Conditions for nonexistence of static or stationary, Einstein-Maxwell, non-inheriting black-holes

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Abstract

We consider asymptotically-flat, static and stationary solutions of the Einstein equations representing Einstein-Maxwell space-times in which the Maxwell field is not constant along the Killing vector defining stationarity, so that the symmetry of the space-time is not inherited by the electromagnetic field. We find that static degenerate black hole solutions are not possible and, subject to stronger assumptions, nor are static, non-degenerate or stationary black holes. We describe the possibilities if the stronger assumptions are relaxed.

1 Introduction

In the study of static or stationary Einstein-Maxwell solutions of Einstein’s equations, it is frequently assumed that the Maxwell field is also static or stationary, in the sense that the Lie derivative of the Maxwell tensor $F_{ab}$ along the Killing vector $K^a$ is zero. One needs the energy-momentum tensor $T_{ab}$ to be static or stationary for the Einstein equations to be consistent, but this does not actually require that $F_{ab}$ be static or stationary. Following custom, (see e.g. [1]) we shall say that $F_{ab}$ does not inherit the symmetry if $T_{ab}$ is static or stationary but $F_{ab}$ is not.

It is not always assumed that $F_{ab}$ is static or stationary, and there are explicit non-inheriting solutions in the literature (see e.g. [1], [2]). However none of these solutions is asymptotically-flat and it may be wondered whether the properties of being asymptotically-flat and non-inheriting are incompatible. As we shall see, this is certainly the case for Maxwell fields in Minkowski space, and it is very likely though not proven to be the case for
static space-times without horizons. However, in the presence of a black hole, it is easier to proceed by a study of the horizon rather than a study of the asymptotic conditions. We consider static and stationary, asymptotically-flat, Einstein-Maxwell black-hole solutions, where a black-hole solution is understood to be an asymptotically-flat solution with a regular Killing horizon bounding the domain of outer communication (d.o.c.). We shall assume that black holes are (topologically) spherical, and we conclude that, subject to conditions we give, indeed there are none which are non-inheriting. (In [3] it is shown that black holes are necessarily topologically spherical if certain regularity conditions are satisfied; by adding these conditions to our assumptions, we could omit the separate assumption of sphericity.)

The first result is the following:

**Theorem 1.1** Let \((M, g)\) be an asymptotically-flat, Einstein-Maxwell, space-time with a Killing vector \(K^a\) which is time-like near infinity.

Then necessarily
\[ \mathcal{L}_K F_{ab} = -a F^*_{ab}, \] (1)
for some \(a\), where \(\mathcal{L}_K\) is the Lie derivative along \(K^a\) and \(F^*_{ab}\) is the dual of \(F_{ab}\). We shall suppose that the right-hand-side in (1) is not everywhere zero, and that any black holes are (topologically) spherical. then:

1. If \(M\) is static, then \(a\) is constant and there are no black holes with degenerate horizons.

2. If \(M\) is static, then there are no black holes for which \(M\) is analytic up to and including the horizons.

3. If \(M\) is stationary with the Killing vector in (1) tangent to the generators of the horizon and either non-null Maxwell field or null Maxwell field with constant \(a\) then there are no black holes for which \(M\) is analytic up to and including the horizons.

4. If \(a\) is a non-zero constant, then the charge of any black hole vanishes.

It is not assumed here that the solutions contain just a single black hole, nor that the horizons are nondegenerate, nor that the Maxwell field is non-null (though, as we shall see, this is necessarily true for the static case). The assumption in part 3, that the Killing vector which is time-like at large distances is also tangent to the generator of the horizon might be thought unreasonable, but we shall see from the proof that this part still holds in an analytic solution of the kind envisaged, provided the null generator of the
horizon satisfies (1) with non-null Maxwell field or with null Maxwell field but a nonzero constant $a$.

One usually expects a stationary black-hole solution also to be axisymmetric and one could then imagine both symmetries not being inherited. Part 3 will apply provided the Killing vector generating the horizon is not inherited, but in the span of two non-inherited symmetries there will be an inherited one and this could be the null generator of the horizon (see the end of Section 3 for this argument). That this can’t happen in the orthogonally-transitive case with a non-null Maxwell field follows from a result of Michalski and Wainwright [4]:

If $(M, ^4g)$ is a stationary, axisymmetric Einstein-Maxwell space-time which is orthogonally-transitive with a non-null Maxwell field, then necessarily the Maxwell field inherits the symmetry in that

$$\mathcal{L}_K F_{ab} = \mathcal{L}_L F_{ab} = 0,$$

where $K$ and $L$ are the stationarity and axisymmetry Killing vectors.

There is no assumption of asymptotic flatness or of analyticity in this result. For vacuum or inheriting Einstein-Maxwell space-times, orthogonal transitivity can be deduced from conditions we have imposed (see e.g. [5]), but this does not seem to be the case for non-inheriting Einstein-Maxwell where it seems to be a definite restriction. The above result of [4] assumes that the Maxwell field is non-null in order to use the ‘already-unified’ formalism [6] and in particular the ‘complexion’ vector field, whose definition we shall recall in Section 3. With the aid of this formalism one can show the following:

**Theorem 1.2** Let $(M, ^4g)$ be an Einstein-Maxwell space-time having a non-degenerate Killing horizon generated by a Killing vector $K^a$ with

$$\mathcal{L}_K F_{ab} = -a F_{ab},$$

and subject to the regularity conditions of [7], so that, without loss of generality, there is a bifurcation surface on the Killing horizon. Suppose that $F_{ab}$ is nowhere null in the d.o.c. and that the complexion vector field $\alpha_a$ extends smoothly to the horizon, then $a = 0$ in the set $\mathcal{B}$ which is the union of the d.o.c. and the horizon.

Thus for a non-null Maxwell field and a non-degenerate horizon, if the complexion vector field extends smoothly to the horizon, one can dispense with the assumption of analyticity up to the horizon. In particular, this would
show that there are no non-degenerate non-inheriting static black holes, even without this assumption of analyticity, since for the static case the Maxwell field is obliged to be non-null. This with Part 1 of Theorem 1.1 would rule out static black holes, but the assumption that \( \alpha \) extends smoothly to the horizon is a strong one.

To prove Theorem 1.2, following the argument of [4], one first establishes that \( K^a \alpha_a = 2a \) which is constant. We do this in Section 3. Then, with the regularity assumptions of [7], which are very natural in this context, the non-degeneracy of the Killing horizon implies that, without loss of generality, there is a bifurcation surface on the Killing horizon where \( K^a \) vanishes. Therefore \( a = 0 \), and the symmetry is inherited. (The same argument incidentally shows that the symmetry is inherited if \( K^a \) tends to a constant time-translation at large distances while \( \alpha \) decays to zero."

In the stationary but non-static case, with null fields, Theorem 1.1 rules out constant, nonzero \( a \) with analyticity at the horizon. We also have the following:

**Theorem 1.3** Let \((M,^{4}g)\) be an Einstein-Maxwell space-time, with a Killing horizon generated by a Killing vector \( K^a \) with 
\[
\mathcal{L}_KF_{ab} = -aF^a_{\ b},
\]
and a null Maxwell field. Then either \( a \) is constant or the space-time admits a non-twisting, shear-free null geodesic congruence.

If there exists a non-twisting, shear-free null geodesic congruence, then the metric lies in the Robinson-Trautman class if the expansion is non-zero, or the Kundt class if the expansion is zero, or is a pp-wave if the generator of the congruence can be chosen to be parallel. It is very unlikely that a metric in either of the last two classes could be stationary and asymptotically-flat, as the curvature does not decay along the congruence. In the Robinson-Trautman class, the metric can be given locally in terms of a few functions, with (in our case) one remaining field equation (see Theorem 28.3 of [2] for this). Explicit non-inheriting metrics can be found (see e.g. the metric of Bartrum [8] given in [2]) but again it seems rather unlikely (as we argue in the Appendix) that these metrics, with the Maxwell field null but non-zero, can be stationary and asymptotically-flat.

**Remark 1.4** A final point worth making is that, even for an inherited symmetry, at a non-degenerate horizon a null Maxwell field vanishes to all orders.
The plan of the paper is as follows. In Section 2, we undertake a ‘near horizon’ analysis, as in [9], for static, non-inheriting, Einstein-Maxwell black holes. We find that the Maxwell field vanishes at the horizon to all orders. This proves part 4 of Theorem 1.1 in the static case, since the charge of the black hole is an integral of the Maxwell field over a cross-section of the horizon. For a static, degenerate horizon, the analysis of [9] together with the vanishing of the Maxwell field at the horizon leads to a contradiction, so that at once there are no non-inheriting solutions. This proves part 1 of Theorem 1.1.

For a (static or stationary) non-degenerate horizon or a stationary degenerate horizon we need something more, and we assume analyticity up to and including the horizon. This is a strong assumption: there are general arguments that static or stationary vacuum solutions are analytic away from the horizon, [10], [11], and the black hole uniqueness theorems have always led to solutions which are analytic at the horizon but, aside from [12] for static vacuum black holes, there are no direct proofs of analyticity up to and including the horizon. With this assumption we are done: the Maxwell field vanishes to all orders at the horizon so by analyticity it vanishes everywhere and we are back to vacuum black holes - there are no static non-inheriting black holes, which is part 2 of Theorem 1.1.

In Section 3, we look at stationary, non-inheriting black-hole space-times. A near-horizon analysis again shows that, for non-null Maxwell field or null Maxwell field with constant $a$, the Maxwell field vanishes at the horizon to all orders, and then the assumption of analyticity forces it to vanish everywhere and we have proved the slightly stronger version of part 3 of Theorem 1.1. This also proves part 4 for stationary solutions. For Theorem 1.2 we review the relevant parts of the already-unified theory of [6]. For Theorem 1.3 we take the analysis of null Maxwell fields further. The repeated spinor of the Maxwell spinor is necessarily geodesic and shear-free, and one is able to conclude that $a$ is constant unless the spinor field is also non-twisting, which characterises the Robinson-Trautman solutions. We conclude by noting the remaining loop-holes for non-inheriting black hole solutions.

In the Appendix we argue, without a complete proof, that there are in fact no Robinson-Trautman solutions with a (non-zero) null Maxwell field which are stationary and asymptotically-flat.

For the rest of this section, we develop some theory of non-inheriting solutions.

We assume that there is a Maxwell field determined by a bivector field
$F_{ab}$ given in terms of a symmetric spinor field $\phi_{AB}$ by

$$F_{ab} = \phi_{AB}\epsilon_{A'B'} + \overline{\phi}_{A'B'}\epsilon_{AB},$$

(2)

The corresponding energy-momentum tensor can be written

$$T_{ab} = 2\phi_{AB}\overline{\phi}_{A'B'}.$$  

(3)

The dual Maxwell field is given by

$$F_{ab}^* = -i\phi_{AB}\epsilon_{A'B'} + i\overline{\phi}_{A'B'}\epsilon_{AB}.$$  

(4)

By assumption, we have a Killing vector $K^a$ with $\mathcal{L}_K T_{ab} = 0$, where $\mathcal{L}_K$ is the Lie-derivative along $K^a$, but the same does not hold for $F_{ab}$ or $\phi_{AB}$. However, by (3) we must have

$$(\mathcal{L}_K \phi_{AB}) \overline{\phi}_{A'B'} + \phi_{AB} (\mathcal{L}_K \overline{\phi}_{A'B'}) = 0$$

which entails

$$\mathcal{L}_K \phi_{AB} = ia\phi_{AB}$$  

(5)

for some real $a$ which, at this stage, could vary with position. From (5) with (2) and (4) we obtain

$$\mathcal{L}_K F_{ab} = -aF_{ab}^* ; \mathcal{L}_K F_{ab}^* = aF_{ab}.$$  

(6)

This is the familiar fact that, if $T_{ab}$ does not change with time, then $F_{ab}$ is allowed to change only by a duality rotation.

From the exterior derivative of both sides in each of (6), for source-free fields we obtain

$$da \wedge F = 0 = da \wedge F^*.$$  

(7)

With the aid of (2) and (4), from (7) we obtain the following equation, written in terms of spinors:

$$\phi^{AB \nabla}_{A'A'}a = 0.$$  

(8)

Multiply this by $\phi_{BC}$ to conclude that for non-null $F_{ab}$, $da = 0$ and so $a$ is constant in the d.o.c. (This goes through for static and stationary space times.) If $a = 0$, we are in the inheriting case, so throughout we shall suppose $a \neq 0$. We cannot conclude that $a$ is constant by this argument if $F_{ab}$ is null and will return to this case in Section 3.

Now we restrict to the static case so that the time-like Killing vector $K^a$ is hypersurface-orthogonal. We introduce the usual coordinates, so that the
metric in the d.o.c. (which we assume is simply-connected; this would follow from the assumptions in [3]) can be written

\[ g = V^2 dt^2 - h_{ij}(x^k) dx^i dx^j, \]  

where the Killing vector is \( K = \partial/\partial t \) and \( g(K, K) = V^2 \). Note also that

\[ K_a dx^a = V^2 dt \]  

We shall assume later that \( V^2 \) has one or more zeroes and that these correspond to regular Killing horizons.

The surfaces of constant time in the metric (9) have zero extrinsic curvature so, from the momentum constraint for the Einstein equations, one learns at once that \( T_{0i} = 0 \), or in an invariant formulation

\[ T_{ab} K^b = f K^a. \]  

Here \( f \) is a function on space-time, independent of \( t \) by assumption and non-negative by the Dominant Energy Condition (which is automatic for an energy-momentum tensor of the form of (3)). In spinors, (11) is

\[ 2 \phi_{AB} \phi_{A'B'} K^{B'B'} = f K_{A'A'}. \]

Multiplying both sides of this by \( \phi^{AC} \) we obtain

\[ (\phi_{CD} \phi^{CD}) \phi_{A'B'} K_{A'}^{B'B'} = -f \phi_{AB} K_{A'}^{B'}, \]

which is only possible if

\[ \phi_{CD} \phi^{CD} = f e^{i \theta} \]
\[ \phi_{AB} K_{A'}^{B'} = -e^{i \theta} \phi_{A'B'} K^{B'}_{A} \]

for some real \( \theta \). Note that \( f = 0 \) iff \( F_{ab} \) is null or zero (recall that \( F_{ab} \) is null iff \( \phi_{CD} \phi^{CD} = 0 \)) and that if \( F_{ab} \) is null and nonzero then \( K^a \) must be null, so that this can only happen at a horizon: \( F_{ab} \) is non-null in the d.o.c. (The incompatibility of a null Maxwell field with a static geometry is an old result, to be found in [13].)

Now (5) applied to (12) gives

\[ \mathcal{L}_K \theta = 2a. \]

We shall find below that \( \theta = 2at \).
Before that, where $V > 0$, we introduce the electric and magnetic field vectors by
\[ E_a = V^{-1} K^b F_{ba} \quad B_a = V^{-1} K^b F_{ba}^* \]
or by combining them:
\[ E_a + iB_a = 2V^{-1} K_A^C \phi_{AC} \]
(15)
From this, (13) and algebra, we find
\[ E_a \cos(\theta/2) = -B_a \sin(\theta/2) \]
while from (5) and (15),
\[ \mathcal{L}_K E_a = -aB_a \quad \mathcal{L}_K B_a = aE_a. \]
We may therefore introduce $W_a$ by
\[ E_a = W_a \sin(\theta/2) \quad B_a = -W_a \cos(\theta/2), \]
(16)
to find
\[ \mathcal{L}_K W_a = 0, \]
(17)
and, with the aid of (12),
\[ W^a W_a = -2f. \]
(18)
We can write the Maxwell equations in terms of $W_a$ by first observing that, since the Killing vector $K^a$ is static
\[ \nabla_a (V^{-1} K_b) = -V^{-2} K_a \nabla_b V. \]
(19)
Now from (6),(19) and the source-free Maxwell equations we obtain the system of equations:
\[
\begin{align*}
\nabla^b (V^{-1} E_b) & = 0 \\
\nabla^b (V^{-1} B_b) & = 0 \\
e^{ab}_{\quad cd} K^d \nabla_a (V E_b) & = -a V E_c \\
e^{ab}_{\quad cd} K^d \nabla_a (V B_b) & = -a V B_c.
\end{align*}
\]
Substituting (16) into these, we find
\[
\begin{align*}
\nabla^b (V^{-1} W_b) & = 0 \\
e^{ab}_{\quad cd} K^d \nabla_a (V W_b) & = -a V W_c 
\end{align*}
\]
(20) (21)
and

\[ W^b \partial_b \theta = 0 = \epsilon^{abcd} K_d W_b \partial_a \theta, \]

which, with (14) implies

\[ \partial_a \theta = 2aV^{-2} K_a \]

so that, by (10) and up to an additive constant, \( \theta = 2at \), as anticipated.

In terms of \( W_a \), the energy-momentum tensor is

\[ T_{ab} = -W_a W_b + \frac{1}{2}(W^c W_c) g_{ab} - \frac{1}{V^2} (W^c W_c) K_a K_b. \]  

(22)

In terms of the (positive-definite) spatial metric \( h_{ij} \), its Levi-Civita derivative \( D_i \), Ricci tensor \( R_{ij} \) and Laplacian \( \Delta = h^{ij} D_i D_j \), where indices from this part of the alphabet run from 1 to 3, the Einstein equations are the system:

\[ \Delta V = \frac{1}{2} (h^{ij} W_i W_j) V, \]  

(23)

\[ \epsilon_i^{jk} D_j (V W_k) = -a W_i, \]  

(24)

\[ R_{ij} - \frac{1}{2} R h_{ij} = V^{-1} D_i D_j V - W_i W_j, \]  

(25)

From (24) it follows that

\[ D_i W^i = 0 \]  

(26)

and from (25) that the (three-dimensional) Ricci scalar is

\[ R = h^{ij} W_i W_j. \]

For asymptotic flatness, at large distances \( R = O(r^{-4}) \) so that \( |W| = O(r^{-2}) \) (for the application following equation (27) below, it would suffice to have \( |W| = O(r^{-\frac{3}{2}} - \epsilon) \) or \( R = O(r^{-3-\epsilon}) \) for \( \epsilon > 0 \)).

As a check on these equations, we note that the contracted Bianchi identity:

\[ D^i (R_{ij} - \frac{1}{2} R h_{ij}) = 0 \]

is indeed an identity given (23)-(25). Also if \( a = 0 \), so that the symmetry is inherited, then from (16), \( V W_i \) is a gradient, say

\[ V W_i = \sqrt{2} D_i \phi \]

in terms of a function \( \phi \), which, by (26), satisfies the equation

\[ D^i (V^{-1} \phi_i) = 0. \]

Now this with (23) and (25) makes up the field equations for an inheriting Einstein-Maxwell solution given e.g. in [2].
It is easy to see that these equations are incompatible with asymptotic flatness in the limit of special relativity i.e. for a non-inheriting Maxwell field in flat space. All that remains of the equations in this limit is the Maxwell equation from (24) which we may write as:

$$\nabla \wedge \mathbf{W} = -a \mathbf{W}$$  \hspace{1cm} (27)

We may see as follows that this is incompatible with asymptotic flatness in the sense of \(|\mathbf{W}|^2 = O(r^{-3-\epsilon})\). Introduce the three-dimensional tensor

$$t_{ij} = W_i W_j - \frac{1}{2} \delta_{ij} |\mathbf{W}|^2,$$

then \(t_{ij}\) is divergence-free by virtue of (27) so that

$$\partial_i (t_{ij} x^j) = \delta_{ij} t_{ij} = -\frac{1}{2} |\mathbf{W}|^2$$

where \(x^i\) is the position vector. We integrate this identity over a large ball and use the divergence theorem and the assumed rate of decay of \(W_i\) to show that the surface term tends to zero, so that the volume integral vanishes and \(W_i\) is zero.

The following argument suggests that the same result holds for a static, asymptotically-flat, noninheriting solution without horizons. Rescale the metric with the conformal factor \(V^{-2}\):

$$\tilde{h}_{ij} = V^{-2} h_{ij},$$  \hspace{1cm} (28)

so that also

$$\tilde{\epsilon}_{ij} = V \epsilon_{ij},$$

and define \(\omega_i = V W_i\). Then (24) becomes

$$\tilde{\epsilon}_{ik} \partial_j \omega_k = -a \omega_j,$$

or in form notation

$$\ast d \omega = -a \omega.$$  \hspace{1cm} (29)

Another derivative of this shows that

$$\Delta \omega := (\ast d * d + d * d *) \omega = a^2 \omega,$$

so that \(\omega\) is a one-form eigenfunction of the Laplacian, and the asymptotic conditions imply that \(\omega\) is square-integrable.
It seems very likely (see e.g. footnote 38 on page 76 in [14]) that there are no non-zero square-integrable eigen-one-forms of the Laplacian for an asymptotically-Euclidean metric like $\tilde{h}$. If this is so, then $\omega$ is zero and there are no static, non-inheriting solutions without a black hole.

To deal with black-hole solutions, where $V$ may have zeroes, we turn to a near-horizon analysis.

## 2 Near-horizon analysis

In this section, we assume that we have a static, non-inheriting Einstein-Maxwell black hole with one or more components of the event horizon. All black holes have spherical topology. We shall find that the Maxwell field vanishes to all orders at the horizon. For a degenerate horizon, this is sufficient, with the help of results in [9] to reach a contradiction. For the non-degenerate or stationary cases we shall need an assumption of analyticity.

As in, for example, [9] we introduce Gaussian null coordinates near a component $\mathcal{N}$ of the event horizon, so that the metric is

$$4g = rAdu^2 - 2dudr - 2r(hd\zeta + \tilde{h}d\zeta)du - m\overline{m},$$

where $m = -\dot{X}d\zeta + O(r)$.

In these coordinates, the Killing vector $K$ is $\partial/\partial u$ with norm

$$g(K, K) = rA$$

and $\mathcal{N}$ is located at $r = 0$. The surface gravity of the horizon is $\kappa = \dot{A}$, where the zero means the value at $r = 0$. By a general argument (see e.g.[15] p.333) $\dot{A}$ is constant on each component of the horizon. If it vanishes, the horizon is degenerate.

We shall investigate the metric (29) in the spin-coefficient formalism [16]. We introduce the null tetrad $(l^a, n^a, m^a, \overline{m}^a)$ by

$$l^a\partial_a = D = \partial_u + \frac{rA}{2}\partial_r,$$
$$n^a\partial_a = \Delta = -\partial_r,$$
$$m^a\partial_a = \delta = \frac{1}{X}\partial_\zeta + \frac{r}{Y}\partial_r - \left(\frac{xb}{X} + \frac{rX}{Y}\right)\partial_r,$$

where $X = \dot{X} + O(r)$.

In this tetrad, the Killing vector is given by

$$K^a = l^a + \frac{rA}{2}n^a$$

(30)
and it is clear that all elements of the tetrad are Lie-dragged by $K^a$, whence so is the spinor dyad $(o^A, i^A)$ which lies behind the tetrad. In this dyad, the Maxwell spinor can be expanded in the standard way as

$$\phi_{AB} = \phi_0 i_{AB} - \phi_1 (o_{AB} + i_{A}o_{B}) + \phi_2 o_{A}o_{B},$$

when (13) with (30) implies

$$\phi_0 = \frac{1}{2} r A e^{i\theta} \phi_2,$$

while (5) implies

$$\mathcal{L}_K \phi_i = i a \phi_i$$

for $i = 0, 1, 2$.

For the spin-coefficients we find

$$\epsilon = \frac{1}{4} \dot{A} + O(r),$$

while $\nu = \kappa = 0$, $\sigma$ and $\rho$ are $O(r)$ and the rest are $O(1)$. We shall need two of the Maxwell equations from [16]

$$D\phi_1 - \delta \phi_0 = (\pi - \alpha) \phi_0 + 2 \rho \phi_1 - \kappa \phi_2$$

$$D\phi_2 - \delta \phi_1 = -\lambda \phi_0 + 2 \pi \phi_1 + (\rho - 2\epsilon) \phi_2.$$ 

Consider first (34). We assume that $\phi_1$ and $\phi_2$ are $O(1)$ then by (31), $\phi_0$ is $O(r)$. Now all terms on the right are $O(r)$; on the left however there is an $O(1)$ term $\partial \phi_1 / \partial u$ which is $i a \phi_1$ by (32). Thus in fact $\phi_1 = O(r)$. Next, from (35) we find

$$\frac{\partial \phi_2}{\partial u} + \frac{\dot{A}}{2} \phi_2 = O(r),$$

making use of (33). Using (32) again, since $\dot{A}$ is real, we find $\phi_2 = O(r)$ (for degenerate or non-degenerate horizons).

By (31) we now have $\phi_0 = O(r^2)$ and we can go round again. By induction, all components $\phi_i$ vanish at the horizon faster than any positive power of $r$.

For a degenerate horizon, we can refer to the result in [9]: there are no static vacuum solutions with degenerate horizons subject to regularity conditions being assumed here; now that $\phi_{AB}$ vanishes at the horizon to all orders, the case under investigation here has the same equations holding at the horizon as a vacuum solution and we obtain the same contradiction.
Thus there are no non-inheriting, Einstein-Maxwell black-holes with a degenerate horizon, which proves part 1 of Theorem 1.1.

For non-degenerate horizons, we need to assume analyticity up to and at the horizon. Then the vanishing of the Maxwell field to all orders at the horizon implies that it is everywhere zero and we are back to the vacuum case, proving part 2 of Theorem 1.1.

3 The stationary case

In this section, we consider the case of stationary, non-inheriting Einstein-Maxwell black holes. With the Maxwell field as before, i.e. (2), we obtain (5), but now we don’t have available the argument that $F_{ab}$ is non-null. If it is then we can deduce that $a$ is necessarily constant; if it isn’t then for Theorem 1.1 we shall just assume that $a$ is constant, and look later at the case when it isn’t. We introduce null Gaussian coordinates as before and for non-degenerate horizons the spin-coefficients behave as before (except that now $\kappa$ is $O(r)$ rather than zero). At this stage in the argument, the Killing vector $K^a$ is by assumption tangent to the generator of the horizon.

We don’t have (11) and therefore not (31) either, which was important in making $\phi_{AB}$ vanish at the horizon. However there is a general argument (see [15] equation (12.5.2)) that $T_{ab}K^aK^b = 0$ on any component of the horizon. This forces $\phi_0$ to vanish at $r = 0$, so that by smoothness $\phi_0 = O(r)$. The argument after (34) and (35) goes through so that $\phi_1$ and $\phi_2$ are both $O(r)$.

To go round again we use another Maxwell equation from [16]:

$$\Delta \phi_0 - \delta \phi_0 = (2\gamma - \mu)\phi_0 - 2\tau \phi_1 + \sigma \phi_2.$$  \hspace{1cm} (37)

Every term is known to be $O(r)$ except for $\Delta \phi_0 = -\frac{\partial \phi_0}{\partial r}$ so we conclude that this is too, and then $\phi_0 = O(r^2)$. Now we can go round again to see by induction that $\phi_{AB}$ vanishes to all orders at the horizon.

For degenerate horizons we find the spin-coefficient $\epsilon$ is $O(r^2)$ rather than $O(r)$, but with a non-zero constant $a$ the argument proceeds just as for non-degenerate horizons and the Maxwell field vanishes to all orders at the horizon.

With the assumption of analyticity up to and including the horizon, we may conclude that the Maxwell field is zero everywhere: there are no analytic, non-inheriting stationary black holes with constant nonzero $a$, which is the third and final part of Theorem 1.1.

□
It is convenient here, since the same arguments as above are used, to prove Remark 1.4, that with a null Maxwell field inheriting the symmetry any horizon must be degenerate. The condition for a null Maxwell field is

\[ \phi_0 \phi_2 = (\phi_1)^2. \] (38)

As noted above, at a horizon we have \( \phi_0 = O(r) \) so that by (38) \( \phi_1 \) vanishes at the horizon and then, by smoothness, is \( O(r) \). We obtain (36) as before but, with the symmetry being inherited, the first term is zero. Provided the horizon is non-degenerate, \( \dot{A} \) is a nonzero constant so that \( \phi_2 = O(r) \).

To go round again we use (37) to make \( \phi_0 \) be \( O(r^2) \), then (38) to find \( (\phi_1)^2 = O(r^3) \) when smoothness forces \( \phi_1 = O(r^2) \). From (36), \( \phi_2 = O(r^2) \) and so, inductively, the Maxwell field vanishes to all orders. We use this observation in the list of possibilities at the end of Section 3.

For Theorem 1.2, we need to review some of the ‘already-unified’ theory of [6] (see also section 5.3 of [17]; their conventions are slightly modified here for consistency with our earlier sections). For an Einstein-Maxwell spacetime with an energy-momentum tensor of the form of (3) with a non-null \( \phi_{AB} \), it is possible algebraically to extract from \( T_{ab} \) a symmetric spinor field \( \Phi_{AB} \), unique up to sign, satisfying

\[ T_{ab} = 2\Phi_{AB} \Phi_{A'B'} \]

and

\[ \Phi^2 := \Phi^{AB} \Phi_{AB} = \overline{\Phi}^{A'B'} \Phi_{A'B'} \geq 0, \]

i.e. \( \Phi^2 \) is real and positive. This is clear by taking components in a frame. (The second condition imposed on \( \Phi_{AB} \) implies that the corresponding bivector field is simple.) Thus \( T_{ab} \), and therefore by Einstein’s equations the Ricci tensor \( R_{ab} \), determine \( \phi_{AB} \) up to a phase (since \( \phi_{AB} \) and \( \Phi_{AB} \) differ only by a phase). To fix the phase, one introduces the complexion vector field by

\[ \alpha_a = 2(R^{pq} R_{pq})^{-1} \epsilon_{abcd} R^d e^c R^b_e \]

\[ = 2i(\Phi^2)^{-1}(\Phi^B_A \nabla^C_{A'} \Phi_{CB} - \overline{\Phi}^B_{A'} \nabla^C_A \overline{\Phi}_{C'B'}). \]

Note that this is only defined for a non-null \( \Phi_{AB} \), which is why the formalism makes that assumption. If \( \phi_{AB} \) satisfies Maxwell’s equations then \( \alpha_a \) is necessarily closed as a one-form and then, in a simply-connected region, the solution of the Maxwell equations is \( \phi_{AB} = e^{i\theta/2} \Phi_{AB} \) where

\[ \partial_a \theta = \alpha_a. \]
If $K^a$ is a Killing vector then, as in Section 1, necessarily

$$\mathcal{L}_K \Phi_{AB} = iA \Phi_{AB}$$

for some real $A$, but now since $\Phi^2$ is everywhere real, $A$ must vanish and so $\Phi_{AB}$ inherits the symmetry. If the Maxwell field $\phi_{AB}$ does not inherit the symmetry, so that

$$\mathcal{L}_K \phi_{AB} = ia \phi_{AB}$$

then, by the argument in Section 1 and the assumption of non-nullness, $a$ is a constant and

$$\mathcal{L}_K \phi^2 = 2ia \phi^2$$

where $\phi^2 := \phi^{AB} \phi_{AB}$, but then $\phi^2 = e^{i\theta} \Phi^2$ so that

$$K^a \alpha_a = K^a \partial_a \theta = 2a = \text{constant}, \quad (39)$$

which is what is needed for the proof of Theorem 1.2.

For Theorem 1.3, suppose we have a null Maxwell field so that the Maxwell spinor takes the form

$$\phi_{AB} = \phi^2 o_A o_B \quad (40)$$

for some function $\phi^2$ and spinor field $o_A$. (We could rescale the function $\phi^2$ and spinor field $o_A$ so as to preserve $\phi_{AB}$, even setting $\phi^2 = 1$ if desired, but it is convenient to leave it like this.)

As is well-known, and in any case easy to see, the Maxwell equations force $o_A$ to be geodesic and shear-free.

We obtain (8) as before but this time this is just equivalent to

$$\phi^A \nabla_{AA'} a = 0,$$

which doesn’t yet force constancy of $a$. We introduce a second spinor field $\iota_A$ to make a normalised basis with $o_A$, and define the Newman-Penrose operators ($D$, $\Delta$, $\delta$) in the usual way (see e.g. [16]). Then $a$ satisfies

$$Da = 0 = \delta a$$

and since $a$ is real, we also have $\overline{\delta a} = 0$.

The geodesic shear-free condition implies the vanishing of the spin-coefficients $\sigma$ and $\kappa$, and then the commutator of $D$ and $\delta$ on $a$ is [16]:

$$(\delta D - D\delta)a = (\rho - \overline{\rho}) \Delta a = 0,$$
making use of \( Da = \delta a = \tilde{\gamma}a = 0 \). Therefore, either \( \Delta a \) vanishes, in which case \( a \) is constant, or \( \rho - \tilde{\tau} \) vanishes, in which case the congruence is twist-free. This completes the proof of Theorem 1.3.

\[ \square \]

As noted above, the Robinson-Trautman (RT) metrics are characterised by admitting a twist-free, shear-free congruence of null geodesics with \( \rho \neq 0 \), while for the Kundt class \( \rho = 0 \). The pp-waves have \( \sigma^A \) parallel (i.e. covariantly constant). It is possible to find (local) RT solutions with a null Maxwell field and a non-inherited symmetry, but, as we describe in the Appendix, it seems very unlikely that they can be asymptotically-flat. In the other cases, the curvature does not decay in the direction of the congruence so they probably can’t be asymptotically-flat either.

The arguments of Theorems 1.1-1.3 rule out many possibilities for stationary, non-inheriting black holes but the following remain open:

- the black hole is static and non-degenerate or stationary, but is not analytic at the horizon (though still with constant \( a \) and zero charge);
  or
- the black hole is stationary and axisymmetric with Killing vectors \( K \) and \( L \) satisfying

\[
\mathcal{L}_KF_{ab} = -aF^a_b, \quad \mathcal{L}_LF_{ab} = -bF^a_b,
\]

with \( a \) and \( b \) nonzero constants, but the null generator of the horizon is the inherited symmetry \( J = bK - aL \), not forbidden by Theorem 1.1. The Maxwell field must be null: by the argument leading to (39), for a non-null Maxwell field we would have \( b = L^a\alpha_a \) which vanishes at the fixed points of the rotation \( L \), a contradiction. By Remark 1.4, the horizon must be degenerate. The charge must be zero: since

\[
\mathcal{L}_L\phi_1 = ib\phi_1
\]

with \( b \neq 0 \); now the charge is an integral of \( \phi_1 \) over the horizon, so that the angular dependence on \( \phi \), where \( L = \partial/\partial \phi \), will force it to vanish; or

- the metric lies in the Robinson-Trautman class with the Maxwell field null, the horizon degenerate, and the generator of the horizon a non-inherited symmetry with non-constant \( a \). (We argue in the Appendix that this case isn’t in fact possible.)
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Appendix: Robinson-Trautman metrics with null Maxwell field

In this appendix, we shall consider Robinson-Trautman metrics with null Maxwell field and see that they are unlikely to provide stationary, asymptotically-flat metrics. The discussion is suggestive rather than conclusive.

A local form for the Einstein-Maxwell Robinson-Trautman metrics is given in Theorem 28.3 of [2]. To make the Maxwell field null, one sets $Q = 0$, to find the Maxwell field as

$$F = h(u, \zeta) du \wedge d\zeta + \text{c.c.},$$

and the metric (with signature switched to accord with our conventions) as

$$g = 2 du (dr + \frac{1}{2} H du) - \frac{2r^2}{P^2} d\zeta d\overline{\zeta},$$

where $h(u, \zeta)$ is arbitrary, $P(u, \zeta, \overline{\zeta})$ is subject to the remaining field equation, $H$ is given by

$$H = \Delta \log P - 2r \frac{P_u}{P} - 2 \frac{m}{r},$$

and $\Delta = 2P^2 \partial_\zeta \partial_{\overline{\zeta}}$ (so $\Delta$ here is not the Newman-Penrose operator with that name).

It is hard to see how the Maxwell field $F$ as above can be asymptotically flat unless $h = 0$: regularity at $\zeta = 0$ would require $h$ finite there, but then regularity at $\zeta = \infty$ would be impossible. However, one doesn’t know exactly how these local coordinates should be related to coordinates on $\text{Scri}$, so we give a second argument as follows.

The existence of a Killing vector imposes further restrictions on the metric: the spinor dyad is fixed implicitly by the equations:

$$o_A \overline{\sigma}_A dx^{AA'} = du; o_A \overline{\tau}_A dx^{AA'} = \frac{r}{P} d\zeta,$$
and the Killing vector $K^a$ must preserve $o_A$ up to scale, so that

$$\mathcal{L}_K o_A = f o_A, \mathcal{L}_K i_A = g o_A - f i_A,$$

for some functions $f$, $g$. We make an ansatz

$$K^a \partial_a = A \partial_u + B \partial_r + C \partial_\zeta + \overline{C} \partial_{\overline{\zeta}}.$$

The Killing equations can then be solved to find the only possibilities as

$$K^a \partial_a = \alpha \partial_u + C(\zeta) \partial_\zeta + \overline{C}(\overline{\zeta}) \partial_{\overline{\zeta}},$$

for a constant $\alpha$ and a function $C$ (this form for the Killing vector can be simplified further, using the allowed coordinate freedom, but there are still some necessary conditions to impose).

For the norm of this Killing vector, we find

$$g(K, K) = \alpha^2 H - \frac{2r^2}{P^2} C \overline{C}.$$

However, we want the Killing vector to be time-like asymptotically, which with our signature means $g(K, K) > 0$. Assuming, as seems most likely, that ‘asymptotically’ means ‘for large $r$’, we therefore need $C = 0$ (since $H = O(r)$ at most). However $\partial/\partial u$, which is what then remains, can only be a Killing vector if $\partial P/\partial u = 0$, when (from the remaining field equation) the metric collapses to the Schwarzschild metric.

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