DEFORMATIONS OF POLARIZED MANIFOLDS WITH TORSION CANONICAL BUNDLE

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Abstract. Let $X$ be a smooth projective variety with torsion canonical bundle over complex numbers $\mathbb{C}$ and $L$ be a line bundle on $X$. We prove the pair $(X, L)$ is unobstructed if one of the following conditions is satisfied,
(1) the line bundle $L$ is ample,
(2) the fundamental group $\pi_1(X)$ of $X$ is finite.

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1. Introduction

The deformation problem of Calabi-Yau manifolds was studied in the paper [Tia87] and [Tod89] by the methods of differential geometry. A little bit later, in the paper [Kaw92] and [Ran92], an approach from algebraic geometry was developed, namely, the $T_1$-lifting criterion to give an algebraic proof.

Theorem 1.1. [Tia87, Tod89] Tian-Todorov] If $X$ is a compact Kähler manifold with trivial canonical bundle, then the Kuranishi space of deformations of complex structures on $X$ is smooth. Similarly, if $X$ is projective with an ample class $w$, then the Kuranishi space of deformations of $X$ with $w$ is also smooth.

In general, for a deformation problem, to prove unobstructedness is sufficient to prove some obstruction groups are zero. Unfortunately, it is not always the case. For a Calabi-Yau manifold $X$, the obstruction group $H^2(X, T_X)$ could be huge. Instead of proving that the obstruction group is vanishing, the only thing we need to prove is the obstruction element is vanishing.

On the other hand, the $T_1$-lifting criterion can be applied to some deformation problems with a hull (see [Sch68] for the definition). For examples, the deformation problem for a projective variety $X$ or the pair $(X, L)$, where $L$ is a line bundle on $X$, see [Ser06]. Applying the $T_1$-lifting criterion, we give an algebraic proof of

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unobstructedness of a Calabi Yau variety, see the nice note [TV] for the details. But, for the pair \((X, L)\) where \(X\) is Calabi-Yau, the \(T_1\)-lifting criterion does not work out directly. In this paper, we use another way to provide an algebraic proof for the unobstructedness for a pair \((X, L)\) with torsion \(K_X\). The main theorem of this paper is the following,

**Theorem 1.2.** Let \(X\) be a smooth projective variety with a torsion canonical bundle over complex numbers \(\mathbb{C}\) and \(L\) be a line bundle on \(X\). We prove the pair \((X, L)\) is unobstructed if one of the following conditions is satisfied,

1. the line bundle \(L\) is ample,
2. the fundamental group \(\pi_1(X)\) of \(X\) is finite.

In Section 2, we provide some preliminary about deformation and obstruction theory of the pair \((X, L)\). In general, this theory could be developed in an abstract way by cotangent complexes [Ill71]. For a smooth variety, we have a concrete way to build up this theory by Čech cocycles, we adopt the presentation of this approach from [Ser06].

In Section 3, we prove some general facts about the compatibility of the obstruction elements up to finite étale covering. It clarifies how to pass unobstructedness of \(X\) with trivial canonical bundle to the unobstructedness of \(X\) with torsion canonical bundle (in the paper [Ran92] just mention with a word). It also gives a similar result for the pair \((X, L)\).

In Section 4, we prove the deformation of a product of some kind of varieties preserves the product structure, moreover, we also prove the compatibility of the unobstructedness up to a product. By the results in Section 3 and 4, we use the Beauville-Bogomolov decomposition theorem to reduce the problem to irreducible holomorphic symplectic manifolds (or Hyperkähler manifolds), we provide a proof to this special case in Section 5.

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2. Atiyah Extension and Obstruction Element

Let \(X\) be a non-singular variety over complex numbers. We have a morphism (see [Har77])

\[ d(\log) : \mathcal{O}_X^* \to \Omega_X^1 \]

by the rule \(u \mapsto du/u\). We have an induced group homomorphism:

\[ c_1 : H^1(X, \mathcal{O}_X^*) \to H^1(X, \Omega_X^1) = Ext^1_{\mathcal{O}_X}(T_X, \mathcal{O}_X) \]

So for a line bundle \(L\) on \(X\), we associate to \(c_1(L)\) an extension class

\[ (2.1) \quad 0 \to \mathcal{O}_X \to E_L \to T_X \to 0. \]

We call this class is Atiyah extension of \(L\).

Let \(U = \{U_a\}\) be an affine open covering of \(X\) such that \(L\) is represented by a system of transition functions \(\{f_{ab}\}, f_{ab} \in \Gamma(U_{ab}, \mathcal{O}_X^*)\). Then the Atiyah extension
class of $L$ is represented by the 1-cocycle
\[
\left( \frac{df_{ab}}{f_{ab}} \right) \in Z^1(U, \Omega^1_X).
\]

Locally, the sheaf $E_L|_{U_a}$ is isomorphic to $O_{U_a} \oplus T_X|_{U_a}$. A section $(g_a, d_a)$ of $O_{U_a} \oplus T_X|_{U_a}$ and a section $(g_b, d_b)$ of $O_{U_b} \oplus T_X|_{U_b}$ are identified on $U_{ab}$ if and only if
\[
d_a = d_b \quad \text{and} \quad g_b - g_a = \frac{df_{ab}}{f_{ab}}.
\]

We formulate our deformation problem.

**Definition 2.1.** An infinitesimal deformation of the pair $(X, L)$ over an $C$-local Artin ring $A$ consists of a pair $(X_A, L_A)$ where we have a cartesian diagram as following

\[
\begin{array}{ccc}
X & \to & X_A \\
\downarrow & & \downarrow \\
\text{Spec}(\mathbb{C}) & \to & \text{Spec}(A)
\end{array}
\]

where $X_A$ is flat over $\text{Spec}(A)$ and $L_A$ is a line bundle on $X_A$ such that $L_A|_X = L$.

There is obvious (isomorphic) equivalence relations among the deformations of the pair $(X, L)$. Hence, we can define a functor of Artin rings as the following,

**Definition 2.2.** We define a functor of infinitesimal deformations of the pair $(X, L)$ as following

\[
\text{Def}_{f(X, L)} : (\text{C-local Artin rings}) \to (\text{Sets})
\]

which associate to a $C$-local Artin ring $A$ a set

\[
\{ \text{deformations of} \ (X, L) \ \text{over} \ A \} / \sim
\]

We call $(X, L)$ is unobstructed if the restriction map

\[
\text{Def}_{f(X, L)}(A') \to \text{Def}_{f(X, L)}(A)
\]

is surjective which is induced by any surjection between $C$-local Artin rings $A' \to A$.

We summarize some useful results from [Ser06] in the following proposition.

**Proposition 2.3.** There is a long exact sequence induced by the short exact sequence $2.1$

\[
\ldots \to H^1(X, T_X) \xrightarrow{\cup c_1(L)} H^2(X, O_X) \to H^2(X, E_L) \xrightarrow{\text{forget}} H^2(X, T_X) \to \ldots
\]

Moreover, for a small extension $0 \to C \to A' \to A \to 0$ and a deformation $(X_A, L_A)$ of the pair $(X, L)$ over $A$, there exists an obstruction element $\text{Obs}(X_A, L_A) \in H^2(X, E_L)$ via forget

\[
\text{Obs}(X_A, L_A) \xrightarrow{\cup c_1(L)} \text{Obs}(X_A).
\]

In particular, the pair $(X, L)$ is unobstructed if and only if the obstruction element $\text{Obs}(X_A, L_A)$ is zero for all small extension of $A$.

**Lemma 2.4.** If a smooth variety $X$ is unobstructed, for a line bundle $L$ on $X$, the map

\[
\cup c_1(L) : H^1(X, T_X) \to H^2(X, O_X)
\]

is surjective, then $(X, L)$ is unobstructed.

**Proof.** It is due to the above proposition. \qed
3. Unobstructedness to Finite Étale Covering

We think we could use cotangent complexes [Ill71] to simplify the argument and give more general results in this section. But it seems the formalism of [Ill71] is a little scared, so we decide to restrict our situation and use the approach of Čech-cocycles.

**Proposition 3.1.** Let $Y$ be a smooth projective variety over complex numbers $\mathbb{C}$. If a map $f : X \to Y$ is finite étale, then we have the following consequences,

1. the pull back map $f^* : H^i(Y, T_Y) \to H^i(X, T_X)$ is injective,
2. if $Y_A$ is a deformation $Y_A$ of $Y$ over a local $\mathbb{C}$-Artinian ring $A$, then there is a deformation $X_A$ of $X$ over a local $\mathbb{C}$-Artinian ring with the following cartesian diagrams,

\[
\begin{array}{ccc}
X & \to & X_A \\
| & | & | \\
\downarrow f & \downarrow f_A & \downarrow \\\nY & \to & Y_A
\end{array}
\]

**Proof.** For the first assertion, by the projection formula we have a map

$$T_Y \to f_* T_X = T_Y \otimes_{O_Y} f_* O_X.$$  

It is an inclusion of summand $T_Y \otimes_{O_Y} f_* O_X$. In fact, it is obvious by the existence of the trace map $Tr$ satisfy the following property

\[
\begin{array}{ccc}
O_Y & \xrightarrow{\text{deg}(f)} & f_* f^* O_Y \\
& & \to f_* O_X \\
& & \xrightarrow{Tr} O_Y
\end{array}
\]

Therefore, it induces injections on cohomology groups.

For the second assertion, it is clear that

$$FEt(Y_A) = FEt(Y)$$

since $Y \subseteq Y_A$ is defined by nilpotent elements, where $FEt(X)$ is the category of finite étale coverings, see [FK88] for the details. □

**Proposition 3.2.** Let $X$ be a reduced local complete intersection algebraic scheme over the complex number $\mathbb{C}$ (e.g smooth varieties). If a morphism

$$f : Y \to X$$

is finite étale, then we have a natural map

$$\text{Ext}^2(\Omega_X, O_X) \xrightarrow{f^*} \text{Ext}^2(\Omega_Y, O_Y)$$

which associates $\text{Obs}(X_A) \to \text{Obs}(Y_A)$ for any small extension

$$0 \to (t) \to A' \to A \to 0.$$  

Here $X_A$ and $Y_A$ are just as in the second assertion of Proposition 3.1, in particular, we have a map $f_A : X_A \to Y_A$ as above.

**Proof.** We recall how to construct the obstruction element $\text{Obs}(X_A)$. For the small extension of $A$, we have the corresponding conormal sequence as following,

$$0 \to (t) \to \Omega_{A'/\mathbb{C}} \otimes_{A'} A \to \Omega_{A/\mathbb{C}} \to 0$$
which is exact by \[\text{[Ser06, Lemma B.10]}\]. We pull it back via the structure morphism \(p : X_A \to \text{Spec}(A)\) to obtain an exact sequence

\[
(3.1) 
0 \to \mathcal{O}_{X_A} \to p^*(\Omega_{A'/C} \otimes_{A'} A) \to p^*\Omega_{A/C} \to 0
\]

By \[\text{[Ser06, Theorem D.28]}\], we know the relative cotangent sequence of \(p\)

\[
(3.2) 
0 \to p^*(\Omega_{A/C}) \to \Omega_{X_A/C} \to \Omega_{X_A/A} \to 0
\]

is exact. Composing \[\text{3.1} and \text{3.2}\] we obtain a 2-extension

\[
0 \to \mathcal{O}_X \to p^*(\Omega_{A'/C} \otimes_{A'} A) \to \Omega_{X_A/C} \to \Omega_{X_A/A} \to 0.
\]

It defines the obstruction element

\[
\text{Obs}(X_A) \in \text{Ext}_C^2(\Omega_{X_A/A}, \mathcal{O}_X) = \text{Ext}_X^2(\Omega_X, \mathcal{O}_X).
\]

Since \(f\) is étale, it is obvious that the natural map \(f^*\) associates to a 2-extension

\[
0 \to \mathcal{O}_X \to E \to F \to \mathcal{O}_X \to 0.
\]

a 2-extension

\[
0 \to f^*\mathcal{O}_X = \mathcal{O}_Y \to f^*E \to f^*F \to f^*\Omega_X = \Omega_Y \to 0.
\]

So, to prove \(f^*\) preserves the obstruction element, it is sufficient to prove

\[
0 \to (p \circ f_A)^*(\Omega_A) \to f_A^*\Omega_{X_A} \to f_{X/A}^*\Omega_{X_A/A} \to 0
\]

is just

\[
0 \to (p \circ f_A)^*(\Omega_A) \to \Omega_{Y_A} \to \Omega_{Y_A/A} \to 0.
\]

This just follows from that \(f_A\) is étale, hence, we have the following exact sequences

\[
0 \to f_A^*\Omega_{X_A} \to \Omega_{Y_A} \to \Omega_{Y_A/X_A} \to 0
\]

and

\[
0 \to f_A^*\Omega_{X_A/A} \to \Omega_{Y_A/A} \to \Omega_{Y_A/X_A} \to 0.
\]

Since \(\Omega_{Y_A/X_A} = 0\), we have

\[
f_A^*\Omega_{X_A} = \Omega_{Y_A} \text{ and } f_A^*\Omega_{X_A/A} = \Omega_{Y_A/A}
\]

It completes the proof. \(\square\)

**Corollary 3.3.** With the hypothesis as Proposition \[\text{3.1}\], the variety \(Y\) is unobstructed if and only if \(X\) is unobstructed. \[\text{[Ram92]}\]In particular, if \(X\) is a smooth projective variety with torsion canonical bundle, then \(X\) is unobstructed.

**Proof.** For any smooth projective variety with torsion canonical bundle, we can find a finite étale cover such that its canonical bundle is trivial, hence, unobstructed. Then, the assertion follows from the previous propositions. \(\square\)

**Proposition 3.4.** Let \(Y\) be a smooth projective variety over complex numbers \(\mathbb{C}\) and \(L\) be a line bundle on \(Y\). If a map \(f : X \to Y\) is finite étale, then we have the following consequences,

1. the Atiyah extension is functorial under \(f\), i.e., \(f^*E_L = E_{f^*L}\),
2. the pull back map \(f^* : H^i(Y, E_L) \to H^i(X, E_{f^*L})\) is injective,
3. if we have a small extension \(0 \to C \to A' \to A \to 0\) and a deformation \((Y_A, L_A)\) of \((Y, L)\), then we have that
   \[
f_{A'}^*(\text{Obs}(Y_A, L_A)) = \text{Obs}(X_A, f_A^*L_A),
   \]
   where the morphism \(f_A\) is given in Proposition \[\text{3.1}\].
Proof. As in section 2, the vector bundle $E_L$ is given by the data trivializations $\mathcal{O}_{U_i} \oplus T_Y|_{U_i}$ and a transition functions as following,

$$
\begin{pmatrix}
g_b \\
g_a
\end{pmatrix} = \begin{pmatrix} 1 & df_{ab} / f_{ab} \\ 0 & \text{Id}
\end{pmatrix}
\begin{pmatrix} g_a \\
g_b
\end{pmatrix}
$$

If we pull back the corresponding Atiyah extension class of $L$, then we get

$$(3.3) \quad 0 \to f^*\mathcal{O}_Y = \mathcal{O}_X \to f^*E_L \to f^*T_Y = T_X \to 0.$$ 

Since the transition function for $f^*L$ is given by

$$
\{f^*(f_{ab})\}, f_{ab} \in \Gamma(f^{-1}U_{ab}, \mathcal{O}_X^*)
$$

Since we know $f^*(df_{ab}) = d(f^*f_{ab})$, therefore, comparing the transition functions of the Atiyah extension classes, we know $E_{f,L}$ is the Atiyah extension classes of $f^*L$.

For the second assertion, the proof is similar to the second assertion of Proposition 3.1 once we prove the first assertion.

For the third assertion, we recall how to construct the obstruction element $\text{Obs}(Y_A, L_A)$. Suppose we have a small extension

$$
0 \to (t) \to A' \to A \to 0.
$$

and an affine covering $U_i$ of $Y$. We can choose a trivialization

$$
\theta_i : U_i \times \text{Spec}(A) \to Y_A|_{U_i}
$$

since $Y_A$ is smooth over $A$. Let $\theta_{ij}$ be $\theta_i^{-1}\theta_j \in \text{Aut}(U_{ij} \times \text{Spec}(A))$. The line bundle $L_A$ is given by transition functions $(F_{ij})$ on $U_{ij} \times A$ such that

$$
F_{ij}\theta_{ij}(F_{jk}) = F_{ik}.
$$

To see whether there is a lifting $(Y_{A'}, L_{A'})$ of $(Y_A, L_A)$ over $\text{Spec}(A')$, we choose a collection $(\theta_{ij}', F_{ij}')$ such that

1. $\theta_{ij}' \in \text{Aut}(U_{ij} \times \text{Spec}(A'))$ and $F_{ij}' \in \mathcal{O}_{U_{ij} \times \text{Spec}(A')}^*$,
2. $\theta_{ij}'|_{U_{ij} \times A} = \theta_{ij}$ and $F_{ij}'|_{U_{ij} \times A} = F_{ij}$.

Since $\theta_{ij}' \theta_{jk}'(\theta_{ik}')^{-1}|_{U_{ij} \times \text{Spec}(A)} = Id$, we have

$$(3.4) \quad \theta_{ij}' \theta_{jk}'(\theta_{ik}')^{-1} = Id + t \cdot d_{ijk},$$

$$(3.5) \quad F_{ij}' \theta_{ij}'(F_{jk})(F_{ik}')^{-1} = 1 + t \cdot g_{ijk}$$

where $d_{ijk} \in \Gamma(U_{ijk}, T_X)$ and $g_{ijk} \in \Gamma(U_{ijk}, \mathcal{O}_X)$. So the obstruction element can be represented by a 2-cocycle

$$(g_{ijk}, d_{ijk}) \in Z^2(U, E_L).$$

Since we have a unique trivialization $\tilde{\theta}_i$ of $X_A|_{f^{-1}U_i}$ in the following way,

$$
\begin{array}{ccc}
f^{-1}(U_i) & \longrightarrow & X|_{U_i} \\
\downarrow & & \downarrow \tilde{\theta}_i \\
\tilde{\theta}_i & & \tilde{\theta}_i
\end{array}
$$

$$
\begin{array}{ccc}
f^{-1}(U_i) \times A & \longrightarrow & U_i \times A \\
\downarrow & \searrow & \downarrow \theta_i \\
\tilde{\theta}_i & \longrightarrow & Y_A|_{U_i}
\end{array}
$$
Let \( \tilde{\theta}_{ij} \) be \( \tilde{\theta}_i^{-1} \tilde{\theta}_j \). Similarly, we have \( \tilde{F}_{ij}, \tilde{d}_{ijk} \) and \( \tilde{g}_{ijk} \). Then we have

\[
f^{-1}(U_{ij}) \times A \xrightarrow{h_A} U_{ij} \times A
\]

\[
f^{-1}(U_{ij}) \times A' \xrightarrow{h_{A'}} U_{ij} \times A'
\]

where \( h_A = f \times \text{Id}_{\text{Spec}(A)} \) and \( h_{A'} = f \times \text{Id}_{\text{Spec}(A')} \). Hence, we have that

1. \( h_{A'}^*(\theta'_{ij})|_{f^{-1}(U_{ij} \times A)} = \tilde{\theta}_{ij} \)
2. \( h_{A'}^*(F'_{ij})|_{f^{-1}(U_{ij} \times A)} = h_A^*(F_{ij}) = \tilde{F}_{ij} \).

Moreover, we know \( h_A^*(\tilde{F}_{ij}) \) defines the line bundle \( f_A^*L_A \) on \( X_A \). Combining these facts and applying \( h_{A'}^*(-) \) to \( 3.4 \) and \( 3.5 \) we get

\[
f^*(g_{ijk}, d_{ijk}) = (\tilde{g}_{ijk}, \tilde{d}_{ijk})
\]

It completes the proof. \( \square \)

4. Unobstructedness Up To Product

**Lemma 4.1.** Let \( Z \) be \( X \times Y \), where \( X \) and \( Y \) are smooth projective varieties. Suppose we have

1. \( H^0(Y, T_Y) = H^1(Y, \mathcal{O}_Y) = 0 \),
2. \( K_X = \mathcal{O}_X \) and \( K_Y = \mathcal{O}_Y \) where \( K_X \) (resp \( K_Y \)) is the canonical bundle of \( X \) (resp \( Y \)).

If \( Z_A \) is a deformation of \( Z \) over a \( \mathbb{C} \)-local Artinian ring \( A \), then we have that

\[
Z_A \simeq X_A \times_A Y_A
\]

where \( X_A \) (resp \( Y_A \)) is a deformation of \( X \) (resp \( Y \)) over \( A \).

**Proof.** By the Schlessinger’s criterion\[Sch68\], we know the deformation functors of \( X, Y \) and \( Z \) have hulls as following

\[
h_R \rightarrow \text{Def}_X, \ h_{R'} \rightarrow \text{Def}_Y \text{ and } h_{R''} \rightarrow \text{Def}_Z.
\]

where \( R, R' \) and \( R'' \) are complete local rings over \( \mathbb{C} \). We consider a natural transformation \( g \) between functors

\[
g : \text{Def}_X \times \text{Def}_Y \rightarrow \text{Def}_Z = \text{Def}_{X \times Y}
\]

which associates to \( (X_A, Y_A) \in (\text{Def}_X \times \text{Def}_Y)(A) \)

a deformation \( (X_A \times_A Y_A) \in \text{Def}_{X \times Y}(A) \) of \( Z \)

By the property of hulls, it induces the following commutative diagram

\[
\begin{array}{ccc}
h_R \times h_{R'} \ar[d] & \xrightarrow{f} & h_{R''} \ar[d] \\
\text{Def}_X \times \text{Def}_Y \ar[d] & \xrightarrow{g} & \text{Def}_{X \times Y} \ar[d] \\
\end{array}
\]
We claim the natural transformation \( f \) induced by \( g \) is surjective, hence, \( g \) is surjective. In fact, we have \( h_{R \times R'} = h_R \times h_{R'} \) and \( X, Y, Z \) are unobstructed (since the canonical bundles are trivial), therefore, we have

\[
R \cong \mathbb{C}[x_1, \ldots, x_n], \quad R' \cong \mathbb{C}[x_1, \ldots, x_n] \quad \text{and} \quad R'' \cong \mathbb{C}[x_1, \ldots, x_n],
\]

i.e., rings \( R, R' \) and \( R'' \) are formal power series rings. We only need to check

\[
g : Def_X(\mathbb{C}[\varepsilon]) \times Def_Y(\mathbb{C}[\varepsilon]) \to Def_{X \times Y}(\mathbb{C}[\varepsilon])
\]
is surjective where \( \varepsilon \) is the dual number. By the identification of first order infinitesimal deformation of a variety \( W \) with \( H^1(W, T_W) \), it is sufficient to prove

\[
H^1(X, T_X) \oplus H^1(Y, T_Y) \to H^1(X \times Y, T_{X \times Y})
\]
is surjective (in fact, it is an isomorphism), where this map is induced by the projections \( \pi_1 \) and \( \pi_2 \)

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{\pi_1} & X \\
\downarrow & & \downarrow \pi_2 \\
Y & &
\end{array}
\]

Since \( T_{X \times Y} = \pi_1^* T_X \oplus \pi_2^* T_Y \), we have that

\[
H^1(X \times Y, T_{X \times Y}) = H^1(X \times Y, \pi_1^* T_X) \oplus H^1(X \times Y, \pi_2^* T_Y)
\]

So it reduces to prove that (under \( \pi_1 \) and \( \pi_2 \))

\[
H^1(X \times Y, \pi_1^* T_X) = H^1(X, T_X) \quad \text{and} \quad H^1(X \times Y, \pi_2^* T_Y) = H^1(Y, T_Y).
\]

By Leray spectral sequence for \( \pi_1 \), we have

\[
0 \to H^1(X, \pi_1^* T_X) \to H^1(X \times Y, \pi_1^* T_X) \to H^0(X, R^1\pi_1_\ast \pi_1^* T_X).
\]

By projection formula and base change theorem, we have

1. \( \pi_1^* T_X = T_X \otimes \pi_1^* \mathcal{O}_{X \times Y} = T_X \), hence, \( H^1(X, \pi_1^* T_X) = H^1(X, T_X) \)
2. \( R^1\pi_1^* T_X = T_X \otimes R^1\pi_1^\ast \mathcal{O}_{X \times Y} \)
3. \( R^1\pi_1^\ast \mathcal{O}_{X \times Y} = 0 \) because of the hypothesis \( H^1(Y, \mathcal{O}_Y) = 0 \)

Summarizing up, we have \( H^1(X, T_X) = H^1(X \times Y, \pi_1^* T_X) \). Similarly, for \( Y \), we have

\[
0 \to H^1(Y, T_Y) \to H^1(X \times Y, \pi^*_2 (T_Y)) \to H^0(Y, T_Y \otimes R^1\pi_2^\ast \mathcal{O}_{X \times Y})
\]

Since \( R^1\pi_2^\ast \mathcal{O}_{X \times Y} \) is trivial bundle of rank \( N \), we have (by the hypothesis)

\[
H^0(Y, T_Y \otimes R^1\pi_2^\ast \mathcal{O}_{X \times Y}) = H^0(Y, T_Y)^{\oplus N} = 0.
\]

We get \( H^1(Y, T_Y) = H^1(X \times Y, \pi^*_2 T_Y) \). It completes the proof.

**Proposition 4.2.** With the same hypothesis as in the above lemma. Let \( L \) be a line bundle on \( Z \) and \( x \) (resp \( y \)) be a closed point on \( X \) (resp \( Y \)). Assume the canonical bundle \( K_Z \) is trivial. Let \( g \) and \( f \) be the natural inclusions via points \( x \) (resp \( y \)).

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{f} & X \\
\downarrow & & \downarrow \wedge \quad \downarrow \g \\
Y & &
\end{array}
\]

If the pairs...
are unobstructed, then the pair \((Z, L)\) is unobstructed.

**Proof.** Given a pair \((Z_A, L_A)\) which is a deformation of the pair \((Z, L)\) over an \(\mathbb{C}\)-local Artin Ring \(A\). Assume we have a small extension

\[ 0 \rightarrow \mathbb{C} \rightarrow A' \rightarrow A \rightarrow 0. \]

By the previous lemma, we have a decomposition \(Z_A = X_A \times_A Y_A\). By the smoothness of \(Y_A \rightarrow \text{Spec}(A)\), we have a lifting,

\[
\begin{array}{ccc}
\{y\} & \longrightarrow & Y_A \\
\downarrow & & \downarrow \\
\text{Spec}(A) & \longrightarrow & \text{Spec}(A)
\end{array}
\]

it induce an embedding

\[
i = (id_{X_A}, \sigma \circ \pi) : X_A \hookrightarrow X_A \times_A Y_A
\]

where \(\pi\) is the structure morphism of \(X_A \rightarrow \text{Spec}(A)\). Similarly, we have an inclusion for \(Y\) and the point \(x\),

\[
j : Y_A \hookrightarrow X_A \times_A Y_A.
\]

Since \((X, L|_{X \times \{y\}})\) and \((Y, L|_{\{x\} \times Y})\) are unobstructed, there exists deformations

\[
(X_A', L')|_{X_A} = (X_A, L_A|_{X_A}) \text{ and } (Y_A', L'')|_{Y_A} = (Y_A, L_A|_{Y_A}).
\]

Let \(Z_A'\) be \(X_A' \times_{A'} Y_A'\). We claim that there is a line bundle \(L_{A'}\) on \(Z_A'\) such that the pair \((Z_A', L_{A'})\) is a deformation of \((Z_A, L_A)\) over \(A'\). Therefore, the pair \((Z, L)\) is unobstructed. Since we have short exact sequences as following,

\[
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{O}_Z & \longrightarrow & \mathcal{O}_{Z_{A'}}^2 & \longrightarrow & \mathcal{O}_{Z_A} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X_{A'}} & \longrightarrow & \mathcal{O}_{X_A} & \longrightarrow & 0.
\end{array}
\]

It induces long exact sequences as following,

\[
\begin{array}{cccccc}
\ldots & \longrightarrow & H^1(Z_{A'}, \mathcal{O}_{Z_{A'}}^2) & \longrightarrow & H^1(Z_A, \mathcal{O}_{Z_A}^2) & \longrightarrow & H^2(Z, \mathcal{O}_Z) & \longrightarrow & \ldots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & H^1(Z_{A'}, \mathcal{O}_{X_{A'}}^2) & \longrightarrow & H^1(Z_A, \mathcal{O}_{X_A}^2) & \longrightarrow & H^2(X, \mathcal{O}_X) & \longrightarrow & \ldots.
\end{array}
\]

The second square above gives the following association,

\[
(Z_A, L_A) \xymatrix{\longrightarrow} \text{Obs}_L(Z_A, L_A) \xymatrix{f^*} (Z_A, L_A|_{X_A}) \text{ where } \text{Obs}_L(\_\_) \text{ is just the obstruction element for line bundles. Similarly, we have}
\]

\[
g^*(\text{Obs}_L(Z_A, L_A)) = \text{Obs}_L(Y_A, L_A|_{Y_A}) = 0.
\]
By the hypothesis $H^1(Y, \mathcal{O}_Y) = 0$ and K"{u}nneth theorem, we have an isomorphism

$$(f^*, g^*) : H^2(Z, \mathcal{O}_Z) \xrightarrow{\sim} H^2(\mathcal{O}_X) \oplus H^2(\mathcal{O}_Y).$$

Therefore, we have $\text{Obs}_L(Z_A, L_A) = 0$, therefore, there exists a line bundle $L'_A$ on $Z_{A'}$ which $(Z_{A'}, L_{A'})$ is a deformation of $(Z_A, L_A)$ we are looking for. It completes the proof. 

\[\square\]

5. Main Theorem

Let us recall some definitions. We focus on the category of algebraic varieties over complex numbers.

**Definition 5.1.** Let $X$ be a projective manifold.

1. $X$ is Calabi-Yau if $\dim(X)$ is at least 3 and $h^0(\wedge^p \Omega_X) = 0$ for $0 < p < \dim X$.
2. $X$ is irreducible holomorphic symplectic if $X$ is simply-connected and $H^{2,0}(X)$ is spanned by the class of a holomorphic symplectic form $\sigma$.

For a irreducible holomorphic symplectic $X$ of dimension $2m$, we have the following facts, (see [GHJ03] for the details),

1. On $H^2(X, \mathbb{R})$, there is a (Beauville-Bogomolov) quadratic form $q_X$, 

$$q_X(\alpha) = \frac{m}{2} \int_X \alpha^2(\sigma \bar{\sigma})^{m-1} + (1 - m) \left( \int_X \alpha \sigma^{m-1} \bar{\sigma} \right) \left( \int_X \alpha \sigma^m \bar{\sigma}^{m-1} \right)$$

2. $H^0(X, \wedge^* \Omega_X) = \mathbb{C}[\sigma]$, 

3. the Beauville-Bogomolov quadratic form $q_X$ is positive definite on $\mathbb{R}[w] \oplus (H^{0,2} \oplus H^{2,0})(X)|_{\mathbb{R}}$, negative definite on the primitive $(1, 1)$-part $H^{1,1}(X)_w$ and these two spaces are orthogonal with respect to $q_X$ where $[w]$ is a K"{o}hler class.

**Lemma 5.2.** Let $Y$ a projective manifold. If $Y$ is simply connected, then any holomorphic line $L$ with vanishing $c_1(L) \in H^2(Y, \mathbb{Q})$ is a trivial line bundle.

**Proof.** Since we have exact sequence

$$H^1(Y, \mathcal{O}_Y) \longrightarrow H^1(Y, \mathcal{O}_Y^*) \xrightarrow{c_1} H^2(X, \mathbb{Z})$$

and $Y$ is simply connected, in particular $H^1(Y, \mathcal{O}_Y) = 0$, we have $L$ is a torsion line bundle, i.e, there exists the smallest $N \in \mathbb{N}$ such that

$$L \otimes^N = \mathcal{O}_Y.$$

Therefore, there is a finite $N$-covering of $X$ associates to $L$ as following,

$$\text{Spec}(\mathcal{O}_Y \oplus L^{-1} \oplus \ldots \oplus L^{-N+1}) \rightarrow Y$$

Since $Y$ is simply connected, $N$ has to be 1. It is done. \[\square\]

**Lemma 5.3.** Let $Y$ be a projective manifold. If $H^1(Y, \mathcal{O}_Y) = 0$ (e.g. $Y$ is simply connected), then any deformation $(Y_A, L_A)$ of pair $(Y, \mathcal{O}_Y)$ is $(Y_A, \mathcal{O}_{Y_A})$.

**Proof.** We prove this lemma by induction on the length $l(A)$ of $A$. In fact, if $l(A) = 1$, then $A$ is $\mathbb{C}$, the lemma is trivial. In general, we consider a small extension (see [Sch68]) and a flat deformation $Y_{A'}$ of $Y$ over $A'$,

$$0 \rightarrow \mathbb{C} \rightarrow A' \rightarrow A \rightarrow 0.$$
We denote \((Y_A', A)\) by \(Y_A\). This short exact sequence induce an exact sequence
\[0 \to \mathcal{O}_Y \to \mathcal{O}_{Y_A}' \to \mathcal{O}_{Y_A}^* \to 0\]
which gives an exact sequence
\[H^1(Y, \mathcal{O}_Y) \to H^1(Y_A', \mathcal{O}_{Y_A}') \to \mathcal{O}_{Y_A}^* \mathcal{O}_{Y_A}^* \to 0\]

Suppose \((Y_A', L_A')\) is a deformation of \((Y, \mathcal{O}_Y)\), by induction, we know \((Y_A, (L_A, A))\) is isomorphic to the deformation \((Y_A, \mathcal{O}_{Y_A})\). It is equivalent to say, under the map \(p,(Y_A', L_A')\) and \((Y_A', \mathcal{O}_{Y_A})\) has the same image. Since \(H^1(Y, \mathcal{O}_Y)\) is trivial, \((Y_A', L_A')\) is isomorphic to \((Y_A', \mathcal{O}_{Y_A})\).

\[\square\]

**Proposition 5.4.** Let \(Y\) be irreducible holomorphic symplectic manifold of dimension \(n\). If \(L\) is a line bundle on \(Y\), then the obstruction element \(\text{Obs}((Y, \mathcal{O}_Y), A)\) is zero for any deformation pair \((Y_A, L_A)\) over a local \(\mathbb{C}\)-Artinian ring \(A\), i.e., \((Y, L)\) is unobstructed.

**Proof.** If \(c_1(L) = 0 \in H^2(Y, \mathbb{Z})\), then, by Lemma 5.3 and 2.2, we know \((Y_A, L_A)\) is isomorphic to \((Y_A, \mathcal{O}_A)\) which is obvious unobstructed.

If \(c_1(L) \neq 0 \in H^2(Y, \mathbb{Z})\), then, by Lemma 2.2 and \(H^2(Y, \mathcal{O}_Y) = \mathbb{C}\), we only need to prove the map
\[\cup c_1(L) : H^1(Y, \Omega_Y) \to H^2(Y, \mathcal{O}_Y)\]
is nonzero. By the triviality of the canonical bundle of \(Y\), we have \(T_Y = \wedge^{n-1} \Omega_Y\). More generally, the map \(\cup c_1(L)\) fits inside the sequences of maps
\[H^0(\wedge^{n-1} \Omega_Y) \to H^0(\wedge^{n} \Omega_Y) \to H^2(\mathcal{O}_Y) \to H^2(\wedge^n \Omega_Y)\]
where \([w]\) is a real \((1, 1)\)-form in \(H^{1, 1}(X)\). To prove the map \(\cup c_1(L)\) is nonzero, it is sufficient to prove the map \(s\) is nonzero. Let \(n = 2m\). Since \(H^0(\wedge^{n-1} \Omega_Y) = \mathbb{C}\) is generated by \(\sigma^{m-1}\), the map \(s\) sends
\[\sigma^{m-1} \to \sigma^{m-1}(c_1(L)) \cdot [w] \in H^2(\wedge^n \Omega_Y)\]

Since the property (3) of \(q_X\) mention at the beginning of this section, if \(c_1(L)\) is not zero, then we can choose a \((1, 1)\)-form \([w] \in H^{1, 1}(Y)\) such that the Beauville-Bogomolov form
\[q_Y([w], c_1(L)) = \frac{m}{2} \int_Y (c_1(L)) \cdot [w](\sigma^{m-1}.\]
is non-zero, see [GHJ03 Corollary 23.11], which implies \(s\) is a nonzero map.

\[\square\]

**Theorem 5.5.** Let \(X\) be a smooth projective variety with a torsion canonical bundle over complex numbers \(\mathbb{C}\) and \(L\) be a line bundle on \(X\). We prove the pair \((X, L)\) is unobstructed if one of the following conditions is satisfied,

1. The line bundle \(L\) is ample,
2. The fundament group \(\pi_1(X)\) of \(X\) is finite.
Proof. For the first assertion, as in the proof of Proposition 5.4, the map
\[ \cup c_1(L) : H^1(Y, T_Y) \to H^2(Y, \mathcal{O}_Y) \]
can be fit into the sequences of maps,
\[ H^0(\Lambda^{n-2}\Omega_Y) \xrightarrow{\cup c_1(L)} H^0(\Lambda^{n-1}\Omega_Y) \xrightarrow{\cup c_1(L)} H^2(\mathcal{O}_Y) \xrightarrow{} H^2(\Lambda^n\Omega_Y) \]
Since \( L \) is ample, by the Hard Lefschetz theorem [GH94], we know it is isomorphism. Therefore, the map \( \cup c_1(L) \) is surjective. By Lemma 2.4, we prove the first assertion.

For the second assertion, by the Beauville-Bogomolov decomposition theorem [Bea83, Bog74], there exists a finite étale cover \( \tilde{X} \to X \) such that \( \pi : \tilde{X} \) is a product of a torus and factors each of which is either an irreducible holomorphic symplectic manifold or a Calabi-Yau manifold. Since \( \pi_1(X) \) is finite, there is no torus factor in this decomposition. So we have that \( \tilde{X} = \Pi_{i=1}^n Y_i \) where \( Y_i \) is either a irreducible holomorphic symplectic manifold or a Calabi-Yau manifold of dimension at least 3. It is easy to see, for a Calabi-Yau projective manifold \( Y \) of dimension at least 3, any pair \((Y, L')\) is unobstructed since \( H^2(Y, \mathcal{O}_Y) = 0 \).

It is easy to see a product of Calabi-Yau manifolds and irreducible holomorphic symplectic manifolds satisfy the hypothesis of Lemma 4.2. By Proposition 5.4, it implies the pair \((\tilde{X}, \pi^*(L))\) is unobstructed. By Proposition 3.4, we get the theorem. \( \square \)

Appendix A. Deformations Via Cotangent Complex

In this appendix, we approach to Proposition 3.2 and the third assertion of Proposition 3.4 by cotangent complexes. The cotangent complex is a very powerful machinery to attack deformation problems. The advantage of cotangent complex is its functoriality. But, in general, it is difficulty to calculate it even though we often just use the truncated cotangent complex. The situation is similar as singular (co)homology which has very good functoriality but hard to calculate. In CW complex or simplicial complex, we use simplicial (co)homology which has not very good functoriality but easier to calculate, the situation is similar as our calculation of deformations by Čech-cocycle.

Lemma A.1. For a small extension
\[ 0 \to (t) \to A' \to A \to 0, \]
and morphisms between algebraic schemes over complex numbers
\[ \begin{array}{ccc}
X_0 & \xrightarrow{g_0} & Y_0 \\
\downarrow & & \downarrow \\
X_A & \xrightarrow{g} & Y_A \\
\end{array} \]
\[ \begin{array}{ccc}
f_0 & : & \text{Spec}(\mathbb{C}) \\
\downarrow & & \downarrow \\
f & : & \text{Spec}(A) \\
\end{array} \]
where \( g \) is finite étale and \( f \) is flat. There is a natural map \( g_0^* \) induced by \( g_0 \)
\[ g_0^* : \text{Ext}^2(L_{Y_0}, \mathcal{O}_{Y_0}) \to \text{Ext}^2(L_{X_0}, \mathcal{O}_{X_0}) \]
which \( \text{Obs}(Y_A) \to \text{Obs}(X_A) \) and \( L_{Y_0} \) (resp. \( L_{X_0} \)) is the cotangent complex of \( Y_0 \) (resp. \( X_0 \)) over \( \mathbb{C} \).

Proof. Since \( f \) and \( g \) are flat, we have that
(L_{X_A/A})|_{X_0} = L_{X_0} which implies Ext^1(L_{X_A/A}, O_{X_0}) = Ext^1(L_{X_0}, O_{X_0}).

Let us recall how to construct obstruction elements Obs(X_A) and Obs(Y_A), see [Ill71] for the details. Since we have a distinguished triangle $\Delta_{Y_A}$ in $D^+_{coh}(Y_A)$ as

\[
\begin{array}{ccc}
\mathbb{L}_{Y_A} & \xrightarrow{+1} & \mathbb{L}_{Y_A} \\
\downarrow & & \downarrow \\
\mathbb{L}_{Y_A/A} & \xrightarrow{f^*} & \mathbb{L}_{Y_A/A'}
\end{array}
\]

Similarly, we also have a distinguished triangle $\Delta_{X_A}$ in $D^+_{coh}(X_A)$. We apply the functors $\text{Ext}^i(-, f^*(t)) = \text{Ext}^i(-, O_{Y_0})$ to the triangle $\Delta_{Y_A}$. We get a long exact sequence as following

\[\ldots \to \text{Ext}^1(f^*\mathbb{L}_{A/A'}, O_{Y_0}) \to \text{Ext}^2(\mathbb{L}_{Y_A/A}, O_{Y_0}) \to \ldots\]

where the first vertical identification is due to

\[(A.2) \quad \tau_{\geq -1} f^*\mathbb{L}_{A/A'} = f^*(t)[1] = O_{Y_0}[1].\]

The obstruction element is the following,

\[h : \text{Id}_{O_{Y_0}} \to \text{Obs}(Y_A).\]

Since $g$ is étale, we have $\mathbb{L}_{X_A/Y_A} = 0$. It implies that $g^*\Delta_{Y_A} = \Delta_{X_A}$ by the triangle property of cotangent complexes. Hence, we have a natural commutative diagram of distinguished triangles

\[
\begin{array}{ccc}
\mathbb{R}\text{Hom}_{Y_A}(\Delta_{Y_A}, f^*(t)) & \xrightarrow{Rg_*} & \mathbb{R}\text{Hom}_{X_A}(g^*\Delta_{Y_A}, g^* f^*(t)) \\
\downarrow & & \downarrow \\
\mathbb{R}\text{Hom}_{Y_A}(\Delta_{Y_A}, O_{Y_0}) & \xrightarrow{g_*} & \mathbb{R}\text{Hom}_{X_A}(g^*\Delta_{Y_A}, g^* f^*(t)) \\
\downarrow & & \downarrow \\
\mathbb{R}\text{Hom}_{X_A}(\Delta_{X_A}, O_{X_0}) & \xrightarrow{Rg_*} & \mathbb{R}\text{Hom}_{X_A}(\Delta_{X_A}, O_{X_0})
\end{array}
\]

where the second vertical identification is due to [Har66, Proposition 5.10] and the row map is induced by the adjoint map $f^*(t) \to g_* g^* f^*(t)$. Since we know the following fact

\[\mathbb{H}^i(Y_A, \mathbb{R}\text{Hom}(F, G)) = \text{Ext}^i(F, G) \text{ and } \mathbb{H}^i(Y_A, Rg_* F) = \mathbb{H}^i(X_A, F),\]

apply the hypercohomology functor $\mathbb{H}^n(Y, -)$ to the morphism $\delta$, we get the following commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(O_{Y_0}, O_{Y_0}) & \xrightarrow{h} & \text{Ext}^2(\mathbb{L}_{Y_0}, O_{Y_0}) \xrightarrow{1} \text{Obs}(Y_A) \\
\downarrow & & \downarrow \\
\text{Hom}(O_{X_0}, O_{X_0}) & \xrightarrow{h} & \text{Ext}^2(\mathbb{L}_{X_0}, O_{X_0}) \xrightarrow{1} \text{Obs}(X_A)
\end{array}
\]
For a pair \((Y_A, L_A)\), we have similar argument once we notice that the obstruction elements are constructed in a similar way. Let us indicate it a little bit. For a pair \((Y, V)\) where \(V\) is a vector bundle on \(Y\), for a deformation pair \((Y_A, V_A)\) and a small extension \(0 \to C \to A' \to A \to 0\), the obstruction element

\[
\text{obs}(Y_A, V_A) \in \text{Ext}^2(At(V), V)
\]

is constructed in the following way

\[
\begin{array}{c}
V_A \otimes L_{Y_A/A'} \longrightarrow At'(V_A) \longrightarrow V_A \\
\downarrow \quad \downarrow \\
V_A \otimes L_{Y_A/A} \longrightarrow At(V_A) \longrightarrow V_A \\
\downarrow \quad \kappa \quad \downarrow \\
V_A \otimes f^*L_{A/A'}[1]
\end{array}
\]

where the first vertical arrows is tensor (A.1) with \(V_A\) and \(At'(V_A), At(V_A)\) are Atiyah extensions [Ill71]. By (A.2), we have a induced map

\[
At(V_A) \to V_A \otimes f^*L_{A/A'}[1]
\]

which is in

\[
\text{Ext}^1(At(V), V_A \otimes f^*L_{A/A'}[1]) = \text{Ext}^2(At(V_A), V_A \otimes \mathcal{O}_Y) = \text{Ext}^2(At(V), V).
\]

This map gives the obstruction element \(\text{obs}(Y_A, V_A) \in \text{Ext}^2(At(V), V)\).

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