A Scaling Limit for $t$-Schur Measures

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Abstract

We introduce a new measure on partitions. We assign to each partition $\lambda$ a probability $S_\lambda(x; t)s_\lambda(y)/Z_t$ where $s_\lambda$ is the Schur function, $S_\lambda(x; t)$ is a generalization of the Schur function defined in [M] and $Z_t$ is a normalization constant. This measure, which we call the $t$-Schur measure, is a generalization of the Schur measure [O] and is closely related to the shifted Schur measure studied by Tracy and Widom [TW3] for a combinatorial viewpoint.

We prove that a limit distribution of the length of the first row of a partition with respect to $t$-Schur measures is given by the Tracy-Widom distribution, i.e., the limit distribution of the largest eigenvalue suitably centered and normalized in GUE.

1 Introduction

Let $\mathcal{P}$ be the set of all partitions $\lambda$ and $s_\lambda$ the Schur function (see [M]) with variables $x = (x_1, x_2, \ldots)$ or $y = (y_1, y_2, \ldots)$. The Schur measure introduced in [O] is a probability measure on $\mathcal{P}$ defined by

\[ P_{\text{Schur}}(\{\lambda\}) := \frac{1}{Z_0}s_\lambda(x)s_\lambda(y), \]

where the normalization constant $Z_0$ is determined by the Cauchy identity

\[ Z_0 := \sum_{\lambda \in \mathcal{P}} s_\lambda(x)s_\lambda(y) = \prod_{i,j=1}^{\infty} \frac{1}{1 - x_i y_j}. \]

We consider a certain specialization of this measure. Let $\alpha$ be a real number such that $0 < \alpha < 1$. We put $x_i = \alpha$ and $y_j = \alpha$ for $1 \leq i \leq m$ and $1 \leq j \leq n$, and let the rest be zero. Fix $\tau = m/n$. This is called the $\alpha$-specialization.

Johansson [J1] (see also [J2], [J3]) showed that when $n \to \infty$ a distribution of the length of the first row $\lambda_1$ of a partition $\lambda$ with respect to the $\alpha$-specialized Schur measure converges to the Tracy-Widom distribution [TW1], which is the limit distribution of the largest eigenvalue suitably centered and normalized in the Gaussian Unitary Ensemble (GUE).
The Tracy-Widom distribution \( F_2(s) \) is explicitly expressed as
\[
F_2(s) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[s,\infty)^k} \det(K_{\text{Airy}}(x_i, x_j))_{i,j=1}^k \, dx_1 \ldots dx_k,
\]
where \( K_{\text{Airy}}(x,y) \) denotes the Airy kernel given by
\[
K_{\text{Airy}}(x,y) = \int_0^\infty \text{Ai}(x+z)\text{Ai}(z+y) \, dz.
\]

On the other hand, Tracy and Widom [TW3] studied an analogue of the Schur measure, which they call the shifted Schur measure, and proved that a limit distribution of \( \lambda_1 \) with respect to the \( \alpha \)-specialized shifted Schur measure is also given by the Tracy-Widom distribution.

In this paper, we introduce a generalization of the Schur measure. In order to define such a new measure, let us recall the symmetric functions \( e_n(x; t) \) with parameter \( t \), given by the generating function
\[
(1.3) \quad E_{x,t}(z) := \prod_{i=1}^{\infty} \frac{1 + tx_i z}{1 + x_i z} = \sum_{n=0}^{\infty} e_n(x; t) z^n.
\]
Then the (generalized) Schur function is given by
\[
(1.4) \quad S_{\lambda}(x; t) := \det(e_{\lambda'_i - i + j}(x; t)) ;
\]
where the partition \( \lambda' \) is the conjugate of a partition \( \lambda \), i.e., \( \lambda'_i \) is the length of the \( i \)-th column of \( \lambda \). These functions satisfy the so-called Cauchy identity (see [M])
\[
(1.5) \quad Z_t := \sum_{\lambda \in P} S_{\lambda}(x; t)s_{\lambda}(y) = \prod_{i,j=1}^{\infty} \frac{1 - tx_i y_j}{1 - x_i y_j}.
\]
In particular, when \( t = 0 \), \( E_x(z) = E_{x,0}(z) \) is the generating function of elementary symmetric functions \( e_n(x) = e_n(x; 0) = \sum_{i_1 < i_2 < \ldots < i_n} x_{i_1} x_{i_2} \ldots x_{i_n} \). Since \( s_{\lambda} = \det(e_{\lambda'_i - i + j}) \) (the dual version of the Jacobi-Trudi identity), we notice that the identity (1.5) becomes (1.2) when \( t = 0 \).

By means of the identity (1.3), we may define a probability measure on partitions \( \lambda \) by
\[
(1.6) \quad P_t(\{\lambda\}) := \frac{1}{Z_t} S_{\lambda}(x; t)s_{\lambda}(y).
\]
We call this measure a \( t \)-Schur measure. It reduces to the Schur measure at \( t = 0 \).

Our main result is as follows. Denote by \( P_\sigma \) the \( \alpha \)-specialized \( t \)-Schur measure, where \( \sigma = (m,n,\alpha,t) \) is the associated set of parameters of the measure.
Theorem 1. Suppose $-\infty < t \leq 0$. Then there exist positive constants $c_1 = c_1(\alpha, \tau, t)$ and $c_2 = c_2(\alpha, \tau, t)$ such that
\[
\lim_{n \to \infty} P_\sigma \left( \frac{\lambda_1 - c_1 n}{c_2 n^{1/3}} < s \right) = F_2(s).
\]

This theorem shows that the fluctuations in $\lambda_1$ are independent of the parameter $t$. The assumption that $t$ is non-positive is required since the right hand side of (1.6) must be non-negative after making the $\alpha$-specialization.

In the case where $t = 0$, this theorem gives the result due to Johansson [J1]. In Remark 2, we shall give the explicit expressions of $c_1(\alpha, \tau, 0)$ and $c_2(\alpha, \tau, 0)$ and explain the connection to the result in [J1]. Note also that Theorem 1 does not imply the result of [TW3] (see Remark 1). The proof of Theorem 1 will be given using the method of Tracy-Widom [TW3]. The key of the proof is the determinantal expression of $S_\lambda(x; t)$.

Further we give a combinatorial interpretation of the $t$-Schur measure. Namely, if we denote by $P$ an ordered set $\{1' < 1 < 2' < 2 < 3' < 3 < \ldots\}$, then by virtue of the Robinson-Schensted-Knuth (RSK) correspondence between matrices with entries in $P \cup \{0\}$ and pairs of tableaux of the same shape $\lambda = (\lambda_1, \lambda_2, \ldots)$, we see that the $t$-Schur measure corresponds to a measure (depending on $t$) on $P$-matrices. According to this correspondence, $\lambda_1$ corresponds to the length of the longest increasing subsequence in the biword $w_A$ associated with a $P$-matrix $A$. Using the RSK correspondence and the shifted RSK correspondence, we find that this measure on $P$-matrices at $t = 0$ and $t = -1$, respectively, corresponds to the (original) Schur measure and the shifted Schur measure, respectively (see [J1] and [TW3]).

2 Schur functions, marked tableaux and the RSK correspondence

In this section, we summarize basic properties of Schur functions and marked tableaux for providing a combinatorial interpretation of the $t$-Schur measure (see [M] and [Sa2] for details).

We denote the Young diagram of a partition $\lambda$ by the same symbol $\lambda$. Let $\mathbb{N}$ be the set of all positive integers and $P$ the totally ordered alphabet $\{1' < 1 < 2' < 2 < 3' < 3 < \ldots\}$. The symbols $1', 2', 3', \ldots$ or $1, 2, 3, \ldots$ are said to be marked or unmarked, respectively. When it is not necessary to distinguish a marked element $k'$ from the unmarked one $k$, we write it by $|k|$. A marked tableau $T$ of shape $\lambda$ is an assignment of elements of $P$ to the squares of the Young diagram $\lambda$ satisfying the two conditions.

T1 The entries in $T$ are weakly increasing along each row and down each column.

T2 For each $k \geq 1$, each row contains at most one marked $k'$ and each column contains at most one unmarked $k$. 
The condition **T2** says that for each \( k \geq 1 \) the set of squares labelled by \( k \) (resp. \( k' \)) is a horizontal (resp. vertical) strip.

For example,

\[
\begin{array}{cccc}
1' & 1 & 1 & 2' \\
1' & 2 \\
3 & 3 \\
\end{array}
\]

is a marked tableau of shape \((6, 2, 2)\).

To each marked tableau \( T \), we associate a monomial \( x^T = \prod_{i \geq 1} x_{m_i(T)}^i \), where \( m_i(T) \) is the number of times that \( |i| \) appears in \( T \). In the example above, we have \( x^T = x_1^4x_2^2x_3^4 \).

By the definition of \( S_\lambda(x; t) \) and Chapter I, §5, Example 23 in [M], it follows that

\[
(2.1) \quad S_\lambda(x; t) = \sum_T (-t)^{\text{mark}(T)} x^T,
\]

where the sum runs over all marked tableaux \( T \) of shape \( \lambda \). Here \( \text{mark}(T) \) is the number of marked entries in \( T \). In particular, we have

\[
s_\lambda(x) = S_\lambda(x; 0) = \sum_T x^T,
\]

where the sum runs over all marked tableaux which have no marked entries (i.e., all semi-standard tableaux) of shape \( \lambda \).

We next explain the RSK correspondence between \( \mathbb{P} \)-matrices and pairs of tableaux (see [K], [Sa1] and [HH]). Here \( \mathbb{P} \)-matrix stands for the matrix whose entries are in \( \mathbb{P}_0 = \mathbb{P} \cup \{0\} \).

To each \( \mathbb{P} \)-matrix \( A = (a_{ij}) \) we associate a biword \( w_A \) as follows. For \( i, j \geq 1 \), the pair \((i, j)\) is repeated \( |a_{ij}| \) times in \( w_A \), and if \( a_{ij} \) is marked, the lower entry \( j \) of the first pair \((i, j)\) appeared in \( w_A \) is marked. For example,

\[
A = \begin{pmatrix}
1' & 0 & 2 \\
2 & 1 & 2' \\
1' & 1' & 0
\end{pmatrix} \quad \mapsto \quad w_A = \begin{pmatrix}
1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 & 3 \\
1' & 3 & 1 & 1 & 2 & 3' & 3 & 1' & 2'
\end{pmatrix}.
\]

Observe that for a biword \( w_A = (\beta_1, \beta_2, \ldots, \beta_n) \) the upper line \((\beta_1, \beta_2, \ldots, \beta_n) \in \mathbb{N}^n \) is a weakly increasing sequence. Furthermore if \( \beta_k = \beta_{k+1} \), then \( \alpha_k < \alpha_{k+1} \), or \( \alpha_k \) and \( \alpha_{k+1} \) are identical and unmarked.

Now we state a generalized RSK algorithm. Let \( S \) be a marked tableau and \( \alpha \) an element in \( \mathbb{P} \). The procedure called an **insertion** of \( \alpha \) into \( S \) is described as follows.

**I1** Set \( R := \) the first row of \( S \).

**I2** If \( \alpha \) is unmarked, then
I2a find the smallest element $\beta$ in $R$ greater than $\alpha$ and replace $\beta$ by $\alpha$ in $R$. (This operation is called the BUMP.)

I2b set $\alpha := \beta$ and $R :=$ the next row down.

I3 If $\alpha$ is marked, then

I3a find the smallest element $\beta$ in $R$ which is greater than or equal to $\alpha$ and replace $\beta$ by $\alpha$ in $R$. (This is called the EQBUMP.)

I3b set $\alpha := \beta$ and $R :=$ the next row down.

I4 If $\alpha$ is unmarked and is greater than or equal to the rightmost element in $R$, or if $\alpha$ is marked and greater than every element of $R$, then place $\alpha$ at the end of the row $R$ and stop.

Write the result of inserting $\alpha$ into $S$ by $I_\alpha(S)$.

For a given biword $w_A = \left( \begin{array}{c} \beta_1 \beta_2 \ldots \beta_n \\ \alpha_1 \alpha_2 \ldots \alpha_n \end{array} \right)$, we construct a sequence of pairs of a marked tableau and a semi-standard tableau as

$$(S_0, U_0) = (\emptyset, \emptyset), \ (S_1, U_1), \ldots, \ (S_n, U_n) = (S, U).$$

Assuming that a pair $(S_{k-1}, U_{k-1})$ of the same shape is given. Then we construct $(S_k, U_k)$ as follows. A marked tableau $S_k$ is $I_{\alpha_k}(S_{k-1})$. A semi-standard tableau $U_k$ is obtained by writing $\beta_k$ into the new cell of $U_k$ created by inserting $\alpha_k$ to $S_{k-1}$. We call $S = S_n$ a *insertion tableau* and $U = U_n$ a *recording tableau*.

For example, for a biword $w_A = \left( \begin{array}{c} 1' \ 1 \ 1 \ 2 \ 2 \ 2 \ 3 \ 3 \\ 1' \ 2' \ 3 \ 2' \ 3 \ 2 \ 3 \ 3 \end{array} \right)$, we obtain

$$(S, U) = \left( \begin{array}{cccc} 1' & 1 & 1 & 2' \\ 1' & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{array} \right).$$

The generalized RSK correspondence is then described as follows.

**Theorem 2.** There is a bijection between $\mathbb{P}$-matrices $A = (a_{ij})$ and pairs $(S, U)$ of a marked tableau $S$ and an unmarked tableau $U$ of the same shape such that $\sum_i |a_{ij}| = s_j$ and $\sum_j |a_{ij}| = u_i$. Here we put $s_k = m_k(S)$ and $u_k = m_k(U)$ for $k \geq 1$, and the number of marked entries in $A$ is equal to $\text{mark}(S)$.

**Proof.** The proof of this theorem can be done in a similar way to the original RSK correspondence between $\mathbb{N}$-matrices and pairs of semi-standard tableaux, or of the shifted RSK correspondence between $\mathbb{P}$-matrices and pairs of shifted marked tableaux (See [HH], [Sa1], [Sa2]). We omit the detail. \qed
We call the tableau $S$ (resp. $U$) to be of type $s = (s_1, s_2, \ldots)$ (resp. $u = (u_1, u_2, \ldots)$).

An important property of this correspondence is its relationship to the length of the longest increasing subsequence in a biword $w_A$. The increasing subsequence in $w_A$ is a weakly increasing subsequence in the lower line in $w_A$ such that a marked $k'$ appears at most one for each positive integer $k$. In the example above, $(1'123'3)$ is one of such increasing subsequences in $w_A$. Let $\ell(w_A)$ denote the length of the longest increasing subsequence in $w_A$. Then we have the

**Theorem 3.** If a $\mathbb{P}$-matrix $A$ is corresponding to the pair of tableaux of shape $\lambda = (\lambda_1, \lambda_2, \ldots)$ by the generalized RSK correspondence, then we have $\ell(w_A) = \lambda_1$.

This theorem follows immediately from the following lemma.

**Lemma 1.** If $\pi = \alpha_1\alpha_2\ldots\alpha_n \in \mathbb{P}^n$ and $\alpha_k$ enters a marked tableau $S_{k-1}$ in the $j$th column (of the first row), then the longest increasing subsequence in $\pi$ ending in $\alpha_k$ has length $j$.

**Proof.** We prove the claim by induction on $k$. The result is trivial for $k = 1$. Suppose that it holds for $k - 1$.

First we need to show the existence of an increasing subsequence of length $j$ ending in $\alpha_k$. Let $\beta$ be the element of $S_{k-1}$ in the cell $(1, j-1)$. Then we have $\beta < \alpha_k$, or $\beta$ and $\alpha_k$ are identical and unmarked, since $\alpha_k$ enters in the $j$th column. By induction, there is an increasing subsequence $\sigma$ of length $j - 1$ ending in $\beta$. Thus $\sigma\alpha_k$ is the desired subsequence.

Now we have to prove that there is no longer increasing subsequence ending in $\alpha_k$. Suppose that such a sequence exists and let $\alpha_i$ be the preceding element of $\alpha_k$ in the subsequence. Then it is satisfied that $\alpha_i < \alpha_k$, or $\alpha_i$ and $\alpha_k$ are identical and unmarked. Since the sequence obtained by erasing $\alpha_k$ is a subsequence whose length is greater than or equal to $j$ and whose ending is $\alpha_i$, by induction, $\alpha_i$ enters in some $j'$th column such that $j' \geq j$ when $\alpha_i$ is inserted. Thus the element $\gamma$ in the cell $(1, j')$ of $S_i$ satisfies $\gamma \leq \alpha_i$, so that $\gamma < \alpha_k$, or $\gamma$ and $\alpha_k$ are identical and unmarked.

But since $\alpha_k$ is the element in the cell $(1, j)$ of $S_k$ and $i < k$, we see that $\gamma > \alpha_k$, or that $\gamma$ and $\alpha_k$ are identical and marked. It is a contradiction. Therefore the lemma follows. \[\square\]

For a given $\mathbb{P}$-matrix $A$, $\lambda_1$, where $\lambda = (\lambda_1, \lambda_2, \ldots)$ is obtained from $A$ by the generalized RSK algorithm above, gives the length of the longest increasing subsequence in a biword $w_A$. On the other hands, $\lambda_1$ obtained by the shifted RSK algorithm between $\mathbb{P}$-matrices and pairs of shifted marked tableaux gives the length of the longest ascent pair for a biword $w_A$ associated with $A$ (see in [TW3], [HH]).

### 3 Measures on $\mathbb{P}$-matrices

We give a combinatorial aspect of the $t$-Schur measure using the facts stated in the preceding section. Let $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ be variables satisfying $0 \leq x_i, y_j \leq 1$ for all
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Let $\mathbb{P}_{m,n}$ denote the set of all $\mathbb{P}$-matrices of size $m \times n$. We abbreviate $\ell(w_A)$ to $\ell(A)$ for a $\mathbb{P}$-matrix $A$. Define a measure depending on a parameter $t$ as follows. Assume that matrix elements $a_{ij}$ in $A$ are distributed independently with the following distributions associated with parameters $x_iy_j$:

$$
\text{Prob}_t(a_{ij} = k) = \frac{1 - x_iy_j}{1 - tx_iy_j} (x_iy_j)^k,
\text{Prob}_t(a_{ij} = k') = \frac{1 - x_iy_j}{1 - tx_iy_j} (-t)(x_iy_j)^k
$$

for $k \geq 1$ and

$$
\text{Prob}_t(a_{ij} = 0) = \frac{1 - x_iy_j}{1 - tx_iy_j}.
$$

This $\text{Prob}_t$ indeed defines a probability measure on $\mathbb{P} \cup \{0\}$. Actually we have

$$
\sum_{k=0}^{\infty} \text{Prob}_t(|a_{ij}| = k) = 1
$$

and $\text{Prob}_t(a_{ij} = k') \geq 0$ for every $k$ since $t \leq 0$.

Let

$$
\mathbb{P}_{m,n,s,u,r} := \{ A \in \mathbb{P}_{m,n} \mid \sum_{1 \leq i \leq m} |a_{ij}| = s_j, \sum_{1 \leq j \leq n} |a_{ij}| = u_i, \text{mark}(A) = r \}
$$

for $s \in \mathbb{Z}_{\geq 0}^n$, $u \in \mathbb{Z}_{\geq 0}^m$, and $0 \leq r \leq mn$. Here mark($A$) is the number of marked entries in $A$. Then we have

$$
(3.1) \quad \text{Prob}_t(\{A\}) = \prod_{1 \leq i \leq m, 1 \leq j \leq n} \left(1 - \frac{x_iy_j}{1 - tx_iy_j}\right) (-t)^r x^s y^u = \frac{1}{Z_t} (-t)^r x^s y^u
$$

for $A \in \mathbb{P}_{m,n,s,u,r}$.

If $\text{Prob}_{t,m,n}$ denotes the probability measure obtained by putting $x_i = y_j = 0$ for $i > m$ and $j > n$, then it follows from (3.1), Theorem 2, Theorem 3, and (2.1) that

$$
\text{Prob}_{t,m,n}(\ell \leq h) = \text{Prob}_t(\{ A \in \mathbb{P}_{m,n} \mid \ell(A) \leq h \})
= \sum_{s,u,r} \text{Prob}_t(\{ A \in \mathbb{P}_{m,n,s,u,r} \mid \ell(A) \leq h \})
= \sum_{s,u,r} \# \{ A \in \mathbb{P}_{m,n,s,u,r} \mid \ell(A) \leq h \} \frac{1}{Z_t} (-t)^r x^s y^u
= \sum_{s,u,r} \# \{ (S,U) \mid \text{type} s \text{ and } u, \text{mark}(S) = r, \lambda_1 \leq h \} \frac{1}{Z_t} (-t)^r x^s y^u
= \frac{1}{Z_t} \sum_{\lambda \in \mathcal{P}} \sum_{\lambda_1 \leq h} S_{\lambda}(x_1, \ldots, x_m; t) s_{\lambda}(y_1, \ldots, y_n).
A set \((S, U)|\) type \(s\) and \(u, \text{mark}(S) = r, \lambda_1 \leq h\) consists of all pairs \((S, U)\) of the same shape \(\lambda\) such that \(\lambda_1 \leq h\), where \(S\) is a marked tableau which is of type \(s\) and \(\text{mark}(S) = r\), and \(U\) is a semi-standard tableau which is of type \(u\).

Observe that the rightmost hand side in the above equality is the value of the \(t\)-Schur measure with respect to a set \(\{\lambda \in \mathcal{P}|\lambda_1 \leq h\}\). Particularly, when \(t = 0\), this measure on \(\mathbb{P}\)-matrices turns to be the measure on \(\mathbb{N}\)-matrices and it corresponds to the (original) Schur measure (see [11], Johansson’s \(q\) is equal to our \(\alpha^2\)). On the other hand, when \(t = -1\), by the shifted RSK correspondence we see that it corresponds to the shifted Schur measure (see [TW3]).

**Remark 1.** The \(t\)-Schur measure at \(t = -1\) does not coincide with the shifted Schur measure since the correspondences between \(\mathbb{P}\)-matrices and partitions are different. In fact, Theorem 1 states that a (centered and normalized) limit distribution of \(\ell(w_A)\) is identical with one of the length \(L(w_A)\) of the longest ascent pair for \(w_A\).

## 4 Proof of Theorem 1

In this section, we prove Theorem 1 using the methods developed in [TW3]. We recall some notations. Denote the Toeplitz matrix \(T(\phi) = (\phi_{i-j})_{i,j \geq 0}\) and the Hankel matrix \(H(\phi) = (\phi_{i+j+1})_{i,j \geq 0}\), where \((\phi_n)_{n \in \mathbb{Z}}\) is the sequence of Fourier coefficients of a function \(\phi\) (see [BS]). These matrices act on the Hilbert space \(\ell^2(\mathbb{Z}_+)\) \((\mathbb{Z}_+ = \mathbb{N} \cup 0)\). Also we put \(T_h(\phi) = (\phi_{i-j})_{0 \leq i,j \leq h-1}\) and \(\tilde{\phi}(z) := \phi(z^{-1})\). Let \(P_h\) be the projection operator from \(\ell^2(\mathbb{Z}_+)\) onto the subspace \(\ell^2(\{0, 1, \ldots, h-1\})\) and set \(Q_h := I - P_h\), where \(I\) is the identity operator on \(\ell^2(\mathbb{Z}_+)\).

The following lemmas are keys in the proof. The first one, Lemma 2 is a generalization of the Gessel identity (see [G], [TW2]).

**Lemma 2.**

\[
(4.1) \quad \sum_{\lambda \in \mathcal{P}, \lambda_1 \leq h} S_{\lambda}(x; t) s_{\lambda}(y) = \det T_h(\tilde{E}_{x,t} E_y)
\]

where \(E_{x,t}\) is defined in (1.3) and \(E_y = E_{y,0}\).

**Proof.** Let \(M(x; t)\) be the \(\infty \times h\) submatrix \((e_{i-j}(x; t))_{i \geq 1, 1 \leq j \leq h}\) of the Toeplitz matrix \(T(E_{x,t})\), and for any subset \(S \subset \mathbb{N}\), let \(M_S(x; t)\) be the submatrix of \(M(x; t)\) obtained from rows indexed by elements of \(S\). In particular, write \(M(x) = M(x; 0)\).

For a partition \(\lambda = (\lambda_1, \lambda_2, \ldots)\) such that \(\lambda_1 \leq h\), let \(\lambda^\prime\) (whose length is smaller than or equal to \(h\)) be the conjugate partition of \(\lambda\) and let \(S = \{\lambda^\prime_{h+1-i} + i|1 \leq i \leq h\}\). Then we have \(\det M_S(x; t) = \det (e_{\lambda^\prime_{h+1-i}+i-j}(x; t))_{1 \leq i,j \leq h}\). Reversing the order of rows and columns in
this determinant, we have \( \det M_S(x; t) = \det(e_{\lambda_1-i+j}(x; t)) = S_\lambda(x; t) \) by (4.1). In particular, \( \det M_S(y) = \det(e_{\lambda_1-i+j}(y)) = s_\lambda(y) \). It follows that

\[
\sum_{\lambda \in P} S_\lambda(x; t)s_\lambda(y) = \sum_S \det M_S(x; t) \det M_S(y)
\]

where the sum is all over \( S \subset \mathbb{N} \) such that \( \#S = h \). Then by the Cauchy-Binet identity, we have

\[
= \det M(x; t)M(y) = \det T_h(\tilde{E}_{x,t}E_y).
\]

Hence the lemma follows.

**Lemma 3.** If we put \( \phi = \tilde{E}_{x,t}E_y \), then we have

\[
(4.2) \quad \det T_h(\phi) = E(\phi) \det(I - H_1H_2) \mid_{\ell^2((h,h+1,\ldots))}
\]

where put \( H_1 = H(\tilde{E}_{x,t}E_y^{-1}) \), \( H_2 = H(E_{x,t}^{-1}\tilde{E}_y) \) and \( E(\phi) := \exp\{\sum_{k=1}^{\infty} k(\log \phi)_k(\log \phi)^{-k}\} \). Here the determinant on the right side in (4.2) is a Fredholm determinant defined by

\[
\det(I - K) \mid_{\ell^2((h,h+1,\ldots))} := \det(Q_h - Q_hKQ_h)
\]

for any trace class operator \( K \).

**Proof.** We observe that both \( H_1 \) and \( H_2 \) are Hilbert-Schmidt operators. We obtain the equality (4.2) by applying directly the relation between the Toeplitz determinant and the Fredholm determinant by Borodin and Okounkov [BoO], [BaW] to \( \phi \). We leave the detail to the reader.

Note that for \( \phi = \tilde{E}_{x,t}E_y \), we have \( E(\phi) = Z_t \).

If we denote by \( J \) the diagonal matrix whose \( i \)-th entry equals \( (-1)^i \), then we have \( \det(I - H_1H_2) = \det(I - JH_1H_2J) \). In general, we note that \( -JH(\phi(z))J = H(\phi(-z)) \) for any function \( \phi(z) \).

From (1.6), (1.1) and (4.2), we obtain

\[
(4.3) \quad P(\lambda_1 \leq h) = \det(Q_h - Q_hJH_1H_2JQ_h).
\]

We make here \( \alpha \)-specialization and scaling. We put \( x_i = y_j = \alpha \) (\( 0 < \alpha < 1 \)) for \( 1 \leq i \leq m \) and \( 1 \leq j \leq n \), and \( x_i = y_j = 0 \) for \( i > m \) and \( j > n \). Put \( \tau = m/n > 0 \). We set \( i = h + n^{1/3}x \) and \( j = h + n^{1/3}y \), where \( h = cn + n^{1/3}s \). The positive constant \( c \) will be determined later.

It is convenient to replace \( \ell^2((h,h+1,\ldots,)) \) by \( \ell^2(\mathbb{Z}+) \). Let \( \Lambda \) be a shift operator on \( \ell^2(\mathbb{Z}+) \), i.e., \( \Lambda e_j = e_{j-1} \) for the canonical basis \( \{e_j\}_{j \geq 0} \) of \( \ell^2(\mathbb{Z}+) \) and \( \Lambda^* \) the adjoint operator.
of $\Lambda$. In $Q_h J H_1 H_2 J Q_h$, we interpret a $Q_h$ appearing on the left as $\Lambda^h$ and a $Q_h$ appearing on the right as $\Lambda^{\ast h}$. Then the $(i, j)$-entry of $-Q_h J H_1 J$ is

$$\frac{1}{2\pi \sqrt{-1}} \int \left( \frac{z - \alpha}{z - t \alpha} \right)^m \left( \frac{1}{1 - \alpha z} \right)^n z^{-cn - n^{1/3} s - i - j} \frac{dz}{z^2}$$

and the $(i, j)$-entry of $-J H_2 J Q_h$ is

$$\frac{1}{2\pi \sqrt{-1}} \int \left( 1 - t \alpha z \right)^m \left( 1 - \alpha z^{-1} \right)^n z^{-cn - n^{1/3} s - i - j} \frac{dz}{z^2},$$

where the contours of the integrals are both the unit circle. For the second integral, if we change the variable $z \to z^{-1}$, we obtain

$$\frac{1}{2\pi \sqrt{-1}} \int \left( \frac{z - t \alpha}{z - \alpha} \right)^m \left( 1 - \alpha z \right)^n \frac{dz}{z^2},$$

Putting

$$\psi(z) = \left( \frac{z - \alpha}{z - t \alpha} \right)^m \left( \frac{1}{1 - \alpha z} \right)^n z^{-cn},$$

we find that two integrals are rewritten as

$$(4.4) \quad \frac{1}{2\pi \sqrt{-1}} \int \psi(z) z^{-n^{1/3} s - i - j} \frac{dz}{z^2},$$

$$(4.5) \quad \frac{1}{2\pi \sqrt{-1}} \int \psi(z)^{-1} z^{n^{1/3} s + i + j} \frac{dz}{z^2}.$$

To estimate these integrals, we apply the steepest descent method. At first we determine the constant $c$. Let $\sigma(z) = n^{-1} \log \psi(z)$, so that

$$(4.6) \quad \sigma'(z) = \frac{\tau \alpha (1 - t)}{(z - \alpha)(z - t \alpha)} + \frac{\alpha}{1 - \alpha z} - \frac{c}{z}.$$ 

We choose a constant $c$ such that $\sigma(z)$ has the point $z$ satisfying the equality $\sigma'(z) = \sigma''(z) = 0$. Then we obtain

$$(4.7) \quad \frac{\tau(1 - t)(z^2 - t \alpha^2)}{(z - \alpha)^2(z - t \alpha)^2} - \frac{1}{(1 - \alpha z)^2} = 0.$$ 

Since we assume $t \leq 0$, it is immediate to see that the function on the left hand side in (4.7) is strictly decreasing from $+\infty$ to $-\infty$ on the interval $(\alpha, \alpha^{-1})$ and it follows that there is a unique point $z_0$ in $(\alpha, \alpha^{-1})$, where the left is equal to zero. This is a saddle point and we set

$$(4.8) \quad c := \alpha z_0 \left( \frac{\tau(1 - t)}{(z_0 - \alpha)(z_0 - t \alpha)} + \frac{1}{1 - \alpha z_0} \right)$$
from (4.6). The constant $c$ is positive since $\alpha < z_0 < \alpha^{-1}$ and $t \leq 0$.

It is clear that the number counting with the multiplicity of zeros of the function $\sigma'(z)$ are three from (4.6) when $t \neq 0$. In the case where $t < 0$, $\sigma'(z)$ has a zero in $(t\alpha, 0)$ because $\lim_{z \to t\alpha} \sigma'(z) = -\infty$ and $\lim_{z \to 0} \sigma'(z) = +\infty$. On the other hand, in the case where $t = 0$, the number of zeros of the function $\sigma'(z)$ are two. Therefore, since $\sigma'(z)$ has a double zero $z_0$, we have $\sigma''(z_0) = 0$. Further, since $\lim_{z \to t\alpha} \sigma'(z) = +\infty$ and $\lim_{z \to t\alpha^{-1}} \sigma'(z) = +\infty$, $\sigma'(z)$ is positive on $(\alpha, \alpha^{-1})$ except $z_0$, so that $\sigma''(z_0)$ is positive.

We call $\Gamma_+$ a steepest descent curve for the first integral (4.3) and $\Gamma_-$ for the second integral (4.5). On $\Gamma_+$, the absolute value $|\psi(z)| = \exp \text{Re} \sigma(z)$ is maximal at $z = z_0$ and strictly decreasing as we move away from $z_0$ on the curve. The first one emanates from $z_0$ at angles $\pm \pi/3$ with branches going to $\infty$ in two directions. The second one emanates from $z_0$ at angles $\pm 2\pi/3$ and is getting close to $z = 0$.

Let $D$ be a diagonal matrix whose $i$-th entry is given by $\psi(z_0)^{-1} z_0^{n^{1/3} s + i}$, and multiply $Q_h J_1 H_2 J_Q = (-Q_h J_1 J) (J H_2 J Q_h)$ by $D$ from the left and by $D^{-1}$ from the right. Note that the determinant of the right hand side in (4.4) is not affected.

By the discussion in Section 6.4.1 in [TW3], we see that the $([n^{1/3} x], [n^{1/3} y])$-entry of $n^{1/3} DQ_h J_1 H_2 J Q_h D^{-1}$ converges to $g K_{\text{Airy}}(g(s + x), g(s + y))$ in the trace norm as $n \to \infty$, where we put

$$g := z_0^{-1} \left( \frac{2}{\sigma''(z_0)} \right)^{1/3}$$

and $K_{\text{Airy}}(x, y)$ is the Airy kernel. Clearly, $g$ is positive.

Hence, by (4.3), we have

$$\lim_{n \to \infty} P_\sigma(\lambda_1 \leq cn + n^{1/3} s)$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[0, \infty)^k} \det \left( (DQ_h J_1 H_2 J Q_h D^{-1})_{i,j} \right)_{1 \leq i, j \leq k} \prod_{i<j} dx_i \cdots dx_k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[0, \infty)^k} \det \left( (DQ_h J_1 H_2 J Q_h D^{-1})_{[n^{1/3} x], [n^{1/3} y]} \right)_{1 \leq i, j \leq k} \prod_{i<j} dx_i \cdots dx_k$$

$$\to \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[0, \infty)^k} \det \left( g K_{\text{Airy}}(g(s + x_i), g(s + x_j)) \right)_{1 \leq i, j \leq k} \prod_{i<j} dx_i \cdots dx_k$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{[g(s), \infty)^k} \det \left( K_{\text{Airy}}(x_i, x_j) \right)_{1 \leq i, j \leq k} \prod_{i<j} dx_i \cdots dx_k$$

$$= F_2(gs).$$
This shows the assertion of the theorem, where the constants $c_1(\alpha, \tau, t)$ and $c_2(\alpha, \tau, t)$ are given by $c$ and $g^{-1}$, respectively. We complete the proof of the theorem.

**Remark 2.** In the case where $t = 0$, we have $z_0 = \frac{\alpha + \sqrt{\tau}}{1 + \sqrt{\tau}}$ by (4.7). Therefore we obtain

$$c_1(\alpha, \tau, 0) = \frac{(1 + \sqrt{\tau})^2}{1 - \alpha^2} - 1$$

by (4.8) and

$$c_2(\alpha, \tau, 0) = g^{-1} = \frac{\alpha^{1/3} \tau^{-1/6}}{1 - \alpha^2} (\alpha + \sqrt{\tau})^{2/3} (1 + \sqrt{\tau})^{2/3}$$

by (4.9). These values give the corresponding values in Theorem 1.2 in [J1]. Note that the relation between our $\alpha$ and Johansson’s $q$ is given by $q = \alpha^2$.

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