Invariants of real symplectic 4-manifolds and lower bounds in 
real enumerative geometry

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Abstract :
We first present the construction of the moduli space of real pseudo-holomorphic curves in a given real symplectic manifold. Then, following the approach of Gromov and Witten \[3, 15, 10\], we construct invariants under deformation of real rational symplectic 4-manifolds. These invariants provide lower bounds for the number of real rational $J$-holomorphic curves in a given homology class passing through a given real configuration of points.

Introduction

Let $(X,\omega, c_X)$ be a real symplectic 4-manifold, that is a triple made of a 4-manifold $X$, a symplectic form $\omega$ on $X$ and an involution $c_X$ on $X$ such that $c_X^*\omega = -\omega$, all of them being of class $C^\infty$. The fixed point set of $c_X$ is called the real part of $X$ and is denoted by $\mathbb{R}X$. It is either empty or a smooth lagrangian submanifold of $(X,\omega)$. Let $d \in H_2(X;\mathbb{Z})$ be a homology class satisfying $c_1(X)d > 0$, where $c_1(X)$ is the first Chern class of the symplectic 4-manifold $(X,\omega)$. From Corollary 1.5 of \[7\], we know that the existence of such a class forces the 4-manifold $X$ to be rational or ruled, as soon as $d$ is not the class of an exceptional divisor. Hence, from now on, we will assume $(X,\omega)$ to be rational. Let $x \subset X$ be a real configuration of points, that is a subset invariant under $c_X$, made of $c_1(X)d - 1$ distinct points. Denote by $r$ the number of such points which are real. Let $\mathcal{J}_\omega$ be the space of almost complex structures of $X$, tamed by $\omega$, and which are of H"older class $C^{l,\alpha}$ where $l \geq 2$ and $\alpha \in ]0,1[$ are fixed. This space is a contractible Banach manifold of class $C^{l,\alpha}$ (see \[1\], p. 42). Denote by $\mathbb{R}\mathcal{J}_\omega \subset \mathcal{J}_\omega$ the subspace consisting of those $J \in \mathcal{J}_\omega$ for which $c_X$ is $J$-antiholomorphic. It is a contractible Banach submanifold of class $C^{l,\alpha}$ of $\mathcal{J}_\omega$ (see proposition \[11\]). If $J \in \mathbb{R}\mathcal{J}_\omega$ is generic enough, then there are only finitely many $J$-holomorphic rational curves in $X$ passing through $x$ in the homology class $d$ (see Theorem \[1.1\]). These curves are all nodal and irreducible. The total number of their double points is $\delta = \frac{1}{2}(d^2 - c_1(X)d + 2)$. Let $C$ be such a curve which is assumed to be real. We define the mass of the curve $C$ to be the number of its real isolated double points (see \[21\] for a definition). For every integer $m$ ranging from $0$ to $\delta$, denote by $n_d(m)$ the total number of real $J$-holomorphic rational curves of mass $m$ in $X$ passing through $x$ and realizing the homology class $d$. Then define :

$$\chi^d_r(x,J) = \sum_{m=0}^{\delta} (-1)^m n_d(m).$$

The main result of this paper is the following (see Theorem \[2.1\]) :

Keywords : Symplectic manifold, real algebraic curve, moduli space, enumerative geometry.
AMS Classification : 14N10, 14P25 , 53D05 , 53D45.
Theorem 0.1 The integer $\chi^d(x, J)$ neither depends on the choice of $J$ nor on the choice of $x$ (provided the number of real points in this configuration is $r$).

For convenience, this integer will be denoted by $\chi^d$, and when $r$ does not have the same parity as $c_1(X)d - 1$, we put $\chi^d$ to be 0. We then denote by $\chi^d(T)$ the polynomial $\sum_{y=0}^{c_1(X)d-1} \chi^d_y r^y \in \mathbb{Z}[T]$. It follows from Theorem 0.1 that the function $\chi : d \in H_2(X; \mathbb{Z}) \to \chi^d(T) \in \mathbb{Z}[T]$ only depends on the real symplectic 4-manifold $(X, \omega, c_X)$ and is invariant under deformation of this real symplectic 4-manifold. This invariant is proved to be non-trivial for degree less or equal than five in the complex projective plane, see Proposition 3.6. As an application of this invariant, we obtain the following lower bounds (see Corollary 2.2):

Corollary 0.2 The integer $|\chi^d|$ gives a lower bound for the total number of real rational $J$-holomorphic curves of $X$ passing through $x$ in the homology class $d$, independently of the choice of a generic $J$ in $\mathbb{R}J_\omega$.

Now, let $y = (y_1, \ldots, y_{c_1(X)d-2})$ be a real configuration of $c_1(X)d - 2$ distinct points of $X$, and $s$ be the number of those which are real. We assume $y_{c_1(X)d-2}$ to be real, so that $s \neq 0$. If $J \in \mathbb{R}J_\omega$ is generic enough, then there are only finitely many $J$-holomorphic rational curves in $X$ passing through $y$ in the homology class $d$ and having a node at $y_{c_1(X)d-2}$. These curves are all nodal and irreducible. For every integer $m$ ranging from 0 to $\delta$, denote by $n^+_d(m)$ (resp. $n^-_d(m)$) the total number of these curves which are real, of mass $m$ and with a non-isolated (resp. isolated) real double point at $y_{c_1(X)d-2}$ (see 3.1). Define then:

$$\theta^d_s(y, J) = \sum_{m=0}^{\delta} (-1)^m (n^+_d(m) - n^-_d(m)).$$

Theorem 0.3 The integer $\theta^d_s(y, J)$ neither depends on the choice of $J$ nor on the choice of $y$ (provided the number of real points in this configuration is $s$).

Once more, for convenience, the integer $\theta^d_s(y, J)$ will be denoted by $\theta^d_s$, and we put $\theta^d_s = 0$ when $s$ does not have the same parity as $c_1(X)d$. This invariant makes it possible to give relations in between the coefficients of the polynomial $\chi^d$, namely (see Theorem 3.2):

Theorem 0.4 Let $d \in H_2(X; \mathbb{Z})$ and $r$ be an integer between 0 and $c_1(X)d - 3$. Then $\chi^d_{r+2} = \chi^d_r + 2\theta^d_{r+1}$.

The text is organized as follows. The first paragraph is devoted to the construction of the moduli space $\mathcal{M}^d_g(x)$ (resp. $\mathbb{R}\mathcal{M}^d_g(x)$) of pseudo-holomorphic curves (resp. real pseudo-holomorphic curves) of genus $g$, in the homology class $d$, and passing through the given real configuration of points $x$. The space $\mathbb{R}\mathcal{M}^d_g(x)$ appears to be the fixed point set of a $\mathbb{Z}/2\mathbb{Z}$-action on $\mathcal{M}^d_g(x)$ induced by $c_X$. The main result of this paragraph is the theorem of regular values (see Theorem 1.11) which states that the set of regular values of the Fredholm projection $\pi : \mathcal{M}^d_g(x) \to J_\omega$ intersects $\mathbb{R}J_\omega$ in a dense set of the second category of $\mathbb{R}J_\omega$. This theorem is proved for $g = 0$ in this paragraph and for $g > 0$ in appendix A. This first paragraph is independant of the other ones and is presented in the framework of real symplectic manifolds of any dimension, since this does not require more work. The second paragraph is
devoted to the definition of the invariant $\chi$ and the proof of theorem 0.1. It involves in particular many genericity arguments which are given in §2.2. Few computations of this invariant and applications to real enumerative geometry are given in §2.1. For this paragraph and the third one, we restrict ourselves to rational curves in real rational symplectic 4-manifolds. Finally, the third paragraph is devoted to the definition of the invariant $\theta$, the statements and proofs of theorems 0.3 and 0.4 and the proof of the non-triviality of $\chi^d$ for $d = 4, 5$ in the complex projective plane. With the exception of this non-triviality, all these results have been announced in [14].

**Acknowledgements:**
I am grateful to J.-C. Sikorav for the fruitful discussions we had on the theory of pseudo-holomorphic curves.

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1 Moduli space of real pseudo-holomorphic curves

1.1 Preliminaries

1.1.1 Teichmüller space $T_{g,m}$ and mapping class group $G$

Let $S$ be a compact connected oriented surface of genus $g$. Let $m \in \mathbb{N}$ and $z = (z_1, \ldots, z_m)$ be an ordered set of $m$ distinct points in $S$. Let $\tau$ be a given order two permutation of the set $\{1, \ldots, m\}$. Denote by $Diff^+(S, z)$ the group of diffeomorphisms of class $C^{k+1,\alpha}$ of $S$, $k \geq 1$, which preserve the orientation of $S$ and are the identity once restricted to $z$. Similarly, let $Diff(S, z)$ be the group of diffeomorphisms of class $C^{k+1,\alpha}$ of $S$, which fix $z$ when they preserve the orientation, or induce the permutation on $z$ associated to $\tau$ otherwise.

Let $Diff_0^+(S, z)$ be the subgroup of $Diff^+(S, z)$ consisting of diffeomorphisms isotopic to the identity, and $\mathcal{J}_S$ be the space of complex structures of $S$ of class $C^{k,\alpha}$ which are compatible with the orientation of $S$. Let $s_*$ be the morphism $Diff(S, z) \to \mathbb{Z}/2\mathbb{Z}$ of kernel $Diff^+(S, z)$. The space $\mathcal{J}_S$ is a contractible Banach manifold of class $C^{k,\alpha}$ equipped with an action of the group $Diff(S, z)$ given by:

$$(\phi, J_S) \in Diff(S, z) \times \mathcal{J}_S \mapsto s_*(\phi)(\phi^{-1})^*J_S,$$

where $(\phi^{-1})^*J_S = d\phi \circ J_S \circ d\phi^{-1}$. Denote by $T_{g,m}$ the Teichmüller space $\mathcal{J}_S/\text{Diff}^+_0(S, z)$, it is a finite dimensional contractible manifold. We fix a complex structure on $T_{g,m}$ such that $T_{g,m}$ is the universal curve over $T_{g,m}$, and denote by $G^+$ (resp. $G$) the group of holomorphic (resp. holomorphic or anti-holomorphic) automorphisms of $U_{g,m}$. When $(g, m) \notin \{(0, 0), (0, 1), (0, 2), (1, 0)\}$, we have $G = Diff(S, z)/Diff_0^+(S, z)$, the mapping class group of $S$. The surjective morphism $G \to \mathbb{Z}/2\mathbb{Z}$ of kernel $G^+$ will also be denoted by $s_*$, and we put $G^- = G \setminus G^+$. Note that the exact sequence $1 \to G^+ \to G \to \mathbb{Z}/2\mathbb{Z}$ splits.

1.1.2 The manifold $\mathbb{R}\mathcal{J}_\omega$

The real structure $c_X$ of $(X, \omega)$ induces a $\mathbb{Z}/2\mathbb{Z}$-action on $\mathcal{J}_\omega$ given by $\overline{c_X}^*: J \in \mathcal{J}_\omega \mapsto \overline{c_X}(J) = -dc_X \circ J \circ dc_X$. Denote by $\mathbb{R}\mathcal{J}_\omega$ the fixed point set of this action. It consists of those $J \in \mathbb{R}\mathcal{J}_\omega$ for which $c_X$ is $J$-antiholomorphic. Let $J_0 \in \mathbb{R}\mathcal{J}_\omega$, the involution $\overline{c_X}^*$ induces an involution $d_{J_0}\overline{c_X}^*$ on the tangent space $T_{J_0}\mathcal{J}_\omega = L^{l,\alpha}(X, \Lambda^{0,1}X \otimes \mathbb{C} T X)$, where $L^{l,\alpha}(X, \Lambda^{0,1}X \otimes \mathbb{C} T X)$ denotes the Banach space of sections of class $C^{l,\alpha}$ of the vector bundle $\Lambda^{0,1}X \otimes \mathbb{C} T X$ over $X$. Denote by $L^{l,\alpha}(X, \Lambda^{0,1}X \otimes \mathbb{C} T X)_{+1}$ (resp. $L^{l,\alpha}(X, \Lambda^{0,1}X \otimes \mathbb{C} T X)_{-1}$) the eigenspace of this involution associated to the eigenvalue $+1$ (resp. $-1$), so that $T_{J_0}\mathcal{J}_\omega = L^{l,\alpha}(X, \Lambda^{0,1}X \otimes \mathbb{C} T X)_{+1} \oplus L^{l,\alpha}(X, \Lambda^{0,1}X \otimes \mathbb{C} T X)_{-1}$.

**Proposition 1.1** The fixed point set $\mathbb{R}\mathcal{J}_\omega$ of $\overline{c_X}^*$ is a Banach submanifold of $\mathcal{J}_\omega$ of class $C^{l,\alpha}$ which is non-empty and contractible. For every $J_0 \in \mathbb{R}\mathcal{J}_\omega$, the tangent space $T_{J_0}\mathbb{R}\mathcal{J}_\omega$ is $L^{l,\alpha}(X, \Lambda^{0,1}X \otimes \mathbb{C} T X)_{+1}$.

Note that in particular, the decomposition $T_{J_0}\mathcal{J}_\omega = L^{l,\alpha}(X, \Lambda^{0,1}X \otimes \mathbb{C} T X)_{+1} \oplus L^{l,\alpha}(X, \Lambda^{0,1}X \otimes \mathbb{C} T X)_{-1}$ is a direct sum of locally trivial Banach sub-bundles.

**Proof:**

Let us prove first that $\mathbb{R}\mathcal{J}_\omega$ is non-empty. Let $g_X$ be a Riemannian metric on $X$ invariant under $c_X$. Let $A \in L^\infty(X, End_{\mathbb{R}}(T X))$ be such that for every $x \in X$, $u, v \in T_xX$, $\omega_x(u, v) = \langle A(x)(u), v \rangle$. Then for every $J_0 \in \mathbb{R}\mathcal{J}_\omega$, we have $T_{J_0}\mathbb{R}\mathcal{J}_\omega = L^{l,\alpha}(X, \Lambda^{0,1}X \otimes \mathbb{C} T X)_{+1}$.
This is a Banach manifold whose tangent space at \( u \) induced by \( \nabla \) set is called a pseudo-holomorphic map from \( TX \) connections on the bundles associated to \( \mathcal{J}_\omega \). Let \( \mathcal{J}_\omega \) be an ordered set of distinct points of \( X \) such that \( \mathcal{J}_\omega \) is the space \( S \). The space \( S_g,m \) consisting of non-multiple maps, that is the space of pseudoholomorphic maps from \( S \) to \( X \) passing through \( x \). Fix some Levi-Civita connection \( \nabla \) on \( TX \) associated to some Riemannian metric \( g_X \) invariant under \( c_X \). All the induced connections on the bundles associated to \( TX \) will also be denoted by \( \nabla \), for convenience. The linearization of \( \sigma_\tau \) at \( (u,J_S,J) \in P^d_g(x) \) is defined by (see \([5]\), formula 1.2.3)

\[
\nabla \sigma_\tau(v,J_S,J) = Dv + J \circ du \circ J_S + J \circ du \circ J_S,
\]

where \( D \) is the Gromov operator defined by

\[
v \in E|_{(u,J_S,J)} \mapsto D(v) = \nabla v + J \circ \nabla v \circ J_S + \nabla v J \circ du \circ J_S \in E|_{(u,J_S,J)}.
\]

Finally, denote by \( P^*(x) \) the subspace of \( P^d_g(x) \) consisting of non-multiple maps, that is the space of triple \( (u,J_S,J) \) such that \( u \) cannot be written \( u' \circ \phi \) where \( \phi : S \to S' \) is a non-trivial ramified covering and \( u' : S' \to X \) is a pseudo-holomorphic map.

**Proposition 1.2** The space \( P^*(x) \) is a Banach manifold of class \( C^{l,\alpha} \) whose tangent space at \( (u,J_S,J) \in P^*(x) \) is the space \( T_{(u,J_S,J)}P^*(x) = \{(v,J_S,J) \in T_{(u,J_S,J)}(S^d_g(x) \times T_{g,m} \times J_\omega) | \nabla_{(v,J_S,J)} \sigma_\tau = 0 \} \). □

This proposition follows from the fact that at \( (u,J_S,J) \in P^*(x) \), the operator \( \nabla \sigma_\tau : T_{(u,J_S,J)}(S^d_g(x) \times T_{g,m} \times J_\omega) \to E|_{(u,J_S,J)} \) is surjective (see, for example, \([11]\) Corollary 2.1.3).
The group $G$ acts on $\mathcal{S}^d_g(x) \times T_{g,m} \times \mathcal{J}_\omega$ by

$$
\phi_*(u, J_S, J) = \begin{cases} 
(u \circ \phi^{-1}, (\phi^{-1})^* J_S, J) & \text{if } s_*(\phi) = +1, \\
(c_X \circ u \circ \phi^{-1}, (\phi^{-1})^* J_S, e_X^*(J)) & \text{if } s_*(\phi) = -1,
\end{cases}
$$

where $\phi \in G$ and $(u, J_S, J) \in \mathcal{S}^d_g(x) \times T_{g,m} \times \mathcal{J}_\omega$. Note that via the fixed identification $U_{g,m} = S \times T_{g,m}$, $\phi$ induces some diffeomorphism of $S$ which depends on $J_S \in T_{g,m}$. The action of $G$ lifts to the following actions on the bundles $\mathcal{E}$ and $\mathcal{E}'$:

\begin{align*}
(\phi, v) \in G \times \mathcal{E}_{(u,J_S,J)} & \mapsto \begin{cases} 
v \circ \phi^{-1} \in \mathcal{E}_{\phi,(u,J_S,J)} & \text{if } s_*(\phi) = +1, \\
dc_X \circ v \circ \phi^{-1} \in \mathcal{E}_{\phi,(u,J_S,J)} & \text{if } s_*(\phi) = -1,
\end{cases} \\
(\phi, \alpha) \in G \times \mathcal{E}'_{(u,J_S,J)} & \mapsto \begin{cases} 
\alpha \circ d\phi^{-1} \in \mathcal{E}'_{\phi,(u,J_S,J)} & \text{if } s_*(\phi) = +1, \\
dc_X \circ \alpha \circ d\phi^{-1} \in \mathcal{E}'_{\phi,(u,J_S,J)} & \text{if } s_*(\phi) = -1.
\end{cases}
\end{align*}

The section $\sigma_\mathcal{E}$ is obviously $G$-equivariant for these actions. As a consequence, the manifold $\mathcal{P}^e(x)$ is invariant under the action of $G$.

**Lemma 1.3** With the exception of the identity, only the order two elements of $G^-$ may have non-empty fixed point set in $\mathcal{P}^e(x)$. In particular, two such involutions have disjoint fixed point sets. \(\square\)

Denote by $\mathbb{R}\mathcal{P}^e(x)$ the disjoint union of the fixed point sets of the non-trivial elements of $G$. From Lemma 1.3 we know that each component of $\mathbb{R}\mathcal{P}^e(x)$ determines uniquely an order two element of $G^- \subset G$. This involution induces bundles homomorphisms on $\mathcal{E}|_{\mathbb{R}\mathcal{P}^e(x)}$ and $\mathcal{E}'|_{\mathbb{R}\mathcal{P}^e(x)}$. We denote by $\mathcal{E}_{+1}$, $\mathcal{E}'_{+1}$ (resp. $\mathcal{E}_{-1}$, $\mathcal{E}'_{-1}$) the eigenspace associated to the eigenvalue +1 (resp. −1) of this homomorphism, so that $\mathcal{E}|_{\mathbb{R}\mathcal{P}^e(x)} = \mathcal{E}_{+1} \oplus \mathcal{E}_{-1}$ and $\mathcal{E}'|_{\mathbb{R}\mathcal{P}^e(x)} = \mathcal{E}'_{+1} \oplus \mathcal{E}'_{-1}$.

**Proposition 1.4** The space $\mathbb{R}\mathcal{P}^e(x)$ is a Banach submanifold of $\mathcal{P}^e(x)$ of class $C^{1,\alpha}$ whose tangent space at $(u, J_S, J) \in \mathbb{R}\mathcal{P}^e(x)$ is the space $T_{(u,J_S,J)}\mathbb{R}\mathcal{P}^e(x) = \{ (v, J_S, J) \in \mathcal{E}_{+1} \times T_{J_S}T_{g,m} \times T_{J}\mathcal{J}_\omega \mid \nabla_{(v,J_S,J)}\sigma_\mathcal{E} = 0 \}$.

**Remark 1.5** To every element $(u, J_S, J) \in \mathbb{R}\mathcal{P}^e(x)$ is associated an order two element of $G^-$. We denote by $\mathbb{R}T_{g,m}$ the fixed point set of the action of this element on $T_{g,m}$. Hence the submanifold $\mathbb{R}T_{g,m}$ of $T_{g,m}$ does depend on the choice of the connected component of $\mathbb{R}\mathcal{P}^e(x)$ and thus the notation is abusive. We will however keep this notation for convenience.

**Proof:**

Let $c_S$ be an element of order two of $G^-$. From Lemma 1.3 it suffices to prove that the fixed point set of $c_S$ is a Banach submanifold of $\mathcal{P}^e(x)$. Note that the action of $c_S$ on the product $\mathcal{S}^d_g(x) \times T_{g,m} \times \mathcal{J}_\omega$ is antiholomorphic for the almost-complex structure defined by :

$$(v, J_S, J) \in T_{(u,J_S,J)}(\mathcal{S}^d_g(x) \times T_{g,m} \times \mathcal{J}_\omega) \mapsto (Jv, J_S, J_S, J_J), (u, J_S, J) \in T_{(u,J_S,J)}(\mathcal{S}^d_g(x) \times T_{g,m} \times \mathcal{J}_\omega)$$

The fixed point set of this action is a Banach manifold denoted by $\mathbb{R}\mathcal{S}^d_g(x) \times \mathbb{R}T_{g,m} \times \mathbb{R}\mathcal{J}_\omega$. The restriction of $\sigma_\mathcal{E}$ to $\mathbb{R}\mathcal{S}^d_g(x) \times \mathbb{R}T_{g,m} \times \mathbb{R}\mathcal{J}_\omega$ takes values in the sub-bundle $\mathcal{E}'_{+1}$ of $\mathcal{E}'$ associated to the eigenvalue +1 of $c_S$, since $\sigma_\mathcal{E}$ is $G$-equivariant. It suffices then to prove that
this restriction vanishes transversally along \( \mathbb{R}P^*(x) \cap (\mathbb{R}S_g^d(x) \times RT_{g,m} \times \mathcal{J}_c) \), meaning that at \((u,J_S,J) \in \mathbb{R}P^*(x) \cap (\mathbb{R}S_g^d(x) \times RT_{g,m} \times \mathcal{J}_c)\), the operator \( \nabla \sigma_{\mathcal{F}} : T(u,J_S,J)(\mathbb{R}S_g^d(x) \times RT_{g,m} \times \mathcal{J}_c) \to \mathcal{E}'_{\mathcal{F}}\) is surjective. But this follows from the surjectivity of \( \nabla \sigma_{\mathcal{F}} : T(u,J_S,J)(\mathbb{R}S_g^d(x) \times RT_{g,m} \times \mathcal{J}_c) \to \mathcal{E}'_{\mathcal{F}}\) and the \(G\)-equivariance of \(\sigma_{\mathcal{F}}\). \(\square\)

1.2.2 The Gromov operators \(D\) and \(D_{\mathbb{R}}\)

Remember that the \(C\)-linear part of the Gromov operator \(D\) defined in the previous paragraph is some \(\overline{\partial}\)-operator which will be denoted by \(\overline{\partial}\), and that its \(C\)-antilinear part is some order 0 operator denoted by \(R\). The latter is given by the formula \(R(u,J_S,J)(v) = N_J(v,du)\) where \(v \in \mathcal{E}\) and \(N_J\) is the Nijenhuis tensor of \(J\). In particular, \(R \circ du = 0\), see [5], Lemma 1.3.1. Let \((u,J_S,J) \in \mathbb{R}P^*(x)\), the operator \(\overline{\partial}\) associated to \(D\) induces a holomorphic structure on the bundle \(E_u = u^*TX\) for which the morphism \(du : TS \to E_u\) is an injective analytic bundle homomorphism (see [2], Lemma 3.3.1). Denote by \(\mathcal{E}_u = E_u \otimes \mathcal{O}(v), TS_{u,v} = TS \otimes \mathcal{O}(v)\) and \(\mathcal{N}_{u,v}\) the quotient sheaf \(E_u \to du(TS_{u,v})\). This sheaf splits under the form \(\mathcal{O}(\mathcal{N}_{u,v}) \oplus \mathcal{N}_{u,v}^{sing}\), where \(\mathcal{N}_{u,v} = \mathcal{N}_u \otimes \mathcal{O}(v), \mathcal{N}_u\) being the normal bundle of \(u(S)\) in \(X\), and \(\mathcal{N}_{u,v}^{sing} = \oplus \mathbb{C}\). In the latter, the sum is taken over all the critical points of \(du\) and \(\mathbb{C}\) denotes the skyscraper sheaf of fiber \(\mathbb{C}\). Denote \(\mathcal{E}_u \otimes \mathcal{O}(v)\) and support \(a_i, \mathcal{N}_{\mathbb{C}}\) being the vanishing order of \(du\) at \(a_i\). The operator \(D\) induces on the quotient an operator \(D^N : L^{k,p}(S,\mathcal{N}_{u,v}) \to L^{k-1,p}(S,\Lambda^{0,1}S \otimes \mathcal{N}_{u,v})\) (see [5], formula 1.3.5). Denote by \(H^0_D(S,\mathcal{E}_u)\) (resp. \(H^1_D(S,\mathcal{E}_u)\)) the kernel of the operator \(D\) (resp. \(D^N\)), and \(H^1_D(S,\mathcal{E}_u)\) (resp. \(H^1_D(S,\mathcal{E}_u)\)) the cokernel of this operator. We also denote by \(H^0_D(S,\mathcal{N}_{u,v}) = H^0_D(S,\mathcal{N}_{u,v}) \oplus H^0_D(S,\mathcal{N}_{u,v}^{sing})\). Note that since the operators \(D\) and \(D^N\) are elliptic, all these spaces are finite dimensional and do not depend on the choice of \(\alpha, \beta\). They satisfy the following long exact sequence (see [11], Corollary 1.5.4):

\[
0 \to H^0(S,TS_{u,v}) \to H^0_D(S,\mathcal{E}_u) \to H^0_D(S,\mathcal{N}_{u,v}) \to H^1(S,TS_{u,v}) \to H^1_D(S,\mathcal{E}_u) \to H^1_D(S,\mathcal{N}_{u,v}) \to 0.
\]

Remember finally that the dual of the operator \(D^N\) is given by some operator \(D^* : L^{k,p}(S,\mathcal{N}_{u,v}) \to L^{k-1,p}(S,\Lambda^{0,1}S \otimes \mathcal{N}_{u,v})\), where \(\mathcal{N}_{u,v} = \Lambda^{1,0}S\) and \(D^* = R^* \overline{\partial}\). Thus, the Serre duality gives isomorphisms \(H^0_D(S,\mathcal{N}_{u,v}) \cong H^1_D(S,\mathcal{N}_{u,v})\) and \(H^1_D(S,\mathcal{N}_{u,v}) \cong H^0_D(S,\Lambda^{0,1}S \otimes \mathcal{N}_{u,v})\). \(\square\)

**Lemma 1.6** The operators \(D : \mathcal{E} \to \mathcal{E}'\) and \(D^N\) are \(G\)-equivariant.

Thus, over \(\mathbb{R}P^*(x)\), the operator \(D\) restricts to some operator \(\mathcal{E}_{\mathcal{F}} \to \mathcal{E}'\) which will be denoted by \(D_{\mathbb{R}}\). Similarly, let \((u,J_S,J) \in \mathbb{R}P^*(x)\) and \(c_S\) be the associated order two element of \(G^-.\) Denote by \(L^{k,p}(S,\mathcal{N}_{u,v})\) (resp. \(L^{k-1,p}(S,\Lambda^{0,1}S \otimes \mathcal{N}_{u,v})\)) the eigenspace associated to the eigenvalue \(\pm 1\) of the action of \(c_S\) on \(L^{k,p}(S,\mathcal{N}_{u,v})\) (resp. \(L^{k-1,p}(S,\Lambda^{0,1}S \otimes \mathcal{N}_{u,v})\)). Denote then by \(D^N_{\mathbb{R}}\) the operator \(L^{k,p}(S,\mathcal{N}_{u,v})_{\pm 1} \to L^{k-1,p}(S,\Lambda^{0,1}S \otimes \mathcal{N}_{u,v})_{\pm 1}\) induced by \(D^N\).

**Lemma 1.7** The operators \(D_{\mathbb{R}}\) and \(D^N_{\mathbb{R}}\) are Fredholm, of indices \(\text{ind}(D_{\mathbb{R}}) = \frac{1}{2} \text{ind}(D) = c_1(X)d + n(1-g)\) and \(\text{ind}(D^N_{\mathbb{R}}) = \frac{1}{2} \text{ind}(D^N) = c_1(X)d + (n-3)(1-g) - (n-1)m\) where \(n = \dim\mathbb{C}(X)\).
Proof:
Fix some component of \( \mathbb{R}P^* (x) \) and the associated element \( c_S \) of order two of \( G^- \). Remember that the decomposition of \( D \) in \( \mathbb{C} \)-linear and antilinear parts writes \( \overline{D} + R \), where \( \overline{D} \) and \( R \) are equivariant under the action of \( c_S \) (in fact under the whole \( G \)). The operator \( \overline{D} \) restricts then to some operator \( \mathcal{E}_{+1} \to \mathcal{E}'_{+1} \) which remains Fredholm, of kernel (resp. cokernel) the eigenspace \( \ker_{+1} (\overline{D}) \) (resp. \( \ker_{+1} (\overline{D}) \)) associated to the eigenvalue +1 of the action of \( c_S \) on \( \ker (\overline{D}) \) (resp. \( \ker (\overline{D}) \)). Since \( R \) is of order 0, it follows that \( D_R \) is Fredholm of index \( \text{ind} (D_R) = \text{ind} (\overline{D}_{+1}) = \frac{1}{2} \text{ind} (D) = c_1 (X) d + n (1 - g) \). The last equality coming from Riemann-Roch theorem and the equality before from the fact that \( \overline{D} \) is \( \mathbb{C} \)-linear. The same arguments applied to \( D^N \) give the result for \( D^N_R \). □

Denote by \( H^0_D (S, E_{u^-} + 1) \) (resp. \( H^0_D (S, N_{u^-} + 1) \)) the kernel of the operator \( D_R \) (resp. \( D^N_R \)), and by \( H^1_D (S, E_{u^-} + 1) \) (resp. \( H^1_D (S, N_{u^-} + 1) \)) the cokernel of this operator. Denote also by \( H^0_D (S, N_{u^-} + 1) = H^0_D (S, N_{u^-} + 1) \oplus H^0_D (S, N_{u^-}^{sing} + 1) \). These spaces satisfy the following long exact sequence:

\[
0 \to H^0 (S, TS_{-1} + 1) \to H^0_D (S, E_{u^-} + 1) \to H^0_D (S, N_{u^-} + 1) \oplus H^0_D (S, N_{u^-}^{sing} + 1) \to \]

\[
\to H^1 (S, TS_{-1} + 1) \to H^1_D (S, E_{u^-} + 1) \to H^1_D (S, N_{u^-} + 1) \to 0.
\]

(2)

Note that \( H^0_D (S, E_{u^-} + 1) \) (resp. \( H^0_D (S, N_{u^-} + 1) \)) coincides with the eigenspace associated to the eigenvalue +1 of the action of \( c_S \) on \( H^0_D (S, E_{u^-}) \) (resp. \( H^0_D (S, N_{u^-}) \)). Denote by \( H^0_D (S, E_{u^-} - 1) \) (resp. \( H^0_D (S, N_{u^-} - 1) \)) the eigenspace associated to the eigenvalue -1.

**Lemma 1.8** Serre duality provides isomorphisms:

\( H^0_D (S, N_{u^-} + 1) \cong H^1_D (S, K_S \otimes_{\mathbb{C}} N_{u^-}^{*} - 1) \) and
\( H^1_D (S, N_{u^-} + 1) \cong H^0_D (S, K_S \otimes_{\mathbb{C}} N_{u^-}^{*} - 1). \)

**Proof:**
The duality between the spaces \( L^{k, -p} (S, A^{0, 1} S \otimes_{\mathbb{C}} N_{u^-}) \) and \( L^{k, p} (S, K_S \otimes_{\mathbb{C}} N_{u^-}^{*}) \) writes \((\psi^*, \alpha) \in L^{k, p} (S, K_S \otimes_{\mathbb{C}} N_{u^-}^{*}) \times L^{k, -p} (S, A^{0, 1} S \otimes_{\mathbb{C}} N_{u^-}) \mapsto \Re \int_S < \psi^*, \alpha > \). Now, fix some component of \( \mathbb{R}P^* (x) \) and the associated element \( c_S \) of order two of \( G^- \). We have

\[
((dc_S^X)^t \circ \psi^* \circ dc_S, dc_X \circ \alpha \circ dc_S) = \Re \int_S < (dc_S^X)^t \circ \psi^* \circ dc_S, dc_X \circ \alpha \circ dc_S >
\]

\[
= \Re \int_S < \psi^*, \alpha > \circ dc_S
\]

\[
= -\Re \int_S < \psi^*, \alpha >,
\]

since \( c_S \) reverses the orientation of \( S \). It follows that the spaces \( H^1_D (S, N_{u^-} + 1) \) and \( H^0_D (S, K_S \otimes_{\mathbb{C}} N_{u^-}^{*} + 1) \) are orthogonal to each other and that the spaces \( H^1_D (S, N_{u^-} - 1) \) and \( H^0_D (S, K_S \otimes_{\mathbb{C}} N_{u^-}^{*} - 1) \) are dual to each other. The same holds for \( H^0_D (S, N_{u^-} + 1) \) and \( H^1_D (S, K_S \otimes_{\mathbb{C}} N_{u^-}^{*} + 1) \). □
1.3 Moduli space of pseudo-holomorphic curves

1.3.1 The manifolds \( \mathcal{M}^d_g(x) \) and \( \mathbb{R}\mathcal{M}^d_g(x) \) and the projections \( \pi \) and \( \pi_R \)

Denote by \( \mathcal{M}^d_g(x) \) the quotient of \( \mathcal{P}^*(x) \) under the action of \( G^+ \). The projection \( \pi : (u, J_S, J) \in \mathcal{P}^*(x) \mapsto J \in \mathcal{J}_\omega \) induces on the quotient a projection \( \mathcal{M}^d_g(x) \to \mathcal{J}_\omega \) still denoted by \( \pi \). Remember the following proposition (see [11], Theorem 2.4.3).

**Proposition 1.9** Denote by \( n \) the dimension of \( X \) over \( \mathbb{C} \).

1) The space \( \mathcal{M}^d_g(x) \) is a Banach manifold of class \( C^{l,\alpha} \) and the projection \( \mathcal{P}^*(x) \to \mathcal{M}^d_g(x) \) is a principal \( G^+ \)-bundle.

2) The projection \( \pi : \mathcal{M}^d_g(x) \to \mathcal{J}_\omega \) is Fredholm of index \( \text{ind} \left( \pi \right) = c_1(X)d + (n - 3)(1 - g) - (n - 1)m \). Moreover, at \( [u, J_S, J] \in \mathcal{M}^d_g(x) \), the kernel of \( \pi \) is isomorphic to \( H^0_D(S, N_{u,-z}) \) and its cokernel to \( H^1_D(S, N_{u,-z}) \). \( \square \)

The manifold \( \mathcal{M}^d_g(x) \) is equipped with an action of the group \( G/G^+ \cong \mathbb{Z}/2\mathbb{Z} \). Let us denote by \( \mathbb{R}\mathcal{M}^d_g(x) \) the fixed point set of this action. The restriction of \( \pi \) to \( \mathbb{R}\mathcal{M}^d_g(x) \) takes value in \( \mathbb{R}\mathcal{J}_\omega \). Denote by \( \pi_R \) the induced projection \( \mathbb{R}\mathcal{M}^d_g(x) \to \mathbb{R}\mathcal{J}_\omega \).

**Proposition 1.10** The projection \( \pi_R : \mathbb{R}\mathcal{M}^d_g(x) \to \mathbb{R}\mathcal{J}_\omega \) is Fredholm of index \( \text{ind} \left( \pi_R \right) = c_1(X)d + (n - 3)(1 - g) - (n - 1)m \). Moreover, at \( [u, J_S, J] \in \mathbb{R}\mathcal{M}^d_g(x) \), the kernel of \( \pi_R \) is isomorphic to \( H^0_D(S, N_{u,-z})+1 \) and its cokernel to \( H^1_D(S, N_{u,-z})+1 \).

**Proof:**

The projection \( \pi \) is \( \mathbb{Z}/2\mathbb{Z} \)-equivariant. Let \( [u, J_S, J] \in \mathbb{R}\mathcal{M}^d_g(x) \) and \( (u, J_S, J) \in \mathcal{P}^*(x) \) mapping to this element. Denote by \( c_\mathbb{R} \) the associated element of order two of \( G^- \). Then \( \text{Im}(d_{[u, J_S, J]}\pi_R) = \text{Im}(d\pi) \cap L^1,X,\omega^1X \otimes \mathbb{C}TX+1 \), this image is thus closed in \( L^1,X,\omega^1X \otimes \mathbb{C}TX \). From Proposition 1.9, we know that the cokernel of \( d_{[u, J_S, J]}\pi_R \) is finite dimensional and isomorphic to \( H^0_D(S, N_{u,-z})+1 \). Similarly, its kernel is finite dimensional and isomorphic to \( H^1_D(S, N_{u,-z})+1 \). The index formula follows from the exact sequence (2), from Lemma 1.4, and from the Riemann-Roch formula applied to the bundle \( TS_{-2} \), since the operator \( \mathcal{J} \) on this bundle is \( \mathbb{C} \)-linear. \( \square \)

1.3.2 The theorem of regular values

The following theorem is the main result of this first paragraph. We only give a proof of it in genus zero and dimension 4 here. The general case is postponed to appendix A.

**Theorem 1.11** The set of regular values of the projection \( \pi : \mathcal{M}^d_g(x) \to \mathcal{J}_\omega \) intersects \( \mathbb{R}\mathcal{J}_\omega \) in a dense subset of the second category of \( \mathbb{R}\mathcal{J}_\omega \).

**Proposition 1.12** The submanifold \( \mathbb{R}\mathcal{J}_\omega \) of \( \mathcal{J}_\omega \) is transversal to the restriction of \( \pi \) to \( \mathcal{M}^d_g(x) \setminus \mathbb{R}\mathcal{M}^d_g(x) \).

**Proof:**

Let \( J \in \mathbb{R}\mathcal{J}_\omega \) and \( [u, J_S, J] \in \mathcal{M}^d_g(x) \setminus \mathbb{R}\mathcal{M}^d_g(x) \). Fix some element \( (u, J_S, J) \in \mathcal{P}^*(x) \) lifting \( [u, J_S, J] \). Then from Proposition 1.9, \( \text{ker}(d_{[u, J_S, J]}\pi) \) is isomorphic to \( H^1_D(S, N_{u,-z}) \). Let \( 0 \neq \psi \in H^1_D(S, KS \otimes C N_{u,-z}^*) \cong H^1_D(S, N_{u,-z}) \), it suffices to prove that there exists \( \dot{J} \in L^1,\omega^1X \otimes \mathbb{C}TX+1 = T_J(\mathbb{R}\mathcal{J}_\omega) \) such that \( \langle \psi, \dot{J} \circ du \circ J_S \rangle \neq 0 \). But \( D^*\psi = 0 \)
and $D^\ast$ is of generalized $\overline{\partial}$-type, thus $\psi$ vanishes only at a finite number of points (see [4]). Since $u$ is neither real, nor multiple, there exists an open subset $U$ of $S$, disjoint from $z \subset S$, such that $u|_U$ is an embedding, $u(U) \cap u(S \setminus U) = \emptyset$, $c_X(u(U)) \cap u(S) = \emptyset$ and such that $\psi$ does not vanish on $U$. Then, there exists a section $\alpha$ of $\Lambda^0 1 \otimes C E_{u, -z}$ with support in $U$ such that $< \psi, \alpha > \neq 0$. Let $J \in L^1, \alpha(X, \Lambda^0 1 \otimes C T_X)$ be a section with support in a neighborhood of $u(U)$ such that $J \circ du \circ J_S = \alpha$. The section $J_R = J + \overline{\psi}(J)$ then belongs to $L^1, \alpha(X, \Lambda^0 1 \otimes C T_X) + 1$ and also satisfies $J_R \circ du \circ J_S = \alpha$, hence the result. □

**Proof of Theorem 1.11 in genus zero and dimension 4:**

From Proposition 1.12 and the theorem of Sard-Smale (see [12]), there exists a dense set of the second category of $\mathbb{R} J_\omega$, denoted by $U_1$, such that every point of $\pi^{-1}(U_1) \setminus \mathbb{R} M_0(x)$ is regular for $\pi$. Similarly, from Proposition 1.10 and the theorem of Sard-Smale, the set of regular values of $\pi_\mathbb{R}$ is a dense subset of the second category of $\mathbb{R} J_\omega$ denoted by $U_2$. Then $U = U_1 \cap U_2$ is suitable. Indeed, let $J \in U$ and $[u, J_S, J] \in \pi^{-1}_R(J)$. Choose some element $(u, J_S, J) \in \mathbb{RP}_t^*(x)$ lifting it and denote by $c_S$ the associated order two element of $G^-$. By hypothesis, $H^+_D(S, N_{u, -z})_+ = 0$. It suffices thus to prove that $H^+_D(S, N_{u, -z})_+ = 0$. If this would not be the case, since $S$ is rational, we would have $H^+_D(S, N_{u, -z}) = 0$ (see [4], Theorem 1'). Since $u$ is real and $H^0(S, N_u^{\text{sing}})$ is carried by the cuspidal points of $u$, we see that $\text{dim} \, H^0(S, N_u^{\text{sing}})_+ = \text{dim} \, H^0(S, N_u^{\text{sing}})_- = \frac{1}{2} \text{dim} \, H^0(S, N_u^{\text{sing}})$. From this we would obtain

$$\text{ind}(\pi) = 2 \text{dim} \, H^0(S, N_u^{\text{sing}})_+ - \text{dim} \, H^1_D(S, N_{u, -z})_+ < 2 \text{ind}(\pi_\mathbb{R}),$$

which contradicts Proposition 1.10 □

2 The invariant $\chi$ of real rational symplectic 4-manifolds

2.1 Statements of the results

Let $(X, \omega, c_X)$ be a real rational symplectic 4-manifold and $J_\omega$ be the space of almost complex structures of $X$ of class $C^\ell, \alpha$ tamed by $\omega$. Let $C$ be a real irreducible rational pseudo-holomorphic curve of $X$ having only ordinary nodes as singularities, and $d \in H_2(X; \mathbb{Z})$ be its homology class. The total number of double points of $C$ is given by adjunction formula and is equal to $\delta = \frac{1}{2}(d^2 - c_1(X)d + 2)$. The real double points of $C$ are of two different natures. They are either the local intersection of two real branches, or the local intersection of two complex conjugated branches. In the first case they are called *non-isolated* and in the second case they are called *isolated*.

![Diagram](image)

We define the *mass* of the curve $C$ to be the number of its real isolated double points, it is denoted by $m(C)$. This integer satisfies the upper and lower bounds $0 \leq m(C) \leq \delta$. Now, let $x \subset X$ be a real configuration of $c_1(X)d - 1$ distinct points and $r$ be the number of such points which are real. Let $J \in \mathbb{R} J_\omega$, if $J$ is generic enough, then there are only finitely many
$J$-holomorphic rational curves in $X$ passing through $x$ in the homology class $d$. Moreover, these curves are all nodal and irreducible. For every integer $m$ ranging from 0 to $\delta$, denote by $n_d(m)$ the number of these curves which are real and of mass $m$. Then define:

$$
\chi^d(x, J) = \sum_{m=0}^{\delta} (-1)^m n_d(m).
$$

The main result of this paper is the following:

**Theorem 2.1** Let $(X, \omega, c_X)$ be a real rational symplectic 4-manifold, and $d \in H_2(X; \mathbb{Z})$. Let $x \subset X$ be a real configuration of $c_1(X)d - 1$ distinct points and $r$ be the cardinality of $x \cap \mathbb{R}X$. Finally, let $J \in \mathbb{R}J_\omega$ be an almost complex structure generic enough, so that the integer $\chi^d(x, J)$ is well defined. Then, this integer $\chi^d(x, J)$ neither depends on the choice of $J$ nor on the choice of $x$ (provided the cardinality of $x \cap \mathbb{R}X$ is $r$).

For convenience, this integer will be denoted by $\chi^d_x$, and when $r$ does not have the same parity as $c_1(X)d - 1$, we put $\chi^d_x$ to be 0. We then denote by $\chi^d(T)$ the polynomial $\sum_{r=0}^{c_1(X)d-1} \chi^d_{tr} \in \mathbb{Z}[T]$. It follows from Theorem 2.1 that the function $\chi : d \in H_2(X; \mathbb{Z}) \mapsto \chi^d(T) \in \mathbb{Z}[T]$ only depends on the real symplectic 4-manifold $(X, \omega)$ and is invariant under deformation of this 4-manifold. As an application of this invariant, we obtain the following lower bounds:

**Corollary 2.2** Under the hypothesis of Theorem 2.1, the integer $|\chi^d_x|$ gives a lower bound for the total number of real rational $J$-holomorphic curves of $X$ passing through $x$ in the homology class $d$, independently of the choice of a generic $J \in \mathbb{R}J_\omega$. □

Note that this number of real curves is always bounded from above by the total number $N_d$ of rational $J$-holomorphic curves of $X$ passing through $x$ in the homology class $d$, which does not depend on the choice of $J$. This number $N_d$ is a Gromov-Witten invariant of the symplectic 4-manifold $(X, \omega)$ and was computed by Kontsevich in [6]. One of the main problems of real enumerative geometry is, in this context, to know if there exists a generic real almost-complex structure $J$ so that all these rational $J$-holomorphic curves are real. The following corollary provides a criteria for the existence of such a structure.

**Corollary 2.3** Under the hypothesis of Theorem 2.1 assume that $\chi^d_x \geq 0$ (resp. $\chi^d_x \leq 0$). Assume that there exists a generic $J \in \mathbb{R}J_\omega$ such that $X$ has $\frac{1}{2}(N_d - |\chi^d_x|)$ real $J$-holomorphic curves of odd (resp. even) mass passing through $x$ in the homology class $d$. Then, all of the rational $J$-holomorphic curves of $X$ passing through $x$ in the homology class $d$ are real. □

**Examples:**

1) Let $(X, \omega, c_X)$ be the complex projective plane equipped with its standard symplectic form and real structure. We denote the homology classes of the complex curves of $\mathbb{C}P^2$ by integers. Then $\chi^1(T) = 1 + T^2$, $\chi^2(T) = T + T^3 + T^5$ and $\chi^3(T) = \sum_{r=0}^{4} 2rT^{2r}$. The latter can be obtained computing the Euler characteristic of the real part of the blown up projective plane at the nine base points of a pencil of elliptic curves, as was noticed by V. Kharlamov (see [2], Proposition 4.7.3 or [13], Theorem 3.6). The non-triviality of the polynomials $\chi^k(T)$ and $\tilde{\chi}^k(T)$ is proved in [3, 12]

2) Let $(X, \omega, c_X)$ be a real smooth cubic surface of $\mathbb{C}P^3$, and $\ell$ be the homology class of a line. Assume that $\mathbb{R}X$ is homeomorphic to the blown-up real projective plane at $2k$ points,
\[ 0 \leq k \leq 3. \text{ Then } \chi^4_0 = 2k^2 + 2k + 3. \]

Note that when \( \mathbb{R}X = \emptyset \), Theorem 2.1 states that the number of real rational \( J \)-holomorphic curves of \( X \) passing through \( x \) in the homology class \( d \) does not depend on the choices of \( x \) and \( J \), as it is the case for the number of complex curves. The following question arise from Corollary 2.2: Are the lower bounds given by Corollary 2.2 sharp? In the Example 1, for the degree 3 and \( r = 8 \), the lower bound is sharp from [2], Proposition 4.7.3. Also, is it possible to define a similar invariant using higher genus curves in real symplectic 4-manifolds, or in real symplectic manifolds of higher dimensions? Note that the straightforward generalization of the integer \( \chi^d_r(x, J) \) using higher genus curves, even taking into account the coherent orientation of the complex moduli space \( \mathcal{M}^d_g(x, J) \), certainly does depend on \( x, J \). This can be noticed for projective curves of degree \( d \geq 4 \) with one nodal point, using the same trick as for the degree 3 curves, see Example 1.

### 2.2 Genericity arguments

From now on, the real symplectic 4-manifold \((X, \omega, c_X)\) is fixed, so that it will not in general be mentioned in the following statements.

Denote by \( B^2 \) (resp. \( \overline{B}^2 \)) the open (resp. closed) unit disk of \( \mathbb{C} \) and by \( J_{st} \) the standard complex structure on this disk. Similarly, denote by \( B^4(\rho) \) the open ball of \( \mathbb{C}^2 \) of radius \( \rho \) and by \( J_{st} \) (resp. conj.) the standard complex (resp. real) structure on this ball.

**Lemma 2.4** Let \((J_\lambda)_{\lambda \in [-1,1]}\) be a family of almost complex structures of class \( C^{l,\alpha} \) on \( B^4(2) \) depending \( C^{l-1,\alpha} \)-smoothly on \( \lambda \) and satisfying \( \text{conj}(J_\lambda) = J_\lambda \). Let \( u_0 : B^2 \to B^4(1) \subset B^4(2) \) be a real \( J_0 \)-holomorphic map having an isolated singularity of order \( \mu \) at \( 0 = u_0(0) \).

Then, for every \( \nu \in \mathbb{R}^2 \) and every integer \( \nu \leq 2\mu + 1 \), there exist \( \epsilon > 0 \) and a family of real maps \( w_\lambda \in L^{k,p}(B^2, \mathbb{C}^2), \lambda \in ]-1,1[, \) such that \( w_0 = 0, \hat{w}_0 = \frac{d}{d\lambda}|_{\lambda=0}(w_\lambda)(0) = 0 \) and for every \( \lambda \in ]-\epsilon, \epsilon[ \), the map \( u_\lambda(t) = u_0(t) + t^\nu(\lambda v + w_\lambda(t)) \) is \( J_\lambda \)-holomorphic and real. \( \square \)

This is a real version of Lemma 3.1.1 of [11]. The proof is readily the same, it suffices to notice that the operators \( \tilde{\partial}, \tilde{R}, \tilde{D} \) and \( T^\nu \) are \( \mathbb{Z}/2\mathbb{Z} \)-equivariants. This proof is not reproduced here.

Denote by \( \mathbb{R}J_B^2 \) the space of complex structures of \( \overline{B}^2 \) compatible with the complex conjugation conj.

**Lemma 2.5** Let \( \mathbb{RP}' = \{(u, J_B, J) \in \mathbb{R}^k \times \mathbb{RP}_B \times \mathbb{RP}_s \mid du + J \circ du \circ J_B^{-1} = 0 \text{ and } c_X \circ u = u \circ \text{conj}\}, \) and \( \mathbb{RP}'_s \) be the subspace of \( \mathbb{RP}' \) consisting of maps having a unique cuspidal point which is a real ordinary cusp interior to \( B^2 \). Moreover, let \((u_\lambda, J_B^\lambda, J_\lambda)_{\lambda \in [0,1]} \) be a path of \( \mathbb{RP}' \) such that \((u_0, J_B^0, J_0) \in \mathbb{RP}'_s \) and \( du_0 \) is not injective at the point \( 0 \in \overline{B}^2 \).

Then:

1) The space \( \mathbb{RP}' \) is a Banach manifold of class \( C^{l,\alpha} \) and \( \mathbb{RP}'_s \) is a Banach submanifold of \( \mathbb{RP}' \) of codimension one.

2) The path \((u_\lambda, J_B^\lambda, J_\lambda)_{\lambda \in [0,1]} \) is transversal to \( \mathbb{RP}' \) at \( \lambda = 0 \) if and only if \( \nabla \hat{u}_0 (T_0 B^2) \) is not the tangent of \( u_0(B^2) \) at the cusp \( u_0(0) \). Under this condition, there exists \( \epsilon > 0 \) such that for every \( \lambda \in ]-\epsilon, 0[ \) (resp. \( \lambda \in [0, \epsilon[ \), the curve \( u_\lambda(\overline{B}^2) \) has a non-isolated (resp. isolated) real double point in the neighborhood of the cusp, or vice-versa.
Proof:

Let us start with the first part of the lemma. Remember that the Gromov operator $D$, being elliptic and defined on the compact surface with non-empty boundary $\overline{B^2}$, is surjective. Thus, $\mathcal{P}' = \{(u, J_{\overline{B^2}}, J) \in L^k, p(\overline{B^2}, X) \times \mathcal{J}_{\overline{B^2}} \times \mathcal{J}_\omega | du + J \circ du \circ J_{\overline{B^2}} = 0\}$ is a Banach manifold of class $C^{1,\alpha}$ and the projection $\mathcal{P}' \to \mathcal{J}_\omega$ is everywhere a submersion. The fact that $\mathbb{RP}'$ is a Banach submifold of class $C^{1,\alpha}$ of $\mathcal{P}'$ can be proven in the same way as Proposition 1.3. Now, let $E$ (resp. $F$) be the vector bundle on $\mathbb{RP}' \times [-1, 1]$ whose fiber over $((u, J_{\overline{B^2}}, J), t)$ is the vector space $T_{u(t)}\mathbb{R}X$ (resp. $T^*_{u(t)}\mathbb{R}X$). Denote by $\Gamma$ the section of the bundle $F \otimes E$ defined by $\Gamma((u, J_{\overline{B^2}}, J), t) = dtu$. It suffices to prove that $\Gamma$ vanishes transversally over $\mathbb{RP}'$. Indeed, $\mathbb{RP}'$ is locally defined as the image of $\Gamma^{-1}(0)$ under the projection $\mathbb{RP}' \times [-1, 1] \to \mathbb{RP}'$, and this projection restricted to $\Gamma^{-1}(0)$ is an embedding (see [11], Lemma 3.2.5). So let $((u_0', J_{\overline{B^2}}', J'), t) \in \mathbb{RP}' \times [-1, 1]$ such that $\mathcal{d}_t u_0' = 0$, and let $v$ be an element of $T_{u_0'(t)}\mathbb{R}X$. There exists a real neighborhood of $Im(u_0')$ in $X$ diffeomorphic to the ball $B^4(2)$ of $\mathbb{C}^2$ via some equivariant diffeomorphism. We can then apply Lemma 2.4 with $\nu = 1$ and with the constant path $J'$ of almost complex structures. Let $(u_\lambda', J_{\overline{B^2}}', J')$ be the path of $\mathbb{RP}'$ given by this lemma. We have $\Gamma((u_\lambda', J_{\overline{B^2}}', J), t) = dt u_\lambda'$, thus

$$ \nabla_{((u_0', 0, 0), \frac{1}{\mathcal{d}t})} \Gamma = \nabla u_0' + \nabla \frac{\partial}{\partial s}(du_0') = v \otimes \frac{\partial}{\partial t} + \nabla \frac{\partial}{\partial s}(du_0'). $$

Choosing $t$ constant in $[-1, 1]$, we deduce the surjectivity of $\nabla \Gamma$ since the vector $v$ has been chosen arbitrarily in $T_{u_0'(t)}\mathbb{R}X$. The first part of the lemma is proved.

Now let us prove the second part of the lemma. The kernel of the projection $\mathbb{RP}' \times [-1, 1] \to \mathbb{RP}'$ is generated by vectors of the form $((0, 0, 0), \frac{\partial}{\partial t})$. From what we have done, $\nabla_{((0, 0, 0), \frac{\partial}{\partial t})} \Gamma = \nabla \frac{\partial}{\partial t}(du_0)$. Since the image of $\nabla \frac{\partial}{\partial t}(du_0)$ is the tangent of the curve $u_0([-1, 1])$ at the real ordinary cusp, and $\nabla_{((u_0, J_{\overline{B^2}}, J), 0)} \Gamma = \nabla u_0$, we deduce the transversality condition. It remains to prove that under this transversality condition, the topology of the real double point of $u_\lambda$ near the cuspidal point of $u_0$ changes when we cross the wall $\mathbb{RP}'$ at $\lambda = 0$. This property is independent of the choice of the point $(u_1, J_{\overline{B^2}})$ of $\mathbb{RP}'$, as soon as this point can be joined to $(u_0, J_{\overline{B^2}})$ by a smooth path of $\mathbb{RP}'$ transversal to the projection $\mathbb{RP}' \to \mathbb{R}J_\omega$. Indeed, this follows from the fact that the projection $\mathbb{RP}' \to \mathbb{R}J_\omega$ is a submersion. Without loss of generality, we can assume that $(X, \omega, c_X)$ is the ball $B^4(2)$ of $\mathbb{C}^2$ equipped with the standard symplectic form and complex conjugation, since the problem is local. We will prove that $(u_0, J_{\overline{B^2}})$ can be joined to the standard real ordinary cusp of the ball $B^4(2) \subset \mathbb{C}^2$. First, we can assume that $J$ is compatible with $\omega$. Indeed, it is easy to construct an almost complex structure $J_0 \in \mathbb{R}J_\omega$ compatible with $\omega$ and such that $u_0$ is $J_0$-holomorphic. Moreover, the space $\{J \in \mathbb{R}J_\omega | u_0 \circ J = J \}$ is contractible, since the Cayley-Sévenne transform identifies this space with the space $\{W \in End(TX) | W J_0 = W J_0, 1 - W W \gg 0, c_X W = W\}$. Now, from the theorem of Micallef and White [8], there exist $C^1$-diffeomorphisms $\phi$ and $\psi$ of $B^4(2)$ and $\overline{B^2}$ respectively, such that $u_0 = \phi \circ u_{st} \circ \psi$ where $u_{st} : t \in \overline{B^2} \mapsto (t^2, t^3) \in B^4(2)$. Let $(\phi_n)_{n \in \mathbb{N}}$ (resp. $(\psi_n)_{n \in \mathbb{N}}$) be a sequence of $C^{1+\alpha}$-diffeomorphisms of $B^4(2)$ (resp. of $\overline{B^2}$) converging to $\phi$ (resp. $\psi$) in $C^1$-topology. Let $(J_n)_{n \in \mathbb{N}}$ be a sequence of elements of $\mathbb{R}J_\omega$ converging to $J_0$ and for which $\phi_n \circ u_{st} \circ \psi_n$ is $J_n$-holomorphic. Such a sequence can be chosen compatible with $\omega$. When $n$ is large enough, $(\phi_n \circ u_{st} \circ \psi_n^{-1}, J_n)$ is close enough to $(u_0, J_{\overline{B^2}})$ in $\mathbb{RP}'$ so that the
topology of the curves being on the two sides of \( \mathbb{R}P'_s \) in \( \mathbb{R}P' \) is the same whether they are near of \((u_0, J_B, J_0)\) or of \((\phi_0 \circ u_{st} \circ \varphi_n^{-1}, J_B^0, J_n)\). Fix such a \( n \), there exists a smooth path \((\varphi_\tau)_{\tau \in [0,1]}\) of \( C^{d+1,\alpha} \) diffeomorphisms of \( B^4(2) \) such that \( \varphi_0 = \phi_n \) and \( \varphi_1 = Id \). We can assume that every such diffeomorphism \( \varphi_\tau \) has a constant differential preserving \( \omega \) at the origin, composing them by a linear transformation of \( \mathbb{C}^2 \) otherwise. The path \((\varphi_\tau \circ u_{st} \circ \varphi_n^{-1}, J_B^\tau, (\varphi_0 \circ \varphi_n^{-1})^\ast J_n)\) of \( \mathbb{R}P'_s \) is then transversal to the projection \( \mathbb{R}P' \rightarrow \mathbb{R}J_\omega \) and joins \((\varphi_0 \circ u_{st} \circ \varphi_n^{-1}, J_B^0, J_n)\) to \((u_{st} \circ \varphi_n^{-1}, J_B^1, \varphi_0 \circ \varphi_n \circ J_n)\). Indeed, restricting ourselves to some ball of smaller radius, all these structures \((\varphi_0 \circ \varphi_n^{-1})^\ast J_n\) are tamed by \( \omega \) since they are at the origin. Since the space \( \{ J \in \mathbb{R}J_\omega | u_{st} \circ \varphi_n^{-1} = J - \text{holomorphic} \} \) is contractible and contains the standard complex structure \( J_{st} \), it suffices to prove the result for curves of \( \mathbb{R}P' \setminus \mathbb{R}P'_s \) in the neighborhood of \((u_{st} \circ \varphi_n^{-1}, J_B^1, J_{st})\), where \( J_B^1 = (u_{st} \circ \varphi_n^{-1})^\ast J_{st} \). Now consider the path \((f_\lambda)_{\lambda \in [-\epsilon,\epsilon]}\) of \( J_{st} \)-holomorphic maps \( t \in \mathbb{T}^2 \mapsto (t^2, t^3 + \lambda t) \in B^4(2) \). This path is transversal to \( \mathbb{R}P'_s \) at \( \lambda = 0 \) since \( \nabla f_\lambda = (0,1) \otimes \frac{d}{dt} \) and the tangent of \( f_0 = u_{st} \) at the cusp is generated by the vector \((1,0)\). Moreover, for \( \lambda < 0 \) (resp. \( \lambda > 0 \)), \( f_\lambda \) has a non-isolated (resp. isolated) real double point at the parameters \( t = \pm \sqrt{-\lambda} \). The same holds for the path \( f_\lambda \circ \varphi_n^{-1} \), hence the result.

\[ \square \]

**Proposition 2.6** Let \((X, \omega, c_X)\) be a real rational symplectic 4-manifold, \( d \in H_2(X; \mathbb{Z}) \) and \( x \) be a real configuration of distinct points of \( X \).

1) The subspace of \( \mathbb{R}M_0^0(x) \) consisting of curves having a real ordinary cusp (resp. a non-ordinary cusp or several cusps) is an immersed submanifold of codimension one (resp. two).

2) The subspace of \( \mathbb{R}M_0^0(x) \) consisting of curves having a real tacnode or a real ordinary triple point (resp. a multiple point of higher order or several such points) is an immersed submanifold of codimension one (resp. two) transversal to the previous one.

3) The subspace of \( \mathbb{R}M_0^0(x) \) consisting of curves having a real ordinary cusp, a real tacnode or a real ordinary triple point at some point of \( x \cap \mathbb{R}X \) is an immersed submanifold of codimension two.

**Proof:**

To begin with, let us prove the first part of the proposition. For this purpose, fix some component of \( \mathbb{R}M_0^0(x) \) and a lift \( C \) of this component in \( \mathbb{R}P^s(x) \). Denote by \( c_S \in G^- \) the associated involution and by \( \mathbb{R}S \) the fixed point set of \( c_S \) in \( S \). Let \((u, J_S, J) \in C \) be a map having an ordinary cusp at \( t \in \mathbb{R}S \). Fix some real neighborhood of \( t \) in \( S \) diffeomorphic to \( \mathbb{R}^2 \) and a real neighborhood of \( u(t) \) in \( X \) diffeomorphic to \( B^4(2) \). We deduce some “restriction map” \( rest : C \rightarrow \mathbb{R}P' \) (see Lemma 2.5). From Lemma 2.4, this map is transversal to \( \mathbb{R}P'_s \). Thus the subspace \( rest^{-1}(\mathbb{R}P'_s) \subset C \) made of curves having a real cuspidal point in a neighborhood of \( u(t) \) is an immersed codimension one submanifold of \( C \). Hence, the subspace of \( \mathbb{R}M_0^0(x) \) consisting of curves having a real ordinary cusp is an immersed submanifold of codimension one. Moreover, it follows from this proof that the condition to have two different cusps is transversal, so that the subspace of these curves is an immersed submanifold of codimension two of \( \mathbb{R}M_0^0(x) \). It remains to prove that the same holds for curves having some cuspidal point of higher order or some non-ordinary cusp. For this, we can assume that the cuspidal point is unique. Denote by \( \mathbb{R}M_0^0(x)_s \) the subspace of \( \mathbb{R}M_0^0(x) \) consisting of curves having a unique ordinary cusp which is thus real. Let us fix some component of \( \mathbb{R}M_0^0(x)_s \) and a lift \( C_s \) of this component in \( \mathbb{R}P^s(x) \). Denote as before by \( c_S \in G^- \) the associated involution and by \( \mathbb{R}S \) the fixed point set of \( c_S \) in \( S \). Let \((u, J_S, J) \in C_s \) and \( t \in \mathbb{R}S \) be the point where \( du \)
vanishes. Remember that the order three jet $j_3^\nu(u)$ of $u$ at $t$ is well defined (see [5], Corollary 1.4.3). It is a polynomial map from $T_u\mathbb{R}S$ to $T_{u(t)}\mathbb{R}X$ whose first order term vanishes since we restrict ourselves to $\mathcal{C}_s$. Denote by $\Gamma_2$ the section of the space of 2-jets over $\mathcal{C}_s$ which maps $(u, J_S, J)$ to the term of order two of $j_3^\nu(u)$. Writing $u$ in a local chart in a neighborhood of $u(t)$, we prove that this section is smooth, of class $C^{l,\alpha}$ (see [11], Lemma 3.2.3). Moreover, it follows as before from Lemma 2.4 with $\nu = 2$ that this section is transversal to the zero section. Thus, the subspace of $\mathbb{R}\hat{M}_0^d(x)$ consisting of curves having a cuspidal point of order $\geq 2$ is an immersed submanifold of codimension three. Let us now restrict ourselves to the open set $V \subset \mathcal{C}_s$ on which $\Gamma_2$ does not vanish. Let $(u, J_S, J) \in V$ and $t \in \mathbb{R}S$ be the cuspidal point. The term of order two of $j_3^\nu(u)$ defines a line in $T_{u(t)}\mathbb{R}X$, it is the tangent line of the curve $u(S)$ at $u(t)$. Denote by $N_u(t)$ the quotient of $T_{u(t)}\mathbb{R}X$ by this line. Projecting the term of order three of $j_3^\nu(u)$ on $N_u(t)$, we define a section $\Gamma_3$ of the bundle of 3-jets from $T_t\mathbb{R}S$ to $N_u(t)$, bundle defined over $V$. As before, this section is smooth of class $C^{l-1,\alpha}$ (the bundle $N_u(t)$ is only of class $C^{l-1,\alpha}$), and it follows from Lemma 2.4 with $\nu = 3$ that it is transversal to the zero section. This ends the proof of the first part of Proposition 2.6.

Now let us prove the second part of the Proposition 2.6. Since all the cases are proved nearly in the same way, we only give a proof in the case of the tacnode. Let us fix $\mathcal{C}$ a component of $\mathbb{R}\hat{P}^s(x)$, $c_S \in G^-$ the associated involution and denote by $\mathbb{R}S$ the fixed point set of $c_S$ in $S$. Let

$$\hat{\mathcal{C}} = \{(u, J_S, J), t_1, t_2) \in \mathcal{C} \times \mathbb{R}S \times \mathbb{R}S \mid t_1 \neq t_2, u(t_1) = u(t_2) \text{ and } d_t u \neq 0 \neq d_{t_2} u\}.$$  

This is a submanifold of class $C^{l,\alpha}$ of codimension two of $\mathcal{C} \times \mathbb{R}S \times \mathbb{R}S$. Denote by $N$ (resp. $F$) the vector bundle of class $C^{l-1,\alpha}$ over $\hat{\mathcal{C}}$ whose fiber over $((u, J_S, J), t_1, t_2)$ is the quotient space $T_{u(t_1)}\mathbb{R}X/d_{t_1} u(T_{u(t)}\mathbb{R}S)$ (resp. $T_{t_2}^*\mathbb{R}S$). Denote by $\Theta$ the section of the bundle $F \otimes N$ defined by $\Theta((u, J_S, J), t_1, t_2) = d_{t_2} u$. As before, the section $\Theta$ is smooth of class $C^{l-1,\alpha}$, and from Lemma 2.4 with $\nu = 1$, it is transversal to the zero section. Moreover, the projection $\hat{\mathcal{C}} \to \mathcal{C}$ restricted to $\Theta^{-1}(0)$ is an immersion, hence the result. The transversality of this submanifold of codimension one of $\mathbb{R}\hat{M}_0^d(x)$ with the one defined in the first part of the proposition once more follows from Lemma 2.4 with $\nu = 1$. The proof of the third part of the proposition is left to the reader. \[\Box\]

Denote by $\mathbb{R}\hat{M}_0^d(x)$ the immersed submanifold of codimension one of $\mathbb{R}\hat{M}_0^d(x)$ consisting of curves having a unique cuspidal point which is a real ordinary cusp. Let $(u, J_S, J) \in \mathbb{R}\hat{M}_0^d(x)$, we have $\dim H_D^0(S, N^s_{u,-z})_{+1} = 1$. Since $\text{ind}(\pi_\mathbb{R}) = 0$, it follows that $\dim H_D^1(S, N_{u,-z})_{+1} \geq 1$. Thus, $S$ being rational, $\dim H_D^0(S, N_{u,-z})_{+1} = 0$ (see [11]) and $\dim H_D^1(S, N_{u,-z})_{+1} = 1$. Let $\psi_u$ be a generator of the vector space $H_D^0(S, KS \otimes N^s_{u,-z})_{-1} = H_D^1(S, N_{u,-z})_{+1}$. Remember that since $\psi_u \neq 0$ and $D^*(\psi_u) = 0$, this section $\psi_u$ vanishes at a finite number of points.

**Proposition 2.7** The subspace of $\mathbb{R}\hat{M}_0^d(x)$ consisting of curves $[u, J_S, J]$ for which the section $\psi_u$ vanishes at the unique real cuspidal point of $u$ is an immersed submanifold of $\mathbb{R}\hat{M}_0^d(x)$ of codimension one.

**Proof:**

The following proof is very analogous to the proof of Lemma 4.4.3 of [11]. Fix some component of $\mathbb{R}\hat{M}_0^d(x)$ and a lift $\hat{\mathcal{C}}_s$ of this component in $\mathbb{R}\hat{P}^s(x)$. Denote by $c_S \in G^-$ the associated involution and by $\mathbb{R}S$ the fixed point set of $c_S$ in $S$. For every $(u, J_S, J) \in \mathcal{C}_s$, we
denote by $t_u \in \mathbb{R}S$ the unique point at which $du$ is not injective. Let then $F$ be the real vector bundle of rank one on $C_s$ whose fiber over $(u, J_S, J)$ is the vector space $(K_s \otimes N^*_u)_{-1}|_u$. It is a vector bundle of class $C^{l-1,\alpha}$. Let $\psi$ be a local section of class $C^{l-1,\alpha}$ of the vector bundle over $C_s$ whose fiber over $(u, J_S, J)$ is the vector space $H^0_{\mathcal{F}}(S, K_S \otimes \mathbb{C} N^*_u)_{-1}$. We assume that $\psi$ does not vanish and we will denote as before by $\psi_u$ the value $\psi(u, J_S, J)$. We then deduce a section $\Gamma_\psi$ of the bundle $F$ defined by $\Gamma_\psi(u, J_S, J) = \psi_u(t_u)$. This section $\Gamma_\psi$ is of class $C^{l-1,\alpha}$ and we have to prove that it vanishes transversally. For this, we fix a Riemannian metric on $S$ and the associated Levi-Civita connection. This connection as well as the connection $\nabla$ on $X$ induce connections on all the associated bundles, like $F$ for instance. For convenience, all these connections will be denoted by $\nabla$. Hence, suppose that $(u, J_S, J) \in C_s$ is such that $\Gamma_\psi(u, J_S, J) = 0$, we have to prove that $\nabla|_{(u, J_S, J)} \Gamma_\psi : T_{(u, J_S, J)} C_s \to F_{(u, J_S, J)}$ is surjective. Let then $j_0 \in (K_S \otimes N^*_u)_{-1}|_u$, we are searching for $(v, J_S, J) \in T_{(u, J_S, J)} C_s$ such that $\nabla_{(v, J_S, J)} \Gamma_\psi = j_0$. For this, it suffices to find $(v, J_S, J) \in T_{(u, J_S, J)} C_s$ and a section $\psi \in L^{k,p}(S, K_S \otimes \mathbb{C} N^*_u)_{-1}$ such that $\psi(t_u) = j_0$ and

$$\forall w \in L^{k,p}(S, N_u)_{-1}, <D^\ast \psi, w> + <\psi_u, \nabla_{(v, J_S, J)} D(w) >= 0.$$  

It is indeed proved in \[1\], lines (4.4.12) to (4.4.15) that this last relation ensures that $\nabla_{(v, J_S, J)} \Gamma_\psi = \psi(t_u)$.

Let us start to construct the section $\dot{\psi}$. Let $\dot{\psi}_1 \in L^{k,p}(S, K_S \otimes \mathbb{C} N^*_u)_{-1}$ be a local section such that $\dot{\psi}_1(t_u) = j_0$ and $D^\ast(\dot{\psi}_1) = 0$ in a neighborhood of $t_u$. Such a section does exist. It suffices to solve locally the equation $D^\ast(j_0 + z\phi(z)) = 0$ where the unknown $\phi$ is defined in the neighborhood of $t_u$. This equation is equivalent to $(z^{-1}D^\ast z)(\phi(z)) = z^{-1}D^\ast j_0$. The operator $z^{-1}D^\ast z$ is equivariant under the action of $c_S$ and has, once restricted to a neighborhood of $t_u$, a right inverse $T$ also $\mathbb{Z}/2\mathbb{Z}$-equivariant. Thus $\phi = T \circ z^{-1}D^\ast j_0$ is a local solution and satisfies $d\phi \circ c_S = -\phi$, which provides the existence of $\dot{\psi}_1$. Using partition of unity, this local section is completed to a global section $\dot{\psi} \in L^{k,p}(S, K_S \otimes \mathbb{C} N^*_u)_{-1}$. It remains to find $(v, J_S, J) \in T_{(u, J_S, J)} C_s$ such that (3) is satisfied. Let us search for such a vector among those for which $v = 0$, $J_S = 0$, and $J \neq 0$ along $u(S)$ and in a neighborhood of the cusp $u(t_u)$. Such a vector is tangent to $C_s$ as soon as $\beta^X(J) = \dot{J}$. From Lemma 4.2.3 of \[1\], at such a vector $(0, 0, J)$, we have $\nabla_{(0, 0, J)} D(w) = \nabla_j J \circ du \circ J_S$ and thus (3) rewrites:

$$\forall w \in L^{k,p}(S, N_u)_{-1}, <D^\ast \psi, w> + <\psi_u, \nabla_j J \circ du \circ J_S >= 0,$$

or

$$-D^\ast \psi = <\psi_u, \nabla_j J \circ du \circ J_S>.$$  

Outside of a neighborhood of $t_u$, this equation determines the value of the derivative of $J$ in the normal direction of the curve $u(S)$. After integration of this condition, we construct a solution $J$ satisfying (4) and $\beta^X(J) = \dot{J}$. Hence the result. \[\square\]

**Proposition 2.8** Let $(X, \omega, c_X)$ be a real rational symplectic 4-manifold, $d_1, d_2 \in H_2(X; \mathbb{Z})$, and $y_1, y_2$ be two finite disjoint subsets of $X$ invariant under $c_X$. Denote by

$$\Delta = \{(u_1, J_S, J), [u_2, J_S, J)] \in \mathbb{R}M_{0}^{d_1}(y_1) \times \mathbb{R}M_{0}^{d_2}(y_2) | u_1(S_1) = u_2(S_2)\}.$$  

Then, the projections $\pi^1 : \mathbb{R}M_{0}^{d_1}(y_1) \to \mathbb{R}J_\omega$ and $\pi^2 : \mathbb{R}M_{0}^{d_2}(y_2) \to \mathbb{R}J_\omega$ are transversal outside of $\Delta$. 

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Note that as soon as \( d_1 \neq d_2 \), the diagonal \( \Delta \) is empty.

**Corollary 2.9** Let \((X, \omega, c_X)\) be a real rational symplectic 4-manifold, \( d_1, \ldots, d_l \in H_2(X; \mathbb{Z}) \) and \( y_1, \ldots, y_l \) be finite disjoint subsets of \( X \) invariant under \( c_X \). Denote by

\[
\Delta = \{([u_1, J_{S_1}, J], \ldots, [u_l, J_{S_l}, J]) \in \pi_1 \mathcal{M}^d_0(y_1) \times \cdots \times \pi_1 \mathcal{M}^d_0(y_l) \, | \, \exists i \neq j \text{ for which } u_i(S_i) = u_j(S_j) \}.
\]

Then the fiber product \( (\prod_{i=1}^l \pi_1 \mathcal{M}^d_0(y_i)) \setminus \Delta \) over \( \mathcal{J}_\omega \) is a Banach manifold of class \( C^{l, \alpha} \). Moreover, the projection \( (\prod_{i=1}^l \pi_1 \mathcal{M}^d_0(y_i)) \setminus \Delta \to \mathcal{J}_\omega \) is Fredholm of index \( c_1(X)d - l - \#y \), where \( d = \sum_{i=1}^l d_i \) and \( y = \cup_{i=1}^l y_i \).

**Proof of Proposition 2.8**

Let \([u_1, J_{S_1}, J] \in \pi_1 \mathcal{M}^d_0(y_1) \) and \([u_2, J_{S_2}, J] \in \pi_1 \mathcal{M}^d_0(y_2) \), so that \( \pi_1^1([u_1, J_{S_1}, J]) = \pi_1^2([u_2, J_{S_2}, J]) = J \) \( \in \mathcal{J}_\omega \). Fix \((u_1, J_{S_1}, J)\) (resp. \((u_2, J_{S_2}, J)\)) a lift of \([u_1, J_{S_1}, J]\) (resp. \([u_2, J_{S_2}, J]\)) in \( \mathbb{R}P^*(y_1) \) (resp. \( \mathbb{R}P^*(y_2) \)), so that \( u_1 \) is a \( J \)-holomorphic map \( (S_1, z_1) \to (X, y_1) \) (resp. \( (S_2, z_2) \to (X, y_2) \)). Assume that \( u_1(S_1) \neq u_2(S_2) \). From Proposition 2.8, \( \text{coker}(\pi_1^1|_{u_1, J_{S_1}, J}) \cong H^1_D(S_1, N_{(u_1,-z_1)})_{\alpha} \). Let \( 0 \neq \psi_1 \in H^0_D(S_1, K_{S_1} \otimes \mathcal{C} N^*_1)_{\alpha} \). It suffices to prove the existence of \( J \in L^{l, \alpha}(X, \Lambda^{0,1} X \otimes \mathcal{C} TX)_{\alpha} \) such that \( \langle \psi_1, J \circ du_1 \circ J_{S_1} \rangle \neq 0 \) and \( \langle \psi_2, J \circ du_2 \circ J_{S_2} \rangle = 0 \) for every \( \psi_2 \in H^0_D(S_2, K_{S_2} \otimes \mathcal{C} N^*_2)_{\alpha} \). Since \( D^*(\psi_1) = 0 \) and \( \psi_1 \neq 0 \), the section \( \psi_1 \) vanishes only at a finite number of points (see [4]). Since \( u_1 \) is not multiple, there exists an open set \( U \subset S_1 \setminus z_1 \) such that \( u_1(U) \) is an embedding, \( u_1(U) \cap u_1(S_1 \setminus U) = \emptyset \), \( c_X(u_1(U)) \cap u_1(U) = \emptyset \) and such that \( \psi_1 \) does not vanish on \( U \). Moreover, since \( u_1(S_1) \neq u_2(S_2) \), the intersection \( u_1(S_1) \cap u_2(S_2) \) consists only of a finite number of points and thus the open set \( U \) can be chosen so that \( u_1(U) \cap u_2(S_2) = \emptyset \).

Denote by \( c_{S_1} \) the order two element of \( G^\gamma_1 \) whose fixed point set in \( \mathbb{R}P^*(y_1) \) contains \((u_1, J_{S_1}, J)\), it is in fact the \( J_{S_1} \)-antiholomorphic involution of \( S_1 \) induced by \( u_1 \) and \( c_X \). Let then \( c_U \) a section of \( \Lambda^{0,1} S_1 \otimes E_{u_1} \) with support in \( U \) such that \( \langle \psi_1, c_U \rangle \neq 0 \). There exists \( J \in L^{l, \alpha}(X, \Lambda^{0,1} X \otimes \mathcal{C} TX)_{\alpha} \), with support in a neighborhood of \( u_1(U) \) in \( X \), such that \( J \circ du_1 \circ J_{S_1} = c_U \). Denote by \( J = -dC_X \circ J \circ dC_X \in L^{l, \alpha}(X, \Lambda^{0,1} X \otimes \mathcal{C} TX)_{\alpha} \). We have

\[
\langle \psi_1, J \circ du_1 \circ J_{S_1} \rangle = \langle \psi_1, J \circ du_1 \circ J_{S_1} \rangle + \langle \psi_1, \overline{C_X}(\bar{J}) \circ du_1 \circ J_{S_1} \rangle = \langle \psi_1, c_U \rangle + \langle \psi_1, dC_X \circ J \circ du_1 \circ J_{S_1} \circ dC_X \rangle = \langle \psi_1, c_U \rangle - \langle dC_X \circ \psi_1 \circ dC_X, c_U \rangle \quad \text{(changing of variables, since } c_{S_1} \text{ reverses the orientation of } S_1) \]

\[
= 2 \langle \psi_1, c_U \rangle \neq 0.
\]

Since the support of \( J \) is disjoint from \( u_2(S_2) \), we have \( J \circ du_2 \circ J_{S_2} = 0 \) and thus \( \langle \psi_2, J \circ du_2 \circ J_{S_2} \rangle = 0 \) for every \( \psi_2 \in H^0_D(S_2, K_{S_2} \otimes \mathcal{C} N^*_2)_{\alpha} \). □

Under the hypothesis of Proposition 2.8, we will denote by \( \mathbb{R}M^{d_1, d_2}_0(y_1, y_2) \) the fiber product \( (\mathbb{R}M^{d_1}_0(y_1) \times_{\mathcal{J}_\omega} \mathbb{R}M^{d_2}_0(y_2)) \setminus \Delta \).

**Proposition 2.10** The subspace of \( \mathbb{R}M^{d_1, d_2}_0(y_1, y_2) \) consisting of couples \(([u_1, J_{S_1}, J], [u_2, J_{S_2}, J])\) for which the union \( u_1(S_1) \cup u_2(S_2) \) is not nodal or has some node at some point of \( y_1 \cup y_2 \), is an immersed submanifold of codimension one.
2.3 Proof of Theorem 2.1

Let \( \nu \) be the number of nodal curves, but the intersection \( u_1(S_1) \cap u_2(S_2) \) is not transverse; or \( u_1(S_1) \cup u_2(S_2) \) is a nodal curve, but having some node at some point of \( y_1 \cup y_2 \).

In the two first cases, the result follows from Proposition 2.6. In the two last cases, the proof is very much analogous to the one of cases 2 and 3 of Proposition 2.6 and mainly follows from Lemma 2.4 with \( \nu = 1 \). It is left to the reader. \( \square \)

The path \( \gamma \) is chosen so that every element of \( \mathbb{R} \mathcal{M}_\gamma \) is a nodal curve, with the exception of a finite number of them which may have a unique real ordinary cusp, a unique real triple point or a unique real tacnode. Moreover, this path is chosen so that when a sequence of elements of \( \mathbb{R} \mathcal{M}_\gamma \) converges in Gromov topology to a reducible curve of \( X \), then this curve has only two irreducible components, both real, and only ordinary nodes as singularities. Finally, this path is chosen so that if \( [u, J_S, J] \in \mathbb{R} \mathcal{M}_\gamma \) has a unique real ordinary cusp at the parameter \( t_u \in \mathbb{R} S \), then the generator \( \psi_u \) of \( H^0_{D_1}(S, K_S \otimes_C N_{u_{-2}}^{\text{sing}})_{-1} \) does not vanish at \( t_u \). Such a choice of \( \gamma \) is possible from Propositions 2.6 and 2.8.

**Lemma 2.11** The critical points of \( \pi_\gamma \) are the curves \( [u, J_S, J] \in \mathbb{R} \mathcal{M}_\gamma \) having an ordinary cusp. Moreover, all these critical points are non-degenerate.

**Proof:** From Proposition 2.10, at a point \( [u, J_S, J] \in \mathbb{R} \mathcal{M}_\gamma \), the cokernel of \( d\pi_\gamma \) is isomorphic to \( H^1_D(S, N_{u_{-2}})_{+1} \) and its kernel to \( H^0_D(S, N_{u_{-2}})_{+1} = H^0_D(S, N_{u_{-2}})_{+1} \oplus H^0(D, N_{u_{-2}}^{\text{sing}})_{+1} \). If \( [u, J_S, J] \) is a critical point of \( \pi_\gamma \), then \( \text{dim} H^0_D(S, N_{u_{-2}})_{+1} = +1 \). Since \( S \) is rational and \( D \) is of generalized \( 1 \)-type, this implies that \( \text{dim} H^0_D(S, N_{u_{-2}})_{+1} = 1 \) and thus \( u \) is not an immersion. From the hypothesis made on \( \gamma \), this implies that \( u \) has a real ordinary cusp. Conversely, if \( [u, J_S, J] \in \mathbb{R} \mathcal{M}_\gamma \) has a real ordinary cusp, then \( \text{dim} H^0(S, N_{u_{-2}}^{\text{sing}})_{+1} = 1 \). Thus, \( \text{ker}(d[u, J_S, J] \pi_\gamma) \neq 0 \) and since \( \text{ind}(d\pi_\gamma) = 0 \), \( \text{coker}(d[u, J_S, J] \pi_\gamma) \neq 0 \), hence the first part of the lemma.
Now let \( [u, J_S, J] \in \mathcal{M}_S \) be a critical point of \( \pi_\gamma \). Fix some lift \((u, J_S, J) \in \mathbb{R}\mathcal{P}^*(x) \) of this element, denote by \( c_S \in G^- \) the associated element of order two and by \( \mathbb{R}S \subset S \) the fixed point set of \( c_S \). Let \( t_u \in \mathbb{R}S \) be the point at which \( du \) is not injective. We have to prove that the second order differential

\[
\nabla|_{[u, J_S, J]} : H^0(S, N_u^{\text{sing}})_{1+1} \times H^0(S, N_u^{\text{sing}})_{1+1} \to H^1_D(S, N_{u,-z})_{1+1}
\]

is non-degenerate. Let \( \psi \) be a generator of \( H^0_D(S, K_S \otimes \mathbb{C} N_u^{\text{sing}})_{1-1} = H^1_D(S, N_u^{\text{sing}})_{1} \) and \((v, \dot{J}_S)\) be a generator of \( H^0(S, N_u^{\text{sing}})_{1+1} \). Then \( v \) can be written \( du(\tilde{v}) \) where \( \tilde{v} \in L^{k,p}(S, TS_{-z} \otimes \mathbb{C} O(t_u))_{1+1} \) (see \([11]\), Lemma 4.3.1), that is \( \tilde{v} \) is a meromorphic real vector field of \( S \) having a simple pole at \( t_u \) and vanishing at \( z \in S \). From \([11]\), Lemma 4.3.3, formula 4.3.9, we have:

\[
< \psi, \nabla|_{[u, J_S, J]} d\pi_\gamma((v, \dot{J}_S), (v, \dot{J}_S)) > = \Re \text{Res} < \psi, \nabla d_{\tilde{v}} u(\tilde{v}, \tilde{v}) > = \Re \lim_{\varepsilon \to 0} \int_{|\xi - t_u| = \varepsilon} < \psi, \nabla d_{\tilde{v}} u(\tilde{v}, \tilde{v}) > .
\]

Now, from the computations done in the proof of Lemma 4.3.4 of \([11]\) and from Lemma 4.3.5 of \([11]\), since by hypothesis \( \psi \) does not vanish at the unique real ordinary cusp \( t_u \) of \( u \), the quadratic form (6) is equivalent to \( w \in \mathbb{R} \leftrightarrow \Re \text{Res}_{z=0}(w^2 dz) \), hence is non-degenerate. □

Let \( C_0 \) be a real \( J_0 \)-holomorphic nodal curve having two irreducible components \( C_1 \) and \( C_2 \), and limit in Gromov topology of a sequence \([u_{\lambda_n}, J^{\lambda_n}_S, J^{\lambda_n}_0] \) of elements of \( \mathcal{M}_{\gamma} \), where \( (\lambda_n)_{n \in \mathbb{N}} \) is a sequence of \([0,1] \) converging to some parameter \( \lambda_\infty \in [0,1] \). Denote by \( d_1 \in H_2(X; \mathbb{Z}) \) (resp. \( d_2 \in H_2(X; \mathbb{Z}) \)) the homology class of \( C_1 \) (resp. \( C_2 \)) and by \( x_1 = x \cap C_1 \) (resp. \( x_2 = x \cap C_2 \)), so that \( d = d_1 + d_2 \) and \( x = x_1 \cup x_2 \). From Propositions 2.8 and 1.10 we see that, exchanging \( C_1 \) and \( C_2 \) if necessary, we can assume that \( \#(x_1) = c_1(X)d_1 - 1 \) and \( \#(x_2) = c_1(X)d_2 \). Let \([u_1, J_{S_1}, J_0]\) and \([u_2, J_{S_2}, J_0]\) be elements of \( \mathbb{R}M_d^{d_1}(x_1) \) and \( \mathbb{R}M_d^{d_2}(x_2) \) representing \( C_1 \) and \( C_2 \) respectively. From Proposition 1.10 we know that \( \dim H^1_D(S_1, N_{u_1,-z})_{1+1} \geq 0 \) and \( \dim H^1_D(S_2, N_{u_2,-z})_{1+1} \geq 1 \) and from Corollary 2.9 we see that these inequalities are equalities. As a consequence, the projection \( \pi^{d_1}_{\mathbb{R}} : \mathbb{R}M_d^{d_1}(x_1) \to \mathcal{J}_\omega \) restricts in a neighborhood of \([u_1, J_{S_1}, J_0]\) to a submersion on a neighborhood \( V_1 \) of \( J_0 \) in \( \mathcal{J}_\omega \). Similarly, in a neighborhood of \([u_2, J_{S_2}, J_0]\), the projection \( \pi^{d_2}_{\mathbb{R}} : \mathbb{R}M_d^{d_2}(x_2) \to \mathcal{J}_\omega \) maps onto a codimension one submanifold on a neighborhood \( V_2 \) of \( J_0 \) in \( \mathcal{J}_\omega \). Denote by \( V = V_1 \cap V_2 \) and by \( H \subset V \) this codimension one submanifold. We can assume that \( V \) is connected and that \( V \setminus H \) has two connected components.

Denote by \( \overline{M_d^0}(x) \) (resp. \( \mathbb{R}\overline{M_d^0}(x) \)) the Gromov compactification of \( M_d^0(x) \) (resp. \( \mathbb{R}M_d^0(x) \)), and by \( \overline{\pi} \) (resp. \( \pi_{\mathbb{R}} \)) the projection \( \overline{M_d^0}(x) \to \mathcal{J}_\omega \) (resp. \( \mathbb{R}\overline{M_d^0}(x) \to \mathbb{R}\mathcal{J}_\omega \)). Restricting \( V \) if necessary, we can assume that there exists a neighborhood \( W \) of \( C_0 \) in \( \mathbb{R}\overline{M_d^0}(x) \) such that \( \overline{\pi}(W) = V \) and such that the image of reducible curves of \( W \) under this projection is exactly
Let $C_0$ be a real reducible $J_0$-holomorphic curve of $X$ passing through $x$ and limit of a sequence of elements of $\mathbb{M}_\omega$. Let $J_0 = \gamma(\lambda_0)$ for $\lambda_0 \in [0, 1]$ and $C_1, C_2$ be the two irreducible components of $C_0$. Let $R$ be the number of real intersection points between $C_1$ and $C_2$. Then there exist a neighborhood $W$ of $C_0$ in the Gromov compactification $\overline{\mathbb{M}}_{0,0}(x)$ and $\eta > 0$ such that for every $\lambda \in [\lambda_0 - \eta, \lambda_0 + \eta \setminus \{\lambda_0\}$, $\pi_{\gamma}^{-1}(\lambda) \cap W$ consists exactly of $R$ real $\gamma(\lambda)$-holomorphic curves, each of them obtained topologically by smoothing a different real double point of $W$ intersect at a finite number of points, each local intersection being of positive multiplicity. If $W$ has been chosen small enough, these curves have a double point in a neighborhood of each double point of $C_0$ except $y_0$. In particular, in a neighborhood of each such double point, these curves intersect each other at least two points. This number of double points is $\frac{1}{2}(d^2 - c_1(X)d + 2)$ from adjunction formula. Moreover, since both $C'$ and $C''$ pass through the configuration of points $x$, they intersect each other at each point of $x$ with multiplicity at least one. Thus, one has

$$d^2 = C' \circ C'' \geq (d^2 - c_1(X)d + 2) + c_1(X)d - 1 = d^2 + 1,$$

which is impossible.

Let us now prove that $\pi_{\gamma}^{-1}(\lambda) \cap W$ actually contains exactly one curve for each intersection point of $C_1 \cap C_2$. Let $y \in X$ be such an intersection point. In a neighborhood of $y$, the curve $C_0$ is biholomorphic to the standard real node $A_0 = \{(z^+, z^-) \in B^2 | z^+z^- = 0\}$. Following the definition 5.4.1 of $[11]$, the cylinders close to $A_0$ are the cylinders $A_\varphi = \{(z^+, z^-) \in B^2 | z^+z^- = \varphi\}$, for $\varphi \in B^2(e), e > 0$. These cylinders form a partition of the real analytic space $A = \{(z^+, z^-) \in B^2 | |z^+z^-| < e\}$. Note that when the parameter $\varphi \in B^2(e) \setminus \{0\}$ is real, the cylinders $A_\varphi$ are real and correspond topologically to the two standard ways to smooth the real node $A_0$, for $\varphi > 0$ and $\varphi < 0$.

From Theorem 5.4.1 of $[11]$ (The map $\Phi$ given in this theorem is $\mathbb{Z}/2\mathbb{Z}$-equivariant for the real structures induced on $\mathcal{U} \times \Delta(e')$ and $\mathcal{P}(A)$), the real embedding of $A_0$ in $X$ given by $C_0$
deforms into a one parameter family of real embeddings of the cylinders $A_\varphi$, for $\varphi \in ]-\epsilon,\epsilon[$.

There exists then a continuous family $(J_\varphi)_{\varphi \in ]-\epsilon,\epsilon[}$ in $R_\omega J$, extending $J_0$, such that for every $\varphi \in ]-\epsilon,\epsilon[\setminus \{0\}$, $J_\varphi$ differs from $J_0$ only in a neighborhood of $\partial A_0 \subset X$, and a continuous family $(C_\varphi)_{\varphi \in ]-\epsilon,\epsilon[}$ of real $J_\varphi$-holomorphic curves extending $C_0$, such that for $\varphi \neq 0$, $C_\varphi$ is obtained topologically from $C_0$ by smoothing the real node $y$. Indeed, restricting a little bit the real embedding of the cylinder $A_\varphi$, the image of this embedding can be glued to the curve $C_0 \setminus A_0$ by adding two real small annuli embedded in a neighborhood of $\partial A_0$. The real curve $C_\varphi$ we thus obtain can be easily made $J_\varphi$-holomorphic for some almost-complex structure $J_\varphi \in R_\omega J$ close to $J_0$ and differing from the latter only in a neighborhood of $\partial A_0$.

Let us fix now $\varphi_+ \in ]0,\epsilon[ \setminus \{0\}$ and $\varphi_- \in ]-\epsilon,0[\setminus \{0\}$ such that $J_{\varphi_+}, J_{\varphi_-} \in V \setminus H$ and $C_{\varphi_+}, C_{\varphi_-} \in W$. Deforming locally $C_{\varphi_+}, C_{\varphi_-}$ if necessary, we can assume that $J_{\varphi_+}, J_{\varphi_-}$ are regular values of $\pi_R$. The almost-complex structures $J_{\varphi_+}$ and $J_{\varphi_-}$ do not belong to the same connected component of $V \setminus H$. Indeed, it would be otherwise possible to join them by a path of $V \setminus H$ transversal to the projection $\pi_R$, and there would be no obstruction to isotop $C_{\varphi_-}$ along this path into a continuous family of $W$ to end up with a $J_{\varphi_+}$-holomorphic curve denoted by $C'_{\varphi_-}$. The absence of such an obstruction follows from the fact that $C_{\varphi_-}$ can neither degenerate into a reducible curve nor into a cuspidal curve along this path. Since these curves are rational and immersed, they are all regular points of the projection (see Proposition 10 and 11). This provides the contradiction since $C_{\varphi_+}$ and $C'_{\varphi_-}$ are two $J_{\varphi}$-holomorphic curves in $W$ obtained topologically by smoothing the same node of $C_0$, which is impossible from the computation done at the begining of this proof. Now the result follows similarly. Let $\lambda_+ \in ]0,\epsilon[ \setminus \{0\}$ and $\lambda_- \in ]-\epsilon,0[\setminus \{0\}$, the almost-complex structures $\gamma(\lambda_+)$ and $\gamma(\lambda_-)$ are not in the same component of $V \setminus H$. Each of them can be joined to $J_{\varphi_+}$ or $J_{\varphi_-}$ by a path of $V \setminus H$ transversal to the projection $\pi_R$. There is then no obstruction to isotop $C_{\varphi_-}$ or $C_{\varphi_+}$ along these paths to get a $\gamma(\lambda_+)$ or $\gamma(\lambda_-)$-holomorphic curve which is obtained topologically by smoothing the real node $y$ of $C_0$. Hence the result.

**Proposition 2.13** Let $C_{\lambda_0} \in R_\gamma M_\gamma$ be a critical point of $\pi_\gamma$ which is a local maximum (resp. minimum). Then there exist a neighborhood $W$ of $C_{\lambda_0}$ in $R_\gamma M_\gamma$ and $\eta > 0$ such that for every $\lambda \in ]0,\lambda_0 - \eta,\lambda_0[ \setminus \{0\}$ (resp. for every $\lambda \in ]0,\lambda_0 + \eta[ \setminus \{0\}$, $\pi^{-1}_\gamma(\lambda) \cap W$ consists of two curves $C^{+}\lambda$ and $C^{-}\lambda$ satisfying $m(C^{+}\lambda) = m(C^{-}\lambda) + 1$, and for every $\lambda \in ]0,\lambda_0 - \eta,\lambda_0[ \setminus \{0\}$, $\pi^{-1}_\gamma(\lambda) \cap W = \emptyset$. □

**Proof:**

Let us assume that $C_{\lambda_0}$ is a local maximum of $\pi_\gamma$, and let us denote $C_{\lambda_0}$ by $[u_{\lambda_0}, J_{S_{\lambda_0}^{+}}, J_{\lambda_0}]$. Since $R_\gamma M_\gamma$ is one dimensional and $C_{\lambda_0}$ is a non-degenerate critical point, it is clear that there exists $\eta > 0$ such that in a neighborhood $W$ of $[u_{\lambda_0}, J_{S_{\lambda_0}^{+}}, J_{\lambda_0}]$, $\pi^{-1}_\gamma(\lambda) \cap W$ consists of two curves if $\lambda \in ]0,\lambda_0 - \eta,\lambda_0[ \setminus \{0\}$, and $\pi^{-1}_\gamma(\lambda) \cap W = \emptyset$ if $\lambda \in ]0,\lambda_0 + \eta[ \setminus \{0\}$. The only thing to prove is that if $\eta$ is small enough, the two curves $C^{+}\lambda$ and $C^{-}\lambda$ of $\pi^{-1}_\gamma(\lambda) \cap W$ satisfy $m(C^{+}\lambda) = m(C^{-}\lambda) + 1$. From the choice of $\gamma$ we know that the only singularities of $C_{\lambda_0}$ are nodes and a unique real ordinary cusp. If $\eta$ is small enough, the two curves $C^{+}\lambda$ and $C^{-}\lambda$ are close enough to $C_{\lambda_0}$ so that they have a node in a neighborhood of each node of $C_{\lambda_0}$ plus a node in a neighborhood of the cusp of $C_{\lambda_0}$. Since the nodes which are close to nodes of $C_{\lambda_0}$ are of the same nature, we have to prove that for one of the curves $C^{+}\lambda$ or $C^{-}\lambda$, the real node close to the cusp of $C_{\lambda_0}$ is non-isolated, and for the other one, it is isolated. The result indeed follows then from the definition of the mass.
So let us fix a parametrization \( \mu \in ]-\epsilon, \epsilon[ \mapsto C_\mu \in \mathbb{R} \mathcal{M}_\gamma \cap W \), such that \( C_0 = C_{\lambda_0} \) and \( \pi_\gamma(C_\mu) = \pi_\gamma(C_{-\mu}) \). Considering the restriction of these curves to a neighborhood of the cusp of \( C_0 \) diffeomorphic to the ball \( B^4(2) \) of \( \mathbb{C}^2 \), we deduce, with the notations of Lemma 2.5, a path \( \left( C'_\mu \right)_{\mu \in ]-\epsilon, \epsilon[} \) of \( \mathbb{R} \mathcal{P}'_s \). We have to prove that this path is transversal to \( \mathbb{R} \mathcal{P}'_s \) at \( \mu = 0 \). From the hypothesis, we know that \( v_{\lambda_0} = \frac{d}{d\mu}(C_\mu)|_{\mu=0} \in H^1_0(D(\mathbb{S}, \mathcal{N}_{u_{z,-z}}^{sing}) + 1) \). Denoting by \( t_0 \in \mathbb{R} \mathcal{S} \) the point at which \( du_{\lambda_0} \) is not injective, we deduce from Lemma 4.3.1 of [11] that \( v_{\lambda_0} = du_{\lambda_0}(w_{\lambda_0}) \) for a section \( w_{\lambda_0} \) of class \( L^{k-1,p} \) of the bundle \( T_{S-z} \otimes \mathcal{O}(t_0) \), that is for a vector field of \( S \) vanishing at \( z \) and with a simple pole at \( t_0 \). Restricting ourselves to the ball \( B_4(2) \) defined above, we can write \( w_{\lambda_0} = \frac{1}{t_0} w'_{\lambda_0} \). Moreover, from Corollary 3.1.3 of [3], the diffeomorphism onto this ball can be chosen in order that \( u_{\lambda_0} \) writes \( t \mapsto (t^2, t^3 u'_{\lambda_0}) \) and \( \nabla|_{t=0} v_{\lambda_0} = (2\nabla|_{t=0} w'_{\lambda_0}, 3w'_{\lambda_0} u'_{\lambda_0} \frac{d}{dt}) \). Hence \( \text{Im}(\nabla|_{t=0} v_{\lambda_0}) \) is not the tangent of \( C_{\lambda_0} \) at the cusp and the transversality condition of Lemma 2.5 is satisfied, which proves the result.

\[ \blacksquare \]

**Proof of Theorem 2.1:**

Let \( J_0, J_1 \in \mathbb{R} \mathcal{J}_\omega \) be two regular values of the projection \( \pi : \mathcal{M}_0^d(x) \to \mathcal{J}_\omega \), and \( \gamma : [0, 1] \to \mathbb{R} \mathcal{J}_\omega \) be the path fixed at the beginning of §2.3 joining them. The integer \( \chi^d(x, \gamma(\lambda)) \) is then well defined for all \( \lambda \in [0, 1] \) but a finite number of values \( 0 < \lambda_1 < \cdots < \lambda_j < 1 \) corresponding either to reducible curves, to cuspidal curves, or to curves having a real triple point or tacnode. Since the function \( \lambda \mapsto \chi^d_r(x, \gamma(\lambda)) \) is obviously constant between these values, we just have to prove that for \( i \in \{1, \ldots, j\} \), \( \chi^d_r(x, \gamma(\lambda^-_i)) = \chi^d_r(x, \gamma(\lambda^+_i)) \) where \( \lambda^-_i \) (resp. \( \lambda^+_i \)) is the left limit (resp. right limit) of \( \lambda \) at \( \lambda_i \). If \( \lambda_i \) corresponds to a curve having a real triple point or tacnode, it is straightforward and illustrated by the following pictures.

![Diagram](image)

or

![Diagram](image)

Passing through a real curve with tacnode

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If $\lambda_i$ corresponds to a reducible curve, it follows from Proposition 2.12, and if $\lambda_i$ corresponds to a cuspidal curve, it follows from Proposition 2.13. Hence, the integer $\chi^d_r(x, J)$ does not depend on the choice of $J \in \mathbb{R}J_\omega$. Note that in contrast to the previous cases, the coefficient $(-1)^m$ in the definition of $\chi^d_r(x, J)$ plays in this last case a crucial role to get the invariance. This integer $\chi^d_r(x, J)$ also does not depend on the choice of $x$, since the group of equivariant diffeomorphisms of $X$ acts transitively on these real configurations of points. Theorem 2.1 is thus proved. □

3 Further study of the polynomial $\chi^d(T)$

In the first three subparagraphs, we will give the relations between the coefficients of the polynomial $\chi^d(T)$ in term of a new invariant $\theta$. In the last subparagraph, we will prove the non-triviality of this polynomial in degrees 4 and 5 in $(\mathbb{C}P^2, \omega_{std}, \text{conj})$ (for degree less than four, it has already been computed in the first example given in §2.1).

3.1 The invariant $\theta$

Let $y = (y_1, \ldots, y_{c_1(X)d-2})$ be a real configuration of $c_1(X)d - 2$ distinct points of $X$, and $s$ be the number of those which are real. We assume that $y_{c_1(X)d-2}$ is real, so that $s$ does not vanish. Let $J \in \mathbb{R}J_\omega$ be generic enough. Then there are only finitely many $J$-holomorphic rational curves in $X$ in the homology class $d$ passing through $y$ and having an ordinary node at $y_{c_1(X)d-2}$. These curves are all nodal and irreducible. For every integer $m$ ranging from 0 to $\delta$, denote by $\hat{n}_d^+(m)$ (resp. $\hat{n}_d^-(m)$) the total number of these curves which are real, of mass $m$ and with a non-isolated (resp. isolated) real double point at $y_{c_1(X)d-2}$. Define then :

$$\theta^d_s(y, J) = \sum_{m=0}^{\delta} (-1)^m (\hat{n}_d^+(m) - \hat{n}_d^-(m)).$$

**Theorem 3.1** Let $(X, \omega, c_X)$ be a real rational symplectic 4-manifold, and $d \in H_2(X; \mathbb{Z})$. Let $y \subset X$ be a real configuration of $c_1(X)d - 2$ distinct points and $s \neq 0$ be the cardinality of $y \cap \mathbb{R}X$. Finally, let $J \in \mathbb{R}J_\omega$ be an almost complex structure generic enough, so that the integer $\theta^d_s(y, J)$ is well defined. Then, this integer $\theta^d_s(y, J)$ neither depends on the choice of $J$ nor on the choice of $y$ (provided the cardinality of $y \cap \mathbb{R}X$ is $s$).
For convenience, this integer \( \theta^d_s(y,J) \) will be denoted by \( \theta^d_s \), and we put \( \theta^d_s = 0 \) when \( s \) does not have the same parity as \( c_1(X)d \). This invariant makes it possible to give relations between the coefficients of the polynomial \( \chi^d \), namely:

**Theorem 3.2** Let \((X,\omega,c_1\mathbb{X})\) be a real rational symplectic 4-manifold, \(d \in H_2(X;\mathbb{Z})\) and \(r\) be an integer between 0 and \( c_1(X)d - 3 \). Then \( \chi^d_{r+2} = \chi^d_{r+1} + 2\theta^d_{r+1} \).

### 3.2 Proof of Theorem 3.1

To begin with, we construct as in [1] the moduli space \( \mathcal{M}_0^d(y) \) of real rational pseudoholomorphic maps \( u : S \to X \), realizing the homology class \( d \), and mapping the marked points \( z_1,\ldots,z_{c_1(X)d-2} \) of \( S \) to the corresponding points \( y_1,\ldots,y_{c_1(X)d-2} \) of \( X \) and mapping \( z_{c_1(X)d-1} \) also to \( y_{c_1(X)d-2} \). This moduli space is obtained by taking the quotient of the space of such maps by the group \( \text{Diff}^+(S,z) \) acting by reparametrization. Since now \( u(z_{c_1(X)d-2}) = u(z_{c_1(X)d-1}) \), there is a degree two extension of this group acting by reparametrization, namely the group of diffeomorphisms of \( S \), preserving the orientation, fixing the points \( z_1,\ldots,z_{c_1(X)d-3} \), and fixing or exchanging the points \( z_{c_1(X)d-2} \) and \( z_{c_1(X)d-1} \). This degree two extension induces a \( \mathbb{Z}/2\mathbb{Z} \)-action on \( \mathcal{M}_0^d(y) \) which has no fixed point, since its effect is to exchange the two local branches at the double point \( y_{c_1(X)d-2} \). Denote by \( \mathcal{M}_0^d(y) \) the orbit space of this action. It is a Banach manifold of class \( C^{l,\alpha} \), equipped with an index zero Fredholm projection \( \tilde{\pi} \) on \( \mathcal{J}_\omega \). Denote by \( \tilde{\pi}_\mathbb{R} : \mathbb{R}\mathcal{M}_0^d(y) \to \mathbb{R}\mathcal{J}_\omega \) the restriction of \( \tilde{\pi} \). The Theorem of regular values [1] applies also in this situation, so that the set of regular values of \( \tilde{\pi} \) intersects \( \mathbb{R}\mathcal{J}_\omega \) in a dense set of the second category.

Let then \( J_0, J_1 \in \mathbb{R}\mathcal{J}_\omega \) be regular values of the projection \( \tilde{\pi} : \mathcal{M}_0^d(y) \to \mathcal{J}_\omega \) such that no reducible \( J_0 \) or \( J_1 \)-holomorphic curve in the class \( d \) passes through \( y \) with a double point at \( y_{c_1(X)d-2} \). Let \( \gamma : [0,1] \to \mathbb{R}\mathcal{J}_\omega \) be a path transversal to the projection \( \tilde{\pi}_\mathbb{R} : \mathbb{R}\mathcal{M}_0^d(y) \to \mathbb{R}\mathcal{J}_\omega \), joining \( J_0 \) to \( J_1 \). Hence, \( \mathbb{R}\mathcal{M}_\gamma = \tilde{\pi}_\mathbb{R}^{-1}(\text{Im}(\gamma)) \) is a submanifold of dimension one of \( \mathbb{R}\mathcal{M}_0^d(y) \), equipped with a projection \( \tilde{\pi}_\gamma : \mathbb{R}\mathcal{M}_\gamma \to [0,1] \) induced by \( \tilde{\pi}_\mathbb{R} \).

![Diagram](image)

The path \( \gamma \) is chosen so that every element of \( \mathbb{R}\mathcal{M}_\gamma \) is a nodal curve, with the exception of a finite number of them which may have a unique real ordinary cusp, a unique real triple point or a unique real tacnode. This path is also chosen so that when a sequence of elements of \( \mathbb{R}\mathcal{M}_\gamma \) converges in Gromov topology to a reducible curve of \( X \), then this curve has only two irreducible components, both real, and only nodal points as singularities. Moreover, this path is chosen so that if \([u,J]\) is a unique real ordinary cusp at the parameter \( t_u \in \mathbb{R}\mathcal{S} \), then the generator \( \psi_u \) of \( H^1_{\text{DR}}(S,K_S \otimes \mathbb{C} N_{u,\mathbb{C}}^*) \) does not vanish at \( t_u \). Finally, it is chosen so that when a sequence of elements of \( \mathbb{R}\mathcal{M}_\gamma \) converges in Gromov topology to an irreducible curve of \( X \) not in \( \mathbb{R}\mathcal{M}_\gamma \), thus a cuspidal curve, then this curve has a real ordinary
cusp at \( y_{c_1(X)d-2} \), and only nodal points as remaining singularities. Such a choice of \( \gamma \) is possible from Propositions \ref{prop:1} and \ref{prop:2}. The integer \( \theta^d_s(y, \gamma(\lambda)) \) is then well defined for all \( \lambda \in [0,1] \) but a finite number of values \( 0 < \lambda_1 < \cdots < \lambda_j < 1 \) corresponding either to reducible curves, to cuspidal curves, or to curves having a real triple point or tacnode. Since the function \( \lambda \mapsto \theta^d_s(y, \gamma(\lambda)) \) is obviously constant between these values, we just have to prove that for \( i \in \{1, \ldots, j\} \), \( \theta^d_s(y, \gamma(\lambda^-_i)) = \theta^d_s(y, \gamma(\lambda^+_i)) \) where \( \lambda^-_i \) (resp. \( \lambda^+_i \)) is the left limit (resp. right limit) of \( \lambda \) at \( \lambda_i \). The only cases to consider is the apparition of a cuspidal curve, the cusp being at \( y_{c_1(X)d-2} \), or the apparition of a curve with a tacnode, the tacnode being at \( y_{c_1(X)d-2} \). Indeed, all the other cases follow along the same lines as in the proof of Theorem \ref{thm:1}.

The only additional thing to remark is that the topology of the node at \( y_{c_1(X)d-2} \) does not change under these moves. We will only consider the case of a cuspidal curve, since the other one can be treated exactly in the same way.

So, let \( (C_\lambda)_{\lambda \in \lambda_1 - \epsilon, \lambda_2 + \epsilon} \) be a continuous family of \( \gamma(\lambda) \)-holomorphic curves in \( \mathbb{R} \bar{\mathcal{M}}_\gamma \) which converges in Gromov topology to a real cuspidal irreducible \( \gamma(\lambda_i) \)-holomorphic curve, the cusp being at \( y_{c_1(X)d-2} \). All these curves are nodal as soon as \( \epsilon \) is small enough. Moreover, such a family is unique. Indeed, if for \( \lambda \in \lambda_1 - \epsilon, \lambda_2 + \epsilon \), there were two \( \gamma(\lambda) \)-holomorphic curves \( C' \) and \( C'' \) close to the cuspidal curve, then they would have two intersection points in the neighborhood of each nodal point of the cuspidal curve, plus four intersection points at \( y_{c_1(X)d-2} \) and moreover, they would intersect each other at each point of \( y \). This would give

\[
(c_1(X)d - 3) + 4 + 2\left(\frac{1}{2}(d^2 - c_1(X)d + 2) - 1\right) = d^2 + 1
\]

intersection points, which is to much since all multiplicities are positive. In particular, this family is made of real curves, and since the parity of the number of real curves does not change, this family does extend to a continuous family \( (C_\lambda)_{\lambda \in \lambda_1 - \epsilon, \lambda_2 + \epsilon} \) of real \( \gamma(\lambda) \)-holomorphic curves. Then, after the transformation, either the topology of the real node at \( y_{c_1(X)d-2} \) is unchanged and then the mass of the curve is also unchanged, or it has changed, but then the mass of the curve also has changed. In both cases, the integer \( \theta^d_s(y, \gamma(\lambda)) \) is left invariant, hence the result. \( \square \)

### 3.3 Proof of Theorem \ref{thm:2}

Let \( y = (y_1, \ldots, y_{c_1(X)d-2}) \) be a real configuration of distinct points of \( X \), such that \( y_{c_1(X)d-2} \in \mathbb{R}X \) and \( \#(y \cap \mathbb{R}X) = r + 1 \). Denote by \( \mathcal{M}^d_0(y) \) the moduli space of rational pseudo-holomorphic curves of \( X \) passing through \( y \) in the homology class \( d \). Similarly, denote by \( \mathcal{M}^d_0(y) \) the moduli space of such curves which have a real ordinary node at \( y_{c_1(X)d-2} \). This space has been introduced in \( \mathcal{M}_\omega \). Denote by \( P(T_{y_{c_1(X)d-2}}X) \) the space of tangent lines of \( X \) at \( y_{c_1(X)d-2} \). Then the projection \( \left[ u, J_\omega, J \right] \in \mathcal{M}^d_0(y) \mapsto (J, d) \in \mathbb{R} \mathcal{M}^d_0(y) \mapsto u(T_{y_{c_1(X)d-2}}u(T_{y_{c_1(X)d-2}}S)) \in \mathcal{J}_\omega \times P(T_{y_{c_1(X)d-2}}X) \) is Fredholm of index zero. Let \( (J, \tau) \in \mathbb{R} \mathcal{J}_\omega \times P(T_{y_{c_1(X)d-2}}X) \) be a regular value of this projection. We also assume that \( J \) is a regular value of the projection \( \mathcal{M}^d_0(y) \rightarrow \mathcal{J}_\omega \) and that there exists no reducible or cuspidal rational \( J \)-holomorphic curve passing through \( y \) in the homology class \( d \) and having a node or \( \tau \) as a tangent at \( y_{c_1(X)d-2} \). There is then only finitely many element of \( \mathbb{R} \mathcal{M}^d_0(y) \) having \( \tau \) as a tangent at \( y_{c_1(X)d-2} \). These curves are all nodal and irreducible. For every integer \( m \) between 0 and \( \delta \), denote by \( \tilde{n}_d(m) \)
the number of such curves which are real and of mass $m$. Denote then by:

$$\tilde{\chi}^d_r(y, J) = \sum_{m=0}^{\delta} (-1)^m \tilde{n}_d(m).$$

**Proposition 3.3** Under the above assumptions, we have the relations:

$$\chi^d_{r+2} = \chi^d_r(y, J) + 2 \sum_{m=0}^{\delta} (-1)^m \tilde{n}_d^{+}(m),$$

$$\chi^d_r = \chi^d_r(y, J) + 2 \sum_{m=0}^{\delta} (-1)^m \tilde{n}_d^{-}(m).$$

The integers $\tilde{n}_d^{+}(m)$ and $\tilde{n}_d^{-}(m)$ have been defined in §3.1. The Theorem 3.2 follows easily from this Proposition 3.3 and the definition of the invariant $\theta$.

**Proof of Proposition 3.3**

Let us first prove the first relation. For this purpose, let us fix a path $\mu : ] - \epsilon, \epsilon[ \to \mathbb{R}X$ of class $C^2$ such that $\mu(0) = y_{c_1(X)d-2}$ and $\mu'(0) \in \tau$. For every $\lambda \in ] - \epsilon, \epsilon[ \setminus \{0\}$, denote by $y_\lambda$ the set $(y_1, \ldots, y_{c_1(X)d-2}, \mu(\lambda))$. Denote then by $\mathbb{R}\mathcal{M}^d_0(y_\lambda)$ the moduli space of real rational pseudo-holomorphic curves of $X$ passing through $y_\lambda$ in the homology class $d$. Then $J$ is a regular value of the projection $\pi^1_\mathbb{R} : \mathbb{R}\mathcal{M}^d_0(y_\lambda) \to \mathbb{R}\mathcal{J}_\omega$ as soon as $\lambda$ is close enough to zero. Indeed, from Gromov compactness theorem, as soon as $\lambda$ is close enough to zero, the elements of $\mathbb{R}\mathcal{M}^d_0(y_\lambda)$ are close, in Gromov topology, either to elements of $\mathbb{R}\mathcal{M}^d_0(y)$ having a non-isolated real node at $y_{c_1(X)d-2}$, or to elements of $\mathbb{R}\mathcal{M}^d_0(y)$ having $\tau$ as a tangency at $y_{c_1(X)d-2}$. As a consequence, these curves are neither cuspidal, nor irreducible, and thus $J$ is a regular value of $\pi^1_\mathbb{R}$ from Proposition 1.10. The set $\{(\pi^1_\mathbb{R})^{-1}(J), \lambda \in ] - \epsilon, \epsilon[ \setminus \{0\}\}$ is thus the union of the images of finitely many continuous functions $C_1(\lambda), \ldots, C_f(\lambda)$. Each of these functions converges as $\lambda$ goes to zero either to an irreducible real $J$-holomorphic curve having a non-isolated real node at $y_{c_1(X)d-2}$, or to an irreducible curve having $\tau$ as a tangency at $y_{c_1(X)d-2}$. We will prove that each curve of the first kind (resp. second kind) is limit of exactly two (resp. one) such functions $C_{i_1}(\lambda), C_{i_2}(\lambda)$. The first relation of Proposition 3.3 follows, since $\chi^d_{r+2} = \chi^d_{r+2}(y_\lambda, J)$ for $\lambda$ close enough to zero, and since the masses of the curves are unchanged while passing to the limit $\lambda \to 0$.

Let then $C_0$ be an element of $\mathbb{R}\mathcal{M}^d_0(y)$ having $\tau$ as a tangency at $y_{c_1(X)d-2}$. Since by hypothesis, $(J, \tau)$ is a regular value of the projection $\mathcal{M}^d_0(y) \to \mathbb{R}\mathcal{J}_\omega \times \mathcal{P}(T_{y_{c_1(X)d-2}}X)$, the $J$-holomorphic curves in a neighborhood of $C_0$ in $\mathbb{R}\mathcal{M}^d_0(y)$ are exactly parametrized by their tangencies at $y_{c_1(X)d-2}$. Once we move this tangency, we see that these curves provide a foliation of an angular neighborhood $A$ of $y_{c_1(X)d-2}$ in $\mathbb{R}X$.

Since the path $\mu : ] - \epsilon, \epsilon[ \to \mathbb{R}X$ satisfies $\mu(0) = y_{c_1(X)d-2}$ and $\mu'(0) \in \tau$, restricting $\epsilon$ if necessary, we can assume that its image is completely included in $A$. For every $\lambda \in ] - \epsilon, \epsilon[ \setminus \{0\}$,
there exists thus one and only one $J$-holomorphic curve in a neighborhood of $C_0$, passing through $y_1$, which was the announced result.

Now let $C_0$ be an element of $\mathbb{R}\mathcal{M}_d^0(y)$. This element lifts into two elements $C_1$ and $C_2$ of the moduli space $\mathbb{R}\mathcal{M}_d^0(\overline{y})$, where $\overline{y} = (y_1, \ldots, y_{c_1(X)d-2}, y_{c_1(X)d-2})$, see the beginning of Section 3.2. Denote by $\mathbb{R}\hat{\mathcal{M}}_d^0(y)$ the moduli space of real rational pseudo-holomorphic maps having $c_1(X)d-1$ distinct marked points $z_1, \ldots, z_{c_1(X)d-1}$ at the source, realizing the homology class $d$ and such that $u(z_i) = y_i$ for $1 \leq i \leq c_1(X)d-2$. The map $\hat{\pi}_\mathbb{R} : [u, J, \lambda] \in \mathbb{R}\hat{\mathcal{M}}_d^0(y) \rightarrow (J, u(z_{c_1(X)d-1})) \in \mathbb{R}\mathcal{J}_\omega \times \mathbb{R}X$ is Fredholm of index zero. The value $(J, y_{c_1(X)d-2})$ is regular for this projection. Thus the curves $C_1$ and $C_2$ in $(\hat{\pi}_\mathbb{R})^{-1}(J, y_{c_1(X)d-2})$ extend in a unique way into two families $C_1(\lambda)$ and $C_2(\lambda)$ of $(\hat{\pi}_\mathbb{R})^{-1}(J, \mu(\lambda)) = (\pi_\mathbb{R})^{-1}(J)$. These are the two families we were looking for.

The second relation of Proposition 3.3 can be proved in a similar way. We choose this time a path $\mu : [-, \epsilon) \rightarrow X$ of class $C^2$ such that $\mu(0) = y_{c_1(X)d-2}$, $\mu'(0) \in J(\tau)$, and for every $\lambda \in [-, \epsilon)$, $c_X(\mu(\lambda)) = \mu(-\lambda)$. For every $\lambda \in [-, \epsilon) \setminus \{0\}$, denote by $y_1$ the set $(y_1, \ldots, y_{c_1(X)d-3}, \mu(\lambda), \mu(-\lambda))$, and by $\mathbb{R}\mathcal{M}_d^0(y_1)$ the corresponding moduli space. Now, from Gromov compactness theorem, as soon as $\lambda$ is close enough to zero, the elements of $\mathbb{R}\mathcal{M}_d^0(y_1)$ are close in $\mathcal{M}_d$ to elements of $\mathbb{R}\hat{\mathcal{M}}_d^0(\overline{y})$ having a real isolated node at $y_{c_1(X)d-2}$ or to elements of $\mathbb{R}\hat{\mathcal{M}}_d^0(y)$ having $\tau$ as a tangency at $y_{c_1(X)d-2}$. Now, each curve of the first kind (resp. second kind) is limit of exactly two (resp. one) families $C_{1_1}(\lambda), C_{1_2}(\lambda)$ of elements of $\mathbb{R}\mathcal{M}_d^0(y_1)$. Since the masses of these curves are unchanged while passing to the limit $\lambda \rightarrow 0$, the second relation follows from the fact that $\chi^d = \chi^d(y_1, J)$. □

### 3.4 Non-triviality of $\chi^4(T)$ and $\chi^5(T)$ for the complex projective plane

#### 3.4.1 Generalization of the invariant $\theta$

The invariant $\theta$ has been defined fixing the position of one of the double points of the pseudo-holomorphic curves in the homology class $d$. More generally, one can define such an invariant fixing the position of $\sigma$ double points of these curves, where $0 \leq \sigma \leq \frac{1}{2}c_1(X)d - 1$. More precisely, let $y = (y_1, \ldots, y_{c_1(X)d-1-\sigma})$ be a real configuration of $c_1(X)d-1-\sigma$ distinct points of $X$, and $s$ be the number of those which are real. We assume that $y_{c_1(X)d-2\sigma}, y_{c_1(X)d-2\sigma+1}, \ldots, y_{c_1(X)d-1-\sigma}$ are real, so that $s \geq \sigma$. Let $J \in \mathbb{R}\mathcal{J}_\omega$ be generic enough. Then there are only finitely many $J$-holomorphic rational curves in $X$ in the homology class $d$ passing through $y$ and having a node at each of the points $y_{c_1(X)d-2\sigma}, y_{c_1(X)d-2\sigma+1}, \ldots, y_{c_1(X)d-1-\sigma}$. These curves are all nodal and irreducible. For every integer $m$ ranging from 0 to $\delta$, denote by $\hat{n}^+\sigma(m)$ (resp. $\hat{n}^-\sigma(m)$) the total number of these curves which are real, of mass $m$ and with an even (resp. odd) number of real isolated double points at $y_{c_1(X)d-2\sigma}, y_{c_1(X)d-2\sigma+1}, \ldots, y_{c_1(X)d-1-\sigma}$. Define then:

$$\theta^\sigma_s(y, J) = \sum_{m=0}^{\delta} (-1)^m (\hat{n}^+\sigma(m) - \hat{n}^-\sigma(m)).$$

These definitions extend the ones given in paragraph 3.1. In particular, $\theta^0_s(y, J) = \chi^d_s(y, J)$ and $\theta^{d-1}_s(y, J) = \theta^d_s(y, J)$.

**Theorem 3.4** Let $(X, \omega, c_X)$ be a real rational symplectic 4-manifold, $d \in H_2(X; \mathbb{Z})$ and $0 \leq \sigma \leq \frac{1}{2}c_1(X)d - 1$. Let $y \subset X$ be a real configuration of $c_1(X)d-1-\sigma$ distinct points and $s \geq \sigma$ be the cardinality of $y \cap \mathbb{R}X$. Finally, let $J \in \mathbb{R}\mathcal{J}_\omega$ be an almost complex structure
generic enough, so that the integer $\theta_s^{d,\sigma}(y,J)$ is well defined. Then, this integer $\theta_s^{d,\sigma}(y,J)$ neither depends on the choice of $J$ nor on the choice of $y$ (provided the cardinality of $y \cap \mathbb{R}X$ is $s$). □

The proof of this theorem is the same as the one of Theorem 3.2. As usual, this integer $\theta_s^{d,\sigma}(y,J)$ will be denoted by $\theta_s^{d,\sigma}$, and we put $\theta_s^{d,\sigma} = 0$ when $s$ does not have the suitable parity.

**Theorem 3.5** Let $(X,\omega,c_X)$ be a real rational symplectic 4-manifold and $d \in H_2(X;\mathbb{Z})$. Let $\sigma$ be an integer such that $0 \leq 2\sigma \leq c_1(X)d - 3$, and $s$ be an integer between $\sigma$ and $c_1(X)d - 3 - \sigma$. Then $\theta_s^{d,\sigma} = \theta_{s+2}^{d,\sigma} + 2\theta_s^{d,\sigma+1}$. □

The proof of this theorem is the same as the one of Theorem 3.2.

### 3.4.2 Non-triviality of $\chi^4(T)$ and $\chi^5(T)$

In this subparagraph, the real symplectic 4-manifold $(X,\omega,c_X)$ is the complex projective plane equipped with its standard symplectic form $\omega_{st}$ and the complex conjugation $\text{conj}$. We use the canonical identification of $H_2(\mathbb{CP}^2;\mathbb{Z})$ with $\mathbb{Z}$. We defined in Example 2.1 an invariant $\chi : d \in \mathbb{Z} \mapsto \chi^d(T) \in \mathbb{Z}[T]$ and have computed it for $d \leq 3$ in Example 1 of this paragraph.

**Proposition 3.6** Let $(X,\omega,c_X)$ be $(\mathbb{CP}^2,\omega_{st},\text{conj})$. Then $\chi^4(T)$ and $\chi^5(T)$ are non-zero polynomials of $\mathbb{Z}[T]$.

**Lemma 3.7** Let $(X,\omega,c_X)$ be $(\mathbb{CP}^2,\omega_{st},\text{conj})$. Then $\theta^4_q = 1$ for every odd $1 \leq q \leq 7$ $\theta^4_r = 1$ for every even $4 \leq r \leq 8$ and $\theta^5_s = 1$ for every even $6 \leq s \leq 8$.

**Proof:**

The proofs are the same in all the cases, so we will prove only the degree 4 case. Let $y$ be a real configuration of 8 distinct points in the plane, $r \geq 3$ of which being real. Let $J \in J_\omega$ be generic enough. There exists then only one $J$-holomorphic rational curve of degree 4 in $\mathbb{CP}^2$, passing through $y$, and having its 3 double points at $y_6, y_7, y_8$. Indeed, if there were two of them, they would intersect at each point $y_1, \ldots, y_5$ with multiplicity at least one, and at each point $y_6, y_7, y_8$ with multiplicity at least four. This would give an intersection index greater than 16 which is impossible. This implies that the corresponding Gromov-Witten invariant is one, since it is obviously not zero. Now let $J \in \mathbb{R}J_\omega$ be generic enough, this unique curve is real. Denote by $m$ its mass, we have $\theta^{4,5}_r(x,J) = (-1)^m(-1)^m = 1$. □

**Proof of Proposition 3.6:**

It is a consequence of Theorem 3.5 and Lemma 3.7. □

For instance, the coefficients of the polynomial $\chi^4(T)$ satisfy the relations

\[
\begin{align*}
\chi_3^4 &= \chi_1^4 + 2\theta_1^4, \\
\chi_5^4 &= \chi_1^4 + 4\theta_1^4 + 4\theta_3^{4,2}, \\
\chi_7^4 &= \chi_1^4 + 6\theta_1^4 + 12\theta_3^{4,2} + 8, \\
\chi_9^4 &= \chi_1^4 + 8\theta_1^4 + 24\theta_3^{4,2} + 32, \text{ and} \\
\chi_{11}^4 &= \chi_1^4 + 10\theta_1^4 + 40\theta_3^{4,2} + 80. \\
\text{Hence, all these coefficients cannot vanish simultaneously. Similarly,} \\
\chi_{14}^5 &= \chi_0^5 + 14\theta_0^5 + 84\theta_2^{5,3} + 280\theta_3^{5,3} + 560\theta_4^{5,3} + 672\theta_5^{5,3} + 448.
\end{align*}
\]
Remark 3.8 It has been observed recently by I. Itenberg, V. Kharlamov and E. Shustin that the invariant $\chi_{d-1}^d$ is in fact positive for every $d > 0$, thanks to the research announcement [9] by G. Mikhalkin.

Appendix

A Proof of Theorem 1.11 in higher genus

Let $(X, \omega, c_X)$ be a real symplectic 4-manifold, $g \in \mathbb{N}$, $d \in H_2(X; \mathbb{Z})$ and $x \subset X$ be a real configuration of $c_1(X)d + (3-n)(g-1) \geq 0$ distinct points. Denote by $\mathcal{M}_g^d(x)_{imm}$ the open subset of $\mathcal{M}_g^d(x)$ made of immersed pseudo-holomorphic curves and by $\mathcal{M}_g^d(x)_s$ the subspace of curves having a unique cuspidal point which is real ordinary. The latter is a codimension $n-1$ Banach submanifold, which can be proved along the same lines as Proposition 2.6.

Proposition A.1 Let $(X, \omega, c_X)$ be a real symplectic manifold of dimension $n$, $g \in \mathbb{N}$, $d \in H_2(X; \mathbb{Z})$ and $x \subset X$ be a real configuration of $c_1(X)d + (3-n)(g-1) \geq 0$ distinct points. Then,

1) The space $\{[u, J_S, J] \in \mathcal{M}_g^d(x)_{imm} | \dim H^1_D(S, N_{u,-}) = 1\}$ is a codimension one Banach submanifold of class $C^{-1,\alpha}$ of $\mathcal{M}_g^d(x)$ (might be empty).

2) The complementary in $\mathcal{J}_\omega$ of $\pi_E(\{[u, J_S, J] \in \mathcal{M}_g^d(x)_{imm} | \dim H^1_D(S, N_{u,-}) = 1\})$ is a dense set of the second category of $\mathcal{J}_\omega$.

Proof:

Let us start with the first part of the proposition. From Lemma 3.2.7 of [11], the fibered spaces over $\mathcal{M}_g^d(x)_{imm}$ whose fiber over $[u, J_S, J]$ are the spaces $L^{k,p}(S, N_{u,-})$ and $L^{k-1,p}(S, \Lambda^{0,1} S \otimes N_{u,-})$ respectively have the structure of Banach vector bundles of class $C^{-1,\alpha}$. Moreover, the normal Gromov operator $D^N$ induces a $\mathbb{Z}/2\mathbb{Z}$-equivariant bundle homomorphism $L^{k,p}(S, N_{u,-}) \to L^{k-1,p}(S, \Lambda^{0,1} S \otimes N_{u,-})$. Denote by $D^N_{R, -}$ the associated morphism $L^{k,p}(S, N_{u,-}) \to L^{k-1,p}(S, \Lambda^{0,1} S \otimes N_{u,-})$ over $\mathcal{M}_g^d(x)_{imm}$. Let $[u, J_S, J] \in \mathcal{M}_g^d(x)_{imm}$ be such that $\dim H^1_D(S, N_{u,-}) = 1$. Since $u$ is immersed and $\text{ind}(D^N_{R, -}) = 0$, it implies that $\dim H^0_D(S, N_{u,-}) + 1 = 1$. From the implicit function theorem, to get the first part of the proposition, it suffices to prove that the operator:

$$\nabla D^N_{R, -} : T_{[u, J_S, J]} \mathcal{M}_g^d(x) \to \text{Hom}(H^0_D(S, N_{u,-}) + 1, H^1_D(S, N_{u,-}) + 1)$$

is surjective. For this purpose, let $\psi_-$ be a generator of $H^0_D(S, K_S \otimes N_{u,-}^*)_{-1} \cong H^1_D(S, N_{u,-})$ and $w_+$ be a generator of $H^0_D(S, N_{u,-}) + 1$. We are searching for $(v, J_S, J) \in T_{[u, J_S, J]} \mathcal{M}_g^d(x)$ such that:

$$\text{Re} \int_S \langle \psi_-, \nabla_{(v, J_S, J)} D^N_{R, -}(w_+) \rangle = 0. \quad (7)$$

Let us fix $v = 0$, $J_S = 0$, and search for a section $J \in L^{k,\alpha}(X, \Lambda^{0,1} X \otimes T^* X)_{+1}$ which vanishes along $u(S)$. From formula (4.2.11) of [11], since under these conditions only the term $[7]$ of this formula is non-zero, the relation (7) becomes:

$$\text{Re} \int_S \langle \psi_-, \nabla w_+ J \circ \partial u \circ J_S \rangle = 0.$$

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Let $U$ be an open subset of $S \setminus z$ small enough such that $u$ restricts to an embedding from $U$ to $X$, $u(U) \cap u(S \setminus U) = \emptyset$, $c_X(U) \cap U = \emptyset$ and such that $\psi_-$ and $u_+$ do not vanish on $U$. Let $\alpha$ be a section of the bundle $\Lambda^{0,1} \otimes U_{\alpha-\perp}$ with support on $U$ such that $\Re \int_S \psi_-, \alpha_\not\equiv 0$. By integration on the tangent direction to $u_+$, and thus normal to $u(S)$, we construct a section $J_1$ of $\Lambda^{0,1} X \otimes TX$ with support in a neighborhood of $u(U)$, such that $\nabla u_+ \circ J_u = \alpha$ and $J_1$ vanishes along $u(S)$. The section $J = J_1 + \tau^*_X J_1$ is suitable, which proves the first part of the proposition.

The same proof leads to the fact that the space $\{[u, J_S, J] \in \mathbb{R} \mathcal{M}^d(x)_{\text{imm}} \mid \dim H^1_D(S, N_{u,-z})_{-1} = 1\}$ is a Banach submanifold of class $C^{l-1, \alpha}$ of $\mathbb{R} \mathcal{M}^d(x)$ of codimension one (or is empty), which proves the second part of the proposition in this case. In the general case, let $\{[u, J_S, J] \in \mathbb{R} \mathcal{M}^d(x)_{\text{imm}} \mid \dim H^1_D(S, N_{u,-z})_{-1} = 1 \}$ be such that $\dim H^1_D(S, N_{u,-z})_{-1} = \dim H^1_D(S, N_{u,-z})_{+1} = h \geq 1$. Denote by $D^N_{\pm 1}$ the operator $L^{k,p}(S, N_{u,-z})_{\pm 1} \to L^{k-1,p}(S, \Lambda^{0,1} S \otimes N_{u,-z})_{\pm 1}$. Repeating the same proof as before, we see that the operator

$$\nabla D^N_{\pm 1} : T_{[u, J_S, J]} \mathbb{R} \mathcal{M}^d(x) \to \text{Hom}(H^1_D(S, N_{u,-z})_{+1}, H^1_D(S, N_{u,-z})_{-1})$$

is non-zero. From the implicit function theorem, there exists then, locally, a submanifold $V$ of codimension at least $h^2$ in $\mathbb{R} \mathcal{M}^d(x) \times \mathbb{R}^{h^2-1}$ which maps onto the subspace $\{[u, J_S, J] \in \mathbb{R} \mathcal{M}^d(x)_{\text{imm}} \mid \dim H^1_D(S, N_{u,-z})_{-1} = h\}$ of $\mathbb{R} \mathcal{M}^d(x)$. The projection $V \to J_\omega$ induced then by $\pi_\mathbb{R}$ is Fredholm of index $1$, and maps onto $\pi_\mathbb{R}(\{[u, J_S, J] \in \mathbb{R} \mathcal{M}^d(x)_{\text{imm}} \mid \dim H^1_D(S, N_{u,-z})_{-1} = h\})$, hence the second part of the proposition. 

**Proof of Theorem 1.11:**

It is a consequence of Proposition 1.12 and Proposition A.1. □

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