Classical Methods to Estimate the Parameters of Exponentiated Weibull Distribution

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Abstract. In this article the Exponentiated Weibull distribution with some of its properties is considered. Classical methods, Maximum likelihood estimator method, ordinary least squares estimator method and rank set sampling estimator method are proposed to estimate all the unknown parameters ($\beta, \Psi, \theta$) of the distribution. Newton –Raphson method is used to solve the above three methods, simulation procedure is used to generated some sample sizes and finally mean squares error measure are used to compare between them. We find Maximum likelihood estimator method has the smallest mean square error.

Keywords: Classical distribution methods, Exponentiated Weibull distribution, Newton Raphson method.

1. Introduction

The Exponentiated Weibull distribution is one of the most widely distributions for analyzing lifetime data. The ordinary distributions cannot always be sufficient to model the real data. Furthermore, in many applied areas such as lifetime analysis there is a strong need for extended forms of the ordinary distributions below\cite{2}. As the extension of weibull family Mudholkar and Srivastava introduced in 1993 the exponentiated Weibull distribution by adding one shape parameter\cite{1}. After two years, in 1995, Mudholkar represented the applications of above distribution\cite{3}. A new generated family of continuous distributions based on exponentiated Weibull is introduced in 2019 by Nadia, Salah and Mohammad, then exponentiated Weibull-exponentiated Weibull is proposed as a special case of this family\cite{4}. In (2019) Mohammed, Amal, Ibrahim and Gholam introduced a new distribution from exponentiated weibull dist. And Rayleigh dist. They called it (EWR)\cite{5}. In (2019) Iden and Maysaa discussed some estimation method for new distribution (mixture distribution is mixed between Exponentiated Weibull and exponential Rayleigh)\cite{6} Iden and Hadeer in (2019) estimated the parameter of Rayleigh distribution by used MLE method, LOS method, RSSE method\cite{7}.

The pdf and cdf are respectively:

$$ f(t;\beta,\Psi,\theta)=\beta\Psi^\theta t^{(\Psi-1)}e^{-(\theta t)^\Psi}(1 - e^{-(\theta t)^\Psi})^{\beta-1} $$

(1)
The following is hazard rate function:
\[ H(t) = \frac{f(t)}{1-F(t)} \quad t > 0 \]  
(3)
And the survival function is:
\[ S(t) = 1 - e^{-\left(\theta t\right)^\psi} \quad t > 0 \]  
(4)
As a special cases of exponentiated Weibull distribution there are many distributions some of them as follows:
When \( \Psi = 2 \) become generalized Rayleigh dist. When \( \left(\beta = 1, \Psi = 1\right) \) become exponential dist. When \( \left(\beta = 1\right) \) become Weibull dist. And when \( \left(\beta = 1, \Psi = 2\right) \) become Rayleigh dist.
The aim of this paper is to estimate the parameters of exponentiated Weibull distribution by utilizing three methods (maximum likelihood estimator method, the ordinary least squared estimator method and rank set sampling estimator method) the random samples were generated by simulation technique with \( (n = 10, 20, 30 \) and 50) and finally the mean squared error measure are used to compare between these three methods.

The paper is included , in section (2) the maximum likelihood method , in section (3) the ordinary least squares method, in section (4) the rank set sampling method , in section (5) the simulation technique finally in section (6) we discussed the main conclusions .

2. Maximum likelihood estimator method estimator

The first method to estimate the parameters of exponentiated Weibull distribution is maximum likelihood. The idea of this method to maximize the parameters for likelihood function.

\[
\begin{align*}
f(t;\beta,\Psi,\theta) &= \beta \Psi t^{(\Psi-1)} e^{-\left(\theta t\right)^\psi} \\ L(\beta,\Psi,\theta; t_i) &= \beta^n \Psi^n \theta^n \prod_{t_i=1}^n \left(1 - e^{-\left(\theta t_i\right)^\psi}\right)^{\beta-1}
\end{align*}
\]
By taking the natural logarithm for (5):

\[
\ln L = n \ln \beta + n \ln \Psi + n \Psi \ln \theta + (\Psi - 1) \sum_{t_i=1}^n \ln t_i - \sum_{t_i=1}^n \left(\theta t_i\right)^\psi + \left(\beta - 1\right) \sum_{t_i=1}^n \ln \left(1 - e^{-\left(\theta t_i\right)^\psi}\right)
\]
(6)
Now, the partial derivatives for the logarithm likelihood function with respect to parameters \( (\beta, \Psi, \theta) \):

\[
\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} + \sum_{t_i=1}^n \ln \left(1 - e^{-\left(\theta t_i\right)^\psi}\right)
\]
(7)
\[
\frac{\partial \ln L}{\partial \Psi} = \frac{n}{\Psi} + n \ln \theta + \sum_{t_i=1}^n \ln t_i - \left(\theta \sum_{t_i=1}^n t_i\right)^\psi \ln \left(\theta \sum_{t_i=1}^n t_i\right) + \left(\beta - 1\right) \sum_{t_i=1}^n \frac{-e^{-\left(\theta t_i\right)^\psi}}{1 - e^{-\left(\theta t_i\right)^\psi}} \ln \theta t_i
\]
(8)
\[
\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - \Psi \left(\theta \sum_{t_i=1}^n t_i\right)^\psi - 1 \sum_{t_i=1}^n t_i + \left(\beta - 1\right) \sum_{t_i=1}^n \frac{-e^{-\left(\theta t_i\right)^\psi}}{1 - e^{-\left(\theta t_i\right)^\psi}} \theta t_i
\]
(9)
Let:
\[
f(\beta) = \frac{n}{\beta} + \sum_{t_i=1}^n \ln \left(1 - e^{-\left(\theta t_i\right)^\psi}\right)
\]
\[
g(\Psi) = \frac{n}{\Psi} + n \ln \theta + \sum_{t_i=1}^n t_i - \left(\theta \sum_{t_i=1}^n t_i\right)^\psi \ln \left(\theta \sum_{t_i=1}^n t_i\right) + \left(\beta - 1\right) \sum_{t_i=1}^n \frac{-e^{-\left(\theta t_i\right)^\psi}}{1 - e^{-\left(\theta t_i\right)^\psi}} \ln \theta t_i
\]
\[
h(\theta) = \frac{n}{\theta} - \Psi \left(\theta \sum_{t_i=1}^n t_i\right)^\psi - 1 \sum_{t_i=1}^n t_i + \left(\beta - 1\right) \sum_{t_i=1}^n \frac{-e^{-\left(\theta t_i\right)^\psi}}{1 - e^{-\left(\theta t_i\right)^\psi}} \theta t_i
\]
Now, we must find the Jacobean matrix \( J \), because the above three equations are nonlinear equations so they are hard to solve where Matrix \( J \) is the first derivatives for each equations \( f(\beta), g(\Psi) \) and \( h(\theta) \) with respect to the parameters \( \beta, \Psi \) and \( \theta \).
\[
J = \begin{bmatrix}
\frac{\partial l_1}{\partial \beta} & \frac{\partial l_1}{\partial \theta} & \frac{\partial l_1}{\partial \sigma} \\
\frac{\partial l_2}{\partial \beta} & \frac{\partial l_2}{\partial \theta} & \frac{\partial l_2}{\partial \sigma} \\
\frac{\partial l_3}{\partial \beta} & \frac{\partial l_3}{\partial \theta} & \frac{\partial l_3}{\partial \sigma}
\end{bmatrix} = \begin{bmatrix}
\frac{\partial^2 l_1}{\partial \beta^2} & \frac{\partial^2 l_1}{\partial \beta \theta} & \frac{\partial^2 l_1}{\partial \beta \sigma} \\
\frac{\partial^2 l_2}{\partial \beta^2} & \frac{\partial^2 l_2}{\partial \beta \theta} & \frac{\partial^2 l_2}{\partial \beta \sigma} \\
\frac{\partial^2 l_3}{\partial \beta^2} & \frac{\partial^2 l_3}{\partial \beta \theta} & \frac{\partial^2 l_3}{\partial \beta \sigma}
\end{bmatrix}
\]

\[
\frac{\partial l_1}{\partial \beta \beta} = f(\beta^2) = -\frac{n}{\beta^2}
\]

\[
\frac{\partial l_1}{\partial \beta \theta} = f(\beta \theta) = \sum_{i=1}^{n} \frac{e^{-((\beta \theta)\psi)(\theta \sum_{i=1}^{n} ti)}}{1-e^{-((\beta \theta)\psi)(\theta \sum_{i=1}^{n} ti)}} = \sum_{i=1}^{n} \frac{e^{-((\beta \theta)\psi)(\theta \sum_{i=1}^{n} ti)}}{1-e^{-((\beta \theta)\psi)(\theta \sum_{i=1}^{n} ti)}} \frac{(1-e^{-((\beta \theta)\psi)(\theta \sum_{i=1}^{n} ti)})^2}{(1-e^{-((\beta \theta)\psi)(\theta \sum_{i=1}^{n} ti)})^2}
\]

\[
\frac{\partial l_1}{\partial \beta \sigma} = f(\psi \beta) = \sum_{i=1}^{n} \frac{e^{-((\psi \beta)\sigma)(\sum_{i=1}^{n} ti)}}{1-e^{-((\psi \beta)\sigma)(\sum_{i=1}^{n} ti)}} = \sum_{i=1}^{n} \frac{e^{-((\psi \beta)\sigma)(\sum_{i=1}^{n} ti)}}{1-e^{-((\psi \beta)\sigma)(\sum_{i=1}^{n} ti)}} \frac{(1-e^{-((\psi \beta)\sigma)(\sum_{i=1}^{n} ti)})^2}{(1-e^{-((\psi \beta)\sigma)(\sum_{i=1}^{n} ti)})^2}
\]

\[
\frac{\partial l_1}{\partial \theta \theta} = f(\theta^2) = \sum_{i=1}^{n} \frac{\psi \sigma e^{-((\theta \sum_{i=1}^{n} ti))\psi \sum_{i=1}^{n} ti}}{1-e^{-((\theta \sum_{i=1}^{n} ti))\psi \sum_{i=1}^{n} ti}} = \sum_{i=1}^{n} \frac{\psi \sigma e^{-((\theta \sum_{i=1}^{n} ti))\psi \sum_{i=1}^{n} ti}}{1-e^{-((\theta \sum_{i=1}^{n} ti))\psi \sum_{i=1}^{n} ti}} \frac{(1-e^{-((\theta \sum_{i=1}^{n} ti))\psi \sum_{i=1}^{n} ti)})^2}{(1-e^{-((\theta \sum_{i=1}^{n} ti))\psi \sum_{i=1}^{n} ti)})^2}
\]

\[
\frac{\partial l_1}{\partial \theta \sigma} = f(\psi \sigma) = \sum_{i=1}^{n} \frac{\psi \sigma e^{-((\psi \sigma)\sigma)(\sum_{i=1}^{n} ti)}}{1-e^{-((\psi \sigma)\sigma)(\sum_{i=1}^{n} ti)}} = \sum_{i=1}^{n} \frac{\psi \sigma e^{-((\psi \sigma)\sigma)(\sum_{i=1}^{n} ti)}}{1-e^{-((\psi \sigma)\sigma)(\sum_{i=1}^{n} ti)}} \frac{(1-e^{-((\psi \sigma)\sigma)(\sum_{i=1}^{n} ti)})^2}{(1-e^{-((\psi \sigma)\sigma)(\sum_{i=1}^{n} ti)})^2}
\]

\[
\frac{\partial l_1}{\partial \sigma \sigma} = f(\sigma^2) = \sum_{i=1}^{n} \frac{\psi \sigma e^{-((\psi \sigma)\sigma)(\sum_{i=1}^{n} ti)}}{1-e^{-((\psi \sigma)\sigma)(\sum_{i=1}^{n} ti)}} = \sum_{i=1}^{n} \frac{\psi \sigma e^{-((\psi \sigma)\sigma)(\sum_{i=1}^{n} ti)}}{1-e^{-((\psi \sigma)\sigma)(\sum_{i=1}^{n} ti)}} \frac{(1-e^{-((\psi \sigma)\sigma)(\sum_{i=1}^{n} ti)})^2}{(1-e^{-((\psi \sigma)\sigma)(\sum_{i=1}^{n} ti)})^2}
\]

3. Ordinary least squares estimator method

The second method to estimate the parameters of Exponentiated Weibull distribution is used. When the model is linear or non-linear in variables, the idea of this method is to minimize the sum of squared differences between observed sample values and the expected values by linear approximation, as follows:

\[
Y_i = \beta_0 + \beta_1x_i + \epsilon_i
\]

\[
\sum_{i=1}^{n} \epsilon_i^2 = \sum_{i=1}^{n} (yi - \beta_0 + \beta_1x_i)^2
\]
By using CDF of exponentiated weibull distribution

\[ F(t_i) = \left( 1 - e^{-\theta t_i} \right)^{\beta} \]

\[ 1 - \left( (F(t_i))^{\frac{1}{\beta-1}} \right) = e^{-\theta t_i} \]

Now, taking the double natural logarithm:

\[ \ln(1 - (F(t_i))^{\frac{1}{\beta-1}}) = -\theta t_i \]

Now, comparing with simple linear model:

\[ c_i = \ln\left( \ln\left(1 - (F(t_i))^{\frac{1}{\beta-1}}\right) \right) - \Psi \ln \theta - \Psi \ln t_i \]

\[ \sum_{i=1}^{n} c_i^2 = \sum_{i=1}^{n} \left[ \ln \left( -\ln \left(1 - (F(t_i))^{\frac{1}{\beta-1}}\right) \right) \right] \]

\[ - \Psi \ln \theta - \Psi \ln t_i \]

\[ \left( \frac{\partial c_i^2}{\partial \beta} \right) = 2 \sum_{i=1}^{n} \left[ \ln \left( -\ln \left(1 - (F(t_i))^{\frac{1}{\beta-1}}\right) \right) \right] \left( \frac{-2}{\beta-1} \ln(F(t_i)) \right) \]

\[ \left( \frac{\partial c_i^2}{\partial \Psi} \right) = 2 \sum_{i=1}^{n} \left[ \ln \left( -\ln \left(1 - (F(t_i))^{\frac{1}{\beta-1}}\right) \right) \right] \left( -\ln \theta - \ln t_i \right) \]

\[ \left( \frac{\partial c_i^2}{\partial \theta} \right) = 2 \sum_{i=1}^{n} \left[ \ln \left( -\ln \left(1 - (F(t_i))^{\frac{1}{\beta-1}}\right) \right) \right] \left( -\frac{\Psi}{\theta} \right) \]

We use Newton Raphson method to solve the three non-linear equations:

\[ f(\beta) = 2 \sum_{i=1}^{n} \left[ \ln \left( -\ln \left(1 - (F(t_i))^{\frac{1}{\beta-1}}\right) \right) \right] \]

\[ - \Psi \ln \theta - \Psi \ln t_i \]

\[ g(\Psi) = 2 \sum_{i=1}^{n} \left( -\ln \theta - \ln t_i \right) \]

\[ h(\theta) = 2 \sum_{i=1}^{n} \left[ \ln \left( -\ln \left(1 - (F(t_i))^{\frac{1}{\beta-1}}\right) \right) \right] \left( -\frac{\Psi}{\theta} \right) \]

Now, we must find the Jacobean matrix J, because the above three equations are nonlinear equations so they are hard to solve where Matrix J is the first derivatives for each equations \( f(\beta), g(\Psi) \) and \( h(\theta) \) with respect to the parameters \( \beta, \Psi \) and \( \theta \).

\[ J = \begin{bmatrix} \frac{\partial f(\beta)}{\partial \beta} & \frac{\partial f(\beta)}{\partial \Psi} & \frac{\partial f(\beta)}{\partial \theta} \\ \frac{\partial g(\Psi)}{\partial \beta} & \frac{\partial g(\Psi)}{\partial \Psi} & \frac{\partial g(\Psi)}{\partial \theta} \\ \frac{\partial h(\theta)}{\partial \beta} & \frac{\partial h(\theta)}{\partial \Psi} & \frac{\partial h(\theta)}{\partial \theta} \end{bmatrix} \]

\[ \left( \frac{\partial c_i^2}{\partial \beta \partial \Psi} \right) = 2 \sum_{i=1}^{n} \left[ \ln \left( -\ln \left(1 - (F(t_i))^{\frac{1}{\beta-1}}\right) \right) \right] \left( -\frac{\Psi}{\theta} \right) \]

\[ \left( \frac{\partial c_i^2}{\partial \beta \partial \theta} \right) = 2 \sum_{i=1}^{n} \left[ \ln \left( -\ln \left(1 - (F(t_i))^{\frac{1}{\beta-1}}\right) \right) \right] \left( -\ln \theta - \ln t_i \right) \]

\[ \left( \frac{\partial c_i^2}{\partial \Psi \partial \theta} \right) = 2 \sum_{i=1}^{n} \left[ \ln \left( -\ln \left(1 - (F(t_i))^{\frac{1}{\beta-1}}\right) \right) \right] \left( -\frac{\Psi}{\theta} \right) \]
\[-((-\beta^{-2} (F(t_i))^{\beta-1} \ln(F(t_i))) (-\beta^{-2} \ln(F(t_i))) (-\beta^{-2} + \ln(1 - (F(t_i))^{\beta-1})) (-\beta^{-2} (F(t_i))^{\beta-1} \ln(F(t_i))) (-\beta^{-2}))
\]
\[+ 2 \left[ \frac{-\beta^{-2} (F(t_i))^{\beta-1} \ln(F(t_i))}{\ln(1-(F(t_i))^{\beta-1})} \right] \left( \ln(1 - (F(t_i))^{\beta-1}) \right)^2 \left( 1 - (F(t_i))^{\beta-1} \right)^2 \]
\( (26) \)

\[\frac{\partial e_i^2}{\partial \beta \gamma} = 2 \sum_{i=1}^{n} \left[ \frac{-\beta^{-2} (F(t_i))^{\beta-1} \ln(F(t_i))}{\ln(1-(F(t_i))^{\beta-1})} \right] (-\ln(t_i)) \] (27)

\[\frac{\partial e_i^2}{\partial \beta \theta} = 2 \sum_{i=1}^{n} \left[ \frac{-\beta^{-2} (F(t_i))^{\beta-1} \ln(F(t_i))}{\ln(1-(F(t_i))^{\beta-1})} \right] \left( -\frac{\psi}{\theta} \right) \] (28)

\[\frac{\partial e_i^2}{\partial \gamma \theta} = 2 \sum_{i=1}^{n} \left[ (-\ln(t_i) - \ln(t)) \left[ \frac{-\beta^{-2} (F(t_i))^{\beta-1} \ln(F(t_i))}{\ln(1-(F(t_i))^{\beta-1})} \right] \right] \] (29)

\[\frac{\partial e_i^2}{\partial \theta \gamma \theta} = 2 \sum_{i=1}^{n} \left[ (-\ln(t_i) - \ln(t)) \right] \] (30)

\[\frac{\partial e_i^2}{\partial \gamma \gamma} = 2 \sum_{i=1}^{n} \left[ (-\ln(t_i) - \ln(t)) \right] \] (31)

\[\frac{\partial e_i^2}{\partial \beta \beta} = 2 \sum_{i=1}^{n} \left[ \frac{-\beta^{-2} (F(t_i))^{\beta-1} \ln(F(t_i))}{\ln(1-(F(t_i))^{\beta-1})} \right] \] (32)

\[\frac{\partial e_i^2}{\partial \beta \gamma} = 2 \sum_{i=1}^{n} \left[ (-\ln(t_i) - \ln(t)) \right] \] (33)

\[\frac{\partial e_i^2}{\partial \gamma \gamma} = 2 \sum_{i=1}^{n} \left[ \ln(1 - (F(t_i))^{\beta-1}) - \Psi \ln(t) - \Psi \ln(t_i) \right] \left( -\frac{\psi}{\theta} \right) 
+ \left( -\frac{\psi}{\theta} \right) \left( -\ln(t_i) - \ln(t) \right) \] (34)

4. Rank set sampling estimator method

The third method to estimate the parameters of exponentiated weibull distribution is rank set sampling which symbolic by (RSS). The idea of this method is to apply order statistical this distribution and finally utilize the maximum likelihood function to find the parameters of this distribution by maximize these parameters.

\[g(yi) = \frac{n_i}{(i-1)(n-i)!} \left[ F(yi) \right]^{i-1} \left[ 1 - F(yi) \right]^{n-i} f(yi) \] (35)

where,
\[f(yi) = \beta \Psi \theta \Psi y_i^{(\psi+1)} e^{-(\theta y_i)^{\psi}} \left( 1 - e^{-(\theta y_i)^{\psi}} \right)^{\beta-1} \]
\[F(yi) = \left( 1 - e^{-(\theta y_i)^{\psi}} \right)^{\beta} \]

Let \[\frac{n_i}{(i-1)(n-i)!} = k \]
\[g(yi) = k \left( 1 - e^{-(\theta y_i)^{\psi}} \right)^{\beta-1} \left( 1 - e^{-(\theta y_i)^{\psi}} \right)^{\beta-1} \left( 1 - e^{-(\theta y_i)^{\psi}} \right)^{\beta-1} \left( 1 - e^{-(\theta y_i)^{\psi}} \right)^{\beta-1} \]

the likelihood function of the sample is:
$$L(t;\beta,\Psi,\theta)=k^n\pi \left(1-e^{-(\psi y_i)\Psi}\right)^{\beta(i-1)} \prod_{i=1}^n \left(1 - e^{-(\psi y_i)\Psi}\right)^{\beta} - \theta \sum_{i=1}^n y_i \psi + (\beta i - 1) \sum_{i=1}^n \ln \left(1 - e^{-(\psi y_i)\Psi}\right)^{\beta}$$

Now, taking the logarithm for the likelihood function, we get the following function:

$$\ln L = \ln k + \ln n + \sum_{i=1}^n \ln y_i \Psi + (\Psi - 1) \sum_{i=1}^n \ln y_i - \theta \sum_{i=1}^n y_i \psi + (\beta i - 1) \sum_{i=1}^n \ln \left(1 - e^{-(\psi y_i)\Psi}\right)^{\beta}$$

Now, taking the partial derivatives for the log-likelihood function with respect to unknown parameters $\beta, \Psi$ and $\theta$ are:

$$\frac{\partial \ln L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^n i \ln \left(1 - e^{-(\psi y_i)\Psi}\right) - \sum_{i=1}^n (n - i) \frac{\left(1 - e^{-(\psi y_i)\Psi}\right) \ln \left(1 - e^{-(\psi y_i)\Psi}\right)}{\left(1 - e^{-(\psi y_i)\Psi}\right)^{\beta}}$$

$$\frac{\partial \ln L}{\partial \Psi} = \frac{n}{\Psi} + \ln \theta + \sum_{i=1}^n \ln y_i - \sum_{i=1}^n (\psi y_i) \ln (\psi y_i) - \sum_{i=1}^n (\beta i - 1) \frac{e^{-(\psi y_i)\Psi} - (\psi y_i)^{\Psi-1} y_i}{(1 - e^{-(\psi y_i)\Psi})^{\beta}}$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - \Psi (\psi y_i)^{\Psi-1} y_i + \sum_{i=1}^n (\beta i - 1) \frac{e^{-(\psi y_i)\Psi} - (\psi y_i)^{\Psi-1} y_i}{(1 - e^{-(\psi y_i)\Psi})^{\beta}}$$

Let:

$$f(\beta) = \frac{n}{\beta} + \sum_{i=1}^n i \ln \left(1 - e^{-(\psi y_i)\Psi}\right) - \sum_{i=1}^n (n - i) \frac{\left(1 - e^{-(\psi y_i)\Psi}\right) \ln \left(1 - e^{-(\psi y_i)\Psi}\right)}{\left(1 - e^{-(\psi y_i)\Psi}\right)^{\beta}}$$

$$g(\Psi) = \frac{n}{\Psi} + \ln \theta + \sum_{i=1}^n \ln y_i - \sum_{i=1}^n (\psi y_i) \ln (\psi y_i) - \sum_{i=1}^n (\beta i - 1) \frac{e^{-(\psi y_i)\Psi} - (\psi y_i)^{\Psi-1} y_i}{(1 - e^{-(\psi y_i)\Psi})^{\beta}}$$

$$h(\theta) = \frac{n}{\theta} - \Psi (\psi y_i)^{\Psi-1} y_i + \sum_{i=1}^n (\beta i - 1) \frac{e^{-(\psi y_i)\Psi} - (\psi y_i)^{\Psi-1} y_i}{(1 - e^{-(\psi y_i)\Psi})^{\beta}}$$

Now, we must find the Jacobean matrix $J$, because the above three equations are nonlinear equations so they are hard to solve where Matrix $J$ is the first derivatives for each equations $f(\beta), g(\Psi)$ and $h(\theta)$ with respect to the parameters $\beta, \Psi$ and $\theta$. 


\[
\frac{\partial \ln L}{\partial \beta \partial \beta} = -\frac{n}{\beta^2} \sum_{i=1}^{n} (n - i) \left( \frac{\ln \left( 1 - e^{-\theta y_i} \right)^\beta \left( 1 - e^{-\theta y_i} \right)^\beta - 1 e^{-\theta y_i} \psi(\theta y_i) \ln(\theta y_i)}{\left( 1 - e^{-\theta y_i} \right)^\beta} \right) + \frac{\ln \left( 1 - e^{-\theta y_i} \right) \psi(\theta y_i) \ln(\theta y_i)}{\left( 1 - e^{-\theta y_i} \right)^\beta} 
\]

(39)

\[
\frac{\partial \ln L}{\partial \psi \partial \beta} = \sum_{i=1}^{n} \frac{e^{-\theta y_i} \psi(\theta y_i) \ln(\theta y_i)}{1 - e^{-\theta y_i}} \left( 1 - (1 - e^{-\theta y_i})^\beta \right) \left( 1 - (1 - e^{-\theta y_i})^\beta \right) - \sum_{i=1}^{n} (n - i) \left( 1 - (1 - e^{-\theta y_i})^\beta \right) \left( 1 - (1 - e^{-\theta y_i})^\beta \right)
\]

(40)

\[
\frac{\partial \ln L}{\partial \psi \partial \psi} = \sum_{i=1}^{n} \frac{e^{-\theta y_i} \psi(\theta y_i) \ln(\theta y_i)}{1 - e^{-\theta y_i}} \left( 1 - (1 - e^{-\theta y_i})^\beta \right) \left( 1 - (1 - e^{-\theta y_i})^\beta \right) - \sum_{i=1}^{n} (n - i) \left( 1 - (1 - e^{-\theta y_i})^\beta \right) \left( 1 - (1 - e^{-\theta y_i})^\beta \right)
\]

(41)
\[ \ln(1 - e^{-(\theta y_i)}) + (1 - (1 - e^{-(\theta y_i))^{\beta}})(1 - e^{-(\theta y_i))^{\beta-1}} e^{-(\theta y_i)^{\beta}}(\theta y_i)^{\gamma} \ln(\theta y_i) \] 
\[ = \beta(1 - e^{-(\theta y_i))^{\beta}} e^{-(\theta y_i)^{\beta}}(\theta y_i)^{\gamma} \ln(\theta y_i) \ln(1 - e^{-(\theta y_i))^{\beta}}) \] 
\[ + \frac{\partial \ln l}{\partial \theta} = -n \frac{\psi}{\theta} - \left( \theta \sum_{i=0}^{n} y_i \right) \left( \ln \left( \theta \sum_{i=0}^{n} y_i \right) \right)^2 \] 
\[ \sum_{i=1}^{n} (\beta_i - 1)(1 - e^{-(\theta y_i))^{\beta}}(\theta y_i)^{\beta-2} (e^{-(\theta y_i))^{\beta}}((\theta y_i)^{\gamma}(\ln(\theta y_i) \right)^2 + e^{-(\theta y_i))^{\beta}}(\theta y_i)^{\gamma} \ln((\theta y_i) \right)^2 \] 
\[ \frac{\partial \ln l}{\partial \theta} = \frac{n}{\theta} - \left( \theta \sum_{i=0}^{n} y_i \right) \left( \ln \left( \theta \sum_{i=0}^{n} y_i \right) \right)^2 \] 
\[ + \frac{\beta(1 - e^{-(\theta y_i))^{\beta}} e^{-(\theta y_i)^{\beta}}(\theta y_i)^{\gamma} (\theta y_i)^{\gamma} \ln((\theta y_i) \right)^2 + \beta^2 (1 - e^{-(\theta y_i))^{\beta}})^2 \] 
\[ \frac{\ln((\theta y_i))^{\gamma} \ln(\theta y_i) \right)^2 - \left( \theta \sum_{i=0}^{n} y_i \right) \left( \ln \left( \theta \sum_{i=0}^{n} y_i \right) \right)^2 \] 
\[ \frac{\partial \ln l}{\partial y_i} = \frac{n}{\theta} - \left( \theta \sum_{i=0}^{n} y_i \right) \left( \ln \left( \theta \sum_{i=0}^{n} y_i \right) \right)^2 \] 
\[ + \frac{\beta(1 - e^{-(\theta y_i))^{\beta}} e^{-(\theta y_i)^{\beta}}(\theta y_i)^{\gamma} (\theta y_i)^{\gamma} \ln((\theta y_i) \right)^2 + \beta^2 (1 - e^{-(\theta y_i))^{\beta}})^2 \] 
\[ \frac{\ln((\theta y_i))^{\gamma} \ln(\theta y_i) \right)^2 - \left( \theta \sum_{i=0}^{n} y_i \right) \left( \ln \left( \theta \sum_{i=0}^{n} y_i \right) \right)^2 \] 
\[ \frac{\partial \ln l}{\partial \theta} = \frac{n}{\theta} - \left( \theta \sum_{i=0}^{n} y_i \right) \left( \ln \left( \theta \sum_{i=0}^{n} y_i \right) \right)^2 \] 
\[ + \frac{\beta(1 - e^{-(\theta y_i))^{\beta}} e^{-(\theta y_i)^{\beta}}(\theta y_i)^{\gamma} (\theta y_i)^{\gamma} \ln((\theta y_i) \right)^2 + \beta^2 (1 - e^{-(\theta y_i))^{\beta}})^2 \] 
\[ \frac{\ln((\theta y_i))^{\gamma} \ln(\theta y_i) \right)^2 - \left( \theta \sum_{i=0}^{n} y_i \right) \left( \ln \left( \theta \sum_{i=0}^{n} y_i \right) \right)^2 \] 
\[ \frac{\partial \ln l}{\partial y_i} = \frac{n}{\theta} - \left( \theta \sum_{i=0}^{n} y_i \right) \left( \ln \left( \theta \sum_{i=0}^{n} y_i \right) \right)^2 \]
\[
\psi(\psi y)^{\psi-1} \left[ \beta \left( 1 - e^{-(\psi y)^{\psi}} \right)^{\beta-1} e^{-(\psi y)^{\psi}} (\psi y_i)^{\psi y_i} - \beta \left( 1 - e^{-(\psi y)^{\psi}} \right)^{\beta-1} e^{-(\psi y)^{\psi}} \right]
\]

\[
\frac{\partial \ln l}{\partial \beta} = \sum_{i=1}^{n} \frac{(- \psi (\psi y)^{\psi-1})}{(1 - e^{-(\psi y)^{\psi}})^2} - (n - i) \sum_{i=1}^{n} \frac{1 - (1 - e^{-(\psi y)^{\psi}})^{\beta}}{(1 - e^{-(\psi y)^{\psi}})^2}
\]

\[
\frac{\beta (1 - e^{-(\psi y)^{\psi}})^{\beta-1} \ln \left( 1 - e^{-(\psi y)^{\psi}} \right) e^{-(\psi y)^{\psi}} \psi (\psi y)^{\psi-1} y_i + (1 - e^{-(\psi y)^{\psi}})^{\beta-1} e^{-(\psi y)^{\psi}} y_i}{(1 - (1 - e^{-(\psi y)^{\psi}})^{\beta})^2}
\]

\[
\frac{\partial \ln l}{\partial \psi} = \frac{- \psi (\psi y)^{\psi-1} \sum_{i=0}^{n} y_i \ln \left( \theta \sum_{i=0}^{n} y_i \right) + \left( \theta \sum_{i=0}^{n} y_i \right)^{\psi-1} \sum_{i=0}^{n} y_i + \left( \beta - 1 \right) \sum_{i=1}^{n} \frac{1 - e^{-(\psi y)^{\psi}}}{(1 - e^{-(\psi y)^{\psi}})^2} e^{-(\psi y)^{\psi}} (\psi y)^{\psi-1} y_i + e^{-(\psi y)^{\psi}} (\psi y)^{\psi-1} y_i + e^{-(\psi y)^{\psi}} \psi (\psi y)^{\psi-1} \ln (\psi y) y_i - e^{-(\psi y)^{\psi}} \psi (\psi y)^{\psi-1} y_i + e^{-(\psi y)^{\psi}} \psi (\psi y)^{\psi-1} \ln (\psi y) y_i - e^{-(\psi y)^{\psi}} \psi (\psi y)^{\psi-1} y_i}{(1 - e^{-(\psi y)^{\psi}})^2} - (n - i) \sum_{i=1}^{n} \frac{1 - (1 - e^{-(\psi y)^{\psi}})^{\beta}}{(1 - e^{-(\psi y)^{\psi}})^2}
\]

\[
\frac{\beta (1 - e^{-(\psi y)^{\psi}})^{\beta-2} (- \psi y)^{(\psi y)^{\psi-1}) \ln (\psi y) (\psi y)^{(\psi y)^{\psi-1}) y_i}{(1 - e^{-(\psi y)^{\psi}})^2}
\]

\[
\frac{\beta (1 - e^{-(\psi y)^{\psi}})^{\beta-1} e^{-(\psi y)^{\psi}} (\psi y_i)^{(\psi y_i)^{\psi y_i}) + (1 - e^{-(\psi y)^{\psi}})^{\beta-1} e^{-(\psi y)^{\psi}}}{(1 - (1 - e^{-(\psi y)^{\psi}})^{\beta})^2}
\]

\[
\frac{\psi (\psi y)^{\psi-1} y_i + \beta (1 - e^{-(\psi y)^{\psi}})^{\beta-1} e^{-(\psi y)^{\psi}} (\psi y_i)^{(\psi y_i)^{\psi y_i}) y_i + (1 - e^{-(\psi y)^{\psi}})^{\beta-1} e^{-(\psi y)^{\psi}}}{(1 - (1 - e^{-(\psi y)^{\psi}})^{\beta})^2}
\]

\[
\frac{e^{-(\psi y)^{\psi}} \psi (\psi y)^{\psi-1} y_i (\psi y_i)^{(\psi y_i)^{\psi y_i}) + (1 - e^{-(\psi y)^{\psi}})^{\beta-1} e^{-(\psi y)^{\psi}} (\psi y_i)^{(\psi y_i)^{\psi y_i}) \ln (\psi y)}{(1 - (1 - e^{-(\psi y)^{\psi}})^{\beta})^2}
\]

\[
\frac{\partial \ln l}{\partial \theta \theta} = - \frac{n \psi}{\theta^2} - \psi (\psi - 1) \left( \theta \sum_{i=0}^{n} y_i \right)^{\psi-2} \left( \sum_{i=0}^{n} y_i \sum_{i=0}^{n} y_i + \frac{\beta (1 - e^{-(\psi y)^{\psi}})^{\beta-1} e^{-(\psi y)^{\psi}} (\psi y_i)^{(\psi y_i)^{\psi y_i}) y_i + (1 - e^{-(\psi y)^{\psi}})^{\beta-1} e^{-(\psi y)^{\psi}}}{(1 - (1 - e^{-(\psi y)^{\psi}})^{\beta})^2} \right)
\]
\[(\beta t - 1) \sum_{i=1}^{n} \frac{(1 - e^{-(\theta y_i)^w})(e^{-(\theta y_i)^w})(\psi - 1)(\theta y_i)^{w-2}y_i^2 + \psi'(\theta y_i)^{w-1}y_i}{(1 - e^{-(\theta y_i)^w})^2} \]
\[
e^{-(\theta y_i)^w}(-\psi'(\theta y_i)^{w-1}y_i) - \frac{e^{-(\theta y_i)^w} \psi'(\theta y_i)^{w-1}y_i}{(1 - e^{-(\theta y_i)^w})^2} \frac{(\beta - 1)(1 - e^{-(\theta y_i)^w})^{\beta - 2}}{\beta (1 - e^{-(\theta y_i)^w})^{\beta - 1}} \]
\[
\frac{(-e^{-(\theta y_i)^w}(-\psi'(\theta y_i)^{w-1}y_i)e^{-(\theta y_i)^w} \psi'(\theta y_i)^{w-1}y_i + \beta (1 - e^{-(\theta y_i)^w})^{\beta - 1}e^{-(\theta y_i)^w}}{(1 - e^{-(\theta y_i)^w})^2} \]
\[
\psi(\psi - 1)(\theta y_i)^{w-2}y_i y_i)(\theta y_i)^{w-2} - (\beta (1 - e^{-(\theta y_i)^w})^{\beta - 1}e^{-(\theta y_i)^w} \psi'(\theta y_i)^{w-1}y_i)
\]

\[\frac{(-\beta (1 - e^{-(\theta y_i)^w})^{\beta - 1}e^{-(\theta y_i)^w}(-\psi'(\theta y_i)^{w-1}y_i)}{(1 - (1 - e^{-(\theta y_i)^w})^\beta)} \] (47)

Where, the Jacobian matrix in three above methods is non-singular matrix, then \(\frac{\partial f(\beta)}{\partial \psi} = \frac{\partial g(\psi)}{\delta \beta}, \frac{\partial f(\beta)}{\partial \theta} = \frac{\partial h(\theta)}{\delta \beta}\) and \(\frac{\partial g(\psi)}{\partial \theta} = \frac{\partial h(\theta)}{\delta \psi}\). Therefore:
\[
\begin{bmatrix}
\beta_{k+1} \\
\Psi_{k+1} \\
\theta_{k+1}
\end{bmatrix} =
\begin{bmatrix}
\beta_k \\
\Psi_k \\
\theta_k
\end{bmatrix} - J^{-1}
\begin{bmatrix}
f(\beta) \\
g(\Psi) \\
h(\theta)
\end{bmatrix}
\]

5. Simulation

Mean square error is calculated here to evaluate the performance of estimates, the numerical process is done by using MATLAB (R2018a) programming language to applied simulation and estimation method.

The simulation processing has been passed the following important steps, we mention them here:
1. Generate 1000 from EW distribution of sizes; \(n = 10, 20, 30, 50\)
2. Choose values of parameters \((\beta, \Psi\) and \(0)\)
3. Used Newton-Raphson method to solve the three non-linear equations in three methods
   Maximum likelihood estimator method, ordinary least squares estimator method and rank set sampling estimator method.
4. The mean square error for each sample for all parameters in Exponentiated Weibull distribution has been estimated.
Table (1). The MSE values where \( (\beta = 2, \Psi = 2, \theta = 2) \)

| n  | MLE \((\beta, \Psi, \theta)\) | OLS \((\beta, \Psi, \theta)\) | RSSE \((\beta, \Psi, \theta)\) |
|----|-------------------------------|-----------------------------|-----------------------------|
| 10 | 0.517522 38.87086 0.023581     | 0.002059 0.003667 0.000334    | 0.005227 0.614606 0.051475 |
| 20 | 0.521078 15.23267 0.41248      | 0.007233 0.039194 0.002407    | 0.022966 0.790349 0.431754 |
| 30 | 0.387334 11.59555 0.185622     | 0.006143 0.0774 0.002149      | 0.02764 0.620817 0.303365 |
| 50 | 0.003478 0.203483 0.084106     | 0.000404 0.000867 0.001551    | 0.000676 0.027581 0.030946 |

Table (2). The MSE values where \( (\beta = 2, \Psi = 2.5, \theta = 2) \)

| n  | MLE \((\beta, \Psi, \theta)\) | OLS \((\beta, \Psi, \theta)\) | RSSE \((\beta, \Psi, \theta)\) |
|----|-------------------------------|-----------------------------|-----------------------------|
| 10 | 0.197024 148.5682 0.932233     | 0.067891 0.235204 0.003923    | 0.017042 14.167 0.506254    |
| 20 | 0.372176 15.03869 0.543889     | 0.019854 0.035503 0.003886    | 0.025975 1.123875 0.280472 |
| 30 | 0.358774 11.86231 0.715672     | 0.017686 0.014611 0.003489    | 0.010285 0.218516 0.224953 |
| 50 | 0.001267 0.00772 0.306757      | 0.001294 0.07473 0.002478     | 0.000119 0.000793 0.061572 |
Table (3). The MSE values where ($\beta = 2.5$, $\Psi = 2$, $\theta = 2$)

| n   | MLE ($\beta$, $\Psi$, $\theta$) | OLS ($\beta$, $\Psi$, $\theta$) | RSSE ($\beta$, $\Psi$, $\theta$) |
|-----|---------------------------------|----------------------------------|----------------------------------|
| 10  | 8.771176                        | 328.6812                         | 0.001217                         |
|     | 0.009208                        | 0.221586                         | 0.003701                         |
|     | 0.066323                        | 4.541116                         | 0.241761                         |
| 20  | 0.149801                        | 49.83201                         | 0.004697                         |
|     | 0.020949                        | 0.042417                         | 0.002871                         |
|     | 0.006264                        | 1.20223                          | 0.160961                         |
| 30  | 0.005386                        | 0.002587                         | 0.063658                         |
|     | 0.001165                        | 0.088844                         | 0.002069                         |
|     | 0.000279                        | 0.000281                         | 0.073748                         |
| 50  | 0.002358                        | 0.041431                         | 0.068042                         |
|     | 0.000659                        | 0.002379                         | 0.002466                         |
|     | 0.000185                        | 0.002974                         | 0.018476                         |

Table (4). The MSE values where ($\beta = 2.5$, $\Psi = 2.5$, $\theta = 2$)

| n   | MLE ($\beta$, $\Psi$, $\theta$) | OLS ($\beta$, $\Psi$, $\theta$) | RSSE ($\beta$, $\Psi$, $\theta$) |
|-----|---------------------------------|----------------------------------|----------------------------------|
| 10  | 2.921874                        | 43.20291                         | 0.094021                         |
|     | 0.015449                        | 0.384606                         | 0.005417                         |
|     | 0.024817                        | 8.363208                         | 0.430476                         |
| 20  | 0.003495                        | 27.61399                         | 0.031992                         |
|     | 0.013884                        | 0.06109                          | 0.004598                         |
|     | 0.000359                        | 0.177448                         | 0.113915                         |
| 30  | 0.004568                        | 0.008441                         | 0.066665                         |
|     | 0.009512                        | 0.021075                         | 0.003313                         |
|     | 0.000335                        | 0.000658                         | 0.036091                         |
| 50  | 0.003653                        | 0.090822                         | 0.113722                         |
|     | 0.004853                        | 0.004509                         | 0.002568                         |
|     | 0.000281                        | 0.004248                         | 0.025534                         |

6. Conclusion

1. From table (1) we found that MLE method and RSSE method have the smallest MSE (6 times for each of them).
2. From table (2) we found that MLE method has the smallest MSE (9 times) and RSSE method (3 times).
3. From table (3) we found that MLE method has the smallest MSE (7 times) and RSSE method (4 times) and OLS method (once).
4. From table (4) we found that MLE method has the smallest MSE (7 times) and RSSE method (5 times).
5. Finally from above results we found that MLE method has the smallest MSE.
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