Local Gravity Constraints and Power Law $f(R)$ Theories

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Abstract

There is a conformal equivalence between power law $f(R)$ theories and scalar field theories in which the scalar degree of freedom evolves under the action of an exponential potential function. In the scalar field representation there is a strong coupling of the scalar field with the matter sector due to the conformal transformation. We use chameleon mechanism to implement constraints on the potential function of the scalar field in order that the resulting model be in accord with Solar System experiments. Investigation of these constraints reveals that there may be no possibility to distinguish between a power law $f(R)$ function and the usual Einstein-Hilbert Lagrangian density.

There are strong observational evidences that the expansion of the universe is accelerating. These observations are based on type Ia supernova [1], cosmic microwave background radiation [2], large scale structure formation [3], weak lensing [4], etc. The standard explanation invokes an unknown component, usually referred to as dark energy. It contributes to energy density of the universe with $\Omega_d = 0.7$ where $\Omega_d$ is the corresponding density parameter, see e.g., [5] and references therein. The simplest dark energy scenario which seems to be both natural and consistent with observations is the $\Lambda$CDM model in which dark energy is identified as a cosmological constant [5] [6] [7]. However, in order to avoid theoretical problems [6], other scenarios have been investigated. Among these scenarios, there are quintessence [8], tachyons [9], phantom [10], quintom [11] and modified gravity models [12]. In the latter, one modifies the laws of gravity whereby a late time acceleration is produced without recourse to a dark energy component. One family of these modified gravity models is obtained by replacing the Ricci scalar $R$ in the usual Einstein-Hilbert Lagrangian density for some function $f(R)$. However, changing gravity Lagrangian have consequences not only in cosmological scales but also

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in galactic ones so that it seems to be necessary to investigate the low energy limit of such \( f(R) \) theories. 

Early works on the weak field limit of \( f(R) \) theories led to negative results. In fact, using the equivalence of \( f(R) \) and scalar-tensor theories \cite{13} \cite{14} \cite{15}, it is originally suggested that all \( f(R) \) theories should be ruled out \cite{17} since they violate the weak field constraints coming from Solar System experiments. This claim was based on the fact that \( f(R) \) theories (in the metric formalism) are equivalent to Brans-Dicke theory with \( \omega = 0 \) while observations set the constraint \( \omega > 40000 \) \cite{18}. In this case the post-Newtonian parameter satisfies \( \gamma = \frac{1}{2} \) instead of being equal to unity as required by observations. Later, it was noted by many authors that for scalar fields with sufficiently large mass it is possible to drive \( \gamma \) close to unity even for null Brans-Dicke parameter. In this case the scalar field becomes short-ranged and has no effect at Solar System scales. Recently, it is shown that there exists an important possibility that the effective mass of the scalar field be scale dependent \cite{19}. In this chameleon mechanism, the scalar field may acquire a large effective mass in Solar System scale so that it hides local experiments while at cosmological scales it is effectively light and may provide an appropriate cosmological behavior.

In the present work we intend to use this criterion to set local gravity constraints on \( f(R) \) theories. There are a number of works concerning these constraints on \( f(R) \) theories \cite{20} \cite{21}. We will focus on power law \( f(R) \) theories and show that the constraints on the parameters space suggest that they are hardly distinguishable from \( \Lambda \)CDM scenario.

Let us begin with the following action\(^\dagger\)

\[
S = \frac{1}{2} \int d^4x \sqrt{-g} \ f(R) + S_m(g_{\mu\nu}, \psi) \tag{1}
\]

where \( g \) is the determinant of \( g_{\mu\nu} \), \( f(R) \) is an unknown function of the scalar curvature \( R \) and \( S_m \) is the matter action depending on the metric \( g_{\mu\nu} \) and some matter field \( \psi \). We may use a new set of variables

\[
\bar{g}_{\mu\nu} = p \ g_{\mu\nu} \tag{2}
\]

\[
\phi = \frac{1}{2\beta} \ln p \tag{3}
\]

where \( p \equiv \frac{df}{dR} = f'(R) \) and \( \beta = \sqrt{\frac{1}{6}} \). This is indeed a conformal transformation which transforms the above action in the Jordan frame to the Einstein frame \cite{13} \cite{14} \cite{15}

\[
S = \frac{1}{2} \int d^4x \sqrt{-g} \ \{ \bar{R} - \bar{g}^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2V(\phi) \} + S_m(\bar{g}_{\mu\nu}e^{2\beta \phi}, \psi) \tag{4}
\]

In the Einstein frame, \( \phi \) is a scalar field with a self-interacting potential which is given by

\[
V(\phi) = \frac{1}{2} e^{-2\beta \phi} \{ r[p(\phi)] - e^{-2\beta \phi} f(r[p(\phi)]) \} \tag{5}
\]

\(^\dagger\)We use the unit \( (8\pi G)^{-1} = 1 \).
where \( r(p) \) is a solution of the equation \( f'[r(p)] - p = 0 \) [13]. One usually states that any \( f(R) \) gravity model is mathematically equivalent with a minimally coupled scalar field theory with an appropriate potential function. In general, this does not imply physical equivalence of the two conformal frames. In fact it is shown that some physical systems can be differently interpreted in different conformal frames [13] [16]. The physical status of the two conformal frames is an open question which we are not going to address here. However we assume that the scalar degree of freedom in the Einstein frame should satisfy stringent constraints from solar system experiments. It is important to note that conformal transformation induces the coupling of the scalar field \( \phi \) with the matter sector. The strength of this coupling \( \beta \), is fixed to be \( \sqrt{\frac{1}{6}} \) and is the same for all types of matter fields. In the case of such a strong matter coupling, the role of the potential of the scalar field is important for consistency with local gravity experiments. When the potential satisfies certain conditions it is possible to attribute an effective mass to the scalar field which has a strong dependence on ambient density of matter. A theory in which such a dependence is realized is said to be a chameleon theory [19]. In such a theory the scalar field \( \phi \) can be heavy enough in the environment of the laboratory tests so that the local gravity constraints suppressed even if \( \beta \) is of the order of unity. Meanwhile, it can be light enough in the low-density cosmological environment to be considered as a candidate for dark energy.

Variation of the action (1) with respect to \( \bar{g}_{\mu\nu} \) and \( \phi \), gives the field equations

\[
\bar{G}_{\mu\nu} = \partial_{\mu}\phi \partial_{\nu}\phi - \frac{1}{2} \bar{g}_{\mu\nu} \partial_{\gamma}\phi \partial^{\gamma}\phi - V(\phi)\bar{g}_{\mu\nu} + \bar{T}_{\mu\nu}
\]

(6)

\[
\Box\phi - \frac{dV}{d\phi} = -\beta \bar{T}
\]

(7)

where

\[
\bar{T}_{\mu\nu} = \frac{-2}{\sqrt{-\bar{g}}} \frac{\delta S_m}{\delta \bar{g}^{\mu\nu}}
\]

(8)

and \( \bar{T} = \bar{g}^{\mu\nu} \bar{T}_{\mu\nu} \). Covariant differentiation of (6) and the Bianchi identities give

\[
\nabla^\mu \bar{T}_{\mu\nu} = \beta \bar{T} \partial_\nu \phi
\]

(9)

which implies that the matter field is not generally conserved and feels a new force due to gradient of the scalar field. Let us consider \( \bar{T}_{\mu\nu} \) as the stress-tensor of dust with energy density \( \bar{\rho} \) in the Einstein frame. In a static and spherically symmetric spacetime the equation (7) gives

\[
\frac{d^2\phi}{d\bar{r}^2} + \frac{2}{\bar{r}} \frac{d\phi}{d\bar{r}} = \frac{dV_{eff}(\phi)}{d\phi}
\]

(10)

where \( \bar{r} \) is distance from center of the symmetry in the Einstein frame and

\[
V_{eff}(\phi) = V(\phi) - \frac{1}{4} \rho e^{-4\phi}
\]

(11)

Here we have used the relation \( \bar{\rho} = e^{-4\phi} \rho \) that relates the energy densities in the Jordan and the Einstein frames. We consider a spherically symmetric body with a radius \( \bar{r}_c \) and a
constant energy density \( \bar{\rho}_{in} \) \((\bar{r} < \bar{r}_c)\). We also assume that the energy density outside the body \((\bar{r} > \bar{r}_c)\) is given by \( \bar{\rho}_{out} \). We will denote by \( \varphi_{in} \) and \( \varphi_{out} \) the field values at two minima of the effective potential \( V_{eff}(\phi) \) inside and outside the object, respectively. They must clearly satisfy \( V'_{eff}(\varphi_{in}) = 0 \) and \( V'_{eff}(\varphi_{out}) = 0 \) where prime indicates differentiation of \( V_{eff}(\phi) \) with respect to the argument. As usual, masses of small fluctuations about these minima are given by \( m_{in} = \left[ V''_{eff}(\varphi_{in}) \right]^{\frac{1}{2}} \) and \( m_{out} = \left[ V''_{eff}(\varphi_{out}) \right]^{\frac{1}{2}} \) which depend on ambient matter density. A region with large mass density corresponds to a heavy mass field while regions with low mass density corresponds to a field with lighter mass. In this way it is possible for the mass field to take sufficiently large values near massive objects in the Solar System scale and to hide the local tests. For a spherically symmetric body there is a thin-shell condition

\[
\frac{\Delta \bar{r}_c}{\bar{r}_c} = \frac{\varphi_{out} - \varphi_{in}}{6\beta \Phi_c} \ll 1
\]

(12)

where \( \Phi_c \) is the Newtonian potential at \( \bar{r} = \bar{r}_c \). In this case, equation (10) with some appropriate boundary conditions gives the field profile outside the object [19]

\[
\phi(\bar{r}) = -\frac{\beta}{4\pi} \frac{3 \Delta \bar{r}_c}{\bar{r}_c} M_c e^{-m_{out}(\bar{r} - \bar{r}_c)} + \varphi_{out}
\]

(13)

where \( M_c \) is mass of the object.

The function \( f(R) \) in the Jordan frame is closely related to the potential function of the scalar degree of freedom of the theory in the Einstein frame. Any functional form for the potential function corresponds to a particular class of \( f(R) \) theories. To find a viable function \( f(R) \) passing Solar System tests one can equivalently work with its corresponding potential function in the Einstein frame and put constraints on the relevant parameters via chameleon mechanism. Taking this as our criterion, we will use a pure exponential potential function. There are two reasons for this choice. Firstly, this class of potentials arises in a number of physical situations. In particular, there are reports that quintessence field with exponential potentials can produce late time acceleration [22]. Secondly, as we will show in the following an exponential potential function for the scalar field corresponds to a power law \( f(R) \) theory. In fact, there are arguments concerning cosmological viability of this class of \( f(R) \) models [23] [24]. Moreover, it is argued that some power law \( f(R) \) theories may have sting/M-theory origin [25]. These arguments make investigation of viability of these models in terms of local experiments be a mandate.

To have a pure exponential potential function, we take

\[
r(p) = \left[p \frac{n}{\alpha (n + 1)}\right]^n
\]

(14)

where \( \alpha \) and \( n \) are constant parameters. It then leads to

\[
V(\phi) = \frac{1}{2(n + 1)} \left[ \alpha \frac{n + 1}{n} \right]^{-n} e^{2(n - 1)\beta \phi}
\]

(15)

On the other hand, this choice of the \( r(p) \) function gives a power law \( f(R) \) theory

\[
f(R) = \alpha R^{1 + \frac{n}{\alpha}}
\]

(16)
Thus there is a correspondence between a power law $f(R)$ theory in the Jordan frame and a minimally coupled scalar field theory with an exponential potential in the Einstein frame. We can now find the solution of $V_{\text{eff}}(\phi) = 0$ by substituting (15) into (11)

$$\varphi = \frac{1}{2\beta(n+1)} \ln\{ \rho \frac{n+1}{1-n} (\alpha \frac{n+1}{n})^n \}$$

(17)

In order that $\varphi$ be a local minimum we should have

$$\frac{n+1}{1-n} (\alpha \frac{n+1}{n})^n > 0$$

(18)

In the following we shall consider thin-shell condition and the constraints set by equivalence principle and fifth force experiments.

1. **Thin – shell condition**

In the chameleon mechanism, the chameleon field is trapped inside large and massive bodies and its influence on the other bodies is only due to a thin-shell near the surface of the body. The criterion for this thin-shell condition is given by (12). If we combine (12) and (17) we obtain

$$\frac{\Delta \bar{r}_c}{\bar{r}_c} = \frac{1}{12\beta^2(n+1)} \frac{1}{\Phi_c} \frac{\ln \rho_{\text{out}}}{\rho_{\text{in}}}$$

(19)

where $\rho_{\text{in}}$ and $\rho_{\text{out}}$ are energy densities inside and outside of the body in the Jordan frame, respectively, and $\Phi_c = M_c/8\pi \bar{r}_c$ with $M_c$ being the mass of the body. In weak field approximation the spherically symmetric metric in the Jordan frame is given by

$$ds^2 = -(1 - 2X(r))dt^2 + [1 + 2Y(r)]dr^2 + r^2 d\Omega^2$$

(20)

where $X(r)$ and $Y(r)$ are some functions of $r$. There is a relation between $r$ and $\bar{r}$ so that $\bar{r} = p^{1/2}r$. If we consider this relation under the assumption $m_{\text{out}} r \ll 1$, namely that the Compton wavelength $m_{\text{out}}^{-1}$ is much larger than Solar System scales, then we have $\bar{r} \approx r$. In this case, the chameleon mechanism gives for the post-Newtonian parameter $\gamma$

$$\gamma = \frac{3 - \frac{\Delta \bar{r}_c}{\bar{r}_c}}{3 + \frac{\Delta \bar{r}_c}{\bar{r}_c}} \approx 1 - \frac{2 \Delta \bar{r}_c}{3 \bar{r}_c}$$

(21)

We can now apply (19) on the Earth and obtain the condition that the Earth has a thin-shell. To do this, we assume that the Earth is a solid sphere of radius $R_e = 6.4 \times 10^8$ cm and mean density $\rho_e \sim 10$ gr/cm$^3$. We also assume that the Earth is surrounded by an atmosphere with homogenous density $\rho_a \sim 10^{-3}$ gr/cm$^3$ and thickness 100 km. Then we rewrite equation (19)

$$\frac{\Delta R_e}{R_e} = \frac{1}{12\beta^2(n+1)} \frac{1}{\Phi_e} \frac{\ln \rho_a}{\rho_e}$$

(22)

It should be remarked that in the end of the present work we became aware of [26] in which a relation between power law $f(R)$ theories and a minimally coupled scalar field with specific exponential potential has been reported in the absence of matter systems.
With $\Phi_e = 6.95 \times 10^{-10}$ [28], Newtonian potential on surface of the Earth, it gives

$$\frac{\Delta R_e}{R_e} = -\frac{4.96 \times 10^9}{n + 1}$$  \hspace{1cm} (23)$$

The tightest Solar System constraint on $\gamma$ comes from Cassini tracking which gives $|\gamma - 1| < 2.3 \times 10^{-5}$ [18]. This together with (21) and (23) yields an upper bound for the parameter $n$

$$|n + 1| > 1.44 \times 10^{14}$$  \hspace{1cm} (24)$$

Combining this result with (16) reveals that this power law $f(R)$ theory hardly deviates from general relativity.

2. Equivalence principle

We now consider constraints coming from possible violation of weak equivalence principle. We assume that the Earth, together with its surrounding atmosphere, is an isolated body and neglect the effect of the other compact objects such as the Sun, the Moon and the other planets. Far away the Earth, matter density is modeled by a homogeneous gas with energy density $\rho_G \sim 10^{-24} \text{gr/cm}^3$. To proceed further, we first consider the condition that the atmosphere of the Earth satisfies the thin-shell condition [19]. If the atmosphere has a thin-shell the thickness of the shell ($\Delta R_a$) must be clearly smaller than that of the atmosphere itself, namely $\Delta R_a < R_a$, where $R_a$ is the outer radius of the atmosphere. If we take thickness of the shell equal to that of the atmosphere itself $\Delta R_a \sim 10^2 \text{km}$ we obtain $\Delta R_a < 1.5 \times 10^{-2}$. It is then possible to relate $\frac{\Delta R_e}{R_e} = \frac{\varphi_e - \varphi_a}{63\Phi_e}$ and $\frac{\Delta R_a}{R_a} = \frac{\varphi_a - \varphi_G}{63\Phi_a}$ where $\varphi_e$, $\varphi_a$ and $\varphi_G$ are the field values at the local minimum of the effective potential in the regions $r < R_e$, $R_a > r > R_e$ and $r > R_a$ respectively. Using the fact that newtonian potential inside a spherically symmetric object with mass density $\rho$ is $\Phi \propto \rho R^2$, one can write $\Phi_e = 10^4 \Phi_a$ where $\Phi_e$ and $\Phi_a$ are Newtonian potentials on the surface of the Earth and the atmosphere, respectively. This gives $\Delta R_e/R_e \approx 10^{-4} \Delta R_a/R_a$. With these results, the condition for the atmosphere to have a thin-shell is

$$\frac{\Delta R_e}{R_e} < 1.5 \times 10^{-6}$$  \hspace{1cm} (25)$$

The tests of equivalence principle measure the difference of free-fall acceleration of the Moon and the Earth towards the Sun. The constraint on the difference of the two acceleration is given by [18]

$$\frac{|a_m - a_e|}{a_N} < 10^{-13}$$  \hspace{1cm} (26)$$

where $a_m$ and $a_e$ are acceleration of the Moon and the Earth respectively and $a_N$ is the Newtonian acceleration. The Sun and the Moon are all subject to the thin-shell condition [19] and the field profile outside the spheres are given by (13) with replacement of corresponding quantities. The accelerations $a_m$ and $a_e$ are then given by [19]

$$a_e \approx a_N \{1 + 18\beta^2 \left(\frac{\Delta R_e}{R_e}\right)^2 \frac{\Phi_e}{\Phi_s}\}$$  \hspace{1cm} (27)$$
\[ a_m \approx a_N \{1 + 18\beta^2 \left( \frac{\Delta R_e}{R_e} \right)^2 \frac{\Phi_e^2}{\Phi_s \Phi_m} \} \]  

(28)

where \( \Phi_e = 6.95 \times 10^{-10} \), \( \Phi_m = 3.14 \times 10^{-11} \) and \( \Phi_s = 2.12 \times 10^{-6} \) are Newtonian potentials on the surfaces of the Earth, the Moon and the Sun, respectively [28]. This gives a difference of free-fall acceleration

\[ \frac{|a_m - a_e|}{a_N} = (0.13) \beta^2 \left( \frac{\Delta R_e}{R_e} \right)^2 \]  

(29)

Combining this with (26) results in

\[ \frac{\Delta R_e}{R_e} < 6.74 \times 10^{-6} \]  

(30)

which is of the same order of the condition (25) that the atmosphere has a thin-shell. Taking this as the constraint coming from violation of equivalence principle, we obtain

\[ |n + 1| > 1.67 \times 10^{15} \]  

(31)

which is not much different from the bound given by (24).

3. Fifth force

The potential energy associated with a fifth force interaction is parameterized by a Yukawa-type potential

\[ U(r) = -\alpha \frac{m_1 m_2}{8\pi} \frac{e^{-r/\lambda}}{r} \]  

(32)

where \( m_1 \) and \( m_2 \) are masses of the two test bodies separating by distance \( r \), \( \alpha \) is strength of the interaction and \( \lambda \) is the range. Thus fifth force experiment constrains regions of \((\alpha, \lambda)\) parameter space. These experiments are usually carried out in a vacuum chamber in which the range of the interaction inside it is of the order of the size of the chamber [19], namely \( \lambda \sim R_{\text{vac}} \). The tightest bound on the strength of the interaction is \( \alpha < 10^{-3} \) [29]. Inside the chamber we consider two identical bodies with uniform densities \( \rho_c \), radii \( r_c \) and masses \( m_c \). If the two bodies satisfy the thin-shell condition, their field profile outside the bodies are given by

\[ \phi(r) = -\frac{\beta}{4\pi} \frac{3\Delta r_c m_c e^{-r/R_{\text{vac}}}}{r_c} + \varphi_{\text{vac}} \]  

(33)

Then the corresponding potential energy of the interaction is

\[ V(r) = -2\beta^2 \left( \frac{3\Delta r_c}{r_c} \right)^2 \frac{m_c^2 e^{-r/R_{\text{vac}}}}{8\pi} \frac{1}{r} \]  

(34)

The bound on the strength of the interaction translates into

\[ 2\beta^2 \left( \frac{3\Delta r_c}{r_c} \right)^2 < 10^{-3} \]  

(35)

One can write for each of the test bodies

\[ \frac{\Delta r_c}{r_c} = \frac{1}{12\beta^2(n + 1)} \frac{1}{\Phi_c} \ln \frac{\rho_{\text{vac}}}{\rho_c} \]  

(36)
where $\rho_{\text{vac}}$ is energy density of the vacuum inside the chamber. In the experiment carried out in [29], one used a typical test body with mass $m_c \approx 40 gr$ and radius $r_c \approx 1 cm$. These correspond to $\rho_c \approx 9.5 gr/cm^3$ and $\Phi_c \sim 10^{-27}$. Moreover, the pressure in the vacuum chamber was reported to be $3 \times 10^{-8}$ Torr which is equivalent to $\rho_{\text{vac}} \approx 4.8 \times 10^{-14} gr/cm^3$. Substituting these into (36) and combining the result with (35) gives the bound

$$| n + 1 | > 3 \times 10^{29}$$

which is much stronger than (24) and (31). As the last point, there are some remarks to do with respect to stability of the model (16). In principle, stability issues should be considered to make sure that an $f(R)$ model is viable [30]. In particular, stability in matter sector (the Dolgov-Kawasaki instability [31]) imposes some conditions on the functional form of $f(R)$ models. The first theories which easily pass this instability have been presented in [20] [32]. These conditions require that the first and the second derivatives of $f(R)$ function with respect to the Ricci scalar $R$ should be positive definite. The positivity of the first derivative ensures that the scalar degree of freedom is not tachyonic and positivity of the second derivative tells us that graviton is not a ghost. For power law $f(R)$ theories of the type (16), we should have $n > -1$ and $n > 0$ to ensure that $f'(R) > 0$ and $f''(R) > 0$, respectively.

In summary, we have discussed viability of power law $f(R)$ theories in terms of local gravity constraints. We have used the correspondence between this class of $f(R)$ theories to scalar field theories with an exponential self-interacting potential. In the scalar field representation of the theory there is a strong coupling of the scalar field with the matter sector. We have considered the conditions that this coupling is suppressed by chameleon mechanism. We have found that in order that the theory be consistent with local gravity experiments the exponent of the curvature scalar hardly deviates from unity. The constraint (37) is much stronger than that reported in [33] which is obtained by considering the perihelion precession of Mercury under the assumption that it follows timelike geodesics. Our results preclude the possibility of regarding power law $f(R)$ models as viable candidates for generalizing general relativity.

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$^8$Note that due to the logarithmic dependence of $\Delta r_c / r_c$ with $\rho_{\text{vac}}$ the value of the vacuum energy density does not effectively change the order of magnitude of (37).
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