On bivariate fundamental polynomials

V. Vardanyan (vahagn.vardanyan94@gmail.com)
Department of Mathematics and Mechanics
Yerevan State University
A. Manukyan St. 1
0025 Yerevan, Armenia

Abstract

An $n$-independent set in two dimensions is a set of nodes admitting (not necessarily unique) bivariate interpolation with polynomials of total degree at most $n$. For an arbitrary $n$-independent node set $\mathcal{X}$ we are interested with the property that each node possesses an $n$-fundamental polynomial in form of product of linear or quadratic factors. In the present paper we show that each node of $\mathcal{X}$ has an $n$-fundamental polynomial, which is a product of lines, if $\#\mathcal{X} \leq 2n + 1$. Next we prove that each node of $\mathcal{X}$ has an $n$-fundamental polynomial, which is a product of lines or conics, if $\#\mathcal{X} \leq 2n + \lfloor n/2 \rfloor + 1$. We have a counterexample in each case to show that the results are not valid in general if $\#\mathcal{X} \geq 2n + 2$ and $\#\mathcal{X} \geq 2n + \lfloor n/2 \rfloor + 2$, respectively.

Key words: Bivariate polynomial, interpolation, fundamental polynomial, conic, $n$-poised, $n$-independent nodes.

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1 Introduction

Let $\Pi_n$ be the space of bivariate polynomials of total degree at most $n$:

$$\Pi_n = \left\{ \sum_{i+j \leq n} a_{ij}x^iy^j : a_{ij} \in \mathbb{R} \right\}.$$

We have that

$$N := N_n := \dim \Pi_n = \begin{pmatrix} n + 2 \\ 2 \end{pmatrix}.$$
Consider a set of distinct nodes (points)
\[ X_s = \{(x_1, y_1), (x_2, y_2), \ldots, (x_s, y_s)\}. \]
The problem of finding a polynomial \( p \in \Pi_n \) which satisfies the conditions
\[ p(x_i, y_i) = c_i, \quad i = 1, 2, \ldots, s, \]  
(1.1)
is called interpolation problem. A polynomial \( p \in \Pi_n \) is called an \( n \)-fundamental polynomial for a node \( A = (x_k, y_k) \in X_s \) if
\[ p(x_i, y_i) = \delta_{ik}, \quad i = 1, \ldots, s, \]
where \( \delta \) is the Kronecker symbol. We denote this fundamental polynomial by \( p^*_k = p^*_A = p^*_A, X_s \). Sometimes we call fundamental also a polynomial that vanishes at all nodes of \( X \) but one, since it is a nonzero constant times a fundamental polynomial.

**Definition 1.1.** A set of nodes \( X \) is called \( n \)-**independent** if all its nodes have fundamental polynomials. Otherwise, \( X \) is called \( n \)-**dependent**.

Fundamental polynomials are linearly independent. Therefore a necessary condition of \( n \)-independence is \( \#X \leq N \). Having fundamental polynomials of all nodes of \( X \) we get a solution of general interpolation problem (1.1) by using the Lagrange formula:
\[ p(x, y) = \sum_{i=1}^{s} c_i p^*_i(x, y). \]  
(1.2)
Thus we get readily that the node set \( X_s \) is \( n \)-independent if and only if it is \( n \)-solvable, meaning that for any data \( \{c_1, \ldots, c_s\} \) there exists a (not necessarily unique) polynomial \( p \in \Pi_n \) satisfying the conditions (1.1).

**Definition 1.2.** The interpolation problem with the set of nodes \( X_s \) is called \( n \)-**poised** if for any data \( \{c_1, \ldots, c_s\} \) there exists a unique polynomial \( p \in \Pi_n \), satisfying the conditions (1.1).

A necessary condition for \( n \)-poisedness is \( s = \#X = N \). We have also that a set \( X_N \) is \( n \)-poised if and only if it is \( n \)-independent. The following proposition is based on an elementary Linear Algebra argument.

**Proposition 1.3.** The interpolation problem with the set of nodes \( X_N \) is \( n \)-poised if and only if the following condition holds:
\[ p \in \Pi_n, \quad p(x_i, y_i) = 0, \quad i = 1, \ldots, N \Rightarrow p = 0. \]
Now let us bring some results on $n$-independence we shall use in the sequel. Let us start with the following simple but important result of Severi (see [5]):

**Theorem 1.4** ([5]). Any set $\mathcal{X}$, with $\#\mathcal{X} \leq n + 1$, is $n$-independent.

**Remark 1.5.** For each node $A \in \mathcal{X}$ here we can find $n$-fundamental polynomial which is a product of $\#\mathcal{X} - 1 \leq n$ lines, each of which passes through a respective node of $\mathcal{X} \setminus \{A\}$ and does not pass through $A$.

Next two results extend the Severi theorem to the cases of sets with no more than $2n + 1$ (see [1], Proposition 1) and $3n - 1$ (see [3], Theorem 5.3) nodes, respectively.

**Theorem 1.6** ([1]). Any set $\mathcal{X}$, with $\#\mathcal{X} \leq 2n + 1$, is $n$-independent, if and only if no $n + 2$ nodes of $\mathcal{X}$ are collinear.

**Theorem 1.7** ([3]). Let $\mathcal{X}$ be set of nodes with $\#\mathcal{X} \leq 3n$. Then the set $\mathcal{X}$ is $n$-dependent if and only if one of the following hold:

i) $n + 2$ nodes of $\mathcal{X}$ are collinear,

ii) $2n + 2$ nodes of $\mathcal{X}$ are lying on a conic,

iii) $\#\mathcal{X} = 3$, there are curves $\gamma \in \Pi_3$ and $p \in \Pi_n$ such that $\gamma \cap p = \mathcal{X}$.

Here we use the same letter, say $p$, to denote the polynomial $p \in \Pi_n \setminus \Pi_0$ and the algebraic curve defined by the equation $p(x, y) = 0$. We denote lines and conics by $\alpha$ and $\beta$, respectively.

Note that, according to Theorem 1.3, the interpolation problem with $X_N$ is $n$-poised if and only if there is no algebraic curve of degree $\leq n$ passing through all the nodes of $\mathcal{X}_N$.

At the end of this section let us discuss the problem we consider. In view of the Lagrange formula (1.2) it is very important to find $n$-independent (i.e., $n$-solvable) sets for which the fundamental polynomials have the simplest possible forms. In Section 2 we characterize $n$-independent sets for which all fundamental polynomials are products of lines. It is worth mentioning that for the natural lattice, introduced by Chung and Yao in [2], the fundamental polynomials have the mentioned forms. But in this case the nodes satisfy very special conditions. Namely, they are intersection points of some $n + 2$ given lines. In our characterization (see forthcoming Theorem 2.1 Proposition 2.2) the restrictions on the node set are much more weak. In Sections 3 we consider a much more involved problem. Here we characterize $n$-independent node sets for which all fundamental polynomials are products of lines or conics.
2 The fundamental polynomials as products of lines

Theorem 2.1. Let $\mathcal{X}$ be an $n$-independent set of nodes with $\#\mathcal{X} \leq 2n + 1$. Then for each node of $\mathcal{X}$ there is an $n$-fundamental polynomial, which is a product of lines. Moreover, this statement is not true in general for $n$-independent node sets $\mathcal{X}$ with $\#\mathcal{X} \geq 2n + 2$ and $n \geq 2$.

The first statement of Theorem follows from the following result which covers more wider setting.

Proposition 2.2. Let $\mathcal{X}$ be a set of nodes with $\#\mathcal{X} \leq 2n + 1$ and $A \in \mathcal{X}$. Then the following three statements are equivalent:
   i) The node $A$ has an $n$-fundamental polynomial,
   ii) The node $A$ has an $n$-fundamental polynomial, which is a product of linear factors,
   iii) No $n + 1$ nodes of $\mathcal{X} \setminus \{A\}$ are collinear together with the node $A$.

3 The fundamental polynomials as products of lines and conics

Theorem 3.1. Let $\mathcal{X}$ be an $n$-independent set of nodes with $\#\mathcal{X} \leq 2n + \lceil n/2 \rceil + 1$. Then for each node of $\mathcal{X}$ there is an $n$-fundamental polynomial, which is a product of lines and conics. Moreover, this statement is not true in general for $n$-independent node sets $\mathcal{X}$ with $\#\mathcal{X} \geq 2n + \lceil n/2 \rceil + 2$ and $n \geq 3$.

The first statement of Theorem follows from the following result which covers more wider setting.

Proposition 3.2. Let $\mathcal{X}$ be a set of nodes with $\#\mathcal{X} \leq 2n + \lceil n/2 \rceil + 1$ and $A \in \mathcal{X}$. Then the following three statements are equivalent:
   i) The node $A$ has an $n$-fundamental polynomial,
   ii) The node $A$ has an $n$-fundamental polynomial, which is a product of lines and conics,
   iii) a) no $n + 1$ nodes of $\mathcal{X} \setminus \{A\}$ are collinear together with $A$,
       b) if $n + 1$ nodes of $\mathcal{X} \setminus \{A\}$ are collinear and are lying in a line $\alpha$ then no $n$ nodes of $\mathcal{X} \setminus (A \cup \alpha)$ are collinear together with $A$,
       c) no $2n + 1$ nodes of $\mathcal{X} \setminus \{A\}$ are lying on an irreducible conic together with $A$. 

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References

[1] D. Eisenbud, M. Green and J. Harris (1996) 
Cayley-Bacharach theorems and conjectures, Bull. Amer. Math. Soc. 
(N.S.), 33(3), 295–324.

[2] Chung, K. C. and Yao, T. H., On lattices admitting unique Lagrange 
interpolation, SIAM J. Numer. Anal. 14 (1977), 735-743.

[3] H. Hakopian and A. Malinyan, Characterization of $n$-independent sets 
with no more than $3n$ points, Jaén J. Approx. 4(2012), 119 – 134.

[4] J. Radon, Zur mechanischen Kubatur, Monatsh. Math. 52 (1948) 286–
300.

[5] F. Severi, Vorlesungen ¨Uber Algebraische Geometrie (Teubner, Berlin, 
1921).