Momentum broadening of heavy quark in a magnetized thermal QCD medium

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Anisotropic momentum diffusion and the drag coefficients of heavy quarks have been computed in a strongly magnetized quark-gluon plasma beyond the static limit within the framework of Langevin dynamics. Depending on the orientation of the motion of the heavy quark with respect to the direction of the magnetic field, five components of the heavy quark momentum diffusion coefficients have been estimated in the magnetized thermal medium. The current focus is on the regime $M \gg \sqrt{eB} \gg T, M$ being the mass of heavy quark. The light quarks/antiquarks follow $1+1$-dimensional Landau level kinematics, and heavy quark dynamics are not directly affected by the magnetic field in the medium. It is observed that for the case of heavy quark motion parallel to the magnetic field, the component of diffusion coefficient transverse both to the field and the heavy quark velocity ($\kappa_{TT}$) turns out to be dominant compared to the component longitudinal to both the field and motion ($\kappa_{LL}$), i.e., $\kappa_{TT} \gg \kappa_{LL}$. On the other hand, for heavy quark moving perpendicular to the magnetic field, the component of diffusion coefficient along the direction of the magnetic field and transverse to the heavy quark velocity is dominant in comparison to the components parallel and perpendicular to the heavy quark velocity in the plane transverse to the magnetic field, i.e., $\kappa_{LT} \gg \kappa_{TT} \gg \kappa_{LL}$. Sensitivity of the various diffusion coefficients on the strength of the magnetic field and velocity of the heavy quark is also explored.

Keywords: Heavy quarks, Langevin dynamics, Quark-gluon plasma, Strong magnetic field

I. INTRODUCTION

It is believed that intense magnetic field has been created in the initial stages of non-central Heavy Ion Collisions (HICs) [1–7]. The created field is estimated to be in the order of $eB \sim m_2^2$ at Relativistic Heavy-Ion Collider (RHIC) and a few tens of pion mass square, $eB \sim 15 m_2^2$ at the Large Hadron Collider (LHC). The intense magnetic field may affect various aspects of the physics of the deconfined hot nuclear matter created in HICs termed as Quark-Gluon Plasma (QGP). One of the major uncertainties of the magnetic field is its lifetime in hot nuclear matter. In vacuum, the magnetic field decays very rapidly. However, in a medium of charged particles, it can be sustained for a longer time due to the finite electrical conductivity ($\sigma$) of the medium [8–12]. This is due to the fact that the magnetic field in the medium satisfies a diffusion equation with diffusion coefficient $(\sigma\mu)^{-1}$, where $\mu$ is the magnetic permeability [13]. Thus, one finds that the time scale over which the magnetic field remains reasonably strong is $\tau = L^2/\sigma$, where $L$ is length scale and $\tau$ is relaxation time. For electrical conductivity $\sigma \approx 0.047$ [14], at temperature $T = 200$ MeV the electrical conductivity $\sigma = 8$ MeV. Therefore, for $L = 10$ fm, the time over which the magnetic field remains reasonably strong is $\tau = 1$ fm. For higher temperatures, $\sigma$ will be higher, leading to larger relaxation time. The investigations on various characteristics of the hot nuclear matter in the presence of the magnetic field have gained attention in recent years [15–18]. The study of the QGP in the magnetic field background opens up new avenues to explore physics in different directions such as Chiral Magnetic Effect (CME) [11, 19], charge-dependent elliptic flow [20–23], magnetic catalysis [24], various transport coefficients of QGP in the magnetic field [25–27], photon-dilepton production [28–30], in medium properties of quarkonia [31, 32] and their suppression [33, 34], transport coefficients of heavy quarks (HQs) in the magnetic field [35–37], etc.

To understand the properties of the QGP, one needs external probes such as highly energetic particles created at a very early stage of HICs. HQs serve as an effective probe to describe the properties of hot QCD medium created in the collision experiments, as they do not constitute the bulk medium, owing to their large mass compared to the temperature scale. The HQ traverses through the QGP medium as a nonthermal degree of freedom and gets random kicks from the thermal partons (light quarks/anti-quarks and gluons) in the bulk medium. Thus, the HQ dynamics could be explored within the scope of the Brownian motion [38–42], and their transport parameters, the drag and the diffusion coefficients, have been estimated in the QGP medium [43–45]. The HQ production and dynamics in the nuclear matter and the associated experimental observables have been well explored in several works [49–50]. The HQ evolution and momentum broadening in terms of momentum diffusion in thermal QGP are explained in Ref. [40]. There are some recent investigations on the HQ momentum diffusion coefficients in a strongly magnetized QGP medium in the weak coupling regime in the static limit of the HQ [35–37].

It turns out that in the static limit, there are two diffusion coefficients of HQ in a magnetic field background, one in the direction of the magnetic field and the other to the perpendicular to the field. This, in turn, generates a magnetic field induced anisotropy in the momentum
diffusion. It would be interesting to investigate the nature of the anisotropy in the momentum diffusion of HQ beyond the static limit.

In the current analysis, we estimate the anisotropic diffusion coefficients of HQ moving with finite velocity \( \mathbf{v} \). To estimate the same, we consider the strong field limit with the soft momentum transfer i.e., we shall be interested in the regime \( |\mathbf{q}| \ll g\sqrt{eB} \ll T \ll \sqrt{eB} \ll M \) and use the resummed gluon propagator at finite temperature and magnetic field. A recent work \cite{57} discussed the collisional energy loss of the HQ using a similar technique, which might throw some insights in the directions of jet quenching. The present calculation is more related to the momentum broadening of HQ depending upon the relative orientation of the magnetic field and the velocity of the HQ. We follow an approach similar to the Refs.\cite{10,58}. The HQ dynamics are described by the Langevin equations for two different cases, \( \nu \), the HQ moving parallel, and perpendicular to the magnetic field.

The manuscript is organized as follows. In Sec.II we discuss the Langevin formalism for HQ diffusion for both the cases, \( i.e. \), HQ moving parallel and perpendicular to the magnetic field. In subsections II A and II B we estimate the contribution to the diffusion coefficients from the gluon and the light quarks/anti-quarks, respectively. In Sec.III we discuss in detail the results of the present investigation. Finally, in Sec.IV we summarise the present work and discuss the implications and future possibilities. In Appendix A we discuss the details of the calculation of gluon self energy in the magnetic field in the LLL approximation.

**Notations:** The magnetic field is considered to be constant and along \( z \)-axis so that \( \mathbf{B} = B\mathbf{\hat{z}} \). The calculations of the quark propagator in Real-Time formalism and the relevant matrix element squared require the following notations, where \( \parallel \) and \( \perp \) represent the components parallel and perpendicular to the magnetic field, respectively. For the metric tensor, we use

\[
 g_{\mu\nu}^\parallel = (1, 0, 0, -1), \quad g_{\mu\nu}^\perp = (0, -1, -1, 0). \quad (1)
\]

The parallel (\( i.e. \), \( a_\mu^\parallel = g_{\mu\nu}a^\nu \)) and perpendicular (\( i.e. \), \( a_\mu^\perp = g_{\mu\nu}a^\nu \)) components of a four-vector \( a^\mu \) are represented as

\[
a^\parallel_\mu = (a_0, 0, 0, -a_3), \quad a^\perp_\mu = (0, -a_1, -a_2, 0). \quad (2)
\]

The four-vector product (\( a^\mu b^\nu = a \cdot b \)) can be written as

\[
a \cdot b = a^\parallel_\mu b^\mu - a^\perp_\mu b^\mu. \quad (3)
\]

Similarly, the square of both the components of the four-vector can be denoted as,

\[
a_\parallel^2 = a_0^2 - a_3^2, \quad a_\perp^2 = a_1^2 + a_2^2. \quad (4)
\]

For four momentum vector we use the notation \( K_\mu = (k_0, -\mathbf{k}) \) with the parallel component \( K^\parallel_\mu = (k_0, 0, 0, -k_z) \) and the perpendicular component \( K^\perp_\mu = (0, -k_x, -k_y, 0) \).

**II. FORMALISM: LANGEVIN DYNAMICS OF HEAVY QUARK IN A MAGNETIZED MEDIUM**

We will work in the strong magnetic field limit with \( \sqrt{eB} \gg T \), indicating that the light quarks/antiquarks occupy the Lowest Landau Level (LLL) while thermal gluons are unaffected by the field. Note that the HQ motion is not Landau quantized as \( M \gg \sqrt{eB} \). To estimate the thermal gluons and thermal light quark/antiquark contributions to the transport coefficients of the HQ for the non-static case, \( i.e. \), when the HQ is moving with velocity \( \mathbf{v} \) in the medium, we consider two cases: when HQ is moving along the direction of the magnetic field (\( \mathbf{v} \parallel \mathbf{B} \)) and HQ motion transverse to the magnetic field (\( \mathbf{v} \perp \mathbf{B} \)). The momentum diffusion coefficients are denoted by three indices. The superscript denotes the HQ velocity with respect to the magnetic field. Of the two indices which are given as subscripts of the diffusion coefficient, the first index describes the momentum diffusion relative to the direction of the HQ while the second index corresponds to the momentum diffusion relative to the direction of the magnetic field.

**Case I: \( \mathbf{v} \parallel \mathbf{B} \)**

The magnetic field \( \mathbf{B} \), and the HQ velocity \( \mathbf{v} \), are considered to be in the same direction as depicted in Fig.1. The general structure of HQ momentum diffusion tensor in this case can be decomposed as follows,

\[
 \kappa^{ij} = R^{ij}\kappa^{\parallel TT} + Q^{ij}\kappa^{\perp LL}, \quad (5)
\]

where \( R^{ij} = (\delta^{ij} - \frac{v_i v_j}{p^2}) \) and \( Q^{ij} = \frac{v_i v_j}{p^2} \) are the transverse and longitudinal projection operators orthogonal to each other, \( i.e. \), \( R^{ij}Q_{ij} = 0 \). Here, \( \kappa^{\parallel TT} \) and \( \kappa^{\perp LL} \) are the two diffusion coefficients, transverse and longitudinal to the direction of HQ motion (which is the same direction of \( \mathbf{B} \)). The symbol \( \parallel \) denotes that the HQ motion is parallel to the direction of the magnetic field. The broadening of the variance of HQ momentum distribution can be described by the macroscopic equation of motion as...
where the coefficient \(\eta_D\) measures the average momentum loss. The variance of the HQ momentum distribution transverse and parallel to the direction of the motion can be respectively defined as \(\langle (\Delta p_T)^2 \rangle \equiv (p_T^2)\) and \(\langle (\Delta p_L)^2 \rangle \equiv (p_L - \langle p_L \rangle)^2\). The factor \(\frac{1}{2}\) in the transverse momentum broadening is due to two perpendicular directions. The HQ transport coefficients \(\eta_D^\parallel, \kappa_{LL}^\parallel\) and \(\kappa_{TT}^\parallel\) can be obtained from the kinetic theory by considering the proper collisional scattering matrix element squared and have the following form for the momentum loss,

\[
\frac{d}{dt} \langle p \rangle = \frac{1}{2v} \int_{k,q} |\mathcal{M}|^2 q^2 \left[ f(k) \left( 1 \pm f(k + \omega) \right) \right],
\]

(7)

where \(v = p/E\) is the velocity of HQ, and \(f\) is the distribution of thermal particles in the magnetized QGP, \(|\mathcal{M}|\) is the HQ-thermal particle scattering matrix element, and \(\omega\) is the transferred energy due to the scattering process. We can write similar expressions for the rate of transverse and longitudinal momentum broadening which are \(\kappa_{TT}^\parallel(v)\) and \(\kappa_{LL}^\parallel(v)\), i.e.,

\[
\kappa_{TT}^\parallel(v) = \int_{k,q} |\mathcal{M}|^2 q^2 \left[ f(k) \left( 1 \pm f(k + \omega) \right) \right],
\]

(8)

\[
\kappa_{LL}^\parallel(v) = \int_{k,q} |\mathcal{M}|^2 q^2 \left[ f(k) \left( 1 \pm f(k + \omega) \right) \right].
\]

(9)

The notation \(\int_{k,q}\) denotes the relevant phase space integration over \(k\) and \(q\) with proper dimensions.

Case II: \(v \perp B\)

When the HQ motion is moving transverse to the direction of the magnetic field, say, \(v = (v_x, 0, 0)\) and \(B = B\hat{z}\), the momentum broadening can be characterized by three diffusion coefficients. Defining \(b = (0, 0, 1)\) to project the direction of magnetic field, the diffusion tensor can be decomposed as follows,

\[
\kappa^{ij} = P^{ij} \kappa_{TT}^\parallel + Q^{ij} \kappa_{LL}^\parallel + R^{ij} \kappa_{TT}^\parallel,
\]

(10)

in which the projection operators take the forms,

\[
P^{ij} = b_i b_j/b^2, \quad Q^{ij} = \frac{p^i p^j}{p^2}, \quad R^{ij} = \left( \delta^{ij} - \frac{p^i p^j}{p^2} - \frac{b_i b_j}{b^2} \right).
\]

(11)

\[
\eta_D^\parallel = \frac{1}{2v} \int_{k,q} |\mathcal{M}|^2 q^2 \left[ f(k) \left( 1 \pm f(k + \omega) \right) \right],
\]

(12)

\[
\kappa_{TT}^\parallel(v) = \int_{k,q} |\mathcal{M}|^2 q^2 \left[ f(k) \left( 1 \pm f(k + \omega) \right) \right],
\]

(13)

\[
\kappa_{LL}^\parallel(v) = \int_{k,q} |\mathcal{M}|^2 q^2 \left[ f(k) \left( 1 \pm f(k + \omega) \right) \right],
\]

(14)

\[
\kappa_{TT}^\parallel(v) = \int_{k,q} |\mathcal{M}|^2 q^2 \left[ f(k) \left( 1 \pm f(k + \omega) \right) \right].
\]

(15)

In the following sections, we discuss the interaction of HQ with the thermal gluon and the light quark/antiquark in detail while considering the thermal particle kinematics in the magnetic field for each case.

A. Gluonic contribution

Gluonic contribution to the diffusion coefficient comes via the Compton scattering, i.e., \(Q(P) + g(K) \rightarrow Q(P') + g(K')\), where \(g\) stands for gluon. Generally, at leading order in the coupling, there are three channels, \(s, t\) and \(u\) that contribute to Compton scattering. In the limit \(M \gg \sqrt{eB} \gg T\), the leading order contribution to...
the diffusion coefficient in the magnetic field background arises from the $t$-channel scattering process. This is because the contribution from the $s$ and $u$-channels of the Compton scattering is negligible in the presence of magnetic field due to the hierarchy in the scales considered here, i.e., in the regime $M \gg \sqrt{eB}$, the HQ propagators in $s$ and $u$ channels are not affected by the magnetic field.

In the $t$-channel scattering, the effect of the magnetic field comes through the resummed gluon propagator. For HQ at rest, i.e., the static limit of HQ, the matrix elements for the $t$-channel scattering is well investigated in the Ref. [38] using the Debye mass screened gluon propagator. In contrast, here we use the resummed retarded gluon propagator and also consider the finite velocity $v$ of the HQ. In this case, the color-averaged $t$-channel scattering amplitude takes the form [57],

$$|\mathcal{M}|^2 = \frac{4g^4[\Pi_{R}(Q)]^2}{Q^4(Q^2 − \Pi_{R}(Q))^2} \left( A − M^2(K, P, K') \right) − \frac{4g^4\Pi_{R}(Q)B}{Q^4(Q^2 − \Pi_{R}(Q))^2} \left( B − 2M^2(K, P, K') \right),$$

$$A = (P.P)(K(P, P'), K') + (P.P)(K, K')(K, P, K'),$$

$$B = (P.K)(P, P, K') + (P.K')(K, K')(K, K'),$$

$$\times (P', K') + (P.P)(K, K') − 2(P, P')(K, K').$$

with

$$P.P = P_{\mu}P_{\mu}K_{\nu} = \frac{(P.q)(K.q)}{q_2^2} − P.k,$$

where $Q = K' − K = P − P'$, is the four momentum vector for the exchange gluon and $q_2^2 = \omega^2 − q_z^2$. For the estimation of the diffusion and the drag coefficients, we restrict the energy transfer to be small, which can be done by assuming $\omega = v \cdot q$, where $q$ is the three momentum transfer.

Now, let us first consider the case in which HQ moves along the direction of the magnetic field as shown in Fig. 1. Assuming that $v = (0, 0, v\hat{z})$ is the velocity of the HQ, initial and final gluon momenta $k$ and $k'$ make the angles respectively $\theta_k, \phi_k$ and $\theta_{k'}, \phi_{k'}$ with the $\hat{z}$ and the $x$-axis. The gluonic contribution to diffusion coefficient along the direction of the magnetic field then takes the form [19],

$$\kappa_{LL,LQg} = \frac{1}{16E^2(2\pi)^5\nu^2} \int d\omega \int \frac{dk}{(2\pi)^3} \int \frac{dk'}{(2\pi)^3} \int \frac{dq_2^2}{(2\pi)^3} \frac{|\mathcal{M}|^2f(|k|)}{(1 + f(|k'|))(1 + f(k'))},$$

where $f(|k|)$ is Bose-Einstein distribution function and $q_2^2 = k_2' − k_2$, with $k_2 = k \cos \theta_k$ and $k_2' = k' \cos \theta_{k'}$ is the $z$-component of the exchange gluon momentum. From now onwards, we shall use the notation $k = |k|$ and $k' = |k'|$. After performing $p'$ integration by using the three momenta Dirac delta function, Eq. (20) reduces to

$$\kappa_{LL,LQg} = \frac{1}{16E^2(2\pi)^5\nu^2} \int \frac{dk}{k} \frac{dk'}{k'} \frac{dq_2^2}{(2\pi)^3} |\mathcal{M}|^2 f(|k|) \times (1 + f(|k'|))(1 + f(k')) \delta(E + k - E' − k').$$

Further, for small momentum transfer, we can write $E − E' = v \cdot q$ and from the energy conservation we have the energy of the exchanged gluon as $\omega = |k'| − |k|$. Hence, the delta function in the above equation can be simplified to

$$\delta(\omega − v \cdot q) = \frac{1}{vk^2} \delta \left( \cos \theta_{k'} − \frac{k \cos \theta_k}{k'} − \frac{\omega}{vk^2} \right).$$

The energy delta function of Eq. (22) can be used to perform the angular integration ($\theta_k$) in Eq. (21). To perform the other angular integration ($\theta_{k'}$ integration), we introduce another delta function, which is the energy transfer to the HQ. Employing Eqs. (22) and (23) for performing the polar angular integrations, and integrating over both the azimuthal angles from 0 to $2\pi$, $\kappa_{LL,LQg}$ takes the form as follows,

$$\kappa_{LL,LQg} = \frac{1}{16E^2(2\pi)^5\nu^2} \int d\omega \int \frac{dk}{(2\pi)^3} \int \frac{dk'}{(2\pi)^3} \int \frac{dq_2^2}{(2\pi)^3} \frac{|\mathcal{M}|^2f(|k|)(1 + f(k'))}{\delta(E + k - E' − k')},$$

and can be solved numerically. Similarly, the other component of the diffusion coefficient perpendicular to the plane containing the magnetic, i.e., $\kappa_{TT,Qg}$ can be described as follows,

$$\kappa_{TT,Qg} = \frac{1}{16E^2(2\pi)^5\nu^2} \int d\omega \int \frac{dk}{(2\pi)^3} \int \frac{dk'}{(2\pi)^3} \int \frac{dq_2^2}{(2\pi)^3} \frac{|\mathcal{M}|^2f(|k|)(1 + f(k'))}{\delta(E + k - E' − k')},$$

where $q_2 = k^2 \sin^2 \theta_k + k'^2 \sin^2 \theta_{k'} − 2kk' \sin \theta_k \sin \theta_{k'} \cos(\phi_k − \phi_{k'})$. Now, let us consider the other case where HQ moves perpendicular to the magnetic field, as shown in Fig. 2. Without losing generality, we choose the HQ motion along the $x$-axis so that the HQ velocity takes the form $v = (v\hat{x}, 0, 0)$. As mentioned earlier, in this case, there
are three diffusion coefficients, $\kappa_{LT}$, $\kappa_{LT}$, and $\kappa_{LT}$. Similar to Eq. (21), the gluonic contribution to the diffusion coefficient $\kappa_{LT,Qg}$ takes the form, 
\[
\kappa_{LT,Qg} = \frac{1}{16E^2(2\pi)^3} \int \frac{dk}{k} \frac{dk'}{k'} q_z^2 \Delta f(k) \times (1 + f(k')) \delta(E + k - E' - k'). 
\] (27)
Instead of introducing the energy delta function identity (see before Eq. (23)) in the case of $v \parallel B$, here it is convenient to introduce the momentum delta function \[dq\delta^3(q + k - k') = 1 \text{ identity so that Eq. (27) reduces to}
\[
\kappa_{LT,Qg} = \frac{1}{16E^2(2\pi)^3} \int \frac{dk}{k} \frac{dk'}{k'} q_z^2 \Delta f(k) \times (1 + f(k')) \delta(E + k - E' - k'). 
\] (28)
Again, in the small momentum transfer limit, one can write $k - k' = k - |k - q| = -q \cdot k$ and with $E - E' = v \cdot q$ (energy conservation), the energy delta function in Eq. (28) can be written as,
\[
\delta(E - E' + k - k') = \delta(v \cdot q - q \cdot k). 
\] (29)
Taking $\theta_q$, $\theta_k$ and $\phi_q, \phi_k$ as polar angles and azimuthal angles of exchanged gluon and initial gluon, one can write
\[
q \cdot k = q \sin \theta_k \sin \theta_q \cos(\phi_k - \phi_q) + \cos \theta_q \cos \theta_k, 
\] (30)
and
\[v \cdot q = v q \sin \theta_q \cos \phi_q. \] (31)
Using Eqs. (30) and (31), the energy delta function in Eq. (28) can be simplified to the following form,
\[
\delta(v \cdot q - q \cdot k) = \frac{1}{q \sin \theta_q \sin \theta_k} \delta\left(\cos(\phi_k - \phi_q) + \cot \theta_q \cot \theta_k - v \frac{\cos \phi_q}{\sin \theta_k}\right). 
\] (32)
To perform the angular integration over azimuthal angle of $k$ ($\phi_k$ integration) in Eq. (28), one can further simplify the delta function of Eq. (32) as,
\[
\delta\left(\cos(\phi_k - \phi_q) + \cot \theta_q \cot \theta_k - \frac{v \cos \phi_q}{\sin \theta_k}\right) = \frac{\delta(\phi_k - \phi_q^0)}{\sin(\phi_k^0 - \phi_q^0)}, 
\] (33)
where,
\[
\phi_q^0 = \phi_q + \cos^{-1}\left(\frac{v \cos \phi_q}{\sin \theta_k} - \cot \theta_q \cot \theta_k\right). 
\] (34)
Eq. (34) sets the integration limit for $\theta_q$ integration, which can be obtained by solving the above equation by imposing the condition $|\cos(\phi_k^0 - \phi_q)| \leq 1$. With these simplifications, Eq. (28) takes the following form,
\[
\kappa_{LT,Qg} = \frac{1}{16E^2(2\pi)^3} \int_0^\infty dk \frac{1}{k} \frac{1}{k'} \int_0^\pi \sin \theta_q \sin \theta_k \frac{d\theta_k}{4\pi} 
\]
\[\times d\phi_q \int d(\cos \theta_k) q_z^2 \Delta f(k)(1 + f(k')) \] (35)
Similar to Eq. (35), the other two components of the diffusion coefficient in the yz-plane can be described as, 
\[
\kappa_{LT,Qg} = \frac{1}{16E^2(2\pi)^3} \int_0^\infty d\phi_q \int_0^\pi d\theta_q \int_0^\pi \sin \theta_q \sin \theta_k \int_0^\pi \sin \theta_q \sin \theta_k \frac{d\theta_k}{4\pi} 
\]
\[\times d\phi_q \int d(\cos \theta_k) q_z^2 \Delta f(k)(1 + f(k')) \] (36)
where $q_z$ is the magnitude of the exchange gluon momentum along the $\hat{z}$ direction and
\[
\kappa_{LT,Qg} = \frac{1}{16E^2(2\pi)^3} \int_0^\infty d\phi_q \int_0^\pi d\theta_q \int_0^\pi \sin \theta_q \sin \theta_k \frac{d\theta_k}{4\pi} 
\]
\[\times d\phi_q \int d(\cos \theta_k) q_z^2 \Delta f(k)(1 + f(k')) \] (37)
with $q_y$ as the magnitude of the exchange gluon momentum along the $\hat{y}$ axis.

### B. Quark contribution

The other contribution to the diffusion coefficients in the LLL approximation arises from the Coulomb scattering, i.e., scattering of HQ to that of LLL light thermal quarks. Let us first consider the case in which HQ moves in the direction of the magnetic field with velocity $v$. In the limit of small momentum transfer, the quark contribution to the diffusion coefficient parallel to the magnetic field can be described as \[35,\]
\[
\kappa_{LL,Qg} = g^2(N^2 - 1) 3 \left(\int \frac{d^4Q}{(2\pi)^3} \frac{2T q^2}{\omega} v^2 (D^{q\parallel}_{0,ij}(Q) - D^{R}_{0,ij}(Q))\right), 
\] (38)
where $\mathcal{3}$ stands for the imaginary part, $N$ is number of colors, and $D^{q\parallel}_{0,ij}(Q)$ is resummed gluon propagator in the LLL approximation which is defined as follows,
\[
D^{q\parallel}_{\mu\nu}(Q) = - \frac{P_{\mu\nu}(Q)}{(\omega + i\epsilon)^2 - q^2} + \frac{\Pi_{q\parallel}^{ij}(\omega + i\epsilon, q) P_{\mu\nu}(Q)}{Q^2 Q^2 - \Pi_{q\parallel}^{ij}(\omega + i\epsilon, q)} + \frac{Q_{\mu} Q_{\nu}}{(\omega + i\epsilon)^2 - q^2}, \]
(39)
where $\Pi_{q\parallel}^{ij}$ is the retarded self energy of the gluon arising from the light quarks loop with light quarks in the LLL and $\xi$ is the gauge parameter. The expression for $\Pi_{q\parallel}^{ij}$ is given in appendix $A$, $D^{R}_{0,ij}$ in Eq. (39) is the gluon propagator in vacuum. The longitudinal projection operator $P_{\mu\nu}(Q)$ is defined as
\[
P_{\mu\nu}(Q) = - g_{\mu\nu} + \frac{q_{\mu} q_{\nu}}{q^2}, \]
(40)
and the other projection operator $P_{\mu\nu}(Q)$ takes the form as,
\[
P_{\mu\nu}(Q) = - g_{\mu\nu} + \frac{Q_{\mu} Q_{\nu}}{(\omega + i\epsilon)^2 - q^2}. \] (41)
In Eq. (38), the imaginary contribution comes from the medium dependent term of the retarded propagator and can be described as,

$$\Im D_{ij}^R(Q) = -\frac{\Im \Pi_{ij}^R}{(Q^2 - \Re \Pi_{ij}^R)^2} P_{ij}^R(Q),$$

(42)

where $\Re \Pi_{ij}^R(Q)$ and $\Im \Pi_{ij}^R(Q)$ are the real and the imaginary parts of retarded self energy described in the Appendix. A. Similar to the case of gluonic contribution, the angular integration in Eq. (38) can be performed by employing the energy delta function, and other integrations can be evaluated numerically. Similarly, the other component of the diffusion coefficient can be defined as follows,

$$\kappa_{TT, Q q} = \frac{q^2(N^2 - 1)}{2N} \Im \left( \int \frac{d^4Q}{(2\pi)^4} 2\mathcal{P}_q \mathcal{P}_q \mathcal{P}_Q \right) v^2 \Im \left( \frac{\mathcal{P}_Q}{\mathcal{P}_Q} D_{ij}^R(Q) - D_{ij}^R(Q) \right) \wp(\omega - q \cdot v),$$

(43)

where $q^2 = q^2 \sin^2 \theta_q$.

Next, we consider the quark contribution to the diffusion coefficient for the case HQ moving perpendicular to the magnetic field. In this case, the diffusion coefficients do not get any contribution from the Coulomb scattering in the LLL approximation for the quark loop in the resummed gluon propagator. This can be understood from the resummed gluon propagator in the LLL. For the magnetic field along the $z$-axis, the medium dependent term in the resummed gluon propagator has only one spatial component, i.e., only $D_{33}^R$ is finite. HQ moving in a plane transverse to the plane containing the magnetic field is orthogonal to $D_{33}^R$ and hence leads to $\kappa_{LT, Q q} = \kappa_{TL, Q q} = \kappa_{TT, Q q} = 0$. Thus, the quark contributions to $\kappa^\perp$ for all the components vanish, and these components get the finite contribution from the Compton scatterings only. The total diffusion coefficients can be obtained by

$$\kappa_{LL}^\perp = \kappa_{LL, Q q} + \kappa_{LL, Q q}^\perp,$$

(44)

$$\kappa_{TT}^\perp = \kappa_{TT, Q q} + \kappa_{TT, Q q}^\perp,$$

(45)

$$\kappa_{LT}^\perp = \kappa_{LT, Q q} + \kappa_{LT, Q q}^\perp,$$

(46)

$$\kappa_{TL}^\perp = \kappa_{TL, Q q} + \kappa_{TL, Q q}^\perp,$$

(47)

$$\kappa_{TT}^\perp = \kappa_{TT, Q q} + \kappa_{TT, Q q}^\perp.$$  

(48)

As we have observed that the diffusion coefficients are highly anisotropic with the inclusion of the magnetic field for the case of finite velocity of HQ.

III. RESULTS AND DISCUSSIONS

We discuss here the variation of different HQ anisotropic diffusion coefficient in a magnetized QGP medium as a function of HQ velocity and temperature. For the quantitative analysis, we consider QCD coupling constant $\alpha_s = 0.3$, HQ mass $M = 1.2$ GeV, light quark mass $m = 0.02$ GeV and quark flavor $N_f = 2$. In Fig. 3,

![FIG. 3: Diffusion coefficient ($\kappa^\perp$) for HQ moving along x-axis i.e., perpendicular to the magnetic field as a function of temperature for $m = 20$ MeV, $v = 0.5$ with $eB = 5 m_\pi^2$ (top) and $eB = 10 m_\pi^2$ (bottom). Blue (dotdashed), red (dashed), and black (solid) lines respectively are $\kappa_{LT}, \kappa_{TL}$, and $\kappa_{TT}$.](image-url)

we have shown the variation of $\kappa^\perp$, i.e., the momentum diffusion for the case of HQ moving perpendicular to the magnetic field as a function of temperature for $eB = 5 m_\pi^2$ and $eB = 10 m_\pi^2$, where $m_\pi$ is pion mass.

Let us note that for this diffusion coefficient, only the Compton scattering contributes in the LLL approximation for the resummed gluon propagator. We have considered here HQ motion to be along the $x$-axis, i.e., transverse to the direction of motion in this case) $\kappa_{LL}^\perp$ is dominant in comparison with diffusion along the transverse direction to the magnetic field and we have $\kappa_{LL}^\perp > \kappa_{LT}^\perp > \kappa_{TT}^\perp$. This behavior persists even for the...
higher magnetic fields. This probably could be due to the fact that in the LLL approximation, Coulomb scattering terms do not contribute to the diffusion coefficients $\kappa^\perp$.

In Fig. 4, the temperature dependence of $\kappa^\parallel$ is depicted for $v = 0.5$ at $eB = 5m^2_\pi$ and $eB = 10m^2_\pi$. Here, both the magnetic field and HQ velocity are along $x$-axis. We observe that the HQ momentum diffusion in the plane transverse to the magnetic field ($xy$ plane) is larger than the diffusion along the magnetic field, i.e., $\kappa^\perp_{TT} \gg \kappa^\parallel_{LL}$ for all ranges of temperature considered here. For the case of the static limit of HQ ($v = 0$), similar results have been obtained in Ref. [38, 39]. The component $\kappa^\parallel_{TT}$ has a strong dependence on temperature, on the other hand, $\kappa^\parallel_{LL}$ shows a weak dependence on the temperature. This behavior is in contrast with the case of $\kappa^\perp$ case where the diffusion along the magnetic field is dominant. Numerically it is observed that the Coulomb scattering terms dominate over the Compton scattering ones for the temperatures and the magnetic field considered here.

In Fig. 5, the variation of $\kappa^\perp$ as a function of HQ velocity is shown. All three diffusion coefficients increase with an increase in the HQ velocity. However, $\kappa^\perp_{TT}$ has a very weak dependence on the HQ velocity. Out of three diffusion coefficients, $\kappa^\perp_{TT}$ is the dominant, i.e., $\kappa^\perp_{TT} \gg \kappa^\perp_{LT}, \kappa^\perp_{TT}$ for all ranges of the HQ velocity.

In Fig. 6, diffusion coefficients, $\kappa^\parallel_{LL}$ and $\kappa^\parallel_{TT}$, are plotted as a function of HQ velocity. Similar to the case of $v = 0$ (static limit) in Ref. [36], diffusion in the transverse direction is always larger than the diffusion along the longitudinal direction for all values of $v$. Apart from this, both $\kappa^\parallel_{TT}$ and $\kappa^\parallel_{LL}$ have a similar dependence on the HQ velocity.

The anisotropic HQ drag coefficients in the non-relativistic (NR) limit can be estimated by using the dissipation fluctuation theorem as,

$$\eta^\parallel_D = \frac{\kappa^\parallel}{2MT}, \quad (49)$$

FIG. 4: Diffusion coefficient ($\kappa^\parallel$) for HQ moving along $x$-axis i.e., perpendicular to the magnetic field as a function of temperature for $m = 20 \text{ MeV}$, $v = 0.5$ at $eB = 5m^2_\pi$ (top) and $eB = 10m^2_\pi$ (bottom). Blue (dotdashed) and black (solid) lines are $\kappa^\parallel_{LL}$ and $\kappa^\parallel_{TT}$.

FIG. 5: Diffusion coefficient ($\kappa^\perp$) for HQ moving along $x$-axis, i.e., perpendicular to the magnetic field as a function of HQ velocity for $m = 20 \text{ MeV}$, $eB = 5m^2_\pi$ and $T = 0.25 \text{ GeV}$. Blue (dotdashed), red (dashed), and black (solid) lines respectively are $\kappa^\perp_{LT}$, $\kappa^\perp_{TT}$ and $\kappa^\perp_{LL}$.

FIG. 6: Diffusion coefficient ($\kappa^\parallel$) for HQ moving along $x$-axis i.e., perpendicular to the magnetic field as a function of HQ velocity for $m = 20 \text{ MeV}$, $eB = 5m^2_\pi$ and $T = 0.25 \text{ GeV}$. Blue (dotdashed) and black (solid) lines respectively are $\kappa^\parallel_{LL}$ and $\kappa^\parallel_{TT}$.
and

$$\eta^D_B = \frac{\kappa^\perp}{2MT}. \quad (50)$$

From Fig. 3, it can be observed that for the case of HQ moving perpendicular to the magnetic field, the drag parallel to $B$ and perpendicular to $v$ is the dominant one, i.e., $\eta^D_{\parallel TL} > \eta^D_{\perp LT}, \eta^D_{\perp TT}$ for a given value of $v$. In this case, the HQ is dragged more in the direction of the magnetic field in comparison with the direction along its motion, and direction transverse to both the magnetic field and HQ motion. This anisotropic nature of drag forces to the HQ in the magnetized medium may generate an additional contribution to the flow coefficients, i.e., directed flow and elliptic flow, of HQs. For the case of HQ moving parallel to the magnetic field, the drag perpendicular to the magnetic field is the dominant one, i.e., $\eta^D_{\perp TT} > \eta^D_{\perp LL}$. This implies that HQ is more dragged in the plane perpendicular to the magnetic field (here, $xy$-plane) in the case of $v \parallel B$, which may generate anisotropic flow coefficients. The relative magnitudes of the drag and the diffusion coefficients quantify the anisotropic nature of the transport coefficients. With an increase in the magnetic field in the LLL approximation, the drag/diffusion coefficients increase in magnitude, while the relative trend of the coefficients still remains the same.

IV. SUMMARY AND OUTLOOK

The anisotropic diffusion and drag coefficients of HQ beyond the static limit have been computed in a constant (along the $z$-axis) magnetic field background, at leading order in the QCD coupling constant. In the medium, the HQ makes multiple collisions with the thermal partons, i.e., light quarks and gluons, and the process is akin to the Brownian motion. The magnetic field is assumed to be strong such that the condition $\sqrt{eB} \gg T$ is satisfied, and the dynamics of light quarks are restricted in the LLL. Further, it is also assumed that $M \gg \sqrt{eB}$, so that HQ is not directly affected by the magnetic field due to its large mass. To study the diffusion of HQ, the momentum transfer in the collision of HQ and the thermal partons is assumed to be small.

It is observed that there can be total five momentum diffusion coefficients of HQ in the medium, depending on the orientation of its motion and magnetic field. In the case of the HQ motion along the direction of the magnetic field, $v \parallel B$, the coefficients $\kappa^\parallel_{LL}$ and $\kappa^\parallel_{TT}$ quantify the momentum diffusion along $\hat{z}$ direction and diffusion in the $xy$ plane, i.e., perpendicular to the plane containing magnetic field, respectively. Out of these two, diffusion along the direction of the magnetic field is smaller than the diffusion transverse to the direction of the magnetic field, i.e.,

$$\frac{\kappa^\parallel_{LL}}{\kappa^\parallel_{TT}} \ll 1. \quad (51)$$

In the strong field limit, the Coulomb scattering contribution to $\kappa^\perp$ is observed to be dominant over the contribution arising from the Compton scattering of HQ and gluon.

Similarly, there are three diffusion coefficients for the case of HQ moving transverse to the direction of the magnetic field i.e., $v \perp B$ denoted as $\kappa^\perp_{\parallel LT}, \kappa^\perp_{\perp TT}, \kappa^\perp_{\perp LL}$. In this case, diffusion in the direction transverse to velocity and parallel to $B$ is dominant in comparison with other components and we observe,

$$\frac{\kappa^\parallel_{\perp LT}}{\kappa^\parallel_{\perp TT}}, \frac{\kappa^\perp_{\perp TT}}{\kappa^\perp_{\perp LL}} \ll 1. \quad (52)$$

In contrast to $\k^\parallel$, the contribution to $\k^\perp$ arises only from the Compton scattering process in the strong field approximation.

The relative magnitudes of the diffusion coefficients suggest the anisotropic behavior of the momentum broadening. Further, in the non-relativistic limit, the drag coefficient can be estimated by using the dissipation-fluctuation theorem. Similar to the diffusion coefficients, there are five drag coefficients, and the relative magnitude of the drag coefficient suggests the anisotropic drag force on HQ. For HQ moving parallel to the magnetic field, our observation is qualitatively consistent with the results of Ref. [9] in the static limit, i.e.,

$$\frac{\eta^D_{\parallel LL}}{\eta^D_{\perp TT}} \ll 1. \quad (53)$$

It seems that the outcome as in Eq. (53) is universal, i.e., true in both the static and non-static limits. Similarly, for the case of HQ moving perpendicular to the magnetic field, the relative magnitudes of the drag coefficients satisfy,

$$\frac{\eta^\perp_{\perp TT}}{\eta^\perp_{\perp LL}}, \frac{\eta^\perp_{\perp LT}}{\eta^\perp_{\perp TT}}, \frac{\eta^\perp_{\perp TT}}{\eta^\perp_{\perp LL}} \ll 1, \quad (54)$$

and indicate the anisotropic drag force in different directions. The trend in Eq. (54) is in line with the results of the drag coefficient in an anisotropic QGP in Ref. [32]. The dependence of the magnetic field and HQ velocity on the momentum diffusion coefficients have been explored in the analysis.

We have investigated the anisotropic nature of the momentum diffusion and drag forces arising from the magnetic field considering the finite velocity of HQ. The anisotropic transport coefficients can be used as input parameters for the estimation of HQ flow coefficients in the magnetized medium. It may be noted that HQ directed flow $v_1$, is identified as a novel observable to probe the
initial electromagnetic field produced in high energy collisions. The recent LHC measurement [7], along with the RHIC findings [6], on the D-meson flow coefficient $v_1$, give the indications of the strong electromagnetic field produced in high energy heavy-ion collisions. However, to compute the HQ directed flow, one needs to take into account the effect of electromagnetic field on HQ transport coefficients as well, which has been ignored in the previous calculations [59, 60]. Heavy meson nuclear suppression factor and elliptic flow are the other experimentally measured observables that can be affected by the anisotropic HQ transport coefficient due to the presence of the electromagnetic field. The present investigation is limited to the strong field case of including LLL for the light quarks. For the case of the magnetic field of the order of the temperature i.e., $eB \sim T^2$ higher Landau levels may give significant contributions to the transport coefficients. We intend to explore these aspects in the near future.

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\appendix{Appendix A: Gluon self energy}

In the LLL approximation, the contribution of the gluon loop is proportional to $T^2$, and that of the quark loop is proportional to $eB$ [57]. Hence, in the LLL approximation, the gluon loop contribution ($\sim T^2$) can be neglected with respect to the quark loop contribution ($\sim eB$). Assuming that the magnetic field $\mathbf{B}$ is along the positive $z$-axis, the gauge-invariant structure of the gluon self energy is given as [61]

$$\Pi_R^{\mu\nu}(P) = \Pi_0^\mu(P)P_0^\nu + \Pi_0^\mu(P)P_0^{\nu\parallel} + \Pi_0^\nu(P)P^{\mu\parallel} + \Pi_0^{\nu\parallel}(P)P^{\mu\parallel},$$

where $P_0^{\mu\parallel}$ are the projection operators defined as,

$$P_0^{\mu\parallel} = -g^{\mu\nu} + \frac{P_0^\mu P^\nu}{P^2}, P_0^{\nu\parallel} = -g^{\mu\nu} + \frac{P_0^{\mu\parallel}P_0^{\nu\parallel}}{P_0^{\parallel}}.$$  \hfill (A2)

In Eq. (A1), $\Pi_0^\parallel$ is vacuum term which is independent of both $T$ and $\mathbf{B}$, we shall drop this term here. In the LLL, the light quark motion is restricted along the direction of the magnetic field only so $\Pi_0^{\parallel}(P) = 0$. Further, the term $\Pi^{\parallel\parallel}(P) = \Pi^{\parallel\parallel}(\mathbf{B}, T) + \Pi^{\parallel\parallel}(\mathbf{B}, T)$ where $\Pi_0^{\parallel}(\mathbf{B}, T) = 0$ depends on the magnetic field only and $\Pi^{\parallel\parallel}(\mathbf{B}, T)$ depends on both magnetic field and temperature. $\Pi^{\parallel\parallel}(\mathbf{B}, T) = 0$ is given as [61]

$$\Pi^{\parallel\parallel}(\mathbf{B}, T) = 0 = \frac{\alpha |q_f eB| m^2}{\pi p^2} e^{-\frac{\alpha p}{|q_f eB|}} \left( I(B) - 2 \right), \hfill (A3)$$

where,

$$I(B) = \begin{cases}
\frac{4m^2}{\sqrt{p^2 (p^2 - 4m^2)}} \ln \left[ \frac{p^2 - \sqrt{(p^2 - 4m^2)}^2}{p^2 + \sqrt{(p^2 - 4m^2)}^2} \right] & p^2 < 0 \\
\frac{2}{\sqrt{p^2 (p^2 - 4m^2)}} \arctan \left( \frac{\sqrt{(p^2 - 4m^2)}}{p^2} \right) & 0 < p^2 < 4m^2 \\
\frac{4m^2}{\sqrt{p^2 (p^2 - 4m^2)}} \left[ \ln \left( \frac{p^2 - \sqrt{(p^2 - 4m^2)}^2}{p^2 + \sqrt{(p^2 - 4m^2)}^2} \right) + i\pi \right] & p^2 > 4m^2
\end{cases} \hfill (A4)$$

Here, $m$ is mass of light quark. Another term that depends on $T$ and $\mathbf{B}$ is

$$\Pi_0^{\parallel\parallel}(\mathbf{B}, T) = \pi g^2 \Omega m^2 \left[ J_0(P) + \frac{2p^2}{p_0^2} J_1(P) \right], \hfill (A5)$$

where

$$J_0 = \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} f(E) k_z^a \hat{f}(E) k_z^a \exp \left( -\frac{p_z^2}{2|q_f eB|} \right). \hfill (A7)$$

Here, $q_f = 1/3, 2/3$ for light quark flavor $N_f = 2$.

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