REGULARITY OF EXTREMAL SOLUTIONS
OF A LIOUVILE SYSTEM

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Abstract. Let \( \Omega \subset \mathbb{R}^n \) be a bounded smooth open set. We prove that the extremal solution of the system
\[
-\Delta u = \mu e^{\theta u + (1-\theta)v}, \quad -\Delta v = \lambda e^{\theta v + (1-\theta)u}
\]
in \( \Omega \),
with \( u = v = 0 \) on \( \partial \Omega \), \( \theta \) in \([0, 1]\) and \( \mu, \lambda \geq 0 \) are smooth if \( n \leq 9 \).

1. Introduction. In this article we consider the issue of regularity of extremal solutions to the Liouville system
\[
\begin{align*}
-\Delta u &= \mu e^{\theta u + (1-\theta)v} = g(u, v) \quad \text{in } \Omega, \\
-\Delta v &= \lambda e^{\theta v + (1-\theta)u} = f(u, v) \quad \text{in } \Omega, \\
u = v &= 0 \quad \text{on } \partial \Omega,
\end{align*}
\]
with \( \Omega \) a bounded smooth open subset of \( \mathbb{R}^n \), \( \lambda, \mu \) nonnegative parameters, and \( \theta \) in \([0, 1]\). In the sequel we write \( g(u, v) = g \) and \( f(u, v) = f \) for the sake of simplicity.

This system is a generalization of the equation
\[
\begin{align*}
-\Delta u &= \nu e^u \quad \text{in } \Omega, \\
u = 0 &= \text{on } \partial \Omega,
\end{align*}
\]
where \( \nu \) denotes a positive parameter. For this single equation, it is well known that there is a maximal parameter \( \nu^* > 0 \) for existence of solutions of (2) and for \( 0 < \nu < \nu^* \) there is a minimal solution \( u_\nu \), i.e. a positive solution which is above any positive supersolution. As \( \nu \to \nu^* \), \( \nu < \nu^* \) the solution \( u_\nu \) converges to the so-called extremal solution, which turns out to be smooth for \( n \leq 9 \), see [3, 10].

The interested reader may find in the book [6] the developments of the theory for the last six decades, with a particular focus on stable solutions. The stable solution approach was introduced in [9] to classify the solutions of the Lane-Emden equation (see also [12, 4]). For the regularity issue for the extremal solutions of the biharmonic equation see [8].

For systems as (1), M. Montenegro [11] proved the existence of a nonempty open set \( \mathcal{U} \) in the quarter plane \( \lambda, \mu > 0 \) such that for a couple of parameters \( (\mu, \lambda) \) in \( \mathcal{U} \) there is a smooth minimal solution \((u, v)\) and no solution exists if the couple is in the complement of \( \overline{\mathcal{U}} \). Minimality means \( u \leq \bar{u} \) and \( v \leq \bar{v} \) in \( \Omega \) for any other smooth solution \((\bar{u}, \bar{v})\) for the same \((\mu, \lambda)\). For each slope \( m > 0 \), \( \mathcal{U} \) intersected with the line \( \mu = m \lambda \) is a segment \( \{(m \lambda, \lambda) : \lambda \in (0, \lambda^*(m))\} \) and at the extremal point \((m \lambda^*(m), \lambda^*(m))\) in \( \partial \mathcal{U} \) there is a solution, called the extremal solution.
defined as the limit as \( \lambda \uparrow \lambda^*(m) \) of the minimal solution with parameters \((m, \lambda)\) and it may be singular.

Now and throughout this article we assume without loss of generality that \( \lambda \geq \mu \). Let us discuss first the case \( \lambda = \mu \). In this case we will see in the sequel that for minimal solution \((u, v)\) then \( u = v \) and we are led to a single equation. In high dimension \( n \geq 10 \) and for \( \lambda = \mu \), where (1) reduces to (2), it is known that extremal solutions are not regular [6]. Besides, for \( n \leq 9 \) the extremal solution is smooth.

In the particular case \( \theta = 1 \), the system decouples

\[-\Delta u = \mu u, \quad -\Delta v = \lambda v \quad \text{in } \Omega,\]

then since \( \lambda \geq \mu \) we have that \( v \geq u \). Therefore we can prove \( \lambda^* = \nu^* \), and we are back to the single equation where extremal solution is smooth if and only if \( n \leq 9 \). In the particular case \( \theta = 0 \), we have that \( \lambda u = \mu v \) and that

\[-\Delta(u + v)^2 = \lambda + \mu u + v \quad \text{in } \Omega, \quad \text{(3)}\]

Then we are back to a single equation and we can prove that \( \frac{\lambda^* + \mu^*}{2} = \nu^* \) and that \((u, v)\) are smooth if \( n \leq 9 \).

In the present article we plan to generalize this result for all \( \theta \) in \([0, 1]\).

**Theorem 1.1.** Fix \( \theta \) in \([0, 1]\). Assume \( n \leq 9 \) and let \((u, v)\) be an extremal solution of the Liouville system (1). Then \((u, v)\) are smooth functions.

Let us recall for the sake of completeness that an extremal solution \((u, v)\) satisfies (1) in the sense that \( u, v \in L^1(\Omega), g dist(\cdot, \partial \Omega), f dist(\cdot, \partial \Omega) \in L^1(\Omega), \) and

\[
\int_\Omega u(-\Delta \varphi) = \int_\Omega g \varphi, \quad \int_\Omega v(-\Delta \varphi) = \int_\Omega f \varphi, \quad \text{for all } \varphi \in C^2(\Omega) \text{ with } \varphi = 0 \text{ on } \partial \Omega.
\]

The rest of the article is devoted to the proof of Theorem 1.1. We first recall a useful inequality which is valid for stable solutions of the system; this was essentially proved in C. Cowan, N. Ghoussoub [2] and L. Dupaigne, A. Farina, B. Sirakov [7]. We then state comparison results between \( u \) and \( v \). We then complete the proof of Theorem 1.1.

### 2. A useful inequality for stable solutions.

From [11] we know that for \((\mu, \lambda) \in \mathcal{U}, \) the associated minimal solution \((u, v)\) of (1), which is smooth, is stable in the sense that there exist \( \varphi_1, \psi_1 : \Omega \to \mathbb{R}, \) smooth and positive in \( \Omega, \) satisfying

\[
\begin{cases}
-\Delta \varphi_1 - \theta g \varphi_1 - (1 - \theta) g \psi_1 = \eta \varphi_1 & \text{in } \Omega, \\
-\Delta \psi_1 - (1 - \theta) f \varphi_1 - \theta f \psi_1 = \eta \psi_1 & \text{in } \Omega, \\
\varphi_1 = \psi_1 = 0 & \text{on } \partial \Omega, \quad \text{(4)}
\end{cases}
\]

for some \( \eta > 0 \). We first prove a stability inequality, that follows the guidelines in [2] and [7].
Lemma 2.1. Let \((u, v)\) be a smooth stable solution of the system (1). For any \(\varphi\) in \(H^1_0(\Omega)\), for any \(\varepsilon\) in \([0, 1]\),
\[
\theta \int_\Omega (\varepsilon f + (1 - \varepsilon)g)\varphi^2 + 2(1 - \theta)\sqrt{\varepsilon(1 - \varepsilon)} \int_\Omega \sqrt{fg}\varphi^2 \leq \int_\Omega |\nabla \varphi|^2.
\]  
(5)

Proof. We carbon copy here the proof in [7]. Consider \(\varphi\) a smooth compactly supported function in \(\Omega\). Consider the scalar product of the second equation in (4) with \(\varphi \psi_1\). Appealing Piccione’s inequality
\[
\int_\Omega -\Delta \psi_1 \varphi^2 \leq \int_\Omega |\nabla \varphi|^2.
\]  
(6)
we then have
\[
\theta \int_\Omega f\varphi^2 + (1 - \theta) \int_\Omega \frac{\varphi_1}{\psi_1}\varphi^2 \leq \int_\Omega |\nabla \varphi|^2.
\]  
(7)
Proceeding analogously for the first equation and summing the resulting inequalities we then have
\[
\theta \int_\Omega (\varepsilon f + (1 - \varepsilon)g)\varphi^2 + (1 - \theta) \int_\Omega (\varepsilon f\frac{\varphi_1}{\psi_1} + (1 - \varepsilon)g\frac{\psi_1}{\varphi_1})\varphi^2 \leq \int_\Omega |\nabla \varphi|^2.
\]  
(8)
To observe that
\[
\varepsilon f\frac{\varphi_1}{\psi_1} + (1 - \varepsilon)g\frac{\psi_1}{\varphi_1} \geq 2\sqrt{\varepsilon(1 - \varepsilon)}\sqrt{fg},
\]  
completes the proof.

As a by-product of the Lemma we have

Corollary 1. Let \((u, v)\) be a smooth stable solution of the system (1). For any \(\varphi\) in \(H^1_0(\Omega)\)
\[
\sqrt{\lambda \mu} \int_\Omega \exp\left(\frac{u + v}{2}\right)\varphi^2 = \int_\Omega \sqrt{fg}\varphi^2 \leq \int_\Omega |\nabla \varphi|^2,
\]  
(9)
\[
\theta \int_\Omega f\varphi^2 \leq \int_\Omega |\nabla \varphi|^2.
\]  
(10)

Proof. These are mere consequences of the previous lemma for respectively \(\varepsilon = \frac{1}{2}\) (using \(f + g \geq 2\sqrt{fg}\)) and respectively \(\varepsilon = 1\).

Remark 1. Balanced inequality (9) that is uniform in \(\theta\) is the original inequality proved in [2] and [7]. This inequality plays a central role in the proof of the result for \(\theta = 0\).

3. Comparison. We recall that here and throughout the remaining of the article that \(\lambda \geq \mu\). Hence it will be useful later to have the following inequalities between the components of a solution of (1).

Lemma 3.1. Assume \(\lambda \geq \mu\). Then for any smooth minimal solution to the Liouville system (1) we have
\[
u \leq u.
\]  
(11)
Proof. The proof is based on the sub and super solution method of Perron (see [6] and the references therein). The minimal solution \((u, v)\) of the Liouville system can
be reached as the limit of the sequence \((u_k, v_k)\) defined recursively by \(u_0 = v_0 = 0\) and by

\[
\begin{aligned}
-\Delta u_{k+1} &= g(u_k, v_k) \quad \text{in } \Omega, \\
-\Delta v_{k+1} &= f(u_k, v_k) \quad \text{in } \Omega, \\
\text{and } u_{k+1} = v_{k+1} = 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]  

(12)

It is standard to prove recursively that both sequences \(u_k\) and \(v_k\) are pointwise nondecreasing and that they remain above any supersolution of the system in \(\Omega\).

Consider the minimal solution \((u, v)\) of the Liouville system. Introduce now the open subset of \(\Omega\) defined as \(\omega = \{x \in \Omega; v(x) < u(x)\}\). For \(x\) in \(\omega\) we have that \(-\Delta v = \lambda e^{\theta v + (1-\theta)u} \geq \lambda e^v\). Therefore the couple \((v, v)\) is a supersolution for the Liouville system restricted to \(\omega\). We now prove recursively in \(k\) that \(u_k \leq v\). This is true for \(k = 0\). Let us observe then that \(-\Delta(v - u_{k+1}) \geq g(v, v) - g(u_k, v_k) \geq 0\) in \(\omega\). At the boundary \(\partial \omega\) we have that \(v = u \geq u_{k+1}\). Then \(u_k \leq v\) for any \(k\) and passing to the limit \(u \leq v\) in \(\omega\). Therefore \(\omega\) is an empty set. 

**Remark 2.** As a consequence of this Lemma we have that if \(\lambda = \mu\) then \(u = v\) and we are back to a single equation.

We state another result that is void if \(\theta \geq \frac{1}{2}\) but useful if \(\theta < \frac{1}{2}\).

**Lemma 3.2.** Assume \(\lambda \geq \mu\). Then for any smooth minimal solution to the Liouville system (1) we have

\[g(u, v) \leq f(u, v).\]  

(13)

Then if \(\theta < \frac{1}{2}\) we have that

\[v \leq u + \frac{1}{1-2\theta} \log \frac{\lambda}{\mu}.\]  

(14)

**Proof.** Since \(v \geq u\) and \(\lambda \geq \mu\) then \(f \geq g\) if \(\theta \geq \frac{1}{2}\). We then focus on \(\theta < \frac{1}{2}\). Consider the function \(w = v - u - \frac{1}{1-2\theta} \log \frac{\lambda}{\mu}\). Introduce the function \(a(x) = f e^{(1-2\theta)w - \frac{1}{1-2\theta} \frac{\lambda}{\mu}}\) that is nonnegative. Then \(w\) is solution to

\[-\Delta w + a(x) w = 0,\]  

(15)

and such that \(w = -\frac{1}{1-2\theta} \log \frac{\lambda}{\mu} \leq 0\) on \(\partial \Omega\). Then \(w \leq 0\) in \(\Omega\) by the maximum principle. 

In the case \(\theta \geq \frac{1}{2}\) we state another comparison result.

**Lemma 3.3.** Assume \(\lambda \geq \mu\). Assume \(\theta \geq \frac{1}{2}\). Then for any smooth minimal solution to the Liouville system (1) we have that

\[
\lambda u \leq \mu v,
\]  

(16)

and that therefore for \(\kappa = \frac{(1-\theta)\lambda + \theta \mu}{(1-\theta)\mu + \theta \lambda}\), then

\[
\lambda^\kappa g(u, v) \leq \mu f(u, v)^\kappa.
\]  

(17)

**Proof.** We observe that \(w = \mu v - \lambda u\) is solution to

\[-\Delta w = 2\lambda e^{\frac{\lambda u}{\mu}} \sinh((\theta - \frac{1}{2})(v - u)).\]  

(18)
Due to Lemma 3.1 $v \geq u$ and the right hand side of this inequality is nonnegative. Since $w = 0$ on $\partial \Omega$ then the maximum principle completes the proof of inequality (16). To prove (17) amounts to prove that
\[
[(1 - \theta)\mu + \theta\lambda)(1 - \theta)v + \theta u)] \leq [(1 - \theta)\lambda + \theta\mu)(1 - \theta)u + \theta v],
\]
Observing that this is valid if $\lambda u \leq \mu v$ and $\theta \geq \frac{1}{2}$ completes the proof.

4. Completing the proof of Theorem 1.1. The strategy of the proof is as follows. We prove a priori $L^\infty$ estimates on smooth minimal solutions $(u, v)$. These $L^\infty$ bounds may depend on $(\mu, \lambda)$ but remains bounded when $(\mu, \lambda)$ converges towards a point $(\mu^*, \lambda^*)$ in the extremal curve. Hence we have that the extremal solutions are bounded in $L^\infty$ and then smooth by classical elliptic results. Since the case $\lambda = \mu$ is obvious, we may consider minimal solutions for $\mu = m\lambda$ with $m < 1$.

We now assume without loss of generality that $0 < \theta < 1$ since the cases $\theta = 0$ and $\theta = 1$ are already done (see the references quoted above). We recall that $(u, v)$ is a smooth stable solution of (1).

To begin with, we multiply $-\Delta v = f$ by $e^{\alpha v} - 1$, that is in $H^1_0(\Omega)$. Integrating by parts we have
\[
\int_\Omega f(e^{\alpha v} - 1) = \int_\Omega -\Delta v(e^{\alpha v} - 1) = \frac{4}{\alpha} \int_\Omega |\nabla(e^{\frac{\alpha}{2} v} - 1)|^2.
\]
Appealing the unbalanced stability inequality (7) we then have
\[
\frac{4\theta}{\alpha} \int_\Omega f(e^{\frac{\alpha}{2} v} - 1)^2 \leq \int_\Omega f e^{\alpha v}.
\]
We easily infer from (21) that for $\alpha < 4\theta$
\[
(\frac{4\theta}{\alpha} - 1)^2 \int_\Omega f e^{\alpha v} \leq (\frac{8\theta}{\alpha})^2 \int_\Omega f.
\]
Appealing Cauchy-Schwarz inequality we then have
\[
(\frac{4\theta}{\alpha} - 1)^2 \int_\Omega f e^{\alpha v} \leq (\frac{8\theta}{\alpha})^2 \int_\Omega f.
\]
We now apply the comparison inequality (11) that leads to $f \leq \lambda e^v$ and then
\[
(\frac{4\theta}{\alpha} - 1)^2 \int_\Omega f^{1+\alpha} \leq \lambda^\alpha (\frac{8\theta}{\alpha})^2 \int_\Omega f.
\]
Therefore by Hölder inequality
\[
(\frac{4\theta}{\alpha} - 1)^2 (\int_\Omega f^{1+\alpha})^{\frac{1+\alpha}{\alpha}} \leq \lambda^\alpha (\frac{8\theta}{\alpha})^2 (\text{Vol} \Omega)^{\frac{1}{1+\alpha}}.
\]
We sum up this computations by the following statement: for $\alpha < 4\theta$, then $f$ belongs to $L^{1+\alpha}(\Omega)$.

First case. $\theta \leq \frac{1}{2}$.

We iterate the result (25). Multiply $-\Delta u = g$ by $e^{\alpha u} - 1$, that is in $H^1_0(\Omega)$. Integrating by parts we have
\[
\int_\Omega g(e^{\alpha u} - 1) = \int_\Omega -\Delta u(e^{\alpha u} - 1) = \frac{4}{\alpha} \int_\Omega |\nabla(e^{\frac{\alpha}{2} u} - 1)|^2.
\]
Appealing the stability inequality (9) we then have
\[
\frac{4}{\alpha} \int_\Omega \sqrt{g} (e^{\frac{\alpha}{2} u} - 1)^2 \leq \int_\Omega g e^{\alpha u}.
\]
Appealing (13) we then have
\[
\frac{4}{\alpha} \int_{\Omega} g(e^{\frac{\alpha}{2}} - 1)^2 \leq \int_{\Omega} ge^{\alpha u}, \tag{28}
\]
and then, for \(\alpha < 4\)
\[
\left(\frac{4}{\alpha} - 1\right) \int_{\Omega} ge^{\alpha u} \leq \frac{8}{\alpha} \int_{\Omega} ge^{\frac{\alpha}{2}}. \tag{29}
\]
Since \(g \leq f\), using once again (13), then \(g\) belongs to \(L^1(\Omega)\) and \(ge^{\alpha u}\) belongs to \(L^{1+\alpha}(\Omega)\) for any \(\alpha < 4\).

Using the comparison (14) we then have that \(g\) belongs to \(L^{1+\alpha}(\Omega)\) for any \(\alpha < 4\).

Using Sobolev embeddings, since \(-\Delta u\) belongs to \(L^{1+\alpha}(\Omega)\) for any \(\alpha < 4\), then if \(n \leq 9\) we have that the solution \(u\) belongs to \(L^\infty(\Omega)\). Then \(v\) is also bounded by comparison inequality (14).

Remark 3. Our proof does not work for \(\theta = 0\). For this special case we refer to [7], [5].

Second case. \(\theta > \frac{1}{2}\).

Let us recall that we already have treated the case \(\lambda^* = \mu^*\). Then here consider a sequence \((m\lambda, \lambda)\) that converges towards \((m\lambda^*, \mu^*)\) for a given \(m < 1\).

Assume then \(\mu = m\lambda < \lambda\). Then if \(\theta > \frac{1}{2}\) we have that \(\kappa = \frac{(1-\theta)\lambda + \theta \mu}{(1-\theta)\mu + \theta \lambda} < 1\).

Multiply \(-\Delta v = f\) by \(f^{\alpha} - \lambda^\alpha\), that is in \(H^1_0(\Omega)\). Integrating by parts we have
\[
\alpha \int_{\Omega} f^{\alpha-1} \nabla f. \nabla v \leq \int_{\Omega} f^{\alpha+1}. \tag{30}
\]

Multiply \(-\Delta u = g\) by \(f^{\alpha} - \lambda^\alpha\), that is in \(H^1_0(\Omega)\). Integrating by parts we have
\[
\alpha \int_{\Omega} f^{\alpha-1} \nabla f. \nabla u \leq \int_{\Omega} gf^\alpha. \tag{31}
\]

Since \(\nabla f = f(\theta \nabla v + (1-\theta)\nabla u)\), then gathering (30) and (31) yields
\[
\frac{4}{\alpha} \int_{\Omega} |\nabla (f^{\frac{\alpha}{2}} - \lambda^{\frac{\alpha}{2}})|^2 = \alpha \int_{\Omega} f^{\alpha-2} |\nabla f|^2 \leq \theta \int_{\Omega} f^{\alpha+1} + (1-\theta) \int_{\Omega} gf^\alpha. \tag{32}
\]

Appealing the unbalanced stability inequality (7) we have that for \(\alpha < 4\)
\[
\left(\frac{4}{\alpha} - 1\right) \theta \int_{\Omega} f^{\alpha+1} \leq \frac{8}{\alpha} \lambda^{\frac{\alpha}{2}} \int_{\Omega} f^{1+\frac{\alpha}{2}} + (1-\theta) \int_{\Omega} gf^\alpha. \tag{33}
\]

We now use (17) and then
\[
\left(\frac{4}{\alpha} - 1\right) \theta \int_{\Omega} f^{\alpha+1} \leq \frac{8}{\alpha} \lambda^{\frac{\alpha}{2}} \int_{\Omega} f^{1+\frac{\alpha}{2}} + (1-\theta) \frac{\mu}{\lambda^\alpha} \int_{\Omega} f^{\alpha+\kappa}. \tag{34}
\]

Due to Hölder inequality, and since we already now that \(f\) belongs to \(L^1(\Omega)\) (see (25)) we infer that for \(\alpha < 4\) then
\[
\int_{\Omega} f^{1+\alpha} \leq C(\alpha, \lambda, \mu)(\int_{\Omega} f)^{1+\alpha}. \tag{35}
\]

Then \(f\) belongs to \(L^{1+\alpha}\) for \(\alpha < 4\) and this completes the proof of the Theorem.

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