INCIDENCE HILBERT SCHEMES AND INFINITE DIMENSIONAL LIE ALGEBRAS

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ABSTRACT. Let \( S \) be a smooth projective surface. Using correspondences, we construct an infinite dimensional Lie algebra that acts on the direct sum

\[
\mathbb{H}_S = \bigoplus_{m=0}^{+\infty} H^*(S^{[m,m+1]})
\]

of the cohomology groups of the incidence Hilbert schemes \( S^{[m,m+1]} \). The algebra is related to an extension of an infinite dimensional Heisenberg algebra. The space \( \mathbb{H}_S \) is a highest weight representation of this algebra. Our result provides a representation-theoretic interpretation of Cheah’s generating function of Betti numbers of the incidence Hilbert schemes. As a consequence, an additive basis of \( H^*(S^{[m,m+1]}) \) is obtained.

1. INTRODUCTION

Let \( S \) be a smooth projective surface. The Hilbert scheme \( S^{[n]} \) of \( n \) points on \( S \) parametrizes all the 0-dimensional closed subschemes of \( S \) with length \( n \). It is a crepant resolution of the \( n \)-fold symmetric product \( S^{(n)} \). The generating function of the Betti numbers of \( S^{[n]} \) has an elegant closed formula due to Göttscbe [Got1]:

\[
\sum_{n=0}^{+\infty} \left( \sum_{i=0}^{4n} (-1)^i b_i(S^{[n]}) z^i \right) q^n = \prod_{n=1}^{+\infty} \prod_{i=0}^{4} \left( 1 - \frac{1}{1 - z^{2n-2+i} q^n} \right)^{(-1)^i b_i(S)},
\]

where \( b_i(\cdot) \) denotes the \( i \)-th Betti number. There is a representation theoretic interpretation discovered by Grojnowski [Gro] and Nakajima [Na1] independently. It says that the infinite dimensional vector space

\[
\mathbb{H}_S = \bigoplus_{n=0}^{+\infty} H^*(S^{[n]})
\]

is the Fock space of an infinite dimensional Heisenberg algebra. The algebra is constructed geometrically via correspondences. This result not only shows a beautiful symmetry for the Hilbert schemes \( S^{[n]} \), but also provides the right platform for the intensive research on the cohomology ring of \( S^{[n]} \) (e.g. see [CG, Got2, Lehn, LS1, LS2, LQW1, LQW2, LQW3, LQW4, LQW5, Mar1, Mar2, OP, QW1, QW2, Vas]).
These Hilbert schemes $S^{[n]}$ play special roles in the recent study of interplay between Donaldson-Thomas theory and Gromov-Witten theory \cite{MNOP1, MNOP2, OP, KLQ, EQ}. When $S$ is embedded in a smooth projective threefold $X$, the moduli spaces for the relative Donaldson-Thomas theory of the pair $(X, S)$ admit natural morphisms to the Hilbert schemes $S^{[n]}$. In some cases \cite{KLQ, EQ}, the moduli spaces for the (ordinary) Donaldson-Thomas theory admit morphisms to the Hilbert schemes $S^{[n]}$ as well. It is natural to speculate whether certain interesting Lie algebra, perhaps somehow related to the previously-mentioned Heisenberg algebra of the Hilbert schemes, acts on the cohomologies of these moduli spaces of the (ordinary and relative) Donaldson-Thomas theory.

We are making preliminary progress in this direction \cite{LQ2}. It turns out that the incidence Hilbert schemes $S^{[n,n+1]}$ are part of the ingredient in our work, where

$$S^{[n,n+1]} = \{ (\xi, \xi') | \xi \subset \xi' \} \subset S^{[n]} \times S^{[n+1]}.$$

These incidence Hilbert schemes $S^{[n,n+1]}$ are smooth. The generating function of their Betti numbers has a closed formula due to Cheah \cite{Ch2}:

$$\sum_{n=0}^{+\infty} \left( \sum_{i=0}^{4(n+1)} (-1)^i b_i(S^{[n,n+1]}) z^i \right) q^n = \left( 4 \sum_{i=0}^{4} (-1)^i b_i(S) z^i \right) \cdot \prod_{n=1}^{+\infty} \prod_{i=0}^{4} \left( \frac{1}{1 - z^{2n-2+i} q^n} \right) \cdot \frac{1}{1 - z^2 q}. \quad (1.1)$$

The infinite product in (1.1) hints that there should be a Heisenberg algebra action, similar to the case of the Hilbert schemes $S^{[n]}$, on the space

$$\widetilde{H}_S = \bigoplus_{n=0}^{+\infty} \widetilde{H}^*(S^{[n,n+1]}).$$

The two other factors in the formula (1.1) suggest that a larger Lie algebra be needed for which the space $\widetilde{H}_S$ is a highest weight module.

In this paper, we show that such an infinite dimensional Lie algebra exists, whose character formula agrees with (1.1), and that the space $\widetilde{H}_S$ is the highest weight representation of this Lie algebra. In the following, we outline the basic ingredients in the construction of this Lie algebra and its representation on $\widetilde{H}_S$.

For $m \geq 0$ and $n > 0$, we define certain closed subset $\widetilde{Q}^{[m+n,m]}$ of

$$S^{[m+n,m+n+1]} \times S \times S^{[m,m+1]}.$$

For $\alpha \in H^*(S)$, we define the operator $\tilde{a}_{-n}(\alpha) \in \text{End}(\widetilde{H}_S)$ by

$$\tilde{a}_{-n}(\alpha)(\widetilde{A}) = \tilde{p}_1*([\widetilde{Q}^{[m+n,m]}] \cdot \tilde{p}^* \alpha \cdot \tilde{p}_2^* \widetilde{A})$$

for $\widetilde{A} \in H^*(S^{[m,m+1]})$, where $\tilde{p}_1, \tilde{p}, \tilde{p}_2$ are the projections of

$$S^{[m+n,m+n+1]} \times S \times S^{[m,m+1]}.$$
to $S^{[n+n,m+n+1]}$, $S^{[m,m+1]}$ respectively. Define $\tilde{a}_n(\alpha) \in \text{End}(\tilde{\mathbb{H}}_S)$ to be $(-1)^n$ times the operator obtained from the definition of $\tilde{a}_{-n}(\alpha)$ by switching the roles of $\tilde{p}_1$ and $\tilde{p}_2$. We define $\tilde{a}_0(\alpha) = 0$ for every $\alpha \in H^*(S)$.

The first result is that the operators $\tilde{a}_n(\alpha)$, $n \in \mathbb{Z}$, $\alpha \in H^*(S)$ satisfy the following Heisenberg algebra commutation relation:

$$[\tilde{a}_n(\alpha), \tilde{a}_k(\beta)] = -n \delta_{n,-k} \int_S (\alpha \beta) \cdot \text{Id}_{\tilde{\mathbb{H}}_S}.$$ 

This explains the appearance of the infinite product in the formula (1.1).

The two other factors in (1.1) correspond to two new features of the space $\tilde{\mathbb{H}}_S$. The first feature comes from an $H^*(S)$-module structure of $\tilde{\mathbb{H}}_S$. We observe that sending a pair $(\xi, \xi') \in S^{[n,n+1]}$ to the support of $I_\xi/I_{\xi'}$ yields a morphism:

$$\rho_n : S^{[n,n+1]} \to S.$$ 

It follows that $H^*(S^{[n,n+1]})$ and hence $\tilde{\mathbb{H}}_S$ are $H^*(S)$-modules. It turns out that the operator $\tilde{a}_n(\alpha) : \tilde{\mathbb{H}}_S \to \tilde{\mathbb{H}}_S$ is an $H^*(S)$-module homomorphism for all $n$. This explains the appearance of the first factor in the formula (1.1).

The second feature is the existence of a new operator $\tilde{t} \in \text{End}(\tilde{\mathbb{H}}_S)$, called translation operator. It is constructed via a correspondence of $\tilde{Q}_n$ which is a closed subset of $S^{[n+1,n+2]} \times S^{[n,n+1]}$. It maps the component $H^*(S^{[n,n+1]})$ to the component $H^*(S^{[n+1,n+2]})$, and is commutative with the Heisenberg algebra, i.e.,

$$[\tilde{t}, \tilde{a}_n(\alpha)] = 0$$

for all $n$ and $\alpha$. In addition, $\tilde{t}$ is an $H^*(S)$-module homomorphism. The translation operator $\tilde{t}$ is responsible for the second factor in (1.1).

Now we can construct the Lie algebra $\tilde{\mathfrak{g}}_S$ and state our main theorem. Let $\tilde{\mathfrak{g}}_S$ be the Heisenberg algebra generated by the operators $\tilde{a}_n(\alpha)$, $n \in \mathbb{Z}$, $\alpha \in H^*(S)$ and the identity operator $\text{Id}_{\tilde{\mathbb{H}}_S}$. Define a Lie algebra structure on

$$\tilde{\mathfrak{g}}_S = \tilde{\mathfrak{g}}_S \oplus H^*(S) \oplus \mathbb{C} \tilde{t}$$

compatible with the Lie algebra structure on $\tilde{\mathfrak{g}}_S$ and the cup product on $H^*(S)$.

**Theorem 1.1.** The space $\tilde{\mathbb{H}}_S$ is a representation of the Lie algebra $\tilde{\mathfrak{g}}_S$ with a highest weight vector being the vacuum vector

$$|0\rangle = 1_S \in H^0(S) = H^0(S^{[0,1]})$$

where $1_S$ denotes the fundamental cohomology class of $S$.

The organization of the paper is as follows. In [2] we review the Heisenberg algebra action on the Fock space $\mathbb{H}_S$ for the Hilbert schemes $S^{[n]}$ and some basics of the incidence Hilbert schemes $S^{[n,n+1]}$. In [3] we construct the Heisenberg operators $\tilde{a}_n(\alpha)$ on $\tilde{\mathbb{H}}_S$ and verify the Heisenberg commutation relation. We also compare $\tilde{a}_n(\alpha)$ with the pullback of the Heisenberg operator $a_n(\alpha)$ on $\mathbb{H}_S$ via the morphism $f_n$ from $S^{[n,n+1]}$ to $S^{[n]}$. In [4] we define the translation operator $\tilde{t}$, and show that it commutes with the Heisenberg operators and is an $H^*(S)$-module homomorphism.
We also compare $a_n(\alpha)$ with the pullback of $a_n(\alpha)$ via the morphism $g_{n+1}$ from $S^{[n,n+1]}$ to $S^{[n+1]}$. It turns out that the understanding of the pullback of the creation operators via $g_{n+1}$ needs the translation operator. In [5] we prove Theorem [4].

**Conventions.** Unless otherwise indicated, all the cohomology groups in this paper are in $\mathbb{C}$-coefficients. For a continuous map $p : Y_1 \to Y_2$ between two smooth compact manifolds and for $\alpha \in H^*(Y_1)$, we define $p_\ast(\alpha)$ to be $\text{PD}^{-1} p_\ast(\text{PD}(\alpha))$ where PD stands for the Poincaré duality. The $\tilde{\ }$ is used for all the notations related to the incidence Hilbert schemes $S^{[n,n+1]}$. When a product $X_1 \times X_2 \times \ldots X_n$ is clear from the context, we use $\pi_{i_1 \ldots i_k}$ to denote the projection from $X_1 \times X_2 \times \ldots X_n$ to the product of the $i_1$-th, $\ldots$, $i_k$-th factors.

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2. Basics on Hilbert schemes of points on surfaces

2.1. Hilbert schemes of points on surfaces. Let $S$ be a complex smooth projective surface, and $S^{[n]}$ be the Hilbert scheme of points in $S$. An element in $S^{[n]}$ is represented by a length-$n$ 0-dimensional closed subscheme $\xi$ of $S$. For $\xi \in S^{[n]}$, let $I_\xi$ be the corresponding sheaf of ideals. It is well known that $S^{[n]}$ is smooth. Sending an element in $S^{[n]}$ to its support in the symmetric product $\text{Sym}^n(S)$, we obtain the Hilbert-Chow morphism $\pi_n : S^{[n]} \to \text{Sym}^n(S)$, which is a resolution of singularities. We have the universal codimension-2 subscheme:

$$Z_n = \{ (\xi, s) \subset S^{[n]} \times S \mid s \in \text{Supp}(\xi) \} \subset S^{[n]} \times S. \quad (2.1)$$

Let $H^*(S^{[n]})$ be the total cohomology of $S^{[n]}$ with $\mathbb{C}$-coefficients. Put

$$\mathbb{H}_S = \bigoplus_{n=0}^{+\infty} H^*(S^{[n]}). \quad (2.2)$$

For $m \geq 0$ and $n > 0$, let $Q^{[m,m]} = \emptyset$ and define $Q^{[m+n,m]}$ to be the closed subset:

$$\{(\xi, s, \eta) \in S^{[m+n]} \times S \times S^{[m]} \mid \xi \supset \eta \text{ and } \text{Supp}(I_\eta/I_\xi) = \{s\}\}. $$

We recall Nakajima’s definition of the Heisenberg operators [Na1, Na2]. Let $n > 0$. The linear operator $a_{-n}(\alpha) \in \text{End}(\mathbb{H}_S)$ with $\alpha \in H^*(S)$ is defined by

$$a_{-n}(\alpha)(A) = p_{1\ast}([Q^{[m+n,m]}] \cdot \rho^\ast \alpha \cdot p_2^\ast A) \quad (2.3)$$

for $A \in H^*(S^{[m]}$, where $p_1, \rho, p_2$ are the projections of $S^{[m+n]} \times S \times S^{[m]}$ to $S^{[m+n]}, S, S^{[m]}$ respectively. Define $a_n(\alpha) \in \text{End}(\mathbb{H}_S)$ to be $(-1)^n$ times the operator obtained from the definition of $a_{-n}(\alpha)$ by switching the roles of $p_1$ and $p_2$. We often refer to $a_{-n}(\alpha)$ (resp. $a_n(\alpha)$) as the creation (resp. annihilation) operator.
We also set $a_0(\alpha) = 0$. A non-degenerate super-symmetric bilinear form $\langle -, - \rangle$ on $H_S$ is induced from the standard one on $H^*(S^{[n]})$ defined by
\[ \langle A, B \rangle = \int_{S^{[n]}} AB, \quad A, B \in H^*(S^{[n]}). \]
This allows us to define the adjoint $f^\dagger \in \text{End}(H_S)$ for $f \in \text{End}(H_S)$. Then,
\[ a_n(\alpha) = (-1)^n \cdot a_{-n}(\alpha)^\dagger, \quad (f \circ g)^\dagger = g^\dagger \circ f^\dagger. \quad (2.4) \]

It was proved in [Na1, Na2] that the operators $a_n(\alpha), n \in \mathbb{Z}, \alpha \in H^*(S)$ satisfy the following Heisenberg algebra commutation relation:
\[ [a_n(\alpha), a_k(\beta)] = -n \delta_{n,-k} \int_S (\alpha \beta) \cdot \text{Id}_{H_S}. \quad (2.6) \]
Moreover, the space $H_S$ is an irreducible module over the Heisenberg algebra generated by the operators $a_n(\alpha)$ with a highest weight vector
\[ |0\rangle = 1 \in H^0(S^{[0]}) \cong \mathbb{C}. \]
It follows that $H_S$ is linearly spanned by all the Heisenberg monomial classes:
\[ a_{-n_1}(\alpha_1) \cdots a_{-n_k}(\alpha_k)|0\rangle \quad (2.7) \]
where $k \geq 0, n_1, \ldots, n_k > 0$, and $\alpha_1, \ldots, \alpha_k$ run over a linear basis of $H^*(S)$. We remark that the Lie brackets in (2.6) are understood in the super sense according to the parity of the degrees of the cohomology classes involved.

2.2. Incidence Hilbert schemes. Define the incidence variety:
\[ S^{[n,n+1]} = \{(\xi, \xi') | \xi \subset \xi' \} \subset S^{[n]} \times S^{[n+1]} \quad (2.8) \]

It is well-known (see [Ch1, ES]) that the incidence Hilbert scheme $S^{[n,n+1]}$ is irreducible, smooth and of dimension $2(n+1)$, and that $S^{[n,n+1]}$ is also the blowup of $S^{[n]} \times S$ along the universal codimension-2 subscheme $Z_n$.

Besides the Hilbert scheme $S^{[n]}$ for a surface $S$, the incidence Hilbert scheme $S^{[n,n+1]}$ for a surface $S$ is the only class of (generalized or nested) Hilbert schemes of points on smooth varieties of dimension bigger than one which are smooth for all $n$ (see [Ch1]). Inspired by the results on the Hilbert schemes $S^{[n]}$, it is natural to consider the sum of the total cohomology group of $S^{[n,n+1]}$ over all $n \geq 0$:
\[ H^*_S = \bigoplus_{n=0}^{+\infty} H^*(S^{[n,n+1]}). \quad (2.9) \]

The space $H^*_S$ has richer structures than $H^*_S$. For instance, $H^*_S$ is an $H^*(S)$-module. The module structure is induced by the morphism:
\[ \rho_n : S^{[n,n+1]} \to S \quad (2.10) \]
sending a pair $(\xi, \xi') \in S^{[n,n+1]}$ to the support of $I_{\xi}/I_{\xi'}$. 

The following facts on punctual Hilbert schemes are useful for later sections. Fix a point \( s \in S \). For \( m \geq 0 \) and \( n > 0 \), we define two closed subsets:

\[
M_m(s) = \{ \xi \in S^m | \text{Supp}(\xi) = \{s\} \},
\]

\[
M_{m,m+n}(s) = \{ (\xi, \xi') | \xi \subset \xi' \} \subset M_m(s) \times M_{m+n}(s).
\]

(2.11) (2.12)

It is known that \( M_{m,m+1}(s) \) and \( M_{m+1}(s) \) are irreducible with

\[
\dim M_{m,m+1}(s) = \dim M_{m+1}(s) = m.
\]

(2.13)

3. Heisenberg algebra actions for incidence Hilbert schemes

3.1. General remarks about correspondences. Given two smooth projective varieties \( X_1 \) and \( X_2 \) and a closed subset \( Z \) of \( X_1 \times X_2 \), we can define a map \([Z]_*\), called the correspondence of \( Z \), from \( H^*(X_2) \) to \( H^*(X_1) \) via

\[
Z_*(A) = p_{1*}([Z] \cdot p_2^*(A))
\]

for \( A \in H^*(X_2) \), where \( p_i \) is the projection from \( X_1 \times X_2 \) to its \( i \)-th factor. Given another closed subset \( Y \subset X_2 \times X_3 \), the composition

\[
[Z]_* \circ [Y]_* : H^*(X_3) \to H^*(X_1)
\]

is given by the correspondence of \( (\pi_{13})_* (\pi_{12}^*[Z] \cap \pi_{23}^*[Y]) \). Note that we have used the notations established in the Conventions.

In this paper, we will exhibit a collection of correspondences for the incidence Hilbert schemes whose induced operators form a Lie algebra. From the paragraph above, we see that to check the commutation relations, we need to study

\[
\pi_{13}(\pi_{12}^{-1}(Z) \cap \pi_{23}^{-1}(Y)).
\]

This subset may not be irreducible, and the intersection \( \pi_{12}^{-1}(Z) \cap \pi_{23}^{-1}(Y) \) may not be transversal. However, in most of the cases,

\[
\pi_{12}^{-1}(Z) \cap \pi_{23}^{-1}(Y)
\]

has only one or two components with the expected dimensions, while other components have smaller dimensions and thus will not contribute to the induced map on cohomologies. In addition, either the transversal property is satisfied, or the multiplicities can be computed. Therefore, most of our work is to calculate the dimensions of various stratum. Also, instead of doing the computations for all the cases, we will only present the most exemplary ones in full detail while lay out what the remaining cases are with details skipped.

Even though a large percentage of our work is on the computation of dimensions of various subsets, one should not be mislead to believe that this is the key. It is free to construct the correspondence map between the cohomologies of \( X \) and \( Y \) for any closed subset of \( X \times Y \). However, very few collections of correspondences provide meaningful operators on cohomologies. To make things worse, there is no guiding principle for choosing the “right” ones in most situations. In fact, the Heisenberg operators to be studied below were found with hints from our study of another type of moduli spaces for Donaldson-Thomas invariants. Once the
“right” correspondences are found, the checking of the commutation relations will be relatively straightforward, although it may be very lengthy.

3.2. Definition of the Heisenberg operators. For $m > 0$ and $n > 0$, let 
\[ \tilde{Q}^{[m,n]} = \emptyset \]
and define \( \tilde{Q}^{[m+n,m]} \) to be the following closed subset of \( S^{[m+n,m+n+1]} \times S \times S^{[m,m+1]} \):
\[ \tilde{Q}^{[m+n,m]} = \{ ((\xi,\xi'), s, (\eta, \eta')) | \xi \supset \eta, \xi' \supset \eta', \text{ Supp}(I_\eta/I_{\xi'}) = \{s\}, \text{ Supp}(I_\xi/I_{\xi'}) = \text{Supp}(I_\eta/I_{\eta'}) \} \]

The dimension of the subset \( \tilde{Q}^{[m+n,m]} \) is given by the following Lemma.

**Lemma 3.1.** For \( m > 0 \) and \( n > 0 \), \( \dim \tilde{Q}^{[m+n,m]} = 2m + n + 3 \).

**Proof.** Take an element \( ((\xi,\xi'), s, (\eta, \eta')) \) in \( \tilde{Q}^{[m+n,m]} \). Let 
\[ \{t\} = \text{Supp}(I_\xi/I_{\xi'}) = \text{Supp}(I_\eta/I_{\eta'}) \]
for some \( t \in S \). First of all, assume that \( s \neq t \). Then \( \eta \) can be decomposed as 
\[ \eta = \eta_0 + \eta_s + \eta_t \]
where \( \eta_s \in M_i(s) \) for some \( i \geq 0 \), \( \eta_t \in M_j(t) \) for some \( j \geq 0 \), \( \eta_0 \in S^{[m-i-j]} \), and \( s,t \not\in \text{Supp}(\eta_0) \). Then, \( \eta', \xi \) and \( \xi' \) can be written as
\[ \eta' = \eta_0 + \eta_s + \eta'_t, \quad \xi = \eta_0 + \xi_s + \eta_t, \quad \xi' = \eta_0 + \xi_s + \eta'_t, \]
\[(\eta, \eta'_t) \in M_{j,j+1}(t), \quad (\eta_s, \xi_s) \in M_{i,i+n}(s). \]

When \( i = j = 0 \), we conclude from (2.13) that the number of moduli of these triples \( ((\xi,\xi'), s, (\eta,\eta')) \) in \( \tilde{Q}^{[m+n,m]} \) is equal to
\[ \#(\text{moduli of } \eta_0) + \#(\text{moduli of } \xi_s) + \#(\text{moduli of } t) \]
\[ = 2m + (n - 1) + 4 \]
\[ = 2m + n + 3. \quad (3.1) \]

In general, when \( i \geq 1 \) or \( j \geq 1 \), we see from (2.13) again that the number of moduli of these triples \( ((\xi,\xi'), s, (\eta,\eta')) \) in \( \tilde{Q}^{[m+n,m]} \) is at most
\[ \#(\text{moduli of } \eta_0) + \#(\text{moduli of } \eta_s) + \#(\text{moduli of } \eta_t) \]
\[ = 2(m - i - j) + \max(i - 1, 0) + j + (i + n - 1) + 4 \]
\[ < 2m + n + 3. \quad (3.2) \]

Next, let \( s = t \). This time we decompose the 0-cycle \( \eta \) into 
\[ \eta = \eta_0 + \eta_s \]
where \( \eta_s \in M_i(s) \) for some \( i \geq 0 \), \( \eta_0 \in S^{[m-i]} \), and \( s \not\in \text{Supp}(\eta_0) \). Then,
\[ \eta' = \eta_0 + \eta'_s, \quad \xi = \eta_0 + \xi_s, \quad \xi' = \eta_0 + \xi'_s, \]
\[ (\eta_s, \eta'_s) \in M_{i,i+1}(s), \quad (\eta_s, \xi_s) \in M_{i,i+n}(s), \]
\[ (\xi_s, \xi'_s) \in M_{i+n,i+n+1}(s), \quad (\eta'_s, \xi'_s) \in M_{i+1,i+n+1}(s). \]
So the number of moduli of these triples \(((\xi, \xi'), s, (\eta, \eta')) \in \widetilde{Q}^{[m+n,m]}\) is at most
\[
\#(\text{moduli of } \eta_0) + \#(\text{moduli of } \eta_s \subset \eta_0') + \#(\text{moduli of } \xi_s \subset \xi_0') = 2(m - i) + i + (i + n) + 2 < 2m + n + 3. \tag{3.3}
\]
Combining (3.1), (3.2) and (3.3), we get \(\dim \widetilde{Q}^{[m+n,m]} = 2m + n + 3\). \qed

We will only prove the Lemma for \(\rho\)

\begin{definition}
(1) Let \(\alpha \in H^*(S)\). Define \(\bar{a}_0(\alpha) = 0\) in \(\text{End}(\widetilde{H}_S)\).

(2) Let \(n > 0\) and \(\alpha \in H^*(S)\). Define \(\bar{a}_{-n}(\alpha) \in \text{End}(\widetilde{H}_S)\) by

\[
\bar{a}_{-n}(\alpha) = \bar{p}_1^*([\bar{Q}^{[m+n,m]}] \cdot \bar{\rho}^* \alpha \cdot \bar{p}_2) \tag{3.4}
\]

for \(\bar{A} \in H^*(S^{[m,m+1]})\), where \(\bar{p}_1, \bar{\rho}, \bar{p}_2\) are the projections of \(S^{[m+n,m+n+1]} \times S \times S^{[m,m+1]}\) to \(S^{[m+n,m+n+1]}, S, S^{[m,m+1]}\) respectively.

(3) Define \(\bar{a}_n(\alpha) \in \text{End}(\widetilde{H}_S)\) to be \((-1)^n\) times the operator obtained from the definition of \(\bar{a}_{-n}(\alpha)\) by switching the roles of \(\bar{p}_1\) and \(\bar{p}_2\).

A non-degenerate super-symmetric bilinear form \(\langle - , - \rangle\) on \(\widetilde{H}_S\) is induced from the standard one on \(H^*(S^{[n,n+1]})\) defined by

\[
\langle \bar{A}, \bar{B} \rangle = \int_{S^{[n,n+1]}} \bar{A} \bar{B}, \quad \bar{A}, \bar{B} \in H^*(S^{[n,n+1]}).
\]

This allows us to define the adjoint \(\bar{f}^\dagger \in \text{End}(\widetilde{H}_S)\) for \(f \in \text{End}(\widetilde{H}_S)\). Then,

\[
\bar{a}_n(\alpha) = (-1)^n \cdot \bar{a}_{-n}(\alpha)^\dagger. \tag{3.5}
\]

Define \(|\alpha| = \ell\) if \(\alpha \in H^\ell(S)\). Let \(n \neq 0\). By (3.1) and Lemma 3.1 we see that \(\bar{a}_{-n}(\alpha)\) has bi-degree \((n, 2n - 2 + |\alpha|)\), i.e.,

\[
\bar{a}_{-n}(\alpha) : H^r(S^{[m,m+1]}) \rightarrow H^{r+2n-2+|\alpha|}(S^{[m+n,m+n+1]}).
\]

3.3. \(H^*(S)\)-linearity. Recall from (3.2) that \(\widetilde{H}_S\) is an \(H^*(S)\)-module.

**Lemma 3.3.** The map \(\bar{a}_{-n}(\alpha) : \widetilde{H}_S \rightarrow \widetilde{H}_S\) is an \(H^*(S)\)-module homomorphism.

**Proof.** We will only prove the Lemma for \(n > 0\) since the proof for \(n < 0\) is similar. Let \(n > 0\), \(\beta \in H^*(S)\), and \(\bar{A} \in H^*(S^{[m,m+1]})\). We need to show that

\[
\bar{a}_{-n}(\alpha)(\rho^*_m \beta \cdot \tilde{A}) = (-1)^{|\alpha||\beta|} \rho^*_m \beta \cdot \bar{a}_{-n}(\alpha)(\tilde{A}) \tag{3.6}
\]

where the morphism \(\rho^*_m\) is defined in (2.10). By definition,

\[
\bar{a}_{-n}(\alpha)(\rho^*_m \beta \cdot \tilde{A}) = \bar{p}_1^*([\bar{Q}^{[m+n,m]}] \cdot \bar{\rho}^* \alpha \cdot \bar{p}_2^* (\rho^*_m \beta \cdot \tilde{A}))
\]

\[
= (-1)^{|\alpha||\beta|} \bar{p}_1^*([\bar{Q}^{[m+n,m]}] \cdot \bar{p}_2^* \rho^*_m \beta \cdot \bar{\rho}^* \alpha \cdot \tilde{A})
\]

\[
= (-1)^{|\alpha||\beta|} \bar{p}_1^* (\tau(\rho_m \circ \bar{p}_2 \circ \tau)^* \beta \cdot \bar{\rho}^* \alpha \cdot \tilde{A})
\]

where \(\tau : \bar{Q}^{[m+n,m]} \rightarrow S^{[m+n,m+n+1]} \times S \times S^{[m,m+1]}\) is the inclusion map.
From the proof of Lemma 3.1, we see that for \((\xi, \xi'), (\eta, \eta')\) \(\in \tilde{Q}^{[m+n, m]}\),
\[
\text{Supp}(I_{\eta'}/I_{\xi'}) = \text{Supp}(I_{\eta}/I_{\xi}) = \{s\}.
\]
Therefore, \(\rho_m \circ \tilde{p}_2 \circ \tau = \rho_{m+n} \circ \tilde{p}_1 \circ \tau\). Hence, we obtain
\[
\tilde{a}_{-n}(\alpha)(\rho_m^* \beta \cdot \tilde{A}) = (-1)^{|\alpha| |\beta|} \tilde{p}_1^*(\tau_*(((\rho_{m+n} \circ \tilde{p}_1 \circ \tau)^* \beta) \cdot \tilde{p}_1^* \alpha \cdot \tilde{p}_2^* \tilde{A})
\]
\[
= (-1)^{|\alpha| |\beta|} \tilde{p}_1^*([\tilde{Q}^{[m+n,m]}] \cdot \tilde{p}_1^* \rho_{m+n}^* \beta \cdot \tilde{p}_1^* \alpha \cdot \tilde{p}_2^* \tilde{A})
\]
\[
= (-1)^{|\alpha| |\beta|} \rho_{m+n}^* \beta \cdot \tilde{p}_1^*([\tilde{Q}^{[m+n,m]}] \cdot \tilde{p}_1^* \alpha \cdot \tilde{p}_2^* \tilde{A})
\]
by the projection formula. It follows immediately that (3.6) holds.

3.4. Comparisons with Heisenberg operators on \(\mathbb{H}_S\). There are two natural morphisms from \(S^{[m+m+1]}\) to \(S^{[m]}\) and \(S^{[m+1]}\) respectively:
\[
S^{[m,m+1]} \xrightarrow{\varphi} S^{[m+1]}
\]
\[
\downarrow f_m \quad \downarrow f_{m+1}
\]
\[
S^{[m]}.
\]
Since there are Heisenberg operators \(\mathfrak{a}_{-n}(\alpha)\) on \(\mathbb{H}_S\), it is natural to ask whether the following diagram is commutative or not:
\[
\begin{array}{ccc}
H^*(S^{[m]}) & \xrightarrow{\mathfrak{a}_{-n}(\alpha)} & H^*(S^{[m+n]}) \\
\downarrow f_m^* & & \downarrow f_{m+n}^* \\
H^*(S^{[m,m+1]}) & \xrightarrow{\tilde{a}_{-n}(\alpha)} & H^*(S^{[m+n,m+n+1]}).
\end{array}
\]

Lemma 3.4. The diagram (3.7) is commutative.

This Lemma not only relates different Heisenberg operators on different spaces, but also is used in the proof of Proposition 3.5 to determine a constant.

Proof. Again, we will only prove the Lemma for \(n > 0\).

Let \(n > 0\) and \(A \in H^*(S^{[m]})\). By (3.8) and (3.4), we obtain
\[
f_{m+n}^* a_{-n}(\alpha)(A) = f_{m+n}^* p_1*([\tilde{Q}^{[m+n,m]}] \cdot \rho^* \alpha \cdot \tilde{p}_2^* A),
\]
\[
\tilde{a}_{-n}(\alpha) f_m^*(A) = \tilde{p}_1^*([\tilde{Q}^{[m+n,m]}] \cdot \tilde{p}_1^* \alpha \cdot \tilde{p}_2^* f_m^*(A)).
\]

Let \(g = \text{Id}_{S^{[m+n,m+n+1]}\times S} \times f_m\) and \(h = f_{m+n} \times \text{Id}_{S^{[m]}}\):
\[
g : S^{[m+n,m+n+1]} \times S^{[m]} \to S^{[m+n,m+n+1]} \times S^{[m+n,m+n+1]} \times S^{[m]},
\]
\[
h : S^{[m+n,m+n+1]} \times S^{[m]} \to S^{[m+n,m+n+1]} \times S^{[m+n,m+n+1]} \times S^{[m]},
\]
and let \(\tilde{p}_1 : S^{[m+n,m+n+1]} \times S^{[m]} \to S^{[m+n,m+n+1]}\) be the first projection. By (3.8) and base change, we conclude that
\[
f_{m+n}^* a_{-n}(\alpha)(A) = (\tilde{p}_1)_* h^*([\tilde{Q}^{[m+n,m]}] \cdot \rho^* \alpha \cdot \tilde{p}_2^* A)
\]
\[
= (\tilde{p}_1)_* h^*([\tilde{Q}^{[m+n,m]}] \cdot (\rho \circ h)^* \alpha \cdot (p_2 \circ h)^* A). \quad (3.10)
\]
Similarly, using (3.9), \( \tilde{p}_1 = p_1 \circ g \), and the projection formula, we obtain
\[
\tilde{a}_{-n}(\alpha)f^*_m(A) = (\tilde{p}_1)_*g_* \left( [\tilde{Q}^{[m+n,m]}] \cdot g^*((\rho \circ h)^*\alpha \cdot (p_2 \circ h)^*A) \right)
\]
\[
= (\tilde{p}_1)_*(g_*[\tilde{Q}^{[m+n,m]}]) \cdot (\rho \circ h)^*\alpha \cdot (p_2 \circ h)^*A.
\] (3.11)

Next, we compare the two cycles \( h^*[Q^{[m+n,m]}] \) and \( g_*[\tilde{Q}^{[m+n,m]}] \). Note that both \( h^{-1}(Q^{[m+n,m]}) \) and \( g(\tilde{Q}^{[m+n,m]}) \) are equal to the subset
\[
W := \{ ((\xi, \xi'), s, \eta) | \xi \supset \eta \text{ and } \text{Supp}(I_\eta/I_\xi) = \{s\} \}
\]
of \( S^{[m+n,m+1]} \times S \times S^{[m]} \). A typical element \((\xi, s, \eta) \in Q^{[m+n,m]}\) is of the form:
\[
(s_1 + \ldots + s_m + \xi_s, s, s_1 + \ldots + s_m)
\]
where \( s_1, \ldots, s_m, s \in S \) are distinct, and \( \xi_s \in M_n(s) \). Hence,
\[
h^*[Q^{[m+n,m]}] = [W].
\] (3.12)

On the other hand, a typical element \((((\xi, \xi'), s, (\eta, \eta')) \in \tilde{Q}^{[m+n,m]}\) is of the form:
\[
((\xi_0 + \xi_s, \xi_0 + s_{m+1} + \xi_s), s, (\xi_0, \xi_0 + s_{m+1}))
\]
where \( s_1, \ldots, s_{m+1}, s \in S \) are distinct, \( \xi_0 = s_1 + \ldots + s_m \), and \( \xi_s \in M_n(s) \). Hence,
\[
g_*[\tilde{Q}^{[m+n,m]}] = [W].
\] (3.13)

The combinations of (3.10), (3.11), (3.12) and (3.13) prove the Lemma. \( \square \)

We remark that in Section 4.4, another comparison of the Heisenberg operators \( a_{-n}(\alpha) \) and \( \tilde{a}_{-n}(\alpha) \) by using the pullbacks \( g^*_{m+1} \) will be presented.

3.5. Heisenberg commutation relation. The proof of the following Proposition is a bit long. It is divided into many cases. If one understands one case, one should be able to follow the proof of remaining cases easily.

**Proposition 3.5.** The operators \( \tilde{a}_{n}(\alpha), n \in \mathbb{Z}, \alpha \in H^*(S) \) satisfy the following Heisenberg algebra commutation relation:
\[
[\tilde{a}_{n}(\alpha), \tilde{a}_{k}(\beta)] = -n \delta_{n,-k} \int_S (\alpha \beta) \cdot \text{Id}_{\tilde{\mathbb{R}}^S}.
\] (3.14)

**Proof.** In view of (3.3), formula (3.14) is equivalent to the formulas:
\[
[\tilde{a}_{-n}(\alpha), \tilde{a}_{-k}(\beta)] = 0,
\] (3.15)
\[
[\tilde{a}_{-n}(\alpha), \tilde{a}_{k}(\beta)] = n \delta_{n,k} \int_S (\alpha \beta) \cdot \text{Id}_{\tilde{\mathbb{R}}^S}
\] (3.16)

where \( n, k > 0 \). We will prove these two formulas separately.

**Proof of (3.15):**
Let \( n, k > 0 \). Then the operator \( \tilde{a}_{-n}(\alpha)\tilde{a}_{-k}(\beta) \) is induced by the class:
\[
w = \pi_{1245}^* \left( \pi_{123}^* [\tilde{Q}^{[m+n+k,m+k]}] \cdot \pi_{345}^* [\tilde{Q}^{[m+k,m]}] \right) \in A_{2m+k+n+4}(W')
\] (3.17)
where \( A_s(\cdot) \) denotes the Chow group, and \( W' = \pi_{1245}(W) \) with
\[
W = \pi_{123}^{-1}(Q^{m+n+k,m+k}) \cap \pi_{345}^{-1}(Q^{m+k,m}).
\]

Note that \( W \) is a closed subset of the ambient space
\[
S^{[m+n+k,m+n+k+1]} \times S \times S^{[m+k,m+k+1]} \times S^{[m,m+1]}.
\]

In the sequel, the ambient spaces in similar situations will not be explicitly presented since they can be written out easily from the context.

From Lemma 3.1, we know that the expected dimension of the intersection \( W \) should be \( 2m + n + k + 4 \). The subset \( W \) may have many irreducible components. Those that are mapped by \( \pi_{1245} \) to subsets of dimension less than \( 2m + n + k + 4 \) will not contribute to the cycle \( w \). The aim of the computations below is to pick out those components with dimension no less than the expected dimension.

Any element \((\xi, \zeta, s, (\eta, \eta'), t, (\zeta', \zeta'')) \) in \( W \) must satisfy the conditions:
\[
\zeta \subset \zeta', \quad \eta \subset \eta', \quad \xi \subset \xi', \quad \zeta \subset \eta \subset \xi, \quad \zeta' \subset \eta' \subset \zeta',
\]
\[
\text{Supp}(I_{n}/I_{\xi}) = \{s\}, \quad \text{Supp}(I_{\zeta}/I_{n}) = \{t\},
\]
\[
\text{Supp}(I_{\zeta'}/I_{\eta'}) = \text{Supp}(I_{\eta'}/I_{\xi'}) = \text{Supp}(I_{\xi'}/I_{\eta}) = \{p\}
\]
for some \( p \in S \). In the following, we consider four different cases separately. We use \( W_i \) to denote the subset of \( W \) consisting of all the points satisfying Case \( i \), put
\[
\mathfrak{w}_i = \dim \pi_{1245}(W_i)
\]
(similar notations such as \( U_i, u_i, V_i, v_i \) will be used throughout the rest of the paper).

**Case 1**: \( s, t, p \) are distinct. We have the following decompositions:
\[
\zeta = \zeta_o + \zeta_s + \zeta_t + \zeta_p, \quad \zeta' = \zeta_o + \zeta_s + \zeta_t + \zeta'_p,
\]
\[
\eta = \zeta_o + \zeta_s + \eta_t + \zeta_p, \quad \eta' = \zeta_o + \zeta_s + \eta_t + \zeta'_p,
\]
\[
\xi = \zeta_o + \xi_s + \eta_t + \zeta_p, \quad \xi' = \zeta_o + \xi_s + \eta_t + \zeta'_p,
\]
where \( \text{Supp}(\zeta_o) \cap \{s, t, p\} = \emptyset, \ell(\zeta_o) = i, \ell(\zeta) = j, \ell(\zeta_p) = \ell, \) and \( \zeta_s, \zeta_t, \ldots \) are supported at \( \{s\}, \{t\}, \ldots \) respectively. Then,
\[
\mathfrak{w}_1 = \#(\text{moduli of } \zeta_o) + \#(\text{moduli of } \zeta_s) + \#(\text{moduli of } \zeta_t) + \#(\text{moduli of } \zeta_p) + \#(\text{moduli of } \eta_t) + \#(\text{moduli of } \xi_s) + \#(\text{moduli of } \zeta_p \subset \zeta'_p)
\]
\[
= 2(m - i - j - \ell) + \max(i - 1, 0) + 2 + \max(j - 1, 0) + 2 + (j + k - 1) + (i + n - 1) + \ell + 2
\]
\[
= (2m + k + n + 4) - i + \max(i - 1, 0) - j + \max(j - 1, 0) - \ell.
\]

**Case 1.1**: If \( i = j = \ell = 0 \), then we have \( \mathfrak{w}_1 = (2m + k + n + 4) \).

**Case 1.2**: If one of the integers \( i, j \) and \( \ell \) is positive, then \( \mathfrak{w}_1 < (2m + k + n + 4) \).

The three remaining cases are listed by:
- **Case 2**: \( s = t \neq p \);
- **Case 3**: \( s = p \neq t \) (by symmetry, this also covers the case \( t = p \neq s \));
- **Case 4**: \( s = p = t \).
For these three cases, we skip the arguments which are in the same style as in Case 1. They all have dimension less than the expected dimension \((2m + n + k + 4)\). Therefore, only Case 1.1 has contribution to the cohomological operation. In this subcase, it is not difficult to show that the intersection \((3.18)\) along \(W_1\) is transversal. Moreover, \(\pi_{1245}(W_1)\) consists of all the points of the form:

\[
\left( (\zeta_0 + \xi_s + \eta_t, \zeta_0 + \xi_s + \eta_t + p), s, t, (\zeta_0, \zeta_0 + p) \right)
\]

in \(S^{(m+n+k,m+n+k+1)} \times S \times S^{(m,m+1)}\). So the contribution of this subcase to the operator \(\tilde{a}_{-n}(\alpha)\tilde{a}_{-k}(\beta)\) coincides with the corresponding contribution for the operator \((-1)^{|\alpha||\beta|} \cdot \tilde{a}_{-k}(\beta)\tilde{a}_{-n}(\alpha)\). In other words, we obtain the identity:

\[
\tilde{a}_{-n}(\alpha)\tilde{a}_{-k}(\beta) = (-1)^{|\alpha||\beta|} \cdot \tilde{a}_{-k}(\beta)\tilde{a}_{-n}(\alpha).
\]

This completes the proof of the commutation relation \((3.15)\).

Proof of \((3.16)\):

Let \(n, k > 0\). Then the operator \(\tilde{a}_{-n}(\alpha)\tilde{a}_{-k}(\beta)\) is induced by the class:

\[
u = \pi_{1245}\left( \pi_{123}^* \left[ \tilde{Q}^{(m+n,m)} \right] \cdot \pi_{345}^* \left[ \tau_{m+k} \left( \tilde{Q}^{(m+k,k)} \right) \right] \right) \in A_{2m+k+n+4}(U')
\]

(3.19)

where \(\tau_{m+k} : S^{(m+k,m+k+1)} \times S^{(m,m+1)} \rightarrow S^{(m,m+1)} \times S^{(m+k,m+k+1)}\) is the isomorphism switching \(S^{(m,m+1)}\) and \(S^{(m+k,m+k+1)}\), and \(U' = \pi_{1245}(U)\) with

\[
U = \pi_{123}^{-1} \left( \tilde{Q}^{(m+n,m)} \right) \cap \pi_{345}^{-1} \left( \tau_{m+k} \left( \tilde{Q}^{(m+k,k)} \right) \right).
\]

(3.20)

Any element \(((\xi, \xi'), s, (\eta, \eta'), t, (\zeta, \zeta'))\) in \(U\) must satisfy the conditions

\[
\zeta \subset \xi', \quad \eta \subset \eta', \quad \xi \subset \zeta', \quad \xi' \supset \eta \subset \zeta, \quad \xi' \supset \eta' \subset \zeta',
\]

\[
\text{Supp}(I_{\eta}/I_{\zeta}) = \{s\}, \quad \text{Supp}(I_{\eta}/I_{\zeta'}) = \{t\},
\]

\[
\text{Supp}(I_{\xi}/I_{\zeta'}) = \text{Supp}(I_{\eta}/I_{\eta'}) = \text{Supp}(I_{\xi}/I_{\xi'}) = \{p\}
\]

for some \(p \in S\). In the following, we consider four different cases separately.

**Case 1:** \(s, t, p\) are distinct. We have the following decompositions:

\[
\eta = \eta_0 + \eta_s + \eta_t + \eta_p, \quad \eta' = \eta_0 + \eta_s + \eta_t + \eta_p',
\]

\[
\xi = \eta_0 + \xi_s + \eta_t + \eta_p, \quad \xi' = \eta_0 + \xi_s + \eta_t + \eta_p',
\]

\[
\zeta = \eta_0 + \xi_s + \zeta_t + \eta_p, \quad \zeta' = \eta_0 + \xi_s + \zeta_t + \eta_p'
\]

where \(\text{Supp}(\eta_0) \cap \{s, t, p\} = \emptyset, \ell(\eta_s) = i, \ell(\eta_t) = j, \text{ and } \ell(\eta_p) = \ell\). Then,

\[
u_1 = \# \text{(moduli of } \eta_0) + \# \text{(moduli of } \xi_s) + \# \text{(moduli of } \zeta_t) + \# \text{(moduli of } \eta_p) + \# \text{(moduli of } \eta_0) + \# \text{(moduli of } \eta_0) + \# \text{(moduli of } \eta_0) + \# \text{(moduli of } \eta_0) + \# \text{(moduli of } \eta_0) + \# \text{(moduli of } \eta_0)
\]

\[
= 2(i - j - \ell) + (i + n - 1) + (j + k - 1) + \max(i - 1, 0) + \max(j - 1, 0) + \ell + 6
\]

\[
= (2m + k + n + 4) - i + \max(i - 1, 0) - j + \max(j - 1, 0) - \ell.
\]

**Case 1.1:** If \(i = j = \ell = 0\), then we have \(\nu_1 = (2m + k + n + 4)\).

**Case 1.2:** If one of the integers \(i, j\) and \(\ell\) is positive, then \(\nu_1 < (2m + k + n + 4)\).

The three remaining cases are listed by:
In these three cases, all the dimensions are smaller than the expected dimension $(2m + n + k + 4)$. So only Case 1.1 contributes to the class $u$ in (3.19).

Next we consider the operator $\tilde{a}_k(\beta)\tilde{a}_n(\alpha)$. This is the only case where there are two components for $V$ below with the expected dimension. One of the components will cancel out with the one from $\tilde{a}_n(\alpha)\tilde{a}_k(\beta)$, and the other is a non-transversal intersection which carries a multiplicity. Using Lemma 3.4 which compares $\tilde{a}_n(\alpha)$ with the Heisenberg operators $a_n(\alpha)$ on $\mathbb{H}_S$, we determine the multiplicity.

More precisely, the operator $\tilde{a}_k(\beta)\tilde{a}_n(\alpha)$ is induced by the class:

$$v = \pi_{1245*} \left( \pi_{123}^* \left( \tau_{m+n,m+k+n}(\tilde{Q}^{[m+k+n,m+n]}) \right) \cdot \pi_{345}^* \left( \tilde{Q}^{[m+k+n,m+k]} \right) \right)$$

in $A_{2m+k+n+4}(V')$, where $V' = \pi_{1245}(V)$ and $V$ is given by

$$V = \pi_{123}^1 \left( \tau_{m+n,m+k+n}(\tilde{Q}^{[m+k+n,m+n]}) \right) \cap \pi_{345}^{-1} \left( Q^{[m+k+n,m+k]} \right).$$

Any element $((\xi, \xi'), (\eta, \eta'), (\ell, (\zeta, \zeta'))$ in (3.22) must satisfy the conditions:

$$\zeta, \eta, \xi, \xi', \eta', \zeta' \in \zeta', \eta, \xi, \xi', \eta', \zeta' \subset \zeta, \eta, \xi, \xi', \eta', \zeta,'$$

$$\text{Supp}(I_\zeta/I_{\eta}) = \{s\}, \text{ Supp}(I_{\xi}/I_{\eta}) = \{t\},$$

$$\text{Supp}(I_{\zeta}/I_{\eta'}) = \text{Supp}(I_{\xi}/I_{\eta'}) = \text{Supp}(I_{\xi}/I_{\xi'}) = \{p\}$$

for some $p \in S$. In the following, we consider four different cases separately.

**Case 1':** $s, t, p$ are distinct. We have the following decompositions:

$$\zeta = \zeta_0 + \zeta_s + \zeta_t + \zeta_p, \quad \zeta' = \zeta_0 + \zeta_s + \zeta_t + \zeta'_p,$$

$$\eta = \zeta_0 + \zeta_s + \eta_t + \zeta_p, \quad \eta' = \zeta_0 + \zeta_s + \eta_t + \zeta'_p,$$

$$\xi = \zeta_0 + \zeta_s + \eta_t + \zeta_p, \quad \xi' = \zeta_0 + \zeta_s + \eta_t + \zeta'_p,$$

where $\text{Supp}(\zeta_0) \cap \{s, t, p\} = \emptyset$, $\ell(\zeta_0) = i$, $\ell(\zeta_t) = j$, and $\ell(\zeta_p) = \ell$. Then,

$$v_1 = \#(\text{moduli of } \zeta_0) + \#(\text{moduli of } \zeta_s) + \#(\text{moduli of } \zeta_t) +$$

$$+ \#(\text{moduli of } \eta_t) + \#(\text{moduli of } \zeta_s) + \#(\text{moduli of } \zeta_p \subset \zeta'_p)$$

$$= 2(m - i - j - \ell) + (i + k - 1) + \max(j - 1, 0) +$$

$$+ (j + n - 1) + \max(i - 1, 0) + \ell + 6$$

$$= (2m + k + n + 4) - i + \max(i - 1, 0) - j + \max(j - 1, 0) - \ell.$$ 

**Case 1'.1:** If $i = j = \ell = 0$, then we have $v_1 = (2m + k + n + 4)$.

**Case 1'.2:** If one of the integers $i, j$ and $\ell$ is positive, then $v_1 < (2m + k + n + 4)$.

**Case 2':** $s = t \neq p$. We have the following decompositions:

$$\zeta = \zeta_0 + \zeta_s + \zeta_p, \quad \zeta' = \zeta_0 + \zeta_s + \zeta'_p,$$

$$\eta = \zeta_0 + \eta_s + \zeta_p, \quad \eta' = \zeta_0 + \eta_s + \zeta'_p,$$

$$\xi = \zeta_0 + \xi_s + \zeta_p, \quad \xi' = \zeta_0 + \xi_s + \zeta'_p.$$
where $\text{Supp}(\zeta_0) \cap \{s, p\} = \emptyset$, $\ell(\zeta_s) = i$, $\ell(\zeta_t) = j$, and $\ell(\zeta_p) = \ell$. Note that
\[ k + i = \ell(\eta_s) = n + j. \] (3.23)

Case 2.1: $i = j = \ell = 0$. By (3.23), we have $k = n$. Thus,
\[ \nu_2 = 2(m + k) + 4 = 2m + k + n + 4. \]

Case 2.2: If one of the integers $i, j$ and $\ell$ is positive, then $\nu_2 < (2m + k + n + 4)$.

The two remaining cases are listed by:

- **Case 3’**: $s = p \neq t$ (by symmetry, this also covers the case $t = p \neq s$);
- **Case 4’**: $s = p = t$.

In these cases, the dimensions are smaller than $(2m + k + n + 4)$. Therefore these two cases have no contributions to the class $v$ in (3.21).

Finally, note that the contribution of Case 1.1 to the class $u$ in (3.19) and the contribution of Case 1’ to the class $v$ in (3.21) cancel out in the commutation:
\[ [\tilde{\mathfrak{a}}_{-n}(\alpha), \tilde{\mathfrak{a}}_{k}(\beta)]. \] (3.24)

So $[\tilde{\mathfrak{a}}_{-n}(\alpha), \tilde{\mathfrak{a}}_{k}(\beta)] = 0$ when $n \neq k$. This proves (3.10) when $n \neq k$.

When $n = k$, Case 2’ also contributes to the operator $\tilde{\mathfrak{a}}_{k}(\beta)\tilde{\mathfrak{a}}_{-n}(\alpha)$, and hence to (3.22). Note from Case 2’ that $\pi_{1245}(V_2)$ consists of all the points of the form:
\[ ((\zeta_0, \zeta_0 + p), s, s, (\zeta_0, \zeta_0 + p)) \]
in $S^{[m+n,m+n+1]} \times S \times S \times S^{[m+n,m+n+1]}$, where $s \neq p$ and $p \notin \text{Supp}(\zeta_0)$. Thus,
\[ [\tilde{\mathfrak{a}}_{-n}(\alpha), \tilde{\mathfrak{a}}_{n}(\beta)]]_{H^*(S[m+n,m+n+1])} = c \cdot \int_{S} (\alpha \beta) \cdot \text{Id}_{H^*(S[m+n,m+n+1])} \]
for some constant $c$. Now we conclude from Lemma 3.3 and (2.6) that $c = n$. This completes the proof of the commutation relation (3.10) when $n = k$. \hfill \Box

### 4. A translation operator for incidence Hilbert schemes

In this section, we will introduce a new operator $\tilde{t}$ on $\tilde{\mathbb{H}}_S$, called the translation operator. The operator $\tilde{t}$ is constructed via a correspondence and has many nice properties. Indeed, we will show that it is an $H^*(S)$-module homomorphism, commutes with the Heisenberg operators, and has a left inverse. These properties imply that it is responsible for the second factor in (1.1).

From Lemma 3.3, we may tend to infer that the Heisenberg operators $\tilde{\mathfrak{a}}_{n}(\alpha)$ are no different from the pullback of the more well-known Heisenberg operators $\mathfrak{a}_{n}(\alpha)$. This is almost the case when we compare these two types of operators via the map
\[ f_m: S^{[m,m+1]} \to S^{[m]}. \]
However, we will see that when we compare them via the map
\[ g_{m+1}: S^{[m,m+1]} \to S^{[m+1]}, \]
they are far from the same. The difference somehow is measured by the new translation operator. Once we are presented with the fact that the space $\tilde{\mathbb{H}}_S$ is a highest weight module of the algebra generated by the operators $\tilde{\mathfrak{a}}_{n}(\alpha), \beta \in H^*(S)$,
and \( \tilde{t} \), we realize that the naive choice of the pullback of Heisenberg algebras on Hilbert schemes either by the map \( f_m \) or \( g_{m+1} \) won’t provide the right algebra. In this regard, the new algebra we are going to construct is subtler and richer than the Heisenberg algebra on the Hilbert scheme \( S^{[m]} \). The fundamental difference between these two algebras is the translation operator \( \tilde{t} \).

### 4.1. Definition of the translation operator.

Let \( \tilde{Q}_m \) be the closed subset:
\[
\tilde{Q}_m = \{(\xi', \xi''), (\xi, \xi') | \text{Supp}(I_{\xi}/I_{\xi'}) = \text{Supp}(I_{\xi'}/I_{\xi''}) \subset S^{[m+1,m+2]}_m \times S^{[m,m+1]}_m \}.
\]

As in Lemma 3.1, we can verify that \( \dim \tilde{Q}_m = 2m + 3 \).

**Definition 4.1.** Define the linear operator \( \tilde{t} \in \text{End}(\tilde{\mathbb{H}}_S) \) by
\[
\tilde{t}(\tilde{A}) = p_{1*}([(\tilde{Q}_m) \cdot p_{2*} \tilde{A})]
\]
for \( \tilde{A} \in H^*(S^{[m,m+1]}) \), where \( p_1, p_2 \) are the two projections of \( S^{[m+1,m+2]}_m \times S^{[m,m+1]}_m \).

The bi-degrees of \( \tilde{t} \) and its adjoint \( \tilde{t}^\dagger \) are \((1,2)\) and \((-1,-2)\) respectively.

### 4.2. \( H^*(S) \)-linearity and the left inverse.

Recall that \( \tilde{\mathbb{H}}_S \) is an \( H^*(S) \)-module.

**Lemma 4.2.** (i) The operator \( (-\tilde{t}) \) is the left inverse of \( \tilde{t} \);

(ii) The maps \( \tilde{t}, \tilde{t}^\dagger : \tilde{\mathbb{H}}_S \to \tilde{\mathbb{H}}_S \) are \( H^*(S) \)-module homomorphisms.

**Proof.** We skip the proof of (ii) since it is similar to the proof of Lemma 3.3. In the following, we prove (i). Note that the operator \( \tilde{t}^\dagger \tilde{t} \) is induced by the class:

\[
w = \pi_{13*} \left( \pi_{12*}^* \left[ \tau(\tilde{Q}_m) \right] \cdot \pi_{23*}^* \left[ \tilde{Q}_m \right] \right) \in A_{2m+2}(W')
\]

where \( \tau : S^{[m+1,m+2]}_m \times S^{[m,m+1]}_m \to S^{[m+1,m+2]}_m \times S^{[m,m+1]}_m \) is the isomorphism switching the two factors, \( W' = \pi_{13}(W) \), and \( W \) is given by

\[
W = \pi_{12}^{-1}(\tau(\tilde{Q}_m)) \cap \pi_{23}^{-1}(\tilde{Q}_m).
\]

The expected dimension of the intersection \( W \) is \( 2m + 2 \).

Any element \( ((\eta, \xi'), (\xi', \xi''), (\xi, \xi')) \) in \( W \) must satisfy the conditions:

\[
\eta, \xi \subset \xi' \subset \xi'',
\]

\[
\text{Supp}(I_{\eta}/I_{\xi'}) = \text{Supp}(I_{\xi'}/I_{\xi''}) = \text{Supp}(I_{\xi'}/I_{\xi''}) = \{p\}
\]

for some \( p \in S \). We have the following decompositions:

\[
\xi = \xi_0 + \xi_p, \quad \xi' = \xi_0 + \xi_p',
\]

\[
\xi'' = \xi_0 + \xi_p'', \quad \eta = \xi_0 + \eta_p,
\]

where \( p \notin \text{Supp}(\xi_0) \) and \( \ell(\xi_p) = i \). Note that \( \ell(\eta_p) = i \). Fix the integer \( i \).

If \( i > 0 \), then the projection of the subset of \( W \), consisting of all the elements

\[
((\eta, \xi'), (\xi', \xi''), (\xi, \xi'))
\]
to \(S^{[m,m+1]} \times S^{[m,m+1]}\) has dimension at most equal to:

\[
\#(\text{moduli of } \xi_0) + \#(\text{moduli of } \xi_p \subset \xi'_p) + \#(\text{moduli of } \eta_p) = 2(m-i) + i + (i-1) + 2 < (2m+2).
\]

So the case \(i > 0\) does not contribute to the class \(w\) in (4.2).

If \(i = 0\), then \(\xi'_p = p, \xi''_p \in M_2(p)\), and \(\xi_p = \eta_p = \emptyset\). So \(\eta = \xi = \xi_0\). The projection of this part of \(W\) to \(S^{[m,m+1]} \times S^{[m,m+1]}\) consists of elements of the form:

\[
\{((\xi_0, \xi_0 + p), (\xi_0, \xi_0 + p))\}.
\]

It follows that \(w = c[\Delta]\) for some constant \(c\), where \(\Delta\) denotes the diagonal in \(S^{[m,m+1]} \times S^{[m,m+1]}\). Hence, we get \(\tilde{u}^\dagger \tilde{t} = c \text{Id}_{\mathcal{H}^s(S^{[m,m+1]})}\).

To determine \(c\), note that we can split off the factor \(\xi_0\) from our consideration, i.e., we can simply consider the case \(m = 0\). Then we have the morphism:

\[
\pi_{13} : S \times S^{[1,2]} \times S \to S \times S.
\]

Fix a point \(p \in S\). Since \([\Delta] \cdot \{(p) \times S\} = 1\), we see from (4.2) that

\[
c = w \cdot \{(p) \times S\} = \pi_{13*} \left( \pi_{12*} [\tau(\tilde{Q}_0)] \cdot \pi_{23*} [\tilde{Q}_0] \right) \cdot \{(p) \times S\} = \pi_{12*} [\tau(\tilde{Q}_0)] \cdot \pi_{23*} [\tilde{Q}_0] \cdot \{(p) \times S^{[1,2]} \times S\} = \{(p) \times U_p \times S\} \cdot \{(p) \times \tilde{Q}_0\}
\]

where \(U_p = \{(p, \xi''_p) | \xi''_p \in M_2(p)\}\). It follows that

\[
c = [U_p \times S] \cdot [\tilde{Q}_0] = [U_p] \cdot [\phi_1* (\tilde{Q}_0)] = [U_p] \cdot [\phi_1 (\tilde{Q}_0)]
\]

(4.4)

where \(\phi_1 : S^{[1,2]} \times S \to S^{[1,2]}\) is the projection. We have

\[
\phi_1 (\tilde{Q}_0) = \{(s, \xi''_s) | s \in S \text{ and } \xi''_s \in M_2(s)\}.
\]

Recall the natural morphism \(g_2 : S^{[1,2]} \to S^{[2]}\). It is known from [ES] that

\[
[\phi_1 (\tilde{Q}_0)] = \frac{1}{2} g^*_2[M_2(S)]
\]

where \(M_2(S) = \cup_{s \in S} M_2(s)\). Hence, we see from (4.4) that

\[
c = [U_p] \cdot \frac{1}{2} g^*_2[M_2(S)] = \frac{1}{2} g_2* [U_p] \cdot [M_2(S)] = \frac{1}{2} [g_2(U_p)] \cdot [M_2(S)] = \frac{1}{2} [M_2(p)] \cdot [M_2(S)] = -1
\]

where we have used the fact that \([M_2(p)] \cdot [M_2(S)] = -2\) from [ES]. \(\square\)
4.3. **Commutativity with the Heisenberg operators.** The translation operator $\tilde{t}$ may look similar to the creation operators at the first glimpse. However, we now show that it differs from the creation operators in the essential way in that it commutes with all the annihilation operators.

**Proposition 4.3.** The translation operator $\tilde{t}$ and its adjoint $\tilde{t}^\dagger$ commute with the Heisenberg operators $\tilde{a}_{-n}(\alpha)$ for all $n$ and $\alpha$, i.e.,

$$[\tilde{t}, \tilde{a}_{-n}(\alpha)] = [\tilde{t}^\dagger, \tilde{a}_{-n}(\alpha)] = 0.$$ 

**Proof.** Let $n > 0$. Then Proposition 4.3 is decomposed into two parts:

$$[\tilde{t}, \tilde{a}_{-n}(\alpha)] = 0, \quad (4.5)$$
$$[\tilde{t}, \tilde{a}_{n}(\alpha)] = 0. \quad (4.6)$$

We prove them separately, and will compare the proof with that of Proposition 3.5.

**Proof of (4.5):**

Let $n > 0$. This part is similar to the proof of (3.15).

The operator $\tilde{a}_{-n}(\alpha)\tilde{t}$ is induced by the class:

$$w = \pi_{124*}^{} \left( \pi_{123}^* \left[ \tilde{Q}[^{m+n+1,m+1}] \cdot \pi_{34}^* \left[ \tilde{Q}_m \right] \right] \right) \in A_{2m+n+4}(W') \quad (4.7)$$

where $W' = \pi_{124}(W)$ and the subset $W$ is defined by

$$W = \pi_{123}^{-1} \left( \tilde{Q}[^{m+n+1,m+1}] \right) \cap \pi_{34}^{-1} \left( \tilde{Q}_m \right). \quad (4.8)$$

The expect dimension of the intersection $W$ is $2m + n + 4$.

Any element $((\eta, \eta'), s, (\xi', \xi''), (\xi, \xi'))$ in (4.8) must satisfy the conditions

$$\eta \subset \eta', \quad \xi \subset \xi' \subset \xi'', \quad \xi' \subset \eta, \quad \xi'' \subset \eta',$$

$$\text{Supp} (I_{\xi}/I_{\eta}) = \{s\},$$

$$\text{Supp} (I_{\eta}/I_{\eta'}) = \text{Supp} (I_{\xi}/I_{\xi'}) = \text{Supp} (I_{\xi}/I_{\xi''}) = \{p\}$$

for some $p \in S$. In the following, we consider two cases separately.

**Case 1:** $s \neq p$. We have the following decompositions:

$$\xi = \xi_0 + \xi_s + \xi_p, \quad \xi' = \xi_0 + \xi_s + \xi'_p, \quad \xi'' = \xi_0 + \xi_s + \xi''_p,$$

$$\eta = \xi_0 + \eta_s + \xi'_p, \quad \eta' = \xi_0 + \eta_s + \xi''_p,$$

where $\text{Supp}(\xi_0) \cap \{s, p\} = \emptyset, \ell(\xi_s) = i$, and $\ell(\xi_p) = j$.

**Case 1.1:** If $i = j = 0$, then we have $w_1 = (2m + n + 4)$.

**Case 1.2:** If $i > 0$ or $j > 0$, then we obtain $w_1 < (2m + n + 4)$.

**Case 2:** $s = p$. We have $w_2 < (2m + n + 4)$.

In summary, only Case 1.1 contributes to the class $w$. In this subcase, the intersection (4.8) along $W_1$ is transversal, and $\pi_{1245}(W_1)$ consists of all the elements:

$$((\xi_0 + \eta_s + p, \xi_0 + \eta_s + \xi''_p), s, (\xi_0, \xi_0 + p))$$
in \( S^{[m+n+1,m+n+2]} \times S^{[m,m+1]} \). So the contribution of this subcase to the operator \( \tilde{a}_{-n}(\alpha) \) coincides with the corresponding contribution for the operator \( \tilde{t} \tilde{a}_{-n}(\alpha) \). This completes the proof of the commutation relation (4.5).

**Proof of (4.6):**

This part looks similar to the proof of (3.16). However, it is fundamentally different. Recall that the subset \( V \) in (3.22) whose correspondence defines \( \tilde{a}_{n}(\alpha)\tilde{a}_{-n}(\beta) \) has one more component with the expected dimension than the subset for the operator \( \tilde{a}_{-n}(\alpha)\tilde{a}_{n}(\beta) \). It doesn’t occur here. If there were two components, they would come from the operator \( \tilde{a}_{n}(\alpha)\tilde{t} \). So pay a special attention to Case 2 below and compare it with Case 2’.1 in the proof of Proposition 3.5. In order to illustrate this subtle difference, we present the full content of the proof.

Let \( n > 0 \). Then the operator \( \tilde{a}_{n}(\alpha)\tilde{t} \) is induced by the class:

\[
\alpha \equiv \pi_{124*} \left( \pi_{123}^{*} \left[ \tau_{m+1,m+n+1} \tilde{(Q^{[m+n+1,m+1]})} \right] \cdot \pi_{34}^{*} \left[ \tilde{Q}_{m+n} \right] \right) \in A_{2m+n+4}(U')
\]

where \( U' = \pi_{124}(U) \) and the subset \( U \) is defined by

\[
U = \pi_{123}^{-1} \left( \tau_{m+1,m+n+1} \tilde{(Q^{[m+n+1,m+1]})} \right) \cap \pi_{34}^{-1} \left( \tilde{Q}_{m+n} \right).
\]

Any element \( ((\eta, \eta'), s, (\xi', \xi''), (\xi, \xi'')) \) in (4.10) must satisfy the conditions

\[
\eta \subset \eta', \quad \xi \subset \xi' \subset \xi'', \quad \eta \subset \xi', \quad \eta' \subset \xi'',
\]

\[
\text{Supp}(I_{\eta}/I_{\xi'}) = \{s\}, \quad \text{Supp}(I_{\eta}/I_{\xi''}) = \text{Supp}(I_{\xi}/I_{\xi'}) = \{p\}
\]

for some \( p \in S \). In the following, we consider two cases separately.

**Case 1:** \( s \neq p \).

We have the following decompositions:

\[
\xi = \xi_{0} + \xi_{s} + \xi_{p}, \quad \xi' = \xi_{0} + \xi_{s} + \xi'_{p}, \quad \xi'' = \xi_{0} + \xi_{s} + \xi''_{p},
\]

\[
\eta = \xi_{0} + \eta_{s} + \xi'_{p}, \quad \eta' = \xi_{0} + \eta_{s} + \xi''_{p},
\]

where \( \text{Supp}(\xi_{0}) \cap \{s, p\} = \emptyset, \ell(\eta_{p}) = i, \) and \( \ell(\xi_{p}) = j \).

**Case 1.1:** If \( i = j = 0 \), then we have \( u_{1} = (2m + n + 4) \).

**Case 1.2:** If \( i > 0 \) or \( j > 0 \), then we obtain

\[
u_{1} \leq \#	ext{(moduli of } \xi_{0}) + \#	ext{(moduli of } \xi_{s}) + \#	ext{(moduli of } \eta_{s}) +
\]

\[
+ \#	ext{(moduli of } \xi_{p}) + \#	ext{(moduli of } \xi'_{p} \subset \xi''_{p})
\]

\[
= 2(m - i - j) + (i + n - 1) + \max(i - 1, 0) +
\]

\[
+ \max(j - 1, 0) + (j + 1) + 4
\]

\[
< (2m + n + 4).
\]

**Case 2:** \( s = p \).

We have the following decompositions:

\[
\xi = \xi_{0} + \xi_{s}, \quad \xi' = \xi_{0} + \xi'_{s}, \quad \xi'' = \xi_{0} + \xi''_{s},
\]

\[
\eta = \xi_{0} + \eta_{s}, \quad \eta' = \xi_{0} + \eta'_{s},
\]
where \( s \not\in \text{Supp}(\xi_0) \) and \( \ell(\eta_s) = i \). In this case, we see that
\[
\begin{align*}
\mu_2 &\leq \#(\text{moduli of } \xi_0) + \#(\text{moduli of } \xi_s \subset \xi'_s) + \#(\text{moduli of } \eta_s \subset \eta'_s) \\
&\leq 2(m + 1 - i) + (i + n - 1) + i + 2 \\
&< (2m + n + 4).
\end{align*}
\]

In summary, only Case 1.1 contributes to the class \( u \). The contribution of this subcase to the operator \( \tilde{a}_n(\alpha) \tilde{i} \) coincides with that of the operator \( i \tilde{a}_n(\alpha) \). This completes the proof of the commutation relation (4.6). \( \Box \)

4.4. Another comparison with Heisenberg operators on \( S^{[m+1]} \). This subsection is not related to the Theorem in \S 5. The main theme here is to compare the Heisenberg operators \( \tilde{a}_n(\alpha) \) with the pull-back of the Heisenberg operators \( a_n(\alpha) \) via the morphism \( g_{m+1}: S^{[m,m+1]} \rightarrow S^{[m+1]} \).

The comparison of annihilation operators is similar to Lemma 3.4. The proof of the following lemma is omitted since it is similar to the proof of Lemma 3.4:

**Lemma 4.4.** Let \( n > 0 \) and \( \alpha \in H^*(S) \). Then, we have a commutative diagram:
\[
\begin{array}{ccc}
H^*(S^{[m+1]}) & \xrightarrow{a_n(\alpha)} & H^*(S^{[m+1]}) \\
\downarrow g_{m+1} & & \downarrow g_{m+1}^* \\
H^*(S^{[m,m+1]}) & \xleftarrow{\tilde{a}_n(\alpha)} & H^*(S^{[m+1,n,m+1]})
\end{array}
\]

Lemma 4.4 will not hold for the creation operators. However, Proposition 4.6 below provides an explicit formula relating the creation operators. In order to prove Proposition 4.6, we begin with a technical lemma.

Let \( m \geq 0 \). Define \( \tilde{Q}_{m,0} \) to be the diagonal of \( S^{[m,m+1]} \times S^{[m,m+1]} \). For \( n \geq 1 \), define \( \tilde{Q}_{m,n} \) to be the closed subset of \( S^{[m,n,m+1]} \times S^{[m,m+1]} \):
\[
\tilde{Q}_{m,n} = \{((\xi, \xi'), (\eta, \eta')) | \xi' \supset \xi \supset \eta' \supset \eta, \\
\text{Supp}(I_\xi/I_{\xi'}) = \text{Supp}(I_\eta/I_{\eta'}) = \text{Supp}(I_{\eta}/I_{\eta'})\}.
\]

Note that \( \tilde{Q}_{m,n} \) has dimension \( (2m + n + 2) \), and contains exactly one irreducible component of dimension \( (2m + n + 2) \) whose generic elements are of the form:
\[
((\eta + \xi_s, \eta + \xi'_s), (\eta, \eta + s)), \quad s \notin \text{Supp}(\eta).
\]

**Lemma 4.5.** The restriction of \( \tilde{\imath}^n \) to \( H^*(S^{[m,m+1]}) \) is given by the cycle \( [\tilde{Q}_{m,n}] \).

*Proof.* Use induction on \( n \). Note that \( \tilde{Q}_{m,1} = \tilde{Q}_m \) by definition. So the Lemma is trivially true when \( n = 0,1 \). Next, assume that \( n \geq 2 \) and the restriction of \( \tilde{\imath}^{n-1} \) to \( H^*(S^{[m,m+1]}) \) is given by the cycle \( [\tilde{Q}_{m,n-1}] \). Then we see that the restriction of \( \tilde{\imath}^n = \tilde{\imath}^{n-1} \) to \( H^*(S^{[m,m+1]}) \) is given by the cycle
\[
w = \pi_{13*} \left( \pi_{12}^* [\tilde{Q}_{m,n+1}] \cdot \pi_{23}^* [\tilde{Q}_{m,n-1}] \right) \in A_{2m+n+2}(W') \tag{4.12}
\]
where \( W' = \pi_{13}(W) \) and the subset \( W \) is given by
\[
W = \pi_{12}^{-1}(\tilde{Q}_{m+n-1}) \cap \pi_{23}^{-1}(\tilde{Q}_{m,n-1}). \tag{4.13}
\]
Any element \( ((\xi, \xi'), (\theta, \theta'), (\eta, \eta')) \) in (4.13) must satisfy:
\[
\xi' \supset \xi = \theta' \supset \theta \supset \eta' \supset \eta,
\]
Supp\( (I_\xi/I_{\xi'}) = \text{Supp}(I_\theta/I_{\theta'}) = \text{Supp}(I_\eta/I_{\eta'}) = \{s\} \) for some point \( s \in S \).
We have the following decompositions:
\[
\eta = \eta_0 + \eta_s, \quad \eta' = \eta_0 + \eta'_s, \quad \theta = \eta_0 + \theta_s, \quad \theta' = \xi = \eta_0 + \xi_s, \quad \xi' = \eta_0 + \xi'_s,
\]
where \( s \notin \text{Supp}(\eta_0) \) and \( \eta_s \subset \eta'_s \subset \theta_s \subset \xi_s \subset \xi'_s \). When \( \ell := \ell(\eta_s) \geq 1 \), the relation \( \eta'_s \subset \xi_s \) imposes a nontrivial condition on the points
\[
((\eta_s, \eta'_s), (\xi_s, \xi'_s)) \in M_{\ell, \ell+1}(s) \times M_{\ell+n, \ell+n+1}(s).
\]
Hence the subset consisting of the images \( ((\xi, \xi'), (\eta, \eta')) \) of those points
\[
(((\xi, \xi'), (\eta, \eta'))
\]
in (4.13) with \( \ell = \ell(\eta_s) \geq 1 \) has dimension at most
\[
2(m - \ell) + 2 + [\ell + (n + \ell) - 1] = (2m + n + 1)
\]
which is less than the expected dimension \((2m + n + 2)\). So the case \( \ell \geq 1 \) does not contribute to the cycle \( w \) in (4.12). When \( \ell = 0 \), we have
\[
\eta = \eta_0, \quad \eta' = \eta_0 + s, \quad \theta = \eta_0 + \theta_s, \quad \theta' = \xi = \eta_0 + \xi_s, \quad \xi' = \eta_0 + \xi'_s
\]
where \( s \notin \text{Supp}(\eta_0) \) and \( \theta_s \subset \xi_s \subset \xi'_s \). The images \( ((\eta_0 + \xi_s, \eta_0 + \xi'_s), (\eta_0, \eta_0 + s)) \) form a subset of dimension \((2m + n + 2)\) in \( S^{[m+n,m+n+1]} \times S^{[m,m+1]} \). From the description (4.11) of the generic points in \( \tilde{Q}_{m,n} \), we conclude that
\[
w = c \cdot \tilde{Q}_{m,n}
\]
where \( c \) is the intersection multiplicity of (4.13) along the generic points.
To determine \( c \), choose a primitive integral class \( \tilde{A} \in H^*(S^{[m,m+1]}) \), i.e.,
\[
\tilde{A} \in H^*(S^{[m,m+1]}; \mathbb{Z})/\text{Tor} \subset H^*(S^{[m,m+1]}).
\]
By (4.14), \( \tilde{\nu}^n(\tilde{A}) = c \tilde{B} \) for some integral class \( \tilde{B} \). By Lemma 4.2 (i), we see that
\[
\tilde{A} = (\tilde{\nu})^n(\tilde{A}) = c(\tilde{\nu})^n(\tilde{B}).
\]
Since \( \tilde{B} \) is primitive and \( (\tilde{\nu})^n(\tilde{B}) \) is an integral class, we must have \( c = 1 \). \( \square \)

Proposition 4.6. Let \( n > 0 \), \( \alpha \in H^*(S) \), and \( A \in H^*(S^{[m+1]}) \). Then,
(i) \( g_n^*a_{n-\alpha}(\alpha)|0n = n \cdot \tilde{\nu}^{n-1} \rho_n^0(\alpha) \);
(ii) \( g_{m+n+1}^*a_{-\alpha}(\alpha) = \tilde{a}_{n-\alpha}(\alpha)(g_{m+1}^*A) + n \cdot \tilde{\nu}^{n-1}(\rho_{m+1}(\alpha) \cdot f_{m+1}^*(A)) \).

Proof. (i) It suffices to prove the formula for \( \alpha = 1_S \). The cohomology class \( a_{-n}(1_S)|0 \) is represented by the subscheme
\[
M_n(S) := \{\xi' \in S^{[n]}| \text{Supp}(\xi') = \{s\} \text{ for some } s \in S\}.
\]
By the Lemma 2.4 in [ES], \( 1/n \cdot g^*_n a_{-n}(1_S)|0 \) is represented by the subscheme
\[
M_{n-1,n}(S) := \{ (\xi, \xi') \in S^{[n-1,n]} | \text{Supp}(\xi') = \{ s \} \text{ for some } s \in S \}.
\]
By Lemma 4.5, \( m^{-1}(1_{S[0,1]}) \) is also represented by \( M_{n-1,n}(S) \). Therefore,
\[
g^*_n a_{-n}(1_S)|0 = n \cdot \tilde{P}^{-1} \rho_0(1_S).
\]
(ii) By the definitions (4.13) and (3.1), we obtain
\[
g^*_{m+1} a_{-n}(\alpha)(A) = g^*_{m+1} p_1^*([Q_{[m+n+1, m+1]} \cdot \rho^* \alpha \cdot p^*_2 A),
\]
\[
\tilde{a}_{-n}(\alpha) g^*_{m+1}(A) = \tilde{p}_1^*([\tilde{Q}_{[m+n, m]} \cdot \tilde{\rho}^* \alpha \cdot \tilde{p}^*_2 g^*_{m+1}(A)]).
\]
Let \( g = \text{Id}_{S^{[m+n, m+n+1]\times S}} \times g_{m+1} \) and \( h = g_{m+n+1} \times \text{Id}_{S \times S^{[m+1]}} \):
\[
g: S^{[m+n, m+n+1]} \times S \times S^{[m+1]} \to S^{[m+n, m+n+1]} \times S \times S^{[m+1]},
\]
\[
h: S^{[m+n, m+n+1]} \times S \times S^{[m+1]} \to S^{[m+n+1]} \times S \times S^{[m+1]}.
\]
Let \( \tilde{p}_1: S^{[m+n, m+n+1]} \times S \times S^{[m+1]} \to S^{[m+n, m+n+1]} \) be the first projection. By (4.15) and base change, we conclude that
\[
g^*_{m+n+1} a_{-n}(\alpha)(A) = (\tilde{p}_1^* h^*([Q_{[m+n+1, m+1]} \cdot \rho^* \alpha \cdot p^*_2 A),
\]
\[
\tilde{a}_{-n}(\alpha) g^*_{m+1}(A) = (\tilde{p}_1^* h^*([Q_{[m+n, m]} \cdot (\rho \circ h)^* \alpha \cdot (p_2 \circ h)^* A]) .
\]
Similarly, using (4.16), \( \tilde{p}_1 = \tilde{p}_1 \circ g \) and the projection formula, we obtain
\[
\tilde{a}_{-n}(\alpha) g^*_{m+1}(A) = (\tilde{p}_1^* h^*([Q_{[m+n, m]} \cdot (\rho \circ h)^* \alpha \cdot (p_2 \circ h)^* A]),
\]
Next, we compare \( h^* [Q_{[m+n+1, m+1]}] \) and \( g_* [\tilde{Q}_{[m+n, m]}] \). Let \( U_1 \) be the closure of the subset of \( S^{[m+n, m+n+1]} \times S \times S^{[m+1]} \) consisting of all the elements of the form:
\[
\left( \sum_{i=1}^{m} s_i + \xi_i', \sum_{i=1}^{m+1} s_i + \xi_i', \sum_{s} s \right)
\]
where \( s_1, \ldots, s_{m+1}, s \in S \) are distinct, and \( \xi'_s \in M_n(s) \) is curvi-linear. Let \( U_2 \) be the closure of the subset of \( S^{[m+n, m+n+1]} \times S \times S^{[m+1]} \) consisting of all the elements:
\[
\left( \sum_{i=1}^{m+1} s_i + \xi_s, \sum_{i=1}^{m+1} s_i + \xi_s', \sum_{s} s \right)
\]
where \( s_1, \ldots, s_{m+1}, s \in S \) are distinct, and \( \xi_s \in M_n(s) \) is curvi-linear, \( \xi'_s \subset \xi_s \). Note that general elements in \( M_n(s) \) are curvi-linear. Moreover, since \( \xi'_s \subset \xi_s \) is curvi-linear, \( \xi'_s \) uniquely determines \( \xi_s \) with \( \xi_s \subset \xi'_s \).
A typical element \( (\xi', s, \eta') \in Q^{[m+n+1, m+1]} \) is:
\[
(s_1 + \ldots + s_{m+1} + \xi'_s, s, s_1 + \ldots + s_{m+1})
\]
where \( s_1, \ldots, s_{m+1}, s \in S \) are distinct, and \( \xi'_s \in M_n(s) \) is curvi-linear. Hence,
\[
h^* [Q_{[m+n+1, m+1]}] = [U_1] + n[U_2].
\]
On the other hand, a typical element \((\xi, \xi', s, (\eta, \eta')) \in \tilde{Q}^{[m+n,m]}\) is of the form:

\[
\left( \sum_{i=1}^{m} s_i + \xi'_s, \sum_{i=1}^{m+1} s_i + \xi'_s, s, \left( \sum_{i=1}^{m} s_i, \sum_{i=1}^{m+1} s_i \right) \right)
\]

where \(s_1, \ldots, s_{m+1}, s \in S\) are distinct, and \(\xi'_s \in M_n(s)\) is curvi-linear. Hence,

\[
g_s[\tilde{Q}^{[m+n,m]}] = [U_1]. \tag{4.21}
\]

Combining (4.17), (4.18), (4.20) and (4.21), we obtain

\[
g^*_{m+n+1} a_n(\alpha)(A) - \tilde{a}_n(\alpha) g^*_{m+1}(A) = n \cdot (\tilde{p}_1)^* \left( [U_2] \cdot \left( \rho \circ h \cdot (\rho_2 \circ h)^* \right) \cdot \alpha \right).
\]

By (4.22), the projection formula and Lemma 4.5, we obtain

\[
\sum_{i=1}^{m} s_i + \xi'_s, \sum_{i=1}^{m+1} s_i + \xi'_s, s, \left( \sum_{i=1}^{m} s_i, \sum_{i=1}^{m+1} s_i \right) \]

where \(\pi_1, \pi_2\) are the two projections of \(S^{[m+n,m+n+1]} \times S^{[m+1,m+2]} \rightarrow S^{[m+n,m+n+1]} \times S \times S^{[m+1]}\).

5. Incidence Hilbert schemes and Lie algebras

5.1. The main Theorem. With all the results obtained in previous sections, we are ready to formulate and prove the main theorem.

Definition 5.1. (i) Define \(\tilde{h}_S\) to be the Heisenberg algebra generated by the operators \(\tilde{a}_n(\alpha), n \in \mathbb{Z}, \alpha \in H^*(S)\) and the identity operator \(\text{Id}_{\tilde{h}_S}\);

(ii) Define a Lie algebra structure on

\[
\tilde{h}_S = \tilde{h}_S \oplus H^*(S) \oplus \mathbb{C} \mathbf{i}
\]

by declaring

\[
[\tilde{a}(\alpha), \beta] = 0, \quad [\beta, \gamma] = 0, \quad [\mathbf{i}, \tilde{a}(\alpha)] = 0, \quad [\mathbf{i}, \beta] = 0 \tag{5.1}
\]

for operators \(\tilde{a}(\alpha) \in \tilde{h}_S\) and cohomology classes \(\beta, \gamma \in H^*(S)\).

Theorem 5.2. The space \(\tilde{H}_S\) is a representation of the Lie algebra \(\tilde{h}_S\) with a highest weight vector being the vacuum vector

\[
|0\rangle = 1_S \in H^0(S) = H^0(S^{[0,1]})
\]

where \(1_S\) denotes the fundamental cohomology class of \(S\).
Proof. By Lemma 3.3, Proposition 3.5, Lemma 4.2 (ii) and Proposition 4.3, we see that the space \( \tilde{H}_S \) is a representation of the Lie algebra \( \tilde{g}_S \).

Next, for each \( 0 \leq i \leq 4 \), we fix a linear basis
\[
\{ \alpha_{i,1}, \ldots, \alpha_{i,b_i(S)} \}
\]
of \( H^i(S) \). By Proposition 3.5 and the existence of the left inverse \( \tilde{t} \) of the operator \( \tilde{t} \) in Lemma 4.2 (i), the following cohomology classes:
\[
\tilde{t}^m \cdot \prod_{i \even} a_{-n_i,i} (\alpha_{i,i})^{m_{i,i}} \cdot \prod_{i \odd} a_{-n_i,i} (\alpha_{i,i})^{m_{i,i}} \cdot \alpha_{k,jk} | 0 \rangle
\]
are linearly independent, where \( 0 \leq i, k \leq 4 \), \( 0 \leq j_i \leq b_i(S) \), \( 0 \leq j_k \leq b_k(S) \), \( n_i,i > 0 \), \( m \geq 0 \), \( m_{i,i} \geq 0 \) for all the \( i \) and \( j_i \), and \( m_{i,j} = 0 \) or \( 1 \) for odd \( i \). Since the bi-degrees of the operators \( \tilde{t} \) and \( \tilde{a}_{-n}(\alpha) \) are \((1,2)\) and \((n,2n-2+|\alpha|)\) respectively, we conclude from (5.2) that cohomology classes in (5.2) form a linear basis of \( \tilde{H}_S \). Therefore, the representation is the highest weight representation. \( \square \)

Example 5.3. We express certain distinguished cohomology classes by using (5.2). First, consider the fundamental cohomology class of \( S^{[n,n+1]} \). For \( n \geq 0 \), let
\[
1_n = \frac{1}{n!} a_{-1}(S)^n, \quad \tilde{1}_n = \frac{1}{n!} \tilde{a}_{-1}(S)^n.
\]
For convenience, we also put \( 1_n = 0 \) and \( \tilde{1}_n = 0 \) when \( n < 0 \). Let \( 1_{S^{[n]}} \in H^0(S^{[n]}) \) and \( 1_{S^{[n,n+1]}} \in H^0(S^{[n,n+1]}) \) be the fundamental cohomology classes of \( S^{[n]} \) and \( S^{[n,n+1]} \) respectively. Then it is well-known that
\[
1_{S^{[n]}} = 1_{n}|0\rangle
\]
where by abusing notations, we also let \( |0\rangle \in H^0(S^{[0]}) \cong \mathbb{C} \) to stand for the cohomology class corresponding to the point \( S^{[0]} \). By Lemma 3.4,
\[
1_{S^{[n,n+1]}} = f_n^* 1_{S^{[n]}} = f_n^* 1_{-n}|0\rangle = \tilde{1}_n f_0^* |0\rangle = \tilde{1}_n |0\rangle.
\]
Next, let \( n \geq 1 \) and \( E_{n,n+1} \) be the exceptional divisor in \( S^{[n,n+1]} \) with respect to the blowing-up morphism \( S^{[n,n+1]} \to S^{[n]} \times S \), i.e., \( E_{n,n+1} \) is given by
\[
E_{n,n+1} = \{ ((\xi, \xi')) \in S^{[n,n+1]} | \text{Supp}(\xi) = \text{Supp}(\xi') \}.
\]
From the definition of the operator \( \tilde{t} \), we see that \([E_{1,2}] = \tilde{t}|0\rangle\). Assume \( n \geq 2 \). Let \( B_n \) be the boundary divisor in the Hilbert scheme \( S^{[n]} \), i.e.,
\[
B_n = \{ \xi \in S^{[n]} | \# \text{Supp}(\xi) \leq n-1 \}.
\]
Then, \([B_n] = a_{-2}(S)1_{-(n-2)}|0\rangle\) and \( g_{n+1}^*[B_{n+1}] = f_n^*[B_n] + 2[E_{n,n+1}] \). Thus,
\[
[E_{n,n+1}] = \frac{1}{2} (g_{n+1}^*[B_{n+1}] - f_n^*[B_n])
\]
\[
= \frac{1}{2} (g_{n+1}^* a_{-2}(S)1_{-(n-1)}|0\rangle - \tilde{a}_{-2}(1S)\tilde{1}_{-(n-2)}|0\rangle).
\]
From Proposition 4.6 (ii), (5.4) and (5.5), we conclude that
\[
\begin{align*}
E_{n,n+1} &= \frac{1}{2} (\tilde{a}_{-2}(1s) g_{n-1}^s I_{S[n-1]} + 2\tilde{t} f_{n-1}^s 1_{S[n-1]} - \tilde{a}_{-2}(1s) \tilde{I}_{S[n-2]} |0\rangle) \\
&= \tilde{t} (1_{S[n-1,n]} I_{S[n-1]}) \\
&= \tilde{t} \tilde{I}_{S[n-1,n]} |0\rangle. \quad (5.6)
\end{align*}
\]

5.2. Quasi-projective case. We make some comments for a smooth quasi-projective surface \( S \). It has been proved in \( \text{LQ1} \) that Cheah’s formula (1.1) holds for \( S = \mathbb{C}^2 \). However, it is unclear whether (1.1) holds for an arbitrary quasi-projective surface \( S \). In the following, we will bypass this issue by assuming that (1.1) holds for the quasi-projective surface \( S \).

Since the surface \( S \) is quasi-projective, some maps involved in the definition of the Heisenberg operators, which are proper in the projective case, are not proper anymore. Hence some modifications are needed. We follow Nakajima’s treatment of this issue in his book \( \text{[Na2]} \). The main remedy is to choose appropriate (co)homology theories so that the Heisenberg operators can still be defined.

Since \( S \) is smooth, the Borel-Moore homology group \( H_{bf}^*(S) \) is dual to \( H_{bf}^{4-i}(S) \). It follows that the morphism (2.10) induces an \( H_{bf}^*(S) \)-module structure on:
\[
\tilde{H} S = \bigoplus_{m=0}^{+\infty} H^*(S^[[m,m+1]]).
\]

For the creation operators \( \tilde{a}_{-n}(\alpha) \), where \( n > 0 \), we take \( \alpha \) to be a class in the Borel-Moore homology group \( H_{bf}^*(S) \). For the annihilation operators \( \tilde{a}_{n}(\beta) \), where \( n > 0 \), we take \( \beta \) to be a class in the ordinary homology group \( H_*(S) \). Once we have this set-up, the Heisenberg operators are well defined and the arguments in the previous sections are still valid. Note that \( H_{bf}^*(S) \) and \( H_*(S) \) are dual to each other. So we have the commutation relation as (3.14).

The translation operator \( \tilde{t} \) and its adjoint \( \tilde{t}^\dagger \) can be defined as usual without any change since the projections from the subset \( \tilde{Q}_m \subset S^{[m+1,m+2]} \times S^{[m,m+1]} \) to both factors \( S^{[m+1,m+2]} \) and \( S^{[m,m+1]} \) are proper.

In summary, with this set-up, we have the same statement as in Theorem 5.2 on \( \tilde{H} S \). In \( \text{LQ1} \), we have worked out this for \( S = \mathbb{C}^2 \) in the equivariant setting.

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