On the Limit as $s \to 0^+$ of Fractional Orlicz–Sobolev Spaces

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Abstract
An extended version of the Maz’ya–Shaposhnikova theorem on the limit as $s \to 0^+$ of the Gagliardo–Slobodeckij fractional seminorm is established in the Orlicz space setting. Our result holds in fractional Orlicz–Sobolev spaces associated with Young functions satisfying the $\Delta_2$-condition, and, as shown by counterexamples, it may fail if this condition is dropped.

Keywords Fractional Orlicz–Sobolev space · Limits of smoothness parameters

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1 Introduction and Main Results

Pivotal instances of spaces of functions endowed with a non-integer order of smoothness are the Besov spaces, defined in terms of norms of differences, the Triebel–Lizorkin spaces, whose notion relies upon the Fourier transform, the Bessel potential spaces, based on representation formulas via potential operators, and the Gagliardo–Slobodeckij spaces, defined in terms of fractional difference quotients. Relations among these families of spaces are known—see e.g. [14, Remark 2.1.1] for a survey of results with this regard. It is also well known that, with the exception of the Bessel potential spaces, they do not agree, in general, with the classical integer-order Sobolev spaces when the order of smoothness is formally set to an integer.

In particular, this drawback affects the Gagliardo–Slobodeckij spaces $W^{s,p}(\mathbb{R}^n)$, which are defined, for $n \in \mathbb{N}, s \in (0,1)$ and $p \in [1,\infty)$, via a seminorm depending on an integral over $\mathbb{R}^n \times \mathbb{R}^n$ of an $s$-th-order difference quotient. However, some twenty years ago it was discovered that a suitably normalized Gagliardo–Slobodeckij seminorm in $W^{s,p}(\mathbb{R}^n)$ recovers, in the limit as $s \to 1^-$ or $s \to 0^+$, its integer-order counterpart. The result was first established at the endpoint $1^-$ by Bourgain, Brezis and Mironescu in [4,5]. In those papers it is shown that the seminorm in $W^{s,p}(\mathbb{R}^n)$ of a function $u$, times $(1-s)^{1/p}$, approaches the $L^p$ norm of $\nabla u$ as $s \to 1^-$ (up to a multiplicative constant depending only on $n$).

The problem concerning the opposite endpoint $0^+$ was solved by Maz’ya and Shaposhnikova. In [11] they proved that
\[
\lim_{s \to 0^+} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left( \frac{|u(x) - u(y)|}{|x-y|^s} \right)^p \frac{dx \; dy}{|x-y|^n} = 2n \omega_n \int_{\mathbb{R}^n} |u(x)|^p \; dx \tag{1.1}
\]
for every function $u$ decaying to 0 near infinity and making the double integral finite for some $s \in (0,1)$. Here, $\omega_n$ denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$. Several related results can be found in [10, Chapter 12].

The present paper deals with a version of property (1.1) in the broader framework of fractional Orlicz–Sobolev spaces. These spaces extend the spaces $W^{s,p}(\mathbb{R}^n)$ in that the role of the power function $t^p$ is played by a more general Young function $A: [0,\infty) \to [0,\infty)$, namely a convex function vanishing at 0. Specifically, we address the problem of the existence of
\[
\lim_{s \to 0^+} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left( \frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{dx \; dy}{|x-y|^n}, \tag{1.2}
\]
and of its value in the affirmative case. The ambient space for $u$ is $\bigcup_{s \in (0,1)} V_{d,A}^s(\mathbb{R}^n)$, where $V_{d,A}^s(\mathbb{R}^n)$ denotes the space of those measurable functions $u$ in $\mathbb{R}^n$ which render the double integral in (1.2) finite, and decay to 0 near infinity, in the sense that
\[
|\{x \in \mathbb{R}^n : |u(x)| > t\}| < \infty \quad \text{for every } t > 0.
\]
Here, $|E|$ stands for the Lebesgue measure of a set $E \subset \mathbb{R}^n$. 

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A partial result in this connection is contained in the recent contribution [6], where bounds for the lim inf$_{s \to 0^+}$ and lim sup$_{s \to 0^+}$ of the expression under the limit in (1.2) are given for Young functions $A$ satisfying both the $\Delta_2$-condition and the $\nabla_2$-condition. Recall that this condition amounts to requiring that there exists a constant $c$ such that

$$A(2t) \leq c A(t) \quad \text{for } t \geq 0.$$ 

Our results provide a full answer to the relevant problem. We prove that, under the $\Delta_1^2$-condition on $A$, the limit in (1.2) does exist, and equals the integral of a function of $|u|$ over $\mathbb{R}^n$. Moreover, we show that the result can fail if the $\Delta_2$-condition is dropped. Interestingly, the function of $|u|$ appearing in the integral obtained in the limit is not $A$, but rather the Young function $A$ associated with $A$ by the formula

$$A(t) = \int_0^t \frac{A(\tau)}{\tau} d\tau \quad \text{for } t \geq 0.$$ 

Notice that $A$ and $\overline{A}$ are equivalent as Young functions, since $A(t/2) \leq \overline{A}(t) \leq A(t)$ for $t \geq 0$, owing to the monotonicity of $A(t)$ and $A(t)/t$.

**Theorem 1.1** Let $n \in \mathbb{N}$ and let $A$ be a Young function satisfying the $\Delta_2$-condition. Assume that $u \in \bigcup_{s \in (0, 1)} V^s_d(A_d)(\mathbb{R}^n)$. Then

$$\lim_{s \to 0^+} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{dx \, dy}{|x - y|^n} = 2n \omega_n \int_{\mathbb{R}^n} \overline{A}(|u(x)|) \, dx.$$ 

Plainly, Eq. (1.3) recovers (1.1) when $A(t) = t^p$ for some $p \geq 1$, inasmuch as $\overline{A}(t) = t^p/p$ in this case.

The indispensability of the $\Delta_2$-condition for the function $A$ is demonstrated via the next result.

**Theorem 1.2** Let $n \in \mathbb{N}$. There exist Young functions $A$, which do not satisfy the $\Delta_2$-condition, and corresponding functions $u : \mathbb{R}^n \to \mathbb{R}$ such that $u \in V^s_d(A_d)(\mathbb{R}^n)$ for every $s \in (0, 1)$,

$$\int_{\mathbb{R}^n} \overline{A}(|u(x)|) \, dx \leq \int_{\mathbb{R}^n} A(|u(x)|) \, dx < \infty,$$

but

$$\lim_{s \to 0^+} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|u(x) - u(y)|}{|x - y|^s}\right) \frac{dx \, dy}{|x - y|^n} = \infty.$$ 

Incidentally, let us mention that an analogue of the Bourgain–Brezis–Mironescu theorem on the limit as $s \to 1^-$ for fractional Orlicz–Sobolev spaces built upon Young functions satisfying both the $\Delta_2$-condition and the $\nabla_2$-condition can be found in [9]. These conditions are removed in a version of this result offered in [2]. Further properties and applications of fractional Orlicz–Sobolev spaces are the subject of [1,3,8,13].
2 Proof of Theorem 1.1

Our approach to Theorem 1.1 is related to that of [11], yet calls into play specific Orlicz space results and techniques. In particular, it makes critical use of a Hardy type inequality for functions in $V_{d}^{s,A}(\mathbb{R}^n)$, with $s \in (0, 1)$, recently established in [1, Theorem 5.1]. This inequality tells us what follows.

Given a Young function $A$, denote by $a : [0, \infty) \to [0, \infty)$ the left-continuous non-decreasing function such that

$$ A(t) = \int_0^t a(\tau)d\tau \quad \text{for } t \geq 0. $$

Assume that

$$ \int_0^\infty \left( \frac{t}{A(t)} \right)^{\frac{s}{n-s}} dt = \infty \quad (2.1) $$

and

$$ \int_0^t \left( \frac{t}{A(t)} \right)^{\frac{s}{n-s}} dt < \infty. \quad (2.2) $$

Call $B$ the Young function defined by

$$ B(t) = \int_0^t b(\tau)d\tau \quad \text{for } t \geq 0, $$

where the (generalized) left-continuous inverse of the function $b$ obeys

$$ b^{-1}(r) = \left( \int_{a^{-1}(r)}^\infty \left( \int_0^t \left( \frac{1}{a(\varrho)} \right)^{\frac{s}{n-s}} d\varrho \right)^{-\frac{n}{s}} \frac{dt}{a(t)^{\frac{n}{n-s}}} \right)^{\frac{s}{s-n}} \quad \text{for } r \geq 0. $$

Then, there exists a constant $C = C(n, s)$ such that

$$ \int_{\mathbb{R}^n} B \left( \frac{|u(x)|}{|x|^s} \right) dx \leq (1 - s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( C \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx}{|x|^n} dy \quad (2.3) $$

for every function $u \in V_{d}^{s,A}(\mathbb{R}^n)$. Moreover, the constant $C$ is uniformly bounded from above in $s$ if $s$ is bounded away from 1.

Proof of Theorem 1.1 For ease of presentation, we split the proof in a few steps. 

Step 1. Here we show that there exists $s_1 \in (0, 1)$ such that, if $u \in V_{d}^{s,A}(\mathbb{R}^n)$ for $s \in (0, s_1)$, then

$$ \int_{\mathbb{R}^n} \overline{A} \left( \frac{|u(x)|}{\lambda |x|^s} \right) dx \leq \int_{\mathbb{R}^n} A \left( \frac{|u(x)|}{\lambda |x|^s} \right) dx < \infty \quad (2.4) $$
for every $\lambda > 0$.

To this purpose, recall that, inasmuch as $A$ satisfies the $\Delta_2$-condition, its upper Matuszewska–Orlicz index $I(A)$, introduced in [12] and defined as

$$I(A) = \lim_{\lambda \to \infty} \frac{\log \left( \sup_{t > 0} \frac{A(\lambda t)}{A(t)} \right)}{\log \lambda},$$

is finite. A standard (and easily verified) consequence of this fact is that there exists a positive constant $C = C(A)$ such that

$$A(\lambda t) \leq C \lambda^{I(A) + 1} A(t) \quad \text{for } t \geq 0 \text{ and } \lambda \geq 1.$$  \hspace{1cm} (2.5)

Thereby, there exists $s_0 \in (0, 1)$ such that conditions (2.1) and (2.2) are fulfilled if $s \in (0, s_0)$. Hence, inequality (2.3) holds for $s \in (0, s_0)$.

On the other hand, $I(A) < \frac{n}{s}$ provided that $s < \frac{n}{I(A)}$. Hence, [7, Proposition 5.2] ensures that the function $B$ is equivalent to $A$ if $s < \frac{n}{I(A)}$. Namely, there exist constants $c_2 > c_1 > 0$ such that $A(c_1 t) \leq B(t) \leq A(c_2 t)$ for $t \geq 0$. Set $s_1 = \min\{s_0, \frac{n}{I(A)}\}$. As a consequence of inequality (2.3), of the equivalence of $A$ and $B$, of the $\Delta_2$-condition for $A$, and of the inequality $\overline{A} \leq A$, property (2.4) holds for every function $u$ such that $u \in V^s_\Delta(\mathbb{R}^n)$ for $s \in (0, s_1)$.

**Step 2.** We next prove that the $\liminf_{t \to 0^+}$ of the expression on the left-hand side of Eq. (1.3) is bounded from below by the right-hand side. Namely, we show that

$$\liminf_{s \to 0^+} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|u(x) - u(y)|}{|x - y|^{s}} \right) \frac{dx \, dy}{|x - y|^n} \geq 2 \omega_n \int_{\mathbb{R}^n} \overline{A} \left( |u(x)| \right) \, dx.$$  \hspace{1cm} (2.6)

In particular, inequality (2.6) implies that, if the integral on the right-hand side diverges, then Eq. (1.3) certainly holds. Thus, in what follows, we may assume that it converges. Hence, since $A$ satisfies the $\Delta_2$-condition and $\overline{A}$ is equivalent to $A$, we may assume that

$$\int_{\mathbb{R}^n} \overline{A} \left( \frac{|u(x)|}{\lambda} \right) \, dx < \infty$$  \hspace{1cm} (2.7)

for every $\lambda > 0$.

In order to establish inequality (2.6), let us begin by observing that

$$\int_{\mathbb{R}^n} \int_{|x - y| > 2|x|} A \left( \frac{|u(x)|}{|x - y|^{s}} \right) \frac{dx \, dy}{|x - y|^n} = n \omega_n \int_{\mathbb{R}^n} \int_{2|x|}^{\infty} A \left( \frac{|u(x)|}{r^s} \right) \frac{dr \, dx}{r} = \frac{n \omega_n}{s} \int_{\mathbb{R}^n} \overline{A} \left( \frac{|u(x)|}{2^s |x|^s} \right) \, dx.$$  \hspace{1cm} (2.8)
Fix $\varepsilon > 0$. Owing to the convexity of $A$,

$$\int_{\mathbb{R}^n} \int_{|x-y|>2|x|} A \left( \frac{|u(x)|}{|x-y|^s} \right) \frac{dx \, dy}{|x-y|^n}$$

$$\leq \frac{1}{1+\varepsilon} \int_{\mathbb{R}^n} \int_{|x-y|>2|x|} A \left( \frac{1+\varepsilon}{\varepsilon} \frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{dx \, dy}{|x-y|^n}$$

$$+ \frac{\varepsilon}{1+\varepsilon} \int_{\mathbb{R}^n} \int_{|x-y|>2|x|} A \left( \frac{1+\varepsilon}{\varepsilon} \frac{|u(y)|}{|x-y|^s} \right) \frac{dx \, dy}{|x-y|^n} = I_1 + I_2. \quad (2.9)$$

Consider the integral $I_2$. If $|x-y| > 2|x|$, then $\frac{2}{3}|y| \leq |x-y| \leq 2|y|$. Therefore,

$$I_2 \leq \frac{\varepsilon}{1+\varepsilon} \int_{\mathbb{R}^n} \int_{|x-y|>2|x|} A \left( \frac{1+\varepsilon}{\varepsilon} \left( \frac{3}{2} \right)^s \frac{|u(y)|}{|y|^s} \right) \left( \frac{3}{2} \right)^n \frac{dy \, dx}{|y|^n}$$

$$= \frac{\varepsilon}{1+\varepsilon} \left( \frac{3}{2} \right)^n \int_{\mathbb{R}^n} \frac{1}{|y|^n} A \left( \frac{1+\varepsilon}{\varepsilon} \left( \frac{3}{2} \right)^s \frac{|u(y)|}{|y|^s} \right) \left( \int_{|x-y|\geq 2|x|} dx \right) dy$$

$$\leq \frac{\varepsilon \omega_n}{1+\varepsilon} \int_{\mathbb{R}^n} A \left( \frac{1+\varepsilon}{\varepsilon} \left( \frac{3}{2} \right)^s \frac{|u(y)|}{|y|^s} \right) dy. \quad (2.10)$$

Note that the last inequality holds since, for each $y \in \mathbb{R}^n$, the set $\{|x-y| > 2|x|\}$ agrees with the ball centered at $-\frac{1}{2}y$, with radius $\frac{2}{3}|y|$.

In order to estimate the integral $I_1$, observe that

$$\int_{|x-y|>2|x|} A \left( \frac{1+\varepsilon}{\varepsilon} \frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{dx \, dy}{|x-y|^n}$$

$$= \int_{|x-y|>2|x|} A \left( \frac{1+\varepsilon}{\varepsilon} \frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{dx \, dy}{|x-y|^n}.$$}

Furthermore, $\{|x-y| > 2|x|\} \cap \{|x-y| > 2|y|\} = \emptyset$, since if there existed $x$, $y$ such that $|x-y| > 2|x|$ and $|x-y| > 2|y|$, then $|x-y| \leq |x|+|y| < \frac{|x-y|}{2} + \frac{|x-y|}{2} = |x-y|$, a contradiction. Thus,

$$I_1 \leq \frac{1}{2(1+\varepsilon)} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{1+\varepsilon}{\varepsilon} \frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{dx \, dy}{|x-y|^n}. \quad (2.11)$$

Consequently,

$$\frac{s}{1+\varepsilon} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{1+\varepsilon}{\varepsilon} \frac{|u(x) - u(y)|}{|x-y|^s} \right) \frac{dx \, dy}{|x-y|^n}$$

$$\geq 2s I_1 \geq 2n \omega_n \int_{\mathbb{R}^n} A \left( \frac{|u(x)|}{2^s|x|^s} \right) dx - 2s I_2$$

$$\geq 2n \omega_n \int_{\mathbb{R}^n} A \left( \frac{|u(x)|}{2^s|x|^s} \right) dx - \frac{2s \varepsilon \omega_n}{1+\varepsilon} \int_{\mathbb{R}^n} A \left( \frac{1+\varepsilon}{\varepsilon} \left( \frac{3}{2} \right)^s \frac{|u(y)|}{|y|^s} \right) dy, \quad (2.12)$$
where the first inequality follows from (2.11), the second one is due to (2.9) and (2.8), and the third one to (2.10). Since \( A(t) \leq A(t, 2t) \) for \( t \geq 0 \), inequality (2.5) implies that

\[
A(\lambda t) \leq C \lambda^{I(A)+1} A(2t) \quad \text{if } t \geq 0 \text{ and } \lambda \geq 1.
\]

(2.13)

From inequalities (2.12) and (2.13) one deduces that

\[
\frac{s}{1 + \varepsilon} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( (1 + \varepsilon) \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^n}
\]

\[
\geq 2n \omega_n \int_{\mathbb{R}^n} \bar{A} \left( \frac{|u(x)|}{2s|x|^s} \right) dx - 2Cs \varepsilon \omega_n \left( \frac{2(1 + \varepsilon)}{ \varepsilon} \right)^{I(A)+1} \int_{\mathbb{R}^n} \bar{A} \left( \frac{|u(y)|}{2s|y|^s} \right) dy
\]

\[
= 2n \omega_n \left[ 1 - \frac{C \varepsilon}{(1 + \varepsilon)n} \left( \frac{2(1 + \varepsilon)}{ \varepsilon} \right)^{I(A)+1} \right] \int_{\mathbb{R}^n} \bar{A} \left( \frac{|u(x)|}{2s|x|^s} \right) dx.
\]

Thus, there exists \( s_2 = s_2(A, n, \varepsilon) \in (0, s_1) \) such that, if \( s \in (0, s_2) \), then

\[
\frac{s}{1 + \varepsilon} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( (1 + \varepsilon) \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^n}
\]

\[
\geq 2n \omega_n (1 - \varepsilon) \int_{\mathbb{R}^n} \bar{A} \left( \frac{|u(x)|}{1 + \varepsilon} \right) dx.
\]

(2.14)

On replacing \( u \) by \( u/(1 + \varepsilon) \) in inequality (2.14), one can infer, via Fatou’s lemma, that

\[
\liminf_{s \to 0^+} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^n} \geq 2n \omega_n (1 - \varepsilon^2) \int_{\mathbb{R}^n} \bar{A} \left( \frac{|u(x)|}{1 + \varepsilon} \right) dx.
\]

By the arbitrariness of \( \varepsilon \), inequality (2.6) follows.

**Step 3.** We conclude by proving that the limit \( \sup_{s \to 0^+} \) of the expression on the left-hand side of Eq. (1.3) is bounded from above by the right-hand side. Namely, we show that

\[
\limsup_{s \to 0^+} s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^n}
\]

\[
\leq 2n \omega_n \int_{\mathbb{R}^n} \bar{A} \left( |u(x)| \right) dx.
\]

(2.15)

As observed in Step 2, we may assume that Eq. (2.7) holds for every \( \lambda > 0 \).

To prove inequality (2.15), notice that

\[
s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^n}
\]

\[
= s \int_{\mathbb{R}^n} \int_{\{ |y| \geq |x| \}} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx dy}{|x - y|^n}
\]
where the inequality holds since $A$ is convex. Let us estimate $J_1$ first. To this purpose, notice that \(|y| \geq 2|x|\) $\subset \{|x-y| \geq |x|\}$, since $|x-y| \geq |y|-|x| \geq 2|x|-|x| = |x|$. Thus,

$$
\int_{|y| \geq 2|x|} A \left(1 + \varepsilon \right) \frac{|u(x)|}{|x-y|^s} \, \frac{dy}{|x-y|^n}
\leq \int_{|x-y| \geq |x|} A \left(1 + \varepsilon \right) \frac{|u(x)|}{|x-y|^s} \, \frac{dy}{|x-y|^n}
= n\omega_n \int_{|x|}^\infty A \left(1 + \varepsilon \right) \frac{|u(x)|}{r^s} \, \frac{dr}{r^n}
$$

for every $x \in \mathbb{R}^n$. A change of variables tells us that

$$
\int_{t}^{\infty} A \left(1 + \varepsilon \right) \frac{|u(x)|}{r^s} \, \frac{dr}{r^s} = \frac{1}{s} \int_{0}^{t (1+\varepsilon) \rho / r^s} A(\tau) \, d\tau
= \frac{1}{s} A \left(1 + \varepsilon \right) \frac{\rho}{r^s}
$$

for $t, \rho \geq 0$. (2.18)

Thanks to Eqs. (2.17) and (2.18),

$$
J_1 \leq \frac{2 n \omega_n}{1 + \varepsilon} \int_{\mathbb{R}^n} A \left(1 + \varepsilon \right) \frac{|u(x)|}{|x|^n} \, dx.
$$

(2.19)
As far as the term $J_2$ is concerned, observe that, if $|y| \geq 2|x|$, then $|x - y| \geq \frac{1}{2}|y|$. Therefore, an application of Fubini’s theorem tells us that

$$J_2 \leq \frac{2^{n+1}s \varepsilon}{1 + \varepsilon} \int_{\mathbb{R}^n} A\left(\frac{1 + \varepsilon}{\varepsilon} 2^s |u(y)| \right) \left(\int_{(|x| \leq \frac{|y|}{2})} dx\right) \frac{dy}{|y|^n}$$

$$= \frac{2\omega_n s \varepsilon}{1 + \varepsilon} \int_{\mathbb{R}^n} A\left(\frac{1 + \varepsilon}{\varepsilon} 2^s |u(y)| \right) dy.$$ 

In order to provide an upper bound for $J_3$, note that, given $r > 3$,

$$J_3 = 2s \int_{\mathbb{R}^n} \int_{(|x| \leq |y| < 2|x|, |x - y| < r)} A\left(\frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx \, dy}{|x - y|^n}$$

$$+ 2s \int_{\mathbb{R}^n} \int_{(|x| \leq |y| < 2|x|, |x - y| \geq r)} A\left(\frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx \, dy}{|x - y|^n}$$

$$= J_{31} + J_{32}.$$

Since we are assuming that $u \in \bigcup_{s \in (0,1)} V^{s,A}_{d}(\mathbb{R}^n)$, there exists $s_3 \in (0, 1)$ such that $u \in V^{s_3,A}_{d}(\mathbb{R}^n).$ Let $s \in (0, s_3).$ Then

$$J_{31} \leq 2s \int_{\mathbb{R}^n} \int_{(|x| \leq |y| < 2|x|, |x - y| < r)} A\left(\frac{|u(x) - u(y)|}{|x - y|^{s_3 - s}} \right) \frac{dx \, dy}{|x - y|^n},$$

(2.20)

since $s_3 - s > 0$ and $u \in V^{s_3,A}_{d}(\mathbb{R}^n).$ If $|x| \leq |y| < 2|x|$ and $|x - y| \geq r$, then

$$3|x| = 2|x| + |x| \geq |y| + |x| \geq |x - y| \geq r,$$

whence $|x| \geq \frac{r}{3}$, and $|x| \leq |y| < 2|x| \leq 2|y|.$ Consequently,

$$3|y| = 2|y| + |y| \geq 2|x| + |y| \geq |x - y| \geq r,$$

and hence $|y| \geq \frac{r}{3}$ as well. Therefore, owing to the convexity of the function $A$,

$$J_{32} \leq s \int_{\mathbb{R}^n} \int_{(|x| \leq |y| < 2|x|, |x - y| \geq r)} A\left(\frac{2|u(x)|}{|x - y|^s} \right) \frac{dx \, dy}{|x - y|^n}$$

$$+ s \int_{\mathbb{R}^n} \int_{(|x| \leq |y| < 2|x|, |x - y| \geq r)} A\left(\frac{2|u(y)|}{|x - y|^s} \right) \frac{dx \, dy}{|x - y|^n}$$

$$\leq s \int_{(|x| \geq \frac{r}{3})} \left(\int_{(|x - y| \geq r)} A\left(\frac{2|u(x)|}{|x - y|^s} \right) \frac{dy}{|x - y|^n} \right) dx$$

$$+ s \int_{(|y| \geq \frac{r}{3})} \left(\int_{(|x - y| \geq r)} A\left(\frac{2|u(y)|}{|x - y|^s} \right) \frac{dx}{|x - y|^n} \right) dy$$

$$= 2s \int_{(|x| \geq \frac{r}{3})} \left(\int_{(|x - y| \geq r)} A\left(\frac{2|u(x)|}{|x - y|^s} \right) \frac{dy}{|x - y|^n} \right) dx.$$
\[= 2 s n \omega_n \int_{|x| \geq \frac{1}{2}} \left( \int_r^\infty A \left( \frac{2|u(x)|}{Q^s} \right) \frac{dQ}{Q} \right) \, dx\]
\[= 2 n \omega_n \int_{|x| \geq \frac{1}{2}} \overline{A} \left( \frac{2|u(x)|}{|x|^s} \right) \, dx.\]

Since we are assuming that \(r > 3\), the latter inequality implies that

\[J_{32} \leq 2 n \omega_n \int_{|x| \geq \frac{1}{2}} \overline{A} (2|u(x)|) \, dx \quad \text{for every } s \in (0, 1).\]

Consequently, if \(r\) is large enough, then

\[J_{32} < \varepsilon \quad \text{for every } s \in (0, 1). \tag{2.21}\]

Combining Eqs. (2.16), (2.19)–(2.20) and (2.21) implies that, for every \(s \in (0, s_3)\),

\[
s \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx \, dy}{|x - y|^n}
\leq 2 n \omega_n \int_{\mathbb{R}^n} A \left( (1 + \varepsilon) \frac{|u(x)|}{|x|^s} \right) \, dx + 2 \omega_n \varepsilon \int_{\mathbb{R}^n} A \left( \frac{1 + \varepsilon}{\varepsilon} 2^s \frac{|u(y)|}{|y|^s} \right) \, dy
+ 2 s \int_{\mathbb{R}^n} \int_{|x| \leq |y| < 2|x|, |x - y| < r_0} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \, dx \, dy + \varepsilon. \tag{2.22}\]

Passage to the limit as \(s \to 0^+\) in inequality (2.22) can be performed as follows. If \(|y| \leq 2\), then the function \((0, 1) \ni s \mapsto A \left( \frac{1 + \varepsilon}{\varepsilon} 2^s \frac{|u(y)|}{|y|^s} \right)\) is non-decreasing. Thus,

\[A \left( \frac{1 + \varepsilon}{\varepsilon} 2^s \frac{|u(y)|}{|y|^s} \right) \leq A \left( \frac{1 + \varepsilon}{\varepsilon} 2^{s_3} \frac{|u(y)|}{|y|^{s_3}} \right) \quad \text{for every } s \in (0, s_3),\]

and, since we are assuming that \(u \in V_{d,A}^{s_3}(\mathbb{R}^n)\), we have that

\[\int_{\mathbb{R}^n} A \left( \frac{1 + \varepsilon}{\varepsilon} 2^{s_3} \frac{|u(y)|}{|y|^{s_3}} \right) \, dy < \infty,\]

owing to (2.4). Inasmuch as

\[\lim_{s \to 0^+} A \left( \frac{1 + \varepsilon}{\varepsilon} 2^s \frac{|u(y)|}{|y|^s} \right) = A \left( \frac{1 + \varepsilon}{\varepsilon} |u(y)| \right) \quad \text{for } y \neq 0,\]

the dominated convergence theorem ensures that

\[\lim_{s \to 0^+} \int_{|y| \leq 2} A \left( \frac{1 + \varepsilon}{\varepsilon} 2^s \frac{|u(y)|}{|y|^s} \right) \, dy = \int_{|y| \leq 2} A \left( \frac{1 + \varepsilon}{\varepsilon} |u(y)| \right) < \infty. \tag{2.23}\]
On the other hand, if \(|y| > 2\), then the function \((0, 1) \ni s \mapsto A \left( \frac{1 + \varepsilon}{\varepsilon} 2^s \frac{|u(y)|}{|y|^s} \right)\) is non-increasing. Consequently, the monotone convergence theorem yields

\[
\lim_{s \to 0^+} \int_{\{|y| > 2\}} A \left( \frac{1 + \varepsilon}{\varepsilon} 2^s \frac{|u(y)|}{|y|^s} \right) \, dy = \int_{\{|y| > 2\}} A \left( \frac{1 + \varepsilon}{\varepsilon} |u(y)| \right) < \infty.
\]

(2.24)

Equations (2.23) and (2.24) imply that

\[
\lim_{s \to 0^+} \frac{2\omega_n \varepsilon s}{1 + \varepsilon} \int_{\mathbb{R}^n} A \left( \frac{1 + \varepsilon}{\varepsilon} 2^s \frac{|u(y)|}{|y|^s} \right) \, dy = 0.
\]

(2.25)

An argument analogous to that of the proofs of Eqs. (2.23) and (2.24) yields

\[
\lim_{s \to 0^+} \int_{\mathbb{R}^n} A \left( \frac{1 + \varepsilon}{\varepsilon} \frac{|u(x)|}{|x|^s} \right) \, dx = \int_{\mathbb{R}^n} A((1 + \varepsilon)|u(x)|) \, dx.
\]

(2.26)

Next, for every \(s \in (0, s_3)\),

\[
\int_{\mathbb{R}^n} \int_{\{|x| \leq |y| < 2|x|, |x-y| < r\}} A \left( \frac{|u(x) - u(y)|}{|x - y|^{s_3}} \right) \, dx \, dy \leq \int_{\mathbb{R}^n} \int_{\{|x| \leq |y| < 2|x|, |x-y| < r\}} A \left( \frac{|u(x) - u(y)|}{|x - y|^{s_3}} \right) \, dx \, dy < \infty.
\]

Observe that the convergence of the last integral is due to the fact that \(u \in V_{d}^{s_3,A}(\mathbb{R}^n)\) and \(A\) satisfies the \(\Delta_2\)-condition. Therefore,

\[
\lim_{s \to 0^+} 2s \int_{\mathbb{R}^n} \int_{\{|x| \leq |y| < 2|x|, |x-y| < r\}} A \left( \frac{|u(x) - u(y)|}{|x - y|^{s_3}} \right) \, dx \, dy = 0.
\]

(2.27)

Thanks to Eqs. (2.22), (2.25), (2.26) and (2.27),

\[
\limsup_{s \to 0^+} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \, dx \, dy \leq \frac{2 n \omega_n}{1 + \varepsilon} \int_{\mathbb{R}^n} A((1 + \varepsilon)|u(x)|) \, dx + \varepsilon.
\]

Hence, inequality (2.15) follows, owing to the arbitrariness of \(\varepsilon\). \(\square\)
3 Proof of Theorem 1.2

Functions $A$ and $u$ as in the statement of Theorem 1.2 are explicitly exhibited in our proof.

**Proof of Theorem 1.2** Since the proof involves rather technical computations, we split it in steps.

*Step 1.* We begin by giving some definitions and making some basic observations. Let $\gamma \geq 1$ and let $A$ be any finite-valued Young function such that

$$A(t) = e^{-\frac{1}{t^\gamma}} \quad \text{for} \quad t \in (0, \frac{1}{2e}).$$

Note that functions $A$ enjoying this property do exist, since $\lim_{t \to 0^+} e^{-\frac{1}{t^\gamma}} = 0$ and the function $e^{-\frac{1}{t^\gamma}}$ is convex on the interval $(0, \frac{1}{2e})$. The fact that $A$ is a Young function ensures that, for every $t_0 > 0$,

$$A(t) \leq t \frac{A(t_0)}{t_0} \quad \text{for} \quad t \in [0, t_0]. \quad (3.1)$$

Also, one can verify that, for each $s \in (0, 1)$, there exists $\bar{t} = \bar{t}(s, n, \gamma) \in (0, \frac{1}{2e})$ such that the function $(0, \bar{t}) \ni t \mapsto A(t^{1-s})$ is increasing. \quad (3.2)

Let $v : \mathbb{R}^n \to \mathbb{R}$ be the function defined as

$$v(x) = \begin{cases} \frac{x_1}{|x| \log^{\frac{1}{\gamma}} (\kappa + |x|)} & \text{if } |x| \geq 1 \\ \frac{x_1}{\log^{\frac{1}{\gamma}} (\kappa + 1)} & \text{if } |x| < 1, \end{cases}$$

where $x = (x_1, \ldots, x_n)$ and $\kappa > 1$ is a sufficiently large constant to be chosen later in such a way that the argument of the function $A$, evaluated at several expressions depending on $v$, belongs to the interval $(0, \frac{1}{2e})$ or $(0, \bar{t})$.

Notice that the function $v$ is Lipschitz continuous in $\mathbb{R}^n$ and continuously differentiable in $\{|x| > 1\}$, and

$$|\nabla v(x)| \leq \frac{\kappa}{|x| \log^{\frac{1}{\gamma}} (\kappa + |x|)} \quad \text{if } |x| > 1.$$ \quad (3.3)

Moreover, if $x, y \in \mathbb{R}^n$ are such that $|(1 - \tau)x + \tau y| > 1$ for $\tau \in [0, 1]$, then there exists $\tau_0 \in [0, 1]$ satisfying

$$|v(x) - v(y)| \leq \frac{3|x - y|}{|(1 - \tau_0)x + \tau_0 y| \log^{\frac{1}{\gamma}} (\kappa + |(1 - \tau_0)x + \tau_0 y|)}.$$
Given $\lambda > 1$, choose $\kappa$ so large that

$$\frac{1}{\lambda \log \frac{\kappa}{(\kappa + 1)}} < \frac{1}{2e}.$$ Therefore, there exists a constant $C$ such that, if $\lambda \gamma > n$, then

$$\int_{\mathbb{R}^n} A\left(\frac{|v(x)|}{\lambda}\right) \, dx \leq C + \int_{\{|x| \geq 1\}} \frac{dx}{(\kappa + |x|)^{\lambda \gamma}} < \infty.$$ 

**Step 2.** Here, we show that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|v(x) - v(y)|}{\lambda |x - y|^s}\right) \, dx \, dy < \infty \quad (3.4)$$

for every $s \in (0, 1)$ and $\lambda \geq 1$. To verify Eq. (3.4), observe that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A\left(\frac{|v(x) - v(y)|}{|x - y|^s}\right) \, dx \, dy = \int \int_{\{|x| \leq 1, |y| \leq 1\}} A\left(\frac{|v(x) - v(y)|}{|x - y|^s}\right) \, dx \, dy$$

$$+ 2 \int \int_{\{|x| < 1, |y| > 1\}} A\left(\frac{|v(x) - v(y)|}{|x - y|^s}\right) \, dx \, dy$$

$$+ \int \int_{\{|x| > 1, |y| > 1\}} A\left(\frac{|v(x) - v(y)|}{|x - y|^s}\right) \, dx \, dy = J_1 + J_2 + J_3. \quad (3.5)$$

Owing to the Lipschitz continuity of $v$ and to property (3.1), if $E \subset \mathbb{R}^n \times \mathbb{R}^n$ is a bounded set, then there exist positive constants $C$ and $C'$ such that

$$\int_E \int_E A\left(\frac{|v(x) - v(y)|}{|x - y|^s}\right) \, dx \, dy \leq \int_E \int_E A\left(C |x - y|^{-1-s}\right) \, dx \, dy$$

$$\leq C' \int_E \int_E \frac{dx \, dy}{|x - y|^{n-1+s}} < \infty. \quad (3.6)$$

Hence,

$$J_1 < \infty. \quad (3.7)$$

Next, let us split $J_2$ as

$$J_2 = \int \int_{\{|x| \leq 1, |y| > 1, |x - y| > 2\}} A\left(\frac{|v(x) - v(y)|}{|x - y|^s}\right) \, dx \, dy$$

$$+ \int \int_{\{|x| \leq 1, |y| > 1, |x - y| \leq 2\}} A\left(\frac{|v(x) - v(y)|}{|x - y|^s}\right) \, dx \, dy = J_{21} + J_{22}. \quad (3.8)$$
Consider $J_{21}$ first. If $|x| \leq 1$ and $|x - y| > 2$, then

\[
|x| + |y| \leq |x| + |y - x| + |x| = 2|x| + |y - x| \\
\leq 2 + |y - x| \leq |x - y| + |y - x| = 2|x - y|.
\]

Thus,

\[
|x - y| \leq |x| + |y| \leq 2|x - y|.
\]

Hence, there exist positive constants $C$, $C'$, $C''$ such that

\[
J_{21} \leq \int \int_{\{ |x| \leq 1, |y| > 1, |x - y| > 2 \}} A \left( C \frac{|v(x)| + |v(y)|}{(|x| + |y|)^S} \right) \frac{dx \, dy}{(|x| + |y|)^n} \\
\leq \int \int_{\{ |x| \leq 1, |y| > 1 \}} A \left( 2C \frac{|v(x)|}{|y|^S} \right) \frac{dx \, dy}{|y|^n} \\
+ \int \int_{\{ |x| \leq 1, |y| > 1 \}} A \left( 2C \frac{|v(y)|}{|y|^S} \right) \frac{dx \, dy}{|y|^n} \\
\leq 2 \int \int_{\{ |x| \leq 1, |y| > 1 \}} A \left( \frac{2C}{\log^\frac{1}{S} (\kappa + 1) |y|^S} \right) \frac{dx \, dy}{|y|^n} \\
= C' \int_1^{\infty} A \left( \frac{2C}{1 \log^\frac{1}{S} (\kappa + 1) r^S} \right) \frac{dr}{r} < \infty,
\]

where the last equality holds provided that the constant $\kappa$ is so large that

\[
\frac{2C}{\log^\frac{1}{S} (\kappa + 1)} < \frac{1}{2e}.
\]

As for $J_{22}$, notice that, if $|x| \leq 1$ and $|x - y| \leq 2$, then $|y| \leq |y - x| + |x| \leq 3$. Thus, by property (3.6), one has that

\[
J_{22} < \infty.
\]

Finally, let us focus on the term $J_3$, that can be split as

\[
J_3 = \int \int_{\{ |x| > 1, |y| > 1, |x - y| \geq \frac{|x| + |y|}{2} \}} A \left( \frac{|v(x) - v(y)|}{|x - y|^S} \right) \frac{dx \, dy}{|x - y|^n} \\
+ \int \int_{\{ |x| > 1, |y| > 1, |x - y| < \frac{|x| + |y|}{2} \}} A \left( \frac{|v(x) - v(y)|}{|x - y|^S} \right) \frac{dx \, dy}{|x - y|^n} = J_{31} + J_{32}.
\]
Consider $J_{32}$. If 

$$|x - y| < \frac{|x| + |y|}{2},$$

(3.12) then $|x| \leq |x - y| + |y| \leq \frac{|x|}{2} + \frac{|y|}{2} + |y|$, whence $|x| \leq 3|y|$. Similarly, $|y| \leq 3|x|$. Thus, 

$$|y| \frac{3}{2} \leq |x| \leq 3|y|,$$

and

$$|x| \geq \frac{|x| + |y|}{6}, \quad |y| \geq \frac{|x| + |y|}{6}.$$ 

(3.13)

Moreover, if $x$ and $y$ fulfill inequality (3.12), then there exists an absolute constant $\beta > 0$ such that

$$|(1 - \tau)x + \tau y| \geq \beta(|x| + |y|) \quad \text{for } \tau \in [0, 1].$$

(3.14)

Indeed, squaring both sides of inequality (3.12) shows that it is equivalent to

$$8x \cdot y > 2(|x|^2 + |y|^2) + (|x| - |y|)^2,$$

where the dot “·” denotes scalar product in $\mathbb{R}^n$. Hence, $x \cdot y > \frac{1}{4}(|x|^2 + |y|^2)$ and, by inequality (3.13), there exists an absolute constant $C$ such that

$$|(1 - \tau)x + \tau y|^2 = (1 - \tau)^2|x|^2 + 2\tau(1 - \tau)x \cdot y + \tau^2|y|^2$$

$$\geq (1 - \tau)^2|x|^2 + \tau(1 - \tau)\frac{|x|^2 + |y|^2}{2} + \tau^2|y|^2$$

$$\geq C \min\left\{|x|^2, |y|^2\right\} \geq C \left(\frac{|x| + |y|}{6}\right)^2 \quad \text{for } \tau \in [0, 1].$$

Inequality (3.14) is thus established. Let us split $J_{32}$ as

$$J_{32} = \int \int_{|x| > 1, |y| > 1, |x - y| \leq \frac{|x| + |y|}{2}} A \left(\frac{|v(x) - v(y)|}{|x - y|^s}\right) \frac{dx \, dy}{|x - y|^n}$$

$$+ \int \int_{|x| > 1, |y| > 1, |x - y| < \frac{|x| + |y|}{2}} A \left(\frac{|v(x) - v(y)|}{|x - y|^s}\right) \frac{dx \, dy}{|x - y|^n}$$

$$= J_{321} + J_{322}.$$ 

(3.15)

By property (3.6),

$$J_{321} < \infty.$$ 

As for $J_{322}$, note that if $x$, $y$ are such that $\sqrt{|x|^2 + |y|^2} \geq \frac{1}{\beta}$, then

$$1 \leq \beta \sqrt{|x|^2 + |y|^2} \leq \beta(|x| + |y|).$$ 

(3.16)
Let us set $E = \{ |x| > 1, |y| > 1, |x - y| < \frac{|x| + |y|}{2}, \sqrt{|x|^2 + |y|^2} \geq \frac{1}{2} \}$. If $\kappa$ is sufficiently large, then the following chain holds for a suitable constant $C$:

\[
J_{322} \leq \int \int_E A \left( \frac{3|x - y|^{1-s}}{\log^{\frac{1}{s}} (\kappa + \beta (|x| + |y|)) \beta (|x| + |y|)} \right) \frac{dx \, dy}{|x - y|^n}
\]

\[
\leq \int \int_E A \left( \frac{3}{\log^{\frac{1}{s}} (\kappa + \beta (|x| + |y|)) \beta (|x| + |y|)} \right) \frac{dx \, dy}{(|x| + |y|)^n}
\]

\[
= \int \int_E e^{-C \log(x+\beta(|x|+|y|))(|x|+|y|)^{1-s}} \frac{dx \, dy}{(|x| + |y|)^n}
\]

\[
\leq \int \int_{\{ |x| > 1, |y| > 1 \}} \kappa^{-C(|x|+|y|)^{1-s}} \frac{dx \, dy}{(|x| + |y|)^n} < \infty, \tag{3.17}
\]

where the first inequality holds owing to (3.14), (3.16), (3.3), and the second one by property (3.2) and the fact that $|x - y| \leq |x| + |y|$. Equations (3.15)–(3.17) ensure that

\[
J_{32} < \infty. \tag{3.18}
\]

It remains to estimate $J_{31}$. The following chain holds, provided that $\kappa$ is sufficiently large:

\[
J_{31} \leq \int \int_{\{ |x| > 1, |y| > 1, |x-y| \geq \frac{|x| + |y|}{2} \}} A \left( \frac{2|v(x)|}{|x-y|^s} \right) \frac{dx \, dy}{|x - y|^n}
\]

\[
+ \int \int_{\{ |x| > 1, |y| > 1, |x-y| \geq \frac{|x| + |y|}{2} \}} A \left( \frac{2|v(y)|}{|x-y|^s} \right) \frac{dx \, dy}{|x - y|^n}
\]

\[
= 2 \int \int_{\{ |x| > 1, |y| > 1, |x-y| \geq \frac{|x| + |y|}{2} \}} A \left( \frac{2|v(x)|}{|x-y|^s} \right) \frac{dx \, dy}{|x - y|^n}
\]

\[
\leq 2 \int \int_{\{ |x| > 1, |y| > 1, |x-y| \geq \frac{|x| + |y|}{2} \}} A \left( \frac{2}{\log^{\frac{1}{s}} (\kappa + 1) |x-y|^s} \right) \frac{dx \, dy}{|x - y|^n}
\]

\[
\leq 2^{n+1} \int \int_{\{ |x| > 1, |y| > 1, |x-y| \geq \frac{|x| + |y|}{2} \}} A \left( \frac{2^{s+1}}{\log^{\frac{1}{s}} (\kappa + 1) (|x| + |y|)^s} \right) \frac{dx \, dy}{(|x| + |y|)^n}
\]
\[
\leq 2^{n+1} \int_{|x|>1,|y|>1} e^{-\log(x+1)(|x|+|y|)^{2s}} \frac{dx
dy}{(|x|+|y|)^n}
\]

\[
= 2^{n+1} \int_{|x|>1,|y|>1} (\kappa + 1)^{-\left(\frac{(|x|+|y|)^s}{2s+1}\right)} \frac{dx
dy}{(|x|+|y|)^n} < \infty. \tag{3.19}
\]

Property (3.4) follows from (3.5), (3.7), (3.8), (3.9), (3.10), (3.11), (3.18) and (3.19).

**Step 3.** We next prove that, if \( \lambda \in (1,2) \), then

\[
\lim_{s \to 0^+} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|v(x) - v(y)|}{\lambda |x - y|^s} \right) \frac{dx
dy}{|x - y|^n} = \infty. \tag{3.20}
\]

To this purpose, define \( G = \{|x| > 1, x_1 > \sigma |x|, |y| > 1, y_1 < -\sigma |y|\} \) and note that, given \( \sigma \in (0,1) \),

\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} A \left( \frac{|v(x) - v(y)|}{\lambda |x - y|^s} \right) \frac{dx
dy}{|x - y|^n}
\geq \int_{G} \int_{G} A \left( \frac{|v(x) - v(y)|}{\lambda |x - y|^s} \right) \frac{dx
dy}{|x - y|^n}
= \int_{G} \int_{G} A \left( \frac{1}{\lambda} \left( \frac{x_1}{|x| \log^{\frac{1}{s}} (\kappa + |x|)} - \frac{y_1}{|y| \log^{\frac{1}{s}} (\kappa + |y|)} \right) \frac{1}{|x - y|^s} \right) \frac{dx
dy}{|x - y|^n}
\geq \int_{G} \int_{G} A \left( \frac{2\sigma}{\lambda \log^{\frac{1}{s}} (\kappa + |x| + |y|)} \frac{1}{(|x| + |y|)^s} \right) \frac{dx
dy}{(|x| + |y|)^n}
= C_{\sigma,n} \int_{1}^{\infty} \int_{1}^{\infty} A \left( \frac{2\sigma}{\lambda \log^{\frac{1}{s}} (\kappa + \rho + r)} \frac{1}{(\rho + r)^s} \right) \frac{\rho^{n-1} r^{n-1}}{(\rho + r)^n} d\rho
dr \tag{3.21}
\]

for some positive constant \( C_{\sigma,n} \) depending on \( \sigma \) and \( n \). Note that the last equality follows on making use of the polar coordinates in the integral with respect to \( x \) and in the integral with respect to \( y \). One has also to make use of the fact that the integrand is a radial function in \( x \) and \( y \), respectively, and that each of the sets \( \{|x| > 1, x_1 > \sigma |x|\} \) and \( \{|y| > 1, y_1 < -\sigma |y|\} \) is the intersection of the exterior of a ball centered at 0 with a cone whose vertex is also 0. Via the change of variables \( \xi = \varrho + r, \eta = \varrho - r \), we obtain that
\[ \int_1^\infty \int_1^\infty A \left( \frac{2\sigma}{\lambda \log^\frac{1}{s} (\kappa + \xi + r)} \frac{1}{(\varrho + r)^s} \right) \frac{(\varrho^{n-1} r^{n-1})}{(\varrho + r)^n} d\varrho dr = \frac{1}{2} \int_2^\infty \int_{2-\xi}^{-2+\xi} A \left( \frac{2\sigma}{\lambda \log^\frac{1}{s} (\kappa + \xi + \eta)} \frac{1}{\xi^s} \right) \left( \frac{\xi^2 - \eta^2}{4n - 1} \right) d\eta d\xi. \] (3.22)

Given \( \alpha \in (0, 2) \), if \( \xi > \frac{4}{2-\alpha} \) and \( 2 - \xi \leq \eta \leq \xi - 2 \), then \( 2\xi - 4 > \alpha \xi \) and \( \xi^2 - \eta^2 \geq \xi^2 - (\xi - 2)^2 = 4\xi - 4 > \alpha \xi \). Thereby, on choosing \( \kappa \) large enough, one has that

\[ \int_2^\infty \int_{2-\xi}^{-2+\xi} A \left( \frac{2\sigma}{\lambda \log^\frac{1}{s} (\kappa + \xi + \eta)} \frac{1}{\xi^s} \right) \left( \frac{\xi^2 - \eta^2}{4n - 1} \right) d\eta d\xi \geq \int_4^{2-\alpha} \int_{2-\xi}^{-2+\xi} A \left( \frac{2\sigma}{\lambda \log^\frac{1}{s} (\kappa + \xi + \eta)} \frac{1}{\xi^s} \right) \left( \frac{\xi^2 - \eta^2}{4n - 1} \right) d\eta d\xi \]

\[ \geq \alpha^n \int_4^{2-\alpha} \int_{2-\xi}^{-2+\xi} A \left( \frac{2\sigma}{\lambda \log^\frac{1}{s} (\kappa + \xi + \eta)} \frac{1}{\xi^s} \right) d\xi \]

\[ = \alpha^n \int_4^{2-\alpha} e^{-\left( \frac{1}{2\sigma} \right) \log(\kappa + \xi + \eta) \gamma} d\xi = \alpha^n \int_4^{2-\alpha} \frac{d\xi}{(\kappa + \xi)^\gamma \xi^\gamma t^\gamma} \]

\[ = \frac{\alpha^n}{s} \int_0^\infty (\frac{1}{t^\gamma} \gamma \frac{\varrho^{n-1} r^{n-1}}{(\varrho + r)^n} d\varrho dr, \] (3.23)

Now, fix any \( \sigma \in (\frac{\xi}{2}, 1) \). Then \( (\frac{\lambda}{2\sigma})^\gamma < 1 \). Also, \( (\frac{\lambda}{2\sigma})^\gamma t^\gamma < 1 \) if \( t < \frac{2\sigma}{\lambda} \). Thus,

\[ X((\frac{4}{2-\alpha})^\gamma \frac{2\sigma}{\lambda}) (t) \frac{1}{(\kappa + \frac{1}{t^\gamma} (\frac{2\sigma}{\lambda})^\gamma t^\gamma)} \rightarrow \infty \] as \( t \searrow 0^+ \) for \( t \in (1, \frac{2\sigma}{\lambda}) \). \] (3.24)

Equation (3.20) follows from (3.21), (3.22), (3.23) and (3.24), via the monotone convergence theorem for integrals.

**Step 4.** Combining Steps 1–3 shows that the conclusions of the theorem hold for any Young function \( A \) obeying condition (2.9) and for \( u = v \lambda \), if \( \lambda \) and \( \gamma \) are chosen in such a way that \( \lambda \in (1, 2) \) and \( \lambda \gamma > n \).

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**Compliance with ethical standards**

**Conflict of interest** The authors declare that they have no conflict of interest.

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