SECOND-ORDER CONSTRAINED VARIATIONAL PROBLEMS
ON LIE ALGEBROIDS: APPLICATIONS TO
OPTIMAL CONTROL

LEONARDO COLOMBO
Department of Mathematics, University of Michigan
530 Church Street, 3828 East Hall
Ann Arbor, Michigan, 48109, USA

(Communicated by Jorge Cortés)

ABSTRACT. The aim of this work is to study, from an intrinsic and geometric
point of view, second-order constrained variational problems on Lie algebroids,
that is, optimization problems defined by a cost function which depends on
higher-order derivatives of admissible curves on a Lie algebroid. Extending
the classical Skinner and Rusk formalism for the mechanics in the context of
Lie algebroids, for second-order constrained mechanical systems, we derive the
corresponding dynamical equations. We find a symplectic Lie subalgebroid
where, under some mild regularity conditions, the second-order constrained
variational problem, seen as a presymplectic Hamiltonian system, has a unique
solution. We study the relationship of this formalism with the second-order
constrained Euler-Poincaré and Lagrange-Poincaré equations, among others.
Our study is applied to the optimal control of mechanical systems.

CONTENTS

1. Introduction 2
2. Lie algebroids and admissible elements 5
   2.1. Lie algebroids, Lie subalgebroids and Cartan calculus on Lie algebroids 5
   2.2. $E$-tangent bundle to a Lie algebroid $E$ 9
   2.3. $E$-tangent bundle of the dual bundle of a Lie algebroid 13
   2.4. Symplectic Lie algebroids 15
   2.5. Admissible elements on a Lie algebroid 16
3. Second-order variational problems on Lie algebroids 17
   3.1. Mechanics on Lie algebroids 17
   3.2. Constraint algorithm for presymplectic Lie algebroids 20
   3.3. Vakonomic mechanics on Lie algebroids 23
   3.4. Second-order variational problems on Lie algebroids 26
4. Application to optimal control of mechanical systems 32
   4.1. Optimal control problems of fully-actuated mechanical systems on Lie
        algebroids 32

2010 Mathematics Subject Classification. Primary: 70H25; Secondary: 70H30, 70H50, 37J15,
58K05, 70H03, 37K05.

Key words and phrases. Lie algebroids, optimal control, higher-order mechanics, higher-order
variational problems.
1. Introduction. Lie algebroids have deserved a lot of interest in recent years. Since a Lie algebroid is a concept which unifies tangent bundles and Lie algebras, one can suspect their relation with mechanics. More precisely, a Lie algebroid over a manifold $Q$ is a vector bundle $\tau_E : E \to Q$ over $Q$ with a Lie algebra structure over the space $\Gamma(\tau_E)$ of sections of $E$ and an application $\rho : E \to TQ$ called anchor map satisfying some compatibility conditions (see [53]). Examples of Lie algebroids are the tangent bundle over a manifold $Q$ where the Lie bracket of vector fields and the anchor map is the identity function; a real finite dimensional Lie algebra as vector bundles over a point, where the anchor map is the null application; action Lie algebroids of the type $\text{pr}_1 : M \times \mathfrak{g} \to M$ where $\mathfrak{g}$ is a Lie algebra acting infinitesimally over the manifold $M$ with a Lie bracket over the space of sections induced by the Lie algebra structure and whose anchor map is the action of $\mathfrak{g}$ over $M$; and, the Lie-Atiyah algebroid $\tau_{TQ/G} : TQ/G \to \tilde{M} = Q/G$ associated with the $G$-principal bundle $p : Q \to \tilde{M}$ [51, 53, 60, 76].

In [76] A. Weinstein developed a generalized theory of Lagrangian mechanics on Lie algebroids and he obtained the equations of motion using the linear Poisson structure on the dual of the Lie algebroid and the Legendre transformation associated with a regular Lagrangian $L : E \to \mathbb{R}$. In [76] also he asked about whether it is possible to develop a formalism similar on Lie algebroids to Klein’s formalism [47] in Lagrangian mechanics. This was obtained by E. Martínez in [60] (see also [59]). The main notion is that of prolongation of a Lie algebroid over a mapping introduced by Higgins and Mackenzie in [53]. A more general situation, the prolongation of an anchored bundle was also considered by Popescu in [69, 70].

The importance of Lie algebroids in mathematics is beyond doubt. In the last years there has been a lot of applications of Lie algebroids in theoretical physics and other related sciences, more precisely in Classical Mechanics, Classical Field Theory and their applications. The main point is that Lie algebroids provide a general framework for systems with different features as systems with symmetries, systems over semidirect products, Hamiltonian and Lagrangian systems, systems with constraints (nonholonomic and vakonomic) and Classical Fields theory [1, 14, 15, 16, 26, 27, 33, 49, 54, 65].

In [51] M. de León, J.C. Marrero and E. Martínez have developed a Hamiltonian description for the mechanics on Lie algebroids and they have shown that the dynamics is obtained solving an equation for the Hamiltonian section (Hamiltonian vector field) in the same way than in Classical Mechanics (see also [59] and [76]). Moreover, they shown that the Legendre transformation $\text{leg}_L : E \to E^*$ associated to a Lagrangian $L : E \to \mathbb{R}$ induces a Lie algebroid morphism and when the Lagrangian is regular both formalisms are equivalent.

Marrero and collaborators also have studied non-holonomic mechanics on Lie algebroids [20]. In other direction, in [42] D. Iglesias, J.C. Marrero, D. Martín de Diego and D. Sosa have studied singular Lagrangian systems and vakonomic
mechanics from the point of view of Lie algebroids obtained through the application of a constrained variational principle. They have developed a constraint algorithm for presymplectic Lie algebroids generalizing the well know constraint algorithm of Gotay, Nester and Hinds [38, 39] and they have also established the Skinner and Rusk formalism on a Lie algebroids. Some of the results given in this work are an extension of this framework in the context of constrained second-order systems.

In this work we choose a framework to study mechanics based in the Skinner-Rusk formalism, which combines simultaneously some features of the Lagrangian and Hamiltonian classical formalisms to study the dynamics associated with optimal control problems as in [4]. The idea of this formulation was to obtain a common description for both regular and singular dynamics, obtaining simultaneously the Hamiltonian and Lagrangian formulations of the dynamics. Over the years, however, Skinner and Rusk’s framework was extended in many directions: It was originally developed for first-order autonomous mechanical systems [74], and later generalized to non-autonomous dynamical systems [2, 25, 72], control systems [4] and, more recently to classical field theories [12, 29, 75].

Briefly, in this formulation, one starts with a differentiable manifold $Q$ as the configuration space, and the Whitney sum $TQ \oplus T^*Q$ as the evolution space (with canonical projections $\pi_1 : TQ \oplus T^*Q \to TQ$ and $\pi_2 : TQ \oplus T^*Q \to T^*Q$). Define on $TQ \oplus T^*Q$ the presymplectic 2-form $\Omega = \pi_2^*\omega_Q$, where $\omega_Q$ is the canonical symplectic form on $T^*Q$, and note that the rank of the presymplectic form is everywhere equal to $2n$. If the dynamical system under consideration admits a Lagrangian description, with Lagrangian $L \in C^\infty(TQ)$, one can obtain a (presymplectic)-Hamiltonian representation on $TQ \oplus T^*Q$ given by the presymplectic 2-form $\Omega$ and the Hamiltonian function $H = \langle \pi_1, \pi_2 \rangle - \pi_1^*L$, where $\langle \cdot, \cdot \rangle$ denotes the natural pairing between vectors and covectors on $Q$. In this Hamiltonian system the dynamics is given by vector fields $X$, which are solutions to the Hamiltonian equation $i_X \Omega = dH$. If $L$ is regular, then there exists a unique vector field $X$ solution to the previous equation, which is tangent to the graph of the Legendre map. In the singular case, it is necessary to develop a constraint algorithm in order to find a submanifold (if it exists) where there exists a well-defined vector field solution.

Recently, higher-order variational problems have been studied for their importance in applications to aeronautics, robotics, computer-aided design, air traffic control, trajectory planning, and in general, problems of interpolation and approximation of curves on Riemannian manifolds [6, 11, 41, 48, 52, 63, 64, 66]. There are variational principles which involve higher-order derivatives by Gay Balmaz et.al., [30, 31, 32], (see also [50]) since from it one can obtain the equations of motion for Lagrangians where the configuration space is a higher-order tangent bundle. More recently, there have been an interest in the study of the geometrical structures associated with higher-order variational problems with the aim of a deepest understanding of those objects [20, 23, 71, 61, 44, 45, 46] as well as the relationship between higher-order mechanics and graded bundles [8, 9, 10].

Optimal control on Lie algebroids has been a subject of study among the last years by extending the Pontryagin maximum principle to control systems on Lie algebroids as it has been shown in [1], [55], [57], [65]. The first two references given before are based on the geometry of prolongations of Lie algebroids (in sense of Higgins and Mackenzie [53]) over the vector bundle projection of a dual bundle; the same approach as we use in this work. The first reference focuses on kinematic
and dynamical control problems of nonholonomic systems by understanding quasi-velocities as coordinates on the non-holonomic distribution induced by a basis of section of the Lie algebroid. The second approach of Pontryagin maximum principle is based on the geometry of almost-Lie algebroids \[37\] and \[68\] studies some distributional systems with positive homogeneous cost, using the Pontryagin Maximum Principle at the level of a Lie algebroid.

Our approach is a mixture between geometric and variational. It starts by considering the controlled Euler-Lagrange equations that comes from a Lagrangian system subject to external forces and a given Lagrangian defined on a Lie algebroid. We consider an optimization problem associated with the controlled dynamics and subject to second-order constraints (i.e. on the acceleration) depending on the degrees of underactuation in the controlled system (a different approach compared with the ones commented before). We reformulate the problem as a truly Hamiltonian problem on a suitable symplectic Lie algebroid. After the integration of Hamilton equations, we will be able to reconstruct the control forces and to solve the original problem. In practice the integration of Hamilton equations as well as the optimization are performed numerically. More precisely, it is possible to integrate numerically the equations of motion, and then, by using a shooting method, optimize the trajectories and find the related control forces.

In this work we develop a geometric formalism in mechanics, know as Skinner-Rusk formalism, for constrained second-order variational problems which are determined by a cost function arising from an optimal control problem and depending on second order derivatives of admissible curves on a Lie algebroid. This formalism is studied by applying the geometric construction described above in combination with an extension of the constraint algorithm developed by Gotay, Nester and Hinds \[38, 39\] in the context of Lie algebroids \[42\] following the point of view of mechanics on Lie algebroids on prolongations as \[61\] instead of the approach given in \[8\] based on the concept of Tulczyjew triple and geometry of graded bundles.

We derive constrained second-order Euler-Lagrange equations, Euler-Poincaré and Lagrange-Poincaré equations into an unified framework by consider the subjacent geometric structures in the constrained second-order variational problem (this approach has never been done before in the literature). We show how the construction based on the geometry associated with the problem can be applied to the problem of finding necessary conditions for optimality in optimal control problems of mechanical system with symmetries, where trajectories are parameterized by the admissible controls and existence of normal extremals (or necessary conditions for extremals) in the optimal control problem are expressed using a pseudo-Hamiltonian formulation based on the Pontryagin maximum principle.

The paper is organized as follows\(^1\). In Section 2 we introduce some known notions concerning Lie algebroids that are necessary for further developments in this work. In section 3 we will use the notion of Lie algebroid and prolongation of a Lie algebroid described in 2 to derive the Euler-Lagrange equations and Hamilton equations on Lie algebroids. Next, after introducing the constraint algorithm for presymplectic Lie algebroids and studying vakonomic mechanics on Lie algebroids, we study the geometric formalism for second-order constrained variational problems using and adaptation of the classical Skinner-Rusk formalism for the second-order

\(^1\)If the reader is familiarized with the notions of Lie algebroids, prolongations, their geometric structures and how apply those in mechanics, it is recommended to start from section 3.3 and move backwards along the paper when it is necessary a more detailed clarifications.
constrained systems on Lie algebroids. In section 4 we study optimal control problems of mechanical systems defined on Lie algebroids. Optimality conditions for the optimal control of the Elroy’s Beanie are derived. Several examples show how to apply the techniques along all the work.

2. Lie algebroids and admissible elements. In this section, we introduce some known notions and develop new concepts concerning Lie algebroids that are necessary for further developments in this work. We illustrate the theory with several examples. We refer the reader to [13, 53] for more details about Lie algebroids and their role in differential geometry.

2.1. Lie algebroids, Lie subalgebroids and Cartan calculus on Lie algebroids.

Definition 2.1. Let $E$ be a vector bundle of rank $n$ over a manifold $M$ of dimension $m$. A Lie algebroid structure on the vector bundle $\tau_E : E \to M$ is a $\mathbb{R}$-linear bracket $[\cdot, \cdot] : \Gamma(\tau_E) \times \Gamma(\tau_E) \to \Gamma(\tau_E)$ on the space $\Gamma(\tau_E)$, the $C^\infty(M)$-module of sections of $E$, and a vector bundle morphism $\rho : E \to TM$, the anchor map, such that:

1. The bracket $[\cdot, \cdot]$ satisfies the Jacobi identity, that is,

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0 \quad \forall X, Y, Z \in \Gamma(\tau_E).$$

2. If we also denote by $\rho : \Gamma(\tau_E) \to \mathfrak{X}(M)$ the homomorphism of $C^\infty(M)$-modules induced by the anchor map then

$$[X, fY] = f[X, Y] + \rho(X)(f)Y \quad \text{for } X, Y \in \Gamma(\tau_E) \text{ and } f \in C^\infty(M). \quad (1)$$

We will say that the triple $(E, [\cdot, \cdot], \rho)$ is a Lie algebroid over $M$. In this context, sections of $\tau_E$, play the role of vector fields on $M$, and the sections of the dual bundle $\tau_{E^\ast} : E^\ast \to M$ of 1-forms on $M$.

We may consider two types of distinguished functions: given $f \in C^\infty(M)$ one may define a function $\hat{f}$ on $E$ by $\hat{f} = f \circ \tau_E$, the basic functions. A section $\theta$ of the dual bundle $\tau_{E^\ast} : E^\ast \to M$ may be regarded as a linear function $\hat{\theta}$ on $E$ as $\hat{\theta}(e) = \langle \theta(\tau_E(e)), e \rangle$ for all $e \in E$. In this sense, $\Gamma(\tau_E)$ is locally generated by the differential of basic and linear functions.

If $X, Y, Z \in \Gamma(\tau_E)$ and $f \in C^\infty(M)$, then using the Jacobi identity we obtain that

$$[[X, Y], fZ] = f[[X, Y], Z]] + [\rho(X), \rho(Y)](f)Z. \quad (2)$$

Also, from (1) it follows that

$$[[X, Y], fZ] = f[[X, Y], Z] + \rho[X, Y](f)Z. \quad (3)$$

Then, using (2) and (3) and the fact that $[\cdot, \cdot]$ is a Lie bracket we conclude that

$$\rho[X, Y] = [\rho(X), \rho(Y)],$$

that is, $\rho : \Gamma(\tau_E) \to \mathfrak{X}(M)$ is a homomorphism between the Lie algebras $(\Gamma(\tau_E), [\cdot, \cdot])$ and $(\mathfrak{X}(M), [\cdot, \cdot])$.

The algebra $\bigoplus_k \Gamma(\Lambda^k E^\ast)$ of multisections of $\tau_{E^\ast}$ plays the role of the algebra of the differential forms and it is possible to define a differential operator as follows:
Definition 2.2. If $(E, [\cdot, \cdot], \rho)$ is a Lie algebroid over $M$, one can define the differential of $E$, $d^E : \Gamma(\wedge^k \tau_{E^*}) \to \Gamma(\wedge^{k+1} \tau_{E^*})$, as follows:

$$d^E \mu(X_0, \ldots, X_k) = \sum_{i=0}^k (-1)^i \rho(X_i)(\mu(X_0, \ldots, \tilde{X}_i, \ldots, X_k))$$

$$+ \sum_{i<j} (-1)^{i+j} \mu([X, Y], X_0, \ldots, \tilde{X}_i, \ldots, \tilde{X}_j, \ldots, X_k),$$

for $\mu \in \Gamma(\wedge^k \tau_{E^*})$ and $X_0, \ldots, X_k \in \Gamma(\tau_E)$.

From the properties of Lie algebroids it follows that $d^E$ is a cohomology operator, that is, $(d^E)^2 = 0$ and $d^E(\alpha \wedge \beta) = d^E \alpha \wedge \beta + (-1)^k \alpha \wedge d^E \beta$, for $\alpha \in \Gamma(\wedge^k E^*)$ and $\beta \in \Gamma(\wedge^*(E^*))$ (see [53] for more details).

Conversely it is possible to recover the Lie algebroid structure of $E$ from the existence of an exterior differential on $\Gamma(\wedge^* \tau_{E^*})$. If $f : M \to \mathbb{R}$ is a real smooth function, one can define the anchor map and the Lie bracket as follows:

1. $d^E f(X) = \rho(X)f$, for $X \in \Gamma(\tau_E)$,
2. $i_{\{X, Y\}} \theta = \rho(X)\theta(Y) - \rho(Y)\theta(X) - d^E\theta(X, Y)$ for all $X, Y \in \Gamma(\tau_E)$ and $\theta \in \Gamma(\tau_{E^*})$.

Moreover, from the last equality, the section $\theta \in \Gamma(\tau_{E^*})$ is a 1-cocycle if and only if $d^E \theta = 0$, or, equivalently,

$$\theta[X, Y] = \rho(X)(\theta(Y)) - \rho(Y)(\theta(X)),$$

for all $X, Y \in \Gamma(\tau_E)$.

We may also define the Lie derivative with respect to a section $X \in \Gamma(\tau_E)$ as the operator $\mathcal{L}_X^E : \Gamma(\wedge^k \tau_{E^*}) \to \Gamma(\wedge^k \tau_{E^*})$ given by

$$\mathcal{L}_X^E \theta = i_X \circ d^E \theta + d^E \circ i_X \theta,$$

for $\theta \in \Gamma(\wedge^k \tau_{E^*})$. One also has the usual identities

1. $d^E \circ \mathcal{L}_X^E = \mathcal{L}_X^E \circ d^E$,
2. $\mathcal{L}_X^E i_Y - i_Y \mathcal{L}_X^E = i_{[X, Y]}$,
3. $\mathcal{L}_X^E \mathcal{L}_Y^E - \mathcal{L}_Y^E \mathcal{L}_X^E = \mathcal{L}_X^E i_Y - i_Y \mathcal{L}_X^E$.

We take local coordinates $(x^i)$ on $M$ with $i = 1, \ldots, m$ and a local basis $\{e_A\}$ of sections of the vector bundle $\tau_E : E \to M$ with $A = 1, \ldots, n$, then we have the corresponding local coordinates on an open subset $\tau_{E^*}^{-1}(U)$ of $E$, $(x^i, y^A)$ ($U$ is an open subset of $Q$), where $y^A(e)$ is the $A$-th coordinate of $e \in E$ in the given basis i.e., every $e \in E$ is expressed as $e = y^A e_A(\tau_E(e)) + \ldots + y^A e_n(\tau_E(e))$.

Such coordinates determine the local functions $\rho^A, e^C_{AB}$ on $M$ which contain the local information of the Lie algebroid structure, and accordingly they are called structure functions of the Lie algebroid. These are given by

$$\rho(e_A) = \rho^A_i \frac{\partial}{\partial x^i} \quad \text{and} \quad [e_A, e_B] = e^C_{AB} e_C.$$  \hspace{1cm} (4)

These functions should satisfy the relations

$$\rho_B^A \frac{\partial \rho^B_A}{\partial x^j} - \rho_A^j \frac{\partial \rho^B_A}{\partial x^i} = \rho^C_{AB} e_C$$  \hspace{1cm} (5)

and

$$\sum_{\text{cyclic}(A, B, C)} \left[ \rho^A_i \frac{\partial e^B_C}{\partial x^i} + e^D_{AF} e^F_{BC} \right] = 0, \hspace{1cm} (6)$$
which are usually called the structure equations.

If \( f \in C^\infty(M) \),
\[
d^E f = \frac{\partial f}{\partial x^i} \rho^i_A e^A,
\]
(7)
where \( \{e^A\} \) is the dual basis of \( \{e_A\} \). If \( \theta \in \Gamma(\tau_M) \) and \( \theta = \theta^C e^C \) it follows that
\[
d^E \theta = \left( \frac{\partial \theta^C}{\partial x^i} \rho^i_B - \frac{1}{2} \theta_A \sigma^A_{BC} \right) e^B \wedge e^C.
\]
(8)
In particular,
\[
d^E x^i = \rho^i_A e^A, \quad d^E e^A = -\frac{1}{2} \sigma^A_{BC} e^B \wedge e^C.
\]

2.1.1. Examples of Lie algebroids.

**Example 1.** Given a finite dimensional real Lie algebra \( \mathfrak{g} \) and \( M = \{m\} \) be a unique point, we consider the vector bundle \( \tau_\mathfrak{g} : \mathfrak{g} \to M \). The sections of this bundle can be identified with the elements of \( \mathfrak{g} \) and therefore we can consider as the Lie bracket the structure of the Lie algebra induced by \( \mathfrak{g} \), and denoted by \([\cdot,\cdot]_{\mathfrak{g}}\). Since \( TM = \{0\} \) one may consider the anchor map \( \rho = 0 \). The triple \((\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},0)\) is a Lie algebroid over a point.

**Example 2.** Consider a tangent bundle of a manifold \( M \). The sections of the bundle \( \tau_{TM} : TM \to M \) are the set of vector fields on \( M \). The anchor map \( \rho : TM \to TM \) is the identity function and the Lie bracket defined on \( \Gamma(\tau_{TM}) \) is induced by the Lie bracket of vector fields on \( M \).

**Example 3.** Let \( \phi : M \times G \to M \) be an action of \( G \) on the manifold \( M \) where \( G \) is a Lie group. The induced anti-homomorphism between the Lie algebras \( \mathfrak{g} \) and \( \mathfrak{X}(M) \) by the action is determined by \( \Phi : \mathfrak{g} \to \mathfrak{X}(M) \), \( \xi \mapsto \xi_M \), where \( \xi_M \) is the infinitesimal generator of the action for \( \xi \in \mathfrak{g} \).

The vector bundle \( \tau_{M \times \mathfrak{g}} : M \times \mathfrak{g} \to M \) is a Lie algebroid over \( M \). The anchor map \( \rho : M \times \mathfrak{g} \to TM \), is defined by \( \rho(m,\xi) = -\xi_M(m) \) and the Lie bracket of sections is given by the Lie algebra structure on \( \Gamma(\tau_{M \times \mathfrak{g}}) \) as
\[
[[\xi,\eta]]_{M \times \mathfrak{g}}(m) = (m,[[\xi,\eta]]_{M \times \mathfrak{g}}) = (\xi,\eta)
\]
for \( m \in M \), where \( \xi(m) = (m,\xi), \ \eta(m) = (m,\eta) \) for \( \xi,\eta \in \mathfrak{g} \). The triple \((M \times \mathfrak{g},\rho,[[\cdot,\cdot]]_{M \times \mathfrak{g}})\) is called Action Lie algebroid.

**Example 4.** Let \( G \) be a Lie group and we assume that \( G \) acts freely and properly on \( M \). We denote by \( \pi : M \to \hat{M} = M/G \) the associated principal bundle. The tangent lift of the action gives a free and proper action of \( G \) on \( TM \) and \( T\hat{M} \) is \( TM/G \) is a quotient manifold. The quotient vector bundle \( \tau_{\hat{T}M} : \hat{T}M \to \hat{M} \) where \( \tau_{\hat{T}M}([v_m]) = \pi(m) \) is a Lie algebroid over \( \hat{M} \). The fiber of \( \hat{T}M \) over a point \( \pi(m) \in \hat{M} \) is isomorphic to \( T_mM \).

The Lie bracket is defined on the space \( \Gamma(\tau_{\hat{T}M}) \) which is isomorphic to the Lie subalgebra of \( G \)-invariant vector fields, that is,
\[
\Gamma(\tau_{\hat{T}M}) = \{ X \in \mathfrak{X}(M) \mid X \text{ is } G\text{-invariant} \}.
\]
Thus, the Lie bracket on \( \hat{T}M \) is the bracket of \( G \)-invariant vector fields. The anchor map \( \rho : \hat{T}M \to T\hat{M} \) is given by \( \rho([v_m]) = T_m \pi(v_m) \). Moreover, \( \rho \) is a Lie algebra homomorphism satisfying the compatibility condition since the \( G \)-invariant vector
fields are $\pi$-projectable. This Lie algebroid is called Lie-Atiyah algebroid associated with the principal bundle $\pi : M \to \hat{M}$.

Let $A : TM \to \mathfrak{g}$ be a principal connection in the principal bundle $\pi : M \to \hat{M}$ and $B : TM \oplus TM \to \mathfrak{g}$ be the curvature of $A$. The connection determines an isomorphism $\alpha_A$ between the vector bundles $T\hat{M} \to \hat{M}$ and $T\hat{M} \oplus \tilde{\mathfrak{g}} \to \hat{M}$, where $\tilde{\mathfrak{g}} = (M \times \mathfrak{g})/G$ is the adjoint bundle associated with the principal bundle $\pi : M \to \hat{M}$ (see [17] for example).

We choose a local trivialization of the principal bundle $\pi : M \to \hat{M}$ to be $U \times G$, where $U$ is an open subset of $\hat{M}$. Suppose that $e$ is the identity of $G$, $(x^i)$ are local coordinates on $U$ and $\{\xi_A\}$ is a basis of $\mathfrak{g}$.

Denote by $\{\xi_A\}$ the corresponding left-invariant vector field on $\mathfrak{g}$, that is,

$$\xi_A(g) = (T_e L_g)(\xi_A)$$

for $g \in G$ where $L_g : G \to G$ is the left-translation on $G$ by $g$. If

$$A \left( \frac{\partial}{\partial x^i} \right) = A_i^A(x)\xi_A, \quad B \left( \frac{\partial}{\partial x^i} \right) = B_i^A(x)\xi_A,$$

for $i, j \in \{1, \ldots, m\}$ and $x \in U$, then the horizontal lift of the vector field $\frac{\partial}{\partial x^i}$ is the vector field on $\pi^{-1}(U) \cong U \times G$ given by

$$\left( \frac{\partial}{\partial x^i} \right)^h = \frac{\partial}{\partial x^i} - A_i^A\xi_A.$$

Therefore, the vector fields on $U \times G$

$$e_i = \frac{\partial}{\partial x^i} - A_i^A\xi_A$$

are $G$-invariant under the action of $G$ over $M$ and define a local basis $\{\hat{e}_i, \hat{e}_B\}$ on $\Gamma(\hat{T}\hat{M})$. The corresponding local structure functions of $\tau_{\hat{T}\hat{M}} : \hat{T}\hat{M} \to \hat{M}$ are

$$\hat{e}_i A = -c_i^A A_i, \quad \hat{e}_i B = -c_i^B B_i,$$

$$c_i^A = 0, \quad c_i^B = -c_i^C A_i B_i,$$

being $\{c_i^A\}$ the constant structures of $\mathfrak{g}$ with respect to the basis $\{\xi_A\}$ (see [51] for more details). That is,

$$[\hat{e}_i, \hat{e}_j]_{\hat{T}\hat{M}} = -c_i^C A_j \hat{e}_C, \quad [\hat{e}_i, \hat{e}_A]_{\hat{T}\hat{M}} = c_i^C A_B \hat{e}_C, \quad [\hat{e}_A, \hat{e}_B]_{\hat{T}\hat{M}} = c_i^C \hat{e}_C,$$

$$\rho_{\hat{T}\hat{M}}(\hat{e}_i) = \frac{\partial}{\partial x^i}, \quad \rho_{\hat{T}\hat{M}}(\hat{e}_A) = 0.$$

The basis $\{\hat{e}_i, \hat{e}_B\}$ induce local coordinates $(x^i, y^i, \hat{g}^B)$ on $\hat{T}\hat{M} = TM/G$.

Next, we introduce the notion of Lie subalgebroid associated with a Lie algebroid.

**Definition 2.3.** Let $(E, [\cdot, \cdot]_E, \rho_E)$ be a Lie algebroid over $M$ and $N$ is a submanifold of $M$. A Lie subalgebroid of $E$ over $N$ is a vector subbundle $B$ of $E$ over $N$
such that $\rho_B = \rho_E|_B: B \to TN$ is well defined and, given $X, Y \in \Gamma(B)$ and $\tilde{X}, \tilde{Y} \in \Gamma(E)$ arbitrary extensions of $X, Y$ respectively, we have that $([\tilde{X}, \tilde{Y}]_E)|_N \in \Gamma(B)$.

2.1.2. Examples of Lie subalgebroids.

Example 5. Let $E$ be a Lie algebroid over $M$. Given a submanifold $N$ of $M$, if $B = E|_N \cap (\rho|_N)^{-1}(TN)$ exists as a vector bundle, it will be a Lie subalgebroid of $E$ over $N$, and will be called Lie algebroid restriction of $E$ to $N$ (see [53]).

Example 6. Let $N$ be a submanifold of $M$. Then, $TN$ is a Lie subalgebroid of $TM$.

Now, let $\mathcal{F}$ be a completely integrable distribution on a manifold $M$. $\mathcal{F}$ equipped with the bracket of vector fields is a Lie algebroid over $M$ since $\tau_E|_{\mathcal{F}}: \mathcal{F} \to M$ is a vector bundle. The anchor map is the inclusion $i_{\mathcal{F}}: \mathcal{F} \to TM$ ($i_{\mathcal{F}}$ is a Lie algebroid monomorphism). If $N$ is a submanifold of $M$ and $\mathcal{F}_N$ is a foliation on $N$, then $\mathcal{F}_N$ is a Lie subalgebroid of the Lie algebroid $\tau_{TM}: TM \to M$.

Example 7. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h}$ be a Lie subalgebra. If we consider the Lie algebroid induced by $\mathfrak{g}$ and $\mathfrak{h}$ over a point, then $\mathfrak{h}$ is a Lie subalgebroid of $\mathfrak{g}$.

Example 8. Let $M \times \mathfrak{g} \to M$ be an action Lie algebroid and let $N$ be a submanifold of $M$. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ such that the infinitesimal generators of the elements of $\mathfrak{h}$ are tangent to $N$; that is, the application

$$\mathfrak{h} \to \mathfrak{X}(N)$$

$$\xi \mapsto \xi_N$$

is well defined. Thus, the action Lie algebroid $N \times \mathfrak{h} \to N$ is a Lie subalgebroid of $M \times \mathfrak{g} \to M$.

Example 9. Suppose that the Lie group $G$ acts freely and properly on $M$. Let $\pi: M \to M/G = \hat{M}$ be the associated $G$–principal bundle. Let $N$ be a $G$–invariant submanifold of $M$ and $\mathcal{F}_N$ be a $G$–invariant foliation over $N$. We may consider the vector bundle $\hat{\mathcal{F}}_N = \mathcal{F}_N/G \to N/G = \hat{N}$ and endow it with a Lie algebroid structure. The sections of $\hat{\mathcal{F}}_N$ are

$$\Gamma(\hat{\mathcal{F}}_N) = \{ X \in \mathfrak{X}(N) \mid X \text{ is } G\text{-invariant and } X(q) \in \mathcal{F}_N(q), \forall q \in N\}.$$  

The standard bracket of vector fields on $N$ induces a Lie algebra structure on $\Gamma(\hat{\mathcal{F}}_N)$. The anchor map is the canonical inclusion of $\hat{\mathcal{F}}_N$ on $T\hat{N}$ and $\hat{\mathcal{F}}_N$ is a Lie subalgebroid of $T\hat{M} \to \hat{M}$.

2.2. $E$-tangent bundle to a Lie algebroid $E$. We consider the prolongation over the canonical projection of a Lie algebroid $E$ over $M$, that is,

$$\mathcal{T}^E E = \bigcup_{e \in E} (E_\rho \times_{T\tau_E} T_e E) = \bigcup_{e \in E} \{(e', v_e) \in E \times T_e E \mid \rho(e') = (T_e \tau_E)(v_e)\},$$

where $T\tau_E: TE \to TM$ is the tangent map to $\tau_E$.

In fact, $\mathcal{T}^E E$ is a Lie algebroid of rank $2n$ over $E$ where $\tau_E^{(1)}: \mathcal{T}^E E \to E$ is the vector bundle projection, $\tau_E^{(1)}(b, v_e) = \tau_{TE}(v_e) = e$, and the anchor map is $\rho_1: \mathcal{T}^E E \to TE$ is given by the projection over the second factor. The bracket of sections of this new Lie algebroid will be denoted by $[\cdot, \cdot]_\tau^{(1)}$ (See [60] for more details).
If we denote by \((e, e', v_e)\) an element \((e', v_e) \in T^r e\) where \(e \in E\) is the point where \(v\) is tangent; we rewrite the definition for the prolongation of the Lie algebroid as the subset of \(E \times E \times TE\) by

\[ T^r E = \{(e, e', v_e) \in E \times E \times TE \mid \rho(e') = (T \tau_E)(v_e), v_e \in T_e E \text{ and } \tau_E(e) = \tau_E(e')\}. \]

Thus, if \((e, e', v_e) \in T^r E\); then \(\rho_1(e, e', v_e) = (e, v_e) \in T_e E\), and \(\tau_E^{(1)}(e, e', v_e) = e \in E\).

Next, we introduce two canonical operations that we have on a Lie algebroid \(E\). The first one is obtained using the Lie algebroid structure of \(E\) and the second one is a consequence of \(E\) being a vector bundle. On one hand, if \(f \in C^\infty(M)\) we will denote by \(f^c\) the complete lift to \(E\) of \(f\) defined by \(f^c(e) = \rho(e)(f)\) for all \(e \in E\). Let \(X\) be a section of \(E\) then there exists a unique vector field \(X^c\) on \(E\), the complete lift of \(X\), satisfying the two following conditions:

1. \(X^c\) is \(\tau_E\)-projectable on \(\rho(X)\) and
2. \(X^c(\hat{\alpha}) = \mathcal{L}_X^E \alpha\),

for every \(\alpha \in \Gamma(\tau_E)\) (see [34]). Here, if \(\beta \in \Gamma(\tau_E)\) then \(\hat{\beta}\) is the linear function on \(E\) defined by

\[
\hat{\beta}(e) = \langle \beta(\tau_E(e)), e \rangle, \quad \text{for all } e \in E. 
\]

We may introduce the complete lift \(X^c\) of a section \(X \in \Gamma(\tau_E)\) as the sections of \(\tau_E^{(1)} : T^r E \to E\) given by

\[
X^c(e) = (X(\tau_E(e)), X^c(e))
\]

for all \(e \in E\) (see [60]).

Given a section \(X \in \Gamma(\tau_E)\) we define the vertical lift as the vector field \(X^v \in \mathfrak{X}(E)\) given by

\[
X^v(e) = X(\tau_E(e))^v_e, \quad \text{for } e \in E,
\]

where \(v : E_q \to T_e E_q\) for \(q = \tau_E(e)\) is the canonical isomorphism between the vector spaces \(E_q\) and \(T_e E_q\).

Finally we may introduce the vertical lift \(X^v\) of a section \(X \in \Gamma(\tau_E)\) as a section of \(\tau_E^{(1)}\) given by

\[
X^v(e) = (0, X^v(e)) \text{ for } e \in E.
\]

With these definitions we have the properties (see [34] and [60])

\[
[X^c, Y^c] = [X, Y]^c, \quad [X^c, Y^v] = [X, Y]^v, \quad [X^v, Y^v] = 0
\]

for all \(X, Y \in \Gamma(\tau_E)\).

If \((x^i)\) are local coordinates on an open subset \(U\) of \(M\) and \(\{e_A\}\) is a basis of sections of \(\tau_E\) then we have induced coordinates \((x^i, y^A)\) on \(E\).

From the basis \(\{e_A\}\) we may define a local basis \(\{e_A^{(1)}, e_A^{(2)}\}\) of sections of \(\tau_E^{(1)}\) given by

\[
e_A^{(1)}(e) = \left( e, e_A(\tau_A(e)), \rho_A^{\frac{\partial}{\partial x^i}} e \right), \quad e_A^{(2)}(e) = \left( e, 0, \rho_A^{\frac{\partial}{\partial y^A}} e \right),
\]

for \(e \in (\tau_E)^{-1}(U)\) with \(U\) an open subset of \(M\) (see [51] for more details).

From this basis we have that the structure of Lie algebroid is determined by...
Moreover, let the Lie algebroid structure on $\tau$ over a single point $TM$ be a standard Lie algebroid structure over $\mathcal{T}e^* E$. In the case of $\mathcal{T}e^* E$, one may introduce local coordinates $(x^i, y^A; z^A, v^A)$ on $\mathcal{T}e^* E$. If $V$ is a section of $\tau^{(1)}_E$, locally it is determined by

$$V(x, y) = (x^i, y^A, z^A(x, y), v^A(x, y));$$

therefore the expression of $V$ in terms of the basis $\{e^{(1)}_A, e^{(2)}_A\}$ is $V = z^A e^{(1)}_A + v^A e^{(2)}_A$ and the vector field $\rho_1(V) \in \mathfrak{X}(E)$ has the expression

$$\rho_1(V) = \rho_1^i z^A(x, y) \frac{\partial}{\partial x^i}(x, y) + v^A(x, y) \frac{\partial}{\partial y^A}(x, y).$$

Moreover, if $\{e^{(1)}_A, e^{(2)}_A\}$ denotes the dual basis of $\{e^{(1)}_A, e^{(2)}_A\}$,

$$d^{\mathcal{T}e^* E} F(x^i, y^A) = \rho_1^i \frac{\partial F}{\partial x^i} e^{(1)}_A + \frac{\partial F}{\partial y^A} e^{(2)}_A,$$

$$d^{\mathcal{T}e^* E} e^{(1)}_A = -\frac{1}{2} C_{AB} e^{(1)}_A \wedge e^{(2)}_A,$$

$$d^{\mathcal{T}e^* E} e^{(2)}_A = 0.$$

**Example 10.** In the case of $E = TM$ one may identify $\mathcal{T}e^* E$ with $TTM$ with the standard Lie algebroid structure over $TM$.

**Example 11.** Let $\mathfrak{g}$ be a real Lie algebra of finite dimension. $\mathfrak{g}$ is a Lie algebroid over a single point $M = \{q\}$. The anchor map of $\mathfrak{g}$ is zero constant function, and from the anchor map we deduce that

$$\mathcal{T}q^* \mathfrak{g} = \{(\xi_1, \xi_2, \nu, \xi_3) \in \mathfrak{g} \times T_{q^*} \mathfrak{g} \} \simeq \mathfrak{g} \times \mathfrak{g} \simeq 3\mathfrak{g}.$$  

The vector bundle projection $\tau^{(1)}_q : 3\mathfrak{g} \to \mathfrak{g}$ is given by $\tau^{(1)}_q(\xi_1, \xi_2, \xi_3) = \xi_1$ with anchor map $\rho_1(\xi_1, \xi_2, \xi_3) = (\xi_1, \xi_{23}) \simeq \xi_1 \in T_{\xi_1} \mathfrak{g}$.

Let $\{e_A\}$ be a basis of the Lie algebra $\mathfrak{g}$, this basis induces local coordinates $y^A$ on $\mathfrak{g}$, that is, $\xi = y^A e_A$. Also, this basis induces a basis of sections of $\tau^{(1)}_q$ as

$$e^{(1)}_A(\xi) = (\xi, e_A, 0), \quad e^{(2)}_A(\xi) = \left(\xi, 0, \frac{\partial}{\partial y^A}\right).$$

Moreover

$$\rho_1(e^{(1)}_A)(\xi) = (\xi, 0), \quad \rho_1(e^{(2)}_A)(\xi) = \left(\xi, \frac{\partial}{\partial y^A}\right).$$

The basis $\{e^{(1)}_A, e^{(2)}_A\}$ induces adapted coordinates $(y^A, z^A, v^A)$ in $3\mathfrak{g}$ and therefore a section $Y$ on $\Gamma(\tau^{(1)}_q)$ is written as $Y(\xi) = z^A(\xi) e^{(1)}_A + v^A(\xi) e^{(2)}_A$. Thus, the vector field $\rho_1(Y) \in \mathfrak{g}$ has the expression $\rho_1(Y) = v^A(\xi) \frac{\partial}{\partial y^A}|_\xi$. Finally, the Lie algebroid structure on $\tau^{(1)}_q$ is determined by the Lie bracket $[[\xi, \xi], (\eta, \eta)] = \sum_{A=1}^n C_{AB} e^{(1)}_A \wedge e^{(2)}_A$.
(\[\{ [\xi, \eta] + \tilde{\xi}(\eta) - \tilde{\eta}(\xi), \tilde{\xi}(\eta) - \tilde{\eta}(\xi) \}\) with \(\xi, \tilde{\xi}, \eta, \tilde{\eta} \in g\) and where \(\tilde{\xi}(\eta)\) should be understood as differentiating \(\eta\) in the direction of \(\xi\) (see [24] for details).

**Example 12.** We consider a Lie algebra \(g\) acting on a manifold \(M\), that is, we have a Lie algebra homomorphism \(g \to X(M)\) mapping every element \(\xi\) of \(g\) to a vector field \(\xi_M\) on \(M\). Then we can consider the action Lie algebroid \(E = M \times g\). Identifying \(T \Sigma E = TM \times Tg = TM \times 2g\), an element of the prolongation Lie algebroid to \(E\) over the bundle projection is of the form \((a, b, v_x) = ((x, \xi), (x, \eta), (v_x, \xi, \chi))\) where \(v_x \in T_x M\) and \((\xi, \eta, \chi) \in 3g\). The condition \(T_Tg(v) = \rho(b)\) implies that \(v_x = -\eta_M(x)\). Therefore we can identify the prolongation Lie algebroid with \(M \times g \times g\) with projection onto the first two factors \((x, \xi)\) and anchor map \(\rho_1(x, \xi, \eta, \chi) = (-\eta_M(x), \xi, \chi)\). Given a base \(\{e_A\}\) of \(g\) the basis \(\{e_A^{(1)}, e_A^{(2)}\}\) of sections of \(\mathcal{T}^{TM_{\times g}}(M \times g)\) is given by

\[
e_A^{(1)}(x, \xi) = (x, \xi, e_A, 0), \quad e_A^{(2)}(x, \xi) = (x, \xi, 0, e_A).
\]

Moreover,

\[
\rho_1(e_A^{(1)}(x, \xi)) = (x, -e_A)_M(x, \xi, 0), \quad \rho_2(e_A^{(2)}(x, \xi)) = (0, 0, \xi, e_A).
\]

Finally, the Lie bracket of two constant sections is given by \([[(\xi, \tilde{\xi}), (\eta, \tilde{\eta})]] = ((\xi, \eta), 0)\).

**Example 13.** Let us describe the \(E\)-tangent bundle to \(E\) in the case of \(E\) being an Atiyah algebroid induced by a trivial principal \(G\)-bundle \(\pi : G \times M \to M\). In such case, by left trivialization we get the Atiyah algebroid, the vector bundle \(\tau_{g \times TM} : g \times TM \to TM\). If \(X \in X(M)\) and \(\xi \in g\) then we may consider the section \(X^\xi : M \to g \times TM\) of the Atiyah algebroid by

\[
X^\xi(q) = (\xi, X(q)) \text{ for } q \in M.
\]

Moreover, in this sense

\[
[X^\xi, Y^\xi]_{g \times TM} = ([X, Y]_{TM}, [\xi, \eta]_g), \quad \rho(X^\xi) = X.
\]

On the other hand, if \((\xi, v_q) \in g \times T_q M\), then the fiber of \(\mathcal{T}^{TM_{\times g}}(g \times TM)\) over \((\xi, v_q)\) is

\[
\mathcal{T}^{TM_{\times g}}((\xi, v_q)) = \left\{ (\eta, u_q), (\tilde{\eta}, X(v_q)) \in g \times T_q M \times g \times T_v g(TM) \mid u_q = T_v g \tau_{g \times TM}(X(v_q)) \right\}.
\]

This implies that \(\mathcal{T}^{TM_{\times g}}(g \times TM)\) may be identified with the space \(2g \times T_q g(TM)\). Thus, the Lie algebroid \(\mathcal{T}^{TM_{\times g}}(g \times TM)\) may be identified with the vector bundle \(g \times 2g \times TTM \to g \times TM\) whose vector bundle projection is

\[
(\xi, ((\eta, \tilde{\eta}), X(v_q))) \mapsto (\xi, v_q)
\]

for \((\xi, ((\eta, \tilde{\eta}), X(v_q))) \in g \times 2g \times TTM\). Therefore, if \((\eta, \tilde{\eta}) \in 2g\) and \(X \in X(TM)\) then one may consider the section \(((\eta, \tilde{\eta}), X)\) given by

\[
((\eta, \tilde{\eta}), X)(\xi, v_q) = (\xi, ((\eta, \tilde{\eta}), X(v_q))) \text{ for } (\xi, v_q) \in g \times T_q M.
\]

Moreover,

\[
[[((\eta, \tilde{\eta}), X), ((\xi, \tilde{\xi}), Y)]_{\mathcal{T}^{TM_{\times g}}}]_{g \times TM}^{(1)} = \frac{1}{2}(([\eta, \xi]_g, 0), [X, Y]_{TM}),
\]
and the anchor map \( \rho_1 : \mathfrak{g} \times 2\mathfrak{g} \times TTM \to \mathfrak{g} \times \mathfrak{g} \times TTM \) is defined as
\[
\rho_1(\xi, ((\eta, \tilde{\eta}), X)) = ((\xi, \tilde{\eta}), X).
\]

2.3. \( E \)-tangent bundle of the dual bundle of a Lie algebroid. Let \((E, \llbracket \cdot, \cdot \rrbracket, \rho)\)
be a Lie algebroid of rank \( n \) over a manifold of dimension \( m \). Consider the projection of the dual \( E^* \) of \( E \) over \( M \), \( \tau_{E^*} : E^* \to M \), and define the prolongation \( T^*E^* \) of \( E \) over \( \tau_{E^*} \); that is,
\[
T^*E^* = \bigcup_{\mu \in E^*} \{ (e, v_\mu) \in E \times T_{\mu}E^* \mid \rho(e) = T_{\tau_{E^*}(v_\mu)} \}.
\]

\( T^*E^* \) is a Lie algebroid over \( E^* \) of rank \( 2n \) with vector bundle projection \( \tau^{(1)}_{E^*} : T^*E^* \to E^* \) given by \( \tau^{(1)}_{E^*}(e, v_\mu) = \mu \), for \((e, v_\mu) \in T^*E^* \).

As before, if we now denote by \((\mu, e, v_\mu)\) an element \((e, v_\mu) \in T^*E^* \) where \( \mu \in E^* \), we rewrite the definition of the prolongation Lie algebroid as the subset of \( E^* \times E \times TE^* \) by
\[
T^*E^* = \{ (\mu, e, v_\mu) \in E^* \times E \times TE^* \mid \rho(e) = T_{\tau_{E^*}(v_\mu)} \mu \in T_{\tau_{E^*}(v_\mu)} \}.
\]

If \((x^i)\) are local coordinates on an open subset \( U \) of \( M \), \( \{e_A\} \) is a basis of sections of the vector bundle \( (\tau_{E^*})^{-1}(U) \to U \) and \( \{e^A\} \) is its dual basis, then \( \{\varepsilon_A^{(1)}, \varepsilon_A^{(2)}\} \) is a basis of sections of the vector bundle \( \tau_{E^*}^{(1)} \), where
\[
\varepsilon_A^{(1)}(\mu) = \left( \mu, e_A(\tau_{E^*}(\mu)), \rho_A^j \frac{\partial}{\partial x^j}(\mu) \right), \quad \varepsilon_A^{(2)}(\mu) = \left( \mu, 0, \frac{\partial}{\partial p_A}(\mu) \right),
\]
for \( \mu \in (\tau_{E^*})^{-1}(U) \). Here, \((x^i, p_A)\) are the local coordinates on \( E^* \) induced by the local coordinates \((x^i)\) and the basis of sections of \( E^* \), \( \{e^A\} \).

Using the local basis \( \{\varepsilon_A^{(1)}, \varepsilon_A^{(2)}\} \), one may introduce, in a natural way, local coordinates \((x^i, p_A, z^A, v_A)\) on \( T^*E^* \). If \( \omega^* \) is a point of \( T^*E^* \) over \((x^i, p_A) \in E^* \), then
\[
\omega^*(x, p) = z^A \varepsilon_A^{(1)}(x, p) + v_A(\varepsilon_A^{(2)}(x, p).
\]

Denoting by \( \rho_{\tau_{E^*}}^{(1)} \) the anchor map of the Lie algebroid \( T^*E^* \to E^* \) locally given by
\[
\rho_{\tau_{E^*}}^{(1)}(x^i, p_A, z^A, v_A) = (x^i, p_A, \rho_A^j z^A, v_A),
\]
we have that
\[
\rho_{\tau_{E^*}}^{(1)}(\varepsilon_A^{(1)}(\mu)) = \left( \mu, \rho_A^j \frac{\partial}{\partial x^j}(\mu) \right), \quad \rho_{\tau_{E^*}}^{(1)}(\varepsilon_A^{(2)}(\mu)) = \left( \mu, 0, \frac{\partial}{\partial p_A}(\mu) \right).
\]

Therefore, we have that the corresponding vector field \( \rho_{\tau_{E^*}}^{(1)}(V) \) for the section determined by \( V = (x^i, p_A, z^A(x, p), v_A(x, p)) \) is given by
\[
\rho_{\tau_{E^*}}^{(1)}(V) = \rho_A^j z^A \frac{\partial}{\partial x^j} + v_A \frac{\partial}{\partial p_A}.
\]

Finally, the structure of the Lie algebroid \((T^*E^*, \llbracket \cdot, \cdot \rrbracket, \rho_{\tau_{E^*}}^{(1)})\), is determined by the bracket of sections
\[
\llbracket \varepsilon_A^{(1)} , \varepsilon_B^{(1)} \rrbracket_{\tau_{E^*}}^{(1)} = \varepsilon_C^{(1)} \llbracket B^{(1)} , C^{(1)} \rrbracket_{\tau_{E^*}}^{(1)}, \quad \llbracket \varepsilon_A^{(1)} , (\varepsilon_B^{(2)}) \rrbracket_{\tau_{E^*}}^{(1)} = \llbracket (\varepsilon_A^{(2)}), (\varepsilon_B^{(2)}) \rrbracket_{\tau_{E^*}}^{(1)} = 0,
\]
for all \( A, B \) and \( C \). Thus, if we denote by \( \{\varepsilon_A^{(1)}, \varepsilon_A^{(2)}\} \) is the dual basis of \( \{e_A^{(1)}, e_A^{(2)}\} \), then
This implies that \( \text{vector bundle } (A) \) for \( f \in C^\infty(E^*) \). We refer to \([51]\) for further details about the Lie algebroid structure of the E-tangent bundle of the dual bundle of a Lie algebroid.

**Example 14.** In the case of \( E = TM \) one may identify \( T\tau^*E \) with \( T(T^*M) \) with the standard Lie algebroid structure.

**Example 15.** Let \( g \) be a real Lie algebra of finite dimension. Then \( g \) is a Lie algebroid over a single point. Using that the anchor map is zero we have that \( T\tau^*g \) may be identified with the vector bundle \( pr_1 : g^* \times (g \times g^*) \to g^* \). Under this identification the anchor map is given by

\[
\rho_{\tau^*g^*} : g^* \times (g \times g^*) \to Tg^* \simeq g^* \times g^*, \quad (\mu, (\xi, \alpha)) \mapsto (\mu, \alpha)
\]

and the Lie bracket of two constant sections \((\xi, \alpha), (\eta, \beta) \in g \times g^*\) is the constant section \((\xi, \eta), 0)\).

**Example 16.** Let \( A = M \times g \to M \) be an action Lie algebroid over \( M \) and \((q, \mu) \in M \times g^*\). It follows that the prolongation may be identified with the trivial vector bundle \((M \times g^*) \times (g \times g^*) \to M \times g^*\) since

\[
T(M \times g^*) \times g = \{(q, \xi, (X_q, \alpha)) \in M \times g \times T_qM \times g^* \mid -\xi_M(q) = X_q \} \simeq g \times g^*
\]

The anchor map \( \rho_{\tau_{M \times g^*}^{(1)}} : (M \times g^*) \times (g \times g^*) \to TM \times (g^* \times g^*)\) is given by

\[
\rho_{\tau_{M \times g^*}^{(1)}}((q, \mu), (\xi, \alpha)) = (-\xi_M(q), \mu, \alpha).
\]

Moreover, the Lie bracket of two constant sections \((\xi, \alpha), (\eta, \beta) \in g \times g^*\) is just the constant section \((\xi, \eta), 0)\).

**Example 17.** Let us describe the \( A \)-tangent bundle to \( A^*\) in the case of \( A \) being an Atiyah algebroid induced by a trivial principal \( G \)-bundle \( \pi : G \times M \to M \). In such case, by left trivialization we have that the Atiyah algebroid is the vector bundle \( \tau_g \times TM : g \times TM \to TM \). If \( X_M \) and \( \xi \in g \) then we may consider the section \( X^\xi : M \to g \times TM \) of the Atiyah algebroid by

\[
X^\xi(q) = (\xi, X(q)) \text{ for } q \in M.
\]

Moreover, in this sense

\[
[X^\xi, Y^\xi]_{g \times TM} = ([X, Y]_{TM}, [\xi, \eta]_g), \quad \rho(X^\xi) = X.
\]

If \((\mu, \alpha_q) \in g^* \times T_q^*M\) then the fiber of \( T^{(g \times TM)^*}(g \times TM)\) over \((\mu, \alpha_q)\) is

\[
T^{(g \times TM)^*}_{(\mu, \alpha_q)}(g \times TM) = \left\{ (\eta, u_q), (\beta, X_{\alpha_q}) \in g \times T_qM \times g^* \times T_{\alpha_q}(T^*M) \right\}
\]

such that \( u_q = T_{\alpha_q} \tau_{(g \times TM)^*}(X_{\alpha_q}) \).

This implies that \( T^{(g \times TM)^*}_{(\mu, \alpha_q)}(g \times TM)\) may be identified with the vector space \((g \times g^*) \times T_{\alpha_q}(T^*M)\). Thus, the Lie algebroid \( T^{(g \times TM)^*}(g \times TM)\) may be identified with
the vector bundle $\mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*) \times T^* M \to \mathfrak{g}^* \times T^* M$ whose vector bundle projection is

$$(\mu, ((\xi, \beta), X_{\alpha_q})) \mapsto (\mu, \alpha_q)$$

for $(\mu, ((\xi, \beta), X_{\alpha_q})) \in \mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*) \times T^* M$. Therefore, if $(\xi, \beta) \in \mathfrak{g} \times \mathfrak{g}^*$ and $X \in \mathfrak{X}(T^* M)$ then one may consider the section $((\xi, \beta), X)$ given by

$$(\xi, \beta, X)(\mu, \alpha_q) = (\mu, ((\xi, \beta), X(\alpha_q)))$$

for $(\mu, \alpha_q) \in \mathfrak{g}^* \times T_q^* M$.

Moreover,

$$(((\xi, \beta), X), ((\tilde{\xi}, \tilde{\beta}), \tilde{X})|_{(x,TM)^*}) = (((\xi, \tilde{\xi}), \tilde{\beta}), [X, \tilde{X}]|_{TM}),$$

and the anchor map $\rho_{\gamma,T^* M^*}^*: \mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*) \times T^* M \to \mathfrak{g}^* \times \mathfrak{g}^* \times T^* M$ is defined as

$$\rho_{\gamma,T^* M^*}^*(\mu, ((\xi, \beta), X)) = ((\mu, \beta), X).$$

### 2.4. Symplectic Lie algebroids

In this subsection we will recall some results given in [51] about symplectic Lie algebroids.

**Definition 2.4.** A Lie algebroid $(E, [[\cdot, \cdot]], \rho)$ over a manifold $M$ is said to be symplectic if it admits a symplectic section $\Omega$, that is, $\Omega$ is a section of the vector bundle $\wedge^2 E^* \to M$ such that:

1. For all $x \in M$, the 2-form $\Omega_x : E_x \times E_x \to \mathbb{R}$ in the vector space $E_x$ is nondegenerate and
2. $\Omega$ is a 2-cocycle, that is, $d^E \Omega = 0$.

#### 2.4.1. The canonical symplectic structure of $T^* E$

Let $(E, [[\cdot, \cdot]], \rho)$ be a Lie algebroid of rank $n$ over a manifold $M$ of dimension $m$ and $T^* E$ be the prolongation of $E$ over the vector bundle projection $\tau_{E^*} : E^* \to M$. We may introduce a canonical section $\lambda_E$ of $(T^* E)^*$ as follows. If $\mu \in E^*$ and $(e, v_\mu)$ is a point on the fibre of $T^* E$ over $\mu$ then

$$\lambda_E(\mu)(e, v_\mu) = \langle \mu, e \rangle.$$  \hspace{1cm} (11)

$\lambda_E$ is called the Liouville section of $T^* E$. Now, in an analogous way that the canonical symplectic form is defined from the Liouville 1-form on the cotangent bundle, we introduce the 2-section $\Omega_E$ on $T^* E$ as

$$\Omega_E = -d^T T^* E \lambda_E.$$  \hspace{1cm} (12)

**Proposition 1.** [51] $\Omega_E$ is a non-degenerate 2-section of $T^* E$ such that

$$d^T T^* E \Omega_E = 0.$$ 

Therefore $\Omega_E$ is a symplectic 2-section on $T^* E$ called canonical symplectic section on $T^* E$.

**Example 18.** If $E$ is the standard Lie algebroid $TM$ then $\lambda_E = \lambda$ and $\Omega_E = \omega_M$ are the usual Liouville 1-form and canonical symplectic 2-form on $T^* M$, respectively.

**Example 19.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then $\mathfrak{g}$ is a Lie algebroid over a single point $M = \{q\}$. If $\xi \in \mathfrak{g}$ and $\mu, \alpha \in \mathfrak{g}^*$ then

$$\lambda_\mathfrak{g}(\mu)(\xi, \alpha) = \mu(\xi)$$

is the Liouville 1-section on $\mathfrak{g}^* \times (\mathfrak{g} \times \mathfrak{g}^*)$. Thus, the symplectic section $\Omega_\mathfrak{g}$ is

$$\Omega_\mathfrak{g}(\mu)(((\xi, \alpha), (\eta, \beta))) = \langle \mu, [\xi, \eta] \rangle - \langle \alpha, \eta \rangle - \langle \beta, \xi \rangle$$

for $\mu \in \mathfrak{g}^*, (\xi, \alpha), (\eta, \beta) \in \mathfrak{g} \times \mathfrak{g}^*$. 

2.5. Admissible elements on a Lie algebroid. Let $E$ be a Lie algebroid over $M$ with fiber bundle projection $\tau : E \to M$ and anchor map $\rho : E \to TM$.

**Definition 2.5.** A tangent vector $v$ at the point $e \in E$ is called admissible if $\rho(e) = \tau E(v)$; and a curve on $E$ is admissible if its tangent vectors are admissible. The set of admissible elements on $E$ will be denoted $E(2)$.

Notice that $v$ is admissible if and only if $(e, e, v)$ is in $\mathcal{T}e E$. We will consider $E(2)$ as the subset of the prolongation of $E$ over $\tau E$, that is, $E(2) \subset E_0 \times_{\tau E} TE$ is given by

$$E(2) = \{(e, v_e) \in E \times TE \mid \rho(e) = \tau E(v_e)\}.$$ 

Other authors call this set $\text{Adm}(E)$ (see [14] and [61]).

A curve $\gamma : I \subset \mathbb{R} \to E$ is said to be an admissible curve on $E$ if it satisfies $\rho(\gamma(t)) = \dot{\gamma}(t)$ where $\gamma = \tau E(\gamma(t))$ is a curve on $M$. Locally, admissible curves on $E$ are characterized by the so-called admissibility condition. A curve $\gamma(t) = (x^i(t), y^A(t))$ on $E$ is admissible if it satisfies the admissibility condition $\dot{x}^i(t) = \rho^{i}e(x^i(t))y^A(t)$. Therefore, locally, $E(2)$ is determined by $(\gamma(0), \dot{\gamma}(0))$ where $\gamma$ is an admissible curve on $E$. Admissible curves on $E$ are also called $E$-paths [61].

We consider $E(2)$ as the substitute of the second order tangent bundle in classical mechanics. If $(x^i)$ are local coordinates on $M$ and $\{e_A\}$ is a basis of sections of $E$ then we denote $(x^i, y^A)$ the corresponding local coordinates on $E$ and $(x^i, y^A, z^A, v^A)$ local coordinates on $\mathcal{T}^e E$ induced by the basis of sections $\{e_A^{(1)}, e_A^{(2)}\}$ of $\mathcal{T}^e E$ (see subsection 2.2). Therefore, the set $E(2)$ is locally characterized by the condition $(x^i, y^A, z^A, v^A) \in \mathcal{T}^e E \mid y^A = z^A$, that is $(x^i, y^A, v^A) := (x^i, y^A, \dot{y}^A)$ are local coordinates on $E(2)$.

We denote the canonical inclusion of $E(2)$ on the prolongation of $E$ over $\tau E$ as

$$i_{E(2)} : E(2) \hookrightarrow \mathcal{T}^e E,$$

$$\begin{align*}
(x^i, y^A, \dot{y}^A) &\mapsto (x^i, y^A, \dot{y}^A, \ddot{y}^A).
\end{align*}$$

**Example 20.** Let $M$ be a differentiable manifold of dimension $m$, if $(x^i)$ are local coordinates on $M$, then $\{\frac{\partial}{\partial x^j}\}$ is a local basis of $\mathfrak{X}(M)$ and then we have fiber local coordinates $(x^i, \dot{x}^i)$ on $TM$. The corresponding local structure functions of the Lie algebroid $\tau TM : TM \to M$ are

$$\xi^i_{ij} = 0 \text{ and } \rho^i_{ij} = \delta^i_j, \text{ for } i, j, k \in \{1, \ldots, m\}.$$ 

In this case, we have seen that the prolongation Lie algebroid over $\tau TM$ is just the tangent bundle $TTM$ where the Lie algebroid structure of the vector bundle $T(TM) \to TM$ is as we have described above as the tangent bundle of a manifold.

The set of admissible elements is given by

$$E(2) = \{(x^i, v^i, \dot{x}^i, w^i) \in T(TM) \mid \dot{x}^i = v^i\}$$

and observe that this subset is just the second-order tangent bundle of a manifold $M$, that is, $E(2) = T^2 M$. Admissible curves on $E(2) = T^2 M$ are given by

$$\sigma(t) = (x^i(t), \dot{x}^i(t), \ddot{x}^i(t)).$$

**Example 21.** Consider a Lie algebra $\mathfrak{g}$ as a Lie algebroid over a point $\{e\}$. Given a basis of section $\{e_A\}$ and element $\xi \in \mathfrak{g}$ can be written as $\xi = e_A \xi^A$ and given that the anchor map is given by $\rho(\xi) \equiv 0$, every curve $\xi(t) \in \mathfrak{g}$ is an admissible curve. The set of admissible elements is described by the cartesian product of two copies of the Lie algebra, $2\mathfrak{g}$. Local coordinates on $2\mathfrak{g}$ are determined by the basis of sections.
of \( \mathfrak{g} \), \( \{ e_A \} \) and \( \{ \xi^{(1)}_A, \xi^{(2)}_A \} \), the basis of the prolongation Lie algebroid introduced in Example 11. They are denoted by \( (\xi^1, \xi^2) \) and also \( (\xi^1, \xi^2) := (\xi(0), \xi(0)) \in 2\mathfrak{g} \) where \( \xi(t) \) is admissible.

**Example 22.** Let \( G \) be a Lie group and we assume that \( G \) acts free and properly on \( M \). We denote by \( \pi : M \rightarrow \hat{M} = M/G \) the associated principal bundle. The tangent lift of the action gives a free and proper action of \( G \) on \( TM \) and \( \hat{T}M = TM/G \) is a quotient manifold. Then we consider the Atiyah algebroid \( \hat{T}M \) over \( \hat{M} \).

According to Example 4, the basis \( \{ \hat{e}_i, \hat{e}_B \} \) induce local coordinates \( (x^i, y^i, \dot{y}^B) \). From this basis one can induces a basis of the prolongation Lie algebroid, namely \( \{ \xi^{(1)}_i, \xi^{(1)}_B \} \). This basis induce adapted coordinates \( (x^i, y^i, \dot{y}^B, \ddot{y}^B) \) on \( T^{(2)}M = (T^{(2)}M)/G \).

### 3. Second-order variational problems on Lie algebroids.

The geometric description of mechanics in terms of Lie algebroids gives a general framework to obtain all the relevant equations in mechanics (Euler-Lagrange, Euler-Poincaré, Lagrange-Poincaré,...). In this section we use the notion of Lie algebroid and prolongation of a Lie algebroid described in [2] to derive the Euler-Lagrange equations and Hamilton equations on Lie algebroids. Next, after introducing the constraint algorithm for presymplectic Lie algebroids and studying vakonomic mechanics on Lie algebroids, we study the geometric formalism for second-order constrained variational problems using and adaptation of the classical Skinner-Rusk formalism for the second-order constrained systems on Lie algebroids.

#### 3.1. Mechanics on Lie algebroids.

In [60] (see also [51]) a geometric formalism for Lagrangian mechanics on Lie algebroids was introduced. It was developed in the prolongation \( \mathcal{T}E \) of a Lie algebroid \( E \) (see [2]) over the vector bundle projection \( \tau_E : E \rightarrow M \). The prolongation of the Lie algebroid is playing the same role as \( TTQ \) in the standard mechanics. We first introduce the canonical geometrical structures defined on \( \mathcal{T}E \) to derive the Euler-Lagrange equations on Lie algebroids.

Two canonical objects on \( \mathcal{T}E \) are the **Euler section** \( \Delta \) and the **vertical endomorphism** \( S \). Considering the local basis of sections of \( \mathcal{T}E \), \( \{ e^{(1)}_A, e^{(2)}_A \} \), \( \Delta \) is the section of \( \mathcal{T}E \rightarrow E \) locally defined by

\[
\Delta = y^A e^{(2)}_A
\]

and \( S \) is the section of the vector bundle \( (\mathcal{T}E) \otimes (\mathcal{T}E)^* \rightarrow E \) locally characterized by the following conditions:

\[
Se^{(1)}_A = e^{(2)}_A, \quad Se^{(2)}_A = 0, \quad \text{for all } A.
\]

Finally, a section \( \xi \) of \( \mathcal{T}E \rightarrow E \) is said to be a **second order differential equation** (SODE) on \( E \) if \( S(\xi) = \Delta \) or, alternatively, \( pr_1(\xi(e)) = e \), for all \( e \in E \) (for more details, see [51]).

Given a Lagrangian function \( L \in C^\infty(E) \) we introduce the **Cartan 1-section** \( \Theta_L \in \Gamma((\mathcal{T}E)^*) \), the **Cartan 2-section** \( \omega_L \in \Gamma(\wedge^2(\mathcal{T}E)^*) \) and the **Lagrangian energy** \( E_L \in C^\infty(E) \) as

\[
\Theta_L = S^* (d^{\mathcal{T}E}E_L), \quad \omega_L = -d^{\mathcal{T}E}E_L \Theta_L \quad E_L = L^{\mathcal{T}E}E - L.
\]

If \( (x^i, y^A) \) are local fibred coordinates on \( E \), \( (\rho_A^i, \partial_A^i) \) are the corresponding local structure functions on \( E \) and \( \{ e^{(1)}_A, e^{(2)}_A \} \) the corresponding local basis of sections of
\[ T^\tau_E \mathcal{E} \]

\[ \omega_L = \frac{\partial^2 L}{\partial y^A \partial y^B} \epsilon^{(1)}_B \wedge \epsilon^{(2)}_A + \frac{1}{2} \left( \frac{\partial^2 L}{\partial x^i \partial y^B} \rho^i_B - \frac{\partial^2 L}{\partial x^i \partial y^B} \rho^i_A + \frac{\partial L}{\partial y^A} \xi^B_{AB} \right) \epsilon^{(1)}_A \wedge \epsilon^{(2)}_B, \]

(15)

\[ E_L = \frac{\partial L}{\partial y^A} y^A - L. \]

(16)

A curve \( t \to c(t) \) on \( E \) is a solution of the Euler-Lagrange equations for \( L \) if

- it is \textit{admissible} (that is, \( \rho(c(t)) = \dot{m}(t) \), where \( m = T_E \circ c \) and
- \( \iota_{c(t),c(t)} \omega_L(c(t)) = d^{\tau_E E} E_L(c(t)) = 0 \), for all \( t \).

If \( c(t) = (x^i(t), y^A(t)) \) then \( c \) is a solution of the Euler-Lagrange equations for \( L \) if and only if

\[ \dot{x}^i = \rho^i_A y^A, \quad \frac{d}{dt} \frac{\partial L}{\partial y^A} + \frac{\partial L}{\partial y^B} \xi^B_{AB} - \rho^i_A \frac{\partial L}{\partial x^i} = 0. \]

(17)

Observe that, if \( E \) is the standard Lie algebroid \( TQ \) then the above equations are the classical Euler-Lagrange equations for \( L : TQ \to \mathbb{R} \).

On the other hand, the Lagrangian function \( L \) is said to be \textit{regular} if \( \omega_L \) is a symplectic section. In such a case, there exists a unique solution \( \xi_L \) verifying \( i_{\xi_L} \omega_L - d^{\tau_E E} E_L = 0 \).

(18)

In addition, one can check that \( i_{\xi_L} \omega_L = i_{\Delta} \omega_L \), which implies that \( \xi_L \) is a SODE section. Thus, the integral curves of \( \xi_L \) (that is, the integral curves of the vector field \( \rho_1(\xi_L) \)) are solutions of the Euler-Lagrange equations for \( L \). \( \xi_L \) is called the Euler-Lagrange section associated with \( L \).

From (15), we deduce that the Lagrangian \( L \) is regular if and only if the matrix

\[ (W_{AB}) = \left( \frac{\partial^2 L}{\partial y^A \partial y^B} \right) \]

is regular. Moreover, the local expression of \( \xi_L \) is

\[ \xi_L = y^A \epsilon^{(1)}_A + f^A \epsilon^{(2)}_A, \]

where the functions \( f^A \) satisfy the linear equations

\[ \frac{\partial^2 L}{\partial y^B \partial y^A} f^B + \frac{\partial^2 L}{\partial x^i \partial y^B} \rho^i_B y^B + \frac{\partial L}{\partial y^C} \xi^C_{AB} y^B - \rho^i_A \frac{\partial L}{\partial x^i} = 0, \quad \forall A. \]

Another possibility is when the matrix \( (W_{AB}) = \left( \frac{\partial^2 L}{\partial y^A \partial y^B} \right) \) is singular. This type of Lagrangian is called \textit{singular} or \textit{degenerate Lagrangian}. In such a case, \( \omega_L \) is not a symplectic section and Equation (18) has no solution, in general, and even if it exists it will not be unique. In the next subsection, we will give the extension of the classical Gotay-Nester-Hinds algorithm \cite{39} for symplectic systems on Lie algebroids given in \cite{12}, which in particular will be applied to optimal control problems.

For an arbitrary Lagrangian function \( L : E \to \mathbb{R} \), we introduce the Legendre transformation associated with \( L \) as the smooth map \( \text{leg}_L : E \to E^* \) defined by

\[ \text{leg}_L(e)(e') = \left. \frac{d}{dt} \right|_{t=0} L(e + te'), \]
for $e, e' \in E_x$. Its local expression is

$$leg_L(x^i, y^A) = (x^i, \frac{\partial L}{\partial y^A}).$$

(19)

The Legendre transformation induces a Lie algebroid morphism

$$\mathcal{T}leg_L: \mathcal{T}E \rightarrow \mathcal{T}E^*$$

over $\text{leg}_L: E \rightarrow E^*$ given by

$$(\mathcal{T}leg_L)(e, v) = (e, (T\text{leg}_L)(v)),$$

where $T\text{leg}_L: TE \rightarrow TE^*$ is the tangent map of $\text{leg}_L: E \rightarrow E^*$. We have that (see [51] for details)

$$(\mathcal{T} leg_L, leg_L)^*(\lambda_E) = \Theta_L, \quad (\mathcal{T} leg_L, leg_L)^*(\Omega_E) = \omega_L.$$ (20)

where $\lambda_E$ is the Liouville section introduced in (11) and $\Omega_E$ is the canonical symplectic section on $\mathcal{T}E^*$. On the other hand, from (19), it follows that the Lagrangian function $L$ is regular if and only if $\text{leg}_L: E \rightarrow E^*$ is a local diffeomorphism.

Next, we will assume that $L$ is hyperregular, that is, $\text{leg}_L: E \rightarrow E^*$ is a global diffeomorphism. Then, the pair $(\mathcal{T} leg_L, leg_L)$ is a Lie algebroid isomorphism. Moreover, we may consider the Hamiltonian function $H: E^* \rightarrow \mathbb{R}$ defined by

$$H = E_L \circ \text{leg}^{-1}_L$$

and the Hamiltonian section $\xi_H \in \Gamma(\mathcal{T}E^*)$ which is characterized by the condition

$$i_{\xi_H} \Omega_E = d^{\mathcal{T}E^*} EH.$$

The integral curves of the vector field $\rho_1(\xi_H)$ on $E^*$ satisfy the Hamilton equations for $H$

$$\frac{dx^i}{dt} = \rho^i_A \frac{\partial H}{\partial p_A}, \quad \frac{dp_A}{dt} = -\rho^i_A \frac{\partial H}{\partial x^i} - p_C \epsilon_{AB}^C \frac{\partial H}{\partial p_B}$$

for $i \in \{1, \ldots, m\}$ and $A \in \{1, \ldots, n\}$ (see [51]).

In addition, the Euler-Lagrange section $\xi_L$ associated with $L$ and the Hamiltonian section $\xi_H$ are $(\mathcal{T} leg_L, leg_L)$-related, that is,

$$\xi_H \circ \text{leg}_L = \mathcal{T}\text{leg}_L \circ \xi_L.$$

Thus, if $\gamma: I \rightarrow E$ is a solution of the Euler-Lagrange equations associated with $L$, then $\mu = \text{leg}_L \circ \gamma: I \rightarrow E^*$ is a solution of the Hamilton equations for $H$ and, conversely, if $\mu: I \rightarrow E^*$ is a solution of the Hamilton equations for $H$ then $\gamma = \text{leg}^{-1}_L \circ \mu$ is a solution of the Euler-Lagrange equations for $L$ (for more details, see [51]).

**Example 23.** Consider the Lie algebroid $E = TQ$, the fiber bundle of a manifold $Q$ of dimension $m$. If $(x^i)$ are local coordinates on $Q$, then $\left\{ \frac{\partial}{\partial x^i} \right\}$ is a local basis of $\mathfrak{X}(Q)$ and we have fiber local coordinates $(x^i, \dot{x}^i)$ on $TQ$. The corresponding structure functions are $\epsilon_{ij}^k = 0$ and $\rho_i^j = \delta_i^j$ for $i, j, k \in \{1, \ldots, m\}$. Therefore given a Lagrangian function $L: TQ \rightarrow \mathbb{R}$ the Euler-Lagrange equations associated to $L$ are

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) = \frac{\partial L}{\partial x^i}, \quad i = 1, \ldots, m.$$
Moreover, given a Hamiltonian function $H : T^*Q \to \mathbb{R}$, the Hamilton equations associated to $H$ are
\[
\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}, \quad i = 1, \ldots, m
\]
where $(x^i, p_i)$ are local coordinates on $T^*Q$ induced by the local coordinates $(x^i)$ and the local basis $\{dx^i\}$ of $T^*Q$ (see [5] for example).

**Example 24.** Consider as a Lie algebroid the finite dimensional Lie algebra $(\mathfrak{g}, [\cdot, \cdot]_g)$ over a point. If $e_A$ is a basis of $\mathfrak{g}$ and $\tilde{C}^C_{AB}$ are the structure constants of the Lie algebra, the structures constant of the Lie algebroid $\mathfrak{g}$ with respect to the basis $\{e_A\}$ are $C^C_{AB} = \tilde{C}^C_{AB}$ and $\rho^i_A = 0$. Denote by $(y^A)$ and $(\mu_A)$ the local coordinates on $\mathfrak{g}^*$ respectively, induced by the basis $\{e_A\}$ and its dual basis $\{e^A\}$ respectively.

Given a Lagrangian function $L : \mathfrak{g} \to \mathbb{R}$ then the Euler-Lagrange equations for $L$ are just the Euler-Poincaré equations
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial y^A} \right) = \frac{\partial L}{\partial y^B} C^B_{A} y^C.
\]
Given a Hamiltonian function $H : \mathfrak{g}^* \to \mathbb{R}$ the Hamilton equations on $\mathfrak{g}^*$ read as the Lie-Poisson equations for $H$

\[
\dot{\mu} = ad^*_{\mu^A} \mu
\]
(see [3] for example).

**Example 25.** Let $G$ be a Lie group and assume that $G$ acts free and properly on $M$. We denote by $\pi : M \to \tilde{M} = M/G$ the associated principal bundle. The tangent lift of the action gives a free and proper action of $G$ on $TM$ and $\tilde{TM} = TM/G$ is a quotient manifold. Then we consider the Atiyah algebroid $\tilde{TM}$ over $\tilde{M}$.

According to Example [4] the basis $\{\tilde{e}_1, \tilde{e}_B\}$ induce local coordinates $(x^i, y^j, \tilde{y}^B)$ on $\tilde{TM}$. Given a Lagrangian function $\ell : \tilde{TM} \to \mathbb{R}$ on the Atiyah algebroid $\tilde{TM} \to \tilde{M}$, the Euler-Lagrange equations for $\ell$ are
\[
\frac{\partial \ell}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial \ell}{\partial y^j} \right) = \frac{\partial \ell}{\partial \tilde{y}^D} \left( B^A_{ij} y^i + C^A_{DB} \tilde{y}^B \right) \quad \forall j,
\]
\[
\frac{d}{dt} \left( \frac{\partial \ell}{\partial \tilde{y}^B} \right) = \frac{\partial \ell}{\partial \tilde{y}^D} \left( C^A_{DB} \tilde{y}^D - C^A_{DB} \tilde{y}^D \right) \quad \forall B,
\]
which are the Lagrange-Poincaré equations associated to a $G$-invariant Lagrangian $L : TM \to \mathbb{R}$ (see [17] and [51] for example) where $C^A_{AB}$ are the structure constants of the Lie algebra according to Example [4].

### 3.2. Constraint algorithm for presymplectic Lie algebroids

In this section we introduce the constraint algorithm for presymplectic Lie algebroids given in [42] which generalizes the well-known Gotay-Nester-Hinds algorithm [39]. First we give a review of the Gotay-Nester-Hinds algorithm and then we introduce the construction given in [42] to the case of Lie algebroids.

#### 3.2.1. The Gotay-Nester-Hinds algorithm of constraints

In this subsection we will briefly review the constraint algorithm of constraints for presymplectic systems (see [38] and [39]).
Take the following triple \((M, \Omega, H)\) consisting of a smooth manifold \(M\), a closed 2-form \(\Omega\) and a differentiable function \(H : M \to \mathbb{R}\). On \(M\) we consider the equation

\[
i_X \Omega = dH. \tag{21}
\]

Since we are not assuming that \(\Omega\) is nondegenerate (that is, \(\Omega\) is not, in general, symplectic) then Equation (21) has no solution in general, or the solutions are not necessarily unique) solution \(X\). In this case, we say that the system admits global dynamics. Otherwise, we select the subset of points of \(M\) where such a solution exists. We denote by \(M_2\) this subset and we will assume that it is a submanifold of \(M = M_1\). Then the equations (21) admit a solution \(X\) defined at all points of \(M_2\), but \(X\) need not be tangent to \(M_2\), hence, does not necessarily induce a dynamics on \(M_2\). So we impose an additional tangency condition, and we obtain a new submanifold \(M_3\) along which there exists a solution \(X\), but, however, such \(X\) needs to be tangent to \(M_3\). Continuing this process, we obtain a sequence of submanifolds

\[\cdots M_s \hookrightarrow \cdots \hookrightarrow M_2 \hookrightarrow M_1 = M\]

where the general description of \(M_{i+1}\) is

\[M_{i+1} = \{ p \in M_i \text{ such that there exists } X_p \in T_p M_i \text{ satisfying } i_{X_p} \Omega(p) = dH(p) \}\].

If the algorithm ends at a final constraint submanifold, in the sense that at some \(s \geq 1\) we have \(M_{s+1} = M_s\). We will denote this final constraint submanifold by \(M_f\). It may still happen that \(\dim M_f = 0\), that is, \(M_f\) is a discrete set of points, and in this case the system does not admit a proper dynamics. But, in the case when \(\dim M_f > 0\), there exists a well-defined solution \(X\) of (21) along \(M_f\).

There is another characterization of the submanifolds \(M_l\) that we will useful in the sequel. If \(N\) is a submanifold of \(M\) then we define

\[TN^\perp = \{ Z \in T_p M, \ p \in N \text{ such that } \Omega(X, Z) = 0 \text{ for all } X \in T_p N \}\].

Then, at any point \(p \in M_l\) there exists \(X_p \in T_p M_l\) verifying \(i_{X_p} \Omega(p) = dH(p)\) if and only if \(\langle TM_l^\perp, dH \rangle = 0\) (see [38, 39]). Hence, we can define the \(l + 1\) step of the constraint algorithm as

\[M_{l+1} := \{ p \in M_l \text{ such that } \langle TM_l^\perp, dH \rangle(p) = 0 \} .\]

3.2.2. Constraint algorithm for presymplectic Lie algebroids. Let \(\tau_E : E \to M\) be a Lie algebroid and suppose that \(\Omega \in \Gamma(\wedge^2 E^*)\). Then, we can define the vector bundle morphism \(\flat_\Omega : E \to E^*\) (over the identity of \(M\)) as follows

\[\flat_\Omega(e) = i(e)\Omega(x), \text{ for } e \in E_x.\]

Now, if \(x \in M\) and \(F_x\) is a subspace of \(E_x\), we may introduce the vector subspace \(F_x^\perp\) of \(E_x\) given by

\[F_x^\perp = \{ e \in E_x | \Omega(x)(e, f) = 0, \forall f \in E_x \}\].

On the other hand, if \(\flat_\Omega_x = \flat_\Omega|_{E_x}\), it is easy to prove that

\[\flat_\Omega_x(F_x) \subseteq (F_x^\perp)^0, \tag{22}\]

where \((F_x^\perp)^0\) is the annihilator of the subspace \(F_x^\perp\). Moreover, using

\[\dim F_x^\perp = \dim E_x - \dim F_x + \dim (E_x^\perp \cap F_x) , \tag{23}\]

we obtain that

\[\dim (F_x^\perp)^0 = \dim F_x - \dim (E_x^\perp \cap F_x) = \dim (\flat_\Omega_x(E_x)).\]
Thus, from (22), we deduce that
\[ b_{\Omega^\epsilon}(F_x) = (F_x^\perp)^\perp. \] (24)

Next, we will assume that \( \Omega \) is a presymplectic 2-section (\( d^E\Omega = 0 \)) and that \( \alpha \in \Gamma(E^\ast) \) is a closed 1-section (\( d^E\alpha = 0 \)). Furthermore, we will assume that the kernel of \( \Omega \) is a vector subbundle of \( E \).

The dynamics of the presymplectic system defined by \( (\Omega, \alpha) \) is given by a section \( X \in \Gamma(E) \) satisfying the dynamical equation
\[ i_X \Omega = \alpha. \] (25)

In general, a section \( X \) satisfying (25) cannot be found in all points of \( E \). First, we look for the points where (25) has sense. We define
\[ M_1 = \{ x \in M \mid \exists e \in E_x : i(e)\Omega(x) = \alpha(x) \}. \] (26)

From (24), it follows that
\[ M_1 = \{ x \in M \mid \alpha(x)(e) = 0, \text{ for all } e \in \ker\Omega(x) = E_x^\perp \}. \] (26)

If \( M_1 \) is an embedded submanifold of \( M \), then we deduce that there exists \( X : M_1 \to E \) a section of \( \tau_E : E \to M \) along \( M_1 \) such that (25) holds. But \( \rho(X) \) is not, in general, tangent to \( M_1 \). Thus, we have to restrict to \( E_1 = \rho^{-1}(TM_1) \). We remark that, provided that \( E_1 \) is a manifold and \( \tau_1 = \tau_E \mid E_1 : E_1 \to M_1 \) is a vector bundle, \( \tau_1 : E_1 \to M_1 \) is a Lie subalgebroid of \( E \to M \).

Now, we must consider the subset \( M_2 \) of \( M_1 \) defined by
\[ M_2 = \{ x \in M_1 \mid \alpha(x)(e) = 0, \text{ for all } e \in (E_1)^\perp = (\rho^{-1}(T_x M_1))^\perp \}. \]

If \( M_2 \) is an embedded submanifold of \( M_1 \), then we deduce that there exists \( X : M_2 \to E_1 \) a section of \( \tau_1 : E_1 \to M_1 \) along \( M_2 \) such that (25) holds. However, \( \rho(X) \) is not, in general, tangent to \( M_2 \). Therefore, we have to restrict to \( E_2 = \rho^{-1}(TM_2) \). As above, if \( \tau_2 = \tau_E \mid E_2 : E_2 \to M_2 \) is a vector bundle, it follows that \( \tau_2 : E_2 \to M_2 \) is a Lie subalgebroid of \( \tau_1 : E_1 \to M_1 \).

Consequently, if we repeat the process, we obtain a sequence of Lie subalgebroids (by assumption)
\[ \ldots \hookrightarrow M_{k+1} \hookrightarrow M_k \hookrightarrow \ldots \hookrightarrow M_2 \hookrightarrow M_1 \hookrightarrow M_0 = M \]
and
\[ E_{k+1} = \rho^{-1}(TM_{k+1}). \]

If there exists \( k \in \mathbb{N} \) such that \( M_k = M_{k+1} \), then we say that the sequence stabilizes. In such a case, there exists a well-defined (but non necessarily unique) dynamics on the final constraint submanifold \( M_f = M_k \). We write
\[ M_f = M_{k+1} = M_k, \quad E_f = E_{k+1} = E_k = \rho^{-1}(TM_k). \]

Then, \( \tau_f = \tau_E \mid E_f : E_f \to M_f = M_k \) is a Lie subalgebroid of \( \tau_E : E \to M \) (the Lie algebroid restriction of \( E \) to \( E_f \)). From the construction of the constraint algorithm, we deduce that there exists a section \( X \in \Gamma(E_f) \), verifying (25). Moreover, if \( X \in \Gamma(E_f) \) is a solution of the equation (25), then every arbitrary solution is of
the form $X' = X + Y$, where $Y \in \Gamma(E_f)$ and $Y(x) \in \ker \Omega(x)$, for all $x \in M_f$. In addition, if we denote by $\Omega_f$ and $\alpha_f$ the restriction of $\Omega$ and $\alpha$, respectively, to the Lie algebroid $E_f \to M_f$, we have that $\Omega_f$ is a presymplectic 2-section and then any $X \in \Gamma(E_f)$ verifying Equation (25) also satisfies

$$i_X \Omega_f = \alpha_f$$

but, in principle, there are solutions of (28) which are not solutions of (25) since $\ker \Omega \cap E_f \subset \ker \Omega_f$.

**Remark 1.** Note that one can generalize the previous procedure to the general setting of implicit differential equations on a Lie algebroid. More precisely, let $\tau_E : E \to M$ be a Lie algebroid and $S \subset E$ be a submanifold of $E$ (not necessarily a vector subbundle). Then, the corresponding sequence of submanifolds of $E$ is

$$S_0 = S$$
$$S_1 = S_0 \cap \rho^{-1}(T\tau_E(S_0))$$
$$\vdots$$
$$S_{k+1} = S_k \cap \rho^{-1}(T\tau_E(S_k))$$
$$\vdots$$

In our case, $S_k = \rho^{-1}(TM_k)$ (equivalently, $M_k = \tau_E(S_k)$).

3.3. Vakonomic mechanics on Lie algebroids. In this section we will develop a geometrical description for second-order mechanics on Lie algebroids in the Skinner and Rusk formalism, given a general geometric framework for the previous results in this chapter and using strongly the results given in [42].

First, we will review the description of vakonomics mechanics on Lie algebroids given by Iglesias, Marrero, Martín de Diego and Sosa in [42]. After it we will introduce the notion of admissible elements on a Lie algebroid and we will particularize the previous construction to the case when the Lie algebroid is the prolongation of a Lie algebroid and the constraint submanifold is the set of admissible elements. Then we will obtain the second-order Skinner and Rusk formulation on Lie algebroids.

Let $\tau_E : \tilde{E} \to Q$ be a Lie algebroid of rank $n$ over a manifold $Q$ of dimension $m$ with anchor map $\rho : \tilde{E} \to TQ$ and $L : \tilde{E} \to \mathbb{R}$ be a Lagrangian function on $\tilde{E}$. Moreover, let $M \subset \tilde{E}$ be an embedded submanifold of dimension $n + m - \bar{m}$ such that $\tau_M = \tau_{\tilde{E}} \big|_M : M \to Q$ is a surjective submersion.

Suppose that $e$ is a point of $M$ with $\tau_M(e) = x \in Q$, $(x^i)$ are local coordinates on an open subset $U$ of $Q$, $x \in U$, and $\{e_A\}$ is a local basis of $\Gamma(\tilde{E})$ on $U$. Denote by $(x^i, y^A)$ the corresponding local coordinates for $\tilde{E}$ on the open subset $\tau_{\tilde{E}}^{-1}(U)$. Assume that

$$M \cap \tau_{\tilde{E}}^{-1}(U) \equiv \{(x^i, y^A) \in \tau_{\tilde{E}}^{-1}(U) \mid \Phi^\alpha(x^i, y^A) = 0, \alpha = 1, \ldots, \bar{m}\}$$

where $\Phi^\alpha$ are the local independent constraint functions for the submanifold $M$.

We will suppose, without loss of generality, that the $(\bar{m} \times n)$-matrix

$$\left(\frac{\partial \Phi^\alpha}{\partial y^B}\right)_{e \in M, \alpha = 1, \ldots, \bar{m}, B = 1, \ldots, n}$$

is of maximal rank.
Now, using the implicit function theorem, we obtain that there exists an open subset $\tilde{V}$ of $(\tilde{E}^*)^{-1}(U)$, an open subset $W \subseteq \mathbb{R}^{m+n}$ and smooth real functions $\Psi^\alpha : W \to \mathbb{R}$, $\alpha = 1, \ldots, \tilde{m}$, such that

$\mathcal{M} \cap \tilde{V} \equiv \{(x^i, y^A) \in \tilde{V} \mid y^\alpha = \Psi^\alpha(x^i, y^a), \text{ with } \alpha = 1, \ldots, \tilde{m} \text{ and } \tilde{m} + 1 \leq a \leq n\}.$

Consequently, $(x^i, y^A)$ are local coordinates on $\mathcal{M}$ and we will denote by $\tilde{L}$ the restriction of $L$ to $\mathcal{M}$.

Consider the Whitney sum of $\tilde{E}^*$ and $\tilde{E}$, that is, $W = \tilde{E} \oplus \tilde{E}^*$, and the canonical projections $pr_1 : \tilde{E} \oplus \tilde{E}^* \to \tilde{E}$ and $pr_2 : \tilde{E} \oplus \tilde{E}^* \to \tilde{E}^*$. Now, let $W_0$ be the submanifold $W_0 = pr_1^{-1}(\mathcal{M}) = \mathcal{M} \times_Q \tilde{E}^*$ and the restrictions $\pi_1 = pr_1|_{W_0}$ and $\pi_2 = pr_2|_{W_0}$. Also denote by $\nu : W_0 \to Q$ the canonical projection of $W_0$ over the base manifold.

Next, we consider the prolongation of the Lie algebroid $\tilde{E}$ over $\tau_{\tilde{E}^*} : \tilde{E}^* \to Q$ (respectively, $\nu : W_0 \to Q$). We will denote this Lie algebroid by $\mathcal{T}^*\tilde{E}$ (respectively, $\mathcal{T}^\nu \tilde{E}$). Moreover, we can prolong $\pi_2 : W_0 \to \tilde{E}^*$ to a morphism of Lie algebroids $\mathcal{T}\pi_2 : \mathcal{T}\tilde{E} \to \mathcal{T}^*\tilde{E}$ defined by $\mathcal{T}\pi_2 = (Id, \mathcal{T}\pi_2)$.

If $(x^i, p_A)$ are the local coordinates on $\tilde{E}^*$ associated with the local basis $\{e^A\}$ of $\Gamma(\tilde{E}^*)$, then $(x^i, p_A, y^a)$ are local coordinates on $W_0$ and we may consider the local basis $\{\tilde{e}^{(1)}_A, (\tilde{e}^A)^{(2)}, \epsilon^{(2)}_a\}$ of $\Gamma(\mathcal{T}^\nu \tilde{E})$ defined by

$$\tilde{e}^{(1)}_A(x, e^*) = \left(e^A(x), \rho^A_i \frac{\partial}{\partial x^i}\right)_{(\tilde{e}, e^*)}, \quad (\tilde{e}^A)^{(2)}(\tilde{e}, e^*) = \left(0, \frac{\partial}{\partial p_A}\right)_{(\tilde{e}, e^*)},$$

$$\epsilon^{(2)}_a(\tilde{e}, e^*) = \left(0, \frac{\partial}{\partial y^a}\right)_{(\tilde{e}, e^*)},$$

where $(\tilde{e}, e^*) \in W_0$ and $\nu(\tilde{e}, e^*) = x$. If $[,]^\nu$, $\rho^\nu$ is the Lie algebroid structure on $\mathcal{T}^\nu \tilde{E}$, we have that

$$[\tilde{e}^{(1)}_A, \tilde{e}^{(1)}_B] = C^{AB}_C \epsilon^{(1)}_C,$$

and the rest of the fundamental Lie brackets are zero. Moreover,

$$\rho^\nu(\tilde{e}^{(1)}_A) = \rho^A_i \frac{\partial}{\partial x^i}, \quad \rho^\nu((\tilde{e}^A)^{(2)}) = \frac{\partial}{\partial p_A}, \quad \rho^\nu(\epsilon^{(2)}_a) = \frac{\partial}{\partial y^a}.$$

The Pontryagin Hamiltonian $H_{W_0}$ is a function defined on $W_0 = \mathcal{M} \times_Q \tilde{E}^*$ given by

$$H_{W_0}(\tilde{e}, e^*) = \langle e^*, \tilde{e} \rangle - \tilde{L}(\tilde{e}),$$

or, in local coordinates,

$$H_{W_0}(x^i, y^a, p_A, y^a) = p_a y^a + p_A \Psi^a(x^i, y^a) - \tilde{L}(x^i, y^a). \quad (29)$$

Moreover, one can consider the presymplectic 2-section $\Omega_0 = (\mathcal{T}\pi_2, \pi_2)^* \Omega_\tilde{E}$, where $\Omega_\tilde{E}$ is the canonical symplectic section on $\mathcal{T}\tilde{E} \cdot \tilde{E}$ defined in Equation (12).

In local coordinates,

$$\Omega_0 = \tilde{e}^{(1)}_A \wedge (\tilde{e}^A)^{(2)} + \frac{1}{2} C^{AC}_{ABP} \tilde{e}^{(1)}_A \wedge (\tilde{e}^A)^{(2)} \wedge \tilde{e}^{(1)}_B, \quad (30)$$

where $\{\tilde{e}^{(1)}_A, (\tilde{e}^A)^{(2)}, \epsilon^{(2)}_a\}$ denotes the dual basis of $\{\tilde{e}^{(1)}_A, (\tilde{e}^A)^{(2)}, \epsilon^{(2)}_a\}$. Therefore, we have the triple $(\mathcal{T}^\nu \tilde{E}, \Omega_0, d^{\mathcal{T}^\nu \tilde{E}} H_{W_0})$ as a presymplectic hamiltonian system.
Definition 3.1. The vakonomic problem on Lie algebroids consists of finding the solutions for the equation

$$i_X \Omega_0 = d^\nu \tilde{E} H_{W_0};$$

that is, to solve the constraint algorithm for $$(\mathcal{T}^\nu \tilde{E}, \Omega_0, d^\nu \tilde{E} H_{W_0})$$.

In local coordinates, we have that

$$d^\nu \tilde{E} H_{W_0} = \left( p_a \frac{\partial \Psi^a}{\partial x^i} - \frac{\partial \tilde{L}}{\partial x^i} \right) \rho_i^A \tilde{e}^a_A + \Psi^a \tilde{e}^a = \left( p_a + p_0 \frac{\partial \Psi^a}{\partial y^a} - \frac{\partial \tilde{L}}{\partial y^a} \right) e^a_a(2).$$

If we apply the constraint algorithm,

$$W_1 = \{ w \in \mathcal{M} \times \tilde{Q} \mid d^\nu \tilde{E} H_{W_0}(w)(Y) = 0, \forall Y \in \ker \Omega_0(w) \}. $$

Since $\ker \Omega_0 = \text{span} \{ e^a_a(2) \}$, we get that $W_1$ is locally characterized by the equations

$$\varphi_a = d^\nu \tilde{E} H_{W_0}(e^a_a) = p_a + p_0 \frac{\partial \Psi^a}{\partial y^a} - \frac{\partial \tilde{L}}{\partial y^a} = 0,$$

or

$$p_a = \frac{\partial \tilde{L}}{\partial y^a} - p_0 \frac{\partial \Psi^a}{\partial y^a}, \quad \tilde{m} + 1 \leq a \leq n.$$

Let us also look for the expression of $X$ satisfying Eq. (31). A direct computation shows that

$$X = \tilde{e}^a_A(1) + \Psi^a \tilde{e}^a(1) + \left( \frac{\partial \tilde{L}}{\partial x^i} - p_a \frac{\partial \Psi^a}{\partial x^i} \right) \rho_i^A - y^a e_A \rho_B - \Psi^a e_A \rho_B \right)(\tilde{e}^A)(2) + \mathcal{T}^a e_a(2).$$

Therefore, the vakonomic equations are

$$\begin{align*}
\dot{x}^i &= y^a \rho^i_a + \Psi^a \dot{\rho}^i_a, \\
\dot{p}_a &= \left( \frac{\partial \tilde{L}}{\partial x^i} - p_\beta \frac{\partial \Psi^\beta}{\partial x^i} \right) \rho^i_a - y^a e_A \rho_B - \Psi^\beta \rho^{A\beta} \rho_B, \\
\frac{d}{dt} \left( \frac{\partial \tilde{L}}{\partial y^a} - p_\alpha \frac{\partial \Psi^\alpha}{\partial y^a} \right) &= \left( \frac{\partial \tilde{L}}{\partial x^i} - p_a \frac{\partial \Psi^a}{\partial x^i} \right) \rho_\alpha - y^a e_A \rho_B - \Psi^\beta \rho^{B\alpha} \rho_B.
\end{align*}$$

Of course, we know that there exist sections $X$ of $\mathcal{T}^\nu \tilde{E}$ along $W_1$ satisfying (31), but they may not be sections of $(\nu^\nu)^{-1}(TW_1) = \mathcal{T}^{\nu_1} \tilde{E}$, in general (here $\nu_1 : W_1 \rightarrow Q$). Then, following the procedure detailed in Section 3.2.2, we obtain a sequence of embedded submanifolds

$$\ldots \hookrightarrow W_{k+1} \hookrightarrow W_k \hookrightarrow \ldots \hookrightarrow W_2 \hookrightarrow W_1 \hookrightarrow W_0 = \mathcal{M} \times \tilde{Q} \tilde{E}^*.$$

If the algorithm stabilizes, then we find a final constraint submanifold $W_f$ on which at least a section $X \in \Gamma(\mathcal{T}^\nu E)$ verifies

$$(i_X \Omega_0 = d^\nu \tilde{E} H_{W_0})|_{W_f}$$

where $\nu_f : W_f \rightarrow Q$.

One of the most important cases is when $W_f = W_1$. The authors of [42] have analyzed this case with the following result: Consider the restriction $\Omega_1$ of $\Omega_0$ to
3.4. Second-order variational problems on Lie algebroids. In this section we will study second-order variational problems on Lie algebroid. First we introduce the geometric objects for the formalism and then we study second-order unconstrained variational problems. After that, we will analyze the constrained case.

3.4.1. Prolongation of a Lie algebroid over a smooth map (cont’d). This subsection is devoted to studying some additional properties of the prolongation of a Lie algebroid over a smooth map (see subsection 2.2).

Let $E$ be a Lie algebroid over $Q$ with fiber bundle projection $\tau_E : \widetilde{E} \to Q$ and anchor map $\rho : \widetilde{E} \to TQ$. Also, let $\tau_E : E \to M$ be a Lie algebroid with anchor map $\rho : E \to TM$ and let $\tau^E$ be the $E$–tangent bundle to $E$. Now we will define the bundle $\tau^E$ over $\tau^E$. This bundle plays the role of $\tau_{TTM} : T(TTM) \to T(TM)$ in ordinary Lagrangian Mechanics.

In what follows we will describe the Lie algebroid structure of the $E$–tangent bundle to the prolongation Lie algebroid over $\tau_E : E \to Q$.

As we know from subsection 2.2, the basis of sections $\{e_A\}$ of $E$ induces a local basis of the sections of $\tau^E$ given by

$$e_A^{(1)}(e) = \left(e, e_A(\tau_E(e)), \rho_A^i \frac{\partial}{\partial x^i}|_e \right), \quad e_A^{(2)}(e) = \left(e, 0, \frac{\partial}{\partial y^A}|_e \right),$$

for $e \in E$. From this basis we can induce local coordinates $(x^i, y^A, z^A, v^A)$ on $\tau^E$.

Now, from this basis, we can induce a local basis of sections of $\tau^E$ in the following way: consider an element $(e, v_b) \in \tau^E$, then define the components of the basis $\{e_A^{(1,1)}, e_A^{(2,1)}, e_A^{(1,2)}, e_A^{(2,2)}\}$ as

$$e_A^{(1,1)}(e, v_b) = \left(e, v_b, e_A^{(1)}(e), \rho_A^i \frac{\partial}{\partial x^i}|_{(e, v_b)} \right), \quad e_A^{(1,2)}(e, v_b) = \left(e, v_b, 0, \frac{\partial}{\partial z^A}|_{(e, v_b)} \right),$$

$$e_A^{(2,1)}(e, v_b) = \left(e, v_b, e_A^{(2)}(e), \frac{\partial}{\partial y^A}|_{(e, v_b)} \right), \quad e_A^{(2,2)}(e, v_b) = \left(e, v_b, 0, \frac{\partial}{\partial v^A}|_{(e, v_b)} \right).$$

The basis $\{e_A^{(1,1)}, e_A^{(2,1)}, e_A^{(1,2)}, e_A^{(2,2)}\}$ induces local coordinates $(x^i, y^A, z^A, v^A, b^A, c^A, d^A, w^A)$ on $\tau^{(1,1)}(\tau^E)$.

If we denote by $\tau^{(1)}(\tau^E), \{\cdot, \cdot\}_{\tau_E}^{(2)}, \rho_2$ the Lie algebroid structure of the fiber bundle $\tau^{(1)}(\tau^E)$, it is characterized by

$$\rho_2(e_A^{(1,1)}(e, v_b) = \left(e, v_b, \rho_A^i \frac{\partial}{\partial x^i}|_{(e, v_b)} \right), \quad \rho_2(e_A^{(2,1)}(e, v_b) = \left(e, v_b, \frac{\partial}{\partial y^A}|_{(e, v_b)} \right),$$

$$\rho_2(e_A^{(1,2)}(e, v_b) = \left(e, v_b, \frac{\partial}{\partial z^A}|_{(e, v_b)} \right), \quad \rho_2(e_A^{(2,2)}(e, v_b) = \left(e, v_b, \frac{\partial}{\partial v^A}|_{(e, v_b)} \right),$$

$$[e_A^{(1,1)}, e_B^{(1,1)}]_{\tau_E} = [e_A^{(1,1)}, e_B^{(1,1)}]_{\tau_E} = [e_A^{(1,1)}, e_B^{(1,2)}]_{\tau_E} = [e_A^{(1,2)}, e_B^{(1,2)}]_{\tau_E} = 0,$$

$$[e_A^{(1,1)}, e_A^{(2,2)}]_{\tau_E} = [e_A^{(2,1)}, e_A^{(2,1)}]_{\tau_E} = [e_A^{(1,2)}, e_A^{(2,1)}]_{\tau_E} = [e_A^{(1,1)}, e_B^{(2,1)}]_{\tau_E} = 0.$$
for all \(A, B\) and \(C\) where \(c_{AB}^C\) are the structure constants of \(E\).

In the same way, from the basis \(\{e_A^{(1)}, (\bar{e}_A)^{(2)}\}\) of sections of \(T^*E\) given by

\[
\bar{e}_A^{(1)}(e^*) = \left( e^*, e_A(\tau_E^*(e^*)), \rho_A^I \frac{\partial}{\partial x^I} \right), \quad \bar{e}_A^{(2)}(e^*) = \left( e^*, 0, \frac{\partial}{\partial p_A^i} \right),
\]

where \(e^* \in E\), we construct the set \(\{e_A^{(1)}, (\bar{e}_A)^{(2)}\}\) of sections of \(\tau_{(\tau^*E)^*} : T^*E\). In what follows \((x^i, y^A, p_A, \bar{p}_A)\) denote local coordinates on \(T^*E\) induced by the basis \(\{e_A^{(1)}, (\bar{e}_A)^{(2)}\}\). This basis is given by

\[
\bar{e}_A^{(1)}(\alpha^*) = \left( \alpha^*, e_A^{(1)}(\tau_{(\tau^*E)^*}(\alpha^*)), \rho_A^I \frac{\partial}{\partial x^I} \right), \quad \bar{e}_A^{(2)}(\alpha^*) = \left( \alpha^*, 0, \frac{\partial}{\partial p_A^i} \right),
\]

\[
(\bar{e}_A)^{(2)}(\alpha^*) = \left( \alpha^*, e_A^{(2)}(\tau_{(\tau^*E)^*}(\alpha^*)), \rho_A^I \frac{\partial}{\partial p_A^i} \right), \quad \bar{e}_A^{(2)}(\alpha^*) = \left( \alpha^*, 0, \frac{\partial}{\partial p_A^i} \right),
\]

where \(\alpha^* \in (T^*E)^*\) and \(\tau_{(\tau^*E)^*} : (T^*E)^* \to E\) is the vector bundle projection.

The Lie algebroid structure \((T^*E)^* : (T^*E) ; [\cdot, \cdot], \rho_2\) is given by

\[
\rho_2(e_A^{(1)}(\alpha^*)) = \left( \alpha^*, \rho_A^I \frac{\partial}{\partial x^I} \right), \quad \rho_2((\bar{e}_A)^{(2)}(\alpha^*)) = \left( \alpha^*, \rho_A^I \frac{\partial}{\partial x^I} \right),
\]

\[
\rho_2((\bar{e}_A)^{(2)}(\alpha^*)) = \left( \alpha^*, \rho_A^I \frac{\partial}{\partial p_A^i} \right), \quad \rho_2(e_A^{(2)}(\alpha^*)) = \left( \alpha^*, \rho_A^I \frac{\partial}{\partial p_A^i} \right),
\]

where the unique non-zero Lie bracket is \([e_A^{(1)}, e_B^{(1)}]_2 = c_{AB}^C e_C^{(1)}\). This basis induces local coordinates \((x^i, y^A, p_A, \bar{p}_A, q^A, \bar{q}^A, \ell_A, \bar{\ell}_A)\) on \(T_{(\tau^*E)^*} : T^*E\).

3.4.2. Second-order unconstrained problem on Lie algebroids. Next, we will study second-order problem on Lie algebroids. Consider the Whitney sum of \((T^*E)^*\) and \(T^*E\), \(W = T^*E \times E (T^*E)^*\) and its canonical projections \(p_r : W \to T^*E\) and \(p_{r2} : W \to (T^*E)^*\). Now, let \(W_0\) be the submanifold \(W_0 = p_{r1}^{-1}(E(2)) = E(2) \times_E (T^*E)^*\) and the restrictions \(\pi_1 = p_1 \mid W_0\) and \(\pi_2 = p_{r2} \mid W_0\). Also we denote by \(\nu : W_0 \to E\) the canonical projection. The diagram in Figure 1 illustrates the situation.

\[
\begin{array}{c}
W_0 = E(2) \times_E (T^*E)^* \\
\pi_1 \quad \nu \quad \pi_2 \\
E \quad (T^*E)^* \\
\end{array}
\]

\[
\begin{array}{c}
E(2) \\
\tau^{(2,1)}_E \\
\end{array}
\]

\[
\begin{array}{c}
\quad T_{(\tau^*E)^*} \quad \\
\end{array}
\]

**Figure 1.** Second order Skinner and Rusk formalism on Lie algebroids.

Consider the prolongations of \(T^*E\) by \(\tau_{(\tau^*E)^*}\) and by \(\nu\), respectively. We will denote these Lie algebroids by \(T^*_{(\tau^*E)^*} : (T^*E)^*\) and \(T^* T^*E\) respectively. Moreover, we can prolong \(\pi_2 : W_0 \to (T^*E)^*\) to a morphism of Lie algebroids \(T^* \pi_2 : T^* T^*E \to T^*_{(\tau^*E)^*} : (T^*E)^*\) defined by \(T^* \pi_2 = (Id, T^* \pi_2)\).

We denote by \((x^i, y^A, p_A, \bar{p}_A)\) local coordinates on \((T^*E)^*\) induced by \(\{e_A^{(1)}, e_A^{(2)}\}\), the dual basis of the basis \(\{e_A^{(1)}, e_A^{(2)}\}\), a basis of \(T^*E\). Then, \((x^i, y^A, p_A, \bar{p}_A, z^A)\) are
local coordinates in $W_0$ and we may consider $\{\tilde{e}_A^{(1,1)}, \tilde{e}_A^{(2,1)}, (\tilde{e}_A^{(1,2)}), (\tilde{e}_A^{(2,2)}), e_A^{(1,2)}\}$, the local basis of $\Gamma(T^*T^E E)$ defined as
\[
\tilde{e}_A^{(1,1)}(\tilde{\alpha}, \alpha^*) = \left( (\tilde{\alpha}, \alpha^*), e_A^{(1)}(\tau_{(T^*E)E}^*, (\alpha^*)), \rho_A^i \frac{\partial}{\partial x^i} \bigg|_{(\tilde{\alpha}, \alpha^*)} \right),
\]
\[
(\tilde{e}_A^{(1,2)}(\tilde{\alpha}, \alpha^*) = \left( (\tilde{\alpha}, \alpha^*), 0, \frac{\partial}{\partial p_A} \bigg|_{(\tilde{\alpha}, \alpha^*)} \right),
\]
\[
\tilde{e}_A^{(2,1)}(\tilde{\alpha}, \alpha^*) = \left( (\tilde{\alpha}, \alpha^*), e_A^{(2)}(\tau_{(T^*E)E}^*, (\alpha^*)), \frac{\partial}{\partial y^A} \bigg|_{(\tilde{\alpha}, \alpha^*)} \right),
\]
\[
(\tilde{e}_A^{(2,2)}(\tilde{\alpha}, \alpha^*) = \left( (\tilde{\alpha}, \alpha^*), 0, \frac{\partial}{\partial z^A} \bigg|_{(\tilde{\alpha}, \alpha^*)} \right),
\]
for $\alpha^* \in (T^*E)^*, \tilde{\alpha} \in E^{(2)}$, $(\tilde{\alpha}, \alpha^*) \in W_0$, and $\tau_{(T^*E)E}^*: (T^*E)^* \rightarrow E$ is the canonical projection.

If $(\cdot, \cdot, \rho)$ is the Lie algebroid structure on $\mathcal{E}^{(1,1)} = e_A^{(1)}(1,1), e_B^{(1,1)}, \epsilon^{(1)} = e^{A\beta\gamma} e_A^{(1,1)}$, and the rest of the fundamental Lie brackets are zero. Moreover,
\[
\rho^\nu(\tilde{e}_A^{(1,1)}(\tilde{\alpha}, \alpha^*)) = \left( (\tilde{\alpha}, \alpha^*), \rho_A^i \frac{\partial}{\partial x^i} \bigg|_{(\tilde{\alpha}, \alpha^*)} \right),
\]
\[
\rho^\nu((\tilde{e}_A^{(1,2)}(\tilde{\alpha}, \alpha^*)) = \left( (\tilde{\alpha}, \alpha^*), \frac{\partial}{\partial p_A} \bigg|_{(\tilde{\alpha}, \alpha^*)} \right),
\]
\[
\rho^\nu(\tilde{e}_A^{(2,1)}(\tilde{\alpha}, \alpha^*)) = \left( (\tilde{\alpha}, \alpha^*), \frac{\partial}{\partial y^A} \bigg|_{(\tilde{\alpha}, \alpha^*)} \right),
\]
\[
\rho^\nu(\tilde{e}_A^{(2,2)}(\tilde{\alpha}, \alpha^*)) = \left( (\tilde{\alpha}, \alpha^*), \frac{\partial}{\partial z^A} \bigg|_{(\tilde{\alpha}, \alpha^*)} \right),
\]

The Pontryagin Hamiltonian $H_{W_0}$ is a function in $W_0$ given by
\[
H_{W_0}(\tilde{\alpha}, \alpha^*) = \langle \alpha^*, \tilde{\alpha} \rangle - L(\tilde{\alpha}),
\]
or in local coordinates
\[
H_{W_0}(x^i, y^A, p_A, p_B, z^A) = p_A z^A + p_A y^A - L(x^i, y^A, z^A).
\]

Moreover, one can consider the presymplectic 2-section $\Omega_0 = (\mathcal{P}_2, \pi_2)^*\Omega_E$, where $\Omega_E$ is the canonical symplectic section on $\mathcal{T}^*E$. In local coordinates,
\[
\Omega_0 = \tilde{e}_A^{(1,1)} \wedge (\tilde{e}_A^{(1,2)}) + e_A^{(2,1)} \wedge (\tilde{e}_A^{(2,2)}) + \frac{1}{2} \epsilon^{A\beta\gamma} e_A^{(1,1)} \wedge e_B^{(1,1)}.
\]

Here, the basis $\{\tilde{e}_A^{(1,1)}, \tilde{e}_A^{(2,1)}(\tilde{e}_A^{(1,2)}), (\tilde{e}_A^{(2,2)}), (\tilde{e}_A^{(1,2)}), (\tilde{e}_A^{(2,2)}, e_A^{(1,2)})\}$ denotes the dual basis of sections for $\mathcal{T}^*(T^*E)\wedge T^E$, denoted by $\{\tilde{e}_A^{(1,1)}, \tilde{e}_A^{(2,1)}, (\tilde{e}_A^{(1,2)}, (\tilde{e}_A^{(2,2)}, e_A^{(1,2)})\}$.

Therefore, the triple $(\mathcal{T}^*T^E E, \Omega_0, d^\mathcal{T}^*T^E E H_{W_0})$ is a presymplectic Hamiltonian system.

The second-order problem on the Lie algebroid $\tau_{E}: E \rightarrow M$ consists on finding the solutions of the equation
\[
i_X \Omega_0 = d^\mathcal{T}^*T^E E H_{W_0},
\]
that is, to solve the constraint algorithm for $(\mathcal{T}^*T^E E, \Omega_0, d^\mathcal{T}^*T^E E H_{W_0})$. 
In adapted coordinates,
\[
d^{T^* T^* E} H_{W_0} = - \rho_A^i dL \frac{\partial A}{\partial x_i} + \left( \tilde{e}_A^{(1,1)} + \left( \tilde{\rho}_A - \frac{\partial L}{\partial y^A} \right) \tilde{e}_A^{(2,1)} \right) + \left( \tilde{\bar{e}}_A - \frac{\partial L}{\partial z^A} \right) \tilde{e}_A^{(2,1)}.
\]

If we apply the constraint algorithm, since \( \ker \Omega_0 = \text{span} \{ \tilde{e}_A^{(2,1)} \} \) the first constraint submanifold \( W_1 \) is locally characterized by the equation
\[
\varphi_A = d^{T^* T^* E} H_{W_0} \left( \tilde{e}_A^{(2,1)} \right) = \tilde{\bar{e}}_A - \frac{\partial L}{\partial z^A} = 0,
\]
or
\[
\tilde{\bar{e}}_A = \frac{\partial L}{\partial z^A}.
\]
Looking for the expression of \( X \) satisfying the equation for the second-order problem we have that the second-order equations are
\[
\dot{x}^i = \rho_A^i y^A,
\]
\[
\dot{\tilde{p}}_A = \rho_A^i \frac{\partial L}{\partial x^i} + \tilde{e}_A^{iC} y^C,
\]
\[
\dot{\tilde{\bar{p}}}_A = -p_A + \frac{\partial L}{\partial y^A},
\]
\[
\tilde{\bar{p}}_A = \frac{\partial L}{\partial z^A}.
\]

After some straightforward computations the last equations are equivalent to the following equations:
\[
0 = \frac{d^2}{dt^2} \frac{\partial L}{\partial z^A} + \tilde{e}_A^{iC} y^C \frac{d}{dt} \left( \frac{\partial L}{\partial z^A} \right) - \frac{d}{dt} \frac{\partial L}{\partial y^A} - \tilde{e}_A^{iC} y^C \left( \frac{\partial L}{\partial y^A} \right) + \rho_A \frac{\partial L}{\partial x^i}.
\]
(32)

As in the previous section, it is possible to apply the constraint algorithm \( 3.2.2 \) to obtain a final constraint submanifold where we have at least a solution which is dynamically compatible. The algorithm is exactly the same but applied to the equation \( i_X \Omega_0 = d^{T^* T^* E} H_{W_0} \). Observe that the first constraint submanifold \( W_1 \) is determined by the conditions
\[
\varphi_A = \tilde{\bar{e}}_A - \frac{\partial L}{\partial z^A} = 0.
\]

If we denote by \( \Omega_{W_1} \) the pullback of the presymplectic 2-section \( \Omega_{W_0} \) to \( W_1 \), then \( \Omega_{W_1} \) is a symplectic section of the Lie algebroid \( T^{\nu_1} T^* E \) if and only if
\[
\left( \frac{\partial^2 L}{\partial z^A \partial z^B} \right)
\]
is nondegenerate along \( W_1 \), where \( \nu_1 = \nu \mid_{W_1} : W_1 \rightarrow E \). This can be deduced as an application of the theorem given in \( 42 \) explained in section \( 3.3 \) to the particular case when the \( M = E^{(2)} \).

**Example 26.** Observe that we can particularize the equations \( 32 \) to the case of Atiyah algebroids to obtain the second-order Lagrange-Poincaré equations.

Let \( G \) be a Lie group and we assume that \( G \) acts free and properly on \( M \). We denote by \( \pi : M \rightarrow \hat{M} = M/G \) the associated principal bundle. The tangent lift of the action gives a free and proper action of \( G \) on \( TM \) and \( T\hat{M} = TM/G \) is a quotient manifold. Then we consider the Atiyah algebroid \( T\hat{M} \) over \( \hat{M} \).
According to example 3, the basis \{\hat{e}_i, \hat{e}_B\} induce local coordinates \((x^i, y^A, \hat{y}^B)\). From this basis one can induces a basis of the prolongation Lie algebroid, namely \(\{\hat{e}^{(1)}_i, \hat{e}^{(1)}_B\}\). This basis induce adapted coordinates \((x^i, y^B, \hat{y}^i, \hat{y}^B)\) on \(T^{(2)}M = (TTM)/G\).

Given a Lagrangian function \(\ell : \hat{TTM} \to \mathbb{R}\) over the set of admissible elements of the Atiyah algebroid \(\hat{TTM} \to T\hat{M}\), where \(\hat{TTM} = (TTM)/G\), the Euler-Lagrange equations for \(\ell\) are

\[
\frac{\partial \ell}{\partial x^i} - \frac{d}{dt} \left( \frac{\partial \ell}{\partial \dot{x}^i} \right) + \frac{d^2}{dt^2} \left( \frac{\partial \ell}{\partial y^A} \right) = \left( \frac{d}{dt} \left( \frac{\partial \ell}{\partial \dot{y}^A} \right) - \frac{\partial \ell}{\partial y^A} \right) (B^{A}_{\bar{j}} y^j + c^{A}_{DB} A^B_{\bar{j}} \hat{y}^B) \quad \forall j,
\]

\[
\frac{d^2}{dt^2} \left( \frac{\partial \ell}{\partial B} \right) - \frac{d}{dt} \left( \frac{\partial \ell}{\partial \dot{y}^B} \right) = \left( \frac{d}{dt} \left( \frac{\partial \ell}{\partial \dot{y}^B} \right) - \frac{\partial \ell}{\partial y^B} \right) (c^{A}_{DB} \hat{y}^D - c^{A}_{DB} A^I_{\bar{i}} y^I) \quad \forall B
\]

which are the second-order Lagrange-Poincaré equations associated to a \(G\)-invariant Lagrangian \(L : T^{(2)}M \to \mathbb{R}\) (see [31] and [32]) where \(c^{A}_{AB}\) are the structure constants of the Lie algebra according to Example 3.

Observe that If \(G = \{e\}\), the identity of \(G\), \(\hat{T}^{(2)}M = T^{(2)}M\) and the second-order Lagrange-Poincaré equations become into the second-order Euler-Lagrange equations [19], [31]

\[
0 = \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \dot{x}^i} \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial y^A} \right) + \frac{\partial L}{\partial x^i}.
\]

If \(G = M\), \(\hat{T}^{(2)}M = 2g\) after a left-trivialization, and the second-order Lagrange-Poincaré equations become into the second-order Euler-Poincaré equations [20], [30], [31]

\[
0 = \frac{d^2}{dt^2} \left( \frac{\partial L}{\partial \dot{x}^i} \right) + c^{A}_{AB} y^B \frac{d}{dt} \left( \frac{\partial L}{\partial y^A} \right) - \frac{d}{dt} \left( \frac{\partial L}{\partial y^A} \right) + c^{A}_{AB} y^B \left( \frac{\partial L}{\partial x^i} \right).
\]

3.4.3. Second-order constrained problem on Lie algebroids. Now, we will consider second-order mechanical systems subject to second-order constraints. Let \(M \subset E^{(2)}\) be an embedded submanifold of dimension \(n + 2m - \hat{m}\) (locally determined by the vanishing of the constraint functions \(\Phi^\alpha : E^{(2)} \to \mathbb{R}, \alpha = 1, \ldots, \hat{m}\)) such that the bundle projection \(\pi^{(2,1)}_E|_M : M \to E\) is a surjective submersion.

We will suppose that the \((\hat{m} \times n)\) matrix \(\left( \frac{\partial \Phi^\alpha}{\partial y^a} \right)\) with \(\alpha = 1, \ldots, \hat{m}\) and \(B = 1, \ldots, n\) of maximal rank. Then, we will use the following notation \(z^A = (z^a, z^A)\) for \(1 \leq A \leq n, 1 \leq \alpha \leq \hat{m}\) and \(\hat{m} + 1 \leq a \leq n\). Therefore, using the implicit function theorem we can write

\[
z^a = \Psi^a(x^i, y^A, z^a).
\]

Consequently we can consider local coordinates on \(M\) by \((x^i, y^A, z^a)\) and we will denote by \(\hat{L}\) the restriction of \(L\) to \(M\).

**Proposition 2** [33]. Let \((E, \{\cdot, \cdot\}, \rho)\) be a Lie algebroid over a manifold \(M\) with projection \(\pi_E : E \to M\) and anchor map with constant rank. Consider a submanifold \(N\) of \(M\). If \(\pi_E|_{\rho^{-1}(TN)} : \rho^{-1}(TN) \to M\) is a vector subbundle, then \(\rho^{-1}(TN)\) is a Lie algebroid over \(N\).

Let us take the submanifold \(\hat{W}_0 = \rho^{-1}(\hat{M}) = M \times_{\hat{E}} (\hat{T}^{(2)}E)^\ast\) and the restrictions of \(\hat{W}_0\) of the canonical projections \(\pi_1\) and \(\pi_2\) given by \(\pi_1 = \rho^{-1}|_{\hat{W}_0}\) and \(\pi_2 = \rho^{-1}|_{\hat{W}_0}\). We will denote local coordinates on \(\hat{W}_0\) by \((x^i, y^A, \hat{p}_A, \hat{z}^a)\).
Therefore, proceeding as in the unconstrained case one can construct the presymplectic Hamiltonian system \((\overline{W}_0, \Omega_{\overline{W}_0}, H_{\overline{W}_0})\), where \(\Omega_{\overline{W}_0}\) is the presymplectic 2-section on \(\overline{W}_0\) and the Hamiltonian function \(H : \overline{W}_0 \to \mathbb{R}\) is locally given by
\[
H_{\overline{W}_0}(x^i, y^A, p_A, \overline{p}_A, z^\alpha) = p_A y^A + \overline{p}_a z^\alpha + \overline{p}_a \Psi^\alpha(x^i, y^A, z^\alpha) - \tilde{L}(x^i, y^A, z^\alpha).
\]

With these two elements it is possible to write the following presymplectic system
\[
i_X \Omega_{\overline{W}_0} = d(\rho')^{-1}(T\overline{W}_0) H_{\overline{W}_0}, \tag{33}
\]
where \((\rho')^{-1}(T\overline{W}_0)\) denotes the Lie subalgebroid of \(T^*E\) over \(\overline{W}_0 \subset W_0\).

To characterize the equations we will adopt an “extrinsic point of view”, that is, we will work on the full space \(W_0\) instead of in the restricted space \(\overline{W}_0\). Consider an arbitrary extension \(L : E^{(2)} \to \mathbb{R}\) of \(L_M : M \to \mathbb{R}\). The main idea is to take into account that Equation (33) is equivalent to
\[
\left\{ \begin{array}{l}
i_X \Omega_{\overline{W}_0} - d^{T^*TE} E H \in \text{ann} (\rho')^{-1}(T\overline{W}_0), \\
x \in (\rho')^{-1}(T\overline{W}_0) \text{ and } x \in \overline{W}_0,
\end{array} \right.
\]
where \(H : W_0 \to \mathbb{R}\) is the function defined in the last section and \text{ann} denotes the set of sections \(\tilde{X} \in \Gamma((T^*TE)^*)\) such that \(\langle \tilde{X}, Y \rangle = 0\) for all \(Y \in (\rho')^{-1}(T\overline{W}_0)\).

Assuming that \(M\) is determined by the vanishing of \(m\)-independent constraints
\[
\Phi^\alpha(x^i, y^A, z^\alpha) = 0, \quad 1 \leq \alpha \leq m,
\]
then, locally, \(\text{ann} (\rho')^{-1}(T\overline{W}_0) = \text{span} \{d^{T^*TE} E \Phi^\alpha\}\), and therefore the previous equations are rewritten as
\[
\left\{ \begin{array}{l}
i_X \Omega_{\overline{W}_0} - d^{T^*TE} E H = \lambda_\alpha d^{T^*TE} E \Phi^\alpha, \\
x(x) \in (\rho')^{-1}(T\overline{W}_0) \quad \text{for all } x \in \overline{W}_0,
\end{array} \right.
\]
where \(\lambda_\alpha\) are Lagrange multipliers to be determined.

Proceeding as in the previous section, one can obtain the following system of equations for \(\tilde{L} = L + \lambda_\alpha \Phi^\alpha\)
\[
0 = \frac{d^2}{dt^2} \frac{\partial \tilde{L}}{\partial z^A} + \epsilon_{ABC} y^B \frac{d}{dt} \left( \frac{\partial \tilde{L}}{\partial z^A} \right) - \frac{d}{dt} \frac{\partial \tilde{L}}{\partial y^B} \epsilon_{ABC} y^B \left( \frac{\partial \tilde{L}}{\partial y^A} \right) + \rho_A \frac{\partial \tilde{L}}{\partial x^A} \tag{34}
\]
\[
0 = \Phi^\alpha(x^i, y^A, z^A).
\]

Here the first constraint submanifold \(\overline{W}_1\) is determined by the condition
\[
0 = \overline{p}_A - \frac{\partial L}{\partial z^A} + \lambda_\alpha \frac{\partial \Phi^\alpha}{\partial z^A}
\]
\[
0 = \Phi^\alpha(x^i, y^A, z^A).
\]

If we denote by \(\Omega_{\overline{W}_1}\), the pullback of the presymplectic section \(\Omega_{\overline{W}_0}\) to \(\overline{W}_1\), it is possible to deduce that \(\Omega_{\overline{W}_1}\) is a symplectic section if and only if
\[
\begin{pmatrix}
\frac{\partial^2 L}{\partial z^A \partial z^B} + \lambda_\alpha \frac{\partial^2 \Phi^\alpha}{\partial z^A \partial z^B} & \frac{\partial \Phi^\alpha}{\partial z^A} \\
\frac{\partial \Phi^\alpha}{\partial z^B} & 0
\end{pmatrix}
\tag{35}
\]
is nondegenerate.
4. **Application to optimal control of mechanical systems.** In this section we study optimal control problems of mechanical systems defined on Lie algebroids. First we discuss fully actuated system and next underactuated systems. Optimality conditions for the optimal control of the controlled Elroy’s Beanie system are derived.

Optimal control problems can be seen as higher-order variational problems (see [5] and [6]). Higher-order variational problems are given by

$$
\min_{q(t)} \int_0^T L(q^i, \dot{q}^i, \ldots, q^{(k)i})dt,
$$

subject to boundary conditions. The relationship between higher-order variational problems and optimal control problems of mechanical systems comes from the fact that Euler-Lagrange equations are represented by a second-order Newtonian system and mechanical control systems have the form

$$
F(q^i, \dot{q}^i, \ddot{q}^i) = u,
$$

where $u$ are the control inputs. Then, if $C$ is a given cost function,

$$
\min_{(q(t), u(t))} \int_0^T C(q^i, \dot{q}^i, u)dt,
$$

is equivalent to a higher-order variational problem with $k = 2$.

4.1. **Optimal control problems of fully-actuated mechanical systems on Lie algebroids.** Let $(E, [\cdot, \cdot], \rho)$ be a Lie algebroid over $Q$ with bundle projection $\tau_E : E \to Q$. The dynamics is specified fixing a Lagrangian $L : E \to \mathbb{R}$. External forces are modeled, in this case, by curves $u_F : \mathbb{R} \to E^*$ where $E^*$ is the dual bundle $\tau_E^* : E^* \to Q$.

Given local coordinates $(q^i)$ on $Q$, and fixing a basis of sections $\{e_A\}$ of $\tau_E : E \to Q$ we can induce local coordinates $(q^i, y^A)$ on $E$; that is, every element $\xi \in E_q = \tau_E^{-1}(q)$ is expressed univocally as $\xi = y^A e_A(q)$.

It is possible to adapt the derivation of the Lagrange-d’Alembert principle to study fully-actuated mechanical controlled systems on Lie algebroids (see [27] and [62]). Let $q_0$ and $q_T$ fixed in $Q$, consider an admissible curve $\xi : I \subset \mathbb{R} \to E$ which satisfies the principle

$$
0 = \delta \int_0^T L(\xi(t))dt + \int_0^T \langle u_F(t), \eta(t) \rangle dt,
$$

where $\eta(t) \in E_{\tau_E(\xi(t))}$ and $u_F(t) \in E^*_{\tau_E(\xi(t))}$ defines the control force (where we are assuming they are arbitrary). The infinitesimal variations in the variational principle are given by $\delta \xi = \eta^C$, for all time-dependent sections $\eta \in \Gamma(\tau_E)$, with $\eta(0) = 0$ and $\eta(T) = 0$, where $\eta^C$ is a time-dependent vector field on $E$, the complete lift, locally defined by

$$
\eta^C = \rho^A_{\cdot} e_A^* \frac{\partial}{\partial q^i} + (\eta + e_B^{\cdot} e_C^* y^C) \frac{\partial}{\partial y^A}
$$

(see [27, 57, 59, 60]). Here the structure functions $e^A_{BC}$ are determined by $[e_B, e_C] = e^A_{BC} e_A$. 


From the Lagrange-d’Alembert principle one easily derives the controlled Euler-Lagrange equations by using standard variational calculus
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial y^A} \right) - \rho_A \frac{\partial L}{\partial q^i} + \mathcal{C}^C_{AB}(q) y^B \frac{\partial L}{\partial y^C} = (u_F)_A, \\
\frac{dq^i}{dt} = \rho_A y^A.
\]
where \((u_F)_A(t) = \langle u_F(t), e_A(q(t)) \rangle\) are the local components of \(u_F\) fixed the system of coordinates \((q^i)\) on \(Q\) and the basis of section \(\{e_A\}\).

The control force \(u_F\) is chosen such that it minimizes the cost functional
\[
\int_0^T C(q^i, y^A, (u_F)_A) dt,
\]
where \(C : E \oplus E^* \to \mathbb{R}\) is the cost function associated with the optimal control problem.

Therefore, the optimal control problem consists on finding an admissible curve \(\xi(t) = (q^i(t), y^A(t))\) solution of the controlled Euler-Lagrange equations, the boundary conditions and minimizing the cost functional for \(C : E \oplus E^* \to \mathbb{R}\). This optimal control problem can be equivalently solved as a second-order variational problem by defining the second-order Lagrangian \(\tilde{L} : E^{(2)} \to \mathbb{R}\) as
\[
\tilde{L}(q^i, y^A, z^A) = C \left( q^i, y^A, \frac{d}{dt} \left( \frac{\partial L}{\partial y^A} \right) - \rho_A \frac{\partial L}{\partial q^i} + \mathcal{C}^C_{AB}(q) y^B \frac{\partial L}{\partial y^C} \right)
\]
where we are considering local coordinates \((q^i, y^A, z^A)\) on \(E^{(2)}\).

Consider \(W_0 = E^{(2)} \times (T^*_{\mathbb{R}} E)^*\) with local coordinates \((q^i, y^A, p_a, \bar{p}_A, z^A)\). The optimality conditions are determined by
\[
\dot{q}^i = \rho_A y^A, \\
\dot{p}_A = \rho_A \frac{\partial C}{\partial q^i} + \mathcal{C}^C_{AB} p_c y^B, \\
\dot{\bar{p}}_A = -p_A + \frac{\partial C}{\partial y^A}, \\
\bar{p}_A = \frac{\partial C}{\partial z^A}.
\]

The constraint submanifold \(W_1\) is determined by \(\bar{p}_A - \frac{\partial C}{\partial z^A} = 0\). If the matrix
\[
\begin{pmatrix}
\frac{\partial^2 C}{\partial z^A \partial z^B}
\end{pmatrix}
\]
is non-singular then we can write the previous equations as an explicit system of ordinary differential equations. This regularity assumption is equivalent to the condition that the constraint algorithm stops at the first constraint submanifold \(W_1\). Proceeding as in the previous section, after some computations, the dynamics associated with the second-order Lagrangian \(\tilde{L} : E^{(2)} \to \mathbb{R}\) (and therefore the optimality conditions for the optimal control problem) is given by the second-order Euler-Lagrange equations on Lie algebroids
\[
\frac{d^2}{dt^2} \left( \frac{\partial \tilde{L}}{\partial z^A} \right) + \mathcal{C}^C_{AB}(q) y^B \frac{d}{dt} \left( \frac{\partial \tilde{L}}{\partial z^C} \right) - \frac{d}{dt} \frac{\partial \tilde{L}}{\partial y^A} - \mathcal{C}^C_{AB}(q) y^B \frac{\partial \tilde{L}}{\partial y^C} + \rho_A \frac{\partial \tilde{L}}{\partial q^i} = 0,
\]
(37)

together with the admissibility condition \(\frac{dq^i}{dt} = \rho_A y^A\).
Remark 2. Alternatively, one can define the Lagrangian \( \overline{L} : E^{(2)} \to \mathbb{R} \) in terms of the Euler-Lagrange operator as
\[
\overline{L} = C \circ (\tau^{E}\overline{L}) : E^{(2)} \to \mathbb{R},
\]
where \( \mathcal{L}(L) : E^{(2)} \to E^* \) is the Euler-Lagrange operator which locally reads as
\[
\mathcal{L}(L) = \left( \frac{d}{dt} \frac{\partial L}{\partial y^A} - \rho_A^i \frac{\partial L}{\partial y^i} + \mathcal{C}^A_{AB} \right) \frac{\partial L}{\partial y^B} \epsilon^A.
\]
Here \( \{ \epsilon^A \} \) is the dual basis of \( \{ e_A \} \), the basis of sections of \( E \) and \( \tau^{E} : E^{(2)} \to E \) is the canonical projection between \( E^{(2)} \) and \( E \) given by the map \( E^{(2)} \equiv (q^i, y^A, z^A) \mapsto (q^i, y^A) \in E \).

Example 27. An illustrative example: optimal control of a fully actuated rigid body and cubic splines on Lie groups

We consider the motion of a rigid body where the configuration space is the Lie group \( G = SO(3) \) and \( \mathfrak{so}(3) \equiv \mathbb{R}^3 \) its Lie algebra. The motion of the rigid body is invariant under \( SO(3) \). The reduced Lagrangian function for this system defined on the Lie algebroid \( E = \mathfrak{so}(3) \), \( \ell : \mathfrak{so}(3) \to \mathbb{R} \) is given by
\[
\ell(\Omega_1, \Omega_2, \Omega_3) = \frac{1}{2} (I_1 \Omega_2^2 + I_2 \Omega_3^2 + I_3 \Omega_1^2).
\]

Denote by \( t \mapsto R(t) \in SO(3) \) a curve. The columns of the matrix \( R(t) \) represent the directions of the principal axis of the body at time \( t \) with respect to some reference system. Consider the following left invariant control problem. First, we have the reconstruction equation:
\[
\dot{R}(t) = R(t) \begin{pmatrix}
0 & -\Omega_3(t) & \Omega_2(t) \\
\Omega_3(t) & 0 & -\Omega_1(t) \\
-\Omega_2(t) & \Omega_1(t) & 0
\end{pmatrix} = R(t) (\Omega_1(t) E_1 + \Omega_2(t) E_2 + \Omega_3(t) E_3)
\]
where
\[
E_1 := \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & -1 \\
0 & 1 & 0
\end{pmatrix}, \quad E_2 := \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}, \quad E_3 := \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]
and the equations for the angular velocities \( \Omega_i \) with \( i = 1, 2, 3 \):
\[
I_1 \dot{\Omega}_1(t) = (I_2 - I_3) \Omega_2(t) \Omega_3(t) + u_1(t)
\]
\[
I_2 \dot{\Omega}_2(t) = (I_3 - I_1) \Omega_3(t) \Omega_1(t) + u_2(t)
\]
\[
I_3 \dot{\Omega}_3(t) = (I_1 - I_2) \Omega_1(t) \Omega_2(t) + u_3(t)
\]
where \( I_1, I_2, I_3 \) are the moments of inertia and \( u_1, u_2, u_3 \) denote the applied torques playing the role of controls of the system.

The optimal control problem for the rigid body consists on finding the trajectories \( (R(t), \Omega(t), u(t)) \) with fixed initial and final conditions \( (R(0), \Omega(0)), (R(T), \Omega(T)) \) respectively and minimizing the cost functional
\[
\mathcal{A} = \int_0^T \mathcal{C}(\Omega, u_1, u_2, u_3) dt = \frac{1}{2} \int_0^T (u_1^2 + u_2^2 + u_3^2) dt.
\]

This optimal control problem is equivalent to solve the following second-order (unconstrained) variational problem
\[
\min \bar{\mathcal{J}} = \int_0^T \bar{L}(\Omega, \Omega) dt
\]
where

\[ \tilde{L}(\Omega, \dot{\Omega}) = C \left( \Omega, I_1 \dot{\Omega}_1 - (I_2 - I_3)\Omega_2 \Omega_3, I_2 \dot{\Omega}_2 - (I_3 - I_1)\Omega_3 \Omega_1, 
I_3 \dot{\Omega}_3 - (I_1 - I_2)\Omega_1 \Omega_2 \right). \]

Next, for simplicity, we consider the particular case \( I_1 = I_2 = I_3 = 1 \). The second order Lagrangian is given by

\[ \tilde{L}(\Omega, \dot{\Omega}) = \frac{1}{2} \left( \dot{\Omega}_1^2 + \dot{\Omega}_2^2 + \dot{\Omega}_3^2 \right). \]

The Pontryagin bundle is \( W_0 = 2so(3) \times 2so(3)^* \) with induced coordinates 
\((\Omega_1, \Omega_2, \Omega_3, \dot{\Omega}_1, \dot{\Omega}_2, \dot{\Omega}_3, p_1, p_2, p_3, \bar{p}_1, \bar{p}_2, \bar{p}_3)\).

The first constraint submanifold is given by

\[ W_1 = \{ \bar{p}_1 - \dot{\Omega}_1 = 0, \quad \bar{p}_2 - \dot{\Omega}_2 = 0, \quad \bar{p}_3 - \dot{\Omega}_3 = 0 \}. \]

Observe that

\[ \left( \frac{\partial^2 \tilde{L}}{\partial \Omega_A \partial \Omega_B} \right) = I_{3 \times 3} \]

where \( I_{3 \times 3} \) denotes the \( 3 \times 3 \) identity matrix. Thus, the constraint algorithm stops at the first constraint submanifold \( W_1 \).

We can write the equations of motion for the optimal control system as:

\[
\begin{align*}
\dot{p}_1 &= p_3 \Omega_2 - p_2 \Omega_3, \\
\frac{d}{dt} \Omega_1 &= \dot{\Omega}_1, \\
\dot{p}_2 &= p_1 \Omega_3 - p_3 \Omega_1, \\
\frac{d}{dt} \Omega_2 &= \dot{\Omega}_2, \\
\dot{p}_3 &= p_2 \Omega_1 - p_1 \Omega_2, \\
\frac{d}{dt} \Omega_3 &= \dot{\Omega}_3, \\
\dot{\bar{p}}_1 &= -p_1, \\
\dot{\bar{\Omega}}_1 &= \bar{p}_1, \\
\dot{\bar{p}}_2 &= -p_2, \\
\dot{\bar{\Omega}}_2 &= \bar{p}_2, \\
\dot{\bar{p}}_3 &= -p_3, \\
\dot{\bar{\Omega}}_3 &= \bar{p}_3.
\end{align*}
\]

After some straightforward computations, previous equations can be reduced to

\[
\begin{align*}
\ddot{\Omega}_1 &= \Omega_3 \ddot{\Omega}_2 - \Omega_2 \ddot{\Omega}_3, \\
\ddot{\Omega}_2 &= \Omega_1 \ddot{\Omega}_3 - \Omega_3 \ddot{\Omega}_1, \\
\ddot{\Omega}_3 &= \Omega_2 \ddot{\Omega}_1 - \Omega_1 \ddot{\Omega}_2.
\end{align*}
\]

or in short notation,

\[ \ddot{\Omega} = -\Omega \times \dot{\Omega}. \]

The previous equations are the equations given by L. Noakes, G. Heinzinger and B. Paden, \cite{66} for cubic splines on \( SO(3) \).

Finally, we would like to comment that the regularity condition provides the existence of a unique solution of the dynamics along the submanifold \( W_1 \). Therefore, there exists a unique vector field \( X \in \mathcal{X}(W_1) \) which satisfies \( i_X \Omega_{W_1} = dH_{W_1} \). In consequence, we have a unique control input which extremizes (minimizes) the objective function \( A \). If we take the flow \( F_t : W_1 \rightarrow W_1 \) of the vector field \( X \) then we have that \( F_t^* \Omega_{W_1} = \Omega_{W_1} \). Obviously, the Hamiltonian function

\[ H_{W_1}(\Omega, \dot{\Omega}, p, \bar{p}) = p_A \dot{\Omega}_A + p_A \dot{\Omega}_A - \frac{1}{2} \left( \dot{\Omega}_1^2 + \dot{\Omega}_2^2 + \dot{\Omega}_3^2 \right) \]
is preserved by the solution of the optimal control problem, that is \( \tilde{H} |_{W_1} \circ F_t = \tilde{H} |_{W_1} \).

4.2. Optimal control problems of underactuated mechanical systems on Lie algebroids. Now, suppose that our mechanical control system is underactuated, that is, the number of control inputs is less than the dimension of the configuration space. The class of underactuated mechanical systems is abundant in real life for different reasons; for instance, as a result of design choices motivated by the search of less cost engineering devices or as a result of a failure regime in fully actuated mechanical systems. Underactuated systems include spacecrafts, underwater vehicles, mobile robots, helicopters, wheeled vehicles and underactuated manipulators. In the general situation, the dynamics is specified fixed a Lagrangian \( L: E \to \mathbb{R} \) where \( (E, [\cdot, \cdot], \rho) \) is a Lie algebroid over a manifold \( Q \) with fiber bundle projection \( \tau_E: E \to Q \).

If we take local coordinates \( (q^i) \) on \( Q \) and a local basis \( \{e_A\} \) of sections of \( E \), then we have the corresponding local coordinates \( (q^i, y^A) \) on \( E \). Such coordinates determine the local structure functions \( \rho_A^i \) and \( C_{AB}^C \) and then the Euler-Lagrange equations on Lie algebroids can be written as

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial y^A} \right) - \rho_A^i \frac{\partial L}{\partial q^i} + C_{AB}^C y^B \frac{\partial L}{\partial y^A} = 0.
\]

These equations are precisely the components of the Euler-Lagrange operator \( \mathcal{E}L : E^{(2)} \to E^* \), which locally reads

\[
\mathcal{E}L = \left( \frac{d}{dt} \left( \frac{\partial L}{\partial y^A} \right) - \rho_A^i \frac{\partial L}{\partial q^i} + C_{AB}^C y^B \frac{\partial L}{\partial y^A} \right) e^A,
\]

where \( \{e^A\} \) is the dual basis of \( \{e_A\} \) (see [27]). In terms of the Euler-Lagrange operator, the equations of motion just read \( \mathcal{E}L = 0 \).

In the underactuated case, we model the set of control forces by the vector subbundle span\( \{e^a\} \subset E^* \) and the forces are given by \( u_F = (u_f)_a e^a \).

Now, we add controls in our picture. Assume that the controlled Euler-Lagrange equations are

\[
\left( \frac{d}{dt} \left( \frac{\partial L}{\partial y^A} \right) - \rho_A^i \frac{\partial L}{\partial q^i} + C_{AB}^C y^B \frac{\partial L}{\partial y^A} \right) e^A = u_a e^a,
\]

where we are denoting as \( \{e^A\} = \{e^a, e^\alpha\} \) the dual basis of \( \{e_A\} \) and \( u_a \) are admissible control parameters. Using the basis of sections of \( E \), equations (38) can be rewritten as

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial y^a} \right) - \rho_a^i \frac{\partial L}{\partial q^i} + C_{AB}^C y^B \frac{\partial L}{\partial y^a} = u_a,
\]

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial y^\alpha} \right) - \rho_\alpha^i \frac{\partial L}{\partial q^i} + C_{AB}^C y^B \frac{\partial L}{\partial y^\alpha} = 0.
\]

The optimal control problem consists on finding an admissible curve \( \gamma(t) = (q^i(t), y^A(t), u(t)) \) of the state variables and control inputs given initial and final boundary conditions \( (q^i(0), y^A(0)) \) and \( (q^i(T), y^A(T)) \), respectively, solving the controlled Euler-Lagrange equations (39) and minimizing

\[
\mathcal{A}(q^i, y^A, u_a) = \int_0^T C(q^i, y^A, u_a) dt,
\]

where \( C: E \times U \to \mathbb{R} \) denotes the cost function.
To solve this optimal control problem is equivalent to solve the following second-order problem:

\[
\min_{\tilde{L}} \tilde{L}(q^i(t), y^A(t), z^A(t))
\]
subject to \(\Phi^\alpha(q^i(t), y^A(t), z^A(t)), \alpha = 1, \ldots, m\)

where \(\tilde{L}, \Phi^\alpha \in C^\infty(E^{(2)})\). Here

\[
\tilde{L}(q^i(t), y^A(t), z^A(t)) = C(q^i(t), y^A(t), F_a(x^i(t), y^A(t), z^A(t)))
\]

where \(F_a(q^i(t), y^A(t), z^A(t)) = \frac{d}{dt} \left( \frac{\partial L}{\partial y^a} \right) - \rho^i_a \frac{\partial L}{\partial q^i} + C^C_a y^B \frac{\partial L}{\partial y^C}\).

The Lagrangian \(\tilde{L}\) is subjected to the second-order constraints:

\[
\Phi^\alpha(q^i(t), y^A(t), z^A(t)) = \frac{d}{dt} \left( \frac{\partial L}{\partial y^\alpha} \right) - \rho^i_{\alpha} \frac{\partial L}{\partial q^i} + C^C_{\alpha B} y^B \frac{\partial L}{\partial y^C},
\]

which determines a submanifold \(M\) of \(E^{(2)}\).

**Remark 3.** Note that the cost function is not completely defined in \(E \oplus E^*\), it is only defined in a smaller subset of this space because \(\frac{d}{dt} \left( \frac{\partial L}{\partial y^\alpha} \right) - \rho^i_{\alpha} \frac{\partial L}{\partial q^i} + C^C_{\alpha B} y^B \frac{\partial L}{\partial y^C}\) only belongs to the vector subbundle span\(\{e^a\}\) \(\subset E^*\). That is, in the case of fully actuated system the cost function would be defined in the full space \(E^*\), and when we are dealing with an underactuated systems, the cost function is defined in a proper subset of \(E^*\). Next, for simplicity, we assume that \(C : E \oplus E^* \to \mathbb{R}\).

Observe that from the constraint equations we have that

\[
\frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} z^\beta + \frac{\partial^2 L}{\partial y^\alpha \partial y^a} z^a - \rho^i_{\alpha} \frac{\partial L}{\partial q^i} + C^C_{\alpha B} y^B \frac{\partial L}{\partial y^C} = 0.
\]

Therefore, assuming that the matrix \(W_{\alpha \beta} = \left( \frac{\partial^2 L}{\partial y^\alpha \partial y^\beta} \right)\) is regular, we can write the equations as

\[
z^\alpha = -W_{\alpha \beta} \left( \frac{\partial^2 L}{\partial y^\alpha \partial y^a} z^a - \rho^i_{\beta} \frac{\partial L}{\partial q^i} + C^C_{\beta B} y^B \frac{\partial L}{\partial y^C} \right) = G^\alpha(q^i, y^A, z^a)
\]

where \(W_{\alpha \beta} = (W_{\alpha \beta})^{-1}\).

Therefore, we can choose coordinates \((q^i, y^A, z^a)\) on \(M\). This choose allows us to consider an intrinsic point of view, that is, to work directly on \(\tilde{W} = \tilde{M} \times (T^* E)^*\) avoiding the use of the Lagrange multipliers.

Define the restricted Lagrangian \(\tilde{L}_M\) by \(\tilde{L}|_M : \tilde{M} \to \mathbb{R}\) and take induced coordinates on \(\tilde{W}\), \((q^i, y^A, z^a, p_A, \tilde{p}_A)\). Applying the same procedure than in section 3.4.3 we derive the following system of equations
\[
\begin{align*}
\dot{q}^i &= \rho^i_A y^A, \\
\dot{y}^a &= z^a, \\
\dot{y}^a &= G^a(q^i, y^A, z^a), \\
\dot{p}_A &= \rho^i_A \left( \frac{\partial L_M}{\partial q^i} - \tilde{p}_\beta \frac{\partial G^\beta}{\partial y^\beta} \right) + \epsilon^C_{ABPC} y^B, \\
\dot{p}_A &= -p_A + \frac{\partial L_M}{\partial y^A} - \tilde{p}_\beta \frac{\partial G^\beta}{\partial y^\beta}, \\
\tilde{p}_a &= \frac{\partial L_M}{\partial z^a} - \bar{p}_\beta \frac{\partial G^\beta}{\partial z^\beta}.
\end{align*}
\]

To shorten the number of unknown variables involved in the previous set of equations, we can write them using as variables \((q^i, y^A, z^a, p_\alpha)\)
\[
\begin{align*}
\dot{q}^i &= \rho^i_A y^A, \\
\dot{y}^a &= G^a(q^i, y^A, z^a), \\
0 &= \frac{d^2}{dt^2} \left( \frac{\partial L_M}{\partial z^a} - \tilde{p}_\beta \frac{\partial G^\beta}{\partial z^\beta} \right) - \epsilon^b_{Aa} y^A \left( \frac{d}{dt} \left[ \frac{\partial L_M}{\partial z^b} - \bar{p}_\beta \frac{\partial G^\beta}{\partial z^\beta} \right] \right) - \epsilon^\gamma_{Aa} y^A \frac{d\tilde{p}_1}{dt} \\
&\quad + \epsilon^C_{Aa} y^A \left( \frac{\partial L_M}{\partial y^C} - \bar{p}_\beta \frac{\partial G^\beta}{\partial y^\beta} \right) + \rho^i_A \left( \frac{\partial L_M}{\partial q^i} - \tilde{p}_\beta \frac{\partial G^\beta}{\partial q^\beta} \right) - \frac{d}{dt} \left[ \frac{\partial L_M}{\partial y^A} - \bar{p}_\beta \frac{\partial G^\beta}{\partial y^\beta} \right], \\
0 &= \frac{d^2 p}_A dt^2 + \epsilon^b_{Aa} y^A \frac{d\bar{p}_\beta}{dt} - \epsilon^C_{Aa} y^A \left[ \frac{\partial L_M}{\partial y^C} - \bar{p}_\beta \frac{\partial G^\beta}{\partial y^\beta} \right] - \frac{d}{dt} \left[ \frac{\partial L_M}{\partial y^A} - \bar{p}_\beta \frac{\partial G^\beta}{\partial y^\beta} \right] \\
&\quad + \rho^i_A \left( \frac{\partial L_M}{\partial q^i} - \tilde{p}_\beta \frac{\partial G^\beta}{\partial q^\beta} \right) + \epsilon^b_{Aa} y^A \left( \frac{d}{dt} \left[ \frac{\partial L_M}{\partial z^b} - \bar{p}_\beta \frac{\partial G^\beta}{\partial z^\beta} \right] \right) \\
&\quad - \epsilon^b_{Aa} y^A \left[ \frac{\partial L_M}{\partial y^b} - \bar{p}_\beta \frac{\partial G^\beta}{\partial y^\beta} \right].
\end{align*}
\]

If the matrix
\[
\begin{pmatrix}
\frac{\partial^2 L_M}{\partial z^a \partial z^b}
\end{pmatrix}
\]
is regular then we can write the previous equations as an explicit system of third-order differential equations. This regularity assumption is equivalent to the condition that the constrain algorithm stops at the first constraint submanifold. In this submanifold there exists a unique solution for the boundary value problem determined by the optimal control problem.

**Example 28. Optimal control of an underactuated Elroy’s beanie:** This mechanical system is probably the simplest example of a dynamical system with a non-Abelian Lie group symmetry. It consists of two planar rigid bodies connected through their centers of mass (by a rotor let’s say) moving freely in the plane (see [5] and [67]). The main (i.e. more massive) rigid body has the capacity to apply a torque to the connected rigid body.
The configuration space is \( Q = SE(2) \times S^1 \) with coordinates \((x, y, \theta, \psi)\), where the first three coordinates describe the position and orientation of the center of mass of the first body and the last one describe the relative orientation between both bodies.

The Lagrangian \( L : TQ \to \mathbb{R} \) is
\[
L = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_2 (\dot{\theta} + \dot{\psi})^2 - V(\psi)
\]
where \( m \) denotes the mass of the system and \( I_1 \) and \( I_2 \) are the inertias of the first and the second body, respectively; additionally, we also consider a potential function of the form \( V(\psi) \). The kinetic energy is associated with the Riemannian metric \( \mathcal{G} \) on \( Q \) given by
\[
\mathcal{G} = m(dx^2 + dy^2) + (I_1 + I_2)d\theta^2 + I_2 d\psi \otimes d\psi + I_2 d\psi \otimes d\theta + I_2 d\psi^2.
\]

The system is \( SE(2) \)-invariant for the action
\[
\Phi_g(q) = (z_1 + x \cos \alpha - y \sin \alpha, z_2 + x \sin \alpha + y \cos \alpha, \alpha + \theta, \psi)
\]
where \( g = (z_1, z_2, \alpha) \).

Let \( \{\xi_1, \xi_2, \xi_3\} \) be the standard basis of \( \mathfrak{se}(2) \),
\[
[\xi_1, \xi_2] = 0, \quad [\xi_1, \xi_3] = -\xi_2 \quad , [\xi_2, \xi_3] = \xi_1
\]
The quotient space \( \hat{Q} = Q/SE(2) = (SE(2) \times S^1)/SE(2) \simeq S^1 \) is naturally parameterized by the coordinate \( \psi \). The Atiyah algebroid \( TQ/SE(2) \to \hat{Q} \) is identified with the vector bundle: \( \tau_{\hat{A}} : \hat{A} = TS^1 \times \mathfrak{se}(2) \to S^1 \). The canonical basis of sections of \( \tau_{\hat{A}} \) is:
\[
\left\{ \frac{\partial}{\partial \psi}, \xi_1, \xi_2, \xi_3 \right\}
\]
Since the metric \( \mathcal{G} \) is also \( SE(2) \)-invariant we obtain a bundle metric \( \hat{\mathcal{G}} \) and a \( \hat{\mathcal{G}} \)-orthonormal basis of sections:
\[
X_1 = \sqrt{\frac{I_1 + I_2}{I_1 I_2}} \left( \frac{\partial}{\partial \psi} - \frac{I_2}{I_1 + I_2} \xi_3 \right), \quad X_2 = \frac{1}{\sqrt{m}} \xi_1, \quad X_3 = \frac{1}{\sqrt{m}} \xi_2, \quad X_4 = \frac{1}{\sqrt{I_1 + I_2}} \xi_3.
\]
In the coordinates \((\psi, v^1, v^2, v^3, v^4)\) induced by the orthonormal basis of sections, the reduced Lagrangian is
\[
\hat{L} = \frac{1}{2} \left( (v^1)^2 + (v^2)^2 + (v^3)^2 + (v^4)^2 \right) - V(\psi).
\]

Additionally, we deduce that
\[
[X_1, X_2]_{\hat{A}} = -\sqrt{\frac{I_2}{I_1(I_1 + I_2)}} X_3, \quad [X_1, X_3]_{\hat{A}} = \sqrt{\frac{I_2}{I_1(I_1 + I_2)}} X_2, \\
[X_1, X_4]_{\hat{A}} = 0, \quad [X_2, X_3]_{\hat{A}} = 0, \\
[X_2, X_4]_{\hat{A}} = -\frac{1}{\sqrt{I_1 + I_2}} X_3, \quad [X_3, X_4]_{\hat{A}} = \frac{1}{\sqrt{I_1 + I_2}} X_2.
\]

Therefore, the non-vanishing structure functions are
\[
C^1_{12} = -\sqrt{\frac{I_2}{I_1(I_1 + I_2)}}, \quad C^3_{12} = \sqrt{\frac{I_2}{I_1(I_1 + I_2)}}, \quad C^3_{24} = -\frac{1}{\sqrt{I_1 + I_2}}, \quad C^3_{24} = \frac{1}{\sqrt{I_1 + I_2}}.
\]

Moreover,
\[
\rho_{\hat{A}}(X_1) = \sqrt{\frac{I_1 + I_2}{I_1 I_2}} \frac{\partial}{\partial \psi}, \quad \rho_{\hat{A}}(X_2) = 0, \quad \rho_{\hat{A}}(X_3) = 0, \quad \rho_{\hat{A}}(X_4) = 0.
\]
The local expression of the Euler-Lagrange equations for the reduced Lagrangian system $\bar{L}: \bar{A} \to \mathbb{R}$ is:

\[
\dot{\psi} = \sqrt{I_1 + I_2} v^1, \\
\dot{v^1} = -\sqrt{\frac{I_1 + I_2}{I_1 I_2}} \frac{\partial V}{\partial \psi} + \frac{1}{\sqrt{I_1 + I_2}} v^3 v^4, \\
\dot{v^2} = -\sqrt{\frac{I_2}{I_1 (I_1 + I_2)}} v^1 v^3 + \frac{1}{\sqrt{I_1 + I_2}} v^3 v^4, \\
\dot{v^3} = \sqrt{\frac{I_2}{I_1 (I_1 + I_2)}} v^1 v^2 - \frac{1}{\sqrt{I_1 + I_2}} v^3 v^4, \\
\dot{v^4} = 0.
\]

Next we introduce controls in our picture. Let $u(t) \in \mathbb{R}$ be a control input that permits to steer the system from an initial position to a desired position by controlling only the variable $\psi$. Therefore the controlled Euler-Lagrange equations are now

\[
\dot{\psi} = \sqrt{I_1 + I_2} v^1, \\
\dot{v^1} = \frac{1}{\sqrt{I_1 I_2}} \frac{\partial V}{\partial \psi} + u, \\
\dot{v^2} = -\sqrt{\frac{I_2}{I_1 (I_1 + I_2)}} v^1 v^3 + \frac{1}{\sqrt{I_1 + I_2}} v^3 v^4, \\
\dot{v^3} = \sqrt{\frac{I_2}{I_1 (I_1 + I_2)}} v^1 v^2 - \frac{1}{\sqrt{I_1 + I_2}} v^3 v^4, \\
\dot{v^4} = 0.
\]

From the second equation we obtain the feedback control law:

\[
u = \dot{v^1} + \sqrt{\frac{I_1 + I_2}{I_1 I_2}} \frac{\partial V}{\partial \psi}.
\]

The optimal control problem consists on finding trajectories of the states variables and controls inputs, satisfying the previous equations subject to given initial and final conditions and minimizing the cost functional,

\[
\min_{(v, \dot{v}, \psi, \dot{\psi}, u)} \int_0^T C(v, \psi, \dot{\psi}, u) dt = \min_{(\psi, \dot{\psi}, \Omega, u)} \int_0^T \frac{1}{2} u^2 dt
\]

where $v = (v^1, v^2, v^3, v^4)$.

Our optimal control problem is equivalent to solving the following second-order variational problem with second-order constraints given by

\[
\min_{(v, \dot{v}, \psi, \dot{\psi})} \bar{L}(v, \dot{v}, \psi, \dot{\psi}) = C \left( v, \psi, \dot{\psi}, \sqrt{\frac{I_1 + I_2}{I_1 I_2}} \frac{\partial V}{\partial \psi} \right).
\]
where $\ddot{L} : T^{(2)}S^1 \times 2\overline{SE}(2) \to \mathbb{R}$, subject the second-order constraints $\Phi^\alpha : T^{(2)}S^1 \times 2\overline{SE}(2) \to \mathbb{R}, \alpha = 1, \ldots, 4$, 

\[
\begin{align*}
\Phi^1(v, \dot{v}, \psi, \dot{\psi}, \ddot{\psi}) &= \dot{\psi} - \sqrt{\frac{I_2 + I_1}{I_2 I_4}} v^1, \\
\Phi^2(v, \dot{v}, \psi, \dot{\psi}, \ddot{\psi}) &= \dot{v}^2 - \frac{1}{\sqrt{I_1 + I_2}} v^3 v^4 + \sqrt{\frac{I_2}{I_1 (I_1 + I_2)}} v^1 v^3, \\
\Phi^3(v, \dot{v}, \psi, \dot{\psi}, \ddot{\psi}) &= \dot{v}^3 + \frac{1}{\sqrt{I_1 + I_2}} v^2 v^4 - \sqrt{\frac{I_2}{I_1 (I_1 + I_2)}} v^1 v^2, \\
\Phi^4(v, \dot{v}, \psi, \dot{\psi}, \ddot{\psi}) &= \dot{v}^4.
\end{align*}
\]

Therefore, the constraint submanifold $M$ of $T^{(2)}S^1 \times 2\overline{SE}(2)$ is given by 

\[
M = \left\{ (v, \dot{v}, \dot{\psi}) \mid \dot{\psi} = \sqrt{\frac{I_2 + I_1}{I_2 I_4}} v^1, \dot{v}^2 = \frac{1}{\sqrt{I_1 + I_2}} v^3 v^4 + \sqrt{\frac{I_2}{I_1 (I_1 + I_2)}} v^1 v^3, \right. \\
\left. \dot{v}^3 = -\frac{1}{\sqrt{I_1 + I_2}} v^2 v^4 + \sqrt{\frac{I_2}{I_1 (I_1 + I_2)}} v^1 v^2, \dot{v}^4 = 0 \right\}.
\]

We consider the submanifold $W_0 = M \times 2\overline{SE}(2)^*$ with induced coordinates 

\[(v^1, v^2, v^3, v^4, \psi, \dot{v}^1, p_1, p_2, p_3, p_4, \dot{\psi}, p_1, \dot{p}_2, \dot{p}_3, \dot{p}_4).\]

Now, we consider the restriction $\tilde{L}_M$ given by 

\[
\tilde{L}_M = \frac{1}{2} \left( \dot{v}^1 + \sqrt{\frac{I_1 + I_2}{I_1 I_2}} \frac{\partial V}{\partial \dot{v}} \right)^2.
\]

Moreover, the first constraint submanifold $W_1$ is determined by 

\[
W_1 = \left\{ z \in W_0 \mid \frac{\partial V}{\partial \dot{v}} + \dot{p}_1 - \sqrt{\frac{I_1 + I_2}{I_1 I_2}} \left( \frac{\partial V}{\partial \dot{v}} + \dot{p}_1 \right) - \sqrt{\frac{I_2}{I_1 (I_1 + I_2)}} (p_3 v^2 - p_2 v^3) = 0 \right\}.
\]

Observe that 

\[
\det \left( \frac{\partial^2 \tilde{L}_M}{\partial \dot{v}^1 \partial \dot{v}^4} \right) \neq 0.
\]

Thus, the constraint algorithm stops at the first constraint submanifold $W_1$. 

Finally, in a similar fashion as the unconstrained situation, we would like to point out that the regularity condition provides the existence of a unique solution of the dynamics along the submanifold $W_1$. 

Then, we can write the equations determining necessary conditions for the optimal control problem:
Leonardo Colombo

42 LEONARDO COLOMBO

L. C would like to thank the anonymous referees for the observations that have comments and discussions and to propose the initial ideas of this work. Finally, SPIRE 1343720. L. C would like to thank David Martín de Diego for fruitful discussions.

Acknowledgments. This work has been partially supported by NSF grant INSPIRE 1343720. L. C would like to thank David Martín de Diego for fruitful comments and discussions and to propose the initial ideas of this work. Finally, L. C would like to thank the anonymous referees for the observations that have improved the manuscript.

REFERENCES

[1] L. Abrunheiro, M. Camarinha, J. Carinena, J. Clemente-Gallardo, E. Martínez and P. Santos, Some applications of quasi-velocities in optimal control, International Journal of Geometric Methods in Modern Physics, 8 (2011), 835–861.

[2] M. Barbero Liñán, A. Echeverría-Enríquez, D. Martín de Diego, M.C. Muñoz-Lecanda and N. Román-Roy, Unified formalism for non-autonomous mechanical systems, J. Math. Phys., 49 (2008), 062902, 14pp.

[3] M. Barbero Liñán, M. de León, JC Marrero, D. Martín de Diego and M. Muñoz Lecanda, Kinematic reduction and the Hamilton-Jacobi equation, J. Geometric Mechanics, 4 (2012), 207–237.

[4] M. Barbero-Liñán, A. Echeverría Enríquez, D. Martín de Diego, M. C. Muñoz-Lecanda and N. Román-Roy, Skinner-Rusk unified formalism for optimal control systems and applications J. Phys. A: Math Theor., 40 (2007), 12071–12093.
A. M. Bloch, [Nonholonomic Mechanics and Control], Interdisciplinary Applied Mathematics Series, 24, Springer-Verlag, New York, 2003.

A. M. Bloch and P. E. Crouch, On the equivalence of higher order variational problems and optimal control problems, Proceedings of 35rd IEEE Conference on Decision and Control, (1996), 1648–1653.

A. Bloch, J. Marsden and D. Zenkov, Quasivelocities and symmetries in non-holonomic systems, Dynamical Systems, 24 (2009), 187–222.

A. J. Bruce, K. Grabowska and J. Grabowski, Higher order mechanics on graded bundles, J. Phys. A, 48 (2015), 205203, 32pp.

A. J. Bruce, Higher contact-like structures and supersymmetry, J. Phys. A: Math. Theor., 45 (2012), 265205, 12pp.

A. J. Bruce, K. Grabowska, J. Grabowski and P. Urbanski, New developments in geometric mechanics, Conference proceedings “Geometry of Jets and Fields” (Bedlewo, 10–16 May, 2015), Banach Center Publ., 110 (2016), 57–72, Polish Acad. Sci., Warsaw, 2016.

M. Camarinha, F. Silva-Leite and P. Crouch, Splines of class \( C^k \) on non-Euclidean spaces, IMA J. Math. Control Info., 12 (1995), 399–410.

C. M. Campos, M. de León, D. Martín de Diego and K. Vankerschaver, Unambiguous formalism for higher-order lagrangian field theories, J. Phys A: Math Theor., 42 (2009), 475207.

A. Cannas da Silva and A. Weinstein, Geometric Models for Noncommutative Algebras, Amer. Math. Soc., Providence, RI, 1999.

J. Cariñena, J. Nunes da Costa and P. Santos, Quasi-coordinates from the point of view of Lie algebroid, J. Phys. A: Math. Theor., 40 (2007), 10031–10048.

J. Cariñena and E. Martínez, Lie Algebroid Generalization of Geometric Mechanics, In Lie Algebroids and Related Topics in Differential Geometry, Banach Center Publications, 54 (2001), p201.

J. Cariñena and M. Rodríguez-Olmos, Gauge equivalence and conserved quantities for Lagrangian systems on Lie algebroids, Journal of Physics A: Mathematical and Theoretical, 42 (2009), 265209.

H. Cendra, J. Marsden and T. Ratiu, Lagrangian reduction by stages, Memoirs of the American Mathematical Society, 152 (2001), x+108 pp.

S. A. Chaplygin, On the Theory of Motion of Nonholonomic Systems. The Theorem on the Reducing Multiplier, Math. Sbornik XXVIII, 303314, (in Russian) 1911.

L. Colombo, D. Martín de Diego D and M. Zuccalli, Optimal control of underactuated mechanical systems: A geometrical approach, Journal Mathematical Physics, 51 (2010), 083519, 24pp.

L. Colombo and D. Martín de Diego, Higher-order variational problems on Lie groups and optimal control applications, J. Geom. Mech., 6 (2014), 451–478.

L. Colombo, M. de León, P. D. Prieto-Martínez and N. Rom an-Roy, Geometric Hamilton-Jacobi theory for higher-order autonomous systems J. Phys. A: Math. Theor., 47 (2014), 235203, 24pp.

L. Colombo, M. de León, P. D. Prieto-Martínez and N. Román-Roy, Unified formalism for the generalized kth-order Hamilton-Jacobi problem Int. J. Geom. Methods Mod. Phys., 11 (2014), 1460037, 9pp.

L. Colombo and P. D. Prieto-Martínez, Unified formalism for higher-order variational problems and its applications in optimal control Int. J. Geom. Methods Mod. Phys., 11 (2014), 1450034, 31pp.

L. A. Cordero, C. T. J. Dodson and M. de León, Differential Geometry of Frame Bundles: Mathematics and Its Applications, Kluwer, Dordrecht, 1989.

J. Cortés, S. Martínez and F. Cantrijn, Skinner-Rusk approach to time-dependent mechanics Phys. Lett. A, 300 (2002), 250–258.

J. Cortés, M. de León, J. C. Marrero and E. Martínez, Nonholonomic Lagrangian systems on Lie algebroids Discrete and Continuous Dynamical Systems - Series A, 24 (2009), 213–271.

J. Cortés and E. Martínez, Mechanical control systems on Lie algebroids SIAM J. Control Optim., 41 (2002), 1389–1412.

M. Crampin, W. Sarlet and F. Cantrijn, Higher order differential equations and higher order Lagrangian Mechanics Math. Proc. Camb. Phil. Soc., 99 (1986), 565–587.

A. Echeverría-Enríquez, C. López, J. Marín-Solano, M. C. Muñoz-Lecanda and N. Román-Roy, Lagrangian-Hamiltonian unified formalism for field theory, J. Math. Phys., 45 (2004), 360–380.
[30] F. Gay-Balmaz, D. D. Holm, D. Meier, T. Ratiu and F. Vialard, Invariant higher-order variational problems. Communications in Mathematical Physics, 309 (2012), 413–458.

[31] F. Gay-Balmaz, D. D. Holm, D. Meier, T. Ratiu and F. Vialard, Invariant higher-order variational problems II. Journal of Nonlinear Science, 22 (2012), 553–597.

[32] F. Gay-Balmaz, D. D. Holm and T. Ratiu, Higher-order Lagrange-Poincaré and Hamilton-Poincaré reductions. Bulletin of the Brazilian Math. Soc., 42 (2011), 579–606.

[33] K. Grabowska and J. Grabowski, Variational calculus with constraints on general algebroids. Journal of Physics A: Mathematical and Theoretical, 41 (2008), 175204, 25pp.

[34] K. Grabowska, P. Urbański and J. Grabowski, Tangent and cotangent lifts and graded Lie algebras associated with Lie algebroids. Ann. Global Anal. Geom., 15 (1997), 447–486.

[35] K. Grabowska and L. Vitagliano, Tulczyjew triples in higher derivative field theory. J. Geometric Mechanics, 7 (2015), 1–33.

[36] K. Grabowska, M. de León, J. C. Marrero and D. Martín de Diego, Linear almost Poisson structures and Hamilton-Jacobi equation. Applications to nonholonomic mechanics. J. Geom. Mech., 2 (2010), 159–198.

[37] M. de León and P. Rodrigues, Generalized Classical Mechanics and Field Theory, North-Holland Mathematical Studies 112, North-Holland, Amsterdam, 1985.

[38] M. de León, J. C. Marrero and E. Martínez, Lagrangian submanifolds and dynamics on Lie algebroids. J. Phys. A, 38 (2005), R241–R308.

[39] L. Machado, F. Silva Leite and K. Krakowski, Higher-order smoothing splines versus least squares problems on Riemannian manifolds. J. Dyn. Control Syst., 16 (2010), 121–148.

[40] K. Mackenzie, General Theory of Lie Groupoids and Lie Algebroids, London Mathematical society Lecture Notes, 213, Cambridge University Press, 2005.

[41] J. C. Marrero, Hamiltonian mechanical systems on Lie algebroids, unimodularity and preservation of volumes. J. Geometric Mechanics, 2 (2010), 243–263.

[42] J. C. Marrero, E. Padrón and M. Rodríguez-Olmos, Reduction of a symplectic-like Lie algebroid with momentum map and its application to fiberwise linear Poisson structures. J. Phys. A: Math. Theor., 45 (2012), 16520, 34pp.

[43] E. Martínez, Classical field theory on Lie algebroids: Variational aspects. J. Phys. A: Math. Gen., 38 (2005), 7145–7160.

[44] E. Martínez, Variational calculus on Lie algebroids. ESAIM: Control, Optimisation and Calculus of Variations, 14 (2008), 356–380.
[58] E. Martínez, Reduction in optimal control theory, Reports in Mathematical Physics, 53 (2004), 79–90.

[59] E. Martínez, Geometric formulation of Mechanics on Lie algebroids, In Proceedings of the VIII Fall Workshop on Geometry and Physics, Medina del Campo, 1999, Publicaciones de la RSME, 2 (2001), 209–222.

[60] E. Martínez, Lagrangian Mechanics on Lie algebroids, Acta Appl. Math., 67 (2001), 295–320.

[61] E. Martínez, Higher-order variational calculus on Lie algebroids, J. Geometric Mechanics, 7 (2015), 81–108.

[62] E. Martínez and J. Cortés, Lie algebroids in classical mechanics and optimal control, SIGMA Symmetry Integrability Geom. Methods Appl., 3 (2007), 17 pp.

[63] J. Maruskin and A. Bloch, The Boltzmann-Hamel equations for the optimal control of mechanical systems with nonholonomic constraints, International Journal of Robust and Nonlinear Control, 21 (2011), 373–386.

[64] D. Meier, Invariant Higher-Order Variational Problems: Reduction, Geometry and Applications, Ph.D Thesis, Imperial College London, 2013.

[65] T. Mestdag and B. Langerock, A Lie algebroid framework for non-holonomic systems, Journal of Physics A: Mathematical and General, 38 (2005), 1097–1111.

[66] L. Noakes, G. Heinzinger and B. Paden, Cubic splines on curved spaces, IMA Journal of Mathematical Control & Information, 6 (1989), 465–473.

[67] J. P. Ostrowski, Computing reduced equations for robotic systems with constraints and symmetries, IEEE Transactions on Robotic and Automation, 15 (1999), 111–123.

[68] L. Popescu, The geometry of Lie algebroids and its applicationsto optimal control, Preprint, arXiv:1302.5212, 2013.

[69] P. Popescu, On higher order geometry on anchored vector bundles, Central European Journal of Mathematics, 2 (2004), 826–839.

[70] M. Popescu and P. Popescu, Geometric objects defined by almost Lie structures, In Proceedings of the Workshop on Lie algebroids and related topics in Differential Geometry, Warsaw 2001, Banach Center Publ., 54 (2001), 217–233.

[71] P. D. Prieto-Martínez and N. Román-Roy, Lagrangian-Hamiltonian unified formalism for autonomous higher-order dynamical systems, J. Phys. A, 44 (2011), 385203, 35pp.

[72] P. D. Prieto-Martínez and N. Román-Roy, Unified formalism for higher-order non-autonomous dynamical systems, J. Math. Phys., 53 (2012), 032901, 38pp.

[73] D. Saunders, Prolongations of Lie groupoids and Lie algebroids, Houston J. Math., 30 (2004), 637–655.

[74] R. Skinner and R. Rusk, Generalized Hamiltonian dynamics I. Formulation on $T^*Q \oplus TQ$, Journal of Mathematical Physics, 24 (1983), 2589–2594.

[75] L. Vitagliano, The Lagrangian-Hamiltonian formalism for higher order field theories, Journal of Geometry and Physics, 60 (2010), 857–873.

[76] A. Weinstein, Lagrangian Mechanics and groupoids, Fields Inst. Comm., 7 (1996), 207–234.

Received June 2016; revised January 2017.

E-mail address: ljcolomb@umich.edu