Conformal Covariantization of Moyal-Lax Operators

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Abstract

A covariant approach to the conformal property associated with Moyal-Lax operators is given. By identifying the conformal covariance with the second Gelfand-Dickey flow, we covariantize Moyal-Lax operators to construct the primary fields of one-parameter deformation of classical W-algebras.

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I. INTRODUCTION

Recently, there has been a great deal of interest to study the Moyal deformation of the KdV equations in variant ways, such as Lax and/or Hamiltonian formulations \[1,5\], zero-curvature formulation \[6\], Bäcklund transformation \[7\] and classical Virasoro and \(W\)-algebras \[3,4\], etc. In these formulations, the ordinary (pseudo-) differential Lax operator \(L = \sum_i u_i(x) \partial^i\) is replaced by (pseudo-) differential symbols \(M(x,p)\), the formal Laurent series in \(p\), which obey a noncommutative but associative algebra with respect to the \(\star\)-product \[8\] defined by

\[
M(x,p) \star N(x,p) = \sum_{s=0}^{\infty} \frac{\theta^s}{s!} \sum_{j=0}^{s} (-1)^j \binom{s}{j} (\partial_x^j \partial_p^{s-j} M)(\partial_x^{s-j} \partial_p^j N),
\]

where \(\theta\) is a dimensionless parameter characterizing the strength of the deformation. On the other hand, by (1.1), the ordinary commutator is thus taken over by the Moyal bracket \[9\]

\[
\{M(x,p), N(x,p)\}_\theta = \frac{M \star N - N \star M}{2\theta},
\]

that possesses the anti-symmetry, bi-linearity and Jacobi identity. The Moyal bracket (1.2) can be viewed as the higher-order derivative (or dispersive) generalization of the canonical Poisson bracket since it recovers the canonical Poisson bracket in the limit \(\theta \to 0\), namely, \(\lim_{\theta \to 0} \{M, N\}_\theta = \partial_p M \partial_x N - \partial_x M \partial_p N\). It turns out that the Moyal formulation of Lax equations reduces to dispersionless Lax equations \[10-14\] in this limit.

In the previous work \[3\] we have studied the \(W\)-algebraic structure associated with the Moyal formulation of the KdV equations. We worked out the Poisson brackets of the second Gelfand-Dickey (GD) structure \[15,16\] defined by the \(\star\)-product and obtained an one-parameter deformation of the classical \(W_n\)-algebra including a Virasoro subalgebra with central charge \(\theta^2(n^3 - n)/3\). In this work, we would like to investigate further the \(W\)-algebraic structure from the point of view of conformal covariance. We shall follow the approach developed by Di Francesco, Itzykson and Zuber (DIZ) \[17\] to covariantize the Moyal-Lax operators \[1,4\] and identify the underlying primary fields in a systematic way.

This paper is organized as follows. In section II, we recall the Moyal-Lax formulation of the KdV equations using pseudo-differential symbols with respect to the \(\star\)-product. We then introduce the second GD structure defined by the Moyal bracket and show that it indeed provides the Hamiltonian structure for the Moyal-type Lax equations. In section III, the diffeomorphism \((S^1)\) is defined and the conformal transformation of the Moyal-Lax operators is investigated. Then in section IV, we show that the infinitesimal diffeomorphism flow defined by conformal covariance is equivalent to that of the Hamiltonian flow defined by the second GD structure. This
enables us to define the primary fields of the diffeomorphism. Following DIZ, in section V, we systematically covariantize the Moyal-Lax operator to decompose the coefficient functions of the Lax operator into the conformal primary fields which satisfy an one-parameter deformation of the classical $W_n$-algebra including a Virasoro subalgebra. In section VI the covariantization is generalized to pseudo-differential symbols to construct more additional primary fields. Section VII is devoted to the conclusions and discussions.

II. LAX EQUATIONS AND HAMILTONIAN STRUCTURES

For the differential symbol $L = p^n + \sum_{i=1}^n u_i \star p^{n-i}$ with coefficients $u_i$ depending on an infinite set of variables $x \equiv t_1, t_2, t_3, \cdots$ one can define the Lax equations \[2.1\]

$$\frac{\partial L}{\partial t_k} = \{(L^{1/n})^k_+, L\}_\theta, \quad (L^{1/n})^k_+ = (L_1^{1/n} \star L_1^{1/n} \star \cdots \star L_1^{1/n})_k,$$

where $L_1^{1/n} = p + \sum_{i=0}^n a_i \star p^{-i}$ is the $n$th root of $L$ in such a way that $L = (L_1^{1/n})^n$ and $(A)_{+/-}$ refer to the non-negative/negative powers in $p$ of the pseudo-differential symbol $A$. Note that the evolution equation for $u_1$ is trivial since the highest order in $p$ on the right-hand side of the Lax equations \[2.1\] is $n - 2$ due to the definition of the Moyal bracket, and hence one can drop $u_1$ in the Lax formulation. However, we will see that this is not the case for the Hamiltonian formulation.

Next let us formulate the Lax equations \[2.1\] in terms of Hamiltonian structure. For the functionals $F[L]$ and $G[L]$ we define the second Gelfand-Dickey bracket \[16\] with respect to the $\star$-product as

$$\{F, G\}_2 = \text{tr}[J^{(2)}(d_L F) \star d_L G] = \int \text{res}[J^{(2)}(d_L F) \star d_L G], \quad (2.2)$$

where $\text{res}(A) = a_{-1}$ and $\text{tr}(A) = \int \text{res}(A)$ denote the residue and trace of $A = \sum_i a_i \star p^i$, and $J^{(2)}$ is the Adler map \[12\] defined by

$$J^{(2)}(d_L F) = \{L, d_L F\}_\theta \star L - \{L, (d_L F \star L)_+\}_\theta,$$

$$\{L, (d_L F \star L)_-\}_\theta - \{L, d_L F\}_\theta \star L, \quad (2.3)$$

with $d_L F \equiv \delta F/\delta L = \sum_{i=1}^n p^{-n+i-1} \star \delta F/\delta u_i$. The bracket defined by $J^{(2)}$ is indeed Hamiltonian since $\{F, G\}_2 = -\{G, F\}_2$ due to the cyclic property of the trace and the Jacobi identity can be verified \[7\] by the Kupershmidt-Wilson (KW) theorem \[19\]. Form \[2.3\] $J^{(2)}(X)$ is linear in $X$ and has order at most $n - 1$. One can use the standard Dirac procedure \[17\] to get rid of $u_1$ so that
\[ \hat{j}^{(2)}(X) = \{L, X\}_{\theta^+} \ast L - \{L, (X \ast L)_{\theta^+}\} + \frac{1}{n} \{L, \int^{x} \text{res}\{L, X\}_{\theta^+}\} \]  

(2.4)

or, in components, \( \hat{j}^{(2)}(X) = \sum_{i,j=2}^{n} (\hat{j}^{(2)}_{ij} \cdot x_j) \ast p^{n-i} \) where \( \hat{j}^{(2)}_{ij} \) are differential operators, and hence the reduced Poisson brackets for \( u_i \) can be expressed as \( \{u_i(x), u_j(y)\}_{D} = \hat{j}^{(2)}_{ij} \ast \delta(x - y) \). From the reduced GD brackets (2.4) the Hamiltonian flows can be expressed as

\[ \frac{\partial L}{\partial t_k} = \{L, H_k\}_{D} = \hat{j}^{(2)}(d_L H_k), \]  

(2.5)

where the Hamiltonians \( H_k \) are defined by

\[ H_k = \frac{n}{k} \int \text{res}\{L^{1/n} \ast\}^k. \]  

(2.6)

Using (2.6) and the fact \( d_L H_k = (L^{1/n} \ast)^{k-n} \) mod \( p^{-n} \) it is straightforward to show that the Hamiltonian flows (2.5) are equivalent to the Lax equations (2.1).

### III. DIFF(S^1) AND CONFORMAL COVARIANCE

A function \( f(x) \) is a primary field with conformal weight \( h \) if under the diffeomorphism \( x \rightarrow t(x) \) it transforms as

\[ f(x) \rightarrow \tilde{f}(t) = \phi^{-h} f(x) = \phi^{-h} \ast f(x), \]  

(3.1)

where \( \phi(x) \equiv dt(x)/dx \). We denote \( F_h \) the space of functions with weight \( h \) (or spin-\( h \) primary fields). For a covariant operator \( \Delta(x, p) \) that maps \( F_h \) to \( F_l \), then it transforms according to

\[ \tilde{\Delta}(t, p_t) = \phi^{-t} \ast \Delta(x, p) \ast \phi^h, \]  

(3.2)

where \( p_t = \phi^{-1} \ast p \) is the conjugate momentum of \( t \) with respect to the Moyal bracket, i.e. \( \{p_t, t\}_\theta = 1 \) and has an inverse \( p_t^{-1} = p^{-1} \ast \phi \).

Let us treat the Lax operator \( L_n(x, p) = p^n + u_2(x) \ast p^{n-2} + \cdots + u_n(x) \) as a covariant operator such that \( L_n(x, p) : F_h \rightarrow F_l \). The corresponding weights \( h \) and \( l \) have to be determined from the transformation law:

\[ \tilde{L}_n(t, p_t) = \phi^{-t} \ast L_n(x) \ast \phi^h, \]  

(3.3)

We note that \( p_t = \phi^{-1} \ast p = (\sqrt{\phi})^{-1} \ast \phi^{-1} p \ast \sqrt{\phi} \) which, by induction, gives

\[ (p_t)^k = \frac{1}{\sqrt{\phi}} \ast [\phi^{-k} p^k + \frac{\theta^2 f_k}{\phi^k} p^{k-2} + \cdots] \ast \sqrt{\phi}, \]  

(3.4)
with
\[ f_k = -\frac{k(k-1)}{2} \left( \frac{\phi'}{\phi} \right)^2 - \frac{k(k-1)(k-2)}{6} \frac{\phi''}{\phi}. \]

Substituting (3.4) into (3.3) we have
\[ h = \frac{n-1}{2}, \quad l = \frac{n+1}{2} \quad \text{and} \quad u_2(x) \]
transforms like an anomalous spin-2 primary field
\[ \tilde{u}_2(t) = \phi^{-2} u_2(x) + \frac{\theta^2(n^3-n)}{3} \{ \{ x, t(x) \} \} , \quad (3.5) \]
where \( \{ \{ x, t(x) \} \} \) is the schwarzian derivative defined by
\[ \{ \{ x, t(x) \} \} = \left( \frac{\phi'''}{\phi^3} \right) - \left( \frac{\phi''}{\phi^2} \right)^2 = \phi''' - \frac{3}{2} \left( \frac{\phi'}{\phi^2} \right)^2. \quad (3.6) \]

Eq. (3.5) indicates that \( u_2 \) can be viewed as the generator of the classical Virasoro algebra with central charge \( c_{n,\theta} = \frac{\theta^2(n^3-n)}{3} \).

**IV. VIRASORO FLOWS AS HAMILTONIAN FLOWS**

As we shown in the previous section that it is quite difficult to obtain the transformation laws for \( u_{i>2} \) under the finite diffeomorphism. However it is manageable to investigate the infinitesimal transformations of \( u_i \). For an infinitesimal diffeomorphism \( x \rightarrow t(x) \approx x - \epsilon(x) \) we have \( \phi(x) \approx 1 - \epsilon'(x) \) and \( p_t = p + \{ p, \epsilon \}_\theta \star p \). In particular, it can be easily proved by induction that \( (p_t^* i = p^i + \{ p^i, \epsilon \}_\theta \star p \). Hence from (3.3) we have
\[ \tilde{L}_n(t) = \sum_i (u_i(x) - \epsilon(x) u_i'(x) + \delta_i u_i(x)) \star (p^i + \{ p^i, \epsilon \}_\theta \star p), \]
\[ = L_n(x) + \{ L_n(x), \epsilon(x) \}_\theta \star p - \epsilon(x) \star L_n(x) + \delta_i L_n(x), \]
\[ = \left( 1 + \frac{n+1}{2} \epsilon'(x) \right) \star L_n(x) \star \left( 1 + \frac{n-1}{2} \epsilon'(x) \right), \]
\[ = L_n(x) + \frac{n+1}{2} \epsilon'(x) \star L_n(x) + \frac{n-1}{2} L_n(x) \star \epsilon'(x) , \]
which leads to the infinitesimal change of the Lax operator
\[ \delta_i L_n(x) = \frac{n+1}{2} \epsilon'(x) \star L_n(x) + \frac{n-1}{2} L_n(x) \star \epsilon'(x) \]
\[ - \{ L_n(x), \epsilon(x) \}_\theta \star p + \epsilon(x) \star \{ p, L_n(x) \}_\theta. \quad (4.1) \]

Next let us consider the Hamiltonian flow generated by the Hamiltonian \( H = \int \epsilon(x) u_2(x) dx \). From the second GD structure (2.4) and Hamiltonian flow (2.5) we have
\[ \delta^{GD} L_n(x) = \{ L_n(x), X \}_\theta * L_n(x) - \{ L_n(x), (X * L_n(x))_+ \}_\theta \]
\[ + \frac{1}{n} \{ L_n(x), \int^x \text{res} \{ L_n(x), X \}_\theta \}_\theta, \]

where \( X \equiv \delta H/\delta L = p^{-n+1} * \epsilon(x) \). A simple algebra shows that
\[ (L_n * X)_+ = p * \epsilon, \]
\[ (X * L_n)_+ = \epsilon * p - 2\theta(n-1)\epsilon', \]
\[ \frac{1}{n} \{ L_n, \int^x \text{res} \{ L_n, X \}_\theta \}_\theta = -\frac{n-1}{2} (L_n * \epsilon' - \epsilon' * L_n), \]

which implies
\[ \delta^{GD} L_n = \frac{1}{2\theta} \left[ p * \epsilon * L_n - L_n * (\epsilon * p - 2\theta(n-1)\epsilon') \right] - \frac{n-1}{2} (L_n * \epsilon' - \epsilon' * L_n), \]
\[ = \delta \epsilon L_n \]

as desired. Comparing the both hand sides of (4.1) we get the infinitesimal variations of \( u_k \) (\( 2 \leq k \leq n \)) as
\[ \delta \epsilon u_k = u_k' \epsilon + k u_k \epsilon' + \frac{(2\theta)^k (k-1)}{2} \left( \frac{n+1}{k+1} \right) \epsilon^{(k+1)} \]
\[ + \sum_{i=2}^{k-1} (2\theta)^{k-i} \left[ \frac{n-1}{2} \left( \frac{n-i}{k-i} \right) - \left( \frac{n-i}{k-i+1} \right) \right] u_i \epsilon^{(k-i+1)}, \]  
(4.2)

where \( \binom{n}{m} \) are the standard binomial coefficients with \( 0 \leq m \leq n \). Let us list the first few \( \delta \epsilon u_k \):

\[ \delta \epsilon u_2 = u_2' \epsilon + 2 u_2 \epsilon' + \frac{\theta^2 (n^3 - n)}{3} \epsilon^{''}, \]
\[ \delta \epsilon u_3 = u_3' \epsilon + 3 u_3 \epsilon' + 2\theta(n-2)u_2 \epsilon'' + \frac{\theta^3 (n^3 - n)(n-2)}{3} \epsilon^{(4)}, \]
\[ \delta \epsilon u_4 = u_4' \epsilon + 4 u_4 \epsilon' + 3\theta(n-3)u_3 \epsilon'' + \frac{\theta^2 (n-2)(n-3)(n+5)}{3} u_2 \epsilon^{''} \]
\[ + \frac{\theta^4 (n-2)(n-3)(n^3 - n)}{5} \epsilon^{(5)}, \]
\[ \delta \epsilon u_5 = u_5' \epsilon + 5 u_5 \epsilon' + 4\theta(n-4)u_4 \epsilon'' + \frac{\theta^3 (n-3)(n-4)(n+7)}{3} u_3 \epsilon^{''} \]
\[ + \frac{\theta^2 (n-2)(n-3)(n-4)(n+3)}{3} u_2 \epsilon^{(4)} + \frac{4\theta^5 (n-2)(n-3)(n-4)(n^3 - n)}{45} \epsilon^{(6)}, \]  
(4.3)

etc. The first equation in (4.3) is just the infinitesimal version of (3.5) which together with the Hamiltonian flow \( \delta \epsilon u_2(x) = \{ u_2(x), H \}^D_2 = \int \{ u_2(x), u_2(y) \}^D_2 \epsilon(y) dy \) implies the classical Virasoro algebra.
\( \{ u_2(x), u_2(y) \}_2^D = [c_{n, \theta} \partial_x^2 + 2u_2 \partial_x + u_2'] \delta(x - y). \) (4.4)

Furthermore, it has a simple interpretation about the other relations in (4.3). We can define a new variable \( w_k = u_k + f(u_i) \), where \( f(u_i) \) is a differential polynomial in \( u_i < k \), such that \( w_k \) is a spin-\( k \) primary field with respect to the generator \( u_2 \), namely,

\[ \{ w_k(x), u_2(y) \}_2 = [kw_k \partial_x + w_k'] \delta(x - y). \]

For instance, let \( w_3 = u_3 + \alpha u_2' \) and demanding the relation \( \delta_\epsilon w_3 = \epsilon w_3' + 3w_3' \epsilon' \) then we get \( \alpha = -\theta(n - 2) \). On the other hand, let \( w_4 = u_4 + \alpha u_3' + \beta u_2'' + \gamma u_2 \) and demanding the relation \( \delta_\epsilon w_4 = \epsilon w_4' + 4w_4' \epsilon' \) we have \( \alpha = -\theta(n - 3) \), \( \beta = 2\theta^2(n - 2)(n - 3)/5 \) and \( \gamma = -(n - 2)(n - 3)(5n + 7)/[10(n^3 - n)] \).

In summary, we can identify the following primary fields

\[
\begin{align*}
   w_3 & = u_3 - \theta(n - 2)u_2', \\
   w_4 & = u_4 - \frac{(n - 2)(n - 3)(5n + 7)}{10(n^3 - n)} u_2'' - \theta(n - 3)u_3' + \frac{2\theta^2(n - 2)(n - 3)}{5} u_2'', \\
   w_5 & = u_5 - \theta(n - 4)u_4' + \frac{3\theta^2(n - 3)(n - 4)}{7} u_3'' - \frac{2\theta^3(n - 2)(n - 3)(n - 4)}{21} u_2''' \\
   & + \frac{(n - 3)(n - 4)(7n + 13)}{7(n^3 - n)} [\theta(n - 2)u_2u_2' - u_2u_3],
\end{align*}
\]

(4.5)

etc. To construct the primary fields \( w_k \) for \( k > 5 \) we shall covariantize the Lax operator in a systematic way.

**V. COVARIANTIZING THE LAX OPERATORS**

For a series of change of variable \( v \rightarrow x \rightarrow t \), the schwarzian derivative obeys the equation

\[
\{ \{ v, t \} \} = \left( \frac{dx}{dt} \right)^2 \{ \{ v, x \} \} + \{ \{ x, t \} \},
\]

(5.1)

which, comparing with (3.3), shows that \( u_2(x) \) transforms as \( c_{n, \theta} \{ \{ v, x \} \} \). Define the variable \( b(x) = \frac{\partial u_2}{\partial x} (\frac{dx}{dt})^{-1} \) it turns out that, for \( n \neq -1, 0, 1 \) and \( \theta \neq 0 \)

\[
\frac{u_2(x)}{c_{n, \theta}} = \{ \{ v, x \} \} = b'(x) - \frac{1}{2} b^2(x),
\]

(5.2)

with \( v \) being the coordinate where \( u_2 \) vanishes, i.e. \( u_2(v) = 0 \). It is easy to show that \( b(x) \) transforms as an anomalous spin-1 primary field.
The purpose for introducing \(b(x)\) is to construct a covariant operator \(D_k = p - 2\theta kb(x)\) which maps \(\mathcal{F}_k\) to \(\mathcal{F}_{k+1}\). Using \(D_k\) the covariant operator \(D_k^l : \mathcal{F}_k \to \mathcal{F}_{k+l}\) can be constructed as \(D_k^l = D_{k+l-1} * D_{k+l-2} * \cdots * D_k(l > 1)\).

Now, following DIZ procedure, the Lax operator \(L_n\) can be decomposed into the sum of the covariant operators \(\Delta_k^{(n)} : \mathcal{F}_{n \frac{n-1}{2}} \to \mathcal{F}_{n \frac{n-1}{2}}\) as

\[
L_n = \Delta_2^{(n)}(u_2) + \Delta_3^{(n)}(w_3, u_2) + \cdots + \Delta_n^{(n)}(w_n, u_2),
\]

where

\[
\Delta_2^{(n)} = D_{n \frac{n-1}{2}} = [p - \theta(n - 1)b(x)] * [p - \theta(n - 3)b(x)] * \cdots * [p + \theta(n - 1)b(x)],
\]

\[
\Delta_k^{(n)} = \sum_{l=0}^{n-k} \alpha_{k,l}^{(n)}(D_k^l * w_k) * D_{n \frac{n-1}{2} - \frac{n-k-l}{2}},
\]

and the coefficients \(\alpha_{k,l}^{(n)}\) are determined from the requirement that the Lax operator \(L_n\) depends on \(u_2\) only through the relation (5.2). Therefore the function \(b(x)\) is defined up to the condition \((\delta b)' - \delta b = 0\) or equivalently, \(D_{k+1} \delta b = \delta b * D_k\). In particular we have

\[
\delta_b D_k^l = \sum_{i=1}^{l} D_{k+l-1} * \cdots * \delta_b D_{k+1} \cdots * D_k,
\]

\[
= \sum_{i=1}^{l} D_{k+l-1} * \cdots * [-2\theta(k + l - i)\delta b] * \cdots * D_k,
\]

\[
= -\theta l(2k + l - 1)\delta b * D_k^{l-1}.
\]

Hence \(\delta_b L_n = 0\) implies

\[
\delta_b D_{n \frac{n-1}{2}}^n + \sum_{k=3}^{n} \sum_{l=0}^{n-k} \left[ \alpha_{k,l}^{(n)}(D_k^l * w_k) * D_{n \frac{n-1}{2} - \frac{n-k-l}{2}} + \alpha_{k,l}^{(n)}(D_k^l * w_k) * \delta_b D_{n \frac{n-1}{2} - \frac{n-k-l}{2}} \right] = 0.
\]

From (5.3) it is easy to show that the first term in (5.6) vanishes. For those terms in summation we get the recursive relation

\[
\alpha_{k,l+1}^{(n)} = \frac{(k + l)(n - k - l)}{(2k + l)(l + 1)} \alpha_{k,l}^{(n)}, \quad k \geq 3
\]

which together with the normalization condition \(\alpha_{k,0}^{(n)} = 1\) yields
\[\alpha_{k,l}^{(n)} = \frac{(k+l-1)(n-k)}{(2k+l-1)}.\]

Let us work out the first few terms for the decomposition (5.4). A straightforward computation yields

\[
(D_k \ast w_k) = 2\theta(w'_k - kbw_k),
\]
\[
(D_k^2 \ast w_k) = 4\theta^2[w''_k - (2k + 1)bw'_k + (k(k + 1)b^2 - kb')w_k],
\]
\[
(D_k^3 \ast w_k) = 8\theta^3[w'''_k - 3(k + 1)bw'''_k - (3k + 1)b'w'_k + (3k^2 + 6k + 2)b^2w'_k - k(k + 1)(k + 2)b^3w_k + k(3k + 4)bb'w_k - kb''w_k],
\]

and

\[
D_{n-1}^n = p^n + u_2 \ast p^{n-2} + \theta(n - 2)u'_2 \ast p^{n-3} + \frac{3\theta^2(n - 2)(n - 3)}{5} u''_2 + \frac{(n - 2)(n - 3)(5n + 7)}{10(n^3 - n)} u'''_2 + \frac{\theta(n - 2)(n - 3)(n - 4)(5n + 7)}{15(n^3 - n)} u'''_2 + \frac{4\theta^3(n - 2)(n - 3)(n - 4)}{6(n^3 - n)} u''''_2 \ast p^{n-4} + \cdots,
\]

\[
D_{n-2}^{n-3} = p^{n-3} + 3\theta(n - 3)b \ast p^{n-4} + \frac{\theta^2(n - 3)(n - 4)(n + 7)}{3} b' - \frac{(n - 3)(n - 4)(n - 29)}{6} b^2 \ast p^{n-5} + \cdots,
\]

\[
D_{n-3}^{n-4} = p^{n-4} + 4\theta(n - 4)b \ast p^{n-5} + \cdots,
\]

\[
D_{n-4}^{n-5} = p^{n-5} + \cdots.
\]

Thus

\[
\Delta_2^{(n)}(u_2) = D_{n-1}^{n-1},
\]
\[
\Delta_3^{(n)}(w_2, u_2) = w_3 \ast p^{n-3} + \theta(n - 3)w'_3 \ast p^{n-4} + \frac{4\theta^2(n - 3)(n - 4)}{7} u''_3 + \frac{(n - 3)(n - 4)(7n + 13)}{7(n^3 - n)} u'''_2 \ast p^{n-5} + \cdots,
\]
\[
\Delta_4^{(n)}(w_4, u_2) = w_4 \ast p^{n-4} + \theta(n - 4)w'_4 \ast p^{n-5} + \cdots,
\]
\[
\Delta_5^{(n)}(w_5, u_2) = w_5 \ast p^{n-5} + \cdots,
\]

which decomposes the coefficient functions \(u_i\) into the primary fields

\[
u_2 = w_2,
\]
\[
u_3 = w_3 + \theta(n - 2)u'_2,
\]

\[9\]
\[
\begin{align*}
    u_4 &= w_4 + \theta(n-3)w'_4 + \frac{3\theta^2(n-2)(n-3)}{5} u''_2 + \frac{(n-2)(n-3)(5n+7)}{10(n^3-n)} u_2, \\
    u_5 &= w_5 + \theta(n-4)w'_4 + \frac{4\theta^2(n-3)(n-4)}{7} w'''_3 + \frac{(n-3)(n-4)(7n+13)}{7(n^3-n)} w_3u_2 \\
    &+ \frac{\theta(n-2)(n-3)(n-4)(5n+7)}{5(n^3-n)} u_2u'_2 + \frac{4\theta^3(n-2)(n-3)(n-4)}{15} u'''_2. \\
\end{align*}
\]

Inverting the above relation we recover the definition (4.3) of the primary fields.

VI. GENERALIZATIONS

In this section we would like to show that the conformal covariantization for the Lax operator (3.3) can be extended to a more general form

\[
\Lambda_n = p^n + u_2 \ast p^{n-2} + \cdots + u_n + u_{n+1} \ast p^{-1} + u_{n+2} \ast p^{-2} + \cdots. 
\]

(6.1)

It is not hard to show that, for the pseudo-differential symbol (6.1), the associated Hamiltonian structure is defined by the reduced Adler map (2.4) as well. Due to the fact that \((\Lambda_n)_+\) and \((\Lambda_n)_-\) are transformed independently under (3.5), the infinitesimal change of \(u_k(2 \le k \le n)\) is the same as (4.3), while that of \(u_k(k \ge n+1)\), governed by (4.4), yields

\[
\delta u_{n+k} = u'_{n+k} \epsilon + (n+k)u_{n+k} \epsilon' + \sum_{i=1}^{k-1} (2\theta)^{k-i} \left[ \frac{n-1}{2} \left( \frac{-i}{k-i} \right) - \left( \frac{-i}{k-i+1} \right) \right] u_{n+i} \epsilon^{(k-i+1)},
\]

where \((-n \choose m) \equiv (-1)^m \binom{n+m-1}{m}\) with \(n, m \ge 0\); from which, the following primary fields can be defined

\[
\begin{align*}
    w_{n+1} &= u_{n+1}, \\
    w_{n+2} &= u_{n+2} + \theta u'_{n+1}, \\
    w_{n+3} &= u_{n+3} + 2\theta u'_{n+2} + \frac{2\theta^2(n+1)}{2n+3} u''_{n+1} - \frac{6(n+1)}{n(n-1)(2n+3)} u_2u_{n+1}, \\
    w_{n+4} &= u_{n+4} + 3\theta u'_{n+3} + \frac{6\theta^2(n+2)}{2n+5} u''_{n+2} + \frac{2\theta^3(n+1)}{2n+5} u''_{n+1} \\
    &- \frac{6(3n+7)}{n(n-1)(2n+5)} u_2u_{n+2} - \frac{6(3n+7)}{n(n-1)(2n+5)} u_2u'_{n+1}, \\
\end{align*}
\]

etc. To covariantize the negative part \((\Lambda_n)_-\) one can define the covariant operator \(D^{-1}_{k}: \mathcal{F}_k \rightarrow \mathcal{F}_{k-1}\) as

\[
D^{-1}_{k} \equiv [D_{k-1}]^{-1} = p^{-1} + 2\theta(k-1) b \ast p^{-2} + \cdots, 
\]

(6.3)
and thus

\[ D^{-l}_k = [D^{-l}_{k-l}]^{-1} = D^{-1}_{k-l-1} \ast D^{-1}_{k-l} \cdots \ast D^{-1}_k, \quad (6.4) \]

with a covariant property determined by that of \( D^{-l}_{k-l} \) as

\[ D^{-l}_k(t) = [D^{-l}_{k-l}(t)]^{-1} = \phi^{l-k} \ast D^{-l}_k(x) \ast \phi^k. \]

Now let us decompose \((\Lambda_n)_-\) as

\[ (\Lambda_n)_- = \sum_{l=1}^{\infty} \Delta(n)_{n+k}(w_{n+k}, u_2), \quad (6.5) \]

where the covariant operator \( \Delta(n)_{n+k}(w_{n+k}, u_2) \) is linear in \( w_{n+k} \) and is defined by

\[ \Delta(n)_{n+k}(w_{n+k}, u_2) = \sum_{l=0}^{\infty} \beta^{(n)}_{n+k,l}(D^l_{n+k} \ast w_{n+k}) \ast D^{-l-1}_{k-1}, \quad k \geq 1. \]

The coefficients \( \beta^{(n)}_{n+k,l} \) can be determined in a similar manner so that \((\Lambda_n)_-\) depends on \( u_2 \) only through \((5.2)\). It turns out that

\[ \beta^{(n)}_{n+k,l} = (-1)^l \binom{k+l-1}{l} \binom{n+k+l-1}{l} \binom{2n+2k+l-1}{l}. \]

Following a similar procedure discussed in the previous section and comparing \((5.1)\) with \((5.5)\) we get

\[ u_{n+1} = w_{n+1}, \]
\[ u_{n+2} = w_{n+2} - \theta w'_{n+1}, \]
\[ u_{n+3} = w_{n+3} - 2\theta w''_{n+2} + \frac{2\theta^2(n+2)}{2n+3} w'''_{n+1} + \frac{6(n+1)}{n(n-1)(2n+3)} u_2 w_{n+1}, \]
\[ u_{n+4} = w_{n+4} - 3\theta w'''_{n+3} + \frac{6\theta^2(n+3)}{(2n+5)} w''''_{n+2} + \frac{6(3n+7)}{n(n-1)(2n+5)} u_2 w_{n+2} \]
\[ - \frac{2\theta^3(n+3)}{(2n+3)} w'''_{n+1} - \frac{18\theta(n+1)}{n(n-1)(2n+3)} (u_2 w_{n+1})'. \quad (6.6) \]

Inverting the above equations yield \((5.2)\) as expected.

VII. CONCLUSION AND DISCUSSIONS

We have discussed the covariance of the Moyal-type Lax operator under the diffeomorphism\((S^1)\). By comparing the infinitesimal \(\text{Diff}(S^1)\)-flow with the GD-flow
we have identified the primary fields with respect to the classical energy-momentum generator which obeys the classical Virasoro algebra with central charge \( c_{n,\theta} = \theta^2(n^3 - n)/3 \). We then follow the DIZ procedure to covariantize Moyal-Lax operators and identify the primary fields in a systematic way.

Few remarks are in order. First, the \( w_k \) shown above form an one-parameter deformation of the primary fields arising from the (pseudo-)differential Lax operator. In particular, the central charge \( c_{n,\theta} \) can be used to characterize the dispersion effect since \( \theta \rightarrow 0 \) corresponds to the dispersionless limit of the Lax equation (2.1). Secondly, for \( \theta = 1/2 \), the primary fields \( w_k \) recover the standard result \([17,20]\) while for \( \theta = 0 \) \([13]\) do not directly reproduce to those result in the dispersionless limit in which the coefficient functions \( u_k \) are already primary fields with respect to \( u_2 \), the generator of the centerless Virasoro algebra. This is due to the fact that the parametrization \([5.2]\) does not work for \( \theta = 0 \) and thus the associated conformal property should be traced back to the GD structure or infinitesimal transformation \([4.2]\). Thirdly, in spite of covariantizing the Lax operator \( L_n = p^n + \sum_{i=2}^n u_i * p^{n-i} \), the conformal property associated with the Lax operator of the form

\[
K_n = p^n + v_2 p^{n-2} + v_3 p^{n-3} + \cdots + v_n
\]

has been investigated \([3]\) as well. In fact, the Lax equations defined by \( K_n \) and \( L_n \) are equivalent up to the following isomorphism

\[
v_j = \sum_{i=1}^j (-\theta)^{j-i} \binom{n-i}{n-j} u_i^{(j-i)},
\]

which can be used to construct the primary fields associated with \( K_n \). For instance, from \([13,3]\) and \([7.2]\), the first few primary fields can be expressed as

\[
\begin{align*}
  w_2 &= v_2, \\
  w_3 &= v_3, \\
  w_4 &= v_4 - \frac{(n-2)(n-3)(5n+7)}{10(n^3-n)} v_2 - \frac{\theta^2(n-2)(n-3)}{10} v_2^2,
\end{align*}
\]

which are just those primary fields obtained in \([3]\).

Finally, based on the algebra of pseudo-differential symbols with respect to the \(*\)-product, it would be intriguing to carry out the covariant approach for reductions, truncations, and even supersymmetrization \([3]\) of the Lax operator \([6.1]\) to construct the corresponding \( W \)-algebras. Works in these directions are now in progress.

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