THE STRANGE WORLD OF TRANSFINITE MELODIES – RECOGNIZABILITY FOR WEAK AND STRONG INFINITE TIME $\alpha$-REGISTER MACHINES

MERLIN CARL

Abstract. For exponentially closed ordinals $\alpha$, we consider recognizability of constructible subsets of $\alpha$ for $\alpha$-(w)ITRMs and their distribution in the constructible hierarchy. In particular, for $\alpha$-ITRMs, we show that, there are lost melodies that are recognizable without parameters for all $\alpha$, that the iterated recognizability is absolute between $L$ and $V$ for most values of $\alpha$ and generalize “all or nothing”-phenomenon known from ITRMs occurs for a proper class of $\alpha$. For $\alpha$-wITRMs, we offer a complete characterization of those $\alpha$ for which lost melodies exist and that the relation between the sets of computable and recognizable subsets of $\alpha$ varies wildly, depending on $\alpha$: The computable sets may be included among the recognizable sets (which is usually the case in ordinal computability), but there are also class many values of $\alpha$ for which the set of recognizable sets is empty and such for which the set of recognizable sets is non-empty, but disjoint from the set of computable sets.

This paper is an extension of our paper in the CiE 2023 proceedings [10].

1. Introduction

In [15], Hamkins and Lewis introduced infinite time Turing machines (ITTTMs), which are Turing machines that compute with transfinite time, but still on a “standard” tape of length $\omega$. Koepke then introduced a number of models of computation in which also memory is extended to transfinite ordinals; these include $\alpha$-ITTTMs [19], which are ITRMs with a tape of length $\alpha$, and also transfinite generalizations of register machines which can store a single ordinal less than a given ordinal $\alpha$ in each of their registers. Depending on their behaviour at limit steps, these are known as (weak) infinite time $\alpha$-register machines, or $\alpha$-(w)ITTTMs for short (a brief explanation of these will be given below).

Associated with each model of computation are a concept of explicit definability – called computability – which concerns the ability of the machine to produce a certain object “from scratch”, and another one of implicit definability, which concerns the ability of the machine to decide whether or not an object given in the oracle is equal to a certain $x$;

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in ordinal computability, the latter is known as “recognizability”. For many models of ordinal computability, the “lost melody phenomenon” occurs, which means that there are objects which are recognizable, but not computable; this phenomenon was first discovered (and named) for ITTM in [15]. Recognizability has been studied in detail for ITTM and ITRM in [15], [6], [7], [8] and for Ordinal Turing Machines (OTMs) in [4]. In [6], we considered recognizability by $\alpha$-ITRMs. All of these works concerned real numbers, i.e., subsets of $\omega$. This has the advantage that the recognizability strength of different models becomes comparable.

However, the natural domain of computation for $\alpha$-(w)ITRM are clearly subsets of $\alpha$, not just subsets of $\omega$. (By analogy, the computability strength of these models (and also of $\alpha$-ITTMs) is studied in terms of subsets of $\alpha$, not of $\omega$.) The recognizability of subsets of arbitrary ordinals by OTMs with ordinal parameters is currently studied in joint work with Philipp Schlicht and Philip Welch; since already the recognizable subsets of $\omega$ can, depending on the set-theoretical background, go far beyond $L$ in this case by [4], it is hardly surprising that the same happens in the more general case. However, interesting phenomena also arise under the assumption $V = L$. This paper studies the recognizability strength of weak and strong $\alpha$-register machines with respect to constructible subsets of $\alpha$. While $\alpha$-ITRM behave rather similarly to ITRMs, $\alpha$-wITRM show a rather interesting behaviour. In particular, while the computable sets are included in the recognizable sets for all models studied so far, we will below show that, for class many values of $\alpha$, the sets of $\alpha$-wITRM-computable and $\alpha$-wITRM-recognizable subsets of $\alpha$ are both non-empty and disjoint, while for other values of $\alpha$, the set of $\alpha$-wITRM-recognizable subsets of $\alpha$ is empty.

This paper expands our contribution to CiE 2023 [10] by several new main results (such as the existence of lost melodies for $\alpha$-ITRM without parameters, Theorem 4, Theorem 5), Corollary 5 and Corollary 6, a systematic reorganization of proofs by isolating reoccurring crucial steps as lemmata such as the existence of nice codes (Lemma 2) (which will hopefully increase the readability) and by more elaborate versions of some arguments for the sake of being self-contained.

2. Basic definitions

In the following, $\alpha$ will denote an exponentially closed ordinal, unless explicitly stated otherwise. We briefly describe $\alpha$-ITRM and $\alpha$-wITRM, which were originally introduced by Koepke in [19]; full definitions can also be found in [3].

$\alpha$-register machines use finitely many registers, which can store a single ordinal strictly less than $\alpha$ each. Programs the $\alpha$-(w)ITRM are finite sequences of program lines, each containing one of the basic
commands to increment the content of a register by 1, to copy the content of one register to another, or to jump to a certain program line when the contents of two registers agree and continue with the next one afterwards. For reasons of technical convenience, the last command was extended a bit in [5] to allow for an instantaneous comparison of two finite sequences of registers; this modification has no influence on the results in this paper. In addition, there is the oracle command: Oracles for \(\alpha\)-register machines are subsets \(x\) of \(\alpha\). Given a register index \(j \in \omega\), the oracle command takes the content of the \(j\)-th register, say \(\iota\), and then writes 1 to the \(j\)th register when \(\iota \in x\) and otherwise 0.

Infinitary register computations are now defined by recursion along the ordinals. At successor ordinals, the command in the active program line is simply carried out. (We shall assume that \(\alpha\) is a limit ordinal, so that there is no question what to do when the incrementation operator increases a register content above \(\alpha\).) At limit stages, the register contents and the active program line are obtained as the inferior limits of the sequences of the register contents and active program lines so far. A difficulty arises when this limit is equal to \(\alpha\), and this can be solved in two ways: Either one regards the computation as undefined in such a way, thus obtaining “weak” or “unresetting” \(\alpha\)-register machines, called \(\alpha\)-wITRMs, or one resets the contents of the overflowing registers to 0, which yields “resetting” or “strong” \(\alpha\)-register machines, called \(\alpha\)-ITRMs. When one lets \(\alpha = \text{On}\), thus allowing arbitrary ordinals as register contents, one obtains Koepke’s Ordinal Register Machines (ORMs), which are equal in computational power to Ordinal Turing Machines (OTMs, [20]) and can compute exactly the constructible sets of ordinals [23]. Thus, \(\alpha\)-wITRMs can be regarded as space-bounded versions of ORMs (with a constant space bound).

Thus, \(\alpha\)-wITRM-programs and \(\alpha\)-ITRM-programs are the same and both are just classical register machine programs as, e.g., introduced in [12]. When we write something like “\(\alpha\)-wITRM-program”, we really mean that the program is intended to be run on an \(\alpha\)-wITRM.

An important observation about \(\alpha\)-(w)ITRM-computations is the following:

**Definition 1.** In an \(\alpha\)-(w)ITRM-computation, a “strong loop” is a pair \((i, \xi)\) of ordinals such that the computation states — i.e., the active program line and the register contents — at times \(i\) and \(\xi\) are identical and such that, for every time in between, the computation states were in each component greater than or equal to these states.

It is easy to see from the liminf-rule that a strong loop will be repeated forever.
Theorem 1 (Cf. [3], generalizing [21], Lemma 3). An $\alpha$-ITRM-program either halts or runs into a strong loop. An $\alpha$-wITRM-program either halts, runs into a strong loop or is undefined due to a register overflow.

We denote by $\chi_x$ the characteristic function of a set $x \subseteq \alpha$.

**Definition 2.** A set $x \subseteq \alpha$ is $\alpha$-(w)ITRM-computable if and only if there is an $\alpha$-(w)ITRM-program $P$ and some parameter $\rho < \alpha$ such that, for each $i < \alpha$, $P(i, \rho) \downarrow = \chi_x(i)$. If $\rho = 0$, we say that $x$ is $\alpha$-ITRM-computable without parameters.

We will now define the concept of decidability of a set of subsets of $\alpha$. For the tape models of transfinite computations, such as ITTMs, one needs to distinguish between recognizability, semirecognizability and co-recognizability, depending on whether $\{x\}$ is decidable, semi-decidable or has a semi-decidable complement. Indeed for ITTMs, these concepts have different extensions, see [4].

For ITRMs, no such distinction had to be introduced, as they exhibit a rather surprising feature: The halting problem for ITRMs with a fixed number of registers is solvable by an ITRM-program with a larger number of registers, uniformly in the oracle (see Koepke and Miller, [21], Theorem 4). Thus, semi-, co- and plain decidability all coincide. For wITRMs, there is even less reason for conceptual differentiation, since for these, recognizability coincides with computability, see [6], [3].

For general $\alpha$-(w)ITRMs, however, the situation is different. It is currently not known whether the bounded halting problem is solvable for $\alpha$-ITRMs unless $\alpha = \omega$ or $L_\alpha \models ZF^-$. Moreover, for $\alpha$-wITRMs, one needs to decide whether, in the definition of the semi-decidability of a set $x \subseteq \mathcal{P}(\alpha)$, one allows undefined computations (i.e., computations in which an oracle overflows) or not.

**Definition 3.** For $X \subseteq \mathcal{P}(\alpha)$, let us denote by $\chi_X$ the characteristic function of $X$ in $\mathcal{P}(\alpha)$.

A set $X \subseteq \mathcal{P}(\alpha)$ is $\alpha$-(w)ITRM-semi-decidable if and only if there are an $\alpha$-(w)ITRM-program $P$ and some $\xi < \alpha$ such that, for all $y \subseteq \alpha$, $P^y(\xi) \downarrow$ if and only if $y \in X$. In the case of $\alpha$-wITRMs, we demand that the computations $P^y(\xi)$ for $y \notin X$ do not halt, but are still defined.

$X$ is called $\alpha$-(w)ITRM-co-semi-decidable if and only if $\mathcal{P}(\alpha) \setminus X$ is $\alpha$-(w)ITRM-semi-decidable.

If $X$ is both $\alpha$-(w)ITRM-semi-decidable and $\alpha$-(w)ITRM-co-semi-decidable, i.e., if there is an $\alpha$-(w)ITRM-program $P$ and some $\xi < \alpha$ such that $P^y(\xi) \downarrow = \chi_X(y)$ for all $y \subseteq \alpha$, we call $X$ $\alpha$-(w)ITRM-decidable.

If there are an $\alpha$-wITRM-program $P$ and some $\xi < \alpha$ such that $P^y(\xi) \downarrow$ for all $y \in X$ and, for all $y \notin X$, $P^y(\xi)$ is either undefined or diverges, we call $X$ “weakly $\alpha$-wITRM-semi-decidable”. The concept of weak $\alpha$-wITRM-co-semidecideability and of $\alpha$-wITRM-decidability are now defined in the obvious way.
If $\xi = 0$, we say that $X$ is $\alpha$-(w)ITRM-(co-)(semi-)decidable etc. without parameters.

**Definition 4.** Let $x \subseteq \alpha$.
- $x$ is called $\alpha$-(w)ITRM-recognizable if and only if \{x\} is $\alpha$-(w)ITRM-decidable.
- $x$ is called $\alpha$-(w)ITRM-semirecognizable if and only if \{x\} is $\alpha$-(w)ITRM-semidecidable.
- $x$ is called $\alpha$-(w)ITRM-cosemirecognizable if and only if \{x\} is $\alpha$-(w)ITRM-co-semi-decidable.

The weak versions of $\alpha$-wITRM-semirecognizability and $\alpha$-wITRM-co-semi-recognition are defined in the obvious way. If the (co-)(semi-)decision algorithm uses only 0 as a parameter, we say that $x$ is $\alpha$-(w)ITRM-(co-)(semi-)recognizable without parameters.

We will use $p$ to denote Cantor’s ordinal pairing function. Moreover, for a set $X$, an $\varepsilon$-formula $\phi$ and a finite tuple $\vec{p}$, we denote by $\text{Def}(X, \phi, \vec{p})$ the set $\{x \in X : (X, \varepsilon) \models \phi(x, \vec{p})\}$.

When $X$ is a set and $E$ is a binary relation on $X$, then the structure $(X, E)$ can be encoded as a subset of an exponentially closed ordinal $\alpha$ (cf., e.g., [3], Def. 2.3.18) by fixing a bijection $f : \alpha \to X$ and letting $c_f(X, E) := \{p(\iota, \xi) : \iota, \xi < \alpha \land f(\iota)E f(\xi)\}$. In general, when $\alpha \subseteq X$, it is computationally nontrivial to identify which $\iota < \alpha$ codes a certain ordinal $\xi < \alpha$. In order to circumvent this problem, we define a class of codes for which this is trivial, as $\iota < \alpha$ is encoded by the $\iota$-th limit ordinal.

**Definition 5.** Let $\alpha$ be exponentially closed, $X$ be a transitive set with $\alpha \subseteq X$, and let $f : \alpha \to X$ be bijective. Then $c_f(X, \varepsilon)$ is called an $\alpha$-code for $X$. If $f$ is additionally such that, for all $\iota < \alpha$, we have $f(\omega \iota) = \iota$, then $c_f(X, \varepsilon)$ is called a nice $\alpha$-code for $X$. We say that $c_f(X, \varepsilon)$ is very nice if additionally $f(1) = \alpha$.

For $\alpha, \gamma \in \text{On}$, we denote by $\text{nice}_\alpha(\gamma)$ the set of nice $\alpha$-codes for $L_\gamma$. If $\text{nice}_\alpha(\gamma)$ contains a constructible element, then we denote by $c_\alpha$ its $<_{L_\gamma}$-minimal element. Moreover, we let $\text{nice}_\alpha := \bigcup_{\beta \in \text{On}} \text{nice}_\alpha(\gamma)$.

When proving that certain subsets of $\alpha$ are not computable, it is often convenient to recall from the folklore that no $L$-level can contain a nice code for itself:

**Lemma 1.** Let $\beta < \alpha$. Then $L_\alpha$ does not contain a nice $\beta$-code for $L_\alpha$.

*Proof.* Suppose for a contradiction that $c \in \mathfrak{P}(\beta) \cap L_\alpha$ codes $L_\alpha$ via $f : \beta \to L_\alpha$. There is some $\delta < \alpha$ such that $c$ is defined over $L_\delta$ by some $\varepsilon$-formula (possibly with parameters in $L_\delta$).

Consider the set $D := \{\iota < \beta : p(\omega \iota, \iota) \notin c\}$. Since $c$ is definable over $L_\delta$, so is $D$, so that $D \in L_\alpha$. Pick $\xi$ such that $f(\xi) = D$. Now $\xi \in D \iff p(\omega \xi, \xi) \notin c \iff f(\omega \xi) \notin f(\xi) \iff \xi \notin D$, a contradiction. \qed
Remark 1. In ordinal computability and in definability considerations, one can often switch back and forth between ordinals and the corresponding $L$-levels, so that their difference seems negligible. This is different with nice codes, since already $L_{\omega+1}$ contains codes for all ordinals below $\omega_1^{CK}$, but not even a code for $L_{\omega+1}$.

Question 1. Is it possible for an $L$-level to contain a code for itself?

As the following shows, there is no trifling difference between codes and nice codes concerning the question where such codes appear in the constructible hierarchy:

Lemma 2. Let $\alpha < \beta < \gamma$ be ordinals, where $\alpha$ is exponentially closed. Then the following are equivalent:

1. $L_{\gamma+1} \setminus L_\gamma$ contains a bijection between $\alpha$ and $L_\beta$.
2. $L_{\gamma+1} \setminus L_\gamma$ contains an $\alpha$-code for $L_\beta$.
3. $L_{\gamma+1} \setminus L_\gamma$ contains a nice $\alpha$-code for $L_\beta$.
4. $L_{\gamma+1} \setminus L_\gamma$ contains a very nice $\alpha$-code for $L_\beta$.

Proof. If there is a bijection $f : \alpha \to L_\gamma$ in $L_{\gamma+1}$, then there is some formula $\phi$ such that $f(x) = y$ if and only if $L_\gamma \models \phi(x, y)$ (we suppress parameters for the sake of simplicity). Then $c_f = \{(a, b) : \exists x, y (\phi(a, x) \land \phi(b, y) \land x \in y) \in L_{\gamma+1}\}$. So (1)$\Rightarrow$(2).

Now suppose that $c \in L_{\gamma+1} \setminus L_\gamma$ is an $\alpha$-code for $L_\beta$. In particular, this means that a new subset of $\alpha$ is generated at the $L$-level $\gamma$, which, by fine-structure, implies that a bijection between $\alpha$ and $L_\gamma$, and hence a bijection $f : \alpha \to L_\beta$, is definable over $L_\gamma$, and hence an element of $L_{\gamma+1}$.

(4)$\Rightarrow$(3) and (3)$\Rightarrow$(2) are trivial. We show that (2) implies (3).

Suppose that (2) holds, and let $c \in L_{\gamma+1} \setminus L_\gamma$ be an $\alpha$-code for $L_\beta$. By what we just showed, we have (1), so there is a bijection $f : \alpha \to L_\beta$ in $L_{\gamma+1} \setminus L_\gamma$. We want to define a new bijection $\hat{f} : \alpha \to L_\beta$ over $L_\gamma$ with the property that $\hat{f}(\omega \xi) = \xi$ for all $\xi < \alpha$. Then $c_{\hat{f}}(L_\beta)$ will be as desired: It is definable over $L_\gamma$ (and hence contained in $L_{\gamma+1}$) because $f$ is, and it is not an element of $L_\gamma$ by Lemma [1]. Let $f = \{(x, y) \in L_\gamma : L_\gamma \models \phi_f(x, y)\}$ (as above, parameters are suppressed).

We start by define a surjection $f' : \alpha \to L_\beta$ by letting $f'(\omega \xi) = \xi$ and $f'(\xi + 1) = f(\xi)$ for all $\xi < \alpha$. Note that $f'$ is still definable over $L_\gamma$ as

$$\phi_{f'}(x, y) :\Leftrightarrow \exists \xi < \alpha(x = \omega \xi \land y = \xi) \lor \exists \xi < \alpha(x = \xi + 1 \land \phi_f(\xi, y)).$$

Although $f'$ is clearly surjective (as $L_\gamma = f[\alpha] = f'(\{\xi + 1 : \xi \in \alpha\})$), it will not be a bijection: Every $\xi < \alpha$ has exactly two different preimages $\omega \xi$ and $f^{-1}(\xi) + 1$ under $f'$, while every $x \in L_\gamma \setminus \alpha$ has precisely one preimage under $f'$.

In order to obtain a bijection, we make a further modification to $f'$. To this end, define, for a given set $x$, $\eta^0(x) := x$ and $\eta^{k+1}(x) = \{\eta^k(x)\}$ for $k \in \omega$. Note that, since $\alpha$ is a limit ordinal, we
have $\eta^k(i) \in L_{i+k+1} \subseteq L_\alpha$ for every $i < \alpha$ and every $k \in \omega$. Moreover, the map $\eta : \omega \times \alpha \to \alpha$ which sends $(k, i) \in \omega \times \alpha$ to $\eta^k(i)$ is definable over $L_\alpha$ as

$$\phi_\eta(k, i, x) : \Leftrightarrow \exists f(f : k \to L_\alpha \land f(0) = \{i\} \land f(k-1) = x \land \forall i < (k - 1) f(i+1) = \{f(i)\}).$$

For every $\xi < \alpha$ such that $f'(\xi + 1) = \xi \in \alpha$, we let $\hat{f}(\xi + 1) := \eta^2(\xi)$ and in general $\hat{f}(f^{-1}(\eta^k(i))) = \eta^{k+1}(i)$ for $k \in \omega$, while for all $\xi < \alpha$ not covered by this modification, we let $\hat{f}(\xi) = f'(\xi)$.

We show that $\hat{f} : \alpha \to L_\beta$ is indeed bijective and that it is definable over $L_\gamma$.

To see that $\hat{f}$ is bijective, let $x \in L_\gamma$. If $x \notin \alpha$ and $x$ is not of the form $\eta^k(i)$ for some $k \in \omega \setminus \{1\}$, $i < \alpha$, then $|\hat{f}^{-1}(x)| = |(f')^{-1}(x)| = |f^{-1}(x)| = 1$. If $x \in \alpha$, then $|\hat{f}^{-1}(x)| = |\omega x| = 1$. If $x = \eta^k(i)$ for some $1 < k \in \omega$, $i < \alpha$, then $|\hat{f}^{-1}(x)| = |(f')^{-1}(\eta^{k-1}(x))| = 1$. So every $x \in L_\alpha$ has exactly one pre-image under $\hat{f}$, as desired.

We now show that $\hat{f}$ is definable over $L_\gamma$. Indeed, we have $\hat{f}(x) = y$ if and only if

1. $x$ is a limit ordinal and $y = f'(x)$ or
2. $\exists \xi < \alpha(x = \xi + 1 \land f'(\xi + 1) \in \alpha \land y = \eta^2(f'(\xi + 1)))$ or
3. $\exists k \in \omega \exists i < (k \geq 2 \land x = (f')^{-1}(i) \land y = \eta^{k+1}(i)).$

Since the uses of $\eta$ and $f'$ can be eliminated using $\phi_\eta$ and $\phi_f$, this is expressible as an $\in$-formula over $L_\gamma$.

By an obvious analogous argument (additionally letting $f'(1) = \alpha$ and then adapting $\hat{f}$ accordingly), we obtain (2)$\Rightarrow$(4).

\[\square\]

We recall a standard definition and two observations.

**Definition 6.** For $\alpha \in On$, $\sigma_\alpha$ is the first stable ordinal above $\alpha$, that is, the smallest ordinal $\beta > \alpha$ such that $L_\beta \prec \Sigma_1 L$.

The next lemmas are part of the folklore.

**Lemma 3.** For an ordinal $\alpha$, $\sigma_\alpha$ is the supremum of all ordinals $\beta$ such that, for some $\in$-formula $\phi$ and some $\xi < \alpha$, $\beta$ is minimal with the property $L_\beta \models \phi(\xi)$.

**Lemma 4.** If $\alpha$ and $\beta$ are ordinals such that $\beta \in [\alpha, \sigma_\alpha)$, then $\sigma_\beta = \sigma_\alpha$.

**Proof.** It is clear that $\sigma_\alpha \leq \sigma_\beta$.

For the converse, let $\rho < \beta$ and pick $\beta' > \rho$ such that, for some formula $\phi$ and some $\xi < \alpha$, $\beta'$ is minimal with $L_{\beta'} \models \phi(\xi)$. By a standard fine-structural argument, $L_{\beta'}$ is the $\Sigma_1$-hull of $\xi + 1$ in $L_\beta$.

\[\footnotesize\text{1That we take $\eta^2(i)$ (i.e., $\{\{i\}\}$) rather than $\eta^1(i)$ (i.e., $\{i\}$) has the sole reason that this does not require a special treatment of 0, as $\{\{0\}\}$ is not an ordinal, while $\{0\} = 1$.}\]
Thus, $\rho$ is the minimal witness for some such formula $\psi$ in some parameter $\zeta < \xi + 1$. It follows that $\rho$ is the unique witness $x$ of the formula “There is a minimal $L$-level $L$, that satisfies $\phi(\xi)$ and in $L_\gamma$, $x$ is minimal such that $\psi(x, \zeta)$”. Consequently, every $\in$-formula in the parameter $\rho$ is equivalent to some $\in$-formula using only parameters less than $\alpha$. Thus $\sigma_\beta \leq \sigma_\alpha$. □

For the parameter-free case, we will also need the following:

**Definition 7.** For an ordinal $\alpha$, we denote by $\sigma^\alpha$ the supremum of ordinals $\beta$ which are minimal with the property that, for some $\Sigma_1$-formula $\phi$, $L_\beta \models \phi(\alpha)$.

**Definition 8.** (Cf. [3], p. 34) An ordinal $\alpha$ is called $\alpha$-(w)ITRM-singular if and only if there are an $\alpha$-(w)ITRM-program $P$ and an ordinal $\xi < \alpha$ such that, for some $\beta < \alpha$, $P$ computes a cofinal function $f : \beta \to \alpha$ in the parameter $\xi$.

We will make use of the following result of Boolos:

**Lemma 5.** ([2], Theorem 1′) There is a parameter-free $\in$-formula $\phi$ such that, for a transitive set $X$, we have $X \models \phi$ if and only if $X$ is of the form $L_\alpha$ for some ordinal $\alpha$. We will call this sentence “I am an $L$-level” from now on.

### 3. $\alpha$-ITRMs

In this section, we will consider recognizability of subsets of $\alpha$ for $\alpha$-ITRMs. In particular, we will prove that lost melodies exist for all exponentially closed $\alpha$.

We recall the following theorem from [5]; here, $\beta(\alpha)$ is the supremum of $\alpha$-ITRM halting times. ($ZF^-$ denotes Zermelo-Fraenkel set theory without the power set axiom; see [14] for a discussion of the axiomatizations.)

**Theorem 2.** For every $\alpha$, there is an ordinal $\gamma$ such that $x \subseteq \alpha$ is $\alpha$-ITRM-computable if and only if $x \in L_\gamma$. For every $\alpha$, $\gamma$ is smaller than the next $\Sigma_2$-admissible ordinal after $\alpha$. If $L_\alpha \models ZF^-$, then $\gamma = \alpha + 1$. If $L_\alpha \not\models ZF^-$, then $\gamma = \beta(\alpha)$.

**Proof.** It is shown in [5] that $\gamma = \alpha + 1$ if and only if $L_\alpha \models ZF^-$ (Theorem 1) and that $\gamma$ is the supremum of the $\alpha$-ITRM-clockable ordinals otherwise (Theorem 42). The upper bound is Corollary 3.4.13 of [3]. □

It was shown in [5] that, for $\alpha$ ITRM-singular, there is a lost melody for $\alpha$-ITRMs, namely the halting set (encoded as a subset of $\alpha$). We start by showing that the extra condition is in fact unnecessary and
that there are constructible lost melodies for all exponentially closed ordinals $\alpha$.

We also recall the following generalization of results by Koepke and Seyfferth [22], which is Theorem 2.3.28(iii) of [3].

**Lemma 6.** For every exponentially closed $\alpha$ and any $n \in \omega$, there is an $\alpha$-ITRM-program $P_{\alpha-ntrth}$ such that, for every formula $\phi$ that starts with $n$ quantifier alternations, followed by a quantifier-free formula and every $x \subseteq \alpha$, $P_{\alpha-ntrth}(\phi, x) \downarrow = 1$ if and only if $\phi$ holds in the structure coded by $x$, and otherwise, $P_{\alpha-ntrth} \downarrow = 0$.

**Lemma 7.** Let $\alpha$ be exponentially closed, $\gamma \in On$, and let $c \subseteq \alpha$ be a nice $\alpha$-code for $L_\gamma$ with corresponding bijection $f : \alpha \to L_\gamma$. Moreover, let $\phi, \psi$ be $\in$-formulas.

1. The set $\{\xi < \alpha : \text{Def}(L_\gamma, \phi, f(\xi)) \models \psi\}$ is $\alpha$-ITRM-decidable relative to $c$ in the parameter $\zeta$.

2. If $X \subseteq \mathcal{P}(\alpha)$ is $\alpha$-ITRM-decidable, then $f^{-1}(\min\{\xi < \gamma : \text{Def}(L_\gamma, \phi, \xi) \in X\})$ is uniformly $\alpha$-ITRM-computable without parameters in the oracle $c$.

**Proof.**

1. By Lemma 6 there is an $\alpha$-ITRM-program $P$, which, for any $\xi < \alpha$, computes $c_\xi := \{\iota < \alpha : f(\iota) \in \text{Def}(L_\gamma, \phi, f(\xi))\}$ in the oracle $c$. By niceness of $c$, it is easy to check whether all elements of $c_\xi$ code elements of $\alpha$ (one only needs to check whether $c_\xi$ consists of limit ordinals). If that is the case, then, from $c_\xi$, one can compute $c'_\xi := \{\iota < \alpha : \omega \iota \in c_\xi\}$, and then, again by Lemma 6, one can check whether the structure coded by $c'_\xi$ is a model of $\psi$.

2. This can be done by two nested searches through $\alpha$: For each $\zeta < \alpha$, first check, using Lemma 6, whether $\zeta$ codes an ordinal in $c$, then compute $c_\zeta := \text{Def}(L_\gamma, \phi, f(\zeta))$ (again using Lemma 6 to evaluate $\phi$), then decide whether $c_\zeta \in X$ If any of these is not the case, we continue with $\zeta + 1$. Otherwise, we run through all $\iota < \alpha$, checking first whether $f(\iota)$ is an ordinal such that $f(\iota) < f(\zeta)$ and then whether $c_\iota \in X$. If such a $\iota$ is found, we continue with $\zeta + 1$. Otherwise, we halt with output $\zeta$. 

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2 The condition of exponential closure is a technical convenience; it allows us, for example, to carry out halting algorithms after each other or run nested loops of algorithms without caring for possible register overflows. We conjecture that dropping this condition would not substantially change most of the results, but merely lead to more cumbersome arguments.

3 What we really mean by this is that we run the decision algorithm for $X$ and, whenever a bit of the oracle is requested, we use the algorithm computing $c_\zeta$ to compute the relevant bit. It is, however, heuristically much easier, and also harmless, to think about this as a two-staged process in which $c_\zeta$ is first computed and then the decision algorithm is applied to it. We will use this manner of speaking in the following without further discussion.
Lemma 8. Let \( \alpha \) be exponentially closed.

(1) If \( \alpha \) is not a regular cardinal in \( L \), then there are a constructible set \( s \subseteq \alpha \) and a program \( P_{\alpha-WO} \) such that, for all \( x \subseteq \alpha \), \( P_{x^{\infty}WO} \downarrow = 1 \) if and only if \( x \) codes a well-ordering, and otherwise, \( P_{\alpha-WO} \downarrow = 0 \). In fact, \( s \) can be taken to be \( \alpha \)-ITRM-recognizable without parameters.

(2) (3, Exercise 2.3.26) If \( \alpha = \omega \) or \( c_f^V(\alpha) > \omega \), then there is such a program \( P_{\alpha-WO} \) that works in the empty oracle.

Proof. By simply searching through \( \alpha \), it is easy to check whether a given \( x \subseteq \alpha \) codes a linear ordering. We are thus left with checking whether this ordering is well-founded.

(1) In [3], Theorem 2.3.25, the well-foundedness test on ITRMs by Koepke and Miller [21] is generalized to \( \alpha \)-ITRMs when \( \alpha \) is ITRM-singular, i.e., there is an \( \alpha \)-ITRM-computable cofinal function \( f : \beta \to \alpha \) for some \( \beta < \alpha \). Since we assume that \( \alpha \) is not a regular cardinal in \( L \), there is a constructible cofinal function \( f : \beta \to \alpha \) for some \( \beta < \alpha \). Let \( s = \{ p(\iota, f(\iota)) | \iota < \beta \} \).

Then \( \alpha \) is ITRM-singular in the oracle \( s \), and one can easily check that the argument in [3] does not depend on whether the program computing the singularization uses an oracle.

To see that we can take \( s \) to be \( \alpha \)-ITRM-recognizable without parameters, let \( \gamma \) be minimal with the property that \( L_\gamma \models \text{“}\alpha \text{ is singular.”} \)

It follows by fine-structure that \( L_{\gamma+1} \) contains a bijection \( f : \alpha \to L_\gamma \) and thus, by Lemma 2 a very nice code \( c \) for \( L_\gamma \). It is thus definable over \( L_\gamma \), and we can assume without loss of generality that the definition uses a single ordinal parameter \( \rho < \gamma \). Let \( c = \{ x \in L_\gamma : L_\gamma \models \phi(x, \rho) \} \). Without loss of generality, assume that \( \rho \) is the minimal ordinal \( \xi \) such that \( \{ x \in L_\gamma : L_\gamma \models \phi(x, \xi) \} \) is a code for \( L_\gamma \). We claim that \( c \) is \( \alpha \)-ITRM-recognizable without parameters.

So let \( x \subseteq \alpha \) be given in the oracle. By two nested searches through \( \alpha \), we check whether, for all \( \iota, \xi < \alpha \), we have \( p(\iota, \omega \xi) \in x \) if and only if \( \iota \) is of the form \( \omega \iota \) for some \( \iota < \xi \). We then check whether, for all \( \iota < \alpha \), \( p(\iota, 1) \) if and only if \( \iota \) is a limit ordinal. If not, \( x \) cannot be a very nice \( \alpha \)-code, and we halt with output 0. Otherwise, we check, using Lemma 6 whether the \( \in \)-structure \( (X, \in) \) coded by \( x \) is a model of \( V = L \), of “\( \alpha \) is singular” and of “There is no \( \delta \) such that \( L_\delta \models \text{“}\alpha \text{ is singular.”} \”), if not, then \( x \) is not a code for \( L_\gamma \), and we halt with output 0. (Note that, at this point, we have not yet checked the well-foundedness of \( (X, \in) \).) By searching through \( \alpha \), we identify the minimal \( \iota \) that codes a cofinal function \( f : \beta \to \alpha \) for some \( \beta < \alpha \) in \( x \). More precisely, we check, for each \( \sigma < \alpha \), whether \( \sigma \) codes a set of ordered pairs, whether all first elements of such pairs are below
some $\beta < \alpha$ (this is possible by searching through all such $\beta$), whether the set of these first elements is downwards closed and whether, for each $\delta < \alpha$, there is a second element of such a pair that is larger (which is again possible by searching through $\alpha$). This search is guaranteed to be successful, as $(X, \in)$ believes $\alpha$ to be singular. Once such a $\sigma$ has been found, we stop the search and continue. Note that at this point we know that $\alpha + 1$ belongs to the well-founded part of $(X, \in)$; it follows that $\sigma$ codes an actual singularization $f$ of $\alpha$. By another search, we can identify the ordinal $\beta$ that is the domain of $f$. Now $f$ is $\alpha$-ITRM-computable from $x$: Namely, given $i < \beta$, we search through $\alpha$ for an element $\xi$ such that $p(p(\omega, \omega_\xi), \sigma) \in x$.

As described above, we can now use $f$ to perform well-foundedness checks on subsets of $\alpha$. In particular, once we have finished the proof of the recognizability of $c$, we know how to perform well-foundedness checks in the oracle $c$.

In particular, we can now use $f$ to perform a well-foundedness check on $x$. If this is successful, we know that $x$ is a very nice code for $L_{\gamma}$ (otherwise, we halt with output 0). Let $g : \alpha \to L_{\gamma}$ be the corresponding bijection. It remains to see whether $x = c$.

Moreover, we can use $f$, along with the above procedure, to decide the set of very nice codes for $L_{\gamma}$. Let $P_{\text{vn}}$ be the $\alpha$-ITRM-program that does this.

By Lemma 7, we can now identify the ordinal $\xi < \alpha$ such that $g(\xi)$ is the minimal ordinal for which $\text{Def}(L_{\gamma}, \phi, g(\xi))$ is a very nice $\alpha$-code for $L_{\gamma}$.

By definition, we have $\text{Def}(L_{\gamma}, \phi, g(\xi)) = c$. Using Lemma 4 and the niceness of $x$, we can now compute $c$ from $x$; by comparing it bitwise to $x$, we finally determine whether $x = c$.

(2) For $\alpha = \omega$, this is proved in Koepke and Miller [21]. We now assume that $\alpha$ is uncountable. By assumption, every countable sequence of elements of $\alpha$ is bounded in $\alpha$. It thus suffices to search through all sequences bounded by $\beta$, for all elements of an unbounded set of ordinals $\beta$ below $\alpha$. Since $\alpha$ is exponentially closed, it suffices to consider multiplicatively closed values of $\beta$. To this end, count upwards to $\alpha$ in some separate register $R$ and consider the multiplicatively closed ordinals $\beta$ occurring in $R$. By [3], Theorem 2.3.25(ii), there is, for each such $\beta$, an $\alpha$-wITRM-program $P$ that checks subsets of $\beta$ for coding well-orderings. Moreover, this program is uniform in $\beta$.
Remark 2. Note that, if one assumes $V = L$, one of the cases of Lemma 5 will be true for any exponentially closed $\alpha \in On$. The only problematic case regular $L$-cardinals that have cofinality $\omega$ in $V$. We do not know whether a well-foundedness test exists for such ordinals.

Corollary 1. Let $\alpha$ be an exponentially closed ordinal.

1. If $\alpha$ that is ITRM-singular, equal to $\omega$ or satisfies $\text{cf}^{V}(\alpha) > \omega$, there is an $\alpha$-ITRM-program $P_{\text{nice}}$ that decides $\text{nice}_{\alpha}$ without parameters.

2. If $\alpha$ is not a regular cardinal in $L$, $\alpha = \omega$ or $\text{cf}^{V}(\alpha) > \omega$, there is a parameter-freely $\alpha$-ITRM-recognizable $s \subseteq \alpha$ such that, for some $\alpha$-ITRM-program $P_{\text{nice}}$, $P_{\text{nice}}^{s}$ decides $\text{nice}_{\alpha}$ without parameters. If $V = L$, this is true for all $\alpha$.

Proof. (1) By Lemma 5(2), it is possible to check whether $x$ codes a well-founded structure $(S, \in)$. By Lemma 6 and Lemma 5, it is possible to check whether $(S, \in)$ is an $L$-level. To check whether $x$ is also nice, run through $\alpha$ and check, for each $\iota, \xi < \alpha$ whether $p(\iota, \omega \xi) \in x$ if and only if $\iota$ is a limit ordinal smaller than $\omega \xi$.

(2) The possibility of a well-foundedness check relative to some $\alpha$-ITRM-recognizable $s \subseteq \alpha$ is now guaranteed by Lemma 8(1). The rest then works as in (1). If $V = L$, then every exponentially closed ordinal is either equal to $\omega$, not a regular cardinal in $L$ or has a cofinality $> \omega$, so one of the conditions is met.

We will freely use the following result:

Corollary 2. Let $\alpha$ be exponentially closed. There is an $\alpha$-ITRM-program $P$ such that, for any $x \subseteq \alpha$ that nicely codes a transitive $\in$-structure containing $\alpha$, $P^{x}$ halts with output $\iota$, where $\iota$ codes $\alpha$ in $x$. In particular, whenever there is a program for deciding the set of nice names, there is also one for deciding very nice names.

Proof. $P$ works by running through $\alpha$, checking, for each $\xi < \alpha$, whether $p(\iota, \xi) \in x$ if and only if $\iota$ is a limit ordinal. Once such $\xi$ has been found, $P$ writes it to the output register and halts.

Lemma 9. Let $\alpha$ be exponentially closed and let $\gamma$ be an ordinal such that $c_{\gamma}$ exists, $c_{\gamma} \in (L_{\gamma+1} \setminus L_{\gamma})$ and such that $\text{nice}_{\alpha}(\gamma)$ is $\alpha$-ITRM-decidable (possibly in some oracle $s$). Then some element of $(L_{\gamma+1} \setminus L_{\gamma}) \cap \text{nice}_{\alpha}(\gamma)$ is $\alpha$-ITRM-recognizable (in the oracle $s$). If $\text{nice}_{\alpha}(\gamma)$ is $\alpha$-ITRM-decidable without parameters, then there is an element of $(L_{\gamma+1} \setminus L_{\gamma}) \cap \text{nice}_{\alpha}(\gamma)$ that is $\alpha$-ITRM-recognizable without parameters.

Proof. Let $P$ be an $\alpha$-ITRM-program that decides $\text{nice}_{\alpha}(\gamma)$. By assumption, there is a formula $\psi'$ such that $c_{\gamma} = \text{Def}(L_{\gamma}, \psi', \vec{p})$ for some finite $\vec{p} \subseteq L_{\gamma}$. It is well-known that this implies the existence of a formula $\psi$ such that $c_{\gamma} = \text{Def}(L_{\gamma}, \psi, \vec{p})$ for some finite $\vec{p} \subseteq \gamma$. To simplify
our notation, we assume that $\vec{p}$ consists of a single element ordinal. Let $\rho < \alpha$ be the minimal ordinal such that $c := \text{Def}(L, \psi, \rho) \in \text{nice}_\alpha(\gamma)$. We claim that $c$ is $\alpha$-ITRM-recognizable.

To see this, let $x \subseteq \alpha$ be given in the oracle. Using $P$, we first decide whether $x \in \text{nice}_\alpha(\gamma)$. If not, we halt with output 0. Otherwise, we know that $x$ nicely codes $L$. We run through $\alpha$ in some register. For each $\iota < \alpha$, let $x_\iota$ be the element of $L$ coded by $\iota$ in $x$. Using Lemma 6 we can check whether $x_\iota$ is an ordinal. If not, we continue with $\iota + 1$. If it is, we again use Lemma 6 to check whether $d_\iota := \text{Def}(L, \psi, x_\iota) \subseteq \alpha$:

More precisely, we use $P_{\alpha-n\text{-truth}}$ for some sufficiently large $n$ to decide, for each $\xi < \alpha$, whether or not $L_\gamma \models \psi(x_\xi, x_\iota)$, and if that is true, we check whether $\xi$ is a limit ordinal. If that is false for any $\xi$, we halt with output 0. Otherwise, we know that $d_\iota \subseteq \alpha$ and we can again use $P$ to check whether $d_\iota \in \text{nice}_\alpha(\gamma)$. If not, we continue with $\iota + 1$. Otherwise, we must have $x_\iota = \rho$, and all that remains to check is whether $d_\iota = x$. If that is the case, we halt with output 1, otherwise, we halt with output 0.

Lemma 10. Let $\alpha$ be an exponentially closed ordinal, and let $n \in \omega$. Let $\Theta$ be an $\alpha$-ITRM-decidable set of $\Sigma_n$-sentences using parameters $\leq \alpha$, and let $\gamma > \alpha$ be minimal such that $L_\gamma \models \Theta$. Let $s \subseteq \alpha$ be such that $P_{\text{nice}}$ works in the oracle $s$ (which includes the possibility that $s = \emptyset$). Then $\text{nice}_\alpha(\gamma)$ is $\alpha$-ITRM-decidable in the oracle $s$. If $\Theta$ only uses the parameter $\alpha$ and $P_{\text{nice}}$ works parameter-freely, then $\text{nice}_\alpha(\gamma)$ is $\alpha$-ITRM-decidable (in the oracle $s$) without parameters.

Proof. Let $x \subseteq \alpha$ be given. Using $P_{\text{nice}}$ (and the oracle $s$, if necessary), we can check whether $x$ is a nice code for some $L$-level $L_\delta$. By running through all $\iota < \alpha$ and checking for each whether, for all $\xi < \alpha$, $p(\xi, \iota) \in x$ if and only if $\xi$ is a limit ordinal, we can identify the ordinal $t_\alpha < \alpha$ which codes $\alpha$ in the sense of $x$ (if no such $t_\alpha$ exists, then $\delta \leq \alpha < \gamma$, so we halt with output 0). Using $P_{\alpha-n\text{-truth}}$, we can now check, for all $\theta \in \Theta$, whether $L_\delta \models \theta$; note that the ordinals encoding parameters $< \alpha$ can easily be found due to $x$ being nice, while the ordinal coding $\alpha$ is at this point known to be $t_\alpha$. If this is not the case for some $\theta$, we halt with output 0. Otherwise, we halt with output 1.

Theorem 3. For every exponentially closed $\alpha$, there is a lost melody for $\alpha$-ITRMs, i.e., a set $x \subseteq \alpha$ such that $x$ is $\alpha$-ITRM-recognizable, but not $\alpha$-ITRM-computable. In fact, we can take $x$ to be $\alpha$-ITRM-recognizable without parameters.

Proof. By Theorem 2 every $\alpha$-ITRM-computable $x \subseteq \alpha$ will be an element of $L_{\beta(\alpha)}$. It thus suffices to find a recognizable subset of $\alpha$ that is not contained in $L_{\beta(\alpha)}$.

We split the proof into two cases, depending on whether or not $\alpha$ is a regular cardinal in $L$. 
Case 1: $\alpha$ is not a regular cardinal in $L$.

By Corollary[1](2), pick a parameter-free $\alpha$-ITRM-recognizable $s \subseteq \alpha$ (possibly empty) and an $\alpha$-ITRM-program $P_{\text{nice}}$ such that $P_{\text{nice}}$ decides nice$_s$. We claim that there is a very nice $\alpha$-ITRM-recognizable $\alpha$-code $c$ for $L_{\beta(\alpha)}$. Let $\phi(\alpha)$ be the $\in$-sentence “For each $\alpha$-ITRM-program $P$, $P$ either halts or runs into a strong loop”. Then $\beta(\alpha)$ is the minimal $\gamma$ with the property that $L_\gamma \models \phi(\alpha)$. By Lemma[10] nice$_s(\gamma)$ is decidable without parameters in the oracle $s$. Moreover, by fine-structure, $L_{\beta(\alpha)}$ is the $\Sigma_2$-Skolem hull of $\alpha + 1$ in itself, so a bijection $f : \alpha \to L_{\beta(\alpha)}$ is definable over $L_{\beta(\alpha)}$ and thus, we have nice$_s(\gamma) \cap (L_{\beta(\alpha)+1} \setminus L_{\beta(\alpha)}) \neq \emptyset$. By Lemma[9] nice$_s(\beta(\alpha)) \cap (L_{\beta(\alpha)+1} \setminus L_{\beta(\alpha)})$ contains an element $c$ that is $\alpha$-ITRM-recognizable without parameters relative to $s$.

But now, $s \oplus c$ is clearly $\alpha$-ITRM-recognizable: Given the input $x \subseteq \alpha$, first decompose $x := x_0 \oplus x_1$, then check whether $x_0 = s$ and, if so, use $s$ to check whether $x_1 = c$. Moreover, since $c \notin L_{\beta(\alpha)}$ by definition, we also have $s \oplus c \notin L_{\beta(\alpha)}$, so $s \oplus c$ is as desired.

Case 2\footnote{This case uses ideas similar to those used for proving the lost melody theorem for infinite time Blum-Shub-Smale machines, see [9].}: $\alpha$ is a regular cardinal in $L$. In this case, we have $L_\alpha \models \text{ZF}^-$, and so it follows from Theorem[2] that the $\alpha$-ITRM-computable subsets of $\alpha$ are exactly those in $L_{\alpha+1}$. By Lemma[1] it suffices to show that there is an $\alpha$-ITRM-recognizable nice $\alpha$-code for $L_{\alpha+1}$. Since $L_{\alpha+2}$ clearly contains a bijection $g : \alpha \to L_{\alpha+1}$, some very nice code for $L_{\alpha+1}$ is contained in $L_{\alpha+2}$ by Lemma[2] and thus definable over $L_{\alpha+1}$. Say such a code is defined over $L_{\alpha+1}$ by the formula $\phi$ in the parameter $\rho \in \alpha + 1$. Without loss of generality, we assume that $\rho$ is minimal with the property that $c := \text{Def}(L_{\gamma+1}, \phi, \rho)$ is a very nice code for $L_{\alpha+1}$. Let $\zeta := f^{-1}(\rho)$ (thus, either $\zeta = \omega \rho$ if $\rho < \alpha$ or $\zeta = 1$ if $\rho = \alpha$). We claim that $c$ is $\alpha$-ITRM-recognizable without parameters.

To see this, let $d \subseteq \alpha$ be given in the oracle. We start by checking, as in the proof of Lemma[8] whether each $\iota < \alpha$ is coded by $\omega \iota$ and whether $\alpha$ is coded by $1$ in $d$.

Using bounded truth predicate evaluation from Lemma[6], we can now run through $\alpha$ and check, for each $\iota < \alpha$, whether the $\in$-structure coded by $d$ believes that $\iota$ codes an ordinal if and only if $\iota$ is either $1$ or a limit ordinal. If not, we halt with output $0$; otherwise, we know that the set of ordinals coded by $d$ is equal to $\alpha + 1$.

Again using bounded truth predicate evaluation, we check whether the structure coded by $d$ is a model of the sentence “I am an $L$-level”. If not, we halt with output $0$. Otherwise, we know that $d$ codes $L_{\alpha+1}$, and it remains to check that $d$ is the “right” code. But this can be checked as in the proof of Lemma[8](1) by searching through $\alpha$ for the
minimal ordinal parameter for which $\phi$ defines a very nice $\alpha$-code for $L_{\alpha+1}$, and, once it is found, using it to compute $c$ and compare it to $d$. □

Remark 3. Note that, in case (2), we have obtained a lost melody at the lowest possible $L$-level, since all elements of $L_{\alpha+1}$ are $\alpha$-ITRM-computable, and hence -recognizable by computing them and then comparing them to the oracle bit by bit.

For reasons explained above, we will be concerned with constructible sets in this paper. It should, however, be noted that the relation between recognizability in $L$ and in $V$ is not trivial: On the one hand, recognizable sets can be non-constructible (see [4]). On the other hand, it is not clear that a set that is recognizable in $L$ is also recognizable in $V$: If $P$ recognizes $x$ in $L$, then $V$ may contain some $y \neq x$ such that $P^y \downarrow = 1$, thus spoiling $P$’s ability to recognize $x$. Even for constructible sets $x$, we thus need to carefully distinguish between being $\alpha$-ITRM-recognizable and being $\alpha$-ITRM-recognizable in $L$.

Question 2. Is there an ordinal $\alpha$ and a set $x \subseteq \alpha$ such that $x$ is $\alpha$-(w)ITRM-recognizable in $L$, but not in $V$?

Passing over to what in [4] is called the “recognizable closure”, we can exclude this possibility, at least in many cases. Let us say that $x \subseteq \alpha$ belongs to the $\alpha$-(w)ITRM-recognizable closure if and only if there is $y \subseteq \alpha$ such that $x \oplus y$ is $\alpha$-(w)ITRM-recognizable, and denote this by $c\text{RECOG}^{\text{(w)ITRM}}_{\alpha}$.

Theorem 4. If $\alpha$ is exponentially closed and either not a regular cardinal in $L$ or satisfies $\text{cf}^V(\alpha) > \omega$, and $M \models \text{ZFC}$ is a transitive class, we have $(c\text{RECOG}^{\text{(w)ITRM}}_{\alpha})^L \subseteq (c\text{RECOG}^{\text{(w)ITRM}}_{\alpha})^M$.

Proof. Let $\alpha$ be an exponentially closed ordinal, and let $x \in (c\text{RECOG}^{\text{(w)ITRM}}_{\alpha})^L$. Thus, there is $y \in L$ such that $x \oplus y$ is $\alpha$-ITRM-recognizable in $L$. We show that, for some $c \in \mathcal{P}^L(\alpha)$, $(x \oplus y) \oplus c$ is $\alpha$-ITRM-recognizable in $V$. It then follows by absoluteness of computations that $(x \oplus y) \oplus c$ is $\alpha$-ITRM-recognizable in any transitive $M \models \text{ZFC}$ with $L \subseteq M \subseteq V$. For simplicity of notation, let us denote $x \oplus y$ by $z$.

So suppose that $z$ is $\alpha$-ITRM-recognizable in $L$ by the program $P$ in the parameter $\rho < \alpha$. Let $\phi(\alpha)$ be the sentence $\exists a P^a(\rho) \downarrow = 1$, and let $\gamma > \alpha$ be minimal such that $L_\gamma \models \phi(\alpha)$. By absoluteness of computations, this implies $z \in L_\gamma$. By Lemma [11] we pick an $\alpha$-ITRM-recognizable $s \subseteq \alpha$ such that $P_{\text{nice}}$ works in the oracle $s$. The only place where $s$ is used in the proof of Lemma [8] is in checking well-foundedness, and the proof of Lemma [8] shows that $s$ is $\alpha$-ITRM-recognizable in $V$.

6 As we will see in Corollary [11] below, this procedure does no longer work for $\alpha$-wITRM.
By Lemma 10, nice_{\alpha}(\gamma) is \alpha-ITRM-decidable. By fine-structure and Lemma 2, there is a nice \alpha-code for \gamma in \gamma_{\alpha+1} \setminus \gamma. By Lemma 9, there is \( c \in \text{nice}_{\alpha}(\gamma) \cap \gamma_{\alpha+1} \) which is \alpha-ITRM-recognizable (in the oracle \( s \)). It follows that \( s \oplus c \) is \alpha-ITRM-recognizable, as in the proof of Theorem 3. Since \( z \in \gamma \), there is some \( \xi < \alpha \) which codes \( z \) in the sense of \( c \). But now, given the parameter \( \xi, z \) is \alpha-ITRM-computable, and hence \alpha-ITRM-recognizable, relative to \( s \oplus c \). Hence \( z \oplus (s \oplus c) \) is \alpha-ITRM-recognizable, so \( s \oplus c \) is as desired.  

□

**Definition 9.** Denote by \( \rho(\alpha) \) the supremum of the set of ordinals \( \beta \) for which \( \gamma_{\sigma+1} \setminus \gamma_{\sigma} \) contains an \alpha-ITRM-recognizable subset of \( \alpha \). Similarly, we write \( \rho^w(\alpha) \) for the analogous concept for \alpha-wITRMs.

Moreover, denote by \( \theta(\alpha) \) and \( \theta^w(\alpha) \) the suprema of ordinals with an \alpha-ITRM-computable and an \alpha-wITRM-computable \alpha-code, respectively.

**Remark 4.** It was shown in [5] that \( \theta(\alpha) = \beta(\alpha) \) unless \( L_{\alpha} \models ZF^- \).

The only property of \( \alpha + 1 \) used in the argument for the existence of a recognizable nice \alpha-code for \( \gamma_{\alpha+1} \) is the existence of an \alpha-ITRM-computable \alpha-code for it. The same argument hence yields:

**Corollary 3.** If \( \beta \) has an \alpha-ITRM-computable \alpha-code, then \( \gamma_{\beta} \) has an \alpha-ITRM-recognizable \alpha-code. In particular, we have \( \rho(\alpha) \geq \theta(\alpha) \).

It is shown in [6] that \( \rho^w(\omega) = \theta^w(\omega) \). We currently do not know whether \( \rho(\alpha) = \beta(\alpha) \) for any exponentially closed \( \alpha \) (it is known to be false for \( \alpha = \omega \)).

In the first case, however, we can also get more precise information about their distribution\(^7\)

**Definition 10.** We say that \( x \subseteq \alpha \) is \alpha-ITRM-quick if and only if \( x \in \gamma_{\sigma(\alpha)} \).

**Theorem 5.** Let \( \alpha \) be an exponentially closed ordinal that is not a regular cardinal in \( L \) or satisfies \( \text{cf}(\alpha) > \omega \).

1. (Cf. [7], Theorem 27(i) for the ITRM-version) The constructible subsets of \( \alpha \) that are \alpha-ITRM-recognizable with parameters are contained in \( \gamma_{\sigma(\alpha)} \), and those that are \alpha-ITRM-recognizable without parameters are contained in \( \gamma_{\sigma(\alpha)}^\omega \).

2. (Cf. [7], Theorem 27(ii) for the ITRM-version) \( \sigma(\alpha) \) (and \( \sigma^\omega \), respectively) are minimal with this property. Thus \( \rho(\alpha) = \sigma(\alpha) \).

3. (Cf. [7], Theorem 27(iii)) For any \( \delta < \sigma(\alpha) \), there is a “gap” of length \( \geq \delta \) in the \alpha-ITRM-recognizables; that is, there are ordinals \( \beta, \gamma, \eta \) such that \( \beta + \delta \leq \gamma < \eta \), \( \gamma - \sigma(\alpha) \) contains no

\(^7\)The following result are the analogues of results obtained about ITRMs in [7] and [8] for ITRM-singular \( \alpha \).
α-ITRM-recognizable subsets of α, \( L_\eta \setminus L_\gamma \) does contain an α-ITRM-recognizable subset of α, and for cofinally in γ many ξ, we have \((L_{\xi+1} \setminus L_\xi) \cap \mathfrak{P}(\alpha) \neq \emptyset\). The same is true for α-ITRM-recognizability without parameters.

(4) (Cf. [8], Theorem 5.2 for an ITRM-version) For all γ, either all α-ITRM-quick elements of \( \mathfrak{P}(\alpha) \cap (L_{\gamma+1} \setminus L_\gamma) \) are α-ITRM-semirecognizable or none is. The same is true for α-wITRMs and for α-ITRM-semirecognizability without parameters.

(5) If α is ITRM-singular or \( c^\alpha(\alpha) > \omega \), then, for each γ such that \( \mathfrak{P}(\alpha) \cap (L_{\gamma+1} \setminus L_\gamma) \neq \emptyset \), \( L_{\gamma+1} \setminus L_\gamma \) contains an α-ITRM-recognizable subset of α if and only if \( c_\gamma \) is α-ITRM-recognizable. The same is true without parameters.

Proof. The proofs are adaptations of those given for ITRMs in [7] and [8]. We only prove the versions in which parameters are allowed; the proofs for the parameter-free versions are entirely analogous.

1. Let \( P \) be an α-ITRM-program that recognizes a constructible set \( X \subseteq \alpha \) in the parameter \( \zeta < \alpha \). The statement “There is \( Y \subseteq \alpha \) such that \( P^Y(\zeta) \downarrow = 1 \)” is \( \Sigma_1 \) in the parameters \( \zeta \) and \( \alpha \). Let us write this statement as \( \exists x, c \phi(x, c) \), where \( \phi(x, c) \) is the \( \Delta_0 \)-statement “\( c \) is a halting computation of \( P \) in the oraced \( x \) with output 1”. By assumption, this statement is true in \( L \). Thus, by definition of \( \sigma_{\alpha+1} \), it holds in \( L_{\sigma_{\alpha+1}} \). So there are \( x, c \in L_{\sigma_{\alpha+1}} \) such that \( \phi(x, c) \). Since computations are absolute between transitive \( \in \)-structures, \( c \) is actually such a computation, so we must have \( x = Y \), so that \( Y \in L_{\sigma_{\alpha+1}} \). By Lemma 4 we have \( \sigma_\alpha = \sigma_{\alpha+1} \), so \( Y \in L_\alpha \).

2. Let \( \beta < \sigma_\alpha \). We will show that there is an α-ITRM-recognizable subset of \( \alpha \) that is not contained in \( L_\beta \). To this end, pick, by definition of \( L_{\sigma_\alpha} \), an ordinal \( \gamma \in (\beta, \sigma_\alpha) \) such that, for some \( \Sigma_1 \)-statement \( \phi(\rho) \) with parameter \( \rho \in \alpha \), we have \( L_\gamma \models \phi(\rho) \) and \( \gamma \) is minimal with this property. By fine-structure, \( L_{\gamma+1} \) then contains a nice \( \alpha \)-code for \( L_\gamma \). By Lemma 9 and Lemma 11 there is an \( \alpha \)-recognizable nice \( \alpha \)-code \( c \) for \( L_\gamma \) in \( L_{\gamma+1} \setminus L_\gamma \). Since \( \gamma > \beta \), we have \( c \notin L_\beta \).

3. Pick \( \beta > \sigma_\alpha \), along with a limit \( \lambda \) of \( \alpha \)-indices greater than \( \beta + \delta \) which is at the same time \( \Sigma_2 \)-admissible. Then \( L_\lambda \) is a model of the statement \( \psi(\alpha, \delta) \), which is “There is an ordinal \( \beta \) such that \( \beta + \delta \) exists and all \( \alpha \)-ITRM-recognizable subsets of \( \alpha \) are contained in \( L_\beta \)”, and also of “There are cofinally many \( \alpha \)-indices”. Thus, the statement that there exists an \( L \)-level with these properties is true in \( L \); moreover, it is easily seen to be \( \Sigma_1 \) in the parameters \( \alpha \) and \( \delta \). Consequently, it will be true in \( L_{\sigma_{\alpha+1}} \), and hence in \( L_{\sigma_\alpha} \). Hence, there is a \( \Sigma_2 \)-admissible ordinal \( \lambda \in L_{\sigma_\alpha} \) which is a limit of \( \alpha \)-indices and believes \( \psi(\alpha, \delta) \). Pick
a witness $\beta$ for this statement. Assume for a contradiction that some element $X$ of $(L_\lambda \setminus L_\delta) \cap \Psi(\alpha)$ is $\alpha$-ITRM-recognizable, say by the program $P$. It follows from $\[5\]$, Theorem 46 by the $\Sigma_2$-admissibility of $\lambda$ that $P^Y$ will either halt or run into a strong loop by time $\lambda$ for all $Y \in L_\lambda \cap \Psi(\alpha)$. If the latter option were true for any such $Y$, then $P^Y$ would actually be looping and hence $P$ could not recognize $X$. Thus $P^Y$ halts in less than $\lambda$ many steps for all $Y \in L_\lambda$. Since $L_\lambda$ does not believe $X$ to be recognizable by $P$, we either must have $P^X \downarrow = 0$ or $P^Y \downarrow = 1$ for some $Y \in L_\lambda \cap \Psi(\alpha)$ different from $X$. But both options contradict the assumption that $P$ recognizes $X$ by absoluteness of computations.

(4) (For simplicity, we drop mentioning of parameters in this proof.) Let $x, y \in \Psi(\alpha) \cap (L_{\gamma+1} \setminus L_\gamma)$ be $\alpha$-ITRM-quick and assume that $x$ is $\alpha$-ITRM-semirecognizable. Pick a program $P$ that semi-recognizes $x$. By assumption, $y \in L_{\gamma+1}$, and because $x$ is $\alpha$-ITRM-quick, we have $\beta^x(\alpha) > \gamma + 1$, and by Lemma $[2]$ we have $y \in L_{\gamma+1} \subseteq L_{\beta^x(\alpha)} \subseteq L_{\beta^y(\alpha)}[x]$. So $y$ is $\alpha$-ITRM-computable from $x$. Let $Q_{xy}$ be an $\alpha$-ITRM-program that computes $y$ in the oracle $x$. By the same argument, let $Q_{yx}$ be an $\alpha$-ITRM-program that computes $x$ in the oracle $y$. To semi-recognize $y$, proceed as follows: Given $z \subseteq \alpha$, first run $P^z_{yx}(\iota)$ for all $\iota < \alpha$. If this does not halt (or if $P^z_{yx}(\iota)$ halts with an output different from 0 or 1 for some $\iota < \alpha$), then $z \neq x$, and so the machine behaves as expected. If it halts, let $z'$ be the subset of $\alpha$ computed by this program. Run $P^{z'}$. If this computation does not halt, then $z' \neq x$, which means (by definition of $Q_{xy}$) that $z \neq y$. If it does halt, then we know that $z' = x$, so we run $Q^{z'}_{yx}(\iota)$ for all $\iota < \alpha$ and compare each bit to $z$. If they disagree in some place, we run into and endless loop. Otherwise, by definition of $Q_{yx}$, we know that $z = y$, so we halt.

(5) By a simple fine-structural argument, we have $c_\gamma \in L_{\gamma+1}$ for every such $\gamma$ (because a new subset of $\alpha$ is definable over $L_\gamma$, namely $x$), while $c_\gamma \notin L_\gamma$ by Lemma $[1]$. Thus, if $c_\gamma$ is $\alpha$-ITRM-recognizable, then $L_{\gamma+1} \setminus L_\gamma$ contains an $\alpha$-ITRM-recognizable element.

On the other hand, suppose that $x \in \Psi(\alpha) \cap (L_{\gamma+1} \setminus L_\gamma)$ is $\alpha$-ITRM-recognizable, and let $P$ be an $\alpha$-ITRM-program that recognizes $x$. We want to show that $c_\gamma$ is $\alpha$-ITRM-recognizable. So let $y \subseteq \alpha$ be given in the oracle.

By Lemma $[1]$ we can check whether $y$ is a nice name for an $L$-level. If not, we halt with output 0. Otherwise, let $L_\delta$ be the $L$-level coded by $y$, with bijection $f : \alpha \to L_\delta$. We run through $\alpha$, and, for each $\xi < \alpha$, we use Lemma $[6]$ to check whether $f(\xi) \subseteq \alpha$; if so, we check whether $P^{f(\xi)} \downarrow = 1$. If no
such element is found, we halt with output 0. If one is found, we check whether \( L_\delta \) contains an \( L \)-level that contains \( f(\xi) \). If that is the case, we halt with output 0. Otherwise, we know that \( y \) codes \( L_\gamma \). We have now shown how to decide nice \( \alpha(\gamma) \).

We know that \( c_\gamma = \text{Def}(L_\gamma, \phi, \rho) \) for some \( \epsilon \)-formula \( \phi \) and some \( \rho < \gamma \). Moreover, by minimality of \( c_\gamma \), we can assume without loss of generality that \( \rho \) is minimal with the property that \( \text{Def}(L_\gamma, \phi, \rho) \) is a nice code for \( L_\gamma \). By Lemma 7, we can compute from \( y \) the ordinal \( \zeta \) that codes \( \rho \). But now, we can use \( \zeta \) to compute \( c_\gamma \) from \( y \) and compare it to \( y \), thus deciding whether \( y = c_\gamma \).

If \( x \) is \( \alpha \)-ITRM-recognizable without parameters, then the argument shows that the same is true for \( c_\gamma \).

\( \square \)

**Question 3.** Is it true that, for all \( \gamma \), either all \( \alpha \)-ITRM-quick elements of \( \mathcal{P}(\alpha) \cap (L_{\gamma+1} \setminus L_\gamma) \) are \( \alpha \)-ITRM-recognizable or none is? (By \([8]\), Theorem 5.2, this is true for ITRMs.)

The Jensen-Karp-theorem ([17], section 5) states that \( \Sigma_1 \)-statements are absolute between \( V_\alpha \) and \( L_\alpha \) when \( \alpha \) is a limit or admissible ordinals. From this, and the fact that \( \beta^x(\omega) = \omega^x_{CK} \) for all \( x \subseteq \omega \) (Koepke, [19]) one obtains that, when \( x \) is \( \omega \)-ITRM-recognizable, then \( x \in L_{\omega^x_{CK}} \).

It is then natural to ask whether we have in general that \( x \in L_{\beta^x(\alpha)} \) when \( x \subseteq \alpha \) is \( \alpha \)-ITRM-recognizable. For the general case, we can only partial information so far.

**Definition 11.** An ordinal \( \alpha \) is called ITRM-countable if and only if there is an \( \alpha \)-ITRM-computable bijection \( f : \omega \to \alpha \).

**Lemma 11.**

1. If \( \alpha \) is ITRM-countable and \( x \subseteq \alpha \) is \( \alpha \)-ITRM-recognizable, then \( x \in L_{\beta^x(\alpha)} \).
2. If \( \alpha \) is a regular cardinal, then there is an \( \alpha \)-ITRM-recognizable set \( x \subseteq \alpha \) such that \( x \notin L_{\beta^x(\alpha)} \).

**Proof.**

1. As in Proposition 46 of [5], \( \beta^x(\alpha) \) is a limit of admissible ordinals. Let \( P \) be a program that recognizes \( x \). Then \( L_{\omega^x(\beta)}[x] = \exists y \exists P^y \models 1 \), which is \( \Sigma_1 \). Thus, by the Jensen-Karp theorem, the same statement holds in \( L_{\beta^x(\alpha)} \), so \( x \in L_{\beta^x(\alpha)} \).
2. By [3], Corollary 16, we have \( \beta^x(\alpha) = \alpha^\omega \) for all \( x \subseteq \omega \) in this case. We will show that \( L_{\alpha^\omega} \) has an \( \alpha \)-ITRM-recognizable very nice \( \alpha \)-code \( c \). By Lemma 1 we then have \( c \notin L_{\alpha^\omega} = L_{\beta^x(\alpha)} \). By Lemma 1 we can check whether a given \( x \subseteq \alpha \) is a very nice
α-code for an L-level $L_\delta$ with $\delta > \alpha$. One can now use Lemma 4 to check whether $L_\delta$ believes that $\alpha^n$ exists for all $n \in \omega$, but $\alpha^\omega$ does not exist. We can thus decide whether a given $x \subseteq \alpha$ is a very nice code for $L_\alpha^\omega$. It is standard that a bijection from $\alpha$ to $\alpha^\omega$, and hence from $\alpha$ to $L_\alpha^\omega$, is definable over $L_\alpha^\omega$, so, by Lemma 2, such a code is contained in $L_{\alpha^\omega+1} \setminus L_{\alpha^\omega}$. It now follows from Lemma 9 that there is an $\alpha$-ITRM-recognizable very nice code for $L_\alpha^\omega$ in $L_{\alpha^\omega+1} \setminus L_{\alpha^\omega}$ (which is in fact $\alpha$-ITRM-recognizable without parameters).

\[ \square \]

**Remark 5.** Note that the program that recognizes $c$ in part (2) uses a fixed natural number $n$ of registers, and will thus halt for any input in time $< \alpha^{n+1}$ by [5], Theorem 14. This observation thus reveals a considerable divergence between two natural notions of complexity for constructible subsets $x$ of $\alpha$, namely the “time complexity” of an $\alpha$-ITRM deciding \{x\} on the one hand and the constructible rank (i.e., the minimal index $\gamma$ of an L-level such that $x \in L_{\gamma+1} \setminus L_{\gamma}$) of $x$ on the other.

**Question 4.** Are there ordinal $\alpha$ such that $L_\alpha \not\models \text{ZF}^-$ such that, for some $\alpha$-ITRM-recognizable $x \subseteq \alpha$, we have $x \not\in L_{\beta^\omega\alpha}$?

This leaves the case of regular cardinals in $L$ somewhat under-explored. We currently do not know the minimal ordinal $\tau$ such that $L_\tau$ contains all $\alpha$-ITRM-recognizable subsets of $\alpha$ in this case. However, there is a concise characterization in this case. Recall from Hamkins and Leahy [16] that, for an $\in$-structure $M$, a set $X \subseteq M$ is called implicitly definable in $M$ if and only if, for some $\in$-formula $\phi$, $X$ is the unique subset of $M$ such that $(M,X) \models \phi(X)$.

**Theorem 6.** Let $\alpha$ be a regular cardinal. Then $X \subseteq \alpha$ is $\alpha$-ITRM-recognizable if and only if $X$ is implicitly definable in $L_\alpha$.

**Proof.** First suppose that $X$ is $\alpha$-ITRM-recognizable, and let $P$ be an $\alpha$-ITRM-program that recognizes $x$. Suppose that $P$ uses $n$ registers. By an easy relativization of the argument in [5], Theorem 14, $P^x$ halts in $< \alpha^n$ many steps. Using the “Pull-Back” technique from [5], Lemma 7, we can express the statement “$P^Y$ halts in $< \alpha^n$ many steps” as an $\in$-formula $\phi(Y)$ using $Y$ as an extra predicate over $(L_\alpha,Y)$. By assumption, $X$ is unique such that $(L_\alpha,X) \models \phi(X)$.

Conversely, suppose that $X$ is implicitly definable in $L_\alpha$ by the $\in$-formula $\phi(Y)$. Pick $n$ such that $\phi$ is $\Sigma_n$. Then, given some $Y \subseteq \alpha$ in the oracle, a slight variant of $P_{\alpha\text{-truth}}$ can be used to evaluate the truth of $\phi(Y)$ in $(L_\alpha,Y)$ relative to the oracle $Y$, where statements of the form $Y(\iota)$ are evaluated by calling the oracle with $\iota$. □

We also note the following. We say that a formula is $\Sigma^1_1$ if and only if it is of the form $\exists X \subseteq \alpha \phi(X)$, and that it is $\Pi^1_1$ if and only if it is
of the form $\forall X \subseteq \alpha \phi(X)$, where $\phi$ is a first-order $\in$-formula. If a set $Y \subseteq \alpha$ is definable both by a $\Sigma_1^{\alpha}$ and by a $\Pi_1^{\alpha}$-formula, where these formulas are evaluated in a $\in$-structure $M$, then $Y$ is called $\Delta_1^{\alpha}(M)$.

**Proposition 1.** If $\alpha$ is an uncountable regular cardinal in $L$, then every $\alpha$-ITRM-recognizable $x \subseteq \alpha$ is $\Delta_1^{\alpha}(L^{\alpha})$.

**Proof.** As in the proof Theorem 6, there is an $\in$-formula $\phi_P$ such that, for all $y \subseteq \alpha$, $L^{\alpha}[x] = \phi(y)$ if and only if $P^y$ halts in $\prec \alpha$ many steps. Thus, we can describe $x$ as “$\iota \in X$ if and only if $\exists Y (P(Y) \downarrow = 1 \land \iota \in Y)$” which is $\Sigma_1^{\alpha}(L^{\alpha})$ – and as $\forall Y (P(Y) \downarrow = 1 \rightarrow \iota \in Y)$ – which is $\Pi_1^{\alpha}(L^{\alpha})$, so $x$ is $\Delta_1^{\alpha}(L^{\alpha})$. □

**Remark 6.** The preceding proposition fails in general. For example, $L^{\omega_1^{CK}}$ contains all $\Delta_1^{\omega_1}$-subsets of $\omega$ (see, e.g., [1], Corollary 3.2), but there are $\omega$-ITRM-recognizable subsets of $\omega$ in $L^{\alpha+1} \setminus L^{\alpha}$ for cofinally in $\sigma$ many ordinals $\alpha$ occur in up to $\sigma$, which is much bigger than $\omega_1^{CK}$.

We have so far little information about where in the constructible hierarchy the first non-recognizable subsets of $\alpha$ appear. This is known to happen at the first possible level $L^{\beta(\omega)}$ when $\alpha = \omega$, in which case a locally Cohen-generic real number over $L^{\beta(\omega)}$ provides an example, see [3], Theorem 3.8. This can be further extended to ITRM-countable values of $\alpha$. In general, however, such generics will not be constructible, and even if they are (namely, if $\alpha$ is countable in $L$), the argument for their non-recognizability no longer works in general. Hence, we ask:

**Question 5.** Is it true for every exponentially closed ordinal $\alpha$ that $L^{\beta(\alpha)+1} \setminus L^{\alpha}$ contains a subset of $\alpha$ that is not $\alpha$-ITRM-recognizable? In the case where $L_{\alpha} \models ZF^-$, the same question can be asked about $L_{\alpha+2}$.

4. $\alpha$-WITRMs

We now consider recognizability for unresetting $\alpha$-register machines; again, we are only interested in the recognizability of constructible subsets of $\alpha$. We recall from [5] that, for any $\alpha$, $\beta^w(\alpha)$ denotes the supremum of $\alpha$-WITRM-halting times.

A convenient feature of $\alpha$-ITRMs is their ability to “search through $\alpha$”, i.e., count upwards in some register from 0 on until it overflows, thus making it possible to check each element of $\alpha$ for a certain property. For unresetting machines, this obvious strategy is not available: Counting upwards in some register would lead to an overflow of that register at time $\alpha$, which results in the computation being undefined. In some cases, however, such a search is still possible. This motivates the next definition.

**Definition 12.** An ordinal $\alpha$ is $w$ITRM-searchable if and only if there is a halting $\alpha$-wITRM-program $P$ such that the first register used by $P$ contains each element of $\alpha$ at least once before $P$ stops. If such
a program exists that works in the oracle \( x \subseteq \alpha \), we call \( \alpha \) wITRM-searchable in \( x \). If such a program exists that works in the parameter 0, we say that \( \alpha \) is wITRM-searchable without parameters.

Based on the results in [5], we can give a full characterization of the wITRM-searchable ordinals. To this end, we recall from [5], Definition 60 that an ordinal is called \( u \)-weak if and only if any halting \( \alpha \)-wITRM in the empty input halts in less than \( \alpha \) many steps. It is shown in [5] that all \( \Pi_3 \)-reflecting ordinals (and hence, in particular, all \( \Sigma_2 \)-admissible ordinals) are \( u \)-weak. Moreover, the following was proved in [5]:

**Theorem 7** ([5], Theorem 60 and 63). An ordinal is \( u \)-weak if and only if it is admissible and not wITRM-singular.

**Lemma 12.**

1. An ordinal \( \alpha \) is wITRM-searchable if and only if it is not \( u \)-weak, i.e., if and only if \( \alpha \) is admissible and wITRM-singular.
2. If \( \alpha \) is wITRM-searchable without parameters, then there is an \( \alpha \)-wITRM-program that halts after at least \( \alpha \) many steps.
3. If a singularization for \( \alpha \) is \( \alpha \)-wITRM-computable without parameters, then \( \alpha \) is wITRM-searchable without parameters.

**Proof.**

1. Suppose first that \( \alpha \) is wITRM-searchable, and let \( P \) be an \( \alpha \)-wITRM-program that halts after writing each element of \( \alpha \) to the first register \( R_1 \) at least once. Consider the slightly modified program \( P' \) that runs \( P \) but, whenever the content of \( R_1 \) changes, uses a separate register to count from 0 upwards to the content of \( R_1 \). Clearly, \( P' \) will run for at least \( \alpha \) many steps before halting, so that \( \alpha \) is not \( u \)-weak.

On the other hand, suppose that \( \alpha \) is not \( u \)-weak. Thus, \( \alpha \) is not admissible or wITRM-singular. In the latter case, it is immediate from the definition of wITRM-singularity that there is an \( \alpha \)-wITRM-computable cofinal function \( f : \beta \to \alpha \) for some \( \beta < \alpha \); in the former case, this is shown in [5, Theorem 63]. For simplicity, let us assume without loss of generality that \( f \) is increasing. Consider the following algorithm, which works in the parameter \( \beta \): Use some register \( R_2 \) to run through \( \beta \). For each \( \iota < \beta \), compute \( f(\iota) \) and \( f(\iota + 1) \) and store them in \( R_3 \) and \( R_4 \). Copy the content of \( R_3 \) to \( R_1 \) and use \( R_1 \) to count upwards until one reaches the content of \( R_4 \). After that, reset the contents of \( R_1, R_3 \) and \( R_4 \) to 0 and increase the content of \( R_2 \) by 1. If \( R_2 \) contains \( \beta \), halt. It is easy to see that, in this way, \( R_1 \) will contain every ordinal less than \( \alpha \) at least once before halting.

2. The proof is the same as that for the first direction of (1).
3. This works as in the reverse direction of (1).

\( \square \)
It was shown in [6], Corollary 9 (see also [3], Corollary 4.2.20) to follow from Kreisel’s basis theorem that there are no lost melodies for \(\omega\)-wITRMs.

If \(\alpha\) is an uncountable regular cardinal in \(L\), then the lost melody theorem fails for \(\alpha\)-wITRMs for rather drastic reasons:

**Lemma 13.** If \(\alpha\) is a regular cardinal in \(L\), then there are no constructible \(\alpha\)-wITRM-recognizable subsets of \(\alpha\).

**Proof.** If \(\alpha\) is a regular cardinal in \(L\), then it is in particular \(\Sigma^2_\text{adm}\) for any constructible set \(x \subseteq \alpha\). It is shown in [3, Lemma 3.4.10(ii)] that this implies that, for every \(x \subseteq \alpha\) and any \(\alpha\)-wITRM-program \(P\), \(P^x\) will either halt in \(<\alpha\) many steps or not at all.

We can now use a standard compactness argument: Suppose for a contradiction that \(\alpha\) is regular in \(L\) and that \(x \subseteq \alpha\) is recognized by the \(\alpha\)-wITRM-program \(P\) in the parameter \(\rho < \alpha\). In particular, this means that \(P^x\) halts in \(\tau < \alpha\) many steps. Since the basic command set for \(\alpha\)-wITRMs allows a register content to increase at most by 1 in each step, all register contents generated by \(P\) during this computation will be smaller than \(\rho + \tau\). Since \(\alpha\) is indecomposable, we have \(\rho + \tau < \alpha\). In particular, the oracle command can only be applied to the first \(\rho + \tau\) many bits of \(x\). Consequently, if we flip the \((\rho + \tau + 1)\)th bit of \(x\) to obtain \(\tilde{x}\), we shall have \(P(\rho)^\tilde{x} \downarrow = 1\) and \(\tilde{x} \neq x\), contradicting the assumption that \(P\) recognizes \(x\) in the parameter \(\rho\). \(\square\)

**Corollary 4.** If \(\alpha\) is \(u\)-weak, then no \(\alpha\)-wITRM-computable subset of \(\alpha\) is \(\alpha\)-wITRM-recognizable.

**Proof.** If \(x \subseteq \alpha\) is \(\alpha\)-wITRM-computable, then any \(\alpha\)-wITRM-program \(P\) running in the oracle \(x\) can be simulated by another \(\alpha\)-wITRM-program in the empty oracle that runs \(P\) and uses the computability of \(x\) to supply the required answers to oracle calls. If \(\alpha\) is \(u\)-weak, then all halting times of \(\alpha\)-wITRMs in \(\alpha\)-wITRM-computable ordinals are thus still strictly below \(\alpha\). Now argue as in the second part of the proof of Lemma 13. \(\square\)

The situation in Corollary 4 is somewhat surprising: One would expect at least the computable objects to be recognizable simply by computing them and then comparing the result to the oracle. As we have seen, this comparison is non-trivial, and sometimes impossible, on \(\alpha\)-wITRMs. Using the notion of searchability, we can give a precise characterization of when this happens.

**Corollary 5.** Let \(\alpha\) be exponentially closed. The following are equivalent:

1. \(\text{COMP}_{\alpha\text{-wITRM}} \subseteq \text{RECOG}_{\alpha\text{-wITRM}}\).
2. \(0 \in \text{RECOG}_{0\text{-wITRM}}\) (i.e., \(0\) is \(\alpha\)-wITRM-recognizable).
3. \(\text{COMP}_{\alpha\text{-wITRM}} \cap \text{RECOG}_{\alpha\text{-wITRM}} \neq \emptyset\) (i.e., some \(\alpha\)-wITRM-computable subset of \(\alpha\) is \(\alpha\)-wITRM-recognizable).
(4) \( \alpha \) is not u-weak.
(5) \( \alpha \) is wITRM-searchable.

Proof. Since 0 is clearly \( \alpha \)-wITRM-computable, it is clear that (1) implies (2). It is also clear that (2) implies (3). That (3) implies (4) is the contraposition of Corollary 4(1) . The equivalence of (4) and (5) is Lemma 12.

It remains to see that (5) implies (1). Suppose that \( \alpha \) is wITRM-searchable, and let \( x \subseteq \alpha \) be \( \alpha \)-ITRM-computable, say by the program \( P \). Moreover, by searchability of \( \alpha \), let \( S \) be an \( \alpha \)-wITRM-program that halts after having written all elements of \( \alpha \) to its first register at least once. We will show that \( x \) is \( \alpha \)-wITRM-recognizable: Given \( y \subseteq \alpha \) in the oracle, run \( S \). Whenever \( S \) changes the content of the first register, say, to \( \iota \), check whether \( \iota \in y \) and whether \( P(\iota) \downarrow = 1 \). If the answers do not agree, halt with output 0. When \( S \) reaches the halting state, halt with output 1. By the definition of \( S \), this will halt with output 1 if and only if \( y = x \).

□

Corollary 6. Let \( \alpha \) be exponentially closed. The following are equivalent:

(1) Every \( x \subseteq \alpha \) that is \( \alpha \)-wITRM-computable without parameters is \( \alpha \)-wITRM-recognizable without parameters.
(2) 0 is \( \alpha \)-wITRM-recognizable without parameters.
(3) Some \( x \subseteq \alpha \) that is \( \alpha \)-wITRM-computable without parameters is \( \alpha \)-wITRM-recognizable without parameters.
(4) \( \alpha \) is wITRM-searchable without parameters.

Proof. The implications between (1), (2), (3) and the implication (4)\( \Rightarrow \) (1) work as in the proof of Corollary 5. We show that (2)\( \Rightarrow \) (4). Let \( P \) be an \( \alpha \)-wITRM-program that recognizes 0. Then \( P \) halts in the empty oracle. If there was some \( \iota < \alpha \) such that \( P^0 \) makes no oracle call with \( \iota \), then we would have \( P(\iota) \downarrow = 1 \), contradicting the assumption that \( P \) recognizes 0. So the sequence of oracle calls performed by \( P^0 \) constitutes a search through \( \alpha \).

□

We recall some results from [3], which in turn are generalizations of results from Koepke [18] (pp. 261f).

Lemma 14. Let \( \alpha \) be exponentially closed, \( \beta < \alpha \) and \( c \) a nice \( \beta \)-code for a (transitive) \( \in \)-structure \( S \supseteq \alpha \) via some bijection \( f : \beta \rightarrow S \).

(1) (Cf. [3], Lemma 2.3.29, generalizing [18], p. 261f.) There is an \( \alpha \)-wITRM-program \( P_{\text{compare}} \) such that, when \( c, c' \subseteq \beta \) are nice \( \beta \)-codes for (transitive) \( \in \)-structures and \( \iota, \iota' < \beta \) are such that \( \iota \) codes an ordinal in \( c \) and \( \iota' \) codes an ordinal in \( c' \), then \( P_{\text{compare}}(\iota, \iota') \) decides whether \( \iota \) codes the same ordinal in \( c \) that \( \iota' \) codes in \( c' \).
2. There is an $\alpha$-wITRM-program $P_{\text{identify}}$ such that, for every $i < \alpha$, $P_{\text{identify}}^c(i, \beta)$ halts with output $f^{-1}(i)$ (i.e., the ordinal that codes $i$ in the sense of $c$).

3. Moreover, there is an $\alpha$-wITRM-program $P_{\text{decode}}$ such that, for every $i < \beta$ such that $f(\omega i) \in \alpha$, $P_{\text{decode}}^c(\omega i, \beta)$ halts with output $f(\omega i)$, i.e., with output $i$.

Proof. In [3], Lemma 2.3.29 and Corollary 2.3.31, it is shown that this can be achieved when $i < \beta$. However, this condition can be met by simply making $\beta$ larger if necessary and regarding $c$ as a subset of the increased $\beta$.

We sketch the (slightly generalized) algorithms described in [3] for the convenience of the reader.

Note that $P_{\text{identify}}$ is easily obtained from $P_{\text{compare}}$: We fix a code $c_{t+1} := \{p(t_1, t_2) : t_1 < t_2 < t + 1\}$ for $t + 1$; $c_i$ is clearly computable on an $\alpha$-wITRM. Then, we can run $P_{\text{compare}}^{\alpha \cdot c_{t+1}}(\xi, i)$ for every $\xi < \alpha$ and output $\xi$ as soon as the answer is positive. (Note that this search is guaranteed to terminate by our assumptions, so that the search does not lead to an overflow.)

Likewise, given the program $P_{\text{identify}}$ of (2), $P_{\text{decode}}$ is easily obtained: Given $i < \beta$, first compute $\omega i$, then use an extra register to run through $\alpha$ and, for each $\xi < \alpha$, apply $P_{\text{identify}}$ to check whether $\omega i$ codes $\xi$. By assumption, such a $\xi$ will eventually be considered, in which case the algorithm stops and outputs $\xi$.

It thus remains to sketch $P_{\text{compare}}$. So let $c, c' \subseteq \beta$ and $i, i' < \beta$ be given. Note that, by assumption, $c$ and $c'$ code well-founded structures, say via bijections $f : \beta \rightarrow S$, $f' : S' \rightarrow S'$.

We work with two main registers $R$ and $R'$, both of which are intended store finite sequences of ordinals below $\max\{\beta, i\}$, encoded via iterated Cantor pairing: let us write $p(i_0, \ldots, i_k)$ for this code. In order to ensure compatibility with inferior limits – i.e., in order to ensure that $\liminf_{\xi < \lambda} p(i_0, \ldots, i_k, \xi) = p(i_0, \ldots, \liminf_{\xi < \lambda} i_k, \xi)$ – we fix $\mu := \max(\beta, i) + 1$ as the first element of these sequences, which will never be changed.

Let $\delta := f(i)$, $\delta' := f(i')$ be the ordinals coded by $i$ in $c$ and by $i'$ in $c'$, respectively.

We now perform two checks, one whether there is an order-preserving embedding of $\delta$ into $\delta'$ and one for the reverse embedding. These checks will recursively call $P_{\text{compare}}$. The well-foundedness of $c$ and $c'$ will ensure that the recursion terminates.

To check whether $\delta$ embeds into $\delta'$, we start with the sequences $(\mu, i)$ in $R$ and $(\mu, i')$ in $R'$. Now, for every $\xi < \beta$, we check whether $p(\xi, i) \in c$, i.e., whether $f(\xi) \in f(i)$. If that is the case, we replace the content of $R$ by $(\mu, i, \xi)$ and do the following: Searching through $\beta$, we test for each $\xi' < \beta$ whether $p(\xi', i') \in c'$ (i.e., whether $f'(\xi') \in f'(i')$.

\footnote{See [3], p. 31-32 or [3] for a detailed explanation of this trick.}
If that is the case, we replace the content of \( R \) with \((\mu, \iota', x')\). If such a \( \xi \), but no such \( \xi' \) is found, or vice versa, we output 0; this means one, but not the other, of \( \iota, \iota' \) codes 0, so that they do not code the same ordinal. Otherwise, we recursively call \( P_{\text{compare}} \) to use \( R \) and \( R' \) to check whether \( f(\xi) = f'(\xi') \) (leaving the first three elements of the sequences stored in this register unchanged). When this check terminates with output 0, we know that \( \xi \) and \( \xi' \) do not code the same ordinal, and so we proceed with the next candidate for \( \xi' \). When no candidate for \( \xi' \) are left – i.e., when the search through \( \beta \) has been completed without success –, we know that \( \delta \) has an element that is not isomorphic to any element of \( \delta' \), so that we must have \( \delta > \delta' \) and output 0. Otherwise, we continue with the next candidate for \( \xi \). When all \( \xi < \beta \) have been checked and the check has not terminated with a negative result, we know that \( \delta \leq \delta' \). We then proceed in exactly the same way to check whether \( \delta' \leq \delta \).

When both of these checks terminate successfully, we halt with output 1.

\[ \square \]

**Lemma 15.** Let \( \alpha \) be an exponentially closed ordinal such that, for some \( \beta \), we have \( \beta < \alpha < \sigma_\beta \). Then there is a lost melody for \( \alpha \)-wITRMs.

**Proof.** By Lemma 4, we have \( \sigma_\alpha = \sigma_\beta \).

The statement \( \psi \) “There is an ordinal \( \tau \) such that every \( \alpha \)-wITRM-program in every parameter \( \rho < \alpha \) either halts, loops or overflows by time \( \tau \)” is \( \Sigma_1 \) (since computations of length \( < \tau \) are contained in \( L_\tau \)) and thus, since such \( \tau \) exists, it is \( < \sigma_\alpha \).

By fine-structure, \( L_{\sigma_\beta} \) contains a bijection \( f : \beta \to \alpha \). Pick \( \gamma \in (\alpha, \sigma_\beta) \) such that \( f \in L_\gamma \), and for some \( \rho \in \beta \) and some \( \in \)-formula \( \phi \), \( \gamma \) is minimal with the property \( L_\gamma \models \phi(\rho) \). Let \( c \) be the \( <_L \)-minimal nice \( \beta \)-code for \( L_\gamma \) and let \( \xi < \beta \) be the ordinal that codes \( f \) in the sense of \( c \).

By the techniques discussed in section 4, \( c \) is \( \alpha \)-ITRM-recognizable in the parameter \( \rho \). More precisely, the well-foundedness check specifically works as described in the proof of \( \mathfrak{R} \), Theorem 2.3.25(ii): One performs a depth-first search for an infinite descending chain through \( \beta \), initially putting \( \beta^2 \) on the stack to ensure that the stack encoding works properly at limits. Store the length of the descending sequence currently built in a separate register, and output 1 if that register contains \( \omega \) at some point, and 0 if the search terminates and that has not happened. The other checks necessary for recognizing \( c \) in the way described in the proof of Lemma 9 such as evaluating truth predicates
in β-coded structures, can now be executed on a β-ITRM, which can be simulated on an α-wITRM in the parameter α when α > β.\footnote{Note that the extra complications in the well-foundedness are only necessary when β is a regular cardinal in L that has cofinality ω in V; otherwise, a well-foundedness check can be performed on a β-ITRM (possible relative to some recognizable oracle), which can be simulated on an α-wITRM.}

We claim that c is α-wITRM-recognizable (as a subset of α) in the parameters β, ρ and ι. Thus, let a set d ⊆ α be given in the oracle.

First, let \( d := d \cap \beta \), which is computable from d in the parameter β. As just described, check whether \( d = c \). If the program returns 0, we halt with output 0.

Otherwise, we know that the first \( \beta \) many bits of d are correct and it remains to check that there d contains no elements that are greater than or equal to β.

Using the parameter ι and the program \( P_{\text{decode}} \) from Lemma \ref{lem:decode}, we can compute the bijection \( f : \beta \to \alpha \) as follows: Given \( \xi < \beta \), search through c to find the (unique) element of the form \( p(p(\xi, \zeta), \iota) \). Then run \( P_{\text{decode}}(\zeta, \beta) \).

Using \( f \), we can now run through \( \beta \) and check, for every \( \xi \in \beta \), whether \( f(\xi) \geq \beta \) and whether \( f(\xi) \in c \). If the answer is positive for some \( \xi \in \beta \), we halt with output 0. Otherwise, we halt with output 1.

We note the following amusing consequence which yields examples of a definability concept for which the sets of explicitly and implicitly definable objects are disjoint:

**Corollary 7.** If \( \alpha \) is \( \Pi_3 \)-reflecting and \( \alpha \in (\beta, \sigma_\beta) \) for some ordinal \( \beta \), then the set of \( \alpha \)-wITRM-computable subsets of \( \alpha \) and the set of constructible \( \alpha \)-wITRM-recognizable subsets of \( \alpha \) are (both non-empty, but) disjoint.

**Proof.** Immediate from Lemma \ref{lem:reflect} and Corollary \ref{cor:reflect}.

It thus remains to consider the cases where \( \alpha \) is either of the form \( \sigma_\beta \) or a limit of ordinals of this form (note that the latter case generalizes the case of regular cardinals in L).

**Lemma 16.** If \( \alpha \) is of the form \( \sigma_\beta \) for some ordinal \( \beta \), then there are no \( \alpha \)-wITRM-recognizable constructible subsets of \( \alpha \) (and thus, in particular, no lost melodies for \( \alpha \)-wITRMs).

**Proof.** Let us first assume that \( \alpha = \sigma_\beta \) for some \( \beta \in \text{On} \). Suppose for a contradiction that \( x \subseteq \alpha \) is \( \alpha \)-wITRM-recognizable by the program \( P \) in the parameter ρ < α. Thus, L believes that there are a set \( x \subseteq \alpha \) and a halting \( \alpha \)-wITRM-computation of \( P^x(\rho) \) with output 1. In particular, L believes that there are a set \( x \subseteq \text{On} \) and a halting ORM-computation of \( P^x(\rho) \) with output 1, which is a \( \Sigma_1 \)-formula in
the parameter $\rho$. By definition of $\sigma_\beta$, and the fact that $\sigma_\beta = \sigma_{\rho+1}$ (see Lemma 4 above), the same holds in $L_\alpha$. Since computations are absolute between transitive $\in$-structures, $L_{\sigma_\beta}$ contains a set $x$ of ordinals and a halting ORM-computation $P^x$ with output 1. Let $\delta$ be the length of this computation. Then $\delta < \alpha$. Consequently, this computation cannot generate register contents $\geq \alpha$ and is thus actually a $\sigma$-wITRM-computation. During this computation, at most the first $\delta$ many bits of $x$ can be considered. It follows that both $P^{(x \cap \delta)}(\rho)$ and $P^{(x \cap \delta) \cup (\delta+1)}(\rho)$ halt in $\delta$ many steps with output 1 without generating register contents $\geq \alpha$; thus, we have found two different oracles $y$ for which the $\alpha$-wITRM-computation $P^y(\rho)$ halts with output 1, a contradiction to the assumption that $P$ recognizes $x$ in the oracle $y$. If $\alpha$ is a limit of ordinals of the form $\sigma_\beta$, pick $\beta \in \text{On}$ large enough such that $\rho \in \sigma_\beta$ and repeat the above argument. □

This settles the question whether lost melodies exist for $\alpha$-wITRMs for all exponentially closed values of $\alpha$. We summarize the possible relations between $\text{COMP}^\alpha_{\text{wITRM}}$ and $L \cap \text{RECOG}^\alpha_{\text{wITRM}}$ that can occur for exponentially closed ordinals $\alpha$:

- $\text{COMP}^\alpha_{\text{wITRM}} = L \cap \text{RECOG}^\alpha_{\text{wITRM}}$ holds if and only if $\alpha = \omega$.
- $\text{COMP}^\alpha_{\text{wITRM}} \subsetneq L \cap \text{RECOG}^\alpha_{\text{wITRM}}$ holds if and only if $\alpha > \omega$ is searchable and, for some $\beta$, $\beta < \alpha < \sigma_\beta$.
- $\text{COMP}^\alpha_{\text{wITRM}} \cap \text{RECOG}^\alpha_{\text{wITRM}} = \emptyset$ (with $\text{COMP}^\alpha_{\text{wITRM}} \neq \emptyset$, $\text{RECOG}^\alpha_{\text{wITRM}} \neq \emptyset$) holds if and only if $\alpha > \omega$ is not searchable and, for some $\beta$, $\beta < \alpha < \sigma_\beta$.
- $\text{RECOG}^\alpha_{\text{wITRM}} = \emptyset \nsupseteq \text{COMP}^\alpha_{\text{wITRM}}$ holds if and only if $\alpha$ is stable.

**Question 6.** Under what conditions are there lost melodies for $\alpha$-wITRMs that are $\alpha$-ITRM-recognizable without parameters? Which relations can occur between the parameter-free versions of $\text{COMP}^\alpha_{\text{wITRM}}$ and $\text{RECOG}^\alpha_{\text{wITRM}}$?

By basically the same arguments, we obtain:

**Corollary 8.** If $\alpha$ is of the form $\sigma_\beta$ for some ordinal $\beta$, then there are no (weakly) $\alpha$-wITRM-semi-recognizable and no $\alpha$-wITRM-co-semi-recognizable constructible subsets of $\alpha$.

**Proof.** This works by the same argument as Lemma 16 noting that “There is $x$ such that $P^x(\rho)$ halts” is a $\Sigma_1$-formula and that “There is $x$ such that $P^x(\rho)$ does not halt (but is defined)” is equivalent to “There is $x$ such that there is a strong loop in the computation of $P^x(\rho)$”, which is again $\Sigma_1$. □

For weak $\alpha$-wITRM-co-semi-recognizability, however, things are different:
**Proposition 2.** For all $\alpha$, each $\alpha$-wITRM-computable subset $x \subseteq \alpha$ is also $\alpha$-wITRM-co-semi-recognizable.

**Proof.** Let $x \subseteq \alpha$ be $\alpha$-wITRM-computable, and pick a program $P$ and an ordinal $\xi < \alpha$ such that $P$ computes $x$ in the parameter $\xi$. Let $Q$ be the program that, for each $i < \alpha$, stored in some register $R$, computes $P(i, \xi)$ and compares the output to the $i$-th bit of the oracle. If they agree, $Q$ continues with $i+1$; otherwise, $Q$ halts. Clearly, $R$ will overflow at time $\alpha$ if and only if the oracle is equal to $x$, and otherwise, $Q$ will halt. □

**Question 7.** Suppose that $x \subseteq \alpha$ is $\alpha-(w)$ITRM-recognizable by the program $P$ and that $P^x$ halts in $\tau$ many steps. Does it follow that $x \in L_{\tau+\omega}$ (where $\tau+\omega$ denotes the next limit of admissible ordinals after $\tau$)? (This is true for $\alpha = \omega$ by Theorem 14 of [8].)

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