A NEW LOOK AT THE VORTEX EQUATIONS
AND DIMENSIONAL REDUCTION

STEVEN BRADLOW\textsuperscript{1}, JAMES GLAZEBROOK\textsuperscript{2} AND FRANZ KAMBER\textsuperscript{3}

ABSTRACT. In order to use the technique of dimensional reduction, it is usually necessary for there to be a symmetry coming from a group action. In this paper we consider a situation in which there is no such symmetry, but in which a type of dimensional reduction is nevertheless possible. We obtain a relation between the Coupled Vortex equations on a closed Kahler manifold, $X$, and the Hermitian-Einstein equations on certain $\mathbb{P}^1$-bundles over $X$. Our results thus generalize the dimensional reduction results of Garcia-Prada, which apply when the Hermitian-Einstein equations are on $X \times \mathbb{P}^1$.

§1. Introduction

Dimensional reduction is a technique for studying special solutions to partial differential equations. The technique is applicable when there is a symmetry i.e. a group action, in which case it makes sense to look for invariant solutions. The term ‘dimensional reduction’ refers to the fact that the invariant solutions to the original equation can be interpreted as ordinary solutions to a related set of equations on the (lower dimensional) orbit space of the group action.

In this paper we will describe a situation in which an effective dimensional reduction of equations is possible, even though there is no global group action. The equations in question are the Hermitian-Einstein (HE) equations for special metrics on holomorphic bundles. As a result of our dimensional reduction procedure, we relate special solutions of the HE equations to solutions of the so-called Coupled Vortex equations. These latter equations are also for special metrics on holomorphic bundles, but involve more data than HE equations. The extra data is in the form of prescribed holomorphic sections.

A dimensional reduction from the Hermitian-Einstein to the Vortex equations is not new. Indeed in [GP], Garcia-Prada described just such a relation between the equations. In the situation considered by Garcia-Prada the vortex equations are defined on bundles over a closed Kahler manifold, say $X$, the Hermitian-Einstein

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equations are on bundles over $X \times \mathbb{P}^1$, and the reduction is with respect to the standard $SU(2)$-action on $\mathbb{P}^1$.

The main difference between Garcia-Prada’s results and those described in this paper is that we have replaced $X \times \mathbb{P}^1$ by certain non-trivial $\mathbb{P}^1$-bundles over $X$. One way to view such a situation is as one in which there is no group action but in which the (group) orbits are still evident. In fact, the group orbits are now the leaves of a foliation structure. Instead of reducing to an orbit space of a group action, we thus reduce to the leaf space of a foliation. Such reduction with respect to a foliation has been investigated in the context of foliations on Riemannian manifolds in [GKPS]. Interestingly, despite the suggestiveness of the foliation aspects of our construction, the techniques we use actually rely more on the holomorphic aspects, and are thus much closer to those of Garcia-Prada than to those of [GKPS].

Indeed, the results we describe in this paper show that the Garcia-Prada techniques readily extend to the case where $X \times \mathbb{P}^1$ is replaced by a projectively flat $\mathbb{P}^1$-bundle over $X$. More precisely, we assume that the $\mathbb{P}^1$-bundle has a flat $PU(2)$ structure. This assumption allows us to extend any $PU(2)$-invariant structure on $\mathbb{P}^1$ to the total space, say $M$, of the $\mathbb{P}^1$-bundle over $X$. In particular, the Kähler form of the Fubini-Study metric can be extended to become a global form on $M$, and the canonical bundle on $\mathbb{P}^1$ extends to become the relative canonical bundle $K_{M/X}$. By carefully exploiting these constructions we are able to identify an interesting class of holomorphic bundles over $M$ on which the Garcia-Prada techniques can be made applicable.

In Section 2 we give some background material on the Hermitian-Einstein equations and the Coupled Vortex equations, and in Section 3 we review the Garcia-Prada dimensional reduction techniques. In section 4 we introduce our Projective Flatness Condition and discuss its implications. Our main result is given in section 5, where it appears as Theorem 5.1. Section 6 contains some computations for the special case that the projective bundle is the projectivization of a vector bundle. Finally, in section 7 we indulge in some discussion of our results, with particular emphasis on possible interpretations from the point of view of foliation theory.

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§2. The Equations

The Hermitian-Einstein equations determine special metrics on holomorphic bundles over Kähler manifolds (cf. [Ko] or [LT] for a complete review). Indeed, if $(M, \omega_M)$ is any closed Kähler manifold, and $\mathcal{E}$ is a holomorphic bundle over $M$, then the Hermitian-Einstein equations for a Hermitian metric $H$ are

$$i\Lambda_M F_H = \lambda I_\mathcal{E}.$$  \hfill (2.1)
Here $E$ denotes the smooth complex bundle underlying $\mathcal{E}$, $F_H$ is the curvature of the metric connection on $\mathcal{E}$ determined by $H$, $\Lambda_M F_H$ is the contraction of $F_H$ with the Kähler form $\omega_M$, $\Omega^0(\text{End} E)$ is the identity and $\lambda$ is a constant determined by $\omega_M$, the rank of $\mathcal{E}$, and its degree. Recall that the degree of a complex bundle over a Kähler manifold is defined by

$$\deg(E) = \int_M c_1(E) \wedge \omega_M^{m-1} = \int_M \Lambda_M c_1(E) \frac{\omega_M^m}{m},$$

where $m$ is the dimension of $M$ and $c_1(E)$ is the first Chern class of $E$. It follows (by the Chern-Weil formula for $c_1(E)$) that the $\lambda$ in (2.1) is given by

$$\lambda = \frac{2\pi}{(m-1)!} \frac{1}{\text{Vol}(M)} \frac{\deg(E)}{\text{rank}(E)}.$$

We remind the reader that the $\deg(E)$ is used to define the slope, $\mu(E)$, of the bundle by

$$\mu(E) = \frac{\deg(E)}{\text{rank}(E)}.$$

For a holomorphic bundle $\mathcal{E}$, these definitions extend to coherent subsheaves, and are used to define the algebro-geometric notion of stability. We record, for the sake of completeness, that a bundle $\mathcal{E}$ is called stable if $\mu(\mathcal{E}') < \mu(\mathcal{E})$ for all coherent subsheaves $\mathcal{E}' \subset \mathcal{E}$.

**Remark 2.1.** The Hermitian-Einstein equations have many interesting features, among which are the following:

(1) They are the Kähler versions of the anti-self-dual equations. What we mean by this is that, like the ASD equations, the HE equations can be obtained as the minimizing condition for the Yang-Mills functional on the space of unitary connections on a complex vector bundle. Furthermore the ASD and HE equations are equivalent in the case where they are both defined, i.e. on unitary bundles with vanishing first Chern class over closed Kähler surfaces.

(2) The existence of solutions is closely related to the stability of a holomorphic bundle. This is known as the Hitchin-Kobayashi correspondence.

The Coupled Vortex Equations also determine special metrics, but on a holomorphic triple over a Kähler manifold (cf. [GP], [BGP]). If $(X, \omega_X)$ is a closed Kähler manifold of dimension $n$, then by definition a holomorphic triple on $X$ is a triple $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ consisting of two holomorphic vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ on $X$ together with a homomorphism $\Phi : \mathcal{E}_2 \rightarrow \mathcal{E}_1$, i.e. an element $\Phi \in H^0(\text{Hom}(\mathcal{E}_2, \mathcal{E}_1))$. The coupled vortex equations on $(\mathcal{E}_1, \mathcal{E}_2, \Phi)$ are defined to be

$$i \Lambda_X F_{H_1} + \Phi \Phi^* = \tau_1 I_{\mathcal{E}_1},$$

$$i \Lambda_X F_{H_2} - \Phi^* \Phi = \tau_2 I_{\mathcal{E}_2}.$$  

(2.3)
In these equations, $\Phi^*$ is the adjoint of $\Phi$ with respect to the metrics on $E_1$ and $E_2$, and $\tau_1$ and $\tau_2$ are real parameters. If the ranks of the bundles are $r_1$ and $r_2$ respectively, and we denote their degrees by $d_1$ and $d_2$, then the parameters $\tau_1$ and $\tau_2$ satisfy the constraint

$$r_1\tau_1 + r_2\tau_2 = \frac{2\pi}{(n-1)!}\frac{1}{\text{Vol}(X)}(\deg E_1 + \deg E_2). \quad (2.4)$$

**Remark 2.2.** The Coupled Vortex equations may be viewed as a natural generalization of the abelian vortex equations (cf. [JT]). Originally introduced in the so-called Ginsburg-Landau theory of superconductivity, these equations were later generalized to describe special Hermitian metrics on holomorphic bundles with prescribed holomorphic sections. In that setting the equations play as important a role as do the Hermitian-Einstein equations in the study of stable holomorphic bundles. Most recently, the abelian vortex equations have appeared in Seiberg-Witten theory, where they emerge as the Kähler version of the Seiberg-Witten equations (cf. [W]).

§3. **Garcia-Prada Dimensional Reduction**

Suppose now that $M = X \times \mathbb{P}^1$. Following Garcia-Prada in [GP], we can consider the situation in which the product $X \times \mathbb{P}^1$ has an $SU(2)$-action in which $SU(2)$ acts trivially on $X$ and via the identification with the homogeneous space $SU(2)/U(1)$ on $\mathbb{P}^1$. We can thus consider $SU(2)$-equivariant bundles on $M$, where a smooth bundle $V \to M$ is called $SU(2)$-equivariant if there is an action of $SU(2)$ on $V$ covering the action on $X \times \mathbb{P}^1$. In particular, we can consider smooth bundles over $M = X \times \mathbb{P}^1$ of the form

$$V = \pi_1^*E_1 \oplus \pi_1^*E_2 \otimes \pi_2^*H \otimes^2. \quad (3.1)$$

Here $\pi_1$ and $\pi_2$ are the projections from $X \times \mathbb{P}^1$ to the first and second factors, $E_i$ is a smooth vector bundle over $X$, and $H$ is the smooth line bundle over $\mathbb{P}^1$ with Chern class 1. The $SU(2)$-action on $V$ is trivial on the $\pi_1^*E_i$ factors and standard on $H \otimes^2$.

Not all the $SU(2)$-equivariant bundles are of the form in (3.1); in general they can be direct sums where each summand is of the form $\pi_1^*E \otimes \pi_2^*H \otimes^k$. The special case represented by (3.1) is chosen with a view towards the dimensional reduction from the Hermitian-Einstein equations to the Coupled Vortex equations. For our purposes, the key feature of such bundles, given in the next Proposition, has to do with the $SU(2)$-equivariant holomorphic structures that they admit. Recall that a holomorphic bundle can be viewed as an underlying smooth bundle with a holomorphic structure, and that a holomorphic bundle $V$ is $SU(2)$-equivariant if it is $SU(2)$-equivariant as a smooth bundle and in addition the action of $SU(2)$ on $V$ is holomorphic.

**Proposition 3.1** [Prop. 3.9 in [GP]]. There is a one-to-one correspondence between $SU(2)$-equivariant holomorphic vector bundles $E$ with underlying $SU(2)$-equivariant $C^\infty$ structure given by (3.1), and holomorphic extensions of the form
0 \to \pi_1^*\mathcal{E}_1 \to \mathcal{E} \to \pi_1^*\mathcal{E}_2 \otimes \pi_2^*\mathcal{O}(2) \to 0 \quad (3.2)

where \(\mathcal{E}_1\) and \(\mathcal{E}_2\) are the bundles in (3.1), equipped with holomorphic structures. Moreover, every such extension is defined by an element \(\Phi \in \text{Hom}(\mathcal{E}_2, \mathcal{E}_1)\).

The characterization of the extension classes in the above Proposition depends on the fact that extensions over \(X \times \mathbb{P}^1\) of the form (3.2) are parametrized by \(H^1(X \times \mathbb{P}^1, \pi_1^*(\mathcal{E}_1 \otimes \mathcal{E}_2^*) \otimes \pi_2^*\mathcal{O}(-2))\), and that by the Kunneth formula this is isomorphic to \(H^0(X, \mathcal{E}_1 \otimes \mathcal{E}_2^*) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-2))\). The result thus follows from the fact that \(H^1(\mathbb{P}^1, \mathcal{O}(-2)) \cong \mathbb{C}\).

It follows immediately from Proposition 3.1 that there is a one-to-one correspondence between extensions over \(X \times \mathbb{P}^1\) of the type given in (3.2), and holomorphic triples \((\mathcal{E}_1, \mathcal{E}_2, \Phi)\) on \(X\). Thus, at the level of holomorphic objects,

a holomorphic triple on \(X\) can be viewed as the dimensional reduction of an \(SU(2)\)-equivariant holomorphic bundle on \(X \times \mathbb{P}^1\).

We now turn to the dimensional reduction at the level of equations, i.e. we examine the relation between the Hermitian-Einstein equations on an extension as in (3.2) and the Coupled Vortex equations on the corresponding triple.

In order to define the equations, we need to fix Kähler structures. We let \(\omega_X\) be the (fixed) Kähler form on \(X\), and let \(\omega_{\mathbb{P}^1}\) denote the Kähler form of the Fubini-Study metric on \(\mathbb{P}^1\) normalized so that

\[
\text{Vol}(\mathbb{P}^1) = \int_{\mathbb{P}^1} \omega_{\mathbb{P}^1} = 1.
\]

On \(X \times \mathbb{P}^1\) there is a natural 1-parameter family of \(SU(2)\)-equivariant Kähler metrics, with Kähler forms

\[
\omega_\sigma = \pi_1^*\omega_X + \sigma \pi_2^*\omega_{\mathbb{P}^1}.
\]

Using these Kähler structures, one finds:

**Proposition 3.2 (Proposition 3.11 [GP]).** Let \(T = (\mathcal{E}_1, \mathcal{E}_2, \Phi)\) be a holomorphic triple. Let \(\mathcal{E}\) be the \(SU(2)\)-equivariant holomorphic bundle over \(X \times \mathbb{P}^1\) associated to \(T\), that is, let \(\mathcal{E}\) be given as an extension

\[
0 \to \pi_1^*\mathcal{E}_1 \to \mathcal{E} \to \pi_1^*\mathcal{E}_2 \otimes \pi_2^*\mathcal{O}(2) \to 0.
\]

Suppose that \(\tau_1\) and \(\tau_2\) are related by (2.4) and let \(\sigma\) be chosen to satisfy

\[
\frac{1}{\sigma} = \frac{(r_1 + r_2) \frac{\tau_1 \text{Vol}(X)}{2\pi} - \frac{(\deg E_1 + \deg E_2)}{(n-1)!} \tau_1}{2r_2 \text{Vol}(X)} = \frac{\tau_1 - \tau_2}{4\pi}.
\]

Fix the Kähler metric on \(X \times \mathbb{P}^1\) such that the Kähler form is \(\omega_\sigma\), as given by (3.3).
Then $\mathcal{E}_1$ and $\mathcal{E}_2$ admit metrics satisfying the Coupled Vortex equations (i.e. equations (2.3)) if and only if $\mathcal{E}$ admits an $SU(2)$-invariant metric which satisfies the Hermitian-Einstein equation (i.e. equation (2.1)) with respect to $\omega_\sigma$.

Remark 3.3. Now by the Hitchin-Kobayashi correspondence, the existence of a solution to the Hermitian-Einstein equation is equivalent (cf. [UY], [Do], [NS]) to the polystability of the holomorphic bundle $\mathcal{E}$. There is a similar correspondence for holomorphic triples, which relates the existence of solutions to the coupled vortex equations to a notion of stability for a holomorphic triple (cf. [BGP]). As one might expect, Proposition 3.2 thus has an analog which relates stable holomorphic extensions on $X \times \mathbb{P}^1$ to stable triples on $X$ (cf. Theorem 4.1 in [BGP]). It should be noted that, with fixed Kähler structures on $X$ and $\mathbb{P}^1$, the notion of stability for the holomorphic triple on $X$ depends explicitly on a parameter $\tau$, while the notion of stability for the holomorphic bundles on $X \times \mathbb{P}^1$ depends on the choice of Kähler structure on $X \times \mathbb{P}^1$. If the Kähler form on $X \times \mathbb{P}^1$ is taken to be $\omega_\sigma$, then the theorem relates the stability of $\mathcal{E}$ to the $\tau$-stability of the corresponding triple on $X$, where $\tau$ is the same as $\tau_1$ in Proposition 3.2. This analog to Proposition 3.2 can be be thought of as a dimensional reduction result for stable holomorphic objects (rather than for equations).

§4. Non-trivial $\mathbb{P}^1$-bundles over $X$

§4.1 Dimensional reduction of bundles.

It is natural to view $X \times \mathbb{P}^1$ as a special case of a $\mathbb{P}^1$-fibration over $X$. One is then lead to consider whether any of the results of the previous section carry over to this more general situation. Henceforth we thus replace $X \times \mathbb{P}^1$ by

$$\mathbb{P}^1 \hookrightarrow M \to X,$$

a holomorphic $\mathbb{P}^1$-bundle over $X$.

In its most general form this means that $M$ has a description as a $\mathbb{P}^1$-bundle associated to a principal $PGL(2, \mathbb{C})$-bundle. Denoting the principal $PGL(2, \mathbb{C})$-bundle by $\tilde{P}$, we can thus write

$$M = \tilde{P} \times_{PGL(2, \mathbb{C})} \mathbb{P}^1.$$

Remark. If $X$ is a Riemann surface, so that $M$ is a ruled surface, then the $PGL(2, \mathbb{C})$ structure always lifts to $GL(2, \mathbb{C})$ (cf. [GH] or [H]) and $M$ can be described as $M = \mathbb{P}(E)$, where $E \to X$ is a rank two holomorphic bundle. If $\dim \mathbb{C}X > 1$, then this is not always possible. In general there is an obstruction to such a lift, with the obstruction located in $H^3(X, \mathbb{Z})$.

Our initial goal is to find a class of holomorphic bundles on $M$ whose structure is determined by data on $X$. In general there is no longer an $SU(2)$-action on $M$, so it does not make sense to study $SU(2)$-equivariant bundles over $M$. Instead, it turns out that the appropriate replacements for the extensions described in (3.2) are extensions of the form
0 \longrightarrow \pi^*\mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \pi^*\mathcal{E}_2 \otimes \mathcal{K}^*_M/X \longrightarrow 0 \ , \quad (4.1)

where \( \pi : M \longrightarrow X \) is the projection, \( \mathcal{E}_i \) are holomorphic bundles over \( X \), and \( \mathcal{K}^*_M/X \) is the dual of the relative canonical bundle on \( M \).

Notice that if \( M = X \times \mathbb{P}^1 \), then \( \mathcal{K}^*_M/X = \pi^*_2 \mathcal{O}(2) \) and we recover the extensions of the form (3.2). Furthermore, an important part of Proposition 3.1 remains true for extensions of this sort, namely that every such extension is defined by an element \( \Phi \in \text{Hom}(\mathcal{E}_2, \mathcal{E}_1) \). More precisely,

**Proposition 4.1.** There is a natural isomorphism

\[
H^{0,1}(M, \pi^*(\mathcal{E}_1 \otimes \mathcal{E}_2^*) \otimes \mathcal{K}^*_M/X) \cong H^0(X, \mathcal{E}_1 \otimes \mathcal{E}_2^*) \ . \quad (4.2)
\]

**Proof.**

This follows by the Leray spectral sequence, plus the fact that the direct image sheaves of the relative canonical bundle satisfy (cf. [H])

\[
R^q\pi_*\mathcal{K}^*_M/X \cong 0 \text{ if } q \neq 1 ,
\]

\[
R^1\pi_*\mathcal{K}^*_M/X \cong \mathcal{O}_X . \quad (4.3)
\]

What this says is:

**Corollary 4.2.** Let \( M \) be any holomorphic \( \mathbb{P}^1 \)-bundle over \( X \). There is a one to one correspondence between holomorphic triples \( (\mathcal{E}_1, \mathcal{E}_2, \Phi) \) on \( X \) and extensions

\[
0 \longrightarrow \pi^*\mathcal{E}_1 \longrightarrow \mathcal{E} \longrightarrow \pi^*\mathcal{E}_2 \otimes \mathcal{K}^*_M/X \longrightarrow 0 \ ,
\]

over \( M \).

§4.2 Projective flatness Condition. To proceed further with the dimensional reduction, i.e. to relate the Hermitian-Einstein and Coupled Vortex equations, we need to consider the Kähler structures on \( M \). In general, it seems to be a difficult matter to find a Kähler metric on \( M \) such that the resulting Hermitian-Einstein equations on an extension (4.1) will ‘dimensionally reduce’ to the Coupled Vortex equations on the corresponding triple over \( X \). We have, however, discovered a sufficient condition on \( M \) in order for this to be possible. It remains an interesting open question whether our condition is also necessary.

The condition we impose on \( M \) is the

**Projective Flatness Condition.** We assume that \( \mathbb{P}^1 \hookrightarrow M \longrightarrow X \) is a flat \( PU(2) \)-bundle.

The real significance of the projective flatness condition lies in the fact, that under this assumption, we are able to construct the following:

1. a class of bundles which correspond to the \( SU(2) \)-equivariant extensions in the Garcia-Prada construction,
(2) a suitable family of Kähler metrics on $M$, generalizing the family corresponding to the Kähler forms given in (3.3), and
(3) explicit descriptions of the forms representing the extension classes in (4.1).

In short, we can construct bundles which are are “locally equivariant” and retain enough symmetry so that the key constructions required for dimensional reduction still apply.

Before describing these constructions, it is useful to elaborate somewhat on the projective flatness condition. There are three equivalent ways of viewing this condition:

(1) Using the definition of a flat bundle in terms of locally constant transition functions, the projective flatness condition means that we can choose a cover $\{U_i\}$ for $X$, and local trivializations $M|_{U_i} \cong U_i \times \mathbb{P}^1$, such that

\[
M = \left( \coprod U_i \times \mathbb{P}^1 \right)/g_{ij} ,
\]

where the transition functions

\[ g_{ij} : U_i \cap U_j \longrightarrow PU(2) , \]

are locally constant maps. In particular, the coordinate transformations from $U_j \times \mathbb{P}^1$ to $U_i \times \mathbb{P}^1$, given by $(x, \lambda) \mapsto (x, g_{ji} \lambda)$, are holomorphic maps. The description (M1) thus describes $M$ as a complex manifold.

(2) Using the cover $\{U_i\}$ and transition functions $g_{ij}$, we can construct a flat principal $PU(2)$–bundle. This is the bundle $\tilde{P}$ in the description given at the beginning of §4.1. Hence we see that a second way to formulate the projective flatness condition, is to say that

\[
M = \tilde{P} \times_{PU(2)} \mathbb{P}^1 ,
\]

where $\tilde{P}$ is a flat principal $PU(2)$–bundle.

(3) Finally, we can use the fact that a flat bundle $\tilde{P}$ can be associated to a representation

\[ \rho : \pi_1(X) \longrightarrow PU(2) . \]

This gives a description of a projectively flat $M$ as

\[
M = \tilde{X} \times_{\rho} \mathbb{P}^1 ,
\]

where $\tilde{X}$ is the universal cover of $X$, acted on by $\pi_1(X)$ via deck transformations.

One way that such flat $PU(2)$–bundles arise, is as the projectivization of rank two holomorphic vector bundles. If $E \longrightarrow X$ is a vector bundle, then the projective bundle $\mathbb{P}(E)$ is defined as follows. Let $E$ be defined by local trivializations $E|_{U_i} \cong U_i \times \mathbb{C}^2$, where $\{U_i\}$ is an open cover of $X$, and transition functions

\[ \hat{g}_{ij} : U_i \cap U_j \longrightarrow GL(n + 1, \mathbb{C}) , \]
i.e.

\[ E = \left( \coprod_i U_i \times \mathbb{C}^{n+1} \right)/\hat{g}_{ij} . \]

Then \( \mathbb{P}(E) \) is defined as

\[ \mathbb{P}(E) = \left( \coprod_i U_i \times \mathbb{P}^n \right)/g_{ij} , \]

where the transition functions \( g_{ij} : U_i \cap U_j \rightarrow PGL(n, \mathbb{C}) \) are obtained from the \( \hat{g}_{ij} \) by the projection from \( GL(n+1, \mathbb{C}) \) to \( PGL(n, \mathbb{C}) \). The proof of the following well known fact can be found in [Ko].

Lemma 4.3. Let \( E \) be a rank \((n + 1)\) holomorphic vector bundle, and let \( H \) be a Hermitian metric on \( E \). If the corresponding metric connection is projectively flat, i.e. if its curvature \( F_H \) is given by

\[ F_H = \alpha I , \]

where \( \alpha \) is a 2-form on \( X \) and \( I \in \Omega^0(\text{EndE}) \) is the identity map, then \( \mathbb{P}(E) \) has a flat \( PU(n + 1) \)–structure.

If \( X \) is a Riemann surface, i.e. \( \text{dim}_\mathbb{C} X = 1 \), then the flat \( PU(n) \)–bundles which arise in this way are characterized by the following:

Proposition 4.4. Let \( E \rightarrow X \) be a rank \((n + 1)\) holomorphic vector bundle over a closed Riemann surface. Then \( E \) admits a projectively flat Hermitian metric if and only if \( E \) is polystable, i.e. if and only if

\[ E = \bigoplus_i E_i , \]

where each summand \( E_i \) is a stable holomorphic bundle with

\[ \frac{\deg(E_i)}{\text{rank}(E_i)} = \frac{\deg(E)}{\text{rank}(E)} . \]

In particular, if \( n = 1 \), then \( \mathbb{P}(E) \) has a flat \( PU(2) \)–structure if and only if either

1. \( E \) is a stable rank 2 holomorphic bundle, or
2. \( E = L_1 \oplus L_2 \), where \( L_1 \) and \( L_2 \) are holomorphic line bundles with

\[ \deg(L_1) = \deg(L_2) = \frac{1}{2}\deg(E) . \]

Proof. On a Riemann surface, the condition \( F_H = \alpha I \) is equivalent (cf. [Ko]) to the Hermitian-Einstein condition \( \Lambda F_H = \lambda I \). The result thus follows from the Hitchin-Kobayashi correspondence (or, in this case, the theorem of Narasimhan and Seshadri). \( \square \)
If \( \dim \mathbb{C} X > 1 \), then the projective flatness condition is in general stronger than the Hermitian-Einstein condition. However, if \( X \) has a Kähler metric with Kähler form \( \omega \), and if the Chern classes of \( E \) satisfy the Bogomolov relation
\[
\int_X ((r - 1)c_1(E)^2 - 2rc_2(E)) \wedge \omega^{n-2} = 0 ,
\]
(\text{where } r = \text{rank}(E)) then the Hermitian-Einstein condition is again equivalent to the projective flatness of the full curvature tensor (cf. [LT], also [Ko], p.114). We thus get:

**Proposition 4.5.** Let \((X, \omega)\) be a closed Kähler manifold, and let \( E \rightarrow X \) be a rank two holomorphic bundle over \( X \). Then \( \mathbb{P}(E) \) has a flat \( \text{PU}(2) \)-structure if and only if either
\begin{enumerate}
  \item \( E \) is stable (with respect to \( \omega \)) and
  \[
  \int_X (c_1(E)^2 - 4c_2(E)) \wedge \omega^{n-2} = 0 ,
  \]
  \text{or}
  \item \( E = L_1 \oplus L_2 \), where \( L_1 \) and \( L_2 \) are holomorphic line bundles with
  \[
  c_1(L_1) = c_1(L_2) .
  \]
\end{enumerate}

**Proof.** Part (1) follows immediately from the Bogomolov relation. For part (2), we use the fact that the first Chern classes are represented by forms of type \((1,1)\). They can thus be expressed as \( c_1(L_i) = \alpha_i \omega + \beta_i \), where (for \( i = 1, 2 \)) \( \alpha_i \in \mathbb{R} \), and the \( \beta_i \) are primitive forms of type \((1,1)\), i.e. \( \int_X (\beta_i \wedge \omega^{n-1}) = 0 \). Furthermore, since \( E \) is polystable, it follows that \( \alpha_1 = \alpha_2 \). The Bogomolov relation thus reduces, in this case, to
\[
\int_X (\beta_1 - \beta_2) \wedge (\beta_1 - \beta_2) \wedge \omega^{n-2} = 0 .
\]
Using the Hodge-Riemann bilinear relations for real primitive \((1,1)\) forms (cf. [GH], p.207), we see that the left hand side of this expression is strictly negative unless \( (\beta_1 - \beta_2) = 0 \). \( \square \)

**Remark.** It should be noted that not all flat \( \text{PU}(2) \)-bundles arise in this way. As pointed out in the Remark at the beginning of §4.1, there is an obstruction to realizing a \( \mathbb{P}^1 \)-bundle as \( \mathbb{P}(E) \), where \( E \) is a vector bundle.

**§4.3 Equivariant bundles.**

Given a \( \text{PU}(2) \)-equivariant holomorphic bundle \( \mathcal{V} \rightarrow \mathbb{P}^1 \), we can use the description of \( M \) as \( M = \tilde{P} \times_{\text{PU}(2)} \mathbb{P}^1 \) to construct a holomorphic bundle
\[
\tilde{\mathcal{V}} = \tilde{P} \times_{\text{PU}(2)} \mathcal{V} \rightarrow M = \tilde{P} \times_{\text{PU}(2)} \mathbb{P}^1 . \quad (4.4)
\]
We call \( \tilde{\mathcal{V}} \) the *extension of \( \mathcal{V} \) to \( M \). In fact this construction applies equally well if \( \text{PU}(2) \) is replaced by \( \text{PGL}(2, \mathbb{C}) \), and does not require \( \tilde{P} \) to have a flat structure. If however, \( \tilde{P} \) does have a flat structure, so that descriptions (M1-3) apply, then we get:
Lemma 4.6. Let \( \tilde{P} \) be a flat \( PU(2) \)-bundle, described as above by either the representation \( \rho : \pi_1(X) \to PU(2) \) or the locally constant transition functions \( g_{ij} : U_i \cap U_j \to PU(2) \). Let \( V \to \mathbb{P}^1 \) be a \( PU(2) \)-equivariant holomorphic bundle. Then the extension of \( V \) to \( M \) can be described in the following two equivalent ways:

\[
(1) \quad \tilde{V} = \tilde{X} \times_{\rho} V,
\]

\[
(2) \quad \tilde{V} = (\prod_i U_i \times V) / g_{ij}.
\]

Remark 4.7. Using the above construction, we can extend any invariant structure on \( V \) to \( \tilde{V} \). In particular, an invariant Hermitian bundle metric on \( V \) extends to define a Hermitian metric on \( \tilde{V} \).

When \( V \) is a line bundle, we can describe the \( PU(2) \)-equivariant bundles over \( \mathbb{P}^1 \). They are precisely the \( SU(2) \)-equivariant bundles on which the element \( -I \in SU(2) \) acts as the identity. The \( PU(2) \)-action is then obtained by lifting to an \( SU(2) \)-action. We conclude therefore that the \( PU(2) \)-equivariant holomorphic line bundles on \( \mathbb{P}^1 \) are precisely the even powers of the tautological bundle \( \mathcal{O}_{\mathbb{P}^1}(-1) \), i.e. are the bundles \( \mathcal{O}_{\mathbb{P}^1}(2k) \), for any integer \( k \). We denote the corresponding extensions to \( M \) by the notation \( \mathcal{O}(2k) \). For our purposes, the most important special case is given by \( \mathcal{O}(-2) \).

Lemma 4.8. Let \( M \) be a projectively flat \( PU(2) \)-bundle, as above. Let \( K_{M/X} \) be the relative canonical bundle. Then we can identify

\[
K_{M/X} = \mathcal{O}(-2).
\]

§4.4 Equivariant metrics.

If \( M \) is projectively flat (and thus has descriptions as in (M1-3)), then we can construct a 1-parameter family of Kähler metrics which generalizes the construction used in [GP]. This is made possible by the fact that \( \mathbb{P}^1 \) has an \( SU(2) \)-equivariant (and thus a \( PU(2) \)-equivariant) Kähler metric, viz. the Fubini-Study metric. The corresponding Kähler form, which we denote by \( \omega_{\mathbb{P}^1} \), is thus a \( PU(2) \)-invariant form of holomorphic type (1, 1) on \( \mathbb{P}^1 \). It therefore extends to a form of type (1, 1) on \( M \). This form, which we denote by \( \tilde{\omega}_{\mathbb{P}^1} \) can be described in two ways (corresponding to the two descriptions (M3) and (M1)):

(1) Let \( \pi_{\mathbb{P}^1}^* \omega_{\mathbb{P}^1} \) be the pull-back of \( \omega_{\mathbb{P}^1} \) to \( \tilde{X} \times \mathbb{P}^1 \), and let

\[
q : \tilde{X} \times \mathbb{P}^1 \to \tilde{X} \times_{\rho} \mathbb{P}^1,
\]

be the quotient map. Then \( \tilde{\omega}_{\mathbb{P}^1} \) is the form on \( M \) such that \( q^* \tilde{\omega}_{\mathbb{P}^1} = \pi_{\mathbb{P}^1}^* \omega_{\mathbb{P}^1} \).
(2) Let \(\omega^{(i)}_{P^1}\) be the pull-back of \(\omega_{P^1}\) to \(U_i \times \mathbb{P}^1\), where \(\{U_i\}\) is the cover of \(X\) used in the description (M1). Because of the \(PU(2)\)-invariance of \(\omega_{P^1}\), and the fact that the transition functions \(g_{ij}\) are locally constant, the \(\omega^{(i)}_{P^1}\) patch together to define a globally defined form. The form they define is \(\tilde{\omega}_{P^1}\).

Notice that \(\tilde{\omega}_{P^1}\) is closed and restricts to \(\omega_{P^1}\) on the \(P^1\) fibers of \(M\). If \(\pi : M \to X\) is the projection map, and \(\omega_X\) denotes the Kähler form on \(X\), then we can define the 1-parameter family

\[
\omega_{\sigma} = \pi^* \omega_X + \sigma \tilde{\omega}_{P^1} \ .
\] (4.6)

This is clearly a family of closed, non-degenerate, positive forms of type \((1,1)\). In fact \(\omega_{\sigma}\) is the Kähler form for the Hermitian metric on \(M\) which (using description (M1)) pulls back to the (weighted) product metric on each neighborhood \(U_i \times \mathbb{P}^1\). This can also be described as the metric which descends from the weighted product metric on \(\tilde{X} \times \mathbb{P}^1\).

In the case where \(M = X \times \mathbb{P}^1\), this is exactly the family of Kähler forms described in [GP]. If \(M\) is not a product, but is projectively flat with a metric of the above sort, then \(M\) is ‘close enough’ to the product case for the following to be true.

**Lemma 4.9.** Let \(f : X \to R\) be a smooth function on \(X\). Then

\[
\int_M \pi^*(f) d\text{vol}_{\sigma} = \sigma \text{Vol}(\mathbb{P}^1) \int_X f \ d\text{vol}_{X} \ ,
\] (4.7a)

where \(d\text{vol}_{\sigma} = \frac{\omega^{n+1}_{X}}{(n+1)!} \) is the volume element on \(M\), and \(d\text{vol}_{X} = \frac{\omega^n_{X}}{n!} \) is the volume element on \(X\). In particular,

1. \(\text{Vol}_{\sigma}(M) = \sigma \text{Vol}(X)\text{Vol}(\mathbb{P}^1) \) ,

2. if \(V \to X\) is a complex bundle on \(X\), then

\[
\text{deg}_{\sigma} (\pi^* V) = n \sigma \text{deg}(V) \ ,
\] (4.7c)

where for any bundle, say \(W\), on \(M\), \(\text{deg}_{\sigma}(W) = \int_M c_1(W) \wedge \omega^n_{\sigma} \) is the degree with respect to the Kähler form in (4.6).

**Proof.** Equation (4.7a) is an immediate consequence of the fact that the metric is locally a product metric on the open sets homeomorphic to \(U_i \times \mathbb{P}^1\), and the fact that

\[
\frac{\omega^{n+1}_{\sigma}}{(n+1)!} = \pi^*(\frac{\omega^n_{X}}{n!}) \wedge \sigma \tilde{\omega}_{P^1} \ .
\]

For (4.7b), take \(f = 1\). For (4.7c), we use the fact that \(c_1(\pi^* V) = \pi^* c_1(V)\), and also the identity

\[
n \alpha \wedge \omega^{n-1}_X = (\Lambda_X \alpha) \omega^n_X \ ,
\]
where $\alpha$ is any complex 2-form on $X$. Using equation (4.6) we then compute
\[
\deg_{\sigma} \pi^* V = n\sigma \int_M \pi^* (c_1(V) \wedge \omega_X^{n-1}) \wedge \tilde{\omega}_{\mathbb{P}^1} \\
= \sigma \int_M \pi^* (n! \Lambda_X c_1(V)) \text{dvol}_{\sigma}.
\]
Applying (4.7a) with $f = n! \Lambda_X c_1(V)$ then gives the result. $\square$

§4.5 Explicit description of extension classes.

In order to adapt the dimensional reduction procedure of Garcia-Prada, we need to have an explicit description of the extension class in (4.1), i.e. we need explicit representatives for the classes in $H^{0,1}(M, \pi^* (\mathcal{E}_1 \otimes \mathcal{E}_2^*) \otimes \mathcal{K}_{M/X})$. Using the Projective Flatness condition, we now show that these can be taken to be of the form $\pi^* \Phi \otimes \tilde{\eta}$, where $\Phi \in \Omega^{0,0}(X, \mathcal{E}_1 \otimes \mathcal{E}_2^*)$ and $\tilde{\eta}$ is the extension to $\mathcal{K}_{M/X}$ of a uniquely determined invariant element $\eta \in \Omega^{(0,1)}(\mathbb{P}^1, \mathcal{O}(-2))$.

We begin with a description of $\eta$. Up to a scale factor, this is uniquely determined by the requirements that

1. $\eta$ is $SU(2)$- (and thus also $PU(2)$-) invariant,
2. $\eta$ represents a generator of $H^{0,1}(\mathbb{P}^1, \mathcal{O}(-2)) \cong \mathbb{C}$.

We can give an explicit description of $\eta$ if we use the identification of $\mathcal{O}(-2)$ with $K_{\mathbb{P}^1}$ i.e. with the holomorphic cotangent bundle. Let $z$ be a local coordinate on $\mathbb{P}^1$, and use $\{dz\}$ as a local frame for $K_{\mathbb{P}^1}$. With respect to this frame, and with respect to the local coordinate $z$, we get
\[
\eta(z) = \eta_0 dz \otimes \frac{dz}{(1 + |z|^2)^2}, \tag{4.8}
\]
where $\eta_0$ is the undetermined scale factor. Now if we interpret the Fubini-Study metric as a bundle metric on the holomorphic tangent bundle $T^{1,0}\mathbb{P}^1$, then with respect to the local coordinate $z$, and the local frame $\frac{\partial}{\partial z}$, this metric is given by
\[
k\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial z}\right) = \frac{1}{(1 + |z|^2)^2}. \tag{4.9}
\]
Thus taking duals with respect to this metric, we can write
\[
\eta(z) = \eta_0 d\bar{z} \otimes \left(\frac{\partial}{\partial z}\right)^*. \tag{4.10}
\]
Or, if $\eta^*$ denotes the ‘conjugate adjoint’, i.e. the section of $\Omega^{(1,0)}(\mathbb{P}^1, \mathcal{O}(2))$ obtained from $\eta$ by complex conjugation on the form part of $\eta$ and taking the adjoint (with respect to the metric induced by $k$) of the section of $\mathcal{O}(-2)$, then $\eta^* = \eta_0 dz \otimes \frac{\partial}{\partial z}$. We thus get that
\[
\eta \wedge \eta^* = i |\eta_0|^2 \omega_{\mathbb{P}^1}. \tag{4.10}
\]
With a view towards the next section, we henceforth fix $\eta_0$ such that
\[
|\eta_0|^2 = \frac{1}{\sigma}, \tag{4.11}
\]
where $\sigma$ is the weighting factor in the Kähler metric on $M$, i.e. in (4.6).
**Definition 4.10.** We define 
\[ \tilde{\eta} \in \Omega^{(0,1)}(M, \mathcal{O}(-2)) \cong \Omega^{(0,1)}(M, \mathcal{K}_{M/X}), \]
to be the extension of \( \eta \) (defined by (4.8) and (4.11)).

A local calculation confirms that this is a \( \overline{\partial} \)-closed form. Furthermore, the cohomology class which it represents in \( H^{(0,1)}(\mathbb{P}^1, \mathcal{O}(-2)) \cong \mathbb{C} \) is non-trivial since, for example, it restricts to the generator of \( H^{(0,1)}(\mathbb{P}^1, \mathcal{O}(-2)) \) on each \( \mathbb{P}^1 \) fiber of \( M \). We thus get

**Proposition 4.11.** The isomorphism
\[ H^0(X, \mathcal{E}_1 \otimes \mathcal{E}_2^*) \cong H^0,1(M, \pi^*(\mathcal{E}_1 \otimes \mathcal{E}_2^*) \otimes \mathcal{K}_{M/X}), \]
is realized by the map
\[ \Phi \longrightarrow \pi^* \Phi \otimes \tilde{\eta} \in \Omega^{0,1}(M, \pi^*(\mathcal{E}_1 \otimes \mathcal{E}_2^*) \otimes \mathcal{K}_{M/X}). \quad (4.12) \]

**Proof.** We need to check that \( \pi^* \Phi \otimes \tilde{\eta} \) is \( \overline{\partial} \)-closed (so that the map is well defined at the level of the cohomology groups), and that the map is injective (and thus an isomorphism). The first issue is clear, since \( \Phi \) is assumed to be a holomorphic section of \( \mathcal{E}_1 \otimes \mathcal{E}_2^* \) and \( \tilde{\eta} \) is also \( \overline{\partial} \)-closed. The injectivity of the map follows from the above comment about the non-triviality of the cohomology class of \( \tilde{\eta} \). \( \square \)

The following property of \( \tilde{\eta} \) will be used in the next section:

**Lemma 4.12.** Let \( \tilde{k} \) be the extension to \( \mathcal{O}(2) \cong \mathcal{K}_{M/X}^* \) of the metric \( k \) on \( \mathcal{O}(2) \). Using this metric, and the metric induced by it on \( \mathcal{O}(-2) \cong \mathcal{K}_{M/X} \), we get
\[ \tilde{\eta} \wedge \tilde{\eta}^* = \frac{i}{\sigma} \tilde{\omega}_{\mathbb{P}^1}. \quad (4.13) \]

**Proof.** This is a pointwise property. But in local coordinates on a neighborhood biholomorphic to \( U_i \times \mathbb{P}^1 \), this corresponds precisely to the fact that \( \eta \wedge \eta^* = \frac{i}{\sigma} \omega_{\mathbb{P}^1} \). \( \square \)

### §5. Dimensional Reduction of the Hermitian-Einstein Equations

We are now ready to examine the Hermitian-Einstein equations for an extension of the form in (4.1) over a projectively flat \( \text{PU}(2) \)–bundle \( M \), with respect to a Kähler metric as above. The setup we consider is the following:

Let \( M \) be a projectively flat \( \text{PU}(2) \)–bundle over \( X \), with descriptions as in (M1), (M2), and (M3). Fix a Kähler metric on \( M \) with Kähler form \( \omega_\sigma \) as in equation (4.6). Let \( \text{Vol}_\sigma(M) \) denote the volume with respect to this metric (cf. Lemma 4.9), and for any vector bundle \( V \longrightarrow M \), set
\[ \deg_\sigma(V) = \int_M c_1(V) \wedge \omega_\sigma^n. \]
Consider a holomorphic extension over $M$ of the form

$$0 \to \pi^*\mathcal{E}_1 \to \mathcal{E} \to \pi^*\mathcal{E}_2 \otimes \mathcal{K}_{M/X}^* \to 0,$$

as in (4.1). The extension class is represented by a holomorphic section in

$$\Omega^{(0,1)}(M, \text{Hom}(\pi^*\mathcal{E}_2 \otimes \mathcal{K}_{M/X}^*, \pi^*\mathcal{E}_1)) \cong \Omega^{(0,1)}(M, \pi^*(\mathcal{E}_1 \otimes \mathcal{E}_2^*) \otimes \mathcal{K}_{M/X}^*).$$

Using the isomorphism given in Proposition 4.11, we can take this to be

$$\beta = \pi^*\Phi \otimes \tilde{\eta}. \quad (5.1)$$

**Theorem 5.1.** Let $\mathcal{E}$ be a holomorphic extension as above, with extension class $\beta$ as in (5.1). Let $h$ be a Hermitian metric on $\mathcal{E}$ given by

$$h = \pi^*h_1 \oplus \pi^*h_2 \otimes \tilde{k},$$

where, for $i = 1, 2$, $h_i$ is a Hermitian metric on $\mathcal{E}_i$ and $\tilde{k}$ is the Hermitian metric on $\mathcal{K}_{M/X}^* = \hat{\mathcal{O}}(2)$ described in Lemma 4.12.

Fix $\sigma > 0$, and set

$$\lambda = \frac{2\pi}{n! \text{Vol}_\sigma(M)} \frac{1}{\text{deg}_\sigma(\mathcal{E})} \text{rank} \mathcal{E}. \quad (5.2)$$

Let parameters $\tau_1$ and $\tau_2$ be given by

$$\tau_1 = \lambda$$

$$\tau_2 = \lambda - \frac{2\pi}{n! \text{Vol}_\sigma(M)} \text{deg}_\sigma(\mathcal{K}_{M/X}^*). \quad (5.3)$$

Then the following are equivalent:

1. The metric $h$ satisfies the Hermitian–Einstein equation

$$i\Lambda_\sigma F_h = \lambda I_\mathcal{E},$$

where $\Lambda_\sigma$ denotes contraction against the Kähler form

$$\omega_\sigma = \pi^*\omega_X + \sigma \hat{\omega}_{\mathbb{P}^1}.$$

2. The metrics $h_1$ and $h_2$ satisfy the coupled vortex equations:

$$i\Lambda_X F_{h_1} + \Phi \Phi^* = \tau_1 I_{\mathcal{E}_1}$$

$$i\Lambda_X F_{h_2} - \Phi^* \Phi = \tau_2 I_{\mathcal{E}_2},$$

where the adjoint in $\Phi^*$ is with respect to the metrics $h_1$ and $h_2$, and $\Lambda_X$ denotes contraction against the Kähler form $\omega_X$ on $X$. 

Proof. Because of the projectively flat structure on $M$ and our choice of Kähler structure, the geometry of our situation is locally indistinguishable from the case of $M = X \times \mathbb{P}^1$. The proof of this theorem is thus essentially the same as that of the corresponding result in [GP]. We proceed as follows.

Given a Hermitian metric $h$ on $\mathcal{E}$, we can analyse the Hermitian-Einstein tensor $i\Lambda_{\sigma} F_h$. With respect to a smooth orthogonal splitting $\mathcal{E} = \pi^* \mathcal{E}_1 \oplus \pi^* \mathcal{E}_2 \otimes K^*_M/X$, we can write

$$i\Lambda_{\sigma} F_h = i\Lambda_{\sigma} \left( F_{h_1} - \beta \wedge \beta^* \begin{pmatrix} D' & D' \beta \\ -D'' \beta^* & F_{h_2} - \beta^* \wedge \beta \end{pmatrix} \right). \quad (5.4)$$

Here $h_1$ and $h_2$ are the metrics induced on the sub- and quotient bundles, the $F_{h_i}$ are the curvatures of the corresponding metric connections, and

$$D = D' + D'' : \Omega^1(\pi^* (\mathcal{E}_1 \otimes \mathcal{E}_2^*) \otimes K^*_M/X) \to \Omega^2(\pi^* (\mathcal{E}_1 \otimes \mathcal{E}_2^*) \otimes K^*_M/X)$$

is constructed in the standard way from the metrics on $\pi^* (\mathcal{E}_1)$ and $\pi^* (\mathcal{E}_2) \otimes K^*_M/X$. The summands $D'$ and $D''$ are the $(1,0)$ and $(0,1)$ parts respectively. The "*" in $\beta^*$ denotes the adjoint on sections (with respect to the bundle metrics) and conjugation on forms, so that $\beta^* \in \Omega^{(1,0)}(M, \pi^* (\mathcal{E}_2 \otimes \mathcal{E}_1^*) \otimes K^*_M/X)$. We can write

$$F_{h_1} = \pi^* F_{h_1}$$
$$F_{h_2} = \pi^* F_{h_2} + \tilde{k} \otimes \mathbf{I}_2. \quad (5.5)$$

Now the $SU(2)$-invariant metric $\tilde{k}$ on $\mathcal{O}_{\mathbb{P}^1}(2)$ (described in (4.9)) is Hermitian-Einstein with respect to the Fubini-Study metric on $\mathbb{P}^1$. Because of our choice of Kähler structure on $M$ it thus follows that $\tilde{k}$ is a Hermitian-Einstein metric on $K^*_M/X$, so that we have

$$i\Lambda_{\sigma} F_{\tilde{k}} = c = \frac{2\pi}{n! \text{Vol}_\sigma(M)} \text{deg}_\sigma(\tilde{\mathcal{O}}(2)). \quad (5.6)$$

Also, with $\beta$ as in (5.1), we get

$$i\Lambda_{\sigma} (\beta \wedge \beta^*) = \pi^* \Phi^* \otimes i\Lambda_{\sigma} (\tilde{\eta} \wedge \tilde{\eta}^*) \quad (5.7)$$

where the adjoint in $\Phi^*$ is with respect to the metrics on $\mathcal{E}_1$ and $\mathcal{E}_2$, and $\tilde{\eta}^*$ is determined by the metric $\tilde{k}$. The proof of the theorem thus depends on the following lemma:

**Lemma 5.2.** With $\tilde{\eta}$, $\beta$, and $\Lambda_{\sigma}$ as above, we have

1. $i\Lambda_{\sigma} (\tilde{\eta} \wedge \tilde{\eta}^*) = -1$,
2. $\Lambda_{\sigma} D' \beta = 0$, and
3. $\Lambda_{\sigma} D'' \beta^* = 0$. 


Furthermore, for \( i = 1, 2 \), we get

\[
(4) \ \Lambda_\sigma \pi^* F_{h_i} = \pi^* \Lambda_X F_{h_i}.
\]

**Proof of Lemma 5.2.**

Part (1) follows immediately from Lemma 4.12.

For part (2), write

\[
D' \beta = D' (\pi^* \Phi) \otimes \tilde{\eta} + \pi^* \Phi \otimes D' \tilde{\eta}.
\]

We are abusing notation here, since the covariant derivatives denoted by \( D' \) are not the same in the three terms in this expression. We can be more precise if we restrict to local coordinates on a neighborhood biholomorphic to \( U_i \times \mathbb{P}^1 \). The formula then becomes

\[
D' \beta = \pi_1^* D'_{\mathcal{E}_1 \otimes \mathcal{E}_2} \Phi \otimes \pi_2^* \eta + \pi_1^* \Phi \otimes \pi_2^* D'_{\mathcal{O}_{\mathbb{P}^1}(-2)} \eta,
\]

where the \( \pi_i \) are the projections onto the first and second factors of \( U_i \times \mathbb{P}^1 \). The covariant derivatives are now those corresponding to the metric connections on \( \mathcal{E}_1 \otimes \mathcal{E}_2^* \) and on \( \mathcal{O}_{\mathbb{P}^1}(-2) \). Furthermore, in these local coordinates,

\[
\omega_\sigma = \pi_1^* \omega_X + \sigma \pi_2^* \omega_{\mathbb{P}^1}.
\]

The rest of the proof is essentially the same as in [GP], with the key points being:

(i) The term \( \Lambda_\sigma \pi_1^* D'_{\mathcal{E}_1 \otimes \mathcal{E}_2} \Phi \otimes \pi_2^* \eta \) vanishes because the Kähler form has no contribution from ‘mixed’ terms, i.e. terms in \( \Omega^{(1,0)}(U_i) \wedge \Omega^{(0,1)}(\mathbb{P}^1) \), and

(ii) the term \( \pi_1^* \Phi \otimes \Lambda_\sigma \pi_2^* D'_{\mathcal{O}_{\mathbb{P}^1}(-2)} \eta \) vanishes because

\[
\Lambda_\sigma \pi_2^* D'_{\mathcal{O}_{\mathbb{P}^1}(-2)} \eta = \pi_2^*(\sigma \Lambda D'_{\mathcal{O}_{\mathbb{P}^1}(-2)} \eta) = 0.
\]

The proof of part (3) is similar to (2) (alternatively, one can simply observe that \( \Lambda_\sigma D'' \beta^* = -(\Lambda_\sigma D' \beta)^* \)).

For part (4), we note that with \( \omega_\sigma = \pi^* \omega_X + \sigma \omega_{\mathbb{P}^1} \), we get

\[
\Lambda_\sigma \pi^* F_{h_i} = \pi^* \Lambda_X F_{h_i} + \sigma (\pi^* F_{h_i}, \omega_{\mathbb{P}^1}),
\]

where the inner product in the second term on the right is on \( \Omega^2(M, \mathbb{C}) \). A computation in local coordinates verifies that for any \( \alpha \in \Omega^2(X, \mathbb{C}) \), we get \( (\pi^* \alpha, \tilde{\omega}_{\mathbb{P}^1}) = 0 \). \( \square \)

**Remark 5.3** The proof of Lemma 5.2 uses the fact that \( M \) is projectively flat and the special properties of the Kähler metrics that we fix on \( M \). In a more general situation, it is not so clear which (if any) Kähler structures are suitable for the carrying out of a dimensional reduction of the Hermitian-Einstein equations. It is tempting to speculate that the properties listed in the lemma should serve as the definition of suitable Kähler structures.
Completion of Proof of Theorem 5.1

Using (5.2), (5.6), and Lemma 5.2, we thus find that the Hermitian-Einstein condition on $\mathbf{h}$ is equivalent to

$$
\begin{pmatrix}
\pi^*(i\Lambda X F_{h_1} + \Phi \Phi^*) \\
0
\end{pmatrix}
\begin{pmatrix}
\pi^*(i\Lambda X F_{h_2} - \Phi^* \Phi) + c \otimes I_2 \\
0
\end{pmatrix}
= \lambda
\begin{pmatrix}
I_1 \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
I_2
\end{pmatrix}
$$

(5.10)

where $c$ is as in (5.6). The results follows from this. □

Remark 5.4 By using (5.3), (5.4), and (4.7c) we can compute

$$
\begin{align*}
\tau_1 + \tau_2 &= \frac{2\pi}{n! \text{Vol}_\sigma(M)} (\deg_\sigma(\mathcal{E}) - \deg_\sigma(\mathcal{K}_{M/X}^*)) \\
&= \frac{2\pi}{n! \text{Vol}_\sigma(M)} (\deg_\sigma(\pi^*(\mathcal{E}_1 \oplus \mathcal{E}_2)) \\
&= \frac{2\pi}{(n-1)! \text{Vol}(X)} \deg(\mathcal{E}_1 \oplus \mathcal{E}_2),
\end{align*}
$$

thus verifying that the relation given by (2.4) holds. Also, by (5.4) and (4.7b), we have

$$
\tau_1 - \tau_2 = \frac{2\pi}{\sigma} \frac{\deg_\sigma(\mathcal{K}_{M/X}^*)}{n! \text{Vol}(X)}. 
$$

(5.11)

We can compute $\deg_\sigma(\mathcal{K}_{M/X}^*)$, using the fact that $\mathcal{K}_{M/X}^* = \hat{O}(2)$:

Lemma 5.5. Let $\hat{O}(2k)$ be the extension to $M$ of the bundle $\mathcal{O}_{\mathbb{P}^1}(2k)$. Then the first Chern class $c_1(\hat{O}(2k))$ can be represented by the $(1,1)$-form $2k\bar{\omega}_{\mathbb{P}^1}$, where $\bar{\omega}_{\mathbb{P}^1}$ is the extension of the Fubini Study Kähler form (as in §4.4). Hence (assuming that the Fubini-Study metric is normalized so that $\text{Vol}(\mathbb{P}^1) = 1$) we get

$$
\deg_\sigma(\hat{O}(2k)) = 2kn! \text{Vol}(X). 
$$

Proof. The first Chern class of $\mathcal{O}_{\mathbb{P}^1}(2k)$ can be represented by the $(1,1)$-form $2k\omega_{\mathbb{P}^1}$. This in turn can be obtained as $\frac{i}{2\pi} \partial \bar{\partial} \log(h)$, where $h$ is an $(SU(2))$-invariant) hermitian metric on $\mathcal{O}_{\mathbb{P}^1}(2k)$. Since $h$ is $PU(2)$-invariant, it extends to a metric $\bar{h}$ on $\hat{O}(2k)$. One can compute that

$$
\frac{i}{2\pi} \partial_M \bar{\partial}_M \log(\bar{h}) = 2k\bar{\omega}_{\mathbb{P}^1},
$$

where $\partial_M$ and $\bar{\partial}_M$ are the holomorphic and anti-holomorphic parts of the exterior derivative on $M$. It follows immediately from this that $c_1(\hat{O}(2k))$ can be represented by $2k\bar{\omega}_{\mathbb{P}^1}$. The formula for $\deg_\sigma(\hat{O}(2k))$ then follows from the form of $\omega_\sigma$ (cf. (4.6)), and by the computation for $\text{Vol}_\sigma(M)$ (cf. (4.7b)). □

Applying Lemma 5.5. to $\mathcal{K}_{M/X}^* = \hat{O}(2)$, we get that $\deg_\sigma(\mathcal{K}_{M/X}^*) = 2n! \text{Vol}(X)$. Equation (5.11), which applies in the case where $M$ is a flat $PU(2)$-bundle, is therefore the same as the relation given in Proposition 3.1, which refers to the case where $M = X \times \mathbb{P}^1$. 
§6. Special case where $M = \mathbb{P}(E)$

As an interesting special case, we consider the situation in which $M$ comes from a holomorphic vector bundle with a projectively flat Hermitian structure. In this case, the parameter computations of the previous section can then be carried out quite explicitly. We thus assume that:

1. $M = \mathbb{P}(E)$, where $E \to X$ is a rank 2 holomorphic bundle, and
2. there is a Hermitian metric $h$ on $E$ such that

$$i \Lambda_X F_h = \text{const.} \quad I = \frac{\pi \deg(E)}{(n-1)!\text{Vol}(X)}.$$

Using the description $M = \mathbb{P}(E)$, we get a canonically defined line bundle on $M$, namely the tautological line bundle on $\mathbb{P}(E)$. This bundle, which we denote by $O_M(-1)$, is a subbundle of $\mathbb{P}(E) \times \pi^* E$ and restricts to the tautological line bundle on each $\mathbb{P}^1$–fiber of $\mathbb{P}(E)$. The main result of this section is:

**Proposition 6.1.**

$$\deg_\sigma(O_M(-1)) = \frac{n \sigma \deg(E)}{2} - n! \text{Vol}(X) \quad (6.1a)$$

$$= \frac{\deg_\sigma(\pi^*(\det E))}{2} - n! \text{Vol}(X). \quad (6.1b)$$

**Proof.** The proof is a computation, using Equation (2.2) and the Chern-Weil formula for $c_1(E)$ in terms of curvature. Since $O_M(-1)$ is a subbundle of $\mathbb{P}(E) \times \pi^* E$, the metric $h$ on $E$ induces a fiber metric, say $H$, on $O_M(-1)$. Let $F_H$ be the curvature of its corresponding metric connection. We thus need to compute $\frac{i}{2} \pi \Lambda_\sigma F_H$ at a point of $\mathbb{P}(E)$.

Any point in $\mathbb{P}(E)$ can be represented by a unit vector $\xi_0$ in a fiber $E_{x_0}$ of $E$. As in [Ko] (p. 89), we choose a normal holomorphic frame $s = (s_1, s_2)$ for $E$ with $s_2(x_0) = \xi_0$. This gives local coordinates $(z^1, \ldots, z^n, \xi^1, \xi^2)$ on $E$, where $(z^1, \ldots, z^n)$ are local coordinates near $x_0$ on $X$, and $(\xi^1, \xi^2)$ are the coordinates with respect to the frame $s$ of a point in the fiber. If we denote the points in $E$ by $\xi(z)$, and the corresponding points in $\mathbb{P}(E)$ by $[\xi(z)]$. Then

$$[\xi(z)] \mapsto (z^1, \ldots, z^n, [\xi^1, \xi^2]),$$

gives a local trivialization $\mathbb{P}(E)|_U \cong U \times \mathbb{P}^1$, and thus defines a local holomorphic frame for $O_M(-1)$. Using these local coordinates, the computation in [Ko2 (p.90 and I.4)] shows that

$$(iF_H)_{[\xi_0]} = i \sum R_{i\bar{k}a\bar{b}} \xi^i_0 \xi^k_0 \, dz^a \wedge d\bar{z}^\bar{b} - (\omega_{\mathbb{P}^1})_{[0,1]}$$

$$= ih(R(\xi_0), \xi_0) - (\omega_{\mathbb{P}^1})_{[0,1]}, \quad (6.2)$$

where $R$ is the curvature of the metric connection for $h$ on $E$. But, since $h$ satisfies the Hermitian–Einstein condition, we get that
in \( h(R(\xi_0), \xi_0) \wedge \omega_X^{n-1} = \frac{\pi \deg(E)}{\Vol(X)(n-1)!} |\xi_0|^2 \omega_X^n \). \hspace{1cm} (6.3)

Also, using the local coordinate system on \( \mathbb{P}(E) \) near \([\xi_0]\), the Kähler form on \( M = \mathbb{P}(E) \) is given by \( \omega_{\sigma} = \omega_X + \sigma \omega_{\mathbb{P}^1} \). Thus, using \(|\xi_0|^2 = 1\), we get

\[
\frac{(i F_H)}{2\pi} \wedge \omega_{\sigma}^n = \frac{(i n \sigma)}{2\pi} h(R(\xi_0), \xi_0) \wedge \omega_X^{n-1} \wedge \omega_{\mathbb{P}^1} - \omega_X^n \wedge \omega_{\mathbb{P}^1}
\]
\[
= \left( \frac{n \sigma \deg(E)}{2\Vol(X)} - n! \right) \left( \frac{\omega_X^n}{n!} \wedge \omega_{\mathbb{P}^1} \right)
\]
\[
= \left( \frac{n \sigma \deg(E)}{2\Vol(X)} - n! \right) \left( \frac{\omega_{\sigma}^{n+1}}{(n+1)!} \right). \hspace{1cm} (6.4)
\]

The first result follows from this, Lemma 4.9, and the fact that \( \int_{\mathbb{P}^1} \omega_{\mathbb{P}^1} = 1 \). The second result then follows by \((4.7c)\) in Lemma 4.9, applied to \( \det E \). \hspace{1cm} \(\blacksquare\)

We can define \( O_M(1) \) to be the dual bundle to \( O_M(-1) \), and set

\[ O_M(\pm k) = O_M(\pm 1)^{\otimes k}. \]

**Proposition 6.2** (cf. [Ko]).

\[ K_{M/X}^* \equiv \tilde{O}(2) = O_M(2) \otimes \pi^*(\det E), \]

where \( \det E \) is the determinant bundle of \( E \), and \( \pi : M \rightarrow X \) is as in the previous section.

Combining Propositions 6.1 and 6.2 we thus get

**Proposition 6.3.**

1. For \( k \in \mathbb{Z} \),

\[
\deg_{\sigma} O_M(k) = k(n!\Vol(X) - \frac{\deg_{\sigma} \pi^*(\det E)}{2})
\]
\[
= k(n!\Vol(X) - \frac{n \sigma \deg(E)}{2}), \hspace{1cm} (6.5)
\]

2. \[ \deg_{\sigma}(K_{M/X}^*) = 2n!\Vol(X). \hspace{1cm} (6.6) \]

**Remark 6.4.** Notice that as a result of \((6.6)\) and \((5.11)\) we see that when \( M = \mathbb{P}(E) \) the parameters \( \tau_1, \tau_2, \) and \( \sigma \) (in Theorem 5.1) are related in precisely the same way as in the case where \( M = X \times \mathbb{P}^1 \), i.e. as in Proposition 3.2.
§7. Discussion of Results

It is clear from the results described above that dimensional reduction should be thought of in more general terms than was previously expected. In particular, the role of a global symmetry in the form of a group action is not as central as we assumed. On the other hand, what makes the reduction possible is certainly some kind of geometric order, even if cannot quite be called a global symmetry. In this section we explore what kind of geometric structure is evident in our situation, and look at how it compares to the Garcia-Prada case.

In the case that \( M = X \times \mathbb{P}^1 \), there is an action of \( SU(2) \) on \( M \) as described in [GP]. The action is trivial on \( X \) and via the identification \( \mathbb{P}^1 = SU(2)/U(1) \) on \( \mathbb{P}^1 \). The group orbits are thus of the form \( x \times \mathbb{P}^1 \), and the orbit space is \( X \).

In order to exploit this symmetry (i.e. the group action), one must consider \textit{invariant} solutions for whatever equations are to be dimensionally reduced. In order for such solutions to exist, one needs to be in an \textit{equivariant} setting, and it is for this reason that the bundles considered in [GP] are \( SU(2) \)-equivariant bundles.

As soon as the structure of \( M \) as a \( \mathbb{P}^1 \)-bundle over \( X \) becomes non-trivial, the \( SU(2) \)-action is destroyed. Notice however that the group orbits survive in the form of fibers of the bundle. As we shall see, it is perhaps more pertinent to describe the fiber \( \mathbb{P}^1 \)'s as the \textit{leaves of a foliation}. Indeed, given a foliation on \( M \), there is a natural replacement for the notion of an equivariant bundle, namely that of a \textit{foliated bundle} (cf. [KT]). Roughly speaking, a foliated bundle has a flat structure along each leaf of the foliation. The relation between such bundles and the equivariant bundles (which can exist when the foliation comes from a global group action) is made even clearer if we introduce the holonomy groupoid, \( G_F \), associated to the foliation \( F \). The foliated bundles we consider can be described as \( G_F \)-equivariant bundles, where an \( G_F \)-equivariant bundle is one which supports an action of \( G_F \) on its fibers. The action in question is in fact via holonomy transport along the leaves of the foliation.

This language of foliations provides a convenient framework for understanding the dimensional reduction we have carried out in §5. It is illuminating even in the case where \( M = X \times \mathbb{P}^1 \). In that case the \( SU(2) \)-equivariant bundles over \( X \times \mathbb{P}^1 \) can indeed be considered as foliated bundles - but in way that is slightly unexpected. The novelty is due to the fact that \( X \times \mathbb{P}^1 \) is foliated in \textit{two} ways; in one foliation the leaves are the copies of \( \mathbb{P}^1 \) in \( X \times \mathbb{P}^1 \), and in the other way the leaves are the copies of \( X \). We will refer to these as the \textit{fiber foliation} and the \textit{base foliation} respectively.

\textbf{Lemma 7.1.} The bundles over \( X \times \mathbb{P}^1 \) that are foliated with respect to the fiber foliation are of the form \( \pi_1^* E \), where \( E \rightarrow X \) is any smooth vector bundle over \( X \) and \( \pi_1 \) denotes the projection onto the first factor of \( X \times \mathbb{P}^1 \).

The line bundles over \( X \times \mathbb{P}^1 \) that are foliated with respect to the base foliation are of the form \( \pi_2^* H^k \), where \( H \rightarrow \mathbb{P}^1 \) is the line bundle of degree 1 over \( \mathbb{P}^1 \) and \( \pi_2 \) denotes the projection onto the second factor of \( X \times \mathbb{P}^1 \).

Thus the \( SU(2) \)-equivariant bundles that appear in the dimensional reduction
construction, i.e. the bundles of the form $V = \pi_1^* E \otimes \pi_2^* H^{\otimes k}$, can be thought of as *doubly foliated bundles*. The next Proposition shows that it is this doubly foliated property that is retained when the trivial fibration $X \times \mathbb{P}^1$ is replaced by a flat projective fibration $M \to X$.

**Proposition 7.2.** Let $M \to X$ be a flat projective $PU(2)$-bundle over $X$, as in (§4). Then $M$ has two natural foliations; one in which the leaves are the $\mathbb{P}^1$-fibers of $M \to X$, and one in which the leaves are copies of $\tilde{X}$, the universal cover of $X$. The leaves of the first foliation intersect the leaves of the second transversally.

**Proof.** This follows immediately from the descriptions (M1-3) of $M$ given in §4.$\square$

We will denote the foliation by the $\mathbb{P}^1$ fibers by $\mathcal{F}_\pi$, and the other foliation by $\mathcal{F}_\alpha$. The next result shows that the bundles with the structure of extensions as in (4.1) are indeed the analogues of the bundles considered in [GP].

**Proposition 7.3.** Let $M \to X$ be a flat projective $PU(2)$-bundle over $X$, as in (§4).

1. There is a one to one correspondence between vector bundles $W$ over $X$ and $\mathcal{G}_{\mathcal{F}_\pi}$-equivariant bundles on $M$, given by $W \to \pi^* W$.
2. There is a one to one correspondence between $\alpha$-equivariant vector bundles $V$ over $\mathbb{P}^1$ and $\mathcal{G}_{\mathcal{F}_\alpha}$-equivariant bundles on $M$, given by $V \to \tilde{V}$, where $\tilde{V}$ is the extension as defined in §4.2

Finally, we remark that $\mathbb{P}^1$-bundles are almost certainly not the only objects on which our techniques will work. In particular, there is good reason to expect that $\mathbb{P}^1 = SU(2)/U(1)$ can be replaced by any Kählerian homogeneous space $G/H$. We will return to this question in a later publication.

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**Department of Mathematics, University of Illinois, Urbana, IL 61801** 1,2,3

**Department of Mathematics, East. Illinois University, Charleston, IL 61920** 2

_E-mail address:_ bradlow@uiuc.edu, glazebro@math.uiuc.edu, kamber@math.uiuc.edu