Mutation graphs of maximal rigid modules over finite dimensional preprojective algebras*

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Abstract Let $Q$ be a finite quiver of Dynkin type and $\Lambda = \Lambda_Q$ be the preprojective algebra of $Q$ over an algebraically closed field $k$. Let $\mathcal{T}_\Lambda$ be the mutation graph of maximal rigid $\Lambda$ modules. Geiss, Leclerc and Schröer conjectured that $\mathcal{T}_\Lambda$ is connected, see [C.Geiss, B.Leclerc, J.Schröer, Rigid modules over preprojective algebras, Invent.Math., 165(2006), 589-632]. In this paper, we prove that this conjecture is true when $\Lambda$ is of representation finite type or tame type. Moreover, we also prove that $\mathcal{T}_\Lambda$ is isomorphic to the tilting graph of $\text{End}_\Lambda T$ for each maximal rigid $\Lambda$-module $T$ if $\Lambda$ is representation-finite.

Key words and phrases: Preprojective algebras; maximal rigid module; mutation graph of maximal rigid modules; tilting graph.

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1 Introduction

Let \( Q \) be a finite quiver without oriented cycles and \( kQ \) be the path algebra of \( Q \) over an algebraically closed field \( k \). The preprojective algebra \( \Lambda = \Lambda_Q \) of \( Q \) was introduced by Gelfand and Ponomarev in [16] such that \( \Lambda \) contains \( kQ \) as a subalgebra, and when considered as a left \( kQ \) module, \( \Lambda \) decomposes as a direct sum of the indecomposable preprojective \( kQ \) modules with one from each isomorphism class. Now, preprojective algebras play important roles in representation theory and other areas of mathematics, such as resolutions of Kleinian singularities, quantum groups, quiver varieties, and cluster theory, see [8, 11, 12, 13, 14, 17, 18] for details.

By using mutations of maximal rigid modules and their endomorphism algebras over preprojective algebras of Dynkin type, Geiss, Leclerc and Schröer studied the cluster algebra structure on the ring \( \mathbb{C}[N] \) of polynomial functions on a maximal unipotent subgroup \( N \) of a complex Lie group of Dynkin type, and obtained that all cluster monomials of \( \mathbb{C}[N] \) belong to the dual semicanonical basis, see [11].

Let \( Q \) be a Dynkin quiver, and \( \Lambda \) be the preprojective algebra of \( Q \). Recall from [11], \( \mathcal{T}_\Lambda \) denotes the mutation graph of maximal rigid modules of \( \Lambda \). Fix a basic maximal rigid \( \Lambda \)-module \( T \), then the contravariant functor \( F^T = \text{Hom}_\Lambda(\cdot, T) : \text{mod} \, \Lambda \to \text{mod} \, \text{End}_\Lambda T \) yields an anti-equivalence of categories

\[
\text{mod} \, \Lambda \to \mathcal{P}(\text{mod} \, \text{End}_\Lambda T)
\]

where \( \mathcal{P}(\text{mod} \, \text{End}_\Lambda T) \subset \text{mod} \, \text{End}_\Lambda T \) denotes the full subcategory of all \( \text{End}_\Lambda T \)-modules of projective dimension at most one. Moreover, the functor \( F^T \) induces an embedding of graphs \( \psi_T : \mathcal{T}_\Lambda \to \mathcal{T}_{\text{End}_\Lambda T} \) whose image is a union of connected components of \( \mathcal{T}_{\text{End}_\Lambda T} \), where \( \mathcal{T}_{\text{End}_\Lambda T} \) is the tilting graph of the algebra \( \text{End}_\Lambda T \). Each vertex of \( \mathcal{T}_\Lambda \) (and therefore each vertex of the image of \( \psi_T \)) has exactly \( r - n \) neighbours.

In [11], Geiss, Leclerc and Schröer conjectured that the graph \( \mathcal{T}_\Lambda \) is connected. In this paper, we prove that this conjecture is true when \( \Lambda \) is of representation finite type or tame type. Moreover, we also prove that \( \psi_T \) is an isomorphism whenever \( \Lambda \) is representation finite. The following theorems are our main results.
Theorem 1. Let $\Lambda$ be a preprojective algebra of type $A_n$ with $n \leq 4$, and $T$ be a maximal rigid $\Lambda$-module. Then the functor $F^T = \text{Hom}_\Lambda(-,T)$ induces an isomorphism of graphs $\psi_T : \mathcal{T}_\Lambda \to \mathcal{T}_{\text{End}_\Lambda T}$.

Corollary 2. Let $\Lambda$ be a preprojective algebra of type $A_n$ with $n \leq 4$. Then for each maximal rigid $\Lambda$-module $T$, the tilting graph $\mathcal{T}_{\text{End}_\Lambda T}$ of $\text{End}_\Lambda T$ is isomorphic to the mutation graph $\mathcal{T}_\Lambda$ of maximal rigid modules of $\Lambda$.

Remarks. Let $\Lambda$ be a preprojective algebra with finite representation type. The above corollary implies that for all maximal rigid $\Lambda$-modules, their endomorphism algebras have same tilting graphs up to isomorphism. However, this kind of algebras are very different, such as some of them is strongly quasi-hereditary and most of them is even not quasi-hereditary, see [14] for details.

Theorem 3. Let $\Lambda$ be a preprojective algebra of representation finite or tame type. Then the mutation graph $\mathcal{T}_\Lambda$ of the maximal rigid $\Lambda$-modules is connected.

This paper is organized as follows: in Section 2, we recall some definitions and facts needed for our research, in Section 3, we prove Theorem 1 and Corollary 2, in Section 4, we prove Theorem 3.

2 Preliminaries

Let $k$ be an algebraically closed field, and let $A$ be a finite dimensional algebra over $k$. We denote by $\text{mod } A$ the category of all finitely generated left $A$-modules, and by $\text{ind } A$ the full subcategory of $\text{mod } A$ consisting of one representative from each isomorphism class of indecomposable modules. For a $A$-module $M$, we denote by $\text{add } M$ the full subcategory of $\text{mod } A$ whose objects are the direct summands of finite direct sums of copies of $M$. The projective dimension of $M$ is denoted by $\text{pd } M$, and the Auslander Reiten translation of $A$ by $\tau_A$.

$T \in \text{mod } A$ is called a classical tilting module if the following conditions are satisfied:

(1) $\text{pd } T \leq 1$;
(2) $\text{Ext}_A^1(T, T) = 0$;

(3) There is an exact sequence $0 \rightarrow A \rightarrow T_0 \rightarrow T_1 \rightarrow 0$ with $T_i \in \text{add } T$ for $0 \leq i \leq 1$.

Let $T_A$ be the set of all basic classical tilting $A$-modules up to isomorphism. According to [11, 15], the tilting graph $T_A$ is the defined as following: the vertices are the non-isomorphic basic tilting modules, there is an edge between $T_1$ and $T_2$ if $T_1 = T' \oplus T'_1$ and $T_2 = T' \oplus T'_2$ for some $A$-module $T'$ and some indecomposable $A$-modules $T'_1$ and $T'_2$ with $T'_1 \not\cong T'_2$.

Let $Q = (Q_0, Q_1)$ be a connected quiver, where $Q_0$ is the set of vertices and $Q_1$ is the set of arrows. Given an arrow $\alpha$, we denote by $s(\alpha)$ the starting vertex of $\alpha$ and by $t(\alpha)$ the ending vertex of $\alpha$. Let $\overrightarrow{Q}$ be the double quiver of $Q$, which is obtained from $Q$ by adding an arrow $\alpha^*: j \rightarrow i$ whenever there is an arrow $\alpha: i \rightarrow j$ in $Q_1$. Let $Q_1^* = \{ \alpha^* | \alpha \in Q_1 \}$ and $\overrightarrow{Q}_1 = Q_1 \cup Q_1^*$. The preprojective algebra of $Q$ is defined as

$$\Lambda = \Lambda_Q = k\overrightarrow{Q}/(\rho)$$

where $\rho$ is the relation with

$$\rho = \sum_{\alpha \in Q_1} [\alpha, \alpha^*],$$

and $k\overrightarrow{Q}$ is the path algebra of $\overrightarrow{Q}$. See [20].

Note that the preprojective algebra $\Lambda$ is independent of the orientation of $Q$, and that $\Lambda$ is finite dimensional if and only if $Q$ is a Dynkin quiver. Moreover, $\Lambda$ is also self-injective if it is finite dimensional. In particular, $\Lambda$ is of finite representation type if and only if $Q$ is of type $A_n$ with $n \leq 4$, and it is of tame representation type if and only if $Q$ is of type $A_5$ or $D_4$, see [9, 12].

Let $d, e \in \mathbb{Z}^n$ be two dimension vectors. The symmetry bilinear form is defined as $(d, e) = 2 \sum_{i \in Q_0} d_i e_i - \sum_{a \in \overrightarrow{Q}_1} d_{s(a)} e_{t(a)}$. The following lemma is proved in [8].

**Lemma 2.1.** Let $\Lambda$ be a preprojective algebra and $X, Y$ be $\Lambda$-modules. Then we have

$$\dim \text{Ext}_\Lambda^1(X, Y) = \dim \text{Hom}_\Lambda(X, Y) + \dim \text{Hom}_\Lambda(Y, X) - (\dim X, \dim Y).$$
In particular, \( \dim \text{Ext}^1_{\Lambda}(X,Y) = \dim \text{Ext}^1_{\Lambda}(Y,X) \).

From now on, we always assume that \( \Lambda \) is a preprojective algebra of Dynkin type. A \( \Lambda \)-module \( T \) is called \textbf{rigid} if \( \text{Ext}^1_{\Lambda}(T,T) = 0 \). \( T \) is called \textbf{Maximal rigid} if for any \( \Lambda \)-module \( M \) with \( \text{Ext}^1_{\Lambda}(T \oplus M,T \oplus M) = 0 \), then we have \( M \in \text{add} \ T \).

Note that each maximal rigid \( \Lambda \)-module \( T \) is also a generator-cogenerator. Let \( F^T = \text{Hom}_{\Lambda}(-,T) \). A short exact sequence \( 0 \to X \to E \to Y \to 0 \) of \( \Lambda \)-modules is called \( F^T \)-exact if \( 0 \to F^T(Y) \to F^T(E) \to F^T(X) \to 0 \) is an exact sequence of \( \text{End}_{\Lambda}T \)-modules. We denote by \( F^T(Y,X) \) the equivalent classes of all the \( F^T \)-exact sequences as above.

Let \( \chi_T \) be a subcategory of \( \text{mod} \ \Lambda \) whose objects admit an \( \text{add} \ T \)-resolution. Namely, \( X \in \chi_T \) if and only if there is an exact sequence
\[
0 \longrightarrow X \longrightarrow T_0 \longrightarrow T_1 \longrightarrow T_2 \longrightarrow \cdots
\]
with all \( T_i \in \text{add} \ T \), which is still exact by applying the functor \( \text{Hom}_{\Lambda}(T,-) \). Let \( \text{Ext}^i_{F^T}(Y,X) \) be the cohomology group by applying the functor \( \text{Hom}_{\Lambda}(Y,-) \) to an \( \text{add} \ T \)-resolution of \( X \).

The following lemma is proved in \[3, 4\].

**Lemma 2.2.** Assume that \( X \in \chi_T \) and \( Y \in \text{mod} \ \Lambda \). Then there are following functorial isomorphisms:

1. \( \text{Ext}^1_{F^T}(Y,X) \cong F^T(Y,X) \);
2. \( \text{Ext}^i_{F^T}(Y,X) \cong \text{Ext}^i_{\text{End}_{\Lambda}(T)}(\text{Hom}_{\Lambda}(X,T),\text{Hom}_{\Lambda}(Y,T)) \) for all \( i \geq 1 \).

Let \( \Lambda \) be a finite dimensional preprojective algebra, and let \( T \) be a maximal rigid \( \Lambda \)-module. Then \( \chi_T = \text{mod} \ \Lambda \) since every \( \Lambda \)-module has an \( \text{add} \ T \)-resolution \[11, \text{Corollary 5.2}\].

Recall from \[11, \text{section 6}\], the \textit{mutation graph} \( T_\Lambda \) of maximal rigid modules is defined as following. The vertex set of \( T_\Lambda \) is the set of the isomorphism classes of basic maximal rigid \( \Lambda \)-modules, and there is an edge between vertices \( T_1 \) and \( T_2 \) if
and only if $T_1 = T \oplus T'_1$ and $T_2 = T \oplus T'_2$ for some $T$ and some indecomposable modules $T'_1$ and $T'_2$ with $T'_1 \ncong T'_2$.

**Lemma 2.3.** Let $T$ be a basic maximal rigid $\Lambda$-module. The functor $F^T : \text{mod } \Lambda \to \text{mod } \text{End}_{\Lambda}(T)$ induces an embedding of graphs $\psi_T : T_\Lambda \to T_{\text{End}_{\Lambda}(T)}$ whose image is a union of connected components of $T_{\text{End}_{\Lambda}(T)}$.

We follow the standard terminology and notation used in the representation theory of algebras, see [1, 2, 19].

### 3 The mutation graph and the tilting graph of representation finite preprojective algebras

In this section, we assume that $\Lambda$ is a preprojective algebra of representation finite type. Namely, $\Lambda$ is of type $A_n$ with $n \leq 4$. For the AR-quivers of this kind of preprojective algebras we refer to [12, section 20.1]. Here we give the stable AR-quivers of $\Lambda_{A_3}$ and $\Lambda_{A_4}$ for convenience.

![The stable quiver of $\Lambda_{A_3}$](image)

![The stable quiver of $\Lambda_{A_4}$](image)
Definition. Two AR-sequences are called centrally connected if they have
common indecomposable summands in the middle terms. A column in the AR-
quiver is a set consist of the indecomposable summands of the middle terms in the
centrally connected AR-sequences.

A path from $X$ to $Y$ in the AR-quiver is a chain of irreducible morphisms
$X = M_0 \to M_1 \to M_2 \to \cdots \to M_{n-1} \to M_n = Y$. We say that $Z$ is between $X$
and $Y$ if there is a chain

$$X = M_0 \to M_1 \to M_2 \to \cdots \to M_{n-1} \to M_n = Y$$

such that all $M_i$ is not in the same column with $Y$ for $0 < i < n$ and that $Z$ is in
the same column with some one of $M_i$ with $0 < i < n$.

A class $\Sigma$ of pairwise non-isomorphic indecomposable $\Lambda$-modules in the stable
quiver above is called a complete slice if it satisfies the following conditions:

1. the indecomposable modules in $\Sigma$ lie in different $\tau$-orbits;
2. $\Sigma$ is convex. Namely, if $X$ and $Y$ belong to $\Sigma$ and there is a path from $X$
to $Z$ and a path from $Z$ to $Y$, then $Z$ belongs to $\Sigma$.

A complete slice is called standard if it lies in two adjacent columns.

For example, in the stable quiver of $\Lambda_{A_3}$, $Z_1$ is between $X$ and $Y$ while $Z_2$ is
between $Y$ and $X$. The complete slice which consists of $\bullet$ in the stable quiver of
$\Lambda_{A_4}$ is standard.

Lemma 3.1. Given a communicative diagram of exact sequences

$$\begin{array}{ccccccccc}
0 & \to & X & \overset{i}{\to} & E & \to & Y & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow h & & \\
0 & \to & X & \overset{f}{\to} & F & \to & Z & \to & 0
\end{array}$$

with the bottom sequence non-split. Then the top sequence is non-split if and only
if $h$ cannot factor through $F$.

Proof. Apply the functor $\text{Hom}_\Lambda(Y, -)$ to the bottom sequence, we get an exact
sequence

$$0 \to \text{Hom}_\Lambda(Y, X) \to \text{Hom}_\Lambda(Y, F) \overset{\alpha}{\to} \text{Hom}_\Lambda(Y, Z) \overset{\beta}{\to} \text{Ext}_\Lambda^1(Y, X).$$
Then $h$ is in the kernel of $\beta$ if and only if it is in the image of $\alpha$. Namely, the top sequence is the zero element in $\text{Ext}_A^1(Y, X)$ if and only if $h$ factors through $F$. This complete the proof. \hfill \Box

**Lemma 3.2.** Let $\Lambda$ be a preprojective algebra of type $A_n$, $n \leq 4$. Let $X$, $Y$ and $Z$ be non-isomorphic indecomposable $\Lambda$-modules with $\text{Ext}_A^1(Y, X) \neq 0$ and $\text{Ext}_A^1(Z, X) \neq 0$. If $Y$ is between $X$ and $Z$ with $\text{Hom}_\Lambda(Y, Z) \neq 0$, then there is a non-split exact sequence

$$0 \to X \to E \to Y \to 0$$

which is induced from a non-split exact sequence

$$0 \to X \xrightarrow{f} F \to Z \to 0.$$

**Proof.** Let $(3) 0 \to X \xrightarrow{i} M \to \tau^{-1}X \to 0$ be the AR-sequence start at $X$. Then we have the following communicative diagram:

$$
\begin{array}{cccc}
0 & \longrightarrow & X & \xrightarrow{i} & M & \longrightarrow & \tau^{-1}X & \longrightarrow & 0
\end{array}
$$

$$
\begin{array}{cccc}
0 & \longrightarrow & X & \xrightarrow{f} & F & \longrightarrow & Z & \longrightarrow & 0
\end{array}
$$

By using AR-formula $\text{Hom}_\Lambda(\tau^{-1}X, Z) \simeq D\text{Ext}_A^1(Z, X)$, we know that different sequences of the form $(2)$ corresponds to different homomorphisms from $\tau^{-1}X$ to $Z$ in the stable category $\text{mod}\Lambda$. According to Lemma 3.1, we know that $h$ can’t factor through $F$.

Let $X$, $Y$ and $Z$ be non-isomorphic indecomposable $\Lambda$-modules with $\text{Ext}_A^1(Y, X) \neq 0 \neq \text{Ext}_A^1(Z, X)$. If $Y$ is between $X$ and $Z$ with $\text{Hom}_\Lambda(Y, Z) \neq 0$, then by reading the pictures given in [12, section 20.4] we know that there is a path from $\tau^{-1}X$ to $Z$ which induces a nonzero morphism from $\tau^{-1}X$ to $Z$ in $\text{mod}\Lambda$ factoring through $Y$. Hence there exists a morphism $g$ from $Y$ to $Z$ which cannot factor through $F$. Then we have a pull-back diagram:

$$
\begin{array}{cccc}
0 & \longrightarrow & X & \longrightarrow & E & \longrightarrow & Y & \longrightarrow & 0
\end{array}
$$

$$
\begin{array}{cccc}
0 & \longrightarrow & X & \xrightarrow{f} & F & \longrightarrow & Z & \longrightarrow & 0
\end{array}
$$
by Lemma 3.1 again, we have a non-split sequence of the form (1) which is induced from (2).

\[ \square \]

**Remark.** We should mention that Lemma 3.2 is not true without the assumption that \( Y \) is between \( X \) and \( Z \). The following example is pointed out to us by C.M. Ringel. Let

\[
\begin{array}{cccc}
1 & \overset{\alpha_1}{\longrightarrow} & 2 & \overset{\alpha_2}{\longrightarrow} 3 & \overset{\alpha_3}{\longrightarrow} 4 \\
\alpha_1' & & \alpha_2' & \alpha_3'
\end{array}
\]

be the quiver of \( \Lambda_{A_4} \). Take \( X = \begin{array}{l} 4 \\ 3 \end{array} \), \( Y = \begin{array}{l} 1 \\ 3 \end{array} \), \( Z = 2 \), \( V = \begin{array}{l} 2 \\ 4 \\ 3 \end{array} \). Then \( \text{Ext}^1_{\Lambda}(Y, X) = \text{Ext}^1_{\Lambda}(Z, X) = k \), \( \text{Hom}_{\Lambda}(Y, Z) = k \). 0 \( \rightarrow X \rightarrow P(2) \rightarrow Y \rightarrow 0 \) and 0 \( \rightarrow X \rightarrow V \rightarrow Z \rightarrow 0 \) are the corresponding exact sequences. But the first sequence cannot be induced by the second one because the inclusion map 0 \( \rightarrow X \rightarrow V \) cannot factor through \( P(2) \), since there is no map from \( P(2) \) to the simple module 4.

**Lemma 3.3.** Let \( \Lambda \) be a preprojective algebra of type \( A_n \), \( n \leq 4 \). Let \( X \) and \( Y \) be non-isomorphic indecomposable \( \Lambda \)-modules with \( \text{Ext}^1_{\Lambda}(X, Y) \neq 0 \). Let \( N \) be an indecomposable non-projective \( \Lambda \)-module which is between \( X \) and \( Y \) or in the same column with \( X \). Then any exact sequence

\[
(\ast) \quad 0 \rightarrow Y \rightarrow M \rightarrow X \rightarrow 0
\]

is \( F_N \)-exact. Moreover, if \( N \) is in the same column with \( X \), then any exact sequence

\[
(\ast\ast) \quad 0 \rightarrow X \rightarrow M \rightarrow Y \rightarrow 0
\]

is also \( F_N \)-exact.

**Proof.** We choose a standard complete slice which contains \( X \) and extend it to a maximal rigid \( \Lambda \) module \( T \) by adding all the indecomposable projective-injective modules. Then it follows from the stable quiver of \( \Lambda \) that every non-zero map from \( Y \) to \( N \) factors through \( T \) since \( Y \not\subset \text{add } T \).

Note that \( \text{Ext}_{\Lambda}(X, T) = 0 \), by applying \( \text{Hom}_{\Lambda}(\cdot, T) \) to the exact sequence (\ast\ast), we get an exact sequence

\[
0 \rightarrow \text{Hom}_{\Lambda}(X, T) \rightarrow \text{Hom}_{\Lambda}(M, T) \rightarrow \text{Hom}_{\Lambda}(Y, T) \rightarrow 0.
\]
Thus every map from $Y$ to $T$ factors through $M$, which implies that every map from $Y$ to $N$ factors through $M$. Namely, the sequence

$$0 \to \text{Hom}_\Lambda(X, N) \to \text{Hom}_\Lambda(M, N) \to \text{Hom}_\Lambda(Y, N) \to 0$$

is exact.

Now, we assume that $N$ is in the same column with $X$. Then any map from $X$ to $N$ in the stable quiver factors through the maximal rigid module obtained from the standard complete slice which contains $X$. Repeat the proof above we see that any exact sequence

$$(**) \quad 0 \to X \to M \to Y \to 0$$

is also $F^N$-exact. This completes the proof. 

Lemma 3.4. Let $\Lambda$ be a preprojective algebra of type $A_n$, $n \leq 4$. Let $X$ and $Y$ be non-isomorphic indecomposable $\Lambda$-modules with $\text{Ext}^1_\Lambda(X, Y) \neq 0$. Let $N$ be an indecomposable non-projective $\Lambda$-module. Then there exists a non-split exact sequence

$$(*) \quad 0 \to Y \to E \to X \to 0$$

or

$$(**) \quad 0 \to X \to M \to Y \to 0$$

which is $F^N$-exact.

Proof. If $N$ is between $X$ and $Y$ or in the same column with $X$, then any exact sequence $(*) 0 \to Y \to M \to X \to 0$ is $F^N$-exact by Lemma 3.3. If $N$ is between $Y$ and $X$ or in the same column with $Y$, then any exact sequence $(**)$ $0 \to X \to E \to Y \to 0$ is $F^N$-exact by Lemma 3.3 again. 

Lemma 3.5. Let $\Lambda$ be a preprojective algebra of type $A_n$, $n \leq 4$. Let $X$ and $Y$ be non-isomorphic indecomposable $\Lambda$-modules with $\text{Ext}^1_\Lambda(X, Y) \neq 0$. Let $N_1$ and $N_2$ be two non-isomorphic indecomposable $\Lambda$-module with $\text{Ext}^1_\Lambda(N_1, N_2) = 0$. Then there exists a non-split exact sequence

$$(*) \quad 0 \to Y \to E \to X \to 0$$

or

$$(**) \quad 0 \to X \to M \to Y \to 0$$
which is both $F^{N_1}$-exact and $F^{N_2}$-exact.

**Proof.** If $N_1$ and $N_2$ are both between $X$ and $Y$ or both between $Y$ and $X$, then the assertion is true by Lemma 3.3.

If $N_1$ is in the same column with $X$, then both $(\ast)$ and $(\ast\ast)$ are exact by Lemma 3.3. Hence the assertion is true by Lemma 3.4.

Now, we assume that $N_1$ is between $X$ and $Y$ while $N_2$ is between $Y$ and $X$.

If $\text{Hom}_\Lambda(Y, N_2) = 0$, then $(\ast)$ is $F^{N_2}$-exact, and by Lemma 3.3, $(\ast)$ is also $F^{N_1}$-exact.

If $\text{Hom}_\Lambda(Y, N_2) \neq 0$, then by Lemma 3.3, any exact sequence of form $(\ast)$ is $F^{N_1}$-exact and any exact sequence of form $(\ast\ast)$ is $F^{N_2}$-exact.

**Case I.** If there exists a non-split sequence of form $(\ast\ast)$ is $F^{N_1}$-exact, then our sequence is true.

**Case II.** Now, we suppose that any non-split exact sequence of the form $(\ast\ast)$ is not $F^{N_1}$-exact. We claim that $\text{Ext}_1^\Lambda(N_2, X) = 0$.

Indeed, if by contrary we assume that $\text{Ext}_1^\Lambda(N_2, X) \neq 0$, then by Lemma 3.2, there exists a non-split exact sequence

$$0 \to X \xrightarrow{j} E \to Y \to 0$$

which is induced from a non-split exact sequence $0 \to X \xrightarrow{f} F \to N_2 \to 0$. Then we have the following commutative diagram:

$$
\begin{array}{c}
0 & \longrightarrow & X & \xrightarrow{j} & E & \longrightarrow & Y & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \xrightarrow{f} & F & \longrightarrow & N_2 & \longrightarrow & 0
\end{array}
$$

Thus $f$ factors through $j$.

Note that $0 \to X \to E \to Y \to 0$ is not $F^{N_1}$-exact, hence there exists a map $\lambda$ from $X$ to $N_1$ which cannot factor through $E$, this forces that there exists a map $g$ from $X$ to $N_1$ such that $g$ cannot factor through $F$.

Then we have following push-out diagram:

$$
\begin{array}{c}
0 & \longrightarrow & X & \xrightarrow{f} & F & \longrightarrow & N_2 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_1 & \longrightarrow & M & \longrightarrow & N_2 & \longrightarrow & 0
\end{array}
$$

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which implies that the exact sequence $0 \rightarrow N_1 \rightarrow M \rightarrow N_2 \rightarrow 0$ is non-split. This is a contradiction with $\text{Ext}^1_{\Lambda}(N_2, N_1) = \text{Ext}^1_{\Lambda}(N_1, N_2) = 0$. Hence our claim is true. Namely, $\text{Ext}^1_{\Lambda}(X, N_2) = \text{Ext}^1_{\Lambda}(N_2, X) = 0$. Therefore $0 \rightarrow Y \rightarrow M \rightarrow X \rightarrow 0$ is $F^{N_1}$-exact and $F^{N_2}$-exact. This completes the proof.

\textbf{Lemma 3.6.} Let $\Lambda$ be a preprojective algebra of type $A_n$, $n \leq 4$. Let $X$ and $Y$ be indecomposable $\Lambda$-modules. Then there exits an dense open orbits in the variety of extensions between $X$ and $Y$.

\textbf{Proof.} It can be proved easily from [5, section 2.1].

\textbf{Remark.} Recall from [5], we say that $M$ degenerate to $N$ and denote by $M \leq_{\text{deg}} N$, if $O_N \subset \overline{O}_M$. Let $\Lambda$ be a preprojective algebra of type $A_n$, $n \leq 4$. Using the AR-formula and hammock algorithm we can see that $\dim \text{Ext}^1_{\Lambda}(X, Y) \leq 2$. In the case of $A_3$, we have that $\dim \text{Ext}^1_{\Lambda}(X, Y) \leq 1$. If $\dim \text{Ext}^1_{\Lambda}(X, Y) = 2$, by Lemma 3.6 we have two non-split exact sequence

$$0 \rightarrow Y \rightarrow M_1 \rightarrow X \rightarrow 0$$

and

$$0 \rightarrow Y \rightarrow M_2 \rightarrow X \rightarrow 0$$

such that $M_1 \leq_{\text{deg}} M_2$. Then by [5], we know that

$$\dim \text{Hom}_{\Lambda}(M_1, T) \leq \dim \text{Hom}_{\Lambda}(M_2, T)$$

for any $\Lambda$-module $T$. Hence if

$$0 \rightarrow Y \rightarrow M_1 \rightarrow X \rightarrow 0$$

is $F^T$-exact, then

$$0 \rightarrow Y \rightarrow M_2 \rightarrow X \rightarrow 0$$

is also $F^T$-exact by comparing dimensions.

\textbf{Lemma 3.7.} Let $\Lambda$ be a preprojective algebra of type $A_n$ with $n \leq 4$. Let $X$ and $Y$ be non-isomorphic indecomposable $\Lambda$-modules with $\text{Ext}^1_{\Lambda}(X, Y) \neq 0$. Let $T$ be a basic maximal rigid $\Lambda$-module. Then there exists a non-split exact sequence

$$(*) \quad 0 \rightarrow Y \rightarrow E \rightarrow X \rightarrow 0$$
or
\[(**)
0 \to X \to M \to Y \to 0\]

which is $F^T$-exact.

**Proof.** According to the Remark after Lemma 3.6, we only need to consider the case that $\dim \text{Ext}^1_{\Lambda}(X,Y) = 1$.

If $\Lambda$ is of type $A_3$, then $T$ has three indecomposable non projective direct summands $T_1$, $T_2$ and $T_3$. We divide them into three combinations $\{T_1, T_2\}$, $\{T_2, T_3\}$ and $\{T_1, T_3\}$. By Lemma 3.5, there is an exact sequence $(*)$ or $(**)$ which is $F^T_i$-exact for at least two combinations, then it is $F^T$-exact.

If $\Lambda$ is of type $A_4$, then $T$ has six indecomposable non projective direct summands $T_1$, $T_2$, $T_3$, $T_4$, $T_5$ and $T_6$. There are twenty combinations say $\{C_i\}_{1 \leq i \leq 20}$, such that each $C_i$ consists of three non isomorphic direct summands. We say an exact sequence is $C_i$-exact if it is $F^T_k$-exact with $T_k \in C_i$. Then as above each $C_i$ has at least one exact sequence $(*)$ or $(**)\) that is $C_i$-exact.

Now we show the assertion that if we cut the set $\{C_i\}$ into two parts, there always exists one part that covers all the six $T_i$. If we choose three elements from five elements, there is ten kind of possibilities. So, if we cut $\{C_i\}$ into two parts $U_1$ and $U_2$ such that the number of $C_i$ in $U_1$ is bigger than ten, then $\bigcup_{C_i \in U_1} C_i$ must contain at least six elements. The assertion is right. Now suppose the number of the $C_i$ in $U_1$ and $U_2$ are both ten. If $\bigcup_{C_i \in U_1} C_i$ contains six elements, then the assertion is right. If not, $\bigcup_{C_i \in U_1} C_i$ contains five elements. Without loss of generality, we may assume that $\bigcup_{C_i \in U_1} C_i = \{T_1, T_2, T_3, T_4, T_5\}$. Then each $C_i$ in $U_2$ contains $T_6$ and $\bigcup_{C_i \in U_2} C_i = \{T_1, T_2, T_3, T_4, T_5, T_6\}$. The assertion is also true.

Now we divide $\{C_i\}$ into two parts according to the $C_i$-exact sequence is of form $(*)$ or of form $(**)\). If $C_i$ can belong to both part, put it in only one part. Then there is an exact sequence that is $F^{Ti}$-exact for all $1 \leq i \leq 6$. \[\square\]

**Proposition 3.8.** Let $\Lambda$ be a preprojective algebra of type $A_n$, $n \leq 4$. Let $T$ be a basic maximal rigid $\Lambda$-module and $B = \text{End} T$. Then every classical tilting $B$-module is of the form $\text{Hom}_\Lambda(T', T)$, where $T'$ is a maximal rigid $\Lambda$-module.

**Proof.** By Proposition 4.4 in [11], we know that any $B$-module with projective dimension at most 1 is of the form $\text{Hom}_\Lambda(M, T)$ with $M$ being a $\Lambda$-module.

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If $M$ is not rigid, then there are indecomposable direct summands $X$ and $Y$ of $M$ such that $\text{Ext}^1_\Lambda(X,Y) = \text{Ext}^1_\Lambda(Y,X) \neq 0$. By Lemma 2.2 and Lemma 3.7, we know that
\[
\text{Ext}^1_{\text{End}_\Lambda(T)}(\text{Hom}_\Lambda(X,T), \text{Hom}_\Lambda(Y,T)) \neq 0
\]
or
\[
\text{Ext}^1_{\text{End}_\Lambda(T)}(\text{Hom}_\Lambda(Y,T), \text{Hom}_\Lambda(X,T)) \neq 0.
\]
Hence $\text{Hom}_\Lambda(M,T)$ is not partial tilting as $B$-module.

In particular, any partial tilting $B$-module is of the form $\text{Hom}_\Lambda(T',T)$ with $T'$ being a rigid $\Lambda$-module. Note that the number of non-isomorphic simple $B$-module is equal to the number of the non-isomorphic indecomposable direct summands of the maximal rigid $\Lambda$-module $T$, hence $\text{Hom}_\Lambda(T',T)$ is a tilting module if and only if $T'$ is a maximal rigid $\Lambda$-module. This completes the proof.

Summarizing above discussions, we have the following theorem which is one of our main results.

**Theorem 3.9.** Let $\Lambda$ be a preprojective algebra of type $A_n$ with $n \leq 4$, and $T$ be a maximal rigid $\Lambda$-module. Then the functor $F_T = \text{Hom}_\Lambda(-,T)$ induces an isomorphism of graphs $\psi_T : T_\Lambda \rightarrow T_{\text{End}_\Lambda T}$.

**Proof.** By Lemma 2.3, we know that $\psi_T$ is injective, and $\psi_T$ is also surjective by Proposition 3.8. Namely, $\psi_T$ is an isomorphism. This completes the proof.

The following Corollary is a direct consequence.

**Corollary 3.10.** Let $\Lambda$ be a preprojective algebra of type $A_n$ with $n \leq 4$. Then for each maximal rigid $\Lambda$-module $T$, the tilting graph $T_{\text{End}_\Lambda T}$ of $\text{End}_\Lambda T$ is isomorphic to the mutation graph $T_\Lambda$ of maximal rigid modules of $\Lambda$.

We illustrate our results by the example $\Lambda_{A_3}$. The AR-quiver of $\Lambda_{A_3}$ is as follows, here we represent $\Lambda_{A_3}$-modules by Lowvey series.
Note that there are exactly 14 basis maximal rigid $\Lambda_{A_3}$-modules up to isomorphism. We list non projective direct summands of every maximal rigid $\Lambda_{A_3}$-modules as follows.

\[
R_1 = \begin{array}{c}
2 \\
2 \\
1
\end{array} \oplus \begin{array}{c}
1 \\
2 \\
3
\end{array}, \quad R_2 = \begin{array}{c}
1 \\
3 \\
1
\end{array} \oplus \begin{array}{c}
1 \\
2 \\
3
\end{array}, \quad R_3 = \begin{array}{c}
2 \\
2 \\
3
\end{array} \oplus \begin{array}{c}
1 \\
2 \\
1
\end{array}, \quad R_4 = \begin{array}{c}
2 \\
3 \\
2
\end{array} \oplus \begin{array}{c}
2 \\
2 \\
3
\end{array}, \\
R_5 = \begin{array}{c}
3 \\
3 \\
2
\end{array} \oplus \begin{array}{c}
2 \\
3 \\
2
\end{array}, \quad R_6 = \begin{array}{c}
3 \\
3 \\
1
\end{array} \oplus \begin{array}{c}
2 \\
1 \\
1
\end{array}, \\
R_7 = \begin{array}{c}
3 \\
2 \\
1
\end{array} \oplus \begin{array}{c}
1 \\
2 \\
1
\end{array}, \quad R_8 = \begin{array}{c}
3 \\
3 \\
1
\end{array} \oplus \begin{array}{c}
1 \\
1 \\
3
\end{array}, \\
R_9 = \begin{array}{c}
1 \\
3 \\
1
\end{array} \oplus \begin{array}{c}
2 \\
1 \\
3
\end{array}, \quad R_{10} = \begin{array}{c}
1 \\
3 \\
2
\end{array} \oplus \begin{array}{c}
2 \\
1 \\
3
\end{array}, \\
R_{11} = \begin{array}{c}
1 \\
3 \\
2
\end{array} \oplus \begin{array}{c}
2 \\
1 \\
3
\end{array}, \quad R_{12} = \begin{array}{c}
3 \\
2 \\
2
\end{array} \oplus \begin{array}{c}
1 \\
3 \\
2
\end{array}, \\
R_{13} = \begin{array}{c}
2 \\
3 \\
1
\end{array} \oplus \begin{array}{c}
2 \\
1 \\
3
\end{array}, \quad R_{14} = \begin{array}{c}
2 \\
2 \\
1
\end{array} \oplus \begin{array}{c}
1 \\
2 \\
1
\end{array}.
\]

The mutation graph of basic maximal rigid $\Lambda_{A_3}$-modules is following.
According to Corollary 3.10, this picture is also the tilting graphs for endomorphism algebras of all maximal rigid $\Lambda_{A_5}$-modules, and every such endomorphism algebras has 14 basic tilting modules up to isomorphism.

Remarks. We conjecture that Theorem 3.7 is also true for preprojective algebras of tame representation type. In this case, the AR-quivers of the preprojective algebras are of tubular type.

4 The connectedness of mutation graphs of maximal rigid modules

In this section, we investigate the connectedness of mutation graphs of maximal rigid modules over preprojective algebras of representation finite type or tame type and prove Theorem 3 promised in the introduction.

It is well known that a preprojective algebra $\Lambda$ is of tame type if and only if it is of type $A_5$ and $D_4$. In this case, their AR-quivers are of tubular type which are the following.

We denote by $\Lambda_5$ the preprojective algebra of type $A_5$, then the ordinary quiver
of \( \Lambda_5 \) is

\[
\overline{A}_5 : 1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\alpha_3} 4 \xrightarrow{\alpha_4} 5,
\]

and \( \Lambda_5 = k\overline{A}_5/I \) with \( I \) generated by relations \( \{\alpha_1\alpha_1^*, \alpha_1^*\alpha_1 + \alpha_2\alpha_2^*, \alpha_2^*\alpha_2 + \alpha_3\alpha_3^*, \alpha_3^*\alpha_3 + \alpha_4\alpha_4^*, \alpha_4^*\alpha_4\} \).

Note that \( \Lambda_5 \) admits a Galois covering \( \widetilde{\Lambda}_5 \):

\[
\begin{array}{c}
1_3 & : & 3_2 & : & 5_1 \\
\downarrow & & \downarrow & & \downarrow \\
1_2 & & 3_1 & & 5_0 \\
\downarrow & & \downarrow & & \downarrow \\
1_1 & & 3_0 & & 4_0 \\
\downarrow & & \downarrow & & \downarrow \\
1_0 & : & 3_{-1} & : & 5_{-2}
\end{array}
\]

with the mesh relations and zero relations. All \( \Lambda_5 \)-module can be obtained by applying the push down functor to the \( \widetilde{\Lambda}_5 \)-modules, and \( \widetilde{\Lambda}_5 \) can be regarded as the repetitive algebras of the tubular algebra \( \Delta \):

\[
\begin{array}{c}
2_2 & \quad & 4_1 \\
\downarrow & & \downarrow \\
2_1 & & 3_1 \\
\downarrow & & \downarrow \\
1_1 & & 3_0 \\
\downarrow & & \downarrow \\
1_0 & & 3_{-1}
\end{array}
\]

of tubular type (6,3,2). We have \( \mod \widetilde{\Lambda}_5 \cong \mathcal{D}^b(\mod \Delta) \cong \mathcal{D}^b(\text{coh}(X)) \) by the theorems of Happel, Geigle and Lenzing, where \( X \) is a weighted projective line of type (6,3,2), see [12, section 9 and section 19] for details.

Let \( \Lambda_{D_4} \) be the preprojective algebra of type \( D_4 \), then the ordinary quiver of \( \Lambda_{D_4} \) is

\[
\overline{D}_4 : 2 \xrightarrow{\alpha_1^*} 1 \xrightarrow{\alpha_3} 4,
\]

\[
\begin{array}{c}
\alpha_1 \\
\downarrow \alpha_1 \\
\alpha_2 \\
\downarrow \alpha_2 \\
3
\end{array}
\]
and $\Lambda_{D_4} = k\overline{D}_4/I$ with $I$ generated by relations $\{\alpha_1^*\alpha_1, \alpha_2^*\alpha_2, \alpha_3^*\alpha_3, \alpha_1\alpha_1^* + \alpha_2\alpha_2^* + \alpha_3\alpha_3^*\}$.

Note that $\Lambda_{D_4}$ has a Galois covering $\widetilde{\Lambda}_{D_4}$ as follows:

```
    / \  \
   /   \  \\
  /     \  \\
 /       \  \\
1_3  2_3  3_3  4_3
  |     |     |
  |     |     |
  |     |     |
  |     |     |
1_2  2_2  3_2  4_2
  |     |     |
  |     |     |
  |     |     |
  |     |     |
1_1  2_1  3_1  4_1
  |     |     |
  |     |     |
  |     |     |
  |     |     |
1_0
```

with the mesh relations and the zero relations. It can be regarded as the repetitive algebra $\Delta$:

```
    / \  \
   /   \  \\
  /     \  \\
 /       \  \\
2_2  3_2  4_2
  |     |     |
  |     |     |
  |     |     |
  |     |     |
2_1  3_1  4_1
  |     |     |
  |     |     |
  |     |     |
  |     |     |
1_0
```

the relations are $\{\alpha_{12}^*\alpha_{12}, \alpha_{22}^*\alpha_{22}, \alpha_{32}^*\alpha_{32}, \alpha_{12}\alpha_{11}^* + \alpha_{22}\alpha_{21}^* + \alpha_{32}\alpha_{31}^*\}$.

It is a tubular algebra obtained through one point extensions of the $\overline{D}_4$ tame
concealed algebra $\Delta^0$:

Thus $\Delta$ is of tubular type $(3,3,3)$. Again, $\text{mod} \tilde{\Lambda}_{D_4} \cong \mathcal{D}^b(\text{mod} \Delta) \cong \mathcal{D}^b(\text{coh}(X))$, where $X$ is a weighted projective line of type $(3,3,3)$, see [12, section 9 and section 19] for details.

Let $G = \mathbb{Z}$ be the Galois group of the Galois covering $F : \tilde{\Lambda} \to \Lambda$. $G$ has an action on the $\tilde{\Lambda}$-modules $X$, that is for every vector space $X_{k_j}$ of $X$ corresponding to the vertex $k_j$, we get $X^{(i)}$ with $X_{k_{j+i}} = X_{k_j}$ and keep the maps between the vector spaces. Let $F$ be the push down functor from $\text{mod} \tilde{\Lambda}$ to $\text{mod} \Lambda$. Then we have $\text{Hom}_\Lambda(F(X), F(X)) = \sum_{i \in \mathbb{Z}} \text{Hom}_X(X, X^{(i)})$.

Let $\mathcal{C}$ be the cluster category of a hereditary abelian category with cluster-tilted objects in the sense of [6, 21]. According to [6, Proposition 3.5], the tilting graph of cluster-tilted objects in $\mathcal{C}$ is connected if $\mathcal{C}$ is the cluster category of a finite dimensional hereditary algebra.

**Theorem 4.1.** Let $\Lambda$ be a preprojective algebra of finite or tame representation type. Then the mutation graph of basic maximal rigid $\Lambda$-modules is connected.

**Proof.** It is well known that $\text{mod} \Lambda$ is 2-Calabi-Yau. And it is clear that the basic maximal rigid modules of $\text{mod} \Lambda$ are in bijection with the basic cluster-tilted objects in $\text{mod} \Lambda$ and the mutation graphs of them are the same by definitions. So we only need to consider the mutation graph of the basic cluster-tilted objects in $\text{mod} \Lambda$.

If $\Lambda$ is of type $A_2$, $A_3$ or $A_4$, then the AR-quiver of $\text{mod} \Lambda$ is the same with the quivers of the cluster category $\mathcal{C}$ of $A_1$, $A_3$, and $D_6$ respectively, see [12, section 20.1]. Hence, the cluster-tilted objects in $\text{mod} \Lambda$ are in bijection with the cluster-tilted objects in $\mathcal{C}$. Hence the mutation graph of the basic cluster-tilted objects in $\text{mod} \Lambda$ is connected by [6] Proposition 3.5].
If \( \Lambda \) is of type \( A_5 \), we know that \( \text{mod} \tilde{\Lambda}_5 \cong \mathcal{D}^b(\text{mod}\Delta) \cong \mathcal{D}^b(\text{coh}(X)) \), and by [12, section 14.5,14.6] we know that \( \text{mod} \Lambda_5 \) is a fundamental domain of \( \text{mod} \tilde{\Lambda}_5 \) under the action of the Galois group \( Z \). So, \( \text{mod} \Lambda_5 \cong \text{mod} \tilde{\Lambda}_5/(1) \) as the orbit category, where \( (1) \) is the generator of the Galois group. The cluster category of \( \mathcal{D}^b(\text{coh}(X)) \) is by definition \( \mathcal{C} = \mathcal{D}^b(\text{coh}(X))/\tau^{-1}[1] \cong \text{mod} \tilde{\Lambda}_5/\tau^{-1}[1] \), where \( \tau^{-1} \) is the inverse of the AR translation and \( [1] \) is the shift functor. By [10, Lemma 6.1], \( (1) \cong \tau^{-1}[1] \), so we have \( \mathcal{C} \cong \text{mod} \Lambda_5 \). By [7, Theorem 8.8], the tilting graph of \( \mathcal{C} \) is connected, so the mutation graph of \( \text{mod} \Lambda_5 \) is connected. The \( D_4 \) case can be proved similarly. This completes the proof. \( \square \)

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