Quantum line bundles on noncommutative sphere

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Abstract

Noncommutative (NC) sphere is introduced as a quotient of the enveloping algebra of the Lie algebra $su(2)$. Following [GS] and using the Cayley-Hamilton identities we introduce projective modules which are analogues of line bundles on the usual sphere (we call them quantum line bundles) and define a multiplicative structure in their family. Also, we compute a pairing between quantum line bundles and finite dimensional representations of the NC sphere in the spirit of the NC index theorem. A new approach to constructing the differential calculus on a NC sphere is suggested. The approach makes use of the projective modules in question and gives rise to a NC de Rham complex being a deformation of the classical one.

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1 Introduction

One of basic notions of the usual (commutative) geometry is that of the vector bundle on a variety. As was shown in [Se] the category of vector bundles over a regular affine algebraic variety $X$ is equivalent to the category of finitely generated projective modules over the algebra $\mathcal{A} = \mathbb{K}(X)$ which is the coordinate ring of the given variety $X$ (a similar statement for smooth compact varieties was shown in [Sw]). Hereafter $\mathbb{K}$ stands for the basic field (always $\mathbb{C}$ or $\mathbb{R}$).

The language of projective modules is perfectly adapted to the case of a noncommutative (NC) algebra $\mathcal{A}$. Any such (say, right) $\mathcal{A}$-module can be identified with an idempotent $e \in M_n(\mathcal{A})$ for some natural $n$. These idempotents play the key role in all approaches to NC geometry, in particular, in a NC version of the index formula of A. Connes ([C], [L]).

The problem of constructing projective modules over physically meaningful algebras is of great interest. The $\mathbb{C}^*$-algebras (apart from commutative ones) are the mostly studied from this viewpoint. Besides, there are known very few examples. As an example let us evoke the paper [K] where projective modules over NC tori are studied (also cf. [KS] and the references therein). In the recent time a number of papers have appeared dealing with some algebras (less standard than NC tori and those arising from the Moyal product) for which certain projective modules are constructed by hand (cf. for example [DL], [LM] and the references therein).

Nevertheless, there exists a natural method suggested in [GS] of constructing projective modules over NC analogues of $\mathbb{K}(\mathcal{O})$, where $\mathcal{O}$ is a generic $SU(n)$-orbit in $su(n)^*$ including its "q-analogues".

1 An orbit is called generic if it contains a diagonal matrix with pairwise distinct eigenvalues.
arising from the so-called reflection equation (RE) algebra. This method is based on the Cayley-
Hamilton (CH) identity for matrices with entries belonging to the NC algebras in question. (However,
it seems very plausible that other interesting examples of ”NC varieties” can be covered by this
method. In particular, by making use of the CH identity for super-matrices, cf. [KT], it is possible to
generalize our approach to certain super-varieties).

The idea of the method consists in the following. Consider a matrix \( L = \begin{bmatrix} l_{ij} \end{bmatrix} \) subject to the RE
related to a Hecke symmetry (cf. [GPS]). Then it satisfies a polynomial identity

\[
L^p + \sum_{i=1}^{p} a_i(L)L^{p-i} = 0 \quad (1.1)
\]

where coefficients \( a_i(L) \) belong to the center of the RE algebra generated by \( l_{ij} \).

Passing to a specific limit in RE algebra we get a version of the CH identity for the matrix whose
entries \( l_{ij} \) commute as follows:

\[
[l_{ij}, l_{kl}] = \hbar (\delta^k_i \delta^j_l - \delta^j_i \delta^k_l), \quad (1.2)
\]
i.e. \( l_{ij} \) generate the Lie algebra \( gl(n)_\hbar \) where \( g_n \) stands for a Lie algebra whose Lie bracket equals \( h[ , ] \),
where \( [ , ] \) is the bracket of a given Lie algebra \( g \). Introducing the parameter \( \hbar \) allows us to consider
the enveloping algebras as deformations of commutative ones. (Let us note that this type of the CH
identity was known since 80’s, cf. [Go]).

Similar to the general case the coefficients of the corresponding CH identity belong to the center
\( Z[U(gl(n)_\hbar)] \) of the enveloping algebra \( U(gl(n)_\hbar) \). Therefore, by passing to a quotient\(^3\)

\[
U(gl(n))/\{z - \chi(z)\}
\]

where \( z \in Z[U(sl(n)_\hbar)] \) and

\[
\chi : Z[U(gl(n)_\hbar)] \to \mathbb{K}
\]
is a character we get a CH identity with numerical coefficients

\[
L^p + \sum_{i=1}^{p} \alpha_i L^{p-i} = 0, \quad \alpha_i = \chi(a_i(L)). \quad (1.3)
\]

Assuming the roots of this equation to be distinct pairwise we can assign an idempotent (or what is
the same one-sided projective module) to each root. For quasiclassical Hecke symmetries (see footnote
2) these projective modules are deformations of line bundles on the corresponding classical variety.
For this reason we call them quantum line bundles (q.l.b.).

In this paper we constrain ourselves to the case arising from the Lie algebra \( gl(2) \). Namely, we
describe a family of projective modules over the algebra

\[
A_\hbar = U(sl(2)_\hbar)/\{\Delta - \alpha\}, \quad (1.4)
\]

where \( \Delta \) stands for the Casimir element in the algebra \( U(sl(2)_\hbar) \). We consider this algebra as a NC
counterpart of a hyperboloid.

\(^2\)A Hecke symmetry is a Hecke type solution of the quantum Yang-Baxter equation. There exist different types of
Hecke symmetries: quasiclassical ones being deformations of the classical flip and non-quasiclassical ones (a big family
of such Hecke symmetries was introduced in [G]). Let us note that a version of the CH identity exists for any of them
independently on the type.

\(^3\)Hereafter \( \{X\} \) stands for the ideal generated by a set \( X \) in the algebra in question.

\(^4\)Note that in fact we deal with the Lie algebra \( sl(2)_\hbar \) instead of \( gl(2)_\hbar \) since the trace of the matrix \( L \) is always
assumed to vanish. The studies of such quotients were initiated by J.Dixmier (cf. [H] and the references therein). In
certain papers compact form of this algebra is called fuzzy sphere.
In order to get a NC counterpart of the sphere we should pass to the compact form of the algebra in question. However, it does not affect the CH identity since it is indifferent to a concrete form (compact or not) of the algebra.

To describe our method in more detail we begin with the classical (commutative) case. Put the matrix

$$L = \begin{pmatrix}
ix & -iy + z \\
-iy - z & -ix
\end{pmatrix}$$

in correspondence to a point \((x, y, z) \in S^2\). This matrix satisfies the CH identity (1.3) where \(p = 2\), \(\alpha_1 = 0\), \(\alpha_2 = x^2 + y^2 + z^2 = \text{const} \neq 0\).

Let \(\lambda_1\) and \(\lambda_2 = -\lambda_1\) be the roots of equation (1.3). To each point \((x, y, z) \in S^2\) we assign the eigenspace of the above matrix \(L\) corresponding to the eigenvalue \(\lambda_l, l = 1, 2\). Thus, we come to a line bundle \(E_l\) which will be called basic.

Various tensor products of the basic bundles \(E_l, l = 1, 2\), give rise to a family of derived or higher line bundles

$$E^{k_1,k_2} = E^{k_1}_1 \otimes E^{k_2}_2, \quad k_1, k_2 = 0, 1, ...$$

Note, that certain line bundles of this family are isomorphic to each other. In particular, we have \(E_1 \otimes E_2 = E^{0,0}\) where \(E^{0,0}\) stands for the trivial line bundle. In general, the line bundles

$$E^{k_1,k_2} \quad \text{and} \quad E^{k_1+l,k_2+l}, \quad l = 1, 2, ...$$

are isomorphic to each other. So, any line bundle is isomorphic either to \(E^k_1\) or to \(E^k_2\) for some \(k = 0, 1, ...\) (we assume that \(E^0_1 = E^0_2 = E^{0,0}\)). Finally, we conclude that the Picard group \(\text{Pic}(S^2)\) of the sphere (which is the set of classes of isomorphic line bundles equipped with the tensor product) is nothing but \(\mathbb{Z}\) since any line bundle can be represented as \(E^k_1\) with a proper \(k \in \mathbb{Z}\), where we put \(E^k_1 = E^{\otimes k}_1\) for \(k > 0\) and \(E^k_2 = E^{\otimes (-k)}_2\) for \(k < 0\).

It is worth emphasizing that we deal with an algebraic setting: the sphere and total spaces of all bundles in question are treated as real or complex affine algebraic varieties (depending on the basic field \(k\)).

Now, let us pass to the NC case. Using the above NC version of the CH identity one can define NC analogues \(E_l(h), l = 1, 2\), of the line bundles \(E_l\). We will call them basic q.l.b. (they are defined in section 2).

Unfortunately, for a NC algebra \(\mathcal{A}\) any tensor product of two or more one-sided (say, right) \(\mathcal{A}\)-modules is not well-defined. So, the construction of derived line bundles cannot be generalized to a NC case in a straightforward way. Nevertheless, using the CH identities for some ”extensions” \(L(k), k = 2, 3, ...\) of the matrix \(L\) to higher spins (in the sequel \(k = 2\times\text{spin}\)) we directly construct NC counterparts \(E^{k_1,k_2}(h)\) of the above derived line bundles. Thus, for any \(k = 2, 3, ...\) (with \(k_1 + k_2 = k\)) we have \(k + 1\) derived q.l.b.

In section 3 we explicitly calculate the CH identity for the matrix \(L(2)\) and state that such an identity exists for any matrix \(L(k), k > 2\).

However, as usual we are interested in projective modules (in particular, q.l.b.) modulo natural isomorphisms. Thus, we show that the q.l.b. \(E^{1,1}(h)\) is isomorphic to the trivial one \(E^{0,0}(h)\) which is nothing but the algebra \(\mathcal{A}_h\) itself. We also conjecture that the q.l.b.

$$E^{k_1,k_2}(h) \quad \text{and} \quad E^{k_1+l,k_2+l}(h)$$

are isomorphic to each other. If it is so, any q.l.b. is isomorphic to \(E^{k,0}(h)\) or \(E^{0,k}(h)\) similar to the commutative case.

Then we define an associative product in the family of q.l.b. over the NC sphere in a natural way. By definition, the product of two (or more) q.l.b. over NC sphere are the NC analogue of the product of their classical counterparts. Otherwise stated, we set by definition

$$E^{k_1,k_2}(h) \cdot E^{l_1,l_2}(h) = E^{k_1+l_1,k_2+l_2}(h)$$
(in particular, we have \( E_1(h) \cdot E_2(h) = E^{1,1}_1(h) \) and in virtue of the above mentioned result this product is isomorphic to \( E^{0,0}_0(h) \)). The family of all modules \( E^{k_1,k_2}(h) \) equipped with this product is a semigroup. It is denoted \( \text{prePic}(A_h) \) and called prePicard. Assuming the above conjecture to be true we get the Picard group \( \text{Pic}(A_h) \) of the NC sphere (it is a group since any its element becomes invertible). All these notions are introduced in the section 4. Moreover, in this section we compute the pairing between q.l.b. in question and irreducible representations of the algebra \( A_h \) in the spirit of the NC index theorem.

In the last section we suggest a method of constructing a version of differential calculus on the NC sphere similar to that on the commutative sphere. The method makes use of the above projective modules instead of the Leibniz rule which is habitually employed in this area. In contrast with the usual approach giving rise to much bigger differential algebra than the classical one our approach leads to the de Rham complex whose terms are projective modules being deformations of the classical ones.

To complete the Introduction we emphasize that our approach is in principle applicable to NC analogue of any generic orbit in \( su(n)^* \) (and even to their ”q-analogues” arising from the RE algebra) but the calculation of the higher CH identities becomes much more difficult as the degree of the CH identity for the initial matrix becomes greater than 2. Actually, a paper \[GLS\] devoted to mentioned ”q-analogues” is in progress.

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2 Basic quantum line bundles on the NC sphere

Consider the Lie algebra \( gl(2)_h \) which is generated by the elements \( a, b, c \) and \( d \) satisfying the following commutation relations

\[
[a, b] = h b, \quad [a, c] = -h c, \quad [a, d] = 0, \quad [b, c] = h(a - d), \quad [b, d] = h b, \quad [c, d] = -h c.
\]

The nonzero numerical parameter \( h \) is introduced into the Lie brackets for the future convenience. This parameter can be evidently equated to one by the renormalization of the generators. In this case we come to the conventional Lie algebra \( gl(2) \).

Let us form a \( 2 \times 2 \) matrix \( L \) whose entries are the above generators

\[
L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]

It is a matter of straightforward checking that this matrix satisfies the following second order polynomial identity

\[
L^2 - (\text{tr} + h)L + (\Delta + h \text{tr}/2)\text{id} = 0
\]

where

\[
\text{tr} = \text{tr} L = a + d, \quad \Delta = ad - (bc + cb)/2.
\]

As \( h \to 0 \) we get the classical CH identity for a matrix with commutative entries.

In order to avoid any confusion we want to stress that we only deal with the enveloping algebra \( U(gl(2)_h) \) (and some its quotients) and we disregard its (restricted) dual object — the algebra of functions on the Lie group \( GL(2) \). Our immediate aim consists in constructing some ”derived” matrices with entries from \( U(gl(2)_h) \) satisfying some ”higher” CH identities. It will be done by some sort of coproduct applied to the matrix \( L \) and restricted onto the symmetric component. However, we do not use any coalgebraic (and hence any Hopf) structure of the algebra \( U(gl(2)_h) \) itself.

\[\text{Let us note that } K_0 \text{ of the NC sphere equipped only with the additive structure was calculated in } [1]. \text{ We are rather interested in quantum line bundles. Once they are defined other projective modules can be introduced as their direct sums.}\]
Remark 1 A version of the CH identity for matrices with entries from $U(gl(n))$ is known for a long time (cf. [GL]). However, traditionally one deals with the CH identity in a concrete representation of the algebra $gl(n)$ while we prefer to work with the above universal form of the CH identity. A way of obtaining the CH identity by means of the so-called Yangians was suggested in [NT]. In [G-T] another NC version of the CH identity was presented. The coefficients of the polynomial relation suggested there are diagonal (not scalar) matrices. But such a form of the CH identity is not suitable for our aims.

Since the elements $\text{tr}$ and $\triangle$ from (2.1) belong to the center $Z[U(gl(2)_h)]$ of the algebra $U(gl(2)_h)$ one can consider the quotient

$$A_h = U(gl(2)_h)/\{\text{tr}, \triangle - \alpha\}, \quad \alpha \in \mathbb{K}.$$ 

Taking in consideration the fact that the trace of the matrix $L$ vanishes we can also treat this algebra as quotient (1.4). In what follows the algebra $A_h$ will be called a NC variety (or more precisely, a NC hyperboloid).

Being restricted to the algebra $A_h$ the CH identity becomes a polynomial relation in $L$ with numerical coefficients

$$L^2 - hL + \alpha \text{id} = 0. \quad (2.2)$$

Denote $\lambda_1$ and $\lambda_2$ the roots of this equation that is

$$\lambda_1 = (h - \sqrt{h^2 - 4\alpha})/2, \quad \lambda_2 = (h + \sqrt{h^2 - 4\alpha})/2.$$ 

Let us suppose that $\lambda_1 \neq \lambda_2$. If $h = 0$ this condition means that the cone corresponding to the case $\triangle = \alpha = 0$ is forbidden. However, if $h \neq 0$ we have $h^2 - 4\alpha \neq 0$.

Abusing the language we say that $\lambda_1$ and $\lambda_2$ are eigenvalues of the matrix $L$. Of course, this does not mean the existence of an invertible matrix $A \in M_2(A_h)$ such that the matrix $A \cdot L \cdot A^{-1}$ becomes diagonal: $\text{diag}(\lambda_1, \lambda_2)$.

Remark 2 If an algebra $A$ is not a field, the invertibility of a matrix $A \in M_n(A)$ is an exceptional situation. As follows from the CH identity for the matrix $L$ it is invertible. However, it is not so even for small deformations of $L$.

It is easy to see that the matrices

$$e_{10} = (\lambda_2 \text{id} - L)/(\lambda_2 - \lambda_1), \quad e_{01} = (\lambda_1 \text{id} - L)/(\lambda_1 - \lambda_2) \in M_2(A_h) \quad (2.3)$$

are idempotents and $e_{10} \cdot e_{01} = 0$.

Denote $E_1(h)$ and $E_2(h)$ the projective modules (also called q.l.b.) corresponding respectively to the idempotents $e_{10}$ and $e_{01}$ in (2.3). Let us explicitly describe these modules.

Let $V(k)$ be the $k$-th homogeneous component $\text{Sym}^k(V)$ of the symmetric algebra $\text{Sym}(V)$ of the space $V$. Thus, $k = 2 \times \text{spin}$ and $\dim(V(k)) = k + 1$. Consider the tensor product $V \otimes A_h$. It is nothing but the free right $A_h$-module $A_h^{\otimes 2}$. We can imagine the matrix $L$ (as well as any polynomial in it) as an operator acting from $V$ to $V \otimes A_h$ which can be presented in a basis $(v_1, v_2)$ as follows:

$$(v_1, v_2) \mapsto (v_1 \otimes a + v_2 \otimes c, v_1 \otimes b + v_2 \otimes d).$$

Following [GS] we define the projective module $E_l(h)$, $l = 1, 2$ as quotient of $V \otimes A_h$ over its submodule generated by the elements

$$v_1 \otimes a + v_2 \otimes c - v_1 \otimes \lambda_l, \quad v_1 \otimes b + v_2 \otimes d - v_2 \otimes \lambda_l \quad (2.4)$$
Also, the module \( E_1(h) \) (resp. \( E_2(h) \)) can be identified with the image of the idempotent \( e_{10} \) (resp \( e_{01} \)) which consists of the elements

\[
(v_1 \otimes a + v_2 \otimes c - v_1 \otimes \lambda_m) f_1 + (v_1 \otimes b + v_2 \otimes d - v_2 \otimes \lambda_m) f_2, \quad \forall f_1, f_2 \in \mathcal{A}_h
\]

where \( m = 2 \) for \( E_1(h) \) and \( m = 1 \) for \( E_2(h) \). (In a similar way we can associate left projective modules to the idempotent in question.)

Now, consider the compact form of the NC variety in question. Changing the basis

\[
x = i(d - a)/2 = -ia, \quad y = i(b + c)/2, \quad z = (b - c)/2
\]

we get the following commutation relations between the new generators

\[
[x, y] = hz, \quad [y, z] = hx, \quad [z, x] = hy
\]

and the defining equation of the NC variety reads now

\[
\Delta = (x^2 + y^2 + z^2) = \alpha.
\]

Thus, assuming \( K = \mathbb{R} \) and \( \alpha > 0 \) we get a NC analogue of the sphere, namely, the algebra

\[
A_h = U(su(2)) / \{ x^2 + y^2 + z^2 - \alpha \}
\]

However, the eigenvalues of equation (2.2) are imaginary (for positive \( \alpha \) and real and small enough \( h \)) and as usual we should consider the idempotents and related projective modules over the field \( \mathbb{C} \).

Completing this section we want to stress that equations (2.4) are covariant w.r.t. the action of the group \( G \) where \( G = SU(2) \) or \( G = SL(2) \) depending on the form (compact or not) we are dealing with.

### 3 Derived quantum line bundles

In this section we discuss the problem of extension of the matrix \( L = L_{(1)} \) to the higher spins and suggest a method of finding the corresponding CH identities.

First, consider the commutative case. Let

\[
\Delta(L) = L \otimes \text{id} + \text{id} \otimes L \in M_4(A_h)
\]

be the first extension of the matrix \( L \) to the space \( V^\otimes 2 \). If \( \lambda_1 \) and \( \lambda_2 \) are (distinct) eigenvalues of \( L \) and \( u_1, u_2 \in V \) are corresponding eigenvectors, then the spectrum of \( \Delta(L) \) is

\[
2\lambda_1 (u_1 \otimes u_1), \quad 2\lambda_2 (u_2 \otimes u_2), \quad \lambda_1 + \lambda_2 \quad (u_1 \otimes u_2 \quad \text{and} \quad u_2 \otimes u_1)
\]

where in brackets we indicate the corresponding eigenvectors.

The commutativity of entries of the matrix \( L \) can be expressed by the relation

\[
L_1 \cdot L_2 = L_2 \cdot L_1 \quad \text{where} \quad L_1 = L \otimes \text{id}, \quad L_2 = \text{id} \otimes L.
\]

Rewriting (3.1) in the form \( \Delta(L) = L_1 + L_2 \) and taking into account the CH identity for \( L \)

\[
0 = (L - \lambda_1 \text{id})(L - \lambda_2 \text{id}) = L^2 - \mu L + \nu \text{id}, \quad \mu = \lambda_1 + \lambda_2, \quad \nu = \lambda_1 \lambda_2
\]

we find

\[
\Delta(L)^2 = \mu \Delta(L) + 2L_1 \cdot L_2 - 2\nu \text{id} \\
\Delta(L)^3 = (\mu^2 - 4\nu)\Delta(L) + 6\mu L_1 \cdot L_2 - 2\mu \nu \text{id}.
\]
Note that $\mu = 0$ if $L \in \mathfrak{sl}(2)$. Upon excluding $L_1 \cdot L_2$ from the above equations we get
\[ \Delta(L)^3 - 3\mu\Delta(L)^2 + (2\mu^2 + 4\nu)\Delta(L) - 4\mu\nu \text{id} = 0. \]
Substituting the values of $\mu$ and $\nu$ we can present this relation as follows
\[ (\Delta(L) - 2\lambda_1)(\Delta(L) - \lambda_1 - \lambda_2)(\Delta(L) - 2\lambda_2) = 0. \]
Thus, the minimal polynomial for $\Delta(L)$ is of the degree 3.

A similar statement is valid for further extensions of the matrix $L$:
\[ \Delta^2(L) = L \otimes \text{id} \otimes \text{id} + \otimes \text{id} \otimes L \otimes \text{id} + \text{id} \otimes \text{id} \otimes L \]
and so on. Namely, the matrix $\Delta^k(L)$ satisfies the CH identity whose roots are $k_1 \lambda_1 + k_2 \lambda_2$ with $k_1 + k_2 = k + 1$ and the multiplicity of each root is $C_{k+1}^{k+1}$. To avoid this multiplicity it suffices to consider the symmetric component (denoted as $L_{(k)}$) of the matrix $\Delta^{k+1}(L)$ (see below). Finally, the matrix $L_{(k)}$ has $k + 1$ pairwise distinct eigenvalues and its characteristic polynomial equals to that of $\Delta^k(L)$. Then by the same method as above we can associate to this matrix $k + 1$ idempotents and corresponding projective modules. Thus, we have realized the line bundles $E^{k_1,k_2}$ under the guise of projective modules.

Now, let us pass to the NC variety in question. With matrices $L_1$ and $L_2$ the commutation relation (1.2) takes the form
\[ L_1 \cdot L_2 - L_2 \cdot L_1 = \hbar(L_1 P - PL_1) \]
where $P$ is the usual flip. So, we cannot apply the commutative binomial formula for calculating the powers of $\Delta^k(L)$. This prevents us from calculating the CH identities for the matrices $\Delta^k(L)$ with the above method.

Instead, we will calculate the CH identities directly for the symmetric components of these matrices, also denoted $L_{(k)}$, $k = 2, 3, ...$ and defined as
\[ L_{(k)} = k S^{(k)} L_1 S^{(k)}, \]
where $S^{(k)}$ is the Young symmetrizer in $V^\otimes k$. Taking in consideration that the element $S^{(k)} L_1$ is already symmetrized w.r.t. all factors apart from the first one we can represent the matrix $L_{(k)}$ as
\[ L_{(k)} = S^{(k)} L_1 (\text{id} + P^{12} + P^{12}P^{23} + ... + P^{12}P^{23}...P^{k-1,k}) \]
where $P^{i,i+1}$ is the operator transposing the $i$-th and $(i+1)$-th factors in the tensor product of spaces.

**Remark 3** We can treat the matrix $L_{(k)}$ as an operator acting from $V^\otimes k$ to $V^\otimes k \otimes \mathcal{A}_h$ assuming it to be trivial on all components except for $V_{(k)} \subset V^\otimes k$.

In case $k = 2$ we have the following proposition.

**Proposition 4** The CH identity for the matrix $L_{(2)}$ restricted to the symmetric component $V_{(2)}$ of the space $V^\otimes 2$ is
\[ L_{(2)}^3 - 4\hbar L_{(2)}^2 + 4(\alpha + \hbar^2)L_{(2)} - 8\alpha \text{id} = 0. \]

**Proof.** By definition the matrix $L_{(2)}$ has the form
\[ L_{(2)} = \frac{1}{2}(\text{id} + P^{12})L_1(\text{id} + P^{12}). \]
Taking in consideration that 
\[ L_2 = P^{12}L_1 P^{12} \]
we rewrite (3.2) as follows
\[ L_1 P^{12}L_1 P^{12} - P^{12}L_1 P^{12}L_1 = \hbar (L_1 P^{12} - P^{12}L_1). \] (3.4)

Now we need some powers of the matrix \( L_2 \). We find \( (L \equiv L_1, P \equiv P^{12}) \):
\[
L_2^{(2)} = \frac{1}{2} (\text{id} + P) L (\text{id} + P)
\]
\[
L_3^{(2)} = \frac{1}{2} (\text{id} + P) L (\text{id} + P) L (\text{id} + P).
\]

Then taking into account (2.2) and (3.4) we have the following chain of identical transformations for \( L_3^{(2)} \):
\[
2 L_3^{(2)} = (\text{id} + P) L (\text{id} + P) L (\text{id} + P) = (\text{id} + P) L \left[ P L P L + P L^2 + L P L + L^2 \right] (\text{id} + P) = 2 (\text{id} + P) L \left[ L P L + \hbar P L - \alpha \text{id} \right] (\text{id} + P) = 4 \hbar (\text{id} + P) L P L (\text{id} + P) - 4 \alpha (\text{id} + P) L (\text{id} + P) = 4 \hbar (\text{id} + P) L (\text{id} + P) - 4 \hbar (\text{id} + P) (\hbar L - \alpha \text{id}) (\text{id} + P) - 4 \alpha (\text{id} + P) L (\text{id} + P) = 8 \hbar L_2^{(2)} - 8 (\hbar^2 + \alpha) L_2 + 8 \alpha (\text{id} + P).
\]

Canceling the factor 2 we come to the result
\[
L_3^{(2)} = 4 \hbar L_2^{(2)} + 4 (\alpha + \hbar^2) L_2 - 4 \alpha (\text{id} + P) = 0. \] (3.5)

To complete the proof it remains to note, that after restriction to the symmetric component \( V^{(2)} \) of the space \( V \otimes 2 \) the last term in (3.5) turns into \( 8 \hbar \alpha \text{id} \) and we come to (3.3).

Now, let us exhibit the matrix \( L_2 \) in a basis form. In the base
\[
v_{20} = v_1^{\otimes 2}, \quad v_{11} = v_1 \otimes v_2 + v_2 \otimes v_1, \quad v_{02} = v_2^{\otimes 2}
\]
of the space \( V^{(2)} \) this matrix has the following form
\[
L^{(2)} = \begin{pmatrix} 2a & 2b & 0 \\ c & a + d & b \\ 0 & 2c & 2d \end{pmatrix} = \begin{pmatrix} 2a & 2b & 0 \\ c & 0 & b \\ 0 & 2c & -2a \end{pmatrix}
\]
(3.6)

(the latter equality holds in virtue of the condition \( a + d = 0 \)).

In the sequel we prefer to deal with another basis in the space \( V^{(2)} \). Namely, by putting
\[
(v_{20}, v_{11}, v_{02}) = (u_{20}, u_{11}, u_{02}) P
\]
with transition matrix
\[
P = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & -1/2 \\ -i/2 & 0 & -i/2 \end{pmatrix}
\]
we transform \( L^{(2)} \) into the form
\[
\overline{L}^{(2)} = P L^{(2)} P^{-1} = 2 \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}.
\] (3.7)

The matrix \( \overline{L}^{(2)} \) is expressed through the generators \( (x, y, z) \) and it is better adapted to the compact form of the NC varieties in question. However, it satisfies the same NC version of CH identity (2.2).
Remark 5 By straightforward checking we can see that the roots of (3.3) are
\[ \lambda_{20} = \hbar - \sqrt{\hbar^2 - 4\alpha} = 2\lambda_1, \quad \lambda_{11} = 2 = 2(\lambda_1 + \lambda_2), \quad \text{and} \quad \lambda_{02} = \hbar + \sqrt{\hbar^2 - 4\alpha} = 2\lambda_2. \] (3.8)
These quantities are "eigenvalues" of the matrix (3.3) which is the restriction of the matrix (3.4) to the symmetric part of the space \( V^\otimes 2 \). A similar restriction of the matrix (3.1) to the skew-symmetric part of the space \( V^\otimes 2 \) gives rise to the operator\[ v_1 \otimes v_2 - v_2 \otimes v_1 \rightarrow (v_1 \otimes v_2 - v_2 \otimes v_1) \otimes (a + d) = 0. \]
Thus, the matrix (3.4) on the whole space \( V^\otimes 2 \) has 4 distinct eigenvalues (those (3.8) and 0) and in contrast with the commutative case its minimal polynomial cannot be of the third degree.

As for higher extensions \( L_k \), \( k > 2 \) of the matrix \( L \) the following holds.

Proposition 6 For any integer \( k > 2 \) there exists a polynomial
\[ P_k(x) = \lambda^k + \sum_{i=1}^{k} a_{k-i} \lambda^{k-i} \]
with numerical coefficients such that \( P_k(L_k) = 0 \). Moreover, its roots are
\[ \lambda_{k_1 k_2} = k_1 \lambda_1 + k_2 (\lambda_1 + \lambda_2) + k_2 \lambda_2, \quad k_1 + k_2 = k. \]

This formula can be deduced from [Ro]. We will present its q-analogue in [GLS].

Similar to the basic case the roots of the polynomial \( P_k \) will be called eigenvalues of the corresponding matrix \( L_k \).

Now, let us assume the eigenvalues \( \lambda_{20}, \lambda_{11}, \lambda_{02} \) to be also pairwise distinct. Then, by using the same method as above we can introduce the idempotent
\[ e_{20} = (\lambda_{11} \text{id} - L_2(\lambda_{02} \text{id} - L_2))/((\lambda_{11} - \lambda_{20})(\lambda_{02} - \lambda_{20}) \]
and similarly \( e_{11} \) and \( e_{02} \) corresponding to the eigenvalues \( \lambda_{11} \), and \( \lambda_{02} \) respectively. The related q.l.b. (projective \( A_k \)-modules) will be denoted \( E^{20}(\hbar), E^{11}(\hbar), \) and \( E^{02}(\hbar) \) respectively.

Assuming the eigenvalues of the polynomials \( P_k, k > 2 \) (see proposition 6) to be also pairwise distinct we can associate to the matrix \( L_k \) \( k + 1 \) idempotents
\[ e_{k_1 k_2}, \quad k_1 + k_2 = k, \quad k > 0 \]
and the corresponding q.l.b. \( E^{k_1 k_2}(\hbar) \). For \( k = 0 \) we set \( e_{00} = 1 \). The corresponding q.l.b. is \( E^{0,0}(\hbar) \).

Remark 7 Let us note that if we do not fix any value of \( \triangle \) we can treat the elements \( \text{tr} e_{ij} \) as those of \( M_2(U(sl(2)_{\hbar})) \otimes R \) where \( R \) is the field of fractions of the algebraic closure \( \mathbb{Z} U(sl(2)_{\hbar}) \).

Remark 8 Note that there exists a natural generalization of the above constructions giving rise to some "braided varieties" and corresponding "line bundles" as follows. Let \( R \) be a Hecke symmetry of rank 2 (cf. [G]). Then the matrix \( L \) satisfying the RE with such \( R \) obeys an equation analogous to (2.7) but with appropriate trace and determinant (cf. [GFS]). Introducing the quotient algebra \( A_k \) in a similar way we treat it as a braided analogue of a NC sphere. Then, by defining the extensions \( L_k \) as above (but with modified meaning of the symmetric powers of the space \( V \)) we can define a family of q.l.b. over such a "NC braided variety" as above. This construction will be presented in details in (GLS).

Now, we pass to computing the quantities \( \text{tr} e_{k_1 k_2} \).

Proposition 9 The following relation holds
\[ \text{tr} e_{k_1 k_2} = 1 + \frac{(k_1 - k_2)\hbar}{\sqrt{\hbar^2 - 4\alpha}}. \]
A proof of this formula will be given in [GLS] in a more general context including its q-analogue.
4 Isomorphic modules and multiplicative structure

First of all we discuss the problem of isomorphism between the projective modules introduced above (namely, q.l.b.). There exists a number of definitions of isomorphic modules over \(\mathbb{C}^*\)-algebras (cf. [W]). However, for the algebras in question we use the following definition motivated by [R].

**Definition 10** We say that two projective modules \(M_1 \subset A^\oplus m\) and \(M_2 \subset A^\oplus n\) over an algebra \(A\) corresponding to the idempotents \(e_1\) and \(e_2\) respectively are isomorphic iff there exist two matrices 
\[A \in \mathbb{M}_{m,n}(A), B \in \mathbb{M}_{n,m}(A)\] such that
\[AB = e_1, \quad BA = e_2, \quad A = e_1A = Ae_2, \quad B = e_2B = Be_1.\]

**Proposition 11** The q.l.b. \(E^{1,1}(\hbar)\) is isomorphic to that \(E^{0,0}(\hbar)\).

**Proof** It is not difficult to see that
\[e_{11} = (L^2_{(2)} - 2\hbar L_{(2)} + 4\alpha \text{id})/(4\alpha).\]

Then, by straightforward calculations we check that for the idempotent \(e_{11}\) the following relation holds
\[e_{11} = (4\alpha)^{-1} \left( \begin{array}{ccc} 2a^2 + 2bc & 4ab & 2b^2 \\ 2ab & 2c^2 & -2ab \\ 4ac & 2b + 4a^2 & \end{array} \right) - 2\hbar \left( \begin{array}{ccc} 2a & 2b & 0 \\ c & 0 & b \\ 0 & 2c & -2a \end{array} \right) + 4\alpha \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right) =
(4\alpha)^{-1} \left( \begin{array}{ccc} -2b \\ 2a \\ 2c \end{array} \right) \cdot \left( \begin{array}{ccc} c & -2a & -b \end{array} \right).\]

Passing to the matrix \(\overline{L}_{(2)}\) we get
\[e_{11} = (4\alpha)^{-1}(\overline{L}_{(2)}^2 - 2\hbar \overline{L}_{(2)} + 4\alpha \text{id}) = \alpha^{-1} \left( \begin{array}{ccc} x \\ y \\ z \end{array} \right) \left( \begin{array}{ccc} x & y & z \end{array} \right).\]

(\text{So, the idempotent } e_{11} \text{ defines the following operator in } A^{\oplus 3}:)
\[\left( \begin{array}{c} f_1 \\ f_2 \\ f_3 \end{array} \right) \mapsto (\alpha)^{-1} \left( \begin{array}{c} x \\ y \\ z \end{array} \right) \left( x \cdot f_1 + y \cdot f_2 + z \cdot f_3 \right).\]

It remains to say that if we put \(A = (\alpha)^{-1} \left( \begin{array}{ccc} x & y & z \end{array} \right)\) and \(B = \left( \begin{array}{c} x \\ y \\ z \end{array} \right)\) we satisfies the definition above with \(e_1 = e_{00}\) and \(e_2 = e_{11}\)

In general, the problem of isomorphism between modules \((\ref{13})\) is open. We can only conjecture that the q.l.b. \((\ref{13})\) are isomorphic to each other.

**Remark 12** If the algebra \(A\) is not commutative the first two relations of the definition do not yield the equality \(\text{tr } e_1 = \text{tr } e_2\). However, proposition 8 implies that
\[\text{tr } e_{k_1,k_2} = \text{tr } e_{k_1+l,k_2+l}.\]
Now, we can introduce an associative product on the set of q.l.b. in a natural way by setting

\[ E^{k_1,k_2}(\hbar) \cdot E^{l_1,l_2}(\hbar) = E^{k_1+l_1,k_2+l_2}(\hbar). \]

This product is evidently associative and commutative. In particular, we have

\[ E_1(\hbar) \cdot E_2(\hbar) = E^{1,1}(\hbar), \quad E_1(\hbar) \cdot E_1(\hbar) = E^{2,0}(\hbar), \quad E_2(\hbar) \cdot E_2(\hbar) = E^{0,2}(\hbar). \]

The family of the modules \( E^{k_1,k_2}(\hbar) \) equipped with this product is denoted \( \text{prePic}(A_\hbar) \) and called \( \text{prePicard semigroup} \) of the NC sphere.

Assuming that the q.l.b. \( \text{(1.5)} \) are indeed isomorphic to each other we can naturally define the Picard group \( \text{Pic}(A_\hbar) \) of the NC sphere as the classes of isomorphic modules \( E^{k_1,k_2}(\hbar) \) equipped with the above product.

So, under this assumption, the Picard group \( \text{Pic}(A_\hbar) \) of the NC sphere is at most \( \mathbb{Z} \) (recall that \( \text{Pic}(S^2) = \mathbb{Z} \)).

Now, consider the problem of computing the pairing

\[ \langle , \rangle : \text{prePic}(A_\hbar) \otimes K^0 \to \mathbb{K}. \quad (4.1) \]

Such a pairing plays the key role in the Connes version of the index formula. (Usually, one considers \( K_0 \) instead of prePic but we restrict ourselves to “quantum line bundles”. Moreover, assuming the conjecture formulated before remark 12 to be true we can replace prePic in formula \( (4.1) \) by Pic.)

Let us remind that \( K^0 \) stands for the Grothendieck ring of the category of irreducible modules of the algebra in question. In the spirit of the NC index (cf. \[ L \]) the paring \( (4.1) \) can be defined as

\[ \langle E^{k_1,k_2}(\hbar), U \rangle = \text{tr} \pi_U(\text{tr}(e_{k_1,k_2})) \quad (4.2) \]

where \( \text{tr}(e_{k_1,k_2}) \in A_\hbar \) and \( \pi_U : A_\hbar \to \text{End}(U) \) is the representation corresponding to the irreducible \( U \).

It is not difficult to see that the result of the pairing of the module \( E^{k_1,k_2}(\hbar) \) with the irreducible \( U_j \) of the spin \( j \) is equal to the quantity \( n \text{ tr } e_{k_1,k_2} \) evaluated at the point

\[ \alpha = -\hbar^2(n^2 - 1)/4 \quad \text{where} \quad n = \dim U_j = 2j + 1. \quad (4.3) \]

In particular, we have

\[ \langle E^{0,0}(\hbar), U_j \rangle = n, \quad \langle E^{1,0}(\hbar), U_j \rangle = n + 1, \quad \langle E^{0,1}(\hbar), U_j \rangle = n - 1, \]

\[ \langle E^{2,0}(\hbar), U_j \rangle = n + 2, \quad \langle E^{0,2}(\hbar), U_j \rangle = n - 2 \]

(here we assume that \( \sqrt{n^2\hbar^2} = nh \)). More generally, if \( n > k_1 + k_2 \) we have

\[ \langle E^{k_1,k_2}(\hbar), U_j \rangle = k_1 - k_2 + n. \]

This formula follows immediately from proposition 9. There exists a q-analogue of this formula which will be considered in \[ GLS \].

\[ ^6\text{By this we mean the Grothendieck ring of the algebra } U(su(2)_\hbar) \text{ or what is the same } U(su(2)) \text{ since } \hbar \text{ does not matter here. However, any irreducible } U(su(2)_\hbar)-\text{module defines a relation between } \hbar \text{ and } \alpha \text{ (see (4.3)) and therefore the factors in the formula (4.2) are not independent.} \]
5 Differential calculus via projective modules

The aim of this section is to develop some elements of differential calculus on the NC sphere in terms of projective modules (namely, the q.l.b. above and their direct sums). Usually, a differential algebra associated to a NC algebra is much bigger than the classical one even if such an algebra is a deformation of the commutative coordinate ring of a given variety (cf. for example [GVF]). Moreover, the components of these differential algebras are not finitely generated modules. This leads to a NC version of de Rham complex which is drastically different from the classical one.

We suggest another NC version of de Rham complex looking like its classical counterpart. The main idea of our approach consists in the following. Instead of using the Leibniz rule and introducing a way of transposing ”functions” and ”differential” we treat any term of the classical de Rham complex as a projective module. Let us consider projective $A_\hbar$-modules which are analogues of the former ones.

By decomposing them into direct sums of irreducible $A_\hbar$-modules we define a differential on each irreducible component following the classical pattern (a detailed description of this decomposition in the classical case is given in [AG]). Then the property $d^2 = 0$ for the deformed differential is preserved automatically and the cohomology of the final complex are just the same as in the classical case. Let us describe this approach in more details.

However, we want to begin with a NC analogue of the tangent bundle $T(S^2)$ on the sphere. Since this bundle is complementary to the normal one and since the latter bundle (treated as a module) is nothing but $E_{1,1}$, it is natural to define NC analogue of $T(S^2)$ as

$$T(A_\hbar) = E^{2,0}(\hbar) + E^{0,2}(\hbar).$$

We call it a tangent module on the NC sphere.

This module can be represented by the equation $\text{Im } e_{11} = 0$. It is equivalent to the relation

$$u_{20} x + u_{11} y + u_{02} z = 0. \quad (5.1)$$

This means that the module $T(A_\hbar)$ is realized as the quotient of the free module $A_\hbar^{3,3}$ generated by the elements

$$u_{20}, u_{11}, u_{02} \quad (5.2)$$

over the submodule $\{C f, f \in A_\hbar\}$ where $C$ is the l.h.s. of (5.1).

In the classical case relation (5.1) is motivated by an operator meaning of the tangent space. Namely, if generators (5.2) are treated as infinitesimal rotations of the sphere and the symbols $x, y, z$ in (5.1) are considered as operators of multiplication on the corresponding functions, then the element $C$ treated as an operator is trivial. This allows us to equip the tangent module $T(S^2)$ with an operator meaning by converting any element of this module into a vector field. Thus, we get the action

$$A \otimes T(S^2) \to A, \ A = \mathbb{K}(S^2)$$

which consists in applying a vector field to a function.

However, if we assign the same meaning to the generators (5.2) and to those $x, y, z$ on the NC sphere (by setting $u_{20} = [x, \cdot]$ and so on and considering $x, y, z$ as operators of multiplication on the corresponding generator) then the element $C$ treated as an operator is no longer trivial. This is the reason why we are not able to provide the tangent module $T(A_\hbar)$ with a similar action on the algebra $A_\hbar$.

**Remark 13** We want to stress that the space of derivations of the algebra $A_\hbar$ often considered as a proper NC counterpart of the usual tangent space does not have any $A_\hbar$-module structure. So, for a NC variety which is a deformation of a classical one we have a choice: which properties of the classical
object we want to preserve. In our approach we prefer to keep the property of the tangent space to be a projective module. In the same manner we will treat the terms of the deformed de Rham complex (see below).

Passing to the cotangent bundle $T^*(S^2)$ or otherwise stated to the space of the first order differentials $\Omega^1(S^2)$ we see that it is isomorphic to $T(S^2)$. Therefore, it is defined by the same relation (5.1) but with another meaning of generators (5.2): now we treat them as the differentials of the functions $x$, $y$ and $z$ respectively:

$$\begin{align*}
    u_{20} &= dx, \\
    u_{11} &= dy, \\
    u_{02} &= dz.
\end{align*}$$

This gives us a motivation to introduce cotangent module $T^*(A_\hbar)$ on the NC sphere similarly to the tangent one but with a new meaning of generators (5.2). We will also use the notation $\Omega^1(A_\hbar)$ for the module $T^*(A_\hbar)$ and call the elements of this space the first order differential forms on the NC sphere.

Up to now we have no modifications in the relations defining the tangent and cotangent objects. Nevertheless, it is no longer so for the NC counterpart of the second order differentials space $\Omega^2(S^2)$.

In the classical case this space is defined by

$$\begin{align*}
    u_{11}z - u_{02}y &= 0, \\
    -u_{20}z + u_{02}x &= 0, \\
    u_{20}y - u_{11}x &= 0
\end{align*}$$

with the following meaning of generators (5.2)

$$\begin{align*}
    u_{20} &= dy \wedge dz, \\
    u_{11} &= dz \wedge dx, \\
    u_{02} &= dx \wedge dy.
\end{align*}$$

In a concise form we can rewrite (5.3) as

$$\begin{align*}
    (u_{20}, u_{11}, u_{02}) \cdot L_{(2)} &= 0
\end{align*}$$

with $L_{(2)}$ given by (3.7).

However, in the NC case we should replace (5.5) by

$$\begin{align*}
    (u_{20}, u_{11} u_{02}) \cdot \overline{L}_{(2)} &= 2\hbar(u_{20}, u_{11} u_{02})
\end{align*}$$

or in more detailed form

$$\begin{align*}
    u_{11}z - u_{02}y &= \hbar u_{20}, \\
    -u_{20}z + u_{02}x &= \hbar u_{11}, \\
    u_{20}y - u_{11}x &= \hbar u_{02}.
\end{align*}$$

This is motivated by the fact that $2\hbar$ becomes an eigenvalue of the matrix $\overline{L}_{(2)}$. Let us denote the corresponding quotient module $\Omega^2(A_\hbar)$. Of course, we can represent generators (5.2) in form (5.4) but now it does not have any sense. Nevertheless, we will call the elements of this module the second order differential forms.

Thus, we have defined all terms of the following de Rham complex on the NC sphere

$$\begin{align*}
    0 &\rightarrow \Omega^0(A_\hbar) = A_\hbar \\
    &\rightarrow \Omega^1(A_\hbar) \\
    &\rightarrow \Omega^2(A_\hbar) \\
    &\rightarrow 0.
\end{align*}$$

These terms are projective $A_\hbar$-modules which are deformations of the corresponding $A$-modules. Now we would like to define a differential $d$ in this complex in such a way that this complex would have just the same cohomology as in the classical case.

Let us decompose each term of the classical de Rham complex into a direct sum of irreducible $SU(2)$-modules (cf. \[AG\]). Since in the classical case the differential $d$ is $SU(2)$-covariant then any such an irreducible module is mapped by the differential either to 0 or to an isomorphic module.

Remark that the terms of (5.6) being a deformation of their classical counterparts consist of just the same irreducible $SU(2)$-modules as in the classical case. This property follows from the fact that the modules in question are projective and the corresponding idempotents smoothly depend on $\hbar$. 

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Now, we define the differential in (5.6) as a mapping which takes any irreducible $SU(2)$-module to 0 or to the isomorphic module following the classical pattern.

As an example we describe how the differential $d$ acts on the algebra $A_\hbar$ itself. It is easier to use the non-compact form of the algebra. The algebra $A_\hbar$ is a multiplicity free direct sum of irreducible $SL(2)$-modules. Their highest (or lowest) weight elements w.r.t. the action of the group $SL(2)$ are $b^k$, $k = 1, 2, ...$. In virtue of $SL(2)$-covariance it suffices to define the differential on these elements. Similar to the classical case we set

$$d b^k = k (d b) b^{k-1}.$$ 

Stress that this relation is not obtained as a result of transposing "functions" and "differentials" mixed up in virtue of the Leibniz rule but it is imposed by definition. In our version of NC de Rham complex we do not use either any form of the Leibniz rule or any transposing "functions" and "differentials".

However, by construction we have $d^2 = 0$ and just the same cohomology as in the classical case. Similarly to the classical case this cohomology is generated by 1 in the term $\Omega^0(A_\hbar)$ and by the element $v_{20} x + u_{11} y + u_{20} z$ in the term $\Omega^2(A_\hbar)$.

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