The Genus One Gromov-Witten Invariants of Calabi-Yau Complete Intersections

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Abstract

We obtain mirror formulas for the genus 1 Gromov-Witten invariants of projective Calabi-Yau complete intersections. We follow the approach previously used for projective hypersurfaces by extending the scope of its algebraic results; there is little change in the geometric aspects. As an application, we check the genus 1 BPS integrality predictions in low degrees for all projective complete intersections of dimensions 3, 4, and 5.

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1 Mirror Symmetry Formulas

Gromov-Witten invariants of projective varieties are counts of curves that are conjectured (and known in some cases) to possess a rich structure. The original mirror prediction of [CaDGP] for the

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genus 0 GW-invariants of a quintic threefold has since been verified and shown to be a special case of mirror formulas satisfied by GW-invariants of complete intersections; see [Gi] and [LLY]. Mirror formulas for the genus 1 GW-invariants of projective Calabi-Yau hypersurfaces are obtained in [Z4] and [Z5], in particular confirming the prediction of [BCOV] for a quintic threefold. In this paper, we obtain mirror formulas for the genus 1 GW-invariants of all projective Calabi-Yau complete intersections following the approach in [Z5], extending [ZaZ], and using [PoZ] in place of [Z3].

Throughout this paper, $n, a_1, a_2, \ldots, a_l \geq 2$ will be fixed integers. Let

\[ a \equiv (a_1, a_2, \ldots, a_l), \quad \langle a \rangle \equiv \prod_{k=1}^l a_k, \quad \text{and} \quad a^a \equiv \prod_{k=1}^l a_k^{a_k}. \]

Let $\varepsilon_0(a)$ and $\varepsilon_1(a)$ be the coefficients of $w^{n-1-l}$ and $w^{n-2-l}$, respectively, in the power series expansion of $\prod_{r=1}^l (1+a_r w)$ around $w=0$. We denote by $X_a$ a smooth complete intersection in $\mathbb{P}^{n-1}$ of multi-degree $a$. This complete intersection is Calabi-Yau if and only if $\sum_{r=1}^l a_r = n$; from now on it will be assumed that this condition holds. Let $N_d^1(X_a)$ denote the degree $d$ genus 1 GW-invariant of $X_a$. Note that $\varepsilon_0(a)$ and $\varepsilon_1(a)$ describe the top two Chern classes of $X_a$:

\[ c_{n-1-l}(X_a) = \varepsilon_0(a)H^{n-1-l}|_{X_a}, \quad c_{n-2-l}(X_a) = \varepsilon_1(a)H^{n-2-l}|_{X_a}, \quad (1.1) \]

where $H \in H^2(\mathbb{P}^{n-1})$ is the hyperplane class.

As in [ZaZ], we denote by

\[ \mathcal{P} \subset 1 + q\mathbb{Q}[[q]] \]

the subgroup of power series in $q$ with constant term 1 whose coefficients are rational functions in $w$ which are holomorphic at $w=0$. Thus, the evaluation map

\[ \mathcal{P} \rightarrow 1 + q\mathbb{Q}[[q]], \quad F(w, q) \mapsto F(0, q), \]

is well-defined. We define a map $M : \mathcal{P} \rightarrow \mathcal{P}$ by

\[ MF(w, q) = \left\{ 1 + \frac{q}{w dq} \right\} F(w, q). \quad (1.2) \]

Let $\tilde{F} \in \mathcal{P}$ be the hypergeometric series

\[ \tilde{F}(w, q) = \sum_{d=0}^{\infty} q^d \prod_{k=1}^l \prod_{r=1}^d \frac{(a_k w + r)^n}{(w + r)^n}. \quad (1.3) \]

For $p = 0, 1, \ldots, n - 1$, set

\[ I_p(q) \equiv M^p \tilde{F}(0, q). \quad (1.4) \]

\[ ^1\text{The assumption that } a_k \neq 1 \text{ is used only to streamline the computations in Section 5. Theorem 1 below is valid as long as } a_k \geq 1. \]
Since any such map is an unramified cover of \( X \), the mirror map
\[
\sum_{d=1}^{\infty} q^d (a_1d)! (a_2d)! \ldots (a_id)! \quad (d!)^n
\]
Let
\[
J(q) = \frac{1}{I_0(q)} \sum_{d=1}^{\infty} q^d \left( \prod_{k=1}^{l} (a_kd)! \left( \sum_{r=1}^{a_r} \sum_{k=1}^{a_k} \frac{a_k}{r} \right) \right) \quad \text{and} \quad Q = q e^{J(q)}.
\]
The map \( q \rightarrow Q \) is a change of variables; it will be called the mirror map.

**Theorem 1.** The genus 1 GW-invariants of a multi-degree \( a \) CY CI \( X_a \) in \( \mathbb{P}^{n-1} \) are given by:
\[
\sum_{d=1}^{\infty} N^d_1(X_a) Q^d = \frac{\langle a \rangle}{24} \varepsilon_0(a) [\log I_0(q)] + \frac{\langle a \rangle}{24} \varepsilon_1(a) J(q)
\]
where \( Q = q e^{J(q)} \).

Since dropping a component of \( a \) equal to 1 has no effect on the power series \( \tilde{F} \) in (1.3), this also has no effect on the right-hand side of the formula in Theorem 1 as expected from the relation
\[
N^d_1(X_{a_1, a_2, \ldots, a_l}) = N^d_1(X_{a_1, a_2, \ldots, a_l}).
\]
If \( l = 1 \) and thus \( a = (n) \), by the Residue Theorem on \( S^2 \)
\[
\varepsilon_0(a) = \mathcal{R}_{w=0} \left\{ \frac{(1+w)^n}{w^{n-1}(1+nw)} \right\} = -\mathcal{R}_{w=1/n} \left\{ \frac{(1+w)^n}{w^{n-1}(1+nw)} \right\} + \mathcal{R}_{z=0} \left\{ \frac{(z+1)^n}{z^2(z+n)} \right\}
\]
\[
= \frac{(n-1)^n}{(-1)^n n^2} + 1 - \frac{1}{n^2} = \frac{n^2 - 1 + (1-n)^n}{n^2},
\]
\[
\varepsilon_1(a) = \mathcal{R}_{w=0} \left\{ \frac{(1+w)^n}{w^{n-2}(1+nw)} \right\} = -\mathcal{R}_{w=1/n} \left\{ \frac{(1+w)^n}{w^{n-2}(1+nw)} \right\} + \mathcal{R}_{z=0} \left\{ \frac{(z+1)^n}{z^3(z+n)} \right\}
\]
\[
= \frac{(n-1)^n}{(-1)^n n^3} + \frac{n-1}{n^3} + \frac{1}{n^3} - \frac{1}{2n} - \frac{1}{2n} = \frac{(n-2)(n+1)}{2n} + \frac{1}{n} \frac{1}{n} - \frac{1}{n^3}.
\]

where \( \mathcal{R}_{w=w_0} \) denotes the residue at \( w = w_0 \). Thus, the \( l = 1 \) case of Theorem 1 reduces to [Z4, Theorem 2]. The cases \( l = n \), \( X_a = \emptyset \) and \( l = n - 1 \) \( X_a \) is \( \langle a \rangle = 2 \) points reduce to the case \( l = 1 \); as explained in [Z5, Section 0.3] the right-hand side of the formula in Theorem 1 vanishes as expected.

If \( l = n - 2 \), \( X_a \) is a torus, either \( X_3 \subset \mathbb{P}^2 \) or \( X_{2,2} \subset \mathbb{P}^3 \). In this case, \( N^d_1(X_a) \) is the number of degree \( d \) maps from genus 1 curves to \( X_a \) modulo automorphisms of such maps; see [KlPa, 0.2]. Since any such map is an unramified cover of \( X_a \) by a torus, it follows that \( N^d_1(X_{2,2}) \) is 0 unless \( d \) is divisible by 4 and \( N^d_1(X_{2,2}) \) is the number of degree \( r \) covers of \( X_{2,2} \) by a torus divided by \( r \).
Thus, using the formula [Z5, (B.12)] for the number of degree $r$ unramified covers of a torus, we obtain:

$$
\sum_{d=1}^{\infty} N^d_1(X_{2,2}) Q^d = - \sum_{r=1}^{\infty} \log \left(1 - Q^{4r}\right).
$$

This identity together with Theorem 1 implies that

$$
\frac{1}{6} J(q) - \frac{1}{24} \log(1 - 16q) - \frac{1}{2} \log I_0(q) = - \sum_{r=1}^{\infty} \log \left(1 - Q^{4r}\right).
$$

The same argument is applied to $X_3$ in [Z5, Section 0.3] to obtain

$$
\frac{1}{8} J(q) - \frac{1}{24} \log(1 - 27q) - \frac{1}{2} \log I_0(q) = - \sum_{r=1}^{\infty} \log \left(1 - Q^{3r}\right).
$$

The latter identity is verified directly in [Sc]; we expect that similar modular-forms techniques can be used to verify the former identity directly as well.

If $l = n - 3$, $X_a$ is a K3 surface, either $X_4 \subset \mathbb{P}^3$, $X_{2,3} \subset \mathbb{P}^4$, or $X_{2,2,2} \subset \mathbb{P}^5$. Since

$$
\langle a \rangle \varepsilon_0(a) = \chi(X_a) = 24 \quad \text{and} \quad \varepsilon_1(a) = 0,
$$

by [11], the right hand-side of the formula in Theorem 1 is zero in all 3 cases, as expected (all GW-invariants of K3 surfaces vanish).

If $l = n - 4$, $X_a \subset \mathbb{P}^{n-1}$ is a CY threefold. Since CY 3-folds are of a particular interest in GW-theory, we restate the $l = n - 4$ case of Theorem 1 as a corollary below. In this case,

$$
\varepsilon_0(a) = \mathcal{N}_{w=0} \left\{ \frac{(1+w)^n}{w^4 \prod_{r=1}^{l} (1+a_r w)} \right\} = \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)}{2} \sum_{r=1}^{l} a_r + \frac{1}{2} \sum_{r_1=1}^{l} \sum_{r_2=r_1}^{l} a_{r_1} a_{r_2} - \sum_{r_1=1}^{l} \sum_{r_2=r_1}^{l} \sum_{r_3=r_2}^{l} a_{r_1} a_{r_2} a_{r_3},
$$

$$
\varepsilon_1(a) = \mathcal{N}_{w=0} \left\{ \frac{(1+w)^n}{w^3 \prod_{r=1}^{l} (1+a_r w)} \right\} = \frac{n(n-1)}{2} - n \sum_{r=1}^{l} a_r + \frac{1}{2} \sum_{r_1=1}^{l} \sum_{r_2=r_1}^{l} a_{r_1} a_{r_2}.
$$

Corollary 2. The genus 1 GW-invariants of a CY CI threefold $X_a \subset \mathbb{P}^{n-1}$ are given by:

$$
\sum_{d=1}^{\infty} N^d_1(X_a) Q^d = \left[ -2 + \frac{\langle a \rangle}{72} \left( n - S_3(a) \right) \right] \log I_0(q) + \frac{\langle a \rangle}{48} \left( S_2(a) - n \right) J(q)
$$

$$
+ \log \left[ I_1(q) - \frac{1}{4} (1 - a^3 q)^{-\frac{1}{72}} \right],
$$

where $S_p(a) \equiv \sum_{r=1}^{l} a_r^p$ and $Q = q e^{-J(q)}$. 

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Table 1: Low-degree genus 1 BPS numbers for all CY CI 3-folds

| d   | 3                        | 4                        | 5                        | 6                        | 7                        |
|-----|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| X_3 | 609250                   | 3721431625               | 12129909700200           | 31147299732677250        | 71578406022880761750     |
| X_4 | 2560                     | 17407072                 | 24834612736              | 23689021707008           | 1907657726517760         |
| X_5 | 3402                     | 5520393                 | 4820744484               | 3163476678678            | 179369482469092          |
| X_6 | 64                       | 2651113                 | 198087264                | 89191834992             | 32343228035424          |
| X_7 | 0                        | 14752                    | 8782848                  | 2672004608              | 615920502784            |

Table 2: Low-degree genus 1 BPS numbers for all CY CI 4-folds

| d   | 3                        | 4                        | 5                        | 6                        | 7                        |
|-----|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| X_3 | 2734099200               | 387176346729900          | 2687329416654597632      | 1418722120880095142462400|
| X_4 | 9058000                  | 845459512250             | 202017164192950520       | 320471504960631822000    |
| X_5 | 2813440                  | 81906297984              | 1006848150400512         | 870717570941649792       |
| X_6 | 47104                    | 4277292544               | 428439214234832          | 249771462364601344       |
| X_7 | 1024                     | 65526084                 | 338199639552             | 923753814135936          |
| X_8 | 0                        | 3779200                  | 15090827264             | 27474707200000           |

Table 3: Low-degree genus 1 BPS numbers for all CY CI 5-folds

| d   | 3                        | 4                        | 5                        | 6                        | 7                        |
|-----|--------------------------|--------------------------|--------------------------|--------------------------|--------------------------|
| X_7 | 26123172457235           | 815458234653841075      | 11749847929567288677099464| 126043741686616181922427686855602 |
| X_8 | 69072837120              | 101190144588682320      | 41238110240324210242768  | 111476403211912149828779296 |
| X_9 | 8597353175               | 4075445624973975        | 72587697608410684840     | 88498079911311785027601450|
| X_10| 3950411776              | 1453445296487936        | 20112996721550639104     | 19073323868053990475971360|
| X_11| 254083200               | 244005174397575         | 33504170048610349120     | 270660538551145151653200|
| X_12| 76664320                | 22674871508976          | 1630795524423750144      | 72802469323317263218688|
| X_13| 39550437                | 5866761130074          | 28943587120690644        | 908636706403558378332   |
| X_14| 1507328                 | 1349735463168          | 75612640693245568        | 22287069498009830400    |
| X_15| 532160                  | 357068201643           | 13410965796358752        | 278702673457074092928   |
| X_16| 32768                   | 21650838624            | 62299658307072           | 8565078597579227136    |
| X_17| 0                       | 1342995456            | 2908032827136           | 264415120930570240      |

Tables 1-3 below show low-degree genus 1 BPS numbers for all CY CI 3, 4 and 5-folds obtained from Theorem 1 using [MirSym, (34.3)], [KlPa, (3)], and [PaZ, (0.5)], respectively. Using computer programs, we verified the predicted integrality of these numbers up to degree 100 for all CY CI 3, 4, and 5-folds. While the degree 1 and 2 genus 1 BPS numbers are 0 as expected, the degree 3 BPS numbers match the classical Schubert calculus on $G(3,n)$. It should be possible to obtain the degree 4 numbers using the approach of [ESt], which provides such numbers for hypersurfaces.

I would like to express my deep gratitude to Aleksey Zinger for explaining [Z4] and [Z5] to me, for proposing the questions answered in this paper, and for his invaluable suggestions.

\(^2\)Genus 1 BPS counts in higher dimensions are yet to be defined.

\(^3\)based on Aleksey Zinger’s programs for hypersurfaces.
2 Outline of the proof

We prove Theorem 1 following the approach used to prove [Z4, Theorem 2]. In particular, we compute the reduced genus 1 GW-invariants $N_1^{d,0}(X_a)$ of $X_a$ defined in [Z1]; these are related to the standard genus 1 invariants by Lemma 2.1 below.

The genus 1 hyperplane theorem of [LiZ] and the desingularization construction of [YaZ] express the reduced genus 1 GW-invariants of $X_a$ in terms of integrals over smooth spaces of maps to $\mathbb{P}^{n-1}$. We use this in Section 3.2 to package the numbers $N_1^{d,0}(X_a)$ into a power series $\mathcal{X}(\alpha, x, Q)$, in a formal variable $Q$ and with coefficients in the equivariant cohomology of $\mathbb{P}^{n-1}$. As $\mathcal{X}(\alpha, x, Q)$ involves integrals on smooth moduli spaces, the Atiyah-Bott Localization Theorem [ABo] can be applied as in [Z5]. This leads to Proposition 3.1 of Section 3.3; the latter expresses $\mathcal{X}(\alpha, x, Q)$ in terms of residues of some genus 0 generating functions.

We extract “the non-equivariant part” of $\mathcal{X}(\alpha, x, Q)$ in Section 5 using [Z5, Lemma 3.3] and mirror formulas for genus 0 generating functions. This reduces the problem of computing the numbers $N_1^{d,0}(X_a)$ to purely algebraic questions concerning the power series $[13]$. These are addressed in Section 6, which significantly extends [ZaZ]; this section can be read independently of the rest of the paper.

All cohomology groups in this paper will be with rational coefficients. We will denote by $[n]$, whenever $n \in \mathbb{Z}_{\geq 0}$, the set of positive integers not exceeding $n$:

$$[n] = \{1, 2, \ldots, n\}.$$

Whenever $g$, $d$, $k$, and $n$ are nonnegative integers and $X$ is a smooth subvariety of $\mathbb{P}^{n-1}$, $\overline{M}_{g,k}(X, d)$ will denote the moduli space of stable degree $d$ maps into $X$ from genus $g$ curves with $k$ marked points

$$ev_i: \overline{M}_{g,k}(\mathbb{P}^{n-1}, d) \mid [C, y_1, \ldots, y_k, f] \mapsto f(y_i), \quad i = 1, 2, \ldots, k,$$

for the evaluation map at the $i$-th marked point; see [MirSym, Chapter 24]. For each $m \in \mathbb{Z}_{>0}$, define

$$\overline{M}_{(m)}(X, d) \equiv \left\{(b_i)_{i \in [m]} \in \prod_{i = 1}^{m} \overline{M}_{0,1}(X, d_i) : d_i \in \mathbb{Z}_{>0}, \sum_{i = 1}^{m} d_i = d, \ ev_1(b_i) = ev_1(b_{i'}) \ \forall \ i, i' \in [m] \right\},$$

$$ev_1: \overline{M}_{(m)}(X, d) \to X, \quad (b_i)_{i \in [m]} \mapsto ev_1(b_i),$$

where $i$ is any element of $[m]$. For each $i \in [m]$, let

$$\pi_i: \overline{M}_{(m)}(X, d) \to \bigsqcup_{d_i \in \mathbb{Z}_{>0}, d_i \leq d} \overline{M}_{0,1}(X, d_i)$$

be the projection onto the $i$-th component. If $p \in \mathbb{Z}_{\geq 0}$, we define $\eta_p \in H^{2p}(\overline{M}_{(m)}(X, d))$ to be the degree $2p$ term of

$$\prod_{i = 1}^{m} \pi_i^* \frac{1}{1 - \psi_1} \in H^*(\overline{M}_{(m)}(X, d)).$$

Thus, $\eta_p$ is the sum of all degree $p$ monomials in $\left\{\pi_i^* \psi_1 : i \in [m]\right\}$.

The symmetric group on $m$ elements, $S_m$, acts on $\overline{M}_{(m)}(X, d)$ by permuting the elements of each $m$-tuple of stable maps. Let

$$Z_{(m)}(X, d) \equiv \overline{M}_{(m)}(X, d)/S_m.$$
Since the map \( \text{ev}_1 \) and the cohomology class \( \eta_p \) on \( \overline{\mathcal{M}}_{(m)}(X, d) \) are \( S_m \)-invariant, they descend to the quotient:

\[
\text{ev}_1 : Z_{(m)}(X, d) \to X \quad \text{and} \quad \eta_p \in H^{2p}(Z_{(m)}(X, d)).
\]

Let \( \mathfrak{U} \) be the universal curve over \( \overline{\mathcal{M}}_{(m)}(\mathbb{P}^{n-1}, d) \), with structure map \( \pi \) and evaluation map \( \text{ev} \):

\[
\begin{array}{c}
\mathfrak{U} \\
\downarrow \pi \\
\overline{\mathcal{M}}_{(m)}(\mathbb{P}^{n-1}, d).
\end{array}
\]

The orbi-sheaf

\[
\pi_* \text{ev}^* \bigoplus_{r=1}^{l} \mathcal{O}_{\mathbb{P}^{n-1}}(a_r) \to \overline{\mathcal{M}}_{(m)}(\mathbb{P}^{n-1}, d)
\]

is locally free; it is the sheaf of (holomorphic) sections of the vector orbi-bundle

\[
\mathcal{V}_{(m)} \equiv \overline{\mathcal{M}}_{(m)}(\mathcal{L}, d) \to \overline{\mathcal{M}}_{(m)}(\mathbb{P}^{n-1}, d),
\]

where \( \mathcal{L} \to \mathbb{P}^{n-1} \) is the total space of the vector bundle corresponding to the sheaf \( \bigoplus_{r=1}^{l} \mathcal{O}_{\mathbb{P}^{n-1}}(a_r) \).

By the (genus-zero) hyperplane-section relation,

\[
\langle \eta_p - 2m \text{ev}_1^* H^{n-1-l-p}[Z_{(m)}(X, d)]^{\vir} \rangle = \frac{1}{m!} \langle \eta_p - 2m \text{ev}_1^* H^{n-1-l-p} \mathcal{V}_{(m)}^{\vir}, [\overline{\mathcal{M}}_{(m)}(\mathbb{P}^{n-1}, d)] \rangle,
\]

(2.1)

where \( H \in H^2(\mathbb{P}^{n-1}) \) is the hyperplane class.

There is a natural surjective bundle homomorphism

\[
\tilde{\text{ev}}_1 : \mathcal{V}_{(1)} \to \text{ev}_1^* \bigoplus_{r=1}^{l} \mathcal{O}_{\mathbb{P}^{n-1}}(a_r), \quad ([\mathcal{C}, u, \xi]) \to \xi(x_1(\mathcal{C})),
\]

\( \overline{\mathcal{M}}_{(1)}(\mathbb{P}^{n-1}, d) \equiv \overline{\mathcal{M}}_{0,1}(\mathbb{P}^{n-1}, d) \), where \( x_1(\mathcal{C}) \) is the marked point. Thus,

\[
\mathcal{V}_{(1)}' \equiv \ker \tilde{\text{ev}}_1 \to \overline{\mathcal{M}}_{(1)}(\mathbb{P}^{n-1}, d)
\]

is a vector orbi-bundle\(^4\) It is straightforward to see that

\[
e(\mathcal{V}_{(m)}) = \langle \mathbf{a} \rangle \text{ev}_1^* \prod_{i=1}^{m} \pi_i^* e(\mathcal{V}_{(1)}'),
\]

(2.2)

If \( f = f(w) \) admits a Laurent series expansion around \( w = 0 \), for any \( p \in \mathbb{Z} \) we denote by \( \llbracket f(w) \rrbracket_{w^p} \) the coefficient of \( w^p \). Let

\[
\left[ \sum_{d=0}^{\infty} f_d(w) Q^d \right]_{w^p} \equiv \sum_{d=0}^{\infty} \left[ f_d(w) \right]_{w^p} Q^d \quad \text{if} \quad f_d \in \mathbb{Q}(w) \forall d \geq 0.
\]

(2.3)

Theorem\(^4\) follows immediately from from (3.9), Theorem\(^3\) stated at the beginning of Section\(^5\) and Lemma\(^2.1\) below, which extends \([Z4, \text{Lemma } 2.2]\) to complete intersections.

\(^4\)In the notation of Section\(^3.2\) \( \mathcal{V}_{(1)} = \mathcal{V}_0 \) and \( \mathcal{V}_{(1)}' = \mathcal{V}_0' \).
Lemma 2.1. If $X_a \subseteq \mathbb{P}^{n-1}$ is a complete intersection of multi-degree $a$,

$$N_1^d(X_a) = N_1^{d;0}(X_a) + \frac{1}{24} \sum_{p=2}^{n-1-l} \sum_{m=1}^{2m \leq p} (-1)^m (m-1)! \langle \eta_{p-2m} \nu_1^*(c_{n-1-l-p}(X_a)), [Z(m)_d(X_a,d)] \rangle.$$ 

Furthermore, for all $p \in \mathbb{Z}^{\geq 0}$

$$c_p(X_a) = \left. \left[ \frac{(1+w)^n}{\prod_{r=1}^{l} (1+a_r w)} \right] \right|_{w:p}^H$$

and for all $p \leq n-1-l$

$$\sum_{d=1}^{\infty} Q^d \left( \sum_{m=1}^{2m \leq p} (-1)^m (m-1)! \langle \eta_{p-2m} \nu_1^* H^{n-1-l-p} |_{X_a}, [Z(m)_d(X_a,d)] \rangle \right) = -\langle a \rangle \left[ \log \frac{\hat{F}(w,q)}{I_0(q)} \right]_{w:p},$$

where $H \in H^2(\mathbb{P}^{n-1})$ is the hyperplane class, $Q$ and $q$ are related by the mirror map (1.6), and $\hat{F}(w,q)$ and $I_0(q)$ are given by by (1.3) and (1.5), respectively.

Proof. The first identity above is a special case of [24 (2.15)]. The second identity is immediate from

$$c(T \mathbb{P}^{n-1}) = (1+H)^n \quad \text{and} \quad c(N_{X_a/\mathbb{P}^{n-1}}) = \prod_{r=1}^{l} (1+a_r H)^n \bigg|_{X_a}.$$ 

It remains to verify the third identity. For each $r \in \mathbb{Z}^{\geq 0}$, let

$$Z_r(Q) \equiv \sum_{d=1}^{\infty} Q^d \left( \sum_{m=1}^{2m \leq p} (-1)^m (m-1)! \langle \eta_{p-2m} \nu_1^* H^{n-1-l-p} |_{X_a}, [Z(m)_d(X_a,d)] \rangle \right).$$ 

By (2.1), (2.2), and the decomposition along the small diagonal in $(\mathbb{P}^{n-1})^m$, the left-hand side of the third identity in Lemma 2.1 above equals

$$\langle a \rangle \sum_{d=1}^{\infty} Q^d \left( \sum_{m=1}^{2m \leq p} (-1)^m (m-1)! \prod_{i=1}^{m} \pi_i^* \nu_1^* (\psi_i^* H^{n-1-p} |_{X_a}, [\overline{\mathcal{M}}(\mathbb{P}^{n-1},d)] ) \right)$$

$$= \langle a \rangle \sum_{m=1}^{2m \leq p} \frac{(-1)^m}{m} \sum_{d=1}^{\infty} Q^d \sum_{i=1}^{m} \left( \psi_i^* H^{n-1-p} |_{X_a}, [\overline{\mathcal{M}}(\mathbb{P}^{n-1},d_i)] \right)$$

$$= \langle a \rangle \sum_{m=1}^{2m \leq p} \frac{(-1)^m}{m} \sum_{i=1}^{m} \left[ Z_{p_i-2}(Q) \right] = -\langle a \rangle \left[ \log \left( 1 + \sum_{r=0}^{n-3} Z_r(Q)w^{r+2} \right) \right]_{w:p}.$$
The third statement of Lemma 2.1 now follows from
\[ 1 + \sum_{r=0}^{n-3} Z_r(Q) w^{r+2} = e^{-J(q)w} \frac{\mathcal{F}(w,q)}{I_0(q)} \in \mathbb{Q}[w][[q]]/w^n; \]
the last identity is obtained from [Gi, Theorem 11.8] using the string relation [MirSym, Section 26.3]. □

3 Equivariant Setup

3.1 Equivariant cohomology

This section reviews the basics of equivariant cohomology following [Z5, Section 1.1] closely and setting up related notation.

The classifying space for the \( n \)-torus \( T \) is \( B_T \equiv (\mathbb{P}^\infty)^n \). Thus, the group cohomology of \( T \) is
\[ H^*_T \equiv H^*(B_T) = \mathbb{Q}[\alpha_1, \ldots, \alpha_n], \]
where \( \alpha_i \equiv \pi^* i_1(\gamma^*), \gamma \rightarrow \mathbb{P}^\infty \) is the tautological line bundle, and \( \pi_i : (\mathbb{P}^\infty)^n \rightarrow \mathbb{P}^\infty \) is the projection to the \( i \)-th component. In the remainder of the paper,
\[ \alpha = (\alpha_1, \ldots, \alpha_n). \]
The field of fractions of \( H^*_T \) will be denoted by
\[ \mathbb{Q}_\alpha \equiv \mathbb{Q}(\alpha_1, \ldots, \alpha_n). \]
We denote the equivariant \( \mathbb{Q} \)-cohomology of a topological space \( M \) with a \( T \)-action by \( H^*_T(M) \). If the \( T \)-action on \( M \) lifts to an action on a complex vector bundle \( V \rightarrow M \), let \( e(V) \in H^*_T(M) \) denote the equivariant Euler class of \( V \). A continuous \( T \)-equivariant map \( f : M \rightarrow M' \) between two compact oriented manifolds induces a pushforward homomorphism
\[ f_* : H^*_T(M) \rightarrow H^*_T(M'), \]
which is characterized by the property that
\[ \int_{M'} (f_* \eta)' \eta' = \int_M \eta (f^* \eta') \quad \forall \eta \in H^*_T(M), \eta' \in H^*_T(M'). \quad (3.1) \]
If \( M' \) is a point, this is the integration-along-the-fiber homomorphism
\[ \int_M : H^*_T(M) \rightarrow H^*_T \]
for the fiber bundle \( E_T \times_T M \rightarrow B_T \).

Throughout this paper, \( T \) will act on \( \mathbb{P}^{n-1} \) in the standard way:
\[ (e^{i\theta_1}, \ldots, e^{i\theta_n}) \cdot [z_1, \ldots, z_n] = [e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n]. \]
This action has \( n \) fixed points:
\[ P_1 = [1, 0, \ldots, 0], \quad P_2 = [0, 1, 0, \ldots, 0], \ldots, \quad P_n = [0, \ldots, 0, 1]. \]
For each \( i = 1, 2, \ldots, n \), let
\[
\phi_i \equiv \prod_{k \neq i} (x - \alpha_k) \in H_T^* (\mathbb{P}^{n-1}).
\]

By the Atiyah-Bott Localization Theorem [ABG],
\[
\eta|_{P_i} = \int_{\mathbb{P}^{n-1}} \eta \phi_i \in \mathbb{Q}_\alpha \quad \forall \eta \in H_T^* (\mathbb{P}^{n-1}) \otimes H_\alpha^* \mathbb{Q}_\alpha, \ i = 1, 2, \ldots, n;
\]
thus, \( \phi_i \) is the equivariant Poincaré dual of \( P_i \).

The standard action of \( T \) on \( \mathbb{P}^{n-1} \) lifts to an action on the tautological bundle
\[
\gamma \equiv \mathcal{O}_{\mathbb{P}^{n-1}} (-1) \subset \mathbb{P}^{n-1} \times \mathbb{C}^n
\]
by restricting the standard diagonal \( T \)-action on \( \mathbb{P}^{n-1} \times \mathbb{C}^n \). The equivariant hyperplane class is defined to be
\[
x \equiv e(\gamma^*) \equiv e(\mathcal{O}_{\mathbb{P}^{n-1}} (1)) \in H_T^2 (\mathbb{P}^{n-1}).
\]

The equivariant cohomology of \( \mathbb{P}^{n-1} \) is given by
\[
H_T^* (\mathbb{P}^{n-1}) = \mathbb{Q}[x, \alpha_1, \ldots, \alpha_n] / (x - \alpha_1) \ldots (x - \alpha_n).
\]

The restriction map on the equivariant cohomology induced by the inclusion \( P_i \hookrightarrow \mathbb{P}^{n-1} \) is given by
\[
H_T^* (\mathbb{P}^{n-1}) = \mathbb{Q}[x, \alpha_1, \ldots, \alpha_n] / \prod_{k=1}^n (x - \alpha_k) \to H_T^* (P_i) = \mathbb{Q}[\alpha_1, \ldots, \alpha_n], \quad x \to \alpha_i,
\]
and so
\[
\eta = 0 \in H_T^* (\mathbb{P}^{n-1}) \iff \eta|_{P_i} = 0 \in H_T^* \forall i = 1, 2, \ldots, n.
\]

### 3.2 Generating function for reduced genus 1 GW-invariants

As in [Z5], the reduced genus 1 GW-invariants \( N_{1,0}^d (X_a) \) of \( X_a \) are packaged into a generating function \( \mathcal{X} \); this is a power series in the formal variable \( Q \) with coefficients in the equivariant cohomology of \( \mathbb{P}^{n-1} \). In this section, we define \( \mathcal{X} \) and explain what its relationship with \( N_{1,0}^d (X_a) \) is; see (3.8) and (3.9).

Let \( \pi : \mathcal{U} \to \overline{\mathcal{M}}_{g,k} (\mathbb{P}^{n-1}, d) \) be the universal curve with evaluation map \( \text{ev} \) as before and
\[
\mathcal{V}_0 \to \overline{\mathcal{M}}_{0,k} (\mathbb{P}^{n-1}, d)
\]
the vector bundle corresponding to the locally free sheaf
\[
\bigoplus_{r=1}^l \pi_* \text{ev}^* \mathcal{O}_{\mathbb{P}^{n-1}} (a_r) \to \overline{\mathcal{M}}_{0,k} (\mathbb{P}^{n-1}, d).
\]

The Euler class \( c(\mathcal{V}_0) \) relates genus 0 GW-invariants of \( X_a \subset \mathbb{P}^{n-1} \) to genus 0 GW-invariants of \( \mathbb{P}^{n-1} \); it also appears in the genus 0 2 point generating functions (3.11)-(3.13) which are used in the proof in Theorem 3.
The genus 1 GW-invariants of $X_a$ are related to the GW-invariants of $\mathbb{P}^{n-1}$ in a more complicated way. This is partly because $\mathcal{M}_{1,k}(\mathbb{P}^{n-1},d)$ is not an orbifold and

$$\bigoplus_{r=1}^{l} \pi_* \text{ev}^* \mathcal{O}_{\mathbb{P}^{n-1}}(a_r) \to \mathcal{M}_{1,k}(\mathbb{P}^{n-1},d)$$

is not locally free. However, it is shown in [VaZ] that there exists a natural desingularization

$$p: \widetilde{\mathcal{M}}^0_{1,k}(\mathbb{P}^{n-1},d) \to \mathcal{M}_{1,k}(\mathbb{P}^{n-1},d)$$

of the main component of $\mathcal{M}_{1,k}(\mathbb{P}^{n-1},d)$, whose generic element is a map from smooth domain. There is also a vector orbi-bundle $V_1$ over $\widetilde{\mathcal{M}}^0_{1,k}(\mathbb{P}^{n-1},d)$ so that the diagram

$$\begin{array}{ccc}
V_1 & \xrightarrow{p_*} & \bigoplus_{r=1}^{l} \pi_* \text{ev}^* \mathcal{O}_{\mathbb{P}^{n-1}}(a_r) \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{M}}^0_{1,k}(\mathbb{P}^{n-1},d) & \xrightarrow{p} & \mathcal{M}_{1,k}(\mathbb{P}^{n-1},d)
\end{array}$$

commutes. By [LZ] Theorem 1.1 and [Z2] Theorem 1.1,

$$d N_{1,0}^d(X_a) = \langle e(V_1) \text{ev}_1^* \mathcal{H}, [\mathcal{M}^0_{1,1}(\mathbb{P}^{n-1},d)] \rangle. \quad (3.7)$$

The standard $T$-action on $\mathbb{P}^{n-1}$ induces $T$-actions on the moduli spaces of $\mathcal{M}_{g,k}(\mathbb{P}^{n-1},d)$ and lifts to an action on $\mathcal{M}_{0,1}^0(\mathbb{P}^{n-1},d)$. The evaluation maps,

$$\text{ev}_i: \mathcal{M}_{g,k}(\mathbb{P}^{n-1},d), \mathcal{M}_{1,k}(\mathbb{P}^{n-1},d) \to \mathbb{P}^{n-1}, \quad [C, y_1, \ldots, y_k, f] \to f(y_i), \quad i \in [k],$$

are $T$-equivariant. The natural $T$-action on $\mathcal{O}_{\mathbb{P}^{n-1}}(-1) \to \mathbb{P}^{n-1}$ induces $T$-actions on the sheafs $\pi_* \text{ev}^* \mathcal{O}_{\mathbb{P}^{n-1}}(a)$ and the vector bundle

$$\mathcal{V}_1 \to \mathcal{M}_{1,1}^0(\mathbb{P}^{n-1},d).$$

With $\text{ev}_{1,d}$ the evaluation map on $\mathcal{M}_{1,1}^0(\mathbb{P}^{n-1},d)$, let

$$X(\alpha, x, Q) \equiv \sum_{d=1}^{\infty} Q^d (\text{ev}_{1,d}^* e(\mathcal{V}_1)) \in \left( H^{n-2}_{n}(\mathbb{P}^{n-1}) \right)[[Q]].$$

By (3.4),

$$X(\alpha, x, Q) = X_0(Q)x^{n-2} + X_1(\alpha, Q)x^{n-3} + \ldots + X_{n-2}(\alpha, Q)x^0, \quad (3.8)$$

for some $X_0 \in \mathbb{Q}[[Q]]$ and power series $X_p \in \mathbb{Q}[\alpha_1, \ldots, \alpha_n][[Q]]$, whose coefficients are symmetric degree $p$ homogeneous polynomials in $\alpha_1, \ldots, \alpha_n$. By (3.7) and (3.1),

$$Q \frac{d}{dQ} \sum_{d=1}^{\infty} N_{1,0}^d(X_a) Q^d = X_0(Q). \quad (3.9)$$
By (3.5), (3.3), and (3.1),

\[
\mathfrak{X}(\alpha, \alpha_i; Q) = \mathfrak{X}(\alpha, x, Q)|_{P_i} = \sum_{d=1}^{\infty} Q^d \int_{\mathbb{P}^{n-1}} (\text{ev}_{1,d*}(\mathfrak{V}_1)) \phi_i
\]

\[
= \sum_{d=1}^{\infty} Q^d \int_{\mathbb{P}_{n-1}^{Q}} e(\mathfrak{V}_1) e(\mathfrak{V}_1^*) \phi_i \in \mathbb{Q}_0[[Q]]
\]

(3.10)

for each \(i = 1, 2, \ldots, n\). Since \(\mathfrak{X}\) is symmetric in \(\alpha_1, \ldots, \alpha_n\), \(\mathfrak{X}_0 \in \mathbb{Q}[[Q]]\) is completely determined by either of the \(n\) power series in (3.10). We use this to obtain the explicit formula for \(\mathfrak{X}_0\) given in Theorem 3.

### 3.3 A localization proposition

As in [Z5], we express \(\mathfrak{X}(\alpha, \alpha_i; Q)\) in terms of residues of genus 0 generating functions. Proposition 3.1 below is the analogue of [Z5] Propositions 1.1, 1.2; its proof is essentially identical to the proof of [Z5] Propositions 1.1, 1.2 in [Z5] Section 2. In this section, we set up the notation needed to state Proposition 3.1 and motivate it, and describe the few minor changes needed in [Z5, Section 2] for a complete proof of this proposition. In the remainder of this paper, we will use Proposition 3.1 to obtain an explicit formula for \(\mathfrak{X}_0\).

If \(f\) is a rational function in \(h\) and possibly other variables and \(h_0 \in S^2\), let \(\mathfrak{R}_{h=h_0} f(h)\) denote the residue of the one-form \(f(h) dh\) at \(h=h_0\); thus,

\[
\mathfrak{R}_{h=\infty} f(h) \equiv -\mathfrak{R}_{w=0}\{ w^{-2} f(w^{-1}) \}.
\]

If \(f\) involves variables other than \(h\), \(\mathfrak{R}_{h=h_0} f(h)\) is a function of the other variables. If \(f\) is a power series in \(Q\) with coefficients that are rational functions in \(h\) and possibly other variables, let \(\mathfrak{R}_{h=h_0} f(h)\) denote the power series in \(Q\) obtained by replacing each of the coefficients by its residue at \(h=h_0\). If \(h_1, \ldots, h_k\) is a collection of points in \(S^2\), not necessarily distinct, we define

\[
\mathfrak{R}_{h=h_1, \ldots, h_k} f(h) \equiv \sum_{z \in \{h_1, \ldots, h_k\}} \mathfrak{R}_{h=z} f(h).
\]

If \(h_0 \in \mathbb{C}\) or \(h_0\) is one of the “other” variables in \(f\), let

\[
\mathfrak{R}_{h=-ah_0} f(h) \equiv \mathfrak{R}_{h=-a_1h_0, \ldots, -a_kh_0} f(h).
\]

For instance, if \(a = (2, 2, 3, 3, 3)\) and \(\alpha_i\) is one of the other variables, then

\[
\mathfrak{R}_{h=-a\alpha_i} f(h) \equiv \mathfrak{R}_{h=-2\alpha_i} f(h) + \mathfrak{R}_{h=-3\alpha_i} f(h).
\]

Since the \(\mathbb{T}\)-equivariant bundle homomorphism

\[
\tilde{\text{ev}}_1 : V_0 \rightarrow \bigoplus_{r=1}^{l} \text{ev}_{1}^* O_{\mathbb{P}^{n-1}}(a_r), \quad \left[C, x_1, \ldots, x_k, f, \xi \right] \rightarrow \left[\xi (x_1(C)) \right],
\]

is surjective, its kernel

\[
\mathfrak{V}_0' \equiv \ker \tilde{\text{ev}}_1 \rightarrow \mathfrak{M}_{0,k}(\mathbb{P}^{n-1}, d),
\]

is
is a $\mathbb{T}$-equivariant vector bundle. Since the $\mathbb{T}$-action on $\mathbb{M}_{g,k}(\mathbb{P}^{n-1}, d)$ lifts naturally to the tautological tangent line bundles $L_i$, there are well-defined equivariant $\psi$-classes

$$\psi_i \equiv c_1(L_i^*) \in H^2_{eq}(\mathbb{M}_{g,k}(\mathbb{P}^{n-1}, d));$$

see [MirSym] Section 25.2. For all $i, j = 1, 2, \ldots, n$, let

$$Z^*_i(h, Q) \equiv \sum_{d=1}^{\infty} Q^d \int_{\mathbb{M}_{0,2}(\mathbb{P}^{n-1}, d)} \frac{e(V_0^i)}{h - \psi_1} ev^*_1 \phi_i;$$  \hspace{0.5cm} (3.11)

$$Z^*_ij(h, Q) \equiv h^{-1} \sum_{d=1}^{\infty} Q^d \int_{\mathbb{M}_{0,2}(\mathbb{P}^{n-1}, d)} \frac{e(V_0^i)}{h - \psi_1} ev^*_1 \phi_i ev^*_2 \phi_j;$$  \hspace{0.5cm} (3.12)

$$\tilde{Z}^*_ij(h_1, h_2; Q) \equiv \frac{1}{2h_1 h_2} \sum_{d=1}^{\infty} Q^d \int_{\mathbb{M}_{0,2}(\mathbb{P}^{n-1}, d)} \frac{e(V_0^i)}{(h_1 - \psi_1)(h_2 - \psi_2)} ev^*_1 \phi_i ev^*_2 \phi_j.$$  \hspace{0.5cm} (3.13)

Explicit formulas for these generating functions are given explicitly in [Gi] Theorem 11.8 and [PoZ] Theorem 4. These theorems show that in particular

$$Z^*_i, Z^*_ij \in Q_\alpha(h)[[Q]] \quad \text{and} \quad \tilde{Z}^*_ij \in Q_\alpha(h_1, h_2)[[Q]] .$$

Thus, the $h$-residues of these power series are well-defined. Since the $Q$ degree 0 term of the power series $Z^*_i(h, Q)$ is 0, the residue

$$\eta_i(Q) \equiv R_{h=0}\left\{ \log (1 + Z^*_i(h, Q)) \right\} \in Q_\alpha[[Q]]$$  \hspace{0.5cm} (3.14)

is well-defined. Let

$$\Phi_0(\alpha_i, Q) \equiv R_{h=0}\left\{ h^{-1} e^{-\eta_i(Q)/h} (1 + Z^*_i(h, Q)) \right\} \in Q_\alpha[[Q]].$$  \hspace{0.5cm} (3.15)

By [Z5] Lemma 2.3, the power series $e^{-\eta_i(Q)/h}(1 + Z^*_i(h, Q))$ is holomorphic at $h = 0$; thus $\Phi_0(\alpha_i, Q)$ is its value at $h = 0$.\footnote{While $l$ is meant to be 1 in [Z5] Section 2.2, the argument goes through for any $a$ without any change.} Note that the degree-zero term of $\Phi_0(\alpha_i, Q)$ is 1.

Proposition 3.1 below is obtained by applying the Atiyah-Bott Localization Theorem [ABo] to the last expression in (3.10). As described in detail in [Z5] Sections 1.3, 1.4, the fixed loci of the $\mathbb{T}$-action on $\mathbb{M}_{0,1}(\mathbb{P}^{n-1}, d)$ are indexed by decorated graphs with one marked point. The vertices are decorated by elements of $[n]$, indicating the $T$-fixed point of $\mathbb{P}^{n-1}$ to which the node or component corresponding to the vertex is mapped to. These graphs have either zero loops and one distinguished vertex (as in Figure 2) or one loop (as in Figure 1), depending on whether the stable maps they describe are constant or not on the principal component of the domain.\footnote{Figures 1 and 2 are Figures 3 and 2 in [Z5]; they are used by permission to indicate what is involved in the proof of (the $l=1$ case of) Proposition 3.1 in [Z5].} The graphs with no loops are called $B$-graphs in [Z5], while the graphs with one loop are called $A$-graphs.

In a $B$-graph, the distinguished vertex corresponds to the contracted principal component. As every graph has a marked point, even the $A$-graphs have a distinguished vertex: the vertex in the loop closest to the vertex to which the marked point is attached. The distinguished vertices are indicated by thick dots in the four graphs in Figures 1 and 2.
Within each of the 2 types, there are 2 sub-types of graphs, depending on whether the marked point is attached to the distinguished vertex or some other vertex. In the former case, a graph has one special vertex label: the number decorating the vertex to which the marked point is attached. In the latter case, a graph has two special vertex labels: the number decorating the vertex to which the marked point is attached and the number decorating the distinguished vertex. Since $\phi_i|_{P_j} = \delta_{ij}$, only the graphs that describe stable maps taking the marked point to $P_i$ contribute to (3.10); in these graphs the first special vertex label is $i$. Thus, the types of graphs that contribute to (3.10) can be described as $A_i$, $\tilde{A}_{ij}$, $B_i$, and $\tilde{B}_{ij}$, with the first subscript describing the label of the vertex to which the marked point is attached and the second describing the label of the distinguished vertex if this vertex is different from the first (the label may still be the same).

The approach of [Z5] to computing the total contribution to (3.10) of all graphs of a fixed type is to break every graph at the distinguished vertex, adding a marked point to each of the resulting “strands” so that each graph is completely encoded by its strands. In the case of $B_i$-graphs, all strands are graphs with one marked point. In the case of $A_i$ and $B_{ij}$-graphs, there is precisely one strand with two marked points; in the former case it contributes to $\tilde{Z}_{ii}^*$, while in the latter it contributes to $Z_{ji}^*$. In the case of $A_{ij}$-graphs, there are two strands with two marked points, one of which contributes to $\tilde{Z}_{jj}^*$, while the other to $Z_{ji}^*$. Each of the one-pointed strands contributes to $Z_j^*$. While the number of one-pointed strands can be arbitrary large, it is possible to sum up over all arrangements of such strands because of a special property of the power series $Z_i^*$ described in [Z5, Section 2.2]. This reduces the total contribution, $A_i$, $\tilde{A}_{ij}$, $B_i$, or $\tilde{B}_{ij}$ of all graphs of a fixed type, $A_i$, $\tilde{A}_{ij}$, $B_i$, or $\tilde{B}_{ij}$, to a fairly simple expression involving $Z_i^*$, $Z_{ij}^*$, and/or $\tilde{Z}_{ij}^*$.

**Proposition 3.1.** For every $i=1,2,\ldots,n$,

$$X(\alpha, \alpha_i, Q) = A_i(Q) + \sum_{j=1}^{n} \tilde{A}_{ij}(Q) + B_i(Q) + \sum_{j=1}^{n} \tilde{B}_{ij}(Q),$$

(3.16)
where

\[
A_i(Q) = \frac{1}{\Phi_0(\alpha_i, Q)} \mathcal{R}_{h_1=0} \left\{ \mathcal{R}_{h_2=0} \left\{ e^{-\eta_i(Q)/h_1} e^{-\eta_i(Q)/h_2} \tilde{Z}^*_{ii}(h_1, h_2, Q) \right\} \right\};
\]

\[
\tilde{A}_{ij}(Q) = \frac{A_{ij}(Q)}{\prod_{k \neq j} (\alpha_j - \alpha_k)} \mathcal{R}_{h=0} \left\{ e^{-\eta_j(Q)/h} \tilde{Z}^*_{ji}(h, Q) \right\};
\]

\[
\mathcal{B}_i(Q) = \frac{(a) \alpha_i^l}{24} \mathcal{R}_{h=0, \infty, -a_{\alpha_i}} \left\{ \prod_{k=1}^n (\alpha_i - \alpha_k + h) \right\} \left\{ \prod_{r=1}^l (a_r \alpha_i + h) \right\} \frac{\tilde{Z}^*_{i}(h, Q)}{1 + \tilde{Z}^*_{i}(h, Q)};
\]

\[
\tilde{B}_{ij}(Q) = \frac{1}{\prod_{k \neq j} (\alpha_j - \alpha_k)} \frac{(a) \alpha_i^l}{24} \mathcal{R}_{h=0, \infty, -a_{\alpha_j}} \left\{ \prod_{k=1}^n (\alpha_j - \alpha_k + h) \right\} \left\{ \prod_{r=1}^l (a_r \alpha_j + h) \right\} \frac{\tilde{Z}^*_{ji}(h, Q)}{1 + \tilde{Z}^*_{ji}(h, Q)}.
\]

This proposition is essentially proved in [Z5] Sections 1.3, 1.4, 2], which treats the \( l = 1 \) case. In the general case, the T-fixed loci and their normal bundles remain the same. The only required changes involve the Euler classes of the bundles \( V'_0 \) and \( V_1 \), which are now products of the Euler classes of the bundles in [Z5]. These changes are:

(i) equation [Z5, (1.37)] becomes

\[
e(V_1) | \tilde{Z}_p = \langle a \rangle \alpha_i^l \prod_{e \in \text{Edg}(V_0)} \prod_{r=1}^l \frac{\tilde{Z}^*_{i}(h_1, h_2, Q)}{1 + \tilde{Z}^*_{i}(h_1, h_2, Q)};
\]

(ii) \( n \alpha_i + h \) is replaced by \( \prod_{r=1}^l (a_r \alpha_i + h) \) in the definition of \( \Psi \) above [Z5, (2.19)], in [Z5] (2.23) and (2.24), and in the last equation in [Z5] Section 2.3, leading to the corresponding modification in the final expressions for \( B_i \) and \( \tilde{B}_{ij} \) above;

(iii) \( n \alpha_i \) is replaced by \( (a) \alpha_i^l \) in [Z5] (2.24) and \( n \alpha_j \) is replaced by \( (a) \alpha_j^l \) in the last equation in [Z5] Section 2.3, leading to the corresponding modification in the final expressions for \( B_i \) and \( \tilde{B}_{ij} \) above.

### 4 Some properties of hypergeometric series \( \tilde{F} \)

In this section we study properties of the hypergeometric series \( \tilde{F} \) of (1.3) which are used in Section 5.3 to deduce Theorem 3 from Proposition 3.1. The results in this section extend most of the statements and proofs in [ZaZ], which treats the \( l = 1 \) case.

Let \( M: \mathcal{P} \rightarrow \mathcal{P} \) be as in (1.2) and define \( \mathcal{F} \in \mathcal{P} \) by

\[
\mathcal{F}(w, q) \equiv \sum_{d=0}^{\infty} q^d \frac{\prod_{k=1}^l \prod_{r=1}^d (a_k w + r)}{\prod_{r=1}^d (w + r)^n - w^n}.
\]
Proposition 4.3. The power series

\[ I_p(q) = M^p \hat{F}(0, q) = M^p F(0, q) \quad \forall p = 0, 1, \ldots, n-1. \]  

(4.2)

Some advantages of the power series \( F \) over \( \hat{F} \) are illustrated by Lemmas 4.1 and 4.2 below.

Lemma 4.1. The hypergeometric series \( F \) satisfies \( M^n F = F \).

Lemma 4.2 ([ZaZ] Lemma 1.3 and its proof). If \( F \in \mathcal{P} \) and \( M^k F = F \) for some \( k > 0 \), then every coefficient of the power series \( \log M^p F(w, q) \in \mathbb{Q}(w)[[q]] \) is \( \mathcal{O}(w) \) as \( w \to \infty \) for all \( p \geq 0 \). Moreover, \( \Re_{w=0} \{ \log M^p F(w^{-1}, q) \} \) does not depend on \( p \).

Applying this lemma to \( F = \mathcal{F} \), we find that \( \mathcal{F}(w, q) \) has an asymptotic expansion

\[ \mathcal{F}(w, q) \sim e^{\mu(q)w} \sum_{s=0}^{\infty} \frac{s}{(s+\delta)} \Phi_s(q) w^{-s} \quad (w \to \infty) \]  

(4.3)

for some power series \( \mu, \Phi_0, \Phi_1, \ldots \) in \( \mathbb{Q}[[[q]] \). Let

\[ L(q) \equiv (1 - a^n q)^{-1/n} \in \mathbb{Q}[[q]]. \]  

(4.4)

Proposition 4.3. The power series \( \mu, \Phi_0, \) and \( \Phi_1 \) in (4.3) are given by

\[ \mu(q) = \int_0^q L(u) - \frac{1}{u} \, du, \quad \Phi_0 = L^{-\frac{1}{2}}, \]  

(4.5)

\[ \Phi_1 = \left\{ \left[ \frac{n-1}{24} - \sum_{r=1}^{l} \frac{1}{12} \frac{1}{a_r} \right] (L-1) - \frac{(n-2)(n+1)}{24n} \left( L - 1 \right)^2 \right\} \Phi_0 \frac{1}{L}. \]  

(4.6)

The last proposition of this section concerns properties of \( \hat{F} \) and \( F \) around \( w=0 \).

Proposition 4.4. The power series \( I_p(q) \) defined by (1.4) satisfy

\[ I_p(q) = 1 \quad \forall p = n-l+1, \ldots, n-1, \]  

(4.7)

\[ I_p(q) = I_{n-l-p}(q) \quad \forall p = 0, 1, \ldots, n-l, \]  

(4.8)

\[ I_0(q) I_1(q) \ldots I_{n-l}(q) = (1 - a^n q)^{-l}, \]  

(4.9)

\[ I_0(q) I_1(q) I_{n-l-1}(q)^{n-l-1} I_{n-l-2}(q)^{n-l-2} \ldots I_{n-l-1}(q)^{n-l-1} I_{n-l}(q)^{n-l} \quad (1 - a^n q)^{-(n-l)/2}. \]  

(4.10)

While (4.8) and (4.9) imply (4.10), (4.10) is simpler to prove directly than (4.8) and will be verified together with (4.9), as is done in [ZaZ].

Lemma 4.1 and Propositions 4.3 and 4.4 are proved in Sections 4.1 and 3 following the approach in [ZaZ]. Let

\[ D = q \frac{d}{dq}, D_w = D + w : \mathbb{Q}(w)[[q]] \to \mathbb{Q}(w)[[q]]. \]

Thus,

\[ D_w \left[ \sum_{d=0}^{\infty} c_d(w) q^d \right] = \sum_{d=0}^{\infty} (w+d) c_d(w) q^d, \]  

(4.11)

\[ MF(w, q) = w^{-1} D_w \left[ F(w, q)/F(0, q) \right] \quad \forall F \in \mathcal{P}. \]
4.1 Proof of Lemma 4.1 and Proposition 4.4

We will repeatedly use the following lemma.

**Lemma 4.5** ([ZaZ, Corollary 2.2]). Suppose\(F(w,q) \in \mathcal{P}\) satisfies

\[
\left( \sum_{r=0}^{m} C_r(q) D_{w}^{r} \right) F(w,q) = A(w,q) \tag{4.12}
\]

for some power series \(C_0(q), \ldots, C_m(q) \in \mathbb{Q}[[q]]\) and \(A(w,q) \in \mathbb{Q}(w)[[q]]\) with \(A(0,q) \equiv 0\). Then

\[
\left( \sum_{s=0}^{m-1} \tilde{C}_s(q) D_{w}^{s} \right) M \mathcal{F}(w,q) = \frac{1}{w} A(w,q), \tag{4.13}
\]

where \(\tilde{C}_s(q) \equiv \sum_{r=s+1}^{m} \binom{r}{s+1} C_r(q) D_{w}^{r-1-s} F(0,q)\).

Define power series \(\mathcal{F}_{-l}, \mathcal{F}_{-l+1}, \ldots \in \mathcal{P}\) by

\[
\mathcal{F}_{-l}(w,q) \equiv \sum_{d=0}^{\infty} q^d \prod_{k=1}^{l} \prod_{r=0}^{l} (a_k w + r)^{\binom{a_k d-1}{d}} \prod_{r=1}^{l} [(w+r)^n - w^n], \tag{4.14}
\]

\[
\mathcal{F}_p \equiv M^{l+p} \mathcal{F}_{-l} \quad \forall p > -l.
\]

Using (4.11), we find that

\[
\mathcal{F}_p(0,q) = 1 \quad \forall p = -l, -l+1, \ldots, -1, \quad w^{-l} D_w \mathcal{F}_{-l} = \mathcal{F}; \tag{4.15}
\]

thus, \(\mathcal{F}_p = M^p \mathcal{F}\) for all \(p \geq 0\).

It is also straightforward to check that \(\mathcal{F}_{-l}\) solves the differential equation

\[
\left\{ D_w^n - q \prod_{k=1}^{l} \prod_{r=0}^{l} (a_k D_w + r) \right\} \mathcal{F}_{-l} = w^n \mathcal{F}_{-l}. \tag{4.16}
\]

This equation is of the form (4.12) with \(F = \mathcal{F}_{-l}\), \(A = w^n \mathcal{F}_{-l}\), \(m = n\),

\[
C_n(q) = 1 - a^n, \quad C_{n-1}(q) = -\frac{n-l}{2} a^n q. \tag{4.17}
\]

Applying Lemma 4.5 repeatedly, we obtain

\[
\sum_{s=0}^{n-l-p} C_s^{(p)}(q) D_w^{s} \mathcal{F}_p(w,q) = w^{n-l-p} \mathcal{F}_{-l}(w,q), \quad -l \leq p \leq n-l, \tag{4.18}
\]

where by the first identity in (4.15) and by (4.2)

\[
C_s^{(p)}(q) = C_{s+l+p}^{(-l)}(q) \quad -l \leq p \leq 0,
\]

\[
C_s^{(p)}(q) = \sum_{r=s+1}^{n-l+p+1} \binom{r}{s+1} C_{r}^{(p-1)}(q) D_{w}^{r-s-1} I_{p-1}(q) \quad p > 0.
\]
Using (4.17) and induction on \( p \), we find that the top two coefficients in (4.18) are given by

\[
C^{(p)}_{n-l-p} = (1 - a^a q) p \prod_{r=0}^{p-1} I_r(q),
\]

(4.19)

\[
C^{(p)}_{n-l-p-1} = \left[ -a^a q \frac{n-l}{2} + (1 - a^a q) \sum_{r=0}^{p-1} (n-l-r) \frac{D I_r(q)}{I_r(q)} \right] p \prod_{r=0}^{p-1} I_r(q).
\]

(4.20)

Setting \( p = n-l \) in (4.18) and (4.19) thus gives

\[
(1 - a^a q) \left( \prod_{r=0}^{n-l-1} I_r(q) \right) F_{n-l}(w, q) = F_{-l}(w, q).
\]

(4.21)

Setting \( w = 0 \) in (4.21) and using \( F_{-l}(0, q) = 1 \) gives (4.9). Substituting (4.9) back into (4.21) gives \( F_{n-l}/I_{n-l} = F_{-l} \) and thus \( F_{n-l+1} = F_{-l+1} \). Applying \( M \) to both sides of the last identity \( l-1 \) times and using (4.15), we obtain Lemma 4.4 and equation (4.7). Similarly, setting \( p = n-l-1 \) in (4.18)-(4.20) and then taking \( w = 0 \) gives

\[
\sum_{r=0}^{n-l-1} \frac{n-l-r}{2} p \prod_{r=0}^{n-l-1} I_r(q) = \frac{n-l}{2} a^a q.
\]

Integrating this identity and then exponentiating, we obtain (4.10).

We next prove the reflection symmetry (4.8). The function \( \tilde{F} \in \mathcal{P} \) defined in (1.3) satisfies the differential equation

\[
\left\{ D_w^{n-l} - \langle a \rangle q \prod_{k=1}^{l} \prod_{r=1}^{a_k-1} (a_k D_w + r) \right\} \tilde{F} = w^{n-l}.
\]

This equation is of the form (4.12) with \( F = \tilde{F}, A = w^{n-l}, m = n-l, \) and

\[
C_{n-l}(q) = 1 - a^a q.
\]

Applying Lemma 4.5 repeatedly, we obtain

\[
\sum_{s=0}^{n-l-p} \tilde{C}_s^{(p)}(q) D_w^s M^p \tilde{F}(w, q) = w^{n-l-p}, \quad 0 \leq p \leq n-l,
\]

(4.22)

with \( \tilde{C}_s^{(p)} = C^{(p)}_s \) given by (4.19). Setting \( p = n-l \) in (4.22) and using (4.19) and (4.9), we find that

\[
M^{n-l} \tilde{F}(w, q) = I_{n-l}(q)
\]

is independent of \( w \). Using (4.11) and downward induction on \( p \), we then find that

\[
w^{l-n} \tilde{F}(w, q) = I_0 D_w^{-1} I_1 D_w^{-1} \ldots I_{n-l-1} D_w^{-1} I_{n-l},
\]

(4.23)

where

\[
D_w^{-1} \left[ \sum_{d=0}^{\infty} c_d(w) q^d \right] = \sum_{d=0}^{\infty} \frac{c_d(w)}{(w+d)} q^d.
\]
Comparing the coefficients of $q^d$ on the two sides of (4.23), we find that

$$\langle a \rangle^{-1} \sum_{k=1}^{l} \prod_{r=0}^{d} (a_k w + r) \prod_{r=0}^{d} (w + d + d_n) = \sum_{d_0 + \ldots + d_{n-l} = d} \prod_{d_0, \ldots, d_{n-l} \geq 0} c_0(d_0) \cdots c_{n-l}(d_{n-l})$$

for all $d \geq 0$, where $c_p(d)$ is the coefficient of $q^d$ in $I_p(q)$. This identity is equivalent to

$$\sum_{p=0}^{n-l} \frac{c_p(d)}{w^{n-l-p}(w+d)^p} = \sum_{r=0}^{d} \prod_{r=0}^{d} (w + r) \prod_{r=0}^{d} (w + d + d_n) = \sum_{d_0 + \ldots + d_{n-l} = d} \prod_{d_0, \ldots, d_{n-l} < d} c_0(d_0) \cdots c_{n-l}(d_{n-l}).$$

We will use this identity to show by induction that

$$c_p(d) = c_{n-l-p}(d) \quad \forall p = 0, 1, \ldots, n-l,$$

thus establishing (4.8). Since $c_p(0) = I_p(0) = 1$ for all $p$, (4.23) holds for $d = 0$. Suppose $d \geq 1$ and (4.25) holds with $d$ replaced by every $d' < d$. The substitution $w \to -w - d$ acts by $(-1)^{n-l}$ on the first term on the right-hand side of (4.24). It acts in the same way on the second term by the induction assumption; this can be seen by the renumbering

$$(d_0, \ldots, d_{n-l}) \to (d_{n-l}, \ldots, d_0).$$

Thus, the substitution $w \to -w - d$ acts by $(-1)^{n-l}$ on the left-hand side of (4.24), and so $c_p(d) = c_{n-l-p}(d)$ for all $0 \leq p \leq n-l$, as needed for the inductive step.

### 4.2 Proof of (4.5)

By Lemmas 4.1 and 4.2, the functions $F_p(w, q) \equiv \mathbf{M}^p F(w, q)$ admit asymptotic expansions

$$F_p(w, q) \sim e^{\mu(q)w} \sum_{s=0}^{\infty} \Phi_{p,s}(q) w^{-s} \quad (w \to \infty),$$

with the same function $\mu(q)$ in the exponent for all $p$. Since $F_0 = F$ and $F_{p+1} = \mathbf{M} F_p$,

$$\Phi_{0,s} = \Phi_s, \quad \Phi_{p+1,s} = \frac{1 + D \mu}{I_p} \Phi_{p,s} + \begin{cases} D \left( \frac{\Phi_{p,s-1}}{I_p} \right), & \text{if } s \geq 1, \\ 0, & \text{if } s = 0. \end{cases}$$

Taking $s=0$ in (4.27), we find by induction that

$$\Phi_{p,0} = \frac{(1 + D \mu)^p}{I_0 I_1 \ldots I_{p-1}} \Phi_0.$$ 

Since $F_n = F_0$ by Lemma 4.1 and $\Phi_0(0) = 1$, setting $p=n$ in the above identity we obtain

$$(1 + D \mu)^n = I_0 \ldots I_{n-1}.$$
The first claim in (4.5) now follows from (4.7) and (4.9).

For each \( p \geq 0 \), let

\[
H_p(q) \equiv \frac{L^p(q)}{I_0(q) \ldots I_{p-1}(q)}.
\]

By definition, (4.7), (4.9), (4.10), (4.28), and the first identity in (4.5),

\[
H_0 = H_n = 1, \quad H_1H_2 \ldots H_n = L^{-\frac{n(n-1)}{2}}, \quad \Phi_{p,0} = H_p\Phi_0.
\]  (4.29)

Taking \( s=1 \) in (4.27) and using the first and last equations above, we find inductively that

\[
\Phi_{p,1} = H_p \left( \Phi_1 + p\frac{D\Phi_0}{L} - p\frac{\Phi_0DL}{L^2} + \frac{\Phi_0}{L} \sum_{r=1}^{p} \frac{DH_r}{H_r} \right) \quad \forall p \geq 0.
\]

Setting \( p=n \) in this relation and using \( \Phi_{n,1} = \Phi_1 \), along with the first and second equations in (4.29), we find that

\[
\frac{D\Phi_0}{\Phi_0} = \frac{l+1}{2} \frac{DL}{L}.
\]

Since \( \Phi_0(0)=1=L(0) \), this confirms the second claim in (4.5).

4.3 Proof of (4.6)

The argument in Section 4.2 can be systematized as in [ZaZ] to obtain an algorithm for computing every \( \Phi_s \) by a differential recursion.

Define \( \xi_s \in \mathbb{Q} \) by

\[
\prod_{k=1}^{l} \prod_{j=1}^{a_k} (a_kD + j) \equiv a^n \sum_{s=0}^{n} \xi_sD^s \in \mathbb{Z}[D];
\]

thus, \( \xi_n = 1, \xi_{n-1} = (n+1)/2, \) and

\[
\xi_{n-2} = \frac{1}{24} \sum_{k=1}^{l} \frac{(a_k - 1)(a_k + 1)(3a_k + 2)}{a_k} + \frac{1}{4} \sum_{1 \leq i < j \leq l} \frac{1}{2} (1 + a_i)(1 + a_j)
\]

\[
= -\frac{1}{12} \sum_{r=1}^{l} \frac{1}{a_r} + \frac{3n^2 + n(6l - 4) + 3l^2 - 6l}{24}.
\]  (4.30)

Let

\[
\tilde{D}_w \equiv D + Lw: \mathbb{Q}(w)[[q]] \rightarrow \mathbb{Q}(w)[[q]].
\]

The series \( \tilde{F}(w, q) \equiv e^{-\mu(q)w} F(w, q) \) admits an asymptotic expansion

\[
\tilde{F}(w, q) \sim \sum_{s=0}^{\infty} \Phi_s(q) w^{-s} \quad (w \to \infty).
\]  (4.31)

Since \( 1 + D\mu = L \) by the first claim in (4.5) and \( \tilde{F}(w, q) \) satisfies the ODE

\[
\left\{ D_w^n - w^n - q \prod_{k=1}^{l} \prod_{r=1}^{a_k} (a_kD_w + r) \right\} \tilde{F} = 0,
\]

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the series \( \overline{F}(w,q) \) satisfies the ODE
\[
\mathcal{L}F = 0, \tag{4.32}
\]
where \( \mathcal{L} \) is the differential operator
\[
\mathcal{L} \equiv L^n \left[ \widehat{D}_w^n - w^n - q \prod_{k=1}^l a_k \widehat{D}_w + r \right] = \widehat{D}_w^n - (Lw)^n - (L^n - 1) \sum_{s=0}^{n-1} \xi_s \widehat{D}_w^s. \tag{4.33}
\]

By (4.35), the first two of these operators are
\[
\mathcal{L}_1 = nD - \frac{l+1}{2}(L^n - 1) = nL^{\frac{l+1}{2}} DL^{-\frac{l+1}{2}},
\]
\[
\mathcal{L}_2 = \left( \begin{array}{c} n \end{array} \right) D^2 - \frac{(l+2)(n-1)}{2}(L^n - 1) D
+ \left[ \frac{(n-1)(n-2)(n-6l-5)}{24n} L^n + \frac{(3n+6l+5)(n-1)(n-2)}{24n} - \xi_{n-2} \right] (L^n - 1). \tag{4.38}
\]

Combining (4.33), (4.32), and (4.36), we obtain the following.

**Proposition 4.6.** The power series \( \Phi_s \in \mathbb{Q}[[q]] \), \( s \geq 0 \), defined by (4.3), are determined by the first-order ODEs
\[
\mathcal{L}_1(\Phi_s) + \frac{1}{L} \mathcal{L}_2(\Phi_{s-1}) + \frac{1}{L^2} \mathcal{L}_3(\Phi_{s-2}) + \ldots + \frac{1}{L^{n-1}} \mathcal{L}_n(\Phi_{s+1-n}) = 0, \quad s \geq 0, \tag{4.39}
\]

**Proposition 4.6.** The power series \( \Phi_s \in \mathbb{Q}[[q]] \), \( s \geq 0 \), defined by (4.3), are determined by the first-order ODEs
\[
\mathcal{L}_1(\Phi_s) + \frac{1}{L} \mathcal{L}_2(\Phi_{s-1}) + \frac{1}{L^2} \mathcal{L}_3(\Phi_{s-2}) + \ldots + \frac{1}{L^{n-1}} \mathcal{L}_n(\Phi_{s+1-n}) = 0, \quad s \geq 0, \tag{4.39}
\]
extended with the initial conditions \( \Phi_s(0) = \delta_{0,s} \) and \( \Phi_s = 0 \) for \( s < 0 \).
The \( s=0 \) case of (4.39) immediately recovers the second claim in (4.5). The \( s=1 \) case of (4.39) then gives

\[
nD(\Phi_1/\Phi_0) = -\frac{1}{L} (L^n - 1) \left( \frac{n^2 - n - 3l^2 + 1}{24n} (n-1)L^n + \frac{1}{12} \sum_{r=1}^{l} \frac{1}{a_r} - \frac{3l^2 - 1}{24n} \right).
\]

Along with \( DL = L(n^2 - 1)/n \) and \( \Phi_1(0) = 0 \), this identity gives (4.6).

5 Computation of reduced genus 1 GW-invariants

In this section, we deduce Theorem 3 below from Proposition 3.1, using Lemmas 5.1 and 5.3 and the properties of the hypergeometric series \( F(w,q) \) described by Proposition 4.3 and 4.4. Lemma 5.1 is used to drop purely equivariant terms from the power series \( \mathcal{X} \), while Lemma 5.3 provides the relevant information about the genus 0 generating functions \( Z_0^\ast, Z_{ij}^\ast, \) and \( Z_{ii}^\ast \).

**Theorem 3.** The generating function \( \mathcal{X}_0(Q) \) defined by (3.8) is given by

\[
\mathcal{X}_0(Q) = Q \frac{d}{dQ} \left( \tilde{A}(q) + \tilde{B}(q) \right),
\]

where \( Q \) and \( q \) are related by the mirror map (1.6) and

\[
\tilde{A}(q) = \frac{n}{48} \left( n - 1 - 2 \sum_{k=1}^{l} \frac{1}{a_k} \right) \mu(q) - \left\{ \frac{n+1}{48} \log (1 - a^q) + \sum_{p=0}^{n=2} \frac{(n-l-2p)^2}{8} \log I_p(q), \right. \]

\[
\left. \left. - \frac{n-2}{48} \log (1 - a^q) + \sum_{p=0}^{n=2} \frac{(n-l-2p)^2-1}{8} \log I_p(q), \right. \right. \text{if } 2 \mid (n-l);
\]

\[
\tilde{B}(q) = -\frac{n}{48} \left( n - 1 - 2 \sum_{k=1}^{l} \frac{1}{a_k} \right) \mu(q) + \frac{\langle a \rangle}{24} \log I_0(q) + \frac{\langle a \rangle}{24} \varepsilon_0(a) J(q) + \frac{l-1}{48} \log (1 - a^q) + \frac{\langle a \rangle}{24} \sum_{p=2}^{n=1} \left[ \frac{(1+w)^n}{\prod_{k=1}^{l} (1 + a_k w)} \right] \log \tilde{F}(w,q) \right|_{w:p},
\]

where \( \tilde{F}, I_p, J, \mu, \) and \( \left[ \right|_{w:p} \) are defined by (1.3), (1.4), (1.6), (2.3) and (2.5), respectively.

We will show that the terms \( \mathcal{A}_i \) and \( \tilde{A}_{ij} \), with \( j \in [n] \), in (3.16) together contribute \( \frac{1}{2} Q \frac{d}{dQ} A(q) \) to \( \mathcal{X}_0(Q) \), where

\[
\mathcal{A}(q) = \frac{n}{24} \left( n - 1 - 2 \sum_{r=1}^{l} \frac{1}{a_r} \right) \mu(q) - \frac{3(n-1-l)^2 + (n-2)}{24} \log (1 - a^q)
\]

\[
- \sum_{p=0}^{n-2} \left( \frac{n-l-p}{2} \right) \log I_p(q), \quad (5.1)
\]
while the terms $B_i$ and $\tilde{B}_{ij}$, with $j \in [n]$, together contribute $Q \frac{d}{dq} \tilde{B}(q)$. Since
\[
\sum_{r=0}^{n-2-\ell} \log I_r(q) + \frac{1}{2} \log I_{n-r-\ell}(q) = -\frac{1}{2} \log (1-a^2q), \quad \text{if } 2|\ell, \quad \sum_{r=0}^{n-1-\ell} \log I_r(q) = -\frac{1}{2} \log (1-a^2q), \quad \text{if } 2 \not{|}\ell,
\]
by (4.8) and (4.9), the expression on the right-hand side of (5.1) equals twice the right-hand side in the first equation in Theorem [3].

5.1 Some algebraic notation and observations

This section recalls the statement of [Z3, Lemma 3.3], which shows that most terms appearing in the computation of $X(\alpha, x, Q)$ have no effect on $X_0(Q)$. We then set up additional related notation and make a few algebraic observations that help streamline computations in the remainder of the paper.

For each $p \in [n]$, let $\sigma_p$ be the $p$-th elementary symmetric polynomial in $\alpha_1, \ldots, \alpha_n$. Denote by
\[
Q[\alpha]^{S_n} \equiv Q[\alpha_1, \ldots, \alpha_n]^{S_n} \subset Q[\alpha_1, \ldots, \alpha_n]
\]
the subspace of symmetric polynomials, by $\mathcal{I} \subset Q[\alpha]^{S_n}$ the ideal generated by $\sigma_1, \ldots, \sigma_{n-1}$, and by
\[
\tilde{Q}[\alpha]^{S_n} \equiv Q[\alpha_1, \ldots, \alpha_n]_{(\alpha_j, (\alpha_j - \alpha_k)) \neq (j, k)} \subset Q[\alpha]
\]
the subalgebra of symmetric rational functions in $\alpha_1, \ldots, \alpha_n$ whose denominators are products of $\alpha_j$ and $(\alpha_j - \alpha_k)$ with $j \neq k$. For each $i = 1, \ldots, n$, let
\[
\tilde{Q}_i[\alpha]^{S_{n-1}} \equiv Q[\alpha_1, \ldots, \alpha_n]_{(\alpha_{i}, (\alpha_j - \alpha_k)) \neq (i, k)} \subset Q[\alpha]
\]
be the subalgebra consisting of rational functions symmetric in $\{\alpha_k : k \neq i\}$ and with denominators that are products of $\alpha_i$ and $(\alpha_i - \alpha_k)$ with $k \neq i$. Let
\[
K_i \equiv \text{Span}_Q \{\mathcal{I} : \tilde{Q}_i[\alpha]^{S_{n-1}}, \alpha_i^{n-2}, \mathcal{I} \tilde{Q}[\alpha]^{S_n}, \{1, \alpha_i, \ldots, \alpha_i^{n-3}, \alpha_i^{n-1}\} : \tilde{Q}[\alpha]^{S_n}\}. \quad (5.2)
\]

Lemma 5.1 ([Z3, Lemma 3.3]). If $n \geq 2$, the linear span of $\alpha_i^{n-2}$ is disjoint from $K_i$:
\[
\text{Span}\{\alpha_i^{n-2}\} \cap K_i = \{0\} \subset Q[\alpha].
\]

For each $i = 1, \ldots, n$, let
\[
\tilde{Q}_i[\alpha]^{S_{n-1}} \equiv Q(h, \alpha_i)[\alpha_i^{S_{n-1}}_{((\alpha_i - \alpha_k) + rh)\in (i, r \in \mathbb{Z}, k \neq i)} \subset Q[\alpha](h)
\]
be the subalgebra consisting of rational functions symmetric in $\{\alpha_k : k \neq i\}$ and with denominators that are a product of a polynomial with rational coefficients in $h$ and $\alpha_i$ and of linear factors of the form $(\alpha_i - \alpha_k + rh), r \in \mathbb{Z}$. Denote by
\[
S_{i,h} \subset \tilde{Q}_i[\alpha]^{S_{n-1}}
\]

\text{The definition of } K_i \text{ in [Z3] is missing } \alpha_i^{n-2} \mathcal{I} \tilde{Q}[\alpha]^{S_n}, \text{ but the proof of [Z3, Lemma 3.3]} \text{ still goes through. This change adds the term } \alpha_i^{n-1} g_{n-1}, \text{ with } g_{n-1} \in \mathcal{I}, \text{ to the second numerator in [Z3] (3.13) and } g_{n-1} \text{ to the right-hand side of [Z3] (3.15)}. \text{ As } g \in \mathcal{I}, \text{ this addition has no effect on the concludeing sentence in the proof of Lemma 3.3 in [Z3].}
the subalgebra consisting of rational functions of the form $A + B \prod_{k=1}^{l} (a_k \alpha_i + h)$ with $A, B \in \tilde{Q}_i[\alpha]_{h_1}^{S_{n-1}}$
both regular at $h = -a_k \alpha_i$ for every $k \in [l]$ and the denominator of $A$ an element of $Q[\alpha_i, h]$. We define
\[ \tilde{Q}_i[\alpha]_{h_1,h_2}^{S_{n-1}} \subset Q_\alpha(h_1, h_2) \]
to be the subalgebra generated by $\tilde{Q}_i[\alpha]_{h_1}^{S_{n-1}}$ and $\tilde{Q}_i[\alpha]_{h_2}^{S_{n-1}}$. If in addition $j \in [n]$, let
\[ K^{(i,j)}_h \equiv \text{Span}_Q \{ \alpha_i^{n-2} I \cdot \tilde{Q}_j[\alpha]^{S_{n-1}}, \{ 1, \alpha_i, \ldots, \alpha_i^{n-3}, \alpha_i^{n-1} \} \tilde{Q}_j[\alpha]^{S_{n-1}} \} \subset Q_\alpha, \]
\[ K^{(i,j)} \equiv \text{Span}_Q \{ \alpha_i^{n-2} I \cdot S_{j,h}, \{ 1, \alpha_i, \ldots, \alpha_i^{n-3}, \alpha_i^{n-1} \} S_{j,h} \} \subset Q_\alpha(h). \]

All statements in the next lemma follow immediately from the definitions.

**Lemma 5.2.** If $i \in [n]$,
\[
F \in \tilde{Q}_i[\alpha]_{h_1}^{S_{n-1}}[[q]] \implies \mathfrak{R}_{h=0} F, \mathfrak{R}_{h=\infty} F \in \tilde{Q}_i[\alpha]^{S_{n-1}}[[q]]; \\
F \in \tilde{Q}_i[\alpha]_{h_1,h_2}^{S_{n-1}}[[q]] \implies \mathfrak{R}_{h_1=0} \mathfrak{R}_{h_2=0} \left\{ \frac{F}{h_1 + h_2} \right\} \in \tilde{Q}_i[\alpha]^{S_{n-1}}[[q]]; \\
F \in S_{i,h}[[q]] \implies \mathfrak{R}_{h=-a_i} \left\{ \frac{F}{\prod_{r=1}^{l} (a_r \alpha_i + h)} \right\} \in \tilde{Q}_i[\alpha]^{S_{n-1}}[[q]].
\]

If in addition $F, G \in q \tilde{Q}_i[\alpha]_{h}^{S_{n-1}}[[q]]$, then
\[
F - G \in \mathcal{I} \cdot \tilde{Q}_i[\alpha]_{h}^{S_{n-1}}[[q]] \implies e^F - e^G, \log(1+F) - \log(1+G) \in \mathcal{I} \cdot q \tilde{Q}_i[\alpha]_{h}^{S_{n-1}}[[q]].
\]

### 5.2 The genus zero generating functions

We will now express the genus 0 generating functions $Z^*_i$, $Z^*_{ij}$, and $\tilde{Z}^*_i$ defined in Section 3.3 in terms of the hypergeometric series $F$ of (4.1) and the operator $M$ of (1.2).

**Lemma 5.3.** The genus 0 generating functions $Z^*_i$, $Z^*_{ij}$, and $\tilde{Z}^*_i$ satisfy

\[
[(\alpha_i + h)^n - \alpha_i^n] \left[ 1 + Z^*_i(h, Q) - e^{-J(q)\alpha_i} \frac{\mathcal{F}(\alpha_i/h, q)}{I_0(q)} \right] \in \mathcal{I} \cdot q S_{i,h}[[q]], \tag{5.3}
\]
\[
[(\alpha_j + h)^n - \alpha_j^n] \left[ \alpha_i^{n-2} \alpha_j + h Z^*_j(h, Q) - \alpha_i^{n-2} \alpha_j e^{-J(q)\alpha_i} \frac{\mathcal{M} \mathcal{F}(\alpha_j/h, q)}{I_1(q)} \right] \in K^{(i,j)}_h[[q]], \tag{5.4}
\]
\[
n \alpha_i^{n-1} + 2(h_1 + h_2) h_1 h_2 \tilde{Z}_i^*(h_1, h_2, Q) \left[ -\alpha_i^{n-1} e^{-J(q)\alpha_i} \frac{\mathcal{F}(\alpha_i/h_1, \alpha_i/h_2, q)}{I_0(q)} \right] \in \mathcal{I} \cdot \tilde{Q}_i[\alpha]_{h_1,h_2}^{S_{n-1}}[[q]], \tag{5.5}
\]

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where

\[
\mathbb{F}(w_1, w_2, q) = \sum_{p=0}^{n-1-l} \frac{M^p F(w_1, q) M^{n-1-l-p} F(w_2, q)}{I_p(q)} I_{n-1-l-p}(q)
\]

and \( Q \) and \( q \) are related by the mirror map (1.7).

**Proof.** By [Gi] Theorem 11.8,

\[
1 + Z_i^*(h, Q) = e^{-J(q)\alpha_i + C(q)\sigma_1} \frac{\mathcal{Y}(h, \alpha_i, q)}{I_0(q)}
\]

for some \( C \in qQ[[q]] \) and

\[
\mathcal{Y}(h, x, q) = \sum_{d=0}^{\infty} q^d \prod_{r=1}^{l} \left( a_k x + rh \right)^{a_k \frac{d}{r}}
\]

There is no term \( \prod_{k=1}^{n} (x - \alpha_k) \) in the generating function used in place of \( \mathcal{Y} \) in [Gi], but putting it does not effect the validity of [Gi] Theorem 11.8 as it vanishes under all evaluations \( x \rightarrow \alpha_i \).

On the other hand, with this extra term in place \( \mathcal{Y} \) becomes a function of \( h, x, \sigma_1, \ldots, \sigma_{n-1} \), and not \( \sigma_n \). Since all denominators in \( \mathcal{Y}(h, \alpha_i, q) \) are products of \( \alpha_i - \alpha_k + rh \) with \( r \in \mathbb{Z} \),

\[
\mathcal{Y}(h, \alpha_i, q) - F(\alpha_i / h, q) \in \mathcal{I} \cdot \mathcal{Q}_i[\mathcal{A}_{h}^{S_{n-1}}[[q]]].
\]

If \( a_k \neq 1, 2 \) or \( n \) is odd, the denominators in the above expression do not vanish at \( h = -a_k \alpha_i \), and so the difference lies in \( \mathcal{I} \cdot q \mathcal{S}_i[[q]] \). Otherwise, the denominators have a simple zero at \( h = -a_k \alpha_i \) (in \( q \)-degree at least 2 if \( a_k = 1 \)). If \( a_k = 2 \) and \( n \) is even, the factor \( [(\alpha_i + h)^n - \alpha_i^n] \) has a zero at \( h = -a_k \alpha_i \) as well, and so

\[
[(\alpha_i + h)^n - \alpha_i^n] [\mathcal{Y}(h, \alpha_i, q) - F(\alpha_i / h, q)] \in \mathcal{I} \cdot q \mathcal{S}_i[[q]].
\]

The case \( a_k = 1 \) is excluded by the assumption on \( a \) in Section 1. Thus, (5.3) follows from (5.6).

By (3.2), [Poz] Theorem 4, and the same reasoning as in the previous paragraph,

\[
\sum_{p+r+s=n-1 \atop p, r, s \geq 0} (-1)^s \sigma_5 \alpha_i^p \alpha_j^r + h Z_i^*(h, Q) = e^{-J(q)\alpha_i + C(q)\sigma_1} \frac{\mathcal{M} F(\alpha_i / h, q)}{I_1(q)} \sum_{p+r+s=n-1 \atop p, r, s \geq 0} (-1)^s \sigma_5 \alpha_i^p \mathcal{Y}_r(h, \alpha_j, q)
\]

\[\text{Note that (5.4) implies that}
\]

\[
\alpha_i^{-2} \alpha_j + h Z_i^*(h, Q) - \alpha_i^{-2} \alpha_j e^{-J(q)\alpha_i^2} \frac{\mathcal{M} F(\alpha_i / h, q)}{I_1(q)} \in \text{Span}_q \left\{ \alpha_i^{-2} \mathcal{I} \cdot \mathcal{Q}_i[\mathcal{A}_{h}^{S_{n-1}}[[q]]], 1, \alpha_i, \ldots, \alpha_i^{-3}, \alpha_i^{-1} \right\} \mathcal{Q}_i[\mathcal{A}_{h}^{S_{n-1}}[[q]]];
\]

this is the only information about \( Z_i^*(Q) \) used in computing the non-equivariant part of \( \mathcal{A}_{ij}(Q) \) in Section 5.3.
Let $D$ be defined by

$$D = \sum_{p+r+s=n-1} \frac{(-1)^s \sigma_s \alpha_i^{p+r} + 2(h_1 + h_2)h_1h_2 \tilde{Z}_{ii}(h_1, h_2, Q)}{p, r, s, \sigma \geq 0}$$

(5.9)

By (5.10) and (5.11),

$$\sum_{s_1+s_2=k} (-1)^s \sigma \gamma_{s_1} \gamma_{s_2}(h, \alpha_i, q) = 0 \quad \text{if} \quad n-l \leq k \leq n-1,$$

(5.10)

where $Y_r \in Q_{\alpha}(x)[[q]]$ is a power series such that

$$Y_r(h, \alpha_i, q) \in S_{j,h}[[q]] \quad \text{and}$$

$$[(\alpha_j + h)^n - \alpha_j^n] Y_r(h, \alpha_j, q) \in I \cdot qS_{j,h}[[q]].$$

(5.8)

The claim (5.4) thus follows from (5.7).

Finally, by [PoZ, Theorem 4],

$$\sum_{p+r+s=n-1} (-1)^s \sigma_s \gamma^{p+r} \gamma^{s-1} + 2(h_1 + h_2)h_1h_2 \tilde{Z}_{ii}(h_1, h_2, Q)$$

(5.11)

where $Y_r \in Q_{\alpha}(x)[[q]]$ is a power series such that

$$Y_r(h, \alpha_i, q) \in \tilde{Q}_i[\alpha]^S_{n-1}[[q]].$$

(5.11)

By (5.10) and (5.11),

$$Y_{-p}(h, \alpha_i, q) - \alpha_i^{-n-1} Y_{n-p}(h, \alpha_i, q) \in I \cdot \tilde{Q}_i[\alpha]^S_{n-1}[[q]] \quad \forall \ p = 1, 2, \ldots, l.$$

Thus, (5.5) follows from (5.9), (5.11), and (5.8). □

### 5.3 Proof of Theorem 3

We will use Lemmas 5.1 and 5.3 to extract the coefficients of $\alpha_i^{n-2}$ from the expressions of Proposition 3.1 modulo $K_i[[q]]$. In the notation of Theorem 3 and Proposition 3.1

$$Q \frac{dA(q)}{dQ} = \text{the coefficient of } \alpha_i^{n-2} \text{ in } \left( A_i(Q) + \sum_{j=1}^{n} A_{ij}(Q) \right) \mod K_i[[q]],$$

(5.12)

$$Q \frac{dB(q)}{dQ} = \text{the coefficient of } \alpha_i^{n-2} \text{ in } \left( B_i(Q) + \sum_{j=1}^{n} B_{ij}(Q) \right) \mod K_i[[q]].$$

(5.13)

Let $D \equiv q \frac{d}{dq}$ as in Section 4. We begin by computing residues of the transforms of $\mathcal{F}$ that appear in the description of the generating functions in Lemma 5.3.

#### Lemma 5.4

With $\mu, \Phi_0, L, \mathcal{K} \in Q[[q]]$ given by (4.3), (4.4), and (5.7),

$$\mathcal{R}_{h=0} \left\{ h^{-1} e^{-\mu(q) \gamma} \frac{\alpha_i}{\gamma} M \mathcal{F}(\alpha_j/h, q) \right\} = \frac{L(q) \Phi_0(q)}{I_0(q)},$$

(5.14)

$$\mathcal{R}_{h_1=0} \mathcal{R}_{h_2=0} \left\{ \frac{e^{-\mu(q) \gamma} \alpha_i}{h_1 h_2 (h_1 + h_2)} F(\alpha_i/h_1, \alpha_i/h_2, q) \right\} = \alpha_i^{-1} L(q)^{-1} D \mathcal{K}(q).$$

(5.15)

In the notation of [PoZ], $Y_r(h, x, q) = Y_i(h, x, q)/x^i$.  

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Proof. By Lemmas 4.1 and 4.2 (4.3), (1.2), and the first statement in (4.5),

\[ M^p F(w, q) \sim e^{\mu(q)w} \sum_{s=0}^{\infty} \Phi_{p,s}(q) w^{-s} \quad (w \to \infty), \]  

where

\[ \Phi_{0,s} = \Phi_s, \quad \Phi_{p+1,s} = \frac{L}{I_p} \Phi_{p,s} + \left\{ \begin{array}{ll} D \left( \frac{\Phi_{p+1,s}}{I_p} \right), & \text{if } s \geq 1, \\ 0, & \text{if } s = 0. \end{array} \right. \]

The \( s=0, 1 \) cases of the recursion give

\[ \Phi_{p,0} = H_p \Phi_0, \quad \Phi_{p,1} = H_p \left( \Phi_1 + pD \left( \frac{\Phi_0}{L} \right) + \Phi_0 \sum_{r=1}^{l} \frac{DH_r}{H_r} \right) \]

where \( H_p = L^p/\left( I_0 \ldots I_{p-1} \right) \) as in Section 4.2. The \( p=1 \) case of the first identity gives (5.14).

On the other hand, for all \( p, r \geq 0 \)

\[ \Re_{h_1=0} \Re_{h_2=0} \left\{ \frac{e^{-\mu(q)\alpha_i h_1^{-1}h_2^{-1}}}{h_1 h_2(h_1+h_2)} M^p F(\alpha_i/h_1, q)M^p F(\alpha_i/h_2, q) \right\} \]

\[ = \Re_{h_1=0} \left\{ \frac{e^{-\mu(q)\alpha_i h_1}}{h_1^2} M^p F(\alpha_i/h_1, q) \cdot \Phi_{r,0}(q) \right\} = \alpha_i^{-1} \Phi_{p,1}(q) \Phi_{r,0}(q). \]

By (4.7) to (4.9),

\[ H_p H_{n-1-l-p} = \frac{I_p I_{n-1-l-p}}{L^{l+1}} \quad \text{if } 0 \leq p \leq n-1-l, \]

\[ H_{n-1-l+p} H_{n-p} = \frac{I_{n-1-l+p} I_{n-p}}{L^{l+1}} \quad \text{if } 1 \leq p \leq l. \]

Thus, by (5.17) and the second statement in (4.5),

\[ \Re_{h_1=0} \Re_{h_2=0} \left\{ \frac{e^{-\mu(q)\alpha_i h_1^{-1}h_2^{-1}}}{h_1 h_2(h_1+h_2)} F(\alpha_i/h_1, \alpha_i/h_2, q) \right\} \]

\[ = \alpha_i^{-1} L^{-\frac{n-1}{3}} \left( n \Phi_1 + \left( \frac{n}{2} \right) D \left( L^{\frac{1}{3}} \right) + L^{\frac{1}{3}} \sum_{r=1}^{n-1} (n-r) \frac{DH_r}{H_r} \right) \]

By (4.9) and (4.10),

\[ \sum_{r=1}^{n-1} (n-r) \frac{DH_r}{H_r} = \sum_{r=1}^{n-1} \frac{DH_r}{H_r} + \sum_{r=1}^{l-1} \frac{DH_r}{H_r} + \sum_{r=1}^{l-1} \frac{DH_{n-l+r}}{H_{n-l+r}} \]

\[ = \left( \frac{n-l+1}{3} \right) \frac{DL}{L} - \sum_{r=1}^{n-1} \sum_{p=0}^{n-r-1} \frac{DI_p}{I_p} - \frac{(l-1)(l(n-l)-DL)}{2} L - \frac{(l-1)(2l-1)}{6} \frac{DL}{L} \]

\[ = \left( \frac{n-l+1}{3} \right) \frac{DL}{L} - \frac{(l-1)(3n-l-1)}{6} \frac{DL}{L} - \sum_{p=0}^{n-l-1} \frac{(n-l-p)}{2} \frac{DI_p}{I_p}. \]

The second identity in the lemma follows from the last two equations along with (4.5), (4.6), \( DL = L(L^n-1)/n \), and (4.4). \( \Box \)
By the Residue Theorem on $S^2$. By Lemma 5.2 (5.3), (3.14), (3.15), and (4.3),
\[ \eta_i(Q) - (\mu(q) - J(q))\alpha_i \in \mathcal{I} \cdot q\mathfrak{Q}_i[\alpha]_{S^2-1}[[q]], \tag{5.18} \]
\[ \Phi_0(\alpha_i, Q) - \frac{\Phi_0(q)}{I_0(q)} \in \mathcal{I} \cdot q\mathfrak{Q}_i[\alpha]_{S^2-1}[[q]]. \tag{5.19} \]
Thus, by Proposition 3.1 Lemma 5.2, (5.5), (5.15), and (5.17),
\[ \mathcal{A}_i(Q) - \alpha_i^{n-2} \frac{I_0(q)DA_i(q)}{2L(q)\Phi_0(q)} \in \mathcal{I} \cdot q\mathfrak{Q}_i[\alpha]_{S^2-1}[[q]]. \tag{5.20} \]
By Proposition 3.1 Lemma 5.2 (5.4), (3.15), (4.3), and (5.18), (5.20),
\[ \sum_{j=1}^{n} \mathcal{A}_{ij}(Q) - \sum_{j=1}^{n} \left( \alpha_j^{n-2} \frac{I_0(q)DA_i(q)}{2L(q)\Phi_0(q)} \prod_{k \neq j}(\alpha_j - \alpha_k) \right) \in \mathcal{K}_i[[q]]. \tag{5.21} \]
By the Residue Theorem on $S^2$,
\[ \sum_{j=1}^{n} \frac{\alpha_j^{n-1}}{\prod_{k \neq j}(\alpha_j - \alpha_k)} = \sum_{j=1}^{n} \mathfrak{R}_{z=\alpha_j} \left\{ \frac{z^{n-1}}{\prod_{k=1}^{n}(z - \alpha_k)} \right\} = -\mathfrak{R}_{z=\infty} \left\{ \frac{z^{n-1}}{\prod_{k=1}^{n}(z - \alpha_k)} \right\} = 1. \tag{5.22} \]
Thus, by (5.20) and (5.21),
\[ \mathcal{A}_i(Q) + \sum_{j=1}^{n} \mathcal{A}_{ij}(Q) - \alpha_i^{n-2} \frac{DA_i(q)}{2I_1(q)} \in \mathcal{K}_i[[q]]. \]
Since \( \frac{1}{I_1(q)}D = Q \frac{d}{dq} \), this proves the claim stated in the sentence after Theorem 3.
We next compute (5.13). Let
\[ \tilde{\mathbb{B}}(q) = -\frac{24}{(a)} \tilde{B}(q) \tag{5.23} \]
with \( \tilde{B}(q) \) as in Theorem 3 and
\[ \alpha_j^{n-2-l}B(q) = \mathfrak{R}_{h=0, \infty, -a\alpha_j} \left\{ \frac{(\alpha_j + h)^n - \alpha_j^n}{h^3 \prod_{r=1}^{l}(a_r \alpha_j + h)} \frac{e^{-J(q)\alpha_j}}{\mathcal{F}(\alpha_j/q, I_0(q))} - 1 \right\} \tag{5.24} \]
\[ \mathfrak{R}_{h=0, \infty, -a\alpha_j} \left\{ \frac{(\alpha_j + h)^n - \alpha_j^n}{h^3 \prod_{r=1}^{l}(a_r \alpha_j + h)} \frac{e^{-J(q)\alpha_j}}{\mathcal{F}(\alpha_j/q, I_0(q))} - 1 \right\} = \alpha_j^{n-2-l} \left( Q \frac{d}{dq} \tilde{\mathbb{B}}(q) + B(q) \right); \tag{5.24} \]
\[ \mathfrak{R}_{h=0, \infty, -a\alpha_j} \left\{ \frac{(\alpha_j + h)^n - \alpha_j^n}{h^2 \prod_{r=1}^{l}(a_r \alpha_j + h)} \frac{\log \frac{e^{-J(q)\alpha_j}}{\mathcal{F}(\alpha_j/q, I_0(q))}}{I_0(q)} \right\} = \alpha_j^{n-1-l} \tilde{\mathbb{B}}(q). \tag{5.25} \]

Lemma 5.5. With notation as above,
Proof. Since \( I_1(q) = 1 + q \frac{d}{dq} J(q) \) and \( Q \frac{d}{dq} = \frac{1}{I_1(q)} q \frac{d}{dq}, \)
\[
e^{-J(q)w} \frac{M F(w, q)}{I_1(q)} = \frac{1}{I_1(q)} \left\{ 1 + q \frac{d}{dq} (q + \frac{q}{w} \frac{d}{dq}) \left( e^{-J(q)w} \frac{F(w, q)}{I_0(q)} \right) \right\} = \left\{ 1 + \frac{Q}{w} \frac{d}{dq} \right\} \left( e^{-J(q)w} \frac{F(w, q)}{I_0(q)} \right).
\]
Along with the definition of \( B \) and \( (5.25) \), this gives \( (5.24) \).

By the Residue Theorem on \( S^2 \), the terms \( e^{-J(q)\eta_j/h} \) and \( I_0(q) \) do not effect the left-hand side of \( (5.25) \). Since the coefficients of the positive powers of \( \left[ (\alpha_j + \mu)^n - \alpha_j^n \right] F(\alpha_j/h, q) \) vanish at \( h = -a_k \alpha_j \) to the same order as \( \prod_{r=1}^{l} (a_r \alpha_j + h) \)^{10}, it follows that
\[
\mathcal{R}_{h=0, \infty, -a\alpha_j} \left\{ \frac{(\alpha_j + h)^n - \alpha_j^n}{h^2 \prod_{r=1}^{l} (a_r \alpha_j + h)} \log \left( e^{-J(q)\eta_j/h} \frac{F(\alpha_j/h, q)}{I_0(q)} \right) \right\} = \mathcal{R}_{h=0, \infty} \left\{ \frac{(\alpha_j + h)^n - \alpha_j^n}{h^2 \prod_{r=1}^{l} (a_r \alpha_j + h)} \log F(\alpha_j/h, q) \right\}.
\]
Expanding \( (\alpha_j + h)^n - \alpha_j^n \) and using \( (4.3) \), we obtain
\[
\mathcal{R}_{h=0} \left\{ \frac{(\alpha_j + h)^n - \alpha_j^n}{h^2 \prod_{r=1}^{l} (a_r \alpha_j + h)} \log F(\alpha_j/h, q) \right\} = \frac{\alpha_j^{n-l}}{\langle \alpha \rangle} \left\{ \left( \frac{n}{2} \right) - n \sum_{r=1}^{l} \frac{1}{a_r} \right\} \mu(q) + n \log \Phi_0(q) \right) \right\}.
\]
On the other hand,
\[
\mathcal{R}_{h=\infty} \left\{ \frac{(\alpha_j + h)^n - \alpha_j^n}{h^2 \prod_{r=1}^{l} (a_r \alpha_j + h)} \log F(\alpha_j/h, q) \right\} = -\mathcal{R}_{w=0} \left\{ \frac{(\alpha_j w + l)^n - \alpha_j^n w^n}{w^{n-l} \prod_{r=1}^{l} (1 + a_r \alpha_j w)} \log F(w \alpha_j, q) \right\} = -\alpha_j^{n-1-l} \sum_{p=0}^{n-l} \left[ \frac{(1+w)^n}{\prod_{k=1}^{l} (1+a_k w)} \right] w^{n-1-l-p} \log F(w, q) \right] w:p.
\]
Since \( [F(w, q)]_{w:0} = I_0(q) \) and \( [F(w, q)]_{w:1} = J(q) \), \( (5.25) \) follows by adding up the last two equations and using \( (2.4) \), the second identity in \( (4.3) \), and \( (4.4) \). \( \square \)

^{10} because \( a_k \neq 1 \) by assumption
We now complete the proof of Theorem 3. By Proposition 3.1, Lemma 5.2, (5.3), and the definition of $B(q)$ above,

$$B_i(Q) - \alpha_i^{n-2}(\frac{a}{24})B(q) \in I \cdot \tilde{Q}_i[a]S_{n-1}[[q]].$$

By Proposition 3.1, Lemma 5.2, (5.4), and (5.24),

$$\sum_{j=1}^{n} \tilde{B}_{ij}(Q) + \frac{\langle a \rangle}{24} \sum_{j=1}^{n} \left( \frac{\alpha_j^{n-2}}{\prod_{k \neq j} (\alpha_j - \alpha_k)} \left[ Q \frac{d\tilde{B}}{dQ} (q) + B(q) \right] \alpha_i^{n-2} \alpha_j \right) \in K_i[[q]].$$

By the last two equations, (5.22), and (5.23),

$$B_i(Q) + \sum_{j=1}^{n} \tilde{B}_{ij}(Q) - \alpha_i^{n-2} Q \frac{d\tilde{B}(q)}{dQ} \in K_i[[q]].$$

This concludes the proof of Theorem 3.

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