An adapted solution of a fractional backward stochastic differential equation

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Abstract

Our aim in this paper is to deal with a new type differential equation so-called Caputo fractional backward stochastic differential equations (for short Caputo fBSDEs) and study the global existence and uniqueness of an adapted solution to Caputo fBSDEs of order $\alpha \in \left(\frac{1}{2}, 1\right)$ whose coefficients satisfy Lipschitz condition by applying fundamental lemma which plays a crucial role in the theory of Caputo fBSDEs. The interesting point here is to use a new weighted norm in square-integrable measurable function space for fundamental lemma and therefore for the whole paper. For this class of systems, we then show the coincidence between the notion of stochastic Volterra integral equation and mild solution.

Keywords: Fractional backward stochastic differential equations, backward stochastic nonlinear Volterra integral equation, existence and uniqueness, adapted process, weighted norm

1 Introduction

Fractional stochastic differential equations (FSDEs) which are a generalization of differential equations by the use of fractional and stochastic calculus are more popular due to their applications in mathematical modelling and finance [1, 2, 3]. Recently, FSDEs are intensively applied to model mathematical problems in finance [4, 5], dynamics of complex systems in engineering [6] and other areas [7, 8]. Several results have been investigated on the qualitative theory and applications of fractional stochastic differential equations (FSDEs) [9]-[16].

Studying backward stochastic differential equations (BSDEs) has necessary applications in stochastic optimal control, stochastic differential game, probabilistic formula for the solutions of quasilinear partial differential equations and financial markets. The adapted solution for a linear BSDE which arises as the adjoint process for a stochastic control problem was first investigated by Bismut [17] in 1973, then by Bensoussan [18], and while Pardoux and Peng [19] first studied the result for the existence and uniqueness of an adapted solution to a continuous general non-linear BSDE which is a terminal value problem for an Itô type stochastic differential equation under the uniform Lipschitz conditions of the following form:

$$\begin{cases}
    dY(t) = h(t, Y(t), Z(t))dt + Z(t)dW(t), t \in [0, T], \\
    Y(T) = \xi.
\end{cases}$$

They established existence and uniqueness of an adapted solution using Bihari’s inequality which is the most important generalization of the Gronwall-Bellman inequality.

Since then, the theory of BSDE became a powerful tool in many fields such as mathematical finance, optimal control, semi-linear and quasi-linear partial differential equations [20-24]. Later Peng and Pardoux developed the theory and applications of continuous BSDEs in a series of papers [19, 24, 25, 26, 27].

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under the assumptions the coefficients satisfy Lipschitz conditions. Tang and Li \[28\] then applied the idea of Peng \[19\] to get the first result on the existence of an adapted solution to a BSDE with Poisson jumps for a fixed terminal time and with Lipschitz conditions. Moreover, Mao \[29\] obtained more general result than that of Pardoux and Peng \[19\] in which he proved existence and uniqueness result under mild assumptions applying Bihari’s inequality which was the key tool in the proof.

A few years later, Lin \[30\] considered the following backward stochastic nonlinear Volterra integral equation

\[
X(t) + \int_t^T f(t, s, X(s), Z(t, s))ds + \int_t^T [g(t, s, X(s)) + Z(t, s)]dW(s) = X,
\]

His aim in \[30\] is to look for a pair \(\{X(s), Z(t, s)\}\) which requires this pair to be \(\mathcal{F}_{\infty s}\)-adapted and \(Z(t, s)\) is related to \(t\). This is the intersection point of our results on linear Caputo fBSDEs with \[30\]. The author also defines \(Z(t, s) = \tilde{Z}(t, s) - g(t, s), (t, s) \in \mathcal{D} = \{(t, s) \in \mathbb{R}_+^2; 0 \leq t \leq s \leq T\}\) which we have defined it for linear Caputo fBSDEs. Another main point of intersection with \[31, 32, 33, 34\] is to use the well-known extended martingale representation theorem in which we consider extended martingale representation to an adapted solution \(\{x(t), y(t, s)\}\), \((t, s) \in \mathcal{D}\) for linear Caputo fBSDEs as well.

In 2005, Mahmudov and McKibben \[31\] studied the existence and uniqueness of the adapted solution to a backward stochastic evolution equation in Hilbert space in the following form of

\[
\begin{align*}
\begin{cases}
  dy(t) = -[Ay(t) + F(t, y(t), z(t))]dt - [G(t, y(t)) + z(t)]dw(t), \\
  y(T) = \xi,
\end{cases}
\end{align*}
\]

where \(A\) is a linear operator which generates a \(C_0\)-semigroup. The authors also applied Bihari’s inequality to show existence and uniqueness under a non-Lipschitz condition. The main idea on this work is to establish fundamental lemma which plays an important role in the theory of BSDEs. Furthermore, they discussed a stochastic maximum principle for optimal control problems in Hilbert space.

After a year, Yong \[33\] introduced backward stochastic Volterra integral equation (BSVIEs) which takes the following form:

\[
Y(t) = f(t) - \int_t^T h(t, s, Y(s), Z(t, s), Z(s, t))ds - \int_t^T Z(t, s)dW(s), \quad t \in [0, T].
\]

The interesting features are \(Z(t, s)\) depends on \(t\) and the drift depends on both \(Z(t, s)\) and \(Z(s, t)\) in general. As applications of BSVIEs, Yong proved comparison theorem for one-dimensional BSVIEs, and a Pontryagin type maximum principle for optimal control of stochastic differential equations.

Recently, Wang and Yong \[34\] considered BSVIEs. By taking a look at special case of BSVIEs they reduced the given BSVIEs to the following form:

\[
Y(t) = \xi + \int_t^T g(s, Y(s), Z(t, s))ds - \int_t^T Z(t, s)dW(s), \quad t \in [0, T]. \tag{1.1}
\]

The unknown pair that they are looking for is \((Y(\cdot), Z(\cdot))\) of stochastic processes. Eq. \[1.1\] is comparable with the integral form of BSDE which takes the following form:

\[
Y(t) = \xi + \int_t^T g(s, Y(s), Z(s))ds - \int_t^T Z(s)dW(s), \quad t \in [0, T]. \tag{1.2}
\]

If \[1.2\] admits a unique \(\mathcal{F}_t\)-adapted solution \((Y(\cdot), Z(\cdot))\), then this solution is also an adapted solution of \[1.1\] with \(Z(t, s) = Z(s)\) and it turns out that BSVIE can be regarded as an extension of BSDE.

To the best of our knowledge, we study a new fractional analogue of BSDEs which is an untreated topic in the present literature and so-called \textit{Caputo fractional backward stochastic differential equations} of order \(\alpha \in (\frac{1}{2}, 1)\) on \([0, T] \) as follows:

\[
\begin{align*}
\begin{cases}
\left(\frac{D^\alpha}{dt}x\right)(t) = Ax(t) - f(t, x(t), y(t, s)) - [g(t, s, x(t)) + y(t, s)]
\end{cases}
\end{align*}
\]

\[
x(T) = \xi. \tag{1.3}
\]
where $A$ is a $n \times n$ constant matrix and $(u(t))_{t \geq 0}$ denote a standard $m$-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t \geq 0}$ satisfying usual conditions. Moreover, $f : [0, T] \times \mathbb{R}^n \times \Omega \to \mathbb{R}^n$ and $g : \mathcal{D} \times \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are assumed to be measurable mappings and terminal value $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R}^n)$ is an $\mathcal{F}_T$-measurable $\mathbb{R}^n$-valued random variable such that $\mathbb{E}\|\xi\|^2 < \infty$.

Let $x(t, \omega) = x(t)$ be a stochastic process on $[0, T]$ and $\omega \in \Omega$. The expectation operator $\mathbb{E}$ is denoted by

$$(\mathbb{E}x)(t) := \int_\Omega x(t)\mathbb{P}(d\omega), \quad t \in [0, T].$$

Our aim in this paper is to look for a pair of stochastic processes $(x(t), y(t), s(t), s(s)) \in \mathcal{F}_{t,v,s}$ adapted and satisfies (1.3) in the usual Itô’s sense. Such a pair is called an adapted solution of the equation (1.3). Our main result will be an existence and uniqueness result for an adapted pair $(x(t), y(t), s(t), s(s)) \in \mathcal{F}_{t,v,s}$ which solves (1.3). We first derive representations of the adapted solution and then we present that the operator is well-defined before proving contraction mapping principle. Next we study existence and uniqueness results to the adapted solution of Eq. (1.3) under Lipschitz condition.

Therefore, the plan of this paper is organized as below: In Section 2 we introduce main definitions from stochastic and fractional calculus, and some necessary inequalities from stochastic calculus which play a key role in certain estimations of the main results. Section 3 is devoted to proving global existence and uniqueness of the adapted solution to Caputo FBSDE (1.3) of order $\alpha \in (\frac{1}{2}, 1)$ using the Banach’s fixed point approach. Section 4 is dedicated to presenting the coincidence between the notion of integral equation and mild solution. Finally, Section 5 is for conclusion and introducing some open problems.

To conclude introductory section, we introduce some notations which will be used throughout this paper. Let $\mathbb{R}^n$ be endowed with the standard Euclidean norm of $x \in \mathbb{R}^n$ and $(x, \bar{x})$ denote the inner product of $x, \bar{x} \in \mathbb{R}^n$. An element $y \in \mathbb{R}^{n \times m}$ will be considered as a $n \times m$ matrix and its Euclidean norm is defined by $\|y\| = \sqrt{\text{tr}(yy^T)}$ and $(x, y) = \text{tr}(xy^T)$. For any given $0 \leq t \leq T$, we denote by $L^2_p(0, T; \mathbb{R}^n)$ (accord. $L^2_p(\mathcal{D}; \mathbb{R}^{n \times m})$) the family of $\mathbb{R}^n$-valued (resp. $\mathbb{R}^{n \times m}$-valued ) $\mathcal{F}_t$-adapted processes (resp. $\mathcal{F}_{t,v,s}$) which are measurable and square integrable on $\Omega \times [0, T]$ with respect to $\mathbb{P} \times \lambda$ where $\lambda$ denotes the Lebesgue measure on $[0, T]$.

Let us recall some spaces. We define:

$$L^2_p(0, T; \mathbb{R}^n) = \{\xi(\cdot) \in L^2(0, T; \mathbb{R}^n) \mid \xi(t) \text{ is } \mathbb{R}^n \text{- valued } \mathcal{F}_T \text{- adapted}\};$$

$$L^2_p(\mathcal{D}; \mathbb{R}^{n \times m}) = \{\xi : \mathcal{D} \times \Omega \to \mathbb{R}^{n \times m} \mid \xi(\cdot) \text{ is } \mathcal{B}([0, T]) \otimes \mathcal{F}_T \text{- measurable}\}$$

with $\mathcal{B}$ be the Borel $\sigma$-field of $[0, T]$.

**Definition 1.1.** Let $\frac{1}{2} < \alpha < 1$ and $1 \leq p < \infty$ be fixed. We say that a measurable function $h : \Omega \times [0, T] \to \mathbb{R}^n$ belongs to $L^{p,\alpha}([0, T], \mathbb{R}^n)$ if and only if the quantity

$$\|h\|_{p,\alpha} := \sup_{0 \leq \tau \leq T} \mathbb{E}\left(\int_\tau^T \frac{\|h(s)\|^p}{(s-\tau)^{1-\alpha}} ds\right)^{\frac{1}{p}} < \infty.$$

**Lemma 1.1.** Let $\frac{1}{2} < \alpha < 1$ and $1 \leq p < \infty$, $L^{p,\alpha}$ is a Banach space.

**Proof.** Obviously, the defined weighted norm satisfies the conditions of norm in $L^{p,\alpha}$ space. It is only left to show that is complete. Hence, a proof of this part is similar to the proof of Theorem 3.4 in [35], so we omit it here. \qed

In particular, we consider $p = 2$ and $\frac{1}{2} < \alpha < 1$ throughout this paper such that the weighted norm is defined by

$$\|h\|_{2,\alpha,t} := \sup_{0 \leq \tau \leq T} \mathbb{E}\left(\int_\tau^T \frac{\|h(s)\|^2}{(s-\tau)^{1-\alpha}} ds\right)^{\frac{1}{2}}. \quad (1.4)$$

To show existence and uniqueness, we consider the following norm:

$$\|h\|_{2,\alpha,0} := \sup_{0 \leq \tau \leq T} \mathbb{E}\left(\int_\tau^T \frac{\|h(s)\|^2}{(s-\tau)^{1-\alpha}} ds\right)^{\frac{1}{2}}. \quad (1.5)$$
We define $M[t, T] := \mathbb{L}_p^2(t, T; \mathbb{R}^n) \times \mathbb{L}_p^2(D; \mathbb{R}^{n \times m})$ to be a Banach space endowed with the norm
\[
\|(x, y)\|_t^2 = \|x\|_{2, \alpha, t}^2 + \|y\|_{2, \alpha, t}^2,
\]
where
\[
\|y\|_{2, \alpha, t}^2 := \sup_{t \leq s \leq T} \mathbb{E} \int_s^T (s - \tau)^{-\alpha - 1} \int_s^T \|y(s, u)\|^2 \, du \, ds, \quad t \leq s \leq u \leq T. \quad (1.6)
\]
Then $M[0, T] := \mathbb{L}_p^2(0, T; \mathbb{R}^n) \times \mathbb{L}_p^2([0, T]^2; \mathbb{R}^{n \times m})$ is also Banach space equipped with the norm as below:
\[
\|(x, y)\|_0^2 = \|x\|_{2, \alpha, 0}^2 + \|y\|_{2, \alpha, 0}^2.
\]

2 Fractional and Stochastic Calculus

We embark on this section by briefly introducing the essential structure of fractional calculus and fractional operators, as well as some important inequalities of stochastic calculus. For the more salient details on these matters, see the textbooks [36, 37, 38, 39, 40] and research paper [41, 42]. We introduce the classical fractional operators.

Since we study backward stochastic fractional differential equation throughout this research paper, we consider the right stochastic Riemann-Liouville and Caputo fractional operators of order $\alpha > 0$ on $[t, T]$.

- the right Riemann-Liouville stochastic fractional derivative of order $\alpha$ is as follows:
\[
\iota D_T^\alpha x(t) = \frac{(-1)^n}{\Gamma(n - \alpha)} \left( \frac{d}{dt} \right)^n \int_t^T (s - t)^{n-\alpha-1} x(s) \, ds, \quad \text{where } n - 1 < \alpha \leq n, \quad t < T.
\]

- The right Riemann-Liouville stochastic fractional integral of order $\alpha$ is given by
\[
\iota I_T^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s - t)^{\alpha-1} x(s) \, ds, \quad \text{for } t < T, \quad (2.1)
\]
where $\Gamma$ is the gamma function.

- The right Caputo stochastic fractional derivative operator of order $\alpha$ is defined by:
\[
\iota I_T^\alpha C_T^\alpha x(t) = \frac{(-1)^n}{\Gamma(n - \alpha)} \int_t^T (s - t)^{n-\alpha-1} x^{(n)}(s) \, ds, \quad \text{where } n - 1 < \alpha \leq n, \quad t < T.
\]
In particular, for $\alpha \in (0, 1)$
\[
\iota I_T^\alpha C_T^\alpha x(t) = x(t) - x(T).
\]

- the relationship between the Riemann–Liouville and Caputo fractional derivatives are as follows:
\[
C_T^\alpha D_T^\alpha x(t) = \iota D_T^\alpha x(t) - \sum_{k=0}^{n-1} \frac{(T - t)^{k-\alpha} x^{(k)}(T)}{\Gamma(k - \alpha + 1)}. \quad (2.2)
\]

- the property of the Riemann–Liouville fractional integral operator and the Caputo fractional derivative of order $\alpha$ is given by:
\[
\iota I_T^\alpha C_T^\alpha x(t) = x(t) - \sum_{k=0}^{n-1} \frac{(T - t)^{k} x^{(k)}(T)}{\Gamma(k + 1)}. \quad (2.3)
\]

The following lemma plays a crucial role on proofs of main results in Section 3 and 4.
Lemma 2.1 (Hölder’s inequality). Suppose that $\alpha_1, \alpha_2 > 1$ and $\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1$. If $|f(t)|^{\alpha_1}, |g(t)|^{\alpha_2} \in L^1(\Omega)$, then $|f(t)g(t)| \in L^1(\Omega)$ and
\[
\int_{\Omega} |f(t)g(t)| dt \leq \left( \int_{\Omega} |f(t)|^{\alpha_1} dt \right)^{\frac{1}{\alpha_1}} \left( \int_{\Omega} |g(t)|^{\alpha_2} dt \right)^{\frac{1}{\alpha_2}},
\]
where $L^1(\Omega)$ represents the Banach space of all Lebesgue measurable functions $f : \Omega \to \mathbb{R}$ with $\int_{\Omega} |f(t)| dt < \infty$. Especially, when $\alpha_1 = \alpha_2 = 2$, the Hölder’s inequality reduces to the Cauchy-Schwartz inequality
\[
\left( \int_{\Omega} |f(t)g(t)| dt \right)^2 \leq \int_{\Omega} |f(t)|^2 dt \int_{\Omega} |g(t)|^2 dt.
\] (2.4)

Lemma 2.2 (Jensen’s inequality). Let $n \in \mathbb{N}$, $q > 1$ and $x_i \in \mathbb{R}_+$, $i = 1, 2, \ldots, n$. Then, the following inequality holds true:
\[
\| \sum_{i=1}^{n} x_i \|^q \leq n^{q-1} \sum_{i=1}^{n} \| x_i \|^q.
\]
In particular, we consider the following inequality with $q = 2$ within the estimations on this paper:
\[
\| \sum_{i=1}^{n} x_i \|^2 \leq n \sum_{i=1}^{n} \| x_i \|^2.
\] (2.5)

The following inequalities are fundamental inequalities of stochastic calculus. First, we introduce Jensen’s inequality in probability setting plays an important role in the stochastic process theory. It has become a standard result which appears in almost every introductory text in this field.

Theorem 2.1 (Jensen’s inequality in probabilistic setting). Let $X$ be an integrable real-valued random variable and $\varphi$ be a convex function. Then:
\[
\varphi \left( E[X] \right) \leq E \left[ \varphi(X) \right].
\]

In this probability setting, $\varphi$ is intended as the integral with respect to an expected value $E$.

Theorem 2.2 (The law of total expectation). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which two sub σ-algebras $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{F}$ are defined. For a random variable $X$ on such a space, the smoothing law states that if $E[X]$ is defined, then
\[
E[E[X \mid \mathcal{G}_2] \mid \mathcal{G}_1] = E[X \mid \mathcal{G}_1], \quad \text{a.s.}
\]
In special case, when $\mathcal{G}_1 = \{\emptyset, \Omega\}$ and $\mathcal{G}_2 = \sigma(Y)$, the smoothing law reduces to
\[
E[E[X \mid Y]] = E[X].
\]

3 Main results for fractional backward stochastic differential equations

In this section, we study existence and uniqueness results to a mild solution of the equation (1.3). To state our main results, we resort the following assumptions which will be considered in Section 3 and 4. More precisely, let us propose the standing assumptions for the function $f$ and $g$ as follows.

Assumption 3.1. For all $x, \bar{x} \in \mathbb{R}^n$, $y, \bar{y} \in \mathbb{R}^{n \times m}$ and $0 \leq t \leq T$, there exists $c > 0$ such that
\[
\| f(t, x, y) - f(t, \bar{x}, \bar{y}) \| \leq c \left( \| x - \bar{x} \|^2 + \| y - \bar{y} \|^2 \right), \quad \text{a.s},
\]
\[
\| g(t, s, x) - g(t, s, \bar{x}) \| \leq c \| x - \bar{x} \|^2, \quad \text{a.s.}
\]
Assumption 3.2. \( f(\cdot, 0, 0) \) is \( L^2(0, T; \mathbb{R}^n) \) integrable i.e.
\[
\int_0^T \|f(r, 0, 0)\|^2 \, dr < \infty,
\]
and \( g(\cdot, \cdot, 0) \) is essentially bounded i.e.
\[
\text{ess sup}_{r \in [0, T]} \|g(r, u, 0)\| < \infty.
\]

Definition 3.1. A stochastic process \((x(t), y(t, s)) \in \mathcal{D}\) is called a mild solution of (1.3) if
- \((x, y)\) is adapted to \(\{\mathcal{F}_t\}_{t \geq 0}\) with \(\int_0^T \|(x, y)\|^2 \, dt < \infty\) almost everywhere;
- \((x, y) \in M[t, T]\) has continuous path on \([0, T]\) a.s. and satisfies the following Volterra integral equation of second kind on \(t \in [0, T]\):
\[
x(t) = \xi - \frac{1}{\Gamma(\alpha)} \int_t^T (s - t)^{\alpha - 1} x(s) \, ds
+ \frac{1}{\Gamma(\alpha)} \int_t^T (s - t)^{\alpha - 1} f(s, x(s), y(t, s)) \, ds
+ \frac{1}{\Gamma(\alpha)} \int_t^T (s - t)^{\alpha - 1} [g(t, s, x(s)) + y(t, s)] \, dw(s).
\]
(3.1)

Definition 3.2. A pair of adapted process \((x, y) \in M[t, T]\) is a mild solution of (1.3) for all \(t \in [0, T]\) if satisfies the following backward stochastic nonlinear Volterra integral equation
\[
x(t) = E_\alpha(A(T - t)\alpha)\xi + \int_t^T (s - t)^{\alpha - 1} E_{\alpha, \alpha}(A(s - t)\alpha) f(s, x(s), y(t, s)) \, ds
+ \int_t^T (s - t)^{\alpha - 1} E_{\alpha, \alpha}(A(s - t)\alpha) [g(t, s, x(s)) + y(t, s)] \, dw(s), \quad \text{P-a.s.}
\]
(3.2)
where \(E_\alpha(\cdot)\) and \(E_{\alpha, \alpha}(\cdot)\) are one and two-parameter Mittag-Leffler functions (see [43]).

We then introduce a fundamental lemma which plays an important role to state and prove existence and uniqueness results.

Lemma 3.1. For any \((x(\cdot), y(\cdot, \cdot)) \in M[t, T]\), the linear Caputo fBSDE equation
\[
x(t) = E_\alpha(A(T - t)\alpha)\xi + \int_t^T (s - t)^{\alpha - 1} E_{\alpha, \alpha}(A(s - t)\alpha) f(s) \, ds
+ \int_t^T (s - t)^{\alpha - 1} E_{\alpha, \alpha}(A(s - t)\alpha) [g(t, s) + y(t, s)] \, dw(s), \quad \text{P-a.s.}
\]
(3.3)
admits a unique solution in \(M[0, T]\) and moreover
\[
\|x\|_2^2,_{\alpha, t} + \|y\|_2^2,_{\alpha, t} \leq 2M_\alpha^2 E\|\xi\|_2^2, \mathcal{F}_t \|E\|_2^2,_{\alpha, t} + 16M_\alpha^2 \frac{(T - t)^\alpha}{\alpha} E\|\xi\|^2
+ 8M_\alpha^2 \left( \frac{T^{2\alpha}}{\alpha^2} + \frac{(2T)^{2\alpha}}{\alpha(2\alpha - 1)} \right) \|f\|_2,_{\alpha, t}^2 + 2\|g\|_2^2,_{\alpha, t},
\]
(3.4)
where \(M_\alpha := \sup \{\|E_\alpha(t^\alpha A)\|, 0 \leq t \leq T\}\) and \(M_{\alpha, \alpha} := \sup \{\|E_{\alpha, \alpha}(t^\alpha A)\|, 0 \leq t \leq T\}\).
Proof. Uniqueness: Let \((x_1, y_1)\) and \((x_2, y_2)\) be two solutions of (3.3).

\[
x_1(t) - x_2(t) = \int_t^T (s-t)^{\alpha-1}E_{\alpha,\alpha}(A(s-t)^{\alpha})[y_1(t,s) - y_2(t,s)] \, dw(s),
\]

Taking \(\mathbb{E}\{\cdot \mid \mathcal{F}_t\}\) from above, we can deduce that

\[
\mathbb{E}\{x_1(t) - x_2(t) \mid \mathcal{F}_t\} = 0, \quad \forall t \in [0, T],
\]

It is obvious that \(x_1(t) = x_2(t)\) and this follows that \(y_1(t,s) = y_2(t,s), \ (t,s) \in \mathcal{D}\).

Existence: Taking a conditional expectation from (3.3), we have

\[
x(t) = E_\alpha(A(T-t)^{\alpha})\mathbb{E}\{\xi \mid \mathcal{F}_t\} + \int_t^T (s-t)^{\alpha-1}E_{\alpha,\alpha}(A(s-t)^{\alpha})\mathbb{E}\{f(s) \mid \mathcal{F}_t\} \, ds.
\]

From extended martingale representation theorem, there exists \(L(\cdot) \in \mathbb{L}_\mathcal{F}^2(0,T;L^2(\mathbb{R}^n))\) and uniquely \(K(t,\cdot) \in \mathbb{L}_\mathcal{F}^2(\mathcal{D};L^2(\mathbb{R}^n \times \mathbb{R}^n))\) which satisfy the following relations:

\[
\mathbb{E}\{\xi \mid \mathcal{F}_t\} = \mathbb{E}\xi + \int_0^t L(u) \, dw(u), \quad (3.5)
\]

\[
\mathbb{E}\{f(s) \mid \mathcal{F}_t\} = \mathbb{E}f(s) + \int_s^t K(s,u) \, dw(u). \quad (3.6)
\]

Note also from (3.6), we can easily deduce that \(\forall s \in [0, T]\)

\[
K(s,u) = 0, \quad \text{a.e.,} \quad u \in [s, T], \quad \text{a.s.}
\]

and that

\[
\mathbb{E}\int_0^T \int_0^s |K(s,u)|^2 \, du \, ds \leq 4\mathbb{E}\int_0^T |f(s)|^2 \, ds. \quad (3.7)
\]

Since \(t \in [0,T]\), it is obvious that

\[
\xi = \mathbb{E}\xi + \int_0^T L(u) \, dw(u)
\]

\[
= \mathbb{E}\xi + \int_0^t L(u) \, dw(u) + \int_t^T L(u) \, dw(u)
\]

\[
= \mathbb{E}\{\xi \mid \mathcal{F}_t\} + \int_t^T L(u) \, dw(u),
\]

and since \(s \geq t\), we have

\[
f(s) = \mathbb{E}f(s) + \int_0^s K(s,u) \, dw(u)
\]

\[
= \mathbb{E}f(s) + \int_0^t K(s,u) \, dw(u) + \int_t^s K(s,u) \, dw(u)
\]

\[
= \mathbb{E}\{f(s) \mid \mathcal{F}_t\} + \int_t^s K(s,u) \, dw(u).
\]

Therefore, we obtain

\[
\mathbb{E}\{\xi \mid \mathcal{F}_t\} = \xi - \int_t^T L(u) \, dw(u), \quad (3.8)
\]

and

\[
\mathbb{E}\{f(s) \mid \mathcal{F}_t\} = f(s) - \int_t^s K(s,u) \, dw(u). \quad (3.9)
\]
Substituting \((3.8)\) and \((3.9)\) into \((3.10)\) and using stochastic Fubini’s theorem, we have
\[
x(t) = E_\alpha(A(T - t)^\alpha) \left( \xi - \int_t^T \frac{d}{dt} L(u)dw(u) \right) + \int_t^T (s - t)^{\alpha - 1} E_\alpha,\alpha(A(s - t)^\alpha) \left( f(s) - \int_t^s K(s, u)dw(u) \right) ds
\]
\[
= E_\alpha(A(T - t)^\alpha)\xi + \int_t^T (s - t)^{\alpha - 1} E_\alpha,\alpha(A(s - t)^\alpha)f(s)ds
\]
\[
- E_\alpha(A(T - t)^\alpha) \int_t^T L(u)dw(u) - \int_t^T (s - t)^{\alpha - 1} E_\alpha,\alpha(A(s - t)^\alpha) \int_u^s K(s, u)dw(u) ds
\]
\[
= E_\alpha(A(T - t)^\alpha)\xi + \int_t^T (s - t)^{\alpha - 1} E_\alpha,\alpha(A(s - t)^\alpha)f(s)ds
\]
\[
- E_\alpha(A(T - t)^\alpha) \int_t^T L(u)dw(u) - \int_t^T \int_u^T (s - t)^{\alpha - 1} E_\alpha,\alpha(A(s - t)^\alpha)K(s, u)dsdw(u).
\]
Thus, we get
\[
x(t) = E_\alpha(A(T - t)^\alpha)\xi + \int_t^T (s - t)^{\alpha - 1} E_\alpha,\alpha(A(s - t)^\alpha)f(s)ds + \int_t^T \tilde{y}(t, u)dw(u).
\]
Then there exists a mild solution \((x, y) \in M[0, T]\) of \((3.8)\) given by
\[
x(t) = E_\alpha(A(T - t)^\alpha)\xi + \int_t^T (s - t)^{\alpha - 1} E_\alpha,\alpha(A(s - t)^\alpha)f(s)ds + \int_t^T \tilde{y}(t, u)dw(u).
\]
and
\[
\tilde{y}(t, u) = -E_\alpha(A(T - t)^\alpha)L(u) - \int_u^T (s - t)^{\alpha - 1} E_\alpha,\alpha(A(s - t)^\alpha)K(s, u)ds.
\]
We finally define \(y(t, u) = \tilde{y}(t, u) - q(t, u)\), \((t, u) \in D = \{(t, u) \in \mathbb{R}_+^2; 0 \leq t \leq u \leq T\}\). It is easily seen that the pair \((x, y)\) solves \((3.3)\). Therefore, the existence is proved.

From \((3.8)\) and \((3.9)\), we invoke the following inequalities for \(0 \leq t \leq s \leq T\):
\[
E \int_t^s \|L(u)\|^2du \leq 4E\|\xi\|^2,
\]
and
\[
E \int_t^s \|K(s, u)\|^2du \leq 4E\|f(s)\|^2.
\]
Now we estimate the solution \((x, y)\) given by \((3.10)\) and \((3.11)\) in \([0, T]\). From \((3.10)\) it follows that
\[
\sup_{t \leq \tau \leq T} E \int_\tau^T (s - \tau)^{\alpha - 1}\|x(s)\|^2ds \leq 2M_2^2 \sup_{t \leq \tau \leq T} E \int_\tau^T (s - \tau)^{\alpha - 1}\|E\{\xi|\mathcal{F}_s\}\|^2ds
\]
\[
+ 2M_2^2 \sup_{t \leq \tau \leq T} E \int_\tau^T (s - \tau)^{\alpha - 1} \left( \int_s^T (r - s)^{\alpha - 1} E\{\|f(r)\| | \mathcal{F}_s\} dr \right)^2 ds
\]
\[
:= I_1 + I_2.
\]
From law of total expectation, it follows that
\[
I_1 := 2M_2^2 \sup_{t \leq \tau \leq T} E \int_\tau^T (s - \tau)^{\alpha - 1}\|E\{\xi|\mathcal{F}_s\}\|^2ds = 2M_2^2E\|\xi | \mathcal{F}_s\|^2_{2,\alpha, t}
\]
and Jensen’s inequality in probabilistic setting imply that
\[
I_2 := 2M_2^2 \sup_{t \leq \tau \leq T} E \int_\tau^T (s - \tau)^{\alpha - 1} \left( \int_s^T (r - s)^{\alpha - 1} E\{\|f(r)\| | \mathcal{F}_s\} dr \right)^2 ds
\]
\[ \begin{aligned}
&= 2M_{\alpha,\alpha}^2 \sup_{t \leq \tau \leq T} \mathbf{E} \left( \int_{\tau}^{T} (s - \tau)^{\alpha - 1} \left( \mathbf{E} \left\{ \int_{s}^{T} (r - s)^{\alpha - 1} \| f(r) \| dr \mid \mathcal{F}_s \right\} \right)^2 \right) ds \\
&\leq 2M_{\alpha,\alpha}^2 \sup_{t \leq \tau \leq T} \int_{\tau}^{T} (s - \tau)^{\alpha - 1} \mathbf{E} \left( \int_{s}^{T} (r - s)^{\alpha - 1} \| f(r) \| dr \right)^2 ds \\
&\leq 2M_{\alpha,\alpha}^2 \sup_{t \leq \tau \leq T} \int_{\tau}^{T} (s - \tau)^{\alpha - 1} \mathbf{E} \left( \int_{s}^{T} (r - s)^{\alpha - 1} \| f(r) \|^2 dr \right) ds \\
&\leq 2M_{\alpha,\alpha}^2 \frac{(T - t)^\alpha}{\alpha} \sup_{t \leq \tau \leq T} \int_{\tau}^{T} (s - \tau)^{\alpha - 1} \mathbf{E} \int_{s}^{T} (r - s)^{\alpha - 1} \| f(r) \|^2 dr ds \\
&= 2M_{\alpha,\alpha}^2 \frac{(T - t)^\alpha}{\alpha} \sup_{t \leq \tau \leq T} \int_{\tau}^{T} (s - \tau)^{\alpha - 1} \| f \|^2_{\alpha,\tau,t} \\
&= 2M_{\alpha,\alpha}^2 \frac{(T - t)^{2\alpha}}{\alpha^2} \| f \|^2_{\alpha,\tau,t}.
\end{aligned} \]

Thus, considering (1.4), we can write

\[ \| x \|^2_{\alpha,\tau,t} \leq 2M_{\alpha,\alpha}^2 \mathbf{E} \{ \xi \mid \mathcal{F} \} \| f \|^2_{\alpha,\tau,t} + 2M_{\alpha,\alpha}^2 \frac{(T - t)^{2\alpha}}{\alpha^2} \| f \|^2_{\alpha,\tau,t}. \] (3.12)

Next we estimate \( \mathbf{\tilde{y}} \) using Hölder’s inequality. We attain,

\[ \begin{aligned}
\| \mathbf{\tilde{y}}(s, u) \|^2 &\leq 2 \| E_{\alpha}(A(T - s)^\alpha L(u)) \|^2 + 2 \left( \int_{u}^{T} (r - s)^{\alpha - 1} E_{\alpha,\alpha}(A(r - s)^\alpha) K(r, u) dr \right)^2 \\
&\leq 2M_{\alpha}^2 \| L(u) \|^2 + 2M_{\alpha,\alpha}^2 \left( \frac{(T - s)^{2\alpha - 1}}{2\alpha - 1} - \frac{(u - s)^{2\alpha - 1}}{2\alpha - 1} \right) \int_{u}^{T} \| K(r, u) \|^2 dr \\
&\leq 2M_{\alpha}^2 \| L(u) \|^2 + 2M_{\alpha,\alpha}^2 \frac{(T + u)^{2\alpha - 1}}{2\alpha - 1} \int_{u}^{T} \| K(r, u) \|^2 dr \\
&\leq 2M_{\alpha}^2 \| L(u) \|^2 + 2M_{\alpha,\alpha}^2 \frac{(2T)^{2\alpha - 1}}{2\alpha - 1} \int_{u}^{T} \| K(r, u) \|^2 dr.
\end{aligned} \]

Taking double integral of above inequality and applying stochastic Fubini’s theorem twice yield that

\[ \begin{aligned}
\sup_{t \leq \tau \leq T} \mathbf{E} \int_{\tau}^{T} (s - \tau)^{\alpha - 1} \int_{s}^{T} \| \mathbf{\tilde{y}}(s, u) \|^2 duds &\leq 2M_{\alpha}^2 \sup_{t \leq \tau \leq T} \mathbf{E} \int_{\tau}^{T} (s - \tau)^{\alpha - 1} \int_{s}^{T} \| L(u) \|^2 duds \\
+ 2M_{\alpha,\alpha}^2 \frac{(2T)^{2\alpha - 1}}{2\alpha - 1} &\sup_{t \leq \tau \leq T} \mathbf{E} \int_{\tau}^{T} (s - \tau)^{\alpha - 1} \int_{s}^{T} \| K(r, u) \|^2 duds \\
&\leq 8M_{\alpha}^2 \sup_{t \leq \tau \leq T} \int_{\tau}^{T} (s - \tau)^{\alpha - 1} \mathbf{E} \| \xi \|^2 ds \\
+ 2M_{\alpha,\alpha}^2 \frac{(2T)^{2\alpha - 1}}{2\alpha - 1} &\sup_{t \leq \tau \leq T} \mathbf{E} \int_{\tau}^{T} (s - \tau)^{\alpha - 1} \int_{s}^{T} \| K(r, u) \|^2 duds \\
&\leq 8M_{\alpha}^2 \frac{(T - t)^\alpha}{\alpha} \mathbf{E} \| \xi \|^2 + 8M_{\alpha,\alpha}^2 \frac{(2T)^{2\alpha - 1}}{2\alpha - 1} \sup_{t \leq \tau \leq T} \mathbf{E} \int_{\tau}^{T} (s - \tau)^{\alpha - 1} \| f(r) \|^2 dr \\
&\leq 8M_{\alpha}^2 \frac{(T - t)^\alpha}{\alpha} \mathbf{E} \| \xi \|^2 + 8M_{\alpha,\alpha}^2 \frac{(2T)^{2\alpha - 1}}{2\alpha - 1} \sup_{t \leq \tau \leq T} \mathbf{E} \int_{\tau}^{T} (s - \tau)^{\alpha - 1} \| f(r) \|^2 dr \\
&\leq 8M_{\alpha}^2 \frac{(T - t)^\alpha}{\alpha} \mathbf{E} \| \xi \|^2 + 8M_{\alpha,\alpha}^2 \frac{(2T)^{2\alpha - 1}(T - t)}{\alpha(2\alpha - 1)} \sup_{t \leq \tau \leq T} \mathbf{E} \int_{\tau}^{T} (r - \tau)^{\alpha - 1} \| f(r) \|^2 dr.
\end{aligned} \]
Thus, we get
\[
\sup_{0 \leq \tau \leq T} E \int_\tau^T (s - \tau)^{\alpha - 1} \int_s^T \|\tilde{y}(s, u)\|^2 du ds \leq 8M_0^2 \frac{(T - t)^\alpha}{\alpha} E\|\xi\|^2 + 8M_0^2 \frac{(2T)^{2\alpha - 1}(T - t)}{\alpha(2\alpha - 1)} \|f\|^2_{2,\alpha,t}.
\]
(3.13)
Since \(\frac{1}{2} < \alpha \leq 1\) and \(0 < 2\alpha - 1 \leq 1\)
\[
\frac{(T - t)^{2\alpha}}{\alpha^2} + 2\frac{2(T)^{2\alpha - 1}(T - t)}{\alpha(2\alpha - 1)} \leq \frac{T^{2\alpha}}{\alpha^2} + \frac{(2T)^{2\alpha}}{\alpha(2\alpha - 1)},
\]
and taking into account (1.6), then the inequalities (3.12) and (3.13) imply that
\[
\|x\|_{2,\alpha,t}^2 + 2\|y\|_{2,\alpha,t}^2 \leq 2M^2_0 E\|\{\xi \mid \mathcal{F}\}\|_{2,\alpha,t}^2 + 16M_0^2 \frac{(T - t)^\alpha}{\alpha} E\|\xi\|^2
\]
\[+ 8M_0^2 \left( \frac{T^{2\alpha}}{\alpha^2} + \frac{(2T)^{2\alpha}}{\alpha(2\alpha - 1)} \right) \|f\|^2_{2,\alpha,t} + 2\|g\|^2_{2,\alpha,t}.
\]
Then for any \((x, y)\) the associated mild solution of (3.3) satisfies the following estimate:
\[
\|x\|^2_{2,\alpha,t} + \|y\|^2_{2,\alpha,t} \leq 2M^2_0 E\|\{\xi \mid \mathcal{F}\}\|_{2,\alpha,t}^2 + 16M_0^2 \frac{(T - t)^\alpha}{\alpha} E\|\xi\|^2
\]
\[+ 8M_0^2 \left( \frac{T^{2\alpha}}{\alpha^2} + \frac{(2T)^{2\alpha}}{\alpha(2\alpha - 1)} \right) \|f\|^2_{2,\alpha,t} + 2\|g\|^2_{2,\alpha,t}.
\]

**Theorem 3.1.** If
\[
8cM^2_{\alpha,\alpha} \left( \frac{T^{2\alpha}}{\alpha^2} + \frac{(2T)^{2\alpha}}{\alpha(2\alpha - 1)} \right) T < 1,
\]
(3.14)
then Caputo fBSDE \((\mathbf{3.3})\) admits a unique solution \((x, y) \in M[0, T]\) under Assumptions 3.1 and 3.2.

**Proof.** For any fixed \((\bar{x}, \bar{y}) \in M[0, T]\), it follows from Assumption 3.2 that
\[
f(\cdot) = f(\cdot, \bar{x}(\cdot), \bar{y}(\cdot, \cdot)) \in L^2_\mathbb{F}([0, T], \mathbb{R}^n).
\]
By Lemma 3.1 the equation
\[
x(t) = E_\alpha(A(T - t)^\alpha)\xi + \int_t^T (s - t)^{\alpha - 1} E_{\alpha,\alpha}(A(s - t)^\alpha)f(s, \bar{x}(s), \bar{y}(t, s)) ds 
\]
\[+ \int_t^T (s - t)^{\alpha - 1} E_{\alpha,\alpha}(A(s - t)^\alpha) \{g(t, s) + y(t, s)\} dw(s), \quad \text{P-a.s}
\]
(3.15)
has a unique solution in \(M[0, T]\).

Thus, the operator \(\Psi : M[0, T] \to M[0, T]\) defined by
\[
\Psi(x, y) = (x, y)
\]
where \((x, y)\) is a solution of \((\mathbf{3.15})\), is well-defined. Now we prove the contractivity of the operator \(\Psi\). To do so, applying Lemma 3.1 implies that
\[
\|\Psi(x, y) - \Psi(\bar{x}, \bar{y})\|_0^2 = \|\Psi(x - \bar{x}, \tilde{y} - \bar{y})\|_0^2
\]
\[\leq 8M_0^2 \frac{T^{2\alpha}}{\alpha^2} + \frac{(2T)^{2\alpha}}{\alpha(2\alpha - 1)} \sup_{0 \leq \tau \leq T} E \int_\tau^T (s - \tau)^{\alpha - 1} \int_s^T \|f(s, \bar{x}(s), \bar{y}(s, u)) - f(s, \bar{x}(s), \bar{y}(s, u))\|^2 du ds 
\]
\[\leq 8cM^2_{\alpha,\alpha} \left( \frac{T^{2\alpha}}{\alpha^2} + \frac{(2T)^{2\alpha}}{\alpha(2\alpha - 1)} \right) \sup_{0 \leq \tau \leq T} E \int_\tau^T (s - \tau)^{\alpha - 1} \int_s^T (\|x(s) - \bar{x}(s)\|^2 + \|\tilde{y}(s, u) - \bar{y}(s, u)\|) du ds.
\]

10
4 A coincidence between the notion of integral equation and mild solution

In this section, we are going to prove the coincidence between the notion of stochastic Volterra integral equation and mild solution by following theorem.

**Theorem 4.1.** The unique mild solution of (1.3) with terminal value \( \xi \) on \([0, T]\) is given by

\[
\psi(t, \xi) = E_\alpha(A(T-t)^\alpha)\xi + \int_t^T (s-t)^{\alpha-1} E_{\alpha, \alpha}(A(s-t)^\alpha) f(s, \psi(s, \xi), y(t, s)) ds
+ \int_t^T (s-t)^{\alpha-1} E_{\alpha, \alpha}(A(s-t)^\alpha) [g(t, s, \psi(s, \xi)) + y(t, s)] dw(s).
\]

**Proof.** Before providing the proof of above theorem, we first need to show preparatory lemma and remark. In doing so, we present martingale representation theorem for any function \( h \in L^2(\Omega, \mathcal{F}_T, \mathbb{R}^m) \), there exists a unique adapted process \( \Theta \in L^2(\Omega, \mathcal{F}_T, \mathbb{R}^m) \) such that

\[
h = \mathbf{E} h + \int_0^T \Theta(s) dw(s).
\]

It is clear that

\[
h = \sum_{i=1}^m h_i e_i = \sum_{i=1}^m \left( \mathbf{E} h_i + \int_0^T \theta_i(s) dw(s) \right) e_i,
\]

where

\[
h_i = \mathbf{E} h_i + \int_0^T \theta_i(s) dw(s), \quad h_i \in L^2(\Omega, \mathcal{F}_T, \mathbb{R}^m).
\]

It is sufficient to show that

\[
\psi(t, \xi) = \tilde{\psi}(t, \xi).
\] (4.1)

To show (4.1), it is enough to prove that for any \( h \in L^2(\Omega, \mathcal{F}_T, \mathbb{R}^m) \),

\[
\mathbf{E}(\psi(t, \xi), h) = \mathbf{E}(\tilde{\psi}(t, \xi), h).
\]
Remark 4.1

In other words, 
\[ \mathbf{E}(\psi(t, \xi) - \tilde{\psi}(t, \xi), h) = \sum_{i=1}^{m} \mathbf{E}(\psi(t, \xi) - \tilde{\psi}(t, \xi)) h_i, e_i). \]

It follows that 
\[ |\mathbf{E}(\psi(t, \xi) - \tilde{\psi}(t, \xi), h)|^2 \leq \left| \sum_{i=1}^{m} \mathbf{E}(\psi(t, \xi) - \tilde{\psi}(t, \xi)) h_i \right|^2 \]
\[ \leq \sum_{i=1}^{m} \int_{0}^{T} \sum_{i=1}^{m} \left| \mathbf{E}(\psi(t, \xi) - \tilde{\psi}(t, \xi)) h_i \right|^2. \]

Before estimating \(|\mathbf{E}(\psi(t, \xi) - \tilde{\psi}(t, \xi), h)|\), define the following functions:

\[ \chi_i(t) = \mathbf{E}\psi(t, \xi)h_i, \quad \kappa_i(t) = \mathbf{E}f(t, \psi(t, \xi), y(t, s))h_i, \]
\[ \tilde{\chi}_i(t) = \mathbf{E}\tilde{\psi}(t, \xi)h_i, \quad \tilde{\kappa}_i(t) = \mathbf{E}f(t, \tilde{\psi}(t, \xi), y(t, s))h_i. \]

**Lemma 4.1.** For all \( t \in [0, T] \) and \( c \in \mathbb{R}^m \), the following statements hold:

\[ \chi_i(t) = cE_\alpha(A(T - t)\alpha) \mathbf{E}x + \int_{t}^{T} (s - t)^{\alpha - 1} E_{\alpha, \alpha}(A(s - t)\alpha) \kappa_i(s)ds \]
\[ + \int_{t}^{T} (s - t)^{\alpha - 1} E_{\alpha, \alpha}(A(s - t)\alpha) \mathbf{E}\theta_i(g(s, u, \psi(s, \xi)) + y(t, s)) dw(s) \]  
(4.2)

and

\[ \tilde{\chi}_i(t) = cE_\alpha(A(T - t)\alpha) \mathbf{E}x + \int_{t}^{T} (s - t)^{\alpha - 1} E_{\alpha, \alpha}(A(s - t)\alpha) \tilde{\kappa}_i(s)ds \]
\[ + \int_{t}^{T} (s - t)^{\alpha - 1} E_{\alpha, \alpha}(A(s - t)\alpha) \mathbf{E}\theta_i(g(s, u, \tilde{\psi}(s, \xi)) + y(t, s)) dw(s), \quad P-a.s. \]  
(4.3)

**Proof.** Since \( \psi(t, \xi) \) is a solution of (1.2), it follows that

\[ \psi(t, \xi) = \xi - \frac{1}{\Gamma(\alpha)} \int_{t}^{T} A(s - t)^{\alpha - 1} \psi(s, \xi)ds \]
\[ + \frac{1}{\Gamma(\alpha)} \int_{t}^{T} (s - t)^{\alpha - 1} f(s, \psi(s, \xi), y(t, s))ds \]
\[ + \frac{1}{\Gamma(\alpha)} \int_{t}^{T} (s - t)^{\alpha - 1} [g(t, s, \psi(s, \xi)) + y(t, s)] dw(s). \]  
(4.4)

Taking product of both sides of (4.2) with \( h_i \) and then taking expectation of both sides yield that

\[ \chi_i(t) = cE\xi - \frac{1}{\Gamma(\alpha)} \int_{t}^{T} A(s - t)^{\alpha - 1} \chi_i(s)ds \]
\[ + \frac{1}{\Gamma(\alpha)} \int_{t}^{T} (s - t)^{\alpha - 1} \kappa_i(s)ds \]
\[ + \frac{1}{\Gamma(\alpha)} \int_{t}^{T} (s - t)^{\alpha - 1} [g(t, s, \psi(s, \xi)) + y(t, s)] dw(s), \int_{0}^{T} \theta_i(s) dw(s). \]
Proof. To proof Remark 4.2, we start with the following inequality:

\[
\chi_i(t) = cE\xi - \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} \left( A\chi_i(s) - \kappa_i(s) - \mathbf{E}\theta_i(s) [g(t,s,\psi(t,\xi)) + y(t,s)] \right) ds
\]

such that \( \chi_i(t) \) is a solution of the following fractional stochastic backward differential equation:

\[
\frac{C}{T} D_t^\alpha x(t) = Ax(t) - \kappa_i(t) - \mathbf{E}\theta_i(t) [g(t,s,\psi(t,\xi)) + y(t,s)], \quad x(T) = cE\xi. \quad \text{(4.5)}
\]

Then, by means of Remark 4.1, (4.2) is proved.

Next, we define

\[
\tilde{\psi}(t,\xi) = E_\alpha(A(T-t)^\alpha)\xi + \int_t^T (s-t)^{\alpha-1} E_\alpha(A(s-t)^\alpha)f(s,\psi(s,\xi),y(t,s))ds \\
+ \int_t^T (s-t)^{\alpha-1} E_\alpha(A(s-t)^\alpha) \left[ g(t,s,\tilde{\psi}(s,\xi)) + y(t,s) \right] dw(s), \quad P - a.s. \quad \text{(4.6)}
\]

Similarly, by taking product of both sides of (4.6) with \( h_i \) and then taking expectation of both sides follow that

\[
\tilde{\chi}_i(t) = cE_\alpha(A(T-t)^\alpha)E\xi + \int_t^T (s-t)^{\alpha-1} E_\alpha(A(s-t)^\alpha)\tilde{\kappa}_i(s)ds \\
+ \left\{ \int_t^T (s-t)^{\alpha-1} E_\alpha(A(s-t)^\alpha) \left[ g(t,s,\tilde{\psi}(s,\xi)) + y(t,s) \right] dw(s), \int_0^T \theta_i(s)dw(s) \right\}.
\]

Applying Itô’s isometry theorem, (4.3) is proved:

\[
\tilde{\chi}_i(t) = cE_\alpha(A(T-t)^\alpha)E\xi + \int_t^T (s-t)^{\alpha-1} E_\alpha(A(s-t)^\alpha)\tilde{\kappa}_i(s)ds \\
+ \int_t^T (s-t)^{\alpha-1} E_\alpha(A(s-t)^\alpha)E\theta_i(s) \left[ g(t,s,\tilde{\psi}(s,\xi)) + y(t,s) \right] dw(s), \quad \text{P-a.s.}
\]

Therefore, \( \tilde{\chi}_i(t) \) is a solution of the following fractional stochastic backward differential equations:

\[
\frac{C}{T} D_t^\alpha x(t) = Ax(t) - \tilde{\kappa}_i(t) - E\theta_i(t) \left[ g(t,s,\tilde{\psi}(t,\xi)) + y(t,s) \right], \quad x(T) = cE_\alpha(A(T-t)^\alpha)E\xi. \quad \text{(4.7)}
\]

Remark 4.2. For any \( h \in L^2(\Omega,\mathcal{F}_T,\mathbb{R}^m) \), we have

\[
|E(\psi(t,\xi) - \tilde{\psi}(t,\xi),h)|^2 = m \sum_{i=1}^m \|E(\psi(t,\xi) - \tilde{\psi}(t,\xi))h_i\|^2 \\
\leq 4mcM_{\alpha,\alpha}^2 (T-t)\frac{2^{\alpha-1}}{2\alpha - 1} \int_t^T \|\psi(s,\xi) - \tilde{\psi}(s,\xi)\|^2 ds E\|h\|^2 \\
+ 4mcM_{\alpha,\alpha}^2 \int_t^T (s-t)^{2\alpha-2} E\|\psi(s,\xi) - \tilde{\psi}(s,\xi)\|^2 ds E\|h\|^2. \quad \text{(4.7)}
\]

Proof. To proof Remark 4.2, we start with the following inequality:

\[
|\langle \psi(t,\xi) - \tilde{\psi}(t,\xi),h \rangle|^2 = \sum_{i=1}^m |E(\psi(t,\xi) - \tilde{\psi}(t,\xi),h_i)|^2 \quad \text{(4.8)}
\]
First, we estimate $\|\chi_i(t) - \tilde{\chi}_i(t)\|$ as below:

$$
\|\chi_i(t) - \tilde{\chi}_i(t)\| \leq M_{\alpha,\alpha} \int_t^T (s-t)^{\alpha-1} \|\kappa_i(s) - \tilde{\kappa}_i(s)\| ds
+ m M_{\alpha,\alpha} \int_t^T (s-t)^{\alpha-1} E(\|\theta_i(s)\|\|\psi(s,\xi) - \tilde{\psi}(s,\xi)\|) ds.
$$

Applying Cauchy-Schwarz inequality yields that

$$
\|\chi_i(t) - \tilde{\chi}_i(t)\| \leq M_{\alpha,\alpha} \sqrt{\frac{(T-t)^{2\alpha-1}}{2\alpha - 1}} \left( \int_t^T (s-t)^{\alpha-1} \|\kappa_i(s) - \tilde{\kappa}_i(s)\|^2 ds \right)^{\frac{1}{2}}
+ m M_{\alpha,\alpha} \left( \int_t^T E(\|\theta_i(s)\|^2 ds \right)^{\frac{1}{2}} \left( \int_t^T (s-t)^{2\alpha-2} E(\|\psi(s,\xi) - \tilde{\psi}(s,\xi)\|^2 ds \right)^{\frac{1}{2}}.
$$

By definition of $\kappa$ and $\tilde{\kappa}$ for all $s \in [t, T]$,

$$
\|\kappa_i(s) - \tilde{\kappa}_i(s)\|^2 = \|E(f(s,\psi(s,\xi),y(t,s)) - f(s,\tilde{\psi}(s,\xi),y(t,s)))h_i\|^2
= \sum_{i=1}^m \|E(f_i(s,\psi(s,\xi),y(t,s)) - f_i(s,\tilde{\psi}(s,\xi),y(t,s)))h_i\|^2
\leq \sum_{i=1}^m E\|f_i(s,\psi(s,\xi),y(t,s)) - f_i(s,\tilde{\psi}(s,\xi),y(t,s))\|^2 E\|h_i\|^2
= E\|f(s,\psi(s,\xi),y(t,s)) - f(s,\tilde{\psi}(s,\xi),y(t,s))\|^2 E\|h_i\|^2
\leq c E\|\psi(s,\xi) - \tilde{\psi}(s,\xi)\|^2 E\|h_i\|^2.
$$

Then, we make use of above estimation in (4.9), we obtain

$$
\|\chi_i(t) - \tilde{\chi}_i(t)\| \leq m M_{\alpha,\alpha} \sqrt{\frac{(T-t)^{2\alpha-1}}{2\alpha - 1}} \left( \int_t^T E(\|\psi(s,\xi) - \tilde{\psi}(s,\xi)\|^2 ds \right)^{\frac{1}{2}} \left( E\|h_i\|^2 \right)^{\frac{1}{2}}
+ m M_{\alpha,\alpha} \left( \int_t^T E(\|\theta_i(s)\|^2 ds \right)^{\frac{1}{2}} \left( \int_t^T (s-t)^{2\alpha-2} E(\|\psi(s,\xi) - \tilde{\psi}(s,\xi)\|^2 ds \right)^{\frac{1}{2}}.
$$

Now we take expectation of (4.8) and plug (4.9) into (4.8), we attain desired result as follows:

$$
E(\psi(t,\xi) - \tilde{\psi}(t,\xi),h_i)^2 = \sum_{i=1}^m E(\psi(t,\xi) - \tilde{\psi}(t,\xi),h_i)^2
\leq m \sum_{i=1}^m \|E(\psi(t,\xi) - \tilde{\psi}(t,\xi))h_i\|^2.
$$
Using Remark \((4.2)\), we attain
\[
\leq m \sum_{i=1}^m \|E(\psi(t, \xi) - \tilde{\psi}(t, \xi))h_i\|^2
\]
\[
\leq 4mcM_{\alpha,a}^2 \frac{(T-t)^{2\alpha-1}}{2\alpha-1} \int_t^T E\|\psi(s, \xi) - \tilde{\psi}(s, \xi)\|^2 ds E\|h\|^2
\]
\[
+ 4cmM_{\alpha,a}^2 \int_t^T (s-t)^{2\alpha-2} E\|\psi(s, \xi) - \tilde{\psi}(s, \xi)\|^2 ds E\|h\|^2,
\]
which completes the proof.

**Proof of Theorem 4.1.** Let \(T^* = \inf \{t \in [0, T] ; \psi(t, \xi) \neq \tilde{\psi}(t, \xi)\}\). Then it is sufficient to show that \(T^* = T\).

Suppose the contrary : \(T^* < T\). Choose and fix an arbitrary \(\delta > 0\) satisfying the following expression:
\[
4mcM_{\alpha,a}^2 \frac{(T-t)^{2\alpha-1}}{2\alpha-1} \delta + 4mcM_{\alpha,a}^2 \frac{\delta^{2\alpha-1}}{2\alpha-1} < 1. \tag{4.11}
\]

To lead contradiction, we show that \(\psi(t, \xi) = \tilde{\psi}(t, \xi)\) for all \(t \in [T^* - \delta, T^*]\). Using Ito’s representation, there exists a unique \(h \in \mathbb{L}^2\) such that \(\psi(t, \xi) - \tilde{\psi}(t, \xi) = h\). Therefore, we have
\[
E\|\psi(t, \xi) - \tilde{\psi}(t, \xi)\|^2 = E\|h\|^2.
\]

Using Remark \((4.2)\), we attain
\[
E\|\psi(t, \xi) - \tilde{\psi}(t, \xi)\|^2 \leq 4mcM_{\alpha,a}^2 \frac{(T-t)^{2\alpha-1}}{2\alpha-1} \int_t^{T^*} E\|\psi(s, \xi) - \tilde{\psi}(s, \xi)\|^2 ds
\]
\[
+ 4mcM_{\alpha,a}^2 \int_t^{T^*} (s-t)^{2\alpha-2} E\|\psi(s, \xi) - \tilde{\psi}(s, \xi)\|^2 ds.
\]

As a consequence,
\[
\sup_{t \in [T^* - \delta, T^*]} E\|\psi(t, \xi) - \tilde{\psi}(t, \xi)\|^2 \leq \left[ 4mcM_{\alpha,a}^2 \frac{(T-t)^{2\alpha-1}}{2\alpha-1} \delta + 4mcM_{\alpha,a}^2 \frac{\delta^{2\alpha-1}}{2\alpha-1} \right]
\]
\[
\times \sup_{t \in [T^* - \delta, T^*]} E\|\psi(t, \xi) - \tilde{\psi}(t, \xi)\|^2.
\]

By selecting \(\delta\) as in \((4.11)\), we have \(\sup_{t \in [T^* - \delta, T^*]} E\|\psi(t, \xi) - \tilde{\psi}(t, \xi)\|^2 = 0\). This leads to a contradiction and the proof is complete.

## 5 Conclusions and future works

In this paper, we first formulated new problem in BSDE theory so-called Caputo fBSDE which is an untreated topic in recent literature. We recalled some definitions from stochastic and fractional calculus and present a new weighted maximum norm in square-integrable measurable space. To derive adapted pair of stochastic processes, we first established fundamental lemma which plays a crucial role in the theory of Caputo fBSDE. The main results in our paper were to show global existence and uniqueness of adapted solution to \((1.3)\) in finite dimensional setting with the help of aforementioned fundamental lemma. The key point in the proof of this lemma was to apply extended martingale representation theorem and some inequalities from stochastic calculus. Last but not least, we proved the coincidence between the notion of integral equation and mild solution using martingale representation theorem.

Since our results are sufficiently new in the theory of BSDEs, there are still open problems to discuss related to fractional stochastic control theory and risk sensitive control problems. If we consider the same statement of our results shown above, an interesting problem in Caputo fBSDE is for multi-dimensional case: \(m \geq 2\). The problem of existence with coefficient \(f\) only continuous in \((x, y)\) becomes very hard. Similar problem appears in situation of the quadratic growth condition. Successfully applied techniques in one-dimensional case fails here due to the lack of comparison theorems. In general, multi-dimensional case has many interesting applications. For instance, one can apply Caputo fBSDE to optimal control theory using stochastic maximum principle.
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