No scalar hair behaviors of static massive scalar fields with nodes

Yan Peng\textsuperscript{1,a}
\textsuperscript{1} School of Mathematical Sciences, Qufu Normal University, Qufu 273165, Shandong, China

Received: 1 May 2020 / Accepted: 16 June 2020 / Published online: 26 June 2020
© The Author(s) 2020

Abstract We study no scalar field hair behavior for spherically symmetric objects in the scalar-Gauss–Bonnet gravity. In this work, we focus on static massive scalar fields with nodes. We analytically obtain a bound on the coupling parameter. Below the bound, the static massive scalar field with nodes cannot exist outside the object. In particular, our conclusion is independent of surface boundary conditions.

1 Introduction

One famous property of classical black holes is the no hair theorem, which states that asymptotically flat spherically symmetric black holes cannot support external static scalar fields, see references [1–8] and reviews [9,10]. Interestingly, no hair behavior also appears for horizonless reflecting stars [11–23]. Recently, it was found that scalar field hairs can exist outside black holes and reflecting stars when considering the non-minimal coupling between scalar fields and the Gauss–Bonnet invariant [24–30]. This scalar-Gauss–Bonnet gravity attracted lots of attentions and other models were constructed [31–45].

The mostly studied scalar configurations are the cases with no nodes. Theoretically, scalar configurations can possess nodes. It is well known that, in the general case, scalar configurations with nodes are usually unstable [46–49]. So it is natural to conjecture that scalar configurations with nodes may finally evolve into the more stable nodeless solutions. However, it is still meaningful to study the solution with nodes. Firstly, the solution with nodes may be sufficiently stable, which means that the perturbation growth time is extremely large [50]. In this case, the unstable node solution stays for a long time and can be observed from physical aspects. And secondly, in some gravity models, it is very surprising that the solution with nodes seems to be the endpoint of the tachyonic instability [51].

In particular, for massless scalar fields non-minimally coupled to the Gauss–Bonnet invariant in the background of black holes, an interesting relation \( \Delta_n = \sqrt{n_{n+1}} - \sqrt{n_n} = \frac{3\sqrt{3}}{4} \pi \) for \( n \to \infty \) was obtained through WKB approach in [52] and this relation is also precisely supported by numerical data in [27], where \( n \) is the number of nodes. For massive scalar fields in black hole spacetimes, with both analytical and numerical methods, we demonstrated that this relation still holds in the large node number limit [53]. On the other side, some quantum-gravity theories suggest that quantum effects may prevent the formation of classical black hole absorbing horizons and the horizonless compact star may serve as an alternative [54–61]. So it is also interesting to search for properties independent of surface boundary conditions.

In this work, we plan to study the existence (or non-existence) of scalar fields with nodes outside general spherically symmetric objects.

In the following, we shall consider a gravity system with a scalar field coupled to the Gauss–Bonnet invariant in the background of general spherically symmetric objects. With analytical methods, we obtain a bound on the scalar-Gauss–Bonnet coupling parameter, below which there is no hair behavior for static massive scalar fields with nodes. We summarize main results in the last section.

2 Bounds on the scalar-Gauss–Bonnet coupling parameter

We consider a gravity with a scalar field \( \Psi \) coupled to the Gauss–Bonnet invariant \( R^2_{GB} \). The general Lagrange density can be written as [26–31]

\[
\mathcal{L} = R - \left| \nabla \Psi \right|^2 - m^2 \Psi^2 + f(\Psi) R^2_{GB}.
\]

Here \( m \) is the scalar field mass and the Gauss–Bonnet invariant is defined as \( R^2_{GB} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 \). In the probe limit, there is \( R^2_{GB} = \frac{48 M^2}{r^6} \). The function \( f(\Psi) \)

\textsuperscript{a} e-mail: yanpengphy@163.com (corresponding author)
describes the nontrivial coupling between scalar fields and the Gauss–Bonnet invariant. In the linearized regime, one can generally put the coupling function in the quadratic form \( f(\Psi) = \eta \Psi^2 \) with \( \eta \) as the model parameter [28–30].

The exterior metric solution of spherically symmetric objects is [29]
\[
ds^2 = -g(r)dt^2 + \frac{dr^2}{g(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2).
\] (2)

In the probe limit, the metric is given by the function \( g(r) = 1 - \frac{2M}{r} \), where \( M \) is the mass of the objects. We define \( r_s \geq 2M \) as the object surface radii. When \( r_s = 2M \), the metric is a black hole.

The scalar field differential equation is
\[
\nabla^\dagger \Psi - m^2 \Psi + \frac{f' \bar{R}^2_{GB}}{2} = 0.
\] (3)

We choose to study a static massive scalar field in the form
\[
\Psi(t, r, \theta, \phi) = \sum_{lm} e^{im\phi} S_{lm}(\theta) R_{lm}(r).
\] (4)

Here \( l \) is the spherical harmonic index and \( l(l+1) \) is the characteristic eigenvalue of the angular scalar eigenfunction \( S_{lm}(\theta) \) [27,62,63]. For simplicity, we label \( R_{lm}(r) \) as \( \psi(r) \). One can obtain the scalar field equation
\[
\psi'' + \left( \frac{2}{r} + \frac{g'}{g} \right) \psi' + \left( \frac{\eta \bar{R}^2_{GB}}{g} - \frac{l(l+1)}{r^2g} - \frac{m^2}{g} \right) \psi = 0,
\] (5)

where \( g = 1 - \frac{2M}{r} \) and \( \bar{R}^2_{GB} = \frac{48M^2}{r^2} \).

In this work, we focus on scalar fields with nodes. That is to say, there is at least a point \( r_0 \) satisfying \( \psi(r_0) = 0 \). The bound-state massive static scalar fields satisfy asymptotically decaying behaviors \( \psi(r \to \infty) \sim \frac{1}{r} e^{-mr} \). In the range \((r_0, \infty)\), the scalar field boundary conditions are
\[
\psi(r_0) = 0, \quad \psi(\infty) = 0.
\] (6)

According to boundary conditions (6), the scalar field \( \psi(r) \) must possess one extremum point \( r = r_{\text{peak}} \) between the vanishing point \( r = r_0 \) and the infinity. It may be a positive maximum extremum point satisfying
\[
\psi(r_{\text{peak}}) > 0, \quad \psi'(r_{\text{peak}}) = 0, \quad \psi''(r_{\text{peak}}) \leq 0,
\] (7)

otherwise it will be a negative minimum extremum point with
\[
\psi(r_{\text{peak}}) < 0, \quad \psi'(r_{\text{peak}}) = 0, \quad \psi''(r_{\text{peak}}) \geq 0.
\] (8)

Relations (7) and (8) yield that
\[
\{ \psi \neq 0, \quad \psi' = 0 \ and \ \psi'' \leq 0 \} \ for \ r = r_{\text{peak}}.
\] (9)

Multiplying both sides of (5) with \( \psi \), we obtain the new equation
\[
\psi'' + \left( \frac{2}{r} + \frac{g'}{g} \right) \psi' + \left( \frac{\eta \bar{R}^2_{GB}}{g} - \frac{l(l+1)}{r^2g} - \frac{m^2}{g} \right) \psi^2 = 0.
\] (10)

At the extremum point, relations (9) and (10) yield the inequality
\[
\frac{\eta \bar{R}^2_{GB}}{g} - \frac{l(l+1)}{r^2g} - \frac{m^2}{g} \geq 0 \ for \ r = r_{\text{peak}}.
\] (11)

The extremum point is outside the gravitational radius \( r_{\text{peak}} > r_s \geq 2M \), which yields
\[
g(r) = 1 - \frac{2M}{r} > 0 \ for \ r = r_{\text{peak}}.
\] (12)

With relations (11) and (12), we obtain the inequality
\[
\eta \bar{R}^2_{GB} - \frac{l(l+1)}{r^2} - \frac{m^2}{g} \geq 0 \ for \ r = r_{\text{peak}}.
\] (13)

From relation (13), one deduces that
\[
\eta \bar{R}^2_{GB} - m^2 \geq 0 \ for \ r = r_{\text{peak}}.
\] (14)

The inequality (14) is equal to
\[
\eta \geq \frac{m^2 r_{\text{peak}}^6}{48M^2}.
\] (15)

The extremum point is outside the object surface \( r_{\text{peak}} > r_s \geq 2M \). It yields the relation
\[
\eta \geq \frac{m^2 r_{\text{peak}}^6}{48M^2} > \frac{m^2 r_s^6}{48M^2} \geq \frac{m^2(2M)^6}{48M^2} = \frac{4m^2M^4}{3}.
\] (16)

For scalar fields with nodes, we obtain a lower bound on the coupling parameter. Below this bound, the static scalar field with nodes cannot exist outside the object surface. It implies that static massive scalar fields with nodes usually cannot exist outside objects of large mass. Here we obtain a no hair behavior for scalar fields with nodes in the region
\[
\eta \leq \frac{4}{3} m^2 M^4.
\] (17)

In particular, for \( \eta = 0 \), (17) always holds. It means minimally coupled massive scalar fields always cannot exist outside general spherically symmetric objects, such as black holes and horizonless stars.

### 3 Conclusions

We studied no scalar field hair behavior for spherically symmetric objects in the scalar-Gauss–Bonnet gravity. We considered a static massive scalar field with nodes. Through ana-
lytical methods, we obtained a bound on the scalar-Gauss–Bonnet coupling parameter as $\eta \leq \frac{4}{3} m^2 M^4$, where $\eta$ is the coupling parameter, $m$ is the scalar field mass and $M$ is the mass of objects. Below the lower bound, static massive scalar fields with nodes cannot exist outside the objects. In particular, our analysis doesn’t depend on the surface condition. So this no hair behavior for scalar fields with nodes is a very general property.

Acknowledgements This work was supported by the Shandong Provincial Natural Science Foundation of China under Grant No. ZR2018QA008. This work was also supported by a grant from Qufu Normal University of China under Grant No. xkjjc201906.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors’ comment: I would like to emphasize that all relevant physical and mathematical calculations are explicitly presented in this paper.]

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/. Funded by SCOAP3.

References

1. J.D. Bekenstein, Transcendence of the law of baryon-number conservation in black hole physics. Phys. Rev. Lett. 28, 452 (1972)
2. J.E. Chase, Event horizons in Static scalar-vacuum space-times. Commun. Math. Phys. 19, 276 (1970)
3. C. Teitelboim, Nonmeasurability of the baryon number of a black-hole. Lett. Nuovo Cim. 3, 326 (1972)
4. R. Ruffini, J.A. Wheeler, Introducing the black hole. Phys. Today 24, 30 (1971)
5. J.D. Bekenstein, Novel “no-scalar-hair” theorem for black holes. Phys. Rev. D 51(12), R6608 (1995)
6. D. Núñez, H. Quevedo, D. Sudarsky, Black holes have no short hair. Phys. Rev. Lett. 76, 571 (1996)
7. S. Hod, Hairy black holes and null circular geodesics. Phys. Rev. D 84, 124030 (2011)
8. Y. Peng, Hair mass bound in the black hole with non-zero cosmological constants. Phys. Rev. D 98, 104041 (2018)
9. J.D. Bekenstein, Black hole hair: 25-years after. arXiv:gr-qc/9605059
10. C.A.R. Herdeiro, E. Radu, Asymptotically flat black holes with scalar hair: a review. Int. J. Mod. Phys. D 24(09), 1542014 (2015)
11. S. Hod, No-scalar-hair theorem for spherically symmetric reflecting stars. Phys. Rev. D 94, 104073 (2016)
12. S. Hod, Charged massive scalar field configurations supported by a spherically symmetric charged reflecting shell. Phys. Lett. B 763, 275 (2016)
13. Y. Peng, No hair theorem for massless scalar fields outside asymptotically flat horizonless reflecting compact stars. Eur. Phys. J. C 79(10), 850 (2019)
14. S. Bhattacharjee, S. Sarkar, No-hair theorems for a static and stationary reflecting star. Phys. Rev. D 95, 084027 (2017)
15. S. Hod, Marginally bound resonances of charged massive scalar fields in the background of a charged reflecting shell. Phys. Lett. B 768, 97–102 (2017)
16. Y. Peng, B. Wang, Y. Liu, Scalar field condensation behaviors around reflecting shells in Anti-de Sitter spacetimes. Eur. Phys. J. C 78(8), 680 (2018)
17. Y. Peng, Scalar field configurations supported by charged compact reflecting stars in a curved spacetime. Phys. Lett. B 780, 144–148 (2018)
18. S. Hod, Charged reflecting stars supporting charged massive scalar field configurations. Eur. Phys. J. C 78, 173 (2017)
19. Y. Peng, Static scalar field condensation in regular asymptotically AdS reflecting star backgrounds. Phys. Lett. B 782, 717–722 (2018)
20. Y. Peng, On instabilities of scalar hairy regular compact reflecting stars. JHEP 10, 185 (2018)
21. Y. Peng, Hair formation in the background of noncommutative reflecting stars. Nucl. Phys. B 938, 143–153 (2019)
22. M. Khodaei, H.M. Sadjadi, No skyrmion hair for stationary spherically symmetric reflecting stars. Phys. Lett. B 797, 134922 (2019)
23. B. Kiczek, M. Rogatko, Ultra-compact spherically symmetric dark matter charged star objects. JCAP 1909(09), 049 (2019)
24. I.Z. Stefanov, S.S. Yazadjiev, M.D. Todorov, Phases of 4D Scalar-tensor black holes linked to Born–Infeld nonlinear electrodynamics. Mod. Phys. Lett. A 23, 2915–2931 (2008)
25. D.D. Doneva, S.S. Yazadjiev, K.D. Kokkotas, I.Z. Stefanov, Quasi-normal modes, bifurcations and non-uniqueness of charged scalar-tensor black holes. Phys. Rev. D 82, 064030 (2010)
26. T.P. Sotiriou, S.-Y. Zhou, Black hole hair in generalized scalar-tensor gravity. Phys. Rev. Lett. 112, 251102 (2014)
27. D.D. Doneva, S.S. Yazadjiev, New Gauss–Bonnet black holes with curvature induced scalarization in the extended scalar-tensor theories. Phys. Rev. Lett. 120, 131103 (2018)
28. H.O. Silva, J. Sakstein, L. Gualtieri, T.P. Sotiriou, E. Berti, Spontaneous scalarization of black holes and compact stars from a Gauss–Bonnet coupling. Phys. Rev. Lett. 120, 131104 (2018)
29. G. Antoniou, A. Bakopoulos, P. Kanti, Evasion of No-Hair theorems and novel black-hole solutions in Gauss–Bonnet theories. Phys. Rev. Lett. 120, 131102 (2018)
30. Y. Peng, Scalarization of compact stars in the scalar-Gauss–Bonnet gravity. JHEP 12, 064 (2019)
31. P.V.P. Cunha, C.A.R. Herdeiro, E. Radu, Spontaneously scalarised Kerr black holes. Phys. Rev. Lett. 123, 011101 (2019)
32. Y. Peng, Analytical investigations on formations of hair neutral reflecting shells in the scalar-Gauss–Bonnet gravity. Eur. Phys. J. C 80(3), 202 (2020)
33. C.A.R. Herdeiro, E. Radu, N. Sanchis-Gual, J.A. Font, Spontaneous scalarisation of charged black holes. Phys. Rev. Lett. 121, 101102 (2018)
34. D.D. Doneva, S.S. Yazadjiev, Neutron star solutions with curvature induced scalarization in the extended Gauss–Bonnet scalar-tensor theories. JCAP 04, 011 (2018)
35. D.D. Doneva, K.V. Stoykov, S.S. Yazadjiev, Gauss–Bonnet black holes with a massive scalar field. Phys. Rev. D 99, 104045 (2019)
36. Y. Brihaye, C. Herdeiro, E. Radu, The scalarised Schwarzschild-NUT spacetime. Phys. Lett. B 788, 295–301 (2019)
37. Y. Brihaye, B. Hartmann, Charged scalar-tensor solitons and black holes with (approximate) Anti-de Sitter asymptotics. JHEP 1901, 142 (2019)
38. C.A.R. Herdeiro, E. Radu, Black hole scalarisation from the breakdown of scale-invariance. Phys. Rev. D 99, 084039 (2019)
39. H. Motohashi, S. Mukohyama, Shape dependence of spontaneous scalarization. Phys. Rev. D 99, 044030 (2019)
40. Y.S. Myung, D.-C. Zou, Quasimormal modes of scalarized black holes in the Einstein–Maxwell-Scalar theory. Phys. Lett. B 790, 400–407 (2019)
41. D.-C. Zou, Y.S. Myung, Scalarized charged black holes with scalar mass term. Phys. Rev. D 100(12), 124055 (2019)
42. S. Hod, Gauss–Bonnet black holes supporting massive scalar field configurations: the large-mass regime. Eur. Phys. J. C 79, 966 (2019)
43. S. Hod, Spontaneous scalarization of charged Reissner–Nordström black holes: Analytic treatment along the existence line. Phys. Lett. B 798, 135025 (2019)
44. S. Hod, Scalarization of horizonless reflecting stars: neutral scalar fields non-minimally coupled to Maxwell fields. Phys. Lett. B 804, 135372 (2020)
45. C.F.B. Macedo, J. Sakstein, E. Berti, L. Gualtieri, H.O. Silva, T.P. Sotiriou, Self-interactions and spontaneous black hole scalarization. Phys. Rev. D 99, 104041 (2019)
46. S.R. Dolan, S. Ponglertsakul, E. Winstanley, Stability of black holes in Einstein-charged scalar field theory in a cavity. Phys. Rev. D 92, 124047 (2015)
47. J.L. Blázquez-Salcedo, D.D. Doneva, J. Kunz, S.S. Yazadjiev, Radial perturbations of the scalarized Einstein-Gauss–Bonnet black holes. Phys. Rev. D 98, 084011 (2018)
48. D.D. Doneva, S. Kiorpelidi, P.G. Nedkova, E. Papantonopoulos, S.S. Yazadjiev, Charged Gauss–Bonnet black holes with curvature induced scalarization in the extended scalar-tensor theories. Phys. Rev. D 98, 104056 (2018)
49. M. Minamitsuji, T. Ikeda, Scalarized black holes in the presence of the coupling to Gauss–Bonnet gravity. Phys. Rev. D 99, 044017 (2019)
50. M. Horbatsch, H.O. Silva, D. Gerosa, P. Pani, E. Berti, L. Gualtieri, U. Sperhake, Tensor-multi-scalar theories: relativistic stars and 3+1 decomposition. Class. Quantum Gravity 32(20), 204001 (2015)
51. R. Kase, M. Minamitsuji, S. Tsujikawa, Neutron stars with a generalized Proca hair and spontaneous vectorization. arXiv:2001.10701 [gr-qc]
52. S. Hod, Spontaneous scalarization of Gauss–Bonnet black holes: analytic treatment in the linearized regime. Phys. Rev. D 100, 064039 (2019)
53. Y. Peng, Spontaneous scalarization of Gauss–Bonnet black holes surrounded by massive scalar fields. arXiv:2004.12566
54. P.O. Mazur, E. Mottola, Gravitational condensate stars: an alternative to black holes. arXiv:gr-qc/0109035
55. C.B.M.H. Chirenti, Luciano Rezzolla How to tell a gravastar from a black hole. Class. Quantum Gravity 24, 4191–4206 (2007)
56. K. Skenderis, M. Taylor, The fuzzball proposal for black holes. Phys. Rept. 467, 117–171 (2008)
57. V. Cardoso, L.C.B. Crispino, C.F.B. Macedo, H. Okawa, P. Pani, Light rings as observational evidence for event horizons: long-lived modes, ergoregions and nonlinear instabilities of ultracompact objects. Phys. Rev. D 90(4), 044069 (2014)
58. M. Saravani, N. Afshordi, R.B. Mann, Empty black holes, firewalls, and the origin of Bekenstein–Hawking entropy. Int. J. Mod. Phys. D 23(13), 1443007 (2015)
59. V. Cardoso, P. Pani, Testing the nature of dark compact objects: a status report. Living Rev. Rel. 22(1), 4 (2019)
60. C. Barcel‘, R. Carballo-Rubio, L.J. Garay, Gravitational wave echoes from macroscopic quantum gravity effects. JHEP 1705, 54 (2017)
61. S. Hod, Stationary bound-state scalar configurations supported by rapidly-spinning exotic compact objects. Phys. Lett. B 770, 186 (2017)
62. S. Hod, Onset of superradiant instabilities in rotating spacetimes of exotic compact objects. JHEP 1706, 132 (2017)
63. S. Hod, Ultra-spinning exotic compact objects supporting static massless scalar field configurations. Phys. Lett. B 774, 582 (2017)