Phase transition for the existence of van Kampen 2-complexes in random groups

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Abstract

Gromov showed that [Gro93] with high probability, every bounded and reduced van Kampen diagram $D$ of a random group at density $d$ satisfies the isoperimetric inequality $|\partial D| \geq (1 - 2d - s)|D|\ell$. In this article, we adapt Gruber-Mackay’s prove [GM18] for random triangular groups, showing a non-reduced 2-complex version of this inequality.

Moreover, for any 2-complex $Y$ of a given geometric form, we exhibit a phase transition: we give explicitly a critical density $d_c$ depending only on $Y$ such that, in a random group at density $d$, if $d < d_c$ then there is no reduced van Kampen 2-complex of the form $Y$; while if $d > d_c$ then there exists reduced van Kampen 2-complexes of the form $Y$.

As an application, we show a phase transition for the $C(p)$ small-cancellation condition: for a random group at density $d$, if $d < 1/(p + 1)$ then it satisfies $C(p)$; while if $d > 1/(p + 1)$ then it does not satisfy $C(p)$.

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1 Introduction

Random groups. The first mention of random group presentations is the density model by M. Gromov in [Gro93, 9.B]. Formally, a random group is a random variable with values in a given set of groups, often constructed by group presentations with a fixed set of generators and a random set of relators. The goal is to study the asymptotic behaviors of a sequence of random groups \( (G_\ell) \) when the maximal relator lengths \( \ell \) goes to infinity. We say that \( G_\ell \) satisfies some property \( Q_\ell \) asymptotically almost surely (a.a.s.) if the probability that \( G_\ell \) satisfies \( Q_\ell \) converges to 1 as \( \ell \) goes to infinity.

Let us consider the permutation invariant density model of random groups introduced by Gromov in [Gro93, p. 272] and developed in [Tsa21]. Fix the set of generators \( X_m = \{x_1, \ldots, x_m\} \) with \( m \geq 2 \) for group presentations. Let \( B_\ell \) be the set of cyclically reduced words of \( X_m \) of length at most \( \ell \). We shall construct random groups by densable and permutation invariant random subsets of \( B_\ell \).

Definition 1.1 ([Gro93, p.272], [Tsa21, Definition 1.5 and Definition 2.5]). A sequence of random subsets \( (R_\ell) \) of the sequence of sets \( (B_\ell) \) is called densable with density \( d \in \{-\infty\} \cup [0, 1] \) if the sequence of random variables \( \text{dens}_{B_\ell}(R_\ell) := \log_{|B_\ell|}(|R_\ell|) \) converges in probability to the constant \( d \).

The sequence \( (R_\ell) \) is called permutation invariant if \( R_\ell \) is a permutation measure-invariant random subset of \( B_\ell \).

Many natural models of random subsets are densable and permutation invariant. For example, the uniform distribution on all subsets of cardinality \( |B_\ell| \) considered in [Oll04], [Oll05] and [Oll07], or the Bernoulli sampling of parameter \( |B_\ell|^{d-1} \) considered in [ALS15] for random triangular groups.

Definition 1.2 ([Gro93, p.273], [Tsa21, Definition 4.1]). A sequence of random groups \( (G_\ell(m, d)) \) with \( m \) generators at density \( d \) is defined by

\[
G_\ell(m, d) = \langle X_m | R_\ell \rangle
\]

where \( (R_\ell) \) is a densable sequence of permutation invariant random subsets of \( (B_\ell) \) with density \( d \).

For detailed surveys on random groups, we refer the reader to [Ghy04] by E. Ghys, [Oll05] by Y. Ollivier, [KS08] by I. Kapovich and P. Schupp, and [BNW20] by F. Bassino, C. Nicaud and P. Weil.

Isoperimetric inequalities. In order to prove the hyperbolicity of a random group at density \( d < 1/2 \), Gromov showed in [Gro93, 9.B] that a.a.s. reduced local van Kampen diagrams of \( G_\ell(m, d) \) satisfy an isoperimetric inequality depending on the density \( d \).

Theorem 1.3 ([Gro93, p.274], [Oll04, Chapter 2]). Let \( (G_\ell(m, d)) \) be a sequence of random groups with \( m \geq 2 \) generators at density \( d \). For any \( \varepsilon > 0 \) and \( K > 0 \), a.a.s. every reduced van Kampen diagram of \( G_\ell(m, d) \) with \( |D| \leq K \) satisfies the isoperimetric inequality

\[
|\partial D| \geq (1 - 2d - \varepsilon)|D|\ell.
\]

Ollivier’s proof in [Oll04] can achieve a slightly stronger\(^1\) inequality

\[
|D^{(1)}| \geq (1 - d - \varepsilon/2)|D|\ell.
\]

\(^1\)Note that every van Kampen diagram composed of relators of lengths at most \( \ell \) satisfies \( 2|D^{(1)}| - |\partial D| \leq |D|\ell \), so the given inequality implies the isoperimetric inequality.
One may expect such an inequality to hold for every reduced van Kampen 2-complex \( Y \) with \( |Y| \leq K \). In [GM18, Section 2], D. Gruber and J. Mackay showed that in the triangular model of random groups\(^2\), the above inequality holds for every non-reduced van Kampen 2-complexes \( Y \) with \( |Y| \leq K \) if the reduction degree (Definition 2.1) \( \text{Red}(Y) \) is added in the left hand side of the inequality.

However, the result fails in the regular Gromov density model: the condition \( |Y| \leq K \) is not enough (see Remark 2.4). In Section 2 of this paper, we introduce the notion of complexity (Definition 2.2) to adapt Gruber-Mackay’s inequality in the Gromov density model, establishing a non-reduced van Kampen 2-complex version of Theorem 1.3.

**Theorem 1.4.** Let \( (G_\ell(m, d)) \) be a sequence of random groups with \( m \geq 2 \) generators at density \( d \). Let \( \varepsilon > 0 \), \( K > 0 \). For any \( d < 1/2 \), a.a.s. every van Kampen 2-complex \( Y \) of complexity \( K \) of \( G_\ell(m, d) \) satisfies
\[
|Y^{(1)}| + \text{Red}(Y) \geq (1 - d - \varepsilon)|Y|\ell.
\]

**Phase transition for the existence of van Kampen 2-complexes.** We are now interested in the converse of Theorem 1.4: Given a 2-complex \( Y \) satisfying the inequality of Theorem 1.4, is it true that a.a.s. there exists a reduced van Kampen 2-complex of \( G_\ell(m, d) \) whose underlying 2-complex is \( Y \)?

A 2-complex \( Y \) is said to be fillable by a group presentation \( G = \langle X | R \rangle \) (or by the set of relators \( R \)) if there exists a reduced van Kampen 2-complex of \( G \) whose underlying 2-complex is \( Y \). An edge of a 2-complex is called isolated if it is not adjacent to any face. Since isolated edges do not affect fillability, we will only consider finite 2-complexes without isolated edges in the following.

To better formulate the problem, we consider a sequence of 2-complexes \( (Y_\ell) \) and introduce the notion of geometric form of 2-complexes \( (Y, \lambda) \) (Definition 3.1), together with its density \( \text{dens}_c(Y) \) and its critical density \( \text{dens}_c(Y) \) (Definition 3.2). The main result of this article is the phase transition at density \( 1 - \text{dens}_c(Y) \), for the fillability of the 2-complex \( Y_\ell \).

**Theorem 1.5.** Let \( (G_\ell(m, d)) \) be a sequence of random groups with \( m \geq 2 \) generators at density \( d \). Let \( (Y_\ell) \) be a sequence of 2-complexes with some geometric form \((Y, \lambda)\).

(i) If \( d < 1 - \text{dens}_c(Y) \), then a.a.s. \( Y_\ell \) is not fillable by \( G_\ell(m, d) \).

(ii) If \( d > 1 - \text{dens}_c(Y) \) and \( Y_\ell \) is fillable by \( B_\ell \), then a.a.s. \( Y_\ell \) is fillable by \( G_\ell(m, d) \).

In Section 3, we prove Theorem 1.5 using the multidimensional intersection formula for random subsets (Theorem 3.6, [Tsa21, Theorem 3.7]), which generalizes the proof for the \( C^*(\lambda) \) phase transition in [Tsa21, Theorem 1.4]. We will see in Remark 3.3 that the second assertion of the theorem is equivalent to the following corollary.

**Corollary 1.6.** Let \( (G_\ell(m, d)) \) be a sequence of random groups with \( m \geq 2 \) generators at density \( d \). Let \( s > 0 \) and \( K > 0 \). Let \( (Y_\ell) \) be a sequence of 2-complexes of the same geometric form such that \( Y_\ell \) is fillable by \( B_\ell \). If every sub-2-complex \( Z_\ell \) of \( Y_\ell \) satisfies
\[
|Z_\ell^{(1)}| \geq (1 - d + s)|Z_\ell|\ell,
\]
\(^2\)A model that the relator length \( \ell = 3 \) is fixed, and we are interested in asymptotic behaviors when the number of generators \( m \) goes to infinity.
then a.a.s. \( Y_\ell \) is fillable by \( G_\ell(m, d) \).

Note that we need \( Y_\ell \) to have at least one filling by the set of all possible relators \( B_\ell \). It is automatically satisfied for planar and simply connected 2-complexes. In addition, if every face boundary length of \( Y_\ell \) is exactly \( \ell \), then the given inequality is equivalent to an isoperimetric inequality similar the inequality of Theorem 1.3. Hence the following corollary.

**Corollary 1.7.** Let \( (G_\ell(m, d)) \) be a sequence of random groups with \( m \geq 2 \) generators at density \( d \). Let \( s > 0 \) and \( K > 0 \). Let \( (D_\ell) \) be a sequence of finite planar 2-complexes of the same geometric form such that every face boundary length of \( D_\ell \) is exactly \( \ell \). If every sub-2-complex \( D'_\ell \) of \( D_\ell \) satisfies

\[
|\partial D'_\ell| \geq (1 - 2d + s)|D'_\ell|\ell,
\]

then a.a.s. \( D_\ell \) is fillable by \( G_\ell(m, d) \).

It is mentioned in [OW11, Proposition 1.8] that when \( d < 1/(p + 1) \), a.a.s. a random group at density \( d \) has the \( C(p) \) small cancellation condition. As an application of Theorem 1.5, we show that there is a phase transition: if \( d > 1/(p + 1) \), then a.a.s. a random group at density \( d \) does not have \( C(p) \) (see Proposition 4.2).

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## 2 Isoperimetric inequality for van Kampen 2-complexes

We shall prove Theorem 1.4 in this section.

**Van Kampen 2-complexes.** We consider oriented combinatorial 2-complexes and van Kampen diagrams as in [LS77].

A 2-complex is a triplet \( Y = (V, E, F) \) where \( V \) is the set of vertices, \( E \) is the set of oriented edges and \( F \) is the set of oriented faces. Every edge \( e \in E \) has a starting point \( \alpha(e) \in V \), an ending point \( \omega(e) \in V \) and an inverse edge \( e^{-1} \in E \), satisfying \( \alpha(e^{-1}) = \omega(e) \), \( \omega(e^{-1}) = \alpha(e) \) and \( (e^{-1})^{-1} = e \). A geometric edge is a pair of inverse edges \( \tau \), denoted by \( \tau \). Every face \( f \in F \) has a boundary \( \partial f \), which is a cyclically reduced loop of the underlying combinatorial oriented graph \( Y^{(1)} = (V, E) \), and an inverse face \( f^{-1} \in F \) satisfying \( \partial(f^{-1}) = (\partial f)^{-1} \) and \( (f^{-1})^{-1} = f \). A geometric face is a pair of inverse faces \( \{f, f^{-1}\} \), denoted by \( \overline{f} \). We denote \( |Y^{(1)}| \) the number of geometric edges and \( |Y| \) the number of geometric faces.

A van Kampen 2-complex with respect to a group presentation \( G = \langle X | R \rangle \) is a 2-complex \( Y = (V, E, F) \) with two compatible labeling functions: labels on edges by generators \( \varphi_1 : E \to X^\pm \), and labels on faces by relators \( \varphi_2 : F \to R^\pm \). Compatible means that \( (V, E, \varphi_1) \) is a labeled graph, \( \varphi_2(f^{-1}) = \varphi_2(f)^{-1} \) and \( \varphi_1(\partial f) = \varphi_2(f) \). The data of the labels \( \varphi_1, \varphi_2 \) on \( Y \) is equivalently given by a combinatorial map \( Y \to K(X, R) \) where \( K(X, R) \) is the standard 2-complex with respect to the group presentation \( G = \langle X | R \rangle \) (with one vertex, an edge for each generator and its inverse, and a face for each relator and its inverse). We denote briefly \( Y = (V, E, F, \varphi_1, \varphi_2) \).
A van Kampen diagram $D$ is a finite, planar (embedded in the plan) and simply connected van Kampen 2-complex. Its boundary length $|\partial D|$ is the length of a boundary path, passing once by every edge adjacent to one face and twice by every edge adjacent to zero face.

A pair of faces in a van Kampen 2-complex is called reducible if they have the same relator label and there is a common edge on their boundaries at the same position. A van Kampen 2-complex is called reduced if there is no reducible pair of faces.

![Figure 1: A reducible pair of faces.](image)

### 2.1 Reduction degree and Complexity

Let us define the reduction degree of a van Kampen 2-complex and the complexity of a 2-complex.

The reduction degree of a non-reduced van Kampen diagram $Y = (V, E, F, \varphi_1, \varphi_2)$ with respect to a group presentation $\langle X | R \rangle$ is the total number of geometric edges causing reducible pair of faces, counted with multiplicity: for any edge $e \in E$, any relator $r \in R$ and any integer $j$, we count the number of faces $f \in F$ labeled by $r$ and having $e$ as the $j$-th boundary edge. If this number is $k$, we add $(k - 1)^+$ to the reduction degree where $(\cdot)^+$ is the positive part function.

**Definition 2.1** (Reduction degree, [GM18, Definition 2.5]). Let $Y = (V, E, F, \varphi_1, \varphi_2)$ be a van Kampen 2-complex of a group presentation $G = \langle X | R \rangle$. Let $\ell$ be the maximal boundary length of faces of $Y$. The reduction degree of $Y$ is

$$\text{Red}(Y) = \sum_{e \in E} \sum_{r \in R} \sum_{1 \leq j \leq \ell} \left( |\{ f \in F \mid \varphi_2(f) = r, e \text{ is the } j\text{-th edge of } \partial f \}| - 1 \right)^+.$$  

It is not hard to see that a van Kampen 2-complex $Y$ is reduced if and only if $\text{Red}(Y) = 0$.

A maximal arc of a 2-complex is a reduced combinatorial path passing only by vertices of degree 2 whose endpoints are not of degree 2. The complexity of a 2-complex encodes the number of maximal arcs with the number of faces.

**Definition 2.2** (Complexity of a 2-complex). Let $Y$ be a 2-complex. Let $K > 0$. We say that $Y$ is of complexity $K$ if the following three conditions hold:

- $|Y| \leq K$.
- The number of maximal arcs of $Y$ is bounded by $K$.
- For any face $f$ of $Y$, the boundary path $\partial f$ is divided into at most $K$ maximal arcs.
Note that if $D$ is a planar and simply connected 2-complex with $|D| \leq K$, then the complexity of $D$ is $6K$. In fact, as the rank of its underlying graph is $K$, the number of its maximal arcs is at most $3K$, and every boundary path is divided into at most $6K$ maximal arcs (an arc may be used twice).

**Lemma 2.3.** Let $K > 0$. For $\ell$ large enough, the number of 2-complexes of complexity $K$ with face boundary lengths at most $\ell$ is bounded by $\ell^{3K}$.

**Proof.** As there are at most $(K^2)^K$ ways to draw an arc connecting two of these vertices, every arc is with length at most $\ell$, which is smaller than $3$.

To attach $K$ faces on a graph, we choose $K$ loops passing by at most $K$ arcs, there are at most $(K^2)^K$ choices. There are at most $(2\ell)^K$ ways to choose a starting point and an orientation for every face. The number of such 2-complexes is hence bounded by

$$(K^2)^K \ell^K \times (K^2)^K \times (2\ell)^K,$$

which is smaller than $\ell^{3K}$ if $\ell$ is large enough.

**Remark 2.4.** While the number of 2-complexes with a bounded complexity grows polynomially with the maximal face boundary length $\ell$, it is not the case for 2-complexes with a bounded number of faces. Hence the related argument in **Proof of Theorem 1.4** does not work. Actually, there exists van Kampen 2-complexes that contradicts the inequality of Theorem 1.4.

For example, in [CW15], D. Calegari and A. Walker proved that at any density $d < 1/2$, there exists a number $K$ depending only on $d$ such that a.a.s. there is a reduced van Kampen 2-complex $Y$ homeomorphic to a surface of genus $O(\ell)$ in $G(\ell, m, d)$ with at most $K$ faces. Since every edge is adjacent to two faces in a surface, we have $|Y^{(1)}| \leq \frac{1}{2} |Y| \ell$, while we expect that $|Y^{(1)}| \geq \left(1 - d - \frac{\ell}{4}\right) > \frac{1}{2} |Y| \ell$.

### 2.2 Abstract van Kampen 2-complexes

Let $(G(\ell, m, d))$ be a sequence of random groups at density $d$, defined by $G(\ell, m, d) = \langle x_1, \ldots, x_n | R_{\ell}\rangle$. Recall that $B_{\ell}$ is the set of all cyclically reduced words of length at most $\ell$ and $|B_{\ell}| = (2m - 1)^{\ell + O(1)}$. Let $0 < \varepsilon < 1 - d$. Since $\log |B_{\ell}|$ converges in probability to the constant $d$, the probability event

$$Q_{\ell} := \left\{(2m - 1)^{(d-\varepsilon)\ell} \leq |R_{\ell}| \leq (2m - 1)^{(d+\varepsilon)\ell}\right\}$$

is a.a.s. true (cf. [Tsa21] Proposition 1.8).

If we consider the Bernoulli density model that the events $\{r \in R_{\ell}\}$ through $r \in B_{\ell}$ are independent of the same probability $(2m - 1)^{(d-1)\ell}$, it is obvious that we have $\Pr(r_1, \ldots, r_k \in R_{\ell}) = (2m - 1)^{k(d-1)\ell}$ for distinct $r_1, \ldots, r_k$ in $B_{\ell}$. In the permutation invariant density model, we have the following corresponding proposition, which is a variant of [Tsa21] Lemma 3.10.

**Proposition 2.5.** Let $r_1, \ldots, r_k$ be pairwise different relators in $B_{\ell}$. We have

$$\Pr(r_1, \ldots, r_k \in R_{\ell} | Q_{\ell}) \leq (2m - 1)^{k(d-1+\varepsilon)\ell}.$$  


Abstract van Kampen 2-complexes, as abstract van Kampen diagrams introduced by Ollivier in [Oll04], is a structure between 2-complexes and van Kampen 2-complexes that helps us solve 2-complex problems in random groups. Recall that since isolated edges do not affect fillability, we will only consider finite 2-complexes without isolated edges.

**Definition 2.6 (Abstract van Kampen 2-complex).** An abstract van Kampen 2-complex \( \tilde{Y} \) is a 2-complex \((V, E, F)\) with a labeling function on faces by integer numbers and their inverses \( \tilde{\varphi}_2 : F \to \{1, 1^-, 2, 2^-, \ldots, k, k^-\} \) such that \( \tilde{\varphi}_2(f^-) = \tilde{\varphi}_2(f)^- \). We denote simply \( \tilde{Y} = (V, E, F, \tilde{\varphi}_2) \).

By convention \((i^-)^- = i\). The integers \(\{1, \ldots, k\}\) are called abstract relators. Similar to a van Kampen diagram, a pair of faces \(f, f' \in F\) is reducible if they are labeled by the same abstract relator, and they share an edge at the same position of their boundaries. An abstract diagram is called reduced if there is no reducible pair of faces. Let \(\ell\) be the maximal boundary length of faces. The reduction degree of \(\tilde{Y}\) 2-complex can be similarly defined as

\[
\text{Red}(\tilde{Y}) = \sum_{e \in E} \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq \ell} \left( |\{f \in F | \tilde{\varphi}_2(f) = i, e \text{ is the } j\text{-th edge of } \partial f\}| - 1 \right)^+.
\]

We say that an abstract van Kampen 2-complex with \(k\) abstract relators \(\tilde{Y} = (V, E, F, \tilde{\varphi}_2)\) is fillable by a group presentation \(G = \langle X | R \rangle\) (or by a set of relators \(R\)) if there exists \(k\) different relators \(r_1, \ldots, r_k \in R\) such that the construction \(\varphi_2(f) := r_\tilde{\varphi}_2(f)\) gives a van Kampen 2-complex \(Y = (V, E, F, \varphi_1, \varphi_2)\) of \(G\). The \(k\)-tuple of relators \((r_1, \ldots, r_k)\), or the van Kampen 2-complex \(Y\), is called a filling of \(\tilde{Y}\). As we picked different relators for different abstract relators, if \(Y\) is a filling of \(\tilde{Y}\), then \(\text{Red}(Y) = \text{Red}(\tilde{Y})\), and \(\tilde{Y}\) is reduced if and only if \(Y\) is reduced.

![Figure 2: Filling an abstract van Kampen 2-complex.](image)

Denote \(\ell_i\), the length of the abstract relator \(i\) for \(1 \leq i \leq k\). Let \(\ell = \max\{\ell_1, \ldots, \ell_k\}\) be the maximal boundary length of faces. The pairs of integers \((i, 1), \ldots, (i, \ell_i)\) are called abstract letters of \(i\). The set of abstract letters of \(\tilde{Y}\) is then a subset of the product set \(\{1, \ldots, k\} \times \{1, \ldots, \ell\}\). The geometric edges of \(\tilde{Y}\) are decorated by abstract letters and directions: Let \(f \in F\) labeled by \(i\) and let \(e \in E\) at the \(j\)-th position of \(\partial f\). The geometric edge \(e\) is decorated, on the side of \(f\), by an arrow indicating the direction of \(e\) and the abstract letter \((i, j)\). The number of decorations on a geometric edge is the number of its adjacent faces with multiplicity (an edge may be attached twice by the same face).

\[1\] Note that the edge labeling \(\varphi_1\) is determined by the face labeling \(\varphi_2\) as there is no isolated edges.
Figure 3: A geometric edge decorated by two abstract letters.

**Definition 2.7** (free-to-fill). An abstract letter \((i, j)\) of \(\tilde{D}\) is **free-to-fill** if, for any edge \(e\) decorated by \((i, j)\), it is the minimal decoration on \(e\).

Denote \(\alpha_i\) the number of faces labeled by the abstract relator \(i\) and \(\eta_i\) the number of free-to-fill edges of \(i\). We have the following estimation.

**Lemma 2.8.** Let \(\tilde{Y} = (V, E, F, \tilde{\varphi}_2)\) be an abstract van Kampen 2-complex with \(k\) abstract relators. Then

\[
\sum_{i=1}^{k} \alpha_i \eta_i \leq |\tilde{Y}(1)| + \text{Red}(\tilde{Y}).
\]

**Proof.** Denote \(E\) the set of geometric edges and \(F\) the set of geometric faces. For any geometric edge \(e\), an adjacent face \(f\) from which the decoration is minimal is called a **preferred face** of \(e\). For any face \(f\), let \(E_f\) be the set of geometric edges \(e\) on its boundary such that \(f\) is a preferred face of \(e\). Note that an edge will never be counted twice as the decorations given by one face are all different. According to Definition 2.7, for any face \(f\) with \(\tilde{\varphi}_2(f) = i\), we have \(\eta_i \leq |E_f|\). Hence,

\[
\sum_{i=1}^{k} \alpha_i \eta_i \leq \sum_{f \in F} |E_f|.
\]

Denote \(\text{Red}(\varpi)\) the reduction degree caused by the edge \(\varpi\). That is,

\[
\text{Red}(\varpi) := \sum_{1 \leq i \leq k} \sum_{1 \leq j \leq \ell} \left( |\{f \in F \mid \tilde{\varphi}_2(f) = i, e \text{ or } e^{-1} \text{ is the } j\text{-th edge of } \partial f\}| - 1 \right)^+,
\]

so that the number of preferred faces of \(\varpi\) is bounded by \(1 + \text{Red}(\varpi)\). Hence,

\[
\sum_{f \in F} |E_f| \leq \sum_{\varpi \in \varGamma} \left( 1 + \text{Red}(\varpi) \right) = |\tilde{Y}(1)| + \text{Red}(\tilde{Y}).
\]

**Probability of filling.** We shall estimate the probability that an abstract van Kampen 2-complex \(\tilde{Y}\) is fillable by a random group \(G_\ell(m, d)\). This step is the key to prove Theorem 1.4. Recall that \(Q_\ell := \{ (2m - 1)^{(d-\frac{2}{\ell})} \leq |R_\ell| \leq (2m - 1)^{(d+\frac{2}{\ell})} \}\) is an a.a.s. true probability event.

**Lemma 2.9.** Let \(\tilde{Y}\) be an abstract van Kampen 2-complex with \(k\) abstract relators. We have

\[
\text{Pr} \left( \tilde{Y} \text{ is fillable by } G_\ell(m, d) \mid Q_\ell \right) \leq \left( \frac{2m}{2m - 1} \right)^k (2m - 1)^{\sum_{i=1}^{k} \eta_i (d-1+\frac{2}{\ell})}.
\]
Proof. Let us estimate the number of fillings of \( \tilde{Y} \). For every free-to-fill abstract letter \((i, j)\), there are at most \(2m\) ways to fill a generator if \( j = 1 \), at most \((2m - 1)\) ways to fill if \( j \neq 1 \) for avoiding reducible word. As there are \( \eta_i \) free-to-fill abstract letters on the \( i \)-th abstract relator, there are at most \(2m(2m - 1)^{\eta_i - 1}\) ways to fill it. So there are at most \(\prod_{i=1}^k (2m(2m - 1)^{\eta_i - 1})\) ways to fill \( \tilde{Y} \).

Let \( Y \) be a van Kampen 2-complex, which is a filling of \( \tilde{Y} \). The 2-complex \( Y \) is labeled by \( k \) different relators in \( B_{\ell} \), denoted \( r_1, \ldots, r_k \). By lemma 2.5,

\[
\Pr(Y \text{ is a 2-complex of } G_{\ell}(m, d) \mid Q_{\ell}) = \Pr(r_1, \ldots, r_k \in R_{\ell} \mid Q_{\ell}) \\
\leq (2m - 1)^{k(d - 1 + \frac{\varepsilon}{2})\ell}.
\]

Hence

\[
\Pr(\tilde{Y} \text{ is fillable by } G_{\ell}(m, d) \mid Q_{\ell}) \leq \sum_{Y \text{ fills } \tilde{Y}} \Pr(Y \text{ is a 2-complex of } G_{\ell}(m, d) \mid Q_{\ell}) \\
\leq \prod_{i=1}^k (2m(2m - 1)^{\eta_i - 1})(2m - 1)^{k(d - 1 + \frac{\varepsilon}{2})\ell} \\
\leq \left( \frac{2m}{2m - 1} \right)^k (2m - 1)^{\sum_{i=1}^k (\eta_i + (d - 1 + \frac{\varepsilon}{2})\ell)}.
\]

Lemma 2.10. Let \( \tilde{Y} \) be an abstract van Kampen 2-complex with \( k \) abstract relators. Suppose that \( \tilde{Y} \) does not satisfy the inequality given in Theorem 1.4, i.e.

\[
|\tilde{Y}(1)| + \text{Red}(\tilde{Y}) < (1 - d - \varepsilon)|\tilde{Y}|\ell,
\]

then

\[
\Pr(\tilde{Y} \text{ is fillable by } G_{\ell}(m, d) \mid Q_{\ell}) \leq \left( \frac{2m}{2m - 1} \right)^{n} (2m - 1)^{-\frac{\varepsilon}{2}\ell}.
\]

Proof. Let \( \tilde{Y}_i \) be the sub-2-complex of \( \tilde{Y} \) consisting of faces labeled by the \( i \) first abstract relators. Denote \( P_i = \Pr(\tilde{Y}_i \text{ is fillable by } G_{\ell}(m, d) \mid Q_{\ell}) \). Apply lemma 2.9 on \( \tilde{Y}_i \), we have

\[
P_i \leq \left( \frac{2m}{2m - 1} \right)^i (2m - 1)^{\sum_{j=1}^i (\eta_j + (d - 1 + \frac{\varepsilon}{2})\ell)}.
\]

Note that if \( \tilde{Y} \) is fillable by \( G_{\ell}(m, d) \) then its sub 2-complex \( \tilde{Y}_i \) is fillable by the same group. So for any \( 1 \leq i \leq k \),

\[
\log_{2m-1}(P_k) \leq \log_{2m-1}(P_i) \leq \sum_{j=1}^i \left( \eta_j + \left( d - 1 + \frac{\varepsilon}{2} \right) \ell + \log_{2m-1} \left( \frac{2m}{2m - 1} \right) \right).
\]
Without loss of generality, suppose that $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$. Note that $\log_{2m-1}(P_k)$ is negative and $\alpha_1 \leq |\tilde{Y}|$, so $|\tilde{Y}| \log_{2m-1}(P_k) \leq \alpha_1 \log_{2m-1}(P_k)$. By Abel’s summation formula, with convention $\alpha_{k+1} = 0$,

$$|\tilde{Y}| \log_{2m-1}(P_k) \leq \alpha_1 \log_{2m-1}(P_k) = \sum_{i=1}^{k} (\alpha_i - \alpha_{i+1}) \log_{2m-1}(P_k)$$

$$\leq \sum_{i=1}^{k} (\alpha_i - \alpha_{i+1}) \sum_{j=1}^{i} \left[ \eta_i + \left( d - 1 + \frac{\epsilon}{2} \right) \ell + \log_{2m-1} \left( \frac{2m}{2m-1} \right) \right]$$

$$= \sum_{i=1}^{k} \alpha_i \left[ \eta_i + \left( d - 1 + \frac{\epsilon}{2} \right) \ell + \log_{2m-1} \left( \frac{2m}{2m-1} \right) \right]$$

$$= \sum_{i=1}^{k} \alpha_i \eta_i + \left( \sum_{i=1}^{k} \alpha_i \right) \left[ \left( d - 1 + \frac{\epsilon}{2} \right) \ell + \log_{2m-1} \left( \frac{2m}{2m-1} \right) \right].$$

Note that $\sum_{i=1}^{k} \alpha_i = |\tilde{Y}|$. By Lemma 2.8 and the hypothesis of the current lemma,

$$\sum_{i=1}^{k} \alpha_i \eta_i \leq |\tilde{Y}(1)| + \text{Red}(\tilde{Y}) < (1 - d - \epsilon)|\tilde{Y}|\ell.$$

Hence,

$$|\tilde{Y}| \log_{2m-1}(P_k) \leq (1 - d - \epsilon)|\tilde{Y}|\ell + |\tilde{Y}| \left[ \left( d - 1 + \frac{\epsilon}{2} \right) \ell + \log_{2m-1} \left( \frac{2m}{2m-1} \right) \right]$$

$$\leq |\tilde{Y}| \left[ \frac{\epsilon}{2} \ell + \log_{2m-1} \left( \frac{2m}{2m-1} \right) \right].$$

\[\Box\]

2.3 Proof of Theorem 1.4

Under the condition $Q_\ell := \{(2m-1)^{d-\ell} \leq |R_\ell| \leq (2m-1)^{d+\ell}\}$, the probability that there exists a van Kampen 2-complex of complexity $K$ of $G_\ell(m,d)$ satisfying the inverse inequality

$$|Y(1)| + \text{Red}(Y) < (1 - d - \epsilon)|Y|\ell$$

is bounded by

$$\sum_{\tilde{Y} \text{ of complexity } K, \text{ satisfying } (*)} \text{Pr} \left( \tilde{Y} \text{ is fillable by } G_\ell(m,d) \mid Q_\ell \right).$$

By Lemma 2.3 and the face that there at most $K^{2K}$ ways to label a 2-complex with $K$ faces by abstract relators $\{1^\pm, \ldots, K^\pm\}$, there are at most $\ell^{3K} \times K^{2K}$ terms in the sum. By Lemma 2.10, every term is bounded by $\left( \frac{2m}{2m-1} \right)^{2m-1} (2m-1)^{-\frac{\ell}{2}}$. So the sum is smaller than

$$\ell^{3K} K^{2K} \left( \frac{2m}{2m-1} \right)^{2m-1} (2m-1)^{-\frac{\ell}{2}},$$

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which converges to $0$ as $\ell \to \infty$.

By definition $\Pr(Q_\ell) \xrightarrow[\ell \to \infty]{} 1$, so the probability that there exists a van Kampen 2-complex of $G_\ell(m, d)$ of complexity $K$ satisfying $(\ast)$ converges to $0$ as $\ell$ goes to infinity. That is to say, a.a.s. every van Kampen diagram of $G_\ell(m, d)$ of complexity $K$ satisfies the inequality

$$|Y^{(1)}| + \text{Red}(Y) \geq (1 - d - \epsilon)|Y|\ell.$$

Closed surfaces. Recall that a 2-complex $Y$ without isolated edges is called contractible if there is an edge of $Y$ that is adjacent to one single face. If a 2-complex $Y$ is not contractible, then each of its edge is adjacent to at least $2$ faces, and we have $|Y^{(1)}| \leq \frac{1}{2}|Y|\ell$ where $\ell$ is the maximal boundary length of faces. As this contradicts the inequality of Theorem 1.4 for any density $d < 1/2$, we have the following proposition.

**Proposition 2.11.** Let $(G_\ell(m, d))$ be a sequence of random groups with $m \geq 2$ generators at density $d$. For any $d < 1/2$ and $K > 0$, a.a.s. every reduced van Kampen 2-complex of complexity $K$ of $G_\ell(m, d)$ can be contracted to a graph.

3 Phase transition for the existence of van Kampen 2-complexes

In this section, we work on the proof of Theorem 1.5.

**Motivation and a counterexample.** Let $(G_\ell(m, d))$ be a sequence of random groups at density $d$. We are interested in the converse of Theorem 1.4 without the reduction part: if a 2-complex $Y_\ell$ with bounded complexity satisfies the inequality

$$|Y^{(1)}_\ell| \geq (1 - d + s)|Y_\ell|\ell$$

with some $s > 0$, does there exist a face labeling by relators and an edge labeling by generators, so that $Y_\ell$ becomes a reduced van Kampen 2-complex of $G_\ell(m, d)$?

The motivation for this question comes from the well-known phase transition at density $d = \frac{\lambda}{2}$, mentioned in [Gro93, p. 274]: if $d < \frac{\lambda}{2}$ then a.a.s. $G_\ell(m, d)$ has the $C^\prime(\lambda)$ small cancellation condition; while if $d > \frac{\lambda}{2}$ then a.a.s. $G_\ell(m, d)$ does not have $C^\prime(\lambda)$. The first assertion is a simple application of Theorem 1.3. For the second assertion, we need to show that a.a.s. there exists a van Kampen 2-complex $D$ of $G_\ell(m, d)$ with exactly $2$ faces of boundary length $\ell$, sharing a common path of length at least $\lambda \ell$ (Figure 4).

The first detailed proof of such an existence is given in [BNW20, Theorem 2.1], using an analog of the probabilistic pigeonhole principle. Another proof is given in [Tsa21, Theorem 1.4]. An intuitive explanation using the "dimension reasoning" is given in [Oll05] p.30: The dimension of the set of couples $R_\ell \times R_\ell$ is $2d\ell$. Sharing a common subword of length $L$ imposes $L$ equations, so the "dimension" of the set of couples of relators sharing a common subword of length $\lambda \ell$ is $2d\ell + \lambda \ell$. If $d > \lambda/2$,
then there will exist such a couple because the dimension will be positive. However, this argument is not true for any 2-complex in general. Here is a counterexample:

At density \( d = 0.4 \), let \((D_\ell)\) be a sequence of 2-complexes where \(D_\ell\) is given in Figure 5. The given inequality is satisfied because \(|D_\ell^{(1)}| = 1.9\ell > 1.8\ell = (1-d)|D_\ell|\ell\). However, the sub-diagram \(D'_\ell\) gives \(|D'_\ell^{(1)}| = 1.1\ell < 1.2\ell = (1-d)|D'_\ell|\ell\), which contradicts the isoperimetric inequality of Theorem 1.4 and can not be a van Kampen diagram of \(G_\ell(m, d)\).

\[ D_\ell \]

\[ D'_\ell \]

Figure 5: A 2-complex that satisfies the isoperimetric inequality with a sub-2-complex that does not.

### 3.1 Geometric form and Critical density

Let us define the geometric form of 2-complexes and the critical density of a geometric form. To simplify the notations, for a 2-complex \(Y = (V, E, F)\), we denote \(\text{Edge}(Y)\) as the set of geometric edges of \(Y\) and \(e\) instead of \(\tau\) for geometric edges.

**Definition 3.1.** A geometric form of 2-complexes is a couple \((Y, \lambda)\) where \(Y = (V, E, F)\) is a finite connected 2-complex without isolated edges, and \(\lambda\) is a length labeled on edges defined by \(\lambda : \text{Edge}(Y) \to [0, 1], e \mapsto \lambda_e\), such that for every face \(f\) of \(Y\), the boundary length \(|\partial f|\) is bounded by 1.

A sequence of 2-complexes \((Y_\ell)\) is called of the geometric form \((Y, \lambda)\) if \(Y_\ell\) is obtained from \(Y\) by dividing every edge \(e\) of \(Y\) into \(\lfloor \lambda_e \ell \rfloor\) edges of length 1.

A sequence of 2-complexes \((Y_\ell)\) is briefly said to be of the same geometric form if the geometric form \((Y, \lambda)\) is not specified. Note that the boundary length of every face

\[ \ell \]

\[ \lambda \ell \]

Figure 4: A van Kampen diagram denying the \(C'(\lambda)\) condition.
\( f \) of \( Y_\ell \) is at most \( \ell \). If \( Z \) is a sub-2-complex of \( Y \), we denote \( Z \leq Y \). By convention, if \((Z_\ell)\) is a sequence of 2-complexes of the geometric form \((Z, \lambda_{|Z})\), we have \( Z_\ell \leq Y_\ell \) for any integer \( \ell \).

**Definition 3.2.** Let \((Y, \lambda)\) be a geometric form of 2-complexes. The **density** of \( Y \) is
\[
dens(Y) := \frac{\sum_{e \in \text{Edge}(Y)} \lambda_e}{|Y|}.
\]

The **critical density** of \( Y \) is
\[
dens_c(Y) := \min_{{Z \subseteq Y}} \{\dens(Z)\}.
\]

The intuition of this definition can be found in Lemma 3.8: the density of \( Y \) is actually the density of all possible van Kampen 2-complexes that fill \( Y \).

**Remark 3.3.** Taking Definition 3.2 and Definition 3.1 together, we have
\[
dens(Y) = \frac{\sum_{e \in \text{Edge}(Y)} \lambda_e}{|Y|} = \lim_{\ell \to \infty} \frac{\sum_{e \in \text{Edge}(Y)} |E|_e \ell}{|Y| \ell} = \lim_{\ell \to \infty} \frac{|Y_\ell^{(1)}|}{|Y| \ell}.
\]

Hence, the condition “\( \dens_c(Y) + d > 1 \)” is equivalent to the following statement: Given \( s > 0 \), for \( \ell \) large enough, every sub-2-complex \( Z_\ell \) of \( Y_\ell \) satisfies
\[
|Z_\ell^{(1)}| \geq (1 - d + s)|Z_\ell| \ell.
\]

This argument shows that the second assertion of Theorem 1.5 is equivalent to Corollary 1.6.

**Proof of Theorem 1.5 (i).** We will use Theorem 1.4 without the reduction part. Let \((G_\ell(m, d))\) be a sequence of random groups with \( m \) generators at density \( d \). Recall that a 2-complex \( Y_\ell \) is said to be fillable by \( G_\ell(m, d) \) if there exists a reduced van Kampen 2-complex of \( G_\ell(m, d) \) whose underlying 2-complex is \( Y_\ell \).

Let \((Y, \lambda)\) be a geometric form of 2-complexes with \( \dens_c(Y) + d < 1 \). Let \((Y_\ell)\) be a sequence of 2-complexes of the geometric form \((Y, \lambda)\). We shall prove that a.a.s. the 2-complex \( Y_\ell \) is not fillable by the random group \( G_\ell(m, d) \). By the definition of critical density, there exists a sub-2-complex \( Z \leq Y \) satisfying \( \dens(Z) + d < 1 \). Let \((Z_\ell)\) be the sequence of 2-complexes of the geometric form \((Z, \lambda_{|Z})\). We shall prove that a.a.s. \( Z_\ell \) is not fillable by \( G_\ell(m, d) \).

Let \( \epsilon > 0 \) such that \( \dens(Z) = 1 - d - 3\epsilon \). By definition,
\[
\lim_{\ell \to \infty} \frac{|Z_\ell^{(1)}|}{|Z_\ell| \ell} = 1 - d - 3\epsilon,
\]
so for \( \ell \) large enough,
\[
|Z_\ell^{(1)}| \leq (1 - d - 2\epsilon)|Z_\ell| \ell < (1 - d - \epsilon)|Z_\ell| \ell.
\]

The complexity of \( Z_\ell \) is \( K = \max\{|Z|, |Z_\ell^{(1)}|, \max\{\frac{1}{|e|} \mid e \in \text{Edge}(Z)\}\} \), independent of \( \ell \). By Theorem 1.4 with \( \epsilon \) and \( K \) given above, a.a.s. every van Kampen 2-complex \( Z_\ell \) of \( G_\ell(m, d) \) of complexity \( K \) should satisfy
\[
|Z_\ell^{(1)}| \geq (1 - d - \epsilon)|Z_\ell| \ell.
\]

Hence, a.a.s. the given 2-complex \( Z_\ell \) is not fillable by \( G_\ell(m, d) \), which implies that a.a.s. \( Y_\ell \) is not fillable by \( G_\ell(m, d) \).
3.2 The multidimensional intersection formula for random subsets

To prove the second assertion of Theorem 1.5, we need the multidimensional intersection formula for random subsets with density, introduced in [Tsa21, Section 3].

Recall that $B_\ell$ is the set of cyclically reduced words of $X_{\ell}^m = \{x_1^{\pm}, \ldots, x_m^{\pm}\}$ of length at most $\ell$, and that $|B_\ell| = (2m-1)^{\ell+o(\ell)}$. Let $k \geq 1$ be an integer. Denote $B_\ell^{(k)}$ as the set of $k$-tuples of pairwise distinct relators $(r_1, \ldots, r_k)$ in $B_\ell$. Such a notation can be used for any set or any random set.

Note that $|B_\ell^{(k)}| = (2m-1)^{k\ell+o(\ell)}$. Recall that a sequence of fixed subsets $(Y_\ell)$ of the sequence $(B_\ell^{(k)})$ is called densable with density $\alpha \in \{-\infty\} \cup [0, 1]$ if the sequence of real numbers $(\log |B_\ell^{(k)}| |Y_\ell|)$ converges to $\alpha$ (see [Gro93, p.272] and [Tsa21, Definition 1.5]). That is to say, $|Y_\ell| = (2m-1)^{\alpha \ell k + o(\ell)}$.

Definition 3.4 (Self-intersection partition, [Tsa21, Definition 3.4]). Let $(Y_\ell)$ be a sequence of fixed subsets of the sequence $(B_\ell^{(k)})$. Let $0 \leq i \leq k$ be an integer. The $i$-th self-intersection of $Y_\ell$ is

$$S_{i,\ell} := \{(x, y) \in Y_\ell^2 \mid |x \cap y| = i\}$$

where $|x \cap y|$ is the number of common elements between the sets $x = (r_1, \ldots, r_k)$ and $y = (r'_1, \ldots, r'_k)$.

The family of subsets $\{S_{i,\ell} \mid 0 \leq i \leq k\}$ is a partition of $Y_\ell^2$, called the self-intersection partition of $Y_\ell$. Note that $(S_{i,\ell})_{\ell \in \mathbb{N}}$ is a sequence of subsets of the sequence $(B_\ell^{(k)})_{\ell \in \mathbb{N}}$, with density smaller than dens$(B_\ell^{(k)})$. Let $(Y_\ell)$ be a sequence of permutation invariant random subsets of the sequence $(B_\ell^{(k)})_{\ell \in \mathbb{N}}$.

Definition 3.5 (d-small self-intersection condition, [Tsa21, Definition 3.5]). Let $(Y_\ell)$ be a sequence of fixed subsets of $(B_\ell^{(k)})$ with density $\alpha$. Let $S_{i,\ell}$ with $0 \leq i \leq k$ be its self-intersection partition. Let $d > 1 - \alpha$. We say that $(Y_\ell)$ satisfies the $d$-small self-intersection condition if, for every $1 \leq i \leq k - 1$,

$$\text{dens}(B_\ell^{(k)}) \cdot S_{i,\ell} < \alpha - (1 - d) \times \frac{i}{2d}.$$

Theorem 3.6 (Multidimensional intersection formula, [Tsa21, Theorem 3.6]). Let $(R_\ell)$ be a sequence of permutation invariant random subsets of $(B_\ell)$ of density $\alpha$. Let $(Y_\ell)$ be a sequence of fixed subsets of $(B_\ell^{(k)})$ of density $\alpha > 1 - d$. If $(Y_\ell)$ satisfies the $d$-small self-intersection condition, then the sequence of random subsets $(Y_\ell \cap R_\ell^{(k)})$ is densable with density $\alpha + d - 1$.

In particular, a.a.s. the random subset $Y_\ell \cap R_\ell^{(k)}$ of $B_\ell^{(k)}$ is not empty.

3.3 Proof of Theorem 1.5 (ii)

Let $(Y_\ell)$ be a sequence of 2-complexes of the same geometric form $(Y, \lambda)$ with $k$ faces. In the following, we denote $Y_\ell$ as the set of pairwise distinct relators in $B_\ell$ that fills $Y_\ell$, which is a subset of $B_\ell^{(k)}$.

Let $(G_\ell(m, d))$ be a sequence of random groups at density $d$, defined by $G_\ell(m, d) = (X_m | R_\ell)$ where $(R_\ell)$ is a sequence of random subsets with density $d$. The intersection $Y_\ell \cap R_\ell^{(k)}$ is hence the set of $k$-tuples of pairwise distinct relators in $R_\ell$ that fills $Y_\ell$. 

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We want to prove that this intersection is not empty, so that $Y_\ell$ is fillable by $G_\ell(m,d)$. According to Theorem 3.6, it remains to prove that if $\text{dens}_c Y > 1 - d$, then the sequence $(Y_\ell)$ is densable and satisfies the $d$-small self intersection condition.

We will prove in Lemma 3.8 that $(Y_\ell)$ is densable with density exactly $\text{dens}(Y)$, and in Lemma 3.9 that it satisfies the $d$-small self intersection condition.

**Lemma 3.7.** Let $\overline{Y}_\ell$ be the set of $k$-tuples of relators in $B_\ell$ that fills $Y_\ell$, not necessarily pairwise distinct. If $Y_\ell$ is fillable by $B_\ell$, then

$$\text{dens}_{(B_\ell^k)}(\overline{Y}_\ell) = \text{dens}(Y).$$

**Proof.** We shall estimate the number $|\overline{Y}_\ell|$ by counting the number of labelings on edges of $Y_\ell$ that produce van Kampen 2-complexes with respect to all possible relators $B_\ell$.

We start by filling edges in the neighborhoods of vertices that are originally vertices of the geometric form $Y$ (before dividing). Consider the set of oriented edges of $Y_\ell$ starting at some vertex that is originally a vertex of $Y$ before dividing. A vertex labeling is a labeling on these edges by $X_m^n$ that does not produce any reducible pair of edges on face boundaries: for every pair of different edges $e_1, e_2$ starting at the same vertex, if they are labeled by the same generator $x \in X_m^n$, then the path $e_1, e_2$ is not cyclically part of any face boundary loop. Since the 2-complex $Y_\ell$ is fillable, the set of vertex labelings is not empty. Denote $C \geq 1$ as the number of vertex labelings of $Y_\ell$.

As $m \geq 2$ and $|\lambda_\ell\ell| \geq 3$ for $\ell$ large enough, if there exists a vertex labeling, then the other edges of $Y_\ell$ can be completed as a van Kampen 2-complex of $B_\ell$, and the number $C$ depends only on the geometric form $Y$.

To label the remaining $|\lambda\ell| - 2$ edges on the arc divided from the edge $e \in \text{Edge}(Y)$, there are $2m - 1$ choices for the first $|\lambda\ell| - 3$ edges, and $2m - 2$ or $2m - 1$ choices for the last edge. So

$$C \prod_{e \in \text{Edge}(Y)} (2m - 1)^{|\lambda_\ell\ell| - 3}(2m - 2) \leq |\overline{Y}_\ell| \leq C \prod_{e \in \text{Edge}(Y)} (2m - 1)^{|\lambda_\ell\ell| - 2}.$$

Recall that $k = |Y| = |Y_\ell|$ and that $|B_\ell^k| = (2m - 1)^{k+\omega(\ell)}$. We have

$$\text{dens}_{(B_\ell^k)}(\overline{Y}_\ell) = \frac{\sum_{e \in \text{Edge}(Y)} \lambda_e}{|Y|} = \text{dens}(Y).$$

□

**Lemma 3.8.** If $\text{dens}_c Y > 1/2$ and $Y_\ell$ is fillable by $B_\ell$, then $(Y_\ell)$ is densable in $(B_\ell^k)$ and

$$\text{dens}_{(B_\ell^k)}(Y_\ell) = \text{dens}(Y).$$

**Proof.** Suppose that $|Y| \geq 2$. The case $|Y| = 1$ is trivial. Let $Z$ be a sub-2-complex of $Y$ with exactly two faces $f_1, f_2$. As $\text{dens}(Z) \geq 1$, by Definition 3.2, we have

$$\sum_{e \in \text{Edge}(Z)} \lambda_e > \frac{1}{2}|Z| = 1 \geq \partial f_1.$$
Let $\mathcal{Y}_\ell$ be the set of fillings of $Y_\ell$ by $B_\ell$ such that the two faces of $Z$ are filled by the same relator. By the same arguments of the previous lemma,

$$|\mathcal{Y}_\ell| \leq C(2m - 1)^{|\partial f_1|} \prod_{e \in \text{Edge}(Y) \setminus \text{Edge}(Z)} (2m - 1)^{|\lambda_e|} - 2,$$

so

$$\text{dens}_{(B_\ell^k)(\mathcal{Y}_\ell)} \leq \frac{1}{|\mathcal{Y}_\ell|} \left[ \sum_{e \in \text{Edge}(Y)} \lambda_e + \left( |\partial f_1| - \sum_{e \in \text{Edge}(Z)} \lambda_e \right) \right]$$

$$< \frac{\sum_{e \in \text{Edge}(Y)} \lambda_e}{|\mathcal{Y}_\ell|}$$

$$= \text{dens}_Y = \text{dens}_{(B_\ell^k)(\mathcal{Y}_\ell)}.$$

Knowing that

$$Y_\ell = \mathcal{Y}_\ell \setminus \bigcup_{Z < Y, |Z| = 2} \mathcal{Y}_\ell^Z,$$

we have

$$|\mathcal{Y}_\ell| - \sum_{Z < Y, |Z| = 2} |\mathcal{Y}_\ell^Z| \leq |\mathcal{Y}_\ell| \leq |\mathcal{Y}_\ell|.$$
is determined. There are at most \( i! \) choices for ordering these \( i \) relators. To fill the remaining \( k - i \) relators in \((r'_1, \ldots, r'_k)\), by the same arguments of Lemma 3.7, we get

\[
|S_\ell(Z, W)| \leq |Y_\ell| \times i! \times C \prod_{e \in \text{Edge}(Y) \setminus \text{Edge}(W)} (2m - 1)^{|\lambda_e|^{-2}}.
\]

Recall that the density of \( Y \) is defined by \( \frac{1}{|Y|} \left( \sum_{e \in \text{Edge}(Y)} \lambda_e \right) \), and that \( \text{dens}_c Y > 1 - d \) by Definition 3.2. Together with the hypothesis \( \text{dens}_c Y > 1 - d \), we have

\[
\text{dens}_{\left( (B^i)_e \right)^2} (S_\ell(Z, W)) \leq \frac{1}{2k} \left( \sum_{e \in \text{Edge}(Y)} \lambda_e + \sum_{e \in \text{Edge}(Y) \setminus \text{Edge}(W)} \lambda_e \right)
\]

\[
= \frac{1}{2k} \left( 2 \sum_{e \in \text{Edge}(Y)} \lambda_e - \sum_{e \in \text{Edge}(W)} \lambda_e \right)
\]

\[
= \text{dens}_c Y - \frac{i}{2k} \text{dens}_c W
\]

\[
< \text{dens}_c Y - \frac{i}{2k} (1 - d).
\]

Note that

\[
S_{i,\ell} = \bigcup_{Z \prec Y, W \prec Y, |Z| = |W| = i} S_\ell(Z, W).
\]

It is a union of \( \left( \binom{k}{2} \right)^2 \) subsets of densities strictly smaller than \( \text{dens}_c Y - \frac{i}{2k} (1 - d) \). According to [Tsa21, Proposition 2.7], we have

\[
\text{dens}_{\left( (B^i)_e \right)^2} (S_{i,\ell}) < \text{dens}_c Y - \frac{i}{2k} (1 - d).
\]

This completes the proof of Theorem 1.5.

### 4 Phase transitions for small cancellation conditions

Let us recall small cancellation notions in [LS77, p.240]. A **piece** with respect to a set of relators is a cyclic sub-word that appears at least twice. A group presentation satisfies the \( C'(\lambda) \) small cancellation condition for some \( 0 < \lambda < 1 \) if the length of a piece is at most \( \lambda \) times the length of any relator that it appears. It satisfies the \( C(p) \) small cancellation condition for some integer \( p \geq 2 \) if no relator is a product of fewer than \( p \) pieces.

**The \( C'(\lambda) \) condition.** Let \( (G_\ell(m, d)) \) be a sequence of random groups at density \( d \). It is known that there is a phase transition at density \( d = \lambda/2 \) for the \( C'(\lambda) \) condition (see [Gro93, p.274], [BNW20, Theorem 2.1] and [Tsa21, Theorem 1.4]). We give here a much simpler proof using Theorem 1.5.
**Proposition 4.1.** Let $0 < \lambda < 1$. Let $(G_t(m, d))$ be a sequence of random groups at density $d$. There is a phase transition at density $d = \lambda/2$:

(i) If $d < \lambda/2$, then a.a.s. $G_t(m, d)$ satisfies $C'(\lambda)$.

(ii) If $d > \lambda/2$, then a.a.s. $G_t(m, d)$ does not satisfy $C'(\lambda)$.

**Proof:**

(i) Let us prove by contradiction. Suppose that a.a.s. $G_t(m, d)$ does not satisfy $C'(\lambda)$. That is to say, a.a.s. there exists a piece $w$ that appears in relators $r_1, r_2$ with $|w| > \lambda |r_1|$. It is possible that $r_1 = r_2$, but the piece should be at different positions.

Construct a van Kampen diagram $D$ by gluing two combinatorial disks with one face, labeled respectively by $r_1$ and $r_2$, along with the paths where the piece $w$ appears (Figure 6). As $r_1 \neq r_2$ or $r_1 = r_2$ but the piece appears at different positions, we obtain a reduced van Kampen diagram. The diagram satisfies $|D^{(1)}| = |r_1| + |r_2| + |w| < \ell + \ell + \lambda \ell < (1 - \lambda/2) |D| \ell$, which contradicts Theorem 1.4.

![Figure 6: A van Kampen 2-complex constructed from a $C'(\lambda)$ group.](image)

(ii) Consider a geometric form $Y$ with two faces sharing a common edge of length $\lambda$, the other two edges are of length $1 - \lambda$ (Figure 7). We have $\text{dens } Y = \frac{2(1-\lambda) + \lambda}{2} > 1 - d$, and every sub 2-complex with one face is with density $1 > 1 - d$. So $\text{dens}_c Y > 1 - d$.

![Figure 7: The geometric form for the $C''(\lambda)$ condition.](image)

Let $(Y_t)$ be a sequence of 2-complexes of the geometric form $Y$. By Theorem 1.5, a.a.s. $Y_t$ is fillable by $G_t(m, d)$, hence a.a.s. $G_t(m, d)$ does not satisfy $C''(\lambda)$. 

\[ \square \]
The $C(p)$ condition. We shall prove by Theorem 1.5 that for random groups with density, there is a phase transition at density $1/(p+1)$ for the $C(p)$ condition.

Proposition 4.2. Let $p \geq 2$ be an integer. Let $(G_{\ell}(m, d))$ be a sequence of random groups at density $d$. There is a phase transition at density $d = 1/(p+1)$:

(i) If $d < 1/(p+1)$, then a.a.s. $G_{\ell}(m, d)$ satisfies $C(p)$.

(ii) If $d > 1/(p+1)$, then a.a.s. $G_{\ell}(m, d)$ does not satisfy $C(p)$.

Proof.

(i) Let us prove by contradiction. Suppose that a.a.s. $G_{\ell}(m, d)$ does not satisfy $C(p)$. That is to say, a.a.s. there exists a reduced van Kampen diagram $D$ with $(p+1)$ faces, one face is placed in the center, attached by the other $p$ faces on the whole boundary, and there is no other attachments (Figure 8).

![Figure 8: A van Kampen 2-complex constructed from a $C(p)$ group.](image)

We have $|D| = p+1$ and $|D^{(1)}| \leq p\ell$. Let $\varepsilon = \left(\frac{1}{p+1} - d\right)/2$, we have $|D^{(1)}| \geq (1 - d - \varepsilon)|D|\ell$, which contradicts Theorem 1.4.

(ii) Consider a geometric form with $p+1$ faces, one of the faces is placed in the center, having $p$ edges of length $1/p$, such that every edge is attached by another face with two edges of lengths $1/p$ and $1 - 1/p$. There are no other attachments (Figure 9).

The density of $Y$ is $rac{\ell + \ell(1-1/p)}{p+1} = \frac{p/(p+1)}{1-d} > 1 - d$. If $Z$ is a sub-2-complex of $Y$ not containing the center face, then $\text{dens } Z = 1$. If $Z$ contains the center face and $i \leq p$ other faces, then $\text{dens } Z = \frac{1+i(1-1/p)}{i+1} > 1 - d$. So $\text{dens }, Y > 1 - d$.

Let $(Y_{\ell})$ be a sequence of 2-complexes of the geometric form $Y$. By Theorem 1.5, a.a.s. $Y_{\ell}$ is fillable by $G_{\ell}(m, d)$, hence a.a.s. $G_{\ell}(m, d)$ does not satisfy $C(p)$.

The $B(2p)$ condition. The same argument holds for the $B(2p)$ condition, introduced in [OW11, Definition 1.7] by Y. Ollivier and D. Wise: half of a relator can not be the product of fewer than $p$ pieces. One can construct a geometric form with $p$ faces, one of the faces is in the center, with half of its boundary attached by the other $p$ faces, each with length $1/p$ (Figure 10). Its critical density is $\frac{p+1}{2p+2}$, so a phase transition occurs at density $d = \frac{1}{2p+2}$.
Proposition 4.3. Let $p \geq 1$ be an integer. Let $(G_\ell(m, d))$ be a sequence of random groups at density $d$. There is a phase transition at density $d = 1/(2p + 2)$:

(i) If $d < 1/(2p + 2)$, then a.a.s. $G_\ell(m, d)$ satisfies $B(2p)$.

(ii) If $d > 1/(2p + 2)$, then a.a.s. $G_\ell(m, d)$ does not satisfy $B(2p)$. 

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