BALANCED DERIVATIVES, IDENTITIES, AND BOUNDS FOR TRIGONOMETRIC AND BESSEL SERIES

BRUCE C. BERNDT, MARTINO FASSINA, SUN KIM, AND ALEXANDRU ZAHARESCU

ABSTRACT. Motivated by two identities published with Ramanujan’s lost notebook and connected, respectively, with the Gauss circle problem and the Dirichlet divisor problem, in an earlier paper, three of the present authors derived representations for certain sums of products of trigonometric functions as double series of Bessel functions. These series are generalized in the present paper by introducing the novel notion of balanced derivatives, leading to further theorems. As we will see below, the regions of convergence in the unbalanced case are entirely different than those in the balanced case. From this viewpoint, it is remarkable that Ramanujan had the intuition to formulate entries that are, in our new terminology, “balanced”. If \( x \) denotes the number of products of the trigonometric functions appearing in our sums, in addition to proving the identities mentioned above, theorems and conjectures for upper and lower bounds for the sums as \( x \to \infty \) are established.

1. INTRODUCTION AND MAIN RESULTS

In a series of papers [5], [6], [7], [8], [11] written by three of the present authors and J. Li, they examined two formulas of Ramanujan in an unpublished fragment found in [17, p. 335]. The two formulas are connected with the famous Gauss circle problem and the equally famous Dirichlet divisor problem. Each of the two formulas has three distinct interpretations. Ramanujan’s formulas and the methods developed to prove them have generated further research, in particular, in [6] and [8]. In this paper, we continue our study by examining “balanced” derivatives of the series representations and making applications to the trigonometric sums studied in [8].

In order to state Ramanujan’s formulas, the Gauss circle problem, and the Dirichlet divisor problem, it is necessary to first define the relevant Bessel functions appearing in Ramanujan’s identities. Let \( J_\nu(z) \) denote the ordinary Bessel function of order \( \nu \). Define

\[
I_\nu(z) := -Y_\nu(z) - \frac{2}{\pi}K_\nu(z),
\]

where \( Y_\nu(z) \) denotes the Bessel function of imaginary argument of order \( \nu \) given by

\[
Y_\nu(z) := \frac{J_\nu(z) \cos(\nu \pi) - J_{-\nu}(z)}{\sin(\nu \pi)}, \quad |z| < \infty,
\]

and \( K_\nu(z) \) denotes the modified Bessel function of order \( \nu \) defined by

\[
K_\nu(z) := -\frac{\pi}{2} \frac{e^{\pi i \nu/2} J_{-\nu}(iz) - e^{-\pi i \nu/2} J_\nu(iz)}{\sin(\nu \pi)}, \quad -\pi < \arg z < -\frac{1}{2} \pi.
\]

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If \( \nu \) is an integer \( n \), it is understood that we define the functions by taking the limits as \( \nu \to n \) in (1.2) and (1.3).

We now describe the Gauss circle problem and the Dirichlet divisor problem. Detailed discussions and references for these two famous problems can be found in [9].

Let \( r_2(n) \) denote the number of ways in which the positive integer \( n \) can be expressed as a sum of two squares, where different orders and different signs of the summands are regarded as distinct representations of \( n \) as a sum of two squares. For example, \( 5 = (\pm 2)^2 + (\pm 1)^2 = (\pm 1)^2 + (\pm 2)^2 \), and so \( r_2(5) = 8 \). Let

\[
R(x) := \sum_{0 \leq n \leq x} r_2(n),
\]

where \( r_2(0) = 1 \) and the prime \(^\prime\) indicates that, if \( n = x \), only \( \frac{1}{2}r_2(x) \) is counted. Then

\[
R(x) =: \pi x + P(x) = \pi x + \sum_{n=1}^{\lfloor \frac{x}{2} \rfloor} r_2(n) \left( \frac{x}{n} \right)^{1/2} J_1(2\pi \sqrt{nx}),
\]

where the representation for \( P(x) \) on the far right side is due to Hardy and Ramanujan [14, p. 265]. Finding the precise order of magnitude of \( P(x) \), as \( x \to \infty \), is known as the Gauss circle problem.

Throughout this paper, for arithmetic functions \( f \) and \( g \), let

\[
\sum_{nm \leq x} f(n)g(m) := \begin{cases} \sum_{nm \leq x} f(n)g(m), & \text{if } x \text{ is not an integer}, \\ \sum_{nm \leq x} f(n)g(m) - \frac{1}{2} \sum_{nm = x} f(n)g(m), & \text{if } x \text{ is an integer}. \end{cases}
\]

In [7], three of the authors proved the following enigmatic identity of Ramanujan from his lost notebook [17].

**Entry 1.1.** [17, p. 335] If \( 0 < \theta < 1 \) and \( x > 0 \), then

\[
\sum_{n \leq x} \left\lfloor \frac{x}{n} \right\rfloor \sin(2\pi n\theta) = \pi x \left( \frac{1}{2} - \theta \right) - \frac{1}{4} \cot(\pi \theta)
+ \frac{1}{2} \sqrt{x} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \left\{ J_1 \left( 4\pi \sqrt{m(n + \theta)x} \right) - \frac{1}{\sqrt{m(n + 1 - \theta)}} \right\},
\]

where, as customary, \( \lfloor x \rfloor \) denotes the greatest integer less than or equal to \( x \).

The identity (1.5) can be seen as a 2-variable analogue of (1.4). If we set \( \theta = \frac{1}{4} \), by an elementary theorem on \( r_2(n) \) [15, p. 313], the left sides of (1.4) and (1.5) are identical.

Let \( d(n) \) denote the number of positive divisors of the positive integer \( n \), and let

\[
D(x) := \sum_{n \leq x} \prime d(n).
\]
Then
\[
D(x) = x (\log x + (2\gamma - 1)) + \Delta(x) = x (\log x + (2\gamma - 1)) + \frac{1}{4} + \sum_{n=1}^{1} d(n) \left(\frac{x}{n}\right)^{1/2} I_1(4\pi \sqrt{n}x),
\]  
(1.6)

where \( x > 0 \), \( \gamma \) denotes Euler’s constant, the identity (1.6) is due to Dirichlet [13] and defines the “error term” \( \Delta(x) \), \( I_1(x) \) is defined in (1.1), and the series representation for \( \Delta(x) \) on the right side of (1.7) is due to Voronoï [18]. Finding the optimal bound for \( \Delta(x) \) as \( x \to \infty \) is the Dirichlet divisor problem. Voronoï [18] proved that
\[
\Delta(x) = O(x^{1/3} \log x).
\]  
(1.8)
The upper bound for \( \Delta(x) \) given in (1.8) is not the best currently known. See [9] for a list of upper bounds that have been obtained for \( \Delta(x) \). Furthermore [14], [16], [12, p. 130],
\[
\Delta(x) = \Omega_\pm (x^{1/4}),
\]  
(1.9)
as \( x \to \infty \). We say that \( f(x) = \Omega_+(x^\theta) \) if there exists a sequence \( \{x_n\} \to \infty \) such that
\[
f(x_n) \leq C_1(x_n)^\theta
\]
fails to hold for every positive constant \( C_1 \). Similarly, \( f(x) = \Omega_-(x^\theta) \) if there exists a sequence \( \{x_n'\} \to \infty \) such that
\[
f(x_n') \geq -C_2(x_n')^\theta
\]
fails to hold for every positive constant \( C_2 \).

A proof of the following second enigmatic identity of Ramanujan from [17, p. 335] has been given by J. Li and two of the present authors [10]. When \( \theta = 0 \), the left-hand side of (1.10) below reduces to the left-hand side of (1.7).

**Entry 1.2.** [17, p. 335] For \( x > 0 \) and \( 0 < \theta < 1 \),
\[
\sum_{n\leq x} \left\lfloor \frac{x}{n} \right\rfloor \cos(2\pi n\theta) = \frac{1}{4} - x \log(2 \sin(\pi \theta)) + \frac{1}{2} \sqrt{x} \sum_{m=1}^{1} \sum_{n=0}^{1} \left\{ I_1 \left( \frac{4\pi \sqrt{m(n+\theta)}x}{\sqrt{m(n+\theta)}} \right) + I_1 \left( \frac{4\pi \sqrt{m(n+1-\theta)}x}{\sqrt{m(n+1-\theta)}} \right) \right\}.
\]  
(1.10)

**Remark 1.3.** Examining (1.7) and (1.10), we observe that “big O” conjectures and theorems about the error term \( \Delta(x) \), which traditionally and frequently involve the series of Bessel functions on the right-hand side of (1.7), pertain to the double series of Bessel functions on the right-hand side of (1.10), although, as of the present, (1.10) has not been employed in deriving “big O” theorems.

In this paper we prove new identities in the spirit of Ramanujan, where on the left sides are sums of products of two trigonometric functions, while on the right sides are double series of Bessel functions. Our formulas involve two parameters \( \sigma, \theta \) in the interval \((0, 1)\) and stem from identities that three of the present authors proved in [8]. The novelty in the current paper consists in the possibility of taking termwise derivatives with respect to \( \sigma \) and \( \theta \). Since we are only allowed (by reasons of convergence) to take the same number of derivatives
in $\sigma$ and in $\theta$, we say that the identities thus obtained are “balanced”. As an interesting application, we consider two identities that were independently proved in [8], and we show that one of them can be obtained as the first balanced derivative of the other (see Section 5).

In the second portion of the paper, our goal is to derive “big O” and $\Omega$ theorems for sums of two trigonometric functions. Unfortunately, in many cases, for lower bounds we are only able to make conjectures. We also extend our study of sine sums to sums of $k$ sines, $k \geq 2$.

2. IDENTITIES FOR TRIGONOMETRIC SUMS IN TERMS OF BESSEL FUNCTIONS

Here is our first main result.

**Theorem 2.1.** Let $\sigma, \theta$ be in the interval $(0, 1)$, and let $x > 0$. Then for every non-negative integer $k$,

$$\frac{\partial^{2k}}{\partial \sigma^k \partial \theta^k} \left\{ \sum_{mn \leq x}^{'} \cos(2\pi m \sigma) \sin(2\pi n \theta) + \frac{\cot(\pi \theta)}{4} \right\}$$

$$= \left[ \frac{4}{\sqrt{x}} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{J_1(4\pi \sqrt{(m + \sigma)(n + \theta)x})}{\sqrt{(m + \sigma)(n + \theta)}}, \right. \left. \frac{J_1(4\pi \sqrt{(m + 1 - \sigma)(n + \theta)x})}{\sqrt{(m + 1 - \sigma)(n + \theta)}}, \right. \left. - \frac{J_1(4\pi \sqrt{(m + \sigma)(n + 1 - \theta)x})}{\sqrt{(m + \sigma)(n + 1 - \theta)}}, \right. \left. - \frac{J_1(4\pi \sqrt{(m + 1 - \sigma)(n + 1 - \theta)x})}{\sqrt{(m + 1 - \sigma)(n + 1 - \theta)}} \right\}. \quad (2.1)$$

The proof of Theorem 2.1 relies on a detailed study of the convergence of a double series more general than the one appearing on the right-hand side of (2.1).

Let $\sigma, \theta, x$ be as in Theorem 2.1. Let $\alpha, \beta$ be non-negative integers and $s, w$ complex numbers. Consider the double series

$$G^{\alpha, \beta}(x, \sigma, \theta, s, w) := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\partial^{\alpha + \beta}}{\partial \sigma^\alpha \partial \theta^\beta} \left\{ \frac{J_1(4\pi \sqrt{(m + \sigma)(n + \theta)x})}{(m + \sigma)^\alpha(n + \theta)^w}, \right. \left. \frac{J_1(4\pi \sqrt{(m + \sigma)(n + 1 - \theta)x})}{(m + \sigma)^\alpha(n + 1 - \theta)^w}, \right. \left. - \frac{J_1(4\pi \sqrt{(m + 1 - \sigma)(n + \theta)x})}{(m + 1 - \sigma)^\beta(n + \theta)^w}, \right. \left. - \frac{J_1(4\pi \sqrt{(m + 1 - \sigma)(n + 1 - \theta)x})}{(m + 1 - \sigma)^\beta(n + 1 - \theta)^w} \right\}. \quad (2.2)$$

We determine values of $\alpha, \beta, s, w$ for which the double series $G^{\alpha, \beta}(x, \sigma, \theta, s, w)$ converges.

**Theorem 2.2.** Let $G^{\alpha, \beta}(x, \sigma, \theta, s, w)$ be defined as above. Assume that

$$\begin{align*}
4 \text{Re}(s) + 2\alpha - 2\beta &> 1, \\
4 \text{Re}(w) + 2\beta - 2\alpha &> 1.
\end{align*}$$
Moreover, if \( x \) is an integer, assume that \( \Re(s) + \Re(w) > \frac{5}{6} \), while if \( x \) is not an integer, assume that \( \Re(s) + \Re(w) > \frac{5}{6} \). Then the double series \( G^{\alpha, \beta}(x, \sigma, \theta, s, w) \) converges uniformly with respect to \( \sigma \) and \( \theta \) in any compact subset of \( (0, 1)^2 \).

Consider the interesting case \( s = \frac{1}{2} = w \), which corresponds to the setting of Theorem 2.1. To meet the conditions of Theorem 2.2 ensuring the convergence of (2.2), the only possibility is to take \( \alpha = \beta = k \) (“balanced” situation).

**Remark 2.3.** Theorem 2.2 also shows, for every choice of non-negative integers \( \alpha, \beta \), that there exists an unbounded region \( D_{\alpha, \beta} \) of \( \mathbb{C}^2 \) such that for every \( (s, w) \in D_{\alpha, \beta} \), the corresponding series (2.2) converges.

With similar methods as the those used to prove Theorem 2.1, we establish other “balanced” identities similar to (2.1). In these new identities the left-hand side contains only cosines or only sines, respectively.

First recall, for each integer \( \nu \), that \( I_\nu \) was defined in (1.1). Let then

\[
T_\frac{\nu}{2}(x) := \int_0^\infty J_\frac{\nu}{2}(u) J_\frac{\nu}{2}\left(\frac{x}{u}\right) du.
\]

(In [8, p. 71], the definition (2.3) is misprinted; replace \( J_\frac{\nu}{2}(x) \) by \( J_\frac{\nu}{2}(\frac{x}{u}) \) there.)

**Theorem 2.4.** Let \( \sigma, \theta \) be in the interval \( (0, 1) \), and let \( x > 0 \). Then for every non-negative integer \( k \),

\[
\frac{\partial^{2k}}{\partial \sigma^k \partial \theta^k} \left\{ \sum_{mn \leq x} \cos(2\pi m\sigma) \cos(2\pi n\theta) - \frac{1}{4} \right\} = \sqrt{x} \int_0^\infty \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{\partial^{2k}}{\partial \sigma^k \partial \theta^k} \left\{ \frac{I_1(4\pi \sqrt{(m + \sigma)(n + \theta)x})}{\sqrt{(m + \sigma)(n + \theta)}} + \frac{I_1(4\pi \sqrt{(m + 1 - \sigma)(n + \theta)x})}{\sqrt{(m + 1 - \sigma)(n + \theta)}} \right. 
\]

\[
+ \left. I_1(4\pi \sqrt{(m + \sigma)(n + 1 - \theta)x}) + \frac{I_1(4\pi \sqrt{(m + 1 - \sigma)(n + 1 - \theta)x})}{\sqrt{(m + 1 - \sigma)(n + 1 - \theta)}} \right\}. \tag{2.4}
\]

**Theorem 2.5.** Let \( \sigma, \theta \) be in the interval \( (0, 1) \), and let \( x > 0 \). Then for every non-negative integer \( k \),

\[
\frac{\partial^{2k}}{\partial \sigma^k \partial \theta^k} \left\{ \sum_{mn \leq x} \sin(2\pi m\sigma) \sin(2\pi n\theta) \right\} = x \sqrt{x} \int_0^\infty \sum_{m=0}^\infty \sum_{n=0}^\infty \frac{\partial^{2k}}{\partial \sigma^k \partial \theta^k} \left\{ \frac{T_\frac{\nu}{2}(4\pi^2 (m + \sigma)(n + \theta)x)}{\sqrt{(m + \sigma)(n + \theta)}} - \frac{T_\frac{\nu}{2}(4\pi^2 (m + 1 - \sigma)(n + \theta)x)}{\sqrt{(m + 1 - \sigma)(n + \theta)}} \right. 
\]

\[
- \left. \frac{T_\frac{\nu}{2}(4\pi^2 (m + \sigma)(n + 1 - \theta)x)}{\sqrt{(m + \sigma)(n + 1 - \theta)}} + \frac{T_\frac{\nu}{2}(4\pi^2 (m + 1 - \sigma)(n + 1 - \theta)x)}{\sqrt{(m + 1 - \sigma)(n + 1 - \theta)}} \right\}. \tag{2.5}
\]
Similarly, for the “error terms” in (3.5),

\[ \frac{\partial^{\alpha+\beta}}{\partial \sigma^\alpha \partial \theta^\beta} J_1(4\pi \sqrt{(m+\sigma)(n+\theta)x}) = \sum_{\nu \in A} c_{\nu} J_\nu(4\pi \sqrt{(m+\sigma)(n+\theta)x}) \]

\[\frac{(m+\sigma)^{s+\frac{\alpha-1}{2} - \frac{\beta}{2} - \frac{\nu-1}{2}} (n+\theta)^{w+\frac{\beta}{2} + \frac{\nu-1}{2}}} + \cdots.\]

(3.1)

Here \( A \) is a finite subset of \( \mathbb{Z} \), the constants \( c_{\nu} \) are non-negative, and the dots \( \cdots \) denote a sum of terms that have the form

\[ r_{\nu} J_\nu(4\pi \sqrt{(m+\sigma)(n+\theta)x}) \]

\[\frac{(m+\sigma)^{s+\frac{\alpha-1}{2} - \frac{\beta}{2} - \frac{\nu-1}{2}} (n+\theta)^{w+\frac{\beta}{2} + \frac{\nu-1}{2}}} \]

\[\nu \in \mathbb{Z}, \ r_{\nu} \in \mathbb{R},\]

where

\[ \begin{cases} \Re(\gamma) \geq \Re(s) + \frac{\alpha}{2} - \frac{\beta}{2} \\ \Re(\delta) \geq \Re(w) + \frac{\beta}{2} - \frac{\alpha}{2} \end{cases}, \]

with at least one of the two inequalities in \((3.3)\) being strict.

**Proof.** Recall from [19, p. 17] the following identity, which holds for every integer \( \nu \):

\[ J_{\nu-1}(z) - J_{\nu+1}(z) = 2J_{\nu}'(z). \]

(4.4)

We argue by induction on the total number of derivatives \( k = \alpha + \beta \). For \( k = 0 \) there is nothing to prove. Assume now that the statement holds for some \( k \geq 0 \). We will prove that it holds for \( k + 1 \). Write \( k + 1 = \alpha + \beta \), and assume without loss of generality that \( \alpha \geq 1 \).

By the inductive hypothesis,

\[ \frac{\partial^{k+1}}{\partial \sigma^\alpha \partial \theta^\beta} J_1(4\pi \sqrt{(m+\sigma)(n+\theta)x}) = \partial \frac{\partial^{k}}{\partial \sigma^\alpha} \left[ \sum_{\nu \in A} c_{\nu} J_\nu(4\pi \sqrt{(m+\sigma)(n+\theta)x}) \right] \]

\[\frac{(m+\sigma)^{s+\frac{\alpha-1}{2} - \frac{\beta}{2} - \frac{\nu}{2} - \frac{\nu-1}{2}} (n+\theta)^{w+\frac{\beta}{2} + \frac{\nu}{2} - \frac{\nu-1}{2}}} + \cdots \].

(3.5)

The dots \( \cdots \) in \((3.5)\) stand for terms of the form \((3.2)\), with the exponents \( \gamma \) and \( \delta \) satisfying

\[ \begin{cases} \Re(\gamma) \geq \Re(s) + \frac{\alpha-1}{2} - \frac{\beta}{2} \\ \Re(\delta) \geq \Re(w) + \frac{\beta}{2} - \frac{\alpha-1}{2} \end{cases}. \]

(3.6)

where at least one of the two inequalities is strict. By the chain rule and \((3.4)\),

\[ \frac{\partial}{\partial \sigma} \left[ J_\nu(4\pi \sqrt{(m+\sigma)(n+\theta)x}) \right] = \pi \sqrt{x} \frac{(J_{\nu-1} - J_{\nu+1})(4\pi \sqrt{(m+\sigma)(n+\theta)x})}{(m+\sigma)^{s+\frac{\alpha-1}{2} - \frac{\beta}{2} - \frac{\nu}{2} - \frac{\nu-1}{2}} (n+\theta)^{w+\frac{\beta}{2} + \frac{\nu}{2} - \frac{\nu-1}{2}}} \]

\[+ \left( \frac{\beta}{2} - \frac{\alpha}{2} - \sigma \right) \frac{J_\nu(4\pi \sqrt{(m+\sigma)(n+\theta)x})}{(m+\sigma)^{s+\frac{\alpha+1}{2} - \frac{\beta}{2} - \frac{\nu}{2} - \frac{\nu-1}{2}}} \].

(3.7)

Similarly, for the “error terms” in \((3.5)\),

\[ \frac{\partial}{\partial \sigma} \left[ J_\nu(4\pi \sqrt{(m+\sigma)(n+\theta)x}) \right] = \pi \sqrt{x} \frac{(J_{\nu-1} - J_{\nu+1})(4\pi \sqrt{(m+\sigma)(n+\theta)x})}{(m+\sigma)^{s+\frac{\alpha-1}{2} - \frac{\beta}{2} - \frac{\nu}{2} - \frac{\nu-1}{2}} (n+\theta)^{w+\frac{\beta}{2} + \frac{\nu}{2} - \frac{\nu-1}{2}}} \]

\[ - \gamma J_\nu(4\pi \sqrt{(m+\sigma)(n+\theta)x}) \]

\[\frac{(m+\sigma)^{s+\frac{\alpha}{2} - \frac{\beta}{2} - \frac{\nu}{2} - \frac{\nu-1}{2}} (n+\theta)^{w+\frac{\beta}{2} + \frac{\nu}{2} - \frac{\nu-1}{2}}} \].

(3.8)
The second term on the right side of (3.7) and both terms on the right side of (3.8) are of the form
\[ r_νJ_ν(4\pi \sqrt{(m + σ)(n + θ)x}) \](m + σ)\(n(n + θ)^δ_1\), \( ν ∈ ℤ, r_ν ∈ ℜ, \)
where by (3.6) the complex numbers \( γ_1 \) and \( δ_1 \) satisfy
\[
\begin{cases}
\text{Re}(γ_1) ≥ \text{Re}(s) + \frac{α}{2} - \frac{β}{2}, \\
\text{Re}(δ_1) ≥ \text{Re}(w) + \frac{β}{2} - \frac{α}{2},
\end{cases}
\]
with at least one of the two inequalities being strict. Substituting the identities (3.7) and (3.8) into (3.5), we obtain (3.1), thus completing the proof.

The same analysis can be repeated for the other summands appearing in (2.2), separating in each case the “main term” from the “error terms.” We then obtain identities analogous to (3.1). For instance,
\[
\frac{∂^{α+β}}{∂σ^α∂θ^β} J_1(4\pi \sqrt{(m + 1 - σ)(n + θ)x}) \]
\[ (m + 1 - σ)^s(n + θ)^w \]
\[ = (-1)^α \sum_{ν∈A} c_ν J_ν(4\pi \sqrt{(m + 1 - σ)(n + θ)x}) \]
\[ (m + 1 - σ)^s+\frac{α}{2} - \frac{β}{2}(n + θ)^s+\frac{β}{2} - \frac{α}{2} + \cdots. \]  
(3.9)

Note that the \( c_ν \) are the same exact constants that appear in (3.1).

We shift our attention to the double series
\[
G_ν(x, σ, θ, s, w) := \sum_{m=0}^{∞} \sum_{n=0}^{∞} \left( J_ν(4\pi \sqrt{(m + σ)(n + θ)x}) \right) \]
\[ (m + σ)^s(n + θ)^w \]
\[ ± J_ν(4\pi \sqrt{(m + 1 - σ)(n + θ)x}) \]
\[ (m + 1 - σ)^s(n + θ)^w \]
\[ ± J_ν(4\pi \sqrt{(m + 1 - σ)(n + 1 - θ)x}) \]
\[ (m + 1 - σ)^s(n + 1 - θ)^w \]
\[ ± J_ν(4\pi \sqrt{(m + 1 - σ)(n + 1 - θ)x}) \]
\[ (m + 1 - σ)^s(n + 1 - θ)^w \],
(3.10)

where \( x, σ, θ, s, w \) are as before, and \( ν \) is an integer. The designation \( ± \) indicates either choice of sign, so that \( G_ν(x, σ, θ, s, w) \) actually represents eight different double series. We need to consider these different combinations of signs because of the powers of \( -1 \) appearing as factors in (3.9) and the analogous formulas for the other terms of (2.2). We will see below that the choice of signs does not affect the convergence of the series.

Theorem 2.2 follows by combining Lemma 3.1 with the following result on the convergence of (3.10).

**Theorem 3.2.** Let \( G_ν(x, σ, θ, s, w) \) be defined as above. Assume that \( 4 \text{Re}(s) > 1 \), and that \( 4 \text{Re}(w) > 1 \). Moreover, if \( x \) is an integer, assume \( \text{Re}(s) + \text{Re}(w) > \frac{25}{26} \), while if \( x \) is not an integer, assume \( \text{Re}(s) + \text{Re}(w) > \frac{5}{6} \). Then the double series \( G_ν(x, σ, θ, s, w) \) converges uniformly with respect to \( σ \) and \( θ \) in any compact subset of \((0, 1)^2\).

The remaining part of this section is devoted to proving Theorem 3.2.
We start by recalling the following fact. Let \( z \) be a complex number, and \( j \) a non-negative integer. Recall the notation for the binomial coefficient

\[
\binom{z}{j} = \frac{z(z-1)(z-2) \cdots (z-j+1)}{j!}.
\]

Then, for every complex number \( \zeta \), with \(|\zeta| < 1\), we have the binomial theorem

\[
(1 + \zeta)^z = \sum_{j=0}^{\infty} \binom{z}{j} \zeta^j.
\]

Using (3), we can easily show that, for every \( \theta \in (0, 1) \),

\[
\frac{1}{(n+\theta)^z} = \frac{1}{n^z} + O\left(\frac{1}{n^{z+1}}\right).
\]  

(3.11)

This simple formula will be used several times in the following discussion.

Now recall the following asymptotic formulas, which hold for any positive integer \( \nu \) [19, p. 199]:

\[
J_{\nu}(z) = \nu \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos \left(z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi\right) + O\left(\frac{1}{z^2}\right),
\]

(3.12)

\[
J_{-\nu}(z) = \nu \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos \left(z + \frac{1}{2} \nu \pi - \frac{1}{4} \pi\right) + O\left(\frac{1}{z^2}\right).
\]

(3.13)

Let \( \nu \) be a fixed non-zero integer, and let \( \beta = -\frac{1}{2} \nu \pi - \frac{1}{4} \pi \). By (3.12), (3.13), and (3.11), to study the convergence of \( G_{\nu}(x, \sigma, \theta, s, w) \), it is sufficient to investigate the convergence of

\[
S_1 := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left( \frac{\cos(a \sqrt{(m+\sigma)(n+\theta)} + \beta)}{(m+\sigma)^{s+\frac{1}{4}}(n+\theta)^{w+\frac{1}{4}}} \pm \frac{\cos(a \sqrt{(m+\sigma)(n+1-\theta)} + \beta)}{(m+\sigma)^{s+\frac{1}{4}}(n+1-\theta)^{w+\frac{1}{4}}} \right).
\]

Here for convenience we have set \( a = 4\pi \sqrt{x} \).

3.1. Large values of \( n \). We examine the double sum \( S_1 \) for large values of \( n \). We follow the arguments in [10, p. 576–577]. Let \( M \) and \( N \) be integers with \( M < N \). By the Euler-Maclaurin summation formula [1, p. 619],

\[
\sum_{n=M+1}^{N} \frac{\cos(a \sqrt{(m+\sigma)(n+\theta)} + \beta)}{(n+\theta)^{w+\frac{1}{4}}} = \int_{M+\theta}^{N+\theta} \frac{\cos(a \sqrt{(m+\sigma)t + \beta})}{t^{w+\frac{1}{4}}} dt + \int_{M+\theta}^{N+\theta} \{t-\theta\} \frac{d}{dt} \left( \frac{\cos(a \sqrt{(m+\sigma)t + \beta})}{t^{w+\frac{1}{4}}} \right) dt,
\]
where \( \{t - \theta\} \) denotes the fractional part of \( t - \theta \). Note that

\[
\frac{d}{dt} \left( \cos(a \sqrt{(m + \sigma)t + \beta}) \right) = \frac{1}{4t^{w+\frac{3}{4}}} \left( -2a \sqrt{(m + \sigma)t} \sin(a \sqrt{(m + \sigma)t + \beta}) - (4w + 1) \cos(a \sqrt{(m + \sigma)t + \beta}) \right)
\]

\[
= O \left( \frac{a \sqrt{m + \sigma}}{t^{\text{Re} w - \frac{1}{4}}} \right).
\]

Hence,

\[
\int_{M+\theta}^{N+\theta} \{t - \theta\} \frac{d}{dt} \left( \cos(a \sqrt{(m + \sigma)t + \beta}) \right) dt = O \left( \frac{a \sqrt{m + \sigma}}{(M + \theta)^{\text{Re} w - \frac{1}{4}}} \right). \tag{3.14}
\]

Now let \( u = a \sqrt{(m + \sigma)t} \). Then \( t = \frac{u^2}{a^2(m + \sigma)} \) and \( dt = \frac{2u}{a^2(m + \sigma)} du \). Thus,

\[
\int_{M+\theta}^{N+\theta} \frac{a \sqrt{(m + \sigma)t + \beta}}{t^{w+\frac{3}{4}}} dt = 2(a^2(m + \sigma))^{w-\frac{3}{2}} \int_{a \sqrt{(m + \sigma)(M+\theta)}}^{a \sqrt{(m + \sigma)(N+\theta)}} \frac{\cos(u + \beta)}{u^{2w-\frac{1}{4}}} du. \tag{3.15}
\]

Let \( c, e \) be real constants. An integration by parts shows that

\[
\int_{A}^{B} \frac{c \sin u + e \cos u}{u^{2w-\frac{1}{2}}} du = \left. \frac{c \cos u + e \sin u}{u^{2w-\frac{1}{2}}} \right|_{A}^{B} + \left( 2w - \frac{1}{2} \right) \int_{A}^{B} \frac{-c \cos u + e \sin u}{u^{2w+\frac{3}{4}}} du
\]

\[
= O_w \left( \frac{1}{A^2 \text{Re} w - \frac{1}{4}} + \frac{1}{B^2 \text{Re} w - \frac{1}{4}} \right), \tag{3.16}
\]

as \( A, B \to \infty \). Recall the identity \( \cos(u + \beta) = \cos u \cos \beta - \sin u \sin \beta \). By (3.15) and (3.16),

\[
\int_{M+\theta}^{N+\theta} \frac{a \sqrt{(m + \sigma)t + \beta}}{t^{w+\frac{3}{4}}} dt = O \left( \frac{1}{a \sqrt{m + \sigma}} \left( \frac{1}{(M + \theta)^{\text{Re} w - \frac{1}{4}}} + \frac{1}{(N + \theta)^{\text{Re} w - \frac{1}{4}}} \right) \right).
\]

Hence, by (3.14), as \( M \to \infty \),

\[
\sum_{n=M}^{\infty} \frac{\cos(a \sqrt{(m + \sigma)(n + \theta) + \beta})}{(n + \theta)^{w+\frac{3}{4}}} = \lim_{N \to \infty} \sum_{n=M}^{N} \frac{\cos(a \sqrt{(m + \sigma)(n + \theta) + \beta})}{(n + \theta)^{w+\frac{3}{4}}}
\]

\[
= O \left( \frac{a \sqrt{m + \sigma}}{(M + \theta)^{\text{Re} w - \frac{1}{4}}} \right).
\]
Analogous results hold when $\sigma$ is replaced by $1 - \sigma$ and when $\theta$ is replaced by $1 - \theta$. We can thus let $M = [m^{1/(\text{Re} s - 1/4)}]$ and conclude that

$$
\sum_{n \geq m^{1/(\text{Re} s - 1/4)}} \left( \frac{\cos(a\sqrt{(m + \sigma)(n + \theta) + \beta})}{(m + \sigma)^{s + \frac{1}{4}}(n + \theta)^{w + \frac{1}{4}}} \pm \frac{\cos(a\sqrt{(m + \sigma)(n + 1 - \theta) + \beta})}{(m + \sigma)^{s + \frac{1}{4}}(n + 1 - \theta)^{w + \frac{1}{4}}} + \frac{\cos(a\sqrt{(m + 1 - \sigma)(n + \theta) + \beta})}{(m + 1 - \sigma)^{s + \frac{1}{4}}(n + \theta)^{w + \frac{1}{4}}} \pm \frac{\cos(a\sqrt{(m + 1 - \sigma)(n + 1 - \theta) + \beta})}{(m + 1 - \sigma)^{s + \frac{1}{4}}(n + 1 - \theta)^{w + \frac{1}{4}}} \right) = O\left( \frac{a}{m^{\text{Re} s + 3/4}} \right).
$$

Recall that $\text{Re} s > \frac{1}{4}$. Hence, in our study of the uniform convergence of the sum $S_1$, we can replace it with the sum $S_2$, defined by

$$
\sum_{m=0}^{\infty} \sum_{0 \leq n \leq m^{1/(\text{Re} s - 1/4)}} \left( \frac{\cos(a\sqrt{(m + \sigma)(n + \theta) + \beta})}{(m + \sigma)^{s + \frac{1}{4}}(n + \theta)^{w + \frac{1}{4}}} \pm \frac{\cos(a\sqrt{(m + \sigma)(n + 1 - \theta) + \beta})}{(m + \sigma)^{s + \frac{1}{4}}(n + 1 - \theta)^{w + \frac{1}{4}}} + \frac{\cos(a\sqrt{(m + 1 - \sigma)(n + \theta) + \beta})}{(m + 1 - \sigma)^{s + \frac{1}{4}}(n + \theta)^{w + \frac{1}{4}}} \pm \frac{\cos(a\sqrt{(m + 1 - \sigma)(n + 1 - \theta) + \beta})}{(m + 1 - \sigma)^{s + \frac{1}{4}}(n + 1 - \theta)^{w + \frac{1}{4}}} \right).
$$

3.2. **Small values of $n$.** Let $\delta > 0$ be a small positive number to be specified later, and let $S_3$ be the same double series as $S_2$ but with the sum on $n$ performed over the interval $0 \leq n \leq m^{1-\delta}$. We now prove convergence of $S_3$, using the following result from [7].

**Lemma 3.3.** [7, p. 31–33] **Consider the sum**

$$
S(\alpha, \beta, \mu, H_1, H_2) = \sum_{H_1 < m \leq H_2} \frac{\cos(\alpha\sqrt{m + \mu + \beta})}{(m + \mu)^{s + \frac{1}{4}}},
$$

**where** $\alpha > 0$, $\beta \in \mathbb{R}$, $\mu \in [0, 1]$, and $H_1 < H_2$, **where** $H_1$ and $H_2$ are large positive integers. **Assume also that**

$$
c_1 \leq \alpha \leq c_2 H_1^{(1-\delta)/2},
$$

**where** $c_1$ and $c_2$ are positive constants and $\delta > 0$ is a fixed small positive real number. **Then**

$$
S(\alpha, \beta, \mu, H_1, H_2) = O\left( \frac{1}{\alpha H_1^{\text{Re} s - 1/4}} \right).
$$

**We write** $S_{3,M}$ **for the partial sum in** $S_3$, **where the summation over** $m$ **is restricted to** $1 \leq m \leq M$. **To prove the convergence of** $S_3$, **we use Cauchy’s criterion. That is, for every** $\epsilon > 0$, **we show that there exists** $M_\epsilon$ **such that** $|S_{3,M_2} - S_{3,M_1}| < \epsilon$ **whenever** $M_1, M_2 > M_\epsilon$. 
Exchanging the order of summation in a term of $S_3$ yields

$$\sum_{m=M_1}^{M_2} \sum_{0 \leq n < m^{1-\delta}} \frac{\cos(a \sqrt{(m+\sigma)(n+\theta) + \beta})}{(m+\sigma)^{s+\frac{1}{4}} (n+\theta)^{w+\frac{1}{4}}} \cos(a \sqrt{(m+\sigma)(n+\theta) + \beta}) (m+\sigma)^{s+\frac{1}{4}} (n+\theta)^{w+\frac{1}{4}}.$$

We now apply Lemma 3.3 with $\alpha = a \sqrt{n + \theta}, \mu = \sigma, H_1 = \max\{n^{1/(1-\delta)}, M_1\}$, and $H_2 = M_2$. Hence,

$$\sum_{m=M_1}^{M_2} \sum_{0 \leq n < m^{1-\delta}} \frac{\cos(a \sqrt{(m+\sigma)(n+\theta) + \beta})}{(m+\sigma)^{s+\frac{1}{4}} (n+\theta)^{w+\frac{1}{4}} \max\{n^{1/(1-\delta)}, M_1\}^{\Re s-\frac{1}{4}}}$$

$$= O_{a,\delta} \left( \sum_{0 \leq n < M_2^{1-\delta}} \frac{1}{(n+\theta)^{w+\frac{1}{4}} \max\{n^{1/(1-\delta)}, M_1\}^{\Re s-\frac{1}{4}}} \right)$$

$$= O_{a,\delta} \left( \sum_{0 \leq n < M_2^{1-\delta}} \frac{1}{(n+\theta)^{w+\frac{1}{4}} \max\{n^{1/(1-\delta)}, M_1\}^{\Re s-\frac{1}{4}}} \right)$$

$$+ O_{a,\sigma} \left( \sum_{M_1^{1-\delta} \leq n < M_2^{1-\delta}} \frac{1}{(n+\theta)^{w+\frac{1}{4}} \max\{n^{1/(1-\delta)}, M_1\}^{\Re s-\frac{1}{4}}} \right)$$

$$= O_{a,\delta} \left( \frac{1}{M_1^{\Re s-\frac{1}{4}}} \right),$$

where in the last step we used (3.11) and the hypothesis $\Re w > \frac{1}{4}$. The same reasoning applies to every term of the sum $S_3$, by replacing $\theta$ with $1 - \theta$ and $\sigma$ with $1 - \sigma$. Since $\Re s > \frac{1}{4}$ by hypothesis, we conclude that $|S_{3,M_2} - S_{3,M_1}| < \epsilon$ when $M_1$ is large enough. We have thus proved the convergence of $S_3$, uniformly for $(\sigma, \theta)$ in a compact subset of $(0, 1)^2$. Note that in our situation we need the full strength of Lemma 3.3, while the corresponding reduction in [7, p. 33] only requires the case $\mu = 0$.

3.3. Further reductions. Assume that $\delta < 1/(\Re w - \frac{1}{4}) - 1$. Let $S_4$ be the same double series as $S_2$ but with the sum on $n$ performed over the interval $m^{1+\delta} < n \leq m^{1/(\Re w-\frac{1}{4})}$. The techniques used above to prove the convergence of $S_3$ also show the convergence of $S_4$. Indeed, Lemma 3.3 applied with $\alpha = a \sqrt{n + \theta}, \mu = \theta, H_1 = m^{1+\delta}$, and $H_2 = m^{1/(\Re w-\frac{1}{4})}$....
By (3.11), it is sufficient to prove the convergence of $S$ and the asymptotic formulas for $S$.

Recall the identities

\[
\sin a \cos b = \frac{1}{2} \left[ \cos(a + b) + \cos(a - b) \right],
\]

\[
\cos a \sin b = \frac{1}{2} \left[ \sin(a + b) - \sin(a - b) \right],
\]

and the asymptotic formulas

\[
\sqrt{n + \theta} - \sqrt{n + 1 - \theta} = \frac{2\theta - 1}{2\sqrt{n}} + O\left( \frac{1}{n^{3/2}} \right),
\]

\[
\sqrt{n + \theta} + \sqrt{n + 1 - \theta} = \sqrt{n + 1} + \frac{1}{2} + O\left( \frac{1}{n^{3/2}} \right).
\]

Hence the convergence of $S_5$ will be proved if we can show, under the hypotheses of our theorem, the convergence of the double series

\[
\sum_{m=0}^{\infty} \sum_{m^1 \leq n \leq m^{1+\delta}} \cos\left( a \sqrt{(m + \sigma)(n + \frac{1}{2})} + \beta \right) \cos\left( \frac{a(\theta - 1)}{4} \sqrt{\frac{m + \sigma}{n}} \right),
\]

(3.18)

as well as the convergence of (3.18) with both occurrences of the function $\cos$ replaced by $\sin$. Now see [10] from Section 4.3 onwards and use the same arguments to complete the proof.
TRIGONOMETRIC AND BESSEL SERIES

4. BALANCED IDENTITIES

Having established the convergence result in Theorem 2.2, we are now ready to prove Theorem 2.1.

Proof of Theorem 2.1 When \( k = 0 \), Equation (2.1) becomes

\[
\sum_{mn \leq x} \cos(2\pi m\sigma) \sin(2\pi n\theta) + \frac{\cot(\pi\theta)}{4} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{J_1(4\pi \sqrt{(m + \sigma)(n + \theta)x})}{\sqrt{(m + \sigma)(n + \theta)}} + \frac{J_1(4\pi \sqrt{(m + 1 - \sigma)(n + \theta)x})}{\sqrt{(m + 1 - \sigma)(n + \theta)}} \right. \\
- \left. \frac{J_1(4\pi \sqrt{(m + \sigma)(n + 1 - \theta)x})}{\sqrt{(m + \sigma)(n + 1 - \theta)}} - \frac{J_1(4\pi \sqrt{(m + 1 - \sigma)(n + 1 - \theta)x})}{\sqrt{(m + 1 - \sigma)(n + 1 - \theta)}} \right\}. 
\]

Note that convergence of the double sum on the right-hand side of (4.1) is a consequence of Theorem 2.2. To prove (4.1) it is therefore sufficient to compute the Fourier coefficients of both sides of the equation and show that they are equal.

The general statement of Theorem 2.1 follows from the case \( k = 0 \) by taking derivatives on both sides of (4.1). Such derivatives can be brought inside the infinite sums because the double series is uniformly convergent by Theorem 2.2 applied in the special case \( s = w = \frac{1}{2} \).

In [6, Theorems 4.1, 4.4], three of the present authors first proved (4.1) by showing that the Fourier sine series of both sides are identical. Secondly, they proved (4.1), but with the order of summation reversed, by demonstrating that the Fourier cosine series of both sides are identical. Thus, it was shown that one could reverse the order of summation by proving that the two iterated sums converge to the same limit. In our analysis above, using uniform convergence, we also demonstrated that the two iterated series converged to the same limit. But now appealing to the aforementioned two theorems in [6], we have completed the proof of (4.1), and consequently of Theorem 2.1.

Theorems 2.4 and 2.5 can be proved similarly to Theorem 2.1. We describe below the necessary modifications to the arguments presented above.

Proof of Theorems 2.4 and 2.5 In the case \( k = 0 \) the two theorems yield known identities (see [8, Theorem 2.1] and [8, Theorem 2.3], respectively). In [8] these identities were proved with the iterated sums replaced by double sums where, for brevity, the products of the indices \( m \) and \( n \) tend to infinity. However, in each case, the same arguments developed there can be used to prove the identities with iterated sums, yielding (2.4) and (2.5) in the case \( k = 0 \). Theorems 2.4 and 2.5 then follow from the case \( k = 0 \) by taking derivatives on both sides of the identities.

To conclude the proofs of Theorems 2.4 and 2.5 we need theorems on the uniform convergence of the two double series appearing on the right sides of (2.4) and (2.5). To this end, we prove that Theorem 2.2 holds even when, in the definition (2.2) of \( G_{\alpha,\beta}(x, \sigma, \theta, s, w) \), the expression inside the brackets is replaced with a double series involving the function \( I_1 \) (respectively \( T_\frac{1}{2} \)) corresponding to the one on the right-hand side of (2.4) (respectively (2.5)).

We denote these two new series by \( G_{I,\alpha,\beta}(x, \sigma, \theta, s, w) \) and \( G_{T,\alpha,\beta}(x, \sigma, \theta, s, w) \), respectively.
In the case of $G^\alpha_{\nu} (x, \sigma, \theta, s, w)$ the proof of Theorem 2.2 carries over with minor modifications. One starts from an analogue of Lemma 3.1, which holds if one simply replaces in the statement every occurrence of $J_\nu$ with $I_\nu$. In the proof, instead of Equation (3.4), one uses the following similar recurrence relation [19, p. 79], which holds for every integer $\nu$:

$$I_{\nu-1}(z) + I_{\nu+1}(z) = 2 I'_\nu(z). \tag{4.2}$$

Later in the proof, the asymptotic formulas (3.12) and (3.13), which hold for every positive integer $\nu$, are replaced by

$$I_\nu(z) = -\nu \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} \sin \left( z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi \right) + O \left( \frac{1}{z^{\frac{3}{2}}} \right), \tag{4.3}$$

$$I_{-\nu}(z) = \nu \left( \frac{2}{\pi z} \right)^{\frac{1}{2}} \sin \left( z + \frac{1}{2} \nu \pi - \frac{1}{4} \pi \right) + O \left( \frac{1}{z^{\frac{3}{2}}} \right). \tag{4.4}$$

These asymptotic formulas arise from combining the definition (1.1) with the asymptotic expressions for $Y_\nu$ and $K_\nu$ given in [19, p. 199] and [19, p. 202], respectively. One can now follow the proof of Theorem 2.2 with the appropriate minor changes due to the fact that the trigonometric function $\cos$ has been replaced with $\sin$. It can be readily checked that this change does not affect the convergence of the corresponding double series.

Let us now consider $G^\alpha_{\nu} (x, \sigma, \theta, s, w)$. In order to prove the convergence of this double series we need the following analogue of Lemma 3.1.

**Lemma 4.1.** The following identity holds:

$$\frac{\partial^{\alpha+\beta}}{\partial \sigma^\alpha \partial \theta^\beta} T_{\frac{\pi}{2}}(4\pi^2 (m + \sigma)(n + \theta)x) = \sum_{\nu \in A} c_\nu Q_\nu \frac{(4\pi \sqrt{(m + \sigma)(n + \theta)x})}{(m + \sigma)^{s+\frac{\alpha}{2}}(n + \theta)^{w+\frac{\beta}{2}}} \cdot \ldots$$

Here $A$ is a finite subset of $\mathbb{Z}$, the $c_\nu$ are constants, and the dots $\ldots$ denote a sum of terms that have the form

$$r_\nu \frac{Q_\nu (4\pi \sqrt{(m + \sigma)(n + \theta)x})}{(m + \sigma)^{s}(n + \theta)^{\delta}}, \quad \nu \in \mathbb{Z}, \ r_\nu \in \mathbb{R},$$

where

$$\begin{cases} \Re(\gamma) \geq \Re(s) + \frac{\alpha}{2} - \frac{\beta}{2}, \\
\Re(\delta) \geq \Re(w) + \frac{\beta}{2} - \frac{\alpha}{2}, \end{cases}$$

with at least one of the two inequalities in (4.1) being strict. With $Q_\nu$ we denote one of the Bessel functions $I_\nu, K_\nu, Y_\nu$.

The first step in the proof of this lemma is to recall a formula for $T_{\frac{\pi}{2}}(y^2)$ as a linear combination of Bessel functions. From [8, p. 90, Equation (5.9)], if $y = 2\pi \sqrt{n x / pq}$, where $p$ and $q$ are primes, then

$$T_{\frac{\pi}{2}}(y^2) = \frac{1}{2y^2} I_1(2y) - \frac{1}{y} I'_1(2y) + Y_1(2y) - \frac{2}{\pi} K_1(2y).$$

Lemma 4.1 can now be proved by induction, similarly to Lemma 3.1. One needs to use, instead of (3.4), the corresponding recurrence relations for the functions $I_\nu, K_\nu, Y_\nu$ [19, p. 79 formula (2); p. 66 formula (2)]. To finish the proof of the convergence of $G^\alpha_{\nu} (x, \sigma, \theta, s, w)$,
one can now follow the same steps as in the proof of the convergence of (3.10), using the
asymptotic expansions given in (4.3), (4.4), [19, p. 199], and [19, p. 202]. □

5. APPLICATION: A BALANCED IDENTITY OF ORDER 1

We record the formulas for $k = 0$ from Theorems 2.4 and 2.5. As indicated above, they
were proved in [8, pp. 70–71, Theorems 2.1, 2.3] under hypotheses that were stronger than
necessary.

Theorem 5.1. Let $I_1(x)$ be defined by (1.1). If $0 < \theta, \sigma < 1$ and $x > 0$, then

$$\sum_{nm \leq x} \cos(2\pi n\theta) \cos(2\pi m\sigma)$$

$$= \frac{1}{4} + \frac{\sqrt{x}}{4} \sum_{n,m \geq 0} \left\{ I_1(4\pi \sqrt{(n+\theta)(m+\sigma)x}) \right. + I_1(4\pi \sqrt{(n+1-\theta)(m+\sigma)x})

$$

$$\left. + \frac{I_1(4\pi \sqrt{(n+\theta)(m+1-\sigma)x})}{\sqrt{(n+\theta)(m+1-\sigma)}} + \frac{I_1(4\pi \sqrt{(n+1-\theta)(m+1-\sigma)x})}{\sqrt{(n+1-\theta)(m+1-\sigma)}} \right\}.$$ 

Theorem 5.2. If $0 < \theta, \sigma < 1$ and $x > 0$, then

$$\sum_{nm \leq x} \sin(2\pi n\theta) \sin(2\pi m\sigma)$$

$$= \frac{x}{4} \sqrt{x} \sum_{n,m \geq 0} \left\{ T_{3/2}(4\pi^2(n+\theta)(m+\sigma)x) \right. - T_{3/2}(4\pi^2(n+1-\theta)(m+\sigma)x)

$$

$$\left. - \frac{T_{3/2}(4\pi^2(n+\theta)(m+1-\sigma)x)}{\sqrt{(n+\theta)(m+1-\sigma)}} + \frac{T_{3/2}(4\pi^2(n+1-\theta)(m+1-\sigma)x)}{\sqrt{(n+1-\theta)(m+1-\sigma)}} \right\}.$$ 

where $T_{3/2}$ is defined in (2.3).

From our remarks above on the uniform convergence of the right-hand side of (2.4), differ-
entiating the identity (2.4) for $k = 1$ yields an identity for the left-hand side of (2.5) for
$k = 0$. However, we only know the precise nature of the right-hand side of (2.5) because of
the independent proof of (2.5) in [8], but under stronger hypotheses, as emphasized above.
Our goal is to show directly that the first mixed partial derivative of (5.1) is equal to (5.2).
The needed termwise differentiation is justified by Theorem 2.4.

Let

$$u = 4\pi \sqrt{(n+\theta)(m+\sigma)x}.$$ 

Then,

$$\frac{\partial}{\partial \theta} \left\{ \frac{I_1(u)}{\sqrt{(n+\theta)(m+\sigma)}} \right\} = I'(u)2\pi \sqrt{x} \frac{I_1(u)}{(n+\theta)} - \frac{I_1(u)}{2(n+\theta)^{3/2}(m+\sigma)^{1/2}}$$

where $T_{3/2}$ is defined in (2.3).
We now use [19, p. 66, formula (3); p. 79, formula (3)], respectively,

\[ uY_1'(u) + Y_1(u) = uY_0(u) \quad \text{and} \quad uK_1'(u) + K_1(u) = -uK_0(u). \]  

(5.5)

Also, [19] p. 66, formula (4); p. 79, formula (4)], respectively,

\[ Y_0'(u) = -Y_1(u) \quad \text{and} \quad K_0'(u) = -K_1(u). \]  

(5.6)

Thus, from (1.1), (5.5), and (5.6),

\[ I_1'(u) = -Y_1'(u) - \frac{2}{\pi}K_1'(u) \]
\[ = \frac{1}{u}Y_1(u) - Y_0(u) + \frac{2}{\pi u}K_1(u) + \frac{2}{\pi}K_0(u), \]  

(5.7)

and from (5.6) and (5.7),

\[ I_1''(u) = \frac{1}{u}Y_1'(u) - \frac{1}{u^2}Y_1(u) - Y_0'(u) + \frac{2}{\pi u}K_1'(u) - \frac{2}{\pi u^2}K_1(u) + \frac{2}{\pi}K_0'(u) \]
\[ = \frac{1}{u} \left( -\frac{1}{u}Y_1(u) + Y_0(u) \right) - \frac{1}{u^2}Y_1(u) + Y_1(u) + \frac{2}{\pi u} \left( -\frac{1}{u}K_1(u) - K_0(u) \right) \]
\[ - \frac{2}{\pi u^2}K_1(u) - \frac{2}{\pi}K_1(u) \]
\[ = \frac{2}{u^2} \left( -Y_1(u) - \frac{2}{\pi}K_1(u) \right) + \frac{1}{u} \left( Y_0(u) - \frac{2}{\pi}K_0(u) \right) + Y_1(u) - \frac{2}{\pi}K_1(u). \]  

(5.8)
Now return to (5.4) and substitute from (5.3), (1.1), (5.7), and (5.8) to deduce that
\[
\frac{\partial^2}{\partial \sigma \partial \theta} \left\{ \frac{I_1(u)}{\sqrt{(n + \theta)(m + \sigma)}} \right\}
\]
\[= \frac{16\pi^3 x^{3/2}}{u} \left\{ - \frac{2}{u^2} Y_1(u) - \frac{4}{u^2} K_1(u) + \frac{1}{u} Y_0(u) - \frac{2}{\pi u} K_0(u) + Y_1(u) - \frac{2}{\pi} K_1(u) \right\}
\]
\[= \frac{16\pi^3 x^{3/2}}{u^2} \left\{ \frac{1}{u} Y_1(u) - Y_0(u) + \frac{2}{\pi u} K_1(u) + \frac{2}{\pi} K_0(u) \right\}
\]
\[= \frac{16\pi^3 x^{3/2}}{u^3} \left\{ Y_1(u) + \frac{2}{\pi} K_1(u) \right\}
\]
\[= x^{3/2} \left\{ - \frac{64\pi^3}{u^3} Y_1(u) - \frac{128\pi^2}{u^3} K_1(u) + \frac{16\pi^3}{u} Y_1(u) - \frac{32\pi^2}{u} K_1(u)
\]
\[+ \frac{32\pi^3}{u^2} Y_0(u) - \frac{64\pi^2}{u^2} K_0(u) \right\},
\]
or
\[\frac{1}{4\pi^2} \frac{\partial^2}{\partial \sigma \partial \theta} \left\{ \frac{I_1(u)}{\sqrt{(n + \theta)(m + \sigma)}} \right\}
\]
\[= x^{3/2} \left\{ - \frac{16\pi}{u^3} Y_1(u) - \frac{32}{u^3} K_1(u) + \frac{4\pi}{u} Y_1(u) - \frac{8}{u} K_1(u) + \frac{8\pi}{u^2} Y_0(u) - \frac{16}{u^2} K_0(u) \right\}. \quad (5.9)
\]

We now turn to \( T_{3/2}(z) \). If we set \( u = 2y \) in (4), we find that
\[ T_{3/2}(\frac{1}{4}u^2) = \frac{2}{u^2} I_1(u) - \frac{2}{u} I'_1(u) + Y_1(u) - \frac{2}{\pi} K_1(u). \quad (5.10)
\]

Appealing to (5.7), we see from (5.10) that
\[ T_{3/2}(\frac{1}{4}u^2) = - \frac{2}{u^2} Y_1(u) - \frac{4}{\pi u^2} K_1(u) - \frac{2}{u^2} Y_1(u) + \frac{2}{u} Y_0(u) - \frac{4}{\pi u^2} K_1(u) - \frac{4}{\pi u} K_0(u)
\]
\[+ Y_1(u) - \frac{2}{\pi} K_1(u)
\]
\[= - \frac{4}{u^2} Y_1(u) - \frac{8}{\pi u^2} K_1(u) + \frac{2}{u} Y_0(u) - \frac{4}{\pi u} K_0(u) + Y_1(u) - \frac{2}{\pi} K_1(u). \quad (5.11)
\]

Recalling the definition (5.3), we can write (5.11) in the form
\[ \frac{T_{3/2}(\frac{1}{4}u^2)}{\sqrt{(n + \theta)(m + \sigma)}} = \frac{4\pi \sqrt{x} T_{3/2}(\frac{1}{4}u^2)}{u}
\]
\[= \sqrt{x} \left( - \frac{16\pi}{w^3} Y_1(u) - \frac{32}{w^3} K_1(u) + \frac{8\pi}{u^2} Y_0(u) - \frac{16}{u^2} K_0(u) + \frac{4\pi}{u} Y_1(u) - \frac{8}{u} K_1(u) \right). \quad (5.12)
\]

Hence, from (5.9) and (5.12), we conclude that the first balanced derivative of the terms in the first series on the right side of (5.1), multiplied by \( 1/(4\pi^2) \), are equal to the terms of
the first series on the right-hand side of (5.2). If we successively set
\[ u = 4\pi \sqrt{(n + 1 - \theta)(m + \sigma)x}, \]
\[ u = 4\pi \sqrt{(n + \theta)(m + 1 - \sigma)x}, \]
\[ u = 4\pi \sqrt{(n + 1 - \theta)(m + 1 - \sigma)x}, \]
we can make analogous conclusions for the second, third, and fourth series terms on the right-hand sides of (5.1) and (5.2). In conclusion, we have shown that taking the first balanced derivative of (5.1) yields (5.2), as expected.

6. General Theorems of Chandrasekharan and Narasimhan

We offer two general theorems of Chandrasekharan and Narasimhan [12], which we employ in the sequel. First, we provide the general setting [12, p. 93–96].

**Definition 6.1.** Let \( a(n) \) and \( b(n) \) be two sequences of complex numbers, where not all terms are equal to 0 in either sequence. Let \( \lambda_n \) and \( \mu_n \) be two sequences of positive numbers, strictly increasing to \( \infty \). Let \( \delta > 0 \). Throughout, \( s = \sigma + it \), where \( \sigma \) and \( t \) are both real. Let

\[
\Delta(s) := \prod_{n=1}^{N} \Gamma(\alpha_n s + \beta_n),
\]

(6.1)

where \( N \geq 1, \beta_n, 1 \leq n \leq N, \) is any complex number, and \( \alpha_n > 0, 1 \leq n \leq N \). Assume that

\[ A := \sum_{n=1}^{N} \alpha_n \geq 1. \]

Let

\[ \varphi(s) := \sum_{n=1}^{\infty} \frac{a(n)}{\lambda_n^s} \quad \text{and} \quad \psi(s) := \sum_{n=1}^{\infty} \frac{b(n)}{\mu_n^s}, \]

converge absolutely in some half-plane, and suppose they satisfy the functional equation

\[ \Delta(s)\varphi(s) = \Delta(\delta - s)\psi(\delta - s). \]

(6.2)

Furthermore, assume that there exists in the \( s \)-plane a domain \( \Omega \), which is the exterior of a compact set \( S \), in which there exists an analytic function \( \chi \) with the properties

\[ \lim_{|\sigma| \to \infty} \chi(s) = 0, \]

uniformly in every interval \(-\infty < \sigma_1 \leq \sigma \leq \sigma_2 < \infty, \) and

\[ \chi(s) = \Delta(s)\varphi(s), \quad \sigma > \alpha, \]

\[ \chi(s) = \Delta(\delta - s)\psi(\delta - s), \quad \sigma < \beta, \]

where \( \alpha \) and \( \beta \) are particular constants.

For \( \rho \geq 0 \), let

\[ A_{\rho}(x) := \frac{1}{\Gamma(\rho + 1)} \sum_{\lambda_n \leq x} a(n)(x - \lambda_n)^\rho, \]

(6.3)
where the prime ′ indicates that if \( x = \lambda_n \) and \( \rho = 0 \), the last term is to be multiplied by \( \frac{1}{2} \).

Furthermore, let

\[
Q_\rho(x) := \frac{1}{2\pi i} \int_C \frac{\Gamma(s) \varphi(s)}{\Gamma(s + \rho + 1)} x^{s+\rho} ds,
\]

where \( C \) is a closed curve enclosing all of the singularities of the integrand to the right of \( \sigma = -\rho - 1 - k \), where \( k \) is chosen such that \( k > |\delta/2-1/(4A)| \), and all of the singularities of \( \varphi(s) \) lie in \( \sigma > -k \). In Sections 7–9, \( \rho = 0 \), and so in these sections we write \( A_\rho(x) = A(x) \) and \( Q_\rho(x) = Q(x) \). In Section 10, we consider the more general case when \( \rho > 0 \).

**Theorem 6.2.** [12, p. 98, Theorem 3.1] Suppose that \( \varphi(s) \) and \( \psi(s) \) satisfy (6.2). Suppose that \( \{\mu_n\} \) contains a subset \( \{\mu_{n_k}\} \) such that no number \( \mu_n^{1/(2A)} \) is represented as a linear combination of the numbers \( \{\mu_{n_k}^{1/(2A)}\} \) with the coefficients \( \pm 1 \), unless \( \mu_n^{1/(2A)} = \mu_{n_r}^{1/(2A)} \) for some \( r \), in which case \( \mu_n^{1/(2A)} \) has no other representation. Suppose furthermore that

\[
\sum_{n=1}^{\infty} \frac{|\text{Re } b(n_k)|}{\mu_n^{A\delta+\rho+1/2}/(2A)} = +\infty.
\]

Set

\[
\theta := \frac{A\delta + \rho(2A - 1) - \frac{1}{2}}{2A}.
\]

Then

\[
\lim_{x \to \infty} \frac{\text{Re}\{A_\rho(x) - Q_\rho(x)\}}{x^\theta} = +\infty,
\]

\[
\lim_{x \to \infty} \frac{\text{Re}\{A_\rho(x) - Q_\rho(x)\}}{x^\theta} = -\infty.
\]

If in assumption (6.5), we replace \( \text{Re } b(n_k) \) by \( \text{Im } b(n_k) \), then (6.7) and (6.8) remain valid.

The conclusion (6.7) is equivalent to the following statement: There exists a sequence \( \{x_n\} \) tending to \( \infty \), such that there does not exist any positive constant \( C \) such that, for all \( n \),

\[
\text{Re}\{A_\rho(x_n) - Q_\rho(x_n)\} \leq Cx_n^\theta.
\]

In such a situation, we write

\[
\text{Re}\{A_\rho(x_n) - Q_\rho(x_n)\} = \Omega_+ (x^\theta).
\]

Similar remarks hold for (6.8).

**Theorem 6.3.** [12, p. 106, Theorem 4.1] Suppose that the functional equation

\[
\Delta(s) \varphi(s) = \Delta(\delta - s) \psi(\delta - s)
\]

is satisfied with \( \delta > 0 \), and that \( \varphi(s) \) is an entire function. Then, if \( A(x) \) is defined by (6.3) and \( Q(x) \) is defined by (6.4), as \( x \to \infty \),

\[
A(x) - Q(x) = O \left( x^{\delta/2-1/(4A)} + 2A\eta \right) + O \left( \sum_{x < \lambda_n \leq x'} |a(n)| \right),
\]

for every \( \eta \geq 0 \), where

\[
u := \beta - \frac{1}{2} \delta - \frac{1}{4A},
\]

(6.10)
and $\beta$ is chosen so that $\sum_{n=1}^{\infty} |b(n)| \mu_n^{-\beta}$ converges. Furthermore,

$$x' = x + O(x^{1-\eta-1/(2A)}). \quad (6.11)$$

7. THE FIRST $\Omega$ AND “BIG O” THEOREMS AND CONJECTURE

Let $\chi_1$ and $\chi_2$ be primitive, non-principal even characters modulo $p$ and $q$, respectively. Let $\tau(\chi_1)$ and $\tau(\chi_2)$ denote their corresponding Gauss sums. Lastly, we use the notation

$$d_{\chi_1,\chi_2}(n) = \sum_{d|n} \chi_1(d)\chi_2(n/d). \quad (7.1)$$

Theorem 7.1. Assume that $\chi_1$ and $\chi_2$ are non-principal even characters modulo the primes $p$ and $q$, respectively. Let

$$D_{\chi_1,\chi_2}(x) := \sum_{n \leq x} d_{\chi_1,\chi_2}(n). \quad (7.2)$$

Then

$$\lim_{x \to \infty} \frac{\text{Re} D_{\chi_1,\chi_2}(x)}{x^{1/4}} = +\infty, \quad (7.3)$$
$$\lim_{x \to \infty} \frac{\text{Re} D_{\chi_1,\chi_2}(x)}{x^{1/4}} = -\infty. \quad (7.4)$$

Both (7.3) and (7.4) remain valid if we replace Re by Im in each of them.

Proof. Recall that [8, p. 74] (if $2s$ is replaced by $s$)

$$\left(\frac{\pi^2}{pq}\right)^{-s/2} \Gamma^2 \left(\frac{1}{2} s\right) L(s, \chi_1) L(s, \chi_2) = \frac{\tau(\chi_1)\tau(\chi_2)}{\sqrt{pq}} \left(\frac{\pi^2}{pq}\right)^{-(1-s)/2} \Gamma^2 \left(\frac{1}{2}(1-s)\right) L(1-s, \overline{\chi_1}) L(1-s, \overline{\chi_2}). \quad (7.5)$$

We now apply Theorem 6.2. The parameters from Definition 6.1 and Theorem 6.2 are:

$$N = 2, \quad \delta = 1, \quad A = 1, \quad \theta = \frac{1}{4}, \quad (7.6)$$
$$a(n) = d_{\chi_1,\chi_2}(n), \quad b(n) = \tau(\chi_1)\tau(\chi_2)\frac{d_{\chi_1,\chi_2}(n)}{\sqrt{pq}}, \quad \lambda_n = \mu_n = \frac{\pi n}{\sqrt{pq}}. \quad (7.7)$$

From the functional equation (7.5), since the analytic continuations of $L(s, \chi_1)$ and $L(s, \chi_2)$ are entire functions, we see that $L(0, \chi_1) = L(0, \chi_2) = 0$. It follows that $Q(x) = 0$. Theorem 7.1 now follows immediately provided that we can show that (6.3) holds, that is, there exists a subset $n_k, 1 \leq n_k < \infty$, such that

$$\sum_{n_k=1}^{\infty} \left|\frac{\text{Re} \{\tau(\chi_1) \tau(\chi_2) d_{\chi_1,\chi_2}(n_k)\}}{n_k^{1/4}}\right| = +\infty. \quad (7.8)$$

Recall that $p$ and $q$ are primes. First, suppose that $\tau(\chi_1)\tau(\chi_2)$ is not purely imaginary. Choose $n$ such that $n \equiv 1 \pmod{p}$ and $n \equiv 1 \pmod{q}$. Thus, we consider the set $S_{pq} := $
\{1 + mpq, 1 \leq m < \infty\}. By Dirichlet’s theorem on primes in arithmetic progressions, \(S_{pq}\) contains an infinite number of primes \(\{P_k\}\). Note that
\[
d_{\chi_1,\chi_2}(P_m) = 2, \quad m \geq 1.
\]
Thus, we have located an infinite subset \(n_k = P_k\), where the series terms in (7.8) are positive and where the series diverges.

Suppose now that \(\tau(\chi_1)\tau(\chi_2)\) is purely imaginary. Choose \(r\) such that \(\chi_1(r) + \chi_2(r)\) is not real. The set \(S_{pq} := \{r + mpq, 1 \leq m < \infty\}\) contains an infinite number of primes \(\{Q_k\}\) by Dirichlet’s theorem. Hence,
\[
d_{\chi_1,\chi_2}(Q_k) = \chi_1(r) + \chi_2(r).
\]
is not real. Now proceed as in the previous case with \(n_k = Q_k\). This completes the proof. \(\square\)

We provide the motivation for studying \(d_{\chi_1,\chi_2}(n)\). We see from [8, p. 78, Equation (3.11)] that
\[
CC(x) := \sum_{nm \leq x} \left\lfloor \frac{2\pi na}{p} \right\rfloor \cos \left(\frac{2\pi mb}{q}\right) \tag{7.9}
\]
\[
= \frac{1}{\phi(p)\phi(q)} \sum_{n \leq x} \left( \sum_{\chi_1 \text{ mod } p, \chi_1 \neq \chi_0, \text{ even}} \sum_{\chi_2 \text{ mod } q, \chi_2 \neq \chi_0, \text{ even}} \chi_1(a)\chi_2(b)\tau(\chi_1)\tau(\chi_2) \sum_{n \leq x} d_{\chi_1,\chi_2}(n) \right)
- \frac{1}{\phi(p)} \sum_{n \leq x} \left( \sum_{[x/pn] \leq x} \cos \left(\frac{2\pi nb}{q}\right) \right) - \frac{1}{\phi(q)} \sum_{n \leq x} \left( \sum_{qn \leq x} \cos \left(\frac{2\pi na}{p}\right) \right)
+ \frac{p}{\phi(p)} \sum_{pm \leq x} \left( \sum_{x/pm \leq x} \cos \left(\frac{2\pi nb}{q}\right) \right) + \frac{q}{\phi(q)} \sum_{qk \leq x} \left( \sum_{x/qk \leq x} \cos \left(\frac{2\pi na}{p}\right) \right)
- \frac{1}{\phi(p)\phi(q)} \left( \sum_{n \leq x} d(n) - q \sum_{n \leq x/q} d(n) - p \sum_{n \leq x/p} d(n) + pq \sum_{n \leq x/pq} d(n) \right), \tag{7.10}
\]
where \(\phi(n)\) denotes Euler’s \(\phi\)-function. The sums
\[
\sum_{n \leq x} d_{\chi_1,\chi_2}(n)
\]
in (7.10) were examined in Theorem 7.1. To examine the next four sums recall the identity [5, p. 2063]
\[
\sum_{n \leq x} \left\lfloor \frac{x}{n} \right\rfloor \cos \left(\frac{2\pi na}{q}\right) = \sum_{n \leq x/q} d(n) + \sum_{d \mid q, d > 1} \phi(d) \sum_{\chi \text{ mod } d, \chi \neq 1} \chi(a)\tau(\chi) \sum_{n \leq dx/q} \left( \left\lfloor \frac{x}{n} \right\rfloor \cos \left(\frac{2\pi na}{q}\right) \right) \tag{7.11}
\]
where \(\phi(d)\) denotes Euler’s \(\phi\)-function, and where
\[
\phi(d) = 1 \quad \text{d}_{\chi}(n) = \sum_{d \mid n} \chi(d). \tag{7.12}
\]
Chandrasekharan and Narasimhan [12, p. 133] established an \(\Omega\) theorem for
\[
D_{\chi}(x) := \sum_{n \leq x} d_{\chi}(n) \tag{7.13}
\]
analogous to Theorem 7.1. For the last four sums on the right-hand side of (7.10), an Ω theorem is found in (1.9). In summary, although we can apply an Ω theorem to each of the divisor sums appearing in (7.10), we cannot apply Theorem 6.2 and the aforementioned analogue from [12] to a sum of these arithmetic sums for which we have Ω theorems. Thus, we formulate the following conjecture.

**Conjecture 7.2.** If $\mathcal{C}(x)$ is defined in (7.9), then
\[
\lim_{x \to \infty} \frac{\mathcal{C}(x)}{x^{1/4}} = +\infty,
\]
\[
\lim_{x \to \infty} \frac{\mathcal{C}(x)}{x^{1/4}} = -\infty.
\]

**Theorem 7.3.** For each $\epsilon > 0$, as $x \to \infty$,
\[
\mathcal{D}_{\chi_1, \chi_2}(x) = O(x^{1/3+\epsilon}).
\]

**Proof.** To prove Theorem 7.3, we apply Theorem 6.3. For the parameters that are needed, we refer to (7.6) and (7.7). First, recall that $\delta = A = 1$. We need the series $\sum_{n=1}^{\infty} |b(n)|n^{-\beta}$ to converge, and so we take $\beta = 1 + \epsilon$, for every $\epsilon > 0$. (The parameter $\epsilon$ will not necessarily be the same with each occurrence.) Thus, from (6.10),
\[
u := \beta - \frac{1}{2} \delta - \frac{1}{4} A = \frac{1}{4} + \epsilon.
\]
The first “Big O” power in (6.9) is then
\[
\frac{1}{4} + \left(\frac{1}{2} + 2\epsilon\right) \eta,
\]
where $\eta \geq 0$ and is yet to be determined. We also need to determine the order of
\[
\sum_{x < a_n \leq x'} |a(n)|,
\]
where $a(n)$ is given in (7.7) and $x' = x + O(x^{1/2-\eta})$ is defined in (6.11). Trivially, $|a(n)| \leq d(n)$. By (1.8), an upper bound for this sum is $O(x^{1/2-\eta} \log x)$. In order to achieve the most effective upper bound, we find the solution to
\[
\frac{1}{4} + \left(\frac{1}{2} + 2\epsilon\right) \eta = \frac{1}{2} - \eta + \epsilon',
\]
where $\epsilon, \epsilon'$ are arbitrarily small positive numbers. Thus, our optimal choice is $\eta = \frac{1}{6}$. Hence, by (7.14), our proof of Theorem 7.3 is complete.

**Theorem 7.4.** For each $\epsilon > 0$, as $x \to \infty$,
\[
\mathcal{C}(x) = O(x^{1/3+\epsilon}).
\]

**Proof.** Again, appearing in (7.10), there are three kinds of divisor sums. For the former sums, we have the bound in Theorem 7.3. For the “middle” four sums, we recall (7.11). Since $|d_{\chi}(n)| \leq d(n)$, we can apply the bound given in (1.8) for each of these four “middle” sums. For the latter four sums, we can invoke (1.8). In summary, each of the sums in (7.10) has the bound expressed in Theorem 7.4 and so the proof is complete.
8. The Second $\Omega$ and “Big O” Theorems and Conjecture

Let $\chi_1$ and $\chi_2$ be non-principal, primitive even and odd characters modulo $p$ and $q$, respectively. Let $\tau(\chi_1)$ and $\tau(\chi_2)$ denote the corresponding Gauss sums. Lastly, recall that $d_{\chi_1,\chi_2}(n)$ is given by (7.1).

**Theorem 8.1.** Assume that $\chi_1$ is a non-principal primitive even character modulo $p$ and that $\chi_2$ is a non-principal primitive odd character modulo $q$, where $p$ and $q$ are primes. Let $D_{\chi_1,\chi_2}(x)$ be defined by (7.2). Then

$$\lim_{x \to \infty} \frac{\text{Re} D_{\chi_1,\chi_2}(x)}{x^{1/4}} = +\infty,$$

(8.1)

$$\lim_{x \to \infty} \frac{\text{Re} D_{\chi_1,\chi_2}(x)}{x^{1/4}} = -\infty.$$  

(8.2)

Both (8.1) and (8.2) remain valid if we replace $\text{Re}$ by $\text{Im}$, respectively, in (8.1) and (8.2).

**Proof.** Recall from [8, p. 82] that

$$\left(\frac{2\pi}{\sqrt{pq}}\right)^{-s} \Gamma(s) L(s, \chi_1) L(s, \chi_2)$$

$$= -\frac{i \tau(\chi_1) \tau(\chi_2)}{\sqrt{pq}} \left(\frac{2\pi}{\sqrt{pq}}\right)^{s-1} \Gamma(1-s) L(1-s, \chi_1) L(1-s, \chi_2).$$

We now apply Theorem 6.3. The parameters from Definition 6.1 and (6.6) are:

$$N = 1, \quad \delta = 1, \quad A = 1, \quad \theta = \frac{1}{4},$$

$$a(n) = d_{\chi_1,\chi_2}(n), \quad b(n) = -i \tau(\chi_1) \tau(\chi_2) \frac{d_{\chi_1,\chi_2}(n)}{\sqrt{pq}}, \quad \lambda_n = \mu_n = \frac{2\pi n}{\sqrt{pq}}.$$ 

Since $L(0, \chi_1) = 0$, and both $L(s, \chi_1)$ and $L(s, \chi_2)$ are entire functions, then $Q(x) = 0$. Theorem 8.1 now follows immediately provided that we can show that (6.5) holds. The proof that (6.5) is valid is exactly the same as in the previous theorem. □

Our motivation for studying $d_{\chi_1,\chi_2}(n)$ is similar to that for Theorem 7.1. From [8, p. 85, Equation (4.5)]

$$\mathbb{C}(x) := \sum_{nm \leq x} \cos \left(\frac{2\pi nm}{p}\right) \sin \left(\frac{2\pi mb}{q}\right)$$

$$= \frac{1}{i\phi(p)\phi(q)} \sum_{\chi_1 \mod p}^{\chi_1 \neq \chi_0, \text{even}} \sum_{\chi_2 \mod q}^{\chi_2 \text{ odd}} \chi_1(a) \chi_2(b) \tau(\chi_1) \tau(\chi_2) \sum_{n \leq x} d_{\chi_1,\chi_2}(n)$$

$$- \frac{1}{\phi(p)} \sum_{m \leq x} \left[\frac{x}{m}\right] \sin \left(\frac{2\pi mb}{q}\right) + \frac{p}{\phi(p)} \sum_{m \leq x} \left[\frac{x}{pm}\right] \sin \left(\frac{2\pi mb}{q}\right).$$

(8.3)

As in our study of $\mathbb{C}(x)$, we observe that multiple sums of the form

$$\sum_{n \leq x} d_{\chi_1,\chi_2}(n)$$
arise. For and \( \Omega \) theorem for the first set of sums on the right-hand side of (8.3) appeal to Theorem 8.1. For the second set of sums, we recall the identity [5, p. 2068, Lemma 11]

\[
\sum_{n \leq x} \frac{x}{n} \sin \left( \frac{2\pi na}{p} \right) = -i \sum_{\substack{d|q \, d > 1}} \frac{\phi(d)}{\chi \mod d} \sum_{\chi \text{odd}} \chi(a) \tau(\overline{\chi}) \sum_{n \leq dx/q} \prime d \chi(n).
\] (8.4)

Recall that the character sums on the far right-hand side of (8.4) were defined in (7.12) and (7.13). An \( \Omega \) theorem analogous to Theorem 8.1 can also be established [12, p. 133]. We cannot appeal directly to Theorem 8.1 and the aforementioned analogue in order to establish an \( \Omega \) theorem for \( \operatorname{CS}(x) \). We therefore must content ourselves to making the following conjecture.

**Conjecture 8.2.** If \( \operatorname{CS}(x) \) is defined on the far left-hand side of (8.3), then

\[
\lim_{x \to \infty} \frac{\operatorname{CS}(x)}{x^{1/4}} = +\infty,
\] (9.2)

\[
\lim_{x \to \infty} \frac{\operatorname{CS}(x)}{x^{1/4}} = -\infty.
\] (9.3)

Both (9.2) and (9.3) remain valid if we replace \( \operatorname{Re} \) by \( \operatorname{Im} \) in each of (9.2) and (9.3).

**Theorem 8.3.** For each \( \epsilon > 0 \), as \( x \to \infty \),

\[
\mathbb{D}_{\chi_1,\chi_2}(x) = O(x^{1/3+\epsilon}).
\]

**Proof.** Because the values of \( A \) and \( \delta \) are identical to those in the proof of Theorem 7.3, the proof of Theorem 8.3 is the same as that for Theorem 7.3. \( \square \)

**Theorem 8.4.** For each \( \epsilon > 0 \), as \( x \to \infty \),

\[
\operatorname{CS}(x) = O(x^{1/3+\epsilon}).
\]

**Proof.** Referring to (8.3), we see that each member of the first set of sums on the right-hand side of (8.3) satisfies the bounds of Theorem 8.3. For the second set of sums, refer to (8.4). Since \( |d_\chi(n)| \leq d(n) \), each of the divisor sums on the right-hand side of (8.3) has an upper bound given by (1.8). Hence, the proof of Theorem 8.4 is complete. \( \square \)

9. **The Third \( \Omega \) and “Big O” Theorems and Conjecture**

The third general \( \Omega \) theorem is similar to Theorems 7.1 and 8.1.

**Theorem 9.1.** Assume that \( \chi_1 \) and \( \chi_2 \) are non-principal primitive odd characters modulo \( p \) and \( q \), respectively. Define

\[
\mathbb{D}^*_{\chi_1,\chi_2}(x) := \sum_{n \leq x} \prime nd_{\chi_1,\chi_2}(n).
\] (9.1)

Then

\[
\lim_{x \to \infty} \frac{\operatorname{Re} \mathbb{D}^*_{\chi_1,\chi_2}(x)}{x^{5/4}} = +\infty,
\] (9.2)

\[
\lim_{x \to \infty} \frac{\operatorname{Re} \mathbb{D}^*_{\chi_1,\chi_2}(x)}{x^{5/4}} = -\infty.
\] (9.3)

Both (9.2) and (9.3) remain valid if we replace \( \operatorname{Re} \) by \( \operatorname{Im} \) in each of (9.2) and (9.3).
Proof. The relevant functional equation is \[8\] top of page 89 (with \(s\) replaced by \(\frac{1}{2}s\)),

\[
\frac{\pi^{-s}}{(pq)^{-s/2}} \Gamma^2 \left( \frac{1}{2} \right) L(s - 1, \chi_1) L(s - 1, \chi_2)
\]

\[
= -\frac{\tau(\chi_1)\tau(\chi_2)}{\sqrt{pq}} \frac{\pi^{-(3-s)}}{(pq)^{-(3-s)/2}} \Gamma^2 \left( \frac{1}{2} (3 - s) \right) L(2 - s, \overline{\chi_1}) L(2 - s, \overline{\chi_2}).
\]

Note that

\[L(s - 1, \chi_1) L(s - 1, \chi_2) = \sum_{n=1}^{\infty} \frac{\chi_1(n) \chi_2(m)}{n^s} = \sum_{n=1}^{\infty} \frac{nd_{\chi_1, \chi_2}(n)}{n^s}.
\]

In the notation of Definition 6.1 and (6.6),

\[N = 2, \quad \delta = 3, \quad A = 1, \quad \theta = \frac{5}{4}, \quad (9.4)
\]

\[a(n) = nd_{\chi_1, \chi_2}(n), \quad b(n) = -\tau(\chi_1)\tau(\chi_2) \frac{nd_{\chi_1, \chi_2}(n)}{\sqrt{pq}}, \quad \lambda_n = \mu_n = \frac{\pi n}{\sqrt{pq}}.
\]

(In \[8\] p. 89] the factor \(n\) is missing from the definition of \(a(n)\).) Since \(\chi_1\) and \(\chi_2\) are odd primitive characters, \(L(2s - 1, \chi_1)\) and \(L(2s - 1, \chi_2)\) are both entire functions of \(s\) which vanish at \(s = 0\). Hence, \(Q(x) = 0\). Theorem 9.1 now follows if we can show that (6.5) is valid. This can be shown in the same way as given in the proof of Theorem 7.1. \(\square\)

Let

\[\text{SS}(x) := \sum_{mn \leq x} \sin(2\pi n a/p) \sin(2\pi mb/q). \quad (9.5)
\]

From \[8\] p. 91, Equation (5.13),

\[\text{SS}(x) = -\frac{1}{\phi(p)\phi(q)} \sum_{\chi_1 \mod p \chi_2 \mod q} \sum_{\chi_1 \text{ odd}} \chi_1(a)\chi_2(b)\tau(\chi_1)\tau(\chi_2) D_{\chi_1, \chi_2}^*(x), (9.6)
\]

where \(D_{\chi_1, \chi_2}^*(x)\) is defined in (9.1). Thus, since \(\text{SS}(x)\) is a linear combination of the sums \(D_{\chi_1, \chi_2}^*(x)\), we cannot appeal directly to Theorem 6.2. Thus, we make the following conjecture.

**Conjecture 9.2.** If \(\text{SS}(x)\) is defined by (9.5), then

\[
\lim_{x \to \infty} \frac{\text{SS}(x)}{x^{5/4}} = +\infty,
\]

\[
\lim_{x \to \infty} \frac{\text{SS}(x)}{x^{5/4}} = -\infty.
\]

We now use Chandrasekharan and Narasimhan’s Theorem 6.3 to obtain an upper bound for \(D_{\chi_1, \chi_2}^*(x)\). From (9.4), \(A = 1\) and \(\delta = 3\). Observe that \(|a(n)| \leq nd(n)|\). Furthermore, for some constant \(C > 0\), we also see that \(|b(n)| \leq Cnd(n)|\). Thus, for each \(\epsilon > 0\), we shall take \(\beta = 2 + \epsilon\). Hence, from (6.10),

\[u = 2 + \epsilon - \frac{3}{2} - \frac{1}{4} = \frac{1}{4} + \epsilon.
\]
With a reference to (6.9), we need to calculate, for \( \eta \geq 0 \),
\[
\frac{1}{2} - \frac{1}{4A} + 2A\eta = \frac{3}{2} - \frac{1}{4} + 2 \left( \frac{1}{4} + \epsilon \right) \eta = \frac{5}{4} + \left( \frac{1}{2} + 2\epsilon \right) \eta.
\] (9.7)

For the second power in (6.9), we use partial summation and (1.8) to deduce that
\[
\sum_{x < \mu \leq x + O( \frac{x^{1/2 - \eta}}{x^{1/2 - \eta}} )} nd(n) = O( x^{3/2 - \eta} \log x ).
\] (9.8)

From (9.7) and (9.8), we seek the optimal power of \( x \) by solving
\[
\frac{5}{4} + \left( \frac{1}{2} + 2\epsilon \right) \eta = \frac{3}{2} - \eta + \epsilon.
\]

Solving this simple equation, we see that \( \eta = \frac{1}{6} + \epsilon \). Therefore, we have established the following theorem.

**Theorem 9.3.** As \( x \to \infty \), for every \( \epsilon > 0 \),
\[
D_{x_1, x_2}^*(x) = O( x^{4/3 + \epsilon} ),
\]
where \( D_{x_1, x_2}^*(x) \) is defined by (9.1).

Using (9.6) and Theorem 9.3, we can immediately deduce the following theorem.

**Theorem 9.4.** As \( x \to \infty \), for every \( \epsilon > 0 \),
\[
SS(x) = O( x^{4/3 + \epsilon} ).
\]

We conclude this section with a special case. Let \( \frac{a}{p} = \frac{b}{q} = \frac{1}{4} \). Then
\[
\sin(2\pi n/4) = \begin{cases} (-1)^{(n-1)/2}, & \text{if } n \text{ odd,} \\ 0, & \text{if } n \text{ even,} \end{cases}
\]
and
\[
SS(\frac{1}{4}, \frac{1}{4}, x) := - \sum_{mn \leq x, m, n \text{ odd}} mn(-1)^{(m+n)/2}
\]
\[
= \sum_{(2j+1)(2k+1) \leq x} (-1)^{j+k}(2j+1)(2k+1),
\]
where we set \( m = 2j + 1, n = 2k + 1 \). This is a rather interesting lattice point problem. We are counting lattice points under the hyperbola \( ab \leq x \), but we require both coordinates to be odd, and we put a weight on them.

We restate Conjecture 9.2 and Theorem 9.4 in this particular case \( \frac{a}{p} = \frac{b}{q} = \frac{1}{4} \).

**Conjecture 9.5.** If \( SS(x) \) is defined by (9.5), then
\[
\lim_{x \to \infty} \frac{SS(\frac{1}{4}, \frac{1}{4}, x)}{x^{5/4}} = +\infty,
\]
\[
\lim_{x \to \infty} \frac{SS(\frac{1}{4}, \frac{1}{4}, x)}{x^{5/4}} = -\infty.
\]
**Theorem 9.6.** As \( x \to \infty \), for every \( \epsilon > 0 \),
\[
\mathbb{S}_S\left(\frac{1}{4}, \frac{1}{4}, x\right) = O(x^{4/3+\epsilon}).
\]

10. Sums with a Product of an Arbitrary Number of \( \sin \)’s

We begin this section with a definition. Let \( J_\nu(x) \) denote the ordinary Bessel function of order \( \nu \). Define
\[
K_\nu(x; \mu; m) := \int_0^\infty \frac{u^{\mu-1}}{\sqrt{u}} \int_0^\infty \frac{u^{\nu-1}}{\sqrt{u}} \frac{J_\mu(u_{m-1})}{u_{m-1}} \frac{J_\mu(u_{m-2})}{u_{m-2}} \cdots \frac{J_\mu(u_1)}{u_1} \frac{J_\nu(x/u_1 u_2 \cdots u_{m-1})}{u_1},
\tag{10.1}
\]
provided that \( \mu, \nu > -3/2 \), so that the integral converges.

In the sequel, we apply Theorems 2 and 4 of \cite{2}, pp. 351, 356, which, for convenience, we offer below.

**Theorem 10.1.** Let \( \varphi(s) \) and \( \psi(s) \) satisfy the functional equation (6.1) with \( \Delta(s) = \Gamma^k(s) \). If \( k \geq 2 \), suppose that \( \delta > -\frac{1}{2} \). Assume that \( \rho > 2k\sigma_a^* - k\delta - \frac{3}{2} \), where \( \sigma_a^* \) is the abscissa of absolute convergence of \( \psi(s) \). If \( x > 0 \), then
\[
\frac{1}{\Gamma(\rho + 1)} \sum_{\lambda_n \leq x} \varphi(n)(x - \lambda_n)^\rho = 2^{\rho(1-k)} \sum_{b(n)} \left( \frac{x}{\mu_n} \right)^{(\delta+\rho)/2} \sum_{K_{\delta+\rho}(2^k(\mu_n x)^{1/2}; \delta - 1; k) + Q_\rho(x),}
\]
where \( Q_\rho(x) \) is defined in (6.4).

The following Theorem 10.2 is an extension of Theorem 10.1. In our application, the hypotheses are readily verified.

**Theorem 10.2.** Suppose that for \( \sigma > \sigma_a^* \),
\[
\sup_{0 \leq \lambda \leq 1} \left| \sum_{r^{2k} \mu_n \leq (r+\lambda)^{2k}} b(n) \mu_n^{\sigma-1/(2k)} \right| = o(1),
\]
as \( r \to \infty \). Then (10.1) is valid for \( q > 2k\sigma_a^* - k\delta - \frac{3}{2} \) and for those positive values of \( x \) such that the left-hand side of (10.1) is defined. The series on the right-hand side of (10.1) converges uniformly on any interval for \( x > 0 \) where the left-hand side is continuous. The convergence is bounded on any interval \( 0 < x_1 \leq x \leq x_2 \) when \( \rho = 0 \).

Let \( k \) be an arbitrary positive integer. Let \( \chi_1, \chi_2, \ldots, \chi_k \) be odd, primitive, non-principal, characters modulo \( p_1, p_2, \ldots, p_k \), respectively. Throughout this section, we assume that \( \chi_1, \chi_2, \ldots, \chi_k \) and \( \chi \) are odd, and so we normally make this assumption without comment. Let \( a_1, a_2, \ldots, a_k \) denote positive integers such that \( (a_j, p_j) = 1, 1 \leq j \leq k \).

**Definition 10.3.** Let \( n_1, n_2, \ldots, n_k \) denote positive integers. Define
\[
d_{\chi_1, \chi_2, \ldots, \chi_k}(n) := \sum_{n_1 n_2 \cdots n_k = n} \chi_1(n_1)\chi_2(n_2) \cdots \chi_k(n_k)
\]
and
\[ D^k(x) := \sum_{n \leq x}^\prime nd_{\chi_1, \chi_2, \ldots, \chi_k}(n), \] (10.2)
where, as customary, the prime \( \prime \) on the summation sign indicates that if \( x \) is an integer, only \( \frac{1}{2} \) of the term is counted.

Note that if \( k = 2 \), then (10.2) is identical to (9.1).

**Theorem 10.4.** Recall the definition of \( K_{\rho + 3/2}(x; \frac{1}{2}; k) \) defined by (10.1), as well as the definitions and notation above. Then for \( \rho > (k - 3)/2 \),
\[ \sum_{n \leq x}^\prime nd_{\chi_1, \chi_2, \ldots, \chi_k}(n)(x^2 - n^2)^\rho = \frac{(-i)^k \rho^{\rho - 1/2} \tau(\chi_1) \tau(\chi_2) \cdots \tau(\chi_k)}{\pi^{k\rho}} \]
\[ \times \sum_{n=1}^\infty nd_{\chi_1, \chi_2, \ldots, \chi_k}(n) \left( \frac{x}{n} \right) \rho^{3/2} K_{\rho + 3/2} \left( \frac{2\pi \rho^{\rho} x}{(p_1p_2 \cdots p_k)^{3/2}} ; \frac{1}{2} ; k \right). \]

The integral
\[ K_{\rho + 3/2} \left( \frac{2\pi \rho^{\rho} x}{(p_1p_2 \cdots p_k)^{3/2}} ; \frac{1}{2} ; k \right) \]
can be represented by a Meijer G-function, but we do not provide it here.

**Proof.** Recall that the Dirichlet \( L \)-function \( L(s, \chi) \) of modulus \( q \) satisfies the functional equation [8 p. 82]
\[ \left( \frac{\pi}{q} \right)^{-2s+1/2} \Gamma \left( s + \frac{1}{2} \right) L(2s, \chi) = -i \frac{\tau(\chi)}{\sqrt{q}} \left( \frac{\pi}{q} \right)^{-1-s} \Gamma(1 - s) L(1 - 2s, \chi). \] (10.3)

We replace \( s \) by \( s - \frac{1}{2} \), \( q \) by \( p_j \), and \( \chi \) by \( \chi_j \) in (10.3), \( 1 \leq j \leq k \). Multiply the \( k \) functional equations together to obtain
\[ \Gamma^k \left( s \right) L(2s - 1, \chi_1) L(2s - 1, \chi_2) \cdots L(2s - 1, \chi_k) \]
\[ = \frac{(-i)^k \tau(\chi_1) \tau(\chi_2) \cdots \tau(\chi_k)}{\sqrt{p_1p_2 \cdots p_k}} \pi^{-k(3-2s)/2} \]
\[ \times L(2-2s, \chi_1) L(2-2s, \chi_2) \cdots L(2-2s, \chi_k). \] (10.4)

Note that \( \delta = \frac{3}{2} \) and that
\[ L(2s - 1, \chi_1) L(2s - 1, \chi_2) \cdots L(2s - 1, \chi_k) \]
\[ = \sum_{n_1=1}^\infty \frac{\chi_1(n_1)}{n_1^{2s-1}} \cdots \sum_{n_k=1}^\infty \frac{\chi_k(n_k)}{n_k^{2s-1}} = \sum_{n=1}^\infty \frac{nd_{\chi_1, \chi_2, \ldots, \chi_k}(n)}{n^{2s}}, \quad \sigma = \text{Re } s > 1. \]

We apply Theorems 10.1 and 10.2. Observe that \( Q_\rho(x) = 0 \), because, for \( 1 \leq j \leq k \), \( L(s, \chi_j) \) is an entire function and \( L(-1, \chi_j) = 0 \), since \( \chi_j \) is odd. Also observe that
\[ a(n) = nd_{\chi_1, \chi_2, \ldots, \chi_k}(n), \quad b(n) = \frac{(-i)^k \tau(\chi_1) \tau(\chi_2) \cdots \tau(\chi_k)}{\sqrt{p_1p_2 \cdots p_k}} \frac{nd_{\chi_1, \chi_2, \ldots, \chi_k}(n)}{n^{2s}}. \]
and
\[ \lambda_n = \mu_n = \frac{\pi^kn^2}{p_1p_2 \cdots p_k}. \]

In Theorem 10.1 the sum is over \( \lambda_n \leq x \). Replace \( x \) by
\[ \frac{\pi^kx^2}{p_1p_2 \cdots p_k}. \]
Thus, the amended sum will be over \( n \leq x \) and
\[ (x - \lambda_n)^\rho \Rightarrow \frac{\pi^k\rho}{(p_1p_2 \cdots p_k)^\rho}(x^2 - n^2)^\rho. \]

Also appearing in Theorem 10.1 is a quotient in the summands on the right side that will be transformed by the change in variable above, i.e.,
\[ \left( \frac{x}{\mu_n} \right)^{(3/2+\rho)/2} \Rightarrow \left( \frac{x}{n} \right)^{3/2+\rho}. \]
Lastly, on the right-hand side of (10.1),
\[ 2^k(\mu_n x)^{1/2} \Rightarrow 2^k \left( \frac{\pi^k n^2}{p_1p_2 \cdots p_k} \cdot \frac{\pi^k x^2}{p_1p_2 \cdots p_k} \right)^{1/2} = \frac{2^k\pi^knx}{p_1p_2 \cdots p_k}. \]

With all of these substitutions, we deduce that
\[ \pi^k \rho \sum_{n \leq x} nd_{\chi_1,\chi_2,\ldots,\chi_k}(n)(x^2 - n^2)^\rho \]
\[ = \frac{(-i)^k\tau(\chi_1)\tau(\chi_2) \cdots \tau(\chi_k)}{\sqrt{p_1p_2 \cdots p_k}} \sum_{n=1}^\infty nd_{\chi_1,\chi_2,\ldots,\chi_k}(n) \left( \frac{x}{n} \right)^{\rho+3/2} K_{\rho+3/2} \left( \frac{2^k\pi^k n x}{p_1p_2 \cdots p_k}; 1/2; k \right), \]
where \( K_{\rho+3/2} \) is defined in (10.1). The identity above is precisely (10.4), and so the proof is complete. \( \square \)

We next prove an identity for the sum
\[ S_\rho(a_1, a_2, \ldots, a_k; p_1, p_2, \ldots, p_k; x) := \sum_{1 \leq n_1 n_2 \cdots n_k \leq x} n_1 n_2 \cdots n_k \sin(2\pi n_1 a_1/p_1) \sin(2\pi n_2 a_2/p_2) \cdots \sin(2\pi n_k a_k/p_k)(x^2 - n^2)^\rho, \]

(10.5)

where \( (a_j, p_j) = 1, 1 \leq j \leq k \) and \( n = n_1 n_2 \cdots n_k \).

**Lemma 10.5.** [8, p. 72] Let \( (a, q) = 1 \) and \( n \in \mathbb{Z} \). Suppose that \( \chi \) is an odd, primitive, non-principal character of order \( q \). Then
\[ \sin(2\pi na/q) = \frac{1}{i\phi(q)} \sum_{\chi \mod q, \chi \text{odd}} \chi(a)\tau(\chi)\chi(n), \]
where \( \phi(q) \) denotes Euler’s \( \phi \)-function.
Lemma 10.6. [3 p. 306, Equation (4.12)] For any primitive character $\chi$ modulo $q$,\n\[\sum_{\chi \equiv \pm b \mod q} \chi(a)\chi(b) = \begin{cases} \pm \frac{1}{2}\phi(q), & \text{if } a \equiv b \mod q \text{ and } (a, q) = 1, \\ 0, & \text{otherwise.} \end{cases} \]
\nonumber

We let\n\[\sum_{\pm n_1, \pm n_2, \ldots, \pm n_k} \]
denote a sum over all $2^k$ pairs\n\[n_1 \equiv \pm a_1 \mod p_1, \quad n_2 \equiv \pm a_2 \mod p_2, \quad n_k \equiv \pm a_k \mod p_k.\]

Theorem 10.7. In the notation above, for $\rho > (k - 3)/2$,\n\[S_\rho(a_1, a_2, \ldots, a_k; p_1, p_2, \ldots, p_k; x) = \frac{x^{\rho+3/2}(p_1 p_2 \cdots p_k)^{\rho+1/2}}{2k\pi k^p} \sum_{\pm n_1, \pm n_2, \ldots, \pm n_k} (-1)^{\text{sgn}} K_{\rho+3/2} \left( \frac{2k\pi k n_1 n_2 \cdots n_k x}{p_1 p_2 \cdots p_k}; \frac{1}{2}; k \right), \]
where sgn denotes the number of minus signs in a particular $k$-tuple, $\pm n_1, \pm n_2, \ldots, \pm n_k$.

Proof. By (10.5) and Lemma 10.6, if $n = n_1 n_2 \cdots n_k$,\n\[S_\rho(a_1, a_2, \ldots, a_k; p_1, p_2, \ldots, p_k; x) = \frac{(-i)^k}{\phi(p_1)\phi(p_2)\cdots\phi(p_k)} \sum_{\chi_1 \equiv 1 \mod p_1, \chi_2 \equiv 1 \mod p_2, \ldots, \chi_k \equiv 1 \mod p_k} \chi(a_1)\chi(a_2)\cdots\chi(a_k) \tau(\chi_1)\tau(\chi_2)\cdots\tau(\chi_k) \chi(n_1)\chi(n_2)\cdots\chi(n_k) \]
\[= \frac{(-i)^k}{\phi(p_1)\phi(p_2)\cdots\phi(p_k)} \sum_{\chi_1 \equiv 1 \mod p_1, \chi_2 \equiv 1 \mod p_2, \ldots, \chi_k \equiv 1 \mod p_k} \chi(a_1)\chi(a_2)\cdots\chi(a_k) \tau(\chi_1)\tau(\chi_2)\cdots\tau(\chi_k) \sum_{n \leq x} n d_{\chi_1, \chi_2, \ldots, \chi_k}(n)(x^2 - n^2)^\rho. \]

On the other hand, by Lemma 10.6\n\[= \frac{x^{\rho+3/2}(p_1 p_2 \cdots p_k)^{\rho+1/2}}{\phi(p_1)\phi(p_2)\cdots\phi(p_k)} \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \cdots \sum_{n_k=1}^{\infty} K_{\rho+3/2} \left( \frac{2k\pi k n_1 n_2 \cdots n_k x}{p_1 p_2 \cdots p_k}; \frac{1}{2}; k \right), \]
\[= \frac{x^{\rho+3/2}(p_1 p_2 \cdots p_k)^{\rho+1/2}}{\phi(p_1)\phi(p_2)\cdots\phi(p_k)} \sum_{\chi_1 \equiv 1 \mod p_1, \chi_2 \equiv 1 \mod p_2, \ldots, \chi_k \equiv 1 \mod p_k} \chi(a_1)\chi(a_2)\cdots\chi(a_k) \tau(\chi_1)\tau(\chi_2)\cdots\tau(\chi_k) \chi(n_1)\chi(n_2)\cdots\chi(n_k) \]
\[= \frac{x^{\rho+3/2}(p_1 p_2 \cdots p_k)^{\rho+1/2}}{\phi(p_1)\phi(p_2)\cdots\phi(p_k)} \sum_{\chi_1 \equiv 1 \mod p_1, \chi_2 \equiv 1 \mod p_2, \ldots, \chi_k \equiv 1 \mod p_k} \chi(a_1)\chi(a_2)\cdots\chi(a_k). \]
Proof. Replacing proof of Theorem 9.1. □

With the use of Theorem 6.2, the remainder of the proof follows along the same lines as the proof of Theorem 10.8.

\[
\sum_{n=1}^{\infty} d_{x_1, x_2, \ldots, x_k}(n) \frac{K_{\rho+3/2} \left( \frac{2^k \pi^k n x}{p_1 p_2 \cdots p_k}, \frac{1}{k} \right)}{n^{\rho+1/2}}
\]

\[
= \frac{(p_1 p_2 \cdots p_k)^{\rho+1/2}}{\phi(p_1) \phi(p_2) \cdots \phi(p_k)} \sum_{\chi_1 \mod p_1} \sum_{\chi_2 \mod p_2} \cdots \sum_{\chi_k \mod p_k} \chi_1(a_1) \chi_2(a_2) \cdots \chi_k(a_k)
\]

\[
\times \sum_{n=1}^{\infty} n d_{x_1, x_2, \ldots, x_k}(n) \left( \frac{x}{n} \right)^{\rho+3/2} K_{\rho+3/2} \left( \frac{2^k \pi^k n x}{p_1 p_2 \cdots p_k}, \frac{1}{k} \right)
\]

\[
= \frac{(p_1 p_2 \cdots p_k)^{\rho+1/2}}{\phi(p_1) \phi(p_2) \cdots \phi(p_k)} \sum_{\chi_1 \mod p_1} \sum_{\chi_2 \mod p_2} \cdots \sum_{\chi_k \mod p_k} \chi_1(a_1) \chi_2(a_2) \cdots \chi_k(a_k)
\]

\[
\times \pi^{k \rho} \left( -i \right)^k \tau(\chi_1) \tau(\chi_2) \cdots \tau(\chi_k) (p_1 p_2 \cdots p_k)^{\rho-1/2} \sum_{n \leq x} n d_{x_1, x_2, \ldots, x_k}(n) (x^2 - n^2)^\rho
\]

\[
= \frac{(p_1 p_2 \cdots p_k)^{\rho+1/2}}{\phi(p_1) \phi(p_2) \cdots \phi(p_k)} \sum_{\chi_1 \mod p_1} \sum_{\chi_2 \mod p_2} \cdots \sum_{\chi_k \mod p_k} \chi_1(a_1) \chi_2(a_2) \cdots \chi_k(a_k)
\]

\[
\times \tau(\chi_1) \tau(\chi_2) \cdots \tau(\chi_k) \sum_{n \leq x} n d_{x_1, x_2, \ldots, x_k}(n) (x^2 - n^2)^\rho,
\]

(10.7)

where in the penultimate step we applied Theorem 10.4 and in the last step, used the fact that for odd \( \chi \) [4, p. 45],

\[\tau(\chi_j) \tau(\chi_j) = -p_j, \quad 1 \leq j \leq k.\]

If we now compare (10.6) with (10.7), we deduce Theorem 10.7 □

Next, we offer a generalization of Theorem 9.1.

**Theorem 10.8.** Assume that \( \chi_1, \chi_2, \ldots, \chi_k \) are non-principal primitive odd characters modulo \( p_1, p_2, \ldots, p_k \), respectively. Recall the definition (10.2) of \( \mathbb{D}^k(x) \). Then

\[
\lim_{x \to \infty} \Re \mathbb{D}^k(x) = +\infty,
\]

(10.8)

\[
\lim_{x \to \infty} \Im \mathbb{D}^k(x) = -\infty.
\]

(10.9)

Both (10.8) and (10.9) remain valid if we replace \( \Re \) by \( \Im \) in each of (10.8) and (10.9).

**Proof.** Replacing \( s \) by \( s/2 \) in (10.4), we see that \( A = \frac{1}{2} k \) and \( \delta = 3 \). Thus, from the definition (6.6),

\[\theta = \frac{1}{2} k \cdot 3 - \frac{1}{2} \frac{k}{2} = \frac{3k - 1}{2k}.\]

With the use of Theorem 6.2, the remainder of the proof follows along the same lines as the proof of Theorem 9.1 □
In analogy with Conjecture 9.2, a similar conjecture can be made for Conjecture 10.9.

Thus, we apply Theorem 10.8 to each of these sums to obtain the following conjecture.

Recall from (10.5) that

$$S(a_1, a_2, \ldots, a_k; p_1, p_2, \ldots, p_k; x) := S_0(a_1, a_2, \ldots, a_k; p_1, p_2, \ldots, p_k; x)$$

$$= \sum_{1 \leq n_1 n_2 \cdots n_k \leq x} n_1 n_2 \cdots n_k \sin(2\pi n_1 a_1/p_1) \sin(2\pi n_2 a_2/p_2) \cdots \sin(2\pi n_k a_k/p_k).$$

From (10.6) with $\rho = 0$, we see that $S(a_1, a_2, \ldots, a_k; p_1, p_2, \ldots, p_k; x)$ is a linear combination of terms of the form

$$D_k(x) = \sum_{n \leq x} \eta \cdot d_{k, \chi_1, \chi_2, \ldots, \chi_k}(n).$$

In analogy with Conjecture 9.2, a similar conjecture can be made for

$$S(a_1, a_2, \ldots, a_k; p_1, p_2, \ldots, p_k; x).$$

Thus, we apply Theorem 10.8 to each of these sums to obtain the following conjecture.

**Conjecture 10.9.** If $S(a_1, a_2, \ldots, a_k; p_1, p_2, \ldots, p_k; x)$ is defined by (10.5), then

$$\lim_{x \to \infty} \frac{S(a_1, a_2, \ldots, a_k; p_1, p_2, \ldots, p_k; x)}{x^{(3k-1)/(2k)}} = +\infty,$$

$$\lim_{x \to \infty} \frac{S(a_1, a_2, \ldots, a_k; p_1, p_2, \ldots, p_k; x)}{x^{(3k-1)/(2k)}} = -\infty.$$

Note that Conjecture 9.2 is the special case $k = 2$ of Conjecture 10.9.

Similarly, we can use (10.6) and Theorem 6.3 to obtain an upper bound for the order of $S(a_1, a_2, \ldots, a_k; p_1, p_2, \ldots, p_k; x)$.

**Theorem 10.10.** For every $\epsilon > 0$, as $x \to \infty$,

$$S(a_1, a_2, \ldots, a_k; p_1, p_2, \ldots, p_k; x) = O(x^{2k/(k+1) + \epsilon}).$$

**Proof.** Note that $\beta = 2 + \epsilon$ for each $\epsilon > 0$. Then, by (6.10),

$$u = 2 + \epsilon - \frac{3}{2} - \frac{1}{2k} = \frac{1}{2} + \epsilon - \frac{1}{2k}.$$

Next, by (6.9), we need to calculate

$$\frac{1}{2} - \frac{1}{4A} + 2Au \eta = \frac{3}{2} - \frac{1}{2k} + k \left(\frac{1}{2} + \epsilon - \frac{1}{2k}\right) \eta = \frac{3}{2} - \frac{1}{2k} + \frac{k-1}{2} \eta + k\epsilon \eta,$$

where $\eta$ is a non-negative number to be determined.

Let $d_k(n)$ denote the number of ways $n$ can be written as a product of $k$ factors. Then, by (1.8) and induction on $k$,

$$\sum_{n \leq x} d_k(n) = xP_{k-1}(\log x) + O(x^{(k-1)/(k+1)} \log^{k-1} x),$$

where $k \geq 2$ and $P_r(\log x)$ is a polynomial of degree $r$ in $\log x$. (See also [12, p. 133, Equation (10.10)]). Next, use (10.12) and partial summation to deduce that

$$\sum_{x < \mu_n \leq x + O(x^{1-1/k-\eta})} \sum_{x < \mu_n \leq x + O(x^{1-1/k-\eta})} nd_{k, \chi_1, \chi_2, \ldots, \chi_k}(n) \leq \sum_{x < \mu_n \leq x + O(x^{1-1/k-\eta})} nd_k(n) = O(x^{2-1/k-\eta} \log^{k-1} x).$$
Appealing to Theorem 6.3, we should find the optimal value of \( \eta \) by equating the powers in (10.11) and (10.13). Thus, we should solve

\[
2 - \frac{1}{k} - \eta = \frac{3}{2} - \frac{1}{2k} + \frac{k-1}{2} \eta.
\]

Hence,

\[
\eta = \frac{k-1}{k(k+1)},
\]

and so the optimal power is, for every \( \epsilon > 0 \),

\[
2 - \frac{k-1}{k(k+1)} + \epsilon = \frac{2k}{k+1} + \epsilon.
\]

This completes the proof of (10.10). \( \square \)

Note that if \( k = 2 \), (10.10) reduces to Theorem 9.4. Note also that

\[
\frac{2(k+1)}{k+2} - \frac{2k}{k+1} = \frac{2}{(k+1)(k+2)}
\]

is the difference in the exponents of (10.10) for successive values of \( k \). Thus, increasing the number of sin’s by 1 in \( \mathbb{S}(a_1, a_2, \ldots, a_k; p_1, p_2, \ldots, p_k; x) \) increases the upper bound for the power in the error term by a “small” amount, i.e., \( O(1/k^2) \).

Suppose that

\[
\frac{a_j}{p_j} = \frac{1}{4}, \quad 1 \leq j \leq k,
\]

so that the terms are equal to 0 if one or more of the \( n_j, 1 \leq j \leq k \) are even. For odd \( n_j \), let \( n_j = 2m_j + 1, \quad 1 \leq j \leq k \). Then,

\[
\sum'_{1 \leq n_1, n_2, \ldots, n_k \leq x} n_1n_2 \cdots n_k \sin(2\pi n_1a_1/p_1) \sin(2\pi n_2a_2/p_2) \cdots \sin(2\pi n_ka_k/p_k)
\]

\[
= \sum'_{1 \leq 2m_1+1, 2m_2+1, \ldots, 2m_k+1 \leq x} (2m_1+1)(2m_2+1) \cdots (2m_k+1)(-1)^{m_1+2m_2+\cdots+m_k}
\]

\[
= \sum'_{1 \leq m_1, m_2, \ldots, m_k \leq x} (2m_1+1)(2m_2+1) \cdots (2m_k+1)(-1)^{m_1+2m_2+\cdots+m_k},
\]

after replacing \( x \) by \( 2x + 1 \). Thus, Theorem 10.7 gives an identity for the weighted sum of products of positive odd lattice points \( 2m_1+1, 2m_2+1, \ldots, 2m_k+1 \) in \( k \)-dimensional space weighted by \( (-1)^{m_1+2m_2+\cdots+m_k} \). However, instead of applying Theorem 10.7 directly, if we apply Equation (10.6) instead, we obtain a “Big O” bound for this sum weighted by \( (-1)^{m_1+2m_2+\cdots+m_k} \). Recall that we addressed the case \( k = 2 \) earlier.

Finding a representation for a sum of \( k \) cos-functions appears to be enormously complicated. Similarly, finding a representation for a sum with a mixture of sin’s and cos’s appears also to be extremely complicated. To make any progress, it would appear that the numbers of sin’s and cos’s should be equal.
REFERENCES

[1] G. E. Andrews, R. A. Askey, and R. Roy, Special functions, Cambridge University Press, Cambridge, 1999.

[2] B. C. Berndt, Identities involving the coefficients of a class of Dirichlet series. I, Trans. Amer. Math. Soc. 137 (1969), 345–359.

[3] B. C. Berndt, A. Dixit, S. Kim, and A. Zaharescu, On a theorem of A. I. Popov on sums of squares, Proceedings of Amer. Math. Soc. 145 (2017), 3795–3808.

[4] B. C. Berndt, R. J. Evans, and K. S. Williams, Gauss and Jacobi Sums, John Wiley, New York, 1998.

[5] B. C. Berndt, A. Dixit, S. Kim, and A. Zaharescu, On a theorem of A. I. Popov on sums of squares, Proceedings of Amer. Math. Soc. 145 (2017), 3795–3808.

[6] B. C. Berndt, S. Kim, and A. Zaharescu, Weighted divisor sums and Bessel function series, II, Adv. Math. 229 (2012), 2055–2097.

[7] B. C. Berndt, S. Kim, and A. Zaharescu, Weighted divisor sums and Bessel function series, III, J. Reine Angew. Math. 683 (2013), 67–96.

[9] B. C. Berndt, J. Li, and A. Zaharescu, The final problem: an identity from Ramanujan’s lost notebook, J. London Math. Soc. 100 (2019)

[10] B. C. Berndt and A. Zaharescu, Weighted divisor sums and Bessel function series, Math. Ann. 335 (2006), 249–283.

[12] K. Chandrasekharan and R. Narasimhan, Functional equations with multiple gamma factors and the average order of arithmetical functions, Ann. Math. 76 (1962), 93–136.

[13] P. G. L. Dirichlet, Recherches sur diverses applications de l’analyse infinitésimale à théorie des nombres, J. Reine Angew. Math. 21 (1840), 1–12.

[14] G. H. Hardy, The average order of the arithmetical functions \( P(x) \) and \( \Delta(x) \), Proc. London Math. Soc. (2) 15 (1916), 192–213.

[15] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, sixth ed., Oxford University Press, 2006.

[16] A. E. Ingham, On two classical lattice point problems, Proc. Cambridge Phil. Soc. 40 (1936), 131–138.

[17] S. Ramanujan, The Lost Notebook and Other Unpublished Papers, Narosa, New Delhi, 1988.

[18] G. F. Voronoï, Sur une fonction transcandante et ses applications à la sommation de quelques séries, Ann. École Norm. Sup. (3) 21 (1904), 207–267, 459–533.

[19] G. N. Watson, Theory of Bessel Functions, second ed., University Press, Cambridge, 1966.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 WEST GREEN STREET, URBANA, IL 61801, USA
Email address: berndt@illinois.edu

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, OSKAR-MORGENSTERN-PLATZ 1, 1090 WIEN, AUSTRIA
Email address: martino.fassina@univie.ac.at

DEPARTMENT OF MATHEMATICS, AND INSTITUTE OF PURE AND APPLIED MATHEMATICS, JEONBUK NATIONAL UNIVERSITY, 567 BAEKJE-DAERO, JEONJU-SI, JEOLLABUK-DO 54896, REPUBLIC OF KOREA
Email address: sunkim@jbnu.ac.kr

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILLINOIS, 1409 WEST GREEN STREET, URBANA, IL 61801, USA; INSTITUTE OF MATHEMATICS OF THE ROMANIAN ACADEMY, P.O. BOX 1-764, BUCHAREST RO-70700, ROMANIA
Email address: zaharesc@illinois.edu