Network bypasses sustain complexity

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Edited by David Weitz, Harvard University, Cambridge, MA; received March 27, 2023; accepted June 24, 2023

Real-world networks are neither regular nor random, a fact elegantly explained by mechanisms such as the Watts–Strogatz or the Barabási–Albert models, among others. Both mechanisms naturally create shortcuts and hubs, which while enhancing the network’s connectivity, also might yield several undesired navigational effects: They tend to be overused during geodesic navigational processes—making the networks fragile—and provide suboptimal routes for diffusive-like navigation. Why, then, networks with complex topologies are ubiquitous? Here, we unveil that these models also entropically generate network bypasses: alternative routes to shortest paths which are topologically longer but easier to navigate. We develop a mathematical theory that elucidates the emergence and consolidation of network bypasses and measure their navigability gain. We apply our theory to a wide range of real-world networks and find that they sustain complexity by different amounts of network bypasses. At the top of this complexity ranking we found the human brain, which points out the importance of these results to understand the plasticity of complex systems.

Significance

We show here that existing mechanisms of network evolution also create structural network bypasses as by-products: alternative routes to shortest paths that enable collateral circulation, making the system naturally resilient against failure of the widespread topological shortcuts or hubs. We develop a mathematical theory that explains the emergence of bypasses and quantify their impact on aspects such as network navigability gain. We finally apply our framework to analyze a large set of real-world networks and rank them according to the amount of network bypasses they show. At the top of this ranking, we find the human brain, what points out the importance of these results, to understand the plasticity of complex systems.

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Author contributions: E.E. designed research; E.E., J.G.-G., and L.L. performed research; E.E. and L.L. analyzed data; and E.E. and L.L. wrote the paper.

The authors declare no competing interest.

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This article contains supporting information online at https://www.pnas.org/lookup/suppl/doi:10.1073/pnas.2305001120/-/DCSupplemental.

Published July 25, 2023.
of some communication efficiency. SW and BA-type mechanisms indeed tend to generate networks with enhanced connectivity (9) (a form of structural deepening) which are robust against random failures (10, 11), what in principle could explain the ubiquity of these mechanisms and the resulting macroscopic patterns, even if quantifying such complexity has proven elusive.

However, observe that the SW mechanism reduces mean path length simply by creating path shortcuts, making enhanced connectivity overly dependent—and thus, fragile—on them. Likewise, in BA-like networks, shortest paths often involve hubs, and these networks are known to be extremely fragile against failure of hubs (12) or jamming (13–16), potentially inducing a failure cascade which can severely harm the macroscopic network’s function.

Why, then, complex networks are ubiquitously observed? First, note that walkers navigating a network do not necessarily have full information of the network structure, and geodesic navigation is indeed a global optimization problem (17) that, accordingly, “blind” walkers cannot perform. Second, such blind walkers typically undergo diffusion-like navigation, and such parsimonious navigation strategy can lead walkers to “diffuse out” and get lost easily if attempting to follow shortest paths, as these tend to have higher degree nodes". Accordingly, nongeodesic navigational strategies have been proposed (18–24), usually providing heuristic recipes based on local network information available [such as the degree (18–20) or the matching index (21)]. Solving the apparent dilemma between the prevalence of complex network architectures—underpinned by WS and BA mechanisms among others—with structural deepening related to enhanced communication capacity requires to find parsimonious mechanisms which can mitigate the undesired effects of geodesic navigability, and this is the second motivation of our work.

Our contention in this work is that as a network complexifies, it is capable to mitigate the impact of the undesired geodesic navigability issues by structural deepening mechanisms which favor the consolidation of network bypasses: alternative routes to mere geodesic navigation that i) decrease the tendency of “getting lost” by blind walkers, and ii) if needed can also be used by nonblind walkers to avoid problematic links and nodes, therefore allowing the overall connectivity to be maintained and the network to be robust against failure of shortcuts and hubs.

In what follows we start from first principles and develop a theory to define and detect the emergence of network bypasses in both synthetic and real-world networks and quantify their associated gain and impact in terms of network navigability. Our theory is based on a network geometrization by which initially unweighted edges and paths acquire an effective weight—an effective length, or cost—induced solely by the topology of the surrounding network’s structure. Network bypasses then emerge as geodesic paths in the geometrized network, i.e., they are the solutions of a topology-induced minimum-cost path optimization problem (25), and in many cases, we show that they do not coincide with the shortest paths of the original network. We also show that i) the emergence of these network bypasses is an unavoidable (entropic) by-product of the WS and BA mechanisms themselves and that ii) the effect of these bypasses is optimally emphasized when networks fall in a specific point of SW regime and an intermediate edge density in the sparse regime for BA-like networks, thus finding a quantitative proxy for structural deepening. We also certify that iii) network bypasses indeed provide source—destination routes with better navigation properties for diffusive-like blind walkers than geodesic routes and finally rank and discuss the emergence of network bypasses and their associated navigability gain in a range of real-world networks.

**Results**

To fix the intuition, let us begin by illustrating two situations in simple graphs that highlight the importance of bypasses in the operation of a network that harbors transportation and propagation of signals and information. To this aim, we initially consider a particle hopping between the nodes of a network created via the WS model (3), and we focus on the propagation of the particle between nodes $i$ and $j$ (Fig. 1A). Starting with rewiring probability $p = 0$, we have a circulant graph $G_1$, and the path $P_1 = \{i, j−1, j\}$ of length $2$ (highlighted in blue) is a shortest path connecting $i$ and $j$. Mimicking the action of WS-like mechanism kicking in, the edge $e = \{(i, i+1)\}$ of $G_1$ is randomly rewired, and subsequently, another edge is also randomly rewired, so that node $j−1$ now receives an edge from a “distant” node. In the resulting graph $G_2$, vertex $i+1$ drops its degree by one, whereas vertex $j−1$ increases its degree. This situation creates a small degree heterogeneity in the graph $G_2$ which did not exist in the circulant graph $G_1$: Node $j−1$ now participates in many more shortest paths starting elsewhere and ending at vertex $j−1$. Accordingly, the length-$2$ path $P_1$, in practice, might not be the “best” route to connect $i$ and $j$, even if it is still the shortest path, topologically speaking. For instance, a random walker choosing $P_1$ has a higher likelihood of “diffusing out” through $j−1$, thus hardly reaching the destination. Likewise, geodesic navigation will make $j−1$ systematically overused, leading to a higher chance of damage or jamming. In turn, the length-$3$ path $P_2 = \{i, i−1, i+1, j\}$ (highlighted in pink), while being topologically longer than $P_1$, contains node $i+1$ whose degree is at the same time lower than the average and also avoids $j−1$: hence, it can be seen as a potentially more ballistic route that avoids a potentially problematic $j−1$ and still connects $i$ and $j$.

A similar situation is depicted in Fig. 1B where node $h$ becomes a hub via a rich-get-richer (i.e., BA-like) mechanism. The shortest path between $i$ and $j$ (highlighted in blue) will again be more prone for the walker to get lost due to the presence

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*Fig. 1. (A) Illustration of the effects of the edge rewiring process in the Watts–Strogatz model on the paths connecting two arbitrary vertices of the resulting graph: the shortest path $P_1$ (blue) can be bypassed by the path $P_2$ (pink), topologically longer but with a lower energetic cost. (B) A similar phenomenon happens when the node $h$ becomes a hub after a rich-get-richer mechanism. The shortest path $P_1$ (blue) typically crosses the hub but, with a sufficiently large mean degree, other paths such as $P_2$ (pink) can bypass the shortest path $P_1$, allowing alternative routes when hubs reach capacity and become saturated or damaged. (C) Navigational dilemma embedded in a network: The blue path $P_1$ is the shortest path, but it turns out that the pink “braquistochronic” path $P_2$ is more advantageous as it avoids congestion and is less resistive (see SI Appendix, section S5 for an explicit calculation).*
of a high-degree node, once the BA mechanism enhances such heterogeneity. Now, if the network supports a sufficiently large\(^1\) mean degree—i.e., if the network allows more edges to be formed than a spanning tree—, then other routes can emerge, bypassing the hub (pink path).

The two examples illustrated in Fig. 1 A and B raise the question of whether a particle would “prefer” to travel from \(i \) to \(j \) via the shortest—albeit with higher uncertainty to reach the destination—path \(P_1 \) or along the slightly longer but more ballistic—smaller uncertainty—alternative path \(P_2 \). In Panel (C) of the same figure, we illustrate such a conundrum where two alternative routes (a shortest path \(P_1 \), in blue, and a topologically longer one \(P_2 \), in pink) are highlighted. It is intuitive to think that there is a trade-off: Sometimes, \(P_1 \) is to be preferred, sometimes \(P_2 \) is a contingently better option. Extending this situation to a network-growth mechanism, this suggests that the creation of shortcuts (SW) and hubs (BA) should be sustained by the emergence of some alternative pathways bypassing these, with structural deepening effects that would reach a maximum impact for a specific rewiring probability \(p \) (SW) as well as specific hub abundance (BA). In what follows, we introduce a formalism that puts these questions and their general solution in a solid grounding.

### The Concept of Resistive Paths

Starting from first principles, the possible trajectories that a hopping particle can perform over a network \(G = (V, E) \) of \(|V| = n \) nodes with binary adjacency matrix \(A = \{A_{ij}\}_{i,j=1}^{n} \) can be enumerated by computing the powers of \(A \). A natural way to penalize longer trajectories connecting the same initial and end nodes is to properly weight them

\[
G(\beta) = e^{\beta A}; \quad G_{ij}(\beta) = \sum_{l=0}^{\infty} \frac{\beta^l (A^l)_{ij}}{l!} = \left( e^{\beta A} \right)_{ij},
\]

where \(\beta \) is an empirical parameter. This expression is known as the communicability function of a graph (26, 27). While originally being a purely combinatorial expression that encapsulates the contributions of different walks in a graph, \(G(\beta) \) indeed emerges as a central matrix when analyzing a wide variety of dynamics on graphs (27–31) (see SI Appendix, S1.1 for details and S1.2 for a derivation of \(G(\beta) \) as the actual propagator in a specific case with Hamiltonian dynamics). Nowadays communicability is applied across a range of disciplines, from neuroscience (32–39) or cancer research (40) to ecology (41) or economics (42), to cite a few.

While this operator naturally emerges in relation to different types of dynamics on networks, in this work, we shall highlight that it is fundamentally a combinatorial one and is not a priori derived from any concrete dynamics running on the network. In other words, while we will consider that there is some kind of generic propagation—let it be information, electrons, or other types of particles hopping through the network—the theory presented hereafter does not require to specify which dynamical equations rule such propagation, as we focus on the structural (topological) constraints which generally affect such propagation. By analogy to the cases discussed in SI Appendix, S1.1 and S1.2 and (27), we call \(G_{ij} \) the structural propagator, which parsimoniously captures the role that the network’s architecture plays in \(j \) receiving particles sent from \(i \). Similarly, \(G_{ij} \) accounts for how much a node \(i \) structurally retains an item at it, as the item returns to \(i \) infinitely often. For a particle initially located at the node \(i \), the difference,

\[
R_{i \rightarrow j}(\beta) = G_{ii}(\beta) - G_{ij}(\beta),
\]

accounts for the opposition offered by the network structure to the directional displacement of a particle sent by node \(i \) to the node \(j \), where the smaller the value of \(R_{i \rightarrow j} \), the higher the probability that the particle does not get trapped at the origin \(i \) and can propagate to node \(j \), i.e., there are more conductive walks between \(i \) and \(j \) than those returning back to the origin. In order to account for the resistance of the displacements between any pair of nodes, we should take into account the two possible directions of their mutual communication (\(i \rightarrow j \) and \(j \rightarrow i \)). To this aim, one can symmetrize (2) to define the communication resistance between nodes \(i \) and \(j \) as 

\[
\xi_{ij}(\beta) := (R_{ij}(\beta) + R_{ji}(\beta))^{1/2}.
\]

From the definition of the communicability function, and setting \(\beta = 1 \) without loss of generality, we obtain that the communication resistance reads:

\[
\xi_{ij}^{2} = \sum_{m=1}^{n} e^{2\mu_{m}} \left( (\psi_{m})_{i} - (\psi_{m})_{j} \right)^{2}.
\]

where \((\psi_{m})_{i} \) is the \(i\)-th entry of the eigenvector associated with the \(m\)-th eigenvalue \((\lambda_{m}) \) of \(A \). We rigorously proved that \(\xi_{ij} \) is an Euclidean distance (see SI Appendix, section S2 for a proof). Conceptually, \(\xi_{ij} \) is a measure of the network resistance to a flow between \(i \) and \(j \). Recently (43), it was proven that this communicability distance—and every spherical Euclidean distance—is the effective resistance between nodes in a network with given edge weights.

### Network Geometrization and Resistive Shortest Paths

Since \(\xi_{ij} \) is an Euclidean distance and particles motion is confined to the network edges, we can proceed to the geometrization of the network (44, 45). To this aim, we first transform every edge of the network into a compact 1-dimensional manifold. That is, for an edge \(e = \{i, j\} \) we consider the boundary of the manifold to be \(\partial e = i \cup j \). Then, each edge \(e \) inherits a metric \(g_{e} \) such that \((e, g_{e}) \) is isometric to a finite interval \([0, L(e)] \) of the real line with the standard metric, where the length \(L(e) \) is given by the communicability distance of the corresponding edge, i.e., \(L(e) \equiv \xi_{e} = \xi_{ij} \). Finally, the distance metric on the edges is extended to the full graph via infima of lengths of curves in the geometrization of \(G \), such that the graph becomes a metrically complete length space (45).

Equipped with this geometrization, we can now define two different types of lengths for any given path \(p(s \rightarrow t) = (s, \ldots, t) \) connecting nodes \(s \) and \(t \) in the network. First, the topological length \(L_{p}(s \rightarrow t) \) of this path is just the number of edges in it. Among all paths \(p(s \rightarrow t) \) connecting \(s \) and \(t \), the one with the minimum length is denoted the shortest path \(SP(s, t) \) as

\[
SP(s, t) = \underset{p(s \rightarrow t)}{\text{argmin}} |L_{p}(s \rightarrow t)|.
\]

Observe that Eq. 4 can have more than one solution, specially for large networks SI Appendix, S4.

Second, and based on the geometrization induced by the communicability resistance above, we also define an effective length \(L_{p}(s \rightarrow t) \) by summing the induced length of each of the links involved in \(p(s \rightarrow t) \):

\[
L_{p}(s \rightarrow t) = \sum_{(i,j) \in p(s \rightarrow t)} \xi_{ij}.
\]
At odds with \( \ell_{p(i \rightarrow j)} \), which blindly assigns the same length (unity) to every edge of the network, \( L_{p(i \rightarrow j)} \) takes into account the topological neighborhoods of each of the nodes in the path and the associated likelihood that the particle might diffuse out of the path, accordingly. Likewise, it penalizes paths for which particles take naturally more time to travel due to the structure of the network in which the path is embedded. The specific path connecting \( s \) and \( t \) that minimizes this effective length is denoted the Shortest Resistive Path SRP\((s, t)\), defined as:

\[
\text{SRP}(s, t) = \arg\min_{p(i \rightarrow j)} [L_{p(i \rightarrow j)}].
\]  

We are now ready to quantify i) the emergence of potential bypasses—i.e., the proliferation of non-SP between any two nodes—and ii) decide in a principled way when this path redundancy becomes relevant to the network function—something that, we advance, will happen when SRPs start to differ from SPs.

**Communicability Entropy.** To address the first question above, we now quantify, both microscopically and then at the network level, the degree by which, as disorder increases, new routes between edges become available. To this aim, let us return to the WS and BA models that we have considered before. As we have discussed, both the rewiring process and the BA mechanism create degree heterogeneities that intuitively make some a priori “inefficient” paths—e.g., long ones—to scale up in a predefined efficiency ranking (that would indeed be the case of path \( P_2 \) connecting nodes \( i \) and \( j \) in Fig. 1). Now, in practice, both WS and BA mechanisms can have heterogeneous effects on this reranking, depending on the particularities of the starting and ending nodes \( i \) and \( j \) (see SI Appendix, section S3 for an in-depth microscopic analysis on the effect of these local mechanisms on \( \xi_j \) and \( L_{p(i \rightarrow j)} \)).

We first start by quantifying how these mechanisms generate a richness of possible trajectories connecting any pair of nodes \( i \) and \( j \). The probability that a randomly intercepted trajectory indeed corresponds to one connecting \( i \) and \( j \) is

\[
q_{ij} = \frac{G_{ij}}{\sum_k G_{ik}}.
\]  

Then, the heterogeneity in the different number of choices for the trajectory of a particle, i.e., the trajectory richness of the network is given by the entropy

\[
S(q) = -\frac{1}{2} \sum_{i < j} q_{ij} \ln q_{ij},
\]  

that we call the communicability entropy. From an information-theoretic perspective, this entropy is a measure of the ignorance we have on who is the sender node and receiver node, when intercepting a message navigating the network. Since \( 0 \leq S(q) \leq \ln(n(n - 1)/2) \), the upper bound only reached when the set of probabilities \( q \) are uniform, we define a normalized version \( \hat{S}(q) := S(q)/\ln(n(n - 1)/2) \).

Let us now analyze how \( \hat{S}(q) \) behaves in our two reference frameworks. Intuitively, for a fixed mean degree \( \langle k \rangle \), \( \hat{S}(q) \) will increase in the WS model as \( p \) increases since rewiring increases trajectory richness. Likewise, in a BA model, one can vary the network’s mean degree: For very small \( k \), the resulting BA network is almost tree-like, with no potential bypasses and thus low trajectory richness, whereas when we allow \( \langle k \rangle \) to increase, additional routes are formed, thus increasing the trajectory richness; hence, \( \hat{S}(q) \) should also increase. Fig. 2A and B (red axis) confirm our intuitive arguments. In particular, in Fig. 2A, we observe that entropy grows rather quickly in a WS model for small rewiring probability \( 0 < p < 0.4 \), reaching a steady maximum afterward. The impact of rewiring is notably stronger for small \( p \), and this effect is emphasized further for SW networks of increasing \( \langle k \rangle \). This behavior is easy to understand: In the small \( p \) region, there are few shortcuts, and each new one makes a difference. On the contrary, for large values of \( p \), the entropy saturates very quickly to \( \hat{S}(q) \simeq 1 \), i.e., the addition of more shortcuts does not make much of a difference beyond a certain \( p \) (see below for further analysis on the influence of the average degree). Fig. 2B reveals a similar behavior of \( \hat{S}(q) \) for the BA model as the mean degree \( \langle k \rangle \) increases (within the sparse regime for the BA preferential attachment mechanism to hold, see below), reaching full trajectory richness very quickly after a sudden increase in the region of small \( \langle k \rangle \) values. In short, rewiring an ordered structure and increasing the link density of a heterogeneous network quickly (nonlinearly) boosts the trajectory richness and, thus, the amount of potential bypasses to any specific shortest path connecting any pair of nodes.

We now need to quantify when some of these new routes actually may become consolidated bypasses to shortest paths, like the situation illustrated in Fig. 1, where a particle traveling between two nodes \( i \) and \( j \) “might prefer” to use \( P_2 \), although being longer (in terms of number of edges to be traversed) than the shortest path \( P_1 \).

**Bypass Consolidation and Associated Navigability Gain.** To evaluate the impact of potential bypasses on the actual navigability, we use Eq. 5 and consider that, for any pair of nodes \( i \) and \( j \), the SRP between \( i \) and \( j \) is a consolidated bypass to the shortest path(s) if the effective length of the SRP is smaller than the effective length of the (potentially many) SPs (i.e., \( L_{SRP}(i,j) < L_{SP}(i,j) \) for all SPs connecting \( i \) and \( j \)). Interestingly, this criterion results to be equivalent to check that \( \ell_{SRP}(i,j) > \ell_{SP}(i,j) \) (see SI Appendix, S4 for details). Once bypass detection is done, we need to quantify its impact. A measure that quantifies the impact of bypasses on the network’s navigability is the topological length excess \( \varepsilon_{(i,j)} \)

\[
\varepsilon_{(i,j)} = \left(1 - \frac{\ell_{SRP}(i,j)}{\ell_{SP}(i,j)}\right) \cdot 100,
\]

which indicates that, for a particle traveling between two arbitrary nodes \( i \) and \( j \), choosing the consolidated bypass SRP over the SP, while beneficial according to the (hidden) network geometry, leads to an apparent excess of \( \varepsilon_{(i,j)}\% \) from the topological distance traveled via the shortest path. It turns out that Eq. 9 also quantifies the effective distance per link and the resulting gain of using SRP over SP (see SI Appendix, S4 for a full derivation of these metrics and their interpretation). To extract a global metric for the whole network, we just average \( \varepsilon_{(i,j)} \) over all pairs of nodes to define the network navigability gain:

\[
\varepsilon = \frac{2}{N(N-1)} \sum_{i < j} \varepsilon_{(i,j)}.
\]  

An illustration of these metrics in a toy network is given in SI Appendix, S5. Observe that \( \varepsilon \) quantifies an improvement of
a function (network navigability) as a result of an innovation (consolidation of bypasses) and is therefore a quantitative proxy of structural deepening.

We can now quantify bypass consolidation and its associated navigability gain on relation to both WS and BA mechanisms. When we apply this formalism to the evolving SW network we obtain the results illustrated in Fig. 2A. Left axis. We observe that the navigability gain factor $\epsilon$ exhibits a clear nonmonotonic shape as a function of the rewiring probability $p$. In fact, our measure detects a maximum for $p \approx 0.15$ at which, on average, traveling through the SRP is much more favorable than doing so through the SP. We call this probability the “good navigational point” (GNP) of the network, $p_{\text{GNP}}$. It is interesting to observe that $p_{\text{GNP}}$ is a precise location inside the so-called small-world regime, which is independent of the network mean degree $\langle k \rangle$. Anecdotally, this value appears close to the saturating point of spectral spacing in SW networks (47).

Now, note that the SW mechanism consolidates bypasses out of a regular-to-random transition, so comparatively speaking the values of $\epsilon$ should be typically higher in more structured networks—e.g., in networks with fat-tailed degree distributions like the BA model—where the presence of hubs makes the existence of bypasses even more necessary. This hypothesis is confirmed in Fig. 2B. Right axis, in which $\epsilon$ reaches roughly values one order of magnitude larger in the BA model than those found in a comparable WS model. In this case, we observe again nonmonotonic behavior of $\epsilon$ with $\langle k \rangle$, displaying a maximum close to $\langle k \rangle \approx 11$, i.e., the BA model also has a good navigational point when mean degree is $\langle k \rangle_{\text{GNP}} \approx 11$, where bypassing shortest paths that include hubs is maximally relevant.

To further analyze the impact of bypasses, we now compare the values of $\epsilon$ obtained in a BA model ($n = 250$ nodes and mean degree $\langle k \rangle$) against i) those obtained for an Erdős-Rényi (ER) graph with the same $n$ and $\langle k \rangle$—this latter being a model with the same number of edges but with a homogeneous (Poisson) degree distribution and thus virtually lacking any hubs—and ii) those of a WS model with the same $n$ and same $\langle k \rangle$, and poised at $p = p_{\text{GNP}}$. Results are shown in Fig. 2C and certify that, in the sparse regime $(\langle k \rangle < 35)$, $\epsilon$ is substantially larger in BA than both ER and SW, i.e., the gain supported by bypasses is considerably more important in heterogeneous networks, as expected (48). When comparing the behavior of $\epsilon$ in ER vs SW networks (both in principle lacking substantial hubs), we observe an interesting effect: For a range of small mean degrees $\langle k \rangle < 11$, SW networks benefit more from bypasses than ER ones. The opposite is true for an intermediate $11 < \langle k \rangle < 30$, and the effect is again changed for very large mean degrees $\langle k \rangle > 30$. This nontrivial behavior can be explained by comparing the degree distributions of both ER networks and SW networks at $p_{\text{GNP}}$ and by realizing the (often overlooked) fact that the degree distribution (in particular, the skewness and kurtosis) of an SW network poised at a fixed $p$ undergoes different shapes as $\langle k \rangle$ increases (see SI Appendix, section S8 for details). Incidentally, this can also explain why $\hat{S}(q)$ initial increase in SW networks is sharper for larger $\langle k \rangle$ (SI Appendix).

In summary, the effect of bypasses is maximized for SW networks at the good navigational point $p_{\text{GNP}} \approx 0.15$, and within that point, this effect appears to be monotonically boosted when these SW networks increase their degree heterogeneity, i.e., increasing $\langle k \rangle$. ER networks have bypassing properties as long as they show degree heterogeneities, and to a small extent (Poisson distribution), this is the case. Such effect is then maximal around $\langle k \rangle \approx 20$ (the fact that bypasses have a nonmonotonic effect also within ER networks can again be explained in terms of the
skewness degree distribution; see SI Appendix). Finally, in BA networks, bypassing effects are substantially larger due to higher degree heterogeneities, as expected.

To close this analysis, one can ask about the theoretical upper bound on $c$. Heuristically, the effect of bypasses would be fully maximized in a situation where we add to a given (connected) network a new node that is linked to every other node. Such a new node would be a "superhub" that makes the network have shortest paths of length $≤ 2$ for all pairs of nodes. In this extreme situation, many of the shortest paths will be systematically bypassed and $c$ would explode (see SI Appendix, S9 for details). Now, is this just a theoretical scenario? It turns out that this situation can take place in an extreme version of the BA model in a finite graph, where $(k)$ is large enough (compared to the initial seed) so that new nodes entering systematically connect to a large portion of the network, leading to so-called ultrashort graphs (49). This explosion is reported in Fig. 2F. Evidently, in this case, there is no preferential attachment anymore, so in some sense, the rationale behind the BA model breaks down in this dense regime.$^3$

Effect on Dynamics. As already anticipated, our theory is purely structural and therefore dynamically agnostic and speaks of the effect of network geometrization on the formation of shortest paths in the geometrized network—the SRPs—which are different from the shortest paths of the original, ungeometrized network. Our contention is that these emergent bypasses have an effect on the network’s navigability, and here, we provide an initial validation of this hypothesis by considering source–destination random walk trajectories navigating a network. Each of these random walk trajectories is then classified as SRP-like or SP-like, depending on the specific sequence of nodes the walker is visiting (see SI Appendix, S6 for details). One can subsequently compare the SRP class and the SP class by computing a number of quantities, such as the average hitting time in each class, or the excess time (i.e., for each class, how much more time than the time spent by a ballistic walker it takes to reach the destination), which yield dynamical proxies for the effective length or the associated navigability gain defined above (SI Appendix, S6). Results for both a synthetic small network and for a large real network (a coactivation brain network, see below) are shown in Fig. 3 and confirm our hypothesis that particles are more prone to "get lost" (and thus spend a significantly longer time) navigating through a SP-like path compared to an SRP-like one. In other words, the presence of SRPs enhances navigability for diffusion-like dynamics (additional details and analysis are provided in SI Appendix, S6). At the same time, this finding further confirms that bypasses induce structural deepening by increasing the efficiency of network navigability.

We have also made some preliminary progress on analyzing how bypasses impact other network functions by considering two additional dynamical processes running on a network: synchronization and epidemic spreading. Results (fully detailed in SI Appendix, section S7) suggest that the prototypical dynamical fingerprints in each case (i.e., eigenratio of the Laplacian matrix for synchronization and epidemic threshold for epidemic spreading) are affected by bypass consolidation, and, in particular, qualitative dynamical changes occur in both types of dynamics close to $p_{GNP}$.

Empirical Networks. To round off, we have considered a total of 177 empirical networks of different nature, including social (4 collaboration networks of different nature, 3 termite mounds), biological (Human brain—70 anatomical, 70 functional at resting-state, one functional at task-driven (extracted and averaged from a meta-analysis of 1,600 works)—, neural network of C. elegans, a protein–protein interaction, a transcription yeast, 15 food webs), and technological ones (air transportation, Internet, 3 electronic circuits, power grid, 5 software networks), see SI Appendix, section S11.1 for details and full references. Results on several metrics are summarized in Table 1, and some scatter plots are visualized in Fig. 4.

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$^3$In the dense regime $(k) > 35$, the calculations of $c$ need to be taken with caution as numerical rounding effects might become important when computing $\exp(A)$. 

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Fig. 3. (A) Toy network with a concrete source–destination pair $(p, q)$, which can be navigated via paths $P_1$ and $P_2$. (B) Computation of the different metrics certifies that $P_2$ is a bypass of the shortest path $P_1$, and the navigability gain associated with this pair is 25% (a lower bound of the actual gain; see SI Appendix, S5). Random walk trajectories starting at $p$ and eventually hitting $q$ can be classified as $P_1$-like or $P_2$-like, depending on their specific trajectories (SI Appendix, S6). The hitting time and excess time of the $P_1$ class is larger than for $P_2$, meaning that random-walk navigability is enhanced in the $P_2$ (that is, the SRP) class. The excess time saving is a diffusion-proxy of the navigability gain (SI Appendix, S6). (C) Coactivation brain network. (D) Excess time saving for several source–destination pairs in the brain network, finding that SRP enhances navigability in all cases.
Table 1. Summary of metrics for empirical networks, depicting the communicability entropy $\hat{S}(q)$, the navigability gain $\epsilon$, the optimal rewiring probability $p^*$, and the navigability ratio $\epsilon/\epsilon_{BA}$ (see the text) across 177 different empirical networks (for many of them, we offer averages; see SI Appendix, S10 for details), where: Human brain (anatomical) provides the averaged results across 70 anatomical networks (using the same parcellation), Human brain (functional, resting-state) provides the averaged results across 70 functional networks (using the same parcellation as the anatomical networks), Software provides the averaged results across the networks MySQL, XMMS, Abi, Digital, and VTK; Food webs is the average of 15 food webs (see SI Appendix, S11.3 for disaggregation); Electronic circuits is the average of three electronic circuits; Termite mounds is the average of three termite mounds

| Network                                  | $\hat{S}(q)$ | $\epsilon$ (%) | $p^*$ | $\epsilon/\epsilon_{BA}$ |
|------------------------------------------|--------------|----------------|------|--------------------------|
| Human brain (functional, task-driven)    | 0.9234       | 51.71          | 0.30 | 0.78                     |
| Collaboration CoGe                      | 0.7776       | 41.50          | 0.21 | 0.93                     |
| Collaboration QcGr                       | 0.4598       | 38.39          | 0.15 | 0.79                     |
| Human brain (anatomical)                | 0.925 ± 0.022| 36.21 ± 1.52   | 0.23 ± 0.01 | 0.86 ± 0.04 |
| C. elegans neurons                      | 0.9312       | 31.69          | 0.34 | 0.87                     |
| USA airports 1997                       | 0.8501       | 28.60          | 0.27 | 0.76                     |
| Internet AS 1997                        | 0.8891       | 25.49          | ∼1   | 0.52                     |
| Yeast PPI                               | 0.8344       | 25.50          | 0.24 | 0.55                     |
| Drugs users                             | 0.7794       | 21.18          | 0.10 | 0.57                     |
| Software                                | 0.8308 ± 0.0263 | 21.11 ± 12.10 | ∼1$^*$ | 0.58                     |
| Human brain (functional, resting-state) | 0.758 ± 0.054 | 20.81 ± 1.42   | 0.17 ± 0.02 | 0.49 ± 0.05 |
| Roget thesaurus                         | 0.9215       | 19.18          | 0.35 | 0.43                     |
| Transcription yeast                     | 0.8128       | 12.26          | ∼1   | 0.38                     |
| Food webs                               | 0.9498 ± 0.0208 | 9.94 ± 7.13   | ∗∗    | 0.64***                  |
| electronic circuits                     | 0.8202 ± 0.0260 | 3.456 ± 2.561 | ∼1   | 0.12                     |
| Termite mounds                          | 0.5707 ± 0.0331 | 3.100 ± 2.12  | ∼1   | 0.11                     |
| Power grid                              | 0.6348       | 2.61           | ∼1   | 0.05                     |

$^*$Except MySQL which has $p^* \approx 0.29$. $^*$Three types of behaviors: i) $p^* \approx 1$ for 8 food webs; ii) $0.43 \leq p^* \leq 0.45$ for El Verde, Shelf, Ythan1, and Ythan2; iii) $0.03 \leq p^* \leq 0.14$ for Bridge Brooks, Coachella, and Little Rock. ***See SI Appendix, section S11 for disaggregated data and additional details.

The first two columns of this table report the normalized communicability entropy $\hat{S}(q)$ and navigability gain $\epsilon$. Interestingly, all of them appear to be entropic enough for potential bypasses to have been formed, as values of $\hat{S}(q)$ are in the region where our analysis on synthetic models show consolidated bypasses.$^5$ We indeed find that essentially all real-world networks harbor consolidated bypasses ($\epsilon > 0$), albeit with different impacts, what allows us to rank them accordingly. At the top of the ranking, the net gain induced by consolidated bypasses reaches over 50% for the (task-driven) functional brain network, followed by many other self-organized networks (collaboration networks, C. elegans, etc.). It is interesting to see that the navigability gain substantially drops for functional brain networks when passing from task-driven activation to resting state. This might be suggesting the possibility that navigability gain in functional brain networks might be task-related, something that deserves further research. Our finding that the navigability gain of anatomic networks is in between those of task-driven and resting-state functional networks is reasonable. On one hand, resting-state function in adults is usually thought to be restricted to a brain module. On the other hand, the specific task-driven network that we analyze here is the outcome of a meta-analysis of over 1,600 works considering different tasks—and thus, in principle, the result of multiple brain modules. These hypotheses await confirmation, and, in any case, further research is needed to elucidate the relation of the topology-induced bypasses studied here with specific cognitive aspects.

At the bottom of the list in Table 1, we find some designed networks, such as electronic circuits or the power grid, the latter having only a discrete 2.6% navigability gain. This can be indicative that the power grid, while having hubs to some extent (4, 50), has not evolved according to mechanisms such as WS or BA, is not self-organized, and, as a consequence, does not hold the necessary preemptive structural bypasses to avoid systemic failures, as we have seen during blackouts (51). Note at this point that the navigability gain $\epsilon$ does not trivially correlate with more standard network metrics, such as network density (linear regression of the scatter plot offers a $R^2 = 0.12$), mean degree ($R^2 = 0.09$), average path length ($R^2 = 0.01$), or average clustering ($R^2 = 0.006$), see SI Appendix, S11.4 for details.

Now, to which extent the observed bypasses are indeed of the SW-type (i.e., bypassing shortest paths consistently generated via a WS-like mechanism), and in such case, how close empirical networks are to their theoretical good navigational point? While this question is difficult to answer, the metric $p^*$ reported in the third column of Table 1 (Fig. 4) provides a first step. Operationally, for a given empirical network $G$ with $n$ nodes and mean degree $\langle k \rangle$, we estimate the closest purely SW-generated network $G^{\ast}(p)$ (with the same $\langle n, \langle k \rangle \rangle$). This is achieved by minimizing the spectral dissimilarity distance

$$D(G, G^{\ast}) = \sum_{j=1}^{\sqrt{n}} (\lambda_j(G) - \lambda_j(G^{\ast}))^2,$$

where $\lambda_j(G)$ is the $j$-th eigenvalue of the adjacency matrix of network $G$ and minimization is over $p$, i.e., $p^* = \text{argmin}_p D(G, G^{\ast}(p))$. This metric indicates that networks can be typically clustered in two types: one (which includes all human brain networks, the neural
network of *C. elegans*, the protein–protein interaction network, collaboration networks, Roget network, and the US air network) where $D(G, G'(p))$ has a nonmonotonic shape with a minimum $p^* \in [0.15, 0.35]$—i.e., close but not exactly at the good navigational point—and another cluster of networks (including electronic circuits, Internet AS97, software networks of termite mounds) where $D(G, G'(p))$ is monotonically decreasing and thus $p^* = 1$ (see SI Appendix, S11.2 for further details and analysis). The former class thus tends to harbor bypasses of the SW-type—avoiding shortcuts—and its network formation includes at least partially some SW ingredient while the second one tends to have a structure which cannot be well explained only by SW mechanisms (this does not mean, however, that such network is random). Incidentally, no clear function-related clustering emerges.

The fourth column of Table 1 finally depicts $\epsilon/\epsilon_{BA}$—where $\epsilon_{BA}$ is the navigability gain of a BA network with the same number of nodes $n$ and mean degree $\langle k \rangle$ of the real network—and quantities whether the observed network bypasses are effectively bypassing hubs. This metric highlights two different groups of networks. The first group is characterized by the relevance of hubs (i.e., $\epsilon/\epsilon_{BA} \sim 1$). In the second group of networks the hubs are not necessarily playing a fundamental role in terms of the consolidated bypasses. That is, either the bypasses are not necessarily skipping hubs, or such networks have not been designed to harbor bypasses. In closing, they do not abide to a BA-like mechanism, so $\epsilon/\epsilon_{BA}$ is closer to zero (see SI Appendix, S11.3 for further discussion).

**Discussion**

The journey of network complexification is supported by basic mechanisms including the celebrated WS and BA, among others. As the network evolves accordingly, we have shown that it naturally increases its communicability entropy $S(q)$ and, in so doing, it allows for new navigational routes to be built, entropically providing bypass “candidates” to the network. Our theory allows to detect when some of these new routes consolidate their bypassing property by subsequently getting to be more favorable than the corresponding shortest paths connecting the same pairs of nodes, and we show that consolidation takes place in both WS and BA models. Interestingly, we find that the role of bypasses is maximized in a small parameter region—which we call the network’s good navigational point—located in a point inside the Small-World regime and for a specific mean degree in the BA model. These findings suggest that the navigation gain offered by the network bypasses is indeed reflecting a form of structural deepening, thus putting the onset of complexity in networks into a solid quantitative footing.

We have certified that bypasses induce clear navigation gain for particles undergoing diffusion-like dynamics and also play an effect on other network functions, including harboring synchronization and epidemic spreading. We have then shown that many empirical networks considered complex, including brain networks, indeed have good navigational point properties, while those that are not cataloged as self-organized but have been designed tend to not include bypasses in their design, with well-known unfortunate consequences (51).

In hindsight, our results could provide a theoretical and mechanistic support for the role of bypasses in, e.g., physiological systems—where plasticity is of utmost importance (52). First, network bypasses naturally relate to the existence of the so-called “collateral circulation”: a system of specialized endogenous bypass vessels present in most tissues providing protection against ischemic injury caused by ischemic stroke, coronary atherosclerosis, peripheral artery disease, and other conditions and diseases (53). Second, in brain networks, there is nowadays enough observational evidence which supports that these are SW in the Watts–Strogatz sense (54) and possess hubs which create skewness of their degree distributions (55). At the same time, recent experiments (56) suggest that propagating signals in the brain using hubs as part of the navigation path might have a large energetic cost, triggering research on nongeodesic information propagation (56–58). Our work indeed supports the concept of nongeodesic navigability (via network bypasses) and reconciles this with the reported network structure. In this context, note that (59) proposed considering networks of neurons as evolving and growing connections in a distributed fashion (via mechanisms different that SW or BA) such that shortest path minimization and robustness maximization (which in general implied to avoid the creation of hubs) was performed at the same time. Note, however, that brain navigation is not likely to occur always geodesically (56–58) (this also would imply that individual neurons perform global optimization and have access to the whole brain structure). The logical conclusion is that the seminal findings in ref. 59 imply that the creation of shortest paths should be accompanied by the proliferation of additional structure that plays a role of structural deepening, in good agreement with our theory.

Third, in another recent work (60), it has been shown that brain function appears to be robust against damage by readapting and repurposing nondamaged links, something that can be interpreted to the brain’s ability to recompute SRPs and thus...
rerank bypasses after network damage. All in all, elucidating the impact of our findings in the context of neuroscience is an exciting avenue for future work.

An aspect not explored in this paper but also of major interest is the implications of our theory to congestion or jamming phenomena in networks and to which extent our proposed measures of topological length excess and navigability gain could anticipate congestion in, e.g., transportation and urban systems. First, we should disclose that conceptually similar problems have been theorized in the mathematics literature, where some authors have studied the so-called “resistance distance” in networks (61)—where some unit resistances are placed at every edge in a network—in the context of congestion (62, 63). Now, while an interesting mathematical concept, this latter distance analytically converges, for large graphs and in high dimensions, to an expression that does not take into account the structure of the graph (64) (i.e., it only depends on the degrees of the source and destination nodes) and thus unfortunately turns useless in real-world scenarios6. Second, recent empirical evidence in urban science indeed suggests that, at rush hours, in different cities worldwide, paths which can be identified as SRPs are supporting more traffic than SPs (65), i.e., they become systematically preferred routes. This constitutes preliminary support in favor of the relevance of SRPs for navigation strategies in networks subject to jamming, and further research is deserved. For instance, we speculate that this strategy can be further refined by, instead of systematically selecting only the SRP as the preferred route, ranking each of the paths connecting any two locations via the computation of its associated topological length excess and rerouting traffic accordingly when needed.

Other important open questions for further research include understanding the role played by network bypasses and their relation to structural deepening in other mechanistic growth models (e.g., assortative/disassortative mixing, triadic closure, etc.), and the extension of our theory to weighted (see a preliminary discussion on this topic in SI Appendix, S10), temporal and higher-order networks (66, 67).

Finally, while network bypass emergence appears to be contingent on the growth mechanism—and thus appears to be a by-product of it—bypass consolidation (structural deepening) is the effect which probably makes those growth mechanisms to be sustainable in the first place. Simply put, we argue, bypasses sustain complexity.

ACKNOWLEDGMENTS. We thank Olaf Sporns, Yasser Aleman-Gomez, and Yasser Iturria-Medina for assistance with the data and analysis of brain networks, Adrian Garcia-Candel for help formatting Fig. 1, and referees for insightful comments. E.E. acknowledges funding from project OLGSA (PID2019-106356GB-I00) funded by Spanish Ministry of Science and Innovation. J.G. acknowledges financial support from the Departamento de Industria e Innovación del Gobierno de Aragón y Fondo Social Europeo (FENOL group grant E36-23R) and from project PID2020-113582GB-I00 funded by the Spanish Ministry of Science and Innovation. L.L. acknowledges funding from project DYNDEEP (EUR2021-122007) and project MISLAND (PID2020-114324GB-C22), both projects funded by Spanish Ministry of Science and Innovation. This work has also been partially supported by the Maria de Maetzu project CEX2021-001164-M funded by the MCIN/AEI/10.13039/501100011033.
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