FLOER HOMOLOGY, NIELSEN THEORY AND SYMPLECTIC ZETA FUNCTIONS

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ABSTRACT. We describe connection between symplectic Floer homology for surfaces and Nielsen fixed point theory. A new zeta functions and asymptotic invariant of symplectic origin are defined. We show that special values of symplectic zeta functions are Reidemeister torsions.

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1. INTRODUCTION

In two dimensions a diffeomorphism is symplectic if it preserves area. As a consequence, the symplectic geometry of surfaces lacks many of the interesting phenomena which are encountered in higher dimensions. For example, two symplectic automorphisms of a closed surface are symplectically isotopic iff they are homotopic, by a theorem of Moser [19]. On other hand symplectic fixed point theory is very nontrivial in dimension 2, as shown by the Poincare-Birkhoff theorem. It is known that symplectic Floer homology on surface is a simple model for the instanton Floer homology of the mapping torus of the surface diffeomorphism [21].

In this article we define new zeta functions related to symplectic Floer homology groups and investigate their analytical properties. We show that special values of these zeta functions are Reidemeister torsions. We also define asymptotic invariant of monotone
symplectomorphism. We prove the following result: Let $M$ be a compact connected surface of Euler characteristic $\chi(M) < 0$. If $\phi$ is a non-trivial orientation preserving periodic diffeomorphism of $M$ or $\phi$ is a diffeomorphism of finite type with only isolated fixed points, then $\phi$ is monotone with respect to some $\phi$-invariant area form on $M$ and $HF_*(\phi) \cong \mathbb{Z}^{N(\phi)}$, $\dim HF_*(\phi) = N(\phi)$ where, $N(\phi)$ denotes the Nielsen number of $\phi$ and $HF_*(\phi)$ denotes symplectic Floer homology group. Due to P. Seidel \[22\] $\dim HF_*(\phi)$ is a new symplectic invariant of a four-dimensional symplectic manifold with nonzero first Betti number. This 4-manifold produced from symplectomorphism $\phi$ by a surgery construction which is a variation of earlier constructions due to McMullen-Taubes, Fintushel-\-Stern and J. Smith. We hope that our symplectic zeta functions and asymptotic invariant also give rise to a new invariants of contact 3- manifolds and symplectic 4-manifolds.

The author came to the idea that Nielsen numbers are connected with Floer homology of surface diffeomorphisms at the Autumn 2000, after conversations with Joel Robbin and Dan Burghelea. Last years I try to find a cohomological theory which lies behind of the Nielsen zeta function \[5\]. The results of this paper were reported in the author talk on the International Conference Topological Methods in Nonlinear Analysis, June 2001 in Bendlewo, Poland. I am very grateful to Dietmar Salomon who send me handwritten notes of Wu-Chung Hsiang paper “A speculation on Floer theory and Nielsen theory” and to Stefan Haller, Wu-Chung Hsiang, Jarek Kedra, Wilhelm Klingenberg, Francois Laudenbach, Kaoru Ono, Yuli Rudjak, Andrei Tyurin and Vladimir Turaev for very useful discussions of the results of this paper. The delay in publication was connected with the absence of the important notion of monotonicity, introduced by P. Seidel\[21\] in the Spring 2001.

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2. The monotonicity condition

In this section we discuss the notion of monotonicity as defined in \[21\] \[10\]. Monotonicity plays important role for Floer homology in two dimensions. For a more detailed account we refer to the original articles of P. Seidel and R. Gautschi.

Throughout this article, $M$ denotes a closed connected and oriented 2-manifold of genus $\geq 2$. Pick an everywhere positive two-form $\omega$ on $M$.

Let $\phi \in \text{Symp}(M, \omega)$, the group of symplectic automorphisms of the two-dimensional symplectic manifold $M, \omega$. The mapping torus of $\phi$, $T_\phi = \mathbb{R} \times M/(t + 1, x) \sim (t, \phi(x))$, is a 3-manifold fibered over $S^1 = \mathbb{R}/\mathbb{Z}$. There are two natural second cohomology classes on $T_\phi$, denoted by $[\omega_\phi]$ and $c_\phi$. The first one is represented by the closed two-form $\omega_\phi$ which is induced from the pullback of $\omega$ to $\mathbb{R} \times M$. The second is the Euler class of the vector bundle

$$V_\phi = \mathbb{R} \times TM/(t + 1, \xi_x) \sim (t, d\phi_x \xi_x),$$

which is of rank 2 and inherits an orientation from $TM$.

$\phi \in \text{Symp}(M, \omega)$ is called monotone, if

$$[\omega_\phi] = (\text{area}_\omega(M)/\chi(M)) \cdot c_\phi$$

in $H^2(T_\phi; \mathbb{R})$; throughout this article $\text{Symp}^m(M, \omega)$ denotes the set of monotone symplectomorphisms.
Now $H^2(T_\phi; \mathbb{R})$ fits into the following short exact sequence \[21\,10\]

\[
0 \longrightarrow \frac{H^1(M; \mathbb{R})}{\text{im}(\text{id} - \phi^*)} \overset{d}{\longrightarrow} H^2(T_\phi; \mathbb{R}) \overset{r^*}{\longrightarrow} H^2(M; \mathbb{R}) \longrightarrow 0.
\]

where the map $r^*$ is restriction to the fiber. The map $d$ is defined as follows. Let $\rho : I \to \mathbb{R}$ be a smooth function which vanishes near 0 and 1 and satisfies $\int_0^1 \rho \, dt = 1$. If $\theta$ is a closed 1-form on $M$, then $\rho \cdot \theta \wedge dt$ defines a closed 2-form on $T_\phi$; indeed $d[\theta] = [\rho \cdot \theta \wedge dt]$. The map $r : M \to T_\phi$ assigns to each $x \in M$ the equivalence class of $(1/2, x)$. Note, that $r^* \omega_\phi = \omega$ and $r^* c_\phi$ is the Euler class of $TM$. Hence, by \[11\], there exists a unique class $m(\phi) \in H^1(M; \mathbb{R})/\text{im}(\text{id} - \phi^*)$ satisfying

\[
d m(\phi) = [\omega_\phi] - (\text{area}_\omega(M)/\chi(M)) \cdot c_\phi,
\]

where $\chi$ denotes the Euler characteristic. Therefore, $\phi$ is monotone if and only if $m(\phi) = 0$.

We recall the fundamental properties of $\text{Symp}^m(M, \omega)$ from \[21\,10\]. Let $\text{Diff}^+(M)$ denotes the group of orientation preserving diffeomorphisms of $M$.

(Identity) $id_M \in \text{Symp}^m(M, \omega)$.

(Naturality) If $\phi \in \text{Symp}^m(M, \omega)$, $\psi \in \text{Diff}^+(M)$, then $\psi^{-1} \phi \psi \in \text{Symp}^m(M, \psi^* \omega)$.

(Isotopy) Let $(\psi_t)_{t \in I}$ be an isotopy in $\text{Symp}(M, \omega)$, i.e. a smooth path with $\psi_0 = \text{id}$. Then $m(\phi \circ \psi_1) = m(\phi) + [\text{Flux}(\psi_t)_{t \in I}]$ in $H^1(M; \mathbb{R})/\text{im}(\text{id} - \phi^*)$; see \[21\, Lemma 6\]. For the definition of the flux homomorphism see \[18\].

(Inclusion) The inclusion $\text{Symp}^m(M, \omega) \hookrightarrow \text{Diff}^+(M)$ is a homotopy equivalence. This follows from the isotopy property, surjectivity of the flux homomorphism and Moser’s isotopy theorem \[19\] which says that each element of the mapping class group admits representatives which preserve $\omega$. Furthermore, the Earl-Eells Theorem \[3\] implies that every connected component of $\text{Symp}^m(M, \omega)$ is contractible.

(Floer homology) To every $\phi \in \text{Symp}^m(M, \omega)$ symplectic Floer homology theory assigns a $\mathbb{Z}_2$-graded vector space $HF_*(\phi)$ over $\mathbb{Z}_2$, with an additional multiplicative structure, called the quantum cap product, $H^*(M; \mathbb{Z}_2) \otimes HF_*(\phi) \longrightarrow HF_*(\phi)$. For $\phi = id_M$ the symplectic Floer homology $HF_*(id_M)$ are canonically isomorphic to ordinary homology $H_*(M; \mathbb{Z}_2)$ and quantum cap product agrees with the ordinary cap product. Each $\psi \in \text{Diff}^+(M)$ induces an isomorphism $HF_*(\phi) \cong HF_*(\psi^{-1} \phi \psi)$ of $H_*(M; \mathbb{Z}_2)$-modules.

(Invariance) If $\phi, \phi' \in \text{Symp}^m(M, \omega)$ are isotopic, then $HF_*(\phi)$ and $HF_*(\phi')$ are naturally isomorphic as $H_*(M; \mathbb{Z}_2)$-modules. This is proven in \[21\, Page 7\]. Note that every Hamiltonian perturbation of $\phi$ (see \[2\]) is also in $\text{Symp}^m(M, \omega)$.

Now let $g$ be a mapping class of $M$, i.e. an isotopy class of $\text{Diff}^+(M)$. Pick an area form $\omega$ and a representative $\phi \in \text{Symp}^m(M, \omega)$ of $g$. Then $HF_*(\phi)$ is an invariant of $g$, which is denoted by $HF_*(g)$. Note that $HF_*(g)$ is independent of the choice of an area form $\omega$ by Moser’s isotopy theorem \[19\] and naturality of Floer homology.

3. SYMPLECTIC FLOER HOMOLOGY

Let $\phi \in \text{Symp}(M, \omega)$. There are two ways of constructing Floer homology detecting its fixed points, $\text{Fix}(\phi)$. Firstly, the graph of $\phi$ is a Lagrangian submanifold of $M \times M, (-\omega) \times \omega$ and its fixed points correspond to the intersection points of $\text{graph}(\phi)$.
with the diagonal $\Delta = \{(x, x) \in M \times M\}$. Thus we have the Floer homology of the Lagrangian intersection $HF_*(M \times M, \Delta, graph(\phi))$. This intersection is transversal if the fixed points of $\phi$ are nondegenerate, i.e. if 1 is not an eigenvalue of $d\phi(x)$, for $x \in Fix(\phi)$.

The second approach was mentioned by Floer in [7] and presented with details by Dostoglou and Salomon in [2]. We follow here Seidel’s approach [21] which, comparable with [2], uses a larger class of perturbations, but such that the perturbed action form is still cohomologous to the unperturbed. As a consequence, the usual invariance of Floer homology under Hamiltonian isotopies is extended to the stronger property stated above.

Let now $\phi \in \text{Symp}^m(M, \omega)$, i.e $\phi$ is monotone. Firstly, we give the definition of $HF_*(\phi)$ in the special case where all the fixed points of $\phi$ are non-degenerate, i.e. for all $y \in Fix(\phi)$, $\text{det}(\text{id} - d\phi_y) \neq 0$, and then following Seidels approach [21] we consider general case when $\phi$ has degenerate fixed points. Let $\Omega_\phi = \{y \in C^\infty(\mathbb{R}, M) | y(t) = \phi(y(t+1))\}$ be the twisted free loop space, which is also the space of sections of $T\phi \to S^1$. The action form is the closed one-form $\alpha_\phi$ on $\Omega_\phi$ defined by

$$\alpha_\phi(y)Y = \int_0^1 \omega(dy/dt, Y(t)) dt.$$ 

where $y \in \Omega_\phi$ and $Y \in T_y \Omega_\phi$, i.e. $Y(t) \in T_{y(t)} M$ and $Y(t) = d\phi_{y(t+1)} Y(t+1)$ for all $t \in \mathbb{R}$.

The tangent bundle of any symplectic manifold admits an almost complex structure $J : TM \to TM$ which is compatible with $\omega$ in sense that $(v, w) = \omega(v, Jw)$ defines a Riemannian metric. Let $J = (J_t)_{t \in \mathbb{R}}$ be a smooth path of $\omega$-compatible almost complex structures on $M$ such that $J_{t+1} = \phi^* J_t$. If $Y, Y' \in T_y \Omega_\phi$, then $\int_0^1 \omega(Y'(t), J_t Y(t)) dt$ defines a metric on the loop space $\Omega_\phi$. So the critical points of $\alpha_\omega$ are the constant paths in $\Omega_\phi$ and hence the fixed points of $\phi$. The negative gradient lines of $\alpha_\omega$ with respect to the metric above are solutions of the partial differential equations with boundary conditions

$$\begin{align*}
\left\{ 
\begin{array}{l}
\quad u(s, t) = \phi(u(s, t+1)), \\
\quad \partial_s u + J_t (u) \partial_t u = 0, \\
\quad \lim_{s \to \pm \infty} u(s, t) \in Fix(\phi)
\end{array}
\right.
\end{align*}$$

These are exactly Gromov’s pseudoholomorphic curves [11].

For $y^\pm \in Fix(\phi)$, let $\mathcal{M}(y^-, y^+; J, \phi)$ denote the space of smooth maps $u : \mathbb{R}^2 \to M$ which satisfy the equations (2). Now to every $u \in \mathcal{M}(y^-, y^+; J, \phi)$ is associated a Fredholm operator $D_u$ which linearizes (2) in suitable Sobolev spaces. The index of this operator is given by the so called Maslov index $\mu(u)$, which satisfies $\mu(u) = \deg(y^+) - \deg(y^-)$ mod 2, where $(-1)^{\deg y} = \text{sign}(\text{det}(\text{id} - d\phi_y))$. We have no bulkling, since for surface $\pi_2(M) = 0$. For a generic $J$, every $u \in \mathcal{M}(y^-, y^+; J, \phi)$ is regular, meaning that $D_u$ is onto.

Hence, by the implicit function theorem, $\mathcal{M}_k(y^-, y^+; J, \phi)$ is a smooth $k$-dimensional manifold, where $\mathcal{M}_k(y^-, y^+; J, \phi)$ denotes the subset of those $u \in \mathcal{M}(y^-, y^+; J, \phi)$ with $\mu(u) = k \in \mathbb{Z}$. Translation of the $s$-variable defines a free $\mathbb{R}$-action on 1-dimensional manifold $\mathcal{M}_1(y^-, y^+; J, \phi)$ and hence the quotient is a discrete set of points. The energy of a map $u : \mathbb{R}^2 \to M$ is given by $E(u) = \int_{\mathbb{R}} \int_0^1 \omega(\partial_s u(s, t), J_t \partial_t u(s, t)) dt ds$ for all $y \in Fix(\phi)$. P. Seidel has proved in [21] that if $\phi$ is monotone, then the energy is constant on each $\mathcal{M}_k(y^-, y^+; J, \phi)$. Since all fixed points of $\phi$ are nondegenerate the set $Fix(\phi)$ is a finite set and the $\mathbb{Z}_2$-vector space $CF_*(\phi) := \mathbb{Z}_2^{\# Fix(\phi)}$ admits a $\mathbb{Z}_2$-grading with $(-1)^{\deg y} = \text{sign}(\text{det}(\text{id} - d\phi_y))$, for all $y \in Fix(\phi)$. The boundness of the energy $E(u)$ for monotone $\phi$ implies that the 0-dimensional quotients $\mathcal{M}_1(y_-, y_+, J, \phi)/\mathbb{R}$ are actually
finite sets. Denoting by \( n(y_-,y_+) \) the number of points mod 2 in each of them, one defines a differential \( \partial_J : CF_\ast(\phi)\to CF_{\ast+1}(\phi) \) by \( \partial_J y_- = \sum_{y_+} n(y_-,y_+)y_+ \). Due to gluing theorem this Floer boundary operator satisfies \( \partial_J \circ \partial_J = 0 \). For gluing theorem to hold one needs again the boundness of the energy \( E(u) \). It follows that \((CF_\ast(\phi),\partial_J)\) is a chain complex and its homology is by definition the Floer homology of \( \phi \) denoted \( HF_\ast(\phi) \). It is independent of \( J \) and is an invariant of \( \phi \).

If \( \phi \) has degenerate fixed points one needs to perturb equations \([2]\) in order to define the Floer homology. Equivalently, one could say that the action form needs to be perturbed. The necessary analysis is given in \([21]\) is essentially the same as in the slightly different situations considered in \([2]\). But Seidel’s approach also differs from the usual one in \([2]\). He uses a larger class of perturbations, but such that the perturbed action form is still cohomologous to the unperturbed.

4. Nielsen numbers and Floer homology

Before discussing the results of the paper, we briefly describe the few basic notions of Nielsen fixed point theory which will be used. We assume \( X \) to be a connected, compact polyhedron and \( f : X \to X \) to be a continuous map. Let \( p : \tilde{X} \to X \) be the universal cover of \( X \) and \( \tilde{f} : \tilde{X} \to \tilde{X} \) a lifting of \( f \), i.e. \( p \circ \tilde{f} = \tilde{f} \circ p \). Two liftings \( \tilde{f} \) and \( \tilde{f}' \) are called conjugate if there is a \( \gamma \in \Gamma \cong \pi_1(X) \) such that \( \tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1} \). The subset \( p(\text{Fix}(\tilde{f})) \subset \text{Fix}(f) \) is called the fixed point class of \( f \) determined by the lifting class \([\tilde{f}]\). Two fixed points \( x_0 \) and \( x_1 \) of \( f \) belong to the same fixed point class iff there is a path \( c \) from \( x_0 \) to \( x_1 \) such that \( c \cong f \circ c \) (homotopy relative endpoints). This fact can be considered as an equivalent definition of a non-empty fixed point class. Every map \( f \) has only finitely many non-empty fixed point classes, each a compact subset of \( X \). A fixed point class is called essential if its index is nonzero. The number of essential fixed point classes is called the Nielsen number of \( f \), denoted by \( N(f) \). The Nielsen number is always finite. \( R(f) \) and \( N(f) \) are homotopy invariants. In the category of compact, connected polyhedra, the Nielsen number of a map is, apart from certain exceptional cases, equal to the least number of fixed points of maps with the same homotopy type as \( f \).

4.1. Periodic diffeomorphisms. The following Lemma was first proven in \([12]\). We repeat here the arguments from \([10]\).

**Lemma 1.** \([12]\) Let \( \phi \) a non-trivial orientation preserving periodic diffeomorphism of a compact connected surface \( M \) of Euler characteristic \( \chi(M) \leq 0 \). Then each fixed point class of \( \phi \) consists of a single point.

**Proof.** First assume that \( M \) is closed. The uniformization theorem states that in every conformal class of metrics on \( M \), there is a unique metric of constant curvature \(-1\) if \( \chi(M) < 0 \) or \( 0 \) if \( \chi(M) = 0 \). This implies that the unique representative of a \( \phi \)-invariant conformal class of metrics (such a class exists since \( \phi \) is finite order) is itself \( \phi \)-invariant. Hence we can pick a \( \phi \)-invariant metric of constant curvature \(-1\) or \( 0 \) on \( M \) and lift \( \phi \) to an isometry \( \tilde{\phi} \) of the universal cover \( \tilde{M} \) of \( M \). \( \tilde{M} \) is either isometric to the hyperbolic plane \( H^2 \) or the Euclidean plane \( \mathbb{R}^2 \).

Let \( x \in \text{Fix}(\phi) \) and let \( \tilde{x}, \tilde{x} \) be lifts of \( \phi, x \) to \( \tilde{M} \), such that \( \tilde{\phi}(\tilde{x}) = \tilde{x} \). Note, that a fixed point of \( \phi \) is in the same class as \( x \) if and only if it can be lifted to a fixed point of \( \tilde{\phi} \). Assume by contradiction that \( \tilde{y} \neq \tilde{x} \) is a fixed point of \( \tilde{\phi} \). It follows that the unique
Lemma 2. \(k\) out by the subgroup \(\ker(id - \phi) \subset H_1(M; \mathbb{Z})\) is represented by a map \(\gamma : S \to \text{Fix}(\phi)\), where \(S\) is a compact oriented 1-manifold. Then \(\phi\) is monotone.

Lemma 3. \(\phi^k\) is monotone for some \(k > 0\), then \(\phi\) is monotone. If \(\phi\) is monotone, then \(\phi^k\) is monotone for all \(k > 0\).

Proof. We repeat Gautschi arguments from [10] here. Recall that \(T_\phi\) is the orbit space of the \(\mathbb{Z}\)-action \(n \cdot (t, x) = (t + n, \phi^{-n}(x))\), where \(n \in \mathbb{Z}\) and \((t, x) \in \mathbb{R} \times \Sigma\). If we only divide out by the subgroup \(k\mathbb{Z}\), for \(k \in \mathbb{N}_{>0}\), we naturally get the mapping torus of \(\phi^k\). Further dividing by \(\mathbb{Z}/k\mathbb{Z}\) defines the \(k\)-fold covering map \(p_k : T_{\phi^k} \to T_\phi\). It is straight forward to check that

\[
p_k^*[\omega_\phi] = [\omega_{\phi^k}] \quad \text{and} \quad p_k^*c_\phi = c_{\phi^k}.
\]

The first equality follows immediately from the definitions. To prove the second, note that \(p_k^*((TM \times \mathbb{R})/\mathbb{Z}) \cong (TM \times \mathbb{R})/k\mathbb{Z} \cong V_{\phi^k}\), where the \(\mathbb{Z}\)-action on \(\mathbb{R} \times T\Sigma\) is given by \(n \cdot (t, \xi) = (t + n, d\phi^{-n}_x \xi)\), for \(n \in \mathbb{Z}\) and \(\xi \in T_xM\). The lemma follows from [10] and the fact that \(p_k^*\) is injective. To prove injectivity, define the map \(a_k^* : H^2(T_{\phi^k}; \mathbb{R}) \to H^2(T_\phi; \mathbb{R})\) by averaging differential forms; \(a_k^*\) is a left inverse of \(p_k^*\), i.e. \(a_k^* \circ p_k^* = id\). This ends the proof of the lemma.

We shall say that \(\phi : M \to M\) is a periodic map of period \(m\), if \(\phi^m = id_M : M \to M\).

Theorem 4. If \(\phi\) is a non-trivial orientation preserving periodic diffeomorphism of a compact connected surface \(M\) of Euler characteristic \(\chi(M) < 0\), then \(\phi\) is monotone with respect to some \(\phi\)-invariant area form and

\[
HF_*(\phi) \cong \mathbb{Z}^{N(\phi)}_2, \quad \dim HF_*(\phi) = N(\phi),
\]

where \(N(\phi)\) denotes the Nielsen number of \(\phi\).

Proof. Let \(\phi\) be a periodic diffeomorphism of least period \(l\). First note that if \(\omega\) is an area form on \(M\), then area form \(\omega := \sum_{i=1}^l (\phi^i)^*\omega\) is \(\phi\)-invariant, i.e. \(\phi \in \text{Symp}(M, \omega)\). By periodicity of \(\phi\), \(\phi^l\) is the identity map \(id_M : M \to M\). Then from Lemmas [10] and [10] it follows that \(\omega\) can be chosen such that \(\phi \in \text{Symp}^m(M, \omega)\).

Lemma [10] implies that every \(y \in \text{Fix}(\phi)\) forms a different fixed point class of \(\phi\), so \(# \text{Fix}(\phi) = N(\phi)\). This has an immediate consequence for the Floer complex \((CF_*(\phi), \partial_J)\) with respect to a generic \(J = (J_t)_{t \in \mathbb{R}}\). If \(y^\pm \in \text{Fix}(\phi)\) are in different fixed point classes,
then $\mathcal{M}(y^-, y^+; J, \phi) = \emptyset$. This follows from the first equation in (2). Then the boundary map in the Floer complex is zero $\partial J = 0$ and $\mathbb{Z}_2$-vector space $CF_*(\phi) := \mathbb{Z}_2^{\# \text{Fix}(\phi)} = \mathbb{Z}_2^{N(\phi)}$. This immediately implies $HF_*(\phi) \cong \mathbb{Z}_2^{N(\phi)}$ and $\dim HF_*(\phi) = N(\phi)$.

4.2. Algebraically finite mapping classes. A mapping class of $M$ is called algebraically finite if it does not have any pseudo-Anosov components in the sense of Thurston’s theory of surface diffeomorphism. The term algebraically finite goes back to J. Nielsen. R. Gautschi [10] defined the notion of a diffeomorphism of finite type for surface diffeomorphisms. These are special representatives of algebraically finite mapping classes adopted to the symplectic geometry.

**Definition 5.** [10] We call $\phi \in \text{Diff}_+(M)$ of finite type if the following holds. There is a $\phi$-invariant finite union $N \subset M$ of disjoint non-contractible annuli such that:

1. $\phi|\text{int}(N)$ is periodic, i.e. there exists $\ell > 0$ such that $\phi^\ell|\text{int}(N) = \text{id}$.
2. Let $N'$ be a connected component of $N$ and $\ell' > 0$ be the smallest integer such that $\phi^{\ell'}$ maps $N'$ to itself. Then $\phi^{\ell'}|N'$ is given by one of the following two models with respect to some coordinates $(q, p) \in I \times S^1$:
   - (twist map) $(q, p) \mapsto (q, p - f(q))$
   - (flip-twist map) $(q, p) \mapsto (1 - q, -p - f(q))$

where $f : I \to \mathbb{R}$ is smooth and strictly monotone. A twist map is called positive or negative, if $f$ is increasing or decreasing.

3. Let $N'$ and $\ell'$ be as in (2). If $\ell' = 1$ and $\phi|\text{int}(N')$ is a twist map, then $\text{im}(f) \subset [0, 1]$, i.e. $\phi|\text{int}(N')$ has no fixed points.

4. If two connected components of $N$ are homotopic, then the corresponding local models of $\phi$ are either both positive or both negative twists.

The term flip-twist map is taken from [14].

By $M_{id}$ we denote the union of the components of $M \setminus \text{int}(N)$, where $\phi$ restricts to the identity.

The next lemma describes the set of fixed point classes of $\phi$. It is a special case of a theorem by B. Jiang and J. Guo [14], which gives for any mapping class a representative that realizes its Nielsen number.

**Lemma 6** (Fixed point classes). [14] Each fixed point class of $\phi$ is either a connected component of $M_{id}$ or consists of a single fixed point. A fixed point $x$ of the second type satisfies $\det(id - d\phi_x) > 0$.

The monotonicity of diffeomorphisms of finite type was investigated in details in the recent preprint by R. Gautschi [10]. We describe now his results. Let $\phi$ be a diffeomorphism of finite type and $\ell$ be as in (1). Then $\phi^\ell$ is the product of (multiple) Dehn twists along $N$. Moreover, two parallel Dehn twists have the same sign, by (4). We say that $\phi$ has uniform twists, if $\phi^\ell$ is the product of only positive, or only negative Dehn twists.

Furthermore, we denote by $\ell$ the smallest positive integer such that $\phi^\ell$ restricts to the identity on $M \setminus N$. 

If ω' is an area form on M which is the standard form dq ∧ dp with respect to the (q, p)-coordinates on N, then ω := ∑_{i=1}^{ℓ}(ϕ^*)ω' is standard on N and ϕ-invariant, i.e. ϕ ∈ Symp(M, ω). To prove that ω can be chosen such that ϕ ∈ Symp^m(M, ω), Gautschi distinguishes two cases: uniform and non-uniform twists. In the first case he proves the following stronger statement.

**Lemma 7.** [10] If ϕ has uniform twists and ω is a ϕ-invariant area form, then ϕ ∈ Symp^m(M, ω).

In the non-uniform case, monotonicity does not hold for arbitrary ϕ-invariant area forms.

**Lemma 8.** [10] If ϕ does not have uniform twists, there exists a ϕ-invariant area form ω such that ϕ ∈ Symp^m(M, ω). Moreover, ω can be chosen such that it is the standard form dq ∧ dp on N.

**Theorem 9.** If ϕ is a diffeomorphism of finite type of a compact connected surface M of Euler characteristic χ(M) < 0 and if ϕ has only isolated fixed points, then ϕ is monotone with respect to some ϕ-invariant area form and

\[ HF_*(ϕ) \cong \mathbb{Z}_2^{N(ϕ)} , \quad \dim HF_*(ϕ) = N(ϕ), \]

where N(ϕ) denotes the Nielsen number of ϕ.

**Proof.** From Lemmas 7 and 8 it follows that ω can be chosen such that ϕ ∈ Symp^m(M, ω). Lemma 6 implies that every y ∈ Fix(ϕ) forms a different fixed point class of ϕ, so # Fix(ϕ) = N(ϕ). This has an immediate consequence for the Floer complex (CF_*(ϕ), ∂_J) with respect to a generic J = (J_t)_{t ∈ \mathbb{R}}. If y^± ∈ Fix(ϕ) are in different fixed point classes, then M(y^−, y^+; J, ϕ) = ∅. This follows from the first equation in (2). Then the boundary map in the Floer complex is zero ∂_J = 0 and \( \mathbb{Z}_2 \)-vector space \( CF_*(ϕ) := \mathbb{Z}_2^{|Fix(ϕ)|} = \mathbb{Z}_2^{N(ϕ)}. \)

This immediately implies \( HF_*(ϕ) \cong \mathbb{Z}_2^{N(ϕ)} \) and \( \dim HF_*(ϕ) = N(ϕ). \)

**Remark 10.** R. Gautschi has proved in preprint [10] that, if ϕ is a diffeomorphism of finite type, then ϕ is monotone with respect to some ϕ-invariant area form and

\[ HF_*(ϕ) = H_*(M_{id}; \partial M_{id}; \mathbb{Z}_2) \oplus \mathbb{Z}_2^{L(ϕ|M \setminus M_{id})}. \]

Here, L denotes the Lefschetz number (see section 6).

In the theorem the set \( M_{id} \) is empty and every fixed point of ϕ has fixed point index 1 [14]. The Lefschetz fixed point formula implies that \# Fix(ϕ) = N(ϕ) = L(ϕ). So, theorem 9 now follows also from result of R. Gautschi.

### 4.3. Hyperbolic diffeomorphisms of 2-dimensional torus.

**Theorem 11.** If ϕ is a hyperbolic diffeomorphism of a 2-dimensional torus \( T^2 \), then ϕ is symplectic and

\[ HF_*(ϕ) \cong \mathbb{Z}_2^{N(ϕ)} , \quad \dim HF_*(ϕ) = N(ϕ) \]

where N(ϕ) = |det(E − ϕ_*)| denotes the Nielsen number of ϕ and ϕ_* is an induced homomorphism on the fundamental group of \( T^2 \).
Proof. Hyperbolicity of \( \phi \) means that the covering linear map \( \tilde{\phi} : R^2 \to R^2, \ det \tilde{\phi} = 1 \) has no eigenvalue of modulus one. The hyperbolic diffeomorphism of a 2-dimensional torus \( T^2 \) is area preserving so symplectic. In fact, the covering map \( \tilde{\phi} \) has a unique fixed point, which is the origin; hence, by the covering homotopy theorem, the fixed points of \( \phi \) are pairwise Nielsen nonequivalent. The index of each Nielsen fixed point class, consisting of one fixed point, coincides with its Lefschetz index, and by the hyperbolicity of \( \phi \), the later is not equal to zero. Thus the Nielsen number \( N(\phi) = \# \text{Fix}(\phi) \). If \( y^\pm \in \text{Fix}(\phi) \) are in different Nielsen fixed point classes, then \( M(y^-, y^+; J, \phi) = \emptyset \). This follows from the first equation in (2). Then the boundary map in the Floer complex is zero \( \partial_J = 0 \) and \( \mathbb{Z}_2 \)-vector space \( CF_*(\phi) = \mathbb{Z}_2^{N(\phi)} \). This immediately implies \( HF_*(\phi) \cong \mathbb{Z}_2^{N(\phi)} \) and \( \dim HF_*(\phi) = N(\phi) \).

\[ \square \]

Remark 12. It is interesting to compare this result with the first computation by Marcin Poźniak of the Floer homology of linear symplectomorphisms in case of torus [20].

5. Symplectic zeta functions

Let \( \Gamma = \pi_0(\text{Diff}^+(M)) \) be the mapping class group of a closed connected oriented surface \( M \) of genus \( \geq 2 \). Pick an everywhere positive two-form \( \omega \) on \( M \). A isotopy theorem of Moser [19] says that each mapping class of \( g \in \Gamma \), i.e. an isotopy class of \( \text{Diff}^+(M) \), admits representatives which preserve \( \omega \). Due to Seidel [21] we can pick a monotone representative \( \phi \in \text{Symp}^a(M, \omega) \) of \( g \). Then \( HF_*(\phi) \) is an invariant of \( g \), which is denoted by \( HF_*(g) \). Note that \( HF_*(g) \) is independent of the choice of an area form \( \omega \) by Moser’s theorem and naturality of Floer homology. By Gautschi lemma [3] symplectomorphisms \( \phi^n \) are also monotone for all \( n > 0 \). Taking a dynamical point of view, we consider the iterates of monotone symplectomorphism \( \phi \) and define the first symplectic zeta function of \( \phi \) as the following power series:

\[
\chi_\phi(z) = \exp \left( \sum_{n=1}^{\infty} \frac{\chi(HF_*(\phi^n))}{n} z^n \right),
\]

where \( \chi(HF_*(\phi^n)) \) is the Euler characteristic of Floer homology group of \( \phi^n \). Then \( \chi_\phi(z) \) is an invariant of \( g \), which we denote by \( \chi_g(z) \).

Let us consider the Lefschetz zeta function

\[
L_\phi(z) := \exp \left( \sum_{n=1}^{\infty} \frac{L(\phi^n)}{n} z^n \right),
\]

where \( L(\phi^n) := \sum_{k=0}^{\dim X} (-1)^k \text{tr} \left[ \phi_{*k}^n : H_k(M; \mathbb{Q}) \to H_k(M; \mathbb{Q}) \right] \) is the Lefschetz number of \( \phi^n \). The Lefschetz zeta function is always a rational function of \( z \) and is given by the formula:

\[
L_\phi(z) = \prod_{k=0}^{\dim X} \det (E - \phi_{*k}^n.z)^{(-1)^{k+1}}.
\]
Theorem 13. Symplectic zeta function $\chi_\phi(z)$ is a rational function of $z$ and

$$\chi_\phi(z) = L_\phi(z) = \prod_{k=0}^{\dim X} \det \left( E - \phi^{*k} \cdot z \right)^{(-1)^{k+1}}.$$ 

Proof: If for every $n$ all the fixed points of $\phi^n$ are non-degenerate, i.e. for all $x \in \text{Fix}(\phi^n)$, $\det(\text{id} - d\phi^n(x)) \neq 0$, then we have (see section 3):

$$\chi(H^*_F(\phi^n)) = \sum_{x=\phi^n(x)} \text{sign}(\det(\text{id} - d\phi^n(x))) = L(\phi^n).$$

If we have degenerate fixed points one needs to perturb equations (2) in order to define the Floer homology. The necessary analysis is given in [21] is essentially the same as in the slightly different situations considered in [2], where the above connection between the Euler characteristic and the Lefschetz number was firstly established. □

Remark 14. Theorem [3] shows that symplectic zeta function $\chi_\phi(z)$ counts symplectic periodic points of $\phi$ algebraically- in the Lefschetz way.

The next issue is a relation of the symplectic zeta function with the Reidemeister torsion, a very important topological invariant. We will show that special value of symplectic zeta function $\chi_\phi(z)$ is a Reidemeister torsion. Let $T_\phi := (X \times I)/(x, 0) \sim (\phi(x), 1)$ be the mapping tori of $\phi$. We shall consider the bundle $p : T_\phi \rightarrow S^1$ over the circle $S^1$ with fibers $M$. We assume here that $E$ is a flat, complex vector bundle with finite dimensional fibre and base $S^1$. We form its pullback $p^*E$ over $T_\phi$. Note that the vector spaces $H^i(p^{-1}(b), \mathbb{C})$ with $b \in S^1$ form a flat vector bundle over $S^1$, which we denote $H^*M$. The integral lattice in $H^i(p^{-1}(b), \mathbb{R})$ determines a flat density by the condition that the covolume of the lattice is $1$. We suppose that the bundle $E \otimes H^iM$ is acyclic for all $i$. Under these conditions D. Fried [2] has shown that the bundle $p^*E$ is acyclic, and we have

$$\tau(T_\phi; p^*E) = \prod_i \tau(S^1; E \otimes H^iM)^{(-1)^i}. \quad (4)$$

Let $g$ be the prefered generator of the group $\pi_1(S^1)$ and let $A = \rho(g)$ where $\rho : \pi_1(S^1) \rightarrow GL(V)$. Then the holonomy around $g$ of the bundle $E \otimes H^iM$ is $A \otimes \phi_i^*$. Since $\tau(S^1; E) = |\det(I - A)|$, it follows from (4) that

$$\tau(T_\phi; p^*E) = \prod_i |\det(I - A \otimes \phi_i^*)|^{(-1)^i}. \quad (5)$$

We now consider the special case in which $E$ is one-dimensional, so $A$ is just a complex scalar $\lambda$ of modulus one. Then in terms of the rational function $L_\phi(z)$ we have [2]:

$$\tau(T_\phi; p^*E) = \prod_i |\det(I - \lambda \cdot \phi_i^*)|^{(-1)^i} = |\chi_\phi(\lambda)|^{-1} \quad (6)$$

This proves the following

Theorem 15. The Reidemeister torsion is the special value of the symplectic zeta function:

$$\tau(T_\phi; p^*E) = |\chi_\phi(\lambda)|^{-1},$$

where $\lambda$ is the holonomy of the one-dimensional flat complex bundle $E$ over $S^1$. 
Now we define the second symplectic zeta function for monotone symplectomorphism \( \phi \) as the following power series:

\[
F_{\phi}(z) = \exp \left( \sum_{n=1}^{\infty} \frac{\dim HF_*(\phi^n)}{n} z^n \right).
\]

Then \( F_{\phi}(z) \) is an invariant of mapping class \( g \), which we denote by \( F_g(z) \).

Motivation for this definition is the theorem 4 and nice analytical properties of the Nielsen zeta function

\[
N_{\phi}(z) = \exp \left( \sum_{n=1}^{\infty} N(\phi^n) \frac{z^n}{n} \right),
\]

see [5, 6]. We denote the numbers \( \dim HF_*(\phi^n) \) by \( N_n \). Let \( \mu(d), d \in \mathbb{N} \), be the Möbius function from number theory. As is known, it is given by the following equations:

\[
\mu(d) = 0 \text{ if } d \text{ is divisible by a square different from one;}
\]

\[
\mu(d) = (-1)^k \text{ if } d \text{ is not divisible by a square different from one,}
\]

where \( k \) denotes the number of prime divisors of \( d \); \( \mu(1) = 1 \).

**Theorem 16.** Let \( \phi \) be a non-trivial orientation preserving periodic diffeomorphism of least period \( m \) of a compact connected surface \( M \) of Euler characteristic \( \chi(M) < 0 \). Then the zeta function \( F_{\phi}(z) \) is a radical of a rational function and

\[
F_{\phi}(z) = \prod_{d \mid m} d \sqrt{(1 - z^d)^{-P(d)}},
\]

where the product is taken over all divisors \( d \) of the period \( m \), and \( P(d) \) is the integer

\[
P(d) = \sum_{d_1 \mid d} \mu(d_1) N_{d_1}.
\]

**Proof.** Since \( \phi^m = id \), then for each \( j \), \( N_j = N_{m+j} \). If \((k, m) = 1\), then there exist positive integers \( t \) and \( q \) such that \( kt = mq + 1 \). So \( (\phi^k)^t = \phi^{kt} = \phi^{mq+1} = \phi^{mq} \phi = (\phi^m)^q \phi = \phi \). Consequently, \( \text{Fix}((\phi^k)^t) = \text{Fix}(\phi) \). We have \( \text{Fix}(\phi) \subseteq \text{Fix}((\phi^k)^t) \) and \( \text{Fix}((\phi^k) \subseteq \text{Fix}((\phi^k)^t) = \text{Fix}(\phi) \). Then \( \text{Fix}(\phi) = \text{Fix}((\phi^k) \) and \( N_1 = N_k \). One can prove completely analogously that \( N_d = N_{d_1} \), if \( (i, m/d) = 1 \), where \( d \) is a divisor of \( m \). Using these sequences of equal numbers, one can regroup the terms of the series in the exponential of the zeta function so as to get logarithmic functions by adding and subtracting missing terms with necessary coefficients. We show how to do this first for period \( m = p^l \), where \( p \) is a prime number. We have the following sequence of equal numbers:

\[
N_1 = N_k, (k, p^l) = 1 \text{ (i.e., no } N_{ip}, N_{ip^2}, \ldots, N_{ip^l}, i = 1, 2, 3, \ldots),
\]

\[
N_p = N_{2p} = N_{3p} = \cdots = N_{(p-1)p} = N_{(p+1)p} = \cdots \text{ (no } N_{ip^2}, N_{ip^3}, \ldots, N_{ip^l})
\]

etc.; finally,

\[
N_{p^l-1} = N_{2p^l-1} = \cdots \text{ (no } N_{ip^l})
\]
and separately the number $N_p$.

Further,

$$
\sum_{i=1}^{\infty} \frac{N_i}{i} z^i = \sum_{i=1}^{\infty} \frac{N_1}{i} z^i + \sum_{i=1}^{\infty} \frac{(N_p - N_1) z^p}{i} + \\
+ \sum_{i=1}^{\infty} \frac{(N_p^2 - (N_p - N_1) - N_1) z^{p^2}}{i} + \\
+ \sum_{i=1}^{\infty} \frac{(N_p - ... - (N_p - N_1) - N_1) z^{p^i}}{i} + ...
$$

$$
= -N_1 \cdot \log(1 - z) + \frac{N_1 - N_p}{p} \cdot \log(1 - z^p) + \\
+ \frac{N_p - N_p^2}{p^2} \cdot \log(1 - z^{p^2}) + ... \\
+ \frac{N_p^{l-1} - N_p^l}{p^l} \cdot \log(1 - z^{p^l}).
$$

For an arbitrary period $m$, we get completely analogously,

$$
F_f(z) = \exp \left( \sum_{i=1}^{\infty} \frac{N_i}{i} z^i \right) \\
= \exp \left( \sum_{d|m} \sum_{i=1}^{\infty} \frac{P(d)}{d} \cdot \frac{z^{d^i}}{i} \right) \\
= \exp \left( \sum_{d|m} \frac{P(d)}{d} \cdot \log(1 - z^d) \right) \\
= \prod_{d|m} \sqrt[d]{(1 - z^d)^{-P(d)}},
$$

where the integers $P(d)$ are calculated recursively by the formula

$$
P(d) = N_d - \sum_{d_1|d; d_1 \neq d} P(d_1).
$$

Moreover, if the last formula is rewritten in the form

$$
N_d = \sum_{d_1|d} \mu(d_1) \cdot P(d_1)
$$

and one uses the Möbius Inversion law for real functions in number theory, then

$$
P(d) = \sum_{d_1|d} \mu(d_1) \cdot N_{d/d_1},
$$

where $\mu(d_1)$ is the Möbius function in number theory. The lemma is proved.\[\square\]
Corollary 17. If in Theorem 2 the period $m$ is a prime number, then
\[
F_f(z) = \frac{1}{(1 - z)^{N_1}} \cdot \sqrt[n]{(1 - z^m)^{N_1 - N_m}}.
\]

We denote by $\zeta_\phi(z)$ the Artin-Mazur zeta function
\[
\zeta_\phi(z) := \exp \left( \sum_{n=1}^{\infty} \frac{I(\phi^n)}{n} z^n \right),
\]
where $I(\phi^n)$ is the number of isolated fixed points of $\phi^n$.

Theorem 18. If $\phi$ is a hyperbolic diffeomorphism of a 2-dimensional torus $T^2$, then the symplectic zeta function $F_\phi(z)$ is a rational function and $F_\phi(z) = N_\phi(z) = \zeta_\phi(z) = (L_\phi(\sigma \cdot z))^{(-1)^r}$, where $r$ is equal to the number of $\lambda_i \in \text{Spec}(\tilde{\phi})$ such that $| \lambda_i | > 1$, $p$ is equal to the number of $\mu_i \in \text{Spec}(\tilde{\phi})$ such that $\mu_i < -1$ and $\sigma = (-1)^p$.

Proof. From theorem [11] and [1] it follows that $\dim HF_\ast(\phi^n) = \# \text{Fix}(\phi^n) = I(\phi^n) = N(\phi^n) = | \det(E - \phi^n) | = | L(\phi^n) |$ [1]. Thus $\dim HF_\ast(\phi^n) = \# \text{Fix}(\phi^n) = I(\phi^n) = N(\phi^n) = (-1)^{r+pm} \cdot \det(E - \phi^n) = (-1)^{r+pm} \cdot L(\phi^n)$. Now a direct computation ends the proof of the theorem.

\[\square\]

6. Asymptotic invariant. Concluding remarks and conjectures

6.1. Topological entropy and the Nielsen numbers. The most widely used measure for the complexity of a dynamical system is the topological entropy. For the convenience of the reader, we include its definition. Let $f : X \to X$ be a self-map of a compact metric space. For given $\epsilon > 0$ and $n \in N$, a subset $E \subset X$ is said to be $(n, \epsilon)$-separated under $f$ if for each pair $x \neq y$ in $E$ there is $0 \leq i < n$ such that $d(f^i(x), f^i(y)) > \epsilon$. Let $s_n(\epsilon, f)$ denote the largest cardinality of any $(n, \epsilon)$-separated subset $E$ under $f$. Thus $s_n(\epsilon, f)$ is the greatest number of orbit segments $x, f(x), ..., f^{n-1}(x)$ of length $n$ that can be distinguished one from another provided we can only distinguish between points of $X$ that are at least $\epsilon$ apart. Now let
\[
h(f, \epsilon) := \limsup_{n} \frac{1}{n} \cdot \log s_n(\epsilon, f)
\]
\[
h(f) := \limsup_{\epsilon \to 0} h(f, \epsilon).
\]
The number $0 \leq h(f) \leq \infty$, which to be independent of the metric $d$ used, is called the topological entropy of $f$. If $h(f, \epsilon) > 0$ then, up to resolution $\epsilon > 0$, the number $s_n(\epsilon, f)$ of distinguishable orbit segments of length $n$ grows exponentially with $n$. So $h(f)$ measures the growth rate in $n$ of the number of orbit segments of length $n$ with arbitrarily fine resolution. A basic relation between Nielsen numbers and topological entropy was found by N. Ivanov [16]. We present here a very short proof of Jiang of the Ivanov’s inequality.

Lemma 19. [16]
\[
h(f) \geq \limsup_{n} \frac{1}{n} \cdot \log N(f^n)
\]
Proof Let $\delta$ be such that every loop in $X$ of diameter $< 2\delta$ is contractible. Let $\epsilon > 0$ be a smaller number such that $d(f^n(x), f^n(y)) < \delta$ whenever $d(x, y) < 2\epsilon$. Let $E_n \subset X$ be a set consisting of one point from each essential fixed point class of $f^n$. Thus $|E_n| = N(f^n)$.

By the definition of $h(f)$, it suffices to show that $E_n$ is $(n, \epsilon)$-separated. Suppose it is not so. Then there would be two points $x \neq y \in E_n$ such that $d(f^i(x), f^i(y)) \leq \epsilon$ for $0 \leq i < n$ hence for all $i \geq 0$. Pick a path $c_i$ from $f^i(x)$ to $f^i(y)$ of diameter $< 2\epsilon$ for $0 \leq i < n$ and let $c_n = c_0$. By the choice of $\delta$ and $\epsilon$, $f \circ c_i \simeq c_{i+1}$ for all $i$, so $f^n \circ c_0 \simeq c_n = c_0$, such that

This means $x, y$ in the same fixed point class of $f^n$, contradicting the construction of $E_n$.

This inequality is remarkable in that it does not require smoothness of the map and provides a common lower bound for the topological entropy of all maps in a homotopy class.

We recall Thurston classification theorem for homeomorphisms of surface $M$ of genus $\geq 2$.

**Theorem 20.** [23] Every homeomorphism $\phi : M \to M$ is isotopic to a homeomorphism $f$ such that either

1. $f$ is a periodic map; or
2. $f$ is a pseudo-Anosov map, i.e. there is a number $\lambda > 1$ (stretching factor) and a pair of transverse measured foliations $(F^s, \mu^s)$ and $(F^u, \mu^u)$ such that $f(F^s, \mu^s) = (F^s, \frac{1}{\lambda} \mu^s)$ and $f(F^u, \mu^u) = (F^u, \lambda \mu^u)$; or
3. $f$ is reducible map, i.e. there is a system of disjoint simple closed curves $\gamma = \{\gamma_1, \ldots, \gamma_k\}$ in $\text{int} M$ such that $\gamma$ is invariant by $f$ (but $\gamma_i$ may be permuted) and $\gamma$ has a $f$-invariant tubular neighborhood $U$ such that each component of $M \setminus U$ has negative Euler characteristic and on each (not necessarily connected) $f$-component of $M \setminus U$, $f$ satisfies (1) or (2).

The map $f$ above is called the Thurston canonical form of $f$. In (3) it can be chosen so that some iterate $f^m$ is a generalised Dehn twist on $U$. Such a $f$, as well as the $f$ in (1) or (2), will be called standard. A key observation is that if $f$ is standard, so are all iterates of $f$.

**Lemma 21.** [4] Let $f$ be a pseudo-Anosov homeomorphism with stretching factor $\lambda > 1$ of surface $M$ of genus $\geq 2$. Then

$$h(f) = \log(\lambda) = \limsup_n \frac{1}{n} \cdot \log N(f^n)$$

**Lemma 22.** [13] Suppose $f$ is a standard homeomorphism of surface $M$ of genus $\geq 2$ and $\lambda$ is the largest stretching factor of the pseudo-Anosov pieces ( $\lambda = 1$ if there is no pseudo-Anosov piece). Then

$$h(f) = \log(\lambda) = \limsup_n \frac{1}{n} \cdot \log N(f^n)$$

6.2. **Asymptotic invariant.** The growth rate of a sequence $a_n$ of complex numbers is defined by

$$\text{Growth}(a_n) := \max \{1, \limsup_{n \to \infty} |a_n|^{1/n}\}$$

which could be infinity. Note that $\text{Growth}(a_n) \geq 1$ even if all $a_n = 0$. When $\text{Growth}(a_n) > 1$, we say that the sequence $a_n$ grows exponentially.
Definition 23. We define the asymptotic invariant $F^\infty(g)$ of mapping class $g \in \Gamma = \pi_0(\Diff^+(M))$ to be the growth rate of the sequence $\{a_n = \dim HF_*(\phi^n)\}$ for a monotone representative $\phi \in \Symp^M(M, \omega)$ of $g$:

$$F^\infty(g) := \text{Growth}(\dim HF_*(\phi^n))$$

Example 24. If $\phi$ is a non-trivial orientation preserving periodic diffeomorphism of a compact connected surface $M$ of Euler characteristic $\chi(M) < 0$, then the periodicity of the sequence $\dim HF_*(\phi^n)$ implies that for the corresponding mapping class $g$ the asymptotic invariant

$$F^\infty(g) := \text{Growth}(\dim HF_*(\phi^n)) = 1$$

Example 25. Let $\phi$ be a monotone diffeomorphism of finite type of a compact connected surface $M$ of Euler characteristic $\chi(M) < 0$ and $g$ a corresponding algebraically finite mapping class. Let $U$ be the open regular neighborhood of the $k$ reducing curves $\gamma_1, \ldots, \gamma_k$ in the Thurston theorem, and $M_j$ be the component of $M \setminus U$. Let $F$ be a fixed point class of $\phi$. Observe from [13] that if $\bar{F} \subset M_j$, then $\text{ind}(F, \phi) = \text{ind}(\bar{F}, \phi_j)$. So if $F$ is counted in $N(\phi)$ but not counted in $\sum_j N(\phi_j)$, it must intersect $U$. But we see from [14] that a component of $U$ can intersect at most 2 essential fixed point classes of $\phi$. Hence we have $N(\phi) \leq \sum_j N(\phi_j)$. For the monotone diffeomorphism of finite type $\phi$ maps $\phi_j$ are periodic. Applying last inequality to $\phi^n$ and using remark [10] we have

$$0 \leq \dim HF_*(\phi^n) = \dim H_*(M_{id}^{(n)}, \partial M_{id}^{(n)}; \mathbb{Z}_2) + N(\phi^n|M \setminus M_{id}^{(n)}) \leq$$

$$\leq \dim H_*(M_{id}^{(n)}, \partial M_{id}^{(n)}; \mathbb{Z}_2) + N(\phi^n) \leq \dim H_*(M_{id}^{(n)}, \partial M_{id}^{(n)}; \mathbb{Z}_2) + \sum_j N((\phi_j)^n) + 2k \leq \text{Const}$$

by periodicity of $\phi_j$. Taking the growth rate in $n$, we get that asymptotic invariant $F^\infty(g) = 1$.

Example 26. Let $\phi$ be an hyperbolic automorphism of 2-dimensional torus defined by an integer matrix with eigenvalues $\lambda_1, \lambda_2, |\lambda_1| > 1$. Then $h(\phi) = \log(|\lambda_1|)$. On the other hand $N(\phi^n) = |\det(I - A^n)| = |(1 - \lambda_1^n)(1 - \lambda_2^n)|$. Hence theorem [14] implies

$$F^\infty(g) := \text{Growth}(\dim HF_*(\phi^n)) = \limsup_{n \to \infty} |N(\phi^n)|^{1/n} = \exp(h(\phi)) = |\lambda_1| > 1$$

6.3. Conjectures.

6.3.1. Pseudo-Anosov mapping class.

Conjecture 27. For pseudo-Anosov mapping class $g \in \Gamma = \pi_0(\Diff^+(M))$ we have

$$HF_*(g) = \mathbb{Z}_2^{N(g)}, \quad \dim HF_*(g) = N(g), \quad F^\infty(g) = \limsup_{n \to \infty} |N(\phi^n)|^{1/n} = h(\psi) = \lambda > 1$$

where $N(g)$ denotes the Nielsen number of $g$ and $\psi$ is a standard (Thurston canonical form) representative of $g$, i.e there is a monotone representative $\phi \in \Symp^M(M, \omega)$ of $g$ such that

$$HF_*(\phi) = \mathbb{Z}_2^{N(\phi)}, \quad \dim HF_*(\phi) = N(\phi), \quad F^\infty(g) = \limsup_{n \to \infty} |N(\phi^n)|^{1/n} = h(\psi) = \lambda > 1$$
Remark 28. For pseudo-Anosov “diffeomorphism” we also have, as in theorems 4, 9 and 11, a topological separation of fixed points [23, 14, 15], i.e. the Nielsen number of pseudo-Anosov diffeomorphism equals to the number of fixed points and there are no connecting orbits between them. But pseudo-Anosov diffeomorphism is a symplectic automorphism only on the complement of his fixed points set.

Fathi and Shub [4] have proved the existence of Markov partitions for a pseudo-Anosov homeomorphism \( f \) representing mapping class \( g \). The existence of Markov partitions implies that there is a symbolic dynamics for \((M, f)\). This means that there is a finite set \( N \), a matrix \( A = (a_{ij})_{(i,j) \in N \times N} \) with entries 0 or 1 and a surjective map \( p: \Omega \to M \), where \( \Omega = \{ (x_n)_{n \in \mathbb{Z}} : a_{x_n x_{n+1}} = 1, n \in \mathbb{Z} \} \) such that \( p \circ \sigma = f \circ p \) where \( \sigma \) is the shift (to the left) of the sequence \( (x_n) \) of symbols. We have firstly:

\[
\# \text{Fix } \sigma^n = \text{tr } A^n.
\]

In general \( p \) is not bijective. The non-injectivity of \( p \) is due to the fact that the rectangles of the Markov partition can meet on their boundaries. To cancel the overcounting of periodic points on these boundaries, we use Manning’s combinatorial arguments [17] proposed in the case of Axiom A diffeomorphism. Namely, we construct finitely many subshifts of finite type \( \sigma_i, i = 0,1,...,m \), such that \( \sigma_0 = \sigma \), the other shifts semi-conjugate with restrictions of \( f \), and signs \( \epsilon_i \in \{-1,1\} \) such that for each \( n \)

\[
\# \text{Fix } f^n = \sum_{i=0}^{m} \epsilon_i \cdot \# \text{Fix } \sigma^n_i = \sum_{i=0}^{m} \epsilon_i \cdot \text{tr } A^n_i,
\]

where \( A_i \) is transition matrix, corresponding to subshift of finite type \( \sigma_i \). For pseudo-Anosov homeomorphism of compact surface \( N(f^n) = \# \text{Fix}(f^n) \) for each \( n > o \) [23]. So we have following trace formula for Nielsen numbers

\[
N(f^n) = \sum_{i=0}^{m} \epsilon_i \cdot \text{tr } A^n_i.
\]

Conjecture 27 and this trace formula immediately imply the following

**Conjecture 29.** For any pseudo-Anosov mapping class \( g \) the symplectic zeta function \( F_g(z) \) is a rational function given by the formula

\[
(7) \quad F_g(z) = N_g(z) = N_f(z) = \prod_{i=0}^{m} \det(1-A_i z)^{-\epsilon_i}
\]

6.3.2. The general case. Concluding remarks.

**Conjecture 30.** For any mapping class \( g \in \Gamma = \pi_0(Diff^+(M)) \) there is a monotone representative \( \phi \in \text{Symp}^m(M, \omega) \) with respect to some \( \phi \)-invariant area form \( \omega \) such that

\[
HF_*(\phi) = H_*(M_{id}, \partial M_{id}; \mathbb{Z}_2) \oplus \mathbb{Z}_2^{\mathbb{N}(\phi|M \setminus M_{id})},
\]

where by \( M_{id} \) we denote the union of the components of \( M \setminus \text{int}(U) \), where \( \phi \) restricts to the identity. Suppose \( \psi \) is a standard (Thurston canonical form) representative of \( g \)
and $\lambda$ is the largest stretching factor of pseudo-Anosov pieces of $\psi$ ($\lambda := 1$ if there is no pseudo-Anosov piece). Then asymptotic invariant

$$F^\infty(g) := \text{Growth}(\dim HF_*(\phi^n)) = \lambda = h(\psi) = \limsup_{n \to \infty} |N(\psi^n)|^{1/n}$$

Remark 31. (i) If $\phi \in \text{Symp}^m(M, \omega)$ has only non-degenerate fixed points, then

$$\#\text{Fix}(\phi) \geq \dim HF_*(\phi)$$

. (ii) Due to P. Seidel [22] $\dim HF_*(\phi)$ is a new symplectic invariant of a four-dimensional symplectic manifold with nonzero first Betti number. This 4-manifold produced from symplectomorphism $\phi$ by a surgery construction which is a variation of earlier constructions due to McMullen-Taubes, Fintushel-Stern and J. Smith. We hope that our symplectic zeta functions and asymptotic invariant also give rise to a new invariants of contact 3-manifolds and symplectic 4-manifolds.

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