Abstract

Graph Convolution Networks (GCNs) are becoming more and more popular for learning node representations on graphs. Though there exist various developments on sampling and aggregation to accelerate the training process and improve the performances, limited works focus on dealing with the dimensional information imbalance of node representations. To bridge the gap, we propose a method named Dimensional Reweighting Graph Convolution Network (DrGCN). We theoretically prove that our DrGCN can guarantee to improve the stability of GCNs via mean field theory. Our dimensional reweighting method is very flexible and can be easily combined with most sampling and aggregation techniques for GCNs. Experimental results demonstrate its superior performances on several challenging transductive and inductive node classification benchmark datasets. Our DrGCN also outperforms existing models on an industrial-sized Alibaba recommendation dataset.

1 Introduction

Deep neural networks (DNNs) have been widely applied in various fields, including computer vision [He et al. 2016, Hu et al. 2018], natural language processing [Devlin et al. 2018], and speech recognition [Abdel-Hamid et al. 2014], among many others. Graph neural networks (GNNs) have been proposed for learning node presentations for networked data [Scarselli et al. 2009], and later have been extended with graph convolutions that can better capture the topological information in a network [Kipf and Welling 2017]. Since then, GCNs begin to attract wide interests. For example, GraphSage [Hamilton et al. 2017] defines the convolutional neural network (CNN) based graph learning framework as sampling and aggregation. Vast follow-up works further enhance the sampling or aggregation process via various techniques such as the attention mechanism [Velickovic et al. 2018] and the adaptive sampling mechanism [Huang et al. 2018].

Our work studies a very important phenomenon, referred to as dimensional information imbalance, where information of different layers in the GCNs are very imbalanced. This phenomenon severely limits the capacity of GCNs. For example, on Reddit, using different weighting schemes for different layers in a GCN would result in quite different performances. A carefully chosen weighting scheme can reduce the error rate by even 40%. To address this problem, we propose Dimensional Reweighting Graph Convolutional Networks (DrGCNs), in which the input of each hidden layer has been reweighted using local receptive field information. We found that with a simple reweighting scheme, the performance of GCNs can be significantly improved on five benchmark data sets. We also give theoretical analyses via the mean field theory [Kadanoff 2009, Yang et al. 2019] and prove that DrGCN can guarantee to improve the stability of GCNs. To further validate its effec-
tiveness, we deployed the proposed DrGCNs on Alibaba company’s recommendation system and clearly demonstrated performance improvements via offline A/B tests.

2 Preliminaries

Notations. In this paper, we mainly focus on the undirected graph $G = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_i\}$ represents the node set, $\mathcal{E} = \{(v_i, v_j)\}$ represents the edge set, and $X$ represents the node features. For a certain layer of the GCN, we use $\mathbf{R}^{in} = (\mathbf{r}^{in}_1, \ldots, \mathbf{r}^{in}_n)$ to denote the input node representations and $\mathbf{R}^o = (\mathbf{r}^o_1, \ldots, \mathbf{r}^o_n)$ to denote the output representation. For the whole layer-stacked GCN structure, we use $\mathbf{H}^0$ to denote the input node representation and $\mathbf{H}^l$ to denote the output node representation of the $l$th layer. We use $\mathbf{A}$ to denote the adjacency matrix, where $a_{ij} = 1$ when $(v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise.

Graph Convolutional Networks (GCNs). A general graph convolutional layer can be viewed as the following, given input node set of $n$ nodes $\mathcal{V}$, the adjacency matrix $\mathbf{A}$, and the input representations of each node $\mathbf{R}^{in}$, we need to generate the output representations $\mathbf{R}^o$ for the GCN layer:

$$\mathbf{R}^o = \sigma(\text{aggregator}(\mathbf{R}^{in}, \mathbf{A})), \quad (1)$$

where the aggregator function can be arbitrarily complex that aggregates information according to neighborhood of each node, and $\sigma$ is the activation function. Although there exists non-linear aggregators like the LSTM aggregator [Hamilton et al. 2017], in most GCN variants the aggregator is a linear function which can be viewed as a weighted sum of node representations among the neighborhood [Kipf and Welling 2017], [Huang et al. 2018], followed by a matrix multiplication with a bias added. We denote $\mathbf{A}$ as the aggregation matrix, $\mathbf{W}$ as the projection matrix and $\mathbf{b}$ as the bias vector. We use equation (2) to formulate such procedure:

$$\mathbf{R}^o = \sigma(\mathbf{W}\mathbf{R}^{in} \hat{\mathbf{A}} + \mathbf{b}). \quad (2)$$

To improve the scalability of GCN, some GCN variants contain a sampling procedure, which samples a subset of neighborhood for aggregation [Chen et al. 2018], [Huang et al. 2018]. Sampling-based GCNs still lie within the framework of equation (2), since we can simply set all unsampled edges to 0 in $\mathbf{A}$. Development on GCNs mainly lies on different ways to generate the aggregation matrix $\hat{\mathbf{A}}$. GCNs [Kipf and Welling 2017] proposed some variants including simply taking $\hat{\mathbf{A}} = \mathbf{A}\mathbf{D}^{-1}$, which is uniform average among neighbors or weighted by degree of each node $\hat{\mathbf{A}} = \mathbf{D}^{-\frac{1}{2}}\mathbf{A}\mathbf{D}^{-\frac{1}{2}}$, or including self-loops $\hat{\mathbf{A}} = \mathbf{I} + \mathbf{D}^{-\frac{1}{2}}\mathbf{A}\mathbf{D}^{-\frac{1}{2}}$. Other methods include using attention [Veličković et al. 2018], or gated attention [Zhang et al. 2018], or even neural architecture search methods [Gao et al. 2019] to generate $\hat{\mathbf{A}}$.

3 DrGCNs: Dimensional reweighting Graph Convolutional Networks

3.1 Model Formulation

Given the node representations of the graph $\mathbf{R}^{in}$, DrGCN tries to learn a dimensional reweighting vector $\mathbf{s} = (s_1, \ldots, s_d)$, where $s_i$ is an adaptive scalar for each dimension $i$. This reweighting vector $\mathbf{s}$ then helps reweight each dimension of the node representation $\mathbf{r}^v$ as $\mathbf{r}^{r v}_v$, $v \in \mathcal{V}$, where $\mathbf{r}^{r v}_v = \mathbf{r}^{in}_v \odot \mathbf{s}$. Here we use $\odot$ to denote component-wise multiplication, i.e.,

$$r^{r v}_{v,j} = s_j r^{in}_{v,j}, \quad \forall 1 \leq j \leq d, \forall v \in \mathcal{V}. \quad (3)$$

Hence for any $\mathbf{x}$, $\mathbf{s} \odot \mathbf{x} = \mathbf{Sx}$, where $\mathbf{S}$ is the diagonal matrix with diagonal entries consisting components of $\mathbf{s}$. Then the dimensional reweighting graph convolutional layer can be formulated as equation (4):

$$\mathbf{R}^o = \sigma(\mathbf{W}\mathbf{R}^{in} \hat{\mathbf{A}} + \mathbf{b}). \quad (4)$$

\footnote{For the convenience of our analysis, we use a column of $\mathbf{R}$ instead of a row, to represent a node representation.}

\footnote{$\mathbf{D}$ is the degree matrix, $D_{ij} = \text{deg}(i)\delta(i,j)$}
Inspired by SENet [Hu et al. 2018], we represent the calculation process of shared dimensional reweighting vector $s$ in two stages. First we calculate a global representation $r$ in $v$, whose value is the expected value of $r$ in $v$. Then we feed $r$ in $v$ into an encoder-decoder network structure to produce $s$ with the same dimension size. Equation (5) denotes the procedure to generate $s$ given node weight $\{w_v|v \in V, \sum_{v \in V} w_v = 1\}$ and node representations $\{r_v\in v|v \in V\}$:

$$
\begin{align*}
    r_v &= E[r_v|v \in V] = \sum_{v \in V} w_v r_v, \\
    g &= \sigma_g(W_g r + b_g), \\
    s &= \sigma_s(W_s g + b_s),
\end{align*}
$$

where $g$ is the bottleneck representation whose dimension is much less than $r$ and $W_g, b_g, W_s, b_s$ are parameters to be learnt. To summarize, Figure 1 illustrates the Dimensional reweighting block in Graph Convolutional Networks.

Combining With Existing GCNs. Our dimensional reweighting method can be implemented as an independent functional process and easily combined with GCNs. As shown in equation (4), our method only works on $R$ and does not involve in the procedure of calculating $\tilde{A}$, $W$ and $b$ explicitly. Thus, our method can be combined with any existing sampling or aggregation methods without causing any contradictions. We combine the DrGCN with multiple types of existing sample and aggregation GCN methods in experiments. Finally, GCNs are usually stacking of multiple GCN layers, so as our DrGCN structure. Suppose that the input features are $X$, DrGCN can be viewed as equation (6), where $H^0 = X$ and use $H^k$ as output representation for a k-layer DrGCN:

$$
H_l^i = \sigma_l(W_l S_l^i H^{i-1}_l \tilde{A}_l + b^l), \forall 1 \leq l \leq k
$$

### 3.2 Theoretical Analyses

In this section, we first demonstrate that our method reduces the learning variance brought by perturbations on the input, or GCN aggregators, making the update more stable in the long run. Then we develop a metric to quantify the improvement for dimensional reweighting.

Mean Field Perspective. Mean field theory is an approximation approach in the case where the dimension of the data representation is extremely high. In this approach, rather than dealing with high dimensional data, we use a continuous parameter to model them. More specifically, we use the mean field approximation [Yang et al. 2019] to analyze fully connected networks. Fully connected networks can be viewed as a special case of GCN with $\tilde{A} = I$ and this analysis can be generalized. Mean field approximation basically takes the average over degrees of freedom, and is valid on relatively large data set. We also assume the mean of the data is 0 in the analysis below, since translation will not affect the covariance structure. For simplicity we only consider the case that our neuron network has a constant width $N$ and assume they all use the same activation function $\sigma^3$. Thus our recursive relation is:

[1] Yang et al. [2019] uses a pre-activation recurrence relation $h_i^l = W_i^l \sigma_i(S_i^l \circ h_{i-1}^l) + b_i^l$, while we use the standard post-activation recurrence relation.
\[ H^l = \phi(W^l S^l H^{l-1} + b^l). \]

In the mean field approximation, we replace the input data by a normal random variable with the same mean and variance. When \( N \to \infty \), this approximation becomes exact. Define:

\[ V_\phi(S^l, C^{l-1}) = E[\phi(W^l S^l h + b^l)\phi(W^l S^l h + b^l)^T], \]

where \( h \sim N(0, C^{l-1}) \), \( \phi \) is the ReLU activation, and \( C^l \) represents the covariance matrix of \( H^l \). Then with mean field approximation, the covariance structure can be updated as:

\[ C^l = V_\phi(S^l, C^{l-1}). \]

The key observation here is, we can take \( S^l h \) as a whole, which follows \( N(0, S^l C^{l-1} S^l) \) distribution. Thus the recurrence relationship can be written as:

\[ C^l = V_\phi(I, S^l C^{l-1} S^l). \]

The derivative of \( C^l \) measures the fluctuation of our updating algorithm when input perturbations exist, hence it characterizes the sensitivity of the algorithm to the data structures and its robustness. We will turn to show that DrGCN can relieve this sensitivity. We will fix the point \( C^l \) where we are taking derivative at, and it is worthwhile to point out that for most common activation functions, this recursive map will have a fixed point, at which this linearization will be most useful. Recall that such a derivative will be a linear map from symmetric matrices to symmetric matrices. We define

\[ J_\phi(C) := \frac{dV_\phi(I, C)}{dC}|_{C = C_1}. \]

Here \( C_1 \) could be intuitively understood as the increment near \( C^l \). We denote by \( H_d \) the space of symmetric matrices of size \( d \times d \). Under these notations, we prove that:

**Theorem 1.** There exist diagonal matrices \( S^l \), constants \( 0 < \gamma^l < 1 \) such that, \( \|J_\phi(S^l C S^l)\|_F \leq \gamma^l \|J_\phi(C)\|_F \|S^l\|_F \) for any fixed general \( C \). By general, we mean there exists a Haar measure on the collection of symmetric matrices \( H_d \) with respect to which the statement fails has measure zero.\[4\]

Detailed proofs and explanations are included in the appendix. For symmetric matrices, with \( \lambda_i \) denoting eigenvalues of \( A \), we have:

\[ \|A\|_F^2 = \sum_{i=1}^d \lambda_i^2. \]

This norm measures the overall magnitude of eigenvalues of the operator. This result demonstrates that our method brings variance reduction and thus stability of the updating algorithms. To summarize, for any input data, there exists a vector \( s \) that improves the stability of the updating algorithms.

**Stability Measure for Dimensional Reweighting.** Next we turn to define a quantified measurement of the improvement of the stability. We define:

\[ K = \frac{\sum_i c_{ii} s_{ii}^2}{\sum_i c_{ii} - \frac{1}{d} \sum_{i,j} c_{ij} s_{ii} s_{jj}} - \frac{1}{d} \sum_{i,j} c_{ij} s_{ij} s_{ij} - \frac{1}{d} \sum_{i,j} c_{ij} + \frac{1}{d^2} \sum_{i} s_{ii}^2, \]

where \( c_{ij} \) is the \((i, j)\)th element of the covariance matrix \( C \), that is \( c_{ii} = \text{Var}(R_i), c_{ij} = \text{cov}(R_i, R_j) \). Notice that, this quantity only involves the covariance structure of the data set; and it is homogeneous of degree 0 with respect to them, thus it is invariant under any linear transformation. Consequently, we could proceed our analyses under the dimensional normalized assumption without loss of generality. We turn to consider the dimensional normalized version of \( V_\phi \) by replacing \( \phi \) with \( d_\phi \), and shift to pre-activation formulation (we move the \( b^l \) in the previous definition of \( V_\phi \) out):

\[ C^l = V_{d_\phi}(S^l, C^{l-1}) + b^l, \quad d_\phi : \mathbb{R}^d \to \mathbb{R}^d, d_\phi(h) = \phi\left(\frac{\sqrt{d}G S^l h}{\|G h\|}\right), \]

where \( G = I - \frac{1}{d} \mathbf{1}\mathbf{1}^T \) i.e. \( G x = x - \mu \mathbf{1} \), \( \mu = \frac{1}{d} \sum_i x_i \).

\[4\] \( \| \cdot \|_F \) is the Frobenius norm, i.e. for a matrix \( A = (A_{ij}) \), \( \|A\|_F^2 = \sum_{i,j} A_{ij}^2 \).

\[5\] \( \mathbf{1} \) is the \( d \)-dimensional vector with all component 1.
Theorem 2. Near the fixed point $C^*$ of $\mathcal{V}_{d_\theta}$, the exponential growth rate of the deviation of $C^i$ from $C^*$ is proportional to $K$.

Here $C^*$ is used to denote the as BSB1 (Batch Symmetry Breaking 1 or 1 Block Symmetry Breaking) fixed point[2019] of $\mathcal{V}_{d_\theta}(I, C)$. Here we take the latter one. Since $S^i h$ has covariance matrix $S^i C S^i$ now, our scaling effect is that $\mathcal{V}_{d_\theta}(S, C) = \mathcal{V}_{d_\theta}(I, S^i C S^i)$. From now on we use the definition $C_S := G C G^T$. Here $S^i$ is the diagonal matrix with entries equaling to components of $s^i$, our reweighting vector of $l$-th layer.

Now we use the notation $K(S, C) = S C S$. From[2019] we know that the derivative of $\mathcal{V}_{d_\theta}(I, C)$ (as a linear map) has a very explicit eigenspace decomposition, which will be described below.

Theorem 3. $J_{d_\theta} := \frac{d\mathcal{V}_{d_\theta}}{dC^*}$ at $C^*$ has eigenspaces and eigenvalues:

1. $V_0 = \{ C_0 : C_G = 0 \}$, with eigenvalue 0.
2. $V_G = \mathbb{R}G$, with eigenvalue $\lambda_G$.
3. A $(d-1)$ dimensional eigenspace $V_L = \{ D^G : D$ diagonal, $trD = 0 \}$.
4. A $\frac{d(d-3)}{2}$-dimensional eigenspace $V_M = \{ C : C_G = C, diag C = 0 \}$ with eigenvalue $\lambda_M$. $\lambda_L, \lambda_M < 1$, whereas $\lambda_G > 1$.

All of these eigenvalues could be computed explicitly[2019]. So the task is to choose $S$ appropriately to reduce the proportion that lies in $V_G$. We prove that the Frobenius norm of the component in $V_G$ will be proportional to $K$ in appendix. Thus, it is natural to consider the orthogonal (in terms of Frobenius norm and corresponding inner product) eigendecomposition (with subindices indicating the corresponding eigenspaces we listed above):

$$K(S, C) = C_0 + C_G + C_L + C_M.$$  \hspace{1cm} (15)

So, the effect of our reweighting is, at each step, we artificially reduce the $\mathbb{R}G$-component to make the dynamic system more stable. Since the decomposition is orthogonal, this is equivalent to reduce $G_{ti} := <S^i CS^i, G>$, recall that $G = I - \frac{1}{d} \bar{t} \bar{t}^T$, i.e. $G x = x - \frac{1}{d} \sum x_i$. In addition, since we take the normalization assumption, so only the relative magnitude of $s_{ti}$ matters, we could put any homogeneous restriction. In order to include the case $s_{ti} = 1$, we consider the restriction $\sum_{i=1}^d s_{ti}^2 = d$. By definition $C_{ij} = E[h_i h_j]$, hence

$$<S^i CS^i, G> = Tr(S^i CS^i(I - \frac{1}{d} \bar{t} \bar{t}^T)^T)$$

$$= Tr(S^i CS^i) - \frac{1}{d} \sum_{i,j} c_{ij}s_{ti}s_{tj}.$$  \hspace{1cm} (17)

In order to measure how close our $S^i$ in practice to the theoretical prediction we consider:

$$K = \frac{\sum_i c_{ii}s_{ti}^2 - \frac{1}{d} \sum_{i,j} c_{ij}s_{ti}s_{tj}}{(\sum_i c_{ii} - \frac{1}{d} \sum_{i,j} c_{ij}) \times \sum_i s_{ti}^2}. \hspace{1cm} (18)$$

The denominator is chosen as the value of the numerator with all $s_{ti}$ equaling to 1. This ratio is 1 when all $s_{ti}$s are 1, or being the same, which is what we can achieve even without the reweighting. From our calculation on the inner product, we know this quantity is proportional to the part in $V_G$ in the orthogonal decomposition. Since this is the only part for $J_{d_\theta}$ with eigenvalue larger than 1, the exponential growth rate will be proportional to this quantity. Therefore, how much this quantity being smaller than 1 measures the amount of improvement our dimensional reweighting makes to the stability of the learning process under perturbation.

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Footnotes:

6 All results involve the BSB1[2019] fixed point are purely linear algebra, and hold for dimensional normalization, too.

7 $G, S$ above are symmetric, so the transpose is only introduced for the sake of notational balance.
4 Experiments

In this section, we evaluate the proposed DrGCN on a variety of datasets compared to several state-of-the-art methods. The detailed description of the experiments, as well as the pseudo code for reproducing the experiments are included in the supplementary materials.

4.1 Experimental Settings

We present the performance of our model on 5 public benchmark node classification datasets, including citation networks (Cora [Ding et al. 1999], Yang et al. 2016], Citeseer [Lawrence et al. 1999], Yang et al. 2016], Pubmed [Sen et al. 2008], Yang et al. 2016], Reddit [Hamilton et al. 2017], PPI [Subramanian et al. 2005], Hamilton et al. 2017] and a large-scale real-world commercial recommendation Alibaba dataset. Table 1 summarizes statistics of the datasets. All of the experiments for citation networks in the main article are under the fully-supervised setting, where the training set contains all node labels except for the nodes in the validation or test set. For another widely adopted semi-supervised setting on citation networks, where training nodes being a small fraction of the whole graph compared with validation or test set, we provide the results of our models together with various baselines including ManiReg [Belkin et al. 2006], SemiEmb [Weston et al. 2012], LP [Zhu et al. 2003], DeepWalk [Perozzi et al. 2014], ICA [Lu and Getoor 2003], Planetoid [Yang et al. 2016], GraphSGAN [Ding et al. 2018], Chebyshev [Defferrard et al. 2016], MoNet [Monti et al. 2017], DPFCNN [Monti et al. 2018], Mix-Hop [Abu-El-Haija et al. 2019] in supplementary material. For PPI we compare our Dr-GAT method with several state-of-the-art methods on that dataset including GraphSAGE [Hamilton et al. 2017], LGCN [Gao et al. 2018], GeniePath [Li et al. 2018], GaAN [Zhang et al. 2018] and GraphNAS [Gao et al. 2019].

Methods Combined with Dr block We combine and compare our Dimensional Reweighting block with four most representative GCN methods on public datasets. Two full GCN methods include the vanilla GCN [Kipf and Welling 2017], and a variant that exploits an attention aggregator GAT [Veličković et al. 2018]. Two sampling GCN methods include FastGCN [Chen et al. 2018] and Adaptive Sampling GCN [Huang et al. 2018]. As for Alibaba dataset, we combine and compare our method with the company’s previous best GraphSAGE model. We change every graph convolutional layer of such models to a DrGCN as in Equation (4) while keeping all other settings the same as the original model. For further implementation details, see appendix.

4.2 Results and Analytics

Table 2 illustrate the performance of our dimensional reweighting methods on four public transductive datasets when being combined with four different variations of GCN models. Our results are averaged among 20 runs with different random seeds. Our Dimensional reweighting method matches SOTA performances on citation networks, and shows great improvement towards the base methods it combines with on Reddit dataset. Our Dr-ASGCN even reduces the error rate by more than 40% (3.6% → 2.1%), compared with previous state-of-the-art method ASGCN.

The performance improvements can be explained and even, to some extent, predicted by our stability measurement proposed in equation 18. Theoretically when $K \approx 1$, we expect the learnt vector to

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Table 1: Dataset statistics.

|          | Cora | Citeseer | Pubmed | PPI      | Reddit | Alibaba |
|----------|------|----------|--------|----------|--------|---------|
| Nodes/Users | 2,708 | 3,327    | 19,717 | 56,944   | 232,965| 35,246,808 |
| Edges     | 5,429 | 4,732    | 44,338 | 818,716  | 11,606,919 | 129,834,116 |
| Classes/Items | 7     | 6        | 3      | 121      | 41     | 6,338,428 |
| Features  | 1,433 | 3,703    | 500    | 50       | 602    | 27      |
| Training Nodes | 1,208 | 1,812    | 18,217 | 44,906   | 152,410| 35,246,808 |
| Validation Nodes | 500   | 500      | 500    | 6,514    | 23,699 |-        |
| Test Nodes  | 1,000 | 1,000    | 1,000  | 5,524    | 55,334 | 35,246,808 |

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\(^{9}\)learning rate, early stop criteria, loss function, hidden representation dimension, batch size, weight decay
Table 2: Summary of classification accuracy on public transductive datasets(\%).

| Category       | Method           | Cora   | Citeseer | Pubmed | Reddit |
|----------------|------------------|--------|----------|--------|--------|
| Full GCNs      | GCN Kipf and Welling \[2017\] | 86.4±0.3 | 77.4±0.2 | 86.4±0.3 | -      |
|                | GAT Velickovic et al. \[2018\] | 87.2±0.4 | 77.8±0.2 | 82.6±0.3 | -      |
| Sampling-based | GaAN Zhang et al. \[2018\] | -      | -        | -      | 96.4   |
| GCNs           | GraphSage Hamilton et al. \[2017\] | 82.2   | 71.4     | 87.1   | 94.3   |
|                | FastGCN Chen et al. \[2018\] | 83.9±0.4 | 78.6±0.4 | 87.6±0.6 | 92.5±0.2 |
|                | ASGCN Huang et al. \[2018\] | 87.2±0.2 | 79.0±0.4 | 89.8±0.3 | 96.4±0.3 |
| Dr GCNs(ours)  | Dr-GCN           | 86.8±0.2 | 77.5±0.3 | 86.8±0.2 | -      |
|                | Dr-GAT           | 87.4±0.2 | 77.8±0.2 | 82.5±0.3 | -      |
| Dr Sampling-based | Dr-FastGCN     | 84.0±0.4 | 78.3±0.3 | 88.0±0.6 | 94.0±0.1 |
| GCNs(ours)     | Dr-ASGCN        | 87.1±0.2 | 79.1±0.4 | 90.3±0.4 | 97.9±0.1 |

Table 3: Summary of performance (micro F1) on inductive PPI dataset.

| Method          | GraphSAGE | LGCN | GeniePath | GAT | GaAN | GraphNAS | Dr-GAT(ours) |
|-----------------|-----------|------|-----------|-----|------|----------|--------------|
| micro F1        | 61.2      | 77.2 | 97.9      | 97.3| 98.7 | 98.6     | **98.8±0.1** |

have limited ability in refining the representation, while when $K \ll 1$ we expect the vector to strengthen the stability of the model by reducing the magnitude of the derivatives of the covariance matrix and improve the performance. To verify our theoretically analysis, we collect the average $K$-value of our learnt reweighting vectors for different layers in our Dr-ASGCN model, which is a two-layer SOTA sampling-based GCN model, see table[5] The $K$-value in the second layer is around 1 for all datasets. However, the $K$-value for the first layer is around 1 for citation datasets, but 0.32 on Reddit dataset, which strongly explains why our Dimensional reweighting method achieves such a big improvement on Reddit dataset.

On the inductive PPI dataset(Table[3], our Dimensional reweighting method increases the micro f1-score of GAT by 1.5% and outperforms all previous methods, including dataset-specific searched neural architecture Gao et al. \[2019\].

Table[4] suggests that our dimensional reweighting method can also achieve solid improvements on the real-world large scale recommendation dataset. Our model shows improvement on industrial measure recall@50, which is the rate of users clicking the top 50 predicted items among 6 million different items in the next day of the training set, from 5.19% (previous best model) to 5.26% (Dr Block added).

4.3 Batch-norm and Layer-norm

We also compare our DrGCN with other variation reduction methods including batch normalization Ioffe and Szegedy \[2015\] and layer normalization Lei Ba et al. \[2016\]. We compare our model and these methods on Reddit dataset for ASGCN in table[6] It can be seen that batch-norm and layer-norm does improve the performance of ASGCN model on reddit dataset, but not as much as our Dimensional Reweighting method, which learns a finer reweighting vector to reweight dimensional representations.

On citation datasets, both layer normalization and batch normalization reduces the performance of models, while our Dr method does not. See appendix for further details.

4.4 Scalability of DrGCNs

The computational time cost of computing our reweighting vector is $O(nd_i)$, where $n$ is the batch size and $d_i$ is the input dimension. This is relatively small compared with the aggregation procedure of a GCN layer, which is at least $O(nd_i d_o)$ for the matrix multiplication. Table[6] also shows that training our Dr-ASGCN consumes as much time as training a plain ASGCN. Working well on industrial-level Alibaba dataset also proves the scalability of our DrGCN method.
Table 4: Performance (Recall@50) on Alibaba online recommendation system. GraphSAGE refers to Alibaba company’s best heterogeneous GraphSAGE model.

| Method          | GraphSAGE  | Dr-GraphSAGE (ours) |
|-----------------|------------|---------------------|
| Recall@50(%)    | 5.19       | 5.26                |

Table 5: Classification accuracy of ASGCN and Dr-ASGCN model, with average K value learnt for each layer.

| Method        | Cora  | Citeseer | Pubmed | Reddit |
|---------------|-------|----------|--------|--------|
| ASGCN         | 87.23 | 78.95    | 89.82  | 96.37  |
| Dr-ASGCN (ours)| 87.07 | 79.06    | 90.34  | 97.95  |
| Improvement Rate(%) | -0.16 | 0.11    | 0.52   | 1.58   |
| Learnt K-value (Layer 1) | 1.04  | 1.01    | 0.98   | 0.32   |
| Learnt K-value (Layer 2)   | 1.00  | 1.00    | 0.98   | 1.14   |

Table 6: Accuracy and average training time per epoch for plain, batchnorm, layernorm and Dimensional Reweighting ASGCN methods on Reddit dataset.

| Method       | ASGCN          | Batch-norm | Layer-norm | Dr             | Dr+LN          |
|--------------|----------------|------------|------------|----------------|----------------|
| Accuracy(%)  | 96.37±0.22     | 96.99±0.15 | 97.68±0.15 | 97.95±0.13     | 98.02±0.12     |
| Time(s/epoch)| 17.99          | 17.90      | 18.74      | 17.46          | 17.93          |

5 Related Works

Graph Neural Network [Scarselli et al. 2009] adopts a neural network structure on graph to learn structures. Graph Convolutional Network [Kipf and Welling 2017] proposes a deep learning based method to embed node on a graph based on gathering information from neighborhood of a node. GraphSage [Hamilton et al. 2017] formulated a sample and aggregation framework of inductive node embedding. The idea of such sample and aggregation frameworks is to incorporate information from neighborhood node to generate node embedding. In short, a graph convolutional neural network is basically a fully connected neural network with sample and aggregation techniques added on top of it. Despite being uniform when first being proposed, both sampling and aggregation can be weighted. These methods, including FastGCN [Chen et al. 2018], GAT [Veličković et al. 2018], LGCN [Gao et al. 2018], ASGCN [Huang et al. 2018], GaAN [Zhang et al. 2018], and GraphNAS [Gao et al. 2019], treat all nodes in the graph unequally and try to figure out more important nodes and assign them higher weights in sampling and aggregation procedure.

Feature imbalance phenomena has long been aware of [Blum and Langley 1997] in machine learning. Different dimensions of the hidden representation generated by neural networks may also share such imbalance behavior. The idea of refining hidden representations in neural networks can be traced back to Network in Network [Lin et al. 2014], whom proposes a fully-connected neural network to refine the pixel-wise hidden representation before each convolutional layer. Known as the 1×1 convolution layer which is widely used in modern convolutional neural networks. Squeeze and Excitation Networks [Hu et al. 2018] proposes a dimensional reweighting method called Squeeze and Excitation block, which involves the techniques of global average pooling and encoder-decoder structure. It works well in computer vision CNNs and won the image classification task of Imagenet 2017. The success attracts our concern that dimensional reweighting methods might also be useful in node representation learning on graphs.

Another natural idea to refine representations of neural networks is normalization. Batch normalization [Ioffe and Szegedy 2015] is a useful technique in neural networks to normalize and reduce the variance of input representations. Layer normalization [Lei Ba et al. 2016] is an improved version, for it normalizes the hidden representations layerwise. Many also try to give theoretical analysis to such normalization techniques. Kohler et al. [2018] explains the efficiency of batch normalization in terms of convergence rate. Björck et al. [2018] shows that batch normalization enables larger learning rate. Yang et al. [2019] shows the gradient explosion behaviors of batch normalization
on fully-connected networks using mean field theory [Kadanoff 2009], in this paper we adopt its conclusion and apply it on dimensional reweighting cases.

6 Conclusion

We propose dimensional reweighting graph convolutional networks (DrGCNs) and prove that DrGCNs can improve the stability of GCN models. We conduct experiments on five benchmark data sets and compare DrGCNs with five GCN variations. Experimental results not only prove the efficiency of our dimensional reweighting method, but also support our theoretical analysis on the effectiveness of the method. Our method is also proved useful on large-scale industrial data set (Alibaba).
Appendices

A Summary of GCN Methods

Table 7 provides a summary of existing GCN methods. We show the properties (sampling, aggregation, dimensional reweighting) of various GCN methods, including our proposed Dr-GCN methods in the table.

| GCN Type | Method | Sampling and Aggregation | Dimensional Reweighting |
|----------|--------|--------------------------|-------------------------|
|          |        | Sampling | Aggregation |                  |
| Full GCNs| GCN    | Kipf and Welling [2017]  | Heuristic | Heuristic |
|          | Chebyshev | Defferrard et al. [2016] | Heuristic | Heuristic |
|          | GAT     | Velickovic et al. [2018] | Attentional | Multi-way |
|          | Mix-Hop | Abu-El-Hana et al. [2019] | Searched | / |
| Sampling GCNs | GraphSAGE | Hamilton et al. [2017] | Uniform | Heuristic |
|          | GaAN   | Zhang et al. [2018] | Weighted | Heuristic |
|          | FastGCN | Chen et al. [2018] | Attentional | Heuristic |
|          | ASGCN  | Huang et al. [2018] | Attentional | / |
| Dimensional Reweighting GCNs | Dr-GCN | Heuristic | Heuristic | Dimensional Reweighting |
|          | Dr-GAT | Heuristic | Heuristic |
|          | Dr-FastGCN | Heuristic | Heuristic |
|          | Dr-ASGCN | Heuristic | Heuristic |

B Implementation Details

B.1 Dr Block configuration

For the dimensional reweighting part of each layer, the dimension of the encoder $g$ is set to the closest integer of the square root of the dimension of the input node representation $r^i$. The pooling weight $w_v$ is set to uniform weight $w_v = \frac{1}{|V|}$, where $|V|$ is the number of nodes on the graph for full GCNs, and the batch size for batch-wise sampling GCNs. We use ELU activation for $\sigma_g$ and sigmoid activation for $\sigma_s$. We do not apply dropout or regularization in the dimensional reweighting part.

B.2 Evaluation Details

All of our methods and compared baseline methods are run 20 times and we report the average accuracy and variation for methods that we run. For methods that share the evaluation setting of ours, we use their reported performance. Otherwise we evaluate their performance based on their released code and paper.

B.3 Method Details

We describe our method implementation details here.

**GCN** [Kipf and Welling, 2017] We use the GCN code provided in AS-GCN[10] which is a 2-layer GCN model with 16 hidden units in layer 1. We run it for 20 times and report the average and variance. For each running time we use the model that performs best within 200 training epochs on validation set for testing.

**GAT** [Velickovic et al., 2018] We use the GAT code provided by the authors[11]. We use the dataset-specific structures described in [Velickovic et al., 2018] and early stopping strategy mentioned in the

[10]https://github.com/huangwb/AS-GCN
[11]https://github.com/PetarV-/GAT
code repo. The original paper uses high dropout rate of 0.6 on semi-supervised citation datasets test setting. We find that for fully-supervised test setting, such a high dropout may have a chance to lead the original GAT model not to converge, so we adjusted the dropout rate to 0.4 (which gives the best performance among all dropout rates from 0 to 0.6) on the fully-supervised setting of citation datasets for both the original and our Dr-GAT. On PPI we simply follow their instructions and use their suggested structure and hyperparameters.

GAT forms a 2 layer structure for citation datasets. For Cora and Citeseer, GAT has 8 hidden units in every attention head in the first layer, and 8 attention heads in the first layer and 1 in the second layer, which has number of hidden units equal to node classes. For Pubmed, GAT has 8 hidden units in every attention head in the first layer, and 8 attention heads in the first layer and 8 in the second layer, each second layer attention head has a number of hidden units equal to node classes.

For PPI, GAT has a three layer structure, with 256 hidden units in every attention head in the first two layers. It has 4 attention heads in the first layer and 4 in the second layer, and 6 attention heads in the third layer, each third layer attention head has a number of hidden units equal to node classes. It also sets dropout equals to 0 and uses residual connection[He et al. 2016].

We simply adopt all these dataset-specific structures for our GAT and Dr-GAT evaluation.

**FastGCN** Chen et al. [2018] We run the code provided by the authors. We use the weighted sampling method described in paper, and use a neighborhood size of 128 for cora, citeseer, 256 for pubmed and 512 for reddit. We run 20 times and generate an average performance and variations.

**ASGCN** Huang et al. [2018] We use the code provided by the authors. We use a neighborhood size of 128 for cora, citeseer, 256 for pubmed and 512 for reddit as the paper suggested. Their original code seems to have a bug causing unnecessary test set information leaking during validation process, we modified their code to avoid such problem. We choose the best model on validation set within 400 epoches for citation datasets and 200 epochs for Reddit, and use that for testing.

**Other Methods** Many of them do not have their codes released, so we simply use their reported performance in their papers, or reported performance of their method by other papers, if we share a similar evaluation setting.

Specifically, the performances of all baseline methods in appendix table 8 are from their original papers (some non-GCN baseline results are from [Kipf and Welling 2017]). The performance of GraphSAGE, GaAN, GeniePath, LGCN, GAT on PPI dataset are from their original papers.

For transductive datasets, we run 20 times and report average and standard deviation like our proposed DrGCN models for GCN, ASGCN, GAT, FastGCN. For GaAN, we use the performance reported in the original paper. As for GraphSAGE, the original code and evaluation setting does not match ours, so we use the reported performance of GraphSAGE in Huang et al. [2018], which shares the same evaluation setting with our model, to ensure fair comparison.

### C Datasets

The details of our datasets is listed in this section. We generally use 6 datasets, including 3 citation datasets, 1 Reddit dataset, 1 inductive PPI dataset, and 1 large online recommendation A* dataset.

### Citation Networks

We evaluate the performance of our DR models on the three citation network datasets, Cora [Ding et al. 1999], Citeseer [Lawrence et al. 1999] and Pubmed [Sen et al. 2008] provided by [Yang et al. 2016].

There are two types of experimental settings, the semi-supervised setting for full GCNs [Kipf and Welling 2017], [Veličković et al. 2018], which uses only a little fraction of the node labels on the graph and all link information for training. And the fully-supervised setting [Chen et al. 2018], [Huang et al. 2018], which uses node labels of the full graph except the validation and test set for training. We do experiments on both settings and provide the results on the fully-supervised setting.

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12https://github.com/matenure/FastGCN

13https://github.com/huangwb/AS-GCN

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Table 8: Dataset statistics for semi-supervised citation datasets.

|            | Cora | Citeseer | Pubmed |
|------------|------|----------|--------|
| Nodes      | 2,708| 3,327    | 19,717 |
| Edges      | 5,429| 4,732    | 44,338 |
| Classes    | 7    | 6        | 3      |
| Features   | 1,433| 3,703    | 500    |
| Training Nodes | 140 | 120      | 60     |
| Validation Nodes | 500 | 500      | 500    |
| Test Nodes | 1,000| 1,000    | 1,000  |

in table 2 for it has lower variance and is more general to GCNs. We also provide our results on semi-supervised setting on these datasets in appendix table 9.

PPI

The protein-protein interaction dataset is collected by SNAP Hamilton et al. [2017] from the Molecular Signatures Database Subramanian et al. [2005], which is an inductive multi-label node classification task. The training set contains 20 protein graphs, while the validation and test set contains two graphs each. We evaluate the performance of different models by micro F1-score.

Reddit

The Reddit dataset is collected by SNAP Hamilton et al. [2017] from Reddit posts. It is a node classification dataset for classifying different communities of each users by their posts.

D A* dataset: Dataset, Baseline and Evaluation

As for the industrial A* dataset we use. It is an item recommendation dataset, with the training set has about 35 million users and 6.3 million items with 120 million edges. Although the target is node-classification like(to find the most likely items that each user may click), instead of simply taken each item as a class, A* uses graph embedding model to generate embedding for both users and items. There are 27 user attributes and 33 item attributes. For every user, we use K nearest neighbor(KNN) with Euclidean distance to calculate the top-N items that the user is most likely to click, and the customer will see these recommended items in A* company’s APP. We use the recall@N to evaluate the model:

\[ recall@N = mean\left(\sum_u \frac{|M_u| \cap |I_u|}{|I_u|}\right) \]  

(19)

where \(M_u\) is the top-N items recommended by the model and \(I_u\) is the items clicked by the customer. The baseline model is the A* online heterogeneous GraphSAGE, and we add Dr block in it to compare Recall@N with the online model.

Recall@50 is the most commonly used metric in A* company. Experimental results show that we reach 5.264% on Recall@50, improving from the original best model’s 5.188%. It is quite a good result, considering random guess will only give less than 0.001% (50/6,300,000).

E Evaluation on Semi-supervised Settings of Citation Datasets

Besides the results of the fully supervised setting on citation networks (Cora, Citeseer and Pubmed) provided in main article, we also evaluate our Dimensional reweighting method on the semi-supervised setting of these citation datasets, and compare with all sorts of baseline methods. The dataset statistics are in Table 8 while the results are listed in Table 9. Notice that in semi-supervised setting, the number of training labels is very small compared with the fully-supervised setting, so models tend to have a higher variation. Like in Table 2 and Table 3, we provide the results of our Dr-GCN and Dr-GAT based on the average among 20 runs with different random seeds.
Table 9: Summary of classification accuracy (%) of semi-supervised labels on citation datasets.

| Category              | Method           | Cora  | Citeseer | Pubmed |
|-----------------------|------------------|-------|----------|--------|
| Non-Graph Convolution | MLP               | 55.1  | 46.5     | 71.4   |
|                       | ManiReg Belkin et al. [2006] | 59.5  | 60.1     | 70.7   |
|                       | SemiEmWeston et al. [2012]   | 59.0  | 59.6     | 71.1   |
|                       | Li Zhu et al. [2003]        | 68.0  | 45.3     | 63.0   |
|                       | DeepWalk Perozzi et al. [2014] | 67.2  | 43.2     | 65.3   |
|                       | ICA [2005]            | 75.1  | 69.1     | 73.9   |
|                       | Planetoid Yang et al. [2016] | 75.7  | 64.7     | 77.2   |
|                       | GraphSGAN Ding et al. [2018] | 83.0  | 73.1     |        |
| Graph Convolution     | Chebyshev Defferrard et al. [2016] | 81.2  | 69.8     | 74.4   |
|                       | GCN Kipf and Welling [2017] | 81.5  | 70.3     | 79.0   |
|                       | MoNe Monti et al. [2017]   | 81.7  | –        | 78.8   |
|                       | DPFCNN Monti et al. [2018] | 83.3  | 72.6     | –      |
|                       | LGCN Gao et al. [2018]     | 83.3  | 73.0     | 79.5   |
|                       | GAT Veličković et al. [2018] | 83.0  | 72.5     | 79.0   |
|                       | GraphNAS Gao et al. [2019] | **84.2** | **73.1** | 79.6   |
|                       | Mix-Hop Abu-El-Haija et al. [2019] | 81.9  | 71.4     | **80.8** |
| DR-GCNs (ours)        | DR-GCN            | 81.6±0.1 | 71.0±0.6 | 79.2±0.4 |
|                       | DR-GAT            | 83.6±0.5 | 72.8±0.8 | 79.1±0.3 |

Table 10: Batchnorm, Layernorm and Dimensional Reweighting AS-GCN methods accuracy(%) on citation datasets.

| Method            | Cora    | Citeseer | Pubmed |
|-------------------|---------|----------|--------|
| ASGCN             | 87.2±0.2 | 79.0±0.4 | 89.8±0.3 |
| Batch-norm        | 86.4±0.6 | 77.9±0.7 | 90.0±0.4 |
| Layer-norm        | 84.2±0.5 | 77.7±0.4 | 89.6±0.5 |
| Dimensional Reweighting | 87.1±0.2 | 79.1±0.4 | 90.3±0.4 |

F Batch-Norm and Layer-norm GCNs on citation networks

In Table 10 we also provide the batch-norm and layer-norm AS-GCN results on publication datasets. It can be seen that although performing good on Reddit, batch-norm and layer-norm does not generalize in publication datasets, and suffer reduced accuracy. While dimensional reweighting does not suffer accuracy loss and performs slightly better than the original model. Also the results are averaged among 20 runs.

G Proof of Theorem 1

We also provide proofs for theorems in the main article.

Now we turn to prove theorem 1. We say an linear operator $T : H_d \rightarrow H_d$ is diagonal-off-diagonal semidirect, or DOS for short if and only if:

\[ \forall C \in H_d, T(C)_{ii} = uc_{ii}, \]
\[ T(C)_{ij} = vc_{ii} + vc_{jj} + wc_{ij}. \]

Here $u, v, w$ are constants, and we will call the set of operators with these parameters $DOS(u, v, w)$. By the definition of $V_\phi$, the $(i, j)$ component of its output will only involve the $i-th$ and $j-th$ components of the input and symmetric with respect to them, hence itself and its derivatives $J_\phi$ will also involve them only and being symmetric with respect to them. Thus it is determined by $c_{ii}, c_{ij}, c_{jj}$. Furthermore, since $J_\phi$ is a linear map, so it will have this form. The result in Theorem 1 should hold in general for DOS operators, and do not require information about the fixed point structure.
Now $J_\phi$ is a DOS operator, hence it will belong to $DOS(u, v, w)$ for some $u, v, w$. Then we know:

$$||J_\phi(C)||_F^2 = \sum_{i,j=1}^{d} (J_\phi(C)_{ij})^2 = \sum_{i=1}^{d} (uc_{ii})^2 + \sum_{i\neq j} (vc_{ii} + vc_{jj} + wc_{ij})^2.$$

Correspondingly we have:

$$||J_\phi(SCS)||_F^2 = \sum_{i,j=1}^{d} (J_\phi(SCS)_{ij})^2 = \sum_{i=1}^{d} (us_i^2c_{ii})^2 + \sum_{i\neq j} (vs_i^2c_{ii} + vs_j^2c_{jj} + ws_is_jc_{ij})^2.$$

Since the inequality we want to prove is homogeneous of degree 2 with respect to $s_i$ on both sides, hence without loss of generality we can assume $\sum_{i=1}^{d} s_i^2 = d$ (this choice of gauge is intended to include the case in which all $s_i = 1$). Consider the function of (nonzero) $C \in \mathcal{H}_d$:

$$K(C) := \min_S \frac{||J_\phi(SCS)||_F^2}{||J_\phi(C)||_F^2}.$$

Here $\min_S$ is minimizing over diagonal $S$ with $\sum_{i=1}^{d} s_i^2 = d$, which is a compact set, hence the minimum is achieved at some point for any fixed $C$. And notice that at $S = Id$, $K = 1$, so $K(C) \leq 1$ and the equality will hold if and only if $S = Id$ is the minimum point of:

$$\tilde{K}(C, S) := ||J_\phi(SCS)||_F^2 - \sum_{i=1}^{d} (us_i^2c_{ii})^2 - \sum_{i\neq j} (vs_i^2c_{ii} + vs_j^2c_{jj} + ws_is_jc_{ij})^2,$$

with this fixed $C$. In particular, we know that at $s_1 = s_2 = \ldots = s_d = 1$ this function will satisfy Karush-Kuhn-Tucker (KKT for short) conditions (which could also be derived from the method of Lagrange multiplier in this case). Next we derive the KKT condition for each component. Define:

$$L(s_1, \ldots, s_d, \lambda) := \sum_{i=1}^{d} (us_i^2c_{ii})^2 + \sum_{i\neq j} (vs_i^2c_{ii} + vs_j^2c_{jj} + ws_is_jc_{ij})^2 - \lambda(\sum_{i=1}^{d} s_i^2 - d),$$

then the KKT conditions (i.e., the extreme value condition for restricted optimization) is:

$$4u^2s_i^3c_{ii} + \sum_{j\neq i} 2(vs_i^2c_{ii} + vs_j^2c_{jj} + ws_is_jc_{ij})(2vs_i^2c_{ii} + ws_is_jc_{ij}) - 2\lambda s_i = 0.$$

Now evaluate at $s_1 = \ldots = s_d = 1$, we have:

$$4u^2c_{ii} + \sum_{j\neq i} 2(vc_{ii} + vc_{jj} + wc_{ij})(2vc_{ii} + wc_{ij}) - 2\lambda = 0.$$

When $v \neq 0$, the coefficient of $C^2$ will be nonzero. Thus this gives a quadratic defining function for those $C$ where our statement may fail. Denote the left hand side of the equation by $F_1$, when $\nabla_C F_1 \neq 0$, it defines a smooth codimension 1 submanifold of $H_d$. When $\nabla_C F_1 = 0$, it give rise to a linear equation, in which $c_{ii}$ has coefficient $4(d - 1)v^2$, hence still gives rise to a codimension one smooth submanifold of $H_d$. In particular, the union of them will be a codimension 1 object (not necessarily smooth after we take union). Thus, those $C$ where $\tilde{K}(C, S)$ could reach 1 will has measure zero (this can be proved rigorously by outer regularity of the Haar measure $\mu$ on $H_d$, but this will distract from the main point here). Thus, for $\mu$-almost every matrix, we could choose an $S$ with $\sum_{i=1}^{d} s_i^2 = d$, such that $||J_\phi(SCS)||_F < ||J_\phi(C)||_F$. That is, the scaling increases the stability of updates.
Characterization of The Operator in Theorem 1

For $T \in DOS(u, v, w)$, its eigenspace has clear characterization, which will be useful in the study of the behavior of $T$. Let $M_d$ be the subspace of $H_d$ spanned by those matrices only has off-diagonal entries, which will have dimension $\frac{d(d-1)}{2}$ and a basis $M_{ij} = E_{ij} + E_{ji}$, where $E_{ij}$ is the matrix with 1 on $(i, j)$ position and 0 anywhere else. And $L_d$ is the spanned of $L_i$ defined as:

$$L_i = \begin{pmatrix}
  \cdots & -v & \cdots \\
  \cdots & -v & \cdots \\
 -v & -v & \cdots & w - u & -v & \cdots \\
  \cdots & -v & \cdots \\
  \cdots & -v & \cdots
\end{pmatrix},$$

where those non-zero entries are on $i$–th row and column.

**Theorem 4.** For $T \in DOS(u, v, w), w \neq u$, $M_d, L_d$ are its eigenspaces, with eigenvalues $w, u$ respectively.

**Proof.** Here the condition $w \neq u$ is to ensure that $L_d$ will be linearly independent with elements in $M_d$ since it will span the diagonal part of $H_d$. The results $TM_{ij} = wM_{ij}, TL_i = uL_i$ could be calculated using the definition equations of $DOS(u, v, w)$ and consider them on component level. Since $T$ is a linear operator, verifying these eigen-properties on the basis is enough for the result. Further, the space we have specified spans a $\frac{d(d-1)}{2} + d = \frac{d(d+1)}{2} = \dim H_d$ dimensional space, hence it will be the whole $H_d$. Thus we have completely characterized the eigenspaces of such $T$. 

\[\square\]
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