ON A PROBLEM OF YAU REGARDING A HIGHER DIMENSIONAL GENERALIZATION OF THE COHN-VOSSEN INEQUALITY

BO YANG

Abstract. We show that a problem by Yau in [15] cannot be true in general. The counterexamples are constructed based on the recent work of Wu and Zheng [14].

1. Introduction

Shing-Tung Yau asked the following question in [15]:

Question 1.1. Given an n-dimensional complete manifold with nonnegative Ricci curvature, let \( B(r) \) be the geodesic ball around some point \( p \). Let \( \sigma_k \) be the \( k \)-th elementary symmetric function of the Ricci tensor. Then is it true that \( r^{-n+2k} \int_{B(r)} \sigma_k \) has an upper bound when \( r \) tends to infinity? This should be considered as a generalization of the Cohn-Vossen inequality.

In the Kähler category one would like to ask the following similar question.

Question 1.2. On a complete Kähler manifold with complex dimension \( n \), if we denote \( \omega \) and \( \text{Ric} \) the Kähler form and the Ricci form respectively, one would like to ask if \( r^{-2n+2k} \int_{B(r)} \text{Ric}^k \wedge \omega^{n-k} \) is bounded for any \( 1 \leq k \leq n \) when \( r \) goes to infinity.

In this note we exhibit counterexamples to Question 1.1 in the case of \( 1 < k \leq n \) via the recent interesting work of Wu and Zheng [14]. We will show that for any complex dimension \( n \geq 2 \) and any \( 2 \leq k < n \), there exists a \( U(n) \) invariant complete Kähler metrics on \( \mathbb{C}^n \) with nonnegative bisectional curvature such that \( r^{-2n+2k} \int_{B(r)} \sigma_k \) is unbounded when \( r \) large (See Theorem 3.4). We also prove that Question 1.2 is true for all \( U(n) \) invariant complete Kähler metrics on \( \mathbb{C}^n \) with nonnegative bisectional curvature (See Theorem 3.6).

2. Results of Wu and Zheng

In the section, we collect some of the results from the recent work of Wu and Zheng [14] since they will be used in our constructions of counterexamples to Question 1.1. Unless stated otherwise all results in this section are due to Wu and Zheng [14].

Wu and Zheng [14] develops a systematic way to construct \( U(n) \) invariant complete Kähler metrics on \( \mathbb{C}^n \) with positive bisectional curvature. One of the motivations behind their work is the uniformization conjecture by Yau [16]. The conjecture states that a complete noncompact Kähler manifold with positive bisectional curvature is biholomorphic to the complex Euclidean space. See [8, 3, 4, 5] and
reference therein for some recent progress towards Yau’s uniformization conjecture. See also [9, 1, and 2] for some earlier works on the construction of rotationally symmetric complete Kähler metrics with positive curvature on \( \mathbb{C}^n \).

We follow the notations in [14]. Let \( z = (z_1, \cdots, z_n) \) be the standard coordinate on \( \mathbb{C}^n \) and \( r = |z|^2 \). A \( U(n) \)-invariant Kähler metric on \( \mathbb{C}^n \) has the Kähler form

\[
\omega = \frac{\sqrt{-1}}{2} \partial \bar{\partial} p(r)
\]

where \( p \in C^\infty[0, +\infty) \). Under the local coordinates, the metric has components:

\[
g_{ij} = f(r) \delta_{ij} + f'(r) \overline{z_i z_j}.
\]

We further denote:

\[
f(r) = p'(r), \quad h(r) = \frac{r f'}{f}.
\]

Then the Kähler form \( \omega \) will give a complete metric if and only if

\[
f > 0, \quad h > 0, \quad \int_0^{+\infty} \frac{\sqrt{h}}{\sqrt{r}} dr = +\infty.
\]

Now if we compute the components of the curvature tensor at \((z_1, 0, \cdots, 0)\) under the orthonormal frame \( \{e_1 = \frac{1}{\sqrt{h}} \partial z_1, e_2 = \frac{1}{\sqrt{f}} \partial z_2, \cdots, e_n = \frac{1}{\sqrt{f}} \partial z_n\} \), then define \( A, B, C \) respectively by:

\[
A = R_{1111} = -\frac{1}{h} \left( \frac{r h'}{h} \right)', \quad B = R_{1j1j} = \frac{f'}{f^2} - \frac{h'}{h f}, \quad C = R_{iijj} = 2 R_{iijj} = -\frac{2 f'}{f^2},
\]

where we assume \( 2 \leq i \neq j \leq n \). It is easy to check all other components of curvature tensor are zero.

Let \( \mathcal{M}_n \) denote the space of all \( U(n) \)-invariant complete Kähler metrics on \( \mathbb{C}^n \) with positive bisectional curvature.

**Theorem 2.1 (Characterization of \( \mathcal{M}_n \) by the ABC function).** Suppose \( n \geq 2 \) and \( h \) is a smooth positive function on \([0, +\infty)\) satisfying (4), then the form defined by (1) gives a complete Kähler metric with positive (nonnegative) bisectional curvature if and only if \( A, B, C \) are positive (nonnegative).

If we define another function \( \xi \in C^\infty[0, +\infty) \) by

\[
\xi(r) = -\frac{r h'(r)}{h},
\]

then \( h \) determines \( \xi \) uniquely. On the other hand, note that \( \xi \) determines \( h \) by \( h(r) = h(0) e^{\int_0^r \frac{\xi}{\sqrt{r}} dt} \), hence \( \omega \) up to scaling. The following interesting theorem in [14] reveals that the space \( \mathcal{M}_n \) is in fact quite large.

**Theorem 2.2 (Characterization of \( \mathcal{M}_n \) by the \( \xi \) function).** Suppose \( n \geq 2 \) and \( h \) is a smooth positive function on \([0, +\infty)\), then the form defined by (1) gives a complete Kähler metric with positive bisectional curvature on \( \mathbb{C}^n \) if and only if \( \xi \) defined by (6) satisfying

\[
\xi(0) = 0, \quad \xi' > 0, \quad \xi < 1.
\]

Fix a metric \( \omega \) in \( \mathcal{M}_n \), the geodesic distance between the origin and a point \( z \in \mathbb{C}^n \) is:

\[
s = \int_0^r \frac{\sqrt{h}}{2 \sqrt{r}} dr.
\]
where $r = |z|^2$. We denote $B(s)$ the ball in $\mathbb{C}^n$ centered at the origin and with the radius $s$ with respect to $\omega$. It is further shown in [13] that:
\[
\text{Vol}(B(s)) = c_n(rf)^n.\]
where $c_n$ is the Euclidean volume of the Euclidean unit ball in $\mathbb{C}^n$.

Using Theorem 2.2 Wu and Zheng further proved the following estimates on volume growth of geodesics ball $B(s)$ and the first Chern number for metrics in $\mathcal{M}_n$. Note that an estimate on volume growth of geodesics ball in the general case has been proved by Chen and Zhu [7].

**Proposition 2.3 (Volume growth estimates for metrics in $\mathcal{M}_n$).** $rf = f(1) + 2\sqrt{h(1)(s - s(1))}$ for $r > 1$ and $rf \leq s^2$ for any $r \geq 0$. So there exists a constant $C$ such that:
\[
Cs^n \leq \text{Vol}(B(s)) \leq c_n s^{2n}.
\]
for $s$ large enough.

**Proposition 2.4 (Bounding the first Chern number for metrics in $\mathcal{M}_n$).** Given any $\omega$ in $\mathcal{M}_n$ with $n \geq 1$, we have
\[
\int_{\mathbb{C}^n} (\text{Ric})^n = c_n \left(\frac{n\xi(\infty)}{\pi}\right)^n \leq c_n \left(\frac{n}{\pi}\right)^n.
\]
while $\text{Vol}(B(s)) = c_n v^n$.

In order to construct more examples and compute the scalar curvature curvature of metrics in $\mathcal{M}_n$ in a more convenient way, Wu and Zheng [14] introduced another function $F$ in the following way: First we define $x = \sqrt{rh}$ and a nonnegative function $y$ of $r$ by
\[
y(0) = 0, \quad x'^2 + y'^2 = \frac{h}{4r}, \quad y' > 0.
\]
One can check that $x(r)$ is strictly increasing and then we may define $F(x)$ a function on $[0, x_0)$ by $y = F(x)$, where
\[
x_0^2 = \lim_{r \to +\infty} rh = h(1)e^{\int_1^{+\infty} \frac{h}{2s} ds}.
\]
Extending $F$ to $(-x_0, x_0)$ by letting $F(x) = F(-x)$, one can check that $F$ is a smooth, even function on $|x| < x_0$. Starting with such a $F$ satisfying certain conditions, one can recover the metric $\omega$ in a geometric way. See Section 5 in [14] for details. This result is summarized as the following theorem.

**Theorem 2.5 (Characterization of $\mathcal{M}_n$ by the $F$ function).** Suppose $n \geq 1$, there is a one to one correspondence of between the set $\mathcal{M}_n$ and the set of $\mathcal{F}$ of smooth, even function $F(x)$ defined on $(-x_0, x_0)$ satisfying
\[
F(0) = 0, \quad F'' > 0, \quad \lim_{x \to x_0} F(x) = +\infty.
\]
Denote $v = rf$, one can rewrite $s$ and $\text{Vol}(B(s))$ in terms of $F$:
\[
s = \int_0^x \sqrt{1 + (F'(\tau))^2} d\tau, \quad \text{Vol}(B(s)) = c_n v^n = c_n \left(\int_0^x 2\tau \sqrt{1 + (F'(\tau))^2} d\tau\right)^n.
\]
Rewrite $A$, $B$, and $C$ defined in (5) in terms of $F$:

\begin{align}
A &= \frac{F''}{2x(1 + F'x)^2}, \quad B = \frac{x^2}{v^2} - \frac{1}{v\sqrt{1 + F'^2}}, \quad C = \frac{2}{v} - \frac{2x^2}{v^2}.
\end{align}

Recall the scalar curvature at the point $z = (z_1, 0, \cdots, 0)$ is given by

\begin{equation}
R = A + 2(n-1)B + \frac{1}{2}n(n-1)C.
\end{equation}

Using (17), (16), (15) and a careful integration by parts, Wu and Zheng [14] proved the following relation between average scalar curvature decay and volume growth of geodesic balls. See also [7] for a related result on any complete Kähler manifold with positive bisectional curvature.

**Proposition 2.6 (Estimates on average scalar curvature for metrics in $\mathcal{M}_n$).** Given any Kähler metric $\omega$ in $\mathcal{M}_n$ with $n \geq 2$, there exists a constant $c > 0$ such that

\begin{equation}
\frac{1}{c(1 + v)} \leq \frac{1}{\operatorname{Vol}(B(s))} \int_{B(s)} R(s) w^n \leq \frac{c}{1 + v},
\end{equation}

while $\operatorname{Vol}(B(s)) = c_n v^n$.

3. **Counterexamples to Question 1.1**

Let $\bar{\mathcal{M}}_n$ denote the space of all $U(n)$ invariant complete Kähler metrics on $\mathbb{C}^n$ with nonnegative bisectional curvature. First we state a generalization of Theorem 2.2 to the space $\bar{\mathcal{M}}_n$.

**Proposition 3.1 (Characterization of $\bar{\mathcal{M}}_n$ by the $\xi$ function).** Suppose $n \geq 2$ and $h$ is a smooth positive function on $[0, +\infty)$, then the form defined by (1) gives a complete Kähler metric with nonnegative bisectional curvature if and only if $\xi$ defined by (6) satisfies

\begin{equation}
\xi(0) = 0, \quad \xi' \geq 0, \quad \xi \leq 1.
\end{equation}

**Proof of Proposition 3.1** The original proof of Theorem 2.2 due to Wu and Zheng [14] works here. Now we only sketch the necessary part. First from (6) we know $\xi(0) = 0$. Note that (6) and Theorem 2.1 imply

\begin{equation}
A = \frac{\xi'}{h} \geq 0
\end{equation}

which leads to $\xi' \geq 0$.

To prove $\xi \leq 1$, argue by contradiction as in [14]. Assume $\lim_{r \to +\infty} b > 1$, then take $\delta_0 > 0$ such that $1 + \delta_0 < b$. It follows that there exists $r_0 > 0$ with $\xi(r_0) \geq 1 + \delta_0$. Thus integrating (6) leads to $h(r) = h(0) \exp \int_0^r \frac{\xi}{h} dr \leq \frac{1}{\pi \delta_0}$ which contradicts to the completeness of the metric (4).

It also follows from the original proof of Proposition 2.3 and 2.4 due to Wu and Zheng that the same conclusion holds for the space $\bar{\mathcal{M}}_n$. Namely, for any metric $\omega$ in $\bar{\mathcal{M}}_n$, $C s^n \leq \operatorname{Vol}(B(s)) \leq c_n s^{2n}$ holds for $s$ sufficiently large, and $\int_{C_n} (\operatorname{Ric})^n \leq c_n (\frac{2}{\pi})^n$ is true. We remark here that the estimate on lower bounds of the volume growth of $B(s)$ here can not be true for an arbitrary complete noncompact Kähler manifolds with nonnegative bisectional curvature. For example, take $\Sigma_1 \times \mathbb{C}P^1 \times \cdots \times \mathbb{C}P^1$ where $\Sigma_1$ is a capped cylinder on one end and $\mathbb{C}P^1$ is the complex projective plane with the standard metric.
Next we state another generalization of Theorem 2.5 to $\overline{M}_n$.

**Theorem 3.2 (Characterization of $\overline{M}_n$ by the $F$ function).** Suppose $n \geq 1$, there is a partition of the set $\overline{M}_n \setminus \{g_e\} = S_1 \cup S_2 \cup S_3$ where $g_e$ is the standard Euclidean metric on $\mathbb{C}^n$ such that:

1. $S_1$ has a one to one correspondence with the set of $F$ of smooth, even function $F(x)$ on $(-\infty, +\infty)$ defined above satisfying

\begin{equation}
F(0) = F'(0) = 0, \quad F''(\infty) = +\infty, \quad F'(\infty) = +\infty.
\end{equation}

$S_1$ consists of nonflat Kähler metrics in $\overline{M}_n$ whose geodesic balls have Euclidean volume growth.

2. $S_2$ has a one to one correspondence with the set of $F$ of smooth, even function $F(x)$ on $(-\infty, +\infty)$ satisfying

\begin{equation}
F(0) = F'(0) = 0, \quad F''(\infty) < +\infty, \quad F'(\infty) = +\infty.
\end{equation}

$S_2$ includes Kähler metrics in $\overline{M}_n$ whose geodesic balls have strictly less than Euclidean volume growth and bisectional curvatures in the radial direction strictly positive along a sequence of points in $\mathbb{C}^n$ tending to infinity.

3. For any metric $\omega \in S_3$, there exists a positive real number $r_0$ such that $r_0 = \inf\{r : \xi(r) = 1\}$ and a corresponding positive real number $x_0$ such that there exists a smooth even function $F(x)$ defined on $(-x_0, x_0)$ such that

\begin{equation}
F(0) = F'(0) = 0, \quad F''(x_0) = +\infty, \quad F'(x_0) < 0,
\end{equation}

$S_3$ is the set of metrics with geodesic balls having half Euclidean volume growth and whose bisectional curvatures in the radial direction vanish outside a compact set. A standard example in complex dimension 1 is a capped cylinder on one end.

**Proof of Theorem 3.2** The proof of Theorem 3.2 is based on a modification of Theorem 2.5 due to Wu and Zheng. From Proposition 2.5, we know for any Kähler metric in $\overline{M}_n$, there exists a corresponding $\xi(r)$ on $[0, +\infty)$ with $\xi(0) = 0, \xi' \geq 0$, and $\xi \leq 1$. Denote $r_0 = \inf\{r : \xi(r) = 1\}$.

Recall the definition of $x$ and $y$ in (12), $x = \sqrt{r h}$ and $x'(r)^2 + y'(r)^2 = \frac{1}{r}$ with $y(0) = 0$ and $y' \geq 0$. It is easy to check:

\begin{equation}
\frac{dx}{dr} = (1 - \xi) \sqrt{\frac{h}{4r}}.
\end{equation}

then we know $x(r)$ and $y(r)$ are both nondecreasing with respect to $r$.

**Case 1** $r_0 = +\infty$. From the definition of $x_0$ in (13) and (24) we know $x(r)$ is strictly increasing on $[0, +\infty)$, then we can define $F(x)$ by $y = F(x)$ on $x \in (-x_0, x_0)$ after an even extension by letting $F(\cdot - x_0) = F(x)$. It is not hard to see that

\begin{equation}
F(0) = 0, \quad F'(x) \geq 0, \quad 1 + [F'(x)]^2 = \frac{1}{(1 - \xi)^2}.
\end{equation}

Recall that $0 \leq \xi(r) \leq 1$ is nondecreasing on $(-\infty, +\infty)$, we conclude that $F'' \geq 0$. 
Moreover, (24) and (25) implies:

\[
\lim_{x \to x_0} F(x) = \int_0^{x_0} \sqrt{\frac{1}{1 - \xi^2}} - 1 \, dx
\]

\[
= \int_0^{+\infty} \sqrt{1 - (1 - \xi^2)} \frac{h}{4r} \, dr
\]

\[
\geq \sqrt{1 - (1 - \xi(+)^2)} \int_0^{+\infty} \sqrt{\frac{h}{4r}} \, dr.
\]

Note that the integral in the last step of (26) is distance function (8). We conclude \( F(x_0) = \infty \) if and only if \( \xi(+) > 0 \). Note that the latter condition is satisfied when \( \omega \) is nonflat.

We further divide our discussion into two subcases:

(Subcase Ia) \( 0 < \xi(+) < 1 \). In this case we have \( F' \) is bounded on \((-x_0, x_0)\) and \( x_0 = +\infty \). Moreover, we will prove that the geodesic balls of \((\mathbb{C}^n, \omega)\) has Euclidean volume growth. We follow the method of Wu and Zheng (See P 528 of [14]). Note that (8), (9), \((rf)'(r) = h\) and \((rh)'(r) = h(1 - \xi)\), it follows from the L'Hospital’s rule that:

\[
\lim_{s \to +\infty} \frac{\text{Vol}(B(s))}{s^{2n}} = \lim_{r \to +\infty} \frac{c_n(rf)^n}{s^{2n}} = \lim_{r \to +\infty} \frac{c_n(\sqrt{rf})^{2n}}{s^{2n}} = c_n(1 - \xi(+)^4)
\]

(Subcase Ib) \( \xi(+) = 1 \). It follows from the (27) that in this case the geodesic balls of \((\mathbb{C}^n, \omega)\) has strictly less than Euclidean volume growth. Since \( A = \frac{\xi}{r} \), \( \xi(0) = 0 \), and \( \xi(+) = 1 \) we also know that bisectional curvatures in the radial direction strictly positive along at least a sequence of points in \( \mathbb{C}^n \) tending to infinity.

(Case II) \( r_0 > 0 \) is finite. Note that \((rh)' = h(1 - \xi)\), we conclude that \( x_0 = \lim_{r \to +\infty} \sqrt{rh} \) is finite and \( x_0^2 = r_0 h(r_0) \). This implies that \( F(x) \) is well defined on \((-x_0, x_0)\) with \( F(x_0) < +\infty \). Since \( A = \frac{\xi}{n} \) we conclude that bisectional curvatures in the radial direction vanishes outside a compact set in \( \mathbb{C}^n \). Next we proceed to show that the geodesic balls of \((\mathbb{C}^n, \omega)\) has half Euclidean volume growth. Again the methods follows from Wu and Zheng (See P528 of [14]).

\[
\lim_{s \to +\infty} \frac{\text{Vol}(B(s))}{s^n} = \lim_{r \to +\infty} \frac{c_n(rf)^n}{s^n} = \lim_{r \to +\infty} \frac{c_n(2\sqrt{rh})^n}{s^n} = 2c_n x_0.
\]

Denote \( S_1 \), \( S_2 \), and \( S_3 \) the sets of metrics in the above three cases (Subcase Ia), (Subcase Ib), and (Case II) respectively, we have proved Theorem 3.2.

\( \square \)
Next we gives some more explicit description of $S_3$. Given any metric $\omega$ in $S_3$, 
\[
\frac{d(rh)}{dr} = (1 - \xi)h, \quad h = (rf)'
\]
and $\xi(r) = 1$ when $r > r_0$, then:

\[
r f |_{r_0} = \int_{r_0}^r \frac{r_0 h(r_0)}{r} dr,
\]

which further implies:

\[
r f = x_0^2 \ln \frac{r}{r_0} + r_0 f(r_0).
\]

Now we compute $A, B, C$ with (5) in Section 2 when $r \geq r_0$:

\[
A = -\frac{1}{h} (\frac{rh'}{h})' = \frac{\xi'}{h} = 0,
\]
\[
B = \frac{f'}{f^2} - \frac{h'}{hf} = \frac{1}{r} \left( \frac{(rf)' - f}{f^2} - \frac{rh'}{hf} \right)
\]
\[
= \frac{1}{r} \left( \frac{h}{f^2} \frac{1 - \xi}{f} \right) = \frac{h}{rf^2} = \frac{x_0^2}{r^2 f^2},
\]
\[
C = -\frac{2f'}{f^2} = (-2) \frac{h - f}{rf^2} = 2 \frac{rf - rh}{(rf)^2} = 2 \frac{x_0^2 (\ln \frac{r}{r_0} - 1) + r_0 f(r_0)
\]
\[
= 2 \frac{x_0^2 (\ln \frac{r}{r_0} + r_0 f(r_0))^2}{x_0^2 \ln \frac{r}{r_0} + r_0 f(r_0)}.
\]

We also see the distance function for metrics in $S_3$:

\[
s(r) = \int_0^{r_0} \sqrt{\frac{h}{4r} dr + \frac{x_0}{2} \ln \frac{r}{r_0}}.
\]

Now one can estimate the average of $A, B, C$ inside $B(s)$ from (17), (31), (32), (33), and (34). Namely, if $n \geq 2$, for any metric in $S_3$ there exists a constant $c > 0$ such that

\[
\frac{1}{c rf} \leq \frac{1}{\text{Vol}(B(s))} \int_{B(s)} R \omega^n \leq \frac{c}{rf},
\]

where $\text{Vol}(B(s)) = c_n(rf)^n$.

If $\omega$ is a nonflat Kähler metric in $S_1 \cup S_2$, we see from Theorem 3.2 that $F$ must have $F'(x_0) > 0$. Then the formula of $A, B, C$ in terms of $F$ is exactly the same as (10) derived in [14] (See P536 in [14]). Follow the proof of Proposition 2.6 in [14], we get the same conclusion. We summarize the above discussion as the following result. Note that $f$ is defined in (3).

**Proposition 3.3.** When $n \geq 2$, given any non flat metric in $M_n$, there exists a constant $C > 0$ such that

\[
\frac{1}{c (1 + rf)} \leq \frac{1}{\text{Vol}(B(s))} \int_{B(s)} R \omega^n \leq \frac{c}{1 + rf},
\]

where $\text{Vol}(B(s)) = c_n(rf)^n$.

Now we state the main theorem of this note.

**Theorem 3.4.** Given any $n \geq 2$, any nonflat Kähler metric in $M_n$ has $\int_{B(s)} \sigma_n \omega^n$ unbounded when $s$ goes to infinity. Moreover, if $2 \leq k < n$ one can construct a complete Kähler metric $\omega$ from $S_1 \subset M_n$ with bounded curvature on $\mathbb{C}^n$ such that

\[
\frac{1}{x^{2k-2}} \int_{B(s)} \sigma_k \omega^n
\]

is unbounded when $s$ tends to infinity.
Proof of Theorem 3.4. It follows from (36) that for any metric in $\overline{\mathcal{M}}_n$ we have Ricci curvature at $z$ given by:

\begin{equation}
\lambda = R_{1\mathbf{T}} = A + (n - 1)B, \quad \mu = R_{\mathbf{n}} = B + \frac{n}{2}C \quad 2 \leq i \leq n.
\end{equation}

Note that we are now working on the Kähler manifolds $\mathbb{C}^n$ and the Ricci tensor is $J$-invariant where $J$ is the standard complex structure on $\mathbb{C}^n$. Therefore the Ricci tensor in the real case has eigenvalue $\lambda$ of multiplicity 2 and $\mu$ of multiplicity $2n - 2$. From now on, let $\sigma_k$ denote the $k$-th elementary symmetric function of the Ricci curvature tensor.

First note that Question 1.1 are true for any metric $\omega \in \mathcal{M}_n$ when $k = 1$. Since $\sigma_1 = 2R$ where $R$ is the scalar curvature in the Kähler case, it follows from Proposition 3.3 and the upper bound of the volume growth of $B$ on Question 1.1 in the case of $2$ linear combination of $\lambda^2 \mu^{k-2}, \lambda^k \mu^{-1}$, and $\mu^k$. To sum up, $\sigma_k$ is a linear combination of three types of quantities:

- (Type I) $A^2 B^i C^j, AB^{1+i} C^j, B^{2+i} C^j$ when $i \geq 0, j \geq 0, \text{and } i + j = k - 2$.
- (Type II) $A^i B^j C^j$ and $B^{1+i} C^j$ when $i \geq 0, j \geq 0, \text{and } i + j = k - 1$.
- (Type III) $B^i C^j$ when $i \geq 0, j \geq 0, \text{and } i + j = k$.

We divide the proof of Theorem 3.4 into two cases.

(\textbf{Case I}) If $k = n$, we only need to look at the term $C^n$ contained in $\sigma_n$. Recall that if for any non flat Kähler metric $\omega$ in $S_1 \cup S_2$, we may assume that there exists $0 < M_1 < x_0$ such that $F'(x) \geq C_0$ where $C_0 = F'(M_1) > 0$ when $x \geq M_1$. we have the expression of $C$ from (38):

\begin{equation}
C = \frac{2v - 2x^2}{v^2} = \int_0^x 2\tau (\sqrt{1 + F'(\tau)^2} - 1)d\tau \geq \int_{M_1}^x 2\tau \sqrt{\frac{F'(\tau)^2}{1 + F'(\tau)^2 + 1}}d\tau
\end{equation}

Note that we have $1 \leq \frac{F'(x)}{C_0}$ when $x \geq M_1$.

\begin{equation}
C \geq \frac{C_0}{1 + \sqrt{M_1^2 + 1} \alpha(x)} \geq \frac{C_1}{v},
\end{equation}

where

\begin{equation}
C_1 = \frac{C_0}{1 + \sqrt{M_1^2 + 1}} I(x), \quad I(x) = \int_{M_1}^x 2\tau F'(\tau)d\tau.
\end{equation}
Since $I(x)$ goes to $\infty$ and $C_1$ is bounded when $x$ tends to $x_0$, we conclude that there exists a $C_2$ and $M_2$ such that when $x > M_2$,

$$C \geq \frac{C_2}{v}.$$  

We remark that (41) is used in the proof of Proposition 2.3 in [14].

There exists a constant $C_3$ only depending on $n$ such that:

(42)  
$$\int_{B(s)} \sigma_n \omega^n \geq C_3 \int_{B(s)} C^n \omega^n = C_3 c_n \int_0^s C^n dv^n \geq C_3 c_n \int_{v(x)}^{v(M_2)} (\frac{C_2}{v})^n dv^n = nC_3 c_n C_2 \ln \frac{v(x)}{v(M_2)}.$$  

Of course (42) is unbounded when $x$ tends to $x_0$ since $v(x_0) = +\infty$.

To sum up, we show that for any non flat Kähler metric $\omega$ in $S_1 \cup S_2$, $\int_{S_1} \sigma_n \omega^n$ is $\infty$. If $\omega \in S_3$, it follows from (30) and (33) that $\lim_{s \rightarrow +\infty} \int_{B(s)} \sigma_n \omega^n$ is unbounded when $s$ goes to infinity. Therefore, Question 1.1 is false when $k = n$ for any non flat Kähler metric $\omega$ in $S_n$.

(Case II) If $2 \leq k < n$, For any fixed nonflat Kähler metric $\omega$ in $S_1$, we may assume that there exist $C_4$ and $M_3$ such that $F'(x) \leq C_4$ for all $x \in (-x_0, 0)$ and $F'(x) \geq \frac{1}{n}$ when $x \geq M_3$. Then it follows from a similar argument as in (41), we may further assume that there exist $C_5$ and $M_3$ such that for any $x \geq M_3$

(43)  
$$C \geq \frac{C_5}{v}.$$  

Since $A = \frac{F''F''}{2(F'(F''))}$, we conclude that $A$ and $\frac{F''(x)}{x}$ are equivalent. If we can construct an Kähler metric $\omega$ in $S_1$ such that

(44)  
$$\frac{1}{s^{2n-2k}} \int_{B(s)} \left( \frac{F''(x)}{x} \right)^2 \left( \frac{C_5}{v} \right)^{k-2} \omega^n$$  

is unbounded when $s$ tends to $\infty$, then so is $\frac{1}{s^{2n-2k}} \int_{B(s)} A^2 C^{k-2} \omega^n$. Note that the term $A^2 C^{k-2}$ is contained in $\sigma_k$, it follows that $\frac{1}{s^{2n-2k}} \int_{B(s)} \sigma_k \omega^n$ will be unbounded when $s$ tends to $\infty$.

Let us rewrite (44):

(45)  
$$\int_{B(s)} \left( \frac{F''(x)}{x} \right)^2 \left( \frac{C_5}{v} \right)^{k-2} \omega^n = c_n C_5^{k-2} \int_0^x \left( \frac{F''(\tau)}{\tau} \right)^2 \left( \frac{1}{v} \right)^{k-2} dv^n = 2nc_n C_5^{k-2} \int_0^x \left( \frac{F''(\tau)}{\tau} \right)^2 v^{n-k+1} \sqrt{1 + (F'(\tau))^2} d\tau. $$  

Since $s = \int_0^x \sqrt{1 + (F'(\tau))^2} d\tau$, $v = \int_0^x 2\tau \sqrt{1 + (F'(\tau))^2} d\tau$ and $F'(x) \leq C_4$ we know $s$ and $x$ are equivalent, $v$ and $x^2$ are equivalent. In order to estimate (45), it
suffices to estimate the following.

\begin{equation}
\int_0^\infty (F''(x))^2 x^{2(n-k)+1} d\tau.
\end{equation}

To sum up, if there exists a function \( \delta(x) \in C^\infty[0, +\infty) \), such that

\begin{equation}
\lim_{x \to x_0} \frac{1}{x^{2n-2k}} \int_0^\infty \delta^2(\tau) \tau^{2(n-k)+1} d\tau = +\infty, \quad \int_0^{+\infty} \delta(x) dx < +\infty.
\end{equation}

Then we can solve \( F(x) \) with \( F''(x) = \delta(x) \) with the initial value \( F(0) = F'(0) = 0 \), it will follow from Theorem 3.2 that we can construct a complete Kähler metric \( \omega \) in \( S_1 \) such that

\begin{equation}
\frac{1}{s^{2n-2k}} \int_{B(s)} \sigma_k \omega^n
\end{equation}

is unbounded when \( s \) tends to infinity. Hence both Question 1.1 and 1.2 can not be true when \( 2 \leq k < n \).

In fact such a \( \delta(x) \) is not hard to construct. Consider \( \tilde{\delta}(x) \) defined by the following with \( q \) an integer to be determined.

\begin{equation}
\tilde{\delta} = \begin{cases} 
2 & x \in [2, 2 + (\frac{1}{2})^q] \\
\vdots & \\
l & x \in [l, l + (\frac{1}{2})^q] \\
\ vdots & \\
0 & x \in [0, +\infty) \setminus (\cup_{l \geq 2}[l, l + (\frac{1}{2})^q])
\end{cases}
\end{equation}

Now set \( q = \frac{5}{2} \), it is easy to verify that \( \psi(x) \) satisfies (47). Choose \( \delta(x) \) as a suitable smoothing of \( \tilde{\delta}(x) \) on \([0, +\infty)\) which also satisfies (47), we will get the desired counterexample. It can be checked that the result metric \( \omega \in S_1 \) has bounded curvature on \( \mathbb{C}^n \).

\[ \square \]

It follows from Theorem 3.2 and Proposition 3.3 that for any Kähler metric \( \omega \in S_1 \), \((\mathbb{C}^n, \omega)\) has quadratic average scalar curvature decay. Note that the same result for any complete Kähler manifolds with bounded nonnegative bisectional curvature and Euclidean volume growth has been proved by Ni (See [11] and [12]).

Now we construct the following example which implies that in general only assuming Euclidean volume growth one can not expect the same rate of decay for \( L^p \) norm of curvature for any \( p > 1 \).

**Proposition 3.5.** For any \( n \geq 2 \) and any \( p > 1 \), there exists a metric \( \omega \in \overline{M}_n \) such that the geodesic balls in \((\mathbb{C}^n, \omega)\) has Euclidean volume growth. Moreover,

\begin{equation}
\frac{s^2}{\text{Vol}(B(s))} \int_{B(s)} (Rm(\frac{\partial}{\partial s}, \frac{\partial}{\partial s}, \frac{\partial}{\partial s}, \frac{\partial}{\partial s}))^p \omega^n
\end{equation}

is unbounded as \( s \) goes to infinity. Here we denote \( \frac{\partial}{\partial s} \) to the unit radial direction on \( \mathbb{C}^n \).
Proof of Proposition 3.3. For a given metric $\omega$ in $S_1$, follow a similar argument in (Case II) of the proof of Theorem 3.4, it suffices to show that we can find a smooth function $\eta(x)$ on $[0, +\infty)$ such that

$$\lim_{x \to +\infty} \frac{1}{x^{2n-2}} \int_0^x \eta^p(\tau) \tau^{2n-1-p} \, d\tau = +\infty,$$

(51) \hfill $\int_0^{+\infty} \eta(x) \, dx < +\infty.$

Consider $\bar{\eta}(x)$ defined by the following where $\alpha$ and $\beta$ are two integers to be determined.

$$\bar{\eta} = \begin{cases} 2^\alpha & x \in [2, 2 + (\frac{1}{2})^\beta] \\
 & \vdots \\
1^\alpha & x \in [l, l + (\frac{1}{2})^\beta] \\
 & \vdots \\
0 & x \in [0, +\infty) \setminus (\cup_{l \geq 2}[l, l + (\frac{1}{2})^\beta]) \end{cases} \tag{52}$$

Pick any $\alpha > 1$ and $1 + \alpha < \beta < p(\alpha - 1) + 2$, then $\bar{\eta}$ defined above satisfies (51). It is not hard to find $\eta(x)$ from a suitable smoothing of $\bar{\eta}(x)$ which will result in the desired metric $\omega$. Note that $(\mathbb{C}^n, \omega)$ we constructed has unbounded curvature on $\mathbb{C}^n$.

We proceed to show that Question 1.2 is true for $\widetilde{\mathcal{M}}_n$. It seems that Question 1.2 should be a more suitable conjecture at least for complete Kähler manifolds with nonnegative bisectional curvature.

**Theorem 3.6.** For any metric $\omega \in \widetilde{\mathcal{M}}_n$, then $s^{-2n+2k} \int_{B(s)} \text{Ric}^k \wedge \omega^{n-k}$ is bounded when $s$ goes to infinity.

**Proof of Proposition 3.6.** First we remark that it directly follows from analogues of Proposition 2.3, 2.4 and 2.6 in [14] for the space $\mathcal{M}_n$ (See Proposition 3.3 and the paragraph after Proposition 3.1) that Question 1.3 is true for $k = 1$ and $k = n$. It suffices to show that $s^{-2n+2k} \int_{B(s)} \text{Ric}^k \wedge \omega^{n-k}$ is bounded for any $2 \leq k < n$.

Note that for $2 \leq k < n$, $\text{Ric}^k \wedge \omega^{n-k}$ is a linear combination of $\lambda \mu^{k-1}$ and $\mu^k$. It turns out that we only need to show that $\frac{1}{s^{2n-2k}} \int_{B(s)} P(A, B, C) \omega^n$ is bounded when $s$ goes to infinity where $P$ is a monomial of the following two types:

(Type I) $AB^iC^j$, and $B^{1+i}C^j$ when $i \geq 0$, $j \geq 0$, and $i + j = k - 1$.

(Type II) $B^pC^q$ when $p \geq 0$, $q \geq 0$, and $p + q = k$.

First we consider any Kähler metric $\omega$ in $S_1 \cup S_2$. Note that (16) implies that

$$B \leq \frac{x^2}{v^2} \leq \frac{1}{v}, \quad C \leq \frac{2}{v} \tag{53}$$

Then we have the following estimate:

$$\frac{1}{s^{2n-2k}} \int_{B(s)} B^pC^q \omega^n \leq 2^q c_n \frac{1}{s^{2n-2k}} \int_0^{v(x)} \frac{1}{v^r} \nu v^{n-1} \, dv \leq 2^q c_n \left( \int_0^{\tau} 2\tau \sqrt{1 + (F'(\tau))^2} \, d\tau \right)^{n-k} \frac{1}{(n-k)(\int_0^x \sqrt{1 + (F'(\tau))^2} \, d\tau)^{2n-2k}} \tag{54}$$
According to the L’Hospital’s rule, (54) has the limit when \( x \) tends to \( x_0 \):

\[
\lim_{x \to x_0} \frac{\int_{x_0}^{x} 2 \tau \sqrt{1 + (F'(\tau))^2} d\tau}{\int_{x_0}^{x} \sqrt{1 + (F'(\tau))^2} d\tau} = \left( \frac{2}{\sqrt{1 + \lim_{x \to x_0} F'(x)}} \right)^{n-k}.
\]

We conclude that \( \frac{1}{s^{2n-2k}} \int_{B(s)} B^p C^q \omega^n \) is bounded when \( s \) goes to infinity.

Next we turn to the term \( AB^i C^j \), integrate by parts as in the original proof of Proposition 2.6 in [14].

\[
\int_{B(s)} AB^i C^j \omega^n = c_n \int_{0}^{x} \frac{F''(\tau) - \frac{1}{(F'(\tau))^2} \nu^k - 1}{\sqrt{1 + (F'(\tau))^2}} d\tau
\]

\[
\leq c_n \nu v^{n-k}.
\]

Using the L’Hospital’s rule again, we conclude that

\[
\frac{1}{s^{2n-2k}} \int_{B(s)} AB^i C^j \omega^n
\]

is bounded when \( s \) tends to infinity.

It remains to verify that Question 1.2 is true when \( 2 \leq k < n \) for any metric \( \omega \in S_3 \). Note that in this case we have (32), (33), (34), and \( A = 0 \) outside a compact set for metrics in \( S_3 \), it follows from a straightforward calculation that \( \frac{1}{s^{2n-2k}} \int_{B(s)} Ric^k \wedge \omega^{n-k} \) is bounded when \( s \) goes to infinity. Hence we finish the proof of Proposition 3.6.

We also have the following result relating the growth of the coordinate function \( z_i \) to the volume growth of the geodesic balls with respect to the metric \( \omega \) in \( M_n \).

**Proposition 3.7.** Given any metric \( \omega \in M_n \), if some coordinate function \( z_i \) for some \( 1 \leq i \leq n \) has polynomial growth with respect to \( \omega \), then the geodesic balls of \((C^n, \omega)\) have Euclidean volume growth.

**Proof of Proposition 3.7.** Assume some coordinate function \( z_i \) for some \( 1 \leq i \leq n \) has polynomial growth with respect to \( \omega \) in \( M_n \), it follows from \( \omega \) being rotationally symmetric that there exists some integer \( \alpha \) and constant \( C_\alpha > 0 \) such that:

\[
r = |z|^2 \leq C_\alpha s^\alpha.
\]

From Theorem 3.2 it suffices to show that \( \omega \in S_1 \), namely \( F'(x) \) bounded when \( x \) goes to \( x_0 \). First we note that \( \omega \) can not be from \( S_3 \) from the explicit formula (3) on the distance with respect to metrics in \( S_3 \) given in Theorem 3.1.
Plugging (15) into (58) leads to:

$$\int_0^x \sqrt{1 + (F'(\tau))^2} d\tau \leq C_6 \int_0^x 1 + (F'(x))^2 d\tau.$$  

Note that:

$$\frac{dr}{dx} = \frac{2r}{(1 - \xi)x} = \frac{2r \sqrt{1 + (F'(x))^2}}{x}.$$  

Solve $r$ in terms of $x$ from (60) and plug into (59):

$$e^{\int_0^x \sqrt{1 + (F'(\tau))^2} d\tau} : C_6 \int_0^x \sqrt{1 + (F'(\tau))^2} d\tau$$

for any $C_7 \leq x < x_0$. Here $C_7$ is the value of $x$ which corresponds to $r = 1$.

It is not hard to show $F'(x)$ is bounded for all $x \in (0, x_0)$ from (61). First we see that $x_0$ must be infinity. Otherwise, the left hand side $e^{\int_0^x \sqrt{1 + (F'(\tau))^2} d\tau}$ can not be controlled by any polynomials when $\int_0^x \sqrt{1 + (F'(\tau))^2} d\tau$ goes to infinity. Next we show that $F'(x)$ is bounded for all $x \in (0, +\infty)$. It follows from (61) that

$$\frac{2 \int_{C_7}^{x} \sqrt{1 + (F'(x))^2} d\tau}{\alpha \ln \int_0^x \sqrt{1 + (F'(\tau))^2} d\tau} + \ln C_6$$

should be bounded when $x$ tends to infinity.

It is easy to see that (62) has a limit when $x$ goes to infinity.

$$\lim_{x \to +\infty} \frac{2 \int_{C_7}^{x} \sqrt{1 + (F'(x))^2} d\tau}{\alpha \ln \int_0^x \sqrt{1 + (F'(\tau))^2} d\tau} + \ln C_6 = \frac{2}{\alpha} \sqrt{1 + (\lim_{x \to +\infty} F'(\tau))^2},$$

which implies that $F'(x)$ is bounded for all $x$ in $[0, +\infty)$. Therefore $\omega \in S_1$. □

**Remark 3.8.** See [10], [6], [7], [11], [13], and [12] for some results on the geometry of general complete noncompact Kähler manifolds with nonnegative bisectional curvature. Some further generalizations will also appear in a separate paper by the author.

**Acknowledgments.** The author thanks Professor Lei Ni for making Wu and Zheng’s paper [14] available to him, bringing Question [1.2] to his attention, as well as many helpful discussions during the preparation of this paper. The author also thanks Professor Bennett Chow and Professor Xiaohua Zhu for helpful comments and Professor Fangyang Zheng for his interest in this work.

**References**

[1] Cao, Huai-Dong. *Existence of gradient Kähler-Ricci solitons*. Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994), 1-16, A K Peters, Wellesley, MA, 1996.

[2] Cao, Huai-Dong. *Limits of solutions to the Kähler-Ricci flow*. Journal of Differential Geometry, 65 (1997), 257-272.

[3] Chau, Albert; Tam, Luen-Fai. *On the complex structure of Kähler manifolds with nonnegative curvature*. Journal of Differential Geometry 73, 2006, no.3, 491-530.

[4] Chau, Albert; Tam, Luen-Fai. *Non-negatively curved Kähler manifolds with average quadratic curvature decay*. Communications in Analysis and Geometry 15, 2007, no.1, 121-146.

[5] Chau, Albert; Tam, Luen-Fai. *On the Steinness of a class of Kähler manifolds*. Journal of Differential Geometry 79, 2008, no.2, 167-183.
[6] Chen, Bing-Long; Zhu, Xi-Ping. On complete noncompact Kähler manifolds with positive bisectional curvature. Mathematische Annalen 327, 2003, no.1, 1-23.

[7] Chen, Bing-Long; Zhu, Xi-Ping. Volume growth and curvature decay of positively curved Kähler manifolds. Quarterly Journal of Pure and Applied Mathematics 1, 2005, no.1, 68-108.

[8] Chen, Bing-Long; Tang, Siu-Hung; Zhu, Xi-Ping. A uniformization theorem for complete non-compact Kähler surfaces with positive bisectional curvature. Journal of Differential Geometry 67, 2004, no.3, 519-570.

[9] Klembeck, Paul F. A complete Kähler metric of positive curvature on $C^n$. Proceedings of the American Mathematical Society, 64, 1977, no.2, 313-316.

[10] Ni, Lei. A monotonicity formula on complete Kähler manifolds with nonnegative bisectional curvature. Journal of the American Mathematical Society 17, 2004, no.4, 909-946.

[11] Ni, Lei. Ancient solutions to Kähler-Ricci flow. Mathematical Research Letters 12, 2005, no.5-6, 633-653.

[12] Ni, Lei. Monotonicity and Holomorphic functions. Geometry and Analysis Volume I, Advanced Lecture in Mathematics 17, Higher Education Press and International Press, Beijing and Boston, 2010, 447-457.

[13] Ni, Lei; Tam, Luen-Fai. Plurisubharmonic functions and the structure of complete Kähler manifolds with nonnegative curvature. Journal of Differential Geometry 64, 2003, no.3, 457-524.

[14] Wu, Hung-Hsi; Zheng, Fangyang. Examples of positively curved complete Kähler manifolds. Geometry and Analysis Volume I, Advanced Lecture in Mathematics 17, Higher Education Press and International Press, Beijing and Boston, 2010, 517-542.

[15] Yau, Shing-Tung. Open problems in geometry. Chern—a great geometer of the twentieth century, International Press, Hong Kong, 1992, 275-319.

[16] Yau, Shing-Tung. A review of complex differential geometry. Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), Proceedings of Symposia in Pure Mathematics, 619-625, Vol 52, Part II, American Mathematical Society, 1991.

Department of Mathematics, University of California San Diego, La Jolla, CA 92093
E-mail address: bsyang@math.ucsd.edu