The Magnetic Mechanism for Hotspot Reversals in Hot Jupiter Atmospheres

A. W. Hindle1 ©, P. J. Bushby1 ©, and T. M. Rogers1,2 ©

1 School of Mathematics, Statistics and Physics, Newcastle University, Newcastle upon Tyne, NE1 7RU, UK; alex.hindle@newcastle.ac.uk
2 Planetary Science Institute, Tucson, AZ 85721, USA

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Abstract

Magnetically driven hotspot variations (which are tied to atmospheric wind variations) in hot Jupiters are studied using nonlinear numerical simulations of a shallow-water magnetohydrodynamic (SWMHD) system and a linear analysis of equatorial SWMHD waves. In hydrodynamic models, mid-to-high-latitude geostrophic circulations are known to cause a net west-to-east equatorial thermal energy transfer, which drives hotspot offsets eastward. We find that a strong toroidal magnetic field can obstruct these energy transporting circulations. This results in winds aligning with the magnetic field and generates westward Lorentz force accelerations in hotspot regions, ultimately causing westward hotspot offsets. In the subsequent linear analysis we find that this reversal mechanism has an equatorial wave analogy in terms of the planetary-scale equatorial magneto-Rossby waves. We compare our findings to three-dimensional MHD simulations, both quantitatively and qualitatively, identifying the link between the mechanics of magnetically driven hotspot and wind reversals. We use the developed theory to identify physically motivated reversal criteria, which can be used to place constraints on the magnetic fields of ultra-hot Jupiters with observed westward hotspots.

Unified Astronomy Thesaurus concepts: Magnetohydrodynamics (1964); Astrophysical fluid dynamics (101); Exoplanet atmospheres (487); Exoplanet dynamics (490); Hot Jupiters (753)

1. Introduction

In recent years the field of exoplanetary research has greatly developed its understanding of exoplanet characterization both observationally and theoretically. The field has now reached the point where light curves, infrared photometry, and spectra from spaced-based telescopes can be used to test, inform, and update our understanding of the atmospheric dynamics of these closely orbiting gas giants.

Generally, observational measurements of hot Jupiters (e.g., Harrington et al. 2006; Cowan et al. 2007; Knutson et al. 2007, 2009; Charbonneau et al. 2008; Swain et al. 2009; Crossfield et al. 2010; Wong et al. 2016) find that these planets have equatorial temperature maxima (hotspots) located eastward of their substellar points. This is consistent with both hydrodynamic simulations (e.g., Showman & Guillot 2002; Shell & Held 2004; Cooper & Showman 2005, 2006; Langton & Laughlin 2007; Dobbs-Dixon & Lin 2008; Menou & Rauscher 2009; Dobbs-Dixon et al. 2010; Rauscher & Menou 2010; Perna et al. 2010a; Heng et al. 2011; Perez-Becker & Showman 2013) and hydrodynamic theory (Showman & Polvani 2011; Debras et al. 2020) of synchronously rotating hot Jupiters, which predict that such hotspots are driven eastward by the interaction between mid-to-high-latitude geostrophic circulations and equatorial jets. This fundamental behavior of the hydrodynamic system can also be described in terms of interactions between the system’s dominant equatorial waves and its mean equatorial flows (Showman & Polvani 2011).

However, recent observations suggest that eastward hotspots may not be found ubiquitously, particularly on the hottest hot Jupiters (ultra-hot Jupiters). Continuous optical Kepler measurements find east–west brightspot oscillations on the ultra-hot Jupiters HAT-P-7b (Armstrong et al. 2016) and Kepler-76b (Jackson et al. 2019), optical phase curve measurements from TESS find westward brightspot offsets on the ultra-hot Jupiter WASP-33b (von Essen et al. 2020), while thermal phase curve measurements from Spitzer find westward hotspots on the ultra-hot Jupiter WASP-12b (Bell et al. 2019) and the cooler hot Jupiter CoRoT-2b (Dang et al. 2018). There are three main explanations for these observations: reflections from cloud asymmetries confounding optical measurements (Demory et al. 2013; Lee et al. 2016; Parmentier et al. 2016; Roman & Rauscher 2017), asynchronous rotation (Rauscher & Kempston 2014), and magnetism (Rogers & Komacek 2014; Rogers 2017; Hindle et al. 2019). Ultra-hot Jupiters generally have near-zero eccentricities and are thought to be tidally locked, so they are expected to be synchronously rotating. They are also expected to have cloud-free daysides, where their atmospheres are too hot for condensates to form. Helling et al. (2019) recently ruled out cloud asymmetries as the explanation for westward brightspots on HAT-P-7b.

Using three-dimensional (3D) magnetohydrodynamic (MHD) studies, Rogers & Komacek (2014) predicted that magnetic fields could cause wind variations that drive east–west hotspot oscillations. Rogers (2017) then showed that the westward venturing hotspot displacements on the ultra-hot Jupiter HAT-P-7b can be well explained by the moderate deep-seated dipolar magnetic field strengths that are expected to be generated in the convective interior of such planets. In Hindle et al. (2019) we used a shallow-water MHD (SWMHD) model to show, first, that the magnetically driven hotspot reversal mechanism is a shallow phenomenon that is driven by the flow’s interaction with the planet’s atmospheric toroidal magnetic field and, second, that the SWMHD model also requires a moderate planetary dipolar magnetic field strength to drive westward hotspot displacements.
on HAT-P-7b but that an excessively strong deep-seated dipolar magnetic field is required to reverse flows within the cooler (and hence less thermally ionized) atmosphere of CoRoT-2b. The westward hotspot offsets on CoRoT-2b are therefore more plausibly explained by nonmagnetic phenomena. Interestingly, the hot Jupiters Kepler-76b, WASP-12b, and WASP-33b are of the ultra-hot type and so are more akin to HAT-P-7b than CoRoT-2b, making magnetically driven reversals plausible for these observations.

While 3D MHD simulations have proved crucial for identifying that magnetism can drive hotspot reversals in ultra-hot Jupiters, their dynamics is often too subtle and complex to glean physical understanding from. The aim of this study is to use a reduced physics model, alongside known features of 3D MHD simulations, to identify the mechanism by which magnetism can reverse hotspots in ultra-hot Jupiters.

In Sections 2 and 3, we use the numerical two-layer Cartesian SWMHD model of Hindle et al. (2019), with an equatorial beta-plane treatment of the Coriolis effect and different purely azimuthal equatorially antisymmetric initial magnetic field treatments, to study the westward transition of hotspots. In Section 4, we examine the link between magnetically driven wind reversals and equatorial wave dynamics. Finally, in Section 5 we collate our findings, compare them to results of 3D MHD simulations, and present a physically motivated hotspot reversal criterion for ultra-hot Jupiters.

2. Nonlinear Shallow-water Model

Three-dimensional models are fundamental to understanding the general features and flow behaviors of planetary atmospheres. However, with so many physical processes in play, it can be difficult to isolate the mechanisms responsible for driving a given flow pattern. In such instances, simplified models can be used to reduce the number of physical processes involved, exposing the underlying physics responsible for specific dynamical features. In this section, we present a detailed description of the reduced-gravity SWMHD model briefly described in Hindle et al. (2019), which we will use to explore the physics of wind reversals.

2.1. Governing Equations

The reduced-gravity SWMHD model is an adaptation of the SWMHD model of Gilman (2000) and is the MHD analog of its hydrodynamic namesake (e.g., Vallis 2006), which has been used extensively in hydrodynamic studies of hot Jupiters (Langton & Laughlin 2007; Showman & Polvani 2010; Perez-Becker & Showman 2013; Showman et al. 2013). It is made up of two constant-density fluid layers: a shallow active upper layer and an infinitely deep inactive lower layer, which has no pressure gradients, velocity fields, or induced magnetic fields in the horizontal direction (see Hindle et al. 2019, for a model schematic). Physically, the upper layer represents the meteorologically active upper atmosphere, and the lower layer represents the deep atmosphere and deep interior of a hot Jupiter. The interface between the two layers is a material surface over which no magnetic flux is permitted to cross. When the system’s length scales approach the shallow-water limit (i.e., if typical active-layer horizontal scales, \( L \), are much larger than the active layer’s thickness, \( H \)), the vertical momentum equation of the full 3D system approaches magnetohydrostatic balance. This limiting approximation may be used together with the model’s interface constraints to vertically integrate the 3D MHD equations over the vertical coordinate, \( z \), to yield a shallow-water model with vertically independent variables (see Gilman 2000; Hindle et al. 2019, for further discussion). Using Cartesian horizontal spatial coordinates, \( (x, y) \), the dynamical behavior of the active layer can be described by the following governing equations:

\[
\frac{du}{dt} + f(z \times u) = -g\nabla h + (B \cdot \nabla)B
\]

\[
- \frac{u}{\tau_{\text{drag}}} + R + D_u,
\]

\[
\frac{\partial h}{\partial t} + \nabla \cdot (hu) = \frac{h_{\text{eq}} - h}{\tau_{\text{rad}}} \equiv Q,
\]

\[
\frac{dA}{dt} = D_\eta,
\]

\[
hB = \nabla \times A\hat{z},
\]

where \( u(x, y, t) \equiv (u_x, v) \) is the horizontal active-layer fluid velocity, \( h(x, y, t) \) is the active-layer thickness that is used as the model’s temperature proxy (see below), \( B(x, y, t) \equiv (B_x, B_y) \) is the horizontal active-layer magnetic field (in velocity units), and \( A(x, y, t) \) is the magnetic flux function of the active layer. We comment that the magnetic flux function definition differs from its two-dimensional definition through the inclusion of \( h \) in Equation (4). This arises because the magnetic flux function describes the vertically integrated horizontal magnetic field over the whole fluid column, rather than simply the horizontal magnetic field at a specific vertical level. We use \( \nabla \equiv (\partial_x, \partial_y) \) to define the horizontal gradient operator, we use \( d/dt \equiv \partial/\partial t + u \cdot \nabla \) to define the Lagrangian time derivative operator, and \( \nabla \times A \hat{z} \equiv (\partial_y A_x - \partial_x A_y) \) is the horizontal curl of the scalar field \( A \) about the vertical coordinate.

Defining the system in terms of the magnetic flux function guarantees that the SWMHD divergence-free condition, \( \nabla \cdot (hB) = 0 \), remains satisfied throughout the domain at all times. This is the shallow-water analog of Gauss’s law of magnetism, which excludes magnetic monopoles. This shallow-water divergence-free condition is obtained by integrating the full 3D form of Gauss’s law over the vertical coordinate, while imposing zero magnetic flux constraints across our model’s layer interfaces. Using this formulation also highlights that in the absence of magnetic diffusion \( (D_\eta = 0) \), \( A \) is a materially conserved quantity (see Equation (3) with \( D_\eta = 0 \)).

For numerical stability, we apply the following explicit diffusion prescriptions (Gilbert et al. 2014, A. D. Gilbert et al. 2021, in preparation):

\[
D_u = h^{-1} \nabla \cdot \left[ \nu h (\nabla u + (\nabla u)') \right],
\]

\[
D_\eta = \eta (\nabla^2 A - h^{-1} \nabla h \cdot \nabla A),
\]

where \( \nu \) is the kinematic viscosity and \( \eta \) is the magnetic diffusivity.
variations in the planetary rotation vector’s vertical component. The approximation also uses the fact that in equatorial regions the Coriolis parameter, \( f \), is approximately linear to set \( f = \beta y \), where the constant \( \beta = 2\Omega / R \) is the local latitudinal variation of the Coriolis parameter at the equator, \( \Omega \) is the planetary rotation rate, and \( R \) is the planetary radius.

The system is driven by a Newtonian cooling treatment, \( Q \), in the continuity equation (Equation (2)), which relaxes the system toward the prescribed radiative equilibrium thickness profile, \( h_{\text{eq}} \), over a radiative timescale, \( \tau_{\text{rad}} \). The Newtonian cooling is implemented with

\[
h_{\text{eq}} = H + \Delta h_{\text{eq}} \cos \left( \frac{x}{R} \right) \cos \left( \frac{y}{R} \right),
\]

where \( H \) is the system’s reference active-layer thickness at radiative equilibrium and \( \Delta h_{\text{eq}} \) is the difference in \( h_{\text{eq}} \) between this reference thickness and the radiative equilibrium layer thickness at the substellar point. This profile is similar to the spherical forcing prescriptions used in comparable hydrodynamic models (e.g., Shell & Held 2004; Langton & Laughlin 2007; Showman & Polvani 2010, 2011; Showman et al. 2012; Perez-Becker & Showman 2013). The transfer of mass caused by \( Q \) generates horizontal pressure gradients, which drive recirculation via the generation of planetary-scale shallow-water waves.

Similarly, in 3D models, pressure gradients caused by heating drive recirculation via internal gravity waves. Using this analogy, mass sources and sinks represent heating and cooling, respectively. This connection has been used extensively in hydrodynamic models of hot Jupiters, with active-layer geopotential, \( gh \), used as a proxy for specific thermal energy (Langton & Laughlin 2007; Showman & Polvani 2010; Perez-Becker & Showman 2013; Showman et al. 2013). Using this physical link, we equate the model’s active-layer reference geopotential, \( gh \), to the reference thermal energy, \( R T_{\text{eq}} \), of the modeled planet’s atmosphere, where \( R \) and \( T_{\text{eq}} \) respectively, denote the specific gas constant and the equilibrium reference temperature.

With the addition of \( Q \), a vertical mass transport term, \( R \), needs to be introduced to enforce specific momentum conservation. In “cooling” regions (\( Q < 0 \)) mass sinks from the active layer to the quiescent layer and causes no active-layer accelerations.\(^4\) However, in “heating” regions (\( Q > 0 \)) mass transport causes deceleration of the active layer as motionless fluid is transferred upward. This deceleration due to heating is calculated by requiring specific momentum conservation in the active layer, yielding

\[
R = \begin{cases} 
0 & \text{for } Q < 0 \\
-\frac{uQ}{h} & \text{for } Q \geq 0,
\end{cases}
\]

which has also been used in the hydrodynamic version of this model (e.g., Shell & Held 2004; Showman & Polvani 2010, 2011; Showman et al. 2012; Perez-Becker & Showman 2013).

We parameterize atmospheric drag with a linear Rayleigh drag treatment, \(-u/\tau_{\text{drag}}\), where \( \tau_{\text{drag}} \) is the timescale of the dominant horizontal drag process in the thin active layer. Previous hydrodynamic studies use this Rayleigh drag to parameterize Lorentz forces (e.g., Perna et al. 2010a; Rauscher & Menou 2013) or basal drag at the bottom of the radiative zone (e.g., Held & Suarez 1994; Liu & Showman 2013; Komacek & Showman 2016). In our study, we include Lorentz forces explicitly. However, due to the geometry of the SWMHD model, we only explicitly include the Lorentz forces caused by the atmospheric toroidal magnetic field (see Section 2.2 for a discussion of the magnetic field geometry in the atmosphere). We hence use the Rayleigh drag treatment to parameterize the Lorentz forces caused by a planet’s deep-seated poloidal magnetic field, which are not included explicitly. This is consistent with the treatment proposed by Perna et al. (2010a), whose \( \tau_{\text{drag}} \) parameterization was based on estimating the direct influence that the planet’s deep-seated poloidal magnetic field has on zonal flows. Though one could argue that in this setting the Rayleigh drag should have no meridional component, for comparison with past hydrodynamic results, we follow the commonly applied treatment of using Rayleigh drag in both horizontal directions (e.g., Perna et al. 2010a; Showman & Polvani 2011; Perez-Becker & Showman 2013; Rauscher & Menou 2013).\(^3\) We also comment that Rogers & Komacek (2014) found that magnetically driven wind variations emerge in the upper radiative atmosphere (where basal drags are negligible), so we do not consider basal drag in this work.

\[\text{2.2. Magnetic Field Profile}\]

The extension of planetary dynamo theory into the hot Jupiter regime is not well understood. That said, from current dynamo theory one would expect hot Jupiters to have planetary dynamos that are sustained within the convective deep interior, generating deep-seated poloidal magnetic fields. The hottest hot Jupiters also have weakly ionized atmospheres. If the atmospheres are sufficiently ionized, the zonally dominated atmospheric flows become sufficiently connected to the planet’s deep-seated poloidal magnetic field to induce a strong toroidal field that dominates the atmospheric magnetic field geometry (Menou 2012). Assuming this picture, and that the planet’s deep-seated magnetic field’s geometry is dominated by an axial dipole, the induction of the toroidal component of the magnetic field can be approximated by

\[
\frac{\partial B_\phi}{\partial t} \approx (B_{\text{dip}} \cdot \nabla_\phi) V_\phi - \nabla_\phi \times (\eta \nabla_\phi \times B_\phi),
\]

where \( B_\phi \equiv B_\phi \phi \) is the toroidal component of the magnetic field, \( B_{\text{dip}} \) is the planetary dipolar field, \( V_\phi \equiv V_\phi \phi \) is the zonal component of the atmospheric flow, \( \nabla_\phi \) is the 3D gradient operator, and the electric currents generating the dipolar planetary field are implicitly assumed to be located far below the atmospheric region of interest (Perna et al. 2010a, 2010b; Batygin et al. 2011; Menou 2012). Therefore, if toroidal field induction dominates toroidal field diffusion, the atmospheric toroidal field profile is expected to be equatorially antisymmetric, as found in the simulations of Rogers & Komacek (2014).

\(^3\) We find that the meridional component of the Rayleigh drag never has a leading-order influence, being 1–2 orders of magnitude smaller than the system’s dominant meridional accelerations, and so does not qualitatively influence any of our results. An example of this can be seen in Figure 2.
There are not enough degrees of freedom in the SWMHD induction equation to simultaneously model the planetary dipolar field and the atmospheric toroidal field, so we only model the dominant atmospheric toroidal field self-consistently. We choose to enforce the simple equatorially antisymmetric, purely azimuthal, initial magnetic field:

\[
B_0 = B_0 \hat{x} = V_A e^{1/2 \tanh(y/L_{eq})} \hat{x},
\]

(10)

where \( V_A \) is the constant parameter that sets the magnitude of the azimuthal magnetic field. This profile may appear an unintuitive choice at first, but London (2017) noted that it has the useful properties for wave dynamics, which we shall exploit in Section 4. It is monotonic, behaves linearly in the equatorial region, and is bounded as \( y/L_{eq} \to \infty \). The approximately linear latitudinal dependence of \( B_0 \) in the equatorial region means that one can choose \( V_A \) in accordance with the first-order Taylor expansion of nonmonotonic equatorially antisymmetric profiles. Upon comparing to other field profiles, we generally find that doing so reproduces similar equatorial dynamics. To illustrate this, in Section 3 we compare some basic results to the profile \( B_0 = V_A (y/L_{eq}) \exp(1/2 - y^2/2L_{eq}^2) \), which is the equatorially antisymmetric profile used in Hindle et al. (2019). This has the same first-order Taylor expansion as Equation (10), has the maximum \( B_0 = V_A \) at \( y = L_{eq} \) (i.e., \( V_A \) is the maximal initial Alfvén speed), and can be motivated from both Equation (9) and the simulations of Rogers & Komacek (2014). We implement the initial magnetic field profile of Equation (10) across an initially flat layer \((h(x, y, 0) = H, \text{ everywhere})\), using the initial magnetic flux function, \( A_0(y) = H V_A L_{eq} e^{1/2 \ln(\cosh(y/L_{eq}))} \).

2.3. Numerical Method and Parameter Choices

Numerical solutions are obtained by evolving Equations (1), (2), (3), and (4) from an initially uniformly flat rest state \((i.e., h(x, 0) = H, u(x, 0) = 0)\), in the presence of a purely azimuthal magnetic field \((A(x, 0) = A_0(y))\). For hydrodynamic solutions we evolve until steady state is achieved, and for MHD solutions we run for a magnetic diffusion timescale. The system is solved on a \( 256 \times 511 \times \gamma \)-grid, using an adaptive third-order Adam–Bashforth time-stepping scheme (Cattaneo et al. 2003), with spatial derivatives taken pseudo-spectrally in \( x \) and using a fourth-order finite difference scheme in \( y \). We use periodic boundary conditions on \( u, h, \) and \( A \) in the \( x \)-direction. On the \( y \) boundaries we impose \( \nu = 0 \) (impermeability), \( \partial h/\partial y = 0 \) (stress-free), and \( \partial A/\partial x = 0 \) (no normal magnetic flux) and maintain the total columnar horizontal magnetic flux of the system. These conditions do not fix values of \( h \) on the \( y \) boundaries, which are updated to satisfy a consistency condition that results from mass conservation and our other boundary conditions.\(^6\)

We choose simulation parameters based on the planetary parameters of HAT-P-7b, an ultra-hot Jupiter with observed east–west brightspot variations (Armstrong et al. 2016) that can be well explained by 3D MHD simulations (Rogers 2017). Relevant planetary parameters are presented in Table 1. As discussed above, we equate the active layer’s reference geopotential with a radiative equilibrium thermal energy reference level. Therefore, the gravity wave speed is set using \( c_g = \sqrt{gH} = \sqrt{R_T}T_{eq} = 3.0 \times 10^3 \text{ m s}^{-1} \), where we use the planet’s orbit-averaged effective temperature for the equilibrium reference temperature and the specific gas constant is calculated using the solar system abundances in Lodders (2010). We assume synchronous orbits, so \( \Omega = \pi/\tau_{orb} \), where \( \tau_{orb} \) is the orbital period. We calculate \( \beta = 2\Omega/R = 6.6 \times 10^{-13} \text{ m}^{-1} \text{ s}^{-1} \), so the equatorial Rossby deformation radius is

\[
L_{eq} \equiv \left( \frac{c_g}{\beta} \right)^{1/2} \approx 6.7 \times 10^7 \text{ m}.
\]

(11)

This is a fundamental length scale over which gravitational and rotational effects balance, and it is the interaction length scale of planetary-scale flows that corresponds to their latitudinal widths.

The characteristic wave travel timescale, \( \tau_{wave} \), is defined by the time a shallow-water gravity wave takes to travel over the distance \( L_{eq} \):

\[
\tau_{wave} \equiv \frac{L_{eq}}{c_g} \approx 2.2 \times 10^4 \text{ s} \approx 0.26 \text{ Earth days}.
\]

(12)

We set the reference thickness of the model’s active layer to the atmospheric pressure scale height, that is, \( H \equiv RT_{eq}R_J^2/GM = 4.3 \times 10^5 \text{ m} \), where \( M \) is the planetary mass and \( G \) is Newton’s gravitational constant.

In hydrodynamic shallow-water models (e.g., Shell & Held 2004; Langton & Laughlin 2007; Showman & Polvani 2010, 2011; Showman et al. 2012; Perez-Becker & Showman 2013), the forcing profile is usually set so that \( \Delta h_{eq}/H \sim (T_{days} - T_{eq})/T_{eq} \), where \( T_{eq} \) is the average reference temperature (for a given atmospheric depth) and \( T_{days} \) is the maximal dayside reference temperature (at that atmospheric depth). For comparison, applying the reference temperatures used for HAT-P-7b in Rogers (2017), this equates to \( \Delta h_{eq}/H \sim 0.22, 0.19, \) and \( 0.14 \) at \( P = 10^{-3}, 10^{-2}, \) and \( 10^{-1} \) bars, respectively. We consider models with \( \Delta h_{eq}/H \in \{0.01, 0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6\} \) to cover forcing parameter regimes within and on either side of the expected range.

The simulations presented in this paper have a viscous diffusion of \( \nu = 4 \times 10^8 \text{ m}^2 \text{ s}^{-1} \). In terms of “true” physical values, this diffusion coefficient is comparatively large; yet, upon checking, we find that viscous components of Equation (1) remain negligibly small. This is to be expected, as we are predominantly modeling large-scale planetary flows, upon which viscous dissipation generally has little direct influence. We set the magnetic diffusivity to \( \eta = 4 \times 10^8 \text{ m}^2 \text{ s}^{-1} \), which is within the expected \( \eta \) range on HAT-P-7b’s nightside. These values of \( \eta \) and \( \nu \) are both small.

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\(^6\) The results we present in Section 3 are also robust to other modeling setups, including initializing \( h(x, 0) \) and \( u(x, 0) \) from hydrodynamic steady-state profiles and applying different boundary treatments (e.g., extended domains and absorbing boundaries like those discussed in Glatzmaier 2015). Regardless of these modeling variations, solutions exhibited similar fundamental behaviors (and reversal thresholds).
enough to make the dynamical timescales of our system much smaller than the diffusion timescales. In 3D geometries, longitudinal variations in smaller than the diffusion timescales. In 3D geometries, longitudinal enough to make the dynamical timescales of our system much smaller than the diffusion timescales. In 3D geometries, longitudinal

as presented in Section 2. First, in Sections 3.1 and 3.2 we respectively highlight the basic flow behaviors of hydrodynamic and magneto-hydrodynamic solutions. Then, in Section 3.3, we discuss detailed force balances of these numerical solutions. In Sections 3.1–3.3, we focus on solutions with $\Delta N_{\text{eq}}/H = 0.2$, which lies within the expected forcing range of our fiducial planet HAT-P-7b (see Section 2). Finally, in Sections 3.4 and 3.5, we discuss the extension of the developed theory to other forcing magnitudes and toroidal field profiles.

We visualize the basic form of our numerical solutions by plotting their (nondimensionalized) geopotential distributions in Figure 1. As discussed in Section 2, we use geopotential energy, $gh$, as a shallow-water proxy of thermal energy, so the geopotential distributions are analogous to those of temperature perturbations. In the hydrodynamic version of our shallow-water model, solutions are known to converge upon a steady state (e.g., Langton & Laughlin 2007; Showman & Polvani 2010; Perez-Becker & Showman 2013; Showman et al. 2013), and we replicate such hydrodynamic steady-state solutions in the left column of Figure 1 for comparison with our MHD simulations, which we plot in the middle and right columns for two difference solution phases (see Section 3.2). In each row of Figure 1 (from top to bottom) we display the solutions for (a) short $\tau_{\text{drag}}$ and strong $\tau_{\text{drag}}$, $\tau_{\text{run}} = \tau_{\text{wave}}$, $\tau_{\text{drag}} = 25\tau_{\text{wave}}$; (b) moderate $\tau_{\text{run}}$ and moderate $\tau_{\text{drag}}$, $\tau_{\text{run}} = 25\tau_{\text{wave}}$, $\tau_{\text{drag}} = 5\tau_{\text{wave}}$; and (c) long $\tau_{\text{run}}$ and weak $\tau_{\text{drag}}$, $\tau_{\text{run}} = 25\tau_{\text{wave}}$, $\tau_{\text{drag}} = 5\tau_{\text{wave}}$. Rogers & Komacek (2014) found magnetically driven reversals to occur in the upper atmospheres of ultra-hot Jupiters, where $\tau_{\text{run}} \sim \tau_{\text{wave}}$ and $\tau_{\text{drag}} \sim \tau_{\text{wave}}$; the conditions are most akin to case (a), though $\tau_{\text{run}}$ and $\tau_{\text{drag}}$ are not generally exactly equal.

The remaining free parameter in our system is $V_A$, which determines the magnitude of the system’s magnetic field. Our general approach is to increase $V_A$, from $V_A = 0$, until we find a change in the nature of the SWMHD system (i.e., hotspot reversals). Here we highlight that, for large enough $V_A$, we always find hotspot reversals in the SWMHD model, regardless of our choices of $\Delta N_{\text{eq}}/H$, $\tau_{\text{run}}$, and $\tau_{\text{drag}}$. In Section 3 we will discuss both hydrodynamic and magnetohydrodynamic solutions over a wide range of parameter choices to illustrate the magnetic mechanism that drives reversals, as well as its robustness to changes in parameter space.

2.4. Model Validity

Here we briefly discuss validity criteria for our model in the context of our parameter choices. First, we comment that $H/L_{\text{eq}} \approx 6 \times 10^{-3} \ll 1$, so the shallow-water approximation is well founded and vertical dependences in the atmosphere are not of leading-order importance. Second, we take $\Omega = \Omega_0$, which is typically known as the traditional approximation and is formally valid in the limit of strongly stable stratification ($N^2/\Omega^2 \gg 1$; e.g., Vallis 2006). For our parameters, $N^2/\Omega^2 \approx 4 \times 10^4 \gg 1$, so this approximation is also well founded. Third, our boundaries are located at $y = \pm R\tau_r/2 \pm 2.3L_{\text{eq}}$, so the impermeable wall at our model’s “poles” has little physical influence on our solutions and does not interact with the equatorial dynamics we wish to study. Finally, the equatorial beta-plane truncation of the Coriolis parameter is $f = \beta y + O((y/R)^3)$, so we are careful not to draw conclusions about the polar flows (with $y \geq R$), where the Coriolis parameter is overestimated and boundary effects can occur. The initial magnetic field choice of Equation (10) is based on an analogous Taylor truncation, so it places no further constraint on our discussion.
severity of zonal geopotential gradients. At mid- to high latitudes the Coriolis force becomes significant and solutions satisfy the aforementioned drag-adjusted geostrophic balance. In a “true” geostrophic balance, without suppression from drags and forcing, pressure gradients are exactly balanced by the Coriolis force, which acts perpendicularly to the velocity, causing flows to rotate (to their right in the northern hemisphere and to their left in the southern hemisphere). This yields large-scale mid-to-high-latitude vortices that are aligned with isobars, similar to those seen in the short-τ_{rad}, weak-drag, hydrodynamic solution (Figure 1(c), left column). However, the slowing of winds from the Rayleigh drag reduces the magnitude of Coriolis deflection. Therefore, in the strong-drag limit large-scale vortices cannot fully develop. Similarly, when τ_{rad} is short, heating/cooling occurs before large-scale vortices fully develop. Comparing the mid-to-high-latitude flows of the hydrodynamic solutions, one finds a transition from the long-τ_{rad}/weak-drag solutions, with fully formed geostrophic vortices, to the short-τ_{rad}/strong-drag solutions, in which the drag-adjusted geostrophic circulations are approximately aligned with the isobars of the equilibrium geopotential (see Figure 1, left column). Aside from an unphysical special case discussed in Showman & Polvani (2011) and Perez-Becker & Showman (2013), for all finite physically relevant choices of τ_{rad} and τ_{drag}, the meridional mass transport into the equator, caused by the drag-adjusted geostrophic circulations, is maximized east of the substellar point.

These solutions always exhibit eastward hotspots. This is because the equatorward (rescaled) geopotential energy transport from the mid-to-high-latitude circulations, −∂(h)/∂y, always has its equatorial maximum located eastward of the substellar point. At the equator, the pressure gradient drives winds that diverge from hotspots, causing equatorial geopotential energy transport away
from the hotspot regions (i.e., \(-\partial(hv)/\partial x < 0\) in hotspot regions). Hence, by Equation (2) (geopotential energy conservation), the hotspots locate themselves at the equatorial point of maximal incoming geopotential energy flux, which is located between the equatorial maxima of \(-\partial(hv)/\partial y\) and \(Q\). The Newtonian cooling \((Q)\) attempts to return a solution to its forcing equilibrium (i.e., with its hotspot at the substellar point), whereas, as stated above, the equatorial maximum of \(-\partial(hv)/\partial y\) is always eastward. The degree of the hotspot’s eastward offset is therefore determined by the location of the equatorial maximum of \(-\partial(hv)/\partial y\) and its relative magnitude compared to \(Q\). In short, the size of the (eastward) hotspot offset is determined by the efficiency over which the drag-adjusted geostrophic circulations can redistribute thermal energy from the western equatorial dayside to the eastern equatorial dayside, by circulating it to and from the higher latitudes.

### 3.2. Basic Magnetohydrodynamic Solutions

In the weakly magnetic limit, shallow-water magnetohydrodynamic solutions behave much like their hydrodynamic counterparts (i.e., solutions reach a steady state that is characterized by eastward hotspots, zonal equatorial winds, and drag-adjusted geostrophic circulations at mid- to high latitudes). However, when the azimuthal magnetic field exceeds a critical magnitude, the nature of the solution changes. Supercritical magnetic solutions have three phases: an initial phase, in which winds and geopotentials resemble their hydrodynamic counterparts but their circulations induce magnetic field evolution; a transient phase, in which mid-to-high-latitude winds align with the azimuthal magnetic field and dayside equatorial winds experience a net westward acceleration, driving an east-to-west hotspot transition; and a reversed quasi-steady phase, in which westward zonally dominated dayside winds maintain westward hotspots (until, after a comparably long period of time, the magnetic field decays via magnetic diffusion). We present geopotential distributions of supercritical magnetic solutions in the transient and quasi-steady phases in the two right columns of Figure 1 (middle and right, respectively). The supercritical magnetic solutions are plotted for the same drag choices as the hydrodynamic solutions that they share a row with (see Section 3.1), but now lines of constant \(A\), which approximately correspond to field lines of the horizontal magnetic field, are also overplotted for visualization of the magnetic field.

After a magnetic solution’s initial phase, in which it behaves similarly to its hydrodynamic counterpart, in mid-to-high-latitude regions there is a competition between the drag-adjusted geostrophic balance and the magnetic tension (i.e., \(B \cdot \nabla B\), the restorative force that acts to straighten bent horizontal magnetic field lines) that the circulating flows generate. Initially, the magnetic field is purely azimuthal, with only latitudinal gradients in its profile, so magnetic tension is zero everywhere. To understand the magnetic field’s evolution, we highlight that, as the magnetic diffusion timescale is large in comparison to the dynamical timescales of the system, \(A\) is approximately materially conserved. This means that lines of constant \(A\) are advected by the mid-to-high-latitude circulations, bending them and causing a growth of magnetic tension.

For subcritical magnetic field strengths, a drag-adjusted magnetoglostric balance can be supported, with winds and geopotential profiles making small adjustments to balance the magnetic contribution (before magnetic diffusion eventually returns the system to a hydrodynamic steady state). In contrast, for supercritical magnetic field strengths, magnetic tension becomes strong enough to obstruct the drag-adjusted geostrophic circulations and solutions enter into a transient phase, which ultimately results in hotspot reversals. In Section 3.3, we shall see that the reversal is driven by a westward Lorentz force acceleration in the region surrounding the hotspot, which is itself generated by this obstruction of geostrophic balance. The westward Lorentz force acceleration causes the point of zonal wind divergence on the equator to shift eastward, so that in hotspot regions geopotential energy flux is westward (i.e., \(ghu < 0\)) rather than zero. This shifts the hotspot westward until the system rebalances into a state with a westward hotspot (again, see Section 3.3).

We find that this reversal mechanism (i.e., westward equatorial-dayside Lorentz force accelerations driven by the obstruction of geostrophic balance) always leads to hotspot reversals in the SWMHD model, regardless of our choice of \(\Delta h_{eq}/H\), \(\tau_{rad}\), and \(\tau_{drag}\). However, since these parameters control pressure gradient magnitudes and recirculation efficiency, they determine the critical magnetic field strength sufficient for reversal. We present bounds on the magnetic field strength’s critical magnitude, \(V_{A, crit}\), for various parameter choices in Figure 5. Generally, \(\Delta h_{eq}/H\) and \(\tau_{drag}\) set the magnitude of a solution’s pressure gradients, and therefore the magnitude of the circulations to be overcome, so shorter \(\tau_{rad}\) and larger \(\Delta h_{eq}/H\) correspond to larger \(V_{A, crit}\) magnitudes. Initially in long-\(\tau_{drag}\) solutions, the fully formed large-scale geostrophic vortices advect the lines of constant \(A\) efficiently until they are resisted by magnetic tension; whereas for short-\(\tau_{drag}\) solutions, the slowing of winds from drags decreases the distance over which winds initially advect the lines of constant \(A\). Therefore, weak-drag solutions generally experience a larger degree of field line bending and hence more magnetic tension (relative to the other accelerations in their solutions for a given \(V_A\)) than strong-drag solutions. Put simply, strong-drag solutions require larger \(V_{A, crit}\) magnitude to reverse. We quantify dependences of \(V_{A, crit}\) on \(\Delta h_{eq}/H\), \(\tau_{rad}\) and \(\tau_{drag}\) in later discussion.

In the quasi-steady phase of supercritical SWMHD solutions, the magnitudes of \(\tau_{rad}\) and \(\tau_{drag}\) determine the efficiency of the westward energy redistribution. For large \(\tau_{rad}\) and \(\tau_{drag}\) timescales, the (westward) hotspot offsets are large, as the equatorial pressure—Lorentz balance is free to redistribute energy toward the point where the zonal winds converge, almost entirely without restriction; conversely, for short \(\tau_{rad}\) and \(\tau_{drag}\) timescales, this equatorial energy redistribution is less efficient and hotspot offsets are smaller. Comparing between rows in Figure 1 (right column) suggests that \(\tau_{drag}\) is the most influential timescale in determining westward hotspot offsets in the SWMHD system.

### 3.3. Force Balances

In this subsection we compare the force balances of Equation (1) for hydrodynamic and supercritical MHD solutions with the parameters of regime (b) in Figure 1 (i.e., for \(\Delta h_{eq}/H = 0.2\), \(\tau_{rad} = \tau_{drag} = 5\tau_{wave}\), with either \(V_A = 0\) or \(V_A = 0.7c_e\)). We highlight how the presence of a strong equatorially antisymmetric azimuthal magnetic field modifies the
force balances of different planetary regions, and we link these modifications to the more general discussions of Sections 3.1 and 3.2.

In Figures 2 and 3 we respectively plot the dominant meridional and zonal acceleration components of Equation (1), for solutions in regime (b). In the left column of Figures 2 and 3, we present the acceleration components for the hydrodynamic steady-state solution, whereas in the middle and right columns of Figures 2 and 3, we present the acceleration components of the transient and quasi-steady phases of its supercritical MHD counterpart. Along each row of Figures 2 (meridional components) and 3 (zonal components), we plot (from top downward) the acceleration contributions due to horizontal pressure gradients (−g∇h), the Coriolis effect (−f × u), the Lorentz force (B · ∇B), Rayleigh drag (−u/τ_{drag}), and advection (−u · ∇u). Additionally, in the bottom row of Figure 2 we plot the total meridional acceleration (∂h/∂t), and, likewise, in the bottom row of Figure 3 we plot the total zonal acceleration (∂u/∂t). For the presented parameter choices the acceleration contributions due to vertical mass transport (R) and viscous diffusion (D_ν) are much weaker and so are not included in the plots.

At mid- to high latitudes, the force balances of hydrodynamic solutions in steady state are well described by the three-way drag-adjusted geostrophic balance discussed in Section 3.1. In particular, Figures 2 and 3 (left column) highlight this for regime (b), showing that in both horizontal directions the mid-to-high-latitude accelerations due to horizontal pressure gradients and the Coriolis force almost exactly cancel, albeit with small Rayleigh drag adjustment and a yet smaller advection contribution. The meridional components of these accelerations remain balanced in equatorial regions, with all of them vanishing at the equator. However, in the zonal direction, the Coriolis force vanishes in equatorial regions but zonally directed pressure gradients do not, so zonal pressure gradients are balanced by the Rayleigh drag, with an advection adjustment. Since hotspots in hydrodynamic solutions are always located where zonal equatorial jets diverge, these three acceleration components are equally zero at hotspots (see cyan markers in Figure 3). As discussed in Section 3.1, hotspots are driven eastward by the net west-to-east equatorial energy transfer that results from the mid-to-high-latitude drag-adjusted geostrophic circulations.

As discussed in Section 3.2, magnetic tension (B · ∇B) is initially zero everywhere, so MHD solutions initially resemble their hydrodynamic counterparts. However, lines of constant A (which closely follow magnetic field lines) are advected by the mid-to-high-latitude circulations that are archetypal of hydrodynamic solutions. This causes them to bend equatorward between the western and eastern dayside (where the initial circulations are poleward and equatorward, respectively; see Figure 1, second row, middle column). Consequently, a restorative Lorentz force that resists meridional winds is produced (see Figure 2, third row, middle column). For subcritical MHD solutions (not plotted) this Lorentz force resists but does not fully obstruct the mid-to-high-latitude circulations, which adjust into a (drag-adjusted) magnetogeostric balace. However, in supercritical MHD solutions, the Lorentz force resists meridional winds strongly enough to zonally align the mid-to-high-latitude winds. Hence, supercritical MHD solutions enter into the transient phase discussed in Section 3.2.

When the magnetic field geometry is azimuthally dominated, understanding Lorentz force accelerations is less intuitive in the zonal direction than in the meridional direction (in which they simply oppose meridional flows). The zonal Lorentz accelerations, B · ∇B_x, are most easily understood geometrically when considered as the directional derivative of B_x along horizontal magnetic field lines, which are approximately equivalent to lines of constant A. When the magnetic field lines bend equatorward, they generally move into regions of smaller |B_x|, and hence the zonal Lorentz force component generally accelerates flows westward (as B · ∇B_x < 0); conversely, when they bend poleward, they generally move into regions of larger |B_x|, and hence the zonal Lorentz force component generally accelerates flows eastward (as B · ∇B_x > 0). One can see this by comparing lines of constant A in mid- to high latitudes of Figure 1 (second row, middle column) with the corresponding mid-to-high-latitude zonal Lorentz force accelerations in Figure 3 (third row, middle column). Since magnetic field lines bend equatorward between the western and eastern dayside at mid- to high latitudes, the Lorentz force accelerates mid-to-high-latitude dayside flows westward (and eastward on the nightside). Similar westward dayside Lorentz force accelerations are generated along the equator by magnetic field lines bending into equatorial regions. To visualize this, in Figure 4 we plot the horizontal magnetic field geometry (top row) and the zonal component of the Lorentz force (bottom row) in the equatorial region, −π/8 < y/R < π/8, for the transient (left) and quasi-steady (right) phases of the supercritical MHD solution (again, for parameter regime (b)). In the initial phase the Lorentz force primarily acts to resist drag-adjusted geostrophic circulations (see above). Therefore, in the early transient phase the magnetic field lines bend equatorward between the western and eastern dayside (where the initial circulations are poleward and equatorward, respectively). For the lowest equatorial regions (|y/R| ≲ π/32 in Figure 4) such equatorward magnetic field line bending causes the lines to move into regions of smaller |B_x|. Consequently, zonal Lorentz force accelerations are westward in regions surrounding the hotspot (see Figure 4, left column). In fact, zonal Lorentz force accelerations are always westward in hotspot regions, regardless of radiative/drag/forcing parameter choices, because in hydrodynamic and weak/early-phase MHD) solutions hotspots are located between the substellar point and the (eastward) maximum of equatorward flow (where lines of constant A are bent most equatorward). The resulting westward accelerations cause an equatorial imbalance in the zonal momentum equation (see Figure 3, bottom row, middle column), which drives the point of zonal equatorial wind divergence eastward of the hotspot and, consequently, shifts the hotspot westward (see discussion in Section 3.2). Finally, as these westward accelerations cause dayside equatorial winds to become more westward, lines of constant A are swept from east to west along the equator, bending them further and thus enhancing equatorial Lorentz force accelerations across all equatorial latitudes (see Figure 4, right column).9

Across radiative/drag/forcing parameter choices, when the hotspots have transitioned westward, the system rebaalances into a quasi-steady state, which is characterized by westward hotspots, zonally aligned winds, and magnetic field lines that have an equatorward bend along the line x = 0 in equatorial regions. The predominantly meridional balance is between pressure gradients, the Coriolis force, and the Lorentz force

9 This is equivalent to saying that the more westwardly oriented dayside winds cause B_x to become more significant in equatorial regions, which in turn enhances the westward Lorentz force accelerations.
whereas the predominant zonal balance is between pressure gradients, the Lorentz force, and the Rayleigh drag (see Figure 3, right column). In these balances the zonally aligned winds cause the meridional Rayleigh drag and the zonal Coriolis force to be small. We comment that as the magnetic field eventually diffuses away, the balance adjusts to the decreasing Lorentz force contribution, eventually restoring the drag-adjusted geostrophic/magnetogostrophic balances associated with hydrodynamic and weakly magnetic solutions (and hence eastward hotspots).

3.4. Forcing Dependence

We find that, when one compares marginally supercritical magnetic solutions with \( \Delta h_{eq}/H = 0.2 \), \( \tau_{rad} = \tau_{drag} = 5 \tau_{wave} \), the qualitative physical behaviors and balances discussed in Sections 3.1–3.3 (and illustrated in Figures 1–4) remain highly similar (in fact, remarkably so). The only discernible changes we observe between marginally supercritical magnetic solutions, upon increasing \( \Delta h_{eq}/H \), are an approximately linear scaling of dependent variable magnitudes and a correction from advection, which generally only provides a lower-order correction. This is to be expected from the theory we have developed so far, as the process that needs to be overcome in order to trigger hotspot reversals (i.e., the drag-adjusted geostrophic balance) is a linear one. Consequently, choices of \( \Delta h_{eq}/H \) do not change the mechanics of the hotspot reversals, though they do determine quantitative features of the system (such as magnitudes and \( V_{\Lambda, crim} \)).

Figure 2. Meridional force balances. In each column, reading from left to right, we plot meridional accelerations corresponding to hydrodynamic steady-state solutions, transient phase supercritical MHD solutions, and quasi-steady supercritical MHD solutions. In rows 1–4, we respectively plot meridional accelerations due to horizontal pressure gradients, the Coriolis effect, the Lorentz force, and Rayleigh drag; the summed meridional accelerations are plotted in row 5. The solutions are presented for \( \Delta h_{eq}/H = 0.2 \), \( \tau_{rad} = \tau_{drag} = 5 \tau_{wave} \), with \( V_A = 0 \) (HD) or \( V_A = 0.7 c_g \) (MHD) (i.e., parameter regime (b) in Figure 1).

\( \Delta h_{eq}/H = 0.2 \), \( \tau_{rad} = \tau_{drag} = 5 \tau_{wave} \)

Steady HD Transient MHD Quasi-steady MHD

Pressure

Lorentz

Rayleigh

Advection

Total

\( \pi/2 \)

\( y/R \)

\( -\pi/2 \)

\( 0 \)

\( \pi \)

\( x/R \)

\( 5e-02 \)

\( 0e+00 \)

\( -5e-02 \)

\( 4e-02 \)

\( 2e-02 \)

\( 0e+00 \)

\( -2e-02 \)

\( -4e-03 \)

\( -5e-02 \)

\( 0e+00 \)

\( -2e-02 \)

\( -4e-03 \)

\( -5e-04 \)

\( 0e+00 \)

\( -2e-04 \)
We can use our developed understanding of the reversal mechanism to predict magnitudes $V_{A,crit}$ with simple scaling arguments based on the respective magnitudes of geostrophic circulations and the restorative Lorentz force. Let $\tau_{A,geo}^{-1} = U/L_{eq}$ be the frequency over which geostrophic flows circulate and $\tau_{A,rad}^{-1} = V/L_A$ be the (Alfvén) frequency over which the azimuthal field attempts to zonally align these circulations, where $U$, $V$, $L_{eq}$, and $L_A$ are the typical velocity and length scales associated with the two opposing processes. Reversals occur when $\tau_{A,rad}^{-1} \gtrsim \tau_{A,geo}^{-1}$ or equivalently when $V \gtrsim UL_A/L_{eq}$ (i.e., when the azimuthal field is strong enough to restrict the geostrophic flows). Perez-Becker & Showman (2013) showed that the velocities of geostrophic circulations in Coriolis-dominated regions scale like

$$\frac{U}{c_g} \sim \left( \frac{\Delta h_{eq}/H}{\tau_{rad}/\tau_{wave}} \right)^{-1} \left( \frac{2 \Omega \tau_{wave}^2}{\tau_{rad}} + 1 \right)^{-1},$$

highlighting that the reversal threshold is expected to have a linear dependence on $\Delta h_{eq}/H$.

In Figure 5 we plot the dependence of $V_{A,crit}$ on $\Delta h_{eq}/H$ from our simulations. For comparison, we overplot the lines

$$\frac{V_{A,crit}}{c_g} = \left( \frac{2 \pi R (\Delta h_{eq}/H)}{\kappa L_{eq} (\tau_{rad}/\tau_{wave})} \right)^{-1} \left( \frac{2 \Omega \tau_{wave}^2}{\tau_{rad}} + 1 \right)^{-1},$$

Figure 3. The zonal force balances corresponding to the meridional force balances of Figure 2 (see Figure 2 caption). As in Figure 2, we present solutions for the parameter choices $\Delta h_{eq}/H = 0.2$, $\tau_{rad} = \tau_{drag} = 5 \tau_{wave}$, with $V_A = 0$ (HD) or $V_A = 0.7 c_g$ (MHD) (i.e., parameter regime (b) in Figure 1). To aid discussion in the text, hotspot locations have been marked with cyan crosses in hydrodynamic solution panels that correspond to zonal acceleration components with a nonzero equatorial contribution.
where, since the circulations bend field lines on the planetary azimuthal scale, we take $L_A = 2\pi R$ and $\kappa$ is a constant of order unity based on the profile of $B_0(y)$.$^{10}$

We generally find reasonable agreement between this simple scaling prediction and numerical simulations, particularly in the realistic regimes of $\tau_{\text{rad}}$ short and $\Delta h_{\text{eq}}/H \sim 0.1-0.3$, but note that $V_{A,crit}$ approaches a minimum as $\Delta h_{\text{eq}}/H \rightarrow 0$, which we shall consider in Section 4. This scaling law approximation deals less favorably in the (less physical) long-$\tau_{\text{rad}}$ cases, where $\tau_{\text{drag}}$ dependencies become important. However, as we shall discuss in Sections 5 and 6, the other uncertainties in atmospheric characteristics are likely to provide much larger uncertainties than those arising from this scaling law approximation.

### 3.5. Linear-Gaussian Magnetic Field Profiles

Upon comparing the discussed results to their equivalents for the initial magnetic field profile $B(x, y, 0) = V_A(y/L_{eq}) \exp(1/2 - y^2/2L_{eq}^2)$, we found the same mechanical features. Namely, subcritical solutions behave similarly to their hydrodynamic counterparts, whereas, for supercritical magnetic solutions, the obstruction of geostrophic circulations by the magnetic field causes zonal wind alignment, a westward Lorentz force acceleration, and therefore reversed hotspots. The only different qualitative flow features arise at the poles, where $V_A(y/L_{eq}) \exp(1/2 - y^2/2L_{eq}^2)$ decays, but our model and aims are not directed toward the polar regions. The quantitative differences between solutions also tend to be minor, with a second-order change in $V_{A,crit}$ as the two profiles cause a slightly different magnitude of Lorentz force to be given for a given $V_A$. To make this comparison, we have marked $V_{A,crit}$ for Linear-Gaussian profiles on Figure 5 with starred markers. We conclude that the choice of a $B_0 \propto \tanh(y/L_{eq})$ profile is a useful simplification when considering reversals. This can be advantageous owing to the properties of the hyperbolic tangent function, which is both monotonic and bounded as $y \rightarrow \infty$.

### 3.6. Summary of Findings

In this section we have identified the mechanism responsible for driving hotspot reversals in our SWMHD model. The reversals are caused by the westward Lorentz force acceleration that is generated when strong equatorially antisymmetric azimuthal magnetic fields obstruct the geostrophic circulation patterns responsible for energy redistribution in the hydrodynamic system. The understanding we have developed explains why such hotspot reversals always emerge in the SWMHD model, regardless of our choices for the free forcing/drag parameters $\Delta h_{eq}/H$, $\tau_{\text{rad}}$, and $\tau_{\text{drag}}$. Moreover, this developed understanding has allowed us to use simple scaling arguments to predict the reversal threshold, $V_{A,crit}$, in terms of planetary parameters, finding reasonable agreement between predictions and numerical simulations in realistic forcing regimes for our fiducial planet HAT-P-7b. However, our simulations also show that $V_{A,crit}$ approaches a minimal threshold in the zero-amplitude limit. In Section 4 we shall probe linear theory to explain this finding. For this, we shall use our finding that, when compared, equatorially antisymmetric azimuthal magnetic field profiles with similar latitudinal

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$^{10}$ $\kappa$ is an estimate of the relative strength of $B_0$ (compared to $V_A$) at low latitudes, $y_0$, where westward Lorentz force accelerations first develop. In Figure 5, we take $\kappa = e^{y_0^2/\tau_{\text{crit}}(y_0/L_{eq})} \approx 0.47$ (using $y_0 \approx R\pi/16$ based on Figure 4).
dependence at equatorial and midlatitudes behave similarly to one another.

4. Linear Theory

4.1. Linearized Steady-state Solutions

First, we seek to establish the features of the reversals that linear theory can capture, as well as its limitations. We do so by linearizing the nondiffusive versions of Equations (1)–(4) about the background state \( \{u_0, v_0, h_0, A_0\} = \{u_0(y), 0, H, A_0(y)\} \), where \( H \) is the (constant) background layer thickness, \( A_0 \) is defined such that \( dA_0/ dy = HB_0 \) for the latitudinally dependent azimuthal background magnetic field, \( B_0 = B_0(y)x \), and \( u_0(y) \) is to be fixed in a manner that balances the zeroth-order zonal momentum equation of the hydrodynamic version of the system that we wish to investigate. To probe the system at the reversal threshold, we assume that steady-state perturbations exist about this background state and apply the plane wave ansatz, \( \{u, v, h, A\} = \{\hat{u}(y), \hat{v}(y), \hat{h}(y), \hat{A}(y)\}e^{ikx} \), where \( k \) denotes the azimuthal wavenumber and subscripts of unity denote perturbations from the background state. Such perturbations satisfy

\[
(iku_0 + \tau_\text{drag}^{-1})\hat{u} = \left( f - \frac{du_0}{dy} \right) \hat{v} - ikg\hat{h} \\
+ iKB_0\hat{B}_x + \frac{dB_0}{dy}\hat{B}_y,
\]

\[
(iku_0 + \tau_\text{drag}^{-1})\hat{v} = -fu + g\hat{h} + ikB_0\hat{B}_y,
\]

\[
(iku_0 + \tau_\text{drag}^{-1})\hat{h} = -H\left(ik\hat{u} + \frac{d\hat{v}}{dy}\right) + HS(y),
\]

\[
iku_0\hat{A} = -HB_0\hat{v},
\]

where \( \hat{B}_x = (d\hat{A}/ dy - B_0\hat{h})/H \), \( \hat{B}_y = -i\hat{A}/H \), \( S(y) = (\Delta h_\text{eq}/H)\tau_\text{drag}^{-1}\exp(-y^2/2L_\text{eq}^2) \) is the first-order forcing contribution in the system based on the equilibrium thickness profile, \( h_\text{eq} = H + \Delta h_\text{eq}\cos(kx)\exp(-y^2/2L_\text{eq}^2) \), and based on our numerical findings we have assumed that \( R \) does not make a first-order contribution to Equations (15) and (16). Before solving, we note that hydrodynamic solutions are never singular, but that Equation (18) causes the magnetic version of the system to be singular if \( u_0 = 0 \). To compare to the simulations of Section 3, we solve the system for \( f = \beta y \) and \( B_0 = V_A e^{1/2}\tanh(y/L_0) \).

For a given \( u_0(y) \), we seek solutions of Equations (15)–(18) on \(-L_\gamma < y < L_\gamma \), with impermeable boundaries at \( y = \pm L_\gamma \), using the shooting method outlined in Appendix A. We take \( L_\gamma = 5L_\text{eq} \) (see Equation (11)), which is large enough to ensure that the outer boundary condition has a negligible influence on solutions. We solve the system for \( u_0(y) = U_0\exp(-y^2/2L_\gamma^2) \), where \( U_0 \) is chosen so that in the hydrodynamic limit the zonally averaged zonal acceleration in Equation (22) of Showman & Polvani (2011) vanishes at the equator. We plot linear solutions for \( \Delta h_\text{eq}/H = 0.01 \) in Figure 6, on the reduced domain \(-\pi/2 < y < \pi/2 \), for three \( \tau_\text{drag} \) and \( \tau_\text{drag} \) choices, comparing hydrodynamic solutions with MHD solutions at the threshold of criticality, as found by simulations.

Hydrodynamic solutions generally resemble those discussed in Showman & Polvani (2011), albeit with an adjustment due to \( u_0 \) (as discussed by Tsai et al., 2014, for \( u_0 \) constant). They are characterized by geostrophic circulations at mid- to high latitudes and zonal-pressure-driven jets at equatorial latitudes. Such solutions closely resemble the nonlinear hydrodynamic steady-state solutions we discussed in Section 3. The characteristic flow patterns of hydrodynamic steady-state solutions can also be directly linked to the forcing responses of specific standing, planetary-scale, equatorial shallow-water waves (Matsumo 1966; Showman & Polvani 2011; Tsai et al., 2014). The geostrophic circulations are linked to the planetary-scale equatorial Rossby waves, which are geostrophic in nature at mid- to high latitudes, while the equatorial jets are linked to the superposition of the planetary-scale equatorial Rossby waves and the equatorial Kelvin wave, which travels eastward about the equator in response to pressure perturbations. The
presented linear hydrodynamic solutions all have eastward hotspots (located at points of zonal wind divergence), as the linearized meridional convergence of geopotential flux into the equator, $-g \partial H \partial y_{\parallel} = 0$, is maximized eastward of the substellar point (due to the form of the geostrophic circulations; further discussion in Section 3.1).

The marginally critical MHD solutions share some common characteristics with their nonlinear simulated counterparts. Specifically, in these solutions the aligning influence of the meridional Lorentz force is strong enough to obstruct geostrophic circulations, which are replaced by zonally aligned winds. However, unlike their simulated nonlinear counterparts, the magnetohydrodynamic solutions do not have westward hotspots. This arises because in this simple linear model one can show that the Lorentz force components, which drive hotspot reversals in nonlinear simulations (see Section 3), vanish at the equator. Instead, marginally critical MHD solutions approach a limit of zero hotspot offset, as the obstruction of geostrophic circulations causes $-g \partial H \partial y_{\parallel} = 0$. This highlights that in simple linear models, with similar linearizations of the Lorentz force and the induction equation (i.e., without more sophisticated treatments of magnetic diffusion and nonlinear effects), one can identify the obstruction of geostrophic circulations that cause hotspot reversals in nonlinear simulations, but not westward hotspot offsets explicitly. This observation is useful in the remainder of this section, where we aim to link the magnetic obstruction of geostrophic circulation patterns to wave dynamics.

11 For $S(y)$ equatorially symmetric and $B_0$ equatorially antisymmetric, $\hat{v}$ is antisymmetric and $[\hat{A}, \hat{v}, \hat{A}]$ are symmetric about the equator (see Appendix A). Hence, $\hat{B} = (\partial A / \partial y - B_0 B_\parallel) / \hat{H}$ is antisymmetric, while, by Equation (18), $A(0) = 0$, so $\hat{B}(0) = -ikA(0) / \hat{H} = 0$. Consequently, $ikB_0 \hat{B} + dB_0 / dy B_\parallel$ and $ikB_0 B_\parallel$ both vanish at the equator.

4.2. Wave Background: Alfvén–Rossby Wave Coupling

Various authors have studied the linear waves present in rotating MHD systems. Early studies, which used quite general (usually uniform) flow/field geometries, focused on the influence that these waves have on the geodynamo (Hide 1966, 1969b; Acheson & Hide 1973). Since the development of SWMHD (Gilman 2000), authors have been able to utilize its reduced geometry to study waves in more specific flow/field geometries. Rotating SWMHD waves have now been studied for a variety of thin-layered astrophysical and geophysical systems, including the geodynamo, the solar tachocline, and neutron star atmospheres (Schecter et al. 2001; Zaqarashvili et al. 2007, 2009; Heng & Spitkovsky 2009; Márquez-Artavia et al. 2017; Zaqarashvili 2018). Of these, the solar tachocline, which is also expected to have an equatorially antisymmetric toroidal dominant magnetic field geometry, can be considered as similar to the hot Jupiter system. Schecter et al. (2001) studied waves in the local regions of the solar tachocline, focusing on regions away from the equator, whereas Zaqarashvili et al. (2007, 2009) studied the global dynamics of these waves for the two extreme cases ($\epsilon \gg 1$ and $\epsilon \ll 1$) of the rotation-stratification parameter, $\epsilon = 4\Omega^2 R^2 / c_s^2$. However, the atmosphere of HAT-P-7b lies in the region of parameter space between these two extremes ($\epsilon \approx 4.8$). Zaqarashvili (2018) studied equatorial SWMHD waves using an equatorial beta-plane model for two purely azimuthal magnetic field geometries: first, uniform, and second, equatorially antisymmetric (latitudinally linear). London (2017) and London (2018) studied some asymptotic solutions of the beta-plane and spherical version of the system, with an equatorially antisymmetric azimuthal field, in certain weak- and strong-field limits, but we wish to study the transition where magnetism becomes
dynamically important. The predictions made in Hindle et al. (2019) were based on the equatorially antisymmetric azimuthal magnetic field results of Zaqarashvili (2018). However, in this section we relax the weakly magnetic assumptions that those analyses take.

Past works of linear waves in rotating MHD systems find that Alfvén waves and Rossby waves are coupled. To illustrate this, we highlight the known local dispersion relations of waves in the nondiffusive, unforced, drag-free SWMHD system, with the uniform azimuthal background magnetic field, \( B_0 = V_{A,0} \hat{x} \), and the generalized plane-treatment \( f = f_0 + \beta y \). In local regions (i.e., if \(|y/R| \ll 1\) and \(|\beta y| \ll |f_0|\)), this linearized system associated with this background state may be approximated solved with the plane wave ansatz: \( \{u, v, h, A\} = \{\hat{u}, \hat{v}, \hat{h}, \hat{A}\} e^{i(kx + ly - \omega t)} \), where hatted variables are constant amplitudes of the plane wave solutions, \( k \) is the azimuthal wavenumber, \( l \) is the latitudinal wavenumber, and \( \omega \) is the oscillation frequency. Seeking solutions that are first order in the Coriolis parameter only yields the following dispersion relation (Zaqarashvili et al. 2007; Heng & Spitkovsky 2009):

\[
\omega^4 - \omega^2(K^2 c_s^2 + 2k^2 V_{A,0} + f_0^2) - \omega \beta kc_s^2 + k^2 V_{A,0}^2 K^2 c_s^2 + k^2 V_{A,0}^2 = 0,
\]

(19)

for arbitrary wave amplitudes, where \( K \equiv (k^2 + l^2)^{1/2} \). For \( V_{A,0} > 0 \), Equation (19) has four solutions, which in the rotation-free limit \( (f_0 = \beta = 0) \) are (Schechter et al. 2001)

\[
\omega^2 = \begin{cases} 
V_{A,0}^2 k^2, \\
\left(c_{m,0}^2 k^2 + c_s^2 l^2\right),
\end{cases}
\]

(20)

where \( c_{m,0} = c_s + V_{A,0}^2 \) is the magnetogravity wave speed for a constant background magnetic field. The first pair of solutions are Alfvén waves, which are driven by magnetic tension and travel parallel to the background magnetic field; the second pair of solutions are magnetogravity waves, which propagate horizontally to restore pressure gradients and magnetic tension. For rotationally modified waves, we follow Schechter et al. (2001) by labeling the rotationally modified Alfvén waves as slow “Alfvén branch” solutions and the rotationally modified magnetogravity waves as fast “magnetogravity branch” solutions.

In the fast-wave limit \((|\omega|/2|\Omega| \gg 1)\), to leading order, fast magnetogravity branch solutions satisfy (Heng & Spitkovsky 2009, but with \( B_0 = V_{A,0} \hat{x} \))

\[
\omega^2 \approx \frac{K^2 c_s^2}{2} + k^2 V_{A,0} + \frac{f_0^2}{2} + \frac{1}{2} \sqrt{K^2 c_s^2 (K^2 c_s^2 + 2f_0^2) + f_0^2 (f_0^2 + 4k^2 V_{A,0}^2)},
\]

(21)

These two magnetogravity branch solutions travel in opposite directions in order to restore pressure gradients and magnetic tension, but with a Coriolis modification. Solutions of this kind are known as magneto-Poincaré waves (as they reduce to Poincaré waves for \( V_{A,0} = 0 \); e.g., Heng & Spitkovsky 2009) or magneto-inertial gravity waves (and inertial gravity waves in hydrodynamics; e.g., Márquez-Artavia et al. 2017; Zaqarashvili 2018). We choose the inertial gravity and magneto-inertial gravity nomenclature (IG and MIG hereafter). The independence of Equation (21) on \( \beta \) highlights that IG/MIG solutions do not generally have a leading-order dependence on \( \beta \) (and exist on the f-plane; e.g., Vallis 2006).

In the slow-wave \((|\omega|/|2\Omega| \ll 1)\) limit, the dispersion relation evaluated at the equator \((f_0 = 0 \text{ and } \beta = 2\Omega/R)\) yields the two Alfvén branch solutions (Heng & Spitkovsky 2009, but with \( B_0 = V_{A,0} \hat{x} \)):

\[
\omega = -\frac{\beta k c_s^2 + \sqrt{(\beta k c_s^2)^2 + M}}{2(c_m^2 + V_{A,0}^2 k^2 + c_s^2 l^2)},
\]

(22)

where \( M = 4k^2 V_{A,0}^2 (c_{m,0}^2 k^2 + c_s^2 l^2)((c_{m,0}^2 + V_{A,0}^2 k^2 + c_s^2 l^2)^2) \) is the magnetic component of the numerator. In the limit where these solutions are dominated by the Alfvén speed, these waves are Alfvénic in nature (see by taking \( V_{A,0} \) dominatingly large). Conversely, in the hydrodynamic limit \((V_{A,0} = 0)\) the Alfvén branch solutions reduce to

\[
\omega = \begin{cases} 
-\beta k/(k^2 + l^2), \\
0,
\end{cases}
\]

(23)

so the eastward Alfvén branch solution vanishes and the westward Alfvén branch solution reduces to a Rossby wave.

This Alfvén–Rossby wave coupling is a well-documented feature of MHD in systems with a latitudinally dependent planetary vorticity (Hinde 1966, 1969b; Acheson & Hinde 1973). However, Alfvén and Rossby waves are fundamentally different in nature. Rossby waves arise owing to potential vorticity conservation and the latitudinal variation of the Coriolis parameter. They behave geostrophically and are highly dispersive, so they can transfer energy and angular momentum to the surrounding system (e.g., Pedlosky 2003; Vallis 2006). Conversely, Alfvén waves are nondispersive and travel in a direction aligned with the dominant azimuthal magnetic field geometry. Comparing the oscillation frequency of Rossby (\( \omega_R \)) and Alfvén (\( \omega_A \)) waves gives

\[
|\omega_R/\omega_A| = \beta/V_{A,0}(k^2 + l^2),
\]

(24)

suggesting that, for given choices of \( \beta \) and \( V_{A,0} \), Rossby wave characteristics dominate at large scales, whereas Alfvén wave characteristics dominate at small scales. In Section 3, we showed that reversals on hot Jupiters are closely tied to the zonal alignment of equatorially adjacent geostrophic circulations by equatorially antisymmetric azimuthal magnetic fields. Therefore, to investigate reversals in the zero-amplitude limit, we examine the behavior of equatorial waves as the Alfvén oscillation frequency approaches \( \omega_R \) in magnitude for an antisymmetric azimuthal background magnetic field.

4.3. Equatorial Magnetohydrodynamic Wave Equations

To study the linear equatorial magnetohydrodynamic waves of the system, we linearize the nondiffusive, unforced, drag-free versions of Equations (1) and (4) about the background state, \( \{u_0, v_0, h_0, A_0\} = \{0, 0, H, A_0(y)\} \), where \( H \) is the constant and \( dA_0/dy = H B_0 \) (for \( B_0 = B_0(y) \hat{x} \) in velocity
units). Applying the plane wave ansatz, \( \{ u(t, y, h, A) = (\hat{u}(y), \hat{v}(y), \hat{h}(y), \hat{A}(y)) e^{i(\omega t - \nu y)} \} \), the evolution of the perturbations is determined by the following linearized SW/H system:

\[
-\i \omega \hat{u} = f \hat{v} - i k g \hat{h} + ik B_0 \hat{B}_y + \frac{dB_0}{dy} \hat{B}_y, \tag{25}
\]

\[
-\i \omega \hat{v} = -f \hat{u} - g \frac{d\hat{h}}{dy} + ik B_0 \hat{B}_y, \tag{26}
\]

\[
-\i \omega \hat{h} = -H \left( ik \hat{u} + \frac{d\hat{v}}{dy} \right), \tag{27}
\]

\[
-\i \omega \hat{A} = -HB_0 \hat{v}, \tag{28}
\]

where \( \hat{B}_y = (dA/\mathrm{d}y - B_0 \hat{h})/H \) and \( \hat{B}_y = -i k \hat{A}/H \). From this we eliminate \( \hat{u}, \hat{h}, \hat{A}, \hat{B}_y, \) and \( \hat{B}_y \) to obtain the single ordinary differential equation:

\[
\mathcal{L} \{ \hat{v} \} \equiv F_1 \frac{d^2 \hat{v}}{dy^2} + F_2 \frac{d\hat{v}}{dy} + F_3 \hat{v} = 0, \tag{29}
\]

for the latitudinal solving domain, \(-L_y < y < L_y\), with

\[
F_1 = (\omega^2 - B_0^2 k^2)(\omega^2 - c_m^2 k^2), \tag{30}
\]

\[
F_2 = 2B_0 \frac{dB_0}{dy} c_n^2 k^4, \tag{31}
\]

\[
F_3 = \frac{(\omega^2 - c_m^2 k^2)^2}{c_n^2} \left[ \frac{(\omega^2 - c_m^2 k^2)(\omega^2 - B_0^2 k^2)}{\omega^2 - c_n^2 k^2} \right] - \omega^2 f^2 + \omega \frac{df^2}{dy} \frac{c_n^2}{c_m^2} - 2 \omega \frac{dB_0}{dy} \frac{dB_0}{dy} k^3, \tag{32}
\]

where \( c_m(y) \equiv (c_n^2 + B_0^2)^{1/2} \) denotes the (rotationless) magnetogravity wave speed. This system can contain singular points at \( y = y_i \), if \( \omega = \pm B_0(y_i)k \) (Alfvén singularity) or \( \omega = \pm c_m(y_i)k \) (magnetogravty singularity), which we label based on the \( \omega \)-regions each singularity is associated with.

If one attempts to write \( \mathcal{L} \) in Sturm–Liouville form, 13 through use of an integrating factor, it is found that the highest-order functional coefficient of the Sturm–Liouville operator, \( p = (\omega^2 - B_0^2 k^2) / (\omega^2 - c_m^2 k^2) \), is not independent of the oscillation frequency. Therefore, the desirable properties of the Sturm–Liouville eigenvalue problem (e.g., real eigenvalues and orthogonality of eigenfunctions) are not generally guaranteed. Zaqarashvili (2018) studied this system in the weakly magnetic limit where singular points do not influence the planetary-scale waves. 14 In this approximation \( \mathcal{L} \) can be reexpressed in terms of the parabolic cylinder Sturm–Liouville operator (the hydrodynamic version of \( \mathcal{L} \); see Matsuno 1966). Therefore, away from singular \( \omega \)-regions, where the approximations of Zaqarashvili (2018) hold, one may expect solutions to conform to Sturm–Liouville properties (which we find in the following analysis).

### 4.4. Equatorial Wave Solving Method

We now examine nontrivial eigenvalue–eigenfunction pairs, \( \{ \omega, \hat{v}(y) \} \), that satisfy \( \mathcal{L} \{ \hat{v} \} = 0 \) everywhere in the latitudinal domain, \(-L_y < y < L_y\), subject to impermeable boundary conditions (i.e., \( \hat{v}(\pm L_y) = 0 \)). We use the planetary parameters discussed in Section 2, \( f = \frac{\partial y}{\partial t} \) and \( B_0 = V_A e^{1/2} \tan(\gamma/\Gamma_{eq}) \). This \( B_0(\gamma) \) choice is useful because it is both monotonic and bounded as \( y \to \infty \) (London 2017), so there is at most one Alfvén singularity in each hemisphere. For this \( B_0(\gamma) \) choice, solutions with \( c_k \leq |\omega| \leq (c_n^2 + V_A^2)^{1/2} k \) have magnetogravity singularities, while solutions with \( |\omega| \leq V_A e^{1/2} k \) have Alfvén singularities. We seek wave-like solutions with the planetary-scale azimuthal wavenumber, \( k = 1/R \). We find that solving this eigenvalue problem, without further approximation on \( \mathcal{L} \), is an analytically intractable problem, so we use a semianalytic approach.

Since \( \mathcal{L} \) is symmetric about the equator, homogeneous solutions will be either symmetric (\( \hat{v} \) symmetric and \( \hat{u}, \hat{h}, \hat{A} \) antisymmetric) or antisymmetric (\( \hat{v} \) antisymmetric and \( \hat{u}, \hat{h}, \hat{A} \) symmetric) about the equator. 15 Although the system we solve here is unforced, we wish to compare solutions to the numerical simulations of Section 3, which had equatorially symmetric forcing on \( h \). Therefore, we only consider antisymmetric homogeneous solutions and solve \( \mathcal{L} \{ \hat{v} \} = 0 \) in the upper-half domain, \( 0 < y < L_y \), with the antisymmetric lower boundary condition \( \hat{v}(0) = 0 \), which replaces \( \hat{v}(-L_y) = 0 \). Eigenfunctions are defined up to a constant factor, so a third and final normalization boundary condition must also be included. We set \( \bar{d}v/\bar{d}y|_{y=0} = N \), where \( N \) is a normalization constant chosen for numerical convenience, and take \( L_y = 5L_{eq} \) to ensure that boundary influences are negligible.

We use a shooting method to seek eigensolutions. The shooting method calculates successive “shots” (or test solutions, \( \hat{v}_T \)) for given test frequencies, \( \omega_T \), where each shot satisfies \( \mathcal{L} \{ \hat{v}_T \} = 0 \), subject to two of the three boundary conditions. The third boundary condition is then satisfied by varying \( \omega_T \) so that the deviation from the third boundary condition, \( G(\omega_T) \), vanishes.

If the system has no singular points, shots are carried out by the inversion of the tridiagonal matrix that corresponds to Equation (29), with finite-difference discretizations, such that the lower boundary conditions are satisfied. We find that magnetogravity singularities are false singularities (i.e., \( \mathcal{L} \) is singular but solutions are not; see Appendix B), so, for \( V_A > 0 \), solutions in the magnetogravity singularity \( \omega \)-range can also be treated as regular everywhere. For solutions in the Alfvén singularity \( \omega \)-range, we construct Frobenius power series solutions in the singular region (see Appendix B) and fix constants of integration by shooting into and matching with the \( y = 0 \) boundary conditions, before finally shooting toward \( y = L_y \) to obtain \( G(\omega_T) \). Solutions are then checked via back-substitution.

As discussed above, Sturm–Liouville theory only guarantees real eigenvalues in the weakly magnetic limit. Therefore, we examine convergence for complex test frequencies, which have \( G = G_r + iG_i = 0 \) for \( G_r, G_i \in \mathbb{R} \). We find that \( G_r(G_i) \) is antisymmetric about \( \omega_i = 0 \), with contours \( G_r = 0 \) and \( G_i = 0 \) crossing exclusively on the real line, so \( \omega \in \mathbb{R} \). We find the position of eigensolutions on the real line using the bracketed Newton–Raphson method discussed in Press et al. (1992).

### 4.5. Free Wave Eigensolutions

We label nonsingular eigensolutions with a meridional mode number, \( n \), based on the hydrodynamic convention. Generally,
when the domain is finite and large enough, magnetic eigenfunctions for solutions without singularities are qualitatively similar to their hydrodynamic counterparts and $n$ is the number of internal points where $\hat{v}(y) = 0$ in $-L_y < y < L_y$. However, hydrodynamic Kelvin solutions have the property $\hat{v} = 0$ everywhere and so represent a special case. They are typically labeled with the meridional mode number $n = -1$, with $\psi_{-1} = 0$ (Matsuno 1966). We find that solutions with $c_gk \lesssim |\omega| \lesssim (c_g^2 + V_A^2)^{1/2}k$ are the magnetic versions of Kelvin solutions, so we label them with $n = -1$ for consistency, although we find that they have small nonzero $\hat{v}$ (see below). For hydrodynamic and weakly magnetic systems there are three solutions for each $n \geq 1$: one equatorial Rossby/magneto-Rossby solution, one westward equatorial IG/MIG solution, and one eastward equatorial IG/MIG solution. When magnetism is included, another two sets of solutions (one east; one west), with $|\omega| \leq V_Ae^{1/2}k$, emerge. These solutions, which have Alfvén singularities (where $\omega^2 = B_0(y)^2k^2$), differ significantly from regular equatorial wave solutions (see below). For convenience, we label these with a meridional mode number, $n$, determined by the scale of latitudinal variations in $\hat{v}$ (for $n = 1, 3, 5$, $\hat{v}$ is plotted in Figure 7). In Table 2 we present oscillation frequencies, $\omega$, for the $n = 1, n = 3$, and $n = -1$ free wave eigensolutions, with each row representing a specific type of equatorial wave (see caption). We present the oscillation frequencies for $V_A = 0$, $V_A = 0.15c_g/R$, and $V_A = 0.2c_g/R$, and, in cases where eigenfunctions are finite everywhere, we plot the corresponding free wave eigenfunctions for the equatorial $n = 1$ and $n = -1$ waves in Figure 8.

Eastward and westward equatorial IG/MIG solutions are the system’s most rapidly oscillating waves (with $|\omega| > c_gk$). The azimuthal background magnetic field slightly increases the phase speed of the IG modes (see Table 2). However, their energy redistribution patterns remain qualitatively similar to their hydrodynamic IG counterparts (see Figure 8, rows 1 and 3).

| $n$ | Solution Type | $\omega/(c_g/R)$ for $V_A = 0$ | $\omega/(c_g/R)$ for $V_A = 0.15c_g$ | $\omega/(c_g/R)$ for $V_A = 0.2c_g$ |
|-----|---------------|-------------------------------|-----------------------------------|-----------------------------------|
| 1   | WIG/          | -2.57                         | -2.61                             | -2.62                             |
|     | WMIG          |                               |                                   |                                   |
| 1   | R/MR          | -0.293                        | -0.326                            |                                   |
| 1   | EIG/EMIG      | 2.89                          | 2.90                              | 2.91                              |
| 1   | WA $^b$       | $^b$                          | -0.117$^b$                        | -0.142$^b$                        |
| 1   | EA $^b$       | $^b$                          | 0.0329$^b$                        | 0.0556$^b$                        |
| 3   | WIG/          | -3.98                         | -3.99                             | -4.00                             |
|     | WMIG          |                               |                                   |                                   |
| 3   | R/MR          | -0.134                        | $^a$                              | $^a$                              |
| 3   | EIG/EMIG      | 4.11                          | 4.12                              | 4.13                              |
| 3   | WA $^b$       | $^b$                          | -0.161$^b$                        | -0.201$^b$                        |
| 3   | EA $^b$       | $^b$                          | 0.0640$^b$                        | 0.102$^b$                        |
| -1  | K/MK          | 1                             | 1.01                              | 1.01                              |
| -1  | BK/BMK        | -1                            | -1.03                             | -1.05                             |

Notes. In the “Solution Type” column we use the following shorthands: R/MR denotes Rossby/magneto-Rossby solutions, WIG/WMIG denotes westward inertial gravity/magneto-inertial gravity solutions, EIG/EMIG denotes eastward inertial gravity/magneto-inertial gravity solutions, WA denotes (singular) westward Alfvén solutions, EA denotes (singular) eastward Alfvén solutions, K/MK denotes equatorial Kelvin/magneto-Kelvin solutions, and BK/BMK denotes boundary Kelvin/magneto-Kelvin solutions.

$^a$ Empty entries indicate that no solution exists for this $V_A$ value.

$^b$ Solutions with Alfvén singularities (see text).

![Figure 7](https://example.com/figure7.png)

**Figure 7.** The velocity profiles of the first few singular free wave eigenfunctions are plotted for $V_A = 0.2c_g/R$ and $k = 1/R$. Magnetic systems have two sets of singular solutions, one westward traveling and one eastward traveling, which have Alfvénic properties (see main text). $\hat{v}$ (blue) and $\hat{u}$ (red) are, respectively, purely real and purely imaginary for the normalization we apply. We mark asymptotes at $y = \pm y_s$ with dotted black lines. The solutions are labeled with the latitudinal mode number, $n$, based on the latitudinal dependence of $\hat{v}$. The corresponding profiles for $V_A = 0.15c_g/R$ are qualitatively identical.

Kelvin/magneto-Kelvin solutions are characterized by zonally dominated winds. The are two hydrodynamic Kelvin solutions: an eastward equatorial Kelvin solution, with $\omega = c_kk$, $\hat{v} = 0$, $\{\hat{u}, \hat{h}\} \propto \exp(-y^2/2L_{eq})$, and a westward boundary Kelvin solution, with $\omega = -c_kk$, $\hat{v} = 0$, $\{\hat{u}, \hat{h}\} \propto \exp(y^2/2L_{eq})$. These

---

16 The westward boundary Kelvin solution is removed when the condition is $|\hat{u}| \to 0$, as $|y| \to \infty$ is imposed (Matsuno 1966).
hydrodynamic solutions are special cases of magneto-Kelvin eigensolutions, which have \( c_m k \leq |\omega| \leq (c_m^2 + \nu A)^{1/2} k \). While hydrodynamic Kelvin solutions have \( \hat{v} = 0 \) everywhere, we find that magneto-Kelvin solutions acquire a nonzero \( \hat{v} \) in order to maintain latitudinally independent oscillation frequencies. This can be understood by combining Equations (25), (27), and (28) to yield

\[
(\omega^2 - c_m^2 k^2) \hat{u} = i\omega \hat{v} - ikc_m^2 \frac{d\hat{v}}{dy}.
\]  

For hydrodynamic Kelvin solutions, the left- and right-handsides of Equation (33) are identically zero throughout the domain, whereas magneto-Kelvin solutions have \( c_m k \leq |\omega| \leq (c_m^2 + \nu A)^{1/2} k \), \( \{\hat{u}, \hat{h}\} \) similar to their hydrodynamic counterparts, and a nonzero \( \hat{v} \) that ensures that Equation (33) remains balanced. Like in the hydrodynamic limit, we find two magneto-Kelvin solutions: an eastward equatorial magneto-Kelvin solution and a westward boundary magneto-Kelvin solution. Magnetism causes both varieties to have a small nonzero meridional velocity component \( (|\hat{v}|/|\hat{u}| \ll 1) \) and an increased \( |\omega| \), but both are characteristically similar to their hydrodynamical counterparts. For the equatorially magneto-Kelvin solution, this is illustrated in Figure 8, which shows that its energy redistribution pattern remains qualitatively similar as \( V_A \) is increased.

In the hydrodynamic version of the system, equatorial Rossby solutions propagate westward and oscillate slowly \( (|\omega| < c_m k) \), with their azimuthal phase speeds, \( |\omega|/k \), successively decreasing for larger \( n \) solutions. In the hydrodynamic limit, the structures of equatorial Rossby solutions are characterized by mid-to-high-latitude geostrophic vortices (see Figure 8, row 2, left column). For weakly magnetic equatorial magneto-Rossby solutions, we find that the presence of the azimuthal background magnetic field has little effect on the form of the waves’ eigensolutions, which are magnetogeostrophic in nature. Weakly magnetic solutions adjust to the contribution of magnetic tension with small increases to their azimuthal phase speeds. However, when their oscillation frequencies are exceeded by the maximal background azimuthal Alfvén frequency (i.e., when \( V_A \geq c_A^{-1/2} |\omega|/k \)), equatorial magneto-Rossby solutions enter the \( \omega \)-range of Alfvén singularities and are removed from the system. Higher-\( n \) equatorial magneto-Rossby solutions are removed for the weakest \( V_A \) values, before successively lower-\( n \) solutions are removed for larger \( V_A \) values (as Alfvénic properties become dynamically important at larger and larger scales). We attribute the removal of the planetary-scale equatorial magneto-Rossby solutions to the breaking of potential vorticity conservation in regions of large Lorentz force.

The shallow-water hydrodynamic definition of potential vorticity is \( q := (\partial \omega/\partial x - \partial \hat{u}/\partial y + f) \) (e.g., Vallis 2006). In the nondiffusive, unforced, drag-free version of the SWMHD model, the potential vorticity evolution satisfies

\[
\frac{dq}{dt} = \frac{1}{h}[\nabla \times (J \times B)] \cdot \hat{z},
\]  

where \( J := (\partial B_x/\partial x - \partial B_y/\partial y) \hat{z} \). Equation (34) shows that the curl of the Lorentz force generated by the horizontal magnetic field component generally prevents potential vorticity

![Figure 8](90x421 to 522x739)

**Figure 8.** The regular equatorial \( n = 1 \) (rows 1–3) and \( n = -1 \) (row 4) free wave eigensolutions (geopotential contours with overlaid velocity vectors) are plotted for \( V_A = 0, V_A = 0.15 c_g \), and \( V_A = 0.2 c_g \), taking \( k = 1/R \). We label rows according to their wave types (see Table 2). Solutions are calculated for \(-5L_{eq} < y < 5L_{eq} \) but are cut off for \(-R \pi/2 < y < R \pi/2 \) \((L_{eq}/R \approx 0.67)\).
conservation in the magnetic limit.\textsuperscript{17} Since the material conservation of potential vorticity is essential to the propagation mechanism of Rossby waves (see, e.g., Vallis 2006), in regions of large Lorentz force their generation is inhibited.

In magnetic systems, two additional sets of solutions emerge. These solutions have $|\omega| \leq V_A^2 k^2$ and so contain singularities, yet present some distinguishable properties of Alfvén waves. Specifically, they arise in both eastward and westward traveling varieties, and $|\omega|$ increases with $V_A$ and $n$. To assess their nature as $y \rightarrow y_c$ (for $\omega^2 = B_0(y_c)^2 k^2$), we use the Frobenius solutions discussed in Appendix B. In singular regions, $\omega = O((y - y_c)/L_{eq})$, so by Equation (33) $u = O((y - y_c)/L_{eq})^{-1}$, meaning that $|u/\omega| \rightarrow \infty$ as $y \rightarrow y_c$. This highlights that Alfvén singularities cause a wave barrier to emerge at $y = y_c$, over which wave-driven meridional energy/momentum transport mechanisms cannot cross. Since they are not finite everywhere, equatorial wave structures with Alfvén singularities cannot determine global energy redistribution in the same way that planetary-scale equatorial waves do in hydrodynamic hot Jupiter models. Hence, in the limit where magnetism becomes significant, dissipative and nonlinear effects become essential for understanding equatorial dynamics. A nonsingular analog of these solutions could be present in systems that include these extra physical processes, but since we are focused on the breakdown of geostrophic balance, we do not investigate solutions of this kind further.\textsuperscript{18}

Thus far, we have discussed magnetic free wave solutions about a flat rest state. However, Tsai et al. (2014) and Debras et al. (2020) find that the redistributing properties of waves can be altered by the presence of a background zonal flow, though the fundamental characteristics of these waves remain unchanged. Compared to the system we have so far explored, taking $u_0 = U_0$ constant (as in Tsai et al. 2014) simply manifests itself in the trivial phase translation $\omega \rightarrow \omega - U_0 k$, where $\omega$ and $\omega'$ are oscillation frequencies for a background at rest and a background with a zonal flow, respectively. For this translation, Alfvén singularities emerge where $B_0(y_c)^2 k^2 = |\omega|^2 - U_0 k^2 \approx \omega^2$, which is the same condition as the rest case. We have also considered solutions about the latitudinally dependent background state, $u_0 = u_0(y)$, finding that Alfvénic singularities, with similar Frobenius solution dependencies, emerge at points where $B_0(y_c)^2 k^2 = (\omega^2 - u_0(y_c) k^2)$.\textsuperscript{18}

4.6. Comparisons with Nonlinear Simulations

Our findings concerning Alfvén–Rossby wave coupling in an equatorial beta-plane model, with an equatorially antisymmetric azimuthal background magnetic field, are consistent with our developed theory of hotspot reversals from the simulations of Section 3. In the hydrodynamic limit, planetary-scale geostrophic circulations associated with equatorial Rossby waves are free to recirculate energy between the equatorial and mid- to high latitudes in a manner described by Showman & Polvani (2011). In the weakly magnetic limit, planetary-scale circulations remain largely unchanged, with equatorial magneto-Rossby waves only altering slightly to account for the magnetic contribution to their magnetogeostrophic circulations. However, at a critical threshold magnetic tension becomes large enough to inhibit the magnetogeostrophic circulations associated with equatorial magneto-Rossby waves. This is the free wave manifestation of the obstruction of geostrophic circulations, which we identified as the trigger for hotspot reversals in Section 3. Here the analogy between global circulations and the standing wave description of linear steady-state solutions described by Showman & Polvani (2011) breaks down and the force balance description used in Section 3 is preferable. In Section 3, we saw that the meridional Lorentz force responsible for obstructing geostrophic circulations always has a corresponding westward component that, ultimately, results in hotspot reversals. Together, the developed theory of Sections 3 and 4 can be used to place a zero-amplitude limit on the reversal threshold, $V_{A,crit}$. In linear theory, magnetic tension inhibits the propagation of equatorial Rossby waves, with the oscillation frequency

$$\omega_{R,n} = \frac{-\beta k}{k^2 + (2n + 1)\beta/e_g},$$

when $\omega_{A,max} \geq |\omega_{R,n}|$, where $\omega_{A,max} = B_{A,max} k = V_A^2 k$ is the maximal Alfvén frequency. Our findings suggest that when the slowest (largest $n$) equatorial Rossby wave that is important for supporting the planetary-scale mid-to-high-latitude geostrophic balance becomes inhibited by magnetic tension, geostrophic circulations are obstructed and hotspots are driven westward by the resulting zonal Lorentz force.

In Figure 5, we have overplotted the theoretical thresholds associated with the obstruction of the $n = 1, 3, 5$ equatorial Rossby solutions, for comparison with the zero-amplitude ($\Delta h_{eq}/H \rightarrow 0$) limits of the simulated reversal thresholds, $V_{A,crit}$. We generally find acceptable agreement between the simulations and these theoretical criteria, noting that in the most physically relatable case, where $\tau_{rad}$ is short, reversals occur at the point where the $n = 1$ equatorial Rossby wave is overcome by magnetic tension. When $\tau_{rad}$ and $\tau_{drag}$ act over longer timescales, Figure 5 suggests that the obstruction of geostrophic circulations is associated with the loss of larger-$n$ equatorial Rossby solutions. This is somewhat consistent with the standing wave description of linear steady-state hydrodynamic solutions, as geostrophic circulations in solutions with longer $\tau_{rad}$ and $\tau_{drag}$ timescales are located at higher latitudes (see, e.g., Figure 1) and so require contributions to their energy recirculation patterns from larger-$n$ equatorial Rossby waves (e.g., Matsumo 1966; Showman & Polvani 2011; Tsai et al. 2014). While a wave analysis with nonlinear effects and diffusion may be able to more precisely define these weakly forced limits, we note that this description provides a vast improvement on scaling predictions of typical toroidal field strengths on hot Jupiters, which have order-of-magnitude (or larger) uncertainties (discussion in Section 5).

5. The Magnetic Reversal Mechanism

In Section 3, we identified the mechanism that drives magnetic hotspot reversals in SWMHD simulations of hot

\textsuperscript{17} Further, Dellar (2002) showed that potential vorticity has no materially invariant counterpart in SWMHD.

\textsuperscript{18} In the very strong field limit, London (2017) identified “outer band” solutions akin to these Alfvénic solutions that were trapped in polar regions in linear nondiffusive beta-plane systems, but they concluded that they do not have a finite global (linear, nondiffusive) counterpart in London (2018). Spherical (linear, nondiffusive) SWMHD waves studies in other geometries have found additional slow magneto-Rossby (Márquez-Artavia et al. 2017) and magnetostricthrophic (Heng & Spitkovsky 2009) type waves at the poles of shallow-water systems, which may be useful in explaining the dynamics of the polar MHD flows. Márquez-Artavia et al. (2017) also found polar trapping of the “fast” magneto-Rossby solutions, which can plausibly be related to the removal of equatorial magneto-Rossby solutions (i.e., magneto-Rossby waves could become confined to regions of the atmosphere less influenced by magnetism).
Jupiters. We provide a schematic and summarized explanation of the mechanism in Figure 9 and its caption. This mechanism is also relevant for other, less idealized, magnetic field geometries. The reversal mechanism requires two features in the azimuthal field geometry:

1. large $|B_x|$ at mid- to high latitudes to block the circulation of the energy transporting geostrophic flows, and
2. smaller or zero $|B_x|$ at equatorial latitudes, so that when magnetic field lines are bent into the equatorial region (by the mid-to-high-latitude circulations), they pass into regions of smaller $|B_x|$, generating a westward Lorentz force acceleration. This suggests that, as long as the profile is characterized by these two features, the developed theory does not depend on exact antisymmetry in the dominant magnetic field geometry. This observation is useful when comparing to the 3D MHD simulations of Rogers & Komacek (2014) and Rogers (2017), which are characterized by antisymmetrically dominant, but not exactly antisymmetric, toroidal magnetic field geometries.

**5.1. Hotspot Reversal Criterion**

In Sections 3 and 4, we identified two physically motivated reversal criteria on the Alfvén speed. The azimuthal Alfvén speed is defined as $V_A = B_0 / \sqrt{\mu_0 \rho}$, where $\mu_0$ and $\rho$ are the permeability of free space and the density, respectively. Taking $c_g = \sqrt{RT}$ (see Section 2) and applying the ideal gas law therefore yields $B_g \sim (V_A/c_g) \sqrt{\mu_0 P}$, where $T$ and $P$ are the temperature and pressure, respectively, at which the reversal occurs. From this, we have the following critical reversal criterion on the toroidal field magnitude:

$$B_{g,crit} \approx \sqrt{\mu_0 P} \max \left[ \frac{\beta/c_g}{k^2 + 3\beta/c_g}, \right]$$

$$2\pi R \left( \frac{\Delta h_{eq}}{H} \right) \left( \frac{\tau_{rad}}{\tau_{wave}} \right)^{-1} \left( \frac{2\pi r^2}{\tau_{rad}} + 1 \right)^{-1},$$

**Figure 9.** A schematic of the magnetic reversal mechanism, with gray temperature contours and white magnetic field lines (solid for $B_x > 0$; dashed for $B_x < 0$). (a) In hydrodynamic steady-state solutions, drag-adjusted geostrophic circulations dominate at mid- to high latitudes, whereas zonal-pressure-driven jets dominate at the equator. Hotspots are shifted eastward as these circulations transport thermal energy from the western equatorial dayside to the eastern equatorial dayside, via higher latitudes. (b) In ultra-hot Jupiters, partially ionized winds flow through the planet’s deep-seated magnetic field, inducing a dominant equatorially antisymmetric atmospheric toroidal magnetic field. When field lines are parallel to the equator, magnetic tension is zero, so flows behave hydrodynamically. (c) As the field and flow couple, the geostrophic circulations bend the magnetic field lines poleward on the western dayside and equatorward on the eastern dayside, generating a Lorentz force, $\mathbf{B} \cdot \nabla \mathbf{B}$. The meridional Lorentz force component acts to resist the geostrophic circulations, whereas, since $|B_x|$ is smallest in equatorial regions, the zonal Lorentz force component, $\mathbf{B} \cdot \nabla B_x$, is westward in hotspot regions, where field lines bend equatorward (and vice versa where field lines bend poleward). (d) Beyond a magnetic threshold, the system’s nature changes. The meridional Lorentz force obstructs the circulating geostrophic winds, causing zonal wind alignment. This confines thermal structures and blocks the hydrodynamic transport mechanism. The zonal Lorentz force accelerates winds westward in the hottest dayside regions, causing a net westward dayside temperature flux. This drives the hottest thermal structures westward, until zonal pressure gradients can balance the zonal Lorentz force.
where \( n = 1 \) (largest-scale Rossby wave) and \( \kappa \approx 1 \) (\( B_{\phi} \) approaches maximal amplitudes close to the equator, as in Rogers & Komacek 2014) have been taken. This criterion quantifies the toroidal field magnitude sufficient to obstruct geostrophic circulations, with the first term in the maximum relating to when the toroidal field inhibits the propagation of the largest-scale equatorial Rossby wave (in the small \( \Delta h_{eq}/H \) limit).

Further, if the electric currents that generate the planet’s assumed deep-seated dipolar field are located far below the atmosphere, Menou (2012) argued that the toroidal and dipolar field magnitudes should be related by the scaling law: \( B_{\phi} \sim R_{m} B_{dip} \), where \( R_{m} = U_{\phi} H / \bar{\gamma} \) is the magnetic Reynolds number and \( U_{\phi} \) is the magnitude of zonal wind speeds. We use the toroidal field criterion and apply \( B_{\phi} \sim R_{m} B_{dip} \) to quantitatively compare the predictions of SWMHD theory to the 3D MHD simulations of Rogers & Komacek (2014) and Rogers (2017).

### 5.2. Comparisons between SWMHD and 3D MHD

#### 5.2.1. Linking Hotspot and Wind Reversals

Thus far, we have considered hotspot reversals, rather than the reversal of zonal-mean zonal winds, \( \bar{u} \). Though time-correlated in 3D MHD models (Rogers 2017), hotspot and wind reversals are not necessarily synonymous. While thermal/wind structures and geopotential/wind structures compare well between hydrodynamic shallow-water and 3D models (e.g., Perez-Becker & Showman 2013; Komacek & Showman 2016), Debras et al. (2020) found that a consistent treatment of the vertical component of the eddy-momentum flux (i.e., the vertical Reynolds stress) is critical to the development of equatorial superrotation (\( \bar{u} > 0 \)).

In hydrodynamic models of hot Jupiters, equatorial superrotation emerges from the momentum transport mechanism of Showman & Polvani (2011). Showman & Polvani (2011) noted that the necessity for such a mechanism is a consequence of an angular momentum conservation theorem arising from Hide (1969a), which implies that equatorial superrotation can only be maintained if driven by an up-gradient angular momentum pumping mechanism. Showman & Polvani (2011) showed that this up-gradient mechanism is provided by the same geostrophic circulations that result in eastward hotspots. Therefore, since we have shown that magnetically driven hotspot reversals are caused by the obstruction of these recirculation patterns, Hide’s theorem provides an anti-theorem, which implies that the magnetically driven hotspot reversals are accompanied by a disruption of superrotation.

The realization of this anti-theorem can be identified in 3D MHD simulations. These found that mid-to-high-latitude vortical structures zonally align and, consequently, the transport of eastward eddy-momentum (horizontal Reynolds stress) from midlatitudes into equatorial regions is reduced at atmospheric depths where reversals occur (compare Figures 2, 9, and 11 in Rogers & Komacek 2014). Rogers & Komacek (2014) found that, when the up-gradient horizontal Reynolds stress component diminishes, westward equatorial zonal-mean zonal accelerations are driven by the remaining down-gradient momentum transport components (i.e., the vertical Reynolds stress and the Maxwell stresses). Thus, the above application of Hide’s theorem provides a meaningful connection between wind reversals and the magnetically driven hotspot reversal mechanism we have presented.

#### 5.2.2. Wave Dynamics and Turbulence

While we have not modeled turbulence in this work, actual planetary flows are expected to be highly turbulent. In hydrodynamic planetary systems, wave arguments have historically proved useful for developing understanding of geostrophic turbulence and how its conservation properties relate to eddies. Specifically, potential vorticity conservation is fundamental for both Rossby wave propagation and geostrophic turbulence, so Rossby wave properties can be used to understand the structures of planetary-scale turbulence (e.g., Rhines 1975; Vallis 2006). Rogers & Komacek (2014) found that the relationship between zonal jets and magnetic fields in 3D MHD simulations shared intermittent features with MHD turbulence on a beta-plane that were identified by Tobias et al. (2007). Hydrodynamic geostrophic turbulence and MHD beta-plane turbulence have very different characteristics. Among them, the wave–wave/flow–flow interactions associated with the inverse cascade of geostrophic turbulence are replaced with interactions that result in a forward MHD cascade, with MHD interactions occurring over scales on (and below) the planetary-scale when the azimuthal Alfvén wave frequencies exceed the planetary-scale Rossby wave frequency (Diamond et al. 2007). This turbulence condition is remarkably similar to the hotspot reversal criterion we identified in the weakly forced regime, which was motivated by wave dynamics and the findings of nonturbulent SWMHD simulations. We attribute this kinship to the breaking of potential vorticity conservation in MHD models in regions of large horizontal Lorentz force, which inhibits geostrophic characteristics such as Rossby wave propagation (as discussed in Section 4). We also highlight that forcing and drag models generate potential vorticity sources/sinks, so potential vorticity conservation is modified when drag and forcing treatments are strong, which is why reversal thresholds deviate from this simple criterion in the strongly forced limit.

#### 5.2.3. Magnetic Field Evolution and Structure

After the initial hotspot transition, long-term temporal differences between SWMHD and 3D MHD models arise because SWMHD can only model the planetary dipolar field or the atmospheric toroidal field self-consistently (see Section 2.2), meaning that it cannot take into account toroidal field induction from reversed zonal winds passing through the planetary dipolar field. If a strong toroidal field can be maintained indefinitely, the shallow-water theory predicts completely reversed winds, even in 3D models. However, at the onset of the wind reversals, the induction caused by the reversed winds flowing through the deep-seated magnetic field will result in a reduction of the atmospheric toroidal field’s magnitude. Hence, while the quasi-steady magnetically driven wind reversals of SWMHD are useful for modeling the reversal process, in reality one would expect to see oscillatory wind variations as toroidal fields successively strengthen and weaken in a wind-up–wind-down cycle of the toroidal magnetic field. Wind variations of this kind can be both observationally inferred from the oscillating peak brightness offsets of HAT-P-7b (Armstrong et al. 2016) and...
directly measured in 3D MHD simulations of the HAT-P-7b parameter space (Rogers 2017). This in itself has the interesting consequence that the reversal mechanism may provide a saturation process for the atmospheric toroidal magnetic field.

Due to the density dependence of the Alfvén speed, $B_{\text{crit}}$ has a $\sim P^{1/2}$ pressure dependence (see Equation (36)). This explains why Rogers & Komacek (2014) and Rogers (2017) found that reversals first emerge in the upper atmosphere, but emerge at deeper depths for stronger field strengths. Furthermore, if the reversal mechanism is a toroidal field saturation process (as discussed above), $B_\phi$ should not greatly exceed $B_{\text{crit}}$. Hence, $B_\phi$ should decrease above the deepest region where reversals occur (since $B_{\phi,\text{crit}}$ decreases upward), which is a feature of the toroidal field profiles found in Rogers & Komacek (2014), though other processes may also cause an upward reduction in $B_\phi$. Comparing the geometry of the toroidal fields in the quasi-steady reversed SWMHD solutions with those in oscillating 3D MHD solutions is difficult. However, when the toroidal field is approaching criticality in strength, we do find similarities between our toroidal field geometries and those of Rogers & Komacek (2014). In both models the equatorially antisymmetric toroidal fields couple to mid-to-high-latitude circulations in a manner that bends them toward the equator from west to east, which we showed is a geometry that results in westward Lorentz force accelerations (see Section 3).

### 5.2.4. Quantitative Comparisons with 3D MHD

In Table 3, we compare predictions of the reversal criterion to magnetic field strengths in three 3D MHD simulations: M7b1 and M7b2 of Rogers & Komacek (2014), and the HAT-P-7b model of Rogers (2017), all of which display wind reversals at some critical pressure depth, $P_{\text{crit}}$. In these estimates, we take $T_{\text{eq}} = T$, $\Delta T = T_{\text{day}} - T$, and $\tau_{\text{rad}} = \tau_{\text{wave}}$ and set $\Delta h_{\text{eq}}/H = \Delta T/T_{\text{eq}}$. For comparisons to the simulations of Rogers & Komacek (2014), we compare $B_{\phi,\text{crit}}$ to $|B_\phi|$, the horizontally averaged toroidal field component at the end of the run; whereas for the HAT-P-7b simulation of Rogers (2017), we estimate the critical dipolar field strength at the atmospheric base, $B_{\text{dip,\text{crit,base}}}$. This is calculated using the $B_\phi \sim R_\oplus B_{\text{dip}}$ scaling law of Menou (2012). We take $\eta = 2 \times 10^6$ m$^2$ s$^{-1}$ and $U_\phi \sim 10^2$ m s$^{-1}$ from 3D simulations, to yield $B_{\text{dip,\text{crit}}} \approx 4.3$ G at $P = 1$ mbar. Then, noting that the 3D simulations have an atmospheric base located at $r = 0.15$, yields $B_{\text{dip,\text{crit,base}}} = 7$ G. We note that the reversal criterion compares reasonably to the magnitude of the horizontally averaged toroidal component field in the simulations of Rogers & Komacek (2014), with uncertainties in $T_{\text{day}}$ bracketing the true $|B_\phi|$ value. This occurs both at $P = P_{\text{crit}}$ and above $P_{\text{crit}}$, supporting the idea of reversals providing a toroidal field saturation process. The prediction of $B_{\text{dip,\text{crit,base}}} = 7$ G lies within the range $3 \leq B_{\text{dip,\text{crit,base}}} < 10$ G identified by Rogers (2017). We note that, while $B_{\phi,\text{crit}}$ has dependencies on $\Delta T/T_{\text{eq}}$ and $\tau_{\text{rad}}$, $\eta$ can vary significantly between the daysides and nightsides of ultra-hot Jupiters (by orders of magnitude). Therefore, current understanding of the connection between toroidal and poloidal fields on hot Jupiters is constrained by large uncertainties in $B_\phi$ which far outweigh uncertainties in the toroidal field criterion that we have developed.

### 6. Discussion

In this work we have explained the atmospheric mechanics of magnetically driven hotspot reversals in hot Jupiters using numerical (Section 3) and semianalytic (Section 4) analyses of an SWMHD model (Section 2), where we have applied parameters based on the ultra-hot Jupiter HAT-P-7b. In Section 5 we used the theory developed throughout this study to identify a criticality criterion and discussed our findings in the context of 3D MHD simulations. This criticality criterion can be used to place physically motivated constraints on the magnetic fields of ultra-hot Jupiters with observed westward hotspots. It also represents the point where hydrodynamic models with Lorentz force mimicking Rayleigh drag treatments should be replaced with self-consistent MHD modeling. In Section 5, we also identified the link between wind reversals and the hotspot reversal mechanism, highlighted relevant shared features between modifications to wave dynamics and atmospheric turbulence, discussed the role of reversals on the magnetic field’s evolution, and made quantitative comparisons between the reversal criterion 3D MHD simulations.

Using the numerical SWMHD simulations, we demonstrated that hotspot reversals occur when equatorially antisymmetric azimuthal components of the magnetic field are strong enough to obstruct the geostrophic circulations that transport energy to the eastern dayside in hydrodynamic models. The magnetic field geometry that results from this obstruction always drives westward Lorentz force accelerations in hotspot regions, causing hotspots to transition from east to west. Using this finding, we identified a reversal criterion for the toroidal field in the strong forcing regime using a simple argument based on the timescales of the two competing processes.

The recent observational drive in exoplanet meteorology provides a timely backdrop around which theories regarding the mechanism of wind/hotspot reversals can be tested and developed. Observational constraints on atmospheric properties continue to improve while a combination of archival data and
differential equation of the form
\[ \mathcal{L}(\hat{v}) = F_1(y) \frac{d^2\hat{v}}{dy^2} + F_2(y) \frac{d\hat{v}}{dy} + F_3(y)\hat{v} = Q(y), \]  
(A1)
where \( F_1(y), F_2(y), \) and \( F_3(y) \) are latitudinally dependent coefficient functions, \( \mathcal{L} \) is the system’s second-order differential operator, and \( Q(y) \) is the system’s source term. We have omitted the exact dependencies of \( F_1(y), F_2(y), F_3(y), \) and \( Q(y) \) for steady forced solutions (due to their cumbersome forms). These can be provided upon reasonable request. If \( S(y) \) and \( u_0(y) \) are symmetric about the equator and \( B_0(y) \) is antisymmetric about the equator, \( \mathcal{L} \) and \( Q \) are respectively symmetric and antisymmetric about the equator.

Solutions of Equation (A1) on \(-L_y < y < L_y\) are obtained by noting that, since \( \mathcal{L} \) and \( Q \) are respectively symmetric and antisymmetric about the equator, inhomogeneous solutions are antisymmetric (i.e., \( \hat{v} \) antisymmetric and \( \hat{u}, \hat{h}, \hat{A} \) symmetric).

Consequently, we solve Equation (A1) in the upper-half domain, \( 0 < y < L_y \), with \( \hat{v}(L_y) = 0 \) (impermeability) and \( \hat{v}(0) = 0 \) (antisymmetry), before reflecting solutions. This reduced boundary value problem is solved by inverting the tridiagonal matrix that corresponds to Equation (A1) with finite-difference discretizations. We fix the equatorial boundary condition \( \hat{v}(0) = 0 \) and vary \( d\hat{v}/dy\big|_{y=0} \) in order to satisfy \( \hat{v}(L_y) = 0 \), converging upon \( d\hat{v}/dy\big|_{y=0} \) with the complex equivalent of the bracketed Newton-Raphson method discussed in Press et al. (1992).

**Appendix B**

**Singular Test Solutions in the Linear Equatorial Wave Solving Method**

For \( V_A > 0 \), we examine the nature of the test solutions of Equation (29) in the upper-half domain, \( 0 < y < L_y \), about the singular points, \( y = y_s \). For \( |\omega| \leq B_{0,\max} k \), \( y_s \) is located where \( B_0(\mathcal{y}) k = |\omega| \) (Alfvén singularities), whereas for \( c_s^2 k^2 + B_{0,\max}^2 k \), \( y_s \) is located where \( B_0(\mathcal{y}) k = (\omega^2 - c_s^2 k^2)^{1/2} \) (magnetogravity singularities). The method of Frobenius
gives
\[
\hat{v} = \mathcal{C}_1 \hat{v}_1 + \mathcal{C}_2 \hat{v}_2, \quad \hat{v}_1 = \sum_{n=0}^{\infty} a_n \mathcal{y}^{n+\mu_1}, \\
\hat{v}_2 = D\hat{v}_1 \ln|\mathcal{y}| + \sum_{n=0}^{\infty} b_n \mathcal{y}^{n+\mu_2}, \quad \text{(B1)}
\]
where \( \mathcal{y} = (y - y_s)/L_{\text{eq}} \); \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are the constants of integration; \( \hat{v}_1 \) and \( \hat{v}_2 \) are the first and second fundamental solutions; \( a_n \), \( b_n \), and \( D \) are constant coefficients to be set or determined; and \( \mu_1 \in \mathbb{Z} \) and \( \mu_2 \in \mathbb{Z} \) are the roots of the indicial equation given by substitution into Equation (29).

About magnetogravity singular points, \( \mu_1 = 2 \) and \( \mu_2 = 0 \), so
\[
\hat{v} = \mathcal{C}_1 \sum_{n=0}^{\infty} a_n \mathcal{y}^{n+2} + \mathcal{C}_2 \left( \sum_{n=0}^{\infty} b_n \mathcal{y}^n + D \ln|\mathcal{y}| \sum_{n=0}^{\infty} a_n \mathcal{y}^{n+2} \right), \quad \text{(B2)}
\]
where one is free to set \( a_0 = 1, b_0 = 1, b_2 = 0 \) (in fact, or \( b_2 \) can be set to any constant) and use Equation (29) to determine \( D \), \( a_n \), and \( b_n \). Solutions of this kind are not singular at \( y = y_s \), so magnetogravity singularities are in fact false singularities where the solution remains finite as \( y \to y_s \).
About Alfvén singular points, $\mu_1 = 0$ and $\mu_2 = 0$, so

$$\hat{\nu} = C_1 \sum_{n=0}^{\infty} a_n \hat{y}^n + C_2 \left( \sum_{n=0}^{\infty} b_n \hat{y}^n + D \ln|\hat{y}| \sum_{n=0}^{\infty} a_n \hat{y}^n \right),$$

where one is free to set $a_0 = 1$, $b_1 = 1$, $b_0 = 0$ (again, or $b_0$ can be set to any constant) and use Equation (29) to determine $D$, $a_n$, and $b_n$. Solutions of this kind are dominated by the $\hat{\nu} = O(\ln|\hat{y}|)$ component as $y \to y_s$, so solutions with Alfvén singularities have infinite discontinuities for $D \neq 0$ (which we always find).

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