Motion Planning and Robust Tracking for the Heat Equation using Boundary Control

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Abstract—Robust output tracking is addressed in this paper for a heat equation with Neumann boundary conditions and anti-collocated boundary input and output. The desired reference tracking is solved using the well-known flatness and Lyapunov approaches. The reference profile is obtained by solving the motion planning problem for the nominal plant. To robustify the closed-loop system in the presence of the disturbances and uncertainties, it is then augmented with PI feedback plus a discontinuous component responsible for rejecting matched disturbances with \textit{a priori} known magnitude bounds. Such control law only requires the information of the system at the same boundary as the control input is located. The resulting dynamic controller globally exponentially stabilizes the error dynamics while also attenuating the influence of Lipschitz-in-time external disturbances and parameter uncertainties. For the case when the motion planning is performed over the uncertain plant, an exponential Input-to-State Stability is obtained, preserving the boundedness of the tracking error norm. The proposed controller relies on a discontinuous term that however passes through an integrator, thereby minimizing the chattering effect in the plant dynamics. The performance of the closed-loop system, thus designed, is illustrated in simulations under different kinds of reference trajectories in the presence of external disturbances and parameter uncertainties.

I. INTRODUCTION

The heat equation is a first-order parabolic partial differential equation used to describe the heat and other diffusion processes such as chemical reactions, population growth, market price fluctuations, and fluids flow to name a few.

The motion planning has been addressed for the heat equation due to the flat output approach capable of parameterising the state dynamics using it as a reference [1], [2], [3], [4], [5]. Such approach provides an open-loop control able to track the desired reference but only for the nominal system with no uncertainties and disturbances and starting at some specific initial condition.

In order to robustify such a tracking control, one may invoke a feedback control. Examples of boundary tracking control of the heat equation utilize output feedback [6], [7] and backstepping designs [3], [4], [5] as well as sliding-modes techniques [8], [9].

Clearly, more realistic models should count for uncertainties and disturbances. A classic approach to deal with constant disturbances in the finite dimension setting is the integral action [10]. Dealing with a wider class of disturbances, sliding-mode control has long been recognized as a powerful control method to counteract non-vanishing external disturbances and unmodelled dynamics, even for infinite dimension systems (see, \textit{e.g.}, [11] and a very recent monograph [12]).

The design of robust controllers in the heat equation has been addressed in [9] using sliding-modes and using $H_\infty$ control in [13]. For the case of the tracking task, in [8] a control has been designed to compensate Lipschitz-in-time disturbances using sliding-modes, but using distributed control, and in [6], [7] robust boundary controls have been designed but only to compensate bounded disturbances and requiring to design two external systems (disturbance estimator and servo system) to fulfil the task. Therefore, the design of boundary controllers capable of the boundary tracking of the heat equation under a wider kind of disturbances calls for further investigation.

In this paper, a simple control strategy is proposed to achieve global exponential tracking of the heat dynamics. Using boundary state feedback only, a PI control, coupled to a discontinuous integral term, is designed. Such a controller is capable of compensating model uncertainties and Lipschitz-in-time disturbances using a continuous control signal. This strategy can be viewed as the merge of the classical integral action and the discontinuous functions commonly used in sliding mode control applications. Using a well-known strategy for motion planning, reference profiles are obtained for the nominal heat equation depending on the choice of the flat output reference. In the presence of parameter uncertainties, Lyapunov analysis yields the control synthesis, resulting in both norm-bounded tracking errors and exponential Input-to-State Stability (eISS) of the closed-loop system. Supporting simulations illustrate the closed-loop performance for different kinds of references in the presence of unbounded but Lipschitz-in-time disturbances and parameter uncertainties.

The outline of this work is as follows. The underlying heat conduction model and the control objective are introduced in Section \textbf{II}. Using the flat approach, the trajectory generation and open-loop tracking control are described in Section \textbf{III}. The feedback control design for the nominal and disturbed error systems is given in Section \textbf{IV}. The reliability of the proposed control strategy is supported in the simulation study of Section \textbf{V}. Finally, some concluding remarks are collected in Section \textbf{VI}.

Notation: The function $|\cdot| \gamma := |\cdot|^\gamma \text{sign}(\cdot)$ is determined for any $\gamma \in \mathbb{R}_{>0}$. Given a differentiable function $r(t)$, notation
$u(t)$ with $i \in \mathbb{Z}_{> 0}$ stands for the $i$-th time derivative of $r(t)$. The Sobolev space of absolutely continuous scalar functions $u(x)$ on $(a, b)$ with square integrable derivatives $u^{(i)}(x)$, $i = 1, \ldots, l$ and the norm $||u(\cdot)||_{H^1(a, b)} = \sqrt{\int_a^b \sum_{i=0}^l |u^{(i)}(x)|^2 \, dx}$ is typically denoted by $H^1(a, b)$ with $a \leq b$ and $l \in \{0, 1, 2, \ldots\}$. For ease of reference, the nomenclature $||u(\cdot)||_{H^1(a, b)} = ||u(\cdot)||_{L^2(a, b)} = ||u(\cdot)||$ is used throughout. The spatial derivatives are denoted by $u_x = \partial u/\partial x$ and $u_{xx} = \partial^2 u/\partial x^2$.

II. PROBLEM STATEMENT

Consider the following heat equation with Neumann boundary conditions (BC)

$$u_t(x, t) = d u_{xx}(x, t),$$
$$u_x(0, t) = 0, \quad u_x(D, t) = q + \varphi(t),$$

where $u(x, t)$ is the state vector evolving in the space $H^1(0, D)$, $x \in [0, D]$ is the space variable, $t \geq 0$ is the time variable, $d$ is the thermal diffusivity, $q$ is the boundary control input and the function $\varphi(t)$ is a disturbance term supposed to satisfy

$$|\varphi| \leq L$$

with an a priori known constant $L$. Furthermore, the positive parameters $d, D$ are uncertain, but bounded as

$$0 < d_m \leq d \leq d_M, \quad 0 < D_m \leq D \leq D_M,$$

by some known constant $d_m, d_M, D_m, D_M$. Apart from these bounds, The nominal values $d_n, D_n$ of $d, D$, satisfy the same bounds \cite{3}.

It is well-known (see, e.g., [12, Chapter 3]) that for arbitrary initial conditions $u(x, 0)$ of class $H^1(0, D)$, there exists a mild solution of the open-loop boundary value problem (BVP) \cite{1}. Throughout the paper, only mild solutions of the corresponding BVP are in play.

The objective of this work is to design a control input $q$ capable of driving the output

$$y(t) = u(0, t)$$

of the underlying BVP \cite{1} to a desired reference $r(t)$, despite the presence of uncertainties and/or disturbances.

III. MOTION PLANNING

Following \cite{1, 3, 12, Chapter 12}, the reference trajectory generation for the heat equation \cite{1} becomes available through its flat output \cite{3}. The trajectory generation is further performed for the unperturbed system with $\varphi(t) \equiv 0$. To begin with, the perfect knowledge of $d, D$ is assumed.

The state trajectory to follow is represented in the form

$$\bar{u}(x, t) = \sum_{i=0}^\infty a_i(t) x^{2i},$$

where the time-varying coefficients $a_i(t)$ have to be determined by substituting the latter sum into \cite{1, 3} and using the desired tracking $r(t) = \bar{u}(0, t)$. Thus, the reference state trajectory is specified as

$$\bar{u}(x, t) = \sum_{i=0}^\infty r^{(i)}(t) x^{2i} \frac{d^i}{d^i((2i)!)},$$

and the nominal input signal from the BC at $x = D$ is defined as

$$\bar{q} = \sum_{i=1}^\infty r^{(i)}(t) \frac{D^{2i-1}}{d^i((2i-1)!)}.\tag{6}$$

In order to guarantee that $u(x, t) \rightarrow \bar{u}(x, t)$ and $y(t) \rightarrow r(t)$, the convergence of \cite{5} to be guaranteed. The next theorem states which conditions should be imposed on the to-be-tracked reference in order to achieve this.

Theorem 1: The series \cite{5} is absolutely convergent provided that the output reference $r(t)$ fulfills

$$\sup_{t \geq 0} r^{(i+1)}(t) \leq \frac{2d}{D^2} (i+1) \sup_{t \geq 0} r^{(i)}(t) \quad \forall \ i \in \mathbb{Z}_{> 0}.\tag{7}$$

Proof: In order to check if under conditions of the theorem, \cite{5} is a convergent series, the ratio test is performed with $S_p = r^{(p)}(t) \frac{d^{2p}}{d^p(2p)!}$. Then

$$\lim_{p \to \infty} \frac{S_{p+1}}{S_p} = \lim_{p \to \infty} \left| \frac{r^{(p+1)}(t)}{r^{(p)}(t)} \right| \frac{x^{2p+2}}{x^{2p}} \frac{d^p}{d^p((2p)!) (2p)!} = \lim_{p \to \infty} \frac{x^2}{d} \left| \frac{r^{(p+1)}(t)}{r^{(p)}(t)} \right| \frac{1}{(2p+1)(2p+2)} < 1$$

for all $x \in [0, D]$ under condition \cite{7}. Theorem 1 is thus proved.

References $r(t)$, which are usually adopted in motion planning \cite{1}, \cite{2}, \cite{4}, \cite{5}, are the Gevrey class defined as follows.

Definition 1: A smooth function $r(t)$ is Gevrey of order $\alpha$ if exist $M, \beta > 0$ such that $\sup_{t \geq 0} |r^{(i)}(t)| \leq M \beta^i t^\alpha$, for all $i \in \mathbb{Z}_{> 0}$.

In the works cited above, the condition to guarantee the series convergence requires $r(t)$ to be Gevrey of order $\alpha < 2$. The next lemma links references $r(t)$ satisfying condition \cite{7} to a certain kind of the Gevrey functions.

Lemma 1: Any smooth function $r(t)$, fulfilling \cite{7}, is Gevrey of order $\alpha = 1$ such that Definition \cite{1} holds for $r(t)$ with $R = \frac{d^2}{D^2}$ and $M > 0$.

Proof: From the Definition \cite{1} one concludes that

$$\sup_{t \geq 0} |r^{(i+1)}(t)| \leq M \left( \frac{i+1}{R^i} \right)^\alpha \leq M \frac{(i+1)^\alpha}{R^i},$$

thereby deriving the inequality

$$\sup_{t \geq 0} |r^{(i+1)}(t)| \leq \frac{(i+1)^\alpha}{R} \sup_{t \geq 0} |r^{(i)}(t)|.$$

Since \cite{7} follows from the latter inequality with the selection of $\alpha, \beta, R, M$ under the lemma conditions, the proof is thus completed.

Although the principle reference, used in the afore-cited works, is the so-called "bump function" (see \cite{1}), its proposed counterpart \cite{7} allows one to exemplify more admissible references among analytical functions such as:

- $F.1 \ r(t) = A$ for any $A \in \mathbb{R}$.
- $F.2 \ r(t) = A t \in [0, t_f]$ for any $A \in \mathbb{R}, t_f \geq \frac{d^2}{2d}$.
- $F.3 \ r(t) = A \sin(\omega t)$, $\omega \in (0, \frac{\pi}{2d})$ for any $A \in \mathbb{R}$, $\omega \leq \frac{d^2}{2d}$.
- $F.4 \ r(t) = A e^{\beta t}$, $\beta \in \mathbb{R}$ for any $A \in \mathbb{R}$, $|\beta| \leq \frac{d^2}{2\beta}$. 


Remark 1: The output (4) matches the reference trajectory \( r(t) \) only if the state trajectory to follow satisfies the same initial condition as that of the plant, i.e., if \( u(x,0) = \bar{u}(x,0) \). Furthermore, the nominal control (6) is not capable of compensating any non-trivial disturbance \( \varphi(t) \neq 0 \). Next the tracking problem of interest is addressed for the arbitrarily initialized state trajectory to follow in the presence of matched disturbances.

IV. TRACKING CONTROL DESIGN

Setting the state deviation
\[
\bar{u}(x,t) = u(x,t) - \bar{u}(x,t),
\]
from the reference trajectory \( \bar{u}(x,t) \), given by (5), the error dynamics (8) are then governed by
\[
\dot{\bar{u}}(x,t) = d\bar{u}_{xx}(x,t),
\]
where the control input
\[
q = \bar{q} + \tilde{q}
\]
is precomposed in terms of the reference input \( \bar{q} \), determined by (4), and the virtual component \( \tilde{q} \). Now the tracking objective for the nominal reference (5) is reduced to the virtual input design \( \tilde{q} \) exponentially stabilizing the error dynamics (9) in the origin.

A. Disturbance-free tracking

First, the disturbance-free case \( \varphi(t) \equiv 0 \) is analysed. Selecting the control \( \tilde{q} \) as
\[
\tilde{q} = -\frac{\lambda_1}{D_n} \bar{u}(D,t),
\]
where \( \lambda_1 \) is a gain to be tuned and \( D_n \) the nominal value of \( D \). The next result is in order.

Theorem 2: Let \( \varphi(t) \equiv 0 \) and let the control parameters in (11) be such that
\[
\lambda_1 > \frac{D_n}{D_m}.
\]
(12)

Then the error dynamics (9), driven by (11), are globally exponentially stable.

Proof: Consider the positive definite Lyapunov functional candidate \( V = \frac{1}{2} ||\bar{u}(x,t)||^2 \). Its derivative along the system (9) reads as
\[
\dot{V} = \int_0^D \bar{u}(x,t)\ddot{u}_x(x,t) \, dx = d \int_0^D \bar{u}(x,t)\bar{u}_{xx}(x,t) \, dx.
\]

Applying the integration by parts and then employing the BC and Poincare’s inequality, it follows that
\[
\dot{V} \leq -\frac{d}{2D^2} ||\bar{u}(x,t)||^2 + \frac{d}{D} \bar{u}^2(D,t) + d\bar{u}(D,t)\bar{q}.
\]

Now, taking into account the control law (11), coupled to (12), the Lyapunov derivative is further estimated as
\[
\dot{V} \leq -\frac{d}{2D^2} ||\bar{u}(x,t)||^2 - \left( \frac{\lambda_1}{D_n} - \frac{1}{D} \right) d\bar{u}^2(D,t)
\leq -\frac{d}{2D^2} ||\bar{u}(x,t)||^2 \leq -\frac{d}{D^2} V,
\]
that guarantees the exponential decay of the system dynamics (9). Since the Lyapunov functional is radially unbounded, (9) is concluded to be globally exponentially stable.

B. Tracking under disturbances

In the presence of the external disturbance \( \varphi(t) \neq 0 \), the proposed control \( \tilde{q} \) in (11) is no longer capable of stabilizing the error dynamics. To robustify the control law in the disturbance-corrupted case, it is modified to
\[
\tilde{q} = -\frac{\lambda_1}{D_n} \bar{u}(D,t) + \nu, \quad \nu = -\lambda_2 \bar{u}(D,t) - \lambda_3 [\bar{u}(D,t)]^0,
\]
where \( \lambda_1, \lambda_2, \lambda_3 \) are gains to be tuned and \( D_n \) is the nominal value of \( D \).

The proposed feedback law (13) is composed by a PI control and a discontinuous term passing through an integrator. It generates a continuous control signal despite having a discontinuous (multi-valued) right-hand side in the manifold \( \bar{u}(D,\cdot) = 0 \). The precise meaning of the solutions of the distributed parameter system (9) driven by this discontinuous controller, are viewed in the Filippov sense [14]. Extension of the Filippov concept towards the infinite-dimensional setting may be found in [12]. The present paper focuses on the tracking synthesis whereas the well-posedness analysis of the closed-loop system (9), (13) is similar to that of [8] and it remains beyond the scope of the paper. Thus, for the closed-loop system in question, it is assumed that it possesses a Filippov solution.

The closed-loop system (9), driven by (13), reads as
\[
\ddot{u}_x(x,t) = d\bar{u}_{xx}(x,t),
\]
where \( \tilde{q} \) is the feedback component
\[
\ddot{u}_x(0,t) = 0, \quad \ddot{u}_x(D,t) = q - \tilde{q} + \varphi(t) = \tilde{q} + \varphi(t),
\]
(9)

with \( \delta = \nu + \varphi \), being substituted into (13) for \( \nu = \delta - \varphi \) for deriving \( \delta \)-dynamics (14).

Theorem 3: Let the error dynamics (9) be driven by the feedback \( \tilde{q} \), governed by (13) and tuned in accordance with
\[
\lambda_1 > \frac{D_n}{D_m}, \quad \lambda_2 > D_M, \quad \lambda_3 > L.
\]

Then the closed-loop error dynamics (14) are globally exponentially stable despite the presence of any boundary disturbance \( \varphi \) of class (2).

Proof: Consider the positive definite Lyapunov functional candidate
\[
V = \frac{1}{2} ||\bar{u}(x,t)||^2 + \frac{1}{2} \delta^2,
\]
(16)
and employing the magnitude bound (2) for \( \dot{\varphi} \), compute its time derivative along the error dynamics (14), thus obtaining
\[
\dot{V} \leq -\frac{d}{2D^2} \| \ddot{u}(x, t) \| ^2 \left( \lambda_1 + \frac{1}{D} \right) + d \ddot{u}(D, t) \delta + (-\lambda_2 \ddot{u}(D, t) - \lambda_3 [\ddot{u}(D, t)] ^0 + \dot{\varphi}) \delta
\]
\[
\leq -\frac{d}{2D^2} \| \ddot{u}(x, t) \| ^2 \left( \lambda_1 + \frac{1}{D} \right) + (\lambda_2 + \lambda_3 [\ddot{u}(D, t)] ^0 + \varphi) \delta
\]
\[
\leq -\alpha V
\]
where \( \alpha = \min \{ \frac{d}{D^2}, \lambda_2 - d + \lambda_3 - L \} \). The latter inequality ensures the exponential stability of (14). Since the Lyapunov functional is radially unbounded, the result holds globally.

Remark 2: Due to the exponential decay of the Lyapunov functional (16) and by virtue of \( \delta = \nu + \dot{\varphi} \), the virtual input \( \nu \) approaches the negative disturbance value, i.e., \( \nu(t) \to -\dot{\varphi}(t) \) as \( t \to \infty \).

Remark 3: The proposed feedback (13) requires only the boundary state information \( u(D, t) \) at \( x = D \) only. Thus, the boundary output feedback is available to perform the tracking task.

C. Tracking under uncertain motion planning

The feedback controller, constructed by now, enforces the heat conduction dynamics to exponentially track an output reference signal. An admissible reference to be tracked is obtained from the motion planning assuming to perfectly know the nominal system (1). This is not however the case in realistic applications where only nominal values \( d_n \), \( D_n \), of respectively \( d, D \) are available. From now on, the reference state trajectory \( \bar{u}(x, t) = \sum_{i=0}^{\infty} r^{(i)}(t) \frac{D^i}{(2i)!} \), and the nominal input signal \( \bar{q} = \sum_{i=1}^{\infty} r^{(i)}(t) \frac{D^i}{(2i-1)!} \), rely on the nominal plant parameter values rather than on their real values.

Using the same error variable (8), the closed-loop error dynamics, enforced by (13), are governed by
\[
\hat{u}_t(x, t) = d \hat{u}_x(x, t) + M(x, t),
\]
\[
\hat{u}_x(0, t) = 0, \quad \hat{u}_x(D, t) = \lambda_1 D_n \hat{u}(D, t) + \delta, \quad \hat{\delta} = -\lambda_2 \hat{u}(D, t) - \lambda_3 [\hat{u}(D, t)] ^0 + \dot{\varphi}(t),
\]
where the perturbation term
\[
M(x, t) = (d - d_n) \sum_{i=1}^{\infty} r^{(i)}(t) \frac{x^{2i-2}}{i!}
\]
is due to the uncertainties on \( d \).

Lemma 2: The uncertain term \( M(x, t) \) is uniformly bounded
\[
\| M(\cdot, t) \| \leq (d_M - d_n)^2 L_1 \quad \forall \quad t \geq 0, \quad (19)
\]
with some \( L_1 \in \mathbb{R}^*_0 \) provided that the output reference \( r(t) \) fulfills the condition
\[
\sup_{t \geq 0} \left| r^{(i)}(t) \right| \leq \begin{cases} d_n \sup_{t \geq 0} r^{(i)}(t), & \forall \quad i \in \mathbb{Z}^*_0. \end{cases}
\]

Proof: With the upper bound \( D_M \) of \( d \), the norm of \( M(x, t) \) is estimated as follows
\[
\| M(x, t) \| = \int_0^D M^2(x, t) \, dx
\]
\[
\leq (d_M - d_n)^2 \sum_{i=1}^{\infty} \left| r^{(i)}(t) \right|^2 \int_0^D \frac{x^{4i-4}}{D^i(2i-2)!} \, dx
\]
\[
= (d_M - d_n)^2 \sum_{i=1}^{\infty} \left| r^{(i)}(t) \right|^2 \frac{D^{4i-3}}{D^i(2i-3)(2i-2)!}. \]

In order to check if the latter series is absolutely convergent, the ratio test is now performed with \( S_p = \frac{x^{4i-4}}{D^i(2i-2)!} \) to conclude that
\[
\lim_{p \to \infty} \frac{S_{p+1}}{S_p} = \lim_{p \to \infty} \frac{r^{(p+1)}(t)}{r^{(p)}(t)} \left( \frac{D^{4p+1}}{D^p 4p - 3} \right)^2 \left( \frac{1}{(2p-1)(2p)} \right)^2 \frac{4p - 3}{4p + 1} \frac{1}{(2p)!} \frac{1}{(2p-1)!} \frac{1}{(2p-2)!} < 1
\]
provided condition (20) holds true. Hence, the series is convergent and approaches a finite value \( L_1 \), thereby ensuring the term \( \| M(x, t) \| \) is bounded according to (19).

For later use, let us specify the eISS concept for solutions \( \bar{u}(x, t), \bar{\delta} \) of the BVP (17) with respect to the \( H^3(0, D) \)-norm
\[
\Gamma(t) = \left( \| \ddot{u}(x, t) \| ^2 + \delta^2 \right)^{1/2} ;
\]
see [15] for ISS details in the infinite-dimensional setting.

Definition 2: System (17) is called eISS in the sense of the norm (21) with respect to \( \bar{M}(x, t) \) if there exist \( \alpha_1, \alpha_2 > 0 \) and a function \( \gamma \) of class \( \mathcal{X} \) such that the inequality
\[
\Gamma(t) \leq \alpha_1 e^{-\alpha_2 t} \Gamma(0) + \gamma(L_1),
\]
holds on arbitrary solution \( \bar{u}(x, t) \) of (17), with \( \Gamma(0) < \infty \) and \( M(x, t) : \|M(\cdot, t)\| \leq L_1 < \infty \) for all \( t \geq 0 \).

The eISS of system (17) is then established.

Theorem 4: Let the controller gains in (13) be such that (15) holds and let the output reference \( r(t) \) respect condition (19). Then system (17) is eISS.

Proof: The norm of the Lyapunov functional (16) is straightforwardly estimated as \( m_1 \Gamma^2(t) \leq V(t) \leq m_2 \Gamma^2(t) \), with some \( m_1, m_2 > 0 \). Following the same steps as before,
its derivative along the trajectories of system (17) is evaluated as
\[ \dot{V} \leq -\frac{d}{2D} \| \dot{u}(x,t) \|^2 + \int_0^D \ddot{u}(x,t)M(x,t) \, dx \\
- (\lambda_2 - d + \lambda_3 |\dot{u}(D,t)|^{-1} - L |\ddot{u}(D,t)|^{-1})^2 \Delta_2. \]

Using the Cauchy-Schwarz inequality and the above bounds of the Lyapunov functional as well as bound (19) of the term \( \| M(x,t) \| \), its derivative reads as
\[ \dot{V} \leq -\alpha_m \Gamma^2(t) + \| \dot{u}(x,t) \| \| M(x,t) \| \\
\leq -\alpha_m (1-\theta) \Gamma^2(t) - \alpha_m \theta \Gamma^2(t) + (d_M - d_n)^2 L_1 \Gamma(t) \\
\leq -\alpha_m (1-\theta) \Gamma^2(t) \quad \forall \quad \Gamma(t) \geq \frac{(d_M - d_n)^2}{\alpha_m \theta} L_1, \]
where \( 0 < \theta < 1, \alpha = \min \{ \frac{d}{D}, \lambda_2 - d + \lambda_3 - L \} \). Last expression results in the norm \( \Gamma(t) \) to be globally ultimately bounded with respect to \( M(x,t) \). Furthermore, applying the comparison lemma (see [10]) one concludes the eISS bounded with respect to \( t \).

V. SIMULATIONS

The proposed control strategy [5], [10], [13] has been implemented in the system (11) using Matlab Simulink with Euler’s integration method of fixed step and a sampling time equal to 50 [ms]. The heat equation was implemented using \( d = 0.05, D = 10 \) and \( \phi(t) = 0.01t + 2\sin(t) \) and the finite-differences approximation technique, discretizing the spatial domain \( x \in [0, D] \) into 51 ordinary differential equations (ODE). The initial condition was set in \( u(x,0) = 0 \).

The tracking reference \( r(t) \) and the corresponding nominal reference \( \bar{u}(x,t) \) and control \( \bar{q} \), are defined depending on the simulation time and expressions (5)-(6):

1) \( t \in [0, 1 \times 10^4] \) [s]:
   \[ r(t) = At, \quad A = 1 \times 10^{-3}, \\\n   \bar{u}(x,t) = At + A\frac{x^2}{2}, \quad \bar{q} = A\frac{D}{2}. \]

2) \( t \in [1 \times 10^4, 2 \times 10^4] \) [s]:
   \[ r(t) = 20, \quad \bar{u}(x,t) = 20, \quad \bar{q} = 0. \]

3) \( t \in [2 \times 10^4, 3 \times 10^4] \) [s]:
   \[ r(t) = Ae^{-b(t-2 \times 10^4)}, \quad A = 20, \quad b = 1 \times 10^{-3}, \\\n   \bar{u}(x,t) = Ae^{-b(t-2 \times 10^4)} \cos \left( D\sqrt{\frac{b}{d}} \right), \\\n   \bar{q} = -A \sqrt{\frac{b}{d}} e^{-b(t-2 \times 10^4)} \sin \left( D\sqrt{\frac{b}{d}} \right). \]

4) \( t \in [3 \times 10^4, 4 \times 10^4] \) [s]:
   \[ r(t) = Asin(\omega t), \quad A = 10, \quad \omega = 1 \times 10^{-3}, \\\n   \bar{u}(x,t) = \frac{1}{2}Ae^{-D\sqrt{\frac{\omega}{d}}} \sin (\omega t + D\sqrt{\frac{\omega}{d}}) + \frac{1}{2}Ae^{-D\sqrt{\frac{\omega}{d}}} \sin (\omega t - D\sqrt{\frac{\omega}{d}}), \\\n   \bar{q} = -\frac{1}{2}Ae^{-D\sqrt{\frac{\omega}{d}}} \cos (\omega t + D\sqrt{\frac{\omega}{d}}) + \frac{1}{2}Ae^{-D\sqrt{\frac{\omega}{d}}} \cos (\omega t - D\sqrt{\frac{\omega}{d}}), \]

All of the proposed output references fulfil the condition (7), and more precisely, the conditions described in F.1-F.4. The respective gains of the error control \( \bar{q} \) have been selected according to (15) as \( \lambda_1 = 1, \lambda_2 = 10, \lambda_3 = 2.5 \) and \( D_n = 9 \). The results are displayed in Figs. 1-4. The solution \( u(x,t) \) of the heat equation is performing a tracking over the space variable \( x \). The main objective of stabilizing the output \( y = u(0,t) \) over the four successive references \( r(t) \) is achieved despite the presence of the unbounded but Lipschitz perturbation \( \varphi(t) \) (see Fig. 2). Furthermore, the designed control strategy is able to stabilize the norm of the error despite the abrupt change between references, as seen in Fig. 3. The boundary control \( q \) shown in Fig. 4 clarifies how such disturbance compensation is performed. This is due to the presence of the discontinuous term on the control design. Nevertheless, the control signal generated is continuous throughout the tracking task.

1See [3, Chapter 12] for more details on how to obtain the analytic nominal expressions for exponential and sinusoidal references.
VI. CONCLUSIONS

The heat equation with boundary control is analysed, and robust output tracking is developed. The proposed boundary control requires the state at the boundary only and it is composed of a PI control and an extra discontinuous term, passing through an integrator. Such a controller is typically used the sliding mode control theory and it is derived from a Lyapunov approach. The controller, thus composed, compensates Lipschitz-in-time disturbances and uncertainties in the system. It generates sliding modes in the actuator dynamics so that after passing through the integrator, a continuous control signal is applied to the underlying system, thereby diminishing the chattering effect. Capabilities of tracking heat conduction dynamics along different kinds of boundary references and good robustness properties of the developed design are illustrated in the simulation study. The reference profiles are obtained for the nominal heat conduction model using the flatness approach. Being applied to the disturbed plant model with uncertain parameters, the closed-loop eISS is additionally established along with the boundedness of the tracking error norm. Robustification of the motion planning to inherit the exponential stability from the nominal plant is among open problems calling for further investigation.

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