Abstract

The Descriptor System Tools (DSTOOLS) is a collection of MATLAB functions for the operation on and manipulation of rational transfer function matrices via their descriptor system realizations. The DSTOOLS collection relies on the Control System Toolbox and several mex-functions based on the Systems and Control Library SLICOT. Many of the implemented functions are based on the computational procedures described in Chapter 10 of the book: "A. Varga, Solving Fault Diagnosis Problems – Linear Synthesis Techniques, Springer, 2017". This document is the User’s Guide for the version V0.6 of DSTOOLS. First, we present the mathematical background on rational matrices and descriptor systems. Then, we give in-depth information on the command syntax of the main computational functions. Several examples illustrate the use of the main functions of DSTOOLS.
## Contents

1 Introduction 7

2 Background Material on Generalized System Representations 8
   2.1 Rational Transfer Function Matrices 8
   2.2 Descriptor Systems 10
   2.3 Linear Matrix Pencils 11
   2.4 Minimal Nullspace Bases 17
   2.5 Range Space Bases 19
   2.6 Poles and Zeros 20
   2.7 Additive Decompositions 25
   2.8 Coprime Factorizations 26
   2.9 Inner-Outer and Spectral Factorizations 27
   2.10 Linear Rational Matrix Equations 31
   2.11 Dynamic Cover-Based Order Reduction 33
   2.12 Hankel Norm 34
   2.13 Balancing-Related Order Reduction 35
   2.14 Solution of the Optimal Nehari Problems 36
   2.15 Solution of Least-Distance Problems 36

3 Description of DSTOOLS 39
   3.1 Quick Reference Tables 39
   3.2 Getting Started 40
      3.2.1 Building Generalized LTI Models 41
      3.2.2 Conversions between LTI Model Representations 42
      3.2.3 Conversion to Standard State-Space Form 43
      3.2.4 Sensitivity Issues for Polynomial-Based Representations 46
   3.3 Functions for System Analysis 47
      3.3.1 gpole 50
      3.3.2 gzero 51
      3.3.3 nrank 53
      3.3.4 ghanorm 55
   3.4 Functions for System Order Reduction 56
      3.4.1 gir 56
      3.4.2 gminreal 58
      3.4.3 gbalmr 60
      3.4.4 gss2ss 61
   3.5 Functions for Operations on Generalized LTI Systems 63
      3.5.1 grnull 63
      3.5.2 glnull 67
      3.5.3 grange 72
      3.5.4 grsol 77
      3.5.5 glsol 81
      3.5.6 gsdec 86
Notations and Symbols

∅ empty set
C field of complex numbers
R field of real numbers
Cs stability domain (i.e., open left complex half-plane in continuous-time or open
unit disk centered in the origin in discrete-time)
∂Cs boundary of stability domain (i.e., extended imaginary axis with infinity in-
cluded in continuous-time, or unit circle centered in the origin in discrete-time)
Cs closure of Cs: Cs = Cs ∪ ∂Cs
Cu open instability domain: Cu := C \ Cs
Cs closure of Cu: Cu := Cu ∪ ∂Cs
Cg “good” domain of C
Cb “bad” domain of C: Cb = C \ Cg
s complex frequency variable in the Laplace transform: s = σ + iω
z complex frequency variable in the Z-transform: z = e^{sT}, T – sampling time
λ complex frequency variable: λ = s in continuous-time or λ = z in discrete-time
λ̄ complex conjugate of the complex number λ
R(λ) field of real rational functions in indeterminate λ
R[λ] set of rational matrices in indeterminate λ with real coefficients and unspecified
dimensions
R[λ]_{p×m} set of p × m rational matrices in indeterminate λ with real coefficients
R[λ] set of polynomial matrices in indeterminate λ with real coefficients and un-
specified dimensions
R[λ]_{p×m} set of p × m polynomial matrices in indeterminate λ with real coefficients
δ(G(λ)) McMillan degree of the rational matrix G(λ)
G~(λ) Conjugate of G(λ) ∈ R(λ): G~(s) = G^T(−s) in continuous-time and G~(z) =
G^T(1/z) in discrete-time
ℓ2 Banach-space of square-summable sequences
L2 Lebesgue-space of square-integrable functions
H2 Hardy-space of square-integrable complex-valued functions analytic in Cu
L∞ Space of complex-valued functions bounded and analytic in ∂Cs
H∞ Hardy-space of complex-valued functions bounded and analytic in Cu
∥G∥_2 H2- or L2-norm of the transfer function matrix G(λ)
∥G∥_∞ H∞- or L∞-norm of the transfer function matrix G(λ)
∥G∥_{2/∞} either the H2- or H∞-norm of the transfer function matrix G(λ)
∥G∥_H Hankel norm of the transfer function matrix G(λ)
coli(M) the i-th column of the matrix M
rowi(M) the i-th row of the matrix M
MT transpose of the matrix M
MP pertranspose of the matrix M
M^{-1} inverse of the matrix M
M^{-T} transpose of the inverse matrix M^{-1}
M^† pseudo-inverse of the matrix M
\( \sigma(M) \) largest singular value of the matrix \( M \)
\( \sigma(M) \) least singular value of the matrix \( M \)
\( \mathcal{N}(M) \) kernel (or right nullspace) of the matrix \( M \)
\( \mathcal{N}_L(G(\lambda)) \) left kernel (or left nullspace) of \( G(\lambda) \in \mathbb{R}(\lambda) \)
\( \mathcal{N}_R(G(\lambda)) \) right kernel (or right nullspace) of \( G(\lambda) \in \mathbb{R}(\lambda) \)
\( \mathcal{R}(M) \) range (or image space) of the matrix \( M \)
\( \Lambda(A) \) set of eigenvalues of the matrix \( A \)
\( \Lambda(A, E) \) set of generalized eigenvalues of the pair \( (A, E) \)
\( \Lambda(A - \lambda E) \) set of eigenvalues of the pencil \( A - \lambda E \)
\( u \) unit roundoff of the floating-point representation
\( O(\epsilon) \) quantity of order of \( \epsilon \)
\( I_n \) or \( I \) identity matrix of order \( n \) or of an order resulting from context
\( e_i \) the \( i \)-th column of the (known size) identity matrix
\( 0_{m \times n} \) or \( 0 \) zero matrix of size \( m \times n \) or of a size resulting from context
\( \text{span } M \) span (or linear hull) of the columns of the matrix \( M \)
### Acronyms

| Acronym | Description                                      |
|---------|--------------------------------------------------|
| GCARE   | Generalized continuous-time algebraic Riccati equation |
| GDARE   | Generalized discrete-time algebraic Riccati equation |
| GRSD    | Generalized real Schur decomposition             |
| GRSF    | Generalized real Schur form                      |
| LCF     | Left coprime factorization                       |
| LDP     | Least distance problem                           |
| LTI     | Linear time-invariant                            |
| MIMO    | Multiple-input multiple-output                   |
| RCF     | Right coprime factorization                      |
| RSF     | Real Schur form                                  |
| SISO    | Single-input single-output                       |
| SVD     | Singular value decomposition                     |
| TFM     | Transfer function matrix                         |
1 Introduction

The Descriptor System Tools (DSTOOLS) is a collection of MATLAB functions for the operation on and manipulation of rational transfer function matrices via their descriptor system realizations. The initial version V0.5 of DSTOOLS covers the main computations encountered in the synthesis approaches of linear residual generation filters for continuous- or discrete-time linear systems, described in the Chapter 10 of the author’s book [58]:

Andreas Varga, *Solving Fault Diagnosis Problems - Linear Synthesis Techniques*, vol. 84 of Studies in Systems, Decision and Control, Springer International Publishing, xxviii+394, 2017.

The functions of the DSTOOLS collection rely on the Control System Toolbox [25] and several mex-functions based on the Systems and Control Library SLICOT [4]. The current release of DSTOOLS is version V0.6, dated June 30, 2017. DSTOOLS is distributed as a free software via the Bitbucket repository.\footnote{https://bitbucket.org/DSVarga/dstools} The codes have been developed under MATLAB 2015b and have been also tested with MATLAB 2016a, 2016b and 2017a. To use the functions of DSTOOLS, the Control System Toolbox must be installed in MATLAB running under 64-bit Windows 7, 8, 8.1 or 10.

This document describes version V0.6 of the DSTOOLS collection. It will be continuously extended in parallel with the implementation of new functions. The book [58] represents an important complementary documentation for the DSTOOLS collection: it describes the mathematical background on rational matrices and descriptor systems, and gives detailed descriptions of many of the underlying procedures. Additionally, the M-files of the functions are self-documenting and a detailed documentation can be obtained online by typing help with the M-file name. Please cite DSTOOLS as follows:

A. Varga. DSTOOLS – The Descriptor System Tools for MATLAB, 2017. 
https://sites.google.com/site/andreasvargacontact/home/software/dstools.
2 Background Material on Generalized System Representations

In this section we give background information on two system representations of linear time-invariant systems, namely the input-output representation via rational transfer function matrices and the generalized state-space representation, also known as descriptor system representation. Since each rational matrix can be interpreted as the transfer function matrix of a descriptor system, the manipulation of rational matrices can be alternatively performed via their descriptor representations, using numerically reliable computational algorithms.

The treatment in depth of most of concepts was not possible in the restricted size of this guide. The equivalence theory of linear matrix pencils is covered in [13]. The material on rational matrices is covered in several textbooks, of which we mention only the two widely cited books of Kailath [22] and Vidyasagar [61]. Linear descriptor systems (also known in the literature as linear differential-algebraic-equations-based systems or generalized state-space systems or singular systems), are discussed, to different depths and with different focus, in several books [5, 7, 24, 9].

2.1 Rational Transfer Function Matrices

Transfer functions are used to describe the input-output behaviour of single-input single-output (SISO) linear time-invariant (LTI) systems by relating the input and output variables via a gain depending on a frequency variable. For a SISO system with input $u(t)$ and output $y(t)$ depending on the continuous time variable $t$, let $\mathbf{u}(s) := \mathcal{L}(u(t))$ and $\mathbf{y}(s) := \mathcal{L}(y(t))$ denote the Laplace transformed input and output, respectively. Then, the transfer function of the continuous-time LTI system is defined as

$$g(s) := \frac{\mathbf{y}(s)}{\mathbf{u}(s)}$$

and relates the input and output in the form

$$\mathbf{y}(s) = g(s)\mathbf{u}(s).$$

The complex variable $s = \sigma + j\omega$, has for $\sigma = 0$ the interpretation of a complex frequency. If the time variable has a discrete variation with equally spaced values with increments given by a sampling-period $T$, then the transfer function of the discrete-time system is defined using the $\mathcal{Z}$-transforms of the input and output variables $\mathbf{u}(z) := \mathcal{Z}(u(t))$ and $\mathbf{y}(z) := \mathcal{Z}(y(t))$, respectively, as

$$g(z) := \frac{\mathbf{y}(z)}{\mathbf{u}(z)}$$

and relates the input and output in the form

$$\mathbf{y}(z) = g(z)\mathbf{u}(z).$$

The complex variable $z$ is related to the complex variable $s$ as $z = e^{sT}$. We will use the variable $\lambda$ to denote either the $s$ or $z$ complex variables, depending on the context, continuous- or discrete-time, respectively. Throughout this guide, bolded variables as $\mathbf{u}(\lambda)$ and $\mathbf{y}(\lambda)$ will be used to denote either the Laplace- or $\mathcal{Z}$-transformed quantities of the corresponding time-variables $u(t)$.
and \( y(t) \). Furthermore, we will restrict our discussion to rational transfer functions \( g(\lambda) \) which can be expressed as a ratio of two polynomials with real coefficients

\[
g(\lambda) = \frac{\alpha(\lambda)}{\beta(\lambda)} = \frac{a_m \lambda^m + a_{m-1} \lambda^{m-1} + \cdots + a_1 \lambda + a_0}{b_n \lambda^n + b_{n-1} \lambda^{n-1} + \cdots + b_1 \lambda + b_0},
\]

with \( a_m \neq 0 \) and \( b_n \neq 0 \). Thus, \( g(\lambda) \in \mathbb{R}(\lambda) \), where \( \mathbb{R}(\lambda) \) is the field of real rational functions.

Transfer function matrices are used to describe the input-output behaviour of multi-input multi-output (MIMO) LTI systems by relating the input and output variables via a matrix of gains depending on a frequency variable. Consider a MIMO system with \( m \) inputs \( u_1(t), \ldots, u_m(t) \), which form the \( m \)-dimensional input vector \( u(t) = [u_1(t), \ldots, u_m(t)]^T \), and \( p \) outputs \( y_1(t), \ldots, y_p(t) \), which form the \( p \)-dimensional output vector \( y(t) = [y_1(t), \ldots, y_p(t)]^T \). For a continuous dependence of \( u(t) \) and \( g(t) \) on the time variable \( t \), let \( u(s) \) and \( y(s) \) be the Laplace-transformed input and output vectors, respectively, while in the case of a discrete dependence on \( t \), we denote \( u(z) \) and \( y(z) \) the \( Z \)-transformed input and output vectors, respectively. We denote with \( \lambda \) the frequency variable, which is either \( s \) or \( z \), depending on the nature of the time variation, continuous or discrete, respectively. Let \( G(\lambda) \) be the \( p \times m \) transfer function matrix (TFM) defined as

\[
G(\lambda) = \begin{bmatrix} g_{11}(\lambda) & \cdots & g_{1m}(\lambda) \\ \vdots & \ddots & \vdots \\ g_{p1}(\lambda) & \cdots & g_{pm}(\lambda) \end{bmatrix},
\]

which relates relates the \( m \)-dimensional input vector \( u \) to the \( p \)-dimensional output vector \( y \) in the form

\[
y(\lambda) = G(\lambda)u(\lambda).
\]

The element \( g_{ij}(\lambda) \) describes the contribution of the \( j \)-th input \( u_j(t) \) to the \( i \)-th output \( y_i(t) \). We assume that each matrix entry \( g_{ij}(\lambda) \in \mathbb{R}(\lambda) \) and thus it can be expressed as ratio of two polynomials \( \alpha_{ij}(\lambda) / \beta_{ij}(\lambda) \) with real coefficients as \( g_{ij}(\lambda) = \alpha_{ij}(\lambda) / \beta_{ij}(\lambda) \) of the form (1).

Each TFM \( G(\lambda) \) belongs to the set of rational matrices with real coefficients, thus having elements in the field of real rational functions \( \mathbb{R}(\lambda) \). Polynomial matrices, having elements in the ring of polynomials with real coefficients \( \mathbb{R}[\lambda] \), can be assimilated in a natural way with special rational matrices with all elements having 1 as denominators. Let \( \mathbb{R}(\lambda)^{p \times m} \) and \( \mathbb{R}[\lambda]^{p \times m} \) denote the sets of \( p \times m \) rational and polynomial matrices with real coefficients, respectively. To simplify the notation, we will also use \( G(\lambda) \in \mathbb{R}(\lambda) \) or \( G(\lambda) \in \mathbb{R}[\lambda] \) if the dimensions of \( G(\lambda) \) are not relevant or are clear from the context.

A rational matrix \( G(\lambda) \in \mathbb{R}(\lambda) \) is called proper if \( \lim_{\lambda \to \infty} G(\lambda) = D \), with \( D \) having a finite norm. Otherwise, \( G(\lambda) \) is called improper. If \( D = 0 \), then \( G(\lambda) \) is strictly proper. An invertible \( G(\lambda) \) is biproper if both \( G(\lambda) \) and \( G^{-1}(\lambda) \) are proper. A polynomial matrix \( U(\lambda) \in \mathbb{R}[\lambda] \) is called unimodular if it is invertible and its inverse \( U^{-1}(\lambda) \in \mathbb{R}[\lambda] \) (i.e., is a polynomial matrix). The determinant of a unimodular matrix is therefore a constant.

The degree of a rational matrix \( G(\lambda) \), also known as the McMillan degree, is defined in Section 2.6. We only give here the definition of the degree of a rational vector \( v(\lambda) \). For this, we express first \( v(\lambda) \) in the form \( v(\lambda) = \tilde{v}(\lambda)/d(\lambda) \), where \( d(\lambda) \) is the monic least common multiple of all denominator polynomials of the elements of \( v(\lambda) \) and \( \tilde{v}(\lambda) \) is the corresponding polynomial vector \( \tilde{v}(\lambda) := d(\lambda)v(\lambda) \). Then, \( \deg v(\lambda) = \max(\deg \tilde{v}(\lambda), \deg v(\lambda)) \).
2.2 Descriptor Systems

A descriptor system is a generalized state-space representation of the form

\[ E\lambda x(t) = Ax(t) + Bu(t), \]
\[ y(t) = Cx(t) + Du(t), \]  

where \( x(t) \in \mathbb{R}^n \) is the state vector, \( u(t) \in \mathbb{R}^m \) is the input vector, and \( y(t) \in \mathbb{R}^p \) is the output vector, and where \( \lambda \) is either the differential operator \( \lambda x(t) = \frac{d}{dt}x(t) \) for a continuous-time system or the advance operator \( \lambda x(t) = x(t+1) \) for a discrete-time system. In all what follows, we assume \( E \) is square and possibly singular, and the pencil \( A - \lambda E \) is regular (i.e., \( \det(A - \lambda E) \neq 0 \)). If \( E = I_n \), we call the representation (2) a standard state-space system. The corresponding input-output representation of the descriptor system (2) is

\[ y(\lambda) = G(\lambda)u(\lambda), \]

where, depending on the system type, \( \lambda = s \), the complex variable in the Laplace transform for a continuous-time system, or \( \lambda = z \), the complex variable in the \( Z \)-transform for a discrete-time system, \( y(\lambda) \) and \( u(\lambda) \) are the Laplace- or \( Z \)-transformed output and input vectors, respectively, and \( G(\lambda) \) is the rational TFM of the system, defined as

\[ G(\lambda) = C(\lambda E - A)^{-1}B + D. \]

We alternatively denote descriptor systems of the form (2) with the quadruple \( (A - \lambda E,B,C,D) \) or a standard state-space system with \( (A,B,C,D) \) (if \( E = I_n \)), and use the notation

\[ G(\lambda) := \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix}, \]

(5)

to relate the TFM \( G(\lambda) \) to a particular descriptor system realization as in (2).

It is well known that a descriptor system representation of the form (2) is the most general description for a linear time-invariant system. Continuous-time descriptor systems arise frequently from modelling interconnected systems containing algebraic loops or constrained mechanical systems which describe contact phenomena. Discrete-time descriptor representations are frequently used to model economic processes.

The manipulation of rational matrices can be easily performed via their descriptor representations. The main result which allows this is the following [60]:

**Theorem 1.** For any rational matrix \( G(\lambda) \in \mathbb{R}(\lambda)^{p \times m} \), there exist \( n \geq 0 \) and the real matrices \( E,A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n} \) and \( D \in \mathbb{R}^{p \times m} \), with \( A - \lambda E \) regular, such that (4) holds.

The descriptor realization \( (A - \lambda E,B,C,D) \) of a given rational matrix \( G(\lambda) \) is not unique. For example, if \( U \) and \( V \) are invertible matrices of the size \( n \) of the square matrix \( E \), then two descriptor realizations \( (A - \lambda E,B,C,D) \) and \( (\tilde{A} - \lambda \tilde{E},\tilde{B},\tilde{C},\tilde{D}) \) related by a system similarity transformation of the form

\[ (A - \lambda E, B, C, D) = (UAV - \lambda UEV, UB, CV, D), \]

(6)

have the same TFM \( G(\lambda) \). Moreover, among all possible realizations of a given \( G(\lambda) \), with different sizes \( n \), there exist realizations which have the least dimension. A descriptor realization
$\begin{bmatrix} A - \lambda E & B \end{bmatrix}$ of the rational matrix $G(\lambda)$ is called minimal if the dimension $n$ of the square matrices $E$ and $A$ is the least possible one. The minimal realization of a given $G(\lambda)$ is also not unique, since two minimal realizations related by a system similarity transformation as in (6) correspond to the same $G(\lambda)$.

A minimal descriptor system realization $(A - \lambda E, B, C, D)$ is characterized by the following five conditions [59].

**Theorem 2.** A descriptor system realization $(A - \lambda E, B, C, D)$ of order $n$ is minimal if the following conditions are fulfilled:

(i) $\text{rank} \begin{bmatrix} A - \lambda E & B \end{bmatrix} = n$, $\forall \lambda \in \mathbb{C}$,

(ii) $\text{rank} \begin{bmatrix} E & B \end{bmatrix} = n$,

(iii) $\text{rank} \begin{bmatrix} A - \lambda E \\ C \end{bmatrix} = n$, $\forall \lambda \in \mathbb{C}$,

(iv) $\text{rank} \begin{bmatrix} E \\ C \end{bmatrix} = n$,

(v) $AN(E) \subseteq \mathcal{R}(E)$.

The conditions (i) and (ii) are known as finite and infinite controllability, respectively. A system or, equivalently, the pair $(A - \lambda E, B)$, is called finite controllable if it fulfills (i), infinite controllable if it fulfills (ii), and controllable if it fulfills both (i) and (ii). Similarly, the conditions (iii) and (iv) are known as finite and infinite observability, respectively. A system or, equivalently, the pair $(A - \lambda E, C)$, is called finite observable if it fulfills (iii), infinite observable if it fulfills (iv), and observable if it fulfills both (iii) and (iv). The condition (v) expresses the absence of non-dynamic modes. A descriptor realization which satisfies only (i) - (iv) is called irreducible (also weakly minimal). The numerical computation of irreducible realizations is addressed in [44] (see also [39] for alternative approaches).

### 2.3 Linear Matrix Pencils

Linear matrix pencils of the form $M - \lambda N$, where $M$ and $N$ are $m \times n$ matrices with elements in $\mathbb{C}$, play an important role in the theory of generalized LTI systems. In what follows we shortly review the equivalence theory of linear matrix pencils. For more details on this topic see [13].

The pencil $M - \lambda N$ is called regular if $m = n$ and $\det(M - \lambda N) \not\equiv 0$. Otherwise, the pencil is called singular. Two pencils $M - \lambda N$ and $\tilde{M} - \lambda \tilde{N}$ with $M, N, \tilde{M}, \tilde{N} \in \mathbb{C}^{m \times n}$ are strictly equivalent if there exist two invertible matrices $U \in \mathbb{C}^{n \times n}$ and $V \in \mathbb{C}^{n \times n}$ such that

$$U(M - \lambda N)V = \tilde{M} - \lambda \tilde{N}. \quad (7)$$

For a regular pencil, the strict equivalence leads to the (complex) Weierstrass canonical form, which is instrumental to characterize the dynamics of generalized systems.

**Lemma 1.** Let $M - \lambda N$ be an arbitrary regular pencil with $M, N \in \mathbb{C}^{n \times n}$. Then, there exist invertible matrices $U \in \mathbb{C}^{n \times n}$ and $V \in \mathbb{C}^{n \times n}$ such that

$$U(M - \lambda N)V = \begin{bmatrix} J_f - \lambda I & 0 \\ 0 & I - \lambda J_\infty \end{bmatrix}, \quad (8)$$

11
where $J_f$ is in a (complex) Jordan canonical form

$$J_f = \text{diag} \left( J_{s_1}(\lambda_1), J_{s_2}(\lambda_2), \ldots, J_{s_k}(\lambda_k) \right), \tag{9}$$

with $J_{s_i}(\lambda_i)$ an elementary $s_i \times s_i$ Jordan block of the form

$$J_{s_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & \cdots & \\ & \lambda_i & \cdots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

and $J_{\infty}$ is nilpotent and has the (nilpotent) Jordan form

$$J_{\infty} = \text{diag} \left( J_{s_1}^{\infty}(0), J_{s_2}^{\infty}(0), \ldots, J_{s_h}^{\infty}(0) \right). \tag{10}$$

The Weierstrass canonical form (8) exhibits the finite and infinite eigenvalues of the pencil $M - \lambda N$. The finite eigenvalues are $\lambda_i$, for $i = 1, \ldots, k$. Overall, by including all multiplicities, there are $n_f = \sum_{i=1}^{k} s_i$ finite eigenvalues and $n_{\infty} = \sum_{i=1}^{h} s_i^{\infty}$ infinite eigenvalues. Infinite eigenvalues with $s_i^{\infty} = 1$ are called simple infinite eigenvalues. We can also express the rank of $N$ as

$$\text{rank } N = n_f + \text{rank } J_{\infty} = n_f + \sum_{i=1}^{h} (s_i^{\infty} - 1) = n_f + n_{\infty} - h = n - h.$$ 

If $M$ and $N$ are real matrices, then there exist real matrices $U$ and $V$ such that the pencil $U(M - \lambda N)V$ is in a real Weierstrass canonical form, where the only difference is that $J_f$ is in a real Jordan form [20, Section 3.4]. In this form, the elementary real Jordan blocks correspond to pairs of complex conjugate eigenvalues.

If $M - \lambda N = A - \lambda I$ (e.g., the pole pencil for a standard state-space system), then all eigenvalues are finite and $J_f$ in the Weierstrass form is simply the (real) Jordan form of $A$. The transformation matrices can be chosen such that $U = V^{-1}$.

The eigenvalue structure of a regular pencil $A - \lambda E$ is completely described by the Weierstrass canonical form (see Lemma 1). However, the computation of this canonical form involves the use of (potentially ill-conditioned) general invertible transformations, and therefore numerical reliability cannot be guaranteed. Fortunately, the computation of Weierstrass canonical form can be avoided in almost all computations, and alternative “less” condensed forms can be employed instead, which can be computed by exclusively employing orthogonal similarity transformations.

The generalized real Schur decomposition (GRSD) of a matrix pair $(A, E)$ reveals the eigenvalues of the regular pencil $A - \lambda E$, by determining the generalized real Schur form (GRSF) of the pair $(A, E)$ (a quasi-triangular–triangular form) using orthogonal similarity transformations on the pencil $A - \lambda E$. The main theoretical result regarding the GRSD is the following theorem.

**Theorem 3.** Let $A - \lambda E$ be an $n \times n$ regular pencil, with $A$ and $E$ real matrices. Then, there exist orthogonal transformation matrices $Q$ and $Z$ such that

$$S - \lambda T := Q^T (A - \lambda E) Z = \begin{bmatrix} S_{11} & \cdots & S_{1k} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & S_{kk} \end{bmatrix} - \lambda \begin{bmatrix} T_{11} & \cdots & T_{1k} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T_{kk} \end{bmatrix}. \tag{11}$$
where each diagonal subpencil $S_{ii} - \lambda T_{ii}$, for $i = 1, \ldots, k$, is either of dimension $1 \times 1$ in the case of a finite real or infinite eigenvalue of the pencil $A - \lambda E$ or of dimension $2 \times 2$, with $T_{ii}$ upper triangular, in the case of a pair of finite complex conjugate eigenvalues of $A - \lambda E$.

The pair $(S, T)$ in (11) is in a GRSF and the eigenvalues of $A - \lambda E$ (or the generalized eigenvalues of the pair $(A, E)$) are given by

$$\Lambda(A - \lambda E) = \bigcup_{i=1}^{k} \Lambda(S_{ii} - \lambda T_{ii}) .$$

If $E = I$, then we can always choose $Q = Z$, $T = I$ and $S$ is the real Schur form (RSF) of $A$.

The order of eigenvalues (and thus of the associated pairs of diagonal blocks) of the reduced pencil $S - \lambda T$ is arbitrary. The reordering of the pairs of diagonal blocks (thus also of corresponding eigenvalues) can be done by interchanging two adjacent pairs of diagonal blocks of the GRSF. For the swapping of such two pairs of blocks orthogonal similarity transformations can be used. Thus, any arbitrary reordering of pairs of blocks (and thus of the corresponding eigenvalues) can be achieved in this way. An important application of this fact is the computation of orthogonal bases for the deflating subspaces of the pencil $A - \lambda E$ corresponding to a particular eigenvalue or a particular set of eigenvalues.

For a general (singular) pencil, the strict equivalence leads to the (complex) Kronecker canonical form, which is instrumental to characterize the zeros and singularities of a descriptor system.

**Lemma 2.** Let $M - \lambda N$ be an arbitrary pencil with $M, N \in \mathbb{C}^{m \times n}$. Then, there exist invertible matrices $U \in \mathbb{C}^{m \times m}$ and $V \in \mathbb{C}^{n \times n}$ such that

$$U(M - \lambda N)V = \begin{bmatrix} K_r(\lambda) & K_{reg}(\lambda) \\ K_{reg}(\lambda) & K_l(\lambda) \end{bmatrix},$$

where:

1) The full row rank pencil $K_r(\lambda)$ has the form

$$K_r(\lambda) = \text{diag}(L_{\epsilon_1}(\lambda), L_{\epsilon_2}(\lambda), \cdots, L_{\epsilon_{\nu_r}}(\lambda)),$$

with $L_i(\lambda)$ ($i \geq 0$) an $i \times (i + 1)$ bidiagonal pencil of form

$$L_i(\lambda) = \begin{bmatrix} -\lambda & 1 & & \\ & \ddots & \ddots & \\ & & -\lambda & 1 \end{bmatrix};$$

2) The regular pencil $K_{reg}(\lambda)$ is in a Weierstrass canonical form

$$K_{reg}(\lambda) = \begin{bmatrix} \tilde{J}_f - \lambda I & \\ I - \lambda \tilde{J}_\infty \end{bmatrix},$$

with $\tilde{J}_f$ in a (complex) Jordan canonical form as in (9) and with $\tilde{J}_\infty$ in a nilpotent Jordan form as in (10);
3) The full column rank \(K_l(\lambda)\) has the form

\[
K_l(\lambda) = \text{diag} \left( L_{n1}^T(\lambda), L_{n2}^T(\lambda), \ldots, L_{n\nu_l}^T(\lambda) \right).
\]  

As it is apparent from (12), the Kronecker canonical form exhibits the right and left singular structures of the pencil \(M - \lambda N\) via the full row rank block \(K_r(\lambda)\) and full column rank block \(K_l(\lambda)\), respectively, and the eigenvalue structure via the regular pencil \(K_{\text{reg}}(\lambda)\). The full row rank pencil \(K_r(\lambda)\) is \(n_r \times (n_r + \nu_r)\), where \(n_r = \sum_{i=1}^{\nu_r} \epsilon_i\), the full column rank pencil \(K_l(\lambda)\) is \((n_l + \nu_l) \times n_l\), where \(n_l = \sum_{j=1}^{\nu_l} \eta_j\), while the regular pencil \(K_{\text{reg}}(\lambda)\) is \(n_{\text{reg}} \times n_{\text{reg}}\), with \(n_{\text{reg}} = \tilde{n}_f + \tilde{n}_\infty\), where \(\tilde{n}_f\) is the number of finite eigenvalues in \(\Lambda(\mathcal{J})\) and \(\tilde{n}_\infty\) is the number of infinite eigenvalues in \(\Lambda(I - \lambda \mathcal{J}_\infty)\) (or equivalently the number of null eigenvalues in \(\Lambda(\mathcal{J}_\infty)\)).

The \(\epsilon_i \times (\epsilon_i + 1)\) blocks \(L_{\epsilon_i}(\lambda)\) with \(\epsilon_i \geq 0\) are the right elementary Kronecker blocks, and \(\epsilon_i\), for \(i = 1, \ldots, \nu_r\), are called the **right Kronecker indices**. The \((\eta_i + 1) \times \eta_i\) blocks \(L_{\eta_i}^T(\lambda)\) with \(\eta_i \geq 0\) are the left elementary Kronecker blocks, and \(\eta_i\), for \(i = 1, \ldots, \nu_l\), are called the **left Kronecker indices**. The normal rank \(r\) of the pencil \(M - \lambda N\) results as

\[
r := \text{rank}(M - \lambda N) = n_r + \tilde{n}_f + \tilde{n}_\infty + n_l.
\]

If \(M - \lambda N\) is regular, then there are no left- and right-Kronecker structures and the Kronecker canonical form is simply the Weierstrass canonical form.

**Remark 1.** By additional column permutations of the block \(K_r(\lambda)\) and row permutations of the block \(K_l(\lambda)\) (which can be included in the left and right transformations matrices \(U\) and \(V\)) we can bring these blocks to the alternative forms

\[
K_r(\lambda) = \begin{bmatrix} B_r & A_r - \lambda I_{n_r} \end{bmatrix}, \quad K_l(\lambda) = \begin{bmatrix} A_l - \lambda I_{n_l} \\ C_l \end{bmatrix},
\]

where the pair \((A_r, B_r)\) is in a Brunovsky controllable form

\[
A_r = \begin{bmatrix} A_{r,1} \\ A_{r,2} \\ \vdots \\ A_{r,\nu_r} \end{bmatrix}, \quad B_r = \begin{bmatrix} b_{r,1} \\ b_{r,2} \\ \vdots \\ b_{r,\nu_r} \end{bmatrix},
\]

with \(A_{r,i}\) an \(\varepsilon_i \times \varepsilon_i\) matrix and \(b_{r,i}\) an \(\varepsilon_i \times 1\) column vector of the forms

\[
A_{r,i} = \begin{bmatrix} 0 \\ I_{\varepsilon_i-1} \\ 0 \end{bmatrix} = J_{\varepsilon_i}(0), \quad b_{r,i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},
\]

and the pair \((A_l, C_l)\) is in a Brunovsky observable form

\[
A_l = \begin{bmatrix} A_{l,1} \\ A_{l,2} \\ \vdots \\ A_{l,\nu_l} \end{bmatrix}, \quad C_l = \begin{bmatrix} c_{l,1} \\ c_{l,2} \\ \vdots \\ c_{l,\nu_l} \end{bmatrix},
\]
with $A_{l,i}$ an $\eta_i \times \eta_i$ matrix and $c_{l,i}$ a $1 \times \eta_i$ row vector of the forms

$$A_{l,i} = \begin{bmatrix} 0 & 0 \\ I_{\eta_i-1} & 0 \end{bmatrix} = J^T_{\eta_i}(0), \quad c_{l,i} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}.$$ 

The computation of the Kronecker-canonical form may involve the use of ill-conditioned transformations and, therefore, is potentially numerically unstable. Fortunately, alternative staircase forms, called *Kronecker-like forms*, allow to obtain basically the same (or only a part of) structural information on the pencil $M - \lambda N$ by employing exclusively orthogonal transformations (i.e., $U^T U = I$ and $V^T V = I$).

The following result concerns with one of the main Kronecker-like forms.

**Theorem 4.** Let $M \in \mathbb{R}^{m \times n}$ and $N \in \mathbb{R}^{m \times n}$ be arbitrary real matrices. Then, there exist orthogonal $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$, such that

$$U(M - \lambda N)V = \begin{bmatrix} M_r - \lambda N_r & * & * \\ 0_{n_r \times (m_r + n_r)} & M_{\text{reg}} - \lambda N_{\text{reg}} & * \\ 0 & 0_{n_l \times n_l} & M_l - \lambda N_l \end{bmatrix},$$

(19)

where

1) The $n_r \times (m_r + n_r)$ pencil $M_r - \lambda N_r$ has full row rank, $n_r$, for all $\lambda \in \mathbb{C}$ and is in a *controllability staircase form*

$$M_r - \lambda N_r = \begin{bmatrix} B_r & A_r - \lambda E_r \end{bmatrix},$$

(20)

with $B_r \in \mathbb{R}^{n_r \times m_r}$, $A_r, E_r \in \mathbb{R}^{n_r \times n_r}$, and $E_r$ invertible.

2) The $n_{\text{reg}} \times n_{\text{reg}}$ pencil $M_{\text{reg}} - \lambda N_{\text{reg}}$ is regular and its eigenvalues are the eigenvalues of pencil $M - \lambda N$. The pencil $M_{\text{reg}} - \lambda N_{\text{reg}}$ may be chosen in a GRSF, with arbitrary ordered diagonal blocks.

3) The $(p_l + n_l) \times n_l$ pencil $M_l - \lambda N_l$ has full column rank, $n_l$, for all $\lambda \in \mathbb{C}$ and is in an *observability staircase form*

$$M_l - \lambda N_l = \begin{bmatrix} A_l - \lambda E_l \\ C_l \end{bmatrix},$$

(21)

with $C_l \in \mathbb{R}^{m_l \times n_l}$, $A_l, E_l \in \mathbb{R}^{m_l \times m_l}$, and $E_l$ invertible.

The generalized controllability staircase form (20) is defined with

$$\begin{bmatrix} B_r & A_r \end{bmatrix} = \begin{bmatrix} A_{1,0} & A_{1,1} & A_{1,2} & \cdots & A_{1,k-1} & A_{1,k} \\ 0 & A_{2,1} & A_{2,2} & \cdots & A_{2,k-1} & A_{2,k} \\ 0 & 0 & A_{3,2} & \cdots & A_{3,k-1} & A_{3,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_{k,k-1} & A_{k,k} \end{bmatrix},$$

(22)
where \( A_{j,j-1} \in \mathbb{R}^{\nu_j \times \nu_{j-1}} \), with \( \nu_0 = m_r \), are full row rank matrices for \( j = 1, \ldots, k \), and the resulting upper triangular matrix \( E_r \) has a similar block partitioned form

\[
E_r = \begin{bmatrix}
E_{1,1} & E_{1,2} & \cdots & E_{1,k-1} & E_{1,k} \\
0 & E_{2,2} & \cdots & E_{2,k-1} & E_{2,k} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & E_{k-1,k-1} & E_{k-1,k} \\
0 & 0 & \cdots & 0 & E_{k,k}
\end{bmatrix},
\]

(23)

where \( E_{j,j} \in \mathbb{R}^{\nu_j \times \nu_j} \). The resulting block dimensions \( \nu_j, j = 0, 1, \ldots, k \), satisfy

\[
m_r = \nu_0 \geq \nu_1 \geq \cdots \geq \nu_k > 0.
\]

The dimensions \( \nu_i, i = 1, \ldots, k \), of the diagonal blocks of \( A_r - \lambda E_r \) completely determine the right Kronecker structure of \( M - \lambda N \) as follows: there are \( \nu_{i-1} - \nu_i \) blocks \( L_{i-1}(\lambda) \) of size \((i-1) \times i\), for \( i = 1, \ldots, k \).

The generalized observability staircase form (21) is

\[
\begin{bmatrix}
A_t \\
C_t
\end{bmatrix} = \begin{bmatrix}
A_{t,\ell} & A_{t,\ell-1} & \cdots & A_{t,2} & A_{t,1} \\
A_{t-1,\ell} & A_{t-1,\ell-1} & \cdots & A_{t-1,2} & A_{t-1,1} \\
0 & A_{t-2,\ell-1} & \cdots & A_{t-2,2} & A_{t-2,1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & A_{1,2} & A_{1,1} \\
0 & 0 & \cdots & 0 & A_{0,1}
\end{bmatrix},
\]

(24)

where \( A_{j-1,j} \in \mathbb{R}^{\mu_{j-1} \times \mu_j} \), with \( \mu_0 = p_t \), are full column rank matrices for \( j = 1, \ldots, \ell \), and the resulting upper triangular matrix \( E_o \) has a similar block partitioned form

\[
E_t = \begin{bmatrix}
E_{\ell,\ell} & E_{\ell,\ell-1} & \cdots & E_{\ell,2} & E_{\ell,1} \\
0 & E_{\ell-1,\ell} & \cdots & E_{\ell-1,2} & E_{\ell-1,1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & E_{2,2} & E_{2,1} \\
0 & 0 & \cdots & 0 & E_{1,1}
\end{bmatrix},
\]

(25)

with \( E_{j,j} \in \mathbb{R}^{\mu_j \times \mu_j} \). The resulting block dimensions \( \mu_j, j = 0, 1, \ldots, \ell \), satisfy

\[
p_t = \mu_0 \geq \mu_1 \cdots \geq \mu_\ell > 0.
\]

The dimensions \( \mu_i, i = 1, \ldots, \ell \) of the diagonal blocks of \( A_t - \lambda E_t \) in the observability staircase form completely determine the left Kronecker structure of \( M - \lambda N \) as follows: there are \( \mu_{i-1} - \mu_i \) blocks \( L_{i-1}^T(\lambda) \) of size \( i \times (i-1) \), \( i = 1, \ldots, \ell \). We have \( n_r = \sum_{i=1}^k \nu_i \) and \( n_t = \sum_{i=1}^\ell \mu_i \), and the normal rank of \( M - \lambda N \) is \( n_r + n_{\text{reg}} + n_t \).

Algorithms for the computation of Kronecker-like forms of linear pencils, using SVD-based rank determinations, have been proposed in [38, 8]. Albeit numerically reliable, these algorithms have a computational complexity \( O(n^4) \), where \( n \) is the minimum of row or column dimensions of the pencil. More efficient algorithms of complexity \( O(n^3) \) have been proposed in [3, 48, 30], which rely on using QR decompositions with column pivoting for rank determinations.
2.4 Minimal Nullspace Bases

A $p$-dimensional rational vector $v(\lambda) \in \mathbb{R}(\lambda)^p$ can be seen as either a $1 \times p$ or a $p \times 1$ rational matrix. A set of rational vectors $V(\lambda) := \{v_1(\lambda), \ldots, v_k(\lambda)\}$ is said to be linearly dependent over the field $\mathbb{R}(\lambda)$ if there exists $k$ rational functions $\gamma_i(\lambda) \in \mathbb{R}(\lambda)$, $i = 1, \ldots, k$, with $\gamma_i(\lambda) \neq 0$ for at least one $i$, such that, the linear combination

$$\sum_{i=1}^{k} \gamma_i(\lambda)v_i(\lambda) = 0. \quad (26)$$

The set of vectors $V(\lambda)$ is linearly independent over $\mathbb{R}(\lambda)$ if (26) implies that $\gamma_i(\lambda) = 0$ for each $i = 1, \ldots, k$. It is important to note, that a linearly dependent set $V(\lambda)$ over $\mathbb{R}(\lambda)$, can be still linearly independent over another field (e.g., the field of reals if $\gamma_i \in \mathbb{R}$).

The normal rank of a rational matrix $G(\lambda) \in \mathbb{R}(\lambda)^{p \times m}$, which we also denote by rank $G(\lambda)$, is the maximal number of linearly independent rows (or columns) over the field of rational functions $\mathbb{R}(\lambda)$. It can be shown that the normal rank of $G(\lambda)$ is the maximally possible rank of the complex matrix $G(\lambda)$ for all values of $\lambda \in \mathbb{C}$ such that $G(\lambda)$ has finite norm. This interpretation provides a simple way to determine the normal rank as the maximum of the rank of $G(\lambda)$ for a few random values of the frequency variable $\lambda$.

It is easy to check that the set of $p$-dimensional rational vectors $\mathbb{R}(\lambda)^p$ forms a vector space with scalars defined over $\mathbb{R}(\lambda)$. If $\mathcal{V}(\lambda) \subset \mathbb{R}(\lambda)^p$ is a vector space, then there exists a set of linearly independent rational vectors $V(\lambda) := \{v_1(\lambda), \ldots, v_{n_b}(\lambda)\} \subset \mathcal{V}(\lambda)$ such that any vector in $\mathcal{V}(\lambda)$ is a linear combination of the vectors in $V(\lambda)$ (equivalently, any set of $n_b + 1$ vectors, including an arbitrary vector from $V(\lambda)$ and the $n_b$ vectors in $V(\lambda)$, is linearly dependent). The set $V(\lambda)$ is called a basis of the vector space $\mathcal{V}(\lambda)$ and $n_b$ is the dimension of $\mathcal{V}(\lambda)$. With a slight abuse of notation, we denote $V(\lambda)$ the matrix formed of the $n_b$ stacked row vectors

$$V(\lambda) = \begin{bmatrix} v_1(\lambda) \\ \vdots \\ v_{n_b}(\lambda) \end{bmatrix}$$

or the $n_b$ concatenated column vectors

$$V(\lambda) = \begin{bmatrix} v_1(\lambda) & \cdots & v_{n_b}(\lambda) \end{bmatrix}.$$ 

Interestingly, as basis vectors we can always use polynomial vectors since we can replace each vector $v_i(\lambda)$ of a rational basis, by $v_i(\lambda)$ multiplied with the least common multiple of the denominators of the components of $v_i(\lambda)$.

The use of polynomial bases allows to define the main concepts related to so-called minimal bases. Let $n_i$ be the degree of the $i$-th polynomial vector $v_i(\lambda)$ of a polynomial basis $V(\lambda)$ (i.e., $n_i$ is the largest degree of the components of $v_i(\lambda)$). Then, $n := \sum_{i=1}^{n_b} n_i$ is, by definition, the degree of the polynomial basis $V(\lambda)$. A minimal polynomial basis of $\mathcal{V}(\lambda)$ is one for which $n$ has the least achievable value. For a minimal polynomial basis, $n_i$, for $i = 1, \ldots, n_b$, are called the row or column minimal indices (also known as left or right Kronecker indices, respectively). Two important examples are the left and right nullspace bases of a rational matrix, which are shortly discussed below.
Let $G(\lambda)$ be a $p \times m$ rational matrix $G(\lambda)$ whose normal rank is $r < \min(p, m)$. It is easy to show that the set

$$\mathcal{N}_L(G(\lambda)) := \{v(\lambda) \in \mathbb{R}(\lambda)^{1 \times p} \mid v(\lambda)G(\lambda) = 0\}$$

is a linear space called the left nullspace of $G(\lambda)$. Analogously,

$$\mathcal{N}_R(G(\lambda)) := \{v(\lambda) \in \mathbb{R}(\lambda)^{m \times 1} \mid G(\lambda)v(\lambda) = 0\}$$

is a linear space called the right nullspace of $G(\lambda)$. The dimension of $\mathcal{N}_L(G(\lambda))$ [$\mathcal{N}_R(G(\lambda))$] is $p - r$ [$m - r$], and therefore, there exist $p - r$ [$m - r$] linearly independent polynomial vectors which form a minimal polynomial basis for $\mathcal{N}_L(G(\lambda))$ [$\mathcal{N}_R(G(\lambda))$]. Let $n_{l,i}$ [$n_{r,i}$] be the left [right] minimal indices and let $n_l := \sum_{i=1}^{p-r} n_{l,i}$ [$n_r := \sum_{i=1}^{m-r} n_{r,i}$] be the least possible degree of the left [right] nullspace basis. The least left and right degrees $n_l$ and $n_r$, respectively, play an important role in relating the main structural elements of rational matrices (see the discussion of poles and zeros in Section 2.6).

Some properties of minimal polynomial bases are summarized below for the left nullspace bases.

**Lemma 3.** Let $G(\lambda)$ be a $p \times m$ rational matrix of normal rank $r$ and let $\tilde{N}_l(\lambda)$ be a $(p - r) \times p$ minimal polynomial basis of the left nullspace $\mathcal{N}_L(G(\lambda))$ with left minimal (or left Kronecker) indices $n_{l,i}, i = 1, \ldots, p - r$. Then the following holds:

1. The left minimal indices are unique up to permutations (i.e., if $\tilde{N}_l(\lambda)$ is another minimal polynomial basis, then $N_l(\lambda)$ and $\tilde{N}_l(\lambda)$ have the same left minimal indices).
2. $\tilde{N}_l(\lambda)$ is irreducible, having full row rank for all $\lambda \in \mathbb{C}$ (i.e., $\tilde{N}_l(\lambda)$ has no finite zeros, see Section 2.6).
3. $\tilde{N}_l(\lambda)$ is row reduced (i.e., the leading row coefficient matrix formed from the coefficients of the highest row degrees has full row rank.)

An irreducible and row-reduced polynomial basis is actually a minimal polynomial basis. Irreducibility implies that any polynomial vector $v(\lambda)$ in the space spanned by the rows of $\tilde{N}_l(\lambda)$, can be expressed as a linear combination of basis vectors $v(\lambda) = \phi(\lambda)\tilde{N}_l(\lambda)$, with $\phi(\lambda)$ being a polynomial vector. In particular, assuming the rows of $N_l(\lambda)$ are ordered such that $n_{l,i} \leq n_{l,i+1}$ for $i = 1, \ldots, p - r - 1$, then for any $v(\lambda)$ of degree $n_{l,i}$, the corresponding $\phi(\lambda)$ has as its $j$-th element a polynomial of degree at most $n_{l,i} - n_{l,j}$ for $j = 1, \ldots, i$, and the rest of components are zero. This property allows to easily generate left polynomial annihilators of given degrees.

Minimal polynomial bases allow to easily build simple minimal proper rational bases. These are proper rational bases having the property that the sum of degrees of the rows [columns] is equal to the least left [right] degree of a minimal polynomial basis $n_l$ [$n_r$]. A simple minimal proper rational left nullspace basis with arbitrary poles can be constructed by forming $\tilde{N}_l(\lambda) := M(\lambda)N_l(\lambda)$ with

$$M(\lambda) = \text{diag}(1/d_1(\lambda), \ldots, 1/d_{p-r}(\lambda)),$$

(27)

where $d_i(\lambda)$ is a polynomial of degree $n_{l,i}$ with arbitrary roots. Since $N_l(\lambda)$ is row reduced, it follows that $D_l := \lim_{\lambda \to \infty} \tilde{N}_l(\lambda)$ has full row rank (i.e., $\tilde{N}_l(\lambda)$ has no infinite zeros, see Section 2.6). A simple minimal proper rational left nullspace basis allows to generate, in a
straightforward manner, left rational annihilators of given McMillan degrees by forming linear combinations of the basis vectors in \( \tilde{N}_l(\lambda) \) using specially chosen rational vectors \( \phi(\lambda) \) (see Section 2.6 for the definition of the McMillan degree of a rational matrix). The concept of simple minimal proper rational basis has been introduced in [42] as the natural counterpart of a minimal polynomial basis.

Let \( G(\lambda) \) be a \( p \times m \) rational matrix of normal rank \( r \) and let \((A - \lambda E, B, C, D)\) be an irreducible descriptor realization of \( G(\lambda) \). To determine a basis \( N_l(\lambda) \) of the left nullspace of \( G(\lambda) \), we can exploit the simple fact (see [60, Theorem 2]) that \( N_l(\lambda) \) is a left minimal nullspace basis of \( G(\lambda) \) if and only if, for a suitable \( M_l(\lambda) \),

\[
Y_l(\lambda) := \begin{bmatrix} M_l(\lambda) & N_l(\lambda) \end{bmatrix}
\tag{28}
\]

is a left minimal nullspace basis of the associated system matrix pencil

\[
S(\lambda) := \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix}.
\tag{29}
\]

Thus, to compute \( N_l(\lambda) \) we can determine first a left minimal nullspace basis \( Y_l(\lambda) \) for \( S(\lambda) \) and then \( N_l(\lambda) \) results as

\[
N_l(\lambda) = Y_l(\lambda) \begin{bmatrix} 0 \\ I_p \end{bmatrix}.
\]

By duality, if \( Y_r(\lambda) \) is a right minimal nullspace basis for \( S(\lambda) \), then a right minimal nullspace basis of \( G(\lambda) \) is given by

\[
N_r(\lambda) = \begin{bmatrix} 0 & I_m \end{bmatrix} Y_r(\lambda).
\]

The Kronecker canonical form (12) of the system matrix pencil \( S(\lambda) \) in (29) allows to easily determine left and right nullspace bases of \( G(\lambda) \). A numerically reliable computational approach to compute proper minimal nullspace bases of rational matrices is described in [50] and relies on using the Kronecker-like form (19) of the system matrix pencil, which can be determined by using exclusively orthogonal similarity transformations. The computation of simple minimal proper nullspace bases is described in [55].

### 2.5 Range Space Bases

For any \( p \times m \) real rational matrix \( G(\lambda) \) of normal rank \( r \), there exists a full rank factorization of \( G(\lambda) \) of the form

\[
G(\lambda) = R(\lambda)X(\lambda),
\tag{30}
\]

where \( R(\lambda) \) is a \( p \times r \) full column rank rational matrix and \( X(\lambda) \) is a \( r \times m \) full row rank rational matrix. This factorization generalizes the full-rank factorization of constant matrices, and, similarly to the constant case, it is not unique. Indeed, for any \( r \times r \) invertible rational matrix \( M(\lambda) \), \( G(\lambda) = \tilde{R}(\lambda)\tilde{X}(\lambda) \), with \( \tilde{R}(\lambda) = R(\lambda)M(\lambda) \) and \( \tilde{X} = M^{-1}(\lambda)X(\lambda) \), is also a full rank factorization of \( G(\lambda) \).

Using (30), it is straightforward to show that \( G(\lambda) \) and \( R(\lambda) \) have the same range space over the rational functions

\[
\mathcal{R}(G(\lambda)) = \mathcal{R}(R(\lambda)).
\]

For this reason, with a little abuse of language, we will call \( R(\lambda) \) the range (or image) matrix of \( G(\lambda) \) (or simply the range of \( G(\lambda) \)). Since \( R(\lambda) \) has normal rank \( r \), its columns form a set of \( r \) basis vectors of \( \mathcal{R}(G(\lambda)) \).
The poles of $R(\lambda)$ can be chosen (almost) arbitrary, by replacing $R(\lambda)$ with $\tilde{R}(\lambda)$, the numerator factor of right coprime factorization $R(\lambda) = \tilde{R}(\lambda)M^{-1}(\lambda)$, where $\tilde{R}(\lambda)$ and $M(\lambda)$ have only poles in a given complex domain $C_g$. From the full-rank factorization (30) results that each zero of $G(\lambda)$ is either a zero of $R(\lambda)$ or of $X(\lambda)$. Therefore, it is possible to construct full-rank factorizations with $R(\lambda)$ having as zeros a symmetric subset of zeros of $G(\lambda)$. A minimal McMillan degree basis results if $R(\lambda)$ has no zeros. Of special interest in some applications are bases which are inner, such that $R(\lambda)$ is stable and satisfies $R^\sim(\lambda)R(\lambda) = I_r$. Applications of such bases are the computation of inner-outer factorizations (see Section 2.9) and of the normalized coprime factorizations (see Section 2.8). Methods to compute various full rank factorizations are discussed in [56], based on techniques developed in [32] and [29].

### 2.6 Poles and Zeros

For a scalar rational function $g(\lambda) \in \mathbb{R}(\lambda)$, the values of $\lambda$ for which $g(\lambda)$ is infinite are called the poles of $g(\lambda)$. If $g(\lambda) = \alpha(\lambda)/\beta(\lambda)$ has the form in (1), then the $n$ roots $\lambda^i, i = 1, \ldots, n$, of $\beta(\lambda)$ are the finite poles of $g(\lambda)$, while if $m < n$, there are also, by convention, $n-m$ infinite poles. The values of $\lambda$ for which $g(\lambda) = 0$ are called the zeros of $g(\lambda)$. The $m$ roots $\lambda^i, i = 1, \ldots, m$, of $\alpha(\lambda)$ are the finite zeros of $g(\lambda)$, while if $n < m$, there are also, by convention, $m - n$ infinite zeros. It follows that the number of poles is equal to the number of zeros and is equal to $\max(m, n)$, the degree of $g(\lambda)$. The rational function $g(\lambda)$ in (1) can be equivalently expressed in terms of its finite poles and zeros in the factorized form

$$g(\lambda) = k_g \prod_{i=1}^{m}(\lambda - \lambda^i),$$

(31)

where $k_g = a_m/b_n$. If $g(\lambda)$ is the transfer function of a SISO LTI system, then we will always assume that $g(\lambda)$ is in a minimal cancelled (irreducible) form, that is, the polynomials $\alpha(\lambda)$ and $\beta(\lambda)$ in (1) have 1 as greatest common divisor. Equivalently, the two polynomials have no common roots, and therefore no pole-zero cancellation may occur in (31). Two such polynomials are called coprime.

In studying the stability of systems, the poles play a primordial role. Their real parts, in the case of a continuous-time system, or moduli, in the case of a discrete-time system, determine the asymptotic (exponential) decay or divergence speed of the system output. A SISO LTI system with the transfer function $g(\lambda)$ is exponentially stable (or equivalently $g(\lambda)$ is stable) if $g(\lambda)$ is proper and has all poles in the appropriate stability domain $C_s$. The system is unstable if it has at least one pole outside of the stability domain and anti-stable if all poles lie outside of the stability domain. Poles inside the stability domain are called stable poles, while those outside of the stability domain are called unstable poles. For continuous-time systems the stability domain is the open left half complex plane $C_s = \{s \in \mathbb{C} : \Re(s) < 0\}$, while for discrete-time systems the stability domain is the open unit disk $C_s = \{z \in \mathbb{C} : |z| < 1\}$. We denote by $\partial C_s$ the boundary of the stability domain. For continuous-time systems, the boundary of the stability domain is the extended imaginary axis (i.e., including the point at infinity) $\partial C_s = \{\infty\} \cup \{s \in \mathbb{C} : \Re(s) = 0\}$, while for discrete-time systems the boundary of the stability domain is the unit circle $\partial C_s = \{z \in \mathbb{C} : |z| = 1\}$. We denote $\overline{C_s} = C_s \cup \partial C_s$ the closure of the stability domain. The instability domain of poles we denote by $\overline{C_u}$ and is the complement of $C_s$ in $\mathbb{C}$, $\overline{C_u} = \mathbb{C} \setminus C_s$. It is also the closure of the set denoted by $C_u$, which for a continuous-time system is the open right-half plane $C_u = \{s \in \mathbb{C} : \Re(s) > 0\}$, while for a discrete-time systems...
is the exterior of the unit circle \( C_u = \{ z \in \mathbb{C} : |z| > 1 \} \). The stability degree of poles is defined as the largest real part of the poles in the continuous-time case, or the largest absolute value of the poles in the discrete-time case.

Let \( \mathbb{R}_s(\lambda) \) be the set of proper stable transfer functions having poles only in \( C_s \). A transfer function \( g(\lambda) \in \mathbb{R}_s(\lambda) \) having only zeros in \( C_s \) is called minimum-phase. Otherwise it is called non-minimum-phase. The zeros of \( g(\lambda) \) in \( C_s \) are called minimum-phase zeros, while those outside \( C_s \) are called non-minimum-phase zeros.

There are no straightforward generalizations of poles and zeros of scalar rational functions to the rational matrix case. Instrumental for a rigorous definition are two canonical forms: the Smith form for polynomial matrices and the Smith-McMillan form for rational matrices. For polynomial matrices we have the following important result.

**Lemma 4.** Let \( P(\lambda) \in \mathbb{R}[\lambda]^{p \times m} \) be any polynomial matrix. Then, there exist unimodular matrices \( U(\lambda) \in \mathbb{R}[\lambda]^{p \times p} \) and \( V(\lambda) \in \mathbb{R}[\lambda]^{m \times m} \) such that

\[
U(\lambda)P(\lambda)V(\lambda) = S(\lambda) := \begin{bmatrix}
\alpha_1(\lambda) & 0 & \cdots & 0 & \cdots & 0 \\
0 & \alpha_2(\lambda) & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \alpha_r(\lambda) & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{bmatrix}
\]

and \( \alpha_i(\lambda) \) divides \( \alpha_{i+1}(\lambda) \) for \( i = 1, \ldots, r - 1 \).

The polynomial matrix \( S(\lambda) \) is called the Smith form of \( P(\lambda) \) and \( r \) is the normal rank of \( P(\lambda) \). The diagonal elements \( \alpha_1(\lambda), \ldots, \alpha_r(\lambda) \) are called the invariant polynomials of \( P(\lambda) \). The roots of the polynomials \( \alpha_i(\lambda) \), for \( i = 1, \ldots, r \), are called the finite zeros of the polynomial matrix \( P(\lambda) \). To each distinct finite zero \( \lambda_z \) of \( P(\lambda) \), we can associate the multiplicities \( \sigma_i(\lambda_z) \geq 0 \) of root \( \lambda_z \) in each of the polynomials \( \alpha_i(\lambda) \), for \( i = 1, \ldots, r \). By convention, \( \sigma_i(\lambda_z) = 0 \) if \( \lambda_z \) is not a root of \( \alpha_i(\lambda) \). The divisibility properties of \( \alpha_i(\lambda) \) imply that

\[
0 \leq \sigma_1(\lambda_z) \leq \sigma_2(\lambda_z) \leq \cdots \leq \sigma_r(\lambda_z).
\]

Any rational matrix \( G(\lambda) \) can be expressed as

\[
G(\lambda) = \frac{P(\lambda)}{d(\lambda)},
\]

where \( d(\lambda) \) is the monic least common multiple of the denominator polynomials of the entries of \( G(\lambda) \), and \( P(\lambda) := d(\lambda)G(\lambda) \) is a polynomial matrix. Then, we have the following straightforward extension of Lemma 4 to rational matrices.

**Lemma 5.** Let \( G(\lambda) \in \mathbb{R}(\lambda)^{p \times m} \) be any rational matrix. Then, there exist unimodular matrices
$U(\lambda) \in \mathbb{R}[\lambda]^{p \times p}$ and $V(\lambda) \in \mathbb{R}[\lambda]^{m \times m}$ such that

$$U(\lambda)G(\lambda)V(\lambda) = H(\lambda) := \begin{bmatrix}
\frac{\alpha_1(\lambda)}{\beta_1(\lambda)} & 0 & \cdots & 0 & \cdots & 0 \\
0 & \frac{\alpha_2(\lambda)}{\beta_2(\lambda)} & \cdots & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\
0 & 0 & \cdots & \frac{\alpha_r(\lambda)}{\beta_r(\lambda)} & \cdots & 0 \\
\vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & \cdots & 0
\end{bmatrix}, \tag{33}$$

with $\alpha_i(\lambda)$ and $\beta_i(\lambda)$ coprime for $i = 1, \ldots, r$ and $\alpha_i(\lambda)$ divides $\alpha_{i+1}(\lambda)$ and $\beta_{i+1}(\lambda)$ divides $\beta_i(\lambda)$ for $i = 1, \ldots, r - 1$.

The rational matrix $H(\lambda)$ is called the Smith–McMillan form of $G(\lambda)$ and $r$ is the normal rank of $G(\lambda)$. The Smith-McMillan form is a powerful conceptual tool which allows to define rigorously the notions of poles and zeros of MIMO LTI systems and to establish several basic factorization results of rational matrices.

The roots of the numerator polynomials $\alpha_i(\lambda)$, for $i = 1, \ldots, r$, are called the finite zeros of the rational matrix $G(\lambda)$ and the roots of the denominator polynomials $\beta_i(\lambda)$, for $i = 1, \ldots, r$, are called the finite poles of the rational matrix $G(\lambda)$. To each finite $\lambda_z$, which is a zero or a pole of $G(\lambda)$ (or both), we can associate its multiplicities $\{\sigma_1(\lambda_z), \ldots, \sigma_r(\lambda_z)\}$, where $\sigma_i(\lambda_z)$ is the multiplicity of $\lambda_z$ either as a pole or a zero of the ratio $\alpha_i(\lambda)/\beta_i(\lambda)$, for $i = 1, \ldots, r$. By convention, we use negative values for poles and positive values for zeros. The divisibility properties of $\alpha_i(\lambda)$ and $\beta_i(\lambda)$ imply that

$$\sigma_1(\lambda_z) \leq \sigma_2(\lambda_z) \leq \cdots \leq \sigma_r(\lambda_z).$$

The $r$-tuple of multiplicities $\{\sigma_1(\lambda_z), \ldots, \sigma_r(\lambda_z)\}$ completely characterizes the pole-zero structure of $G(\lambda)$ in $\lambda_z$.

The relative degrees of $\alpha_i(\lambda)/\beta_i(\lambda)$ do not provide the correct information on the multiplicity of infinite zeros and poles. This is because the used unimodular transformations may have poles and zeros at infinity. To overcome this, the multiplicity of zeros and poles at infinity are defined in terms of multiplicities of poles and zeros of $G(1/\lambda)$ in $\lambda_z = 0$. This allows to define the multiplicities of infinite zeros of a polynomial matrix $P(\lambda)$ as the multiplicities of the null poles in the Smith-McMillan for of $P(1/\lambda)$.

The McMillan degree of a rational matrix $G(\lambda)$, usually denoted by $\delta(G(\lambda))$, is the number $n_p$ of its poles, both finite and infinite, counting all multiplicities. If $n_z$ is the number of zeros (finite and infinite, counting all multiplicities), then we have the following important structural relation for any rational matrix [60]

$$n_p = n_z + n_l + n_r, \tag{34}$$

where $n_l$ and $n_r$ are the least degrees of the minimal polynomial bases for the left and right nullspaces of $G(\lambda)$, respectively.

For a given rational matrix $G(\lambda)$, any (finite or infinite) pole of its elements $g_{ij}(\lambda)$, is also a pole of $G(\lambda)$. Therefore, many notions related to poles of SISO LTI systems introduced previously can be extended in a straightforward manner to MIMO LTI systems. For example,
the notion of properness of $G(\lambda)$ can be equivalently defined as the nonexistence of infinite poles in the elements of $G(\lambda)$ (i.e., $G(\lambda)$ has only finite poles). The notion of exponential stability of a LTI system with a proper TFM $G(\lambda)$ can be defined as the lack of unstable poles in all elements of $G(\lambda)$ (i.e., all poles of $G(\lambda)$ are in the stable domain $\mathbb{C}_s$). A TFM $G(\lambda)$ with only stable poles is called stable. Otherwise, $G(\lambda)$ is called unstable. A similar definition applies for the stability degree of poles.

The zeros of $G(z)$ are also called the transmission zeros of the corresponding LTI system. A proper and stable $G(z)$ is minimum-phase if all its finite zeros are stable. Otherwise, it is called non-minimum-phase.

Consider the irreducible descriptor system $(A - \lambda E, B, C, D)$ with the corresponding TFM $G(\lambda) \in \mathbb{R}(\lambda)^{p \times m}$. Two pencils play a fundamental role in defining the main structural elements of the rational matrix $G(\lambda)$. The regular pole pencil

$$P(\lambda) := A - \lambda E$$

characterizes the pole structure of $G(\lambda)$, exhibited by the Weierstrass canonical form of the pole pencil $P(\lambda)$. The (singular) system matrix pencil

$$S(\lambda) := \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix}$$

characterizes the zero structure of $G(\lambda)$, as well as the right- and left-singular structures of $G(\lambda)$, which are exhibited by the Kronecker canonical form of the system matrix pencil $S(\lambda)$.

The main structural results for the rational TFM $G(\lambda)$ can be stated in terms of its irreducible descriptor system realization $(A - \lambda E, B, C, D)$ of order $n$. The following facts rely on the Weierstrass canonical form (8) of the pole pencil $P(\lambda)$ in (35) (see Lemma 1) and the Kronecker canonical form (12) of the system matrix pencil $S(\lambda)$ in (36) (see Lemma 2):

1) The finite poles of $G(\lambda)$ are the finite eigenvalues of the pole pencil $P(\lambda)$ and are the eigenvalues (counting multiplicities) of the matrix $J_f$ in the Weierstrass canonical form (8) of the pencil $P(\lambda)$.

2) The infinite poles of $G(\lambda)$ have multiplicities defined by the multiplicities of the infinite eigenvalues of the pole pencil $P(\lambda)$ minus 1 and are the dimensions minus 1 of the nilpotent Jordan blocks in the matrix $J_{\infty}$ in the Weierstrass canonical form (8) of the pencil $P(\lambda)$.

3) The finite zeros of $G(\lambda)$ are the $n_f$ finite eigenvalues (counting multiplicities) of the system matrix pencil $S(\lambda)$ and are the eigenvalues (counting multiplicities) of the matrix $\tilde{J}_f$ in the Kronecker canonical form (12) of the pencil $S(\lambda)$.

4) The infinite zeros of $G(\lambda)$ have multiplicities defined by the multiplicities of the infinite eigenvalues of the system matrix pencil $S(\lambda)$ minus 1 and are the dimensions minus 1 of the nilpotent Jordan blocks in the matrix $\tilde{J}_{\infty}$ in the Kronecker canonical form (12) of the pencil $S(\lambda)$.

5) The left minimal indices of $G(\lambda)$ are pairwise equal to the left Kronecker indices of $S(\lambda)$ and are the row dimensions $\epsilon_i$ of the blocks $L_{\epsilon_i}(\lambda)$ for $i = 1, \ldots, \nu_r$ in the Kronecker canonical form (12) of the pencil $S(\lambda)$. 

23
6) The right minimal indices of $G(\lambda)$ are pairwise equal to the right Kronecker indices of $S(\lambda)$ and are the column dimensions $\eta_i$ of the blocks $I^T_{\eta_i}(\lambda)$ for $i = 1, \ldots, \nu_l$ in the Kronecker canonical form (12) of the pencil $S(\lambda)$.

7) The normal rank of $G(\lambda)$ is $r = \text{rank} S(\lambda) - n = n_r + \tilde{n}_f + \tilde{n}_\infty + n_l - n$.

These facts allow to formulate simple conditions to characterize some pole-zero related properties, such as, properness, stability or minimum-phase of an irreducible descriptor system $(A - \lambda E, B, C, D)$ in terms of the eigenvalues of the pole and system matrix pencils. The descriptor system $(A - \lambda E, B, C, D)$ is proper if all infinite eigenvalues of the regular pencil $A - \lambda E$ are simple (i.e., the system has no infinite poles). It is straightforward to show using the Weierstrass canonical form of the pencil $A - \lambda E$, that any irreducible proper descriptor system can be always reduced to a minimal order descriptor system, with the descriptor matrix $E$ invertible, or to a standard state-space representation with $E = I$. The irreducible descriptor system $(A - \lambda E, B, C, D)$ is improper if the regular pencil $A - \lambda E$ has at least one infinite eigenvalue which is not simple (i.e, has at least one infinite pole). A polynomial descriptor system is one for which $A - \lambda E$ has only infinite eigenvalues of which at least one is not simple (i.e, has only infinite poles). The concept of stability involves naturally the properness of the system. The irreducible descriptor system $(A - \lambda E, B, C, D)$ is exponentially stable if it has only finite poles and all poles belong to the stable region $C_s$ (the pencil $A - \lambda E$ still can have simple infinite eigenvalues). The irreducible descriptor system $(A - \lambda E, B, C, D)$ is unstable if it has at least one finite pole outside of the stability domain or at least one infinite pole. The finite poles (or finite eigenvalues) inside the stability domain are called stable poles (stable eigenvalues), while the poles lying outside of the stability domain are called unstable poles. The irreducible descriptor system $(A - \lambda E, B, C, D)$ is minimum-phase if it has only finite zeros and all finite zeros belong to the stable region $C_s$.

To check the finite controllability condition (i) and finite observability condition (iii) of Theorem 2, it is sufficient to check that

$$\text{rank} \left[ \begin{array}{cc} A - \lambda_i E & B \\ \end{array} \right] = n$$

(37)

and, respectively,

$$\text{rank} \left[ \begin{array}{cc} A - \lambda_i E \\ C \end{array} \right] = n$$

(38)

for all distinct finite eigenvalues $\lambda_i$ of the regular pencil $A - \lambda E$. A finite eigenvalue $\lambda_i$ is controllable if (37) is fulfilled, and uncontrollable otherwise. Similarly, a finite eigenvalue $\lambda_i$ is observable if (38) is fulfilled, and unobservable otherwise. If the rank conditions (37) are fulfilled for all $\lambda_i \in \overline{C}_u$ we call the descriptor system $(A - \lambda E, B, C, D)$ (or equivalently the pair $(A - \lambda E, B)$) finite stabilizable. Finite stabilizability guarantees the existence of a state-feedback matrix $F \in \mathbb{R}^{m \times n}$ such that all finite eigenvalues of $A + BF - \lambda E$ lie in $C_s$. If the rank conditions (38) are fulfilled for all $\lambda_i \in \overline{C}_u$ we call the descriptor system $(A - \lambda E, B, C, D)$ (or equivalently the pair $(A - \lambda E, C)$) finite detectable. Finite detectability guarantees the existence of an output-injection matrix $K \in \mathbb{R}^{n \times p}$ such that all finite eigenvalues of $A + KC - \lambda E$ lie in $C_s$. The notion of strong stabilizability is related to the existence of a state-feedback matrix $F$ such that all finite eigenvalues of $A + BF - \lambda E$ lie in $C_s$ and all infinite eigenvalues of $A + BF - \lambda E$ are simple. The necessary and sufficient conditions for the existence of such an $F$
is the strong stabilizability of the pair \((A - \lambda E, B)\), that is: (1) the finite stabilizability of the pair \((A - \lambda E, B)\); and (2) \(\text{rank}[E \ AN_\infty \ B] = n\), where the columns of \(N_\infty\) form a basis of \(\mathcal{N}(E)\). Similarly, strong detectability is related to the existence of an output-injection matrix \(K\) such that all finite eigenvalues of \(A + KC - \lambda E\) lie in \(C_s\) and all infinite eigenvalues of \(A + KC - \lambda E\) are simple. The necessary and sufficient conditions for the existence of such a \(K\) is the strong detectability of the pair \((A - \lambda E, C)\), that is: (1) the finite detectability of the pair \((A - \lambda E, C)\); and (2) \(\text{rank}[E^T \ AN_\infty \ C^T] = n\), where the columns of \(L_\infty\) for a basis of \(\mathcal{N}(E^T)\).

### 2.7 Additive Decompositions

Let \(G(\lambda)\) be the transfer function matrix of a LTI system and let \(C_g\) be a domain of interest of the complex plane \(C\) for the poles of \(G(\lambda)\) (e.g., a certain stability domain). Define \(C_b := C \setminus C_g\), the complement of \(C_g\) in \(C\). Since \(C_g\) and \(C_b\) are disjoint, each pole of any element \(g_{ij}(\lambda)\) of \(G(\lambda)\) lies either in \(C_g\) or in \(C_b\). Therefore, using the well-known partial fraction decomposition results of rational functions, \(G(\lambda)\) can be additively decomposed as

\[
G(\lambda) = G_1(\lambda) + G_2(\lambda), \tag{39}
\]

where \(G_1(\lambda)\) has only poles in \(C_g\), while \(G_2(\lambda)\) has only poles in \(C_b\). For such a decomposition of \(G(\lambda)\) we always have that

\[
\delta \ G(\lambda)\ = \ \delta \ G_1(\lambda)\ + \ \delta \ G_2(\lambda). \tag{40}
\]

For example, if \(C_g = C \setminus \{\infty\}\) and \(C_b = \{\infty\}\), then (39) represents the additive decomposition of a possibly improper rational matrix as the sum of its proper and polynomial parts. This decomposition, in general, is not unique, because an arbitrary constant term can be always added to one term and subtracted from the other one. Another frequently used decomposition is the stable-unstable decomposition of proper rational matrices, when \(C_g = C_s\) (stability region) and \(C_b = C_u\) (instability region).

Let \(G(\lambda) = (A - \lambda E, B, C, D)\) be a descriptor system representation of \(G(\lambda)\). Using a general similarity transformation using two invertible matrices \(Q\) and \(Z\), we can determine an equivalent representation of \(G(\lambda)\) with partitioned system matrices of the form

\[
G(\lambda) = \begin{bmatrix}
QAZ - \lambda QEZ & QB \\
CZ & D
\end{bmatrix} = \begin{bmatrix}
A_1 - \lambda E_1 & 0 & B_1 \\
0 & A_2 - \lambda E_2 & B_2 \\
C_1 & C_2 & D
\end{bmatrix}, \tag{40}
\]

where \(\Lambda(A_1 - \lambda E_1) \subset C_1\) and \(\Lambda(A_2 - \lambda E_2) \subset C_2\). It follows that \(G(\lambda)\) can be additively decomposed as

\[
G(\lambda) = G_1(\lambda) + G_2(\lambda), \tag{41}
\]

where

\[
G_1(\lambda) = \begin{bmatrix}
A_1 - \lambda E_1 & B_1 \\
C_1 & D
\end{bmatrix}, \quad G_2(\lambda) = \begin{bmatrix}
A_2 - \lambda E_2 & B_2 \\
C_2 & 0
\end{bmatrix}, \tag{42}
\]

and \(G_1(\lambda)\) has only poles in \(C_g\), while \(G_2(\lambda)\) has only poles in \(C_b\). For the computation of additive spectral decompositions, a numerically reliable procedure is described in [21].
2.8 Coprime Factorizations

Consider a disjunct partition of the complex plane \( \mathbb{C} \) as

\[
\mathbb{C} = \mathbb{C}_g \cup \mathbb{C}_b, \quad \mathbb{C}_g \cap \mathbb{C}_b = \emptyset,
\]

(43)

where both \( \mathbb{C}_g \) and \( \mathbb{C}_b \) are symmetrically located with respect to the real axis, and such that \( \mathbb{C}_g \) has at least one point on the real axis. Any rational matrix \( G(\lambda) \) can be expressed in a left fractional form

\[
G(\lambda) = M^{-1}(\lambda)N(\lambda),
\]

(44)
or in a right fractional form

\[
G(\lambda) = N(\lambda)M^{-1}(\lambda),
\]

(45)

where both the denominator factor \( M(\lambda) \) and the numerator factor \( N(\lambda) \) have only poles in \( \mathbb{C}_g \). These fractional factorizations over a “good” domain of poles \( \mathbb{C}_g \) are important in various observer, fault detection filter, or controller synthesis methods, because they allow to achieve the placement of all poles of a TFM \( G(\lambda) \) in the domain \( \mathbb{C}_g \) simply, by a premultiplication or postmultiplication of \( G(\lambda) \) with a suitable \( M(\lambda) \).

Of special interest are the so-called coprime factorizations, where the factors satisfy additional conditions. A fractional representation of the form (44) is a left coprime factorization (LCF) of \( G(\lambda) \) with respect to \( \mathbb{C}_g \), if there exist \( U(\lambda) \) and \( V(\lambda) \) with poles only in \( \mathbb{C}_g \) which satisfy the Bezout identity

\[
M(\lambda)U(\lambda) + N(\lambda)V(\lambda) = I.
\]

A fractional representation of the form (45) is a right coprime factorization (RCF) of \( G(\lambda) \) with respect to \( \mathbb{C}_g \), if there exist \( U(\lambda) \) and \( V(\lambda) \) with poles only in \( \mathbb{C}_g \) which satisfy

\[
U(\lambda)M(\lambda) + V(\lambda)N(\lambda) = I.
\]

An important class of coprime factorizations is the class of coprime factorizations with minimum-degree denominators. Recall that \( \delta(G(\lambda)) \), the McMillan degree of \( G(\lambda) \), is defined as the number of poles of \( G(\lambda) \), both finite and infinite, counting all multiplicities. It follows that for any \( G(\lambda) \) we have \( \delta(G(\lambda)) = n_g + n_b \), where \( n_g \) and \( n_b \) are the number of poles of \( G(\lambda) \) in \( \mathbb{C}_g \) and \( \mathbb{C}_b \), respectively. The denominator factor \( M(\lambda) \) has the minimum-degree property if \( \delta(M(\lambda)) = n_b \).

A square TFM \( G(\lambda) \) is all-pass if \( G^{-1}(\lambda)G(\lambda) = I \). If \( G(\lambda) \) is a stable TFM and satisfies \( G^{-1}(\lambda)G(\lambda) = I \) then it is called an inner TFM, while if it satisfies \( G(\lambda)G^{-1}(\lambda) = I \) it is called a co-inner TFM. Note that an inner or co-inner TFM must not be square, but must have full column rank (injective) or full row rank (surjective), respectively. It is remarkable, that each proper TFM \( G(\lambda) \) without poles on the boundary of stability domain \( \partial \mathbb{C}_s \) has a stable LCF of the form (44) or a stable RCF of the form (45) with the denominator factor \( M(\lambda) \) inner. As before, the minimum McMillan degree of the inner denominator \( M(\lambda) \) is equal to the number of the unstable poles of \( G(\lambda) \).

A special class of coprime factorizations consists of the so-called normalized coprime factorizations. For an arbitrary \( p \times m \) rational matrix \( G(\lambda) \), the normalized left coprime factorization of \( G(\lambda) \) is

\[
G(\lambda) = M^{-1}(\lambda)N(\lambda),
\]

(46)
where $N(\lambda)$ and $M(\lambda)$ are stable TFMs and $[N(\lambda) \ M(\lambda)]$ is co-inner, that is
\begin{equation}
N(\lambda)N^\sim(\lambda) + M(\lambda)M^\sim(\lambda) = I_p. \tag{47}
\end{equation}

Similarly, the normalized right coprime factorization of $G(\lambda)$ is
\begin{equation}
G(\lambda) = N(\lambda)M^{-1}(\lambda), \tag{48}
\end{equation}
where $N(\lambda)$ and $M(\lambda)$ are stable TFMs and $[N(\lambda) \ M(\lambda)]$ is inner, that is
\begin{equation}
N^\sim(\lambda)N(\lambda) + M^\sim(\lambda)M(\lambda) = I_m. \tag{49}
\end{equation}

For the computation of coprime factorizations with minimum degree denominators, descriptor system representation based methods have been proposed in [49, 57], by using iterative pole dislocation techniques developed in the spirit of the approach described in [40]. Alternative, non-iterative approaches to compute minimum degree coprime factorizations with inner denominators have been proposed in [31, 29]. For the computation of normalized coprime factorizations computational methods have been proposed in [28] (see also [56]).

2.9 Inner-Outer and Spectral Factorizations

A proper and stable TFM $G(\lambda)$ is outer if it is minimum-phase and full row rank (surjective), and is co-outer if it is minimum-phase and full column rank (injective). A full row rank (full column rank) proper and stable TFM $G(\lambda)$ is quasi-outer (quasi-co-outer) if it has only zeros in $\mathbb{C}_s$ (i.e., in the stability domain and its boundary). Any stable TFM $G(\lambda)$ without zeros in $\partial\mathbb{C}_s$ has an inner–outer factorization
\begin{equation}
G(\lambda) = G_i(\lambda)G_o(\lambda), \tag{50}
\end{equation}
with $G_i(\lambda)$ inner and $G_o(\lambda)$ outer. Similarly, $G(\lambda)$ has a co-outer–co-inner factorization
\begin{equation}
G(\lambda) = G_{co}(\lambda)G_{ci}(\lambda), \tag{51}
\end{equation}
with $G_{co}(\lambda)$ co-outer and $G_{ci}(\lambda)$ co-inner. Any stable TFM $G(\lambda)$ has an inner–quasi-outer factorization of the form (50), where $G_i(\lambda)$ is inner and $G_o(\lambda)$ is quasi-outer, and also has a quasi-co-outer–co-inner factorization of the form (51), where $G_{ci}(\lambda)$ is co-inner and $G_{co}(\lambda)$ is quasi-co-outer.

In some applications, instead of the (compact) inner-outer factorization (50), an alternative (extended) factorization with square inner factor is desirable. The extended inner-outer factorization and extended inner–quasi-outer factorization have the form
\begin{equation}
G(\lambda) = \begin{bmatrix} G_i(\lambda) & G_i^\perp(\lambda) \end{bmatrix} \begin{bmatrix} G_o(\lambda) \\ 0 \end{bmatrix} = U(\lambda) \begin{bmatrix} G_o(\lambda) \\ 0 \end{bmatrix}, \tag{52}
\end{equation}
where $G_i^\perp(\lambda)$ is the inner orthogonal complement of $G_i(\lambda)$ such that $U(\lambda) := [G_i(\lambda) \ G_i^\perp(\lambda)]$ is square and inner. Similarly, the extended co-outer–co-inner factorization and extended quasi-co-outer–co-inner factorization have the form
\begin{equation}
G(\lambda) = \begin{bmatrix} G_{co}(\lambda) & 0 \end{bmatrix} \begin{bmatrix} G_{ci}(\lambda) \\ G_{ci}^\perp(\lambda) \end{bmatrix} = [G_{co}(\lambda) \ 0] V(\lambda), \tag{53}
\end{equation}
where \( G_{ci}^o(\lambda) \) is the co-inner orthogonal complement of \( G_i(\lambda) \) such that \( V(\lambda) := \begin{bmatrix} G_{ci}(\lambda) \\ G_{ci}^o(\lambda) \end{bmatrix} \) is square and co-inner (thus also inner).

The extended inner-outer factorization (52) of a TFM \( G(\lambda) \) can be interpreted as a generalization of the orthogonal QR-factorization of a real matrix. The inner factor \( \hat{U}(\lambda) = [G_i(\lambda) \ G_i^o(\lambda)] \) can be seen as the generalization of an orthogonal matrix. Its role in an inner-outer factorization is twofold: to compress the given \( G(\lambda) \) to a full row rank TFM \( G_o(\lambda) \) and to dislocate all zeros of \( G(\lambda) \) lying in \( \mathbb{C}_u \) into positions within \( \mathbb{C}_s \), which are symmetric (in a certain sense) with respect to \( \partial \mathbb{C}_s \). A factorization of \( G(\lambda) \) as in (52), where only the first aspect (i.e., the row compression) is addressed is called an extended QR-like factorization of the rational matrix \( G(\lambda) \) and produces a compressed full row rank \( G_o(\lambda) \), which contains all zeros of \( G(\lambda) \). The similar factorization in (53) with \( V(\lambda) \) inner, which only addresses the column compression aspect for \( G(\lambda) \), is called an extended RQ-like factorization of \( G(\lambda) \). Applications of these factorizations are in computing the pseudo-inverse of a rational matrix [32] (see also Example 6).

The outer factor \( G_o(\lambda) \) of a TFM \( G(\lambda) \) without zeros in \( \partial \mathbb{C}_s \) satisfies
\[ G^o(\lambda)G(\lambda) = G_o^o(\lambda)G_o(\lambda) \]
and therefore, it is a solution of the minimum-phase right spectral factorization problem. Similarly, the co-outer factor \( G_{co}(\lambda) \) of a TFM \( G(\lambda) \) without zeros in \( \partial \mathbb{C}_s \) satisfies
\[ G(\lambda)G^o(\lambda) = G_{co}(\lambda)G_{co}^o(\lambda) \]
and therefore, it is a solution of the minimum-phase left spectral factorization problem.

Combining the LCF with inner denominator and the inner-outer factorization, we have for an arbitrary \( G(\lambda) \), without poles and zeros on the boundary of the stability domain \( \partial \mathbb{C}_s \), that
\[ G(\lambda) = M_i^{-1}(\lambda)N(\lambda) = M_i^{-1}(\lambda)N_i(\lambda)N_o(\lambda), \]
where \( M_i(\lambda) \) and \( N_i(\lambda) \) are inner and \( N_o(\lambda) \) is outer. It follows that the outer factor \( N_o(\lambda) \) is the solution of the stable minimum-phase right spectral factorization problem
\[ G^o(\lambda)G(\lambda) = N_o^o(\lambda)N_o(\lambda). \]
Similarly, by combining the RCF with inner denominator and the co-outer–co-inner factorization we obtain
\[ G(\lambda) = N(\lambda)M_i^{-1}(\lambda) = N_{co}(\lambda)N_{ci}(\lambda)M_i^{-1}(\lambda), \]
with \( M_i(\lambda) \) inner, \( N_{co}(\lambda) \) co-inner and \( N_{co}(\lambda) \) co-outer. Then, \( N_{co}(\lambda) \) is the solution of the stable minimum-phase left spectral factorization problem
\[ G(\lambda)G^o(\lambda) = N_{co}(\lambda)N_{co}^o(\lambda). \]

If \( G(\lambda) \) has poles or zeros on the boundary of the stability domain \( \partial \mathbb{C}_s \), then we can still achieve the above factorizations by including all poles and zeros of \( G(\lambda) \) in \( \partial \mathbb{C}_s \) in the resulting spectral factors \( N_o(\lambda) \) or \( N_{co}(\lambda) \).

Assume that the stable proper TFM \( G(\lambda) \) has an irreducible descriptor realization
\[ G(\lambda) = \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix}. \]
For illustration of the inner-outer factorization approach, we only consider in detail the standard problem with $E$ invertible and when $G(\lambda)$ has full column rank, in which case explicit formulas for both the inner and outer factors can be derived using the results presented in [64]. Similar dual results can be obtained if $G(\lambda)$ has full row rank. The factorization procedures to compute inner–quasi-outer factorizations in the most general setting (i.e., for $G(\lambda)$ possibly improper with arbitrary normal rank and arbitrary zeros), are described in [32] for continuous-time systems and in [29] for discrete-time systems. These methods, essentially reduce the factorization problems to the solution of standard problems, for which straightforward extensions of the results for standard systems in [64] apply.

We have the following result for a continuous-time system:

**Theorem 5.** Let $G(s)$ be a proper and stable, full column rank TFM without zeros in $\partial\mathcal{C}_s$, with the descriptor system realization $G(s) = (A - sE, B, C, D)$ and $E$ invertible. Then, $G(s)$ has an inner–outer factorization $G(s) = G_i(s)G_o(s)$, with the particular realizations of the factors

$$G_i(s) = \begin{bmatrix} A + BF - sE & BH^{-1} \\ C + DF & DH^{-1} \end{bmatrix}, \quad G_o(s) = \begin{bmatrix} A - sE & B \\ -HF & H \end{bmatrix},$$

where $H$ is an invertible matrix satisfying $D^T D = H^T H$, $F$ is given by

$$F = -(D^T D)^{-1}(B^T X_s E + D^T C),$$

with $X_s \geq 0$ being the stabilizing solution of the generalized continuous-time algebraic Riccati equation (GCARE)

$$A^T X E + E^T X A - (E^T X B + C^T D)(D^T D)^{-1}(B^T X E + D^T C) + C^T C = 0.$$

The similar result for a discrete-time system is:

**Theorem 6.** Let $G(z)$ be a proper and stable, full column rank TFM without zeros in $\partial\mathcal{C}_s$, with the descriptor system realization $G(z) = (A - zE, B, C, D)$ and $E$ invertible. Then, $G(z)$ has an inner–outer factorization $G(z) = G_i(z)G_o(z)$, with the particular realizations of the factors

$$G_i(z) = \begin{bmatrix} A + BF - zE & BH^{-1} \\ C + DF & DH^{-1} \end{bmatrix}, \quad G_o(z) = \begin{bmatrix} A - zE & B \\ -HF & H \end{bmatrix},$$

where $H$ is an invertible matrix satisfying $D^T D + B^T X_s B = H^T H$, $F$ is given by

$$F = -(H^T H)^{-1}(B^T X_s A + D^T C),$$

with $X_s \geq 0$ being the stabilizing solution of the generalized discrete-time algebraic Riccati equation (GDARE)

$$A^T X A - E^T X E - (A^T X B + C^T D)(D^T D + B^T X B)^{-1}(B^T X A + D^T C) + C^T C = 0.$$

When computing the extended inner-outer factorization (52), the inner orthogonal complement $G_i^\perp(\lambda)$ of $G_i(\lambda)$ is also needed. In the continuous-time case, a descriptor realization of $G_i^\perp(\lambda)$ is given by

$$G_i^\perp(s) = \begin{bmatrix} A + BF - sE & -X_i^\perp E^T C^T D^\perp \\ C + DF & D^\perp \end{bmatrix},$$
where $D^\perp$ is an orthogonal complement chosen such that $\begin{bmatrix} DH^{-1} & D^\perp \end{bmatrix}$ is square and orthogonal. In the discrete-time case we have

$$G^\perp_I(z) = \begin{bmatrix} A + BF - zE & Y \\ C + DF & W \end{bmatrix},$$

where $Y$ and $W$ satisfy

$$A^T X_s Y + C^T W = 0,$$
$$B^T X_s Y + D^T W = 0,$$
$$W^T W + Y^T X_s Y = I.$$

The similar results for the co-outer–co-inner factorization (or the extended co-outer–co-inner factorization) can be easily obtained by considering the inner–outer factorization (or its extended version) for the dual system with the TFM $G^T(\lambda)$ having the descriptor realization $(A^T - \lambda E^T, C^T, B^T, D^T)$.

A special factorization problem encountered when solving the approximate model-matching problem by reducing it to a least distance problem (see equation (82) in Section 2.15) is the following special left spectral factorization problem: for a given TFM $G(\lambda)$ without poles in $\partial C_s$ and a given bound $\gamma > \|G(\lambda)\|_\infty$, compute a stable and minimum-phase TFM $G_o(\lambda)$ such that

$$\gamma^2 I - G(\lambda)G^\perp(\lambda) = G_o(\lambda)G^\perp_o(\lambda).$$

This computation can be addressed in two steps. In the first step, we compute a RCF $G(\lambda) = N(\lambda)M^{-1}(\lambda)$, with the denominator factor $M(\lambda)$ inner. It follows that

$$\gamma^2 I - G(\lambda)G^\perp(\lambda) = \gamma^2 I - N(\lambda)N^\perp(\lambda),$$

where $N(\lambda)$ is proper and has only poles in $C_s$. In the second step, we determine the stable and minimum-phase $G_o(\lambda)$ which satisfies

$$\gamma^2 I - N(\lambda)N^\perp(\lambda) = G_o(\lambda)G^\perp_o(\lambda).$$

(55)

The first step has been already discussed in Section 2.8, and therefore we assume that for an irreducible descriptor realization $(A - \lambda E, B, C, D)$ of $G(\lambda)$, we determined a stable $N(\lambda)$ with a descriptor realization $(\tilde{A} - \lambda \tilde{E}, \tilde{B}, \tilde{C}, \tilde{D})$.

In the continuous-time case, we can compute the spectral factor $G_o(s)$ by using the following result, which extends to proper descriptor systems the formulas developed in [64, Corollary 13.22].

**Lemma 6.** Let $N(s)$ be a stable TFM and let $(\tilde{A} - s \tilde{E}, \tilde{B}, \tilde{C}, \tilde{D})$ be its descriptor system realization. For $\gamma > \|N(s)\|_\infty$, a realization of a stable and minimum-phase spectral factor $G_o(s)$, satisfying (55) for $\lambda = s$, is given by

$$G_o(s) = \begin{bmatrix} \tilde{A} - s \tilde{E} & -K_s R^{1/2} \\ \tilde{C} & R^{1/2} \end{bmatrix},$$

where

$$R = \gamma^2 I - \tilde{D} \tilde{D}^T,$$
$$K_s = (\tilde{E} Y_s \tilde{C}^T + \tilde{B} \tilde{D}^T) R^{-1},$$

and $Y_s$ is the stabilizing solution of the GCARE

$$\tilde{A} Y_s \tilde{E}^T + \tilde{E} Y_s \tilde{A}^T + (\tilde{E} Y_s \tilde{C}^T + \tilde{B} \tilde{D}^T) R^{-1}(\tilde{C} Y_s \tilde{E}^T + \tilde{D} \tilde{B}^T) + \tilde{B} \tilde{B}^T = 0.$$
We have the following analogous result in the discrete-time case, which extends to proper descriptor system the formulas developed in [64, Theorem 21.26] for the dual special right spectral factorization problem (see below).

**Lemma 7.** Let \( N(z) \) be a stable TFM and let \((\tilde{A} - z\tilde{E}, \tilde{B}, \tilde{C}, \tilde{D})\) be its descriptor realization. For \( \gamma > \|N(z)\|_\infty \), a realization of a stable and minimum-phase spectral factor \( G_o(z) \), satisfying (55) for \( \lambda = z \), is given by

\[
G_o(z) = \begin{bmatrix}
-\lambda \tilde{E}
\tilde{C}
-\tilde{K}_s R^{1/2}
\end{bmatrix},
\]

where

\[
R_D = \gamma^2 I - \tilde{D}\tilde{D}^T,
\]

\[
R = R_D - \tilde{C}Y_s \tilde{C}^T,
\]

\[
K_s = (\tilde{A}Y_s \tilde{C}^T + \tilde{B}\tilde{D}^T)R^{-1},
\]

and \( Y_s \) is the stabilizing solution of the GDARE

\[
\tilde{A}Y\tilde{A}^T - \tilde{E}Y\tilde{E}^T - (\tilde{A}Y\tilde{C}^T + \tilde{B}\tilde{D}^T)(-R_D + \tilde{C}Y\tilde{C}^T)^{-1}(\tilde{C}Y\tilde{A}^T + \tilde{D}\tilde{B}^T) + \tilde{B}\tilde{B}^T = 0.
\]

Similar formulas to those provided by Lemma 6 and Lemma 7 can be derived for the solution of the dual special right spectral factorization problem

\[
\gamma^2 I - N^\sim(\lambda)N(\lambda) = G_o^\sim(\lambda)G_o(\lambda).
\]

### 2.10 Linear Rational Matrix Equations

For \( G(\lambda) \in \mathbb{R}(\lambda)^{p\times m} \) and \( F(\lambda) \in \mathbb{R}(\lambda)^{p\times q} \) consider the solution of the linear rational matrix equation

\[
G(\lambda)X(\lambda) = F(\lambda),
\]

where \( X(\lambda) \in \mathbb{R}(\lambda)^{m\times q} \) is the solution we seek. The existence of a solution is guaranteed if the compatibility condition for the linear system (56) is fulfilled.

**Lemma 8.** The rational equation (56) has a solution if and only if

\[
\text{rank } G(\lambda) = \text{rank } [G(\lambda) F(\lambda)].
\]

Let \( r \) be the rank of \( G(\lambda) \). In the most general case, the solution of (56) (if exists) is not unique and can be expressed as

\[
X(\lambda) = X_0(\lambda) + N_r(\lambda)Y(\lambda),
\]

where \( X_0(\lambda) \) is a particular solution of (56), \( N_r(\lambda) \in \mathbb{R}(\lambda)^{m\times(m-r)} \) is a rational matrix representing a basis of the right nullspace \( \mathcal{N}_R(G(\lambda)) \) (is empty if \( r = m \)), while \( Y(\lambda) \in \mathbb{R}(\lambda)^{(m-r)\times q} \) is an arbitrary rational matrix.

An important aspect in control related applications is to establish conditions which ensure the existence of a solution \( X(\lambda) \) which has only poles in a “good” domain \( \mathcal{C}_g \), or equivalently, \( X(\lambda) \) has no poles in the “bad” domain \( \mathcal{C}_b := \mathcal{C} \setminus \mathcal{C}_g \). Such a condition can be obtained in terms of the pole-zero structures of the rational matrices \( G(\lambda) \) and \( [G(\lambda) F(\lambda)] \) at a particular value \( \lambda_z \) of the frequency parameter \( \lambda \) (see, for example, [22]).
Lemma 9. The rational equation (56) has a solution without poles in $\mathbb{C}_b$ if and only if (57) is fulfilled and the rational matrices $G(\lambda)$ and $[G(\lambda) \ F(\lambda)]$ have the same pole-zero structure for all $\lambda_z \in \mathbb{C}_b$.

The characterization provided by Lemma 9 is relevant when solving synthesis problems of fault detection filters and controllers using an exact model-matching approach, where the physical realizability requires the properness and stability of the solutions (i.e., $\mathbb{C}_g = \mathbb{C}_s$). For example, if $G(\lambda)$ has unstable zeros in $\lambda_z$, then $F(\lambda)$ must be chosen to have the same (or richer) zero structure in $\lambda_z$, in order to ensure the cancellation of these zeros (appearing now as unstable poles of any particular solution $X_0(\lambda)$). The fixed poles in $\mathbb{C}_b$ of any particular solution $X_0(\lambda)$ correspond to those zeros of $G(\lambda)$ for which the above condition is not fulfilled, and thus no complete cancellations take place.

In the case when $\text{rank} \ G(\lambda) < m$, an important aspect is the exploitation of the non-uniqueness of the solution in (56) by determining a solution with the least possible McMillan degree. This problem is known in the literature as the minimum design problem (MDP) and primarily targets the reduction of the complexity of real-time burden when implementing filters or controllers. Of particular importance are proper and stable solutions which are suitable for a physical (causal) realization. If the minimal degree solution is not proper and stable, then it is of interest to find a proper and stable solution with the least McMillan degree. Surprisingly, this problem does not have a satisfactory procedural solution, most of proposed approaches involves parametric searches using suitably parameterized solutions of given candidate degrees.

An equivalent solvability condition to that of Lemma 8 can be derived in terms of descriptor system representations of $G(\lambda)$ and $F(\lambda)$, which we assume to be of the form

$$ G(\lambda) = \begin{bmatrix} A - \lambda E & B_G \\ C & D_G \end{bmatrix}, \quad F(\lambda) = \begin{bmatrix} A - \lambda E & B_F \\ C & D_F \end{bmatrix}. $$

Such representations, which share the pair $(A - \lambda E, C)$, can be easily obtained by determining a descriptor realization of the compound rational matrix $[G(\lambda) \ F(\lambda)]$. It is easy to observe that any solution of (56) is also part of the solution of the linear polynomial equation

$$ \begin{bmatrix} A - \lambda E & B_G \\ C & D_G \end{bmatrix} Y(\lambda) = \begin{bmatrix} B_F \\ D_F \end{bmatrix}, $$

where $Y(\lambda) = \begin{bmatrix} W(\lambda) \\ X(\lambda) \end{bmatrix}$. Therefore, alternatively to solving (56), we can solve instead (60) for $Y(\lambda)$ and compute $X(\lambda)$ as

$$ X(\lambda) = \begin{bmatrix} 0 & I_m \end{bmatrix} Y(\lambda). $$

Define the system matrix pencils corresponding to $G(\lambda)$ and the compound $[G(\lambda) \ F(\lambda)]$ as

$$ S_G(\lambda) := \begin{bmatrix} A - \lambda E & B_G \\ C & D_G \end{bmatrix}, \quad S_{G,F}(\lambda) := \begin{bmatrix} A - \lambda E & B_G & B_F \\ C & D_G & D_F \end{bmatrix}. $$

We have the following result similar to Lemma 8.

Lemma 10. The rational equation (56) with $G(\lambda)$ and $F(\lambda)$ having the descriptor realizations in (59) has a solution if and only if

$$ \text{rank} \ S_G(\lambda) = \text{rank} \ S_{G,F}(\lambda). $$
Let $\mathbb{C}_b$ be the “bad” domain of the complex plane, where the solution $X(\lambda)$ must not have poles. We have the following result similar to Lemma 9.

**Lemma 11.** The rational equation (56) with $G(\lambda)$ and $F(\lambda)$ having the descriptor realizations in (59) has a solution without poles in $\mathbb{C}_b$ if and only if the matrix pencils $S_G(\lambda)$ and $S_{G,F}(\lambda)$ defined in (62) fulfill (63) and have the same zero structure for all zeros of $G(\lambda)$ in $\mathbb{C}_b$.

The general solution of (56) can be expressed as in (58) where $X_0(\lambda)$ is any particular solution of (56), $N_\tau(\lambda)$ is a rational matrix whose columns form a basis for the right nullspace of $G(\lambda)$, and $Y(\lambda)$ is an arbitrary rational matrix with compatible dimensions. A possible approach to compute a solution $X(\lambda)$ of least McMillan degree is to determine a suitable $Y(\lambda)$ to achieve this goal. A computational procedure to determine a least McMillan degree solution $X(\lambda)$ is described in [58, Section 10.3.7] and relies on the Kronecker-like form of the associated system matrix pencil $S_G(\lambda)$ in (62), which is used to determine the particular solution $X_0(\lambda)$, the nullspace basis $N_\tau(\lambda)$, and, combined with the generalized minimum cover algorithm of [53], to obtain the least McMillan degree solution.

### 2.11 Dynamic Cover-Based Order Reduction

Let $X_1(\lambda)$ and $X_2(\lambda)$ be rational transfer function matrices. For $X_1(\lambda)$ and $X_2(\lambda)$ with the same row dimension, the **right minimal cover problem** is to find $X(\lambda)$ and $Y(\lambda)$ such that

$$X(\lambda) = X_1(\lambda) + X_2(\lambda)Y(\lambda),$$

and the McMillan degree of $[X(\lambda) Y(\lambda)]$ is minimal [1]. Similarly, for $X_1(\lambda)$ and $X_2(\lambda)$ with the same column dimension, the **left minimal cover problem** is to find $X(\lambda)$ and $Y(\lambda)$ such that

$$X(\lambda) = X_1(\lambda) + Y(\lambda)X_2(\lambda),$$

and the McMillan degree of $[X(\lambda) Y(\lambda)]$ is minimal [1]. Any method to solve a right minimal cover problem can be used to solve the left minimal proper problem, by applying it to the transposed matrices $X_1^T(\lambda)$ and $X_2^T(\lambda)$ to determine the transposed solution $X^T(\lambda)$ and $Y^T(\lambda)$. Therefore, in what follows we will only discuss the solution of right minimal cover problems.

By imposing additional conditions on $X_1(\lambda)$ and $X_2(\lambda)$, as well as on the class of desired solutions $X(\lambda)$ and $Y(\lambda)$, particular cover problems arise, which may require specific methods for their solution. In what follows, we only discuss the proper case, when $X_1(\lambda), X_2(\lambda), X(\lambda)$ and $Y(\lambda)$ are all proper. Various applications involving the solution of minimal cover problems are discussed in [1], as for example—observer and controller synthesis using model-matching based approaches. Other problems, as for example, imposing $X(\lambda) = 0$, are equivalent to solve linear rational equations for the least McMillan degree solutions (see Section 2.10).

Assume $X_1(\lambda)$ and $X_2(\lambda)$ have the descriptor realizations

$$[ X_1(\lambda) \mid X_2(\lambda) ] = \left[ \begin{array}{c|cc} A - \lambda E & B_1 & B_2 \\ \hline C & D_1 & D_2 \end{array} \right],$$

with the descriptor pair $(A - \lambda E, [B_1 B_2])$ controllable and $E$ invertible. The methods we describe, use for $Y(\lambda)$ the descriptor realization

$$Y(\lambda) = \left[ \begin{array}{c|cc} A + B_2 F - \lambda E & B_1 + B_2 G \\ \hline F & G \end{array} \right],$$

33
where $F$ and $G$ are state-feedback gain and feedforward gains, respectively, to be determined. It is straightforward to check that $X(\lambda) = X_1(\lambda) + X_2(\lambda) Y(\lambda)$ has the descriptor system realization

$$X(\lambda) := \begin{bmatrix} A + B_2F - \lambda E & B_1 + B_2G \\ C + D_2F & D_1 + D_2G \end{bmatrix}. \tag{68}$$

If the gains $F$ and $G$ are determined such that the pair $(A + B_2F - \lambda E, B_1 + B_2G)$ is maximally uncontrollable, then the resulting realizations of $X(\lambda)$ and $Y(\lambda)$ contain a maximum number of uncontrollable eigenvalues which can be eliminated using minimal realization techniques. In some applications, the use of a strictly proper $Y(\lambda)$ (i.e., $G = 0$) is sufficient to achieve the maximal order reduction, therefore, this case will be treated explicitly in what follows.

The problem to determine the matrices $F$ and $G$, which make the descriptor system pair $(A + B_2F - \lambda E, B_1 + B_2G)$ maximally uncontrollable, is essentially equivalent [27] to compute a subspace $\mathcal{V}$ having the least possible dimension and satisfying

$$(\overline{A} + \overline{B}_2F)\mathcal{V} \subset \mathcal{V}, \quad \text{span}(\overline{B}_1 + \overline{B}_2G) \subset \mathcal{V}, \tag{69}$$

where $\overline{A} := E^{-1}A$, $\overline{B}_1 := E^{-1}B_1$, and $\overline{B}_2 := E^{-1}B_2$. If we denote $\overline{B}_1 = \text{span}(\overline{B}_1)$ and $\overline{B}_2 = \text{span}(\overline{B}_2)$, then the above condition for $G = 0$ can be equivalently rewritten as the condition defining a Type 1 minimum dynamic cover [11, 23]:

$$\overline{A}\mathcal{V} \subset \mathcal{V} + \overline{B}_2, \quad \overline{B}_1 \subset \mathcal{V}. \tag{70}$$

If we allow for $G \neq 0$, then (69) can be equivalently rewritten as the condition defining a Type 2 minimum dynamic cover [11, 23]:

$$\overline{A}\mathcal{V} \subset \mathcal{V} + \overline{B}_2, \quad \overline{B}_1 \subset \mathcal{V} + \overline{B}_2. \tag{71}$$

The underlying computational problems to solve these minimum dynamic cover problems is the following: given the controllable descriptor system pair $(A - \lambda E, B)$ with $A, E \in \mathbb{R}^{n \times n}$ and $E$ invertible, $B \in \mathbb{R}^{n \times m}$, and $B$ partitioned as $B = [B_1 \ B_2]$ with $B_1 \in \mathbb{R}^{n \times m_1}$, $B_2 \in \mathbb{R}^{n \times m_2}$, determine the matrices $F$ and $G$ (either with $G = 0$ or $G \neq 0$) such that the descriptor system pair $(A + B_2F - \lambda E, B_1 + B_2G)$ has the maximal number of uncontrollable eigenvalues. Computational procedures to solve these problems have been proposed in [53] for descriptor systems and in [51] for standard systems (see also Procedures GRMCOVER1 and GRMCOVER2, described in [58]).

2.12 Hankel Norm

The Hankel norm of a stable TFM $G(\lambda)$ is a measure of the influence of past inputs on future outputs and is denoted by $\|G(\lambda)\|_H$. The precise mathematical definition of the Hankel norm involves some advanced concepts from functional analysis (i.e., it is the norm of the Hankel operator $\Gamma_G$ associated to $G(\lambda)$). For further details, see [64].

For the evaluation of the Hankel norm of a stable TFM $G(\lambda)$, the state-space representation allows to use explicit formulas.
Lemma 12. Let $G(s)$ be a proper and stable TFM of a continuous-time system and let $(A - sE, B, C, D)$ be an irreducible descriptor realization with $E$ invertible. Then, the Hankel norm of $G(s)$ can be evaluated as

$$\|G(s)\|_H = \sigma(\text{RES}),$$

where $P = SS^T$ is the positive semi-definite controllability Gramian and $Q = R^T R$ is the positive semi-definite observability Gramian, which satisfy the following generalized Lyapunov equations

\begin{align*}
APE^T + EPA^T + BB^T &= 0, \\
A^T Q E + E^T Q A + C^T C &= 0.
\end{align*}

\[(72)\]

Lemma 13. Let $G(z)$ be a proper and stable TFM of a continuous-time system and let $(A - zE, B, C, D)$ be an irreducible descriptor realization with $E$ invertible. Then, the Hankel norm of $G(z)$ can be evaluated as

$$\|G(z)\|_H = \sigma(\text{RES}),$$

where $P = SS^T$ is the positive semi-definite controllability Gramian and $Q = R^T R$ is the positive semi-definite observability Gramian, which satisfy the following generalized Stein equations

\begin{align*}
APA^T - EPE^T + BB^T &= 0, \\
A^T QA - E^T QE + C^T C &= 0.
\end{align*}

\[(73)\]

For the solution of the generalized Lyapunov equations (72) and the generalized Stein equations (73), computational procedures have been proposed in [33], which allow to directly determine the Cholesky factors $S$ and $R$ of the Gramians. These methods extend the algorithm of [18] proposed for standard systems with $E = I$.

2.13 Balancing-Related Order Reduction

Consider a stable TFM $G(\lambda)$ with a state-space descriptor system realization $(A - \lambda E, B, C, D)$ with $E$ invertible and $\Lambda(A, E) \subset \mathbb{C}_s$. The controllability and observability properties of the state-space realization can be characterized by the controllability and observability gramians. The controllability gramian $P$ and observability gramian $Q$ satisfy appropriate generalized Lyapunov and Stein equations. In the continuous-time case, $P$ and $Q$ satisfy the generalized Lyapunov equations (72), while in the discrete-time case, $P$ and $Q$ satisfy the generalized Stein equations (73). For a stable system, both gramians $P$ and $Q$ are positive semi-definite matrices and fully characterize the controllability and observability properties. The controllability gramian $P > 0$ if and only if the pair $(A - \lambda E, B)$ is controllable, and the observability gramian $Q > 0$ if and only if the pair $(A - \lambda E, C)$ is observable. Besides these qualitative characterizations, the eigenvalues of $P$ and $Q$ provides quantitative measures of these properties. Thus, the degree of controllability can be defined as the least eigenvalue of $P$ and represents a measure of the nearness of the state-space realization to an uncontrollable one. Analogously, the degree of observability, defined as the least eigenvalue of $Q$, represents a measure of the nearness of the state-space realization to an unobservable one. These measures are not invariant to coordinate transformations and, therefore, it is of interest to have state-space representations for which the controllability and observability properties are balanced. In fact, for a controllable and observable system, it is possible to find a particular state-space representation, called a balanced realization, for which the two gramians are equal and even diagonal.
The eigenvalues of the gramians of a balanced system are called the \textit{Hankel singular values}. The largest singular value represents the \textit{Hankel norm} $\|G(\lambda)\|_H$ of the corresponding TFM $G(\lambda)$, while the smallest one can be interpreted as a measure of the nearness of the system to a non-minimal one. Important applications of balanced realizations are to ensure minimum sensitivity to roundoff errors of real-time filter models or to perform model order reduction, by reducing large order models to lower order approximations. The order reduction can be performed by simply truncating the system state to a part corresponding to the “large” singular values, which significantly exceed the rest of “small” singular values.

A procedure to compute minimal balanced realizations of stable descriptor systems is \textbf{Procedure GBALMR} described in [58, Section 10.4.4]. This procedure is instrumental in solving the Nehari approximation problem for descriptor systems (see Section 2.14). The reduction of a linear state-space model to a balanced minimal realization may involve the usage of ill-conditioned coordinate transformations (or projections) for systems which are nearly non-minimal or nearly unstable. This is why, for the computation of minimal realizations or of lower order approximations, the so-called \textit{balancing-free} approaches, as proposed in [46] for standard systems and in [35] for descriptor systems, are generally more accurate.

2.14 Solution of the Optimal Nehari Problems

In this section we consider the solution of the following optimal Nehari problem: Given an anti-stable $G(\lambda)$ (i.e., such that $G^\sim(\lambda)$ is stable), find a stable $X(\lambda)$ which is the closest to $G(\lambda)$ and satisfies

$$\|G(\lambda) - X(\lambda)\|_\infty = \|G^\sim(\lambda)\|_H .$$

This computation is encountered in the solution of the least-distance problem formulated in Section 2.15. As shown in [15], to solve the optimal Nehari approximation problem (74), we can solve instead for $X^\sim(\lambda)$ the optimal zeroth-order Hankel-norm approximation problem

$$\|G^\sim(\lambda) - X^\sim(\lambda)\|_\infty = \|G^\sim(\lambda)\|_H .$$

A computational method for continuous-time systems, can be devised using the method proposed in [15], which relies on computing first a balanced minimal order state-space realization of $G^\sim(\lambda)$. The corresponding procedure for discrete-time systems is much more involved (see [17]) and therefore, a preferred alternative, suggested in [15], is to use the procedure for continuous-time systems in conjunction with bilinear transformation techniques. This approach underlies \textbf{Procedure GNEHARI} described in [58, Section 10.4.5], which can be employed even for improper discrete-time systems.

If $\rho$ is a given value satisfying $\rho > \|G^\sim(\lambda)\|_H$, then the suboptimal Nehari approximation problem is to determine a stable $X(\lambda)$ such that

$$\|G(\lambda) - X(\lambda)\|_\infty < \rho .$$

To solve this problem, a suboptimal Hankel-norm approximation $X^\sim(\lambda)$ of $G^\sim(\lambda)$ can be computed using the method of [15].

2.15 Solution of Least-Distance Problems

A possible solution method of the $\mathcal{H}_\infty$-model-matching problem [12] is to reduce this problem to a \textit{least-distance problem} (LDP), which can be solved using Nehari-approximation techniques.
The optimal LDP is the problem of computing a stable solution \( X(\lambda) \) such that
\[
\| [ G_1(\lambda) - X(\lambda) \ G_2(\lambda) ] \|_\infty = \min ,
\] (77)
where \( G_1(\lambda) \) and \( G_2(\lambda) \) are TFMs without poles in \( \partial \mathcal{C}_s \). Therefore, the use of the \( L_\infty \)-norm in (77) is assumed. The suboptimal LDP is to find a stable \( X(\lambda) \) such that
\[
\| [ G_1(\lambda) - X(\lambda) \ G_2(\lambda) ] \|_\infty < \gamma ,
\] (78)
where \( \gamma > \| G_2(\lambda) \|_\infty \). If \( G_2(\lambda) \) is present, we have a 2-block LDP, while if \( G_2(\lambda) \) is not present we have an 1-block LDP. In what follows, we discuss shortly the solution approaches for the optimal 1- and 2-block problems.

**Solution of the 1-block \( H_\infty \)-LDP.** In the case of the \( H_\infty \)-norm, the stable optimal solution \( X(\lambda) \) of the 1-block problem can be computed by solving an optimal Nehari problem. Let \( L_s(\lambda) \) be the stable part and let \( L_u(\lambda) \) be the unstable part in the additive decomposition
\[
G_1(\lambda) = L_s(\lambda) + L_u(\lambda) .
\] (79)
Then, for the optimal solution we have successively
\[
\| G_1(\lambda) - X(\lambda) \|_\infty = \| L_u(\lambda) - X_s(\lambda) \|_\infty = \| L_u(\lambda) \|_H ,
\]
where \( X_s(\lambda) \) is the stable optimal Nehari solution and
\[
X(\lambda) = X_s(\lambda) + L_s(\lambda) .
\]

**Solution of the 2-block \( H_\infty \)-LDP.** A stable optimal solution \( X(\lambda) \) of the 2-block LDP can be approximately determined as the solution of the suboptimal 2-block LDP
\[
\| [ G_1(\lambda) - X(\lambda) \ G_2(\lambda) ] \|_\infty < \gamma ,
\] (80)
where \( \gamma_{\text{opt}} < \gamma \leq \gamma_{\text{opt}} + \varepsilon \), with \( \varepsilon \) an arbitrary user specified (accuracy) tolerance for the least achievable value \( \gamma_{\text{opt}} \) of \( \gamma \). The standard solution approach is a bisection-based \( \gamma \)-iteration method, where the solution of the 2-block problem is approximated by successively computed \( \gamma \)-suboptimal solutions of appropriately defined 1-block problems [12].

Let \( \gamma_l \) and \( \gamma_u \) be lower and upper bounds for \( \gamma_{\text{opt}} \), respectively. Such bounds can be computed, for example, as
\[
\gamma_l = \| G_2(\lambda) \|_\infty , \quad \gamma_u = \| [ G_1(\lambda) \ G_2(\lambda) ] \|_\infty .
\] (81)
For a given \( \gamma > \gamma_l \), we solve first the stable minimum-phase left spectral factorization problem
\[
\gamma^2 I - G_2(\lambda)G_2^\sim(\lambda) = V(\lambda)V^\sim(\lambda),
\] (82)
where the spectral factor \( V(\lambda) \) is biproper, stable and minimum-phase. Further, we compute the additive decomposition
\[
V^{-1}(\lambda)G_1(\lambda) = L_s(\lambda) + L_u(\lambda) .
\] (83)
where \( L_s(\lambda) \) is the stable part and \( L_u(\lambda) \) is the unstable part. If \( \gamma > \gamma_{\text{opt}} \), the suboptimal 2-block problem (80) is equivalent to the suboptimal 1-block problem
\[
\| V^{-1}(\lambda)(G_1(\lambda) - X(\lambda)) \|_\infty \leq 1
\] (84)
and $\gamma_H := \|L_u(\lambda)\|_H < 1$. In this case we readjust the upper bound to $\gamma_u = \gamma$. If $\gamma \leq \gamma_{opt}$, then $\gamma_H \geq 1$ and we readjust the lower bound to $\gamma_l = \gamma$. For the bisection value $\gamma = (\gamma_l + \gamma_u)/2$ we redo the factorization (82) and decomposition (83). This process is repeated until $\gamma_u - \gamma_l \leq \varepsilon$.

At the end of iterations, we have either $\gamma_{opt} < \gamma \leq \gamma_u$ if $\gamma_H < 1$ or $\gamma_l < \gamma \leq \gamma_{opt}$ if $\gamma_H \geq 1$, in which case we set $\gamma = \gamma_u$. We compute the stable solution of (84) as

$$X(\lambda) = V(\lambda)(L_s(\lambda) + X_s(\lambda)),$$

(85)

where, for any $\gamma_1$ satisfying $1 \geq \gamma_1 > \gamma_H$, $X_s(\lambda)$ is the stable solution of the optimal Nehari problem

$$\|L_u(\lambda) - X_s(\lambda)\|_\infty = \|L_u(\lambda)\|_H.$$  

(86)
3 Description of DSTOOLS

This user’s guide is intended to provide basic information on the DSTOOLS collection for the operation on and manipulation of rational transfer function matrices via their descriptor system realizations as described in Section 2. The notations and terminology used throughout this guide have been introduced and extensively discussed in Chapter 9 of the accompanying book [58], while Chapter 10 also represents the main reference for the implemented computational methods in DSTOOLS. Information on the requirements for installing DSTOOLS are given in Appendix A.

In this section, we present first a short overview of the existing functions of DSTOOLS and then, we give in-depth information on the command syntax of the functions of the DSTOOLS collection. To execute the examples presented in this guide, simply copy and paste the presented code sequences into the MATLAB command window.

3.1 Quick Reference Tables

The current release of DSTOOLS is version V0.6, dated July 31, 2017. The corresponding Contents.m file is listed in Appendix B. This section contains quick reference tables for the functions of the DSTOOLS collection. The main M- and MEX-files available in the current version of DSTOOLS are listed below by category, with short descriptions.

| Demonstration       |
|---------------------|
| DSToolsdemo         | Demonstration of Descriptor System Tools |

| System analysis     |
|---------------------|
| gpole               | Poles of a LTI descriptor system. |
| gzero               | Zeros of a LTI descriptor system. |
| nrank               | Normal rank of a transfer function matrix of a LTI system. |
| ghanorm             | Hankel norm of a proper and stable LTI descriptor system. |

| Order reduction     |
|---------------------|
| gir                 | Reduced order realizations of LTI descriptor systems. |
| gminreal            | Minimal realization of a LTI descriptor system. |
| gbalmr              | Balancing-based model reduction of a stable LTI descriptor system. |
| gss2ss              | Conversions to SVD-like coordinate forms without non-dynamic modes. |

| Operations on transfer function matrices |
|------------------------------------------|
| grnull                    | Right nullspace basis of a transfer function matrix. |
| glnull                    | Left nullspace basis of a transfer function matrix. |
| grange                    | Range space basis of a transfer function matrix. |
| grsol                     | Solution of the linear rational matrix equation $G(\lambda)X(\lambda) = F(\lambda)$. |
| glsol                     | Solution of the linear rational matrix equation $X(\lambda)G(\lambda) = F(\lambda)$. |
| gsdec                     | Generalized additive spectral decompositions. |
| grmcover1                 | Right minimum dynamic cover of Type 1 based order reduction. |
| glmcover1                 | Left minimum dynamic cover of Type 1 based order reduction. |
3.2 Getting Started

In this section we shortly recall how to construct generalized LTI system objects using commands of the Control System Toolbox (CST) of MATLAB [25] and discuss some basic model conversion techniques and operations with rational matrices via their descriptor system representations. We also illustrate and compare the functionality of some functions available in the CST and DSTOOLS to perform model conversions and manipulations.
3.2.1 Building Generalized LTI Models

To describe generalized LTI systems via their TFM representations, the CST supports two model objects (or classes) called \( \text{tf} \) and \( \text{zpk} \). The model class \( \text{tf} \) represents the elements of the TFM as ratios of two polynomials, in the form (1). The corresponding constructor command \( \text{tf} \) can be used to build transfer functions for SISO or TFM\`s for MIMO LTI systems. The model class \( \text{zpk} \) represents the elements of the TFM in a zero-pole-gain (factorized) form as in (31) and the corresponding constructor command is \( \text{zpk} \).

A straightforward method to enter TFM based models relies on a powerful feature of the CST to connect subsystems. For example, to enter the continuous-time improper TFM

\[
G_c(s) = \begin{bmatrix} s^2 & s \\ s & \frac{s+1}{s+1} \\ 0 & \frac{1}{s} \end{bmatrix},
\]

the following commands can be used

\[
s = \text{tf}('s'); \quad \% \text{define the complex variable } s
G_c = [s^2 s/(s+1); 0 1/s] \quad \% \text{define the 2-by-2 improper } G_c(s)
\]

Similarly, the discrete-time improper TFM

\[
G_d(z) = \begin{bmatrix} z^2 & z \\ z & \frac{z-2}{z} \\ 0 & \frac{1}{z} \end{bmatrix},
\]

with sampling time equal to 0.5, can be entered using the commands

\[
z = \text{tf}('z',0.5); \quad \% \text{define the complex variable } z
G_d = [z^2 z/(z-2); 0 1/z] \quad \% \text{define the 2-by-2 improper } G_d(z)
\]

To describe LTI systems in state-space form, the model class \( \text{ss} \) is provided jointly with the class constructor commands \( \text{dss} \), for descriptor systems and \( \text{ss} \) for standard state-space systems (with \( E = I \)). For example, a descriptor system model of the form (2) can be constructed by entering the system matrices \( E, A, B, C \) and \( D \) and using the command \( \text{dss} \). The descriptor system with the state-space realization

\[
\begin{pmatrix} A - sE \\ C \end{pmatrix} B = \begin{bmatrix} 1 & -s & 0 & 0 & 0 \\ 0 & 1 & -s & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & -1 & -s \\ 1 & 0 & 0 & -1 & 0 \end{bmatrix}
\]

can be entered using the following commands:

\[
A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0; 0 & 1 & 0 & 0 & 0; 0 & 0 & 1 & 0 & 0; 0 & 0 & 0 & -1 & 0; 0 & 0 & 0 & 0 & 0 \end{bmatrix};
B = \begin{bmatrix} 0 & 0 & -1 & 0 & 0; 0 & 0 & 1 & 0 \end{bmatrix}';
\]
C = [ 1 0 0 -1 0; 0 0 0 0 1];
D = [ 0 1; 0 0 ];
E = [ 0 1 0 0; 0 0 1 0; 0 0 0 0; 0 0 0 1 ];
sys = dss(A,B,C,D,E);

If \( E = I \), the system model is a standard state-space model, which can be constructed using the class constructor command \texttt{ss}.

A less known aspect of building descriptor system models is the available “freedom” in the CST to allow the construction of descriptor system models with a singular pole pencil \( A - \lambda E \). This “freedom” is very questionable and may lead to potential conceptual and computational difficulties. This can be easily illustrated with the following trivial non-regular descriptor system, for which many of the functions of the CST produce misleading warnings or questionable results, as shown by the following sequence of commands:

\[
A = 0; E = 0; B = 1; C = 1; D = 0;
\]

\[
syst = \text{dss}(A,B,C,D,E);
\]

\[
\text{pole(syst)} \quad \% \text{the pole must be NaN, similar to eig(A,E), and not empty!}
\]

\[
\text{tf(syst)} \quad \% \text{the transfer function is infinite and not NaN!}
\]

\[
\text{evalfr(syst,1)} \quad \% \text{this evaluation of an infinite frequency response is correct}
\]

\[
\text{isproper(syst)} \quad \% \text{this test is wrong, because the system is not proper}
\]

The function \texttt{gpole} of \texttt{DSTOOLS} can be used to check the regularity of the pole pencil. Both commands \texttt{gpole} and \texttt{eig} below

\[
\text{gpole(syst)}
\]

\[
\text{eig(A,E)}
\]

compute the “correct” value of the pole, which, in this case, is \( \text{NaN} \).

The \texttt{DSTOOLS} collection exclusively deals with regular descriptor models, for which the pole pencil \( A - \lambda E \) is regular. All functions of \texttt{DSTOOLS} guarantee that the computed results are regular descriptor systems, provided the input systems are regular. We have to stress, that for efficiency reasons, in most of functions of \texttt{DSTOOLS}, the regularity condition for the input system data is only \textit{assumed}, but it is not explicitly checked. Therefore, it is likely that some functions, even if they perform without issuing error messages, may still deliver nonsense results if the regularity assumption is not fulfilled by the input system descriptions.

### 3.2.2 Conversions between LTI Model Representations

Most of functions of \texttt{DSTOOLS} accept only state-space system objects as inputs and therefore the input-output models must be converted to a standard or descriptor state-space form. For the above defined TFMs \( G_c(s) \) and \( G_d(z) \), this conversion can be done simply using

\[
sysc = \text{ss}(Gc)
\]

\[
sysd = \text{ss}(Gd)
\]

The resulting state-space realization are usually non-minimal. Incidentally, both of the resulting state-space realizations \texttt{sysc} and \texttt{sysd} have order 5 and are minimal.

For visualization purposes, often the more compact TFM models are better suited than state-space models. Both class constructor commands \texttt{tf} and \texttt{zpk} can also serve to explicitly convert
a LTI model to `tf` or `zpk` forms, respectively. The resulting transfer-function models computed from state-space models, usually contains uncancelled common factors in the numerator and denominator polynomials of the rational matrix elements, and, therefore, are non-minimal.

To compute minimal realizations, the function `minreal` is available in the CST. This function, applied to transfer-function models, enforces the cancellation of common factors in the numerator and denominator polynomials of each element. However, this function is only applicable to proper descriptor systems (also including the case of systems with singular $E$). When applied to an improper descriptor system, an error message is issued:

```matlab
order(minreal(sysc))
Error using DynamicSystem/minreal (line 53)
The "minreal" command cannot be used for models with more zeros than poles.
```

Unfortunately, the above error message is misleading, since for the system `sysc` with an invertible TFM $G_c(s)$, the number of poles and zeros (counting also the infinite poles and zeros), must coincide in accordance with (34). This can be checked with the functions `gpole` and `gzero` of DSTOOLS:

```matlab
POLES = gpole(sysc)
ZEROS = gzero(sysc)
```

which produce the following results:

```matlab
POLES =
  0.0000 + 0.0000i
 -1.0000 + 0.0000i
   Inf + 0.0000i
   Inf + 0.0000i

ZEROS =
 -1.0000 + 0.0000i
  0.0000 + 0.0000i
  0.0000 + 0.0000i
   Inf + 0.0000i
```

These results also indicate that the McMillan degree of the system is 4 (recall that the order of the minimal descriptor state-space realization is 5). In contrast, the functions `pole` and `tzero` of the CST, compute only the finite poles and finite zeros, respectively, and provide no hint on the actual McMillan degree.

Alternative functions to compute irreducible and minimal realizations, are, respectively, the functions `gir` and `gminreal` available in DSTOOLS. For example, the minimality of the above computed realizations can be checked with the function `gminreal` of DSTOOLS:

```matlab
order(gminreal(sysc)) % computes the order of the minimal realization
```

3.2.3 Conversion to Standard State-Space Form

A useful conversion which we discuss separately is the conversion of a descriptor system model of a proper system into a standard state-space model. Specifically, we discuss the conversion of
a proper descriptor system of the form

\[
\begin{align*}
E \lambda x(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t),
\end{align*}
\]  

(87)

with \(x(t) \in \mathbb{R}^n\), to a standard state-space system of the form

\[
\begin{align*}
\lambda \tilde{x}(t) &= \tilde{A} \tilde{x}(t) + \tilde{B} u(t), \\
y(t) &= \tilde{C} \tilde{x}(t) + \tilde{D} u(t),
\end{align*}
\]  

(88)

with \(\tilde{x}(t) \in \mathbb{R}^\tilde{n}\) and \(\tilde{n} \leq n\), and such that the two systems have the same transfer function matrices, i.e.

\[C(\lambda E - A)^{-1}B + D = \tilde{C}(\lambda I - \tilde{A})^{-1}\tilde{B} + \tilde{D}.\]

This conversion is usually necessary, to obtain for the designed controllers and filters simpler representations, which are better suited for real-time processing. However, we cautiously recommend to avoid such conversions at early steps of the synthesis procedures, unless it is possible to guarantee that no significant loss of accuracy takes place due to ill-conditioned transformations.

For simplicity, we consider only the case when the descriptor realization \((A - \lambda E, B, C, D)\) is already irreducible. Such realizations can be obtained, for example, using the DSTOOLS function \textsc{gir}. We further assume that the regular pencil \(A - \lambda E\) has \(r\) finite eigenvalues and \(n - r\) simple eigenvalues at infinity, where \(r\) is the rank of \(E\). When \(E\) is nonsingular, we can simply choose \(\tilde{x}(t) = x(t)\) and

\[\tilde{A} = E^{-1}A, \quad \tilde{B} = E^{-1}B, \quad \tilde{C} = C, \quad \tilde{D} = D,
\]

or alternatively choose \(\tilde{x}(t) = Ex(t)\) and

\[\tilde{A} = AE^{-1}, \quad \tilde{B} = B, \quad \tilde{C} = CE^{-1}, \quad \tilde{D} = D.\]

In these conversion formulas, the inverse of \(E\) is explicitly involved and, therefore, severe loss of accuracy can occur if the condition number \(\kappa(E) := \|E\|_2\|E^{-1}\|_2\) is large.

A numerically better conversion approach is to use the singular value decomposition (SVD) \(E = UV^T\), with \(U\) and \(V\) orthogonal matrices and \(\Sigma\) a diagonal matrix whose diagonal elements are the decreasingly ordered singular values \(\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0\). We can choose \(\tilde{x}(t) = \Sigma^{\frac{1}{2}}V^Tx(t)\) and

\[\tilde{A} = \Sigma^{-\frac{1}{2}}UA\Sigma^{-\frac{1}{2}}, \quad \tilde{B} = \Sigma^{-\frac{1}{2}}U^TB, \quad \tilde{C} = CV\Sigma^{-\frac{1}{2}}, \quad \tilde{D} = D.\]

(89)

From the SVD of \(E\), we can easily compute the condition number \(\kappa(E) = \sigma_1/\sigma_n\), and thus have a rough estimation of potential loss of accuracy induced by using the above transformation.

It is also possible to use the QR-decomposition \(E = QR\), with \(Q\) orthogonal and \(R\) upper-triangular and with positive diagonal elements (this can always be arranged using an orthogonal diagonal scaling matrix). Let \(R^\frac{1}{2}\) be the upper-triangular square root of \(R\), which can be computed using the method described in [19, Algorithm 6.7], which is also implemented in the MATLAB function \textsc{sqrtm}. We can choose \(\tilde{x}(t) = R^\frac{1}{2}x(t)\) and

\[\tilde{A} = R^{-\frac{1}{2}}Q^TAR^{-\frac{1}{2}}, \quad \tilde{B} = R^{-\frac{1}{2}}Q^TB, \quad \tilde{C} = CR^{-\frac{1}{2}}, \quad \tilde{D} = D.\]

(90)
From the QR-decomposition of $E$, we can easily compute the condition number $\kappa(E) = \kappa(R)$ (e.g., by using `rcond` to estimate the reverse condition number $1/\kappa(R)$). In this way, we can have a rough estimation of potential loss of accuracy induced by using the above transformation. Although this method can be generally used, its primary use is when the original pair $(A,E)$ is already in a GRSF, with $A$ upper quasi-triangular and $E$ upper triangular. In this case, the resulting $\tilde{A}$ is in a RSF.

More involved transformation is necessary when $E$ is singular, with rank $E = r < n$. In this case, we can employ the singular value decomposition of $E$ in the form

$$E = U\Sigma V^T := \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix}^T,$$

where $\Sigma$ is a nonsingular diagonal matrix of order $\tilde{n} := r$ with the nonzero singular values of $E$ on the diagonal, and $U$ and $V$ are compatibly partitioned orthogonal matrices. If we apply a system similarity transformation with the transformation matrices

$$\tilde{U} = \text{diag} \left( \Sigma^{-\frac{1}{2}} , I_{n-r} \right) U^T, \quad \tilde{V} = V \text{diag} \left( \Sigma^{-\frac{1}{2}} , I_{n-r} \right)$$

we obtain

$$\tilde{U}(A - \lambda E)\tilde{V} = \begin{bmatrix} A_{11} - \lambda I_r & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad \tilde{U}B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C\tilde{V} = \begin{bmatrix} C_1 & C_2 \end{bmatrix},$$

where $A_{22}$ is nonsingular, due to the assumption of only simple infinite eigenvalues of the regular pencil $A - \lambda E$. The above transformed matrices correspond to the coordinate transformation $\pi = \tilde{V}^{-1}\pi(t)$ and lead to the partitioned system representation

$$\begin{array}{ll}
\lambda \pi_1(t) & = A_{11} \pi_1(t) + A_{12} \pi_2(t) + B_1 u(t), \\
0 & = A_{21} \pi_1(t) + A_{22} \pi_2(t) + B_2 u(t), \\
y(t) & = C_1 \pi_1(t) + C_2 \pi_2(t) + Du(t),
\end{array}$$

where $\pi(t) = \begin{bmatrix} \pi_1(t) \\ \pi_2(t) \end{bmatrix}$ is partitioned such that $\pi_1(t) \in \mathbb{R}^r$ and $\pi_2(t) \in \mathbb{R}^{n-r}$. We can solve the second (algebraic) equation for $\pi_2(t)$ to obtain

$$\pi_2(t) = -A_{22}^{-1} A_{21} \pi_1(t) - A_{22}^{-1} B_2 u(t)$$

and arrive to a standard system representation with $\tilde{\pi}(t) = \pi_1(t)$ and the corresponding matrices

$$\tilde{A} = A_{11} - A_{12} A_{22}^{-1} A_{21}, \quad \tilde{B} = B_1 - A_{12} A_{22}^{-1} B_2,$$

$$\tilde{C} = C_1 - C_2 A_{22}^{-1} A_{21}, \quad \tilde{D} = D - C_2 A_{22}^{-1} B_2.$$  \hspace{1cm} (93)

In this case, if any of the condition numbers $\kappa(\tilde{\Sigma})$ or $\kappa(A_{22})$ is large, potential accuracy losses can be induced by the conversion to a standard state-space form.

The conversion formulas (89) and (91)-(93) underly the implementation of the function `gss2ss` of DSTOOLS (used as default options). The formulas (90) are used, by default, if the input pair $(A,E)$ is in a GRSF, with $E$ invertible. The CST function `isproper` has a hidden input flag (‘explicit’), which allows to convert proper descriptor models to a standard state-space form.
3.2.4 Sensitivity Issues for Polynomial-Based Representations

The ill-conditioning of polynomial-based representations was a constant discussion subject in the control literature to justify the advantage of using state-space realization-based models for numerical computations. For an authoritative discussion see [39]. More details on these issues are provided in Chapter 6 of [41].

The extreme sensitivity of roots of polynomials with respect to small variations in the coefficients, illustrated in the following example, is well known in the literature and is inherent for polynomial-based representations above a certain degree (say \( n > 10 \)). Therefore, all algorithms which involve rounding errors are doomed to fail by giving results of extremely poor accuracy when dealing with an ill-conditioned polynomial. This potential loss of accuracy is one of the main reasons why polynomial-based system representations with rational or polynomial matrices are generally not suited for numerical computations.

It is well known that polynomials with multiple roots are very sensitive to small variations in the coefficients. However, it is less known that this large sensitivity may be present even in the case of polynomials with well separated roots, if the order of the polynomial is sufficiently large. This will be illustrated by the following example.

**Example 1.** The simple transfer function

\[
g(s) = \frac{1}{(s+1)(s+2) \cdots (s+25)} = \frac{1}{s^{25} + 325s^{24} + \cdots + 25!}
\]

has the exact poles \(-1, -2, \ldots, -25\). The denominator is a modification coined by Daniel Kressner (private communication) of the famous Wilkinson polynomial analyzed in [63] (originally of order 20 and with positive roots). This polynomial has been used in many works to illustrate the pitfalls of algorithms for computing eigenvalues of a matrix by computing the roots of its characteristic polynomial.

If we explicitly construct the transfer function \( g(s) \) and compute its poles using the MATLAB commands

\[
g = \text{tf}(1, \text{poly}(-25:1:-1));
\]

\[
\text{sys} = \text{ss}(g);
\]

\[
\text{pole(sys)}
\]

inaccurate poles with significant imaginary parts result, as can be observed from Fig. 1. For example, instead the poles at \(-19\) and \(-20\), two complex conjugate poles at \(-19.8511 \pm 3.2657i\) result.\(^2\)

The main reason for these inaccurately computed poles is the high sensitivity of the polynomial roots to small variations in the coefficients. In this case, inherent truncations take place in representing the large integer coefficients due to the finite representation with 16 accurate digits of double-precision floating-point numbers. For example, the constant term in the denominator \(25! \approx 1.55 \cdot 10^{25}\) has 25 decimal digits, thus can not be exactly represented with 16 digits precision. While the relative error in representing \(25!\) is of the order of the machine-precision \(\varepsilon_M \approx 10^{-16}\), the absolute error is of the order of \(10^9\)!

For this particular example, the zero-pole-gain based representation is possibly better suited as starting point for building a state-space realization. For example, a state-space realization of

\(^2\)Computed with MATLAB R2015b Version, running under 64-Bit Microsoft Windows 10
$g(s)$ can be constructed, which still preserves the full accuracy of poles, as shown in the following example:

```matlab
s = zpk('s'); g = 1; for i=1:20, g = g/(s+i); end
sys = ss(g);
pole(sys)
```

Interestingly, the following command sequence builds a 60-th order descriptor system realization of $g(s)$, which contains 40 non-dynamics modes. After eliminating the non-dynamic modes and converting the realization to a standard state-space model, the full accuracy of poles is still preserved to machine precision.

```matlab
s = ss(tf('s')); g = 1; for i=1:20, g = g/(s+i); end
sys = gss2ss(g);
pole(sys)
```

This example illustrates that using descriptor system models for model building may occasionally alleviate the numerical difficulties which are inherent when using polynomial based models.

### 3.2.5 Operations with Rational Matrices

In this section we succinctly discuss the operations with rational matrices via their descriptor system realizations. These operations play an important role in implementing many of the functions available in DSTOOLS and therefore, we present some illustrations of the usage of these operations. Let $G(\lambda)$ be a rational TFM having the descriptor system realization $(A - \lambda E, B, C, D)$. We assume that two system objects have been defined to represent $G(\lambda)$:
the transfer-function model \( g \) (e.g., defined with either the \texttt{tf} or \texttt{zpk} constructors) and the state-space model \( \text{sys} \) (e.g., defined either with the \texttt{ss} or the \texttt{dss} constructors).

The transpose \( G^T(\lambda) \) and the corresponding dual descriptor system realization \( G^T(\lambda) = (A^T \! - \! \lambda E^T, C^T, B^T, D^T) \) can be simply computed as \( \text{g.'} \) and \( \text{sys.'} \), respectively. This operation can be useful to implement, for example, a left oriented function by using an already available right oriented function. For example, to compute a left coprime factorization of \( G(\lambda) \) as \( G(\lambda) = M^{-1}(\lambda)N(\lambda) \), the function \text{glcf} calls the function \text{grcf} to compute a right coprime factorization of the transpose as \( G^T(\lambda) = \tilde{N}(\lambda)\tilde{M}^{-1}(\lambda) \). An interesting aspect of computing right coprime factorizations with the function \text{grcf} is that both resulting descriptor realizations of \( \tilde{N}(\lambda) = (\tilde{A}_N - \lambda\tilde{E}_N, \tilde{B}_N, \tilde{C}_N, \tilde{D}_N) \) and \( M(\lambda) = (\tilde{A}_M - \lambda\tilde{E}_M, \tilde{B}_M, \tilde{C}_M, \tilde{D}_M) \) have the resulting pairs \((\tilde{A}_N, \tilde{E}_N)\) and \((\tilde{A}_M, \tilde{E}_M)\) in a GRSF (i.e., with the pole pencils upper quasi-triangular). This condensed form of the pole pencils may be useful for further computations, where this structure can be efficiently exploited (e.g., the eigenvalues can be computed at no cost using the function \text{ordeig} or the function \text{grcf} can be applied a second time, with a significantly less computational burden). Therefore, it is highly desirable that the left oriented function \text{glcf} preserves the upper quasi-triangular shape of the pole pencils of the factors. However, by simply forming the transposed matrices of the dual descriptor systems, this useful property is lost. To preserve the upper quasi-triangular form of the pole pencil of the dual system, an alternative realization of the dual system can be constructed as \( G^T(\lambda) = (P^T\! A^T \! - \! \lambda P^T \! E^T P^T, PC^T, B^T, D^T) \), where \( P \) is the (orthogonal) permutation matrix with ones down on the secondary diagonal

\[
P = \begin{bmatrix} 0 & \cdots & 0 \\ 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \end{bmatrix}.
\]

The following sequence of commands is used in the function \text{glcf} to efficiently build permuted dual realizations using the function \text{xperm} of the CST:

\[
\begin{align*}
\% \text{ apply grcf to the dual system, by trying to preserve upper-triangular shapes}
[syns,sysm] &= \text{grcf}(\text{xperm(sys,order(sys):-1:1).'},\text{options}); \\
\% \text{ build dual factors, by preserving upper-triangular shapes}
\text{sysn} &= \text{xperm(sysn,order(sysn):-1:1).'}; \\
\text{sysm} &= \text{xperm(sysm,order(sysm):-1:1).'};
\end{align*}
\]

If \( G(\lambda) \) is invertible, then an inversion free realization of the inverse TFM \( G^{-1}(\lambda) \) is given by

\[
G^{-1}(\lambda) = \begin{bmatrix} A - \lambda E & B & 0 \\ C & D & I \\ 0 & -I & 0 \end{bmatrix}.
\]

This realization is not minimal, even if the original realization is minimal. If \( D \) is invertible, then an alternative realization of the inverse is

\[
G^{-1}(\lambda) = \begin{bmatrix} A - BD^{-1}C - \lambda E & -BD^{-1} \\ D^{-1}C & D^{-1} \end{bmatrix},
\]

which is minimal if the original realization is minimal.

To compute inverse systems, the overloaded function \text{inv} can be used to compute the inverse systems as \text{inv}(g) or \text{inv}(sys). The function \text{inv} applied to a descriptor state-space model
always builds the realization (94) of the inverse (even if \( E = I \)), while for standard state-space systems (with \( E = I \)), the realization (95) is used, provided \( D \) is reasonably well-conditioned with respect to the inversion. For transfer-function models, an automatic conversion to state-space form is performed, and the computed inverse is converted back to the transfer-function form. These conversions often lead to non-minimal representations containing uncanceled factors between the numerator and denominators of the matrix elements.

Operations involving inverses can often be performed using the overloaded matrix operators \( \backslash \) (left divide) or \( / \) (right divide), as for example, to compute \( \text{sys1} \backslash \text{sys2} \) or \( \text{sys1} / \text{sys2} \) for two systems \( \text{sys1} \) and \( \text{sys2} \). However, these operations do not avoid the explicit building of the inverses, as can be seen by performing \( \text{sys1} / \text{sys1} \) or \( \text{sys2} \backslash \text{sys2} \), which, instead of producing a non-dynamic system with the direct feedthrough gain equal to the identity matrix, determine realizations of orders (at least) double of the orders of \( \text{sys1} \) or \( \text{sys2} \). Alternative computation of \( \text{sys1} \backslash \text{sys2} \) can be done using the function \text{glsol} of \text{DSTOOLS}, while for the evaluation of \( \text{sys1} / \text{sys2} \), the function \text{grsol} of \text{DSTOOLS} can be used. These functions are generally applicable to solve compatible systems of linear rational matrix equations and always determine minimal order realizations of the solutions.

The conjugate (or adjoint) TFM \( G^\sim(\lambda) \) is defined in the continuous-time case as \( G^\sim(s) = G^T(-s) \) and has the realization

\[
G^\sim(s) = \begin{bmatrix}
-A^T - sE^T & C^T \\
-B^T & D^T
\end{bmatrix},
\]

while in the discrete-time case \( G^\sim(z) = G^T(1/z) \) and has the realization

\[
G^\sim(z) = \begin{bmatrix}
E^T - zA^T & 0 & -C^T \\
zB^T & I & D^T \\
0 & I & 0
\end{bmatrix}.
\]

If \( G(z) \) has a standard state-space realization \((A, B, C, D)\) with \( A \) invertible, then an alternative realization of \( G^\sim(z) \) is

\[
G^\sim(z) = \begin{bmatrix}
A^{-T} - zI & -A^{-T}C^T \\
B^TA^{-T} & D^T - B^TA^{-T}C^T
\end{bmatrix}.
\]

This realization is only recommended if \( A \) is well-conditioned with respect to the inversion.

The conjugate of a transfer function model \( g \) can be simply computed as \( g' \), while for a state-space model \( \text{sys} \) with \( \text{sys}' \). For a discrete-time state-space model, the conjugate system is always a descriptor system, which is usually non-minimal (frequently contains non-dynamic modes).

### 3.3 Functions for System Analysis

The system analysis functions cover the computation of poles and zeros, of normal rank and Hankel norm of the transfer function matrix of a LTI descriptor system.
3.3.1 gpole

Syntax

[POLES,MI,REGULAR] = gpole(SYS)
[POLES,MI,REGULAR] = gpole(SYS,TOL)

Description

gpole computes for a LTI descriptor system, the finite and infinite zeros of the pole pencil, and provides information on its regularity.

Input data

SYS is a LTI system in a descriptor system state-space form

\[
E\lambda x(t) = Ax(t) + Bu(t), \\
y(t) = Cx(t) + Du(t).
\]

(96)

TOL is a relative tolerance used for rank determinations. If TOL is not specified as input or if TOL = 0, an internally computed default value is used.

Output data

POLES is a complex column vector which contains the zeros (finite and infinite) of the pole pencil \( A - \lambda E \). These are the poles of the TFM of SYS if the descriptor realization \((A - \lambda E, B, C, D)\) is irreducible. If the pencil \( A - \lambda E \) is not regular, some components of POLES are set to NaN.

MI returns additional information on the multiplicities of infinite zeros of \( A - \lambda E \) as follows:

\( A - \lambda E \) has \( MI(i) \) infinite zeros of multiplicity \( i \). MI results empty if \( A - \lambda E \) has no infinite zeros.

REGULAR is a logical variable which returns information on the regularity of the pencil \( A - \lambda E \) as follows:

REGULAR = true, if the pencil \( A - \lambda E \) is regular;

REGULAR = false, if the pencil \( A - \lambda E \) is singular.

Method

Let \( G(\lambda) \) be the TFM \( G(\lambda) = C(\lambda E - A)^{-1}B + D \) of the LTI system SYS. For the definition of the poles of \( G(\lambda) \) in terms of the descriptor realization (96), see Section 2.6. If the descriptor system realization \((A - \lambda E, B, C, D)\) of SYS has a regular pole pencil \( A - \lambda E \) and is irreducible (i.e., controllable and observable), then the computed finite poles in POLES are simply the finite generalized eigenvalues of the pair \((A, E)\), and the multiplicities of the infinite generalized eigenvalues of \((A, E)\) are in excess with one with respect to the multiplicities of the infinite poles. For the computation of the eigenvalues of \( A - \lambda E \) and its Kronecker structure, the zeros computation algorithm of [26] (see also [36]) is applied to the particular system matrix pencil \( S(\lambda) := A - \lambda E \) (i.e., of a system without inputs and outputs), by calling the MEX-function sl_gzero. This
algorithm determines the $n_f$ finite zeros of $A - \lambda E$, $\lambda_i$, $i = 1, \ldots, n_f$; the $n_{\infty} = \sum_{i=1}^{h} m_i^\infty$ infinite zeros of $A - \lambda E$, with their multiplicities $m_i^\infty$, for $i = 1, \ldots, h$; and also the $\nu_r$ right Kronecker indices $\epsilon_i = 0$, for $i = 1, \ldots, \nu_r$ of the pencil $A - \lambda E$ (corresponding to the $\nu_r$ Kronecker blocks $L_{\epsilon_i}(\lambda)$ of the form $\epsilon_i \times (\epsilon_i + 1)$ which are part of the Kronecker-form of the pencil $A - \lambda E$, as in (13)). The presence of the right Kronecker indices indicates that $A - \lambda E$ is singular, and a number of $\nu_r$ eigenvalues of the pair $(A, E)$ are set to NaN.

The outputs variables are set as follows. The components of the vector POLES are: $n_f$ finite values $\lambda_i$, $i = 1, \ldots, n_f$; $n_{\infty}$ infinite values; and $\nu_r$ values set to NaN. The vector MI contains the multiplicities $m_i^\infty$, for $i = 1, \ldots, h$. Finally, REGULAR is set to false if $\nu_r > 0$, otherwise is set to true.

### 3.3.2 gzero

**Syntax**

\[
[Z,MI,KRONS] = gzero(SYS)
\]

\[
[Z,MI,KRONS] = gzero(SYS,TOL)
\]

**Description**

*gzero* computes for a LTI descriptor system, the finite and infinite zeros of the system matrix pencil, and provides information on its normal rank and Kronecker structure.

**Input data**

SYS is a LTI system in a descriptor system state-space form

\[
E \dot{x}(t) = Ax(t) + Bu(t),
\]

\[
y(t) = Cx(t) + Du(t).
\]

(97)

TOL is a relative tolerance used for rank determinations. If TOL is not specified as input or if TOL = 0, an internally computed default value is used.

**Output data**

Z is a complex column vector which contains the invariant zeros (finite and infinite) of the system matrix pencil

\[
S(\lambda) = \begin{bmatrix}
A - \lambda E & B \\
C & D
\end{bmatrix}.
\]

(98)

These are also called the transmission zeros of the TFM $G(\lambda)$ of SYS if the descriptor realization $(A - \lambda E, B, C, D)$ is irreducible.

MI returns additional information on the multiplicities of infinite zeros as follows: $S(\lambda)$ has MI(i) infinite zeros of multiplicity $i$. MI results empty if $S(\lambda)$ has no infinite zeros.
Krons is a MATLAB structure, whose fields contain the following information:

- **Krons.nrank** – normal rank of the pencil $S(\lambda)$;
- **Krons.kronr** – right Kronecker indices of the pencil $S(\lambda)$;
- **Krons.infe** – orders of the elementary infinite blocks in the Kronecker form of the pencil $S(\lambda)$;
- **Krons.kronl** – left Kronecker indices of the pencil $S(\lambda)$.

**Method**

Let $G(\lambda)$ be the TFM $G(\lambda) = C(\lambda E - A)^{-1}B + D$ of the LTI system $\text{SYS}$. For the definition of the zeros of $G(\lambda)$ in terms of the descriptor realization (97), see Section 2.6. If the descriptor system realization $(A - \lambda E, B, C, D)$ of $\text{SYS}$ is irreducible (i.e., controllable and observable), then the computed finite zeros in $\mathbb{Z}$ are simply the finite generalized eigenvalues of the system matrix pencil $S(\lambda)$, and the multiplicities of the infinite generalized eigenvalues of $S(\lambda)$ are in excess with one with respect to the multiplicities of the infinite zeros. For the computation of the eigenvalues of $S(\lambda)$ and its Kronecker structure, the zeros computation algorithm of [26] is applied to the system matrix pencil $S(\lambda)$ in (36), by calling the MEX-function $\text{sl}_gzero$. In the case of a standard state-space model with $E = I$, $\text{sl}_gzero$ uses the algorithm of [10] in conjunctions with the extension proposed in [36]. These algorithms determine:

- the $n_f$ finite zeros of $S(\lambda)$, $\lambda_i$, $i = 1, \ldots, n_f$;
- the $n_{\infty} = \sum_{j=1}^{k} m_{j}^{\infty}$ infinite zeros of $S(\lambda)$, with their multiplicities $m_{j}^{\infty}$, for $j = 1, \ldots, k$;
- the $\nu_r$ right Kronecker indices $\epsilon_i$, for $i = 1, \ldots, \nu_r$, of the pencil $S(\lambda)$ (corresponding to the $\nu_r$ Kronecker blocks $L_{\epsilon_i}(\lambda)$ of the form $\epsilon_i \times (\epsilon_i + 1)$ which are part of the Kronecker canonical form of the pencil $S(\lambda)$, as in (13));
- the $\nu_l$ left Kronecker indices $\eta_i$, for $i = 1, \ldots, \nu_l$, of the pencil $S(\lambda)$ (corresponding to the $\nu_l$ Kronecker blocks $L_{\eta_i}(\lambda)$ of the form $\eta_i \times (\eta_i + 1)$ which are part of the Kronecker canonical form of the pencil $S(\lambda)$, as in (16));
- the orders of the elementary infinite blocks $s_i^{\infty}$, for $i = 1, \ldots, h$, (i.e., the dimensions of the elementary infinite Jordan blocks in (10) in the Weierstrass canonical form of the regular part (15)).

The multiplicities of the infinite zeros are related to the orders of the elementary infinite blocks, as follows: to each elementary infinite block of order $s_i^{\infty} > 1$ corresponds an infinite zeros of multiplicity $s_i^{\infty} - 1$.

The outputs variables are set as follows. The components of the vector $Z$ are the $n_f$ finite values $\lambda_i$, $i = 1, \ldots, n_f$ and the $n_{\infty}$ infinite values. The vector $\text{MI}$ contains the multiplicities of the infinite zeros $m_{j}^{\infty}$, for $j = 1, \ldots, k$. The fields of $\text{Krons}$ are set as follows: $\text{Krons.nrank}$ is set to the normal rank of $S(\lambda)$, which is given by $n_f + \sum_{i=1}^{h} s_i^{\infty} + \sum_{i=1}^{\nu_r} \epsilon_i + \sum_{i=1}^{\nu_l} \eta_i$; $\text{Krons.kronr}$ contains the $\nu_r$ right Kronecker indices $\epsilon_i$, for $i = 1, \ldots, \nu_r$, of the pencil $S(\lambda)$; $\text{Krons.infe}$ contains the orders of the elementary infinite blocks $s_i^{\infty}$, for $i = 1, \ldots, h$; and, $\text{Krons.kronl}$ contains the $\nu_l$ left Kronecker indices $\eta_i$, for $i = 1, \ldots, \nu_l$, of the pencil $S(\lambda)$.
Application examples

The input decoupling zeros are defined as the finite zeros of the particular system matrix pencil

\[ S(\lambda) := [A - \lambda E \ B], \]

which corresponds to a system with empty outputs. These zeros are also known as the finite uncontrollable eigenvalues of the pencil \( A - \lambda E \). The function \texttt{gzero} can be used to compute the input decoupling zeros of a LTI state-space system \texttt{sys} using

\[ [Z, M, Krons] = \texttt{gzero}(\texttt{sys}([],:), \texttt{tol}) \]

For a finite controllable pair \((A - \lambda E, B)\), \( Z, M \) and \texttt{Krons.kronl} result empty.

Similarly, the output decoupling zeros are defined as the invariant zeros of the particular system matrix

\[ S(\lambda) := \begin{bmatrix} A - \lambda E \\ C \end{bmatrix}, \]

which corresponds to a system with empty inputs. These zeros are also known as the finite unobservable eigenvalues of the pencil \( A - \lambda E \). The function \texttt{gzero} can be used to compute the output decoupling zeros of a LTI state-space system \texttt{sys} using

\[ [Z, M, Krons] = \texttt{gzero}(\texttt{sys}(:,[]), \texttt{tol}) \]

For a finite observable pair \((A - \lambda E, C)\), \( Z, M \) and \texttt{Krons.kronr} result empty.

Finally, the zeros of the pole pencil can be computed as the invariant zeros of the particular system matrix

\[ S(\lambda) := A - \lambda E, \]

which corresponds to a system with empty inputs and empty outputs. The function \texttt{gzero} can be used to compute the zeros of the pole pencil of a LTI state-space system \texttt{sys} using

\[ [Z, M, Krons] = \texttt{gzero}(\texttt{sys}([],[]), \texttt{tol}) \]

This calling form of \texttt{gzero} corresponds to the functionality implemented in the function \texttt{gpole}. For a descriptor system with a regular pole pencil, both \texttt{Krons.kronr} and \texttt{Krons.kronl} result empty.

3.3.3 \texttt{nrank}

Syntax

NR = \texttt{nrank}(SYS)
NR = \texttt{nrank}(SYS,TOL)

Description

\texttt{nrank} computes for the LTI system \texttt{sys}, the normal rank of its transfer function matrix.
Input data

SYS is a LTI system, which can be specified in a descriptor system state-space form

\[ \begin{align*}
E \lambda x(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t)
\end{align*} \tag{99} \]

or in an input-output form

\[ y(\lambda) = G(\lambda)u(\lambda), \tag{100} \]

where \(G(\lambda)\) is the rational transfer function matrix of the system.

TOL is a relative tolerance used for rank determinations. If TOL is not specified as input or if TOL = 0, an internally computed default value is used.

Output data

NR is the normal rank of the transfer function matrix \(G(\lambda)\) of the LTI system SYS.

Method

If SYS is specified in the descriptor form (99), then its transfer-function matrix is \(G(\lambda) = C(\lambda E - A)^{-1} B + D\). If SYS is specified in the input-output form (100), then a (possibly non-minimal) state-space realization of the form (99) is automatically constructed. For the definition of the normal rank \(r\) of a rational TFM \(G(\lambda)\), see Section 2.4. For the calculation of the normal rank \(r\) of \(G(\lambda)\) in terms of the descriptor representation, we use the relation

\[ r = \text{rank} S(\lambda) - n, \]

where \(\text{rank} S(\lambda)\) is the normal rank of the system matrix pencil \(S(\lambda)\) defined as

\[ S(\lambda) = \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix} \tag{101} \]

and \(n\) is the order of the descriptor state-space realization. For the computation of the normal rank of \(S(\lambda)\), the structural elements of its Kronecker structure are determined using the zeros computation algorithm of [26], by calling the MEX-function sl_gzero. In the case of a standard state-space model with \(E = I\), sl_gzero uses the algorithm of [10] in conjunctions with the extension proposed in [36]. These algorithms determine:

- \(n_f\), the number of finite eigenvalues of \(S(\lambda)\);
- the \(\nu_r\) right Kronecker indices \(\epsilon_i\), for \(i = 1, \ldots, \nu_r\) of the pencil \(S(\lambda)\) (corresponding to the \(\nu_r\) Kronecker blocks \(L_{\epsilon_i}(\lambda)\) of the form \(\epsilon_i \times (\epsilon_i + 1)\) which are part of the Kronecker canonical form of the pencil \(S(\lambda)\), as in (13));
- the \(\nu_l\) left Kronecker indices \(\eta_i\), for \(i = 1, \ldots, \nu_l\) of the pencil \(S(\lambda)\) (corresponding to the \(\nu_l\) Kronecker blocks \(L_{\eta_i}(\lambda)\) of the form \(\eta_i \times (\eta_i + 1)\) which are part of the Kronecker canonical form of the pencil \(S(\lambda)\), as in (16));

54
• the orders of the elementary infinite blocks $s_i^\infty$, for $i = 1, \ldots, h$, (i.e., the dimensions of the elementary infinite Jordan blocks in (10) in the Weierstrass canonical form of the regular part (15)).

The normal rank of $S(\lambda)$ is determined as

$$\text{rank } S(\lambda) = n_f + \sum_{i=1}^{h} s_i^\infty + \sum_{i=1}^{\nu_r} \epsilon_i + \sum_{i=1}^{\nu_l} \eta_i.$$

### Alternative computation of normal rank

The normal rank of a LTI system $\text{sys}$ can be alternatively computed by evaluating the rank of the TFM $G(\lambda)$ at a random frequency (or taking the maximum rank for a few random frequencies). The following command can be used for this purpose:

$$\text{nr} = \text{rank} (\text{evalfr}(\text{sys}, \text{rand}), \text{tol})$$

#### 3.3.4 \texttt{ghanorm}

**Syntax**

$$[\text{HANORM}, \text{HS}] = \text{ghanorm}(\text{SYS})$$

**Description**

\texttt{ghanorm} computes for a proper and stable LTI state-space system $\text{SYS}$ with the transfer function matrix $G(\lambda)$, the Hankel norm $\|G(\lambda)\|_H$ and the Hankel singular values of the system.

**Input data**

$\text{SYS}$ is a LTI system, in a descriptor system state-space form

$$E\lambda x(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$

(102)

**Output data**

$\text{HANORM}$ is the Hankel norm $\|G(\lambda)\|_H$ of the transfer function matrix $G(\lambda)$ of the LTI system $\text{SYS}$.

$\text{HS}$ is a column vector which contains the decreasingly ordered Hankel singular values of $\text{SYS}$.

**Method**

Let $G(\lambda)$ be the TFM of the LTI $\text{SYS}$. For the definition and computation of the Hankel norm of a proper and stable rational TFM $G(\lambda)$ see Section 2.12 and for the definition and computation of the Hankel singular values, see Section 2.13. If the original descriptor system is proper, but $E$ is singular, then an automatic conversion is performed using the function $\text{gss2ss}$, to a reduced order descriptor state-space representation with $E$ invertible and upper triangular. For the solution of the intervening generalized Lyapunov equation (72) or generalized Stein equation (73), the MEX-function $\text{sl\_glme}$ is called.
3.4 Functions for System Order Reduction

The system order reduction functions cover the computation of irreducible or minimal realizations, the balancing-related model reduction, and the conversion of descriptor systems representation to various SVD-like coordinate forms without non-dynamic modes, including the conversion to standard state-space forms.

3.4.1 gir

Syntax

SYSR = gir(SYS)
SYSR = gir(SYS,TOL)
SYSR = gir(SYS,TOL,JOBOPT)

Description

gir computes for a LTI descriptor state-space system \((A - \lambda E, B, C, D)\), a reduced order (e.g., controllable, observable, or irreducible) descriptor realization \((\tilde{A} - \lambda \tilde{E}, \tilde{B}, \tilde{C}, D)\), such that the corresponding transfer function matrices are equal.

Input data

SYS is a LTI system, in a descriptor system state-space form

\[
\begin{align*}
E\lambda x(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t).
\end{align*}
\] (103)

TOL is a relative tolerance used for rank determinations. If TOL is not specified as input or if TOL = 0, an internally computed default value is used.

JOBOPT is a character option variable to specify various order reduction options, as follows:

- 'irreducible' – compute an irreducible descriptor realization (default);
- 'finite' – compute a finite controllable and finite observable realization
- 'infinite' – compute an infinite controllable and infinite observable realization
- 'contr' – compute a controllable realization
- 'obs' – compute an observable realization
- 'finite_contr' – compute a finite controllable realization
- 'infinite_contr' – compute an infinite controllable realization
- 'finite_obs' – compute a finite observable realization
- 'infinite_obs' – compute an infinite observable realization

56
Output data

SYSR contains the resulting reduced order system in a descriptor system state-space form

\[
\tilde{E}\lambda\tilde{x}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t), \\
y(t) = \tilde{C}\tilde{x}(t) + Du(t).
\]

SYSR has the same TFM as SYS and the resulting order of SYSR depends on the order reduction option selected via JOBOPT. If no order reduction takes place, then SYSR has the same realization as SYS.

Remark on input and output data

The function gir accepts an array of LTI systems SYS as input parameter. In this case, SYSR is also an array of LTI systems of the same size as SYS. To each component system in SYS(:,:,i) corresponds a component system SYSR(:,:,i) with a reduced order.

Method

Let \(G(\lambda)\) be the TFM of the LTI SYS with the descriptor realization (103). The conditions for finite and infinite controllability and observability are given by Theorem 2, as conditions (i) – (iv). The concepts of finite controllable and observable eigenvalues of the pencil \(A - \lambda E\) are discussed in Section 2.6. The elimination of uncontrollable or unobservable eigenvalues is done by determining the so-called Kalman controllability or Kalman observability forms, which exhibit explicitly these eigenvalues. For the computation of the controllability Kalman form for a descriptor system representation, the Procedure GCSF described in [58] is employed, which employs the orthogonal reduction technique proposed in [44]. This algorithm computes for a pair \((A - \lambda E, B)\), orthogonal transformation matrices \(Q\) and \(Z\) such that the matrices of the transformed triple \((\tilde{A} - \lambda \tilde{E}, \tilde{B}, \tilde{C}) := (Q^T AZ - \lambda Q^T EZ, Q^T B, CZ)\) have the form

\[
\tilde{A} - \lambda \tilde{E} = \begin{bmatrix}
A_c - \lambda E_c & * \\
0 & A_e - \lambda E_e
\end{bmatrix}, \quad \tilde{B} = \begin{bmatrix}
B_c \\
0
\end{bmatrix}, \quad \tilde{C} = [C_c *],
\]

where the pair \((A_c - \lambda E_c, B_c)\) is finite controllable, and \(\Lambda(A_e - \lambda E_e)\) contains the finite uncontrollable eigenvalues (as well as possible some infinite uncontrollable eigenvalues too). The reduced order system \((A_c - \lambda E_c, B_c, C_c, D)\) is finite controllable and has the same TFM \(G(\lambda)\) as the original system. In this way, all uncontrollable finite eigenvalues contained in \(A_e - \lambda E_e\) have been removed from the system model. By applying the same algorithm to the dual realization \((A^T - \lambda E^T, C^T, B^T, D^T)\), the finite unobservable eigenvalues can be removed (as the finite uncontrollable eigenvalues of the dual system). If the matrices \(A\) and \(E\) are interchanged, by forming a model with \((E - \lambda A, B, C, D)\), then the uncontrollable or unobservable null eigenvalues can be removed using the same algorithms. However, by removing the uncontrollable or unobservable null eigenvalues of this model, we remove in fact the infinite uncontrollable or unobservable eigenvalues of the original model. An irreducible (controllable and observable) realization can be thus computed in four steps, which form the Procedure GIR described in [58]. The computational method to compute an irreducible realization or to perform only a specific step of the Procedure GIR is implemented in the MEX-function sl_gminr, which is called, with appropriately set options, by the function gir.
Application example

The function `gir` displays messages indicating the number of uncontrollable and the number of unobservable eigenvalues removed from the model. This verbose output (especially when applied to array of systems) can be disabled as shown in the example below:

```matlab
warning off
sysr = gir(sys);
warning on
```

3.4.2 gminreal

Syntax

```matlab
[SYSMIN,INFO] = gminreal(SYS)
[SYSMIN,INFO] = gminreal(SYS,TOL)
[SYSMIN,INFO] = gminreal(SYS,TOL,NDONLYFLAG)
```

Description

gminreal computes for a LTI descriptor state-space system \((A - \lambda E, B, C, D)\), a minimal order descriptor realization \((A_m - \lambda E_m, B_m, C_m, D_m)\) (i.e., controllable, observable, without non-dynamic modes), such that the corresponding transfer function matrices are equal.

Input data

SYS is a LTI system, in a descriptor system state-space form

\[
E\lambda x(t) = Ax(t) + Bu(t),
\]
\[
y(t) = Cx(t) + Du(t).
\]

TOL is a relative tolerance used for rank determinations. If TOL is not specified as input or if TOL = 0, an internally computed default value is used.

NDONLYFLAG is a character option variable to be set to ‘ndonly’ to remove only the non-dynamic modes (i.e., simple infinite eigenvalues). By default, the non-dynamic modes are jointly removed with the uncontrollable and unobservable eigenvalues.

Output data

SYSMIN contains the resulting minimal order system in a descriptor system state-space form

\[
E_m\lambda x_m(t) = A_m x_m(t) + B_m u(t),
\]
\[
y_m(t) = C_m x_m(t) + D_m u(t).
\]

SYSMIN has the same TFM as SYS. If NDONLYFLAG = ‘ndonly’ is specified, then the resulting SYSMIN contains a reduced order system without non-dynamic modes. If no order reduction takes place, then SYSMIN has the same realization as SYS.
INFO(1:3) contains information on the number of removed eigenvalues, as follows:
- INFO(1) – the number of removed uncontrollable eigenvalues;
- INFO(2) – the number of removed unobservable eigenvalues;
- INFO(3) – the number of removed non-dynamic (infinite) eigenvalues.

**Method**

The conditions for minimality are given by Theorem 2, as conditions $(i) - (v)$. The concepts of finite controllable and observable eigenvalues of the pencil $A - \lambda E$ are discussed in Section 2.6. The elimination of uncontrollable and unobservable eigenvalues is done in several steps, by determining appropriate Kalman controllability and observability forms, which explicitly exhibit these eigenvalues. The basic computation is the reduction of the descriptor system matrices to the controllability Kalman form using the orthogonal transformation based reduction technique proposed in [44] (see also Procedure GCSF described in [58]). For more details on the computation of irreducible realizations, see the description of Method for the function `gir`.

To remove the non-dynamic infinite eigenvalues of a descriptor system $(A - \lambda E, B, C, D)$, the system matrices are reduced to a SVD-like coordinate form

$$\tilde{A} - \lambda \tilde{E}, \tilde{B}, \tilde{C}, D) := (Q^T AZ - \lambda Q^T E Z, Q^T B, C Z, D),$$

where

$$\tilde{A} - \lambda \tilde{E} = \begin{bmatrix}
A_{11} - \lambda E_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & 0 \\
A_{31} & 0 & 0
\end{bmatrix}, \quad \tilde{B} = \begin{bmatrix}
B_1 \\
B_2 \\
B_3
\end{bmatrix},$$

$$\tilde{C} = \begin{bmatrix}
C_1 \\
C_2 \\
C_3
\end{bmatrix},$$

with $E_{11}$ and $A_{22}$ upper triangular invertible matrices. Then, the reduced descriptor system without non-dynamic modes $(A_m - \lambda E_m, B_m, C_m, D_m)$ is computed as

$$A_m - \lambda E_m = \begin{bmatrix}
A_{11} - A_{12} A_{22}^{-1} A_{21} - \lambda E_{11} & A_{13} \\
A_{31}
\end{bmatrix},$$

$$B_m = \begin{bmatrix}
B_1 - A_{12} A_{22}^{-1} B_2 \\
B_3
\end{bmatrix},$$

$$C_m = \begin{bmatrix}
C_1 - C_2 A_{22}^{-1} A_{21} \\
C_3
\end{bmatrix},$$

$$D_m = D - C_2 A_{22}^{-1} B_2.$$

The computational method to determine a minimal order descriptor system realization or a reduced order realization without non-dynamic modes is implemented in the MEX-function `s1_gminr`, which is called, with appropriately set options, by the function `gminreal`.

**Application example**

The function `gminreal` displays messages indicating the number of eigenvalues (uncontrollable, unobservable, non-dynamic) removed from the model. This verbose output can be disabled as shown in the example below:

```matlab
warning off
sysmin = gminreal(sys);
warning on
```
3.4.3 gbalmr

Syntax

SYSR, HS = gbalmr(SYS)
SYSR, HS = gbalmr(SYS, TOL)
SYSR, HS = gbalmr(SYS, TOL, BALANCE)

Description

gbalmr performs model reduction of a stable LTI state-space system using balancing-related methods.

Input data

SYS is a stable LTI system, in a descriptor system state-space form

\[
\begin{align*}
E \lambda x(t) &= A x(t) + B u(t), \\
y(t) &= C x(t) + D u(t).
\end{align*}
\]  

(107)

TOL is a relative tolerance used to determine the order of the reduced model. If TOL is not specified as input, or if TOL is empty, or if TOL = 0, then the value TOL = sqrt(eps) is internally used.

BALANCE is a character option variable to be set to 'balance' to compute a balanced realization of the reduced order model. By default, the state-space realization of the computed reduced order model is not balanced.

Output data

SYSR contains the minimal realization of the resulting reduced order system in a descriptor system state-space form

\[
\begin{align*}
E_r \lambda x_r(t) &= A_r x_r(t) + B_r u(t), \\
y_r(t) &= C_r x_r(t) + D_r u(t).
\end{align*}
\]  

(108)

The realization of SYSR is balanced (i.e., \(E_r = I\) and the controllability and observability Gramians are equal and diagonal) if the balancing option BALANCE = 'balance' has been specified. The order of SYSR is the number of Hankel singular values of SYS, which are greater than TOL \|G(\lambda)\|_H, where \(G(\lambda)\) is the TFM of the system (107).

HS is a column vector which contains the decreasingly ordered Hankel singular values of SYS.

Method

For the order reduction of a standard system (i.e., with \(E = I\)), the balancing-free method of [45] or the balancing-based method of [37] are used. For a descriptor system the balancing related order reduction methods of [35] are used. For the solution of the intervening Lyapunov and Stein equation for the Cholesky factors of the solution, the mex-function sl_glme is used.
3.4.4 gss2ss

Syntax

\[
[\text{SYSR, RANKE}] = \text{gss2ss}(\text{SYS}) \\
[\text{SYSR, RANKE}] = \text{gss2ss}(\text{SYS}, \text{TOL}) \\
[\text{SYSR, RANKE}] = \text{gss2ss}(\text{SYS}, \text{TOL}, \text{ESHAPE})
\]

Description

gss2ss performs the conversion of a LTI descriptor state-space systems \((A - \lambda E, B, C, D)\) to a SVD-like coordinate form \((A_r - \lambda E_r, B_r, C_r, D_r)\) without non-dynamic modes, such that the corresponding transfer function matrices are equal.

Input data

SYS is a LTI system, in a descriptor system state-space form

\[
\begin{align*}
E\lambda x(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t).
\end{align*}
\]  

(109)

TOL is a relative tolerance used for rank determinations. If TOL is not specified as input or if \(\text{TOL} = 0\), an internally computed default value is used.

ESHAPE is a character option variable to specify the shape of the leading invertible diagonal block \(E_{11}\) of the resulting descriptor matrix \(E_r = \text{diag}(E_{11}, 0)\) (see Method). The following options can be used for ESHAPE:

- ‘diag’ – diagonal (the nonzero diagonal elements are the decreasingly ordered nonzero singular values of \(E\));
- ‘triu’ – upper triangular;
- ‘ident’ – identity (default).

Output data

SYSR contains the resulting reduced order system without non-dynamic modes, in a descriptor system state-space form

\[
\begin{align*}
E_r\lambda x_r(t) &= A_r x_r(t) + B_r u(t), \\
y_r(t) &= C_r x_r(t) + D_r u(t),
\end{align*}
\]  

(110)

where \(E_r\) has a block-diagonal form \(E_r = \text{diag}(E_{11}, 0)\), with \(E_{11}\) invertible. The resulting shape of \(E_{11}\) is in accordance with the specified option by ESHAPE. SYSR has the same TFM as SYS and the resulting order of SYSR is the order of SYS minus the number of simple infinite (non-dynamic) eigenvalues of the pole pencil \(A - \lambda E\).

RANKE is the rank of \(E\) (and also the order of \(E_{11}\)).
Method

To remove the non-dynamic eigenvalues of the descriptor system \((A - \lambda E, B, C, D)\), the system matrices are first reduced using non-orthogonal transformation matrices \(Q\) and \(Z\) to a SVD-like coordinate form

\[
(\tilde{A} - \lambda \tilde{E}, \tilde{B}, \tilde{C}, D) := (QAZ - \lambda QEZ, QB, CZ, D),
\]

where

\[
\tilde{A} - \lambda \tilde{E} = \begin{bmatrix}
A_{11} - \lambda E_{11} & A_{12} & A_{13} \\
A_{21} & I & 0 \\
A_{31} & 0 & 0
\end{bmatrix}, \\
\tilde{B} = \begin{bmatrix}
B_1 \\
B_2 \\
B_3
\end{bmatrix}, \\
\tilde{C} = \begin{bmatrix}
C_1 \\
C_2 \\
C_3
\end{bmatrix},
\]

with \(E_{11}\) invertible and either upper triangular (if \(\text{ESHAPE} = \text{'triu'}\)) or diagonal (if \(\text{ESHAPE} = \text{'diag'}\) or \(\text{ESHAPE} = \text{'identity'}\)). To compute the above SVD-like form, the MEX-function \(\text{sl_gstra}\) is called by the function \(\text{gss2ss}\), with appropriately set options.

If \(\text{ESHAPE} = \text{'triu'}\) or \(\text{ESHAPE} = \text{'diag'}\), the reduced descriptor system without non-dynamic modes \((A_r - \lambda E_r, B_r, C_r, D_r)\) is computed as

\[
A_r - \lambda E_r = \begin{bmatrix}
A_{11} - A_{12} A_{21} - \lambda E_{11} & A_{13} \\
A_{31} & 0
\end{bmatrix}, \\
B_r = \begin{bmatrix}
B_1 - A_{12} B_2 \\
B_3
\end{bmatrix}, \\
C_r = \begin{bmatrix}
C_1 - C_2 A_{21} & C_3
\end{bmatrix}, \\
D_r = D - C_2 B_2.
\]

If \(\text{ESHAPE} = \text{'identity'}\), then the above matrices are computed as

\[
A_r - \lambda E_r = \begin{bmatrix}
E_{11}^{-1/2}(A_{11} - A_{12} A_{21})E_{11}^{-1/2} - \lambda I & E_{11}^{-1/2} A_{13} \\
A_{31} E_{11}^{-1/2} & 0
\end{bmatrix}, \\
B_r = \begin{bmatrix}
E_{11}^{-1/2}(B_1 - A_{12} B_2) \\
B_3
\end{bmatrix}, \\
C_r = \begin{bmatrix}
(C_1 - C_2 A_{21}) E_{11}^{-1/2} & C_3
\end{bmatrix}, \\
D_r = D - C_2 B_2.
\]

The particular case of an invertible \(E\) and with the pair \((A, E)\) in a generalized Hessenberg form (i.e., with \(A\) in upper Hessenberg form and \(E\) upper triangular) is handled separately, in order to preserve the Hessenberg form of \(A\). In this case, with the additional assumption that all diagonal elements of \(E\) are positive (this can be easily arranged by changing the signs of the corresponding rows of \(E, A\) and \(B\)), the matrices of the standard state-space realization \((A_r - \lambda I, B_r, C_r, D_r)\) are computed with

\[
A_r = E^{-1/2} A E^{-1/2}, \quad B_r = E^{-1/2} B, \quad C_r = C E^{-1/2}, \quad (111)
\]

where \(E^{-1/2}\) is the square root of \(E\) computed using the method described in [19, Algorithm 6.7] (implemented in the MATLAB function \(\text{sqrtm}\)).
3.5 Functions for Operations on Generalized LTI Systems

These functions cover the computation of rational nullspace and range space bases, the solution of linear rational equations, the computation of additive spectral decompositions and order reductions using minimal dynamic cover based techniques.

3.5.1 grnull

Syntax

\[ [\text{SYSRNULL}, \text{INFO}] = \text{grnull} (\text{SYS}, \text{OPTIONS}) \]

Description

\text{grnull} computes a proper rational basis \( N_r(\lambda) \) of the right nullspace of the transfer function matrix \( G_1(\lambda) \) of a LTI descriptor system, such that

\[ G_1(\lambda)N_r(\lambda) = 0 \]

and determines \( G_2(\lambda)N_r(\lambda) \), where \( G_2(\lambda) \) is a TFM having the same number of columns as \( G_1(\lambda) \).

Input data

\text{SYS} is an output concatenated compound LTI system, \( \text{SYS} = [ \text{SYS1}; \text{SYS2} ] \), in a descriptor system state-space form

\[
\begin{align*}
E \lambda x(t) &= Ax(t) + Bu(t), \\
y_1(t) &= C_1 x(t) + D_1 u(t), \\
y_2(t) &= C_2 x(t) + D_2 u(t),
\end{align*}
\]

where \( \text{SYS1} \) has the transfer function matrix \( G_1(\lambda) \) with the descriptor system realization \((A - \lambda E, B, C_1, D_1)\), \( \text{SYS2} \) has the transfer function matrix \( G_2(\lambda) \) with the descriptor system realization \((A - \lambda E, B, C_2, D_2)\), and \( y_1(t) \in \mathbb{R}^{p_1} \) and \( y_2(t) \in \mathbb{R}^{p_2} \) are the outputs of \( \text{SYS1} \) and \( \text{SYS2} \), respectively.

\text{OPTIONS} is a MATLAB structure to specify user options and has the following fields:

| OPTIONS fields | Description |
|----------------|-------------|
| tol | relative tolerance for rank computations (Default: internally computed) |
| p2 | \( p_2 \), the number of outputs of \( \text{SYS2} \) (Default: \( p_2 = 0 \)) |
| simple | option to compute a simple proper basis: |
| true | compute a simple basis; the orders of the basis vectors are provided in \text{INFO}.degs; |
| false | no simple basis computed (default) |
| tcond | maximum allowed value for the condition numbers of the employed non-orthogonal transformation matrices (Default: \( 10^4 \)) (only used if \text{OPTIONS}.simple = true) |
### sdeg
prescribed stability degree for the resulting right nullspace basis
(Default: [ ])

### poles
a complex conjugated set of desired poles to be assigned for the
resulting right nullspace basis (Default: [ ])

## Output data

SYSRNULL contains the output concatenated compound system \([ \text{NR; SYS2*NR} ]\), in a descriptor system state-space form

\[
\begin{align*}
\bar{E}_r \lambda x_r(t) &= \bar{A}_r x_r(t) + \bar{B}_r v(t), \\
y_{r,1}(t) &= \bar{C}_{r,1} x_r(t) + \bar{D}_{r,1} v(t), \\
y_{r,2}(t) &= \bar{C}_{r,2} x_r(t) + \bar{D}_{r,2} v(t),
\end{align*}
\]

where NR is the descriptor realization \((\bar{A}_r - \lambda \bar{E}_r, \bar{B}_r, \bar{C}_{r,1}, \bar{D}_{r,1})\) of the right nullspace basis \(N_r(\lambda)\) of the transfer function matrix \(G_1(\lambda)\), and SYS2*NR (the series coupling of SYS2 and NR) is the descriptor system realization \((\bar{A}_r - \lambda \bar{E}_r, \bar{B}_r, \bar{C}_{r,2}, \bar{D}_{r,2})\) of the transfer function matrix \(G_2(\lambda)N_r(\lambda)\).

INFO is a MATLAB structure containing additional information, as follows:

| INFO fields | Description |
|-------------|-------------|
| nrank       | normal rank of the transfer function matrix of SYS1; |
| stdim       | dimensions of the diagonal blocks of \(\bar{A}_r - \lambda \bar{E}_r\):  
if OPTIONS.simple = false, these are the row dimensions of the full row rank subdiagonal blocks of the pencil \([\bar{B}_r \bar{A}_r - \lambda \bar{E}_r]\) in controllability staircase form;  
if OPTIONS.simple = true, these are the orders of the state-space realizations of the proper rational vectors of the computed simple proper rational right nullspace basis of SYS1; |
| degs        | increasingly ordered degrees of the vectors of a polynomial right nullspace basis of the transfer function matrix \(G_1(\lambda)\) of SYS1, representing the right Kronecker indices of \(G_1(\lambda)\); also the orders of the realizations of the proper rational vectors of a simple proper rational right nullspace basis. If OPTIONS.simple = true, INFO.deg(i) is the dimension of the i-th diagonal blocks of \(\bar{A}_r\) and \(\bar{E}_r\). |
| tcond       | maximum of the condition numbers of the employed non-orthogonal transformation matrices; a warning is issued if INFO.tcond \(\geq\) OPTIONS.tcond. |
| fnorm       | the norm of the employed state-feedback used for stabilization; is zero if both OPTION.sdeg and OPTIONS.pole are empty. |
Method

For the definitions related to minimal nullspace bases see Section 2.4. In what follows, we sketch the approach to compute minimal proper rational right nullspace bases proposed in [50] (see also [58, Section 10.3.2] for more details). This approach is employed if OPTIONS.simple = false.

Let $G_1(\lambda)$ be the $p_1 \times m$ TFM of SYS1 and let $G_2(\lambda)$ be the $p_2 \times m$ TFM of SYS2. Assume $r = \text{rank} G_1(\lambda)$ is the normal rank of $G_1(\lambda)$. Let $N_r(\lambda)$ be a $m \times (m-r)$ rational left nullspace basis of $G_1(\lambda)$ satisfying

\[ G_1(\lambda)N_r(\lambda) = 0. \]

Such a basis can be computed as

\[ N_2(\lambda) = [0 I_m]Y_r(\lambda), \]

where $Y_r(\lambda)$ is a rational basis of the right nullspace of the system matrix pencil

\[ S(\lambda) = \left[ \begin{array}{cc} A - \lambda E & B \\ C_1 & D_1 \end{array} \right]. \]

The right nullspace $Y_r(\lambda)$ is determined using the Kronecker-like staircase form of $S(\lambda)$ computed as

\[ Q^T S(\lambda) Z = \left[ \begin{array}{ccc} B_r & A_r - \lambda E_r & * \\ 0 & 0 & A_l - \lambda E_l \end{array} \right], \]

where $Q$ and $Z$ are orthogonal transformation matrices, the subpencil $A_l - \lambda E_l$ contains the right Kronecker structure and the regular part of $S(\lambda)$, and the subpencil $[B_r \ A_r - \lambda E_r]$ contains the right Kronecker structure of $S(\lambda)$. $[B_r \ A_r]$ is obtained in a controllability staircase form with $[B_r \ A_r]$ as in (22) and $E_r$ upper triangular and nonsingular, as in (23). $Y_r(\lambda)$ results as

\[ Y_r(\lambda) = Z \left[ \begin{array}{c} I \\ (\lambda E_r - A_r - B_r F)^{-1} B_r \\ 0 \end{array} \right], \]

where $F$ is a stabilizing state feedback ($F = 0$ if both OPTION.sdeg and OPTIONS.pole are empty). The rational basis $N_r(\lambda)$ results using (114) with the controllable state-space realization

\[ (\tilde{A}_r - \lambda \tilde{E}_r, \tilde{B}_r, \tilde{C}_{r,1}, \tilde{D}_{r,1}) := (A_r + B_r F - \lambda E_r, B_r, C_{r,1} + D_{r,1} F, D_{r,1}), \]

where

\[ \begin{bmatrix} * & C_{r,1} \\ D_{r,1} \end{bmatrix} := [0 \ I_m] Z. \]

The resulting basis is column proper, that is, $D_{r,1}$ has full column rank. The descriptor realization of $G_2(\lambda)N_r(\lambda)$ is obtained as

\[ (\tilde{A}_r - \lambda \tilde{E}_r, \tilde{B}_r, \tilde{C}_{r,2}, \tilde{D}_{r,2}) := (A_r + B_r F - \lambda E_r, B_r, C_{r,2} + D_{r,2} F, D_{r,2}), \]

where

\[ \begin{bmatrix} * & C_{r,2} \\ D_{r,2} \end{bmatrix} := [C_2 \ D_2] Z. \]

For the computation of the Kronecker-like form (116), the mex-function $\text{sl_klf}$, based on the algorithm proposed in [2], is called by the function $\text{grnull}$. INFO.stdim contains the dimensions $\nu_j$, $j = 1, \ldots, k$ of the diagonal blocks in the staircase form (22). INFO.degs contains the degrees
of a minimal polynomial left nullspace basis. These are the right Kronecker indices of the system matrix pencil $S(\lambda)$ in (115) and are determined as follows: there are $\nu_{i-1} - \nu_i$ vectors of degree $i - 1$, for $i = 1, \ldots, k$, where $\nu_0 := m - r$.

If OPTIONS.simple = true, a simple proper left nullspace basis is computed, using the method of [54] to determine a simple basis from a proper basis as computed above. The employed dynamic covers based algorithm relies on performing non-orthogonal similarity transformations. The estimated maximum condition number used in these computations is provided in INFO.tcond. For a simple proper basis, $\tilde{A}_r$, $\tilde{E}_r$ and $\tilde{B}_r$ are block diagonal

$$\tilde{A}_r - \lambda \tilde{E}_r = \text{diag}(A^1_r - \lambda E^1_r, \ldots, A^k_r - \lambda E^k_r), \quad \tilde{B}_r = \text{diag}(B^1_r, \ldots, B^k_r),$$

with $\tilde{E}_r$ upper triangular. The state-space realization of the $i$-th basis (column) vector $v_i(\lambda)$ can be explicitly constructed as $(A^i_r - \lambda E^i_r, B^i_r, \tilde{C}_r, 1, D^i_r, 1)$, where $D^i_r, 1$ is the $i$-th column of $D^i_r, 1$. INFO.stdim contains the dimensions of the diagonal blocks $A^i_r - \lambda E^i_r$, $i = 1, \ldots, k$ and are equal to INFO.degs. The corresponding realization for $G_2(\lambda)v_i(\lambda)$ is constructed as $(A^i_r - \lambda E^i_r, B^i_r, \tilde{C}_r, 2, D^i_r, 2)$, where $D^i_r, 2$ is the $i$-th column of $D^i_r, 2$.

The resulting realization of SYSRNULL is minimal provided the realization of SYS is minimal. However, NR is a minimal proper basis only if the realization $(A - \lambda E, B, C_1, D_1)$ of SYS1 is minimal. In this case, INFO.degs are the degrees of the vectors of a minimal polynomial basis or, if OPTIONS.simple = true, of the resulting minimal simple proper basis.

Example

Example 2. Consider the transfer function matrix used in [22, p. 459]

$$G(s) = \begin{bmatrix} 1 & 0 & 1 & s \\ s & 0 & s & s \\ 0 & (s+1)^2 & (s+1)^2 & 0 \\ -1 & (s+1)^2 & s^2 + 2s & -s^2 \end{bmatrix},$$

which has normal rank $r = 2$ and the right Kronecker indices $\nu_1 = 0$ and $\nu_2 = 2$. A simple proper minimal right nullspace basis $N_r(s)$ with the poles assigned in $-1$ has been computed with the function grnull as

$$N_r(s) = \begin{bmatrix} -0.5547 & 0.70165s^2 \\ -0.5547 & -0.43853s^2 \\ 0.5547 & 0.43853s^2 \\ 0 & -1.14 \end{bmatrix} \begin{bmatrix} (s+1)^2 \\ (s+1)^2 \\ (s+1)^2 \\ (s+1)^2 \end{bmatrix}.$$
and has two basis vectors of McMillan degrees 0 and 2. A polynomial basis results simply by taking the numerator polynomial vectors

\[
\tilde{N}_r(s) = \begin{bmatrix}
-0.5547 & 0.70165s^2 \\
-0.5547 & -0.43853s^2 \\
0.5547 & 0.43853s^2 \\
0 & -1.14
\end{bmatrix}
\]

To compute \(N_r(s)\) and \(\tilde{N}_r(s)\), the following sequence of commands can be used:

```matlab
% Kailath (1980), page 459: rank 2 matrix
s = tf('s');
G = [1/s 0 1/s s;
    0 (s+1)^2 (s+1)^2 0;
    -1 (s+1)^2 s^2+2*s -s^2];
sys = gir(ss(G),1.e-7);
% set options for simple basis and pole assignment
options = struct('tol',1.e-7,'simple',true,'poles',[-1,-1]);
% compute a simple right nullspace basis Nr(s)
[Nr,info] = grnull(sys,options);
tf(Nr), rki = info.degs % right Kronecker indices
% check nullspace condition G(s)*Nr(s) = 0
gminreal(sys*Nr,1.e-7)
% minimal polynomial basis computation
Nrtf = tf(Nr);
Nrp = tf(Nrtf.num,1)
% check nullspace condition G(s)*Nrp(s) = 0
gminreal(sys*Nrp,1.e-7)
```

For examples illustrating the computational details of determining simple and polynomial bases, see Example 12 and Example 13, respectively.

### 3.5.2 \texttt{glnull}

**Syntax**

```
[SYSLNULL,INFO] = glnull(SYS,OPTIONS)
```
Description

glnull computes a proper rational basis $N_l(\lambda)$ of the left nullspace of the transfer function matrix $G_1(\lambda)$ of a LTI descriptor system, such that

$$N_l(\lambda)G_1(\lambda) = 0$$

and determines $N_l(\lambda)G_2(\lambda)$, where $G_2(\lambda)$ is a TFM having the same number of rows as $G_1(\lambda)$.

Input data

SYS is an input concatenated compound LTI system, $SYS = \begin{bmatrix} SYS_1 & SYS_2 \end{bmatrix}$, in a descriptor system state-space form

$$
E_\lambda x(t) = A x(t) + B_1 u_1(t) + B_2 u_2(t),
$$
$$
y(t) = C x(t) + D_1 u_1(t) + D_2 u_2(t),
$$

where $SYS_1$ has the transfer function matrix $G_1(\lambda)$ with the descriptor system realization $(A - \lambda E, B_1, C, D_1)$, $SYS_2$ has the transfer function matrix $G_2(\lambda)$ with the descriptor system realization $(A - \lambda E, B_2, C, D_2)$, and $u_1(t) \in \mathbb{R}^{m_1}$ and $u_2(t) \in \mathbb{R}^{m_2}$ are the inputs of $SYS_1$ and $SYS_2$, respectively.

OPTIONS is a MATLAB structure to specify user options and has the following fields:

| OPTIONS fields | Description |
|----------------|-------------|
| tol            | relative tolerance for rank computations (Default: internally computed); |
| m2             | $m_2$, the number of inputs of $SYS_2$ (Default: $m_2 = 0$); |
| simple         | option to compute a simple proper basis: |
|                | true – compute a simple basis; the orders of the basis vectors are provided in INFO.degs; |
|                | false – no simple basis computed (default); |
| tcond          | maximum allowed value for the condition numbers of the employed non-orthogonal transformation matrices (Default: $10^4$) |
|                | (only used if OPTIONS.simple = true); |
| sdeg           | prescribed stability degree for the resulting left nullspace basis |
|                | (Default: [ ]); |
| poles          | a complex conjugated set of desired poles to be assigned for the resulting left nullspace basis |
|                | (Default: [ ]). |

Output data

SYSLNULL contains the input concatenated compound system $\begin{bmatrix} NL & NL\cdot SYS_2 \end{bmatrix}$, in a descriptor system state-space form

$$
E_\lambda x_l(t) = \tilde{A}_l x_l(t) + \tilde{B}_{l,1} v_1(t) + \tilde{B}_{l,2} v_2(t),
$$
$$
y_l(t) = \tilde{C}_l x_l(t) + \tilde{D}_{l,1} v_1(t) + \tilde{D}_{l,2} v_2(t),
$$

(119)
where NL is the descriptor system realization \((\tilde{A}_l - \lambda \tilde{E}_l, \tilde{B}_{l,1}, \tilde{C}_l, \tilde{D}_{l,1})\) of the left nullspace basis \(N_l(\lambda)\) of the transfer function matrix \(G_1(\lambda)\), and NL*SYS2 (the series coupling of NL and SYS2) is the descriptor system realization \((\tilde{A}_l - \lambda \tilde{E}_l, \tilde{B}_{l,2}, \tilde{C}_l, \tilde{D}_{l,2})\) of the transfer function matrix \(N_l(\lambda)G_2(\lambda)\).

INFO is a MATLAB structure containing additional information, as follows:

| FIELD | DESCRIPTION |
|-------|-------------|
| nrank | normal rank of the transfer function matrix of SYS1; |
| stdim | dimensions of the diagonal blocks of \(\tilde{A}_l - \lambda \tilde{E}_l\): if OPTIONS.simple = false, these are the column dimensions of the full column rank subdiagonal blocks of the pencil \(\begin{bmatrix} \tilde{A}_l - \lambda \tilde{E}_l \\ \tilde{C}_l \end{bmatrix}\) in observability staircase form; if OPTIONS.simple = true, these are the orders of the state-space realizations of the proper rational vectors of the computed simple proper rational left nullspace basis of SYS1; |
| degs | increasingly ordered degrees of the vectors of a polynomial left nullspace basis of the transfer function matrix \(G_1(\lambda)\) of SYS1, representing the left Kronecker indices of \(G_1(\lambda)\); also the orders of the realizations of the proper rational vectors of a simple proper rational left nullspace basis. If OPTIONS.simple = true, INFO.deg(i) is the dimension of the \(i\)-th diagonal blocks of \(\tilde{A}_l\) and \(\tilde{E}_l\). |
| tcond | maximum of the condition numbers of the employed non-orthogonal transformation matrices; a warning is issued if INFO.tcond \(\geq\) OPTIONS.tcond. |

Method

For the definitions related to minimal nullspace bases see Section 2.4. In what follows, we sketch the approach to compute minimal proper rational left nullspace bases, which is the dual version of the method proposed in [50] (see also [58, Section 10.3.2] for more details). This approach is employed if OPTIONS.simple = false.

Let \(G_1(\lambda)\) be the \(p \times m_1\) TFM of SYS1 and let \(G_2(\lambda)\) be the \(p \times m_2\) TFM of SYS2. Assume \(r = \text{rank} G_1(\lambda)\) is the normal rank of \(G_1(\lambda)\). Let \(N_l(\lambda)\) be a \((p - r) \times p\) rational left nullspace basis of \(G_1(\lambda)\) satisfying

\[N_l(\lambda)G_1(\lambda) = 0.\]

Such a basis can be computed as

\[N_l(\lambda) = Y_l(\lambda) \begin{bmatrix} 0 \\ I_p \end{bmatrix},\] (120)

where \(Y_l(\lambda)\) is a rational basis of the left nullspace of the system matrix pencil

\[S(\lambda) = \begin{bmatrix} A - \lambda E & B_1 \\ C & D_1 \end{bmatrix}.\] (121)
The left nullspace $Y_l(\lambda)$ is determined using the Kronecker-like staircase form of $S(\lambda)$ computed as

$$Q^T S(\lambda) Z = \begin{bmatrix} A_r - \lambda E_r & * \\ 0 & A_l - \lambda E_l \end{bmatrix}, \quad (122)$$

where $Q$ and $Z$ are orthogonal transformation matrices, the subpencil $A_r - \lambda E_r$ contains the right Kronecker structure and the regular part of $S(\lambda)$, and the subpencil $\begin{bmatrix} A_l - \lambda E_l \\ C_l \end{bmatrix}$ contains the left Kronecker structure of $S(\lambda)$. $\begin{bmatrix} A_l - \lambda E_l \\ C_l \end{bmatrix}$ is obtained in an observability staircase form with $\begin{bmatrix} A_l \\ C_l \end{bmatrix}$ as in (24) and $E_l$ upper triangular and nonsingular, as in (25). $Y_l(\lambda)$ results as

$$Y_l(\lambda) = \begin{bmatrix} 0 \\ C_l (\lambda E_l - A_l - FC_l)^{-1} \end{bmatrix} Q^T,$$

where $F$ is a stabilizing output injection matrix. The rational basis $N_l(\lambda)$ results using (120) with the observable state-space realization

$$(\tilde{A}_l - \lambda \tilde{E}_l, \tilde{B}_{l,1}, \tilde{C}_l, \tilde{D}_{l,1}) := (A_l + FC_l - \lambda E_l, B_{l,1} + FD_{l,1}, C_l, D_{l,1}),$$

where

$$\begin{bmatrix} * \\ B_{l,1} \\ D_{l,1} \end{bmatrix} := Q^T \begin{bmatrix} 0 \\ I_p \end{bmatrix}. $$

The resulting basis is row proper, that is, $D_{l,1}$ has full row rank. The descriptor realization of $N_l(\lambda) G_2(\lambda)$ is obtained as

$$(\tilde{A}_l - \lambda \tilde{E}_l, \tilde{B}_{l,2}, \tilde{C}_l, \tilde{D}_{l,2}) := (A_l + FC_l - \lambda E_l, B_{l,2} + FD_{l,2}, C_l, D_{l,2}),$$

where

$$\begin{bmatrix} * \\ B_{l,2} \\ D_{l,2} \end{bmatrix} := Q^T \begin{bmatrix} B_2 \\ D_2 \end{bmatrix}. $$

For the computation of the Kronecker-like form (122), the mex-function sl_klf, based on the algorithm proposed in [2], is called by the function glnull. INFO.stdim contains the dimensions $\mu_j, j = 1, \ldots, \ell$ of the diagonal blocks in the staircase form (24). INFO.degs contains the degrees of a minimal polynomial left nullspace basis. These are the left Kronecker indices of the system matrix pencil $S(\lambda)$ in (121) and are determined as follows: there are $\mu_{i-1} - \mu_i$ vectors of degree $i - 1$, for $i = 1, \ldots, \ell$, where $\mu_0 := p - r$.

If OPTIONS.simple = true, a simple proper left nullspace basis is computed, using the method of [54] to determine a simple basis from a proper basis as computed above. The employed dynamic covers based algorithm relies on performing non-orthogonal similarity transformations. The estimated maximum condition number used in these computations is provided in INFU.tcond. For a simple proper basis, $A_l$, $E_l$ and $C_l$ are block diagonal

$$\tilde{A}_l - \lambda \tilde{E}_l = \text{diag}(A^1_1 - \lambda E^1_1, \ldots, A^\ell_1 - \lambda E^\ell_1), \quad \tilde{C}_l = \text{diag}(C^1_l, \ldots, C^\ell_l),$$

with $E_l$ upper triangular. The state-space realization of the $i$-th basis (row) vector $v_i(\lambda)$ can be explicitly constructed as $(A^i_l - \lambda E^i_l, B_{l,1}, C^i_l, D^i_{l,1})$, where $D^i_{l,1}$ is the $i$-th row of $D_{l,1}$.  

70
INFO.stdim contains the dimensions of the diagonal blocks $A_i^l - \lambda E_i^l$, $i = 1, \ldots, \ell$ and are equal to INFO.degs. The corresponding descriptor system realization for $v_i(\lambda)G_2(\lambda)$ is constructed as $(A_i^l - \lambda E_i^l, B_i^2, C_i^l, D_i^l)$, where $D_i^l$ is the $i$-th row of $D_i^2$.

The resulting realization of SYSLNULL is minimal provided the realization of SYS is minimal. However, NL is a minimal proper basis only if the realization $(A - \lambda E, B_1, C, D_1)$ of SYS1 is minimal. In this case, INFO.degs are the degrees of the vectors of a minimal polynomial basis or, if OPTIONS.simple = true, of the resulting minimal simple proper basis.

Example

Example 3. Consider the $3 \times 4$ transfer function matrix (117) from [22, p. 459] used in Example 2. $G(s)$ has normal rank $r = 2$ and a left Kronecker index $\mu_1 = 1$. A proper minimal left nullspace basis $N_l(s)$ with the poles assigned in $-1$ has been computed with the function glnull as

$$N_l(s) = \begin{bmatrix} -\frac{s}{s+1} & 1 & -1 \\ s+1 & s+1 & s+1 \end{bmatrix}$$

and consists of a single basis vector of McMillan degree 1. To compute $N_l(s)$, the following sequence of commands can be used:
% Kailath (1980), page 459: rank 2 matrix
s = tf('s');
G = [1/s 0 1/s 0;
     0 (s+1)^2 (s+1)^2 0;
     -1 (s+1)^2 s^2+2*s -s^2];
sys = gir(ss(G),1.e-7);

% set options for simple basis and pole assignment
options = struct('tol',1.e-7,'simple',true,'poles',[-1]);

% compute a simple left nullspace basis Nl(s)
[Nl,info] = glnull(sys,options);
minreal(tf(Nl))

% check the nullspace condition Nl(s)*G(s) = 0
gminreal(Nl*sys,1.e-7)

3.5.3    grange

Syntax

[SYSR,SYSX] = grange(SYS,OPTIONS)

Description

grange computes a proper rational basis \( R(\lambda) \) of the range space of the transfer function matrix \( G(\lambda) \) of a LTI descriptor system, and a full row rank \( X(\lambda) \) such that

\[
G(\lambda) = R(\lambda)X(\lambda)
\]

is a full-rank factorization of \( G(\lambda) \).

Input data

SYS is a LTI system in a descriptor system state-space form

\[
\begin{align*}
E\lambda x(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t),
\end{align*}
\]

whose transfer function matrix is \( G(\lambda) \).

OPTIONS is a MATLAB structure to specify user options and has the following fields:

| OPTIONS fields | Description |
|---------------|-------------|
| tol           | relative tolerance for rank computations (Default: internally computed) |
**zeros**

Option for the selection of zeros to be included in the computed range space basis:
- 'none' – include no zeros (default)
- 'all' – include all zeros of SYS
- 'unstable' – include all unstable zeros of SYS
- 'stable' – include all stable zeros of SYS
- 'finite' – include all finite zeros of SYS
- 'infinite' – include all infinite zeros of SYS

**inner**

Option to compute an inner basis:
- true – compute an inner basis (only if OPTIONS.zeros = 'none' or OPTIONS.zeros = 'unstable');
- false – no inner basis is computed (default)

**balance**

Balancing option for the Riccati equation solvers (see functions `care` and `dare` of the Control System Toolbox):
- true – perform balancing (default);
- false – disable balancing.

---

**Output data**

`SYSR` contains the descriptor system state-space realization of the full column rank proper range basis matrix $R(\lambda)$ in the form

$$
E_R \lambda x_R(t) = A_R x_R(t) + B_R v(t),
$$

$$
y_R(t) = C_R x_R(t) + D_R v(t),
$$

(125)

where $E_R$ is invertible. The resulting $R(\lambda)$ contains the selected zeros of $G(\lambda)$ via the option parameter OPTIONS.zeros. $R(\lambda)$ is inner if OPTIONS.inner = true was selected. The dimension $r$ of the input vector $v(t)$ is the normal rank of $G(\lambda)$.

`SYSX` contains the descriptor system state-space realization of the full row rank transfer function matrix $X(\lambda)$ in the form

$$
E \lambda \tilde{x}(t) = A \tilde{x}(t) + B u(t),
$$

$$
\tilde{y}(t) = C \tilde{x}(t) + D u(t),
$$

(126)

where the dimension $r$ of the output vector $\tilde{y}(t)$ is the normal rank of $G(\lambda)$.

**Method**

Consider a disjunct partition of the complex plane $\mathbb{C}$ as

$$
\mathbb{C} = \mathbb{C}_g \cup \mathbb{C}_b, \quad \mathbb{C}_g \cap \mathbb{C}_b = \emptyset,
$$

(127)

where $\mathbb{C}_g$ and $\mathbb{C}_b$ are symmetric with respect to the real axis. $\mathbb{C}_g$ and $\mathbb{C}_b$ are associated with the “good” and “bad” domains of the complex plane $\mathbb{C}$ for the poles and zeros of $G(\lambda)$. Assume $G(\lambda)$ is a $p \times m$ real rational matrix of normal rank $r$, with a $\mathbb{C}_b$-stabilizable descriptor system.
realization \((124)\). Then, there exist two orthogonal matrices \(U\) and \(Z\) such that
\[
\begin{bmatrix}
U & 0 \\
0 & I
\end{bmatrix}
\begin{bmatrix}
A - \lambda E & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
U_r g - \lambda E_r g & * & * \\
0 & A_b l - \lambda E_b l & B_b l & * \\
0 & 0 & 0 & B_n
\end{bmatrix}
\]
\[
= \begin{bmatrix}
\text{Arg} - \lambda E & \text{Arg}^* & * & * \\
A_b l - \lambda E_b l & B_b l & * & * \\
0 & 0 & 0 & B_n
\end{bmatrix},
\]
\[(128)\]
where
(a) The pencil \(A_r g - \lambda E_r g\) has full row rank for \(\lambda \in C\) and \(E_r g\) has full row rank.
(b) \(E_b l\) and \(B_n\) are invertible, the pencil
\[
\begin{bmatrix}
A_b l - \lambda E_b l & B_b l \\
C_b l & D_b l
\end{bmatrix}
\]
has full column rank \(n_{b l} + r\) for \(\lambda \in C\) and the pair \((A_b l - \lambda E_b l, B_{b l})\) is \(C_b\)-stabilizable.

The range matrix of \(G(\lambda)\), which includes the zeros of \(G(\lambda)\) in \(C_b\), has the proper descriptor system realization
\[
R(\lambda) = \begin{bmatrix}\begin{array}{c}
A_{b l} - \lambda E_{b l} & B_{b l} \\
C_{b l} & D_{b l}
\end{array}\end{bmatrix}.
\]
\[(130)\]
If \(\text{OPTIONS.inner = true}\) was selected, the inner range matrix is determined in the form
\[
R(\lambda) = \begin{bmatrix}
A_{b l} + B_{b l} F - \lambda E_{b l} & B_{b l} W \\
C_{b l} + D_{b l} F & D_{b l} W
\end{bmatrix},
\]
\[(131)\]
where \(W\) is a suitable invertible matrix and \(F\) is a stabilizing state-feedback matrix. \(X(\lambda)\) is determined with a descriptor realization of the form
\[
X(\lambda) = \begin{bmatrix}
A - \lambda E & B \\
C & D
\end{bmatrix}.
\]
\[(132)\]
where \([\bar{C} \bar{D}] = W^{-1} [0 \quad -F \quad I_r \quad 0] Z^T\).

The overall factorization approach is described in [56]. The reduction of the system matrix pencil to the special Kronecker-like form \((128)\) is described in [29] and involves the use of the mex-function \texttt{sl_klf} to compute the appropriate Kronecker-like form. For the computation of an inner basis, extensions of the standard inner-outer factorization methods of [64] are used. These methods involve the solution of appropriate (continuous- or discrete-time) generalized algebraic Riccati equations. For additional details, see [32] for continuous-time systems and [29] for discrete-time systems.

**Examples**

**Example 4.** This is Example 1 from [32] of the transfer function matrix of a continuous-time proper system:
\[
G(s) = \begin{bmatrix}
\frac{s - 1}{s + 2} & \frac{s}{s + 2} & \frac{1}{s + 2} \\
0 & \frac{s - 2}{(s + 1)^2} & \frac{s - 2}{(s + 1)^2} \\
\frac{s - 1}{s + 2} & \frac{s^2 + 2s - 2}{(s + 1)(s + 2)} & \frac{2s - 1}{(s + 1)(s + 2)}
\end{bmatrix}.
\]
\[(133)\]
$G(s)$ has zeros at $\{1, 2, \infty\}$, poles at $\{-1, -1, -2, -2\}$, and normal rank $r = 2$.

A minimum proper basis of $\mathcal{R}(G(s))$, computed with \texttt{grange}, is

$$R(s) = \frac{1}{s + 1.374} \begin{bmatrix} 1.552s + 2.124 & 1.314s + 1.817 \\ 0.593s + 1.186 & -0.758s - 1.516 \\ 2.145s + 2.717 & 0.555s + 1.059 \end{bmatrix},$$

has McMillan-degree 1 and no zeros. The full row rank factor $X(s)$, satisfying $G(s) = R(s)X(s)$, has McMillan degree 4, and zeros at $\{1, 2, -1.374, \infty\}$. The zero at $-1.374$ is equal to the pole of $R(s)$.

The full-rank factorization of $G(s)$ has been computed using the following sequence of commands:

% Oara and Varga (2000), Example 1
s = tf('s'); % define the complex variable s
% enter G(s) and determine a minimal state-space realization
G = [(s-1)/(s+2) s/(s+2) 1/(s+2);
     0 (s-2)/(s+1)^2 (s-2)/(s+1)^2;
     (s-1)/(s+2) (s^2+2*s-2)/(s+1)/(s+2) (2*s-1)/(s+1)/(s+2)];
sys = minreal(ss(G));
gpole(sys) % the system is stable
gzero(sys) % the system has 2 unstable zeros and an infinite zero
nrank(sys) % the normal rank of G(s) is 2

% compute the full-rank factorization G(s) = R(s)*X(s)
[sysr,sysx] = grange(sys,struct('tol',1.e-7));

% check the factorization
norm(sysr*sysx-sys,inf) % ||R(s)*X(s)-G(s)||_inf = 0

gzero(sysr) % R(s) has no zeros
gzero(sysx) % X(s) has all zeros of G(s)

\begin{example}
This example illustrates a straightforward application of the inner range computation in determining a \textit{normalized right coprime factorization} of an arbitrary rational matrix $G(\lambda)$ as

$$G(\lambda) = N(\lambda)M^{-1}(\lambda),$$

such that $N(\lambda)$ and $M(\lambda)$ are stable and $\begin{bmatrix} N(\lambda) \\ M(\lambda) \end{bmatrix}$ is inner. The factors $N(\lambda)$ and $M(\lambda)$ can be computed from a minimal inner basis $R(\lambda)$ of the range of $\begin{bmatrix} G(\lambda) \\ I_m \end{bmatrix}$ satisfying

$$\begin{bmatrix} G(\lambda) \\ I_m \end{bmatrix} = R(\lambda)X(\lambda),$$

with

$$R(\lambda) = \begin{bmatrix} N(\lambda) \\ M(\lambda) \end{bmatrix}, \quad X(\lambda) = M^{-1}(\lambda).$$
\end{example}
For the polynomial transfer function matrix $G(s)$ considered in the Example 3 of [32] with
\[
G(s) = \begin{bmatrix}
  s^2 + s + 1 & 4s^2 + 3s + 2 & 2s^2 - 2 \\
  s & 4s - 1 & 2s - 2 \\
  s^2 & 4s^2 - s & 2s^2 - 2s
\end{bmatrix},
\]
the following MATLAB code can be used to compute a normalized right coprime factorization of $G(s)$:

```matlab
% Oara and Varga (2000), Example 3
s = tf('s'); % define the complex variable s
% enter G(s) and determine a minimal state-space realization
G = [s^2+s+1 4*s^2+3*s+2 2*s^2-2; 
     s 4*s-1 2*s-2; 
     s^2 4*s^2-s 2*s^2-2*s];
sys = gir(ss(G));

% compute the extended inner-outer factorization [G(s);I] = Gi(s)*Go(s)
[p,m]=size(sys);
sysr = grange([sys;eye(m)],struct('tol',1.e-7,'inner',true));

% extract the factors
N = sysr(1:p,1:m); M = sysr(p+1:end,1:m);

% check the coprime factorization
norm(gir(N*inv(M)-sys),inf) % ||N*inv(M)-G||_inf = 0

% checking the innerness of [N;M]
norm(sysr'*sysr-eye(m),inf) % ||conj([N;M])*[N;M]-I||_inf = 0
gpole(sysr) % [N;M] is stable
```

Example 6. This is Example 2 from [32], and concerns with the transfer function matrix (133) of a continuous-time proper system, already considered in Example 4. The Moore-Penrose pseudoinverse $G^\#$ of the rational matrix $G(s)$ can be computed in three steps, using a simplified version of the approach described in [32]:

1. Compute a full-rank factorization $G(s) = U(s)G_1(s)$, with $U(s)$, a minimal inner range matrix, and $G_1(s)$ full row rank.
2. Compute the dual full-rank factorization $G_1(s) = G_2(s)V(s)$, with $V(s)$, a minimal co-inner coimage (i.e., $V(s)V^\sim(s) = I$), and $G_2(s)$ invertible.
3. Compute
\[
G^\#(s) = V^\sim(s)G_2^{-1}(s)U^\sim(s).
\]

The dual full-rank factorization at Step 2 can be simply determined by computing the full-rank factorization $G_1^T(s) = V^T(s)G_2^T(s)$, with $V^T(s)$ a minimal inner range matrix.

These computational steps are implemented in the following MATLAB code:
% Oara and Varga (2000), Example 2
s = tf('s'); % define the complex variable s
% enter G(s) and determine a minimal state-space realization
Gs = [(s-1)/(s+2) s/(s+2) 1/(s+2);
0 (s-2)/(s+1)^2 (s-2)/(s+1)^2;
(s-1)/(s+2) (s^2+2*s-2)/(s+1)/(s+2) (2*s-1)/(s+1)/(s+2)];
G = minreal(ss(Gs));

% use tolerance 1.e-7 for rank determinations
grange_opt = struct('tol',1.e-7,'inner',true);

% compute the full-rank factorization G(s) = U(s)*G1(s) with U(s) inner
[U,G1] = grange(G,grange_opt);

% compute the full-rank factorization G1(s) = G2(s)*V(s) with V(s) co-inner
[V,G2] = grange(G1.',grange_opt); V = V.'; G2 = G2.);

% compute the pseudo-inverse Gpinv
Gpinv = gir(V'*(G2\(U')));

% check the four axioms defining the Moore-Penrose pseudoinverse
norm(G*Gpinv*G-G,inf) % (i) ||G*Gpinv*G-G||_inf = 0
norm(Gpinv*G*Gpinv-Gpinv,inf) % (ii) ||Gpinv*G*Gpinv-Gpinv||_inf = 0
norm(G*Gpinv-(G*Gpinv)',inf) % (iii) ||G*Gpinv-(G*Gpinv)||_inf = 0
norm(Gpinv*G-(Gpinv*G)',inf) % (iv) ||Gpinv*G-(Gpinv*G)||_inf = 0

3.5.4 grsol

Syntax

[SYSX,INFO,SYSGEN] = grsol(SYSG,SYSF,OPTIONS)
[SYSX,INFO,SYSGEN] = grsol(SYSGF,MF,OPTIONS)

Description

grsol computes the solution X(\lambda) of the linear rational matrix equation

\[ G(\lambda)X(\lambda) = F(\lambda), \]  \hspace{1cm} (135)

where G(\lambda) and F(\lambda) are the (rational) transfer function matrices of LTI descriptor systems.

Input data

For the usage with

[SYSX,INFO,SYSGEN] = grsol(SYSG,SYSF,OPTIONS)
the input parameters are as follows:

**SYSG** is a LTI system, whose transfer function matrix is \( G(\lambda) \), and is in a descriptor system state-space form

\[
E_G\lambda x_G(t) = A_G x_G(t) + B_G u(t),
\]
\[
y_G(t) = C_G x_G(t) + D_G u(t),
\]
(136)

where \( y_G(t) \in \mathbb{R}^p \).

**SYSF** is a LTI system, whose transfer function matrix is \( F(\lambda) \), and is in a descriptor system state-space form

\[
E_F\lambda x_F(t) = A_F x_F(t) + B_F v(t),
\]
\[
y_F(t) = C_F x_F(t) + D_F v(t),
\]
(137)

where \( y_F(t) \in \mathbb{R}^p \).

**OPTIONS** is a MATLAB structure to specify user options and has the following fields:

| OPTIONS fields | Description |
|----------------|-------------|
| tol            | relative tolerance for rank computations (Default: internally computed); |
| sdeg           | prescribed stability degree for the free poles of the solution \( X(\lambda) \) (Default: [ ], i.e., no stabilization performed); |
| poles          | a complex conjugated set of desired poles to be assigned for the free poles of the solution \( X(\lambda) \) (Default: [ ]); |
| mindeg         | option to compute a minimum degree solution: \texttt{true} – determine a minimum order solution; \texttt{false} – determine a particular solution which has possibly non-minimal order (default). |

For the usage with

\[
\text{[SYSX,INFO,SYSGEN]} = \text{grsol(SYSGF,MF,OPTIONS)}
\]

the input parameters are as follows:

**SYSGF** is an input concatenated compound LTI system, \( \text{SYSGF} = [\ \text{SYSG} \ \text{SYSF} \ ] \), in a descriptor system state-space form

\[
E\lambda x(t) = Ax(t) + B_G u(t) + B_F v(t),
\]
\[
y(t) = C x(t) + D_G u(t) + D_F v(t),
\]
(138)

where \( \text{SYSG} \) has the transfer function matrix \( G(\lambda) \), with the descriptor system realization \((A - \lambda E, B_G, C, D_G)\), and \( \text{SYSF} \) has the transfer function matrix \( F(\lambda) \), with the descriptor system realization \((A - \lambda E, B_F, C, D_F)\).

**MF** is the dimension of the input vector \( v(t) \) of the system **SYSF**.

**OPTIONS** is a MATLAB structure to specify user options and has the same fields as described previously.
Output data

SYSX contains the descriptor system state-space realization of the solution $X(\lambda)$ in the form

$$
\tilde{E} \lambda \tilde{x}(t) = \tilde{A} \tilde{x}(t) + \tilde{B} v(t),
$$

$$
u(t) = \tilde{C} \tilde{x}(t) + \tilde{D} v(t).
$$

INFO is a MATLAB structure containing additional information, as follows:

| INFO fields | Description |
|-------------|-------------|
| rankG       | normal rank of the transfer function matrix $G(\lambda)$; |
| rdeg        | vector which contains the relative column degrees of $X(\lambda)$ (i.e., the numbers of integrators/delays needed to make each column of $X(\lambda)$ proper); |
| tcond       | maximum of the condition numbers of the employed non-orthogonal transformation matrices (large values indicate possible loss of numerical stability); |
| fnorm       | the norm of the employed state-feedback/feedforward used for dynamic cover computation if OPTIONS.mindeg = true, or for stabilization of free poles if OPTION.sdeg is not empty (large values indicate possible loss of numerical stability); |
| nr          | the order of $A_r - \lambda E_r$, also the row dimension of $B_r$ and also the number of freely assignable poles of the solution (see Method); |
| nf          | the order of $A_f - \lambda E_f$ (see Method); |
| ninf        | the order of $A_{\infty} - \lambda E_{\infty}$ (see Method). |

SYSGEN contains the input concatenated compound system $[\text{SYSX0} \ \text{SYSNR}]$, in a descriptor system state-space form

$$
E_g \lambda x_g(t) = A_g x_g(t) + B_0 v_1(t) + B_N v_2(t),
$$

$$
y_g(t) = C_g x_g(t) + D_0 v_1(t) + D_N v_2(t),
$$

where the transfer function matrix $X_0(\lambda)$ of SYSX0, with the descriptor system realization $(A_g - \lambda E_g, B_0, C_g, D_0)$, is a particular solution satisfying $G(\lambda)X_0(\lambda) = F(\lambda)$, and the transfer function matrix $N_r(\lambda)$ of SYSNR, with the descriptor system realization $(A_g - \lambda E_g, B_N, C_g, D_N)$, is a proper right nullspace basis of $G(\lambda)$, satisfying $G(\lambda)N_r(\lambda) = 0$. The transfer function matrices $X_0(\lambda)$ and $N_r(\lambda)$ can be used to generate all solutions of the system (135) as $X(\lambda) = X_0(\lambda) + N_r(\lambda)Y(\lambda)$, where $Y(\lambda)$ is an arbitrary rational matrix with suitable dimensions.

Method

The employed solution approach of the linear rational matrix equation (135) is sketched in Section 2.10. This approach is based on a realization of the form (138) of the input concatenated compound system $\text{SYSGF} = [\text{SYSG} \ \text{SYSF}]$. A detailed computational procedure to solve (135)
is described in [52] (see also [58, Section 10.3.7] for more details). For the intervening computation of the Kronecker-like form of the system matrix pencil of the descriptor system $\text{SYSG}$, the mex-function $\text{sl\_kif}$, based on the algorithm proposed in [2], is employed. If the descriptor realizations of $\text{SYSG}$ and $\text{SYSF}$ are separately provided, then an irreducible realization of the input concatenated compound system $[\text{SYSG} \text{SYSF}]$ is internally computed. For orthogonal transformation based order reduction purposes, the mex-function $\text{sl\_gminr}$ is employed. Furthermore, the mex-function $\text{sl\_gstra}$ is employed to count controllable infinite poles (needed to determine the relative column degrees) and for reduction to SVD-like form (needed for the elimination of non-dynamic modes).

For the computation of the solution, a so-called generator of all solutions is determined in the form (140), where the resulting matrices have the forms

$$A_g - \lambda E_g = \begin{bmatrix} A_r + B_r F - \lambda E_r & * & * \\ 0 & A_f - \lambda E_f & * \\ 0 & 0 & A_\infty - \lambda E_\infty \end{bmatrix}, \quad [B_0 \mid B_N] = \begin{bmatrix} B_1 & B_r \\ B_2 & 0 \\ B_3 & 0 \end{bmatrix},$$

$$C_g = \begin{bmatrix} C_r + D_N F & * & * \end{bmatrix},$$

with the descriptor pair $(A_r - \lambda E_r, B_r)$ controllable and $E_r$ invertible, $A_f - \lambda E_f$ regular and $E_f$ invertible (i.e., $A_f - \lambda E_f$ has only finite eigenvalues), and $A_\infty - \lambda E_\infty$ regular and upper triangular with $A_\infty$ invertible and $E_\infty$ nilpotent (i.e. $A_\infty - \lambda E_\infty$ has only infinite eigenvalues), and $D_N$ full row rank. If $G(\lambda)$ is a $p \times m$ TFM of normal rank $r$, then the matrices $B_r$ and $D_N$ have $m - r$ columns. The pair $(A_r - \lambda E_r, B_r)$ being controllable, the eigenvalues of $A_r + B_r F - \lambda E_r$ can be freely assigned by using a suitably chosen state-feedback matrix $F$.

The descriptor system realization (140) is usually not minimal, being uncontrollable, or having non-dynamic modes, or both. The generator contains the particular solution $X_0(\lambda)$ with the descriptor realization $(A_g - \lambda E_g, B_0, C_g, D_0)$ and a right nullspace basis $N_r(\lambda)$ of $G(\lambda)$ with the (non-minimal) descriptor system realization $(A_g - \lambda E_g, B_r, C_g, D_N)$. A minimal order descriptor system realization of $N_r(\lambda)$ is $(A_r + B_r F - \lambda E_r, B_r, C_r + D_N F, D_N)$. All solutions of the system (135) can be expressed as $X(\lambda) = X_0(\lambda) + N_r(\lambda) Y(\lambda)$, where $Y(\lambda)$ is an arbitrary rational matrix with suitable dimensions.

The resulting generator $\text{SYSGEN}$ contains the descriptor system realization

$$[X_0(\lambda) \mid N_r(\lambda)] = (A_g - \lambda E_g, [B_0 \ B_N], C_g, [D_0 \ D_N]).$$

The orders of the diagonal blocks of $A_g - \lambda E_g$ are provided in the $\text{INFO}$ structure as follows: $\text{INFO.nr}$ contains the order of $A_r - \lambda E_r$, $\text{INFO.nf}$ contains the order of $A_f - \lambda E_f$, and $\text{INFO.ninf}$ contains the order of $A_\infty - \lambda E_\infty$. $\text{INFO.rankG}$ contains the normal rank $r$ of $G(\lambda)$. The relative column degrees, provided in $\text{INFO.rdeg}$, are the numbers of infinite poles of the successive columns of $X_0(\lambda)$.

If $\text{OPTIONS.mindeg} = \text{false}$, the computed solution $\text{SYSX}$ represents a minimal realization of the particular solution $X_0(\lambda)$, computed by eliminating the uncontrollable eigenvalues and non-dynamic modes of the realization $(A_g - \lambda E_g, B_0, C_g, D_0)$, where $F$ is determined such that the (free) eigenvalues of $A_r + B_r F - \lambda E_r$ are moved to the stability domain specified via $\text{OPTIONS.sdeg}$ or to locations specified in $\text{OPTIONS.poles}$. If both $\text{OPTIONS.sdeg}$ and $\text{OPTIONS.poles}$ are empty, then $F = 0$ is used.

If $\text{OPTIONS.mindeg} = \text{true}$, the computed solution $\text{SYSX}$ represents a minimal realization of $X(\lambda) = X_0(\lambda) + N_r(\lambda) Y(\lambda)$, where $Y(\lambda)$ is determined such that $X(\lambda)$ has the least achievable
McMillan degree. For this computation, order reduction based on computing minimum dynamic covers is employed (see Procedure GRMCOVER2 in [58, Section 10.4.3]).

Example

Example 7. This example is taken from [14], where an one-sided model matching problem is solved, which involves the computation of a stable and proper solution of $G(s)X(s) = F(s)$ with

$$G(s) = \begin{bmatrix} \frac{s-1}{s(s+1)} & \frac{s-1}{s(s+2)} \end{bmatrix}, \quad F(s) = \begin{bmatrix} \frac{s-1}{(s+1)(s+3)} & \frac{s-1}{(s+1)(s+4)} \end{bmatrix}.$$ 

Both $G(s)$ and $[G(s)\ F(s)]$ have rank equal to 1 and zeros $\{1, \infty\}$. It follows, according to Lemma 9, that the linear rational matrix equation $G(s)X(s) = F(s)$ has a stable and proper solution. A fourth order stable and proper solution has been computed in [14]. A least order solution, with McMillan degree equal to 2, has been computed with grsol as

$$X(s) = \begin{bmatrix} 0.48889(s - 1.045) & 0.41333(s - 1.419) \\ s + 3 & s + 4 \\ 0.51111(s + 2) & 0.58667(s + 2) \\ s + 3 & s + 4 \end{bmatrix}.$$ 

To compute a least order solution $X(s)$, the following sequence of commands can be used:

```matlab
% Gao & Antsaklis (1989)
s = tf('s');
G = [(s-1)/(s*(s+1)) (s-1)/(s*(s+2))];
F = [(s-1)/((s+1)*(s+3)) (s-1)/((s+1)*(s+4))];

% build a minimal realization of [G(s) F(s)]
sysgf = minreal(ss([G F]));

% solve G(s)*X(s) = F(s) for the least order solution
[X,info] = grsol(sysgf,2,struct('mindeg',true)); info
minreal(zpk(X))

% check solution
minreal(G*X-F)
```

3.5.5 glsol

Syntax

$[\text{SYSX,INFO,SYSGEN}] = \text{glsol}([\text{SYSG,SYSF,OPTIONS})$

$[\text{SYSX,INFO,SYSGEN}] = \text{glsol}([\text{SYSGF,OPTIONS})$
Description

glsol computes the solution $X(\lambda)$ of the linear rational matrix equation

$$X(\lambda)G(\lambda) = F(\lambda), \quad (141)$$

where $G(\lambda)$ and $F(\lambda)$ are the (rational) transfer function matrices of LTI descriptor systems.

Input data

For the usage with

$$[\text{SYSX}, \text{INFO}, \text{SYSGEN}] = \text{glsol}(\text{SYSG}, \text{SYSF}, \text{OPTIONS})$$

the input parameters are as follows:

**SYSG** is a LTI system, whose transfer function matrix is $G(\lambda)$, and is in a descriptor system state-space form

$$E_G \lambda x_G(t) = A_G x_G(t) + B_G u(t),$$
$$y_G(t) = C_G x_G(t) + D_G u(t), \quad (142)$$

where $u(t) \in \mathbb{R}^m$.

**SYSF** is a LTI system, whose transfer function matrix is $F(\lambda)$, and is in a descriptor system state-space form

$$E_F \lambda x_F(t) = A_F x_F(t) + B_F v(t),$$
$$y_F(t) = C_F x_F(t) + D_F v(t), \quad (143)$$

where $v(t) \in \mathbb{R}^m$.

**OPTIONS** is a MATLAB structure to specify user options and has the following fields:

| OPTIONS fields | Description |
|----------------|-------------|
| **tol**       | relative tolerance for rank computations (Default: internally computed); |
| **sdeg**      | prescribed stability degree for the free poles of the solution $X(\lambda)$ (Default: [ ], i.e., no stabilization performed); |
| **poles**     | a complex conjugated set of desired poles to be assigned for the free poles of the solution $X(\lambda)$ (Default: [ ]); |
| **mindeg**    | option to compute a minimum degree solution: true – determine a minimum order solution; false – determine a particular solution which has possibly non-minimal order (default). |

For the usage with

$$[\text{SYSX}, \text{INFO}, \text{SYSGEN}] = \text{glsol}(\text{SYSGF}, \text{MF}, \text{OPTIONS})$$

the input parameters are as follows:
SYSGF is an output concatenated compound LTI system, $\text{SYSGF} = [\text{SYSG}; \text{SYSF}]$, in a descriptor system state-space form

$$
E\lambda x(t) = Ax(t) + Bu(t),
$$

$$
y_G(t) = C_G x(t) + D_G u(t),
$$

$$
y_F(t) = C_F x(t) + D_F u(t),
$$

where $\text{SYSG}$ has the transfer function matrix $G(\lambda)$, with the descriptor system realization $(A - \lambda E, B, C_G, D_G)$, and $\text{SYSF}$ has the transfer function matrix $F(\lambda)$, with the descriptor system realization $(A - \lambda E, B, C_F, D_F)$.

$\text{MF}$ is the dimension of the output vector $y_F(t)$ of the system $\text{SYSF}$.

$\text{OPTIONS}$ is a MATLAB structure to specify user options and has the same fields as described previously.

Output data

$\text{SYSX}$ contains the descriptor system state-space realization of the solution $X(\lambda)$ in the form

$$
\tilde{E}\lambda \tilde{x}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}v(t),
$$

$$
u(t) = \tilde{C}\tilde{x}(t) + \tilde{D}v(t).
$$

$\text{INFO}$ is a MATLAB structure containing additional information, as follows:

| INFO fields | Description |
|-------------|-------------|
| rankG       | normal rank of the transfer function matrix $G(\lambda)$; |
| rdeg        | vector which contains the relative row degrees of $X(\lambda)$ (i.e., the numbers of integrators/delays needed to make each row of $X(\lambda)$ proper); |
| tcond       | maximum of the condition numbers of the employed non-orthogonal transformation matrices (large values indicate possible loss of numerical stability); |
| fnorm       | the norm of the employed state-feedback/feedforward used for dynamic cover computation if $\text{OPTIONS.mindeg} = \text{true}$, or for stabilization of free poles if $\text{OPTION.sdeg}$ is not empty (large values indicate possible loss of numerical stability); |
| ninf        | the order of $A_\infty - \lambda E_\infty$ (see Method); |
| nf          | the order of $A_f - \lambda E_f$ (see Method); |
| nl          | the order of $A_l - \lambda E_l$, also the column dimension of $C_l$, and also the number of freely assignable poles of the solution (see Method), if $\text{SYSGEN}$ is computed; otherwise, empty. |

$\text{SYSGEN}$ contains the output concatenated compound system $[\text{SYSX0}; \text{SYSNL}]$, in a descriptor
system state-space form
\[
E_g \lambda x_g(t) = A_g x_g(t) + B_g v(t), \\
y_0(t) = C_0 x_g(t) + D_0 v(t), \\
y_N(t) = C_N x_g(t) + D_N v(t),
\]
where the transfer function matrix \(X_0(\lambda)\) of SYSX0, with the descriptor system realization \((A_g - \lambda E_g, B_g, C_0, D_0)\), is a particular solution satisfying \(X_0(\lambda)G(\lambda) = F(\lambda)\), and the transfer function matrix \(N_1(\lambda)\) of SYSNL, with the descriptor system realization \((A_g - \lambda E_g, B_g, C_N, D_N)\), is a proper left nullspace basis of \(G(\lambda)\), satisfying \(N_1(\lambda)G(\lambda) = 0\). The transfer function matrices \(X_0(\lambda)\) and \(N_1(\lambda)\) can be used to generate all solutions of the system (141) as \(X(\lambda) = X_0(\lambda) + Y(\lambda)N_1(\lambda)\), where \(Y(\lambda)\) is an arbitrary rational matrix with suitable dimensions.

**Method**

To solve the linear rational equation (141), the function glsol calls grsol to solve the dual system \(G^T(\lambda)X(\lambda) = F^T(\lambda)\) and obtain the solution as \(X(\lambda) = X^T(\lambda)\). Thus, the employed solution method corresponds to the dual approach sketched in Section 2.4 (see also [52] and [58, Section 10.3.7] for more details). The function grsol relies on the mex-functions sl_klf, sl_gminr, and sl_gstra.

For the computation of the solution, a so-called *generator* of all solutions is determined in the form (146), where the resulting matrices have the forms

\[
\begin{bmatrix}
A_g - \lambda E_g \\
C_0 \\
C_N
\end{bmatrix} = \begin{bmatrix}
A_\infty - \lambda E_\infty & * & * \\
0 & A_f - \lambda E_f & * \\
0 & 0 & A_l + KC_l - \lambda E_l
\end{bmatrix}
\]

\[
\begin{bmatrix}
B_g \\
B_l + KD_N
\end{bmatrix} = \begin{bmatrix}
* & * \\
\end{bmatrix}
\]

with \(A_\infty - \lambda E_\infty\) regular and upper triangular with \(A_\infty\) invertible and \(E_\infty\) nilpotent (i.e. \(A_\infty - \lambda E_\infty\) has only infinite eigenvalues), \(A_f - \lambda E_f\) regular and \(E_f\) invertible (i.e., \(A_f - \lambda E_f\) has only finite eigenvalues), the descriptor pair \((A_l - \lambda E_l, C_l)\) observable and \(E_l\) invertible, and \(D_N\) full column rank. If \(G(\lambda)\) is a \(p \times m\) TFM of normal rank \(r\), then the matrices \(C_l\) and \(D_N\) have \(p - r\) rows. The pair \((A_l - \lambda E_l, C_l)\) being observable, the eigenvalues of \(A_l + KC_l - \lambda E_l\) can be freely assigned by using a suitably chosen output-injection matrix \(K\).

The descriptor system realization (146) is usually not minimal, being unobservable, or having non-dynamic modes, or both. The generator contains the particular solution \(X_0(\lambda)\) with the descriptor realization \((A_g - \lambda E_g, B_g, C_0, D_0)\) and a left nullspace basis \(N_1(\lambda)\) of \(G(\lambda)\) with the (non-minimal) descriptor system realization \((A_g - \lambda E_g, B_g, C_N, D_N)\). A minimal order descriptor system realization of \(N_1(\lambda)\) is \((A_l + KC_l - \lambda E_l, B_l + KD_N, C_l, D_N)\). All solutions of the system (135) can be expressed as \(X(\lambda) = X_0(\lambda) + Y(\lambda)N_1(\lambda)\), where \(Y(\lambda)\) is an arbitrary rational matrix with suitable dimensions.

The resulting generator SYSGEN contains the descriptor system realization
\[
\begin{bmatrix}
X_0(\lambda) \\
N_1(\lambda)
\end{bmatrix} = \begin{pmatrix}
A_g - \lambda E_g, B_g, [C_0 \\
C_N]
\end{pmatrix}
\begin{pmatrix}
D_0 \\
D_N
\end{pmatrix}
\]
The orders of the diagonal blocks of $A_g - \lambda E_g$ are provided in the INFO structure as follows: INFO.ninf contains the order of $A_\infty - \lambda E_\infty$, INFO.nf contains the order of $A_f - \lambda E_f$, and INFO.nl contains the order of $A_l - \lambda E_l$. INFO.rankG contains the normal rank $r$ of $G(\lambda)$. The relative row degrees, provided in INFO.rdeg, are the numbers of infinite poles of the successive rows of $X_0(\lambda)$.

If OPTIONS.mindeg = false, the computed solution SYSX represents a minimal realization of the particular solution $X_0(\lambda)$, computed by eliminating the unobservable eigenvalues and non-dynamic modes of the realization $(A_g - \lambda E_g, B_g, C_0, D_0)$, where $K$ is determined such that the (free) eigenvalues of $A_l + KC_l - \lambda E_l$ are moved to the stability domain specified via OPTIONS.sdeg or to locations specified in OPTIONS.poles. If both OPTIONS.sdeg and OPTIONS.poles are empty, then $K = 0$ is used.

If OPTIONS.mindeg = true, the computed solution SYSX represents a minimal realization of $X(\lambda) = X_0(\lambda) + Y(\lambda)N_l(\lambda)$, where $Y(\lambda)$ is determined such that $X(\lambda)$ has the least achievable McMillan degree. For this computation, order reduction based on computing minimum dynamic covers is employed (see Procedure GRMCOVER2 in [58, Section 10.4.3]).

Examples

Example 8. This example illustrates the computation of a left inverse of least McMillan degree used in [62]. The transfer function matrices $G(s)$ and $F(s)$ are

\[
G(s) = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s + 1 & s + 2 \\ s + 3 & s^2 + 2s \\ s^2 + 3s & 0 \end{bmatrix}, \quad F(s) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

The solution $X(s)$ of the rational equation $X(s)G(s) = I$ is a left inverse of $G(s)$. A third order minimal state-space realization of $G(s)$ can be computed using the function gir (or minreal). To compute a least McMillan order proper left inverse, the following MATLAB commands can be used:

```matlab
% Wang and Davison Example (1973)
s = tf('s');
g = [ s+1 s+2; s+3 s^2+2*s; s^2+3*s 0 ]/(s^2+3*s+2); f = eye(2);
sysg = gir(ss(g)); sysf = ss(f);

% compute a least order solution of X(s)G(s) = I
[sysx,info] = glsol(sysg,sysf,struct('mindeg',true)); info
gpole(sysx) % the left inverse is unstable
```

The computed solution is unstable. In fact, it turns out that all least order left inverses are unstable.

Example 9. This example illustrates the computation of a stable left inverse for the example of [62], used previously. Since both $G(s)$ and $\frac{G(s)}{I_2}$ have no zeros (in particular no unstable zeros), a stable solution of equation $X(s)G(s) = I$ exists, in accordance with Lemma 9. In Example
we computed \( \text{INFO.nr} = 3 \), and, therefore, we can assign three free poles of the inverse to \( \{-1, -2, -3\} \) to obtain a stable left inverse. This can be done by calling \( \text{glsol} \) as follows:

\[
\text{sysx} = \text{glsol}(\text{sysg}, \text{sysf}, \text{struct('poles',[-1 -2 -3]))};
\]

\( \text{gpole(sysx)} \)

The resulting minimal realization of left inverse \( \text{sysx} \) has the poles equal to \( \{-1, -2, -3\} \).

3.5.6 \( \text{gsdec} \)

Syntax

\[
[\text{SYS1, SYS2, Q, Z}] = \text{gsdec}(\text{SYS, OPTIONS})
\]

Description

\( \text{gsdec} \) computes, for the transfer function matrix \( G(\lambda) \) of a LTI descriptor system, additive spectral decompositions in the form

\[
G(\lambda) = G_1(\lambda) + G_2(\lambda),
\]

where \( G_1(\lambda) \) has only poles in a certain domain of interest \( C_g \subset \mathbb{C} \) and \( G_2(\lambda) \) has only poles in the complementary domain \( \mathbb{C} \setminus C_g \).

Input data

\( \text{SYS} \) is a LTI system, whose transfer function matrix is \( G(\lambda) \), and is in a descriptor system state-space form

\[
E\lambda x(t) = Ax(t) + Bu(t),
\]

\[
y(t) = Cx(t) + Du(t).
\]

\( \text{OPTIONS} \) is a MATLAB structure to specify user options and has the following fields:

| OPTIONS fields | Description |
|----------------|-------------|
| \( \text{tol} \) | relative tolerance for rank computations (Default: internally computed) |
| \( \text{smarg} \) | stability margin for the stable poles of the transfer function matrix \( G(\lambda) \) of \( \text{SYS} \), such that, in the continuous-time case, the stable eigenvalues have real parts less than or equal to \( \text{OPTIONS.smarg} \), and in the discrete-time case, the stable eigenvalues have moduli less than or equal to \( \text{OPTIONS.smarg} \).
( Default: \( -\sqrt{\text{eps}} \) for a continuous-time system \( \text{SYS} \); \( 1-\sqrt{\text{eps}} \) for a discrete-time system \( \text{SYS} \). ) |
| \( \text{job} \) | option for specific spectral separation tasks:

\[
'\text{finite}' \quad - \quad G_1(\lambda) \text{ has only finite poles and } G_2(\lambda) \text{ has only infinite poles (default)};
\]

\[
'\text{infinite}' \quad - \quad G_1(\lambda) \text{ has only infinite poles and } G_2(\lambda) \text{ has only finite poles};
\]

\[
'\text{stable}' \quad - \quad G_1(\lambda) \text{ has only stable poles and } G_2(\lambda) \text{ has only unstable and infinite poles};
\]

\[
'\text{unstable}' \quad - \quad G_1(\lambda) \text{ has only unstable and infinite poles and } G_2(\lambda) \text{ has only stable poles}.
\]
Output data

SYS1 contains the descriptor system state-space realization of the transfer function matrix $G_1(\lambda)$ in the form

$$E_1\lambda x_1(t) = A_1 x_1(t) + B_1 u(t),$$

$$y_1(t) = C_1 x_1(t) + D u(t).$$

(149)

The pair $(A_1, E_1)$ is in a GRSF.

SYS2 contains the descriptor system state-space realization of the transfer function matrix $G_2(\lambda)$ in the form

$$E_2\lambda x_2(t) = A_2 x_2(t) + B_2 u(t),$$

$$y_2(t) = C_2 x_2(t).$$

(150)

The pair $(A_2, E_2)$ is in a GRSF.

$Q$ is the employed left transformation matrix used to reduce the pole pencil $A - \lambda E$ to a block-diagonal form (see Method).

$Z$ is the employed right transformation matrix used to reduce the pole pencil $A - \lambda E$ to a block-diagonal form (see Method).

Method

The employed additive decomposition approach of the transfer function matrix $G(\lambda)$ in the form (147) is described in Section 2.7, and is based on the method proposed in [21]. For the computation of spectral separations of the descriptor system SYS, the mex-function s1_gsep is employed.

For a certain domain of interest $C_g$, the basic computation consist in determining two invertible matrices $Q$ and $Z$, to obtain an equivalent descriptor system representation of $G(\lambda)$ with a block-diagonal pole pencil, in the form

$$G(\lambda) = Q A Z - \lambda Q E Z = \begin{bmatrix}
A_1 - \lambda E_1 & 0 & B_1 \\
0 & A_2 - \lambda E_2 & B_2 \\
C_1 & C_2 & D
\end{bmatrix},$$

(151)

where $\Lambda(A_1 - \lambda E_1) \subset C_g$ and $\Lambda(A_2 - \lambda E_2) \subset C \setminus C_g$. This leads to the additive decomposition of $G(\lambda)$ as

$$G(\lambda) = G_1(\lambda) + G_2(\lambda),$$

(152)

where

$$G_1(\lambda) = \begin{bmatrix}
A_1 - \lambda E_1 & B_1 \\
C_1 & D
\end{bmatrix}, \quad G_2(\lambda) = \begin{bmatrix}
A_2 - \lambda E_2 & B_2 \\
C_2 & 0
\end{bmatrix}.$$

(153)

The computation of the descriptor realization (151), with a block-diagonal pole pencil, is achieved in two steps. The first step involves the separation of the spectrum of $A - \lambda E$, using orthogonal transformations, in accordance with the selected option in OPTIONS.job, which defines the domain of interest $C_g$. In all cases, a preliminary finite-infinite or infinite-finite separation of the eigenvalues of the pole pencil $A - \lambda E$ is performed (only for descriptor systems) using the orthogonal staircase reduction algorithm proposed in [26] (for the computation of system zeros).
The finite part of the reduced pole pencil is further reduced to a GRSF, and if necessary, the finite eigenvalues are additionally separated into stable-unstable or unstable-stable blocks using eigenvalue reordering techniques (see [16]). In the second step, the block-diagonalization of the reduced pole pencil is performed using the approach of [21].

Four spectral separations of the poles of $G(\lambda)$ can be selected via the option parameter \texttt{OPTIONS.job}. The resulting pole pencils of the corresponding descriptor system realizations (149) of $G_1(\lambda)$ and (150) of $G_2(\lambda)$, exhibit several particular features, which are shortly addressed below:

If \texttt{OPTIONS.job = 'finite'}, the resulting $A_2$ is nonsingular and upper triangular, while the resulting $E_2$ is nilpotent and upper triangular.

If \texttt{OPTIONS.job = 'infinite'}, the resulting $A_1$ is nonsingular and upper triangular, while the resulting $E_1$ is nilpotent and upper triangular. In this case, \texttt{SYS2} contains the strictly proper part of \texttt{SYS}.

If \texttt{OPTIONS.job = 'stable'}, the resulting pair $(A_2, E_2)$, in a GRSF, contains also all infinite eigenvalues of the pole pencil $A - \lambda E$ (the trailing part of $E_2$ contains a nilpotent matrix).

If \texttt{OPTIONS.job = 'unstable'}, the resulting pair $(A_1, E_1)$, in a GRSF, contains also all infinite eigenvalues of the pole pencil $A - \lambda E$ (the leading part of $E_1$ contains a nilpotent matrix).

\textbf{Example}

\textit{Example 10.} Consider the $2 \times 2$ improper transfer function matrix $G(s)$ of a continuous-time system

$$ G(s) = \begin{bmatrix} s^2 & s \\ s + 1 & 1 \\ 0 & \frac{1}{s} \end{bmatrix}. $$

To compute the proper-polynomial additive decomposition of $G(s)$, we can use the following command sequence:

```matlab
s = tf('s'); % define the complex variable s
Gc = [s^2 s/(s+1); 0 1/s] % define the 2-by-2 improper Gc(s)
sysc = ss(Gc); % build continuous-time descriptor system realization

% compute the separation of proper and polynomial parts of Gc(s)
[sysf,sysi] = gsdec(sysc);

% for checking the results, convert the terms to zeros/poles/gain form
Gf = zpk(sysf) % proper part
Gp = zpk(sysi) % polynomial part
```

The resulting transfer function matrices of the proper part $G_f(s)$ and polynomial part $G_p(s)$ are

$$ G_f(s) = \begin{bmatrix} 0 & s \\ s + 1 & 1 \\ 0 & \frac{1}{s} \end{bmatrix}, \quad G_p(s) = \begin{bmatrix} s^2 & 0 \\ 0 & 0 \end{bmatrix}. $$
Example 11. For the transfer function matrix employed in Example 10, we can compute a polynomial-strictly proper additive decomposition of $G(s)$, using the following command sequence:

```matlab
s = tf('s'); % define the complex variable s
Gc = [s^2 s/(s+1); 0 1/s] % define the 2-by-2 improper Gc(s)
syssc = ss(Gc); % build continuous-time descriptor system realization

% compute the separation of polynomial and strictly proper parts of Gc(s)
[syspol, syssp] = gsdec(syssc, struct('job','infinite'));

% for checking the results, convert the terms to zeros/poles/gain form
Gsp = zpk(syssp) % strictly proper part
Gpol = zpk(syspol) % polynomial part
```

The resulting strictly proper part $G_{sp}(s)$ and polynomial part $G_{pol}(s)$ are

\[
G_{sp}(s) = \begin{bmatrix} 0 & -1/s + 1/s + 1 \\ 0 & 1/s \end{bmatrix}, \quad G_{pol}(s) = \begin{bmatrix} s^2 & 1 \\ 0 & 0 \end{bmatrix}.
\]

3.5.7 grmcover1

Syntax

```matlab
[SYSX, INFO, SYSY] = grmcover1(SYS1, SYS2, TOL)
[SYSX, INFO, SYSY] = grmcover1(SYS, M1, TOL)
```

Description

grmcover1 computes, for given proper transfer function matrices $X_1(\lambda)$ and $X_2(\lambda)$, a proper $X(\lambda)$ and a strictly proper $Y(\lambda)$, both with minimal McMillan degrees, which satisfy

\[
X(\lambda) = X_1(\lambda) + X_2(\lambda)Y(\lambda)
\]

and represent the solution of a right minimum cover problem. An approach based on the computation of a minimum dynamic cover of Type 1 is used to determine $X(\lambda)$ and $Y(\lambda)$, such that $\delta(X(\lambda)) \leq \delta(X_1(\lambda))$. If $X_1(\lambda)$ is improper, then a maximum reduction of $\delta(X(\lambda))$ is achieved, by maximally reducing the order of the proper part of $X_1(\lambda)$ and keeping unaltered its polynomial part.

Input data

For the usage with

```matlab
[SYSX, INFO, SYSY] = grmcover1(SYS1, SYS2, TOL)
```
the input parameters are as follows:

**SYS1** is a LTI system, whose transfer function matrix is $X_1(\lambda)$, and is in a descriptor system state-space form

\[
E_1 \lambda x_1(t) = A_1 x_1(t) + B_1 u(t), \\
y_1(t) = C_1 x_1(t) + D_1 u(t),
\]

where $y_1(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^{m_1}$.

**SYS2** is a proper LTI system, whose transfer function matrix is $X_2(\lambda)$, and is in a descriptor system state-space form

\[
E_2 \lambda x_2(t) = A_2 x_2(t) + B_2 v(t), \\
y_2(t) = C_2 x_2(t) + D_2 v(t),
\]

where $y_2(t) \in \mathbb{R}^p$.

**TOL** is a relative tolerance used for rank determinations. If **TOL** is not specified as input or if **TOL** = 0, an internally computed default value is used.

For the usage with

\[
[SYSX,INFO,SYSY] = grmcover1(SYS,M1,TOL)
\]

the input parameters are as follows:

**SYS** is an input concatenated compound LTI system, $SYS = [ SYS1 \ SYS2 ]$, in a descriptor system state-space form

\[
E \lambda x(t) = Ax(t) + B_1 u(t) + B_2 v(t), \\
y(t) = C x(t) + D_1 u(t) + D_2 v(t),
\]

where $SYS1$ has the transfer function matrix $X_1(\lambda)$, with the descriptor system realization $(A - \lambda E, B_1, C, D_1)$, and $SYS2$ has the proper transfer function matrix $X_2(\lambda)$, with the descriptor system realization $(A - \lambda E, B_2, C, D_2)$.

**M1** is the dimension $m_1$ of the input vector $u(t)$ of the system **SYS1**.

**TOL** is a relative tolerance used for rank determinations. If **TOL** is not specified as input or if **TOL** = 0, an internally computed default value is used.

**Output data**

**SYSX** contains, in the case when both $X_1(\lambda)$ and $X_2(\lambda)$ are proper, the descriptor system state-space realization of the resulting reduced order $X(\lambda)$ in (154), in the form

\[
E_r \lambda x_{r,1}(t) = A_r x_{r,1}(t) + B_r v(t), \\
y_{r,1}(t) = C_{r,1} x_{r,1}(t) + D_1 v(t),
\]

with the pair $(A_r - \lambda E_r, B_r)$ in a controllability staircase form with $[B_r \ A_r]$ as in (22) and $E_r$ upper triangular and nonsingular, as in (23). If $X_1(\lambda)$ is improper, but $X_2(\lambda)$ is proper, then **SYSX** contains the descriptor system realization corresponding to the sum of the reduced proper part of $X_1(\lambda)$ (in the form (158)) and the polynomial part of $X_1(\lambda)$.

**INFO** is a MATLAB structure containing additional information, as follows:
INFO fields | Description
---|---
``stdim`` | vector which contains the dimensions of the diagonal blocks of \( A_r - \lambda E_r \), which are the row dimensions of the full row rank diagonal blocks of the pencil \([ B_r A_r - \lambda E_r ]\) in controllability staircase form;
``tcond`` | maximum of the Frobenius-norm condition numbers of the employed non-orthogonal transformation matrices (large values indicate possible loss of numerical stability);
``fnorm`` | the norm of the employed state-feedback \( F \) used for minimum dynamic cover computation (see Method) (large values indicate possible loss of numerical stability).

**SYSY** contains the descriptor system state-space realization of the resulting (strictly proper) \( Y(\lambda) \) in (154), in the form

\[
\begin{align*}
E_r x_r,2(t) &= A_r x_r,2(t) + B_r v(t), \\
y_r,2(t) &= C_r x_r,2(t),
\end{align*}
\]

(159)

with the pair \((A_r - \lambda E_r, B_r)\) in a controllability staircase form with \([ B_r A_r ]\) as in (22) and \( E_r \) upper triangular and nonsingular, as in (23).

**Method**

The approach to determine \( X(\lambda) \) and \( Y(\lambda) \), with least McMillan degrees, satisfying (154) with both \( X_1(\lambda) \) and \( X_2(\lambda) \) proper, is based on computing a Type 1 minimum dynamic cover [23] and is described in Section 2.11. This approach is based on a realization of the form (157) of the input concatenated proper compound system \( SYS = \left[ SYS1 \  SYS2 \right] \), with invertible \( E \). A detailed computational procedure to determine \( X(\lambda) \) in (154) is described in [53] (see also Procedure \texttt{GRMCOVER1} in [58, Section 10.4.2] for more details). For the intervening reduction of the partitioned pair \(( A-\lambda E, \left[ B_1 \ B_2 \right] \) to a special controllability staircase form (see Procedure \texttt{GSCSF} in [58, Section 10.4.1]), the mex-function \texttt{sl\_gstra} is employed. If the descriptor realizations of \( SYS1 \) and \( SYS2 \) are separately provided, then an irreducible realization of the input concatenated compound system \([ SYS1 \  SYS2 \] is internally computed. For orthogonal transformation based order reduction purposes, the mex-function \texttt{sl\_gminr} is employed.

For the given descriptor system realization of the transfer function matrix \( \left[ X_1(\lambda) \ X_2(\lambda) \right] \) in the form (157), a state-feedback matrix \( F \) is determined to obtain the resulting reduced order \( X(\lambda) \) and \( Y(\lambda) \) in (154) with the descriptor realization

\[
\begin{bmatrix}
X(\lambda) \\
Y(\lambda)
\end{bmatrix} :=
\begin{bmatrix}
A + B_2 F - \lambda E & B_1 \\
C + D_2 F & D_1 \\
F & 0
\end{bmatrix}.
\]

(160)

The matrix \( F \) is determined such that the pair \(( A + B_2 F - \lambda E, B_1) \) is maximally uncontrollable. Then, the resulting realizations of \( X(\lambda) \) and \( Y(\lambda) \) contain a maximum number of uncontrollable eigenvalues, which are eliminated by the Type 1 dynamic cover computation algorithm [53]. This algorithm involves the use of non-orthogonal similarity transformations, whose maximum
Frobenius-norm condition number is provided in INFO.tcond. The resulting minimal realizations of $X(\lambda)$ and $Y(\lambda)$ have the forms (165) and (166), respectively.

As already mentioned, the underlying Procedure GRMCOVER1 of [58, Section 10.4.2] is based on the algorithm proposed in [53] for descriptor systems with invertible $E$. However, the function grmcover1 also works if the original descriptor system realization has $E$ singular, but SYS2 is proper. In this case, the order reduction is performed working only with the proper part of SYS1. The polynomial part of SYS1 is included, without modification, in the resulting realization of SYSX. To compute the intervening finite-infinite spectral separation, the mex-function sl_gsep is employed.

Examples

Example 12. For a given transfer function matrix $G(\lambda)$, a simple minimal proper right nullspace basis can be computed in two steps: first, compute a minimal proper right nullspace basis $N_r(\lambda)$ of $G(\lambda)$, and then in a second step, determine successively the basis vectors of a minimal simple proper basis $N_s(\lambda)$ by solving a sequence of right minimal cover problems. Specifically, if $\text{col}(N_r(\lambda))$ is the $i$-th basis vector (i.e., column) of $N_r(\lambda)$ and $V_i^c(\lambda)$ is the matrix formed from the rest of basis vectors, then the $i$-th basis vector $	ext{col}(N_s(\lambda))$ of a minimal simple proper basis $N_s(\lambda)$ is the solution of the right cover problem

$$
\text{col}(N_s(\lambda)) = \text{col}(N_r(\lambda)) + V_i^c(\lambda)Y_i(\lambda),
$$

where $Y_i(\lambda)$ is a suitable strictly proper vector of least McMillan degree.

The following MATLAB commands illustrate this approach:

```matlab
% computation of a minimal simple proper right nullspace basis
sys = rss(6,2,6); % generate a random system with a 2x6 TFM

% compute a minimal proper right nullspace basis Nr with a 6x4 TFM
[Nr,info] = grnull(sys);

% the expected orders of vectors of a minimal simple basis
info.degs % number of basis vectors
% let Vi be the i-th basis vector and Vci the rest of basis vectors in Nr
% to compute the i-th basis vector Vsi of a simple basis Nrs,
% solve the right minimum cover problem:
% Vsi = Vi + Vci*Yi (for a suitable strictly proper Yi)

% solve the right minimum cover problem:
% lets = [1;2;3;4;5;6]; % expected orders

Nrs = ss(zeros(size(Nr))); % initialize Nrs
for i = 1:size(Nr,2);
    % apply the cover computation to Nr with permuted columns
    Vsi = grmcover1(Nr(:,[i,1:i-1,i+1:size(Nr,2)]),1);
    % check orders
    if order(Vsi) ~= info.degs(i)
        warning('Expected order not achieved')
    end
    Nrs(:,i) = Vsi;
end
```
Example 13. For a given transfer function matrix \( G(\lambda) \), a minimal polynomial right nullspace basis can be computed in three steps: (1) compute a minimal proper right nullspace basis \( N_r(\lambda) \) of \( G(\lambda) \); (2) determine successively the basis vectors of a minimal simple proper basis \( N_s(\lambda) \) by solving a sequence of right minimal cover problems; and (3) determine the polynomial numerator of each vector of the simple basis. The first two steps have been already described in Example 12.

The following MATLAB commands illustrates the three step approach to compute polynomial bases:

```matlab
% minimal polynomial nullspace computation
sys = rss(6,2,6); % generate a random system with a 2x6 TFM

[Nr,info] = grnull(sys); % compute a proper right nullspace basis Nr with a 6x4 TFM
info.degs % the degrees of vectors of a minimal polynomial basis
.nb = size(Nr,2); % number of basis vectors

% let Vi the i-th basis vector and Vci the rest of basis vectors in Nr
% to compute the i-th basis vector Vsi of a simple basis Nrs,
% solve the right minimum cover problem:
% Vsi = Vi + Vci*Yi (for a suitable strictly proper Yi)
% to obtain the corresponding polynomial basis vector, cancel all finite poles
% of Vsi

Nrp = tf(zeros(6,nb)); % initialize Nrs
for i = 1:nb;
    % apply the cover computation to Nr with permuted columns
    Vsi = grmcover1(Nr(:,[i,1:i-1,i+1:nb]),1);
    % check orders
    if order(Vsi) ~= info.degs(i)
        warning('Expected order not achieved')
    end
    Nrp(:,i) = minreal(tf(Vsi*tf(poly(eig(Vsi)),1)));
end

% check nullspace condition sys*Nrp = 0
gminreal(sys*Nrp,1.e-7)
```

The same approach can be used by calling the function `grmcover2` instead `grmcover1`. ♦

### Syntax

```matlab
[SYSX,INFO,SYXY] = glmcover1(SYS1,SYS2,TOL)
```

3.5.8 glmcover1
[SYSX,INFO,SYSY] = glmcover1(SYS,P1,TOL)

Description
glmcover1 computes, for given proper transfer function matrices \( X_1(\lambda) \) and \( X_2(\lambda) \), a proper \( X(\lambda) \) and a strictly proper \( Y(\lambda) \), both with minimal McMillan degrees, which satisfy

\[
X(\lambda) = X_1(\lambda) + Y(\lambda)X_2(\lambda),
\]

and represent the solution of a left minimum cover problem. An approach based on the computation of a minimum dynamic cover of Type 1 is used to determine \( X(\lambda) \) and \( Y(\lambda) \), such that \( \delta(X(\lambda)) \leq \delta(X_1(\lambda)) \). If \( X_1(\lambda) \) is improper, then a maximum reduction of \( \delta(X(\lambda)) \) is achieved, by maximally reducing the order of the proper part of \( X_1(\lambda) \) and keeping unaltered its polynomial part.

Input data

For the usage with

[SYSX,INFO,SYSY] = glmcover1(SYS1,SYS2,TOL)

the input parameters are as follows:

SYS1 is a LTI system, whose transfer function matrix \( X_1(\lambda) \) has a descriptor system state-space realization of the form

\[
\begin{align*}
E_1 \lambda x_1(t) &= A_1 x_1(t) + B_1 u(t), \\
y_1(t) &= C_1 x_1(t) + D_1 u(t),
\end{align*}
\]

where \( y_1(t) \in \mathbb{R}^{p_1} \) and \( u(t) \in \mathbb{R}^{m} \).

SYS2 is a proper LTI system, whose transfer function matrix \( X_2(\lambda) \) has a descriptor system state-space realization of the form

\[
\begin{align*}
E_2 \lambda x_2(t) &= A_2 x_2(t) + B_2 v(t), \\
y_2(t) &= C_2 x_2(t) + D_2 v(t),
\end{align*}
\]

where \( v(t) \in \mathbb{R}^{m} \).

TOL is a relative tolerance used for rank determinations. If TOL is not specified as input or if TOL = 0, an internally computed default value is used.

For the usage with

[SYSX,INFO,SYSY] = glmcover1(SYS,P1,TOL)

the input parameters are as follows:

SYS is an output concatenated compound LTI system, \( SYS = [ SYS1; SYS2 ] \), in a descriptor system state-space form

\[
\begin{align*}
E \lambda x(t) &= Ax(t) + Bu(t), \\
y_1(t) &= C_1 x(t) + D_1 u(t), \\
y_2(t) &= C_2 x(t) + D_2 u(t),
\end{align*}
\]
where \( \text{SYS1} \) has the transfer function matrix \( X_1(\lambda) \), with the descriptor system realization \((A - \lambda E, B, C_1, D_1)\), and \( \text{SYS2} \) has the proper transfer function matrix \( X_2(\lambda) \), with the descriptor system realization \((A - \lambda E, B, C_2, D_2)\).

\( P_1 \) is the dimension \( p_1 \) of the output vector \( y_1(t) \) of the system \( \text{SYS1} \).

\( \text{TOL} \) is a relative tolerance used for rank determinations. If \( \text{TOL} \) is not specified as input or if \( \text{TOL} = 0 \), an internally computed default value is used.

**Output data**

\( \text{SYSX} \) contains, in the case when both \( X_1(\lambda) \) and \( X_2(\lambda) \) are proper, the descriptor system state-space realization of the resulting reduced order \( X(\lambda) \) in (161), in the form

\[
E_l \lambda x_{l,1}(t) = A_l x_{l,1}(t) + B_{l,1} v(t),
\]
\[
y_{l,1}(t) = C_l x_{l,1}(t) + D_{1} v(t),
\]

(165)

with the pair \((A_l - \lambda E_l, C_l)\) in an observability staircase form with \([A_l \ C_l] \) as in (24) and \( E_l \) upper triangular and nonsingular, as in (25). If \( X_1(\lambda) \) is improper, but \( X_2(\lambda) \) is proper, then \( \text{SYSX} \) contains the descriptor system realization corresponding to the sum of the reduced proper part of \( X_1(\lambda) \) (in the form (165)) and the polynomial part of \( X_1(\lambda) \).

\( \text{INFO} \) is a MATLAB structure containing additional information, as follows:

| INFO fields | Description |
|-------------|-------------|
| stdim       | vector containing the dimensions of the diagonal blocks of \( A_l - \lambda E_l \), which are the column dimensions of the full column rank diagonal blocks of the pencil \([A_l - \lambda E_l \ C_l] \) in observability staircase form; |
| tcond       | maximum of the Frobenius-norm condition numbers of the employed non-orthogonal transformation matrices (large values indicate possible loss of numerical stability); |
| fnorm       | the norm of the employed state-feedback \( F \) used for minimum dynamic cover computation (see Method) (large values indicate possible loss of numerical stability). |

\( \text{SYSY} \) contains the descriptor system state-space realization of the resulting (strictly proper) \( Y(\lambda) \) in (161), in the form

\[
E_l \lambda x_{l,2}(t) = A_l x_{l,2}(t) + B_{l,2} v(t),
\]
\[
y_{l,2}(t) = C_l x_{l,2}(t),
\]

(166)

with the pair \((A_l - \lambda E_l, C_l)\) in an observability staircase form with \([A_l \ C_l] \) as in (24) and \( E_l \) upper triangular and nonsingular, as in (25).
Method

To determine \( X(\lambda) \) and \( Y(\lambda) \), with least McMillan degrees, satisfying (161), the function \texttt{glmcover1} calls \texttt{grmcover1} to determine the dual quantities \( \tilde{X}(\lambda) \) and \( \tilde{Y}(\lambda) \) satisfying

\[
\tilde{X}(\lambda) = X^T_1(\lambda) + X^T_2(\lambda)\tilde{Y}(\lambda)
\]

and obtain \( X(\lambda) = \tilde{X}^T(\lambda) \) and \( Y(\lambda) = \tilde{Y}^T(\lambda) \). Thus, the employed solution method corresponds to the dual of the approach sketched in Section 2.11 (see also [53] and \textbf{Procedure GRMCOVER1} in [58, Section 10.4.2] for more details). The function \texttt{grmcover1} relies on the mex-functions \texttt{sl_gstra} to compute a special controllability staircase form (see \textbf{Procedure GSCSF} in [58, Section 10.4.1]) and \texttt{sl_gminr} to compute irreducible realizations.

For the given descriptor system realization of the transfer function matrix \[
\begin{bmatrix}
X_1(\lambda) & X_2(\lambda)
\end{bmatrix}
\]

in the form (164), an output injection \( F \) is determined to obtain the resulting reduced order \( X(\lambda) \) and \( Y(\lambda) \) in (161) with the descriptor realization

\[
\begin{bmatrix}
X(\lambda) & Y(\lambda)
\end{bmatrix} := \begin{bmatrix}
A + FC_2 - \lambda E & B + FD_2 & F \\
C_1 & D_1 & 0
\end{bmatrix}.
\] (167)

The matrix \( F \) is determined such that the pair \((A + FC_2 - \lambda E, C_1)\) is maximally unobservable. Then, the resulting realizations of \( X(\lambda) \) and \( Y(\lambda) \) contain a maximum number of unobservable eigenvalues, which are eliminated by the Type 1 dynamic cover computation algorithm [53]. This algorithm involves the use of non-orthogonal similarity transformations, whose maximum Frobenius-norm condition number is provided in \texttt{INFO.tcond}. The resulting minimal realizations of \( X(\lambda) \) and \( Y(\lambda) \) have the forms (165) and (166), respectively.

The underlying \textbf{Procedure GRMCOVER1} of [58, Section 10.4.2], used in a dual setting, is based on the algorithm proposed in [53] for descriptor systems with invertible \( E \). However, the function \texttt{glmcover1} also works if the original descriptor system realization has \( E \) singular, but \( \text{SYS2} \) is proper. In this case, the order reduction is performed working only with the proper part of \( \text{SYS1} \). The polynomial part of \( \text{SYS1} \) is included, without modification, in the resulting realization of \( \text{SYSX} \).

3.5.9 \texttt{grmcover2}

Syntax

\[
\texttt{[SYSX,INFO,SYS] = grmcover2(SYS1,SYS2,TOL)}
\]

\[
\texttt{[SYSX,INFO,SYS] = grmcover2(SYS,M1,TOL)}
\]

Description

\texttt{grmcover2} computes, for given proper transfer function matrices \( X_1(\lambda) \) and \( X_2(\lambda) \), a proper \( X(\lambda) \) and a proper \( Y(\lambda) \), both with minimal McMillan degrees, which satisfy

\[
X(\lambda) = X_1(\lambda) + X_2(\lambda)Y(\lambda)
\] (168)

and represent the solution of a right minimum cover problem. An approach based on the computation of a minimum dynamic cover of Type 2 is used to determine \( X(\lambda) \) and \( Y(\lambda) \), such that \( \delta(X(\lambda)) \leq \delta(X_1(\lambda)) \). If \( X_1(\lambda) \) is improper, then a maximum reduction of \( \delta(X(\lambda)) \) is achieved, by maximally reducing the order of the proper part of \( X_1(\lambda) \) and keeping unaltered its polynomial part.
**Input data**

For the usage with

\[
\text{[SYSX,INFO,SYSy]} = \text{grmcover2(SYS1,SYS2,TOL)}
\]

the input parameters are as follows:

**SYS1** is a LTI system, whose transfer function matrix \(X_1(\lambda)\) has a descriptor system state-space realization of the form

\[
\begin{align*}
E_1\lambda x_1(t) &= A_1 x_1(t) + B_1 u(t), \\
y_1(t) &= C_1 x_1(t) + D_1 u(t),
\end{align*}
\]

where \(y_1(t) \in \mathbb{R}^p\) and \(u(t) \in \mathbb{R}^{m_1}\).

**SYS2** is a proper LTI system, whose transfer function matrix \(X_2(\lambda)\) has a descriptor system state-space realization of the form

\[
\begin{align*}
E_2\lambda x_2(t) &= A_2 x_2(t) + B_2 v(t), \\
y_2(t) &= C_2 x_2(t) + D_2 v(t),
\end{align*}
\]

where \(y_2(t) \in \mathbb{R}^p\).

**TOL** is a relative tolerance used for rank determinations. If **TOL** is not specified as input or if **TOL** = 0, an internally computed default value is used.

For the usage with

\[
\text{[SYSX,INFO,SYSy]} = \text{grmcover2(SYS,M1,TOL)}
\]

the input parameters are as follows:

**SYS** is an input concatenated compound LTI system, \(SYS = [\text{SYS1 \text{SYS2}}]\), in a descriptor system state-space form

\[
\begin{align*}
E\lambda x(t) &= Ax(t) + B_1 u(t) + B_2 v(t), \\
y(t) &= Cx(t) + D_1 u(t) + D_2 v(t),
\end{align*}
\]

where **SYS1** has the transfer function matrix \(X_1(\lambda)\) with the descriptor system realization \((A - \lambda E, B_1, C, D_1)\) and **SYS2** has the proper transfer function matrix \(X_2(\lambda)\) with the descriptor system realization \((A - \lambda E, B_2, C, D_2)\).

**M1** is the dimension \(m_1\) of the input vector \(u(t)\) of the system **SYS1**.

**TOL** is a relative tolerance used for rank determinations. If **TOL** is not specified as input or if **TOL** = 0, an internally computed default value is used.
Output data

SYSX contains, in the case when both \( X_1(\lambda) \) and \( X_2(\lambda) \) are proper, the descriptor system state-space realization of the resulting reduced order \( X(\lambda) \) in (168), in the form

\[
E_r \lambda x_{r,1}(t) = A_r x_{r,1}(t) + B_r v(t),
\]
\[
y_{r,1}(t) = C_{r,1} x_{r,1}(t) + D_1 v(t),
\]

with the pair \((A_r - \lambda E_r, B_r)\) in a controllability staircase form with \([B_r \ A_r]\) as in (22) and \(E_r\) upper triangular and nonsingular, as in (23). If \( X_1(\lambda) \) is improper, but \( X_2(\lambda) \) is proper, then SYSX contains the descriptor system realization corresponding to the sum of the reduced proper part of \( X_1(\lambda) \) (in the form (172)) and the polynomial part of \( X_1(\lambda) \).

INFO is a MATLAB structure containing additional information, as follows:

| INFO fields | Description |
|-------------|-------------|
| stdim       | vector which contains the dimensions of the diagonal blocks of \( A_r - \lambda E_r \), which are the row dimensions of the full row rank diagonal blocks of the pencil \([B_r \ A_r - \lambda E_r]\) in controllability staircase form; |
| tcond       | maximum of the Frobenius-norm condition numbers of the employed non-orthogonal transformation matrices (large values indicate possible loss of numerical stability); |
| fnorm       | the norm of the employed state-feedback \( F \) used for minimum dynamic cover computation (see Method) (large values indicate possible loss of numerical stability); |
| gnorm       | the norm of the employed feedforward matrix \( G \) used for minimum dynamic cover computation (see Method) (large values indicate possible loss of numerical stability); |

SYSY contains the descriptor system state-space realization of the resulting \( Y(\lambda) \) in (168), in the form

\[
E_r \lambda x_{r,2}(t) = A_r x_{r,2}(t) + B_r v(t),
\]
\[
y_{r,2}(t) = C_{r,2} x_{r,2}(t) + D_r v(t),
\]

with the pair \((A_r - \lambda E_r, B_r)\) in a controllability staircase form with \([B_r \ A_r]\) as in (22) and \(E_r\) upper triangular and nonsingular, as in (23).

Method

The approach to determine \( X(\lambda) \) and \( Y(\lambda) \), with least McMillan degrees, satisfying (154), with both \( X_1(\lambda) \) and \( X_2(\lambda) \) proper, is based on computing a Type 2 minimum dynamic cover [23] and is described in Section 2.11. This approach is based on a realization of the form (171) of the input concatenated compound proper system \( \text{SYS} = [\text{SYS1 SYS2}] \), with invertible \( E \). A detailed computational procedure to determine \( X(\lambda) \) in (168) is described in [53] (see also Procedure GRMCOVER2 in [58, Section 10.4.2] for more details). For the intervening reduction of the partitioned pair \((A - \lambda E_r, [B_2 \ B_1]\) to a special controllability staircase form (see Procedure GSCSF in [58, Section 10.4.1]), the mex-function \texttt{sl_gstra} is employed. If the descriptor
realizations of SYS1 and SYS2 are separately provided, then an irreducible realization of the input concatenated compound system \([\text{SYS1} \ \text{SYS2}]\) is internally computed. For orthogonal transformation based order reduction purposes, the mex-function `sl_gminr` is employed.

For the given descriptor system realization of the transfer function matrix \([X_1(\lambda) \ X_2(\lambda)]\) in the form (171), a state-feedback matrix \(F\) and a feedforward gain \(G\) are determined to obtain the resulting reduced order \(X(\lambda)\) and \(Y(\lambda)\) in (168) with the descriptor realization

\[
\begin{bmatrix}
    X(\lambda) \\
    Y(\lambda)
\end{bmatrix} :=
\begin{bmatrix}
    A + B_2F - \lambda E & B_1 + B_2G \\
    C + D_2F & D_1 + D_2G \\
    F & G
\end{bmatrix}.
\]  

(174)

The feedback matrix \(F\) and feedforward matrix \(G\) are determined such that the descriptor pair \((A + B_2F - \lambda E, B_1 + B_2G)\) is maximally uncontrollable. Then, the resulting realizations of \(X(\lambda)\) and \(Y(\lambda)\) contain a maximum number of uncontrollable eigenvalues, which are eliminated by the Type 2 dynamic cover computation algorithm of [53] for descriptor systems and of [51] for standard systems. This algorithm involves the use of non-orthogonal similarity transformations, whose maximum Frobenius-norm condition number is provided in `INFO.tcond`. The resulting minimal realizations of \(X(\lambda)\) and \(Y(\lambda)\) have the forms (172) and (173), respectively.

As already mentioned, the underlying Procedure `GRMCOVER2` of [58, Section 10.4.2] is based on the algorithm proposed in [53] for descriptor systems with invertible \(E\). However, the function `grmcover2` also works if the original descriptor system realization has \(E\) singular, but SYS2 is proper. In this case, the order reduction is performed working only with the proper part of SYS1. The polynomial part of SYS1 is included, without modification, in the resulting realization of SYSX. To compute the intervening finite-infinite spectral separation, the mex-function `sl_gsep` is employed.

### 3.5.10 glmcover2

**Syntax**

\[
\text{[SYSX,INFO,SYSY]} = \text{glmcover2(SYS1,SYS2,TOL)}
\]

\[
\text{[SYSX,INFO,SYSY]} = \text{glmcover2(SYS,P1,TOL)}
\]

**Description**

`glmcover2` computes, for given proper transfer function matrices \(X_1(\lambda)\) and \(X_2(\lambda)\), a proper \(X(\lambda)\) and a proper \(Y(\lambda)\), both with minimal McMillan degrees, which satisfy

\[
X(\lambda) = X_1(\lambda) + Y(\lambda)X_2(\lambda),
\]  

(175)

and represent the solution of a left minimum cover problem. An approach based on the computation of a minimum dynamic cover of Type 2 is used to determine \(X(\lambda)\) and \(Y(\lambda)\), such that \(\delta(X(\lambda)) \leq \delta(X_1(\lambda))\). If \(X_1(\lambda)\) is improper, then a maximum reduction of \(\delta(X(\lambda))\) is achieved, by maximally reducing the order of the proper part of \(X_1(\lambda)\) and keeping unaltered its polynomial part.

**Input data**

For the usage with
\[ [\text{SYSX}, \text{INFO}, \text{SYSY}] = \text{glmcover2}(\text{SYS1}, \text{SYS2}, \text{TOL}) \]

the input parameters are as follows:

\textbf{SYS1} is a LTI system, whose transfer function matrix \( X_1(\lambda) \) has a descriptor system state-space realization of the form
\[
\begin{align*}
E_1\lambda x_1(t) &= A_1 x_1(t) + B_1 u(t), \\
y_1(t) &= C_1 x_1(t) + D_1 u(t),
\end{align*}
\]
where \( y_1(t) \in \mathbb{R}^{p_1} \) and \( u(t) \in \mathbb{R}^m \).

\textbf{SYS2} is a proper LTI system, whose transfer function matrix \( X_2(\lambda) \) has a descriptor system state-space realization of the form
\[
\begin{align*}
E_2\lambda x_2(t) &= A_2 x_2(t) + B_2 v(t), \\
y_2(t) &= C_2 x_2(t) + D_2 v(t),
\end{align*}
\]
where \( v(t) \in \mathbb{R}^m \).

\textbf{TOL} is a relative tolerance used for rank determinations. If \text{TOL} is not specified as input or if \text{TOL} = 0, an internally computed default value is used.

For the usage with
\[ [\text{SYSX}, \text{INFO}, \text{SYSY}] = \text{glmcover2}(\text{SYS}, \text{P1}, \text{TOL}) \]

the input parameters are as follows:

\textbf{SYS} is an output concatenated compound LTI system, \( \text{SYS} = [\text{SYS1}; \text{SYS2}] \), in a descriptor system state-space form
\[
\begin{align*}
E \lambda x(t) &= Ax(t) + Bu(t), \\
y_1(t) &= C_1 x(t) + D_1 u(t), \\
y_2(t) &= C_2 x(t) + D_2 u(t),
\end{align*}
\]
where \text{SYS1} has the transfer function matrix \( X_1(\lambda) \) with the descriptor system realization \((A - \lambda E, B, C_1, D_1)\) and \text{SYS2} has the proper transfer function matrix \( X_2(\lambda) \) with the descriptor system realization \((A - \lambda E, B, C_2, D_2)\).

\textbf{P1} is the dimension \( p_1 \) of the output vector \( y_1(t) \) of the system \text{SYS1}.

\textbf{TOL} is a relative tolerance used for rank determinations. If \text{TOL} is not specified as input or if \text{TOL} = 0, an internally computed default value is used.

**Output data**

\textbf{SYSX} contains, in the case when both \( X_1(\lambda) \) and \( X_2(\lambda) \) are proper, the descriptor system state-space realization of the resulting reduced order \( X(\lambda) \) in (175), in the form
\[
\begin{align*}
E_1\lambda x_{1,1}(t) &= A_1 x_{1,1}(t) + B_{1,1} v(t), \\
y_{1,1}(t) &= C_{1} x_{1,1}(t) + D_{1} v(t),
\end{align*}
\]
with the pair \( (A_l - \lambda E_l, C_l) \) in an observability staircase form with \( \begin{bmatrix} A_l \\ C_l \end{bmatrix} \) as in (24) and \( E_l \) upper triangular and nonsingular, as in (25). If \( X_1(\lambda) \) is improper, but \( X_2(\lambda) \) is proper, then \( SYSX \) contains the descriptor system realization corresponding to the sum of the reduced proper part of \( X_1(\lambda) \) (in the form (179)) and the polynomial part of \( X_1(\lambda) \).

**INFO** is a MATLAB structure containing additional information, as follows:

| INFO fields | Description |
|-------------|-------------|
| stdim       | vector containing the dimensions of the diagonal blocks of \( A_l - \lambda E_l \), which are the column dimensions of the full column rank diagonal blocks of the pencil \( \begin{bmatrix} A_l - \lambda E_l \\ C_l \end{bmatrix} \) in observability staircase form; |
| tcond       | maximum of the Frobenius-norm condition numbers of the employed non-orthogonal transformation matrices (large values indicate possible loss of numerical stability); |
| fnorm       | the norm of the employed state-feedback \( F \) used for minimum dynamic cover computation (see Method) (large values indicate possible loss of numerical stability). |

\( SYSY \) contains the descriptor system state-space realization of the resulting \( Y(\lambda) \) in (175), in the form

\[
E_l \lambda x_{l,2}(t) = A_l x_{l,2}(t) + B_{l,2} v(t), \quad y_{l,2}(t) = C_l x_{l,2}(t) + D_{l,2} v(t),
\]

with the pair \( (A_l - \lambda E_l, C_l) \) in an observability staircase form with \( \begin{bmatrix} A_l \\ C_l \end{bmatrix} \) as in (24) and \( E_l \) upper triangular and nonsingular, as in (25).

**Method**

To determine \( X(\lambda) \) and \( Y(\lambda) \), with least McMillan degrees, satisfying (161), the function \texttt{glmcover2} calls \texttt{grmcover2} to determine the dual quantities \( \tilde{X}(\lambda) \) and \( \tilde{Y}(\lambda) \) satisfying

\[
\tilde{X}(\lambda) = X^T_1(\lambda) + X^T_2(\lambda) \tilde{Y}(\lambda)
\]

and obtain \( X(\lambda) = \tilde{X}^T(\lambda) \) and \( Y(\lambda) = \tilde{Y}^T(\lambda) \). Thus, the employed solution method corresponds to the dual of the approach sketched in Section 2.11 (see also [53] and Procedure \texttt{GRMCOVER2} in [58, Section 10.4.2] for more details). The function \texttt{grmcover2} relies on the mex-functions \texttt{s1_gstra} to compute a special controllability staircase form (see Procedure \texttt{GSCSF} in [58, Section 10.4.1]) and \texttt{s1_gminr} to compute irreducible realizations.

For the given descriptor system realization of the transfer function matrix \( \begin{bmatrix} X_1(\lambda) \\ X_2(\lambda) \end{bmatrix} \) in the form (178), an output injection \( F \) is determined to obtain the resulting reduced order \( \tilde{X}(\lambda) \) and \( \tilde{Y}(\lambda) \) in (175) with the descriptor realization

\[
[ X(\lambda) \ Y(\lambda) ] = \begin{bmatrix} A + FC_2 - \lambda E \\ C_1 + GC_2 \end{bmatrix} \begin{bmatrix} B + FD_2 \\ D_1 + GD_2 \end{bmatrix} F.
\]
The matrices $F$ and $G$ are determined such that the pair $(A + FC_2 - \lambda E, C_1 + GC_2)$ is maximally unobservable. Then, the resulting realizations of $X(\lambda)$ and $Y(\lambda)$ contain a maximum number of unobservable eigenvalues, which are eliminated by the Type 2 dynamic cover computation algorithm [53]. This algorithm involves the use of non-orthogonal similarity transformations, whose maximum Frobenius-norm condition number is provided in $\text{INFO.tcond}$. The resulting minimal realizations of $X(\lambda)$ and $Y(\lambda)$ have the forms (179) and (180), respectively.

The underlying Procedure GRMCOVER2 of [58, Section 10.4.2], used in a dual setting, is based on the algorithm proposed in [53] for descriptor systems with invertible $E$. However, the function glmcover2 also works if the original descriptor system realization has $E$ singular, but $\text{SYS2}$ is proper. In this case, the order reduction is performed working only with the proper part of $\text{SYS1}$. The polynomial part of $\text{SYS1}$ is included, without modification, in the resulting realization of $\text{SYSX}$.

Example 14. This example, taken from [62], has been already considered in Example 8 to illustrate the computation of a left inverse of least McMillan degree of a full column rank transfer function matrix $G(s)$, using the function glmcover2. In this example, we illustrates an alternative way to compute a left inverse of least McMillan degree, using the generators of all solution of the equation $X(\lambda)G(\lambda) = I$. A generator of all solutions is formed from a pair $(X_0(s), N_l(s))$, where $X_0(\lambda)$ is any particular solution (i.e., any particular left inverse) and $N_l(s)$ is a proper basis of the left nullspace of $G(s)$. Such a generator can be also computed using glmcover2.

The determination of a least order left inverse can be alternatively formulated in terms of the solution of the following left minimal cover problem: for a given generator $(X_0(s), N_l(s))$, compute least order $X(s)$ and $Y(s)$ such that

$$X(s) = X_0(s) + N_l(s)Y(s).$$

Recall transfer function matrix used in Example 8

$$G(s) = \frac{1}{s^2 + 3s + 2} \begin{bmatrix} s + 1 & s + 2 \\ s + 3 & s^2 + 2s \\ s^2 + 3s & 0 \end{bmatrix},$$

which has a third order minimal state-space realization, which can be computed using the function gir (or minreal). This realization is used to compute, using glmcover2, a pair $(X_0(s), N_l(s))$ representing a generator of all solutions of $X(s)G(s) = I$. The function glmcover2 is then used to solve the above left minimal cover problem, to determine a least McMillan order proper left inverse $X(s)$. The following MATLAB commands illustrate this approach:

```
% Wang and Davison Example (1973)
s = tf('s');
g = [ s+1 s+2; s+3 s^2+2*s; s^2+3*s 0 ]/(s^2+3*s+2);
sysg = gir(ss(g));

% compute a generator (X0,N1) of all solutions of X(s)G(s) = I
[~,~,sysgen] = glmcover2(sysg,ss(eye(2)));
% X0 = sysgen(:,1:2), N1 = sysgen(:,3)
gpole(sysgen) % the generator is proper (no infinite poles)
```
% compute a least order inverse as $X(s) = X_0(s) + NL(s) * Y(s)$, by using
% order reduction based on a left minimal dynamic cover
[sysx,~,sysy] = glmcover2(sysgen,2);
gpole(sysx) % resulting 2nd (least order) left inverse is unstable

% check solution applying gminreal to sysx*sysg-eye(2) or computing its norm
rez = gminreal(sysx*sysg-eye(2))
norm(sysx*sysg-eye(2),inf)

The computed solution is unstable. Since the resulting $Y(s)$ is strictly proper, the same
approach can be used by calling the function glmcover1 instead glmcover2.

3.6 Functions for Factorizations

These functions cover the computation of several factorizations of rational matrices, such as the
coprime factorizations, inner-outer factorizations, and special spectral factorizations.

3.6.1 grcf

Syntax

[SYSN,SYSM] = grcf(SYS,OPTIONS)

Description

grcf computes, for the transfer function matrix $G(\lambda)$ of a LTI descriptor state-space system, a
right coprime factorization in the form

$$G(\lambda) = N(\lambda) M^{-1}(\lambda),$$

(182)
such that $N(\lambda)$ and $M(\lambda)$ are proper transfer function matrices with poles in a specified stability
region $C_g \subset \mathbb{C}$.

Input data

SYS is a LTI system, whose transfer function matrix is $G(\lambda)$, and is in a descriptor system
state-space form

$$E\lambda x(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

(183)

with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$.

OPTIONS is a MATLAB structure to specify user options and has the following fields:

| OPTIONS fields | Description |
|----------------|-------------|
| tol            | tolerance for the singular values based rank determination of $E$
                | (Default: $n^2\|E\|\cdot\text{eps}$) |
| tolmin | tolerance for the singular values based controllability tests  
|        | (Default: $nm \| B \|_1 \epsilon$) |
| smarg  | stability margin which specifies the stability region $C_g$ of the eigenvalues of the pole pencil as follows: in the continuous-time case, the stable eigenvalues have real parts less than or equal to \texttt{OPTIONS.smarg}, and in the discrete-time case, the stable eigenvalues have moduli less than or equal to \texttt{OPTIONS.smarg}.  
|        | (Default: $-\sqrt{\epsilon}$ for a continuous-time system \texttt{SYS}; $1-\sqrt{\epsilon}$ for a discrete-time system \texttt{SYS}). |
| sdeg   | prescribed stability degree for the poles of the factors assigned within $C_g$  
|        | (Default: [ ] ) |
| poles  | complex conjugated set of desired poles to be assigned for the factors  
|        | (Default: [ ] ) |
| mindeg | option to compute a minimum degree denominator:  
|        | \texttt{true} – determine a minimum degree denominator;  
|        | \texttt{false} – determine both factors with the same order (default) |
| mininf | option for the removal of simple infinite eigenvalues (non-dynamic modes) of the factors:  
|        | \texttt{true} – remove simple infinite eigenvalues;  
|        | \texttt{false} – keep simple infinite eigenvalues (default) |

Output data

SYSN contains the descriptor system state-space realization of the numerator factor $N(\lambda)$ in the form

$$E_N \lambda x_N(t) = A_N x_N(t) + B_N v(t),$$

$$y_N(t) = C_N x_N(t) + D_N v(t),$$  \hspace{1cm} (184)

where the pair $(A_N, E_N)$ is in a GRSF. The eigenvalues of $A_N - \lambda E_N$ include all eigenvalues of $A - \lambda E$ in $C_g$, and, additionally, all assigned eigenvalues in accordance with the specified stability degree in \texttt{OPTIONS.sdeg} and poles in \texttt{OPTIONS.poles}. The resulting $E_N$ is invertible if \texttt{OPTIONS.mininf} = \texttt{true}. If \texttt{OPTIONS.mindeg} = \texttt{true} then the eigenvalues of $A_M - \lambda E_M$ include only the assigned eigenvalues in accordance with the specified stability degree in \texttt{OPTIONS.sdeg} and poles in \texttt{OPTIONS.poles}. In this case, the resulting $E_M$ is always invertible.
Method

For the definitions related to coprime factorizations of transfer function matrices see Section 2.8. The implemented computational methods to compute the right coprime factorizations of general rational matrices rely on a preliminary orthogonal reduction of the pole pencil $A - \lambda E$ to a special GRSF, which allows to preserve all eigenvalues of $A - \lambda E$ in $\mathbb{C}_g$ in the resulting factors. The underlying reduction is described in [57]. The function grcf implements the Procedure GRCF of [57], which represents an extension of the recursive factorization approach of [49] to cope with infinite poles. In this procedure, all infinite poles are first assigned to finite real values.

If OPTIONS.poles is empty, then a stabilization oriented factorization is performed (see [57]), where the infinite poles are assigned to real values specified by OPTIONS.sdeg and all finite poles lying outside $\mathbb{C}_g$ are assigned to the nearest values having a stability margin specified by OPTIONS.sdeg. If OPTIONS.poles is not empty, then a pole assignment oriented factorization is performed, by assigning first all infinite poles to real values specified in OPTIONS.poles. If OPTIONS.poles does not contain a sufficient number of real values, then a part or all of infinite poles are assigned to the value specified by OPTIONS.sdeg. Then, all finite poles lying outside $\mathbb{C}_g$ are assigned to values specified in OPTIONS.poles or assigned to the values having a stability margin specified by OPTIONS.sdeg.

Example

**Example 15.** Consider the continuous-time improper TFM

$$G(s) = \begin{bmatrix} s^2 & s \\ \frac{s}{s+1} & 0 \\ 0 & 1 \\ \frac{1}{s} \end{bmatrix},$$

which has the following set of poles: $\{-1, 0, \infty, \infty\}$. To compute a stable and proper right coprime factorization of $G(s)$, we employed the pole assignment oriented factorization, with a stability degree of $-1$, a desired set of poles $\{-1, -2, -3\}$, and with the option to eliminate the simple infinite eigenvalues. The resulting factors have the transfer function matrices

$$N(s) = \begin{bmatrix} -\frac{s^2}{(s+1)(s+2)} & \frac{s^2}{(s+1)(s+3)} \\ 0 & \frac{1}{s+3} \end{bmatrix}, \quad M(s) = \begin{bmatrix} -\frac{1}{(s+1)(s+2)} & 0 \\ 0 & \frac{s}{s+3} \end{bmatrix}. $$

The McMillan degree of $M(s)$ is three, thus the least possible one. The above factors have been computed with the following sequence of commands:

```plaintext
% Varga (2017), Example 1  
s = tf('s'); % define the complex variable s  
% enter G(s) and determine a minimal state-space realization  
G = [s^2 s/(s+1); 0 1/s];  
sys = ss(G);  
gpole(sys) \% the system is unstable and improper
```
% compute the right coprime factorization \( G(s) = N(s)\cdot \text{inv}(M(s)) \)
[sysn,sysm] = grcf(sys,struct('poles',[-1,-2,-3,-4],'sdeg',-1,'mininf',true));

% check the factorization \( ||G(s)\cdot M(s) - N(s)||_{\infty} = 0 \)
norm(gminreal(sys*sysm-sysn),inf) = 0

% check the poles of the factors
gpole(sysm), gpole(sysn)

% check coprimeness of the factors
gzero(gir([sysn;sysm])) % \([N(s); M(s)]\) has no zeros

\[3.6.2\] \textbf{glcf}

\textbf{Syntax}

[SYSN,SYSM] = glcf(SYS,OPTIONS)

\textbf{Description}

\textbf{glcf} computes, for the transfer function matrix \( G(\lambda) \) of a LTI descriptor state-space system, a left coprime factorization in the form

\[
G(\lambda) = M^{-1}(\lambda)N(\lambda),
\]

such that \( N(\lambda) \) and \( M(\lambda) \) are proper transfer function matrices with poles in a specified stability region \( C_g \subset \mathbb{C} \).

\textbf{Input data}

\textbf{SYS} is a LTI system, whose transfer function matrix is \( G(\lambda) \), and is in a descriptor system state-space form

\[
E\lambda x(t) = Ax(t) + Bu(t),
\]

\[
y(t) = Cx(t) + Du(t),
\]

with \( x(t) \in \mathbb{R}^n \) and \( y(t) \in \mathbb{R}^p \).

\textbf{OPTIONS} is a MATLAB structure to specify user options and has the following fields:

| OPTIONS fields | Description |
|----------------|-------------|
| \textbf{tol}   | tolerance for the singular values based rank determination of \( E \) (Default: \( n^2\|E\|_1\epsilon_p \)) |
| \textbf{tolmin}| tolerance for the singular values based observability tests (Default: \( np\|C\|_\infty\epsilon_p \)) |
smarg stability margin which specifies the stability region $C_g$ of the eigenvalues of the pole pencil as follows: in the continuous-time case, the stable eigenvalues have real parts less than or equal to OPTIONS.smarg, and in the discrete-time case, the stable eigenvalues have moduli less than or equal to OPTIONS.smarg. (Default: $-\sqrt{\text{eps}}$ for a continuous-time system SYS; $1-\sqrt{\text{eps}}$ for a discrete-time system SYS.)

sdeg prescribed stability degree for the poles of the factors assigned within $C_g$ (Default: [ ])

poles complex conjugated set of desired poles to be assigned for the factors (Default: [ ])

mindeg option to compute a minimum degree denominator:
true - determine a minimum degree denominator;
false - determine both factors with the same order (default)

mininf option for the removal of simple infinite eigenvalues (non-dynamic modes) of the factors:
true - remove simple infinite eigenvalues;
false - keep simple infinite eigenvalues (default)

Output data

SYSN contains the descriptor system state-space realization of the numerator factor $N(\lambda)$ in the form

$$
E_N \lambda x_N(t) = A_N x_N(t) + B_N u(t),
$$

$$
y_N(t) = C_N x_N(t) + D_N u(t),
$$

where the pair $(A_N, E_N)$ is in a GRSF. The eigenvalues of $A_N - \lambda E_N$ include all eigenvalues of $A - \lambda E$ in $C_g$, and, additionally, all assigned eigenvalues in accordance with the specified stability degree in OPTIONS.sdeg and poles in OPTIONS.poles. The resulting $E_N$ is invertible if OPTIONS.mininf = true.

SYSM contains the descriptor system state-space realization of the denominator factor $M(\lambda)$ in the form

$$
E_M \lambda x_M(t) = A_M x_M(t) + B_M w(t),
$$

$$
y_M(t) = C_M x_M(t) + D_M w(t),
$$

where the pair $(A_M, E_M)$ is in a GRSF. If OPTIONS.mindeg = false, then $N(\lambda)$ and $M(\lambda)$ have realizations of the same order with $E_M = E_N$, $A_M = A_N$, and $C_M = C_N$ and the eigenvalues of $A_M - \lambda E_M$ include all eigenvalues of $A - \lambda E$ in $C_g$, which are however uncontrollable. Additionally, the eigenvalues of $A_M - \lambda E_M$ include the assigned eigenvalues in accordance with the specified stability degree in OPTIONS.sdeg and poles in OPTIONS.poles. The resulting $E_M$ is invertible if OPTIONS.mininf = true. If OPTIONS.mindeg = true then the eigenvalues of $A_M - \lambda E_M$ include only the assigned eigenvalues in accordance with the specified stability degree in OPTIONS.sdeg and poles in OPTIONS.poles. In this case, the resulting $E_M$ is always invertible.
Method

For the definitions related to coprime factorizations of transfer function matrices see Section 2.8. To compute the left coprime factorization (187), the function `glcf` calls `grcf` to compute the right coprime factorization of $G^T(\lambda)$ in the form

$$G^T(\lambda) = \tilde{N}(\lambda)\tilde{M}^{-1}(\lambda)$$

and obtain the factors as $N(\lambda) = \tilde{N}^T(\lambda)$ and $M(\lambda) = \tilde{M}^T(\lambda)$. The function `grcf` implements the Procedure GRCF of [57], which represents an extension of the recursive factorization approach of [49] to cope with infinite poles.

If `OPTIONS.poles` is empty, then a stabilization oriented factorization is performed (see [57]), where the infinite poles are assigned to real values specified by `OPTIONS.sdeg` and all finite poles lying outside $C_g$ are assigned to the nearest values having a stability margin specified by `OPTIONS.sdeg`. If `OPTIONS.poles` is not empty, then a pole assignment oriented factorization is performed, by assigning first all infinite poles to real values specified in `OPTIONS.poles`. If `OPTIONS.poles` does not contain a sufficient number of real values, then a part or all of infinite poles are assigned to the value specified by `OPTIONS.sdeg`. Then, all finite poles lying outside $C_g$ are assigned to values specified in `OPTIONS.poles` or assigned to the values having a stability margin specified by `OPTIONS.sdeg`.

Example

Example 16. Consider the continuous-time improper TFM

$$G(s) = \begin{bmatrix} s^2 & \frac{s}{s+1} \\ 0 & \frac{1}{s} \end{bmatrix}, \quad (191)$$

which has the following set of poles: $\{-1, 0, \infty, \infty\}$. To compute a stable and proper left coprime factorization of $G(s)$, we employed the pole assignment oriented factorization, with a stability degree of $-1$, a desired set of poles $\{-1, -2, -3\}$, and with the option to eliminate the simple infinite eigenvalues. The resulting factors have the transfer function matrices

$$N(s) = \begin{bmatrix} -\frac{s^2}{(s+1)(s+2)} & -\frac{s}{(s+1)^2(s+2)} \\ 0 & \frac{1}{s+3} \end{bmatrix}, \quad M(s) = \begin{bmatrix} -\frac{1}{(s+1)(s+2)} & 0 \\ 0 & \frac{s}{s+3} \end{bmatrix}. $$

The McMillan degree of $M(s)$ is three, thus the least possible one. The above factors have been computed with the following sequence of commands:

```
% Varga (2017), Example 1
s = tf('s'); % define the complex variable s
% enter G(s) and determine a minimal state-space realization
G = [s^2 s/(s+1); 0 1/s];
```
sys = ss(G);
gpole(sys)    % the system is unstable and improper

% compute the left coprime factorization G(s) = inv(M(s))*N(s)
[sysn,sysm] = glcf(sys,struct('poles',[-1,-2,-3,-4],'sdeg',-1,'mininf',true));

% check the factorization ||M(s)*G(s)-N(s)||_inf = 0
norm(gminreal(sysm*sys-sysn,1.e-7),inf)

% check the poles of the factors
gpole(sysm), gpole(sysn)

% check coprimeness of the factors
gzero(gir([sysn sysm])) % [N(s) M(s)] has no zeros

3.6.3 grcfid

Syntax
[SYSN,SYSM] = grcfid(SYS,OPTIONS)

Description
grcfid computes, for the transfer function matrix $G(\lambda)$ of a LTI descriptor state-space system, a right coprime factorization with inner denominator in the form

$$G(\lambda) = N(\lambda)M^{-1}(\lambda),$$

such that $N(\lambda)$ and $M(\lambda)$ are proper and stable transfer function matrices, and $M(\lambda)$ is inner.

Input data
SYS is a LTI system, whose transfer function matrix is $G(\lambda)$, and is in a descriptor system state-space form

$$E\lambda x(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. $G(\lambda)$ must not have poles in $\partial C_s$.

OPTIONS is a MATLAB structure to specify user options and has the following fields:

| OPTIONS fields | Description |
|----------------|-------------|
| tol            | tolerance for the singular values based rank determination of $E$ (Default: $n^2\|E\|_1\epsilon$) |
| tolmin         | tolerance for the singular values based controllability tests (Default: $nm\|B\|_1\epsilon$) |
**mindeg** | option to compute a minimum degree denominator:
---|---
true | determine a minimum degree denominator;
false | determine both factors with the same order (default)

**mininf** | option for the removal of simple infinite eigenvalues (non-dynamic modes) of the factors:
---|---
true | remove simple infinite eigenvalues;
false | keep simple infinite eigenvalues (default)

### Output data

**SYSN** contains the descriptor system state-space realization of the numerator factor \(N(\lambda)\) in the form

\[
E_N \lambda x_N(t) = A_N x_N(t) + B_N v(t),
\]

\[
y_N(t) = C_N x_N(t) + D_N v(t),
\]

(194)

where the pair \((A_N, E_N)\) is in a GRSF. The eigenvalues of \(A_N - \lambda E_N\) include all stable eigenvalues of \(A - \lambda E\) (i.e., eigenvalues located in \(C_s\)). Additionally, to each unstable eigenvalue of \(A - \lambda E\) corresponds a stable eigenvalue of \(A_N - \lambda E_N\) located in a symmetric location with respect to the imaginary axis, in the continuous-time case, or with respect to the unit circle centered in the origin, in the discrete-time case. The resulting \(E_N\) is invertible if OPTIONS.mininf = true.

**SYSM** contains the descriptor system state-space realization of the inner denominator factor \(M(\lambda)\) in the form

\[
E_M \lambda x_M(t) = A_M x_M(t) + B_M w(t),
\]

\[
y_M(t) = C_M x_M(t) + D_M w(t),
\]

(195)

where the pair \((A_M, E_M)\) is in a GRSF. The resulting \(E_M\) is invertible if OPTIONS.mininf = true. If OPTIONS.mindeg = false, then \(N(\lambda)\) and \(M(\lambda)\) have realizations of the same order with \(E_M = E_N\), \(A_M = A_N\), and \(B_M = B_N\) and the eigenvalues of \(A_M - \lambda E_M\) include all stable eigenvalues of \(A - \lambda E\), which are however unobservable. Additionally, to each unstable eigenvalue of \(\lambda_u \in \Lambda(A - \lambda E)\) corresponds a stable eigenvalue of \(A_M - \lambda E_M\) located in a symmetric location with respect to the boundary of the appropriate stability domain (i.e., \(-\lambda_u \in \Lambda(A_M - \lambda E_M)\), in the continuous-time case, or \(1/\lambda_u \in \Lambda(A_M - \lambda E_M)\) in the discrete-time case). If OPTIONS.mindeg = true, only the latter eigenvalues are present and the resulting \(E_M\) is always invertible.

### Method

For the definitions related to coprime factorizations of transfer function matrices see Section 2.8. The implemented computational methods to compute the right coprime factorizations with inner denominators of transfer function matrices rely on a preliminary orthogonal reduction of the pole pencil \(A - \lambda E\) to a special GRSF, which allows to isolate all eigenvalues of \(A - \lambda E\) lying outside of the stability domain \(C_s\). The underlying reduction is described in [57]. The function grcfid implements the Procedure GRCFID of [57], which represents an extension of the corresponding recursive factorization approach of [49] to cope with infinite poles in the discrete-time case. In this procedure, all infinite poles of a discrete-time system are reflected into poles in the origin of the factors.
Consider the discrete-time improper TFM
\[ G(z) = \begin{bmatrix} z^2 & \frac{z}{z - 2} \\ 0 & \frac{1}{z} \end{bmatrix}, \] (196)
which has the following set of poles: \( \{2, 0, \infty, \infty\} \) and therefore, the right coprime factorization with inner denominator exists. With the option to eliminate the simple infinite eigenvalues, the function \texttt{grcfid} computes the following factors having the transfer function matrices
\[ N(z) = \begin{bmatrix} 1 & \frac{1}{2z - 1} \\ 0 & \frac{z - 2}{z(2z - 1)} \end{bmatrix}, \quad M(z) = \begin{bmatrix} 1 & 0 \\ \frac{z^2}{2z - 1} & \frac{z - 2}{2z - 1} \end{bmatrix}. \]

The McMillan degree of \( M(z) \) is three, thus the least possible one. Interestingly, the McMillan degree of \( N(z) \) is only two, because two unobservable eigenvalues in 0 have been removed. These eigenvalues are the zeros of the (improper) all-pass factor \( \text{diag}(z^2, 1) \) with two infinite poles, which is contained in \( G(z) \).

The above factors have been computed with the following sequence of commands:

```matlab
% Varga (2017), Example 2
z = tf('z'); % define the complex variable z
% enter G(z) and determine a minimal state-space realization
G = [z^2 z/(z-2); 0 1/z];
sys = ss(G);
gpole(sys) % the system is unstable and improper

% compute the right coprime factorization G(z) = N(z)*inv(M(z)),
% with inner denominator M(z)
[sysn,sysm] = grcfid(sys,struct('mininf',true));

% check the factorization \( \|G(z)*M(z)-N(z)\|_\text{inf} = 0 \)
norm(gminreal(sys*sysm-sysn),inf)

% check the innerness of M(z): \( \|\text{conj}(M(z))*M(z)-I\|_\text{inf} = 0 \)
norm(sysm'*sysm-eye(2),inf)

% check the poles of the factors
gpole(sysm), gpole(sysn)

% check coprimeness of the factors
gzero(gir([sysn;sysm])) % [N(z); M(z)] has no zeros
```

\*

Example

\textit{Example 17.} Consider the discrete-time improper TFM

\[ G(z) = \begin{bmatrix} z^2 & \frac{z}{z - 2} \\ 0 & \frac{1}{z} \end{bmatrix}, \] (196)

which has the following set of poles: \( \{2, 0, \infty, \infty\} \) and therefore, the right coprime factorization with inner denominator exists. With the option to eliminate the simple infinite eigenvalues, the function \texttt{grcfid} computes the following factors having the transfer function matrices

\[ N(z) = \begin{bmatrix} 1 & \frac{1}{2z - 1} \\ 0 & \frac{z - 2}{z(2z - 1)} \end{bmatrix}, \quad M(z) = \begin{bmatrix} 1 & 0 \\ \frac{z^2}{2z - 1} & \frac{z - 2}{2z - 1} \end{bmatrix}. \]

The McMillan degree of \( M(z) \) is three, thus the least possible one. Interestingly, the McMillan degree of \( N(z) \) is only two, because two unobservable eigenvalues in 0 have been removed. These eigenvalues are the zeros of the (improper) all-pass factor \( \text{diag}(z^2, 1) \) with two infinite poles, which is contained in \( G(z) \).

The above factors have been computed with the following sequence of commands:

```matlab
% Varga (2017), Example 2
z = tf('z'); % define the complex variable z
% enter G(z) and determine a minimal state-space realization
G = [z^2 z/(z-2); 0 1/z];
sys = ss(G);
gpole(sys) % the system is unstable and improper

% compute the right coprime factorization G(z) = N(z)*inv(M(z)),
% with inner denominator M(z)
[sysn,sysm] = grcfid(sys,struct('mininf',true));

% check the factorization \( \|G(z)*M(z)-N(z)\|_\text{inf} = 0 \)
norm(gminreal(sys*sysm-sysn),inf)

% check the innerness of M(z): \( \|\text{conj}(M(z))*M(z)-I\|_\text{inf} = 0 \)
norm(sysm'*sysm-eye(2),inf)

% check the poles of the factors
gpole(sysm), gpole(sysn)

% check coprimeness of the factors
gzero(gir([sysn;sysm])) % [N(z); M(z)] has no zeros
```

\*



111
3.6.4 glcfid

Syntax

\[ [\text{SYSN}, \text{SYSM}] = \text{glcfid}(\text{SYS}, \text{OPTIONS}) \]

Description

glcfid computes, for the transfer function matrix \( G(\lambda) \) of a LTI descriptor state-space system, a left coprime factorization with inner denominator in the form

\[
G(\lambda) = M^{-1}(\lambda)N(\lambda),
\]

such that \( N(\lambda) \) and \( M(\lambda) \) are proper and stable transfer function matrices, and \( M(\lambda) \) is inner.

Input data

SYS is a LTI system, whose transfer function matrix is \( G(\lambda) \), and is in a descriptor system state-space form

\[
\begin{align*}
E\lambda x(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t),
\end{align*}
\]

(198)

with \( x(t) \in \mathbb{R}^n \) and \( y(t) \in \mathbb{R}^p \). \( G(\lambda) \) must not have poles in \( \partial C_s \).

OPTIONS is a MATLAB structure to specify user options and has the following fields:

| OPTIONS fields | Description |
|----------------|-------------|
| tol            | tolerance for the singular values based rank determination of \( E \) (Default: \( n^2\|E\|_1\text{eps} \)) |
| tolmin         | tolerance for the singular values based observability tests (Default: \( np\|C\|_\infty\text{eps} \)) |
| mindeg         | option to compute a minimum degree denominator: \text{true} – determine a minimum degree denominator; \text{false} – determine both factors with the same order (default) |
| mininf         | option for the removal of simple infinite eigenvalues (non-dynamic modes) of the factors: \text{true} – remove simple infinite eigenvalues; \text{false} – keep simple infinite eigenvalues (default) |

Output data

SYSN contains the descriptor system state-space realization of the numerator factor \( N(\lambda) \) in the form

\[
\begin{align*}
E_N\lambda x_N(t) &= A_Nx_N(t) + B_Nu(t), \\
y_N(t) &= C_Nx_N(t) + D_Nu(t),
\end{align*}
\]

(199)

where the pair \((A_N, E_N)\) is in a GRSF. The eigenvalues of \( A_N - \lambda E_N \) include all stable eigenvalues of \( A - \lambda E \) (i.e., eigenvalues located in \( C_s \)). Additionally, to each unstable eigenvalue of \( A - \lambda E \) corresponds a stable eigenvalue of \( A_N - \lambda E_N \) located in a symmetric
location with respect to the imaginary axis, in the continuous-time case, or with respect to the unit circle centered in the origin, in the discrete-time case. The resulting $E_N$ is invertible if \texttt{OPTIONS.mininf} = true.

\textbf{SYSM} contains the descriptor system state-space realization of the inner denominator factor $M(\lambda)$ in the form

$$
E_M \lambda x_M(t) = A_M x_M(t) + B_M w(t),
\quad
y_M(t) = C_M x_M(t) + D_M w(t),
$$

(200)

where the pair $(A_M, E_M)$ is in a GRSF. The resulting $E_M$ is invertible if \texttt{OPTIONS.mininf} = true. If \texttt{OPTIONS.mindeg} = false, then $N(\lambda)$ and $M(\lambda)$ have realizations of the same order with $E_M = E_N$, $A_M = A_N$, and $B_M = B_N$ and the eigenvalues of $A_M - \lambda E_M$ include all stable eigenvalues of $A - \lambda E$, which are however unobservable. Additionally, to each unstable eigenvalue of $\lambda_u \in \Lambda(A - \lambda E)$ corresponds a stable eigenvalue of $A_M - \lambda E_M$ located in a symmetric location with respect to the boundary of the appropriate stability domain (i.e., $-\bar{\lambda}_u \in \Lambda(A_M - \lambda E_M)$, in the continuous-time case, or $1/\lambda_u \in \Lambda(A_M - \lambda E_M)$ in the discrete-time case). If \texttt{OPTIONS.mindeg} = true, only the latter eigenvalues are present and the resulting $E_M$ is always invertible.

\textbf{Method}

For the definitions related to coprime factorizations of transfer function matrices see Section 2.8. To compute the left coprime factorization (197), the function \texttt{glcfid} calls \texttt{grcfid} to compute the right coprime factorization with inner denominator of $G^T(\lambda)$ in the form

$$
G^T(\lambda) = \tilde{N}(\lambda)\tilde{M}^{-1}(\lambda)
$$

and obtain the factors as $N(\lambda) = \tilde{N}^T(\lambda)$ and $M(\lambda) = \tilde{M}^T(\lambda)$. The function \texttt{grcfid} implements the Procedure GRCFID of [57], which represents an extension of the corresponding recursive factorization approach of [49] to cope with infinite poles in the discrete-time case.

\textbf{Example}

\textit{Example 18.} Consider the discrete-time improper TFM

$$
G(z) = 
\begin{bmatrix}
  z^2 & \frac{z}{z - 2} \\
  0 & \frac{1}{z}
\end{bmatrix},
$$

(201)

which has the following set of poles: $\{2, 0, \infty, \infty\}$ and therefore, the left coprime factorization with inner denominator exists. With the option to eliminate the simple infinite eigenvalues, the function \texttt{glcfid} computes the following factors having the transfer function matrices

$$
N(z) = 
\begin{bmatrix}
  \frac{z - 2}{2z - 1} & \frac{1}{z(2z - 1)} \\
  0 & \frac{1}{z}
\end{bmatrix},
\quad
M(z) = 
\begin{bmatrix}
  \frac{z - 2}{z^2(2z - 1)} & 0 \\
  0 & 1
\end{bmatrix}.
$$
The McMillan degree of $M(z)$ is three, thus the least possible one. Interestingly, the McMillan degree of $N(z)$ is only two, because two unobservable eigenvalues in 0 have been removed. These eigenvalues are the zeros of the (improper) all-pass factor $\text{diag}(z^2, 1)$ with two infinite poles, which is contained in $G(z)$.

The above factors have been computed with the following sequence of commands:

```matlab
% Varga (2017), Example 2
z = tf('z'); % define the complex variable z
% enter G(z) and determine a minimal state-space realization
G = [z^2 z/(z-2);
     0 1/z];
sys = ss(G);
gpole(sys) % the system is unstable and improper

% compute the right coprime factorization $G(z) = \text{inv}(M(z))*N(z)$,
% with inner denominator $M(z)$
[sysn,sysm] = glcfid(sys,struct('mininf',true));

% check the factorization $\|M(z)*G(z)-N(z)\|_{\infty} = 0$
norm(gminreal(sysm*sys-sysn),inf)

% check the innerness of $M(z)$: $\|\text{conj}(M(z))*M(z)-I\|_{\infty} = 0$
norm(sysm'*sysm-eye(2),inf)

% check the poles of the factors
gpole(sysm), gpole(sysn)

% check coprimeness of the factors
gzero(gir([sysn sysm])) % [N(z) M(z)] has no zeros

3.6.5 giofac

Syntax

[SYSI,SYSO] = giofac(SYS,OPTIONS)

Description

giofac computes, for the transfer function matrix $G(\lambda)$ of a LTI descriptor state-space system, the extended inner–quasi-outer or the extended QR-like factorization in the form

$$G(\lambda) = G_i(\lambda) \begin{bmatrix} G_o(\lambda) \\ 0 \end{bmatrix},$$

(202)

where $G_i(\lambda)$ is square and inner, and $G_o(\lambda)$ is quasi-outer or full row rank, respectively.
Input data

SYS is a LTI system, whose transfer function matrix is $G(\lambda)$, and is in a descriptor system state-space form

\[ E\lambda x(t) = Ax(t) + Bu(t), \]
\[ y(t) = Cx(t) + Du(t). \]  \hspace{1cm} (203)

OPTIONS is a MATLAB structure to specify user options and has the following fields:

| OPTIONS fields | Description |
|----------------|-------------|
| tol | relative tolerance for rank computations and observability tests (Default: internally computed) |
| minphase | option to compute a minimum-phase quasi-outer factor: 
true – compute a minimum phase quasi-outer factor, with all zeros stable, excepting possibly zeros on the boundary of the appropriate stability domain (default); 
false – compute a full row rank factor, which includes all zeros of $G(\lambda)$. |
| balance | balancing option for the Riccati equation solvers (see functions care and dare of the Control System Toolbox): 
true – apply balancing (default); 
false – disable balancing. |

Output data

SYSI contains the descriptor system state-space realization of the square inner transfer function matrix $G_i(\lambda)$ in the form

\[ E_i\lambda x_i(t) = A_ix_i(t) + B_iv(t), \]
\[ y_i(t) = C_ix_i(t) + D_iv(t), \]  \hspace{1cm} (204)

where $E_i$ is invertible and $\Lambda(A_i - \lambda E_i) \subset \mathbb{C}_s$. The realization of the inner factor is a standard system with $E_i = I$ if the original system (203) is a standard system with $E = I$.

If OPTIONS.minphase = false, then $G_i(\lambda)$ has the least possible McMillan degree.

SYSO contains the descriptor system state-space realization of the transfer function matrix $G_o(\lambda)$ in the form

\[ E\lambda x_o(t) = Ax_o(t) + Bu(t), \]
\[ y_o(t) = C_o x_o(t) + D_o u(t), \]  \hspace{1cm} (205)

where the dimension $r$ of $y_o(t)$ is the normal rank of $G(\lambda)$. If OPTIONS.minphase = true, then $G_o(\lambda)$ is quasi-outer and all zeros of $G_o(\lambda)$ lie in $\mathbb{C}_s$. If OPTIONS.minphase = false, then $G_o(\lambda)$ is full row rank and contains all zeros of $G(\lambda)$.

Method

For the definitions related to inner-outer and QR-like factorizations of transfer function matrices see Section 2.9. Assume that the transfer function matrix $G(\lambda)$ has normal rank $r$ and the inner
factor $G_i(\lambda)$ in (202) is partitioned as $G_i(\lambda) = [G_{i,1}(\lambda) \ G_{i,2}(\lambda)]$, where $G_{i,1}(\lambda)$ has $r$ columns and is inner. Then $G_{i,2}(\lambda)$ represents the complementary inner factor $G_{i,1}^\perp(\lambda)$, which is the inner orthogonal complement of $G_{i,1}(\lambda)$. Thus, the full rank inner–quasi-outer or QR-like factorization of $G(\lambda)$ has the form

$$G(\lambda) = G_{i,1}(\lambda) G_o(\lambda). \quad (206)$$

If OPTIONS.minphase = true, the resulting factor $G_o(\lambda)$ has full row rank $r$ and is minimum phase, excepting possibly zeros in $\partial C_s$, the boundary of the appropriate stability domain. If OPTIONS.minphase = false, $G_o(\lambda)$ has full row rank $r$ and contains all zeros of $G(\lambda)$. In this case, the resulting inner factor has the least possible McMillan degree. If $G(\lambda)$ contains a so-called free inner factor, then the resulting realization (205) of $G_o(\lambda)$ is unobservable. The unobservable eigenvalues of the pencil $[A - \lambda E \ C_o]$ are precisely the (stable) poles of the free inner factor and can be readily eliminated (e.g., by using the function `gir`).

The implemented computational methods to determine the inner–quasi-outer or QR-like factorizations of general rational matrices rely on a preliminary orthogonal reduction of the system matrix pencil

$$S(\lambda) = \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix}$$

to a special Kronecker-like form (see (128)), which allows to reduce the original computational problem to a standard inner-outer factorization problem for a system with full column rank transfer function matrix and without zeros in $\partial C_s$. The underlying reduction is described in [32] and involves the use of the mex-function `sl_klf` to compute the appropriate Kronecker-like form. For the computation of the inner and outer factors for the reduced problem, extensions of the standard factorization methods of [64] are used. These methods involve the solution of appropriate (continuous- or discrete-time) generalized algebraic Riccati equations. The overall factorization procedures are described in [32] for continuous-time systems and in [29] for discrete-time systems. The formulas for the determination of the complementary inner factors have been derived extending the results of [64]. The function `giofac` employs the mex-function `sl_gminr` to obtain the inner factor with a standard state-space realization with $E_i = I$.

### Examples

**Example 19.** This is Example 1 from [32] of the transfer function matrix of a continuous-time proper system:

$$G(s) = \begin{bmatrix} s - 1 & s & 1 \\ s + 2 & s + 2 & s + 2 \\ 0 & s - 2 & s - 2 \\ s - 1 & s^2 + 2s - 2 & 2s - 1 \\ s + 2 & (s + 1)(s + 2) & (s + 1)(s + 2) \end{bmatrix}. \quad (207)$$

$G(s)$ has zeros at $\{1, 2, \infty\}$, poles at $\{-1, -1, -2, -2\}$, and normal rank $r = 2$. The extended inner–quasi-outer factorization of $G(s)$ can be computed with the following command sequence:
% Oara and Varga (2000), Example 1
s = tf('s'); % define the complex variable s
% enter G(s) and determine a minimal state-space realization
G = [(s-1)/(s+2) s/(s+2) 1/(s+2);
     0 (s-2)/(s+1)^2 (s-2)/(s+1)^2;
     (s-1)/(s+2) (s^2+2*s-2)/(s+1)/(s+2) (2*s-1)/(s+1)/(s+2)];
sys = minreal(ss(G));
gpole(sys)    % the system is stable
gzero(sys)   % the system has 2 unstable zeros and an infinite zero
nrank(sys)   % the normal rank of G(s) is 2

% compute the extended inner-quasi-outer factorization G(s) = Gi(s)*[Go(s);0]
[sysi,syso] = giofac(sys,struct('tol',1.e-7)); % use tolerance 1.e-7

% check the factorization
nr = size(syso,1); % nr = 2 is also the normal rank of G(s)
norm(sysi(:,1:nr)*syso-sys,inf) % ||Gi(:,1:nr)(s)*Go(s)-G(s)||_inf = 0

% checking the innerness of Gi(s)
norm(sysi'*sysi-eye(3),inf) % ||conj(Gi(s))*Gi(s)-I||_inf = 0

syso = gir(syso); % a free inner factor is present in G(s)
gzero(syso)     % Go(s) has no zeros in open right-half plane

3.6.6  goifac

Syntax

[SYSI,SYSO] = goifac(SYS,OPTIONS)

Description

goifac computes, for the transfer function matrix $G(\lambda)$ of a LTI descriptor state-space system, the extended quasi-co-outer–inner or the extended RQ-like factorization in the form

$$G(\lambda) = [G_o(\lambda) \ 0]G_i(\lambda),$$  \hspace{1cm} (208)

where $G_i(\lambda)$ is square and inner, and $G_o(\lambda)$ is quasi-co-outer or full column rank, respectively.

Input data

SYS is a LTI system, whose transfer function matrix is $G(\lambda)$, and is in a descriptor system state-space form

$$E\lambda x(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$  \hspace{1cm} (209)

OPTIONS is a MATLAB structure to specify user options and has the following fields:
| Options fields | Description |
|----------------|-------------|
| **tol** | relative tolerance for rank computations and observability tests (Default: internally computed) |
| **minphase** | option to compute a minimum-phase quasi-co-outer factor:  
true – compute a minimum phase quasi-co-outer factor, with all zeros stable, excepting possibly zeros on the boundary of the appropriate stability domain (default);  
false – compute a full row rank factor, which includes all zeros of $G(\lambda)$. |
| **balance** | balancing option for the Riccati equation solvers (see functions care and dare of the Control System Toolbox):  
true – apply balancing (default);  
false – disable balancing. |

**Output data**

**SYSI** contains the descriptor system state-space realization of the square inner transfer function matrix $G_i(\lambda)$ in the form

$$\begin{align*}
E_i\lambda x_i(t) &= A_i x_i(t) + B_i u(t), \\
y_i(t) &= C_i x_i(t) + D_i u(t),
\end{align*}$$  \(\text{(210)}\)

where $E_i$ is invertible and $\Lambda(A_i - \lambda E_i) \subset C_s$. The realization of the inner factor is a standard system with $E_i = I$ if the original system (209) is a standard system with $E = I$. If **OPTIONS.minphase = false**, then $G_i(\lambda)$ has the least possible McMillan degree.

**SYSO** contains the descriptor system state-space realization of the transfer function matrix $G_o(\lambda)$ in the form

$$\begin{align*}
E\lambda x_o(t) &= Ax_o(t) + B_o v(t), \\
y_o(t) &= C x_o(t) + D_o v(t),
\end{align*}$$  \(\text{(211)}\)

where the dimension $r$ of $v(t)$ is the normal rank of $G(\lambda)$. If **OPTIONS.minphase = true**, then $G_o(\lambda)$ is quasi-co-outer and all zeros of $G_o(\lambda)$ lie in $\overline{C_s}$. If **OPTIONS.minphase = false**, then $G_o(\lambda)$ is full column rank and contains all zeros of $G(\lambda)$.

**Method**

For the definitions related to co-outer–co-inner or RQ-like factorizations of transfer function matrices see Section 2.9. Assume that the transfer function matrix $G(\lambda)$ has normal rank $r$ and the inner factor $G_i(\lambda)$ in (208) is partitioned as $G_i(\lambda) = \begin{bmatrix} G_{i,1}(\lambda) \\ G_{i,2}(\lambda) \end{bmatrix}$, where $G_{i,1}(\lambda)$ has $r$ rows and is co-inner. Then $G_{i,2}(\lambda)$ represents the complementary co-inner factor $G_{i,1}^\perp(\lambda)$, which is the co-inner orthogonal complement of $G_{i,1}(\lambda)$. Thus, the full rank quasi-co-outer–co-inner or full rank RQ-like factorization of $G(\lambda)$ has the form

$$G(\lambda) = G_o(\lambda)G_{i,1}(\lambda).$$  \(\text{(212)}\)

If **OPTIONS.minphase = true**, the resulting factor $G_o(\lambda)$ has full row rank $r$ and is minimum phase, excepting possibly zeros in $\partial C_s$, the boundary of the appropriate stability domain. If
OPT\text{IONS\_minphase} = \text{false}, \ G_o(\lambda) \text{ has full row rank } r \text{ and contains all zeros of } G(\lambda). \text{ In this case, the resulting inner factor has the least possible McMillan degree. If } G(\lambda) \text{ contains a so-called free inner factor, then the resulting realization (211) of } G_o(\lambda) \text{ is uncontrollable. The uncontrollable eigenvalues of the pencil } [A - \lambda E \ B_o] \text{ are precisely the (stable) poles of the free inner factor and can be readily eliminated (e.g., by using the function gir).}

To compute the extended quasi-co-outer–inner or the extended RQ-like factorization (208), the function \text{goifac} calls \text{giofac} to compute the extended inner–quasi-outer or extended QR-like factorization of \ G_T(\lambda) \text{ in the form}

\[
G_T(\lambda) = \tilde{G}_i(\lambda) \begin{bmatrix} \tilde{G}_o(\lambda) & 0 \end{bmatrix}
\]

and obtain the inner and quasi-co-outer/full column rank factors as \ G_i(\lambda) = \tilde{G}_T^i(\lambda) \text{ and } \ G_o(\lambda) = \tilde{G}_T^o(\lambda), \text{ respectively. The factorization procedures underlying the function \text{giofac} are described in [32] for continuous-time systems and in [29] for discrete-time systems. The function \text{giofac} relies on the mex-functions \text{sl\_klf} and \text{sl\_gminr}.}

**Examples**

*Example 20.* This is Example 1 from [29] of the transfer function matrix of a discrete-time proper system:

\[
G(z) = \begin{bmatrix}
z^4 - \frac{z^3}{2} - 16 z^2 - \frac{29 z}{2} + 18 & z^4 + 5 z^3 - z^2 - 11 z + 6 & \frac{11 z^3}{2} + 15 z^2 + \frac{7 z}{2} - 12 \\
z^4 + \frac{5 z^3}{2} + 2 z^2 + \frac{z}{2} - 3 z^2 + 12 & z^4 + \frac{5 z^3}{2} + 2 z^2 + \frac{z}{2} - 3 z^2 + 4 & z^3 - z^2 + 2 z^2 \\
z^4 + \frac{5 z^3}{2} + 2 z^2 + \frac{z}{2} - 19 z^2 - 23 z + 24 & z^4 + \frac{5 z^3}{2} + 2 z^2 + \frac{z}{2} - 12 z + 8 & \frac{13 z^3}{2} + 16 z^2 - \frac{z}{2} - 16 \\
z^4 + \frac{5 z^3}{2} + 2 z^2 + \frac{z}{2} - 16 & z^4 + \frac{5 z^3}{2} + 2 z^2 + \frac{z}{2} - 4 z - 8 & z^3 + 2 z^2 - 4 z - 8
\end{bmatrix}
\]

\(G(z)\) has the zeros \ \{1, 2, \infty\}, \text{ the poles } \ \{0, -0.5, -1, -1\}, \text{ and normal rank } r = 2. \text{ The extended quasi-co-outer–inner factorization of } G(z) \text{ can be computed with the following sequence of commands:}

\[
\begin{align*}
\text{% Oara (2005), Example 1} \\
z = \text{tf(’z’)}; & \quad \text{ % define the complex variable } z \\
\text{% enter G(z) and determine a minimal state-space realization} \\
\text{G} = 1/(z^4+5/2*z^3+2*z^2+z/2)*… \\
& \quad \text{ [z^4-z^3/2-16*z^2-29/2*z+18 \ z^4+5*z^3-z^2-11*z+6 11/2*z^3+15*z^2+7/2*z-12} \\
& \quad \quad -3*z^2+12 z^3-3*z^2-4*z+4 z^3+2*z^2-4*z-8; \\
& \quad \quad -3*z^2+12 z^3-3*z^2-4*z+4 z^3+2*z^2-4*z-8; \\
& \quad \quad z^4-z^3/2-19*z^2-23/2*z+24 \ z^4+6*z^3-3*z^2-12*z+8 13/2*z^3+16*z^2-3*z^2-12*z+8 13/2*z^3+16*z^2-3*z^2-12*z+8] \\
\text{sys} = \text{gir(ss(G),1.e-7);} \\
\text{gpole(sys)} & \quad \text{ % the system is marginally stable} \\
\text{gzero(sys)} & \quad \text{ % the system has an unstable zero and an infinite zero}
\end{align*}
\]
nrank(sys) % the normal rank of G(z) is 2

% compute the extended quasi-co-outer-inner factorization
% G(z) = [Go(z) 0]*Gi(z)
[sysi,syso] = goifac(sys,struct('tol',1.e-7)); % use tolerance 1.e-7

% check the factorization
nr = size(syso,2); % nr = 2 is also the normal rank of G(z)
% check that ||Go(z)*Gi(1:nr,:) - G(z)||_inf = 0
norm(gir(syso*sysi(1:nr,:)-sys,1.e-7),inf)

% check the innerness of Gi(z): ||conj(Gi(z))*Gi(z)-I||_inf = 0
norm(sysi'*sysi-eye(3),inf)

gzero(syso) % Go(z) has no zeros outside the unit circle

3.6.7 grsfg

Syntax

SYSF = grsfg(SYS,GAMMA,OPTIONS)

Description

grsfg solves, for the transfer function matrix $G(\lambda)$ of a LTI descriptor state-space system, and a given $\gamma$ satisfying $\gamma > \|G(\lambda)\|_{\infty}$, the right stable and minimum-phase spectral factorization problem

$$\gamma^2 I - G^{-}(\lambda)G(\lambda) = F^{-}(\lambda)F(\lambda),$$

such that the resulting spectral factor $F(\lambda)$ is proper, stable and minimum-phase.

Input data

SYS is a LTI system, whose transfer function matrix is $G(\lambda)$, and is in a descriptor system state-space form

$$E \lambda x(t) = Ax(t) + Bu(t),$$

$$y(t) = Cx(t) + Du(t),$$

with $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^p$. $G(\lambda)$ must not have poles in $\partial C_s$.

GAMMA is a given scalar $\gamma$, which must satisfy $\gamma > \|G(\lambda)\|_{\infty}$.

OPTIONS is a MATLAB structure to specify user options and has the following fields:

| OPTIONS fields | Description |
|----------------|-------------|
| tol            | tolerance for the singular values based rank determination of $E$  |
|                | (Default: $n^2\|E\|_{\text{eps}}$) |
tolmin  |  tolerance for the singular values based observability tests  
| (Default: \texttt{np||C||}_\infty \texttt{eps})

stabilize | stabilization option:  
| \texttt{true} – perform a preliminary stabilization using a left coprime factorization with inner denominator of \( G(\lambda) \) (see \textbf{Method}) (default);  
| \texttt{false} – no preliminary stabilization is performed.

\textbf{Output data}

\texttt{SYSF} contains the descriptor system state-space realization of the spectral factor \( F(\lambda) \) in the form

\[
E_F x_F(t) = A_F x_F(t) + B_F v(t), \quad y_F(t) = C_F x_F(t) + D_F v(t).
\]

(215)

\textbf{Method}

For the computation of the right spectral factorization (213) the dual of the two-step approach sketched in Section 2.9 is employed. In the first step, a preliminary left coprime factorization with inner denominator of \( G(\lambda) \) is computed such that \( G(\lambda) = M^{-1}(\lambda) N(\lambda) \), with both \( N(\lambda) \) and \( M(\lambda) \) stable, and \( M(\lambda) \) inner. For this purpose, the dual algorithm to compute right coprime factorizations with inner denominators, given in Procedure \textbf{GRCFID} of [57], is employed. This step is not performed if \texttt{OPTIONS.stabilize} = \texttt{false}, in which case \( N(\lambda) := G(\lambda) \) and \( M(\lambda) = I_p \). In the second step, the spectral factorization problem is solved

\[
\gamma^2 I - G^\sim(\lambda) G(\lambda) = \gamma^2 I - N^\sim(\lambda) N(\lambda) = F^\sim(\lambda) F(\lambda)
\]

for the minimum-phase phase factor \( F(\lambda) \). For this computation, the formulas provided by the dual versions of Lemma 6 and Lemma 7 are employed. These lemmas extend to proper descriptor system the formulas developed in [64].

\textbf{Example}

\textit{Example 21.} Consider the discrete-time improper TFM

\[
G(z) = \begin{bmatrix}
z^2 + z + 1 & 4 z^2 + 3 z + 2 & 2 z^2 - 2 \\
z & 4 z - 1 & 2 z - 2 \\
z^2 & 4 z^2 - z & 2 z^2 - 2 z
\end{bmatrix},
\]

(216)

which has two infinite poles (i.e., McMillan-degree of \( G(z) \) is equal to 2) and has a minimal descriptor state-space realization of order 4. Therefore, the spectral factorization problem (213) has a solution for all \( \gamma > \|G(z)\|_\infty = 10.4881 \). With \( \gamma = 1.1 \|G(z)\|_\infty \), the function \texttt{grsfg} computes the proper minimal-phase spectral factor \( F(z) \), having two poles in 0 and two stable zeros in \(-0.2908, 0.4188\).

The spectral factor \( F(z) \) can be computed with the following sequence of commands:

\begin{verbatim}
z = tf('z');  \% define the complex variable z  
\% enter G(z) and determine a minimal state-space realization
\end{verbatim}
G = [z^2+z+1 4*z^2+3*z+2 2*z^2-2;  
z 4*z-1 2*z-2;  
z^2 4*z^2-z 2*z^2-2*z];
sys = gir(ss(G));

% the system is unstable and improper

gpole(sys)  % gamma = 1.1*norm(sys,inf) % set gamma = 1.1*||G||_inf

% compute the minimum-phase stable spectral factor F(z) satisfying
% gamma^2*I-conj(G(z))*G(z) = conj(F(z))*F(z)
sysf = grsfg(sys,gamma);  

% check the factorization ||F'(z)*F(z)+G'(z)*G(z)-gamma^2*I||_inf = 0
norm(gir(sysf'*sysf+sys'*sys,1.e-7)-gamma^2*eye(size(sys,2)),inf)

% check the stability of poles and zeros of F(z)
gpole(sysf), gzero(sysf)  % F(z) is stable and minimum-phase

3.6.8  glsfg

## Syntax

SYSF = glsfg(SYS,GAMMA,OPTIONS)

## Description

`glsfg` solves, for the transfer function matrix $G(\lambda)$ of a LTI descriptor state-space system, and a given $\gamma$ satisfying $\gamma > \|G(\lambda)\|_{\infty}$, the left stable and minimum-phase spectral factorization problem

$$\gamma^2 I - G(\lambda)G^\sim(\lambda) = F(\lambda)F^\sim(\lambda),$$

such that the resulting spectral factor $F(\lambda)$ is proper, stable and minimum-phase.

## Input data

`SYS` is a LTI system, whose transfer function matrix is $G(\lambda)$, and is in a descriptor system state-space form

$$E\lambda x(t) = Ax(t) + Bu(t),$$

$$y(t) = Cx(t) + Du(t),$$

with $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$. $G(\lambda)$ must not have poles in $\partial \mathbb{C}_s$.

`GAMMA` is a given scalar $\gamma$, which must satisfy $\gamma > \|G(\lambda)\|_{\infty}$.

`OPTIONS` is a MATLAB structure to specify user options and has the following fields:

| OPTIONS fields | Description |
|----------------|-------------|
| tol            | tolerance for the singular values based rank determination of $E$  
(Default: $n^2\|E\|_1\epsilon$) |
tolmin tolerance for the singular values based controllability tests
(Default: $nm\|B\|_1\text{eps}$)

stabilize stabilization option:
true – perform a preliminary stabilization using a right coprime factor-
ization with inner denominator of $G(\lambda)$ (see Method) (default);
false – no preliminary stabilization is performed.

Output data
SYSF contains the descriptor system state-space realization of the spectral factor $F(\lambda)$ in the
form
\[
E_F \dot{x}_F(t) = A_F x_F(t) + B_F v(t),
\]
\[
y_F(t) = C_F x_F(t) + D_F v(t).
\]
(219)

Method
For the computation of the left spectral factorization (217) the two-step approach sketched in
Section 2.9 is employed. In the first step, a preliminary right coprime factorization with inner
denominator of $G(\lambda)$ is computed such that $G(\lambda) = N(\lambda)M^{-1}(\lambda)$, with both $N(\lambda)$ and $M(\lambda)$
stable, and $M(\lambda)$ inner. For this purpose, the algorithm to compute right coprime factorizations
with inner denominators, given in Procedure GRCFID of [57], is employed. This step is not
performed if OPTIONS.stabilize = false, in which case $N(\lambda) := G(\lambda)$ and $M(\lambda) = I_m$. In
the second step, the left spectral factorization problem is solved
\[
\gamma^2 I - G(\lambda)G^\sim(\lambda) = \gamma^2 I - N(\lambda)N^\sim(\lambda) = F(\lambda)F^\sim(\lambda)
\]
for the minimum-phase phase factor $F(\lambda)$. For this computation, the formulas provided by
Lemma 6 and Lemma 7 are employed. These lemmas extend to proper descriptor system the
formulas developed in [64].

Example
Example 22. Consider the discrete-time improper TFM
\[
G(z) = \begin{bmatrix}
z^2 + z + 1 & 4 z^2 + 3 z + 2 & 2 z^2 - 2 \\
z & 4 z - 1 & 2 z - 2 \\
z^2 & 4 z^2 - z & 2 z^2 - 2 z
\end{bmatrix},
\]
(220)
which has two infinite poles (i.e., McMillan-degree of $G(z)$ is equal to 2) and has a minimal
descriptor state-space realization of order 4. Therefore, the spectral factorization problem (217)
has a solution for all $\gamma > \|G(z)\|_\infty = 10.4881$. With $\gamma = 1.1\|G(z)\|_\infty$, the function gisfg
computes the proper minimal-phase spectral factor $F(z)$, having two poles in 0 and two stable
zeros in \{-0.2908, 0.4188\}.

The spectral factor $F(z)$ can be computed with the following sequence of commands:

```matlab
z = tf('z'); % define the complex variable z
% enter G(z) and determine a minimal state-space realization
```
\begin{verbatim}
G = [z^2+z+1 4*z^2+3*z+2 2*z^2-2;
    z 4*z-1 2*z-2;
    z^2 4*z^2-z 2*z^2-2*z];
sys = gir(ss(G));
gpole(sys) % the system is unstable and improper
gamma = 1.1*norm(sys,inf) % set gamma = 1.1*||G||_inf

% compute the minimum-phase stable spectral factor F(z) satisfying
% gamma^2*I-G(z)*conj(G(z)) = F(z)*conj(F(z))
sysf = glsfg(sys,gamma);

% check the factorization ||F(z)*F'(z)+G(z)*G'(z)-gamma^2*I||_inf = 0
norm(gir(sysf*sysf'+sys*sys',1.e-7)-gamma^2*eye(size(sys,1)),inf)

% check the stability of poles and zeros of F(z)
gpole(sysf), gzero(sysf) % F(z) is stable and minimum-phase
\end{verbatim}

### 3.7 Functions for Approximations

These functions cover the computation of Nehari approximations and the solution of the 1-block and 2-block least distance problems.

#### 3.7.1 gnehari

**Syntax**

\[
\begin{align*}
\text{[SYSX,S1]} & = \text{gnehari}(\text{SYS}) \\
\text{[SYSX,S1]} & = \text{gnehari}(\text{SYS},\text{GAMMA})
\end{align*}
\]

**Description**

*gnehari* computes an optimal or suboptimal stable Nehari approximation of the transfer function matrix \( G(\lambda) \) of a LTI descriptor state-space system. The optimal Nehari approximation \( X(\lambda) \) satisfies

\[
\|G(\lambda) - X(\lambda)\|_\infty = \|G_u(\lambda)\|_H,
\]

where \( G_u(\lambda) \) is the anti-stable part of \( G(\lambda) \). For a given \( \gamma > \|G_u(\lambda)\|_H \), the suboptimal approximation satisfies

\[
\|G(\lambda) - X(\lambda)\|_\infty < \gamma.
\]

**Input data**

SYS is a LTI system, whose transfer function matrix is \( G(\lambda) \), and is in a descriptor system state-space form

\[
\begin{align*}
E\lambda x(t) & = Ax(t) + Bu(t), \\
y(t) & = Cx(t) + Du(t).
\end{align*}
\]

124
\( G(\lambda) \) must not have poles in \( \partial C_s \).

**GAMMA**, if specified, is the desired suboptimality level \( \gamma \) for the suboptimal Nehari approximation problem (222) and must satisfy \( \gamma > \| G_u(\lambda) \|_H \), where \( G_u(\lambda) \) is the anti-stable part of \( G(\lambda) \).

**Output data**

SYSX contains the descriptor system state-space realization of the optimal or suboptimal stable Nehari approximation \( X(\lambda) \) in the form
\[
E_X X(t) = A_X X(t) + B_X u(t), \\
y_X(t) = C_X X(t) + D_X u(t).
\] (224)

S1 is the Hankel-norm of the anti-stable part of \( G(\lambda) \) (also the \( L_\infty \)-norm of the optimal approximation error).

**Method**

The case when \( G(\lambda) \) is antistable, is discussed Section 2.14. For a general \( G(\lambda) \) without poles in \( \partial C_s \), a preliminary spectral separation is performed as
\[
G(\lambda) = G_s(\lambda) + G_u(\lambda),
\] (225)

where \( G_s(\lambda) \) is the stable part (i.e., all poles of \( G_s(\lambda) \) are in \( C_s \)) and \( G_u(\lambda) \) is the anti-stable part (i.e., all poles of \( G_u(\lambda) \) are in \( C_u \)). The optimal or suboptimal Nehari approximation of \( G_u(\lambda) \) is then computed, by determining \( Y^\sim(\lambda) \), the optimal zeroth-order Hankel-norm approximation or the suboptimal Hankel-norm approximation of \( G_u^\sim(\lambda) \), respectively, using the methods proposed in [15] (see also [34]) with straightforward extensions for proper descriptor systems. For the computation of the optimal Nehari approximation, the system balancing-based **Procedure GNEHARI** in [58] (extended to non-square systems) is used. The solution of the Nehari approximation problem for the original problem is obtained as
\[
X(\lambda) = G_s(\lambda) + Y(\lambda).
\]

Explicit approximation formulas developed in [15] are employed for continuous-time systems, while for discrete-time systems, the bilinear transformation based approach suggested in [15] is used. For the computation of additive spectral separation (225) of the descriptor system SYS, the mex-function **sl_gsep** is employed. In the computation of the balancing transformations, the intervening Lyapunov and Stein equations, satisfied by the gramians, have been solved directly for the Cholesky factors of the gramians using the mex-function **sl_glme**.

**Example 23.** Consider the discrete-time improper TFM
\[
G(z) = \begin{bmatrix}
z^2 + z + 1 & 4 z^2 + 3 z + 2 & 2 z^2 - 2 \\
z & 4 z - 1 & 2 z - 2 \\
z^2 & 4 z^2 - z & 2 z^2 - 2 z
\end{bmatrix},
\] (226)

which has two infinite poles (i.e., \( G(z) \) is antistable and its McMillan-degree is equal to 2) and has a minimal descriptor state-space realization of order 4. The Hankel-norm of \( G^\sim(z) \) is
\[ \|G^*(z)\|_H = 8.6622, \] and therefore the optimal stable Nehari approximation \( X(z) \) must satisfy 
\[ \|G(z) - X(z)\|_\infty = 8.6622. \] Indeed, the optimal stable Nehari approximation, computed with the function \texttt{gnehari}, achieves exactly this approximation error, with \( X(z) \) having the McMillan degree equal to 1. In contrast, the suboptimal stable Nehari approximation computed for \( \gamma = 10 \), has McMillan degree 2 and the achieved approximation error is 9.8207.

The optimal and suboptimal Nehari approximations can be computed with the following sequence of commands:

```matlab
z = tf('z'); % define the complex variable z
G = [z^2+z+1 4*z^2+3*z+2 2*z^2-2; z 4*z-1 2*z-2; z^2 4*z^2-z 2*z^2-2*z];
sys = gir(ss(G));
gpole(sys) % the system is unstable and improper
ghanorm(sys') % the achievable optimal approximation error

% compute the optimal Nehari approximation
[sysx,s1] = gnehari(sys);

% check the approximation error \( \|G(z)-X(z)\|_\infty \) = s1
norm(gminreal(sys-sysx,1.e-7),inf) - s1

% check the stability of poles of X(z)
gpole(sysx) % X(z) is stable and has order 1

% compute the suboptimal Nehari approximation for gamma = 10
[sysxsub,s1] = gnehari(sys,10);

% compute the approximation error \( \|G(z)-Xsub(z)\|_\infty \)
norm(gminreal(sys-sysxsub,1.e-7),inf)

% check the stability of poles of Xsub(z)
gpole(sysxsub) % Xsub(z) is stable and has order 2
```

3.7.2 \texttt{glinfldp}

\textbf{Syntax}

\begin{verbatim}
[SYSX,MINDIST] = glinfldp(SYS1,SYS2,OPTIONS)
[SYSX,MINDIST] = glinfldp(SYS,M2,OPTIONS)
\end{verbatim}
**Description**

glinfldp solves the 2-block optimal least distance problem to find a stable $X(\lambda)$ such that

$$\| [G_1(\lambda) - X(\lambda) G_2(\lambda)] \|_{\infty} = \min,$$

or the 2-block suboptimal least distance problem to find a stable $X(\lambda)$ such that

$$\| [G_1(\lambda) - X(\lambda) G_2(\lambda)] \|_{\infty} < \gamma,$$

where $G_1(\lambda)$ and $G_2(\lambda)$ are the transfer function matrices of LTI descriptor state-space systems and $\gamma > \|G_2(\lambda)\|_{\infty}$.

**Input data**

For the usage with 

$$[SYSX, MINDIST] = \text{glinfldp}(SYS1, SYS2, OPTIONS)$$

the input parameters SYS1 and SYS2 are as follows:

SYS1 is a LTI system, whose transfer function matrix is $G_1(\lambda)$, and is in a descriptor system state-space form

$$E_1\lambda x_1(t) = A_1 x_1(t) + B_1 u(t),$$

$$y_1(t) = C_1 x_1(t) + D_1 u(t),$$

where $y_1(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$. $G_1(\lambda)$ must not have poles in $\partial \mathbb{C}_s$.

SYS2 is a LTI system, whose transfer function matrix is $G_2(\lambda)$, and is in a descriptor system state-space form

$$E_2\lambda x_2(t) = A_2 x_2(t) + B_2 v(t),$$

$$y_2(t) = C_2 x_2(t) + D_2 v(t),$$

where $y_2(t) \in \mathbb{R}^p$ and $v(t) \in \mathbb{R}^m$. If SYS2 is empty, a 1-block least distance problem is solved. $G_2(\lambda)$ must not have poles in $\partial \mathbb{C}_s$.

For the usage with

$$[SYSX, MINDIST] = \text{glinfldp}(SYS, M2, OPTIONS)$$

the input parameters SYS and M2 are as follows:

SYS is an input concatenated compound LTI system, $SYS = [SYS1 \ SYS2]$, in a descriptor system state-space form

$$E\lambda x(t) = Ax(t) + B_1 u(t) + B_2 v(t),$$

$$y(t) = Cx(t) + D_1 u(t) + D_2 v(t),$$

where SYS1 has the transfer function matrix $G_1(\lambda)$ with the descriptor system realization $(A - \lambda E, B_1, C, D_1)$ and SYS2 has the transfer function matrix $G_2(\lambda)$ with the descriptor system realization $(A - \lambda E, B_2, C, D_2)$. $[G_1(\lambda) \ G_2(\lambda)]$ must not have poles in $\partial \mathbb{C}_s$. 

127
\[ M_2 \text{ is the dimension } m_2 \geq 0 \text{ of the input vector } v(t) \text{ of the system SYS2. If } m_2 = 0, \text{ a 1-block least distance problem is solved.} \]

For both usages:

**OPTIONS** is a MATLAB structure to specify user options and has the following fields:

| OPTIONS fields | Description |
|----------------|-------------|
| tol            | tolerance for rank determinations (Default: internally computed) |
| reltol         | specifies the relative tolerance \( rtol \) for the desired accuracy of the gamma-iteration. The iterations are performed until the current estimates of the maximum distance \( \gamma_u \) and minimum distance \( \gamma_l \), which bound the optimal distance \( \gamma_{opt} \) (i.e., \( \gamma_l \leq \gamma_{opt} \leq \gamma_u \)), satisfy \( \gamma_u - \gamma_l < rtol(\|[G_1(\lambda) G_2(\lambda)]\|_\infty - \|G_2(\lambda)\|_\infty) \). (Default: \( rtol = 10^{-4} \)) |
| gamma          | desired suboptimality level \( \gamma \) for the suboptimal least distance problem (228) (Default: \( [ ] \), i.e., the optimal least distance problem (227) is solved) |

**Output data**

SYSX contains the descriptor system state-space realization of the optimal or suboptimal solution \( X(\lambda) \) in the form

\[
\begin{align*}
E_X x_X(t) &= A_X x_X(t) + B_X u(t), \\
y_X(t) &= C_X x_X(t) + D_X u(t).
\end{align*}
\]

(232)

MINDIST is the achieved distance by the computed (optimal or suboptimal) solution \( X(\lambda) \).

**Method**

The solution approach is sketched in Section 2.15 and corresponds to extensions of the method proposed in [6] to descriptor system state-space representations.

**Example**

Example 24. The formulation of the approximate model-matching problem can be done as an error minimization problem, where the approximate solution of the rational equation \( X(\lambda)G(\lambda) = F(\lambda) \) involves the minimization of the \( L_\infty \)-norm of the error \( \mathcal{E}(\lambda) := F(\lambda) - X(\lambda)G(\lambda) \). For example, the standard formulation of the optimal \( \mathcal{H}_\infty \) model-matching problem (\( \mathcal{H}_\infty \)-MMP) is: given \( G(\lambda), F(\lambda) \in \mathcal{H}_\infty \), find \( X(\lambda) \in \mathcal{H}_\infty \) which minimizes \( \|\mathcal{E}(\lambda)\|_{\infty} \). The optimal solution is typically computed by solving a sequence of suboptimal \( \mathcal{H}_\infty \) model-matching problems, such that \( \|\mathcal{E}(\lambda)\|_{\infty} < \gamma \), for suitably chosen \( \gamma \) values.

For the solution of the \( \mathcal{H}_\infty \)-MMPs we assume that \( G(\lambda) \) has full row rank and employ the (extended) outer–inner factorization of \( G(\lambda) \) to reduce this problem to a \( \mathcal{H}_\infty \) least distance problem (LDP). Consider the extended factorization

\[
G(\lambda) = \begin{bmatrix} G_o(\lambda) & 0 \end{bmatrix} G_i(\lambda) = \begin{bmatrix} G_o(\lambda) & 0 \end{bmatrix} \begin{bmatrix} G_{i,1}(\lambda) \\
G_{i,2}(\lambda) \end{bmatrix} = G_o(\lambda)G_{i,1}(\lambda),
\]

128
where $G_i(\lambda) := \begin{bmatrix} G_{i,1}(\lambda) \\ G_{i,2}(\lambda) \end{bmatrix}$ is square and inner and $G_o(\lambda)$ is square and outer (therefore invertible in $H_\infty$). This allows to write successively

$$
\|E(\lambda)\|_\infty = \|F(\lambda) - X(\lambda)G(\lambda)\|_\infty
= \left\| \begin{bmatrix} F(\lambda)G_i(\lambda) - X(\lambda) \begin{bmatrix} G_o(\lambda) & 0 \end{bmatrix} G_i(\lambda) \end{bmatrix} \right\|_\infty,
$$

where $Y(\lambda) := X(\lambda)G_o(\lambda) \in H_\infty$ and

$$
F(\lambda)G_o^-(\lambda) = \begin{bmatrix} F(\lambda)G_o^{1-}(\lambda) \\ F(\lambda)G_o^{2-}(\lambda) \end{bmatrix} := \begin{bmatrix} \tilde{F}_1(\lambda) \\ \tilde{F}_2(\lambda) \end{bmatrix}.
$$

Thus, the problem of computing a stable $X(\lambda)$ which minimizes the error norm $\|E(\lambda)\|_\infty$ has been reduced to a LDP to compute the stable solution $Y(\lambda)$ which minimizes $\|[\tilde{F}_1(\lambda) - Y(\lambda) \tilde{F}_2(\lambda)]\|_\infty$. A $\gamma$-iteration based approach is used for this purpose, as described in [12]. The solution of the original MMP is given by

$$
X(\lambda) = Y(\lambda)G_o^{-1}(\lambda).
$$

We apply this approach to solve a MMP discussed in the book of Francis [12, Example 1, p. 112] with

$$
G(s) = \begin{bmatrix} -\frac{s-1}{s^2+s+1} \\ \frac{s(s-1)}{s^2+s+1} \end{bmatrix} W(s), \quad F(s) = \begin{bmatrix} W(s) & 0 \end{bmatrix},
$$

where $W(s) = (s + 1)/(10s + 1)$ is a suitable weighting factor. The optimal solution

$$
X(s) = \frac{2.3144(s + 0.4569)(s + 2.189)(s^2 + s + 1)}{(s + 3.095)(s + 2.189)(s + 1)(s + 0.4569)}
$$

leads to the optimal error norm $\|F(\lambda) - X(\lambda)G(\lambda)\|_\infty = 0.2521$, which fully agrees with the computed minimum distance $\text{mindist}$ (see bellow). The solution in Francis’ book [12, p. 114] corresponds to a suboptimal solution for $\gamma = 0.2729$ and is

$$
X_{\text{sub}}(s) = \frac{2.2731(s + 0.4732)(s + 2.183)(s^2 + s + 1)}{(s + 3.108)(s + 2.189)(s + 1)(s + 0.4569)}.
$$

The corresponding suboptimal error norm $\|F(\lambda) - X_{\text{sub}}(\lambda)G(\lambda)\|_\infty = 0.2536$. The solutions of the optimal and suboptimal $H_\infty$-MMPs can be computed using the following MATLAB code:

```matlab
s = tf('s'); % define the complex variable s
W = (s+1)/(10*s+1); % weighting function
G = [ -(s-1)/(s^2+2*s+1) (s^2-2*s)/(s^2+2*s+1)]*W;
F = [ W 0 ];

% compute the extended outer-co-inner factorization
```
[Gi,Go] = goifac(sys,struct('tol',1.e-7));

% define the LDP
Fbar = F*Gi'; m2 = 1;

% compute the optimal solution of the LDP
[Y,mindist] = glinfldp(Fbar,m2); mindist

% compute the optimal solution of the MMP
X = minreal(zpk(Y/Go))

% compute the error norm of the optimal solution
norm(X*G-F,inf)

% compute the suboptimal solution of the LDP
[Ysub,mindistsub] = glinfldp(Fbar,m2,struct('gamma',0.2729)); mindistsub

% compute the suboptimal solution of the MMP
Xsub = minreal(zpk(Ysub/Go))

% compute the error norm of the suboptimal solution
norm(Xsub*G-F,inf)

3.8 Functions for Matrix Pencils and Stabilization

These functions cover the reduction of linear matrix pencils to several Kronecker-like forms, the computation of specially ordered generalized real Schur forms, and the stabilization using state feedback.

3.8.1 gklf

Syntax

[AT,ET,INFO,Q,Z] = gklf(A,E,TOL,JOBOPT)
[AT,ET,INFO,Q,Z] = gklf(A,E,TOL,...,QOPT)

Description

gklf computes several Kronecker-like forms $\tilde{A} - \lambda \tilde{E}$ of a linear matrix pencil $A - \lambda E$.

Input data

A,E are the $m \times n$ real matrices A and E, which define the linear matrix pencil $A - \lambda E$.

TOL is a relative tolerance used for rank determinations. If TOL is not specified as input or if TOL = 0, an internally computed default value is used.
JOBOPT is a character option variable to specify various options to compute Kronecker-like forms. The valid options are:

- `'standard'` – compute the standard Kronecker-like form (234), with infinite-finite ordering of the blocks of the regular part (see Method) (default);
- `'reverse'` – compute the standard Kronecker-like form (235), with reverse ordering of the blocks of the regular part (see Method);
- `'right'` – compute the Kronecker-like form (236), which exhibits the right and infinite Kronecker structures (see Method);
- `'left'` – compute the Kronecker-like form (237), which exhibits the left and infinite Kronecker structures (see Method).

QOPT is a character option variable to specify the options to accumulate or not the left orthogonal transformations in $Q$:

- `'Q'` – accumulate $Q$ (default);
- `'noQ'` – do not accumulate $Q$.

**Output data**

$AT, ET$ contain the matrices $\tilde{A}$ and $\tilde{E}$ which define the resulting pencil $\tilde{A} - \lambda \tilde{E}$ in a Kronecker-like form, satisfying

$$
\tilde{A} - \lambda \tilde{E} = Q^T (A - \lambda E) Z,
$$

where $Q$ and $Z$ are orthogonal transformation matrices.

INFO is a MATLAB structure, which provides information on the structure of the pencil $\tilde{A} - \lambda \tilde{E}$ (see Method) as follows:

- `INFO.mr, INFO.nr` – the dimensions of the full row rank diagonal blocks of $[B_r \ A_r - \lambda E_r]$, which characterize the right Kronecker structure (see Method); `INFO.mr` and `INFO.nr` are empty if no right structure exists; if JOBOPT = `'left'`, `INFO.mr` and `INFO.nr` are set to the negative values of the row and column dimensions of the pencil $A_{r,r} - \lambda E_{r,r}$ in (237);
- `INFO.minf` – the dimensions of the square diagonal blocks of $A_{\infty} - \lambda E_{\infty}$, which characterize the infinite Kronecker structure; `INFO.minf` is empty if no infinite structure exists;
- `INFO.mf` – the dimension of the square regular pencil $A_f - \lambda E_f$, which contains the finite eigenvalues;
- `INFO.ml, INFO.nl` – the dimensions of the full column rank diagonal blocks of $[A_l - \lambda E_l]$, which characterize the left Kronecker structure (see Method); `INFO.ml` and `INFO.nl` are empty if no left structure exists; if JOBOPT = `'right'`, `INFO.ml` and `INFO.nl` are set to the negative values of the row and column dimensions of the pencil $A_{l,l} - \lambda E_{l,l}$ in (236).

$Q, Z$ contain the orthogonal matrices $Q$ and $Z$ used to compute the Kronecker-like form $\tilde{A} - \lambda \tilde{E}$ in (233). $Q$ and $Z$ are not accumulated if both output variables $Q$ and $Z$ are not specified.

If QOPT = `'noQ'`, $Q$ is not accumulated and, if specified, set to $Q = [\ ]$. 

131
Method

The Kronecker-like forms, obtainable by using orthogonal transformations, contains most of the relevant structural information provided by the potentially highly sensitive Kronecker canonical form (see Section 2.3). The computation of Kronecker-like forms is based on the method proposed in [3], which underlies the implementation of the mex-function \( \text{sl}_\text{klf} \), called by \( \text{gklf} \).

The (standard) Kronecker-like form has the following structure

\[
\tilde{A} - \lambda \tilde{E} = \begin{bmatrix}
B_r & A_r - \lambda E_r & * & * & * \\
0 & 0 & A_\infty - \lambda E_\infty & * & * \\
0 & 0 & 0 & A_f - \lambda E_f & * \\
0 & 0 & 0 & 0 & A_l - \lambda E_l \\
0 & 0 & 0 & 0 & C_l
\end{bmatrix},
\]

(234)

where

(1) \( [B_r, A_r - \lambda E_r] \), with \( E_r \) invertible and upper triangular, contains the right Kronecker structure and is in a controllability staircase form with \( [B_r, A_r] \) as in (22) and \( E_r \) as in (23); the dimensions of the full row rank diagonal blocks of \( [B_r, A_r - \lambda E_r] \) are provided in \( \text{INFO.mr and INFO.nr} \);

(2) \( A_\infty - \lambda E_\infty \), with \( A_\infty \) invertible and upper triangular, and \( E_\infty \) nilpotent and upper triangular, contains the infinite structure and is in a block upper triangular form; the dimensions of the square diagonal blocks of \( A_\infty - \lambda E_\infty \) are provided in \( \text{INFO.minf} \);

(3) \( A_f - \lambda E_f \), with \( E_f \) invertible, contains the finite structure; the dimension of the square regular pencil \( A_f - \lambda E_f \) is provided in \( \text{INFO.gf} \).

(4) \( \begin{bmatrix} A_r - \lambda E_r \\ C_l \end{bmatrix} \), with \( E_l \) invertible and upper triangular, contains the left Kronecker structure and is in an observability staircase form with \( \begin{bmatrix} A_l \\ C_l \end{bmatrix} \) as in (24) and \( E_l \) as in (25); the dimensions of the full column rank subdiagonal blocks of \( \begin{bmatrix} A_l - \lambda E_l \\ C_l \end{bmatrix} \) are provided in \( \text{INFO.ml and INFO.nl} \).

Depending on the option selected via the option parameter \( \text{JOBOPT} \), several Kronecker-like forms can be determined, which contains basically the same blocks, however ordered differently, or contains only a part of the main structural blocks, which are relevant for particular applications (e.g., nullspace computation).

The following Kronecker-like forms can be computed:

1. \( \text{JOBOPT} = \text{’standard’} \): to determine the (standard) Kronecker-like form (234).

2. \( \text{JOBOPT} = \text{’reverse’} \): to determine a Kronecker-like form with a reversed order of the regular blocks

\[
\tilde{A} - \lambda \tilde{E} = \begin{bmatrix}
B_r & A_r - \lambda E_r & * & * & * \\
0 & 0 & A_f - \lambda E_f & * & * \\
0 & 0 & 0 & A_\infty - \lambda E_\infty & * \\
0 & 0 & 0 & 0 & A_f - \lambda E_l \\
0 & 0 & 0 & 0 & C_l
\end{bmatrix}.
\]

(235)
3. **JOBOPT = 'right':** to determine a Kronecker-like form emphasizing the right and infinite structures

\[
\tilde{A} - \lambda \tilde{E} = \begin{bmatrix}
B_r & A_r - \lambda E_r & * & * \\
0 & 0 & A_\infty - \lambda E_\infty & * \\
0 & 0 & 0 & A_{f,l} - \lambda E_{f,l}
\end{bmatrix},
\]  

(236)

where \( A_{f,l} - \lambda E_{f,l} \) contains the finite and left Kronecker structures. In this case, \( \text{INFO.ml} \) and \( \text{INFO.nl} \) are set to the negative values of the row and column dimensions of the pencil \( A_{f,l} - \lambda E_{f,l} \), respectively, and \( \text{INFO.mf} = 0 \).

4. **JOBOPT = 'left':** to determine a Kronecker-like form emphasizing the left and infinite structures

\[
\tilde{A} - \lambda \tilde{E} = \begin{bmatrix}
A_{r,f} - \lambda E_{r,f} & * & * \\
0 & A_\infty - \lambda E_\infty & * \\
0 & 0 & A_l - \lambda E_l
\end{bmatrix},
\]  

(237)

where \( A_{r,f} - \lambda E_{r,f} \) contains the right and finite Kronecker structures. In this case, \( \text{INFO.mr} \) and \( \text{INFO.nr} \) are set to the negative values of the row and column dimensions of the pencil \( A_{r,f} - \lambda E_{r,f} \), respectively, and \( \text{INFO.mf} = 0 \).

### 3.8.2 gsklf

#### Syntax

\[
[\text{AT,ET,DIMSC,Q,Z}] = \text{gsklf} (\text{SYS,TOL,ZEROSEL})
\]

\[
[\text{AT,ET,DIMSC,Q,Z}] = \text{gsklf} (\text{SYS,TOL,...,QOPT})
\]

#### Description

gsklf computes several special Kronecker-like forms of the system matrix pencil of a LTI descriptor system.

#### Input data

- **SYS** is a LTI system, in a descriptor system state-space form

\[
E \dot{x}(t) = Ax(t) + Bu(t),
\]

\[
y(t) = Cx(t) + Du(t).
\]

(238)

- **TOL** is a relative tolerance used for rank determinations. If **TOL** is not specified as input or if \( \text{TOL} = 0 \), an internally computed default value is used.

- **ZEROSEL** is a character option variable to specify various zero selection options for the computation of the special Kronecker-like form (241) using various partitions of the form (240) of the extended complex plane (see Method). The valid options are:

  - 'none' – use \( C_g = C \cup \{ \infty \} \) and \( C_b = \emptyset \) (default);
  - 'unstable' – use \( C_g = \overline{C}_s \) and \( C_b = C_u \);
  - 'stable' – use \( C_g = \overline{C}_u \) and \( C_b = C_s \);
'all' – use $C_g = \emptyset$ and $C_b = \mathbb{C} \cup \{\infty\}$;
'finite' – use $C_g = \{\infty\}$ and $C_b = \mathbb{C}$;
'infinite' – use $C_g = \mathbb{C}$ and $C_b = \{\infty\}$.

QOPT is a character option variable to specify the options to accumulate or not the left orthogonal transformations in $Q$:

'Q' – accumulate $Q$ (default);
'noQ' – do not accumulate $Q$.

Output data

$AT, ET$ contain the matrices $\tilde{A}$ and $\tilde{E}$ which define the resulting system matrix pencil $\tilde{A} - \lambda \tilde{E}$ in a special Kronecker-like form (241), satisfying

$$\tilde{A} - \lambda \tilde{E} = \begin{bmatrix} Q^T & 0 & A - \lambda E & B \\ 0 & I_p & C & D \end{bmatrix} Z,$$  \hspace{1cm} (239)

where $Q$ and $Z$ are orthogonal transformation matrices.

$\text{DIMSC}(1:4)$ contain the column dimensions of the matrices $A_{\text{rg}}, A_{\text{bl}}, B_{\text{bl}}$ and $B_n$, respectively, in the special Kronecker-like form (241). The column dimension $\text{DIMSC}(3)$ of $B_{\text{bl}}$ represents the normal rank of the transfer function matrix $G(\lambda)$ of the system (238).

$Q, Z$ contain the orthogonal matrices $Q$ and $Z$ used to compute the special Kronecker-like form $\tilde{A} - \lambda \tilde{E}$ in (241). $Q$ and $Z$ are not accumulated if both outputs $Q$ and $Z$ are not specified.

If QOPT = 'noQ', $Q$ is not accumulated and, if specified, set to $Q = [\ ]$.

Method

Consider a disjunct partition of the extended complex plane $C_e = \mathbb{C} \cup \{\infty\}$ as

$$C_e = C_g \cup C_b, \quad C_g \cap C_b = \emptyset,$$  \hspace{1cm} (240)

where $C_g$ and $C_b$ are symmetric with respect to the real axis. Assume that the descriptor system realization (238) is $C_b$-stabilizable and let $G(\lambda)$ be the $p \times m$ transfer function matrix of the system (238) having normal rank $r$. Then, there exist two orthogonal matrices $Q$ and $Z$ such that

$$\begin{bmatrix} Q^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} A - \lambda E & B \\ C & D \end{bmatrix} Z = \begin{bmatrix} A_{\text{rg}} - \lambda E_{\text{rg}} & * & * & * \\ 0 & A_{\text{bl}} - \lambda E_{\text{bl}} & B_{\text{bl}} & * \\ 0 & 0 & 0 & B_n \\ 0 & C_{\text{bl}} & D_{\text{bl}} & * \end{bmatrix},$$  \hspace{1cm} (241)

where

(a) The pencil $A_{\text{rg}} - \lambda E_{\text{rg}}$ has full row rank for $\lambda \in C_g$ and $E_{\text{rg}}$ has full row rank.

(b) $E_{\text{bl}}$ and $B_n$ are invertible, the pencil

$$S_{\text{bl}}(\lambda) := \begin{bmatrix} A_{\text{bl}} - \lambda E_{\text{bl}} & B_{\text{bl}} \\ C_{\text{bl}} & D_{\text{bl}} \end{bmatrix}$$  \hspace{1cm} (242)

has full column rank $n_{\text{bl}} + r$ in $C_g$ and the pair $(A_{\text{bl}} - \lambda E_{\text{bl}}, B_{\text{bl}})$ is $C_b$-stabilizable.
The pencil $S_{bl}(\lambda)$ is the system matrix pencil of a descriptor system $(A_{bl} - \lambda E_{bl}, B_{bl}, C_{bl}, D_{bl})$, whose proper transfer function matrix $R(\lambda)$ is a range space basis matrix of $G(\lambda)$ [56]. By construction, the zeros of $R(\lambda)$ are those zeros of $G(\lambda)$ which are contained in $C_{bl}$, and can be selected via the option parameter ZEROSEL.

The computation of the special Kronecker-like form (241) is based on the method proposed in [29]. The function gsklf calls the more general function gklf to compute standard Kronecker-like forms, which is based on the mex-function sl_klf.

### 3.8.3 gsorsf

**Syntax**

$$[AT,ET,Q,Z,DIMS,NI] = \text{gsorsf}(A,E,\text{OPTIONS})$$

**Description**

$\text{gsorsf}$ computes, for a pair of square matrices $(A, E)$, an orthogonally similar pair $(\tilde{A}, \tilde{E})$ in a specially ordered GRSF.

**Input data**

$A, E$ are $n \times n$ real matrices $A$ and $E$, which define a regular linear matrix pencil $A - \lambda E$.

$\text{OPTIONS}$ is a MATLAB structure to specify user options and has the following fields:

| OPTIONS fields | Description |
|----------------|-------------|
| tol            | tolerance for the singular values based rank determination of $E$ (Default: $n^2 \|E\|_1 \text{eps}$) |
| disc           | disc option for the “good” region $C_g$: true – $C_g$ is a disc centered in the origin; false – $C_g$ is a left half complex plane (default). |
| smarg          | stability margin $\beta$, which defines the stability region $C_g$ of the eigenvalues of $A - \lambda E$, as follows: if $\text{OPTIONS.disc} = \text{false}$, the stable eigenvalues have real parts less than or equal to $\beta$, and if $\text{OPTIONS.disc} = \text{true}$, the stable eigenvalues have moduli less than or equal to $\beta$. (Default: $-\text{sqrt(eps)}$ if $\text{OPTIONS.disc} = \text{false}$; $1-\text{sqrt(eps)}$ if $\text{OPTIONS.disc} = \text{true}$.) |
| reverse        | option for reverse ordering of diagonal blocks: true – the diagonal blocks are as in (246) (see Method); false – the diagonal blocks are as in (245) (see Method) (default). |
| sepinf         | option for the separation of higher order infinite eigenvalues: true – separate higher order generalized infinite eigenvalues in the trailing positions if $\text{OPTIONS.reverse} = \text{false}$, or in the leading positions if $\text{OPTIONS.reverse} = \text{true}$ (default); false – no separation of higher order infinite eigenvalues. |
| fast           | option for fast separation of higher order infinite eigenvalues (to be used in conjunction with the $\text{OPTIONS.sepinf} = \text{true}$): |
| true  | fast separation of higher order infinite eigenvalues using orthogonal pencil manipulation techniques with QR decomposition based rank determinations (default); |
| false | separation of higher order infinite eigenvalues, by using SVD-based rank determinations (potentially more reliable, but slower). |

Output data

\( \mathbf{A}, \mathbf{E} \) contain the transformed matrices \( \tilde{\mathbf{A}} \) and \( \tilde{\mathbf{E}} \), obtained as

\[
\tilde{\mathbf{A}} = Q^T \mathbf{A} Z, \quad \tilde{\mathbf{E}} = Q^T \mathbf{E} Z, \tag{243}
\]

where \( Q \) and \( Z \) are orthogonal transformation matrices. The pair \( (\tilde{\mathbf{A}}, \tilde{\mathbf{E}}) \) is in a specially ordered GRSF, with the regular pencil \( \tilde{\mathbf{A}} - \lambda \tilde{\mathbf{E}} \) in the form (245), if \( \text{OPTIONS.reverse} = \text{false} \), or in the form (246), if \( \text{OPTIONS.reverse} = \text{true} \).

\( Q, Z \) contain the orthogonal matrices \( Q \) and \( Z \) used to compute the pair \( (\tilde{\mathbf{A}}, \tilde{\mathbf{E}}) \) in (243), in a specially ordered GRSF.

\( \text{DIMS(1:4)} \) contain the orders of the diagonal blocks \( (A_{\infty}, A_g, A_{b,f}, A_{b,\infty}) \), if \( \text{OPTIONS.reverse} = \text{false} \), or of the diagonal blocks \( (A_{b,\infty}, A_{b,f}, A_g, A_{\infty}) \), if \( \text{OPTIONS.reverse} = \text{true} \) (see Method).

\( \text{NI} \) contains the dimensions of the square diagonal blocks of \( A_{b,\infty} - \lambda E_{b,\infty} \), which characterize the higher order infinite eigenvalues of the pair \( (\mathbf{A}, \mathbf{E}) \). \( \text{NI} \) is empty if no higher order infinite eigenvalues exist.

Method

Consider a disjunct partition of the complex plane \( \mathbb{C} \) as

\[
\mathbb{C} = \mathbb{C}_g \cup \mathbb{C}_b, \quad \mathbb{C}_g \cap \mathbb{C}_b = \emptyset, \tag{244}
\]

where both \( \mathbb{C}_g \) and \( \mathbb{C}_b \) are symmetrically located with respect to the real axis, and \( \infty \in \mathbb{C}_b \). The complex domains \( \mathbb{C}_g \) and \( \mathbb{C}_b \) are typically associated with the “good” and “bad” generalized eigenvalues of the matrix pair \( (\mathbf{A}, \mathbf{E}) \), respectively. Using orthogonal similarity transformations as in (243), the pair \( (\mathbf{A}, \mathbf{E}) \) can be reduced to the specially ordered GRSF

\[
\tilde{\mathbf{A}} - \lambda \tilde{\mathbf{E}} = \begin{bmatrix}
A_{\infty} & * & * & * \\
0 & A_g - \lambda E_g & * & * \\
0 & 0 & A_{b,f} - \lambda E_{b,f} & * \\
0 & 0 & 0 & A_{b,\infty} - \lambda E_{b,\infty}
\end{bmatrix}, \tag{245}
\]

where: (i) \( A_{\infty} \) is an \( (n - r) \times (n - r) \) invertible (upper triangular) matrix, with \( r = \text{rank} \mathbf{E} \); the leading pair \( (A_{\infty}, 0) \) contains all infinite eigenvalues of \( \mathbf{A} - \lambda \mathbf{E} \) corresponding to first-order eigenvectors; (ii) \( A_g \) and \( E_g \) are \( n_g \times n_g \) matrices, such that the pair \( (A_g, E_g) \), with \( E_g \) invertible, is in a GRSF (i.e., \( A_g \) upper quasi-triangular and \( E_g \) upper triangular) and \( \Lambda(A_g - \lambda E_g) \) are the finite eigenvalues lying in \( \mathbb{C}_g \); (iii) \( A_{b,f} \) and \( E_{b,f} \) are \( n'_b \times n'_b \) matrices, such that the
pair \((A_{b,f}, E_{b,f})\), with \(E_{b,f}\) invertible, is in a GRSF and \(\Lambda(A_{b,f} - \lambda E_{b,f})\) are the finite eigenvalues lying in \(\mathbb{C}_b\); and (iv) \(A_{b,\infty}\) and \(E_{b,\infty}\) are \(n_b^\infty \times n_b^\infty\) upper triangular matrices, with \(A_{b,\infty}\) invertible and \(E_{b,\infty}\) nilpotent, and \(\Lambda(A_{b,\infty} - \lambda E_{b,\infty})\) are the higher order infinite eigenvalues. The orders \([n-r, n_g, n_f^l, n_f^r]\) of the diagonal blocks \((A_{\infty}, A_g, A_{b,f}, A_{b,\infty})\), respectively, are provided in \(\text{DIMS}(1:4)\). \(A_{b,\infty} - \lambda E_{b,\infty}\) has a block upper triangular form with \(k\) diagonal blocks, whose increasingly ordered dimensions are provided in \(\text{NI}(1:k)\). The dimensions \(\text{NI}(1:k)\) define the multiplicities of the higher order infinite eigenvalues as follows: for \(i = 1, \ldots, k\), there are \(\text{NI}(k-i+1) - \text{NI}(k-i)\) infinite eigenvalues of multiplicity \(i\), where \(\text{NI}(0) := 0\). The multiplicity of simple infinite eigenvalues is \(\text{DIMS}(4) - \text{NI}(1)\).

To obtain the specially ordered GRSF (245), the Procedure GSORSF, described in [57], can be used. For the separation of the higher order infinite generalized eigenvalues, the mex-function \text{sl_klf} is called by \text{gsorsf}, if \text{OPTIONS.fast} = \text{true}. This function employs rank determinations based on QR-decompositions with column pivoting and, therefore, is more efficient than an alternative, potentially more reliable approach, based on SVD-based rank determination. This latter approach is performed if \text{OPTIONS.fast} = \text{false}.

The same procedure, applied to the transposed pair \((A^T, E^T)\), is used to obtain the specially ordered GRSF with a reverse ordering of the blocks

\[
\tilde{A} - \lambda \tilde{E} = \begin{bmatrix}
A_{b,\infty} - \lambda E_{b,\infty} & * & * & *

0 & A_{b,f} - \lambda E_{b,f} & * & *

0 & 0 & A_g - \lambda E_g & *

0 & 0 & 0 & A_{\infty}
\end{bmatrix}.
\] (246)

The orders \([n_b^\infty, n_f^l, n_g, n-r] \) of the diagonal blocks \((A_{b,\infty}, A_{b,f}, A_g, A_{\infty})\), respectively, are provided in \(\text{DIMS}(1:4)\). \(A_{b,\infty} - \lambda E_{b,\infty}\) has a block upper triangular form with \(k\) diagonal blocks, whose decreasingly ordered dimensions are provided in \(\text{NI}(1:k)\). The dimensions \(\text{NI}(1:k)\) define the multiplicities of the higher order infinite eigenvalues as follows: for \(i = 1, \ldots, k\), there are \(\text{NI}(i) - \text{NI}(i+1)\) infinite eigenvalues of multiplicity \(i\), where \(\text{NI}(k+1) := 0\). The multiplicity of simple infinite eigenvalues is \(\text{DIMS}(4) - \text{NI}(1)\).

### 3.8.4 gsfstab

**Syntax**

\[
[F, \text{INFO}] = \text{gsf stab}(A, E, B, \text{POLES}, \text{SDEG}, \text{OPTIONS})
\]

**Description**

\text{gsf stab} computes, for a descriptor system pair \((A - \lambda E, B)\), a state-feedback matrix \(F\) such that the controllable finite generalized eigenvalues of the closed-loop pair \((A + BF, E)\) lie in the stability domain \(\mathbb{C}_s\).

**Input data**

\(A, E\) are \(n \times n\) real matrices \(A\) and \(E\), which define a regular linear pencil \(A - \lambda E\). \(E = I\) is assumed, for \(E = [ \, ]\).

\(B\) is a \(n \times m\) real matrix \(B\).
POLES specifies a complex conjugated set of desired eigenvalues to be assigned for the pair 
\((A + BF, E)\). The specified values are intended to replace the finite generalized eigenvalues 
of \((A, E)\) lying outside of the specified stability domain \(C_s\). If the number of specified 
eigenvalues in POLES is less than the number of controllable generalized eigenvalues of 
\((A, E)\) outside of \(C_s\), then the rest of generalized eigenvalues of \((A + BF, E)\) are assigned 
to the nearest values on the boundary of \(C_s\) (defined by the parameter SDEG, see below) 
or are kept unmodified if SDEG = [ ]. (Default: POLES = [ ])

SDEG specifies a prescribed stability degree for the eigenvalues of the pair \((A + BF, E)\). For 
a continuous-time setting, with \(SDEG < 0\), the stability domain \(C_s\) is the set of complex 
numbers with real parts at most \(SDEG\), while for a discrete-time setting, with \(0 \leq SDEG < 1\), 
\(C_s\) is the set of complex numbers with moduli at most \(SDEG\) 
(Default: \(-0.2\), used if SDEG = [ ] and POLES = [ ]).

OPTIONS is a MATLAB structure to specify user options and has the following fields:

| OPTIONS fields | Description                                      |
|----------------|--------------------------------------------------|
| tol            | tolerance for rank determinations (Default: internally computed) |
| sepinf         | option for a preliminary separation of the infinite eigenvalues:  |
|                | true – perform preliminary separation of the infinite generalized eigenvalues from the finite ones; |
|                | false – no separation of infinite generalized eigenvalues (default) |

Output data

\(F\) contains the resulting \(m \times n\) state-feedback gain \(F\).

INFO is a MATLAB structure, which provides additional information on the closed-loop matrices 
\(A_{cl} := Q(A + BF)Z\), \(E_{cl} := QEZ\), and \(B_{cl} := QB\), where \(Q\) and \(Z\) are the orthogonal 
matrices used to obtain the pair \((A_{cl}, E_{cl})\) in a GRSF. The fields of the INFO structure are:

| INFO fields | Description                                      |
|-------------|--------------------------------------------------|
| Acl         | contains \(A_{cl}\) in a quasi-upper triangular form; |
| Ecl         | contains \(E_{cl}\) in an upper triangular form; |
| Bcl         | contains \(B_{cl}\); |
| Q,Z         | contains the orthogonal transformation matrices \(Q\) and \(Z\). |
| ninf        | number of infinite generalized eigenvalues of \((A, E)\); |
| nfg         | number of finite generalized eigenvalues of \((A, E)\) lying in the stability domain \(C_s\); |
| naf         | number of assigned finite generalized eigenvalues in \(C_s\); |
| nuf         | number of uncontrollable finite generalized eigenvalues lying outside \(C_s\). |
Method

For a standard system pair \((A,I)\) (for \(E = [\ ]\)), the Schur method of [43] is used, while for a generalized system pair \((A,E)\) the generalized Schur method of [47] is used. The resulting closed-loop matrices

\[ A_{cl} = Q(A + BF)Z, \quad E_{cl} = QEZ, \quad B_{cl} = QB, \]

where \(Q\) and \(Z\) are the orthogonal matrices used to obtain the pair \((A_{cl}, E_{cl})\) in a GRSF, have the forms

\[
A_{cl} = \begin{bmatrix} A_{\infty} & * & * & * \\ 0 & A_{f,g} & * & * \\ 0 & 0 & A_{f,a} & * \\ 0 & 0 & 0 & A_{f,u} \end{bmatrix}, \quad E_{cl} = \begin{bmatrix} E_{\infty} & * & * & * \\ 0 & E_{f,g} & * & * \\ 0 & 0 & E_{f,a} & * \\ 0 & 0 & 0 & E_{f,u} \end{bmatrix}, \quad B_{cl} = \begin{bmatrix} * \\ * \\ * \\ 0 \end{bmatrix},
\]

(247)

where: (i) the pair \((A_{\infty}, E_{\infty})\), with \(A_{\infty}\) upper triangular and invertible and \(E_{\infty}\) upper triangular and nilpotent, contains the \texttt{INFO.ninf} infinite generalized eigenvalues of \((A,E)\); (ii) the pair \((A_{f,g}, E_{f,g})\), in GRSF, contains the \texttt{INFO.nfg} finite generalized eigenvalues of \((A,E)\) in \(\mathbb{C}_s\); (iii) the pair \((A_{f,a}, E_{f,a})\), in GRSF, contains the \texttt{INFO.naf} assigned finite generalized eigenvalues in \(\mathbb{C}_s\); and, (iv) the pair \((A_{f,u}, E_{f,u})\), in GRSF, contains the uncontrollable finite generalized eigenvalues of \((A,E)\) lying outside \(\mathbb{C}_s\). For the separation of the infinite generalized eigenvalues, the mex-function \texttt{sl_klf} is called by \texttt{gsfstab}.
A Installing DSTOOLS

DSTOOLS runs with MATLAB R2015b (or later versions) under 64-bit Windows 7 (or later). Additionally, the Control System Toolbox (Version 9.10 or later) is necessary to be installed. To install DSTOOLS, perform the following steps:

- download DSTOOLS as a zip file from Bitbucket\(^3\)
- create on your computer the directory dstools
- extract, using any unzip utility, the functions of the DSTOOLS collection in the corresponding directory dstools
- start MATLAB and put the directory dstools on the MATLAB path, by using the pathtool command; for repeated use, save the new MATLAB search path, or alternatively, use the addpath command to set new path entries in startup.m
- try out the installation by running the demonstration script DSToolsdemo.m

\(^3\)https://bitbucket.org/DSVarga/dstools
B Current Contents.m File

The M-functions available in the current version of FDITOOLS are listed in the current version of the Contents.m file, given below:

```matlab
% DSTOOLS - Descriptor System Tools.
% Version 0.6 31-July-2017
% Copyright (c) 2016-2017 by A. Varga
%
% Demonstration.
% DSToolsdemo - Demonstration of DSTOOLS.
%
% System analysis.
% gpole - Poles of a LTI descriptor system.
% gzero - Zeros of a LTI descriptor system.
% nrank - Normal rank of the transfer function matrix of a LTI system.
% ghanorm - Hankel norm of a proper and stable LTI descriptor system.
%
% Order reduction.
% gir - Reduced order realizations of LTI descriptor systems.
% gminreal - Minimal realization of a LTI descriptor system.
% gbalmr - Balancing-based model reduction of a LTI descriptor system.
% gss2ss - Conversions to SVD-like forms without non-dynamic modes.
%
% Operations on transfer function matrices.
% grnull - Right nullspace basis of a transfer function matrix.
% glnull - Left nullspace basis of a transfer function matrix.
% grange - Range space basis of a transfer function matrix.
% grsol - Solution of the linear rational matrix equation G*X = F.
% glsol - Solution of the linear rational matrix equation X*G = F.
% gsdec - Generalized additive spectral decompositions.
% glmcover1 - Right minimum dynamic cover of Type 1 based order reduction.
% glmcover1 - Left minimum dynamic cover of Type 1 based order reduction.
% glmcover2 - Right minimum dynamic cover of Type 2 based order reduction.
% glmcover2 - Left minimum dynamic cover of Type 2 based order reduction.
%
% Factorizations of transfer function matrices.
% grcf - Right coprime factorization with proper and stable factors.
% glcf - Left coprime factorization with proper and stable factors.
% grcfid - Right coprime factorization with inner denominator.
% glcfid - Left coprime factorization with inner denominator.
% giofac - Inner-outer/QR-like factorization.
% goifac - Co-outer-inner/RQ-like factorization.
% grsfg - Right spectral factorization of gamma^2*I-G'*G.
% glsfg - Left spectral factorization of gamma^2*I-G*G'.
```

141
% Approximations of transfer function matrices.
% gnehari - Generalized Nehari approximation.
% glinfldp - Solution of the least distance problem min||G1-X G2||_inf.
%
% Feedback stabilization.
% gsfstab - Generalized state-feedback stabilization.
% eigselect1 - Selection of a real eigenvalue to be assigned.
% eigselect2 - Selection of a pair of eigenvalues to be assigned.
% galoc2 - Generalized pole allocation for second order systems.
%
% Pencil similarity transformations
% gklf - Kronecker-like staircase forms of a linear matrix pencil.
% gsklf - Special Kronecker-like form of a system matrix pencil.
% gsorsf - Specially ordered generalized real Schur form.
%
% SLICOT-based mex-functions.
% sl_gstra - Descriptor system coordinate transformations.
% sl_gminr - Minimal realization of descriptor systems.
% sl_gsep - Descriptor system additive spectral decompositions.
% sl_gzero - Computation of system zeros and Kronecker structure.
% sl_klf - Pencil reduction to Kronecker-like forms.
% sl_glme - Solution of generalized linear matrix equations.
C DSTOOLS Release Notes

The DSTOOLS Release Notes describe the changes introduced in the successive versions of the DSTOOLS collection, as new features, enhancements to functions, or major bug fixes.

C.1 Release Notes V0.5

This is the initial version of the DSTOOLS collection of M- and MEX-functions, which accompanies the book [58]. All numerical results presented in this book have been obtained using this version of DSTOOLS.

C.1.1 New Features

The M-functions and MEX-functions available in the Version 0.5 of DSTOOLS are listed below, where we kept the originally employed descriptions of the functions:

```matlab
% DSTOOLS - Descriptor System Tools.
% Version 0.5 31-Dec-2016
% Copyright (c) 2016 by A. Varga
%
% Demonstration.
% DSToolsdemo - Demonstration of DSTOOLS.
%
% System analysis.
% gpole - Poles of a LTI descriptor system.
% gzero - Invariant zeros and Kronecker structure of a system pencil.
% nrank - Normal rank of a transfer function matrix of a LTI system.
% ghanorm - Hankel norm of a proper and stable LTI descriptor system.
%
% Order reduction.
% gir - Irreducible realizations of LTI descriptor systems.
% gminreal - Minimal realization of a LTI descriptor system.
% gbalmr - Balancing-based model reduction of a stable descriptor system.
% gss2ss - Conversions to SVD-like forms without non-dynamic modes.
%
% Operations on generalized LTI systems.
% gnull - Left nullspace basis of a rational matrix.
% grnull - Right nullspace basis of a rational matrix.
% glsol - Solution of linear rational equation X(s)*G(s)=F(s).
% grsol - Solution of linear rational equation G(s)*X(s)=F(s).
% gsdec - Generalized additive spectral decompositions.
% glmcover1 - Left minimum dynamic cover of Type 1 based order reduction.
% grmcover1 - Right minimum dynamic cover of Type 1 based order reduction.
% glmcover2 - Left minimum dynamic cover of Type 2 based order reduction.
% grmcover2 - Right minimum dynamic cover of Type 2 based order reduction.
```
% Factorizations.
% giofac - Generalized inner-outer factorization of descriptor systems.
% golfac - Generalized co-outer-inner factorization of descriptor systems.
% glcf - Generalized left coprime factorization.
% grcf - Generalized right coprime factorization.
% glcfd - Generalized left coprime factorization with inner denominator.
% grcfd - Generalized right coprime factorization with inner denominator.
% gisfg - Generalized left spectral factorization of $g^2-G*G'$.
% grsfg - Generalized right spectral factorization of $g^2-G'*G$.

% Approximations.
% gnehari - Generalized Nehari approximation.
% glinfdp - Solution of the least distance problem $\min ||F1-X F2||_\text{inf}$.

% Feedback stabilization.
% gsfstab - Generalized state-feedback stabilization.
% eigeval - Selection of a real eigenvalue to be assigned.
% eigeval2 - Selection of a pair of eigenvalues to be assigned.
% galoc2 - Generalized pole allocation for second order systems.

% Pencil similarity transformations
% gkklf - Generalized Kronecker-like staircase form of a linear pencil.
% gosrsf - Specially ordered generalized real Schur form.

% SLICOT-based mex-functions.
% sl_gstra - Descriptor system coordinate transformations.
% sl_gminr - Minimal realization of descriptor systems.
% sl_gsep - Descriptor system spectral separations.
% sl_gzero - Computation of system zeros and Kronecker structure.
% sl_kklf - Pencil reduction to Kronecker-like forms.
% sl_glme - Solution of generalized linear matrix equations.

C.2 Release Notes V0.6
This version of the DSTOOLS includes minor revisions of most functions, by adding exhaustive input parameter checks and performing several simplifications in the codes. Besides this, two new functions have been implemented and the underlying mex-functions have been replaced with new ones, which use an enhanced rank determination strategy.

C.2.1 New Features
Two new functions have been added to DSTOOLS:

% Operations on generalized LTI systems.
% grange - Range space basis of a transfer function matrix.
% Pencil similarity transformations
% gskklf - Special Kronecker-like form of a system matrix pencil.
The function `grange` to compute a range space basis of a transfer function matrix is based on a special Kronecker-like form of the system matrix pencil, which is computed by the function `gsklf`.

Several additional changes have been performed in the following functions:

- **gpole**: the computations for standard state-space realizations have been separated from those for descriptor system realizations;
- **nrank**: functionality extended to handle all LTI system objects (i.e., `ss`, `tf`, `zpk`);
- **gir**:
  - explicit handling of standard state-space representations added;
  - option structure removed and replaced with a character string option parameter;
- **gminreal**:
  - explicit handling of standard state-space representations added;
  - logical option parameter replaced with a character string option parameter;
- **gbalmr**:
  - original realization preserved if no order reduction and no balancing take place;
  - checking the invertibility of $E$ added;
- **gss2ss**:
  - relies entirely on `sl_gstra` to compute the SVD-like coordinate forms;
  - exploits the generalized Hessenberg form of the pair $(A, E)$ if $E$ is invertible;
- **grnull**: stabilization and pole assignment options added;
- **glnull**: stabilization and pole assignment options added;
- **grsol**: `INFO` provides always full structural information in `INFO.nr`, `INFO.nf` and `INFO.ninf`;
- **glsol**: `INFO` provides always full structural information in `INFO.ninf`, `INFO.nf` and `INFO.nl`;
- **gsdec**: explicit handling of standard state-space representations added;
- **giofac**:
  - `OPTIONS` structure input parameter introduced to specify several user options;
  - new functionality added to compute the QR-like factorization of rational matrices;
  - the code sequence to compute a special Kronecker-like form replaced by a call of `gsklf`;
- **goifac**:
  - `OPTIONS` structure input parameter introduced to specify several user options;
  - new functionality added to compute the RQ-like factorization of rational matrices;
- **gnehari**: functionality restricted to systems without poles on the stability domain boundary;
- **gsorsf**:
  - the output parameter `DIMS` contains now the orders of all four diagonal blocks;
  - the default option for `OPTIONS.sepinf` has been redefined;
C.2.2 Bug Fixes

Several minor bug fixes have been performed in the following functions:

\textbf{grnull}:
- nullspace computation fixed for zero input dimensions;

\textbf{glnull}:
- nullspace computation fixed for zero output dimensions;
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Index

canonical form
  Jordan, 11
  Kronecker, 13
  Weierstrass, 11
condensed form
  controllability staircase form, 15, 132
  generalized real Schur (GRSF), 12, 104, 107, 110, 113
  specially ordered, 135
  Kronecker-like, 15, 130
  special, 133
  observability staircase form, 15, 132
  real Schur (RSF), 13
descriptor system
  additive decomposition, 25, 86
  conjugate, 49
  controllability, 11
  controllable eigenvalue, 24
  coprime factorization, 26
  exponential stability, 24
  finite controllability, 11
  finite detectable, 24
  finite observability, 11
  finite stabilizable, 24
  Hankel norm, 34, 55
  improper, 24
  infinite controllability, 11
  infinite observability, 11
  inverse, 48
  irreducible realization, 11, 56, 58
  left minimal cover problem, 33
  Type 1, 94
  Type 2, 99
  linear rational matrix equation, 32, 77, 82
  minimal nullspace basis, 19, 63, 68
  minimal realization, 11, 58
  normal rank, 24
  observability, 11
  observable eigenvalue, 24
  poles, 24
  polynomial, 24
  proper, 24
  right minimal cover problem, 33
  Type 1, 34, 89
  Type 2, 34, 96
  similarity transformation, 10
  strongly detectable, 24
  strongly stabilizable, 24
  uncontrollable eigenvalue, 24
  unobservable eigenvalue, 24
  zeros, 24
factorization
  co-outer–co-inner, 27, 28
  extended, 27
  fractional, 26
  full rank, 19, 72
  inner–outer, 27, 28
  extended, 27
  inner–quasi-outer, 27
  extended, 27, 114
  left coprime (LCF), 26, 106
  minimum-degree denominator, 26, 106
  normalized, 27
  with inner denominator, 26, 112
  QR-like
  extended, 28, 114
  quasi-co-outer–co-inner, 27
  extended, 27, 117
  right coprime (RCF), 26, 103
  minimum-degree denominator, 26, 103
  normalized, 27
  with inner denominator, 26, 109
  RQ-like
  extended, 28, 117
  spectral, 28
  minimum-phase left, 28
  minimum-phase right, 28
  special, stable minimum-phase left, 30, 37, 122
  special, stable minimum-phase right, 31, 120
  stable left, 28
  stable minimum-phase left, 28
  stable minimum-phase right, 28
stable right, 28

linear matrix pencil
eigenvalues, 11
finite eigenvalues, 11
infinite eigenvalues, 11
Kronecker canonical form, 13
Kronecker indices, 14
strict equivalence, 11, 23
Weierstrass canonical form, 11
zeros, 50
finite, 50
infinite, 50

M-functions
gbalmr, 60
ghanorm, 55
giofac, 114
gir, 56
gklf, 130
glc fid, 112
glcf, 106
glinfldp, 126
glmcover1, 93
glmcover2, 99
glnull, 67
glsfg, 122
glsol, 81
gminreal, 58
gnehari, 124
goifac, 117
gpole, 50
grange, 72
grcfid, 109
grcf, 103
grmcover1, 89, 96, 101
grmcover2, 96
grnul1, 63
grsfg, 120
grsol, 77, 84
gsdec, 86
gsf stab, 137
gsklf, 133
gsorsf, 135
gss2ss, 61
gzero, 51

nrank, 53

matrix equation
generalized algebraic Riccati
continuous-time (GCARE), 29, 30
discrete-time (GDARE), 29, 31
generalized Lyapunov, 35
generalized Stein, 35

MEX-functions
sl_glme, 55, 60, 125
sl_gminr, 57, 59, 80, 84, 91, 96, 99, 101,
116, 119
sl_gsep, 87, 92, 99, 125
sl_gstra, 62, 80, 84, 91, 96, 98, 101
sl_gzero, 51, 52, 54
sl_klf, 65, 70, 74, 80, 84, 116, 119, 132,
135, 137, 139

minimal basis
polynomial, 18
proper rational, 18
simple, proper rational, 18

model-matching problem, 128

nullspace
basis, 17
minimal polynomial, left, 18
minimal proper, left, 19, 68
minimal proper, right, 19, 63
minimal rational, left, 19
minimal rational, right, 19
simple minimal proper, left, 19, 68
simple minimal proper, right, 19, 63
left, 18
right, 18

polynomial basis
irreducible, 18
minimal, 17
row reduced, 18

polynomial matrix
invariant polynomials, 21
normal rank, 21
Smith form, 21
unimodular, 9
zeros, 21
finite, 21
infinite, 22
range space
  basis
    inner, 19
    minimal proper, 19
rational basis
  minimal proper, 18
  simple minimal proper, 18
transfer function
  anti-stable, 20
  exponential stability, 20
  minimum-phase, 20
  poles, 20
    finite, 20
    infinite, 20
    stability degree, 20
    stable, 20
    unstable, 20
  stable, 20
  zeros, 20
    finite, 20
    infinite, 20
    minimum-phase, 20
    non-minimum-phase, 20
transfer function matrix (TFM)
  additive decomposition, 25, 37, 86
  biproper, 9
  co-inner, 26
  co-outer, 27
  conjugate, 26
  Hankel norm, 34, 36, 55
  improper, 9
  inner, 26
  left minimal cover problem, 33, 94, 99
  linear rational matrix equation, 31, 77, 82
  McMillan degree, 22
  minimum-phase, 23
  model-matching problem
    approximate, 30, 36
    exact, 32
  Nehari approximation
    optimal, 36–38, 124
    suboptimal, 36, 124
  non-minimum-phase, 23
  normal rank, 17, 51, 53, 55
  outer, 27
  poles, 22, 50
    finite, 50
    infinite, 22, 50
    proper, 9
    quasi-co-outer, 27
    quasi-outer, 27
  right minimal cover problem, 33, 89, 96
  Smith-McMillan form, 21
  stable, 23
  strictly proper, 9
  unstable, 23
  zeros, 22, 50, 51
    finite, 22, 50, 51
    infinite, 22, 50, 51

153