Surfaces of infinite-type are non-Hopfian

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Abstract

We show that finite-type surfaces are characterized by a topological analog of the Hopf property. Namely, an oriented surface $\Sigma$ is of finite-type if and only if every proper map $f : \Sigma \to \Sigma$ of degree one is homotopic to a homeomorphism.

1 Introduction

All surfaces will be assumed to be connected and orientable throughout this note. We will say a surface is of finite-type if its fundamental group is finitely generated; otherwise, we will say it is of infinite-type.

Recall that a group $G$ is said to be Hopfian if every surjective homomorphism $\varphi : G \to G$ is an isomorphism. It is well known that a finitely generated free group is Hopfian, for instance, as a consequence of Grushko’s theorem. On the other hand, a free group generated by an infinite set $S$ is not Hopfian as a surjective function $f : S \to S$ that is not injective extends to a surjective homomorphism on the free group generated by $S$ which is not injective.

In this note, we show that there is an analogous characterization for orientable surfaces of finite-type. The natural topological analog of a surjective group homomorphism is a proper map of degree one, and that of an isomorphism is a homotopy equivalence.

One-half of this characterization is classical, namely that any proper map of degree one from a surface of finite-type to itself is a homotopy equivalence. For instance, a theorem of Olum (see [2, Corollary 3.4]) says that every proper map of degree one between two oriented manifolds of the same dimension is $\pi_1$-surjective. Now, the fundamental group of any surface is residually finite (see [4]). Also, any finitely generated residually finite group is Hopfian. Thus, every degree one self map of a finite-type surface is a weak homotopy equivalence, hence a homotopy equivalence by Whitehead’s theorem.

Our main result is that infinite-type surfaces are not Hopfian.

**Theorem 1.1** Let $\Sigma$ be any infinite-type surface. Then there exists a proper map $f : \Sigma \to \Sigma$ of degree one such that $\pi_1(f) : \pi_1(\Sigma) \to \pi_1(\Sigma)$ is not injective. In particular, $f$ is not a homotopy equivalence.

2 Background

A surface is a connected, orientable two-dimensional manifold without boundary, and a bordered surface is a connected, orientable two-dimensional manifold with a non-empty boundary. A (possibly bordered) subsurface $\Sigma'$ of a surface $\Sigma$ is an embedded submanifold of codimension zero.

Let $\Sigma$ be a non-compact surface. A boundary component of $\Sigma$ is a nested sequence $P_1 \supseteq P_2 \supseteq \cdots$ of open, connected subsets of $\Sigma$ such that the followings hold:
• the closure (in \( \Sigma \)) of each \( P_n \) is non-compact,
• the boundary of each \( P_n \) is compact, and
• for any subset \( A \) with compact closure (in \( \Sigma \)), we have \( P_n \cap A = \emptyset \) for all large \( n \).

We say that two boundary components \( P_1 \supseteq P_2 \supseteq \cdots \) and \( P'_1 \supseteq P'_2 \supseteq \cdots \) of \( \Sigma \) are equivalent if for any positive integer \( n \) there are positive integers \( k_n, \ell_n \) such that \( P_{k_n} \subseteq P'_n \) and \( P'_{\ell_n} \subseteq P_n \). For a boundary component \( \mathcal{P} = P_1 \supseteq P_2 \supseteq \cdots \), we let \( [\mathcal{P}] \) to denote the equivalence class of \( \mathcal{P} \).

The space of ends \( \text{Ends}(\Sigma) \) of \( \Sigma \) is the topological space having equivalence classes of boundary components of \( \Sigma \) as elements, i.e., as a set \( \text{Ends}(\Sigma) := \{ [\mathcal{P}] | \mathcal{P} \) is a boundary component \}; with the following topology: For any set \( X \) with compact boundary, at first, define

\[
X^\dagger := \{ [\mathcal{P} = P_1 \supseteq P_2 \supseteq \cdots ] | X \supseteq P_n \supseteq P_{n+1} \supseteq \cdots \text{ for some large } n \}.
\]

Now, take the set of all such \( X^\dagger \) as a basis for the topology of \( \text{Ends}(\Sigma) \). The topological space \( \text{Ends}(\Sigma) \) is compact, separable, totally disconnected, and metrizable, i.e., homeomorphic to a non-empty closed subset of the Cantor set.

For a boundary component \([\mathcal{P}]\) with \( \mathcal{P} = P_1 \supseteq P_2 \supseteq \cdots \), we say \([\mathcal{P}]\) is planar if \( P_n \) are homeomorphic to open subsets \( \mathbb{R}^2 \) for all large \( n \). Define \( \text{Ends}_{\text{np}}(\Sigma) := \{ [\mathcal{P}] : [\mathcal{P}] \) is not planar \}; Thus, \( \text{Ends}_{\text{np}}(\Sigma) \) is a closed subset of \( \text{Ends}(\Sigma) \). Also, define the genus of \( \Sigma \) as \( g(\Sigma) := \sup g(S) \), where \( S \) is a compact bordered subsurface of \( \Sigma \).

**Theorem 2.1** (Kerékjártó’s classification theorem [7, Theorem 1]) Let \( \Sigma_1, \Sigma_2 \) be two non-compact surfaces. Then \( \Sigma_1 \) is homeomorphic to \( \Sigma_2 \) if and only if \( g(\Sigma_1) = g(\Sigma_2) \), and there is a homeomorphism \( \Phi: \text{Ends}(\Sigma_1) \rightarrow \text{Ends}(\Sigma_2) \) with \( \Phi(\text{Ends}_{\text{np}}(\Sigma_1)) = \text{Ends}_{\text{np}}(\Sigma_2) \).

Let \( \Sigma \) be a non-compact surface, and let \( \mathcal{E}_{\text{np}}(\Sigma) \subseteq \mathcal{E}(\Sigma) \) be two closed, totally-disconnected subsets of \( S^2 \) such that the pair \( \text{Ends}_{\text{np}}(\Sigma) \subseteq \text{Ends}(\Sigma) \) is homeomorphic to the pair \( \mathcal{E}_{\text{np}}(\Sigma) \subseteq \mathcal{E}(\Sigma) \). Consider a pairwise disjoint collection \( \{ D_i \subseteq S^2 \setminus \mathcal{E}(\Sigma) : i \in \mathcal{A} \} \) of closed disks, where \( |\mathcal{A}| = g(\Sigma) \), such that the following holds: For \( p \in S^2 \), any open neighborhood (in \( S^2 \)) of \( p \) contains infinitely many \( D_i \) if and only if \( p \in \mathcal{E}_{\text{np}}(\Sigma) \). The proof of [7, Theorem 2] describes constructing such a collection of disks.

Now, let \( M := (S^2 \setminus \mathcal{E}(\Sigma)) \cup \bigcup_{i \in \mathcal{A}} \text{int}(D_i) \) and \( N := \bigcup_{i \in \mathcal{A}} S_{1,1} \), where \( S_{1,1} \) is the genus one compact bordered surface with one boundary component. Define a non-compact surface \( \Sigma_{\text{handle}} \) as follows: \( \Sigma_{\text{handle}} := M \sqcup_{\partial M = \partial N} N \). Then we have the following theorem.

**Theorem 2.2** (Richards’ representation theorem [7, Theorems 2 and 3]) The surface \( \Sigma_{\text{handle}} \) is homeomorphic to \( \Sigma \).

### 3 Proof of Theorem 1.1

Let \( M \) and \( N \) be two non-compact, oriented, connected, boundaryless smooth \( n \)-manifolds. Then the singular cohomology groups with compact support \( H_c^k(M; \mathbb{Z}) \) and \( H_c^k(N; \mathbb{Z}) \) are infinite cyclic with preferred generators \([M]\) and \([N]\). If \( f: M \rightarrow N \) is a proper map then the degree of \( f \) is the unique integer \( \deg(f) \) defined as follows: \( H^k_c(f)([N]) = \deg(f) \cdot [M] \). Note that \( \deg \) is proper-homotopy invariant and multiplicative. See [2, Section 1] for more details.

We will use the following well-known characterization of degree.
Lemma 3.1 [2, Lemma 2.1b.] Let \( f : M \to N \) be a proper map between two non-compact, oriented, connected, boundaryless smooth \( n \)-manifolds. Let \( D \) be a smoothly embedded closed disk in \( N \) and suppose \( f^{-1}(D) \) is a smoothly embedded closed disk in \( M \) such that \( f \) maps \( f^{-1}(D) \) homeomorphically onto \( D \). Then \( \operatorname{deg}(f) = \pm 1 \) or \( -1 \) according as \( f|f^{-1}(D) \to D \) is orientation-preserving or orientation-reversing.

We will prove Theorem 1.1 by considering the following three cases:

1. \( \Sigma \) has infinite genus.

2. \( \Sigma \) has finite genus and the set of isolated points \( \mathcal{I}(\Sigma) \) of \( \mathcal{E}(\Sigma) \) is finite.

3. \( \Sigma \) has finite genus and the set of isolated points \( \mathcal{I}(\Sigma) \) of \( \mathcal{E}(\Sigma) \) is infinite.

Remark 3.2 If \( \Sigma \) is an infinite-type surface of a finite genus, then \( \mathcal{E}(\Sigma) \) is an infinite set.

Our first result proves Theorem 1.1 in the case with infinite genus.

Theorem 3.3 Let \( \Sigma \) be a surface of the infinite genus. Then there exists a degree one map \( f : \Sigma \to \Sigma \) which is not \( \pi_1 \)-injective.

Proof. Since \( \Sigma \) has infinite genus, there exists a compact bordered subsurface \( S \subset \Sigma \) such that \( S \) has genus one and one boundary component. Define \( \Sigma' \) as \( \Sigma' := \Sigma/S \), i.e., \( \Sigma' \) is the quotient of \( \Sigma \) with \( S \) pinched to a point. Let \( q : \Sigma \to \Sigma' \) be the quotient map. Thus, \( \Sigma' \) is also an infinite genus surface. Further, there are compact sets in \( K \subset \Sigma \) and \( K' \subset \Sigma' \) whose complements are homeomorphic, so the pair \( (\mathcal{E}(\Sigma), \mathcal{E}_{\text{np}}(\Sigma)) \) is homeomorphic to the pair \( (\mathcal{E}(\Sigma'), \mathcal{E}_{\text{np}}(\Sigma')) \). Hence, by Theorem 2.1, there is a homeomorphism \( \varphi : \Sigma' \to \Sigma \).

Let \( f : \Sigma \to \Sigma \) be the composition \( f = \varphi \circ q \). By Lemma 3.1, the quotient map \( q : \Sigma \to \Sigma' \) is of degree \( \pm 1 \). Thus, \( \operatorname{deg}(f) = \pm 1 \) as homeomorphisms have degree \( \pm 1 \). Notice that \( f \) sends \( \partial S \) to a point. But \( \partial S \) does not bound any disk in \( \Sigma \), i.e., \( \partial S \) represents a primitive element of \( \pi_1(\Sigma) \), see [1, Theorem 1.7. and Theorem 4.2.]. Hence, \( f \) is not \( \pi_1 \)-injective. If \( \operatorname{deg}(f) = 1 \), then we are done. Otherwise, we replace \( f \) by \( f \circ f \) to get a map that has degree one and is not injective on \( \pi_1 \).

For the remaining two cases, we use a map from the sphere to the sphere, which has degree \( \pm 1 \) but with some disks identified. We will replace these disks with appropriate surfaces to get \( \Sigma \).

Lemma 3.4 There exist pairwise disjoint closed disks \( D_0, D_1 \subseteq S^2 \) and a map \( f : S^2 \to S^2 \) such that the following hold:

- \( f^{-1}(D_0) = D_0 \) and \( f|_{D_0} : D_0 \to D_0 \) is the identity map.

- \( f^{-1}(D_1) \) is the union of pairwise-disjoint closed disks \( D_{1,1}, D_{1,2}, \text{ and } D_{1,3} \) in \( S^2 \); and \( f|_{D_{1,k}} : D_{1,k} \to D_1 \) is a homeomorphism for each \( k \in \{1, 2, 3\} \).

Further, there is a loop \( \gamma \) in \( S^2 \setminus \text{int}(D_0 \cup D_{1,1} \cup D_{1,2} \cup D_{1,3}) \) which is not homotopically trivial in \( S^2 \setminus \text{int}(D_0 \cup D_{1,1} \cup D_{1,2} \cup D_{1,3}) \), but such that \( f(\gamma) \) is null-homotopic in \( S^2 \setminus \text{int}(D_0 \cup D_1) \).
Proof. For each $k \in \{0, 1, 2, 3\}$, choose $(a_k, b_k) \in \mathbb{R}^2$ such that if we define
\[ B_k := \{(x, y) \in \mathbb{R}^2 : (x - a_k)^2 + (y - b_k)^2 \leq 1\}, \]
then $\{B_0, B_1, B_2, B_3\}$ is a pairwise-disjoint collection of closed disks.

Define $X := S^2 \setminus \bigcup_{j=0}^3 \text{int}(B_j)$ and $Y := S^2 \setminus \bigcup_{j=0}^1 \text{int}(B_j)$. Next, define a map $f: \partial X \to Y$ as follows:

- $f|_{\partial B_k}: \partial B_k \to \partial B_k$ is the identity map for each $k \in \{0, 1\}$;
- $f|_{\partial B_2}: \partial B_2 \to \partial B_1$ is defined as $f(x, y) := (-x + a_2 + a_1, y - b_2 + b_1)$ for all $(x, y) \in \partial B_2$.
- $f|_{\partial B_3}: \partial B_3 \to \partial B_1$ is defined as $f(x, y) := (x - a_3 + a_1, y - b_3 + b_1)$ for all $(x, y) \in \partial B_3$.

For each $k \in \{0, 1, 2\}$, let $\gamma_k: [0, 1] \to X$ be an embedding such that $\text{im}(\gamma_k) \cap \partial X$ consists of $\gamma_k(0) = (a_k + 1, b_k) \in \partial B_k$ and $\gamma_k(1) = (a_{k+1} - 1, b_{k+1}) \in \partial B_{k+1}$.

Define $\Gamma_0: [0, 1] \to Y$ as $\Gamma_0(t) := \gamma_0(t)$ for all $t \in [0, 1]$. Let $\Gamma_1, \Gamma_2: [0, 1] \to Y$ be the constant loops based at the points $(a_1 + 1, b_1) \in \partial Y$ and $(a_1 - 1, b_1) \in \partial Y$, respectively.

Next, define $X^{(1)} := \partial X \cup \text{im}(\gamma_0) \cup \text{im}(\gamma_1) \cup \text{im}(\gamma_2)$. Extend $f: \partial X \to Y$ to a map $X^{(1)} \to Y$, which we again denote by $f$: $X^{(1)} \to Y$, by mapping $\gamma_0$ onto $\Gamma_0$ by the identity, and, for each $k = 1, 2$, mapping $\gamma_k$ to the constant loop $\Gamma_k$.

Let $\theta_0$ (resp. $\theta_3$) be the simple loop that traverses $\partial B_0$ (resp. $\partial B_3$) in the counter-clockwise direction starting from $(a_0 + 1, b_0)$ (resp. $(a_3 - 1, b_3)$).

Let $\theta_{1,t}$ (resp. $\theta_{1,u}$) be the simple arc that traverses $\partial B_1 \cap \{y \leq b_1\}$ (resp. $\partial B_1 \cap \{y \geq b_1\}$) counter-clockwise direction. Similarly, define $\theta_{2,t}$ and $\theta_{2,u}$.

Now, $X \cong X^{(1)} \cup_{\varphi} \mathbb{D}^2$, (see Figure 1) where the attaching map $\varphi: S^1 \to X^{(1)}$ can be described as
\[ \varphi := \theta_0 * \gamma_0 * \theta_{1,t} * \gamma_1 * \theta_{2,t} * \gamma_2 * \theta_{3} * \gamma_3 * \theta_{1,u} * \gamma_1 * \theta_{1,u} * \gamma_0. \]

Notice that $f(\gamma_1) = \Gamma_1$ and $f(\gamma_2) = \Gamma_2$ are constant loops. Also, as in Figure 2, $\overline{f \circ \theta_{1,t}} = f \circ \theta_{2,t}$ and $\overline{f \circ \theta_{1,u}}$ is homotopic to $f \circ \theta_{2,u}$. Thus, $f \circ \varphi$ is homotopic to $(f \circ \theta_0) * \Gamma_0 * (f \circ \theta_3) * \Gamma_0$.

If $r: Y \cong S^1 \times [0, 1] \to S^1$ is the projection then $r \circ f \circ \theta_0$ and $r \circ f \circ \theta_3$ traverse $S^1$ in opposite directions. Since $r$ is a strong deformation retract, $(f \circ \theta_0) * \Gamma_0 * (f \circ \theta_3) * \Gamma_0$, and hence $f \circ \varphi$ is null-homotopic. Now, the null-homotopic map $f \circ \varphi: S^1 \to Y$ can be extended to a map $\mathbb{D}^2 \to Y$. Thus $f: X^{(1)} \to Y$ can be extended to a map $X \cong X^{(1)} \cup_{\varphi} \mathbb{D}^2 \to Y$, which will be again denoted by $f: X \to Y$.
Note that every homeomorphism $\mathbb{S}^1 \to \mathbb{S}^1$ can be extended to a homeomorphism $\mathbb{D}^2 \to \mathbb{D}^2$ naturally. Thus, we can extend $f': X \to Y$ to a map $\mathbb{S}^2 \to \mathbb{S}^2$, which will be again denoted by $f: \mathbb{S}^2 \to \mathbb{S}^2$. Let $D_0$ (resp. $D_1$) be any closed disk, which is contained in $\text{int}(B_0)$ (resp. $\text{int}(B_1)$).

Finally, observe that if $\gamma = \theta_{1u} \ast \theta_{1l} \ast \gamma_1 \ast \theta_{2l} \ast \theta_{2u} \ast \gamma_2$, then $\gamma$ is a loop in $\mathbb{S}^2 \setminus \text{int}(D_0 \cup D_{1,1} \cup D_{1,2} \cup D_{1,3})$ which is not homotopically trivial in $\mathbb{S}^2 \setminus \text{int}(D_0 \cup D_{1,1} \cup D_{1,2} \cup D_{1,3})$, but $f(\gamma)$ is null-homotopic in $\mathbb{S}^2 \setminus \text{int}(D_0 \cup D_1)$, as claimed.

We now prove Theorem 1.1 in the two remaining cases, in both of which we have a finite genus surface. Note that for a finite genus surface, all ends are planar, so in applying Theorem 2.1, it suffices to consider the genus and the space of ends.

**Theorem 3.5** Let $\Sigma$ be a finite genus infinite-type surface such that $\mathcal{E}(\Sigma)$ has finitely many isolated points. Then there is a degree one map $f: \Sigma \to \Sigma$ which is not $\pi_1$-injective.

**Proof.** Let $\mathcal{E}(\Sigma)$ be the set of all isolated points of $\mathcal{E}(\Sigma)$, let $k \in \mathbb{N} \cup \{0\}$ be the cardinality of $\mathcal{E}(\Sigma)$, and let $g$ be the genus of $\Sigma$. Then $\mathcal{E}(\Sigma) \subset \mathcal{E}(\Sigma) \setminus \mathcal{E}(\Sigma)$ is a non-empty, perfect, compact, totally-disconnected, metrizable space as it is infinite (by Remark 3.2) and has no isolated points. Thus $\mathcal{E}(\Sigma)$ is a Cantor space (see [5, Theorem 8 of Chapter 12]).

Let $D_0, D_1, D_{1,1}, D_{1,2}, D_{1,3} \subseteq \mathbb{S}^2, f: \mathbb{S}^2 \to \mathbb{S}^2$, and $\gamma$ be as in the conclusion of Lemma 3.4. Also, let $C_1 \subset \text{int}(D_1)$ be a subset homeomorphic to the Cantor set, and let $\mathcal{I} \subset \text{int}(D_0)$ be a set consisting of $k$ points (hence homeomorphic to $\mathcal{E}(\Sigma)$). Now, define $C_{1,j} := f^{-1}(C_1) \cap D_{1,j}$ for $j = 1, 2, 3$. Note that each $C_{1,j}$ is homeomorphic to the Cantor set. See Figure 3.

As $f^{-1}(D_0) = D_0$ and $f|_{D_0}: D_0 \to D_0$ is the identity map, we can say that $f^{-1}(\mathcal{I}) = \mathcal{I}$. Let $\Sigma_1$ be the surface obtained from $\mathbb{S}^2 \setminus (\mathcal{I} \cup C_1)$ by attaching $g$ handles along disjoint disks $\Delta_k \subset \text{int}(D_0) \setminus \mathcal{I}$, $1 \leq k \leq g$, and let $\Sigma_2$ be the surface obtained from $\mathbb{S}^2 \setminus (\mathcal{I} \cup C_{1,1} \cup C_{1,2} \cup C_{1,3})$ by attaching $g$ handles along the (same) disks $\Delta_k$, $1 \leq k \leq g$. Then $f$ induces a proper map, which we also call $f$, from $\Sigma_2$ to $\Sigma_1$.

By Lemma 3.1, $\deg(f) = \pm 1$. 

Fig. 2: The map on the $\mathcal{I}^{(1)}$
Further, we claim that \( f : \Sigma_2 \to \Sigma_1 \) is not injective on \( \pi_1 \). Namely, the fundamental group of \( \Sigma_2 \) is the amalgamated free product of four groups, one of which is \( \pi_1(S^2 \setminus \text{int}(D_0 \cup D_{1,1} \cup D_{1,2} \cup D_{1,3})) \). As \( \gamma \) is not homotopic to the trivial loop in \( S^2 \setminus \text{int}(D_0 \cup D_{1,1} \cup D_{1,2} \cup D_{1,3}) \), and components of an amalgamated free product inject, \( \gamma \) is not homotopic to the trivial loop in \( \Sigma_2 \). However, \( f(\gamma) \) is homotopic to the trivial loop in \( S^2 \setminus \text{int}(D_0 \cup D_1) \) and hence in \( \Sigma_1 \). Therefore, \( f \) is not injective on \( \pi_1 \).

Both \( \Sigma_1 \) and \( \Sigma_2 \) have genus the same as \( \Sigma \), and the space of ends homeomorphic to that of \( \Sigma \) (as a finite disjoint union of Cantor spaces is a Cantor space by the universality of the Cantor set) with all ends planar. Hence, by Theorem 2.1, both \( \Sigma_1 \) and \( \Sigma_2 \) are homeomorphic to \( \Sigma \).

Identifying \( \Sigma_1 \) and \( \Sigma_2 \) with \( \Sigma \) by homeomorphisms, we get a proper map \( f : \Sigma \to \Sigma \) which is not \( \pi_1 \)-injective. As homeomorphisms have degree \( \pm 1 \), it follows that \( \deg(f) = \pm 1 \). Replacing \( f \) by \( f \circ f \) if necessary, we obtain a proper map of degree one that is not injective on \( \pi_1 \).

**Theorem 3.6** Let \( \Sigma \) be a finite genus surface such that \( E(\Sigma) \) has infinitely many isolated points. Then there is a degree one map \( f : \Sigma \to \Sigma \) which is not \( \pi_1 \)-injective.

**Proof.** Let \( \mathcal{I}(\Sigma) \) be the set of all isolated points of \( E(\Sigma) \), and let \( g \) be the genus of \( \Sigma \). Also, let \( D_0, D_1, D_{1,1}, D_{1,2}, D_{1,3} \subseteq \mathbb{S}^2, f : \mathbb{S}^2 \to \mathbb{S}^2 \), and \( \gamma \) be as in the conclusion of Lemma 3.4. Now, consider a subset \( \mathcal{E} \) of \( \text{int}(D_0) \) such that \( \mathcal{E} \) is homeomorphic to \( E(\Sigma) \). Also, consider points \( p_1 \in \text{int}(D_1) \) and \( p_{1,i} \in \text{int}(D_{1,i}), i = 1, 2, 3 \) such that \( f(p_{1,i}) = p_1 \) for each \( i = 1, 2, 3 \). See Figure 4.

Recall that \( f^{-1}(D_0) = D_0 \) and \( f|_{D_0} : D_0 \to D_0 \) is the identity map. Thus \( f^{-1}(\mathcal{E}) = \mathcal{E} \). Now, let \( \Sigma_1 \) be the surface obtained from \( \mathbb{S}^2 \setminus (\mathcal{E} \cup \{p_1\}) \) by attaching \( g \) handles along disjoint disks \( \Delta_k \subset \text{int}(D_0) \setminus \mathcal{I} \).
1 \leq k \leq g$, and let $\Sigma_2$ be the surface obtained from $S^2 \setminus (E \cup \{p_{1,1}, p_{1,2}, p_{1,3}\})$ by attaching $g$ handles along the same disks $\Delta_k$, $1 \leq k \leq g$. Then $f$ induces a proper map, which we also call $f$, from $\Sigma_2$ to $\Sigma_1$. By Lemma 3.1, deg($f$) = ±1.

Further, we claim that $f: \Sigma_2 \to \Sigma_1$ is not injective on $\pi_1$. Namely, the fundamental group of $\Sigma_2$ is the amalgamated free product of four groups, one of which is $\pi_1(S^2 \setminus \text{int}(D_0 \cup D_{1,1} \cup D_{1,2} \cup D_{1,3}))$. As $\gamma$ is not homotopic to the trivial loop in $S^2 \setminus \text{int}(D_0 \cup D_{1,1} \cup D_{1,2} \cup D_{1,3})$, and components of an amalgamated free product inject, $\gamma$ is not homotopic to the trivial loop in $\Sigma_2$. However, $f(\gamma)$ is homotopic to the trivial loop in $S^2 \setminus \text{int}(D_0 \cup D_1)$ and hence in $\Sigma_1$. Therefore, $f$ is not injective on $\pi_1$.

Both $\Sigma_1$ and $\Sigma_2$ have genus the same as $\Sigma$ and, by Lemma 3.7 below, $\mathcal{E}(\Sigma_1)$ and $\mathcal{E}(\Sigma_2)$ are homeomorphic to $\mathcal{E}(\Sigma)$. Further, all ends of $\Sigma$, $\Sigma_1$ and $\Sigma_2$ are planar. Hence, by Theorem 2.1 both $\Sigma_1$ and $\Sigma_2$ are homeomorphic to $\Sigma$.

Identifying $\Sigma_1$ and $\Sigma_2$ with $\Sigma$ by homeomorphisms, we get a proper map $f: \Sigma \to \Sigma$ which is not injective on $\pi_1$. As homeomorphisms have degree ±1, it follows that deg($f$) = ±1. Replacing $f$ by $f \circ f$ if necessary, we obtain a proper map of degree one that is not injective on $\pi_1$. \qed

**Lemma 3.7** Let $\mathcal{E}$ be a closed totally disconnected subset of $S^2$. Let $\mathcal{I}$ be the set of all isolated points of $\mathcal{E}$. Assume $\mathcal{I}$ is infinite. If $\mathcal{I}$ is a finite subset of $S^2 \setminus \mathcal{E}$, then $\mathcal{E} \cup \mathcal{I}$ is homeomorphic to $\mathcal{E}$.

**Proof.** Let $\mathcal{A} \coloneqq \{a_1, a_2, \ldots\}$ be a subset of $\mathcal{I}$ such that $a_n \to \ell \in \mathcal{E}$ (\(\mathcal{A}\) exists as $\mathcal{E}$ is compact and...}
Define $B := \mathcal{A} \cup \mathcal{F}$. Write $B$ as $B = \{b_1, b_2, \ldots\}$. Then the map $g: \mathcal{E} \cup \mathcal{F} \to \mathcal{E}$ defined by

$$g(z) := \begin{cases} 
  z & \text{if } z \in (\mathcal{E} \cup \mathcal{F}) \setminus B, \\
  a_n & \text{if } z = b_n \in B,
\end{cases}$$

is a homeomorphism. To prove this, note that $g$ is a bijection from a compact space to a Hausdorff space, so it suffices to show that $g$ is continuous. But observe that $g$ restricted to the closed set $(\mathcal{E} \cup \mathcal{F}) \setminus B$ is the identity, so $g$ is continuous on $(\mathcal{E} \cup \mathcal{F}) \setminus B$. Also $g$ restricted to the closed set $B \cup \{\ell\}$ is continuous as $b_n \to \ell$ and $g(b_n) = a_n \to \ell = g(\ell)$, and all other points of $B \cup \{\ell\}$ are isolated. Thus $g$ is continuous, as required.

\begin{remark}
In the paper [3], the authors have proved that for every infinite-type surface $\Sigma$, there exists a subsurface homeomorphic to $\Sigma$ such that the inclusion map is not homotopic to a homeomorphism. As our surfaces are connected, this type of inclusion map can’t be proper because of the following two facts:

- Any injective map between two boundaryless topological manifolds of the same dimension is an open map. This follows from the invariance of domain.

- Any proper map between two topological manifolds is a closed map, as manifolds are compactly generated spaces, see [6].

Also, notice that all our results are related to proper maps.
\end{remark}

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