VANISHING OF CERTAIN EQUIVARIANT DISTRIBUTIONS ON
p-ADIC SPHERICAL SPACES, AND NON-VANISHING OF
SPHERICAL BESSEL FUNCTIONS

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Abstract. We prove vanishing of distribution on p-adic spherical spaces that are
equivariant with respect to a generic character of the nilradical of a Borel subgroup
and satisfy a certain condition on the wave-front set. We deduce from this non-
vanishing of spherical Bessel functions for Galois symmetric pairs.

1. Introduction

Let $G$ be a reductive group, quasi-split over a non-Archimedean local field $F$ of
characteristic zero. Let $B$ be a Borel subgroup of $G$, and let $U$ be the unipotent
radical of $B$. Let $H$ be a closed subgroup of $G$. Let $G, B, U, H$ denote the $F$-points
of $G, B, U, H$ respectively. Suppose that $H$ is an $F$-spherical subgroup of $G$, i.e. that
there are finitely many $B \times H$-double cosets in $G$. Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of
$G, H$ respectively. Let $\psi$ be a non-degenerate character of $U$ and let $\chi$ be a (locally
constant) character of $H$. For $x \in G$ denote $H^x := xHx^{-1}$ and denote by $\chi^x$ the
character of $H^x$ defined by conjugation of $\chi$. For a $B \times H$-double coset $O \subset G$ define

$$O_c := \left\{ x \in O \mid \psi|_{H^x \cap U} = \chi^x|_{H^x \cap U} \right\}.$$

Let

$$Z := \bigcup_{O \text{ s.t. } O \neq O_c} O_c.$$

Identify $T^*G$ with $G \times \mathfrak{g}^*$ and let $N_{\mathfrak{g}^*}$ be the set of nilpotent elements in $\mathfrak{g}^*$.

Consider the action of $U \times H$ on $G$ given by $(u, h)x = uxh^{-1}$. This gives rise to
an action of $U \times H$ on the space $S(G)$ of Schwartz (i.e. locally constant compactly
supported) functions on $G$ and the dual action on the space of distributions $S^*(G)$.

In this paper we prove the following theorem.

Theorem A (see Section 3). Let $\xi \in S^*(G)^{(U \times H, \psi \times \chi)}$ be an equivariant distribution
on $G$, i.e. $(u, h)\xi = \psi(u)\chi(h)\xi$. Suppose that the wave-front set (see section 2.2)
$WF(\xi)$ lies in $G \times N_{\mathfrak{g}^*}$ and $\text{Supp}(\xi) \subset Z$. Then $\xi = 0$.

In the case when $H$ is a subgroup of Galois type we can prove a stronger statement.
By a subgroup of Galois type we mean a subgroup $H \subset G$ such that

$$(G \times_{\text{Spec}F} \text{Spec}E, H \times_{\text{Spec}F} \text{Spec}E) \simeq (H \times_{\text{Spec}F} H \times_{\text{Spec}F} \text{Spec}E, \Delta H \times_{\text{Spec}F} \text{Spec}E)$$
for some field extension $E$ of $F$, where $\Delta H$ is the diagonal copy of $H$ in $H \times_{\text{Spec} F} H$.

**Corollary** B (see Section 4). Let $H \subset G$ be a subgroup of Galois type, and let $\chi$ be a character of $H$. Let $S$ be the union of all non-open $B \times H$-double cosets in $G$. Let $\xi \in \mathcal{S}^*(G)((U \times H, \psi \times \chi))^\vee$. Suppose that $WF(\xi) \subset G \times \mathcal{N}_{g^*}$ and $\text{Supp}(\xi) \subset S$. Then $\xi = 0$.

Note that if $\chi$ is trivial, we can consider the distribution $\xi$ as a distribution on $G/H$. Considering $\tilde{G} := G \times G$ and taking $H$ to be the diagonal copy of $G$ we obtain the following corollary for the group case.

**Corollary** C (see Section 4). Let $\psi_1$ and $\psi_2$ be non-degenerate characters of $U$. Let $B \times B$ act on $G$ by $(b_1, b_2)g := b_1gb_2^{-1}$. Let $S$ be the complement to the open $B \times B$-orbit in $G$. For any $x \in G$, identify $T_xG$ with $g$ and $T_x^*G$ with $g^*$. Let

$$\xi \in \mathcal{S}^*(G)^{U \times U ; \psi_1 \times \psi_2}$$

and suppose that $WF(\xi) \subset S \times \mathcal{N}_{g^*}$. Then $\xi = 0$.

### 1.1. Applications to non-vanishing of spherical Bessel functions.

Let $\pi$ be an admissible representation of $G$ (of finite length), and $\tilde{\pi}$ be the smooth contragredient representation. Let $H \subset G$ be an algebraic spherical subgroup and let $\chi$ be a character of $H$. Let $\phi \in (\pi^*)^U(\psi \chi)$ be a $(U, \psi)$-equivariant functional on $\pi$ and $v$ be an $(H, \chi)$-equivariant functional on $\tilde{\pi}$. For any function $f \in \mathcal{S}(G)$, we have $\pi^*(f)\phi \in \tilde{\pi} \subset \pi^*$.

This enables us to define the spherical Bessel distribution corresponding to $v$ and $\phi$ by

$$\xi_{v, \phi}(f) := \langle v, \pi^*(f)\phi \rangle.$$

By [AGS, Theorem A] we have $WF(\xi_{v, \phi}) \subset G \times \mathcal{N}_{g^*}$.

The spherical Bessel function is defined to be the restriction $j_{v, \phi} := \xi_{v, \phi}|_{G-S}$, where $S$ is the union of all non-open $B \times H$-double cosets in $G$. One can easily deduce from [AGS, Theorem A] and Lemma 3.1 that $j_{v, \phi}$ is a smooth function. Theorem A and Corollary B imply the following corollary.

**Corollary** D. Suppose that $\pi$ is irreducible and $v, \phi$ are non-zero. Then

(i) For any open subset $U \subset G$ that includes $G \setminus Z$ we have $\xi_{v, \phi}|_U \neq 0$.

(ii) If $H$ is a subgroup of Galois type then $j_{v, \phi} \neq 0$.

For the group case this corollary was proven in [LM, Appendix A].

### 1.2. Related results.

In [AG] a certain Archimedean analog of Theorem A is proven (see [AG, Theorem A]). This analog implies that the Archimedean analog of Corollary B holds for any spherical pair $(G, H)$ (see [AG, Corollary B]).

Corollary C together with [AGS, Theorem A] can replace [GK75, Theorem 3] in the proof of uniqueness of Whittaker models [GK75, Theorem C].

Theorem A can be used in order to study the dimensions of the spaces of $H$-invariant functionals on irreducible generic representations of $G$ (see [AG, §1.3] for more details). It can also be used in the study of analogs of Harish-Chandra’s density theorem (see [AGS, §1.7] for more details).

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2. Preliminaries

2.1. Conventions.

- We fix $F, G, B, U, X$ and $\psi$ as in the introduction.
- All the algebraic groups and algebraic varieties that we consider are defined over $F$. We will use capital bold letters, e.g. $G, X$ to denote algebraic groups and varieties defined over $F$, and their non-bold versions to denote the $F$-points of these varieties, considered as $l$-spaces or $F$-analytic manifolds (in the sense of [Ser64]).
- When we use a capital Latin letter to denote an $F$-analytic group or an algebraic group, we use the corresponding Gothic letter to denote its Lie algebra.
- We denote by $G_x$ the stabilizer of $x$ and by $g_x$ its Lie algebra.
- For an $F$-analytic manifold $X$, a submanifold $Y \subset X$ and a point $y \in Y$ we denote by $CN_Y^X \subset T^*_X$ the conormal bundle to $Y$ in $X$, and by $CN_{Y,y}^X$ the conormal space at $y$ to $Y$ in $X$.
- By a smooth measure on an $F$-analytic manifold we mean a measure which in a neighborhood of any point coincides (in some local coordinates centered at the origin) with some Haar measure on a closed ball centered at 0. A Schwartz measure is a compactly supported smooth measure.
- The space of generalized functions $G(X)$ on an $F$-analytic manifold $X$ is defined to be the dual of the space of Schwartz measures. One can identify $G(X)$ with $\mathcal{S}^*(X)$ by choosing a smooth measure with full support.
- Let $\phi : X \to Y$ be a submersion of analytic manifolds. Note that the pushforward of a Schwartz measure with respect to $\phi$ is a Schwartz measure. By dualizing the pushforward map we define the pullback map $\phi^* : G(Y) \to G(X)$.

2.2. Wave front set. In this section we give an overview of the theory of the wave front set as developed by D. Heifetz [Hei85], following L. Hörmander (see [Hör90, §8]). For simplicity we ignore here the difference between distributions and generalized functions.

**Definition 2.1.**

1. Let $V$ be a finite-dimensional vector space over $F$. Let $f \in C^\infty(V^*)$ and $w_0 \in V^*$. We say that $f$ vanishes asymptotically in the direction of $w_0$ if there exists $\rho \in \mathcal{S}(V^*)$ with $\rho(w_0) \neq 0$ such that the function $\phi \in C^\infty(V^* \times F)$ defined by $\phi(w,\lambda) := f(\lambda w) \cdot \rho(w)$ is compactly supported.

2. Let $U \subset V$ be an open set and $\xi \in \mathcal{S}^*(U)$. Let $x_0 \in U$ and $w_0 \in V^*$. We say that $\xi$ is smooth at $(x_0,w_0)$ if there exists a compactly supported non-negative function $\rho \in \mathcal{S}(V)$ with $\rho(x_0) \neq 0$ such that the Fourier transform $\mathcal{F}^*(\rho \cdot \xi)$ vanishes asymptotically in the direction of $w_0$.

3. The complement in $T^*U$ of the set of smooth pairs $(x_0,w_0)$ of $\xi$ is called the wave front set of $\xi$ and denoted by $WF(\xi)$.

4. For a point $x \in U$ we denote $WF_x(\xi) := WF(\xi) \cap T^*_x U$.

**Remark 2.2.**

1. Heifetz defined $WF_\Lambda(\xi)$ for any open subgroup $\Lambda$ of $F^\times$ of finite index. Our definition above differs slightly from the definition in [Hei85]. They relate by

$$WF(\xi) - (U \times \{0\}) = WF_{F^\times}(\xi).$$
(2) Though the notion of Fourier transform depends on a choice of a non-degenerate additive character of $F$, this dependence effects the Fourier transform only by dilution, and thus does not change our notion of wave front set.

**Proposition 2.3** (see [Har90] Theorem 8.2.4 and [He75] Theorem 2.8). Let $U \subset F^m$ and $V \subset F^n$ be open subsets, and suppose that $\phi : U \to V$ is an analytic submersion. Then for any $\xi \in S^*(V)$, we have

$$WF(\phi^*(\xi)) \subset \phi^*(WF(\xi)) := \{(x, v) \in T^*U | \exists w \in WF(\phi(x))(\xi) \text{ s.t. } d_{\phi(x)}^v \phi(w) = v \}.$$ 

**Corollary 2.4.** Under the assumption of Proposition 2.3 we have

$$WF(\phi^*(\xi)) = \phi^*(WF(\xi)).$$

**Proof.** The case when $\phi$ is an analytic diffeomorphism follows immediately from Proposition 2.3. This implies the case of open embedding. It is left to prove the case of linear projection $\phi : F^{n+m} \to F^n$. In this case the assertion follows from the fact that $\phi^*(\xi) = \xi \boxtimes 1_{F^m}$ where $1_{F^m}$ is the constant function 1 on $F^m$.

This corollary enables to define the wave front set of any distribution on an $F$-analytic manifold, as a subset of the cotangent bundle. The precise definition follows.

**Definition 2.5.** Let $X$ be an $F$-analytic manifold and $\xi \in S^*(X)$. We define the wave front set $WF(\xi)$ as the set of all $(x, \lambda) \in T^*X$ which lie in the wave front set of $\xi$ in some local coordinates. In other words, $(x, \lambda) \in WF(\xi)$ if there exist open subsets $U \subset X$ and $V \subset F^n$, an analytic diffeomorphism $\phi : U \simeq V$ and $(y, \beta) \in T^*V$ such that $x \in U$, $\phi(x) = y$, $d_x^y \phi^*(\beta) = \lambda$, and $(y, \beta) \in WF((\phi^{-1})^*(\xi|_V))$.

**Theorem 2.6.** (Corollary from [A13] Theorem 4.1.5) Let an $F$-analytic group $H$ act on an $F$-analytic manifold $Y$ and let $\chi$ be a character of $H$. Let $\xi \in S^*(Y)^{(H, \chi)}$. Then

$$WF(\xi) \subset \{(x, v) \in T^*Y | v(T_x(Hx)) = 0 \}.$$ 

**Theorem 2.7** ([A13] Theorem 4.1.2). Let $Y \subset X$ be $F$-analytic manifolds and let $y \in Y$. Let $\xi \in S^*(X)$ and suppose that $\text{Supp}(\xi) \subset Y$. Then $WF_y(\xi)$ is invariant with respect to shifts by the conormal space $CN^X_{y,y}$.

**Corollary 2.8.** Let $M$ be an $F$-analytic manifold and $N \subset M$ be a closed algebraic submanifold. Let $\xi$ be a distribution on $M$ supported in $N$. Suppose that for any $x \in N$, we have $CN^M_{N,x} \not\subseteq WF_x(\xi)$. Then $\xi = 0$.

**Proof.** Suppose $\xi \neq 0$ and let $x \in \text{Supp}(\xi)$. Then $(x, 0) \in WF_x(\xi)$. But then from Theorem 2.7 we have $CN^M_{N,x} \subseteq WF_x(\xi)$ which contradicts our assumption on $\xi$.

2.3. Vanishing of equivariant distributions. The following criterion for vanishing of equivariant distributions follows from [BZ76] §6 and [Ber83] §1.5.

**Theorem 2.9** (Bernstein-Gelfand-Kazhdan-Zelevinsky). Let an algebraic group $H$ act on an algebraic variety $X$, both defined over $F$. Let $\chi$ be a character of $H$. Let $Z \subset X$ be a closed $H$-invariant subset. Suppose that for any $x \in Z$ we have

$$\chi|_{H_x} \neq \Delta_H|_{H_x}, \Delta_H^{-1}|_{H_x},$$

where $\Delta_H$ and $\Delta_H^{-1}$ denote the modular functions of the groups $H$ and $H_x$. Then there are no non-zero $(H, \chi)$-equivariant distributions on $X$ supported in $Z$. 


2.4. Characters of unipotent groups. The following lemma is standard.

Lemma 2.10. Let $V$ be a unipotent algebraic group defined over $F$, let $\alpha$ be a (locally constant, complex) character of $V$ and $\beta$ be a non-trivial character of $F$. Then there exists an algebraic group morphism $\phi : V \to G_a$ such that $\alpha = \beta \circ \phi$.

For completeness we include a proof in Appendix [A]. In the case when $V$ is a maximal unipotent subgroup of a reductive group and $F$ is an arbitrary field (of an arbitrary characteristic) this lemma is [BH02 Theorem 4.1].

3. Proof of Theorem [A]

Lemma 3.1. Let $x \in G$. Let $\xi$ be a $(U, \psi)$-left equivariant and $(H, \chi)$-right equivariant distribution on $G$ such that $WF(\xi) \subset G \times N^*_g$. Then $WF_x(\xi) \subset CN^G_{BxH,x}$.

Proof. Let $t$ be the Lie algebra of a maximal torus contained in $B$, and let $\mathfrak{h}, u$ be the Lie algebras of $H, U$ respectively. Identify $T^*_x G$ with $\mathfrak{g}^*$ using the right multiplication by $x^{-1}$. We have $CN^G_{BxH,x} = (t + u + ad(x)h)^+$. Since $\xi$ is $u$-equivariant, by Theorem 2.6 we have $WF_x(\xi) \subset u^\perp$. Similarly, since $\xi$ is $h$-equivariant on the right, we have $WF_x(\xi) \subset (ad(x)h)^\perp$. By our assumption $WF_x(\xi) \subset N^*_g$. Now, $u^\perp \cap N^*_g = (t + u)^\perp$ and thus

$WF_x(\xi) \subset (ad(x)h)^\perp \cap u^\perp \cap N^*_g = (ad(x)h)^\perp \cap (u + t)^\perp = (t + u + ad(x)h)^\perp = CN^G_{BxH,x}$.

Now we would like to describe the structure of the varieties $O_c$. For this we will use the following notation.

Notation 3.2. For a $B \times H$ double coset $O = BxH \subset G$ define

$$\tilde{O}_c = \bigcup_{O' = ByH \subset (BxH)(F)} O'_c$$

Lemma 3.3. For any double coset $O = BxH \subset G$ there exists a closed algebraic subvariety $\tilde{O}_c \subset BxH$ s.t. $\tilde{O}_c = \tilde{O}_c(F)$.

Proof. Note that

$$O_c = \{ x \in O \mid \psi^{-1} x^{-1} \big|_{H \cap U^{-1}} = \chi \big|_{H \cap U^{-1}} \}.$$  

Let $H_x := H \cap U^{-1}$. Since $U$ is normal in $B$, for any $y \in (BxH)(F)$ we have $H_x = H_y$. Thus we will denote $H_\tilde{O} := H_x$.

By Lemma 2.10 there exist an additive character $\beta$ of $F$ and algebraic group homomorphisms $\psi' : U \to G_a, \chi' : H_\tilde{O} \to G_a$ such that $\psi = \beta \circ \psi'$ and $\chi \big|_{H_\tilde{O}} = \beta \circ \chi'$. Let us show that

$$\tilde{O}_c = \{ y \in (BxH)(F) \mid \psi' y^{-1} \big|_{H_\tilde{O}} = \chi' \}.$$  

Indeed, if $y \in \tilde{O}_c$ then $\beta \circ (\psi')^{-1} \big|_{H_\tilde{O}} = \beta \circ \chi'$, hence $\beta \circ (\chi' - (\psi')^{-1} \big|_{H_\tilde{O}} = 1$, thus $\chi' - (\psi')^{-1} \big|_{H_\tilde{O}}$ is bounded on $H_\tilde{O}$, and thus $\chi' - (\psi')^{-1} \big|_{H_\tilde{O}}$ is trivial. We obtain

$$\tilde{O}_c = \{ y \in (BxH)(F) \mid \forall u \in H_\tilde{O} \text{ we have } \psi'(uy^{-1}) = \chi'(u) \},$$

which is clearly the set of $F$-points of a closed algebraic subvariety of $BxH$.  

Corollary 3.4.

(1) There exists a stratification of $O_c$ into a union of smooth $F$-analytic locally closed submanifolds $O_i^c$ s.t. $\bigcup_{i \leq i_0} O_i^c$ is open in $O_c$.

(2) Moreover, if $O_c \neq \emptyset$ then the dimensions of $O_i^c$ are strictly smaller than the dimension of $O_c$.

Proof of Theorem A. Suppose that there exists a non-zero right $(U, \psi)$-equivariant and left $(H, \chi)$-equivariant distribution $\xi$ supported on $Z$ such that $WF(\xi) \subset G \times N_g^\times$. We decompose $G$ into $B \times H$-double cosets and prove the required vanishing coset by coset. For a $B \times H$-double coset $O \subset G$ define $O_s := O \setminus O_c$ and stratify $O_c$, using Corollary 3.4, to a union of smooth locally closed $F$-analytic subvarieties $O_i^c$. The collection

$$\{O_c^i | O \text{ is a } B \times H\text{-double coset}\} \cup \{O_s | O \text{ is a } B \times H\text{-double coset}\}$$

is a stratification of $G$. Order this collection to a sequence $\{S_i\}_{i=1}^N$ of smooth locally closed $F$-analytic submanifolds of $G$ such that $U_k := \bigcup_{i=1}^k S_i$ is open in $G$ for any $1 \leq k \leq N$. Let $k$ be the maximal integer such that $\xi|_{U_{k-1}} = 0$. Suppose $k \leq N$ and let $\eta := \xi|_{U_k}$. Note that $\text{Supp}(\eta) \subset S_k$. We will now show that $\eta = 0$, which leads to a contradiction.

Case 1. $S_k = O_s$ for some orbit $O$. Then $\eta = 0$ by Theorem 2.9 since $\eta$ is $(U \times H, \psi \times \chi)$-equivariant.

Case 2. $S_k \subset O = O_c$ for some orbit $O$. Then $S_k \subset G \setminus Z$ and $\eta = 0$ by the conditions.

Case 3. $S_k \subset O_c \subseteq O$ for some orbit $O$. In this case, by Corollary 3.4 $\dim S_k < \dim O$ and thus

$$CN_{S_k, x}^G \supset CN_{O_c, x}^G.$$ 

By Lemma 3.1 we have, for any $x \in S_k$,

$$WF_x(\eta) \subset CN_{O_c, x}^G$$

and thus $CN_{S_k, x}^G \nsubseteq WF_x(\eta)$.

By Corollary 2.8 this implies $\eta = 0$. 

□

4. Proof of Corollaries B and C

Let $U'$ denote the derived group of $U$.

Lemma 4.1. Let $\overline{W}$ be the Weyl group of $G$. Let $\overline{w} \in \overline{W}$ and let $w \in G$ be a representative of $\overline{w}$. Suppose that $wUw^{-1} \cap U \subset U'$. Then $\overline{w}$ is the longest element in $\overline{W}$.

Proof. Let $\mathfrak{u}$ be the Lie algebra of $U$. On the level of Lie algebras the condition $wUw^{-1} \cap U \subset U'$ means that $(Ad(w)\mathfrak{u}) \cap \mathfrak{u} \subset \mathfrak{u}'$. The algebra $\mathfrak{u}$ can be decomposed as

$$\mathfrak{u} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha.$$ 

It is easy to see that

$$(Ad(w)\mathfrak{u}) \cap \mathfrak{u} = \sum_{\alpha \in \Phi^+, \overline{w}^{-1}(\alpha) \in \Phi^+} \mathfrak{g}_\alpha.$$
Let $\Delta \subset \Phi^+$ be the set of simple roots in $\Phi^+$. Then from the condition of the lemma we obtain that $\overline{w}(\Delta) \subset \Phi^-$, and as a consequence $\overline{w}(\Phi^+) \subset \Phi^-$. Let $\overline{w_0}$ be the longest element in $\overline{W}$. Then $\overline{w_0w}(\Phi^+) \subset \Phi^+$. Since $\Phi^+$ is a finite set and $\overline{w_0w}$ acts by an invertible linear transformation, we get $\overline{w_0w}(\Phi^+) = \Phi^+$. Since simple roots are the indecomposable ones, it follows that $\overline{w_0w}(\Delta) = \Delta$. This implies that $\overline{w_0w} = 1$ (see e.g. [Hum72, §10.3]), and thus $\overline{w_0} = \overline{w}$.

**Corollary 4.2.** Let $H$ be a reductive group. Assume $G = H \times H$ and let $\Delta H \subset G$ be the diagonal copy of $H$. Denote $X = G/\Delta H$ and let $x \in X$ be such that $U_x \subset U'$. Then the orbit $Bx$ is open in $X$.

**Proof.** We can identify $X$ with $H$ using the projection on the first coordinate. We can assume that $B = B_H \times B_H$ where $B_H$ is a Borel subgroup of $H$. Let $\overline{W}$ be the Weyl group of $H$ and $W$ be a set of its representatives. By the Bruhat decomposition,

$$H = \bigsqcup_{w \in W} B_H w B_H$$

It is well-known that the only open $B_H \times B_H$ orbit in $H$ is $B_H w_0 B_H$, where $w_0 \in W$ is the representative of the longest Weyl element. Let $w \in W$. Let $U_H$ be the nilradical of $B_H$. Then

$$U_w = \{(u_1, u_2) \mid u_1 w_2 = w, \ u_1, u_2 \in U_H\},$$

and we see that for a pair $(u_1, u_2) \in U_w$ we have $u_1 = w u_2 w^{-1} \in w U_H w^{-1}$. Therefore,

$$U_w \cong U_H \cap w U_H w^{-1}.$$

Let

$$R = \{x \in X \mid U_x \subset U'\} = \{x \in H \mid U_H \cap x U_H x^{-1} \subset U_H' = [U_H, U_H]\},$$

and let $R$ be the corresponding algebraic variety. Since $U$ and $U'$ are normal in $B$, we obtain that $R$ is $B$-invariant. The corollary follows now from Lemma 4.1. □

**Corollary 4.3.** Let $H \subset G$ be a subgroup of Galois type. Then for every non-open $B$-orbit $O \subset G/H$ there exists $y \in O$ such that $\psi(U_y) \neq 1$.

**Proof.** Let $O \subset G/H$ be a non-open $B$-orbit and $x \in O$. Then the map $B \to G$ given by the action on $x$ is not submersive and thus $Bx$ is not Zariski open in $G/H$. By Corollary 4.2 this implies $U_x \not\subset U'$. Thus, there exists a non-degenerate character $\varphi$ of $U$ such that $\varphi(U_x) \neq 1$. For a fixed $x \in O$, the set of characters $\varphi'$ of $U$ such that $\varphi'(U_x) \neq 1$ is Zariski-open, thus dense in the $l$-space topology and thus intersects the $B$-orbit of $\psi$. Thus there exists $y \in Bx = O$ such that $\psi(U_y) \neq 1$. □

**Proof of Corollary 13.** By Theorem 10 it is enough to show that $S \subset Z$. Let $O \subset S$ be a $B \times H$ double coset. Corollary 1.3 implies that there exists $x \in O$ such that $\psi|_{U \cap H^x} \neq 1$. Since $H^x$ is reductive and $U$ is unipotent, we have $\chi^x|_{U \cap H^x} = 1$, and thus $O \subset Z$. □

**Proof of Corollary 1.** Define $\tilde{G} = G \times G$, $\tilde{H} = \Delta(G) \subset \tilde{G}$ and $\tilde{B} = B \times B$. The non-degenerate characters $\psi_1, \psi_2$ define a non-degenerate character of the nilradical $\tilde{U}$ of $\tilde{B}$. Note that $\tilde{H} \subset \tilde{G}$ is a subgroup of Galois type and that $\tilde{G}/\tilde{H}$ is naturally isomorphic to $G$. Let $\eta$ be the pull-back of $\xi$ to $\tilde{G}$ under the projection $\tilde{G} \to \tilde{G}/\tilde{H} \cong G$. Then we have $\text{Supp} \eta \subset \tilde{S}$, where $\tilde{S}$ is the union of all non-open $\tilde{B} \times \tilde{H}$-double cosets.
in \( \tilde{G} \). Also, by Corollary 2.4 we have \( WF(\eta) \subset \tilde{G} \times N_{\tilde{g}^r} \). By Corollary 13 we obtain \( \eta = 0 \) and thus \( \xi = 0 \).

\[ \square \]

**Remark 4.4.** Corollary 13 can not be generalized literally to arbitrary symmetric pairs. The reason is that neither can Corollary 4.3. For example consider the pair \( G = GL_{2n}, H = GL_n \times GL_n \), where the involution is conjugation by the diagonal matrix with first \( n \) entries equal to 1 and others equal to \(-1\). Let \( x \) be the coset of the permutation matrix given by the product of transpositions

\[
\prod_{i=0}^{\lceil (n-1)/2 \rceil} (2i + 1, 2n - 2i),
\]

and let \( B \) consist of upper-triangular matrices. Then \( U_x \subset U' \), while \( Bx \) is of middle dimension in \( G/H \). It can be shown that there exists a \((U, \psi)\)-left equivariant, \( H\)-right invariant distribution \( \xi \) on \( G \) supported in \( BxH \) and satisfying \( WF(\xi) \subset G \times N_{\tilde{g}^r} \).

However, Corollary 13 might hold for any spherical subgroup \( H \). In fact, this is the case over the archimedean fields, see [AG, Corollary B].

**Appendix A. Proof of Lemma 2.10**

**Lemma A.1.** Let \( \mathfrak{v} \) be the \((F\text{-points of})\) the Lie algebra of \( V \). Then the exponential map \( \exp : \mathfrak{v} \to V \) maps the commutant \([\mathfrak{v}, \mathfrak{v}]\) of \( \mathfrak{v} \) onto the subgroup \([V, V]\) of \( V \) generated (set-theoretically) by all the commutators in \( V \).

**Proof.** Let \( \mathfrak{v}_i \) be the sequence of subalgebras of \( \mathfrak{v} \) defined by \( \mathfrak{v}_0 := \mathfrak{v} \), \( \mathfrak{v}_{i+1} := [\mathfrak{v}, \mathfrak{v}_i] \). The Baker-Campbell-Hausdorff formula implies that for any \( X \in \mathfrak{v} \) and \( Y \in \mathfrak{v} \), there exist \( A, B \in \mathfrak{v}_{i+2} \) and \( C \in \mathfrak{v}_{i+1} \) such that

1. \( \log(e^X e^Y) = X + Y + \frac{1}{2}[X, Y] + A \)
2. \( \log(e^X e^Y e^{-X} e^{-Y}) = [X, Y] + B \)
3. \( e^X e^Y = e^C e^X e^Y \)

By (12) we have \([V, V] \subset \exp([\mathfrak{v}, \mathfrak{v}]]) \). To prove the opposite inclusion we prove by descending induction on \( i \) that \( \exp(\mathfrak{v}_i) \subset [V, V] \) for any \( i > 0 \). Since \( \exp(\mathfrak{v}_i) \) is a group, it is enough to show that for any \( X \in \mathfrak{v} \) and \( Y \in \mathfrak{v}_{i-1} \) we have \( \exp([X, Y]) \in [V, V] \). Let \( B \) be as in (2), and \( C \) be as in (3) applied to \( \log(e^X e^Y e^{-X} e^{-Y}) \) and \(-B\). Then \( B, C \in \mathfrak{v}_{i+1} \), the induction hypothesis implies that \( e^B, e^C \in [V, V] \) and thus

\[
\exp([X, Y]) = \exp(\log(e^X e^Y e^{-X} e^{-Y}) - B) = e^C(e^X e^Y e^{-X} e^{-Y}) e^{-B} \in [V, V].
\]

\[ \square \]

**Corollary A.2.** Let \( V/[V, V] \) denote the abelization of \( V \). Then the natural map \( V/[V, V] \to (V/[V, V])(F) \) is an isomorphism.

**Proof.** Let \( \mathfrak{v} \) be \( \mathfrak{v} \) considered as an algebraic variety. By (12), the quotient \( V/\exp([\mathfrak{v}, \mathfrak{v}]) \) is an abelian group. Hence \([V, V] \subset \exp([\mathfrak{v}, \mathfrak{v}]) \). Thus, by Lemma A.1 we have \([V, V] \subset [V, V](F) \subset \exp([\mathfrak{v}, \mathfrak{v}]) = [V, V] \). Therefore \([V, V] = [V, V](F) \). Since unipotent groups have trivial Galois cohomologies (see [Ser97, §III.2.1, Proposition 6]), \( V(F)/[V, V](F) = (V/[V, V])(F) \) and the statement follows. \[ \square \]
By this corollary Lemma 2.10 reduces to the case when $V$ is commutative. Since any commutative unipotent group over $F$ is a power of $G_a$, this case follows from the isomorphism of $F$ to its Pontryagin dual. ∎

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