REACTION-DIFFUSION SYSTEMS AND NONLINEAR WAVES

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Abstract. The authors investigate the solution of a nonlinear reaction-diffusion equation connected with nonlinear waves. The equation discussed is more general than the one discussed recently by Manne, Hurd, and Kenkre (2000). The results are presented in a compact and elegant form in terms of Mittag-Leffler functions and generalized Mittag-Leffler functions, which are suitable for numerical computation. The importance of the derived results lies in the fact that numerous results on fractional reaction, fractional diffusion, anomalous diffusion problems, and fractional telegraph equations scattered in the literature can be derived, as special cases, of the results investigated in this article.

1 Introduction

Reaction-diffusion models have found numerous applications in pattern formation in biology, chemistry, and physics, see Smoller (1983), Grindrod (1991), Gilding and Kersner (2004), and Wilhelmsson and Lazzaro (2001). These systems show that diffusion can produce the spontaneous formation spatio-temporal patterns. For details, refer to the work of Nicolis and Prigogine (1977), and Haken (2004). A general model for reaction-diffusion systems is discussed by Henry and Wearne (2000, 2002), and Henry, Langlands, and Wearne (2005). A piecewise linear approach in connection with the diffusive processes has been developed by Strier, Zanette, and Wio (1995) which leads to analytic results in reaction-diffusion systems. A similar approach
was recently used by Manne, Hurd, and Kenkre (2000) to investigate effects on the propagation of nonlinear wave fronts.

The simplest reaction-diffusion models can be described by an equation

\[
\frac{\partial N}{\partial t} = D \frac{\partial^2 N}{\partial x^2} + \gamma F(N),
\]

(1)

where \( D \) is the diffusion coefficient and \( F(N) \) is a nonlinear function representing reaction kinetics. It is interesting to observe that for \( F(N) = \gamma N(1 - N) \), eq.(1) reduces to Fisher-Kolmogorov equation and if we set \( F(N) = \gamma N(1 - N^2) \), it reduces to the real Ginsburg-Landau equation.

A generalization of (1) has been considered by Manne, Hurd, and Kenkre (2000) in the form

\[
\frac{\partial^2 N}{\partial t^2} + a \frac{\partial N}{\partial t} = \nu^2 \frac{\partial^2 N}{\partial x^2} + \xi^2 N(x,t),
\]

(2)

where \( \xi \) indicates the strength of the nonlinearity of the system. In this article, we present a straightforward method for the systematic derivation of the solution of nonlinear reaction-diffusion equations connected with nonlinear waves, which is more general than the equation (2). The results are derived in a closed-form, by the application of Laplace and Fourier transforms, which are suitable for numerical computation. The present study is in continuation of our investigations reported earlier in the articles (Saxena, Mathai, and Haubold, 2002, 2004, 2004a, 2004b, 2005).

2 Mathematical Prerequisites

A generalization of the Mittag-Leffler function (Mittag-Leffler 1903, 1905)

\[
E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}, \quad \alpha \in \mathbb{C}, \text{Re}(\alpha) > 0,
\]

(3)

was introduced by Wiman (1905) in the general form

\[
E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}, \quad \alpha, \beta \in \mathbb{C}, \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0.
\]

(4)
The main results of these functions are available in the handbook of Erdélyi, Magnus, Oberhettinger, and Tricomi (1955, Section 18.1) and the monographs by Dzherbashyan (1966, 1993). Prabhakar (1971) introduced a generalization of (4) in the form

\[ E_{\alpha,\beta}^\gamma(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(n\alpha + \beta)(n)!} \quad \alpha, \beta, \gamma \in \mathbb{C}; \quad \Re(\alpha), \Re(\beta), \Re(\gamma) > 0, \tag{5} \]

where \((\gamma)_r\) is Pochhammer’s symbol, defined by

\[ (\gamma)_0 = 1, \quad (\gamma)_r = \gamma(\gamma+1)(\gamma+2)\ldots(\gamma+r-1), \quad r = 1, 2, \ldots, \gamma \neq 0. \tag{6} \]

It is an entire function with \(\rho = [\Re(\alpha)]^{-1}\) (Prabhakar, 1971). The solution of generalized Volterra-type differ-integral equations associated with this function as a kernel is derived by Kilbas, Saigo and Saxena (2004). A general theory of generalized fractional calculus based on this function has been developed by Kilbas, Saigo and Saxena (2004), generalizing the results for Riemann-Liouville fractional integrals and derivatives, which form the backbone of fractional differ-integral equations.

For \(\gamma = 1\), this function coincides with (4), while for \(\beta = \gamma = 1\) with (3)

\[ E_{\alpha,\beta}^1(z) = E_{\alpha,\beta}(z), \quad E_{\alpha,1}^1(z) = E_{\alpha}(z). \tag{7} \]

We also have

\[ \Phi(\alpha, \beta; z) = \Gamma(\beta) E_{1,\beta}^{\alpha}(z), \tag{8} \]

where \(\Phi(\alpha, \beta; z)\) is Kummer’s confluent hypergeometric function defined in Erdélyi, Magnus, Oberhettinger, and Tricomi (1953, p.248, eq.(1)). Prabhakar (1971, p.8, eq.(2.5)) has shown that

\[ \int_0^{\infty} t^{\gamma-1} e^{-st} E_{\beta,\gamma}^\delta(\omega t^\beta)dt = s^{-\gamma}(1 - \omega s^{-\beta})^{-\delta}, \tag{9} \]

where \(\Re(\beta) > 0, \Re(\gamma) > 0, \Re(s) > 0 \) and \(s > |\omega|\).\(\sqrt{\frac{1}{\Re(\beta)}}\).

The Riemann-Liouville fractional integral of order \(\nu\) is defined by Miller and Ross (1993, p.45)

\[ _0D_t^{-\nu}f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} f(u)du, \quad \Re(\nu) > 0. \tag{10} \]
Following Samko, Kilbas, and Marichev (1990, p.37), we define the fractional derivative for $Re(\alpha) > 0$ in the form

$$0_D^\alpha f(t) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dt^n} \int_0^t \frac{f(u)du}{(t-u)^{n+1}}; \quad n = [\alpha] + 1,$$

(11)

where $[\alpha]$ means the integral part of the number $\alpha$. In particular, if $0 < \alpha < 1$,

$$0_D^\alpha f(t) = \frac{d}{dt} \frac{1}{\Gamma(1 - \alpha)} \int_0^t \frac{f(u)du}{(t-u)^\alpha},$$

(12)

and if $\alpha = n \in N = \{1, 2, \ldots\}$, then

$$0_D^\alpha f(t) = D^n f(t), \quad (D = d/dt),$$

(13)

which is the standard derivative of order $n$. From Erdélyi, Magnus, Oberhettinger, and Tricomi (1954a, 1954b, p.182), we have

$$L \{0_D^{-\nu} f(t); s\} = s^{-\nu} \tilde{f}(s),$$

(14)

where $\tilde{f}(s)$ is the Laplace transform of $f(t)$, defined by

$$L \{f(t); s\} = \tilde{f}(s) = \int_0^{\infty} \exp(-st) f(t) dt, \quad Re(s) > 0,$$

(15)

which may be written symbolically, as follows

$$\tilde{f}(s) = L \{f(t); s\} \text{ or } f(t) = L^{-1} \{\tilde{f}(s); t\},$$

(16)

provided that the function $f(t)$ is continuous for $t \geq 0$ and of exponential order as $t \to \infty$. The Laplace transform of the fractional derivative is given by Oldham and Spanier (1974, p.134, Eq. (8.1.3))

$$L \{0_D^\alpha f(t); s\} = s^\alpha \tilde{f}(s) + \sum_{r=1}^{n} s^{r-1} 0_D^{\alpha-r} f(t)|_{t=0}.$$

(17)

In certain boundary-value problems, the following fractional derivative of order $\alpha > 0$ of a causal function $f(t)$ (that is, $f(t) = 0$, for $t < 0$), is introduced by Caputo (1969) in the form

$$D_t^\alpha f(t) = \frac{1}{\Gamma(m - \alpha)} \int_0^t \frac{f^{(m)}(\tau)d\tau}{(t-\tau)^{\alpha-m+1}}; \quad m - 1 < \alpha \leq m, \quad Re(\alpha) > 0,\quad m \in N;,$$

$$= \frac{d^m}{dt^m}, \quad \text{if } \alpha = m,$$

(18)
where $\frac{d^m}{dx^m} f$ is the $m^{th}$ derivative of $f$.

Caputo (1969) has given the Laplace transform of the fractional derivative as

$$L\{D_t^\alpha f(t); s\} = s^\alpha F(s) - \sum_{r=0}^{m-1} s^{\alpha-r-1} f^{(r)}(0^+), \ m - 1 < \alpha \leq m, \ (19)$$

where $F(s)$ is the Laplace transform of $f(t)$.

The above formula is useful in deriving the solution of differ-integral equations of fractional order governing certain physical problems of reaction and diffusion. We also need the Weyl fractional operator defined by

$$-\infty D_x^\mu f(t) = \frac{1}{\Gamma(n - \mu)} \frac{d^n}{dt^n} \int_{-\infty}^{t} \frac{f(u)du}{(t - u)^{\mu-n+1}}, \ (20)$$

where $n = [\mu]$ is an integral part of $\mu > 0$. Its Fourier transform is given by Metzler and Klafter (2000, p.59, A.11, 2004)

$$F\{-\infty D_x^\mu f(x)\} = (ik)^\mu f^*(k), \ (21)$$

where we define the Fourier transform by the integral equation

$$h^*(q) = \int_{-\infty}^{\infty} h(x) exp(irq) dx. \ (22)$$

Following the convention initiated by Compte (1996), we suppress the imaginary unit in Fourier space by adopting a slightly modified form of the result (21) in our investigations (Metzler and Klafter 2000, p. 59, A.12, 2004)

$$F\{-\infty D_x^\mu f(x)\} = -|k|^\mu f^*(k). \ (23)$$

Now we will establish the following results, which provide the inverse Laplace transforms of certain algebraic functions and are directly applicable in the analysis of reaction-diffusion systems that follows.

It will be shown here that

$$(A) \ L^{-1}\left\{\frac{s^{\alpha-1}}{s^\alpha + \alpha s^\beta + b}; t\right\} = \sum_{r=0}^{\infty} (-a)^r t^{(\alpha-\beta)r} E_{\alpha,(\alpha-\beta)r+1}^{\alpha-\beta+1}(-bt^\alpha), \ (24)$$
where \( \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(s) > 0, \left| \frac{as}{s^2 + b} \right| < 1 \), and \( E_{\beta,\gamma}^\delta(x) \) is the generalized Mittag-Leffler function defined in eq. (5).

**Proof.** We have

\[
\frac{s^{\alpha-1}}{s^\alpha + as^\beta + b} = \frac{s^{\alpha-\beta-1}}{(s^{\alpha-\beta} + bs^\beta)(1 + \frac{a}{s^{\alpha-\beta} + bs^\beta})}
\]

\[
= s^{\alpha-\beta-1} \sum_{r=0}^{\infty} \frac{(-a)^r}{(s^{\alpha-\beta} + bs^\beta)^{r+1}}
\]

\[
= \sum_{r=0}^{\infty} \frac{(-a)^r s^{\alpha+(r+1)\beta-1}}{(s^\alpha + b)^{r+1}}.
\]

Taking the inverse Laplace transform of the result (25) and using eq. (9), we obtain the result (24). The term by term inversion is justified by virtue of a theorem given by Doetsch (1956, 22). In a similar manner, the following results can be established

\[(B) \quad L^{-1} \left\{ \frac{s^{\beta-1}}{s^\alpha + as^\beta + b} ; t \right\}
= t^{\alpha-\beta} \sum_{r=0}^{\infty} (-a)^r t^{(\alpha-\beta)r} E_{\alpha,(\alpha-\beta)(r+1)+1}^{r+1}(-bt^\alpha),
\]

where \( \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(s) > 0, \left| \frac{as}{s^2 + b} \right| < 1 \), and \( \alpha > \beta \).

\[(C) \quad L^{-1} \left\{ \frac{1}{s^\alpha + as^\beta + b} ; t \right\}
= t^{\alpha-1} \sum_{r=0}^{\infty} (-a)^r t^{(\alpha-\beta)r} E_{\alpha,\alpha+(\alpha-\beta)r}^{r+1}(-bt^\alpha),
\]

where \( \text{Re}(\alpha) > 0, \text{Re}(\beta) > 0, \text{Re}(s) > 0, \) and \( \left| \frac{as}{s^2 + b} \right| < 1 \).

We now show that

\[(D) \quad L^{-1} \left\{ \frac{s^{2\alpha-1} + as^{\alpha-1}}{s^{2\alpha} + as^\alpha + b} ; t \right\}
= \frac{1}{\sqrt{(a^2 - 4b)}} [(\lambda + a)E_{\alpha}(\lambda t^\alpha) - (\mu + a)E_{\alpha}(\mu t^\alpha)],
\]

6
where $a^2 - 4b > 0$ and $E_{\alpha}(x)$ is the Mittag-Leffler function defined in eq. (3), $Re(\alpha) > 0, Re(s) > 0$, and $\lambda$ and $\mu$ are the real and distinct roots of the quadratic equation, $x^2 + ax + b = 0$, namely $\lambda = \frac{1}{2}(-a + \sqrt{(a^2 - 4b)})$ and $\mu = \frac{1}{2}(-a - \sqrt{(a^2 - 4b)})$.

**Proof.** We have

$$\frac{s^{2\alpha-1} + as^{\alpha-1}}{s^{2\alpha} + as^{\alpha} + b} = \frac{1}{\lambda - \mu} \left[ \frac{s^{2\alpha-1} + as^{\alpha-1}}{s^{\alpha} - \lambda} - \frac{s^{2\alpha-1} + as^{\alpha-1}}{s^{\alpha} - \mu} \right] = \frac{1}{\lambda - \mu} \left[ (\lambda + a)s^{\alpha-1} - (\mu + a)s^{\alpha-1} \right].$$

(29)

Taking the inverse Laplace transform of (29) gives

$$L^{-1} \left\{ \frac{s^{2\alpha-1} + as^{\alpha-1}}{s^{2\alpha} + as^{\alpha} + b} \right\} = \frac{1}{\lambda - \mu} \left[ (\lambda + a)E_{\alpha}(\lambda t^\alpha) - (\mu + a)E_{\alpha}(\mu t^\alpha) \right].$$

This completes the proof of eq. (28).

In a similar manner, it can be shown that

$$(E) \ L^{-1} \left\{ \frac{1}{s^{2\alpha} + as^{\alpha} + b} \right\} = \frac{t^{\alpha-1}}{\lambda - \mu} [E_{\alpha,\alpha}(\lambda t^{\alpha}) - E_{\alpha,\alpha}(\mu t^{\alpha})],$$

(30)

where $Re(\alpha) > 0, Re(s) > 0$, and $\lambda$ and $\mu$ are given along with (28).

## 3 Solution of Fractional Reaction-Diffusion Equation

In this section, it is proposed to derive the solution of the fractional-diffusion system connected with nonlinear waves governed by the eq. (31). This system is a generalized form of the reaction-diffusion equation recently studied by Manne, Hurd, and Kenkre (2000). The result is given in the form of the following theorem.

**Theorem.** Consider the fractional reaction-diffusion equation

$$0D_t^\alpha N(x, t) + a_0 0D_t^\beta N(x, t) = \nu_2 \Delta D_x^\gamma N(x, t) + \xi^2 N(x, t) + \phi(x, t),$$

(31)
with initial conditions

\[ N(x, 0) = f(x), \text{ for } (x \in \mathbb{R}), \]  

(32)

where \( \nu^2 \) is a diffusion coefficient, \( \varphi \) is a constant which describes the nonlinearity in the system, and \( \varphi(x, t) \) is a nonlinear function for reaction kinetics, then there holds the following formula for the solution of (31)

\[
N(x, t) = \sum_{r=0}^{\infty} \frac{(-a)^r}{2\pi} \int_{-\infty}^{\infty} t^{(\alpha-\beta)r} f^*(k)e^{\text{exp}(-kx)} \times \\
\times \left[ E^{r+1}_{\alpha, (\alpha-\beta)r+1}(-bt^\alpha) + t^{\alpha-\beta} E^{r+1}_{\alpha, (\alpha-\beta)(r+1)+1}(-bt^\alpha) \right] dk \\
+ \sum_{r=0}^{\infty} \frac{(-a)^r}{2\pi} \int_0^t \xi^{(\alpha-\beta)r-1} \int_{-\infty}^{\infty} \varphi(k, t-\xi)e^{\text{exp}(-ikx)} \times \\
\times E^{r+1}_{\alpha, \alpha+(\alpha-\beta)r}(-b\xi^\alpha) dkd\xi,
\]

(33)

where \( \alpha > \beta \) and \( E^\delta_{\beta, \gamma}(.) \) is the generalized Mittag-Leffler function, defined in (5) and \( b = \nu^2|k|^\gamma - \xi^2 \).

**Proof.** Applying the Laplace transform with respect to the time variable \( t \) and using the boundary conditions, we find that

\[
s^\alpha \tilde{N}(x, s) - s^{\alpha-1} f(x) + as^\beta \tilde{N}(x, s) - as^{\beta-1} f(x) \\
= \nu^2 \int_{-\infty}^{\infty} D^*_{x} \tilde{N}(x, s) + \xi^2 \tilde{N}(x, s) + \tilde{f}(x, s).
\]

(34)

If we apply the Fourier transform with respect to the space variable \( x \), it yields

\[
s^\alpha \tilde{N}^*(k, s) - s^{\alpha-1} f^*(k) + as^\beta \tilde{N}^*(k, s) - as^{\beta-1} f^*(k) \\
= -\nu^2|k|^\gamma \tilde{N}^*(k, s) + \xi^2 \tilde{N}^*(k, s) + \tilde{f}^*(k, s).
\]

(35)

Solving for \( \tilde{N}^*(k, s) \), it gives

\[
\tilde{N}^*(k, s) = \frac{(s^{\alpha-1} + as^{\beta-1})f^*(k) + \tilde{f}^*(k, s)}{s^\alpha + as^\beta + b},
\]

(36)

where \( b = \nu^2|k|^\gamma - \xi^2 \). To invert eq. (36), it is convenient to first invert the Laplace transform and then the Fourier transform. Inverting the Laplace
transform with the help of the results (28) and (30), yields

\[ N^*(k, t) = \sum_{r=0}^{\infty} (-a)^r t^{(\alpha-\beta)r} f^*(k) [E_{\alpha,(\alpha-\beta)r+1}^{r+1}(-bt^\alpha) + t^{\alpha-\beta} \times \\
\times E_{\alpha,(\alpha-\beta)r+1+1}^{r+1}(-bt^\alpha)] \\
+ \sum_{r=0}^{\infty} (-a)^r \int_0^t \varphi^*(k, t-\xi) \xi^{\alpha+(\alpha-\beta)r-1} \times \\
\times E_{\alpha,(\alpha-\beta)r+\alpha}^{r+1}(-b\xi^\alpha) d\xi. \quad (37) \]

Finally, the inverse Fourier transform gives the desired solution in the form

\[ N(x, t) = \sum_{r=0}^{\infty} \frac{(-a)^r}{2\pi} \int_{-\infty}^{\infty} t^{(\alpha-\beta)r} f^*(k) [E_{\alpha,(\alpha-\beta)r+1}^{r+1}(-bt^\alpha) + t^{\alpha-\beta} \times \\
\times E_{\alpha,(\alpha-\beta)r+1+1}^{r+1}(-bt^\alpha)] \exp(-ikx) dk \\
+ \sum_{r=0}^{\infty} \frac{(-a)^r}{2\pi} \int_0^t \xi^{\alpha+(\alpha-\beta)r-1} \int_{-\infty}^{\infty} \exp(-ikx) \varphi^*(k, t-\xi) \times \\
\times E_{\alpha,\alpha+(\alpha-\beta)r}^{r+1}(-b\xi^\alpha) dkd\xi. \quad (38) \]

This completes the proof of the theorem.

4 Special cases

When \( f(x) = \delta(x) \), where \( \delta(x) \) is the Dirac delta function, the theorem reduces to the following

**Corollary 1.** Consider the fractional reaction-diffusion system

\[ _0D_t^\alpha N(x, t) + a_0D_t^\beta N(x, t) = \nu^2 -\infty D_x^2 N(x, t) + \xi^2 N(x, t) + \varphi(x, t), \quad (39) \]

subject to the initial conditions

\[ N(x, 0) = \delta(x) \text{ for } 0 \leq a \leq 1 \text{ and } 0 \leq \beta \leq 1, \quad (40) \]

where \( \delta(x) \) is the Dirac delta function. Here \( \xi \) is a constant that describes the nonlinearity in the system, and \( \varphi(x, t) \) is a nonlinear function which belongs
to the reaction kinetics. Then there exists the following eq. for the solution of (39), subject to the initial conditions (40)

\[ N(x, t) = \sum_{r=0}^{\infty} \frac{(-a)^r}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \left[ t^{(\alpha-\beta)r} E_{\alpha,\alpha}^{r+1}(-bt^\alpha) + t^{(\alpha-\beta)(r+1)} \times \right] \]

\[ \times E_{\alpha(\alpha-\beta)(r+1)+1}(-bt^\alpha) \] \]

\[ \times \sum_{r=0}^{\infty} \frac{(-a)^r}{2\pi} \int_{0}^{t} \xi^{\alpha+(\alpha-\beta)r-1} \int_{-\infty}^{\infty} \phi^*(k, t-\xi) \]

\[ \times \exp(-ikx) E_{\alpha,\alpha}^{r+1}(-b\xi^\alpha) dk d\xi, \]

(41)

where \( b = \nu^2|k|^\gamma - \xi^2 \).

Now if we set \( f(x) = \delta(x), \gamma = 2, \alpha \) is replaced by \( 2\alpha \) and \( \beta \) by \( \alpha \), the following result is obtained.

**Corollary 2.** Consider the following reaction-diffusion system

\[ \frac{\partial^{2\alpha} N(x, t)}{\partial t^{2\alpha}} + a \frac{\partial^\alpha N(x, t)}{\partial t^\alpha} = \nu^2 \frac{\partial^2 N(x, t)}{\partial x^2} + \xi^2 N(x, t) + \varphi(x, t) \]

with the initial conditions

\[ N(x, 0) = \delta(x), \quad N_t(x, 0) = 0, \quad 0 \leq \alpha \leq 1, \]

(42)

\( \varphi(x, t) \) is a nonlinear function belonging to the reaction kinetics. Then for the solution of (39) subject to the initial conditions (40), there holds the formula

\[ N(x, t) = \frac{1}{2\pi \sqrt{(a^2 - 4b)}} \int_{-\infty}^{\infty} \exp(-ikx) \times \]

\[ \times \{ (\lambda + a) E_{\alpha} (\lambda t^\alpha) - (\mu + a) E_{\alpha} (\mu t^\alpha) \} dk \]

\[ + \frac{1}{2\pi} \int_{0}^{t} \xi^{\alpha-1} \int_{-\infty}^{\infty} \exp(-ikx) \phi^*(k, t-\xi) \times \]

\[ \times [ E_{\alpha,\alpha} (\mu \xi^\alpha) - E_{\alpha,\alpha} (\mu \xi^\alpha) ] dk d\xi, \]

(43)

where \( \lambda \) and \( \mu \) are the real and distinct roots of the quadratic equation

\[ y^2 + ay + b = 0, \]

(44)

given by

\[ \lambda = \frac{1}{2} \left( -a + \sqrt{(a^2 - 4b)} \right) \quad \text{and} \quad \mu = \frac{1}{2} \left( -a - \sqrt{(a^2 - 4b)} \right), \]

(45)
where $b^2 = \nu^2k^2 - \xi^2$.

**Proof.** In order to prove (43), we replace $\alpha$ by $2\alpha$ and $\beta$ by $\alpha$, then eq. (36) becomes

$$N^*(k, s) = \frac{s^{2\alpha-1} + as^{\alpha-1} + \tilde{\varphi}(k, s)}{s^{2\alpha} + as^\alpha + b}.$$  \hfill (46)

Taking the inverse Laplace transform and using the results (26) and (30), yields

$$N^*(k, t) = \frac{1}{\lambda - \mu} \left[ (\lambda + a)E_\alpha(\lambda t^\alpha) - (\mu + a)E_\alpha(\mu t^\alpha) \right]$$

$$+ \int_0^t \varphi^*(k, t - \xi)\xi^{\alpha-1}[E_{\alpha,\alpha}(\lambda\xi^\alpha) - E_{\alpha,\alpha}(\mu\xi^\alpha)]d\xi, \quad \lambda \neq \mu,$$

where $\lambda$ and $\mu$ are given in (45). The application of the inverse Fourier transform to the above equation gives the desired result (43).

Next, if we set $\varphi(x, t) = 0$, $\gamma = 2$, replace $\alpha$ by $2\alpha$, and $\beta$ by $\alpha$ in (31), we then obtain the following result, which includes many known results on the fractional telegraph equations including the one recently given by Orsingher and Beghin (2004).

**Corollary 3.** Consider the following reaction-diffusion system

$$\frac{\partial^{2\alpha} N(x, t)}{\partial t^{2\alpha}} + a\frac{\partial^\alpha N(x, t)}{\partial t^\alpha} = \nu^2\frac{\partial^2 N(x, t)}{\partial x^2} + \xi^2 N(x, t),$$

with the initial conditions

$$N(x, 0) = \delta(x), \quad N_t(x, 0) = 0, \quad 0 \leq \alpha \leq 1.$$  \hfill (49)

Then for the solution of (48), subject to the initial conditions (49), there holds the formula

$$N(x, t) = \frac{1}{2\pi \sqrt{(a^2 - 4b)}} \times$$

$$\times \left[ \int_{-\infty}^{+\infty} \exp(-ikx) \{(\lambda + a)E_\alpha(\lambda t^\alpha) - (\mu + a)E_\alpha(\mu t^\alpha)\} \, dk \right],$$

where $\lambda$ and $\mu$ are defined in (45), $b = \nu^2k^2 - \xi^2$ and $E_\alpha(x)$ is the Mittag-Leffler function defined by (3). If we set $\xi^2 = 0$, then corollary 3 reduces to
the result, which states that the reaction-diffusion system
\[ \frac{\partial^{2\alpha} N(x,t)}{\partial t^{2\alpha}} + a \frac{\partial^\alpha N(x,t)}{\partial t^\alpha} = \nu^2 \frac{\partial^2 N(x,t)}{\partial x^2} \]  
(51)
with the initial conditions
\[ N(x,0) = \delta(x), \quad N_t(x,0) = 0, \quad 0 \leq \alpha \leq 1, \]  
(52)
has the solution, given by
\[ N(x,t) = \frac{1}{2\pi \sqrt{(a^2 - 4b)}} \times \left[ \int_{-\infty}^{\infty} \exp(-ikx) \left\{ (\lambda + a)E_\alpha(\lambda t^\alpha) - (\mu + a)E_\alpha(\mu t^\alpha) \right\} dk \right], \]  
(53)
where \( \lambda \) and \( \mu \) are defined in (45), \( b = \nu^2k^2 \) and \( E_\alpha(x) \) is the Mittag-Leffler function defined by (3). Eq. (53) can be rewritten in the form
\[ N(x,t) = \frac{1}{4\pi} \left[ \int_{-\infty}^{\infty} \exp(-ikx) \left\{ (1 + \frac{a}{\sqrt{(a^2 - 4\nu^2k^2)}})E_\alpha(\lambda t^\alpha) + (1 - \frac{a}{\sqrt{(a^2 - 4\nu^2k^2)}})E_\alpha(\mu t^\alpha) \right\} dk \right], \]  
(54)
where \( \lambda \) and \( \mu \) are defined in (45) and \( E_\alpha(x) \) is the Mittag-Leffler function, defined by (3). The equation (54) represents the solution of the time-fractional telegraph equation (51), subject to the initial conditions (52), recently solved by Orsingher and Beghin (2004). It may be remarked here that the solution as given by Orsingher and Beghin (2004) is in terms of the Fourier transform of the solution in the form given below. The Fourier transform of the solution of the equations (51) and (52) can be expressed in the form
\[ N^*(x,t) = \frac{1}{2} \left\{ (1 + \frac{a}{\sqrt{(a^2 - 4\nu^2k^2)}})E_\alpha(\lambda t^\alpha) + (1 - \frac{a}{\sqrt{(a^2 - 4\nu^2k^2)}})E_\alpha(\mu t^\alpha) \right\}, \]  
(55)
where \( \lambda \) and \( \mu \) are defined in (45) and \( E_\alpha(x) \) is the Mittag-Leffler function defined by (3). Finally, it is interesting to observe that the solution of various fractional-reaction and fractional diffusion and fractional telegraph equations scattered in the literature can be derived as special cases of the theorem established in this article.
5 Conclusions

There is a host of reaction-diffusion equations such as eqs. (1) and (2) that allow the formation of wave fronts which maintain their shape despite the diffusive element of evolution contrary to linear expectation. Such generation of waves plays a particularly important role in spatio-temporal processes in physical systems, including astrophysical fusion plasmas (Kulsrud, 2005; Wilhelmsson and Lazarro, 2001; Gilding and Kersner 2004). Much work has been done for the numerical treatment of such equations.

The motivation for the research reported in this paper is the derivation of analytic closed-form solutions of fractional reaction-diffusion equations (31), (39), (42), and (48) that give rise to nonlinear waves in a respective physical medium. For this purpose the paper summarizes specific techniques for Laplace, Fourier, and Mellin transforms, results for Mittag-Leffler functions, as well as the applications of Riemann-Liouville, Weyl, and Caputo fractional calculus for tackling fractional reaction-diffusion equations. Closed-form solutions of the equations are given in terms of Fox’s function and their behavior for small and large values of the respective parameter is derived.

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