A model of eddy current nondestructive evaluation for a current leaking crack

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ABSTRACT

This paper presents a model-based study of undesirable signal loss problems, often encountered by eddy current inspections of tight cracks. Possible current leakage through a portion of the crack face is the focus here as the principal mechanism of signal reduction. Two problematic types of cracking conditions are studied particularly, i.e., a surface-smeared crack and a crack under compressive stress. With a newly developed model and its numerical results, the paper shows quantitative analyses of the signal loss for the above two cases, compared to the response of a pristine surface-breaking crack in the similar geometrical configuration.

1. Introduction

There is a long-standing issue of NDE, i.e., cracks of nominally equal sizes and configurations do not necessarily yield consistent responses to the inspection. Particularly risky is a possible loss of signals from cracks that should be detectable. Eddy current (EC) inspections of tight cracks experience undesirable signal reduction, when a portion of the crack allows currents to leak through. This paper develops a model of EC NDE that is capable of describing the current leakage phenomenon, thus providing a quantitative analysis tool against the possible signal loss.

Two problematic cracking conditions are studied specifically, i.e., a surface-smeared crack and a crack under compressive stress. A smeared crack is modeled as a slightly submerged crack, where a thin layer of host metal is covering over the top edge. The host metal layer forms a bridge to let the current flow through. In contrast, a crack under compression may experience current leakage through asperity contacts formed across the crack face. In this case, the electrical connections are made by effective materials of a higher resistivity, because oxides tend to form at the contacts.

Under EC excitation, a crack develops potential gaps, $\Delta \phi$, across its face [1]. It is important to recognize that the on-face $\Delta \phi$ distribution changes drastically when leakages exist, depending on the leak location and condition. With the leakages under consideration, the behavior of $\Delta \phi$ is both non-trivial and sometimes even counter-intuitive. First, consider a surface-breaking crack. The potential $\Delta \phi$ is finite, and in fact large, on the top edge (the “mouth”) of the crack. Then, suppose that the crack submerges below the part surface by a minute depth, resembling a smeared crack. One might think intuitively that $\Delta \phi$ would vanish suddenly on the top edge because the covering layer would short-circuit the ligament. However, this intuition should be rejected because such a large discontinuous decrease is unphysical and unrealistic to occur. As the crack submerges further, the potential at the top will eventually vanish, even if the burial depth is within the skin depth. The implication is that the potential on the top edge undergoes a continuous but rapid decrease from the large (breaking) to the null (finitely buried) values, where the smeared-crack condition falling in between. Second, consider a crack under compression. In this case, the asperity contacts provide the leakage path, and the potential need not vanish across, because of the contact resistance. Instead, the potential is considered proportional to the leak current intensity. In summary, these are the intricacies of the $\Delta \phi$ distributions and their drastic deviations which a successful model must be able to keep track, given the different current leakage conditions. The objective of the work is to develop such a model of the required capabilities, and then to predict the degree of signals losses as a function of the current leakage in question.

This paper presents an EC NDE model that meets the objective, being uniquely capable of solving the above-described problems involving non-trivial potential behaviors. Mathematically, the model is formulated around surface integral representations, expressing fields everywhere by independent boundary fields. The key is to take the field dynamics into account everywhere, particularly within the defect region, from the outset. In the tight crack limit, the defect region collapses to the crack face which is a planar geometrical surface with a perimeter contour. Among the boundary variables, two independent dynamical variables...
remain on the crack, i.e., the potential \( \Delta \phi \) on the perimeter and the magnetic surface current densities on the face. The magnetic currents are directly related to the EC around the crack, and coupled to the on-face potential \( \Delta \phi \) through the governing integral equations. Having the explicit dynamical variable on the perimeter is the strength of this model, enabling the description of the intricate potential variations under the leakage conditions. The resulting model is applicable to both insulating and leaky cracks on an equal footing.

A number of papers appeared to study the tight-crack modeling issue [2–14], none of which is adequately equipped to study the above-mentioned local nontrivial behaviors of the potential. The papers have not described the said near-surface potential behavior. Indeed, they would have technical difficulties in keeping track of the continuous but rapid disappearance of the potential in the vicinity of the part surface. The majority of the approaches use volumetric elements (the volume integral methods and FEM), where they would face a difficulty in discretizing the infinitesimally thin ligament which would require doubly small volumetric elements. The volumetric approaches are also awkward at best for handling of the internal leakage problem. They were forced to consider a fictitious material to fill the defect volume, where its conductivity must be considered to diminish as the defect width decreases. A few of the quoted papers [7,8] addressed the internal leakage problem using the electric dipole moment approach. Although a non-volumetric approach, the method has its own difficulty, having to do with the hyper singular integral equation that must be solved. Artificial regularizations are needed to solve the singular equation, and no process has been developed to extract sensible finite results for the intricate near-surface potential behavior.

The remainder of this paper is organized as follows: The following Section 2 formulates the governing equations to solve for on-surface field variables. The solution procedure is established there for the tight crack problem, presenting a set of the boundary integral equations (BIEs) for general part geometry. The subsequent Section 3 applies the developed solution procedure to specific flat part geometry, where the part surface variables are eliminated analytically via the Fourier transformation. The reduced set of the BIEs determines the principal unknowns, i.e., the magnetic currents and the potential gap on the crack face. The BIEs are soluble numerically as described in Sec. 4, followed by Sec. 5 where several numerical results are shown for impedance predictions in the specific half-space geometry. Needs for empirical laboratory-based studies are discussed in Sec. 6, followed by Conclusions and Discussions in Sec. 7.

\[
\Theta(\vec{x}_p) \begin{bmatrix} \vec{E}(\vec{x}) \\ -\vec{H}(\vec{x}) \end{bmatrix} = \int_{S} K^{\nu} \begin{bmatrix} \vec{x} - \vec{x}_p \\ \vec{x} \end{bmatrix} \nabla \times \begin{bmatrix} \vec{j}(\vec{x}) \\ -\vec{J}(\vec{x}) \end{bmatrix} dS - \int_{S} K^{\nu k} \begin{bmatrix} \vec{x} - \vec{x}_p \\ \vec{x} \end{bmatrix} \nabla \times \begin{bmatrix} \vec{j}(\vec{x}) \\ -\vec{J}(\vec{x}) \end{bmatrix} dS
\]

2. Formulation in general part geometry

2.1. Stratton-Chu formula for defect problem

The mathematical formulation starts with the version of the Stratton-Chu (S–C) Formulas written for the dynamical problem, where the electromagnetic fields, \( \vec{E} \) and \( \vec{H} \), are given the integral representations in terms of the tensor Green’s functions and the surface currents [15]. In what follows, we introduce a compact notation where the fields and the surface currents are combined into the column vectors as

\[
\begin{bmatrix} \vec{E}(\vec{x}) \\ -\vec{H}(\vec{x}) \end{bmatrix} = \Theta(\vec{x}_p) \begin{bmatrix} \vec{E}(\vec{x}) \\ -\vec{H}(\vec{x}) \end{bmatrix} = \int_{S} K^{\nu} \begin{bmatrix} \vec{x} - \vec{x}_p \\ \vec{x} \end{bmatrix} \nabla \times \begin{bmatrix} \vec{j}(\vec{x}) \\ -\vec{J}(\vec{x}) \end{bmatrix} dS - \int_{S} K^{\nu k} \begin{bmatrix} \vec{x} - \vec{x}_p \\ \vec{x} \end{bmatrix} \nabla \times \begin{bmatrix} \vec{j}(\vec{x}) \\ -\vec{J}(\vec{x}) \end{bmatrix} dS
\]

Accordingly, the integration kernels are combined into a 2-by-2 matrix \( K^{\nu}(\vec{x}) \) where its components are the tensor Green’s functions. These and related notations are used throughout the paper, and all defined in Appendix A, including \( K^{\nu}(\vec{x}) \) and \( \left\{ U, D, R, L, \mu, \sigma \right\} \) as the basis matrices and vectors. Now, the two S–C formulas can be expressed in a compact form

\[
\Theta(\vec{x}_p) \begin{bmatrix} \vec{E}(\vec{x}_p) \\ \vec{H}(\vec{x}_p) \end{bmatrix} = \Theta(\vec{x}_p) \begin{bmatrix} \vec{E}(\vec{x}_p) \\ \vec{H}(\vec{x}_p) \end{bmatrix} = \int_{S} K^{\nu} \begin{bmatrix} \vec{x} - \vec{x}_p \\ \vec{x} \end{bmatrix} \nabla \times \begin{bmatrix} \vec{j}(\vec{x}) \\ -\vec{J}(\vec{x}) \end{bmatrix} dS
\]

written for a finite region of the volume \( V \) with the enclosing surface \( S \), and with the wave number \( k \). In Eq. (2), the field point variable \( \vec{x}_p \) can occur anywhere in space, while the integration point \( \vec{x} \) runs over the integration region \( S \). By convention, the surface normal \( \vec{n} \) points outward from \( V \), unless specified otherwise, and \( \Theta(\vec{x}) \) is the characteristic function of \( V \). The term \( F^{\nu}(\vec{x}_p) \) represents the incident fields, and can be written similarly as

\[
F^{\nu}(\vec{x}_p) = \int_{V} K^{\nu} \begin{bmatrix} \vec{x} - \vec{x}_p \\ \vec{x} \end{bmatrix} \nabla \times \begin{bmatrix} \vec{j}(\vec{x}) \\ -\vec{J}(\vec{x}) \end{bmatrix} dV
\]

which is present when a current source \( \vec{j} \) exists in the volume \( V \) if contained by \( V \).

A prototypical EC NDE model system is illustrated in Fig. 1, involving a metal part of the volume \( V \) with the total enclosing surface \( S \), placed in the empty space (air). The air region contains a probe coil with the current density \( \vec{j} \). To start with, consider a volumetric defect of the volume \( V_{\text{def}} \) with the surface \( S_{\text{def}} \), embedded in \( V \). The S–C formula can be written in the three regions respectively, i.e., for the outside of \( S \) (in air), inside the defect volume \( V_{\text{def}} \) (also in air), and in the metal region excluding the defect. For instance, the outside formula has the expression

\[
\left[ 1 - \Theta(\vec{x}) \right] \begin{bmatrix} \vec{E}(\vec{x}_p) \\ \vec{H}(\vec{x}_p) \end{bmatrix} = F^{\nu}(\vec{x}_p) - \int_{S} K^{\nu k} \begin{bmatrix} \vec{x} - \vec{x}_p \\ \vec{x} \end{bmatrix} \nabla \times \begin{bmatrix} \vec{j}(\vec{x}) \\ -\vec{J}(\vec{x}) \end{bmatrix} dS
\]

where the in-air wavenumber \( k_{\text{air}} \) vanishes in the quasi-static limit. The other two formulas in the metal and in the defect can be written down similarly, except that we here work with the following linear combinations of the Mueller type [15]. One combination is the sum,

\[
\begin{bmatrix} \vec{B}(\vec{x}) \\ \vec{J}(\vec{x}) \end{bmatrix} = \left\{ \begin{array}{c} \frac{\rho_{0}}{\mu} \vec{H}(\vec{x}) \\ \frac{k_{0}^{2}}{\omega \mu} \vec{E}(\vec{x}) \end{array} \right\}, \quad \vec{E}(\vec{x}) \in \begin{cases} V_{\text{air}} \\ V_{\text{metal}} \end{cases}
\]

The second linear combination is a weighted sum that represents the flux densities, defined as

\[
\begin{bmatrix} \vec{B}(\vec{x}) \\ \vec{J}(\vec{x}) \end{bmatrix} = \left\{ \begin{array}{c} \frac{\rho_{0}}{\mu} \vec{H}(\vec{x}) \\ \frac{k_{0}^{2}}{\omega \mu} \vec{E}(\vec{x}) \end{array} \right\}, \quad \vec{x} \in \begin{cases} V_{\text{air}} \\ V_{\text{metal}} \end{cases}
\]
Fig. 1. Illustrations of an EC NDE model configuration, involving a metal part of the volume $V$ that contains an air-filled defect indicated by the volumetric region $V^{cr}$. The volumes are enclosed by the surfaces $S$ and $S^{cr}$, respectively.

$$(-i\omega\mu_0)\vec{E} \text{ in the air. The combined formula reads}$$

$$\Theta_i\left(\vec{x}_p\right) \begin{bmatrix} \vec{J}\left(\vec{x}_p\right) \\ -i\omega\vec{B}\left(\vec{x}_p\right) \end{bmatrix} = \int_S \left( \frac{k^2}{i\omega\mu_0} U + io\mu D \right) K^3\left(\vec{x} - \vec{x}_p\right) \vec{J}^m(\vec{x}) dS$$

$$- \int_{S^{cr}} \left( \frac{k^2}{i\omega\mu_0} U + io\mu D \right) K^3\left(\vec{x} - \vec{x}_p\right) \left( \frac{k^2}{i\omega\mu_0} U + io\mu D \right) K^0(\vec{x} - \vec{x}_p) \vec{J}^m(\vec{x}_p) dS.$$

(7)

Notice that no characteristic function of the crack volume $V^{cr}$ appears in Eqs. (5) and (7), indicating that the corresponding integral representations are continuous across the air-metal interface of $S^{cr}$. Physically, however, continuous are the tangential fields and the normal components of the fluxes across the interface. Consequently, Equation (5) provides numerically stable representations everywhere in $V$, including the vicinity of $S^{cr}$, for the tangential fields. Similar statement applies to the normal components of the flux densities, applicable to Eq. (7).

Several remarks are in order: (i) In the direct term of Eq. (7), the double derivative terms cancel each other, thus removing the hyper-singularity from the resulting integral equation. (ii) Being quasi-static electromagnetism, the EC problem is free from spurious resonance problems. Thus, we can use arbitrary Mueller combinations. However, we must pay attention not to encounter any $k^2_{\perp}$ denominator in order to avoid the low-frequency breakdown. (iii) Equations (5) and (7) apply equally to the buried and surface-breaking defects. For a surface-breaking defect, the surfaces $S$ and $S^{cr}$ share the overlapping surface $S^m$, which is the opening portion of $S^{cr}$ (the “mouth” of the crack). It is possible to write the equations without overlapping $S^m$ integration, by splitting the part surface $S$ and the surface current as

$$\vec{J}(\vec{x}) = \begin{cases} \vec{J}_m(\vec{x}) & \vec{x} \in \hat{S} = S - S^m. \end{cases}$$

(8)

However, Equations (5) and (7) can be recovered formally, after rearranging terms involving $S^m$ integrations.

2.2. Field behaviors around closed crack

This sub-section explores the field behavior around a tightly closed crack that may permit current leakage, based on physics intuitions. After starting with a finitely open defect, we examine the limit $w \to 0$ where $w$ is the opening width. Let $S^-_f$ be the two mutually approaching sides, which we assume equal in area and shape, planar, and separated by $w$, with a constant normal vector $\vec{N}$ (Fig. 2). This paper makes an additional simplifying assumption that the defect occurs perpendicularly to the part surface. The enclosing surface $S^{cr}$ may be divided as

$$S^{cr} = S_f + S^{cr} + S^m,$$

(9)

where $S_f$ is the perimeter surface. For a surface-breaking defect, the surface $S^m$ may be subdivided into the mouth and edge surfaces, $S^m$ and $S^{cr}$.

Geometrically, the volume $V^{cr}$ collapses to the crack face $S'$ (the average of $S_f$) in the limit $w \to 0$, while $S^m$ collapses to the perimeter contour $C'$. For a given point $\vec{x}$ on $S'$, consider the pair of the neighboring points $\vec{x}_+ \in S^m_f$ defined as

$$\vec{x}_+ \equiv \vec{x} \pm \frac{w}{2} \vec{N}, \quad \vec{x}_- \in S^{cr}_f, \quad \vec{x} \in S'$$

(10)

when $w \neq 0$. On each of $S^m_f$, there exists a surface magnetic current

Fig. 2. Illustrations of a surface-breaking defect in the (a) perspective and (b) cross-sectional views. The defect volume is air-filled, containing no fictitious fill material.

Fig. 3. Illustration of the closed contour $C$, (a) encompassing a part of the defect volume in the cross section, while (b) projected onto the crack face $S'$ shown as the path from $\vec{x}_0$ to $\vec{x}$.
density
\[
\vec{M} = \hat{\vec{N}} \times \vec{E}.
\] (11)

As illustrated in Fig. 2, the eddy current (\(\sigma \vec{E}\)) flows discontinuously between \(\Sigma_1\), indicating that there is a discontinuity between the tangential \(\vec{E}\) fields across. Hence, it follows that
\[
\Delta \vec{M}(\vec{x}) \equiv \vec{M}(\vec{x}_+) - \vec{M}(\vec{x}_-) \neq 0,
\] (12)

which is one of the defining characteristics of the near-crack field behavior. Here, we demand that this characteristics remains intact in the infinitesimal distance. The discontinuous tangential \(\vec{E}\) fields are continuous across the interface, i.e.
\[
\Delta \vec{E} = 0\quad \text{on} \quad \Sigma,
\] (13)

where \(\Sigma = \vec{N} \cdot \vec{E}\) in \(\mathbb{V}^r\). To prove that \(\Delta \vec{E} = 0\), consider the contour integral of the \(\vec{E}\) field along the closed rectangular path shown in Fig. 3, encircling a portion of \(\mathbb{V}^r\). The integral is proportional to the total magnetic flux through the rectangle, which vanishes when \(w \rightarrow 0\) because there is no reason for the magnetic field to diverge in \(\mathbb{V}^r\).

The other field components are assumed continuous across the infinitesimal distance. The discontinuous tangential \(\vec{E}\) fields imply that there exists a finite potential gap \(\Delta \phi\) across the crack face, calculated as
\[
\Delta \phi(\vec{x}) = \lim_{w \rightarrow 0} \left[ \vec{E}(\vec{x}_+) - \vec{E}(\vec{x}_-) \right] = \left( \vec{E}_N \right),
\] (14)

where \(\vec{E}_N = \vec{N} \cdot \vec{E}\) in \(\mathbb{V}^r\). From Eqs. (16) and (18), it follows that the crack surface integrals can be replaced by the magnetic source integrals in Eqs. (5) and (7). Explicitly, Equation (5) reduces to
\[
\vec{M}_f^N = \int_{\Sigma} K^N(\vec{x}_f - \vec{x}) \vec{J}(\vec{x}) d\vec{x},
\]

while the upper component (E-equation) of Eq. (7) reduces to
\[
\Theta_f(\vec{x}_f) \vec{E}(\vec{x}_f) = \int_A U \vec{K}(\vec{x}_f - \vec{x}) \vec{J}(\vec{x}) d\vec{x} - \int_{\Sigma} U \vec{K}^k(\vec{x}_f - \vec{x}) \vec{u} \Delta \vec{M}(\vec{x}) d\vec{x} - \int_{\Sigma} U \vec{K}^N(\vec{x}_f - \vec{x}) \vec{u} \Delta \phi(\vec{x}) d\vec{x},
\] (21)

For a surface-breaking crack, Equations (20) and (21) hold as they are, except that the mouth term admits dual expressions of the form
\[
\int_{\Sigma} K^N(\vec{x}_f - \vec{x}) \vec{J}(\vec{x}) d\vec{x} = \int_{\Sigma} K^N(\vec{x}_f - \vec{x}) \vec{J}^N(\vec{x}) d\vec{x} = \int_{\Sigma} K^N(\vec{x}_f - \vec{x}) \vec{u} ||\vec{P}_N|| \Delta \phi(\vec{x}) d\vec{x},
\] (22)

resulting from the mouth surface overlap [Eq. (8)]. In Eq. (22), the superscript \(A\) stands for \(k, k_0, \) or \(k_0, k\). The tangential projection is included in the last term because the mouth contour \(C_m\) is the intersection between \(S\) and \(\Sigma\).

It is not proper to start with the simple hypothesis that the crack acts as magnetic currents. Such assumption would result in the expression that resembles Eq. (20) but fails to share the following important features. In contrast, we take the field dynamics into account inside the crack volume from the outset, via Eqs. (5) and (7). The positive features of Eq. (20) arise as the remnant of the internal dynamics: First, the
weakly singular kernel appears in the direct terms of Eqs. (20) which generate the \( E \) field from the magnetic current sources. Thanks to the weak singularity, the generated \( E \) field remains finite even when the field and source points coincide. The crack-generated \( E \) field can counteract the incident field finitely on the crack face, so as to divert the EC around. Second, the same kernel contains the non-damping Green’s function \( G^\sigma \). The naïve expression without the internal dynamics will fail to account for the possible long-range correlation between the top and bottom regions of the crack, which will play a role under exceptional circumstances.

The description in terms of the magnetic surface current, Eq. (20), replaces the earlier electric dipole field description \([1, 7, 8]\). Their integration kernel contains hyper-singularity, while ours is only weakly singular. Hyper-singular integral equations require artificial regularization where the crack face must be given a complex internal structure, not being a simple planar object. Regularized solutions of singular integral equations are ambiguous, because the solutions will depend on how the regularization is made.

It should be remarked also that no denominator of the form \( k_n^2 \) appears in the corresponding kernels in Eqs. (20). Therefore, these representa-

tions do not suffer from the low-frequency breakdown issue in the limit \( k_n \to 0 \).

In the thick crack limit, the potential gap \( \Delta \phi \) appears explicitly in Eqs. (20) and (21). To complete the governing equation for \( \Delta \phi \) is needed. Here, we derive the 2D integral representation of \( \Delta \phi \) on \( S' \), first by taking the tangential derivatives of Eq. (15),

\[
\nabla_x \Delta \phi(x) = - \vec{N} \times \Delta M^N(x), \quad \nabla_x \Delta \phi(x) = - \vec{N} \times \Delta M^N(x),
\]

and then by turning the results into the integral formula

\[
\theta_{\nu}(\vec{x}_p) \Delta \phi(\vec{x}_p) = \left[ \theta_{\nu}(\vec{x}_p) \right] \Delta \phi(\vec{x}_p) = \left[ \theta_{\nu}(\vec{x}_p) \right] \Delta \phi(\vec{x}_p) + \int_{\partial S} \left[ -\delta_\nu(\vec{x}_p - \vec{x}_s) \right] \Delta \phi(\vec{x}_s) dS,
\]

\[
\nabla x \Delta \phi(x) = - \vec{N} \times \Delta M^N(x),
\]

and then by turning the results into the integral formula

\[
\theta_{\nu}(\vec{x}_p) \Delta \phi(\vec{x}_p) = \left[ \theta_{\nu}(\vec{x}_p) \right] \Delta \phi(\vec{x}_p) + \int_{\partial S} \left[ -\delta_\nu(\vec{x}_p - \vec{x}_s) \right] \Delta \phi(\vec{x}_s) dS.
\]

We state, without derivation, that Equation (24) is also derivable directly from the S–C formula in \( V^\nu \).

Given these equations, the on-crack unknown variables can be determined in the following steps: (i) Select the H equations from Eqs. (4) and (20). (ii) Turn them into simultaneous surface integral equations, and solve them for \( J^\nu \) on the part surface \( S \) in terms of the coil and crack sources. (iii) Insert the resulting \( J^\nu \) on \( S \) into Eq. (21) after taking its normal component, so that \( \Delta \phi \) on \( S' \) is expressed in terms of the sources. (iv) Cast Eq. (24) into the contour integral equation to solve \( \Delta \phi \) on \( C' \) in terms of \( \Delta M^N \) as the source. (v) Turn the current leakage condition, Eq. (19), into the integral equation over \( S' \), and finally solve the equations to determine the unknown crack sources, i.e. \( \Delta M^N \) on \( S' \) and \( \Delta \phi \) on \( C' \).

The solution process involves the computation of \( \Delta \phi \) everywhere on \( S' \). The representation (24) is unsuitable for this computation near the contour, because such formulas are known to exhibit numerical instability near the boundary. A stable computational method is described in the subsequent section where explicit solutions are given.

3. Electric fields in flat surface geometry

This section presents the detail of the solution procedure developed in Sec. 2, applied to specific flat surface part geometry. The description is given primarily for the uniform half space, while the results only for a plate problem. Suppose that a metal of the conductivity \( \sigma \) and the permeability \( \mu \) is occupying the lower half space \( \sigma \leq 0 \). In this case, the part surface \( S \) is the xy-plane \( Sv \sigma \), and the lower components (i.e. H equations of Eqs. (4) and (20) read

\[
\frac{1}{2} D \nabla^\nu_x \left( \vec{x}_s, 0 \right) + \int_{S'} D \nabla^\nu_x \times \vec{\varphi} \cdot K^\nu \left( \vec{x}_s, 0 \right) J^\nu \left( \vec{x}_s, 0 \right) dS = D \nabla^\nu_x \left( \vec{x}_s, 0 \right),
\]

\[
\frac{1}{2} R \nabla^\nu_x \left( \vec{x}_s, 0 \right) - \int_{S'} R \nabla^\nu_x \times \vec{\varphi} \cdot K^\nu \left( \vec{x}_s, 0 \right) J^\nu \left( \vec{x}_s, 0 \right) dS = R R^\nu_x \left( \vec{x}_s, 0 \right),
\]

respectively, where the crack surface term is treated as the source.
expression for the coil-induced in-air field that reads
\[
D \frac{n}{\sigma} \times \mathbf{F}_{\text{ind}}(\hat{B}, \varphi_c) = \frac{d}{dV} \int e^{-\hat{r}\cdot \vec{E}} \left[ P_{\varphi} + P_{\varphi q} - \frac{P_q}{t_0} \right] \frac{1}{2} e^{-\hat{r}\cdot \vec{E}} \frac{\vec{f}^c(x)}{dV}
\]
\[
= \frac{d}{dV} \int e^{-\hat{r}\cdot \vec{E}} \left[ -\frac{k_i^2 P_{\varphi}}{t_0} + P_{\varphi q} \right] \left[ \frac{1}{2} e^{-\hat{r}\cdot \vec{E}} \frac{\vec{f}^c(x)}{dV} \right],
\]
where the first line comes directly from the Biot–Savart law, while the second line follows after the \( j_c \) component is eliminated by the use of the closed-loop current condition \( \nabla \cdot \vec{J} = 0 \). From Eqs. 30–32, the coil-induced electric field in the metal body can be derived in the form
\[
\vec{E}^{(cl)}(x) = \int \phi(\hat{r}\cdot \vec{E}) \{ -\frac{k_i^2 P_{\varphi}}{t_0} + P_{\varphi q} \} \left[ \frac{1}{2} e^{-\hat{r}\cdot \vec{E}} \frac{\vec{f}^c(x)}{dV} \right].
\]
The kernel function \( K^{(cl)} \) is given explicitly in Appendix B. The crack-induced reflection field follows similarly from Eqs. (27), (30) and (31), with the result
\[
\vec{E}^{(ir)}(x) = -\left[ \frac{d}{dS} \int K^{(ir)}(\hat{r}\cdot \vec{E}) \left[ \Delta \vec{M}(x) - \Delta \vec{M}^{cr}(x) \right] \left( \frac{1}{2} e^{-\hat{r}\cdot \vec{E}} \frac{\vec{f}^c(x)}{dV} \right) \right],
\]
where the kernel function \( K^{(ir)} \) is also given in Appendix B.

The derivation for the half space geometry is straightforwardly extendable to a single uniform plate of the infinite extent with a finite thickness. In particular, Equations (33) and (34) remain the same in form, except for the different kernel functions. Their results are included in Appendix B as well.

It should be remarked that the non-trivial functions, \( \{ \alpha(x), \beta(x), \gamma(x) \} \), appear in the kernel expressions as a remnant of accounting for the crack internal dynamics explicitly. These functions would appear as unity if one would naively assume that the crack acts as a magnetic current source.

4. Boundary integral equations on crack face

Given the electric field representations (33) and (34), the current leakage condition (19) now reads
\[
\left[ 1 - \theta_\varphi \left( \frac{\chi}{\hat{B}} \right) \right] \chi(\frac{\chi}{\hat{B}}) = -\int_{S_\varphi} \left\{ -\partial_s g^\alpha \left( \frac{\chi}{\hat{B}} \right) \Delta \varphi \left( \frac{\chi}{\hat{B}} \right) + g^\alpha \left( \frac{\chi}{\hat{B}} \right) \partial_s g^\alpha \left( \frac{\chi}{\hat{B}} \right) \right\} dS,
\]
\[
g^\alpha \left( \frac{\chi}{\hat{B}} \right) \equiv \frac{1}{2\pi} \kappa_0 (m|\bar{\chi}^c)
\]

where the characteristic function is represented in terms of the Green’s function \( g^\alpha \). The solution of Eq. (37) yields \( \Delta \varphi \) on \( S_\varphi \) in terms of \( \Delta \vec{M}^{cr} \) on \( S_\varphi \). The current leakage term of Eq. (35) further requires the computation of \( \Delta \varphi \) on \( S_\varphi \), for which one might consider to use Eq. (24) itself. However, formulas of this type are known to be unstable against numerical computation near the boundary contour. A method to stabilize the computation was presented in Ref. [18]. Here, we need an auxiliary scalar field \( \chi(\bar{\chi}) \) and the infinite plane extending \( S_\varphi \). Suppose that the field \( \chi(\bar{\chi}) \) exists on the plane outside \( S_\varphi \) where it satisfies the field equation
\[
(\nabla^2 - m^2) \chi(\bar{\chi}) = 0,
\]
as well as the boundary condition
\[
\chi(\bar{\chi}) = \Delta \varphi(\bar{\chi}),
\]
on \( S_\varphi \). If so, then the field \( \chi(\bar{\chi}) \) obeys the 2D Green’s formula
\[
\Delta \varphi(\bar{\chi}) = \int_{S_\varphi} \left\{ -\partial_s g^\alpha \left( \frac{\chi}{\hat{B}} \right) \Delta \varphi \left( \frac{\chi}{\hat{B}} \right) - g^\alpha \left( \frac{\chi}{\hat{B}} \right) \partial_s g^\alpha \left( \frac{\chi}{\hat{B}} \right) \right\} dS = 0.
\]
which can be solved for \( \partial \phi^C \) in terms of \( \Delta \phi \) on \( \mathcal{O} \). Finally, the sum of Eqs. (24) and (40) yields the smooth representation of \( \Delta \phi \) as

\[
\Delta \phi(\mathbf{x}_p) = \int_{\mathcal{O}} \left\{ -\partial_n \left[ g^p(\mathbf{x}_s - \mathbf{x}_p) - g^\phi(\mathbf{x}_s - \mathbf{x}_p) \right] \Delta \phi(\mathbf{x}_s) - g^p(\mathbf{x}_s - \mathbf{x}_p) \partial_n \phi(\mathbf{x}_p) \right\} d\mathbf{x}_s + \int_{\mathcal{S}} \left[ \nabla g^\phi(\mathbf{x}_s - \mathbf{x}_p) \right] \cdot \mathbf{N} \cdot \Delta \mathbf{M}(\mathbf{x}_s) dS
\]

which is valid and numerically stable everywhere on \( \mathcal{S}' \), thanks to the weak kernel singularity. There is a freedom of choice for the auxiliary parameter \( m \), such as \( m = |k| \).

\[
\Delta \phi(\mathbf{x}_s) = \int_{\mathcal{S}} \frac{\nabla \times \mathbf{G}^{-\alpha}(\mathbf{x}_s - \mathbf{x}_p)}{\mathbf{N} \cdot \Delta \mathbf{M}(\mathbf{x}_s)} dS,
\]

\[
\Delta \phi(\mathbf{x}_s) = -\int_{\mathcal{S}} \mathbf{G}^{-\alpha}(\mathbf{x}_s - \mathbf{x}_p) \cdot \nabla \times \Delta \mathbf{M}(\mathbf{x}_s) dS + \int_{\mathcal{O}} \mathbf{G}^{-\alpha}(\mathbf{x}_s - \mathbf{x}_p) \cdot \Delta \mathbf{M}(\mathbf{x}_s) dS + \int_{\mathcal{S}} \mathbf{G}^{-\alpha}(\mathbf{x}_s - \mathbf{x}_p) \cdot d\mathbf{x}_s.
\]

The complete set of the equations, i.e., Eqs. (35), (37), (41) and (42), determines all the unknowns, i.e., \( \Delta \mathbf{M} \) on \( \mathcal{S}' \) as well as \( \Delta \phi \) and \( \partial \phi^C \) on \( \mathcal{O} \). Given the solutions, we can finally compute the impedance response, as well as the leak percentage \( \lambda \) defined below, as the functions of the input parameters including the leakage parameter \( \kappa \). The impedance deflection \( \Delta \mathbf{Z} \) can be computed according to the formula

\[
\Delta \mathbf{Z} = \frac{1}{\mathbf{f}_s} \int_{\mathcal{S}} \sigma \mathbf{E}_s^{(\phi)}(\mathbf{x}_s) \Delta \phi(\mathbf{x}_s) dS,
\]

where \( \sigma = \kappa^2 / i \omega \mu \). Equation (43) follows the volumetric Auld reciprocity formula [19] after Eq. (14) for the tight crack. The leak percentage \( \lambda \) is defined as

\[
\lambda \equiv \int_{\mathcal{S}} \phi^C(\mathbf{x}) dS \left/ \int_{\mathcal{S}} \Delta \phi(\mathbf{x}) dS \right. \left/ \int_{\mathcal{S}} \mathbf{E}_s^{(\phi)}(\mathbf{x}) dS \right. \left/ \int_{\mathcal{S}} \mathbf{E}_s^{(\phi)}(\mathbf{x}) dS \right.
\]

Physically, it is the ratio between the total leakage current through \( \mathcal{S}' \) and the total incident eddy current acting on \( \mathcal{S}' \). The minus sign reflects the fact that the leakage current occurs anti-parallel to the incident, effectively weakening the incoming eddy current.

It is important to note that all the above BIEs have the local kernel singularities which are integrable at worst. These well-behaving BIEs are soluble numerically by the point-matching method. In addition, the discretization is not burdened by any requirements of artificial singularity regularization. The direct term of Eq. (35) can be made particularly mild by the use of the Stokes theorem

5. Numerical analysis of half space model

As stated, the set of the BIEs admits numerical solution by the point-matching method without special singularity regularization. In particular, the standard discretization method by the node-oriented shape function can be applied to expand \( \Delta \phi \) and \( \partial \phi^C \) on \( \mathcal{O} \). For the expansion of \( \Delta \mathbf{M} \) on \( \mathcal{S}' \), flux-generating edge elements are preferred for use. Here, we introduce a special type of rooftop elements which resembles the RWG basis geometrically [20], while being flux-generating and divergence-free. The basis is constructed so that it describes closed-loop currents locally, where no current flows out of the element region. This feature is shared neither by the mRWG basis [21], nor by the conventional edge element (e.g. Whitney 1-form [22–24]). As illustrated in Fig. 4, the proposed basis has the supports of a pair of triangles sharing a side segment (the rooftop ridge), except that each triangle is subdivided into two sub-triangles separated along the median. Unlike the RWG basis, the current flows parallel to the external edge in each sub-triangle, where the intensity is the strongest on the edge and diminishes linearly toward the mid-ridge point, at which it vanishes.

Let \( \mathcal{T}_n \) be the n-th expansion function supported by the triangles \( T_n \). To be explicit, consider one of the triangle, e.g. \( \{ \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \} \), and introduce the standard position parameterization

\[
\mathbf{r}_n(\alpha) = \sum_{i=1}^{3} \mathbf{a}_i \mathbf{x}_i, \quad \sum_{i=1}^{3} \mathbf{a}_i = 1, \quad \mathbf{a}_i \geq 0.
\]

Then, the stated current flow can be expressed mathematically as

\[
\mathbf{J}_n(\mathbf{r}_n(\alpha)) = \left\{ \begin{array}{ll}
(1 - 2 \mathbf{a}_1) \mathbf{a}_1, & \mathbf{a}_1 < \mathbf{a}_2 \\
(1 - 2 \mathbf{a}_2) \mathbf{a}_2, & \mathbf{a}_2 > \mathbf{a}_1
\end{array} \right.
\]

where \( \mathbf{a} \) denotes the unit vector in the direction of \( \mathbf{a}^\top \), and where the total current is normalized to unity. The current assignment can be given similarly to the other triangle.

It is straightforward to show that the surface divergence vanishes everywhere on the element. In contrast, the surface curl is non-vanishing, piece-wise constant, and pointing the normal direction,

![Fig. 4. Illustration for the proposed rooftop basis that is flux-generating and divergence-free, which shares the geometrical configuration with the RWG basis.](image-url)
\[ \vec{\nabla} \times \vec{J}_n (\vec{r}_n) = \begin{cases} \frac{2}{N^+} \begin{cases} [\alpha_1] \quad \alpha_1 < \alpha_2 \\ [\alpha_2] \quad \alpha_1 > \alpha_2 \end{cases} \quad \text{for} \quad \alpha_1 < \alpha_2 \\ \frac{2}{N^-} \begin{cases} [\alpha_1'] \quad \alpha_1' < \alpha_2 \\ [\alpha_2'] \quad \alpha_1' > \alpha_2 \end{cases} \end{cases} + \{ \text{delta function terms} \}, \]

where \( A^\pm \) are the areas of the triangles \( T^\pm_n \). The delta function terms appear along the internal boundaries because the current directions are discontinuous across. When applying to Eq. (45), the delta-function contribution can be properly taken into account by applying the Stokes theorem in each sub-triangle, and by retaining all the contour integrals along the internal edges, while retaining those along the external edge only when it is a part of the crack perimeter \( C^p \).

For discretization along the perimeter \( C^p \), we use the node-based shape functions, where the potential \( \Delta \phi \), for example, is represented by its discrete values \( \Delta \phi(\vec{x}_k) \) at the nodal points \( \vec{x}_k \) along \( C^p \). The magnetic current density \( \Delta M \) is expanded as

\[ \Delta M \cdot \vec{n}_{uni} \in \in - A \in \in \in \in \in . \]

### Table 1

| Table 1 | Input parameters used for the case study by computation. |
|---------|----------------------------------------------------------|
| Probe Coil | Cylindrically symmetric solenoid |
| Shape | Height | 1 mm |
| Height | 1 mm |
| Inner [outer] diameter | 2 [4] mm |
| Number of turns | 100 |
| Frequency | 100 kHz |
| Material Property | Conductivity | 1.02 MS/m |
| Conductivity | 1.02 MS/m |
| Relative permeability | 1 |
| Crack Mesh | Skin depth @ 100 kHz | 1.58 mm |
| Shape | Semi-circle |
| Length/Depth | Number of triangles | 63 |
| Number of triangles | 63 |
| Relative leak strength | 0–0.16 mm \(^{-1}\) |
| Simulated Scan | Pattern | Line scan along the crack length |
| Pattern | Scan length | 5 mm from crack center |
| Scan length | Lift off | 0.1 mm |

**Fig. 5.** Computed impedance (solid lines) compared to the benchmark data (dots) for the resistance (left) and reactance (right) components.

**Fig. 6.** Peak impedance signals as the function of the tip depth, the relative magnitude (solid line) and phase shift (dotted line), both relative to the signal at the surface, i.e. depth = 0 mm.

**Fig. 7.** Peak impedance signals as the function of the covering layer thickness, the relative magnitude (solid line) and phase shift (dotted line), both relative to the signal of the uncovered crack.

**Fig. 8.** Impedance plane plots of the computed impedance deflections for varying leakage parameter. \( a, b, c, \ldots, h \) correspond to \( k/\sigma = 0, 0.02, 0.04, \ldots, 0.14 \) [1/mm].
distribution along the edge (b) or the mouth (c).

Options of leak parameter assignments on the crack face, based on Fig. 11.

Fig. 9. Predicted peak impedance (left) plotted against the relative leakage parameter, in terms of the amplitude reduction (solid line) and the phase shift (dotted line) relative to the insulator crack. Also plotted is the corresponding leakage fraction (right) against the leakage, magnitude (solid line) and phase (dotted line).

Fig. 10. The computed impedance amplitude reduction plotted against the computed leak percentage in magnitude.

Fig. 11. Options of leak parameter assignments on the crack face, based on intuitive expectation. A uniform \( \kappa \) distribution over the face (a), or partial distribution along the edge (b) or the mouth (c).

In terms of computational efficiency, this model claims no superiority to others, except that it runs as easily as most Green’s-function-based models do.

5.1. Validation against benchmark data

As a basic validation, impedance output was calculated and compared with the published benchmark data [25]. The model calculation was made with the input parameters mostly matching the experiment, except that the model assumes a tight, insulating crack, while the experiment used an open rectangular notch. The results of the comparison are shown in Fig. 5. The prediction nominally tracks the measurement. The observed discrepancy is likely due to the prediction ambiguity coming from crack mesh dependence that is recognizable. It is unlikely that the finite notch width may account for the difference. The plate thickness effect is too small to account for the difference, as confirmed by computations. In the spirit of the numerical approximation, we state that there exists a discretization mesh which leads to predictions of the indicated agreement.

5.2. Case studies of prediction capabilities

Case studies are presented in what follows, showing how the model predicts impedance, etc. The following examples use a circular solenoidal coil, scanning over a semi-circular tight crack contained by a metal object in half-space geometry. The common input parameters are listed in Table 1.

Case Study 1. Sub-surface Crack Approaching Surface: This example demonstrates the sub-surface crack modeling capability, intending to simulate the response of a submerged crack, emanating from an interior position and approaching the inspection surface. A series of computations was made for an upside-down semi-elliptical crack, immersed with the varying tip depth ranging between \(-0.05\) and \(1.5\) mm. Here, the “0 mm” depth means the crack tip touching the surface, while “-0.05 mm” indicates that the crack is shifted upward by 0.05 mm from the surface, thus breaking the surface. Each simulated line-scan signal traces a locus in the impedance plane, from which the peak impedance value can be extracted. Fig. 6 plots the peak impedance thus obtained, or its magnitude and phase, as the function of the tip depth, relative to the value of the 0-mm depth. As expected intuitively, the signal decreases exponentially, as much as the order-of-magnitude reduction at the skin depth.

Case Study 1-1. Sub-surface Crack Approaching Surface: This example demonstrates the sub-surface crack modeling capability, intending to simulate the response of a submerged crack, emanating from an interior position and approaching the inspection surface. A series of computations was made for an upside-down semi-elliptical crack, immersed with the varying tip depth ranging between \(-0.05\) and \(1.5\) mm. Here, the “0 mm” depth means the crack tip touching the surface, while “-0.05 mm” indicates that the crack is shifted upward by 0.05 mm from the surface, thus breaking the surface. Each simulated line-scan signal traces a locus in the impedance plane, from which the peak impedance value can be extracted. Fig. 6 plots the peak impedance thus obtained, or its magnitude and phase, as the function of the tip depth, relative to the value of the 0-mm depth. As expected intuitively, the signal decreases exponentially, as much as the order-of-magnitude reduction at the skin depth.

Case Study 1-2. Surface Smearing Effect: Another typical sub-surface cracking problem occurs when a crack is covered with a host metal layer on top. Such configuration can occur, e.g., after surface treatments such as shot-peening. To simulate, a series of computations was made for a typical semi-elliptical crack, with the covering layer thicknesses ranging between 0.0 and 1.5 mm. The relative peak impedance signals are obtained similarly from the simulated scans, and the results are plotted in
Fig. 7, relative to the peak impedance of the uncovered crack.

Notice that the signal reduces much faster than exponential near the surface, compared to the behavior of Fig. 6. This implies a serious detection problem, i.e., the smearing can easily hide a dangerous crack. To understand this rapid decrease near surface, recall the discussion given in Introduction. As stated, it is incorrect to assert intuitively that the potential gap $\Delta \varphi$ on the mouth edge vanishes suddenly when the crack submerges infinitesimally below the surface. However, the intuition is not entirely unjustified. In fact, the near-surface behavior of Fig. 7 shows that, instead of dropping discontinuously, the crack signal decays precipitously, i.e., undergoes a continuous but rapid decrease, faster than exponential, when submerging from the surface.

Case Study 2. Current Leakage Effect: This example demonstrates the model capability to calculate the current leakage effect. In this case, there are two kinds of output, i.e., the impedance deflection $\Delta Z$ in Eq. (43) and the leak percentage $\lambda$ in Eq. (44). Fig. 8 shows predicted coil impedance deflections, plotted in the impedance plane, for simulated line scans taken along the length of the crack. The impact of the leakage effect is clearly shown, in terms of the signal reduction in the sequence of the increasing leak parameter. The predicted peak impedance and corresponding leak percentage are shown in Fig. 9, where their relative changes, in both amplitude and phase, are plotted against the relative leakage parameter. Since the leak percentage is more intuitive physically than the leakage parameter, the computed results are re-plotted in Fig. 10 in terms of the impedance reduction versus leak percentage.

The result of Fig. 10 shows a linear trend between the peak impedance reduction and the leak current fraction, up to the fraction of $\sim 35\%$, beyond which the trend flattens. The reason for the flattening has not been understood, although it is unlikely that this is a physical effect. In terms of the integral equation theory, the large leakage parameter corresponds to the strong coupling regime, where the solution process can become unstable. Besides, it is possible that the leakage condition, Eq. (19), may break down for strong leakage.

The model predicted here takes account of the current leakage phenomenon through cracks. The development was motivated by the need to explain the possible causes of the complicated response signal behavior, experienced in real-world EC inspections. The crack morphology effect is a complicated subject that requires dedicated and extensive empirical studies to be conducted, beyond the scope of the existing work [11,26]. Efforts should include studies of the current leakage through the asperity contacts across the crack face, for which this model provides an analysis tool, as long as the leakage is not too intensive.

Our model quantifies the leakage effect in terms of the conductance per unit area $\kappa$ introduced in Eq. (19). However, not much is known about this parameter, except that it is not an intrinsic material parameter, but rather a phenomenological descriptor being dependent on the process of how the crack being formed and on what surrounding conditions it is exposed to. Given controlled experimental data that exhibit morphology effects, our task is to solve a sort of inverse problem to determine the $\kappa$ parameter, in both intensity and distribution, from the data. Since there is no intrinsic guiding principle available to control the $\kappa$ parameter space, we may set experience-based guidelines for $\kappa$ in order to keep the inverse problem manageable.

In terms of the distribution, the earlier publication gave thoughts on conceivable $\kappa$ assignments over the crack face [16]. When the crack is small and/or it is under uniform stress, it is plausible to assign a constant value of $\kappa$ uniformly over the face, as illustrated in Fig. 11(a). For larger cracks, closures may occur at a portion of the face, sometimes (b) along the edge when the crack is still evolving, or else (c) along the mouth when there is surface smearing. For each case, there is a portion of the crack face where leakage may occur, to which a single value of $\kappa$ may be assigned. A more sophisticated assignment is conceivable when on-surface stress distribution is known independently. In this situation, the parameter $\kappa$ can be given a varying spatial distribution, dictated by the given stress distribution with an overall proportionality coefficient.

When the data exhibit any hysteresis behavior [27], one may replace the stress distribution by the strain distribution.

In terms of the magnitude of $\kappa$, empirical determination by fitting the data is a possible approach, subject to the data availability. Alternatively, one may consider the leakage percentage $\lambda$ [Eq. (44)] to determine from the data, because it is more intuitive physically than $\kappa$. However, since $\lambda$ is a computed quantity, the process amounts to solving a double-inverse problem, which is complicated inevitably. It is not known whether these inverse problems are well-posed.

7. Conclusions and discussions

This paper presents a model of eddy current nondestructive evaluation, applicable to simulate EC inspections of tightly closed cracks. The feature of the model is that it possesses a computational capability to predict impedance responses when there is a possible current leakage through the cracking. The foundation of the model is established in Sec. 2, where the model is formulated so that both current-leaking and perfectly insulating cracks are treated on an equal footing. The general governing equations are obtained there, and reduced for specific part geometry in Sec. 3. The resulting set of the non-singular BIEs is described in Sec. 4, to solve the specific leaky crack problems in flat-surface geometry. Actual numerical analyses were conducted in Sec. 5 by means of mostly conventional discretization methods, and the impedance predictions are presented for a number of example problems, which involve half space part geometry and a tight crack.

The impedance predictions of Sec. 5 show quantitatively how much the computed signal decreases, when a portion of the incoming EC leaks through the crack face. For a smeared crack, the predictions show how fast the crack indication disappears from the inspection. The model can generate this result because it is capable of describing the continuous but rapid decrease of the response signal, thanks to the potential variable explicitly assigned to the crack perimeter. For a crack with leakage through asperity contacts, the model has demonstrated its quantitative capability to correlate the signal reduction with the leakage percentage. Indeed, the case study predicted the reduction rate, e.g., $35\%$ signal reduction for $35\%$ fractional leakage, shown in the linear trend between the signal reduction and the fraction of the incoming eddy current leaking through the crack face.

The model encounters a few recognizable problems, one being the mesh dependence of the predictions, and the other is the apparent breakdown of the leaky crack modeling for strong leakage. These issues are subjects of further investigation.

Managing the crack morphology effects has been a long-standing problem to address for decades. For example, asperity contacts have been known to exist within the morphology, and anticipated to cause loss of signals. The phenomenon is so complex that it still requires a significant amount of research to conduct. The current leakage must be a part of morphology studies, in which the present model may serve as an analysis tool. As described in Sec. 6, it is important to know the mechanics of crack behaviors, in order to advance our understanding of the current leakage mechanism, especially under various stress conditions. Combined efforts of mechanics and electromagnetics are called for, in order to shed a light on this complex problem.

Looking back more broadly at the efforts of the last half century, the NDE methodology has been elevated to an established engineering discipline that has the firm scientific ground and demonstrable reliability. NDE modeling has been playing an important role in advancing the quantitative NDE methods [28-29], culminating, for example, in the so-called model-assisted probability-of-detection (MAPOD) approach [30,31]. The approach is both promising and challenging. The principal challenge has to do with where to draw the line. Namely, a consensus is yet to develop as to what extent model predictions can be incorporated...
and what else should be handled empirically. The more capable a model may be, the more predictive and extensible the MAPOD can become. Generally, models may be regarded as too idealistic, and the empirical portion of the MAPOD procedure may include significant redundancy to compensate for the possible discrepancy between model predictions and reality.

Specific to EC NDE models, response modeling of pristine cracks is considered established after a number of pioneering works [1,17,19,32–36]. Extended models such as the present one aim to expand the scope of applicability beyond idealized cracks. For those attempting to solve inverse problems, such forward model extension may create additional burdens by expanding the parameter space beyond the usual sizes and shapes. In contrast, the model extension can have a positive impact on MAPOD efforts. Possible signal loss by current leakage is a case in point. Once the effects of the signal reduction are predetermined and well-understood in separate dedicated studies, then the information can be brought into the MAPOD analysis with the extended modeling capability. As a result, a portion of empirical compensation factors can be traded to model-based estimations, thus reducing the redundancy built into MAPOD.

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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**Appendix A. Notation for the Stratton-Chu formulas**

This Appendix introduces various notations used in the text, particularly a compact notation to express the Stratton-Chu Formulas with the tensor Green’s functions. In the conventional notation, they read

\[
\Theta_s(x_r) E(x_r) = \oint_{x_r} -i\omega \mathcal{G}(x-x_r) \frac{\partial \mathcal{E}(x)}{\partial x} \, dV + \oint_{S} \left\{ \nabla \mathcal{G}(x-x_r) \times \left[ \nabla \times \mathcal{E}(x) - \nabla \times \mathcal{H}(x) \right] \right\} \, dS, \quad (A-1)
\]

\[
\Theta_h(x_r) H(x_r) = -\oint_{x_r} \nabla \mathcal{G}(x-x_r) \times \mathcal{E}(x) \, dV - \oint_{S} \left\{ \nabla \mathcal{G}(x-x_r) \times \left[ -\nabla \times \mathcal{H}(x) \right] + \frac{k^2}{\omega \mu} \right\} \, dS, \quad (A-2)
\]

where \( \Theta_s(x_r) \) is the characteristic function of the volume \( V \) with its surface \( S \), and where

\[
\mathcal{G}(x) = \frac{1}{4\pi R}, \quad \mathcal{G}(x) = \left( \frac{1}{k} + \frac{1}{k} \frac{\nabla \times \nabla}{\nabla \times \nabla} \right) \mathcal{G}(x), \quad x \neq 0, \quad k = (1+i)\sqrt{\omega \mu / \epsilon}. \quad (A-3)
\]

For the compact expression, we combine the fields and currents into the two-component vectors as

\[
\mathbf{F}(x) = \begin{bmatrix} \mathcal{E}(x) \\ -\mathcal{H}(x) \end{bmatrix}, \quad \mathbf{J}(x) = \begin{bmatrix} 0 \\ \frac{1}{k} \nabla \times \mathcal{E}(x) \end{bmatrix}, \quad \mathbf{J}(x) = \begin{bmatrix} \nabla \times \mathcal{E}(x) \\ -\nabla \times \mathcal{H}(x) \end{bmatrix}. \quad (A-4)
\]

Correspondingly, we form a 2-by-2 matrix, \( \mathbf{K} \), of the tensor Green’s functions,

\[
\mathbf{K}(x) = (U + D) \left\| \nabla \mathcal{G}(x) \right\| + \left( \frac{1}{i\omega R} + \frac{k^2}{\omega \mu} L \right) \left\| \nabla \mathcal{G}(x) \right\|. \quad (A-5)
\]

where the double-lined bracket emphasizes that the entry is a 2nd-rank tensor, and where

\[
U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \quad (A-6)
\]

In the text, we also use the notation

\[
\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (A-7)
\]

In terms of this notation, the two Stratton-Chu Formulas can be combined into a single expression

\[
\Theta_s(x_r) \mathbf{F}(x_r) = \oint_{x_r} \mathbf{K}(x-x_r) \mathbf{J}(x) \, dV + \oint_{S} \mathbf{K}(x-x_r) \mathbf{J}(x) \, dS. \quad (A-8)
\]

For the half space and plate problems, we use the cylindrical coordinates and 2-dimensional Fourier transform with respect to the x-y variables. Accordingly, the following coordinate notations are used.

\[
\begin{align*}
\{ x_r \} &= \{ x \hat{e}_x + y \hat{e}_y + z \hat{e}_z \} = \{ x \hat{e}_r + z \hat{e}_z \} = \rho \hat{r} + z \hat{z}, \\
\{ \rho \} &= \{ p_x \hat{e}_x + p_y \hat{e}_y + p_z \hat{e}_z \} = \{ p_x \hat{e}_r + z \hat{z} \} = \rho \hat{r} + z \hat{z}, \\
\{ \rho \} &= \{ \rho \hat{r} \} = \{ \rho \hat{r} \} = \rho \hat{r}, \\
\{ \theta \} &= \{ \theta \} = \{ \theta \} = \theta. \quad (A-9)
\end{align*}
\]

The most basic is the 2D transform of \( \mathcal{G}(x) \), i.e.
\[ G^\theta(\vec{x}) = \int \frac{d^2 p}{(2\pi)^2} e^{-i \vec{p} \cdot \vec{x}} G^\theta(p, z), \quad G^\theta(p, z) = \frac{1}{2} e^{-i \gamma}, \quad \gamma = (p^2 - k^2)^{1/2}, \text{Re} \gamma \geq 0. \]  

which leads to the 2D transform of \( K^\theta(\vec{x}) \),
\[
K^\theta(\vec{p}, z) = \left( U + D \right) \left\{ A^\theta(\vec{p}, z) + \frac{i \omega}{k^2} \left( R + \frac{1}{i \omega} L \right) \right\} |S^\theta(\vec{p}, z)| \right| \frac{d^2 p}{(2\pi)^2},
\]

Here, the tensor factors, \( A^\theta \) and \( S^\theta \), are obtained explicitly as
\[
\begin{align*}
A^\theta(\vec{p}, z) &= \epsilon(z) \left( -P_{\beta p} + P_{\rho p} \right) + \frac{L}{\gamma} (P_{\rho p} - P_{\theta p}) \\
S^\theta(\vec{p}, z) &= -i P_{\rho p} + k^2 \frac{P_{\theta p}}{\gamma} + \frac{1}{\gamma} \frac{d^2 p}{(2\pi)^2} \rho \epsilon(z)(-i p)(P_{\rho p} + P_{\theta p})
\end{align*}
\]

where \( \epsilon(z) \) is the signature function, \( \epsilon(z) = \pm 1, \forall z \neq 0 \), and where
\[
P_{\rho p} = \bar{\omega} \nu_\rho \bar{\nu}_\rho,
\]

Notice that \( P_{\rho p} \) acts as the projection operator to the \( p \) component, etc., while \( P_{\rho p} \) as the conversion operator from the \( \theta \) to \( p \) component, etc. Frequently used is the tangential projection
\[
\bar{n}_z \times K^\theta(-\bar{p}, \pm |z|) = \left[ \left( U + D \right) \left\{ \pm P_{\beta p} - \frac{L}{\gamma} P_{\theta p} \right\} - \left( \frac{i \omega}{k^2} R + \frac{1}{i \omega} L \right) \right] |P_{\rho p} + k^2 \frac{P_{\theta p}}{\gamma} + \frac{1}{\gamma} \frac{d^2 p}{(2\pi)^2} \rho \epsilon(z)(-i p)(P_{\rho p} + P_{\theta p})| \frac{1}{2} \xi^{\gamma}.
\]

The solution process also uses its principal value part, i.e. the terms continuous across \( z = 0 \),
\[
\bar{n}_z \times \rho \bar{K}^\theta(-\bar{p}, z) \left[ \left( U + D \right) \left\{ \frac{1}{\gamma} \frac{d^2 p}{(2\pi)^2} \rho \epsilon(z)(-i p)(P_{\rho p} + P_{\theta p}) \right\} \right] \frac{1}{2} \xi^{\gamma}.
\]

**Appendix B. Explicit formulas of the integration kernels**

In the text, several formulas include the reflection kernel functions. This Appendix compiles their explicit expressions, in terms of the spatial components and the Fourier relationships are
\[
K^{(\theta)}(\vec{x}, \vec{x}_p) = \mathcal{F} \left[ \frac{\omega}{k^2} K^{(\theta)}(\rho, \rho, z, \zeta_p) + P_{\rho \rho} K^{(\theta)}(\rho, \rho, \zeta, \zeta_p) + P_{\rho \rho} K^{(\theta)}(\rho, \rho, \zeta, \zeta_p) \right],
\]

\[
K^{(r,\rho)}(\vec{x}, \vec{x}_p) = \mathcal{F} \left[ \frac{\omega}{k^2} K^{(r,\rho)}(\rho, \rho, z, \zeta_p) + P_{\rho \rho} K^{(r,\rho)}(\rho, \rho, \zeta, \zeta_p) + P_{\rho \rho} K^{(r,\rho)}(\rho, \rho, \zeta, \zeta_p) \right]
\]

where \( \bar{x} - \bar{x}_p = \rho \bar{n}_p \), and where
\[
K^{(r)}(\rho, \rho, \zeta, \zeta_p) + K^{(r)}(\rho, \rho, \zeta, \zeta_p) = \frac{1}{2\pi} \int_0^\infty \! d\rho p J_0(p \rho) \left[ K^{(r)}(\rho, \rho, \zeta, \zeta_p) + K^{(r)}(\rho, \rho, \zeta, \zeta_p) \right]
\]

\[
K^{(r)}(\rho, \rho, \zeta, \zeta_p) - K^{(r)}(\rho, \rho, \zeta, \zeta_p) = \frac{1}{2\pi} \int_0^\infty \! d\rho p J_1(p \rho) \left[ K^{(r)}(\rho, \rho, \zeta, \zeta_p) - K^{(r)}(\rho, \rho, \zeta, \zeta_p) \right]
\]

\[
K^{(r)}(\rho, \rho, \zeta, \zeta_p) = \frac{1}{2\pi} \int_0^\infty \! d\rho p J_0(p \rho) K^{(r)}(\rho, \rho, \zeta, \zeta_p)
\]
For the half space problem, the explicit formulas are given for completeness, not assuming the quasi-static condition ($k_0 \neq 0$).

$$K_{mp}^{(\alpha)}(p, z, z) = \frac{i \omega a}{r} e^{-r/(\omega \mu)} + \sum_{\nu=0}^{\infty} e^{-r/(\omega \mu)} \left( A_{\nu} e^{-r/(\omega \mu)} + B_{\nu} e^{-r/(\omega \mu)} \right)$$

(B-5)

$$\Delta \equiv 1 - \left( A_{\nu} e^{-r/(\omega \mu)} + B_{\nu} e^{-r/(\omega \mu)} \right)$$

(B-13)

where $d$ is the plate thickness.

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