Exact Dynamics of Quantum Dissipative System in Constant External Field

Chang-Pu Sun ♣ and Li-Hua Yu ♠

♣ Institute for Theoretical Physics, State University of New York, Stony Brook, NY 11794-3840

♠ 725C, National Synchrotron Light Source, Brookhaven National Laboratory, N.Y.1197

Abstract

The quantum dynamics of a simplest dissipative system, a particle moving in a constant external field, is exactly studied by taking into account its interaction with a bath of Ohmic spectral density. We apply the main idea and methods developed in our recent work [1] to quantum dissipative system with constant external field. Quantizing the dissipative system we obtain the simple and exact solutions for the coordinate operator of the system in Heisenberg picture and the wave function of the composite system of system and bath in Schroedinger picture. An effective Hamiltonian for the dissipative system is explicitly derived from these solutions with Heisenberg picture method and thereby the meaning of the wavefunction governed

♣Department of Physics, Northeast Normal University, Changchun 130024, P.R.China
by it is clarified by analyzing the effect of the Brownian motion. Especially, the general effective Hamiltonian for the case with arbitrary potential is directly derived with this method for the case when the Brownian motion can be ignored. Using this effective Hamiltonian, we show an interesting fact that the dissipation suppresses the wave packet spreading.
1. Introduction

This paper is mainly devoted to the application and generalization of the idea and methods developed in our recent work [1] to a dissipative system with constant external field. Though most discussions in this paper are proceeded based on ref.[1], the context is written in a self-contained form. In ref.[1], we work on the case of harmonic oscillator moving in a bath with the Ohmic spectral density presented by Caldeira-Leggett [2]. It has been shown there that the wavefunction of system plus bath is precisely described by a direct product of two independent Hilbert space, one of which is described by an effective Hamiltonian while the other represents the effect of the bath, i.e., the Brownian motion. Therefore, this study clarifies the structure of the wavefunction of the system dissipated by its interaction with the bath and thereby reveals the relationship between the different approaches for quantum dissipative systems before. No path integral technology is needed in this treatment. Notice that the study of dissipative quantum systems(DQS), especially for the damped harmonic quantum oscillator(DHQO), has a rather long history [3-18] and has been paid much attention more recently due to the work by Caldeira and Leggett [2].

Now let us briefly describe two main approaches for quantum dissipative systems before the work in ref.[1]. To reproduce and quantize the phenomenological dissipative equation

\[
M \ddot{q}(t) = -\eta \dot{q}(t) - \frac{\partial V(q)}{\partial q},
\]

(1.1)
for one-dimensional dissipative system S with coordinate q, mass M and potential V(q), one approach is to embed it into an environment, a bath B of N harmonic oscillators interacting with the system S through certain coupling [3-12]. Since the bath
and system constitute a closed composite system \( C \) equals to \( S \) plus \( B \), the quantization of \( C \) is naturally direct. Then, its corresponding Heisenberg equation results in the above phenomenological dissipative equation in the operator form by eliminating the variables of the bath through proper approximations, such as the Markovian approximation and the Wigner-Wisskopf approximation [9]. This approach provides the motion equation with both the friction force in dissipation process and the fluctuation force in Brownian motion. The path integral technique [2,13] and field theory method [14] were used in this approach sometimes. Along this direction, the work of Caldeira and Leggett reveals a remarkable fact that the dissipation can occur exactly, instead of approximately, if the spectral density of the bath is Ohmic (to be described later). It enlightens people to reconsider many problems of dissipation in an exact version.

Another approach, for DHQO, is the use of an effective Hamiltonian [15-17]

\[
H_E = H_E(t) = \frac{1}{2M}e^{-\eta t/M}p^2 + \frac{1}{2}M\omega^2e^{\eta t/M}q^2.
\]  

(1.2)

which is now called Caldirora-Kani (CK) Hamiltonian [13]. With the canonical commutation relation

\[
[q, p] = i\hbar.
\]  

(1.3)

this Hamiltonian automatically yields the dissipation equation (1.1) through the Heisenberg equation. Notice that the alternative forms of the effective Hamiltonian have been by many authors [17] and an elegant example can be found in ref.[18]. Though this approach is very convenient to treat some dynamical problems of dissipation process for both classical and quantum cases, such as tunnelling and motion of wave packet, it is only a purely phenomenological method without the microscopic mechanism constructing \( H_E \) and thus the Brownian motion can not be analysed by
making its use. Especially, the meaning of wavefunction of the evolution governed the CK Hamiltonian is much ambiguous.

The present paper applies the method and idea developed in reference [1] to dissipative system with a constant external field, to establish the connection between the above two approaches and to consider how the interaction between the bath and the system leads to an explicit description for the dissipative system in terms of the effective Hamiltonian. All the discussions in this paper is proceeded with a simplest model, a particle moving in one dimension with constant potential field, but the main idea and methods can be generalized for other cases. The present work is completed in an unified framework in accompany with the answers to the entangling questions listed as follows:

i. How to quantize the dissipative system with the effective Hamiltonian in an exact context?

ii. What is the meaning of the wavefunction governed by the effective Hamiltonian?

iii. How to construct the propagator for the dissipative system in the Heisenberg representation?

iv. What happens to the spreading of the wave packet in present of dissipation?

2. Exact Classical Langevin Equation

In this section we reformulate the first approach about quantum dissipative system in Ohmic case with a simplest example that the dissipative system S is considered as a charged particle with unit mass, unit negative charge and coordinate q in a constant electric field E. It interacts with a bath B of N harmonic oscillators $B_i$ of coordinates
Let $q_j$, mass $m_j$ and frequency $\omega_j$. Let $p$ and $p_j$ be the corresponding momentums to $q$ and $x_j$ respectively. The Hamiltonian of the composite system of S plus B is written as

$$H = \frac{1}{2}p^2 - Eq + \sum_{j=1}^{N}[\frac{p_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2(x_j - q)^2]$$

$$= \frac{1}{2}p^2 - Eq + H_B - \sum_{j=1}^{N}C_jx_jq + \Delta V.$$  \hspace{0.5cm} (2.1)

where

$$H_B = \sum_{j=1}^{N}[\frac{p_j^2}{2m_j} + \frac{1}{2}m_j\omega_j^2x_j^2]$$

The problem of renormalization of interaction for $N = \infty$ mentioned in ref.[2] is naturally enjoyed by the frequency-dependent coupling constant $C_j = m_j\omega_j^2$ and the renormalized potential

$$\Delta V = \sum_{j=1}^{N} \frac{C_j^2}{2m_j\omega_j^2}q^2 = \sum_{j=1}^{N} \frac{1}{2}m_j\omega_j^2q^2$$

in the Hamiltonian (2.1).

Through the canonical equation, the Hamiltonian (2.1) defines a system of classical motion equations

$$\ddot{q} = E - \sum_{j=1}^{N}C_jx_j - \sum_{j=1}^{N} \frac{C_j^2}{m_j\omega_j^2}q, \hspace{0.5cm} (2.2a)$$

$$\ddot{x}_j = -\omega_j^2x_j - \frac{C_j}{m_j}q. \hspace{1cm} (2.2b)$$

Notice that the canonical equation for the closed composite system C of S plus B determines the ordinary momentum-velocity relations

$$p(t) = \dot{q}(t), \hspace{0.5cm} p_j(t) = m_j\dot{x}_j(t).$$

By making use of the Laplace transformations of eqs.(2.2), a direct of substitution of eq.(2.2b) into eq.(2.2a) yields an exact motion equation for the system S

$$\ddot{q} = E + L(q) + G(t), \hspace{1cm} (2.3)$$
\[ G(t) = \sum_{j=1}^{N} C_j [x_j(0) \cos \omega_j t + \frac{\dot{x}_j(0)}{\omega_j} \sin \omega_j t], \]  
\[ L(s) = \varphi^{-1} [-s^2 \sum_{j=1}^{N} \frac{C_j^2 s^2}{m_j \omega_j^2 (s^2 + \omega_j^2)} q(s)] \]  
where

and the second term in the right hand side in eq.(2.3)

\[ L(s) = \varphi^{-1} [-s^2 \sum_{j=1}^{N} \frac{C_j^2 s^2}{m_j \omega_j^2 (s^2 + \omega_j^2)} q(s)] \]  
is determined by the inverse of Laplace transformation \( \varphi \)

\[ \varphi[q(t)] = \bar{q}(s) = \int_{0}^{\infty} q(t) e^{-st} dt \]

Usually, for the finite N or the general spectral distribution \( \rho(\omega_j) \) of infinite oscillators in bath, the dissipative term \(-\eta \dot{q}\) for a positive number \( \eta \) does not appear exactly. However, according to Caldeira and Leggett, a specific spectral distribution of the bath

\[ \rho(\omega_j) = \frac{2\eta \omega_j^2 m_j}{\pi C_j^2} = \frac{2\eta}{\pi m_j \omega_j^2}, \]  
which is called Ohmic distribution, enable the sum over index j

\[ \sum_{j=1}^{N} \frac{C_j^2 s^2}{m_j \omega_j^2 (s^2 + \omega_j^2)} \]

to become an integral

\[ \int_{0}^{\infty} \rho(\omega_j) \frac{C_j^2 s^2}{m_j \omega_j^2 (s^2 + \omega_j^2)} d\omega_j = \frac{2\eta s^2}{\pi} \int_{0}^{\infty} \frac{d\omega_j}{(\omega_j^2 + s^2)} = s\eta. \]

and thus results in

\[ L(q) = -\eta \dot{q} - \eta q(0) \delta(t) \]

immediately through an inverse Laplace transformation. Then, the dissipative equation - the classical Langevin equation

\[ \ddot{q} = E - \eta \dot{q} + G(t), \]
occurs as an exact evolution based on the elementary Hamiltonian dynamics. Because the dissipative process is invertable, it is significant to pay our attention only to the process with $t > 0$ as follows. In this sense the impact $\delta(t)$ in the dissipative equation does not play role in dynamical problems.

Notice that the above Ohmic distribution (2.6) is only an alternative, but explicit and convenient reformulation of the Caldeira-Leggett’s constraint

$$J(\omega) = \eta \omega$$
on spectral density

$$J(\omega) = \frac{\pi}{2} \sum_{j=1}^{N} \frac{C_j^2}{m_j \omega_j} \delta(\omega - \omega_j). \quad (2.8)$$

It is also remarked that the fluctuating external force $G(t)$ acting on system $S$ is due to the effect of the bath and depends on the initial states of the oscillators in the bath. For the classical statistical thermal average $< >_{classical}$, $G(t)$ obeys the dissipation-fluctuation relation at temperature $T$

$$< G(t)G(t') >_{classical} = \frac{1}{2} \eta K T \delta(t - t') \quad (2.9)$$

and $< G(t) >_{classical} = 0$. It actually is the classical Brownian force in dissipation process. In concluding this section it is pointed out that the classical Langevin equation has been studied for a long time by many authors by certain approximations, but here our emphasis is that it can appear exactly with Ohmic spectral density for the bath.

3. Quantization of Dissipation with Effective Hamiltonian

In the following sections, we first detail the description in ref.[1] of the the system plus bath with the present example. Then, we develop this theory to derive a general effective Hamiltonian for arbitrary potential.
It can be observed from eq.(2.7) that, in Ohmic case, the action of bath B on the system S can be exactly described as two parts, the dissipative term $-\eta \dot{q}$ only depending on the state of S and the Brownian force $G(t)$ depending on the initial state of bath. In order to study various dynamical problems in dissipative process, it is necessary to determine in what sense the dissipative system can be isolated from the environment as a ‘quasi-closed’ system only depending on its own variables. Such a ‘quasi-closed’ system can evolve independently and the effect of the bath is totally enjoyed by the friction coefficient $\eta$. For the classical case, it is quite clear that at zero temperature the Brownian force can be neglected for the vanishing dissipation-fluctuation relation. According to the fluctuation-dissipation relation eq.(2.9), the classical statistical fluctuation of the bath is proportional to $\eta T$. Only when $T = 0$, the system is isolated with an effective dissipation equation

$$\ddot{q} = E - \eta \dot{q},$$

and the whole effect of bath on the system is only characterized through the friction constant $\eta$. However, for the quantum case, the story is not so direct. Now, we consider the composite system C equals S plus B. Since C is closed its quantization process is well build in elementary quantum mechanics.

Let us start with the exact solution to eqs.(2.2b) and (2.7),

$$q(t) = Q(t) + X(t) :$$

$$Q(t) = a(t)\dot{q}(0) + q(0) + g(t)$$

$$X(t) = \sum_{j=1}^{N} X_j(t) :$$

$$X_j(t) = \alpha_k(t)x_k(0) + \beta_k(t)\dot{x}_k(0),$$
where

\[ a(t) = t_\eta = \frac{1 - e^{-\eta t}}{\eta}, \quad g(t) = \frac{E}{\eta} (t - t_\eta) \]

\[ \alpha_k(t) = \frac{C_\eta}{\eta^2 + \omega_k^2} \left[ \frac{\eta}{\omega_k} \sin[\omega_k t] - \frac{\omega_k}{\eta} (\cos[\eta t] - 1) - \eta a(t) \right] \]

\[ \beta_k(t) = \frac{C_\eta}{(\eta^2 + \omega_k^2) \omega_k} \left[ \frac{\eta}{\omega_k} (\cos[\omega_k t] - 1) - \frac{\omega_k}{\eta} \sin[\eta t] + \omega_k a(t) \right]. \]

The canonical commutation relations at \( t = 0 \) for the closed system of \( S \) plus \( B \)

\[ [q(0), \dot{q}(0)] = i\hbar, \quad [x_j(0), \dot{x}_j(0)] = \frac{i\hbar}{m_j} \]  

(3.3)

quantize \( q(t) \) and \( \dot{q}(t) \) at any instant \( t \) so that

\[ [q(t), p(t)] = i\hbar, \quad [x_j(t), p_j(t)] = i\hbar \]  

(3.4)

for the ordinary momentum-velocity relations

\[ p(t) = \dot{q}(t), \quad p_j(t) = m_j \dot{x}_j(t). \]

Notice that the quantized coordinate \( q(t) \) is separated into two commuting part \( Q(t) \)

and \( X(t) \) depending on the system and bath respectively.

Now, we consider the quantum statistical problem of the dissipation and fluctuation

for the above linear system. The quantum statistical average \(< \quad > = < \quad >_{\text{quantum}} \)

defined by

\[ < A > = Tr[A e^{-\beta H_B}] \]

\[ Tr[e^{-\beta H_B}] \]

over the bath is defined by

\[ \beta = \frac{1}{KT} \]

for an observable \( A \). Here, \( H_B \) is the Hamiltonian for bath. A direct calculation gives

the quantum fluctuation - dissipation relation

\[ < G(t) > = 0, \]

\[ D(t - t')_{t \to 0} = \frac{1}{2} < \{ G(t), G(t') \} > = \frac{\eta \hbar}{\pi} \int_{0}^{\infty} \omega_j \coth\left( \frac{\beta \hbar \omega_j}{2} \right) \cos[\omega_j(t - t')] d\omega_j. \]  

(3.4)
At high temperature limit, i.e., $\beta \to 0$ or $T \to \infty$, the above results approach the classical dissipation-fluctuation relation (2.9). In zero temperature limit, it becomes

$$D(t - t')_{T \to 0} = \left( \frac{\eta h}{\pi} \right) \int_{0}^{\infty} \omega_j \cos[\omega_j(t - t')] d\omega_j \right.$$ 

$$= \left( \frac{\eta h}{\pi} \right) \lim_{\mu \to \infty} \left\{ \mu^2 \left[ \frac{\sin[\mu(t - t')]}{\mu(t - t')} - \frac{1}{2} \frac{\sin^2[\frac{\mu(t - t')}{2}]}{(\frac{\mu(t - t')}{2})^2} \right] \right\}$$

This equation concludes that the Brownian force can hardly be neglected even at zero temperature for quantum case, but the corresponding fluctuation is proportional to $\eta h$. So this is a quantum fluctuation. The effective coordinate $Q(t)$ can be used to approach the physical coordinate $q(t)$ only when this quantum fluctuation is neglected in certain sense. In next section we will clarify further the exact meaning of this argument in terms of the wavefunction description.

When the quantum fluctuation can be ignored, the evolution of the dissipative system may be approached by the variable $Q(t)$ independent of the bath. Then, the system is isolated from bath to produce a “closed” dynamics. Now, let us derive the effective Hamiltonian governing this kind of dynamics. Our derivation is valid for both the classical and quantum case.

By making an observation that the explicit expression (3.2) for $Q(t)$ determines the commutator

$$[Q(t), \dot{Q}(t)] = ihe^{-\eta t},$$

it is quite natural to define the canonical momentum-velocity relation

$$P(t) = e^{\eta t} \dot{Q}(t)$$

for the effective variable $Q$ so that the basic commutators for the Hamiltonian dy-
dynamics is

\[ [Q(t), P(t)] = i\hbar. \] (3.8)

Notice that the definition of the canonical momentum based on eq.(3.6) is not unique and the different definitions such as in ref.[18] may give different forms of the effective Hamiltonian. For the given \( Q(t) \) changing as eq.(3.2), one can explicitly obtain expressions of \( \dot{Q}(t) \) and \( \dot{P}(t) \) in terms of \( Q(t) \) and \( P(t) \)

\[ \dot{Q}(t) = -Ee^{\eta t}Q(t), \quad \dot{P}(t) = e^{-\eta t}P(t), \] (3.9)

Associated with Heisenberg equation

\[ \dot{Q}(t) = \frac{1}{i\hbar}[Q(t), H_e(t)], \]
\[ \dot{P}(t) = \frac{1}{i\hbar}[P(t), H_e(t)], \]

the eqs.(3.9) lead to a simple system of partial differential equations about the effective Hamiltonian \( H_e(t) \)

\[ \frac{\partial H_e(t)}{\partial Q} = -Ee^{\eta t}Q, \quad \frac{\partial H_e(t)}{\partial P} = e^{-\eta t}P(t). \] (3.10)

A solution to eq.(3.10) determines an effective Hamiltonian \( H_e(t) \)

\[ H_e = H_e(t) = \frac{1}{2}e^{-\eta t}P^2 - EQe^{\eta t} + F(t). \] (3.11)

up to an arbitrary function \( F(t) \) independent of \( P \) and \( Q \). It can be regarded a generalization of CK Hamiltonian [14-16] of damped harmonic oscillator.

In spite of the above discussions, for an arbitrary potential \( V(Q) \), one guesses the general effective Hamiltonian for dissipation problem

\[ H_E = H_E(t) = \frac{1}{2}e^{-\eta t}P^2 + V(Q)e^{\eta t}. \] (3.12)
Indeed, the Heisenberg equation of $H_E = H_E(t)$ with the basic commutator (3.8) can result in the dissipation equation

$$\ddot{Q}(t) = -\eta\dot{Q}(t) - \frac{\partial V(Q)}{\partial Q}.$$  

(3.13)

for arbitrary potential.

In fact, the generalized CK Hamiltonian (3.12) can be directly derived in the explicit equations satisfied by $Q(t)$ and $P(t)$ as Heisenberg equation. It follows from eq.(3.13) that the commutator $[Q(t), \dot{Q}(t)]$ at time $t$ satisfies an equation

$$\frac{d}{dt}[Q(t), \dot{Q}(t)] = -\eta[Q(t), \dot{Q}(t)].$$  

(3.14)

Even in present of an arbitrary potential $V(q)$, it still leads to the same commutator

$$[Q(t), \dot{Q}(t)] = i\hbar e^{-\eta t}.$$  

(3.6')

as that for case of harmonic oscillator and the case of constant external field. Then, eq.(3.6) suggests the same canonical momentum-velocity relation

$$P(t) = e^{\eta t} \dot{Q}(t)$$  

(3.7')

as eq.(3.7) for the two above mentioned cases. Using the dissipative equation eq.(3.13) and eq.(3.7'), we have

$$\dot{P}(t) = -e^{-\eta t} \frac{\partial V(Q)}{\partial Q}.$$  

(3.15)

Equations (3.15, 3.7') and the Heisenberg equation determine the equations of commutators about unknown-Hamiltonian $H_E$

$$[Q(t), H_E(t)] = i\hbar e^{\eta t} P(t) / M,$$  

(3.16)

$$[P(t), H_E(t)] = -i\hbar e^{-\eta t} \frac{\partial V(Q)}{\partial Q},$$
Obviously, the generalized CK Hamiltonian (3.12) is just a solution of above equations. Therefore, we derive out the generalized CK Hamiltonian for the arbitrary potential $V(q)$. Notice that the derivation of the generalized CK Hamiltonians (3.12) and its special case (3.11) here can also make sense for the classical case without Brownian motion so long as one use the commutators $y$ insteading of the Poisson brackets.

4. Meaning of Wave Function for Dissipation

In the last section we derived the effective Hamiltonian in terms of the Heisenberg equations. It can serve as a starting point studying some dynamical problems of quantum dissipation under certain sense. To understand the problem completely, one must investigate the physical meaning of the (effective) wavefunction defined by the effective Hamiltonian (3.11) and thereby clarify in what sense this effective Hamiltonian can be used correctly. Thus, we turn to the Schroedinger picture and detail the main ideas and methods developed in ref. [1].

Let us first show that the wavefunction of the composite system C equals S plus B can be reduced to a direct product of wavefunctions in in two independent Hilber spaces. To this end, it is observed that the coordinate operator

$$q(t) = Q(t) + X(t)$$

of S is the sum of two commuting parts $Q(t)$ and $X(t)$. So the eigenfunction of $q(t)$ with eigenvalue $q$ is expressed as a direct product

$$|q, t> = |Q, t > \otimes |\xi_1, t > \otimes |\xi_2, t > \otimes \ldots \otimes |\xi_N, t >$$

(4.1)

of the the eigenstates $|Q, t >$ of $Q(t)$ and $|\xi_j, t >$ of $X_j(t)$ with the eigenvalues $Q$ and
ξ_j respectively. Notice that these eigenvalues satisfy

\[ q = Q + \sum_{j=0}^{N} \xi_j. \] (4.2)

Due to eq.(4.2), for a given q, there exist many different sets of \( \{\xi_1, \xi_2, \ldots, \xi_N\} \) corresponding to q, that is to say, these eigenstates \( |q, \xi\rangle \) are degenerate. So, the new notation \( |q, \{\xi_j\}, \xi\rangle \) with additional index \( \{\xi_j\} \) is needed, instead of \( |q, \xi\rangle \), to distinguish among the different degenerate eigenstate with the same eigenvalues q. Correspondingly, the following notation for arbitrary complex number \( \xi_j \)

\[ |Q, \{\xi_j\} \rangle = |Q\rangle \otimes |\xi_1\rangle \otimes |\xi_2\rangle \otimes \ldots \otimes |\xi_N\rangle \] (4.3)

is used to represent the degenerate eigenstate of q(0) with eigenvalue Q, which belong to the Hilbert space

\[ V = V_S \otimes V_B = V_S \otimes \prod_{j=1}^{N} V_j \]

of the composite system C. Here, \( V_S \) and \( V_B = \prod_{j=1}^{N} V_j \) is the Hilbert spaces of S and B respectively; \( V_j \) is the Hillbert space for the j'th oscillator in the bath B; \( |Q\rangle \) is the eigenstates with eigenvalue Q and \( |\xi_j\rangle \ (\in V_j) \) the eigenstates of \( x_j(0) \) with eigenvalue \( \xi_j \).

Let the composite system be initially in a product state

\[ |\psi(0)\rangle = |\phi\rangle \otimes |W\rangle = |\phi\rangle \otimes \prod_{i=1}^{N} |W_i\rangle, \] (4.4)

at t=0 where \( |\phi\rangle \), \( |W\rangle \) and \( |W_i\rangle \) belong to \( V_S \), \( V_B \) and \( V_j \) respectively. The central problem we should face is whether the product form similar to eq.(4.4), where the wavefunction is a direct product of two wavefunctions in two independent Hilbert spaces, is persevered in the Schrodinger evolution. The positive answer will implies that the system can be isolated from the bath by making use of an effective Hamiltonian. To consider this problem, it is necessary to calculate the evolution operator
U(t) or its matrix elements. Because the coordinate operator q(t), for general value of t, can formally be regarded as an unitary transformation

\[ q(t) = U(t)\dagger q(0)U(t) \]

of q(0), the eigenstate

\[ |q, \{\xi_j\}, t\rangle = U(t)\dagger |q, \{\xi_j\}\rangle \]

of q(t) with eigenvalue q can be constructed in terms of the evolution operator and the eigenstate |Q = q, \{\xi_j\}\rangle of q(0) with the same eigenvalue q. Then, the coordinate component of the evolution state

\[ |\psi(t)\rangle = U(t)|\psi(0)\rangle \]

can be calculated as

\[ \Psi(Q, \{\xi_j\}) = <Q, \{\xi_j\}|\psi(t)\rangle = <Q, \{\xi_j\}|U(t)|\psi(0)\rangle \]

\[ = [<\psi(0)|U(t)\dagger|Q, \{\xi_j\}\rangle]^* = [<\psi(0)|Q, \{\xi_j\}, t\rangle]^* \]

according to the eigenstate |Q, \{\xi_j\}, t\rangle of q(t) for general t. In fact, this eigenstate can be directly solved in the coordinate representation with

\[ q(0) = Q, \dot{q}(0) = -i\hbar\frac{\partial}{\partial Q}, x_j(0) = \xi_j, \dot{x}_j(0) = -i\hbar/m_j \frac{\partial}{\partial \xi_j} \]

Obviously, the eigenstates |Q, t\rangle of Q(t) and |\xi_j, t\rangle of X_j(t) with eigenvalues x_i_j :

\[ \phi_Q(Q', t) = <Q'|Q, t\rangle = e^{\frac{1}{\hbar a(t)}[-\frac{1}{2}Q'^2 + (Q - g(t))Q' + \lambda(Q)]} \]

\[ u_{\xi_j}(\xi_j', t) = <\xi_j'|\xi_j, t\rangle = e^{\frac{1}{\hbar a(t)}[-\frac{1}{2}\beta_j(t)\xi_j'^2 + \xi_j\xi_j' + \mu(\xi_j)]} \]

are explicitly obtained from the differential equations

\[ [-i\hbar\alpha(t)\frac{\partial}{\partial Q'} + Q' + g(t)]\phi_Q(Q', t) = Q\phi_Q(Q', t) \]
\[ [-i\hbar \alpha_j(t) \frac{\partial}{\partial \xi_j} + \beta_j(t)] u_{\xi_j}(\xi'_j, t) = \xi_j u_{\xi_j}(\xi'_j, t). \quad (4.8) \]

where \( \lambda(Q) \) and \( \mu(\xi_j) \) are the functions independent of \( Q' \) and \( \xi'_j \) respectively.

Now, we can write
\[
\langle Q', \{\xi'_j\}|q, \{\xi_j\}, t \rangle = \langle Q'|q = q - \sum_{j=1}^{N} \xi_j, t \rangle \prod_{j=1}^{N} \langle \xi'_j|\xi_j, t \rangle
\]
\[
= \phi_{q-\sum \xi_j}(Q', t) \prod_{j=1}^{N} u_{\xi_j}(x'_j, t), \quad (4.9)
\]

which results in
\[
\Psi(q, \{\xi'_j\}) = [\langle \psi(0)|q, \{\xi'_j\}, t \rangle]^* = W(q - \sum \xi_j, t)^* \prod_{j=1}^{N} W'_j(\xi_j, t)^* \quad (4.10)
\]

where
\[
W(q - \sum \xi_j) = \int dQ' <\phi(0)|Q'| \phi_{q-\sum \xi_j}(Q', t)>
\]
\[
W'_j(\xi_j, t) = \int d\xi'_j <w'_j|\xi'_j > u_{\xi_j}(\xi'_j, t)
\]

Notice that the variables \( \xi_j \) are related to the bath, but they are not the coordinates \( x_j \) of bath. Therefore, applying the above analysis to a practical problem, one should distinguish between \( \xi_j \) and \( x_j \).

It is observed from eq.(4.10) that, if the variable \( q \) is so highly excited that the Brownian motion contribution \( \sum \xi_j \) is small enough in comparison with \( q \) in certain sense, the first factor in the right hand side of eq.(4.10) is approximately independent of \( \xi_j \). In this sense, the whole wavefunction for composite system \( C \) is factorized into two independent parts belong to \( V_S \) and \( V_B \) respectively. The first part \( W \) represents the wave function of the dissipative system isolated to evolve according to the effective Hamiltonian (3.11). Because of the Brownian motion, the physical variable \( q = Q + \sum \xi_j \) fluctuates about \( Q \) through the mean value of \( (\sum \xi_j)^2 \)
\[
\langle (\sum \xi_j(t))^2 \rangle = \sum_{j=0}^{N} \frac{\hbar}{2m_j \omega_j} (|\alpha_j(t)|^2 + \omega_j^2 |\beta_j(t)|^2) \coth \frac{\hbar \omega_j}{2Kt} \quad (4.11)
\]
at temperature $T$. It is zero at $t = 0$ and approach its final value

$$< \left[ \sum \xi_j(t = \infty) \right]^2 > = \frac{\hbar}{2m_j \omega_j} \left( \frac{\pi}{2} + \arctg \left[ \frac{\omega_0}{\eta \omega} \right] \right).$$

(4.12)

with low temperature limit in the time of the order of $1/\eta$. Obviously, whether the effective Hamiltonian can work well or not mainly depends on whether the values

$$< (\sum \xi_j(t))^2 >$$

can be neglected in practical problems or not.

For the further study in the dynamics of the dissipative system, we need to calculate its propagator. There are usually two ways to do that in principle: 1). By solving the effective Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} |\Psi(t)\rangle = H_E(t) |\Psi(t)\rangle,$$

directly for the obtained effective Hamiltonian $H_E$. 2). By using reduced density matrix in terms of the path integral for the composite system $C$ equals $S$ plus $B$. Now, we will deal with this problem along a shorter road, as the third way in which we need not to know what is the effective Hamiltonian $H_E$. Having known the time-evolution of the observable $Q(t)$ explicitly, we can derive the propagator from $Q(t)$ directly without use of the effective Hamiltonian $H_E$. $H_E$ will thereby be derived in a purely quantum mechanical way. This approach not only avoids the complexity in calculations in the first two approaches, but also give us new insight for the understanding of quantum dissipation.

According to the definition of propagator, we have

$$G(q_2, \{\xi_{j,2}\}, t_2, q_1, \{\xi_{j,1}\}, t_1) = < q_2, \{\xi_{j,2}\}, t_2 | q_1, \{\xi_{j,1}\}, t_1 >$$

$$= < Q_2, t_2 | Q_1, t_1 > \prod_{j=1}^{N} < \xi_{j,2}, t_2 | \xi_{j,1}, t_1 >$$

$$= G(Q_2, t_2, Q_1, t_1) \prod_{j=1}^{N} G_j(\xi_{j,2}, t_2 | \xi_{j,1}, t_1).$$

(4.13)
Because of the linearity of the Heisenberg equation in the variables \( Q(t), \dot{Q}(t) \) and \( \xi_j(t), \dot{\xi}_j(t) \), the operators \( O(t_2)(O = Q, \dot{Q}, \xi_j, \dot{\xi}_j) \) at time \( t_2 \) must be a linear combination of the operators \( O(t_1) \) at \( t_1 \). For example,

\[
Q(t_1) = Q(t_2) + a(t_1, t_2)P(t_2) + b(t_1, t_2)
\]

\[
Q(t_2) = Q(t_1) + a(t_2, t_1)P(t_1) + b(t_2, t_1)
\]

where

\[
b(t_2, t_1) = g(t_2) - g(t_1) - E a(t_1) e^{-\eta t_1} a(t_2, t_1)
\]

\[
a(t_2, t_1) = -a(t_1, t_2) = a(t_2) - a(t_1).
\]

Then, the definitions of eigenstates of \( Q_i(t) \)

\[
Q(t_i)|Q_i, t_i = Q_i|Q_i, t_i >, i = 1, 2
\]

and its \( Q_j, (j \neq i) \)-representation leads to the partial differential equations for the propagator

\[
[Q_2 + a(t_1, t_2).i\hbar \frac{\partial}{\partial Q_2} + b(t_1, t_2)]G(Q_2, t_2; Q_1, t_1) = Q_2G(Q_2, t_2; Q_1, t_1)
\]

\[
[Q_1 - a(t_1, t_2).i\hbar \frac{\partial}{\partial Q_1} + b(t_2, t_1)]G(Q_2, t_2; Q_1, t_1) = Q_1G(Q_2, t_2; Q_1, t_1)^*
\]

Solving the above equations, we obtain the propagator

\[
G(Q_2, t_2; Q_1, t_1) = \frac{1}{\sqrt{2\pi \eta |e^{-\eta t_1} - e^{-\eta t_2}|}} \times \exp\left[\frac{i}{a(t_1, t_2)\hbar} \left\{ \frac{1}{2} Q_1^2 + \frac{1}{2} Q_2^2 - Q_1 Q_2 + b(t_2, t_1)Q_1 + b(t_1, t_2)Q_2 + \theta(t) \right\} \right]
\]

where \( \theta(t) \) is an arbitrary function of time independent of \( Q_i, (i = 1, 2) \) Similarly, we can also calculate the factors \( G_j(\xi_j, t_2; \xi_j, t_1) \), but here we need not to explicitly write them out for our present purpose. For highly-excited q-system where the Brownian
motion is ignored, the first factor \( G(Q_2, t_2; Q_1, t_1) \) can be regarded as the effective propagators for the dissipative system. By taking into account that the propagator, in fact, is the certain matrix elements of the evolution operator \( U_Q(t) \), i.e.,

\[
< Q_2 | U_Q(t) | Q_1 > = < Q_1 | U_Q(t)^\dagger | Q_2 >^* = < Q_2, t | Q_1 > = G(Q_2, t; Q_1, 0)
\]

the effective Hamiltonian (3.11) can also be derived again from

\[
H_E = i\hbar \frac{\partial U_Q(t)}{\partial t} U_Q(t)^{-1}
\]

(4.17)

Notice that this derivation of the effective Hamiltonian here is purely quantum mechanical.

### 5. Spreading of Wave Packet Suppressed by Dissipation

In this section the effective Hamiltonian (3.12) for the dissipative system is applied to study a quite simple dynamical problem: the motion of the wave packet of a ‘free’ (\( E=0 \)) particle of mass \( M \) in one dimension. This study will show an interesting fact that the dissipation must suppress the spreading of the wave packet if the breadth of initial wave packet is so wide that the effect of Brownian motion can be ignored. Usually, without dissipation, a Gaussian wave packet will infinitely spreads into the whole space and the localization of wave will lose completely in evolution process. In future direction its breadth increases as \( t \) until infinity while its height decreases from its initial value to zero. Notice that the height and breadth of a wave packet are correlated through its normalization. However, for the present case with dissipation, there appears a quite different picture about the wave packet spreading. It will be proved that the final breadth and height have the finite limit value as \( t \to \infty \). In the following we use \( M \neq 1 \) and denote by \( x \) the operator \( Q \) approaching the physical coordinate \( q \).
Starting with the effective Schrodinger equation
\[ i\hbar \frac{\partial}{\partial t} \Psi(x, t) = -\frac{\hbar^2}{2M} e^{-\eta t/M} \frac{\partial^2}{\partial x^2} \Psi(x, t) \] (5.1)
governed by the effective Hamiltonian in absent of the external field, the evolution of wave function is generally expressed by
\[ \Psi(x, t) = \sum_k <k|\Psi(x, 0) > e^{iE_k t - i\frac{E_k t \eta}{\hbar}} \] (5.2)
where
\[ E_k = \frac{\hbar^2 k^2}{2M} \]
is the energy of the momentum eigenstate |k >:
\[ <k|x> = \frac{1}{\sqrt{2\pi}} e^{ikx}. \]
Notice that, for the definition
\[ t_\eta = a(t) = \frac{M(1 - e^{-\eta t/M})}{\eta} \]
we have \( t_\eta \to t \) when \( \eta/M \to 0 \). So one can regard \( t_\eta \) as a \( \eta \)-deformation of time t. Especially, \( t_\eta \) approaches a limit \( \frac{M}{\eta} \) as \( t \to \infty \). This fact will enjoys the physical features of wave packet spreading in present of dissipation.

Take the Gaussian wave packet
\[ \Psi(x, 0) = <x|\Psi(0) > = \frac{1}{[2\pi d^2]^{1/4}} e^{ik_0 x - \frac{x^2}{4d^2}} \] (5.3)
as an initial state, in which
\[ <\Psi(0)|x|\Psi(0) > = 0, <\Psi(0)|P|\Psi(0) > = \hbar k_0, \]
\[ <\Delta x > = \sqrt{<\Psi(0)|x^2|\Psi(0) > - <\Psi(0)|x|\Psi(0) >^2} = d. \] (5.4)
The above equation (5.4) shows that the wave packet is centered at \( x = 0 \) and has an average momentum \( \hbar k_0 \). Its breadth is \( d \). According to the wave equation (5.1), the Fourier transformation

\[
\Psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(k) e^{ikx} \, dk
\]

\[
\psi(k) = \frac{2d^2}{\pi} e^{-d^2(k-k_0)^2}
\]

for the initial wave packet (5.3) determines the wavefunction at \( t \)

\[
\Psi(x,t) = \frac{\exp[ik_0x - iE_k t_\eta]}{(2\pi)^{1/4}} \sqrt{\frac{d}{d + i(t_\eta \hbar/(2Md^2)}} e^{\exp[-\frac{1}{4}(x - k_0 t_\eta \hbar/M)^2 - \frac{1 - it_\eta \hbar/2Md^2}{d^2 + (t_\eta \hbar/2Md)^2}]} (5.5)
\]

To understand the physical meaning represented by the wavefunction (5.5), we write down the corresponding position probability:

\[
|\Psi(x,t)|^2 = \frac{1}{\sqrt{2\pi}[d^2 + (t_\eta \hbar)^2/(2Md)^2]} e^{\exp[-\frac{(x - k_0 t_\eta \hbar/M)^2}{2[d^2 + (t_\eta \hbar)^2/(2Md)^2]}]. (5.6)
\]

The above formula tells us that the motion of the wave packet in dissipation process starts with initial velocity

\[
v_0 = < P/M > = \hbar k_0/M
\]

and its center is initially at the position \( x = 0 \). Subjected to time evolution, the center of the Gaussian wave packet will stop at a limit position

\[
x_{\text{limit}} = \frac{\hbar k_0}{\eta}.
\]

as \( t \to \infty \). In this process, the velocity of the center

\[
v(t) = \frac{d}{dt} < \Psi(x,t)|x|\Psi(x,t) > = \frac{\hbar k_0}{M} e^{-\eta t/M}
\]

decrease from \( \frac{\hbar k_0}{M} \) to zero. The above fact means that the motion of the center of wave packet is as the same as the that of a dissipative classical particle. However, a
purely quantum picture is manifested by the finite change of its breadth

\[ B(t) = \sqrt{d^2 + (\hbar \nu_0 / 2Md)^2} \]

from \( d \) at \( t=0 \) to a limit value

\[ B_{\text{limit}} = \sqrt{d^2 + (\hbar/2\eta d)^2} \]

as \( t \to \infty \). Both the position at which the center of wave packet finally stop and its final breadth are independent of the mass \( M \)! It defines the limit shape of wave packet at which the spreading wave packet finally takes. These physical features are illustrated by the figures 1 and 2.

It is finally pointed out that this suppressing of wave packet spreading by dissipation possibly provides a mechanism to localize a quantum particle.

**Acknowledgements**

The authors wish to express their sincere thanks to Professor C.N. Yang for drawing our attention to the problem of dissipative systems, for spending his valuable time in many sessions of stimulating discussions on this subject, and for many suggestions which are critically important for the ideas of this paper. The part of work by Chang-Pu Sun is supported in part by the Cha Chi-Ming fellowship through the CEEC program at the State University of New York at Stony Brook, and in part by the NSF of China through the Northeast Normal University. The part of work by Li-Hua Yu is performed under the auspices of the U.S. Department of Energy under Contract No. DE-AC0276CH00016.
References

1. L.H.Yu, C.P.Sun, *Wavefunction Evolution of a Dissipative System*, NSLS Brookhaven National Laboratory and SUNY.SB.ITP preprint, May 1993, submitted for publication in Phys.Rev.Lett.

2. A.O. Caldeira, A.J. Leggett, Ann. Phys. 149, 374 (1983), and Physica 121A, 587 (1983)

3. R.Zwanzig, J. Chem. Phys. 33, 1338 (1960)

4. M.J.Lax, J. Phys. Chem. Solid 25, 487 (1964); Phys. Rev. A145, 110 (1966)

5. P.Ullersma, Physica, 32, 27 (1966)

6. H.Haken, Rev. Mod. Phys. 47, 67 (1975)

7. Senitzky, Phys. Rev. 119, 670 (1960)

8. G.W.Ford, J.T.Lewis, R.F. O’comell, Phys. Rev. A.37, 4419, (1988)

9. W.H. Louisell, "Quantum Statistical Properties of Radiation", John Wiley and Sons (1973)

10. G.W. Ford, M. Kac, P. Mazur, Jour. Math. Phys., 6, 504 (1965)

11. S. Nakajima, Prog. Theor. Phys. 20, 948 (1958)

12. M. D. Kostin, J. Chem. Phys. 57, 3589 (1972)

13. R.P. Feynman, F.L. Vernon, Ann. Phys. 24, 118 (1963)

14. K. Fujikawa, S. Iso, M. Sasaki, H. Suzuki, Phys. Rev. Lett. 68, 1093 (1992)

15. P. Caldirola, Nuovo Cimento, 18, 393, (1941)
16. E. Kanai, Prog. Theor. Phys. 3, 440 (1948)

17. H. Dekker, Phys. Report, 80, 1 (1981)

18. H.W. Peng, in *Guangzhou Particle Physics Conference of 1980*, p.57-67, Science Press, 1981.

**Figure Captions**

**Figure 1**

The Motion of Wave Packet in Present of Dissipation: \( F(x, t) = |\Psi(x, t)|^2 \). The dissipation not only restrict the motion of the center of the Gaussian wave packet like a classical particle, but also suppress its spreading so that it takes a limit Gaussian wave packet with finite breadth and height as \( t \to \infty \).

**Figure 2**

The projection of Figure 1 on the \( |\Psi|^2 - x \) Plane. It actually represent the figures \( F(x, t) = |\Psi(x, t)|^2 \) at \( t=0, t=T, t=2T, t=3T, ..., t=NT \). When \( N \to \infty \), there is a lower Gaussian wave packet \( |\Psi(x, t = \infty)|^2 = F(x, 1/\eta) \).