Abstract: A Kakeya set in $\mathbb{F}_q^n$ is a set containing a line in every direction. We show that every Kakeya set in $\mathbb{F}_q^n$ has density at least $1/2^n - 1$, matching the construction by Dvir, Kopparty, Saraf and Sudan.

Key words and phrases: Kakeya problem, lines, directions.

1 Introduction

Kakeya sets in finite fields. Let $\mathbb{F}_q$ be the finite field of size $q$. A subset $K \subseteq \mathbb{F}_q^n$ is Kakeya if, for every direction $b \in \mathbb{F}_q^n \setminus \{0\}$, it contains a line of the form $\{a + bt : t \in \mathbb{F}_q\}$. In a 2008 breakthrough, Dvir [2] proved that every Kakeya set in $\mathbb{F}_q^n$ is of size at least $\frac{1}{n!}q^n$. This was the first lower bound of the order $\Omega(n(q^n))$. The constant $1/n!$ was improved to $e^{-n}$ for some $c \approx 2.5$ by Saraf and Sudan [5], who also presented a construction (due to Dvir, with modifications by themselves and Kopparty) of a Kakeya set of size $2^{-n+1}q^n + O_n(q^{n-1})$. The best general lower bound on the size of Kakeya sets, which was found shortly thereafter by Dvir, Kopparty, Saraf, and Sudan [3], is

$$|K| \geq (2 - 1/q)^n q^n \quad \text{for every Kakeya set } K \subseteq \mathbb{F}_q^n.$$  \hspace{1cm} (1)

Motivated by the applications to randomness extractors, they also proved an extension of this bound to sets obtained by replacing the notion of a ‘line’ in the definition of the Kakeya set by that of an ‘algebraic curve of bounded degree’.

The only result breaching the factor-of-two gap between the construction in [5] and the lower bound in [3] is that of Lund, Saraf and Wolf [4] who proved that $|K| \geq 0.2107q^2$ for Kakeya sets in $\mathbb{F}_q^3$.

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Our result. We improve the lower bound (1) by a factor of \(2 - 1/q\), thereby closing the factor-of-two gap in all dimensions.

Theorem 1. The size of every Kakeya set \(K \subseteq \mathbb{F}_q^n\) satisfies

\[
|K| \geq (2 - 1/q)^-(n-1)q^n.
\]

Paper organization. We begin by presenting a proof of a slightly weaker bound in dimension \(n = 3\) in Section 2. Though this proof does not seem to generalize to the \(n > 3\), it illustrates one of the ideas used in the general case.

After presenting the proof of Theorem 1 in Section 3, we finish with a brief discussion of the remaining gap between our bound and the known constructions in Section 4.

2 Simple argument in dimension 3

In this section we shall prove the following.

Theorem 2. The size of every Kakeya set \(K \subseteq \mathbb{F}_q^3\) satisfies \(|K| \geq \frac{1}{4}(q^3 + q^2)\).

Let

\[
A \overset{\text{def}}{=} \{(\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_{\geq 0}^3 : \alpha_1 + \alpha_2 + \alpha_3 < 2q, \text{ and } \alpha_1, \alpha_2 < q\},
\]

and consider the vector space of polynomials in \(\mathbb{F}_q[x_1, x_2, x_3]\) whose monomials are indexed by \(A\),

\[
V \overset{\text{def}}{=} \left\{ \sum_{\alpha \in A} c_{\alpha}x^\alpha : c_{\alpha} \in \mathbb{F}_q \right\},
\]

where \(x^\alpha \overset{\text{def}}{=} x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\). We say that a polynomial \(P \in \mathbb{F}_q[x_1, x_2, x_3]\) vanishes at \(p \in \mathbb{F}_q^3\) to order 2 if \(P(p) = 0\) and \(\nabla P(p) = 0\).

Lemma 3. Let \(K\) be a Kakeya set in \(\mathbb{F}_q^3\). If a polynomial \(P \in V\) vanishes to order 2 at every point of \(K\), then \(P\) is the zero polynomial.

Before proving the lemma, let us see how to derive Theorem 2 from it. For any \(p \in \mathbb{F}_q^3\), the polynomials vanishing at \(p\) to order 2 form a subspace of codimension 4 in \(V\). So, the polynomials vanishing to order 2 at all points of \(K\) form a subspace of codimension at most \(4|K|\) in \(V\). According to the lemma, the latter subspace is trivial, and so

\[
4|K| \geq \dim V = |A| = \sum_{\alpha_1, \alpha_2 = 0}^{q-1} (2q - \alpha_1 - \alpha_2) = q^3 + q^2,
\]

as desired.
We say that whose definition and properties we recall. For more extensive discussion of Hasse derivatives, including Theorem 1, we define the univariate polynomial \( P(t) \equiv P(a + bt) \). Since \( P \) vanishes at every point of \( t \) to order 2, the polynomial \( P_t \) vanishes at all points of \( \mathbb{F}_q \) to order 2. Because \( \deg P_t \leq \deg P < 2q \), this implies that \( P_t \) is the zero polynomial. Since the coefficient of \( t^d \) in \( P_t \) is \( P_d(b) \), it follows \( P_d(b) = 0 \).

Since \( K \) is Kakeya, this means that \( P_d(b) = 0 \) for every \( b \neq 0 \). In particular, \( P_d(b_1, b_2, 1) = 0 \) for all \( b_1, b_2 \in \mathbb{F}_q \). The polynomial \( Q(x_1, x_2) \equiv P_d(x_1, x_2, 1) \) is of degree less than \( q \) in each of \( x_1 \) and \( x_2 \). Write \( Q \) as \( Q(x_1, x_2) = \sum_{i=0}^{q-1} Q_i(x_2)x_1^i \) where \( \deg Q_i < q \). Since \( Q \) vanishes identically on \( \mathbb{F}_q^2 \), the polynomial \( Q(x_1, c) \in \mathbb{F}_q[x_1] \) vanishes on \( \mathbb{F}_q \), for every choice of \( c \in \mathbb{F}_q \). As this polynomial is of degree less than \( q \), this means that \( Q_i(c) = 0 \) for every \( i \) and every \( c \). Since \( Q_i \)'s are themselves of degree less than \( q \), it follows that they are zero as polynomials, and so is \( Q \). Because \( P_d \) is homogeneous and \( Q(x_1, x_2) = P_d(x_1, x_2, 1) \), this implies that \( P_d \) is zero as well, contrary to \( \deg P = d \).

\[ \square \]

3 Proof of Theorem 1

Stronger result. The result we prove is in fact slightly stronger than Theorem 1. We call any line of the form \( \{a + bt : t \in \mathbb{F}_q\} \) with \( b = (b_1, \ldots, b_{n-1}, 1) \) non-horizontal. A set \( K \subseteq \mathbb{F}_q^n \) is almost Kakeya if it contains a line in every non-horizontal direction.

Theorem 1'. The size of every almost Kakeya set \( K \subseteq \mathbb{F}_q^n \) satisfies \( |K| \geq (2 - 1/q)^{-(n-1)}q^n \).

Proof outline. Like the proof of the case \( n = 3 \), we shall use polynomials built out of the monomials \( x^\alpha \) in which the exponent of \( x_1 \) is less constrained than the exponents of \( x_1, \ldots, x_{n-1} \). As is common in the other proofs in the area we shall use polynomials vanishing to high order at points of \( K \), and not merely to order 2. However, this is not enough to obtain the factor-of-two improvement we seek, and to bridge the gap we use the idea of Ruixiang Zhang. Like in his work on multijoints [7], our vanishing conditions at a point \( p \in K \) depend on the lines through \( p \). The actual conditions are quite different from those in [7] though. A similar general idea was used in [6] to obtain the sharp constant in the joints problem.

Hasse derivatives and high-order vanishing along lines. Over finite fields, all derivatives of order higher than the field’s characteristic vanish. The standard workaround is to employ Hasse derivatives, whose definition and properties we recall. For more extensive discussion of Hasse derivatives, including proofs of their properties, see [3, Section 2].

Definition 4 (Hasse derivatives). The Hasse derivatives of a polynomial \( P(x) \in \mathbb{F}_q[x_1, \ldots, x_n] \) are the polynomials \( P^{(i)}(x), i \in \mathbb{Z}_{\geq 0} \) such that

\[ P(x + y) = \sum_i P^{(i)}(x)y^i. \]

We say that \( P^{(i)} \) is the \( i \)-th Hasse derivative of \( P \).

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Definition 5 (Multiplicities). The multiplicity \( \text{mult}(P, p) \) of a polynomial \( P(x) \in \mathbb{F}_q[x_1, \ldots, x_n] \) at point \( p \in \mathbb{F}^n_q \) is the largest integer \( m \) such that \( P^{(i)}(p) = 0 \) for all \( i = (i_1, \ldots, i_n) \) such that \( |i| \overset{\text{def}}{=} i_1 + \cdots + i_n < m \). We say that the polynomial \( P \) vanishes to order \( \text{mult}(P, p) \) at \( p \).

Given a line \( \ell = \{ a + bt : t \in \mathbb{F}_q \} \) and a polynomial \( P \in \mathbb{F}_q[x_1, \ldots, x_n] \), we say that \( P \) vanishes to order \( m \) at the point \( p = a + bt \) along \( \ell \) if the univariate polynomial \( P(a + bt) \) vanishes to order \( m \) at \( t = t_0 \). We write \( \text{mult}_\ell(P, p) \) for the order of vanishing of \( P \) along \( \ell \) at the point \( p \).

Note that, by the third part of the following proposition, \( \text{mult}_\ell(P, p) \) does not depend on the parameterization of the line \( \ell \).

Proposition 6 (Properties of Hasse derivatives).

- The map \( P \mapsto P^{(i)} \) is a linear operator on \( \mathbb{F}_q^n[x_1, \ldots, x_n] \), for every \( i \in \mathbb{Z}_{\geq 0}^n \).
- If we write the polynomial \( P \) as \( P(x) = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} c_\alpha x^\alpha \), then the Hasse derivatives of \( P \) are given by
  \[
  P^{(i)}(x) = \sum_{\alpha} \binom{\alpha}{i} c_\alpha x^{\alpha - i},
  \]
  where \( \binom{\alpha}{i} \overset{\text{def}}{=} \prod_{k=1}^n \binom{\alpha_k}{i_k} \).
- If \( A \) is an invertible affine transformation of \( \mathbb{F}_q^n \), then
  \[
  \text{mult}(P \circ A, p) = \text{mult}(P, Ap).
  \]

Lemma 7 (Generalized Schwartz–Zippel lemma). If a non-zero polynomial \( P(x) \in \mathbb{F}_q[x_1, \ldots, x_n] \) vanishes to order at least \( r \) at every point of \( \mathbb{F}_q^n \), then \( \deg P \geq rq \).

Proof of Theorem 1’. For a non-horizontal direction \( b \), there might be several lines in direction \( b \) contained in \( K \). We select one such line for each \( b \), and let \( L \) be the resulting set of lines. Note that \( |L| = q^{n-1} \). From now on we shall work exclusively with the lines in \( L \), ignoring the other lines that might be contained in \( K \). For each point \( p \in K \), let \( L_p \overset{\text{def}}{=} \{ \ell \in L : \ell \ni p \} \) be the lines containing \( p \).

For each line \( \ell \in L \), and each point \( p \) of \( \ell \), we shall impose vanishing conditions at \( p \) described in the following definition.

Definition 8. Let \( \ell \) be a non-horizontal line, and \( p \in \ell \). We say that a polynomial \( P \) vanishes to order \( (r, r') \) at \( p \) along \( \ell \) if \( \text{mult}_\ell(P^{(i,0)}, p) \geq r' - |i|/q \) for all \( i \in \mathbb{Z}_{\geq 0}^{n-1} \) such that \( |i| < r \).

The definition is motivated by the proof in [3], which implicitly uses that \( P \) vanishes to order \( (r, r) \) at \( p \) along \( \ell \) whenever \( P \) vanishes to order \( (2 - 1/q)r \) at \( p \).

Throughout this proof, the integer \( r' \) will be equal to \( r' = (2 - 1/q)r \), where \( r \) is some multiple of \( q \), which eventually will tend to infinity.

Let
\[
A = \{ \alpha \in \mathbb{Z}_{\geq 0}^n : |\alpha| < (2 - 1/q)rq, \text{ and } \alpha_1 + \cdots + \alpha_{n-1} < rq \},
\]
and consider the vector space of polynomials with the monomials indexed by \( A \),
\[
V \overset{\text{def}}{=} \left\{ \sum_{\alpha \in A} c_\alpha x^\alpha : c_\alpha \in \mathbb{F}_q \right\}.
\]
Lemma 9. If $P \in V$ vanishes to order $(r, (2-1/q)r)$ at $p$ along $\ell$, for every $p \in K$ and every $\ell \in L_p$, then $P$ is the zero polynomial.

Proof. Assume that, on the contrary, $P$ is not the zero polynomial. Let $\ell \in L$ be an arbitrary line, and $|i| < r$. By the lemma’s assumption, $P^{(i,0)}$ vanishes to order $(2-1/q)r - |i|/q$ at $p$ along $\ell$. Since $\deg P^{(i,0)} < (2-1/q)r - |i|/q \leq q((2-1/q)r - |i|/q)$, we have $P^{(i,0)}(\ell) = \sum_{p \in \ell} \text{mult}_P(P^{(i,0)}(p))$. It follows from the one-dimensional case of Lemma 7 that the univariate polynomial obtained by restricting $P^{(i,0)}$ to the line $\ell$ is the zero polynomial.

Write $P = P_0 + P_1 + \cdots + P_d$ where $P_k$ is the degree-$k$ homogeneous component of $P$ and $d \overset{\text{def}}{=} \deg P$, and also write the line $\ell$ as $\ell = \{a + bt : t \in \mathbb{F}_q\}$, where $b = (b_1, \ldots, b_{n-1}, 1)$. Note that, by the second part of Proposition 6, $P^{(i,0)}_d$ is the homogeneous part of $P^{(i,0)}$ of degree $d - |i|$, and $\deg P^{(i,0)}_d \leq d - |i|$. Thus, $P^{(i,0)}(\ell) = P^{(i,0)}_d(\ell)$ is the coefficient of $t^{d-|i|}$ in the univariate polynomial $P^{(i,0)}(a + bt)$. Therefore, $P^{(i,0)}_d(b) = 0$ for all $|i| < r$ and for all $b = (b_1, \ldots, b_{n-1}, 1)$.

Define the polynomial $Q \in \mathbb{F}_q[x_1, \ldots, x_{n-1}]$ by $Q(x_1, \ldots, x_{n-1}) \overset{\text{def}}{=} P_d(x_1, \ldots, x_{n-1}, 1)$. Note that $Q \neq 0$ since $P_d \neq 0$ and $P_d$ is homogeneous. For every $b \in \mathbb{F}_q^{n-1}$ and every $i \in \mathbb{Z}_{\geq 0}^{n-1}$ such that $|i| < r$, we have $Q^{(i)}(b) = P^{(i,0)}(b, 1) = 0$. Hence, mult$(Q, b) \geq r$ for all $b \in \mathbb{F}_q^{n-1}$, and so the generalized Schwartz–Zippel lemma implies that $\deg Q \geq rq$, which contradicts the definition of the space $V$. \hfill \Box

The next task is to estimate the number of independent linear conditions that different lines impose at the point $p$. To that end, for any $p \in K$ and line $\ell \in L_p$, we set $W_{p,\ell} \subseteq \mathbb{F}_q[x_1, \ldots, x_n]$ to be the subspace consisting of all polynomials vanishing to order $(r, (2-1/q)r)$ at $p$ along $\ell$.

Lemma 10. Let $p \in K$ be arbitrary. Then the codimension of $W_p \overset{\text{def}}{=} \bigcap_{\ell \in L_p} W_{p,\ell}$ in $\mathbb{F}_q[x_1, x_2, \ldots, x_n]$ satisfies

$$\text{codim} W_p \leq \left((2-1/q)^n + m(n-1)(1-1/q)\right) \frac{r^n}{n!} + O_{n,q}(r^{n-1}),$$

where $m = |L_p|$.

Proof. Suppose $\ell = \{a + bt : t \in \mathbb{F}_q\}$ and $p = a + bt_0$ is a point on $\ell$. For $i \in \mathbb{Z}_{\geq 0}^{n-1}$ and $j \in \mathbb{Z}_{\geq 0}$, write $W_{p,\ell}(i, j)$ for the space of polynomials $P$ such that the univariate polynomial $Q = P^{(i,0)}(a + bt)$ satisfies $Q^{(j)}(t_0) = 0$. Note that $W_{p,\ell}(i, j)$ is a subspace of codimension 1. Since $\bigcap_{j < r'} W_{p,\ell}(i, j)$ consists of polynomials $P$ satisfying mult$(P^{(i,0)} + P^{(i,0)}(bt)) \geq r'$, it follows that

$$W_{p,\ell} = \bigcap_{j < r'} W_{p,\ell}(i, j).$$

For the purpose of proving the lemma, we may assume that $p$ is the origin and so $\ell = \{bt : t \in \mathbb{F}_q\}$. Given a polynomial $P$, write it as $P(x) = \sum_{\alpha} c_\alpha x_\alpha$. Observe that, by the second part of Proposition 6, the $t^{-|i|}$ coefficient of $P^{(i,0)}(bt)$ is a linear combination of the coefficients $c_\alpha$ whose $\alpha = |i| + j$. Thus, the linear condition $P \in W_{p,\ell}(i, j)$ involves only such $c_\alpha$. 

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We use this observation to separate the linear conditions into those affecting coefficients of degree $|\alpha| < (2 - 1/q)r$ from the rest. To that end we define

$$W_{p, \ell}^- \overset{\text{def}}{=} \bigcap_{j < (2 - 1/q)r - |i|/q} W_{p, \ell}(i, j) \quad \text{and} \quad W_{p, \ell}^+ \overset{\text{def}}{=} \bigcap_{j < (2 - 1/q)r - |i|/q} W_{p, \ell}(i, j). \quad (2)$$

Set $W_p^- \overset{\text{def}}{=} \bigcap_{\ell \in L_p} W_{p, \ell}^-$ and $W_p^+ \overset{\text{def}}{=} \bigcap_{\ell \in L_p} W_{p, \ell}^+$. Clearly,

$$\text{codim} W_p = \text{codim} W_p^- + \text{codim} W_p^+. \quad (3)$$

The $\text{codim} W_p^-$ is easy to bound: the linear conditions in the definition of $W_p^-$ involve only the coefficients $c_\alpha$ of monomials of degree $|\alpha| < (2 - 1/q)r$, and so we may bound $\text{codim} W_p^-$ by the number of such monomials, i.e.,

$$\text{codim} W_p^- \leq \binom{(2 - 1/q)r + n - 1}{n} = \frac{(2 - 1/q)r^n}{n!} + O_{n,q}(r^{n-1}). \quad (4)$$

To estimate $\text{codim} W_p^+$, we upper bound $\text{codim} W_{p, \ell}^+$, for each $\ell \in L_p$, by the number of linear conditions appearing in (2), i.e.,

$$\text{codim} W_{p, \ell}^+ \leq m \cdot |\{(i, j) \in \mathbb{Z}_{\geq 0}^{n-1} \times \mathbb{Z}_{\geq 0} : |i| < r, j < (2 - 1/q)r - |i|/q, |i| + j \geq (2 - 1/q)r\}|. \quad (5)$$

Define the polytope

$$\square \overset{\text{def}}{=} \{(i, j) \in \mathbb{R}_{\geq 0}^{n-1} \times \mathbb{R}_{\geq 0} : |i| < 1, j < (2 - 1/q) - |i|/q, |i| + j \geq 2 - 1/q\}. \quad (6)$$

Since the set on the right side of (5) is the set of lattice points in $r \cdot \square$, we may write the bound (5) as

$$\text{codim} W_{p, \ell}^+ \leq m \cdot r^n \text{vol}(\square) + O_{n,q}(r^{n-1}).$$

The $\square$ can be expressed as a Boolean combination of three simpler polytopes,

$$\square \overset{\text{def}}{=} \{(i, j) \in \mathbb{R}_{\geq 0}^{n-1} \times \mathbb{R}_{\geq 0} : |i| < 1, 1 - 1/q < j \leq 2 - 2/q\},$$

$$\triangle_1 \overset{\text{def}}{=} \{(i, j) \in \mathbb{R}_{\geq 0}^{n-1} \times \mathbb{R}_{\geq 0} : |i| < 1, j > 1 - 1/q, |i| + j < 2 - 2/q\},$$

$$\triangle_2 \overset{\text{def}}{=} \{(i, j) \in \mathbb{R}_{\geq 0}^{n-1} \times \mathbb{R}_{\geq 0} : |i| < 1, 2 - 2/q < j < (2 - 1/q) - |i|/q\}.$$

The $\triangle_1$ and $\triangle_2$ are $n$-simplices, whereas $\square$ is a cylinder over an $(n - 1)$-simplex. We can depict these polytopes as follows.
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Since $\Box + \triangle_1 = \square + \triangle_2$, with pluses denoting disjoint unions, it follows that

$$\text{vol}(\square) = \text{vol}(\Box) - \text{vol}(\triangle_1) + \text{vol}(\triangle_2) = \frac{1 - 1/q}{n - 1} - \frac{1}{n!} + \frac{4}{nq}.$$ 

We thus obtain

$$\text{codim} W_p^+ \leq m \left( \frac{(n-1)(1 - 1/q)}{n!} \right)^p r^n + O_{n,q}(r^{n-1}).$$

Combining (3), (4) and (7) we obtain the desired result.

To get a lower bound on $|K|$ it remains to estimate $\dim V$. On one hand, it is equal to

$$\dim V = |A| = \sum_{\alpha_e = 0}^{(1 - 1/q)r - 1} \left( \begin{array}{c} rq + n - 2 \\ n - 1 \end{array} \right) + \sum_{\alpha_e = (1 - 1/q)r}^{(2 - 1/q)r - 1} \left( \begin{array}{c} 2 - 1/q \\ n - 1 \end{array} \right) \left( \begin{array}{c} 2 - 1/q \\ n - 1 \end{array} \right) \left( \begin{array}{c} rq - \alpha_n + n - 2 \\ n - 1 \end{array} \right)$$

$$= (1 - 1/q)r \left( \begin{array}{c} r - 2 \\ n - 1 \end{array} \right) + \left( \begin{array}{c} r + n - 1 \\ n \end{array} \right)$$

$$= \left( \frac{1 - 1/q}{n - 1} \right) r^n q^n + \frac{r^n q^n}{n!} - O_{n,q}(r^{n-1}).$$

(8)

On the other hand, from Lemma 9 we know that $\bigcap_{p \in K} W_p \cap V = \{0\}$, and hence

$$\dim V \leq \text{codim} \bigcap_{p \in K} W_p \leq \sum_{p \in K} \text{codim} W_p \leq \sum_{p \in K} \left( \left( 2 - 1/q \right)^n + |L_p| (n-1) (1 - 1/q) \right) \frac{r^n}{n!} + O_{n,q}(r^{n-1})$$

by Lemma 10.

The sum above can be computed by noting that $\sum |L_p| = q |L| = q^n$. We then let $r \to \infty$ and compare the resulting upper bound on $\dim V$ with the asymptotics in (8) to obtain

$$n (1 - 1/q) q^n + q^n \leq |K| (2 - 1/q)^n + q^n (n-1) (1 - 1/q).$$

By rearranging the inequality, we see that this is equivalent to

$$|K| \geq \frac{q^n}{(2 - 1/q)^{n-1}}.$$

4 Lower-order terms

In dimension 2, one can use the simple fact that two distinct lines intersect at most once to derive the lower bound of $|K| \geq \frac{q(q+1)}{2}$ for every 2-dimensional Kakeya set. Though this bound is sharp for even $q$, the sharp bound for odd $q$ is $|K| \geq \frac{q(q+1)}{2} + q^{-1}$, as shown by Blokhuis and Mazzocca [1].
Recall that we defined a set to be almost Kakeya if it contains a line in every non-horizontal direction. The paper [5] presents two constructions of higher-dimensional Kakeya sets, one construction for each possible parity of \( q \). The construction for odd \( q \) (due to Dvir) relies on an auxiliary construction of almost Kakeya sets, whereas the construction for even \( q \) (due to Kopparty, Saraf, and Sudan) is direct.

For odd \( q \), the almost Kakeya sets in [5] are of size \( \frac{q^n}{2} \left( 1 + \frac{n-1}{q} + O_n(q^{-2}) \right) \) whereas the lower bound in Theorem 1 is \( \frac{q^n}{2} \left( 1 + \frac{n-1}{2q} + O_n(q^{-2}) \right) \). It would be interesting to close the gap.

To turn an almost Kakeya set into a genuine Kakeya set, one must take care of the horizontal directions. In [5] this was achieved by adding a horizontal hyperplane. This can be done more efficiently by adding a lower-dimensional Kakeya set instead.

**Proposition 11.** There is a Kakeya set in \( \mathbb{F}_q^n \) of size \( \frac{q^n}{2} \left( 1 + \frac{n-1}{q} + O_n(q^{-2}) \right) \) if \( q \) is odd.

**Proof.** By induction on \( n \), with the base case \( n = 1 \) being trivial. Let \( K'_n \) be an almost Kakeya set in \( \mathbb{F}_q^n \) of size \( \frac{q^n}{2} \left( 1 + \frac{n-1}{q} + O_n(q^{-2}) \right) \). Let \( K_{n-1} \) be the inductively-constructed \((n-1)\)-dimensional Kakeya set of size \( 2^{-n-1}q^n + O_n(q^{-2}) \). We think of \( K_{n-1} \) as lying inside a horizontal hyperplane in \( \mathbb{F}_q^n \). Then the set \( K_n \) is a Kakeya set for any choice of \( x \in \mathbb{F}_q^n \). For a random choice of \( x \),

\[
\mathbb{E}[|K_n|] = |K'_n| + |K_{n-1}| \left( 1 - \frac{|K'_n|}{q^n} \right) = 2^{-n-1}q^n \left( 1 + \frac{n-1}{q} + O_n(q^{-2}) \right). \]

For even values of \( q \), the Kakeya sets constructed in [5] are of size \( 2^{-n-1}q^n + (1 - 2^{-n+1})q^{n-1} \).

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