Symmetries of the Schrödinger-Pauli equation for neutral particles

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Abstract

With using the algebraic approach the Lie symmetries of Schrödinger equations with matrix potentials are classified. Thirty three inequivalent equations of such type together with the related symmetry groups are specified, the admissible equivalence relations are clearly indicated. In particular the Boyer results concerning kinematical invariance groups for arbitrary potentials (C. P. Boyer, Helv. Phys. Acta, 47, 450–605 (1974)) are clarified and corrected.

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I. INTRODUCTION

In addition to its dominant position in quantum physics, Schrödinger equation is a very important mathematical subject which stimulated the development (or even creation and development) of fundamental research fields of the quin of sciences. A well known example of such research fields is the inverse problem approach and some special branches of functional analysis.

Schrödinger equation is also a very important base for application of various symmetry approaches to mathematical physics. Its symmetry with respect to the eleven parameter continuous group in fact was established long time ago by Sophus Lie. More exactly, Lie discovered the symmetries of the linear heat equation, but the SE is nothing but its complex form. Then the Lie results were recovered and developed in papers [1], [2] and [3]. It was Niederer [1] who had found the maximal invariance group of the free Schrödinger equation. He was the first who shows for physicists that in addition to the Galilei group, this group includes also dilations and conformal transformations.

Symmetries of the one dimensional Schrödinger equation with non-trivial potential were described in paper [2]. Boyer [3] extends these results for two- and three-dimensional systems. The mentioned results occupy a place of honor in modern physics. In fact they form the group-theoretical grounds of quantum mechanics. The group classification presented in [3] also is a necessary step in investigations of higher symmetries of Schrödinger equation which starts with the papers of Winternitz with collaborators [4], [5], and in the search for the coordinate systems which can be used for separation of variables [6].

The higher symmetries of Schrödinger equations are nothing but integrals of motion realized by differential operators whose order is higher than one. The complete description of 2d quantum mechanical systems admitting second order integrals of motion was presented in [4]. And it had needed as much as twenty four years to extend this result to the case of the 3d systems, see papers [7] and [8].

The higher symmetries give rise to such nice properties of SE as superintegrability and supersymmetry, see surveys [9] and [10]. We will not discuss them here but mention that searching for such symmetries is still a very popular business, and the modern trends in this field are related to the third order and even arbitrary order integrals of motion [11], see also paper [12] where the determining equations for such symmetries were presented.
But there are important generalizations of SE which also play the outstanding roles in theoretical physics. They are Schrödinger-Pauli (SP) equation and Schrödinger equation with position dependent mass (PDM equation). Some of them are superintegrable and supersymmetric and admit various types of Lie symmetries. And it would be natural (and desirable) to extend the results concerning symmetry and superintegrability properties of the SE to the case of its mentioned generalizations.

It happens that the group classification of Schrödinger equations with PDM has been waited for a long time. There is a lot of papers devoted to PDM Schrödinger equations with particular symmetries, see, e.g., [13–16]. Superintegrability aspects of such equations (with trivial potentials) are discussed in [17] and [18], see also the references cited therein. But the complete group classification of these equations appears rather recently in papers [19] and [20, 21] for the stationary and time dependent equations correspondingly. In paper [22] we start the systematic search for the higher order symmetries in the PDM systems, but this program is not completed yet.

Symmetries of the SP equations also are studied only partially. We can mention few papers devoted to its supersymmetries [23–25], extended supersymmetries, higher order symmetries [26–28] and Fock symmetries [29–31]. There are also papers [32] and [33] where the relativistic aspects of such symmetries are discussed. However, in contrast with the standard SE, we have no general group classification of these equations. This circumstance have to cause the blame for experts in group analysis, taking into account the fundamental role played by this equation in quantum physics.

In the present paper we give the completed description of all inequivalent continuous symmetries which can be accepted by time dependent SP equation. However, we restrict ourselves to SP equation for neutral particles with spin, which have zero charge but non-zero dipole momentum. A perfect example of such particle is the neutron.

We also present the corrected version of the Boyer classification of continuous symmetries of the standard SE. First, these symmetries form a subclass of symmetries of SP equation, and we need them to formulate the results of our research. Secondly, the Boyer classification appears to be incomplete [21, 34].

We believe that the physical community can pretend to a conveniently presented and correct information on continuous symmetries which can be admitted by the main equation of quantum mechanics, and use the occasion to present the completed list of symmetries of both Schrödinger
and SP equations together with a clear definition of the equivalence relations and notification of the corresponding symmetry groups.

To solve the classification problem we use the so-called algebraic approach whose main idea is the a priori analysis of possible symmetry algebras which are nothing but subalgebras of generic invariance algebra of equations of interest. This approach makes it possible to simplify the procedure of solving of the determining equations and makes the classification results credible.

II. SCHRÖDINGER-PAULI EQUATION AND ITS GENERALIZATIONS

We will consider SP equations of the following generic form

\[
\left( i \frac{\partial}{\partial t} - H \right) \psi(t, x) = 0
\]  

(1)

where \( H \) is the Hamiltonian given by the following relation:

\[
H = -\frac{1}{2} \partial_a \partial_a + V(x)
\]  

(2)

where

\[
\partial_a = ip_a = \frac{\partial}{\partial x_a}, \quad x = (x_1, x_2, x_3)
\]

and summation is imposed over the repeating indices \( a \) over the values \( a = 1, 2, 3 \). Moreover, in contrast with the standard SE, \( V = V(x) \) is not a scalar, but matrix potential which we expand via Pauli matrices:

\[
V = V_0 + \sigma_a V^a.
\]  

(3)

If \( V_0 = 0 \) than equations (2) and (3) define the standard SP equation for particles with trivial electric charge (in his case \( V_a \) should be proportional to components of the external magnetic field). We reserve the possibility of nontrivial \( V_0 \) to be able to recover the case of the standard SE (in this case \( V_a = 0 \)) and the case of presence of a more generic scalar potential term not necessary the electric one.

We will search for symmetries of equations (1) with respect to continuous groups of transformations of dependent and independent variables. We will not apply the generic Lie approach whose perfect presentation can be found in [35] but restrict ourselves to its simplified version.
adopted to linear equation (1). Let us write the generator of the searched transformation group in the form

\[ Q = \xi^0 \partial_t + \xi^a \partial_a + \tilde{\eta} \equiv \xi^0 \partial_t + \frac{1}{2} (\xi^a \partial_a + \partial_a \xi^a) + i\eta, \]  

where \( \tilde{\eta} = \frac{1}{2} \xi^a \partial_a + i\eta \), \( \xi^0 \), \( \xi^a \) and \( \eta \) are functions of \( t, x \) and \( \partial_t = \frac{\partial}{\partial t} \). Moreover, \( \eta \) is a 2 × 2 matrix which, in analogy with (3), we represent in the following form:

\[ \eta = \eta^0 + \sigma_a \eta^a. \]

In contrary, \( \xi^0 \) and \( \xi^a \) are scalar functions which can be treated as multipliers for the unit matrix.

Generator (4) transforms solutions of equation (1) into solutions if it satisfies the following operator equation

\[ [Q, L] \equiv QL - LQ = \alpha L \]

where \( L = i\partial_t - H \) and \( \alpha \) is one more unknown function of \( t \) and \( x \).

Evaluating the commutator in the l.h.s. of (6) and equating coefficients for the linearly independent differentials we obtain the following system of equations for unknowns \( \xi^0, \xi^a, \eta, V \) and \( \alpha \):

\[ \dot{\xi}^0 = -\alpha, \quad \dot{\xi}_a^0 = 0, \]  

\[ \xi^b_a + \xi^a_b = \frac{2}{n} \delta_{ab} \xi^i = 0, \]  

\[ \dot{\xi}^i = -\frac{n}{2} \alpha, \]  

\[ \dot{\xi}^a + \eta_a = 0, \]  

\[ \xi^a V_a = \alpha V + \dot{\eta} + i[\eta, V] \]

where \( \dot{\eta} = \frac{\partial \eta}{\partial t} \) and \( \eta_a = \frac{\partial \eta}{\partial x_a} \).

Formally speaking, system of the determining equations (7) – (11) is rather complicated since it includes four arbitrary elements \( V^0, V^1, V^2 \) and \( V^3 \) whose form should be fixed from the compatibility condition of this system. However, the major part of this system, i.e., equations (7), (8), (9) and (10), do not include these arbitrary elements. The immediate consequences of equation (10) are the following conditions:

\[ \frac{\partial \eta^a}{\partial x_b} = 0 \]
\[ \dot{\xi}^a + \dot{\eta}_a^0 = 0. \]  

(13)

Equation (11) in its turn is decoupled to the scalar and vector parts:

\[ \xi^a V^0_a = \alpha V^0 + \dot{\eta}^0, \]  

(14)

and

\[ \xi^a V^b_a = \alpha V^b + \dot{\eta}^b - 2\varepsilon^{bcd} \eta^c V^d \]  

(15)

where \( \varepsilon^{bcd} \) is the absolutely antisymmetric unit tensor.

In other words, system (7)–(11) includes the autonomous subsystem formed by equations (7), (8), (9), (13) and (14). Solving this subsystem we recover symmetries of the standard SE describing a spinless particle. Then, to find symmetries of the SP equation it is sufficient to solve equations (13) and (15) where \( \xi^b \) and \( \alpha \) are functions found at the previous step.

Thus the description of symmetries of the standard SE is the necessary step in the group classification of the SP equations.

### III. SYMMETRIES OF THE STANDARD SCHRÖDINGER EQUATION

In this section we specify inequivalent continuous symmetries of the standard SE with the following Hamiltonian:

\[ H = -\frac{1}{2} \partial_a \partial_a + V^0 \]  

(16)

where \( V^0 = V^0(x) \) is a scalar potential. To achieve this goal it is sufficient to find inequivalent solutions of the determining equations (7), (8), (9), (13) and (14).

#### A. Analysis of the determining equation

It follows from (7) that both \( \xi^0 \) and \( \alpha \) do not depend on \( x \). Equations (8) and (9) specify the dependence of coefficients \( \xi^a \) on \( x \):

\[ \xi^a = -\frac{\alpha}{2} x_a + \theta^{ab} x_b + \nu_a \]  

(17)
where $\alpha, \theta^{ab} = -\theta^{ba}$ and $\nu^a$ are arbitrary parameters. If $\alpha \neq 0$ then, up to shifts of spatial variables $x_a$ we can set $\nu^a = 0$, and so equation (17) is decoupled to two versions: either

$$\xi^a = -\frac{\alpha}{2} x_a + \theta^{ab} x_b$$  \hspace{1cm} (18)$$

or

$$\xi^a = \theta^{ab} x_b + \nu_a.$$  \hspace{1cm} (19)$$

Moreover, in accordance with (10), $\theta^{ab}$ are time independent, and so we have to specify the dependence on $t$ only for $\alpha$ or $\nu^a$.

Substituting (17) into (13) and integrating the resultant equation we obtain the generic form of function $\eta^0$:

$$\eta^0 = \frac{\dot{\alpha}}{4} x^2 - \dot{\nu}_a x_a + f(t), \quad \alpha \nu^a = 0$$  \hspace{1cm} (20)$$

and so the generic symmetry operator (4) is reduced to the following form:

$$Q = \xi^0 \partial_t + \left( \frac{\dot{\xi}^0}{2} x_a + \theta^{ab} x_b + \nu_a \right) \partial_a - \frac{\ddot{\xi}^0}{4} x^2 - \dot{\nu}_a x_a + f(t), \quad \dot{\xi}^0 = 0.$$  \hspace{1cm} (21)$$

In addition, equation (11) takes one of the following forms:

$$\left( \frac{\alpha}{2} x_a - \theta^{ab} x_b \right) V^0_a + \alpha V^0 + \frac{\ddot{\alpha}}{4} x^2 - \dot{f} = 0$$  \hspace{1cm} (22)$$

if $\alpha \neq 0$, and

$$\left( \nu_a + \theta^{ab} x_b \right) V^0_a + \dot{\nu}_a x_a + \dot{f} = 0$$  \hspace{1cm} (23)$$

for $\alpha$ zero. Thus to make the group classification of SEs (16) are supposed to find non-equivalent solutions of equations (22) and (23) for $V^0$.

Let us specify the possible dependence of functions $\alpha, \eta$ and $\nu^0$ on $t$. Differentiating equations (22) and (23) w.r.t. $x_c$ we obtain

$$2\alpha (3V^0_c + x_b V^0_{cb}) + \ddot{\alpha} x_c = \theta^{ab} x_b V^0_{ac}$$  \hspace{1cm} (24)$$

for $\alpha$ nonzero, and

$$\ddot{\nu}^c + \nu^b V^0_{cb} = -\theta^{ab} x_b V^0_{ac}.$$  \hspace{1cm} (25)$$
if $\alpha = 0$.

In accordance with (24) and (25) and in view of the time independence of $V$ and $\theta^{ab}$ there are the following conditions for functions $\alpha$ and $\nu^a$

\begin{align*}
a &= 0, \quad \ddot{\nu} = \mu \nu, \quad \text{if} \quad V_{bc} = -\delta_{bc}\mu, \quad (26) \\
a &= 0, \quad \ddot{\nu} = 0, \quad \text{if} \quad V_{bc} \neq -\delta_{bc}\mu, \quad (27) \\
\ddot{\alpha} &= \mu \alpha \quad \text{if} \quad V_{bc} = -\delta_{bc}\mu, \quad (28) \\
\ddot{\alpha} &= 0 \quad \text{if} \quad V_{bc} \neq -\delta_{bc}\mu. \quad (29)
\end{align*}

where $\mu$ is a constant.

In other words, functions functions $\alpha$ and $\nu^a$ should be trigonometric, hyperbolic, linear or constant.

B. Equivalence transformations

By definition the equivalence transformations of the dependent and independent variables keep the generic form of equation (1) but can change the potential $V = V(x)$. The obvious examples of such transformations is given by the following formulae:

\begin{align*}
x \rightarrow \tilde{x} &= x, \quad t \rightarrow \tilde{t} = t, \\
\psi(t, x) \rightarrow \tilde{\psi}(t, x) &= \tilde{M}\psi(t, x) \quad (30)
\end{align*}

where $\tilde{M}$ is a constant non-degenerated matrix, and

\begin{align*}
x \rightarrow \tilde{x} &= x, \quad t \rightarrow \tilde{t} = t, \\
\psi(t, x) \rightarrow \tilde{\psi}(t, x) &= \exp(iMt)\psi(t, x) \quad (31)
\end{align*}

where $M$ is a numeric matrix commuting with the potential. Up to equivalence transformations (35), this matrix is diagonal, i.e., $M = \mu + \nu\sigma_3$ with real parameters $\mu$ and $\nu$.

The equivalence transformations include all continuous (Lie) symmetries of equation (1) which do not change the potential $V$, and also transformations changing the potential. It is possible to show that for generic $V$ such (continuous) transformations include (30), (31), and transformations belonging to the extended Euclid group $\tilde{E}$, i.e., shifts, rotations and scalings of independent variables see Section 5.
In addition, for some particular potentials there exist additional equivalence transformations. In the case of the trivial potential they have the following form:

\[
\begin{align*}
    x & \rightarrow \tilde{x} = \frac{x}{\sqrt{1 + t^2}}, \quad t \rightarrow \tilde{t} = \frac{1}{\omega} \arctan(t), \\
    \psi(t, x) & \rightarrow \tilde{\psi}(\tilde{t}, \tilde{x}) = (1 + t^2)^{\frac{3}{4}} e^{-\frac{i}{2(1 + t^2)} \omega^2 r^2} \psi(t, x),
\end{align*}
\]

(32)

\[
\begin{align*}
    x & \rightarrow \tilde{x} = \frac{x}{\sqrt{1 - t^2}}, \quad t \rightarrow \tilde{t} = \frac{1}{\omega} \arctanh(t), \\
    \psi(t, x) & \rightarrow \tilde{\psi}(\tilde{t}, \tilde{x}) = (1 - t^2)^{\frac{3}{4}} e^{-\frac{i}{2(1 - t^2)} \omega^2 r^2} \psi(t, x),
\end{align*}
\]

(33)

and

\[
\begin{align*}
    x_a & \rightarrow x'_a = x_a - \frac{1}{2} \kappa_a t^2, \quad t \rightarrow t' = t, \\
    \psi(t, x) & \rightarrow \psi'(t', x') = \exp \left( -it\kappa_a x_a + \frac{i}{3} \kappa^2 t^3 \right) \psi(t, x).
\end{align*}
\]

(34)

where \( \omega \) and \( \kappa_a \) are arbitrary parameters, and \( \kappa^2 = \kappa_1^2 + \kappa_2^2 + \kappa_3^2 \).

Transformations (31), (32), (33) and (34) keep the generic form of the related equation (1) but change the trivial potential to

\[
\begin{align*}
    V &= \mu + \nu \sigma_3, \\
    V^0 &= \frac{1}{2} \omega^2 r^2, \\
    V^0 &= -\frac{1}{2} \omega^2 r^2,
\end{align*}
\]

(35) (36) (37)

and

\[
V^0 = \kappa_a x_a
\]

(38)
correspondingly.

The equivalence of the isotropic harmonic and repulsive oscillators to the free particle Schrödinger equation was discovered in [36]. Formulae (32) and (33) present transformations for wave functions dependent on tree spatial variables while in [36] we can find them only for one dimensional case. For the equivalence transformations with arbitrary number of spatial variables see [34].

Notice that transformation (32) and (33) are valid for any equation (1) with potential \( V = V^0 \) being a homogeneous function of degree -2. In this case this equation is invariant with respect to dilatation transformations.
Mapping (34) connects the systems with trivial and free fall potentials [37]. But it is valid also for potentials being functions of one or two spatial variables, say $x_1$ or $x_1$ and $x_2$. In other words, it is valid provided the related equation (11) is invariant w.r.t. shifts of independent variables along the third coordinate axis. In this case we have to set in (34) $a = 2$, 3 or $a = 3$ correspondingly.

C. Symmetries for SE with trivial potentials

Let us present the symmetries accepted by the Schrödinger equations with the trivial, isotropic oscillator and free fall potentials. They are known, but the related publications are not necessary easy accessible, and we fix them for the readers convenience. In addition, the specific combinations of just these symmetries are accepted by the other systems classified below, and we need them to formulate the classification results.

Setting in (22) and (23) $V^0 = 0$ we easy solve the obtained equation and find the corresponding admissible symmetries (4). They are linear combinations of the following symmetry operators:

\[
\begin{align*}
P_a &= -i \partial_a, \\
M_{ab} &= x_a P_b - x_b P_a, \\
D &= 2t P_0 - x_a P_a + \frac{3i}{2}, \\
P_0 &= i \partial_t, \\
G_a &= t P_a - x_a, \\
A &= t D - t^2 P_0 - \frac{r^2}{2}.
\end{align*}
\]

(39)

Let us remind that operators $P_0, P_a$ and $M_{ab}$ generate shifts and rotations of the independent variables and leave the wave function invariant. Operators $G_a, D$ and $A$ generate Galilei, dilatation and conformal transformations correspondingly which act on dependent and independent variables. For the explicit form of these transformation see, e.g., [38].

Operators (39), (40) together with the unit operator $I$ form the 13-dimensional Lie algebra sometimes called Schrödinger algebra. Operators (39) and operator $I$ form a central extension of the Lie algebra of Schrödinger group.

The additional identities satisfied by operators (39) and (40) are [38]:

\[
\begin{align*}
P_a G_b - P_b G_a &= M_{ab}, \\
P_a G_a + G_a P_a &= 2D + 2t(P^2 - 2P_0), \\
G_a G_a &= 2A + t^2(P^2 - 2P_0).
\end{align*}
\]

(41)
On the set of solutions of equation (2) the term in brackets is equal to \(-2V^0\) which in our case is equal to zero, and so relations (41) express generators \(M_{ab}\), \(D\) and \(A\) via bilinear combination of \(P_a\) and \(G_a\). Thus the invariance of the free Schrödinger equation with respect to rotation, dilatation and conformal transformations appears to be a consequence of the symmetry with respect to the displacement and Galilei transformations.

Transformations (33) can be used to obtain symmetries of equation (2) with the harmonic oscillator potential. They include \(P_0\), \(M_{ab}\) and the following generators:

\[
\begin{align*}
A^+(\omega) &= \sin(2\omega t)(P_0 - \omega^2 r^2) - \frac{\omega}{2} \cos(2\omega t) (x_a P_a + P_a x_a), \\
\hat{A}^+(\omega) &= \cos(2\omega t)(P_0 - \omega^2 r^2) + \frac{\omega}{2} \sin(2\omega t) (x_a P_a + P_a x_a), \\
B^+_a(\omega) &= \sin(\omega t) P_a - \omega x_a \cos(\omega t), \\
\hat{B}^+_a(\omega) &= \cos(\omega t) P_a + \omega x_a \sin(\omega t)
\end{align*}
\]

where the upper marks ”+” indicate the sign of the related potential (36).

Symmetries for the repulsive oscillator potential (37) can be obtained from (42) by the change \(\omega \rightarrow i\omega\). As a result we obtain generators \(P_0\), \(M_{ab}\) in the same form as in (39), and the following operators:

\[
\begin{align*}
A^- &= \exp(2\omega t)(P_0 + \omega^2 r^2 - \frac{i\omega}{2} (x_a P_a + P_a x_a)), \\
\hat{A}^- &= \exp(-2\omega t)(P_0 + \omega^2 r^2 + \frac{i\omega}{2} (x_a P_a + P_a x_a)), \\
B^-_a &= \exp(\omega t)(P_a - \omega x_a), \\
\hat{B}^-_a &= \exp(-\omega t)(P_a + \omega x_a)
\end{align*}
\]

where the upper marks ”-” indicate the sign of the related potential (37).

Analogously, starting with realization (39), (40) and making transformations (34) we find symmetries for equation (2) with the free fall potential. To this effect it is sufficient to make the following changes:

\[
P_0 \rightarrow P_0 + \kappa_a G_a + \frac{1}{2}\kappa^2 t^2, \quad P_a \rightarrow P_a + \kappa_a t, \quad x_a \rightarrow x_a + \frac{1}{2}\kappa_a t^2
\]

in all generators (39) and (40).

The presented symmetries appear partly for the case of other particular potentials presented below. However, for generic potential the equivalence relations (32)-(34) are not valid.

D. Classification results for SE with arbitrary potential

Consider equations (22), (23) and their differential consequences (24)–(29) for arbitrary potential \(V\). Their solution is is a rather complicated procedure. We will use the algebraic
approach which presupposes to use the basic property of symmetry operators: that they should form a basis of a Lie algebra. This algebra by definition includes operator \( P_0 \) and the unit operator.

Using the mentioned differential consequences it is possible to show that the generic symmetry (4) with coefficients (17), (20) is a linear combination of symmetries (39), (40), (42), (43) and yet indefinite function \( f \). Thus to find all non-equivalent solutions of equation (22) and (23) we have to go over these combinations, restricting ourselves to the cases when they are non-equivalent.

Let one of conditions (27) or (29) is satisfied. In this case we come to a linear combinations of generators \( P_a, L_a = \frac{1}{2} \varepsilon_{abc} \) and \( D \) given by equation (39). They form a basis of the extended Euclid algebra \( \tilde{e}(3) \) whose non-equivalent subalgebras has been classified in [39]. And just these subalgebras generate non-equivalent linear combinations of symmetries which we have to consider.

In particular, algebra \( \tilde{e}(3) \) has four non-equivalent one-dimensional subalgebras whose generators are [39]:

\[
L_3 = M_{12}, \quad L_3 + P_3, \quad D + \mu L_3, \quad P_3.
\]

The related parameters in (22) and (23) are \( \theta^{12} = 1 \) for \( L_3 \), \( \theta^{12} = \nu^3 = 1 \) for \( L_3 + P_3 \), \( \alpha = -2 \), \( \theta^{12} = \mu \) for \( D + \mu L_3 \), \( \nu^3 = 1 \) for \( P_3 \), and in all cases \( f = \kappa t \). In particular, for generator \( Q = L_3 + \kappa t \) equation (22) is reduced to the following form:

\[
L_3 V = -i \kappa
\]

and so

\[
V = \kappa \varphi + G(\theta, r)
\]

where \( \varphi = \arctan \left( \frac{x_2}{x_1} \right) \) and \( \theta = \arctan \left( \frac{x_3}{x_2} \right) \) are Euler angles. Just this solution is missing in the Boyer classification.

Solving the subclass of equations (22) and (23) corresponding to one-dimensional subalgebras specified in (45) we obtain the results presented in Items 1–5 of Table 1. In Item 4 we set \( \kappa = 0 \) since this parameter can be reduced to zero using mapping inverse to (34). Symmetry \( G_3 \) presented there generates the same equation for potential as \( P_3 \) does.

It is possible to fix the following pairs of "friendly symmetries"

\[
\langle P_a, G_a \rangle, \quad \langle A, D \rangle
\]
which have the following property: any symmetry induces the other symmetry from this pair. This phenomena is caused by the similarity of the determining equations corresponding to these symmetries.

A more extended set of "friendly symmetries" looks as follows:

$$\langle (P_2, P_3), (G_2, G_3, L_1) \rangle, \quad \langle (G_2, G_3), (P_2, P_3, L_1) \rangle$$

(48)

and any pair from the first bracket induces the triplet from the second bracket.

The next step is to use the non-equivalent two-dimensional subalgebras of $\tilde{e}(3)$ spanned on the following basis elements [39]:

$$\langle L_3 + \kappa t, P_3 \rangle, \quad \langle D + \kappa L_3, P_3 \rangle, \quad \langle P_2, P_3 \rangle, \quad \langle D, L_3 \rangle.$$  

(49)

Any sets (49) includes at least one element from (45). Thus we have to solve equations (22) or (23) generated by the second element for potentials presented in Items 1–5 of Table 1. As a result we obtain the results in Items 6–9. In fact the corresponding equations (11) admit three (or even four) dimensional symmetry algebras since generators $P_3$ and $D$ are automatically attended by $G_3$ and $A$, see equations (47), (48) and the discussion nearby.

Analogously, considering the non-equivalent three dimensional subalgebras of $\tilde{e}(3)$, whose basis elements are presented in the following formulae

$$\langle D, P_3, L_3 \rangle, \quad \langle D, P_1, P_2 \rangle, \quad \langle L_1, L_2, L_3 \rangle, \quad \langle L_3, P_1, P_2 \rangle$$

(50)

$$\langle P_1, P_2, P_3 \rangle, \quad \langle L_3 + P_3, P_1, P_2 \rangle, \quad \langle D + \mu L_3, P_1, P_2 \rangle, \quad \mu > 0$$

(51)

we obtain potentials presented in Items 11–14. In the cases 12–14 we again have more extended symmetries thanks to the presence of the "friendly" elements.

In Table 1 as well as in the following Tables 2–4 $G(.)$ are arbitrary function of variables given in the brackets, $\mu$, $\kappa$ and $\omega_a$ are arbitrary real parameters, $\varepsilon_1$, $\varepsilon_2$ and $\varepsilon_3$ independently take values $\pm 1$, subindexes $a$ and $k$ take all values 1, 2, 3 and 1, 2 correspondingly. In addition, we denote $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $\tilde{r} = \sqrt{x_1^2 + x_2^2}$ and $\varphi = \arctan (x_2/x_1)$.

All presented systems by construction admit symmetries $P_0$ and $I$, the latter is the unit operator. The additional symmetries are presented in Columns 3, where $P_a, L_a = \frac{1}{2} \varepsilon_{abc} M^{bc}$, $D$, $A$, $B^a_\alpha(\omega_a)$ and $\hat{B}^a_\alpha(\omega_a)$ are generators (39), (40), (42) and (43). The related symmetry algebras are fixed in the fourth columns, where $n_{a,b}$ and $s_{a,b}$ are nilpotent and solvable Lie algebras correspondingly, whose dimension is $a$ and the identification number is $b$. To
identify these algebras for \( a \leq 6 \) we use the notations proposed in [40]. The symbol \( 2n_{1,1} \) (or \( 3n_{1,1} \)) denotes the direct sum of two (or three) one-dimension algebras. In addition, \( g(1,2) \) and \( \text{shcr}(1,2) \) are Lie algebras of Galilei and Schrödinger groups in \( (1+2) \) dimensional space.

Table 1. Non-equivalent symmetries of 3d Schrödinger equations whose potentials do not include quadratic terms.

| No | Potential \( V \) | Symmetries | Invariance algebras |
|----|------------------|------------|---------------------|
| 1  | \( G(\tilde{r}, x_3) + \kappa \varphi \) | \( L_3 + kt \) | \( n_{3,1} \) if \( \kappa \neq 0 \), \( 3n_{1,1} \) if \( \kappa = 0 \) |
| 2  | \( G(\tilde{r}, x_3 - \varphi) + \kappa \varphi \) | \( L_3 + P_3 + kt \) | \( n_{3,1} \) if \( \kappa \neq 0 \), \( 3n_{1,1} \) if \( \kappa = 0 \) |
| 3  | \( \frac{1}{\tilde{r}}G(\tilde{r}, \tilde{r}e^{-\varphi}) \) | \( D + L_3 \) | \( s_{2,1} \oplus n_{1,1} \) |
| 4* | \( G(x_1, x_2) \) | \( G_3, P_3 \) | \( n_{4,1} \) |
| 5* | \( \frac{1}{\tilde{r}}G(\varphi, \tilde{r}) \) | \( A, D \) | \( \mathfrak{sl}(2,\mathbb{R}) \oplus n_{1,1} \) |
| 6* | \( \frac{1}{\tilde{r}}G(\tilde{r}) \) | \( A, D, L_3 \) | \( \mathfrak{sl}(2,\mathbb{R}) \oplus 2n_{1,1} \) |
| 7* | \( G(\tilde{r}) + \kappa \varphi \) | \( L_3 + kt, G_3, P_3 \) | \( s_{5,14} \) if \( \kappa \neq 0 \), \( n_{4,1} \oplus n_{1,1} \) if \( \kappa = 0 \) |
| 8  | \( \frac{1}{\tilde{r}}G(\tilde{r}e^{-\varphi}) \) | \( D + L_3, G_3, P_3 \) | \( s_{5,38} \) |
| 9**| \( \frac{1}{\tilde{r}}G(\varphi) \) | \( A, D, G_3, P_3 \) | \( s_{0,242} \) |
| 10*| \( G(x_1) \) | \( G_3, P_3, P_2, G_2, L_1 \) | \( g(1,2) \) |
| 11 | \( G(r) \) | \( L_1, L_2, L_3 \) | \( \mathfrak{so}(3) \oplus 2n_{1,1} \) |
| 12*| \( \kappa \tilde{r} \) | \( A, D, L_1, L_2, L_3 \) | \( \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{so}(3) \oplus n_{1,1} \) |
| 13**| \( \kappa \tilde{r} \) | \( A, D, G_3, P_3, L_3 \) | \( s_{0,242} \oplus n_{1,1} \) |
| 14**| \( \kappa \tilde{x} \) | \( A, D, G_2, G_3, P_2, P_3, L_1 \) | \( \text{shcr}(1,2) \) |

In the tables we specify also the admissible equivalence transformations additional to ones belonging to the extended Euclid group. Namely, the star near the item number indicates that the corresponding Schrödinger equation admits additional equivalence transformation (34) for independent variables \( x_a \) provided \( \frac{\partial V}{\partial x_a} = 0 \). The asterisk marks the items which specify equations admitting transformation (32) and (33).

In other words, up to the equivalence transformations (32) and (33) the potentials presented
in items 5, 6, 9, 12, 13 and 14 can be generalized to include the additional quadratic term \( \frac{\xi}{2} \omega^2 r^2 \).

Analogously, the potentials presented in items 4, 7, 9, 10, 13, and 14 can be transformed to the equivalent forms applying transformations (34). As a result the additional terms linear in \( x_a \) will appear.

Thus we have described all symmetry algebras including generators with time independent coefficients \( \xi^a \). In fact some of them include the coefficients linear in \( t \), but such generators appear automatically, since they belong to ”friendly symmetries”. And just these algebras are presented in Table 1.

The next step is to consider the versions corresponding to symmetries dependent on time in a more complicated manner. To do it we use the versions presented in (26) and (28), which correspond to symmetries of the oscillator type presented by formulae (12) and (13) and enumerate the possibilities with one, two, or three pairs of operators \( B^e_a, \hat{B}^e_a \) with the same or different frequency parameters \( \omega_a \). In other words, we again exploit the subalgebras of the extended Euclid algebra \( \hat{e}(3) \), but the related basis elements are now given by equations (42) and (43). As a result we come to the list of inequivalent potentials and symmetries presented in Table 2. Five of them include arbitrary functions, but the related number of symmetries is rather restricted. The remaining five versions include arbitrary parameters, but the number of symmetries is more extended and equal to seven, nine or even eight. More exactly, the algebras of symmetries presented in Items 6-10 of Table 2 are solvable and have dimension \( d = 8 \) and \( d = 9 \). We denote them formally as \( s_{d,a}(.) \) without refereing to any data base, since the classification such dimension algebras is unknown.

Let us present commutation relations for basic elements of these algebras:

\[
\begin{align*}
[P_0, B^e_a] &= i\omega \hat{B}^e_a, \\
[P_0, \hat{B}^e_a] &= i\varepsilon \omega B^e_a, \\
[P_0, G_3] &= iP_3, \\
[B^e_a, B^e_b] &= i\delta_{ab}I, \\
[B^e_1, L_3] &= iB^e_1, \\
[B^e_2, L_3] &= -iB^e_2,
\end{align*}
\] (52)

where only non-trivial commutators are presented.

Thus, using the algebraic approach we classify all non-equivalent Lie symmetries admitted by 3d Schrödinger equations. We recover and complete the classical Boyer results but present them in more convenient form with explicit specification of the admissible symmetries and equivalence transformations. Moreover, some of the presented results are new, see Section 6.
### Table 2. Non-equivalent symmetries of 3d Schrödinger equations whose potentials include quadratic terms.

| No | Potential $V$                                      | Symmetries                                      | Invariance algebras |
|----|---------------------------------------------------|-------------------------------------------------|---------------------|
| 1  | $\varepsilon \frac{x_1^2 - x_2^2}{2} + G(x_1, x_2)$ | $B_3^\varepsilon(\omega), \hat{B}_3^\varepsilon(\omega)$ | $s_{4,6}$ if $\varepsilon = -1$, $s_{4,7}$ if $\varepsilon = 1$ |
| 2  | $\varepsilon \frac{\omega^2 x_2^2}{2} + G(\vec{r}) + \mu \varphi$ | $B_3^\varepsilon(\omega), \hat{B}_3^\varepsilon(\omega)$, $L_3 + \mu \upsilon$ | $s_{4,6} \oplus n_{1,1}$ if $\varepsilon = -1, \mu \neq 0$, $s_{5,15}$ if $\varepsilon = -1, \mu \neq 0$, $s_{5,16}$ if $\varepsilon = 1, \mu \neq 0$ |
| 3* | $\varepsilon \frac{\omega^2 x_2^2}{2} + G(x_1)$ | $B_2^\varepsilon(\omega), \hat{B}_2^\varepsilon(\omega)$, $P_3$, $G_3$ | $s_{6,160}$ if $\varepsilon = -1$, $s_{6,161}$ if $\varepsilon = 1$ |
| 4  | $\varepsilon_1 \frac{\omega^2 x_1^2}{2} + \varepsilon_2 \frac{\omega^2 x_2^2}{2} + G(x_3)$ | $B_1^\varepsilon_1(\omega_1), \hat{B}_1^\varepsilon_1(\omega_1)$, $B_2^\varepsilon_2(\omega_2), \hat{B}_2^\varepsilon_2(\omega_2)$ | $s_{6,162}$ if $\varepsilon_1 = \varepsilon_2 = -1$, $s_{6,164}$ if $\varepsilon_1 \varepsilon_2 = -1$, $s_{6,166}$ if $\varepsilon_1 = \varepsilon_2 = 1$ |
| 5  | $\varepsilon \frac{\omega^2 x_2^2}{2} + G(x_3)$ | $B_1^\varepsilon_1(\omega_1), \hat{B}_1^\varepsilon_1(\omega_1)$, $B_2^\varepsilon_2(\omega_2), \hat{B}_2^\varepsilon_2(\omega_2)$, $L_3$ | $s_{7,1}(\varepsilon_1, \varepsilon_2)$ |
| 6  | $\varepsilon_1 \frac{\omega^2 x_1^2}{2} + \varepsilon_2 \frac{\omega^2 x_2^2}{2} + \varepsilon_3 \frac{\omega^2 x_3^2}{2}$ | $B_a^\varepsilon_a(\omega_a), \hat{B}_a^\varepsilon_a(\omega_a)$, $a = 1, 2, 3$ | $s_{8,1}(\varepsilon_1, \varepsilon_2, \varepsilon_3)$ |
| 7* | $\varepsilon_1 \frac{\omega^2 x_1^2}{2} + \varepsilon_2 \frac{\omega^2 x_2^2}{2}$ | $B_1^\varepsilon_1(\omega_1), \hat{B}_1^\varepsilon_1(\omega_1)$, $P_3$, $B_2^\varepsilon_2(\omega_2), \hat{B}_2^\varepsilon_2(\omega_2)$, $G_3$ | $s_{8,2}(\varepsilon_1, \varepsilon_2)$ |
| 8  | $\varepsilon \frac{\omega^2 x_2^2}{2} + \varepsilon_3 \frac{\omega^2 x_3^2}{2}$ | $B_3^\varepsilon_3(\omega_3), \hat{B}_3^\varepsilon_3(\omega_3)$ | $s_{9,1}(\varepsilon, \varepsilon_3)$ |
| 9* | $\varepsilon \frac{\omega^2 x_2^2}{2}$ | $G_3, P_3, L_3, B_1^\varepsilon(\omega), \hat{B}_1^\varepsilon(\omega), B_2^\varepsilon(\omega), \hat{B}_2^\varepsilon(\omega)$ | $s_{9,2}(\varepsilon)$ |
| 10* | $\varepsilon \frac{\omega^2 x_2^2}{2}$ | $B_3^\varepsilon(\omega), \hat{B}_3^\varepsilon(\omega)$, $L_3$, $P_1, P_2, G_1, G_2$ | $s_{9,3}(\varepsilon)$ |
IV. SYMMETRIES OF SP EQUATIONS WHICH DO NOT INCLUDE OSCILLATOR TERMS

We have solved the subproblem of our classification problem, i.e., classified a reduced versions of SP equations with diagonal matrix potentials. In the present section we consider the general case with non-trivial external magnetic field. In our notations it means that $V^a$ are not identically zero and the related potential is a generic $2 \times 2$ matrix.

Since we have in hands all inequivalent solutions of equations (8)-(10) which are found in the previous section, the only thing we need is to find the corresponding solutions of equations (15) where $\xi^a$ and $\alpha$ are known and, in accordance with (12), $\eta^a$ are not dependent on $x$. In other words, it is necessary to consider all cases indicated in Table 1 and extend them to the case of non-trivial $V^a$ solving the corresponding equation (15) for potential components $V^a$.

First we consider the cases when SP equation admit one dimension algebras whose generators are presented in (45). The related scalar potentials are enumerated in items 1-4 of Table 1. In item 4 we can find one more symmetry, namely, $G_3$, but it can be treated as induced by $P_3$.

Let us start with the case which is not included into Table 1: no symmetry, all coefficients $\xi^a$ and $\eta^0$ are trivial. The corresponding equation (15) is reduced to the following form:

$$\dot{\eta}^b - 2 \varepsilon^{bcd} \eta^c V^d = 0$$

Up to constant matrix transformation the generic solution of (53) is:

$$V^1 = V^2 = 0, \quad V^3 = \lambda,$$

$$\eta^1 = \cos(t), \quad \eta^2 = \sin(t), \quad \eta^3 = \rho, \quad \text{and}$$

$$\eta^1 = \sin(t), \quad \eta^2 = -\cos(t), \quad \eta^3 = \rho$$

where $\lambda$ and $\rho$ are arbitrary constants.

Thus we find the symmetry of SP equation for a neutral particle interacting with the constant magnetic field which without loss of generality can be directed along the third coordinate axis. However, like in the case of the harmonic oscillator, the related SP equation (11) can be reduced to the equation with trivial potential, which can be done using transformation (30).

Let us consider consequently the matrix extensions of all potentials and symmetries presented in Table 1. For the first symmetry specified in Item 4, i.e., for $P_3$ we have $\xi = 1$ while $\xi^1$, $\xi^2$ and $\alpha$ are trivial. Substituting these data into (15) we obtain:

$$V^a_3 = \dot{\eta}^a - 2 \varepsilon^{abc} \eta^b V^c.$$
Differentiating (55) w.r.t. \( t \) we obtain the condition
\[ \ddot{\eta}^a = \frac{1}{2} \frac{\partial}{\partial t} (\dot{\eta}^a \dot{\eta}^a) = 0. \] (56)

In accordance with (56) vector components \( \dot{\eta}^a \) should be time independent, and so
\[ \eta^a = k^a t + n^a \] (57)
with some constants \( k^a \) and \( n^a \). Moreover, up to constant matrix transformations, \( k^1 = k^2 = n^2 = 0 \). As a result equation is reduced to the following system:
\begin{align*}
V_3^1 &= 2n^3 V^2, \quad V_3^2 = 2(n^1 V^3 - n^3 V^1) \\
V_3^3 &= k^3 - 2n^1 V^2, \quad k^3 V^1 = k^3 V^2 = 0
\end{align*}
(58) (59)
which has two classes of solutions defined up to constant matrix transformations:
\begin{align*}
V^1 &= V^2 = 0, \quad V^3 = k^3 x_3 + \Phi(x_1, x_2), \quad \eta^1 = \eta^2 = 0, \quad \eta^3 = k^3 t + n^3, \\
V^3 &= \Phi(x_1, x_2), \quad V^1 = V^2 = 0, \quad \eta^1 = \eta^2 = \eta^3 = 0,
\end{align*}
(60)
and
\begin{align*}
V^1 &= \Phi \cos(2n x_3) + \tilde{\Phi}(x_1, x_2) \sin(2n x_3), \\
V^2 &= \Phi(x_1, x_2) \sin(2n x_3) - \tilde{\Phi}(x_1, x_2) \cos(2n x_3), \quad V^3 = \tilde{G}(x_1, x_2), \\
\eta^1 &= \eta^2 = 0, \quad \eta^3 = n^3 = n
\end{align*}
(61)
where \( \Phi(x_1, x_2), \tilde{\Phi}(x_1, x_2), G(x_1, x_2) \) and \( \tilde{G}(x_1, x_2) \) are arbitrary functions of \( x_1 \) and \( x_2 \).

Solutions (60) are not interesting since they correspond to the direct sum of ordinary SEs considered in the previous section. However, solutions (61) generate the matrix potential which cannot be diagonalized. Namely, these solutions generate the following potential (3):
\[ V = N(x_1, x_2) + F(x_1, x_2) M(n, x_3) \] (62)
where \( N(x_1, x_2) \), \( F(x_1, x_2) \) and \( M(n, x_3) \) are matrices of the following generic form:
\begin{align*}
N(x_1, x_2) &= G(x_1, x_2) + \sigma_3 \tilde{G}(x_1, x_2), \\
F(x_1, x_2) &= \Phi(x_1, x_2) + i \sigma_3 \tilde{\Phi}(x_1, x_2), \\
M(n, x_3) &= \sigma_1 \cos(2n x_3) + \sigma_2 \sin(2n x_3)
\end{align*}
(63)
with arbitrary functions \( G, \tilde{G}, \Phi \) and \( \tilde{\Phi} \).
In the case of standard SE the symmetry $P_3$ induces the symmetry $G_3$. It can be verified by the direct calculation that for matrix potential [3], (61) it is not the case, and so the related SP equation is not Galilei invariant.

Thus we have generalized the potential $V = G(x_1, x_2)$ presented in Item 4 of Table 1 which admit symmetry $P_3$ to the case of matrix potential given by equation (62). However the SE with potential $V = G(x_1, x_2)$ admits equivalence transformation (34) while in the case of matrix potential (63) it loses this property. It means that the term $\kappa x_3$ which we omit in the case of scalar potential (since it can be reduced to zero by the equivalence transformation (34)) now cannot be omitted, and so we should add it to potential (62) as it is done in Item 1 of Table 3. The same situation appears in all cases when potentials include matrix $M(n, x_3)$ given by equation (63).

In complete analogy with the above we solve equations (15) corresponding to the first symmetries presented in Items 2 and 3, i.e., to $L_3$ and $L_3 + P_3$. The only new element is the change of variables of $V^a$ from $x_3$ to $\varphi$ and $\varphi + x_3$. As a result we obtain the following solutions

$$V = N(\tilde{r}, x_3) + F(\tilde{r}, x_3)M(n, \varphi) + \kappa \varphi,$$

$$Q_1 = L_3 + \sigma_3 n$$

and

$$V = N(\tilde{r}, x_3 - \varphi) + F(\tilde{r}, x_3 - \varphi)M(n, \varphi),$$

$$Q_2 = L_3 + P_3 + \sigma_3 n$$

where $Q_1$ and $Q_2$ are symmetries admitted by the SP equation with the given potentials.

The meaning of symbols $N(., .)$, $F(., .)$ and $M(., .)$ in equations (64) and (65) is the same as in equation (62). They denote matrices (63) depending on the arguments indicated in the brackets:

$$N(\tilde{r}, x_3) = G(\tilde{r}, x_3) + \sigma_3 \tilde{G}(\tilde{r}, x_3),$$

$$F(\tilde{r}, x_3) = \Phi(\tilde{r}, x_3) + i\sigma_3 \tilde{\Phi}(\tilde{r}, x_3),$$

$$M(n, \varphi) = \sigma_1 \cos(2n\varphi) + \sigma_2 \sin(2n\varphi),$$

etc.

A bit more efforts are requested for solution of equations (15) corresponding to the symmetry operator present in Item 4 of Table 1 since in this case parameter $\alpha$ is not trivial but equal to 2. However up to this small generalization the solution procedure is the same and gives the results presented in Item 4 of Table 3.
Table 3. Symmetries for SP equation without quadratic potential.

| No Potential $V$ | Symmetries | Algebras |
|------------------|-------------|----------|
| 1 $N(x_1, x_2) + F(x_1, x_2)M(n, x_3) + n k x_3$ | $P_3 + \sigma_3 n + n k t$ and $G_3$ if $n = 0$ | $n_{3,1}$ if $n k \neq 0$, $3n_{1,1}$ if $k = 0, n \neq 0$, $n_{4,1}$ if $n = 0$ |
| 2 $N(\tilde{r}, x_3) + F(\tilde{r}, x_3)M(n, \varphi) + n k \varphi$ | $L_3 + n k t + \sigma_3 n$ | $n_{3,1}$ if $n k \neq 0$, $3n_{1,1}$ if $k = 0$ |
| 3 $N(\tilde{r}, x_3 - \varphi) + F(\tilde{r}, x_3 - \varphi)M(n, \varphi)$ | $L_3 + P_3 + \sigma_3 n + n k t$ | $n_{3,1}$ if $k \neq 0$, $3n_{1,1}$ if $k = 0$ |
| 4 $\frac{1}{r^2}N(\theta, r^n \kappa e^{-\varphi}) + \frac{1}{r^2}F(\theta, r^n \kappa e^{-\varphi})M(n, y)$ | $D + k L_3 + n(k + \nu)\sigma_3$ | $s_{2,1} \oplus n_{1,1}$ |
| | $y = \varphi + \nu \ln(r)$, $\nu + k \neq 0$ | |
| 5 $N(\tilde{r}) + n k \varphi + F(\tilde{r})M(n, x_3) + \nu x_3$, $n \neq 0$ | $L_3 + n k t$, $P_3 + n \sigma_3 + \nu t$ | $n_{3,1} \oplus n_{1,1}$ if $k^2 + \nu^2 \neq 0$, $4n_{1,1}$ if $k = \nu = 0$ |
| 6 $N(\tilde{r}) + n k \varphi + F(\tilde{r})M(n, \varphi)$ | $L_3 + n k t + n \sigma_3$ | $n_{3,1} \oplus n_{1,1}$ if $n k \neq 0$, $4n_{1,1}$ if $k = 0, n \neq 0$, $n_{4,1} \oplus n_{1,1}$ if $n = 0$ |
| 7 $\frac{1}{r^2}N(r^n \kappa e^{-\varphi}) + \frac{1}{r^2}F(\tilde{r}^n \kappa e^{-\varphi})M(n, \varphi)$ | $D + k(L_3 + n \sigma_3)$, $P_3$ | $s_{2,1} \oplus 2n_{1,1}$ |
| 8 $\frac{1}{r^2}N(\theta) + \frac{1}{r^2}F(\theta)M(n, \varphi)$, $n \neq 0$ | $D$, $L_3 + n \sigma_3$ | $s_{2,1} \oplus 2n_{1,1}$ |
| 9 $\frac{1}{r^2}N(\theta) + \frac{1}{r^2}F(\theta)M(n, \ln(r))$ | $D + n \sigma_3$, $L_3$ and $A$ if $n = 0$ | $s_{2,1} \oplus 2n_{1,1}$ if $n \neq 0$, $s_{3,1} \oplus n_{1,1}$ if $n k \neq 0$, $s_{3,1} \oplus 2n_{1,1}$ if $n = 0$ |
| 10 $N(x_2) + F(x_2)M(n, x_3) + n k x_3$ | $P_1$, $P_3 + n \sigma_3 + n k t$ and $G_1$, $G_3$, $L_2$ if $n = 0$ | $4n_{1,1}$ if $k = 0$, $g(1,2)$ if $n = 0$ |
| 11 $G(r) + \Phi(r)\sigma_a x_a$ | $L_a + \frac{1}{2}\sigma_a$, $a = 1, 2, 3$ | $so(3) \oplus 2n_{1,1}$ |
| 12 $\frac{\nu}{r^2} + \sigma_3 \frac{\nu}{r^2} + \frac{\sigma_3}{r^2}M(n, \ln(r))$ | $D + n \sigma_3$, $L_1$, $L_2$, $L_3$ and $A$ if $n = 0$ | $s_{2,1} \oplus n_{1,1} \oplus so(3)$ if $n \neq 0$, $sl(2, R) \oplus so(3) \oplus n_{1,1}$ if $n = 0$ |
| 13 $\frac{\nu}{r^2} + \sigma_3 \frac{\mu}{r^2} + \frac{\sigma_3}{r^2}M(n, \varphi)$, $n \neq 0$ | $D$, $L_3 + n \sigma_3$, $P_3$ | $s_{3,1} \oplus 2n_{1,1}$ |
| 14 $\frac{\nu}{r^2} + \sigma_3 \frac{\mu}{r^2} + \frac{\sigma_3}{r^2}M(n, \ln(\tilde{r}))$ | $D + n \sigma_3$, $P_3$, $L_3$ and $A$ if $n = 0$ | $s_{3,1} \oplus 2n_{1,1}$ if $n \neq 0$, $sl(2, R) \oplus 3n_{1,1}$ if $n = 0$ |
| 15 $\frac{\nu}{r^2} + \sigma_3 \frac{\mu}{r^2} + \frac{\sigma_3}{r^2}M(n, \ln(x_3))$ | $D + n \sigma_3$, $P_1$, $P_2$, $L_3$ and $A$, $G_1$, $G_2$ if $n = 0$ | $s_{5,43} \oplus n_{1,1}$ if $n \neq 0$, $schr(1,2)$ if $n = 0$ |
| 16 $\frac{\nu}{r^2} + \frac{\mu}{r^2}\sigma_a x_a$ | $D$, $A$, $L_a + \frac{1}{2}\sigma_a$, $a = 1, 2, 3$ | $s_{2,1} \oplus n_{1,1} \oplus so(3)$ |
In Table 3 the symbols $M(\cdot), N(\cdot), F(\cdot)$ denote matrices defined in equations (63) and (66) where $n$ can take arbitrary values including zero, $G(r)$ and $\Phi(r)$ are arbitrary functions.

The potentials presented in Items 1–4 of Table 3 include arbitrary functions, and the related SP equations admit one dimensional symmetry algebras generated by operators (45). For some particular functions $G, \tilde{G}, \Phi$ and $\tilde{\Phi}$ these symmetries can be extended to two dimensional algebras presented by equation (49). To fix these particular functions we can start with the first of two symmetries presented in some brackets given in (49) and use the corresponding potential presented in one of the items 1-4. Then, asking for existence of the second symmetry we specify the mentioned arbitrary functions. The other way which in general leads to another result is to start with the second symmetry with the related potential and ask for existence of the first symmetry. In this way we find the versions presented in Items 5-9 of Table 3.

Analogously, asking for extensions of the two dimension symmetry algebras we obtain the potentials and the related generators presented in Items 10-16. The potential in Item 10 includes two arbitrary functions while the remaining potentials are defined up to arbitrary parameters.

Thus we have generalized all symmetries and potentials proportional to the unit matrix presented in Table 1, to the case of generic matrix potentials. The related classification results are summarized in Table 3.

V. SYMMETRIES OF SP EQUATIONS WITH OSCILLATOR TERMS

The final step of our classification consists in the generalization of the scalar potentials which include the harmonic oscillator terms. Such potentials are presented in Table 2. In addition, we are supposed to analyze all versions presented in Table 1 and marked by the asterisk. The related potentials can be generalized to include the isotropic harmonic oscillator term. Moreover, such term cannot be removed using the equivalence transformations (32) and (33) provided the potential includes non-diagonal matrix terms.

Starting with the analogous reasons, it is necessary to generalize the potentials including linear terms, which can be generated using equivalence transformations (34) starting with the data of Table 1 marked by the star.

Let us consider consequently all symmetries presented in Table 2 and find matrix potentials compatible with them. To do it we are supposed solve equations (15) for $V^a$ where functions $\xi^a$ can be found comparing definition (4) and explicit expressions for symmetry operators presented...
in the Table.

Considering symmetries presented in Item 1 of Table 2 we find that neither \( B_3 \) nor \( \hat{B}_3 \) generate non-trivial solutions of equation (15) for components \( V^a \) of matrix potential (3). However, the linear combinations \( Q^\pm = B_3 \pm \hat{B}_3 \) are compatible with non-trivial \( V^a \). Indeed, in this case the only nonzero components \( \xi^a \) and \( \eta^0 \) are

\[
\xi^3 = \exp(\pm \omega t), \quad \eta^0 = \mp \exp(\pm \omega t) \omega x_3.
\]

The related components \( \eta^a \) should have the same dependence on \( t \) as \( \xi^3 \), and so in analogy with (60) we have to set

\[
\eta_1 = \eta_2 = 0, \quad \eta^3 = n \exp(\pm \omega t).
\]

In the following we omit signs \( \pm \) but reserve the possibilities for parameter \( \omega \) be positive or negative, and write the corresponding symmetry as

\[
Q = \exp(\omega t)(P_3 - \omega x_3 + n\sigma_3).
\]

Substituting these data into equation (15) we come to the following system:

\[
V_3^1 = 2nV^2, \quad V_3^2 = -2nV^1, \quad V_3^3 = \omega n.
\]

Equation (71) is easy solved. System (70) coincides with (58) where \( k_1 = k_2 = k_3 = n_1 = n_2 = 0 \). Thus its solution is given by formula (61). Just this solution is presented in Item 1 of Table 4 together with the scalar term \(-\frac{1}{2} \omega^2 x_3^2\). Moreover, the potential presented here generalizes the scalar potential presented in Item 1 of Table 2 to the matrix case.

Thus we have found an example of matrix potential which include the repulsive oscillator term such that equation SP equation (11) admits a one parametric Lie group additional to shifts of the time variable. This potential includes three arbitrary matrices of special form fixed in (63) and dependent on two or one spatial variables. To classify the potentials which admit symmetry (15) and are compatible with more extended symmetry groups we should apply the additional conditions (53) where \( \xi^a \) and \( \eta^a \) are functions specifying the additional symmetries (4), (5). Since all admissible inequivalent symmetries are presented in Table 2, we can easily fix these functions and then solve the related equations (53).
Let the SP equation with generic potential presented in Item 1 of Table 4 admits the additional symmetry $L_3 + \kappa t$ fixed in Item 2 of Table 2. The corresponding functions $N(x_1, x_2)$ and $F(x_1, x_2)$ should be rotationally invariant, i.e., to depend on $\tilde{r} = \sqrt{x_1^2 + x_2^2}$, and we come to the potential presented in Item 2 of Table 4. For the case of additional symmetry $P_1$ these functions can depend only on $x_2$, and we come to the potential fixed in Item 3 of Table 4, and so on and so on.

In this way we find all potentials presented in Items 1 - 7 of Table 4. Notice that this list does not include matrix extensions of scalar potentials presented in Items 5, 8 and 10 of Table 2 since the related conditions (53) appears to be incompatible.

Consider now the case when the potential includes the 3d oscillator term $V^0 = \frac{\varepsilon}{2}\omega^2 r^2$. This term cannot be reduced to zero provided the other components $V^a$ are functions of $x$. Moreover, it is nothing but a particular case of the potential presented in Item 6 of Table 2, whose possible extensions to the case of matrix potential have been already classified in the above, see Items 3-7 of Table 3. The only specific point is that now we have to consider the additional symmetries $A^\pm$ and $\tilde{A}^\pm$ whose explicit form is given in (42) and (43).

It is easy to show that neither $A^+, \tilde{A}^+$ nor their linear combinations are compatible with equations (53). However, the linear combinations $\tilde{Q}^\pm = A^- \pm \tilde{A}^-$ are the only symmetries including $A^+$ and $\tilde{A}^+$ which solve this equation with the corresponding $V^a$. These symmetries can be represented as:

$$\tilde{Q} = \exp(2\omega t) \left( P_0 + \omega x_a P_a - \frac{3\omega i}{2} + \omega^2 r^2 \right)$$  \hspace{1cm} (72)

where we omit the signs $\pm$ but reserve the possibilities for parameter $\omega$ to be positive or negative. Comparing (72) with (4) we find the corresponding functions $\xi^a$ and $\eta^0$ in the following form:

$$\xi^a = \exp(2\omega t) \omega x_a, \quad \eta^0 = \exp(2\omega t) \left( \omega^2 r^2 - \frac{3\omega i}{2} \right)$$  \hspace{1cm} (73)

The corresponding functions $\eta^a$ should have the same dependence on $t$ as $\xi^a$, and so up to matrix transformations and in analogy with (60) we can set

$$\eta^3 = \exp(2\omega t) \omega n \sigma_3, \quad \eta^1 = \eta^2 = 0$$  \hspace{1cm} (74)

Substituting (74) into equation (15) we obtain the following system:

$$x_a V^1_a = -2V^1 + 2nV^2, \quad x_a V^2_a = -2V^2 - 2nV^1, \quad x_a V^3_a = -2V^3 + 2n$$  \hspace{1cm} (75)
whose generic solution is presented in Item 8 of Table 4. This solution is compatible with the only symmetry \( Q = \dot{Q} + \eta_3 \) additional to \( P_0 \).

In analogy with the above we can specify arbitrary function \( N(\varphi, \theta) \) and \( F(\varphi, \theta) \) in such way that symmetries of the corresponding SP equation became more extended. All such inequivalent specifications are enumerated in Items 9–13 of Table 4.

Table 4. Potentials with oscillator terms and symmetries for Schrödinger-Pauli equation.

| No | Potential \( V \) | Symmetries | Algebras |
|----|-----------------|-------------|----------|
| 1  | \( N(x_1, x_2) + \sigma_3 \omega n x_3 - \frac{1}{2} \omega^2 x_3^2 + F(x_1, x_2) M(n, x_3) \) | \( Q \) | \( s_{2,1} \oplus n_{1,1} \) |
| 2  | \( N(\tilde{r}) + \sigma_3 \omega n x_3 + F(\tilde{r}) M(n, x_3) + n k \varphi - \frac{1}{2} \omega^2 x_3^2 \) | \( Q, L_3 + nk \tau \) | \( 3n_{1,1} \) |
| 3  | \( N(x_2) + \sigma_3 \omega n x_3 - \frac{1}{2} \omega^2 x_3^2 + F(x_2) M(n, x_3) \) | \( Q, P_1, G_1 \) | \( s_{2,1} \oplus 2n_{1,1} \) |
| 4  | \( N(x_1) + F(x_1) M(n, x_3) + \frac{s_1}{2} \omega^2 x_2^2 - \frac{1}{2} \omega^2 x_3^2 + n \omega \sigma_3 x_3 \) | \( Q, B_2^\varepsilon, \tilde{B}_2^\varepsilon \) | \( s_{5,17} \) |
| 5  | \( \mu M(n, x_3) + \frac{s_1}{2} \omega^2 x_2^2 - \frac{1}{2} \omega^2 x_3^2 + \sigma_3 n \omega x_3 \) | \( Q, B_2^\varepsilon, \tilde{B}_2^\varepsilon, P_1 \) | \( s_{5,17} \oplus n_{1,1} \) |
| 6  | \( \mu M(n, x_3) + \frac{s_1}{2} \omega^2 x_1^2 + \frac{s_1}{2} \omega^2 x_2^2 - \frac{1}{2} \omega^2 x_3^2 + \sigma_3 n \omega x_3 \) | \( Q, B_1^\varepsilon, \tilde{B}_1^\varepsilon, B_2^\varepsilon, \tilde{B}_2^\varepsilon \) | \( s_{7,3} \) |
| 7  | \( \mu M(n, x_3) + \frac{s_1}{2} \omega^2 x_1^2 - \frac{1}{2} \omega^2 x_3^2 + \sigma_3 n \omega x_3 \) | \( Q, B_1^\varepsilon, \tilde{B}_1^\varepsilon, \) | \( s_{7,3} \oplus n_{1,1} \) |
| 8  | \( \sigma_3 \omega n - \frac{1}{2} \omega^2 r^2 + \frac{1}{r^2} N(\varphi, \theta) + \frac{1}{r} F(\varphi, \theta) M(n, \ln(r)) \) | \( \tilde{Q} \) | \( s_{2,1} \oplus n_{1,1} \) |
| 9  | \( \sigma_3 \omega n - \frac{1}{2} \omega^2 r^2 + \frac{k}{r} N(\varphi) + \frac{1}{r} F(\varphi) M(n, \ln(r)) \) | \( \tilde{Q}, L_3 \) | \( s_{2,1} \oplus 2n_{1,1} \) |
| 10 | \( \sigma_3 \omega n - \frac{1}{2} \omega^2 r^2 + \frac{s_1}{s_2} N(\varphi) + \frac{1}{s_2} F(\varphi) M(n, \ln(\tilde{r})) \) | \( \tilde{Q}, B_3^\varepsilon, \tilde{B}_3^\varepsilon \) | \( s_{6,1} \oplus n_{1,1} \) |
| 11 | \( \sigma_3 \omega n - \frac{s_1}{s_2} \omega^2 r^2 + \frac{s_1}{s_2} M(n, \ln(r)) \) | \( \tilde{Q}, L_1, L_2, L_3 \) | \( \text{so}(3) \oplus s_{2,1} \oplus n_{1,1} \) |
| 12 | \( \sigma_3 \omega n - \frac{s_1}{s_2} \omega^2 r^2 + \frac{s_1}{s_2} + \frac{1}{s_2} M(n, \ln(\tilde{r})) \) | \( \tilde{Q}, L_3, B_3^\varepsilon, \tilde{B}_3^\varepsilon \) | \( s_{6,1} \oplus n_{1,1} \) |
| 13 | \( \sigma_3 \omega n - \frac{s_1}{s_2} \omega^2 r^2 + \frac{s_1}{s_2} + \frac{1}{s_2} M(n, \ln(x_3)) \) | \( \tilde{Q}, L_3, B_3^\varepsilon, \tilde{B}_3^\varepsilon \) | \( s_{6,1} \oplus n_{1,1} \) |
| 14 | \( N(\tilde{r}) + F(\tilde{r}) M(n, \varphi) + \frac{s_1}{2} \omega^2 x_3^2 \) | \( L_3 + \sigma_3 n, B_3^\varepsilon, \tilde{B}_3^\varepsilon \) | \( s_{4,6} \oplus n_{1,1} \) |
| 15 | \( N(x_2) + \frac{s_1}{2} \omega^2 x_1^2 + F(x_2) M(n, x_3) + n \kappa x_3 \) | \( P_3 + n \sigma_3 + nk \tau, B_3^\varepsilon, \tilde{B}_3^\varepsilon \) | \( s_{4,6} \oplus n_{1,1} \) |
| 16 | \( \mu M(n, x_3) + \frac{s_1}{2} \omega^2 x_1^2 + \frac{s_1}{2} \omega^2 x_2^2 + n \kappa x_3 \) | \( B_3^\varepsilon, \tilde{B}_3^\varepsilon, B_3^\varepsilon, \tilde{B}_3^\varepsilon \) | \( s_{7,2} \oplus n_{1,1} \) |
| 17 | \( \mu M(n, x_3) + \frac{s_1}{2} \omega^2 \tilde{r}^2 + n \kappa x_3 \) | \( L_3, P_3 + n \sigma_3 + nk \tau \) | \( s_{8,1} \) if \( \kappa \neq 0 \) |
The remaining items 14–17 of Table 4 present the potentials which also include the oscillator or repulsive oscillator terms, but the corresponding symmetries do not include either $Q$ or $\tilde{Q}$. Notice that these potentials are analytical in $\omega$ and so they are well defined for trivial $\omega$ also.

In the table $Q$ and $\tilde{Q}$ are generators fixed in (62) and (72), algebras $s_{6,n}$ are specified in Item 4 of Table 2. The symbols $N(\cdot), F(\cdot), M(n, \cdot)$ are used to denote matrices (66) depending on the arguments fixed in the brackets, $\mu$ and $\lambda$ are arbitrary real parameters.

Symmetries specified in Table 4 correspond to non-zero values of parameter $n$. However, the corresponding equations are well defined also for $n = 0$. In this case symmetry algebras are more extended, namely, symmetries $Q$ and $\tilde{Q}$ are replaced by the pairs $< B^{-3}, \tilde{B}^{-3} >$ and $< A^{-}, \hat{A}^{-} >$ correspondingly. In addition, symmetry $G_3$ appears in Items 15, 16, 17 and symmetry $G_1$ should be included into Item 5.

VI. DISCUSSION

The main goal of the present paper was to give the group classification of SP equations for neutral particles. This program has been realized, the classification results are summarized in Tables 3 and 4. In accordance with these results there exist 33 inequivalent equations of this type which admit different symmetry groups. The most extended groups are eight parametrical and are presented in Items 7 and 13 of Table 4.

The group classification presents a priori information about all admissible symmetries of the considered class of equations and the explicit form of the corresponding arbitrary elements which in our case are matrix potentials. Such information is useful for construction of physical models with requested symmetries. Moreover, it is an important and in fact the necessary step in search for systems with different kinds of generalized symmetries, in particular, superintegrable systems, since it supplies us by important equivalence relations.

A particular and important case of matrix potentials is the case of diagonal matrices, when we have a direct sum of standard Shcrödinger equations with position dependent potential. The group classification of such SEs is a subproblem of our more general problem, and we present its solutions in Tables 1 and 2. In contrast with the well known presentation in paper [3] our classification results are completed and include four case missing in [3], see Items 1, 2, 7 of Table 1 and Item 2 of Table 2. In addition, we make our best to specify clearly the equivalence relations which are different for different potentials.
The potential terms $\varphi = \kappa \arctan (x_2/x_1)$ missing in the Boyer classification belongs to the class of harmonic potentials which find many interesting applications including such exotic ones as the robot navigation.

One more new feature of our presentation is the clear specification of the invariance algebras using notations proposed in [40]. We believe that this information is important and useful. In particular, the reader interested in the Casimir operators of the symmetry algebras can easy find them in book [40].

Notice that the low dimension algebras of dimension $d \leq 5$ and some class of the algebras of dimension 6 had been classified by Mubarakzianov [41], see also more contemporary and accessible papers [42] and [43] were his results are slightly corrected.

The presented list of symmetries does not include the infinite symmetry group of transformations $\psi \rightarrow \psi + \tilde{\psi}$ where $\tilde{\psi}$ is an arbitrary solution of equation (2). In accordance with the superposition rule, such evident symmetries are valid for all linear equations.

Let us note that in general our matrix potentials cannot be interpreted as a sum of scalar potentials and Pauli terms dependent on a purely magnetic field. Indeed, the found potentials $V^a$ are not supposed to be divergent less. However, many of these potentials are compatible with the condition $\partial_a V^a = 0$ which can be added additionally. Without this condition the found potentials represent more generic external fields.

Symmetries of SE with matrix potential have some specific features in comparison with the case of a scalar potential. which will be fixed in the following comments.

It is well known that the standard SE is invariant w.r.t. Galilei transformation of space variable $x_a$ iff its potential does not depend on this variable or is equivalent to such potential. For example, it is the case for the oscillator potential since the corresponding SE is equivalent to the free one, and the generators of Galilei transforms have exotic form (43).

For the case of matrix potentials this observation does not pay. Indeed, potentials fixed in Items 1–7 of Table 4 are not either trivial or equivalent to trivial. Nevertheless, they admit symmetries being linear combinations of the exotic generators of Galilei transformations.

In contrast with the standard SE for which exotic conformal generators $A^-$ and $\hat{A}^-$ (43) are valid only for the repulsive oscillator potential, there exist such matrix potentials which are not equivalent to linear ones and admit linear combinations of such generators. Just these potentials are presented in Items 8–13 of Table 4.

Let us note that the list potentials and symmetries presented in Table 4 can be generalized
to the case of imaginary parameters $\omega$. In this case the repulsive oscillator term is transformed to the harmonic oscillator one. However, the symmetry operators became non-hermitian with respect to the standard scalar product used in quantum mechanics. This paradox in principle can be overcame in frames of quantum mechanics with indefinite metrics, for example, CP-symmetric quantum mechanics \[44\].

In this paper we restrict ourselves to searching for symmetries of SP equations for neutral particles. A more generic problem of group classification of SP equations for charged particles can to be a subject of our following work. For group classification of nonlinear Schrödinger equations and their conditional symmetries see papers \[45, 46\] and \[47\].

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