A method for proving Ramanujan’s series for $1/\pi$

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Abstract
In a famous paper of 1914 Ramanujan gave a list of 17 extraordinary formulas for the number $1/\pi$. In this paper we explain a general method to prove them, based on some ideas of James Wan and some of our own ideas.

Keywords
Hypergeometric series · Ramanujan series for $1/\pi$ · Legendre’s relation

Mathematics Subject Classification
33E05 · 33C20 · 11F03 · 33C75

1 Introduction

In his famous paper [16] of 1914 Ramanujan gave a list of 17 extraordinary formulas for the number $1/\pi$, which are of the following form:

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{3})_n (1 - \frac{1}{s})_n}{(1)_n} (a + bn) z^n = \frac{1}{\pi}, \quad (c)_0 = 1, \quad (c)_n = \prod_{j=1}^{n} (c + j - 1), \quad (1)$$

where $s \in \{2, 3, 4, 6\}$, and $z, b, a$ are algebraic numbers. Five of his formulas are

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n^3}{(1)_n^3} \frac{42n + 5}{64^n} = \frac{16}{\pi}, \quad (2)$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{3})_n (\frac{2}{3})_n}{(1)_n^2} (33n + 4) \left(\frac{4}{125}\right)^n = \frac{15\sqrt{3}}{2\pi}, \quad (3)$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{6})_n (\frac{2}{6})_n}{(1)_n^3} (11n + 1) \left(\frac{4}{125}\right)^n = \frac{5\sqrt{15}}{6\pi}, \quad (4)$$

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However Ramanujan wrote few details of his proofs and the first rigorous deductions were made by the Borwein brothers in 1985, see [7]. Other general proofs based on the modular theory are for example in [1–3,6–8,18], and we believe that the method that we use in this paper is an interesting alternative one. Other kinds of proofs are in [9–11,14,15], and two beautiful surveys are in [3,18].

2 A method for proving Ramanujan series for $1/\pi$

We will use the following notation:

$$F_s(\alpha) = 2 F_1(\frac{1}{s}, 1 - \frac{1}{s} | \alpha), \quad G_s(\alpha) = \alpha \frac{d F_s(\alpha)}{d \alpha},$$

and the following version of the Legendre’s relation:

$$\alpha F_s(\alpha) G_s(\beta) + \beta F_s(\beta) G_s(\alpha) = \frac{1}{\pi} \sin \frac{\pi}{s}, \quad \beta = 1 - \alpha. \quad (7)$$

We will show that this relation explains why $\pi$ appears in the Ramanujan series. The other ingredients we need to prove the series are: a transformation of modular origin and the well-known Clausen’s identity

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{4})_n (\frac{3}{4})_n}{(1)^3_n} \frac{26390n + 1103}{994n+2} = \frac{\sqrt{2}}{4\pi}, \quad (5)$$

$$\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{4})_n (\frac{3}{4})_n}{(1)^3_n} \frac{21460n + 1123}{8822n+1} = \frac{4}{\pi}. \quad (6)$$

Our method consists in a variant of a Wan’s original idea [17]. We explained it in [12,13] proving some new Ramanujan–Orr series for $1/\pi$. In this paper we show how to apply it to prove Ramanujan series in a simple way when we know the required transformation, or what is equivalent: when we know the required modular equation, because the multiplier is given by the formula (24). We explain our technique with two examples: one corresponding to a series of positive terms and the other one to an alternating series.
2.1 Example for series of positive terms

We reprove below the following series for $1/\pi$ of level $\ell = 2$ ($s = 4$) due to Ramanujan:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^3} \frac{1}{3^{4n}} (10n + 1) = \frac{9\sqrt{2}}{4\pi}. \quad (9)$$

We begin with the following theorem:

**Theorem 1** For $s = 4$ (level $\ell = 2$), we have

$$F_4(\alpha_0) = \frac{\sqrt{5}}{5} F_4(\beta_0), \quad (10)$$

$$G_4(\alpha_0) = \frac{16\sqrt{5} - 36}{5} F_4(\beta_0) + \frac{161\sqrt{5} - 360}{5} G_4(\beta_0), \quad (11)$$

where

$$\alpha_0 = \frac{1}{2} - \frac{2\sqrt{5}}{9}, \quad \beta_0 = 1 - \alpha_0 = \frac{1}{2} + \frac{2\sqrt{5}}{9}. \quad (12)$$

**Proof** In [9, p. 608] we see the following transformation of level 2 and degree $1/d$ with $d = 5$:

$$F_4(\alpha) = m F_4(\beta), \quad (13)$$

where

$$\alpha = \frac{64x^5(1 + x)}{(1 + 4x^2)(1 - 2x - 4x^2)^2}, \quad \beta = \frac{64x(1 + x)^5}{(1 + 4x^2)(1 + 22x - 4x^2)^2},$$

and

$$m = \frac{\sqrt{1 - 2x - 4x^2}}{\sqrt{1 + 22x - 4x^2}}.$$ 

If we take $\beta = 1 - \alpha$, then we get the following solution:

$$x_0 = \frac{\sqrt{5} - 2}{2}, \quad \alpha_0 = \frac{1}{2} - \frac{2\sqrt{5}}{9}, \quad \beta_0 = \frac{1}{2} + \frac{2\sqrt{5}}{9}, \quad m_0 = \frac{1}{\sqrt{5}}.$$ 

In addition, we have

$$\alpha_0' = \frac{\sqrt{5} + 2}{27}, \quad \beta_0' = \frac{\sqrt{5} + 2}{27}, \quad m_0' = \frac{-8\sqrt{5} - 16}{15}.$$ 

Differentiating (12) with respect to $\alpha$ we have

$$G_4(\alpha) = \alpha \frac{m'}{\alpha'} F_4(\beta) + \alpha \frac{m}{\beta} \frac{\beta'}{\alpha'} G_4(\beta), \quad (13)$$
where the ′ stands for the derivative with respect to \( x \). Finally, substituting \( x = x_0 \), in (12) and (13) we arrive at the results stated by the theorem. \( \square \)

We are ready to prove (9).

**Proof** Applying to both sides of (8) with \( s = 4 \) the operator

\[
\frac{2}{9} + \frac{20}{9} z \frac{d}{dz} \bigg|_{z=z_0} = \frac{2}{9} + \frac{20}{9} z \frac{d}{d\alpha} \bigg|_{\alpha=\alpha_0},
\]

where here the ′ means the derivative with respect to \( \alpha \), we obtain

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)^n} \frac{1}{3^{4n}} \left(\frac{20}{9} n + \frac{2}{9}\right)
\]

\[
= \frac{2}{9} F_4(\alpha_0) F_4(\alpha_0) + \left(\frac{\sqrt{5}}{2} + \frac{10}{9}\right) F_4(\alpha_0) G_4(\alpha_0)
\]

\[
+ \left(\frac{\sqrt{5}}{2} + \frac{10}{9}\right) G_4(\alpha_0) F_4(\alpha_0).
\]

Observe that we have intentionally repeated two equal terms without simplifying the sum. Then, if we use the relations (10) and (11) to replace one factor \( F_4(\alpha_0) \) of the two first terms and to replace \( G_4(\alpha_0) \) in the last term, we arrive at

\[
\alpha_0 F_4(\alpha_0) G_4(\beta_0) + \beta_0 F_4(\beta_0) G_4(\alpha_0),
\]

which in view of (7) is equal to

\[
\frac{1}{\pi} \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2\pi},
\]

and we are done. \( \square \)

### 2.2 Example for alternating series

Here we prove with our method the following alternating series of level \( \ell = 2 \) due to Ramanujan:

\[
\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)^n} \left(\frac{-1}{48}\right)^n (28n + 3) = \frac{16\sqrt{3}}{3\pi}.
\]

**Theorem 2** For \( s = 4 \) (level \( \ell = 2 \)), we have

\[
F_4(\alpha_0) = \frac{3 + i}{10} F_4(\beta_0),
\]

\( \square \) Springer
and

\[ G_4(\alpha_0) = \left( \frac{-63}{20} \sqrt{2} + \frac{9 \sqrt{6}}{5} \right) F_4(\beta_0) \]

\[ + \left[ \left( \frac{-291}{10} \sqrt{2} + \frac{84}{5} \sqrt{6} \right) + \left( \frac{-28}{5} \sqrt{6} + \frac{97}{10} \sqrt{2} \right) i \right] G_4(\beta_0), \quad (17) \]

where

\[ \alpha_0 = \frac{1}{2} - \frac{7 \sqrt{3}}{24}, \quad \beta_0 = \frac{1}{2} + \frac{7 \sqrt{3}}{24}. \]

**Proof** We use again the transformation of degree 1/d with \( d = 5 \):

\[ F_4(\alpha) = m F_4(\beta), \quad (18) \]

where

\[ \alpha = \frac{64x^5(1 + x)}{(1 + 4x^2)(1 - 2x - 4x^2)^2}, \quad \beta = \frac{64x(1 + x)^5}{(1 + 4x^2)(1 + 22x - 4x^2)^2}, \]

and

\[ m = \frac{\sqrt{1 - 2x - 4x^2}}{\sqrt{1 + 22x - 4x^2}}. \]

Another solution of \( \beta = 1 - \alpha \) is the following one:

\[ x_0 = \frac{2 \sqrt{3} - 3}{4} - \frac{2 - \sqrt{3}}{4} i, \quad \alpha_0 = \frac{1}{2} - \frac{7 \sqrt{3}}{24}, \]

\[ \beta_0 = \frac{1}{2} + \frac{7 \sqrt{3}}{24}, \quad m_0 = \frac{(3 + i) \sqrt{2}}{10}. \]

It is interesting to note that \( |m_0| = 1/\sqrt{5} \). We also get

\[ m_0' = \left( \frac{-27}{40} \sqrt{2} - \frac{69}{200} \sqrt{6} \right) - \left( \frac{33}{200} \sqrt{6} + \frac{9}{40} \sqrt{2} \right) i, \quad (19) \]

\[ \alpha_0' = \left( \frac{-23}{240} - \frac{\sqrt{3}}{16} \right) - \left( \frac{\sqrt{3}}{48} + \frac{11}{240} \right) i, \quad (20) \]

\[ \beta_0' = \left( \frac{-5}{48} - \frac{\sqrt{3}}{16} \right) + \left( \frac{1}{48} + \frac{\sqrt{3}}{48} \right) i. \quad (21) \]
Differentiating (18) with respect to $\alpha$ we have

$$G_4(\alpha) = \alpha \frac{m'}{\alpha'} F_4(\beta) + \alpha \frac{\beta'}{\alpha'} G_4(\beta), \quad (22)$$

where the $'$ stands for the derivative with respect to $x$. Finally, substituting $x = x_0$, in (18) and (22) we arrive at the results stated by the theorem.

We are ready to prove (15).

**Proof** Applying to both sides of (8) with $s = 4$ the operator

$$\frac{3}{32} \sqrt{6} + \frac{28}{32} \sqrt{6} z \frac{d}{dz} \bigg|_{z = z_0} = \frac{3}{32} \sqrt{6} + \frac{28}{32} \sqrt{6} z \frac{d}{d\alpha} \bigg|_{\alpha = \alpha_0},$$

where here the $'$ means the derivative with respect to $\alpha$, we obtain

$$\sum_{n=0}^{\infty} \left( \frac{5}{4} \right)_n \left( \frac{3}{4} \right)_n \left( \frac{1}{n} \right)_n \left( \frac{-1}{48} \right)^n \left( \frac{28}{32} \sqrt{6n} + \frac{3}{32} \sqrt{6} \right) = \frac{3}{32} \sqrt{6} F_4(\alpha_0) F_4(\alpha_0) + (1 - Ci) \left( \frac{3}{4} \sqrt{2} + \frac{7}{16} \sqrt{6} \right) F_4(\alpha_0) G_4(\alpha_0)$$

$$+ (1 + Ci) \left( \frac{3}{4} \sqrt{2} + \frac{7}{16} \sqrt{6} \right) G_4(\alpha_0) F_4(\alpha_0). \quad (23)$$

Observe that we have intentionally introduced the factors $(1 - Ci)$ and $(1 + Ci)$, and that the expression holds for all values of $C$ because the terms with $C$ cancels. Then, we use the relations (16) and (17) to replace one factor $F_4(\alpha_0)$ of the two first terms and to replace $G_4(\alpha_0)$ in the last term. Finally, if we choose $C = 1/3$, we arrive at

$$\alpha_0 F_4(\alpha_0) G_4(\beta_0) + \beta_0 F_4(\beta_0) G_4(\alpha_0),$$

which in view of (7) is equal to

$$\frac{1}{\pi} \sin \frac{\pi}{4} = \frac{\sqrt{2}}{2\pi},$$

and we are done.

**3 Explicit formulas**

We already know that $z = 4\alpha(1 - \alpha)$. Here, we give explicit formulas for $b$ and $a$. In addition, we relate the modular variable $q$ to $d$ and $\ell$. But first we will show how the multiplier is related to the modular equation.
3.1 On modular equations and multipliers

Transformations of modular origin can be used to prove Ramanujan-type series for $1/\pi$. Those proved in [9,10] are written like the one used in this paper. For the great quantity of them given by Ramanujan, see [4, Chaps. 19, 20] and [5, Chaps. 33, 36]. For the modular equations $P_s(\alpha, \beta) = 0$ corresponding to $s = 2, 3, 4, 6$ (levels $\ell = 4, 3, 2, 1$), the multiplier is given by the following formula:

$$m_s(\alpha, \beta) = \frac{1}{\sqrt{d}} \left( \frac{\beta(1 - \beta) \, d\alpha}{\alpha(1 - \alpha) \, d\beta} \right)^{1/2},$$  \hspace{1cm} (24)

where $1/d$ is the degree of the modular equation. Hence the associated transformation of level $\ell$ and degree $1/d$ reads

$$F_s(\alpha) = m_s(\alpha, \beta) F_s(\beta), \quad P_s(\alpha, \beta) = 0.$$ 

You can see a proof of (24) for the case $s = 2$ (level $\ell = 4$) in [4, Entry 24 (vi)], but a similar proof can be given for the four hypergeometric levels.

3.2 Explicit formulas

To get the explicit formulas for $a$ and $b$ we will use our method and the identity

$$\frac{\beta_0'}{\alpha_0'} = \frac{1}{d m_0^2},$$  \hspace{1cm} (25)

which comes from (24). First, observe that the level is related to $s$ in the following way:

$$\ell = 4 \sin^2 \frac{\pi s}{s}.$$  \hspace{1cm} (26)

It is not a coincidence [15, Eq. 27]. Using our method for the case of general series and cases $z_0 > 0$ or $z_0 < 0$, that is, applying the operator

$$a + bz \frac{d}{dz} \bigg|_{z_0},$$

to both sides of (8), using the substitutions

$$F_s(\alpha_0) = m F_s(\beta_0), \quad G_s(\alpha_0) = \alpha_0 \frac{m_0}{\alpha_0} F_s(\beta_0) + \frac{\alpha_0}{d m_0 \beta_0} G_s(\beta_0),$$

in the way that we have explained in the examples, and equating the coefficients of $F(\alpha_0)F(\beta_0)$, $F(\alpha_0)G(\beta_0)$ and $F(\beta_0)G(\alpha_0)$ to 0, $\alpha_0$ and $\beta_0$ respectively, we deduce...
the following explicit formulas:

\[ b = (1 - 2\alpha_0) \frac{\text{Re}(m_0)}{\sin \frac{\pi}{s}} d, \quad (27) \]

and

\[ a = -(1 + Ci) \frac{\alpha_0 \beta_0}{\alpha_0} \frac{m_0'}{m_0} \frac{b}{1 - 2\alpha_0}, \quad C = \frac{\text{Im}(m_0)}{\text{Re}(m_0)}. \quad (28) \]

For the case \( z > 0 \), we have \( m_0 = 1/\sqrt{d} \). Hence, we can write the above formulas in the way

\[ b = 2(1 - 2\alpha_0) \sqrt{\frac{d}{\ell}}, \quad a = -2\alpha_0 \beta_0 \frac{m_0'}{\alpha_0} \frac{d}{\sqrt{\ell}}. \quad (29) \]

For the case \( z < 0 \) (alternating series), we have observed experimentally that

\[ m_0 = \frac{\sqrt{4d - \ell}}{2d} + \frac{\sqrt{\ell}}{2d} i. \quad (30) \]

Hence, assuming it, we can write (27) and (28) in the following form:

\[ b = 2(1 - 2\alpha_0) \sqrt{\frac{d}{\ell} - \frac{1}{4}}, \quad a = -2\alpha_0 \beta_0 \frac{m_0'}{\alpha_0} \frac{d}{\sqrt{\ell}}. \quad (31) \]

Finally, we relate the modular variable \( q \) with \( d \) and \( \ell \) assuming (30). Let \( q = e^{-2\pi \sqrt{r}} \) and \( q = -e^{-2\pi \sqrt{r}} \), the modular variable corresponding to the cases \( z > 0 \) and \( z < 0 \), respectively. From the known formula

\[ 4r = \frac{b^2}{1 - z} = \frac{b^2}{(1 - 2\alpha)^2}, \]

we deduce that

\[ r = \frac{d}{4 \sin^2 \frac{\pi}{s}} = \frac{d}{\ell}, \quad r = \frac{d}{4 \sin^2 \frac{\pi}{s} - \frac{1}{4}} = \frac{d}{\ell} - \frac{1}{4}, \]

for the cases \( z > 0 \) and \( z < 0 \), respectively. Hence,

\[ q = e^{-2\pi \sqrt{\frac{d}{\ell}}}, \quad q = -e^{-2\pi \sqrt{\frac{d}{\ell} - \frac{1}{4}}}, \quad (32) \]

for all the series of positive terms and for all the alternating series respectively.
3.3 An experimental test

The test which consists of evaluating numerically

\[ \frac{F_s(\alpha_0)}{F_s(\beta_0)} = m_0, \quad \beta_0 = 1 - \alpha_0, \tag{33} \]

has been very useful in discovering that \(|m_0^2|\) but not \(m_0^2\) (algebraic) is a positive rational number, and that \(|m_0^2| = 1/d\). For example, for the series of level \(\ell = 3\) \((s = 3)\) and \(z_0 = -1/500^2\), as \(z_0 = 4\alpha_0(1 - \alpha_0)\) we obtain

\[
\alpha_0 = \frac{1}{2} - \frac{53\sqrt{89}}{1000}, \quad \beta_0 = 1 - \alpha_0 = \frac{1}{2} + \frac{53\sqrt{89}}{1000},
\]

and evaluating numerically (33), we get with an approximation of 20 digits that

\[
\left| \frac{F_s(\alpha_0)}{F_s(\beta_0)} \right| \approx 0.20851441405707476267,
\]

which we identify as \(1/\sqrt{23}\). Hence, for proving with our method that alternating series, we need a transformation of degree \(1/d\) with \(d = 23\) for the level \(\ell = 3\), and with such a transformation we can prove it rigorously. The data in Tables 1, 2, 3, and 4 give rise to exactly the same 36 series as the tables in Cooper’s book [9, Chap. 14]. If we label each series with \((\sigma, \ell, d)\), where \(\sigma = P\) for series of positive terms and \(\sigma = A\) for alternating series, then, the series (2), (3), (4), (5) and (6) correspond respectively to \((P, 4, 7), (P, 3, 5), (P, 1, 3), (P, 2, 29), \) and \((A, 2, 19), \) and the series (9) and (15) to \((P, 2, 5)\) and \((A, 2, 5)\) respectively. An unexpected but very pleasant result, that can be used to simplify the proofs of alternating series for \(1/\pi\) of levels \(\ell = 1, 2, 3\), is that for those series the degree \(d\) is lower than the \(N\) in the tables in Cooper’s book.

| Table 1 | Rational Ramanujan-type series for \(1/\pi\) \((\ell = 1)\) |
|---------|---------------------------------|
| \(d\) | \(a\) | \(b\) | \(z < 0\) | \(z > 0\) |
| 2 | \(\frac{8}{5\sqrt{15}}\) | \(\frac{63}{5\sqrt{15}}\) | \(-\frac{3^3}{5^2}\) | 2 | \(\frac{3}{5\sqrt{5}}\) | \(\frac{28}{5\sqrt{5}}\) | \(\frac{3^3}{5^2}\) |
| 3 | \(\frac{15}{32\sqrt{2}}\) | \(\frac{154}{32\sqrt{2}}\) | \(-\frac{3^3}{8^2}\) | 3 | \(\frac{6}{5\sqrt{15}}\) | \(\frac{66}{5\sqrt{15}}\) | \(\frac{4}{5^2}\) |
| 5 | \(\frac{25}{32\sqrt{6}}\) | \(\frac{342}{32\sqrt{6}}\) | \(-\frac{1}{8^2}\) | 4 | \(\frac{20}{11\sqrt{33}}\) | \(\frac{252}{11\sqrt{33}}\) | \(\frac{23}{11^2}\) |
| 7 | \(\frac{279}{160\sqrt{30}}\) | \(\frac{4554}{160\sqrt{30}}\) | \(-\frac{9}{40^2}\) | 7 | \(\frac{144\sqrt{3}}{85\sqrt{85}}\) | \(\frac{2394\sqrt{3}}{85\sqrt{85}}\) | \(\frac{43}{85^2}\) |
| 11 | \(\frac{526\sqrt{15}}{80^2}\) | \(\frac{10836\sqrt{15}}{80^2}\) | \(-\frac{1}{80^2}\) | 4 | \(\frac{1}{80^2}\) |
| 17 | \(\frac{10177\sqrt{330}}{3440^2}\) | \(\frac{261702\sqrt{330}}{3440^2}\) | \(-\frac{1}{440^3}\) | 4 | \(\frac{1}{440^3}\) |
| 41 | \(\frac{27182818\sqrt{10005}}{3\cdot33360^2}\) | \(\frac{109020268\sqrt{10005}}{3\cdot33360^2}\) | \(-\frac{1}{33360^2}\) | 4 | \(\frac{1}{33360^2}\) |
Table 2  Rational Ramanujan-type series for $1/\pi$ ($\ell = 2$)

| $d$ | $a$ | $b$ | $z < 0$ | $d$ | $a$ | $b$ | $z > 0$ |
|-----|-----|-----|---------|-----|-----|-----|---------|
| 3   | $\frac{3}{8}$ | $\frac{20}{8}$ | $-\frac{1}{4}$ | 2   | $\frac{2}{9}$ | $\frac{14}{9}$ | $\frac{32}{87}$ |
| 4   | $\frac{8}{9\sqrt{7}}$ | $\frac{65}{9\sqrt{7}}$ | $\frac{1}{16}$ | 3   | $\frac{1}{2\sqrt{3}}$ | $\frac{8}{2\sqrt{3}}$ | $\frac{1}{9}$ |
| 5   | $\frac{3\sqrt{2}}{16}$ | $\frac{28\sqrt{2}}{16}$ | $-\frac{1}{48}$ | 5   | $\frac{4}{9\sqrt{2}}$ | $\frac{40}{9\sqrt{2}}$ | $\frac{1}{81}$ |
| 7   | $\frac{23}{72}$ | $\frac{260}{72}$ | $-\frac{1}{18^2}$ | 9   | $\frac{27}{49\sqrt{3}}$ | $\frac{360}{49\sqrt{3}}$ | $\frac{1}{77}$ |
| 13  | $\frac{41\sqrt{5}}{288}$ | $\frac{644\sqrt{5}}{288}$ | $-\frac{1}{572^2}$ | 11  | $\frac{19}{18\sqrt{11}}$ | $\frac{280}{18\sqrt{11}}$ | $\frac{1}{99^2}$ |
| 19  | $\frac{1123}{3528}$ | $\frac{21460}{3528}$ | $-\frac{1}{882^2}$ | 29  | $\frac{4412}{9801\sqrt{2}}$ | $\frac{105560}{9801\sqrt{2}}$ | $\frac{1}{99^4}$ |

Table 3  Rational Ramanujan-type series for $1/\pi$ ($\ell = 3$)

| $d$ | $a$ | $b$ | $z < 0$ | $d$ | $a$ | $b$ | $z > 0$ |
|-----|-----|-----|---------|-----|-----|-----|---------|
| 3   | $\frac{\sqrt{3}}{4}$ | $\frac{\sqrt{3}}{4}$ | $-\frac{9}{16}$ | 2   | $\frac{1}{3\sqrt{3}}$ | $\frac{6}{3\sqrt{3}}$ | $\frac{1}{2}$ |
| 5   | $\frac{7}{12\sqrt{3}}$ | $\frac{51}{12\sqrt{3}}$ | $-\frac{1}{16}$ | 4   | $\frac{8}{27}$ | $\frac{60}{27}$ | $\frac{2}{27}$ |
| 7   | $\frac{7\sqrt{2}}{12}$ | $\frac{9\sqrt{2}}{12}$ | $-\frac{1}{80}$ | 5   | $\frac{8}{15\sqrt{3}}$ | $\frac{66}{15\sqrt{3}}$ | $\frac{4}{125}$ |
| 11  | $\frac{106}{192\sqrt{3}}$ | $\frac{1230}{192\sqrt{3}}$ | $-\frac{1}{216}$ | 11  | $\frac{307}{307}$ | $\frac{307}{307}$ | $\frac{1}{3024}$ |
| 13  | $\frac{26\sqrt{7}}{216}$ | $\frac{330\sqrt{7}}{216}$ | $-\frac{1}{3024}$ | 23  | $\frac{827}{1500\sqrt{3}}$ | $\frac{14151}{1500\sqrt{3}}$ | $\frac{1}{500^2}$ |

Table 4  Rational Ramanujan-type series for $1/\pi$ ($\ell = 4$)

| $d$ | $a$ | $b$ | $z < 0$ | $d$ | $a$ | $b$ | $z > 0$ |
|-----|-----|-----|---------|-----|-----|-----|---------|
| 3   | $\frac{1}{2}$ | $2$ | $-1$ | 3   | $\frac{1}{4}$ | $\frac{6}{4}$ | $\frac{1}{4}$ |
| 5   | $\frac{1}{2\sqrt{2}}$ | $\frac{6}{2\sqrt{2}}$ | $-\frac{1}{8}$ | 7   | $\frac{5}{16}$ | $\frac{42}{16}$ | $\frac{1}{64}$ |

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