On basic 2-arc-transitive graphs

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Abstract
A connected graph $\Gamma = (V, E)$ of valency at least 3 is called a basic 2-arc-transitive graph if its full automorphism group has a subgroup $G$ with the following properties:
(i) $G$ acts transitively on the set of 2-arcs of $\Gamma$, and (ii) every minimal normal subgroup of $G$ has at most two orbits on $V$. Based on Praeger’s theorems on 2-arc-transitive graphs, this paper presents a further understanding on the automorphism group of a basic 2-arc-transitive graph.

Keywords 2-arc-transitive graph · Stabilizer · Quasiprimitive permutation group · Almost simple group

Mathematics Subject Classification 05C25 · 20B25

1 Introduction

All graphs considered in this paper are assumed to be finite, simple and undirected.

Let $\Gamma = (V, E)$ be a graph with vertex set $V$ and edge set $E$. Denote by $\text{Aut}(\Gamma)$ the full automorphism group of the graph $\Gamma$. A subgroup $G$ of $\text{Aut}(\Gamma)$, written as $G \leq \text{Aut}(\Gamma)$, is called a group of $\Gamma$. For a vertex $\alpha \in V$, let $G_\alpha = \{ g \in G \mid \alpha^g = \alpha \}$ and $\Gamma(\alpha) = \{ \beta \in V \mid \{\alpha, \beta\} \in E \}$, called the stabilizer of $\alpha$ in $G$ and the neighborhood of $\alpha$ in $\Gamma$, respectively. A group $G$ of $\Gamma$ is call locally primitive on $\Gamma$ if for each $\alpha \in V$ the stabilizer $G_\alpha$ acts primitively on $\Gamma(\alpha)$, that is, $\Gamma(\alpha)$ has no nontrivial $G_\alpha$-invariant partition. Recall that an arc of $\Gamma$ is an ordered pair of adjacent vertices, and a 2-arc is a triple $(\alpha, \beta, \gamma)$ of vertices with $\{\alpha, \beta\}, \{\beta, \gamma\} \in E$ and $\alpha \neq \gamma$. A group $G$ of $\Gamma$
is said to be vertex-transitive, edge-transitive, arc-transitive or 2-arc-transitive on $\Gamma$ if $G$ acts transitively on the vertices, edges, arcs or 2-arcs of $\Gamma$, respectively. A graph is called vertex-transitive, edge-transitive, arc-transitive or 2-arc-transitive if it has a vertex-transitive, edge-transitive, arc-transitive or 2-arc-transitive group, respectively.

A connected regular graph $\Gamma = (V, E)$ of valency at least 3 is called a basic 2-arc-transitive graph if it has a 2-arc-transitive group $G$ such that every minimal normal subgroup of $G$ has at most two orbits on $V$. Praeger [17, 18] observed that a connected 2-arc-transitive graph of valency at least 3 is a normal cover of some basic 2-arc-transitive graph. Based on the O’Nan–Scott theorem for quasiprimitive permutation groups established in [17], Praeger [17, 18] characterized the group-theoretic structures for basic 2-arc-transitive graphs. She proved that, except for complete bipartite graphs and another case about bipartite graphs, basic 2-arc-transitive graphs are associated with quasiprimitive groups of type I, II, IIIb(i) or III(c) described as in [17, Section 2], which is named HA, AS, PA or TW in [19], respectively.

Praeger’s framework for 2-arc-transitive graphs stimulated a wide interest in classification or characterization of basic 2-arc-transitive graphs. For example, a construction of the graphs associated with quasiprimitive permutation groups of type TW is given in [2], the graphs associated with Suzuki simple groups, Ree simple groups and 2-dimensional projective linear groups are classified in [5, 6, 9], respectively, the graphs of order a prime power are classified in [10]. Besides, Li [11] proved that all basic 2-arc-transitive graphs of odd order can be constructed from almost simple groups, which inspires the ongoing project to classify basic 2-arc-transitive graphs of odd order, see [12] for some progress in this topic.

In this paper, we have a further understanding on the automorphism groups of basic 2-arc-transitive graphs, which may be helpful to study the Praeger’s problem proposed in [18]: *Classify all finite basic 2-arc-transitive graphs.* Assume that $\Gamma = (V, E)$ is a basic 2-arc-transitive graph with respect to $G$. Fix an edge $\{\alpha, \beta\} \in E$, and set $G^* = \langle G_\alpha, G_\beta \rangle$. It is well-known that $|G : G^*| \leq 2$, $G^*$ is edge-transitive on $\Gamma$, and $\Gamma$ is bipartite if and only if $|G : G^*| = 2$, refer to [21, Exercise 3.8]. If $\Gamma$ is not bipartite, then $G$ is a quasiprimitive permutation group on $V$ of type HA, AS, PA or TW, refer to [17, Theorem 2] or [19, Theorem 6.1]. In this case, it is easily deduced that $G$ has a unique minimal normal subgroup, the socle $\text{soc}(G)$ of $G$. Somewhat surprisingly, this is almost true for the bipartite case. If $\Gamma$ is bipartite, that is, $|G : G^*| = 2$, then Praeger [18] proved that either $\Gamma$ is a complete bipartite graph, or $G^*$ acts faithfully on both parts of $\Gamma$ and one of the following holds:

(I) $G^*$ is quasiprimitive on both parts of $\Gamma$ with a same type HA, AS, PA or TW;

(II) $G$ has a normal subgroup $N$ which is a direct product of two intransitive minimal normal subgroups of $G^*$.

For (I) and (II), we prove in Sect. 3 that $\text{soc}(G^*)$ is the unique minimal normal subgroup of $G$, and so $\text{soc}(G) = \text{soc}(G^*)$. Thus, in general, $\text{soc}(G)$ is the unique minimal normal subgroup of $G$, provided that $\Gamma$ is not a complete bipartite graph. Based on this observation and the description of types HA, AS, PA or TW, we investigate in Sect. 4 the action of $\text{soc}(G)$ on the graph $\Gamma$, including the structure of vertex-stabilizers and the semiregularity of simple direct factors of $\text{soc}(G)$. Then, we formulate the following result, which is finally proved in Sect. 4.
Theorem 1.1 Assume that $\Gamma = (V, E)$ is a basic 2-arc-transitive graph with respect to a group $G$. Let $G^* = \langle G_\alpha, G_\beta \rangle$ and $N = \text{soc}(G^*)$, where $\{\alpha, \beta\} \in E$. Then, either $\Gamma$ is a complete bipartite graph, or the following statements hold:

1. $N$ is the unique minimal normal subgroup of $G$, in particular, $N = \text{soc}(G)$;
2. either $N$ is simple, or every simple direct factor of $N$ is semiregular on $V$;
3. either $N$ is locally primitive on $\Gamma$, or $N_\alpha$ is given as follows:
   
   (i) $N_\alpha = 1$; or
   (ii) $N_\alpha = \mathbb{Z}_p^d : (\mathbb{Z}_{m_1} \times \mathbb{Z}_{m^2})$, where $m_1 | m$, $m | (p^d - 1)$ for some divisor $d$ of $k$ with $d < k$; or
   (iii) $N_\alpha = \mathbb{Z}_3^4 : (Q_8 \times Q_8) = (\mathbb{Z}_3^4 \times Q_8)$ and $|\Gamma(\alpha)| = 3^4$, where $Q_8$ is the quaternion group and $Q$ is isomorphic to a subgroup of $Q_8$.

It is well known that the order of a finite nonabelian simple group is divisible by 4 and two distinct odd primes. By (2) of Theorem 1.1, we have the following corollaries.

Corollary 1.2 Assume that $\Gamma = (V, E)$ is a basic 2-arc-transitive graph with respect to a group $G$, and $G^* = \langle G_\alpha, G_\beta \rangle$ for an edge $\{\alpha, \beta\} \in E$. Assume that one of $G^*$-orbits on $V$ has length $p^a q^b$, where $a$ and $b$ are positive integers, $p$ and $q$ are distinct primes. If $\Gamma$ is not a complete bipartite graph, then $G$ is almost simple.

Corollary 1.3 Assume that $\Gamma = (V, E)$ is a basic 2-arc-transitive graph with respect to a group $G$, and $G^* = \langle G_\alpha, G_\beta \rangle$ for an edge $\{\alpha, \beta\} \in E$. Assume that one of $G^*$-orbits on $V$ has length $n$ or $2n$, where $n$ is either an odd integer or a power of 2. If $\Gamma$ is not a complete bipartite graph, then either $G$ is almost simple, or $|G^* : G_\alpha| = p^k$ and $\text{soc}(G^*) \cong \mathbb{Z}_p^k$, where $p$ is a prime and $k \geq 1$.

Another consequence of Theorem 1.1 is stated as follows.

Theorem 1.4 Let $\Gamma = (V, E)$ be a connected graph, $G \leq \text{Aut}(\Gamma)$ and $G^* = \langle G_\alpha, G_\beta \rangle$ for an edge $\{\alpha, \beta\} \in E$. Assume that $G$ is 2-arc-transitive on $\Gamma$, and $G^*$ acts primitively on each $G^*$-orbit on $V$. Then, one of the following holds:

1. $\Gamma$ is a complete bipartite graph;
2. $\text{soc}(G) = \text{soc}(G^*)$, and $\text{soc}(G^*)$ is either simple or regular on each $G^*$-orbit;
3. $\Gamma$ is bipartite, $\text{soc}(G) = \text{soc}(G^*) \times M$ with $|M| = 2$, and $\text{soc}(G^*)$ is either simple or regular on each $G^*$-orbit.

2 Some observations on 2-transitive permutation groups

This section gives some simple results about 2-transitive permutation groups, which serve to analyze the structures of vertex-stabilizers of 2-arc-transitive graphs.

Let $X$ be a transitive permutation group on a finite set $\Omega$. Recall that the socle $\text{soc}(X)$ is generated by all minimal normal subgroups of $X$. It is easily shown that $\text{soc}(X)$ is a characteristic subgroup of $X$. Assume that $X$ is a 2-transitive permutation group on $\Omega$. Then, $\text{soc}(X)$ is either elementary abelian and regular on $\Omega$, or simple and primitive on $\Omega$, refer to [3, p. 101, Theorem 4.3] and [4, p. 107, Theorem 4.1B]. In particular, $X$
is either affine or almost simple. Inspecting the lists of finite 2-transitive permutation groups (refer to [3, pp. 195–197, Tables 7.3 and 7.4]), we have the following basic fact, see also [14, Corollary 2.5].

**Lemma 2.1** Let $X$ be a 2-transitive permutation group on a finite set $\Omega$, and $\alpha \in \Omega$. Assume that $K$ is an insoluble normal subgroup of $X_\alpha$. Then, $K$ has a unique insoluble composition factor say $S$, and $S$ is isomorphic to a composition factor of $X$ if and only if $X$ is affine.

Recall that a transitive permutation group $X$ on $\Omega$ is a Frobenius group if $X$ is not regular on $\Omega$ and, for $\alpha \in \Omega$, the point-stabilizer $X_\alpha$, called a Frobenius complement of $X$, is semiregular on $\Omega \setminus \{\alpha\}$.

By Frobenius’ theorem (refer to [1, pp. 190–191, (35.23) and (35.24)]), for a Frobenius group $X$ on $\Omega$, the identity and the elements without fixed-point form a normal subgroup of $X$, which is called the Frobenius kernel of $X$.

**Lemma 2.2** Let $X = KH$ be an imprimitive Frobenius group on $\Omega$ with the Frobenius kernel $K \cong \mathbb{Z}_p^k$ and a Frobenius complement $H$, where $p$ is a prime and $k \geq 2$. Then, $H$ is isomorphic to an irreducible subgroup of the general linear group $GL_d(p)$, and $|H|$ is a divisor of $p^d - 1$, where $2l \leq k$ and $d$ is a common divisor of $k$ and $l$.

**Proof** Note that $H$ acts faithfully and semiregularly on $K \setminus \{1\}$ by conjugation, see [1, p. 191, (35.25)]. Then, $|H|$ is a divisor of $p^k - 1$. Recall that $X$ is imprimitive on $\Omega$. Then, $K$ is not a minimal normal subgroup of $X$. By Maschke’s theorem (refer to [1, p. 40, (12.9)]), $K$ is a direct product of two $H$-invariant proper subgroups. Thus, we may choose a minimal $H$-invariant subgroup $L$ of $K$ with $|L|^2 \leq |K|$. It is easily shown that $LH$ is a primitive Frobenius group (on an $L$-orbit), which has the Frobenius kernel $L$. Set $|L| = p^l$. Then, $|H|$ is a divisor of $p^l - 1$, $2l \leq k$, and $H$ is isomorphic to an irreducible subgroup of $GL_d(p)$.

Choose a minimal positive integer $d$ such that $|H|$ is a divisor of $p^d - 1$. Then, $d \leq l$. Set $k = xd + y$ for integers $x \geq 1$ and $0 \leq y < d$. Then, $p^k - 1 = p^y(p^{xd} - 1) + (p^y - 1)$, and thus $|H|$ is a divisor of $p^y - 1$. By the choice of $d$, we have $y = 0$, and so $d$ is a divisor of $k$. Similarly, $d$ is a divisor of $l$. Then, the lemma follows. \hfill \Box

**Lemma 2.3** Let $X$ be a 2-transitive permutation group on a finite set $\Omega$. Assume that $1 \neq N \subseteq X$. Then, $\text{soc}(N) = \text{soc}(X)$, and either $N$ is primitive on $\Omega$ or one of the following holds:

1. $N = \mathbb{Z}_p^k; \mathbb{Z}_m$ and $|\Omega| = p^k$, where $p$ is a prime, $k \geq 2$, $m|(p^d - 1)$ for some divisor $d$ of $k$ with $d < k$;
2. $N = \mathbb{Z}_3^4; \mathbb{Q}_8$ and $|\Omega| = 3^4$.

**Proof** Since $X$ is 2-transitive on $\Omega$, by [4, p. 107, Theorem 4.1B], $\text{soc}(X)$ is either abelian or nonabelian simple. By [4, p. 114, Theorem 4.3B], the centralizer $C_X(\text{soc}(X)) = \text{soc}(X)$ or 1, respectively. In particular, $\text{soc}(X)$ is the unique minimal normal subgroup of $X$. Noting that $\text{soc}(N)$ is characteristic in $N$, it follows that $\text{soc}(N)$ is a normal subgroup of $X$, and thus $\text{soc}(X) \leq \text{soc}(N)$. Suppose that...
soc(X) ≠ soc(N). Then, soc(N) has a simple direct factor T with T ∩ soc(X) = 1. Since both T and soc(X) are normal in soc(N), we deduce that T centralizes soc(X), and so T ≤ C_X(soc(X)) = soc(X) or 1, a contradiction. Therefore, soc(N) = soc(X).

Next we assume that N is imprimitive on Ω, and show that one of (1) and (2) holds. By [4, pp. 215–217, Theorems 7.2C and 7.2E], soc(N) = soc(X) ≅ \mathbb{Z}_p^k for a prime p and integer k ≥ 2 with |Ω| = p^k, and either N = soc(X) or N is a Frobenius group with the Frobenius kernel soc(X). In particular, by Lemma 2.2, we write N = KH, where K ≅ \mathbb{Z}_p^k and |H| is a divisor of p^d − 1 for a divisor d of k with d < k. Note that X is an affine 2-transitive permutation group. Inspecting the finite affine 2-transitive permutation groups listed in [3, p. 197, Table 7.4], we conclude that either H is cyclic, or one of the following holds:

(i) H ≤ X_0 ≤ ΓL_1(p^k), where X_0 is a point-stabilizer in X;
(ii) p^k = 3^4, yielding d ∈ {1, 2}, and so |H| is a divisor of 8.

If H is cyclic, then N is described as in part (1) of this lemma. In the following, we assume further that H is not cyclic.

Suppose that (i) holds. If k = 2, then H is isomorphic to a subgroup of GL_1(p) by Lemma 2.2, and so H is cyclic, which is not the case. If p^k = 2^9, then |H| is a divisor of 2^d − 1 with d ∈ {1, 2, 3} by Lemma 2.2, which yields that H is cyclic, a contradiction. Thus, k > 2 and p^k ≠ 2^6. By the Zsigmondy theorem, there exists a prime r such that p^k − 1 ≡ 0 mod r but p^l − 1 ≠ 0 mod r for 1 < l < k. In particular, p has order k modulo r, and so k is a divisor of r − 1. Recall that |H| is a divisor of p^d − 1, where d < k. It follows that r is not a divisor of |H|, and so H contains no element of order r. Since X is a 2-transitive group of degree p^k, the order of X_0 is divisible by p^k − 1. Pick an element x ∈ X_0 with order r. Write ΓL_1(p^k) = ⟨a, τ | a^{p^k−1} = 1, τ^k = 1, τ^{-1}aτ = a^p⟩. Clearly, ⟨a⟩ is normal in ΓL_1(p^k), and so ⟨a⟩x ≤ ΓL_1(p^k). In particular, |⟨a⟩x| is a divisor of |ΓL_1(p^k)| = (p^k − 1)k. Noting that |⟨a⟩x| = |⟨[a]|[x]|⟩/|[a]| = (p^k − 1)r/|⟨a⟩∩[x]|, it follows that

\[ r \frac{r}{|⟨a⟩∩[x]|} \]

is a divisor of k. Since r > k and r is a prime, we have |⟨a⟩∩[x]| = r, yielding x ∈ ⟨a⟩. Then, τ_1^{-1}xτ_1 = x^p. Since H is not cyclic, we take an element a^{i}τ_j ∈ H\{a⟩, where 1 < j < k. We have x^{-1}a^iτ_jx ∈ H as H ≦ X_0. Noting that x^{-1}a^iτ_jx = x^{p^k−j−1}a^iτ_j = x^{p^k−j−1}a^iτ_j, we deduce that x^{p^k−j−1} ∈ H. Since 1 < k − j < k, by the choice of r, we have p^k−j−1 ≠ 0 mod r. Thus, H contains an element x^{p^k−j−1} of order r, a contradiction.

Suppose that (ii) holds. By Lemma 2.2, H is isomorphic to an irreducible subgroup of GL_2(3). Choose a minimal H-invariant subgroup L of K with L ≅ \mathbb{Z}_3^2. Then, LH is a primitive Frobenius group of degree 9 and of order a divisor of 72. Confirmed by GAP [20], up to permutation isomorphism, there are four affine primitive groups of degree 9 which have order a divisor of 72, say \mathbb{Z}_3^2;\mathbb{Z}_4, \mathbb{Z}_3^2;\mathbb{Z}_8, \mathbb{Z}_3^2;\mathbb{D}_8 and \mathbb{Z}_3^2;\mathbb{Q}_8. In addition, the group \mathbb{Z}_3^2;\mathbb{D}_8 is not a Frobenius group. Since H is not cyclic, we have H ≦ \mathbb{Q}_8, and thus part (2) of this lemma follows. This completes the proof. □

Lemma 2.4 Let X be an affine 2-transitive permutation group, and soc(X) = K_1 × ⋯ × K_l where 1 < K_i < soc(X) for 1 ≤ i ≤ l. Then, there exist x ∈ X and i such that K_i^x ≠ {K_i | 1 ≤ i ≤ l}.
Proof} Clearly, \( \cup_i (K_i \setminus \{1\}) \neq \text{soc}(X) \setminus \{1\} \). Let \( H \) be a point-stabilizer in \( X \). Then, \( H \) acts transitively on \( \text{soc}(X) \setminus \{1\} \) by conjugation. Thus, \( H \) does not fix \( \cup_i (K_i \setminus \{1\}) \) set-wise by conjugation, and the lemma follows. \( \square \)

3 The uniqueness of minimal normal subgroup

In this section, we assume that \( \Gamma = (V, E) \) is a connected regular graph, and \( G \leq \text{Aut}(\Gamma) \). Denote by \( G_\alpha^\Gamma \) the permutation group induced by \( G_\alpha \) on \( \Gamma(\alpha) \). Let \( G_\alpha^{[1]} \) be the kernel of \( G_\alpha \) acting on \( \Gamma(\alpha) \). Then,

\[
G_\alpha^\Gamma \cong G_\alpha / G_\alpha^{[1]}.
\]

Let \( \beta \in \Gamma(\alpha) \), and set \( G_\alpha^{[1]} = G_\alpha^{[1]} \cap G_\beta^{[1]} \). Then, \( G_\alpha^{[1]} \) is the kernel of the arc-stabilizer \( G_\alpha^{[1]} \beta \) acting on \( \Gamma(\alpha) \cup \Gamma(\beta) \). Noting that \( G_\alpha^{[1]} \cong G_\alpha^{[1]} \beta \), we have

\[
G_\alpha^{[1]} / G_\alpha^{[1]} \cong (G_\alpha^{[1]} \beta)^{\Gamma(\beta)} \leq G_\alpha^{\Gamma(\beta)} = (G_\beta^{\Gamma(\beta)})_\alpha.
\]

Assume that \( G \) is arc-transitive on \( \Gamma \), and \( N \) is an arbitrary normal subgroup of \( G \). Then,

\[
N_\alpha \leq G_\alpha, N_\alpha^{[1]} \leq G_\alpha^{[1]}, N_\alpha^{[1]} \leq G_\alpha^{[1]}, N_\alpha^{[1]} \leq G_\alpha^{[1]}.
\]

Taking \( x \in G \) with \( (\alpha, \beta)^x = (\beta, \alpha) \), we have

\[
N_\beta = N_\alpha^x, N_\alpha^{[1]} = N_\alpha^{[1]} \beta, \Gamma(\beta) = \Gamma(\alpha)^x.
\]

It follows that \( N_\alpha^{\Gamma(\beta)} \cong N_\beta^{\Gamma(\alpha)} \). Since \( N_\alpha^{[1]} \leq G_\alpha^{[1]} \), we have \( N_\alpha^{[1]} \cong G_\alpha^{[1]} \), and so

\[
N_\alpha^{[1]} / N_\alpha^{[1]} \cong (N_\alpha^{[1]})^{\Gamma(\beta)} \leq N_\alpha^{\Gamma(\beta)} \cong N_\alpha^{\Gamma(\alpha)} \cong (N_\alpha^{\Gamma(\alpha)})_\beta \leq (G_\alpha^{\Gamma(\alpha)})_\beta. \quad (3.1)
\]

In particular, \( (N_\alpha^{[1]})^{\Gamma(\beta)} \) is isomorphic to a normal subgroup of \( (N_\alpha^{\Gamma(\alpha)})_\beta \).

Assume that \( G \) is 2-arc-transitive on \( \Gamma \). Then, \( G_\alpha^{[1]} \) has order a prime power, see [7, Corollary 2.3]. In particular, \( G_\alpha^{[1]} \) is soluble. Then, \( (G_\alpha^{[1]})^{\Gamma(\beta)} \) is soluble if and only if \( G_\alpha^{[1]} \) is soluble. Noting that \( G_\alpha^{\Gamma(\alpha)} \) is a 2-transitive group on \( \Gamma(\alpha) \), by Lemma 2.1 and (3.1), we have the following fact.

Lemma 3.1 Assume that \( G \) is 2-arc-transitive on \( \Gamma = (V, E) \), \( N \leq G \) and \( N_\alpha \) is insoluble, where \( \alpha \in V \). Then, \( N_\alpha^{\Gamma(\alpha)} \) has a unique insoluble composition factor, and \( N_\alpha^{[1]} \) has at most one insoluble composition factor. If \( N_\alpha^{[1]} \) and \( N_\alpha^{\Gamma(\alpha)} \) have isomorphic insoluble composition factors, then \( G_\alpha^{\Gamma(\alpha)} \) is an affine 2-transitive permutation group.
Proof Let \( \beta \in \Gamma(\alpha) \). Then, \( N_{\alpha}^{[1]} \) is soluble as \( N_{\alpha}^{[1]} \leq G_{\alpha}^{[1]} \). By (3.1), we may write \( N_{\alpha} = N_{\alpha}^{[1]}(N_{\alpha}^{[1]})^{\Gamma(\beta)} \cdot N_{\alpha}^{\Gamma(\alpha)} \). In addition, \((N_{\alpha}^{[1]})^{\Gamma(\beta)} \) is isomorphic to a normal subgroup of \((N_{\alpha}^{[1]})_\beta \). If \( N_{\alpha}^{\Gamma(\alpha)} \) is soluble, then \((N_{\alpha}^{[1]})^{\Gamma(\beta)} \) is soluble, and so \( N_{\alpha} \) is soluble, a contradiction. Thus, \( N_{\alpha}^{\Gamma(\alpha)} \) is an insoluble normal subgroup of the 2-transitive permutation group \( G_{\alpha}^{\Gamma(\alpha)} \). Inspecting the 2-transitive permutation groups listed in [3, pp. 195–197, Tables 7.3 and 7.4], it follows that \( N_{\alpha}^{\Gamma(\alpha)} \) has a unique insoluble composition factor, which is the unique insoluble composition factor of \( G_{\alpha}^{\Gamma(\alpha)} \).

Since \((N_{\alpha}^{\Gamma(\alpha)})_\beta \leq (G_{\alpha}^{\Gamma(\alpha)})_\beta \), by Lemma 2.1, \((N_{\alpha}^{\Gamma(\alpha)})_\beta \) has at most one insoluble composition factor. Recall that \( N_{\alpha}^{[1]} \) has at most one insoluble composition factor. Suppose that \( N_{\alpha}^{[1]} \) and \( N_{\alpha}^{\Gamma(\alpha)} \) have isomorphic insoluble composition factors. Then, \((N_{\alpha}^{\Gamma(\alpha)})_\beta \) and \( G_{\alpha}^{\Gamma(\alpha)} \) have isomorphic insoluble composition factors. By Lemma 2.1, \( G_{\alpha}^{\Gamma(\alpha)} \) is an affine 2-transitive group. This completes the proof. \( \square \)

Lemma 3.2 Assume that \( G \) is 2-arc-transitive on \( \Gamma = (V, E) \), and \( N \leq G \). Suppose that, for \( \alpha \in V \), the stabilizer \( N_{\alpha} \) has a normal subgroup \( K \cong T^{k} \) for an integer \( k \geq 1 \) and a nonabelian simple group \( T \). Then, \( k = 1 \).

Proof Note that every normal subgroup of \( K \) is isomorphic to \( T^{l} \) for some \( l \leq k \), where \( T^{0} = 1 \). Set \( K \cap G_{\alpha}^{[1]} \cong T^{l} \). Then,

\[
K^{\Gamma(\alpha)} \cong KG_{\alpha}^{[1]} / G_{\alpha}^{[1]} \cong K / (K \cap G_{\alpha}^{[1]}) \cong T^{k-l}.
\]

Since \( K^{\Gamma(\alpha)} \leq N_{\alpha}^{\Gamma(\alpha)} \leq G_{\alpha}^{\Gamma(\alpha)} \), by Lemma 3.1, we conclude that \( l, k - l \in \{0, 1\} \). If \( G_{\alpha}^{\Gamma(\alpha)} \) is of affine type, then \( k - l = 0 \), and so \( k = l = 1 \). If \( G_{\alpha}^{\Gamma(\alpha)} \) is almost simple, then either \( k = l = 1 \) or \( k - l = 1 \) and \( l = 0 \), and so \( k = 1 \). This completes the proof. \( \square \)

Theorem 3.3 Assume that \( \Gamma = (V, E) \) is a basic 2-arc-transitive graph with respect to \( G \), and \( G^* = \langle G_{\alpha}, G_{\beta} \rangle \) for \( \{\alpha, \beta\} \in E \). Then, either \( \Gamma \) is a complete bipartite graph, or \( \text{soc}(G^*) = \text{soc}(G) \) is the unique minimal normal subgroup of \( G \) and one of the following holds:

1. \( \text{soc}(G) \) is semiregular on \( V \);
2. \( \text{soc}(G) \) is a nonabelian simple group;
3. \( G^* \) is a quasiprimitive permutation group of type PA on each \( G^* \)-orbit on \( V \);
4. \( \Gamma \) is a bipartite graph, \( G^* \) is faithful on each part of \( \Gamma \), \( \text{soc}(G) = M_1 \times M_2 \) for minimal normal subgroups \( M_1 \) and \( M_2 \) of \( G^* \), and both \( M_1 \) and \( M_2 \) are semiregular and intransitive on each part of \( \Gamma \).

Proof If \( \Gamma \) is not bipartite, then \( G = G^* \) and, by [17, Theorem 2], \( G \) has a unique minimal normal subgroup, and one of parts (1)–(3) follows. Thus, we assume that \( \Gamma \) is a bipartite graph with two parts \( U \) and \( W \). In particular, \( |G : G^*| = 2 \). By [18, Theorem 2.1], either \( \Gamma \) is a complete bipartite graph, or \( G^* \) is faithful on each of \( U \) and \( W \). In the following, we assume that the latter case occurs.
Let $K$ be an arbitrary minimal normal subgroup of $G$. Suppose that $K \not< G^*$. Then, $K \cap G^* = 1$ and $G = G^*K$, yielding $|K| = 2$. Since $K$ has at most two orbits on $V$, we have $|V| \leq 4$, which is impossible as $\Gamma$ is bipartite and of valency at least 3. Therefore, $K \leq G^*$. Let $K_1$ be a minimal normal subgroup of $G^*$ with $K_1 \leq K$, and let $x \in G \setminus G^*$. Then, $K_1^x$ is also a minimal normal subgroup of $G^*$. Noting that $x^2 \in G^*$, we have $(K_1^x)^x = K_1^{x^2} = K_1$. This implies that $K_1K_1^x$ is normal in $G$. Since $K_1 \leq K \leq K_1K_1^x \leq \text{soc}(G^*)$. It follows that $\text{soc}(G) \leq \text{soc}(G^*)$.

**Case 1.** Assume that $G^*$ is quasiprimitive on both $U$ and $W$. Then, by [18, Theorem 2.3], $\text{soc}(G^*)$ is the unique minimal normal subgroup of $G^*$, and one of parts (1)-(3) of Theorem 3.3 occurs. Noting that $G^* \leq G$ and $\text{soc}(G^*)$ is characteristic in $G^*$, we have $\text{soc}(G^*) \leq G$, and hence $\text{soc}(G^*)$ is a minimal normal subgroup of $G$. Then, $\text{soc}(G^*) \leq \text{soc}(G)$. Recalling that $\text{soc}(G) \leq \text{soc}(G^*)$, we have $\text{soc}(G) = \text{soc}(G^*)$, and hence $\text{soc}(G)$ is the unique minimal normal subgroup of $G$.

**Case 2.** Assume that $G^*$ is not quasiprimitive on one of $U$ and $W$, say $U$. Then, $G^*$ has a minimal normal subgroup $M$ which is intransitive on $U$. Let $x \in G \setminus G^*$. Then, $M^x$ is a minimal normal subgroup of $G^*$, and $M^x$ is intransitive on $W$. Note that $MM^x$ is normal in $G$. Then, $MM^x$ is transitive on both $U$ and $W$. It follows that $M \neq M^x$, and so $M \cap M^x = 1$. Then, $MM^x = M \times M^x$. If $M$ is transitive on $W$, then $M^x$ is semiregular on $W$ by [4, Theorem 4.2A], and thus both $M$ and $M^x$ are regular on $W$, a contradiction. Therefore, $M$ is intransitive on $W$. Similarly, $M^x$ is intransitive on $U$.

It follows from [8, Lemma 5.1] that $M$ and $M^x$ are semiregular on both $U$ and $W$.

Set $N = MM^x$, and write $M = T_1 \times \cdots T_k$, where $T_i$ are isomorphic simple groups. Then,

$$N = T_1 \times \cdots T_k \times T_1^x \times \cdots T_k^x.$$
Then, $2k = 1$ by Lemma 3.2, a contradiction. Therefore, $N = \operatorname{soc}(G^*)$. Recall that $\operatorname{soc}(G) \leq \operatorname{soc}(G^*)$ and $N$ is a minimal normal subgroup of $G$. We have $\operatorname{soc}(G) = N = \operatorname{soc}(G^*)$, and the result follows. □

4 Semiregular direct factors

Let $\Gamma = (V, E)$ be a connected graph, and $G \leq \operatorname{Aut}(\Gamma)$.

Assume that $G$ is a 2-arc-transitive group of $\Gamma$. Then, $G_{\alpha}^{\Gamma(\alpha)}$ is a 2-transitive permutation group on $\Gamma(\alpha)$, where $\alpha \in V$. Let $N \leq G$ with $N_{\alpha} \neq 1$. It is easily shown that $N_{\alpha}$ acts transitively on $\Gamma(\alpha)$, see [13, Lemma 2.5] for example. Thus, $N_{\alpha}^{\Gamma(\alpha)}$ is a transitive normal subgroup of $G_{\alpha}^{\Gamma(\alpha)}$. By Lemma 2.3, $\operatorname{soc}(N_{\alpha}^{\Gamma(\alpha)}) = \operatorname{soc}(G_{\alpha}^{\Gamma(\alpha)})$ and one of the following holds:

(i) $N_{\alpha}^{\Gamma(\alpha)}$ is a primitive permutation group on $\Gamma(\alpha)$;
(ii) $N_{\alpha}^{\Gamma(\alpha)} = \mathbb{Z}_{p}^{k}$, $\mathbb{Z}_{m}$ and $|\Gamma(\alpha)| = p^k$, where $k \geq 2, m|(p^d - 1)$ for some divisor $d$ of $k$ with $d < k$;
(iii) $N_{\alpha}^{\Gamma(\alpha)} = \mathbb{Z}_{3}^{4}Q_{8}$ and $|\Gamma(\alpha)| = 3^4$.

Lemma 4.1 Assume that $G$ is 2-arc-transitive on $\Gamma$, and $N \leq G$ with $N_{\alpha} \neq 1$ for $\alpha \in V$. Suppose that $N_{\alpha}^{\Gamma(\alpha)}$ is not primitive on $\Gamma(\alpha)$. Then, one of the following holds:

1. $N_{\alpha} = \mathbb{Z}_{p}^{k}:(\mathbb{Z}_{m_{1}} \times \mathbb{Z}_{m_{2}})$, $|\Gamma(\alpha)| = p^k$ and $N_{\alpha}^{[1]} \cong \mathbb{Z}_{m_{1}}$, where $m_{1}|m, m|(p^d - 1)$ for some divisor $d$ of $k$ with $d < k$;
2. $N_{\alpha} = \mathbb{Z}_{3}^{4}:(Q, Q_{8}) = (\mathbb{Z}_{3}^{4} \times Q)Q_{8}$, $|\Gamma(\alpha)| = 3^4$ and $Q \cong N_{\alpha}^{[1]}$, where $Q$ is isomorphic to a subgroup of $Q_{8}$.

Proof By the foregoing argument, we may let $N_{\alpha}^{\Gamma(\alpha)} = KH$, where $K = \operatorname{soc}(N_{\alpha}^{\Gamma(\alpha)}) \cong \mathbb{Z}_{p}^{k}$, and either $H \cong \mathbb{Z}_{m}$ or $p^k = 3^4$ and $H \cong Q_{8}$. Without loss of generality, let $H = (N_{\alpha}^{\Gamma(\alpha)})_{\beta}$ for some $\beta \in \Gamma(\alpha)$. Then, $N_{\alpha}^{[1]} / N_{\alpha}^{[1]}_{\beta}$ is isomorphic to a normal subgroup of $H$, see (3.1) given in Sect. 3.

Assume first that $p^k = 4$. In this case, we have $H = 1$ and $N_{\alpha}^{\Gamma(\alpha)} = \mathbb{Z}_{2}^{2}$, and so $N_{\alpha}$ acts faithfully on $\Gamma(\alpha)$, refer to [13, Lemma 2.3]. Then, $N_{\alpha} = \mathbb{Z}_{2}^{2}$, desired as in part (1) of this lemma.

Now assume that $p^k \neq 4$. Then, $|\Gamma(\alpha)| = p^k > 5$. By [21, Theorem 4.7], $G_{\alpha}^{[1]} = 1$, and so $N_{\alpha}^{[1]} = 1$, where $\beta \in \Gamma(\alpha)$. Then, $N_{\alpha}^{[1]}$ is isomorphic to a normal subgroup of $H$, in particular, $(p, |N_{\alpha}^{[1]}|) = 1$. It is easily shown that $|\operatorname{Aut}(N_{\alpha}^{[1]})| < p^k$. Let $P$ be a Sylow $p$-subgroup of $N_{\alpha}$. Then, $P \cong \mathbb{Z}_{p}^{k}$, and $PN_{\alpha}^{[1]} / N_{\alpha}^{[1]}$ is the unique Sylow $p$-subgroup of $N_{\alpha} / N_{\alpha}^{[1]}$, in particular, $PN_{\alpha}^{[1]} \leq N_{\alpha}$. Noting that $PN_{\alpha}^{[1]} / C_{P_{N_{\alpha}^{[1]}}}^{N_{\alpha}^{[1]}}$ is isomorphic to a subgroup of $\operatorname{Aut}(N_{\alpha}^{[1]})$, it follows that $p$ is a divisor of $|C_{P_{N_{\alpha}^{[1]}}}^{N_{\alpha}^{[1]}}|$ and that $Q$ is a Sylow $p$-subgroup of $C_{P_{N_{\alpha}^{[1]}}}^{N_{\alpha}^{[1]}}$. Then, $Q$ is characteristic in $C_{P_{N_{\alpha}^{[1]}}}^{N_{\alpha}^{[1]}}$, and hence, $Q$ is normal in $N_{\alpha}$. This implies that $O_{p}(N_{\alpha}) \neq 1$, where $O_{p}(N_{\alpha})$ is the maximal normal $p$-subgroup of $N_{\alpha}$. Since $N_{\alpha} \leq G_{\alpha}$, we have $O_{p}(N_{\alpha}) \leq G_{\alpha}$. Recalling that $(p, |N_{\alpha}^{[1]}|) = 1$, we
deduce that $O_p(N_\alpha)$ acts faithfully on $\Gamma(\alpha)$. Since $G$ acts 2-transitively on $\Gamma(\alpha)$, the action of $O_p(N_\alpha)$ on $\Gamma(\alpha)$ is transitive. Noting that $O_p(N_\alpha) \leq P$ is abelian, it follows that $O_p(N_\alpha)$ is regular on $\Gamma(\alpha)$. Then, $|O_p(N_\alpha)| = |\Gamma(\alpha)| = p^k$, and hence $O_p(N_\alpha) = P \cong \mathbb{Z}_p^k$. We have $N_\alpha = P : N_{\alpha\beta} = (P \times N_\alpha^{[1]}). H$. Then, part (1) or (2) of the lemma follows.

\begin{flushright}$\square$\end{flushright}

**Theorem 4.2** Assume that $\Gamma = (V, E)$ is a basic 2-arc-transitive graph with respect to $G$, and $G^* = (G_\alpha, G_\beta)$ for an edge $[\alpha, \beta] \in E$. If $\Gamma$ is not a complete bipartite graph, then either $\text{soc}(G)$ is a nonabelian simple group, or every simple direct factor of $\text{soc}(G)$ is semiregular on $V$.

\begin{flushright}$\square$\end{flushright}

**Proof** Assume that $\Gamma$ is not a complete bipartite graph. Then, $G^*$ is faithful on each of its orbits on $V$. Let $N = \text{soc}(G)$. By Theorem 3.3, $N = \text{soc}(G)$ is the unique minimal normal subgroup of $G$. If part (1), (2) or (4) of Theorem 3.3 occurs, then our result is true. Thus, in the following, we suppose that part (3) of Theorem 3.3 occurs, that is, $G^*$ is a quasiprimitive permutation group of type PA on each $G^*$-orbit on $V$. By [17, \text{III}(b)(i)], $N$ is the unique minimal normal subgroup of $G^*$.

Write $N = T_1 \times \cdots \times T_l$, where $l \geq 2$ and $T_i$ are isomorphic nonabelian simple groups. Then, $N_\alpha \neq 1$, and $N_\alpha$ has no composition factor isomorphic to $T_1$, see [17, \text{III}(b)(i)]. We next show that every $T_i$ is semiregular on $V$.

Let $U$ be the $G^*$-orbit on $V$ with $\alpha \in U$, and let $W = V \setminus U$ if $\Gamma$ is bipartite. Clearly, $U$ is an $N$-orbit, and if $\Gamma$ is bipartite, then $W$ is also an $N$-orbit. Recall that $N$ is a minimal normal subgroup of both $G$ and $G^*$. Since $G^* = NG_\gamma$ for $\gamma \in V$, it follows that both $G$ and $G_\gamma$ act transitively on $\Omega := \{T_1, \ldots, T_l\}$ by conjugation. Let

$$C_\gamma = \{(T_i)_\gamma \mid 1 \leq i \leq l\}, \ C = \cup_{\gamma \in V} C_\gamma.$$  

For $1 \leq i \leq l$ and $x \in G$, we have $T_i^x \in \Omega$, and so

$$(T_i)_\gamma^x = (T_i \cap G_\gamma)^x = T_i^x \cap G_\gamma^x = (T_i^x)_\gamma^x \in \mathcal{C}, \ \forall \gamma \in V.$$  

We deduce that $G_\gamma$ acts transitively on $C_\gamma$ by conjugation, and $C$ is a conjugacy class of subgroups in $G$. In particular, all orbits of each $T_i$ on $V$ have the same length $|T_1 : (T_i)_{\alpha}|$. Thus, if $T_1$ is semiregular on $V$, then every $T_i$ is semiregular on $V$.

**Case 1.** Assume that $N_\alpha^{\Gamma(\alpha)}$ is primitive on $\Gamma(\alpha)$. For any $\gamma \in V$, letting $\gamma = \alpha^g$ for some $g \in G$, we have

$$\Gamma(\gamma) = \Gamma(\alpha)^g, \ N_\gamma = N \cap G_{\alpha^g} = (N \cap G_\alpha)^g = N_\alpha^g.$$  

It follows that $N_\gamma$ acts primitively on $\Gamma(\gamma)$. Thus, $N$ is locally primitive on $\Gamma$. Suppose that $T_1$ is transitive on one of the $G^*$-orbits, say $U$. Since $T_1$ centralizes $T_1$, by [4, Theorem 4.2A], $T_1$ is semiregular on $U$. This implies that both $T_1$ and $T_l$ are regular on $U$. Then, $N = T_1 N_\alpha$, and so

$$T_1 \times \cdots \times T_{l-1} \cong N/T_l = T_1 N_\alpha/T_l \cong N_\alpha/(T_l \cap N_\alpha) = N_\alpha/(T_l)_\alpha.$$  

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It follows that $N_\alpha$ has a composition factor isomorphic to $T_1$, a contradiction. Therefore, $T_1$ is intransitive on every $G^*$-orbit, and hence $T_1$ is semiregular on $V$, see [13, Lemma 2.6]. Then, every $T_i$ is semiregular on $V$, and our result is true.

**Case 2.** Assume that $(T_1)_\alpha \leq G^{[1]}_\alpha$. Then, $(T_1)_\alpha \leq (T_1)_\beta$, where $\beta \in \Gamma(\alpha)$. Recalling that $\mathcal{C}$ is a conjugacy class in $G$, it follows that $|(T_1)_\gamma| = |(T_1)_\alpha|$ for all $\gamma \in V$. In particular, $|(T_1)_\alpha| = |(T_1)_\beta|$, and so $(T_1)_\alpha = (T_1)_\beta$. Note that $N_\beta$ acts transitively on $\Gamma(\beta)$, see [13, Lemma 2.5] for example. Since $(T_1)_\alpha = (T_1)_\beta \leq N_\beta$, all $(T_1)_\alpha$-orbits on $\Gamma(\beta)$ have the same length. It follows that $(T_1)_\alpha$ fixes $\Gamma(\beta)$ point-wise, i.e., $(T_1)_\beta = (T_1)_\alpha \leq G^{[1]}_\alpha$. We deduce from the connectedness of $\Gamma$ that $(T_1)_\gamma = (T_1)_\alpha$ for all $\gamma \in V$. This forces that $(T_1)_\alpha = 1$. Then, our result is true in this case.

**Case 3.** Now we suppose that $(T_1)_\alpha \leq G^{[1]}_\alpha$ and $N_\alpha^{\Gamma(\alpha)}$ is not primitive on $\Gamma(\alpha)$, and produce a contradiction. Recall that $G_\alpha$ acts transitively on $\mathcal{C}_\alpha$ by conjugation. This implies that $G_\alpha$ acts transitively on $\{(T_1)^{[1]}_\alpha, \ldots, (T_l)^{[1]}_\alpha\}$, $(T_1)_\alpha \times \cdots \times (T_l)_\alpha \leq G_\alpha$, and $(T_1)_\alpha \leq G^{[1]}_\alpha$ for $1 \leq i \leq l$. By Lemma 2.3, we have that

$$\text{soc}(((T_1)_\alpha \times \cdots \times (T_l)_\alpha)^{\Gamma(\alpha)}) = \text{soc}(G^{\Gamma(\alpha)}_\alpha) = \text{soc}(N^{\Gamma(\alpha)}_\alpha) \cong \mathbb{Z}_p^k,$$

and a Sylow $p$-subgroup of $N_\alpha$ has order $p^k$, where $p$ is a prime and $k \geq 2$. By Lemma 4.1, $N^{[1]}_\alpha$ has order coprime to $p$, and thus $(p, |(T_i)_\alpha^{[1]}|) = 1$ for $1 \leq i \leq l$.

Let $P_i$ be a Sylow $p$-subgroup of $(T_i)_\alpha$, where $1 \leq i \leq l$. Then, $P = P_1 \times \cdots \times P_l$ is a Sylow $p$-subgroup of $N$, and thus

$$P \cong P^{\Gamma(\alpha)} = \text{soc}(N^{\Gamma(\alpha)}_\alpha) = \text{soc}(G^{\Gamma(\alpha)}_\alpha),$$

and $\mathcal{O}_p((T_i)_\alpha^{\Gamma(\alpha)}) = P_i^{\Gamma(\alpha)} \cong P_i$ for each $i$. It follows that

$$\text{soc}(G^{\Gamma(\alpha)}_\alpha) = P_1^{\Gamma(\alpha)} \times \cdots \times P_l^{\Gamma(\alpha)}.$$

Let $K_i$ be the preimage of $P_i^{\Gamma(\alpha)}$ in $(T_i)_\alpha \times \cdots \times (T_l)_\alpha$. Then, $K_i = (T_i)_\alpha^{[1]} P_i$ for $1 \leq i \leq l$. It is easily shown that $G_\alpha^{\Gamma(\alpha)}$ acts transitively on $\{K_1, \ldots, K_l\}$ by conjugation. Then, $G_\alpha^{\Gamma(\alpha)}$ acts transitively on $\{P_1^{\Gamma(\alpha)}, \ldots, P_l^{\Gamma(\alpha)}\}$ by conjugation, which is impossible by Lemma 2.4. This completes the proof of the theorem. □

We are now ready to give a proof of Theorem 1.1.

**Proof of Theorem 1.1** Let $\Gamma = (V, E)$ be a basic 2-arc-transitive graph with respect to a group $G$. Assume that $\Gamma$ is not a complete bipartite graph. Fix an edge $\{\alpha, \beta\} \in E$, and let $G^* = \langle G_\alpha, G_\beta \rangle$ and $N = \text{soc}(G^*)$. By Theorem 3.3, $N = \text{soc}(G)$ is the unique minimal normal subgroup of $G$, desired as in part (1) of Theorem 1.1. By Theorem 4.2, we have part (2) of Theorem 1.1.

Let $\gamma$ be an arbitrary vertex of $\Gamma$. Since $G$ acts transitively on $V$, we write $\gamma = \alpha^g$ for some $g \in G$. Then, $\Gamma(\gamma) = \Gamma(\alpha)^g$. Since $N$ is normal in $G$, we deduce that $N_\gamma = N^{\Gamma(\gamma)}_\alpha$. It follows that $N^{\Gamma(\gamma)}_\alpha$ and $N^{\Gamma(\alpha)}_\alpha$ are permutation isomorphic. Then, $N$
is locally primitive on $\Gamma$ if and only if $N_\alpha^\Gamma(\alpha)$ is primitive on $\Gamma(\alpha)$. If $N_\alpha^\Gamma(\alpha)$ is not primitive on $\Gamma(\alpha)$, then either $N_\alpha \neq 1$, or $N_\alpha$ is described as in part (1) or (2) of Lemma 4.1. Thus, we obtain part (3) of Theorem 1.1. This completes the proof. $\Box$

Finally, we give a proof of Theorem 1.4.

**Proof of Theorem 1.4** Assume that $G$ is a 2-arc-transitive group of $\Gamma = (V, E)$. Let $G^* = \langle G_\alpha, G_\beta \rangle$ for $\{\alpha, \beta\} \in E$. If $\Gamma$ is not bipartite and $G$ is primitive on $V$, then $\text{soc}(G)$ is either simple or regular on $V$ by [16, Theorem A], and the result is true.

Assume next that $\Gamma$ is a bipartite graph with two parts $U$ and $W$, and that $G^*$ acts primitively on both $U$ and $W$. If $G^*$ is unfaithful on $U$ or $W$, then $\Gamma$ is a complete bipartite graph. Thus, we assume further that $G^*$ is faithful on both $U$ and $W$. Let $\alpha \in U$ and $\beta \in W$.

**Case 1.** Assume that $\text{soc}(G) \leq G^*$. If $\Gamma$ has valency 2, then $\Gamma$ is a cycle of length $2p$ for some prime $p$, and $G \cong D_{4p}$; in this case, the center of $G$ is not contained in $G^*$, and so $\text{soc}(G) \not\leq G^*$. Thus, $\Gamma$ has valency at least 3, and hence $\Gamma$ is a basic 2-arc-transitive graph with respect to $G$. By Theorem 3.3, $\text{soc}(G) = \text{soc}(G^*)$, and either part (2) of Theorem 1.4 holds or $G^*$ is a primitive permutation group of type PA on $U$. For the latter case, every simple direct factor of $\text{soc}(G^*)$ is not semiregular on $U$, refer to [15, p. 391, III(b)(i)]. Then, part (2) of Theorem 1.4 occurs by Theorem 4.2.

**Case 2.** Assume that $\text{soc}(G) \not\leq G^*$. Let $M$ be a minimal normal subgroup of $G$ with $M \not\leq G^*$. Then, noting that $|G : G^*| = 2$, we have $G = G^* \times M$ and $|M| = 2$. This implies that $\text{soc}(G) = \text{soc}(G^*) \times M$. Set $M = \langle x \rangle$. Then, $G_\alpha^x = G_\alpha^* = G_\alpha$, and so $G_\alpha$ acts 2-transitively on $\Gamma(\alpha^x)$. Note that $\Gamma(\alpha^x) \subset U$. Considering the (faithful) action of $G^*$ on $U$, by [16, Theorem A], $\text{soc}(G^*)$ is either simple or regular on $U$. Similarly, $\text{soc}(G^*)$ is either simple or regular on $W$. Then, part (3) of Theorem 1.4 follows. This completes the proof. $\Box$

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**Declarations**

**Conflict of interest** We declare that we do not have any commercial or associative interests that represents a conflict of interests in connection with the work reported in this paper.

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