A new method to generate superoscillating functions and supershifts

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Superoscillations are band-limited functions that can oscillate faster than their fastest Fourier component. These functions (or sequences) appear in weak values in quantum mechanics and in many fields of science and technology such as optics, signal processing and antenna theory. In this paper, we introduce a new method to generate superoscillatory functions that allows us to construct explicitly a very large class of superoscillatory functions.

1. Introduction

Superoscillating functions are band-limited functions that can oscillate faster than their fastest Fourier component. Physical phenomena associated with superoscillatory functions have been known for a long time and in more recent years there has been a wide interest both from the physical and the mathematical point of view. These functions (or sequences) appeared in weak values in quantum mechanics, see [1–3], in antenna theory this phenomenon was formulated in [4]. The literature on superoscillations is large, and without claiming completeness we mention the papers [5–13]. This class of functions has been investigated also from
the mathematical point of view, as function theory, but a large part of the results are associated
with the study of the evolution of superoscillations by quantum field equations with particular
attention to the Schrödinger equation. We give a quite complete list of papers [14–30] where one
can find an up-to-date panorama of this field. In order to have an overview of the techniques
developed in recent years to study the evolution of superoscillations and their function theory,
we refer the reader to the introductory papers [22,31–33]. Finally, we mention the Roadmap
on superoscillations, see [34], where the most recent advances in superoscillations and their
applications to technology are well explained by the leading experts in this field.

A fundamental problem is to determine how large the class of superoscillatory functions is.
The prototypical superoscillating function that is the outcome of weak values is given by

\[ F_n(x,a) = \left( \cos \left( \frac{x}{n} \right) + ia \sin \left( \frac{x}{n} \right) \right)^n = \sum_{j=0}^{n} C_j(n,a) e^{i(1-2j/n)x}, \quad x \in \mathbb{R}, \]  

(1.1)

where \( a > 1 \) and the coefficients \( C_j(n,a) \) are given by

\[ C_j(n,a) = \binom{n}{j} \left( \frac{1+a^2}{2} \right)^{n-j} \left( \frac{1-a^2}{2} \right)^{j}. \]  

(1.2)

If we fix \( x \in \mathbb{R} \) and we let \( n \) go to infinity, we obtain that

\[ \lim_{n \to \infty} F_n(x,a) = e^{iax}. \]

Clearly, the name superoscillations comes from the fact that in Fourier’s representation of
the function (1.1) the frequencies \( 1 - 2j/n \) are bounded by 1, but the limit function \( e^{iax} \) has a frequency
\( a \) that can be arbitrarily larger than 1. A precise definition of superoscillating functions follows.

We call \textit{generalized Fourier sequence} a sequence of the form

\[ \tilde{f}_n(x) := \sum_{j=0}^{n} X_j(n,a) e^{ih_j(n)x}, \quad n \in \mathbb{N}, \ x \in \mathbb{R}, \]  

(1.3)

where \( a \in \mathbb{R}, X_j(n,a) \) and \( h_j(n) \) are complex and real-valued functions of the variables \( n,a \) and
\( n \), respectively. A generalized Fourier sequence of the form (1.3) is said to be a \textit{superoscillating
sequence} if \( \sup_{j,n} |h_j(n)| \leq 1 \) and there exists a compact subset of \( \mathbb{R} \), which will be called a
\textit{superoscillation set}, on which \( \tilde{f}_n(x) \) converges uniformly to \( e^{ig(a)x} \), where \( g \) is a continuous
real-valued function such that \( |g(a)| > 1 \).

The classical Fourier expansion is obviously not a superoscillating sequence since its
frequencies are not, in general, bounded. Using infinite-order differential operators we can
define a class of superoscillatory function applying them to the functions (1.1) and we obtain
superoscillating functions of the form

\[ \tilde{Y}_n(x,a) = \sum_{j=0}^{n} C_j(n,a) e^{ig(1-2j/n)x}, \]  

(1.4)

where \( C_j(n,a) \) are the coefficients in (1.2), \( g \) are given entire functions, monotone increasing and
\( x \in \mathbb{R} \). We have shown that

\[ \lim_{n \to \infty} \tilde{Y}_n(x,a) = e^{ig(a)x} \]

under suitable conditions on \( g \) and the simplest, but important, example is

\[ \tilde{Y}_n(x,a) = \sum_{j=0}^{n} C_j(n,a) e^{i(1-2j/n)^m x}, \quad \text{for fixed } m \in \mathbb{N}. \]

From the above considerations, we deduce that there exists a large class of superoscillating
functions taking different functions \( g \).
In this paper, we further enlarge the class of superoscillating functions enlarging both the class of the coefficients \( C_j(n, a) \) and of the sequence of frequencies \( 1 - 2j/n \) that are bounded by 1. A large class of superoscillating functions can be determined solving the following problem.

**Problem 1.1.** Let \( h_j(n) \) be a given set of points in \([-1, 1]\), \( j = 0, 1, \ldots, n \), for \( n \in \mathbb{N} \) and let \( a \in \mathbb{R} \) be such that \(|a| > 1\). Determine the coefficients \( X_j(n) \) of the sequence

\[
 f_n(x) = \sum_{j=0}^{n} X_j(n) e^{ih_j(n)x}, \quad x \in \mathbb{R}
\]

in such a way that

\[
 f_n^{(p)}(0) = (ia)^p, \quad \text{for } p = 0, 1, \ldots, n.
\]

**Remark 1.2.** The conditions \( f_n^{(p)}(0) = (ia)^p \) mean that the functions \( x \mapsto e^{iax} \) and \( f_n(x) \) have the same derivatives at the origin, for \( p = 0, 1, \ldots, n \), so they have the same Taylor polynomial of order \( n \).

Under the condition that the points \( h_j(n) \) for \( j = 0, \ldots, n \), (often denoted by \( h_j \)) are distinct we obtain an explicit formula for the coefficients \( X_j(n, a) \) given by

\[
 X_j(n, a) = \prod_{k=0, k \neq j}^{n} \left( \frac{h_k(n) - a}{h_k(n) - h_j(n)} \right),
\]

so the superoscillating sequence \( f_n(x) \), that solves problem 1.1, takes the explicit form

\[
 f_n(x) = \sum_{j=0}^{n} \prod_{k=0, k \neq j}^{n} \left( \frac{h_k(n) - a}{h_k(n) - h_j(n)} \right) e^{ih_j(n)x}, \quad x \in \mathbb{R},
\]

as shown in theorem 2.2. Observe that, by construction, this function is band limited and it converges to \( e^{iax} \) with arbitrary \(|a| > 1\), so it is superoscillating.

Observe that different sequences \( X_j(n) \) can be explicitly computed when we fix the points \( h_j(n) \). See, for example, the case of the sequence

\[
 h_j(n) = 1 - \frac{2j}{n^p} \quad \text{for } j = 0, \ldots, n, \quad n \in \mathbb{N} \text{ and for fixed } p \in \mathbb{N},
\]

in §3.

We consider now the frequencies \( h_j(n) = (1 - 2j/n)^m \), for fixed \( m \in \mathbb{N} \), to explain some facts.

(I) If we consider the sequence (1.4), with coefficients \( C_j(n, a) \) given by (1.2), we obtain

\[
 \lim_{n \to \infty} \sum_{j=0}^{n} C_j(n, a) e^{i(1 - 2j/n)^m x} = e^{ia^m x}, \quad \text{for fixed } m \in \mathbb{N}.
\]

Note that, in this case, we could have used the frequencies \( h_j(n) = (1 - 2j/n) \) and the coefficients \( \tilde{C}_j(n, a) := C_j(n, a^m) \) to get as limit function \( e^{ia^m x} \). Thus the same limit function \( e^{ia^m x} \) can be obtained by tuning the frequencies and the coefficients.

(II) By solving problem 1.1 with the frequencies \( h_j(n) = (1 - 2j/n)^m \), we can determine the coefficients \( X_j(n) = X_j(n, a) \) such that we obtain as limit function \( e^{iax} \), namely

\[
 \lim_{n \to \infty} \sum_{j=0}^{n} X_j(n, a) e^{i(1 - 2j/n)^m x} = e^{i a x}, \quad \text{for fixed } m \in \mathbb{N}. \tag{1.5}
\]

Changing the coefficients \( \tilde{X}_j(n, a) \) we can get, as limit function, \( e^{ia^m x} \).

(III) The coefficients \( X_j \) and \( C_j \) in the procedures (I) and (II) are different from each other because the two methods to generate superoscillations are different, as explained in §3.
In §4, we will also discuss how to generalize this method to obtain analogous results in the case of the supershift property of functions, a mathematical concept that generalizes the notion of superoscillating function.

2. A new class of superoscillating functions

In this section, we show the main procedure to determine the coefficients $X_j(n)$ and so to construct explicitly the superoscillating functions solving problem 1.1.

**Theorem 2.1 (Existence and uniqueness of the solution of problem 1.1).** Let $h_j(n)$ be a given set of points in $[-1, 1], j = 0, 1, \ldots, n$ for $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that $|a| > 1$. If $h_j(n) \neq h_i(n)$, for every $i \neq j$, then there exists a unique solution $X_j(n)$ of the linear system

$$f_n^{(p)}(0) = (ia)^p, \quad \text{for } p = 0, 1, \ldots, n,$$

in problem 1.1.

**Proof.** For the sake of simplicity, we denote $h_j(n)$ by $h_j$. Observe that the derivatives of order $p$ of $f_n(x)$ are

$$f_n^{(p)}(x) = \sum_{j=0}^{n} X_j(n)(ih_j)^p e^{ih_j x}, \quad x \in \mathbb{R},$$

so if we require that these derivatives are equal to the derivatives of order $p$ for $p = 0, 1, \ldots, n$ of the function $x \mapsto e^{iax}$ at the origin we obtain the linear system

$$\sum_{j=0}^{n} X_j(n)(ih_j)^p = (ia)^p, \quad p = 0, 1, \ldots, n \quad (2.1)$$

from which we deduce

$$\sum_{j=0}^{n} X_j h_j^p = a^p, \quad p = 0, 1, \ldots, n \quad (2.2)$$

where we have written $X_j$ instead of $X_j(n)$. Now we write explicitly the linear system (2.2) of $(n+1)$ equations and $(n+1)$ unknowns $(X_0, \ldots, X_n)$

$$X_0 + X_1 + \ldots + X_n = 1$$
$$X_0h_0 + X_1h_1 + \ldots + X_nh_n = a$$
$$\ldots$$
$$X_0h_0^n + X_1h_1^n + \ldots + X_nh_n^n = a^n$$

and, in matrix form, it becomes

$$H(n)X = B(a), \quad (2.3)$$

where $H$ is the $(n+1) \times (n+1)$ matrix

$$H(n) = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ h_0 & h_1 & \ldots & h_n \\ \vdots & \vdots & \ddots & \vdots \\ h_0^n & h_1^n & \ldots & h_n^n \end{pmatrix} \quad (2.4)$$

and

$$X = \begin{pmatrix} X_0 \\ X_1 \\ \vdots \\ X_n \end{pmatrix} \quad \text{and} \quad B(a) = \begin{pmatrix} 1 \\ a \\ \ldots \\ a^n \end{pmatrix} \quad (2.5)$$
Observe that the determinant of $H$ is the Vandermonde determinant, so it is given by
\[
\det(H(n)) = \prod_{0 \leq i < j \leq n} (h_j(n) - h_i(n)).
\]
Thus, if $h_j(n) \neq h_i(n)$ for every $i \neq j$, there exists a unique solution of the system.

**Theorem 2.2 (Explicit solution of problem 1.1).** Let $h_j(n)$ be a given set of points in $[-1, 1]$, $j = 0, 1, \ldots, n$ for $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that $|a| > 1$. If $h_j(n) \neq h_i(n)$, for every $i \neq j$, the unique solution of system (2.3) is given by
\[
X_j(n, a) = \prod_{k=0, k \neq j}^{n} \left( \frac{h_k(n) - a}{h_k(n) - h_j(n)} \right).
\]
As a consequence, the superoscillating function takes the form
\[
f_n(x) = \sum_{j=0}^{n} \prod_{k=0, k \neq j}^{n} \left( \frac{h_k(n) - a}{h_k(n) - h_j(n)} \right) e^{ih_kx}, \quad x \in \mathbb{R}.
\]

**Proof.** In theorem 2.1, we proved that, if $h_j \neq h_i$ for every $i \neq j$, there exists a unique solution of the system (2.3). The solution is given by
\[
X_j(n, a) = \frac{\det(H_j(n, a))}{\det(H(n))}
\]
for
\[
H_j(n, a) = \begin{pmatrix}
1 & 1 & \ldots & 1 & \ldots & 1 \\
\vdots & \\
1 & 1 & \ldots & a & \ldots & h_n \\
h_0 & h_1 & \ldots & a & \ldots & h_n \\
h_0^n & h_1^n & \ldots & a^n & \ldots & h_n^n
\end{pmatrix},
\]
where the $j$th-column contains $a$ and its powers. The explicit form of the determinant of the matrix $H$ is given by:
\[
\det(H(n)) = (h_1 - h_0) \cdot (h_2 - h_0)(h_2 - h_1) \cdot (h_3 - h_0)(h_3 - h_1)(h_3 - h_2) \\
\cdot (h_4 - h_0)(h_4 - h_1)(h_4 - h_2)(h_4 - h_3) \cdot \ldots \\
\cdot (h_n - h_0)(h_n - h_1)(h_n - h_2)(h_n - h_3) \cdot \ldots \cdot (h_n - h_{n-1}).
\]

The matrix $H_j(n, a)$ is still of the Vandermonde type and its determinant can be computed similarly. So we have that the solution $(X_0(n, a), \ldots, X_n(n, a))$ is such that
\[
X_0(n, a) = \frac{(h_1 - a) \cdot (h_2 - a) \cdot (h_3 - a) \cdot (h_4 - a) \cdot \ldots \cdot (h_n - a)}{(h_1 - h_0) \cdot (h_2 - h_0) \cdot (h_3 - h_0) \cdot (h_4 - h_0) \cdot \ldots \cdot (h_n - h_0)}
\]
\[
= \prod_{k=0, k \neq 0}^{n} (h_k - a) \prod_{k=0, k \neq 0}^{n} (h_k - h_0),
\]
and
\[
X_1(n, a) = \frac{(a - h_0) \cdot (h_2 - a) \cdot (h_3 - a) \cdot (h_4 - a) \cdot \ldots \cdot (h_n - a)}{(h_1 - h_0) \cdot (h_2 - h_1) \cdot (h_3 - h_1) \cdot (h_4 - h_1) \cdot \ldots \cdot (h_n - h_1)}
\]
\[
= \prod_{k=0, k \neq 1}^{n} (h_k - a) \prod_{k=0, k \neq 1}^{n} (h_k - h_1),
\]
and so on, up to
\[
X_n(n, a) = \frac{1 \cdot 1 \cdot 1 \cdot \ldots \cdot (a - h_0)(a - h_1)(a - h_2)(a - h_3) \ldots (a - h_{n-1})}{1 \cdot 1 \cdot 1 \cdot \ldots \cdot (h_1 - h_0)(h_1 - h_1)(h_1 - h_2)(h_1 - h_3) \ldots (h_1 - h_{n-1})}
\]
\[
= \prod_{k=0, k \neq n}^{n} (h_k - a) \prod_{k=0, k \neq n}^{n} (h_k - h_n).
\]
So we get the statement with the recursive formula. ■
3. The methods to generate superoscillations and examples

Below we compare the superoscillating functions obtained by solving problem 1.1 and the superoscillating functions obtained via the sequence $F_n(x,a)$ and infinite order differential operators. For different methods, see also [12].

(I) Observe that the limit
\[ \lim_{n \to \infty} \left( \cos \left( \frac{X}{n} \right) + ia \sin \left( \frac{X}{n} \right) \right)^n = e^{iax} \]
is a direct consequence of
\[ \lim_{n \to \infty} \left( 1 + ia \frac{X}{n} \right)^n = e^{iax}, \]
while the construction method to generate superoscillations in theorem 2.2 has a different nature because we require that the linear system (2.1) in the $n+1$ unknowns $X_j(n)$ are determined in such a way that
\[ \sum_{j=0}^{n} X_j(n)(ih_j)^p = (ia)^p, \quad p = 0, 1, \ldots, n \] (3.1)
so the derivatives
\[ f_n^{(p)}(x) = \sum_{j=0}^{n} X_j(n)(ih_j)^p e^{ih_j(n)x}, \quad x \in \mathbb{R}, \]
at $x = 0$, are equal to the derivatives of the exponential function $e^{iax}$ at the origin. This means that the sequence of functions
\[ f_n(x) = \sum_{j=0}^{n} X_j(n)e^{ih_j(n)x}, \quad x \in \mathbb{R} \]
has $n$ derivatives equal to the derivatives of exponential function $e^{iax}$ at the origin so the limit
\[ \lim_{n \to \infty} f_n(x) = e^{iax} \]
follows by construction of the $f_n(x)$.

(II) In the definition of the superoscillating function (1.1), the derivatives are given by
\[ F_n^{(p)}(x,a) = \sum_{j=0}^{n} C_j(n,a) \left( i \left( 1 - \frac{2j}{n} \right) \right)^p e^{i(1-2j/n)x}, \quad x \in \mathbb{R} \]
and it is just in the limit that we get the derivatives of order $p$ of the exponential function $e^{iax}$ at the origin, namely we have
\[ \lim_{n \to \infty} \sum_{j=0}^{n} C_j(n,a) \left( i \left( 1 - \frac{2j}{n} \right) \right)^p = (ia)^p, \quad p \in \mathbb{N}. \]

(III) With the new procedure proposed in this paper, we impose the conditions
\[ f_n^{(p)}(0) = (ia)^p, \quad p = 0, 1, 2, \ldots, n \]
(where we have genuine equalities, not in the limit) and we link $n$ with the derivatives of $f_n(x)$ in order to determine the coefficients $X_j(n)$ in (1.3), so we have that the Taylor
polynomials of the two functions $f_n(x)$ and $e^{i\alpha x}$ are the same up to order $n$, i.e.
\[ e^{i\alpha x} = 1 + i\alpha x + \frac{(i\alpha x)^2}{2!} + \cdots + \frac{(i\alpha x)^n}{n!} + R_n(x), \]
so we get
\[
 f_n(x) = \sum_{j=0}^{n} X_j(n) e^{i\beta_j x} = f_n(0) + f_n(1)(0)x + f_n(2)(0)\frac{x^2}{2!} + \cdots + f_n(n)(0)\frac{x^n}{n!} + \tilde{R}_n(x), \quad x \in \mathbb{R}
\]
\[
 = 1 + i\alpha x + \frac{(i\alpha x)^2}{2!} + \cdots + \frac{(i\alpha x)^n}{n!} + \tilde{R}_n(x), \quad x \in \mathbb{R},
\]
where $R_n(x)$ and $\tilde{R}_n(x)$ are the errors.

It is now easy to generate a very large class of superoscillatory functions. We write a few examples to further clarify the generality of our new construction of superoscillating sequences: given the sequence $h_j(n)$ we determine the coefficients accordingly for the
\[
 f_n(x) = \sum_{j=0}^{n} \prod_{k=0, k \neq j}^{n} \left( \frac{h_k(n) - a}{h_k(n) - h_j(n)} \right) e^{i\beta_j(n)x}, \quad x \in \mathbb{R}.
\]

Example 3.1. Let $n \in \mathbb{N}$ and set
\[
 h_j(n) = 1 - \frac{2}{n}j,
\]
where $j = 0, \ldots, n$. We have
\[
 h_k(n) - a = 1 - \frac{2}{n}k - a
\]
and
\[
 h_k(n) - h_j(n) = 1 - \frac{2}{n}k - \left( 1 - \frac{2}{n}j \right) = \frac{2}{n} (j - k). 
\]
Thus, we obtain
\[
 f_n(x) = \sum_{j=0}^{n} \prod_{k=0, k \neq j}^{n} \frac{n}{2} \left( \frac{1 - (2/n)k - a}{j - k} \right) e^{i(1 - \frac{2}{n})jx}, \quad x \in \mathbb{R}.
\]

Example 3.2. Let $n \in \mathbb{N}$, and set
\[
 h_j(n) = 1 - \frac{2}{np}j,
\]
where $j = 0, \ldots, n$, for a fixed $p \in \mathbb{N}$. We have
\[
 h_k(n) - a = 1 - \frac{2}{np}k - a
\]
and
\[
 h_k(n) - h_j(n) = 1 - \frac{2}{np}k - \left( 1 - \frac{2}{np}j \right) = \frac{2}{np} (j - k). 
\]
So, we obtain:
\[
 f_n(x) = \sum_{j=0}^{n} \prod_{k=0, k \neq j}^{n} \frac{np}{2} \left( \frac{1 - \frac{2}{np}k - a}{j - k} \right) e^{i(1 - \frac{2}{np})jx}, \quad x \in \mathbb{R}.
\]
Example 3.3. Let $n \in \mathbb{N}$, and set
\[ h_j(n) = 1 - \left( \frac{2j}{n} \right)^p, \]
where $j = 0, \ldots, n$, for a fixed $p \in \mathbb{N}$. We have
\[ h_k(n) - a = 1 - \left( \frac{2k}{n} \right)^p - a \]
and
\[ h_k(n) - h_j(n) = 1 - \left( \frac{2k}{n} \right)^p - \left( \frac{2j}{n} \right)^p = \frac{2p}{n^p} (j^p - k^p). \]
So, we obtain
\[ f_n(x) = \sum_{j=0}^{n} \prod_{k=0, k \neq j}^{n} \frac{n^p}{2^p} \frac{1 - (2k/n) - a}{j^p - k^p} e^{i(1 - (2j/n)x), \ x \in \mathbb{R}.} \]

4. A new class of supershifts

The procedure to define superoscillatory functions can be extended to supershift. We recall that the supershift property of a function extends the notion of superoscillations and this turned out to be the crucial concept behind the study of the evolution of superoscillatory functions as the initial conditions of the Schrödinger equation or of any other field equation. We recall the definition before stating our result.

Definition 4.1 (Supershift). Let $I \subseteq \mathbb{R}$ be an interval with $[-1, 1] \subseteq I$ and let $\varphi : I \times \mathbb{R} \to \mathbb{R}$, be a continuous function on $I$. We set
\[ \varphi_\lambda(x) := \varphi(\lambda, x), \ \lambda \in I, \ x \in \mathbb{R} \]
and we consider a sequence of points $(\lambda_{j,n})$ such that
\[ (\lambda_{j,n}) \in [-1, 1] \quad \text{for } j = 0, \ldots, n \text{ and } n = 0, \ldots, +\infty. \]
We define the functions
\[ \psi_n(x) = \sum_{j=0}^{n} \varphi_{\lambda_{j,n}}(x), \quad (4.1) \]
where $(\epsilon_j(n))$ is a sequence of complex numbers for $j = 0, \ldots, n$ and $n = 0, \ldots, +\infty$. If \( \lim_{n \to +\infty} \psi_n(x) = \varphi_\lambda(x) \) for some $a \in I$ with $|a| > 1$, we say that the function $\lambda \to \varphi_\lambda(x)$, for $x$ fixed, admits a supershift in $\lambda$.

Remark 4.2. We observe that the definition of supershift of a function given above is not the most general one, but it is useful to explain our new procedure for the supershift case. In the following, we will take the interval $I$, in the definition of the supershift, to be equal to $\mathbb{R}$.

Remark 4.3. Let us stress that the term supershift comes from the fact that the interval $I$ can be arbitrarily large (it can also be $\mathbb{R}$) and so also the constant $a$ can be arbitrarily far away from the interval $[-1, 1]$ where the function $\varphi_{\lambda_{j,n}}(\cdot)$ is computed, see (4.1).

Remark 4.4. Superoscillations are a particular case of supershift. In fact, for the sequence $(F_n)$ in (1.1), we have $\lambda_{j,n} = 1 - 2j/n$, $\varphi_{\lambda_{j,n}}(t, x) = e^{i(1 - 2j/n)x}$ and $\epsilon_j(n)$ are the coefficients $C_j(n, a)$ defined in (1.2).

Problem 1.1, for the supershift case, is formulated as follows.

Problem 4.5. Let $h_j(n)$ be a given set of points in $[-1, 1]$, $j = 0, 1, \ldots, n$, for $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that $|a| > 1$. Suppose that for every $x \in \mathbb{R}$ the function $\lambda \mapsto G(\lambda x)$ is the restriction to $\mathbb{R}$ of a
function holomorphic and entire in $\lambda$. Consider the function

$$f_n(x) = \sum_{j=0}^{n} Y_j(n)G(h_j(n)x), \quad x \in \mathbb{R},$$

where $\lambda \mapsto G(\lambda x)$ depends on the parameter $x \in \mathbb{R}$. Determine the coefficients $Y_j(n)$ in such a way that

$$f_n^{(p)}(0) = (a)^p G^{(p)}(0) \quad \text{for } p = 0, 1, \ldots, n. \quad (4.2)$$

The solution of the above problem is obtained in the following theorem.

**Theorem 4.6.** Let $h_j(n)$ be a given set of points in $[-1, 1], j = 0, 1, \ldots, n$ for $n \in \mathbb{N}$ and let $a \in \mathbb{R}$ be such that $|a| > 1$. If $h_j(n) \neq h_i(n)$ for every $i \neq j$ and $G^{(p)}(0) \neq 0$ for all $p = 0, 1, \ldots, n$, then there exists a unique solution $Y_j(n, a)$ of the linear system (4.2) and it is given by

$$Y_j(n, a) = \prod_{k=0, k \neq j}^{n} \left( \frac{h_k(n) - a}{h_k(n) - h_j(n)} \right),$$

so that

$$f_n(x) = \sum_{j=0}^{n} \prod_{k=0, k \neq j}^{n} \left( \frac{h_k(n) - a}{h_k(n) - h_j(n)} \right) G(h_j(n)x), \quad x \in \mathbb{R}$$

and, by construction, it is

$$\lim_{n \to \infty} f_n(x) = G(ax), \quad x \in \mathbb{R}.$$

**Proof.** Observe that we have

$$f_n^{(p)}(x) = \sum_{j=0}^{n} Y_j(n)(h_j(n))^p G^{(p)}(h_j(n)x), \quad x \in \mathbb{R},$$

where $G^{(p)}$ are the derivatives of order $p$, for $p = 0, \ldots, n$ with respect to $x$ of the function $G(\lambda x)$, for $\lambda \in \mathbb{R}$ considered as a parameter. So we get the system

$$f_n^{(p)}(0) = \sum_{j=0}^{n} Y_j(n)(h_j(n))^p G^{(p)}(0) = a^p G^{(p)}(0).$$

Now, since we have assumed that $G^{(p)}(0) \neq 0$ for all $p = 0, 1, \ldots, n$, the system becomes

$$\sum_{j=0}^{n} Y_j(n)(h_j(n))^p = a^p$$

and we can solve it as in theorem 2.2 to get

$$Y_j(n, a) = \prod_{k=0, k \neq j}^{n} \left( \frac{h_k(n) - a}{h_k(n) - h_j(n)} \right).$$

Finally, we get

$$f_n(x) = \sum_{j=0}^{n} \prod_{k=0, k \neq j}^{n} \left( \frac{h_k(n) - a}{h_k(n) - h_j(n)} \right) G(h_j(n)x), \quad x \in \mathbb{R}$$

and by construction it is

$$\lim_{n \to \infty} f_n(x) = G(ax), \quad x \in \mathbb{R}.$$
5. Concluding remarks

In this section, we provide some concluding remarks and we compare our results with the recent literature on superoscillations.

— Our methodology can be useful in applications, for example, to generate superoscillatory light fields in super-resolution imaging. In fact, when a super-resolution imaging technique is based on superoscillations, it is common to consider a light beam with a specific set of wavenumbers in the form of a Fourier sum. On both the theoretical and technical levels, the restriction of taking specific wavenumbers may influence the accuracy of results and may give a difficulty at the technical level, as one should generate a specific set of wavenumbers in order to achieve superoscillations. In the proposed model, we are able to generate superoscillations for generic waves with arbitrary wavenumbers for suitable amplitudes. Thus, for example, we are able to consider a set of wavenumbers where each of them is arbitrary small, i.e. $k_n \sim \varepsilon$ for arbitrary small $\varepsilon > 0$, allowing to display details smaller than the shortest period of their Fourier components, which is of order $\varepsilon$. Moreover, the flexibility in choosing the wavenumbers may also help when one is able to control the amplitudes for each of the Fourier components properly.

— The method presented here is similar to the one presented in [35,36], but the difference is that in this paper we require the superoscillating function to match a Taylor polynomial of a given fast-oscillating function, whereas in [35] the author requires the superoscillatory function to pass through some specified finite number of points. Although in both cases a Vandermonde determinant appears in the computations, the two starting points are different.

Moreover, the merit of the new method is that it presents the solution for the coefficients explicitly, using the fact that the Vandermonde matrix determinants are expressible in a compact product form. Also, the new method allows adding plane waves of any frequency within the bandwidth. By comparison, paper [36] makes use of an analogous superposition of plane waves but requiring that their frequencies are equally spaced.

Thus, the main result in our paper gives flexibility in choosing the bounded set of wavenumbers a signal would have and thus provides new insights about the nature of bounded signals with arbitrary bounded wavenumbers.

— Finally, we note that the idea to use a band-limited function to imitate a fast-oscillating function in some interval by matching their Taylor coefficients up to some order was already introduced in [30]. However, the method of [30] is based on multiplying band-limited functions while in this paper the functions are added.

— As we said before, in our model, it is required that the superoscillating function would match a Taylor polynomial of some fast-oscillating function, whereas in [35] the superoscillating function passes through some pre-specified finite number of points. From the physical point of view, we note that our proposed model focuses on Fourier sums, which are important when describing wavefunctions such as a polychromatic packet. In that case, we wish to find the amplitudes that would be suitable for a pre-specified set of wavenumbers where the wavefunction describes a particle in a superposition state and each state corresponds to a specific wavenumber. Thus, on the physical level, it is more convenient to work with a specific set of wavenumbers suitable to the physical system one describes, rather than with a set of pre-specified points interpolated by function.

— The newly presented method to construct supershifts is analogous to the method to construct superoscillations while in effect generalizing from plane waves to more generic functions. This generalization is new and it is suggested by the literature on the evolution of superoscillations, in which taking a superoscillatory function as the initial condition of the Schrödinger equation, the evolution function becomes a supershift.
Data accessibility. This article has no additional data.

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