Acoustic response in a one-dimensional layered pseudo-Hermitian metamaterial containing defects

D. Psiachos\(^\dagger\) and M. M. Sigalas\(^\ddagger\)

\(^\dagger\)Department of Materials Science, University of Patras, 26504, Rio, Greece

Abstract

Using transfer-matrix methods, we investigate the response of a multilayered metamaterial system containing defects to an incident acoustic plane wave at normal or oblique incidence. The transmission response is composed of pass-bands with oscillatory behaviour, separated by band gaps and covers a wide frequency range. The presence of gain and loss in the layers leads to the emergence of symmetry breaking and re-entrant phases. In the general case, a system containing defects will display a more general property, pseudo-Hermiticity (PH), of which \(\mathcal{PT}\) systems are a subset. In the PH-symmetric phase, unidirectional responses of the reflection, accomplished by reversing the parity \(\mathcal{P}\), can be found but the response sometimes deviates from the predictions of simple scattering theory which call for a pseudo-unitarity relation relating the transmission and the two directions of reflections to hold. The converse of reversing the parity, reversing the time operator \(\mathcal{T}\) in a spatially-asymmetric system within the PH-symmetric regime can lead to different transmissions: a pass-band versus a stop-band. As regions of stable PH-symmetric pass-band transmission oscillations occur over a wide spectral range, there is a large flexibility in system parameters such as layer thicknesses, for leading to the desired unidirectional traits. In addition, we find that while defects in general lead to a near or complete loss of PH symmetry at all frequencies, they can be exploited to produce highly-sensitive responses, making such systems good candidates for sensor applications.

Keywords: acoustic wave propagation, metamaterials, PT-symmetry

\(^\dagger\)Electronic address: dpsiacos@gmail.com
\(^\ddagger\)Electronic address: sigalas@upatras.gr
I. INTRODUCTION

Elastic wave propagation in layered media has parallels with wave propagation in a variety of physical systems: e.g. electrons in atomic lattices, electromagnetic waves in photonic lattices. The latter in particular has many parallels with elastic-wave propagation in phononic lattices, as the general form of the field equations describing the wave propagation in the respective media and also much of the formalism and methods of solution are common to both types of systems, as are some of the proposed applications.

Phononic lattices are particularly interesting as they may exhibit partial or total band gaps in some energy regions, enabling the transmission of one type of polarization or none at all. Unlike atomic crystals, homogeneous phononic materials do not exhibit total band gaps, necessitating the construction of complex composites in order to access regimes where no transmission occurs. The tunability of the characteristics of these highly-complex composite phononic materials results in behaviour completely unlike any material heretofore known, leading to these materials being termed “metamaterials”. Elastic metamaterials exhibiting for example a negative effective mass density and bulk modulus have been constructed and some proposed applications include systems such as superlenses[1,2] and the redirection of sound: acoustic cloaking[3] or illusions[4]. Recently, the term ‘metamaterials’ has come to encompass a more general class of systems composed of materials, often artificial, which combined lead to phenomena not found in the constituents themselves nor in any material found in nature[5,6]. The correspondence with photonic systems is closer in the case of acoustic-wave propagation - in for example a multilayer immersed in fluid - thereby enabling the equations of transformation acoustics to retain their form under coordinate transformation[7], a property particularly useful for the design of acoustic cloaks.

Metamaterials may also be made “active” leading to the production of gain or allowing absorption within them. However, because such materials have large amounts of losses owing to various types of material imperfections, combining gain (G) and loss (L) materials constitutes a approach to achieving reduced losses[8]. Parity-time (PT) or the more general category of pseudo-Hermitian (PH)-symmetric systems, despite being non-Hermitian, can have real eigenvalues, as is necessary for propagation, as well as bound
states. The $\mathcal{P}$ operator refers to reversing spatial coordinates, while the $\mathcal{T}$ operator performs complex conjugation. Thus broken $\mathcal{P}$ in a system of multilayers with G/L can be induced by defects, while broken $\mathcal{T}$ can result from a reversal of the gain/loss elements. However, the combination of $\mathcal{P}$ and $\mathcal{T}$ may still be retained even though one or both are broken separately. In a $\mathcal{PT}$ symmetric system the transfer matrix is equal to its complex inverse$^{10}$ while the more general case, of a PH system, is characterised by the transfer matrix not obeying the above property. For a general PH system, the Hamiltonian obeys the property $H^\dagger = \eta H \eta^{-1}$ where $\eta$ is a Hermitian linear and invertible operator$^{9,11}$, equal to one when $H$ is equal to its Hermitian conjugate $H^\dagger$ while in $\mathcal{PT}$ symmetry, we have $(\mathcal{PT})H(\mathcal{PT})^{-1} = H$. In the more general, PH symmetry, as in our system which displays imperfect periodicity and/or localized defects, the operator $\eta$ is an arbitrary quantity. By modifying the system parameters, the system may migrate from the $\mathcal{PT}$ or PH - symmetric phase into the broken phase and vice versa. The propagation in PH systems differs from that in conventional ones in that it displays unidirectional properties, such as invisibility$^{10}$. In addition, the total energy is not conserved, oscillating about a mean value, owing to the non-orthogonality of the eigenmodes. A consequence of the latter is an asymmetry in the propagation, known as non-reciprocal propagation$^{12}$. In the broken phase, the propagation is unstable - greatly amplified or attenuated - and at these values of the wavevector the energy dispersion relation takes on imaginary components. PH systems not displaying $\mathcal{PT}$ symmetry have recently started to be studied in depth because of the greater flexibility$^{13}$ afforded by removing the restrictions around the spatial arrangement of the components and the presence of defects.

A recent realization of acoustic metamaterials uses a dual microphone setup$^{14}$, one absorbing, the other with gain, to achieve unidirectional propagation of sound. Unidirectional wave propagation in photonic gratings has been demonstrated at the $\mathcal{PT}$ exceptional point$^{15,16}$, the transition point between the $\mathcal{PT}$-symmetric and broken regime. Further, a $\mathcal{PT}$ system which on its own displays lasing, can become a coherent potential absorber$^{17}$ when a second signal is injected in the reverse side and direction from the first signal, but coherent with it. A system which behaves as a coherent potential absorber in the $\mathcal{PT}$—symmetric phase, has also been realized$^{18}$. Some related unidirectional applications in acoustics would include for example the suppression of echoes. Christensen et
propose the use of electrically-biased piezoelectric semiconductors in order to generate amplification or attenuation, demonstrating theoretically or through simulations that exceptional points and unidirectional reflection-suppression may be achieved conveniently by this means. \( \mathcal{PT} \)-symmetric metamaterials have also been proposed for unidirectional acoustic cloaking applications \(^{20} \) or for the construction of superlenses, capable of overcoming the diffraction limit with no losses \(^{20} \).

In this study, we investigate the propagation of acoustic waves through a one-dimensional multilayer system including gain/loss (G/L) in order to establish the criteria for achieving unidirectional and/or orientation-dependent reflection and transmission, for normal or oblique incidence. Previous theoretical investigations, albeit relying on simple scattering models, based on two-port networks \(^{17,20,22–24} \), predict unidirectional reflection at the exceptional point \(^{19,25} \). Most previous studies focus on balanced gain and loss, and find that a gap is necessary between the gain and loss components firstly in order for \( \mathcal{PT} \)-symmetry breaking and reentrant phases to be present and secondly that the thickness of the spacing should be tuned in order to maximize the degree of directional asymmetry \(^{24,25} \). By considering multilayers, we exploit the numerous bands and wide ranges of pass bands in order to examine unidirectional behaviour and responses to the presence of defects over multiple and wide-ranging of \( \mathcal{PT} \)-symmetric regions. The defects or symmetry-breaking features we study consist of an asymmetry in the reflection symmetry and in a modified thickness of some of the layers.

Many methods for determining the transmittance and reflectance properties of phononic systems exist \(^{26} \); from multiple-scattering methods, to the finite-difference-time-domain (FDTD) method, to transfer-matrix methods, etc. The transfer-matrix technique in particular has been applied to multilayered systems in order to study elastic-wave propagation at normal and oblique incidence, where transverse modes are activated, in systems with defects or absorption \(^{27} \).

II. THEORETICAL METHODS

While there are many parallels with optical waves, the motion of elastic waves is more complicated - requiring two equations of motion to be satisfied for the time-harmonic
field - even at normal incidence, where both longitudinal and transverse modes within the elastic media are excited despite being independent.

In the following, we describe our implementation of the elastic-wave propagation. We consider a system of \( n - 1 \) layers with normal the \( \hat{z} \) direction, extending along the negative \( \hat{z} \) direction as in Fig. 1. Each layer is described by homogeneous mass density \( \rho \) and elastic properties: shear modulus \( \mu \) and Lamé constant \( \lambda \). The longitudinal and transverse wave speeds are thus \( c = \sqrt{(\lambda + 2\mu)/\rho} \) and \( b = \sqrt{\mu/\rho} \) respectively. We further assume isotropic elasticity where the Lamé constant \( \lambda = \frac{2\mu\nu}{1-2\nu} \) and the shear modulus is \( \mu = \frac{E}{2(1+\nu)} \) are expressed in terms of the Young’s modulus \( E \) and Poisson ratio \( \nu \). A longitudinally-polarized plane wave of frequency \( \omega \) is incident on the multilayer system from a liquid ambient medium either normally or at an angle \( \theta \) in the \( xz \) plane and exits again in liquid. No other type of elastic wave can propagate through liquids. For such a polarization, the particle displacement is in the \( xz \) plane and there may be both shear and longitudinal modes present but only the latter in the liquid which terminates both ends of the layer. The displacement field may be split into longitudinal \( \phi \) and transverse \( \psi \) potentials

\[
\vec{u} = \vec{\nabla} \phi + \vec{\nabla} \times \vec{\psi}
\]

and we set \( \vec{\psi} = \psi \hat{y} \). The wave equations for the potentials, assuming time-harmonic plane-waves are

\[
\nabla^2 \phi + k^2 \phi = 0, \ k \equiv \omega/c,
\]

\[
\nabla^2 \psi + \kappa^2 \psi = 0, \ \kappa \equiv \omega/b,
\]

and have the solutions

\[
\phi_j = \phi' e^{i\alpha_j z} + \phi'' e^{-i\alpha_j z}, \ \alpha_j = (k_j^2 - \xi^2)^{1/2}, \ \xi = k_j \sin \theta
\]

\[
\psi_j = \psi' e^{i\beta_j z} + \psi'' e^{-i\beta_j z}, \ \beta_j = (\kappa_j^2 - \chi^2)^{1/2}, \ \chi = \kappa_j \sin \theta
\]

for each layer \( j \), including the terminating ambient media \( j = 1 \) and \( j = n + 1 \), where the primed quantities are amplitudes. For example, for a longitudinal wave incident at the liquid half-space \( n + 1 \), \( r = \phi''_{n+1} \) and \( t = \phi'_{n+1} \) are the expressions for the longitudinal-mode reflection and transmission coefficients in an outgoing liquid layer \( j = 1 \).
Through transformations from the \{\phi', \phi'', \psi', \psi''\} basis, a transfer matrix for the passage of a wave through one, and then by repeated application, through the whole system of \(n - 1\) layers may be constructed in terms of the displacements Eq.1 and the stresses

\[
Z_x = \mu \left( \partial u_x / \partial z + \partial u_z / \partial x \right) \quad \text{and} \quad Z_z = \lambda \left( \partial u_x / \partial x + \partial u_z / \partial z \right) + 2\mu \partial u_z / \partial z
\]

as

\[
\begin{pmatrix}
  u_x^{(n)} \\
  u_z^{(n)} \\
  Z_x^{(n)} \\
  Z_z^{(n)}
\end{pmatrix} = A
\begin{pmatrix}
  u_x^{(1)} \\
  u_z^{(1)} \\
  Z_x^{(1)} \\
  Z_z^{(1)}
\end{pmatrix}
\]

where \(A\) is the transfer matrix through the entire multilayer (see ch.1, sec.8 in Ref.28). In the system depicted in Fig. 1 the transfer matrix is not equal to its complex inverse, and this is why we will refer to it as “PH symmetric” rather than “\(\mathcal{PT}\) symmetric”. Otherwise, the transfer matrix of a PH-symmetric system displays the same key property as in a “\(\mathcal{PT}\) symmetric” system - the system displays regions of stable propagation (symmetric phase) when the eigenvalues of the transfer matrix are unimodular, separated by regions of unstable propagation (broken phase) when the eigenvalues deviate from this condition.

![Diagram of a metamaterial system](image.png)

**FIG. 1:** Metamaterial system under study: impinging oblique acoustic wave onto a multilayer stack composed of alternating gain (G) and loss (L) layers, separated by passive material. The materials are discussed in Sec. III: WC (tungsten carbide) and Ep (epoxy thermoset).

The boundary conditions applied are the continuity of the displacements \(u_x\) and \(u_z\), and that of the stresses \(Z_x, Z_z\) across the \((n, n+1)\) boundary. In the case of a liquid
ambient medium, only the latter two are required, with $Z_x$ additionally being zero as shear stresses are not supported in the liquid. When this is done, we arrive at expressions for the reflection and transmission from the incident and outgoing sides respectively of the multilayer system.

Thus, the analytical approach to calculating the response of a multilayered stack embedded in fluid to an incident plane acoustic wave may be summarized as follows: Beginning from the potentials satisfying the two-dimensional wave equations Eq. 2, we construct the state variables - displacement (Eq. 1) and stress (Eq. 4). The coefficients of the state variables then comprise the transfer matrix in a given layer. By repeated multiplication, the transfer matrix through the entire multilayer system is constructed. Then, subject to the boundary conditions at the edges of the stack according to which the transverse stresses are zero at the fluid-stack interfaces, and the other components are continuous, the transmission and reflection coefficients may be determined by inverting the system of equations for the stress and displacement to obtain the amplitudes in Eq. 3 which comprise them, as noted above. The formalism is well-documented, for example in Refs. 28, 29. The formalism is fully general so as to work also with solid ambient media and to transverse incident waves. In the general case, there are interconversions between transverse and longitudinal reflections and transmissions which will be interesting to study.

III. RESULTS

The response of multilayer system composed of alternating layers with the elastic properties of tungsten carbide (WC) and epoxy thermoset (Ep) respectively to an impinging acoustic wave was examined for various incidence angles. The schematic is as in Fig. 1. These particular elastic properties (rather than the materials themselves, which may well end up being artificial in nature in some future implementation) were chosen in order to investigate a typical phononic system, namely with drastically different elastic properties in order to achieve the requisite opening up of spectral gaps. The parameters of these materials are given in Table 1. The layer thicknesses have been all set to the same value unless otherwise noted. The frequencies at which the responses occur scale inversely with the value of the thickness $l$ used (viz. units on graphs’ axes). When including
| Material          | $E$ (GPa) | Im($E$) | $\nu$ | $\rho$ (Mg/m$^3$) | layer thickness |
|-------------------|-----------|---------|-------|-------------------|----------------|
| WC (tungsten carbide) | 550       | ±30     | 0.21  | 15.5              | $l$            |
| Ep (epoxy thermoset)      | 3.5       | 0       | 0.25  | 1.2               | $l$            |

TABLE I: Parameters of the WC/Ep multilayer system. $E$ (Im$E$) is the real (imaginary - when activated) part of the Young’s modulus, $\nu$ the Poisson Ratio, $\rho$ the mass density, $l$ the layer thickness (uniform in this study unless otherwise noted). The parameters of the fluid (water) at both ends are $c = 1.5$ m/s, $\rho = 1$ Mg/m$^3$.

G/L, through the addition of an imaginary component of positive (G) or negative (L) sign to the Young’s modulus, we work in the region of balanced gain/loss, by imposing alternating G/L on the WC layers. While loss can be imparted using dissipation, gain requires the use of active elements. The amount of G/L imparted here is in line with that achievable for the tunable effective bulk modulus in a recent implementation of a metamaterial.$^{30}$

For small incidence angles we find regions of $\mathcal{PT}$ or more generally PH-symmetry separated by broken symmetry. The characterization of $\mathcal{PT}$ or PH-symmetry was done in two ways. In Fig. 2 we show the dispersion relation $k(\omega)$, albeit with inverted axes for simpler interpretation, for $\theta = 0$, $\theta = \pi/16$, $\theta = \pi/8$, and $\theta = \pi/3$ for systems including G/L and in the first two cases we find regions of purely real wavevectors $k$, which is essential for the propagation of these modes with no amplification or attenuation. The dispersion relation was calculated using the Floquet theorem, in an approach modified from Ref. $^{31}$, viz.

$$
\begin{pmatrix}
  u_x^{(n)} \\
  u_z^{(n)} \\
  Z_z^{(n)} \\
  Z_x^{(n)} 
\end{pmatrix}
= \exp{(i k L)}
\begin{pmatrix}
  u_x^{(n-1)} \\
  u_z^{(n-1)} \\
  Z_z^{(n-1)} \\
  Z_x^{(n-1)} 
\end{pmatrix}
$$

for $L$ the total thickness of one multilayer period. By using the analog of the transfer-matrix $A$ but for one period, called $a$, the solutions for the wavevector $k$ are found to satisfy the equation

$$
\det[a - I \exp{(i k L)}] = 0
$$
where $I$ is the identity matrix.

In the same ranges that all four wavevector solutions were purely real, the eigenvalues of the transfer matrix were all simultaneously unimodular, ensuring stable propagation up to infinite stack length. Without G/L the band structures remained to a large extent exactly the same, although for some higher $\omega$ values, deviations can be seen, as in Fig. 2b. When G/L is absent however, the systems display none of the characteristics of PH systems, only those of conventional materials.

In Fig. 2a ($\theta = 0$), we show the longitudinal (L) and transverse (T) modes separately. The imaginary part of the transverse mode is non-zero beginning at $\omega = 0.593$. However, as seen in the calculation of the transmission response, albeit for a finite system, in Fig. 3a-b, oscillations in the pass-band still occur in the region up until $\omega = 1.027$, the point at which the imaginary part of the longitudinal mode becomes non-zero since the transverse mode is not excited. For $\theta = \pi/16$, there is full PH symmetry up until $\omega = 0.948$ (Fig. 2b), reduced from the ‘effective’ 1.027 in the $\theta = 0$ case due to the mixing of the L and T modes. In Fig. 3c-d we see that a greater number of bands appear for $\theta = \pi/16$ compared with the $\theta = 0$ case. Comparing with the cases of no gain/loss, we find that the locations of the bands are mostly unaltered in Fig. 3. However, in some cases, some of the pass-bands disappear upon going from the passive to the G/L system. In Fig. 3, there are bands e.g. at $19.29 < \omega < 19.46$ and $20.02 < \omega < 20.31$, amongst others, which are not present when G/L are present (Fig. 3l). Referring to the band structure in Fig. 2b, where this frequency range is displayed in the second and third panels, we see that in these ranges the real part of the wavevector consists of two nearly coincident bands with small curvature in each of these ranges only in the system with no G/L. In these figures, one out of the four complex wavevector solutions is highlighted as an example. In the case of no G/L, the real part of each wavevector does not overlap with its imaginary part and the mode propagates. In contrast, with G/L included, such a solution does not exist and the mode is not propagating. In the cases shown in Fig. 3b,d which are not classified as PH-symmetric, each wavevector solution is complex, but propagates because the real and imaginary parts of the same solution do not overlap as explained above.

In Fig. 3, regions classified as PH-symmetric often display transmission greater than one. However, the material displacements and stresses ($u_x, u_z, Z_z, Z_x$) remain finite for
FIG. 2: Dispersion relation $k(\omega)$ (albeit with inverted axes) of a WC$^-$/Ep/WC$^+$/Ep infinite system for different incidence angles. Real (imaginary) parts of $k$ are displayed with thick (thin) symbols. In (a), we use a different colour scheme in order to highlight the longitudinal (L) and transverse (T) modes. (b) is comprised of three panels. In (b) and (d) we superimpose as pink (real) and turquoise (imaginary) colours the components of one of the four solutions for $k$. The lattice constant $a$ is the thickness of the tetralayer.

arbitrary multilayer repetition, in contrast with regions not classified as PH-symmetric, where these parameters blow up. For large incidence angles, such as in Fig. 2c for $\theta = \pi/8$, and Fig. 2d for $\theta = \pi/3$ we do not find any regions of real wavevectors; even at the low $\omega$ region, the imaginary parts have moved off the $k = 0$ axis and any propagating solutions as a result of nonoverlapping real and imaginary parts of the solutions for $k(\omega)$ (see Fig. 2d and 3e) are not PH-symmetric and the material parameters blow up.
FIG. 3: Transmission amplitude $T$ for 15 tetralayers $\text{WC}^-/\text{Ep}/\text{WC}^+/\text{Ep}$ for (a)-(b), $\theta = 0$ with and without gain/loss (G/L), (c)-(d), $\theta = \pi/16$. In (e) we show the system for $\theta = \pi/3$ for G/L included. Regions of PH-symmetry corresponding to the infinite system are delineated by the pink bars and/or the triangles for when the region is very narrow (the entire band is PH symmetric).

Signature features of systems with balanced gain and loss - whether in the broken or the PH-symmetric phase - include reflection ($R = |r|^2$) and transmission ($T = |t|^2$) amplitudes achieving values greater than one and the reflection being direction-dependent: $R$ from the right side bring unequal to that from the left side. We did not find a directionality occurring for the $T$, consistent with what the theory from models of $\mathcal{PT}$-symmetric scattering, based on two-port networks, predicts. However, the same theory predicts that the product of the $R$ is zero when $T = 1$, which we find to usually, but not always be the case in a PH system. More generally, the above models predict that $T$
and the forwards and backwards reflections $R_F, R_B$ are related by the pseudo-unitarity relation $|T - 1| = \sqrt{R_F R_B}$. However, the two-port scattering formalism and the aforementioned pseudo-unitary relation hold only for idealized $\mathcal{PT}$ systems, not for general PH systems which arise through the inclusion of defects such as in Ref. 32. This is entirely due to the fact that the forward and backward transfer matrices do not obey the required relations in such systems. In Fig. 4 we show how the pseudo-unitarity relation is broken near strong T or R resonances, something which holds in the PH-symmetric regime (shown) as well as in the broken regime, as well as for different incidence angles and other frequency ranges.

![Figure 4: Reflection and transmission amplitudes for 15 tetralayers WC$^-$/Ep/WC$^+$/Ep for $\theta = 0$, whether approached from the gain side for WC, i.e. Ep/WC$^+$... (backwards) or loss i.e. WC$^-$/Ep... (forwards) direction. In this example the system is in the PH-symmetric regime. The thick blue curve is the expression $T \pm \sqrt{R_F R_B}$ according to whether $T < 1$ (plus) or $T > 1$ (minus). The figure on the bottom is a close-up of the top figure.](image)

Additionally we find that the inclusion of defects in general reduces the range of or completely destroys the PH symmetry. This is evidenced by the eigenvalues of the transfer
matrix no longer being unimodular at all, or being so in a reduced range. Defects we have studied include having a layer with modified thickness. We focus here mainly on systems in which we calculate the response with respect to a reversal of spatial coordinates i.e. $\mathcal{P}$. However, in some special systems not containing defects per se but having pre-existing $\mathcal{P}$ asymmetry, such as an even number of layers WC/Ep, reversing the $\mathcal{T}$ (index of refraction, or G/L ordering here), has the effect of breaking the symmetry in not only $R$ but $T$ as well. We indeed find a dramatic effect of orientation: in which one G/L ordering is close to or greater than one while the other is zero for both $R$ and $T$. Such a system might be implemented with piezo-electric induced imaginary components - demonstrated to be able to tune the G/L character\textsuperscript{19}. We further find that dramatic differences in the two responses is only possible for $T$ when $\theta \neq 0$; at normal incidence the two $T$ differ but only very slightly. Only the $R$ have dramatic differences at normal incidence. In Fig. 5 we show the $R$ and $T$ for $\theta = \pi/16$ where at the point indicated, one ordering has a very large $R$ (19.26) and $T=1$ and the other has low $R$ and $T$ (0.06 and 0.1 respectively).

FIG. 5: Reflection and transmission amplitudes within a PH-symmetric region for 15 tetralayers WC/Ep/WC/Ep... for $\theta = \pi/16$ whether the ordering of the WC begins with gain or with loss. The two different orientations are as depicted as an inset. The vertical line corresponds to one of the $T$ being equal to 1. The green dotted line is a guide to the eye.

We find, in general, that this interesting behaviour, where the response is either “uni-directional” or specific to the G/L ordering, doesn’t occur just at a single point as in
prior work\textsuperscript{19,20} but in many places, leading to a flexible manipulation of resonances. Other resonances are not all as sharp as the one in Fig. 5 and it would be also interesting to investigate the resonances produced by different choices of material parameters. The spacing of the Ep layer can be tuned so that the desired unidirectional behaviour occurs at other frequencies - in contrast with a single gain / loss component, the multilayer system can access many such candidate frequencies owing to the presence of multiple pass bands and the tuning of the spacing is not critically important.

![Defect Bilayer Relative Thicknesses: (WC,Ep) (in terms of l)](image)

**FIG. 6:** Transmission amplitudes $T$ with defects. Region shown is near the highest resonance (1.23 at $\omega=5.7267$, marked with arrow) of the undefected system (black dotted line). The system consists of 25 tetralayers WC+/Ep/WC-/Ep... and $\theta = \pi/8$. The defect consists of different relative thicknesses of a (WC,Ep) bilayer near the central region (49th-50th layer), compared with the undefected value of (WC,Ep)=(1,1) - units in terms of $l$.

Defects of thickness of the components were also examined. Out of the entire 25-tetralayer (WC+/Ep/WC-/Ep...) system one WC-Ep bilayer was chosen to have variable thickness, either half or double the thickness of the rest of the layers. Although the system is in the broken phase owing to its sensitivity to defects, giant increases in transmission resonances, sometimes of several orders of magnitude greater than the uni-
form system, were observed. In Fig. 6 we show that for an incidence angle of $\pi/8$, a (WC,Ep)=(2,1) defect bilayer thickness - henceforth all thickness units are in terms of $l$ - can lead to an enormous resonance in the $\omega=5.65-5.8$ band of the undefected system. This frequency range was studied because it is transmissive at all incidence angles studied (see Fig. 3 but also for $\theta = \pi/8$ and $\pi/4$). Specifically, a doubling of the WC thickness in one of the central bilayers led to an enormous resonance in the transmission of 115417, versus 1.23 the highest point in the undefected system in this range. Similar results were found for other incidence angles, including $\theta = 0$. However, we found that there is a very large variation with position - the largest resonances for when the thickness defect was located in neighbouring bilayers varied by orders of magnitude and sometimes corresponded to different values of the defect thicknesses. There was a variation amongst angles as well: for $\theta = 0$ the preferred defect thickness for achieving the highest resonance was (WC,Ep)=(2,2) while for $\pi/16$ and $\pi/8$ it was fairly inconsistent but often (0.5,2) was preferred for $\pi/16$ and (0.5,1) for $\pi/8$. For $\theta = \pi/4$ however, the preferred thickness was most usually (0.5,1). This means that this particular defect-layer thickness responds to transverse modes better than the (2,2) combination. For all angles and generally for all defect locations, the thickness of 0.5 for the Ep defect layer was found to lead to the smallest resonances.

FIG. 7: Resonances of the transmission amplitude $T$ for one percent changes in the defect-layer thicknesses, about the optimal thickness (WC,Ep)=(2,2) for the 55th-56th layers for a system of 25 tetralayers WC+/Ep/WC-/Ep... for $\theta = 0$. The three curves with the same Ep thickness lie on top of each other. Shown are only the highest resonances for each case.

In recent work, Zhao et al\textsuperscript{24} have shown that small defect-layer thickness variations can dramatically alter the left/right reflectance ratio. We have examined how sensitive
the giant transmission resonances are to small (one percent) variations in the thickness.
In Fig. 7 we examine a one percent variation in the defect thickness which led to the
largest resonance for the 55th-56th (WC-Ep) layers for $\theta = 0$, i.e. $(WC,Ep)=(2,2)$. As
shown in the figure, the response is highly sensitive to changes in thickness. In addition,
for changes in thickness of Ep, the resonance frequency displays a shift, while for changes
in WC thickness with fixed Ep thickness it does not. These giant resonances in the
transmission and general sensitivity to defects can be exploited to fabricate ultra-sensitive
biochemical sensors [33,34]. Another interesting application of the present system would be
for ultrasensitive touch-screens, since very small variations in thickness would result in
extreme changes in transmission or reflection.

IV. CONCLUSIONS

We calculate the reflection and transmission response to obliquely-incident acoustic
waves in a pseudo-Hermitian (PH) system of multilayered gain/loss components sepa-
rated by a passive spacer by solving the elastodynamic equations in combination with
the transfer-matrix technique. We find transmission oscillations occurring in bands, cov-
ering a wide frequency range and in many cases the propagation is stable (PH-symmetric
phase) while in other cases it is unstable (broken phase) leading to a blowing-up of the
material parameters. We also note some deviations from the simple scattering models,
namely that the pseudo-unitary condition relating the transmission to the reflections in
the two spatial propagation directions does not exactly hold, particularly near strong
transmission or reflection resonances. On the other hand, for fixed propagation direc-
tion but reversed time, we find a breaking of the reflection and transmission symmetry,
sometimes simultaneously and at one ordering severely suppressed. As our system is
multilayered in nature, it can access a wide range of frequency bands, thus calibrating
the thickness of the passive spacer is not crucial for its operation, neither is its operation
at a single exceptional point. The presence of defects as well as their location within
the system was found to have a profound effect on the transmission response: changing
the thickness of one passive layer shifts transmission resonances to different frequencies,
while even very small changes in thickness were found to produce great sensitivity in the
responses.

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