THE OBSTACLE PROBLEM FOR SUBELLIPTIC NON-DIVERGENCE FORM OPERATORS ON HOMOGENEOUS GROUPS

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Abstract. The main result established in this paper is the existence and uniqueness of strong solutions to the obstacle problem for a class of subelliptic operators in non-divergence form. The operators considered are structured on a set of smooth vector fields in $\mathbb{R}^n$, $X = \{X_0, X_1, ..., X_q\}$, $q \leq n$, satisfying Hörmander’s finite rank condition. In this setting, $X_0$ is a lower order term while $\{X_1, ..., X_q\}$ are building blocks of the subelliptic part of the operator. In order to prove this, we establish an embedding theorem under the assumption that the set $\{X_0, X_1, ..., X_q\}$ generates a homogeneous Lie group. Furthermore, we prove that any strong solution belongs to a suitable class of Hölder continuous functions.

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1. Introduction

Obstacle problems form an important class of problems in analysis and applied mathematics as they appear, in particular, in the mathematical study of variational inequalities and free boundary problems. The classical obstacle problem involving the Laplace operator is to find the equilibrium position of an elastic membrane, whose boundary is held fixed, and which is constrained to lie above a given obstacle. This problem is closely related to the study of minimal surfaces and to inverse problems in potential theory. Other applications where obstacle problems occur, involving the Laplace operator or more general operators, include control theory and optimal stopping, financial mathematics, fluid filtration in porous media, constrained heating and elasto-plasticity. As classical references for obstacle problems and variational inequalities, as well as their applications, we mention Frehse [14], Kinderlehrer-Stampacchia [21], [22] and Friedman [19]. For an outline of the modern approach to the regularity theory of the free boundary, in the context of the obstacle problem, we refer to Caffarelli [7].

In this paper we continue to develop a theory for the obstacle problem for a general class of second order subelliptic partial differential equations in non-divergence form modeled on a system of vector fields satisfying Hörmander’s finite rank condition. In particular, we consider operators

$$\mathcal{H} = \sum_{i,j=1}^{q} a_{ij}(x)X_iX_j + \sum_{i=1}^{q} b_i(x)X_i - a_0(x)X_0, \quad x \in \mathbb{R}^n, \quad n \geq 3,$$

(1.1)

where $q \leq n$ is a positive integer. In Section 2 we will state the assumptions in detail. To formulate the obstacle problem, let $\mathcal{H}$ be as in (1.1), and let $f, g, \varphi, \gamma : \overline{\Omega} \to \mathbb{R}^n$ be continuous and bounded functions such that $g \geq \varphi$ on $\overline{\Omega}$. We consider the problem

$$\begin{cases}
\max\{\mathcal{H}u(x) + \gamma(x)u(x) - f(x), \varphi(x) - u(x)\} = 0, & \text{in } \Omega, \\
u(x) = g(x), & \text{on } \partial\Omega.
\end{cases}$$

(1.2)

We say that $u$ is a strong solution to (1.2) if $u \in S_{X,loc}^1(\Omega) \cap C(\overline{\Omega})$ satisfy the differential inequality (1.2) almost everywhere in $\Omega$, while the boundary datum is attained at all points of $\partial\Omega$. Here $S_{X,loc}^1$ denotes certain intrinsic Sobolev spaces, defined in Subsection 3.2. The main result is the following.
Theorem 1.1. Under the assumptions in Subsection 2 there exists a unique strong solution to the obstacle problem in (1.2). Furthermore, given $p, 1 \leq p < \infty$, and an open subset $\Omega' \subset \subset \Omega$ there exists a positive constant $c$, depending on $H, \Omega', \Omega, p, ||f||_{L^\infty(\Omega)}, ||\gamma||_{L^\infty(\Omega)}, ||g||_{L^\infty(\Omega)}$ and $||\varphi||_{L^\infty(\Omega)}$, such that
\begin{equation}
||u||_{S^p_H(\Omega)} \leq c. \tag{1.3}
\end{equation}

To briefly put Theorem 1.1 into context we first consider the parabolic case, that is when $q = n - 1$ and $X = \{X_0, X_1, \ldots, X_q\}$ is identical to $\{\partial_t, \partial_{x_1}, \ldots, \partial_{x_n}\}$. Then there is an extensive literature on the existence of generalized solutions to the obstacle problem in (1.2) in Sobolev spaces, starting with the pioneering papers [18], [26], [27] and [28]. We note that the results presented here are new due to the presence of the general lower order term $X$ and that is lacking. Existence, uniqueness and regularity results for solutions in the special case when $X_0 = \partial_t$ are contained in [15] and [16]. Similar results, but in the case of second order differential operators of Kolmogorov type, are contained in [13], [17] and [30]. We note that the results presented here are new due to the presence of the general lower order term $X_0$, and neither of the above mentioned results cover the class of operators studied here, as demonstrated in Section 7.

The proof of Theorem 1.1 is based on the classical penalization technique introduced by Lewy and Stampacchia in [24]. In particular, we consider a family $(\beta_\varepsilon)_{\varepsilon \in (0,1)}$ of smooth functions such that, for fixed $\varepsilon \in (0,1)$, $\beta_\varepsilon$ is an increasing function,
\begin{equation}
\beta_\varepsilon(0) = 0, \quad \beta_\varepsilon(s) \leq \varepsilon, \text{ whenever } s > 0, \tag{1.4}
\end{equation}
and such that
\begin{equation}
\lim_{\varepsilon \to 0} \beta_\varepsilon(s) = -\infty, \text{ whenever } s < 0. \tag{1.5}
\end{equation}
A key step in the proof of Theorem 1.1 is to consider the penalized problem
\begin{equation}
\begin{cases}
H^\delta u_{\varepsilon, \delta} + \gamma^\delta u_{\varepsilon, \delta} = f^\delta + \beta_\varepsilon(u_{\varepsilon, \delta} - \varphi^\delta) & \text{ in } \Omega, \\
u_{\varepsilon, \delta} = g^\delta & \text{ on } \partial \Omega,
\end{cases} \tag{1.6}
\end{equation}
where the superscript $\delta, \delta \in (0,1)$, indicate certain mollified versions of the objects at hand. The subscripts $\varepsilon, \delta$ in $u_{\varepsilon, \delta}$ indicate that the solution depends on $\varepsilon$ through the penalizing function $\beta_\varepsilon$ and on $\delta$ through the mollifier. We first prove that a classical solution to the problem in (1.6) exists. By a classical solution we mean that $u_{\varepsilon, \delta} \in C_\varepsilon^{2, \alpha}(\Omega) \cap C(\overline{\Omega})$, where the Hölder space $C_\varepsilon^{2, \alpha}(\Omega)$ is defined in terms of the intrinsic distance induced by the vector fields. In particular, this imply that (1.6) is in fact satisfied pointwise.

Thereafter, a monotone iterative method is used to prove that $u_{\varepsilon, \delta}$ is the limit of a sequence $\{u_{\varepsilon, \delta}^j\}_{j=1}^\infty$ where $u_{\varepsilon, \delta}^j \in C_\varepsilon^{2, \alpha}(\Omega) \cap C(\overline{\Omega})$. A key step in the argument is to ensure compactness in $C_{X, \text{loc}}^{\alpha}(\Omega) \cap C(\overline{\Omega})$ of the sequence constructed, which requires the use of certain a priori estimates. In particular, we use interior Schauder estimates to conclude that there exists a solution $u_{\varepsilon, \delta}$ to the problem in (1.6) such that $u_{\varepsilon, \delta} \in C_{X, \text{loc}}^{2, \alpha}(\Omega) \cap C(\overline{\Omega})$.

The final step is then to consider limits as $\varepsilon$ and $\delta$ tend to 0 and to prove that $u_{\varepsilon, \delta} \to u$ where $u$ is a strong solution to the obstacle problem in (1.2). However, the penalization technique only allows us to establish quite weak bounds on $u_{\varepsilon, \delta}$ given that those bounds should be independent of $\varepsilon$ and $\delta$. Hence, to prove that as $\varepsilon, \delta \searrow 0$ the function $u_{\varepsilon, \delta} \to u$ weakly in $S_{X, \text{loc}}^p$, $p \in [1, \infty)$, we use a priori interior $S_X^p$ estimates. To be able to subsequently conclude that in fact $u_{\varepsilon, \delta} \to u$ in $C_{X, \text{loc}}^{1, \alpha}(\Omega) \cap C(\overline{\Omega})$, we also prove the following theorem.

Theorem 1.2. Under the assumptions in Subsection 2 let $\Omega' \subset \subset \Omega$. If $u \in S^p(\Omega)$, for some $p \in (Q/2, Q)$, then
for $\alpha = (p - Q)/p$. Moreover, the constant $c$ only depend on $\mathcal{G}$, $\mu$, $p$, $s$, $\Omega$ and $\Omega'$.

In the context of the circle of techniques and ideas used in this paper it also fair to mention [2], [5], [16] and [13].

Throughout the paper, when we write that a constant $c$ depends on the operator $\mathcal{H}$, $c = c(\mathcal{H})$, we mean that the constant $c$ depends on $n$, $q$, $X = \{X_0, X_1, \ldots, X_q\}$, $\{a_{ij}\}_{i,j=1}^q$, $\{b_i\}_{i=1}^q$ and $\lambda$. Furthermore, if $\alpha$ and $\Omega$ are given, then $c$ only depends on $\|a_{ij}\|_{C^{0,\alpha}(\Omega)}$, $\|b_i\|_{C^{0,\alpha}(\Omega)}$, and not on any other properties of these coefficients.

The remainder of this paper is organized as follows. Subsection 2 contains assumptions on the vector fields, the operator, and the domain for which our results hold. In Section 3 which is of preliminary nature, we introduce some notation as well as some basic facts about homogeneous groups and subelliptic metrics, in particular, we account for the proper function spaces. In Section 4 we present some estimates for subelliptic operators. Section 5 is devoted to the proof of the main theorem, and in Section 6 we prove the embedding theorem. Finally, in Section 7, we give some examples of operators to which our results apply. In particular, we demonstrate when and how our results overlap with known ones and provide the reader with examples for which our results are new, and not previously considered in the literature.

2. Assumptions

Here we present the assumptions made to be able to prove Theorem 1.1.

The vector fields. $X = \{X_0, X_1, \ldots, X_q\}$ is a system of smooth vector fields in $\mathbb{R}^n$ satisfying two main conditions. The first of which is Hörmander’s finite rank condition. To further explain this, recall that the Lie-bracket between two vector fields $X_i$ and $X_j$ is defined as $[X_i, X_j] = X_i X_j - X_j X_i$. For an arbitrary multiindex $\alpha = (\alpha_1, \ldots, \alpha_q)$, $\alpha_k \in \{0, 1, \ldots, q\}$, we define weights

$$w_0 = 2 \quad \text{and} \quad w_i = 1 \quad \text{for} \quad i = 1, \ldots, q.$$  

Using this we set

$$|\alpha| = \sum_{i=1}^{\ell} w_{\alpha_i}$$  

and define the commutator $[X]_\alpha$ of length $|\alpha|$ by

$$[X]_\alpha = [X_{\alpha_1}, [X_{\alpha_2}, [X_{\alpha_3}, \ldots [X_{\alpha_{\ell-1}}, X_{\alpha_\ell}]]]].$$

$X = \{X_0, X_1, \ldots, X_q\}$ is said to satisfy Hörmander’s finite rank condition, introduced in [20], if there exists an integer $s$, $s < \infty$, such that

$$\text{Lie}(X_0, X_1, \ldots, X_q) = \{[X]_\alpha : \alpha_i \in \{1, \ldots, q\}, \ |\alpha| \leq s \} \text{ spans } \mathbb{R}^n \text{ at every point.}$$  

Moreover, we assume that there exists a family of dilations, $\{D_\lambda\}_{\lambda > 0}$ in $\mathbb{R}^n$, such that $X_1$, $X_2$, $\ldots$, $X_q$ are of $D_1$-homogeneous degree one, and $X_0$ is of $D_1$-homogeneous degree 2.

These two conditions are enough to ensure the existence of a composition law $\circ$ such that the triplet $(\mathbb{R}^n, \circ, D_\lambda)$ is a homogeneous Lie group where the $X_i$’s are left invariant, see Subsection 3.1. However, this homogeneity assumption is only essential to proving the embedding theorem, Theorem 1.2. This means that, should the embedding theorem become available for general Hörmander vector fields, then this proof carries over directly to this more general case.

The coefficients. Concerning the $q \times q$ matrix-valued function $A = A(x) = \{a_{ij}(x)\} = \{a_{ij}\}$ and $a_0$ we assume that $A = \{a_{ij}\}$ is real symmetric, with bounded and measurable entries and that there exists $\lambda \in [1, \infty)$ such that

$$\lambda^{-1}\|\xi\|^2 \leq \sum_{i,j=1}^{q} a_{ij}(x)\xi_i\xi_j \leq \lambda\|\xi\|^2, \quad \lambda^{-1} \leq a_0(x) \leq \lambda, \quad \text{whenever } x, \xi \in \mathbb{R}^n, \xi \in \mathbb{R}^q.$$  

(2.3)
Concerning the regularity of $a_{ij}$ and $b_i$ we will assume that $a_{ij}$ and $b_i$ have further regularity beyond being only bounded and measurable. In fact, we assume that

$$a_{ij}, b_i \in C^{0,\alpha}_{X,\text{loc}}(\mathbb{R}^n)$$

whenever $i, j \in \{1, ..., q\}$, \hfill (2.4)

for $\alpha \in (0, 1)$, where $C^{0,\alpha}_{X,\text{loc}}(\mathbb{R}^n)$ is the space of functions which are bounded and Hölder continuous on every compact subset of $\mathbb{R}^n$. Here the subscript $X$ indicates that Hölder continuity is defined in terms of the Carnot-Carathéodory distance induced by the set of vector fields $X$, see Section 3.2. In particular, by (2.3) we may divide the entire equation by $a_0$, and hence consider (1.1) with $a_0 = 1$.

**The domain.** $\Omega$ is assumed to be a bounded domain such that there exists, for all $\zeta \in \partial \Omega$ and in sense of Definition 3.1, an exterior normal $v$ to $\Omega$ relative $\Omega$, such that $C(\zeta)v \neq 0$. Here $\tilde{\Omega}$ is a neighborhood of $\Omega$ and $C(\cdot)$ is the matrix valued function given by $(X_1, ..., X_q)^T = C(x)\cdot(\partial_1, ..., \partial_n)^T$. The assumption $C(\zeta)v \neq 0$ assures that (4.1) holds, and thus, that we can use Theorem 4.2.

**The equation.** Let $f, \gamma, g, \varphi : \Omega \to \mathbb{R}^n$ be such that $g \geq \varphi$ on $\Omega$ and assume that $f, \gamma, g, \varphi$ are continuous and bounded on $\Omega$, with $\gamma \leq \gamma_0 < 0$.

Concerning the obstacle $\varphi$ we assume that $\varphi$ is Lipschitz continuous on $\Omega$, where Lipschitz continuity is defined in terms of the intrinsic homogeneous distance. We also assume that there exists a constant $c \in \mathbb{R}^+$ such that

$$\sum_{i,j=1}^q \zeta_i \zeta_j \int_{\Omega} X_i X_j \psi(x) \varphi(x) dx \geq c|\zeta|^2 \int_{\Omega} \psi(x) dx$$

(2.5)

for all $\zeta \in \mathbb{R}^q$ and for all nonnegative test functions $\psi \in C^\infty_0(\Omega)$. The reader might want to think of this as a convexity assumption.

### 3. Preliminaries

In this section we introduce notations and concepts to be used throughout the paper. For a more detailed exposition we refer to the monograph [6] written by Bonfiglioli, Lanconelli and Uguzzoni.

In the following we assume that $X = \{X_0, X_1, ..., X_q\}$ satisfies (2.2). From now on we will write $Xf$ when a vector field $X$ acts on a function $f$ as a differential operator. We begin by defining the Carnot-Carathéodory distance, also known as the control distance, see [3] and [29].

**Definition 3.1.** For any $\delta > 0$, let $\Gamma(\delta)$ be the class of all absolutely continuous mappings $\gamma : [0, 1] \to \Omega$ such that

$$\gamma'(t) = \sum_{i=0}^q \lambda_i(t) X_i(\gamma(t)) \quad \text{a.e. } t \in (0, 1)$$

with $|\lambda_0(t)| \leq \delta^2$ and $|\lambda_i(t)| \leq \delta$ for $i = 1, ..., q$. Then we define the Carnot-Carathéodory distance between two points $x, y \in \Omega$ to be

$$d(x, y) = \inf \{\delta : \exists \gamma \in \Gamma(\delta) \text{ with } \gamma(0) = x \text{ and } \gamma(1) = y\}.$$  

It is a non-trivial result that any two points in $\Omega$ can be connected by such a curve, and the proof relies on a connectivity result of Chow, [9]. We remark that the Carnot-Carathéodory distance $d$ is in fact a quasi-distance because the triangle inequality does not hold. Instead, the inequality has the form

$$d(x, z) \leq C(d(x, y) + d(y, z))$$

where the constant $C$ depends on the vector fields. Moreover, there exist constants $c_1, c_2$, depending on $\Omega$, such that

$$c_1 |x - y| \leq d(x, y) \leq c_2 |x - y|^{1/s} \text{ for all } x, y \in \Omega,$$

(3.1)

where $s$ is the rank in the Hörmander condition, see Proposition 1.1 in [29]. This is not immediate, but follows from [3] Section 5].
3.1. **Homogeneous groups.** Let ◦ be a given group law on \( \mathbb{R}^n \) and suppose that the map \((x, y) \mapsto y^{-1} \circ x\) is smooth. Then \( G = (\mathbb{R}^n, \circ) \) is called a Lie group. \( G \) is said homogeneous if there exists a family of dilations \((D_\lambda)_{\lambda \geq 0}\) on \( G \), which are also automorphisms, of the form

\[
D_\lambda(x) = D_\lambda(x^{(1)},...,x^{(l)}) = (\lambda x^{(1)},...,\lambda^l x^{(l)}) = (\lambda^{\sigma_1} x_1,...,\lambda^{\sigma_n} x_n),
\]

where \( 1 \leq \sigma_1 \leq \ldots \leq \sigma_n \). Note that in (3.2) we have that \( x^{(i)} \in \mathbb{R}^{n_i} \) for \( i \in \{1,...,l\} \) and \( n_1 + \ldots + n_l = n \).

On \( G \) we define a homogeneous norm \(|| \cdot ||\) as follows; for \( x \in \mathbb{R}^n \), \( x \neq 0 \), set

\[
||x|| = \rho \quad \text{if and only if} \quad |D_{1/\rho}(x)| = 1,
\]

where \(| \cdot |\) denotes the standard Euclidean norm, and set \(||0|| = 0\). This norm satisfies the following:

i): \(||D_\lambda(x)|| = \lambda||x|| \) for all \( x \in \mathbb{R}^n \), \( \lambda > 0 \).

ii): The set \( \{x \in \mathbb{R}^n : ||x|| = 1\} \) coincides with the Euclidean unit sphere.

iii): There exist \( c(G) \geq 1 \) such that for every \( x, y \in \mathbb{R}^n \)

\[
||x \circ y|| \leq c(||x|| + ||y||) \quad \text{and} \quad ||x^{-1}|| \leq c||x||.
\]

We also define a quasidistance \( d \) on \( \mathbb{R}^n \) through

\[
d(x, y) = ||y^{-1} \circ x||.
\]

For this quasidistance there exist \( c = c(G) \) such that for all \( x, y, z \in \mathbb{R}^n \) the following holds;

iv): \( d(x, y) \geq 0 \) and \( d(x, y) = 0 \) if and only if \( x = y \).

v): \( c^{-1}d(y, x) \leq d(x, y) \leq cd(y, x) \).

vi): \( d(x, y) \leq c(d(x, z) + d(z, y)) \).

The previously mentioned Carnot-Caratheodory distance is one example of an appropriate distance function. Alternatively, one could begin by defining \( ||x|| = \sum_{j=1}^{n} |x_j|^{1/\sigma_j} \), with the induced distance \( d(x, y) = ||x^{-1} \circ y|| \) satisfying the properties above as well.

We define balls with respect to \( d \) by

\[
B(x, r) = \{y \in \mathbb{R}^n : d(x, y) < r\}.
\]

In particular, we note that \( D_r(B(0,1)) = B(0, r) \). Moreover, in [31, p. 619] it is proved that the Lebesgue measure in \( \mathbb{R}^n \) is the Haar measure of \( G \) and that

\[
|B(x, r)| = |B(0,1)|r^Q,
\]

where \( Q \) is the natural number

\[
Q := n_1 + 2n_2 + \ldots + ln_l,
\]

also called the **homogeneous dimension** of \( G \).

The convolution of two functions \( f, g \), defined on \( G \), is defined as

\[
(f * g)(\xi) = \int_{\mathbb{R}^n} f(\zeta \circ \xi^{-1})g(\xi) d\xi
\]

whenever the integral is well defined. Let \( P \) be a differential operator and let \( \tau_\xi \) be the left translation operator, i.e., \( (\tau_\xi f)(\xi) = f(\xi \circ \zeta) \) whenever \( f \) is a function on \( G \). A differential operator \( P \) is said to be left invariant if

\[
P(\tau_\xi f) = \tau_\xi(Pf).
\]

Further, we say that the differential operator \( P \) is homogeneous of degree \( \delta \) if, for every test function \( f, \lambda > 0 \) and \( \xi \in \mathbb{R}^N \),

\[
P(f(D(\lambda)\xi)) = \lambda^\delta(Pf)(D(\lambda)\xi).
\]

Similarly, a function \( f \) is homogeneous of degree \( \delta \) if

\[
f(D(\lambda)\xi) = \lambda^\delta f(\xi) \quad \text{whenever} \quad \lambda > 0, \quad \xi \in \mathbb{R}^n.
\]

Note that if \( P \) is a differential operator homogeneous of degree \( \delta_1 \) and if \( f \) is a function homogeneous of degree \( \delta_2 \) then \( Pf \) is a differential operator homogeneous of degree \( \delta_1 - \delta_2 \) and \( Pf \) is a function
homogeneous of degree $\delta_2 - \delta_1$. We conclude this section with a proposition which will be used to prove the embedding theorem, see \cite{12} Proposition 1.15.

**Proposition 3.2.** Let $f \in C^1(\mathbb{R}^n \setminus \{0\})$ be homogeneous of degree $\delta$. Then there exist $c = c(\mathcal{G}, f) > 0$ and $M = M(\mathcal{G}) > 1$ such that

$$|f(x \circ y) - f(x)| + |f(y \circ x) - f(x)| \leq c|y| \cdot ||x||^{\delta-1},$$

for every $x, y$ such that $||x|| \geq M||y||$.

**3.2. Function spaces.** Let $U \subset \mathbb{R}^n$ be a bounded domain and let $\alpha \in (0, 1]$. Given $U$ and $\alpha$ we define the Hölder space $C^{0,\alpha}_X(U)$ as $C^{0,\alpha}_X(U) = \{u : U \to \mathbb{R} : ||u||_{C^{0,\alpha}(U)} < \infty\}$, where

$$||u||_{C^{0,\alpha}_X(U)} = |u|_{C^{0,\alpha}_X(U)} + ||u||_{L^\infty(U)},$$

$$|u|_{C^{0,\alpha}_X(U)} = \sup \left\{ \frac{|u(x,t) - u(y,t)|}{d(x,y)^\alpha} : x, y \in U \text{ and } x \neq y \right\}.$$

Given a multiindex $I = (i_1, i_2, \ldots, i_m)$, with $0 \leq i_j \leq q$, $1 \leq j \leq m$, we define the weighted length of the multiindex, $|I|$, as in \cite{21} and we set $X^I u = X_{i_1} X_{i_2} \cdots X_{i_m} u$. Now, given a domain $U$, an exponent $\alpha$ and an arbitrary non-negative integer $k$ we define $C^{k,\alpha}_X(U) = \{u : U \to \mathbb{R} : ||u||_{C^{k,\alpha}(U)} < \infty\}$, where

$$||u||_{C^{k,\alpha}_X(U)} = \sum_{|I| \leq k} ||X^I u||_{C^{0,\alpha}_X(U)}.$$

Sobolev spaces are defined as

$$\mathcal{S}^p_X(U) = \{u \in L^p(U) : X_0 u, \ X_i u, \ X_i X_j u \in L^p(U) \text{ for } i,j = 1, \ldots, q\}$$

and we define the Sobolev norm of a function $u$ by

$$||u||_{\mathcal{S}^p_X(U)} = ||u||_{L^p(U)} + \sum_{i=0}^q ||X_i u||_{L^p(U)} + \sum_{i,j=1}^q ||X_i X_j u||_{L^p(U)}.$$

Above the $L^p$-norms are taken with respect to the standard Euclidean metric, in particular, we integrate with respect to the Lebesgue measure. Let $U \subset \mathbb{R}^n$ be a domain, not necessarily bounded. If $u \in C^{k,\alpha}_X(V)$ for every compact subset $V$ of $U$, then we say that $u \in C^{k,\alpha}_X(U)$. Similarly, if $u \in \mathcal{S}^p_X(V)$ for every compact subset $V$ of $U$, then we say that $u \in \mathcal{S}^p_X(U)$.

An important result about compactly supported test functions multiplied by Sobolev functions is the following lemma \cite{2} Corollary 1.

**Lemma 3.3.** If $u \in \mathcal{S}^p(\Omega)$, $1 \leq p < \infty$, and $\phi \in C_0^\infty(\Omega)$, then $u \phi \in \mathcal{S}^p(\Omega)$.

This lemma will be used when $\phi$ is a cutoff function. The existence of smooth cutoff functions is not immediate, but by \cite{2} Lemma 5, we have the following.

**Lemma 3.4.** For any $\sigma \in (0, 1)$, $r > 0$, $k \in \mathbb{Z}_+$, there exists $\phi \in C_0^\infty(\mathbb{R}^n)$ with the following properties:

$$B_{\sigma r} \prec \phi \prec B_{\sigma' r} \quad \text{with } \sigma' = (1 + \sigma)/2;$$

$$|X^\alpha \phi| \leq \frac{c(\mathcal{G}, j)}{|\sigma|^{-1}(1 - \sigma)^{j r}}$$

for all multiindices $|\alpha| = j \in \{1, \ldots, k\}$.

4. Estimates for subelliptic operators with drift

Here we collect a number of theorems which concern subelliptic operators with drift, all of which are important tools in the proof of the obstacle problem. We begin with a result of Bony \cite{1} Theoreme 5.2 which is both a comparison principle and a result on solvability of the Dirichlet problem. Before we state the theorem we introduce the notion of an exterior normal.
Definition 4.1. A vector $v$ in $\mathbb{R}^n$ is an exterior normal to a closed set $S \subset \mathbb{R}^n$ relative an open set $U$ at a point $x_0$ if there exists an open standard Euclidean ball $B_E$ in $U \setminus S$ centered at $x_1$ such that $x_0 \in \overline{B_E}$ and $v = \lambda(x_1 - x_0)$ for some $\lambda > 0$.

Theorem 4.2. (Bony) Let $U \subset \mathbb{R}^n$ be a bounded domain and let $H := \sum_{i=1}^{r} Y_i^2 + Y_0 + \gamma = \sum_{i,j=1}^{n} a_{ij}^* \partial_{x,x_j} + \sum_{i=1}^{n} a_i^* \partial_{x} + \gamma$. Assume that the set of vector fields $Y = \{Y_0, Y_1, ..., Y_r\}$ satisfies Hörmander’s finite rank condition, that $\gamma(x) \leq \gamma_0 < 0$ for all $x \in U$ and that $a_{ij}^*, a_i^*, \gamma \in C^\infty(U)$. In addition, assume that for all $x \in U$ and for all $\xi \in \mathbb{R}^n$ the quadratic form $\sum_{i,j=1}^{n} a_{ij}^*(x) \xi_i \xi_j \geq 0$. Further, assume that $D$ is a relatively compact subset of $U$ and that at every point $x_0 \in \partial D$ there exists an exterior normal $v$ such that

$$\sum_{i,j=1}^{n} a_{ij}^*(x_0) v_i v_j > 0. \quad (4.1)$$

Then, for all $g \in C(\partial D)$ and $f \in C(\overline{D})$, the Dirichlet problem

$$\begin{cases}
Hu = -f, & \text{in } D, \\
u = g, & \text{on } \partial D,
\end{cases}$$

has a unique solution $u \in C(\overline{D})$. Furthermore, if $f \in C^\infty(D)$, then $u \in C^\infty(D)$ and if $f$ and $g$ are both positive then so is $u$.

We remark that we cannot use this theorem directly since we only assume that our coefficients $a_{ij}$ and $b_i$ are Hölder continuous. However, for smooth coefficients and using linear algebra, our operator $\mathcal{H}$ in (1.1) can be rewritten as a Hörmander operator in accordance with Bony’s assumptions. We will also use a Schauder type estimate, the particular one we use can be found in [5, Theorem 2.1].

Theorem 4.3. (Schauder estimate) Assume that the operator $\mathcal{H}$ is structured on a set of smooth Hörmander vector fields and that the coefficients $a_{ij}, b_i \in C^{0,\alpha}_X(\Omega)$ for some $\alpha \in (0,1)$, $a_0 \in L^\infty(\Omega)$. Then for every domain $\Omega' \subset \subset \Omega$ there exists a constant $c$, depending on $\Omega'$, $\Omega$, $X$, $\alpha$, $\lambda$, $||a_{ij}||_{C^{0,\alpha}_X(\Omega)}$, $||b_i||_{C^{0,\alpha}_X(\Omega)}$ and $||a_0||_{C^{0,\alpha}_X(\Omega)}$ such that for every $u \in C^{2,\alpha}_X(\Omega)$ one has

$$||u||_{C^{2,\alpha}_X(\Omega)} \leq c \left\{ ||Hu||_{C^{0,\alpha}_X(\Omega)} + ||u||_{L^\infty(\Omega)} \right\}. \quad (4.2)$$

We emphasize that in [5] this is only proved when the lower order terms $b_i \equiv 0$. However, by arguing as in the proof of Theorem 10.1 in [4] this also hold for $b_i \in C^{0,\alpha}_X(\Omega)$. This Schauder estimate will be used together with an a priori $S^p$ interior estimate to assure proper convergence of a constructed sequence, converging to a solution to the obstacle problem. The proof is to be found in [5, Theorem 2.2]

Theorem 4.4. (A priori $S^p$ interior estimate) Assume that the operator $\mathcal{H}$ is structured on a set of smooth Hörmander vector fields and that the coefficients $a_{ij} \in C^{0,\alpha}_X$ for some $\alpha \in (0,1)$. Then for every domain $\Omega' \subset \subset \Omega$ there exists a constant $c$, depending on $\Omega'$, $\Omega$, $X$, $\alpha$, $\lambda$, $||a_{ij}||_{C^{0,\alpha}_X(\Omega)}$, $||b_i||_{C^{0,\alpha}_X(\Omega)}$ and $||a_0||_{C^{0,\alpha}_X(\Omega)}$ such that for every $u \in S^p_X(\Omega)$ one has

$$||u||_{S^p_X(\Omega')} \leq c \left\{ ||Hu||_{L^p(\Omega)} + ||u||_{L^\infty(\Omega)} \right\}. \quad (4.3)$$

Also here, we can generalize the results in [5] to hold for $b_i \in C^{0,\alpha}_X$, this time arguing as in Section 5.5 in [16].

5. Proof of Theorem 1.1

To prove Theorem 1.1 we will, as outlined in the introduction, use the classical penalization technique and we let $(\beta_\varepsilon)_{\varepsilon \in (0,1)}$ be a family of smooth functions satisfying (1.4) and (1.5). For $\delta \in (0,1)$ we
let $\mathcal{H}^\delta$ denote the operator obtained from $\mathcal{H}$ by regularization of the coefficients $a_{ij}$, $b_i$, $i, j = 1, \ldots, q$, using a smooth mollifier,

$$
\mathcal{H}^\delta = \sum_{i,j=1}^q a^\delta_{ij}(x)X_iX_j + \sum_{i=1}^q b^\delta_i(x)X_i - X_0, \quad x \in \mathbb{R}^n.
$$

We also regularize $\varphi$, $\gamma$ and $f$ and denote the regularizations $\varphi^\delta$, $\gamma^\delta$ and $f^\delta$ respectively. Especially, we are able to extend these functions by continuity to a neighborhood of $\Omega$. As stated in the introduction, see the discussion above (2.5), we assume that $\varphi$ is Lipschitz continuous on $\overline{\Omega}$ and we denote its Lipschitz norm on $\overline{\Omega}$ by $\mu$. Then, since $g \geq \varphi$ on $\partial\Omega$ we see that

$$
g^\delta := g + \mu \delta \geq \varphi^\delta \text{ on } \partial\Omega.
$$

Note that since $g$ is continuous, $g^\delta$ is also continuous and can thus be used as boundary value function. As a first step we consider the penalized problem

$$
\begin{align*}
\mathcal{H}^\delta u + \gamma^\delta u &= f^\delta + \beta \varepsilon (u - \varphi^\delta) \\
\varepsilon u &= g^\delta
\end{align*}
$$

and we prove that there exists a classical solution to this problem. This is achieved in two steps, the first being:

**Theorem 5.1.** Assume that $\mathcal{H}$ satisfies (2.2), (2.3) and (2.4), let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Assume that at every point $x_0 \in \partial\Omega$ there exists an exterior normal satisfying condition (4.1) in Theorem 4.2. Let $g \in C(\partial\Omega)$ and let $h = h(x, u)$ be a smooth Lipschitz continuous function, in the standard Euclidean sense, on $\overline{\Omega}$. Then there exists a classical solution $u \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})$ to the problem

$$
\begin{align*}
\mathcal{H}^\delta u &= h(\cdot, u) \quad \text{in } \Omega, \\
\varepsilon u &= g \quad \text{on } \partial\Omega.
\end{align*}
$$

Furthermore, there exists a positive constant $c$, only depending on $h$ and $\Omega$, such that

$$
\sup_{\Omega} |u| \leq c \left( 1 + \|g\|_{L^\infty(\partial\Omega)} \right).
$$

**Proof.** To prove Theorem 5.1 we will use the same technique as in the proof of Theorem 3.2 in [13], i.e., a monotone iterative method. To start the proof we note that, since $h = h(x, u)$ is a Lipschitz continuous function in the standard Euclidean sense, there exists a constant $\mu$ such that $|h(x, u)| \leq \mu(1 + |u|)$ for $x \in \overline{\Omega}$. We let

$$
u_0(x) = c(1 + \|g\|_{L^\infty(\partial\Omega)}) - 1,
$$

for some constant $c$ to be chosen later, and we recursively define, for $j = 1, 2, \ldots,$

$$
\begin{align*}
\mathcal{H}^\delta u_j - \mu u_j &= h(\cdot, u_{j-1}) - \mu u_{j-1} \\
 u_j &= g
\end{align*}
$$

in $\Omega$, on $\partial\Omega$. The linear Dirichlet problem in (5.4) has been studied by Bony in [11] and since the coefficients of the operator $\mathcal{H}^\delta$ are smooth in a neighborhood of $\Omega$ it follows that $\mathcal{H}^\delta$ can be rewritten as a Hörmander operator in line with Theorem 4.2. Hence, using Theorem 4.2 we can conclude that a classical solution $u_j \in C^\infty(\Omega)$ exists. In particular $u_j \in C(\overline{\Omega})$ and combining Theorem 4.2 with (5.1) it follows that $u_j \in C^{2,\alpha}_{\text{loc}}(\Omega)$. We prove, by induction, that $(u_j)_{j=1}^\infty$ is a decreasing sequence. By definition $u_1 < u_0$ on $\partial\Omega$ and we can choose the constant $c$ appearing in the definition of $u_0$, depending on $h$, so that

$$
\mathcal{H}^\delta (u_1 - u_0) - \mu (u_1 - u_0) = h(\cdot, u_0) - \mathcal{H}^\delta u_0 = h(\cdot, u_0) + c(1 + u_0) \geq 0
$$

holds. Thus, by the maximum principle, stated at the end of Theorem 4.2 we conclude that $u_1 < u_0$ on $\overline{\Omega}$. Assume, for fixed $j \in \mathbb{N}$, that $u_j < u_{j-1}$. Then by the inductive hypothesis we see that

$$
\begin{align*}
\mathcal{H}^\delta (u_{j+1} - u_j) - \mu (u_{j+1} - u_j) &= h(\cdot, u_j) - h(\cdot, u_{j-1}) - \mu (u_j - u_{j-1}) \\
 &= h(\cdot, u_j) - h(\cdot, u_{j-1}) + \mu |u_j - u_{j-1}| \geq 0.
\end{align*}
$$
Hence, by the maximum principle \( u_{j+1} < u_j \) which proves that \( \{u_j\}_{j=1}^\infty \) is a decreasing sequence. By repeating this calculation for \( u_j + u_0 \), we get the following bounds
\[
-u_0 \leq u_{j+1} \leq u_j \leq u_0.
\] (5.5)
As \( u_j \in C^{2,\alpha}_\text{loc}(\Omega) \cap C(\overline{\Omega}) \) we can now use Theorem 4.3 to conclude that
\[
\|u_j\|_{C^{2,\alpha}(U)} \leq c \left( \sup_{\Omega} |u_j| + \|\mathcal{H}^\delta u_j\|_{C^{0,\alpha}(\Omega)} \right) 
\leq c \left( |u_0| + \|h(\cdot, u_{j-1})\|_{C^{0,\alpha}(\Omega)} + \|\mu(u_j - u_{j-1})\|_{C^{0,\alpha}(\Omega)} \right),
\] (5.6)
whenever \( U \) is a compact subset of \( \Omega \). Thus \( \|u_j\|_{C^{2,\alpha}(U)} \) is clearly bounded by some constant \( c \) independent of \( j \) due to (5.5)-(5.6) and the fact that \( h \) is Lipschitz. Thus \( \{u_j\}_{j=1}^\infty \) has a convergent subsequence in \( C^{2,\alpha}_\text{loc}(\Omega) \) and in the following we will denote the convergent subsequence \( \{u_j\}_{j=1}^\infty \). As \( j \to \infty \) in (5.4) we have that
\[
\left\{ \begin{array}{ll}
\mathcal{H}^\delta u = h(\cdot, u) & \text{in } \Omega, \\
u = g & \text{on } \partial \Omega.
\end{array} \right.
\]
We next prove that \( u \in C(\overline{\Omega}) \) by a barrier argument. For fixed \( \zeta \in \partial \Omega \) and \( \varepsilon > 0 \), let \( V \) be an open neighborhood of \( \zeta \) such that
\[
|g(x) - g(\zeta)| \leq \varepsilon \text{ whenever } x \in V \cap \partial \Omega.
\]
Let \( w : V \cap \overline{\Omega} \to \mathbb{R} \) be a function with the following properties:
\[
(i) \mathcal{H}^\delta w \leq -1 \text{ in } V \cap \Omega,
\]
\[
(ii) w > 0 \text{ in } V \cap \Omega \setminus \{\zeta\} \text{ and } w(\zeta) = 0.
\]
That such a function \( w \) exists follows from the assumption that there exists an exterior normal for all points on \( \partial \Omega \), see Definition 4.1 and Remark 5.2 below. We define
\[
v^\pm(x) = g(\zeta) \pm (\varepsilon + kw(x)) \text{ whenever } x \in V \cap \partial \Omega
\]
for some constant \( k > 0 \) large enough to ensure that
\[
\mathcal{H}^\delta (u_j - v^+) \geq h(\cdot, u_{j-1}) - \mu(u_{j-1} - u_j) + k \geq 0
\]
and that \( u_j \leq v^+ \) on \( \partial(V \cap \Omega) \). Thus, the maximum principle asserts that \( u_j \leq v^+ \) on \( V \cap \Omega \) and likewise \( u_j \geq v^- \) on \( V \cap \Omega \). Note that \( k \) can be chosen to depend on the Lipschitz constant of \( h \), \( \mu \) and \( u_0 \) only and, in particular, \( k \) can be chosen independent of \( j \). Passing to the limit we see that
\[
g(\zeta) - \varepsilon - kw(x) \leq u(x) \leq g(\zeta) + \varepsilon + kw(x), \quad x \in V \cap \Omega,
\]
and hence
\[
g(\zeta) - \varepsilon \leq \liminf_{x \to \zeta} u(x) \leq \limsup_{x \to \zeta} u(x) \leq g(\zeta) + \varepsilon
\]
where the limit \( x \to \zeta \) is taken through \( x \in V \cap \Omega \). Since \( \varepsilon \) can be chosen arbitrarily we can conclude that \( u \in C(\overline{\Omega}) \). Finally, (5.2) follows from an application of the maximum principle. \( \square \)

**Remark 5.2.** In the proof above we used barrier functions, plainly stating that proper barrier functions exists. To see that this is actually the case, let \( \zeta \in \partial \Omega \), then using our assumption on the domain \( \Omega \), see also Definition 4.1, we see that there exists a standard Euclidean ball in \( \mathbb{R}^n \), \( B_E(x_0, \rho) \), with center \( x_0 \in \Omega \setminus \Omega \) and radius \( \rho \), such that \( B_E(x_0, \rho) \subset \Omega \) and \( \overline{B_E(x_0, \rho)} \cap \overline{\Omega} = \{\zeta\} \). Using \( x_0 \) we define, for \( K \gg 1 \),
\[
w(x) = e^{-K|x-x_0|^2} - e^{-K|y-x_0|^2}.
\]
Then, \( w(\zeta) = 0 \) and \( w(x, t) > 0 \) for \( x \in V \cap \overline{\Omega} \setminus \{\zeta\} \). To see that \( \mathcal{H}^\delta w \leq -1 \), we note that since the coefficients of the operator \( \mathcal{H}^\delta \) are smooth in a neighborhood of \( V \cap \Omega \), \( \mathcal{H}^\delta \) can be rewritten as a
Hörmander operator in line with Theorem 4.2. In particular, using the notation of Theorem 4.2 we have
\[
\mathcal{H}^\delta w(x) = -e^{-K|x-x_0|^2} \left( 4K^2 \sum_{i,j=1}^n a_{ij}^*(x)(x^i - x_0^i)(x^j - x_0^j) 
- 2K \sum_{i=1}^n (a_{ii}^*(x) + a_{ii}^*(x)(x^i - x_0^i)) + \gamma(x)w(x) \right),
\]
where \(a_{ij}^*, a_{ii}^*\) and \(\gamma\) denote the coefficients of the Hörmander operator \(\mathcal{H}^\delta\) as stated in Theorem 4.2.

Hence, for \(V\) small and choosing \(K\) large enough, \(\mathcal{H}^\delta w(x) \leq -1\) on \(V \cap \overline{\Omega}\). Thus, \(w\) is indeed a proper barrier function.

**Proof of Theorem 1.1.** We first note, using Theorem 5.1, that the problem in (5.1) has a classical solution \(u_{\varepsilon,\delta} \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})\). The assumption \(\gamma < 0\) enable us to use the maximum principle. To proceed we first prove that
\[
|\beta_\varepsilon(u_{\varepsilon,\delta} - \varphi^\delta)| \leq c
\]
for some constant \(c\) independent of \(\varepsilon\) and \(\delta\). By definition \(\beta_\varepsilon \leq \varepsilon\) and hence we only need to prove the estimate from below. Since \(\beta_\varepsilon(u_{\varepsilon,\delta} - \varphi^\delta) \in C(\overline{\Omega})\) this function achieves a minimum at a point \((\varsigma, \tau) \in \Omega\). Assume that \(\beta_\varepsilon(u_{\varepsilon,\delta}(\varsigma) - \varphi^\delta(\varsigma)) \leq 0\), otherwise we are done. If \(\varsigma \in \partial \Omega\), then, since \(g \geq \varphi\)
\[
\beta_\varepsilon(u_{\varepsilon,\delta}(\varsigma) - \varphi^\delta(\varsigma)) = \beta_\varepsilon(g^\delta(\varsigma) - \varphi^\delta(\varsigma)) \geq 0.
\]
On the other hand, if \(\varsigma \in \Omega\), then the function \(u_{\varepsilon,\delta} - \varphi^\delta\) also reaches its (negative) minimum at \(\varsigma\) since \(\beta_\varepsilon\) is increasing. Now, due to the maximum principle,
\[
\mathcal{H}^\delta u_{\varepsilon,\delta}(\varsigma) - \mathcal{H}^\delta \varphi^\delta(\varsigma) \geq 0 \geq -\gamma^\delta(\varsigma) (u_{\varepsilon,\delta}(\varsigma) - \varphi^\delta(\varsigma)).
\]
Because of (2.5) and the assumption that \(a_0, b_1 \in L^\infty(\Omega)\) we conclude that \(\mathcal{H}^\delta \varphi^\delta \geq \eta\) for some constant \(\eta\) independent of \(\delta\). Now, since \(\gamma, f \in L^\infty(\Omega)\) and using (5.8), we obtain
\[
\beta_\varepsilon(u_{\varepsilon,\delta} - \varphi^\delta) = \mathcal{H}^\delta u_{\varepsilon,\delta}(\varsigma) + \gamma^\delta(\varsigma) u_{\varepsilon,\delta}(\varsigma) - f^\delta(\varsigma)
\geq \mathcal{H}^\delta \varphi^\delta(\varsigma) + \gamma^\delta(\varsigma) \varphi^\delta(\varsigma) - f^\delta(\varsigma) \geq c,
\]
for some constant \(c\) independent of \(\varepsilon\) and \(\delta\) and hence (5.7) holds. We next use (5.7) to prove that \(u_{\varepsilon,\delta} \to u\) for some function \(u \in C^{2,\alpha}(\Omega) \cap C(\overline{\Omega})\) and that \(u\) is a solution to the obstacle problem (1.2).

To do this we first prove that there exist constants \(c_1\) and \(c_2\) such that
\[
||u_{\varepsilon,\delta}||_{L^\infty(\Omega)} \leq c_2 \left(||g||_{L^\infty(\Omega)} + ||f||_{L^\infty(\Omega)} + c_1\right).
\]
In fact, this follows by considering solutions to
\[
\begin{cases}
\mathcal{H}^\delta v_{\varepsilon,\delta} - ||\gamma^\delta||_{L^\infty(\Omega)} v_{\varepsilon,\delta} = -2(||f^\delta||_{L^\infty(\Omega)} + ||\beta_\varepsilon(u_{\varepsilon,\delta} - \varphi^\delta)||_{L^\infty(\Omega)}) & \text{in } \Omega, \\
u = ||g^\delta||_{L^\infty(\Omega)} & \text{on } \partial \Omega.
\end{cases}
\]
Using the maximum principle on \(v_{\varepsilon,\delta} - u_{\varepsilon,\delta}\), we see that \(u_{\varepsilon,\delta} < v_{\varepsilon,\delta}\). Moreover, since \(||\beta_\varepsilon(u_{\varepsilon,\delta} - \varphi^\delta)||_{L^\infty(\Omega)}\) is bounded uniformly for \(\varepsilon, \delta\), and since the \(L^\infty\)-norm of the regularized version of a function is bounded by the \(L^\infty\)-norm of the function itself, (5.9) follows. Then we use (5.7) and (5.9) together with Theorem 4.1 to conclude that for every \(U \subset \subset \Omega\) and \(p \geq 1\) the norm \(||u_{\varepsilon,\delta}||_{S^p(U)}\) is bounded uniformly in \(\varepsilon\) and \(\delta\). Consequently \(\{u_{\varepsilon,\delta}\}\) converges weakly to a function \(u\) on compact subsets of \(\Omega\) as \(\varepsilon, \delta \to 0\) in \(S^p\), and by Theorem 1.2 in \(C^{1,\alpha}\). Also, by construction,
\[
\limsup_{\varepsilon,\delta \to 0} \beta_\varepsilon(u_{\varepsilon,\delta} - \varphi^\delta) \leq 0
\]
and therefore \(\mathcal{H}u + \gamma \leq f\) a.e. in \(\Omega\). In the set \(\{u \geq \varphi\} \cap \Omega\) equality holds. Together with the estimate (5.7) this shows that \(\max\{\mathcal{H}u + \gamma u - f, \varphi - u\} = 0\) on \(\Omega\). Proceeding as in the end of the proof of Theorem 5.1 using barrier functions, we conclude that \(u \in C(\overline{\Omega})\) and \(u = g\) on \(\partial \Omega\), hence \(u\) is a
strong solution to the obstacle problem (1.2). The bound (1.3) is a direct consequence of the above calculations. Altogether, this completes the proof. □

6. Proof of Theorem 1.2

The embedding theorem we aim to prove is not as general as we would have hoped, and actually, when we began working on this paper we did believe that the proof was already out there. Despite several attempts on finding a proper reference we were unable to find one, and in the end, we decided to add the assumption that we are working on a homogeneous group and that the vector fields $X_1, \ldots, X_q$ are left invariant and homogeneous of degree one while $X_0$ is left invariant and homogeneous of degree two. This enables us to prove the necessary embedding, that is that the $C^{1,\alpha}$-norm of solutions are bounded by the $S^p$-norm. In the case of stratified groups this was proved by Folland in [12] Theorem 5.15, and no assumption on $u$ solving a particular equation had to be made. In the pure subelliptic content, that is, when there is no lower order term, this has been extensively investigated, see for instance Lu [25] Theorem 1.1 and the references therein. In the subelliptic parabolic case, that is, when $X_0 = \partial_t$, this was proved in [16] Theorem 1.4. The approach used to the case $X_0 = \partial_t$ cannot be applied to this case since we lack enough information about the fundamental solution. Finally, a slightly less general formulation of the embedding theorem was proved in [2, Theorem 7], where the $C^{0,\alpha}$-norm is bounded by the $S^p$-norm. 

Proof of Theorem 1.2. First, we note that by [2, Theorem 4] we have, for $\alpha = 2 - Q/p$,

$$\|u\|_{C^{0,\alpha}(\Omega)} \leq c \left( \|\mathcal{H}u\|_{L^p(\Omega)} + \|u\|_{L^p} \right),$$

for some $c$ depending only on $G$, $\mu$, $p$, $s$, $\Omega$ and $\Omega'$ (it is stated in a slightly different way, but restricted to our choice of $p$ and $s$ this is what is actually proved). It remains to show that the same holds when $u$ on the left hand side is replaced by $X_i u$ for $i = 1, \ldots, q$. Let $H = \sum_{i=1}^q X_i^2 + X_0$ and let $\Gamma$ be the corresponding fundamental solution. Such a fundamental solution exists by a classical result of Folland [12] Theorem 2.1. Moreover, $\Gamma$ is homogeneous of degree $2 - Q$. This means that, for $u \in C_0^\infty(B_R)$, we can write

$$u = H u * \Gamma.$$

Let $\phi$ be a cutoff function with $B_{R/2}(x_0) \prec \phi \prec B_R(x_0)$, for some $x_0 \in \Omega$, $R \in \mathbb{R}$ such that $B_{2R}(x_0) \subset \Omega$. That such a cutoff function exists follows from Lemma 3.4. By Lemma 3.3, $u\phi \in S_0^2(B_R)$. Since Hölder continuity is a local property, we can restrict ourselves to balls, and by a density argument we can look at smooth functions $u$. Therefore, assume that $u \in C_0^\infty(\Omega)$ and let $M$ be as in Proposition 3.2, then

$$X_i u(x) = X_i \int_{\mathbb{R}^n} \Gamma(y^{-1} \circ x) Hu(y) dy.$$

Since $u$ is smooth with compact support we may differentiate inside integral, and we obtain

$$|X_i u(x) - X_i u(y)| \leq \int_{\mathbb{R}^n} |X_i \Gamma(z^{-1} \circ x) - X_i \Gamma(z^{-1} \circ y)| |Hu(z)| dz$$

$$\leq \int_{||z^{-1}ox|| \geq M||y^{-1}ox||} \ldots dz + \int_{||z^{-1}ox|| \leq M||y^{-1}ox||} \ldots dz = I + II. \quad (6.1)$$

Above, it is implicitly understood that the vector fields act on $\Gamma$ as a function of $x$ respectively $y$ (hence, do not differentiate with respect to the $z$-variable). Since $\Gamma$ is homogeneous of degree $2 - Q$ and $X_i$, $i = 1, \ldots, q$, is homogeneous of degree 1, $X_i \Gamma$ is homogeneous of degree $1 - Q$. By Proposition 3.2 we get

$$I \leq c(G,p) \||y^{-1}o x|| \int_{||z^{-1}ox|| \geq M||y^{-1}ox||} \frac{|Hu(z)|}{||z^{-1}o x||^Q} dz.$$

Further, we introduce the sets

$$\sigma_k = \{ z \in \mathbb{R}^n : 2^k M ||y^{-1}o x|| \leq ||z^{-1}o x|| \leq 2^{k+1} M ||y^{-1}o x|| \},$$
for \(k = 0, 1, \ldots\), and note that the Euclidean volume of the set \(\sigma_k\), by (3.3), is equal to
\[
|B(0, 2^{k+1}M||y^{-1} \circ x||)| - |B(0, 2^{k}M||y^{-1} \circ x||)| = |B(0, 1)| \left( \left(2^{k+1}M||y^{-1} \circ x||\right)^Q - \left(2^{k}M||y^{-1} \circ x||\right)^Q \right) \tag{6.2}
\]
\[
= |B(0, 1)| \left(2^Q - 1\right) 2^{Qk}M^Q||y^{-1} \circ x||^Q. \tag{6.3}
\]
By assumption \(u \in S^p(\Omega)\) for some \(p\). Let \(q\) be such that \(\frac{1}{p} + \frac{1}{q} = 1\). Then, we obtain
\[
I \leq c(G, p) \|y^{-1} \circ x\| \int_{||z^{-1} \circ x|| \geq M||y^{-1} \circ x||} \frac{|Hu(z)|}{||z^{-1} \circ x||^Q} dz 
\leq c(G, p) \|y^{-1} \circ x\| \sum_{k=0}^{\infty} \left(2^{k}M||y^{-1} \circ x||\right)^{-Q} \int_{\sigma_k} |Hu(z)| dz 
\leq c(G, p) \|y^{-1} \circ x\|^{1-Q} \sum_{k=0}^{\infty} 2^{-kQ} \left[(2^Q - 1) 2^{Qk}M^Q||y^{-1} \circ x||^Q\right]^{1/q} \|Hu\|_{L^p(\sigma_k)} 
\leq c(G, p) \|y^{-1} \circ x\|^{1-Q+Q/q} \|Hu\|_{L^p(\Omega)} \sum_{k=0}^{\infty} 2^{-k(Q-Q/q)}.
\]
This sum converges, and for \(Q < p\) we have that the exponent of \(\|y^{-1} \circ x\|\) is larger than zero.
Next step is to look at \(II\) in (6.1). In a similar way we define the sets
\[
\overline{\sigma_k} = \{z \in \mathbb{R}^n : 2^{-(k+1)}M||y^{-1} \circ x|| \leq ||z^{-1} \circ x|| \leq 2^{-k}M||y^{-1} \circ x||\},
\]
for \(k = 0, 1, \ldots\), and in this case we get
\[
II \leq \int_{\|z^{-1} \circ x\| < M \|y^{-1} \circ x\|} \frac{|Hu(z)|}{\|z^{-1} \circ x\|^{Q-1}} dz + \int_{\|z^{-1} \circ x\| < M \|y^{-1} \circ x\|} \frac{|Hu(z)|}{\|z^{-1} \circ y\|^{Q-1}} dz.
\]
To begin with, we deal with the first term above, which by the compact support of \(u\), is bounded by
\[
II_1 \leq c(G, p) \sum_{k=0}^{\infty} \int_{\overline{\sigma_k}} \frac{|Hu(z)|}{2^{-(k+1)}M \||y^{-1} \circ x||} Q^{-1} dz 
\leq c(G, p) \||y^{-1} \circ x||^{-(Q-1)} \sum_{k=0}^{\infty} 2^{(k+1)(Q-1)} \left( \int_{\overline{\sigma_k}} 1 dz \right)^{1/q} \|Hu\|_{L^p(\Omega)} 
\leq c(G, p) \||y^{-1} \circ x||^{-(Q-1)} \|Hu\|_{L^p(\Omega)} \sum_{k=0}^{\infty} 2^{(k+1)(Q-1)} \left[2^{-kM \||y^{-1} \circ x||}\right]^{Q^{1/q}} 
= c(G, p) \||y^{-1} \circ x||^{-(Q-1-Q/q)} \|Hu\|_{L^p(\Omega)} \sum_{k=0}^{\infty} 2^{k(Q-1-Q/q)}.
\]
The sum converges for \(Q < p\), and in that case
\[
II_1 \leq c(G, p) \||y^{-1} \circ x||^{(p-Q)/p} \|Hu\|_{L^p(\Omega)}.
\]
To bound \(II_2\), note that if \(\|z^{-1} \circ x\| < M \|y^{-1} \circ x\|\), then \(\|z^{-1} \circ y\| \leq c(\|z^{-1} \circ x\| + \|y^{-1} \circ x\|) \leq c(1 + M) \|y^{-1} \circ x\|\). This means that we can argue as for \(II_1\), to find that \(II_2 \leq c(G, p) \||y^{-1} \circ x||^{(p-Q)/p} \|Hu\|_{L^p(\Omega)}\). Put together, we have shown that
\[
|X_i u(x) - X_i u(y)| \leq c(G, p) \||y^{-1} \circ x||^{(p-Q)/p} \|Hu\|_{L^p(\Omega)}.
\]
That is, (1.7) hold for functions $u \in S^p(\Omega) \cap C_0^\infty(\Omega)$. The general case follows, as previously mentioned, by using a density argument and cutoff functions. Note that, we proved this for Hölder spaces defined by means of the distance $d_b$, however, this carries over directly to our case. 

7. Homogeneous Hörmander operators

We will now give some examples as to when our results apply. The two first examples shows operators for which our results overlap with the existing literature, while the third and fourth example shows that our results covers equations previously not considered for obstacle problems.

Example 7.1. (Subelliptic parabolic equations) When we replace $a_0X_0$ with $\partial_t$ we get a subelliptic parabolic operator;

$$H = \sum_{i,j=1}^q a_{ij}(x,t)X_iX_j + \sum_{i=1}^{q} b_i(x,t)X_i - \partial_t, \quad x \in \mathbb{R}^n, t \in (0,T), n \geq 3.$$  

In this case, by [16], we need not assume that we have a homogeneous group.

Example 7.2. (Kolmogorov equations) Let

$$H = \sum_{i,j=1}^q a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{q} b_i(x,t) \frac{\partial}{\partial x_i} + \sum_{i,j=1}^{n} c_{ij}x_i \frac{\partial}{\partial x_j} + \partial_t,$$  

(7.1)

where $(x,t) \in \mathbb{R}^n \times \mathbb{R}$, $q < n$, with the usual assumptions on $a_{ij}$ and $b_i$, while $C = \{c_{ij}\}$ is a matrix of constant real numbers. For $(x_0,t_0)$, fixed but arbitrary, we introduce the vector fields

$$X_0 = \sum_{i,j=1}^{n} c_{ij}x_i \frac{\partial}{\partial x_j} + \partial_t, \quad X_i = \frac{1}{\sqrt{2}} \sum_{j=1}^{q} a_{ij}(x_0,t_0) \frac{\partial}{\partial x_j}, \quad i \in \{1,...,q\}.$$  

(7.2)

A condition which assures that $H$ in (7.1) is a Hörmander operator is that $\{X_0,X_1,...,X_q\}$ in (7.2) satisfy the Hörmander condition. An equivalent condition is that the matrix $C$ has the following block structure

$$
\begin{pmatrix}
* & C_1 & 0 & \cdots & 0 \\
* & * & C_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
* & * & * & \cdots & C_k \\
* & * & * & \cdots & * 
\end{pmatrix}
$$

where $C_j$, for $j \in \{1,...,k\}$, is a $q_j-1 \times q_j$ matrix of rank $q_j$, $1 \leq q_k \leq \cdots \leq q_1 \leq q = q_0$. Further, $q + q_1 + \cdots + q_k = n$, while $*$ represents arbitrary matrices with constant entries. In the case of Kolmogorov equations, results on existence of solutions was proved in [13].

Example 7.3. For $(x,y,z,w,t) \in \mathbb{R}^5$, consider the vector fields

$$X = \partial_x - xy\partial_t, \quad Y = \partial_y + x\partial_w, \quad Z = \partial_z + x\partial_t.$$  

These vector fields satisfy Hörmanders condition since

$$W = [X,Y] = \partial_w + x\partial_t, \quad T = [X,Z] = \partial_t.$$  

We note that the Lie algebra generated by these vector fields are nilpotent of step 4, but we do not have a stratified group since $\partial_t = [X,Z] = [X,W]$. Moreover, the group law $\circ$ is given by

$$(x,y,z,w,t) \circ (\xi,\eta,\zeta,\omega,\tau) = (x + \xi,y + \eta, z + \zeta, w + \omega + x\eta,t + \tau - \frac{1}{2}y\xi^2 - x\xi\eta + x\zeta + x\omega),$$

and we can define (non-unique) translations

$$D_\lambda(x,y,z,w,t) = (\lambda x, \lambda y, \lambda^2 z, \lambda^2 w, \lambda^3 t).$$

This is neither a subelliptic parabolic equation, nor is it a Kolmogorov type equation, and the results presented here is therefore new.
Example 7.4. (Link of groups) Following [23], we can link groups together. The simplest example is obtained if we define the vector fields

\[ X_0 = x \partial_w - \partial_t, \quad X_1 = \partial_x + y \partial_s, \quad X_2 = \partial_y - x \partial_s, \]

for \((x, y, s, w, t) \in \mathbb{R}^5\). Then, in the variables \((x, y, s, t)\) we get the heat operator on the Heisenberg group, while in the variables \((x, y, s, w)\) we get a Kolmogorov operator. This again defines a homogeneous Hörmander operator, which previously have not been studied in the setting of obstacle problems.

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