On some collinear configurations in the planar three-body problem

Alexei Tsygvintsev

UMPA, Ecole Normale Supérieure de Lyon, Lyon, France
E-mail: alexei.tsygvintsev@ens-lyon.fr

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Abstract

In this paper, we further investigate the planar Newtonian three-body problem with a focus on collinear configurations, where either the three bodies or their velocities are aligned. We provide an independent proof of Montgomery’s result (Montgomery 2007 *Ergod. Theory Dyn. Syst.* 27 311–40), stating that apart from the Lagrange’s solution, all negative energy solutions to the zero angular momentum case result in syzygies, i.e. collinear configurations of positions. The concept of generalised syzygies, inclusive of velocity alignments, was previously explored by the author for bounded solutions in Tsygvintsev (2023 *C. R. Acad. Sci., Paris* 361 331–5). In this study, we broaden our scope to encompass negative energy cases and provide new bounds. Our methodology builds upon the elementary Sturm–Liouville theory and the Wintner-Conley ‘linear’ form of the three-body problem, as previously explored in the works of Albouy and Chenciner (1997 *Invent. Math.* 131 151–84); Albouy (2004 *Mutual Distances in Celestial Mechanics* (Lectures at Nankai Institute)); Chenciner (2013 *Acta Math. Vietnam.* 38 165–86).

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1. Introduction

We consider the motion of three points with masses $m_1$, $m_2$, and $m_3$, denoted $P_1$, $P_2$, and $P_3$, respectively, in the plane. These points have positions $(x_i, y_i) \in \mathbb{R}^2$, where $i = 1, 2, 3$.

The Newtonian three-body problem [9] involves finding the motion of three point masses under the influence of their mutual gravitational attraction.
The equations of motion can be written in the complex compact form as follows
\begin{align}
\ddot{z}_1 &= m_2 \frac{z_{21}}{|z_{21}|^3} - m_3 \frac{z_{13}}{|z_{13}|^3}, \\
\ddot{z}_2 &= m_3 \frac{z_{32}}{|z_{32}|^3} - m_1 \frac{z_{21}}{|z_{21}|^3}, \\
\ddot{z}_3 &= m_1 \frac{z_{13}}{|z_{13}|^3} - m_2 \frac{z_{32}}{|z_{32}|^3},
\end{align}
(1.1)
where $z_k = x_k + iy_k \in \mathbb{C}$, $k = 1, 2, 3$ and $z_{kl} = z_k - z_l$. We assume that the total linear momentum is zero:
\begin{equation}
\sum_k m_k \dot{z}_k = \sum_k m_k z_k = 0,
\end{equation}
(1.2)
by placing the centre of mass at the origin of the coordinate system.

The word *syzygy* (from Late Latin *syzygia* = ‘conjunction’) historically been used by astronomers to describe the alignment of celestial bodies, and in this context, it refers to a configuration where all three points lie on a straight line. In general, a solution of the $N$-body problem is said to have a syzygy at $t = t_0$ if at that moment all bodies belong to a certain straight line.

Let $t \mapsto z_i(t)$, $i = 1, 2, 3$ be any solution of the equations (1.1) defined for $t \in I = [0, a]$, $a > 0$. In our work [8] we proposed to study the natural generalisation of syzygies by adding the supplementary condition of collinearity of velocities.

**Definition 1.1.** The three bodies $P_1, P_2, P_3$ form a generalised syzygy at the moment $t_0 \in I$ if at least one of the complex triplets $(z_1, z_2, z_3)(t_0)$ (positions) or $(\dot{z}_1, \dot{z}_2, \dot{z}_3)(t_0)$ (velocities) belongs to the same straight line passing through the origin (see figure 1).

In his work [6], Montgomery demonstrated that aside from the Lagrange solution, every solution to the Newtonian three-body problem with zero angular momentum and negative energy inevitably encounters a syzygy, defined as a collinear configuration of positions. Non-syzygy solutions with nonzero angular momentum are still relatively unexplored. Diacu demonstrated in [5] that the set of initial conditions leading to syzygy solutions in the planar three-body problem is non-empty and open.

In particular, the orbit of one particle crosses the line of the other two and can not be tangent to this line in the transition point. Generalised syzygies may occur more frequently, as they include collinear configurations of velocities as well.

The paper is organised as follows.

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**Figure 1.** A generalised syzygy: (A) – the positions are collinear (eclipse), (B) – the velocities (dashed arrows) are collinear.
In chapter 2, we provide a new independent proof of Montgomery’s result [6] on the existence of syzygies in the zero angular momentum case. The general approach involves formulating a second-order ‘linear’ matrix equation \( \ddot{X} = AX \) (see [1–3] for more detail), where the matrix \( X \) characterises the configuration. The term ‘linear’ is used since the matrix \( A \) also depends on the configuration. However, despite this dependence, we can draw certain conclusions based on the general properties of \( A \).

In chapter 3, we impose a certain algebraic restriction on the mutual distances between three bodies. In the case of periodic behaviour, this restriction guarantees the existence of a syzygy.

Finally, in chapter 4, we present a simple algebraic condition that defines an open set of initial conditions leading to a generalised syzygy, which holds for arbitrary angular momentum.

2. Zero angular momentum case

Let \( \Gamma : t \mapsto (z_1(t), z_2(t), z_3(t)) \) be a zero angular momentum solution with negative energy of the three-body problem (1.1). In this section we prove that if \( \Gamma \) is collision free for a sufficiently long period of time, it will always encounter a syzygy. A similar result was first established by Montgomery in [6] using the shape sphere approach. Our proof is essentially algebraic and is more elementary.

Before proceeding further, we need to establish some preliminary results.

After introducing the new variables \( w_i = m_i z_i, \ i = 1, 2, 3 \) the relations (1.2) yield

\[
\sum_i w_i = \sum_i \dot{w}_i = 0. \tag{2.1}
\]

Writing

\[
w_k = X_k + iY_k, \quad X_k = m_k x_k, \quad Y_k = m_k y_k, \quad k = 1, 2, 3, \tag{2.2}
\]

and using (2.1), one derives from the equation (1.1) the following \( 2 \times 2 \) matrix equation

\[
\ddot{X} = AX, \quad X = \begin{bmatrix} X_1 & Y_1 \\ X_2 & Y_2 \end{bmatrix}, \quad A = \begin{bmatrix} -m_2 \rho_3 - m_1 \rho_2 & m_1 \rho_2 \\ m_2 \rho_3 & -m_1 \rho_3 - m_2 \rho_1 \end{bmatrix}, \tag{2.3}
\]

where

\[
\rho_1 = 1/|z_{23}|^3, \quad \rho_2 = 1/|z_{13}|^3, \quad \rho_3 = 1/|z_{21}|^3, \quad m_{ij} = m_i + m_j, \quad \rho_{ij} = \rho_i - \rho_j. \tag{2.4}
\]

The matrix \( A \) is related to the Wintner-Conley endomorphism encoding the forces. Its higher dimension version was investigated in [1–3] in the study of \( n \)-body configurations and the same equation (2.3) was used (up to reduction, transposition and some scaling factor).

**Theorem 2.1.** Let \( t \mapsto (z_1(t), z_2(t), z_3(t)), \quad t \in [0, T_1] \) be a zero angular momentum collision-free solution to the three-body problem (1.1) with negative energy \( H = -\alpha, \ \alpha > 0 \) where

\[
T_1 (\alpha) = \frac{\sqrt{2\pi} \Sigma}{\alpha^{3/2}}, \tag{2.5}
\]
Introducing

By introducing

Then there exists \( t_0 \in [0, T_1] \) such that the three bodies form a syzygy at the moment \( t = t_0 \).

Proof. Introducing \( x = [X_1, X_2]^T, y = [Y_1, Y_2]^T \) and using the equation (2.3), we obtain \( \ddot{x} = Ax, \ \dot{y} = Ay \). Therefore, \( \Delta_1 \) can be written as follows

A simple algebraic computation shows that det\((Ax, y)\) + det\((x, Ay)\) = det\((Ax, y)\) + det\((x, Ay)\) + 2\( \Delta_2 \).

Thus, as follows from (2.7),

\[ \ddot{\Delta}_1 = \text{Tr}(A) \Delta_1 + 2\Delta_2, \]

where

\[ \text{Tr}(A) = -(m_{32}\rho_1 + m_{13}\rho_2 + m_{21}\rho_3). \]

Lemma 2.1. Let \( t \rightarrow z(t), i = 1, 2, 3 \) be any solution of the three-body problem (1.1) with negative energy \( H = -\alpha \), where \( \alpha > 0 \). Then \( \text{Tr}(A) \leq -\alpha^3/\Sigma^2 \) along this solution with \( \Sigma \) defined by (2.6).

Proof. By introducing \( r_i = \rho_i^{1/3}, i = 1, 2, 3 \) the total energy of the three-body problem (1.1) can be written as

\[ H = K - U(r), \quad U(r) = m_3m_2r_1 + m_1m_3r_2 + m_2m_1r_3, \quad r = (r_1, r_2, r_3), \]

where \( K \) is the kinetic energy.

Since \( K - U = -\alpha \) and \( K \geq 0 \), one obtains \( U \geq \alpha \).

Writing \( F(r) = -\text{Tr}(A)(r) = m_{32}r_1^3 + m_{13}r_2^3 + m_{21}r_3^3 \), we shall calculate, for any \( s > 0 \), the minimum of the function \( r \rightarrow F(r) \) on the compact set

\[ K_s = \{ r \in \mathbb{R}^3 | U(r) = s, r_i \geq 0, i = 1, 2, 3 \}. \]

Because \( K_s \) is convex (triangle) and \( F \) is a convex function on \( K_s \) (since \( d^2F \geq 0 \) on \( K \)), it is sufficient to determine its local minimum. One computes:

\[ \nabla U = (m_3m_2m_1, m_1m_3m_2, m_2m_1m_3), \quad \nabla F = 3(m_{32}r_1^2, m_{13}r_2^2, m_{21}r_3^2). \]

The Lagrange multiplier \( \lambda \) found from the equations \( \nabla F = \lambda \nabla U, U = s \) is given by

\[ \lambda = \frac{3s^2}{\Sigma^2}, \]

with \( \Sigma \) defined in (2.6).

The corresponding extremum point is

\[ r^* = (r_1^*, r_2^*, r_3^*) = \frac{s}{\Sigma} \left( \sqrt{\frac{m_3m_2}{m_{32}}}, \sqrt{\frac{m_1m_3}{m_{13}}}, \sqrt{\frac{m_2m_1}{m_{21}}} \right) \in K_s. \]
which is a local minimum of $F$ because $d^2F(r^*)$ is positive definite and $d^2U = 0$. Thus, by substitution:

$$\min_{r \in K} F(r) = F(r^*) = \frac{s^3}{\Sigma^2}.$$  \hfill(2.15)

Hence, considering $s \geqslant \alpha$, one shows that $U(r) \geqslant \alpha$ implies $F(r) = -\text{Tr}(A)(r) \geqslant \frac{\alpha^3}{\Sigma^2}$. The proof of the lemma 2.1 is finished. \hfill\square

The function $t \mapsto C(t) = \dot{X} X^{-1}$ satisfies, using (2.3), the following matrix Riccati equation

$$\dot{C} + C^2 = A.$$  \hfill(2.16)

Combined with the Cayley-Hamilton identity

$$C^2 - \text{Tr}(C) C + \det(C) I_2 = 0,$$  \hfill(2.17)

the equation (2.16) yields

$$\dot{C} + \text{Tr}(C) C = A + \det(C) I_2.$$  \hfill(2.18)

Applying Liouville's formula to the equation $\dot{X} = CX$, we obtain:

$$\dot{\Delta}_1 = \text{Tr}(C) \Delta_1.$$  \hfill(2.19)

Multiplying both sides of equation (2.18) by $\Delta_1$, we get

$$\frac{d}{dt}(\Delta_1 C) = \Delta_1 A + \Delta_1 \text{det}(C) I_2 = \Delta_1 A + \Delta_2 I_2,$$  \hfill(2.20)

where we have used $\det(C) = \Delta_2/\Delta_1$.

Introducing the adjugate matrix $\tilde{X}$ of $X$ and using $C = \dot{X} X^{-1}$, we compute

$$\Delta_1 C = \dot{XX} = \begin{bmatrix} \dot{X}_1 & \dot{Y}_1 \\ \dot{X}_2 & \dot{Y}_2 \end{bmatrix} \begin{bmatrix} Y_2 & -Y_1 \\ -X_2 & X_1 \end{bmatrix}. $$  \hfill(2.21)

The matrix $A(\rho_1, \rho_2, \rho_3)$, defined in (2.3) can be written as a linear combination

$$A = \rho_1 A_1 + \rho_2 A_2 + \rho_3 A_3,$$  \hfill(2.22)

where $A_1$, $A_2$, and $A_3$ are $2 \times 2$ constant matrices that depend on the masses and are defined by

$$A_1 = \begin{bmatrix} 0 & 0 \\ -m_2 & -m_3 \end{bmatrix}, A_2 = \begin{bmatrix} -m_1 & -m_1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} -m_2 & m_1 \\ m_2 & -m_1 \end{bmatrix}.$$  \hfill(2.23)

Let $E = M_3(\mathbb{R})$ be the Euclidean space of real $2 \times 2$ matrices equipped with the inner product $\langle , \rangle$ defined by $\langle A, B \rangle = \text{Tr}(A^T B)$ for $A, B \in E$.

We check that $A_1, A_2, A_3 \in E$ are linearly independent and span a 3-dimensional vector subspace of $E$. Furthermore, we have the identity

$$-\frac{A_1 + A_2 + A_3}{M} = I_2,$$  \hfill(2.24)

where $M = \sum m_i$. 

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Combining equations (2.22), (2.24) and (2.20), we obtain

\[
\frac{d}{dt}(\dot{X}\ddot{X}) = \left(\Delta_1\rho_1 - \frac{\Delta_2}{M}\right)A_1 + \left(\Delta_1\rho_2 - \frac{\Delta_2}{M}\right)A_2 + \left(\Delta_1\rho_3 - \frac{\Delta_2}{M}\right)A_3. \tag{2.25}
\]

Let us check, with help of the equation (2.25), constancy of the angular momentum. We introduce the matrix

\[
L = \begin{bmatrix}
-m_3^{-1} & m_1^{-1} + m_3^{-1} \\
-m_2^{-1} - m_3^{-1} & m_3^{-1}
\end{bmatrix}. \tag{2.26}
\]

It can be checked that \(\langle L, A_i \rangle = 0, i = 1, 2, 3\).

Taking the inner product of both sides of the equation (2.25) with \(L\), we find that

\[
\frac{d}{dt}(\dot{X}\ddot{X}), L = 0, \tag{2.27}
\]

i.e. the function \(I = \langle \dot{X}\ddot{X}, L \rangle = \text{const}\) is time independent.

One can easily verify that \(I, \text{written as a function of variables } X_i, Y_j, 1 \leq i, j \leq 2\) (by excluding \((X_3, Y_3)\) with help of (2.1)) is the angular momentum of the three-body problem (1.1):

\[
I = \sum_{i=1}^{3} m_i R_i \times \dot{R}_i = \sum_{i=1}^{3} \frac{S_i}{m_i}, \tag{2.28}
\]

where \(R_i = (x_i, y_i), S_i = X_i \dot{Y}_i - Y_i \dot{X}_i, 1 \leq i \leq 3\).

The equation (2.20) can be written, by replacing \(\Delta_2\) using equation (2.8), in the following form

\[
\frac{d}{dt}\left(\dot{X}\ddot{X} - \frac{\hat{A}_1}{2} I_2\right) = \Delta_1 \left(A - \frac{\text{Tr}(A)}{2} I_2\right). \tag{2.29}
\]

Applying (2.22), we can transform it to

\[
\frac{d}{dt}\left(\dot{X}\ddot{X} - \frac{\hat{A}_1}{2} I_2\right) = \Delta_1 \sum_{i=1}^{3} \rho_i \hat{A}_i, \tag{2.30}
\]

where \(\hat{A}_i = A_i - \frac{\text{Tr}(A_i)}{2} I_2, \text{Tr}(\hat{A}_i) = 0, 1 \leq i \leq 3\).

It can be shown, using (2.23), (2.24), that \(\hat{A}_1 + \hat{A}_2 + \hat{A}_3 = 0\). Substituting \(\hat{A}_3 = -\hat{A}_1 - \hat{A}_2\) in (2.30) we obtain finally

\[
\frac{d}{dt}\left(\dot{X}\ddot{X} - \frac{\hat{A}_1}{2} I_2\right) = \Delta_1 (\rho_1 - \rho_3) \hat{A}_1 + \Delta_1 (\rho_2 - \rho_3) \hat{A}_2, \tag{2.31}
\]

where

\[
\hat{A}_1 = \begin{bmatrix}
m_{32}/2 & 0 \\
-m_2 & -m_{32}/2
\end{bmatrix}, \hat{A}_2 = \begin{bmatrix}
-m_{23}/2 & -m_1 \\
0 & m_{23}/2
\end{bmatrix}, \quad m_{ij} = m_i + m_j. \tag{2.32}
\]
Moreover, as follows from (2.19), (2.21), one always has

\[ \text{Tr} (\dot{XX}) = \Delta_1, \quad \det (\dot{XX}) = \Delta_1 \Delta_2. \]

(2.33)

Let \( I = \langle \dot{XX}, L \rangle = k \) where \( k \in \mathbb{R} \) is the constant value of the angular momentum. Computing \( \langle \dot{XX}, L \rangle \) using (2.21), it is easy to derive the following equality:

\[ \dot{XX} = \frac{\Delta_1}{2} I_2 = \frac{b}{m_2} \dot{A}_1 - \frac{a}{m_1} \dot{A}_2 - k \frac{m_3}{2} J, \]

(2.34)

where

\[ a = \begin{vmatrix} X_1 & Y_1 \\ \dot{X}_1 & \dot{Y}_1 \end{vmatrix}, \quad b = \begin{vmatrix} X_2 & Y_2 \\ \dot{X}_2 & \dot{Y}_2 \end{vmatrix}, \quad J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \]

(2.35)

For \( k = 0 \), the formula (2.34) is given by

\[ \dot{XX} = \frac{\Delta_1}{2} I_2 = \frac{b}{m_2} \dot{A}_1 - \frac{a}{m_1} \dot{A}_2 = \begin{bmatrix} \beta a + \gamma b \\ -b \\ -\beta a - \gamma b \end{bmatrix} = R, \]

(2.36)

where

\[ \beta = \frac{1}{2} \left( \frac{m_3}{m_1} + 1 \right), \quad \gamma = \frac{1}{2} \left( \frac{m_3}{m_2} + 1 \right), \]

(2.37)

and

\[ -\det(R) = \beta^2 a^2 + (2 \beta \gamma - 1) ab + \gamma^2 b^2. \]

(2.38)

As follows from (2.37), one always has \( \beta, \gamma > 1/2 \). Using these inequalities, it is easy to show that the quadratic form in (2.38), as a function of \( a \) and \( b \), is positive and therefore

\[ \det(R)(a,b) \leq 0, \quad \forall a, b \in \mathbb{R}. \]

(2.39)

Since \( \text{Tr}(R) = 0 \), the eigenvalues of \( R \) are real. Hence, according to (2.36), the eigenvalues \( \lambda_1, \lambda_2 \) of \( \dot{XX} \) are also real. Thus, using (2.33), we have the following inequality holding for arbitrary \( a, b \in \mathbb{R} \)

\[ \text{Tr} (\dot{XX})^2 - 4 \det (\dot{XX}) = \Delta_1^2 - 4 \Delta_1 \Delta_2 \geq 0. \]

(2.40)

Suppose that the solution \( t \mapsto (z_1(t), z_2(t), z_3(t)) \) is syzygy free in the interval \([0, T_1]\). In particular, we can suppose

\[ \Delta_1 (t) > 0, \quad \forall t \in [0, T_1], \]

(2.41)

so that the function \( \delta : t \mapsto \sqrt{\Delta_1(t)} \) is differentiable in this interval.

Differentiating \( \delta \) twice using (2.8), we obtain

\[ \dot{\delta} = \eta \dot{\delta}, \quad \eta = \left( \frac{\text{Tr}(A)}{2} - \frac{\dot{\Delta_1} - 4 \Delta_1 \Delta_2}{4 \Delta_1^2} \right), \quad t \in [0, T_1]. \]

(2.42)
According to (2.40) and lemma 2.1:
\[ \eta(t) \leq \frac{\text{Tr}(A)}{2} \leq -\zeta^2, \quad \zeta^2 = \frac{\alpha^3}{2^{1/2}}, \quad t \in [0, T_1]. \] (4.33)

As follows from the zero comparison theorem of the Sturm–Liouville theory [4], the solution \( \delta \) of (2.42) always has a zero between any two consecutive zeros of any solution \( y \) of the equation \( \ddot{y} = -\zeta^2 y \) whose general solution is \( y(t) = A\cos(\zeta t + \phi_0), A, \phi_0 \in \mathbb{R} \). For a nonzero \( A \), every two consecutive zeros of \( y \) are separated by an interval of the length \( \pi/\zeta \). Hence, \( \delta(t_0) = \Delta_1(t_0) = 0 \) for some \( 0 < t_0 < T_1 = \pi/\zeta \) with \( T_1 \) given by the formula (2.5). This contradicts our hypothesis (2.41). The proof of theorem 2.1 is finished.

\[ \square \]

3. Existence of syzygies for periodic solutions

In this section we will show that any periodic solution of the three-body problem will have a syzygy, i.e. exhibits a collinear configuration of the bodies as soon as some geometrical constraints are imposed on the shape of the triangle formed by the bodies. Not every periodic solution has a syzygy; for example, the Lagrange periodic solution has the three bodies situated at the vertices of an equilateral triangle and never aligned.

**Definition 3.1.** A periodic solution of the three-body problem (1.1) is called \( \theta \)-rigid if there exists a non-zero vector \( \theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3 \) such that
\[ \theta_1(\rho_3(t) - \rho_2(t)) + \theta_2(\rho_1(t) - \rho_3(t)) + \theta_3(\rho_2(t) - \rho_1(t)) \geq 0, \quad \forall t \in \mathbb{R}, \] (3.1)
where the sum is strictly positive for at least one \( t_0 \in \mathbb{R} \).

For instance, let us consider a periodic solution for which
\[ |z_{32}(t)| > |z_{13}(t)| \quad \Leftrightarrow \quad \rho_1(t) < \rho_2(t), \quad \forall t \in \mathbb{R}. \] (3.2)

Thus, it is \( \theta \)-rigid with respect to the vector \( (\theta_1, \theta_2, \theta_3) = (0, 0, 1) \).

An example of this is Euler’s collinear periodic solution, in which all bodies are perpetually collinear and each body describes an elliptical orbit, implying that a syzygy occurs at every moment.

We note that the Lagrange equilateral solution is not \( \theta \)-rigid for any choice of \( \theta \). Indeed, in this case \( \rho_1 = \rho_2 = \rho_3 \) and, therefore, the sum in (3.1) is always zero.

One might ask whether it is possible to have a periodic solution without syzygies, in which one side of the triangle is always smaller than the other and thus satisfies one of the conditions of the form (3.2).

Our next theorem provides a negative response to this question.

**Theorem 3.1.** Every \( \theta \)-rigid periodic solution to the three-body problem (1.1) admits a syzygy.

**Proof.** Let \( t \mapsto z_i(t), i = 1, 2, 3 \) be a periodic solution of the three-body problem (1.1) with period \( \tau > 0 \).

Let us suppose that it is \( \theta \)-rigid and has no syzygies i.e. \( \Delta_1(t) = \det(X(t)) \neq 0, \forall t \in [0, \tau] \). Without loss of generality, we can assume that
\[ \Delta_1(t) > 0, \quad \forall t \in [0, \tau]. \] (3.3)
Let \( S_i = X_i \dot{Y}_i - X_i Y_i, i = 1, 2, 3 \) be three oriented areas of parallelograms formed by the vectors \((X_i, Y_i)\) and \((\dot{X}_i, \dot{Y}_i)\). Then, using (2.3) and (2.1), one derives the following equations

\[
\dot{S}_1 = m_1 \Delta_1 (\rho_3 - \rho_2), \quad \dot{S}_2 = m_2 \Delta_1 (\rho_1 - \rho_3), \quad \dot{S}_3 = m_3 \Delta_1 (\rho_2 - \rho_1). \tag{3.4}
\]

Thus,

\[
\frac{d}{dt} \left( \sum_{i=1}^{3} \frac{\theta_i S_i}{m_i} \right) = \Delta_1 S, \quad S = \theta_1 (\rho_3 - \rho_2) + \theta_2 (\rho_1 - \rho_3) + \theta_3 (\rho_2 - \rho_1), \quad t \in [0, \tau]. \tag{3.5}
\]

Integrating (3.5) and using the periodicity of \( S_i, i = 1, 2, 3 \) we find

\[
\int_0^\tau \Delta_1 S \, dt = 0, \tag{3.6}
\]

which obviously contradicts (3.3) and that \( S(t_0) > 0 \) for some \( t_0 \in [0, \tau] \). The proof is finished. \( \square \)

As pointed out by Richard Montgomery to the author, in the case of the \( \theta = (0, 0, 1) \) rigid solution, the corresponding orbit, when mapped to the shape sphere \([7]\), belongs to the half-sphere bounded by the isosceles circle \([z_2] = [z_3]\). According to theorem 3.1, it should thus always intersect the collinear plane of the shape space.

4. Sufficient condition for the existence of generalised syzygies

In this section, we revisit a simple geometric condition—first introduced in our previous study \([8]\)—which, based on initial positions and velocities, guarantees the occurrence of generalised syzygies (see definition 1.1).

**Definition 4.1.** The configuration of the bodies \( P_1, P_2 \) and \( P_3 \) at the moment \( t_0 \in I \) is called antisymmetric, if the oriented areas of parallelograms spanned by the vectors \((\dot{z}_j(t_0), z_j(t_0))\) and \((\dot{z}_j(t_0), \dot{z}_k(t_0))\) are nonzero and have opposite signs for some \( j \neq k \) (see figure 2). This condition is equivalent algebraically to

\[
\begin{vmatrix}
  x_j & y_j \\
  x_k & y_k \\
  (t_0) \\
\end{vmatrix}
\begin{vmatrix}
  \dot{x}_j & \dot{y}_j \\
  \dot{x}_k & \dot{y}_k \\
  (t_0) \\
\end{vmatrix}
= (x_j y_k - y_j x_k) (t_0) \cdot (\dot{x}_j \dot{y}_k - \dot{y}_j \dot{x}_k) (t_0) < 0. \tag{4.1}
\]

**Proposition 4.1.** The property of being antisymmetric is independent of the choice of the pair of the bodies: once verified for a particular pair of bodies \( P_j, P_k, j \neq k \), the condition (4.1) will be also satisfied for all other possible choices of pairs \( P_n, P_m, n \neq m \).

**Proof.** We observe that the condition (4.1) is invariant under the permutation of \( j \) and \( k \). Without loss of generality, we can assume that (4.1) holds for \((j, k) = (1, 2)\) and show that it is true also for the case where \((j, k) = (2, 3)\). The proof for other cases is similar.

The product of two determinants in (4.1) for \( j = 2, k = 3 \) can be written as

\[
\delta = \begin{vmatrix}
  x_2 & y_2 \\
  x_3 & y_3 \\
\end{vmatrix}
\begin{vmatrix}
  \dot{x}_2 & \dot{y}_2 \\
  \dot{x}_3 & \dot{y}_3 \\
\end{vmatrix}
= \frac{1}{m_2^2 m_3^2} \begin{vmatrix}
  m_2 x_2 & m_2 y_2 \\
  m_3 x_3 & m_3 y_3 \\
\end{vmatrix}
\begin{vmatrix}
  m_2 \dot{x}_2 & m_2 \dot{y}_2 \\
  m_3 \dot{x}_3 & m_3 \dot{y}_3 \\
\end{vmatrix}. \tag{4.2}
\]
As can be seen from the equalities (1.2):
\[
\begin{align*}
\begin{bmatrix} m_3 x_3, m_3 y_3 \end{bmatrix} &= \begin{bmatrix} m_1 x_1, m_1 y_1 \end{bmatrix} = \begin{bmatrix} m_2 x_2, m_2 y_2 \end{bmatrix}, \\
\begin{bmatrix} m_3 \dot{x}_3, m_3 \dot{y}_3 \end{bmatrix} &= \begin{bmatrix} m_1 \dot{x}_1, m_1 \dot{y}_1 \end{bmatrix} = \begin{bmatrix} m_2 \dot{x}_2, m_2 \dot{y}_2 \end{bmatrix}.
\end{align*}
\]
(4.3)

Therefore, by using (4.2), (4.3), and the elementary properties of determinants, we have:
\[
\delta = \frac{1}{m_2^2 m_3^2} \begin{vmatrix} m_2 x_2 & m_2 y_2 \\ -m_1 x_1 & -m_1 y_1 \end{vmatrix} \begin{vmatrix} m_2 \dot{x}_2 & m_2 \dot{y}_2 \\ -m_1 \dot{x}_1 & -m_1 \dot{y}_1 \end{vmatrix} = \frac{m_1}{m_2^2} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \begin{vmatrix} \dot{x}_1 & \dot{y}_1 \\ \dot{x}_2 & \dot{y}_2 \end{vmatrix}.
\]
(4.4)

Obviously, the sign of \( a \) in (4.2) and \( b \) in (4.1) are the same. That finishes the proof. \( \square \)

**Proposition 4.2.** Both definitions of a generalised syzygy and an antisymmetric configuration, are invariant when replacing \((x_i, y_i)\) by \((X_i, Y_i)\) and \((\dot{x}_i, \dot{y}_i)\) by \((\dot{X}_i, \dot{Y}_i)\) for \(i = 1, 2, 3\).

**Proof.** It follows immediately from (2.2) and the identity below, verified for any \(j \neq k\):
\[
\begin{vmatrix} x_j & y_j \\ x_k & y_k \end{vmatrix} \begin{vmatrix} \dot{x}_j & \dot{y}_j \\ \dot{x}_k & \dot{y}_k \end{vmatrix} = \frac{1}{m_j^2 m_k^2} \begin{vmatrix} X_j & Y_j \\ X_k & Y_k \end{vmatrix} \begin{vmatrix} \dot{X}_j & \dot{Y}_j \\ \dot{X}_k & \dot{Y}_k \end{vmatrix}.
\]
(4.5)

\( \square \)

The author has shown in [8] that the existence of a generalised syzygy can be guaranteed by the antisymmetry condition (4.1), provided that the mutual distances of the three bodies remain bounded. The following theorem generalises this result to the case of negative energy, which includes unbounded (escape) solutions in particular. Moreover, the proof presented here is simpler and more straightforward.

**Theorem 4.1.** Let \(t \mapsto (z_1(t), z_2(t), z_3(t))\) be a solution of the three-body problem (1.1) with negative energy \(H = -\alpha, \alpha > 0\). We assume that the initial configuration of the bodies \(P_1, P_2, P_3\) at \(t = 0\) is antisymmetric and the solution is collision free for \(t \in [0, T]\) where
\[
T(\alpha) = \frac{\pi \Sigma}{\alpha^{3/2}},
\]
(4.6)
and \(\Sigma\) is defined by (2.6). Then there exists \(t \in [0, T]\) such that the three bodies attain a generalised syzygy at time \(t = t_0\).
Proof. Writing \( d = \Delta_2/\Delta_1 \), the equation (2.8) can be transformed into the second-order Hill’s linear differential equation:

\[
\ddot{\Delta}_1 = (\text{Tr}(A) + 2d) \Delta_1.
\] (4.7)

We suppose now that the solution of (1.1) \( t \mapsto z_i(t), i = 1, 2, 3 \) is defined in the interval \([0, T]\), with \( T \) given in (4.6). Assuming it starts at \( t = 0 \) from an antisymmetric configuration and that no generalised syzygy happens for any \( 0 < t \leq T \) we obtain

\[
d(t) = \frac{\Delta_2}{\Delta_1}(t) < 0, \quad \Delta_1(t) \neq 0, \quad \forall t \in [0, T].
\] (4.8)

Therefore, according to (4.7), (4.8) and the lemma 2.1:

\[
\ddot{\Delta}_1 = \phi \Delta_1, \quad \phi = \text{Tr}(A) + 2d \leq -\theta^2, \quad \theta^2 = \alpha^3/\Sigma^2, \quad t \in [0, T].
\] (4.9)

Comparing (4.7) with \( \ddot{y} = -\theta^2 y \), and using the same Sturm–Liouville argument as in the proof of theorem 2.1, we conclude that \( \Delta_1 \) admits at least one zero in the interval \([0, \pi/\theta] = [0, T]\). This contradicts our hypothesis (4.8). The proof of theorem 4.1 is finished.

5. Conclusion

Montgomery [6] showed the existence of syzygies in the three-body problem for the case of zero angular momentum and negative energy, except for the Lagrange homothetic solutions. Our theorem 4.1 is free from the restriction on the angular momentum and contains both an easy-to-check sufficient condition and an upper bound on the time instant when the generalised syzygy occurs. Theorem 3.1 deals with the collinear configurations (syzygies) in the periodic case. Under the assumption that the triangle formed by the bodies obeys a geometric restriction called \( \theta \)-rigidity, we show that the bodies become aligned at some instant, resulting in a syzygy in the corresponding solution. We also conjecture that a similar result holds for bounded non-periodic solutions. A more in-depth analysis of equations (2.25) and (2.16) may reveal additional interesting properties regarding collinear configurations in the three-body problem.

Data availability statement

No new data were created or analysed in this study.

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ORCID ID

Alexei Tsygvintsev ♦ https://orcid.org/0000-0002-8744-4100
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