Geometric Phase and Fidelity of The One-Dimensional Extended Quantum Compass Model in a Transverse Field

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(Dated: February 6, 2014)

We study the geometric phase of the ground state in the extended quantum compass model in presence of a transverse field. The exact solution is obtained by using the Jordan-Wigner transformation which maps the Hamiltonian on a fermionic system and applying the Fourier transformation to realize the diagonalization and an analytical expression for the ground state and geometric phase in the momentum space. Furthermore, the scaling behavior of the extremum of geometric phase and its universality are investigated due to the divergence of geometric phase at the critical point. We also study the fidelity of the ground state to confirm the results obtained using the geometric phase.

PACS numbers: 75.10.Pq, 03.65.Vf, 42.50.Vk, 75.10.Jm

I. INTRODUCTION

Quantum phase transitions (QPTs) are essential phenomena in quantum many-body correlated systems which is induced by the ground-state (GS) transition. This can be driven by tuning an external parameters at zero temperature, for instance, the magnetic field or the amount of disorder. Traditionally, QPTs are described in terms of order parameter and symmetry breaking by the Landau-Ginzburg paradigm. The problem in these traditional ways is that there is no general method to find the order parameter for a common system. In the last few years a big effort has been assigned to the analysis of the QPTs from the perspective of Quantum Information [1–9]. Entanglement and fidelity have been accepted as new notion to characterize quantum phase transitions. Entanglement, referring to quantum correlations between subsystems is a good indicator of quantum phase transitions, because the correlation length diverge at the quantum critical points [1–3]. The fidelity which is a measure of distance between quantum states, could be also a nice tool to study the drastic change in the ground states in quantum phase transitions [10]. In view of some difficulties [11], attention has been transferred to include other, potentially related, means of characterizing QPTs [10]. One such approach centers around the concept of geometric phase (GP). Geometric phases which display the curvature of the Hilbert space, and especially has direct relation to the property of the degeneracy in quantum systems, were introduced in quantum mechanics by Berry in 1984 [12]. Thus the GP is a powerful tool to find out the quantum phase transitions [13]. The attention has also increased due to their applicability in quantum-information processing [14]. In other words, GP becomes additive for product states of composite systems since the uncorrelated subsystems set independent geometric phase factors. However, due to their global properties, the GP is propitious to make fault tolerant quantum gates. In this point of view, many physical systems have been studied to realize geometric quantum computation, such as NMR (Nuclear Magnetic Resonance), Josephson junction, ion trap and semiconductor quantum dots. The quantum computation scheme for the geometric phase has been proposed based on the Abelian or non-Abelian geometric phase, in which geometric phase has been used to be robust against faults in the presence of some kind of external noise due to the geometric nature of Berry phase. It was therefore seen that interactions play an important role for the understanding of some specific operations.

For the first time A. Carollo at al., show the GP connection to the quantum phase transitions, where they have been shown that the geometric phase of the spin chains diverge at the critical point [15]. The critical exponents were evaluated from the scaling behavior of geometric phases has been obtained by S. L. Zhu at al. [16]. Then the geometric phase have been considered as a topological test to manifest quantum phase transitions [17].

In the present work, we shall use the geometric phase and fidelity of the ground state to detect the QPTs for the one-dimensional extended quantum compass model [20, 21, 26] in a transverse filed to confirm the results obtained in our previous works [18, 19]. This model cover a group of well-known spin models as its special cases and shows a rich phase diagram. It is worthy to mention that not only this inhomogeneous system exhibits rich phase diagram, it would be interesting to investigate whether the geometric phase is able to determine the quantum phase transition in these more complicated systems. However, an explicit analytical expression of the ground state, which is necessary for studying the geometric phase and fidelity, has not been calculated so far. The exact solution is obtained by using Jordan-Wigner (JW) transformation and because of nice scaling proper-
ties of GP, phase transition can be pinpointed from small systems with considerable accuracy.

II. HAMILTONIAN AND EXACT SOLUTION

Consider the Hamiltonian

\[
H = \sum_{n=1}^{N'} \left[ J_1 \sigma_2^{n-1} \sigma_{2n}^+ + J_2 \sigma_2^{n-1} \sigma_{2n}^- + L_1 \sigma_2^{2n} \sigma_{2n+1}^+ + L_2 \sigma_2^{2n} \sigma_{2n+1}^- + h(\sigma_2^{n-1} + \sigma_2^{2n}) \right],
\]

where \( J_1 \) and \( J_2 \) are the odd bonds exchange couplings, \( L_1 \) is the even bond exchange coupling and \( N = 2N' \) is the number of spins. We assume periodic boundary conditions (ground state for periodic and antiperiodic boundary conditions are identical in the thermodynamic limit and the essential features in finite size are also not altered qualitatively). This model embraces a group of other familiar spin models as its special cases, such as quantum Ising model in a transverse field for \( J_2 = 0 \), the transverse field XY model for \( J_1 = J_2 = 0 \) and the transverse field XX model for \( J_1 = J_2 = 0 \). The above Hamiltonian (Eq. 1) can be exactly diagonalized by standard Jordan-Wigner transformation \[23, 24\] as defined below,

\[
\sigma_j^+ = b_j^+ + b_j^-, \quad \sigma_j^- = b_j^+ - b_j^-, \quad \sigma_j^z = 2b_j^+ b_j^- - 1
\]

\[
b_j^+ = c_j^+ e^{\pi \Sigma_{m=1}^{j-1} i \epsilon_m}, \quad b_j^- = e^{-\pi \Sigma_{m=1}^{j-1} i \epsilon_m} c_j,
\]

which transforms spins into fermion operators \( c_j \).

The crucial step is to define independent Majorana fermions \[23, 24\] at site \( n \), \( c_j^\dagger \equiv c_{2n-1} \) and \( c_j \equiv c_{2n} \). This can be regarded as quasiparticles’ spin or as splitting the chain into bi-atomic elementary cells \[23\].

Substituting for \( \sigma_j^+, \sigma_j^- \) and \( \sigma_j^z \) \((j = 2n, 2n-1)\) in terms of Majorana fermions with antiperiodic boundary condition (subspace with even number of fermions) followed by a Fourier transformation, Hamiltonian Eq. (1) (apart from additive constant), can be written as

\[
H^+ = \sum_k \left[ J_{c_k} c_k^\dagger c_{-k} + L_{c_k} c_k^\dagger c_k + 2h(c_k^\dagger c_{-k} + c_k c_k^\dagger) + h.c. \right],
\]

where \( J = (J_1 - J_2) - (L_1 - L_2) c^{ik} L = (J_1 + J_2) + (L_1 + L_2) c^{ik} \) and \( k = \pm \frac{\pi}{N'} \), \((j = 1, 3, \cdots, N' - 1)\).

By grouping together terms with \( k \) and \( -k \), the Hamiltonian is decoupled into a sum of independent terms acting in the 4-dimensional Hilbert spaces generated by \( k \) and \( -k \) \((H^+ \oplus H^-_k)\), in the other word \([H_k, H_{-k}] = 0\) in which

\[
H_k = \left( J_{c_k} c_k^\dagger c_{-k} + L_{c_k} c_k^\dagger c_k \right) + \left( J_{c_{-k}} c_{-k}^\dagger c_k - J_{c_k} c_{-k}^\dagger c_k \right) - L_{c_{-k}} c_{-k} c_k^\dagger \text{ and } k = \pm \frac{\pi}{N'} \text{ for } j = 1, 3, \cdots, N' - 1.
\]

Hamiltonian Eq. (2) can be written in the diagonal block form

\[
H = \sum_k \Gamma_k^+ A(k) \Gamma_k
\]

where \( \Gamma_k^+ = (c_k^\dagger, c_{-k}^\dagger, c_k, c_{-k}) \) and

\[
A(k) = \begin{pmatrix}
2h & 0 & L & J \\
0 & -2h & -J & -L \\
L^* & -J^* & 2h & 0 \\
J^* & -L^* & 0 & -2h
\end{pmatrix}
\]

By using the element of the new vector \( \Gamma_k^\dagger = (\gamma_k^R, \gamma_k^L, \gamma_k^P, \gamma_k^Q) \) could be described by unitary transformation \( \Gamma_k^\dagger = L_k \Gamma_k \) (see Appendix A) the matrix \( A(k) \) can be diagonalized easily and we find the Hamiltonian Eq. (2) in a diagonal form.

\[
H = \sum_k \left[ E_k^R(\gamma_k^R, \gamma_k^L) + E_k^P(\gamma_k^P, \gamma_k^Q) \right],
\]

where \( E_k^R = \sqrt{a + \sqrt{b}} \) and \( E_k^P = \sqrt{a - \sqrt{b}} \), in which

\[
a = 4h^2 + |J|^2 + |L|^2, \quad b = (16h^2 + 2|J|^2)L^2 + J^2 L^2 + J^2 L^2
\]

The ground state \((E_G)\) and the lowest excited state \((E_E)\) energies are obtained from Eq. (4).

\[
E_G = -\frac{1}{2} \sum_k (E_k^R - E_k^P), \quad E_E = \frac{1}{2} \sum_k (E_k^R - E_k^P)
\]

It is straightforward to show that the energy gap vanishes at \( h_0 = \sqrt{(J_1 + L_2)(J_2 + L_1)} \) and \( h_\pi = \sqrt{(J_1 + L_2)(J_2 - L_1)} \) in the thermodynamic limit.

So, the quantum phase transition (QPT) which could be driven by the transverse-field, depending on exchange couplings, occurs at \( h_0 \) and \( h_\pi \).

By tedious calculation on the unitary transformation the unnormalized ground state has been written in the vacuum \( k \)th mode of \( c_k^\dagger \) and \( c_k \),

\[
|G(h)\rangle = \prod_k \left[ v_1|0\rangle + v_2 c_k^\dagger c_{-k}^\dagger|0\rangle + v_3 c_k^\dagger c_k|0\rangle + v_4 c_{-k}^\dagger c_{-k}^\dagger|0\rangle + v_5 c_k^\dagger c_{-k}^\dagger|0\rangle + v_6 c_{-k}^\dagger c_k |0\rangle \right],
\]

where \( v_i \) \((i = 1, \cdots, 6)\) is functions of the coupling constant which are given in Appendix B. We should point out that there is a flaw in Ref. \[25\] in the basic diagonalization results of the Hamiltonian which is the same as the present model in k-space.
There are four gapped phases in the exchange couplings’ (QPT) as a function of exchange couplings \( [18] \). We have obtained the analytic expressions for except at the critical surfaces where the energy gap dis-
evolving block decimation. This model is always gapful using the exact diagonalization method and infinite time-
fidelity \([29]\) of this model has been studied numerically quantum correlation \([27]\), bipartite entanglement \([28]\) and relation functions \([18]\). However the phase diagram \([26]\),
compass model in homogenous transverse field by use of
investigated the phase diagram of the extended quantum
field for zero temperature (J

\[ J_1 \]

extended quantum compass model in a transverse magnetic
FIG. 1: (Color online) Two-dimensional phase diagram of the
extended quantum compass model in a transverse magnetic
field for zero temperature (\( J_1 = 1 \)).

III. PHASE DIAGRAM

It is useful to recall the zero-temperature phase dia-
gram of the extended compass model in homogenous transverse field, see Figs. \([1] [2] [18] [19]\) (For simplicity we take \( L_1 = 1, L_2 = 0 \). In our recent work, we have inves-
tigated the phase diagram of the extended quantum
compass model in homogenous transverse field by use of
the gap analysis and universality of derivative of the cor-
relation functions \([18]\). However the phase diagram \([26]\),
quantum correlation \([27]\), bipartite entanglement \([28]\) and fidelity \([29]\) of this model has been studied numerically using the exact diagonalization method and infinite time-
evolving block decimation. This model is always gapful except at the critical surfaces where the energy gap dis-
appears. We have obtained the analytic expressions for all critical fields which drive quantum phase transitions (QPT) as a function of exchange couplings \([18]\),
\[ h_0 = \sqrt{J_1(J_2 + L_1)}, \quad h_\pi = \sqrt{J_1(J_2 - L_1)} \] for \( L_2 = 0 \).
There are four gapped phases in the exchange couplings’ space:

- Region (I) \( J_1 = 1, \; 0 < J_2 < 1 \): In this region for small magnetic field \( h < h_0 \) the ground state is in the spin-flop phase (Fig. \([1]\)).

- Region (II) \( J_1 = 1, \; J_2 > 1 \): In this case there is antiparallel ordering of spin y component on odd bonds for \( h < h_\pi \). In this region tuning the magnetic field forces the system goes into a spin-flop phase (region (I)) for \( h_\pi < h < h_0 \). For \( h > h_0 \) the system fall into region (III) (Fig. \([1]\)).

- Region (III) \( J_1 = \pm 1 \): In this region the ground state is the ferromagnetically polarized state along the magnetic field (Figs. \([1]\) [2]).

- Region (IV) \( J_1 = -1, \; J_2 < 1 \): In this region the ground state is in the strip antiferromagnetic (SAF) phase (Fig. \([2]\)).

The fidelity and geometric phase of the ground state properties could reflect these different zero-temperature
regions, in particular the quantum phase transitions at \( h_0 \) and \( h_\pi \).

IV. GEOMETRIC PHASE AND FIDELITY

In order to investigate the geometric phase in this system, we use a new family of Hamiltonians that can be described by applying a rotation of \( \phi \) around the z direction to each spin as in the method in Refs. \([15]\) and \([16]\), i.e.,

\[ H_\phi = g_\phi H g_\phi^\dagger, \quad g_\phi = \prod_{j=1}^N \exp(i\phi \sigma_j^z/2). \]

The energy spectrum does not depend on the angle \( \phi \), and the critical behavior is independent of \( \phi \). The cor-
responding ground state of the new Hamiltonian may be obtained as \( |G_\phi(h)\rangle = g_\phi |G(h)\rangle \) which is described by

\[
|G_\phi(h)\rangle = \prod_k \left[ |v_1|0\rangle + v_2 e^{2i\phi} c_{k}^{\dagger} c_{-k}^{\dagger} |0\rangle + e^{2i\phi} v_3 c_{k}^{\dagger} c_{-k}^{\dagger} |0\rangle \right. \\
\left. \quad + v_4 e^{2i\phi} c_{k}^{\dagger} c_{-k}^{\dagger} |0\rangle \right. \\
\left. \quad + v_5 e^{2i\phi} c_{k}^{\dagger} c_{-k}^{\dagger} |0\rangle \right].
\]

The geometric phase of the ground state will be accumu-
lated when the system finish a cyclic evolution, corre-
sponding to varying the angle \( \phi \) from 0 to \( \pi \),

\[
\beta_\phi = -\frac{i}{N} \int_0^\pi \langle G_\phi(h) \rangle \frac{\partial}{\partial \phi} |G_\phi(h)\rangle.
\]
and is found to be

$$\beta_g = \frac{\pi}{2} \left[ 1 - \frac{2(\sum_{j=2}^{5} |v_j|^2) + 4|v_0|^2}{\sum_{j=1}^{n} |v_j|^2} \right]. \quad (6)$$

By this expression, we could study the geometric phase of the ground state and its derivation to investigate the QPTs of the model.

However a sudden drop of the fidelity, caused by the ground-state level-crossing is too obvious to be interesting enough, to detect the existence of the QPTs. Those ground-state wavefunctions are interested in which are differentiable in parameter space. Therefore, the overlap between two ground states at $h$ and $h+\delta h$ can be defined as

$$F(h, \delta h) = |\langle G(h) | G(h + \delta h) \rangle|, \quad (7)$$

in which $\delta h$ is a small deviation in a transverse field.

Three-dimensional panorama of the geometric phase of the model has been plotted as a functions of a transverse field and $J_2$ in Figs. 3 and 4 for $J_1 = 1$ and $J_1 = -1$ respectively, and system size set as $N = 100$. In Fig. 3 it has been observed that the geometric phase has its minimum value for $0 < h \leq h_2$ and $J_2 > 1$, which corresponds to the region (II) in the phase diagram (Fig. 1). It manifests that the geometric phase stay quite unchanged in region (II). The geometric phase undergo a strong qualitative change in spin-flop phase (region (I)) and enhances by increasing the magnetic field. In region (III) which corresponds to the saturate ferromagnetic phase ($h > h_0$) increasing the magnetic field saturates the geometric phase too. The strip antiferromagnetic phase (region (IV) in Fig. (2)) has been distinguished from the saturate ferromagnetic phase by the concave surface in Fig. 4. However, Figs. 3 and 4 survives that the geometric phase enhances with increasing the magnetic field. We should point out that the two critical lines ($h_0(J_2, h)$ and $h_2(J_2, h)$) at which the quantum phase transitions occur can be described by two assumed lines on the convex parts of the surfaces in Figs. 3 and 4. It is expected that the derivative of the geometric phase probes the QPTs. Then it will be helpful to study the first derivative of the GP with respect to the magnetic field. three dimensional plot of the derivative of the GP with respect to $h$ has been shown in Figs. 5 and 6 as a functions of $h$ and the Hamiltonian parameter $J_2$ for $N = 100$. There are clear peaks for the derivative of geometric phase near the critical lines $h_0(J_2, h)$ and $h_2(J_2, h)$ for finite size lattice. Although there is no real divergence for finite lattice size, but the curves exhibit marked anomalies with height of peak increasing with the system size. However, similar to the geometric phase, the notation of fidelity which also extract from the field of quantum information theory can be used to characterize the QPTs. The basic idea is that near a QPTs point there is a intensive enhancement in the degree of distinguishability between two ground states, corresponding to different values of the parameter space which defines the Hamiltonian. This distinguishability can be specified by the fidelity, which for pure states reduces to the amplitude of inner product or overlap. The behavior can be ascribed to a dramatic change in the structure of the ground state of the system during the quantum phase transition. Therefore, it should contain all the information that describes QPTs and topological order.

The ground state fidelity of the model has been depicted in Figs. 7 and 8 for $J_1 = 1$ and $J_1 = -1$, where we have set $N = 100$ and $\delta h = 0.001$. Obviously, there is a sudden drop in the ground state fidelity at the QPTs lines. Comparing the divergence of the geometric phase
FIG. 5: (Color online) Three-dimensional panorama of the derivative $d\beta_g/dh$ as a function of a transverse field and $J_2$ for lattice sizes $N = 100$ and $J_1 = 1$.

FIG. 6: (Color online) Three-dimensional plot of the derivative of geometric phase versus $h$ and $J_2$ for $N = 100$ and $J_1 = -1$.

FIG. 7: (Color online) Three-dimensional of the ground state fidelity as a function of magnetic field and Hamiltonian parameter $J_2$ for $N = 100$ and $\delta h = 0.001$.

FIG. 8: (Color online) The ground state fidelity versus $h$ and $J_2$. The parameters set as $N = 100$, $\delta h = 0.001$ and $J_1 = -1$.

and fidelity in thermodynamics limit specifies that the QPTs lines that come from the fidelity approach is consistent with the geometric phase results which we do not show here.

V. UNIVERSALITY AND SCALING OF GEOMETRIC PHASE

To better understand the properties of the geometric phase, and the relation between GP and quantum criticality, in this section we investigate the scaling behavior of GP by the finite size scaling approach. In Fig. 6 the derivative of $\beta_g$ with respect to the magnetic field has been shown for different system sizes for $J_1 = 1, J_2 = 2$. In thermodynamics limit $d\beta_g/dh$ diverges as the critical point is touched, while there is no divergence for finite lattice sizes. As the size of system becomes large, the derivative of GP tends to diverge close to the critical point. More information can be obtained when the maximum values of each plot and their positions are examined.

As it manifests the divergences of $d\beta_g/dh$ occur at $h_{c_1} = 1$ and $h_{c_2} = \sqrt{3}$ where exactly correspond to the critical points that were obtained using the energy gap analysis ($h_{c_1} = h_\pi, h_{c_2} = h_0$). The position of the first maximum point ($h_{MAX_1}$) of $d\beta_g/dh$ tends toward the critical point like $h_{MAX_1} = h_{c_1}^N - \theta_1$ ($\theta_1 = 1.72 \pm 0.01$) which has been plotted in Fig. 10. Moreover, we
have shown that the position of the second maximum $(h_{M \text{Max}2})$ of $d\beta_g/dh$ goes to the second critical point, such as $h_{M \text{Max}2} = h_{c2} - N^{-\theta_2}$ with $\theta_2 \approx 1.72 \pm 0.01$ (Fig. 10, inset).

A more detailed analysis manifest the scaling behavior of $d\beta_g/dh$ at the maximum points versus N. In this way we have plotted the scaling behavior of $d\beta_g/dh|_{h_{M \text{Max}i}, i = 1, 2}$ in Fig. 11 and inset of Fig. 11 respectively, which show a linear behavior of $d\beta_g/dh|_{h_{M \text{Max}i}}$ versus $\ln(N)$ with the exponents $\tau_i \approx 0.32 \pm 0.01$.

To study the scaling behavior of $\beta_g$ around the critical points, we perform finite-scaling analysis, since the maximum value of derivative of $\beta_g$ scales logarithmic. Then, by choosing a proper scaling function and taking into account the distance of the maximum of $\beta_g$ from the critical point, it is possible to make all the data for the value of $1 - \exp\left[\frac{d\beta_g}{dh} - \frac{d\beta_g}{dh}|_{h_{M \text{Max}i}}\right]$ as a function of $N^{1/\nu}(h - h_{M \text{Max}i})$ for different N collapse onto a single curve. The manifestation of the finite-size scaling is shown in Fig. 12 and its inset for several typical lattice sizes. It is clear that the different curves which are resemblance of vari-

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**FIG. 9:** (Color online) The derivative of $\beta_g$ with respect to magnetic field as a function of $h$ for various system size for $J_1 = 1, J_2 = 2$.

**FIG. 10:** (Color online) The scaling behavior of $h_{M \text{Max}1}$ in terms of system size ($N$), where $h_{M \text{Max}1}$ is the position of first maximum in Fig. 9. Inset: Scaling of the position $h_{M \text{Max}2}$ of $d\beta_g/dh$ for different-length chains ($J_1 = 1, J_2 = 2$).

**FIG. 11:** (Color online) The logarithm of the first maximum of $d\beta_g/dh$ versus the logarithm of chain size, $\ln(N)$, which is linear and shows a scaling behavior ($J_1 = 1, J_2 = 2$). Inset: The scaling behavior of the second maximum point of $d\beta_g/dh$ for different-length chain in region (II).

**FIG. 12:** (Color online) The finite-size scaling analysis for the case of logarithmic divergence around the first maximum point ($h_{M \text{Max}1}$) for $J_1 = 1, J_2 = 2$. The GP, considered as a function of system size and coupling, collapses on a single curve for different lattice sizes. Inset: A manifestation of finite-size scaling of $d\beta_g/dh$ around the second maximum point ($h_{M \text{Max}2}$) for various system sizes in region (II).
ous system sizes collapse to a single universal curve as expected from the finite size scaling ansatz. Our result shows that $\nu = 1$ is exactly correspond to the correlation length exponent of Ising model in transverse field ($\nu = 1$).

However, we have plotted the derivative of $\beta_g$ for $J_1 = 1, J_2 = 0.8$. versus $h$ in Fig. (13) for different lattice sizes which shows the singular behavior as the size of the system becomes large. As it manifests the divergences of $d\beta_g$ occurs at $h_c = \sqrt{1.8}$. The similar analysis shows the scaling behavior of the position of the maximum point ($h_{Max}$) tends toward the critical point like $h_{Max} = h_c - N^{-\theta}$ ($\theta \approx 1.728 \pm 0.01$) which has been plotted in Fig. (14). The linear behavior of $\frac{d\beta_g}{dh}$ at the maximum point ($h_{Max}$) versus $\ln(N)$ has been plotted in inset of Fig. (14). The exponent for this behavior is $\tau \approx 0.32 \pm 0.01$. We illustrate the finite-size scaling behaviors of $\beta_g$ around its maximum points in Fig. (15). It shows that the GP can be approximately collapsed to a single curve. These results show that all the key ingredients of the finite-size scaling are present in these cases too. In these cases scaling is fulfilled with the critical exponent $\nu = 1$, in agreement with the previous results and the universality hypothesis. We have also investigated the GP behavior for $J_1 = -1, J_2 < 1$. Our calculations show that the non-analytic and scaling behavior of GP are the same as the former results with the same finite size scaling ($\nu = 1$). The relation between the ground state fidelity and Berry phase has been studied in terms of geometric tensors [30]. A similar analysis show that the universality and scaling behavior of the ground state fidelity are the same as geometric phase counterparts in this model.

VI. SUMMARY AND CONCLUSIONS

In this work we have studied the geometric phase and ground fidelity of the one-dimensional extended quantum compass model in a transverse field. We use the Jordan-Wigner transformation to construct an explicit analytic expressions of geometric phase and ground state of the model. We show how the geometry phase and fidelity how could detect the quantum phase transitions in the inhomogeneous system. Moreover, we have investigated the universality and scaling properties of derivatives of geometric phase to confirm the results were obtained using the corelation functions analysis. The results show that the derivatives of the geometric phase diverge close to the critical points and exhibit beautiful scaling law. Comparing the scaling behavior of geometric phase and
correlation functions [18] shows that the scaling behavior of GP is the same as scaling behavior of correlation functions, and obtained exponent ($\nu = 1$) and universality behaviors (logarithmic divergence) confirm our previous results. Finally, there is a good agreement between the theoretical prediction and numerical data [29].

**Acknowledgments**

The author would like to thank A. G. Moghadam for reading the manuscript and valuable comments.

**VII. APPENDIX**

**A. Unitary Transformation**

The unitary transformation matrix $U$ which can transform the Hamiltonian Eq. [1] into a diagonal form, has the following form

$$U = \begin{pmatrix} U_{1, E_k^q} & U_{2, E_k^q} & U_{3, E_k^q} & 1 \\ U_{1, -E_k^q} & U_{2, -E_k^q} & U_{3, -E_k^q} & 1 \\ U_{1, E_k^p} & U_{2, E_k^p} & U_{3, E_k^p} & 1 \\ U_{1, -E_k^p} & U_{2, -E_k^p} & U_{3, -E_k^p} & 1 \end{pmatrix},$$

where

$$U_{1, \pm E_k^\alpha} = \frac{2h \pm E_k^\alpha}{J^*} - \frac{L^*}{L^* - J^2} \left[ J^* \left( 2h \mp E_k^\alpha \left( L(J^* L^2 - J^* L^2) - L^* (2h \pm E_k^\alpha)^2 \right) \right) \right],$$

$$U_{2, \pm E_k^\alpha} = \frac{-L^* (2h \mp E_k^\alpha)}{L^* - J^2} + J^* \left( 2h \mp E_k^\alpha \left( L(J^* L^2 - J^* L^2) - L^* (2h \pm E_k^\alpha)^2 \right) \right),$$

$$U_{3, \pm E_k^\alpha} = \frac{L(J^* L^2 - J^* L^2) - L^* (2h \pm E_k^\alpha)^2}{J^* \left( \gamma a - |L|^2 \right) - JL^2},$$

$$\alpha = q, p, \gamma = 1 \text{ for } \alpha = q \text{ and } \gamma = -1 \text{ for } \alpha = p.$$ 

**B. Ground State**

By using the unitary transformation the unnormalized ground state is obtained as

$$|G(h)\rangle = \prod_k [v_1|0\rangle + v_2 c_{-k}^{q\dagger} c_{k}^{q\dagger}|0\rangle + v_3 c_{-k}^{q\dagger} c_{-k}^{p\dagger}|0\rangle + v_4 c_{k}^{q\dagger} c_{p}^{p\dagger}|0\rangle + v_5 c_{k}^{p\dagger} c_{-k}^{q\dagger}|0\rangle + v_6 c_{-k}^{q\dagger} c_{p}^{q\dagger} c_{-k}^{p\dagger}|0\rangle],$$
\[
v_1 = \frac{64h^2 - J^2L^* + 2J^2(2J^2 - L^2) + 2|J|^2C - 16E_Gh^3 - 4h(|J|^2 - |L|^2 + C)E_G + 8h(3|J|^2 - 2|L|^2 + 2C)}{8h^2|J|^2 + 2|J|^2C + 2|J|^4 - J^2L^2 - J^2L^2},
\]
\[
v_2 = v_5 = \frac{E_G(4h - E_G)(JL^* - JL^*)/2}{8h^2|J|^2 + 2|J|^2C + 2|J|^4 - J^2L^2 - J^2L^2},
\]
\[
v_3 = \frac{(4h - E_G)(4hJ + J(J^2 - L^2) + JC)}{8h^2|J|^2 + 2|J|^2C + 2|J|^4 - J^2L^2 - J^2L^2},
\]
\[
v_4 = \frac{(4h - E_G)(4hJ + J(J^2 - L^2) + JC)}{8h^2|J|^2 + 2|J|^2C + 2|J|^4 - J^2L^2 - J^2L^2},
\]
\[
v_6 = 1,
\]
in which \( C = \sqrt{a^2 - b}. \)

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