Extremal graphs for edge blow-up of lollipops

Yanni Zhai¹, Xiying Yuan¹*, Zhenyu Ni²

¹ Department of Mathematics, Shanghai University, Shanghai 200444, P.R. China
² School of Science, Hainan University, Haikou 570228, P.R. China

Abstract

Given a graph $H$ and an integer $p$ ($p \geq 2$), the edge blow-up $H^{p+1}$ of $H$ is the graph obtained from replacing each edge in $H$ by a clique of order $(p + 1)$, where the new vertices of the cliques are all distinct. The Turán numbers for edge blow-up of matchings were first studied by Erdős and Moon. Very recently some substantial progress of the extremal graphs for $H^{p+1}$ of larger $p$ has been made by Yuan. The range of Turán numbers for edge blow-up of all bipartite graphs when $p \geq 3$ and the exact Turán numbers for edge blow-up of all non-bipartite graphs when $p \geq \chi(H) + 1$ has been determined by Yuan (2022), where $\chi(H)$ is the chromatic number of $H$. A lollipop $C_{k, \ell}$ is the graph obtained from a cycle $C_k$ by appending a path $P_{\ell+1}$ to one of its vertices. In this paper, we consider the extremal graphs for $C_{k, \ell}^{p+1}$ of the rest cases $p = 2$ and $p = 3$.

Keywords: Extremal graph, Turán number, Edge blow-up, Lollipop

1. Introduction

In this paper, we consider undirected graphs without loops and multiedges. For a graph $G$, denote by $E(G)$ the set of edges and $V(G)$ the set of vertices of $G$. The order of a graph is the number of its vertices and the size of a graph is the number of its edges. The number of edges of $G$ is denoted by $e(G) = |E(G)|$. For a vertex $v$ of graph $G$, the neighborhood of $v$ in $G$ is denoted by $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of the vertex $v$, written as $d_G(v)$ or simply $d(v)$, is the number of edges incident with $v$. Usually, a path of order $n$ is denoted by $P_n$, a star of order $n + 1$ is denoted by $S_n$ and a cycle of order $n$ is denoted by $C_n$. An independent set of order $n$ is denoted by $I_n$. A matching in $G$ is a set of vertex disjoint edges from $E(G)$, denoted by $M_k$ a matching of size $k$. For $U \subseteq V(G)$, let $G[U]$ be the subgraph of $G$ induced by $U$, $G - U$ be the graph obtained by deleting all the vertices

*Corresponding author.
Email address: xiyingyuan@shu.edu.cn (Xiying Yuan).
yannizhai2022@163.com(Yanni Zhai).
1051466287@qq.com(Zhenyu Ni).

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in $U$ and their incident edges. The graph $K_p(i_1, i_2, \ldots, i_p)$ denote the complete $p$-partite graph with parts of order $i_1, i_2, \ldots, i_p$. Denoted by $T_p(n)$, the $p$-partite Turán graph is the complete $p$-partite graph on $n$ vertices with the order of each partite set as equal as possible.

For two graphs $G$ and $H$, the union of graphs $G$ and $H$ is the graph $G \cup H$ with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. In particular, $G = kH$ is the vertex-disjoint union of $k$ copies of $H$. The join of $G$ and $H$, denoted by $G \vee H$, is the graph obtained from $G \cup H$ by adding all edges between $V(G)$ and $V(H)$. Let $H(n, p, q)$ be the graph $K_{q-1} \vee T_p(n - q + 1)$, and $H'(n, p, q)$ be any of the graphs obtained by putting one extra edge in any class of $T_p(n - q + 1)$. Let $H^*(n)$ be graphs obtained by putting (almost) perfect matchings in both classes in $K_2([\frac{n}{2}], [\frac{n}{2}])$.

A graph is $H$-free if it does not contain a copy of $H$ as a subgraph. The Turán number $ex(n, H)$ is the maximum number of edges in a graph of order $n$ which is $H$-free. Denote by $EX(n, H)$ the set of $H$-free graphs of order $n$ with $ex(n, H)$ edges and call a graph in $EX(n, H)$ an extremal graph for $H$.

Given a graph $H$, the blow-up of $H$, denoted as $H^{p+1}$, is obtained from $H$ by replacing each edge in $H$ by a clique of order $(p + 1)$, where the new vertices of the cliques are all different. In 1959, Erdős and Gallai [2] characterized the extremal graphs for $M_{2k}$. Later, Erdős [4] studied the Turán numbers of $M_{2k}^p$. Moon [9] and Simonovits [11] determined the extremal graphs for $M_{2k}^p$ when $p \geq 2$. Erdős, Füredi, Gould and Gunderson [5] determined the Turán number of $S_{k+1}^3$. Chen, Gould, Pfender and Wei [1] determined the Turán number of $S_{k+1}^{p+1}$ for general $p \geq 3$. Glebov [6] determined the extremal graphs for edge blow-up of paths. Later, Liu [8] generalized Glebov's result to edge blow-up of paths, cycles and a class of trees. Very recently, Wang, Hou, Liu and Ma [14] determined the Turán numbers for edge blow-up of a large family of trees. A keyring is a $(k + s)$-edge graph obtained from a cycle of order $k$ by appending $s$ leaves to one of its vertices. In some way it is a generalization of cycle and star. Ni, Kang and Shan [10] determined the extremal graphs for edge blow-up of keyrings. A lollipop $C_{k, \ell}$ is the graph obtained from a cycle of order $k$ by appending a path $P_{\ell+1}$ to one of its vertices, and the vertex $v \in V(C_{k, \ell})$ of degree 3 is called the center of the lollipop. In this paper we will consider the extremal graphs for edge blow-up of lollipops.

The range of Turán numbers for edge blow-up of all bipartite graphs when $p \geq 3$ and the exact Turán numbers for edge blow-up of all non-bipartite graphs when $p \geq \chi(H) + 1$ has been determined by Yuan in [15]. The following Theorem 1.1 (i) and (ii) are implied by Theorem 2.3 in [15] by taking $\mathcal{B} = \{K_t\}$ when $\ell$ is odd, and $\mathcal{B} = \{K_{t+1}\}$ when $\ell$ is even; and Theorem 1.1 (iii) and (iv) are implied by Theorem 2.4 in [15] by taking $\mathcal{B} = \{K_t\}$ when $\ell$ is odd, and $\mathcal{B} = \{K_{t+1}\}$ when $\ell$ is even.

**Theorem 1.1.** [15] Suppose $k \geq 3$, $\ell \geq 2$, $t = \lfloor \frac{k-1}{2} \rfloor + \lfloor \frac{\ell-1}{2} \rfloor$, $n$ is sufficiently large.

(i) When $k$ is even, $\ell$ is odd, $p \geq 3$, $EX(n, C_{k, \ell}^{p+1}) = H'(n, p, t + 1)$. 


(ii) When \( k \) is even, \( \ell \) is even, \( p \geq 3 \), \( \text{EX}(n, C_{k, \ell}^{p+1}) = H(n, p, t + 2) \).

(iii) When \( k \) is odd, \( \ell \) is odd, \( p \geq 4 \), \( \text{EX}(n, C_{k, \ell}^{p+1}) = H(n, p, t + 1) \).

(iv) When \( k \) is odd, \( \ell \) is even, \( p \geq 4 \), \( \text{EX}(n, C_{k, \ell}^{p+1}) = H(n, p, t + 2) \).

How about the extremal graphs for \( C_{k, \ell}^{p+1} \) for small \( p \)? In this paper we will consider the rest cases of Theorem 1.1. More precisely, the extremal graph for \( C_{k, \ell}^4 \) when \( k \geq 3 \) is odd, \( \ell \geq 2 \) is determined in Theorem 3.1; the extremal graph for \( C_{k, \ell}^3 \) when \( k \geq 4, \ell \geq 2 \) is odd is determined in Theorem 3.2; the extremal graph for \( C_{k, \ell}^3 \) when \( k \geq 4, \ell \geq 2 \) is even is determined in Theorem 3.3; the extremal graph for \( C_{3, \ell}^3 \) when \( \ell \geq 2 \) is determined in Theorem 3.4.

Simonovits proposed the following problem.

**Problem 1.1.** Characterize graphs whose unique extremal graph is of the form \( H(n, p, q) \), where \( q \geq 1, p \geq 2 \).

Combining Theorems 1.1, 1.2, 3.1 - 3.4, the extremal graph for \( C_{k, 1}^{p+1} \) is of the form \( H(n, p, q) \). In this way an additional family of forbidden graphs of Problem 1.1 is provided in this paper.

We would like to point out that in [10] Ni, Kang and Shan have determined the extremal graphs for \( C_{k, 1}^{p+1} \) (see Theorem 1.2). In this paper we suppose \( \ell \geq 2 \).

**Theorem 1.2.** ([10]) When \( k \geq 3, p \geq 2, n \) is sufficiently large, let \( G \) be the extremal graph for \( C_{k, 1}^{p+1} \).

(i) When \((p, k) \neq (2, 3)\), \( H(n, p, \lceil \frac{k-1}{2} \rceil + 1) \) (or \( H(n, p, \lfloor \frac{k-1}{2} \rfloor + 1) \) resp.) is the unique extremal graph for \( C_{k, 1}^{p+1} \) when \( k \) is odd (even resp.).

(ii) When \((p, k) = (2, 3)\)

\[
G \in \left\{ \begin{array}{ll}
\{(\frac{1}{4} \lceil \frac{n}{2} \rceil K_3) \cup I_{\lceil \frac{n}{2} \rceil}, H^*(n)\}, & \text{if } 12|n,
\{H^*(n)\}, & \text{if } 6 \nmid n \text{ but } 4|n,
\{((\frac{1}{8} \lceil \frac{n}{2} \rceil K_3) \cup I_{\frac{n}{2}}\}, & \text{if } 4 \nmid n \text{ but } 3|\frac{n}{2},
\{(k_1 K_3 \cup k_1^2 P_2 \cup k_1^2 P_1) \cup I_{\lceil \frac{n}{2} \rceil}, H^*(n), (S_{k_2} \cup k_2 K_3) \cup I_{\lceil \frac{n}{2} \rceil}\}, & \text{otherwise},
\end{array} \right.
\]

where \( 1 \leq \lceil \frac{n}{2} \rceil - 3k_1 \leq 2, k_1 = \lceil \frac{n}{2} \rceil - 3k_1 - 1, k_1^2 = 3k_1 + 2 - \lceil \frac{n}{2} \rceil, 0 \leq 3k_2 \leq \lceil \frac{n}{2} \rceil \) and \( 3k_2 + 1 + k_2' = \lceil \frac{n}{2} \rceil \).

2. Preliminaries

The key idea of our proofs is using a result of Simonovits (Theorem 2.1) to get a good vertex partition of an extremal graph for \( C_{k, \ell}^{p+1} \). This was recommended by Liu in [8]. The vertex split graphs family and the decomposition family result was introduced by Liu, which is also very crucial in the proofs.
Given a graph $H$ and a vertex $v \in V(H)$ with $d_H(v) \geq 2$, a vertex split on the vertex $v$ is defined as follows: replace $v$ with an independent set of size $|N_H(v)|$ and each vertex is adjacent to exactly one distinct vertex in $N_H(v)$. Let $U$ be a vertex subset $U \subseteq V(H)$, a vertex split on $U$ means applying vertex split on the vertices in $U$ one by one. Apparently, the order of vertices we apply vertex split does not matter. $\mathcal{H}(H)$ is defined as the family of all the graphs which can be obtained by applying vertex split on any vertex subset $U \subseteq V(H)$. It is easy to see that $U$ can be empty, therefore, $H \in \mathcal{H}(H)$. $\mathcal{H}_p(H)$ is defined as the family of all the graphs obtained from $H$ by applying vertex split on the vertex subset $U \subseteq V(H)$, which satisfies $\chi(H[U]) \leq p$. In particularly, $\mathcal{H}^*(H)$ is the family of all the graphs obtained from $H$ by applying vertex split on any independent set of $H$. It is not difficult to see that when $p \geq 2$ we have $\mathcal{H}^*(H) \subseteq \mathcal{H}_{p-1}(H) \subseteq \mathcal{H}(H)$.

**Definition 2.1.** ([13]) Given a family $\mathcal{L}$, define $p = p(\mathcal{L}) = \min_{L \in \mathcal{L}} \chi(L) - 1$. Let $\mathcal{M} := \mathcal{M}(\mathcal{L})$ be the family of minimal graphs $M$ for which exist an $L \in \mathcal{L}$ and $t = t(L)$ satisfy that $L \subseteq M' \lor K_{p-1}(t, t, \ldots, t)$ where $M' = M \cup I_t$. We call this the decomposition family of $\mathcal{L}$.

The following characterization of $\mathcal{M}(H^{p+1})$ was provided in [10].

**Lemma 2.1.** ([10]) Let $H$ be any graph and $p \geq 2$ be any integer.

(i) If $p \geq 3$ and $\chi(H) \leq p - 1$, then $\mathcal{M}(H^{p+1}) = \mathcal{H}(H)$.

(ii) If $p \geq 3$ and $\chi(H) = p$, then $\mathcal{M}(H^{p+1}) = \mathcal{H}_{p-1}(H)$.

(iii) If $p = 2$ and $\chi(H) = 2$ or 3, then $\mathcal{M}(H^{p+1}) = \mathcal{H}^*(H)$.

**Definition 2.2.** ([12]) Let $G$ be a graph, $H_1$, $H_2$ be subgraphs of $G$. They are called symmetric if $H_1 = H_2$ or satisfy the following conditions:

(i) $V(H_1) \cap V(H_2) = \emptyset$.

(ii) $xy \notin E(G)$ if $x \in V(H_1)$, $y \in V(H_2)$.

(iii) There exists an isomorphism $\omega: H_1 \rightarrow H_2$ such that for every $x \in V(H_1)$ and $u \in G - V(H_1) - V(H_2)$, $x$ is adjacent to $u$ if and only if $\omega(x)$ is adjacent to $u$.

**Definition 2.3.** ([12]) Let $\mathcal{D}(n, p, r)$ be the family of graph $G$ of order $n$ with following conditions:

(i) It is possible to omit at most $r$ vertices of $G$ so that the remaining graph $G'$ is a product of almost equal order: $G' = \bigvee_{i \leq p} G^i$, where $|V(G^i)| = n_i$ and $|n_i - \frac{n}{p}| \leq r$ ($1 \leq i \leq p$).

(ii) For every $1 \leq i \leq p$, there exist connected graphs $H_i$ such that $G^i = k_i H_i$, where $k_i = \frac{n_i}{|V(H_i)|}$ and any two copies $H^j_i$, $H^l_i$ in $G^i$ ($1 \leq j < l \leq k_i$) are symmetric subgraphs of $G$. 
The graphs $H_i$ in Definition 2.3 will be called the blocks, the vertices in $G - G'$ will be called exceptional vertices. Let $A_1, \cdots, A_p$ be the $p$ classes in $G'$, where $A_i = V(G^i)$ for any $i \in \{1, \cdots, p\}$. Let $W$ be the set of vertices in $G - G'$ that are adjacent to all vertices in $G'$ and let $B_i$ be the set of vertices in $G - G' - W$ that are not adjacent to any vertex in $A_i$.

**Theorem 2.1.** (12) Assume that a finite family $\mathcal{L}$ of forbidden graphs with $p(\mathcal{L}) = p$ is given. If there exists some $L \in \mathcal{L}$ with $m := |V(L)|$ such that

$$L \subseteq P_m \lor K_{p-1}(m, m, \cdots, m), \quad (2.1)$$

then there exist $r = r(L)$ and $n_0 = n_0(r)$ such that $\mathbb{D}(n, p, r)$ contains an $\mathcal{L}$-extremal graph for every $n > n_0$. Furthermore, if this is the only extremal graph in $\mathbb{D}(n, p, r)$, then it is the unique extremal graph for every sufficiently large $n$.

Let $Y_{k+1, \ell+1}$ be the family of graphs obtained from a path $P_{k+1}$ by appending a path $P_{\ell+1}$ to one of its vertices except the end points. The vertex of degree 3 is called branching vertex.

**Lemma 2.2.** For any integers $k \geq 3$, $\ell \geq 0$, $p \geq 2$ and $m = |V(C_{k, \ell}^{p+1})|$, 

(i) if $Y$ is a graph in $Y_{k+1, \ell+1}$, then we have $C_{k, \ell}^{p+1} \subseteq Y \lor K_{p-1}(m, m, \cdots, m)$;

(ii) we have $C_{k, \ell}^{p+1} \subseteq (P_k \cup P_{\ell+1}) \lor K_{p-1}(m, m, \cdots, m)$.

**Proof.** The fact $\chi(C_{k, \ell}) = 2$ or 3, and Lemma 2.1 (ii) and (iii) imply that $\mathcal{H}^*(C_{k, \ell}) \subseteq \mathcal{H}_{p-1}(C_{k, \ell}) = \mathcal{M}(C_{k, \ell}^{p+1})$.

(i) Note that $Y$ can be obtained by applying vertex split on the one of the vertices except the center vertex of the cycle of $C_{k, \ell}$. Thus, $Y \in \mathcal{H}^*(C_{k, \ell})$ and then $Y \in \mathcal{M}(C_{k, \ell}^{p+1})$. By the definition of $\mathcal{M}(C_{k, \ell}^{p+1})$, we have $C_{k, \ell}^{p+1} \subseteq Y \lor K_{p-1}(m, m, \cdots, m)$.

(ii) By applying vertex split on the center of $C_{k, \ell}$, the resulting graph is $P_{k+1} \cup P_{\ell+1}$. Thus, $(P_k \cup P_{\ell+1}) \in \mathcal{H}^*(C_{k, \ell})$ and then $(P_k \cup P_{\ell+1}) \in \mathcal{M}(C_{k, \ell}^{p+1})$. By the definition of $\mathcal{M}(C_{k, \ell}^{p+1})$, we have $C_{k, \ell}^{p+1} \subseteq (P_k \cup P_{\ell+1}) \lor K_{p-1}(m, m, \cdots, m)$. □

Obviously $(P_k \cup P_{\ell+1}) \subseteq P_m$, hence $C_{k, \ell}^{p+1} \subseteq P_m \lor K_{p-1}(m, m, \cdots, m)$. By Theorem 2.1 we have $EX(n, C_{k, \ell}^{p+1}) \in \mathbb{D}(n, p, r)$.

3. Main Results

Let $G$ be an extremal graph for $C_{k, \ell}^{p+1}$, then we have $G \in \mathbb{D}(n, p, r)$. In the rest part, we always let $A_i, H_i, B_i, W$ be the decompositions of $G$ as defined in Definition 2.3. In this section we mainly further characterize the structure of them to find the extremal graph for $C_{k, \ell}^{p+1}$. Since $G \in \mathbb{D}(n, p, r)$, we may have the following upper bound for $e(G)$.

**Lemma 3.1.** Suppose $p \geq 2$, $n$ is sufficiently large, each block $H_i$ in $G^i$ is a single vertex, then we have

$$e(G) \leq e(T_p(n)) + \frac{n|W|}{p} + o(n). \quad (3.1)$$
The following Lemma 3.2 implies a lower bound for $e(G)$ (see Corollary 3.1). We always write $t = \lfloor \frac{k-1}{2} \rfloor + \lfloor \frac{\ell-1}{2} \rfloor$, $m = |V(C_{k, \ell}^{p+1})|$ in this section.

**Lemma 3.2.** Suppose $k \geq 3$, $\ell \geq 2$, $p \geq 2$, $n$ is sufficiently large.

(i) $H(n, p, t + 1)$ is $C_{k, \ell}^{p+1}$-free when $k$ is odd, $\ell$ is odd.

(ii) $H(n, p, t + 2)$ is $C_{k, \ell}^{p+1}$-free when $k$ is odd, $\ell$ is even.

(iii) $H'(n, p, t + 1)$ is $C_{k, \ell}^{p+1}$-free when $k$ is even, $\ell$ is odd.

(iv) $H(n, p, t + 2)$ is $C_{k, \ell}^{p+1}$-free when $k$ is even, $\ell$ is even.

**Proof.** Denote by $Q$ the vertex set $V(K_{q-1})$ in graph $H(n, p, q)$. Any $(p + 1)$-clique in $H(n, p, q)$ contains at least one vertex of $Q$ and there is at most one $(p + 1)$-clique in $H'(n, p, q)$ has no vertex in $Q$. On the other hand, in $C_{k, \ell}^{p+1}$ there are only three $(p + 1)$-cliques which share one vertex, and any other pairs of $(p + 1)$-cliques share at most one vertex.

(i) When $k$ is odd, $\ell$ is odd, we have $t = \frac{k+\ell-2}{2}$. Note that $e(C_{k, \ell}) = k + \ell$. In $H(n, p, t + 1)$, we have $|Q| = t$, and then the number of $(p + 1)$-cliques of $C_{k, \ell}^{p+1}$ in $H(n, p, t + 1)$ is at most $2(t - 1) + 3 = k + \ell - 1 < k + \ell$. Hence, $H(n, p, t + 1)$ is $C_{k, \ell}^{p+1}$-free.

(ii) When $k$ is odd, $\ell$ is even, we have $t = \frac{k+\ell-3}{2}$. If $C_{k, \ell}^{p+1} \subseteq H(n, p, t + 2)$, then the vertex set $Q$ is a vertex cover of $C_{k, \ell}$. The minimum size of vertex cover of $C_{k, \ell}$ is $\frac{k+\ell}{2}$, while we have $|Q| = t + 1 = \frac{k+\ell-1}{2} < \frac{k+\ell+1}{2}$. Hence, $H(n, p, t + 2)$ is $C_{k, \ell}^{p+1}$-free.

(iii) When $k$ is even, $\ell$ is odd, we have $t = \frac{k+\ell-3}{2}$. In $H'(n, p, t + 1)$, the number of $(p + 1)$-cliques of $C_{k, \ell}^{p+1}$ is at most $2(t - 1) + 3 + 1 = k + \ell - 1 < k + \ell$. Hence, $H'(n, p, t + 1)$ is $C_{k, \ell}^{p+1}$-free.

(iv) When $k$ is even, $\ell$ is even, we have $t = \frac{k+\ell}{2} - 2$. In $H(n, p, t + 2)$ the number of $(p + 1)$-cliques of $C_{k, \ell}^{p+1}$ is at most $2t + 3 = k + \ell - 1 < k + \ell$. Hence, $H(n, p, t + 2)$ is $C_{k, \ell}^{p+1}$-free.

\[\Box\]

**Corollary 3.1.** Suppose $p \geq 2$, $n$ is sufficiently large.

(i) If $\ell$ is odd, then we have

\[e(G) \geq e(T_p(n)) + \frac{tn}{p} + o(n). \quad (3.2)\]

(ii) If $\ell$ is even, then we have

\[e(G) \geq e(T_p(n)) + \frac{(t+1)n}{p} + o(n). \quad (3.3)\]

If each block $H_i$ in $G^i$ ($1 \leq i \leq p$) is a single vertex, then we may give some characterizations for the set $W$ and $B_i$ (see Lemma 3.3 and Lemma 3.4) of the extremal graph $G$. 

6
Lemma 3.3. Suppose \( k \geq 3, \ell \geq 2, p \geq 2, n \) is sufficiently large and each block \( H_i \) in \( G^i \) \((1 \leq i \leq p)\) is a single vertex. Then we have

(i) \(|W| = t\) when \( \ell \) is odd;

(ii) \(|W| = t + 1\) when \( \ell \) is even.

Proof. (i) When \( \ell \) is odd, by (3.1) and (3.2), it is easy to see \(|W| \geq t\). Furthermore, if \(|W| \geq t + 1\), then we may suppose \( \{w_1, w_2, \ldots, w_{t+1}\} \subseteq W\). The assumption that \( n \) is sufficiently large ensures \( |A_i| \) is sufficiently large, and then we may suppose \( \{u_1, u_2, \ldots, u_{t+3}\} \subseteq A_1\). When \( k \) is odd, we may have a graph \( Y \in Y_{k+1, \ell+1} \) with \( w_{k+1} \) as the branching vertex and

\[
P_{k+1} = u_1 w_1 u_2 w_2 \cdots w_{k-1} u_{k+1} w_{k+1},
\]

\[
P_{\ell+1} = w_{k+1} u_{k+2} w_k u_{k+3} \cdots w_{t+1} u_{t+2}.
\]

When \( k \) is even, we may have a graph \( Y \in Y_{k+1, \ell+1} \) with \( w_{k+1} \) as the branching vertex and

\[
P_{k+1} = u_1 w_1 u_2 w_2 \cdots u_k w_k u_{t+3},
\]

\[
P_{\ell+1} = w_k u_{k+2} w_{k+2} u_{k+4} \cdots w_{t+1} u_{t+2}.
\]

Then \( Y \subseteq G[A_1 \cup W]\). Furthermore \( Y \lor K_{p-1}(m, m, \ldots, m) \subseteq G\). On the other hand, by Lemma 2.2 (i) \( C_{k, \ell}^{p+1} \subseteq Y \lor K_{p-1}(m, m, \ldots, m) \). Therefore, we have a \( C_{k, \ell}^{p+1} \) in \( G \) and this contradiction shows \(|W| = t\).

(ii) When \( \ell \) is even, by (3.1) and (3.3), it is easy to see \(|W| \geq t + 1\). Furthermore, if \(|W| \geq t + 2\), we may suppose \( \{w_1, w_2, \ldots, w_{t+2}\} \subseteq W\), \( \{u_1, u_2, \ldots, u_{t+4}\} \subseteq A_1\). When \( k \) is odd, we have two paths in \( G \) with

\[
P_{k+1} = u_1 w_1 u_2 w_2 \cdots u_{k+1} w_{k+1},
\]

\[
P_{\ell+1} = u_{k+1} w_{k+1} \cdots w_{t+2} u_{t+3}.
\]

When \( k \) is even, we have two paths in \( G \) with

\[
P_{k+1} = u_1 w_1 u_2 w_2 \cdots u_k w_k u_{t+4},
\]

\[
P_{\ell+1} = u_k w_k w_{k+2} \cdots w_{t+2} u_{t+3}.
\]

Then \( (P_{k+1} \cup P_{\ell+1}) \subseteq G[A_1 \cup W]\). Furthermore \( (P_{k+1} \cup P_{\ell+1}) \lor K_{p-1}(m, m, \ldots, m) \subseteq G\). By Lemma 2.2 (ii) we may obtain a \( C_{k, \ell}^{p+1} \) in \( G \) and this contradiction shows \(|W| = t + 1\). \( \square \)

Lemma 3.4. Suppose \( k \geq 3, \ell \geq 2, p \geq 2, n \) is sufficiently large and each block \( H_i \) in \( G^i \) \((1 \leq i \leq p)\) is a single vertex. Then each vertex in \( B_i \) is adjacent to all the vertices in \( V(G^i) \setminus A_i \).
Proof. By Lemma 3.3, we have $|W| = t$ when $\ell$ is odd, $|W| = t + 1$ when $\ell$ is even. Without loss of generality, we may suppose to the contrary that there is a vertex $v$ in $B_2$ which is not adjacent to some vertex in $A_1$. Since the blocks in $G^i$ are symmetric, then $v$ is not adjacent to any vertex in $A_1$. Then when $\ell$ is odd, we have
\[
e(G) \leq e(T_p(n)) + \frac{nt}{p} - |A_1| + o(n)
\]
\[= e(T_p(n)) + \frac{n(t - 1)}{p} + o(n),
\]
which contradicts to (3.2).

When $\ell$ is even
\[
e(G) \leq e(T_p(n)) + \frac{n(t + 1)}{p} - |A_1| + o(n)
\]
\[= e(T_p(n)) + \frac{nt}{p} + o(n),
\]
which contradicts to (3.3). □

Lemma 3.5. Suppose $k \geq 3$ is odd, $\ell \geq 2$, $p \geq 2$, $n$ is sufficiently large. If the extremal graph $G$ for $C_{k, \ell}^{p+1}$ satisfies the following conditions:

(i) when $\ell$ is odd, $|W| = t$; when $\ell$ is even, $|W| = t + 1$,
(ii) each vertex in $B_i$ is adjacent to all the vertices in $V(G') \setminus A_i$,

then we have $e(G[B_i]) = 0$ ($1 \leq i \leq p$).

Proof. Suppose $W = \{w_1, \ldots, w_t\}$ ($W = \{w_1, \ldots, w_{t+1}\}$ resp.) when $\ell$ is odd ($\ell$ is even resp.) and $\{u_1, \ldots, u_{t+1}\} \subseteq A_1$. We first show that $G[B_i]$ is $P_t$-free. Suppose not, without loss of generality, let $P_t = x_1y_1x_2y_2 \subseteq G[B_2]$, then when $\ell$ is odd, we have a lollipop with $w_{k-3}$ as center vertex and
\[
C_k = y_1u_1w_1u_2w_2 \cdots w_{k-3}u_{k-1}x_2y_1,
\]
\[
P_{t+1} = w_{k-3}u_{k+1} \cdots w_{t-1}u_{t+1}.
\]
When $\ell$ is even, we have a lollipop with $w_{k-3}$ as center vertex and
\[
C_k = y_1u_1w_1u_2w_2 \cdots w_{k-3}u_{k-1}x_2y_1,
\]
\[
P_{t+1} = w_{k-3}u_{k+1} \cdots w_{t-1}u_{t+1}w_t.
\]
Each edge between $W$ and $A_1$ can be blown up into a $(p + 1)$-clique by using vertices in $A_i$ ($2 \leq i \leq p$). The edge $y_1u_1$ can be blown up by using vertex $x_1$ and vertices in $A_i$ ($3 \leq i \leq p$) and the edge $u_{k-1}x_2$ can be blown up by using vertex $y_2$ and vertices in $A_i$ ($3 \leq i \leq p$). The
edge $x_2y_1$ can be blown up by using a vertex in $A_1$ and using vertices in $A_i$ ($3 \leq i \leq p$). Then there is a $C_{k, \ell}^{p+1}$ in $G$. Thus $G[B_i]$ is $P_4$-free. 

If $e(G[B_i]) \neq 0$, then there is an edge $e = xy \in E(G[B_i])$ which satisfies that there exists a vertex $w \in W$ with $wx \in E(G)$ or $wy \in E(G)$. Otherwise, let $G_1$ be the graph obtained from $G$ by deleting all edges of $\cup_{i \leq p} G[B_i]$ and adding all missing edges between $W$ and $\cup_{i \leq p} B_i$, then $G_1$ is $C_{k, \ell}^{p+1}$-free. In fact if there is a $C_{k, \ell}^{p+1}$ in $G_1$ then we may choose some vertices in $\cup_{i \leq p} A_i$ as the substitutes for the vertices in $\cup_{i \leq p} B_i$ to obtain a $C_{k, \ell}^{p+1}$ in $G$. So $G_1$ is $C_{k, \ell}^{p+1}$-free. Since $G[B_1]$ is $P_4$-free and by Gallai Theorem (see §2), we have $e(G[B_i]) \leq |B_i|$. Furthermore, $|W| \geq 2$, hence

$$e(G_1) \geq e(G) - \sum_{i \leq p} e(G[B_i]) + 2 \sum_{i \leq p} |B_i| > e(G),$$

which is a contradiction to the definition of $G$. 

So if there is an edge $xy$ in $G[B_1]$, then we may suppose $w_{\frac{k-1}{2}}$ is adjacent to $x$ and find two paths. In fact when $\ell$ is odd, we may have a graph $Y \in \mathcal{Y}_{k+1, \ell+1}$ with $w_{\frac{k-1}{2}}$ as branching vertex and

$$P_{k+1} = u_1 w_1 u_2 w_2 \cdots u_{\frac{k-1}{2}} w_{\frac{k-1}{2}} xy,$$

$$P_{\ell+1} = w_{\frac{k-1}{2}} u_{\frac{k+1}{2}} \cdots w_{t} u_{t+1}.$$ 

When $\ell$ is even, we may have a graph $Y \in \mathcal{Y}_{k+1, \ell+1}$ with $w_{\frac{k-1}{2}}$ as branching vertex and

$$P_{k+1} = u_1 w_1 u_2 w_2 \cdots u_{\frac{k-1}{2}} w_{\frac{k-1}{2}} xy,$$

$$P_{\ell+1} = w_{\frac{k-1}{2}} u_{\frac{k+1}{2}} \cdots w_{t} u_{t+1}.$$ 

Then $Y \subseteq G[W \cup A_1 \cup B_1]$. Furthermore, $Y \cup K_p(m, m, \cdots, m) \subseteq G$. Therefore by Lemma 2.2 (i) we have $C_{k, \ell}^{p+1} \subseteq G$ which is a contradiction. So we have $e(G[B_i]) = 0$ ($1 \leq i \leq p$). 

Let the set of vertices of the cycle of $C_{k, \ell}$ be $\{a_1, a_2, \cdots, a_k\}$, the set of vertices of the path of $C_{k, \ell}$ be $\{a_1, b_2, \cdots, b_{\ell+1}\}$. 

Theorem 3.1. When $k \geq 3$ is odd, $\ell \geq 2$, $n$ is sufficiently large, $H(n, 3, t+1)$ ($H(n, 3, t+2)$ resp.) is the unique extremal graph for $C_{k, \ell}^4$ when $\ell$ is odd (even resp.).

Proof. Suppose $\ell$ is odd, then $t = \frac{k+\ell-2}{2}$. Now we will prove each block $H_i$ is a single vertex ($i = 1, 2, 3$).

If $P_3 \subseteq H_1$, then $(\frac{k-1}{2} P_3 \cup (\ell+1) P_2) \subseteq G[A_1]$, and $(\frac{k-1}{2} P_3 \cup (\ell+1) P_2) \vee K_2(m, m) \subseteq G$. On the other hand, by applying vertex split on the vertex set $U = \{a_1, a_3, \cdots, a_k, b_2, b_3, \cdots, b_\ell\}$, the resulting graph is $\frac{k-1}{2} P_3 \cup (\ell+1) P_2$, since $\chi(C_{k, \ell}[U]) = 2$, we have $(\frac{k-1}{2} P_3 \cup (\ell+1) P_2) \in \mathcal{H}_2(C_{k, \ell})$. By the fact $\chi(C_{k, \ell}) = 3$ and Lemma 2.1 (ii), we have $\mathcal{H}_2(C_{k, \ell}) = \mathcal{M}(C_{k, \ell}^4)$, so $(\frac{k-1}{2} P_3 \cup (\ell+1) P_2) \in \mathcal{M}(C_{k, \ell}^4)$. By using the definition of $\mathcal{M}$, we have $C_{k, \ell}^4 \subseteq (\frac{k-1}{2} P_3 \cup (\ell+1) P_2) \vee K_2(m, m) \subseteq G$. This contradiction implies that $H_i$ is $P_4$-free.
Lemma 3.6. When \( e \mid e \) we have \( k \subseteq (i) \) We claim that \( H \)

Now suppose \( H_1 = H_2 = P_2 \) and then \( (k + \ell)P_2 \vee ((k + \ell)P_2 \vee I_{k+\ell}) \subseteq G \). Note that for any graph \( F, F^4 \subseteq e(F)P_2 \vee F \) holds. The fact \( C_{k, \ell} \subseteq ((k + \ell)P_2 \vee I_{k+\ell}) \) implies that \( C_{k, \ell}^4 \subseteq (k + \ell)P_2 \vee ((k + \ell)P_2 \vee I_{k+\ell}) \subseteq G \).

If \( H_1 = P_2, H_2 = P_1, H_3 = P_1 \), then

\[
e(G) \leq e(T_3(n)) + \frac{n|W|}{3} + \frac{|A_1|}{2} + o(n)
\]

\[
\leq e(T_3(n)) + \frac{n|W|}{3} + \frac{n}{6} + o(n).
\]

From Corollary 3.1, we have \( e(G) \geq e(T_3(n)) + \frac{m_2}{3} + o(n) \), so \( |W| \geq t \). Indeed, if \( |W| \geq t \), then \( (\ell t P_3 \cup (\ell + 1)P_2) \subseteq G[W \cup A_1] \), and then \( (\ell t P_3 \cup (\ell + 1)P_2) \vee K_2(m, m) \subseteq G \). So we have \( C_{k, \ell}^4 \subseteq (\ell t P_3 \cup (\ell + 1)P_2) \vee K_2(m, m) \subseteq G \), which is a contradiction.

Therefore we have each block \( H_i \) is a single vertex \((i = 1, 2, 3)\), and then by Lemma 3.3, 3.4, 3.5, we have (i) \( |W| = t \); (ii) each vertex in \( B_i \) is adjacent to all the vertices in \( V(G') \setminus A_i \); (iii) \( e(G[B_i]) = 0 \) for \( i = 1, 2, 3 \). By the maximality of \( G \) we have \( G = H(n, p, t + 1) \). By using the similar arguments, when \( \ell \) is even, we have \( G = H(n, p, t + 2) \). \( \square \)

**Lemma 3.6.** When \( k \geq 4, \ell \geq 2, p = 2 \), each block \( H_i \) in \( G^i \) is a single vertex \((i = 1, 2)\).

**Proof.** Suppose \( \ell \) is odd, then when \( k \) is odd, \( t = \frac{k + \ell - 2}{2} \); when \( k \) is even, \( t = \frac{k + \ell - 3}{2} \).

(i) We claim that \( H_i \) is \( P_3 \)-free. Suppose to the contrary that \( P_3 \subseteq H_1 \).

When \( k \) is even, we have \( ((t + 1)P_3 \cup P_2) \subseteq G[A_1] \), and then \( ((t + 1)P_3 \cup P_2) \vee I_m \subseteq G \). On the other hand, when \( k \) is even, we apply vertex split on the vertex set \( U = \{a_1, a_3, \ldots, a_{k-1}, b_3, b_5, \ldots, b_{\ell}\} \), the resulting graph is \( (t+1)P_3 \cup P_2 \), since \( \chi(C_{k, \ell}(U)) = 1 \), we have \( ((t + 1)P_3 \cup P_2) \in \mathcal{H}^*(C_{k, \ell}) \). By the fact \( \chi(C_{k, \ell}) = 2 \) and Lemma 2.1 (iii), we have \( \mathcal{M}(C_{k, \ell}^3) = \mathcal{H}^*(C_{k, \ell}) \), so \((t + 1)P_3 \cup P_2 \) and \( M(C_{k, \ell}^3) \) is odd, \( (t + 1)P_3 \cup P_2 \) and \( M(C_{k, \ell}^3) \) is even. By using the definition of \( \mathcal{M} \), we have \( C_{k, \ell}^3 \subseteq ((t + 1)P_3 \cup P_2) \vee I_m \subseteq G \). So \( H_1 \) is \( P_3 \)-free when \( k \) is even.

Now suppose \( k \) is odd. If \( P_3 \subseteq H_1 \) and \( P_2 \subseteq H_2 \) then \( (mP_3 \vee mP_2) \subseteq G \) while \( C_{k, \ell}^3 \subseteq (mP_3 \vee mP_2) \), so \( H_2 = P_1 \). On the other hand, \( (P_4 \cup (\frac{k-3}{2} + \frac{1}{2})P_3 \cup P_2) \in \mathcal{H}^*(C_{k, \ell}) \). By Lemma 2.1 (iii) \( \mathcal{H}^*(C_{k, \ell}) = \mathcal{M}(C_{k, \ell}^3) \), we have \( (P_4 \cup (\frac{k-3}{2} + \frac{1}{2})P_3 \cup P_2) \in \mathcal{M}(C_{k, \ell}^3) \). If \( |W| \neq 0 \), then we have \( (P_4 \cup (\frac{k-3}{2} + \frac{1}{2})P_3 \cup P_2) \subseteq G[W \cup A_1] \) and \( C_{k, \ell} \subseteq (P_4 \cup (\frac{k-3}{2} + \frac{1}{2})P_3 \cup P_2) \vee I_m \subseteq G \). If \( P_4 \subseteq H_1 \), then we have \( (P_4 \cup (\frac{k-3}{2} + \frac{1}{2})P_3 \cup P_2) \subseteq G[A_1] \) and \( C_{k, \ell} \subseteq G \). Therefore, we have \( |W| = 0 \) and \( H_1 \) is \( P_4 \)-free. Since \( H_1 \) is \( P_4 \)-free, by Gallai Theorem (see [2]), the size of \( G \) is maximized when \( H_1 = K_3 \). Hence,

\[
e(G) \leq e(T_2(n)) + |A_1| + o(n) \leq e(T_2(n)) + \frac{n}{2} + o(n).
\]

While it contradicts (3.2) when \( k \geq 4 \) and \( \ell \geq 2 \). Thus \( H_i \) is \( P_3 \)-free.

(ii) If \( H_1 = H_2 = P_2 \), then \( |W| = 0 \) holds. Otherwise, let \( w \in W, u_iu'_i \subseteq G[A_1], v_iv'_i \subseteq G[A_2] (1 \leq i \leq t + 1) \) and then we may find a \( C_{k, \ell}^3 \) in \( G \). In fact when \( k \) is odd, we have
a lollipop with $v_{\frac{k}{2}}$ as center vertex and
\[ C_k = wu_1v_1 \cdots u_{\frac{k-1}{2}}v_{\frac{k-1}{2}}w, \]
\[ P_{\ell+1} = v_{\frac{k-1}{2}}u_{\frac{k+1}{2}} \cdots v_{\ell}u_{\ell+1}. \]
When $k$ is even, we have a lollipop with $u_{\frac{k}{2}}$ as center vertex and
\[ C_k = wu_1v_1 \cdots u_{\frac{k}{2}}w, \]
\[ P_{\ell+1} = u_{\frac{k}{2}}v_{\frac{k}{2}} \cdots u_{\ell+1}v_{\ell+1}. \]
The edges between $W$ and $A_1$ can be blown up into a triangle by using one vertex in $A_2$ and the edges between $W$ and $A_2$ can be blown up by using one vertex in $A_1$. The edge $u_iv_i$ can be expanded by using $u_i'$ and the edge $v_iu_{i+1}$ can be expanded by using $v_i'$ ($1 \leq i \leq t$). Therefore $|W| = 0$. Furthermore, we have
\[ e(G) \leq e(T_2(n)) + \frac{|A_1|}{2} + \frac{|A_2|}{2} + o(n) \leq e(T_2(n)) + \frac{n}{2} + o(n), \]
while it contradicts (3.2).
(iii) If $H_1 = P_2$, $H_2 = P_1$, then
\[ e(G) \leq e(T_2(n)) + \frac{n|W|}{2} + \frac{n}{4} + o(n). \]
Recall Corollary 3.1, $e(G) \geq e(T_2(n)) + \frac{tn}{2} + o(n)$, so we have $|W| \geq t$. On the other hand, if $|W| \geq t$, then $(P_{k+1} \cup P_{\ell+1}) \subseteq G \cup A_1$. The fact that $(P_{k+1} \cup P_{\ell+1}) \cup I_m \subseteq G$ and Lemma 2.2 (ii) imply that $C_{k,\ell}^{m+1} \subseteq G$ which is a contradiction. Therefore we have $H_i = P_1$ ($i = 1, 2$) when $\ell$ is even.

By using the similar arguments, we may prove $H_i = P_1$ ($i = 1, 2$) when $\ell$ is even.

**Theorem 3.2.** When $k \geq 4$, $\ell \geq 2$ is odd, $H(n, 2, t+1)$ ($H'(n, 2, t+1)$ resp.) is the unique extremal graph for $C_{k,\ell}^3$ when $k$ is odd (even resp.).

**Proof.** From Lemma 3.3, Lemma 3.4 and Lemma 3.6, we know that (i) $|W| = t$; (ii) each vertex in $B_i$ is adjacent to all vertices in $V(G') \setminus A_i$; (iii) each block $H_i$ is a single vertex ($i = 1, 2$). To find the extremal graph, we only need to characterize the subgraph $G[B_i]$. When $k$ is odd, by Lemma 3.5, we have $e(G[B_i]) = 0$ ($i = 1, 2$). When $k$ is even, we have the following claim.

**Claim.** $e(G[B_1]) + e(G[B_2]) = 1$.

**Proof of Claim.** Set $\{w_1, \cdots, w_i\} = W$, $\{u_i, \cdots, u_{t+2}\} \subseteq A_1$. We first show that $G[B_i]$ is $P_3$-free for $i = 1, 2$. Suppose not, without loss of generality, let $P_3 = xyz \subseteq G[B_2]$, then we have a lollipop with $w_{\frac{k-2}{2}}$ as center vertex and
\[ C_k = yu_1w_1 \cdots w_{\frac{k-2}{2}}u_{\frac{k}{2}}y, \]
$P_{t+1} = w_{k-2}u_{k+2} \cdots w_t u_{t+2}.$

Each edge between $W$ and $A_1$ can be blown up into a triangle by using one vertex in $A_2$. The edge $yu_1$ can be blown up by using vertex $x$ and the edge $u_2 y$ can be blown up by using vertex $z$. Then there is a $C_{k, \ell}^3$ in $G$. Thus $G[B_i]$ is $P_3$-free for $i = 1, 2$.

Suppose to the contrary that there exist edges $e_1, e_2$ in $G[B_1 \cup B_2]$, say $e_1 = x_1 y_1, e_2 = x_2 y_2$.

(i) Suppose that $\{e_1, e_2\} \subseteq E(G[B_1])$. Since $G[B_1]$ is $P_3$-free, then $e_1$ and $e_2$ are independent. Note that for any vertices $w_i, w_j \in W$, we have $|\{x_1, y_1\} \cap (N(w_i) \cup N(w_j))| \geq 3$ otherwise, if $|\{x_1, y_1\} \cap (N(w_i) \cup N(w_j))| \leq 2$, we can obtain a graph $G_1$ by deleting edge $e_1$ and adding missing edges between $\{w_i, w_j\}$ and $\{x_1, y_1\}$. $G_1$ is still $C_{k, \ell}^3$-free. In fact if there is a $C_{k, \ell}^3$ in $G_1$, since $G[B_1]$ is $P_3$-free, then we may choose two vertices in $A_1$ as the substitutes for $x_1, y_1$ to obtain a $C_{k, \ell}^3$ in $G$. While $e(G_1) > e(G)$, which is a contradiction. Hence $|\{x_1, y_1\} \cap (N(w_i) \cup N(w_j))| \geq 3$. Similarly, $|\{x_2, y_2\} \cap (N(w_i) \cup N(w_j))| \geq 3$. We may suppose $w_1$ is adjacent to $y_1$ and $x_2, w_2$ is adjacent to $x_1$. Further, we have a graph $Y \in Y_{k+1, \ell+1}$ with $w_{k-2}$ as branching vertex and

\[
P_{k+1} = y_2 x_2 w_1 y_1 x_1 w_2 u_1 \cdots w_{k-2} u_{k+2},
\]

\[
P_{\ell+1} = w_{k-2} u_{k+2} w_k \cdots w_\ell u_\ell,
\]

then $Y \subseteq G[W \cup A_1 \cup B_1]$ and then we have $Y \cup I_m \subseteq G$. On the other hand, by Lemma 2.2 (i) we have $C_{k, \ell}^3 \subseteq Y \cup I_m$, hence we have $C_{k, \ell}^3 \subseteq G$.

(ii) Suppose $e_1 \in E(G[B_1])$ and $e_2 \in E(G[B_2])$. Then there are at least three edges between $\{x_1, y_1\}$ and $\{x_2, y_2\}$. Otherwise the graph $G_1$ obtained by deleting edge $x_2 y_2$ and adding the missing edges between $\{x_1, y_1\}$ and $\{x_2, y_2\}$ is still $C_{k, \ell}^3$-free. In fact if there is a $C_{k, \ell}^3$ in $G_1$ then we may choose two vertices in $A_2$ as the substitutes for $x_2, y_2$ to obtain a $C_{k, \ell}^3$ in $G$, while $e(G_1) > e(G)$. So we may suppose $x_2$ is adjacent to $x_1$ and $y_1$. Furthermore, as the proof of Lemma 3.5, we may suppose that $w_{k-2}$ is adjacent to $x_1$ and we have a lollipop with $w_{k-2}$ as center vertex and

\[
C_k = u_1 w_1 u_2 w_2 \cdots w_{k-2} x_1 x_2 u_1,
\]

\[
P_{\ell+1} = w_{k-2} u_{k+2} u_2 \cdots w_\ell u_\ell+1.
\]

The edge between $W$ and $A_1 \cup B_1$ can be expanded into a triangle by using one vertex in $A_2$. The edge $x_1 x_2$ can be expanded by using vertex $y_1$ and the edge $x_2 u_1$ can be expanded by using vertex $y_2$. Then there is a $C_{k, \ell}^3$ in $G$ which is a contradiction.

Thus $e(G[B_1]) + e(G[B_2]) = 1$. Therefore, when $k$ is odd, $G = H(n, 2, t + 1)$. When $k$ is even, $G = H'(n, 2, t + 1)$. \qed
Theorem 3.3. When \( k \geq 4, \ell \geq 2 \) is even, \( H(n, 2, t + 2) \) is the unique extremal graph for \( C_{k, \ell}^3 \).

Proof. From Lemma 3.3, Lemma 3.4 and Lemma 3.6, we know that (i) \( |W| = t + 1 \); (ii) each vertex in \( B_i \) is adjacent to all vertices in \( V(G') \setminus A_i \); (iii) each block \( H_i \) in \( A_i \) is a single vertex \((i = 1, 2)\). To finish the proof, we only need to prove \( e(G[B_i]) = 0, i = 1, 2 \). When \( k \) is odd, by Lemma 3.5 we have \( e(G[B_i]) = 0 \). When \( k \) is even, we have the following claim.

Claim. \( e(G[B_i]) = 0, i = 1, 2 \).

Proof of Claim. Set \( \{w_1, \ldots, w_{t+1}\} = W, \{u_1, \ldots, u_{t+2}\} \subseteq A_1 \). Firstly, \( G[B_i] \) is \( P_3 \)-free for \( i = 1, 2 \). Suppose not, without loss of generality, let \( P_3 = xyz \subseteq G[B_2] \), then we have a lollipop with \( w_{t+2} \) center vertex and

\[
C_k = yu_1w_1 \cdots w_{k-2}u_{k-2}y, \quad P_{t+1} = w_{t+2}u_{t+2} \cdots u_{t+2}w_{t+1}.
\]

The edge between \( W \) and \( A_1 \) can be blown up into a triangle by using one vertex in \( A_2 \). The edge \( yu_1 \) can be blown up by using vertex \( x \) and the edge \( u_{k-2}y \) can be blown up by using vertex \( z \). Then there is a \( C_{k, \ell}^3 \) in \( G \). Thus \( G[B_i] \) is \( P_3 \)-free for \( i = 1, 2 \). Suppose to the contrary that there is an edge \( xy \) in \( G[B_1] \). Then we may suppose there exists a vertex \( w \in W \) with \( wx \in e(G) \) or \( wy \in e(G) \), otherwise we may have a graph \( G_1 \) obtained from \( G \) by deleting all edges of \( G[B_1] \cup G[B_2] \) and adding all missing edges between \( W \) and \( B_1 \cup B_2 \), while \( e(G_1) > e(G) \). Then we may suppose \( w_{t+1} \) is adjacent to \( x \) and have a graph \( Y \in \mathcal{Y}_{k+1, \ell+1} \) with \( w_{t+1} \) as branching vertex and

\[
P_{k+1} = u_1w_1u_2w_2 \cdots w_{k-2}u_{t+2},
\]

\[
P_{t+1} = w_{t+2}u_{t+2}w_{t+2} \cdots w_{t+1}xy.
\]

Then \( Y \subseteq G[W \cup A_1 \cup B_1] \). Furthermore, \( Y \cup I_m \subseteq G \). By Lemma 2.2 (i) we have \( C_{k, \ell}^{t+1} \subseteq G \) which is a contradiction. So we have \( e(G[B_1]) = 0 \).

By the maximality of \( G \), we have \( G = H(n, 2, t + 2) \). \( \square \)

Theorem 3.4. When \( \ell \geq 2, H(n, 2, t + 1) \) (\( H(n, 2, t + 2) \) resp.) is the unique extremal graph for \( C_{3, \ell}^3 \) when \( \ell \) is odd (even resp.).

Proof. We first characterize the subgraph of \( H_i \) and \( G[B_i] \) \((i = 1, 2)\).

Claim. Each block \( H_i \) is a single vertex \((i = 1, 2)\).

Proof of Claim.

(i) We first show that \( H_1 \) and \( H_2 \) are \( P_4 \)-free. Without loss of generality, if \( P_4 \subseteq H_1 \), then we have \( P_4 \cup \lfloor \frac{\ell}{3} \rfloor P_4 \cup P_{\ell-3\lfloor \frac{\ell}{3} \rfloor + 1} \subseteq G[A_1] \). By Lemma 2.1 (iii) and \( \chi(C_{k, \ell}) = 3 \), we have \( M(C_{k, \ell}^3) = \mathcal{H}^*(C_{k, \ell}) \). Since \( P_4 \cup \lfloor \frac{\ell}{3} \rfloor P_4 \cup P_{\ell-3\lfloor \frac{\ell}{3} \rfloor + 1} \in \mathcal{H}^*(C_{3, \ell}) \) and \( M(C_{3, \ell}^3) = \mathcal{H}^*(C_{3, \ell}) \), we have \( C_{3, \ell}^3 \subseteq G[A_1] \cup I_m \subseteq G \), which is a contradiction.
(ii) Secondly we show that $H_1$ and $H_2$ are $P_3$-free. Suppose to the contrary that $P_3 \subseteq H_1$. We will prove $H_2 = P_1$ and $|W| = 0$. Otherwise, if $P_2 \subseteq H_2$, then we may let $u_i'v_iu_i'' \subseteq G[A_1]$, $v_i'v_i' \subseteq G[A_2]$ $(1 \leq i \leq \ell)$. Then we have $C_3 = u_1v_1'v_1u_1$. When $\ell$ is odd, we have

$$P_{\ell+1} = v_1u_2v_2 \cdots u_{\ell+1},$$

and when $\ell$ is even, we have

$$P_{\ell+1} = v_1u_2v_2 \cdots u_{\ell+2}v_{\ell+2}.$$ 

The edge $v_1v_1'$ may be expanded into a triangle by using one vertex in $A_1$; the edge $u_1v_1$ may be expanded by using vertex $u_1'$; the edge $u_1v_1'$ may be expanded by using vertex $u_1''$. The edge $v_iu_{i+1}$ may be expanded by using vertex $u_i'$ and the edge $u_iv_i$ may be expanded by using vertex $u_i''$. Then there is a $C_{3, \ell}^3 \subseteq G[A_1 \cup A_2] \subseteq G$. So we have $H_2 = P_1$.

If $|W| \neq 0$, then there is a $w \in W$. When $\ell$ is odd, it is easy to see $(P_4 \cup \frac{\ell-1}{2}P_3 \cup P_2) \subseteq G[W \cup A_1]$. On the other hand, $(P_4 \cup \frac{\ell-1}{2}P_3 \cup P_2) \in \mathcal{H}^*(C_{3, \ell})$. By Lemma 2.1 (iii) $\mathcal{H}^*(C_{3, \ell}) = \mathcal{M}(C_{3, \ell}^3)$, so $(P_4 \cup \frac{\ell-1}{2}P_3 \cup P_2) \in \mathcal{M}(C_{3, \ell}^3)$. Then we have $C_{3, \ell}^3 \subseteq (P_4 \cup \frac{\ell-1}{2}P_3 \cup P_2) \cup I_1 \subseteq G$. By using the same arguments we may prove the results for even $\ell$. So we have $H_2 = P_1$ and $|W| = 0$ and from (i) we know that $G^i$ is $P_4$-free. By Gallai Theorem (see [2]), we have

$$e(G) \leq e(T_2(n)) + |A_1| + o(n) \leq e(T_2(n)) + \frac{n}{2} + o(n).$$

While it contradicts Corollary 3.1. So $H_1$ and $H_2$ are $P_3$-free.

(iii) Now suppose $H_1 = P_2$. We may set $u_iu_i' \subseteq G[A_1]$ $(1 \leq i \leq \ell)$. We distinguish the following two cases according to $\ell$.

**Case 1:** $\ell = 2, 3$. Set $\{v_1, v_2, \cdots, v_l\} \subseteq A_2$. We claim $|W| = 0$. If not, suppose $w \in W$. Let $C_3 = wu_1v_1'w$, $P_4 = wu_2v_2u_3$. Each edge of $C_3$ and the edge $wu_2$ can be blown up into a triangle by employing one vertex in $A_2$, the edge $u_2v_2$ can be blown up by using vertex $u_2'$, the edge $v_2u_3$ can be blown up by using vertex $u_3'$, then there is a $C_{3, \ell}^3$ in $G$. So $|W| = 0$, and then we have

$$e(G) \leq e(T_2(n)) + \frac{|A_1|}{2} + \frac{|A_2|}{2} + o(n) \leq e(T_2(n)) + \frac{n}{2} + o(n),$$

while it contradicts Corollary 3.1. So $H_1$ is a single vertex.

**Case 2:** $\ell = 4a+b$ $(a \geq 1)$, we may claim $|W| \leq a$. If $|W| \geq a+1$, let $\{w_1, w_2, \cdots, w_{a+1}\} \subseteq W$ and $\{v_1, v_2, \cdots, v_l\} \subseteq A_2$, then we have a $C_3 = w_1u_1v_1'w_1$, each edge of $C_3$ can be blown up into a triangle.

When $\ell = 4a$, set

$$P_{\ell+1} = w_1u_2v_2u_3w_2u_4v_4u_5 \cdots w_{a+1}.$$
When $\ell = 4a + 1$, set
\[ P_{\ell+1} = w_1u_2v_2u_3w_2u_4v_4u_5 \cdots w_{a+1}u_{2a+2}. \]

When $\ell = 4a + 2$, set
\[ P_{\ell+1} = w_1u_2v_2u_3w_2u_4v_4u_5 \cdots w_{a+1}u_{2a+2}v_{2a+2}. \]

When $\ell = 4a + 3$, set
\[ P_{\ell+1} = w_1u_2v_2u_3w_2u_4v_4u_5 \cdots w_{a+1}u_{2a+2}v_{2a+2}u_{2a+3}. \]

The edges $w_iu_{2i}$ and $u_{2i-1}w_i$ can be blown up into a triangle by using one vertex in $A_2$, the edge $u_iv_i$ can be blown up by using vertex $u'_i$, the edge $v_iu_{i+1}$ can be blown up by using vertex $u'_{i+1}$. Then there is a $C^3_{3, \ell}$ in $G$ and it contradicts the definition of $G$. Therefore, we have $|W| \leq a$. Next we will prove that if $|W| \neq 0$, then $H_2 = P_1$. Otherwise, let $v_iw_i \subseteq G[A_2]$ $(1 \leq i \leq \ell)$ and $w \in W$, then we have a $C_3 = wu_1v'_1w$.

When $\ell$ is odd, we have
\[ P_{\ell+1} = u_1v_1 \cdots u_{\frac{\ell+1}{2}}v_{\ell+1}, \]

when $\ell$ is even, we have
\[ P_{\ell+1} = u_1v_1 \cdots u_{\frac{\ell}{2}}v_{\ell}u_{\ell+2}. \]

Each edge of $C_3$ can be blown up by employing one vertex in $A_2$, the edge $u_iv_i$ can be blown up by using vertex $v'_i$, and the edge $v_ju_{j+1}$ can be blown up by using vertex $u'_{j+1}$. Then there is a $C^3_{3, \ell}$ in $G$ which is a contradiction.

When $|W| = 0$, from (ii) we know that $H_i$ is $P_3$-free, so we have
\[ e(G) \leq e(T_2(n)) + \frac{|A_1|}{2} + \frac{|A_2|}{2} + o(n) \leq e(T_2(n)) + \frac{n(\lfloor \frac{\ell}{4} \rfloor + 1)}{2} + o(n). \]

When $0 < |W| \leq a$, we have
\[ e(G) \leq e(T_2(n)) + \frac{n|W|}{2} + \frac{|A_1|}{2} + o(n) \leq e(T_2(n)) + \frac{n(\lfloor \frac{\ell}{4} \rfloor + 1)}{2} + o(n). \]

When $\ell \geq 4$ is odd, $\lfloor \frac{\ell}{4} \rfloor + 1 < \frac{\ell+1}{2}$, when $\ell \geq 4$ is even, $\lfloor \frac{\ell}{4} \rfloor + 1 < \frac{\ell}{2} + 1$ and these contradict Corollary 3.1. Therefore $H_1 = H_2 = P_1$.

Now we have proved that $H_1 = H_2 = P_1$. Combining Lemma 3.3 - 3.5, we have $G = H(n, 2, t + 1)$ when $\ell$ is odd; $G = H(n, 2, t + 2)$ when $\ell$ is even. ∎
References

[1] G. Chen, R. J. Gould, F. Pfender, B. Wei, Extremal graphs for intersecting cliques, J. Comb. Theory, Ser. B 89 (2003) 159-171.

[2] P. Erdős, T. Gallai, On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hung. 10, (1959) 337-356.

[3] P. Erdős, M. Simonovits, A limit theorem in graph theory, Studia Sci. Math. Hung. 1 (1966) 51-57.

[4] P. Erdős, Über ein Extremalproblem in der Graphentheorie (German), Arch. Math. (Basel) 13 (1962) 122-127.

[5] P. Erdős, Z. Füredi, R. J. Gould, D. S. Gunderson, Extremal graphs for intersecting triangles, J. Comb. Theory, Ser. B 63 (1995) 89-100.

[6] R. Glebov, Extremal graphs for clique-paths, arXiv: 1111.7029v1, 2011.

[7] X. Hou, Y. Qiu, B. Liu, Extremal graph for intersecting odd cycles, Electron. J. Comb. 23 (2) (2016) P2. 29.

[8] H. Liu, Extremal graphs for blow-ups of cycles and trees, Electron. J. Comb. 20 (1) (2013) P65.

[9] J. W. Moon, On independent complete subgraphs in a graph, Can. J. Math. 20 (1968) 95-102.

[10] Z. Ni, L. Kang, E. Shan, Extremal graphs for blow-ups of keyrings, Graphs Comb. 36 (2020) 1827-1853.

[11] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in: Theory of Graphs, Proc. Colloq., Tihany, 1996, Academic Press, New York, 1968, pp. 279-319.

[12] M. Simonovits, Extremal graph problems with symmetrical extremal graphs, additional chromatic conditions, Discrete Math. 7 (1974) 349-376.

[13] P. Turán, On the theory of graphs, Colloquium Mathematicum. 3 (1) (1954) 19-30.

[14] A. Wang, X. Hou, B. Liu, The Turán number for the edge blow-up of trees, Discrete Math. 344 (2021) 112627.

[15] L. Yuan, Extremal graphs for edge blow-up of graphs, J. Comb. Theory, Ser. B 152 (2022) 379-398.

[16] H. Zhu, L. Kang, E. Shan, Extremal graphs for odd-ballooning of paths and cycles, Graphs Comb. 36 (2020) 755-766.