Fields of Moduli of Hyperelliptic Curves

Bonnie Huggins

Abstract

Let $F$ be an algebraically closed field with $\text{char}(F) \neq 2$, let $F/K$ be a Galois extension, and let $X$ be a hyperelliptic curve defined over $F$. Let $\iota$ be the hyperelliptic involution of $X$. We show that $X$ can be defined over its field of moduli relative to the extension $F/K$ if $\text{Aut}(X)/\langle \iota \rangle$ is not cyclic. We construct explicit examples of hyperelliptic curves not definable over their field of moduli when $\text{Aut}(X)/\langle \iota \rangle$ is cyclic.

1 Introduction

Let $X$ be a curve of genus $g$ defined over a field $F$, let $F/L$ be a Galois extension, and let $K$ be the field of moduli relative the the extension $F/L$. (See Section 2 for the definition of “field of moduli”.) It is well known that if $g$ is 0 or 1 then $X$ admits a model defined over $K$. It is also well known that if the group of automorphism of $X$ is trivial then $X$ can be defined over $K$; for example, see Example 1.7 in [6]. However, if $g \geq 2$ and $|\text{Aut}(X)| > 1$, the curve $X$ may not be definable over its field of moduli.

We examine the case where $X$ is hyperelliptic and $F$ is an algebraically closed field of characteristic not equal to 2. In this case $\text{Aut}(X)$ is always non-trivial since it contains the hyperelliptic involution $\iota$. Examples of hyperelliptic curves not definable over their field of moduli are given on page 177 in [8]. In [5] it is shown that $X$ can be defined over $K$ if $g = 2$ and $|\text{Aut}(X)| > 2$. In Theorem 4.2 and Corollary 4.4 of [7] it is shown that $X$ is definable over $K$ if $\text{char}(F) = 0$, $g \geq 2$, and $\text{Aut}(X)/\langle \iota \rangle$ has at least two involutions. In Section 1 of [7] it is conjectured that $X$ is definable over $K$ if $\text{char}(F) = 0$ and $|\text{Aut}(X)| > 2$. In this paper, we refute this conjecture and show that $X$ can be defined over $K$ if $\text{Aut}(X)/\langle \iota \rangle$ is not a cyclic group.

2 Fields of Moduli

Let $K$ be a field, let $F/K$ be a Galois extension and let $X$ be a hyperelliptic curve defined over $F$. Let $\sigma \in \text{Gal}(F/K)$. The curve $\sigma X$ is the base
extension $X \times_{\text{Spec } F} \text{Spec } F$ of $X$ by the morphism $\text{Spec } F \xrightarrow{\text{Spec } \sigma} \text{Spec } F$. The field of moduli relative to the extension $F/K$ is defined as the fixed field $F^H$ of $H := \{ \sigma \in \text{Gal}(F/K) \mid X \cong \sigma X \text{ over } F \}$.

A subfield $E$ of $F$ is a field of definition for $X$ if there exists a curve $X_E$ defined over $E$ such that $X \cong X_E \times_{\text{Spec } E} \text{Spec } F$.

**Proposition 2.1.** Let $K_m$ be the field of moduli of $X$. Then the subgroup $H$ is a closed subgroup of $\text{Gal}(F/K)$ for the Krull topology. That is, $H = \text{Gal}(F/K_m)$.

The field of $K_m$ is contained in each field of definition between $K$ and $F$ (in particular, $K_m$ is a finite extension of $K$). Hence if the field of moduli is a field of definition, it is the smallest field of definition between $F$ and $K$. Finally, the field of moduli of $X$ relative to the extension $F/K_m$ is $K_m$.

**Proof.** See Proposition 2.1 in [4].

### 3 Finite Subgroups of 2-Dimensional Projective General Linear Groups

Throughout this section let $\overline{K}$ be an algebraically closed field of characteristic $p$ with $p = 0$ or $p > 2$. In the following two lemmas we identify a matrix in $\text{GL}_2(\overline{K})$ with its image in $\text{PGL}_2(\overline{K})$.

**Lemma 3.1.** Any finite subgroup $G$ of $\text{PGL}_2(\overline{K})$ is conjugate to one of the following groups:

**Case I:** when $p = 0$ or $|G|$ is relatively prime to $p$.

(a) $G_{C_n} = \left\{ \begin{pmatrix} \zeta^r & 0 \\ 0 & 1 \end{pmatrix} : r = 0, 1, \ldots, n-1 \right\} \cong C_n, \ n \geq 1$

(b) $G_{D_{2n}} = \left\{ \begin{pmatrix} \zeta^r & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \zeta^r \\ 1 & 0 \end{pmatrix} : r = 0, 1, \ldots, n-1 \right\} \cong D_{2n}, \ n > 2$

(c) $G_{V_4} = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix} \right\} \cong V_4 := \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

(d) $G_{A_4} = \left\{ \begin{pmatrix} \pm 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} i^\nu & i^\nu \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} i^\nu & -i^\nu \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & i^\nu \\ 1 & -i^\nu \end{pmatrix}, \begin{pmatrix} -1 & -i^\nu \\ 1 & -i^\nu \end{pmatrix} : \nu = 1, 3 \right\} \cong A_4$
(e) $G_{S_4} = \{ \begin{pmatrix} \nu & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \nu \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \nu & -\nu^2 \\ 1 & \nu \end{pmatrix} : \nu, \nu' = 0, 1, 2, 3 \} \cong S_4$

(f) $G_{A_5} = \{ \begin{pmatrix} \varepsilon^r & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \varepsilon^r \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} \varepsilon^r \omega & \varepsilon^r s \\ 1 & -\varepsilon^r s \omega \end{pmatrix} : r, s = 0, 1, 2, 3, 4 \} \cong A_5$

where $\omega = -\frac{1+\sqrt{5}}{2}, \overline{\omega} = -\frac{1-\sqrt{5}}{2}$, $\zeta$ is a primitive $n^{th}$ root of unity, $\varepsilon$ is a primitive $5^{th}$ root of unity, and $i$ is a primitive $4^{th}$ root of unity.

Case II: when $|G|$ is divisible by $p$.

(g) $G_{\beta,A} = \{ \begin{pmatrix} \beta^k & a \\ 0 & 1 \end{pmatrix} : a \in A, k \in \mathbb{Z} \}$, where $A$ is a finite additive subgroup of $\overline{K}$ containing 1 and $\beta$ is a root of unity such that $\beta A = A$

(h) $\text{PSL}_2(\mathbb{F}_{p^r})$

(i) $\text{PGL}_2(\mathbb{F}_{p^r})$

where $\mathbb{F}_{p^r}$ is the finite field with $p^r$ elements.

Proof. See §§55-58 in [10] and Chapter 3 in [9].

Lemma 3.2. Let $N(G)$ be the normalizer of $G$ in $\text{PGL}_2(\overline{K})$. Then

(a) $N(G_{C_n}) = \{ \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \alpha \\ 1 & 0 \end{pmatrix} : \alpha \in \overline{K}^{*} \} \text{ if } n > 1,$

(b) $N(G_{D_{2n}}) = G_{D_{4n}} \text{ if } n > 2,$

(c) $N(G_{V_4}) = G_{S_4},$

(d) $N(G_{A_4}) = G_{S_4},$

(e) $N(G_{S_4}) = G_{S_4},$

(f) $N(G_{A_5}) = G_{A_5},$

(h) $N(\text{PSL}_2(\mathbb{F}_{p^r})) = \text{PGL}_2(\mathbb{F}_{p^r}),$ and

(i) $N(\text{PGL}_2(\mathbb{F}_{p^r})) = \text{PGL}_2(\mathbb{F}_{p^r}).$

Proof.

(a) See §55 in [10].
(b) See §55 in [10].

c) Since \( G_{V_4} \) is a normal subgroup of \( G_{S_4} \), \( G_{S_4} \subseteq N(G_{V_4}) \). Conjugation of \( G_{V_4} \) by \( G_{S_4} \) gives a homomorphism \( G_{S_4} \to \text{Aut}(V_4) \cong S_3 \). A computation shows that the centralizer \( Z \) of \( G_{V_4} \) in \( \text{PGL}_2(K) \) is \( G_{V_4} \). The kernel of this homomorphism is \( Z \cap G_{S_4} = Z \). Since \( G_{S_4}/Z \cong S_3 \), every automorphism of \( G_{V_4} \) is given by conjugation by an element of \( G_{S_4} \). Let \( U \in N(G_{V_4}) \). Then \( UV \in Z = G_{V_4} \) for some \( V \in G_{S_4} \), so \( U \in G_{S_4} \).

d) Since \( G_{V_4} \) is a characteristic subgroup of \( G_{A_4} \), \( N(G_{A_4}) \subseteq N(G_{V_4}) = G_{S_4} \). As \( G_{A_4} \) is normal in \( G_{S_4} \), we get \( N(G_{A_4}) = G_{S_4} \).

e) Since \( G_{A_4} \) is a characteristic subgroup of \( G_{S_4} \), \( N(G_{S_4}) \subseteq N(G_{A_4}) = G_{S_4} \). Thus \( N(G_{S_4}) = G_{S_4} \).

f) Conjugation of \( G_{A_5} \) by \( N(G_{A_5}) \) gives a homomorphism \( N(G_{A_5}) \to \text{Aut}(A_5) \). The kernel of this homomorphism is the centralizer of \( G_{A_5} \) in \( N(G_{A_5}) \), which is just the centralizer \( Z \) of \( G_{A_5} \) in \( \text{PGL}_2(K) \). A computation shows that \( Z \) is just the identity. Since \( \text{Aut}(A_5) \) is finite, \( N(G_{A_5}) \) is a finite subgroup of \( \text{PGL}_2(K) \). Since \( G_{A_5} \subseteq N(G_{A_5}) \), by Lemma 3.1 we must have \( N(G_{A_5}) = G_{A_5} \).

h) We first show that \( N(\text{PSL}_2(\mathbb{F}_{p^r})) \) is finite. Conjugation of \( \text{PSL}_2(\mathbb{F}_{p^r}) \) by \( N(\text{PSL}_2(\mathbb{F}_{p^r})) \) gives a homomorphism \( N(\text{PSL}_2(\mathbb{F}_{p^r})) \to \text{Aut}(\text{PSL}_2(\mathbb{F}_{p^r})) \). The kernel of this homomorphism is the centralizer \( Z \) of \( \text{PSL}_2(\mathbb{F}_{p^r}) \) in \( \text{PGL}_2(K) \). A computation shows that \( Z \) is just the identity. Since \( \text{Aut}(\text{PSL}_2(\mathbb{F}_{p^r})) \) is finite, so is \( N(\text{PSL}_2(\mathbb{F}_{p^r})) \). By Lemma 3.1 any finite subgroup of \( \text{PGL}_2(K) \) containing \( \text{PSL}_2(\mathbb{F}_{p^r}) \) must be isomorphic to either \( \text{PGL}_2(\mathbb{F}_q) \) or \( \text{PSL}_2(\mathbb{F}_q) \) for some \( q \). Since \( \text{SL}_2(\mathbb{F}_{p^r}) \) is normal in \( \text{GL}_2(\mathbb{F}_{p^r}) \), \( \text{PSL}_2(\mathbb{F}_{p^r}) \) is a normal subgroup of \( \text{PGL}_2(\mathbb{F}_{p^r}) \). So \( \text{PGL}_2(\mathbb{F}_{p^r}) \subseteq N(\text{PSL}_2(\mathbb{F}_{p^r})) \), in particular \( \text{PSL}_2(\mathbb{F}_{p^r}) \) is strictly contained in \( N(\text{PSL}_2(\mathbb{F}_{p^r})) \). By the corollary on page 80 of [9], \( \text{PSL}_2(\mathbb{F}_q) \) is simple for \( q > 3 \). It follows that \( N(\text{PSL}_2(\mathbb{F}_{p^r})) \neq \text{PSL}_2(\mathbb{F}_q) \) for \( q \geq 3 \). By Theorem 9.9 on page 78 of [10], the only nontrivial normal subgroup of \( \text{PGL}_2(\mathbb{F}_q) \) is \( \text{PSL}_2(\mathbb{F}_q) \) if \( q > 3 \). Therefore \( N(\text{PSL}_2(\mathbb{F}_{p^r})) = \text{PGL}_2(\mathbb{F}_{p^r}) \).

i) Clear from the proof of the previous case.

\( \square \)
4 Isomorphisms of Hyperelliptic Curves

Throughout this section let $K$ be a perfect field of characteristic $p$ with $p = 0$ or $p > 2$ and let $X$ be a hyperelliptic curve defined over an algebraic closure $\overline{K}$ of $K$ with $K$ as its field of moduli. In particular, $X$ admits a degree-2 morphism to $\mathbb{P}^1$ and the genus of $X$ is at least 2. Each element of $\text{Aut}(X)$ induces an automorphism of $\mathbb{P}^1$ fixing the branch points. The number of branch points is $\geq 3$ (in fact $\geq 6$), so $\text{Aut}(X)$ is finite. We get a homomorphism $\text{Aut}(X) \rightarrow \text{Aut}(\mathbb{P}^1) = \text{PGL}_2(\overline{K})$ with kernel generated by the hyperelliptic involution $\iota$. Let $G \subset \text{PGL}_2(\overline{K})$ be the image of this homomorphism. Replacing the original map $X \rightarrow \mathbb{P}^1$ by its composition with an automorphism $g \in \text{Aut}(\mathbb{P}^1) = \text{PGL}_2(\overline{K})$ has the effect of changing $G$ to $gGg^{-1}$, so we may assume that $G$ is one of the groups listed in Lemma 3.1.

Fix an equation $y^2 = f(x)$ for $X$ where $f \in K[x]$ and $\text{disc}(f) \neq 0$. So the function field $K(X)$ equals $K(x,y)$.

**Proposition 4.1.** Let $X$ be as above and let $X'$ be a hyperelliptic curve defined over $K$ given by $y^2 = f'(x)$, where $f'(x)$ is another squarefree polynomial in $K[x]$. Every isomorphism $\varphi : X \rightarrow X'$ is given by an expression of the form:

$$(x,y) \mapsto \left(\frac{ax+b}{cx+d}, \frac{ey}{(cx+d)^{g+1}}\right),$$

for some $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\overline{K})$ and $e \in \overline{K}$. The pair $(M, e)$ is unique up to replacement by $(\lambda M, e\lambda^{g+1})$ for $\lambda \in \overline{K}$. If $\varphi' : X' \rightarrow X''$ is another isomorphism, given by $(M', e')$, then the composition $\varphi' \varphi$ is given by $(M'M, e'e)$.

**Proof.** See Proposition 2.1 in [1].

Let $\Gamma = \text{Gal}(\overline{K}/K)$ and let $\sigma \in \Gamma$. Then $\sigma X$ is the smooth projective model of $y^2 = f^\sigma(x)$, where $f^\sigma(x)$ is the polynomial obtained from $f(x)$ by applying $\sigma$ to the coefficients.

**Lemma 4.2.** Following the notation used above, let $\sigma \in \Gamma$ and suppose that $\varphi : X \rightarrow \sigma X$ is given by $(M, e)$. Let $M = \text{image of } M$ in $\text{PGL}_2(\overline{K})$. If $G \neq G_{B,A}$ then $M$ is in the normalizer $N(G)$ of $G$ in $\text{PGL}_2(\overline{K})$. If $G = G_{B,A}$ then $M$ is an upper triangular matrix.

**Proof.** Since $\text{Aut}(\sigma X) = \{\psi^\sigma \mid \psi \in \text{Aut}(X)\}$, the group of automorphisms of $\mathbb{P}^1$ induced by $\text{Aut}(\sigma X)$ is $G^\sigma := \{U^\sigma \mid U \in G\}$. 

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Let \( \psi \) be an automorphism of \( X \) given by \( (V,v) \). Since \( \psi \) is an automorphism, \( V \in \text{GL}_2(\overline{K}) \) is a lift of some element \( \overline{V} \in G \). Then \( \varphi \psi \varphi^{-1} \) is an automorphism of \( \tau^*X \) given by \( (M \tau^*V M^{-1}, v) \). We have \( M \tau^*V M^{-1} = \overline{M} \overline{V} \overline{M}^{-1} \in G^\sigma \).

It follows that \( M \tau^* \in G \). If \( G \neq G_{\beta,A} \), by Lemma 5.1 \( G^\sigma = G \). So \( M \tau^* \in N(G) \). If \( G = G_{\beta,A} \), then since \( G^\sigma \) has an elementary abelian subgroup of the same form as \( G \), a simple computation shows that \( M \tau^* \) is an upper triangular matrix.

**Lemma 4.3.** Following the above notation, suppose that for every \( \tau \in \Gamma \) there exists an isomorphism \( \varphi_\tau : X \to \tau^*X \) given by \( (M_\tau, e) \) where \( M_\tau \in G^\tau \). Then \( X \) can be defined over \( K \).

**Proof.** Let \( P_1, \ldots, P_n \) be the hyperelliptic branch points of \( X \to \mathbb{P}^1 \). Let \( \tau \in \Gamma \). The isomorphism \( \varphi_\tau : X \to \tau^*X \) induces an isomorphism on the canonical images \( \mathbb{P}^1 \to \mathbb{P}^1 \) which is given by \( M_\tau \). Then \( M_\tau \) sends \( \{P_1, \ldots, P_n\} \) to \( \{\tau(P_1), \ldots, \tau(P_n)\} \). Since \( M_\tau \in G^\tau \) it merely permutes the set \( \{P_1, \ldots, P_n\} \).

Since \( \tau \) is arbitrary we have
\[
\prod_{P_i \neq \infty} (x - P_i) \in K[x].
\]

It follows that \( X \) can be defined over \( K \).  

**Corollary 4.4.** Suppose that \( N(G) = G \) and \( G \neq G_{\beta,A} \). Then \( X \) can be defined over \( K \).

**Proof.** By Lemma 3.1 \( G^\sigma = G \) for all \( \sigma \in \Gamma \). Let \( \tau \in \Gamma \). By Lemma 4.2 any isomorphism \( X \to \tau^*X \) is given by \( (M, e) \) where \( M \in N(G) = G = G^\tau \).

**5 The Main Result**

The following two results of Dèbes and Emsalem will be used in the proof of our main result. They rely on the notions of a cover and the field of moduli of a cover, for which we refer the reader to § 2.4 in [3].

**Theorem 5.1.** Let \( F/K \) be a Galois extension and \( X \) be a hyperelliptic curve defined over \( F \) with \( K \) as field of moduli. Let \( B = X/\text{Aut}(X) \). Then there exists a model \( B_K \) of the curve \( B = X/\text{Aut}(X) \) defined over \( K \) such that the cover \( X \to B \) with \( K \)-base \( B_K \) is of field of moduli \( K \).

**Proof.** See Theorem 3.1 in [4]. The authors make the additional assumption that \( \text{char}(K) \) does not divide \( |\text{Aut}(X)| \) but do not use it in their proof.

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**Corollary 5.2.** Suppose that $F$ is algebraically closed. If $B_K$ has a $K$-rational point, then $K$ is a field of definition of $X$.

**Proof.** It suffices to show that the cover $X \to B$ with $K$-base $B_K$ can be defined over $K$, since a field of definition of the cover is automatically a field of definition of $X$. By Theorem 5.1, the field of moduli of the cover $X \to B$ with $K$-base $B_K$ is $K$. If $K$ is a finite field then $\text{Gal}(F/K)$ is a projective profinite group. In this case, by Corollary 3.3 of [B] the cover $X \to B$ can be defined over $K$. If $K$ is not a finite field then since $B_K \cong_K \mathbb{P}^1_K$, $B_K$ has a rational point off the branch point set of $X \to B_K \times F$. Then by Corollary 3.4 and § 2.9 of [B], the cover can be defined over $K$. \hfill \square

The curve $B_K$ is called the canonical model of $X/\text{Aut}(X)$ over the field of moduli of $X$. Let $\Gamma = \text{Gal}(F/K)$. In the proof of Theorem 5.1 Debes and Emsalem show the canonical model exists by using the following argument. For all $\sigma \in \Gamma$ there exists an isomorphism $\varphi_\sigma: X \to \sigma X$ defined over $F$. Each induces an isomorphism $\bar{\varphi}_\sigma: X/\text{Aut}(X) \to \sigma X/\text{Aut}(\sigma X)$ that makes the following diagram commute:

$$
\begin{array}{ccc}
X & \xrightarrow{\varphi_\sigma} & \sigma X \\
\rho \downarrow & & \downarrow \rho^\sigma \\
X/\text{Aut}(X) & \xrightarrow{\bar{\varphi}_\sigma} & \sigma X/\text{Aut}(\sigma X)
\end{array}
$$

Composing $\bar{\varphi}_\sigma$ with the canonical isomorphism

$$i_\sigma: \sigma X/\text{Aut}(\sigma X) \to \sigma(X/\text{Aut}(X))$$

we obtain an isomorphism

$$\overline{\varphi}_\sigma: X/\text{Aut}(X) \to \sigma(X/\text{Aut}(X)).$$

The family $\{\overline{\varphi}_\tau\}_{\tau \in \Gamma}$ satisfy Weil’s cocycle condition $\overline{\varphi}_\tau^\sigma \overline{\varphi}_\sigma = \overline{\varphi}_{\sigma \tau}$ given in Theorem 1 of [B]. This shows that $B_K$ exists.

Let $F(B)$ be the function field of $B$. Since $B \cong \mathbb{P}^1$, $F(B) = F(t)$ for some element $t$. We use $t$ as a coordinate on $B$. Suppose that $\overline{\varphi}_\sigma$ is given by

$$t \mapsto \frac{at + b}{ct + d}.$$  

Define $\sigma^* \in \text{Aut}(F(t)/K)$ by

$$\sigma^*(t) = \frac{at + b}{ct + d}, \quad \sigma^*(\alpha) = \sigma(\alpha), \quad \alpha \in F.$$
One can verify that \((\sigma\tau)^*(w) = \sigma^*(\tau^*(w))\) for all \(w \in F(t)\). So we get a homomorphism \(\Gamma \to \text{Aut}(F(B)/K)\), \(\sigma \to \sigma^*\). The curve \(B_K\) is the variety over \(K\) corresponding to the fixed field of \(\Gamma^* = \{\sigma^*\}_{\sigma \in \Gamma}\).

The following lemma will be used in the proof of the main theorem.

**Lemma 5.3.** Let \(L/K\) be a field extension of odd degree. Let \(C\) be a curve of genus 0 defined over \(K\) and suppose that \(C(L) \neq \emptyset\). Then \(C(K) \neq \emptyset\).

**Proof.** Let \(P \in C(L)\) and let \(n = [L : K]\). Let \(\tau_1, \ldots, \tau_n\) be the distinct embeddings of \(L\) into an algebraic closure of \(L\). Then \(D = \Sigma \tau_i(P)\) is a divisor of degree \(n\) defined over \(K\). Let \(\omega\) be a canonical divisor on \(C\). Since \(\deg(\omega) = -2\), we can take a linear combination of \(D\) and \(\omega\) to obtain a divisor \(D'\) of degree 1. Since \(\deg(\omega - D') < 0\), by the Riemann-Roch theorem \(l(D') > 0\). So there exists an effective divisor \(D''\) linearly equivalent to \(D'\) defined over \(K\). Since \(D''\) is effective and of degree 1 it consists of a point in \(C(K)\).

**Theorem 5.4.** Let \(K\) be a field of characteristic not equal to 2. Let \(X\) be a hyperelliptic curve defined over \(\overline{K}\), an algebraic closure of \(K\). Let \(G = \text{Aut}(X)/\langle \iota \rangle\) where \(\iota\) is the hyperelliptic involution of \(X\). Suppose that \(G\) is not cyclic or that \(G\) is cyclic of order divisible by the characteristic of \(K\). Then \(X\) can be defined over its field of moduli relative to the extension \(\overline{K}/K\).

**Proof.** Let \(\Gamma = \text{Gal}(\overline{K}/K)\). By Proposition 2.1 we may assume that \(K\) is the field of moduli of \(X\). By Proposition 4.1 we may assume that \(G\) is given by one of the groups in Lemma 3.1. Fix an equation \(y^2 = f(x)\) for \(X\) where \(f \in \overline{K}[x]\) and \(\text{disc}(f) \neq 0\). So the function field \(\overline{K}(X)\) equals \(\overline{K}(x, y)\). There are eight cases.

(b) \(G \cong D_{2n}, n > 2\). The function field of \(X/\text{Aut}(X)\) equals the subfield of \(\overline{K}(X)\) fixed by \(G_{D_{2n}}\) acting by fractional linear transformations. Then \(t := x^n + x^{-n}\) is fixed by \(G_{D_{2n}}\) and is a rational function of degree 2\(n\) in \(x\), so the function field of \(X/\text{Aut}(X)\) equals \(\overline{K}(t)\). Therefore we use \(t\) as coordinate on \(X/\text{Aut}(X)\). The map \(\rho: X \to X/\text{Aut}(X)\) is given by \((x, y) \mapsto (x^n + x^{-n})\). Let \(\sigma \in \Gamma\). By Lemmas 4.2 and 3.2 \(\varphi_\sigma: X \to \sigma X\) is given by \((M, e)\) where \(\overline{M} \in D_{4n}\). Then the map \(\rho^*\varphi_\sigma: X \to \sigma X/\text{Aut}(\sigma X)\) is given by \((x, y) \mapsto \pm(x^n + x^{-n})\). So \(\sigma^*(t) = \pm t\). The curve \(B_K\) corresponds to the fixed field of \(\overline{K}(t)\) under \(\Gamma^*\). Then \(t = 0\) corresponds to a point \(P \in B_K(K)\).

(c) \(G \cong V_4\). The element \(t := x^2 + x^{-2}\) is fixed by \(G_{V_4}\) and is a rational function of degree 4 in \(x\). So the function field of \(X/\text{Aut}(X)\) equals \(\overline{K}(t)\).
We use \( t \) as a coordinate on \( X / \text{Aut}(X) \). The map \( \rho \colon X \to X / \text{Aut}(X) \) is given by \((x, y) \mapsto (x^2 + x^{-2})\). Let \( \sigma \in \Gamma \). By Lemmas 3.2 and 3.2, \( \varphi_\sigma : X \to \sigma^* X \) is given by \((M, e) \) where \( M \in G_{S_4} \). A computation shows that \( \sigma^*(t) \) is one of the following:

\[ \begin{align*}
\text{i. } & t \\
\text{ii. } & -t \\
\text{iii. } & \frac{2t+12}{t-2} \\
\text{iv. } & \frac{2t-12}{t-2} \\
\text{v. } & \frac{2t-12}{t+2} \\
\text{vi. } & \frac{2t+12}{t+2}.
\end{align*} \]

Since \( \varphi : \Gamma \to \Gamma / \Lambda \) is defined over \( K \) for all \( \tau \in \Gamma \), we have \( \varphi_\sigma \varphi_\tau = \varphi_{\tau \sigma} \) for all \( \tau \in \Gamma \). The fractional linear transformations i through vi form a group under composition isomorphic to \( S_3 \). The map \( \tau \mapsto \tau^*(t) \) defines a homomorphism from \( \Gamma \) to this group. The kernel of this homomorphism is \( \Lambda := \{ \tau \in \Gamma \mid \tau^*(t) = t \} \). So \( |\Gamma / \Lambda| = 1, 2, 3, \) or 6.

Case 1: \( |\Gamma / \Lambda| = 1 \). In this case the fixed field of \( \Gamma^* \) is \( K(t) \) and \( B_K = \mathbb{P}^1_K \).

Case 2: \( |\Gamma / \Lambda| = 2 \). Let \( \sigma \) be a representative of the nontrivial coset. There are three cases.

\[ \begin{align*}
\text{i. } & \sigma^*(t) = -t. \text{ Then } t = 0 \text{ corresponds to a point } P \in B_K(K). \\
\text{ii. } & \sigma^*(t) = \frac{2t+12}{t-2}. \text{ Then } t = 6 \text{ corresponds to a point } P \in B_K(K). \\
\text{iii. } & \sigma^*(t) = \frac{2t-12}{t-2}. \text{ Then } t = -6 \text{ corresponds to a point } P \in B_K(K).
\end{align*} \]

Case 3: \( |\Gamma / \Lambda| = 3 \). Since the fixed field of \( \Lambda^* \) is \( K^\Lambda(t) \), \( B_K \) has a \( K^\Lambda \)-rational point. By Lemma 5.3, since \( [K^\Lambda : K] \) is odd, \( B_K \) has a \( K \)-rational point.

Case 4: \( |\Gamma / \Lambda| = 6 \). Let \( \Pi \) be a subgroup of \( \Gamma \) containing \( \Lambda \) such that \( \Pi / \Lambda \) is a subgroup of \( \Gamma / \Lambda \) of order 2. By Case 2, \( B_K \) has a \( K^\Pi \)-rational point. Since \( [K^\Pi : K] = 3 \) is odd, by Lemma 5.3, \( B_K \) has a \( K \)-rational point.

(d) \( G \cong A_4 \). The element \( t' := x^2 + x^{-2} \) is fixed by the normal subgroup \( G_{V_4} \). From (c), we see that the element

\[ t := \frac{1}{4} t' \left( \frac{2t'-12}{t'+2} \right) \left( \frac{2t'+12}{-t'+2} \right) = \frac{x^{12} - 33x^8 - 33x^4 + 1}{-x^{10} + 2x^6 - x^2}. \]

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is fixed by \( G_{A_4} \) and is a rational function of degree 12 in \( x \). So the function field of \( X/\text{Aut}(X) \) equals \( \overline{K}(t) \). We use \( t \) as coordinate on \( X/\text{Aut}(X) \). The map \( \rho: X \to X/\text{Aut}(X) \) is given by

\[
(x, y) \mapsto (x^{12} - 33x^8 - 33x^4 + 1)/(-x^{10} + 2x^6 - x^2).
\]

Let \( \sigma \in \Gamma \). By Lemmas 4.2 and 3.2 \( \varphi_\sigma: X \to \sigma X \) is given by \((M, e)\) where \( M \in G_{S_4} \). A computation shows that \( \sigma^*(t) = \pm t \). Then \( t = 0 \) corresponds to a point \( P \in B_K(K) \).

(e) \( G \cong S_4 \). By Lemma 3.2 \( N(G) = G \). So by Corollary 4.1 \( X \) can be defined over \( K \).

(f) \( G \cong A_5 \). By Lemma 3.2 \( N(G) = G \). So by Corollary 4.1 \( X \) can be defined over \( K \).

(g) \( G = G_{\beta,A} \). Let \( d \) be the order of \( \beta \) and let \( t = g(x) := \prod_{\alpha \in A}(x - \alpha)^d \). Then \( t \) is a rational function of degree \( |G| \) fixed by \( G_{\beta,A} \) acting by fractional linear transformations. So the function field of \( X/\text{Aut}(X) \) equals \( \overline{K}(t) \). We use \( t \) as a coordinate function of \( X/\text{Aut}(X) \). Let \( \sigma \in \Gamma \). By Lemma 4.2 \( \varphi_\sigma: X \to \sigma X \) is given by \((M, e)\) where \( M \) is an upper diagonal matrix. So \( \sigma^*(t) = g^\alpha(ax + b) \) for some \( a \neq 0 \) and \( b \). Let \( P \) be the point of \( X/\text{Aut}(X) \) corresponding to \( x = \infty \). Then since \( g^\alpha(a\infty + b) = g(\infty) \), \( P \) corresponds to a point in \( B_K(K) \).

(h) \( G = \text{PSL}_2(\mathbb{F}_{p^r}) \). Let \( q = p^r \). It can be deduced from Theorem 6.21 on page 409 of [9] that \( \text{PSL}_2(\mathbb{F}_q) \) is generated by the image in \( \text{PGL}_2(\overline{K}) \) of the following matrices

\[
\left\{ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in \mathbb{F}_{p^r} \right\}.
\]

Let

\[
g(x) = \frac{((x^q - x)^{q-1} + 1)^{\frac{q+1}{2}}}{(x^q - x)^{\frac{q^2 - q}{2}}}.
\]

One can verify that \( g(\frac{1}{x}) = g(x) \) and \( g(x + a) = g(x) \) for all \( a \in \mathbb{F}_{p^r} \). Since \( g \) is a rational function of \( x \) of degree \( \frac{q^2 - q}{2} = |\text{PSL}_2(\mathbb{F}_q)| \), the function field of \( X/\text{Aut}(X) \) is \( \overline{K}(t) \) where \( t = g(x) \). We use \( t \) as a coordinate function on \( X/\text{Aut}(X) \). The map \( \rho: X \to X/\text{Aut}(X) \) is given by

\[
(x, y) \mapsto \frac{((x^q - x)^{q-1} + 1)^{\frac{q+1}{2}}}{(x^q - x)^{\frac{q^2 - q}{2}}}.
\]
Let $\sigma \in \Gamma$. By Lemmas 4.2 and 3.2, $\varphi_\sigma : X \to \sigma X$ is given by $(M, e)$ where $\sigma \in \text{PGL}_2(\mathbb{F}_q)$. A computation shows that $\sigma^* (t) = \pm t$. Then $t = 0$ corresponds to a point $P \in B_K(K)$.

(i) $G = \text{PGL}_2(\mathbb{F}_{p^r})$. By Lemma 3.2, $N(G) = G$. So by Corollary 4.4, $X$ can be defined over $K$.

Specific examples of hyperelliptic curves not definable over their field of moduli are given on page 177 of [8]; these examples have $|G| = 1$. Adjusting these examples, we now construct others with $|G| > 5$.

Let $n > 5$, let $m$ be odd, and consider the polynomial $f(x) \in \mathbb{C}[x]$ given by

$$f(x) = a_0 x^{nm} + \sum_{r=1}^{m} (a_r x^{n(r+r)} + (-1)^r a_r^c x^{n(m-r)}) ,$$

with $a_m = 1$, $a_0 \in \mathbb{R}^*$, and where $z^c$ is the complex conjugate of $z$ for any $z \in \mathbb{C}$. Assume that for $r = 1, \ldots, m-1$ we have $a_r \neq (-1)^r \beta^{-nr} a_r^c$ for any $2mn$th root of unity $\beta$ and that $f(x)$ is square free.

**Lemma 5.5.** Following the above notation, let $X$ by the hyperelliptic curve over $\mathbb{C}$ given by $y^2 = f(x)$. Let $\iota$ be the hyperelliptic involution of $X$ and let $\nu$ be the automorphism of $X$ defined by $\nu(x, y) = (\zeta x, y)$, where $\zeta$ is a primitive $n$th root of unity. Then $\text{Aut}(X) = \langle \iota \rangle \oplus \langle \nu \rangle$.

**Proof.** Let $G = \text{Aut}(X)/\langle \iota \rangle$. The image of $\nu$ in $G$ under the quotient map $\text{Aut}(X) \to G$ has order $n$. Since $n > 5$, by Lemma 3.1, $G$ is either cyclic or dihedral. In either case the image of $\nu$ in $G$ generates a cyclic normal subgroup of $G$.

Suppose that $G$ is cyclic of order $n' > n$. Since the only elements in $\text{PGL}_2(\mathbb{C})$ that commute with the image of diagonal matrices are the images of diagonal matrices, by Lemma 4.1, there exists an element $u \in \text{Aut}(X)$ defined by

$$u(x, y) = (\zeta' x, e y)$$

where $e \in \mathbb{C}^*$ and $\zeta'$ is a primitive $(n')$th root of unity. It follows that $f(\zeta' x)$ is a scalar multiple of $f(x)$. This is a contradiction by our choice of coefficients for $f$.

Suppose that $G$ is dihedral. By Lemma 5.2 (a) and Lemma 4.1, there exists an element $v \in \text{Aut}(X)$ defined by

$$v(x, y) = (\alpha / x, \epsilon y / x^{mn})$$

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where $e', \alpha \in \mathbb{C}^*$. It follows that $x^{2mn} f(\alpha/x)$ is a scalar multiple of $f(x)$. Since
\[
x^{2mn} f(\alpha/x) = \alpha^{nm} (a_0 x^{nm} + \sum_{r=1}^{m} ((-1)^r \alpha^{-nr} a_r x^{n(m+r)} + \alpha^r a_r x^{n(m-r)})
\]
and $a_0 \neq 0$, we must have $x^{2mn} f(\alpha/x) = f(x)$. Since $a_m = 1$, we must have $\alpha^{mn} = -1$ and $a_r = (-1)^r \alpha^{-nr} a_r^c$ for $r = 1, \ldots, m-1$. This is a contradiction. Therefore $G$ is cyclic of order $n$.

The function field of $X$ is $\mathbb{C}(x,y)$ and the function field of $X/\text{Aut}(X)$ is $\mathbb{C}(x^n)$. Since the places in $\mathbb{C}(x^n)$ corresponding to $x^n = 0$ and $x^n = \infty$ do not ramify completely in $\mathbb{C}(x,y)$, by Theorem 5.1 of [2] we have $\text{Aut}(X) = \langle \iota \rangle \oplus \langle \nu \rangle$.

Proposition 5.6. Following the above notation, let $X$ be the hyperelliptic curve of genus $g = mn - 1$ over $\mathbb{C}$ given by $y^2 = f(x)$. The field of moduli of $X$ relative to the extension $\mathbb{C}/\mathbb{R}$ is $\mathbb{R}$ and is not a field of definition for $X$.

Proof. By Lemma 5.5, $\text{Aut}(X) = \langle \iota \rangle \oplus \langle \nu \rangle$ where $\iota$ is the hyperelliptic involution of $X$, and $\nu(x,y) = (\zeta x, y)$ where $\zeta$ is a primitive $n^{th}$ root of unity. The map $\mu$ defined by
\[
\mu(x,y) = ((\omega x)^{-1}, ix^{-nm} y),
\]
where $\omega^n = -1$, is an isomorphism between the curve $X$ and the complex conjugate curve $\overline{X}$. Any isomorphism $X \to \overline{X}$ is given by $\mu \nu^k$, or $\mu \nu^k$ for some $0 \leq k \leq n - 1$. We have $\mu \iota = \iota \mu$,
\[
\mu \nu(x,y) = ((\omega \zeta x)^{-1}, i(\zeta x)^{-nm} y) = \nu^c \mu(x,y),
\]
and
\[
\mu^c \mu(x,y) = ((\omega^{-1} (\omega x)^{-1})^{-1}, -i(\omega x)^{nm} (ix^{-nm} y)) = (\omega^2 x, -y) = \nu^l \iota(x,y)
\]
for some $l$. Then
\[
(\mu \nu^k)^c \mu \nu^k = \mu^c \nu^{-k} \mu \nu^k = \mu^c \mu \nu^{2k} = \nu^{2k+l} \neq Id
\]
and
\[
(\mu \nu^k)^c \mu \nu^k = \mu^c \nu^{-k} \mu \nu^k = \mu^c \mu \nu^{2k} = \nu^{2k+l} \neq Id.
\]
Therefore Weil’s cocycle condition from Theorem 1 of [3] does not hold. So $X$ cannot be defined over $\mathbb{R}$. \qed
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Department of Mathematics, University of California, Berkeley, CA 94720
E-mail address: bhuggins@math.berkeley.edu