On the Concavity of Expected Shortfall

Mikhail Tselishchev

Abstract

It is well known that Expected Shortfall (also called Average Value-at-Risk) is a convex risk measure, i.e. Expected Shortfall of a convex linear combination of arbitrary risk positions is not greater than a convex linear combination with the same weights of Expected Shortfalls of the same risk positions. In this short paper we prove that Expected Shortfall is a concave risk measure with respect to probability distributions, i.e. Expected Shortfall of a finite mixture of arbitrary risk positions is not lower than the linear combination of Expected Shortfalls of the same risk positions (with the same weights as in the mixture).

Keywords: Expected Shortfall, AVaR, quantile function, spectral risk measure, convexity, concavity, risk management, mixture of distributions.

1 Introduction

Expected Shortfall (ES) is a standard risk measure used by financial institutions and regulators to determine capital requirements. The notion of Expected Shortfall was introduced in [1][2]. Expected Shortfall at level $\alpha$ is defined as an average loss in the worst $\alpha \cdot 100\%$ cases for a given risk position.

In order to give mathematical definitions we will need some basic notation. We assume that risk positions are random variables on some probability space $(\Omega, \mathcal{F}, P)$, they denote profits (or losses when negative) of some asset or portfolio at the end of the trading period. We will denote by $F_X(\cdot)$ the cumulative distribution function (cdf) of risk position $X$. A lower quantile function (also known as an inverse cdf) for the distribution of $X$ is defined as

$$q_{\alpha}(X) \equiv F_X^{-1}(\alpha) := \inf \{ x \in \mathbb{R} : F_X(x) \geq \alpha \}, \quad \alpha \in [0, 1].$$

(1)

Note that this definition is independent of whether one takes left-continuous or right-continuous cdf’s. In the following definition of ES, one can replace lower quantiles with the upper quantiles

$$q^{(\alpha)}(X) := \inf \{ x \in \mathbb{R} : F_X(x) > \alpha \}, \quad \alpha \in [0, 1],$$

(2)

since ES is an integral characteristic of quantile function, and the lower quantile function coincides with the upper quantile function almost everywhere w.r.t. Lebesgue.
measure on $[0, 1]$. More precisely, they differ only on at most countable set of points $\alpha \in [0, 1]$, where $F_X^{-1}(\{\alpha\})$ consists of at least two different points (and thus the whole interval of constancy of $F_X$). One might check that the lower quantile function is left-continuous, while the upper quantile function is right-continuous.

Expected Shortfall of risk position $X$ at level $\alpha \in (0, 1]$ is defined as

$$ES_\alpha(X) := -\frac{1}{\alpha} \int_0^\alpha q_{\gamma}(X) \, d\gamma,$$

which makes clear why ES is sometimes called as Average Value-at-Risk (AVaR). It is well known that $q_\gamma(X)$, as a function of $\gamma$ on the unit interval with Lebesgue measure, has the same probability distribution as risk position $X$ itself. This remark clarifies the fact that $ES_\alpha(X)$ shows the average loss in the worst $\alpha \cdot 100\%$ cases, since “cases” are naturally ordered on the unit interval, and the worst $\alpha \cdot 100\%$ cases for $q_\gamma(X)$ are located on the interval $(0, \alpha)$.

Alternatively, Expected Shortfall can be defined a bit trickier:

$$ES_\alpha(X) := -\frac{1}{\alpha} \left( E \left[ X 1\{X < q_\alpha(X)\} \right] + q_\alpha(X) \cdot \left( \alpha - P(X < q_\alpha(X)) \right) \right).$$

We will use both representations (3) and (4) in our work. The proof of their equivalence can be found in [2].

Note that $ES_\alpha(X)$ is correctly defined for risk positions $X$ with $E[X^-] < +\infty$. We will assume that this condition is satisfied for all risk positions presented in the paper.

According to (3) (or (4)), $ES_1(X)$ coincides with $-E[X]$. Expected Shortfall at level 0 is often defined as

$$ES_0(X) := -\text{ess inf} X.$$

Here and later we will use the notation of a standard $(n-1)$-simplex:

$$S_{n-1} := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0 \text{ for all } i = 1, \ldots, n, \text{ and } \sum_{i=1}^n x_i = 1 \right\}.$$

Expected Shortfall is known to be convex, i.e. for random vector $X = (X_1, \ldots, X_n)$, weights $\beta \in S_{n-1}$ and arbitrary level $\alpha \in [0, 1]$, the following inequality holds true:

$$ES_\alpha \left( \sum_{j=1}^n \beta_j X_j \right) \leq \sum_{j=1}^n \beta_j ES_\alpha(X_j),$$

which states that the risk of a convex linear combination of risk positions cannot exceed the convex linear combination (with the same weights) of marginal risks. Due to positively homogeneity of Expected Shortfall (that is $ES_\alpha(\lambda X) = \lambda ES_\alpha(X)$ for any $\lambda \geq 0$, this fact can be proved directly), inequality (6) is equivalent to the subadditivity of Expected Shortfall: $ES_\alpha(X + Y) \leq ES_\alpha(X) + ES_\alpha(Y)$. One can find several proofs of subadditivity in [3].
The equality in (6) is attained when “worst cases” for all risk positions coincide and have the same ordering up to level $\alpha$. In particular, this is the case when all $X_j$ are comonotone, i.e., their joint copula has the form of $\min(x_1, \ldots, x_n)$ for $x_1, \ldots, x_n \in [0, 1]$.

We will denote a mixture of random vector $X = (X_1, \ldots, X_n)$ with weights $\beta \in S_n$ as $\text{mix}_\beta X$. This random variable, whose cdf is $\sum_{j=1}^{n} \beta_j F_{X_j}(\cdot)$, is not uniquely defined (or even could not be defined at all on $(\Omega, \mathcal{F}, P)$ if this probability space is too poor), but since Expected Shortfall and other quantile-based risk measures are characteristics of distribution of random variable, there should be no confusion in what follows.

2 Main Result

We will use the following lemma to prove the concavity of Expected Shortfall.

**Lemma.** Let $X = (X_1, \ldots, X_n)$ be a random vector and $\beta \in S_n$. Then for every $\alpha \in (0, 1)$ there exist $\alpha_1, \ldots, \alpha_n \in [0, 1]$, such that $\alpha = \sum_{j=1}^{n} \alpha_j \beta_j$ and

$$\text{ES}_\alpha(\text{mix}_\beta X) = \sum_{j=1}^{n} \frac{\alpha_j \beta_j}{\alpha} \text{ES}_{\alpha_j}(X_j).$$

(7)

**Proof.** Let $\xi = \text{mix}_\beta X$, and $q_\alpha = q_\alpha(\xi)$ be an $\alpha$-lower quantile of this mixture. Without loss of generality we assume all $\beta_j > 0$, $j = 1 \ldots n$. We are going to show that the following $\alpha_j$ satisfy the statement of lemma:

$$\alpha_j := P(X_j < q_\alpha) + P(X_j = q_\alpha) \cdot \frac{\alpha - P(\xi < q_\alpha)}{P(\xi = q_\alpha)}.$$

(8)

Note that if $P(\xi = q_\alpha) = 0$, then the cdf of $\xi$ is continuous at $q_\alpha$, and thus the cdf of each $X_j$ is also continuous at that point, i.e. $P(X_j = q_\alpha) = 0$, and the second term in the right-hand side of (8) should be interpreted as zero. From the definition of the lower quantile function, one has $\alpha_j \in [P(X_j < q_\alpha), P(X_j \leq q_\alpha)]$. Next, if $P(\xi = q_\alpha) > 0$, then

$$\sum_{j=1}^{n} \alpha_j \beta_j = P(\xi < q_\alpha) + P(\xi = q_\alpha) \cdot \frac{\alpha - P(\xi < q_\alpha)}{P(\xi = q_\alpha)} = \alpha.$$

If $P(\xi = q_\alpha) = 0$, then $P(\xi < q_\alpha) = \alpha$, and $\sum_{j=1}^{n} \alpha_j \beta_j = P(\xi < q_\alpha) = \alpha$ as well. It only remains to prove (7). According to (4), the latter equality and properties of the
mixture of distributions, one has

\[
ES\alpha(\xi) = -\frac{1}{\alpha} \left( E \left[ \xi 1_{\{\xi < q_\alpha\}} \right] + q_\alpha \cdot \left( \alpha - P(\xi < q_\alpha) \right) \right) =
\]

\[
= -\frac{1}{\alpha} \sum_{j=1}^{n} \beta_j E \left[ X_j 1_{\{X_j < q_\alpha\}} \right] + \sum_{j=1}^{n} \beta_j q_\alpha \cdot \left( \alpha_j - P(X_j < q_\alpha) \right)
\]

\[
= -\frac{1}{\alpha} \sum_{j=1}^{n} \beta_j \left( E \left[ X_j 1_{\{X_j < q_\alpha\}} \right] + q_\alpha \cdot \left( \alpha_j - P(X_j < q_\alpha) \right) \right).
\]

(9)

If some \( \alpha_j = 0 \), then \( P(X_j < q_\alpha) = 0 \), which implies \( ES_0(X_j) < +\infty \) and

\[
E \left[ X_j 1_{\{X_j < q_\alpha\}} \right] + q_\alpha \cdot \left( \alpha_j - P(X_j < q_\alpha) \right) = 0 = -\alpha_j ES_{\alpha_j}(X_j) \quad (10)
\]

We want to show that for every \( j \) with \( \alpha_j > 0 \) the following equality holds true:

\[
E \left[ X_j 1_{\{X_j < q_\alpha\}} \right] + q_\alpha \cdot \left( \alpha_j - P(X_j < q_\alpha) \right) =
\]

\[
= E \left[ X_j 1_{\{X_j < q_{\alpha_j}(X_j)\}} \right] + q_{\alpha_j}(X_j) \cdot \left( \alpha_j - P(X_j < q_{\alpha_j}(X_j)) \right). \quad (11)
\]

Since \( P(X_j < q_\alpha) \geq \alpha_j \), then \( q_{\alpha_j}(X_j) \leq q_\alpha \). If \( q_{\alpha_j}(X_j) = q_\alpha \), then (11) is trivial. If, however, \( q_{\alpha_j}(X_j) < q_\alpha \), then one has a chain of inequalities:

\[
P(X_j < q_\alpha) \geq P(X_j \leq q_{\alpha_j}(X_j)) \geq \alpha_j \geq P(X_j < q_\alpha).
\]

Hence, \( P(X_j \leq q_{\alpha_j}(X_j)) = \alpha_j \) and \( P(q_{\alpha_j}(X_j) < X_j < q_\alpha) = 0 \). Given that, consider the difference between the left- and the right-hand sides of (11):

\[
E \left[ X_j 1_{\{X_j < q_\alpha\}} \right] + q_\alpha \cdot \left( \alpha_j - P(X_j < q_\alpha) \right) -
\]

\[
- E \left[ X_j 1_{\{X_j < q_{\alpha_j}(X_j)\}} \right] - q_{\alpha_j}(X_j) \cdot \left( \alpha_j - P(X_j < q_{\alpha_j}(X_j)) \right) =
\]

\[
= E \left[ X_j 1_{\{q_{\alpha_j}(X_j) \leq X_j < q_\alpha\}} \right] + q_\alpha \cdot \left( \alpha_j - P(X_j < q_\alpha) \right) -
\]

\[
- q_{\alpha_j}(X_j) \cdot \left( \alpha_j - P(X_j < q_{\alpha_j}(X_j)) \right) =
\]

\[
= q_{\alpha_j}(X_j) \cdot \left( P(X_j \leq q_{\alpha_j}(X_j)) - \alpha_j \right) + q_\alpha \cdot \left( \alpha_j - P(X_j \leq q_{\alpha_j}(X_j)) \right) =
\]

\[
= \left( q_\alpha - q_{\alpha_j}(X_j) \right) \cdot \left( \alpha_j - P(X_j \leq q_{\alpha_j}(X_j)) \right) = 0.
\]

Equation (11) is proved. Finally, (9), together with (10), (11) and (4), gives

\[
ES\alpha(\xi) = -\frac{1}{\alpha} \sum_{j=1}^{n} \beta_j E \left[ X_j 1_{\{X_j < q_{\alpha_j}(X_j)\}} \right] + q_{\alpha_j}(X_j) \left( \alpha_j - P(X_j < q_{\alpha_j}(X_j)) \right)
\]

\[
= \sum_{j=1}^{n} \frac{\beta_j \alpha_j}{\alpha} ES_{\alpha_j}(X_j).
\]
Lemma is proved.

Now we are ready to formulate and prove the main result of the paper.

**Theorem (Concavity of Expected Shortfall).** Let $X = (X_1, \ldots, X_n)$ be a random vector and $\beta \in S_{n-1}$. Then for every $\alpha \in [0, 1]$ one has

$$\text{ES}_\alpha \left( \text{mix} \beta \right) \geq \sum_{j=1}^{n} \beta_j \text{ES}_\alpha(X_j). \quad (12)$$

**Proof.** Again, we assume all $\beta_j > 0$ without loss of generality. If $\alpha = 0$, then

$$\text{ES}_0(\text{mix} \beta X) = - \text{ess inf} \left( \text{mix} \beta X \right) = - \min_{j=1}^{n} \text{ess inf}(X_j) \geq - \sum_{j=1}^{n} \beta_j \text{ess inf}(X_j) = \sum_{j=1}^{n} \beta_j \text{ES}_0(X_j).$$

If $\alpha = 1$, then

$$\text{ES}_1(\text{mix} \beta X) = - \mathbb{E} \left[ \text{mix} \beta X \right] = - \sum_{j=1}^{n} \beta_j \mathbb{E}[X_j] = \sum_{j=1}^{n} \beta_j \text{ES}_1(X_j).$$

In case $\alpha \in (0, 1)$ we will take advantage of the previous lemma. Taking $q_\alpha$ and $\alpha_j$ from its proof and using representation (3), one has

$$\text{ES}_\alpha \left( \text{mix} \beta \right) = \sum_{j=1}^{n} \frac{\beta_j \alpha_j}{\alpha} \text{ES}_\alpha(X_j) = - \sum_{j=1}^{n} \frac{\beta_j}{\alpha} \int_0^{\alpha_j} q_\gamma(X_j) \, d\gamma = \sum_{j=1}^{n} \beta_j \text{ES}_\alpha(X_j) - \sum_{j \in J_+} \frac{\beta_j}{\alpha} \int_0^{\alpha_j} q_\gamma(X_j) \, d\gamma + \sum_{j \in J_-} \frac{\beta_j}{\alpha} \int_{\alpha_j}^{\alpha} q_\gamma(X_j) \, d\gamma, \quad (13)$$

where $J_+ = \{ j : \alpha_j > \alpha \}$ and $J_- = \{ j : \alpha_j < \alpha \}$. For $j \in J_+$ we have

$$\int_{\alpha}^{\alpha_j} q_\gamma(X_j) \, d\gamma \leq q_{\alpha_j}(X_j) \cdot (\alpha_j - \alpha) \leq q_\alpha \cdot (\alpha_j - \alpha). \quad (14)$$

Now consider $j \in J_-$. If $q_\alpha(X_j) < q_\alpha$ for some $\gamma > \alpha_j$, then $\mathbb{P}(X_j < q_\alpha) \geq \gamma > \alpha_j$, which leads to contradiction with the definition of $\alpha_j$. Thus, $q_\gamma(X_j) \geq q_\alpha$ for all $\gamma > \alpha_j$, and

$$\int_{\alpha_j}^{\alpha} q_\gamma(X_j) \, d\gamma \geq q_\alpha \cdot (\alpha - \alpha_j). \quad (15)$$

By substituting inequalities (14) and (15) into (13), we obtain

$$\text{ES}_\alpha \left( \text{mix} \beta \right) \geq \sum_{j=1}^{n} \beta_j \text{ES}_\alpha(X_j) + \sum_{j=1}^{n} \frac{\beta_j}{\alpha} q_\alpha \cdot (\alpha - \alpha_j) = \sum_{j=1}^{n} \beta_j \text{ES}_\alpha(X_j),$$

where the last equality holds due to $\beta \in S_{n-1}$ and $\sum_{j=1}^{n} \beta_j \alpha_j = \alpha$. The theorem is proved.
We have to notice that the proofs of both lemma and theorem are much simpler and more straightforward when all cdfs $F_{X_j}$ are continuous and strictly increasing.

Also note that inequality (12) turns into equality in case all $\alpha_j$ are equal to $\alpha$. This means that $\alpha$-quantiles for all risk positions $X_1, \ldots, X_n$ coincide.

Proposed result can be naturally extended to a wider class of spectral risk measures. Indeed, according to a spectral representation theorem (see [4] and [5]), under suitable conditions of integrability, a spectral risk measure may be represented as a weighted combination of Expected Shortfalls

$$\rho_\nu(X) = \int_{[0,1]} ES_\alpha(X) \, d\nu(\alpha)$$

for some probability measure $\nu$ on $([0,1], B_{[0,1]})$. Hence, using (12),

$$\rho_\nu\left(\text{mix}_j \beta_j X\right) = \int_{[0,1]} ES_\alpha\left(\text{mix}_j \beta_j X\right) \, d\nu(\alpha) \geq \sum_{j=1}^n \beta_j \int_{[0,1]} ES_\alpha(X_j) \, d\nu(\alpha) = \sum_{j=1}^n \beta_j \rho_\nu(X_j),$$

which means that the spectral risk measures are also concave with respect to probability distributions.

Combining both (6) and (12), one gets

$$ES_\alpha\left(\sum_{j=1}^n \beta_j X\right) \leq ES_\alpha\left(\text{mix}_j \beta_j X\right),$$

i.e. a risk of a convex linear combination can not exceed a risk of a mixture of the same positions with the same weights. That useful property of ES (and spectral risk measures as well) might be called a principle of diversification. This principle seems quite reasonable (unless one chooses a beverage in the bar). When $\alpha$-quantiles of all risk positions coincide, and the "worst cases" up to level $\alpha$ are also the same for all of them, then inequality (16) turns into equality.

3 Conclusions

In this paper we have shown a neat property of Expected Shortfall, which is concavity with respect to probability distributions. We have extended this property to a class of spectral risk measures. Finally, we have discussed the implications of concavity, that led us to the principle of diversification. In our subsequent paper we are going to show that the principle of diversification is somewhat necessary for risk positions to be comparable with Expected Shortfall.

4 Final words

After finishing this paper, we found that the concavity of Expected Shortfall has already been proved recently in [6, Proposition 3.2]. Their proof is even more elegant.
and compact. It uses another representation of Expected Shortfall, called Conditional Value-at-Risk (CVaR), which is a solution to a specific minimization problem. We decided to share our proof as well, since it differs internally. Moreover, our lemma might be useful in some other cases.

References

[1] C. Acerbi and D. Tasche, "Expected Shortfall: A natural coherent alternative to Value at Risk," Economic Notes, vol. 31, pp. 379–388, 05 2002.

[2] C. Acerbi and D. Tasche, “On the coherence of Expected Shortfall,” Journal of Banking & Finance, vol. 26, pp. 1487–1503, 07 2002.

[3] P. Embrechts and R. Wang, “Seven proofs for the subadditivity of Expected Shortfall,” Dependence Modeling, vol. 3, 01 2015.

[4] C. Acerbi, “Risk aversion and coherent risk measures: a spectral representation theorem,” arXiv:cond-mat/0107190, 07 2001.

[5] S. Kusuoka, “On law invariant coherent risk measures,” RIMS Kokyuroku, vol. 3, 01 2001.

[6] G. Pertaia and S. Uryasev, “Fitting mixture models with CVaR constraints.” https://www.isc.ufl.edu/uryasev/files/2019/01/CVarMixture.pdf.