SCALABLE FIRST-ORDER METHODS FOR ROBUST MDPs

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ABSTRACT

Markov Decision Processes (MDPs) are a widely used model for dynamic decision-making problems. However, MDPs require precise specification of model parameters, and often the cost of a policy can be highly sensitive to the estimated parameters. Robust MDPs ameliorate this issue by allowing one to specify uncertainty sets around the parameters, which leads to a non-convex optimization problem. This non-convex problem can be solved via the Value Iteration algorithm, but Value Iteration requires repeatedly solving convex programs that become prohibitively expensive as MDPs grow larger. We propose an algorithmic framework based on first-order methods, where we interleave approximate value iteration updates with a first-order-based computation of the robust Bellman update. Our algorithm relies on having a proximal setup for the uncertainty sets. We go on to instantiate this proximal setup for s-rectangular ellipsoidal uncertainty sets and Kullback-Leibler uncertainty sets. By carefully controlling the warm-starts of our first-order method and the increasing approximation rate at each Value Iteration update, our algorithm achieves a convergence rate of $O(\lambda^k S^3 \log(S) \log(\epsilon^{-1}) \epsilon^{-1})$ for the best choice of parameters, where $S, A$ are the numbers of states and actions. Our dependence on the number of states and actions is significantly better than that of Value Iteration algorithms. In numerical experiments on ellipsoidal uncertainty sets we show that our algorithm is significantly more scalable than state-of-the-art approaches. In the class of s-rectangular robust MDPs, to the best of our knowledge, our algorithm is the first to address Kullback-Leibler uncertainty sets. It is also the only one to solve ellipsoidal uncertainty sets to optimality when the state and actions spaces become on the order of several hundreds.

1 Introduction

Markov Decision Processes. In this paper we focus on solving robust Markov decision processes (MDPs). Markov decision process models are widely used in dynamic pricing, stochastic optimization and decision-making [Bertsekas 2007, Puterman 1994]. An MDP is described by a set of states $S$, a set of actions $A$, a transition kernel $y$ which gives transition probabilities $y_{sa} \in \mathbb{R}^S_+$ for all state-action pair $(s, a)$, costs $c_{sa}$ for each state-action pair $(s, a)$ and a discount factor $\lambda \in (0, 1)$. The goal of the decision maker is to choose a policy $x$ which assigns a probability distribution over the set of actions for each state (i.e. $x \in \Pi = (\Delta(|A|))^{|S|}$, where $\Delta(|A|)$ is the simplex of dimension $|A|$) in order to minimize the infinite horizon discounted expected cost $R(x, y)$, defined as

$$R(x, y) = E^{x,y} \left[ \sum_{t=0}^{\infty} \lambda^t c_{x_t a_t} \right] s_0 \sim p_0,$$

where $s_t$ is the state at period $t \in \mathbb{N}$ and $a_t$ is the action chosen at period $t$ following the probability distribution $(x_{s_t a_t})_a \in \mathbb{R}^S_+$. The vector $p_0 \in \mathbb{R}^S_+$ is a given initial probability distribution over the set of states $S$.

For regular MDPs, an optimal policy can be found in the set $\Pi$ of stationary, Markovian and deterministic policies. Several algorithms have been studied including Policy Iteration, Value Iteration (VI) and linear-programming-based algorithms [Puterman 1994, Bertsekas 2007].

Robust Markov Decision Processes. In many applications, statistical errors in the estimation of the transition kernel $y$ are unavoidable. This is a problem because the cost of a policy can be highly sensitive to the exact kernel parameters. In such settings, it is important to address the uncertainty in parameter estimates when optimizing over the set of policies.

We consider a robust approach where we model the uncertainty in $y$ as an adversarial selection from some safety region $P$ centered around our nominal estimation $y^0$ of the true transition kernel. We refer to this convex, compact set $P$ as the
uncertainty set. Our goal is to find a policy that minimizes the worst-case expected cost over the choices of $y$ in the uncertainty set $P$, i.e., our goal is to solve

$$
\min_{x \in \Pi} \max_{y \in P} R(x, y)
$$

(1.2)

Note that $(x, y) \mapsto R(x, y)$ is not a convex-concave function. Among others, the robust optimization approach has been specifically considered to address parameter uncertainty in MDPs in [Iyengar, 2005], [Nilim and Ghaoui, 2005], [Wiesemann et al., 2013], [Mannor et al., 2016], [Goh et al., 2018] and [Goyal and Grand-Clement, 2018]. Robust MDPs have found applications in decision-making for healthcare [Steimle and Denton, 2017], [Steimle et al., 2018], [Goh et al., 2018] and [Grand-Clement et al., 2020].

In particular, Wiesemann et al. [2013] consider $s$-rectangular uncertainty set, first introduced in Epstein and Schneider [2003]. In this model, the uncertainty on transition probabilities $y_s = (y_{sas'})_{a' \in A}$ corresponding to different states are uncoupled:

$$
P = \times_{s \in S} P_s, \text{ where } P_s \subseteq \mathbb{R}_{+}^{A|\times|S}.
$$

(1.3)

For $s$-rectangular uncertainty sets, an optimal policy exists in the class of stationary and Markovian policies. However, there may not be any deterministic optimal policy. Wiesemann et al. [2013] show that computing an optimal pair $x, y$ in (1.2) is equivalent to computing the fixed point of a contracting operator, the Bellman operator $F$, leading to a value iteration algorithm (described in Section 2).

Kullback-Leibler (KL) uncertainty sets are constructed from approximations of the confidence regions associated with their tractability and the probabilistic guarantees of the optimal solutions of the robust problems [Ben-Tal and Nemirovski, 2000, Bertsimas et al., 2019]. Additionally, they can be understood as a conservative approximation of KL uncertainty sets [Iyengar, 2005]. Section 4.2) When the uncertainty set $P$ of (1.3) is ellipsoidal, the value iteration algorithm of Wiesemann et al. [2013] for solving (1.2) involves solving a convex program with a quadratic constraint at every epoch. While this can be done in polynomial time with modern interior point methods (IPMs, Lobo et al. 1998), this requires inverting matrices at every step of the IPM which can be intractable for large MDP instances. Typically, for $S = |S|, A = |A|$, the number of arithmetic operations for the value iteration algorithm to return an $\epsilon$-solution to (1.2) with ellipsoidal uncertainty sets is $O\left(4.5 S^{1.5} \log^2(\epsilon^{-1})\right)$. This may prove prohibitive for large state or action sets. Additionally, while KL uncertainty sets are well understood in Distributionally Robust Optimization [Hu and Hong, 2013], we are not aware of any tractable algorithm for solving $s$-rectangular robust MDP with KL uncertainty sets.

First Order Methods for Saddle Point Optimization. Many problems in machine learning and game theory can be written in the form

$$
\min_{x \in X} \max_{y \in Y} \mathcal{L}(x, y),
$$

(1.4)

where $\mathcal{L}$ is a general convex-concave function and $X, Y$ are reflexive Banach spaces. For instance, regularized finite-sum loss minimization, imaging models, and sequential two-player zero-sum games have natural saddle-point formulations [Chambolle and Pock, 2011, Kroer et al., 2018b]. Even though convex duality often allows reformulation of (1.4) as a single convex program, first-order methods (FOMs) such as Chambolle & Pock’s Primal-Dual Algorithm (PDA, Chambolle and Pock, 2011), Mirror Descent [Nemirovski and Yudin, 1983] or Mirror Prox [Nemirovski, 2004] are typically preferred for large instances. As with the convex programs mentioned for the value iteration algorithm for RMDPs, this is due to the expensive matrix calculations involved in IPMs or the simplex algorithm. Naively, one may hope to apply FOMs directly to the min-max problem (1.2), which looks superficially similar. However, since this problem is not convex-concave FOMs may fail to converge in this case.

1.1 Our Contributions

A First-Order Method for Robust MDP. We present a new algorithmic framework for solving robust MDPs that is significantly more scalable than previous methods in terms of state and action-space sizes. Our algorithm adapts FOMs for solving static min-max zero-sum games to a dynamic setting with varying payoff matrices. Only cheap proximal mappings need to be computed at each iteration. We prove theoretical convergence guarantees for our algorithm, and show that the dependence on instance size is better than for prior methods. In order to turn the robust MDP problem (1.2) into a dynamic sequence of changing zero-sum games, we interleave FOM updates with occasional approximate VI updates. By carefully controlling the pace of VI updates, and developing bounds on how much the payoff matrices of the zero-sum games change, we show that it is possible to nearly get a $\frac{1}{T}$ convergence rate in terms of the number of FOM steps $T$. 

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Novel proximal setups. Our algorithmic framework is general and works for any uncertainty set for which a suitable proximal setup exists. We show how to instantiate our algorithm on Kullback-Leibler (KL) and ellipsoidal uncertainty sets. To enable these uncertainty sets, we study four proximal setups of independent interest. In particular, we show how to efficiently compute proximal updates over the intersection of a Cartesian product of simplexes and an $\ell_2$ ball, both for the $\ell_2$ proximal setup and the $\ell_1$ proximal setup. We also show how to compute proximal updates over the intersection of a Cartesian product of simplexes and a KL ball, both for the $\ell_2$ and the $\ell_1$ proximal setup. To the best of our knowledge, our algorithm is the only one addressing $s$-rectangular KL uncertainty sets for robust MDPs and is the most scalable for ellipsoidal $s$-rectangular uncertainty sets.

Empirical performance. We focus our numerical experiments on ellipsoidal uncertainty sets. We try several proximal setups numerically, and find that an $\ell_2$ setup empirically performs better than an $\ell_1$ setup, despite better theoretical guarantees for the $\ell_1$ setup. A similar discrepancy between theory and practice has been noted for stationary saddle-point optimization Chambolle and Pock [2016], Cao et al. [2019]. We then compare our algorithms to state-of-the-art VI setups, and show that our approach is significantly more scalable. To the best of our knowledge, our algorithm is the most scalable algorithm for solving Robust MDPs with $s$-rectangular ellipsoidal uncertainty sets.

1.2 Related work

Approximate value iteration and Regularized MDP. For the nominal MDP setting, Scherrer et al. [2015] consider approximate Value Iteration (and approximate Policy Iteration), where the inexact updates arise from either model-free update (i.e. sample-based algorithms) or function approximation. Alternatively, value function approximation (Tsitsiklis and Van Roy [1997], De Farias and Van Roy [2003], Petrik [2010], Konidas et al. [2011]) copes with large state spaces by constraining the value vector to belong to a small-dimensional subspace of $\mathbb{R}^{|s|}$ and making exact Bellman updates on the orthogonal projection of the value vector onto that subspace. In contrast, we consider full-dimensional value vectors, and perform inexact Bellman updates via first-order methods. Therefore, we can control the desired accuracy of our inexact updates, contrary to value function approximation once the basis on the subspace is fixed. Additionally, Geist et al. [2019] add a KL regularization term in Value Iteration for nominal MDP and show a connection the Mirror Descent algorithm in convex optimization, which is also a first-order method. In contrast to Geist et al. [2019], we focus on solving robust MDP, for which our operator is the robust Bellman operator $F$, and we focus our use of FOMs on efficiently computing the min-max Bellman updates $F(v)$, not penalizing the deviation from the previous policy visited by VI. Additionally, we would like to note that our proofs and our algorithm, while based on the Primal-Dual Algorithm (PDA), Chambolle and Pock [2011], Chambolle and Pock [2016], can easily be extended to a setting based on Mirror Descent and Mirror Prox (see Remark 4.3 in Section 4).

Faster value iteration algorithms. For nominal MDPs, several algorithms have been proposed to accelerate the convergence of the Value Iteration algorithm, including Gauss-Seidel and Jacobi iterations (Puterman [1994], Section 6.3.3), $k$-step look-ahead (Herzberg and Yechiali [1996]), Anderson mixing (Zhang et al. [2018], Geist and Scherrer [2018]), and acceleration and momentum schemes (Goyal and Grand-Clement [2019]). However, all of these require the computation of the min-max Bellman updates $F(v)$, which can be very expensive for large robust MDP instances as explained in the previous sections. Furthermore, none of these algorithms specifically aim at solving robust MDPs.

Faster Bellman updates. In the case of $s$, $a$-rectangular uncertainty set, quasi-linear robust Bellman updates were studied for uncertainty sets defined by balls for the $\ell_1$ and weighted $\ell_1$ norms (Iyengar [2005], Ho et al. [2018], $\ell_2$ norm (Iyengar [2005]), KL-divergence (Nilim and Ghaoui [2005]) and $\ell_\infty$ norm (Givan et al. [1997])). To the best of our knowledge, the only paper on accelerating the computation of the robust update $F(v)$ (without value function approximation) for $s$-rectangular uncertainty set is [Ho et al. 2018]. In particular, the authors in [Ho et al. 2018] consider weighted $\ell_1$-ball for $P_s$ and attains a complexity of $O(S^2A\log(S^2A))$ for the update $F(v)$ on each state $s$. Note that their algorithm solves $F(v)_s$ to optimality and does not depend on an accuracy $\epsilon$. Therefore, using their algorithm for the updates, the number of arithmetic operations to find an $\epsilon$-approximation to $v^*$ is $O(S^2A\log(S^2A)\log(\epsilon^{-1}))$. While this is significantly faster than the complexity results from Wiesemann et al. [2013] for ellipsoidal $s$-rectangular uncertainty set, the results in [Ho et al. 2018] rely on linear programming theory and on the assumption that the sets $P_s$ are $\ell_1$-balls centered around a nominal estimation of the transition kernel. Therefore, their complexity results cannot easily be extended to other settings for $P_s$ (e.g. ellipsoidal or KL uncertainty sets).

FOMs for zero-sum games. Our work is motivated by, but distinct from, the application of fast FOMs to computing Nash equilibrium of large two-player zero-sum extensive-form games (EFGs). For EFGs, the sequential decision space of each player is tree-structured, and this can be exploited to formulate a Nash equilibrium in the form of (1.4). The question at that point becomes one of constructing appropriate distance measures for the resulting strategy spaces, typically achieved by carefully combining simplex distance measures Hoda et al. [2010], Kroer et al. [2018a].
setting is very different, because (1.2) is not bilinear (or even convex-concave), and thus these methods are not applicable. Instead, we solve a series of changing bilinear saddle-point problems, where each individual problem does not have a sequential component.

Another class of algorithms, regret-minimization methods, have also been applied to EFG solving, both via appropriate distance measures [Farina et al. 2019b], and via the CFR framework [Zinkevich et al. 2008], which has also been extended to more general structured sequential decision spaces [Farina et al. 2019a]. All of these approaches rely on a tree or directed acyclic graph structure of the decision space, and thus are not applicable to robust MDPs, just as with the FOM methods for EFGs.

1.3 Outline

The remainder of the paper is organized as follows. In Section 2 we present the robust MDP problem in more detail, and present the Value Iteration (VI) algorithm. In Section 3 we present the primal-dual algorithm of Chambolle and Pock [2011], the first-order method used to solve saddle-point problems of the form (1.4). In Section 4 we present our new algorithm for solving robust MDPs, Algorithm VI and its theoretical convergence guarantees. In Section 5 we derive explicit expressions for the complexity of Algorithm VI in terms of number of states $S$, number of actions $A$, and accuracy $\epsilon$. In Section 6 we present our detailed numerical comparisons of Algorithm VI and Algorithm I.

2 Preliminaries on Robust MDP

Notation. We write $|S| = S, |A| = A$ and we assume $S < +\infty, A < +\infty$. Given a policy $x \in \Pi$ and a kernel $y \in \mathcal{P}$, we can define the one-step cost vector $c_x \in \mathbb{R}^S$ as $c_{x,s} = \sum_{a=1}^{A} x_{sa} c_{sa}, \forall s \in S$. For each pair $(x, y) \in \Pi \times \mathcal{P}$, there is a unique value vector $v^{x,y} \in \mathbb{R}^S$, defined by $v^{x,y} = E^{x,y} \left[ \sum_{t=0}^{\infty} \Lambda^t c_{s,a_t} \right], \forall s \in S$.

Value Iteration. Let us define the (robust) Bellman operator $F : \mathbb{R}^S \to \mathbb{R}^S$, where for $v \in \mathbb{R}^S$,

$$F(v)_s = \min_{x_s \in \Delta(A)} \max_{y_s \in \mathcal{P}_s} \left\{ \sum_{a=1}^{A} x_{sa} (c_{sa} + \lambda \cdot y_{sa} v) \right\}, \forall s \in S.$$  

Note that with the notation $F^{x,y}(v)_s = \sum_{a=1}^{A} x_{sa} (c_{sa} + \lambda \cdot y_{sa} v)$, we can also write

$$F(v)_s = \min_{x_s \in \Delta(A)} \max_{y_s \in \mathcal{P}_s} F^{x,y}(v)_s,$$  

which shows that the robust VI update is a stationary saddle-point problem of the form (1.4). The Value Iteration (VI) Algorithm is defined as follow:

$$v_0 \in \mathbb{R}^S, v^{\ell+1} = F(v^{\ell}), \forall \ell \geq 0.$$  

Note that $F$ is a contraction of factor $\lambda$ [Wiesemann et al. 2013]. Moreover, solving (1.2) is equivalent to computing $v^*$, the unique fixed-point of $F$:

$$v^*_s = \min_{x_s \in \Delta(A)} \max_{y_s \in \mathcal{P}_s} \left\{ \sum_{a=1}^{A} x_{sa} (c_{sa} + \lambda \cdot y_{sa} v^*) \right\}, \forall s \in S.$$  

VI returns a sequence $(v^{\ell})_{\ell \geq 0}$ such that

$$\|v^{\ell+1} - v^*\|_\infty \leq \lambda \cdot \|v^{\ell} - v^*\|_\infty, \forall \ell \geq 0.$$  

An optimal pair $(x^*, y^*)$ can be computed as any pair attaining the min max in $F(v^*)$. Additionally, an $\epsilon$-optimal pair can be computed as the pair attaining the min max in $F(v)$, if $\|v - F(v)\|_\infty < 2\lambda(1 - \lambda)^{-1}$ [Puterman 1994].

In this paper we focus on KL s-rectangular uncertainty sets

$$\mathcal{P} = \times_{s \in S} \mathcal{P}_s, \mathcal{P}_s = \{ (y_{sa})_{a \in A} : \sum_{a \in A} KL(y_{sa}, y^0_{sa}) \leq \alpha, y_{sa} \in \Delta(S), \forall a \in A \},$$  

and ellipsoidal s-rectangular uncertainty sets

$$\mathcal{P} = \times_{s \in S} \mathcal{P}_s, \mathcal{P}_s = \{ (y_{sa})_{a \in A} : \sum_{a \in A} \frac{1}{2} \|y_{sa} - y^0_{sa}\|_2^2 \leq \alpha, y_{sa} \in \Delta(S), \forall a \in A \}.$$  

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Note that (2.5) is different from the ellipsoidal uncertainty sets considered in Ben-Tal and Nemirovski [2000], which also adds box constraints. However, Bertsimas et al. [2019] shows that the same probabilistic guarantees exist for (2.5) as in the case of the uncertainty sets considered in Ben-Tal and Nemirovski [2000].

One can instantiate Value Iteration with ellipsoidal uncertainty sets as follows. Using min-max convex duality twice, we can reformulate each of the $S$ min-max programs (2.1) into a larger convex program with linear objective, linear constraints and one quadratic constraint (see (4.12) in Appendix H). Using Interior Point Method, each program can be solved up to $\epsilon$ accuracy in $O(A^{3.5}S^{3.5} \log(1/\epsilon))$ arithmetic operations (Ben-Tal and Nemirovski [2001], Section 4.6.1-4.6.2), resulting in a total number of operations for (VI) to return an $\epsilon$-solution in a number of arithmetic operations of

$$O \left( A^{3.5} S^{4.5} \log^2(\epsilon^{-1}) \right).$$

(2.6)

As mentioned earlier, this becomes intractable as soon as the number of states becomes on the order of hundreds, as highlighted in our numerical experiments of Section 6.

**Approximate Value Iteration.** We now examine a variant of Value Iteration where each sub-problem $F(\nu)_s$ is solved approximately.

**Proposition 2.1.** Suppose that for every VI epoch $\ell \geq 1$, we solve the min-max problem (2.1) up to precision $\epsilon_\ell > 0$, i.e. we compute $(x^\ell, y^\ell)$ such that

$$v^{\ell+1} = Fx^\ell, y^\ell(v^\ell), \|v^{\ell+1} - F(v^\ell)\|_\infty \leq \epsilon_\ell.$$ 

Then we have, for any $\ell \geq 1$,

$$\|v^{\ell+1} - v^*\|_\infty \leq \lambda \|v^\ell - v^*\|_\infty + \epsilon_\ell,$$

$$\|v^{\ell+1} - v^\ell\|_\infty \leq \lambda \|v^\ell - v^{\ell-1}\|_\infty + \epsilon_\ell + \epsilon_{\ell-1}.$$ 

In particular, this implies

$$\|v^\ell - v^*\|_\infty \leq \lambda^\ell \left( \|v^0 - v^*\|_\infty + \sum_{t=0}^{\ell-1} \frac{\epsilon_t}{\lambda^t} \right),$$

$$\|v^{\ell+1} - v^\ell\|_\infty \leq \lambda^\ell \left( \|v^1 - v^0\|_\infty + \sum_{t=0}^{\ell} \frac{\epsilon_t + \epsilon_{\ell-1}}{\lambda^t} \right).$$

We present a proof of in Appendix A. Note that our analysis is close to the case of approximate policy iteration for non-robust MDPs (Gabillon et al. [2013], Scherrer et al. [2015]). While we treat the term $\epsilon_\ell$ as a (chosen) error term in our algorithm, we would like to note that we can think of the term $\epsilon_\ell$ as some random noise, coming from either function approximations or sample-based estimations (see Section 4 in Scherrer et al. [2015]).

### 3 Primal-Dual Algorithm (PDA)

In this section we introduce the primal-dual algorithm PDA Chambolle and Pock [2016], using the notation of Gao et al. [2019].

**Notations.** We consider solving the following min-max problem:

$$\min_{x \in X} \max_{y \in Y} \mathcal{L}(x, y),$$

where $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ are reflexive Banach spaces and $\mathcal{L} : X \times Y \to \mathbb{R}$ is such that

$$\mathcal{L}(x, y) = \langle Kx, y \rangle + f(x) + g(x) - h^*(y),$$

where $K : X \to Y^*$ is a bounded linear operator and $f : X \to \mathbb{R}$ is a proper lower semi-continuous convex function with $L_f$-Lipschitz gradient $\nabla f : X \to X^*$. Moreover, let $\psi_X$ be 1-strongly convex for $\| \cdot \|_X$ and $\psi_Y$ be 1-strongly convex for $\| \cdot \|_Y$. Let $D_X, D_Y$ the Bregman divergence defined by $\psi_X, \psi_Y$:

$$D_X(x, x') = \psi_X(x') - \psi_X(x) - \langle \nabla \psi_X(x), x' - x \rangle, \forall (x, x') \in X \times X,$$

$$D_Y(y, y') = \psi_Y(y') - \psi_Y(y) - \langle \nabla \psi_Y(y), y' - y \rangle, \forall (y, y') \in Y \times Y.$$ 

Let $g : X \to \mathbb{R}, h : Y^* \to \mathbb{R}$ be proper lower semi-continuous convex functions, such that $\text{dom } g \subset \text{dom } \psi_X, \text{dom } h^* \subset \text{dom } \psi_Y$. Let $\Theta_X, \Theta_Y$ be the maximum of the Bregman divergences $D_X$ and $D_Y$ on $\text{dom } g$ and $\text{dom } h$. Let $R_X$ the maximum of $\| \cdot \|_X$ on $\text{dom } g$, and define $R_Y$ similarly.
Algorithm. For $\sigma, \tau \in \mathbb{R}_+$, $x', y'' \in X, y', y'' \in Y$, we write

\begin{align*}
(\hat{x}, \hat{y}) &= PD_{\sigma, \tau}(x', y'', y', y'') \text{ (Primal-Dual update)}, \\
\hat{x} &= \arg\min_{x \in X} f(x') + \nabla f(x') \cdot (x - x') + g(x) + \langle Kx, y'' \rangle + \frac{1}{\tau} D_{X}(x, x'), \\
\hat{y} &= \arg\min_{y \in Y} h^*(y) - \langle Ky'', y \rangle + \frac{1}{\sigma} D_{Y}(y, y').
\end{align*}

Given some step sizes $\tau, \sigma \in \mathbb{R}$, the Primal-Dual Algorithm (PDA) runs as

\begin{equation}
(x^{t+1}, y^{t+1}) = PD_{\sigma, \tau}(x^t, y^t, 2x^{t+1} - x^t, y^t), \; t \geq 0.
\end{equation}

Convergence of PDA. We have the following proposition.

Proposition 3.1 (Chambolle and Pock [2016], Gao et al. [2019]). Fix a scalar $\Omega \geq 0$ and some step sizes $\tau, \sigma$ such that for all $(x, y), (x', y') \in X \times Y$,

\begin{equation}
0 \leq A(x, y, x', y') = \frac{1}{\tau} D_{X}(x, x') + \frac{1}{\sigma} D_{Y}(y, y') - \langle K(x - x'), y - y' \rangle \leq \Omega.
\end{equation}

Consider running PDA on the associated min-max problem for $T$ iterations. Consider weights $\omega_1, ..., \omega_T$ and $S_T = \sum_{t=1}^{T} \omega_t$. Then we have the critical inequality:

\begin{equation}
\mathcal{L}(x^{T+1}, y) - \mathcal{L}(x, y^{T+1}) \leq A(x, y, x^T, y^T) - A(x, y, x^{T+1}, y^{T+1}), \forall (x, y) \in X \times Y.
\end{equation}

Additionally, summing up (3.2), we have, for all $(x, y) \in X \times Y$,

\begin{equation}
\sum_{t=1}^{T} \omega_t \left( \mathcal{L}^{K}(x^T, y) - \mathcal{L}^{K}(x^t, y^t) \right) \leq \omega_0 A[x, y, x^0, y^0] + \omega_T \Omega - \omega_1 \Omega - \omega_T A[x, y, x^T, y^T].
\end{equation}

In particular, for $(\bar{x}^T, \bar{y}^T) = (1/S_T) \sum_{t=1}^{T} \omega_t (x_t, y_t)$, for all $(x, y) \in X \times Y$,

\begin{equation}
\mathcal{L}^{K}(\bar{x}^T, y) - \mathcal{L}^{K}(\bar{x}, \bar{y}^T) \leq \frac{1}{S_T} \sum_{t=1}^{T} \omega_t \left( \mathcal{L}^{K}(x^t, y) - \mathcal{L}^{K}(x^t, y^t) \right) \leq \Omega \frac{\omega_T}{S_T}.
\end{equation}

Choice of step sizes. From Chambolle and Pock [2016], in order to satisfy (3.1), we can choose $\Omega = 2 \left( \frac{\Theta_X}{\tau} + \frac{\Theta_Y}{\sigma} \right)$ and $\tau, \sigma > 0$ such that, for $L_K \geq \sup_{\|x\|, \|y\| \leq 1} \langle Kx, y \rangle$, we have

\begin{equation}
\left( \frac{1}{\tau} - L_f \right) \frac{1}{\sigma} \geq L_K^2.
\end{equation}

4 First-order Methods for Robust MDPs

In this section we introduce our algorithmic framework to compute an optimal pair of policy and transition kernel $(x^*, y^*)$ for the robust MDP problem (1.2). Our algorithmic framework is based on the observation that there exists a collection of operators $\mathcal{L}^s_x : \Delta(A) \times X \rightarrow \mathbb{R}$, for $s \in S$, such that computing an optimal solution $x^*$, $y^*$ to the robust MDP problem boils down to solving $S$ zero-sum games, each with operator $\mathcal{L}^s_x$. This is a straightforward consequence of the Bellman equation (2.1) and its reformulation using $L_{x,y}^s$. Therefore, if we knew the optimal value vector $v^*$, our problem could be solved using known methods for static zero-sum games. Since we do not know $v^*$, we instead construct a sequence $\{v^\ell\}$ that converges to $v^*$. Our algorithm builds upon VI. At every VI epoch $\ell \geq 1$ (we refer to VI iterations as epochs to distinguish from FOM iterations), we have a value vector $v^\ell$ and we use a first-order method (e.g. PDA) to compute an $\epsilon$-approximation of the Bellman update $F(v^\ell)$. At VI epoch $\ell + 1$, we use our approximate solution to $F(v^\ell)$ to warm-start the computation of an approximation to $F(v^{\ell+1})$. Taking the (weighted) average of the FOM strategies across all epochs converges to a solution to the robust MDP problem.
4.1 Notations

We now set up the computation of $F(v)$ as a saddle-point problem. Fix $s \in S$. Let $X = \mathbb{R}^A$, $Y = \mathbb{R}^{A \times S}$. Let $g(x) = 1\{x \in \Delta(A)\}$, $h^*(y) = 1\{y \in \mathbb{R}_{+}^S\}$, where $1\{x \in E\}$ denotes the indicator function of the set $E$. We define $K = K[v]$ and $c_a$ such that

$$f(x) = c^T_a x = \sum_{a=1}^{A} x_a c_{sa}, \quad \langle Kx, y \rangle = \lambda \sum_{a=1}^{A} x_a y_a^T v.$$  

With this notation and $\mathcal{L}(x, y)$ defined as in the PDA setup, $\mathcal{L}(x, y) = F^{x, y}(v)_s$ and

$$\min_{x \in X} \max_{y \in Y} \mathcal{L}(x, y) = \min_{x \in \Delta(A)} \max_{y \in \mathbb{R}_{+}^S} \sum_{a=1}^{A} x_{sa} (c_{sa} + \lambda \cdot y_{sa}^T v) = \min_{x \in \Delta(A)} \max_{y \in \mathbb{R}_{+}^S} F^{x, y}(v)_s = F(v)_s.$$  

Therefore, we can use a first-order method to compute an approximate solution to $F(v)_s$ for each $s$. In this paper we use PDA. In particular, writing $c'_a = c_a + y_{sa}^T v$ and $d \in \mathbb{R}^{A \times S}, d_{as'} = -\lambda x_{sa} v_s', \forall (a, s') \in \mathbb{A} \times \mathbb{S}$, the PD update for $s$ becomes

$$\hat{x} = \arg \min_{x \in \Delta(A)} \sum_{a=1}^{A} x_{sa} c_{sa} + \lambda \sum_{a=1}^{A} x_{sa} y_{sa}^T v + \frac{1}{\tau} D_X(x, x') = \arg \min_{x \in \Delta(A)} \langle x, c_{sa} \rangle + \frac{1}{\tau} D_X(x, x'), \quad (4.1)$$

$$\hat{y} = \arg \min_{y \in \mathbb{R}_{+}^S} -\lambda \sum_{a=1}^{A} x_{sa} y_{sa}^T v + \frac{1}{\sigma} D_Y(y, y') = \arg \min_{y \in \mathbb{R}_{+}^S} \langle y_s, d_{s} \rangle + \frac{1}{\sigma} D_Y(y, y'). \quad (4.2)$$

Corresponding norms of $f$ and $K$. Note that $f$ is a linear function and therefore $L_f = 0$ since $L_f$ is the Lipschitz constant of the gradient of $f$. Let $L_K \in \mathbb{R}_{+}, L_K = \sup_{\|x\|_X \leq 1, \|y\|_Y \leq 1} \langle K x, y \rangle$. We prove in Appendix B the following lemma.

Lemma 4.1. For $(\| \cdot \|_X, \| \cdot \|_Y) = (\| \cdot \|_2, \| \cdot \|_2)$, $L_K = \lambda \|v\|_2$.

For $(\| \cdot \|_X, \| \cdot \|_Y) = (\| \cdot \|_1, \| \cdot \|_1)$, $L_K = \lambda \|v\|_\infty$.

Choice of step sizes. We can choose the step sizes according to (3.3) and Lemma 4.1. In particular, choose $\sigma = \tau = 1/L_K$. Note that $X$ is of dimension $A$ while $Y$ is of dimension $A \times S$. Therefore, we could modify $\sigma, \tau$ to account for the larger dimensions of $Y$. Gao et al. [2019]. While this may provide improved empirical performances, for the sake of simplicity we do not include these changes in our theoretical analysis.

4.2 Our algorithm

Our algorithm performs $k$ approximate VI iterations. At every VI iteration $\ell$ (which we refer to as epoch $\ell$), each state $s$ has its approximate Bellman update computed using a first-order method, PDA, for $T_\ell$ iterations. We present the details below.
Algorithm 1 First-order Method for Robust MDP with \( \sigma \)-rectangular uncertainty set. 

1: **Input** Set of policy \( \Pi \), set of transition kernels \( \mathbb{P} \), an integer \( k \) and some integers \( T_1, \ldots, T_k \), some weights \( \omega_1, \ldots, \omega_T \) for \( T = T_1 + \ldots + T_k \), some step sizes \( \tau, \sigma \) satisfying (3.3) and following Lemma 4.1.

2: **Initialize** \( \ell = 1, v^0 = 0 \), and \( \bar{x}^0, \bar{y}^0 \) at random.

3: for epoch \( \ell = 1, \ldots, k \) do

4: for \( s \in S \) do

5: Set

\[
K^\ell = K[v^\ell], g(x) = 1_{\{x \in \Delta(A)\}}, h = 1_{\{y \in \mathbb{P}_s\}}, f : x \mapsto \sum_{a=1}^{A} x_a c_{sa}.
\]

6: For \( \tau_\ell = T_1 + \ldots + T_{\ell-1} \), compute \( (x_{\tau_\ell+1}, y_{\tau_\ell+1}), \ldots, (x_{\tau_\ell+T_\ell}, y_{\tau_\ell+T_\ell}) \) by running PDA for \( T_\ell \) iterations with step sizes \( \tau, \sigma \) and initialized at \( (x_{\tau_{\ell-1}}, y_{\tau_{\ell-1}}) \).

7: Compute \( \bar{x}^\ell_s, \bar{y}^\ell_s \) the averages for the iterates \( (x_{\tau_\ell+1}, y_{\tau_\ell+1}), \ldots, (x_{\tau_\ell+T_\ell}, y_{\tau_\ell+T_\ell}) \), with weights \( \omega_{\tau_\ell+1}, \ldots, \omega_{\tau_\ell+T_\ell} \).

8: Update \( v^{\ell+1} = F\bar{x}^\ell_s, \bar{y}^\ell_s(v^\ell)_s \).

9: end for

10: end for

In the next proposition we present our convergence analysis of Algorithm 1. Our convergence guarantee depends on several constants of the problem. In particular, recall that \( R_\mathbb{Y} \) is the maximum of \( \|\cdot\|_\mathbb{X} \) on \( \text{dom} \ g = \Delta(A) \), and that \( R_\mathbb{Y} \) is defined similarly. Our convergence guarantee also depends on the proximal setups of each player. In particular, we call \((\|\cdot\|_\mathbb{X}, \|\cdot\|_\mathbb{Y}) = (\ell_1, \ell_2)\) the \( \ell_1 \) setup and we call \((\|\cdot\|_\mathbb{X}, \|\cdot\|_\mathbb{Y}) = (\ell_2, \ell_2)\) the \( \ell_2 \) setup. For \( r_\infty = \max_{s,a} c_{sa} \), in the \( \ell_1 \) setup, we define the constants \( a, \Omega \) as \( a = \lambda, \Omega = \frac{2r_\infty}{1 - \lambda} (\Theta_\mathbb{X} + \Theta_\mathbb{Y}) \). In the \( \ell_2 \) setup, we define the constants \( a, \Omega \) as \( a = \lambda \sqrt{S}, \Omega = \frac{2r_\infty \sqrt{S}}{1 - \lambda} (\Theta_\mathbb{X} + \Theta_\mathbb{Y}) \). Finally, we write \( err_{s,0} = \|v^* - v^0\|_\infty, err_{1,0} = \|v^1 - v^0\|_\infty \).

We have the following proposition. We present a detailed proof in Appendix C.

**Proposition 4.2.** Let \( v^* \) the value vector of \( x^*, y^* \) a pair of optimal solution to the robust MDP problem (1.2).

Let \( \bar{x}^T, \bar{y}^T \) be the averages of the iterates visited by our algorithm for the weights \( w_1, \ldots, w_T \), and \( S_T = \sum_{t=1}^{T} \omega_t \). Then for all states \( s \in S \),

\[
\max_{y \in \mathbb{P}_s} F\bar{x}^T,y(v^*)_s - \min_{x \in \Pi} F\bar{x},\bar{y}^T(v^*)_s \leq c_1 + c_2 + c_3 + c_4 + c_5,
\]

where we have

\[
e_1 = \frac{\omega_T \Omega}{S_T}, \quad e_2 = \frac{4err_{1,0} R_X R_Y a}{S_T} \sum_{t=1}^{k} \omega_{t} \lambda^t, \quad e_3 = \frac{4R_X R_Y \Omega a}{S_T} \sum_{t=1}^{k} \omega_{t} \lambda_\ell \left( \sum_{t=0}^{\ell-1} \left( \frac{1}{T_t} + \frac{1}{T_{t-1}} \right) \right),
\]

\[
e_4 = \frac{2err_{s,0} R_X R_Y \Omega a}{S_T} \sum_{t=1}^{k} \lambda^t, \quad e_5 = \frac{2R_X R_Y \Omega a}{S_T} \sum_{t=1}^{k} \lambda^t \left( \sum_{t=1}^{\ell-1} \frac{1}{T_t} \right) \left( \sum_{t=1}^{\tau_{t+1}} \omega_t \right).
\]

The term \( e_1 \) is the upper bound to that would obtain for a static zero-sum game as in Proposition 3.1, i.e. if we had known the matrix \( K^* = K[v^*] \) from the start. The \( e_2 \) term comes from updating the value vector \( v^\ell \) to \( v^{\ell+1} \) at the end of the epoch \( \ell \), while the \( e_3 \) term comes from \( v^{\ell+1} \) being only an \( O(1/T_\ell) \) approximation of \( F(v^\ell) \). The \( e_4 \) and \( e_5 \) terms are related to the error between \( v^\ell \) and \( v^* \).

We emphasize the importance of warm-starting for Algorithm 1. The proof of Proposition 3.1 involves a telescopic sum by summing up (3.2). Crucially, by warm-starting the PDA algorithm at VI epoch \( \ell + 1 \) using the last iterate of the PDA algorithm at VI epoch \( \ell \), we are able maintain a telescopic sum from \( t = 0 \) to \( T = T_1 + \ldots + T_k \). If we were not using warm-starts, we would end up with \( k \) independent telescopic sums (one per VI epoch). This would give an \( e_1 \) term of \( \left( \sum_{t=1}^{k} \omega_{t+T_t} \right) \Omega/S_T \) which is significantly worse than \( \omega_T \Omega/S_T \), the \( e_1 \) term of Proposition 4.2 (and of Proposition 3.1).

**Remark 4.3.** Similar convergence results as Proposition 4.2 could be obtained using the Mirror Descent or Mirror Prox algorithms, instead of PDA, to compute approximate solutions to \( F(v) \). In particular, the key to Proposition 4.2 is
We consider a polynomial weight-epoch scheme. We present the proof of the next proposition in Appendix D. For the
Moreover, (Theorem 5.3.4, Ben-Tal and Nemirovski [2001]) and Mirror Prox (Theorem 5.6.1, Ben-Tal and Nemirovski [2001]).

Proposition 5.1.

5 Complexity analysis of Algorithm [1]

In this section we present some tractable proximal setups for Algorithm [1]. In particular, we present two results of
independent interests, namely how to compute proximal updates (4.1) and (4.2), both for the
ellipsoidal and KL uncertainty sets and all the proximal setups (2.5). Finally, in Section 5.2 we present the complexity of Algorithm [1] for the
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4.3 Convergence rate

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Proposition 4.4. Let \( p, q \in \mathbb{N} \). At time step \( t \geq 0 \), let \( \omega_t = t^p \), \( T_t = \ell^q \). Then \( T_t = \ell^{q+1}, S_T = k^{q+1}(q+1) \).

Moreover, \( e_1 = O\left(\frac{1}{T}\right), e_2 = O\left(\frac{\lambda^{T^{1/(q+1)}}}{T}\right), e_3 = O\left(\frac{1}{T^{2/(q+1)}}\right), e_4 = O\left(\frac{\lambda^{T^{1/(q+1)}}}{T^{1/(q+1)}}\right), \)

\( e_5 = O\left(\frac{1}{T^{(q+1)}}\right) \).

\( e_5 \) is the term with the slowest theoretical convergence rate. Note that the theoretical convergence rate does not depend of \( p \); however, we notice in our simulations (Section 6) that increasing the weights achieves faster convergence than uniform weights, see Figure 5-6. Similarly, the theoretical convergence is improving for the \( e_3 \) and \( e_4 \) terms for larger \( q \); however, this is to the detriment of the exponent of the discount factor \( \lambda \) in the terms \( e_2 \) and \( e_5 \). We present our detailed numerical experiments in Section 6.

5 Complexity analysis of Algorithm [1]

In this section we present some tractable proximal setups for Algorithm [1]. In particular, we present two results of
independent interests, namely how to compute proximal updates (4.1) and (4.2), both for the \( \ell_1 \) and the \( \ell_2 \) setup, for both the KL uncertainty set and the ellipsoidal uncertainty set. For the sake of conciseness, we present our results for KL uncertainty sets in Appendix E.3. Finally, in Section 5.2 we present the complexity of Algorithm [1] for the
ellipsoidal and KL uncertainty sets and all the proximal setups (\( \ell_1, \ell_2 \)) considered.

5.1 Tractable setups

Tractable updates for min player (4.1). The proximal updates for the \( x \)-player are the classical proximal updates for the simplex (Ben-Tal and Nemirovski [2001]). For the \( \ell_2 \) setup, the proximal update (4.1) boils down to computing the Euclidean projection of a vector onto the simplex of dimension \( A \), and for the \( \ell_1 \) setup, it boils down to optimizing the sum of a linear form and a KL divergence onto the simplex. We present a summary in Table [1].

Tractable updates for max player - (4.2). We now show details about the proximal updates for the ellipsoidal uncertainty set. We present our results for KL uncertainty set in Appendix E.3. We consider an \( \ell_2 \)-based and an \( \ell_1 \)-based setup for the \( y \)-player in Proposition 5.1. We present a detailed proof in Appendix E.

Proposition 5.1.

1. (\( \ell_2 \) setup) Let \( \| \cdot \|_Y = \| \cdot \|_2, \psi_Y = (1/2)\| \cdot \|_2^2, D_Y(y, y') = (1/2)\| y - y' \|_2^2, \forall (y, y') \in \Delta(S) \times \Delta(S) \).

The proximal update (4.2) with uncertainty set (2.5) can be approximated up to \( \epsilon \) in a number of arithmetic operations of \( O\left(AS \log(S) \log(\epsilon^{-1})\right) \).

2. (\( \ell_1 \) setup) Let \( \| \cdot \|_Y = \| \cdot \|_1, \psi_Y (y) = (A/2) \sum_{a=1}^A \text{ENTROPY}(y_a), D_Y(y, y') = (A/2) \sum_{a=1}^A \text{KL}(y_a, y'_a) \).

where

\[
\text{ENTROPY}(y) = \sum_{s'=1}^S y_{s'} \log(y_{s'}), \quad \text{KL}(y, y') = \sum_{s'=1}^S y_{s'} \log\left(\frac{y_{s'}}{y'_{s'}}\right), \forall (y, y') \in \Delta(S) \times \Delta(S).
\]

The proximal update (4.2) with uncertainty set (2.5) can be approximated up to \( \epsilon \) in a number of arithmetic operations of \( O\left(AS \log^2(\epsilon^{-1})\right) \).
The following table summarizes the previous discussion.

Table 1: Proximal setups considered in this paper and the associated complexity. We call $R$ the maximum of the considered norm on the considered set $Z$: $R = \max_{z \in Z} \|z\|_Z$. We call $\Theta$ the maximum of the Bregman divergence $D$ on the considered set $Z$: $\Theta = \max_{z, z' \in Z} D(z, z')$. The complexity of computing the proximal update up to $\epsilon$ is $C_\epsilon$.

| Set $\mathcal{X} = \Delta(A)$ | Norm $\|\cdot\|_Z = \|\cdot\|_1$ | $\psi$ | $R$ | $\Theta$ | $C_\epsilon$ |
|--------------------------------|----------------|--------|----|---------|-----------|
| $X = \Delta(A)$              | $\|\cdot\|_X = \|\cdot\|_2$ | ENTROPY | $O(1)$ | $O(\log(A))$ | $O(A)$ |
| $Y = (\Delta(S))^A \cap B_2(y_0, \alpha)$ | $\|\cdot\|_Y = \|\cdot\|_2$ | SUM-ENTROPY | $O(A)$ | $O(A^2 \log(S))$ | $O(\alpha S \log^2(1/\alpha))$ |

5.2 Summary of proximal setups considered

In this section we present the convergence rate of our algorithm for the different tractable proximal setups of the previous section. The following table summarizes our result. We present details about the computation in Appendix F.

Table 2: Proximal setups considered in this paper and the associated complexity. $e_1, \ldots, e_5$ refer to the terms defined in Proposition 4.2. Here, ‘$\epsilon$-complexity’ refers to the number of arithmetic operations to obtain an $\epsilon$-solution to (1.2). For the sake of clarity, we omit the $O(\cdot)$ notation and any constants that do not depend of $S, A$ and $T$.

| Setup | $e_1$ | $e_2$ | $e_3$ | $e_4$ | $e_5$ | $\epsilon$-Complexity ($q = 2$) |
|-------|-------|-------|-------|-------|-------|-------------------------------|
| $\ell_1$ Setup | $\frac{A^2 \log(S)}{T}$ | $\frac{A X^{1/(q+1)}}{T}$ | $\frac{A^3 \log(S)}{T^{2q/(q+1)}}$ | $\frac{A^3 \log(S)}{T^{(q+1)/q/(q+1)}}$ | $\frac{A^3 \log(S)}{T^{q/(q+1)}}$ | $A^{5.5} S^2 \log(S)^{1.5} \log^2(\epsilon^{-1}) \epsilon^{-1.5}$ |
| $\ell_2$ Setup | $\sqrt{AS} \frac{T}{T}$ | $\sqrt{AS} \frac{T^{1/(q+1)}}{T}$ | $\frac{AS}{T^{2q/(q+1)}}$ | $\frac{AS A^{1/(q+1)}}{T^{1/(q+1)}}$ | $\frac{AS}{T^{q/(q+1)}}$ | $A^{2.5} S^{1.5} \log(S) \log(\epsilon^{-1}) \epsilon^{-1.5}$ |

As can be seen in Table 2, for $s$-rectangular ellipsoidal uncertainty set our algorithm (for $q = 2$) has better theoretical guarantees than [VI] in terms of dependence on the number of states, both for the $\ell_1$ and the $\ell_2$ setup. However, the complexity in terms of the desired accuracy $\epsilon$ is worse than [VI] as expected since we use a First-Order Method to compute approximation of $F(v)$. Note that $S = A$, then the $\ell_2$ setup has better dependence on number of states and actions than both the $\ell_1$ setup and [VI]. If the number of actions $A$ is considered a constant, then the $\ell_1$ has better convergence guarantees than the $\ell_2$ setup. However, each proximal update in the $\ell_1$ setup requires two interwoven binary searches over Lagrange multipliers, which can prove time-consuming in practice, as we show in our numerical experiments in Section C.

Remark 5.2 (Large $q$). In the limit as $q \to +\infty$, we have $q/(q+1) \to 1$. In that case the $\ell_1$ setup has $\epsilon$-complexity approaching $O(A^4 S^2 \log(S) \log^2(\epsilon^{-1}) \epsilon^{-1})$, while the $\ell_2$ setup has $\epsilon$-complexity approaching $O(A^2 S^2 \log(S) \log(\epsilon^{-1}) \epsilon^{-1})$. This last complexity results is significantly better than [6] in terms of number of states and number of actions. Additionally, we would like to emphasize that while for $q = 2$ the worst-case complexity of our algorithm is better than the complexity of [VI] but not substantially better, we still expect our algorithm to outperform [VI] (which is the case for as soon as $S, A \geq 70$, see Section 6). In particular, IPM method may spend a large amount of time computing the inverse of matrices; meanwhile our algorithm is making progress by computing cheaper proximal updates.

Remark 5.3 (Spatial complexity). We can store a running weighted average of the iterates in order to reduce the spatial complexity of Algorithm 1. At VI epoch $\ell$ we only need to store the current value vector $\mathbf{v}^\ell \in \mathbb{R}^S$, and the running weighted average of all the iterates visited $(\mathbf{v}^\ell, \mathbf{g}^\ell) \in \Pi \times P \subset \mathbb{R}^{A \times S} \times \mathbb{R}^{S \times A \times S}$. In total, we need to store $O(S^2 A)$ coefficients. Thus the spatial complexity of our approach is the same as the number of decision variables of a solution.

Remark 5.4. In Distributionally Robust MDP with Wasserstein distance (based on an $\ell_2$ metric) from a nominal density with finite support, the Bellman update (Yang [2017], equation (9)) is very similar to the Bellman update in our
Our FOM-based algorithmic framework can be extended to solve this Distributionally Robust MDP with complexity similar to our ellipsoidal setting, see Appendix E for more details.

While our main focus is ellipsoidal uncertainty, we give a brief overview of our results for KL uncertainty below:

**Remark 5.5** (Complexity for KL uncertainty set). For this class of uncertainty sets, the proximal updates (4.2) can be computed efficiently (see Proposition E.4 in Appendix E). In that case, for $q = 2$, the $\ell_1$ setup has complexity $O\left(A^{2.5}S^2\log(S)^{1.5}\log(\epsilon^{-1})\epsilon^{-1.5}\right)$ for computing an $\epsilon$-optimal pair, while the $\ell_2$ setup has complexity $O\left(A^{2.5}S^3\log(\epsilon^{-1})\epsilon^{-1.5}\right)$. For $q \to +\infty$, we obtain complexity $O\left(A^4S^2\log(S)\log(\epsilon^{-1})\epsilon^{-1}\right)$ for the $\ell_1$ setup and $O\left(A^2S^3\log(\epsilon^{-1})\epsilon^{-1}\right)$ for the $\ell_2$ setup. We present the details of our analysis in Appendix E. In the next section, we choose to present our numerical experiments on the ellipsoidal uncertainty set (2.5) as we can compare Algorithm 1 with the Value Iteration algorithm of Wiesemann et al. [2013], which specifically focus on ellipsoidal uncertainty sets.

6 Numerical experiments

In this section we study the performance of our approach numerically. We focus on ellipsoidal uncertainty sets as we can compare to the algorithms in Wiesemann et al. [2013]. In order to do this, we need to measure the performance of a candidate solution pair $(x, y) \in \Pi \times \mathcal{P}$. We propose two alternative performance measures for the pair $(x, y)$. Ideally we would like to compute the duality gap from (1.2):

$$\max_{y' \in \mathcal{P}} R(x, y') - \min_{x' \in \Pi} R(x', y).$$

(DG)

However, computing $\max_{y' \in \mathcal{P}} R(x, y')$ is itself a heavy computational task which requires iteratively solving convex quadratic programs (which is what we were trying to avoid with Algorithm 1). In Appendix C we investigate the running time to compute this gap and show that it becomes intractable for more than about 100 states and actions. In our experiments, we show this measure when feasible, i.e. for small to medium size instances (Figure 12, Figure 3, Figure 5 and Figure 7).

**Duality gap in (2.3).** A second optimality measure is to consider how close $(x, y)$ is to being a solution to the min-max game at $v^*$ defined in (2.3) (we know that a solution is optimal if and only if it is a solution to this problem):

$$\epsilon^*(x, y) = \max_{v \in \mathbb{R}^S} \{\max_{y' \in \mathcal{P}} F^x y'(v)_{s} - \min_{x' \in \Pi} F^{x'} y(v)_{s}\}.\] However, we do not know $v^*$, so we can not compute $\epsilon^*(x, y)$. Instead, we prove the following upper bound in Appendix C.

**Lemma 6.1.** For any $v \in \mathbb{R}^S$, the residual $\epsilon^*(x, y)$ can be bounded by

$$\frac{2\lambda}{1 - \lambda} ||v - F(v)||_\infty + \max_{s \in \mathbb{R}^S} \{\max_{y' \in \mathcal{P}} F^{x'} y'(v)_{s} - \min_{x' \in \Pi} F^{x'} y(v)_{s}\}.\]$$

(UB-1)

We are free to pick any $v$ in (UB-1); in our experiments we use $v^\ell$ at each epoch $\ell$. (UB-1) is significantly faster to compute than (DG) (see Appendix C for more details). The tradeoff is that (UB-1) is a significantly looser bound. That said, it still converges to 0, and the condition (UB-1) $\leq \epsilon$ is enough to ensure that the policy $x$ considered is an $\epsilon$-optimal policy to the robust MDP problem (1.2). Moreover, (DG) $\leq \epsilon$ is enough to ensure that the policy considered is a $2\epsilon$-optimal solution.

6.1 Empirical performances of Algorithm 1

We start by studying various setups for our approach, in order to identify the best one, which we will then compare to other VI approaches.

**Empirical setup.** All the simulations are implemented in Python 3.7.3, and were performed on a laptop with 2.2 GHz Intel Core i7 and 8 GB of RAM. We use Gurobi 8.1.1 to solve any linear or quadratic optimization problems involved. We generate Garnet MDPs (Archibald et al. [1995]), which are an abstract class of MDPs parametrized by a branching factor $n_{\text{branch}}$, equal to the number of reachble next states from each state-action pair $(s, a)$. We consider $n_{\text{branch}} = 0.5$ in our simulations. We draw the rewards parameters at random uniformly in $[0, 10]$. We fix a discount factor $\gamma = 0.8$. The radius $\alpha$ of the $\ell_2$ ball from the uncertainty set (2.5) is set to $\alpha = \sqrt{n_{\text{branch}} \times A}$. All of the figures in this section show the logarithm of the performance measures (DG) or (UB-1) in terms of the number of VI iterations performed in Algorithm PDA. Apart from Figures 12, these performance measures are averaged across 10 randomly generated Garnet MDPs.
Impact of proximal setup. We fix $S, A = 30$ and we present in Figure 1-2 the Duality Gap (DG) of the current weighted average of the iterates of our algorithm, for three different proximal setups $\left(\|\cdot\|_X, \|\cdot\|_Y\right) \in \{\left(\ell_1, \ell_1\right), \left(\ell_1, \ell_2\right), \left(\ell_2, \ell_2\right)\}$. The $\left(\ell_2, \ell_2\right)$ setup performs the best, even though its theoretical guarantees are worse than the $\left(\ell_1, \ell_1\right)$ setup (as seen in Table 2). This disparity between theory and practice is analogous to the case of stationary bilinear min-max problems [Gao et al., 2019]. In the rest of the simulations we focus on the $\left(\ell_2, \ell_2\right)$ setup. Note that Figure 1-2 shows performance for a single instance. This is because the $\left(\ell_1, \ell_1\right)$ setup takes almost a day to run on a single instance of size $(S, A) = (30, 30)$ (compared to minutes for the $\left(\ell_2, \ell_2\right)$ setup), most likely because of the two interwoven binary searches (see also Appendix E).

Impact of epoch scheme. We now investigate the impact of the epoch length $T_q = \ell_q$, parametrized by $q \in \mathbb{N}$. We fix $(S, A) = (30, 30)$ and we focus on the $\left(\ell_2, \ell_2\right)$ setup. We fix the averaging scheme at $p = 1$ and we compare epoch lengths $q = 0, 1, 2$. The results are shown in Figures 3 and 4. For the performance measure (DG), we find that $q = 0, q = 1$ and $q = 2$ yield comparable convergence rates (in terms of number of PD iterations), with $q = 2$ being slightly better than $q = 0, q = 1$. On the other hand, $q = 0$ performs much better on our upper bound (UB-1) (note that our theory does not even guarantee convergence for $q = 0$). Note that for $q = 0$, our algorithm performs only one PD update at each epoch, before updating the value vector $v$, which has a cost of $O(AS^2)$. This may make $q = 0$ significantly slower in practice for large $S, A$, since the value vector updates have a negligible computational cost for $q > 0$ (compared to the numerous PD updates computational costs).

While the upper bound (UB-1) also converges to 0, the convergence rates for $q = 0, 1, 2$ are qualitatively very different than for the duality gap (DG). We emphasize that our goal is to minimize (DG); (UB-1) is a loose upper bound on $\epsilon^*$ that we only use because (DG) eventually becomes too expensive to compute.
Impact of weight scheme. We now investigate the impact of the weight scheme $\omega_t = t^p$ used to average iterates. We fix $(S, A) = (30, 30)$, $q = 2$ and use the $(\ell_2, \ell_2)$ setup. We compare $p = 0, 1, 2$. Figure 5 and 6 shows that increasing averages ($p = 1, p = 2$) perform better than uniform average ($p = 0$), even though our convergence guarantees are independent of $p$. Similar observations have been made in zero-sum games and other convex-concave saddle-point problems [Gao et al. 2019].

**Remark 6.2.** In our simulations we set $\lambda = 0.8$. Of course, our algorithm works for any $\lambda \in (0, 1)$. However, the performance guarantees of Algorithm 1 may degrade for $\lambda \rightarrow 1$, as some of the constants in the $O(\cdot)$ notations of Table 2 depend on $1/(1 - \lambda)$, a situation similar to the complexity of Value Iteration [VI]. Moreover, when $\lambda \rightarrow 1$, computing the duality gap $\text{DG}$ becomes very slow: computing the minimizer and maximizer requires iterating contraction mappings, each with improvement factor $\lambda$ (see Appendix G). Additionally, the upper bound $\text{UB-1}$ becomes looser. These last two limitations are not a particular shortcoming of our algorithm but are inherent to MDPs.

### 6.2 Comparison with Value Iteration

We present our comparisons with Algorithm VI in Figures 7-9. All figures present the runtime to obtain an $\epsilon$-optimal solution to (1.2) with $\epsilon = 0.1$. We restrict our comparison to efficient implementations of VI with warm-start and using Gurobi 8.1.1, as we are unaware of other methods for fast computation of the Bellman update when $P_s$ is an $\ell_2$ ball.

**Setup for Value Iteration.** We compare our methods to VI, as well as Accelerated Value Iteration (AVI, Goyal and Grand-Clement [2018]), an algorithm extending the acceleration scheme from convex optimization [Nesterov 1983, 2013] to the fixed-point iteration scheme of Value Iteration. In order to obtain an $\epsilon$-solution to (1.2) with VI and AVI, we use the stopping condition $\|v_{\ell+1} - v_{\ell}\|_{\infty} \leq \epsilon \cdot (1 - \lambda) \cdot (2\lambda)^{-1}$ (Chapter 6.3 in Puterman [1994]). We initialize the algorithms with $v_0 = 0$ and, for Accelerated Value Iteration, $v_1 = F(v_0)$. At epoch $\ell$ of VI and AVI, we warm-start each computation of $F(v^\ell)$ with the optimal solution obtained from the previous epoch $\ell - 1$. We present details about our implementation of VI and AVI in Appendix H.

**Performance measure.** For small to medium-size instances (Figure 7), we are able to compute $\text{DG}$ in less than 1800 second. However, we have seen in the previous section and in Appendix G that computing $\text{DG}$ is slow for larger instances. For large and very large instances, in Figure 8 and Figure 9 we only compute $\text{UB-1}$, our performance guarantee that is fastest to compute.

**Maximum running time.** For Figure 8 we set a maximum running time for the algorithms of $\text{time\_max} = 5$ hours. Note that $\text{time\_max}$ does not include the (significant) time spent computing $\text{UB-1}$ at the end of every epoch of Algorithm 1 VI and AVI, since this is only used for performance measurement. There is no $\text{time\_max}$ for Figure 7 since the instances have small to medium sizes ($S, A \leq 100$) we are able to compute $\text{DG}$ Additionally, there is no $\text{time\_max}$ for Figure 9 as we try to push our algorithm to its limit in terms of number of states/actions.

**Small and medium instances.** In Figure 7 we present our simulations for $S = A$ in $\{30, 50, 70, 80, 100\}$ (small and medium instances) using the true Duality Gap $\text{DG}$ as a performance measure in the robust MDP problem (1.2). For small instances (up to 50 states/actions), the performance of our algorithm for $(p, q) = (2, 2)$ is similar to both VI
and AVI, while other choices of \((p, q)\) perform worse. This is to be expected, as our algorithm has worse convergence guarantees in terms of the dependence in \(\epsilon\), but better guarantees in terms of the number of state-actions \(S, A\). When the number of states and actions grows larger, our algorithm with \((p, q) = (2, 2)\) clearly outperforms both VI and AVI. At \((S, A) = (100, 100)\), our algorithm outperforms VI for \((p, q) = (2, 2), (1, 1)\) and \((p, q) = (1, 0.5)\), while AVI is outperformed by our algorithm with both \((p, q) = (2, 2)\) and \((1, 1)\). The runtime trend for \((p, q) = (2, 2)\) looks significantly better than every other method.

**Medium and large instances.** For our larger instances we resort to \((\text{UB-1})\) as a performance measure. Figure 8 shows that our algorithm (with \((p, q) = (1, 1)\) and \((p, q) = (1, 0.5)\)) performs similar to VI and AVI for small instances (up to 70 states/actions). For larger instances our algorithm with the setting \((p, q) = (1, 1)\) or \((p, q) = (1, 0.5)\), outperforms both VI and AVI. In particular, VI times out for \(S, A \geq 200\). For AVI and Algorithm 1 with \((p, q) = (2, 2)\) the timeout happens at \(S, A \geq 250\). For \((p, q) = (1, 1)\) and \((p, q) = (1, 0.5)\) in Algorithm 1, we solve instances up to \(S, A = 300\). Note that following Figure 7 (which shows the true Duality Gap \((\text{DG})\)) and Figure 4, we know that \((\text{UB-1})\) is less tight for \((p, q) = (2, 2)\) than for \((p, q) = (1, 1)\) and \((p, q) = (1, 0.5)\); this may explain the relatively poor performance of \((p, q) = (2, 2)\) in Figure 8 (which uses \((\text{UB-1})\)) compared to its good performance in Figure 7 (which uses \((\text{DG})\)).

**Very large instances.** Since VI, AVI and our algorithm with \((p, q) = (2, 2)\) takes longer than \(\text{time\_max}\) seconds for \(S, A \geq 250\), in Figure 9, we only focus on the performances of our algorithm for \((p, q) = (1, 1)\) and \((p, q) = (1, 0.5)\). Additionally, computing the upper bound \((\text{UB-1})\) takes at least 2 hours for each epoch, for \(S, A \geq 250\). While this time is not counted as actual running time of our algorithm, \((\text{UB-1})\) would still have to be computed at the end of every epoch \(\ell\) of our algorithm. In order to test our algorithms for even larger instances, we change our performance measure for Figure 9. In particular, we consider the following performance measure \((\text{PM})\)

\[
\frac{1}{1 - \lambda} \|v^\ell - F^{x^\ell, y^\ell}(v^\ell)\|_\infty
\]

where \(x, y\) are the current running average of our algorithm at epoch \(\ell\). Note that \((\text{PM})\) relies on both the performance of \((x, y)\) to approximate \(F(v)\) by \(F^{x, y}(v)\), and the intrinsic performance of \(v\) as a potential fixed point of \(F\), measured in term of \(\|\cdot\|_\infty\). Computing \((\text{PM})\) only involves matrix multiplication and component-wise maximum, and is therefore significantly faster to compute than \((\text{DG})\) and \((\text{UB-1})\). Figure 9 shows the running time to achieve \((\text{PM}) \leq \epsilon\). We notice that the setting \((p, q) = (1, 0.5)\) becomes almost twice as fast as the setting \((p, q) = (1, 1)\) for very large instances \((S, A \geq 600)\).

**Remark 6.3.** It should be noted that in the above timing comparisons, we are affording VI and AVI a heavily optimized C implementation (Gurobi) for solving the optimization problem at each epoch. In contrast, our FOM is implemented in python, which is substantially slower. Thus, one may expect even better performance with an optimized implementation of Algorithm 1.

6.3 Numerical take-away and discussion

As we have presented various norms and parameter settings for our algorithm in the previous sections, we present here some conclusions and limitations of our numerical experiments.

**Proximal setup (Figure 12).** The proximal setup with the best empirical performances is the \(\ell_2\) setup where \((\|\cdot\|_X, \|\cdot\|_Y) = (\ell_2, \ell_2)\), even though its theoretical guarantees may be worse than the \(\ell_1\) setup (for large state space); this is similar to the matrix-game setting in Cao et al. [2019].
Parameter tuning (Figure 3-6). For averaging the PD iterates, an increasing weight scheme, i.e. \( p \geq 1 \) in \( \omega_t = t^p \), is clearly stronger (this is again similar to the matrix-game setting). We also recommend setting \( q = 2 \) (or even larger), as this leads to better empirical performance for the true duality gap (DG) in the settings where we could compute that duality gap.

Comparison with VI (Figure 7-9). While performing similarly to VI and AVI for small instances \( (S, A \leq 50) \), our algorithm significantly outperforms both VI and AVI when the state and action sets grow larger. We reiterate that we believe that \( q = 2 \) is the best choice for our algorithm; \( q = 2 \) is performing worse than \( q = 1 \) and \( q = 0.5 \) in Figures 8-9 because we are using the performance measures (UB-1) and (PM), which seem to be less tight than (DG) for \( q = 2 \).

7 Conclusion

We propose a novel first-order algorithmic framework for solving robust MDPs. Our algorithm is based on adapting first-order methods for solving static convex-concave min-max problems to a dynamically changing infinite sequence of zero-sum games. In the case of ellipsoidal and KL uncertainty sets, our algorithm has a theoretical convergence rate which improves upon the Value Iteration algorithm in terms of the dependence on the number of states and actions. In the case of ellipsoidal uncertainty sets, our numerical experiments highlight the significant speedup of our algorithm compared to classical methods, when the instance size becomes large. To the best of our knowledge, our algorithm is the only one able to solve \( s \)-rectangular robust MDPs to optimality when the state and actions spaces become on the order of several hundreds, and when the uncertainty set is either described by an ellipsoidal or a KL constraint. Several interesting questions arise from this work. Our algorithmic framework relies on some choice of uncertainty sets, which leads to tractable proximal setups, and is dependent upon some weights and epochs schemes. In particular, our algorithm can benefit from a better understanding of the choice of the epoch scheme (parametrized by \( q \in \mathbb{N} \)). While we focus on fixed epoch schemes, one could imagine varying the parameter \( q \) as the algorithm makes progress. Additionally, it would be interesting to extend other setups for solving stationary zero-sum games (e.g. regret minimization as in CFR [Zinkevich et al. [2008]]) to solving robust MDPs. Also, even though the approximate Value Iteration scheme considered in Section 2 can be extended to incorporate sampling errors, another interesting question includes a model-free approach, where the decision-maker does not start with a nominal estimation of the transition kernel but is only able to sample from it at every step of the algorithm. Incorporating variations of value iteration (e.g. Anderson Acceleration or Accelerated Value Iteration) in our first-order framework is a promising next step. Finally, first-order methods may be used to improve the convergence of Distributionally Robust MDP algorithms.

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A Proofs of Section 2

A.1 Some useful lemmas

The next lemmas give bounds on some sums that appear in the proof of Proposition 4.4.

Lemma A.1. Let $\lambda \in (0, 1)$ and $k, n \in \mathbb{N}$. Then

$$
\sum_{\ell=1}^{k} \lambda^{\ell} \ell^{n} \leq O\left(\frac{k^{n} \lambda^{k}}{(1 - \lambda)^{n+1}}\right).
$$

Proof of Lemma A.1. Let us define $f: x \mapsto \frac{1 - x^k}{1 - x} = \sum_{\ell=1}^{k} x^{\ell}$. Then $f^{(n)}(x) = \sum_{\ell=1}^{k} x^{\ell}(\ell - 1)\cdots(\ell - n + 1)$, and

$$
\sum_{\ell=1}^{k} \lambda^{\ell} \ell^{n} = O\left(\sum_{\ell=1}^{k} x^{\ell}(\ell - 1)\cdots(\ell - n + 1)\right).
$$

We can conclude by computing the $n$-th derivative of $f$ as the $n$-th derivative of $x \mapsto \frac{1 - x^k}{1 - x}$.

Lemma A.2. Let $\lambda \in (0, 1)$ and $q \geq 0$. Then there exists a constant $M_{\lambda, q}$ such that

$$
\sum_{t=1}^{\ell} \frac{1}{t^q \lambda^t} \leq M_{\lambda, q} \frac{1}{\ell^q \lambda^\ell}.
$$

Proof of Lemma A.2. Let $\lambda^+ = \frac{1 + \lambda}{2}$. Note that we always have $\lambda < \lambda^+ < 1$. For $x = 1/\lambda$, we have $x > x^+ = 1/\lambda^+ > 1$. For $f_\ell(x) = \sum_{t=1}^{\ell} \frac{x^{t}}{t^q}$, we have

$$
f_\ell'(x) = \sum_{t=1}^{\ell} \frac{x^{t-1}}{t^q - 1} 
\leq \ell^{1-q} \sum_{t=1}^{\ell} x^{t-1} 
\leq \ell^{1-q} \frac{x^\ell - 1}{x - 1}.
$$

This proves that

$$
f_\ell(x) - f_\ell(x^+) = \ell^{1-q} \int_{u=x^+}^{x} \frac{u^\ell - 1}{u - 1} du 
\leq \ell^{1-q} \frac{1}{x^+ - 1} \int_{u=x^+}^{x} (u^\ell - 1) du 
\leq \ell^{1-q} \frac{1}{x^+ - 1} \frac{1}{\ell + 1} [u^{\ell+1} - u]_{x^+},
$$

and finally that

$$
f_\ell(x) = f_\ell(x^+) + \ell^{1-q} \frac{1}{x^+ - 1} \frac{1}{\ell + 1} (x^{\ell+1} - x - x^+ \ell+1 + x^+). \quad (A.1)
$$

We will prove that the right-hand side of (A.1) is itself a $O\left(\frac{x^{\ell}}{\ell^q}\right)$ as $\ell \to +\infty$. The proof relies on the fact that $x > x^+$, and therefore that $\left(\frac{x^+}{x}\right)^{\ell} \ell^m = o(1)$, for any $m \geq 0$.\]
First, since $x^+ < x$, we note that

$$\frac{\ell^q}{x^t} \left( \frac{\ell^{1-q}}{x^t - 1} \frac{1}{\ell + 1} \left( x^{\ell + 1} - x - x^+ \ell + 1 + x^+ \right) \right) = \frac{1}{x^+ - 1} \frac{\ell}{\ell + 1} (x - x^+) \left( x^+ \right)^\ell + x^+ = O(1).$$

Now for $\frac{\ell^q}{x^t} f_\ell(x^+)$ we need to distinguish between the potential values of $q \in \mathbb{R}_+$. 

**Proof for** $q = 0$.

$$\frac{\ell^q}{x^t} f_\ell(x^+) = \frac{1}{x^t} \sum_{t=1}^\ell x^+ t = O \left( \frac{x^+}{x^t} \right) = o(1).$$

**Proof for** $q \in (0, 1)$.

$$\frac{\ell^q}{x^t} f_\ell(x^+) = \frac{\ell^q}{x^t} \sum_{t=1}^\ell x^+ t \leq \frac{\ell^q}{x^t} \sum_{t=1}^\ell x^+ t \leq O \left( \frac{\ell x^+}{x^t} \right) = o(1).$$

**Proof for** $q = 1$.

$$\frac{\ell^q}{x^t} f_\ell(x^+) = \frac{\ell}{x^t} \sum_{t=1}^\ell x^+ t \leq \frac{\ell}{x^t} x^+ \log(\ell) = o(1).$$

**Proof for** $q \geq 1$.

$$\frac{\ell^q}{x^t} f_\ell(x^+) = \frac{\ell^q}{x^t} \sum_{t=1}^\ell x^+ t \leq \frac{\ell^q}{x^t} x^+ t \sum_{t=1}^\ell \frac{1}{t^q} \leq O \left( \frac{\ell q x^+}{x^t} \right) = o(1).$$

**A.2 Proofs of Section 2**

**Proof of Proposition 2.7.** We have

$$\|v^* - v^{t+1}\|_\infty = \|F(v^*) - v^{t+1}\|_\infty$$

$$= \|F(v^*) - F(v^t) + F(v^t) - v^{t+1}\|_\infty$$

$$\leq \|F(v^*) - F(v^t)\|_\infty + \|F(v^t) - v^{t+1}\|_\infty$$

$$\leq \lambda \|v^* - v^t\|_\infty + \|F(v^t) - v^{t+1}\|_\infty$$

$$\leq \lambda \|v^* - v^t\|_\infty + \epsilon_t.$$

Similarly,

$$\|v^{t+1} - v^t\|_\infty \leq \|v^{t+1} - F(v^t) + F(v^t) - v^t\|_\infty$$

$$\leq \|v^{t+1} - F(v^t)\|_\infty + \|F(v^t) - v^t\|_\infty$$

$$\leq \epsilon_t + \|F(v^t) - v^t\|_\infty$$

$$\leq \epsilon_t + \|F(v^t) - F(v^{t-1})\|_\infty + \|F(v^{t-1}) - v^t\|_\infty$$

$$\leq \epsilon_t + \lambda \|v^t - v^{t-1}\|_\infty + \|F(v^{t-1}) - v^t\|_\infty$$

$$\leq \epsilon_t + \lambda \|v^t - v^{t-1}\|_\infty + \epsilon_{t-1}.$$

The rest of the lemma follows directly from iterating the recursions on $\|v^* - v^{t+1}\|_\infty$ and on $\|v^{t+1} - v^t\|_\infty$. 

**B Proof of Lemma 4.1**

Recall that $K : \mathbb{R}^A \rightarrow \mathbb{R}^{A \times S}$ is defined as

$$(Kx, y) = \lambda \sum_{a=1}^A x_a y_a^\top v = \lambda \sum_{a=1}^A \sum_{s' = 1}^S x_a y_{as'} v_{s'}, \forall (x, y) \in \mathbb{R}^A \times \mathbb{R}^{A \times S}.$$
Therefore, by definition of matrix-vector product,
\[(K^\top K)x_a = \sum_{a''} (K^\top)_{a,a''} (Kx)_{a''} = \sum_{a''} (K^\top)_{a,a''} \lambda x_a v_{a''} = \sum_{a''} 1\{a=a''\} \lambda x_a v_{a''} = \lambda^2 \left( \sum_{a''} v_{a''}^2 \right) x_a = \lambda^2 \|v\|_2^2 x_a.
\]

This directly implies that \(L_K = \lambda \|v\|_2\).

2. \(\ell_1\) setup. We can use the definition \(\sup\|x\|_x \leq 1, \|y\|_y \leq 1 \langle Kx, y \rangle\), as well as the fact that \(v \geq 0\).

### C Proof of Proposition 4.2

**Upper bound \(\Omega\).** From Remark 2 of Chambolle and Pock [2016], a possible choice for \(\Omega_\ell\) of Proposition 3.1 with matrix \(K^\ell\) is
\[\Omega_\ell \leq 2 \left( \Theta_X \frac{\tau_\ell}{\tau_\ell} + \Theta_Y \frac{\sigma_\ell}{\sigma_\ell} \right).\]

For the \(\ell_1\) setup, following the condition (3.3) and Lemma 4.1 at epoch \(\ell \in \{1, \ldots, k\}\) we can choose \(\sigma_\ell = \tau_\ell \leq \frac{1}{\lambda \|v^\ell\|_\infty}\). Note that at epoch \(\ell\), by construction, the vector \(v^\ell\) corresponds to the reward obtained after \(\ell\) periods by the sequence \((\bar{x}^{\tau_1}, \bar{y}^{\tau_1}, \ldots, \bar{x}^0, \bar{y}^0)\). This implies that \(\|v^\ell\|_\infty \leq \frac{r_\infty}{1 - \lambda} (1 - \lambda^{\ell + 1}) \leq \frac{r_\infty}{1 - \lambda}\), where \(r_\infty = \max_{s,a} \epsilon_{sa}\).

Therefore in the \(\ell_1\) setup a uniform upper bound \(\Omega\) on each \(\Omega_\ell, \ell = 1, \ldots, k\) can be found as \(\Omega = \frac{2\lambda r_\infty}{1 - \lambda} (\Theta_X + \Theta_Y)\).

For the \(\ell_2\) setup, following the condition (3.3) and Lemma 4.1 at epoch \(\ell \in \{1, \ldots, k\}\) we can choose \(\sigma_\ell = \tau_\ell \leq \frac{1}{\lambda \|v^\ell\|_2}\). But from the equivalence of \(\ell_2\) and \(\ell_1\) we have \(\|x\|_\infty \leq \sqrt{S} \|x\|_2\) for any vector \(x \in \mathbb{R}^S\). Therefore, in the \(\ell_2\) setup we can choose \(\sigma_\ell = \tau_\ell\) such that \(\sigma_\ell = \tau_\ell \leq \frac{1}{\lambda \sqrt{S} \|v^\ell\|_\infty}\) and we can choose \(\Omega = \frac{2\lambda r_\infty \sqrt{S}}{1 - \lambda} (\Theta_X + \Theta_Y)\).

**Proof of Proposition 4.2** Let us focus on the error for \(s \in S\). Then
\[
\mathcal{L}^{K^\ell}(\bar{x}^T, y) - \mathcal{L}^{K^\ell}(x, y^T) \leq \frac{1}{S_T} \left( \sum_{t=1}^k \sum_{t=t_{\tau_{\ell}}}^{k_{t+T_{\ell}}} \omega_t (\mathcal{L}^{K^\ell}(x^t, y) - \mathcal{L}^{K^\ell}(x, y^t)) \right) \\
\leq \frac{1}{S_T} \left( \sum_{t=1}^k \sum_{t=t_{\tau_{\ell}}}^{k_{t+T_{\ell}}} \omega_t (\mathcal{L}^{K^\ell}(x^t, y) - \mathcal{L}^{K^\ell}(x, y^t)) \right) + \frac{1}{S_T} \left( \sum_{t=1}^k \sum_{t=t_{\tau_{\ell}}}^{k_{t+T_{\ell}}} \omega_t (\mathcal{L}^{K^\ell}(x^t, y) - \mathcal{L}^{K^\ell}(x, y^t)) \right).
\]
Bounds on (C.1). Let us first focus on the term at (C.1). By applying Lemma [3.1] we obtain
\[
\frac{1}{S_T} \left( \sum_{\ell=1}^{k} \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} \omega_t (L^{K^\ell}(x^t, y) - L^{K^\ell}(x, y^t)) \right) = \frac{1}{S_T} \left( \sum_{\ell=1}^{k} \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} (\omega_{t+1} - \omega_t) (A^{K^\ell}(x, y, x^t, y^t)) \right) 
\]
\[
\leq \frac{1}{S_T} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \tau_{\ell} \Omega + \frac{1}{S_T} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \|

L^{K^\ell - K^{\ell-1}}(x - x^t, y - y^t) \| \tag{C.3}
\]
\[
\leq \frac{1}{S_T} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \|K^\ell - K^{\ell-1}\|
\]
\leq \frac{1}{S_T} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \|v^\ell - v^{\ell-1}\|_{\infty} \tag{C.4}
\]
\[
\leq \frac{1}{S_T} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \left( \sum_{t=0}^{\ell-1} \left( \frac{\Omega_t}{T_t} + \frac{\Omega_{t-1}}{T_{t-1}} \right) \frac{1}{\lambda^t} \right) \tag{C.5}
\]
\leq e_1 + e_2 + e_3, \tag{C.6}
\]
where \(\Omega\) is a uniform bound on \(A^{K^\ell}\), where (C.3) follows from telescoping, (C.4) follows from Lemma [4.1]. Inequality (C.5) follows from Proposition [2.1] and \(e_\ell = O(\Omega_t/T_t)\) in Proposition [3.1] and
\[
e_1 = \frac{1}{S_T} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \tau_{\ell} \Omega + \frac{1}{S_T} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \|K^\ell - K^{\ell-1}\|
\]
\[
= \frac{4err_{1,0} R X Y a}{S_T} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \|v^\ell - v^{\ell-1}\|_{\infty} + \frac{4 R X Y a}{S_T} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \left( \sum_{t=0}^{\ell-1} \left( \frac{\Omega_t}{T_t} + \frac{\Omega_{t-1}}{T_{t-1}} \right) \frac{1}{\lambda^t} \right).
\]

Bounds on (C.2). Note that
\[
\frac{1}{S_T} \left( \sum_{\ell=1}^{k} \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} \omega_t (L^{K^\ell - K^\ell}(x^t, y) - L^{K^\ell - K^\ell}(x, y^t)) \right) \leq \sum_{\ell=1}^{k} \|K^\ell - K^\ell\| \cdot \sum_{\ell=1}^{k} \omega_{\tau_{\ell}} \left( \sum_{t=0}^{\ell-1} \left( \frac{\Omega_t}{T_t} + \frac{\Omega_{t-1}}{T_{t-1}} \right) \frac{1}{\lambda^t} \right).
\]
Note that by definition of \(a\) in Proposition [4.2] and Lemma [4.1] we have \(\|K^\ell - K^\ell\| \leq a\|v^\ell - v^{\ell-1}\|_{\infty}\) and from Proposition [2.1] we have
\[
\|v^\ell - v^0\|_{\infty} \leq \lambda^\ell \|v^0 - v^*\|_{\infty} + \lambda^\ell \sum_{t=1}^{\ell-1} \frac{\Omega_t}{T_t \lambda^t}
\]
where the \(e_t\) is replaced by \(\Omega_t/T_t\), the precision attained after \(T_t\) steps of PDA with payoff matrix \(K^\ell\). This implies that we can have the following upper bound:
\[
\|K^\ell - K^\ell\| \leq a \lambda^\ell \|v^0 - v^*\|_{\infty} + a \lambda^\ell \sum_{t=1}^{\ell-1} \frac{\Omega_t}{T_t \lambda^t}.
\]
Overall, (C.2) satisfies
\[
\frac{1}{S_T} \left( \sum_{\ell=1}^{k} \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} \omega_t (L^{K^\ell - K^\ell}(x^t, y) - L^{K^\ell - K^\ell}(x, y^t)) \right) \leq \sum_{\ell=1}^{k} \left( \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} \omega_t \right) \cdot \left( \lambda^\ell \sum_{t=1}^{\ell-1} \frac{\Omega_t}{T_t \lambda^t} \right)
\]
\[
\leq e_4 + e_5,
\]
where
\[
e_4 = \frac{2err_{*,0} R X Y a}{S_T} \sum_{\ell=1}^{k} \lambda^\ell \cdot \left( \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} \omega_t \right), e_5 = \frac{2 R X Y a}{S_T} \sum_{\ell=1}^{k} \lambda^\ell \left( \sum_{t=1}^{\ell-1} \frac{1}{T_t \lambda^t} \right) \cdot \left( \sum_{t=\tau_{\ell}}^{\tau_{\ell}+T_t} \omega_t \right).
\]
### D Proof of Proposition 4.4

Let $\omega_t = t^p, T_t = \ell^q$, for $t \geq 1$ and $p, q \in \mathbb{N}$. We have

\[
T = \sum_{t=1}^{k} T_t = \sum_{t=1}^{k} t^q = k^{q+1},
\]

\[
\tau_t = \sum_{t=1}^{\ell} T_t^{q+1},
\]

\[
S_T = \sum_{t=1}^{T} \omega_t = \sum_{t=1}^{T} t^p = k^{(q+1)(p+1)} = T^{p+1}.
\]

Recall that

\[
e_1 = \frac{1}{S_T} \omega_T \Omega, e_2 = \frac{4err_{1,0} R_X R_y a}{S_T} \sum_{t=1}^{k} \omega_t \tau_t \lambda^t, e_3 = \frac{4R_X R_y a}{S_T} \sum_{t=1}^{k} \omega_t \tau_t \lambda^t \left( \sum_{t=0}^{\ell-1} \left( \frac{\Omega_t}{T_t} + \frac{\Omega_{t+1}}{T_{t+1}} \right) \right) \frac{1}{\lambda^t},
\]

\[
e_4 = 2err_{1,0} R_X R_y c \frac{1}{S_T} \sum_{t=1}^{k} \left( \sum_{t=t_\ell}^{T_t} \omega_t \right) \left( \sum_{t=1}^{\ell} \Omega_t \right), e_5 = 2R_X R_y c \frac{1}{S_T} \sum_{t=1}^{k} \left( \sum_{t=1}^{\ell-1} \Omega_t \right) \left( \sum_{t=t_\ell}^{T_t} \omega_t \right).
\]

For the sake of readability in the next bounds the $O(\cdot)$ notation hides the dependency in $S, A$.

**Bounds on $e_1$.** We have

\[
e_1 = O \left( \frac{\omega_T}{S_T} \right) = O \left( \frac{T^p}{T^{p+1}} \right) = O \left( \frac{1}{T} \right).
\]

**Bounds on $e_2$.** We have

\[
e_2 = \frac{1}{T^{p+1}} \sum_{t=1}^{k} \tau_t \lambda^t = \frac{1}{T^{p+1}} \sum_{t=1}^{k} t^{p(q+1)} \lambda^t = \frac{1}{T^{p+1}} k^{p(q+1)} \lambda^k = \frac{1}{T^{p+1}} T^p \lambda^{T^1/(q+1)} = \frac{\lambda^{T^1/(q+1)}}{T}.
\]

**Bounds on $e_3$.** We have

\[
e_3 = \frac{1}{k^{(p+1)(q+1)}} \sum_{t=1}^{k} \lambda^{t} \left( \frac{1}{k^{(p+1)(q+1)}} \right) = \frac{1}{k^{(p+1)(q+1)}} \sum_{t=1}^{k} \lambda^{t} = \frac{1}{k^{(p+1)(q+1)}} k^{(p+1)(q+1) - q} + 1 = \frac{1}{k^{2q}} = \frac{1}{T^2/(q+1)}.
\]

**Bounds on $e_4$.** First we need to compute the term $\sum_{t=t_\ell}^{T_t} \omega_t$. We have

\[
\sum_{t=t_\ell}^{T_t} \omega_t = \sum_{t=t_\ell}^{T_t} t^p = \sum_{t=0}^{T_t - T_{t+1}} (\ell + t)^p = \sum_{t=0}^{T_t - T_{t+1}} \left( \sum_{u=0}^{\ell} \left( \binom{p}{u} \right) t^p \right) = \sum_{t=0}^{T_t - T_{t+1}} \sum_{u=0}^{\ell} \left( \binom{p}{u} \right) t^{p(u+1)} = \sum_{t=0}^{T_t - T_{t+1}} \sum_{u=0}^{\ell} \left( \binom{p}{u} \right) t^{p(u+1)} = \ell^{(p+1)q} = \ell^{(p+1)(q+1) - 1}.
\]

Now we have

\[
e_4 = \frac{1}{k^{(p+1)(q+1)}} \sum_{t=1}^{k} \lambda^{t} = \frac{1}{k^{(p+1)(q+1)}} k^{(p+1)(q+1) - 1} = \frac{\lambda^k}{k} = \frac{\lambda^{T^1/(q+1)}}{T^{1/(q+1)}}.
\]
Therefore, we can reduce \( \arg\min \).

The proximal update (4.2) becomes

\[
\text{Upper bound on the Lagrange multiplier.}
\]

**E Proof of Proposition 5.1**

Let \( B_2(y_0, \alpha) = \{ y \in \mathbb{R}^{A \times S} \mid \frac{1}{2} \| y - y_0 \|_2^2 \leq \alpha \} \).

### E.1 \( \ell_2 \) setup for \( y \)-player

The proximal update (4.2) becomes

\[
\min \langle y_s, d_s \rangle + \frac{1}{2\sigma} \| y - y' \|_2^2 + \frac{\mu}{2} \left( \| y - y_0 \|_2^2 - 2\alpha \right).
\]

Let us show that we can compute \( \arg\min_{y \in (\Delta(S))^A} F(y, \mu) \) in complexity \( O(S \log(S)) \). Indeed,

\[
\arg \min_{y \in (\Delta(S))^A} F(y, \mu) = \arg \min_{y \in (\Delta(S))^A} \langle y_s, d_s \rangle + \frac{1}{2\sigma} \| y - y' \|_2^2 + \frac{\mu}{2} \left( \| y - y_0 \|_2^2 - 2\alpha \right)
\]

\[
= \arg \min_{y \in (\Delta(S))^A} \sum_{a=1}^{A} \sum_{s' = 1}^{S} d_{as'} y_{as'} + \frac{1}{2\sigma} \left( y_{as'} - y_{as'}' \right)^2 + \frac{\mu}{2} \left( y_{as'} - y_{0,as'} \right)^2
\]

\[
= \arg \min_{y \in (\Delta(S))^A} \sum_{a=1}^{A} \sum_{s' = 1}^{S} \left( 1 + \sigma \mu \right) y_{as'} - \left( \frac{1}{\sigma} y_{as'}' + \mu y_{0,as'} - d_{as'} \right) y_{i}
\]

\[
= \arg \min_{y \in (\Delta(S))^A} \frac{1}{2} \| y - \frac{1}{1 + \sigma \mu} \left( \frac{1}{\sigma} y_{as'}' + \mu y_{0,as'} - d_{as'} \right) \|_2^2.
\]

Therefore, we can reduce \( \arg\min_{y \in (\Delta(S))^A} F(y, \mu) \) to solving \( A \) Euclidean projections on the simplex \( \Delta(S) \). Each Euclidean projection on the simplex \( \Delta(S) \) can be done in \( O(S \log(S)) \) [Duchi et al., 2008].

**Binary search for optimal Lagrange multiplier \( \mu^* \).** Note that by definition, \( q : \mu \mapsto F(x^*(\mu), \mu) \) is a concave function on \( \mathbb{R}_+ \). Therefore, if we have an upper bound \( \bar{\mu} \) on \( \mu^* \), an optimal Lagrange multiplier, we can binary search the interval \([0, \bar{\mu}]\) to find a maximum of \( q \).

**Upper bound on the Lagrange multiplier.** Note that

\[
q(\mu) = -\mu \alpha + \min_{y \in (\Delta(S))^A} \langle y, d \rangle + \frac{1}{2\sigma} \| y - y' \|_2^2 + \frac{\mu}{2} \| y - y_0 \|_2^2
\]

\[
\leq -\mu \alpha + \langle y_0, d \rangle + \frac{1}{2\sigma} \| y_0 - y' \|_2^2.
\]

(E.1)
Note that \( q : \mu \mapsto q(\mu) \) is concave on \( \mathbb{R}^+ \). Therefore if we found \( \mu^* \) such that \( q(\mu^*) \leq q(0) \), we can claim that \( \mu^* \in [0, \bar{\mu}] \), where \( \mu^* \) attains the maximum of \( q \). Using our upper bound (E.1) on \( q(\cdot) \) we know that we can choose any \( \bar{\mu} \) such that \[-\mu^* + \langle y_0, d \rangle + \frac{1}{2\sigma} \| y_0 - y' \|_2^2 \leq q(0), \] i.e. we choose an upper bound \( \bar{\mu} \) as
\[
\bar{\mu} = \frac{1}{\alpha} \left( \langle y_0, d \rangle + \frac{1}{2\sigma} \| y_0 - y' \|_2^2 - q(0) \right).
\]

### E.2 \( \ell_1 \) setup for \( y \)-player

Let us fix \( \beta \in \mathbb{R} \). For \( d' \in \mathbb{R}^{A \times S}, d'_{as'} = \left( \beta / \sigma \right) \log(\phi_{as'}(y_{as'})) \), we can write the proximal update (4.2) as
\[
\arg \min_y \begin{pmatrix} y, d' \end{pmatrix} + \frac{\beta}{\sigma} \sum_{a=1}^{A} \sum_{s'=1}^{S} y_{as'} \log y_{as'} + \frac{\mu}{2} \left( \| y - y_0 \|_2^2 - \alpha \right)
\]
\[
= \sum_{a=1}^{A} \left( \langle y_a, d'_{sa} \rangle + \frac{\beta}{\sigma} \sum_{s'=1}^{S} y_{as'} \log y_{as'} + \frac{\mu}{2} \left( \| y_{as} - y_{0,as} \|_2^2 - \alpha \right) \right).
\]
The key observation is that \( F(y, \mu) \) is separable over the actions \( a \). Therefore, in order to solve \( \arg \min_{y \in (\Delta(S))^A} F(y, \mu) \) we can solve \( A \) subproblems, where for each \( a = 1, \ldots, A \) we solve the problem
\[
\arg \min_{y_a \in \Delta(S)} \langle y_a, d'_{sa} \rangle + \frac{\beta}{\sigma} \sum_{s'=1}^{S} y_{as'} \log y_{as'} + \frac{\mu}{2} \left( \| y_{as} - y_{0,as} \|_2^2 - \alpha \right).
\]

### Introduce Lagrange multiplier for ball constraint.

Let us write the Lagrangian function \( F(y, \mu) \), where we introduce a Lagrange multiplier \( \mu \geq 0 \) for the ball constraint, but we leave the simplex constraint unchanged.

\[
F(y, \mu) = \langle y, d' \rangle + \frac{\beta}{\sigma} \sum_{a=1}^{A} \sum_{s'=1}^{S} y_{as'} \log y_{as'} + \frac{\mu}{2} \left( \| y - y_0 \|_2^2 - \alpha \right)
\]
\[
= \sum_{a=1}^{A} \left( \langle y_a, d'_{sa} \rangle + \frac{\beta}{\sigma} \sum_{s'=1}^{S} y_{as'} \log y_{as'} + \frac{\mu}{2} \left( \| y_{as} - y_{0,as} \|_2^2 - \alpha \right) \right).
\]

We now arrive at a problem that decomposes into simple variable-wise updates: the negative entropy proximal mapping. For each variable \( y_{as} \) the update (E.4) is known to be equal (Combettes and Pesquet [2011]) to
\[
y_{as}' = \frac{\beta}{\sigma \mu} W \left( \frac{\sigma \mu}{\beta} \exp \left( \frac{\sigma \mu}{\beta} \left( y_{0,as} - \frac{1}{\mu} (d'_{as} + \nu) \right) \right) - 1 \right)
\]

where \( W \) is the principal branch of the Lambert \( W \) function, which is defined as the inverse of \( w \mapsto w \log w \). The inverse is unique for \( w \in [0, \infty) \). This function is not simple, but it can be computed quickly, and has standard implementations in the major numerical computing languages (e.g. in SciPy). As a heuristic benchmark, evaluating \( W(a), a \in \mathbb{R}_{++} \) using SciPy takes about twice as long as evaluating \( \exp(a) \) (using numpy libraries for all function evaluations), based on generating 1000 random numbers in \([0,1000]\). Now we may find the appropriate \( \nu^* \) such that the sum-to-one constraint is satisfied by binary search \( \nu \).
Binary search for $ν^*$. Let $ν \in \mathbb{R}$ and $y_a(ν)$ the associated solution obtained from (E.5). If $\sum_ν y_a(ν) > 1$, then $ν$ is a lower bound on $ν^*$. Similarly, if $\sum_ν y_a(ν) < 1$, then $ν$ is an upper bound on $ν^*$. Since we know that $ν^* > −∞$, we can explore the set $\{-2^ℓ \mid ℓ ≥ 0\}$ until we found a lower bound on $ν^*$. If in this set we also found $ν$ such that $\sum_ν y_a(ν) > 1$ then we also obtain an upper bound on $ν^*$. Otherwise, we can explore the set $\{2^ℓ \mid ℓ ≥ 0\}$ to find an upper bound on $ν^*$.

Finally we get that we can reduce $\arg\min_{ν \in (Δ(S))^A} F(y, ν)$ to a problem that can be solved in $\log(1/ε)$ time, when treating evaluations of the Lambert W function as a constant.

Now that we have a method for computing $\arg\min_{ν \in (Δ(S))^A} F(y, ν)$, we can now binary search the Lagrange multiplier $μ$ in order to find a feasible solution to (E.2).

Upper bound on $μ^*$. We know that $μ^* \in [0, +∞)$, where $μ^*$ is an argmax of

\[ q : μ \mapsto \min_{y_a \in Δ(S)} \langle y_a, d_{sa} \rangle + \frac{β}{σ} KL(y_a, y'_a) + μ (\|y_a - y'_a\|_2^2 - α). \]

- There is a closed form solution for $q(0)$ since this is the proximal update for the relative entropy.
- We know that

\[ q(μ) ≤ −μα + \langle y_a^0, d_{sa} \rangle + \frac{β}{σ} KL(y_a^0, y'_a) \]

- Therefore an upper bound $μ$ for $μ^*$ is

\[ μ = \frac{1}{α} \left( \langle y_a^0, d_{sa} \rangle + \frac{β}{σ} KL(y_a^0, y'_a) - q(0) \right). \]

We can then perform a binary search for $μ^*$ in $[0, μ]$ exactly as for the $ℓ_2$ setup.

Choice of the parameter $β$. Now we need to choose $β$ such that $ψ$ becomes strongly convex modulus 1. If we set $β = \frac{4}{2}$ then we get strong convexity modulus 1 with respect to the $ℓ_1$ norm. To show this, we use the second-order definition of strong convexity:

\[ \langle ∇^2 ψ(y)h, h \rangle ≥ \|h\|_1^2, \forall y \in Y, h \in \mathbb{R}^{AS} \]

Taking an arbitrary $h \in \mathbb{R}^{AS}$ we get from Cauchy-Schwarz:

\[
\left( \sum_a \sum_{s'} h_{as'} \right)^2 \leq \sum_a \left( \sum_{s'} h_{as'} \right)^2 \leq \sum_a A^2 \left( \sum_{s'} h_{as'}^2 \right) \leq \sum_a A^2 \frac{1}{\sqrt{y_{a}}} \left( \sum_{s'} h_{as'}^2 \sqrt{y_{a}} \right) \leq \sum_a \frac{A}{2} \left( \sum_{s'} h_{as'}^2 \right) \leq \sum_a \frac{A}{2} \left( \sum_{s'} h_{as'}^2 \right) \leq \sum_a \frac{A}{2} \left( \sum_{s'} h_{as'}^2 \right) ,
\]

which shows strong convexity modulus 1 with respect to the $ℓ_1$ norm.

Remark E.1. It may be possible to choose a stronger constant $β$, following [Juditsky et al. 2011], Chapter 5, pages 23-24. However, this would require to introduce a modified norm for element $(y_{sa})_{a \in A}$ of the set $\mathbb{P}_s$. We leave this (potential) improvement for future work.

Remark E.2. We briefly describe how our FOM-based algorithmic framework can be used to solve Distributionally Robust MDPs with a Wasserstein ball (based on an $ℓ_2$ metric) around a finite-support nominal distribution [Yang, 2017]. In this setting the Bellman update (for a given vector $ν$) boils down to computing, for a state $s$,

\[
\max_{π \in Δ(S)} \min_{y_1, \ldots, y_N} \frac{1}{N} \sum_{i=1}^N x_a \left( c_{sa} + λ y_{i,sa} \nu \right) \quad (E.6)
\]

\[
y_1, \ldots, y_N \in (Δ(S))^A,
\]

\[
\frac{1}{N} \sum_{i=1}^N \|y_i - \tilde{y}_i\|_2^2 \leq θ^2,
\]

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for some fixed kernels $\hat{y}_1, \ldots, \hat{y}_N$ in the support of the nominal distribution and a radius $\theta$ for the Wasserstein ball. Applying our framework, for a given choice of Bregman divergence we would obtain a proximal update for the minimizer player as follows

$$
\min \frac{1}{N} \sum_{i=1}^{N} \left( \langle y_i, c_i \rangle + \frac{1}{\sigma} D(y_i, y'_i) \right) \\
y_1, \ldots, y_N \in (\Delta(S))^A, \\
\frac{1}{N} \sum_{i=1}^{N} \| y_i - \hat{y}_i \|_2^2 \leq \theta^2.
$$

It is straightforward to extend Proposition (5.1) to this setting. The $\ell_2$ setup would have complexity $O(NA^2 \log(S) \log(\epsilon^{-1}))$ while the $\ell_1$ setup would have complexity $O(NA^2 \log^2(\epsilon^{-1}))$. With this proximal setup, our FOM-based setup can compute an $\epsilon$-optimal solution to the distributionally-robust MDP problem in time $O(NA^2 S^3 \log(S) \log(\epsilon^{-1})(\epsilon^{-1}))$.

### E.3 KL uncertainty set

We present here our complexity result for the KL uncertainty set. Recall consider that the KL uncertainty set is defined as

$$
\mathbb{P} = \times_{s \in \mathbb{S}} \mathbb{P}_s, \mathbb{P}_s = \{ (y_{sa})_{a \in A} \mid \sum_{a \in A} KL(y_{sa}, y^0_{sa}) \leq \alpha, y_{sa} \in \Delta(S), \forall a \in A \}.
$$

Consider proximal updates (4.2) for this choice of uncertainty set. We have the following proposition.

**Proposition E.3.**

1. (\ell_2 setup) Let $\| \cdot \|_Y = \| \cdot \|_2, \psi_Y = (1/2) \| \cdot \|_2^2, D_Y(y, y') = (1/2) \| y - y' \|_2^2, \forall (y, y') \in \Delta(S) \times \Delta(S)$.

   The proximal update (4.2) with uncertainty set (2.4) can be approximated up to $\epsilon$ in a number of arithmetic operations in $O(AS \log^2(\epsilon^{-1}))$.

2. (\ell_1 setup) Let $\| \cdot \|_Y = \| \cdot \|_1, \psi_Y(y) = (A/2) \sum_{a=1}^{A} \text{ENTROPY}(y_a), D_Y(y, y') = (A/2) \sum_{a=1}^{A} KL(y_a, y'_a), \forall (y, y') \in \Delta(S) \times \Delta(S)$.

   The proximal update (4.2) with uncertainty set (2.4) can be approximated up to $\epsilon$ in a number of arithmetic operations in $O(AS \log^2(\epsilon^{-1}))$.

As the proof follows closely the lines of the proofs for the proximal updates on the ellipsoidal uncertainty set, for the sake of conciseness we only present an outline here.

**Proof.** $\ell_2$ setup. We introduce a Lagrange multiplier for the KL constraint, and the proximal update boils down to solving $A$ subproblems, each consisting of optimizing the sum of a linear form, an entropy function and an $\ell_2$ distance. This is equivalent to solving subproblems of the form (E.3). Therefore, the $\ell_2$ proximal update for a KL uncertainty set can be approximated within accuracy $\epsilon$ in $O(AS \log^2(\epsilon^{-1}))$.

$\ell_1$ setup. For the $\ell_1$ setup, we can introduce a Lagrange multiplier for the KL constraint; the objective becomes separable into $A$ subproblems, each requiring to optimize (over the simplex of size $\Delta(S)$) the sum of a linear form and two KL terms, which brings down to optimizing, over the simplex, the sum of a linear form and a KL term. This can be computed in closed-form, and the $\ell_2$ proximal update boils down to a bisection search onto the Lagrange multiplier. Therefore, the $\ell_2$ proximal update for a KL uncertainty set can be approximated within accuracy $O(AS \log^2(\epsilon^{-1}))$.

**Remark E.4.** We would like to highlight the difference between our results of Propositions 5.1-E.3 and the results of [Nilit and Ghavari 2005], Section 6, who introduce a bisection algorithm to compute the Bellman update for $(s, a)$-rectangular, KL uncertainty set, which reduces to computing the minimum over $a \in A$ of the optimization programs $\max_{y \in \Delta(S)} y^T v$, for $U_{sa} = \{ p \in \Delta(S) \mid KL(y, y^0_{sa}) \leq \alpha \}$. Also, [Iyengar 2005] introduces a bisection algorithm for the Bellman update for $(s, a)$-rectangular, ellipsoidal uncertainty set, i.e. for $U_{sa} = \{ y \in \Delta(S) \mid (1/2) \| y - y^0_{sa} \|_2^2 \leq \alpha \}$. We would like to note that their objective function is a linear form, while the objective of the proximal update (4.2) involves the sum of a linear form and a Bregman divergence. Moreover, we consider $s$-rectangular uncertainty set, for which the Bellman update remains a min-max formulation optimization program (and does not boil down to an enumeration of maximization optimization programs).
F Details on the complexities of Table 2

F.1 Overall complexity analysis for \((\| \cdot \|_X, \| \cdot \|_Y) = (\| \cdot \|_1, \| \cdot \|_1)\)

Convergence rate of PD. For \(q = 2\), the error bounds of Proposition 4.4 become:

\[
e_1 = O \left( \frac{A^2 \log(S)}{T} \right), e_2 = O \left( \frac{A \lambda^{T^{1/3}}}{T} \right), e_3 = O \left( \frac{A^3 \log(S)}{T^{4/3}} \right),
\]

\[
e_4 = O \left( \frac{A^3 \log(S) \lambda^{T^{1/3}}}{T^{1/3}} \right), e_5 = O \left( \frac{A^3 \log(S)}{T^{2/3}} \right).
\]

Complexity of PD update. For each epoch \(\ell = 1, \ldots, k\), solving each proximal update (4.2) with accuracy \(\varepsilon > 0\), the complexity of epoch \(\ell\) is as follows.

\[
comp_{\ell} = O \left( S \left( A + AS \log^2(\varepsilon^{-1}) \right) T_{\text{el}} + T_{\ell} A + T_{\ell} AS + AS^2 \right)
\]

\[
= O \left( \left( SA + AS^2 \log^2(\varepsilon^{-1}) \right) T_{\text{el}} \right)
\]

The overall complexity after \(T = T_1 + \cdots + T_k\) iterations is

\[
comp = O \left( \left( AS^2 \log^2(\varepsilon^{-1}) \right) T \right).
\]

Since the \(e_5\) term is the slowest to converge, for \(q = 2\) the number of arithmetic operations in order to obtain a \(\varepsilon\)-optimal pairs in (1.2) is \(O \left( A^{5.5} S^2 \log(S) 1.5 \log^2(\varepsilon^{-1}) \varepsilon^{-1.5} \right)\).

F.2 Overall complexity analysis for \((\| \cdot \|_X, \| \cdot \|_Y) = (\| \cdot \|_2, \| \cdot \|_2)\)

Convergence rate of PD. The error bounds of Proposition 4.4 become, for \(q = 2\):

\[
e_1 = O \left( \frac{\sqrt{AS}}{T} \right), e_2 = O \left( \frac{\sqrt{AS} \lambda^{T^{1/3}}}{T} \right), e_3 = O \left( \frac{AS}{T^{4/3}} \right),
\]

\[
e_4 = O \left( \frac{AS \lambda^{T^{1/3}}}{T^{1/3}} \right), e_5 = O \left( \frac{AS}{T^{2/3}} \right).
\]

Complexity of PD update. For each epoch \(\ell = 1, \ldots, k\), the complexity of epoch \(\ell\) is as follows.

\[
comp_{\ell} = O \left( S \left( A \log(A) + AS \log(S) \log(\varepsilon^{-1}) \right) T_{\text{el}} + T_{\ell} A + T_{\ell} AS + AS^2 \right)
\]

\[
= O \left( \left( SA \log(A) + AS^2 \log(S) \log(\varepsilon^{-1}) \right) T_{\text{el}} \right).
\]

The overall complexity after \(T = T_1 + \cdots + T_k\) iterations is

\[
comp = O \left( \left( SA \log(A) + AS^2 \log(S) \log(\varepsilon^{-1}) \right) T \right).
\]

Typically, \(\log(A) \leq S\), and we have \(comp = O \left( AS^2 \log(S) \log(\varepsilon^{-1}) \right) T\). Therefore, for \(q = 2\), the number of arithmetic operations in order to obtain a \(\varepsilon\)-optimal pairs in (1.2) is \(O \left( A^{2.5} S^{3.5} \log(S) \log(\varepsilon^{-1}) \varepsilon^{-1.5} \right)\).

G Performance measures; Proof of Lemma 6.1

First, we note that for any \(v \in \mathbb{R}^S\), we have \(\|v^* - v\|_\infty = \|F(v^*) - v\|_\infty\) and

\[
\|F(v^*) - v\|_\infty \leq \|F(v^*) - F(v)\|_\infty + \|F(v) - v\|_\infty \leq \lambda \|v^* - v\|_\infty + \|F(v) - v\|_\infty.
\]

This readily implies that for any \(v \in \mathbb{R}^S\), we have

\[
\|v^* - v\|_\infty \leq \frac{\lambda}{1 - \lambda} \|F(v) - v\|_\infty.
\]
Proof of Lemma 6.1 Recall that \( \epsilon^*(x, y) = \max_{s \in \mathcal{S}} \max_{y' \in \mathbb{P}} F^{x, y'} (v^*)_s - \min_{x' \in \Pi} F^{x', y}(v^*)_s \). We have

\[
\max_{s \in \mathcal{S}} \max_{y' \in \mathbb{P}} F^{x, y'} (v^*)_s = \max_{s \in \mathcal{S}} \max_{y' \in \mathbb{P}} F^{x, y'} (v)_s + \lambda \sum_{a=1}^{A} x_{sa} y_{sa}^\top (v^* - v) \\
\leq \max_{s \in \mathcal{S}} \max_{y' \in \mathbb{P}} F^{x, y'} (v)_s + \lambda \|v^* - v\|_\infty \\
\leq \max_{s \in \mathcal{S}} \max_{y' \in \mathbb{P}} F^{x, y'} (v)_s + \frac{\lambda}{1 - \lambda} \|F(v) - v\|_\infty.
\]

We can prove an analogous inequality for \( \max_{s \in \mathcal{S}} \min_{y' \in \mathbb{P}} F^{x, y'} (v^*)_s \) and therefore

\[
\epsilon^*(x, y) \leq \frac{2\lambda}{1 - \lambda} \|v - F(v)\|_\infty + \max_{s \in \mathcal{S}} \max_{y' \in \mathbb{P}} F^{x, y'} (v)_s - \min_{x' \in \Pi} F^{x', y}(v)_s.
\]

Upper bound on \( \|v^* - F^{x, y}(v^*)\|_\infty \). Another measure of the performance of a pair \((x, y)\) is \( \|v^* - F^{x, y}(v^*)\|_\infty \). In particular, we present the following upper bound; the proof is in Appendix G.

Lemma G.1 \( \|v^* - F^{x, y}(v^*)\|_\infty = \|F(v^*) - F^{x, y}(v^*)\|_\infty \) and we have the following upper bound on \( \|v^* - F^{x, y}(v^*)\|_\infty \): for any \( v \in \mathbb{R}^\mathcal{S} \),

\[
\frac{1 + \lambda}{1 - \lambda} \|v - F(v)\|_\infty + \|v - F^{x, y}(v)\|_\infty.
\]

Proof of Lemma G.1 We have

\[
\|v^* - F^{x, y}(v^*)\|_\infty = \|v^* - v + v - F^{x, y}(v) + F^{x, y}(v) - F^{x, y}(v^*)\|_\infty \\
\leq \|v^* - v\|_\infty + \|v - F^{x, y}(v)\|_\infty + \|F^{x, y}(v) - F^{x, y}(v^*)\|_\infty \\
\leq \frac{\lambda}{1 - \lambda} \|v - F(v)\|_\infty + \|v - F^{x, y}(v)\|_\infty + \lambda \|v^* - v\|_\infty \\
\leq \frac{1 + \lambda}{1 - \lambda} \|v - F(v)\|_\infty + \|v - F^{x, y}(v)\|_\infty.
\]

Computing (DG). In order to compute (DG) for a pair \((x, y)\), we need to evaluate \( \max_{y' \in \mathbb{P}} R(x, y') \) and \( \min_{x' \in \Pi} R(x', y) \). Following [Wiesemann et al. 2013], \( \max_{y' \in \mathbb{P}} R(x, y') \) can be computed by finding the fixed point of the following operator, which is a contraction of factor \( \lambda \):

\[
F^x(v)_s = \max_{y' \in \mathbb{P}} \sum_{a=1}^{A} x_{sa} \left( c_{sa} + \lambda y^\top y' \right), \forall s \in \mathcal{S}.
\]

Moreover, computing \( \min_{x' \in \Pi} R(x', y) \) is equivalent to solving the (nominal) MDP with fixed kernel \( y' \in \mathbb{P} \). Following [Puterman 1994], Chapter 6.3, this can be solved by iterating the following contraction of factor \( \lambda \):

\[
F^y(v)_s = \min_{x' \in \Pi} \sum_{a=1}^{A} x_{sa} \left( c_{sa} + \lambda y'^\top y' \right), \forall s \in \mathcal{S}.
\]

Each of these iterative algorithms can be stopped as soon \( \|v^{t+1} - v^t\|_\infty < 2\lambda\epsilon(1 - \lambda)^{-1} \), which ensures \( \epsilon \)-optimality of the current iterates [Puterman 1994], Chapter 6.3.

We present in the next figure the running times to compute (DG), both with Algorithm VI and Algorithm AVI, and to compute (UB-1) and (UB-2). In particular, we generate 10 random Garnet MDP instances (see Section 6), some random policies in \( \Pi \), kernels in \( \mathbb{P} \) and vectors in \( \mathbb{R}^\mathcal{S} \) and we compute (DG), (UB-1) and (UB-2). We present the logarithm of the average running times to obtain \( \epsilon \)-approximations of the quantities of interest, for \( \epsilon = 0.25 \) and \( \epsilon = \sqrt{\text{branch} \times A} \). We present our results for \( \lambda = 0.6 \) in Figure 10 and for \( \lambda = 0.8 \) in Figure 11. We notice that computing (DG) quickly becomes very expensive, even using Algorithm AVI, while computing (UB-1) and (UB-2) is one order of magnitude faster. Therefore, in Section 6 we focus on computing (DG) and (UB-1).
H Details on numerical implementations

All the simulations are coded in Python 3.7.3, and were performed on a laptop with 2.2 GHz Intel Core i7 and 8 GB of RAM. We use Gurobi 8.1.1 to solve any linear or quadratic optimization problems involved.

Value Iteration. At every epoch of Value Iteration [VI] we need to compute $F(v)$ for the current value vector $v \in \mathbb{R}^S$, where

$$F(v)_s = \min_{x_s \in \Delta(A)} \max_{y_s \in \mathcal{P}_s A} \sum_{a=1}^{A} x_{sa} \left( c_{sa} + \lambda \cdot y_{sa}^\top v \right), \forall s \in S.$$ 

In order to solve this program, we could use duality in the inner maximization program, and turn the computation of $F(v)_s$ into a large (minimization) convex program with linear objective, some constraints and a conic quadratic constraint (see Corollary 3 in Wiesemann et al. [2013]). However, we decide to take an alternate approach which results in a simpler optimization program, namely, a convex program with some linear constraints and a quadratic constraint. In particular, from convex duality we have, for any $s \in S$,

$$F(v)_s = \min_{x_s \in \Delta(A)} \max_{y_s \in \mathcal{P}_s A} \sum_{a=1}^{A} x_{sa} \left( c_{sa} + \lambda \cdot y_{sa}^\top v \right)$$

$$= \max_{y_s \in \mathcal{P}_s A} \min_{x_s \in \Delta(A)} \sum_{a=1}^{A} x_{sa} \left( c_{sa} + \lambda \cdot y_{sa}^\top v \right). \tag{H.1}$$

Now we have

$$\min_{x_s \in \Delta(A)} \sum_{a=1}^{A} x_{sa} \left( c_{sa} + \lambda \cdot y_{sa}^\top v \right) = \min_{x_s \geq 0, \mu} \sum_{a=1}^{A} x_{sa} \left( c_{sa} + \lambda \cdot y_{sa}^\top v \right)$$

$$= \min_{x_s \geq 0} \max_{\mu \in \mathbb{R}} \sum_{a=1}^{A} x_{sa} \left( c_{sa} + \lambda \cdot y_{sa}^\top v + \mu \left( 1 - \sum_{a=1}^{A} x_{sa} \right) \right)$$

$$= \min_{x_s \geq 0} \max_{\mu \in \mathbb{R}} \mu + \sum_{a=1}^{A} x_{sa} \left( c_{sa} + \lambda \cdot y_{sa}^\top v - \mu \right)$$

$$= \max_{\mu \in \mathbb{R}} \min_{x_s \geq 0} \mu + \sum_{a=1}^{A} x_{sa} \left( c_{sa} + \lambda \cdot y_{sa}^\top v - \mu \right).$$
and therefore

\[
\min_{x_s \in \Delta(A)} \sum_{a=1}^{A} x_{sa}(c_{sa} + \lambda y_{sa}^\top v) = \max \mu
\]

\[
\mu \in \mathbb{R},
\]

\[
c_{sa} + \lambda y_{sa}^\top v \geq \mu, \forall a \in A.
\]

Overall, we have proved that

\[
F(v)_s = \max \mu
\]

\[
\mu \in \mathbb{R}, y \in \mathbb{P}_s,
\]

\[
c_{sa} + \lambda y_{sa}^\top v \geq \mu, \forall a \in A.
\]

(H.2)

In our simulations, we use the formulation (H.2) in order to obtain the value of \(F(v)_s\). Given the definition of \(\mathbb{P}_s\) as (2.5), formulation (H.1) is a linear program with linear constraints and one quadratic constraint. Following [Ben-Tal and Nemirovski 2001], we can solve (H.2) up to accuracy \(\epsilon\) in a number of arithmetic operations in \(O \left(S^{1.5}A^{3.5}\log(1/\epsilon)\right)\).

We warm-start each of this optimization problem with the optimal solution found in the previous epoch of VI.

We would like to note that a priori, the optimal pair in \(F(v)_s\) in the min-max formulation as in (2.1) may not be the same pair attaining the max-min formulation as in (H.1). However, we are only interested in the scalar value of \(F(v)_s\), in order to run VI and obtain \(v^*\), the fixed-point of the operator \(T\) defined in (2.3). Once we have obtained the vector \(v^*\), we can eventually solve \(F(v^*)\) in its min-max form only once, in order to obtain the pair \((x^*, y^*)\) in \(F(v^*)\) in its min-max formulation. Alternately, the authors in [Ho et al. 2018] provide a method to recover the optimal solution of the min-max problem (2.1) from the optimal solution of the max-min problem (H.1), in the case where \(\mathbb{P}\) is a weighted \(\ell_1\) ball centered around \(\mathbb{P}_0\).

**Accelerated Value Iteration.** [Goyal and Grand-Clement 2018] interpret the vector \((I - T)(v)\) as the gradient of some function at the vector \(v\). Adapting the acceleration scheme from convex optimization [Nesterov 1983, Nesterov 2013] to an accelerated iterative algorithm for computing \(v^*\) leads to Accelerated Value Iteration, which significantly outperforms Value Iteration, Gauss-Seidel Value Iteration and Jacobi Value Iteration when the discount factor is close to 1 [Goyal and Grand-Clement 2018]. In particular, for any sequences of scalar \((\alpha_s)_{s \geq 0}\) and \((\gamma_s)_{s \geq 0} \in \mathbb{R}^N\), Accelerated Value Iteration (AVI) is defined as

\[
v_0, v_1 \in \mathbb{R}^S, \left\{ \begin{array}{l}
h_t = v_t + \gamma_t \cdot (v_t - v_{t-1}) \\
v_{t+1} \leftarrow h_t - \alpha_t (h_t - T(h_t)), \quad \forall \ t \geq 1.\end{array} \right.
\]

(AVI)

Following [Goyal and Grand-Clement 2018], we choose step sizes as

\[
\alpha_s = \alpha = 1/(1 + \lambda), \gamma_s = \gamma = \frac{1 - \sqrt{1 - \lambda^2}}{\lambda}, \forall s \geq 1.
\]

We use (H.2) in order to compute \(F(h)\) for AVI.