What kind of noncommutative geometry for quantum gravity?

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Abstract

We give a brief account of the description of the standard model in noncommutative geometry as well as the thermal time hypothesis, questioning their relevance for quantum gravity.

I Introduction

First of all let us emphasize the importance of the question mark in the title. Our aim is of course not to answer the question but rather to underline that depending on the community one is talking to, "noncommutative geometry" does not have the same meaning. In most of the applications related to quantum gravity (whatever candidate for a quantum theory of gravity is taken into account) noncommutative geometry is understood as the geometry of a quantum space, that is to say a space (spacetime or a phase space) whose coordinates do not commute, either in a canonical way

$$[x^\mu, x^\nu] = \theta_{\mu\nu} \tag{1}$$

where $\theta_{\mu\nu}$ is a constant, or in a Lie-algebraic way

$$[x^\mu, x^\nu] = C^\beta_{\mu\nu} x^\beta. \tag{2}$$

However the physical and/or geometrical meaning of those quantum spaces is not entirely clear. For instance, passing into the noncommutative realm, what happens to the most intuitive notions in geometry such as points, distances, or to more elaborated geometrical tools such as differential structure, homology or spin?

From a more fundamental point of view Noncommutative geometry is an extension of geometry beyond the scope of riemannian spin manifold. The later is encompassed as a particular case, commutative, of the general theory. The price for this generalization is a more abstract approach to geometry in terms of spectral datas. Connes has first observed that the geometrical information of a riemannian spin manifold can be recovered from algebraic datas, the so called spectral triple consisting in an algebra $A$, an Hilbert space $\mathcal{H}$ and an operator $D$ satisfying precise conditions. The involved algebra is the algebra of smooth functions on the manifold, which is commutative. Conversely a spectral triple with $A$ commutative is
associated to a spin manifold,

\[
\text{Riemannian spin geometry} \iff \text{Commutative algebra.} \quad (3)
\]

Now the tools allowing to go from the right side of the arrow (algebra) to the left side (geometry) do not rely on the commutativity of the algebra. They are still available when the algebra is not commutative. Hence a noncommutative geometry is the mathematical object that one obtains starting from a spectral triple \((\mathcal{A}, \mathcal{H}, D)\) in which the algebra \(\mathcal{A}\) is non necessarily commutative,

\[
\text{Noncommutative geometry} \iff \text{Noncommutative algebra} \quad (4)
\]

It is not obvious that all quantum spaces considered in physics literature can be described in spectral terms. However one can expect that the further the physics of those spaces is investigated, the deeper should be the mathematical coherence of the underlying geometry. Hence it is likely that the physics of quantum spaces be confronted sooner or later to some mathematical questions addressed by Connes theory. For instance, as far as I know, the notion of distance is not always available in quantum spaces (a two-form metric is available, but its interpretation as a line element and integration as a distance is not always possible \(\text{[13]}\)). Nevertheless there are now more and more point of contacts between various approaches to noncommutativity. Recently several spectral triples have been proposed for quantum spaces well known in deformation quantization (the Moyal plane \(\text{[11]}\)) or in quantum groups (the Podles sphere \(\text{[5]}\)). Our point here is not to be exhaustive but only to give a brief account of Connes’ theory as well as some of its interest for physics. We shall focus on the description of the standard model of elementary particles on the one hand, and on the beautiful idea (related to the issue of time in relativity) that ”Von Neumann algebras naturally evolve with time” on the other hand. As a conclusion we go back to the title of the talk and question the interface of noncommutative geometry with quantum gravity.

II Tools

II.1 Topology

The degree zero of geometry is the ability to determine whether two points are close to each other or not. This is the subject of topology. A famous theorem by Gelfand, Naimark and Segal establishes that the topological information of a compact space \(X\) is entirely contained within the algebra \(C(X)\) of complex continuous functions on \(X\). As an algebra \(C(X)\) is commutative \(fg(x) = f(x)g(x) = g(x)f(x) = gf(x), \quad (5)\)
equipped with an involution\(^*\)

\[
f^*(x) = \overline{f(x)} \quad (6)
\]

and a norm

\[
\|f\| = \sup_{x \in X} |f(x)| \quad (7)
\]
which make of $C(X)$ a $C^*$-algebra (i.e. closed in the norm topology and such that $\|f\|^2 = \|f^*\|\|f\|$). Conversely given a commutative $C^*$-algebra (with unit) $\mathcal{A}$ it is always possible to build a compact space such that $\mathcal{A}$ interprets as the algebra of continuous functions on $X$. Hence

\[
\text{Commutative $C^*$-algebra with unit} \iff \text{Compact topological space}
\]

\[
\mathcal{A} \iff X \quad (8)
\]

Strictly speaking the equation above is an equivalence of categories. For our purpose it is enough to understand how one goes from one side to the other. We already know half of the bridge

\[
C(X) \iff X. \quad (9)
\]

The other half is built on characters of the algebra, that is to say homorphisms $\mu$ from $\mathcal{A}$ to $\mathbb{C}$

\[
\mu(ab) = \mu(a)\mu(b) \quad \forall a,b \in \mathcal{A}. \quad (10)
\]

The set $K(\mathcal{A})$ of characters of a commutative $C^*$-algebra with unit is a compact space, hence the other half of the bridge

\[
\mathcal{A} \to K(\mathcal{A}). \quad (11)
\]

The two halves of the bridge (9) and (11) fit well together since $K(C(X))$ is homeomorphic to $X$ while $C(K(\mathcal{A}))$ is isomorphic to $\mathcal{A}$.

From a physics point of view the shift from space to algebra is important. A point $x$ of $X$ can be seen as the object on which a function $f$ is evaluated or equivalently, seen as characters, points are objects to be evaluated on functions in order to give numbers

\[
x(f) = f(x). \quad (12)
\]

The right-hand-side of (12) refers to classical physics (first is space) while the left-hand-side is closer to quantum mechanics (first are observables). But it is not quantum mechanics since for the moment we are dealing only with commutative observables.

Viewing points as a characters-of-the-algebra-of-continuous-functions may sound a bit complicated. But the algebraic point of view has the advantage to be adaptable to the noncommutative framework. Namely starting from a noncommutative algebra $\mathcal{A}$, it is possible to build an object, call it a non commutative space $Y$, such that $\mathcal{A}$ plays the role of functions on $Y$. Of course characters are not the suitable tools to extract the geometric information from a noncommutative algebra since characters precisely forget about the noncommutativity. Instead one considers the states of the algebra, that is to say the linear applications $\psi$ from $\mathcal{A}$ to $\mathbb{C}$ which are positive ($\psi(a^*a) \in \mathbb{R}$) and of norm one (which is equivalent to $\psi(I) = 1$ where $I$ is the unit of $\mathcal{A}$). The set $\mathcal{S}(\mathcal{A})$ of states of a $C^*$-algebra with unit is convex, which means that any state $\psi$ decomposes as

\[
\psi = \lambda \omega_1 + (1 - \lambda)\omega_2 \quad (13)
\]

where $\omega_1$, $\omega_2$ are states and $\lambda \in [0, 1]$. The extremal points of $\mathcal{S}(\mathcal{A})$, i.e. the states for which the only convex combination is trivial ($\lambda = 1$), are called pure states of $\mathcal{A}$. When the algebra is commutative characters and pure states coincide. When
the algebra is not commutative, characters are not meaningful but pure states are available. That is why from a topological point of view we consider the pure states of \( \mathcal{A} \) as the "points" of the noncommutative space.

This is only topology and to do physics one needs much more than topology. Especially dynamics requires a differential structure. Is it possible to describe in an algebraic manner the differential structure of a space? The answer in general is no, the knowledge of the spectrum of a differential operator is not enough to recover the geometry of the underlying manifold ("one cannot here the shape of a drum"). Connes has shown that to recover geometry from spectral datas, one needs more than the differential structure, one has to consider also the spin structure.

II.2 Spectral geometry

A noncommutative geometry is given by a spectral triple

\[
\mathcal{A}, \mathcal{H}, D
\]

(14)

where \( \mathcal{A} \) is an involutive algebra (commutative or not) with unit, \( \mathcal{H} \) is an Hilbert space that carries a representation \( \pi \) of \( \mathcal{A} \) and \( D \) is a selfadjoint operator acting on \( \mathcal{H} \), generally unbounded. By definition these three elements compose a spectral triple if and only if they satisfy a set of 7 properties that pick out necessary and sufficient conditions allowing i. an axiomatic definition of riemannian spin geometry in commutative algebraic terms ii. an extension of this definition to the noncommutative framework. Without entering the details that can be found in the literature, let us simply indicate that the first three conditions concern the analytical properties of \( D \) and deal with 1. the dimension of the space, 2. the smoothness of the coordinates, 3. the "bundle nature" of a spin manifold. Then comes 4. a condition on the commutation of \( D \) with the representation that translates the fact that the Dirac operator is a first order differential operator. Condition 5. is the algebraic formulation of Poincare duality (duality between the \( r^{\text{th}} \) and the \( (n-r)^{\text{th}} \) homology group of a \( n \) dimensional manifold). 6. concerns chirality and corresponds, in the commutative case, to the orientation of the manifold. The last condition 7. is the existence of a real structure, which allows the lift of the frame bundle to the spin bundle.

One then checks that

\[
\mathcal{A} = C^\infty(M), \mathcal{H} = L_2(M,S), D = -i\gamma_\mu \partial_\mu
\]

(15)

(where \( L_2(M,S) \) is the space of square integrable spinors over a compact Riemannian spin manifold \( M \) and \( D \) is the usual Dirac operator) satisfy the axioms of a spectral triple, hence a Riemannian spin manifold is a noncommutative geometry. Conversely a spectral triple with \( \mathcal{A} \) commutative fully determines a Riemannian spin manifold (see Ref.6,7 for a detailed proof) whose geodesic distance is

\[
d(x, y) = d(\omega_x, \omega_y) = \sup_{a \in \mathcal{A}} \left\{ |\omega_x(a) - \omega_y(a)| / \| [D, \pi(a)] \| \leq 1 \right\}
\]

(16)

where \( \omega_x \) is the point \( x \) seen as a character of \( C^\infty(M) \). Thus Riemannian spin geometry is a particular case, commutative, of an extended theory of geometry in
spectral terms. In the general case points are recovered as (pure) states $\omega, \omega'$ of $\mathcal{A}$ and formula (16) provides the metric

$$d(\omega, \omega') = \sup_{a \in \mathcal{A}} \left\{ |\omega(a) - \omega'(a)| / \|[D, \pi(a)]\| \leq 1 \right\}.$$  

Note that this formula is coherent for pure as well as non pure states. For explicit computation, in the following we restrict to pure states although one must be aware that, as shown by Rieffel [8], the knowledge of the distance function on the state space is not determined in general by its restriction to the set of pure states.

II.3 Distance

This is not difficult to check that the distance (16) associated to the geometry (15) coincides with the geodesic distance of the Riemannian structure of $M$. To understand what is going on, one can look at the most basic example by taking for $\mathcal{A}$ the continuous functions on $\mathbb{R}$ represented by multiplication on the space $\mathcal{H}$ of square integrable functions, and for $D$ the derivative with respect to $x$. Then the operator $[D, \pi(f)]$ acts on $\psi \in \mathcal{H}$ by multiplication by the derivative $f'$ of $f$,

$$[D, \pi(f)]\psi = \frac{d}{dx} f\psi - f \frac{d}{dx} \psi = (f')\psi$$

hence

$$\|[D, \pi(f)]\| = \sup_{x \in \mathbb{R}} |f'(x)|.$$  

The distance (16) writes

$$d(x, y) = d(\omega_x, \omega_y) = \sup_{f \in C(\mathbb{R})} \left\{ |f(x) - f(y)| \right\}$$

which is nothing that $|x - y|$, i.e. the geodesic distance in $\mathbb{R}$. The proof is identical for a manifold, except that the norm of the commutator is given by the norm of the gradient of $f$.

Formula (17) becomes interesting in situation where the classical definition of the distance as the length of the shortest path is no longer available. For instance noncommutative geometry allows to describe a manifold whose disconnected components are at finite distance from another. The simplest example is given by the geometry

$$\mathcal{A} = \mathbb{C}^2, \mathcal{H} = \mathbb{C}^2, D = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}$$

where $m$ is a complex number and the representation of $(z_1, z_2) \in \mathbb{C}^2$ is diagonal

$$\pi(z_1, z_2) = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix}.$$  

The two states of $\mathcal{A}$ are

$$\omega_1(z_1, z_2) \doteq z_1, \quad \omega_2(z_1, z_2) \doteq z_2$$

so we are dealing with a two point space. The computation of (17) is straightforward and yields

$$d(\omega_1, \omega_2) = \frac{1}{|m|}.$$  

5
Next interesting example is $\mathcal{A} = M_2(\mathbb{C})$ represented over itself and
\[ D\pi(a) = \Delta a + a\Delta \quad \text{with} \quad a \in M_2(\mathbb{C}), \quad \Delta \doteq \begin{pmatrix} 0 & m \\ \overline{m} & 0 \end{pmatrix}. \quad (25) \]

A pure state $\omega_\psi$ of $M_2(\mathbb{C})$ is determined by a normalized vector $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ of $\mathbb{C}^2$,
\[ \omega_\psi(a) = \langle \psi, a\psi \rangle. \quad (26) \]

By Hopf fibration any such vector is in one to one correspondence with points of $S^2$
\[ \psi \leftrightarrow \begin{cases} x_\psi = \text{Re}(\overline{\psi_1}\psi_2) \\ y_\psi = \text{Im}(\overline{\psi_1}\psi_2) \\ z_\psi = |\psi_1|^2 - |\psi_2|^2 \end{cases}. \quad (27) \]

One then finds that the distance $d$ between states at different altitude (different value for $z$) is infinite while it coincides with the euclidean distance of $S^2$ (up to a constant factor $\frac{1}{|m|}$) for states at the same altitude.

It is also possible to describe spaces which are made of continuous and discrete parts as, for instance, the product of the spin geometry (15) by the two point geometry (21). Indeed given two spectral triples $(\mathcal{A}_E, \mathcal{H}_E, D_E), (\mathcal{A}_I, \mathcal{H}_I, D_I)$ the product 
\[ \mathcal{A} = \mathcal{A}_E \otimes \mathcal{A}_I, \mathcal{H} = \mathcal{H}_E \otimes \mathcal{H}_I, D = D_E \otimes \mathbb{I}_I + \Gamma_E \otimes D_I \quad (28) \]
($\Gamma_E$ is the chirality of the first geometry) is again a spectral triple. In the case of a spin manifold $M$ multiplied by the two point space (21), the space of pure states is a two sheet models, two copies of $M$, and the distance coincides with the geodesic distance of $M' = M \times [0,1]$ equipped with the metric 
\[ g_{ab} = \begin{pmatrix} g_{\mu\nu} & 0 \\ 0 & \frac{1}{m^2} \end{pmatrix} \quad (29) \]
where $g_{\mu\nu}$ is the metric on $M$. Note that although the distance coincides with a geodesic distance, there is no geodesic between two points on different sheets.

III Physics

III.1 Symmetry

Trading spaces for algebras, one expects the symmetries of spaces to have an algebraic translation. Inspired by the commutative case in which
\[ \text{Diff}(M) = \text{Aut}(C^\infty(M)), \quad (30) \]
one will trade diffeomorphisms for automorphisms
\[ \text{Diffeomorphisms of the noncommutative space} \quad \Longleftrightarrow \quad \text{Automorphisms of the algebra}. \quad (31) \]

An inner automorphism $\alpha_u$ is an automorphism implemented by a unitary element $(u^*u = uu^* = \mathbb{I})$ of $\mathcal{A}$
\[ \alpha_u(a) = uau^*. \quad (32) \]
Inner automorphisms form a group denoted $\text{In}(A)$. An outer automorphism is a class in the quotient
\[
\text{Out}(A) \equiv \text{Aut}(A)/\text{In}(A).
\] (33)

When $A$ is commutative, $\text{In}(A)$ is trivial. Hence inner automorphisms are a specificity of the noncommutative case that remains hidden in the commutative case. They can be interpreted as the noncommutative part of the "diffeomorphism group" of the noncommutative space. In products of a manifold by a finite dimensional geometry, like $\mathcal{C}^\infty(M) \otimes M_2(\mathbb{C})$ in the precedent section, the action of inner automorphism naturally yields a scalar field which, as we will see later, can be identified as the Higgs field of the standard model.

Inner automorphisms also makes the metric fluctuate. Given a geometry $(A, \mathcal{H}, D)$ with representation $\pi$ and real structure $J$, one defines the action of $\text{In}(A)$ as
\[
\pi \to \pi' = \pi \circ \alpha_u.
\] (34)

This defines a new geometry $(A, \mathcal{H}, D)$ where $\pi$ is replaced by $\pi'$. A bit of algebra shows that this new geometry is in fact equivalent to the geometry, with old representation $\pi$, $(A, \mathcal{H}, D)$ where
\[
D = D + A + JAJ^{-1}
\] (35)

with
\[
A \doteq u[D, u].
\] (36)

$D$ is called the covariant Dirac operator because the action of a unitary $u'$ on $(A, \mathcal{H}, D)$ yields
\[
D' = D + A' + JAJ^{-1}
\] (37)

where
\[
A' = u' Au'^* + u'[D, u'^*]
\] (38)

which is similar to the transformation law of the potential in gauge theory. The metric is said to fluctuate for, given two states $\omega, \omega'$ of $A$, their distance $d$ in the geometry $(A, \mathcal{H}, D)$ does not equal in general their distance $d_A$ in the geometry $(A, \mathcal{H}, D)$ (there is no reason that $\|[[D, a]]\|$ equals $\|[[\omega, a]]\|$). However the fluctuation is covariant in the sense that
\[
d_A(\omega, \omega') = d_A'(\omega \circ \alpha_{u'}, \omega' \circ \alpha_{u'}).
\] (39)

Such fluctuations are a particular example of inner pertubations given by
\[
D = D + A + JAJ^{-1}
\] (40)

where
\[
A = A' = \sum_i a_i[D, b_i] \quad a_i, b_i \in A
\] (41)

belongs to the set $\Omega_1$ of 1-forms of the geometry. Without entering the details (see Ref.1) let us just mention that the replacement of $D$ by $D$ corresponds to the introduction of a connection. Trading the sections of a vector bundle over $M$ for a finite projective $\mathcal{C}(M)$-module (via Serre-Swann theorem) one defines a connection on a $\mathcal{A}$-module $\mathcal{E}$ as an application
\[
\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega_1
\] (42)
satisfying the Leibniz rule as well as an hermitian condition (the algebraic equivalent to the preservation of the metric by the Levi-Civita connection). Taking for \( \mathcal{E} \) the algebra \( \mathcal{A} \) itself, the introduction of the connexion is then equivalent to the replacement of \( D \) by (41).

### III.2 Standard model

Taking the product of a manifold by a finite dimensional geometry

\[
\mathcal{A}_I = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \quad \mathcal{H}_I = \mathbb{C}^{90}, \quad D_I
\]

where 90 is the number of elementary fermions (6 quarks \( \times 3 \) colors \( \times 2 \) chiralities + 3 leptons \( \times 2 \) chiralities + 3 neutrinos = 45 particles to which one adds 45 antiparticles) and \( D_I \) is a matrix whose entries are the masses of the elementary fermions and the coefficient of the unitary Cabibbo-Kobayashi-Maskawa matrix, one can describe the geometry of the standard model of elementary particles in spectral terms. We refer to Ref.\(^\text{13}\) for an updated presentation of the subject. The inner fluctuations of the metric decompose in two parts

\[
A = -i\gamma^\mu \otimes A_\mu - \gamma^5 \otimes H
\]

where \( A_\mu \) is a vector field with value in the skew-adjoint element of \( \mathcal{A}_I \) (i.e. in the Lie algebra of the unitaries of \( \mathcal{A}_I \) which identifies with the gauge group of the standard model) while \( H \) is a scalar field with value in the internal 1-forms. Computing the asymptotic development of the spectral action\(^\text{11}\)

\[
\text{Tr}(\theta(\frac{D^2}{\Lambda^2}))
\]

where \( \theta \) is the characteristic function of the interval \([0, 1]\) and \( \Lambda \) a cut-off, one finds the Einstein-Hilbert action (with euclidean signature) + a Weyl term + the bosonic part of the standard model. The Higgs field identifies to \( H \) and get a geometrical meaning in terms of the scalar part of the fluctuation of the metric. The metric interpretation of the Higgs fields has been fully elucidated in Ref.\(^\text{12}\). Pure states \( \mathcal{A} \) defines a two sheet model, two copies of \( M \), one indexed by the pure state of \( \mathbb{C} \), the other one by the pure state of \( \mathbb{H} \) (algebra of quaternions). The pure states of \( M_3(\mathbb{C}) \) are degenerated from a metric point of view. Writing \( x \in \mathbb{C}, y \in \mathbb{H} \) two points on different sheets, it turns out that for a pure scalar fluctuation \( (A_\mu = 0) \) the noncommutative distance\(^\text{17}\) coincide with the geodesic distance of \( M \times [0, 1] \) with metric

\[
g^{ab} = \begin{pmatrix}
g^{\mu\nu} & 0 \\
0 & g^{tt}(x)
\end{pmatrix}
\]

where the extra metric component \( g^{tt} \) is given by the Higgs doublet \( \begin{pmatrix} 1 + h_1(x) \\ h_2(x) \end{pmatrix} \) and the norm of the mass matrix, i.e. the mass of the quark top,

\[
g^{tt}(x) = (|1 + h_1(x)|^2 + |h_2(x)|^2)m_{\text{top}}^2.
\]

Once again let us emphasize that there is nothing between the two sheets (so no geodesic between points on different sheets). The distance is finite although the internal space is discrete.
III.3 Thermal time hypothesis

We have seen in the precedent subsection that inner perturbations yield the bosonic part of the standard model. Outer automorphisms also have a nice physical interpretation in terms of dynamics. The starting point is the observation that a Von Neumann algebra $\mathcal{A}$ (acting on a Hilbert space $\mathcal{H}$) is a dynamical object, in the sense that it comes equipped with a canonical one-parameter group of outer automorphism, the modular group of Tomita-Takesaki

$$ s \mapsto \sigma^s \in \text{Aut}(\mathcal{A}) $$

$$ \sigma^s(a) = \Delta^is a \Delta^{-is}. $$

$\Delta$ is given by the polar decomposition of the closure of the operator $S$

$$ Sa\Omega = a^*\Omega $$

where $\Omega$ is a vector cyclic and separating for the action of $\mathcal{A}$. $S$, hence $\Delta$ hence $\sigma$ depends on the initial choice of $\Omega$. The remarkable point (co-cycle Radon-Nikodym theorem[13]) is that $\sigma^s$ depends on $\Omega$ only modulo inner automorphisms. Hence there is a unique one parameter group of outer automorphisms associated to $\mathcal{A}$ via the modular theory. Let us fix one representant $\sigma$ in this unique class of equivalence, and write $\Omega$ the corresponding vector. Then $\sigma$ has the remarkable property that it satisfies with respect to $\Omega$ the same properties as the time evolution $\alpha$ with respect to a thermal equilibrium state $\omega$ at inverse temperature $\beta$, namely the KMS condition[14]

$$ \omega(A \alpha_t^i(B)) = \omega(\alpha^{t+i\beta}(B)A). $$

Here $A, B$ are observables of a thermodynamical system with Hamiltonian $H$, $\omega$ is a Gibbs state $\omega(A) = Z^{-1}\text{Tr}(Ae^{-\beta H})$ with $Z$ the partition function and $\alpha^t(A) = e^{-iHt}Ae^{iHt}$. In fact one has

$$ \langle \Omega, a\sigma^s b\Omega \rangle = \langle \Omega, (\alpha^{s-i}b) a \rangle $$

which yields the KMS condition[14] if we put

$$ \sigma^s = \alpha^{-\beta t}. $$

Hence an equilibrium state at inverse temperature $\beta$ is a state such that its modular group $\sigma^s$ coincides with the time flow $\alpha^t$, the parameter $s$ being related to the time $t$ by

$$ s = -\beta t. $$

The modular group is a formal time evolution for the state defined by the vector $\Omega$. Connes and Rovelli[15] have suggested that this evolution may have a physical meaning. The thermal time hypothesis demands that the modular flow determined by the statistical state of a real physical system coincides with what we perceive as the physical flow of time. This hypothesis was initially motivated by the problem of time in quantum gravity[16]. For the time being it has been tested in semi-classical situations where a geometrical background already provides an independent notion of temporal flow. In this case the hypothesis demands that the ratio of the rates of the two flows (geometrical and modular) be identified as the temperature of the state.
The Unruh effect is an example in which the thermal time hypothesis is realized. Let us recall that Unruh effect is the theoretical observation that the vacuum state $\Omega$ of a quantum field theory on Minkovski spacetime looks like a thermal equilibrium state for an uniformly accelerated observer $O$ with acceleration $a$. The observed temperature is the Davis-Unruh temperature $T_U = \hbar a / 2 \pi k_b c$. Among the many derivations of Unruh effect, one is based on the geometrical properties of the region causally connected to the world line of $O$, namely the Rindler wedge $W$ ($|t| < \|x\|$). The modular group defined by $\Omega$ on the algebra of local observables on $W$ has a geometrical action which coincides with the proper time flow of $O$. The proportionality constant between the two flows is $T_U$ and is interpreted as the temperature of the vacuum seen by $O$. Note that a similar analysis has been developed in Ref.20 for the modular flow associated to the causal horizon of a non eternal uniformly accelerated observer, namely a diamond region $D_L$ ($|t| + \|x\| < L$ with $L$ a constant).

Here a time flow is given (the proper time flow of $O$) as well as a state (the vacuum $\Omega$) and the coincidence between the time flow and the modular flow of $W$ yields the interpretation of the ratio as a temperature

\[
\begin{cases}
\text{state} \quad \rightarrow \quad \text{temperature}.
\end{cases}
\]

The thermal time hypothesis makes the opposite analysis: assuming the vacuum is thermal with temperature $T_U$, then physical time is given by the modular flow and it turns out that this time coincides with the proper time of $O$.

\[
\begin{cases}
\text{state} \quad \rightarrow \quad \text{time}.
\end{cases}
\]

This shift in the point of view makes the thermal time hypothesis a interesting tool in quantum gravity for it may allow to extract from a fully covariant quantum formulation of gravity our (strongly non covariant!) intuition of flow of time. Indeed assuming that covariance of general relativity is preserved at the quantum level, then one has the freedom to pick out from the surrounding superposition of states of the gravitational field any particular direction as the time direction. The thermal time hypothesis gives a way to make this freedom compatible with our local intuition of physical flow of time, by making time a state dependant notion. For the moment the hypothesis has not been applied on this context, because we are still lacking of a definite algebra of quantum gravity observables to begin the analysis (i.e. try to compute the modular flow and possibly interpret it in dynamical terms). The situation may change since there are now some candidates as algebras of observables in loop quantum gravity.

\section{IV Quantum gravity ?}

One generally assumes that quantum gravity should yield a non continuous structure of spacetime at Planck scale. Recently it has been underlined that such a structure may not be out of the reach of experimental measurement first because it induces a modification of the usual relativistic dispersion relation which may have a significant effect on the propagation of high energy cosmic rays, second because
the "fuzzyness" of space may yield a characteristic source of noise in gravitational waves experiment. However discreteness of spacetime might not be only a quantum gravity effect since noncommutative geometry provides the spacetime of the standard model with a discrete structure already at the classical level (i.e. classical gravitation). Therefore it could be interesting to adapt to the two sheet model of the standard model the analysis of quantum gravity phenomenology developed for quantum spaces. For instance one could study the propagation of some signal on the two sheet model, or export the distance formula in quantum spaces so that to make a quantitative analysis of the "fuzzyness" of spacetime.

The fact that discreteness of spacetime is not necessarily a quantum gravitational effect makes the link quantum gravity/quantum spaces/noncommutative geometry still more complicated. It is likely that quantum gravity will have to do with a noncommutative structure of spacetime. It is certainly to soon to know whether it should be via the discreteness of spacetime, via the richness of the "diffeomorphism group" of a noncommutative space, or via something else.

References

1 A. Connes, Noncommutative geometry, 1st ed. (Academic Press, London, 1994).
2 A. Connes, Commun. Math. Phys. 182, 155–176 (1996).
3 J. Madore, An introduction to noncommutative differential geometry and its physical interpretation (Cambridge University Press, 1995).
4 V. Gayral, Lett.Math.Phys. 65 147-157 (2003). V. Gayral, J. M Gracia Bondia, B. Iochum, T. Schucker, J.C. Varilly, Commun. Math. Phys. 569-623 (2004).
5 L. Dabrowski, A. Sitarz, Dirac Operator on the Standard Podles Quantum Sphere, math-QA/0209048. L. Dabrowski, G. Landi, M. Paschke, A. Sitarz, The Spectral Geometry of the Equatorial Podles Sphere, math.QA/0408034.
6 A. Rennie, math-ph/9903021.
7 J. M. Gracia-Bondia, J. C. Varilly, H. Figueroa, Elements of noncommutative geometry, Birkhauser (2001).
8 M. A. Rieffel, Metrics on state spaces, math.OA/9906151.
9 B. Iochum, T. Krajewski, P. Martinetti, J. Geom. Phys. 37 100-125 (2001).
10 T. Schucker, hep-th/0409077 to be published in Int. Journ. Mod. Phys. A.
11 A. H. Chamseddine, A. Connes, Commun. Math. Phys. 186 (1996) 737-750.
12 P. Martinetti, R. Wulkenhaar, J.Math.Phys. 43 182-204 (2002).
13 A. Connes, Ann. Sci. Ecole Normale Superieure, 6 133-252 (1973).
14 R. Haag, Local Quantum Physics”, 2nd ed. (Springer, Berlin, 1996).
15 A. Connes, C. Rovelli, Class. Quantum Grav 11 2899-2917 (1994).
16 C. Isham, Integrable systems, quantum groups, and quantum field theories, Proceedings of GIFT, Salamanca, Spain, 1992, eds. L.A. Ibort and M.A. Rodriguez, Kluwer (1993).

17 G. L. Sewell, Phys. Lett. 79 A n. 1, 23 (1980); Annals Physics 141, 201-224 (1982).

18 J. J. Bisognano, E. H. Wichman, J. Math. Phys. 16 984 (1975); J. Math. Phys. 17 303 (1976).

19 P. D. Hislop, R. Longo, Comm. Math. Phys. 84 71-85 (1982).

20 P. Martinetti, C. Rovelli, Class. Quant. Gravity 20 4919-4932 (2003).

21 G. Amelino-Camelia, Introduction to Quantum-Gravity Phenomenology, gr-qc/0412136.