The critical groups for $K_m \lor P_n$ and $P_m \lor P_n$ 

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Abstract

Let $G_1 \lor G_2$ denote the graph obtained from $G_1 + G_2$ by adding new edges from each vertex of $G_1$ to every vertex of $G_2$. In this paper, the critical groups of the graphs $K_m \lor P_n (n \geq 4)$ and $P_m \lor P_n (m \geq 4, n \geq 5)$ are determined.

Keywords Graph; Laplacian matrix; Critical group; Invariant factor; Smith normal form; Spanning tree number.

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1. Introduction

Let $G = (V, E)$ be a finite connected graph without self-loops, but with multiple edges permitted. Then the Laplacian matrix of $G$ is the $|V| \times |V|$ matrix defined by

$$L(G)_{uv} = \begin{cases} d(u), & \text{if } u = v, \\ a_{uv}, & \text{if } u \neq v, \end{cases}$$

where $a_{uv}$ is the number of the edges joining $u$ and $v$, and $d(u)$ is the degree of $u$.

Thinking of $L(G)$ as representing an abelian group homomorphism: $Z^{|V|} \to Z^{|V|}$, its cokernel has the form

$$Z^{|V|}/\text{im}(L(G)) \cong Z \oplus Z^{|V|-1}/\text{im}\left(\overline{L(G)_{uv}}\right),$$

(1.1)

where $\overline{L(G)_{uv}}$ is the matrix obtained from $L(G)$ by striking out row $u$ and column $v$, and $\text{im}(\cdot)$ refers to the integer span of the columns of the argument. The critical group $K(G)$ is defined to be $Z^{|V|-1}/\text{im}\left(\overline{L(G)_{uv}}\right)$. It is not hard to see that this definition is independent of the choice of $u$ and $v$. The critical group $K(G)$ is a finite abelian group, whose order is equal to the absolute value of $\det \overline{L(G)_{uv}}$. By the well known Kirchhoff’s Matrix-Tree Theorem [6, Theorem 13.2.1], the order $|K(G)|$ is equal to the spanning tree number of $G$. For the general theory of the critical group, we refer the reader to Biggs [1, 2], and Godsil [6, Chapter 14].

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Recall that an $n \times n$ integral matrix $P$ is unimodular if $\det P = \pm 1$. So, the unimodular matrices are precisely those integral matrices with integral inverses, and of course form a multiplicative group. Two integral matrices $A$ and $B$ of order $n$ are equivalent (written by $A \sim B$) if there are unimodular matrices $P$ and $Q$ such that $B = PAQ$. Equivalently, $B$ is obtainable from $A$ by a sequence of elementary row and column operations: (1) the interchange of two rows or columns, (2) the multiplication of any row or column by $-1$, (3) the addition of any integer times of one row (resp. column) to another row (resp. column). The Smith normal form (Snf) is a diagonal canonical form for our equivalence relation: every $n \times n$ integral matrix $A$ is equivalent to a unique diagonal matrix $\text{diag}(s_1(A), \ldots, s_n(A))$, where $s_i(A)$ divides $s_{i+1}(A)$ for $i = 1, 2, \ldots, n-1$. The $i$–th diagonal entry of the Smith normal form of $A$ is usually called the $i$–th invariant factor of $A$.

It is easy to see that $A \sim B$ implies that $\text{coker}(A) \cong \text{coker}(B)$. Given any $n \times n$ unimodular matrices $P$ and $Q$ and any integral matrix $A$ with $PAQ = \text{diag}(a_1, \ldots, a_n)$, it is easy to see that $\mathbb{Z}^{|V|}/\text{im}(A) \cong (\mathbb{Z}/a_1\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/a_n\mathbb{Z})$. Assume the Snf of $L(G)_{uv}$ is $\text{diag}(t_1, \ldots, t_{|V|-1})$ (In fact, every such submatrix of $L(G)$ shares the same Snf.), and then it induces an isomorphism

$$K(G) \cong (\mathbb{Z}/t_1\mathbb{Z}) \oplus (\mathbb{Z}/t_2\mathbb{Z}) \oplus \cdots \oplus (\mathbb{Z}/t_{|V|-1}\mathbb{Z}). \quad (1.2)$$

The nonnegative integers $t_1, t_2, \ldots, t_{|V|-1}$ are also called the invariant factors of $K(G)$, and they can be computed in the following way: for $1 \leq i < |V|$, $t_i = \Delta_i/\Delta_{i-1}$ where $\Delta_0 = 1$ and $\Delta_i$ is the greatest common divisor of the determinants of the $i \times i$ minors of $L(G)_{uv}$. Since $|K(G)| = \kappa$, the spanning tree number of $G$, it follows that $t_1t_2\cdots t_{|V|-1} = \kappa$. So the invariant factors of $K(G)$ can be used to distinguish pairs of non-isomorphic graphs which have the same $\kappa$, and so there is considerable interest in their properties. If $G$ is a simple connected graph, the invariant factor $t_1$ of $K(G)$ must be equal to 1, however, most of them are not easy to be determined.

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are graphs on disjoint sets of $r$ and $s$ vertices, respectively, their union is the graph $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and their join $G_1 \vee G_2$ is the graph on $n = r + s$ vertices obtained from $G_1 + G_2$ by inserting new edges joining every vertex of $G_1$ to every vertex of $G_2$. If we use $G^c$ denote the complement graph of $G$, then $G_1 \vee G_2 = (G_1^c + G_2^c)^c$.

Compared to the number of the results on the spanning tree number $\kappa$, there are relatively few results describing the critical group structure of $K(G)$ in terms of the structure of $G$. There are also very few interesting infinite family of graphs for which the group structure has been completely determined (see [3, 4, 5, 7, 8, 9, 11, 12] and the references therein). The aim of this paper is to describe the structure of the critical groups of two families of graphs $K_m \vee P_n$ and $P_m \vee P_n$, where $K_m$ is the complete graph with $m$ vertices, $P_n$ is the path with $n$ vertices.

2. The critical group of $K_m \vee P_n$
Lemma 2.1 If the graph $G$ has $n$ vertices, then

$$L(K_m \vee G) \sim ((m + 1)I_n - L(G^c)) \oplus (m + n)I_{m-2} \oplus I_1 \oplus 0_1.$$  \hfill (2.1)

Proof Note that

$$L(K_m \vee G) = \begin{pmatrix} (m + n)I_m - J_m & -J_{m \times n} \\ -J_{n \times m} & mI_n + L(G) \end{pmatrix}.$$ 

Let

$$P_1 = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 & 1 \\ 1 & 0 & 0 & \ldots & 0 & 0 \\ -1 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & 0 & \ldots & 1 & 0 \\ -1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & -1 & -1 & \ldots & -1 & -1 \\ 1 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \ldots & 1 & 0 \\ 1 & 0 & 0 & \ldots & 0 & 1 \end{pmatrix}.$$ 

Then a direct calculation can show that $P_1L(K_m \vee G)Q_1 = I_1 \oplus (m + n)I_{m-2} \oplus ((m + n)I_n - L(G^c)) \oplus 0_1$. Note that both the matrices $A$ and $B$ are unimodular, so this Lemma holds.\flushright{□}

In order to work out the critical group of graph $K_m \vee P_n$ ($n \geq 4$), we only need to work on the Smith normal form of the matrix $(m + n)I_n - L(P_n^c)$.

Lemma 2.2

$$(m + n)I_n - L(P_n^c) \sim I_{n-2} \oplus \begin{pmatrix} m + n & b_n \\ 0 & a_n \end{pmatrix},$$

where

$$\begin{cases} a_n = \frac{1}{\sqrt{m^2 + 4m}} \left( \frac{m + 2 + \sqrt{m^2 + 4m}}{2} \right)^n - \left( \frac{m + 2 - \sqrt{m^2 + 4m}}{2} \right)^n, \\ b_n = e \left( \frac{m + 2 + \sqrt{m^2 + 4m}}{2} \right)^n - f \left( \frac{m + 2 - \sqrt{m^2 + 4m}}{2} \right)^n, \end{cases}$$

and

$$e = \frac{(m^2 - m - m\sqrt{m^2 + 4m} - \sqrt{m^2 + 4m} + 2mn)(m + 4 - \sqrt{m^2 + 4m})}{4m^2(m + 4)},$$

$$f = \frac{(m - m^2 - m\sqrt{m^2 + 4m} - \sqrt{m^2 + 4m} - 2mn)(m + 4 + \sqrt{m^2 + 4m})}{4m^2(m + 4)}.$$
Proof Note that

\[(m + n)I_n - L(P_n^c) = \begin{pmatrix}
  (m + 2) & 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\
  0 & m + 3 & 0 & 1 & 1 & \cdots & 1 & 1 \\
  1 & 0 & m + 3 & 0 & 1 & \cdots & 1 & 1 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  1 & 1 & \cdots & 1 & 0 & m + 3 & 0 & 1 \\
  1 & 1 & \cdots & 1 & 1 & 0 & m + 3 & 0 \\
  1 & 1 & \cdots & 1 & 1 & 1 & 0 & m + 2
\end{pmatrix}.
\]

Let

\[P_2 = \begin{pmatrix}
  1 & 0 & 0 & \cdots & 0 \\
  -1 & 1 & 0 & \cdots & 0 \\
  0 & -1 & 1 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & \cdots & -1 & 1
\end{pmatrix},
\]

\[Q_2 = \begin{pmatrix}
  1 & 0 & \cdots & 0 \\
  1 & 1 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots \\
  1 & 1 & \cdots & 1
\end{pmatrix},
\]

and

\[A_2 = P_2((m + n)I_n - L(P_n^c))Q_2.\]

Then a direct calculation can show

\[A_2 = \begin{pmatrix}
  m + n & n - 2 & n - 2 & n - 3 & n - 4 & \cdots & 1 \\
  0 & m + 2 & -1 & 0 & 0 & \cdots & 0 \\
  0 & -1 & m + 2 & -1 & 0 & \cdots & 0 \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  0 & \cdots & 0 & -1 & m + 2 & -1 & 0 \\
  0 & \cdots & \cdots & 0 & -1 & m + 2 & -1 \\
  0 & \cdots & \cdots & \cdots & 0 & -1 & m + 2
\end{pmatrix}.
\]

For \(i = 0, \cdots, n - 3\), let

\[M_{i+1} = \begin{pmatrix}
  I_i & 0_{i \times 1} \\
  0_{1 \times i} & 1 & m + 2 & -1 & 0_{1 \times (n-i-3)}
\end{pmatrix}.
\]

and

\[M_{n-1} = \begin{pmatrix}
  I_{n-2} & 0_{(n-2) \times 1} \\
  0_{1 \times (n-2)} & 1 & m + 2 \\
  0_{1 \times (n-1)} & 0_{(n-2) \times 1}
\end{pmatrix}.
\]

Let \(M = M_1 \cdots M_{n-1}\), then

\[A_2 M = \begin{pmatrix}
  m + n & b_2 & b_3 & b_4 & \cdots & b_{n-1} & b_n \\
  0 & a_2 & a_3 & a_4 & \cdots & a_{n-1} & a_n \\
  0_{(n-2) \times 1} & -I_{n-2} & 0_{(n-2) \times 1}
\end{pmatrix},
\]

where \(0_{i \times j}\) is an \(i \times j\) zero matrix, and the numbers \(a_l, b_l\) satisfy the following recurrence relations and initial values

\[
\begin{align*}
  a_l &= (m + 2)a_{l-1} - a_{l-2}, & l \geq 3, \\
  a_1 &= 1, & a_2 &= m + 2; \\
  b_l &= (m + 2)b_{l-1} - b_{l-2} + (n - l + 1), & l \geq 3, \\
  b_1 &= 0, & b_2 &= n - 2.
\end{align*}
\]
Let \( P_3 = \begin{pmatrix} I_2 & b_2 & b_3 & \cdots & b_{n-2} & b_{n-1} \\ a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} \end{pmatrix}_n \), then

\[ P_3 A_2 M = \begin{pmatrix} m + n & 0 & b_n \\ 0_{(n-2) \times 2} & 0_{2 \times (n-2)} & a_n \end{pmatrix}_n \sim I_{n-2} \oplus \begin{pmatrix} m + n & b_n \\ 0_{(n-2)} & a_n \end{pmatrix}. \quad (2.3) \]

From (2.2), we can get that

\[ a_l = \frac{1}{\sqrt{m^2 + 4m}} \left[ \left( \frac{m + 2 + \sqrt{m^2 + 4m}}{2} \right)^l - \left( \frac{m + 2 - \sqrt{m^2 + 4m}}{2} \right)^l \right], \]

and

\[ b_l = \left( e \left( \frac{m + 2 + \sqrt{m^2 + 4m}}{2} \right)^l - f \left( \frac{m + 2 - \sqrt{m^2 + 4m}}{2} \right)^l \right) - \frac{n - l}{m}, \]

where

\[ e = \frac{(m^2 - m - (m + 1)\sqrt{m^2 + 4m} + 2mn)(m + 4 - \sqrt{m^2 + 4m})}{4m^2(m + 4)}, \]

\[ f = \frac{(m - m^2 - (m + 1)\sqrt{m^2 + 4m} - 2mn)(m + 4 + \sqrt{m^2 + 4m})}{4m^2(m + 4)}. \]

\[ \square \]

**Theorem 2.3**

(1) The spanning tree number of \( K_m \lor P_n \) is

\[ \frac{(m + n)^{m-1}}{2^n \sqrt{m^2 + 4m}} \left( \left( m + 2 + \sqrt{m^2 + 4m} \right)^n - \left( m + 2 - \sqrt{m^2 + 4m} \right)^n \right). \]

(2) The critical group of \( K_m \lor P_n \) is

\[ Z/(m + n, a_n, b_n)Z \oplus (Z/(m + n)Z)^{m-2} \oplus Z/\frac{(m + n)a_n}{(m + n, a_n, b_n)}Z, \]

where the parameters \( a_n \) and \( b_n \) are given in the above Lemma 2.2.

**Proof**

Note that every line sum of the Laplacian matrix of a graph is 0, so we have

\[ L(G) \sim \text{Snf}(L(G))_{uv} \oplus 0_1, \] for every \( u, v \in V(K_m \lor P_n) \). \quad (2.4)

It follows from (2.1) and (2.3) that

\[ L(K_m \lor P_n) \sim I_{n-1} \oplus \begin{pmatrix} m + n & b_n \\ 0 & a_n \end{pmatrix} \oplus (m + n)I_{m-2} \oplus 0_1. \quad (2.5) \]
Therefore by (2.4) and (2.5), we have
\[
\text{Snf}(L(G)) \sim I_{n-1} \oplus \begin{pmatrix} m + n & b_n \\ 0 & a_n \end{pmatrix} \oplus (m + n)I_{m-2}.
\]
(2.6)

It is easy to see that the invariant factors of the matrix \((m + n, b_n, a_n, b_n, a_n, b_n) \oplus (m + n)I_{m-2}\) are:
\[
\begin{align*}
(m + n, a_n, b_n),
(m + n, a_n, b_n),
\end{align*}
\]

(2.6)

It stands for the greatest common divisor of \(m + n, a_n, b_n\). So this theorem holds. □

\section*{Remark 2.4}
It is known that the Laplacian eigenvalues of \(P_n\) are: \(0, 2 + 2\cos\left(\frac{j\pi}{n}\right)\) \((1 \leq j \leq n - 1)\); and the Laplacian eigenvalues of \(K_m\) are: \(0, m\) (with multiplicity \(m - 1\)). Then it follows from Theorem 2.1 in [10] that the Laplacian eigenvalues of \(K_m \lor P_n\) are:
\[
0, m + n \text{ (with multiplicity } m - 1),
\]

(2.6)

then by the well known Kirchhoff Matrix-Tree Theorem we know that the spanning tree number of \(K_m \lor P_n\) is \(\kappa(K_m \lor P_n) = (m + n)^{m-1} \prod_{j=1}^{n-1} \left(m + 2 + 2\cos\left(\frac{j\pi}{n}\right)\right)\). Recall the first part of Theorem 2.3, we have
\[
\prod_{j=1}^{n-1} \left(m + 2 + 2\cos\left(\frac{j\pi}{n}\right)\right) = \frac{1}{2^n \sqrt{m^2 + 4m}} \left((m + 2 + \sqrt{m^2 + 4m})^n - (m + 2 - \sqrt{m^2 + 4m})^n\right). \quad (2.7)
\]

\section*{Example 2.5}
If \(m = 3, n = 4\), then \(a_4 = 115, b_4 = 59\). If \(m = 4, n = 4\), then \(a_4 = 204, b_4 = 83\). If \(m = 4, n = 5\), then \(a_5 = 1189, b_5 = 730\). So it follows from Theorem 2.3 we have the following
\[
\begin{align*}
\text{Snf}(K_3 \lor P_4) &= I_4 \oplus \text{diag}(7, 805, 0); \\
\text{Snf}(K_4 \lor P_4) &= I_4 \oplus \text{diag}(8, 8, 1632, 0); \\
\text{Snf}(K_4 \lor P_5) &= I_5 \oplus \text{diag}(9, 9, 10701, 0).
\end{align*}
\]
(2.8)

(2.9)

(2.10)

Note that one can use maple to check the results of (2.8), (2.9) and (2.10).

\section*{3. The critical group of \(P_m \lor P_n\)}

In this section we will work on the critical group of \(P_m \lor P_n(m \geq 4, n \geq 5)\). Let \(L'\) be the submatrix of \(L(P_m \lor P_n)\) resulting from the deletion of the last row and the \((m + 1)\)-th column. Thus \(L' = \begin{pmatrix} nI_m + L(P_m) & -J_{m \times (n-1)} \\ -J_{(n-1) \times m} & U \end{pmatrix}\), where \(J_{m \times (n-1)}\) is an \(m \times (n - 1)\) matrix having all entries equal to 1, and \(U\) is the submatrix obtained from \(mI_n + L(P_n)\) by deleting its first column and last row. Now we discuss the Smith normal form of the
matrix $L'$.

Let $M = \begin{pmatrix} T_m & 0_{m \times (n-1)} \\ 0_{(n-1) \times m} & I_{n-1} \end{pmatrix}$ and $N = \begin{pmatrix} T_m^{-1} & 0_{m \times (n-1)} \\ 0_{(n-1) \times m} & T_{n-1}^{-1} \end{pmatrix}$, where $T_m = (t_{ij})$ is an $m \times m$ matrix with its entries $t_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 1, & \text{if } j = 1, \\ 0, & \text{otherwise.} \end{cases}$ Then a direct calculation shows that

$$PNL'M = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where $P$ is an integral matrix with the left multiplication can imply an interchange of row 1 and $-1$ times row $m+1$ of $NL'M$, and

$$B_{11} = \begin{pmatrix} m & 1 & 1 & \cdots & \cdots & \cdots & 1 \\ 0 & n+3 & -1 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & n+2 & -1 & \cdots & \cdots & 0 \\ 0 & 1 & -1 & n+2 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & -1 & n+2 & -1 \\ 0 & 1 & 0 & \cdots & -1 & n+1 & \end{pmatrix}_{m \times m},$$

$$B_{12} = \begin{pmatrix} 1 & 0_{1 \times (n-2)} \\ 0_{(m-1) \times (n-1)} & I_{n-1} \end{pmatrix}_{m \times (n-1)}, \quad B_{21} = \begin{pmatrix} n & 0_{1 \times (m-2)} \\ 0_{(n-2) \times m} & I_{m} \end{pmatrix}_{(n-1) \times m},$$

$$B_{22} = \begin{pmatrix} -1 & -1 & -1 & \cdots & \cdots & -1 & -1 \\ m+3 & -1 & 0 & \cdots & \cdots & 0 & 0 \\ 0 & m+2 & -1 & \cdots & \cdots & 0 & 0 \\ 1 & -1 & m+2 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & -1 & m+2 & -1 & 0 \\ 1 & 0 & \cdots & -1 & m+2 & -1 & \end{pmatrix}_{(n-1) \times (n-1)}.$$

The sequences $p_i, q_i, c_i, d_i$ will be used in the following Lemma 3.1, where

$$\begin{align*}
 p_k &= (n+2)p_{k-1} + q_{k-1}, \quad k \geq 1 \\
 q_k &= 1 - p_{k-1}, \\
 p_0 &= n + 3, \quad q_0 = 0.
\end{align*} \tag{3.1}$$

and

$$\begin{align*}
 c_k &= (m+2)c_{k-1} + d_{k-1}, \quad k \geq 1 \\
 d_k &= 1 - c_{k-1}, \\
 c_0 &= m + 3, \quad d_0 = 0.
\end{align*} \tag{3.2}$$

Lemma 3.1

$$L' \sim F = \begin{pmatrix} I_{(m+n-3)} & 0_{(m+n-3) \times 2} \\ p'_{m-2} & 0 \\ 0_{2 \times (m+n-3)} & \alpha \beta - 1 & m\beta + n \end{pmatrix}.$$
where
\[ p_{m-2}' = \frac{1}{\sqrt{n^2 + 4n}} \left( \left( \frac{n + 2 + \sqrt{n^2 + 4n}}{2} \right)^m - \left( \frac{n + 2 - \sqrt{n^2 + 4n}}{2} \right)^m \right), \]
\[ \alpha = \frac{1}{np} p_{m-2}' - m, \]
\[ \beta = \frac{1}{m\sqrt{m^2 + 4m}} \left( \left( \frac{m + 2 + \sqrt{m^2 + 4m}}{2} \right)^n - \left( \frac{m + 2 - \sqrt{m^2 + 4m}}{2} \right)^n \right) - \frac{n}{m}. \]

**Proof** Let
\[
M_1 = \begin{pmatrix}
  m & 1 & 1 & 1 & \cdots & \cdots & 1 \\
  0 & p_0 & -1 & 0 & \cdots & \cdots & 0 \\
  0 & q_0 & n + 2 & -1 & \cdots & \cdots & 0 \\
  0 & 1 & -1 & n + 2 & -1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
  0 & 1 & 0 & \cdots & -1 & n + 2 & -1 \\
  0 & 1 & 0 & \cdots & \cdots & -1 & n + 1
\end{pmatrix}_{m \times m},
\]
and
\[
M_2 = \begin{pmatrix}
  -1 & -1 & -1 & \cdots & \cdots & -1 & -1 \\
  c_0 & -1 & 0 & \cdots & \cdots & 0 & 0 \\
  d_0 & m + 2 & -1 & \cdots & \cdots & 0 & 0 \\
  1 & -1 & m + 2 & -1 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
  1 & 0 & \cdots & -1 & m + 2 & -1 & 0 \\
  1 & 0 & \cdots & \cdots & -1 & m + 2 & -1
\end{pmatrix}_{(n-1) \times (n-1)},
\]
then we can rewrite \( L' \) as
\[
L' = \begin{pmatrix}
  M_1' & B_{12} \\
  B_{21} & M_2
\end{pmatrix}.
\]

Let \( B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \) where
\[
B_1 = \begin{pmatrix}
  1 & 0 & 0 & 0 & \cdots & 0 \\
  0 & 1 & 0 & 0 & \cdots & 0 \\
  0 & p_0 & 1 & 0 & \cdots & 0 \\
  0 & p_1 & 0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & p_{m-3} & 0 & 0 & \cdots & 1
\end{pmatrix}_{m \times m},
\]
\[
B_2 = \begin{pmatrix}
  1 & 0 & 0 & \cdots & 0 \\
  c_0 & 1 & 0 & \cdots & 0 \\
  c_1 & 0 & 1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_{n-3} & 0 & 0 & \cdots & 1
\end{pmatrix}_{(n-1) \times (n-1)}.
\]

Then it is easy to check that
\[
L'B = \begin{pmatrix}
  M_1' & B_{12} \\
  B_{21} & M_2
\end{pmatrix},
\]
where

\[
M'_1 = \begin{pmatrix}
  m & \alpha & 1 & 1 & \cdots & \cdots & 1 \\
  0 & 0 & -1 & 0 & \cdots & \cdots & 0 \\
  0 & 0 & n+2 & -1 & \cdots & \cdots & 0 \\
  0 & 0 & -1 & n+2 & -1 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & -1 & n+2 & -1 \\
  p'_{m-2} & 0 & \cdots & \cdots & -1 & n+1 & \\
\end{pmatrix}_{m \times m},
\]

and \(p'_{m-2} = (n+1)p_{m-3} - p_{m-4} + 1, \alpha = 1 + \sum_{k=0}^{m-3} p_k.\)

From (3.1), we get

\[
p_k = (x+y) \left( \frac{n+2 + \sqrt{n^2 + 4m}}{2} \right)^k + (x-y) \left( \frac{n+2 - \sqrt{n^2 + 4m}}{2} \right)^k - \frac{1}{n},
\]

where \(x = \frac{n^2 + 3n + 1}{2n}, y = \frac{n^2 + 5n + 5}{2\sqrt{n^2 + 4m}}.\) And now we can easily get the expression of \(p'_{m-2}\) and \(\alpha\) by a direct calculation.

\[
M'_2 = \begin{pmatrix}
  -\beta & -1 & -1 & -1 & \cdots & -1 & -1 \\
  0 & -1 & 0 & 0 & \cdots & 0 & 0 \\
  0 & m+2 & -1 & 0 & \cdots & 0 & 0 \\
  0 & -1 & m+2 & -1 & \cdots & 0 & 0 \\
  \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  0 & 0 & \cdots & -1 & m+2 & -1 & 0 \\
  0 & 0 & \cdots & \cdots & -1 & m+2 & -1 \\
\end{pmatrix}_{(n-1) \times (n-1)}
\]

and \(\beta = 1 + \sum_{k=0}^{n-3} c_k.\)

From (3.2), we can obtain that

\[
c_k = (u+v) \left( \frac{m+2 + \sqrt{m^2 + 4m}}{2} \right)^k + (u-v) \left( \frac{m+2 - \sqrt{m^2 + 4m}}{2} \right)^k - \frac{1}{m},
\]

where \(u = \frac{m^2 + 3m + 1}{2m}, v = \frac{m^2 + 5m + 5}{2\sqrt{m^2 + 4m}}.\) Thus we get \(\beta\) by a direct calculation.

Now we deal with the matrix \(L'B.\) In the following, we will use \(r_i\) to denote the \(i-th\) row of matrix \(L'B.\)

For \(i = 2, \ldots, m-2,\) we first add \((n+2)r_i\) to \(r_{i+1}\) and add \(-r_i\) to \(r_{i+2};\) then add \(r_i\) to \(r_1\), and add \((n+1)r_{m-1}\) to \(r_m.\) After that we have

\[
M'_1 \sim M''_1 = \begin{pmatrix}
  m & \alpha & 0_{1 \times (m-2)} \\
  0_{(m-2) \times 2} & -I_{(m-2)} \\
  0 & p'_{m-2} & 0_{1 \times (m-2)} \\
\end{pmatrix}_{m \times m}.
\]
Remark 3.3
It is known that the Laplacian eigenvalues of $P_m$ are $0, 2 + 2\cos\left(\frac{i\pi}{m}\right)$, $(1 \leq i \leq m - 1)$. Then it follows from Theorem 2.1 in [10] that the Laplacian eigenvalues of $P_m \vee P_n$ are: $0, m + n + 2 + 2\cos\left(\frac{i\pi}{m}\right)(1 \leq i \leq m - 1), m + 2 + 2\cos\left(\frac{i\pi}{n}\right)(1 \leq j \leq n - 1)$. Then by the well known Kirchhoff Matrix-Tree Theorem we know that the spanning tree number of $P_m \vee P_n$ is $\kappa(P_m \vee P_n) = \prod_{i=1}^{m-1} \left( n + 2 + 2\cos\left(\frac{i\pi}{m}\right) \right) \prod_{j=1}^{n-1} \left( m + 2 + 2\cos\left(\frac{j\pi}{n}\right) \right)$.

Then we have

$$(m + 2 + 2\cos\left(\frac{i\pi}{m}\right))\prod_{j=1}^{n-1} \left( m + 2 + 2\cos\left(\frac{j\pi}{n}\right) \right)$$

$$= \frac{(m + 2 + 2\cos\left(\frac{i\pi}{m}\right))\prod_{j=1}^{n-1} \left( m + 2 + 2\cos\left(\frac{j\pi}{n}\right) \right)}{(m - 2 + 2\cos\left(\frac{i\pi}{m}\right))\prod_{j=1}^{n-1} \left( m + 2 - 2\cos\left(\frac{j\pi}{n}\right) \right)}.$$

(3.3)

Example 3.4
If $m = 4, n = 5$, then $p'_{3} = 329, \alpha = 65, \beta = 296$. If $m = 4, n = 6$, then $p'_{4} = 496, \alpha = 82, \beta = 1731$. If $m = 5, n = 5$, then $p'_{3} = 2255, \alpha = 450, \beta = 450$. So it follows from Theorem 3.2 that we have the following

$${\text{Snf}}(P_{4} \vee P_{5}) = I_{7} \oplus \text{diag}(391181, 0); \quad (3.4)$$

$${\text{Snf}}(P_{4} \vee P_{6}) = I_{8} \oplus \text{diag}(3437280, 0); \quad (3.5)$$

$${\text{Snf}}(P_{5} \vee P_{5}) = I_{7} \oplus \text{diag}(451, 11275, 0). \quad (3.6)$$
Here we also note that one can use Maple to check the results of (3.4), (3.5) and (3.6).

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