A SIMPLIFIED CALCULATION FOR THE FUNDAMENTAL SOLUTION TO THE HEAT EQUATION ON THE HEISENBERG GROUP

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Abstract. Let $L_{\gamma} = -\frac{1}{4} \left( \sum_{j=1}^{n} (X_j^2 + Y_j^2) + i\gamma T \right)$ where $\gamma \in \mathbb{C}$, and $X_j, Y_j$ and $T$ are the left-invariant vector fields of the Heisenberg group structure for $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$. We explicitly compute the Fourier transform (in the spatial variables) of the fundamental solution of the heat equation $\partial_s \rho = -L_{\gamma} \rho$. As a consequence, we have a simplified computation of the Fourier transform of the fundamental solution of the $\Box_b$-heat equation on the Heisenberg group and an explicit kernel of the heat equation associated to the weighted $\overline{\partial}$-operator in $\mathbb{C}^n$ with weight $\exp(-\tau P(z_1, \ldots, z_n))$, where $P(z_1, \ldots, z_n) = \frac{1}{2} (|\text{Im} z_1|^2 + \cdots + |\text{Im} z_n|^2)$ and $\tau \in \mathbb{R}$.

0. Introduction

The purpose of this note is to present a simplified calculation of the Fourier transform of the fundamental solution of the $\Box_b$-heat equation on the Heisenberg group. The Fourier transform of the fundamental solution has been computed by a number of authors [Gav77, Hul76, CT00, Tie06]. We use the approach of [CT00, Tie06] and compute the heat kernel using Hermite functions but differ from the earlier approaches by working on a different, though biholomorphically equivalent, version of the Heisenberg group. The simplification in the computation occurs because the differential operators on this equivalent Heisenberg group take on a simpler form. Moreover, in the proof of Theorem 1.2, we reduce the $n$-dimensional heat equation to a 1-dimensional heat equation, and this technique would also be useful when analyzing the heat equation on the nonisotropic Heisenberg group (e.g., see [CT00]). We actually use the same version of the Heisenberg group as Hulanicki [Hul76], but he computes the fundamental solution of the heat equation associated to the sub-Laplacian and not the Kohn Laplacian acting on $(0, q)$-forms.

A consequence of our fundamental solution computation is that we can explicitly compute the heat kernel associated to the weighted $\overline{\partial}$-problem in $\mathbb{C}^n$ when the weight is given by $\exp(-\tau P(z_1, \ldots, z_n))$, where $\tau \in \mathbb{R}$ and $P(z_1, \ldots, z_n) = \frac{1}{2} (|\text{Im} z_1|^2 + \cdots + |\text{Im} z_n|^2)$. When $n = 1$ and $p(z_1)$ is a subharmonic, nonharmonic

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polynomial, the weighted \(\overline{\partial}\)-problem (with weight \(\exp(-p(z))\)) and the explicit construction of Bergman and Szegö kernels have been studied by a number of authors in different contexts (for example, see [Chr91, Has94, Has95, Has98, FS91, Ber92]). In addition, Raich has estimated the heat kernel and its derivatives [Rai06a, Rai06b, Rai07, Rai].

1. The Heisenberg group and the \(\Box_b\)-heat equation

Definition 1.1. The Heisenberg group is the set \(\mathbb{H}^n = \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\) with the following group structure:

\[
g \ast g' = (x, y, t) \ast (x', y', t') = (x + x', y + y', t + t' + x \cdot y'),
\]

where \((x, y, t), (x', y', t') \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\) and \(\cdot\) denotes the standard dot product in \(\mathbb{R}^n\).

The left-invariant vector fields for this group structure are:

\[
X_j^g = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial t}\quad \text{and} \quad Y_j^g = \frac{\partial}{\partial y_j}, \quad 1 \leq j \leq n, \quad \text{and} \quad T^g = \frac{\partial}{\partial t}.
\]

The Heisenberg group can also be identified with the following hypersurface in \(\mathbb{C}^{n+1}\):

\[H^n = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} : \text{Im} z_{n+1} = (1/2) \sum_{j=1}^n (\text{Im} z_j)^2\},\]

where \((z_1, \ldots, z_n, t + i(1/2) \sum_{j=1}^n (\text{Im} z_j)^2) \in H^n\) with \((z_1, \ldots, z_n, t) = (x_1, \ldots, x_n, y_1, \ldots, y_n, t)\) where \(z_j = x_j + iy_j \in \mathbb{C}\). With this identification, the left-invariant vector fields of types \((0,1)\) and \((1,0)\), respectively, are:

\[
Z_j^g = \frac{1}{2} (X_j + iY_j) = \frac{\partial}{\partial z_j} + \frac{y_j}{2} \frac{\partial}{\partial t}, \quad Z_j^g = \frac{1}{2} (X_j - iY_j) = \frac{\partial}{\partial z_j} - \frac{y_j}{2} \frac{\partial}{\partial t}
\]

for \(g = (x, y, t) \in \mathbb{H}^n\) and \(1 \leq j \leq n\).

The heat equation. The Kohn Laplacian \(\Box_b\) acting on \((0,q)\)-forms on \(H^n \approx \mathbb{H}^n\) can be easily described in terms of these left-invariant vector fields. Suppose \(f = \sum_{j \in I_q} f_J d\bar{z}_J\) is a \((0,q)\)-form where \(I_q\) is the set of all increasing \(q\)-tuples \(J = (j_1, \ldots, j_q), 1 \leq j_k \leq n\). Then

\[
\Box_b f = \sum_{J \in I_q} \mathcal{L}_{n-2q} f_J d\bar{z}_J,
\]

where

\[
\mathcal{L}_\gamma = -\frac{1}{4} \left( \sum_{j=1}^n (X_j^2 + Y_j^2) + i\gamma T \right).
\]

See Stein ([Ste93, XIII §2]) for details on computing \(\Box_b\). For comparison, the box operator (or Laplacian) in Hulanicki ([Hul76]) is

\[-\frac{1}{2} \sum_{j=1}^n (X_j^2 + Y_j^2)\]

The heat equation is defined on \((0,q)\)-forms \(\rho\) on \(\mathbb{H}^n\) with coefficient functions that depend on \(s \in (0, \infty)\) and \((x, y, t) \in \mathbb{H}^n\). It is

\[
\frac{\partial \rho}{\partial s} = -\Box_b \rho
\]

(note that here \(s\) is the “time” variable and \(t\) is a spatial variable). Since \(\Box_b\) acts diagonally, we can restrict ourselves to a fixed component and look for a fundamental
solution $\rho$ that satisfies
\[
\begin{cases}
\frac{\partial \rho}{\partial s} = -\mathcal{L}_\gamma \rho & \text{for } s > 0, \ (x, y, t) \in \mathbb{H}^n, \\
\rho(s = 0, x, y, t) = \delta_0(x, y, t)
\end{cases}
\]
(i.e., the delta function at the origin in the spatial variables).

**Fourier transformed variables.** We will use a Fourier transform in the spatial $(x, y, t)$ variables (i.e., not the $s$-variable): let $(\alpha, \beta, \tau)$ be the transform variables corresponding to $(x, y, t)$, and define:
\[
\hat{f}(\alpha, \beta, \tau) = \int_{\mathbb{H}^n} f(x, y, t) e^{-i(\alpha x + \beta y + \tau t)} \, dx \, dy \, dt.
\]

Our main result is the following:

**Theorem 1.2.** For any $\gamma \in \mathbb{C}$, the spatial Fourier transform of the fundamental solution to the heat equation (2) is given by
\[
\hat{\rho}(s, \alpha, \beta, \tau) = \frac{e^{-\gamma s^2/4}}{(\cosh(s/2))^{n/2}} e^{-A(|\alpha|^2 + |\beta|^2)/2 + iB\alpha \beta},
\]
where
\[
A = \frac{\sinh(s/2)}{\tau \cosh(s/2)}, \quad B = \frac{2\sinh^2(s/4)}{\tau \cosh(s/2)}.
\]
Note that $\gamma$ may be any complex number, but $\gamma = n - 2q$ is the value where $\mathcal{L}_\gamma$ corresponds to $\Box_b$ on $(0, q)$-forms.

We also seek the fundamental solution to the heat equation associated to the weighted $\partial$ operator in $(s, x, y)$-space. Given a function $f$ on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$, let
\[
\hat{f}_\tau(x, y) = \int_{\mathbb{R}} e^{-i\tau t} f(x, y, t) \, dt
\]
be the partial Fourier transform in $t$. Define
\[
\mathcal{L}_j = \frac{\partial}{\partial x_j} + \frac{i}{2} y_j \tau = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + \frac{i}{2} y_j \tau \right), \quad L_j = \frac{\partial}{\partial z_j} + \frac{i}{2} y_j \tau = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - \frac{i}{2} \frac{\partial}{\partial y_j} + iy \tau \right).
\]
Note that these operators are just the Fourier transform of $\overline{Z_j}$ and $Z_j$ in the $t$-direction. If $\Delta_{x,y}$ is the Laplacian in both the $x$ and $y$ variables, the partial $t$-Fourier transform of $\mathcal{L}_\gamma$ is
\[
\hat{\mathcal{L}}_\gamma = -\frac{1}{4} (\Delta_{x,y} + 2i \tau y \cdot \nabla_x - (\tau^2 y \cdot y + \gamma \tau)).
\]
The operator $\hat{\mathcal{L}}_\gamma$ acts on functions, but it can be extended to $(0, q)$-forms by acting on each component function of the form. If $\gamma = n - 2q$, then $\hat{\mathcal{L}}_\gamma$ is the higher dimensional analog of the $\Box_{\tau^2}$-operator from [Rai06a, Rai07, Rai] associated to the weighted $\partial$ operator in $\mathbb{C}^n$ with weight $\exp(-\tau P(z_1, \ldots, z_n))$, where $P(z_1, \ldots, z_n) = \frac{1}{2} (|\Im z_1|^2 + \cdots + |\Im z_n|^2)$ and $\tau \in \mathbb{R}$. As a corollary to our main theorem, we compute the fundamental solution to the heat operator associated to this weighted $\partial$. 

Fourier transform \((3)\), note that \(\gamma\) is a solution to the following initial value problem for the heat equation:

\[
\frac{\partial}{\partial s} \tilde{\gamma} + \tilde{\Delta}_\gamma \tilde{\gamma} = 0
\]

with \(\tilde{\gamma}(s = 0, x, y) = \delta_{(0,0)}(x, y)\).

Finally, we use \(\tilde{\gamma}\) to derive the heat kernel, as studied in [Rai06a, Rai07, RaiNS01].

**Corollary 1.3.** For any \(\gamma \in \mathbb{C}, \tau \in \mathbb{R}\), the function

\[
\tilde{\rho}_\gamma^\tau(s, x, y) = \frac{e^{-\gamma s \tau/4}}{(2\pi)^n (\cosh(s \tau/2))^{n/2}} e^{-\frac{x^2 + y^2}{2 (\cosh(s \tau/2))^{n/2}}} e^{i \frac{\gamma}{4} x \cdot y}
\]

is the fundamental solution to the weighted \(\mathcal{F}\) heat equation: \((\frac{\partial}{\partial s} + \tilde{\Delta}_\gamma) \tilde{\rho}_\gamma^\tau(s, x, y) = 0\) with \(\tilde{\rho}_\gamma^\tau(s = 0, x, y) = \delta_{(0,0)}(x, y)\).

We first reduce the problem down to dimension one. Define \(\tilde{\rho}_\gamma^\tau(s, x, y)\) as given in \(3\), but for dimension one (i.e., with initial condition \(\tilde{\rho}_\gamma^\tau(s = 0, x, y) = \delta_{(0,0)}(x, y)\)).

Corollary 1.4. For any \(\gamma \in \mathbb{C}, \tau \in \mathbb{R}\), let

\[
H_\gamma^\tau(s, x, y, x', y') = \frac{\tau^n e^{-\gamma s \tau/4}}{(4\pi)^n \sinh^{n}(s \tau/4)} e^{-\frac{1}{2} \coth(s \tau/4)(|x-x'|^2 + |y-y'|^2) - i \pi (x-x') \cdot (y+y')}.
\]

Then \(H_\gamma^\tau\) is the heat kernel which satisfies the following property: if \(f \in L^2(\mathbb{C})\), then

\[
H_\gamma^\tau[f](s, x, y) = \int_{\mathbb{R}^n} H_\gamma^\tau(s, x, y, x', y') f(x', y') \, dx' \, dy'
\]

is a solution to the following initial value problem for the heat equation:

\[
\begin{cases}
\frac{\partial}{\partial s} H_\gamma^\tau[f] = 0 \\
H_\gamma^\tau[f](s = 0, x, y) = f(x, y).
\end{cases}
\]

(4)

Note that \(H_\gamma^\tau\) is conjugate symmetric in \(z = x + iy\) and \(z' = x' + iy'\) (i.e., switching \(z\) with \(z'\) results in a conjugate).

2. Proof of Theorem 1.2

It is easy to verify the following calculations. Recall that \(\tilde{\mathcal{F}}\) refers to the spatial Fourier transform

\[
\begin{align*}
\tilde{X}_j^2 f(\alpha, \beta, \tau) & = \left(-\alpha_j^2 - 2i \alpha_j \tau \frac{\partial}{\partial \beta_j} + \tau^2 \frac{\partial^2}{\partial \beta_j^2}\right) \tilde{f} \\
\tilde{Y}_j^2 f(\alpha, \beta, \tau) & = -\beta_j^2 \tilde{f} \\
\tilde{T} f(\alpha, \beta, \tau) & = i \tau \tilde{f}.
\end{align*}
\]

We first reduce the problem down to dimension one. Define \(\tilde{\rho}^{\gamma,1}\) by the same formula as given in \(3\), but for dimension one (i.e., \(n = 1\) and \(\alpha, \beta \in \mathbb{R}\)). From \(3\), note that

\[
\tilde{\rho}^{\gamma}(s, \alpha, \beta, \tau) = \prod_{j=1}^n \tilde{\rho}^{\gamma/n,1}(s, \alpha_j, \beta_j, \tau), \quad \alpha = (\alpha_1, \ldots, \alpha_n), \quad \beta = (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n
\]

(note the \(\gamma\) on the left and the \(\gamma/n\) on the right). Once we show that \(\rho^{\gamma,1}\) satisfies the transformed heat equation in dimension one, i.e.,

\[
\left(\frac{\partial}{\partial s} - (1/4)(\tilde{X}^2 + \tilde{Y}^2 + i \gamma \tilde{T})\right) \{\tilde{\rho}^{\gamma,1}(s, \cdot, \cdot)\} = 0
\]

with initial condition \(\tilde{\rho}^{\gamma,1}(s = 0, \cdot, \cdot) = 1\) (the Fourier transform of the delta function), then by using \(3\), it is an easy exercise to show that \(\tilde{\rho}^{\gamma}\) in dimension \(n\) satisfies Theorem 1.2.
From now on, we assume the dimension $n$ is one and so $x, y, \alpha$ and $\beta$ are all real variables. Also, $\gamma$ will be suppressed as a superscript. Define

$$u(s, \alpha, \beta, \tau) = \hat{\rho}(s, \alpha, \beta, \tau)e^{-i\alpha\beta}. \tag{7}$$

Then, the following equations are easily verified:

$$u(s = 0, \alpha, \beta, \tau) = e^{-i\alpha\beta} \tag{8}$$

$$\frac{\partial u}{\partial s} = \frac{1}{4}(\tau^2 \frac{\partial^2}{\partial \beta^2} - \beta^2 - \gamma\tau)u. \tag{9}$$

The first equation follows from the fact that the Fourier transform of the delta function is the constant one. The second equation follows from the heat equation for $\hat{\rho}$ (from (6)) and the above formulas for the transformed differential operators $\hat{X}, \hat{Y}$ and $\hat{T}$. We will refer to the above differential equation as the transformed heat equation.

**Solution of heat equation using Hermite special functions.** For $m = 0, 1, 2, \ldots$ and $x \in \mathbb{R}$, let

$$\psi_m(x) = \frac{(-1)^{m}e^{x^2/2}}{\sqrt{2^{m}m!\sqrt{\pi}}} \frac{d^{m}}{dx^{m}}\{e^{-x^2}\}. \tag{10}$$

For $\tau \in \mathbb{R}$, let

$$\Psi_m^\tau(x) = |\tau|^{-1/4}\psi_m(x/\sqrt{\tau}).$$

It is a fact that $\psi_m$ and hence $\Psi_m^\tau$ form an orthonormal system for $L^2(\mathbb{R})$ (see [Tha93], pp. 1-7). It is also a fact (again see [Tha93], (1.1.28)) that

$$\psi_m''(x) = x^2\psi_m(x) - (2m + 1)\psi_m(x).$$

We first assume that $\tau > 0$ and later indicate the minor changes needed in the case that $\tau \leq 0$. Replacing $x$ by $\beta/\sqrt{\tau}$ in the previous equation yields:

$$(\tau^2 \frac{\partial^2}{\partial \beta^2} - \beta^2 - \gamma\tau)\{\Psi_m^\tau(\beta)\} = -(2m + 1 + \gamma)\tau\Psi_m^\tau(\beta). \tag{11}$$

In other words, $\Psi_m^\tau$ is an eigenfunction of the differential operator on the right side of (10) with eigenvalue $-(2m + 1 + \gamma)\tau$.

Since $\{\Psi_m^\tau\}$ are an orthonormal basis for $L^2(\mathbb{R})$, $u$ can be expressed as

$$u(s, \alpha, \beta, \tau) = \sum_{m=0}^{\infty} a_m(\alpha, \tau)e^{-\frac{1}{4}(2m+1+\gamma)s\tau}\Psi_m^\tau(\beta), \tag{12}$$

where $a_m(\alpha, \tau)$ will be determined later. Differentiating this with respect to $s$ and using (11) gives

$$\frac{\partial}{\partial s}u(s, \alpha, \beta, \tau) = \sum_{m=0}^{\infty} a_m(\alpha, \tau)e^{-\frac{1}{4}(2m+1+\gamma)s\tau}\left(-\frac{1}{4}(2m + 1 + \gamma)\tau\Psi_m^\tau(\beta)\right)$$

$$= \frac{1}{4}(\tau^2 \frac{\partial^2}{\partial \beta^2} - \beta^2 - \gamma\tau)\{u(t, \alpha, \beta, \tau)\}. \tag{13}$$
So, \( u \) satisfies the transformed heat equation (9). To satisfy the initial condition (8), we must have
\[
e^{-i\alpha \beta / \tau} = u(s = 0, \alpha, \beta, \tau) = \sum_{m=0}^{\infty} a_m(\alpha, \tau) \Psi_m(\beta).
\]
Using the fact that the \( \Psi_m(\beta) \) form an orthonormal system, we have
\[
a_m(\alpha, \tau) = \int_{\mathbb{R}} e^{-i\alpha \beta / \tau} \Psi_m(\beta) \, d\beta = \tau^{1/4} \int_{\mathbb{R}} e^{-i\sqrt{\tau} \beta} \psi_m(\beta) \, d\beta.
\]
The integral on the right is just the Fourier transform of \( \psi_m \) at the point \( \alpha/\sqrt{\tau} \). From Thangavelu (\cite{Tha93}, Lemma 1.1.3), the Fourier transform of \( \psi_m \) equals \( \psi_m \) up to a constant factor of \((-i)^m \sqrt{2\pi}\). Therefore,
\[
a_m(\alpha, \tau) = (-i)^m (2\pi)^{\frac{3}{2}} \tau \frac{1}{2} \psi_m(\alpha/\sqrt{\tau}).
\]
Substituting this value of \( a_m \) into the expression for \( u \) and rearranging gives:
\[
u(s, \alpha, \beta, \tau) = (2\pi)^{1/2} e^{-\frac{i}{2}(1+\gamma)s\tau} \sum_{m=0}^{\infty} (-i)^m \psi_m(\frac{\alpha}{\sqrt{\tau}}) \psi_m(\frac{\beta}{\sqrt{\tau}}) e^{-\frac{1}{2} m s \tau}.
\]
Now solving for \( \hat{\nu} \) (see equation (12) yields
\[
\hat{\nu}(s, \alpha, \beta, \tau) = e^{i\alpha \beta / \tau} u(s, \alpha, \beta, \tau)
\]
\[
= (2\pi)^{\frac{3}{2}} e^{-\frac{i}{2}(1+\gamma)s\tau} \sum_{m=0}^{\infty} (-i)^m \psi_m(\frac{\alpha}{\sqrt{\tau}}) \psi_m(\frac{\beta}{\sqrt{\tau}}) e^{-\frac{1}{2} m s \tau} e^{i\alpha \beta / \tau}.
\]
Now let \( S = e^{-s \tau / 2}, x = \alpha / \sqrt{\tau}, y = \beta / \sqrt{\tau} \). Since \(|S| < 1\), we obtain (see (1.1.36))
\[
\hat{\nu}(s, \alpha, \beta, \tau) = (2\pi)^{\frac{3}{2}} S^{\frac{1}{2}}(1+\gamma) \left( \sum_{m=0}^{\infty} (-i S)^{-m} \psi_m(x) \psi_m(y) \right) e^{i xy}
\]
\[
= \sqrt{2} S^{\frac{1}{2}}(1+\gamma) e^{-\frac{i}{2} \frac{2}{1+S^2} \tau (x^2+y^2)} e^{i xy(\frac{\sqrt{\gamma}}{1+S^2}+1)}.
\]
Now substituting in for \( S, x \) and \( y \), a short calculation finishes the proof for \( \tau > 0 \).

Note that \( \hat{\nu}(s = 0, \alpha, \beta, \tau) = 1 \) (the Fourier transform of the delta function at the origin).

When \( \tau = 0 \), the solution in (13) becomes \( \hat{\nu}(s, \alpha, \beta) = e^{-s(\alpha^2+\beta^2)/4} \), which is easily shown to satisfy (9).

If \( \tau < 0 \), then \( \tau \) is replaced by \(|\tau|\) on the right side of (11), which slightly changes the subsequent calculations. However the formula for the solution given Theorem (12) remains valid for \( \tau < 0 \).

3. Proof of the corollaries

*Proof of Corollary (13).* Again, we assume the dimension is \( n = 1 \). The fundamental solution to this heat operator must satisfy
\[
\frac{\partial}{\partial s} \hat{\nu}_\tau(s, x, y) + \mathcal{L}_s \hat{\nu}_\tau = 0
\]
with the initial condition \( \hat{\nu}_\tau(s = 0, x, y) = \delta_0(x, y) \). Now since \( \hat{\nu} \) is the Fourier transform of the fundamental solution to the original heat operator, clearly \( \hat{\nu}_\tau \) can
be obtained by taking the inverse Fourier transform of $\hat{\rho}$ in the $\alpha, \beta$ variables. This is a standard calculation involving Gaussian integrals and will be left to the reader. □

**Proof of Corollary** [13] If $L_j$ and $\overline{L}_j$, $1 \leq j \leq n$, had constant coefficients, then the heat kernel would just be $\tilde{\rho}_{\tau}(s, x - x', y - y')$, an ordinary convolution. However, we must multiply by a “twist” factor $e^{-i\tau(x-x')\cdot y'}$ to account for the fact that $L_j$ and $\overline{L}_j$ have variable coefficients. Let

$$H_{\tau}(s, x, y, x', y', \tau) = \tilde{\rho}_{\tau}(s, x - x', y - y')e^{-i\tau(x-x')\cdot y'}.$$  

Note that $H_{\tau}(f)$ satisfies the initial condition given in [1] in view of the initial condition satisfied by $\tilde{\rho}_{\tau}$ and noting that the twist term is 1 at $x' = x$. Showing that $H_{\tau}$ satisfies the heat equation in the $s$, $x$, $y$ variables is a short calculation that uses the equation

$$\left( \frac{\partial}{\partial s} - \frac{1}{4}(\triangle_{x,y} + 2i\tau(y - y') \cdot \nabla_x - (\tau^2(y - y') \cdot (y - y') + \gamma \tau)) \right) e^{-i\tau(x-x')\cdot y'}.$$  

which is just the equation $(\frac{\partial}{\partial s} + \tilde{L}_{\tau})\tilde{\rho}_{\tau} = 0$ at the point $(s, x - x', y - y')$.

**Simplification of the formula for $H_{\tau}$.** Note that the coefficient of the imaginary part of the exponent of $\tilde{\rho}_{\tau}$ is

$$\frac{-B}{A^2 + B^2} \quad \text{where} \quad A = \frac{\sinh(s\tau/2)}{\tau \cosh(s\tau/2)}, \quad B = \frac{2\sinh^2(s\tau/4)}{\tau \cosh(s\tau/2)}.$$  

An easy calculation with cosh and sinh identities shows that

$$\frac{B}{A^2 + B^2} = \frac{\tau}{2} \quad \text{and} \quad \frac{A}{B} = \frac{\cosh(s\tau/4)}{\sinh(s\tau/4)}.$$  

Consequently, the fundamental solution $H_{\tau}$, from (11) and Corollary [13] can be rewritten

$$H_{\tau}(s, x, y, x', y') = \frac{\pi^n e^{-\gamma s\tau/4}}{(4\pi)^n \sinh^n(s\tau/4)} e^{-\frac{\tau}{2} \cosh(s\tau/4)(|x-x'|^2 + |y-y'|^2) - i\frac{\tau}{2} (x-x') \cdot (y+y')}.$$  

□

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944 ALBERT BOGGESS AND ANDREW RAICH

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