Fluctuation theorem and natural time analysis

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Upon employing a natural time window of fixed length sliding through a time series, an explicit interrelation between the variability $\beta$ of the variance $\kappa_1 = \langle \chi^2 \rangle - \langle \chi \rangle^2$ of natural time $\chi$ and events’ correlations is obtained. In addition, we investigate the application of the fluctuation theorem, which is a general result for systems far from equilibrium, to the variability $\beta$. We consider for example, major earthquakes that are nonequilibrium critical phenomena. We find that four (out of five) mainshocks in California during 1979-2003 were preceded by $\beta$ minima lower than the relative thresholds deduced from the fluctuation theorem, thus signalling an impending major event.

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Entropy production is a measure of the irreversibility of a thermodynamic process: the difficulty, even impossibility, of reversing the observed often macroscopic behavior of a system that exchanges heat or matter with a complex environment (e.g., see Ref. [1] and references therein). The breakage of time reversal symmetry associated with thermodynamic irreversibility has focused enormous discussion for more than a century. Despite of such concerns, however, the concept of entropy generation in the thermodynamics of large systems has been applied widely. From microscopic point of view, efforts towards understanding the nature of the entropy and its production - mainly focused on the one way character of the second law- have been attempted. They modelled the microscopic evolution of a system and its environment in the frame of stochastic dynamics [2] and stochastic thermodynamics [3-5], but interpretations based on deterministic dynamics (e.g., see Ref. [6]) were also forwarded.

An intense interest towards the latter interpretations has been renewed when Evans, Cohen and Morris in 1993 considered the fluctuations of the entropy production rate in a shearing fluid, and proposed the so-called fluctuation relation or the first fluctuation theorem [7]. This is considered [8] to represent a general result concerning systems arbitrarily far from equilibrium. The proof of the fluctuation [8] and related theorems [10] shows how irreversible macroscopic behavior arises from time reversible microscopic equations of motion. The two theoretical results that illustrate this clearly are the second law inequality [11] and the very recent mechanical proof [12] of Clausius’ inequality without the prior assumption of the second “law” of thermodynamics. These two results have been obtained without treating the nonequilibrium entropy, but used instead a quantity termed dissipation function first defined [13] in 2000. On the basis of this function, being a path function and not a state function, the relaxation of a system to equilibrium, which is inherently a nonequilibrium process, can be quantified [14].

It has been emphasized in Ref. [6] that, unlike linear irreversible thermodynamics, the fluctuation and related theorems are exact for systems of arbitrary size as well as for systems arbitrarily near to, or far from equilibrium, as mentioned. This is why we shall employ here the fluctuation theorem for the purpose of the present study.

This theorem [7, 9, 15-19] gives a general formula for the probability ratio that in a thermostated dissipative system, the time average entropy production $\overline{\Sigma}_t$ takes a value $A$ to minus the value $-A$,

$$Pr(\overline{\Sigma}_t/k_B = A)/Pr(\overline{\Sigma}_t/k_B = -A) = \exp[At] \tag{1}$$

from which it is obvious that as the averaging time or system size increases, it becomes exponentially likely that the entropy production will be positive. The theorem was initially proposed [7] for nonequilibrium steady states that are thermostated in such a way that the total energy of the system is constant. Subsequently, it was shown [15-19] that this theorem can be proved for sufficiently chaotic, iso-energetic nonequilibrium systems using the Sinai-Ruelle-Bowen measure, as well as for purely Hamiltonian systems with or without applied dissipative fields [20] and for a wide class of stochastic nonequilibrium systems [21, 22].

It is one of the two basic aims of this paper to investi-gate for the first time the application of the fluctuation theorem to the case of earthquakes which may be considered (e.g. [23, 24]) as nonequilibrium critical phenomena (the mainshock being the new phase). They exhibit complex correlations in time, space and magnitude (the mainshock being the new phase). They exhibit complex correlations in time, space and magnitude which have been recently studied by several workers (e.g., see Refs. [25-29]). In particular, the present investigation will be made by applying the fluctuation theorem to the order parameter fluctuations that result from the analysis of the time series in a new time domain termed [30] as natural time $\chi$. This is so, because natural time analysis allows us to identify [31] when a complex system approaches a critical point (for a review see Ref. [32]) and in addition enables the introduction of an order parameter for seismicity. The present study has been motivated by the following two findings related to the variability $\beta$ (defined below) of the order parameter of seismicity [33]:

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First, it captures the events’ correlations, as shown here (see Appendix), which constitutes the other basic aim of this paper. Second, the quantity \( \beta \) exhibits characteristic minima \( [33] \) before the occurrence of major events.

In a time series comprising \( N \) earthquakes, the natural time \( \chi_k = k/N \) serves as an index for the occurrence of the \( k \)-th earthquake. In natural time analysis the pair \( (\chi_k, Q_k) \) is studied, where \( Q_k \) is the energy released during the \( k \)-th earthquake of magnitude \( M_k \). One may alternatively study the pair \( (\chi_k, p_k) \), where \( p_k = Q_k/\sum_{n=1}^{N} Q_n \) is the normalized energy released during the \( k \)-th earthquake, and \( Q_k \) -and hence \( p_k \) - is estimated through the relation \( [35] \ Q_k \propto 10^{1.5M_k}. \) The variance \( \kappa_1 = \langle \chi^2 \rangle - \langle \chi \rangle^2 \) of \( \chi \) weighted for \( p_k \), is given by \( [30, 33, 36, 37] \)

\[
\kappa_1 = \sum_{k=1}^{N} p_k (\chi_k)^2 - \left( \sum_{k=1}^{N} p_k \chi_k \right)^2
\]

This quantity, as shown in Ref. \([33]\), can be also considered as an order parameter for seismicity.

The fluctuations of \( \kappa_1 \), are studied by applying the following procedure \([32]\). Taking an excerpt of a seismic catalog comprising \( W \) (\( \geq 100 \)) successive events, we start from the first EQ and calculate the first 35 \( \kappa_1 \) values for 6 to 40 consecutive EQs. Then we proceed to the second EQ, and calculate again 35 values of \( \kappa_1 \) from the 7-th to the 41-st event. Thus, scanning event by event the whole excerpt of \( W \) earthquakes, we calculate the average value \( \mu(\kappa_1) \) and the standard deviation \( \sigma(\kappa_1) \) of the \( \kappa_1 \) values. The quantity

\[
\beta \equiv \sigma(\kappa_1)/\mu(\kappa_1)
\]

is defined \([33]\) as the variability \( \beta \) of \( \kappa_1 \) for this excerpt of length \( W \). In some occasions, as in the present case, it is of prominent importance to know what happens to the \( \beta \) value until just before the occurrence of each EQ, \( e_i \), in the seismic catalog. We then calculate first the \( \kappa_1 \) values using the previous \( l=6 \) to 40 consecutive EQs. These 35 \( \kappa_1 \) values are associated with the EQ \( e_i \), but we clarify that EQ \( e_i \) has not been employed for their calculation. The \( \beta \) value -corresponding to the EQ \( e_i \)- for a natural time window length \( W \) is computed using all the \( (35 \times W) \) \( \kappa_1 \) values associated with the EQs \( e_{i-W+1} \) to \( e_i \). The resulting value is denoted by \( \beta_{W} \), where the subscript \( W \) shows the natural time window length, and the corresponding minimum is designated by \( \beta_{W, \text{min}} \).

It is shown that the quantity \( \beta \) when using \( l \) consecutive events is interrelated with the event’s correlations through

\[
\beta = \sqrt{- \sum_{\text{all pairs}} \left[ (\frac{m}{T} - \langle \chi \rangle_M)^2 - (\frac{m}{T} - \langle \chi \rangle_M)^2 \right]^2 \frac{\text{Cov}(p_j, p_m)}{\sum_{\text{all pairs}} \left[ (\frac{m}{T} - \langle \chi \rangle_M)^2 \right]^2 \text{Cov}(p_j, p_m)}},
\]

where \( \langle \chi \rangle_M \) and \( \kappa_{1,M} \) correspond to the average value of \( \chi \) and \( \kappa_1 \), respectively, obtained when substituting for \( p_k \) the average -within an excerpt of \( W \) events- values \( \mu_k \) of \( p_k \); the symbol \( \text{Cov}(p_j, p_m) \) stands for covariance, i.e., the average value of \( (p_j - \mu_j)(p_m - \mu_m) \) within the excerpt of \( W \) events. The details of the derivation of Eq.\([4]\) are given in the Appendix.

The selection of the \( W \) value used for the purpose of our study is of crucial importance. It is taken equal to the number of the events that would occur in a few months, or so, in view of the following: Low frequency (\( \leq 1 \text{ Hz} \)) electric signals, termed Seismic Electric Signals (SES), appear before earthquakes \([33, 40]\). They are emitted from the future focal region \( [41] \) (see also Ref. \([42]\)) when in the focal region the stress reaches a critical value \( \sigma_{cr} \), and then a cooperative orientation of the electric dipoles occurs. This leads to the emission of a transient electric signal that constitutes an SES. Several such signals within a short time are termed SES activity \([30, 37, 42, 43]\). For example, the three lower channels in Fig.\([1]\)b show three SES activities that preceded major earthquakes in western, southwestern and southern Greece, respectively, as depicted in the map of Fig.\([1]\)a. Furthermore, for the sake of comparison, the upper channel in Fig.\([1]\)b shows a very recent SES activity initiated on 8 January 2013 at a station labelled LAM in Fig.\([1]\)a in central Greece. The following important fact has just been identified \([44]\): At the initiation of an SES activity, which usually occurs a few months (with an upper limit of around 5 months) before a major EQ, a clearly detectable change in seismicity appears, manifested by a minimum \( \beta_{W, \text{min}} \) in the fluctuations of the order parameter of seismicity. Hence, in the case that geoelectrical data are lacking, once we identify the date of \( \beta_{W, \text{min}} \) (by analyzing solely seismic data) this reveals also the date of an SES activity that would have been recorded.

Along these lines, Table \([1]\) shows the dates of the minima \( \beta_{W, \text{min}} \) of seismicity before major mainshocks in California during the 25 year period 1 January 1979 to 1 January 2004. We used the United States Geological Survey Northern California Seismic Network catalog available from the Northern California Earthquake Data Center, at the http address: \( \text{www.ncedc.org/ncedc/catalog-search.html} \), hereafter called NCEDC. The seismic moment \( M_0 \), which
FIG. 1: (color online) (a) Major earthquakes in Greece on 8 June 2008 (green, magnitude $M_w = 6.4$), 14 February 2008 (red, $M_w = 6.9$ and 6.4) and 8 January 2006 (blue, $M_w = 6.7$). (b) Their preceding SES activities recorded at Pirgos (PIR) measuring station located in western Greece are shown (with the corresponding color) in the lower three channels. Furthermore, an SES activity initiated recently on 8 January 2013 at a station in central Greece labelled LAM in (a) is depicted in the upper channel of (b).

TABLE I: The minimum DFA exponent $\alpha_{min}$ along with the values of the minima observed for the variability $\beta$ together with the dates of their observation in parentheses before all major EQs in California with $M \geq 7.0$ within $N_{31.7}W_{127.5}$ during the period 1979-2003. The $M_{6.9}$ Northridge earthquake is also added in italics. The lead time $\Delta t$ for each case, estimated from the difference in the dates between the EQ occurrence and the appearance of $\beta_{300,min}$ is shown in the last column. The values for $\beta_{300,min}$ and $\alpha_{min}$ are taken from Ref. [34].

| EQ Date | EQ Name | $M$ | $\beta_{300,min}$ | $\beta_{200,min}$ | $\alpha_{min}$ | $\Delta t$ (months) |
|---------|---------|-----|-------------------|-------------------|---------------|---------------------|
| 1980-11-08 | Eureka | 7.2 | 0.444 | 0.432 | 0.445 | $\approx 3$ |
| 1989-10-18 | Loma Prieta | 7.0 | - | - | - | | |
| 1992-06-28 | Landers | 7.4 | 0.378 | 0.377 | 0.383 | $\leq 5$ |
| 1994-01-17 | Northridge | 6.9 | 0.459 | 0.324 | 0.431 | $\approx 2$ |
| 1994-09-01 | Mendocino | 7.0 | 0.472 | 0.474 | 0.458 | $\approx 1$ |
| 1999-10-16 | Hector Mine | 7.0 | 0.444 | 0.432 | 0.422 | $\approx 5$ |

| EQ Date | EQ Name | $M$ | $\beta_{300,min}$ | $\beta_{200,min}$ | $\alpha_{min}$ | $\Delta t$ (months) |
|---------|---------|-----|-------------------|-------------------|---------------|---------------------|
| 1999-05-14 | N34.60°W116.34° | 7.0 | 0.444 | 0.432 | 0.422 | $\approx 5$ |

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is proportional to the energy release during an earthquake and hence to the quantity $Q_k$ used in natural time analysis, is calculated [32] from the relation $\log_{10}(M_0) = 1.5M + \text{const}$, where the earthquake magnitudes reported in this catalog are labelled with $M$. The earthquakes with $M \geq 2.5$ reported by NCEDC, within the area $N_{31.7}W_{127.5}$ have been considered. We have on average $\sim 10^2$ EQs per month since 31832 earthquakes occurred for the 25 year period from 1 January 1979 to 1 January 2004. Thus, we adopted natural time window lengths $W = 200$ and $W = 300$.

The results of this analysis are depicted in Fig. 2(a),(b) where we plot the variability $\beta$ (in red for $W = 200$ and in blue for $W = 300$) versus the conventional time for the periods (a) 1 January 1979 to 1 January 1990 and (b) 1 January 1990 to 1 January 2004. An inspection of
these results lead to the \( \beta_{W,\text{min}} \) values inserted in Table I. In five out of the six mainshocks we find values of \( \beta_{300,\text{min}} \) and \( \beta_{200,\text{min}} \) that appear 1 to 5 months before mainshocks. In these five cases \( \beta_{200,\text{min}} \) varies between 0.324 to 0.474 and \( \beta_{300,\text{min}} \) between 0.378 and 0.472. We note that the key criterion to distinguish the true precursory \( \beta_{W,\text{min}} \) from the non precursory ones is the following [33]: The minimum should be followed (before the occurrence of the mainshock) by a period during which the exponent \( \alpha \) of the Detrended Fluctuation Analysis (DFA) [46] -calculated for a length \( W=300 \) events in the magnitude time series- reaches a minimum \( \alpha_{\text{min}} \) slightly smaller than 0.5 (thus, indicating anticorrelated behavior, but close to random) and then \( \beta_{200} > \beta_{300} \). This inequality means that when the system approaches closer to the critical point -which is the case when considering \( W=200 \) events compared to \( W=300 \) events- the fluctuations of the order parameter become more intense.

We now proceed to the investigation of Eq. [1] in the case of seismicity and analyze the statistical distribution of the experimentally determined \( \beta_W \) for \( W=200 \) or \( 300 \), which is clearly path depended. The quantity \( \beta_W \) can be considered as an entropic measure (see Appendix), but its sign is by definition always positive. Thus, in order to apply Eq. [1], we need to define a threshold value \( \beta_{W,0} \) above which the entropy production may be considered positive whereas when below negative. For this reason, we employ the relation

\[
\frac{Pr(\beta_W < \beta_{W,0})}{Pr(\beta_W > \beta_{W,0})} = \exp\left[\tau'(\beta_W - \beta_{W,0})\right],
\]

which results from Eq. [1] when considering \( A = \beta_W - \beta_{W,0} \) and experimentally determine \( Pr(\beta_W < \beta_{W,0}) \) by using bins of width \( \Delta \beta_W = 0.01 \). Figure 3 depicts the natural logarithm of the left hand side of Eq. [4] as a function of \( (\beta_W - \beta_{W,0}) \) for \( W=200 \) and \( W=300 \). In each case, the threshold \( \beta_{W,0} \) is the one that maximizes the linear correlation coefficient (Pearson’s) \( r \), thus pointing to optimal linearity. We find the threshold values of \( \beta_{200,0} = 0.45 \) and \( \beta_{300,0} = 0.46 \). Moreover, the relative ‘time-scale’ \( \tau' \), which corresponds to the slope of
Similarly for the second order moments of $p$ of heavy tails in $Q$ in Table I) led to (1994, the other four mainshocks (including the strongest in Table I) to be considered as point probabilities. We can then define as usual (30, 47) the moments of the natural time $\chi_j = j/l$ as $\langle \chi^q \rangle = \sum_{j=1}^{l} (j/l)^q p_j(k_0)$ and hence $\kappa_1(k_0) = \sum_{j=1}^{l} \left( \frac{j}{l} \right)^2 p_j(k_0) - \left[ \sum_{j=1}^{l} j p_j(k_0) \right]^2$. (A.4)

Note that $\kappa_1$ is a non-linear functional of $\{p_j\}$.

1. The mean value $\mu \equiv \mathcal{E}(\kappa_1)$ of $\kappa_1$

In a window of length $l$ starting at $k = k_0$, the quantities

$$p_j(k_0) = \frac{Q_{k_0+j-1}}{\sum_{m=1}^{l} Q_{k_0+m-1}}, \quad j = 1, 2, \ldots, l$$

(A.1)

representing the normalized energy are obtained, which satisfy the necessary conditions

$$p_j(k_0) > 0,$$

(A.2)

$$\sum_{j=1}^{l} p_j(k_0) = 1$$

(A.3)

to be considered as point probabilities. We can then define as usual (30, 47) the moments of the natural time $\chi_j = j/l$ as $\langle \chi^q \rangle = \sum_{j=1}^{l} (j/l)^q p_j(k_0)$ and hence $\kappa_1(k_0) = \sum_{j=1}^{l} \left( \frac{j}{l} \right)^2 p_j(k_0) - \left[ \sum_{j=1}^{l} j p_j(k_0) \right]^2$. (A.4)

Now that $\kappa_1$ is a non-linear functional of $\{p_j\}$.

Let us consider the average value $\mu_j$ of $p_j$ obtained when the (natural time) window of length $l$ slides through a time series of $Q_k > 0, k = 1, 2, \ldots, W$. Once these quantities have been evaluated, the variability $\beta$ of $\kappa_1$ can then be estimated by $\beta \equiv \sigma/\mu$.

$$\mu_j = \mathcal{E}(p_j) = \frac{1}{W-l+1} \sum_{k_0=1}^{W-l+1} p_j(k_0) = \frac{1}{W-l+1} \sum_{k_0=1}^{W-l+1} \frac{Q_{k_0+j-1}}{\sum_{m=1}^{l} Q_{k_0+m-1}}.$$  

(A.5)

It is obvious that the definition of Eq. (A.5) is consistent with Eq. (A.3), thus we have

$$\sum_{j=1}^{l} \mu_j = 1. \quad (A.6)$$

Similarly for the second order moments of $p_j$, one can estimate (48) the variance of $p_j$ by

$$\text{Var}(p_j) = \mathcal{E}\left[ (p_j - \mu_j)^2 \right] = \frac{1}{W-l+1} \sum_{k_0=1}^{W-l+1} \left( \frac{Q_{k_0+j-1}}{\sum_{m=1}^{l} Q_{k_0+m-1}} - \mu_j \right)^2$$

(A.7)

as well as the covariance

$$\text{Cov}(p_j, p_i) = \mathcal{E}\left[ (p_j - \mu_j) (p_i - \mu_i) \right] = \frac{1}{W-l+1} \sum_{k_0=1}^{W-l+1} \left( \frac{Q_{k_0+j-1}}{\sum_{m=1}^{l} Q_{k_0+m-1}} - \mu_j \right) \left( \frac{Q_{k_0+i-1}}{\sum_{m=1}^{l} Q_{k_0+m-1}} - \mu_i \right).$$

(A.8)

In view of Eqs. (A.2) and (A.3), the quantities $\mu_j, \text{Var}(p_j)$ and $\text{Cov}(p_j, p_i)$ are always finite irrespective of the existence of heavy tails in $Q$ which is for example the case of seismicity. Moreover, for the purpose of our calculations the
relation between the variance of \( p_j \), \( \text{Var}(p_j) \), and the covariance of \( p_j \) and \( p_m \), \( \text{Cov}(p_j, p_m) \), is important. Equations (A.3) and (A.6) lead to

\[
p_j - \mu_j = \sum_{m \neq j} (\mu_m - p_m),
\]

(A.9)

which when multiplied by \((p_j - \mu_j)\) and averaged (cf. \( \hat{E} \equiv \frac{1}{W-l+1} \sum_{k_0=1}^{W-l+1} \)) results in

\[
\text{Var}(p_j) = - \sum_{m \neq j} \text{Cov}(p_j, p_m).
\]

(A.10)

We now turn to the evaluation of the mean value \( \mu \) of \( \kappa_1 \) obtained when the (natural time) window of length \( l \) slides through a time series of \( Q_k > 0, k = 1, 2, \ldots W \),

\[
\mu \equiv \mathcal{E}(\kappa_1) = \frac{1}{W-l+1} \sum_{k_0=1}^{W-l+1} \kappa_1(k_0),
\]

(A.11)

by studying its difference from the one that corresponds to the time series of the averages \( \mathcal{M} = \{ \mu_k \} \) which is labelled \( \kappa_{1,\mathcal{M}} \),

\[
\kappa_{1,\mathcal{M}} = \sum_{j=1}^{I} \left( \frac{j}{l} \right)^2 \mu_j - \left[ \sum_{j=1}^{I} \frac{j}{l} \mu_j \right]^2.
\]

(A.12)

Hence,

\[
\mu - \kappa_{1,\mathcal{M}} = \frac{1}{W-l+1} \sum_{k_0=1}^{W-l+1} \left\{ \left( \frac{l}{m} \right)^2 \kappa_1(k_0) - \mu_m \right\}^2 \sum_{m=1}^{I} \frac{m}{l} \mu_m \}
\]

(A.13)

In view of the definition of \( \mu_m \), the first term in square brackets in the right hand side of Eq.(A.13) vanishes, whereas the latter two terms reduce to the opposite of the variance of

\[
\langle \chi \rangle_{\mathcal{M}} = \sum_{m=1}^{I} \frac{m}{l} \mu_m,
\]

(A.14)

leading to

\[
\mu - \kappa_{1,\mathcal{M}} = - \frac{1}{W-l+1} \sum_{k_0=1}^{W-l+1} \left\{ \sum_{m=1}^{I} \frac{m}{l} \mu_m \right\}^2.
\]

(A.15)

Expanding the term within the curly brackets and interchanging the summations, we get

\[
\kappa_{1,\mathcal{M}} - \mu = \sum_{m=1}^{I} \frac{m^2}{l^2} \text{Var}(p_m) + 2 \sum_{j=1}^{l-1} \sum_{m=j+1}^{I} \frac{jm}{l^2} \text{Cov}(p_j, p_m).
\]

(A.16)

which, upon using Eq.(A.10), leads to

\[
\mu - \kappa_{1,\mathcal{M}} = \sum_{m=1}^{I} \sum_{j=m+1}^{l} \frac{(j-m)^2}{l^2} \text{Cov}(p_j, p_m) = \frac{1}{2} \sum_{j=1}^{l} \sum_{m=1}^{I} \frac{(j-m)^2}{l^2} \text{Cov}(p_j, p_m).
\]

(A.17)

The latter relation turns to

\[
\mu = \kappa_{1,\mathcal{M}} + \sum_{\text{all pairs}} \frac{(j-m)^2}{l^2} \text{Cov}(p_j, p_m)
\]

(A.18)
where $\sum_{\text{all pairs}} \equiv \sum_{j=1}^{l-1} \sum_{m=j+1}^{l}$.

Equation (A.18) shows that the mean value $\mu$ itself is a measure of the correlations between successive earthquake magnitudes. The practical use of this equation, however, in order to estimate the strength of these correlations between seismic events requires the construction of a large number of shuffled copies of the original earthquake catalog and a comparison of $\mu$ with the relevant distribution obtained from the shuffled copies. Obviously, this task becomes cumbersome when the (natural time) window of length $l$ is sliding through a long time series of $Q_k$.

When $Q_k$ are independent and identically distributed positive random variables, Eq. (A.18) results in

$$\mu = \kappa_u \left( 1 - \frac{1}{l^2} \right) - \kappa_u (l + 1) \text{Var}(p),$$

(A.19)

where $\kappa_u = 1/12$ -corresponding to the $\kappa_1$ value for the uniform distribution- and $\text{Var}(p)$ the variance of any $p_j$.

2. The standard deviation $\sigma$ of the $\kappa_1$ values

Let us now investigate the standard deviation $\sigma$ of the $\kappa_1$ values obtained when the (natural time) window of length $l$ slides through a time series of $Q_k$. This is obtained from the variance

$$\sigma^2 = \text{Var}(\kappa_1) \equiv E \left[ (\kappa_1 - \mu)^2 \right] = \frac{1}{W - l + 1} \sum_{k_0=1}^{W-l+1} [\kappa_1(k_0) - \mu]^2. \quad \text{(A.20)}$$

Numerically, the above quantity can be evaluated almost as easily as $\mu$ when $\kappa_1(k_0)$ are available.

In order to obtain an analytical expression, by inserting Eq. (A.18) into (A.20), we obtain

$$\sigma^2 = \frac{1}{W - l + 1} \sum_{k_0=1}^{W-l+1} \left\{ \sum_{m=1}^{l} \frac{m^2}{l^2} [p_m(k_0) - \mu_m] - \left[ \sum_{m=1}^{l} \frac{m}{l} p_m(k_0) \right]^2 \right\}$$

$$+ \left( \sum_{m=1}^{l} \frac{m}{l} \mu_m \right)^2 - \sum_{\text{all pairs}} \frac{(j - m)^2}{l^2} \text{Cov}(p_j, p_m) \right)^2. \quad \text{(A.21)}$$

Rearranging the terms

$$\left[ \sum_{m=1}^{l} \frac{m}{l} p_m(k_0) \right]^2 - \left( \sum_{m=1}^{l} \frac{m}{l} \mu_m \right)^2 = \left\{ \sum_{m=1}^{l} \frac{m}{l} [p_m(k_0) - \mu_m] \right\} \left\{ \sum_{m=1}^{l} \frac{m}{l} [p_m(k_0) + \mu_m] \right\}$$

$$= \left\{ \sum_{m=1}^{l} \frac{m}{l} [p_m(k_0) - \mu_m] \right\}^2 + 2(\chi)_M \left\{ \sum_{m=1}^{l} \frac{m}{l} [p_m(k_0) - \mu_m] \right\}$$

(A.22)

we get

$$\sigma^2 = \frac{1}{W - l + 1} \sum_{k_0=1}^{W-l+1} \left[ \sum_{m=1}^{l} \left( \frac{m^2}{l^2} - 2(\chi)_M \frac{m}{l} \right) [p_m(k_0) - \mu_m] - \left\{ \sum_{m=1}^{l} \frac{m}{l} [p_m(k_0) - \mu_m] \right\}^2 \right]$$

$$- \sum_{\text{all pairs}} \frac{(j - m)^2}{l^2} \text{Cov}(p_j, p_m) \right)^2. \quad \text{(A.23)}$$
Upon expanding the square over the square brackets in Eq. (A.23) we obtain six terms:

\[
\sigma^2 = \frac{1}{W - l + 1} \sum_{k_0=1}^{W-l+1} \left\{ \sum_{m=1}^{l} \left( \frac{m^2}{l^2} - 2\langle \chi \rangle_{\mathcal{M}} \frac{m}{l} \right) [p_m(k_0) - \mu_m] \right\}^2
\]  
\[\]  
(A.24a)

\[
- \frac{2}{W - l + 1} \sum_{k_0=1}^{W-l+1} \left\{ \sum_{m=1}^{l} \left( \frac{m^2}{l^2} - 2\langle \chi \rangle_{\mathcal{M}} \frac{m}{l} \right) [p_m(k_0) - \mu_m] \right\} \left\{ \sum_{m=1}^{l} \frac{m}{l} [p_m(k_0) - \mu_m] \right\}^2
\]  
\[\]  
(A.24b)

\[
- \left[ \sum_{\text{all pairs}} \left( \frac{j-m}{l^2} \right)^2 \text{Cov}(p_j, p_m) \right] \frac{2}{W - l + 1} \sum_{k_0=1}^{W-l+1} \left\{ \sum_{m=1}^{l} \frac{m}{l} [p_m(k_0) - \mu_m] \right\}^2
\]  
\[\]  
(A.24c)

\[
+ \frac{1}{W - l + 1} \sum_{k_0=1}^{W-l+1} \left\{ \sum_{m=1}^{l} \frac{m}{l} [p_m(k_0) - \mu_m] \right\}^4
\]  
\[\]  
(A.24d)

\[
+ \left[ \sum_{\text{all pairs}} \left( \frac{j-m}{l^2} \right)^2 \text{Cov}(p_j, p_m) \right] \frac{2}{W - l + 1} \sum_{k_0=1}^{W-l+1} \left\{ \sum_{m=1}^{l} \frac{m}{l} [p_m(k_0) - \mu_m] \right\}^2
\]  
\[\]  
(A.24e)

\[
+ \left[ \sum_{\text{all pairs}} \left( \frac{j-m}{l^2} \right)^2 \text{Cov}(p_j, p_m) \right] ^2
\]  
\[\]  
(A.24f)

The following comments are in order: First, the term in (A.24c) vanishes due to Eq. (A.5). Second, the terms in (A.24b) and (A.24d) clearly depend on moment correlations higher than the second, thus they should be neglected when restricting ourselves to second order correlations. Third, the second term in (A.24e) can be evaluated using Eqs. (A.15) and (A.17), leading to a partial cancellation with the term in (A.24b). Hence, restricting ourselves to second order correlations, we finally obtain

\[
\sigma^2 = \frac{1}{W - l + 1} \sum_{k_0=1}^{W-l+1} \left\{ \sum_{m=1}^{l} \left( \frac{m^2}{l^2} - 2\langle \chi \rangle_{\mathcal{M}} \frac{m}{l} \right) [p_m(k_0) - \mu_m] \right\}^2 - \left[ \sum_{\text{all pairs}} \left( \frac{j-m}{l^2} \right)^2 \text{Cov}(p_j, p_m) \right] \]  
\[\]  
(A.25)

The first term in Eq. (A.25) can be evaluated by expanding the square over the curly brackets and using Eq. (A.10), in a way similar to Eqs. (A.16) and (A.17), so that we obtain

\[
\sigma^2 = - \sum_{\text{all pairs}} \left[ \left( \frac{m}{l} - \langle \chi \rangle_{\mathcal{M}} \right)^2 - \left( \frac{j}{l} - \langle \chi \rangle_{\mathcal{M}} \right) \right] \text{Cov}(p_j, p_m) - \left[ \sum_{\text{all pairs}} \left( \frac{j-m}{l^2} \right)^2 \text{Cov}(p_j, p_m) \right] \]  
\[\]  
(A.26)

Equation (A.26) reveals that \( \sigma^2 \)-like the mean value \( \mu \) in Eq. (A.18)- is a measure of the correlations, but \( \sigma^2 \) is almost proportional (see also below) to these correlations whereas in \( \mu \) they appear as an additive term in Eq. (A.18).

3. The variability \( \sigma/\mu \)

By combining Eqs. (A.18) and (A.26) we find:

\[
\beta = \sqrt{\sum_{\text{all pairs}} \left[ \left( \frac{m}{l} - \langle \chi \rangle_{\mathcal{M}} \right)^2 - \left( \frac{j}{l} - \langle \chi \rangle_{\mathcal{M}} \right) \right]^2 \text{Cov}(p_j, p_m) - \left[ \sum_{\text{all pairs}} \left( \frac{j-m}{l^2} \right)^2 \text{Cov}(p_j, p_m) \right]^2} \]  
\[\]  
(A.27)

This equation, which is just Eq. (4) of the main text, provides in general the interrelation between the variability \( \beta \) and the event correlations.

Additional insight on the physical meaning of \( \sigma/\mu \) may be obtained when adopting the paradigm of the uniform distribution [32, 36, 37], which corresponds to a simple system operating at stationarity, i.e., when \( Q_k \) are independent and identically distributed positive random variables. In this case, we have [32]

\[
\mu_j = \frac{1}{l}
\]  
\[\]  
(A.28)
\[ \langle \chi \rangle_{\mathcal{M}} = \sum_{m=1}^{l} \frac{m}{l^2} = \frac{1}{2} + \frac{1}{2l}, \]  

and due to Eq. (A.10)

\[ \text{Cov}(p_j, p_m) = -\frac{\text{Var}(p)}{(l - 1)} \]  

thus we obtain

\[ \sigma^2 = \frac{\text{Var}(p)}{(l - 1)} \left\{ \sum_{\text{all pairs}} \left[ \left( \frac{m}{l} - \frac{1}{2} - \frac{1}{2l} \right)^2 - \left( \frac{j}{l} - \frac{1}{2} - \frac{1}{2l} \right)^2 \right]^2 - \frac{\text{Var}(p)}{(l - 1)} \left[ \sum_{\text{all pairs}} \frac{(j - m)^2}{l^2} \right]^2 \right\}. \]  

For large \( l \) the summations over all pairs can be effectively, e.g. \( l > 10 \), approximated by integrations

\[ \sum_{\text{all pairs}} \left[ \left( \frac{m}{l} - \frac{1}{2} - \frac{1}{2l} \right)^2 - \left( \frac{j}{l} - \frac{1}{2} - \frac{1}{2l} \right)^2 \right]^2 \approx \frac{l^2}{2} \int_0^1 \int_0^1 \left[ \left( \chi - \frac{1}{2} \right)^2 - \left( \psi - \frac{1}{2} \right)^2 \right]^2 d\chi d\psi = \frac{l^2}{180}, \]  

\[ \sum_{\text{all pairs}} \frac{(j - m)^2}{l^2} \approx \frac{l^2}{2} \int_0^1 \int_0^1 (\chi - \psi)^2 d\chi d\psi = \frac{l^2}{12}. \]  

Equation (A.31) simplifies to

\[ \sigma^2 \approx l \text{Var}(p) \kappa_n^2 \left[ \frac{4}{5} - l \text{Var}(p) \right]. \]  

and Eq. (A.19) becomes

\[ \mu \approx \kappa_n \left[ 1 - l \text{Var}(p) \right]. \]  

Thus, the variability simply results in

\[ \beta = \frac{\sigma}{\mu} = \sqrt{l \text{Var}(p)} \left[ \frac{\sqrt{\frac{4}{5} - l \text{Var}(p)}}{1 - l \text{Var}(p)} \right]. \]  

When \( Q_k \) do not exhibit heavy tails, which is not of course the case of seismicity, the quantity \( l \text{Var}(p) \) is simply related\(^2\) to the mean \( \mu_0 \) and the standard deviation \( \sigma_0 \) of \( Q_k \):

\[ l \text{Var}(p) = \frac{\sigma_0^2}{\mu_0}. \]  

Assuming that \( \sigma_0/\mu_0 \) is of the order of unity, \( l \text{Var}(p) \) becomes small compared to unity when \( l > 10 \), and Eq. (A.36) becomes

\[ \beta = \frac{\sigma}{\mu} = 2 \frac{\sigma_0}{\sqrt{5} \mu_0} \left( \frac{1}{\sqrt{l}} \right). \]  

i.e., the variability of \( \kappa_1 \) is directly proportional to the variability of the data \( Q_k \). Note that the same holds for the standard deviation of the natural time entropy\(^2\) as well as for change \( \Delta S \) of the entropy in natural time under time reversal\(^2\) (cf. for the analysis in natural time under time reversal, see also Refs.\(^4\) and \(^5\)). Thus, one could alternatively view \( \beta \) as an entropic measure.

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[1] I. J. Ford and R. E. Spinney, Phys. Rev. E 86, 021127 (2012).

[2] C. Gardiner, Stochastic Methods: A Handbook for the Natural and Social Sciences (Springer, New York, 2009).
