \frac{1}{g^2} - \frac{\tilde{g}_{tt}}{g^2} = \frac{1 - \frac{2A}{g^2}}{g^2} \geq \frac{1 - \left( \frac{\gamma A}{\tilde{g}} \right)^2}{g^2} > 0,
\]

provided that \(t_i \geq t_i(e, r, \gamma, C_3, N)\). It holds that

\[
K(\Theta_k, \Sigma_t, \Sigma_t, \Theta_k) = -\frac{\bar{u}_t \tilde{g}_t}{u} \geq \frac{1 - C_3 C_3^2 t_i^2}{n - 1} \geq O \left( \frac{1}{t_i^2} \right).
\]

Note that the above estimate will be verified after the calculation of \(\text{Ric}(\Theta_k, \Theta_k)\) in the following.

Thus, the condition of quadratically asymptotically nonnegative sectional curvature again holds in this interval.

For Ricci curvature,

\[
\text{Ric}(T, T) = -m \frac{\tilde{g}_{tt}}{u} - (n - 1) \frac{\tilde{g}_{tt}}{g}
\]
\[
\sum_{b} \frac{C_{3}}{(1 - C_{3})^{2}l^{2}} + 2(n - 1) \frac{C_{3}}{(1 - C_{3})^{2}l^{2}} > 0
\]

and

\[
\text{Ric}(\Theta_{k}, \Theta_{k}) = (m - 1) \left( \frac{1}{u^{2}} - \frac{u_{t}^{2}}{u^{2}} \right) - \frac{\bar{u}_{t}}{u} - (n - 1) \frac{\bar{u}_{i} \theta_{t}}{\bar{u} \theta}
\]

\[
= (m - 1) \left( \frac{1}{u^{2}} - \frac{u_{t}^{2}}{u^{2}} \right) - C_{3} \frac{u_{t}^{2}}{u} - (n - 1) \frac{\bar{u}_{i} \theta_{t}}{\bar{u} \theta}
\]

\[
\geq 1 - (1 + C_{3})C_{2} \frac{2C_{3}}{C_{2}} \left( \frac{t_{i}}{C_{1}C_{2}} \right)^{2} \left( \frac{t_{i}}{C_{1}C_{2}} \right) \frac{2C_{3}}{C_{2}} \left( \frac{t_{i}}{C_{1}C_{2}} \right) \frac{2C_{3}}{C_{2}} \left( \frac{t_{i}}{C_{1}C_{2}} \right)
\]

\[
- 2(n - 1) \gamma \left( (1 - C_{3}) \frac{t}{l} + \frac{N}{l_{i}} - (1 - C_{3}) b_{i} \right) \frac{1 + C_{3}}{1 + C_{3}}
\]

Let

\[
H(t) \triangleq \frac{1 - (1 + C_{3})C_{2} \frac{2C_{3}}{C_{2}}}{C_{2}} \left( \frac{t_{i}}{C_{1}C_{2}} \right)^{2} \frac{2C_{3}}{C_{2}} \left( \frac{t_{i}}{C_{1}C_{2}} \right)^{2} \left( \frac{t_{i}}{C_{1}C_{2}} \right) \frac{2C_{3}}{C_{2}} \left( \frac{t_{i}}{C_{1}C_{2}} \right)
\]

\[
- 2(n - 1) \gamma \left( (1 - C_{3}) \frac{t}{l} + \frac{N}{l_{i}} - (1 - C_{3}) b_{i} \right) \frac{1 + C_{3}}{1 + C_{3}}
\]

Note that

\[
H(b_{i}) = \frac{1 - (1 + C_{3})C_{2} \frac{2C_{3}}{C_{2}}}{C_{2}} \left( \frac{t_{i}}{C_{1}C_{2}} \right)^{2} \frac{2C_{3}}{C_{2}} \left( \frac{t_{i}}{C_{1}C_{2}} \right)^{2} \left( \frac{t_{i}}{C_{1}C_{2}} \right) \frac{2C_{3}}{C_{2}} \left( \frac{t_{i}}{C_{1}C_{2}} \right)
\]

\[
- 2(n - 1) \gamma \left( (1 - C_{3}) \frac{t}{l} + \frac{N}{l_{i}} - (1 - C_{3}) b_{i} \right) \frac{1 + C_{3}}{1 + C_{3}}
\]

provided that \( N > N(c, r, \gamma, C_{3}) \).

On the other hand,

\[
H_{t}(t) = \frac{1 - (1 + C_{3})C_{2} \frac{2C_{3}}{C_{2}}}{C_{2}} \left( \frac{t_{i}}{C_{1}C_{2}} \right)^{2} \frac{2C_{3}}{C_{2}} \left( \frac{t_{i}}{C_{1}C_{2}} \right)^{2} \left( \frac{t_{i}}{C_{1}C_{2}} \right) \frac{2C_{3}}{C_{2}} \left( \frac{t_{i}}{C_{1}C_{2}} \right)
\]

\[
- 2(n - 1) \gamma \left( (1 - C_{3}) \frac{t}{l} + \frac{N}{l_{i}} - (1 - C_{3}) b_{i} \right) \frac{1 + C_{3}}{1 + C_{3}}
\]

\[
\geq 0,
\]
provided that $C_3 \leq \frac{1}{2}(\frac{1}{t_i^2} - 1)$ and $\gamma \leq \frac{1}{8(n-1)}(\frac{1}{t_i^2} - 1)$.

Thus, for $b_i \leq t \leq R_i$,

$$H(t) \geq H(b_i) > 0.$$ 

So Ric($\Theta_\kappa$, $\Theta_\kappa$) > 0. In addition, we also get

$$- \frac{\ddot{u}_i \dddot{u}_i}{\ddot{u}} \geq - \frac{1 - (1 + C_3)C_2^2}{n - 1} \ddot{u}^2 \geq - \frac{1 - (1 + C_3)C_2^2}{n - 1} t_i^2,$$

which gives the required estimate of $K(\Theta_\kappa, \Sigma_i, \Sigma_i, \Theta_\kappa)$ before. Thus,

$$\text{Ric}(\Sigma_i, \Sigma_i) = (n - 2) \left( \frac{1}{t_i^2} - \frac{\dddot{u}_i}{\ddot{u}} - m \frac{\dddot{u}_i}{\ddot{u}} \right) \geq (n - 2) \left( \frac{1}{t_i^2} - \frac{\dddot{u}_i}{\ddot{u}} - m \frac{\dddot{u}_i}{\ddot{u}} \right) \geq \frac{1}{2\gamma} - 2(1 - (1 + C_3)C_2^2) t_i^2 > 0,$$

provided that $t_i$ is sufficiently large.

### 4 Gluing along the boundary and smoothing near the boundary

In this section, we first check that $Q \setminus (B_{\frac{1}{2}r_i}(a_i) \times \gamma, S^{n-1})$ and $P_i$ have isometric boundaries so that we can glue them together along the boundaries to get a $C^0$ metric. Next, by means of the normal curvature properties of the boundaries, we are able to construct a $C^1$ metric to replace the above $C^0$ metric near the glued boundaries and to verify that the quadratically asymptotic nonnegativeness of curvature and positiveness of Ricci curvature are still preserved.

#### 4.1 Gluing along the boundary

First, we consider the boundary of $Q \setminus (B_{\frac{1}{2}r_i}(a_i) \times \gamma, S^{n-1})$, i.e., $\partial B_{\frac{1}{2}r_i}(a_i) \times \gamma, S^{n-1}$. By the previous construction, $[t_i + \frac{r}{6}, t_i + \frac{11r}{6}] \times \mathcal{U}(t) \times S^m$ can be considered as part of $S^{m+1}(\frac{1}{\sqrt{K}})$, so the removed geodesic ball $B_{\frac{1}{2}r_i}(a_i)$ is a geodesic ball of radius $\frac{1}{2}r_i$ in $S^{m+1}(\frac{1}{\sqrt{K}})$. On the other hand, for $t_i + \frac{r}{6} < t < t_i + \frac{11r}{6}$,

$$g(t) = g_i = \left( t_i + \frac{r}{6} \right)^\gamma = \left( t_i + \frac{r}{6} \right)^\gamma t_i^\gamma.$$

So after a suitable rotation, the restricted metric on the boundary $\partial B_{\frac{1}{2}r_i}(a_i) \times \gamma, S^{n-1}$ is

$$\left( \frac{\sin \frac{1}{2}\sqrt{K}r}{\sqrt{K}} t_i \right)^2 d\Omega^2_{S^m} + \left( \left( 1 + \frac{r}{6} \right)^\gamma t_i^\gamma \right)^2 d\sigma^2_{S^{n-1}}.$$

Next, we consider the boundary of $P_i$, which is exactly the hypersurface $t = R_i$ with the restricted metric

$$\left( \frac{\sin \frac{1}{2}\sqrt{K}r}{\sqrt{K}} t_i \right)^2 d\Omega^2_{S^m} + \left( \left( 1 + \frac{r}{6} \right)^\gamma t_i^\gamma \right)^2 d\sigma^2_{S^{n-1}}.$$

Thus, both boundaries are isometric.
4.2 Smoothing near the boundary

As seen above, a $C^0$ metric $h_0$ is already constructed on the manifold $M^{m+n}$, i.e.,

$$(M^{m+n}, h_0) = \left( Q \setminus \bigcup_{i=1}^{+\infty} (B_{\frac{1}{4}r_i}(a_i) \times S^{n-1}), ds^2 \right) \cup \bigcup_{i=1}^{+\infty} \left( P_i, ds^2_i \right);$$

$h_0$ is smooth on $M^{m+n}$ except the glued boundary part $\partial B_{\frac{1}{4}r_i}(a_i) \times S^{n-1}$ on which $h_0$ is $C^0$ and has positive Ricci curvature and quadratically asymptotically nonnegative sectional curvature on the smooth part. Furthermore, the restricted metric of $h_0$ on the $2\varepsilon_i$-neighbourhood ($\varepsilon_i \ll 1 \ll r_i$) of the boundary (i.e., $h_0 |[-2\varepsilon_i, 2\varepsilon_i] \times S^n \times S^{n-1}$) can be rewritten as

$$\begin{align*}
&\begin{cases}
  dt^2 + \left( \frac{\sin \sqrt{K_i}(t + \frac{4}{5}r_i)}{\sqrt{K_i}} \right)^2 d\bar{\theta}_S^2 + \left( t_i + \frac{r_i}{6} \right) \gamma d\bar{\sigma}_S^2, & 0 \leq t \leq 2\varepsilon_i, \\
  dt^2 + \bar{u}^2(t + R_i) d\bar{\theta}_S^2 + \bar{g}^2(t + R_i) d\bar{\sigma}_S^2, & -2\varepsilon_i \leq t < 0,
\end{cases}
\end{align*}$$

which, as mentioned above, is $C^0$ at $t = 0$ since

$$\bar{u}(R_i) = \frac{\sin \frac{4}{5} \sqrt{K_i} r_i}{\sqrt{K_i}}, \quad \bar{g}(R_i) = \left( t_i + \frac{r_i}{6} \right) \gamma.$$

In the following, we construct a $C^1$ metric $h_1$ on $[-2\varepsilon_i, 2\varepsilon_i] \times S^n \times S^{n-1}$, which is the same as $h_0$ on $[-2\varepsilon_i, -\varepsilon_i] \times S^n \times S^{n-1}$ and $[\varepsilon_i, 2\varepsilon_i] \times S^n \times S^{n-1}$, and verify that it also has positive Ricci curvature and quadratically asymptotically nonnegative curvature on this part (for the similar construction, see also [4], but our case will be more complicated due to the condition of quadratically asymptotically nonnegative curvature).

As before, we introduce some constants for convenience, i.e.,

$$a = \frac{\sin \sqrt{K_i}(\frac{4}{5}r_i + \varepsilon_i)}{\sqrt{K_i}}, \quad b = \cos \sqrt{K_i}(\frac{4}{5}r_i + \varepsilon_i),$$

$$c = \bar{u}(R_i - \varepsilon_i), \quad d = \bar{u}(R_i - \varepsilon_i),$$

$$e = \bar{g}(R_i - \varepsilon_i), \quad f = \bar{g}(R_i - \varepsilon_i).$$

The $C^1$ metric is then

$$h_1 = \begin{cases}
  dt^2 + \left( \frac{\sin \sqrt{K_i}(t + \frac{4}{5}r_i)}{\sqrt{K_i}} \right)^2 d\bar{\theta}_S^2 + \left( t_i + \frac{r_i}{6} \right) \gamma d\bar{\sigma}_S^2, & \varepsilon_i \leq t \leq 2\varepsilon_i, \\
  dt^2 + U^2(t) d\bar{\theta}_S^2 + G^2(t) d\bar{\sigma}_S^2, & -\varepsilon_i < t < \varepsilon_i, \\
  dt^2 + \bar{u}^2(t + R_i) d\bar{\theta}_S^2 + \bar{g}^2(t + R_i) d\bar{\sigma}_S^2, & -2\varepsilon_i \leq t \leq -\varepsilon_i,
\end{cases}$$

where

$$U^2(t) = \frac{\left( \frac{c^2 - a^2}{4 \varepsilon_i} + 2(ab + cd) \varepsilon_i t^3 + \frac{ab - cd}{2 \varepsilon_i} t^2 - \frac{3(c^2 - a^2) + 2(ab + cd) \varepsilon_i t}{4 \varepsilon_i} \right)}{2} + \frac{(a^2 + c^2) + (cd - ab) \varepsilon_i}{4 \varepsilon_i},$$

$$G^2(t) = \frac{\left( \frac{c^2 - g^2}{4 \varepsilon_i} + 2ef \varepsilon_i t^3 - ef \varepsilon_i t^2 - \frac{3(c^2 - g^2) + 2ef \varepsilon_i t}{4 \varepsilon_i} + \frac{(c^2 + g^2) + ef \varepsilon_i}{2} \right)}{2}.$$

It is clear that $h_1$ is the same as $h_0$ on $[-2\varepsilon_i, -\varepsilon_i] \times S^n \times S^{n-1}$ and $[\varepsilon_i, 2\varepsilon_i] \times S^n \times S^{n-1}$, respectively.

By direct computation, we have

$$U_t U = \frac{3(c^2 - a^2) + 6(ab + cd) \varepsilon_i t^2 + \frac{ab - cd}{2 \varepsilon_i} t - \frac{3(c^2 - a^2) + 2(ab + cd) \varepsilon_i t}{8 \varepsilon_i}}{8 \varepsilon_i}.$$
\[ U_t^2 + U_{tt}U = \frac{3(e^2 - a^2) + 6(ab + cd)e_t}{4\varepsilon_i^4} t + \frac{ab - cd}{2\varepsilon_i} \]

and

\[ G_t G = \frac{3(e^2 - g_t^2)}{8\varepsilon_i^4} t^2 - \frac{ef}{2\varepsilon_i} t - \frac{3(e^2 - g_t^2) + 2ef\varepsilon_i}{8\varepsilon_i}, \]

\[ G_t^2 + G_{tt}G = \frac{3(e^2 - g_t^2) + 6ef\varepsilon_i}{4\varepsilon_i^4} t - \frac{ef}{2\varepsilon_i}. \]

Putting \( t = \pm \varepsilon_i \) into the above equations gives

\[ U(\varepsilon_i) = a = \frac{\sin \sqrt{K_i}(\frac{4}{5}r_i + \varepsilon_i)}{\sqrt{K_i}}, \quad U_t(\varepsilon_i) = b = \cos \sqrt{K_i}(\frac{4}{5}r_i + \varepsilon_i), \]

\[ U(-\varepsilon_i) = c = \bar{a}(R_i - \varepsilon_i), \quad U_t(-\varepsilon_i) = d = \bar{a}_t(R_i - \varepsilon_i), \]

\[ G(\varepsilon_i) = g_t, \quad G_t(\varepsilon_i) = 0, \]

\[ G(-\varepsilon_i) = e = \bar{g}(R_i - \varepsilon_i), \quad G_t(-\varepsilon_i) = f = \bar{g}_t(R_i - \varepsilon_i). \]

Thus, the metric \( h_1 \) on \([-2\varepsilon_i, 2\varepsilon_i] \times S^m \times S^{n-1} \) is smooth for \( t \neq \pm \varepsilon_i \) and \( C^1 \) at \( t = \pm \varepsilon_i \). Using this \( C^1 \) metric \( h_1 |[-2\varepsilon_i, 2\varepsilon_i] \times S^m \times S^{n-1} \), to replace the \( C^0 \) metric \( h_0 |[-2\varepsilon_i, 2\varepsilon_i] \times S^m \times S^{n-1} \), we obtain a new \( C^1 \) metric on the whole manifold \( M^{m+n} \), still denoted by \( h_1 \), which actually is smooth except for \( t = \pm \varepsilon_i \).

Next, we verify that the new metric \( h_1 \) still has positive Ricci curvature and quadratically asymptotically nonnegative curvature. For this purpose, we only need to consider the curvature terms of

\[ dt^2 + U^2(t)d\theta^2_{S^m} + G^2(t)d\sigma^2_{S^{n-1}}, \quad -\varepsilon_i < t < \varepsilon_i. \]

Similar to what we have done, we have

\[ K(T, \Theta_k, \Theta_k, T) = -\frac{U_{tt}}{U}, \]

\[ K(\Theta_k, \Theta_p, \Theta_p, \Theta_k) = \frac{1}{U^2} - \frac{U_t^2}{U^2}, \]

\[ K(T, \Sigma_l, \Sigma_l, T) = -\frac{G_{tt}}{G}, \]

\[ K(\Sigma_l, \Sigma_q, \Sigma_q, \Sigma_l) = \frac{1}{G^2} - \frac{G_t^2}{G^2}, \]

\[ K(\Theta_k, \Theta_l, \Sigma_l, \Theta_k) = -\frac{U_t G_{tt}}{U G}. \]

The other terms of curvature tensors are zero.

Note that when \( \varepsilon_i \ll 1 \ll r_i \),

\[ e^2 - a^2 = -2 \frac{\sin \frac{4}{5}\sqrt{K}r}{\sqrt{K_i}} \left( \cos \frac{3}{5}\sqrt{K}r + \cos \frac{4}{5}\sqrt{K}r \right) \varepsilon_i + o(\varepsilon_i), \]

\[ ab + cd = \frac{\sin \frac{4}{5}\sqrt{K}r}{\sqrt{K_i}} \left( \cos \frac{3}{5}\sqrt{K}r + \cos \frac{4}{5}\sqrt{K}r \right) + o(1), \]

\[ a^2 + e^2 = 2 \left( \frac{\sin \frac{4}{5}\sqrt{K}r}{\sqrt{K_i}} \right)^2 + o(1), \]

\[ ab - cd = -\frac{\sin \frac{4}{5}\sqrt{K}r}{\sqrt{K_i}} \left( \cos \frac{3}{5}\sqrt{K}r - \cos \frac{4}{5}\sqrt{K}r \right) + o(1), \]

\[ e^2 - g_t^2 = -2g_t\bar{g}_t(R_i)\varepsilon_i + o(\varepsilon_i), \]

\[ ef = g_t\bar{g}_t(R_i) + o(1), \]

\[ e^2 + g_t^2 = 2g_t^2 + o(1). \]
gluing criterion, i.e., “the normal curvature of conditions in our case.

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Of the terms with $\frac{1}{\varepsilon_i}$ and $\frac{1}{\varepsilon_i}$, that is why we can construct an explicit $C^1$ smoothing and preserve the corresponding curvature conditions in our case.

Remark 4.1. As mentioned in Section 1, we should remark that in the $C^1$ smoothing from the $C^0$ metric near the glued boundaries, the positiveness of Ricci curvature (equivalently, positive lower bounds of curvature terms $-\frac{U}{\partial M}$ and $-\frac{G}{\partial \tau}$) is controlled by the coefficients “$\cos \frac{2}{5} \sqrt{Kr} - \cos \frac{4}{5} \sqrt{Kr}$” and “$\bar{g}(R_i)$” of the terms with $\frac{1}{\varepsilon_i}$. In fact, the positiveness of these terms is just the condition in Perelman’s gluing criterion, i.e., “the normal curvature of $\partial M_1$ is larger than the negative of the normal curvature of $\partial M_2$”; that is why we can construct an explicit $C^1$ smoothing and preserve the corresponding curvature conditions in our case.

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