ASYMPTOTIC DYNAMIC FOR DIPOLAR QUANTUM GASES BELOW THE GROUND STATE ENERGY THRESHOLD

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Abstract. We consider the Gross-Pitaevskii equation describing a dipolar Bose-Einstein condensate without external confinement. We first consider the unstable regime, where the nonlocal nonlinearity is neither positive nor radially symmetric and standing states are known to exist. We prove that under the energy threshold given by the ground state, all global in time solutions behave as free waves asymptotically in time. The ingredients of the proof are variational characterization of the ground states energy, a suitable profile decomposition theorem and localized virial estimates, enabling to carry out a Concentration/Compactness and Rigidity scheme. As a byproduct we show that in the stable regime, where standing states do not exist, any initial data in the energy space scatters.

1. Introduction

The interest concerning the asymptotic dynamics of equations describing a condensate of particles at very low temperatures speedily increased since the first experimental observation in 1995 of Bose-Einstein condensate (BEC), see e.g. [1, 8]. In the recent years, the so-called dipolar Bose-Einstein condensate, namely a condensate made out of particles possessing a permanent electric or magnetic dipole moment, see e.g. [5, 6, 24, 26], has been attracting much attention. At temperature much smaller than the critical one, such a model is well described by the wave function \( u = u(t, x) \) whose evolution is governed by the Gross-Pitaevskii equation (GPE),

\[
\frac{i\hbar}{\partial t} u = -\frac{\hbar^2}{2m} \nabla^2 u + W(x)u + U_0|u|^2u + (V_{\text{dip}} * |u|^2)u, \quad x \in \mathbb{R}^3, \quad t > 0,
\]

where \( t \) is the time variable, \( x = (x_1, x_2, x_3) \) is the space coordinate, \( \hbar \) is the Planck constant, \( m \) is the mass of a dipolar particle and \( W(x) \) is an external, trapping, real potential. In this paper we consider the case when the trapping potential \( W \) is not active, i.e. assuming \( W(x) = 0 \). The coefficient \( U_0 = 4\pi \hbar^2a_s/m \) describes the local interaction between dipoles in the condensate with \( a_s \) the \( s \)-wave scattering length (positive for repulsive interactions and negative for attractive interactions).

The long-range dipolar interaction potential between two dipoles is given by

\[
V_{\text{dip}}(x) = \frac{\mu_0\mu^2_{\text{dip}}}{4\pi} \frac{1 - 3\cos^2(\theta)}{|x|^3}, \quad x \in \mathbb{R}^3,
\]

where \( \mu_0 \) is the vacuum magnetic permeability, \( \mu_{\text{dip}} \) is the permanent magnetic dipole moment and \( \theta \) is the angle between the dipole axis and the vector \( x \). For simplicity, we fix the dipole axis as the vector \( n = (0, 0, 1) \). The wave function is normalized according to

\[
\int_{\mathbb{R}^3} |u(x, t)|^2 \, dx = N,
\]

where \( N \) is the total number of dipolar particles in the dipolar BEC.
In order to simplify the mathematical analysis we rescale (1.1) into the following dimensionless GPE,

\begin{equation}
\begin{cases}
i\partial_t u + \frac{1}{2} \Delta u = \lambda_1 |u|^2 u + \lambda_2 (K \ast |u|^2) u, & (t, x) \in \mathbb{R} \times \mathbb{R}^3, \\
u(0, x) = u_0(x) \in H^1(\mathbb{R}^3)
\end{cases}
\end{equation}

The corresponding normalization now reads

\[ N(u(\cdot, t)) := \|u(\cdot, t)\|_2^2 = \int_{\mathbb{R}^3} |u(x, t)|^2 \, dx = \int_{\mathbb{R}^3} |u(x, 0)|^2 \, dx = 1, \]

and the kernel \( K \) is given by

\[ K(x) = \frac{x_1^2 + x_2^2 - 2x_3^2}{|x|^6}. \]

The physical real parameters \( \lambda_{1,2} \), which describe the strength of the two nonlinearities, are given by

\[ \lambda_1 = 4\pi a_s N\gamma, \quad \lambda_2 = \frac{m N\mu_0 \mu_{dip}^2}{4\pi \hbar^2} \gamma. \]

In this paper we consider the case when the two real parameters \( \lambda_{1,2} \) range in the so called unstable regime:

\begin{equation}
\begin{cases}
\lambda_1 - \frac{4\pi}{3} \lambda_2 < 0 & \text{if } \lambda_2 > 0 \\
\lambda_1 + \frac{8\pi}{3} \lambda_2 < 0 & \text{if } \lambda_2 < 0
\end{cases}
\end{equation}

Solutions \( u(t) \in C((-T_{\min}, T_{\max}); H^1(\mathbb{R}^3)) \) to (1.2) have been proved to exist, at least locally in time, by Carles, Markovich and Sparber in [9], and not only in the unstable regime but also in the complement region

\begin{equation}
\begin{cases}
\lambda_1 - \frac{4\pi}{3} \lambda_2 \geq 0 & \text{if } \lambda_2 > 0 \\
\lambda_1 + \frac{8\pi}{3} \lambda_2 \geq 0 & \text{if } \lambda_2 < 0
\end{cases}
\end{equation}

which is called stable regime.

We recall since now that solutions to (1.2) have preserved mass and energy, respectively defined as

\begin{equation}
\mathcal{M}(t) = \mathcal{M}(u(t)) := \int_{\mathbb{R}^3} |u(t)|^2 \, dx
\end{equation}

and

\begin{equation}
\mathcal{E}(t) = \mathcal{E}(u(t)) := \frac{1}{2} \left( \int_{\mathbb{R}^3} |\nabla u(t)|^2 + \lambda_1 |u(t)|^4 + \lambda_2 (K \ast |u(t)|^2)|u(t)|^2 \, dx \right),
\end{equation}

therefore \( \mathcal{M}(t) = \mathcal{M}(0) \) and \( \mathcal{E}(t) = \mathcal{E}(0) \) for any \( t \in (-T_{\min}, T_{\max}) \), where \( T_{\min}, T_{\max} \in (0, \infty] \) are the minimal and maximal time of existence, respectively.

The unstable regime is of particular interest since stationary solution are allowed in this region. More precisely, stationary states are solutions of the type

\[ u(x, t) = e^{-i\kappa t} u(x), \]

where \( \kappa \in \mathbb{R} \) is the chemical potential, \( u(x) \) is a time-independent function solving the stationary equation

\begin{equation}
-\frac{1}{2} \Delta u + \lambda_1 |u|^2 u + \lambda_2 (K \ast |u|^2) u + \kappa u = 0
\end{equation}
constrained on the manifold $S(1)$, where
\begin{equation}
S(1) = \{ u \in H^1(\mathbb{R}^3; \mathbb{C}) \text{ s.t. } \| u \|^2_2 = 1 \}.
\end{equation}

There are two different approaches to show the existence of standing states. The first one is due to Antonelli and Sparber, see [3], where existence is proved by means of Weinstein method, i.e. as minimizers of the following scaling invariant functional

$$J(v) := \frac{\| \nabla v \|^3_{L^2(\mathbb{R}^3)} \| v \|^2_{L^2(\mathbb{R}^3)}}{-\lambda_1 \| v \|^4_{L^4(\mathbb{R}^3)} - \lambda_2 \int_{\mathbb{R}^3} (K * |v|^2)|v|^2 \, dx}.$$ 

The second strategy, due to the first author and Jeanjean, see [7], relies on geometric analysis methods, more precisely by proving the existence of critical point of the energy functional under the mass constraint depicted in (1.8). In this approach the parameter $\kappa$ is found as Lagrange multiplier. Despite the fact the energy is unbounded from below on $S(1)$, if one restricts to states that are stationary for the evolution equation, i.e. fulfilling (1.7), then the energy is bounded from below by a positive constant; furthermore, this constant, corresponding to the mountain pass level, is reached. The mountain pass solutions hence correspond to least energy states, also called ground states. As a direct consequence of this variational characterization and using a virial approach, the associated standing waves are proved to be orbitally unstable.

In [7] is also proved that for sufficiently small initial data in the $H^1(\mathbb{R}^3)$ topology, then (global) solutions to (1.2) scatter, no matter if the equation is considered in the unstable regime (1.3) or not.

Our aim is to study the scattering property for global solutions to (1.2) subject to condition (1.3). Our strategy follows the Kenig & Merle scheme, developed in the well celebrated papers [18, 19] to solve the global existence and scattering problems for the energy critical, focusing, radial nonlinear Schrödinger and Wave Equations in low space dimensions, respectively.

Before stating our main result we recall the rigorous definition of scattering. In the following $U(t)f = e^{it\hat{\Delta}}f$ will denote, with standard notation, the linear evolution driven by the free Schrödinger propagator of an initial datum $f$, namely $L(t, x) := U(t)f$ satisfies $i\partial_t L + \frac{1}{2} \Delta L = 0$, $L(0) = f$.

**Definition 1.1.** Let $u_0 \in H^1(\mathbb{R}^3)$ be given and $u(t, x) \in C(\mathbb{R}; H^1(\mathbb{R}^3))$ be the corresponding unique global solution (if it exists) to (1.2). Then we say that $u(t, x)$ satisfies the *scattering property* (or *scatters*, simply), provided

$$\lim_{t \to \pm \infty} \| u(t, x) - e^{it\hat{\Delta}}u^\pm \|_{H^1(\mathbb{R}^3)} = 0,$$

for suitable $u^\pm \in H^1(\mathbb{R}^3)$.

We point out that is not guaranteed that solutions to (1.2) do exist globally in time, so in the analysis below we shall also give sufficient conditions such that local in time solutions to (1.2) (whose existence has been shown in [9]) can be extended globally in time. Let us recall some notation introduced in [7]: the quantity defined in (1.6) can be rewritten as

$$\mathcal{E}(t) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \left( \lambda_1 + \lambda_2 \hat{K}(\xi) \right) (|u|^2)^2(\xi) \, d\xi$$

by means of Plancherel identity, where the Fourier transform of $K$ is given by

\begin{equation}
\hat{K}(\xi) = \frac{4\pi}{3} \frac{2 \xi_1^2 - \xi_2^2 - \xi_3^2}{|\xi|^2}
\end{equation}

(see [9] for a proof of the explicit form of $\hat{K}$.) A trivial computation leads to

$$\hat{K} \in \left[ \frac{4}{3} \pi, \frac{8}{3} \pi \right].$$
We split the energy as sum of the following kinetic and potential energies, respectively defined by

\[ T(u) = \int_{\mathbb{R}^3} |\nabla u|^2 \, dx \]

\[ P(u) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left( \lambda_1 + \lambda_2 K(\xi) \right) (|u|^2(\xi)) \, d\xi, \]

and we introduce the quantity (suggested by the Pohozaev identities)

\[ G(u) = T(u) + \frac{3}{2} P(u). \]

Despite the fact that we are primarily interested in solutions satisfying (1.8), for the mathematical treatment of the problem it is convenient to consider the generic set of constraints

\[ S(c) = \{ u \in H^1(\mathbb{R}^3) \text{ s.t. } ||u||_2^2 = c \}. \]

Here \( c > 0 \) and the case \( c = 1 \) trivially corresponds to the normalization (1.8). Given \( c > 0 \), \( \mathcal{E}(u) \) has a mountain pass geometry on \( S(c) \) (see the monograph [2] for a detailed treatment of this topic). More precisely, there exists \( \beta > 0 \) such that

\[ \gamma(c) := \inf_{g \in \Gamma(c)} \max_{t \in [0,1]} \mathcal{E}(g(t)) \geq \max \left\{ \max_{g \in \Gamma(c)} \mathcal{E}(g(0)), \max_{g \in \Gamma(c)} \mathcal{E}(g(1)) \right\} \]

holds in the set

\[ \Gamma(c) = \{ g \in C([0,1]; S(c)) \text{ s.t. } g(0) \in A_\beta, \mathcal{E}(g(1)) < 0 \}, \]

where

\[ A_\beta = \left\{ u \in S(c) \text{ s.t. } ||\nabla u||_2^2 \leq \beta \right\}. \]

It it standard, see [2], that the mountain pass geometry induces the existence of a Palais-Smale sequence at the level \( \gamma(c) \), namely a sequence \( \{u_n\}_{n \in \mathbb{N}} \subset S(c) \) such that, as \( n \to \infty \),

\[ \mathcal{E}(u_n) = \gamma(c) + o(1), \quad ||\mathcal{E}'_{|S(c)}(u_n)||_{H^{-1}} = o(1). \]

If one can show in addition the compactness of \( \{u_n\}_{n \in \mathbb{N}} \), namely that up to a subsequence, \( u_n \to u \) in \( H^1 \), then a critical point is found at the level \( \gamma(c) \). This is exactly what happens under the assumptions (1.3). We can summarize the last paragraph in the following.

**Theorem 1.2.** [7, Theorem 1.1] Let \( c > 0 \) and assume that (1.3) holds. Then \( \mathcal{E}(u) \) has a mountain pass geometry on \( S(c) \) and there exists a couple \( \{u_n, \kappa_n\} \in H^1 \times \mathbb{R}^+ \) solution of (1.7) with \( ||u_n||_{L^2}^2 = c \) and \( E(u_n) = \gamma(c) \). In addition \( u_n \in S(c) \) is a ground state.

We moreover recall that the energy level \( \gamma(c) \) has the following variational characterization that will be crucial to establish the scattering result below:

\[ \gamma(c) = \inf \{ \mathcal{E}(u) : u \in V(u) \}, \quad V(u) = \{ u \in H^1, ||u||_{L^2}^2 = c : \mathcal{G}(u) = 0 \} \]

In [7], among the other results, the following sufficient conditions are given in order to have global existence of solution to (1.2).

**Theorem 1.3.** [7, Theorem 1.3] If \( \mathcal{E}(u_0) < \gamma(c) \), with \( c = ||u_0||_{L^2}^2 \) and \( \mathcal{G}(u_0) > 0 \), then the (local in time) solution \( u \in \mathcal{C}((-T_{min}, T_{max}); H^1(\mathbb{R}^3)) \) to (1.2) can be extended globally in time, i.e. \( T_{min} = T_{max} = \infty \), and

\[ \mathcal{G}(u(t)) > 0 \]

for all \( t \in \mathbb{R} \).

Our aim in this paper is to show that something more can actually be said about solution to (1.2) under the conditions of Theorem 1.3. In fact, in that region all solutions scatter. The main theorem of the paper is then as follows.
Theorem 1.4. For any initial datum $u_0 \in H^1(\mathbb{R}^3)$ satisfying $\mathcal{E}(u_0) < \gamma(c)$, with $c = \|u_0\|_{L^2}^2$ and $\mathcal{G}(u_0) > 0$, then the corresponding global solution to (1.2) scatters.

Remark 1.5. It is worth mentioning that we do not assume neither finite variance nor spherical symmetry of the solutions. Indeed in the radial setting, the equation would reduce to a classical cubic NLS due to the fact that the nonlocal nonlinearity can be defined as a Calderón-Zigmund operator with kernel $|x|^{-3}\mathcal{O}(x)$ where $\mathcal{O}$ is a zero-order function having zero average on the sphere (see [9]).

Remark 1.6. It shall be emphasised that the fact that $\lambda_1, \lambda_2$ are in the unstable regime (1.3) does not imply that the potential energy $\mathcal{P}$ defined in (1.11) is negative for any function in $H^1(\mathbb{R}^3)$, see Lemma 2.4 below. Despite the fact that when the potential energy $\mathcal{P}$ is positive the nonlinear term acts as a defocusing nonlinearity, we are not able to exclude that along the time evolution $\mathcal{P}(u(x,t))$ changes sign. For this reason the conditions $\mathcal{G}(u_0) > 0$ and $\mathcal{E}(u_0) < \gamma(c)$ are necessary even when $\mathcal{P}(u_0) > 0$.

The proof of Theorem 1.4 is based on the Concentration/Compactness and Rigidity argument. We recall briefly the general strategy (based on a contradiction argument) of the Kenig & Merle road map: suppose that the threshold for scattering is strictly below the claimed one. The tool called Profile Decomposition based on Concentration/Compactness principles proves the existence of a global but non-scattering solution (the so called minimal element or soliton-like solution) at the threshold between scattering and non-scattering. Secondly, it is proved that the flow of this minimal element is (up to some symmetries) a precompact subset of $H^1(\mathbb{R}^3)$ and that therefore it remains spatially localized uniformly in time along a continuous path $x(t) \in \mathbb{R}^3$. This uniform localization enables the use of a local virial identity to establish a strictly positive lower bound on the convexity (in time) of the localized variance. More precisely, once defines the localized variance has been defined as $z_R(t) = R^2 \int_{\mathbb{R}^3} \chi \left( \frac{x}{R} \right) |u|^2 \, dx$, where $\chi \in C_\infty^\infty(\mathbb{R}^3)$ is a suitable cut-off function and $R > 1$ is a real rescaling parameter, the goal it to connect the second derivative of this quantity with the function $\mathcal{G}$, introduced in (1.12), as follows

\begin{equation}
\frac{d^2}{dt^2} z_R(t) = 4\mathcal{G}(u(x,t)) + o(1),
\end{equation}

where the error term decays, uniformly in time, as $R$ increases. A lower positive bound on $\mathcal{G}(u(x,t))$, following from the variational characterization of the mountain pass energy and from the properties of the minimal element, finally permits to exhibit the contradiction. Moreover we underline that for any initial datum $u_0 \in H^1(\mathbb{R}^3)$ satisfying $\mathcal{E}(u_0) < \gamma(c)$, with $c = \|u_0\|_{L^2}^2$ and $\mathcal{G}(u_0) > 0$, using our variational approach we are able to derive an explicit lower bound on $\mathcal{G}(u(x,t))$ for the corresponding global solution given by

\begin{equation}
\mathcal{G}(u(x,t)) \geq \min \{ \gamma(c) - \mathcal{E}(u_0), \mathcal{E}(u_0) \}.
\end{equation}

Remark 1.7. From the identity $\mathcal{E} - \frac{1}{2} \mathcal{G} = \frac{1}{2} \mathcal{T}$, it is evident that $\mathcal{E} > 0$ if $\mathcal{G} > 0$. Our lower bound on $\mathcal{G}(u(x,t))$ follows only from the variational characterization of the mountain pass energy and not from the fact that this critical energy level is achieved by the ground state. Despite the fact that in Proposition 2.6 we show that our conditions for scattering coincide with the one in Kenig & Merle approach, we never use the fact that the ground state exists.

The main difficulty concerning (1.14) is clearly the presence of the nonlocal dipolar term that makes the analysis more delicate with respect to local nonlinearities. In particular, for the dipolar interaction term, despite the nice identity $x \cdot \nabla K(x) = -3K(x)$, one cannot use the brutal estimate $|x \cdot \nabla K(x)| = 3|K(x)|$ due to the singularity of R.H.S. This argument is, as a matter of fact, the one that simplifies the computations for nonlocal nonlinearities with positive kernel like for the Coulomb kernel of the form $K(x) = \frac{1}{|x|^3}$. In our case this rough estimate is not allowed.
We conclude this introduction by pointing out how in the stable regime (1.4) the potential energy $\mathcal{P}$ is nonnegative for any function in $H^1(\mathbb{R}^3)$. Indeed let us the consider $\lambda_2 > 0$, $\lambda_1 - \frac{4\pi}{3}\lambda_2 > 0$, then we have

$$\mathcal{P}(u) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left( \lambda_1 + \lambda_2 \mathcal{K}(\xi) \right) |\hat{u}|^2(\xi) \, d\xi \geq \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \left( \lambda_1 - \frac{4\pi}{3}\lambda_2 \right) |\hat{u}|^2(\xi) \, d\xi \geq 0.$$ 

From this fact it is clear that standing states are not allowed in this regime due to the fact that $V(c) = 0$. Hence following verbatim the argument for the unstable regime without any other assumptions, we have the following result (see [12, Section 7] for an analogous remark about the cubic defocusing NLS).

**Corollary 1.8.** In the stable regime for any initial datum in $H^1(\mathbb{R}^3)$, the corresponding global solution to (1.2) scatters.

1.1. **Notations.** In what follows, we will use the notations below.

Given two quantities $A$ and $B$, we denote $A \lesssim B$ ($A \gtrsim B$, respectively) if there exists a positive constant $C$ such that $A \leq CB$ ($A \geq CB$, respectively). If both the relations hold true, we write $A \sim B$. For $1 \leq p \leq \infty$, the $L^p = L^p(\mathbb{R}^3; \mathbb{C})$ are the classical Lebesgue spaces endowed with norm $\|f\|_{L^p} = \left( \int_{\mathbb{R}^3} |f(x)|^p \, dx \right)^{1/p}$ if $p \neq \infty$ or $\|f\|_{L^\infty} = \sup_{x \in \mathbb{R}^3} |f(x)|$ for $p = \infty$. $W^{1,r} = W^{1,r}(\mathbb{R}^3; \mathbb{C})$ is defined as the space of function in $L^p$ with distributional derivatives in $L^p$, with the usual norm $\|f\|_{W^{1,r}} = \|f\|_{L^p} + \|\nabla f\|_{L^p}$. When $r = 2$, we set $H^1 = H^1(\mathbb{R}^3; \mathbb{C}) := W^{1,2} := (1 - \Delta)^{-1/2}L^2$ and its homogeneous version $\dot{H}^1 = \dot{H}^1(\mathbb{R}^3; \mathbb{C}) := (-\Delta)^{-1/2}L^2$. Since we work on $\mathbb{R}^3$, we simply denote $\hat{f} = \int_{\mathbb{R}^3} f(x)$. For a normed (Banach) space $(X, \|\cdot\|)$ we denote by $B_R(X)$ the open ball of radius $R$ with center at the origin, i.e. $B_R(X) = \{ f \in X; \|f\| < R \}$. If $X = \mathbb{R}^3$, then $B(x_0, R)$ is the ball of radius $R$ centered at $x_0$. Given an interval $I \subseteq \mathbb{R}$, bounded or unbounded, we define by $L^p_I(X) = L^p(I; X)$ the Bochner space of vector-valued functions $f : I \to X$ endowed with the norm $(\int_I \|f(s)\|^p_X \, ds)^{1/p}$ for $1 \leq p < \infty$, with similar modification as above for $p = \infty$. In case $I = \mathbb{R}$ it will be simply written $L^p X$. (In what follows, $f \in L^p X$ means that $f = f(t, x)$ is a function depending on the time variable $t \in \mathbb{R}$ and the space variable $x \in \mathbb{R}^3$, and $\|f\|_{L^p_I(X)} = \|f\|_{L^p(I; X)}$).

The operator $\mathcal{F} f(\xi) = \hat{f}(\xi)$ is the standard Fourier Transform, $\mathcal{F}^{-1}$ being its inverse. For $1 \leq p \leq \infty$, $p' := \frac{p}{p-1}$. The operator $\tau_y$ will denote the translation operator $\tau_y f(x) := f(x - y)$, while $f * g$ is the convolution operator between $f$ and $g$. $\mathbb{R}^d$ and $\mathbb{C}^2$ are the common notations for the real and imaginary parts of a complex number $z$. Finally, given a measurable set $\mathcal{A} \subseteq \mathbb{R}^d$, $1_{\mathcal{A}}(x)$ is the indicator function of $\mathcal{A}$.

2. **Strichartz Estimates and Variational Estimates**

We recall the Strichartz estimates which are the basic tool to study nonlinear dispersive equation of Schrödinger-type (but not only them), whose proof can be found in the classical monographs [10, 21] and [17] for the endpoint case ($r = 6$ below). We refer the reader to these already mentioned works for more accurate treatments on these kind of a priori estimates for the Schrödinger propagator. We just point out here that they are essentially consequences of the so-called dispersive estimate

$$\|U(t)f\|_{L^\infty} \leq C|t|^{-3/2}\|f\|_{L^1}, \quad \forall t \neq 0, \quad \forall f \in L^1,$$

which also holds for general dimensions, namely in the whole space $\mathbb{R}^d$ with $L^1 - L^\infty$ decay rate given by $|t|^{-d/2}$. More generally, conservation of the $L^2$-norm along the linear propagation (which for the nonlinearity in (1.2) also holds true for the nonlinear flow, see (1.5)) together with (2.1) implies, as a trivial application of the Riesz-Thorin interpolation theorem, that for any $p \in [2, \infty]$ the $L^p - L^{p'}$ bound below is satisfied:

$$\|U(t)f\|_{L^p} \leq C|t|^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{p'})}\|f\|_{L^{p'}}, \quad \forall t \neq 0, \quad \forall f \in L^{p'}.$$

Let us state the Strichartz estimates we will need along the paper.
Lemma 2.1. Let \((q, r), (\gamma, \rho)\) be two arbitrary \(3D\)–admissible pairs, namely they satisfy the algebraic conditions

\[
\frac{2}{q} = 3 \left( \frac{1}{2} - \frac{1}{r} \right), \quad 2 \leq r \leq 6.
\]

Then for any interval \(I \ni t_0, \) bounded or unbounded,

\[
\|U(t)f\|_{L^q_I L^r} \leq C_1 \|f\|_{L^2}, \quad \forall f = f(x) \in L^2,
\]

\[
\left\| \int_{t_0}^t U(t-s)F(s)ds \right\|_{L^q_I L^r} \leq C_2 \|F\|_{L^p_I L^{p'}}, \quad \forall F = F(t, x) \in L^p_I L^{p'},
\]

where the constant \(C_1, C_2\) depend only on the structural parameter and not on the functions \(f, F\) themselves.

Remark 2.2. Due to the commutativity property between derivatives and the linear flow, the previous estimates extend to Sobolev spaces:

\[
\|U(t)f\|_{L^q_I W^{1,r}} \leq \tilde{C}_1 \|f\|_{H^1}, \quad \forall f = f(x) \in H^1,
\]

\[
\left\| \int_{t_0}^t U(t-s)F(s)ds \right\|_{L^q_I W^{1,r}} \leq \tilde{C}_2 \|F\|_{L^p_I W^{1,p'}}, \quad \forall F = F(t, x) \in L^p_I W^{1,p'}.
\]

We will also use an extension for not admissible pairs for the inhomogeneous Strichartz estimates, see [13] and [27] for a general treatment.

Lemma 2.3. For any interval \(I \ni t_0, \) bounded or unbounded,

\[
\left\| \int_{t_0}^t U(t-s)F(s)ds \right\|_{L^q_I L^r} \leq C_3 \|F\|_{L^q_I L^r L^{q'} L^{r'}}, \quad \forall F = F(t, x) \in L^q_I L^{q'} L^{r'} L^{r'}.
\]

We shall notice that the fact that \(\lambda_1, \lambda_2\) belong to the unstable regime does not guarantee that the potential energy \(\mathcal{P}(u)\) fulfils the condition \(\mathcal{P}(u) < 0.\) As an example of this fact we show the following.

Lemma 2.4. Let \(\lambda_1, \lambda_2,\) belong to the unstable regime (1.3) with \(\lambda_2 > 0\) (without loss of generality). Assume moreover the additional restriction \(\lambda_1 + \frac{8}{3} \pi \lambda_2 > 0.\) Then there exists \(u \in H^1\) with \(\|u\|_{L^2}^2 = c\) such that

\[
\mathcal{P}(u) > 0.
\]

Proof. We will show the existence of a function \(u \in H^1\) with \(\|u\|_{L^2}^2 = c\) such that \(\mathcal{P}(u) > 0\) by scaling argument. Let us consider

\[
u^\mu = \mu u(\mu^{1/2} x_1, \mu^{1/2} x_2, \mu x_3).
\]

This transformation preserves the \(L^2\)-norm and is such that the potential energy rescales straightforwardly as

\[
\mathcal{P}(u^\mu) = c \mu^2 \int \left( \lambda_1 + \frac{4}{3} \pi \lambda_2 \frac{2 \mu^2 \xi_3^2 - \mu \xi_1^2 - \mu \xi_2^2}{\mu \xi_1^2 + \mu \xi_2^2 + \mu^2 \xi_3^2} \right) \left| \left( |u|^2 \right| dx, \right.
\]

where we used the scaling property of Fourier transform for rescaled functions. Now, with our assumptions, we have

\[
\lim_{\mu \to \infty} \lambda_1 + \frac{4}{3} \pi \lambda_2 \frac{2 \mu^2 \xi_3^2 - \mu \xi_1^2 - \mu \xi_2^2}{\mu \xi_1^2 + \mu \xi_2^2 + \mu^2 \xi_3^2} = \lambda_1 + \frac{8}{3} \pi \lambda_2 > 0,
\]

which implies that \(\lim_{\mu \to \infty} \mathcal{P}(u^\mu) = +\infty\) for all \(u\) thanks to Lebesgue theorem.

It is important to notice that if the potential energy is negative it is possible to introduce a Weinstein-like functional analogous to the one arising from the Gagliardo-Nirenberg inequality, whose maximizers correspond to the ground states. In [3] is proved the following result ensuring existence of minimizer for the functional below:

\[
J(u) = \frac{\|\nabla u\|_{L^2}^2}{-\lambda_1 \|u\|_{L^4}^4 - \lambda_2 \int (K * |u|^2)|u|^2} dx.
\]
More precisely the result is as follows.

**Proposition 2.5.** [3, Proposition 3.2] Under the hypothesis (1.3) there exists a minimizer $v_m \in H^1$ to (2.3), namely $\inf \{ J(v), v \in H^1, v \neq 0, \mathcal{P}(v) < 0 \}$ is attained at $v_m$, i.e., $J(v_m) = m := \inf \{ J(v), v \in H^1, v \neq 0, \mathcal{P}(v) < 0 \}$. Moreover, by scaling invariance property of the functional $J$, it can be assumed that $\|v_m\|_{L^2} = \|\nabla v_m\|_{L^2} = 1$. Furthermore $v_m$ solves

$$-3\Delta v_m + 4m (\lambda_1 |v_m|^2 v_m + \lambda_2 (K * |v_m|^2) v_m) + v_m = 0. \quad (2.4)$$

By plugging $Q(x) = 2\sqrt{mv_m(x)}(6^{-1/2}x)$ into (2.4) one finds that such $Q$ satisfies

$$-\frac{1}{2} \Delta Q + (\lambda_1 |Q|^2 Q + \lambda_2 (K * |Q|^2) Q) + Q = 0.$$

Let us consider the Cauchy problem for the focusing cubic NLS in three dimension

$$i\partial_t w + \frac{1}{2} \Delta w = -|w|^2 w, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3; \quad w(0, x) = w_0(x) \in H^1 \quad \text{(2.5)}$$

in [16] sufficient conditions to have global existence (and scattering) for (2.5) has been shown. They are given in term of the energy and mass of initial datum with respect to the same quantities associated to the ground state $S$ for (2.5), the latter being the solution to

$$-\frac{1}{2} \Delta S + S = |S|^2 S.$$

The two conditions are precisely

$$M(w_0)E(w_0) < M(S)E(S) \quad \text{(2.6)}$$

and

$$\|w_0\|_{L^2} \|w_0\|_{H^1} < \|S\|_{L^2} \|S\|_{H^1}, \quad \text{(2.7)}$$

where

$$M(w) = \|w\|_{L^2}^2, \quad E(w) = \frac{1}{2} \int (|\nabla w(t)|^2 - |w(t)|^4) \, dx$$

are conserved along the NLS flow (notice that they are exactly the analogous quantities defined in (1.5) and (1.6) for (1.2)) Conditions (2.6) and (2.7) ensure a uniform bound on the $H^1$ norm of the solution $w(t)$ to (2.5) on its lifespan, hence it exists for every time according to the well-known blow-up alternative criterium.

Let $Q$ be the minimizer for the Weinstein functional (2.3) which fulfills

$$-\frac{1}{2} \Delta Q + (\lambda_1 |Q|^2 Q + \lambda_2 (K * |Q|^2) Q) + Q = 0.$$

We have the following results that shows that our assumptions $\mathcal{G}(u) > 0$ and $\mathcal{E}(u) < \gamma(c)$ are equivalent to (2.6) and (2.7) in the context of focusing and cubic NLS.

**Proposition 2.6.** Under the hypothesis of Theorem 1.3 the initial datum $u_0$ satisfies

$$M(u_0)\mathcal{E}(u_0) < M(Q)\mathcal{E}(Q) \quad \text{(2.8)}$$

and

$$\|u_0\|_{L^2} \|u_0\|_{H^1} < \|Q\|_{L^2} \|Q\|_{H^1}. \quad \text{(2.9)}$$

Moreover, with an analogous reasoning, conditions expressed in (2.8) and (2.9) imply that the initial datum falls into the hypothesis of Theorem 1.3.
Proof. From the definition of the quantities in (1.6), (1.10) and (1.11) related to (1.2), we straightforwardly have
\[ \mathcal{E}(u_0) - \frac{1}{3} G(u_0) = \frac{1}{6} \mathcal{T}(u_0). \]
We notice that \( Q_\mu = \mu Q(\mu x) \) is again a minimizer for the Weinstein functional with
\[
\begin{align*}
\| Q_\mu \|_{L^2}^2 & = \mu^{-1} \| Q \|_{L^2}^2, \\
\| \nabla Q_\mu \|_{L^2}^2 & = \mu \| \nabla Q \|_{L^2}^2.
\end{align*}
\]
We notice that \( Q(x)e^{it} \) is a standing wave solution to the evolution equation and by the symmetry of the equation it is well known that \( Q_\mu e^{i\mu^2 t} = \mu Q(\mu x)e^{i\mu^2 t} \) is another standing wave solution to
\[
-\frac{1}{2} \Delta Q_\mu + (\mu |Q_\mu|^2 Q_\mu + \mu (K * |Q_\mu|^2)Q_\mu) + \mu^2 Q_\mu = 0,
\]
that necessarily satisfies \( G(Q_\mu) = 0 \). Hence \( \mathcal{E}(Q_\mu) = \frac{1}{\mu} \| \nabla Q_\mu \|_{L^2}^2 \). From the condition \( \mathcal{E}(u) < \gamma(c) = \mathcal{E}(Q_\mu) \) with \( c = \| u \|_{L^2}^2 = \| Q_\mu \|_{L^2}^2 \), we get
\[
\| u \|_{L^2}^2 \mathcal{E}(u) < \mathcal{E}(Q) \| Q \|_{L^2}^2,
\]
which corresponds to (2.8). Moreover if \( G(u) > 0 \) and \( \mathcal{E}(u) < \gamma(c) = \mathcal{E}(Q_\mu) \), then we have
\[
\frac{1}{6} \| \nabla Q_\mu \|_{L^2}^2 = \mathcal{E}(Q_\mu) > \mathcal{E}(u) > \mathcal{E}(u) - \frac{1}{3} G(u) = \frac{1}{6} \| \nabla u \|_{L^2}^2
\]
and hence
\[
\| u \|_{L^2}^2 \| u \|_{H^1}^2 < \| Q \|_{L^2}^2 \| Q \|_{H^1}^2,
\]
which is the analogous of condition (2.7) in our setting. \( \square \)

To understand the geometry of \( \mathcal{E}(u) \) on \( S(c) \) we now introduce the scaling
\[
u_\mu(x) = \mu^{3/2} u(\mu x), \quad \mu > 0.
\]
The next lemma is contained in [7].

Lemma 2.7. [7, Lemma 3.3] Let \( u \in S(c) \) be such that \( \int (\lambda_1 + \lambda_2 \hat{K}(\xi))|\hat{u}(\xi)|^2 \, d\xi < 0 \) then we have:
\begin{enumerate}
\item there exists a unique \( \mu^*(u) > 0 \), such that \( u^{\mu^*} \in V(c) \) (defined in (1.13));
\item the map \( \mu \mapsto \mathcal{E}(u^\mu) \) is concave on \( [\mu^*, \infty) \);
\item \( \mu^*(u) < 1 \) if and only if \( G(u) < 0 \);
\item \( \mu^*(u) = 1 \) if and only if \( G(u) = 0 \);
\item the functional \( G \) satisfies
\[
G(u^\mu) \begin{cases}
> 0, & \forall \mu \in (0, \mu^*(u)) \\
< 0, & \forall \mu \in (\mu^*(u), +\infty)
\end{cases}
\]
\item \( \mathcal{E}(u^\mu) < \mathcal{E}(u^{\mu^*}) \), for any \( \mu > 0 \) and \( \mu \neq \mu^* \);
\item \( \frac{\partial}{\partial \mu} \mathcal{E}(u^\mu) = \frac{1}{\mu} G(u^\mu), \forall \mu > 0. \)
\end{enumerate}

Proof. Since
\[
\mathcal{E}(u^\mu) = \frac{\mu^2}{2} \mathcal{T}(u) + \frac{\mu^3}{2} \mathcal{P}(u)
\]
we have that
\[
\frac{\partial}{\partial \mu} \mathcal{E}(u^\mu) = \mu \mathcal{T}(u) + 3 \mu^2 \mathcal{P}(u) = \frac{1}{\mu} G(u^\mu).
\]
Now we denote
\[
y(\mu) = \mu \mathcal{T}(u) + 3 \mu^2 \mathcal{P}(u),
\]
and we observe that $G(u^\mu) = \mu y(\mu)$ which proves (7). After direct calculations, we see that:

$$y'(\mu) = T(u) + 3\mu P(u),$$
$$y''(\mu) = 3P(u).$$

From the expression of $y'(\mu)$ and the assumption $P(u) < 0$ we know that $y'(\mu)$ has a unique zero that we denote $\mu_0 > 0$ such that $\mu_0$ is the unique maximum point of $y(\mu)$. Thus in particular the function $y(\mu)$ satisfies:

(i) $y(\mu_0) = \max_{\mu > 0} y(\mu)$;
(ii) $\lim_{\mu \to +\infty} y(\mu) = -\infty$;
(iii) $y(\mu)$ decreases strictly in $[\mu_0, +\infty)$ and increases strictly in $(0, \mu_0]$.

By the continuity of $y(\mu)$, we deduce that $y(\mu)$ has a unique zero $\mu^* > 0$. Then $G(u^\mu) = 0$ and point (1) follows. Points (2)–(4) and (5) are also easy consequences of (i)–(iii). Finally, since $y(\mu) > 0$ on $(0, \mu^*(u))$ and $y(\mu) < 0$ on $(\mu^*(u), \infty)$ we get (6).

We are now in the position to state a lower bound for (1.12).

**Proposition 2.8.** Under the hypothesis of Theorem 1.3

$$4 \int |\nabla u|^2 \, dx + 6\lambda_1 \int |u|^4 \, dx + 6\lambda_2 \int (K * |u|^2)|u|^2 \, dx \geq 4 \min \{ \gamma(c) - E(u_0), E(u_0) \} := \alpha.$$

**Proof.** We shall distinguish two cases which depends on the sign of the nonlinear term. At a given time $t$ we have either $P(u(t)) > 0$ or $P(u(t)) < 0$. Let $\tilde{t}$ therefore be an arbitrary but fixed time and simply denote $u = u(x, \tilde{t})$. We consider both cases:

**Case 1:** $P(u) > 0$. In this case the estimate is trivial. Indeed, $G(u) = 2E(u) + \frac{1}{4} P(u) > 2E(u) > E(u)$.

**Case 2:** $P(u) < 0$. In this case we argue using the scaling of Lemma 2.7. First let us notice that if $\frac{P(u)}{2} < E(u)$ then $G(u) = 2E(u) + \frac{P(u)}{2} > E(u)$ then the lower bound is achieved. We assume hence that $\frac{P(u)}{2} \geq E(u)$. Let us rescale the function $u$ according to the scaling of Lemma 2.7 such that $u^{\mu^*} \in V(c)$ and let us express $E(u^{\mu^*}) - E(u)$ as

$$E(u^{\mu^*}) - E(u) = (\mu^* - 1) \frac{\partial}{\partial \mu} E(u^{\mu}) \big|_{\mu = \mu_0}$$

with $1 < \mu_0 < \mu^*$, according to the previous Lemma. Now we claim that

$$\mu^* < 2 \quad \text{and} \quad \frac{\partial}{\partial \mu} E(u^{\mu}) \big|_{\mu = \mu_0} < \frac{\partial}{\partial \mu} E(u^{\mu}) \big|_{\mu = 1} = G(u).$$

From the claim we get the desired estimate. Indeed,

$$\gamma(c) - E(u_0) \leq E(u^{\mu^*}) - E(u) \leq G(u),$$

and this concludes the proof.

Thus, it remains to prove (2.10). Since $E(u^{\mu}) = \frac{\mu^2}{2} T(u) + \frac{\mu^4}{2} P(u)$ we get that $E(u^{\mu}) < 0$ for $\mu > \frac{\mu^2}{|P(u)|}$, hence $\mu^* < \frac{\mu^4}{|P(u)|}$. Now we notice that the assumption $\frac{1}{2} |P(u)| \geq E(u)$ implies that $\frac{\mu^4}{|P(u)|} = \frac{2E(u) + |P(u)|}{|P(u)|} \leq 2$. To prove $\frac{\partial}{\partial \mu} E(u^{\mu}) \big|_{\mu = \mu_0} < G(u)$ it is sufficient to show that $\frac{\partial}{\partial \mu} E(u^{\mu})$ is monotone decreasing when $\mu > 1$. Direct computation gives in fact

$$\frac{\partial^2}{\partial \mu^2} E(u^{\mu}) = T(u) + 3\mu P(u) < 0.$$
provided that \( \mu > \frac{T(u)}{3|P(u)|} = \frac{2\mathcal{E}(u) + |P(u)|}{3|P(u)|} \). Now the condition \( \frac{1}{2} |P(u)| \geq \mathcal{E}(u) \) implies that
\[
\frac{2\mathcal{E}(u) + |P(u)|}{3|P(u)|} \leq \frac{2}{3}.
\]

The fact that \( \mu_0 > 1 \) proves that claim. Summing up all the estimates we get
\[
\mathcal{G}(u) > \min \{ \gamma(c) - \mathcal{E}(u_0), \mathcal{E}(u_0) \}.
\]

3. Perturbative nonlinear results

We collect here some perturbative results for (1.2). To lighten the exposition, we just give here the statements, postponing the proofs in Appendix A. The next two Lemmas are in the framework of the so-called Small data Theory, which is actually the first cornerstone on which the Kenig & Merle approach is built. The first one ensures that if the initial datum is sufficiently small in the \( H^1 \)-norm, then its nonlinear evolution under the Gross-Pitaevskii flow (1.2) is global.

**Lemma 3.1.** There exists a radius \( \rho > 0 \) such that if \( u_0 \in B_\rho(H^1) \) then the corresponding solution \( u \in \mathcal{C}(-T_{\text{min}}, T_{\text{max}}); H^1) \) to (1.2) with \( u_0 \) as initial datum is global, i.e. \( T_{\text{min}} = T_{\text{max}} = +\infty \).

The second one claims that if the initial datum is small enough (possibly smaller than the previous one), still in the \( H^1 \) space, then the global solution to (1.2) actually behaves like a free wave asymptotically in time.

**Lemma 3.2.** There exists \( \delta > 0 \) such that for any \( u_0 \in B_\delta(H^1) \), the solution \( u(t, x) \) to the Cauchy problem (1.2) scatters to a linear solution in \( H^1 \).

The following states that if a global solution to (1.2) enjoys some uniform spacetime control, then it scatters.

**Lemma 3.3.** If \( u(t, x) \in \mathcal{C}(\mathbb{R}; H^1) \cap L^8 L^4 \) is a solution to (1.2), then it scatters.

**Remark 3.4.** It is worth mentioning that if a \( \mathcal{C}(\mathbb{R}; H^1) \) solution to (1.2) scatters, then it belongs to \( L^8 L^4 \).

We conclude this preliminary tools section with a small perturbation result.

**Lemma 3.5.** For every \( M > 0 \) there exist \( \varepsilon = \varepsilon(M) > 0 \) and \( C = C(M) > 0 \) such that: if \( u(t, x) \in \mathcal{C}(\mathbb{R}; H^1) \) is the unique global solution to (1.2) and \( w \in \mathcal{C}(\mathbb{R}; H^1) \cap L^8 L^4 \) is a global solution to the perturbed problem
\[
\begin{cases}
  i\partial_t w + \frac{1}{2}\Delta w = \lambda_1 |w|^2 w + \lambda_2 (K \ast |w|^2) w + e(t, x) \\
  w(0, x) = w_0 \in H^1
\end{cases}
\]
satisfying the conditions \( \|w\|_{L^8 L^4} \leq M, \| \int_0^t U(t-s)e(s) ds \|_{L^8 L^4} \lesssim \varepsilon \) and \( \|U(t-t_0)(u(t_0)-w(t_0))\|_{L^8 L^4} \leq \varepsilon \), then \( u \in L^8 L^4 \) and \( \|u-w\|_{L^8 L^4} \lesssim \varepsilon \).

**Remark 3.6.** The Profile Decomposition Theorem below is the fundamental tool to construct a minimal (with respect to the energy) global but not scattering solution to (1.2). As it will be clear in the next section, that is a linear result. Therefore the perturbation lemma above will be crucial to absorb the error once some nonlinear profiles are associated to the linear ones of Theorem 4.1. We refer the reader to the next sections for more rigorous statements.
4. Linear Profile Decomposition and Nonlinear Profiles

The main ingredient in the construction of the minimal element is a suitable Profile Decomposition Theorem. It is worth mentioning that this kind of result goes back to the work of Gérard, see [14], where is given an explicit characterization of the defect of compactness for the Sobolev embeddings. Pioneering results for evolution equations are the works by Bahouri and Gérard, see [4], for the critical wave equation, by Keraani, see [20], about the defect of compactness for the Strichartz embeddings, and by Merle and Vega, see [23], how to concerns concentration phenomena for the two dimensional mass critical nonlinear Schrödinger equation.

If it were not for the presence of the nonlocal term, the next linear result in the non-radial setting would be exactly the same given by Duyckaerts, Holmer and Roudenko in [12], which in turn extended the one in Holmer and Roudenko [16] removing the spherical symmetry assumption of the this last mentioned paper. For (1.2), an additional term must be dealt with, so at first we state the theorem in [12], then we will show how to manage with the nonlocal term of (1.2).

**Theorem 4.1.** Given a bounded sequence \( \{v_n\}_{n \in \mathbb{N}} \subset H^1, \forall J \in \mathbb{N} \) and \( \forall 1 \leq j \leq J \) there exist sequences of time and space translation parameters \( \{t_n^j\}_{n \in \mathbb{N}} \subset \mathbb{R}, \{x_n^j\}_{n \in \mathbb{N}} \subset \mathbb{R}^3 \) and profiles \( \psi^j \in H^1 \) for \( j = 1, \ldots, J \) and \( R_n^j \) such that, up to subsequences,

\[
v_n = \sum_{1 \leq j \leq J} U(-t_n^j)^{\tau}_{x_n^j} \psi^j + R_n^j
\]

with the following properties:

- **[Dichotomy of the parameters]** for any fixed \( j \in \{1, \ldots, J\} \)
  
  \[ \text{either} \quad t_n^j = 0 \quad \forall n \in \mathbb{N} \quad \text{or} \quad t_n^j \to \pm \infty, \]

  \[ \text{either} \quad x_n^j = 0 \quad \forall n \in \mathbb{N} \quad \text{or} \quad x_n^j \to \pm \infty; \]

- **[Divergence property]** for any \( j \neq k \in \{1, \ldots, J\} \)

\[
|x_n^j - x_n^k| + |t_n^j - t_n^k| \overset{n \to \infty}{\to} \infty;
\]

- **[Smallness of the remainder]** \( \forall \varepsilon > 0 \quad \exists \hat{J} = \hat{J}(\varepsilon) \) such that, for any \( J \geq \hat{J} \)

\[
\limsup_{n \to \infty} \|U(t)R_n^j\|_{L^\infty L^1 \cap L^4} \leq \varepsilon;
\]

- **[Pythagorean expansion of mass and kinetic energy]** the mass and the quadratic energy term are almost orthogonal, namely as \( n \to \infty \)

\[
\|v_n\|_{L^2}^2 = \sum_{1 \leq j \leq J} \|\psi^j\|_{L^2}^2 + \|R_n^j\|_{L^2}^2 + o(1), \quad \forall J \in \mathbb{N},
\]

\[
\|v_n\|_{L^4}^4 = \sum_{1 \leq j \leq J} \|U(-t_n^j)^{\tau}_{x_n^j} \psi^j\|_{L^4}^4 + \|R_n^j\|_{L^4}^4 + o(1), \quad \forall J \in \mathbb{N};
\]

- **[Pythagorean expansion of the local potential energy]** \( \forall J \in \mathbb{N} \), as \( n \to \infty \)

\[
\|v_n\|_{L^4}^4 = \sum_{1 \leq j \leq J} \|U(-t_n^j)^{\tau}_{x_n^j} \psi^j\|_{L^4}^4 + \|R_n^j\|_{L^4}^4 + o(1).
\]

If the nonlocal term in (1.2) were not present, namely \( \lambda_2 = 0 \), summing up (4.4) and (4.5) would lead to the so called orthogonal decomposition of the energy. But in our case we must deal with the nonlocal interaction term \( \lambda_2 \int (K * |v|^2)|v|^2 \). The next proposition aims to show exactly that what is inferred in (4.5) has its counterpart also for the dipolar interaction energy.
Proposition 4.2. Under the same hypothesis of Theorem 4.1, the following Orthogonal Expansion of the nonlocal energy can be claimed: for any $J \in \mathbb{N}$, as $n \to \infty$

\[
\int (K \ast |v_n|^2)|v_n|^2 \, dx = \sum_{1 \leq j \leq J} \int (K \ast |U(-t_n^j)\tau_{x_n^j}\psi^j|^2)|U(-t_n^j)\tau_{x_n^j}\psi^j|^2 \, dx
\]
\[
+ \int (K \ast |R_n^j|^2)|R_n^j|^2 \, dx + o(1).
\]

Proof. Up to reordering the indexes, we may suppose that there exists $J$ such that:

Case 1: $t_n^j = 0$ for any $n \in \mathbb{N}$, if $1 \leq j \leq J$;

Case 2: $|t_n^j| \to \infty$ as $n \to \infty$ if $J + 1 \leq j \leq J$.

Due to the divergence property in Theorem 4.1, in the situation of Case 1, given two different indexes $j \neq k \in \{1, \ldots, J\}$ then $|x_n^j - x_n^k| \to \infty$ as $n \to \infty$ and this implies the weak interaction for the cross term in the expression below:

\[
\int \int K(x - y) \left| \sum_{j=1}^J U(-t_n^j)\psi^j(y - x_n^j) \right|^2 \left| \sum_{j=1}^J U(-t_n^j)\psi^j(x - x_n^j) \right|^2 \, dy \, dx
\]
\[
= \int \int K(x - y) \left| \sum_{j=1}^J \psi^j(y - x_n^j) \right|^2 \left| \sum_{j=1}^J \psi^j(x - x_n^j) \right|^2 \, dy \, dx.
\]

More precisely, since $K \ast \tau_z g = \tau_z (K \ast g)$, then as $|z - z'| \to \infty$

\[
\int (K \ast \tau_z g)(x)\tau_{z'} h(x) \, dx = \int \tau_z (K \ast g)(x)\tau_{z'} h(x) \, dx = o(1).
\]

Hence (4.7) implies that in the situation delineated in Case 1

\[
\int \int K(x - y) \left| \sum_{j=1}^J \psi^j(y - x_n^j) \right|^2 \left| \sum_{j=1}^J \psi^j(x - x_n^j) \right|^2 \, dy \, dx
\]
\[
= \sum_{j=1}^J \int \int K(x - y) \left| \psi^j(y - x_n^j) \right|^2 \left| \psi^j(x - x_n^j) \right|^2 \, dy \, dx
\]
\[
= \sum_{j=1}^J \int \int K(x - y) \left| U(-t_n^j)\psi^j(y - x_n^j) \right|^2 \left| U(-t_n^j)\psi^j(x - x_n^j) \right|^2 \, dy \, dx.
\]

Under condition illustrated in Case 2 instead, the continuity property of the operator $K \ast f$, mapping continuously $L^p$ into itself for any $p \in (1, \infty)$, yields, by using the Cauchy-Schwartz inequality and the continuity property with $p = p' = 2$,

\[
\int (K \ast (|U(-t_n^j)\psi^j(\cdot - x_n^j)|^2))(x)|U(-t_n^j)\psi^j(\cdot - x_n^j)|^2 \, dx
\]
\[
\lesssim \|K \ast (|U(-t_n^j)\psi^j(\cdot - x_n^j)|^2)\|_{L^2} \|U(-t_n^j)\psi^j(\cdot - x_n^j)|^2\|_{L^2}
\]
\[
\lesssim \|U(-t_n^j)\psi^j\|_{L^4}^{n-\infty} 0.
\]

The last decay property follows by the dispersive estimate (2.2) of the Schrödinger free propagator and the fact that $U(t)$ is an isometry on any $H^s$ space, and concluding with a density argument by considering
at first $\psi^j \in L^{4/3} \cap H^1$. Summing up the results in (4.8) and (4.9) we claim that for $n \to \infty$

$$\int \int K(x-y) \left| \sum_{j=1}^{J} U(-t_{ij}^n) \psi^j(y-x_i^*) \right|^2 \left| \sum_{j=1}^{J} U(-t_{ij}^n) \psi^j(x-x_i^*) \right|^2 \ dy \ dx$$

$$= \sum_{j=1}^{J} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} K(x-y) \left| U(-t_{ij}^n) \psi^j(y-x_i^*) \right|^2 \left| U(-t_{ij}^n) \psi^j(x-x_i^*) \right|^2 \ dy \ dx + o(1).$$

Recall that we aim to prove that the quantity

$$\int (K * |v_n|^2) |v_n|^2 \ dx - \sum_{j=1}^{J} \int (K * |U(-t_{ij}^n) \tau_x \psi^j|^2) |U(-t_{ij}^n) \tau_x \psi^j|^2 \ dx - \int (K * |R_n^J|^2) |R_n^J|^2 \ dx$$

(4.10)

$$= \int (K * |v_n - R_n^L|^2) |v_n - R_n^L|^2 \ dx + \int (K * |R_n^J - R_n^L|^2) |R_n^J - R_n^L|^2 \ dx$$

goes towards zero as $n \to \infty$, where $L$ is a fixed positive integer. To shorten the notation we define

$$\begin{align*}
g_n^L &= v_n - R_n^L \\
r_n^{L,J} &= R_n^L - R_n^J \\
u_n^j &= U(-t_{ij}^n) \tau_x \psi^j
\end{align*}$$

therefore the L.H.S. of (4.10) can be estimated as

$$L.H.S.(4.10) \leq \left| \int (K * |v_n|^2) |v_n|^2 \ dx - \int (K * |g_n^L|^2) |g_n^L|^2 \ dx \right|$$

$$+ \left| \int (K * |r_n^{L,J}|^2) |r_n^{L,J}|^2 \ dx - \int (K * |R_n^J|^2) |R_n^J|^2 \ dx \right|$$

$$+ \left| \int (K * |g_n^L|^2) |g_n^L|^2 \ dx - \sum_{j=1}^{J} \int (K * |u_n^j|^2) |u_n^j|^2 \ dx \right| - \left| \int (K * |r_n^{L,J}|^2) |r_n^{L,J}|^2 \ dx \right|$$

$$= I + II + III$$

and we show that this three quantities go towards zero as $n \to \infty$. Let us begin with the first term $I$.

$$I \leq \left| \int \int K(x-y) \left( |v_n(y)|^2 |v_n(x)|^2 - |(g_n^L(y))^2 |(g_n^L(x))|^2 \right) \ dy \ dx \right|$$

$$\leq \left| \int \int K(x-y) \Pi^4 \left( v_n(x), \tilde{v}_n(x), v_n(y), \tilde{v}_n(y), R_n^L(x), \bar{R}_n^L(x), R_n^L(y), \bar{R}_n^L(y) \right) \ dy \ dx \right|,$n

where $\Pi^4 \left( v_n(x), \tilde{v}_n(x), v_n(y), \tilde{v}_n(y) \right)$ is an homogeneous polynomial of order 4 not involving any monomial consisting in only $v_n, \tilde{v}_n$’s terms, hence it can be estimated by using the continuity property of the convolution with the kernel $K$, obtaining

$$I \lesssim \| R_n^L \|^2_{L^4} + \| v_n \|^3_{L^4} \| R_n^L \|_{L^4} + \| v_n \|^2_{L^4} \| R_n^L \|^2_{L^4}$$

$$\lesssim \| R_n^L \|^2_{L^4} + \| R_n^L \|^2_{L^4} + \| R_n^L \|_{L^4},$$
where we used that \( \{v_n\}_{n \in \mathbb{N}} \) is uniformly bounded in \( L^4 \) since it is bounded in \( H^1 \), by Sobolev embedding. Since
\[
\|R_n^L\|_{L^4} \leq \|U(t)R_n^L\|_{L^\infty L^4} \leq \|U(t)R_n^L\|_{L^{\infty L^2}}^{1/2} \|U(t)R_n^L\|_{L^{\infty L^6}}^{1/2} \\
\leq \|U(t)R_n^L\|_{L^{\infty L^2}}^{1/2} \|U(t)R_n^L\|_{L^{\infty L^6}}^{1/2}
\]
combining (4.2) and (4.4) one obtains
\[
\lim_{J \to \infty} \limsup_{n \to \infty} \|R_n^J\|_{L^\infty L^4} = 0.
\]
Hence, for any \( \varepsilon > 0 \) there exists \( L \) large enough and \( \bar{n} \) such that for any \( n > \bar{n} \) the term \( I \leq \varepsilon/3 \). The same analysis can be carried out for \( II \), therefore also \( II \leq \varepsilon/3 \). Let turn the attention on the last term
\[
III = \left| \int (K * |g_n^L|^2)|g_n^L|^2 \, dx - \sum_{j=1}^{J} \int (K * |u_n^j|^2)|u_n^j|^2 \, dx - \int (K * |r_n^{L,j}|^2)|r_n^{L,j}|^2 \, dx \right|.
\]
By definition
\[
\begin{align*}
&\left\{ g_n^L = v_n - R_n^L = \sum_{j=1}^{J} u_n^j \right. \\
&\left. g_n^J = v_n - R_n^J = \sum_{j=1}^{J} u_n^j \right. \\
&R_n^L - R_n^J = \sum_{j=L+1}^{J} u_n^j
\end{align*}
\]
then
\[
III \leq \left| \int K(x-y) \sum_{j=1}^{L} u_n^j(y)^2 \sum_{j=1}^{L} u_n^j(x)^2 \, dy \right| \left| \sum_{j=1}^{L} \int K(x-y)|u_n^j(y)|^2|u_n^j(x)|^2 \, dy \right| \\
+ \left| \int K(x-y) \sum_{j=L+1}^{J} u_n^j(y)^2 \sum_{j=L+1}^{J} u_n^j(x)^2 \, dy \right| \left| \sum_{j=L+1}^{J} \int K(x-y)|u_n^j(y)|^2|u_n^j(x)|^2 \, dy \right| \\
\leq \varepsilon/6 + \varepsilon/6 = \varepsilon/3.
\]
Hence (4.6) is proved. \( \square \)

As immediate corollary we have:

**Corollary 4.3.** Under the hypothesis of Theorem 4.1, for the decomposition in (4.1) the Pythagorean expansion of the energy holds true, namely: \( \forall J \in \mathbb{N} \) as \( n \to +\infty \)
\[
\mathcal{E}(v_n) = \sum_{1 \leq j \leq J} \mathcal{E}(U(-t_n^j)x_n^j \psi^j) + \mathcal{E}(R_n^J) + o(1).
\]

As already pointed out in Remark 3.6, we will need to associate nonlinear profiles to the linear ones of Theorem 4.1. The existence of such nonlinear waves basically comes from the local well-posedness theory for (1.2) and the existence of wave operators. More precisely, once we consider a pair \( (\psi^j, t_n^j) \) as above, a nonlinear profile corresponding to it is given by means of a solution \( \Psi^j \) to (1.2) satisfying
\[
\|\Psi^j(t_n^j) - U(t_n^j)\psi^j\|_{H^1} \xrightarrow{n \to \infty} 0.
\]
Since the dichotomy property of the parameters in Theorem 4.1 allows us to restrict the situation only on the cases \( t_n^j = 0 \) or \( t_n^j \to \infty \), finding such \( \Psi^j \) reduces to solve the Cauchy problem for the GPE at \( t_0 = 0 \) or at \( t_0 = \infty \), namely it suffices to solve
\[
\Psi^j = U(t)\psi^j + \int_0^t U(t-s) \left( \lambda_1|\Psi^j|^2\Psi^j + \lambda_2(K * |\Psi^j|^2)\Psi^j \right) (s) \, ds
\]
and
\[ \Psi_j = U(t)\psi^j + i \int_0^t U(t-s) \left( \lambda_1 |\Psi^j|^2 \Psi^j + \lambda_2 (K * |\Psi^j|^2) \Psi^j \right) (s) \, ds, \]
where the equations above are referred as the integral formulation of the solution to (1.2), also known as Duhamel's representation.

5. Existence of the Minimal Element and its Properties

Once the Profile Decomposition is proved, combinations of arguments in Section 2 and Section 3 give the following.

**Theorem 5.1.** There exists a not trivial initial profile \( u_{sl,0} \in H^1 \), such that \( E(u_{sl,0}) < \gamma(\|u_{sl,0}\|^2_{L^2}) \) and \( G(u_{sl,0}) > 0 \) then the corresponding solution \( u_{sl} \) to (1.2) is globally defined but it does not scatter.

**Definition 5.2.** It is standard to refer to this solution as the minimal element or critical solution or soliton-like solution. Along the remaining part of the paper we follow this conventions and we will denote it as \( u_{sl} \), justifying therefore the subscripts in Theorem 5.1.

**Proof.** We just sketch the proof which relies on a contradiction argument. Let us define the threshold
\[ \gamma_{sl}(c) = \sup \left\{ \gamma > 0 \text{ such that if } u_0 \in S(c) \text{ with } E(u_0) < \gamma \text{ and } G(u_0) > 0 \right\} \]
then the solution to (1.2) with initial data \( u_0 \) is in \( L^8 L^4 \).

Since small data scattering holds, see Lemma 3.1 and Lemma 3.2, the set above is well-defined and \( \gamma_{sl} > 0 \).

The aim is to show that \( \gamma_{sl}(c) = \gamma(c) \) and we suppose that this is not the case, namely we assume that \( \gamma_{sl}(c) < \gamma(c) \). Consider a minimizing sequence of initial data \( \{u_n(0)\}_{n \in \mathbb{N}} \) with \( E(u_n(0)) \downarrow \gamma_{sl} \) and such that the corresponding sequence of solutions \( \{u_n(t)\}_{n \in \mathbb{N}} \) to (1.2) satisfy
\[ \limsup_{n \to \infty} \|u_n\|_{L^8 L^4} \to \infty. \]

The sequence \( \{u_n(0)\}_{n \in \mathbb{N}} \) can be decomposed in
\[ u_{0,n} = \sum_{1 \leq j \leq J} U(-t_n^j) \tau_{x_n^j} \psi^j + R_n^j, \]
by means of Theorem 4.1, Proposition 4.2 and Corollary 4.3. In particular, Corollary 4.3 implies that, in the limit \( n \to \infty \),
\[ \gamma_{sl}(c) = \sum_{1 \leq j \leq J} E(U(-t_n^j)\psi^j) + E(R_n^j) \]
while the orthogonal expansion of the mass (4.3) gives, as \( n \to \infty \), (since \( \|u_{0,n}\|^2_{L^2} = c \) for any \( n \in \mathbb{N} \))
\[ c^j := \|\psi^j\|^2_{L^2} \leq c. \]

We claim the following: there exists only one non trivial profile in the expansion (5.2).

Suppose by the absurd that at least two profiles are non-trivial, i.e. \( \psi^{j_h} \neq 0 \) for \( \{j_h\} \subseteq \{1, \ldots, J\} \) and the cardinality \#\( \{j_h\} \geq 2 \). Let us keep the notation \( \psi^j \) instead of \( \psi^{j_h} \). This implies that \( c^j < c \) and \( E(U(-t_n^j)\psi^j) < \gamma_{sl}(c) \).

We recall that the equation (1.2) is invariant under the transformation \( u \mapsto u_\mu := \mu u(\mu^2 t, \mu x) \), the latter moreover leaving invariant the \( L^8 L^4 \)-norm, and we split the situation in two cases.

**Case 1.** In this first situation we assume that the time translation parameter of the profile decomposition above is diverging, namely \( t_n^j \to \infty \) for some \( j \). In this case we have that \( \lim_{n \to \infty} G(U(-t_n^j)\tau_{x_n^j} \psi^j) > 0 \) and \( \lim_{n \to \infty} E(U(-t_n^j)\tau_{x_n^j} \psi^j) > 0 \). Since we have \( E(U(-t_n^j)\psi^j) < \gamma_{sl}(c) \) and the scaling of the equation guarantees that
\[ c_j \gamma_{sl}(c_j) = c \gamma_{sl}(c) \implies \gamma_{sl}(c_j) > \gamma_{sl}(c) \]
we can infer that \( \mathcal{E}(U(-t_n^j)\tau_{x_n^j}\psi_j) < \gamma_{sl}(c_j) \). With the nonlinear profiles constructed at the end of Section 4, therefore mapping \( (U_n, \psi_j) \mapsto \Psi_n \), we get \( \mathcal{E}(\Psi_j) < \gamma_{sl}(c_j) \), \( \Psi_j \in S(c_j) \), \( \mathcal{G}(\Psi_j) > 0 \) and hence we obtain

\[
\|\Psi_j\|_{L^8L^4} < +\infty.
\]

**Case 2.** We consider the the remaining situation, namely when the time translation sequence is the trivial one.

We argue as before using that the convergence

\[
U(-t_n^j)\tau_{x_n^j}\psi_j \to \tau_{x_n^j}\psi_j
\]

as \( n \to \infty \) strongly holds in \( H^1 \) topology.

We first show that \( \mathcal{G}(U(-t_n^j)\tau_{x_n^j}\psi_j) > 0 \). Notice that thanks to \( \mathcal{G}(u_n) > 0 \)

\[
\frac{c_1}{6} \|\nabla U(-t_n^j)\tau_{x_n^j}\psi_j\|_{L^2}^2 < \frac{c}{6} \|\nabla u_n\|_{L^2}^2 < c\mathcal{E}(u_n) = c\gamma_{sl}(c) + o(1) = c\gamma_{sl}(c_j) + o(1).
\]

Let us suppose \( \mathcal{G}(U(-t_n^j)\tau_{x_n^j}\psi_j) < 0 \) and let us choose \( 0 < \mu^* \leq 1 \) such that \( \mathcal{G}(\nu\nu) = 0 \) where \( \nu = U(-t_n^j)\tau_{x_n^j}\psi_j \) and \( \nu = \mu^{3/2}v(\mu x) \) (see Lemma 2.7 for the properties of the functional \( \mathcal{G} \) when evaluated on such rescaled functions). We have

\[
\mathcal{E}(\nu\nu) = \frac{\mu^*}{6} \|\nabla U(-t_n^j)\tau_{x_n^j}\psi_j\|_{L^2}^2 < \frac{1}{6} \|\nabla U(-t_n^j)\tau_{x_n^j}\psi_j\|_{L^2}^2
\]

and therefore

\[
c_2\mathcal{E}(\nu\nu) \leq \frac{c_1}{6} \|\nabla U(-t_n^j)\tau_{x_n^j}\psi_j\|_{L^2}^2 < c_2\gamma_{sl}(c_j) + o(1) < c_2\gamma(c_j)
\]

which leads to the contradiction.

Now, recalling that \( \mathcal{E}(U(-t_n^j)\tau_{x_n^j}\psi_j) > \frac{1}{6} \|\nabla U(-t_n^j)\tau_{x_n^j}\psi_j\|_{L^2}^2 \) we get \( \mathcal{E}(U(-t_n^j)\tau_{x_n^j}\psi_j) < \gamma_{sl}(c) < \gamma_{sl}(c_j) \). As for the previous case, we can associate to the linear profiles their corresponding nonlinear ones \( \Psi_j \) having good properties to belong to \( S(c_j) \), to satisfy \( \mathcal{G}(\Psi_j) > 0 \) and having finite \( L^8L^4 \)-norm, hence they scatter.

All ingredients to show that there can be only a nontrivial profile in the decomposition (5.2) are established, therefore the existence of the minimal element \( u_{sl} \) as stated in Theorem 5.1 can be proved following [12, 16].

By substituting the linear profiles in (5.2) with the nonlinear ones coming from Case 1 and Case 2 above, we write

\[
u_{0,n} = \sum_{1 \leq j \leq J} \Psi_j(-t_n^j) + \rho_n^J,
\]

with

\[
(5.3) \limsup_{n \to \infty} \|U(t)\rho_n^J\|_{L^8L^4} \leq \varepsilon
\]

for any \( J \geq J = \tilde{J}(\varepsilon) \).

The aim is now to give an approximation of \( \{u_n(t)\}_{n \in \mathbb{N}} \) in terms of the scattering nonlinear profiles above to reach a contradiction by showing that \( \{u_n(t)\}_{n \in \mathbb{N}} \) has uniformly bounded \( L^8L^4 \)-norm by means of the perturbation result of Lemma 3.5. Therefore we define

\[
w_n(t) = \sum_{1 \leq j \leq J} \Psi_j(t - t_n^j, x - x_n^j)
\]

and by the very definition of the involved term we get

\[
i \partial_t w_n + \frac{1}{2} \Delta w_n - \lambda_1 |w_n| w_n - \lambda_2(K \ast |w_n|^2)w_n = \lambda_1 |w_n| w_n + \lambda_2(K \ast |w_n|^2)w_n + e_n
\]
with
\[ e_n = \sum_{1 \leq j \leq J} \lambda_1 \left( |\Psi^j(t - t_n^j, x - x_n^j)|^2 \Psi^j(t - t_n^j, x - x_n^j) - |u_n|^2 w_n \right) 
+ \sum_{1 \leq j \leq J} \lambda_2 \left( (K * |\Psi^j(t - t_n^j, x - x_n^j)|^2) \Psi^j(t - t_n^j, x - x_n^j) - (K * |w_n|^2)w_n \right). \]

First of all, one observes that \( w_n(0) - u_n(0) = w_0 - u_{0,n} = p_n \), and therefore by (5.3)
\[ \limsup_{n \to \infty} \left( \lim_{j \to \infty} \|U(t)(w_n(0) - u_{0,n})\|_{L^8 L^4} \right) = 0. \]
Moreover it can be claimed that \( \| \int_0^1 U(t - s)e_n(s) ds \|_{L^8 L^4} \leq \varepsilon \) uniformly in in \( n \), for \( n \) large enough depending on \( \varepsilon \) and \( J \). For this details we refer to [16]. All these ingredients allow to apply the perturbation Lemma 3.5 obtaining therefore that
\[ \sup_{n \in \mathbb{N}} \|u_n\|_{L^8 L^4} \leq C < \infty, \]
which is a contradiction with respect to (5.1).

We eventually arrive to the existence of only one nontrivial profile and again by proceed in the same spirit of [16] we get the existence of a global non-scattering solution as in the statement of Theorem 5.1. □

**Proposition 5.3.** The translation path \( x(t) : \mathbb{R}^+ \to \mathbb{R}^3 \) has a sub-linear growth at infinity, namely as \( t \to \infty \)
\[ \frac{|x(t)|}{t} = o(1). \]

**Proof.** The proof of this spatial control is contained in [12], once one notices that also for (1.2) the momentum of the critical solution \( u_{sl} \) given in Theorem 5.1 is zero, i.e. \( P(u_{sl}) = \exists \int_{\mathbb{R}^3} \bar{u}_{sl} \nabla u_{sl} \ dx = 0 \). In fact, if we consider the Galilean transformation of \( u_{sl} \) given by
\[ T_v(u_{sl}(x, t)) = e^{i(v \cdot x - |v|^2 t)} u_{sl}(x - 2vt, t), \quad v \in \mathbb{R}^3 \]
and we assume that \( P(u_{sl}) \neq 0 \), then by selecting
\[ v = -\frac{P(u_{sl})}{M(u_{sl})} \]
we will have \( P(T_v(u_{sl})) = 0 \) and \( \mathcal{E}(T_v(u_{sl})) < \mathcal{E}(u_{sl}) < \gamma(c) \); moreover the function
\[ \mu \to \mathcal{E}(T_{\mu v}(u_{sl})) \]
is decreasing for \( 0 < \mu < 1 \). Let us suppose therefore that \( \mathcal{G}(T_v(u_{sl})) < 0 \); by continuity there exists \( 0 < \mu < 1 \) such that
\[ \mathcal{G}(T_{\mu v}(u_{sl})) = 0, \quad \mathcal{E}(T_{\mu v}(u_{sl})) < \mathcal{E}(u_{sl}) < \gamma(c) \]
that it is impossible, therefore \( \mathcal{G}(T_v(u_{sl})) > 0 \). But this implies that \( T_v(u_{sl}) \) does scatter and therefore \( u_{sl} \) cannot be the minimal element. □

Finally, we state the uniform localization of \( \{ u_{sl}(t, x - x(t)), t \geq 0 \} \), which is a standard consequence of the precompactness property of the minimal element.

**Proposition 5.4.** For any \( \varepsilon > 0 \) there exists a radius \( \rho = \rho(\varepsilon) > 0 \) such that
\[ \int_{|x + x(t)| > \rho} |u_{sl}(t)|^2 + |\nabla u_{sl}(t)|^2 + |\lambda_1|u_{sl}|^4 + \lambda_2(K * |u_{sl}(t)|^2)|u_{sl}(t)|^2\ dx \leq C(\lambda_1, \lambda_2) \int_{|x + x(t)| > \rho} |u_{sl}(t)|^2 + |\nabla u_{sl}(t)|^2 + |u_{sl}(t)|^4\ dx < \varepsilon. \]
6. Extinction of the Minimal Element

This section is devoted to the conclusion of the proof of Theorem 1.4 with the so-called rigidity part of the Kenig & Merle road map: the minimal global non-scattering solution built in Theorem 5.1 can be only the trivial one, obtaining therefore a contradiction with respect to the fact that its $L^8 L^4$-norm is not finite. It is based on a convexity argument on the localized variance of the minimal element. For the infinite-variance NLS equation, this method of considering a localized version of the variance was pioneered by Ogawa and Tsutsumi, see [25], in order to show finite time singularity formation as an extension of the result by Glassey in a framework with finite variance, see [15].

6.1. Localized Virial Identities. To lighten the notation, since now on we simply write $u$ instead of $u_{sl}$.

Define $z_R(t) = R^2 \int \chi \left( \frac{x}{R} \right) |u|^2 \, dx$ where $\chi \in C^\infty_c(\mathbb{R}^3)$ is a cut-off function. Standard computations yield

\[
\frac{d}{dt} z_R(t) = 2 \mathcal{A} - 2 \lambda_2 R \mathcal{B}
\]

where

\[
\mathcal{A} = 4 \int_{|x| \leq R} |\nabla u|^2 \, dx + 2 \int_{R \leq |x| \leq 2R} \left( \nabla^2 \chi \left( \frac{x}{R} \right) \nabla u \right) \cdot \nabla \bar{u} \, dx
\]

\[
- \frac{1}{2R^2} \int_{R \leq |x| \leq 2R} \Delta^2 \chi \left( \frac{x}{R} \right) |u|^2 \, dx
\]

\[
+ 6 \lambda_1 \int_{|x| \leq R} |u|^4 \, dx + \lambda_1 \int_{R \leq |x| \leq 2R} \Delta \chi \left( \frac{x}{R} \right) |u|^4 \, dx
\]

and

\[
\mathcal{B} = \mathcal{B}(|u|^2) := \int \nabla \chi \left( \frac{x}{R} \right) \cdot \nabla \left( K * |u|^2 \right) |u|^2 \, dx.
\]

If we choose $\chi(x) = |x|^2$ on $|x| \leq 1$ and $\text{supp}(\chi) \subset B(0, 2)$, then by direct computations we get

\[
\frac{d^2}{dt^2} z_R(t) = \mathcal{A} - 2 \lambda_2 R \mathcal{B}
\]
Now we focus on the more delicate term $N$.

Let us consider the second term appearing in the localized virial identity above, i.e. $B$ defined in (6.3). Before starting with the analysis we recall some preliminary tools introduced by Lu and Wu in [22], where the authors study the Davey-Stewartson equation. The nonlocal nonlinearity in that case is given by a convolution with a kernel having symbol $\frac{1}{|x|^2}$ instead of dipolar kernel.

Let $R(f) = \mathcal{F}^{-1} \left(-i \frac{\xi^2}{|\xi|^2} \hat{f}\right)$ the Riesz transform of $f$, defined via the zero-order symbol $-i \frac{\xi^2}{|\xi|^2}$. It is well-known that it maps $L^p$ into itself for any $p \in (1, +\infty)$. One recognizes that the symbol $-\frac{\xi^2}{|\xi|^2}$ is the one defining $R^2$. By remembering the expression in Fourier variable of the dipolar kernel $K$, see (1.9), we get

$$\mathcal{F}(K * f) = (\hat{K} \hat{f}) = \frac{4\pi}{3} \left(\frac{2\xi^2}{|\xi|^2} - \frac{\xi^2}{|\xi|^2} - \frac{\xi^2}{|\xi|^2}\right) \hat{f}$$

and therefore

$$(K * f) = -\frac{8\pi}{3} R^2 f + \frac{4\pi}{3} R^2 f + \frac{4\pi}{3} R^2 f.$$
and since the support of $\nabla \chi (\frac{x}{R})$ is contained in $B(0, 2R)$ while the one of $v_2$ in $B(0, 10R)^c$, which are disjoint sets, it follows that

$$B = \int \nabla \chi \left( \frac{x}{R} \right) \cdot \nabla (K * |v_1|^2) |v_1|^2 \, dx + \int \nabla \chi \left( \frac{x}{R} \right) \cdot \nabla (K * |v_2|^2) |v_1|^2 \, dx$$

$$= B_0 + B_1$$

with

$$B_1 = \int \nabla \chi \left( \frac{x}{R} \right) \cdot \nabla (K * ((1 - 1_{\{|x| \leq 10R\}})|u|^2)) |v_1|^2 \, dx$$

$$= -R^{-1} \int \Delta \chi \left( \frac{x}{R} \right) (K * ((1 - 1_{\{|x| \leq 10R\}})|u|^2)) |v_1|^2 \, dx$$

$$- 2 \int_{B(0, 2R)} (K * ((1 - 1_{\{|x| \leq 10R\}})|u|^2)) \nabla \chi \left( \frac{x}{R} \right) \cdot \Re \{\bar{v}_1 \nabla v_1\} \, dx$$

$$= B_{1,1} + B_{1,2}.$$  

We observe that

$$|B_{1,1}| \lesssim R^{-1} \|K * ((1 - 1_{\{|x| \leq 10R\}})|u|^2)\|_L^2 \|v_1\|_L^4$$

$$\lesssim R^{-1} \|u\|_{L^4(B(0,10R)^c)}^2 \|v_1\|^2$$

$$\lesssim R^{-1} \|u\|_{L^4(B(0,10R)^c)}^2 \|v_1\|^2,$$

while, by using (6.5),

$$|B_{1,2}| \lesssim \|1_{B(0,2R)}K * ((1 - 1_{\{|x| \leq 10R\}})|u|^2)\|_L^\infty \|\bar{v}_1 \nabla v_1\|_L^1$$

$$\lesssim R^{-3} \|v\|_{L^2(B(0,10R)^c)}^2 \|\nabla v_1\|_L^2 \|v_1\|_L^2$$

$$\lesssim R^{-3} \|v\|_{L^2(B(0,10R)^c)}^2 \|u\|_{H^1}^2.$$  

By gluing everything together we eventually obtain

$$|B_1| \lesssim R^{-1} \left( \|u\|_{L^4(B(0,10R)^c)}^2 + R^{-2} \|v\|_{L^2(B(0,10R)^c)}^2 \right) \|u\|_{H^1}^2.$$  

Then it remains to properly estimate the first integral in the R.H.S. of (6.8), namely $B_0$. Following the strategy in [22] we introduce the function $\tilde{\chi}_R = R^2 \chi (\frac{x}{R}) - |x|^2$ and it straightforwardly yields the equality

$$RB_0 = R \int \nabla \chi \left( \frac{x}{R} \right) \cdot \nabla (K * |v_1|^2) |v_1|^2 \, dx$$

$$= \int \nabla \tilde{\chi}_R \cdot \nabla (K * |v_1|^2) |v_1|^2 \, dx + \int \nabla (|x|^2) \cdot \nabla (K * |v_1|^2) |v_1|^2 \, dx = B_{0,1} + B_{0,2}.$$  

By localizing again, by setting $v_1 = w_1 + w_2$ with $w_2 = 1_{\{|x| \leq R/10\}}v_1$ and noticing that the supports of $w_1$ and $w_2$ are disjoint, alike the one of $\nabla \tilde{\chi}_R$ and $w_2$, we can split $B_{0,1}$ in two further terms

$$B_{0,1} = \int \nabla \tilde{\chi}_R \cdot \nabla (K * |v_1|^2) |v_1|^2 \, dx = \int \nabla \tilde{\chi}_R \cdot \nabla (K * |v_1|^2) |w_1|^2 \, dx + B'_{0,1} = B'_{0,1} + B''_{0,1}$$

where

$$B'_{0,1} = \int \nabla \tilde{\chi}_R \cdot \nabla (K * (1_{\{|x| \leq R/10\}}|v_1|^2)) |w_1|^2 \, dx$$

$$= - \int \Delta \tilde{\chi}_R (K * (1_{\{|x| \leq R/10\}}|v_1|^2)) |w_1|^2 \, dx - \int_{2R \leq |x| \leq 10R} (K * (1_{\{|x| \leq R/10\}}|v_1|^2)) \nabla \tilde{\chi}_R \cdot \nabla (|w_1|^2) \, dx.$$
Similarly to the term $B_1$,\
\[
\left| \int \Delta \tilde{\chi}_R (K * (1_{\{|x| \leq R/10\}} |v_1|^2)) |w_1|^2 \, dx \right| \lesssim \|K * (1_{\{|x| \leq R/10\}} |v_1|^2)\|_{L^2} \|w_1\|^2_{L^4} \\
\lesssim \|u\|^2_{H^1} \|w_1\|^2_{L^4} \lesssim \|u\|^2_{H^1} \|u\|^2_{L^4(B(0,R/10)^c)}
\]
while, by means of (6.6)\
\[
\left| \int_{2R \leq |x| \leq 10R} (K * (1_{\{|x| \leq R/10\}} |v_1|^2)) \nabla \tilde{\chi}_R \cdot \nabla (|w_1|^2) \, dx \right| \\
\lesssim R \|L_{2R \leq |x| \leq 10R} K * (1_{\{|x| \leq R/10\}} |v_1|^2)\|_{L^\infty} \|\nabla w_1\|_{L^2} \|w_1\|_{L^2} \\
\lesssim R^{-2} \|u\|^2_{L^2} \|u\|_{H^1} \|u\|^2_{L^4(B(0,R/10)^c)}.
\]
By summing up the two terms we end up with\
\[(6.11) \quad \|B'_{0,1}\| \lesssim \|u\|_{H^1} \left( \|u\|_{H^1} \|u\|^2_{L^4(B(0,R/10)^c)} + R^{-2} \|u\|^2_{L^2} \|u\|^2_{L^4(B(0,R/10)^c)} \right) \cdot
\]
It is left to estimate the term $B'_{0,1} = \int \nabla \tilde{\chi}_R \cdot \nabla (K * |w_1|^2) |w_1|^2 \, dx$. By setting $g = |w_1|^2 = |1_{\{|x| \leq 10R\}}(1 - 1_{\{|x| \leq R/10\}})u|^2$ and making use of the Parseval identity,
\[
\int \nabla \tilde{\chi}_R \cdot \nabla (K * |w_1|^2) |w_1|^2 \, dx = i \int \nabla \tilde{\chi}_R g(\xi) \cdot \xi \tilde{K_g} \, d\xi \\
= \frac{i4\pi}{3} \int \nabla \tilde{\chi}_R g(\xi) \cdot \xi \tilde{K_g}(\xi) \, d\xi + \int \nabla \tilde{\chi}_R g(\xi) \cdot \xi \tilde{K_g}(\xi) \, d\xi
\]
Consider the generic term $\int \nabla \tilde{\chi}_R g(\xi) \cdot \xi \tilde{K_g}(\xi) \, d\xi$; it is explicitly given, up to (complex) constants, by (6.12)
\[
\int \nabla \tilde{\chi}_R g(\xi) \cdot \xi \frac{\xi^2}{|\xi|^2} \tilde{g}(\xi) \, d\xi = \int (\nabla \tilde{\chi}_R * \tilde{g})(\xi) \cdot \xi \frac{\xi^2}{|\xi|^2} \tilde{g}(\xi) \, d\xi \\
= \int \int \tilde{g}(\eta) \nabla \tilde{\chi}_R(\eta - \xi) \cdot \left( \frac{\xi_1 \xi_2}{|\xi|^2} \frac{\xi_1}{|\xi|^2} \right) \tilde{g}(\xi) \, d\eta d\xi \\
= \int \nabla \tilde{\chi}_R \cdot \mathcal{R}_j(\nabla g)(x) \mathcal{R}_j \tilde{g}(x) \, dx + \int \int \tilde{g}(\eta) \nabla \tilde{\chi}_R(\eta - \xi) \cdot \left( \frac{\xi_1 \xi_2}{|\xi|^2} \frac{\xi_1}{|\xi|^2} \right) \tilde{g}(\xi) \, d\eta d\xi \\
= \frac{1}{2} \int \Delta \tilde{\chi}_R |\mathcal{R}_j g(x)|^2 \, dx + \int \int \tilde{g}(\eta) \nabla \tilde{\chi}_R(\eta - \xi) \cdot \left( \frac{\xi_1 \xi_2}{|\xi|^2} \frac{\xi_1}{|\xi|^2} \right) \tilde{g}(\xi) \, d\eta d\xi
\]
since derivatives and Riesz transform commute. The first term in the R.H.S. of (6.12) is simply estimated by $\|u\|^2_{L^4(B(0,R)^c)}$ due to the continuity property of the Riesz transform. For the second term we have
\[
\left| \int \int \tilde{g}(\eta) \nabla \tilde{\chi}_R(\eta - \xi) \cdot \left( \frac{\xi_1 \xi_2}{|\xi|^2} \frac{\xi_1}{|\xi|^2} \right) \tilde{g}(\xi) \, d\eta d\xi \right| \leq \left| \int \int \tilde{g}(\eta) \nabla \tilde{\chi}_R(\eta - \xi) \cdot \left( \frac{\xi_1 \xi_2}{|\xi|^2} \frac{\xi_1}{|\xi|^2} \right) \tilde{g}(\xi) \, d\eta d\xi \right| \\
+ \left| \int \int \tilde{g}(\eta) \nabla \tilde{N}(\eta - \xi) \cdot \left( \frac{\xi_1 \xi_2}{|\xi|^2} \frac{\xi_1}{|\xi|^2} \right) \tilde{g}(\xi) \, d\eta d\xi \right|
\]
where $N = |x|^2$. Now, since $\left| \frac{\nabla \eta}{|\eta|} - \frac{\xi}{|\xi|} \right| \lesssim |\eta - \xi|$, 

\[
\left| \int \int \hat{g}(\eta) \nabla \chi_R(\eta - \xi) \cdot \left( \frac{\xi_1}{|\xi|} \right) \hat{\xi} \eta \frac{\xi_1}{|\xi|} \hat{\xi} \right| \lesssim \int \int |\hat{g}(\xi)||\hat{g}(\eta)|||\eta - \xi|^2 |\nabla \chi_R(\eta - \xi)| \, d\eta \, d\xi 
= \int |\hat{g}(\xi)||\hat{g}(\eta)|||\eta - \xi|^2 |\chi_R(\eta - \xi)| \, d\eta \, d\xi 
= \int |\hat{g}(\xi)||\hat{g}(\eta)|||\eta - \xi|^2 \chi_R(\eta - \xi)| \, d\eta \, d\xi 
= \int |\hat{g}(\xi)||\hat{g}(\eta)|||\eta - \xi|^2 \chi_R(\eta - \xi)| \, d\eta \, d\xi 
= \int |\hat{g}(\xi)||\hat{g}(\eta)||\mathcal{F}(h(\frac{\eta}{R}))| \, d\xi 
\]

where we defined $h(\cdot) = -\Delta \chi(\cdot)$, and continue in this way

\[
\left| \int |\hat{g}(\xi)||\hat{g}(\eta)||\mathcal{F}(h(\frac{\eta}{R}))| \, d\xi \right| \leq \int |\hat{g}(\xi)||\hat{g}(\eta)||\mathcal{F}(h(\frac{\eta}{R}))| \, d\xi 
\lesssim \|g\|_{L^2} \|\hat{g} * \mathcal{F}(h(\frac{\eta}{R}))\|_{L^2} 
\lesssim \|g\|_{L^2}^2 \|\mathcal{F}(h(\frac{\eta}{R}))\|_{L^1} 
\lesssim R^3 \|g\|_{L^2}^2 \|\hat{h}(R)\|_{L^1} 
= \|g\|_{L^2}^2 \|\hat{h}\|_{L^1} 
\lesssim \|g\|_{L^2}^2 = \|u\|_{L^2}^2 \leq \|u\|_{L^2(B(0,R/10)^c)}^2. 
\]

since $\hat{h} \in L^1$ (being $\chi$ in the Schwartz class, hence $\Delta \chi$ is in the Schwartz class, so it is integrable).

In the same way

\[
\left| \int \int \hat{g}(\eta) \nabla \hat{N}(\eta - \xi) \cdot \left( \frac{\xi_1}{|\xi|} \right) \hat{\xi} \eta \frac{\xi_1}{|\xi|} \hat{\xi} \right| \lesssim \int \int |\hat{g}(\xi)||\hat{g}(\eta)|||\eta - \xi|^2 |\nabla \hat{N}(\eta - \xi)| \, d\eta \, d\xi 
= \int |\hat{g}(\xi)||\hat{g}(\eta)|||\eta - \xi|^2 |\hat{N}(\eta - \xi)| \, d\eta \, d\xi 
= \int |\hat{g}(\xi)||\hat{g}(\eta)|||\eta - \xi|^2 \hat{N}(\eta - \xi)| \, d\eta \, d\xi 
\lesssim \|g\|_{L^2}^2 \leq \|u\|_{L^2(B(0,R/10)^c)}^2. 
\]

Therefore

\[
(6.13) \quad |B_{0,1}| \lesssim \|u\|_{L^2(B(0,R/10)^c)}^2. 
\]

Now we finish with the estimate of $B_{0,2} = \int \nabla(|x|^2) \cdot \nabla (K * |v_1|^2) \, |v_1|^2 \, dx$. We observe that a direct application of the Parseval identity gives

\[
B_{0,2} = \int x \cdot \nabla (K * |v_1|^2) \, |v_1|^2 \, dx 
= -3 \int \nabla \cdot (x |v_1|^2) (K * |v_1|^2) \, dx 
\]

(above $\nabla \cdot$ stands for the divergence operator) then, by writing $|v_1| = |v_1 \pm u|$, it is with a direct computation to produce, using the Cauchy-Schwarz inequality and the continuity property of the dipolar kernel,

\[
(6.14) \quad B_{0,2} \geq -3 \int (K * |u|^2) |u|^2 \, dx + \varepsilon(r) 
\]
where
\[ \varepsilon(r) \sim \|u\|_{L^4(B(0,r))}^2, \quad r \sim R. \]

Now summing up (6.2), (6.8), (6.9), (6.10), (6.11), (6.13) with (6.14) we get the desired results stated in (6.7) of Lemma 6.2 and in Proposition 6.3, straightforwardly.

6.2. **Death of the soliton-like solution.** In this section we can close the Kenig & Merle scheme by showing, through a convexity argument, that the soliton-like solution built in Section 5 is the trivial one, clearly reaching a contradiction with respect to its infinite spacetime norm. We still keep the convention \( u = u_{sl} \).

By gluing the estimate in Proposition 6.3 with the bound in Proposition 2.8 we get
\[ \frac{d^2}{dt^2} z_R(t) \geq \alpha - \varepsilon_1(R) - \varepsilon_2(R). \]
and we can finally conclude if we are able to show that also \( \varepsilon_{1,2}(R) \to 0 \) as \( R \to \infty \), uniformly in time. Since they have qualitatively the same form, let us control just \( \varepsilon_1(R) \).

At this point the method developed in [12] allows us to conclude. In fact, consider two times \( 0 < \tau < \tau_1 \) and the interval \( I = [\tau, \tau_1] \) and a radius \( R \geq \sup_I |x(t)| + \rho \) where \( \rho \) is as in Proposition 5.4. Then \( \{|x| > R\} \subset \{|x + x(t)| > \rho\} \) and so \( \varepsilon_1(R) \to 0 \) as \( R \to \infty \), which in turn implies, with the choice of \( \varepsilon_{1,2} = \alpha/4 \)
\[ \frac{d^2}{dt^2} z_R(t) \geq \alpha \geq 0 \]
for \( R \) sufficiently large. Integrating on \( I \), we get
\[ R \geq R \|u\|_{L^2} \|\nabla u\|_{L^2} \geq \left| \frac{d}{dt} z_R(\tau_1) - \frac{d}{dt} z_R(\tau) \right| \geq \frac{\alpha}{2}(\tau_1 - \tau) \]
and by choosing \( R = \rho + \delta \tau_1 \) we get
\[ \rho + \delta \tau_1 \geq \beta(\tau_1 - \tau), \]
for some \( \beta > 0 \).

**Remark 6.4.** Since (5.4) holds, it is always possible, once \( \delta > 0 \) has been selected, to find \( \tau = \tau(\delta) \) such that \( |x(t)| \leq \delta t \) for any \( t \geq \tau \).

Therefore by choosing \( \delta = \beta/2 \) we get
\[ \frac{\beta}{2} \tau_1 \leq \rho + \beta \tau = \rho + \beta \left( \frac{\beta}{2} \right) \]
which is a contradiction since the RHS of the above inequality is a finite constant, while the LHS diverges as \( \tau_1 \to \infty \). We have eventually proved the following.

**Proposition 6.5.** Let \( u_{sl} \) be the precompact solution to (1.2) constructed in the previous section. Then \( u_{sl} \equiv 0 \).

This last Proposition closes the Concentration/Compactness and Rigidity method, since the trivial solution cannot have a divergent spacetime norm.

**Appendix A.**

In this appendix we prove Lemma 3.1, Lemma 3.2, Lemma 3.3 and Lemma 3.5.

**Proof of Lemma 3.1.** The proof in the stable regime can be shown as consequence of the coercivity of the energy, see [9]. In the unstable regime, the proof is contained in [7]. We sketch it. First of all, it is worth mentioning that under condition (1.3) the energy could be negative, then a classical Glassey’s argument would yield finite time blowing-up solutions. Fix now \( \lambda_2 > 0, \lambda_1 - \frac{2}{\pi^2} \lambda_2 < 0 \). Thanks to Theorem 1.3 it is
sufficient to show that for initial data $u_0$ small enough in the $H^1$ space, then $\mathcal{G}(u_0) > 0$ and $E(u_0) < \gamma(u_0)$. Let us recall that the potential energy can be written as

$$
\mathcal{P}(u) = \left( \lambda_1 - \frac{4\pi}{3} \lambda_2 \right) \|u\|^4 + \frac{1}{(2\pi)^3} \int \frac{\xi_1^2}{|\xi|^2} \left( |u|^2 \right)^2 \, d\xi \geq \left( \lambda_1 - \frac{4\pi}{3} \lambda_2 \right) \|u\|^4.
$$

Therefore by using in order the Plancherel identity and the Sobolev embedding and moreover recalling we are working on $\lambda_1 - \frac{4\pi}{3} \lambda_2 < 0$ we have

$$
\mathcal{G}(u) \geq T(u) + \frac{3}{2} \left( \lambda_1 - \frac{4\pi}{3} \lambda_2 \right) \|u\|^4 
\geq \|\nabla u\|^2_{L^2} - C\|u\|_{L^1} \|\nabla u\|^3_{L^2} > 0
$$

provided $\|u\|_{H^1}$ is small enough.

**Proof of Lemma 3.2.** The proof is contained in [7], where it is shown that if the initial datum is small enough (and so the solution is global), this yields uniform bound on the Strichartz norm $L^{8/3}W^{1,4}$ and this in turn leads to the scattering property (see the monographs [10, 21]). The Duhamel's formulation of (1.2) is

$$
u(t, x) = U(t)u_0 + i \int_0^t U(t-s) \left( \lambda_1 |u|^2 u + \lambda_2 (K * |u|^2) u \right)(s) \, ds$$

and by using the Strichartz estimates with $(q, r) = (8/3, 4)$ then $(q', r') = (8/5, 4/3)$, by using the Hölder inequality and the continuity property of the dipolar kernel, it is easy to get

$$
\|u\|_{L^{8/3}W^{1,4}} \leq C\|u_0\|_{H^1} + C\|u\|^5_{L^{8/3}W^{1,4}}.
$$

Let $\delta = \|u_0\|_{H^1}$; noticing that the set $S := \{ s : s - C\delta - C s^{5/3} \leq 0 \}$ decomposes in two disjoint connected components, the continuity of the flow implies that if $\delta$ is sufficiently small, the $L^{8/3}W^{1,4}$-norm of $u$ is uniformly bounded for (positive) times. Scattering for (positive) times is an easy consequence of this uniform control and the definition of the scattering state. In fact, by defining $v(t) = U(-t)u$ and making use of the Duhamel’s representation formula above, it is straightforward to check that

$$
\|v(t_1) - v(t_2)\|_{H^1} \xrightarrow{t_1, t_2 \to +\infty} 0.
$$

Definition of $v$ and the unitary property of the linear propagator $U(t)$ eventually gives the result. The analysis for negative times is the same.

**Proof of Lemma 3.3.** If the solution $u$ to (1.2) is global and such that $u(t, x) \in L^8L^4$, then a perturbative argument shows that $u(t, x) \in L^{8/3}W^{1,4}$, therefore concluding as in the proof of Lemma 3.2. It is worth mentioning that in its generality this result was established by Cazenave and Weissler in their paper on the so-called rapidly decaying solutions, see [11]. Since for any fixed $T$ the $u \in L^4_T W^{1,r}$ with $I = (0, T)$, let us consider $v(t) = u(t + T)$. It follows that

$$
v(t, x) = U(t)u(T) + i \int_0^t U(t-s) \left( \lambda_1 |v|^2 v + \lambda_2 (K * |v|^2) v \right)(s) \, ds$$

and by means of the Strichartz estimates, for $I = (0, T)$

$$
\|v\|_{L_t^{12/5}W^{1,4}} \leq C\|u(T)\|_{H^1} + C\|v\|_{L_t^8L^4}^2 \|v\|_{L_t^{12/5}W^{1,4}} 
\leq C + C\|u\|_{L_t^{12/5}W^{1,4}}^2 \|v\|_{L_t^{12/5}W^{1,4}}
$$

since $u$ is uniformly bounded in time in $H^1$. It suffices to select $T >> 1$ such that $C\|u\|_{L_t^{12/5}W^{1,4}}^2 \leq \frac{1}{2}$ to obtain

$$
\sup_{T>0} \|u\|_{L_t^{12/5}W^{1,4}} < \infty \implies u \in L_t^{8/3}W^{1,4}.
$$

For negative times the analysis is exactly the same.
Proof of Lemma 3.5. If the equation were reduced to the classical NLS equation (2.5), then the proof would be contained in [16]. Since we are in the presence of the dipolar interaction term we will sketch the proof for sake of clarity. Let \( z = u - w \); then \( z \) satisfies

\[
(i\partial_t + \frac{1}{2} \Delta) z = \lambda_1 |w|^2 u + \lambda_1 w^2 w + \lambda_2 (K * |w|^2) u - \lambda_2 (K * |w|^2) w - e
\]

subject to initial condition \( z_0 = z(0, x) = u_0 - w_0 \). Since \( \|w\|_{L^8 L^4} \leq M \) we can partition \([t_0, \infty)\) into \( m = m(M) \) intervals \( I_j = [t_j, t_{j+1}] \) such that \( \|w\|_{L^8_j L^4} \leq \delta \) for each \( j \), where \( \delta \) is small enough (to be chosen later on). The integral formulation of (A.1) is

\[
z = U(t-t_j)z(t_j) + i \int_{t_j}^t U(t-s)(Z_1 + Z_2)(s) \, ds
\]

where, as in [16] for NLS

\[
Z_1 = |w|^2 u - |w|^2 w = |z + w|^2 (z + w) - |w|^2 w = w^2 z + 2 |w|^2 z + 2 w |z|^2 + |z|^2 z + e
\]

while the nonlocal nonlinearity splits as

\[
Z_2 = (K * |w|^2) u - (K * |w|^2) w = (K * |z + w|^2) (z + w) - (K * |w|^2) w = (K * (|z + w|^2 - |w|^2)) w - (K * |z + w|^2) z,
\]

and due to Lemma 2.3 on \( I_j \)

\[
\left\| \int_{t_j}^t U(t-s) \left[ ((K * (|z + w|^2 - |w|^2)) w \right] (s) \, ds \right\|_{L^8_{t_j} L^4} \lesssim \| (K * (|z + w|^2 - |w|^2)) w \|_{L^{8/3} L^{4/3}}
\]

\[
\lesssim \| |z + w|^2 - |w|^2 \|_{L^1_{t_j} L^2} \| w \|_{L^4_{t_j} L^4}
\]

\[
\lesssim \| |z + w|^2 - |w|^2 \|_{L^1_{t_j} L^2} \| w \|_{L^4_{t_j} L^4} + \| \bar{w} \| \| L_{t_j}^4 \| \| w \|_{L^4_{t_j} L^4}
\]

\[
\lesssim \| |z + w|^2 - |w|^2 \|_{L^1_{t_j} L^2} \| w \|_{L^4_{t_j} L^4} + \| \bar{w} \| \| L_{t_j}^4 \| \| w \|_{L^4_{t_j} L^4}
\]

and similarly

\[
\left\| \int_{t_j}^t U(t-s) \left[ (K * |z + w|^2) z \right] (s) \, ds \right\|_{L^8_{t_j} L^4} \lesssim \| (K * |z + w|^2) z \|_{L^{8/3} L^{4/3}}
\]

\[
\lesssim \| |z + w|^2 - |w|^2 \|_{L^1_{t_j} L^2} \| z \|_{L^4_{t_j} L^4}
\]

\[
\lesssim \left( \| |z^2_{t_j} L^4 \| + \| |w| |_{t_j} L^8 \| \right) \| z \|_{L^4_{t_j} L^4}
\]

\[
\lesssim \| |z^2_{t_j} L^4 \| + \| |w| |_{t_j} L^8 \| \| z \|_{L^4_{t_j} L^4}
\]

hence, by using the hypothesis,

\[
\left\| \int_{t_j}^t U(t-s)Z_2(s) \, ds \right\|_{L^8_{t_j} L^4} \lesssim \delta \| z \|_{t_j}^{2} L^4 + \delta^2 \| z \|_{t_j} L^4 + \| z \|_{t_j}^{3} L^4
\]

and therefore the nonlocal interaction term leads to the same estimate for \( Z_1 \) contained in [16]. By gluing up everything together we get

\[
\| z \|_{L^8_{t_j} L^4} \leq \| U(t-t_j) z(t_j) \|_{L^8_{t_j} L^4} + c \delta \| z \|_{t_j}^{2} L^4 + c \delta^2 \| z \|_{t_j} L^4 + c \| z \|_{t_j}^{3} L^4 + c \varepsilon,
\]
thus the proof can be concluded in the same way of [16, Proposition 2.3]. We report here the strategy for sake of clarity. For δ small enough,
\begin{equation}
\|z\|_{L^8_t L^4_z} \leq 2\|U(t-t_j)z(t_j)\|_{L^8_t L^4_z} + 2C\varepsilon
\end{equation}
and choosing \(t = t_{j+1}\) in the integral representation of \(z(t)\) one obtains
\begin{equation}
U(t-t_{j+1})z(t_{j+1}) = U(t-t_j)z(t_j) + i \int_{t_j}^{t_{j+1}} U(t-s)(Z_1 + Z_2)(s) \, ds,
\end{equation}
and so analogously to the estimates above it follows that
\[\|U(t-t_{j+1})z(t_{j+1})\|_{L^8_t L^4_z} \leq \|U(t-t_j)z(t_j)\|_{L^8_t L^4_z} + C\delta^2 \|z\|_{L^8_t L^4_z} + C\delta \|z\|_{L^8_t L^4_z} + C\varepsilon.\]
Summing up (A.2) and (A.3) we eventually obtain
\[\|U(t-t_j)z(t_j)\|_{L^8_t L^4_z} \leq 2\|U(t-t_j)z(t_j)\|_{L^8_t L^4_z} + 2C\varepsilon\]
and iterating on \(j \in \mathbb{N}\) it can be concluded that
\[\|U(t-t_j)z(t_j)\|_{L^8_t L^4_z} \leq 2^j \|U(t-t_0)z(t_0)\|_{L^8_t L^4_z} + 2(2^j - 1)C\varepsilon \lesssim 2^{j+2}\varepsilon.\]
The smallness assumption on \(\delta\) is now done if \(2^{m+2}\varepsilon\) is sufficiently small (depending on the absolute constants of the a priori estimates and of course depending on \(m\) which in turn is depending on \(M\) of the statement). \(\square\)

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