Anti-\(\mathcal{PT}\)-symmetric harmonic oscillator and its relation to the inverted harmonic oscillator

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Abstract

We treat the quantum dynamics of a harmonic oscillator as well as its inverted counterpart in the Schrödinger picture. Generally in the most papers of the literature, the inverted harmonic oscillator is formally obtained from the harmonic oscillator by the replacement of \(\omega\) to \(i\omega\), this leads to unbounded eigenvectors. This explicitly demonstrates that there are some unclear points involved in redefining the variables in the harmonic oscillator inversion. To remedy this situation, we introduce a scaling operator (Dyson transformation) by connecting the inverted harmonic oscillator to an anti-\(\mathcal{PT}\)-symmetric harmonic oscillator, we obtain the standard quasi-Hermiticity relation which would ensure the time invariance of the eigenfunction’s norm. We give a complete description for the eigenproblem. We show that the wavefunctions for this system are normalized in the sense of the pseudo-scalar product. A Gaussian wave packet of the inverted oscillator is investigated by using the ladder operators method. This wave packet is found to be associated with the generalized coherent state that can be crucially utilized for investigating the mean values of the space and momentum operators. We find that these mean values reproduce the classical motion.

Keywords: Inverted oscillator; harmonic oscillator; anti-\(\mathcal{PT}\)-symmetry; inverted coherent states.

This paper is dedicated to the memory of ours colleagues: Brahim Bouzerafa and Rabah Zegadi who died due to covid 19. And to Ali-Sahraoui Ferhat due to cardiac arrest.

1 Introduction

The inverted harmonic oscillator described by the following operator:

\[
H^i = \frac{1}{2m} p^2 - \frac{1}{2} m\omega^2 x^2, \tag{1}
\]
where the index $r$ specify inverted or repulsive, has attracted great attention over the years in quantum mechanics [1]-[4]. It is of course not positive definite but symmetric on the domain $D \subset$ in the Hilbert space $\mathcal{H}$ endowed with scalar product $\langle \cdot, \cdot \rangle$ and related norm $\| \cdot \|$. It is well known that this system gives rise to a complex eigenvalues and a non physical eigenvectors although $H^r$ is self-adjoint - the physical reason is that its potential is unbounded from below.

The inverted harmonic oscillator, obtained by changing $\omega$ to $\pm i\omega$ in the harmonic oscillator, is often encountered as a phenomenological model in dissipative processes and has been used among others, as a model of instability in quantum mechanics [5]-[9]. In fact, its associated time-dependent solutions of the Schrödinger equation can be mapped to those of the Schrödinger equation for the harmonic potentials [10]-[15]. It yield some physical informations on scales that span classical mechanics to quantum field theory [16]-[18]. Recently in the Ref [19], the inverted harmonic oscillator Hamiltonian is used to understand the quantum mechanics of scattering and time-decay in a diverse set of physical systems.

It have a wide range of application in different branches of physics [20]-[28] such as, the tunneling effects, the mechanism of matter-wave bright solitons, the cosmological model, and the quantum theory of measurements.

The fact is that we are able to ”invert” a standard oscillator by redefining the frequency $\omega$. We will expect both of the normal and the inverted oscillator representations to possess the same observable consequences. However, it is not clear how this result truly comes about. It is important to note that the regular and the inverted harmonic oscillators are genuinely different. As a result, one cannot take for granted the known formulae of the regular oscillator and extrapolate them to the inverted oscillators by simply replacing $\omega$ with $\pm i\omega$. Among the goals of this paper is to define the transformation which connects the harmonic oscillator to the inverted one and describes the physical situation, there by clarifying several questions.

The analytic continuation of the angular velocity $\omega \rightarrow \pm i\omega$ leads to an imaginary energy spectrum which eventually could be obtained as an eigenvalue of the anti-$\mathcal{PT}$-symmetric non-Hermitian Hamiltonian ($\pm iH^{os}$). The question that quickly comes to mind is can we relate $H^r$ and ($\pm iH^{os}$)? or precisely, which method should be used to obtain, from the anti-$\mathcal{PT}$ non-Hermitian oscillator ($\pm iH^{os}$), the inverted oscillator $H^r$ without recourse to the substitution of $\omega$ by $\pm i\omega$? We will try to answer these questions in the next section.

Another important branch in studying quantum systems is the so-called non-Hermitian quantum mechanics. In non-Hermitian quantum mechanics, it was found that the criteria for a quantum Hamiltonian to have a real spectrum is that it possesses an unbroken $\mathcal{PT}$-symmetry ($\mathcal{P}$ is the space-reflection operator or parity operator, and $\mathcal{T}$ is the time-reversal operator) [29, 30]. The concept of $\mathcal{PT}$-symmetry has found applications in several areas of physics [31]-[39]. Due to the fact that the energy spectrum of ($\pm iH^{os}$) being completely imaginary that changes the physical structure of the system. We recall that a $\mathcal{PT}$-symmetric system can be transformed to an anti-$\mathcal{PT}$-symmetric one by the transformation $H^{os} \rightarrow ((\pm iH^{os})$ [40]-[43].

In analogy with the $\mathcal{PT}$-symmetric case, we call the anti-$\mathcal{PT}$-symmetry of Hamiltonian $H$ unbroken if all of the eigenfunctions of $H$ are eigenfunctions of the $\mathcal{PT}$ operator, i.e. when the energy spectrum of $H$ is entirely imaginary [44]. An alternative approach exploring the basic structure responsible of a non-Hermitian Hamiltonian is the notion of the pseudo-Hermiticity introduced in Ref. [45]. Mostafazadeh [46] pointed out that the condition for a Hamiltonian $H$ to be $\mathcal{PT}$-symmetric can be understood more generally as a special case of pseudo-Hermiticity. An operator $H$ is said to be pseudo-Hermitian or quasi-Hermitian if it satisfies the following
relation
\[ H^\dagger = \eta H \eta^{-1}, \tag{2} \]
where the Hermitian operator \( \eta = \rho^+ \rho \) and the Dyson operator \( \rho \) are linear and invertible. The pseudo-Hermiticity links as well the pseudo-Hermitian Hamiltonian \( H \) with an equivalent Hermitian Hamiltonian \( h \)
\[ h = \rho H \rho^{-1}. \tag{3} \]

In what follows, we only consider the anti-\( \mathcal{P} \mathcal{T} \)-symmetric Hamiltonian case \((iH^{os})\). Similarly, the case \((-iH^{os})\) would serve equally well.

In the present paper, we generate from anti-\( \mathcal{P} \mathcal{T} \)-symmetric Hamiltonian \((iH^{os})\) an inverted harmonic oscillator-type Hamiltonian \( H^r \) which is a Hermitian Hamiltonian and thus its solution. The paper is organized as follows. In Section 2, we recall briefly some properties of the standard harmonic and inverted oscillators. We then introduce an appropriate quantum scaling operator \( \eta \) which link the anti-\( \mathcal{P} \mathcal{T} \)-symmetric Hamiltonian oscillator \((iH^{os})\) to the inverted oscillator Hamiltonian \( H^r \). We obtain the set of solutions of the inverted harmonic oscillator and also to define the full orthonormalization relation of the eigenstates of \( H^r \). This procedure allows us to construct the ladder operators in the inverted harmonic oscillator, in Section 3, where we will address the problem of construction of generalized coherent states associated with the inverted oscillator \( H^r \). We obtain the mean values of the space and momentum operators in the generalized coherent states and furthermore we calculate the corresponding Heisenberg uncertainty. An outlook over the main results is given in the conclusion.

2 Generalized eigenfunction of the inverted harmonic oscillator

Let us first recall some properties of the harmonic oscillator which are useful to introduce the inverted oscillator. The Hamiltonian of the harmonic oscillator is:
\[ H^{os} = \frac{1}{2m} p^2 + \frac{1}{2} m \omega^2 x^2, \tag{4} \]
and its the normalized eigenfunctions are given by means of Hermite polynomial
\[ \psi^{os}_n(x) = \frac{1}{\sqrt{2^n n!}} \left( \frac{\omega m}{\pi \hbar} \right)^{\frac{1}{4}} \exp \left[ -\frac{\omega m}{2\hbar} x^2 \right] H_n \left( \frac{\sqrt{\omega m}}{\hbar} x \right), \tag{5} \]
therefore the spectrum is discrete and the eigenvalues of \((4)\) are non-negative real numbers which correspond to the quantized energy levels
\[ E_n^r = \hbar \omega (n + \frac{1}{2}), \quad \omega > 0, \tag{6} \]
and the orthonormal condition is well preserved
\[ \langle \psi^{os}_m(x) | \psi^{os}_n(x) \rangle = \int \psi^{*os}_m(x) \psi^{os}_n(x) dx = \delta_{mn}, \tag{7} \]
naturally, the linear combination $\sum c_n \psi_n^{os}(x)$ is a square integrable function for $\sum |c_n|^2 < \infty$.

The Hamiltonian (1) is formally obtainable from the harmonic oscillator by the following change $\omega \rightarrow \pm i\omega$ and it corresponds to the Hamiltonian of the harmonic oscillator with purely imaginary frequency. Therefore, this replacement transforms the eigenfunctions of the harmonic oscillator (5) into generalized eigenvectors of the inverted harmonic oscillator (1)

$$\psi_n^r(x) = \frac{1}{\sqrt{2^n n!}} \left( \frac{i\omega m}{\pi\hbar} \right)^{\frac{1}{2}} \exp \left[ -\frac{i\omega m}{2\hbar} x^2 \right] H_n \left( \sqrt{\frac{i\omega m}{\hbar}} x \right), \quad (8)$$

$$\tilde{\psi}_n^r(x) = \frac{1}{\sqrt{2^n n!}} \left( -\frac{i\omega m}{\pi\hbar} \right)^{\frac{1}{2}} \exp \left[ \frac{i\omega m}{2\hbar} x^2 \right] H_n \left( \sqrt{-\frac{i\omega m}{\hbar}} x \right), \quad (9)$$

which should be proved as eigenvectors of the inverted harmonic oscillator (1) with the discrete purely imaginary spectrum

$$E_n = \pm iE_n^{os} = \pm i\hbar \omega \left( n + \frac{1}{2} \right). \quad (10)$$

From the mathematical point of view, the solution of the inverted Hamiltonian in (1) is similar to that of the harmonic oscillator. But these transformations, applied to the potential of the harmonic oscillator, turn it into the potential $-\frac{1}{2}m\omega^2x^2$ and the same happens with the eigenvalues: they are transformed from discrete real eigenvalues to discrete imaginary ones. Once again we see that these generalized eigenfunctions cannot represent physical states. When calculating the squared norm of these functions

$$\langle \psi_n^r | \psi_n^r \rangle = \frac{1}{2^n n!} \left( \frac{\omega m}{\pi\hbar} \right)^{\frac{1}{2}} \int H_n \left( \sqrt{-\frac{i\omega m}{\hbar}} x \right) H_n \left( \sqrt{\frac{i\omega m}{\hbar}} x \right) dx \neq 1, \quad (11)$$

$$\langle \tilde{\psi}_n^r | \tilde{\psi}_n^r \rangle = \frac{1}{2^n n!} \left( \frac{\omega m}{\pi\hbar} \right)^{\frac{1}{2}} \int H_n \left( \sqrt{\frac{i\omega m}{\hbar}} x \right) H_n \left( \sqrt{-\frac{i\omega m}{\hbar}} x \right) dx \neq 1, \quad (12)$$

we see that they diverge in the limit $|x| \rightarrow \infty$ as $x^{2n}$. Therefore any non-trivial linear combination does not yield a square integrable function. Clearly, $\psi_n^r (\tilde{\psi}_n^r)$ are not elements of $L^2(R)$.

Finally, we are left with the following question: what is the relation between the eigenfunctions $\psi_n^{os}(x)$ (5) and those of the inverted oscillator $\psi_n^r(x)$ (8) (or (11))?. In order to establish a connection between them, we write $\psi_n^r(x)$ (8) as

$$\psi_n^r(x) = \exp \left[ -\frac{n}{8} (xp + px) \right] \frac{1}{\sqrt{2^n n!}} \left( \frac{\omega m}{\pi\hbar} \right)^{\frac{1}{2}} \exp \left[ -\frac{\omega m}{2\hbar} x^2 \right] H_n \left( \sqrt{\frac{\omega m}{\hbar}} x \right)$$

$$= \exp \left[ -\frac{n}{8} (xp + px) \right] \psi_n^{os}(x e^{i\pi}), \quad (13)$$

and $\tilde{\psi}_n^r(x)$ (11) in the form

$$\tilde{\psi}_n^r(x) = \exp \left[ \frac{n}{8} (xp + px) \right] \frac{1}{\sqrt{2^n n!}} \left( \frac{\omega m}{\pi\hbar} \right)^{\frac{1}{2}} \exp \left[ \frac{\omega m}{2\hbar} x^2 \right] H_n \left( \sqrt{\frac{\omega m}{\hbar}} x \right)$$

$$= \exp \left[ \frac{n}{8} (xp + px) \right] \psi_n^{os}(x e^{-i\pi}), \quad (14)$$
the operator \( \exp \left[ \frac{\pi}{8}(xp + px) \right] \) defines a complex squeezed operator, because if \(|\psi\rangle\) is the state vector of a system then \( \exp \left[ \frac{\pi}{8}(xp + px) \right] |\psi\rangle \) represents the same system compressed in position space by the factor \( e^{-i\frac{\pi}{8}} \) and expanded in momentum space by the factor \( e^{+i\frac{\pi}{8}} \).

It can be easily shown that under this transformation the coordinate and momentum operators change according to

\[
\exp \left[ \frac{\pi}{8}(xp + px) \right] x \exp \left[ -\frac{\pi}{8}(xp + px) \right] = xe^{-i\frac{\pi}{8}},
\]

\[
\exp \left[ \frac{\pi}{8}(xp + px) \right] p \exp \left[ -\frac{\pi}{8}(xp + px) \right] = pe^{i\frac{\pi}{8}},
\]

thus

\[
\exp \left[ \frac{\pi}{8}(xp + px) \right] H^* \exp \left[ -\frac{\pi}{8}(xp + px) \right] = (iH^\text{os}).
\]

We now return to Eqs. (9), (13), (14) and notice that

\[
\tilde{\psi}_n^r(x) = \exp \left[ \frac{\pi}{8}(xp + px) \right] \psi_n^\text{os}(x) = \exp \left[ \frac{\pi}{4}(xp + px) \right] \psi_n^r(x).
\]

We observe that these two families of generalized eigenvectors \( \psi_n^r(x) \) and \( \tilde{\psi}_n^r(x) \) have the remarkable properties:

i) Formulas (13) and (14) imply that they are conjugated to each other:

\[
\tilde{\psi}_n^r(x) = \psi_n^r(x),
\]

which implies that \( \psi_n^r(x) \) and \( \tilde{\psi}_n^r(x) \) are related by the time-reversal operator \( T\psi_n^r(x) = \tilde{\psi}_n^r(x) \).

ii) They are bi-orthonormal

\[
\langle \tilde{\psi}_m^r | \psi_n^r \rangle = \langle \psi_m^\text{os} | \tilde{\psi}_n^r \rangle = \delta_{mn},
\]

the generalized orthonormal relations (19) involve the eigenvectors \( |\psi_n^r\rangle \) as well as \( |\tilde{\psi}_n^r\rangle \).

iii) they are bi-complete

\[
\sum_{0}^{\infty} |\psi_n^r(x)\rangle \langle \tilde{\psi}_n^r(x')| = \sum_{0}^{\infty} |\tilde{\psi}_n^r(x)\rangle \langle \psi_n^r(x')| = e^{\frac{1}{8}(xp + px)} \sum |\psi_n^\text{os}\rangle \langle \psi_n^\text{os}| e^{-\frac{1}{8}(xp + px)} = \delta(x - x').
\]

The proof follows immediately from orthonormality and completeness of the oscillator’s eigenfunctions \( \psi_n^\text{os} \). Indeed, for the harmonic oscillator, the completeness relation is \( \sum_n |\psi_n^\text{os}(x)\rangle \langle \psi_n^\text{os}(x')| = \delta(x - x'), \) where the kets \( |\psi_n^\text{os}\rangle = \exp \left[ \frac{\pi}{8}(xp + px) \right] |\psi_n\rangle \) form a complete set of bases, consequently the bi-completeness relation (20) is verified. Notice that, the generalized bi-orthonormal relation (19) can be rewritten as:

\[
\langle \tilde{\psi}_m^r | \psi_n^r \rangle = \langle \psi_n^r | \eta | \psi_n^r \rangle = \delta_{mn},
\]

(21)
which introduces the notion of the $\eta$-pseudo-scalar product where the operator $\eta$ is defined as

$$\eta = \exp\left[\frac{\pi}{4}(xp + px)\right] = \rho^+ \rho,$$

(22)

and

$$\rho = \exp\left[\frac{\pi}{8}(xp + px)\right].$$

(23)

Now, the condition (11) yields

$$\langle \psi^r_m | \eta | \psi^r_n \rangle = \int \psi^{os}_n(x) \exp\left[-\frac{\pi}{8}(xp + px)\right] [\eta] \exp\left[-\frac{\pi}{8}(xp + px)\right] \psi^{os}(x) dx = \delta_{mn},$$

(24)

under this observation, we deduce that to insure the normalization condition concerning the inverted eigenfunctions, one must clearly apply the $\eta$ pseudo-inner product $\langle \cdot, \cdot \rangle_\eta$. This proves that one of the fundamental basis in the study of the inverted oscillator is the pseudo-Hermicity concept.

Therefore, the operator $\eta$ links $(iH^{os})$ to its hermitian conjugate

$$(-iH^{os}) = \eta(iH^{os})\eta^{-1},$$

(25)

which is nothing other than the quasi-Hermiticity relation. It then follows immediately, that the two Hamiltonians $H^r$ and $(iH^{os})$ are related to each other as

$$H^r = \rho^{-1}(iH^{os})\rho.$$ 

(26)

Therefore, Eq. (24) stands for the $\eta$ inner product $\langle \cdot, \cdot \rangle_\eta$ in the pseudo-Hermitian case. The eigenfunctions of the inverted harmonic oscillator $\psi^r_n(x)$ are related to the eigenfunctions of the harmonic oscillator $\psi^{os}_n(x)$ via the transformation operator $\rho$ (23) as

$$\psi^r_n(x) = \rho^{-1}\psi^{os}_n(x),$$

(27)

thus any non-trivial linear combination $\sum_n c_n \psi^r_n(x)$ yields a square integrable function for $\sum_n |c_n|^2 < \infty$.

As advertised before, let us back to the properties (19) and (20) of the generalized eigenvectors $\tilde{\psi}^r_n(x)$ and $\tilde{\psi}_n^r(x)$. The unbounded operator $\eta = \exp\left[\frac{\pi}{4}(xp + px)\right]$, that cannot be defined on all of Hilbert space $\mathcal{H}$, act on $\mathcal{H}$ with domain $\mathcal{D}(\eta)$ and let $\mathcal{D}$ be a dense subspace of $\mathcal{H}$ such that $\eta\mathcal{D} \subseteq \mathcal{D}$, where $\mathcal{D} \subseteq \mathcal{D}(\eta)$. We can define in $\mathcal{D}$ the vectors $|\psi_n\rangle$ and $|\phi_n\rangle = \eta |\psi_n\rangle$ and the related sets $F_{\psi_n} = \{ |\tilde{\psi}_n^r\rangle \}$, $n \geq 0$, $F_{\psi_n} = \{ |\tilde{\psi}_n^r\rangle \}$, $n \geq 0$. In particular, this lead to Eq. (21) so that $F_{\psi_n}$ and $F_{\tilde{\psi}_n}$ are biorthogonal sets and consequently $F_{\psi_n} = \{ |\psi_n^r\rangle \}$ and $F_{\tilde{\psi}_n} = \{ |\tilde{\psi}_n\rangle \}$ are bases for $\mathcal{H}$.

We can assert that the generalized orthonormal condition (19) or rather (21) can be interpreted as a biorthonormalization condition. Thus, from the Eqs. (13) and (17), the completeness relation (20) can be immediately deduced from that of the oscillator eigenfunction $\psi^{os}_n \langle \psi^{os}_n | = \mathbf{I}$. 

6
Before abording, in the next section the coherent states of the inverted oscillator $H^r$, let us introduce the dimensionless annihilation and creation operators of the quantum harmonic oscillator $H^{os}$ as

$$a = \sqrt{\frac{m\omega}{2\hbar}} x + i \frac{p}{\sqrt{2m\hbar\omega}}, \quad a^+ = \sqrt{\frac{m\omega}{2\hbar}} x - i \frac{p}{\sqrt{2m\hbar\omega}}$$

(28)

these ladder operators (28) can be represented in terms of position $x$ and momentum $p$ operators as

$$x = \sqrt{\frac{\hbar}{2m\omega}} (a^+ + a), \quad p = i \sqrt{\frac{\hbar m\omega}{2}} (a^+ - a)$$

(29)

while their commutation relation is

$$[a, a^+] = 1$$

(30)

The best way to present the inverted coherent states is to translate their definitions into the language of the coherent states of the harmonic oscillator. Coherent states, or semi-classic states, are remarkable quantum states that were originally introduced in 1926 by Schrödinger for the Harmonic oscillator [47] where the mean values of the position and momentum operators in these states have properties close to the classical values of the position $x_c(t)$ and the momentum $p_c(t)$. In particular, we can construct coherent states of the harmonic oscillator $|\alpha\rangle$ [48]-[50] as eigenstates of the annihilation operator $a$

$$a |\alpha\rangle = \alpha |\alpha\rangle, \quad \alpha \in \mathbb{C}.$$  

(31)

They can be also obtained from the vacuum state $|0\rangle$ by the action of the unitary displacement operator $D(\alpha)$

$$D(\alpha) |0\rangle = \exp(\alpha a^* - \alpha^* a) |0\rangle.$$  

(32)

3 Generalized coherent states for the inverted harmonic oscillator

One can verify that in the case of the inverted oscillator, the form of Hamiltonian (11) reads

$$H^r = \frac{i\hbar}{2} (\mathcal{A} \mathcal{A} + \mathcal{A} \mathcal{A})$$

(33)

where the ladder operators $(\mathcal{A}, \mathcal{A})$ are linked to those in Eq.(28) through the transformation

$$\mathcal{A} = \rho^{-1} a \rho = \exp\left[i \frac{\pi}{4} \left(\frac{\sqrt{m\omega}}{2\hbar} x + \frac{p}{\sqrt{2m\hbar\omega}}\right)\right] = \left(\frac{i\sqrt{m\omega}}{2\hbar} x + \frac{ip}{\sqrt{2m\hbar\omega}}\right),$$

(34)

$$\mathcal{A}^+ = \rho^{-1} a^+ \rho = \exp\left[i \frac{\pi}{4} \left(\frac{\sqrt{m\omega}}{2\hbar} x - \frac{p}{\sqrt{2m\hbar\omega}}\right)\right] = \left(\frac{i\sqrt{m\omega}}{2\hbar} x - \frac{ip}{\sqrt{2m\hbar\omega}}\right),$$

(35)

and satisfy the following commutation relation $[\mathcal{A}, \mathcal{A}^+] = 1$. As the case of the harmonic oscillator, the coherent states for the inverted oscillator $|\varphi^r_\alpha\rangle^\mathcal{A}$ can be defined as eigenstates of the annihilation operator $\mathcal{A}$

$$\mathcal{A} |\varphi^r_\alpha\rangle^\mathcal{A} = \alpha |\varphi^r_\alpha\rangle^\mathcal{A}, \quad \alpha \in \mathbb{C}.$$  

(36)
It should be noted that Eqs. (34) and (35) indicate that the eigenvalue $\alpha$ can be considered as being a real eigenvalue of the Hermitian operator $\left(\sqrt{\frac{m \omega}{2\hbar}} x + \frac{p}{\sqrt{2m\omega\hbar}} \right)$ multiplied by the phase factor $e^{+i\frac{\pi}{4}}$:

$$\alpha = |\alpha| e^{+i\frac{\pi}{4}}. \quad (37)$$

In the $x$ representation, Eq. (36) can be written as

$$\left(\sqrt{\frac{mi\omega}{2\hbar}} x + \frac{\hbar}{\sqrt{2mi\omega\hbar}} \frac{\partial}{\partial x} \right) \varphi_\alpha^r(x) = \alpha \varphi_\alpha^r(x), \quad (38)$$

and can be explicitly solved; we obtain

$$\varphi_\alpha^r(x) = \left(\frac{i m \omega}{2 \hbar \pi^2} \right)^{\frac{1}{4}} e^{-\frac{i}{2} |\alpha|^2} \exp \left[ \left(\sqrt{\frac{2im \omega}{\hbar}} \alpha x - \frac{m \omega}{2\hbar} x^2 \right) \right]$$

$$= \left(\frac{i m \omega}{2 \hbar \pi^2} \right)^{\frac{1}{4}} e^{-\frac{i}{2} |\alpha|^2} \exp \left[ \alpha (A + \bar{A}) \right] \exp \left[ -\frac{i m \omega}{2\hbar} x^2 \right]$$

$$= \left(\frac{i m \omega}{2 \hbar \pi^2} \right)^{\frac{1}{4}} \exp \left[ \alpha \bar{A} \right] \exp \left[ \alpha A \right] \exp \left[ -\frac{i m \omega}{2\hbar} x^2 \right]$$

$$= \exp \left[ \alpha \bar{A} \right] \varphi_0^r(x), \quad (39)$$

where the vacuum state of the inverted oscillator $\varphi_0^r(x) = \left(\frac{i m \omega}{2 \hbar \pi^2} \right)^{\frac{1}{4}} \exp \left[ -\frac{i m \omega}{2\hbar} x^2 \right]$ is not square integrable. The vacuum states $\{|\varphi_0^r\rangle, |\varphi_0^\bar{r}\rangle\} \in D$ are related to each other as $|\varphi_0^\bar{r}\rangle = \eta |\varphi_0^r\rangle$. Thus, the biorthonormalization condition $\langle \varphi_0^\bar{r}(x) | \varphi_0^r(x) \rangle = \langle \varphi_0^r(x) | \varphi_0^\bar{r}(x) \rangle = I$ is verified.

When the inverted oscillator is in a particular state $\varphi_\alpha^r(x)$ (39) at the instant $t = 0$, how do its physical properties evolve over time?

Suppose that a system is in the state $\varphi_\alpha^r(x)$ at $t = 0$, and when the Hamiltonian is not time-dependent then its state at all $t$ will be given by:

$$\psi_\alpha^r(x,t) = \exp \left[ -\frac{i}{\hbar} H^r t \right] \varphi_\alpha^r(x)$$

$$= \exp \left[ -\frac{i}{\hbar} H^r t \right] \exp \left[ \alpha \bar{A} \right] \varphi_0^r(x),$$

therefore we write

$$\exp \left[ -\frac{i}{\hbar} H^r t \right] \varphi_\alpha^r(x) = \exp \left[ -\frac{i}{\hbar} H^r t \right] \exp \left[ \alpha \bar{A} \right] \exp \left[ \frac{i}{\hbar} H^r t + \frac{\omega}{2} t \right] \varphi_0^r(x), \quad (40)$$

where we used $(H^r - \frac{i m \omega}{2 \hbar}) \varphi_0^r(x) = 0$.

Knowing that

$$\left[ -\frac{i}{\hbar} H^r t, \alpha \bar{A} \right] = \omega t \alpha \bar{A},$$

we can thus use a special case of the Baker-Campbell-Hausdorff formula which states that if $[a,B] = cB$ with $c \in C$, then

$$\exp[a] \exp[B] \exp[-a] = \exp[\exp(c)B]. \quad (41)$$
We apply eq. (41) with \( a = -\frac{i}{\hbar} H' t \), \( B = \alpha (\bar{A}) \) and \( c = \omega t \) to obtain

\[
\psi_{\alpha(t)}^r (x, t) = \exp \left[ \frac{\omega}{2} t \right] \exp \left[ e^{\omega t} \alpha \bar{A} \right] \varphi_{\alpha}^r (x) = \exp \left[ \frac{\omega}{2} t \right] \varphi_{\alpha}^r (x),
\] (42)

if we compare this result with (39), we see that, to go from \( \psi_{\alpha}^r (x, 0) = \varphi_{\alpha}^r (x) \) to \( \psi_{\alpha(t)}^r (x, t) \), all we must do is to change \( \alpha \) to \( \alpha(t) = \alpha e^{\omega t} \) and multiply the obtained state by \( \exp \left[ \frac{\omega}{2} t \right] \) (which is a global amplitude factor). We already know, for the harmonic oscillator, that the mean values \( \langle x \rangle^{os} (t) = x^{os} c(t) \) and \( \langle p \rangle^{os} (t) = -m \omega x^{os} c(t) \) always remain equal to the corresponding classical values. What about the mean values of physical quantities of the inverted oscillator?

The mean values \( \langle x \rangle_{\eta}^r \) and \( \langle p \rangle_{\eta}^r \) can be obtained by expressing \( x \) and \( p \) in terms of \( \mathcal{A} \) and \( \bar{A} \) [Eqs. (34) and (35)]

\[
x = \sqrt{\frac{\hbar}{2m\omega}} (\mathcal{A} + \bar{A}), \quad p = i \sqrt{\frac{\hbar m \omega}{2} (\bar{A} - \mathcal{A})},
\] (43)

we see that

\[
\langle x \rangle_{\eta}^r = \sqrt{\frac{\hbar}{2m\omega}} \langle \psi_{\alpha}^r (x, t) | \eta (\mathcal{A} + \bar{A}) | \psi_{\alpha}^r (x, t) \rangle = \sqrt{\frac{2\hbar}{m\omega} |\alpha|} e^{\omega t} = x_{\eta}^r (t),
\] (44)

\[
\langle p \rangle_{\eta}^r = i \sqrt{\frac{\hbar m \omega}{2}} \langle \psi_{\alpha}^r (x, t) | \eta (\bar{A} - \mathcal{A}) | \psi_{\alpha}^r (x, t) \rangle = \sqrt{2\hbar m \omega} |\alpha| e^{\omega t} = p_{\eta}^r (t),
\] (45)

the time dependence of the expectation value matches the classical one of the inverted oscillator.

An analogous calculation yields:

\[
\langle x^2 \rangle_{\eta}^r = \frac{\hbar}{2m\omega} \left[ (\alpha(t) + \alpha^*(t))^2 + 1 \right], \quad \langle p^2 \rangle_{\eta}^r = \frac{\hbar m \omega}{2} \left[ 1 - (\alpha(t) - \alpha^*(t))^2 \right],
\] (46)

and therefore:

\[
(\Delta x)_{\eta}^r = \sqrt{\frac{\hbar}{2m\omega}}, \quad (\Delta p)_{\eta}^r = \sqrt{\frac{\hbar m \omega}{2}}.
\] (47)

Neither \( (\Delta x)_{\eta}^r \) nor \( (\Delta p)_{\eta}^r \) depends on \( \alpha \). Note also that \( (\Delta x)_{\eta}^r \) \( (\Delta p)_{\eta}^r \) takes on its minimum value:

\[
(\Delta x)_{\eta}^r \cdot (\Delta p)_{\eta}^r = \frac{\hbar}{2}
\]

4 Conclusion

It is well known that the system described by the inverted oscillator gives rise to the generalized complex eigenvalues. The physical reason for that is the potential being unbounded from below. The corresponding energy eigenstates for (1) have been found in terms of the parabolic cylinder functions \( D_\nu(x) \) [2].

Using a nonconventional technique, i.e. a quasi-Hermiticity in quantum mechanics, we have solved the problem relative to the inverted oscillator. In fact, we find that the system can be entirely expressed in terms of idealized states that are related to those of the harmonic oscillator. The connection with a harmonic oscillator may be established by the scaling operator \( \rho \) [23].
Ideal physical systems are conceptually Hermitian, but realistic systems are sometimes non-Hermitian because of their interactions with their environments. We have given the principal properties of the harmonic oscillator $H^\text{os}$ formulation in quantum mechanics and we introduced the inverted oscillator $H^r$. Using the notion of anti-$\mathcal{PT}$-symmetric non-Hermitian Hamiltonian, this led us to show that $H^r$, $(iH^\text{os})$ and their two sets of eigenfunctions are connected in a simple manner. Therefore, the eigenfunctions of the inverted oscillator are pseudo-orthonormal.

The coherent states of the ordinary oscillator are special wave groups giving probability distributions whose shapes never change, and follow classical trajectories. They are obtained as eigenstates of the annihilation operator, or equivalently by acting on vacuum eigenstate with the unitary displacement operator $D(\alpha)$ introduced in section 2. One might think that the coherent states for the inverted oscillator seem less important, largely because, unlike groups of waves, they are not fully integrable which is not at all appropriate.

So, to show their importance, we addressed the problem of construction of ladder operators for $H^r$ and their associated integrable coherent states. To do this, we took as reference state the vacuum state of the inverted oscillator and, as in the case of the harmonic oscillator, the generalized coherent states for inverted oscillator are defined as eigenstates of the annihilation operator $\hat{A} = \rho^{-1} a \rho$ where $a$ is the annihilation operator associated to the harmonic oscillator. These generalized coherent states can be also obtained by the action on the vacuum state of the inverted oscillator of a non unitary displacement operator.

Then, we showed that the mean values of the position and momentum operators in these coherent states have properties close to the classical values of the position $x^r_\text{c}(t)$ and the momentum $p^r_\text{c}(t)$. The corresponding Heisenberg uncertainty relation is minimum.

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