WEIGHTED HOMOLOGY THEORY OF ORBIFOLDS AND WEIGHTED POLYHEDRA

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Abstract. We introduce two new homology theories of orbifolds from some special type of triangulations adapted to orbifolds, called AW-homology and DW-homology. The main idea in the definitions of these two homology theories is that we use divisibly weighted simplices as building blocks of an orbifold and encode the orders of the local groups of the orbifold in the boundary maps of the chain complexes so that these two theories can reflect some information of the singular points. We prove that AW-homology and DW-homology are invariants of orbifolds under isomorphisms and more generally under certain type of homotopy equivalences of orbifolds. Moreover, we find that there exists a natural graded commutative product in the cohomology theory associated to DW-homology, which generalizes the cup product of the ordinary simplicial cohomology. In addition, we introduce a wider class of objects called weighted polyhedra and develop the whole theory of AW-homology and DW-homology in this wider setting. Our goal is to generalize the whole simplicial homology theory to all triangulizable topological spaces with a weight at each point where the weights are compatible with the triangulations in a certain sense.

1. Introduction

Orbifolds were first introduced by I. Satake in [16] where they were originally called V-manifolds. Roughly speaking, an n-dimensional orbifold is a topological space locally modeled on the quotient spaces of open subsets in the Euclidean space $\mathbb{R}^n$ by some finite group actions. Orbifolds have arisen naturally in many ways in mathematics. For example, the orbit space of any proper action by a discrete group on a manifold has the structure of an orbifold, this applies in particular to moduli spaces (see Thurston [19] and Scott [18] for more examples).

Homology and cohomology theories of orbifolds have been defined and studied in many different ways. For example, equivariant cohomology of global quotients, Čech and de Rham cohomology in [16], simplicial cohomology in Moerdijk and Pronk [12], Chen-Ruan cohomology in [5], t-singular homology in Takeuchi and

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Yokoyama [20], ws-singular cohomology in Takeuchi and Yokoyama [21], string homology in Lupercio, Uribe and Xicotencatl [11, 24] and so on.

Here we introduce two new homology theories of orbifolds called AW-homology and DW-homology. These two theories can be considered as generalizations of the classical simplicial homology of simplicial complexes in the category of orbifolds. Our definitions of AW-homology and DW-homology are inspired by the fact that any orbifold admits an appropriate triangulation which is a divisibly weighted simplicial complex with respect to the natural weights induced from the orbifold structure. Then using the idea of weighted simplicial homology from Dawson [8], we encode the orders of the local groups of an orbifold in the boundary maps of the AW-homology and DW-homology so that these two theories may reflect more information of the singular points of the orbifold.

Moreover, we find that the weighted simplicial chain complexes of a divisibly weighted simplex and its barycentric subdivision are both acyclic. So divisibly weighted simplices play an analogous role in weighted simplicial homology as the ordinary simplices in the usual simplicial homology. This analogy allows us to prove a generalization of simplicial approximation theorem for morphisms in the category of divisibly weighted simplicial complexes using the algebraic method of acyclic carrier (see Munkres [13, Section 13]). Based on this, we can prove the invariance of AW-homology and DW-homology under isomorphisms of orbifolds.

The AW-homology and DW-homology of an orbifold are essentially different from the $t$-singular homology defined in [20] since the boundary map in $t$-singular homology is the usual (unweighted) boundary map. We will show some concrete examples in Section 8 to demonstrate their differences (see Remark 8.12). In addition, neither the simplicial cohomology of orbifolds defined in [12] nor the ws-singular cohomology of orbifolds defined in [21] is the cohomology theory dual to our AW-homology and DW-homology.

Moreover, the definitions of AW-homology and DW-homology can be extended to a much wider class of objects called weighted polyhedra. Roughly speaking, a weighted polyhedron is a topological space $X$ with a weight at each point and $X$ admits a triangulation consisting of (divisibly) weighted simplices such that the weight of $X$ at any point $x$ agrees with the weight of the carrier of $x$ (i.e. the unique simplex containing $x$ in its relative interior). In fact, we will develop the whole theory of AW-homology and DW-homology in the category of weighted polyhedra, which may be considered as the natural generalization of ordinary simplicial homology theory.

The paper is organized as follows. In Section 2, we will review some basic definitions and constructions in the theories of orbifolds and weighted simplicial homology. In addition, we will introduce a new notion called descending weighted
simplicial complex and compare it with the usual weighted simplicial complex. In Section 3, we introduce the definitions of AW-homology and DW-homology of an orbifold. In Section 4, we introduce the notion of weighted polyhedron and extend the definitions of AW-homology and DW-homology to weighted polyhedra. In Section 5, we prove some basic properties of a divisibly weighted simplex. In Section 6, we prove that AW-homology and DW-homology are invariants of weighted polyhedra under isomorphisms and more generally under W-homotopy equivalences of weighted polyhedra. In Section 7, we study the basic relations between AW-homology, DW-homology and the usual simplicial homology under some special coefficients. It turns out that the free abelian part of AW-homology and DW-homology of a weighted polyhedron is isomorphic to that of the ordinary simplicial homology. So it is the torsion part of these two homology theories that can really reveal some new informations of a weighted polyhedron. In Section 8, we compute AW-homology and DW-homology of some concrete examples such as circles, disks and closed surfaces with isolated singular points. In Section 9, we construct a product on the DW-cohomology groups with integral coefficients and prove that it is graded commutative (see Theorem 9.14). This construction generalizes the cup product in the ordinary simplicial cohomology. Moreover, this weighted cup product on DW-cohomology groups can also reflect some structural information of a weighted polyhedron. But it seems to us that there is no such a product in AW-cohomology. In addition, we can define a weighted cap product between DW-homology classes and DW-cohomology classes, which generalizes the ordinary cap product.

2. Preliminaries

2.1. Definition of orbifolds.
We first briefly review some basic definitions concerning orbifolds. The reader is referred to [16, 17, 1, 6] for more detailed discussion of these definitions and some other notions in the study of orbifolds (e.g. orbifold fundamental group).

Let $M$ be a paracompact Hausdorff space.

- An orbifold chart on $M$ is given by a connected open subset $\tilde{U}$ of $\mathbb{R}^n$ for some integer $n \geq 0$, a finite group $G$ acting smoothly and effectively on $\tilde{U}$, and a map $\varphi : \tilde{U} \to M$, such that $\varphi$ is $G$-invariant ($\varphi \circ g = \varphi$ for all $g \in G$) and induces a homeomorphism from $\tilde{U}/G$ onto an open subset $U = \varphi(\tilde{U})$ of $M$.
- An embedding $\lambda : (\tilde{U}, G, \varphi) \hookrightarrow (\tilde{V}, H, \psi)$ between two orbifold charts on $M$ is a smooth embedding $\lambda : \tilde{U} \hookrightarrow \tilde{V}$ with $\psi \circ \lambda = \varphi$.
- An orbifold atlas on $M$ is a family $\mathcal{U} = \{(\tilde{U}, G, \varphi)\}$ of such charts, which cover $M$ and are locally compatible in the following sense: given any two
orbifolds charts \((\tilde{U}, G, \varphi)\) for \(U = \varphi(\tilde{U}) \subseteq M\) and \((\tilde{V}, H, \psi)\) for \(V = \psi(\tilde{V}) \subseteq M\), and any point \(x \in U \cap V\), there exists an open neighborhood \(W \subseteq U \cap V\) of \(x\) and a chart \((\tilde{W}, K, \chi)\) with \(W = \chi(\tilde{W})\) such that there are embeddings \((\tilde{W}, K, \chi) \hookrightarrow (\tilde{U}, G, \varphi)\) and \((\tilde{W}, K, \chi) \hookrightarrow (\tilde{V}, H, \psi)\). Since \(M\) is paracompact, we can choose an orbifold atlas \(U = \{(\tilde{U}, G, \varphi)\}\) on \(M\) such that \(\{\varphi(U)\}\) is a locally finite open cover of \(M\).

- An atlas \(U\) on \(M\) is said to refine another atlas \(V\) if for every chart in \(U\) there exists an embedding into some chart of \(V\).
- Two orbifold atlases on \(M\) are said to be equivalent if they have a common refinement. An orbifold (of dimension \(n\)) is such a space \(M\) with an equivalence class of atlases \(U\).

We will generally write \(\mathcal{M} = (M, U)\) for the orbifold represented by the space \(M\) and a chosen atlas \(U\) and say that \(\mathcal{M}\) is based on \(M\). In addition, we call \(\mathcal{M}\) compact if \(M\) is compact. By the fact that smooth actions are locally smooth (see Bredon [3, p. 308]), any orbifold of dimension \(n\) has an atlas consisting of “linear” charts, i.e. charts of the form \((\mathbb{R}^n, G, \varphi)\) where the finite group \(G\) acts on \(\mathbb{R}^n\) via orthogonal linear transformations.

Let \(\mathcal{M} = (M, U)\) be an orbifold of dimension \(n\) where \(U\) is an atlas consisting of linear charts. For each point \(x \in M\), choose a linear chart \((\mathbb{R}^n, G, \varphi)\) around \(x\), with \(G\) a finite subgroup of the orthogonal group \(O(n, \mathbb{R})\). Let \(\tilde{x}\) be a point with \(\varphi(\tilde{x}) = x\), and \(G_x = \{g \in G \mid g \cdot \tilde{x} = \tilde{x}\}\) the isotropy subgroup at \(\tilde{x}\). Up to conjugation, \(G_x\) is a well defined subgroup of \(O(n, \mathbb{R})\), called the local group at \(x\).

- The order \(|G_x|\) of \(G_x\) is called the weight of \(x\), denoted by \(w(x)\), which is independent on the local chart we choose around \(x\).
- The set \(\{x \in M \mid G_x \neq 1\}\) is called the singular set of \(M\), denoted by \(\Sigma M\). A point in \(\Sigma M\) is called a singular point, and a point in \(M \setminus \Sigma M\) is called a regular point.
- The space \(M\) carries a natural stratification whose strata are the connected components of the sets \(\Sigma H(M) := \{x \in M \mid (G_x) = (H)\}\), where \(H\) is any finite subgroup of \(O(n, \mathbb{R})\) and \((H)\) is its conjugacy class.

**Definition 2.1** (Isomorphism of orbifolds). Two orbifolds \(\mathcal{M} = (M, U)\) and \(\mathcal{N} = (N, \mathcal{V})\) are called isomorphic if there exists a homeomorphism \(f : M \rightarrow N\) such that: for any \(x \in M\), there are orbifold charts \((\tilde{U}, G, \varphi)\) around \(x\) and \((\tilde{V}, H, \psi)\) around \(y = f(x)\) where \(f\) maps \(U = \varphi(\tilde{U})\) onto \(V = \psi(\tilde{V})\) and can be lifted to an equivariant homeomorphism \(\tilde{f} : U \rightarrow V\), i.e. there exists a group isomorphism \(\rho : G \rightarrow H\) so that \(\tilde{f}(g \cdot z) = \rho(g) \cdot \tilde{f}(z)\) for all \(z \in \tilde{U}\) and \(g \in G\). Such a homeomorphism \(f\) is called an isomorphism from \(\mathcal{M}\) to \(\mathcal{N}\), denoted by \(f : \mathcal{M} \rightarrow \mathcal{N}\).
In the above definition, if the map \( f \) is only assumed to be continuous with an equivariant lifting and \( w(f(x)) = w(x) \) for all \( x \in M \), then \( f \) is called a \textit{weight-preserving continuous map}.

**Definition 2.2.** The product of two orbifolds \( \mathcal{M} = (M, U) \) and \( \mathcal{N} = (N, V) \), denoted by \( \mathcal{M} \times \mathcal{N} = (M \times N, U \times V) \), is an orbifold whose underlying space is \( M \times N \) and atlas given by \( U \times V = \{(\tilde{U} \times \tilde{V}, G \times H, \phi \times \psi)\} \) where \( U = \{(U, G, \varphi)\} \) and \( V = \{(\tilde{V}, H, \psi)\} \).

Note that the projection \( p : M \times N \to M, (x, y) \mapsto x \) is not a weight-preserving map unless \( \mathcal{N} \) has no singular points.

**2.2. Homology of weighted simplicial complexes.**

Weighted simplicial complexes and weighted homology groups were first studied by Dawson in [8], which were used to construct nonstandard homology theories for categories of a combinatorial nature, such as preconvexity spaces. In the recent years, many interesting applications of weighted simplicial homology are found in computational topology and topological data analysis; see Ren, Wu and Wu [14, 15], Wu, Ren, Wu and Xia [23] and Baccini, Geraci and Bianconi [2].

If not specified otherwise, the coefficients of homology groups in our following discussions are always assumed to be integers \( \mathbb{Z} \).

A (positively) \textit{weighted simplicial complex} is a pair \((K, w)\) where \( K \) is a simplicial complex and \( w \) is a function assigning a positive integer to each simplex of \( K \), which satisfy: for any simplices \( \sigma, \sigma' \) in \( K \),

\[
\sigma' \text{ is a face of } \sigma \Rightarrow w(\sigma') | w(\sigma).
\]

The function \( w \) is called a \textit{weight} on \( K \). In the following, we also use the vertex set of a simplex \( \sigma \) to denote the simplex.

For each \( n \geq 0 \), let \( C_n(K) \) denote the free abelian group generated by all the oriented \( n \)-simplices of \( K \). By abuse of notation, we will use the symbol \( \sigma \) to denote a simplex or an oriented simplex in different occasions. Then the weight function \( w \) determines a boundary operator \( \partial^w \) on \( C_\ast(K) \) by:

\[
\partial^w : C_n(K) \to C_{n-1}(K), \quad \sigma \mapsto \sum_{j=0}^{n} \frac{w(\sigma)}{w(\partial_j(\sigma))} (-1)^j \partial_j \sigma
\]

where the face operators \( \partial_j \) are defined as in the standard simplicial homology, i.e. for an oriented simplex \( \sigma = [v_0, \cdots, v_n] \), \( \partial_j \sigma = [v_0, \cdots, \hat{v}_j, \cdots, v_n] \). It is easy to check that

\[
\partial^w \circ \partial^w = 0.
\]
Then \((C_*(K), \partial^w)\) is called the \textit{weighted simplicial chain complex} of \((K, w)\), whose homology group is called the \textit{weighted simplicial homology} of \((K, w)\), denoted by \(H_*(K, \partial^w)\).

Moreover, for a simplicial subcomplex \(A\) of \(K\), the map \(\partial^w\) induces a boundary map \(\overline{\partial}^w\) on the relative chain complex \(C_*(K, A) = C_*(K)/C_*(A)\). Then similarly, we can define the \textit{relative weighted simplicial homology} \(H_*(K, A, \partial^w)\) as

\[
H_*(K, A, \partial^w) := H_*(C_*(K, A), \overline{\partial}^w).
\]

Given two weighted simplicial complexes \((K, w)\) and \((K', w')\), a simplicial map \(\varrho : K \to K'\) is called a \textit{morphism} from \((K, w)\) to \((K', w')\) if \(w'(\varrho(\sigma)) \mid w(\sigma)\) for any simplex \(\sigma\) in \(K\) (see [8]). We also denote such a map by \(\varrho : (K, w) \to (K', w')\).

In particular, \(\varrho\) is called \textit{weight-preserving} if \(w'(\varrho(\sigma)) = w(\sigma)\) for any simplex \(\sigma\) in \(K\). It is easy to check that a morphism \(\varrho\) induces a chain map

\[
\varrho_# : (C_*(K), \partial^w) \to (C_*(K'), \partial^{w'}), \quad \varrho_#(\sigma) = \frac{w(\sigma)}{w'(\varrho(\sigma))} \varrho(\sigma), \quad \forall \sigma \in K.
\]

Then \(\varrho\) further induces a homomorphism on the weighted homology denoted by

\[
\varrho_* : H_*(K, \partial^w) \to H_*(K', \partial^{w'}). \quad \text{The reader is referred to [8] for more discussion of the categorical properties of weighted simplicial complexes and weighted homology groups.}
\]

### 2.3. Descending weighted simplicial complexes.

In the definition of weighted simplicial complex, the weights of the simplices are ascending when their dimensions go up. But we find that another type of weight functions on simplicial complexes are more suitable for our study of orbifolds, which are defined as follows.

**Definition 2.3.** A \textit{descending weighted simplicial complex} is a pair \((L, \overline{\pi})\) where \(L\) is a simplicial complex and \(\overline{\pi}\) is a function assigning a positive integer to each simplex of \(L\), which satisfy: for any simplices \(\sigma, \sigma'\) in \(L\),

\[
\sigma' \text{ is a face of } \sigma \Rightarrow \overline{\pi}(\sigma) \mid \overline{\pi}(\sigma').
\]

We call \(\overline{\pi}\) a \textit{descending weight} on \(L\). There is a natural boundary operator

\[
\partial^\overline{\pi} : C_n(L) \to C_{n-1}(L), \quad n \geq 0,
\]

defined by:

\[
\partial^\overline{\pi} : \sigma \mapsto \sum_{j=0}^n \frac{\overline{\pi}(\partial_j(\sigma))}{\overline{\pi}(\sigma)} (-1)^j \partial_j \sigma.
\]

It is easy to prove \(\partial^\overline{\pi} \circ \partial^\overline{\pi} = 0\). We call \((C_*(L), \partial^\overline{\pi})\) the \textit{weighted simplicial chain complex} of \((L, \overline{\pi})\), whose homology group is called the \textit{weighted simplicial homology group} of \((L, \overline{\pi})\), denoted by \(H_*(L, \partial^\overline{\pi})\).
A morphism from \((L, \overline{w})\) to \((L', \overline{w}')\) is defined to be a simplicial map \(\overline{\varrho} : L \to L'\) that satisfies: \(\overline{w}(\sigma) \mid \overline{w}'(\overline{\varrho}(\sigma))\) for any simplex \(\sigma\) in \(L\). In particular, \(\overline{\varrho}\) is called weight-preserving if \(\overline{w}'(\overline{\varrho}(\sigma)) = \overline{w}(\sigma)\) for any simplex \(\sigma\) in \(L\). It is easy to see that a morphism \(\overline{\varrho}\) induces a chain map \(\overline{\varrho}_\# : (C_*(L), \partial \overline{w}) \to (C_*(L'), \partial \overline{w}')\) defined by:

\[
\overline{\varrho}_\#(\sigma) = \frac{\overline{w}'(\overline{\varrho}(\sigma))}{\overline{w}(\sigma)}, \forall \sigma \in L.
\]

Then \(\overline{\varrho}\) further induces a homomorphism on homology groups, denoted by \(\overline{\varrho}_* : H_*(L, \partial \overline{w}) \to H_*(L', \partial \overline{w}')\).

The following are some conventions that will be used in the rest of the paper.

(CV-1) To avoid ambiguity, we call a weighted simplicial complex \((K, w)\) defined by Dawson [8] an ascending weighted simplicial complex and call \(w\) an ascending weight.

(CV-2) When we say \((K, \mu)\) is a weighted simplicial complex, \(\mu\) could be either an ascending weight or a descending one.

(CV-3) We say that an ascending weighted simplicial complex and a descending weighted one are of opposite type, so do we say their weight functions.

**Definition 2.4 (Finite Weight).** A weighted simplicial complex \((K, \mu)\) is called finitely weighted if the range of the weight function \(\mu\) is a finite set of integers, and \(\mu\) is called a finite weight. Denote by \(N_\mu\) the least common multiple of the weights of all the simplices in \((K, \mu)\).

Note that if \(K\) is a finite simplicial complex, then \((K, \mu)\) is always finitely weighted. But conversely, if \((K, \mu)\) is finitely weighted, it is possible that \(K\) has infinitely many simplices.

**Lemma 2.5.** For a finitely weighted simplicial complex \((K, \mu)\), there exists a weighted simplicial complex \((K, \mu^*)\) of opposite type such that

\[
H_i(K, \partial^\mu) \cong H_i(K, \partial^\mu^*) \quad \text{for all } i \geq 0.
\]

**Proof.** We define another weight \(\mu^*\) on \(K\) by

\[
\mu^*(\sigma) = \frac{N_\mu}{\mu(\sigma)}, \forall \sigma \in K.
\]

By the definitions of \(\partial^\mu\) and \(\partial^\mu^*\), \((C_*(K), \partial^\mu)\) and \((C_*(K), \partial^\mu^*)\) are in fact isomorphic (abstract) chain complexes. So \(H_*(K, \partial^\mu^*) \cong H_*(K, \partial^\mu)\). \(\square\)

**Definition 2.6 (Adjoint of a Weight).** For a finitely weighted simplicial complex \((K, \mu)\), we call the weight function \(\mu^*\) defined in (4) the adjoint of \(\mu\).
By Lemma 2.5, descending weighted simplicial complexes can be considered as mirror objects of ascending weighted ones. But not all properties of these two categories of objects are the same. We will see later in Section 9 that there is a natural product structure on the weighted cohomology groups on a descending weighted simplicial complex. But there is no such a product for an ascending weighted simplicial complex.

2.4. Augmentation of weighted simplicial complex.

A (abstract) chain complex $C_\ast = (C_p, \partial_p)$ is called non-negative if $C_p = 0$ for all $p < 0$. In this paper, all chain complexes are assumed to be non-negative if not specified otherwise.

The augmentation of a non-negative chain complex $C_\ast = \{\partial_n : C_n \to C_{n-1}\}_{n \geq 1}$ is a map $\varepsilon : C_0 \to \mathbb{Z}$ such that $\varepsilon \circ \partial_1 = 0$. The augmented chain complex $(C_\ast, \varepsilon)$ is the chain complex obtained from $C_\ast$ by adjoining the group $\mathbb{Z}$ in dimension $-1$ and using $\varepsilon$ as the boundary operator in dimension $0$.

Let $(C_\ast, \varepsilon)$ and $(C'_\ast, \varepsilon')$ be two augmented chain complexes. We call a chain map $\phi = \{\phi_n : C_n \to C'_n\}$ augmentation-preserving if $\varepsilon' \circ \phi_0 = \varepsilon$.

Let $(K, \mu)$ be a weighted simplicial complex. We define an augmentation of $(C_\ast(K), \partial^\mu)$, denoted by $\varepsilon^\mu : C_0(K, \partial^\mu) \to \mathbb{Z}$, as follows:

- If $\mu$ is ascending, define
  \begin{equation}
  \varepsilon^\mu(v_i) := \mu(v_i) \text{ for any vertex } v_i \text{ of } K.
  \end{equation}

- If $\mu$ is descending and finite, its adjoint $\mu^*$ is an ascending weight on $K$. Define
  \begin{equation}
  \varepsilon^\mu(v_i) := \mu^*(v_i) = \frac{N_\mu}{\mu(v_i)} \text{ for any vertex } v_i \text{ of } K.
  \end{equation}

The reason why we use $N_\mu$ as the numerator is because we want to use integral coefficients for our weighted homology. So we need to require $(K, \mu)$ to be finitely weighted in this case.

It is easy to check that the above defined $\varepsilon^\mu$ is an augmentation on $(C_\ast(K), \partial^\mu)$ in both cases. Then we call the homology of the augmented chain complex $(C_\ast(K), \partial^\mu, \varepsilon^\mu)$ the reduced weighted simplicial homology of $(K, \mu)$, denoted by $\tilde{H}_\ast(K, \partial^\mu)$. The following lemma is easy to see from our definitions.

**Lemma 2.7.** Let $(K, \mu)$ and $(K', \mu')$ be two weighted simplicial complexes of the same type. If $\varphi : (K, \mu) \to (K', \mu')$ is a morphism, then $\varphi$ extends to an augmentation-preserving chain map $\tilde{\varphi}$ between the augmented chain complexes $(C_\ast(K), \partial^\mu, \varepsilon^\mu)$ and $(C_\ast(K'), \partial'^\mu, \varepsilon'^\mu)$ where at degree $-1$, $\tilde{\varphi} : \mathbb{Z} \to \mathbb{Z}$ is defined...
by: for any \( n \in \mathbb{Z} \),
\[
\bar{\varphi}(n) = \begin{cases} 
  n, & \text{if } (K, \mu) \text{ and } (K', \mu') \text{ are both ascending}; \\
  \frac{n_1}{\text{lcm} n}, & \text{if } (K, \mu) \text{ and } (K', \mu') \text{ are both descending and finitely weighted.}
\end{cases}
\]

### 2.5. Cartesian product of weighted simplicial complexes.

Let \((K, \mu)\) and \((K', \mu')\) be two weighted simplicial complexes and let
\[
v_1 \prec \cdots \prec v_m, \quad v'_1 \prec \cdots \prec v'_n
\]
be some total orderings of the vertices of \(K\) and \(K'\), respectively. Then the Cartesian product \(K \times K'\) is a simplicial complex whose vertex set is
\[
\{(v_i, v'_j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}.
\]

Moreover, all the simplices in \(K \times K'\) are of the form
\[
(7) \quad \{(v_{i_1}, v'_{j_1}), \ldots, (v_{i_s}, v'_{j_s})\}, \quad v_{i_1} \preceq \cdots \preceq v_{i_s}, \quad v'_{j_1} \preceq \cdots \preceq v'_{j_s},
\]
where \(\{v_{i_1}, \ldots, v_{i_s}\}\) is a simplex in \(K\) and \(\{v'_{j_1}, \ldots, v'_{j_s}\}\) is a simplex in \(K'\).

**Definition 2.8** (Cartesian product of weighted simplicial complexes). Let \((K, \mu)\) and \((K', \mu')\) be weighted simplicial complexes which are both ascending or both descending. The **Cartesian product** of \((K, \mu)\) and \((K', \mu')\) with respect to some total orderings of the vertices of \(K\) and \(K'\) is a weighted simplicial complex \((K \times K', \mu \times \mu')\), where for a simplex \(\{(v_{i_1}, v'_{j_1}), \ldots, (v_{i_s}, v'_{j_s})\}\) of \(K \times K'\),
\[
(8) \quad \mu \times \mu'\{(v_{i_1}, v'_{j_1}), \ldots, (v_{i_s}, v'_{j_s})\} := \mu(\{v_{i_1}, \ldots, v_{i_s}\}) \cdot \mu'(\{v'_{j_1}, \ldots, v'_{j_s}\}).
\]

It is easy to see that \((K \times K', \mu \times \mu')\) has the type as \((K, \mu)\) and \((K', \mu')\).

Notice that the simplicial complex structure of \(K \times K'\) depends on the ordering of vertices of \(K\) and \(K'\), so does \((K \times K', \xi \times \xi')\). But we usually omit the orderings of vertices in our notation.

**Remark 2.9.** There is another meaningful weight function \(\bar{\mu} \times \bar{\mu}'\) on \(K \times K'\) defined by: for each simplex \(\{(v_{i_1}, v'_{j_1}), \ldots, (v_{i_s}, v'_{j_s})\}\) of \(K \times K'\) as in (7),
\[
\bar{\mu} \times \bar{\mu}'\{(v_{i_1}, v'_{j_1}), \ldots, (v_{i_s}, v'_{j_s})\} := \text{lcm}(\bar{\mu}(\{v_{i_1}, \ldots, v_{i_s}\}), \bar{\mu}'(\{v'_{j_1}, \ldots, v'_{j_s}\})),
\]
where “lcm” is the abbreviation for least common multiple. In [8], \((K \times K', \bar{\mu} \times \bar{\mu}')\) is called the product of \((K, \mu)\) and \((K', \mu')\) (see [8, Proposition 1.1]). But the Cartesian product in Definition 2.8 is a more suitable notion for our study of orbifolds. The reason is that the local group of the product of two orbifolds \(\mathcal{M} = (M, \mathcal{U})\) and \(\mathcal{N} = (N, \mathcal{V})\) at a point \((x, y) \in M \times N\) is the product of the local group of \(\mathcal{M}\) at \(x\) and the local group of \(\mathcal{N}\) at \(y\). Therefore, the weight of \((x, y)\) in the product orbifold is the product (not the least common multiple) of the weights of \(x\) in \(\mathcal{M}\) and \(y\) in \(\mathcal{N}\).
Proposition 2.10. Let \((K, \mu)\) be a weighted simplicial complex. Then there is an isomorphism \(H_*(K, \partial^\mu) \cong H_*(K \times [0,1], \partial^{\mu \times 1})\).

Proof. Let \(i_0 : K \hookrightarrow K \times \{0\} \subset K \times [0,1]\) be the inclusion and \(p : K \times [0,1] \to K\) be the projection. It is easy to check that they induce chain maps \((i_0)_\# : (C_*(K), \partial^\mu) \to (C_*(K \times [0,1]), \partial^{\mu \times 1})\), \(p_\# : (C_*(K \times [0,1]), \partial^{\mu \times 1}) \to (C_*(K), \partial^\mu)\).

Claim: \((i_0)_\#\) and \(p_\#\) are chain homotopy inverse of each other.

First of all, clearly \(p_\# \circ (i_0)_\# = \text{id}_{C_*(K)}\). It remains to prove that \((i_0)_\# \circ p_\#\) is chain homotopic to the identity map on \(C_*(K \times [0,1])\). Let the vertices of \(K \times [0,1]\) be

\[
v_1, \ldots, v_m \in K \times \{0\}; \quad v'_1, \ldots, v'_m \in K \times \{1\}\end{equation}

where \(p(v'_j) = v_j, 1 \leq j \leq m\).

Then any simplex of \(K \times [0,1]\) can be written as one of the following forms:

\[
\{v_{j_1}, \ldots, v_{j_r}\} \in K \times \{0\}, \quad \{v'_{j_1}, \ldots, v'_{j_r}\} \in K \times \{1\}, \quad 1 \leq j_1 < \cdots < j_r \leq m;
\]

\[
\{v_{j_1}, \ldots, v_{j_s}, v'_{j_{s+1}}, \ldots, v'_{j_r}\}, \quad 1 \leq j_1 < \cdots < j_s \leq j_{s+1} < \cdots < j_r \leq m
\]

where \(\{v_{j_1}, \ldots, v_{j_r}\}\) is a simplex in \(K\). Note that this is a simplified way to write the simplices of \(K \times [0,1]\) given by (7).

Note that \(i_0\) and \(p\) are both weight-preserving by the definition of \(\mu \times 1\) in (8). Moreover,

\[
i_0 \circ p (\{v_{j_1}, \ldots, v_{j_r}\}) = i_0 \circ p (\{v'_{j_1}, \ldots, v'_{j_r}\}) = \{v_{j_1}, \ldots, v_{j_r}\};
\]

\[
i_0 \circ p (\{v_{j_1}, \ldots, v_{j_s}, v'_{j_{s+1}}, \ldots, v'_{j_r}\}) = \begin{cases} \{v_{j_1}, \ldots, v_{j_r}\}, & \text{if } j_s < j_{s+1}; \\ \{v_{j_1}, \ldots, v_{j_s}, v_{j_{s+2}}, \ldots, v_{j_r}\}, & \text{if } j_s = j_{s+1}. \end{cases}
\]

So the chain map \((i_0)_\# \circ p_\#\) vanishes at \([v_{j_1}, \ldots, v_{j_s}, v'_{j_{s+1}}, \ldots, v'_{j_r}]\) because of the dimension reason.

Next, define \(\{P_n : C_n(K \times [0,1]) \to C_{n+1}(K \times [0,1])\}_{n \geq 0}\) by

\[
P_{r-1}([v_{j_1}, \ldots, v_{j_r}]) = 0;
\]

\[
P_{r-1}([v'_{j_1}, \ldots, v'_{j_r}]) = \sum_{t=1}^r (-1)^{t-1} [v_{j_1}, \ldots, v_{j_t}, v'_{j_t}, \ldots, v'_{j_r}], \quad r \geq 1;
\]

\[
P_{r-1}([v_{j_1}, \ldots, v_{j_s}, v'_{j_{s+1}}, \ldots, v'_{j_r}])
\]

\[
= \begin{cases} \sum_{s+1 \leq t \leq r} (-1)^{t-1} [v_{j_1}, \ldots, v_{j_s}, v_{j_{s+1}}, \ldots, v'_{j_t}, \ldots, v'_{j_r}], & \text{if } j_s < j_{s+1}; \\ 0, & \text{if } j_s = j_{s+1}. \end{cases}
\]
It is routine to check that \( \{ P_n \}_{n \geq 0} \) is a chain homotopy between \( (i_0)_\# \circ p_\# \) and the identity map \( \text{id}_{C_*(K \times [0,1])} \).

### 2.6. Divisibly weighted simplicial complex.

Let \( (K, w) \) be an ascending weighted simplicial complex. A simplex \( \sigma \) in \( K \) is called *divisibly weighted* (see Dawson [8]) if all the vertices of \( \sigma \) can be ordered as \( \{ v_0, \ldots, v_k \} \) such that
\[
\tag{9} w(v_0) | \cdots | w(v_k) = w(\sigma).
\]

If all the simplices in \( (K, w) \) are divisibly weighted, the weight function \( w \) is called *divisible*.

Similarly, for a descending weighted simplicial complex \( (L, \overline{w}) \) as well, we call a simplex \( \sigma \) in \( L \) *divisibly weighted* if all the vertices of \( \sigma \) can be ordered as \( \{ v_0, \ldots, v_k \} \) such that
\[
\tag{10} \overline{w}(\sigma) = w(v_0) | \cdots | w(v_k).
\]

If all the simplices in \( (L, \overline{w}) \) are divisibly weighted, the weight function \( \overline{w} \) is also called *divisible*.

In general, we call a weighted simplicial complex \( (K, \mu) \) *divisibly weighted* if its weight function \( \mu \) (either ascending or descending) is divisible. Clearly, a divisible weight on a simplicial complex \( K \) is completely determined by its value on the vertices of \( K \).

The following convention will be used in the rest of paper.

\textbf{(CV-4)} We will use Greek letters \( \xi \) and \( \eta \) to refer to a divisible weight on a simplicial complex \( K \) to distinguish it from a general weight function \( \mu \).

\textbf{Lemma 2.11.} Suppose \( (K, \xi) \) and \( (K', \xi') \) are two divisibly weighted simplicial complexes of the same type. Then a simplicial map \( \varphi : K \to K' \) is a morphism from \( (K, \xi) \) to \( (K', \xi') \) if and only if for every vertex \( v \) of \( K \),
\[
\begin{cases} 
\xi'(\varphi(v)) | \xi(v), & \text{if } \xi \text{ and } \xi' \text{ are ascending}; \\
\xi(v) | \xi'(\varphi(v)), & \text{if } \xi \text{ and } \xi' \text{ are descending}.
\end{cases}
\]

\textbf{Proof.} This is obvious from the definitions of divisibly weighted simplicial complex and the morphism of weighted simplicial complex. \( \square \)

\textbf{Proposition 2.12.} The Cartesian product of two divisibly weighted simplicial complexes of the same type is also divisibly weighted.

\textbf{Proof.} Let \( (K, \xi) \) and \( (K', \xi') \) be two divisibly weighted simplicial complexes of the same type. Let \( v_1 < \cdots < v_m \) and \( v'_1 < \cdots < v'_n \) be total orderings of the
vertices of $K$ and $K'$, respectively, such that
\[ (11) \quad \xi(v_1) \leq \cdots \leq \xi(v_m), \quad \xi'(v_1) \leq \cdots \leq \xi'(v_{m'}). \]

Any simplex in $K \times K'$ is of the form
\[ \{(v_{i_1}, v'_{j_1}), \cdots, (v_{i_s}, v'_{j_s})\}, \quad v_{i_1} \preceq \cdots \preceq v_{i_s}, \quad v'_{j_1} \preceq \cdots \preceq v'_{j_s} \]
where $\{v_{i_1}, \cdots, v_{i_s}\}$ is a simplex of $K$ and $\{v'_{j_1}, \cdots, v'_{j_s}\}$ is a simplex of $K'$. Then since $(K, \xi)$ and $(K', \xi')$ are both divisibly weighted, both $\{(v_{i_1}, \cdots, v_{i_s}), \xi\}$ and $\{(v'_{j_1}, \cdots, v'_{j_s}), \xi'\}$ are divisibly weighted simplices.

If $(K, \xi)$ and $(K', \xi')$ are both ascending, then we have
\[ (12) \quad \xi(v_{i_1}) \cdots \xi(v_{i_s}) = \xi(\{(v_{i_1}, \cdots, v_{i_s})\}), \quad \xi'(v'_{j_1}) \cdots \xi'(v'_{j_s}) = \xi'(\{(v'_{j_1}, \cdots, v'_{j_s})\}). \]

So by Definition 2.8, the weight $\xi \times \xi'$ on $K \times K'$ is given by
\[ (\xi \times \xi')(\{(v_{i_1}, v'_{j_1}), \cdots, (v_{i_s}, v'_{j_s})\}) = \xi(\{(v_{i_1}, \cdots, v_{i_s})\}) \cdot \xi'(\{(v'_{j_1}, \cdots, v'_{j_s})\}) = \xi(v_{i_1}) \cdot \xi'(v'_{j_1}). \]

In particular,
\[ (\xi \times \xi')(\{v_i, v'_j\}) = \xi(v_i) \cdot \xi'(v'_j), \quad 1 \leq i \leq m, \quad 1 \leq j \leq n. \]

Then from (12), we obtain
\[ (\xi \times \xi')(\{v_{i_1}, v'_{j_1}\}) \cdots (\xi \times \xi')(\{v_{i_s}, v'_{j_s}\}) = (\xi \times \xi')(\{(v_{i_1}, v'_{j_1}), \cdots, (v_{i_s}, v'_{j_s})\}), \]
which means that $\{(v_{i_1}, v'_{j_1}), \cdots, (v_{i_s}, v'_{j_s})\}, \xi \times \xi'$ is a divisibly weighted simplex. So $(K \times K', \xi \times \xi')$ is a divisibly weighted simplicial complex.

Similarly, if $(K, \xi)$ and $(K', \xi')$ are both descending, the weight $\xi \times \xi'$ on $K \times K'$ is given by
\[ (\xi \times \xi'')(\{(v_{i_1}, v'_{j_1}), \cdots, (v_{i_s}, v'_{j_s})\}) = \xi(v_{i_1}) \cdot \xi'(v'_{j_1}). \]

It is also easy to check that $(K \times K', \xi \times \xi')$ is divisibly weighted. \qed

**Definition 2.13** (Inversion of a Divisible Weight). If $\xi$ is an ascending divisible weight on a simplicial complex $K$, then $\xi$ canonically determines a descending divisible weight $\hat{\xi}$ on $K$ by: for any simplex $\sigma$ in $K$,
\[ (13) \quad \hat{\xi}(\sigma) = \xi(v_0) \cdots \xi(v_k) = \xi(\sigma) \]
where $\{v_0, \cdots, v_k\}$ are all the vertices of $\sigma$. We call $\hat{\xi}$ the inversion of $\xi$.

Similarly, if $\eta$ is a descending divisible weight on $K$, we obtain an ascending divisible weight $\hat{\eta}$ on $K$ defined by
\[ (14) \quad \eta(\sigma) = \eta(v_0) \cdots \eta(v_k) = \hat{\eta}(\sigma). \]

We call $\hat{\eta}$ the inversion of $\eta$. 

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The following lemma is immediate from our definition.

**Lemma 2.14.** For a divisible weight $\xi$ on a simplicial complex, the adjoint of the inversion of $\xi$ agrees with the inversion of the adjoint of $\xi$, i.e. $(\hat{\xi})^* = \hat{\xi}^*$.

Generally speaking, the inversion of a divisible weight $\xi$ is different from its adjoint (see Figure 1 for example). In addition, the inversion of the inversion of $\xi$ always goes back to $\xi$, while the adjoint of the adjoint of $\xi$ may differ from $\xi$ by a constant factor.

![Figure 1](image)

**Figure 1.** The adjoint and the inversion of a divisible weight $\xi$

A divisible weight $\xi$ on $K$ and its inversion $\hat{\xi}$ clearly determine each other since they have the same values on the vertices. But generally speaking, the chain complexes $(C_*(K), \partial^\xi)$ and $(C_*(K), \partial^{\hat{\xi}})$ are not isomorphic, neither are the weighted homology $H_*(K, \partial^\xi)$ and $H_*(K, \partial^{\hat{\xi}})$. Moreover, it is possible that two divisible weighted simplicial complexes $(K, \xi)$ and $(L, \eta)$ of the same type satisfy $H_*(K, \partial^\xi) \cong H_*(L, \partial^\eta)$ while $H_*(K, \partial^{\hat{\xi}}) \not\cong H_*(L, \partial^{\hat{\eta}})$ (see Example 8.4).

### 2.7. Barycentric subdivision of Weighted Simplicial Complex.

For any weighted simplicial complex $(K, \mu)$, there is a natural divisible weight on the barycentric subdivision $Sd(K)$ of $K$.

**Definition 2.15 (Barycentric Subdivision of Weighted Simplicial Complex).**

For a weighted simplicial complex $(K, \mu)$, define the weight of the barycenter $b_\sigma$ of any simplex $\sigma$ in $K$ to be $\mu(\sigma)$. Since any simplex in $Sd(K)$ is of the form $\{b_{\sigma_0}, \ldots, b_{\sigma_l}\}$ where $\sigma_0 \subset \cdots \subset \sigma_l \in K$, we define a weight function $Sd(\mu)$ on $Sd(K)$ by:

\[
Sd(\mu)(\{b_{\sigma_0}, \ldots, b_{\sigma_l}\}) = \mu(\sigma_l), \quad \sigma_0 \subset \cdots \subset \sigma_l \in K.
\]

(15)

Note that we have either $\mu(\sigma_0) | \cdots | \mu(\sigma_l)$ or $\mu(\sigma_l) | \cdots | \mu(\sigma_0)$ depending on whether $\mu$ is ascending or descending. Then it is easy to check that $Sd(\mu)$ is a divisible weight on $Sd(K)$ which is ascending (or descending) if so is $\mu$. We call $(Sd(K), Sd(\mu))$ the barycentric subdivision of $(K, \mu)$.
Another useful fact of \((Sd(K), Sd(\mu))\) is that for an \(n\)-simplex \(\tau\) in \(K\), every \(n\)-simplex \(\sigma\) in \(Sd(\sigma)\) is of the form \(\{b_{\sigma_0}, \ldots, b_{\sigma_n}\}\) where \(\sigma_0 \subseteq \cdots \subseteq \sigma_n = \sigma\). So we always have
\[
Sd(\mu)(\tau) = \mu(\sigma).
\]

For any \(m \geq 1\), let \((Sd^m(K), Sd^m(\mu))\) denote the \(m\) repeated barycentric subdivision of \((K, \mu)\).

**Remark 2.16.** Given a simplicial subcomplex \(K_0\) in \(K\), [13, p. 89] defines a notion of barycentric subdivision of \(K\) holding \(K_0\) fixed, denoted by \(Sd(K/K_0)\). This notion is used to in [13, p. 90] to define generalized barycentric subdivisions of \(K\). Roughly speaking, \(Sd(K/K_0)\) is just iteratively taking the cone of the barycenter \(b_\sigma\) of a simplex \(\sigma\) in \(K - K_0\) with all the simplices in the subdivision of \(\partial \sigma\) by the previous steps. A general simplex in \(Sd(K/K_0)\) is of the form
\[
\{v_0, \ldots, v_p, b_{\sigma_0}, \ldots, b_{\sigma_q}\}
\]
where \(\tau = \{v_0 \cdots v_p\}\) is a simplex of \(K_0\) and \(\tau \subsetneq \sigma_0 \subsetneq \cdots \subsetneq \sigma_q \in K - K_0\). Similarly to the definition of \(Sd(\mu)\) in (15), we can define a weight \(Sd_{K_0}(\mu)\) on \(Sd(K/K_0)\) from \((K, \mu)\) by:
\[
Sd_{K_0}(\mu)(\{v_0, \ldots, v_p, b_{\sigma_0}, \ldots, b_{\sigma_q}\}) = \mu(\sigma_q).
\]
Note that if \(\mu\) is a divisible weight on \(K\), then \(Sd_{K_0}(\mu)\) is a divisible weight on \(Sd(K/K_0)\) as well. But generally speaking, \(Sd_{K_0}(\mu)\) may not be a divisible weight.

### 2.8. Contractible weighted simplicial complex.

The following definitions are introduced by Dawson [8] for ascending weighted simplicial complexes. But clearly they can be defined for all weighted simplicial complexes as below.

**Definition 2.17** (See [8, p. 236]). Suppose \((K, \mu)\) and \((K', \mu')\) are two weighted simplicial complexes. Then two morphisms \(g_0, g_1 : (K, \mu) \to (K', \mu')\) are called **contiguous** if there exists a morphism \(H : (K \times [0, 1], \mu \times 1) \to (K', \mu')\) with
\[
H(x, 0) = g_0(x), \quad H(x, 1) = g_1(x), \quad \forall x \in K.
\]
Here \(([0, 1], 1)\) is a weighted simplicial complex where the weight of every simplex of \([0, 1]\) is 1, and \((K \times [0, 1], \mu \times 1)\) is the Cartesian product \((K, \mu)\) with \(([0, 1], 1)\).

We warn that the term “contiguous” is also used in Munkres’s textbook [13] but with different meaning.

**Theorem 2.18** ([8, Theorem 2.5]). If two morphisms \(g_0, g_1 : (K, \mu) \to (K', \mu')\) of weighted simplicial complexes are contiguous, then
\[
(g_0)_* = (g_1)_* : H_*(K, \partial^\mu) \to H_*(K', \partial'^\mu).
\]
The following notion is introduced in [8, p. 236].

**Definition 2.19** (Contractible weighted simplicial complex).

A weighted simplicial complex \((K, \mu)\) is called contractible if there exists a sequence \(\varrho_0, \varrho_1, \ldots, \varrho_n : (K, \mu) \to (K, \mu)\) of morphisms of weighted simplicial complexes such that \(\varrho_0\) is the identity on \((K, \mu)\), \(\varrho_i\) is contiguous to \(\varrho_{i-1}\) for each \(1 \leq i \leq n\), and \(\varrho_n\) is a constant map.

The following proposition is very useful for our discussion later.

**Proposition 2.20** ([8, Corollary 2.5.1]). If a weighted simplicial complex \((K, \mu)\) is contractible, then

\[
H_j(K, \partial^\mu) \cong \begin{cases} 
\mathbb{Z}, & \text{if } j = 0; \\
0, & \text{if } j \geq 1.
\end{cases}
\]

The proofs of Theorem 2.18 and Proposition 2.20 in [8] are for ascending weighted simplicial complexes. But it is completely parallel to write the proofs for descending weighted simplicial complexes.

### 3. AW-homology and DW-homology of an orbifold

Let \(\mathcal{M} = (M, \mathcal{U})\) be an orbifold. It is well known that (see Goresky [9] and Verona [22]) there exists a triangulation \(\mathcal{T}\) of \(M\) such that the closure of every stratum of \(M\) is a simplicial subcomplex of \(\mathcal{T}\). Then the relative interior of each simplex of \(\mathcal{T}\) is contained in a single stratum of \(M\). A detailed proof of this result can be found in Choi [6, Section 4.5]. By replacing \(\mathcal{T}\) by a stellar subdivision, one can assume that the cover of closed simplices in \(\mathcal{T}\) refines the cover of \(M\) induced by the atlas \(\mathcal{U}\). For a simplex \(\sigma\) in such a triangulation, the isotropy groups of all interior points of \(\sigma\) are the same, and are subgroups of the isotropy groups of the boundary points of \(\sigma\). By taking a further subdivision of \(\mathcal{T}\), we may assume that for any simplex \(\sigma\) in \(\mathcal{T}\),

\((\star)\) there is one face \(\tau\) of \(\sigma\) such that the local group is constant on \(\sigma \setminus \tau\), and possibly larger on \(\tau\).

We call such a triangulation \(\mathcal{T}\) adapted to \(\mathcal{U}\) (see [12]). By abuse of notation, we also use \(\mathcal{T}\) to refer to the simplicial complex defined by \(\mathcal{T}\). The property \((\star)\) of \(\mathcal{T}\) is crucial for us to define weighted simplicial homology of an orbifold.

**Proposition 3.1** (Theorem 4.5.4 in [6]). For any orbifold \(\mathcal{M} = (M, \mathcal{U})\), there always exists an adapted triangulation.

Note that any simplex \(\sigma\) in a triangulation \(\mathcal{T}\) adapted to \(\mathcal{M}\) will have a vertex \(v \in \sigma\) with maximal local group, i.e. \(G_x \subseteq G_v\), for all \(x \in \sigma\). Let

\[
w(\sigma) = \max\{|G_v|; \text{ } v \text{ is a vertex of } \sigma\}.
\]
Lemma 3.2. For any simplex \( \sigma \) in a triangulation \( T \) adapted to \( M \), we can order the vertices of \( \sigma \) to be \( \{v_0, \cdots, v_k\} \) so that \( w(v_0) \mid \cdots \mid w(v_k) = w(\sigma) \). So for any face \( \sigma' \) of \( \sigma \) in \( T \), we have \( w(\sigma') \mid w(\sigma) \).

Proof. If the local group of \( M \) on the simplex \( \sigma \) is constant, the lemma clearly holds. Otherwise, there exists a proper face \( \tau \) of \( \sigma \) such that the local group is constant on \( \sigma \setminus \tau \), and larger on \( \tau \). Let \( \{v_0, \cdots, v_{s-1}\} \) be all the vertices of \( \sigma \setminus \tau \) and \( \{v_s, \cdots, v_k\} \) be all the vertices of \( \tau \). If the local group is constant on \( \tau \), then we have \( G_{v_0} = \cdots = G_{v_{s-1}} \not\subseteq G_{v_s} = \cdots = G_{v_k} \). Otherwise, there exists a proper face \( \tau' \) of \( \tau \) such that the local group is constant on \( \tau \setminus \tau' \), and larger on \( \tau' \). By reordering the vertices of \( \tau \) if necessary, we can assume that \( v_s, \cdots, v_s' \) are all the vertices of \( \tau \setminus \tau' \) where \( s \leq s' < k \). So \( G_{v_0} = \cdots = G_{v_{s-1}} \not\subseteq G_{v_s} = \cdots = G_{v_{s'}} \). By iterating the above argument, we can order the vertices of \( \sigma \) to be \( \{v_0, \cdots, v_k\} \) so that \( G_{v_0} \subseteq \cdots \subseteq G_{v_k} \). Then we have \( w(v_0) \mid \cdots \mid w(v_k) = w(\sigma) \) since \( v_k \) is the vertex with the maximal local group.

If \( \sigma' \) is a face of \( \sigma \), then the vertex set of \( \sigma' \) is a subset of \( \{v_0, \cdots, v_k\} \), say \( \{v_{i_0}, \cdots, v_{i_s}\} \), \( 0 \leq i_0 < \cdots < i_s \leq k \). Then \( w(\sigma') = w(v_{i_s}) \mid w(v_k) = w(\sigma) \). \( \square \)

By Lemma 3.2, \( w \) is an ascending weight function on \( T \) and \( w \) is divisible. Let \( \hat{w} \) be the inversion of \( w \) (see (13)). Then for any simplex \( \sigma \) in \( T \), we have

\[
\hat{w}(\sigma) = \min\{|G_v|; v \text{ is a vertex of } \sigma\}.
\]

Definition 3.3. For an orbifold \( M = (M, \mathcal{U}) \) with an adapted triangulation \( T \),

- We call \( H_s(T, \partial^w) \) the AW-homology of \( M \), denoted by \( H^AW_s(M) \).
- We call \( H_s(T, \partial^\hat{w}) \) the DW-homology of \( M \), denoted by \( H^DW_s(M) \).

Here “AW-homology” and “DW-homology” are the abbreviations for ascending weighted homology and descending weighted homology, respectively.

We will prove in Section 6 that \( H^AW_s(M) \) and \( H^DW_s(M) \) are independent on the adapted triangulation \( T \) (see Corollary 6.15). In addition, one should consider divisibly weighted simplices (not arbitrary simplices) as the building blocks of an orbifold when computing the AW-homology and DW-homology.

For any point \( x \in M \), there exists a unique simplex in \( T \), denoted by \( \text{Car}_T(x) \), which contains \( x \) in its relative interior.

Lemma 3.4. Let \( T \) be a triangulation adapted to an orbifold \( M = (M, \mathcal{U}) \). For any point \( x \in M \), the order \( |G_x| \) of the local group \( G_x \) of \( x \) is equal to the smallest weight of the vertices of \( \text{Car}_T(x) \), i.e. \( |G_x| = \hat{w}(\text{Car}_T(x)) \).

Proof. Let the vertex set of \( \text{Car}_T(x) \) be \( \{v_0, \cdots, v_k\} \) where \( w(v_0) \mid \cdots \mid w(v_k) \). If the local group is constant on \( \text{Car}_T(x) \), the lemma clearly holds. Otherwise, by the property (\( \star \)) of \( T \), there exists a proper face \( \tau \) of \( \text{Car}_T(x) \) such that...
the local group is constant on $\text{Car}_T(x) \setminus \tau$. This implies $v_0 \notin \tau$. On the other hand, since $x$ is in the relative interior of $\text{Car}_T(x)$, we also have $x \notin \tau$. Hence $|G_x| = w(v_0) = \hat{w}(\text{Car}_T(x))$ by the definition of $\hat{w}$. The lemma is proved. □

Lemma 3.4 tells us that the order of the local group of each point of an orbifold is determined by the weights of the vertices of a triangulation adapted to the orbifold. This property motivates us to define the notion of weighted polyhedron in the next section.

4. Weighted polyhedron

Definition 4.1 (Weighted Space). A weighted space is a pair $(X, \lambda)$ where $X$ is a topological space and $\lambda$ is a function from $X$ to the set of positive integers $\mathbb{Z}_+$. Note that if $X$ is connected, $\lambda$ is a continuous function if and only if it is constant. So in most cases, $\lambda$ is not a continuous function.

The Cartesian product of two weighted spaces $(X, \lambda)$ and $(X', \lambda')$ is a weighted space $(X \times X', \lambda \times \lambda')$ where $(\lambda \times \lambda')(x, x') = \lambda(x) \cdot \lambda'(x')$ for all $x \in X$, $x' \in X'$.

Let $K$ be a simplicial complex. We have the following standard notations.

- For any $n \geq 0$, denote by $K^{(n)}$ the $n$-skeleton of $K$. In particular, $K^{(0)}$ is the vertex set of $K$.
- Let $|K| \subseteq \mathbb{R}^N$ denote a geometrical realization of $K$ where each simplex $\sigma \in K$ determines a geometric simplex $|\sigma|$ in $|K|$. So $|K| = \bigcup_{\sigma \in K} |\sigma|$.
- For any vertex $v$ of $K$, let $\text{St}(v, K)$ be the open star of $v$ in $|K|$, i.e.
  \[ \text{St}(v, K) = \bigcup_{\sigma \subset v} |\sigma|^{\circ} \]
  where $|\sigma|^{\circ}$ is the relative interior of $|\sigma|$.
- For any point $x \in |K|$, let $\text{Car}_K(x)$ denote the unique simplex of $K$ such that $x$ is contained in the relative interior $|\text{Car}_K(x)|^{\circ}$ of $|\text{Car}_K(x)|$. Then $x \in \text{St}(v, K) \iff v \in \text{Car}_K(x)$.
- For a simplicial map $\varphi : K \to L$, let $\overline{\varphi} : |K| \to |L|$ denote the continuous map determined by $\varphi$.

Definition 4.2 (Geometrical Realization of Weighted Simplicial Complex).

Any weighted simplicial complex $(K, \mu)$ determines a map $\lambda_\mu : |K| \to \mathbb{Z}_+$ by:
\begin{equation}
\lambda_\mu(x) = \mu(\text{Car}_K(x)), \quad \forall x \in |K|.
\end{equation}

We call $(|K|, \lambda_\mu)$ the geometrical realization of $(K, \mu)$.

Example 4.3. If $(K, \xi)$ is a divisibly weighted simplicial complex, we obtain two weighted spaces $(|K|, \lambda_\xi)$ and $(|K|, \lambda_{\hat{\xi}})$, where $\hat{\xi}$ is the inversion of $\xi$ (see (13)).
The definition below is inspired by the property of the adapted triangulations of an orbifold shown in Lemma 3.4.

**Definition 4.4 (Weighted Polyhedron).** A weighted space \((X, \lambda)\) is called a weighted polyhedron if there exists a weighted simplicial complex \((K, \mu)\) such that \((X, \lambda)\) is isomorphic to \((|K|, \lambda_\mu)\), and we call \((K, \mu)\) a weighted triangulation of \((X, \lambda)\). Moreover, if the weight function \(\mu\) is divisible, we call \((K, \mu)\) a divisibly weighted triangulation. In addition, we say that a weighted polyhedron \((X, \lambda)\) is of ascending type (or descending type) if its weighted triangulation is an ascending (or descending) weighted simplicial complex.

A weighted polyhedron \((X, \lambda)\) is called compact if the space \(X\) is compact. It is clear that if \((X, \lambda)\) is compact, the range of \(\lambda\) is a finite set.

In the definition of weighted polyhedron, we can actually require the weight function \(\mu\) on \(K\) to be divisible. This is suggested by the following lemma.

**Lemma 4.5.** For any weighted simplicial complex \((K, \mu)\), the weighted spaces \((|K|, \lambda_\mu)\) and \((|Sd(K)|, \lambda_{Sd(\mu)})\) are isomorphic.

**Proof.** For any nonempty simplex \(\sigma\) of \(K\), we can write \(|\sigma| = \bigcup_{\{b_{\sigma_0} \cdots b_{\sigma_l}\} \in Sd(K)} |\{b_{\sigma_0} \cdots b_{\sigma_l}\}|_\sigma\).

For any point \(x \in |\sigma|\), it follows from Definition 4.2 of \(\lambda_\mu\) that
\[
\lambda_\mu(x) = \mu(\sigma) = \lambda_\mu(b_\sigma).
\]

Meanwhile, \(x\) must lie in the relative interior of some simplex \(\{b_{\sigma_0} \cdots b_{\sigma_l}\}\) of \(Sd(K)\) with \(\sigma_0 \subset \cdots \subset \sigma_l = \sigma\). So by the definition of \(Sd(\mu)\) (see (15)),
\[
\lambda_{Sd(\mu)}(x) = Sd(\mu)(\{b_{\sigma_0} \cdots b_{\sigma_l}\}) = \mu(\sigma_l) = \mu(\sigma).
\]

So the two weighted spaces \((|K|, \lambda_\mu)\) and \((|Sd(K)|, \lambda_{Sd(\mu)})\) are isomorphic. \(\square\)

By Definition 2.15, the barycentric subdivision of a weighted simplicial complex is always a divisibly weighted simplicial complex. So we obtain the following corollary immediately from Lemma 4.5.

**Corollary 4.6.** Any weighted polyhedron has a divisibly weighted triangulation.

**Lemma 4.7.** If \((X, \lambda)\) and \((X', \lambda')\) are two weighted polyhedra, their Cartesian product \((X \times X', \lambda \times \lambda')\) is also a weighted polyhedron.

**Proof.** Let \((K, \mu)\) and \((K', \mu')\) be weighted triangulations of \((X, \lambda)\) and \((X', \lambda')\), respectively. We claim that the Cartesian product \((K \times K', \xi \times \xi')\) is a weighted triangulation of \((X \times X', \lambda \times \lambda')\). Indeed, suppose a point \((x, x') \in K \times K'\)

\[\text{Footnotes:} YIN WEI, LISU WU, AND LI YU\]
is carried by a simplex $\{(v_{i_1}, v_{j_1}), \ldots, (v_{i_s}, v_{j_s})\}$ of $K \times K'$ where $\{v_{i_1}, \ldots, v_{i_s}\}$ is a simplex in $K$ and $\{v'_{j_1}, \ldots, v'_{j_s}\}$ is a simplex in $K'$. Then $x$ is carried by $\{v_{i_1}, \ldots, v_{i_s}\}$ while $x'$ is carried by $\{v'_{j_1}, \ldots, v'_{j_s}\}$. So by our definitions,

$$
\mu \times \mu'(\{(v_{i_1}, v'_{j_1}), \ldots, (v_{i_s}, v'_{j_s})\}) \overset{(8)}{=} \mu(\{v_{i_1}, \ldots, v_{i_s}\}) \cdot \mu'(\{v'_{j_1}, \ldots, v'_{j_s}\}) \overset{(19)}{=} \lambda(x)\lambda'(x') = \lambda \times \lambda'((x, x')).
$$

This proves the lemma.

**Example 4.8.** Any orbifold $\mathcal{M} = (M, \mathcal{U})$ canonically defines a weighted space $(M, \lambda_M)$ where for any point $x$ in $M$,

$$\lambda_M(x) = \text{the order } |G_x| \text{ of the local group } G_x \text{ of } x.$$

We claim that $(M, \lambda_M)$ is a weighted polyhedron. Indeed, we can first take an adapted triangulation $\mathcal{T}$ of $\mathcal{M}$ along with the induced ascending weight $\mathbf{w}$ defined by (17). Then since $\mathbf{w}$ is divisible, we can consider the inversion $\hat{\mathbf{w}}$ of $\mathbf{w}$ (see (18)). By Definition 4.2 and Lemma 3.4, we have

$$\lambda_{\hat{\mathbf{w}}}(x) = \hat{\mathbf{w}}(\text{Car}_T(x)) = |G_x| = \lambda_M(x), \forall x \in M.$$

Therefore, $(M, \lambda_M)$ is isomorphic to $(|\mathcal{T}|, \lambda_{\hat{\mathbf{w}}})$ as weighted spaces. Hence $(\mathcal{T}, \hat{\mathbf{w}})$ is a divisibly weighted triangulation of $(M, \lambda_M)$.

The above example shows that an orbifold is naturally a weighted polyhedron of descending type. So it suggests us to consider weighted polyhedra of descending type as a generalization of orbifolds (but without the counterpart of local groups). So we introduce the following term.

**Definition 4.9** (Pseudo-orbifold). A weighted polyhedron $(X, \lambda)$ is called a pseudo-orbifold if it is of descending type and $X_{\text{reg}} = \{x \in X \mid \lambda(x) = 1\}$ is a dense open subset of $X$. We call any point in $X_{\text{reg}}$ a regular point and any point in $X - X_{\text{reg}}$ a singular point of $(X, \lambda)$.

**Definition 4.10** (W-continuous map). Suppose $(X, \lambda)$ and $(X', \lambda')$ are ascending (or descending) type weighted polyhedra. A continuous map $f : X \to X'$ is called a W-continuous map from $(X, \lambda)$ to $(X', \lambda')$ if

$$\lambda'(f(x)) \mid \lambda(x) \text{ (or } \lambda(x) \mid \lambda'(f(x))) \text{, for all } x \in X.$$

We also denote such a map by $f : (X, \lambda) \to (X', \lambda')$ to indicate its W-continuity. In particular, $f$ is called weight-preserving if $\lambda'(f(x)) = \lambda(x)$ for all $x \in X$.

We say that $(X, \lambda)$ is isomorphic to $(X', \lambda')$ if there exists a weight-preserving homeomorphism from $(X, \lambda)$ to $(X', \lambda')$. 

If \( \varrho : (K, \mu) \to (K', \mu') \) is a morphism (or weight-preserving) between two weighted simplicial complexes, then the induced map \( \overline{\varrho} : (|K|, \lambda_\mu) \to (|K'|, \lambda_{\mu'}) \) is clearly \( W \)-continuous (or weight-preserving).

**Definition 4.11** (W-Homotopy). Let \( (X, \lambda) \) and \( (X', \lambda') \) be weighted polyhedra and \( f, g : (X, \lambda) \to (X', \lambda') \) be two \( W \)-continuous maps. A **W-homotopy** from \( f \) to \( g \) relative to \( A \subseteq X \) is a \( W \)-continuous map \( H : (X \times [0, 1], \lambda \times 1) \to (X', \lambda') \) which satisfies:

- \( H(x, 0) = f(x) \) for all \( x \in X \);
- \( H(x, 1) = g(x) \) for all \( x \in X \);
- \( H(a, t) = f(a) = g(a) \) for all \( a \in A \) and \( t \in [0, 1] \).

Here \( ([0, 1], 1) \) is considered to be a weighted polyhedron, and \( (X \times [0, 1], \lambda \times 1) \) is the product \( (X, \lambda) \) with \( ([0, 1], 1) \).

If there exists a W-homotopy from \( f \) to \( g \), we say that \( f \) is **\( W \)-homotopic** to \( g \).

**Remark 4.12.** "W-continuous map" and "W-homotopy" have been defined by Takeuchi and Yokoyama [21, Sec. 5] for orbifolds but requiring all the maps to be weight-preserving. Here we redefine these two terms for the purpose of our study.

**Definition 4.13** (W-Homotopy Equivalence). Two weighted polyhedra \( (X, \lambda) \) and \( (X', \lambda') \) are called **W-homotopy equivalent** if there exist \( W \)-continuous maps

\[ f : (X, \lambda) \to (X', \lambda'), \quad g : (X', \lambda') \to (X, \lambda) \]

such that \( g \circ f \) and \( f \circ g \) are \( W \)-homotopic to \( \text{id}_X \) and \( \text{id}_{X'} \), respectively. The map \( f \) is called a **W-homotopy inverse** of \( f \).

**Example 4.14.** A weighted polyhedron \( (X, \lambda) \) is always \( W \)-homotopy equivalent to \( (X \times [0, 1], \lambda \times 1) \).

**Definition 4.15** (Stratum). For a weighted polyhedron \( (X, \lambda) \) and an integer \( n \geq 1 \), we call

\[ S_n := \{ x \in X \mid \lambda(x) = n \} \subseteq X \]

a **stratum** of \( (X, \lambda) \). Then \( X \) is the disjoint union of all its strata:

\[ X = \bigcup_{n \geq 1} S_n. \]

If \( (X, \lambda) \) is of ascending (or descending) type, then the closure of \( S_n \) in \( X \) is

\[ \overline{S}_n = \bigcup_{k \leq n} S_k \quad \text{or} \quad \overline{S}_n = \bigcup_{k \geq n} S_k. \]
Definition 4.16 (AW-homology and DW-homology of Weighted Polyhedra).

Let \((X, \lambda)\) be a weighted polyhedron. Choose a divisibly weighted triangulation \((K, \xi)\) of \((X, \lambda)\).

- If \(\xi\) is ascending, then its inversion \(\hat{\xi}\) is descending and we define
  \[ H^{\text{AW}}_*(X, \lambda) = H_*(K, \partial \xi), \quad H^{\text{DW}}_*(X, \lambda) = H_*(K, \partial \hat{\xi}). \]

- If \(\xi\) is descending, then its inversion \(\hat{\xi}\) is ascending and we define
  \[ H^{\text{AW}}_*(X, \lambda) = H_*(K, \partial \hat{\xi}), \quad H^{\text{DW}}_*(X, \lambda) = H_*(K, \partial \xi). \]

We call \(H^{\text{AW}}_*(X, \lambda)\) and \(H^{\text{DW}}_*(X, \lambda)\) the AW-homology and DW-homology of \((X, \lambda)\), respectively.

In Section 6, we will prove that \(H^{\text{AW}}_*(X, \lambda)\) and \(H^{\text{DW}}_*(X, \lambda)\) are independent on the divisibly weighted triangulation \((K, \xi)\) in their definitions (see Corollary 6.15). Moreover, we will prove in Theorem 6.17 that \(H^{\text{AW}}_*(X, \lambda)\) and \(H^{\text{DW}}_*(X, \lambda)\) are invariants under W-homotopy equivalences of \((X, \lambda)\). In addition, the categorial properties of \(H^{\text{AW}}_*(X, \lambda)\) and \(H^{\text{DW}}_*(X, \lambda)\) follow from the categorial properties of weighted simplicial complexes proved in [8].

5. Weighted homology of a divisibly weighted simplex

According to Definition 4.16, we should consider divisibly weighted simplices as the building blocks of a weighted polyhedron when we compute the AW-homology and DW-homology groups. So in this section, we first study the properties of divisibly weighted simplices before studying general weighted polyhedra. In the following, we consider a simplex \(\sigma\) along with all its faces as a simplicial complex.

Lemma 5.1. Let \(\sigma\) be a simplex and \(\xi\) be a divisible weight on \(\sigma\). Then

\[ H_j(\sigma, \partial \xi) \cong \begin{cases} \mathbb{Z}, & \text{if } j = 0; \\ 0, & \text{if } j \geq 1. \end{cases} \]

Proof. If the weight \(\xi\) is ascending, the lemma follows from [8, Theorem 3.1]. The argument in [8] is to show that \((\sigma, \xi)\) is contractible (see Definition 2.19). Indeed, suppose \(\sigma = \{v_0, \ldots, v_n\}\) with \(\xi(v_0) | \cdots | \xi(v_n)\). It is easy to see that the identity map \(\text{id}_\sigma : \sigma \to \sigma\) is contiguous to the constant map sending the whole \(\sigma\) to the vertex \(v_0\). If \(\xi\) is descending, then the adjoint \(\xi^*\) of \(\xi\) is an ascending divisible weight on \(\sigma\). So the lemma follows from the isomorphism \(H_*(\sigma, \partial \xi) \cong H_*(\sigma, \partial \xi^*)\) in (3). \(\square\)

Remark 5.2. For a general weight function \(\mu\) on a simplex \(\sigma\), \(H_*(\sigma, \partial \mu)\) may have torsion (even in dimensions greater than 0).
We want to say a bit more about $H_0(\sigma, \partial^\xi)$ in the following. For any face $\sigma'$ of a simplex $\sigma$, let $i_{\sigma\sigma'} : \sigma' \hookrightarrow \sigma$ be the inclusion. Clearly, the restriction of $\xi$ to $\sigma'$ defines a divisible weight on $\sigma'$. Then $i_{\sigma\sigma'}$ induces a homomorphism 

$$(i_{\sigma\sigma'})_* : H_*(\sigma', \partial^\xi) \to H_*(\sigma, \partial^\xi).$$

**Lemma 5.3.** Let $\xi$ be an ascending (descending) divisible weight on $\sigma$.

(a) $H_0(\sigma, \partial^\xi) \cong \mathbb{Z}$ is generated by the vertex of $\sigma$ with the minimal (maximal) weight.

(b) For any face $\sigma'$ of $\sigma$, the homomorphism $(i_{\sigma\sigma'})_* : H_0(\sigma', \partial^\xi) \to H_0(\sigma, \partial^\xi)$ is injective.

**Proof.** In the following we assume the weight $\xi$ to be ascending. The proof of the descending case is completely parallel (or use the isomorphism in Lemma 2.5).

(a) Let the vertices of $\sigma$ be $\{v_0, \ldots, v_n\}$ where $\xi(v_0) \geq \cdots \geq \xi(v_n) = \xi(\sigma)$. For any $0 \leq a < b \leq n$, let $v_av_b$ denote the 1-simplex in $\sigma$ oriented from $v_a$ to $v_b$. Since $\partial^\xi(v_av_b) = \xi(v_b) - \xi(v_a)$, we have

$$\partial^\xi(v_av_b) = v_b - \frac{\xi(v_b)}{\xi(v_a)} v_a.$$

So for any $0 \leq a < b < c \leq n$, we obtain

$$\partial^\xi(v_av_c) = \frac{\xi(v_c)}{\xi(v_b)} \partial^\xi(v_av_b) + \partial^\xi(v_bv_c).$$

This implies:

$$H_0(\sigma, \partial^\xi) = \langle v_0 \rangle \oplus \cdots \oplus \langle v_n \rangle / \langle \partial^\xi(v_av_b), 0 \leq a < b \leq n \rangle = \langle v_0 \rangle \oplus \cdots \oplus \langle v_n \rangle / \langle \partial^\xi(v_av_{a+1}), 0 \leq a \leq n - 1 \rangle \cong \mathbb{Z}.$$ 

The generator of $H_0(\sigma, \partial^\xi)$ is $v_0$ which has the minimal weight in $\sigma$.

(b) Suppose the vertex set of $\sigma'$ is $\{v_{i_0}, \ldots, v_{i_s}\}$ where $0 \leq i_0 < \cdots < i_s \leq n$. Then by the conclusion in (a), $H_0(\sigma, \partial^\xi)$ and $H_0(\sigma', \partial^\xi)$ are generated by $v_0$ and $v_{i_0}$, respectively. So

$$(i_{\sigma\sigma'})_* : H_0(\sigma', \partial^\xi) \longrightarrow H_0(\sigma, \partial^\xi)$$

$$[v_{i_0}] \mapsto [v_{i_0}] = \left[\frac{\xi(v_{i_0})}{\xi(v_0)} v_0\right].$$

Clearly, this homomorphism is injective. $\square$

**Lemma 5.4.** Let $\xi$ be a divisible weight on a simplex $\sigma$. Then the reduced weighted simplicial homology $\tilde{H}_i(\sigma, \partial^\xi) = 0$ for all $i \geq 0$. 
Proof. Let \( \partial_i^\xi : C_i(\sigma, \partial^\xi) \to C_{i-1}(\sigma, \partial^\xi) \) be the \( i \)-th weighted boundary map. By Lemma 5.3, we only need to verify \( \ker(\varepsilon^\xi) \subset \text{Im}(\partial_1^\xi) \) where \( \varepsilon^\xi \) is the augmentation defined by (5) or (6). Since \( \xi \) is a divisible weight, we can order all the vertices of \( \sigma \) to be \( \{ v_0, \ldots, v_n \} \) such that \( \xi(v_0) \mid \cdots \mid \xi(v_n) \). Let \( \alpha = \sum_{i=0}^n m_i v_i \in C_0(\sigma, \partial^\xi) \) be an element in \( \ker(\varepsilon^\xi) \).

- If \( \xi \) is ascending, by the definition of \( \varepsilon^\xi \) in (5), we have
  \[
  \varepsilon^\xi(\alpha) = \sum_{i=0}^n m_i \xi(v_i) = 0 \Rightarrow m_0 \xi(v_0) = -\sum_{i=1}^n m_i \xi(v_i).
  \]
  Since \( \xi(v_0v_i) = \xi(v_i) \) for any \( 1 \leq i \leq n \), we have
  \[
  \partial^\xi \left( \sum_{i=1}^n m_i v_0v_i \right) = \sum_{i=1}^n m_i \left( v_i - \frac{\xi(v_i)}{\xi(v_0)} v_0 \right) = \sum_{i=1}^n m_i v_i - \left( \sum_{i=1}^n m_i \xi(v_i) \right) v_0 = \alpha.
  \]

- If \( \xi \) is descending, by the definition of \( \varepsilon^\xi \) in (6), we have \( N = \xi(v_n) \) and
  \[
  \varepsilon^\xi(\alpha) = \sum_{i=0}^n m_i \frac{N}{\xi(v_i)} = 0 \Rightarrow m_n \xi(v_n) = -\sum_{i=0}^{n-1} m_i \xi(v_i).
  \]
  In this case, \( \xi(v_nv_i) = \xi(v_i) \) for any \( 0 \leq i \leq n-1 \). So we have
  \[
  \partial^\xi \left( \sum_{i=0}^{n-1} m_i v_nv_i \right) = \sum_{i=0}^{n-1} m_i \left( v_i - \frac{\xi(v_n)}{\xi(v_i)} v_n \right) = \sum_{i=0}^{n-1} m_i v_i - \left( \sum_{i=0}^{n-1} m_i \xi(v_n) \right) v_n = \alpha.
  \]
  So in both cases, \( \alpha \) belongs to \( \text{Im}(\partial^\xi) \). Hence \( \ker(\varepsilon^\xi) \subset \text{Im}(\partial^\xi) \).

\[\square\]

Remark 5.5. In Lemma 5.4, \( \tilde{H}_{-1}(\sigma, \partial^\xi) \) may not be trivial. For example, if \( \xi \) is an ascending divisible weight, it follows from the definition of the augmentation \( \varepsilon^\xi \) in (5) that \( \tilde{H}_{-1}(\sigma, \partial^\xi) \cong \mathbb{Z}/k_0\mathbb{Z} \) where
\[
k_0 = \min\{ \xi(v) \mid v \text{ is a vertex of } \sigma \}.
\]

Lemma 5.6. Let \( \sigma \) be a simplex and \( \xi \) be a divisible weight on \( \sigma \). Then
\[
H_j(Sd(\sigma), \partial^{Sd(\xi)}) \cong \begin{cases} 
\mathbb{Z}, & \text{if } j = 0; \\
0, & \text{if } j \geq 1.
\end{cases}
\]
Moreover, \( \tilde{H}_j(Sd(\sigma), \partial^{Sd(\xi)}) = 0 \) for all \( j \geq 0 \).

Proof. By Proposition 2.20, it is sufficient to show that the weighted simplicial complex \( (Sd(\sigma), \partial^{Sd(\xi)}) \) is contractible (see Definition 2.19). Let \( \{ v_0, \ldots, v_n \} \) be the vertex set of \( \sigma \) and assume \( \xi(v_0) \mid \cdots \mid \xi(v_n) \). Then the vertex set of \( Sd(\sigma) \) is
\[
\{ b_\tau \mid \tau \text{ is a face of } \sigma \}
\]
So \( Sd(\xi)(b_\tau) = \xi(\tau) \) has the same weight as one of the vertex of \( \tau \) since \( \xi \) is divisible. So for any vertex \( b_\sigma \) of \( Sd(\sigma) \), we have

\[
\begin{cases}
Sd(\xi)(b_\tau) = \xi(v_n) = \xi(b_\sigma), & \text{if } \xi \text{ is ascending;} \\
\xi(b_\sigma) = \xi(v_0) \mid Sd(\xi)(b_\tau), & \text{if } \xi \text{ is descending.}
\end{cases}
\]

\( (20) \)

For each \( 0 \leq i \leq n \), let \( \varrho_0 : Sd(\sigma) \to Sd(\sigma) \) denote the constant map sending the whole \( Sd(\sigma) \) to the vertex \( b_\sigma \). It is easy to check that the identity map \( \text{id}_{Sd(\sigma)} \) is contiguous to \( \varrho_0 \). So \( (Sd(\sigma), \partial^{Sd(\xi)}) \) is contractible. The proof of the second claim is identical to Lemma 5.4, hence omitted.

In the proof of Lemma 5.6, the weight function \( \xi \) being divisible is crucial for the argument. Indeed, for a non-divisible weight \( \mu \) on a simplex \( \sigma \), the relations in \( (20) \) may not hold and \( (\sigma, \mu) \) may not be contractible.

**Remark 5.7.** Let \( \sigma \) be a simplex and \( K_0 \) be a subcomplex of \( \partial \sigma \), we obtain a generalized barycentric subdivision \( Sd(\sigma/K_0) \) of \( \sigma \) (see Remark 2.16). Then using the same argument as the proof of Lemma 5.6, we can show that for any divisible weight \( \xi \) on \( \sigma \), the weighted simplicial complex \( (Sd(\sigma/K_0), \partial^{Sd(\xi)}) \) is contractible and hence

\[
H_j(Sd(\sigma/K_0), \partial^{Sd(\xi)}) \cong \begin{cases} 
\mathbb{Z}, & \text{if } j = 0; \\
0, & \text{if } j \geq 1.
\end{cases}
\]

**6. Invariance of AW-homology and DW-homology**

In this section, we will prove that the AW-homology and DW-homology of a weighted polyhedron are invariant under isomorphisms (more generally under weighted homotopy equivalences) of weighted spaces. Our argument proceeds along the same line as the proof of the topological invariance of the ordinary simplicial homology of a simplicial complex in [13].

We will first review some basic notions and constructions in the theory of simplicial homology. The standard reference of these contents is [13, Chapter 1-2] which is highly recommended to the reader. Then we will show how to modify these constructions to adapt to our study of weighted polyhedra.

**6.1. Simplicial approximations of continuous maps.**

**Definition 6.1.** Let \( K \) and \( L \) be two simplicial complexes and \( f : |K| \to |L| \) be a continuous map. If \( \varrho : K \to L \) is a simplicial map such that:

\[
f(\text{St}(v, K)) \subseteq \text{St}(\varrho(v), L)
\]

for every vertex \( v \) of \( K \), then \( \varrho \) is called a *simplicial approximation* of \( f \).

The following are some standard facts on simplicial approximation:
A simplicial map \( \varphi : K \to L \) is a simplicial approximation of \( f \) if and only if \( \varphi(x) \in |\text{Car}_L(f(x))| \) for all \( x \in |K| \).

A continuous map \( f : |K| \to |L| \) has a simplicial approximation if and only if \( f \) satisfies the star condition relative to \( K \) and \( L \): for every vertex \( v \) of \( K \), there exists a vertex \( u \) of \( L \) such that \( f(\text{St}(v, K)) \subseteq \text{St}(u, L) \).

If \( \varphi, \varphi' : K \to L \) are two simplicial approximations of \( f \), then the chain maps \( \varphi_\# \) and \( \varphi'_\# \) are chain homotopic and so \( \varphi_* = \varphi'_* : H_*(K) \to H_*(L) \).

**Theorem 6.2** (Existence of Simplicial Approximation [13, Theorem 16.1, 16.5]). For two simplicial complexes \( K \) and \( L \) and a continuous map \( f : |K| \to |L| \), there always exists a subdivision \( K' \) of \( K \) such that \( f \) has a simplicial approximation \( \varphi : K' \to L \).

In Theorem 6.2, if the simplicial complex \( K \) is compact, the subdivision \( K' \) can be taken to be \( St^m(K) \) for some large enough integer \( m \). But if \( K \) is not compact, generalized barycentric subdivisions (see [13, p. 90]) are needed to construct the complex \( K' \).

### 6.2. Acyclic carrier.

A chain complex \( C_* = (C_p, \partial_p) \) is called free if each \( C_p \) is a free abelian group. Besides, \( C_* \) is called acyclic if its homology group \( H_*(C_*) \) satisfies

\[
H_i(C_*) = \begin{cases} 
0, & \text{if } i \geq 1; \\
\mathbb{Z}, & \text{if } i = 0.
\end{cases}
\]

**Definition 6.3** (Acyclic Carrier). Let \( (C_*, \varepsilon) = (C_p, \partial_p, \varepsilon) \) be an augmented chain complex. Suppose \( C_* \) is free; let \( \{\sigma^\alpha_p\} \) be a basis for \( C_p \), as \( \alpha \) ranges over some index set \( J_p \). Let \( (C'_*, \varepsilon') = (C'_p, \partial'_p, \varepsilon') \) be an arbitrary augmented chain complex. An acyclic carrier from \( C_* \) to \( C'_* \), relative to the given bases, is a function \( \Phi \) that assigns to each basis element \( \sigma^\alpha_p \), a subchain complex \( \Phi(\sigma^\alpha_p) \) of \( C'_* \), satisfying the following conditions:

- The chain complex \( \Phi(\sigma^\alpha_p) \) is augmented by \( \varepsilon' \) and is acyclic.
- If \( \sigma^\beta_{p-1} \) appears in the expression for \( \partial_p \sigma^\alpha_p \) in terms of the preferred basis for \( C_{p-1} \), then \( \Phi(\sigma^\beta_{p-1}) \) is a subchain complex of \( \Phi(\sigma^\alpha_p) \).

A homomorphism \( f : C'_p \to C'_q \) is said to be carried by \( \Phi \) if \( f(\sigma^\alpha_p) \) belongs to the \( q \)-dimensional group of the subchain complex \( \Phi(\sigma^\alpha_p) \) of \( C'_* \), for each \( \alpha \). Moreover, a chain map \( \phi = \{\phi_p\} : C_* \to C'_* \) is said to be carried by \( \Phi \) if each \( \phi_p : C_p \to C'_p \) is carried by \( \Phi \).

**Theorem 6.4** (Algebraic Acyclic Carrier Theorem [13, Theorem 13.4]). Suppose \( (C_*, \varepsilon) \) and \( (C'_*, \varepsilon') \) are augmented chain complexes where \( C_* \) is free. Let \( \Phi \) be an
acyclic carrier from $C_\ast$ to $C'_\ast$ relative to a set of preferred bases for $C_\ast$. Then there is an augmentation-preserving chain map $\phi : C_\ast \to C'_\ast$ carried by $\Phi$. Moreover, any two such chain maps are chain homotopic and the chain homotopy is also carried by $\Phi$.

Theorem 6.4 is particularly useful in the situation when one wants to prove that two chain maps are chain homotopic but do not need to write down the explicit formula of a chain homotopy.

If we let the chain complexes $C_\ast$ and $C'_\ast$ in the above theorem be the simplicial chain complexes of two simplicial complexes, we obtain the geometric version of acyclic carrier theorem (see [13, Theorem 13.3]).

### 6.3. Properties of Barycentric subdivision.

For a simplicial complex $K$, let $Sd_\# : C_\ast(K) \to C_\ast(Sd(K))$ be the usual chain map determined by the barycentric subdivision $Sd$ of $K$. By definition, $Sd_\#$ sends any oriented $n$-simplex $\sigma$ to the sum of all the $n$-simplices in $Sd(\sigma)$ with the same orientation. More precisely, any $n$-simplex in $Sd(\sigma)$ can be written as $\{b_{\sigma_0} \cdots b_{\sigma_n}\}$ where $\sigma_0 \subsetneq \cdots \subsetneq \sigma_n = \sigma$ is an ascending sequence of faces of $\sigma$. So

\[
Sd_\#(\sigma) = \sum_{\sigma_0 \subsetneq \cdots \subsetneq \sigma_n = \sigma} \varepsilon(\sigma_0, \cdots, \sigma_n) \cdot [b_{\sigma_0} \cdots b_{\sigma_n}].
\]

where $\varepsilon(\sigma_0, \cdots, \sigma_n) \in \{-1, 1\}$ is properly chosen to make $Sd_\# \circ \partial = \partial \circ Sd_\#$.

For brevity, we omit the sign $\varepsilon(\sigma_0, \cdots, \sigma_n)$ in (22) and write the definition of $Sd_\#$ inductively as

\[
Sd_\#(\sigma) = b_\sigma \cdot Sd_\#(\partial \sigma), \quad Sd_\#(v) = v, \quad \forall v \in K^{(0)},
\]

where for any $(n-1)$-simplex $\tau$ in $Sd_\#(\partial \sigma)$, $b_\sigma \cdot \tau$ is the cone of $\tau$ with $b_\sigma$.

Let $Sd_\ast : H_\ast(K) \to H_\ast(Sd(K))$ denote the homomorphism induced by $Sd_\#$.

**Theorem 6.5** (see [13, §17]). $Sd_\ast : H_\ast(K) \to H_\ast(Sd(K))$ is an isomorphism.

Next, we study the barycentric subdivision of a weighted simplicial complex. Suppose $(K, \mu)$ is a weighted simplicial complex. By Definition 2.15, $\mu$ induces a divisible weight $Sd(\mu)$ on $Sd(K)$. In particular, for a simplex $\{b_{\sigma_0}, \cdots, b_{\sigma_l}\}$ in $Sd(K)$ where $\sigma_0 \subsetneq \cdots \subsetneq \sigma_l \in K$, we have

\[
Sd(\mu)(\{b_{\sigma_0}, \cdots, b_{\sigma_l}\}) = \mu(\sigma_l).
\]

Note that for an $n$-simplex $\sigma$ of $K$, $Sd_\#(\sigma)$ in (22) consists of a collection of $n$-simplices which have the same weight as $\sigma$.

**Lemma 6.6.** $Sd_\# : (C_\ast(K), \partial^\mu) \to (C_\ast(Sd(K)), \partial^{Sd(\mu)})$ is a chain map.
Proof. We assume the weight function $\mu$ to be descending in the argument below. The proof of the ascending weight case is completely parallel. For an oriented $n$-simplex $\sigma$ of $K$, by definition

$$\partial \sigma = \sum_{j=0}^{n} (-1)^j \partial_j \sigma, \quad \partial^\mu \sigma \overset{(2)}{=} \sum_{j=0}^{n} (-1)^j \frac{\mu(\partial_j \sigma)}{\mu(\sigma)} \partial_j \sigma.$$

Then we have

$$\partial^{Sd(\mu)} \circ Sd_#(\sigma) \overset{(23)}{=} \partial^{Sd(\mu)}(b_\sigma \cdot Sd_#(\partial \sigma)) = \partial^{Sd(\mu)} \left( b_\sigma \cdot \sum_{j=0}^{n} (-1)^j Sd_#(\partial_j \sigma) \right)$$

$$= \sum_{j=0}^{n} (-1)^j \partial^{Sd(\mu)} \left( b_\sigma \cdot Sd_#(\partial_j \sigma) \right)$$

$$= \sum_{j=0}^{n} (-1)^j \left( \frac{\mu(\partial_j \sigma)}{\mu(\sigma)} Sd_#(\partial_j \sigma) - b_\sigma \cdot \partial Sd_#(\partial_j \sigma) \right)$$

$$= \left( \sum_{j=0}^{n} (-1)^j \left( \frac{\mu(\partial_j \sigma)}{\mu(\sigma)} Sd_#(\partial_j \sigma) \right) \right) - b_\sigma \cdot \partial \left( Sd_#(\partial \sigma) \right)$$

(by induction) $$Sd_# \left( \sum_{j=0}^{n} (-1)^j \frac{\mu(\partial_j \sigma)}{\mu(\sigma)} \partial_j \sigma \right) - b_\sigma \cdot Sd_#(\partial \sigma)$$

$$= Sd_# \circ \partial^\mu \sigma.$$

The equality $\overset{\ast}{=} \overset{\ast}{=}$ follows from the facts that for any $(n-1)$-simplex $\tau$ in $Sd(\partial \sigma)$:

- $Sd(\mu)(\tau) = \mu(\partial \sigma)$ by (16), and $Sd(\mu)(b_\sigma \cdot \tau) = \mu(\sigma)$ by (24);
- for each $(n-2)$-face $\theta$ of $\tau$, $Sd(\mu)(b_\sigma \cdot \theta) = \mu(\sigma)$ by (24). $\square$

The following theorem is a generalization of Theorem 6.5 in the category of divisibly weighted simplicial complexes.

**Theorem 6.7.** For a divisibly weighted simplicial complex $(K, \xi)$, the chain map $Sd_# : (C_*(K), \partial^K) \rightarrow (C_*(Sd(K)), \partial^{Sd(\xi)})$ induces an isomorphism $Sd_* : H_*(K, \partial^K) \rightarrow H_*(Sd(K), \partial^{Sd(\xi)})$.

**Proof.** We assume the weight $\xi$ to be descending in the following argument. The proof of the ascending weight case is completely parallel.

We first construct a special simplicial approximation of $id_K$. Choose a total ordering $\prec$ of all the vertices of $K$ such that $\xi(v) \leq \xi(v')$ for any $v \prec v'$. Let $\sigma = \{v_0, \cdots, v_n\}$ be a simplex in $K$ where $v_0 \prec \cdots \prec v_n$. Since $(K, \xi)$ is divisibly
weighted and $\xi$ is descending, we have
\[\xi(\sigma) = \xi(v_0) \mid \cdots \mid \xi(v_n).\]

Then we define a simplicial map $\pi^\xi : Sd(K) \to K$ by
\[
\pi^\xi : Sd(K)^{(0)} \longrightarrow K^{(0)}
\]
\[b_\sigma \mapsto v_0 \in \sigma.\]

Note that $\pi^\xi : Sd(K) \to K$ is a simplicial approximation of the identity map $\text{id}_{|K|} : |K| = |Sd(K)| \to |K|$. Moreover, for each $\sigma$ of $K$,
\[Sd(\xi)(b_\sigma) = \xi(\sigma) = \xi(v_0) = \xi(\pi^\xi(b_\sigma)).\]
So $\pi^\xi$ preserves the weight of every vertex of $Sd(K)$. Then since $(K, \xi)$ and $(Sd(K), Sd(\xi))$ are both divisively weighted, $\pi^\xi$ is a weight-preserving simplicial map.

**Claim-1:** The chain map $Sd_\# \circ \pi^\xi_\# : (C_*(Sd(K)), \partial^{Sd(\xi)}) \to (C_*(Sd(K)), \partial^{Sd(\xi)})$ is carried by an acyclic carrier $\Phi^\xi$ on $C_*(Sd(K), \partial^{Sd(\xi)})$ defined by
\[
\Phi^\xi([b_{\sigma_0} \cdots b_{\sigma_l}]) = (C_*(Sd(\sigma_l)), \partial^{Sd(\xi)}), \sigma_0 \subset \cdots \subset \sigma_l \in K.
\]
Indeed, $\Phi^\xi$ is an acyclic carrier since each $(C_*(Sd(\sigma_l)), \partial^{Sd(\xi)})$ is acyclic by Lemma 5.6. In addition, each vertex $b_{\sigma_l}$ of the simplex $[b_{\sigma_0} \cdots b_{\sigma_l}]$ in (26) is clearly mapped to a vertex of $\sigma_l$. This implies that $Sd_\# \circ \pi^\xi_\#$ is carried by $\Phi^\xi$. So Claim-1 is proved.

Moreover, since $\pi^\xi$ is weight-preserving and $Sd(v) = v$ for every vertex $v$ of $K$, the chain map $Sd_\# \circ \pi^\xi_\#$ extends to an augmentation-preserving map on the augmented chain complexes $(Sd(K), \partial^{Sd(\xi)}, \varepsilon^{Sd(\xi)})$. Then since $\text{id}_{C_*(Sd(K))}$ is also an augmentation-preserving chain map carried by $\Phi^\xi$, Theorem 6.4 implies
\[Sd_* \circ \pi^\xi_* = \text{id} : H_*(Sd(K), \partial^{Sd(\xi)}) \to H_*\big(Sd(K), \partial^{Sd(\xi)}\big).
\]

Note that we may have $\widetilde{H}_{-1}(Sd(\sigma_l), \partial^{Sd(\xi)}) \neq 0$ (see Remark 5.5). But since $\widetilde{H}_j(Sd(\sigma_l), \partial^{Sd(\xi)}) = 0$ for all $j \geq 0$ (by Lemma 5.6), the two chain maps carried by $\Phi$ being augmentation-preserving is sufficient for us to construct a chain homotopy between them.

**Claim-2:** $\pi^\xi_\# \circ Sd_\# = \text{id}_{C_*(K)} : (C_*(K), \partial^\xi) \to (C_*(K), \partial^\xi)$. 

\[\bbox\]
Obviously, $\pi_\#^\xi \circ Sd_\#$ agrees with $\text{id}_{C_\#(K)}$ on $C_0(K)$. Assume that $\pi_\#^\xi \circ Sd_\#$ agrees with $\text{id}_{C_\#(K)}$ on $C_{<n}(K)$, $n \geq 1$. For an oriented $n$-simplex $\sigma = [v_0, \cdots, v_n]$ of $K$,

\begin{equation}
\pi_\#^\xi \circ Sd_\#(\sigma) \overset{(23)}{=} \pi_\#^\xi (b_\sigma \cdot (Sd_\#(\partial \sigma))) \overset{(25)}{=} v_0 \cdot (\pi_\#^\xi \circ Sd_\#(\partial \sigma))
\end{equation}

(by induction) $= v_0 \cdot \partial \sigma = v_0 \cdot \left( \sum_{j=0}^n (-1)^j[v_0, \cdots, \hat{v}_j, \cdots, v_n] \right)$

$= v_0 \cdot [v_1, \cdots, v_n] = \sigma.$

So Claim-2 is proved. Then $\pi_\#^\xi$ is a chain homotopy inverse of $Sd_\#$ and so

\begin{equation}
\pi_\#^\xi \circ Sd_\# = \text{id} : H_\#(K, \partial^\xi) \to H_\#(K, \partial^\xi).
\end{equation}

Therefore, $Sd_\#$ is an isomorphism. \hfill \square

6.4. Invariance of AW-homology and DW-homology.

In this section, we prove the invariance of AW-homology and DW-homology of a weighted polyhedron under isomorphisms and more generally under W-homotopy equivalences. To do that, we first define the following notion.

**Definition 6.8 (W-simplicial Approximation).** Suppose $(X, \lambda)$ and $(X', \lambda')$ are two weighted polyhedra of the same type. Let $(K, \xi)$ and $(K', \xi')$ be divisibly weighted triangulations of $(X, \lambda)$ and $(X', \lambda')$, respectively. For a continuous map $f : X \to X'$, if $\varrho : K \to K'$ is a simplicial approximation of $f$ and moreover $\varrho$ is a morphism from $(K, \xi)$ to $(K', \xi')$, then we call $\varrho$ a $W$-simplicial approximation of $f$.

The following lemma is a bit surprising at first look. But it follows naturally from the definition of W-continuous map.

**Lemma 6.9.** Let $(X, \lambda)$ and $(X', \lambda')$ be two weighted polyhedra of the same type. Let $(K, \xi)$ and $(K', \xi')$ be divisibly weighted triangulations of $(X, \lambda)$ and $(X', \lambda')$, respectively. For a W-continuous map $f : (X, \lambda) \to (X', \lambda')$, if $\varrho : K \to K'$ is a simplicial approximation of $f : X \to X'$, then $\varrho$ must be a $W$-simplicial approximation of $f$ and $\overline{\varrho} : X \to X'$ is W-homotopic to $f$.

**Proof.** We assume that $(X, \lambda)$ and $(X', \lambda')$ are of ascending type below. The proof of the descending type case is completely parallel.

First of all, since $\varrho : K \to K'$ is a simplicial approximation of $f : X \to X'$,

\begin{equation}
\overline{\varrho}(x) \in |\text{Car}^\xi_{K'}(f(x))|, \text{ for all } x \in X = |K|.
\end{equation}

Besides, since $f$ is W-continuous, by Definition 4.10 we obtain

\begin{equation}
\lambda'(f(x)) \| \lambda(x), \text{ for all } x \in X.
\end{equation}
So for any vertex \( v \) of \( K \), we deduce from (29) that \( \varrho(v) \) is a vertex of \( \text{Car}_{K'}(f(v)) \) and \( \lambda'(f(v)) | \lambda(v) = \xi(v) \). Moreover since \((K', \xi')\) is an ascending weighted simplicial complexes by our assumption, we have
\[
\xi'(\varrho(v)) | \xi'(\text{Car}_{K'}(f(v))) = \lambda'(f(v)).
\]
So further by (30), we obtain \( \xi'(\varrho(v)) | \lambda(v) = \xi(v) \). This implies that \( \varrho \) is a morphism from \((K, \xi)\) to \((K', \xi')\) since \((K, \xi)\) and \((K', \xi')\) are both divisibly weighted (see Lemma 2.11).

Next, we prove that \( \overline{\varrho} \) is W-homotopic to \( f \). Indeed, the line segment between \( \overline{\varrho}(x) \) and \( f(x) \) in \( |\text{Car}_{K'}(f(x))| \) for any \( x \in X \) determines a homotopy \( H \) from \( \overline{\varrho} \) to \( f \), that is
\[
H(x, t) = (1 - t)\overline{\varrho}(x) + tf(x), \quad t \in [0, 1], \quad x \in X.
\]
Moreover, by (29), \( \text{Car}_{K'}(\overline{\varrho}(x)) \) is a face of \( \text{Car}_{K'}(f(x)) \). This implies
\[
\lambda'(\overline{\varrho}(x)) = \xi'(\text{Car}_{K'}(\overline{\varrho}(x))) | \xi'(\text{Car}_{K'}(f(x))) = \lambda'(f(x)).
\]
For \( 0 < t \leq 1 \), observe that \( H(x, t) \) lies in the relative interior of \( |\text{Car}_{K'}(f(x))| \), which implies that \( \text{Car}_{K'}(H(x, t)) = \text{Car}_{K'}(f(x)) \) and so \( \lambda'(H(x, t)) = \lambda'(f(x)) \).

So for any \( x \in X \) and \( t \in [0, 1] \),
\[
\lambda'(H(x, t)) | \lambda'(f(x)) | \lambda(x) = \lambda'(x, t)).
\]
Therefore, \( H \) is a W-continuous map and so \( H \) is a W-homotopy from \( \overline{\varrho} \) to \( f \). \( \square \)

**Theorem 6.10** (Existence of W-simplicial Approximation). Suppose \((X, \lambda)\) and \((X', \lambda')\) are two weighted polyhedra of the same type where \( X \) is compact and \( f : (X, \lambda) \rightarrow (X', \lambda') \) is a W-continuous map. Let \((K, \xi)\) and \((K', \xi')\) be divisibly weighted triangulations of \((X, \lambda)\) and \((X', \lambda')\), respectively.

(a) There always exists a large enough integer \( m \) such that \( f \) has a W-simplicial approximation \( \varrho : (Sd^m(K), \partial Sd^m(\xi)) \rightarrow (K', \xi') \).

(b) If \( \varrho' : (Sd^m(K), \partial Sd^m(\xi)) \rightarrow (K', \xi') \) is another W-simplicial approximation of \( f \), then the induced chain maps
\[
\varrho_\#, \varrho'_\# : C_*(Sd^m(K), \partial Sd^m(\xi)) \rightarrow (C_*(K'), \partial \xi')
\]
are chain homotopic. So \( \varrho_* = \varrho'_* : H_*(Sd^m(K), \partial Sd^m(\xi)) \rightarrow H_*(K', \partial \xi') \).

**Proof.** (a) It follows immediately from Theorem 6.2 and Lemma 6.9.

(b) For a simplex \( \sigma = \{v_0, \cdots, v_n\} \) of \( Sd^m(K) \), let \( x \in |\sigma|^\circ \) be a relative interior point of \(|\sigma|\). Then since \( \varrho \) and \( \varrho' \) are both W-simplicial approximations of \( f \), the vertices \( \varrho(v_0), \varrho'(v_0), \cdots, \varrho'(v_n) \) must all belong to \( \text{Car}_{K'}(f(x)) \). Let \( \Phi(\sigma) \) be the face of \( \text{Car}_{K'}(f(x)) \) spanned by \( \varrho(v_0), \varrho'(v_0), \cdots, \varrho'(v_n) \). Moreover, since \( \xi' \) is a divisible weight, the chain complex \( C_*(\Phi(\sigma), \xi') \) is acyclic by Lemma 5.1. Then algebraically, \( \Phi \) defines an acyclic carrier that carries both
\[ \varrho_{\#} \text{ and } \varrho'_{\#}. \] So by Lemma 2.7 and Theorem 6.4, there exists a chain homotopy between \( \varrho_{\#} \) and \( \varrho'_{\#} \) which is also carried by \( \Phi \).

Note that in this section, we often assume a weighted polyhedron to be compact in our discussion just for better display of the main ideas in the proof. Later in section 6.5, we will outline the proof of the non-compact case.

**Definition 6.11.** Let \( f : (X, \lambda) \to (X', \lambda') \) be a W-continuous map between two weighted polyhedra \( (X, \lambda) \) and \( (X', \lambda') \) where \( X \) is compact. Let \( (K, \xi) \) and \( (K', \xi') \) be divisibly weighted triangulations of \( (X, \lambda) \) and \( (X', \lambda') \), respectively. If \( \varrho : Sd^m(K) \to K' \) is a W-simplicial approximation of \( f \), define

\[
\varrho_* := \varrho_* \circ Sd_* : H_*(K, \partial \xi) \to H_*(Sd^m(K), \partial Sd(\xi)) \to H_*(K', \partial \xi').
\]

By the following lemma, \( \varrho_* \) is well-defined (i.e. independent on the W-simplicial approximation \( \varrho \)).

**Lemma 6.12.** Let \( (X, \lambda) \) and \( (X', \lambda') \) be two weighted polyhedra where \( X \) is compact. Suppose \( f : (X, \lambda) \to (X', \lambda') \) is a W-continuous map. If \( \varrho : (Sd^m(K), Sd^m(\xi)) \to (K', \xi') \), \( \kappa : (Sd^{m+r}(K), Sd^{m+r}(\xi)) \to (K', \xi') \) are both W-simplicial approximations of \( f \), then \( \varrho_* \circ Sd_*^m = \kappa_* \circ Sd_*^{m+r} \).

**Proof.** By the proof of Theorem 6.7, we have the following diagram

\[
\begin{array}{c}
H_*(K, \xi) \\
\downarrow Sd^m \\
H_*(Sd^m(K), Sd^m(\xi)) \\
\downarrow Sd^r \\
H_*(Sd^{m+r}(K), Sd^{m+r}(\xi))
\end{array}
\]

where \( \pi^\xi \) is defined as in (25) which is weight-preserving. Clearly,

\[
\varrho \circ (\pi^u)^r : (Sd^{m+r}(K), Sd^{m+r}(\xi)) \to (K', \xi')
\]

is also a W-simplicial approximation of \( f \). So by Theorem 6.10 (b),

\[
\kappa_* = \varrho_* \circ (\pi^\xi)^r.
\]

Then we obtain

\[
\kappa_* \circ Sd_*^{m+r} = \varrho_* \circ (\pi^u)^r \circ Sd_*^r \circ Sd_*^{m} \overset{(28)}{=} \varrho_* \circ Sd_*^{m}.
\]

The lemma is proved.

**Lemma 6.13.** Let \((X, \lambda), (X', \lambda')\) and \((X'', \lambda'')\) be compact weighted polyhedra.
(a) For any divisibly weighted triangulation \((K, \xi)\) of \((X, \lambda)\) and the identity map \(\text{id}_X : X \to X\), the induced map \((\text{id}_X)_* : H_*(K, \partial K) \to H_*(K, \partial K)\) is the identity.

(b) For \(W\)-continuous maps \(f : (X, \lambda) \to (X', \lambda')\) and \(g : (X', \lambda') \to (X'', \lambda'')\), \((g \circ f)_* = g_* \circ f_* : H_*(K, \partial K) \to H_*(K, \partial K)\), where \((K, \xi), (K', \xi')\) and \((K'', \xi'')\) are divisibly weighted triangulations of \((X, \lambda), (X', \lambda')\) and \((X'', \lambda'')\), respectively.

Proof. (a) Take \(\text{id}_K : (K, \xi) \to (K, \xi)\) as the W-simplicial approximation of \(\text{id}_X\).

(b) Let \(\psi : (Sd^m(K'), Sd^m(\xi')) \to (K'', \xi'')\) be a W-simplicial approximation of \(g\). Then using \((Sd^m(K'), Sd^m(\xi'))\) as the triangulation of \((X', \lambda')\), we can further construct a W-simplicial approximation \(\phi : (Sd^m(K), Sd^m(\xi)) \to (Sd^m(K'), Sd^m(\xi'))\) for \(f\). Then since \(\pi^\xi\) is weight-preserving, \((\pi^\xi)_r \circ \phi : (Sd^m(K), Sd^m(\xi)) \to (K', \xi')\) is also a W-simplicial approximation of \(f\) (see the following diagram).

\[
\begin{array}{ccc}
H_*(K, \partial K) & \xrightarrow{f_*} & H_*(K', \partial K') \\
Sd^m(K) & \xrightarrow{(\pi^\xi)_r} & Sd^m(K') \\
\downarrow{Sd^m(\xi)} & & \downarrow{Sd^m(\xi')} \\
H_*(Sd^m(K), \partial Sd^m(\xi)) & \xrightarrow{\phi_*} & H_*(Sd^m(K'), \partial Sd^m(\xi')) \\
& \xrightarrow{\psi_*} & \\
& & H_*(K'', \partial K'')
\end{array}
\]

Since \(\psi \circ \phi : (Sd^m(K), Sd^m(\xi)) \to (K'', \xi'')\) is a W-simplicial approximation of \(g \circ f\), we have

\[
(g \circ f)_* = (\psi \circ \phi)_* \circ Sd^m_*,
\]
\[
g_* \circ f_* = (\psi_* \circ Sd^m_*) \circ ((\pi^\xi)_r \circ \phi_* \circ Sd^m_*)
\]
\[
= (\psi_* \circ \phi_*)_* \circ Sd^m_*, \quad \text{(28)}
\]

So \((g \circ f)_* = g_* \circ f_*\). The lemma is proved. \(\square\)

**Theorem 6.14.** Suppose \(f : (X, \lambda) \to (X', \lambda')\) is an isomorphism between two compact weighted polyhedra \((X, \lambda)\) and \((X', \lambda')\). If \((K, \xi)\) and \((K', \xi')\) are divisibly weighted triangulations of \((X, \lambda)\) and \((X', \lambda')\), respectively, then the induced map \(f_* : H_*(K, \partial K) \to H_*(K', \partial K')\) is an isomorphism.

Proof. Let \(g = f^{-1} : (X', \lambda') \to (X, \lambda)\). Then by Lemma 6.13,

\[
g_* \circ f_* = (g \circ f)_* = (\text{id}_X)_*, \quad f_* \circ g_* = (f \circ g)_* = (\text{id}_{X'})_*
\]

are both identity maps. So \(f_*\) is an isomorphism. \(\square\)

We can obtain the following corollary immediately from Theorem 6.14.
Corollary 6.15. If \((K, \xi)\) and \((K', \xi')\) are divisibly weighted triangulations of the same compact weighted polyhedron \((X, \lambda)\), then the identity map \(\text{id}_X : X \to X\) determines an isomorphism \((\text{id}_X)_* : H_*(K, \partial w) \to H_*(K', \partial w')\).

This corollary implies that AW-homology and DW-homology of a compact weighted polyhedron in Definition 4.16 are independent on the divisibly weighted triangulation we choose. So these two notions are both well-defined. Moreover, we can conclude from Theorem 6.14 that AW-homology and DW-homology are invariants of compact weighted polyhedra under isomorphisms.

Next, we prove the invariance of AW-homology and DW-homology with respect to W-homotopy equivalences of weighted polyhedra.

Lemma 6.16. Suppose \((X, \lambda)\) and \((X', \lambda')\) are compact weighted polyhedra. If two \(W\)-continuous maps \(f, g : (X, \lambda) \to (X', \lambda')\) are \(W\)-homotopic, then

\[ f_* = g_* : H_*^{AW}(X, \lambda) \to H_*^{AW}(X', \lambda'), \quad f_* = g_* : H_*^{DW}(X, \lambda) \to H_*^{DW}(X', \lambda'). \]

Proof. Let \(H : (X \times [0, 1], \lambda \times 1) \to (X', \lambda')\) be a \(W\)-homotopy from \(f\) to \(g\). Suppose \((K, \xi)\) and \((K', \xi')\) are divisibly weighted triangulations of \((X, \lambda)\) and \((X', \lambda')\), respectively. Let \(i_0, i_1 : X \to X \times [0, 1]\) be the maps:

\[ i_0(x) = (x, 0), \quad i_1(x) = (x, 1), \quad x \in X. \]

Then \(f = H \circ i_0\) and \(g = H \circ i_1\). Clearly, \(i_0\) and \(i_1\) are both morphisms of weighted simplicial complexes from \((K, \xi)\) to \((K \times [0, 1], \xi \times 1)\).

Claim: \((i_0)_*, (i_1)_* : (C_*(K), \partial \xi) \to (C_*(K \times [0, 1]), \partial \xi \times 1)\) are chain homotopic.

Consider the function \(\Phi\) assigning, to each simplex \(\sigma\) of \(K\), the subchain complex \((C_*(\sigma \times [0, 1]), \partial \xi \times 1)\) of \((C_*(K \times [0, 1]), \partial \xi \times 1)\). By Proposition 2.10 and Lemma 5.1,

\[ H_*(\sigma \times [0, 1], \partial \xi \times 1) \cong H_*(\sigma, \partial \xi) \cong \begin{cases} \mathbb{Z}, & \text{if } j = 0; \\ 0, & \text{if } j > 1. \end{cases} \]

So \(\Phi\) is an (algebraic) acyclic carrier from \((C_*(K), \partial \xi)\) to \((C_*(K \times [0, 1]), \partial \xi \times 1)\). Moreover, both \((i_0)_*\) and \((i_1)_*\) are carried by \(\Phi\) since both \(i_0(\sigma) = \sigma \times \{0\}\) and \(i_1(\sigma) = \sigma \times \{1\}\) belong to \(\Phi(\sigma)\). Then it follows from Theorem 6.4 that \((i_0)_*\) is chain homotopic to \((i_1)_*\). The claim is proved.

By the above claim, we obtain \((i_0)_* = (i_1)_* : H_*(K, \partial \xi) \to H_*(K \times [0, 1], \partial \xi \times 1)\). So by the definitions of AW-homology and DW-homology and Lemma 6.13,

\[ f_* = H_* \circ (i_0)_* = H_* \circ (i_1)_* = g_* \]

The lemma is proved. \(\square\)
Theorem 6.17. Let \((X, \lambda)\) and \((X', \lambda')\) be compact weighted polyhedra. If \((X, \lambda)\) and \((X', \lambda')\) are \(W\)-homotopy equivalent, then there are isomorphisms
\[ H^w_*(X, \lambda) \cong H^w_* (X', \lambda'), \quad H^c_*(X, \lambda) \cong H^c_* (X', \lambda'). \]

Proof. This follows easily from Lemma 6.13 and Lemma 6.16.

6.5. The non-compact case.

If a weighted polyhedron \((X, \lambda)\) is not compact, we have to use generalized barycentric subdivisions to obtain a simplicial approximation of a \(W\)-continuous map \(f : (X, \lambda) \to (X', \lambda')\). Then we need to prove the following results parallel to Lemma 6.6 and Theorem 6.7: for a subcomplex \(K_0\) of a simplicial complex \(K\), the subdivision of \(K\) holding \(K_0\) fixed, denoted by \(Sd_{K_0}\), satisfies (see Remark 2.16):

(R1) For any weight function \(\mu\) on \(K\), the map
\[ (Sd_{K_0})_{\#} : \left( C_*(K), \partial \right) \to \left( C_*(Sd(K/K_0)), \partial^{Sd_{K_0}(\mu)} \right), \]
which sends any oriented \(k\)-simplex \(\sigma\) of \(K\) to the sum of all the \(k\)-simplices of \(Sd_{K_0}(\sigma)\) with the same orientation, is a chain map.

(R2) For any divisible weight \(\xi\) on \(K\), \((Sd_{K_0})_{\#}\) induces an isomorphism
\[ (Sd_{K_0})_{\#}: H_*(K, \partial) \to H_*(Sd(K/K_0), \partial^{Sd_{K_0}(\xi)}). \]

The proof of (R1) is almost identical to Lemma 6.6 except that we only take cones at the barycenters \(b_\sigma\) of simplices \(\sigma\) in \(K - K_0\). The proof of (R2) is also similar to the proof of Theorem 6.7. We can first construct a weight-preserving simplicial map \(\pi_{K_0}^\xi : Sd(K/K_0) \to K\) which is a simplicial approximation of the identity map \(id_K\) and satisfies \((\pi_{K_0}^\xi)_{\#} \circ (Sd_{K_0})_{\#} = id_{C_*(K)}\). Then we can use the result in (21) to define an acyclic carrier (analogous to \(\Phi^\xi\) in (26)) on \((C_*(Sd(K/K_0)), \partial^{Sd_{K_0}(\xi)})\) to prove that \((Sd_{K_0})_{\#} \circ (\pi_{K_0}^\xi)_{\#}\) is chain homotopic to \(id_{C_*(Sd(K/K_0))}\).

Using (R1) and (R2), we can prove the following theorem which generalizes Theorem 6.10. We leave the details of the proof as an exercise to the reader.

Theorem 6.18 (Existence of W-simplicial Approximation – General Version). Suppose \((X, \lambda)\) and \((X', \lambda')\) are two weighted polyhedra of the same type and \(f : (X, \lambda) \to (X', \lambda')\) is a \(W\)-continuous map. Let \((K, \xi)\) and \((K', \xi')\) be divisibly weighted triangulations of \((X, \lambda)\) and \((X', \lambda')\), respectively.

(a) There always exists a generalized barycentric subdivision \(\tilde{Sd}(K)\) of \(K\) and a \(W\)-simplicial approximation \(\varrho : (\tilde{Sd}(K), \tilde{Sd}(\xi)) \to (K', \xi')\) of \(f\).

(b) If \(\varrho' : (\tilde{Sd}(K), \tilde{Sd}(\xi)) \to (K', \xi')\) is another \(W\)-simplicial approximation of \(f\), then the chain maps \(\varrho_{\#}, \varrho'_{\#} : C_*(\tilde{Sd}(K), \partial^{\tilde{Sd}(\xi)}) \to C_*(K', \partial')\) are chain homotopic. Hence \(\varrho_{\#} = \varrho'_{\#} : H_*(\tilde{Sd}(K), \partial^{\tilde{Sd}(\xi)}) \to H_*(K', \partial')\).
The remaining proofs of the well-definedness of AW-homology of DW-homology for non-compact weighted polyhedra and their invariance under isomorphisms and W-homotopy equivalences are completely parallel to the compact case. Indeed, we can just mimic the proof of the topological invariance of ordinary simplicial homology in [13, §18] and remember that we only use generalized barycentric subdivisions in our constructions since we need to keep the weight function always divisible (see Remark 2.16). Then any simplicial approximation of \( f \) from such a subdivision of \( K \) to \( K' \) is automatically a W-simplicial approximation of \( f \) (see Lemma 6.9).

Another issue we need to clarify is: if \( \mu \) is a descending weight on a non-compact simplicial complex \( K \), it is possible that \( \mu \) is not finite and so the augmentation \( \varepsilon^\mu \) of \( \mu \) (see (6)) is not well-defined globally. On the other hand, simplicial homology theory is compactly supported, which means that any simplicial chain is contained in a compact subcomplex of \( K \). So when we use the acyclic carrier theorem to show two chain maps are chain homotopic in our proofs of (R2) and Theorem 6.18, we only need to construct the chain homotopy in compact subcomplexes which do have (local) augmentations of \( \mu \). This is enough for us to construct all the desired chain homotopies and carry out the proofs.

7. AW-homology and DW-homology with coefficients

Let \( (K, \mu) \) be a weighted simplicial complex. Given any abelian group \( G \), we can apply the tensor product functor \( \otimes G \) to \((C_*(K), \partial^\mu)\) to obtain a chain complex denoted by

\[
(C_*(K; G), \partial^\mu) := (C_*(K) \otimes G, \partial^\mu \otimes \text{id}_G).
\]

Let \( H_*(K, \partial^\mu; G) \) denote the homology group of \((C_*(K; G), \partial^\mu)\). By the algebraic universal coefficient theorem of homology (see Hatcher [10, §3.A]),

\[
H_j(K, \partial^\mu; G) \cong (H_j(K, \partial^\mu) \otimes G) \oplus \text{Tor}(H_{j-1}(K, \partial^\mu), G), \ j \in \mathbb{Z}.
\]

If \( A \) is a simplicial subcomplex of \( K \), we can similarly define \( H_*(K, A, \partial^\mu; G) \) which is the homology group of the chain complex

\[
(C_*(K, A; G), \partial^\mu) := ((C_*(K)/C_*(A)) \otimes G, \partial^\mu \otimes \text{id}_G).
\]

For a weighted polyhedron \((X, \lambda)\), the AW-homology and DW-homology of \((X, \lambda)\) with \( G \)-coefficients are defined by the corresponding notions of a divisibly weighted triangulation \((K, \xi)\) of \((X, \lambda)\), respectively, denoted by \( H_*^{AW}(X, \lambda; G) \) and \( H_*^{DW}(X, \lambda; G) \). Then by the universal coefficient theorem,

\[
H_j^{AW}(X, \lambda; G) \cong (H_j^{AW}(X, \lambda) \otimes G) \oplus \text{Tor}(H_{j-1}^{AW}(X, \lambda), G), \ j \in \mathbb{Z};
\]

\[
H_j^{DW}(X, \lambda; G) \cong (H_j^{DW}(X, \lambda) \otimes G) \oplus \text{Tor}(H_{j-1}^{DW}(X, \lambda), G), \ j \in \mathbb{Z}.
\]
The following theorem tells us that with coefficients in some special fields, the AW-homology and DW-homology of a weighted polyhedron \((X, \lambda)\) are isomorphic to the ordinary simplicial (or singular) homology of \(X\).

**Theorem 7.1.** Let \((X, \lambda)\) be a weighted polyhedron. If the character of a field \(\mathbb{F}\) is relatively prime to the weights of all points of \(X\), then \(H_{\ast}^{AW}(X, \lambda; \mathbb{F})\) and \(H_{\ast}^{DW}(X, \lambda; \mathbb{F})\) are both isomorphic to \(H_{\ast}(X; \mathbb{F})\).

**Proof.** Let \((K, \xi)\) be a divisibly weighted triangulation of \((X, \lambda)\). For any \(n \geq 0\), we can define a linear map for the \(n\)-skeleton \(K^{(n)}\) of \(K\), as follows

\[
\Xi : \left(\left(\left(C_{\ast}(K^{(n)}; \mathbb{F}), \partial^{\xi}\right) \to C_{\ast}(K^{(n)}; \mathbb{F})\right)\right)\]

\[
\sigma \mapsto \begin{cases} 
  w(\sigma) \cdot \sigma & \text{if } \xi \text{ is ascending;} \\
  \frac{1}{w(\sigma)} \cdot \sigma & \text{if } \xi \text{ is descending.}
\end{cases}
\]

It is easy to check that \(\Xi\) is a chain map by the definition of \(\partial^{\xi}\) (see (1) and (2)). Note that here \(\frac{1}{w(\sigma)}\) is valid because of the assumption on the character of \(\mathbb{F}\).

By the long exact sequence of weighted simplicial homology ([8, Theorem 2.3]), we obtain a commutative diagram as follows (the coefficients \(\mathbb{F}\) are omitted).

\[
\begin{array}{c}
H_{j+1}(K^{(n)}, K^{(n-1)}, \mathbb{F}) \xrightarrow{\Xi} H_j(K^{(n-1)}, \mathbb{F}) \xrightarrow{\Xi} H_j(K^{(n)}, \mathbb{F}) \xrightarrow{\Xi} H_j(K^{(n)}, K^{(n-1)}, \mathbb{F}) \xrightarrow{\Xi} H_{j-1}(K^{(n-1)}, \mathbb{F}) \\
H_{j+1}(K^{(n)}, K^{(n-1)}) \xrightarrow{\Xi} H_j(K^{(n-1)}) \xrightarrow{\Xi} H_j(K^{(n)}) \xrightarrow{\Xi} H_j(K^{(n)}, K^{(n-1)}) \xrightarrow{\Xi} H_{j-1}(K^{(n-1)}).
\end{array}
\]

For any \(n\)-simplex \(\sigma\) in \(K\), we can directly compute from the definition that

\[
\Xi_{\ast} : H_n(\sigma, \partial \sigma, \overline{\partial}^{\xi}; \mathbb{F}) \cong \mathbb{F} \to \mathbb{F} \cong H_n(\sigma, \partial \sigma; \mathbb{F})
\]

is given by \(\mathbb{F} \xrightarrow{\xi(\sigma)} \mathbb{F}\) if \(\xi\) is ascending and \(\mathbb{F} \xrightarrow{\overline{\xi}(\sigma)} \mathbb{F}\) if \(\xi\) is descending. So the homomorphism \(\Xi_{\ast} : H_n(\sigma, \partial \sigma, \overline{\partial}^{\xi}; \mathbb{F}) \to H_n(\sigma, \partial \sigma; \mathbb{F})\) is an isomorphism. It follows that \(\Xi_{\ast} : H_\ast(K^{(n)}, K^{(n-1)}, \overline{\partial}^{\xi}; \mathbb{F}) \to H_\ast(K^{(n)}, K^{(n-1)}; \mathbb{F})\) is an isomorphism. Then by the five-lemma (see [10, p. 129]), we can inductively prove that \(H_\ast(K^{(n)}, \overline{\partial}^{\xi}; \mathbb{F})\) is isomorphic to \(H_\ast(K^{(n)}; \mathbb{F})\) for all \(n \geq 0\). So \(\Xi_{\ast} : H_\ast(K, \overline{\partial}^{\xi}; \mathbb{F}) \to H_\ast(K; \mathbb{F})\) is an isomorphism. Applying the same argument to the inversion \(\xi\) of \(\xi\), we also obtain an isomorphism from \(H_\ast(K, \partial^{\xi}; \mathbb{F})\) to \(H_\ast(K; \mathbb{F})\). Then the theorem follows. \(\square\)

In particular, for a weighted polyhedron \((X, \lambda)\) we have isomorphisms with respect to the rational coefficients \(\mathbb{Q}\):

\[
H_{\ast}^{AW}(X, \lambda; \mathbb{Q}) \cong H_{\ast}^{DW}(X, \lambda; \mathbb{Q}) \cong H_{\ast}(X; \mathbb{Q}).
\]
The above isomorphisms imply that the free part of $H^*_AW(X, \lambda)$ and $H^*_DW(X, \lambda)$ is actually a homotopy invariant of the topological space $X$. More specifically,

$$H^*_AW(X, \lambda) \cong \mathbb{Z}^{\beta_j(X)} \oplus T^*_AW(X, \lambda), \quad H^*_DW(X, \lambda) \cong \mathbb{Z}^{\beta_j(X)} \oplus T^*_DW(X, \lambda)$$

where $\beta_j(X)$ is the $j$-th Betti number of $X$ and $T^*_AW(X, \lambda)$ and $T^*_DW(X, \lambda)$ are torsion groups. So it is the torsion parts of $H^*_AW(X, \lambda)$ and $H^*_DW(X, \lambda)$ that can really give us new information of the weighted polyhedron structure of $(X, \lambda)$.

### 8. Examples

#### 8.1. Computation strategies.

First of all, the excision and Mayer-Vietoris sequence in weighted simplicial homology (see [8, Theorem 2.3 and Corollary 2.3.1]) naturally induce the excision and Mayer-Vietoris sequence in AW-homology and DW-homology of weighted polyhedra, which are the basic tools for the computation. In addition, it is shown in [23] that the method of discrete Morse theory can also be used to compute weighted simplicial homology.

Note that the definition of weighted homology groups also makes sense for any $\Delta$-complex with a weight function (either ascending or descending). The reader is referred to [10] for the definition of $\Delta$-complex. So a very useful strategy for our computation is: when we compute the AW-homology and DW-homology of a weighted polyhedron $(X, \lambda)$ via a divisibly weighted triangulation $(K, \xi)$, we can replace the simplicial complex $K$ by a $\Delta$-complex $L$ as long as $(L, \xi)$ also consists of divisibly weighted simplices. We readily call such $(L, \xi)$ a *divisibly weighted $\Delta$-triangulation* of $(X, \lambda)$. In many cases, using $\Delta$-complexes can significantly reduce the number of generators of a weighted simplicial chain complex and hence simplify the computation.

Another useful notation for our computation is algebraic mapping cone (see Davis and Kirk [7, Section 11]).

**Definition 8.1** (Algebraic Mapping Cone). Let $C_* = (C_p, \partial_p)$ and $C'_* = (C'_p, \partial'_p)$ be chain complexes. For a chain map $\phi : C_* \to C'_*$, the algebraic mapping cone of $\phi$ is the chain complex $\text{Cone}(\phi)$ where

$$\text{Cone}(\phi)_p = C_{p-1} \oplus C'_p, \quad \forall p \in \mathbb{Z},$$

with boundary map $d_p = \begin{pmatrix} -\partial_{p-1} & 0 \\ \phi & \partial'_p \end{pmatrix}$.

In particular, the algebraic cone on a chain complex $C_*$ is

$$\text{Cone}(C_*) := \text{Cone}(\text{id}_{C_*} : C_* \to C_*).$$

**Lemma 8.2** (see [7, Lemma 11.26]). *The algebraic cone on a free chain complex is always acyclic.*
Remark 8.3. For a simplex $\sigma$, it is easy to see that the ordinary simplicial chain complex $C_*(Sd(\sigma))$ is isomorphic to the algebraic cone of $C_*(Sd(\partial \sigma))$. But given a weight function $\mu$ on $\sigma$, the weighted simplicial chain complex $\left(C_*(Sd(\sigma)), \partial^{Sd(\mu)}\right)$ may not be acyclic. So in general, $\left(C_*(Sd(\sigma)), \partial^{Sd(\mu)}\right)$ may not be isomorphic to the algebraic cone on $\left(C_*(\partial \sigma), \partial^{Sd(\mu)}\right)$. If we examine the definition carefully, we can see that for a simplex $\tau$ of $Sd(\partial \sigma)$, $Sd(\mu)(b_\sigma \cdot \tau) = \mu(\sigma)$ may not agree with $Sd(\mu)(\tau)$. For this reason, $\left(C_*(Sd(\sigma)), \partial^{Sd(\mu)}\right)$ could be different from the algebraic cone on $\left(C_*(\partial \sigma), \partial^{Sd(\mu)}\right)$.

8.2. Examples of Computation of AW-homology and DW-homology.

In this section, we compute the AW-homology and DW-homology of some concrete examples of orbifolds and pseudo-orbifolds in dimension one and two. The reader is referred to [19, 18, 6] for more information on two-dimensional orbifolds.

In the following, “gcd” and “lcm” are the abbreviations for greatest common divisor and least common multiple, respectively.

Example 8.4. It is well known that any one-dimensional compact connected orbifold is isomorphic to one of the following cases (see [6]).

(a) $S^1$ or $[0, 1]$ without singular points.

(b) An interval $I = [x_1, x_2] \subset \mathbb{R}^1$ with only one singular point $x_1$ where $G_{x_1} = \mathbb{Z}_2$.

(c) An interval $I' = [x_1, x_2] \subset \mathbb{R}^1$ with two singular points $x_1, x_2$ where $G_{x_1} = G_{x_2} = \mathbb{Z}_2$.

Note that $I$ is a descending divisibly weighted 1-simplex while $I'$ is not. By Definition 3.3, we can easily obtain

$$H^j_{AW}(I) \cong H^j_{AW}(I') \cong \begin{cases} \mathbb{Z}, & \text{if } j = 0; \\ 0, & \text{if } j \geq 1. \end{cases}$$

$$H^j_{DW}(I) \cong \begin{cases} \mathbb{Z}, & \text{if } j = 0; \\ 0, & \text{if } j \geq 1. \end{cases} \quad H^j_{DW}(I') \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}, & \text{if } j = 0; \\ 0, & \text{if } j \geq 1. \end{cases}$$

There are much more one-dimensional pseudo-orbifolds based on an interval than orbifolds because the weights of the singular points in a pseudo-orbifold could be arbitrary positive integers. Indeed, let $I_{(k_1, k_2)}$ denote the pseudo-orbifold $([x_1, x_2], \lambda)$ (see Figure 2) with only two singular points $x_1$ and $x_2$ where

$$\lambda(x_1) = k_1 \geq 2, \quad \lambda(x_2) = k_2 \geq 2.$$ 

We can add a vertex $x_0$ to the interval to obtain a divisibly weighted triangulation of $I_{(k_1, k_2)}$ with two 1-simplices $\sigma_1$ and $\sigma_2$. Then the weighted chain complexes for
the AW-homology and DW-homology of \( I_{(k_1, k_2)} \) are, respectively:

\[
\begin{pmatrix}
\sigma_1 & \sigma_2 \\
 x_0 & k_1 \\
 x_1 & 1 \\
 x_2 & 0 \\
\end{pmatrix}
\begin{pmatrix}
\sigma_1 & \sigma_2 \\
 x_0 & 1 \\
 x_1 & k_1 \\
 x_2 & 0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
\sigma_1 & \sigma_2 \\
 x_0 & k_2 \\
 x_1 & 0 \\
 x_2 & 1 \\
\end{pmatrix}
\begin{pmatrix}
\sigma_1 & \sigma_2 \\
 x_0 & 1 \\
 x_1 & 0 \\
 x_2 & k_2 \\
\end{pmatrix}
\]

Then it is easy to compute

\[
H^A_W (I_{(k_1, k_2)}) \cong \begin{cases} \mathbb{Z}, & \text{if } j = 0; \\ 0, & \text{if } j \geq 1. \end{cases}
\]

\[
H^D_W (I_{(k_1, k_2)}) \cong \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/\gcd(k_1, k_2)\mathbb{Z}, & \text{if } j = 0; \\ 0, & \text{if } j \geq 1. \end{cases}
\]

**Example 8.5.** In Figure 3, we have a 1-dimensional pseudo-orbifold \((S^1, \lambda)\), denoted by \( S^1_{(k_1,\cdots,k_n)} \), which is a circle with \( n \) singular points \( v_1, \cdots, v_n, n \geq 2 \), whose weights \( \lambda(v_i) = k_i, k_i \geq 2, i = 1, \cdots, n \). By adding a vertex \( u_i \) between \( v_i \) and \( v_{i+1} \) for each \( i \), we obtain a divisibly weighted triangulation \((L, \eta)\) of \( S^1_{(k_1,\cdots,k_n)} \).

Then using the triangulation \((L, \eta)\) and the information from the isomorphisms in (32), we can compute

\[
H^A_W (S^1_{(k_1,\cdots,k_n)}) \cong \begin{cases} \text{cokernel of } \mathbb{Z}^n \xrightarrow{\alpha} \mathbb{Z}^n, & \text{if } j = 0; \\ \mathbb{Z}, & \text{if } j = 1; \\ 0, & \text{if } j \geq 2. \end{cases}
\]
where $\alpha$ is the linear map represented by the $n \times n$ matrix $A_{(k_1, \ldots, k_n)}$ defined by

$$A_{(k_1, \ldots, k_n)} = \begin{pmatrix}
-k_1 & k_2 & \cdots & 0 & 0 \\
0 & -k_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -k_{n-1} & k_n \\
k_1 & 0 & \cdots & 0 & -k_n
\end{pmatrix}.$$ 

This matrix is simplified from the $2n \times 2n$ matrix representing the boundary map $C_1(L, \partial^n) \to C_0(L, \partial^n)$. Moreover, we can similarly compute

$$H_j^{DW}(S^1_{(k_1, \ldots, k_n)}) \cong \begin{cases} 
\text{cokernel of } \mathbb{Z}^n \xrightarrow{\alpha^t} \mathbb{Z}^n, & \text{if } j = 0; \\
\mathbb{Z}, & \text{if } j = 1; \\
0, & \text{if } j \geq 2.
\end{cases}$$

where $\alpha^t$ is the linear map represented by the transpose $A^t$ of $A$. Therefore, there is an isomorphism

$$H_j^{DW}(S^1_{(k_1, \ldots, k_n)}) \cong H_j^{AW}(S^1_{(k_1, \ldots, k_n)}), \forall j \geq 0.$$ 

Next, we do some calculation on $A_{(k_1, \ldots, k_n)}$. First of all, by adding the first row until the $(n-1)$-th row to the $n$-th row of $A_{(k_1, \ldots, k_n)}$, we obtain

$$\begin{pmatrix}
-k_1 & k_2 & \cdots & 0 & 0 \\
0 & -k_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -k_{n-1} & k_n \\
k_1 & 0 & \cdots & 0 & -k_n
\end{pmatrix} \rightarrow \begin{pmatrix}
-k_1 & k_2 & \cdots & 0 & 0 \\
0 & -k_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -k_{n-1} & k_n \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}.$$ 

From this we can assert that the rank of $A_{(k_1, \ldots, k_n)}$ is $n-1$, which implies

$$H_0^{AW}(S^1_{(k_1, \ldots, k_n)}) \cong H_0^{DW}(S^1_{(k_1, \ldots, k_n)}) \cong \mathbb{Z} \oplus G_{(k_1, \ldots, k_n)}.$$ 

where $G_{(k_1, \ldots, k_n)}$ is a finite abelian group. Hence $G_{(k_1, \ldots, k_n)}$ is independent on the ordering of $k_1, \ldots, k_n$. That is why we use "\{k_1, \ldots, k_n\}" in the subscript of "$G$".

To determine the order of $G_{(k_1, \ldots, k_n)}$, observe that we can use some elementary row and column operations over integers to transform the following matrix

$$\begin{pmatrix}
-k_1 & k_2 & 0 & \cdots & 0 \\
0 & -k_2 & k_3 & \cdots & 0
\end{pmatrix} \rightarrow \begin{pmatrix}
\gcd(k_1, k_2) & 0 & 0 & \cdots & 0 \\
0 & \text{lcm}(k_1, k_2) & k_3 & \cdots & 0
\end{pmatrix}$$

where we do not need the operation that adds the second row to the first row. Then by iteratively using such kind of row and column operations, we can turn
the right matrix in (34) into the following diagonal matrix

\[
\begin{pmatrix}
\frac{k_1 k_2}{\text{lcm}(k_1, k_2)} & 0 & \cdots & 0 & 0 \\
0 & \frac{\text{lcm}(k_1, k_2) k_3}{\text{lcm}(k_1, k_2, k_3)} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{\text{lcm}(k_1, k_2, \ldots, k_{n-1}) k_n}{\text{lcm}(k_1, k_2, \ldots, k_n)} & 0 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

(36)

where we use the identity \(\text{lcm}(k_1, \ldots, k_n) = \text{lcm}(\text{lcm}(k_1, \ldots, k_{n-1}), k_n)\). From this matrix, we can deduce that the order of \(G_{\{k_1, \ldots, k_n\}}\) is

\[|G_{\{k_1, \ldots, k_n\}}| = \frac{k_1 k_2 \cdots k_n}{\text{lcm}(k_1, k_2, \ldots, k_n)}.\]

We warn that a priori the matrix in (36) is not the Smith normal form of \(A_{\{k_1, \ldots, k_n\}}\).

**Remark 8.6.** Although \(H^A_{0W}(S^1_{\{k_1, \ldots, k_n\}})\) is isomorphic to \(H^D_{0W}(S^1_{\{k_1, \ldots, k_n\}})\), their generators are different. In fact, \(H^A_{0W}(S^1_{\{k_1, \ldots, k_n\}})\) is always generated by the regular points in \(S^1_{\{k_1, \ldots, k_n\}}\) while \(H^D_{0W}(S^1_{\{k_1, \ldots, k_n\}})\) is always generated by singular points. For example when \(n = 2\) (see Figure 4), \(H^A_{0W}(S^1_{\{k_1, k_2\}})\) is generated by regular points \(u_1\) and \(u_1 - u_2\), while \(H^D_{0W}(S^1_{\{k_1, k_2\}})\) is generated by singular points \(v_1\) and \(\frac{k_1}{\text{gcd}(k_1, k_2)} v_1 - \frac{k_2}{\text{gcd}(k_1, k_2)} v_2\).

**Figure 4.** A circle with two singular points

In addition, let \(S^1_{\{k_1\}}\) denote the pseudo-orbifold which is a circle with only one singular point whose weight is \(k_1 \geq 2\). Then we can easily compute:

\[
H^A_{jW}(S^1_{\{k_1\}}) \cong H^D_{jW}(S^1_{\{k_1\}}) = \begin{cases} 
\mathbb{Z}, & \text{if } j = 0, 1; \\
0, & \text{if } j \geq 2.
\end{cases}
\]

**Example 8.7.** In Figure 5, we have a 2-dimensional pseudo-orbifold \((D^2, \lambda)\), denoted by \(D^2_{\{k_1, \ldots, k_n\}}\), which is a 2-disk \(D^2\) with \(n\) singular points \(v_1, \ldots, v_n\), \(n \geq 2\), on the boundary of \(D^2\) whose weights \(\lambda(v_i) = k_i \geq 2\), \(i = 1, \ldots, n\).
Let \((K, \xi)\) denote the descending divisibly weighted triangulation of \(D^2_{(k_1, \ldots, k_n)}\) as shown in Figure 5. The inversion \(\hat{\xi}\) is an ascending divisible weight on \(K\). It is not hard to check that the weighted simplicial chain complex \((C_*(K), \partial^{\hat{\xi}})\) is isomorphic to the algebraic cone of \((C_*(K|_{\partial D^2}), \partial^{\hat{\xi}})\) (see Definition 8.1). So by Lemma 8.2, \((C_*(K), \partial^{\hat{\xi}})\) is acyclic and hence
\[
H^j_{AW}(D^2_{(k_1, \ldots, k_n)}) \cong \begin{cases} 
\mathbb{Z}, & \text{if } j = 0; \\
0, & \text{if } j \geq 1.
\end{cases}
\]
Meanwhile, \((C_*(K), \partial^{\xi})\) may not be acyclic. A direct calculation shows that
\[
H^j_{DW}(D^2_{(k_1, \ldots, k_n)}) \cong \begin{cases} 
\mathbb{Z} \oplus G_{\{k_1, \ldots, k_n}\}, & \text{if } j = 0; \\
0, & \text{if } j \geq 1.
\end{cases}
\]
Moreover, a simple analysis of the weighted simplicial chain complexes of \(S^1_{(k_1, \ldots, k_n)}\) and \(D^2_{(k_1, \ldots, k_n)}\) shows that the inclusion map \(S^1_{(k_1, \ldots, k_n)} \hookrightarrow D^2_{(k_1, \ldots, k_n)}\) induces an isomorphism \(H^0_{DW}(S^1_{(k_1, \ldots, k_n)}) \cong H^j_{DW}(D^2_{(k_1, \ldots, k_n)})\).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{A 2-disk with \(n\) singular points on the boundary}
\end{figure}

In addition, let \(D^2_{(k_1)}\) denote the pseudo-orbifold based on a 2-disk \(D^2\) with only one singular point \(v_1\) on the boundary of \(D^2\) whose weight is \(k_1 \geq 2\). The right picture of Figure 5 gives a divisibly weighted \(\Delta\)-triangulation of \(D^2_{(k_1)}\). We can easily compute
\[
H^j_{AW}(D^2_{(k_1)}) \cong H^j_{DW}(D^2_{(k_1)}) \cong \begin{cases} 
\mathbb{Z}, & \text{if } j = 0; \\
0, & \text{if } j \geq 1.
\end{cases}
\]
Note that \(H^0_{AW}(D^2_{(k_1)})\) is generated by any regular point of \(D^2_{(k_1)}\) while \(H^0_{DW}(D^2_{(k_1)})\) is generated by the singular point \(v_1\).
Example 8.8. The *teardrop* orbifold $S^2_{(k)}$ is the 2-sphere with only one singular point $v_0$ (see Figure 6) whose local group $G_{v_0} = \mathbb{Z}/k\mathbb{Z}$ ($k \geq 2$). A chart around $v_0$ consists of an open disk $\tilde{U} \subset \mathbb{R}^n$ and an action of $\mathbb{Z}/k\mathbb{Z}$ on $\tilde{U}$ by rotations. $S^2_{(k)}$ is a “bad” orbifold in the sense that it is not isomorphic to the quotient space of a group action on a manifold.

Using the triangulation adapted to $S^2_{(k)}$ given in Figure 6, we can compute

$$H_j^{AW}(S^2_{(k)}) \cong H_j^{DW}(S^2_{(k)}) \cong \begin{cases} 
\mathbb{Z}, & \text{if } j = 0; \\
0, & \text{if } j = 1; \\
\mathbb{Z}, & \text{if } j = 2; \\
0, & \text{if } j \geq 3.
\end{cases}$$

Figure 6. Teardrop orbifold

Example 8.9. In Figure 7, we have an orbifold $S^2_{(k_1,k_2)}$ which is a 2-sphere with two isolated singular points $v_1$, $v_2$ with local groups $G_{v_1} = \mathbb{Z}/k_1\mathbb{Z}$, $G_{v_2} = \mathbb{Z}/k_2\mathbb{Z}$. Usually, $S^2_{(k_1,k_2)}$ is called a *football* if $k_1 = k_2$ and a *spindle* if $k_1 \neq k_2$.

Using the triangulation adapted to $S^2_{(k_1,k_2)}$ in the middle picture of Figure 7, we can compute its AW-homology groups:

$$H_j^{AW}(S^2_{(k_1,k_2)}) \cong \begin{cases} 
\mathbb{Z}, & \text{if } j = 0; \\
\mathbb{Z}/\gcd(k_1, k_2)\mathbb{Z}, & \text{if } j = 1; \\
\mathbb{Z}, & \text{if } j = 2; \\
0, & \text{if } j \geq 3.
\end{cases}$$

If $\gcd(k_1, k_2) \geq 2$, the generator of the factor $\mathbb{Z}/\gcd(k_1, k_2)\mathbb{Z}$ in $H_1^{AW}(S^2_{(k_1,k_2)})$ can be represented by $[u_1u_2 + u_2u_3 + u_3u_4 + u_4u_1]$. 
Notice that the right picture of Figure 7 is a $\Delta$-complex triangulation of $S^2_{(k_1,k_2)}$ which can also be used for our calculation. Moreover, we can compute

$$H^j_{\text{DW}}(S^2_{(k_1,k_2)}) \cong \begin{cases} 
\mathbb{Z} \oplus \mathbb{Z}/\gcd(k_1, k_2)\mathbb{Z}, & \text{if } j = 0; \\
0, & \text{if } j = 1; \\
\mathbb{Z}, & \text{if } j = 2; \\
0, & \text{if } j \geq 3.
\end{cases}$$

The generators of $H^0_{\text{DW}}(S^2_{(k_1,k_2)})$ can be taken to be $v_1$ and $\frac{k_1}{\gcd(k_1,k_2)}v_1 - \frac{k_2}{\gcd(k_1,k_2)}v_2$.

**Example 8.10.** Let $S^2_{(k_1,\ldots,k_n)}$, $n \geq 2$, denote the pseudo-orbifold $(S^2, \lambda)$ with $n$ isolated singular points $v_1, \ldots, v_n$ whose weights are $\lambda(v_i) = k_i \geq 2$, $i = 1, \ldots, n$. We can think of $S^2_{(k_1,\ldots,k_n)}$ as the gluing of two copies of $D^2_{(k_1,\ldots,k_n)}$ along their boundaries, that is

$$S^2_{(k_1,\ldots,k_n)} = D^2_{(k_1,\ldots,k_n)} \cup_{S^1_{(k_1,\ldots,k_n)}} D^2_{(k_1,\ldots,k_n)}.$$  

Then by Mayer-Vietoris sequence and the result from (35), we can easily compute

$$H^j_{\text{AW}}(S^2_{(k_1,\ldots,k_n)}) \cong \begin{cases} 
\mathbb{Z}, & \text{if } j = 0; \\
\mathbb{G}_{\{k_1,\ldots,k_n\}}, & \text{if } j = 1; \\
\mathbb{Z}, & \text{if } j = 2; \\
0, & \text{if } j \geq 3.
\end{cases}$$

$$H^j_{\text{DW}}(S^2_{(k_1,\ldots,k_n)}) \cong \begin{cases} 
\mathbb{Z} \oplus \mathbb{G}_{\{k_1,\ldots,k_n\}}, & \text{if } j = 0; \\
0, & \text{if } j = 1; \\
\mathbb{Z}, & \text{if } j = 2; \\
0, & \text{if } j \geq 3.
\end{cases}$$

Moreover, from the Mayer-Vietoris sequence we can see that
The connecting homomorphism

\[ H_1^{AW}(S_{(k_1, \ldots, k_n)}^2) \xrightarrow{d_1} H_0^{AW}(S_{(k_1, \ldots, k_n)}^1) \cong \mathbb{Z} \oplus G_{\{k_1, \ldots, k_n\}} \]

maps \( H_1^{AW}(S_{(k_1, \ldots, k_n)}^2) \) isomorphically onto the torsion subgroup of \( H_0^{AW}(S_{(k_1, \ldots, k_n)}^1) \).

The inclusion map \( S_{(k_1, \ldots, k_n)}^1 \hookrightarrow S_{(k_1, \ldots, k_n)}^2 \) induces an isomorphism

\[ H_0^{DW}(S_{(k_1, \ldots, k_n)}^1) \cong H_0^{DW}(S_{(k_1, \ldots, k_n)}^2). \]

From Example 8.4, Example 8.8, Example 8.9 and Example 8.10, we can see that changing the weight of a singular point on a pseudo-orbifold may cause drastic changes to its AW-homology and DW-homology.

More generally, we can compute the AW-homology and DW-homology of any pseudo-orbifold with finitely many singular points whose underlying space is a closed connected surface.

**Example 8.11.** Let \( X_{(k_1, \ldots, k_n)} \), \( n \geq 2 \), denote the pseudo-orbifold \((\Sigma, \lambda)\) where \( \Sigma \) is a closed connected surface with \( n \) singular points \( v_1, \ldots, v_n \) whose weights are \( \lambda(v_i) = k_i \geq 2 \), \( i = 1, \ldots, n \). By thinking of \( X_{(k_1, \ldots, k_n)} \) as the connected sum of \( S_{(k_1, \ldots, k_n)}^2 \) with the surface \( \Sigma \) (with no singular points) along a regular circle, we can easily compute the AW-homology and DW-homology of \( X_{(k_1, \ldots, k_n)} \) from the previous results on \( S_{(k_1, \ldots, k_n)}^2 \) using Mayer-Vietoris sequence:

\[
H_j^{AW}(X_{(k_1, \ldots, k_n)}) \cong \begin{cases} 
\mathbb{Z}, & \text{if } j = 0; \\
H_1(\Sigma) \oplus G_{\{k_1, \ldots, k_n\}}, & \text{if } j = 1; \\
\mathbb{Z}, & \text{if } j = 2; \\
0, & \text{if } j \geq 3. 
\end{cases}
\]

\[
H_j^{DW}(X_{(k_1, \ldots, k_n)}) \cong \begin{cases} 
\mathbb{Z} \oplus G_{\{k_1, \ldots, k_n\}}, & \text{if } j = 0; \\
H_1(\Sigma), & \text{if } j = 1; \\
\mathbb{Z}, & \text{if } j = 2; \\
0, & \text{if } j \geq 3. 
\end{cases}
\]

where \( H_1(\Sigma) \) is the singular homology group of \( \Sigma \) at dimension 1.

**Remark 8.12.** By comparing \( H_\ast^{AW}(X_{(k_1, \ldots, k_n)}) \) and \( H_\ast^{DW}(X_{(k_1, \ldots, k_n)}) \) with those examples computed in [20, Theorem 11.1], we can see that AW-homology and DW-homology of an orbifold are both different from the \( t \)-singular homology of the orbifold.

**Example 8.13.** Let \( \widehat{D}_{(l_1, \ldots, l_m)}^2 \), \( m \geq 2 \), denote the pseudo-orbifold \((D^2, \lambda)\) with \( m \) isolated singular points \( z_1, \ldots, z_m \) in the interior of \( D^2 \) whose weights are \( \lambda(z_j) = l_j \geq 2 \), \( j = 1, \ldots, m \). Since the boundary of \( \widehat{D}_{(l_1, \ldots, l_m)}^2 \) is a regular circle,
we can glue a regular 2-disk $D^2$ to $\hat{D}_{(l_1,\ldots,l_m)}^2$ along the boundary circle which gives $S_{(l_1,\ldots,l_m)}^2$. Then by Mayer-Vietoris sequence, we can easily compute

$$H_j^{AW}(\hat{D}_{(l_1,\ldots,l_m)}^2) \cong \begin{cases} \mathbb{Z}, & \text{if } j = 0; \\ G_{\{l_1,\ldots,l_m\}}, & \text{if } j = 1; \\ 0, & \text{if } j \geq 2. \end{cases}$$

$$H_j^{DW}(\hat{D}_{(l_1,\ldots,l_m)}^2) \cong \begin{cases} \mathbb{Z} \oplus G_{\{k_1,\ldots,k_n\}}, & \text{if } j = 0; \\ 0, & \text{if } j \geq 1. \end{cases}$$

By comparing the above computations for $D_{(k_1,\ldots,k_n)}^2$ and $\hat{D}_{(l_1,\ldots,l_m)}^2$, we see that moving the singular points from the boundary of a 2-disk to its interior may change the AW-homology while preserving the DW-homology.

Let $\hat{D}_{(l_1)}^2$ denote the pseudo-orbifold based on a 2-disk $D^2$ with only one singular point $v_1$ with weight $l_1$ in the interior of $D^2$. We can similarly compute

$$H_j^{AW}(\hat{D}_{(l_1)}^2) \cong H_j^{DW}(\hat{D}_{(l_1)}^2) \cong \begin{cases} \mathbb{Z}, & \text{if } j = 0; \\ 0, & \text{if } j \geq 1. \end{cases}$$

Note that $H_0^{AW}(\hat{D}_{(l_1)}^2)$ is generated by any regular point of $\hat{D}_{(l_1)}^2$ while $H_0^{DW}(\hat{D}_{(l_1)}^2)$ is generated by the singular point $v_1$.

It is also an interesting problem to compute AW-homology and DW-homology of a pseudo-orbifold based on $D^2$ which has singular points both in the interior and on the boundary of $D^2$. But it is difficult to write an explicit formula for the answer.

9. Product structure on DW-cohomology

9.1. AW-cohomology and DW-cohomology.

Let $(K, \mu)$ be a weighted simplicial complex. For an abelian group $G$, the weighted simplicial cochain complex with $G$-coefficients is obtained by applying the $\text{Hom}(\cdot, G)$ functor to the weighted simplicial chain complex $(C_*(K), \partial^\mu)$, denoted by $(C^*(K; G), \delta^\mu)$, where $\delta^\mu$ is the coboundary map determined by $\partial^\mu$. Then the weighted simplicial cohomology group of $(K, \mu)$ with $G$-coefficients is

$$H^*(K, \delta^\mu; G) := H^*((C^*(K; G), \delta^\mu)).$$

In practice, we often use $G = \mathbb{Z}$ as the coefficients because of Theorem 7.1.

If $\varrho : (K, \mu) \to (K', \mu')$ is a a morphism of weighted simplicial complexes, define

$$\varrho^* : C^n(K'; G) \to C^n(K; G), \ n \geq 0,$$
by: for any \( \phi \in C^n(K'; G) \) and any oriented \( n \)-simplex \([v_0, \ldots, v_n] \in C_n(K)\),
\[
\varrho^\#(\phi)([v_0, \ldots, v_n]) = \begin{cases} 
\phi([g(v_0), \ldots, g(v_n)]), & \text{if } g(v_0), \ldots, g(v_n) \text{ are distinct;} \\
0, & \text{otherwise}.
\end{cases}
\]
It is easy to see that \( \varrho^\# \) is a cochain map from \((C^*(K'; G), \delta')\) to \((C^*(K; G), \delta)\) and then induces a map on the weighted simplicial cohomology groups
\[
\varrho^*: H^*(K', \delta'; G) \to H^*(K, \delta; G).
\]

**Definition 9.1** (AW-cohomology and DW-cohomology). Suppose \((X, \lambda)\) is a weighted polyhedron with a divisibly weighted triangulation \((K, \xi)\). Then we define the **AW-cohomology** and **DW-cohomology** of \((X, \lambda)\) with \( G \)-coefficients to be \( H^*(K, \delta^\xi; G) \) and \( H^*(K, \delta^\xi; G) \), respectively if \( \xi \) is ascending and contrariwise if \( \xi \) is descending. Denote the AW-cohomology of \((X, \lambda)\) by \( H^*_{AW}(X, \lambda; G) \) and the DW-cohomology by \( H^*_{DW}(X, \lambda; G) \).

By the universal coefficient theorem for cohomology (see [10, Theorem 3.2]), we can compute the AW-cohomology and DW-cohomology groups of \((X, \lambda)\) from its AW-homology and DW-homology groups, respectively.

If \( f : (X, \lambda) \to (X', \lambda') \) is a \( W \)-continuous map of weighted polyhedra, then by the above definitions \( f \) induces group homomorphisms
\[
f^*: H^*_{AW}(X', \lambda'; G) \to H^*_{AW}(X, \lambda; G), \quad f^*: H^*_{DW}(X', \lambda'; G) \to H^*_{DW}(X, \lambda; G).
\]

### 9.2. Weighted cup product.

We discover that there is a natural product structure on the DW-cohomology of a weighted polyhedron defined below, which generalizes the cup product \( \cup \) of ordinary simplicial cochains induced by Alexander-Whitney diagonal. Moreover, we will prove in Section 9.4 that this product is graded commutative on the DW-cohomology.

**Definition 9.2** (Weighted Cup Product). Let \((K, \xi)\) be a descending divisibly weighted simplicial complex and \( R \) be a commutative ring with unit. Choose a total ordering \( \prec \) of the vertices of \( K \). Then for any cochains \( \phi \in C^p(K; R) \) and \( \psi \in C^q(K; R) \), let \( \phi \cup \psi \in C^{p+q}(K; R) \) be the cochain whose value on an oriented \((p + q)\)-simplex \( \sigma = [v_0, \ldots, v_{p+q}] \) of \( K \) with \( v_0 \prec \ldots \prec v_{p+q} \) is defined by
\[
(37) \quad \phi \cup \psi(\sigma) := \frac{\xi([v_0, \ldots, v_p]) \xi([v_p, \ldots, v_{p+q}])}{\xi([v_0, \ldots, v_p])} \phi([v_0, \ldots, v_p]) \cdot \psi([v_p, \ldots, v_{p+q}]).
\]
The coefficient on the right hand side of (37) is always integral because \( \xi \) is descending. Besides, the product of an integer with an element of \( R \) is defined by the abelian group structure of \( R \). We call \( \cup \) the **weighted cup product** on \((C^*(K; R), \delta^\xi)\) with respect to \( \prec \).
Note that the above definition depends on the ordering \( \prec \) of the vertices of \( K \) because \( \mu, \phi \) and \( \psi \) may take different values on different faces of \([v_0, \ldots, v_{p+1}]\). So we should have use a notation like \( \bowtie \) instead of \( \cup \) to indicate the dependence on \( \prec \). But later we will prove in Theorem 9.10 that the ring structure induced by \( \bowtie \) on the weighted cohomology group \( H^*(K, \delta^\xi; R) \) is actually independent on the choice of \( \prec \) up to ring isomorphisms. So we will not put \( \prec \) into our notation and just remember that we need to fix a total ordering of the vertices of \( K \) when computing the weighted cup product. A convenient choice of \( \prec \) for \((K, \xi)\) is to order the vertices of \( K \) according to their weights.

**Lemma 9.3.** Let \((K, \xi)\) be a descending divisibly weighted simplicial complex. For any \( \phi \in C^p(K; R) \) and \( \psi \in C^q(K; R) \), we have

\[
\delta^\xi(\phi \bowtie \psi) = \delta^\xi \phi \bowtie \psi + (-1)^p \phi \bowtie \delta^\xi \psi.
\]

**Proof.** For any \((p+q+1)\)-simplex \( \sigma = [v_0, \ldots, v_{p+q+1}] \) of \( K \) with \( v_0 \prec \cdots \prec v_{p+q} \),

\[
\begin{align*}
\delta^\xi(\phi \bowtie \psi)(\sigma) &= \phi \bowtie \psi \left( \sum_{i=0}^{p+q+1} (-1)^i \frac{\xi([v_0, \ldots, v_{p+q+1}])}{\xi([v_0, \ldots, v_{p+q+1}])} \phi([v_0, \ldots, \hat{v}_i, \ldots, v_{p+q+1}]) \cdot \psi([v_{p+1}, \ldots, v_{p+q+1}]) \right) \\
&= \sum_{i=0}^p (-1)^i \frac{\xi([v_0, \ldots, \hat{v}_i, \ldots, v_{p+q+1}]) \xi([v_{p+1}, \ldots, v_{p+q+1}])}{\xi([v_0, \ldots, v_{p+q+1}])} \phi([v_0, \ldots, \hat{v}_i, \ldots, v_{p+1}]) \cdot \psi([v_{p+1}, \ldots, v_{p+q+1}]) \\
&\quad + \sum_{i=p+1}^{p+q+1} (-1)^i \frac{\xi([v_0, \ldots, v_{p+q+1}])}{\xi([v_0, \ldots, v_{p+q+1}])} \phi([v_0, \ldots, v_{p+q+1}]) \cdot \psi([v_{p+1}, \ldots, v_{p+q+1}]).
\end{align*}
\]

On the other side, we have

\[
(\delta^\xi \phi \bowtie \psi)(\sigma) = \frac{\xi([v_0, \ldots, v_{p+1}]) \xi([v_{p+1}, \ldots, v_{p+q+1}])}{\xi([v_0, \ldots, v_{p+q+1}])} \delta^\xi \phi([v_0, \ldots, v_{p+1}]) \cdot \psi([v_{p+1}, \ldots, v_{p+q+1}])
\]

where

\[
\delta^\xi \phi([v_0, \ldots, v_{p+1}]) = \sum_{i=0}^{p+1} (-1)^i \frac{\xi([v_0, \ldots, \hat{v}_i, \ldots, v_{p+1}])}{\xi([v_0, \ldots, v_{p+1}])} \phi([v_0, \ldots, \hat{v}_i, \ldots, v_{p+1}]).
\]

Then we have

\[
(\delta^\xi \phi \bowtie \psi)(\sigma) = \sum_{i=0}^{p+1} (-1)^i \frac{\xi([v_0, \ldots, \hat{v}_i, \ldots, v_{p+1}]) \xi([v_{p+1}, \ldots, v_{p+q+1}])}{\xi([v_0, \ldots, v_{p+q+1}])} \phi([v_0, \ldots, \hat{v}_i, \ldots, v_{p+1}]) \cdot \psi([v_{p+1}, \ldots, v_{p+q+1}]).
\]
Similarly, we have
\[ (-1)^p (\phi \, \Omega \, \delta \, \psi)(\sigma) \]
\[ = \sum_{i=p}^{p+q+1} (-1)^i \xi([v_0, \cdots, v_p]) \xi([v_p, \cdots, v_{p+q+1}]) \phi([v_0, \cdots, v_p]) \cdot \psi([v_p, \cdots, v_{p+q+1}]). \]

If we add the above two expressions up, the last term of the first sum cancels the first term of the second sum, and the remaining terms give exactly \( \delta^\xi (\phi \, \Omega \, \psi)(\sigma) \). The lemma is proved.

By Lemma 9.3, the above product \( \Omega \) on cochains induces the weighted cup product on \( H^\ast(K, \delta \xi; R) \):
\[ H^p(K, \delta \xi; R) \times H^q(K, \delta \xi; R) \xrightarrow{\Omega} H^{p+q}(K, \delta \xi; R), \quad p,q \in \mathbb{Z}. \]

Then by Definition 4.16, this determines the weighted cup product \( \Omega \) on the DW-cohomology \( H_{DW}^\ast(X, \lambda; R) \) of a weighted polyhedron \((X, \lambda)\), that is
\[ H_p^{DW}(X, \lambda; R) \times H_q^{DW}(X, \lambda; R) \xrightarrow{\Omega} H_{p+q}^{DW}(X, \lambda; R), \quad p,q \in \mathbb{Z}. \]

Moreover, it is easy to check that the weighted cup product \( \Omega \) is natural in the sense that for any morphism \( \varphi : (K, \xi) \to (K', \xi') \) between two descending divisibly weighted simplicial complexes \((K, \xi)\) and \((K', \xi')\), the following diagram commutes
\[ H^p(K', \delta \xi'; R) \times H^q(K', \delta \xi'; R) \xrightarrow{\Omega} H^{p+q}(K', \delta \xi'; R). \]

Correspondingly, if \( f : (X, \lambda) \to (X', \lambda') \) is a W-continuous map of weighted polyhedra, the following diagram commutes:
\[ H_p^{DW}(X', \lambda'; R) \times H_q^{DW}(X', \lambda'; R) \xrightarrow{\Omega} H_{p+q}^{DW}(X', \lambda'; R). \]

**Remark 9.4.** It seems to us that there is no meaningful product structure on AW-cohomology with integral coefficients. Indeed, the naive extension of the formula in (37) to an ascending weighted simplicial complex cannot guarantee the coefficients in the formula to be integral and make the coboundary relation in (38) hold simultaneously.
9.3. Ordered simplex and ordered simplicial chain.

To prove that the product $\otimes$ in Definition 9.2 always defines isomorphic ring structures on $H^*(K, \delta; R)$ with respect to different choices of the total ordering of the vertices of $K$, we need to quote the following notions from [13, §13] and define parallel notions for weighted simplicial complexes.

**Definition 9.5 (Ordered Simplex and Ordered Simplicial Chain Complex).** Let $K$ be a simplicial complex. An ordered $n$-simplex of $K$ is an $n+1$ tuple $\langle v_0, \cdots, v_n \rangle$ of vertices of $K$, where $v_i$ are vertices of a simplex of $K$ but need not be distinct. Let $\hat{C}_n(K)$ denote the free abelian group generated by the ordered $n$-simplices of $K$; it is called the $n$-dimensional ordered simplicial chain group of $K$. Define the boundary map $\hat{\partial}: \hat{C}_n(K) \to \hat{C}_{n-1}(K)$ by

$$\hat{\partial}(\langle v_0, \cdots, v_n \rangle) = \sum_{i=0}^{n} (-1)^i \langle v_0, \cdots, \hat{v}_i, \cdots, v_n \rangle.$$  

It is easy to check that $\hat{\partial}$ is well-defined and $\hat{\partial} \circ \hat{\partial} = 0$ for all $n \geq 1$. We call $(\hat{C}_*(K), \hat{\partial})$ the ordered simplicial chain complex of $K$ and let $\hat{H}_*(K)$ = the homology group of $(\hat{C}_*(K), \hat{\partial})$.

We call $\hat{H}_*(K)$ the ordered simplicial homology of $K$. In addition, let $\sigma_{v_0 \cdots v_n} = the simplex of $K$ spanned by all the different vertices among $v_0, \cdots, v_n$.

Note that the ordinary simplicial chain group $C_*(K)$ can be considered as the quotient group of the ordered simplicial chain group $\hat{C}_*(K)$. Indeed, for all $n \geq 0$ there is a natural epimorphism

$$\Psi_n: \hat{C}_n(K) \to C_n(K)$$

defined by

$$\Psi_n(\langle v_0, \cdots, v_n \rangle) = \begin{cases} [v_0, \cdots, v_n], & \text{if } v_0, \cdots, v_n \text{ are all distinct;} \\ 0, & \text{otherwise.} \end{cases}$$  

On the other hand, given a total ordering $\prec$ of all the vertices of $K$, we can define a monomorphism $\Theta_n: C_n(K) \to \hat{C}_n(K)$ for all $n \geq 0$ by: for any simplex $\sigma = \{v_0, \cdots, v_n\}$ of $K$ with $v_0 \prec \cdots \prec v_n$,

$$\Theta_n([v_0, \cdots, v_n]) = \langle v_0, \cdots, v_n \rangle.$$  

It is easy to check that $\Psi = \{\Psi_n\}_{n \geq 0}$ and $\Theta = \{\Theta_n\}_{n \geq 0}$ are both chain maps. Moreover, according to [13, Theorem 13.6], $\Psi$ and $\Theta$ are chain homotopy inverses of each other. So there is a group isomorphism

$$H_j(K) \cong \hat{H}_j(K), \ \forall j \geq 0.$$
Given any abelian group $G$, it is natural to define ordered simplicial chain complex and ordered simplicial cochain complex of $K$ by applying the tensor product functor $\otimes G$ and $\text{Hom}(-, G)$ functor to $\hat{C}_*(K)$, respectively, denoted by $(\hat{\mathcal{C}}_*(K; G), \hat{\partial}) = (\hat{\mathcal{C}}_*(K) \otimes G, \hat{\partial} \otimes \text{id}_G), \ (\hat{C}^*(K; G), \hat{\delta}) = \text{Hom}((\hat{\mathcal{C}}_*(K), \hat{\partial}), G)$. Moreover, let

$$\hat{H}_*(K; G) := \text{homology group of } (\hat{\mathcal{C}}_*(K; G), \hat{\partial});$$

$$\hat{H}^*(K; G) := \text{cohomology group of } (\hat{\mathcal{C}}^*(K; G), \hat{\delta}).$$

We call $\hat{H}_*(K; G)$ and $\hat{H}^*(K; G)$ the ordered simplicial homology and ordered simplicial cohomology of $K$ with $G$-coefficients, respectively. By the isomorphism in (42) and the universal coefficient theorems, we obtain isomorphisms

$$H_j(K; G) \cong \hat{H}_j(K; G), \ H^j(K; G) \cong \hat{H}^j(K; G), \ \forall j \geq 0. \quad (43)$$

Next, we extend the above definitions to weighted simplicial complexes. Let $(K, \mu)$ be a weighted simplicial complex. Define the weight of any ordered simplex $\langle v_0, \cdots, v_n \rangle$ of $K$ by

$$\mu(\langle v_0, \cdots, v_n \rangle) = \mu(\sigma_{v_0 \cdots v_n}).$$

In addition, we can modify the boundary map $\hat{\partial}$ on $\hat{\mathcal{C}}_*(K)$ into a boundary map $\hat{\partial}^\mu : \hat{\mathcal{C}}_n(K) \to \hat{\mathcal{C}}_{n-1}(K)$ using the similar formula as $\partial^\mu$ given by (1) and (2) (depending on $\mu$ is ascending or descending). So we obtain a chain complex $(\hat{\mathcal{C}}_*(K), \hat{\partial}^\mu)$. Note that the augmentation $\varepsilon^\mu$ of $(C_*(K), \partial^\mu)$ defined in Section 2.4 gives the augmentation of $(\hat{\mathcal{C}}_*(K), \hat{\partial}^\mu)$. Moreover, $\hat{\partial}^\mu$ induces a coboundary map $\hat{\delta}^\mu$ on the cochain complex $\hat{\mathcal{C}}^*(K; G)$. Let

$$\hat{H}_*(K, \hat{\partial}^\mu; G) := \text{the homology group of } (\hat{\mathcal{C}}_*(K; G), \hat{\partial}^\mu);$$

$$\hat{H}^*(K, \hat{\delta}^\mu; G) := \text{the cohomology group of } (\hat{\mathcal{C}}^*(K; G), \hat{\delta}^\mu).$$

As usual, we will omit the coefficient $G$ if $G = \mathbb{Z}$.

The functorial properties of (weighted) simplicial homology naturally extend to the ordered (weighted) simplicial homology. In particular, we have the following theorem which is parallel to Theorem 2.18.

**Theorem 9.6.** If two morphisms $\varrho_0, \varrho_1 : (K, \mu) \to (K', \mu')$ of weighted simplicial complexes are contiguous, then the induced chain maps

$$(\varrho_0)_\# : (\hat{\mathcal{C}}_*(K), \hat{\partial}^\mu) \to (\hat{\mathcal{C}}_*(K'), \hat{\partial}^\mu')$$

are chain homotopic, and hence $(\varrho_0)_* = (\varrho_1)_* : \hat{H}_*(K, \hat{\partial}^\mu) \to \hat{H}_*(K', \hat{\partial}^\mu').$

**Proof.** The proof is almost identical to the proof of Theorem 2.18 in [8] if we replace ordinary simplices by ordered simplices there. \[\Box\]
Lemma 9.7. Let $\sigma$ be a simplex and $\xi$ be a divisible weight on $\sigma$. Then
\[
\hat{H}_j(\sigma, \hat{\partial}^\xi) \cong \begin{cases} 
\mathbb{Z}, & \text{if } j = 0; \\
0, & \text{if } j \geq 1.
\end{cases}
\]

Proof. The proof is parallel to that of Lemma 5.1. The identity map $\text{id}_\sigma : \sigma \to \sigma$ is contiguous to a constant map, and so the lemma follows from Theorem 9.6. \qed

Moreover, we have the following theorem which generalizes \cite[Theorem 13.6]{13}. 

Theorem 9.8. Let $(K, \xi)$ be a divisibly weighted simplicial complex. Then the two maps $\Psi : (\hat{C}_s(K), \hat{\partial}^\xi) \to (C_s(K), \partial^\xi)$ and $\Theta : (C_s(K), \partial^\xi) \to (\hat{C}_s(K), \hat{\partial}^\xi)$ are augmentation-preserving chain maps that are chain homotopy inverses to each other.

Proof. It is routine to check from the definitions that $\Psi$ and $\Theta$ are both chain maps with respect to the weighted boundary $\partial^\xi$ and $\hat{\partial}^\xi$. In addition, they are both augmentation-preserving since they are the identity map on 0-chains. Moreover, it is obvious that
\[
\Psi \circ \Theta = \text{id}_{C_*(K)} : (C_*(K), \partial^\xi) \to (C_*(K), \partial^\xi).
\]

So it remains to prove that $\Theta \circ \Psi$ is chain homotopic to $\text{id}_{\hat{C}_*(K)}$

Let $\prec$ be a total ordering of all the vertices of $K$. Then for any ordered simplex $\langle v_0, \cdots, v_n \rangle$ of $K$,
\[
\Theta_n \circ \Psi_n(\langle v_0, \cdots, v_n \rangle) = \begin{cases} 
(-1)^{t(v_0 \cdots v_n)} \langle v_{i_0}, \cdots, v_{i_n} \rangle, & \text{if } v_0, \cdots, v_n \text{ are all distinct}; \\
0, & \text{otherwise},
\end{cases}
\]

where the sequence $v_{i_0} \cdots v_{i_n}$ is a permutation of $v_0 \cdots v_n$ with $v_{i_0} \prec \cdots \prec v_{i_n}$, and $t(v_0, \cdots, v_n)$ is the number of swaps that are needed to turn $v_0 \cdots v_n$ to $v_{i_0} \cdots v_{i_n}$. In addition, since $(K, \xi)$ is divisibly weighted, $(\sigma_{v_0 \cdots v_n}, \xi)$ is a divisibly weighted simplex. So by Lemma 9.7, $(\hat{C}_s(\sigma_{v_0 \cdots v_n}), \hat{\partial}^\xi)$ is acyclic. Then we obtain an algebraic acyclic carrier $\Phi$ on $(\hat{C}_s(K), \hat{\partial}^\xi)$ defined by
\[
\Phi(\langle v_0, \cdots, v_n \rangle) := (\hat{C}_s(\sigma_{v_0 \cdots v_n}), \hat{\partial}^\xi).
\]

Obviously, $\Phi$ carries the chain map $\Theta \circ \Psi$. Then since $\Phi$ also carries $\text{id}_{\hat{C}_*(K)}$, it follows from Theorem 6.4 that $\Theta \circ \Psi$ is chain homotopic to $\text{id}_{\hat{C}_*(K)}$. \qed

Corollary 9.9. Let $(K, \xi)$ be a divisibly weighted simplicial complex. For any abelian group $G$, the chain maps $\Psi$ and $\Theta$ induce group isomorphisms:
\[
\Psi_* : \hat{H}_*(K, \hat{\partial}^\xi; G) \to H_*(K, \partial^\xi; G), \quad \Theta_* : H_*(K, \partial^\xi; G) \to \hat{H}_*(K, \hat{\partial}^\xi; G);
\]
\[
\Psi^* : H^*(K, \partial^\xi; G) \to \hat{H}^*(K, \hat{\partial}^\xi; G), \quad \Theta^* : \hat{H}^*(K, \hat{\partial}^\xi; G) \to H^*(K, \partial^\xi; G).
\]
The definition of weighted cup product \( \psi \) in (37) can be naturally extended to \((\hat{C}^*(K; R), \hat{\delta}^\xi)\) where \((K, \xi)\) is a descending divisibly weighted simplicial complex and \(R\) is a commutative ring with unit. Indeed, for any cochains \(\hat{\phi} \in \hat{C}^p(K; R)\) and \(\hat{\psi} \in \hat{C}^q(K; R)\), let \(\hat{\phi} \hat{\psi} \in \hat{C}^{p+q}(K; R)\) be the cochain whose value on an ordered \((p+q)\)-simplex \(\langle v_0, \ldots, v_{p+q}\rangle\) of \(K\) is defined by

\[
\hat{\phi} \hat{\psi}(\langle v_0, \ldots, v_{p+q}\rangle) := \xi(\langle v_0, \ldots, v_p\rangle) \xi(\langle v_p, \ldots, v_{p+q}\rangle) \hat{\phi}(\langle v_0, \ldots, v_p\rangle) \cdot \hat{\psi}(\langle v_p, \ldots, v_{p+q}\rangle).
\]

Note that we do not need to order the vertices of \(K\) to define \(\hat{\psi}\). So this is a more intrinsic way to define the weighted cup product as suggested by the following theorem.

**Theorem 9.10.** Let \((K, \xi)\) be a descending divisibly weighted simplicial complex. Then given any total ordering \(\prec\) of the vertices of \(K\), the chain map \(\Theta\) induces a ring isomorphism \(\Theta^* : (\hat{H}^*(K, \hat{\delta}^\xi; R), \hat{\psi}) \to (H^*(K, \delta^\xi; R), \psi)\).

**Proof.** By Corollary 9.9, \(\Theta\) already induces an additive isomorphism. So we only need to verify that \(\Theta\) is a ring homomorphism. Denote by

\[\Theta^\# : (\hat{C}^*(K; R), \hat{\delta}^\xi) \to (C^*(K; R), \delta^\xi)\]

the cochain map determined by \(\Theta\).

For any \(\hat{\phi} \in \hat{C}^p(K; R), \hat{\psi} \in \hat{C}^q(K; R)\) and any oriented \((p+q)\)-simplex \([v_0, \ldots, v_{p+q}]\) of \(K\) with \(v_0 \prec \cdots \prec v_{p+q}\), we have

\[
\Theta^\#(\hat{\phi} \hat{\psi})([v_0, \ldots, v_{p+q}]) = \hat{\phi} \hat{\psi}([v_0, \ldots, v_{p+q}]) = (41) \hat{\phi} \hat{\psi}(\Theta([v_0, \ldots, v_{p+q}]))
\]

\[
= \frac{\mu(\langle v_0, \ldots, v_p\rangle) \mu(\langle v_p, \ldots, v_{p+q}\rangle)}{\mu(\langle v_0, \ldots, v_{p+q}\rangle)} \hat{\phi}(\langle v_0, \ldots, v_p\rangle) \cdot \hat{\psi}(\langle v_p, \ldots, v_{p+q}\rangle)
\]

\[
= \frac{\mu(\langle v_0, \ldots, v_p\rangle) \mu(\langle v_p, \ldots, v_{p+q}\rangle)}{\mu(\langle v_0, \ldots, v_{p+q}\rangle)} \Theta^\#(\hat{\phi})([v_0, \ldots, v_p]) \cdot \Theta^\#(\hat{\psi})([v_p, \ldots, v_{p+q}])
\]

\[
= \Theta^\#(\hat{\phi} \hat{\psi})([v_0, \ldots, v_{p+q}]).
\]

So \(\Theta^*\) is a ring homomorphism and the theorem is proved. \(\square\)

By the above theorem, the ring structure induced by \(\psi\) on \(H^*(K, \delta^\xi; R)\) is independent on the choice of the ordering \(\prec\) up to ring isomorphisms.
Remark 9.11. The cochain map $\Psi^\#$ induced by $\Psi$ is not a ring homomorphism from $(H^*(K, \delta^\xi; R), \psi)$ to $(\check{H}^*(K, \check{\delta}^\xi; R), \check{\psi})$. So the map $\Psi^*$ is only an additive isomorphism.

9.4. Graded commutativity of the weighted cup product.

Now we are ready to prove that the weighted cup product $\psi$ on DW-cohomology is graded commutative. The proof is parallel to the proof of graded commutativity of singular cohomology in [10, Theorem 3.11] where ordered simplices here play the role of singular simplices.

First of all, for a divisibly weighted simplicial complex $(K, \xi)$, we introduce an auxiliary map $\rho$ on $\check{C}_*(K)$ as follows: for any ordered simplex $\langle v_0, \cdots, v_n \rangle$ of $K$,

$$(47) \quad \rho(\langle v_0, \cdots, v_n \rangle) := \varepsilon_n \langle v_n, \cdots, v_0 \rangle \text{ where } \varepsilon_n = (-1)^{\frac{n(n+1)}{2}}.$$  

Note that we have to use ordered simplices instead of oriented simplices to define $\rho$. This is because if we use an oriented simplex $[v_0, \cdots, v_n]$ in (47), the map $\rho$ would be the identity map. This is another reason why we need to introduce ordered simplices.

Lemma 9.12. Suppose $(K, \xi)$ is a divisibly weighted simplicial complex. Then the map $\rho : (\check{C}_*(K), \check{\delta}^\xi) \to (\check{C}_*(K), \check{\delta}^\xi)$ is a chain map and $\rho$ is chain homotopic to the identity map $\text{id}_{\check{C}_*(K)}$.

Proof. Assume that $\xi$ is descending since the ascending case is completely parallel. For any ordered simplex $\langle v_0, \cdots, v_n \rangle$ of $K$, we have

$$\check{\delta}^\xi \circ \rho(\langle v_0, \cdots, v_n \rangle) = \varepsilon_n \check{\delta}^\xi(\langle v_n, \cdots, v_0 \rangle)$$

$$= \varepsilon_n \sum_{i=0}^{n} (-1)^i \frac{\xi(\langle v_n, \cdots, \hat{v}_{n-i}, \cdots, v_0 \rangle)}{\xi(\langle v_n, \cdots, v_0 \rangle)} (v_n, \cdots, \hat{v}_{n-i}, \cdots, v_0);$$

$$\rho \circ \check{\delta}^\xi(\langle v_0, \cdots, v_n \rangle) = \rho \left( \sum_{i=0}^{n} (-1)^i \frac{\xi(\langle v_0, \cdots, \hat{v}_i, \cdots, v_n \rangle)}{\xi(\langle v_0, \cdots, v_n \rangle)} (v_0, \cdots, \hat{v}_i, \cdots, v_n) \right)$$

$$= \varepsilon_{n-1} \sum_{i=0}^{n} (-1)^{n-i} \frac{\xi(\langle v_0, \cdots, \hat{v}_{n-i}, \cdots, v_n \rangle)}{\xi(\langle v_0, \cdots, v_n \rangle)} (v_0, \cdots, \hat{v}_{n-i}, \cdots, v_n).$$

Since $\varepsilon_n = (-1)^n \varepsilon_{n-1}$, we obtain $\check{\delta}^\xi \circ \rho = \rho \circ \check{\delta}^\xi$, i.e. $\rho$ is a chain map.

Moreover, $\rho$ is clearly augmentation-preserving and the acyclic carrier $\Phi$ defined in (45) carries both $\rho$ and $\text{id}_{\check{C}_*(K)}$. So by Theorem 6.4, $\rho$ is chain homotopic to $\text{id}_{\check{C}_*(K)}$. \qed
For a commutative ring $R$ with unit, by Lemma 9.12 and the cohomology universal coefficient theorem, $\rho$ induces a cochain map

$$\rho^\#: (\tilde{C}^*(K; R), \tilde{\delta}^\xi) \to (\tilde{C}^*(K; R), \tilde{\delta}^\xi)$$

which is cochain homotopic to $\text{id}_{\tilde{C}^*(K; R)}$. So the map $\rho^*$ induced by $\rho$ on the cohomology group is the identity map, that is

$$\rho^* = \text{id} : \check{H}^p(K, \tilde{\delta}^\xi; R) \to \check{H}^p(K, \tilde{\delta}^\xi; R).$$

**Lemma 9.13.** Let $(K, \xi)$ be a descending divisibly weighted simplicial complex. For any cochains $\hat{\phi} \in \tilde{C}^p(K; R)$, $\hat{\psi} \in \tilde{C}^q(K; R)$,

$$\rho^\#(\hat{\phi} \hat{\psi}) = (-1)^{pq} \rho^\#(\hat{\psi}) \hat{\phi}. $$

**Proof.** For any ordered simplex $\langle v_0, \ldots, v_{p+q} \rangle$ of $K$, we have

$$\rho^\#(\hat{\phi} \hat{\psi})(\langle v_0, \ldots, v_{p+q} \rangle) = \varepsilon_{p+q} \cdot \hat{\phi}(\langle v_{p+q}, \ldots, v_0 \rangle) \check{H}^\psi(\langle v_{p+q}, \ldots, v_q \rangle) \cdot \hat{\psi}(\langle v_q, \ldots, v_0 \rangle).$$

Then the lemma follows from the simple identity $\varepsilon_{p+q} = (-1)^{pq} \varepsilon_p \varepsilon_q$. \hfill \Box

**Theorem 9.14.** Let $(K, \xi)$ be a descending divisibly weighted simplicial complex. For any cohomology classes $[\hat{\phi}] \in \check{H}^p(K, \tilde{\delta}^\xi; R)$ and $[\hat{\psi}] \in \check{H}^q(K, \tilde{\delta}^\xi; R)$,

$$[\hat{\phi}] \hat{\psi}^* = (-1)^{pq} [\hat{\psi}] \check{H}^\phi(\hat{\phi}).$$

**Proof.** By the property of $\rho$ in (48) and Lemma 9.13, we obtain

$$[\hat{\phi}] \hat{\psi}^* = \rho^*([\hat{\phi}] \hat{\psi}^*) = [\rho^\#(\hat{\phi} \check{H}^\psi)] = (-1)^{pq} [\rho^\#(\hat{\psi}) \check{H}^\phi(\hat{\phi})] = (-1)^{pq} [\rho^\#(\hat{\psi}) \check{H}^\phi(\hat{\phi})] = (-1)^{pq} [\hat{\psi}] \hat{\phi}^*.$$

So the product $\hat{\psi}$ on $\check{H}^*(K, \tilde{\delta}^\xi; R)$ is graded commutative. \hfill \Box

Then by the ring isomorphism $\Theta^*$ (see Theorem 9.10), we obtain the following corollary immediately.
Corollary 9.15. For a descending divisibly weighted simplicial complex \((K, \xi)\), the weighted cup product \(\cup\) on \(H^\ast(K, \delta^\ast; R)\) is graded commutative.

It follows that for a weighted polyhedron \((X, \lambda)\), the weighted cup product \(\cup\) on the DW-cohomology \(H^\ast_{DW}(X, \lambda; R)\) is graded commutative.

Next, we give an example to show the difference between the DW-cohomology ring and the ordinary simplicial cohomology ring of a weighted polyhedron.

Example 9.16. Let \(\Sigma_2\) denote a closed orientable surface of genus 2 which is glued from an octagon as shown in Figure 8. For any integer \(k \geq 2\), let \(\lambda_k\) be a weight function on \(\Sigma_2\) which assigns 1 to all the interior point of the octagon and assigns \(k\) to those points from gluing the boundary edges of the octagon. Then the middle picture in Figure 8 is a divisibly weighted \(\Delta\)-triangulation of the pseudo-orbifold \((\Sigma_2, \lambda_k)\), from which we can compute the DW-cohomology group of \((\Sigma_2, \lambda_k)\):

\[
H^p_{DW}(\Sigma_2, \lambda_k) \cong H^p(\Sigma_2) \cong \begin{cases} 
\mathbb{Z}, & p = 0, 2; \\
\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}, & p = 1; \\
0, & \text{otherwise}.
\end{cases}
\]

Figure 8. A pseudo-orbifold based on a genus 2 closed orientable surface

The DW-cohomology ring \((H^\ast_{DW}(\Sigma_2, \lambda_k), \cup)\), however, is not isomorphic as a graded ring to the ordinary simplicial cohomology ring \((H^\ast(\Sigma_2), \cup)\) of \(\Sigma_2\). Indeed, let \(\phi_1, \phi_2, \psi_1\) and \(\psi_2\) denote the generators of \(H^1_{DW}(\Sigma_2, \lambda_k)\) that are dual to \(a_1, a_2, b_1\) and \(b_2\), respectively. More specifically, \(\phi_i\) (or \(\psi_i\)) has the value 1 on \(a_i\) (or \(b_i\)) and the value \(k\) on the two edges meeting the dotted arc that intersects \(a_i\) (or \(b_i\)) (see the right picture in Figure 8), and has the value 0 on all other edges. Let \(\zeta\) denote the generator of \(H^2_{DW}(\Sigma_2, \lambda_k)\) which has the value 1 on all the 2-simplices of \(\Sigma_2\). Then we can directly compute from the \(\Delta\)-triangulation and the definition of \(\cup\) that the only nontrivial relations among the generators of
$H^1_{DW}(\Sigma_2, \lambda_k)$ and $H^2_{DW}(\Sigma_2, \lambda_k)$ are:

$$\phi_1 \cup \psi_1 = \phi_2 \cup \psi_2 = k^2 \cdot \zeta.$$ 

So $\zeta$ is a multiplicative generator of $(H^*(\Sigma_2, \lambda_k), \cup)$ since $k \geq 2$. So there is no graded ring isomorphism between $(H^*(\Sigma_2), \cup)$ and $(H^*(\Sigma_2), \cup)$.

9.5. Weighted cap product.

For a descending divisibly weighted simplicial complex $(K, \xi)$, we can also define a product between elements of $H_\ast(K, \partial^2; R)$ and $H^\ast(K, \delta^\xi; R)$ for any coefficient ring $R$. To avoid choosing a total ordering of the vertices of $K$, we give the definition of the product through ordered chains and cochains first.

**Definition 9.17 (Weighted Cap Product).** Let $(K, \xi)$ be a descending divisibly weighted simplicial complex. Define $R$-bilinear weighted cap product

$$\hat{\wedge} : \hat{C}_n(K; R) \times \hat{C}^p(K; R) \to \hat{C}_{n-p}(K; R), \quad 0 \leq p \leq n$$

by: for any cochain $\hat{\phi} \in \hat{C}^p(K; R)$ and any ordered $n$-simplex $\langle v_0, \ldots, v_n \rangle$ of $K$,

$$\langle v_0, \ldots, v_n \rangle \hat{\wedge} \hat{\phi} := \frac{\xi(\langle v_0, \ldots, v_p \rangle) \xi(\langle v_p, \ldots, v_n \rangle) \hat{\phi}(\langle v_0, \ldots, v_p \rangle)}{\xi(\langle v_0, \ldots, v_n \rangle)} \langle v_p, \ldots, v_n \rangle.$$ 

The coefficient on the right hand side is integral because $\xi$ is a descending weight.

For any chain $\hat{\alpha} \in \hat{C}_n(K; R)$, it is routine to check from the definitions that

$$(49) \quad \hat{\partial}^\xi(\hat{\alpha} \hat{\wedge} \hat{\phi}) = (-1)^p \left( (\hat{\partial}^\xi \hat{\alpha}) \hat{\wedge} \hat{\phi} - \hat{\alpha} \hat{\wedge} \hat{\delta}^\xi \hat{\phi} \right).$$

This implies that there is an induced product

$$\hat{H}_n(K, \delta^\xi; R) \times \hat{H}^p(K, \partial^\xi; R) \xrightarrow{\hat{\wedge}} \hat{H}_{n-p}(K, \delta^\xi; R).$$

Then by the isomorphisms $\Theta_\ast$ and $\Theta^\ast$ in Corollary 9.9, we obtain a product

$$(50) \quad H_n(K, \delta^\xi; R) \times H^p(K, \partial^\xi; R) \xrightarrow{\hat{\wedge}} H_{n-p}(K, \delta^\xi; R).$$

called the weighted cap product of $(K, \xi)$.

Moreover, the following lemma tells us that the two products $\cup$ and $\hat{\wedge}$ are compatible just as the ordinary cup product and cap product do.

**Lemma 9.18.** Let $(K, \xi)$ be a descending divisibly weighted simplicial complex. Then for any cochains $\hat{\phi} \in \hat{C}^p(K; R)$, $\hat{\psi} \in \hat{C}^q(K; R)$ and any chain $\hat{\alpha} \in \hat{C}_n(K; R)$ with $p + q \leq n$,

$$(\hat{\alpha} \hat{\wedge} \hat{\phi}) \hat{\wedge} \hat{\psi} = \hat{\alpha} \hat{\wedge} (\hat{\phi} \cup \hat{\psi}).$$
Proof. For any ordered $n$-simplex $\langle v_0, \cdots, v_n \rangle$ of $K$, we obtain
\[
\xi(\langle v_0, \cdots, v_p \rangle) \xi(\langle v_{p+q}, \cdots, v_n \rangle) = \xi(\langle v_0, \cdots, v_n \rangle) \xi(\langle v_{p+q}, \cdots, v_n \rangle).
\]
Then it is easy see that
\[
(\langle v_0, \cdots, v_n \rangle) \cap (\hat{\phi} \cup \hat{\psi}) = (\langle v_0, \cdots, v_n \rangle) \cap (\hat{\phi} \cup \hat{\psi}).
\]

By Lemma 9.18 and the isomorphisms $\Theta_*$ and $\Theta^*$ in Corollary 9.9, we obtain the following corollary immediately.

**Corollary 9.19.** Let $(K, \xi)$ be a descending divisibly weighted simplicial complex. Then for any cohomology classes $[\phi] \in H^p(K, \delta^\xi; R)$, $[\psi] \in H^q(K, \delta^\xi; R)$ and a homology class $[\alpha] \in H_n(K, \partial^\xi; R)$ with $p + q \leq n$,
\[
([\alpha] \cap [\phi]) \cap [\psi] = [\alpha] \cap ([\phi] \cup [\psi]).
\]
By the above corollary, the product $\cap$ induces a right $H^*(K, \delta^\xi; R)$-module structure on $H_*(K, \partial^\xi; R)$.

By choosing a divisibly weighted triangulation of a weighted polyhedron $(X, \lambda)$, we can define the weighted cap product $\cap$ of DW-homology and DW-cohomology $(X, \lambda)$, that is: for any $0 \leq p \leq n$,
\[
H^p_{DW}(X, \lambda; R) \times H^n_{DW}(X, \lambda; R) \xrightarrow{\cap} H^p_{DW}(X, \lambda; R).
\]

**Remark 9.20.** We cannot define any $R$-bilinear product analogous to $\cap$ between a AW-homology class and a DW-cohomology class (or between a DW-homology class and a AW-cohomology class). This is because the relation in (49) would fail if the chain and the cochain were of different type.

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