Global existence and large time asymptotic behavior of strong solutions to the Cauchy problem of 2D density-dependent Navier–Stokes equations with vacuum

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Abstract
We are concerned with the Cauchy problem of the two-dimensional (2D) nonhomogeneous incompressible Navier–Stokes equations with vacuum as far-field density. It is proved that if the initial density decays not too slow at infinity, the 2D Cauchy problem of the density-dependent Navier–Stokes equations on the whole space $\mathbb{R}^2$ admits a unique global strong solution. Note that the initial data can be arbitrarily large and the initial density can contain vacuum states and even have compact support. Furthermore, we also obtain the large time decay rates of the spatial gradients of the velocity and the pressure, which are the same as those of the homogeneous case.

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1. Introduction

The motion of a two-dimensional (2D) nonhomogeneous incompressible fluid is governed by the following Navier–Stokes equations:

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) - \mu \Delta u + \nabla P &= 0, \\
\text{div}u &= 0.
\end{aligned}
\]

Here, \( t \geq 0 \) is time, \( x \in \mathbb{R}^2 \) is the spatial coordinates, and the unknown functions \( \rho = \rho(x,t) \), \( u = (u^1, u^2)(x,t) \), and \( P = P(x,t) \) denote the density, velocity, and pressure of the fluid, respectively; \( \mu > 0 \) stands for the viscosity constant.

We consider the Cauchy problem of (1.1) with \((\rho, u)\) vanishing at infinity (in some weak sense) and the initial conditions:

\[
\begin{aligned}
\rho(x,0) &= \rho_0(x), & \rho u(x,0) &= \rho_0 u_0(x), & x \in \mathbb{R}^2,
\end{aligned}
\]

for given initial data \( \rho_0 \) and \( u_0 \).

The study of the system (1.1) has a long history. Since the pioneering work of Leray [16], there is much literature on the studies of the large time existence and behavior of solutions to the homogeneous Navier–Stokes equations (i.e. the density in (1.1) is constant), see for example [5, 13, 15, 19, 25] and references therein. When the density is not constant, the system (1.1) is so-called the nonhomogeneous incompressible Navier–Stokes equations or the density-dependent Navier–Stokes equations, whose mathematical study goes back to the 1970s. Next, we briefly recall some well-posedness results in multi-dimensional space which are more relative with our problem. When the initial density is strictly away from vacuum (i.e. the initial density is strictly positive), Kazhikov [14] established the global existence of weak solutions (see also [1]). Later, Antontsev–Kazhikov–Monakhov [2] gave the first result on local existence and uniqueness of strong solutions. Moreover, they also proved that the unique local strong solution is a global one in two dimensions. Recently, the global existence result of (1.1) with small initial data belonging to certain scale invariant spaces was obtained in [7].

On the other hand, when the initial density allows for vacuum (i.e. the initial density may vanish in some open sets), the issue of the existence of solutions becomes much more complicated due to the possible degeneracy near vacuum. Simon [23] first proved the global existence of weak solutions with finite energy, which was extended later by Lions [19] to the case of density-dependent viscosity. Under some additional compatibility conditions on the initial data, Choe–Kim [3] obtained the local existence of strong solutions with large data to the initial boundary value problem of (1.1) in two or three-dimensional bounded domains. Recently, Huang–Wang [11] showed that the local strong solution obtained in [3] was indeed a global one in the 2D case. In particular, the global well-posedness of strong solutions was also proved in Huang–Wang [11] for density-dependent viscosity \( \mu(\rho) \) provided that the \( L^p \)-norm of \( \nabla \mu(\rho_0) \) is suitably small. Very recently, motivated by [17, 21] dealing with the compressible flow, Lü–Xu–Zhong [20] established the local (with generally large data) existence of strong solutions to the 2D Cauchy problem of the nonhomogeneous incompressible magnetohydrodynamic equations with vacuum as the far-field density, which in particular yields the local existence of strong solutions to the 2D Cauchy problem of (1.1). However, the global existence of strong solution to the 2D Cauchy problem of (1.1) with vacuum and general initial data is still open. In fact, this is the main aim of this paper.

Before stating the main results, we first explain the notations and conventions used throughout this paper. For \( R > 0 \), set
\[ B_R \triangleq \left\{ x \in \mathbb{R}^2 \mid |x| < R \right\}, \quad \int dx \triangleq \int_{\mathbb{R}^2} dx. \]

Moreover, for \( 1 \leq r \leq \infty \) and \( k \geq 1 \), the standard Lebesgue and Sobolev spaces are defined as follows:

\[ L^r = L^r(\mathbb{R}^2), \quad W^{k,r} = W^{k,r}(\mathbb{R}^2), \quad H^k = W^{k,2}. \]

Now we define precisely what we mean by strong solutions.

**Definition 1.1.** If all derivatives involved in (1.1) for \((\rho, u, P)\) are regular distributions, and equation (1.1) hold almost everywhere in \( \mathbb{R}^2 \times (0, T) \), then \((\rho, u, P)\) is called a strong solution to (1.1).

Without loss of generality, we assume that the initial density \( \rho_0 \) satisfies

\[ \int_{\mathbb{R}^2} \rho_0 dx = 1, \quad (1.3) \]

which implies that there exists a positive constant \( N_0 \) such that

\[ \int_{B_{N_0}} \rho_0 dx \geq \frac{1}{2} \int \rho_0 dx = \frac{1}{2}, \quad (1.4) \]

Our main result can be stated as follows:

**Theorem 1.1.** In addition to (1.3) and (1.4), assume that the initial data \((\rho_0, u_0)\) satisfy for any given numbers \( a > 1 \) and \( q > 2 \),

\[ \rho_0 \geq 0, \quad \rho_0 \bar{x}^a \in L^1 \cap H^1 \cap W^{1,q}, \quad \text{div} u_0 = 0, \quad \nabla u_0 \in L^2, \quad \sqrt{\rho_0} u_0 \in L^2, \quad (1.5) \]

where

\[ \bar{x} \triangleq (e + |x|^2)^{1/2} \log^2(e + |x|^2) \quad (1.6) \]

Then the problem (1.1) and (1.2) has a unique global strong solution \((\rho, u, P)\) satisfying that for any \( 0 < T < \infty \),

\[
\begin{align*}
0 \leq & \rho \in C([0, T]; L^1 \cap H^1 \cap W^{1,q}), \\
\rho \bar{x}^a & \in L^\infty(0, T; L^1 \cap H^1 \cap W^{1,q}), \\
\rho \sqrt{u}, \sqrt{\nabla u}, \sqrt{\nabla \sqrt{u}}, \sqrt{\nabla^2 u} & \in L^\infty(0, T; L^2), \\
\nabla u & \in L^2(0, T; H^1) \cap L^2(0, T; W^{1,q}), \\
\nabla P & \in L^2(0, T; L^2) \cap L^{(q+1)/q}(0, T; L^q), \\
\sqrt{\nabla u} & \in L^2(0, T; W^{1,q}), \\
\sqrt{\rho u}, \sqrt{\nabla u}, \sqrt{\bar{x}^{-1} u} & \in L^2(\mathbb{R}^2 \times (0, T)), \\
\end{align*}
\]

and

\[ \inf_{0 \leq t \leq T} \int_{B_{N_0}} \rho(x, t) dx \geq \frac{1}{4}, \quad (1.8) \]

for some positive constant \( N_1 \) depending only on \( \|\sqrt{\rho_0} u_0\|_{L^2}, N_0 \), and \( T \). Moreover, \((\rho, u, P)\) has the following decay rates, that is, for \( t \geq 1 \),
\begin{equation}
\| \nabla u(t) \|_{L^2}^2 + \| \nabla^2 u(t) \|_{L^2} + \| \nabla P(t) \|_{L^2} \leq Ct^{-1},
\end{equation}

where \( C \) depends only on \( \mu, \| \rho_0 \|_{L^{1} \cap L^{\infty}}, \| \sqrt{\rho_0} u_0 \|_{L^2}, \) and \( \| \nabla u_0 \|_{L^2}. \)

The initial density \( \rho_0 \geq 0 \) and \( \rho_0 \bar{x}^a \in L^1 \cap H^1 \cap W^{1,4} \) in (1.5) show that the initial density can contain vacuum (even have compact support) and decays not too slow at infinity. Since one cannot impose initial conditions on the velocity \( u \) in case the density \( \rho \) vanishes (i.e. there is initial vacuum), we thus impose the initial condition on the momentum \( \rho_0 u_0 \) instead of the velocity \( u_0 \) in problem (1.1) and (1.2), see also Lions [19] for more details. Without any smallness conditions and additional compatibility conditions on the initial data, theorem 1.1 tells us the Cauchy problem (1.1) and (1.2) admits a unique global strong solution satisfying (1.7) and the large time decay rates (1.9). It is worth mentioning that the high order estimates on solutions, such as \( \| \sqrt{\rho} u \|_{L^2}, \) \( \| \sqrt{\rho} \nabla u \|_{L^2}, \) and \( \| \sqrt{\nabla u} \|_{L^2} \) in (1.7), are obtained by considering the time weighted type due to the lack of compatibility conditions. Nevertheless, this is enough to derive the desired regularity of strong solutions and thus extend the local solution to global. In particular, by (1.7) and the Aubin–Lions lemma, one can derive directly the time-continuity of \( \rho u \) up to the initial time.

Next, some remarks and the key ideas of this paper are given below.

**Remark 1.1.** Our theorem 1.1 holds for arbitrarily large initial data which is in sharp contrast to Li–Xin [18] where the smallness conditions on the initial energy is needed in order to obtain the global existence of strong solutions to the 2D compressible Navier–Stokes equations.

**Remark 1.2.** It should be noted here that although the equations (1.1) degenerate near vacuum, our large time decay rates (1.9) are the same as those of the homogeneous case [13].

**Remark 1.3.** Compared with [11] requiring that the initial data satisfies

\begin{equation}
-\mu \Delta u_0 + \nabla P_0 = \rho_0^{1/2} g
\end{equation}

with \( (P_0, g) \in H^1 \times L^2 \), there is no need to impose the additional compatibility conditions (1.10) in theorem 1.1. Our theorem 1.1 indicates that we can reduce the compatibility conditions and thus essentially weaken the requirements on the initial data for obtaining the global existence of a strong solution. Indeed, this is achieved by deriving the time weighted estimates on the solution, see lemma 3.3. More precisely, using the time weighted factor \( t \), there exists a positive constant \( C \) independent of the initial data \( \rho u_t \) such that

\begin{equation}
\sup_{t \in [0, T]} t \| \sqrt{t} u \|_{L^2}^2 + \int_0^T t \| \sqrt{t} u \|_{L^2}^2 \, dt \leq C,
\end{equation}

with \( \dot{u} \triangleq u_t + u \cdot \nabla u \). And, this is enough for bounding the \( L^1_t L^\infty_x \)-norm of \( \nabla u \) and thus estimating the higher order derivatives of the solutions (see lemmas 3.5–3.6).

We now make some comments on the key ingredients of the analysis in this paper. Note that for initial data satisfying (1.5), Lü–Xu–Zhong [20] established the local existence and uniqueness of strong solutions to the Cauchy problem (1.1) and (1.2) (see lemma 2.1). Thus, in order to extend the local strong solution to a global one, one needs to obtain global a priori estimates on strong solutions to (1.1) and (1.2) in suitable higher norms. It should be pointed out that the crucial techniques of proofs in [11] cannot be adapted directly to the situation treated...
here, since their arguments depend crucially on the boundedness of the domains. Moreover, it seems difficult to bound the $L^4(\mathbb{R}^2)$-norm of $u$ just in terms of $\|\sqrt{\rho}u\|_{L^2(\mathbb{R}^2)}$ and $\|\nabla u\|_{L^2(\mathbb{R}^2)}$. To overcome these difficulties mentioned above, some new ideas are needed. First, motivated by [8, 12, 18], multiplying (1.1)$_2$ by the material derivatives of the velocity $\dot{u}$ instead of the usual $u_t$ (see [11]), we find that the key point to obtain the estimate on the $L^\infty(0, T; L^2(\mathbb{R}^2))$-norm of the gradient of the velocity is to bound the term

$$I_2 \triangleq \int P(\partial_i u^j) \partial_i u^j \, dx.$$  

Motivated by [6], the term $I_2$ in fact can be bounded by $\|\nabla P\|_{L^2} \|\nabla u\|_{L^2}^2$ (see (3.5)) since $\partial_i u^j \partial_i u^j \in H^1$ due to the facts that $div u = 0$ and that $\nabla^\perp \cdot \nabla u = 0$ (see lemma 2.5). Next, to obtain the estimates on the gradient of the density, motivated by [8–10, 18], we apply the operator $\partial_t + u \cdot \nabla$ to (1.1)$_2$ and multiply the resultant equality by $\dot{u}^i$ to get the time-independent estimates on both the $L^\infty(0, T; L^2(\mathbb{R}^2))$-norm of $t^{1/2} \rho^{1/2} \dot{u}$ and the $L^2(\mathbb{R}^2 \times (0, T))$-norm of $t^{1/2} \nabla \dot{u}$ (see (3.13)). This combined with some careful analysis on the spatial weighted estimate of the density (see (3.21)) thus yield the bound on the $L^1(0, T; L^\infty)$-norm of the gradient of the velocity (see (3.32)), which in particular implies the desired bound on the $L^\infty(0, T; L^q)$-norm of the gradient of the density. Finally, with these a priori estimates on the gradients of the density and the velocity at hand, one can estimate the higher order derivatives by using the same arguments as in [11, 18] to obtain the desired results.

The rest of this paper is organized as follows. In section 2, we collect some elementary facts and inequalities that will be used later. Section 3 is devoted to the $a$ priori estimates. Finally, we will give the proof of theorem 1.1 in section 4.

2. Preliminaries

In this section we shall enumerate some auxiliary lemmas used in this paper.

We start with the local existence of strong solutions whose proof can be found in [20, theorem 1.2].

**Lemma 2.1.** Assume that $(\rho_0, u_0)$ satisfies (1.5). Then there exists a small time $T > 0$ and a unique strong solution $(\rho, u, P)$ to the problem (1.1) and (1.2) in $\mathbb{R}^2 \times (0, T)$ satisfying (1.7) and (1.8).

The following Gagliardo–Nirenberg inequality (see [22]) will be used later.

**Lemma 2.2 (Gagliardo–Nirenberg).** For $q \in [2, \infty)$, $r \in (2, \infty)$, and $s \in (1, \infty)$, there exists some generic constant $C > 0$ which may depend on $q$, $r$, and $s$ such that for $f \in H^1(\mathbb{R}^2)$ and $g \in L^r(\mathbb{R}^2) \cap D^{1, s}(\mathbb{R}^2)$, we have

$$\|f\|_{L^q(\mathbb{R}^2)}^q \leq C \|f\|_{L^2(\mathbb{R}^2)}^q \|\nabla f\|_{L^2(\mathbb{R}^2)^2}^2,$$

$$\|g\|_{C(\mathbb{R}^2)} \leq C \|g\|_{L^r(\mathbb{R}^2)} r^{1/(2r+s(r-2))} \|\nabla g\|_{L^r(\mathbb{R}^2)}^{2r/(2r+s(r-2))}.$$  

The following weighted $L^m$ bounds for elements of the Hilbert space $\dot{D}^{1, 2}(\mathbb{R}^2) \triangleq \{ v \in H^1_{\text{loc}}(\mathbb{R}^2) | \nabla v \in L^2(\mathbb{R}^2) \}$ can be found in [19, theorem B.1].

**Lemma 2.3.** For $m \in [2, \infty)$ and $\theta \in (1 + m/2, \infty)$, there exists a positive constant $C$ such that for all $v \in \dot{D}^{1, 2}(\mathbb{R}^2)$,
\[
\left( \int_{\mathbb{R}^2} \frac{|v|^m}{e + |x|^2} \left( \log \left( e + |x|^2 \right) \right)^{-\theta} \, dx \right)^{1/m} \leq C \|v\|_{L^2(B_1)} + C \|\nabla v\|_{L^2(B_2)}. \tag{2.1}
\]

The combination of lemma 2.3 and the Poincaré inequality yields the following useful results on weighted bounds, whose proof can be found in [18, lemma 2.4].

**Lemma 2.4.** Let  $\bar{x}$ be as in (1.6). Assume that $\rho \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is a non-negative function such that
\[
\|\rho\|_{L^1(B_{N_1})} \geq M_1, \quad \|\rho\|_{L^1(\mathbb{R}^2)} \cap L^\infty(\mathbb{R}^2) \leq M_2,
\]
for positive constants $M_1, M_2,$ and $N_1 \geq 1$. Then for $\varepsilon > 0$ and $\eta > 0$, there is a positive constant $C$ depending only on $\varepsilon, \eta, M_1, M_2,$ and $N_1$, such that every $v \in \tilde{D}^{1,2}(\mathbb{R}^2)$ satisfies
\[
\|\tilde{x}^{-\eta} v\|_{L^{(1+\varepsilon)/\eta}(\mathbb{R}^2)} \leq C \|\rho^{1/2} v\|_{L^2(\mathbb{R}^2)} + C \|\nabla v\|_{L^2(\mathbb{R}^2)}, \tag{2.2}
\]
with $\tilde{\eta} = \min\{1, \eta\}$.

Finally, let $H^s(\mathbb{R}^2)$ and BMO $(\mathbb{R}^2)$ stand for the usual Hardy and BMO spaces (see [24, section 4]). Then the following well-known facts play a key role in the proof of lemma 3.2 in the next section.

**Lemma 2.5.**

(a) There is a positive constant $C$ such that
\[
\|E \cdot B\|_{H^1(\mathbb{R}^2)} \leq C \|E\|_{L^2(\mathbb{R}^2)} \|B\|_{L^2(\mathbb{R}^2)},
\]
for all $E \in L^2(\mathbb{R}^2)$ and $B \in L^2(\mathbb{R}^2)$ satisfying
\[
\text{div} E = 0, \quad \nabla \perp \cdot B = 0 \quad \text{in} \quad D'(\mathbb{R}^2).
\]

(b) There is a positive constant $C$ such that
\[
\|v\|_{\text{BMO}(\mathbb{R}^2)} \leq C \|\nabla v\|_{L^2(\mathbb{R}^2)}, \tag{2.3}
\]
for all $v \in \tilde{D}^{1,2}(\mathbb{R}^2)$.

**Proof.**

(a) For the detailed proof, see [4, theorem 2.1].

(b) It follows from the Poincaré inequality that for any ball $B \subset \mathbb{R}^2$
\[
\frac{1}{|B|} \int_B v(x) \cdot \frac{1}{|B|} \int_B v(y) \, dy \, dx \leq C \left( \int_B |\nabla v|^2 \, dx \right)^{1/2},
\]
which directly gives (2.3).

\[
\Box
\]

3. **A priori estimates**

In this section, we will establish some necessary *a priori* bounds for strong solutions $(\rho, u, P)$ to the Cauchy problem (1.1) and (1.2) to extend the local strong solution. Thus, let $T > 0$ be a fixed time and $(\rho, u, P)$ be the strong solution to (1.1) and (1.2) on $\mathbb{R}^2 \times (0, T]$ with initial data $(\rho_0, u_0)$ satisfying (1.3)–(1.5).
In what follows, we will use the convention that $C$ denotes a generic positive constant depending on $\mu$, $a$, and the initial data, and use $C(\alpha)$ to emphasize that $C$ depends on $\alpha$.

3.1. Lower order estimates

First, since $\text{div} u = 0$, we state the following well-known estimate on the $L^\infty(0,T;L^p)$-norm of the density.

**Lemma 3.1 ([19]).** There exists a positive constant $C$ depending only on $\|\rho_0\|_{L^\infty}$ such that

$$\sup_{t \in [0,T]} \|\rho\|_{L^\infty} \leq C.$$  (3.1)

Next, the following lemma concerns the key time-independent estimates on the $L^\infty(0,T;L^2)$-norm of the gradient of the velocity.

**Lemma 3.2.** There exists a positive constant $C$ depending only on $\mu$, $\|\rho_0\|_{L^\infty}$, $\|\sqrt{\rho_0} u_0\|_{L^2}$, and $\|\nabla u_0\|_{L^2}$ such that

$$\sup_{t \in [0,T]} \|\nabla u\|_{L^2}^2 + \int_0^T \int \rho |\dot{u}|^2 \, dx \, dt \leq C.$$  (3.2)

where $\dot{u} \triangleq u_t + u \cdot \nabla u$ is the material derivatives of the velocity. Furthermore, one has

$$\sup_{t \in [0,T]} t \|\nabla u\|_{L^2}^2 + \int_0^T t \int \rho |\dot{u}|^2 \, dx \, dt \leq C.$$  (3.3)

**Proof.** First, applying standard energy estimate to (1.1) gives

$$\sup_{t \in [0,T]} \|\sqrt{\rho} u\|_{L^2}^2 + \int_0^T \|\nabla u\|_{L^2}^2 \, dt \leq C.$$  (3.4)

Next, multiplying (1.1)$_2$ by $\dot{u}$ and then integrating the resulting equality over $\mathbb{R}^2$ lead to

$$\int \rho |\dot{u}|^2 \, dx = \int (\mu \Delta u \cdot \dot{u} - \nabla P \cdot \dot{u}) \, dx \triangleq I_1 + I_2.$$  (3.5)

It follows from integration by parts and Garliardo–Nirenberg inequality that

$$I_1 = \int \mu \Delta u \cdot (\partial u + u \cdot \nabla u) \, dx$$

$$= -\frac{\mu}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + \mu \int \partial_i u' \partial_i (u^k u^l) \, dx$$

$$\leq -\frac{\mu}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^3$$

$$\leq -\frac{\mu}{2} \frac{d}{dt} \|\nabla u\|_{L^2}^2 + C \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2}.$$  (3.6)

We deduce from integration by parts and (1.1)$_3$ that
\[ I_2 = - \int \nabla P \cdot (\partial_t u + u \cdot \nabla u) \, dx \]
\[ = \int P \partial_t u' \partial_t u' \, dx \]
\[ \leq C \| P \|_{\text{BMO}} \| \partial_t u' \partial_t u' \|_{H^1}, \quad (3.7) \]

where one has used the duality of $H^1$ space and the BMO one (see [24, section 4]) in the last inequality. Since $\text{div}(\partial_t u) = \partial_j \text{div} u = 0$ and $\nabla \cdot (\nabla u') = 0$, lemma 2.5 yields
\[ |I_2| = |\int P \partial_j u_i \partial_i u_j | \leq C \| \nabla P \|_{L^2} \| \nabla u \|_{L^2}^2 . \quad (3.8) \]

Then, substituting (3.6) and (3.8) into (3.5) gives
\[ \mu \frac{d}{dt} \| \nabla u \|_{L^2}^2 + \| \sqrt{\rho} \dot{u} \|_{L^2}^2 \leq C (\| \nabla^2 u \|_{L^2} + \| \nabla P \|_{L^2}) \| \nabla u \|_{L^2}^2 . \quad (3.9) \]

On the other hand, since $(\rho, u, P)$ satisfies the following Stokes system
\[ \begin{cases} -\mu \Delta u + \nabla P = -\rho \dot{u}, & x \in \mathbb{R}^2, \\ \text{div} u = 0, & x \in \mathbb{R}^2, \\ u(x) \to 0, & |x| \to \infty, \end{cases} \quad (3.10) \]

applying the standard $L^p$-estimate to (3.10) (see [25]) yields that for any $r \in (1, \infty)$,
\[ \| \nabla^2 u \|_{L^r} + \| \nabla P \|_{L^r} \leq C(r) \| \rho \dot{u} \|_{L^r}. \quad (3.11) \]

It thus follows from (3.9), (3.11) and (3.1) that
\[ \frac{\mu}{2} \frac{d}{dt} \| \nabla u \|_{L^2}^2 + \| \sqrt{\rho} \dot{u} \|_{L^2}^2 \leq C \| \sqrt{\rho} \dot{u} \|_{L^2} \| \nabla u \|_{L^2}^2 \leq \frac{1}{2} \| \sqrt{\rho} \dot{u} \|_{L^2}^2 + C \| \nabla u \|_{L^2}^4, \quad (3.12) \]

which implies that
\[ \frac{d}{dt} \left( \mu \| \nabla u \|_{L^2}^2 \right) + \| \sqrt{\rho} \dot{u} \|_{L^2}^2 \leq C \| \nabla u \|_{L^2}^4 . \quad (3.13) \]

This combined with (3.4) and Gronwall’s inequality gives (3.2).

Finally, multiplying (3.12) by $t$ leads to
\[ \frac{d}{dt} \left( t \mu \| \nabla u \|_{L^2}^2 \right) + t \| \sqrt{\rho} \dot{u} \|_{L^2}^2 \leq Ct \| \nabla u \|_{L^2}^4 \]

which together with (3.4) and Gronwall’s inequality yields (3.3) and completes the proof of lemma 3.2.

Next, motivated by [8, 12, 18] dealing with the compressible Navier–Stokes equations, we have the following estimates on the material derivatives of the velocity which are important for the higher order estimates of both the density and the velocity.

**Lemma 3.3.** There exists a positive constant $C$ depending only on $\mu$, $\| \rho_0 \|_{L^{1, \infty}}$, $\| \sqrt{\rho_0} u_0 \|_{L^q}$ and $\| \nabla u_0 \|_{L^2}$ such that for $i = 1, 2$,
\[ \sup_{r \in [0, T]} t' \| \sqrt{\rho} \dot{u} \|_{L^2}^2 + \int_0^T t' \| \nabla \dot{u} \|_{L^2}^2 \, dt \leq C. \quad (3.13) \]
and

\[ \sup_{t \in [0,T]} \left( \| \nabla^2 u \|_{L^2}^2 + \| \nabla P \|_{L^2}^2 \right) \leq C. \]  

(3.14)

**Proof.** First, operating \( \partial_t + u \cdot \nabla \) to (1.1) \( j \) yields that

\[
\partial_t (\rho \dot{u}^j) + \text{div}(\rho \dot{u} \dot{u}^j) - \mu \Delta \dot{u}^j = -\mu \partial_i (\partial_i u \cdot \nabla u^j) - \mu \text{div}(\partial_i u \partial_i u^j) - \partial_j \partial_i P - (u \cdot \nabla) \partial_i P.
\]

Now, multiplying the above equality by \( \dot{u}^j \), we obtain after integration by parts that

\[
\frac{1}{2} \frac{d}{dt} \int \rho |\dot{u}|^2 dx + \mu \int |\nabla \dot{u}|^2 dx
\]

\[
= - \int \mu \partial_j (\partial_i u \cdot \nabla u^j) \dot{u}^j dx - \int \mu \text{div}(\partial_i u \partial_i u^j) \dot{u}^j dx + J
\]

\[
\leq C \| \nabla u \|_{L^2}^2 + \frac{\mu}{4} \| \nabla \dot{u} \|_{L^2}^2 + J,
\]

(3.15)

where

\[ J \triangleq - \int \dot{u}^j \partial_j P dx - \int \dot{u}^j u \cdot \nabla \partial_j P dx \]

satisfies

\[
J = \int P \text{div} \dot{u} dx + \int u \cdot \nabla P \text{div} \dot{u} dx + \int \partial_j u^j \partial_j u^4 dx
\]

\[
= \int (P_t + u \cdot \nabla P) \partial_j u^j \partial_j u^4 dx
\]

(3.16)

due to \( \text{div} u = 0 \).

Then, it follows from \( \text{div} u = 0 \) and integration by parts that

\[
\int (P_t + u \cdot \nabla P) \partial_j \partial_j u^4 dx
\]

\[
= \frac{d}{dt} \int P \partial_j u^j \partial_j u^4 dx - \int P \partial_j u^j \partial_j u^4 dx - \int P \partial_j u^j \partial_j u^4 dx
\]

\[
= \frac{d}{dt} \int P \partial_j u^j \partial_j u^4 dx - \int P \partial_j u^j \partial_j u^4 dx - \int P \partial_j u^j \partial_j u^4 dx
\]

\[
= \frac{d}{dt} \int P \partial_j u^j \partial_j u^4 dx - \int P \partial_j u^j \partial_j u^4 dx - \int P \partial_j u^j \partial_j u^4 dx
\]

\[
= \frac{d}{dt} \int P \partial_j u^j \partial_j u^4 dx - \int P \partial_j u^j \partial_j u^4 dx - \int P \partial_j u^j \partial_j u^4 dx
\]

\[
= \frac{d}{dt} \int P \partial_j u^j \partial_j u^4 dx + \int P \partial_j u^j \partial_j u^4 dx - \int P \partial_j u^j \partial_j u^4 dx,
\]
which together with (3.16), Hölder’s and Young’s inequalities yields
\[
J \leq \frac{d}{dt} \int P \partial_t \partial_t u^i \partial_x^i \, dx + C \int |P| \|
abla \dot{u} \| \| \nabla u \| \, dx + C \int |P| \| \nabla u \|^3 \, dx \leq \frac{d}{dt} \int P \partial_j u_i \partial_x^j \, dx + C \int |P| \|
abla \dot{u} \| \| \nabla u \|_L^2 \| + \frac{d}{dt} \| \nabla \dot{u} \|^2_2 . \tag{3.17}
\]

Next, substituting (3.17) into (3.15) gives
\[
\Psi'(t) + \frac{4}{d} \int |\nabla \dot{u}|^2 \, dx \leq C \| P \|^2_{L^4} + C \| \nabla u \|^2_{L^2} . \tag{3.18}
\]
where
\[
\Psi(t) \triangleq \frac{1}{2} \int \rho |\dot{u}|^2 \, dx - \int P \partial_j u_i \partial_x^j \, dx
\]
satisfies
\[
\frac{1}{4} \int \rho |\dot{u}|^2 \, dx - C \| \nabla u \|^2_{L^2} \leq \Psi(t) \leq \int \rho |\dot{u}|^2 \, dx + C \| \nabla u \|^2_{L^2} \tag{3.19}
\]
due to (3.8) and (3.11). Moreover, it follows from Sobolev’s inequality, (3.11) and (3.1) that
\[
\| P \|^2_{L^4} + \| \nabla u \|^2_{L^2} \leq C \left( \| \nabla P \|^2_{L^{4/3}} + \| \nabla^2 u \|^2_{L^{4/3}} \right) \leq C \| \rho \|^2_{L^2} \| \sqrt{\rho} \dot{u} \|^4_{L^2} \leq C \| \sqrt{\rho} \|^4_{L^2} . \tag{3.20}
\]

Multiplying (3.18) by \( t (i = 1, 2) \) and using (3.19) and (3.20), we thus obtain (3.13) from Gronwall’s inequality, (3.2) and (3.3).

Finally, it is easy to deduce (3.14) directly from (3.13) and (3.11). The proof of lemma 3.3 is finished.

\[\square\]

### 3.2. Higher order estimates

The following spatial weighted estimate on the density plays an important role in deriving the bounds on the higher order derivatives of the solutions \((\rho, u, P)\).

**Lemma 3.4.** There exists a positive constant \( C \) depending only on \( \mu \), \( \| \rho_0 \|_{L^1 \cap L^\infty} \), \( \| \nabla \rho_0 \|_{L^2} \), \( \| \sqrt{\rho_0} u_0 \|_{L^2} \), \( N_0 \), and \( T \) such that
\[
\sup_{t \in [0, T]} \| \rho \dot{\varphi}_N \|_{L^1} \leq C . \tag{3.21}
\]

**Proof.** First, for \( N > 1 \), let \( \varphi_N \in C_0^\infty(\mathbb{R}^2) \) satisfy
\[
0 \leq \varphi_N \leq 1, \quad \varphi_N(x) = \begin{cases} 1, & |x| \leq N/2, \\ 0, & |x| \geq N, \end{cases} \quad |\nabla \varphi_N| \leq CN^{-1} . \tag{3.22}
\]
It follows from (1.1) that
\[
\frac{d}{dt} \int \rho \varphi_N \, dx = \int \rho u \cdot \nabla \varphi_N \, dx \geq -CN^{-1} \left( \int \rho \, dx \right)^{1/2} \left( \int \rho |u|^2 \, dx \right)^{1/2} \geq -\tilde{C}N^{-1}, \tag{3.23}
\]

where in the last inequality one has used (3.1) and (3.4). Integrating (3.23) and choosing \( N = N_1 \triangleq 2N_0 + 4\tilde{C}T \), we obtain after using (1.4) that

\[
\inf_{0 \leq t \leq T} \int_{B_{\rho_0}} \rho \, dx \geq \int_{B_{\rho_0}} \rho_0 \, dx - \tilde{C}T \geq 1/4. \tag{3.24}
\]

Hence, it follows from (3.24), (3.1), (2.2), (3.4) and (3.2) that for any \( \eta \in (0, 1] \) and any \( s > 2 \),

\[
\| \bar{u} \cdot x^{-\eta}_{L^{s/\eta}} \| \leq C \left( \| \rho^{1/2} u \|_{L^2} + \| \nabla u \|_{L^2} \right) \leq C. \tag{3.25}
\]

Multiplying (1.1) by \( \bar{x}^a \) and integrating the resulting equality by parts over \( \mathbb{R}^2 \) yield that

\[
\frac{d}{dt} \int \rho \bar{x}^a \, dx \leq C \int \rho |\bar{x}^{a-1} \log (e + |x|^2) | \, dx \leq C \| \rho \bar{x}^{a-1} \|_{L^1} \| \bar{x}^{-1/2} \|_{L^{s+1}} \leq C \int \rho \bar{x}^a \, dx + C,
\]

which along with Gronwall’s inequality gives (3.21) and then finishes the proof of lemma 3.4.

\[\square\]

**Lemma 3.5.** There exists a positive constant \( C \) depending on \( T \) such that

\[
\sup_{t \in [0, T]} \| \rho \|_{W^{1, q}} + \int_0^T \left( \| \nabla^2 u \|_{L^2} + \| \nabla^2 u \|_{L^{2+q}} + t \| \nabla^2 u \|_{L^{2+q}} \right) \, dt \leq C(T). \tag{3.26}
\]

**Proof.** First, it follows from the mass equation (1.1) that \( \nabla \rho \) satisfies for any \( r \geq 2 \),

\[
\frac{d}{dt} \| \nabla \rho \|_{L^r} \leq C(r) \| \nabla u \|_{L^{\infty}} \| \nabla \rho \|_{L^r}. \tag{3.27}
\]

Next, one gets from Gagliardo–Nirenberg inequality, (3.2) and (3.11) that for \( q > 2 \) as in theorem 1.1,

\[
\| \nabla u \|_{L^{\infty}} \leq C \| \nabla u \|_{L^2}^{q-1} \| \nabla^2 u \|_{L^2}^{q-2} \leq C \| \rho \bar{u} \|_{L^2}^{q-1}. \tag{3.28}
\]
It follows from (3.24), (3.1), (2.2) and (3.21) that for any η ∈ (0, 1] and any s > 2,

\[
\|\rho^{\eta/2}v\|_{L^s} \leq C\|\rho^{\eta/2}x\|_{L^\infty} \frac{\|v\|_{L^s}}{\|x\|_{L^s}} \\
\leq C\|\rho\|_{L^\infty}^{\frac{2s-4\eta}{s}}\|\rho^{1/2}v\|_{L^s} + \|\nabla v\|_{L^s}
\leq C \left(\|\rho^{1/2}v\|_{L^2} + \|\nabla v\|_{L^2}\right),
\]

(3.29)

which together with the Gagliardo–Nirenberg inequality shows that

\[
\|\rho\dot{u}\|_{L^1} \leq C\|\rho\dot{u}\|_{L^{2d-2}} \frac{\sigma(s-2)}{\sigma^{(1)}} \\
\leq C\|\rho\dot{u}\|_{L^{2d-2}} \left(\|\sqrt{\rho}\dot{u}\|_{L^2} + \|\nabla \dot{u}\|_{L^2}\right) \\
\leq C \left(\|\sqrt{\rho}\dot{u}\|_{L^2} + \|\nabla \dot{u}\|_{L^2}\right). 
\]

(3.30)

This combined with (3.2) and (3.13) leads to

\[
\int_0^T \left(\|\rho\dot{u}\|_{L^2}^2 + t\|\rho\dot{u}\|_{L^2}^2\right) dt \\
\leq C \int_0^T \left(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + t\|\nabla \dot{u}\|_{L^2}^2 + t \frac{s-2\sigma-2\eta}{s-2\sigma-2\eta} + 1\right) dt \\
\leq C,
\]

(3.31)

which along with (3.28) in particular implies

\[
\int_0^T \|\nabla u\|_{L^\infty} dt \leq C.
\]

(3.32)

Thus, applying Gronwall’s inequality to (3.27) gives

\[
\sup_{t \in [0,T]} \|\nabla \rho\|_{L^2} \leq C.
\]

(3.33)

Furthermore, it is easy to deduce from (3.11), (3.31) and (3.1)–(3.3) that

\[
\int_0^T \left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^{2d+1} + t\|\nabla^2 u\|_{L^2}^2\right) dt \\
+ \int_0^T \left(\|\nabla P\|_{L^2}^2 + \|\nabla P\|_{L^2}^{2d+1} + t\|\nabla P\|_{L^2}^2\right) dt \leq C,
\]

(3.34)

which together with (3.1) and (3.33) yields (3.26) and completes the proof of lemma 3.5. □

**Lemma 3.6.** There exists a positive constant C depending on T such that

\[
\sup_{t \in [0,T]} \|\rho^{1/2}v\|_{L^1} \leq C(T).
\]

(3.35)
\textbf{Proof.} One derives from (1.1) that $\rho \bar{x}$ satisfies
\begin{equation}
\partial_t (\rho \bar{x}) + u \cdot \nabla (\rho \bar{x}) - a \rho \bar{x} u \cdot \nabla \log \bar{x} = 0.
\end{equation}

Taking the $x_i$-derivative on both sides of (3.36) gives
\begin{equation}
0 = \partial_i \partial_t (\rho \bar{x}) + u \cdot \nabla \partial_i (\rho \bar{x}) + \partial_i u \cdot \nabla (\rho \bar{x}) - a \rho \bar{x} \partial_i u \cdot \nabla \log \bar{x} - a \rho \bar{x} u \cdot \nabla \log \bar{x}.
\end{equation}

For any $r \in [2, q]$, multiplying (3.37) by $|\nabla (\rho \bar{x})|^{-2} \partial_t (\rho \bar{x})$ and integrating the resulting equality over $\mathbb{R}^2$, we obtain after integration by parts that
\begin{align*}
\frac{d}{dt} \|\nabla (\rho \bar{x})\|_{L^r} &\leq C (1 + \|\nabla u\|_{L^\infty} + \|\rho u \cdot \nabla \log \bar{x}\|_{L^\infty}) \|\nabla (\rho \bar{x})\|_{L^r} \\
&\quad + C \|\rho \bar{x}\|_{L^\infty} \left( \|\nabla u\|_{L^r} + \|\rho u \cdot \nabla \log \bar{x}\|_{L^r} \right) \\
&\quad + C \|\rho \bar{x}\|_{L^\infty} \left( \|\nabla u\|_{L^r} + \|\rho u \cdot \nabla \log \bar{x}\|_{L^r} \right) \\
&\leq C (1 + \|\nabla u\|_{L^r}) (1 + \|\nabla (\rho \bar{x})\|_{L^r} + \|\nabla (\rho \bar{x})\|_{L^r}),
\end{align*}
where in the second and the last inequalities, one has used (3.25) and (3.21), respectively. Choosing $r = q$ in (3.38), one obtains after using (3.26) that
\begin{equation}
\sup_{t \in [0, T]} \|\nabla (\rho \bar{x})\|_{L^r} \leq C.
\end{equation}

Setting $r = 2$ in (3.38), we deduce from (3.26) and (3.39) that
\begin{equation}
\sup_{t \in [0, T]} \|\nabla (\rho \bar{x})\|_{L^2} \leq C.
\end{equation}

This combined with (3.39) and (3.21) gives (3.35). The proof of lemma 3.6 is completed. \hfill \Box

\textbf{Lemma 3.7.} \textit{There exists a positive constant $C$ depending on $T$ such that}
\begin{equation}
\sup_{t \in [0, T]} t \|\sqrt{\rho} u_t\|_{L^2}^2 + \int_0^T t \|\nabla u_t\|_{L^2}^2 \, dt \leq C(T).
\end{equation}

\textbf{Proof.} Differentiating (1.1) with respect to $t$ leads to
\begin{equation}
\rho u_t + \rho \cdot \nabla u_t - \mu \Delta u_t + \nabla P_t = -\rho u_t - (\rho u_t) \cdot \nabla u.
\end{equation}

Multiplying (3.42) by $u_t$ and integrating the resulting equation by parts over $\mathbb{R}^2$, we obtain after using (1.1) and (1.1) that
\begin{align*}
\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} u_t\|_{L^2}^2 + \mu \|\nabla u_t\|_{L^2}^2 &= -\int \rho |u_t|^2 \, dx - \int (\rho u_t) \cdot \nabla u \cdot u_t \, dx \\
&\leq C \int \rho |u_t| (|\nabla u_t| + |\nabla u|^2 + |u| |\nabla^2 u|) \, dx \\
&\quad + C \int \rho |u_t|^2 |\nabla u| \, dx + C \int \rho |u_t|^2 \, dx \\
&\triangleq K_1 + K_2 + K_3.
\end{align*}
We will estimate each term on the right-hand side of (3.43) as follows. First, the combination of (3.25), (3.29), (3.4) and (3.2) gives that for any \( \eta \in \{0, 1\} \) and any \( s > 2 \),

\[
\|\rho^\eta u\|_{L^{s/n}} + \|u^{s-\eta}\|_{L^{s/n}} \leq C,
\]

(3.44)

which together with (3.29) and (3.2) Hölder’s inequality yields that

\[
K_1 \leq C\|\sqrt{\rho}u\|_{L^1} \|\sqrt{\rho}u\|_{L^2}^{1/2} \|\sqrt{\rho}u\|_{L^2}^{1/2} (\|\nabla u\|_{L^2} + \|\nabla u\|_{L^2})
\]

\[
+ C\|\rho^{1/4}u\|_{L^2}^{1/2} \|\nabla u\|_{L^4}^{1/2} \|\nabla u\|_{L^2} \leq C\|\sqrt{\rho}u\|_{L^2}^{1/2} (\|\sqrt{\rho}u\|_{L^2} + \|\nabla u\|_{L^2})^{1/2} (\|\nabla u\|_{L^2} + \|\nabla^2 u\|_{L^2})
\]

\[
\leq \frac{\mu}{4} \|\nabla u\|_{L^2}^2 + C (1 + \|\sqrt{\rho}u\|_{L^2}^{1/2} + \|\nabla^2 u\|_{L^2}^2).
\]

Next, Hölder’s inequality combined with (3.29), (3.2) and (3.44) leads to

\[
K_2 + K_3 \leq C\|\sqrt{\rho}u\|_{L^2} \|\nabla u\|_{L^2} \|\nabla u\|_{L^2} + \|\nabla u\|_{L^2} \|\sqrt{\rho}u\|_{L^2}^{1/2} \|\sqrt{\rho}u\|_{L^2}^{1/2}
\]

\[
\leq \frac{\mu}{4} \|\nabla u\|_{L^2}^2 + C (1 + \|\sqrt{\rho}u\|_{L^2}^{1/2} + \|\nabla^2 u\|_{L^2}^2).
\]

Substituting (3.45) and (3.46) into (3.43) gives

\[
\frac{d}{dt}\|\sqrt{\rho}u\|_{L^2}^2 + \mu \|\nabla u\|_{L^2}^2 \leq C\|\sqrt{\rho}u\|_{L^2}^2 + C\|\nabla^2 u\|_{L^2}^2 + C.
\]

(3.47)

Finally, it deduces from (3.2), (3.44) and (3.26), Garliardo–Nirenberg inequality that

\[
\int_0^T \|\sqrt{\rho}u\|_{L^2} \, dt \leq \int_0^T (\|\sqrt{\rho}u\|_{L^2} + \|\sqrt{\rho}u\|_{L^2}) \, dt
\]

\[
\leq C + \frac{C}{\mu} \int_0^T \|\nabla u\|_{L^2}^2 \, dt
\]

\[
\leq C + \frac{C}{\mu} \int_0^T (\|\nabla u\|_{L^2}^2 + \|\nabla^2 u\|_{L^2}^2) \, dt
\]

\[
\leq C.
\]

(3.48)

Then, multiplying (3.47) by \( t \), one derives (3.41) from Gronwall’s inequality, (3.26) and (3.48). This completes the proof of lemma 3.7.

\[\square\]

4. Proof of theorem 1.1

With the \textit{a priori} estimates in section 3 at hand, we are now in a position to prove theorem 1.1.

By lemma 2.1, there exists a \( T_* > 0 \) such that the problem (1.1) and (1.2) has a unique local strong solution \((\rho, u, P)\) on \( \mathbb{R}^2 \times (0, T_*] \). We plan to extend the local solution to all time.

Set

\[
T^* = \sup \{ T \mid (\rho, u, P) \text{ is a strong solution on } \mathbb{R}^2 \times (0, T] \}.
\]

(4.1)
First, for any \( 0 \leq r < T < T^* \leq T \) with \( T \) finite, one deduces from (3.2), (3.14) and (3.41) that for any \( q \geq 2 \),
\[
\nabla u \in C([r,T]; L^2 \cap L^q),
\]
where one has used the standard embedding
\[
L^\infty(\tau,T; H^1) \cap H^1(\tau,T; H^{-1}) \hookrightarrow C(\tau,T; L^q) \quad \text{for any} \quad q \in [2, \infty).
\]
Moreover, it follows from (3.26) and (3.35), [19, lemma 2.3] that
\[
\rho \in C([0,T]; L^1 \cap H^1 \cap W^{1,q}).
\]
Finally, if \( T^* < \infty \), it follows from (4.2), (4.3), (3.2), (3.4) and (3.35) that
\[
(\rho, u)(x, T^*) = \lim_{t \to T^*} (\rho, u)(x,t)
\]
satisfies the initial conditions (1.5) at \( t = T^* \). Thus, taking \( (\rho, u)(x, T^*) \) as the initial data, lemma 2.1 implies that one can extend the strong solutions beyond \( T^* \). This contradicts the assumption of \( T^* \) in (4.1). The proof of theorem 1.1 is completed. \( \square \)

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