Propagation of ultra-short dark soliton pulses in an isotropic medium under the influence of third-order linear dispersion and nonlinearity dispersion

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Abstract. The nonlinear Schrödinger equation (NSE) is well-known and is one of the most commonly used in the field of nonlinear optics. Pulses with picosecond and femtosecond duration are very well described by the NSE, but it does not work for attosecond and phase-modulated femtosecond laser pulses, where the spectral width of the pulse is of the order of the carrying frequency. In such cases, it is more convenient to use the general nonlinear amplitude equation (NAE). During the propagation of ultra-short pulses in optical fibers, the effects of the third-order linear dispersion and dispersion of nonlinearity become significant and have to be taken into account. We expanded the NAE by including these two effects. In this paper, we present an analytical study on the influence of third-order linear dispersion and dispersion of nonlinearity on the evolution of ultra-short optical pulses and on the possibility of dark soliton formation under such conditions in a medium with normal dispersion.

1. Introduction

An interesting phenomenon associated with the nonlinear properties of laser pulses in optical fibers is the formation of the so-called solitary waves, or solitons. The soliton regime of propagation of laser pulses is a well-known effect and has been studied by the scientists for decades. Different types of solitons exist depending on the dispersion and nonlinearity of the optical fibers [1-8]. A main subject of many studies has been the propagation of a particular class of solitary waves, called dark solitons. They are characterized by a deep gap in the intensity [9-11]. The dark solitons are more stable in the presence of noises and are less affected by the factors that have impact on the bright solitons [8].

One of the most commonly used equations that describe the nonlinear propagation of laser pulses in single mode fibers is the nonlinear Schrödinger equation [1, 8, 12, 13]. Although it describes very well the behavior of long optical pulses, in the field of ultra-short optics (T₀ < 1 ps) it is usually modified by adding terms which describe the third order of the linear dispersion (TOD) and the dispersion of nonlinearity [8, 13]. It is important to mention that for short laser pulses the influence of TOD is considerable, even when the group velocity dispersion (GVD) is not zero. The combined effects of GVD and TOD make the shape of the pulse’s intensity profile asymmetrical. An oscillatory structure appears...
on one of its edges, depending on the sign of TOD. The sign of GVD shows which edge of the pulses is affected. When the optical pulse propagates near the zero-dispersion wavelength, the effects of TOD lead to deep oscillations and the intensity falls to zero at the trailing edge for positive TOD and at its leading edge when the TOD is negative. In the case of soliton regime of propagation, the TOD shifts the peak position of the optical pulses linearly at a distance \( z \) [8]. The shift is significant in the attosecond and femtosecond regions. Depending on the TOD sign, the soliton peak speeds up or slows down. The TOD effects on the evolution of solitons have been discussed in detail in [14, 15]. Another important nonlinear effect connected with the propagation of broadband optical pulses in fibers is the self-steepening. It leads again to an asymmetry in the shape and spectrum of the short laser pulses. Self-steepening is a higher-order nonlinear effect which shifts the pulse’s peak toward the trailing edge, moving at lower speed than the wings. This phenomenon could create an optical shock wave. In soliton regime of propagation, self-steepening causes generation of new frequencies under the pulse envelope. That leads to significant spectral and temporal shifts [15-18].

A large number of studies have dealt with the dynamics of optical solitons in nonlinear dispersive media [9-11]. The analytical and numerical investigations have usually been based on the standard NSE. As we already mentioned, it describes very well the propagation of narrow-band pulses in waveguides; however, in the framework of broad-band optics (phase-modulated femtosecond and attosecond laser pulses) it has to be modified. Nevertheless, higher order nonlinear and dispersive effects can be taken into account if we work with the more general non-paraxial nonlinear amplitude equation (NAE) [19-21]. NAE differs from NSE with two additional terms. The biggest advantage of NAE is that it can be applied to pulses with both broad-band and narrow-band spectrum. The first numerical results using NAE, including TOD and the dispersion of nonlinearity, were presented in [10]. In our previous investigations we showed analytically the possibility for a broad-band femtosecond pulse to form a stable soliton in the presence of these phenomena [22]. Under certain initial conditions, the pulse maintains its shape at longer distances. This is a result of the balance between the higher-order nonlinear and dispersive effects. The results obtained describe the behavior of bright solitons. In [23], we found an exact solution of the modified NSE in the form of a dark soliton propagating in an isotropic nonlinear dispersive single-mode optical fiber. Its stability is due to the balance between normal dispersion and nonlinearity. The TOD and self-steepening effects were neglected.

In the present paper we propose a theoretical model based on the evolution of laser pulses with a broad-band spectrum in optical fibers with normal dispersion under the effects of TOD and dispersion of nonlinearity. Losses and Raman scattering are neglected. A new exact analytical dark-soliton solution of NAE is found by using the mathematical method described in [19, 20]. A number of numerical calculations are presented on the soliton solution obtained.

2. Basic equation

The normalized amplitude equation that describes the propagation of broadband pulses in optical fibers with normal dispersion \((k'' > 0)\) written in “local time coordinates” has the form:

\[
\frac{\partial A}{\partial \xi} + \frac{1}{2a} \left( \frac{\partial^2 A}{\partial \xi^2} - \frac{2\partial^2 A}{\partial \xi \partial \tau} \right) - g_2 \frac{\partial A}{\partial \tau} - \frac{ig}{2a} \frac{\partial A}{\partial \tau^2} - \frac{|A|^2 A}{2a} + \frac{\gamma}{\sigma} \frac{\partial}{\partial \tau} (|A|^2 A) = 0, \tag{1}
\]

where

\[
\xi = z/L_D, \quad \tau = \frac{T}{T_0}, \quad \tau_0 = v_{gr} T_0, \quad \beta_2 = k_0 \kappa'' \nu_{gr}, \quad \gamma = \frac{\alpha n_2 \rho A_0^2}{2}, \quad \alpha = k_0 z_0 = k_0 T_0 v_{gr}, \tag{2}
\]

\[
\sigma = \beta_2^2 \left( 1 + \theta \right), \quad \theta = \frac{k_0 c n}{n_0 k''}, \quad s = \left[ \frac{2}{\omega_0 T_0} + \frac{1}{\chi''(\omega_0 T_0)} \right] \approx \frac{2}{\alpha}, \quad \beta_3 = \frac{k_0 \kappa'' \nu_{gr}}{3T_0}.
\]

In Eq. (1), \( A \) is the pulse envelope, \( \xi \) is the propagation distance, \( \tau \) is the local time; \( k \), \( v_{gr} \) and \( n \) are respectively the wave number, group velocity and linear refractive index of the medium. The amplitude equation (1) is written in a dimensionless form, using the following substitutions: \( T_0 \) is the time duration of the pulse, \( A_0 \) is its initial amplitude and \( \alpha \) characterizes the number of harmonic oscillations under the pulse envelope at level \( 1/e \) from the maximum of the pulse amplitude. The coefficients \( \beta_2 \) and \( \beta_3 \)
correspond to the second and third order of the linear dispersion. The nonlinear parameters $\gamma$ and $s$ are connected with Kerr-type nonlinearity and self-steepening effect. It is important to mention that the nonparaxial (mixed) linear term, TOD and self-steepening are important when the amplitude function of the electric field admits few optical cycles inside or the pulses are phase modulated.

3. Soliton solution of the nonlinear amplitude equation

We search for a solution of the NAE (1) in the form:

$$A(\xi, \tau) = \Phi(x) e^{\imath \alpha x + \imath b \xi}, \quad x = \tau - U \xi,$$

(3)

where $\Phi(x)$ is a real amplitude function, and $a$, $b$ and $U$ are constants that are to be found. The constant $u$ has the meaning of inverse velocity, which shifts the pulse peak as it propagates along $\xi$. The expression (3) represents a grey soliton. As a first step, we calculate the derivatives and modules of $\Phi$ and then, by using equation (3), we substitute the obtained expressions in equation (1). In the next step, the real and imaginary parts on both sides of the equation are equalized. As a result, the following two differential equations are obtained:

$$\Re: \quad \Phi'' + \Phi '\left(U + \frac{2b U}{2a} - \frac{2a}{2a} \frac{2a b U}{2a} \frac{2a}{2a} - \frac{2a}{2a} \frac{2a}{2a} + \frac{3a^2 r}{2a}\right) + \sigma y \Phi^3 \Phi' = 0. \quad (4)$$

$$\Im: \quad - \frac{\sigma}{2a} \Phi'' + \Phi '\left(U + \frac{2b U}{2a} - \frac{2a}{2a} \frac{2a b U}{2a} \frac{2a}{2a} - \frac{2a}{2a} \frac{2a}{2a} + \frac{3a^2 r}{2a}\right) + 3 \sigma y \Phi^2 \Phi' = 0. \quad (5)$$

To lower the order of equation (5) it can be integrated with respect to the variable $x$. Thus, it takes the following form:

$$- \frac{\sigma}{2a} \Phi'' + \Phi '\left(U + \frac{2b U}{2a} - \frac{2a}{2a} \frac{2a b U}{2a} \frac{2a}{2a} - \frac{2a}{2a} \frac{2a}{2a} + \frac{3a^2 r}{2a}\right) + \sigma y = C = \text{const.} \quad (6)$$

As it can be noticed, equations (4) and (6) are of the same type. They are referred to the same unknown function, so they should match. Moreover, the coefficients in front of the corresponding derivatives and degrees of $\Phi$ must be the same. Thus, the integration constant $C$ is zero. By equalizing these coefficients we obtain expressions for the constants $a$, $b$ and $U$:

$$a = 1 + \frac{\alpha}{2}, \quad b_{1,2} = 1 - \frac{\alpha}{2} \pm \sqrt{1 - \beta_2 \left(\frac{1}{\alpha} + \frac{2}{\alpha} \theta \left(\frac{2}{\alpha} + \frac{3}{\alpha} \right)\right) \pm \frac{\alpha}{2} \sqrt{1 + \beta_2 \left(1 - \theta\right)}, \quad (7)$$

$$U_{1,2} = 1 \pm \sqrt{1 - \beta_2 \left(\frac{1}{\alpha} + \frac{2}{\alpha} \theta \left(\frac{2}{\alpha} + \frac{3}{\alpha} \right)\right).$$

Once the three constants are defined in a way that equations (4) and (6) match, for the unknown real function $\Phi = \Phi(x)$ the following ordinary nonlinear differential equation of second order is obtained:

$$\Phi'' + 2B \Phi - 2\Omega \Phi^3 = 0, \quad (8)$$

where

$$\Omega = \text{const} = \frac{\sigma y}{\sigma}, \quad 2B = \text{const} = \frac{1}{\sigma} \left(2a \beta_2 \sigma + 2a U \sigma + 2b - 2b U - 2a U - 3a^2 \sigma\right). \quad (9)$$

Thus, the soliton solution of equation (8) has the form:

$$\Phi = \left(\frac{a}{2} - 1\right) \frac{1}{\sqrt{2}} \imath h(x).$$

(10)
Bearing in mind the expressions (7) and by substituting (10) in (3), we obtain a new analytical solution of NAE (1), describing the behavior of dark solitons propagating in single-mode fibers. The effects of self-steepening and TOD are taken into account:

$$A(\xi, \tau) = \left(\frac{a}{Z} - 1\right) \frac{1}{\sqrt{2}} e^{i(\alpha \tau + U \xi)} e^{i(\alpha \tau + b \xi)}.$$  

(11)

While the standard dark soliton solution of NSE describes the evolution of narrow-band pulses only, our soliton solution (11) can be used for narrow-band as well as broad-band optical pulses propagating in isotropic one-dimensional nonlinear media with normal dispersion. The additional phase term found in solution (11) leads to a significant temporal shift of the soliton peak position.

4. Numerical calculations

In order to emphasize the applicability of our analytical solution, we performed the following numerical simulations for black solitons with different values of the parameters and time durations. The results obtained are presented in the figures below.

**Figure 1.** Time evolution of a broad-band black soliton in a single-mode fiber. The number of oscillations under the pulse envelope is $\alpha = 4$. The blue line presents the pulse at $\xi = 0$; the purple line, at $\xi = 500$; the grey line corresponds to $\xi = 1000$.

**Figure 2.** Time evolution of a narrow-band black soliton in a single-mode fiber. The number of oscillations under the pulse envelope is $\alpha = 20$. The blue line presents the pulse at $\xi = 0$; the purple line, at $\xi = 500$; the grey line corresponds to $\xi = 1000$.

Figure 1 presents the behavior of a dark soliton propagating in a single-mode fiber with the following parameters: $n = 1.44$; $k_0 = 5.23 \times 10^6$ m$^{-1}$; $\alpha = 10^{-26}$ s$^2$/m; $\beta = 4 \times 10^{-3}$. As expected, for ultra-short broad-band pulses, we observe a significant temporal shift. The dark soliton has a small number of oscillations under its envelope ($\alpha = 4$). The pulse wavelength is $\lambda = 1.2$ μm. The evolution of these pulses is correctly described by our soliton solution (11).

The propagation of a long dark soliton with large number of oscillations under its envelope ($\alpha = 20$) is shown in figure 2. Such kind of laser pulses are narrow-band. Their evolution in single-mode fibers can be described properly by both the classical NS and the NAE. The parameters of the waveguide and the wavelength of the soliton are the same as those in the previous figure.

Here, it is important to mention that our soliton solution (11) works in both cases – for long and ultra-short optical pulses in single mode fibers propagating under the influence of TOD and dispersion of nonlinearity. It is clearly seen that the intensity of the long optical pulse is much higher than that of the ultra-short one. In addition, the long optical pulses are much narrower and exhibit a smaller temporal shift.
5. Conclusions
In the present paper we investigated the evolution of broad-band laser pulses propagating in single mode fibers with normal dispersion, where dark solitons can be observed under certain conditions. A new analytical soliton solution of NAE (1) was found with respect to the effects up to the third order of linear dispersion and dispersion of nonlinearity. The expression (11) differs significantly from the standard soliton solution of NSE. The constant $U$, connected with the velocity of the temporal shift, depends on the coefficients characterizing the second and third order of the linear dispersion and the nonlinearity of the medium, as well as the number of harmonic oscillations under the pulse’s envelope. The numerical calculations demonstrated considerable temporal shift for ultra-short optical pulses.

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