Extra symmetries in the effective theory of heavy quarks

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Abstract: Extra symmetries are shown to exist in the effective theory of heavy quarks when both quarks and anti-quarks with the same velocity are included. These symmetries mix the quark with the anti-quark sector and they resemble axial-type of symmetries. Together with the known flavor and spin symmetries they form a $u(4)$ algebra when a single flavor is considered. It is shown that the full $U(4)$ set of symmetries breaks spontaneously down to $U(2) \otimes U(2)$. The Goldstone modes corresponding to the spontaneously broken currents are identified. Finally, the precise connection of this theory with the fundamental QCD is derived and it is investigated under some approximations. Some physical processes where these extra symmetries may be relevant are pointed out.

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1. Introduction

The physical properties of hadrons with a single heavy quark \( Q \) \((m_Q \gg \Lambda_{QCD})\) are largely independent of the precise value of the heavy quark mass \( m_Q \) \([1,2]\). The heavy quark sector of the full QCD lagrangian can be approximated by the so-called Heavy Quark Effective Theory (HQET), in which the trivial leading dependence on the heavy quark mass is removed \([3-6]\) (see \([7]\) for a review). The most striking feature of the HQET lagrangian is that it enjoys a number of symmetries which are absent in the original QCD lagrangian. Moreover, finite mass corrections can be incorporated systematically \([8,9]\).

The flavor and spin symmetries have been known since the early papers \([2,5]\) and extensively used in phenomenological applications \([9-11]\). These symmetries relate properties of hadrons containing a single heavy quark of different spin and flavor, which move with the same velocity with respect to a given reference frame.

Since the HQET lagrangian has a natural separation in quark and anti-quark sectors \([5]\), the above symmetries are also realized separately in both sectors. Consequently, one can restrict oneself to either sector, as it has been done in most of the applications studied by now. This is physically reasonable since the HQET is a low energy effective theory which is not able to account for heavy quark pair production. (That would require an infinite momentum transfer in the HQET framework.) While, however, the lagrangian does not describe quark anti-quark pair production it may well describe quarks and anti-quarks away from the production point. Indeed, phenomenological applications of the HQET in systems involving a heavy quark and a heavy anti-quark, have recently been studied in the context of \( D - \bar{D} \) and \( B - \bar{B} \) mixing \([12]\).

In the present work we carry out a theoretical study of the HQET with both heavy quark and heavy anti-quark fields included in it. We are concerned with the case of a single flavor, although the case of \( N_f \) flavors is briefly discussed in the last section.

In the approach of HQET the heavy quark field is characterized by a distinct fixed velocity and the Hilbert space decomposes into superselection sectors labeled by the velocity of the heavy quark. In this paper we consider both quark and anti-quark in one of these superselection sectors labeled by a fixed velocity \( v \).
In a previous work [13] we showed that several symmetries, in particular the flavor and spin symmetry, are anomaly free in this approximation. In the same paper we pointed out that an extra symmetry, which we called $\gamma_5$-symmetry, exists in the HQET when heavy quark and anti-quark with the same velocity are included in it. This is an unexpected symmetry because it mixes quark and anti-quark fields while particles and antiparticles are not supposed to know about each other in the HQET. In fact a larger set of symmetries of the same nature exist in this theory in that case, that is, when quarks and anti-quarks are considered with the same velocity. We devote this work to its discussion.

In section 2 we show that the HQET describing quarks and anti-quarks with the same velocity has a $U(4)$ symmetry if only one flavor is considered. In section 3 we analyze the realization of this symmetry and show that it is spontaneously broken down to $U(2) \otimes U(2)$, i.e., to the flavor $\dagger$ and spin symmetries. In section 4 we discuss the implications of the Goldstone theorem and identify the Goldstone modes. In section 5 we make the connection of this theory with the fundamental QCD using the generating functional formalism. We conclude with a brief discussion on possible phenomenological applications of these symmetries in section 5.

2. Extra symmetries in the HQET

Consider the HQET describing heavy quarks and anti-quarks with the same velocity $v_\mu$ ($v_\mu v^\mu = 1$) [5]. It is given by

$$L_v = i\bar{h}_v \not{v} \mu D^\mu h_v = i\bar{h}_v^+ v \cdot D h_v^+ - i\bar{h}_v^- v \cdot D h_v^-,$$

where $h_v = h_v^+ + h_v^-$ and $h_v^\pm = \frac{1+\not{v}}{2} h_v$. $h_v^+$ contains creation operators of quarks with small momentum about $mv_\mu$ and $h_v^-$ contains annihilation operators of anti-quarks again with small momentum about $mv_\mu$. $D_\mu$ is the covariant derivative containing the gluon field.

It is well-known that this theory is symmetric under rotations in the spin and flavor space [2, 5, 7]. That is, the action is invariant under the following transformations:

$$h_v^\pm \to e^{i\epsilon^i S_i^\pm} h_v^\pm \quad \text{and} \quad \bar{h}_v^\pm \to \bar{h}_v^\pm e^{-i\epsilon^i S_i^\pm},$$

$\dagger$ Since we restrict ourselves to a single flavor, flavor symmetry is to be understood as the $U(1) \otimes U(1) \subset U(2) \otimes U(2)$ symmetry corresponding to the separate conservation of number particles and antiparticles all over the paper.
where $S^\pm_i = i\epsilon_{ijk}[\varphi_j, \varphi_k](1 \pm \psi)/2$, with $e^\mu_j, j = 1, 2, 3$ being an orthonormal set of space like vectors orthogonal to $v_\mu$, and

$$h^\pm_v \rightarrow e^{i\vartheta \pm}h^\pm_v \quad \text{and} \quad \bar{h}^\pm_v \rightarrow \bar{h}^\pm_v e^{-i\vartheta \pm}.$$  \hspace{1cm} (3)

$\epsilon^\pm_i$ and $\vartheta \pm$ are arbitrary real numbers corresponding to the parameters of the transformations.

It is important for what follows to emphasize that the above symmetries are realized separately for the quark and anti-quark sectors of the theory. Namely, for every velocity $v$ there exists a $U(2)$ symmetry for the quark sector of the lagrangian and a $U(2)$ for the anti-quark sector, being the total symmetry $U(2) \otimes U(2)$. These symmetries are enlarged into $U(2N_f) \otimes U(2N_f)$ when $N_f$ flavors are included in the theory [5]. Note also that in terms of the field $h_v$ the last transformations can be expressed as

$$h_v \rightarrow e^{i\bar{\psi} \pm}h_v; \quad \bar{h}_v \rightarrow \bar{h}_v e^{-i\bar{\psi} \pm}$$ \hspace{1cm} (4)

for the flavor symmetry and

$$h_v \rightarrow e^{iS_i \pm}h_v; \quad \bar{h}_v \rightarrow \bar{h}_v e^{-iS_i \pm}$$ \hspace{1cm} (5)

for the spin symmetry, where now the spin operator is $S_i = i\epsilon_{ijk}[\varphi_j, \varphi_k]$. There are two generators $(i, i\bar{\psi})$ for the flavor symmetry and six $(iS_i, iS_i \bar{\psi})$ for the spin symmetry. Note that the projection operator $(1 \pm \psi)/2$, which was originally responsible for the separation of the lagrangian in + and − component terms, is hidden in the generators when the symmetries are expressed in this basis.

Apart from the above symmetries the lagrangian (1) is invariant under the following set of transformations:

$$h_v \rightarrow e^{i\gamma_5 \epsilon}h_v; \quad \bar{h}_v \rightarrow \bar{h}_v e^{i\gamma_5 \epsilon},$$ \hspace{1cm} (6)

$$h_v \rightarrow e^{\gamma_5 \bar{\psi} \epsilon}h_v; \quad \bar{h}_v \rightarrow \bar{h}_v e^{\gamma_5 \bar{\psi} \epsilon},$$ \hspace{1cm} (7)

$$h_v \rightarrow e^{i\bar{\psi} \epsilon}h_v; \quad \bar{h}_v \rightarrow \bar{h}_v e^{i\bar{\psi} \epsilon},$$ \hspace{1cm} (8)

$$h_v \rightarrow e^{i\epsilon \bar{\psi} \epsilon}h_v; \quad \bar{h}_v \rightarrow \bar{h}_v e^{i\epsilon \bar{\psi} \epsilon}.$$ \hspace{1cm} (9)
There are two observations to be made about these symmetries. Firstly, they all mix quark and anti-quark sectors. Indeed, in terms of $h^\pm_v$ fields these last transformations can be written as

$$h^\pm_v \rightarrow \cos \epsilon h^\pm_v + i \gamma_5 \sin \epsilon h^\mp_v,$$

$$h^\pm_v \rightarrow \cos \epsilon h^\pm_v + \gamma_5 \phi \sin \epsilon h^\mp_v,$$

$$h^\pm_v \rightarrow \cos |\epsilon| h^\pm_v + \frac{\phi_i e^i}{|\epsilon|} \sin |\epsilon| h^\mp_v,$$

$$h^\pm_v \rightarrow \cos |\epsilon| h^\pm_v + i \frac{\phi_i e^i}{|\epsilon|} \sin |\epsilon| h^\mp_v$$

(10) correspondingly, where $|\epsilon| := \sqrt{\epsilon^i \epsilon^i}$. Secondly, they appear in sets, as in the case of the flavor (3) and spin (2) symmetries. There are also eight generators for this new set of symmetries, given by $(i\gamma_5, \gamma_5 / v)$ and $(\phi_i, i\phi_i \phi)$. The fact that these symmetries transform quark fields into anti-quark fields might suggest at first sight that they must be implemented anti-unitarily at the level of the Hilbert space. However, since these symmetries are continuous this possibility is ruled out.

In what follows we prove that the transformations (6) - (9) together with (4) - (5) can be accommodated in a $u(4)$ algebra. The explicit commutations relations of all symmetry generators are given by

$$[i, i\phi] = [i, i\gamma_5] = [i, \gamma_5 \phi] = [i, iS_i] = [i, iS_i \phi] = [i, \phi_i] = [i, i\phi_i \phi] = 0,$$

$$[i\phi, i\gamma_5] = 2\gamma_5 \phi, \quad [i\phi, \gamma_5 \phi] = -2i\gamma_5, \quad [i\phi, iS_i] = 0, \quad [i\phi, iS_i \phi] = 0,$$

$$[i\phi, \phi_i] = -2i\phi_i \phi, \quad [i\phi, i\phi_i \phi] = 2\phi_i$$

$$[i\gamma_5, \gamma_5 \phi] = 2i\phi, \quad [i\gamma_5, iS_i] = 0, \quad [i\gamma_5, iS_i \phi] = 8\phi_i,$$

$$[i\gamma_5, \phi_i] = -\frac{i}{2} S_i \phi, \quad [i\gamma_5, i\phi_i \phi] = 0$$

$$[\gamma_5 \phi, iS_i] = 0, \quad [\gamma_5 \phi, iS_i \phi] = -8i\phi_i \phi, \quad [\gamma_5 \phi, \phi_i] = 0, \quad [\gamma_5 \phi, i\phi_i \phi] = \frac{i}{2} S_i \phi$$

$$[iS_i, iS_j] = -8ie^{ijk} S_k, \quad [iS_i, iS_j \phi] = -8ie^{ijk} S_k \phi, \quad [iS_i, \phi_j] = -8e^{ijk} \phi_k,$$

$$[iS_i, i\phi_j \phi] = -8ie^{ijk} \phi_k \phi$$

$$[iS_i \phi, iS_j \phi] = -8ie^{ijk} S_k, \quad [iS_i \phi, \phi_j] = 8i\delta_{ij}\gamma_5, \quad [iS_i \phi, i\phi_j \phi] = -8\delta_{ij}\gamma_5 \phi.$$
\[
[\phi_i, \phi_j] = -\frac{i}{2} \epsilon^{ijk} S_k, \quad [\phi_i, i\phi_j \gamma^k] = -2i \delta_{ij} \gamma^k
\]

\[
[i\phi_i \gamma^k, i\phi_j \gamma^k] = -\frac{i}{2} \epsilon^{ijk} S_k
\]  

(14)

In order to prove that the last commutation relations obey the \(u(4)\) algebra, it is more convenient to go to the rest frame. We choose the following basis:

\(v^\mu = (1, 0, 0, 0),\ e_1^\mu = (0, 1, 0, 0),\ e_2^\mu = (0, 0, 1, 0)\) and \(e_3^\mu = (0, 0, 0, 1)\).

(15)

Then the generators are reduced to the following set of 4x4 matrices:

\(i,\ i\gamma_0,\ -\epsilon^{ijk}[\gamma^j, \gamma^k],\ -\epsilon^{ijk}[\gamma^j, \gamma^k]\gamma_0 = -4i\gamma_5 \gamma^i,\ i\gamma_5,\ \gamma_5\gamma_0,\ \gamma^i,\ i\gamma^i\gamma_0.\)  

(16)

These are the 16 independent Dirac matrices, which are antihermitian in the representation where \(\gamma_0\) and \(\gamma_5\) are hermitian, and the \(\gamma^i\) antihermitian. This set of matrices define the \(u(4)\) compact algebra. With this we conclude that our generators satisfy the \(u(4)\) algebra in the rest frame. The fact that the algebra remains the \(u(4)\) in any other frame of reference follows from the commutation relations. The structure constants are preserved in any reference frame. This completes the proof. (A more physical argument on the fact that our theory indeed has a \(u(4)\) algebra will be given later.)

Note at this point that not all transformations (6)-(9) are unitary in an arbitrary reference frame. Only in the rest frame the generators of the \(u(4)\) algebra have definite hermiticity properties. This may seem to be in contradiction with the fact that \(u(4)\) is a compact algebra. However, what is guaranteed by general theorems is that finite dimensional representations of compact Lie algebras (groups) are equivalent to unitary representations, but not necessarily unitary themselves. Therefore, the above generators in an arbitrary frame constitute a non-unitary representation of \(u(4)\), which should certainly be equivalent to a unitary one. (In fact, it is not difficult to explicitly construct such a unitary representation.) Moreover, it follows from (10)-(13) that \(e^{2\pi X} = 1\), for all generators \(X\) of the algebra. This proves that the group obtained by exponentiating the algebra is compact, and hence it must be identified with \(U(4)\).

A straightforward consequence of these extra symmetries in this theory, is the following: In principle one could include a residual mass term of the form \(\delta m \bar{h}_v h_v\) in the effective lagrangian since spin and flavor symmetry allow it [14]. Such a mass term, however,
would break explicitly the extra symmetries that we pointed out above. Enforcing these symmetries forbids such a residual mass term and, more important, guarantees that it will not be induced by radiative corrections.

Next we give the conserved currents for this new set of symmetries. They are given by

\[ J^\mu_5 = \bar{h}_v \gamma^\mu \gamma_5 h_v \quad \text{and} \quad J^\mu_{\gamma_5} = i \bar{h}_v \gamma_5 h_v \]  

(17)

and

\[ J^\mu_{\gamma_5 i} = i \bar{h}_v \gamma^\mu \gamma_i h_v \quad \text{and} \quad J^\mu_{\gamma_5 i} = \bar{h}_v \gamma^\mu \gamma_i h_v \]  

(18)

for the \( \gamma_5 \) and \( \gamma_i \) sets of symmetries correspondingly.

It is not difficult to prove, following ref. [13], that none of these currents has an anomaly. Notice that, due to the presence of the operators \( \gamma_5 \) and \( \gamma_i \) which anticommute with \( \gamma_5 \), these currents mix quark and anti-quark fields in contrast with the currents associated to the flavor and spin symmetries:

\[ J^\mu = \bar{h}_v \gamma^\mu h_v \quad \text{and} \quad J^\mu_{\gamma_5} = \bar{h}_v \gamma^\mu h_v \]  

(19)

\[ J^\mu_{\gamma_5 i} = \bar{h}_v \gamma^\mu \gamma_i h_v \quad \text{and} \quad J^\mu_{\gamma_5 i} = \bar{h}_v \gamma^\mu \gamma_i h_v , \]  

(20)

which contain either quark or anti-quark fields. Since all the currents are multiplied by \( \gamma^\mu \), it is convenient to define \( j \) in such a way that \( j \gamma^\mu := J^\mu \) for all the currents (17)-(20). In the rest of the paper we will use \( j \) when referring to the currents.

In nature the flavor and spin symmetries are realized `a la Wigner-Weyl : \( B \) and \( B^* \) as well as \( D \) and \( D^* \) can be accommodated in the dimension 4 representation of spin \( SU(2) \) whereas \( B \) and \( D \) can be accommodated in a dimension 2 representation of flavor \( U(2) \). The \( U(1) \) factors just keep track of the separate quark and anti-quark number conservation. No obvious \( U(4) \) multiplet is observed which suggests that the extra generators must be spontaneously broken in nature. In the next section, we argue that the extra symmetries are actually broken spontaneously.

3. Realization of the \( U(4) \) symmetry in the vacuum of the HQET

Firstly, we argue that the \( \gamma_5 \)-symmetry, given by (6), is spontaneously broken using phenomenological information.
Consider a meson $M$ made up from a heavy quark $Q$ and a light anti-quark $\bar{q}$. The meson decay constant $f_M$ is defined by

$$
<0|\bar{q}\gamma^{\mu}\gamma_5Q|M \bar{q}Q> = if_M p^\mu
$$

and is known from phenomenology to be different than zero. Next consider the following matrix element in the heavy quark approximation:

$$
<0|\bar{q}\gamma^{\mu}h_{v^-}|a_v^\dagger d^\dagger>
$$

where by $a_v^\dagger, d^\dagger$ we denote the creation operators of the heavy and light quarks correspondingly. Recall that $h_{v^-}$ is the field which contains the creation operator for the heavy anti-quark. $|0>$ denotes the vacuum of the full QCD. The matrix element (22) is zero, since the operator $h_{v^-}$ annihilates the vacuum on the left. Performing an infinitesimal $\gamma_5$-transformation

$$
\delta h_{v^\pm} = i\gamma_5 \epsilon h_{v^\mp}
$$

on the matrix element (22) we obtain

$$
<0|\bar{q}\gamma^{\mu}h_{v^-} |\delta a_v^\dagger d^\dagger> + \epsilon <0|\bar{q}\gamma^{\mu}i\gamma_5h_{v^+} |a_v^\dagger d^\dagger> + <\delta 0|\bar{q}\gamma^{\mu}h_{v^-} |a_v^\dagger d^\dagger> = 0
$$

where $\delta$ means variation with respect to $\gamma_5$. The first term of the last expression is obviously zero, while the second term is equal to the matrix element (21) in the HQET (up to anomalous dimensions which are not important for the argument). Since the meson decay constant $f_M$ goes like $1/\sqrt{m}$ for $m$ large [2,7], $f_M p^\mu$, that is the right hand side of (21), is different than zero for $m$ large. Then in order for the last equation to make sense the variation of the vacuum must be different from zero. With this we conclude that the vacuum of the full QCD must not be invariant under the $\gamma_5$-symmetry.

The question now is whether the breaking of the $\gamma_5$ generator is due to non-perturbative QCD effects, as it is the case for the chiral symmetry, or it can be understood in simpler terms. In order to disentangle this issue let us analyze the transformation properties of the vacuum of the HQET, when QCD is switched off, under the new generators (6) - (9).

Let us begin by analysing (1) in first quantization. The equations of motion for $h_{v^\pm}$ read

$$
i\phi v \cdot \partial h_{v^-} = 0 \quad \leftrightarrow \quad iv \cdot \partial h_{v^\pm} = 0
$$

where $\phi$ is the field for the heavy quark.
If we restrict ourselves to fields with well-defined energy, i.e. \( h^\pm_v(\vec{x}, t) = e^{ik_0t}h^\pm_v(\vec{x}, 0) \) we obtain the following eigenvalue equation

\[
\hat{H}h^\pm_v = k_0h^\pm_v ; \quad \hat{H} := i\frac{v^i\partial_i}{v^0}
\]  

(26)

where \( i\frac{v^i\partial_i}{v^0} \) is the first quantized Hamiltonian operator, which is diagonal in the Dirac matrix space. The eigenfunctions of \( \hat{H} \) are plain waves of momentum \( \vec{k} \) leading to the dispersion relation

\[
k^0 = \frac{\vec{v} \cdot \vec{k}}{v^0}
\]  

(27)

Consider next the infinitesimal \( \gamma_5 \)-transformation (23). Since the \( \gamma_5 \) matrix commutes with the Hamiltonian operator \( \hat{H} \), it follows that if \( h^+_v \) is an eigenstate of \( \hat{H} \) with energy \( k_0 \) the transformed field \( i\gamma_5h^-_v \) is also an eigenstate of \( \hat{H} \) with the same energy \( k_0 \). The same statement is true for any of the generators of the transformations (6)-(9).

In the original Dirac theory the transformation (23) amounts to taking a state of positive energy to a state of energy negative. Indeed, the equations (\( (1 \mp \not{\!p})/2 \))\( h^\pm_v = 0 \), which define the fields \( h^\pm_v \) in this approximation, correspond to the positive and negative energy solution respectively in the original theory. In the massive Dirac theory, then, the present transformation cannot be a symmetry, since in this theory the positive energy states are separated from the negative energy states by an amount of energy \( E \geq 2m \). Naïvely, therefore, one would expect that in the infinite mass limit this mass gap becomes infinitely large and no symmetry which would relate the positive with the negative energy spectrum would exist. In the HQET, however, redefining the field as \( \Psi(x) = e^{-imv^0x}h_v(x) \) we have removed the mass gap from the spectrum and instead of infinitely separating the positive and negative sectors we have brought them close together. Indeed, the extra time dependence of the new field is \( e^{imv^0t} \) and the field redefinition amounts to redefining the zero level of the energy spectrum by an amount equal to \( mv^0 \). The correct picture of how the Dirac vacuum is modified in this approximation is given in fig. (1).

It is worth noticing here that since the Hamiltonian operator (26) is diagonal in the Dirac matrix space, we can multiply a solution \( h^\pm_v \) of the eigenvalue equation by any 4x4 matrix and it still remains a solution with the same energy and momentum. In other words, the transformation of the 4x4 matrices is a symmetry of the equations of motion, and hence of the eigenvalue equation (26). If we associate solutions of the eigenvalue equation (26) to
one-particle states, then the 4x4 matrix transformations conserve the energy of the latter. If we further constrain these transformations to preserve the norm of the first quantized Hilbert space we obtain the $U(4)$ group. This is the physical argument promised before. The invariance of the action (1) restricts automatically these 4x4 matrices to those of the non-unitary representation of $U(4)$, as explained in the previous section.

The Hamiltonian of the system can be read off (1)

$$H = - \int d^3\bar{x}(i\bar{h}_v^+ v^i \partial_i h_v^+ - i\bar{h}_v^- v^i \partial_i h_v^-)$$

(28)

where $\pm i\bar{h}_v^+ v^0$ are the canonical momenta of $h_v^\pm$. At the second quantized level the field variables $h_v^\pm$ are most conveniently expressed in terms of annihilation and creation operators. They are given by

$$h_v^+ (x) = \int \frac{d^3\vec{k}}{(2\pi)^3 v^0} \sum_{\sigma=1,2} u^\sigma_v a^\sigma_v (\vec{k}) e^{-i\vec{k} \cdot \vec{x}}$$

$$h_v^- (x) = \int \frac{d^3\vec{k}}{(2\pi)^3 v^0} \sum_{\sigma=1,2} v^\sigma_v b^{i\sigma}_v (\vec{k}) e^{i\vec{k} \cdot \vec{x}}$$

(29)

where $a^\sigma_v (\vec{k})$ annihilates heavy quark and $b^{i\sigma}_v (\vec{k})$ creates heavy anti-quark respectively of small momentum $\vec{k}$ about $mv^\mu$. The constant Dirac spinors $u^\sigma_v$ and $v^\sigma_v$ are taken with the following normalization:

$$\bar{u}^\sigma_v u^\sigma' = \delta_{\sigma\sigma'}, \quad \bar{v}^\sigma_v v^\sigma' = -\delta_{\sigma\sigma'}$$

$$\bar{u}^\sigma_v v^\sigma' = 0, \quad \bar{v}^\sigma_v u^\sigma' = 0.$$

(30)

Using the anticommutation relations of the fields $h_v^\pm$, given by

$$\{h_v^\pm (\vec{x}), h_v^\pm (\vec{y})\} = \pm i \frac{1 \pm \frac{\v}{v^0}}{2} \frac{1}{v^0} \delta^3 (\vec{x} - \vec{y})$$

(31)

we obtain the following anticommutation relations for the creation and annihilation operators:

$$\{a^\sigma_v (\vec{k}), a^{i\sigma'}_v (\vec{k'})\} = v^0 (2\pi)^3 \delta^3 (\vec{k} - \vec{k'}) \delta_{\sigma\sigma'},$$

and

$$\{b^\sigma_v (\vec{k}), b^{i\sigma'}_v (\vec{k'})\} = v^0 (2\pi)^3 \delta^3 (\vec{k} - \vec{k'}) \delta_{\sigma\sigma'}$$

(32)

All other anticommutation relations are zero.

Substituting now the expressions (29) into (28) we obtain for the second quantized Hamiltonian

$$H = - \int \frac{d^3\vec{k}}{(2\pi)^3 v^0} \sum_{\sigma=1,2} \frac{v^i k_i}{v^0} \left[ a^{i\sigma}_v (\vec{k}) a^\sigma_v (\vec{k}) + b^{i\sigma}_v (\vec{k}) b^\sigma_v (\vec{k}) \right].$$

(33)
The first thing to notice about this Hamiltonian is that its spectrum is unbounded from both above and below. Indeed, since the momentum fluctuation $k^i$ can take any value around zero the Hamiltonian (33) can be negative. However, the full Hamiltonian is always positive definite. The full energy of the system $E = \sqrt{m^2 + \vec{k}^2}$ takes the form $mv^0 + \frac{E^2}{2m}$ in first order in the $1/m$ expansion, where $v^i k_i$ is much smaller than the $mv^0$. In the HQET we count the energy of the states above the value $mv^0$ and the effective Hamiltonian (33) is a small correction of the full Dirac Hamiltonian. Its negative value does not disturb the positiveness requirement of the latter.

Next we build the Hilbert space starting from the vacuum of the theory defined as the state which is annihilated from both $a_{\sigma\nu}(\vec{k})$ and $b_{\sigma\nu}^\dagger(\vec{k})$. It is given by

$$a_{\nu}(\vec{k}) b_{\nu}^\dagger(\vec{k}) |0; 0> = 0. \quad (34)$$

The Hilbert space is a direct product of the Hilbert space of quarks and the Hilbert space of anti-quarks and we denote the vacuum by $|0; 0>$. Excited states are denoted by the momentum and helicities of their quarks and anti-quarks. This is a convenient notation since in this theory quarks and anti-quarks coexist and each can occupy states of either positive or negative energy (see fig. 1 and 2).

The so defined vacuum has zero energy

$$H |0; 0> = 0. \quad (35)$$

Notice, however, that this is not a state of minimum energy in the effective theory. This is only the ‘vacuum’ in the sense that it corresponds to the vacuum of the original Dirac theory. All excited states can now be constructed, as usually, by operating with the creation operators on the vacuum.

The transformation (10) in the second quantized picture takes the following form

$$a_{\nu}^\sigma(\vec{k}) \rightarrow \cos \epsilon a_{\nu}^\sigma(\vec{k}) + \sum_{\sigma'} i\bar{u}_{\nu}^\sigma \gamma_5 v_{\nu}^{\sigma'} \sin \epsilon b_{\nu}^{\dagger \sigma'}(-\vec{k})$$

$$b_{\nu}^\sigma(\vec{k}) \rightarrow \cos \epsilon b_{\nu}^\sigma(\vec{k}) + \sum_{\sigma'} i\bar{u}_{\nu}^{\sigma'} \gamma_5 v_{\nu}^{\sigma} \sin \epsilon a_{\nu}^{\dagger \sigma'}(-\vec{k}). \quad (36)$$

The expression for the creation operators are obtained by taking the hermitian conjugate in equation (36).
The charge operator associated to the symmetry transformation (6) reads (from (17))

\[ Q = \int d^3x (\bar{h}_v v^0 \gamma_5 h_v - \bar{h}_v v^0 \gamma_5 h_v^+) , \]  

which expressed in terms of creation and annihilation operators takes the form

\[ Q = \int \frac{d^3k}{(2\pi)^3 v^0} \sum_{\sigma,\sigma'} [ \alpha_{\sigma\sigma'} v_0^* (\bar{k}) b_{\sigma'} (\bar{-k}) - \beta_{\sigma\sigma'} b_0^* (\bar{-k}) a_{\sigma'} (\bar{k})] , \]  

where \( \alpha_{\sigma\sigma'} = \bar{u}_{\sigma} v_{\sigma'} \gamma_5 u_{\sigma'} \) and \( \beta_{\sigma\sigma'} = \bar{v}_{\sigma} v_{\sigma'} \gamma_5 u_{\sigma'} \). It is, then, straightforward to show that

\[ [Q, H] = 0 \]  

where we have used the anticommutation relations (32). Notice next that by acting with expression (38) on the vacuum we obtain

\[ Q|0; 0\rangle = \int \frac{d^3k}{(2\pi)^3 v^0} \sum_{\sigma,\sigma'} \alpha_{\sigma\sigma'} \bar{\alpha}_{\sigma} |\bar{k}, \sigma; -\bar{k}, \sigma' \rangle \neq 0 . \]  

This shows that the charge operator does not annihilate the vacuum. The meaning of this is that the symmetry is not realized à la Wigner-Weyl but à la Nambu-Goldstone, i.e. there is spontaneous symmetry breaking. We devote the rest of the section to analyzing this phenomenon.

The symmetry (6) is realized on the Hilbert space by the unitary operator \( U = e^{ieQ} \) (recall that \( Q \) is hermitian). The fact that \( U \) is unitary guarantees that the symmetry transformations preserve the normalization of the states even if this symmetry is spontaneously broken. We then have

\[ U|0; 0\rangle = |0; 0\rangle + ie \int \frac{d^3k}{(2\pi)^3 v^0} \sum_{\sigma,\sigma'} \alpha_{\sigma\sigma'} |\bar{k}, \sigma; -\bar{k}, \sigma' \rangle \]

\[ + \frac{e^2}{2} \int d^3k \delta^3(0) \sum_{\sigma,\sigma'} \beta_{\sigma\sigma'} \alpha_{\sigma\sigma'} |0; 0\rangle \]

\[ - \frac{e^2}{2} \int \int \frac{d^3k}{(2\pi)^3 v^0} \frac{d^3k_1}{(2\pi)^3 v^0} \sum_{\sigma,\sigma',\sigma_1,\sigma_1'} \alpha_{\sigma\sigma'} \alpha_{\sigma_1\sigma_1'} |\bar{k}, \sigma, \bar{k}_1, \sigma_1; -\bar{k}, \sigma', -\bar{k}_1, \sigma_1' \rangle + ... \]

The meaning of the last expression is that the \( \gamma_5 \)-symmetry operator acting on the vacuum gives states with so many quarks of momentum \( \bar{k} \) as many anti-quarks of momentum \( -\bar{k} \).
All these states have energy zero since a quark of momentum $\vec{k}$ adds energy equal to $v^i k_i/v^0$ in the system and an anti-quark of momentum $-\vec{k}$ adds an amount of energy equal to $-v^i k_i/v^0$. This shows that the vacuum of the HQET is infinitely degenerate and the $\gamma_5$-symmetry mixes these degenerate states between them. A picture of how all these degenerate vacua look is given in fig. 2.

This degeneracy of the vacuum is, then, the mechanism through which the breaking of the symmetry manifests itself in this formalism. The symmetry is spontaneously broken whenever a specific vacuum state is chosen. This leaves us with the question of how this result reconciles with the Goldstone theorem. The answer to this question lies in the non-relativistic nature of the theory and it is analyzed in the next section.

Notice at this point that we have provided a simple theoretical explanation in terms of the free theory (no gluon fields) of the phenomenological observation made at the beginning of this section by which the $\gamma_5$-symmetry must be spontaneously broken in nature.

The analysis above extends trivially to the rest of the transformations (7)-(9). That is, all these symmetries are spontaneously broken. On the other hand the normal ordered charges corresponding to the flavor and spin symmetries (2) and (3) do annihilate the vacuum and hence they are realized à la Wigner-Weyl.

4. Goldstone modes

Before going into the physical implications of having a symmetry spontaneously broken in the HQET, let us briefly recall what is generically known for such a situation. Whenever we have spontaneous symmetry breaking in a local quantum field theory with short range interactions, the Goldstone theorem applies. For relativistic theories, it implies that massless particles arise in the spectrum [15]. For non-relativistic theories the theorem is less restrictive. It only implies that there exist collective excitations such that their energy vanishes when their momentum does so exist. The particular way in which the energy vanishes, i.e., the dispersion relation at low momentum, depends however on every particular theory. It is not fixed to be $E = |\vec{k}|$ by Lorentz covariance as it is in relativistic theories. (See [16] for a discussion in condensed matter physics and [17] for a general discussion.)

Since the HQET is a non-relativistic theory, we should not expect any detailed information on the low momentum dispersion relation from the Goldstone theorem. A Goldstone
mode of momentum $p$ is created when the Fourier transform of the time component of the current associated to the broken generator acts on the vacuum.

In order to study the Goldstone modes in this theory, we firstly analyze the properties of the currents (17) and (18) corresponding to the broken symmetries under the unbroken generators. We have a freedom in the choice of the currents since any linear combination of them is also conserved. We choose the following combination:

$$j_{5\pm} := \bar{h}_v i \gamma_5 p \pm h_v \quad \text{and} \quad j^{j}_{5\pm} := \bar{h}_v i \gamma_5 p \pm h_v,$$

where by $p_\pm$ we denote the projection operator $(1 \pm \hat{p})/2$. This suitable combination of currents corresponding to the spontaneously broken generators can be accommodated into two dimension 4 irreducible representations of $U(2) \otimes U(2)$. Indeed, they transform as follows under the unbroken flavor and spin symmetries:

$$\delta_{\theta_\pm} j_{5+} = \pm i \theta \pm j_{5+} \quad \delta_{\theta_\pm} j_{5-} = \mp i \theta \pm j_{5-}$$

$$\delta_{\epsilon_\pm} j_{5+} = \mp \epsilon \pm 4 j_{5+} \quad \delta_{\epsilon_\pm} j_{5-} = \pm \epsilon \pm 4 j_{5-}$$

$$\delta_{\epsilon_\pm} j^{j}_{5+} = 4 \epsilon \pm \epsilon j^{j}_{5+} \quad \delta_{\epsilon_\pm} j^{j}_{5-} = 4 \epsilon \pm \epsilon j^{j}_{5-}$$

where the flavor and spin parameters of the transformations $\theta_\pm$ and $\epsilon_\pm$ are defined as in (3) and (2). The linearity of the last transformation assures that this is a proper representation of the full unbroken subgroup acting on the 8-dimensional space of the currents corresponding to the broken generators. Since, however, the $+$ and $-$ sectors of the broken generators factorize under the action of the unbroken ones, this representation is reducible. The 2 irreducible representations are now 4-dimensional. The Goldstone space is spanned by the broken currents.

Next we construct the Goldstone modes corresponding to the currents (42). For this purpose and what follows it is convenient to introduce the following notation:

$$j_{\Gamma A \pm} := \bar{h}_v \Gamma A \pm h_v, \quad \text{where} \quad \Gamma A \pm = i \gamma_5 p \pm, \quad i \phi_\pm p \pm.$$

We generically consider the Fourier transform of the time component of the current $J_{\Gamma A \pm}$ acting on the vacuum. It is given by

$$\int d^3 \bar{x} e^{i \bar{p} \cdot \bar{x}} J_{\Gamma A \pm}^0(x) |0; 0 > = \int \frac{d^3 \bar{k}}{(2\pi)^3} v^0 \sum_{\sigma, \sigma'} \alpha_{\Gamma A \pm}^{\sigma \sigma'} | \bar{k}, \sigma; -\bar{k} - \bar{p}, \sigma' >.$$
where we have substituted (29) in the last step of (45). Here $\alpha_{\sigma'}^{\sigma} = \bar{u}_{\sigma'} v^0 \Gamma_{-} A_{\sigma'} v_{\sigma}$.

It is now straightforward to identify the last expression as a state whose energy goes to zero (being the energy of a heavy quark of momentum $\vec{k}$ and of a heavy anti-quark of momentum $-\vec{k}$) whenever the spatial momentum $\vec{p}$ goes to zero. This is consistent with the Goldstone theorem for non-relativistic theories.

Notice at this point that only the $J_{-}^{0}$ components of the broken currents can create a Goldstone state when acting on the vacuum ($J_{+}^{0}$ annihilate the vacuum). Hence, only four independent Goldstone modes can be created in this theory. This result is related to the fact that the irreducible representations are 4-dimensional.

Let us next take a simple minded point of view and suppose that this picture holds even when the full QCD is switched on. This is very plausible at the level of the HQET since the gluons are blind to all symmetries (4)-(5) and (6)-(9). Since the Goldstone states contain a heavy quark and a heavy anti-quark, they may be identified with $\bar{c}c$ or $\bar{b}b$ states where the $2mv^{0}$ energy dependence corresponding to the mass of the heavy quark and anti-quark has been removed. They would correspond to the $\eta_{c}$, $J/\Psi$ or $\eta_{b}$, $\Upsilon$ particles for the $c$ and $b$ quark respectively. $J_{5-}^{0}$ would create the pseudoscalar mesons whereas $J_{5-}^{0}$ the vector mesons. This, then, would have the immediate physical consequence that $\eta_{c}$ and $J/\Psi$ must have the same mass, since under the symmetry transformations they belong to the same multiplet. (The same holds for $\eta_{b}$ and $\Upsilon$.) Current data tells us that the former fit the bill quite well. ($\eta_{b}$ has not been found yet.)

Let us emphasize at this point that the identification of $\bar{c}c$ or $\bar{b}b$ states with the Goldstone modes must be done after the $2mv^{0}$ dependence of the energy on the heavy quark masses is substracted. In a non-relativistic theory a Goldstone mode does not mean a zero mass particle. In our case, it only means that the residual energy of the $\bar{c}c$ or $\bar{b}b$ states, that is the energy once the the $2mv^{0}$ has been substracted, must go to zero when the three momentum goes to zero (see the discussion below). Whether this actually occurs or not in nature is a separate question we shall comment upon in the last section.

In order to find the dispersion relation of the Goldstone mode, it is enough to calculate the following current-current correlator in the HQET:

$$\Pi_{\Gamma_{-}^{A}}^{HQET}(p) := \int d^{4} xe^{ipx} < 0; 0 | T(j^{+}_{\Gamma_{-}^{A}}(x)j_{\Gamma_{-}^{A}}(0)) | 0; 0 >$$

(46)
We obtain

$$\Pi_{\Gamma^A_{\pm}}^{HQET}(p) = -\int \frac{d^4k}{(2\pi)^4} \text{tr}(\bar{\Gamma}^A_{\pm}) \frac{i}{\pm v \cdot k + i\epsilon \pm v \cdot (k - p) + i\epsilon}$$

(47)

where by $\bar{\Gamma}^A_{\pm}$ we denote $\gamma^0 (\Gamma^A_{\pm})^\dagger \gamma^0$. The last expression is ill-defined. However, using a spherical cut-off in the $\tilde{k}_i = k \cdot e_i$ space, which respects all the symmetries of the HQET, at least when QCD is switched off, it becomes

$$\Pi_{\Gamma^A_{\pm}}^{HQET}(p) = -i\text{tr}(\bar{\Gamma}^A_{\pm} \Gamma^A_{\pm}) \frac{1}{\pm v \cdot p + i\epsilon} \int^A \frac{d^3\tilde{k}_i}{(2\pi)^3}$$

(48)

which is well-defined. Notice that $\text{tr}(\bar{\Gamma}^A_{\pm} \Gamma^A_{\pm})$ is negative definite and hence the residue of the pole has the right sign. The cut-off dependence can be removed by a wave function renormalization of the current.\(^{\dagger}\)

The meaning of the last expression is that the correlator (46) has a pole at $v \cdot p = 0$, which corresponds to the possibility of virtual exchange of this Goldstone mode. The dispersion relation of this mode is identical to the one of the fields $h^A_{\pm}$ appearing in the lagrangian of the HQET (1). The dispersion relation that we find for the Goldstone mode is then the generic dispersion relation for a field describing heavy particles (27), the leading mass dependence of which has been removed [7,18]. Notice that since the energy of this mode goes to zero when the momentum does so (i.e., $p^0 = \vec{p} \cdot \vec{v}$) it is perfectly compatible with the Goldstone theorem for non-relativistic theories. If this simple minded picture is correct one should be able to apply to this case all the well-known machinery of low energy effective lagrangians for Goldstone bosons [19] (see also [20]).

Before closing this section we would like to point out the role of the $i\epsilon$ in the propagators (47) as a symmetry breaking parameter. If the $i\epsilon$ prescription is incorporated in the lagrangian, this becomes

$$L = \bar{h}_v (i\slashed{v} \cdot D + i\epsilon) h_v.$$  

(49)

Therefore, it amounts to an infinitesimal source which breaks the U(4) symmetry. The result of the current-current correlator, which is formally covariant under the U(4) transformations without the inclusion of the $i\epsilon$, becomes well defined only when the $i\epsilon$ is included.

\(^{\dagger}\) If the integrals (47) and (48) are evaluated by using dimensional regularization, the result is zero (since there is no scale in (48)). Dimensional regularisation however breaks the $U(4)$ symmetry and hence it is not suitable to address questions which are intimately related to this symmetry.
and the limit $\epsilon \to 0$ is taken. This is a well-known fact of the symmetry breaking phenomenon, which manifests itself in this subtle way in our calculation of the current-current correlator.

5. Connection with the fundamental theory

In order to have a more precise interpretation of the physical meaning of the currents corresponding to the broken generators, it is convenient to have their representation at the level of the fundamental theory. By naively applying the HQET rule, that is, redefining the heavy quark field as

$$\Psi \longrightarrow e^{-i\frac{m v}{2} \cdot x} h_v$$

we obtain the following expressions

$$j_{\Gamma A}^\pm = \bar{\Psi} \Gamma^\pm_{\mp} \Psi e^{\pm 2imv \cdot x}$$

for the currents (44) in terms of the original field $\Psi$. Below we present a more careful derivation of this connection which in fact brings in some new features. Notice that (50) are local sources with suitable momentum insertions.

Let us next consider the Dirac lagrangian with the most general bilinear sources, given by

$$L = \bar{\Psi} (i\slashed{D} - m) \Psi + S \bar{\Psi} \Psi + P \bar{\Psi} \gamma_5 \Psi + V_\mu \bar{\Psi} \gamma^\mu \Psi + A_\mu \bar{\Psi} \gamma^\mu \gamma_5 \Psi + T_{\mu\nu} \bar{\Psi} \sigma^{\mu\nu} \Psi$$

(51)

Then since our symmetry generators $ip, i\gamma_5 p_\pm, i\gamma_i p_\pm, iS_i p_\pm$ form a basis of the 4x4 matrices (see discussion on section 2), the last lagrangian can be written as

$$L = \bar{\Psi} (i\slashed{D} - m) \Psi + i\bar{\Psi} \gamma_5 p_+ \Psi A_{5+} + i\bar{\Psi} \gamma_5 p_+ \Psi A_{5+}^j + i\bar{\Psi} S_{j\mu} \gamma_5 \Psi + T_{\mu\nu} \bar{\Psi} \sigma^{\mu\nu} \Psi$$

(52)

where we have defined

$$A_{\pm} = -iS \mp iV_\mu v^\mu$$
$$A_{5\pm} = -iP \pm iA_\mu v^\mu$$
$$A_{i\pm} = -\frac{1}{2} T_{\mu\nu} e_i^{\mu\nu} e_\mu^{\nu} \pm \frac{i}{4} A_\mu e_i^\mu$$
$$A_{5i}^{\pm} = iV_\mu e_i^\mu \pm 2iT_{\mu\nu} e_i^\mu v^\nu$$

(53)
Next if we restrict the last sources such that they are given in terms of the slowly varying $a_{5\pm}$ and $a_{9\pm}^j$, as

$$A_{5\pm} = a_{5\pm} e^{\pm i2mv\cdot x}, \quad A_{9\pm}^j = a_{9\pm}^j$$

because of (50), the current-current correlators of the HQET such as the (46), can be generated from the fundamental theory defined by (52). Indeed, following a derivation similar to ref. [13] we obtain

$$Z(a^A, a^V, \bar{\eta}_v, \eta_v) = \frac{det\mathcal{D}}{det\mathcal{D}_v} Z_{HQET}(a^A, a^V, \bar{\eta}_v, \eta_v)$$

(55)

where

$$Z = \int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi e^{i\int d^4x (\bar{\Psi}\mathcal{D}\Psi + \bar{\Psi}\eta + \eta\Psi)}$$

$$Z_{HQET} = \int \mathcal{D}\bar{h}_v\mathcal{D}h_v e^{i\int (\bar{h}_v\mathcal{D}_v\mathcal{h}_v + \bar{\eta}_v\eta_v + \bar{\eta}_v\eta_v)}$$

$$\mathcal{D} = iD - m + a^V_+ \Gamma^V_+ + a^V_- \Gamma^V_- + a^A_+ \Gamma^A_+ e^{2imv\cdot x} + a^A_- \Gamma^A_- e^{-2imv\cdot x}$$

$$\mathcal{D}_v = i\not{v} \cdot D + a^V_+ \Gamma^V_+ + a^V_- \Gamma^V_- + a^A_+ \Gamma^A_+ + a^A_- \Gamma^A_-$$

(56)

and we have defined

$$a^A_\pm = (a_{5\pm}, a_{9\pm}^j), \quad a^V_\pm = (a_{7\pm}, a_{9\pm}^j), \quad \text{and} \quad \eta_v = e^{im\not{v}\cdot x} \eta, \quad \bar{\eta}_v = \bar{\eta} e^{-im\not{v}\cdot x}.$$
On the other hand, because of the terms of the type $\Gamma^A a^A$, $det D_v$ is not a constant and does not admit a local derivative expansion either. In fact they give qualitatively different contributions so there is no a priori reason why they should be neglected. This can be seen, for instance, by calculating the correlator corresponding to (46) in terms of the original fields $\Psi$ when QCD is switched off. It is given by

$$\Pi_{\Gamma^A \pm} (p) = \int d^4 x e^{ip \cdot x} <0|T(\bar{\Psi}(x)\Gamma_{\pm}^A \Psi(x)\bar{\Psi}(0)\Gamma_{\pm}^A \Psi(0))e^{\mp i2mv \cdot x}|0> .$$

This expression, except for the exponential factor, describes a one loop fermion diagram and for finite fermion mass has a branch point singularity corresponding to a pair creation and not a pole singularity as found in (48). The branch point singularity at the value of the external momentum $q^2 = 4m^2$ amounts to a non zero contribution in the imaginary part of the matrix element starting from this value of the momentum. In the large $m$ limit with the external momentum $q^\mu$ fixed we expect zero contribution from this diagram, since the creation of two heavy particles requires infinite external energy.

The result obtained from (57) for large $m$, when we have switched off QCD, is

$$Im(i\Pi_{\Gamma^A \pm}(p)) = \frac{1}{2} tr(\bar{\Gamma}_{\pm}^A \Gamma^A_{\pm}) \left( m^2 \frac{1}{2\pi} \sqrt{\mp \frac{v \cdot p}{m} \mp m \frac{p^2}{16\pi v \cdot p} \sqrt{\mp \frac{v \cdot p}{m}}} \right) + \text{subleading terms in } m$$

which does not have a well-defined limit when $m \to \infty$.

The last result may seem paradoxical but it can be understood by looking at the expression (57). The exponential factor in this expression amounts to putting heavy particles in the external lines (equivalently the external momentum $q^\mu$ takes the value $p^\mu \mp 2mv^\mu$, where $p^\mu$ is very small). Thus, in the large $m$ limit the external momentum is enough to produce a pair and in principle one would expect to have a branch point singularity in the $\Pi_{\Gamma^A \pm}(p)$ given by (57). By taking, however, $p^\mu$ small we are making an expansion exactly on the singular point. Although for $m$ large the singularity moves to infinity, our expansion moves with it and thus it retains a big mass dependence. Notice that the expression (58) corresponds to an imaginary part only for $\mp v \cdot p/m > 0$. This simply reflects the fact that for values of $q^2$ below the threshold there is no contribution from the imaginary part of the correlator (57) to the matrix element. Indeed, when we take $q^\mu = p^\mu \mp 2mv^\mu$ for $p^\mu$ small we obtain $q^2 \sim 4m^2(1 \mp v \cdot p/m)$. 

19
We conclude then that the correlation functions of operators of the type (50) do not have a smooth large $m$ limit to a local HQET describing particles and antiparticles with the same velocity. In spite of this, let us point out some interesting features of the determinants in (55) (keeping QCD switched off). At second order in the currents we have

$$\text{tr} \log \mathcal{D} = \text{tr} \log \mathcal{D}^0 - \frac{1}{2} \text{tr} \left[ (\mathcal{D})^{-1} \left( (\Gamma^V a^V_+ + \Gamma^V a^V_-) (\mathcal{D}^0)^{-1} (\Gamma^V a^V_+ + \Gamma^V a^V_-) \right) \right]$$

$$- \frac{1}{2} \text{tr} \left[ (\mathcal{D}^0)^{-1} \Gamma^A a^A_+ e^{2i m V \cdot x} (\mathcal{D}^0)^{-1} \Gamma^A a^A_- e^{-2i m V \cdot x} \right]$$

$$+ (\mathcal{D}^0)^{-1} \Gamma^A a^A_+ e^{-2i m V \cdot x} (\mathcal{D}^0)^{-1} \Gamma^A a^A_- e^{2i m V \cdot x}$$

$$\text{tr} \log \mathcal{D}_v = \text{tr} \log \mathcal{D}_v^0 - \frac{1}{2} \text{tr} \left[ (\mathcal{D}_v^0)^{-1} \right.$$

$$\left. (\Gamma^V a^V_+ + \Gamma^V a^V_-) (\mathcal{D}_v^0)^{-1} (\Gamma^V a^V_+ + \Gamma^V a^V_-) \right]$$

$$- \frac{1}{2} \text{tr} \left[ (\mathcal{D}_v^0)^{-1} \Gamma^A a^A_+ (\mathcal{D}_v^0)^{-1} \Gamma^A a^A_- \right]$$

$$+ (\mathcal{D}_v^0)^{-1} \Gamma^A a^A_+ (\mathcal{D}_v^0)^{-1} \Gamma^A a^A_-$$

(59)

where $\mathcal{D}^0 = i \mathcal{D} - m$ and $\mathcal{D}_v^0 = i \mathcal{D}_v \cdot D$. The linear terms in the sources which would appear in the expansion of $\text{tr} \log \mathcal{D}$ are dropped because they contain oscillating exponentials that cannot cancel between themselves and hence they do not contribute [13]. Exactly the same occurs with quadratic terms containing $a^A_\pm$ twice.

For the case of the $\text{det}\mathcal{D}_v$ the terms involving $\Gamma^V$ type of currents give zero after the integration over the momentum, since the two poles of the two propagators appear in the same complex half plane. The analogous terms for the $\text{det}\mathcal{D}$ reduce to local counterterms and we shall disregard them. Let us then concentrate on the terms containing $\Gamma^A$ type of currents. From the previous results (48) and (58) we see that these terms have different singularity structure. However, they do have the same structure as far as the $U(4)$ symmetry is concerned. Indeed, except from the exponential factors in $\mathcal{D}$, the terms which contribute in both determinants are the same combinations of $+$ or $-$ components in the currents. Furthermore they are both invariant under the unbroken subgroup $U(2) \otimes U(2)$. Therefore one could use the fact that, even if $\text{det}\mathcal{D}/\text{det}\mathcal{D}_v$ is different from one (and non-local), symmetrywise the properties of $Z$ are the same as the properties of $Z_{\text{HQET}}$. The key question is whether this nice feature survives when QCD is switched on. Unfortunately, it is not difficult to convince oneself that in this case the expansion $B_{\mu\nu}/m^2 << 1$, i.e., the usual expansion in this framework, is singular. (The term proportional to $B_{\mu\nu}B^{\mu\nu}$ has a singularity of higher order than the one in (58) and terms of higher orders in the gauge field have even higher order singularities.) This means that infrared QCD effects
are crucial to answer this question and hence it cannot be addressed in a reliable way by using the standard derivative expansion techniques [8,13]. Speculation on this issue is left to the following section.

6. Summary and discussion

We have pointed out that the HQET describing a quark and an anti-quark with the same velocity, as first written down in [5], enjoys an invariance larger than the known flavor and spin symmetry. The extra symmetries are of axial type and mix quarks and anti-quarks. The full invariance of the theory for a single flavor corresponds to a $U(4)$ group.

We have, then, given a phenomenological argument to show that this symmetry must be spontaneously broken in nature. We have also shown at the level of the HQET that the symmetry breaking takes place even when QCD is neglected. Consequently, unlike chiral symmetry breaking, it must not be regarded as a non-trivial feature of the QCD vacuum but rather as an intrinsic feature of the HQET formalism for Dirac fermions. Next we have analyzed the Goldstone theorem in this approximation and identified the Goldstone modes corresponding to the broken generators as states containing a heavy quark and a heavy anti-quark. In nature they should correspond to $\bar{b}b$ and $\bar{c}c$ mesons, in which the heavy quark mass dependence has been removed. By calculating the current-current correlator (for the broken currents) in this theory we show that the dispersion relation of these Goldstone modes are consistent with the Goldstone theorem for non-relativistic theories.

In order to connect this theory of the quark and anti-quark in the large mass limit with the fundamental one we have, firstly, calculated the axial-type current-current correlators starting from the fundamental theory as well. For this purpose, we have identified the currents in the fundamental theory which correspond to the conserved currents of the $U(4)$ symmetry at the level of the HQET. We concluded that the correlator of the axial type of currents studied from the original theory does not have a smooth limit when $m$ goes to infinity. That shows that at the level of current correlators the large mass limit of the fundamental theory are not reproduced by the HQET †.

† In a relatively different context anomalous dependences on large masses have been pointed out in [21]
Secondly, we have derived the generating functional of the HQET including sources for all currents from the generating functional of the fundamental theory under the standard HQET assumptions [8,13]. The two generating functionals differ by a non-trivial non-local ratio of determinants. The determinant of the denominator corresponds to the heavy quark limit and it is mass independent while the determinant of the numerator corresponds to the original theory and has a non-trivial dependence on the mass. With QCD switched off, both determinants enjoy the same symmetry properties, which shows that at the symmetry level and without gluons present the two theories are equivalent. In the presence of QCD the calculation of the determinant of the original theory is plagued with infrared singularities, which signals that a non-perturbative analysis is needed in order to draw any solid conclusion.

When full QCD is switched on the singularity structure of current-current correlators such as (57) are known from phenomenology. With the assumption that poles and cuts of light neutral mesons are suppressed by powers of $1/m$ the first singularity encountered when rising the energy are the $\eta_b$, $\Upsilon$, $\eta_c$ and $J/\Psi$ poles depending on the currents and heavy flavors one considers. By quantum numbers any of the $\eta_b$ or $\Upsilon$ ($J/\Psi$ or $\eta_c$) states with the mass of the quark and antiquark subtracted could play the role of our Goldstone modes. The residual mass (binding energy) of these states may be considered as due to an explicit breaking of the $U(4)$ symmetry caused by hard gluons, the effect of which we have disregarded.

Let us briefly comment on the case of having $N_f$ flavors. In that case formula (1) is still correct, where now $h_v$ is taking values in the flavor space. In the rest frame it is obvious that (1) enjoys a $U(4N_f)$ symmetry. One may use the argument after (16) in order to prove that the $U(4N_f)$ symmetry holds in any frame. When QCD is switched off, the analysis carried out in sect. 4 is not substantially modified: $U(4N_f)$ breaks spontaneously down to $U(2N_f) \otimes U(2N_f)$. At the level of HQET, this implies that particles like the $B_c$ meson could also be regarded as Goldstone modes, once the $m_b + m_c$ mass dependence is removed. However, if we take into account the ratio of determinants in (55), which appears when making the precise connection between the fundamental theory and the HQET, the implication above desintegrates. When QCD is switched off, the strong mass dependence of the $\Gamma^A$-type current correlators calculated from the fundamental theory (57)-(58) breaks explicitly any symmetry relating $\Gamma^A$-type currents of different flavors. As
mentioned before, the QCD effects in these correlators are difficult to estimate in a reliable way, but it would be very surprising that they conspire to restore the $U(2N_f) \otimes U(2N_f)$ symmetry.

Finally, we would like to emphasize that our analysis is based on a theory which includes both quark and anti-quark with the same velocity. The HQET originally was designed to describe hadrons with a single heavy quark and in the applications studied in the literature by now has been mainly used to describe either the quark or the anti-quark sector but not both. The present analysis could prove useful in describing physical processes which involve a heavy quark and a heavy anti-quark. Indeed, the HQET has been recently applied to such physical processes in the study of the $B_c$ meson [22] or $D - \bar{D}$ and $B - \bar{B}$ mixing [12]. The fact that there is a spontaneously broken $U(4)$ symmetry in HQET describing a quark and an anti-quark may tell us something interesting about these processes.

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**Figure Captions**

Fig. 1. Modification of the Dirac vacuum in the HQET. By the field redefinition $\Psi = e^{-im\mathbf{v} \cdot x} h_v(x)$, the extra time dependence of the new field is $e^{imv^0 t}$ for the $h^+_v(x)$ component and $e^{-imv^0 t}$ for the $h^-_v(x)$ component. The field redefinition, thus, amounts to lowering the positive Dirac sea energy by an amount $mv^0$ and raising the negative Dirac sea by the same amount. Accordingly, the energy levels $mv^0$ and $-mv^0$ of the original theory become zero energy levels in the effective theory.

Fig. 2. The infinite degeneracy of the vacuum of this HQET, where quarks and anti-quarks are included. The $\gamma_5$-symmetry mixes these degenerate states, since the symmetry operator on the state $|0; 0 >$ creates the state $|\vec{k}, \sigma; -\vec{k}, \sigma' >$ without energy cost.