Abelian bosonization approach to quantum impurity problems

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Abstract

Using Abelian Bosonization, we develop a simple and powerful method to calculate the correlation functions of the two channel Kondo model and its variants. The method can also be used to identify all the possible boundary fixed points and their maximum symmetry, to calculate straightforwardly the finite size spectra, to demonstrate the physical picture at the boundary explicitly. Comparisons with Non-Abelian Bosonization method are made. Some fixed points corresponding to 4 pieces of bulk fermions coupled to $s = 1/2$ impurity are listed.
In general quantum impurity problems, a local quantum mechanical degree of freedom couples locally to some extended degree of freedoms which can be described by some critical theory in the continuum limit. Some noted examples are the multichannel Kondo effects \cite{1}, quantum dissipation problems \cite{3}, an impurity in one-dimensional Luttinger liquid \cite{2} and the catalysis of proton decay \cite{4}. For general quantum impurity model, Affleck and Ludwig (AL) \cite{5}, using CFT, pointed out that the impurity degree of freedoms completely disappear from the description of the low temperature fixed point and leave behind conformally invariant boundary conditions. For the multichannel Kondo problems, AL calculated the finite size spectrum at the fixed point which are consistent with the results of Numerical Renormalization Group \cite{6,9}. AL calculated not only all the thermodynamic quantities, but also all the correlation functions \cite{7,8}.

For the special case of 4 pieces of bulk fermions such as the two channel Kondo effect (2CK), the one channel two impurity Kondo models \cite{9} and the four flavor Callan-Rubakov effect \cite{4}, the non-interacting theory possesses \textit{SO}(8) symmetry, Maldacena and Ludwig (ML) \cite{10} showed that the boundary conditions at the fixed point turn out to be linear in the basis which separates charge, spin and flavor. The linear boundary conditions can also be transformed into the non trivial boundary conditions in the original fermion basis by the triality transformation \cite{10}. By using the linear boundary conditions, ML reduced the problem of calculating all the correlation functions to free field exercises. However, ML deduced the boundary conditions by using explicitly the results of AL (for example, the single particle S matrices vanish), therefore their method explicitly depends on AL’s results.

Emery and Kivelson (EK) \cite{11}, using Abelian Bosonization, found an alternative solution to the 2CK. Although EK’s method is applicable only when the number of channels takes some special values, it is simpler and more widely accessible than AL’s method. The operator contents of AL’s and EK’s approaches have been shown to be exactly the same by the author \cite{12}. However, so far, EK’s method cannot be used to calculate the \textit{single-electron} properties.

In this paper, using Abelian Bosonization, we develop a simple, systematic and powerful method to study the 2CK and its variants. The method can identify \textit{all} the possible boundary
fixed points, their corresponding boundary conditions and their maximum symmetry in a very straightforward way. It can also calculate the finite size spectrum and any correlation functions. We explicitly point out that at the fixed point, the impurity degree of freedoms act as Lagrangian multipliers, therefore impose boundary conditions on the conduction electrons, the impurity themselves turn into the corresponding scaling fields at the fixed point. The method has also been successfully applied to study the two channel spin-flavor Kondo model [14], the two channel flavor anisotropic and one channel compactified Kondo models [15] and the problem of a non-magnetic impurity hopping between two sites in a metal [16]. Although demonstrating the method explicitly only by solving the 2CK, we suggest that our method maybe applicable to other quantum impurity problems. Comparisons with AL’s and ML’s results are made. Finally, some fixed points corresponding to 4 pieces of bulk fermions coupled to $s = 1/2$ impurity are listed.

The Hamiltonian of the well-studied 2-channel Kondo model is:

$$H = i v_F \int_{-\infty}^{\infty} dx \psi_{i \alpha}^\dagger(x) \frac{d\psi_{i \alpha}(x)}{dx} + \sum_{a=x,y,z} \lambda^a J^a(0) S^a + h(\int dx J^z_s(x) + S^z)$$

where $J^a(x) = \frac{1}{2} \psi_{i \alpha}^\dagger(x) \sigma^a_{\alpha \beta} \psi_{i \beta}(x)$ is the spin currents of the conduction electrons.

In this paper, for simplicity, we take $\lambda^x = \lambda^y = \lambda$, the symmetry in the spin sector is $U(1) \times Z_2 \sim O(2)$. Abelian-bosonizing the four bulk Dirac fermions separately:

$$\psi_{i \alpha}(x) = \frac{P_{i \alpha}}{\sqrt{2\pi a}} e^{-i \Phi_{i \alpha}(x)}$$

Where $\Phi_{i \alpha}(x)$ are the real chiral bosons, the cocyle factors have been chosen as: $P_{1 \uparrow} = P_{1 \downarrow} = e^{i\pi N_{1 \uparrow}}, P_{2 \uparrow} = P_{2 \downarrow} = e^{i\pi (N_{1 \uparrow} + N_{1 \downarrow} + N_{2 \uparrow})}$.

Following the three standard steps of the EK’s solution [11], (1) Introduce charge, spin, flavor, spin-flavor bosons (2) Make the canonical transformation $U = \exp[i S^z \Phi_s(0)]$ (3) Make the following refermionization

$$S^x = \frac{\hat{a}}{\sqrt{2}} e^{i\pi N_{sf}}, \quad S^y = \frac{\hat{b}}{\sqrt{2}} e^{i\pi N_{sf}}, \quad S^z = -i \hat{a} \hat{b}$$

$$\psi_{sf} = \frac{1}{\sqrt{2}} (a_{sf} - i b_{sf}) = \frac{1}{\sqrt{2\pi a}} e^{i\pi N_{sf}} e^{-i \Phi_{sf}}$$

$$\psi_{s,i} = \frac{1}{\sqrt{2}} (a_{s,i} - i b_{s,i}) = \frac{1}{\sqrt{2\pi a}} e^{i\pi d \dagger d} e^{i\pi N_{sf}} e^{-i \Phi_{s}}$$
The transformed Hamiltonian \( H' = UHU^{-1} = H'_{sf} + H'_{s} + \delta H' \) can be written in terms of the Majorana fermions as \[17\]:

\[
H_{sf} = \frac{iv_F}{2} \int dx (a_{sf}(x) \frac{\partial a_{sf}(x)}{\partial x} + b_{sf}(x) \frac{\partial b_{sf}(x)}{\partial x}) - i \frac{\lambda}{\sqrt{2\pi a}} \hat{a}b_{sf}(0)
\]

\[
H_{s} = \frac{iv_F}{2} \int dx (a_{s}(x) \frac{\partial a_{s}(x)}{\partial x} + b_{s}(x) \frac{\partial b_{s}(x)}{\partial x}) - i \hbar \int dx a_{s}(x)b_{s}(x)
\]

\[
\delta H = -\lambda'_z \hat{a}b_{s}(0)b_{s}(0)
\] (4)

where \( \lambda'_z = \lambda^z - 2\pi v_F \).

From the above equation, it is evident that along the solvable line (EK line) \( \lambda'_z = 0 \), half of the impurity spin \( \hat{b} \) and \( \psi_s(x) \) completely decouple \[11\]. However, the canonical transformation \( U \) is a boundary condition changing operator \[13\], the transformed field \( \psi'_s(x) \) is related to the original field \( \psi_s(x) \) by

\[
\psi'_s(x) = U^{-1} \psi_s, i(x)U = e^{i\pi d^d e^{i\pi S^c sgnx} \psi_s(x)} = -isgnx \psi_s(x)
\] (5)

It is important to observe that the impurity spin completely disappear from the above equation.

The boundary condition \( \psi'_{s,L}(0) = \psi'_{s,R}(0) \) in \( H' \) leads to

\[
a_{s,L}(0) = -a_{s,R}(0), \quad b_{s,L}(0) = -b_{s,R}(0)
\] (6)

In order to identify the fixed point along the solvable line, it is convenient to write \( H_{sf} \) in the action form

\[
S = S_0 + \gamma \frac{1}{2} \int d\tau \hat{a}(\tau) \frac{\partial \hat{a}(\tau)}{\partial \tau} - i \lambda \frac{1}{\sqrt{2\pi a}} \int d\tau \hat{a}(\tau)b_{sf}(0, \tau)
\] (7)

The simple power countings of the action \( S \) leads to the following R. G. flow equations \[18\]

\[
\frac{d\gamma}{dl} = 0
\]

\[
\frac{d\lambda}{dl} = \frac{1}{2} \lambda
\] (8)

The above R. G. equations are equivalent to
\[ \frac{d\gamma}{dl} = -\gamma \]
\[ \frac{d\lambda}{dl} = 0 \]  \hspace{1cm} (9)

It is easy to see the fixed point is located at \( \gamma = 0 \) where \( \hat{a} \) loses its kinetic energy and becomes a Grassmann Lagrangian multiplier, it can be integrated out, therefore the impurity degree of freedoms completely disappear from the effective Hamiltonian and leave behind the boundary conditions \[19\]

\[ b_{sf,L}(0) = -b_{sf,R}(0) \]  \hspace{1cm} (10)

The boundary conditions Eqs.8,10 can be expressed in terms of bosons \[20\]

\[ \Phi_{s,L}(0) = \Phi_{s,R}(0) + \pi, \quad \Phi_{sf,L}(0) = -\Phi_{sf,R}(0) \]  \hspace{1cm} (11)

The three Majorana fermions in the spin sector being twisted, the fixed point possesses the symmetry \( O(3) \times O(5) \). Using totally different method, ML \[10\] identified the symmetry. By using the triality transformation, they showed that Eqs. 8,10 imply that the original fermions scatter into the collective excitations which fit into the spinor representation of \( SO(8) \).

This symmetry enable us to work out easily the finite size spectrum at the fixed point in Table I. Although the conformal embedding is highly non-trivial in the original fermion basis \[4,5\], it is completely trivial in the basis of Eqs.4,5, namely the finite size energy spectrum are simply constructed by the direct sum of 3 Majorana fermions in R sector and 5 Majorana fermions in NS sector or vice versa. Table I is more compact than, but compatible with those listed in Ref. 5,6.

By using Dyson equations, we can get all the correlation functions in \((k,\omega)\) space from the fixed point action where \( \gamma = 0 \)

\[ \langle \hat{a}(\omega)\hat{a}(-\omega) \rangle = \left[ \frac{t^2}{2} \int \frac{dk}{2\pi i\omega} \frac{1}{v_F k} \right]^{-1} \]
\[ \langle b_{sf}(k_1,\omega)b_{sf}(k_2,-\omega) \rangle = \frac{2\pi \delta(k_1 + k_2)}{i\omega - v_F k_1} \]
\[ + \left( \frac{t}{i\omega - v_F k_1} \right) \left( \frac{t}{-i\omega - v_F k_2} \right) \langle \hat{a}(\omega) \hat{a}(-\omega) \rangle \]

\[ \langle b_{sf}(k, \omega) \hat{a}(-\omega) \rangle = \frac{t}{i\omega - v_F k} \langle \hat{a}(\omega) \hat{a}(-\omega) \rangle \tag{12} \]

where \( t = i \frac{\lambda}{2\sqrt{2\pi a}} \).

Paying special attention to the convergence factors when performing contour integrals, we get all the correlation functions in \((x, \tau)\) space.

\[ \langle b_{sf}(x_1, \tau_1) \hat{a}(\tau_2) \rangle \sim \begin{cases} 
\langle b_{sf}(x_1, \tau_1) b_{sf}(0, \tau_2) \rangle_0 & \text{if } x_1 > 0 \\
-\langle b_{sf}(x_1, \tau_1) b_{sf}(0, \tau_2) \rangle_0 & \text{if } x_1 < 0 
\end{cases} \]

\[ \langle b_{sf}(x_1, \tau_1) b_{sf}(x_2, \tau_2) \rangle = \begin{cases} 
\langle b_{sf}(x_1, \tau_1) b_{sf}(x_2, \tau_2) \rangle_0 & \text{if } x_1 x_2 > 0 \\
-\langle b_{sf}(x_1, \tau_1) b_{sf}(x_2, \tau_2) \rangle_0 & \text{if } x_1 x_2 < 0 
\end{cases} \tag{13} \]

where \( \langle \cdots \rangle_0 \) means the non-interacting correlation functions. Any multi-point correlation functions can be calculated by Wick theorem.

Integrating over \( k \), the correlation functions along the boundary follow [19]

\[ \langle \hat{a}(\tau_1) \hat{a}(\tau_2) \rangle = \frac{1}{\tau_1 - \tau_2} \]

\[ \langle b_{sf}(0, \tau_1) b_{sf}(0, \tau_2) \rangle = 0 \]

\[ \langle b_{sf}(0, \tau_1) \hat{a}(\tau_2) \rangle = \frac{1}{\ell} \delta(\tau_1 - \tau_2) \tag{14} \]

Because of the relations \( \psi(x) = \psi_L(x) \) if \( x > 0; \psi(x) = \psi_R(-x) \) if \( x < 0 \) [19], the above equations imply that at the fixed point, the impurity completely disappear and turns into one of the non-interacting primary fields of the fixed point

\[ \hat{a}(\tau) \sim b_{sf,0}(0, \tau) \tag{15} \]

The left moving and right moving parts are related by the boundary condition Eq. [19] \( b_{sf,R}(-x, \tau) = -b_{sf,L}(x, \tau) \) if \( x < 0 \), any correlation functions can be mapped to the corresponding free fermion correlation functions.

From the boundary conditions Eqs [3, 10], it is easy to see that at the fixed point, L-L and R-R moving correlation functions are exactly the same with non-interacting case, however
\( \langle \psi_L \psi_R^\dagger \rangle = 0 \), namely the single particle \( S_1 \) matrix vanishes, this "Unitarity Puzzle" was comprehensively explained by ML \[10\].

Away from the fixed point, there is only one leading irrelevant operator moving away from EK line \(-\lambda' \hat{a} \hat{b} a_s(0)b_s(0) \sim \cos \Phi_{sf}(0)\partial \Phi_s(0) \) \[12\], the first order correction to the single particle L-R Green function \((x_1 > 0, x_2 < 0)\) due to this operator is \[10\]

\[
\langle \psi_1^\dagger(x_1, \tau_1) \psi_1^\dagger(x_2, \tau_2) \rangle \sim \lambda'_s (z_1 - \bar{z}_2)^{-3/2}
\]

where \( z_1 = \tau_1 + ix_1 \) is in the upper half plane, \( \bar{z}_2 = \tau_2 + ix_2 \) is in the lower half plane.

By conformal transformation to finite temperature, the above Eq. turns to Eq.(3.23) of Ref. \[7\] where AL derived the low temperature \( \sqrt{T} \) behavior of the electron resistivity. Higher order corrections can be similarly performed, since they are just the evaluations of the multi-point correlation functions of the free chiral bosons; the low temperature expansion of the electron resistivity is

\[
\rho(T) \sim \frac{\rho_u}{2} (1 + T^{1/2} + T^{3/2} + T^2 + \cdots)
\]

There are two subleading operators with dimension 2 \[12\],

\[
: \hat{a}(\tau) \frac{\partial \hat{a}(\tau)}{\partial \tau} : \sim : b_{sf,0}(\tau) \frac{\partial b_{sf,0}(\tau)}{\partial \tau} : =: \cos 2\Phi_{sf}(0, \tau) : - \frac{1}{2} : (\partial \Phi_{sf}(0, \tau))^2 :
\]

\[
: a_s(\tau) \frac{\partial a_s(\tau)}{\partial \tau} + b_s(\tau) \frac{\partial b_s(\tau)}{\partial \tau} : =: (\partial \Phi_s(0))^2 :
\]

The first one moves along the EK line, second order perturbation in this dimension 2 operator leads to \( \langle b_{sf}(0, \tau)b_{sf}(0, 0) \rangle \sim \frac{\tau^2}{\tau^2} \), for large \( \tau \) \[21\]. The second one moves away from EK line. It is easy to see that both operators do not contribute to the one particle Green function. Straightforward extensions of CFT analysis in Ref. \[12\] leads to the same conclusion. The contributions of these two operators to thermodynamic quantities have been discussed in detail in Ref. \[12\].

The boundary OPE of the spin and flavor singlet pairing operator is

\[
\psi_{1\uparrow}(z_1)\psi_{2\downarrow}(\bar{z}_2) = i(z_1 - \bar{z}_2)^{-1/2} : e^{-i \Phi_c(0, \tau)} : + \cdots
\]
The correlation function of the pairing operator $\mathcal{O} = e^{-i\Phi_c(0)}$ is:

$$\langle \mathcal{O}^\dagger(\tau)\mathcal{O}(0) \rangle = \frac{1}{\tau}$$  \hspace{1cm} (20)

It implies a $\ln T$ divergent pairing susceptibility at the impurity site.

In this paper, we discuss the 2CK fixed point with the symmetry $O(3) \times O(5)$. The fixed point of two impurity, one channel Kondo model (2IK) possesses $O(7) \times O(1)$ \cite{10,9,22}. Gan mapped the fixed point Hamiltonian of the 2IK to that of the 2CK \cite{22}. However, different canonical transformations are employed in the two models. In the 2IK, the impurity spin $S^z_+ = S^z_1 + S^z_2$ takes integer values $-1, 0, 1$, therefore the canonical transformation $U = e^{iS^z_+\Phi_s(0)}$ employed in Ref. \cite{22} does not change the boundary conditions of $a_s, b_s$, this explains why the fixed point symmetry of the 2IK is $O(7) \times O(1)$ instead of $O(3) \times O(5)$. The two channel spin-flavor Kondo model discussed in Ref. \cite{14} has the symmetry $O(2) \times O(6)$. The two channel flavor anisotropic Kondo model always flows to the fixed point with the symmetry $O(4) \times O(4)$ \cite{15}. Most recently, in reanalyzing the problem of an non-magnetic impurity hopping between two sites in a metal, the author found a line of non-fermi liquid fixed points with the symmetry $U(1) \times O(1) \times O(5)$ which continuously interpolates between the 2CK and the 2IK \cite{16} and an additional NFL fixed point which is the same as 2IK. The common feature of the models mentioned above is that the boundary interaction terms can be expressed in terms of Majorana fermions. This series has classified some of boundary fixed points corresponding to four pieces of bulk fermions. In general, twisting even (odd) no. of Majorana fermions, we get fermi (non-fermi) liquid fixed points. From Table 4, it is also easy to conclude that the finite size spectrum of twisting even (odd) no. of Majorana fermions is (not) equally spacing, therefore can (not) be fitted by a phase shift. Ref. \cite{23} studied a model where the one channel conduction electrons transforming as spin 3/2 representation of $SU(2)$ (therefore also 4 pieces of bulk fermions) couple to the spin $s = 1/2$ impurity, the fixed point symmetry of this model remains an open question, but it certainly does not fall into the above series, because the boundary interaction term of this model cannot be expressed in terms of Majorana fermions.
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TABLE I. The finite size spectrum at the 2CK fixed point with the symmetry $O(3) \times O(5)$. R is the Ramond (periodic) sector, NS is the Neveu-Schwarz (anti-periodic) sector; R+1st (NS+1st) is the first excited state in R (NS) sector, et. al. When counting the degeneracy of a energy level, we use the fact that the ground state degeneracy of $N$ Majorana fermions in R sector is $2^{[N/2]}$. 

|        |        | $\frac{t}{v_F R} (E - \frac{3}{10})$ | Degeneracy |
|--------|--------|--------------------------------------|------------|
| R      | NS     | 0                                    | 2          |
| NS     | R      | $\frac{1}{8}$                        | 4          |
| R      | NS+1st | $\frac{1}{2}$                        | 10         |
| NS+1st | R      | $\frac{5}{8}$                        | 12         |
| R+1st  | NS     | 1                                    | 6          |
| R      | NS+2nd | 1                                    | 20         |
| NS     | R+1st  | $1 + \frac{1}{8}$                    | 20         |
| NS+2nd | R      | $1 + \frac{1}{8}$                    | 12         |
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For simplicity, in the following, we don’t distinguish between $\psi_{s,i}(x)$ and $\psi_s(x)$. The primes will also be omitted.

The similar R. G. analysis was also performed by A. L. Moustakas and D. S. Fisher, Phys. Rev. B53, 4300 (1996).

The boundary conditions should be $b_{sf}(0,\tau) = 0$, however $b_{sf}(0,\tau) = \frac{1}{2}(b_{sf,L}(0,\tau) + b_{sf,R}(0,\tau))$, curious readers should check the derivation of mapping three dimensional Kondo problem to a one dimensional problem in Ref. [5].

However, for more complicated cases, see Ref. [16].

see [19] for the correct understanding of this correlation function.

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