ON RECIPROcity FORMULA OF APOSTOL-DEDEKIND SUM WITH QUASI-PERIODIC EULER FUNCTIONS

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Abstract. The Apostol-Dedekind sum with quasi-periodic Euler functions is an analogue of Apostol’s definition of the generalized Dedekind sum with periodic Bernoulli functions. In this paper, using the Boole summation formula, we shall obtain the reciprocity formula for this sum.

1. Introduction

The Euler polynomials $E_k(x), k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, are defined by the generating function

$$\frac{2e^{xt}}{e^t + 1} = \sum_{k=0}^{\infty} E_k(x) \frac{t^k}{k!}. \quad (1.1)$$

The integers $E_k = 2^k E_k(1/2), k \in \mathbb{N}_0$, are called Euler numbers. For example, $E_0 = 1, E_2 = -1, E_4 = 5,$ and $E_6 = -61$. The Euler numbers and polynomials (so called by Scherk in 1825) appear in Euler’s famous book, Institutiones Calculi Differentials (1755, pp.487–491 and p.522). Notice that the Euler numbers with odd subscripts vanish, that is, $E_{2m+1} = 0$ for all $m \in \mathbb{N}_0$. The Euler polynomials can also be expressed in terms of the Euler numbers in the following way (see [18, p. 25]):

$$E_k(x) = \sum_{i=0}^{k} \binom{k}{i} \frac{E_i}{2^i} \left(x - \frac{1}{2}\right)^{k-i} \quad (1.2)$$

which holds for all nonnegative integers $m$ and all real $x$, and which was obtained by Raabe [19] in 1851.

Some properties of Euler polynomials can be easily derived from their generating functions, for example, from (1.1), we have

$$x^k = \frac{1}{2} (E_k(x + 1) + E_k(x)) \quad \text{and} \quad E_k(1 - x) = (-1)^k E_k(x) \quad (1.3)$$

(also see [23, p. 530, (23) and (24)]).

For further properties of the Euler polynomials and numbers including their applications, we refer to [1, 3, 12, 18, 19, 23]. It may be interesting to point out that there is also a connection between the generalized Euler numbers and the ideal class group of the $p^{n+1}$th cyclotomic field when $p$ is a prime number. For details, we refer to a recent paper [14], especially [14, Proposition 3.4].

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The $k$-th quasi-periodic Euler function $\overline{E}_k(x)$ is defined by (see [9, p. 661])
\begin{equation}
\overline{E}_k(x + 1) = -\overline{E}_k(x)
\end{equation}
for all $x$, and
\begin{equation}
\overline{E}_k(x) = E_k(x) \quad \text{for} \quad 0 \leq x < 1,
\end{equation}
where $E_k(x), k \in \mathbb{N}_0$, denotes the $k$th Euler polynomials. It can be shown that $\overline{E}_k(x)$ has continuous derivatives up to the $(k-1)$st order. For $x \in \mathbb{R}$, $[x]$ denotes the greatest integer not exceeding $x$ and \{x\} denotes the fractional part of real number $x$, thus
\begin{equation}
\{x\} = x - [x].
\end{equation}
Then, for $r \in \mathbb{Z}$ and $k \in \mathbb{N}_0$, we have (see [3, (1.2.9)] and [9, (3.3)])
\begin{equation}
\overline{E}_k(x) = (-1)^{[x]}E_k([x]), \quad \overline{E}_k(x + r) = (-1)^r\overline{E}_k(x).
\end{equation}
For further properties of the quasi-periodic Euler functions, we refer to [3, 9, 15].
Recall that, for $p \in \mathbb{N}_0$, the generalized Dedekind sum is defined by
\begin{equation}
S_p(a, b) = \sum_{j=0}^{b-1} B_p \left( \frac{aj}{b} \right) B_1 \left( \frac{j}{b} \right),
\end{equation}
where $B_p(x)$ is the $p$-th Bernoulli function defined by
\begin{equation}
B_k(x) = B_k(\{x\}) \quad \text{for} \quad k > 1 \quad \text{and} \quad B_1(x) = ((x)).
\end{equation}
Here $B_n(x)$ denotes the $n$-th Bernoulli polynomial and $((x))$ denotes
\begin{equation}
((x)) = \begin{cases} 
  x - [x] - \frac{1}{2} & \text{if} \ x \notin \mathbb{Z}, \\
  0 & \text{otherwise}
\end{cases}
\end{equation}
(see [5, 8, 13, 20, 22, 24]). More than 60 years ago, Apostol [2] proved a reciprocity formula for this sum.

The Apostol-Dedekind sum $T_p(a, b)$ with quasi-periodic Euler functions is defined by
\begin{equation}
T_p(a, b) = 2 \sum_{j=0}^{b-1} (-1)^j E_p \left( \frac{aj}{b} \right) E_1 \left( \frac{j}{b} \right),
\end{equation}
which is an analogue of the generalized Dedekind sums (1.8) for quasi-periodic Euler functions.
Notice that, as indicated by Carlitz in [9, p.661, 2nd paragraph], the Bernoulli function are periodic, but the Euler functions are just quasi-periodic (also comparing with Eqs. (1.7) and (1.9) above), thus the signs $(-1)^j$ in the definition of the Apostol-Dedekind sum $T_p(a, b)$ with quasi-periodic Euler functions (1.10) are necessary.
In this paper, we shall prove the following reciprocity formula for the sum $T_p(a, b)$.

**Theorem 1.1** (Reciprocity formula). Let $a$ and $b$ be positive odd integers with $(a, b) = 1$. 
(1) For a even positive integer $p$, we have

$$ab^{p+1}T_p(a, b) + a^{p+1}bT_p(b, a) = 2E_{p+1}(0) - ab\sum_{k=0}^{p-1}\binom{p}{k}b^kE_k(0)a^{p-k}E_{p-k}(0)$$

$$= 2E_{p+1}(0) - ab(bE(0) + aE(0))^p \text{ (symbolically)}.$$

(2) For an odd positive integer $p$, we have

$$b^pT_p(a, b) - a^pT_p(b, a) = (a^p - b^p)E_p(0).$$

**Remark 1.2.** By applying the Euler-MacLaurin summation formula and its generalization for Bernoulli functions [4], Dağlı and Can [11, 10] gave new proofs for the reciprocity formulas of the generalized Dedekind sums and the generalized Hardy-Berndt sums. In this note, we shall modify their methods to prove our results by using the Boole summation formula.

The Boole summation formula is the Euler polynomial version of the well-known Euler-MacLaurin summation formula (see Lemma 2.5 below), as pointed out by N.E. Nörlund [18], this formula is also due to Euler.

**Remark 1.3.** The proof of reciprocity formula for another type of Dedekind sum was given in [16, Theorem 9] and [21, Theorem 13]. For a general treatment of reciprocity formula for the sum $T_p(a, b)$, we refer to [15, Theorem 1.1].

In the last section, server examples will be shown.

2. **Proof of Theorem 1.1**

We need the following lemmas.

**Lemma 2.1** (Fourier expansion, [11 p. 805, 23.1.16] and [21, Lamma 5]).

We have

$$E_p(x) = \frac{4p!}{\pi^{p+1}}\sum_{k=0}^{\infty}\frac{\sin((2k + 1)\pi x - \frac{1}{2}\pi p)}{(2k + 1)^{p+1}},$$

where $0 \leq x < 1$ if $p \in \mathbb{N}$ and $0 < x < 1$ if $p = 0$.

**Lemma 2.2** ([3 p. 355, (1.2.13)]). If $b$ is an odd positive integer, we have

$$\sum_{j=0}^{b-1}(-1)^jE_p\left(\frac{x + j}{b}\right) = b^{-p}E_p(x)$$

for $p \in \mathbb{N}_0$ and arbitrary real numbers $x$.

**Proposition 2.3.** For even $a$ and odd $b, q$, we have

$$T_p(qa, qb) = q^{-1}T_p(a, b).$$
Proof. For even \( a \) and odd \( b, q \), by (1.7) and Lemma 2.2, we have

\[
T_p(qa, qb) = 2 \sum_{j=0}^{q-1} (-1)^j E_p \left( \frac{qa}{qb} \right) E_1 \left( \frac{j}{qb} \right) = 2 \sum_{r=0}^{b-1} \sum_{l=0}^{q-1} (-1)^r (-1)^l \frac{ar}{b} + al \left( \frac{r/b + l}{q} \right) \]

\[
= 2 \sum_{r=0}^{b-1} (-1)^r \frac{ar}{b} \sum_{l=0}^{q-1} (-1)^l \left( \frac{r/b + l}{q} \right) = q^{-1} T_p(a, b).
\]

This completes our proof. \( \square \)

**Lemma 2.4.** For odd positive integers \( a \) and \( b \) with \( (a, b) = 1 \), we have

\[
\sum_{j=0}^{b-1} (-1)^j E_p \left( \frac{x + aj}{b} \right) = b^{-p} E_p(x)
\]

for \( p \in \mathbb{N}_0 \) and arbitrary real numbers \( x \).

**Proof.** For \( n \in \mathbb{N}_0 \), \( \{n\}_N \) denotes by be the integer between 0 and \( N - 1 \) which is congruent to \( n \) modulo \( N \). Let \( a \) and \( b \) be odd positive integers with \( (a, b) = 1 \). Note that

\[
a j \equiv \{aj\}_b \pmod b
\]

and

\[
\frac{\{aj\}_b}{b} = \frac{aj}{b} - \left[ \frac{aj}{b} \right] \quad ([] = \text{greatest integer function}).
\]

Hence, by (1.7) and Lemma 2.2 we have

\[
\sum_{j=0}^{b-1} (-1)^j E_p \left( \frac{x + aj}{b} \right) = \sum_{j=0}^{b-1} (-1)^j E_p \left( \frac{x + \{aj\}_b}{b} + \left[ \frac{aj}{b} \right] \right)
\]

\[
= \sum_{j=0}^{b-1} (-1)^j \left[ \frac{aj}{b} \right] E_p \left( \frac{x + \{aj\}_b}{b} \right)
\]

\[
= \sum_{j'=0}^{b-1} (-1)^j \left[ \frac{aj}{b} \right] \frac{j'}{b} = b^{-p} E_p(x),
\]

where the following fact has been used “there exist unique \( j' \in \{0, 1, \ldots, b-1\} \) such that \( j' = \{aj\}_b \) for each \( j = 0, 1, \ldots, b-1 \), and \( j + \left[ \frac{aj}{b} \right], j' \) have the same parity.” \( \square \)

The following is the Boole summation formula (see, for example, [17, 24.17.1–2]).
Lemma 2.5 (Boole summation formula). Let $\alpha, \beta$ and $m$ be integers such that $\alpha < \beta$ and $0 < m$. If $f^{(m)}(x)$ is absolutely integrable over $[\alpha, \beta]$. Then

$$2 \sum_{j=\alpha}^{\beta-1} (-1)^j f(j) = \sum_{k=0}^{m-1} \frac{E_k(0)}{k!} \left( (-1)^{\beta-1} f^{(k)}(\beta) + (-1)^{\alpha} f^{(k)}(\alpha) \right)$$

$$+ \frac{1}{(m-1)!} \int_{\alpha}^{\beta} f^{(m)}(x) E_{m-1}(-x) dx.$$

The above formula is proved by Boole [7], but a similar one may be known by Euler as well (see [18]).

Lemma 2.6. For odd positive integers $a$ and $b$ with $(a, b) = 1$ and $p, n \in \mathbb{N}_0$, we have

$$\int_{0}^{1} E_p(ax) E_n(bx) dx = \frac{2(-1)^{n+1}}{(n+1)(p+n+1)} \frac{1}{a^{n+1} b^{p+1}} E_{p+n+1}(0).$$

Proof. First, we consider the function $f(x) = E_p(xy)$, where $y \in \mathbb{R}$. The property

$$\frac{d}{dx} (E_p(x)) = pE_{p-1}(x), \quad p \in \mathbb{N}$$

implies

$$\frac{d^j}{dx^j} (f(x)) = \frac{d^j}{dx^j} (E_p(xy)) = y^j \frac{p!}{(p-j)!} E_{p-j}(xy)$$

for $1 \leq j \leq p$. Set $\alpha = 0$ and $\beta = b$ in Lemma 2.5. From (2.3), Lemma 2.5 can be written as

$$2 \sum_{j=0}^{b-1} (-1)^j E_p(jy) = \sum_{k=0}^{m-1} E_k(0) y^k \binom{p}{k} \left( (-1)^{b-1} E_{p-k}(by) + E_{p-k}(0) \right)$$

$$+ m \left( \frac{p}{m} \right) y^m \int_{0}^{b} E_{p-m}(xy) E_{m-1}(-x) dx.$$

Letting $a, b$ be odd positive integers with $(a, b) = 1$, and $y = a/b$ in (2.4). Since $E_{p-k}(a) = (-1)^a E_{p-k}(0) = -E_{p-k}(0)$, we obtain

$$2 \sum_{j=0}^{b-1} (-1)^j E_p \left( \frac{aj}{b} \right) = m \left( \frac{a}{b} \right)^m \left( \frac{p}{m} \right) \int_{0}^{b} E_{p-m} \left( \frac{ax}{b} \right) E_{m-1}(-x) dx.$$

Let $k = 0$ in (1.5), we obtain

$$\lim_{x \to 0^+} E_0(1-x) = 1 = \lim_{x \to 0^+} (-1)^0 E_0(x).$$

By Lemma 2.1, we have $E_k(1-x) = (-1)^k E_k(x)$, for $k \in \mathbb{N}$. Thus, $E_k(1-x) = (-1)^k E_k(x)$, for $k \in \mathbb{N}_0$. So, for $m \in \mathbb{N}$, we have

$$\int_{0}^{b} E_{p-m} \left( \frac{ax}{b} \right) E_{m-1}(-x) dx = - \int_{0}^{b} E_{p-m} \left( \frac{ax}{b} \right) E_{m-1}(1-x) dx$$

(2.6) $$= (-1)^m \int_{0}^{b} E_{p-m} \left( \frac{ax}{b} \right) E_{m-1}(x) dx$$
by (1.4). Letting $x = bt$ in (2.6), we obtain

\[(2.7) \quad \int_0^b E_{p-m} \left( \frac{ax}{b} \right) E_{m-1}(-x) dx = (-1)^m b \int_0^1 E_{p-m}(at) E_{m-1}(bt) dt.\]

Substituting the above into (2.5), we have

\[(2.8) \quad \int_0^1 E_{p-m}(ax) E_{m-1}(bx) dx = \frac{2(-1)^m b^{m-1}}{m(p_m)} \sum_{j=0}^{b-1} (-1)^j E_p \left( \frac{aj}{b} \right).\]

Therefore, using Lemma 2.4 with $x = 0$, (2.8) can be written as

\[(2.9) \quad \int_0^1 E_{p-m}(ax) E_{m-1}(bx) dx = \frac{2(-1)^m b^{m-1}}{m(p_m)} E_p(0).\]

Setting $p = p + m$ in (2.9), we obtain the desired result. \qed

**Proof of Theorem 1.1.** Set $f(x) = x E_p(xy)$, where $y \in \mathbb{R}$.

From (2.2) and Leibniz’s rule for the derivative, we have

\[(2.10) \quad \frac{d^j}{dx^j} f(x) = xy^j \frac{p!}{(p-j)!} E_{p-j}(xy) + y^{j-1} \frac{p!}{(p+1-j)!} E_{p+1-j}(xy),\]

for $1 \leq j \leq p$.

Set $\alpha = 0$ and $\beta = b$ in Lemma 2.3. Let $a, b$ be positive odd integers with $(a, b) = 1$. From (2.10) with $y = a/b$, Lemma 2.5 can be written as

\[(2.11) \quad 2 \sum_{j=0}^{b-1} (-1)^j j E_p \left( \frac{aj}{b} \right) = \sum_{k=0}^{m-1} E_k(0) \left[ \left( \frac{a}{b} \right)^k \binom{p}{k} b E_{p-k}(a) \right.

\[\left. + \left( \frac{a}{b} \right)^{k-1} \binom{p+1}{k} \frac{k}{p+1} E_{p+1-k}(a) \right]

\[+ \left( \frac{a}{b} \right)^{k-1} \binom{p+1}{k} \frac{k}{p+1} E_{p+1-k}(0) \right]

\[+ \int_0^b \left[ \left( \frac{a}{b} \right)^m \binom{p}{m} mx E_{p-m} \left( \frac{ax}{b} \right) \right.

\[\left. + \left( \frac{a}{b} \right)^{m-1} \binom{p}{m-1} m E_{p+1-m} \left( \frac{ax}{b} \right) \right] E_{m-1}(-x) dx.\]

Notice that $E_{p+1-k}(a) = (-1)^a E_{p+1-k}(0) = -E_{p+1-k}(0)$, (2.11) becomes

\[(2.12) \quad 2 \sum_{j=0}^{b-1} (-1)^j j E_p \left( \frac{aj}{b} \right) = -b \sum_{k=0}^{m-1} \binom{p}{k} \left( \frac{a}{b} \right)^k E_k(0) E_{p-k}(0) \]

\[+ \left( \frac{a}{b} \right)^m \binom{p}{m} m \int_0^b x E_{p-m} \left( \frac{ax}{b} \right) E_{m-1}(-x) dx \]

\[+ \left( \frac{a}{b} \right)^{m-1} \binom{p}{m-1} m \int_0^b E_{p+1-m} \left( \frac{ax}{b} \right) E_{m-1}(-x) dx.\]
From Lemma 2.6, the second integral term in (2.12) can be written as
\[(2.13)\]
\[\int_0^b \frac{E_{p+1-m}}{b} \frac{E_{m-1}(-x)}{dx} = (-1)^m b \int_0^1 \frac{E_{p+1-m}(ax)E_{m-1}(bx)}{dx} \]
\[= \frac{2}{m+1} \frac{b^{m-p-1}}{a^m} E_{p+1}(0).\]

We also note that
\[(2.14)\]
\[\int_0^b x \frac{E_{p-m}}{b} \frac{E_{m-1}(-x)}{dx} = (-1)^m b^2 \int_0^1 x \frac{E_{p-m}(ax)E_{m-1}(bx)}{dx}.\]

Therefore, by (2.13) and (2.14), (2.12) can be written as
\[(2.15)\]
\[2 \sum_{j=0}^{b-1} (-1)^j \frac{E_p}{j} \left(\frac{a_j}{b}\right) = -b \sum_{k=0}^{m-1} \left(\frac{p}{k}\right) \left(\frac{a}{b}\right)^k E_k(0) E_{p-k}(0)\]
\[+ \frac{2m}{(p+1)ab^p} E_{p+1}(0)\]
\[+ \frac{(-1)^m m a^m}{b^{m-2}} \left(\frac{b}{a}\right)^k \frac{1}{E_{p-m}(ax)E_{m-1}(bx)} dx.\]

Let \(m = 2\) in (2.15), since \(E_1(0) = -1/2\), we obtain
\[(2.16)\]
\[2 \sum_{j=0}^{b-1} (-1)^j \frac{E_p}{j} \left(\frac{a_j}{b}\right) = -b \left(E_p(0) - \frac{ap}{2b} E_{p-1}(0)\right) + \frac{4}{(p+1)ab^p} E_{p+1}(0)\]
\[+ 2a^2 \left(\frac{p}{2}\right) \int_0^1 x \frac{E_{p-2}(ax)E_1(bx)}{dx}.\]

On the other hand, setting \(m = p - 1\), and interchanging \(a\) and \(b\) in (2.15), we obtain
\[(2.17)\]
\[2 \sum_{j=0}^{a-1} (-1)^j \frac{E_p}{j} \left(\frac{b_j}{a}\right) = -a \sum_{k=0}^{p-2} \left(\frac{p}{k}\right) \left(\frac{b}{a}\right)^k E_k(0) E_{p-k}(0)\]
\[+ \frac{2(p-1)}{(p+1)ap^b} E_{p+1}(0)\]
\[+ \frac{(-1)^{p-1}(p-1)b^{p-1}}{a^{p-3}} \left(\frac{p}{p-1}\right) \int_0^1 x E_1(bx)E_{p-2}(ax)dx.\]

Particularly, since \(E_1(x) = x - 1/2\) for \(0 < x < 1\), we have
\[(2.18)\]
\[T_p(a, b) = \frac{2}{b} \sum_{j=0}^{b-1} (-1)^j \frac{E_p}{j} \left(\frac{a_j}{b}\right) - \sum_{j=0}^{b-1} (-1)^j \frac{E_p}{j} \left(\frac{a_j}{b}\right).\]
Thus, with the help of Lemma 2.4, we have

\begin{align}
T_p(a, b) &= 2 \sum_{j=0}^{b-1} (-1)^j \cdot j \mathcal{E}_p \left( \frac{aj}{b} \right) - b^{-p} \mathcal{E}_p(0). 
\end{align}

For Part (1), if \( p \) is a positive even integer, then from (1.10), (2.16), (2.17) and (2.19), we obtain the reciprocity formula

\begin{align}
ab^{p+1}T_p(a, b) + a^{p+1}bT_p(b, a) &= \frac{a^{2bp}}{2}E_{p-1}(0) + 2E_{p+1}(0) \\
&- a^{p+1}b \sum_{k=0}^{p-2} \binom{p}{k} \left( \frac{b}{a} \right)^k E_k(0)E_{p-k}(0) \\
&= 2E_{p+1}(0) - ab \sum_{k=0}^{p-1} \binom{p}{k} b^k E_k(0) a^{p-k} E_{p-k}(0),
\end{align}

which may be written symbolically in the form

\begin{align}
ab^{p+1}T_p(a, b) + a^{p+1}bT_p(b, a) &= 2E_{p+1}(0) - ab(bE(0) + aE(0))^p,
\end{align}

since \( E_1(0) = -1/2 \), and \( E_p(0) = 0 \) if \( p \) is even. Thus, we have Part (1).

When \( p = 1 \), consider the sum

\begin{align}
T_1(a, b) &= 2 \sum_{j=0}^{b-1} (-1)^j \mathcal{E}_1 \left( \frac{aj}{b} \right) \mathcal{E}_1 \left( \frac{j}{b} \right).
\end{align}

Set \( m = p = 1 \) in (2.15). Since \( E_2(0) = 0 \), we have

\begin{align}
2 \sum_{j=0}^{b-1} (-1)^j \mathcal{E}_1 \left( \frac{aj}{b} \right) &= -bE_1(0) - ab \int_0^1 x \mathcal{E}_0(ax) \mathcal{E}_0(bx) dx,
\end{align}

thus from (2.19) with \( p = 1 \), we obtain

\begin{align}
T_1(a, b) &= -E_1(0) - \frac{1}{b} E_1(0) - a \int_0^1 x \mathcal{E}_0(ax) \mathcal{E}_0(bx) dx.
\end{align}

Also setting \( m = p = 1 \), and interchanging \( a \) and \( b \) in (2.15), we have:

\begin{align}
2 \sum_{j=0}^{a-1} (-1)^j \mathcal{E}_1 \left( \frac{bj}{a} \right) &= -aE_1(0) - ba \int_0^1 x \mathcal{E}_0(bx) \mathcal{E}_0(ax) dx,
\end{align}

thus from (2.19) with \( p = 1 \), we obtain

\begin{align}
T_1(b, a) &= -E_1(0) - \frac{1}{a} E_1(0) - b \int_0^1 x \mathcal{E}_0(ax) \mathcal{E}_0(bx) dx.
\end{align}

Therefore, by (2.22) and (2.24), we have

\begin{align}
bT_1(a, b) - aT_1(b, a) &= (a - b)E_1(0).
\end{align}

Note that, since \( E_k(0) = 0 \) for even \( k \), we obtain

\begin{align}
\sum_{k=0}^{p-2} \binom{p}{k} \left( \frac{b}{a} \right)^k E_k(0) E_{p-k}(0) &= E_p(0) \quad \text{if odd } p > 2.
\end{align}
If \( p \) is a positive odd integer with \( p > 2 \), then form (1.10), (2.16), (2.17), (2.19) and (2.26), we obtain the reciprocity formula

\[
b^p T_p(a, b) - a^p T_p(b, a) = (a^p - b^p) E_p(0).
\]

This completes the proof of Part (2) for odd integer \( p \geq 1 \). \(\Box\)

**Remark 2.7.** Let \( a \) and \( b \) be positive odd integers with \( (a, b) = 1 \). From (1.7), (1.6) and Lemma 2.4, we have

\[
\sum_{j=0}^{b-1}(-1)^{j+[\frac{a}{b}]} = \sum_{j=0}^{b-1}(-1)^j(-1)^{[\frac{a}{b}]}E_0 \left( \left\{ \frac{aj}{b} \right\} \right)
= \sum_{j=0}^{b-1}(-1)^jE_0 \left( \left\{ \frac{aj}{b} \right\} + \left[ \frac{aj}{b} \right] \right)
= \sum_{j=0}^{b-1}(-1)^jE_0 \left( \frac{aj}{b} \right)
= E_0(0) = 1.
\]

Thus, let \( \varrho(a, b) = \sum_{j=0}^{b-1}(-1)^{j+[\frac{a}{b}]} \), we have

\[
\varrho(a, b) + \varrho(b, a) = 2
\]

(also see [6, Theorem 4.2]).

**Remark 2.8.** Let \( I = \int_0^1 (-1)^{[ax]+[bx]} dx \), where \( a \) and \( b \) are positive odd integers with \( (a, b) = 1 \). From Lemma 2.6, we have

\[
I = \int_0^1 (-1)^{[ax]+[bx]} dx = \int_0^1 E_0(ax)E_0(bx)dx = \frac{1}{ab},
\]

for \( (a, b) = 1 \). This integral can be calculated by Stieltjes integrals. For \( (a, b) = 1 \), we see that

\[
I = \sum_{j=0}^{a-1} \int_{j/a}^{(j+1)/a} E_0(ax)E_0(bx)dx.
\]

Thus by substituting \( x = y/a + j/a \), and \( dx = dy/a \), we get

\[
I = \frac{1}{a} \sum_{j=0}^{a-1} \int_{0}^{1} E_0(y+j)E_0 \left( \frac{by}{a} + \frac{bj}{a} \right) dy
= \frac{1}{a} \sum_{j=0}^{a-1} (-1)^j \int_{0}^{1} E_0(y)E_0 \left( \frac{by}{a} + \frac{bj}{a} \right) dy
= \frac{1}{a} \int_{0}^{1} \sum_{j=0}^{a-1} (-1)^j E_0 \left( \frac{by}{a} + \frac{bj}{a} \right) dy
= \frac{1}{a} \int_{0}^{1} E_0(by)dy.
\]
Here we use Lemma 2.4. Repeating the procedure on \(b\), we obtain
\[
\int_0^1 E_0(by)dy = \frac{1}{b}
\]
and \(I = 1/ab\) as claimed.

**Remark 2.9.** From (2.20), we have \(T_1(a, 1) = 1/2\), so when \(b = 1\), (2.22) may be written in the form
\[
\frac{1}{2} = 1 - a \int_0^1 x E_0(ax)dx,
\]
since \(E_1(0) = -1/2\). An immediate consequence of this formula is (2.27)
\[
\int_0^1 (-1)^{[ax]}xdx = \int_0^1 x E_0(ax)dx = \frac{1}{2a}.
\]
On the other hand, from (2.22), we see that
\[
T_1(1, a) = \frac{1}{2} + \frac{1}{2a} - \int_0^1 x E_0(x)E_0(ax)dx
\]
\[
= \frac{1}{2} + \frac{1}{2a} - \int_0^1 x E_0(ax)dx
\]
\[
= \frac{1}{2},
\]
where (2.27) has been used. Therefore, we obtain \(T_1(a, 1) - aT_1(1, a) = (a - 1)E_1(0)\) (also see Theorem 1.1(2) above).

3. SOME EXAMPLES

We have the following concrete examples of the reciprocity formula (Theorem 1.1).

**Example 3.1.** Set \(a = 5\) and \(b = 3\) in (1.10), by (1.7), we have
\[
T_p(5, 3) = 2 \left( E_p(0) E_1(0) - E_p \left( \frac{5}{3} \right) E_1 \left( \frac{1}{3} \right) + E_p \left( \frac{10}{3} \right) E_1 \left( \frac{2}{3} \right) \right)
\]
\[
= 2 \left( E_p(0) E_1(0) + E_p \left( \frac{2}{3} \right) E_1 \left( \frac{1}{3} \right) - E_p \left( \frac{1}{3} \right) E_1 \left( \frac{2}{3} \right) \right)
\]
and
\[
T_p(3, 5) = 2 \left( E_p(0) E_1(0) - E_p \left( \frac{3}{5} \right) E_1 \left( \frac{1}{5} \right) + E_p \left( \frac{6}{5} \right) E_1 \left( \frac{2}{5} \right) \right)
\]
\[
- E_p \left( \frac{9}{5} \right) E_1 \left( \frac{3}{5} \right) + E_p \left( \frac{12}{5} \right) E_1 \left( \frac{4}{5} \right)
\]
\[
= 2 \left( E_p(0) E_1(0) - E_p \left( \frac{3}{5} \right) E_1 \left( \frac{1}{5} \right) - E_p \left( \frac{1}{5} \right) E_1 \left( \frac{2}{5} \right) \right)
\]
\[
+ E_p \left( \frac{4}{5} \right) E_1 \left( \frac{3}{5} \right) + E_p \left( \frac{2}{5} \right) E_1 \left( \frac{4}{5} \right)
\]
Case (I): $p = 1$. Letting $p = 1$ in (3.1) and (3.2), we have

$$T_1(5, 3) = \frac{1}{2}, \quad T_1(3, 5) = \frac{1}{2},$$

so the left-hand side of Theorem 1.1(2) reduces to

$$3T_1(5, 3) - 5T_1(3, 5) = -1$$

and if $p = 1$, then the right-hand side of Theorem 1.1(2) equals to

$$5 - 3E_1(0) = -1.$$

Therefore, (3.3) and (3.4) yield the result of Theorem 1.1(2) when $p = 1$, $a = 5$ and $b = 3$.

Case (II): $p = 2$. Letting $p = 2$ in (3.1) and (3.2), we have

$$T_2(5, 3) = \frac{4}{27}, \quad T_2(3, 5) = -\frac{44}{125},$$

so the left-hand side of Theorem 1.1(1) reduces to

$$5 \cdot 3^3T_2(5, 3) + 5^3 \cdot 3T_2(3, 5) = -112$$

and if $p = 2$, then the right-hand side of Theorem 1.1(1) equals to

$$2E_3(0) - 5 \cdot 3 \sum_{k=0}^{1} \binom{2}{k} 5^k E_k(0) 3^{2-k} E_{2-k}(0)$$

$$= 2E_3(0) - 3^2 \cdot 5^2 \binom{2}{1} E_1(0) E_1(0)$$

$$= -112,$$

since $E_2(0) = 0$. Therefore, (3.5) and (3.6) yields the result of Theorem 1.1(1) when $p = 2$, $a = 5$ and $b = 3$.

Case (III): $p = 3$. Letting $p = 3$ in (3.1) and (3.2), we have

$$T_3(5, 3) = -\frac{1}{4}, \quad T_3(3, 5) = -\frac{1}{4},$$

so the left-hand side of Theorem 1.1(2) reduces to

$$3^3T_3(5, 3) - 5^3T_3(3, 5) = \frac{49}{2}$$

and if $p = 3$, the right-hand side of Theorem 1.1(2) equals to

$$(5^3 - 3^3) E_3(0) = \frac{49}{2}.$$

Therefore, (3.7) and (3.8) yield the result of Theorem 1.1(2) when $p = 3$, $a = 5$ and $b = 3$.

Case (IV): $p = 4$. Letting $p = 4$ in (3.1) and (3.2), we have

$$T_4(5, 3) = -\frac{44}{243}, \quad T_4(3, 5) = \frac{1348}{3125},$$

so the left-hand side of Theorem 1.1(1) reduces to

$$5 \cdot 3^5T_4(5, 3) + 5^5 \cdot 3T_4(3, 5) = 3824.$$
and if \( p = 4 \), then the right-hand side of Theorem 1.1(1) equals to
\[
2E_5(0) - 5 \cdot 3 \sum_{k=0}^{3} \binom{4}{k} 5^k E_k(0) 3^{4-k} E_{4-k}(0)
\]
(3.10)
\[
= 2E_5(0) - 5 \cdot 3(4 \cdot 5 \cdot 3^3 + 4 \cdot 5^3 \cdot 3)E_1(0)E_3(0)
\]
\[
= 3824,
\]
since \( E_2(0) = E_4(0) = 0 \). Therefore, (3.9) and (3.10) yield the result of Theorem 1.1(1) when \( p = 4 \), \( a = 5 \) and \( b = 3 \).

Case (V): \( p = 5 \). By letting \( p = 5 \) in (3.1) and (3.2), we have
\[
T_5(5, 3) = \frac{1}{2}, \quad T_5(3, 5) = \frac{1}{2},
\]
so the left-hand side of Theorem 1.1(2) reduces to
\[
3^5 T_5(5, 3) - 5^5 T_5(3, 5) = -1441
\]
and if \( p = 5 \), then the right-hand side of Theorem 1.1(2) for \( p = 5 \) equals to
\[
(5^5 - 3^5) E_5(0) = -1441.
\]
Therefore, (3.11) and (3.12) yield the result of Theorem 1.1(2) when \( p = 5 \), \( a = 5 \) and \( b = 3 \).

**Example 3.2.** Set \( a = 7 \) and \( b = 11 \) in (1.10). From (1.7), we have
(3.13)
\[
T_p(7, 11) = 2 \left( E_p(0) E_1(0) - E_p \left( \frac{7}{11} \right) E_1 \left( \frac{1}{11} \right) + E_p \left( \frac{14}{11} \right) E_1 \left( \frac{2}{11} \right) \right)
\]
\[
- E_p \left( \frac{21}{11} \right) E_1 \left( \frac{3}{11} \right) + E_p \left( \frac{28}{11} \right) E_1 \left( \frac{4}{11} \right)
\]
\[
- E_p \left( \frac{35}{11} \right) E_1 \left( \frac{5}{11} \right) + E_p \left( \frac{42}{11} \right) E_1 \left( \frac{6}{11} \right)
\]
\[
- E_p \left( \frac{49}{11} \right) E_1 \left( \frac{7}{11} \right) + E_p \left( \frac{56}{11} \right) E_1 \left( \frac{8}{11} \right)
\]
\[
- E_p \left( \frac{63}{11} \right) E_1 \left( \frac{9}{11} \right) + E_p \left( \frac{70}{11} \right) E_1 \left( \frac{10}{11} \right)
\]
\[
= 2 \left( E_p(0) E_1(0) - E_p \left( \frac{7}{11} \right) E_1 \left( \frac{1}{11} \right) - E_p \left( \frac{3}{11} \right) E_1 \left( \frac{2}{11} \right) \right)
\]
\[
+ E_p \left( \frac{10}{11} \right) E_1 \left( \frac{3}{11} \right) + E_p \left( \frac{6}{11} \right) E_1 \left( \frac{4}{11} \right)
\]
\[
+ E_p \left( \frac{2}{11} \right) E_1 \left( \frac{5}{11} \right) - E_p \left( \frac{9}{11} \right) E_1 \left( \frac{6}{11} \right)
\]
\[
- E_p \left( \frac{5}{11} \right) E_1 \left( \frac{7}{11} \right) - E_p \left( \frac{1}{11} \right) E_1 \left( \frac{8}{11} \right)
\]
\[
+ E_p \left( \frac{8}{11} \right) E_1 \left( \frac{9}{11} \right) + E_p \left( \frac{4}{11} \right) E_1 \left( \frac{10}{11} \right)
\]
and

\[ T_p(11, 7) = 2 \left( E_p(0) E_1(0) - E_p \left( \frac{11}{7} \right) E_1 \left( \frac{1}{7} \right) + E_p \left( \frac{22}{7} \right) E_1 \left( \frac{2}{7} \right) \right. \]
\[ \left. - E_p \left( \frac{33}{7} \right) E_1 \left( \frac{3}{7} \right) + E_p \left( \frac{44}{7} \right) E_1 \left( \frac{4}{7} \right) \right. \]
\[ \left. - E_p \left( \frac{55}{7} \right) E_1 \left( \frac{5}{7} \right) + E_p \left( \frac{66}{7} \right) E_1 \left( \frac{6}{7} \right) \right) \]
\[ = 2 \left( E_p(0) E_1(0) + E_p \left( \frac{4}{7} \right) E_1 \left( \frac{1}{7} \right) - E_p \left( \frac{1}{7} \right) E_1 \left( \frac{2}{7} \right) \right. \]
\[ \left. - E_p \left( \frac{5}{7} \right) E_1 \left( \frac{3}{7} \right) + E_p \left( \frac{2}{7} \right) E_1 \left( \frac{4}{7} \right) \right. \]
\[ \left. + E_p \left( \frac{6}{7} \right) E_1 \left( \frac{5}{7} \right) - E_p \left( \frac{3}{7} \right) E_1 \left( \frac{6}{7} \right) \right). \]

Case (I): $p = 1$. Letting $p = 1$ in (3.13) and (3.14), we have

\[ T_1(7, 11) = \frac{1}{2}, \quad T_1(11, 7) = \frac{1}{2}, \]

so the left-hand side of Theorem 1.1(2) reduces to

\[ 11T_1(7, 11) - 7T_1(11, 7) = 2 \]

and if $p = 1$, then the right-hand side of Theorem 1.1(2) equals to

\[ (7 - 11)E_1(0) = 2. \]

Therefore, (3.15) and (3.16) yield the result of Theorem 1.1(2) when $p = 1$, $a = 7$ and $b = 11$.

Case (II): $p = 2$. Letting $p = 2$ in (3.13) and (3.14), we have

\[ T_2(7, 11) = -\frac{524}{1331}, \quad T_2(11, 7) = \frac{64}{343}, \]

so the left-hand side of Theorem 1.1(1) reduces to

\[ 7 \cdot 11^3T_2(7, 11) + 7^3 \cdot 11T_2(7, 11) = -2964 \]

and if $p = 2$, then the right-hand side of Theorem 1.1(1) equals to

\[ 2E_3(0) - 7 \cdot 11 \sum_{k=0}^{1} \binom{2}{k} 11^k E_k(0) 7^{2-k} E_{2-k}(0) \]
\[ = 2E_3(0) - 7^2 \cdot 11^2 \binom{2}{1} E_1(0) E_1(0) \]
\[ = -2964, \]

since $E_2(0) = 0$. Therefore, (3.17) and (3.18) reduce to yield the result of Theorem 1.1(1) when $p = 2$, $a = 7$ and $b = 11$.

Case (III): $p = 3$. Letting $p = 3$ in (3.13) and (3.14), we have

\[ T_3(7, 11) = -\frac{1}{4}, \quad T_3(11, 7) = -\frac{1}{4}. \]
so the left-hand side of Theorem 1.1(2) reduces to
\begin{equation}
11^3T_3(7, 11) - 7^3T_3(11, 7) = -247
\end{equation}
and if \( p = 3 \), then the right-hand side of Theorem 1.1(2) equals to
\begin{equation}
(7^3 - 11^3)E_3(0) = -247.
\end{equation}
Therefore, (3.19) and (3.20) yield the result of Theorem 1.1(2) when \( p = 3, a = 7 \) and \( b = 11 \).

Case (IV): \( p = 4 \). Letting even \( p = 4 \) in (3.13) and (3.14), we have
\begin{align*}
T_4(7, 11) &= \frac{78532}{161051}, \quad T_4(11, 7) = -\frac{4160}{16807},
\end{align*}
so the left-hand side of Theorem 1.1(1) reduces to
\begin{equation}
7 \cdot 11^5T_4(7, 11) + 7^5 \cdot 11T_4(7, 11) = 503964
\end{equation}
and if \( p = 3 \), then the right-hand side of Theorem 1.1(1) equals to
\begin{equation}
2E_5(0) - 7 \cdot 11 \sum_{k=0}^{3} \binom{4}{k} 11^k E_k(0) 7^{4-k} E_{4-k}(0)
\end{equation}
\begin{equation}
= 2E_5(0) - 7^2 \cdot 11^2 \left( \left(\frac{4}{1}\right) 7^2 E_1(0) E_3(0) + \left(\frac{4}{3}\right) 11^2 E_3(0) E_1(0) \right)
\end{equation}
\begin{equation}
= 503964,
\end{equation}
since \( E_2(0) = E_4(0) = 0 \). Therefore, (3.21) and (3.22) yield the result of Theorem 1.1(1) when \( p = 4, a = 7 \) and \( b = 11 \).

Case (II): \( p = 5 \). Letting \( p = 5 \) in (3.13) and (3.14), we have
\begin{align*}
T_5(7, 11) &= \frac{1}{2}, \quad T_5(11, 7) = \frac{1}{2},
\end{align*}
so the left-hand side of Theorem 1.1(2) reduces to
\begin{equation}
11^5T_5(7, 11) - 7^5T_5(11, 7) = 72122
\end{equation}
and if \( p = 5 \), then the right-hand side of Theorem 1.1(2) equals to
\begin{equation}
(7^5 - 11^5)E_5(0) = 72122.
\end{equation}
Therefore, (3.23) and (3.24) yield the result of Theorem 1.1(2) when \( p = 5, a = 7 \) and \( b = 11 \).

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