Spontaneous Symmetry Breaking in Hyperbolic Field Theory

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Abstract. We study a non-compact analog to the U(1) symmetry group. The Higgs potential is obtained as a transversal section of the $\lambda\phi^4$ potential possessing symmetry under the action of this group. Both the spontaneous symmetry breaking and the uniform exact symmetry scenarios are obtained as particular cases. We then study an extension of the U(1) group by taking the direct product of it with this non-compact generator. In particular we obtain an expression for the mass terms of the potential after spontaneous symmetry breaking.

1. Introduction
Hyperbolic numbers appeared in the literature for the first time in a James Cockle article in the year of 1848 [1]. Despite having appeared since so long, applications of them in physical contexts haven’t been developed until recent years. The progress done however has shown (see [2]) that a lot of structures used in physics, usually built in terms of complex numbers, admit definitions in terms of hyperbolic numbers. In particular, studies of relativistic quantum mechanics and relativistic wave equations [3], [4], and even a formulation of a gravito-electromagnetism theory [5], have been done systematically. The present work is based on the more detailed study [6].

Hyperbolic numbers can be introduced rigorously as one of the three possible (up to isomorphism) real commutative unital algebras [2]. More general hypercomplex systems are defined as finite-dimensional (not necessarily commutative) real unital algebras [7]. In this work we do not take this approach; we make practical definitions and present only the properties that will be needed. In section 2 we introduce the hyperbolic number system, the usual arithmetic operations are presented and the hyperbolic phases, or versors, are defined. In section 3 we study the potential of the $\lambda\phi^4$ theory over the hyperbolic numbers, and its respective spontaneous symmetry breaking (SSB) realization. In section 4 we extend the number system of hyperbolic numbers by combining them with the usual complex numbers, thus obtaining the bicomplex number system. Finally, in section 5 we present the case of the bicomplex $\lambda\phi^4$ and one of its possible SSB scenarios. Conclusions are presented in section 6.

2. Hyperbolic numbers
Complex numbers can be constructed heuristically by taking all the formal combinations $x + iy$ with $i$, called the imaginary unit, is such that $i^2 = -1$, and $x, y \in \mathbb{R}$, but obviously $i \notin \mathbb{R}$. In a similar way we define hyperbolic numbers as the set of all numbers of the form $x + jy$, with $x, y \in \mathbb{R}$, and $j$ is a new quantity, called hyperbolic imaginary unit, with the property $j^2 = +1$. 
but \( j \notin \mathcal{R} \), in particular \( j \neq \pm 1 \). We will denote this set as \( \mathcal{D} \), i.e. \( \mathcal{D} = \{ x + j y | x, y \in \mathcal{R} \} \).

We define sum, subtraction and multiplication in the natural way: for any \( z_1, z_2 \in \mathcal{D} \), with \( z_1 = x_1 + j y_1 \) and \( z_2 = x_2 + j y_2 \),

\[
z_1 \pm z_2 \equiv (x_1 \pm x_2) + j(y_1 \pm y_2) \quad (1)
\]

\[
z_1 z_2 \equiv (x_1 x_2 + y_1 y_2) + j(x_1 y_2 + y_1 x_2). \quad (2)
\]

Analogously to the usual complex case, we define a hyperbolic conjugation for any \( z = x + j y \in \mathcal{D} \), \( \bar{z} \equiv x - j y \). The modulus or norm is \( |z|^2 \equiv z \bar{z} = x^2 - y^2 \). Note that this “norm” is not positive definite; in particular \( |z|^2 = 0 \) whenever \( y = x \); these numbers of the form \( z = x + j x \) are called null hyperbolic numbers. The (multiplicative) inverse of \( z \) is denoted by \( z^{-1} \) and defined as

\[
z^{-1} = \frac{\bar{z}}{|z|^2}. \quad (3)
\]

We can see that null numbers do not possess an inverse. Finally we define hyperbolic phases or “versors” in the hyperbolic analogous to the Euler formula:

\[
e^{j \alpha} \equiv \cosh \alpha + j \sinh \alpha. \quad (4)
\]

### 3. The hyperbolic \( \lambda \varphi^4 \) model

In this section we analyze the potential of a massive self-interacting hyperbolic scalar field. We propose the following lagrangian

\[
\mathcal{L} = \partial^\mu \varphi \partial_\mu \varphi - V(\varphi, \bar{\varphi}), \quad (5)
\]

\[
V = m^2 \bar{\varphi} \varphi + \frac{\lambda}{2} (\bar{\varphi} \varphi)^2, \quad (6)
\]

where \( \varphi(x) = \varphi_1(x) + j \varphi_2(x) \in \mathcal{D} \). This potential is invariant under rescaling by hyperbolic phases, i.e. under the transformations

\[
\varphi \to e^{j \alpha}, \quad \bar{\varphi} \to e^{-j \alpha} \bar{\varphi}. \quad (7)
\]

The graph of the potential (as a function of the real and hyperbolic parts of \( \varphi \), denoted by \( \text{Re}(\varphi) \) and \( \text{Hy}(\varphi) \), respectively) is shown in figure 1. The solid black lines correspond to the SSB (for \( \text{Hy}(\varphi) = 0 \)) and non-SSB (for \( \text{Re}(\varphi) = 0 \)) scenarios of a real scalar field.

One interesting thing to notice is that the full hyperbolic potential \( \lambda \varphi^4 \) always presents SSB, independent of the sign of the (squared) mass parameter. This can be seen in figure 2, where the potential is plotted for both positive (figure 3) and negative (figure 3) mass term. Both figures are qualitatively equal, the only difference is a 90 degree rotation in field space, i.e., the change from a real to a tachyonic mass amounts to make the discrete transformation \( (\varphi_1, \varphi_2) \to (\varphi_2, -\varphi_1) \).

We now proceed to make explicit the SSB of this potential. First we write it in terms of \( \varphi_1 \) and \( \varphi_2 \),

\[
V(\varphi_1, \varphi_2) = m^2 (\varphi_1^2 - \varphi_2^2) + \frac{\lambda}{2} (\varphi_1^2 - \varphi_2^2)^2. \quad (8)
\]

The extrema of this function are located at \( \varphi_1 = \varphi_2 = 0 \) and at the hyperbola defined by \( m^2 + \lambda (\varphi_1^2 - \varphi_2^2) \). The former is a saddle point, which can be seen from the figure or by evaluating the trace of the Hessian matrix of \( V \) at \( \varphi_1 = \varphi_2 = 0 \), while the latter are the true minima of the potential. We have then

\[
|\varphi|^2_{\text{min}} = \varphi_{1\text{min}}^2 - \varphi_{2\text{min}}^2 = -\frac{m^2}{\lambda} \equiv K^2. \quad (9)
\]
Figure 1. Potential for a massive self-interacting hyperbolic field.

Figure 2. Potentials for real and imaginary mass.

For $m^2 < 0$ the simplest choice is $\varphi_{1\text{min}} = K$ and $\varphi_{2\text{min}} = 0$, so we define the shifted fields $\chi_1$ and $\chi_2$ by

$$\varphi_1 \equiv \chi_1 + K, \quad \varphi_2 \equiv \chi_2.$$  \hspace{1cm} (10)

In terms of these the potential reads

$$V = 2m^2\chi_1^2 + \frac{\lambda}{2}\chi_1^4 + \frac{\lambda}{2}\chi_2^4 - \lambda\chi_1^2\chi_2^2 + 2\lambda K \chi_1(\chi_1^2 - \chi_2^2).$$ \hspace{1cm} (11)

We can see that $\chi_2$ is now a Goldstone boson which has lost its mass, while $\chi_1$ doubled the value of its mass term. Both have quartic self-interactions and cubic and quartic interactions with each other. This is not very different from the usual complex case.

4. Bicomplex numbers

We now introduce the bicomplex number system, denoted as $\mathcal{H}$, which is basically the direct product of complex and hyperbolic numbers. We define $z \in \mathcal{H}$ as a number of the form

$$z = x + iy + jv + iwj,$$ \hspace{1cm} (12)
with $x, y, v, w \in \mathcal{R}$. Addition, subtraction and multiplication of bicomplex numbers are defined in the natural way. Complex conjugation now acts both on $i$ and $j$: 

$$z = x - iy - jv + ijw,$$

so the first and last terms do not change sign. The “norm” of $z$ is

$$|z|^2 = \bar{z}z = x^2 + y^2 - v^2 - w^2 + 2ij(xw - yv). \quad (13)$$

Now note that $|z|^2$ is now, in general, not even a real number, but it is hermitian in the sense that it is invariant under bicomplex conjugation; we call this type of numbers (the ones of the form $a + ijb$) hybrid numbers. The set of all hybrid numbers has a couple of nice features: it is closed under sum and multiplication, and any number $z = a + ijb$ always has a multiplicative inverse (as long as $a^2 + b^2 \neq 0$). We can also define the bicomplex phases:

$$e^{i\alpha + j\beta} \equiv e^{i\alpha}e^{j\beta} = \cos \alpha \cosh \beta + i \sin \alpha \cosh \beta + j \cos \alpha \sinh \beta + ij \sin \alpha \sinh \beta. \quad (14)$$

In general a bicomplex number has 4 “degrees of freedom”, however in the following sections we will be interested in the simplest generalization of a complex field, which has only two. To reduce the number of components of the bicomplex field, then, we will assume the following relations of proportionality:

$$x = \beta w \ , \ y = \beta v, \quad (15)$$

where $\beta \in \mathcal{R}$ is a constant which in principle could take any value; we will nevertheless see below that in some cases we will have to restrict its range of values to a certain interval. Using (15) we can write

$$z = (\beta + ij)w + (i\beta + j)v \ , \ \bar{z} = (\beta + ij)w - (i\beta + j)v. \quad (16)$$

The norm is now given by

$$|z|^2 = (\beta^2 - 1)(v^2 + w^2) + 2ij\beta(w^2 - v^2). \quad (17)$$

The relations (15) are the only ones consistent with both (circular and hyperbolic) invariances of this norm. For instance, the identifications $x = \beta v$ and $y = \beta w$ would be inconsistent with the circular symmetry.

5. The bicomplex $\lambda \psi^4$ model

We now study the lagrangian

$$\mathcal{L} = \partial^\mu \bar{\psi} \partial_\mu \psi - V(\psi, \bar{\psi}), \quad (18)$$

$$V = \frac{a^2}{2} \bar{\psi} \psi + \frac{\lambda}{4!} (\bar{\psi} \psi)^2, \quad (19)$$

where $a = \pm 1$ is just a constant that allows us to control the sign of the mass term, and $\psi$ is a bicomplex field with the restrictions described in the previous section, i.e. it has the form (16). Also, to have a bounded potential (here with “bounded” we mean that both the real and the $ij$ components of the potential are bounded), we have to assume that the mass and self-coupling parameters are hybrid:

$$m^2 = m_R^2 + ijm_H^2, \quad \lambda = \lambda_R + i\lambda_H. \quad (20)$$

The potential (19) only depends on the combination $\bar{\psi} \psi$, so it takes only hybrid values. Then we can write it as $V = V_R + iV_H$, where

$$V_R = a \left( \frac{\beta^2 - 1}{2} - m_R^2 + \beta m_H^2 \right) v^2 + a \left( \frac{\beta^2 - 1}{2} - m_R^2 - \beta m_H^2 \right) w^2 + \frac{\lambda_R}{6} \left[ \frac{(\beta^2 - 1)^2}{4} - (v^2 + w^2)^2 - \beta^2 (v^2 - w^2)^2 \right]$$

$$-\frac{\lambda_H}{6} \beta (\beta^2 - 1)(w^4 - v^4), \quad (21)$$

$$0,$
The first option corresponds to the origin of field space and is not very interesting to us. The second can be rewritten in terms of the components of $\psi$ as

\begin{equation}
(1 - \beta^2)(\lambda_R + \lambda_H)(v_0^2 + w_0^2) = 6a(\lambda_R m_R^2 + \lambda_H m_H^2),
\end{equation}

\begin{equation}
\beta(\lambda_R + \lambda_H)(v_0^2 - w_0^2) = 3a(\lambda_R m_H^2 - \lambda_H m_R^2).
\end{equation}

We can solve these two equations for $v_0$ and $w_0$, however from the analysis of the hessian matrix of the potentials $V_R$ and $V_H$ we see that the points $(v_0, w_0)$ determined by (25) and (26) are saddle points unless one of the expectation values $v_0$ or $w_0$ vanishes. We assume then that $w_0 = 0$ (the other choice, $v_0 = 0$, is completely analogous), which implies

\begin{equation}
m_H^2 = \frac{(1 - \beta^2)\lambda_H + 2\beta\lambda_R}{(1 - \beta^2)\lambda_R - 2\beta\lambda_H}, \quad v_0^2 = \frac{6am_R^2}{(1 - \beta^2)\lambda_R - 2\beta\lambda_H}.
\end{equation}

In this work we will only analyze the case with the simplifying condition $\lambda_R = \lambda_H \equiv \lambda$. We also assume $\beta^2 \neq 1$. With these assumptions (27) simplifies to

\begin{equation}
m_H^2 = \frac{\beta^2 - 2\beta - 1}{\beta^2 + 2\beta - 1}, \quad v_0^2 = \frac{6m_R^2}{a\lambda(1 - 2\beta - \beta^2)}.
\end{equation}

The latter equation implies that

\begin{equation}
a\lambda(1 - 2\beta - \beta^2) > 0.
\end{equation}

In the following we assume $a = +1$, $\lambda > 0$, and $-1 - \sqrt{2} < \beta < -1 + \sqrt{2}$ so the above inequality holds. (Actually, from the analysis of the hessian, the allowed interval for producing a stable vacuum is $\beta \in (-0.2, 0.2)$, which is within the interval on which the condition (29) holds.)

We can now rewrite the potentials as

\begin{equation}
V_R = \frac{a}{2}P_R m_R^2 v^2 + \frac{a}{2}P_R m_R^2 w^2 + \frac{\lambda}{6}\left[\frac{(\beta^2 - 1)^2}{4}(v^2 + w^2)^2 - \beta^2(v^2 - w^2)^2 - \beta(\beta^2 - 1)(w^4 - v^4)\right],
\end{equation}

\begin{equation}
V_H = \frac{a}{2}P_H m_H^2 v^2 + \frac{a}{2}P_H m_H^2 w^2 + \frac{\lambda}{6}\left[\frac{(\beta^2 - 1)^2}{4}(v^2 + w^2)^2 - \beta^2(v^2 - w^2)^2 + \beta(\beta^2 - 1)(w^4 - v^4)\right],
\end{equation}

where

\begin{equation}
P_R = \frac{\beta^4 + 4\beta^3 - 6\beta^2 - 4\beta + 1}{\beta^2 + 2\beta - 1}, \quad P_R = \frac{(\beta^2 + 1)^2}{\beta^2 + 2\beta - 1},
\end{equation}

\begin{equation}
P_H = \frac{\beta^4 - 4\beta^3 + 6\beta^2 - 4\beta + 1}{\beta^2 + 2\beta - 1}, \quad P_H = P_R.
\end{equation}
Figure 3. The plots shown here correspond to $V_R/m_R^2$ and $V_H/m_H^2$.

$V_R$ and $V_H$ are qualitatively equal; their plots are shown in figure 3. We can see that they have more or less the shape of the traditional mexican hat potential, however now the minima are just two points, located at the bottom of the red valleys.

Once we choose a specific value for the vacuum and expand the lagrangian around that point, we obtain that the quadratic terms (i.e. the mass terms) reduce to

$$-i j P_H^r m_R^2 v^2. \quad (32)$$

So the field $w$ has lost both of its mass components, while $v$ still has a mass in the hybrid sense, though its real part vanishes.

6. Conclusions
We have developed the $\lambda\phi^4$ theory for both hyperbolic and a bicomplex fields. We have shown that the hyperbolic potential contains both the SSB and non-SSB scenarios of a single real scalar field. Also, it is qualitatively insensitive to the sign of the mass term, leading to a hyperbolic SSB scenario whether mass is real or tachyonic. In the more general bicomplex case, we saw that in some cases the incorporation of the new imaginary unit $j$ leads to a deformation of the mexican hat potential, reducing its vacuum manifold to a set of only two points.

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