THE 3D SPIN GEOMETRY OF THE QUANTUM TWO-SPHERE

SIMON BRAIN AND GIOVANNI LANDI

Abstract. We study a three-dimensional differential calculus \( \Omega^1 S^2_q \) on the standard Podleś quantum two-sphere \( S^2_q \), coming from the Woronowicz 4D+ differential calculus on the quantum group \( SU_q(2) \). We use a frame bundle approach to give an explicit description of \( \Omega^1 S^2_q \) and its associated spin geometry in terms of a natural spectral triple over \( S^2_q \). We equip this spectral triple with a real structure for which the commutant property and the first order condition are satisfied up to infinitesimals of arbitrary order.

Contents

1. Introduction 1
2. Preliminaries on Quantum Principal Bundles 3
   2.1. Differential structures 3
   2.2. Quantum principal bundles 4
   2.3. Framed quantum manifolds 5
3. The Standard Podleś Sphere 6
   3.1. The quantum group \( SU_q(2) \) 6
   3.2. The quantum universal enveloping algebra \( \mathcal{U}_q(\mathfrak{su}(2)) \) 7
   3.3. Line bundles on the quantum sphere \( S^2_q \) 9
4. Differential Structure of the Quantum Hopf Fibration 10
   4.1. Differential structure on \( SU_q(2) \) 10
   4.2. Framed manifold structure of \( S^2_q \) 11
5. The Spectral Geometry of \( S^2_q \) 16
   5.1. Background on spectral triples 16
   5.2. A Dirac operator on \( S^2_q \) 17
   5.3. Spectral properties of \( S^2_q \) 21
Acknowledgments 24
References 24

1. Introduction

The standard quantum two-sphere \( S^2_q \) has proven to be one of the most important and useful examples in trying to understand the relationship between the geometric/analytic world of noncommutative geometry and the algebraic setting of quantum group theory. At the algebraic level, it is known that \( S^2_q \) has a unique left-covariant two-dimensional differential calculus \( \mathcal{U}_q(\mathfrak{su}(2)) \). On the other hand, it is known that this same calculus is recovered \textit{via} analytic techniques by means of a noncommutative spin geometry \cite{17,18}. This compatibility has led to the discovery of other noncommutative two-dimensional
geometries on $S^2_q$ with a range of interesting properties [7]. In this paper, we extend the investigation to the noncommutative spin geometry of a differential calculus on $S^2_q$ whose dimension is equal to three.

Quantum two-spheres were constructed and classified by Podleś in [16]. The standard sphere $S^2_q$ is unique amongst the Podleś family in that it also appears as the base space of the noncommutative Hopf fibration $SU_q(2) \to S^2_q$ constructed in [1] as a basic example of a quantum principal bundle. By equipping the total space $SU_q(2)$ with the 3D differential calculus of [22], one finds that the two-dimensional differential calculus on $S^2_q$ appears as an associated vector bundle. This ‘quantum frame bundle’ approach to noncommutative geometry, developed in [13, 14], has been applied successfully to study a host of examples, not least the two-dimensional geometry of the quantum sphere $S^2_q$ itself.

The present paper also uses the frame bundle approach to study the geometry of $S^2_q$, but this time starting with the 4D+ differential calculus on $SU_q(2)$ of [22]. This calculus has the advantage of being bicovariant under both left and right translation, in contrast with the 3D calculus, which is only left-covariant. Using the framing theory we recover the three-dimensional differential calculus $\Omega^1 S^2_q$ of [17, 9, 10] on $S^2_q$. The methods we use are well-adapted to the principal bundle structure and as a consequence we immediately find an explicit description of the bimodule relations in $\Omega^1 S^2_q$, including a decomposition into irreducible components. We do not discuss the deeper aspects of the Riemannian geometry such as Hodge structure and connection theory: these will be developed elsewhere [12].

Our main results concern the spin geometry of the three-dimensional calculus $\Omega^1 S^2_q$. Remarkably, we find that the spinor bundle of $S^2_q$ is unchanged from the one used in [4, 14, 20] for the two-dimensional calculus. We construct a Dirac operator $D$ which implements the exterior derivative in $\Omega^1 S^2_q$, finding that the eigenvalues of $|D|$ grow not faster than $q^{-2j}$ for large $j$ and hence that the associated spectral triple has metric dimension zero.

Moreover, we equip this spectral triple with a $\mathbb{Z}_2$-grading operator and a real structure which is defined ‘up to compact operators’, in the sense that the ‘commutant property’ and the ‘first order condition’ for a real spectral triple [3] are satisfied up to infinitesimals of arbitrary order. As we shall see, this is in contrast with [1], where a ‘true’ real structure for the ‘two-dimensional’ calculus on $S^2_q$ was given (cf. also [20]), but is parallel to the results of [7] for the sphere $S^2_q$. We also find that the ‘KO-theoretic’ dimension of this real spectral triple is equal to the classical value, just two.

The paper is organised as follows. In §2 we give a brief overview of the construction of quantum differential calculi on quantum groups and their homogeneous spaces, followed by the general quantum frame bundle construction itself. Following this, §3 recalls the elementary geometry of the Hopf fibration $SU_q(2) \to S^2_q$ and the Hopf algebra $\mathcal{U}_q(\mathfrak{su}(2))$ which describes its symmetries. In §4 we describe the differential structure of the Hopf fibration. We start from the 4D quantum differential calculus on the total space $SU_q(2)$ from which we derive the calculus on the bundle fibre $U(1)$. The structure of the calculus $\Omega^1 S^2_q$ is then obtained as a ‘framed quantum manifold’ in the sense of [14]. Finally, in §5 we construct our spectral triple $(\mathcal{A}[S^2_q], \mathcal{H}, D)$ over $S^2_q$, which in addition we equip with a $\mathbb{Z}_2$-grading $\Gamma$ of the spinor bundle $\mathcal{H}$ and a real structure $J : \mathcal{H} \to \mathcal{H}$.

**Notation.** In this paper we make frequent use of the ‘$q$-numbers’ defined by

$$[x] := \frac{q^x - q^{-x}}{q - q^{-1}}$$

(1.1)
for each \( x \in \mathbb{R} \) and \( q \neq 1 \). Furthermore, for the sake of brevity we introduce the constants (1.2) \[ \mu := q + q^{-1}, \quad \nu := q - q^{-1} \]
to be used throughout the paper. Our convention is that \( \mathbb{N} = \{0, 1, 2, \ldots\} \).

2. Preliminaries on Quantum Principal Bundles

We start with some generalities on differential calculi and quantum principal bundles. These will be endowed both with universal and non-universal compatible calculi.

2.1. Differential structures. Let \( P \) be a complex \(*\)-algebra with unit. A first order differential calculus over \( P \) is a pair \((\Omega^1 P, d)\) where \( \Omega^1 P \) is a \( P\)-\( P \)-bimodule (the one-forms) and \( d : P \to \Omega^1 P \) is a linear map obeying the Leibniz rule

\[ d(ab) = a(db) + (da)b, \quad a, b \in P, \]

and such that the map \( P \otimes P \to \Omega^1 P \) defined by \( a \otimes b \mapsto adb \) is surjective.

The universal differential calculus over \( P \) is the pair \((\Omega^1 P, d)\), where \( \bar{\Omega}^1 P := \ker m \) is the kernel of the product map \( m : P \otimes P \to P \) on \( P \), with obvious bimodule structure

\[ p \cdot (a \otimes b) = pa \otimes b, \quad (a \otimes b) \cdot p = a \otimes bp, \quad a, b, p \in P \]

and \( \tilde{d} \) is defined by \( \tilde{d}p := 1 \otimes p - p \otimes 1 \), for each \( p \in P \). It is so-called because any other differential calculus \((\Omega^1 P, d)\) over \( P \) arises as a quotient \( \Omega^1 P = \bar{\Omega}^1 P/N_P \), where \( N_P \) is some \( P\)-\( P \)-sub-bimodule of \( \bar{\Omega}^1 P \). With the projection \( \pi_P : \bar{\Omega}^1 P \to \Omega^1 P \) one has

\[ d = \pi_P \circ \tilde{d}. \]

If \( H \) is a Hopf algebra, we write \( m_H : H \otimes H \to H \) and \( 1_H \) for its product and unit, \( \Delta_H : H \to H \otimes H \) and \( \epsilon_H : H \to \mathbb{C} \) for its coproduct and counit and \( S_H : H \to H \) for its antipode (when there is no possibility of confusion, we omit the subscript \( H \)). We use Sweedler notation \( \Delta(h) = h_{(1)} \otimes h_{(2)} \) for the coproduct. A differential calculus \( \Omega^1 H \) over a Hopf algebra \( H \) is said to be left-covariant if the coproduct \( \Delta \), viewed as a left coaction of \( H \) on itself, extends to a left coaction \( \Delta_L : \Omega^1 H \to H \otimes \Omega^1 H \) such that \( d \) is an intertwiner and \( \Delta_L \) is a bimodule map:

\[ \Delta_L(dh) = (id \otimes d)\Delta_L(h), \quad \Delta_L(h\omega) = \Delta(h) \cdot \Delta_L(\omega), \quad \Delta_L(\omega h) = \Delta_L(\omega) \cdot \Delta(h) \]

for all \( h \in H, \omega \in \Omega^1 H \). A similar definition holds for a right-covariant calculus, now with a right coaction \( \Delta_R : \Omega^1 H \to \Omega^1 H \otimes H \). A calculus is said to be bicovariant if it is both left and right covariant with commuting coactions. The universal calculus \( \bar{\Omega}^1 H \) is bicovariant when equipped with the left and right tensor product coactions on \( H \otimes H \).

Left-covariant differential calculi on a Hopf algebra \( H \) are classified as follows after [22]. First, it may be shown that the linear map

\[ r : H \otimes H \to H \otimes H, \quad r(a \otimes b) := ab_{(1)} \otimes b_{(2)}, \tag{2.1} \]

is an isomorphism with inverse

\[ r^{-1} : H \otimes H \to H \otimes H, \quad r^{-1}(a \otimes b) = aS(b_{(1)}) \otimes b_{(2)}. \tag{2.2} \]

Upon restricting \( r \) to the universal calculus \( \bar{\Omega}^1 H \) we obtain an isomorphism

\[ r : \bar{\Omega}^1 H \to H \otimes H^+, \]
where $H^+ := \ker \epsilon_H$ denotes the augmentation ideal of $H$. This is in fact an isomorphism of $H$-$H$ bimodules if we equip $H \otimes H^+$ with the bimodule structure

$$a \cdot (b \otimes \omega) = ab \otimes \omega, \quad (a \otimes \omega) \cdot b = ab_{(1)} \otimes \omega b_{(2)},$$

and an isomorphism of $H$-$H$-bicomodules if we equip $H \otimes H^+$ with the bicomodule structure

$$\Delta_L(a \otimes \omega) = a_{(1)} \otimes (a_{(2)} \otimes \omega), \quad \Delta_R(a \otimes \omega) = (a_{(1)} \otimes \omega_{(1)}) \otimes a_{(2)} \omega_{(2)},$$

Any left-covariant sub-bimodule $N_H$ of $\tilde{\Omega}^1 H$ is carried to a right ideal $I_H$ of $H^+$ by the map $r$ in $(2.1)$. Conversely, any right ideal $I_H$ arises in this way from a left-covariant sub-bimodule of $\tilde{\Omega}^1 H$. It follows that the left-covariant differential calculi on $H$ are in one-to-one correspondence with right ideals $I_H \subset H^+$; indeed, given such an $I_H$, one has $\Omega^1 H \cong H \otimes \Lambda^1$, where $\Lambda^1 \cong H^+/I_H$ are the left-invariant one-forms. We also write $\Omega^1_{\text{inv}} H := r^{-1}(\Lambda^1)$.

A left-covariant sub-bimodule $N_H$ is also right-covariant if and only if the corresponding ideal $I_H$ is stable under the right adjoint coaction

$$\text{Ad}_R : H \to H \otimes H, \quad \text{Ad}_R(a) = a_{(2)} \otimes S(a_{(1)})a_{(3)},$$

in the sense that $\text{Ad}_R(I_H) \subset I_H \otimes H$. It follows that bicovariant calculi on $H$ are in one-to-one correspondence with right ideals $I_H$ of $H^+$ which are $\text{Ad}_R$-stable.

Given a left-covariant differential calculus $\Omega^1 H$ over $H$, the quantum tangent space of $\Omega^1 H$ is the vector space

$$\mathcal{T}_H := \{ X \in H' \mid X(1) = 0 \text{ and } X(a) = 0 \text{ for all } a \in I_H \},$$

where the vector space $H'$ is the linear dual of $H$. This tangent space admits many properties analogous to the classical case, in particular there exists a unique bilinear form $\langle \cdot, \cdot \rangle : \mathcal{T}_H \times \Omega^1 H \to \mathbb{C}$ such that

$$\langle X | a db \rangle = \epsilon_H(a)X(b), \quad a, b \in H, \ X \in \mathcal{T}_H.$$

With respect to this bilinear form, the vector spaces $\Omega^1_{\text{inv}} H$ and $\mathcal{T}_H$ are non-degenerately paired, so that

$$\dim \Omega^1_{\text{inv}} H = \dim \mathcal{T}_H = \dim \Lambda^1.$$

This number is said to be the dimension of the left-covariant differential calculus $\Omega^1 H$.

### 2.2. Quantum principal bundles.

The general set-up for a principal fibration of noncommutative spaces is an algebra $P$ (playing the role of the algebra of functions on the total space) which is a right comodule algebra for a Hopf algebra $H$ with coaction $\delta_P : P \to P \otimes H$. The algebra of functions on the base space of the fibration is the subalgebra $M$ of $P$ consisting of coinvariant elements under $\delta_R$,

$$M := P^H = \{ p \in P : \delta_R(p) = p \otimes 1 \}.$$

For a well-defined bundle structure at the level of universal differential calculi, one requires exactness of the following sequence $\Pi$,

$$0 \to P(\tilde{\Omega}^1 M)P \xrightarrow{i} \tilde{\Omega}^1 P \xrightarrow{\text{ver}} P \otimes H^+ \to 0,$$

with $H^+$ the augmentation ideal, as before. The algebra inclusion $M \hookrightarrow P$ extends to an inclusion $\tilde{\Omega}^1 M \hookrightarrow \tilde{\Omega}^1 P$ of universal differential calculi, hence $P(\tilde{\Omega}^1 M)P$ are the analogues...
of the horizontal one-forms (classically this corresponds to the space of one-forms which have been pulled back from the base of the fibration). The map \( \text{ver} \) is defined by

\[
\text{ver}(p \otimes p') = p \delta_R(p');
\]

the generator of the vertical one-forms. We say that the inclusion \( M \hookrightarrow P \) is a quantum principal bundle with universal calculi and structure quantum group \( H \). Requiring exactness of the sequence (2.6) is equivalent to requiring that the induced canonical map

\[
(2.7) \quad \chi : P \otimes_M P \to P \otimes H, \quad p \otimes_M p' \mapsto p \delta_R(p')
\]

be bijective. If this is the case, one also says that the inclusion \( M \hookrightarrow P \) is a \( H \)-Hopf-Galois extension. This bijection condition is enough for a principal bundle structure at the level of universal differential calculi.

For a principal bundle with non-universal calculi extra conditions are required that we briefly recall. Assume then that \( P \) and \( M \) are equipped with differential calculi \( \Omega^1 P = \tilde{\Omega}^1 P/N_P \) and \( \Omega^1 M = \tilde{\Omega}^1 M/N_M \), where \( N_P \) and \( M_M \) are sub-bimodules of \( \tilde{\Omega}^1 P \) and \( \tilde{\Omega}^1 M \) respectively. Assume further that \( H \) is equipped with a left-covariant calculus \( \Omega^1 H \) corresponding to a right ideal \( I_H \).

Compatibility of the differential structures means that the calculi satisfy the conditions

\[
(2.8) \quad N_M = N_P \cap \tilde{\Omega}^1 M \quad \text{and} \quad \delta_R(N_P) \subset N_P \otimes H.
\]

The role of the first condition is to ensure that \( \Omega^1 M \) is spanned by elements of the form \( m \Delta n \) with \( m, n \in M \) and is hence obtained by restricting the calculus on \( P \). The second condition in (2.8) is sufficient to ensure covariance of \( \Omega^1 P \). Finally, we need the sequence

\[
(2.9) \quad 0 \to P(\Omega^1 M)P \to \Omega^1 P \xrightarrow{\text{ver}} P \otimes \Lambda^1 \to 0
\]

to be exact. This sequence is the analogue of the sequence (2.6) but now at the level of non-universal calculi. The \( P \)-\( P \)-bimodule \( P(\Omega^1 M)P \) once again makes up the horizontal one-forms and \( \text{ver}(p \otimes p') = p \delta_R(p') \) is the canonical map which generates the vertical one-forms. The condition

\[
(2.10) \quad \text{ver}(N_P) = P \otimes I_H
\]

ensures that the map

\[
\text{ver} : \Omega^1 P \to P \otimes \Lambda^1, \quad \Lambda^1 \simeq H^+/I_H
\]

is well-defined and yields that the sequence (2.9) is indeed exact.

2.3. Framed quantum manifolds. Suppose that the total space \( P \) of the bundle is itself a Hopf algebra equipped with a Hopf algebra surjection \( \pi : P \to H \). Here we have a coaction of \( H \) on \( P \) by coproduct and projection to \( H \),

\[
\delta_R : P \to P \otimes H, \quad \delta_R = (\text{id} \otimes \pi) \Delta.
\]

The base is then the quantum homogeneous space \( M = P^H \) of coinvariants and the algebra inclusion \( M \hookrightarrow P \) is automatically an \( H \)-Hopf-Galois extension, i.e. a quantum principal bundle with universal calculi. To impose non-universal differential structure we suppose that \( \Omega^1 P \) is left-covariant for \( P \) and \( \Omega^1 H \) is left-covariant for \( H \), so that they are defined by right ideals \( I_P \) and \( I_H \) of \( P^+ \) and \( H^+ \) respectively. We ensure the first of
by taking it as a definition of $\Omega^1 M$; in the case at hand, the remaining compatibility conditions in (2.8)--(2.10) reduce to

\[(2.11) \quad (\text{id} \otimes \pi)\text{Ad}_R(I_P) \subset I_P \otimes H, \quad \pi(I_P) = I_H.\]

Thus a choice of left-covariant calculus on $P$ satisfying these conditions automatically gives a principal bundle with non-universal calculi [14].

We say that an algebra $M$ is a framed quantum manifold if it is the base of a quantum principal bundle, $M = PH$, to which $\Omega^1 M$ is an associated vector bundle. To give $M$ as a framed quantum manifold we therefore require not only a quantum principal bundle $\delta_R : P \to P \otimes H$ as above but also a right $H$-comodule $V$, so that $E := (P \otimes V)^H$ plays the role of the sections of the corresponding associated vector bundle (the space $P \otimes V$ is equipped with the tensor product coaction). Moreover, we require a ‘soldering form’ $\theta : V \to P \Omega^1 M$ such that the map

\[s_\theta : E \to \Omega^1 M, \quad p \otimes v \mapsto p\theta(v)\]

is an isomorphism.

For a general $M$ it is usually not obvious how to go about looking for a framing. However in the case of a quantum homogeneous space with compatible calculi one has a ‘standard’ framing in the following way [14]. If the conditions in (2.11) are satisfied then the algebra $M = PH$ is automatically framed by the bundle $(P, H, M)$. The $H$-comodule $V$ and soldering form $\theta$ are given explicitly by the formulæ

\[(2.12) \quad V = (P^+ \cap M)/(I_P \cap M), \quad \Delta_R v = \tilde{v}_{(2)} \otimes S\pi(\tilde{v}_{(1)}), \quad \theta(v) = S\tilde{v}_{(1)} d\tilde{v}_{(2)},\]

with $\tilde{v}$ any representative of $v$ in $P^+ \cap M$ and $\Delta(\tilde{v}) = \tilde{v}_{(1)} \otimes \tilde{v}_{(2)}$ is the coproduct on $P$.

3. The Standard Podleś Sphere

We recall here some of the basic geometry of the so-called standard Podleś quantum two-sphere $S^2_q$ of [16]. We begin with the quantum group $A[SU_q(2)]$ and its symmetries $U_q(su(2))$, from which we obtain the quantum sphere $S^2_q$ as the base space of the quantum Hopf fibration $SU_q(2) \to S^2_q$. Finally we sketch the construction of a family of quantum line bundles over $S^2_q$ which shall prove useful in what is to follow.

3.1. The Quantum Group $SU_q(2)$. Recall that the coordinate algebra $A[M_q(2)]$ of functions on the quantum matrices $M_q(2)$ is the associative unital algebra generated by the entries of the matrix

\[x = (x_{ij}) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\]

obeying the relations

\[(3.1) \quad ab = qba, \quad ac = qca, \quad bd = qdb, \quad cd = qdc, \quad bc = cb, \quad ad - da = (q - q^{-1})bc,\]

with $0 \neq q \in \mathbb{C}$ a deformation parameter. The algebra $A[M_q(2)]$ has a coalgebra structure given by $\Delta(x_{ij}) = x_{ij} \otimes x_{ij}$ and $\epsilon(x_{ij}) = \delta_{ij}$. From $A[M_q(2)]$ we obtain a Hopf algebra
\( \mathcal{A}[\text{SL}_q(2)] \) upon quotienting by the determinant relation \( ad = 1 + qbc \) (equivalently \( da = 1 + q^{-1}bc \)) and defining an antipode by

\[
S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -q^{-1}b \\ -qc & a \end{pmatrix}.
\]

When the deformation parameter \( q \) is taken to be real \( \mathcal{A}[M_q(2)] \) is made into a \(*\)-algebra by defining the anti-linear involution

\[
x^* = \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix} = \begin{pmatrix} d & -qc \\ -q^{-1}b & a \end{pmatrix},
\]

It is not difficult to see that \( \mathcal{A}[\text{SL}_q(2)] \) inherits this \(*\)-structure. Without loss of generality we take \( 0 < q < 1 \). The compact quantum group \( \mathcal{A}[\text{SU}_q(2)] \) is defined to be the quotient of \( \mathcal{A}[\text{SL}_q(2)] \) by the additional relations \( S(x_k^l) = (x_k^l)^* \). Thus in \( \mathcal{A}[\text{SU}_q(2)] \) we have

\[
x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix}.
\]

The algebra relations become

\[
ac = qca, \quad ac^* = qc^*a, \quad cc^* = c^*c, \quad aa^* + q^2cc^* = 1, \quad a^*a + c^*c = 1,
\]

together with their conjugates. On generators, the counit is \( \epsilon(a) = \epsilon(a^*) = 1, \epsilon(c) = \epsilon(c^*) = 0 \) and the antipode is now \( S(a) = a^*, S(a^*) = a, S(c) = -qc, S(c^*) = -q^{-1}c^* \), while the coproduct now reads \( \Delta(a) = a \otimes a - qc^* \otimes c \), \( \Delta(c) = c \otimes a + a^* \otimes c \) and \( \Delta(a^*) = a^* \otimes a^* - qc \otimes c^* \), \( \Delta(c^*) = c^* \otimes a^* + a \otimes c^* \).

3.2. The quantum universal enveloping algebra \( \mathcal{U}_q(\mathfrak{su}(2)) \). The quantum universal enveloping algebra \( \mathcal{U}_q(\mathfrak{su}(2)) \) is the unital \(*\)-algebra generated by the four elements \( K, K^{-1}, E, F \), with \( KK^{-1} = K^{-1}K = 1 \), subject to the relations

\[
K^{\pm 1}E = q^{\pm 1}EK^{\pm 1}, \quad K^{\pm 1}F = q^{\mp 1}FK^{\pm 1}, \quad [E, F] = (q - q^{-1})^{-1}(K^2 - K^{-2})
\]

and the \(*\)-structure

\[
K^* = K, \quad E^* = F, \quad F^* = E.
\]

It becomes a Hopf \(*\)-algebra when equipped with the coproduct \( \Delta \) and counit \( \epsilon \) defined on generators by

\[
\Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}, \quad \Delta(E) = E \otimes K + K^{-1} \otimes E, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F,
\]

\[
\epsilon(K) = 1, \quad \epsilon(E) = 0, \quad \epsilon(F) = 0,
\]

and with antipode \( S \) defined by \( S(K) = K^{-1}, S(E) = -qE, S(F) = -q^{-1}F \) on generators. The maps \( \Delta, \epsilon \) are extended as \(*\)-algebra maps, whereas \( S \) extends as a \(*\)-anti-algebra map. From the relations (3.3), one finds that the quadratic Casimir element

\[
C_q := FE + (q - q^{-1})^{-2}(qK^2 - 2 + q^{-1}K^{-2}) - \frac{1}{4}
\]

generates the centre of the algebra \( \mathcal{U}_q(\mathfrak{su}(2)) \).

The finite-dimensional irreducible \(*\)-representations \( \pi_j \) of \( \mathcal{U}_q(\mathfrak{su}(2)) \) are indexed by a half-integer \( j = 0, 1/2, 1, 3/2, \ldots \) called the spin of the representation. Explicitly, these
representations are given by
\begin{align}
\pi_j(K)|j, m\rangle &= q^m|j, m\rangle, \\
\pi_j(F)|j, m\rangle &= (|j - m|)[j + m + 1]^{1/2}|j, m + 1\rangle, \\
\pi_j(E)|j, m\rangle &= (|j - m + 1|)[j + m]^{1/2}|j, m - 1\rangle,
\end{align}
where the vectors \(|j, m\rangle\) for \(m = -j, -j + 1, \ldots, j - 1, j\) form an orthonormal basis of the \((2j + 1)\)-dimensional irreducible \(U_q(\mathfrak{su}(2))\)-module \(V^j\). Moreover, \(\pi_j\) is a \(*\)-representation with respect to the Hermitian inner product on \(V^j\) for which the vectors \(|j, m\rangle\) are orthonormal. In each representation, the Casimir \(C_q\) of (3.6) acts as a multiple of the identity, with constant given by
\begin{equation}
\pi_j(C_q) = |j + \frac{1}{2}|^2 - \frac{1}{4}
\end{equation}
as one may easily verify by direct computation.

The Hopf \(*\)-algebras \(A(\mathfrak{su}(2))\) and \(U_q(\mathfrak{su}(2))\) are dually paired \textit{via} a bilinear pairing
\begin{equation}
(\cdot, \cdot) : U_q(\mathfrak{su}(2)) \times A[\mathfrak{su}(2)] \to \mathbb{C}
\end{equation}
which is non-degenerate. It is defined on generators by
\begin{align}
(K, a) &= q^{-1/2}, \\
(K^{-1}, a) &= q^{1/2}, \\
(K, d) &= q^{1/2}, \\
(K^{-1}, d) &= q^{-1/2}, \\
(E, c) &= 1, \\
(F, b) &= 1,
\end{align}
with all other combinations of generators pairing to give zero. The pairing is extended to products of generators \textit{via} the requirements
\begin{align}
(\Delta(X), p_1 \otimes p_2) &= (X, p_1 p_2), \\
(X_1 X_2, p) &= (X_1 \otimes X_2, \Delta(p)), \\
(X, 1) &= \epsilon(X), \\
(1, p) &= \epsilon(p)
\end{align}
for all \(X, X_1, X_2 \in U_q(\mathfrak{su}(2))\) and all \(p, p_1, p_2 \in A[\mathfrak{SU}_q(2)]\). It is compatible with the antipode and the \(*\)–structures in the sense that, for all \(X \in U_q(\mathfrak{su}(2))\), \(p \in A[\mathfrak{SU}_q(2)]\),
\begin{equation}
(S(X), p) = (X, S(p)), \quad (X^*, p) = S((X, (S(p))^*)^*), \quad (X, p^*) = S((X)^*, p).
\end{equation}
Using the pairing, there is a canonical left action of \(U_q(\mathfrak{su}(2))\) on \(A[\mathfrak{SU}_q(2)]\) defined by
\begin{equation}
\triangleright : U_q(\mathfrak{su}(2)) \times A[\mathfrak{SU}_q(2)] \to A[\mathfrak{SU}_q(2)], \quad X \triangleright p := p_{(1)}(X, p_{(2)}),
\end{equation}
where \(X \in U_q(\mathfrak{su}(2)), p \in A[\mathfrak{SU}_q(2)]\) and \(\Delta(p) = p_{(1)} \otimes p_{(2)}\) denotes the coproduct on \(A[\mathfrak{SU}_q(2)]\). In particular, this action works out on generators to be
\begin{align}
E \triangleright a &= b, & E \triangleright c &= d, & F \triangleright b &= a, & F \triangleright d &= c, \\
K^{\pm 1}a &= q^{\pm 1/2}a, & K^{\pm 1}c &= q^{\pm 1/2}c, & K^{\pm 1}b &= q^{\mp 1/2}b, & K^{\pm 1}d &= q^{\mp 1/2}d, \\
E \triangleright b &= 0, & E \triangleright d &= 0, & F \triangleright a &= 0, & F \triangleright c &= 0.
\end{align}
This action makes \(A[\mathfrak{SU}_q(2)]\) into a left \(U_q(\mathfrak{su}(2))\)-module \(*\)-algebra, in the sense that \(X \triangleright (p_{(1)} p_{(2)}) = (X_{(1)} \triangleright p_{(1)})(X_{(2)} \triangleright p_{(2)}), X \triangleright 1 = 1, X \triangleright p^* = ((S(X))^* \triangleright p)^*\) for all \(p, p_1, p_2 \in A[\mathfrak{SU}_q(2)], X \in U_q(\mathfrak{su}(2))\). There is also a canonical right action of \(U_q(\mathfrak{su}(2))\) on \(A[\mathfrak{SU}_q(2)]\), defined by
\begin{equation}
\triangleleft : A[\mathfrak{SU}_q(2)] \times U_q(\mathfrak{su}(2)) \to A[\mathfrak{SU}_q(2)], \quad p \triangleleft X := (X, p_{(1)}) p_{(2)}
\end{equation}
for $X \in \mathcal{U}_q(\mathfrak{su}(2))$ and $p \in \mathcal{A}[\mathcal{U}_q(2)]$, with properties similar to those for the left action. These two canonical actions commute amongst one another.

3.3. **Line bundles on the quantum sphere $S_q^2$.** The coordinate algebra $H := \mathcal{A}[\mathcal{U}(1)]$ of the group $\mathcal{U}(1)$ is the commutative unital $*$-algebra generated by $t, t^*$, subject to the relations $tt^* = t^*t = 1$. It is a Hopf algebra when equipped with the coproduct, counit and antipode

$$\Delta(t) = t \otimes t, \quad \epsilon(t) = 1, \quad S(t) = t^*,$$

extended as $*$-algebra maps. There is a canonical Hopf algebra projection given on generators by

$$\pi : \mathcal{A}[\mathcal{U}_q(2)] \to \mathcal{A}[\mathcal{U}(1)], \quad \pi \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) := \left( \begin{array}{cc} t & 0 \\ 0 & t^* \end{array} \right).$$

Using this projection a right coaction of $H = \mathcal{A}[\mathcal{U}(1)]$ on $P := \mathcal{A}[\mathcal{U}_q(2)]$ is defined by

$$\delta_R : \mathcal{A}[\mathcal{U}_q(2)] \to \mathcal{A}[\mathcal{U}_q(2)] \otimes \mathcal{A}[\mathcal{U}(1)], \quad \delta_R(x^i) := x^i \otimes \pi(x^i).$$

In fact this coaction is the same thing as a $\mathbb{Z}$-grading on $\mathcal{A}[\mathcal{U}_q(2)]$ for which the generators have degrees

$$\deg(a) = \deg(c) = 1, \quad \deg(b) = \deg(d) = -1.$$ 

The subalgebra of coinvariants under this coaction is denoted $\mathcal{A}[S^2_q]$,

$$\mathcal{A}[S^2_q] := \{ m \in \mathcal{A}[\mathcal{U}_q(2)] \mid \delta_R(m) = m \otimes 1 \}. $$

We shall frequently write $M := \mathcal{A}[S^2]$. This algebra is precisely the subalgebra generated by elements of degree zero: it is the unital $*$-algebra generated by the elements

$$b_+ := cd, \quad b_- := ab, \quad b_0 := bc$$

subject to the relations

$$b_0b_+ = q^{+2}b_+b_0, \quad q^{-2}b_-b_+ = q^2b_+b_- + (1 - q^2)b_0,$$

$$b_+b_- = b_0(1 + q^{-1}b_0)$$

inherited from those of $\mathcal{A}[\mathcal{U}_q(2)]$. In the classical limit $q \to 1$, the first line of relations becomes the statement that the algebra is commutative, whereas the second line becomes the sphere relation for the classical two-sphere $S^2$. The quantum sphere $S_q^2$ is precisely the standard Podleś sphere of [16]. The canonical algebra inclusion $M \hookrightarrow P$ is well-known to be a Hopf-Galois extension [17] and hence a quantum principal bundle with universal differential calculi whose typical fibre is determined by $H := \mathcal{A}[\mathcal{U}(1)]$.

The coaction (3.16) of $H$ on $\mathcal{A}[\mathcal{U}_q(2)]$ is also used to define a family of line bundles over the quantum sphere $S_q^2$, indexed by $n \in \mathbb{Z}$:

$$\mathcal{L}_n := \{ x \in \mathcal{A}[\mathcal{U}_q(2)] \mid \delta_R(x) = x \otimes t^{-n} \}. $$

One has the decomposition [15]

$$\mathcal{A}[\mathcal{U}_q(2)] = \bigoplus_{n \in \mathbb{Z}} \mathcal{L}_n.$$ 

In particular $\mathcal{L}_0 = \mathcal{A}[S^2_q]$ and one finds that $\mathcal{L}_n \cong \mathcal{L}_{-n}$ and $\mathcal{L}_n \otimes_{\mathcal{A}[S^2_q]} \mathcal{L}_m \cong \mathcal{L}_{n + m}$ for each $n, m \in \mathbb{Z}$. Moreover,

$$E \triangleright \mathcal{L}_n \subset \mathcal{L}_{n+2}, \quad F \triangleright \mathcal{L}_n \subset \mathcal{L}_{n-2}, \quad K^{\pm 1} \triangleright \mathcal{L}_n \subset \mathcal{L}_n.$$
for all $n \in \mathbb{Z}$, as can be checked directly using (3.13) and (5.10).

It is known that each $L_n$ is a finitely generated projective (say) left $\mathcal{A}[S^2_q]$-module of rank one [21]. In this way, we think of the module $L_n$ as the space of sections of a line bundle over $S^2_q$ with winding number $-n$.

4. Differential Structure of the Quantum Hopf Fibration

In this section we equip the quantum group $SU_q(2)$ with a four-dimensional bicovariant differential calculus, originally described in [22]. Using this, the base space $S^2_q$ of the Hopf fibration inherits a three-dimensional differential calculus which was originally described in [17], although we describe it here in terms which are more compatible with the principal bundle structure. Finally we show that $S^2_q$ is a framed quantum manifold, in the sense that its cotangent bundle is a vector bundle associated to the Hopf fibration $SU_q(2) \to S^2_q$.

4.1. Differential structure on $SU_q(2)$. In the following we write $\epsilon_P$ for the counit of the Hopf algebra $P := \mathcal{A}[SU_q(2)]$. In terms of the matrix elements in (3.3), we define $I_P$ to be the right ideal of $P^+ := \text{Ker} \epsilon_P$ generated by the nine elements

\begin{equation}
(4.1) \quad b^2, \ c^2, \ b(a - d), \ c(a - d), \ a^2 + q^2d^2 - (1 + q^2)(ad + q^{-1}bc),
zb, \ zc, \ z(a - d), \ z(q^2a + d - (q^2 + 1)),
\end{equation}

where $z := q^2a + d - (q^3 + q^{-1})$. As discussed in [21], this ideal defines a left-covariant first order differential calculus on $SU_q(2)$, which we denote by $\Omega^1 P$. In fact, one checks that $I_P$ is stable under the right adjoint coaction $\text{Ad}_R$ and so this calculus is bicovariant under left and right coactions of $\mathcal{A}[SU_q(2)]$. It is precisely the $4D_+$ calculus on $SU_q(2)$ introduced in [22]; indeed, one may check that the space $\Lambda^1 \cong P^+/I_P$ of left-invariant one-forms is a four-dimensional vector space.

Following [11], we define elements $L_-, L_0, L_+, L_z$ of $U_q(\mathfrak{su}(2))$ by

\begin{equation}
(4.2) \quad q(q - q^{-1})^2 \left( C_q + \frac{1}{2} - [\frac{1}{2}]^2 \right) = qL_0 + q^{-1}L_z.
\end{equation}

The vectors $L_0$ and $L_z$ are related to the quantum Casimir [3.10] by

\begin{equation}
(4.3) \quad \omega_-(\begin{pmatrix} a & b \\
\end{pmatrix}) = \begin{pmatrix} a & b \\
\end{pmatrix} \omega_+ + \nu^2q^{-1}(\begin{pmatrix} b & 0 \\
\end{pmatrix} + \begin{pmatrix} 0 & q^{-1}b \\
\end{pmatrix}) \omega_0;
\end{equation}

\begin{equation}
\omega_+(\begin{pmatrix} a & b \\
\end{pmatrix}) = \begin{pmatrix} a & b \\
\end{pmatrix} \omega_+ + \nu^2q^{-1}(\begin{pmatrix} 0 & a \\
\end{pmatrix} \omega_0; \quad \omega_0(\begin{pmatrix} a & b \\
\end{pmatrix}) = (\begin{pmatrix} q^{-1}a & qb \\
\end{pmatrix} \omega_0;
\end{equation}

\begin{equation}
\omega_z(\begin{pmatrix} a & b \\
\end{pmatrix}) = \begin{pmatrix} a & b \\
\end{pmatrix} \omega_+ + \nu^2q^{-1}(\begin{pmatrix} 0 & a \\
\end{pmatrix} \omega_0 + \begin{pmatrix} b & 0 \\
\end{pmatrix} \omega_+ + \begin{pmatrix} qa & q^{-1}b \\
\end{pmatrix} \omega_z.
\end{equation}

Let $\{\omega_-, \omega_0, \omega_+, \omega_z\}$ be a basis of the space of left-invariant one-forms $\Lambda^1$ such that $(L_j, \omega_k) = \delta_{jk}$ for $j, k = -, 0, +, z$. As given in [19], the bimodule relations in the calculus $\Omega^1 P$ with respect to these one-forms are:

\begin{equation}
(4.3) \quad \omega_-(\begin{pmatrix} a & b \\
\end{pmatrix}) = \begin{pmatrix} a & b \\
\end{pmatrix} \omega_+ + \nu^2q^{-1}(\begin{pmatrix} b & 0 \\
\end{pmatrix} \omega_0; \quad \omega_0(\begin{pmatrix} a & b \\
\end{pmatrix}) = (\begin{pmatrix} q^{-1}a & qb \\
\end{pmatrix} \omega_0;
\end{equation}

\begin{equation}
\omega_z(\begin{pmatrix} a & b \\
\end{pmatrix}) = \begin{pmatrix} a & b \\
\end{pmatrix} \omega_+ + \nu^2q^{-1}(\begin{pmatrix} 0 & a \\
\end{pmatrix} \omega_0 + \begin{pmatrix} b & 0 \\
\end{pmatrix} \omega_+ + \begin{pmatrix} qa & q^{-1}b \\
\end{pmatrix} \omega_z.
\end{equation}
In these terms, the exterior derivative \( d : \mathcal{A}[\text{SU}_q(2)] \rightarrow \Omega^1 P \) has the form
\[
d p = (L_- \triangleright p) \omega_- + (L_0 \triangleright p) \omega_0 + (L_+ \triangleright p) \omega_+ + (L_z \triangleright p) \omega_z, \quad p \in \mathcal{A}[\text{SU}_q(2)],
\]
where \( \triangleright \) is the left action of \( \mathcal{U}_q(\mathfrak{su}(2)) \) on \( \mathcal{A}[\text{SU}_q(2)] \) defined in (3.12). By using the formulæ (3.13) to compute the action of \( L_0, L_z, L_+ - L_- \) on the generators of \( \mathcal{A}[\text{SU}_q(2)] \) and then substituting into (4.4), one obtains the explicit expressions
\[
\begin{align*}
da &= (q^{-1} - 1 + b^2 q^{-1}) a \omega_0 + b \omega_+ + (q - 1) a \omega_z, \\
db &= a \omega_- + (q - 1) b \omega_0 + (q^{-1} - 1) b \omega_z, \\
dc &= (q^{-1} - 1 + b^2 q^{-1}) c \omega_0 + d \omega_+ + (q - 1) c \omega_z, \\
dd &= c \omega_- + (q - 1) d \omega_0 + (q^{-1} - 1) d \omega_z
\end{align*}
\]
for the differentials of the matrix generators of \( \mathcal{A}[\text{SU}_q(2)] \) in terms of these left-invariant one-forms.

### 4.2. Framed manifold structure of \( S^2_q \)

Next we use (2.3) to compute the cotangent bundle \( \Omega^1 S^2_q \) of the base space \( S^2_q \) of the Hopf fibration as an associated vector bundle. As before, we write \( P = \mathcal{A}[\text{SU}_q(2)] \) for the algebra of functions on the total space of the Hopf fibration, \( M = \mathcal{A}[S^2_q] \) for the algebra of functions on the base and \( H = \mathcal{A}[\text{U}(1)] \) for the structure quantum group. Recall the right coaction \( \delta_R : P \rightarrow P \otimes H \) defined in (3.16) and the canonical projection \( \pi : P \rightarrow H \) defined in (3.15).

The differential calculus on \( P \) is taken to be the four-dimensional bicovariant calculus \( \Omega^1 P \) defined in the previous section; it is defined in terms of the \( \text{Ad}_R \)-invariant ideal \( I_P \) generated by the elements in (4.1). Now writing \( \epsilon_H \) for the counit of \( H \), we obtain a bicovariant differential calculus \( \Omega^1 H \) on \( H = \mathcal{A}[\text{U}(1)] \) by projecting the ideal \( I_P \) to obtain an ideal \( I_H := \pi(I_P) \) of \( \text{Ker} \epsilon_H \). As such, \( I_H \) is generated by the three elements
\[
t^2 + q^2 t^* - (1 + q^2), \quad z(t - t^*), \quad z(q^2 t + t^* - (q^2 + 1)),
\]
again with \( z = q^2 t + t^* - (q^2 + q^{-1}) \), where \( t, t^* \) are the generators of \( H \).

**Lemma 4.1**. The calculus \( \Omega^1 H \) is one-dimensional. It is spanned as a left module by the left-invariant one-form \( \omega_t := t^* dt \) and has bimodule relations
\[\omega_t t = q t \omega_t, \quad \omega_t t^* = q^{-1} t^* \omega_t,\]
where \( t, t^* \) are the generators of \( H = \mathcal{A}[\text{U}(1)] \).

**Proof.** We define an equivalence relation \( \sim \) on \( H^+ \) by \( x \sim y \) if and only if \( x - y \in I_H \). By taking a linear combination of the generators in (4.6), one finds in particular that \( (t - 1) + q(t^* - 1) \sim 0 \), which is our key equivalence. Using it, one deduces that
\[
\begin{align*}
t^2 &= (t + 1)(t - 1) + 1 \sim -q(t + 1)(t^* - 1) + 1 = -q(t^* - t) + 1 \sim (q + 1)(t - 1) + 1, \\
t^* t^2 &= (t^* + 1)(t^* - 1) + 1 \sim -q^{-1}(t^* + 1)(t - 1) + 1 \sim -q^{-1}(t - t^*) + 1 \sim -q^{-1}(1 + q^{-1})(t - 1) + 1,
\end{align*}
\]
so that every quadratic polynomial in \( t, t^* \) and 1 is equivalent to a linear combination of \( t - 1 \) and \( t^* - 1 \). By induction any polynomial in \( t \) is equivalent to such a linear combination. Applying the key equivalence once more tells us that we can always eliminate \( t^* - 1 \). Thus we take \( t - 1 \) as a representative of the quotient space \( H^+/I_H \) and \( \omega_t := r^{-1}(1 \otimes (t - 1)) \)
as the corresponding left-invariant one-form, which spans the calculus $\Omega^1 H$ as a left $H$-
module. To obtain the bimodule relations, we compute for example that 
\[
\omega t = ((t^* - 1) \otimes [t - 1])t = (1 - t) \otimes [t^2 - t] = qt(t^* - 1) \otimes [t - 1] = qt\omega_t,
\]
where $[ \cdot ]$ denotes an equivalence class modulo $I_H$. The first and last equalities use the definition of the map $r$ and the middle equality uses the bimodule structure $[2.3]$. \hfill \square

The differential calculus $\Omega^1 M$ on the base of the fibration is defined by restricting the
calculus $\Omega^1 P$ to $M$. This means that it is defined as the quotient $\Omega^1 M := \Omega^1 M/N_M$, where
$N_M$ is the $M$-$M$-bimodule $N_M := N_P \cap \Omega^1 M$. We postpone the computation of generators and
relations for $\Omega^1 M$ and observe that for now we have the following expressions for the exterior
derivative on $M$ in terms of the left-invariant one-forms $\omega_\pm, \omega_0$.

**Lemma 4.2.** The exterior derivative $d$ acts on $M = \mathcal{A}[S^2_q]$ as
\[
(4.7) \quad \begin{pmatrix} db_+ \\ db_0 \\ db_- \end{pmatrix} = \begin{pmatrix} d^2 & \mu \nu^2 q^{-1} cd & q c^2 \\ db & \nu^2 q^{-1} (1 + \mu bc) & ac \\ b^2 & \mu \nu^2 q^{-1} ab & qa^2 \end{pmatrix} \begin{pmatrix} \omega_+ \\ \omega_0 \\ \omega_- \end{pmatrix}
\]
in terms of the generators $b_\pm, b_0$ of $M$ given in $(3.18)$.

**Proof.** This follows from direct computation. For example, to compute $db_+$ the Leibniz rule yields
\[
d b_+ = d(cd) = (dc)d + c(dd).
\]
One uses the expressions $(4.5)$ to rewrite $dc, dd$ in terms of $\omega_\pm$ and $\omega_0$, then the bimodule
relations in Eqs. $(4.3)$ to collect all coefficients to the left. Combining together alike terms
yields the expression as stated. The same method works for computing $db_0$ and $db_-$. \hfill \square

**Lemma 4.3.** With $P$, $H$ and $M$ as above, the differential calculus $\Omega^1 P$, $\Omega^1 H$ and $\Omega^1 M$
satisfy the compatibility conditions of $(2.11)$.

**Proof.** The relation $\pi(I_P) = I_H$ holds by definition of the calculus on $H$. It is sufficient to verify the $\text{Ad}_R$-condition in $(2.11)$ on generators: one finds that
\[
(id \otimes \pi)\text{Ad}_R(c^2) = c^2 \otimes t^4, \quad (id \otimes \pi)\text{Ad}_R(c(a - d)) = c(a - d) \otimes t^2,
\]
\[
(id \otimes \pi)\text{Ad}_R(b^2) = b^2 \otimes t^{4}, \quad (id \otimes \pi)\text{Ad}_R(b(a - d)) = b(a - d) \otimes t^{2},
\]
\[
(id \otimes \pi)\text{Ad}_R(zc) = zc \otimes t^2, \quad (id \otimes \pi)\text{Ad}_R(zb) = zb \otimes t^{-2},
\]
with all other generators coinvariant under the map $(id \otimes \pi)\text{Ad}_R$. \hfill \square

This means that we may apply $(2.3)$ to express $S^2_q$ as a framed quantum manifold. The
framing comodule $V$ is computed as follows. Clearly $P^+ \cap M$ is equal to $M^+ = \text{Ker } \epsilon_M$, the
restriction of the counit $\epsilon_P$ to the subalgebra $M$. In our case, with $M = \mathcal{A}[S^2_q]$ being
generated by $b_\pm, b_0$, we have that $M^+ = \langle b_0, b_\pm \rangle$ as a right ideal. To compute $I_P \cap M$ we
note that, since the generators $b(a - d), c(a - d), a^2 + q^2 d^2 - (1 + q^2)(ad + q^{-1} bc), zb, zc,$
$z(a - d), z(q^2 a + d - (q^2 + 1))$ are not of homogeneous degree, the ideal that each of them
generates has no intersection with $M$. Thus we concentrate on the generators $b^2, c^2$ of $I_P$. The
elements of degree zero in $I_P$ include $b^2 \{a^2, ac, c^2\}$ and so we see that $b^2_+, b^2_-, b^2_0$ all
lie in $I_P \cap M$. Similarly, from the ideal $\langle c^2 \rangle$ we see that $b^2_+$ and $b_+ b_0$ are also in $I_P \cap M$. From this discussion we obtain
\[
(4.8) \quad V = \langle b_0, b_\pm \rangle/\langle b^2_+, b^2_-, b^2_0 \rangle.
\]
Hence $V$ is three-dimensional with representatives $b_\pm$ and $b_0$. We compute the right coaction of $H$ on $V$ from \(^{(2,12)}\) as
\[
\Delta_R(b_+) = cd \otimes S\pi(d^2) = b_+ \otimes t^2, \quad \Delta_R(b_-) = ab \otimes S\pi(a^2) = b_- \otimes t^{*2}, \quad \Delta_R(b_0) = bc \otimes 1 = b_0 \otimes 1.
\]
Hence $V = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ and the associated bundle
\[
E = L_{-2} \oplus L_0 \oplus L_{+2} = A[S\mu_2(2)]_2 \oplus A[S\mu_2(2)]_0 \oplus A[S\mu_2(2)]_{-2}
\]
is the direct sum of the line bundles over $S^2_q$ with winding numbers $-2, 0$ and $2$. This yields the following theorem.

**Theorem 4.4.** The homogeneous space $S^2_q$ is a framed quantum manifold with cotangent bundle
\[
\Omega^1 S^2_q \cong L_{-2} \oplus L_0 \oplus L_{+2}.
\]
The isomorphism is given by the soldering form
\[
\theta(b_+) = q^2 c^2 db_- - q \mu ac db_0 + a^2 db_+ = \omega_+, \quad \theta(b_0) = -qdcd db_- + (1 + \mu bc) db_0 - q^{-1} bad b_+ = \nu^2 q^{-1} \omega_0, \quad \theta(b_-) = d^2 db_- - q^{-1} \mu bd db_0 + q^{-2} b^2 db_+ = q \omega_-
\]
and makes $\Omega^1 S^2_q$ projective as a left $A[S^2_q]$-module.

**Proof.** The only remaining part is to compute the soldering form $\theta(b_\pm), \theta(b_0)$. We find the left coaction on $M = A[S^2_q]$ inherited from the coproduct on $A[S\mu_2(2)]$ to be
\[
\Delta_L(b_+) = \Delta_L(cd) = c^3 \otimes b_- + cd \otimes (1 + \mu b_0) + d^2 \otimes b_+,
\]
\[
\Delta_L(b_0) = \Delta_L(bc) = ca \otimes b_- + 1 \otimes b_0 + bc \otimes (1 + \mu b_0) + db \otimes b_+, \quad \Delta_L(b_-) = \Delta_L(ab) = a^2 \otimes b_- + ab \otimes (1 + \mu b_0) + b^2 \otimes b_+.
\]
In fact these coproducts were already used in computing $\Delta_R$ above. This time we apply the antipode $S$ to the first tensor factor to obtain
\[
\theta(b_+) = S(b_{+(1)}) d(b_{+(2)}) = q^2 c^2 db_- - q \mu ac db_0 + a^2 db_+,
\]
similarly for $\theta(b_-)$ and $\theta(b_0)$. This yields the middle expressions as stated. We then insert the expressions from Lemma \(^{(4,2)}\) to obtain \(\{\omega_+, \nu^2 q^{-1} \omega_0, q \omega_-\}\) for the values of the map $\theta$. According to \(^{(2,3)}\) the map $\theta : V \to P\Omega^1 M$ is well-defined on $V$. In order to get one-forms on $A[S^2_q]$, one must multiply $\theta(b_-)$ by an element of degree 2, $\theta(b_+)$ by an element of degree $-2$ and $\theta(b_0)$ by an element of degree zero. Moreover, every one-form is obtained in this way. This yields the isomorphism as stated. Since all line bundles $L_n$ are projective, so is $\Omega^1 S^2_q$. \hfill \square

The above also shows that the exterior derivative $d$ in the calculus $\Omega^1 S^2_q$ is given by restriction of the expression in \(^{(4,4)}\), namely
\[
dm = (L_- \triangleright m) \omega_- + (L_0 \triangleright m) \omega_0 + (L_+ \triangleright m) \omega_+, \quad m \in A[S^2_q].
\]
We stress that $L_\pm \triangleright m \in L_{\pm 2}$ rather than being element in $A[S^2_q]$. Of course, from \(^{(4,4)}\) combined with the fact that the vertical vector field $L_z$ obeys $L_z m = 0$ for all $m \in A[S^2_q]$,
we already expected this to be the case. From Theorem 4.4 we know that $\Omega^1 S_q^2$ is spanned as a left module by

$$\{d^2, db, b^2\} \omega_+ := \{\partial_+ b_+, \partial_+ b_0, \partial_+ b_-\},$$

$$\nu_+ q^{-1}\{\mu cd, 1 + \mu bc, \mu ab\} \omega_0 := \{\partial_0 b_+, \partial_0 b_0, \partial_0 b_-\},$$

$$\{qe^2, ac, qa^2\} \omega_- := \{\partial_- b_+, \partial_- b_0, \partial_- b_-\}.$$

The bimodule relations in the calculus $\Omega^1 S_q^2$ are in general quite complicated to compute directly, but we can use the expressions in Eqs. (4.10) to break them into smaller pieces which are much easier to work with.

**Corollary 4.5.** The cotangent bundle $\Omega^1 S_q^2$ has first order differential sub-calculi

$$\Omega^1_+ \cong \mathcal{L}_{-2} \oplus \mathcal{L}_0, \quad \Omega^0_0 \cong \mathcal{L}_0, \quad \Omega^1_- \cong \mathcal{L}_0 \oplus \mathcal{L}_{+2}$$

with differentials given by $d_+ := \partial_+ + \partial_0$, $d_0 := \partial_0$ and $d_- := \partial_0 + \partial_-$ respectively. These calculi obey the bimodule relations

$$\partial_+ b_+ \begin{cases} b_+ & \left\{ q^{-2}b_+(\partial_+ b_0) + q^{-3} \mu^{-1}b_+(\partial_0 b_+), \\ b_0 & \left\{ q^{-4}b_0(\partial_+ b_0) + \mu^{-1}q^{-2}(1 + q^{-3}b_0)(\partial_0 b_+) \\ b_- & \left\{ q^{-2}b_-(\partial_+ b_0) - (q^2 - q^{-2})b_+(\partial_+ b_-) + \partial_0 b_0 \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right. \\
\right.
From the bimodule relations in Eqs. (4.3) one finds that

\[
\begin{align*}
\omega_+ & = \begin{cases} 
  b_+ + b_+ \omega_+ + \nu^2 \frac{q^{-1} c^2}{2} \omega_0 \\
  b_0 \omega_+ + \nu^2 q^{-1} a \omega_0 \\
  b_- \omega_+ + \nu^2 q^{-1} a^2 \omega_0 
\end{cases}, \\
\omega_- & = \begin{cases} 
  b_+ \omega_- + \nu^2 d^2 \omega_0 \\
  b_0 \omega_- + \nu^2 d b \omega_0 \\
  b_- \omega_- + \nu^2 q b \omega_0 
\end{cases},
\end{align*}
\]

with \(\omega_0\) commuting with each of \(b_+, b_0\). Combining these with the algebra relations in \(A[\text{SU}_q(2)]\) yields the bimodule relations as stated, together with

\[
\begin{align*}
\partial_0 b_+ & = \begin{cases} 
  b_+ \partial_0 b_+ \\
  q^{-2} b_0 (\partial_0 b_+) \\
  q^{-2} b_- (\partial_0 b_+) - q^{-2} b_- (\partial_0 b_+) + b_+ (\partial_0 b_-),
\end{cases} \\
\partial_0 b_0 & = \begin{cases} 
  q^2 b_+ (\partial_0 b_0) - q^{-1} \mu \nu (\partial_0 b_+) \\
  b_0 (\partial_0 b_0) \\
  q^{-2} b_- (\partial_0 b_0) + q^{-1} \mu^{-1} \nu (\partial_0 b_-),
\end{cases} \\
\partial_0 b_- & = \begin{cases} 
  q^2 b_+ (\partial_0 b_-) + b_- (\partial_0 b_+) - q^{-2} b_+ (\partial_0 b_-) \\
  q^2 b_0 (\partial_0 b_-) \\
  b_- (\partial_0 b_-).
\end{cases}
\end{align*}
\]

The fact that \(\Omega^1_p = \mathcal{L}_{-2} \oplus \mathcal{L}_0, \Omega^1_0 = \mathcal{L}_0\) and \(\Omega^1_p = \mathcal{L}_0 \oplus \mathcal{L}_{+2}\) close as sub-bimodules is now clear by inspection. The Leibniz rules for the differentials \(d_+, d_0\) and \(d_-\) follow from the Leibniz rule for \(d\) and the direct sum decomposition of \(\Omega^1 S^2_q\).

**Corollary 4.6.** The one-forms in the calculus \(\Omega^1 S^2_q\) enjoy the relations

\[
\begin{align*}
\partial_0 b_0 & = q^{-2} b_- (\partial_+ b_+) - q^2 b_+ (\partial_- b_-), \\
\partial_- b_0 & = b_+ (\partial_- b_-) - q^{-4} b_- (\partial_- b_+), \\
b_+ (\partial_0 b_0) & = q^3 (1 + q b_0) b_+ (\partial_+ b_+), \\
b_0 (\partial_0 b_0) & = q^3 (1 + q^{-1} b_0) b_0 (\partial_+ b_+), \\
b_0 \partial_0 b_0 & = -q \mu \nu^{-1} b_- (\partial_0 b_+) + q^{-1} \mu^{-1} \nu b_+ (\partial_0 b_-), \\
b_+ (\partial_0 b_0) & = (\mu^{-1} + q^{-2} b_0) \partial_0 b_+, \\
b_- (\partial_0 b_0) & = (\mu^{-1} + q^2 b_0) \partial_0 b_-.
\end{align*}
\]

**Proof.** These are obtained in analogy with the proof of Cor. 4.5 from the relations in \(A[\text{SU}_q(2)]\) acting on \(\omega_\pm\) and \(\omega_0\). One finds the relations as stated, together with

\[
\begin{align*}
\partial_0 b_+ & = q^{-2} b_- (\partial_+ b_+) - q^2 b_+ (\partial_- b_-), \\
\partial_0 b_- & = b_+ (\partial_- b_-) - q^{-4} b_- (\partial_- b_+), \\
b_+ (\partial_0 b_+) & = q^3 (1 + q b_0) b_+ (\partial_+ b_+), \\
b_0 (\partial_0 b_+) & = q^3 (1 + q^{-1} b_0) b_0 (\partial_+ b_+), \\
b_0 \partial_0 b_+ & = -q \mu \nu^{-1} b_- (\partial_0 b_+) + q^{-1} \mu^{-1} \nu b_+ (\partial_0 b_-), \\
b_+ (\partial_0 b_+) & = (\mu^{-1} + q^{-2} b_0) \partial_0 b_+, \\
b_- (\partial_0 b_+) & = (\mu^{-1} + q^2 b_0) \partial_0 b_-.
\end{align*}
\]

There are other relations involving the differential \(\partial_0\), but they are quite complicated (since the sphere relation in \(A[S^2_q]\) does not explicitly involve the unit) and are not particularly illuminating, so we shall not give them here.

Finally, we use Theorem 4.3 to compute the differentials \(\partial_\pm\) and \(\partial_0\) in terms of the exterior derivative \(d\). Using the algebra relations in \(A[\text{SU}_q(2)]\) and the expressions in
Eqs. (4.10) we find that
\[
\partial_+ b_+ = q^{-1}b_+^2 db_+ - \mu b_+ (1 + q^{-1}b_0) db_0 + (1 + q^{-1}b_0)^2 db_+ + q^{-2} \nu b_+ b_- db_+,
\]
\[
\partial_0 b_0 = q b_+ b_- db_0 + \mu b_+ b_- db_0 + q^{-2} (1 + q^{-1}b_0) b_- db_+,
\]
\[
\partial_+ b_- = q^2 b_0^2 db_- - q^{-1} \mu b_- db_0 + q^{-3} b_-^2 db_+,
\]
\[
\partial_0 b_+ = -\mu b_0 db_- + \mu b_+ (1 + \mu b_0) db_0 - q^{-2} \mu b_+ b_- db_+,
\]
\[
\partial_0 b_0 = (1 + \mu b_0) (-b_- db_0 + (1 + \mu b_0) db_0 - q^{-2} b_- db_+),
\]
\[
\partial_0 b_- = -\mu b_- b_0 db_- + \mu b_- (1 + \mu b_0) db_0 - q^{-2} \mu b_-^2 db_+,
\]
\[
\partial_- b_+ = q b_0^2 db_- - q^{-1} \mu b_+ db_0 + q^{-2} b_0^2 db_+,
\]
\[
\partial_- b_0 = (1 + q b_0) b_+ db_- - q \mu b_0 (1 + q b_0) db_0 + q^{-2} b_- db_+,
\]
\[
\partial_- b_- = (1 + q b_0)^2 + \nu b_- b_+) db_- - \mu b_- (1 + q b_0) db_0 + q^{-2} b_0^2 db_+.
\]

These expressions may now be used to compute the full bimodule structure of the calculus \(\Omega^1 S_q^2\) in terms of the differential \(d\), as well as the deeper structure of the noncommutative Riemannian geometry of this calculus, along similar lines to [14]. However, since our objective is to study the spin geometry of the calculus, we have all we need and so we shall not pursue these directions here.

5. The Spectral Geometry of \(S_q^2\)

In this section we give the ‘three-dimensional’ differential calculus \(\Omega^1 S_q^2\) by a spectral triple on \(S_q^2\). This means equipping \(S_q^2\) with a spinor bundle \(S\) and a Dirac operator \(D\) which together implement the exterior derivative \(d\) for \(\Omega^1 S_q^2\). We then equip this spectral triple with a real structure for which the commutant property and the first order condition for the Dirac operator are satisfied up to infinitesimals of arbitrary order, in parallel with the results of [7] for the ‘two-dimensional’ calculus on \(S_q^2\).

5.1. Background on spectral triples. We recall briefly the notion of a spectral triple [2].

**Definition 5.1.** A unital spectral triple \((A, \mathcal{H}, D)\) consists of a complex unital \(*\)-algebra \(A\), faithfully \(*\)-represented by bounded operators on a (separable) Hilbert space \(\mathcal{H}\), and a self-adjoint operator \(D : \mathcal{H} \to \mathcal{H}\) (the Dirac operator) with the following properties:

(i) the resolvent \((D - \lambda)^{-1}, \lambda \notin \mathbb{R}\), is a compact operator on \(\mathcal{H}\);

(ii) for all \(a \in A\) the commutator \([D, \pi(a)]\) is a bounded operator on \(\mathcal{H}\).

A spectral triple \((A, \mathcal{H}, D)\) is called **even** if there exists a \(\mathbb{Z}_2\)-grading of \(\mathcal{H}\), i.e. an operator \(\Gamma : \mathcal{H} \to \mathcal{H}\) with \(\Gamma = \Gamma^*\) and \(\Gamma^2 = 1\), such that \(\Gamma D + D\Gamma = 0\) and \(\Gamma a = a \Gamma\) for all \(a \in A\). Otherwise the spectral triple is said to be **odd**.

With \(0 < n < \infty\), the Dirac operator \(D\) is said to be \(n^+\)-summable if \((D^2 + 1)^{-1/2}\) is in the Dixmier ideal \(\mathcal{L}^{n^+}(\mathcal{H})\). The **metric dimension** of the spectral triple \((A, \mathcal{H}, D)\) is defined to be the infimum of the set of all \(n\), such that \(D\) is \(n^+\)-summable.

Given a spectral triple \((A, \mathcal{H}, D)\), one associates to it a canonical first order differential calculus \((\Omega^1_D A, \mathrm{d}_D)\). In particular, the \(A\)-\(A\) bimodule \(\Omega^1_D A\) is defined to be

\[
\Omega^1_D A := \{ \omega = \sum a_0^i [D, a_1^i] \mid a_0^i, a_1^i \in A \},
\]

with the differential \(\mathrm{d}_D\) given by \(\mathrm{d}_Da = [D, a]\) for \(a \in A\).
The original definition \[3\] of a real structure on a spectral triple \((A, \mathcal{H}, D)\) was given by an anti-unitary operator \(J : \mathcal{H} \to \mathcal{H}\) with the properties \(J^2 = \pm 1\), \(JD = \pm DJ\) and
\[
[\pi(a), J\pi(b)J^{-1}] = 0, \quad [[D, \pi(a)], J\pi(b)J^{-1}] = 0, \quad a, b \in A.
\]
These are called the \textit{commutant property} and the \textit{first order condition} respectively.

However, in many examples involving quantum spaces, one needs to modify these conditions in order to obtain non-trivial spin geometries \[5, 6, 7, 8\]. Following the approach there, we impose the weaker assumption that (5.2) holds only up to infinitesimals of arbitrary order (i.e. up to compact operators \(T\) with the property that the singular values \(s_k(T)\) satisfy \(\lim_{k \to \infty} k^p s_k(T) = 0\) for all \(p > 0\)).

**Definition 5.2.** A real structure on a spectral triple \((A, \mathcal{H}, D)\) is an anti-unitary operator \(J : \mathcal{H} \to \mathcal{H}\) such that
\[
J^2 = \pm 1, \quad JD = \pm DJ,
\]
(5.3) \([\pi(a), J\pi(b)J^{-1}] \in \mathcal{I}, \quad [[D, \pi(a)], J\pi(b)J^{-1}] \in \mathcal{I}, \quad a, b \in A,
\]
where \(\mathcal{I}\) is an operator ideal of infinitesimals of arbitrary order. We say that the datum \((A, \mathcal{H}, D, J)\) is a real spectral triple (up to infinitesimals). If \((A, \mathcal{H}, D, \Gamma)\) is even and \(J\Gamma = \Gamma J\), we call the datum \((A, \mathcal{H}, D, \Gamma, J)\) an even real spectral triple (up to infinitesimals).

5.2. \textbf{A Dirac operator on} \(S^2_q\). In order to define a spectral triple on \(S^2_q\), we need a spinor bundle over \(S^2_q\) and an associated Dirac operator, which we require should recover the differential calculus \(\Omega^1 S^2_q\) via the commutator representation defined in (5.1). Since the differential calculus \(\Omega^1 S^2_q\) constructed in Theorem \[14\] is equivariant under a left coaction of \(\mathcal{A}[\text{SU}_q(2)]\) and hence a right action of \(\mathcal{U}_q(\text{su}(2))\), we are led to consider spinor bundles and Dirac operators which are right \(\mathcal{U}_q(\text{su}(2))-\)equivariant.

Guided by this principle, as well as by the spin structure of the classical two-sphere \(S^2\), for the \(\mathcal{A}[S^2_q]\)-module of spinors we take
\[
S = S_+ \oplus S_- := \mathcal{L}_{-1} \oplus \mathcal{L}_{1}.
\]
As right \(\mathcal{U}_q(\text{su}(2))-\)modules, the vector spaces \(S_\pm\) are both isomorphic to the direct sum
\[
V := \bigoplus_{j \in \mathbb{N} + \frac{1}{2}} V^j
\]
over all irreducible \(\mathcal{U}_q(\text{su}(2))-\)modules \(V^j\) with spin \(j \in \mathbb{N} + \frac{1}{2}\) a half-odd integer. A corresponding basis for \(V\) is then given by
\[
\{|j, m\} \mid j \in \mathbb{N} + \frac{1}{2}, m = -j, \ldots, j,\}
\]
where the vectors \(|j, m\rangle\) span the irreducible \(\mathcal{U}_q(\text{su}(2))-\)module \(V^j\) in Eqs. \[3.7\]. We denote the orthonormal bases of the two different copies \(S_\pm\) of \(V\) respectively by
\[
|j, m\rangle_\pm, \quad j \in \mathbb{N} + \frac{1}{2}, \ m = -j, \ldots, j.
\]
We equip \(S\) with the inner product which makes this basis orthonormal and write \(\mathcal{H}\) for the corresponding Hilbert space completion of \(S\).
As \(A[S^2_q]\)-modules, the vector spaces \(S_\pm\) each carry one of two inequivalent \(U_q(\mathfrak{su}(2))\)-equivariant representations of \(A[S^2_q]\),

\[
\pi_\pm : A[S^2_q] \to \text{End}(S_\pm).
\]

Recall that \(S_\pm\) are just the subspaces of \(A[SU_q(2)]\) with overall degrees \(\pm 1\) with respect to the \(\mathbb{Z}\)-grading (3.17), so the representations \(\pi_\pm\) on \(S_\pm\) are simply given by restricting the multiplication in \(A[SU_q(2)]\) to the appropriate degrees. However, it is possible to describe these representations explicitly in terms of the basis (5.5) in the following way.

Indeed, the \(U_q(\mathfrak{su}(2))\)-equivariant representations of \(A[S^2_q]\) on \(V\) were already described in [7, 21]. To be able to simply quote them we make a change of generators, now writing

\[
x_1 = -q^{1/2} \mu b_+ , \quad x_0 - 1 = \mu b_0 , \quad x_{-1} = -q^{-3/2} \mu b_- ,
\]

where \(b_\pm, b_0\) are the generators of \(A[S^2_q]\) defined in (3.18), and \(\mu = q + q^{-1}\). With respect to these new generators, the algebra relations of \(A[S^2_q]\) now read

\[
x_{-1}(x_0 - 1) = q^2 (x_0 - 1)x_{-1}, \quad x_1(x_0 - 1) = q^{-2}(x_0 - 1)x_1, \\
(q^2 x_0 + 1)(x_0 - 1) = (q + q^{-1})x_{-1}x_1, \quad (q^{-2}x_0 + 1)(x_0 - 1) = (q + q^{-1})x_1x_{-1}.
\]

Then, with \(N = \pm 1/2\), the two representations \(\pi_\pm = \pi_{\pm 1/2}\) of \(A[S^2_q]\) on \(S_\pm\) have the form

\[
\pi_N(x_i)[j, m]_\pm = \alpha_i^-(j, m; N)[j - 1, m + i]_\pm + \alpha_i^0(j, m; N)[j, m + i]_\pm + \alpha_i^+(j, m; N)[j + 1, m + i]_\pm,
\]

where the coefficients are determined by

\[
\begin{align*}
\alpha_i^+(j, m; N) &= q^{-j+m} \left( \frac{[j + m + 1][j + m + 2]}{[j + 1][2j + 2]} \right)^{1/2} \alpha_N(j + 1), \\
\alpha_i^0(j, m; N) &= -q^{m+2} ([2][j - m][j + m + 1])^{1/2} [2j]^{-1} \beta_N(j), \\
\alpha_i^-(j, m; N) &= -q^{j+m+1} \left( \frac{[j - m - 1][j - m]}{[2j - 1][2j]} \right)^{1/2} \alpha_N(j), \\
\alpha_0^+(j, m; N) &= q^m \left( \frac{[2][j - m + 1][j + m + 1]}{[2j + 1][2j + 2]} \right)^{1/2} \alpha_N(j + 1), \\
\alpha_0^0(j, m; N) &= [2j]^{-1} ( [j - m + 1][j + m] - q^{-2}[j - m][j + m + 1] ) \beta_N(j), \\
\alpha_0^-(j, m; N) &= q^m \left( \frac{[2][j - m + 1][j + m + 1]}{[2j + 1][2j + 2]} \right)^{1/2} \alpha_N(j), \\
\alpha_1^+(j, m; N) &= q^{j+m} \left( \frac{[j - m + 1][j - m + 2]}{[2j + 1][2j + 2]} \right)^{1/2} \alpha_N(j + 1), \\
\alpha_1^0(j, m; N) &= q^m ([2][j - m + 1][j + m])^{1/2} [2j]^{-1} \beta_N(j), \\
\alpha_1^-(j, m; N) &= -q^{-j+m-1} \left( \frac{[j + m - 1][j + m]}{[2j - 1][2j]} \right)^{1/2} \alpha_N(j).
\end{align*}
\]
(with the convention that $\alpha_i(\frac{1}{2}, \pm \frac{1}{2}, N) = 0$) and the real numbers $\alpha_N(j)$, $\beta_N(j)$ are

$$
\alpha_N(j) = ([2j + 1][2j])^{-1/2} ([2j + N][j - N])^{1/2} ([2j + 1][2j])^{1/2} q^N,
$$

$$
\beta_N(j) = q^{-1}[2j + 2]^{-1} (\varepsilon q^\varepsilon - (q - q^{-1})([j][j + 1] - [\frac{1}{2}][\frac{3}{2}])),
$$

with $\varepsilon = \text{sign}(N)$.

Next we come to the Dirac operator. With the $2 \times 2$ Pauli matrices

$$
\sigma_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
$$

one has the relations

$$
\sigma_+ \sigma_- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_0^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_- \sigma_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},
$$

$$
\sigma_0 \sigma_+ = \sigma_+, \quad \sigma_+ \sigma_0 = -\sigma_+, \quad \sigma_- \sigma_0^2 = \sigma_- = 0, \quad \sigma_- \sigma_0 = \sigma_-, \quad \sigma_0 \sigma_- = -\sigma_-.
$$

Further, we use the differential operators $D_\pm, D_0$,

$$
D_\pm := L_\pm, \quad D_0 := L_0 + q^{-2} L_z = q^{-1}(q - q^{-1})^2 (C_q + \frac{1}{4} - [\frac{1}{2}]^2),
$$

having used the expression (L.2) for the last equality. As will be clearly momentarily, the use of $D_0$ instead of $L_0$ (the extra $L_z$ vanishing identically on $A[S^2_q]$) will lead to a Dirac operator whose square is diagonal. We define a Dirac operator $D : S \to S$ by

$$
D = D_+ \sigma_+ + D_0 \sigma_0 + D_- \sigma_-,
$$

where the $2 \times 2$ Pauli matrices $\sigma_{\pm}$, $\sigma_0$ act upon the column vector of $S$ by left multiplication and the vector fields $D_\pm, D_0$ operate via the left action of $U_q(su(2))$ (using the symbol $\triangleright$, which we omit from now on). As mentioned above, elements $a \in A[S^2_q]$ act as multiplicative operators on $S$ via the representations $\pi_{\pm}$:

$$
\pi : A[S^2_q] \to \text{End}(S), \quad \pi(a) := \begin{pmatrix} \pi_+(a) & 0 \\ 0 & \pi_-(a) \end{pmatrix}
$$

although we will not always explicitly denote the representation $\pi$.

**Proposition 5.3.** The Dirac operator $D : S \to S$ obeys

$$
[D, a] = (L_+ a) \sigma_+ + (L_0 a) \sigma_0 + (L_- a) \sigma_-,
$$

for each $a \in A[S^2_q]$.

**Proof.** For $\psi = (\psi_+, \psi_-)^t \in S_+ \oplus S_-$, using the derivation property of the vector fields $D_\pm, D_0$, the commutator $[D, a]$ works out to be

$$
[D, a] \psi = \begin{pmatrix} (D_+ a) \psi_- \\ 0 \end{pmatrix} + \begin{pmatrix} (D_0 a) \psi_+ \\ -(D_0 a) \psi_- \end{pmatrix} + \begin{pmatrix} 0 \\ (D_- a) \psi_+ \end{pmatrix}
$$

$$
= ((D_+ a) \sigma_+ + (D_0 a) \sigma_0 + (D_- a) \sigma_-) \psi.
$$

To obtain the desired result, one simply substitutes $D_\pm = L_\pm$ and $D_0 = L_0 + q^{-2} L_z$, observing that $L_2 a = 0$ for all $a \in A[S^2_q]$. \qed
This also shows that for all \( a \in \mathcal{A}[S^2_q] \) the commutator \([D,a]\) recovers the one-form \(da\), acting on the spinors \(\mathcal{S}\) by ‘Clifford multiplication’.

The summand \(D_+\sigma_+ + D_-\sigma_-\) in the operator \(\Omega\), corresponding \([20]\) to the ‘two-dimensional’ differential calculus on the sphere \(S^2_q\). The extra term \(D_0\) in our Dirac operator is the origin of the extra ‘direction’ in the calculus \(\Omega^1S^2_q\). It is clear from \((4.12)\) that \(D_0\) vanishes when \(q \rightarrow 1\), whence the classical limit of our construction is just the canonical spectral triple on the classical two-sphere \(S^2\).

Next, we compute the spectrum of the Dirac operator. We shall use the identities

\[
L_+ L_- = qEFK^{-2} = q\left(C_q + \frac{1}{4} - \frac{qK^2 - 2 + qK^{-2}}{(q - q^{-1})^2}\right)K^{-2},
\]

\[
L_- L_+ = q^{-1}EFK^{-2} = q^{-1}\left(C_q + \frac{1}{4} - \frac{qK^2 - 2 + q^{-1}K^{-2}}{(q - q^{-1})^2}\right)K^{-2},
\]

each obtained using the expression \((3.6)\) for the quantum Casimir \(C_q\). Moreover, we know from \((3.13)\) that for all \(\psi_\pm \in \mathcal{S}_\pm\) we have

\[
K^2\psi_\pm = q^{\pm 1}\psi_\pm, \quad K^{-2}\psi_\pm = q^{\mp 1}\psi_\pm.
\]

These facts lead to the following result.

**Proposition 5.4.** The Dirac operator \(D\) obeys

\[
D^2 = q^{-2}\nu^4 ((C_q + \frac{1}{4} - [\frac{1}{2}]^2))^2 + (C_q + \frac{1}{4}),
\]

where \(C_q\) is the quantum Casimir.

**Proof.** Using the Pauli relations \((5.8)\) one computes that, for \(\psi = (\psi_+ \quad \psi_-)^{\text{tr}} \in \mathcal{S}\),

\[
D^2\psi = D_0^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \psi + D_+D_- \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \psi + D_-D_+ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \psi.
\]

The crucial fact in this calculation is that \(D_0\) is a function of the Casimir \(C_q\) and therefore commutes with \(D_\pm\). Next, using the relations \((5.11)\) and \((5.12)\) we find

\[
D_\pm D_\mp \psi_\pm = (C_q + \frac{1}{4}) \psi_\pm
\]

for each \(\psi_\pm \in \mathcal{S}_\pm\). Furthermore, we have that

\[
D_0^2 = q^{-2}\nu^4 ((C_q + \frac{1}{4} - [\frac{1}{2}]^2))^2.
\]

Substituting these expressions into \((5.13)\) yields the formula as claimed.

As an immediate consequence we obtain the spectrum of our Dirac operator \(D\).

**Corollary 5.5.** The Dirac operator \(D\) defined in \((5.10)\) has spectrum

\[
\text{Spec}(D) = \left\{ \pm (q^{-2}\nu^4 [j]^2[j + 1]^2 + [j + \frac{1}{2}]^2)^{1/2} \mid j \in \mathbb{N} + \frac{1}{2} \right\}
\]

with multiplicities \(2j + 1\).

**Proof.** The eigenvalues of \(C_q\) are given in \((3.8)\): each \([j, m]_\pm\) is an eigenvector with eigenvalue \([j + \frac{1}{2}]^2 - \frac{1}{4}\), whence the multiplicity of the \(j\)-th eigenvalue is \(2(2j + 1)\). From the expression for \(D^2\) in Prop. 5.4, we read off its eigenvalues using those for \(C_q\), yielding

\[
\text{Spec}(D^2) = \left\{ \lambda_j := q^{-2}\nu^4 [j]^2[j + 1]^2 + [j + \frac{1}{2}]^2 \mid j \in \mathbb{N} + \frac{1}{2} \right\},
\]
each having multiplicity $2(2j+1)$. Here we have used the identity $[j+\frac{1}{2}]^2 - [\frac{1}{2}]^2 = [j][j+1]$. The eigenvalues of $D$ are therefore just $\pm \lambda_j^{1/2}$ with multiplicities $2j + 1$. \hfill \Box

By inspection, we see that the eigenvalues of $|D|$ grow not faster than $q^{-2j}$ for large $j$, in contrast with the Dirac operator of $\mathbb{H}$, whose eigenvalues diverge not faster than $q^{-j}$. It is the extra term $D_0$ which accounts for this behaviour.

This result immediately gives us an expression for $D$ in terms of an orthonormal basis of eigenspinors $|j, m; \uparrow\rangle$, $|j, m; \downarrow\rangle$ defined by

$$D|j, m; \uparrow\rangle = \mu_j |j, m; \uparrow\rangle, \quad D|j, m; \downarrow\rangle = -\mu_j |j, m; \downarrow\rangle$$

with eigenvalues

$$\mu_j := \left(q^{-2j} |j|^2[j+1]^2 + [j+\frac{1}{2}]^2\right)^{1/2}.$$  

To proceed further, it will be necessary to have an explicit description of these eigenspinors in terms of the basic spinors $|j, m\rangle$. By evaluating the actions of $D_{\pm}$, $D_0$ on $\mathcal{S}$ one finds that the Dirac operator is

$$\mu_j := \left(q^{-2j} |j|^2[j+1]^2 + [j+\frac{1}{2}]^2\right)^{1/2}.$$  

for $m = j, j+1, \ldots, j-1, j$ and $j \in \mathbb{N} + \frac{1}{2}$, where we have written

$$\zeta_j^{\pm} = \sqrt{\mu_j + q^{-1} |j|} |j+1|; \zeta_j^- = \sqrt{\mu_j - q^{-1} |j|} |j+1|.$$  

On the two-dimensional subspace $V_{j,m}$ spanned by $|j, m\rangle_+, |j, m\rangle_-$ for fixed values of $j, m$, the operator which diagonalises $D$ is just the orthogonal matrix

$$W_j := \frac{1}{\sqrt{2\mu_j}} \begin{pmatrix} -\zeta_j^+ & -\zeta_j^- \\ -\zeta_j^- & \zeta_j^+ \end{pmatrix}.$$  

We write $W : \mathcal{H} \to \mathcal{H}$ for the closure of the operator defined by the matrices $W_j, j \in \mathbb{N} + \frac{1}{2}$.

5.3. Spectral properties of $S^2_q$. We now show that the datum $(\mathcal{A}[S^2_q], \mathcal{H}, D)$ fulfils the conditions required of a spectral triple, which we then equip with a real structure in the sense of Definition 5.2.

**Theorem 5.6.** The datum $(\mathcal{A}(S^2_q), \mathcal{H}, D)$ constitutes a unital spectral triple over the sphere $S^2_q$ with metric dimension zero.

**Proof.** For each $a \in \mathcal{A}[S^2_q]$ the commutator $[D, a]$ acts on $\mathcal{S}$ by multiplication operators and is therefore itself a bounded operator. In fact, for the summand $D_{+}\sigma_{+} + D_{-}\sigma_{-}$ this goes as in $[\mathbb{H}$, whereas for the term $D_0$ one gets multiplication by $L_0a$ which belongs to $\mathcal{A}[S^2_q]$ itself. The operator $D$ clearly satisfies $D = D^*$ on the dense domain $\mathcal{S}$ of $\mathcal{H}$. From Cor. 5.5 it is clear that the only accumulation points of the spectrum of $D$ are at infinity, so the resolvent of $D$ is compact. Since the eigenvalues of $D$ grow exponentially with $j \in \mathbb{N} + \frac{1}{2}$, the metric dimension is just zero. \hfill \Box
Proposition 5.7. With the $\mathbb{Z}_2$-grading $\Gamma : \mathcal{H} \to \mathcal{H}$ defined by
\[
\Gamma|j, m; \uparrow\rangle := |j, m; \downarrow\rangle, \quad \Gamma|j, m; \downarrow\rangle := |j, m; \uparrow\rangle
\]
on the orthonormal basis (5.15) and extended by $\mathcal{A}[S_q^2]$-linearity, the datum $(\mathcal{A}[S_q^2], \mathcal{H}, D, \Gamma)$ constitutes an even spectral triple.

Proof. It is obvious that $\Gamma^2 = 1$ and $\Gamma = \Gamma^*$. The property $\Gamma D + D \Gamma = 0$ follows from the fact that $\Gamma$ interchanges the $+\mu_j$ and $-\mu_j$ eigenspaces of $D$, as may be verified directly on the basis vectors (5.15).

Next a real structure. Since we have made the same choice for the spinors as in [4], it is tempting to take the same real structure as well. However, one quickly finds that this choice is unsuitable, since it neither commutes nor anti-commutes with our Dirac operator $D$. The reason for this lies mainly in the fact that the term $D_0$ in our Dirac operator (5.10) is proportional to the Casimir operator, which is rather a ‘second order differential operator’, if anything. Instead, we define an anti-unitary operator $J : \mathcal{H} \to \mathcal{H}$ in terms of its action on the orthonormal basis (5.15) by
\[
J|j, m; \uparrow\rangle = (-1)^{m+1/2}|j, -m; \uparrow\rangle, \quad J|j, m; \downarrow\rangle = (-1)^{m+1/2}|j, -m; \downarrow\rangle
\]
and seek to show that this $J$ equips the datum $(\mathcal{A}[S_q^2], \mathcal{H}, D, \Gamma)$ with a real structure. It is not difficult to check that the $J$ above is equivariant under the right action of $\mathcal{U}_q(\mathfrak{su}(2))$ on $\mathcal{H}$, making it a particularly natural choice.

Proposition 5.8. The operator $J$ satisfies $J^2 = -1$, $DJ = JD$ and $\Gamma J = -J \Gamma$.

Proof. The fact that $J^2 = -1$ is immediate. We find that
\[
(DJ - JD)|j, m; \uparrow\rangle = (-1)^{m+1/2}D|j, -m; \uparrow\rangle - \mu_j D|j, m; \uparrow\rangle
\]
\[
= (-1)^{m+1/2}\mu_j|j, -m; \uparrow\rangle - (-1)^{m+1/2}\mu_j|j, -m; \uparrow\rangle = 0,
\]
\[
(J \Gamma + \Gamma J)|j, m; \uparrow\rangle = J|j, m; \downarrow\rangle - (-1)^{m+1/2}\Gamma|j, -m; \uparrow\rangle
\]
\[
= (-1)^{m+1/2}|j, -m; \downarrow\rangle - (-1)^{m+1/2}|j, -m; \downarrow\rangle = 0,
\]
where we have used anti-linearity of $J$. Similar computations hold on $|j, m; \downarrow\rangle$.

Aiming at (modified) commutant and first order conditions as in Definition 5.2 and having in mind the strategy of [7], we denote by $L_q$ the positive trace-class operator defined by
\[
L_q|j, m\rangle := q^j|j, m\rangle, \quad j \in \mathbb{N} + \frac{1}{2},
\]
on $\mathcal{H}$ and let $\mathcal{K}_q$ be the two-sided ideal of $\mathcal{B}(\mathcal{H})$ generated by the operators $L_q$. The ideal $\mathcal{K}_q$ is an ideal of infinitesimals of arbitrarily high order and so we take $\mathcal{I} = \mathcal{K}_q$ as our operator ideal in Definition 5.2. Thus, to prove that $J$ defines a real structure, it remains to check that the commutant property and first order condition in (5.3) are satisfied.

The strategy of [7] is based on the fact that the operators $\pi(x_i)$, $i = -1, 0, 1$, can be ‘approximated’ by operators acting diagonally on the Hilbert space of spinors. Specifically, these operators $z_i$, $i = -1, 0, 1$, on $\mathcal{H}$ are defined by
\[
z_i|j, m\rangle = a^-_i(j, m; 0)|j - 1, m + i\rangle \pm a^0_j(j, m; 0)|j, m + i\rangle \pm a^+_i(j, m; 0)|j + 1, m + i\rangle.
\]
The coefficients are exactly the ones used in (5.1), unless \(|m + i| > j + \nu\) for \(\nu = -1, 0, 1\), in which case we set \(\alpha^+_k(j, m; 0) = 0\). Momentarily we shall show that the operators \(z_i\) approximate the operators \(\pi(x_i)\) modulo the ideal \(\mathcal{K}_q\), but to do this we first need the following technical lemma.

**Lemma 5.9.** With \(W_j, j \in \mathbb{N} + \frac{1}{2}\), the operators in (5.19), there exists a constant \(C\) (independent of \(j\)) such that
\[
||W_j W_j^* - 1|| < C q^j
\]
for all \(j \in \mathbb{N} + \frac{1}{2}\).

**Proof.** One evaluates the norm \(||W_j W_j^* - 1||\) by computing the eigenvalues of the 2 \(\times\) 2 matrix \(W_j W_j^* - 1\) and choosing the larger of the two, finding it to be
\[
||W_j W_j^* - 1|| = \frac{\xi^+ j_{j+1} + \xi^- j_{j+1} - \xi^- j_{j+1} + \xi^+ j_{j+1} - 2\sqrt{\mu_j \mu_{j+1}}}{2\sqrt{\mu_j \mu_{j+1}}}
\]
Using the inequalities \([j] < (q - q^{-1})^{-1}q^{-j}\) and \([j]^{-1} < q^{j-1}\), elementary estimates for each of the terms in this expression yield that \(\xi^+ j < C' q^{-j}\) and \(\sqrt{\mu_j \mu_{j+1}} < C'' q^{-2j}\) for real constants \(C', C''\), so it appears at first glance that the above norm has an \(O(1)\) behaviour. However, a more detailed analysis shows that the coefficient of \(q^{-2j}\) in the numerator is in fact zero; the behaviour of the numerator is therefore \(O(q^{-j})\) and we have our result. \(\Box\)

**Proposition 5.10.** There exist bounded operators \(A_i, B_i, i = -1, 0, 1\), such that
\[
\pi(x_i) - z_i = A_i L_q = L_q B_i
\]
when acting upon the basis vectors \([j, m; \uparrow \downarrow]\). In particular, \(\pi(x_i) - z_i \in \mathcal{K}_q\) for \(i = -1, 0, 1\).

**Proof.** From [7, Lem. 4.4], there exist bounded operators \(A_i, B_i, i = -1, 0, 1\) such that
\[
\pi(x_i) - z_i = A_i L_q = L_q B_i
\]
with respect to the basis \([j, m]_\pm\) of \(\mathcal{H}\), and so the operators \(\pi(x_i)\) are approximated by the operators \(z_i\) modulo the ideal \(\mathcal{K}_q\) of infinitesimals. We need to check that using the operator \(W\) to change the basis vectors from \([j, m]_\pm\) to \([j, m; \uparrow \downarrow]\) does not spoil this approximation property. Evaluating \(W_j z_i W_j^* - z_i\) on \([j, m; \uparrow \downarrow]\) gives
\[
(W_j z_i W_j^* - z_i)[j, m; \uparrow \downarrow] = \alpha_i^-(j, m; 0)(W_{j-1} W_j^* - 1)[j - 1, m + i; \uparrow \downarrow] + \alpha_i^+(j, m; 0)(W_j W_j^* - 1)[j + 1, m + i; \uparrow \downarrow].
\]
This and Lemma 5.9 yield that \(W_j z_i W_j^* - z_i \in \mathcal{K}_q\) for all \(i = -1, 0, 1\) and all \(j \in \mathbb{N} + \frac{1}{2}\). \(\Box\)

As a consequence, we immediately get the commutant property, the first of the two conditions in (5.3).

**Proposition 5.11.** For all \(a, b \in \mathcal{A}[S^2_q]\) we have \([\pi(a), J\pi(b)J^{-1}] \in \mathcal{K}_q\).

**Proof.** From the derivation property of commutators, it suffices to check this only for the generators \(x_{-1}, x_0, x_1\) of \(\mathcal{A}[S^2_q]\). With the operators \(z_{-1}, z_0, z_1\) defined in (5.20), we have
\[
(5.21) \quad J z_k J^{-1}[j, m]_\pm = (-1)^k \left( \alpha^-_k(j, -m; 0)[j - 1, m - k]_\pm + \alpha^+_k(j, -m; 0)[j, m - k]_\pm + \alpha^-_k(j, -m; 0)[j + 1, m - k]_\pm \right).
\]
Using this, one computes as in \cite{7} Lem. 6.2 that
\begin{equation}
[z_i, Jz_k J^{-1}] = 0, \quad i, k = -1, 0, 1.
\end{equation}

It is straightforward to check that
\[
\pi(x_i), J\pi(x_k) J^{-1}] = \pi(x_i) - z_i, J\pi(x_k) J^{-1}] + [z_i, J (\pi(x_k) - z_k) J^{-1}] + [z_i, Jz_k J^{-1}],
\]
whence the assertion follows from Prop. 5.10. \qed

We are now ready for our main theorem regarding the differential structure of $S_q^2$.

**Theorem 5.12.** The datum $(\mathcal{A}(S_q^2), \mathcal{H}, D, \Gamma, J)$ constitutes a real even unital spectral triple (up to infinitesimals) with KO-dimension equal to two.

**Proof.** Having already established Props. 5.8 and 5.11 it remains to verify the first order condition for $D$, namely that $[[D, a], J\sigma J^{-1}] \in \mathcal{K}_q$ for all $a \in \mathcal{A}[S_q^2]$. For this, we split the Dirac operator into two pieces, $D = D_\Delta + D_\Omega$, where $D_\Delta = D_0\sigma_0$ and $D_\Omega = D_\sigma_+ + D_\sigma_-$. By linearity it suffices to check the first order condition for $D_\Delta$ and $D_\Omega$ individually.

Since $D_0$ is a function of the Casimir, each $a \in \mathcal{A}[S_q^2]$ is an eigenfunction for the derivation $[D_\Delta, \cdot]$, whence the first order condition for $D_\Delta$ follows immediately from the commutant property in Prop. 5.11. On the other hand, the component $D_\Omega$ has eigenvalues $\pm \gamma_j$, $\gamma_j := [j + \frac{1}{2}]$, whose growth with $j$ obeys $\gamma_j < Cq^{-j}$ for $C$ a real constant (as already mentioned, $D_\Omega$ is precisely the Dirac operator considered in \cite{4}). It is easy to compute that
\begin{equation}
[D_\Omega, z_i]|j, m\rangle_\pm = (\gamma_{j-1} - \gamma_j)\alpha^-_\pm (j, m; 0)|j - 1, m + i\rangle_\mp + (\gamma_{j+1} - \gamma_j)\alpha^+_\pm (j, m; 0)|j + 1, m + i\rangle_\mp.
\end{equation}

Using this expression, together with (5.21), one calculates the action of the commutators $[[D_\Omega, z_i], Jz_k J^{-1}]$ for $i, k = -1, 0, 1$ and finds them to be a sum of five independent weighted shift operators with weights $S^\nu_{i,k}(j, m)$, $\nu = -2, \ldots, 2$, i.e.
\[
[[D_\Omega, z_i], Jz_k J^{-1}]|j, m\rangle_\pm = \sum_{\nu = -2}^2 S^\nu_{i,k}(j, m)|j + \nu, m + i - k\rangle_\pm.
\]

These weights $S^\nu_{i,k}(j, m)$ are estimated using exactly the same method as in \cite{7} Prop. 6.5. In our case, the growth condition for $\gamma_j$ is sufficient to guarantee that $|S^\nu_{i,k}(j, m)| < C'q^j$ for some real constant $C'$. We conclude that $[[D_\Omega, z_i], Jz_k J^{-1}] \in \mathcal{K}_q$ for all $i, k = -1, 0, 1$. Since the $z_i$ approximate the operators $\pi(x_i)$ modulo $\mathcal{K}_q$, the proof is complete. \qed

**Acknowledgments.** Both authors were partially supported by the Italian Project ‘Cofin08–Noncommutative Geometry, Quantum Groups and Applications’. SB is grateful to INdAM–GNSAGA for support and the Department of Mathematics at the University of Trieste for its hospitality. We thank Francesco D’Andrea for very useful comments.

**References**

[1] Brzeźniński T., Majid S.: Quantum Group Gauge Theory on Quantum Spaces. Commun. Math. Phys. 157, 591–638 (1993) Erratum *ibid.* 167, 235 (1995)

[2] Connes A.: *Noncommutative Geometry*. New York Academic Press, 1994

[3] Connes A.: Gravity Coupled with Matter and the Foundation of Noncommutative Geometry. Commun. Math. Phys. 182, 155–176 (1996)
[4] Dąbrowski L., Sitarz A.: Dirac Operator on the Standard Podleś Quantum Sphere. Noncommutative
Geometry and Quantum groups (Warsaw, 2001), 49–58, Banach Center Publ. 61, Polish Acad. Sci.,
Warsaw, 2003

[5] Dąbrowski L., Landi G., Paschke M., Sitarz A.: The Spectral Geometry of the Equatorial Podleś
Sphere. C. R. Math. Acad. Sci. Paris 340, 819–822 (2005)

[6] Dąbrowski L., Landi G., Sitarz S., van Suijlekom W.D., Varilly J.C.: The Dirac Operator on SU_q(2).
Commun. Math. Phys. 259, 729–759 (2005)

[7] Dąbrowski L., D’Andrea F., Landi G., Wagner E.: Dirac Operators on All Podleś Spheres. J. Non-
commut. Geom. 1, 213–239 (2007)

[8] D’Andrea F., Dąbrowski L., Landi G.: The Isospectral Dirac Operator on the 4-dimensional Ortho-
thogonal Quantum Sphere. Commun. Math. Phys. 279, 77–116 (2008)

[9] Dušević, M: Geometry of Quantum Principal Bundles. I. Commun. Math. Phys. 175, 457–520
(1996)

[10] Dušević, M: Geometry of Quantum Principal Bundles. II. Rev. Math. Phys. 9, 531–607 (1997)

[11] Klimyk A., Schmüdgen K.: Quantum Groups and their Representations. Springer Verlag, Berlin
Heidelberg, 1997

[12] Landi G., Zampini A.: Calculi, Hodge Operators and Laplacians on a Quantum Hopf Fibration.
arXiv:math.qa/1009.3738

[13] Majid S.: Quantum and Braided Group Riemannian Geometry. J. Geom. Phys. 30, 113–146 (1999)

[14] Majid S.: Noncommutative Riemannian and Spin Geometry of the Standard q-Sphere. Commun.
Math. Phys. 256, 255–285, (2005)

[15] Masuda T., Mimachi K., Nakagami Y., Noumi M., Ueno K.: Representations of the Quantum Group
SU_q(2) and the Little q-Jacobi Polynomials. J. Func. Anal. 99, 357–387 (1991)

[16] Podleś P.: Quantum Spheres. Lett. Math. Phys. 14, 193–202 (1987)

[17] Podleś P.: Differential Calculus on Quantum Spheres. Lett. Math. Phys. 18, 107–119 (1989)

[18] Podleś P.: The Classification of Differential Structures on Quantum Two-Spheres. Commun. Math.
Phys. 150, 167–179 (1992)

[19] Schmüdgen K.: Commutator Representations of Differential Calculi on the Quantum Group SU_q(2).
J. Geom. Phys. 31, 241–264 (1999)

[20] Schmüdgen K., Wagner E.: Dirac operator and a twisted cyclic cocycle on the standard Podleś
quantum sphere. J. Reine Angew. Math. 574 (2004) 219–235.

[21] Schmüdgen K., Wagner E.: Representations of Crossed Product Algebras of Podleś Quantum
Spheres. J. Lie Theory 17, 751–790 (2007)

[22] Woronowicz S.L.: Differential Calculus on Compact Matrix Pseudogroups (Quantum Groups). Com-
mun. Math. Phys. 122, 125–170 (1989)

Dipartimento di Matematica e Informatica, Università di Trieste, Via A. Valerio 12/1,
34127 Trieste, Italia
E-mail address: brain@sissa.it

Dipartimento di Matematica e Informatica, Università di Trieste, Via A. Valerio 12/1,
34127 Trieste, Italia, and INFN, Sezione di Trieste, Trieste, Italia
E-mail address: landi@univ.trieste.it