Some Common Mistakes in the Teaching and Textbooks of Modal Logic

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Abstract

We discuss four common mistakes in the teaching and textbooks of modal logic. The first one is missing the axiom $$\Diamond \varphi \leftrightarrow \neg \Box \neg \varphi$$, when choosing $$\Diamond$$ as the primitive modal operator, misunderstanding that $$\Box$$ and $$\Diamond$$ are symmetric. The second one is forgetting to make the set of formulas for filtration closed under subformulas, when proving the finite model property through filtration, neglecting that $$\Box \varphi$$ and $$\Diamond \varphi$$ may be abbreviations of formulas. The third one is giving wrong definitions of canonical relations in minimal canonical models that are unmatched with the primitive modal operators. The final one is misunderstanding the rule of necessitation, without knowing its distinction from the rule of modus ponens. To better understand the rule of necessitation, we summarize six ways of defining deductive consequence in modal logic: omitted definition, classical definition, ternary definition, reduced definition, bounded definition, and deflationary definition, and show that the last three definitions are equivalent to each other.

Keywords: modal logic; deductive consequence; deduction theorem; axiomatic systems; filtration; minimal canonical models; finite model property

1 Primitive Modal Operators and the d/Dual Axiom

In modal logic, we can set $$\Box$$ as the primitive modal operator, and define $$\Diamond$$ to be $$\neg \Box \neg$$ as a derived operator, as in [3]. We can also set $$\Diamond$$ as the primitive operator, defining $$\Box$$ to be $$\neg \Diamond \neg$$, as in [2]. Though the two options seem totally symmetric, they are actually not. In constructing axiomatic systems, the choice of different primitive modal operators may lead to results that are not totally symmetric.

For example, consider the minimal normal modal logic $$\mathbf{K}$$. When $$\Box$$ is the primitive modal operator, apart from the axioms and rules for classical propositional logic (PC henceforth), the axiomatic system needs only include the following axiom (schema) and rule of inference,

$$\mathbf{K} \quad \Box(\varphi \to \psi) \to (\Box \varphi \to \Box \psi)$$
as well as the definition $\Diamond \varphi =_{df} \neg \Box \neg \varphi$, which amounts to adding the following axiom.

dual $\Diamond \varphi \leftrightarrow \neg \Box \neg \varphi$

But when $\Diamond$ is the primitive modal operator, apart from PC, K, RN and the following axiom used as a definition,

Dual $\Box \varphi \leftrightarrow \neg \Diamond \neg \varphi$

we still need to augment dual. From another perspective, if we do not take definitions as a part of the axiomatic system, when choosing $\Box$ as primitive, apart from PC and RN, we need only one more axiom, namely K; when choosing $\Diamond$ as primitive, however, we need two axioms, namely K and dual. Why is there such an asymmetry? The reason is that, the axiom K and the rule RN are intrinsically using $\Box$ rather than $\Diamond$. This makes the following rule derivable from K, RN, and dual (plus PC).

RE $\varphi \leftrightarrow \psi$

$\Box \varphi \leftrightarrow \Box \psi$

The derivation is as follows.

(1) $\varphi \leftrightarrow \psi$ hypothesis
(2) $\varphi \rightarrow \psi$, $\psi \rightarrow \varphi$ (1), PC
(3) $\Box (\varphi \rightarrow \psi)$, $\Box (\psi \rightarrow \varphi)$ (2), RN
(4) $\Box (\varphi \rightarrow \psi)$ $\rightarrow$ ($\Box \varphi \rightarrow \Box \psi$), $\Box (\psi \rightarrow \varphi)$ $\rightarrow$ ($\Box \psi \rightarrow \Box \varphi$) K
(5) $\Box \varphi \rightarrow \Box \psi$, $\Box \psi \rightarrow \Box \varphi$ (3), (4), PC
(6) $\Box \varphi \leftrightarrow \Box \psi$ (5), PC

With RE, Dual can be derived from dual, as shown below.

(1) $\neg \Diamond \neg \varphi \leftrightarrow \neg \Box \neg \neg \varphi$ dual
(2) $\neg \Diamond \neg \varphi \leftrightarrow \neg \Box \neg \neg \varphi$ (1), PC
(3) $\varphi \leftrightarrow \neg \neg \varphi$ PC
(4) $\Box \varphi \leftrightarrow \neg \neg \neg \varphi$ (3), RE
(5) $\Box \varphi \leftrightarrow \neg \Diamond \neg \varphi$ (2), (4), PC

Hence, when $\Box$ is primitive, there is no need to add Dual when dual is available. Dually, if we want to derive dual from Dual, we need the following rule.

re $\varphi \leftrightarrow \psi$

$\Diamond \varphi \leftrightarrow \Diamond \psi$

With re, we can analogously derive dual from Dual as follows.

(1) $\Box \neg \varphi \leftrightarrow \neg \Diamond \neg \varphi$ Dual
(2) $\neg \Box \neg \varphi \leftrightarrow \Diamond \neg \neg \varphi$ (1), PC
(3) $\varphi \leftrightarrow \neg \neg \varphi$ PC
(4) $\Diamond \varphi \leftrightarrow \Diamond \neg \neg \varphi$ (3), re
(5) $\Diamond \varphi \leftrightarrow \neg \Box \neg \varphi$ (2), (4), PC
The problem, however, is that with only K, RN, and PC, even with Dual, we can not obtain re. With RE (which can be obtained from K, RN, and PC) and Dual, we can only obtain

\[ \varphi \leftrightarrow \psi, \quad \lozenge \neg \varphi \leftrightarrow \lozenge \neg \psi, \quad \lozenge \neg \neg \varphi \leftrightarrow \lozenge \neg \neg \psi \]  

but not re. Thus we can not derive dual from Dual. Thereby when \lozenge is primitive, apart from Dual used as a definition, we have to add dual too. Since we have proved that Dual can be derived from dual, the axiom Dual used as a definition can be omitted. This implies that whichever of \Box and \lozenge is chosen as primitive, dual is indispensable (either as a definition, or as an axiom), while Dual can be omitted.

Let \( K \) be the axiomatic system consisting of PC, K, RN, and Dual. We will prove rigorously that dual is not derivable in \( K \).

**Proposition 1.** \( \not\vdash_K \lozenge p \leftrightarrow \Box \neg p \)

**Proof.** Consider \( \mathcal{M} = (W, R, V) \) and the following non-standard semantics.

- \( \mathcal{M}, w \models \lozenge \varphi \) iff \( \varphi = \neg \psi \) and there exists \( u \in W \) such that \( wRu \) and \( \mathcal{M}, u \not\models \psi \);
- \( \mathcal{M}, w \models \Box \varphi \) iff for all \( u \in W \), \( wRu \) implies \( \mathcal{M}, u \models \varphi \).

The semantics of propositional connectives is defined as usual. It is easily verified that PC and K are valid under this semantics (with respect to the class of all frames), and MP and RN preserves validity. Notice that \( \mathcal{M}, w \models \lozenge \neg \varphi \) iff there exists \( u \in W \) such that \( wRu \) and \( \mathcal{M}, u \not\models \varphi \). Thus Dual, namely \( \Box \varphi \leftrightarrow \neg \neg \neg \varphi \) is also valid under this semantics. Hence, if \( \vdash_K \lozenge p \leftrightarrow \neg \Box \neg p \), then \( \lozenge p \leftrightarrow \neg \Box \neg p \) is valid too. But consider the counter-model \( \mathcal{N} = (\{w\}, \{(w, w)\}, V) \), where \( V(p) = \{w\} \). Then \( \mathcal{N}, w \models \neg \lozenge p \) but \( \mathcal{N}, w \not\models \lozenge p \). Therefore, \( \not\vdash_K \lozenge p \leftrightarrow \neg \Box \neg p \). \( \square \)

Early textbooks in modal logic (such as [16, 13, 12]) usually take \Box to be primitive, with dual a definition rather than an axiom added to the axiomatic systems. Moreover, Dual is not required as an axiom. The prevalence of [2] makes more and more people choose \lozenge to be primitive. They may take for granted from duality that only Dual should be added as a definition and dual is not required as an axiom. The above analysis shows that this thought is incorrect. If \lozenge is taken as primitive, the axiomatic system with only Dual and without dual is incomplete. A newly published textbook [14] on neighborhood semantics just made this mistake. On page 54, the author claims that the minimal modal logic \( E \) under neighborhood semantics can be axiomatized by PC, RE, and Dual. But a slight modification of the proof of Proposition 1 will show that this axiomatic system is incomplete, as dual is not derivable in the system. The correct axiomatization is to take dual instead of Dual as an axiom.

In semantics, the choice of primitive modal operators will affect the definition of filtration and minimal canonical models, as well as the syntax of subformulas. Without caution, some subtle mistakes are likely to be made, which will be shown in Section 2 and 3.

Given a set of propositional variables \( PV \), without other specification, we assume the language of modal logic is defined as follows.

\[ \mathcal{L} \equiv \varphi ::= p \mid \neg \varphi \mid (\varphi \rightarrow \varphi) \mid \lozenge \varphi, \]
where \( p \in PV \). The other logical connectives (\( \top, \lor, \land, \leftrightarrow \)) are defined as usual.

## 2 Filtration and Finite Model Property

The basic idea of filtration is as follows. Given a formula \( \varphi \) and its counter-model \( \mathcal{M} \), the satisfiability of \( \varphi \) only depends on the satisfiability of the subformulas of \( \varphi \). Since the subformulas are finite, the possibilities of the satisfiability of them are also finite. Thus, if we take those points in \( \mathcal{M} \) that satisfy the same subformulas of \( \varphi \) to be the same, we obtain a finite model \( \mathcal{M}^f \). If we define in \( \mathcal{M}^f \) an accessibility relation that is closely related to \( \mathcal{M} \) such that the satisfiability of the subformulas in \( \mathcal{M}^f \) is equivalent to that in \( \mathcal{M} \) for all subformulas of \( \varphi \), then we obtain a finite counter-model of \( \varphi \).

Given a model \( \mathcal{M} = (W, R, V) \) and a set of formulas \( \Sigma \subseteq L_\varphi \), we first define an equivalence relation \( \sim_{\Sigma} \subseteq W \times W \) as follows.

\[
\text{iff for all } \varphi \in \Sigma, \mathcal{M}, w \models \varphi \iff \mathcal{M}, w' \models \varphi.
\]

Define the equivalence class \( |w|_{\Sigma} \) of \( w \) as follows.\(^1\)

\[
|w|_{\Sigma} = \{ w' \in W \mid w \sim_{\Sigma} w' \}
\]

When \( \Sigma \) is finite, the set of equivalence classes induced by \( \Sigma \) is also finite. When \( \Sigma \) is used for filtration, it is required to be subformula closed, i.e., every subformula of every formula in \( \Sigma \) is also in \( \Sigma \). In the sequel, we denote by \( Sub\Sigma \) the set of all subformulas of all formulas in \( \Sigma \). If \( \Sigma \) is a singleton \( \{ \varphi \} \), we denote \( Sub\Sigma \) by \( Sub\varphi \).

Now recall the definition of filtration.

**Definition 2 (filtration).** \( \mathcal{M}^f = (W^f, R^f, V^f) \) is a filtration of \( \mathcal{M} = (W, R, V) \) through \( \Sigma \), if the following conditions are satisfied.

1. \( W^f = \{ |w|_{\Sigma} \mid w \in W \} \);  
2. for all \( w, u \in W \), if \( wRu \) then \( |w|_{R^f} = |u|_{R^f} \);  
3. for all \( w, u \in W \), if \( |w|_{R^f} = |u|_{R^f} \), then for all \( \diamond \varphi \in \Sigma \), if \( \mathcal{M}, u \models \varphi \) then \( \mathcal{M}, w \models \diamond \varphi \);  
4. for all \( w \in W \) and \( p \in PV \cap \Sigma \), \( |w|_{V^f(p)} \) iff \( w \in V(p) \).

Such an \( R^f \) is also called a filtration of \( R \) (through \( \Sigma \)).

A few remarks about filtration.

**Remark 3.** If \( \Sigma \) is subformula closed and the primitive modal operator is \( \diamond \), then (3) above implies

\( \text{(3')} \) for all \( w, u \in W \), if \( |w|_{R^f} = |u|_{R^f} \), then for all \( \Box \varphi \in \Sigma \), if \( \mathcal{M}, w \models \Box \varphi \) then \( \mathcal{M}, u \models \varphi \).

But (3') does not imply (3). The reason is that \( \Box \varphi \) is actually an abbreviation of \( \neg \diamond \neg \varphi \). When \( \Box \varphi \in \Sigma \), by the subformula closure of \( \Sigma \), we also have \( \neg \varphi \in \Sigma \), and thus we could prove (3') using (3) by contraposition. But conversely, when \( \diamond \varphi \in \Sigma \), we can not obtain \( \neg \varphi \in \Sigma \), and thus can not prove (3) using (3') by contraposition. On this point, filtrations are different from canonical models and

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\(^1\)We often omit the subscript \( \Sigma \) if no confusion occurs.
ultrafilter extensions. In canonical models and ultrafilter extensions, the definitions using □ can dually be replaced by ◊, and vice versa. More precisely, the canonical relation $R^\Lambda$ for the logic $\Lambda$ can be defined by: $R^\Lambda wu$ if for all $\varphi \in u$, $\varphi \in w$. It can also be defined by: $R^\Lambda wu$ if for all $\square \varphi \in w$, $\varphi \in u$. The two definitions are completely equivalent (assuming $\Lambda$ contains dual). Analogously, the accessibility relation $R^{ue}$ of the ultrafilter extension of $M = (W, R, V)$ can be defined by: $R^{ue} wu$ if for all $X \in u$, $\Diamond_R X \in u$.\footnote{If both $\square$ and $\Diamond$ are primitive modal operators, for the inductive proof of the cases $\square \varphi$ and $\Diamond \varphi$ for the filtration theorem below, (3) in Definition 2 should be replaced by (3’). Then (3’) implies (3) but (3) does not imply (3’). The reasons are as above.} It can also be defined by: $R^{ue} wu$ if for all $\square_R X \in u$, $X \in v$. The two definitions are also equivalent. But for filtrations, (3) and (3’) are not equivalent.

Remark 4. If the primitive modal operator is $\square$, for the inductive proof of the case $\square \varphi$ for the filtration theorem below, (3) in Definition 2 should be replaced by (3’).

Remark 5. If both $\square$ and $\Diamond$ are primitive modal operators, for the inductive proof of the cases $\square \varphi$ and $\Diamond \varphi$ for the filtration theorem below, (3) in Definition 2 should be replaced by (3’).

\begin{itemize}
    \item For all $w, u \in W$, if $|w| R_f |u|$, then for all $\Diamond \varphi \in \Sigma$, if $M, u \models \varphi$ then $M, w \models \Diamond \varphi$, and for all $\square \varphi \in \Sigma$, if $M, w \models \square \varphi$ then $M, u \models \varphi$.
\end{itemize}

i.e., the conjunction of (3) and (3’).

Note that the definition of $R^f$ in Definition 2 is not constructive. The following two particular $R^f$’s are often used.

**Definition 6** (smallest filtration, largest filtration). Given a model $M = (W, R, V)$ and a set of formulas $\Sigma \subseteq \mathcal{L}_\Diamond$, the smallest filtration $R^s$ and the largest filtration $R^l$ of $R$ through $\Sigma$ is defined respectively as follows. For all $w, u \in W$,

1. $|w| R^s |u|$ if there exist $w' \in |w|$ and $u' \in |u|$ such that $w' R a'$;
2. $|w| R^l |u|$ if for all $\Diamond \varphi \in \Sigma$, if $M, u \models \varphi$ then $M, w \models \Diamond \varphi$.

Similarly, when the primitive modal operator is $\square$, (2) should be replaced by (2’) below.

1. $|w| R^l |u|$ if for all $\square \varphi \in \Sigma$, if $M, w \models \square \varphi$ then $M, u \models \varphi$.

If both $\square$ and $\Diamond$ are primitive, then (2) should be replaced by (2”) below.

1. $|w| R^l |u|$ if for all $\Diamond \varphi \in \Sigma$, if $M, u \models \varphi$ then $M, w \models \Diamond \varphi$ and for all $\square \varphi \in \Sigma$, if $M, w \models \square \varphi$ then $M, u \models \varphi$.

The equivalence of satisfiability of related formulas between a model and its filtration is ensured by the following filtration theorem.

**Theorem 7** (Filtration Theorem). Let $\mathcal{M}^f = (W^f, R^f, V^f)$ be a filtration of $\mathcal{M} = (W, R, V)$ through $\Sigma$, which is subformula closed. Then for all $w \in W$ and $\varphi \in \Sigma$,

$M, w \models \varphi$ if $\mathcal{M}^f, |w| \models \varphi$.

It should be emphasized that the subformula closed property of $\Sigma$ is critical in the proof of the filtration theorem; otherwise, we could not use the inductive hypothesis to complete the proof.
Sometimes using the subformulas of the formula in question directly will not achieve our goal. We have to supplement some other formulas for filtration. It should be noticed that after adding more formulas we still have to keep the set of formulas to be subformula closed. At this place, both [3] and [15] made an incautious mistake in the proof of the finite model property of \( \mathbf{K5} \).

The proof in [3, p. 145] goes as follows.

First, \( \mathbf{K5} = \mathbf{K} \oplus \diamond \square p \rightarrow \square p \) is characterized by the class of Euclidean frames. Let \( \mathfrak{M} \) be a countermodel for a formula \( \varphi \) based on a Euclidean frame. Again, a filtration of \( \mathfrak{M} \) through \( \text{Sub} \varphi \) need not be Euclidean. So let us try a bigger filter, say,

\[
\Sigma = \text{Sub} \varphi \cup \{ \diamond \square \varphi \mid \square \psi \in \text{Sub} \varphi \}.
\]

Let \( \mathfrak{N} \) be the largest filtration of \( \mathfrak{M} \) through \( \Sigma \). We show that its underlying frame \( \mathfrak{G} = (V, S) \) is Euclidean.

Suppose \( |x| S |y| \) and \( |x| S |z| \), for some \( |x|, |y|, |z| \in V \), and prove that \( |y| S |z| \). By the definition of \( S \), we need to show that \( \mathfrak{N}, |y| \models \square \psi \) implies \( \mathfrak{N}, |z| \models \psi \), for every \( \square \psi \in \Sigma \). So let \( \square \psi \in \Sigma \) and \( \mathfrak{N}, |y| \models \square \psi \). Then \( \mathfrak{N}, |x| \models \diamond \square \psi \) and, by the filtration theorem, \( \mathfrak{M}, |x| \models \diamond \square \psi \), from which \( \mathfrak{N}, |x| \models \square \psi \), since \( \mathfrak{M} \) is a model for \( \mathbf{K5} \). Therefore, \( \mathfrak{N}, |x| \models \square \psi \) and \( \mathfrak{N}, |z| \models \psi \).

Notice that since [3] takes \( \square \) as primitive, the largest filtration \( R^f \) is defined by (2') in Definition 6 instead of (2). In the proof, the conclusion \( \mathfrak{N}, |x| \models \diamond \square \psi \) is obtained from \( \diamond \square \psi \in \Sigma \) and the claim in Remark 3: when \( \square \) is primitive, (3') implies (3). The whole proof seems innocuous and is much shorter than the standard proof using finite bases\(^3\). Unfortunately, the proof is incorrect!

What is incorrect is that \( \Sigma \) is not subformula closed as it appears. The author thought wrongly that \( \text{Sub} \diamond \square \psi = \{ \diamond \square \psi, \square \psi \} \cup \text{Sub} \psi \). If this is case, then the proof does go through. The problem is that \( \diamond \) is not a primitive operator, but the abbreviation of \( \neg \square \neg \). Hence,

\[
\text{Sub} \diamond \square \psi = \text{Sub} \neg \square \neg \square \psi = \{ \neg \square \neg \square \psi, \square \neg \square \psi, \neg \square \psi, \square \psi \} \cup \text{Sub} \psi.
\]

But \( \neg \square \neg \square \psi \) and \( \neg \square \psi \) are not in \( \Sigma \). Can the problem be solved by supplementing these two formulas? No, because the proof requires that if \( \diamond \neg \square \psi \in \Sigma \) then \( \diamond \neg \square \neg \square \psi \) must also be in \( \Sigma \). But then it will introduce more formulas of the form \( \square \psi \), and the proof requires the formulas with \( \diamond \) prefixed to these formulas must also be in \( \Sigma \). Repeating this process, \( \Sigma \) will become an infinite set. To complete the proof, we have to prove that \( \Sigma \) has finite base with respect to \( \mathfrak{M} \). Can we solve the problem by just taking both \( \square \) and \( \diamond \) to be primitive? No. Though \( \Sigma \) is now subformula closed, the definition for \( R^f \) has to be revised as (2'') in Definition 6. Then to prove that \( \mathfrak{M} \) is Euclidean we have to consider both (2) and (2''), which requires \( \Sigma \) to satisfy not only that \( \square \psi \in \Sigma \) implies \( \diamond \square \psi \in \Sigma \), but also that \( \square \psi \in \Sigma \) implies \( \diamond \square \psi \in \Sigma \). Thereby \( \Sigma \) becomes infinite again.

\(^3\)See Definition 8.
\[15, \text{p. 176}\] uses an infinite set for filtration to prove the finite model property of \(K5\). The author first defines \(\square \Gamma = \{ \square \varphi \mid \varphi \in \Gamma \}\) and \(\Diamond \Gamma = \{ \Diamond \varphi \mid \varphi \in \Gamma \}\), where \(\Gamma = \text{Sub} \varphi\). Then he defines \(\Gamma^* = \Gamma \cup \square \Gamma \cup \Diamond \Gamma\) and \(\Gamma_n\) below.

\[
\Gamma_0 = \Gamma \\
\Gamma_{n+1} = (\Gamma_n)^*
\]

Finally he defines \(\Gamma^* = \bigcup_{n \geq 0} \Gamma_n\), which is just the smallest set that contains \(\text{Sub} \varphi\) and is closed under prefixing \(\square\) and \(\Diamond\). Then \(\Gamma^*\) is used for filtration. Though \(\Gamma^*\) is an infinite set, it has finite base with respect to Euclidean models. The proof is almost right, except that \(\Gamma^*\) is not subformula closed. If both \(\square\) and \(\Diamond\) are primitive, then \(\Gamma^*\) is indeed subformulas closed. But the textbook takes only \(\square\) to be primitive. Then \(\Diamond \varphi\) is just the abbreviation of \(\neg \Box \neg \varphi\), whose subformulas include not only \(\varphi\) but also \(\neg \varphi\). By the definition of \(\Gamma^*\), we can not ensure that \(\neg \varphi\) is also in \(\Gamma^*\) if \(\varphi \in \Gamma^*\). This is a very elusive mistake. Compared to the mistake in [3, p. 145] above, this mistake is more easily to be corrected: just take \(\text{Sub} \Gamma^*\) instead of \(\Gamma^*\) for filtration.

Another option is to take both \(\square\) and \(\Diamond\) to be primitive in the language. Then \(\Gamma^*\) is subformula closed. Though in verifying that the filtration model is Euclidean, we have to consider both (2) and (2') in Definition 6, the proof still goes through, since \(\Gamma^*\) is closed under prefixing both \(\square\) and \(\Diamond\).

A better proof is to take the hint in Exercise 2.3.8 on Page 83 of [2]: using the smallest subformula closed set that includes \(\varphi\) and is closed under prefixing \(\square\) for filtration (assuming \(\mathcal{L}_\Diamond\) is our formal language). We give the complete proof as follows for reference of teaching.

**Definition 8.** Given a set of formulas \(\Sigma \subseteq \mathcal{L}_\Diamond\) and a model \(\mathcal{M} = (W, R, V)\), \(\Sigma\) has finite base with respect to \(\mathcal{M}\), if there exists a finite set \(\Delta \subseteq \mathcal{L}_\Diamond\) such that for every \(\varphi \in \Sigma\) there exists \(\psi \in \Delta\) such that for all \(w \in W\), \(\mathcal{M}, w \models \varphi\) iff \(\mathcal{M}, w \models \psi\).

The following two propositions are easily verified.

**Proposition 9.** If \(\Sigma\) has finite base with respect to \(\mathcal{M}\), then all filtrations of \(\mathcal{M}\) through \(\Sigma\) is a finite model.

**Proposition 10.** The following formulas are valid in Euclidean frames.

1. \(\Diamond \Box \Diamond \varphi \iff \Diamond \Diamond \varphi\)
2. \(\Box \Diamond \Box \varphi \iff \Box \Box \varphi\)
3. \(\Box \Diamond \Box \varphi \iff \Box \Diamond \varphi\)
4. \(\Diamond \Box \Diamond \varphi \iff \Diamond \Diamond \varphi\)

**Proposition 11.** \(K5\) has finite model property.

Proof. It suffices to prove that a filtration of any Euclidean model is also Euclidean. Given \(\varphi \notin K5\), let \(\Sigma\) be the the smallest subformula closed set that includes \(\varphi\) and is closed under prefixing \(\Box\). By Proposition 10, \(\Sigma\) has finite base with respect to Euclidean models. Suppose \(\mathcal{M} = (W, R, V)\) is Euclidean. Consider the largest filtration \(\mathcal{M}^f = (W^f, R^f, V^f)\) of \(\mathcal{M}\) through \(\Sigma\). We prove that \(\mathcal{M}^f\) is also Euclidean. Suppose \(|w| R^f |u|\) and \(|w| R^f |v|\). We prove \(|u| R^f |v|\). Suppose \(\Diamond \psi \in \Sigma\) and \(\mathcal{M}, v \models \psi\). It suffices to prove \(\mathcal{M}, u \models \Diamond \psi\). By \(\mathcal{M}, v \models \psi\) and \(|w| R^f |v|\), we have \(\mathcal{M}, w \models \Diamond \psi\). Since \(\mathcal{M}\) is Euclidean, it follows that \(\mathcal{M}, w \models \Box \Diamond \psi\). By the construction of \(\Sigma\), we have \(\Box \Diamond \psi \in \Sigma\). Then by Remark 3 and \(|w| R^f |u|\), we obtain \(\mathcal{M}, u \models \Diamond \psi\). \(\square\)
3 Minimal Canonical Model and Finite Model Property

Given any normal modal logic \( \Lambda \), if \( \varphi \notin \Lambda \) then it is easily seen that the canonical model of \( \Lambda \) is a counter-model of \( \varphi \). But standard canonical models are not finite models. By constructing canonical models using maximal consistent sets relative to a finite set of formulas, we can prove the finite model property of some logics. Such canonical models are called minimal canonical models. The accessibility relations of minimal canonical models for different logics are usually defined differently. In the sequel, we will show an easily made mistake in defining minimal canonical models, by taking the logic KL as an example.

Given any \( \varphi \), let \( \text{Sub}^- \varphi = \{\neg \psi \mid \psi \in \text{Sub} \varphi \} \). Let \( \text{Sub}^+ \varphi = \text{Sub} \varphi \cup \text{Sub}^- \varphi \). Given a logic \( \Lambda \) and a formula \( \varphi \), \( \Gamma \) is \( \Lambda \)-maximal-consistent relative to \( \varphi \), if \( \Gamma \subseteq \text{Sub}^+ \varphi \) is \( \Lambda \)-consistent, and for all \( \psi \in \text{Sub} \varphi \), \( \psi \in \Gamma \) or \( \neg \psi \in \Gamma \).

**Definition 12.** Given a formula \( \varphi \), define the minimal canonical model \( \mathfrak{M}^{\text{KL}}_{\varphi} = (W^{\varphi}_{\text{KL}}, R^{\varphi}_{\text{KL}}, V^{\varphi}_{\text{KL}}) \) relative to \( \varphi \) for KL as follows.

1. \( W^{\varphi}_{\text{KL}} = \{ \Gamma \subseteq \text{Sub}^+ \varphi \mid \Gamma \) is \( \Lambda \)-maximal-consistent relative to \( \varphi \} \).
2. For all \( w, u \in W^{\varphi}_{\text{KL}} \), \( wR^{\varphi}_{\text{KL}} u \) iff for all \( \psi \in \mathcal{L}_\diamond \), if \( \diamond \psi \in u \) or \( \psi \in u \) then \( \psi \in w \), and there exists \( \psi \in u \) but \( \diamond \chi \notin u \).
3. For all \( w \in W^{\varphi}_{\text{KL}} \), if \( p \in \text{PV} \cap \text{Sub}^+ \varphi \), then \( w \in V^{\varphi}_{\text{KL}}(p) \) iff \( p \in w \); if \( p \in \text{PV} - \text{Sub}^+ \varphi \) then \( V^{\varphi}_{\text{KL}}(p) \) is an arbitrary subset of \( W^{\varphi}_{\text{KL}} \) (for instance, the empty set).

**Remark 13.** Unlike canonical models, in defining accessibility relations of minimal canonical models, we can not freely choose \( \Box \) or \( \diamond \), but have to choose it according to which is primitive in the formal language. Definition 12 only applies to the case that \( \diamond \) is primitive. If the primitive modal operator is \( \Box \), to prove the truth lemma of minimal canonical models, (2) above should be replaced by (2').

(2') For all \( w, u \in W^{\varphi}_{\text{KL}} \), \( wR^{\varphi}_{\text{KL}} u \) iff for all \( \psi \), if \( \Box \psi \in w \) then \( \psi, \Box \psi \in u \) and there exists \( \Box \chi \notin u \).

Neither (2) nor (2') implies the other (and hence they are not equivalent). The reason is that, \( w \) and \( u \) here are not maximal consistent sets, but maximal consistent sets relative to \( \varphi \). They have maximality only for the subformulas of \( \varphi \), not for all formulas. Thus they do not have some properties of maximal consistent sets. For instance, from \( \psi \in w \) and the logical equivalence of \( \psi \) and \( \psi' \) it does not follow that \( \psi' \in w \), because \( \psi' \) may not be in \( \text{Sub}^+ \varphi \). In particular, from \( \diamond \psi \notin w \) we can only obtain \( \neg \diamond \psi \in w \) but not \( \neg \Box \neg \psi \in w \).

Just at this place, [2] made an incautious mistake. Exercise 4.8.7 on Page 246 of this book is to prove the finite model property of KL. It gives a hint for proof: define the canonical relation \( R^{\text{KL}}_{\varphi} \) by: \( wR^{\text{KL}}_{\varphi} u \) iff for all \( \psi \), if \( \Box \psi \in w \) then \( \psi, \Box \psi \in u \), and there exists \( \Box \chi \in u \) but \( \Box \chi \notin w \). This definition is just (2') above instead of (2). But [2] takes \( \diamond \) to be primitive rather than \( \Box \). This implies that the correct hint should be (2) rather than (2'). We guess that the hint was copied from other sources which take \( \Box \) to be primitive and was forgotten to be revised accordingly. This might be a careless mistake. But it could also be that the authors thought (2)
and \( (2') \) are just equivalent. If the latter is the case, then our correction could be of more significance.

Of course, we could take both \( \Box \) and \( \Diamond \) to be primitive in the formal language. Then subformulas are clearer, which will cause less mistakes. But then in defining filtration and (minimal) canonical models, we have to consider both \( \Box \) and \( \Diamond \). Moreover, in proving some critical results (such as the truth lemma of canonical models), we have to consider both the cases \( \varphi = \Box \psi \) and \( \varphi = \Diamond \psi \). Though it does not increase difficulty, the proof will become more tedious. From the perspective of writing textbooks, taking only one modal operator as primitive is definitely more convenient (but requires more care).

4 Deductive Consequence and the Deduction Theorem

Another easily made mistake is regarding the rule of necessitation RN as having the same effect as the rule of modus ponens. Though both preserve validity, they are different in preserving truth. Modus ponens preserves local truth, i.e., for every world \( w \), \( \psi \) is truth at \( w \) as long as \( \varphi \) and \( \varphi \rightarrow \psi \) are true at \( w \). Necessitation does not preserve local truth: from the truth of \( \varphi \) at \( w \) it does not follow that \( \Box \varphi \) is true at \( w \). Necessitation only preserves global truth, i.e., for every model \( \mathcal{M} \), if \( \varphi \) is true at all worlds in \( \mathcal{M} \), then \( \varphi \) is true at all worlds in \( \mathcal{M} \).\(^4\)

As the rule of necessitation does not preserve local truth, it can not be applied to inferences with premises. This implies that when defining the deductive consequence \( \vdash_{\Lambda} \) for a modal logic \( \Lambda \), we can not follow the definition used for classical propositional logic, which is defined by: \( \Gamma \vdash_{\Lambda} \varphi \) iff there exists a finite sequence of formulas \( \varphi_1, \ldots, \varphi_n \) such that \( \varphi_n = \varphi \) and for each \( \varphi_i \), either \( \varphi_i \in \Gamma \), or \( \varphi_i \) is an axiom of \( \Lambda \), or \( \varphi_i \) is obtained from proceeding formulas in the sequence using inference rules of \( \Lambda \). If this definition is adopted for modal logic, then as long as \( \Lambda \) contains the rule of necessitation, we have \( p \vdash_{\Lambda} \Box p \). But the inference from \( p \) to \( \Box p \) is usually invalid: from it is raining today, we can not infer that it is necessary that it is raining today.

The understanding of the rule of necessitation is also related to the deduction theorem in modal logic. Indeed, whether the deduction theorem holds for modal logic had caused debate in the literature.\(^{10}\) To get the rule of necessitation right, there are six ways of defining deductive consequence in modal logic in the literature.\(^5\)

The first way is called omitted definition (as in \(^{12}\)). This definition does not consider inferences with premises, and considers only weak soundness and weak completeness. Then the rule of necessitation can only be applied to axioms and theorems derived from axioms, and will thus avoid the incorrect application of it. The practice which considers only theorems and not consequences was reasonable early when logicians cared only about the relation between logic and mathematics. But with the development of more and more non-classical logics, theorems of an axiomatic system can not completely characterize a logic any more.\(^6\) The practice

\(^4\)See \(^{1}, \, ^{7}\) and \(^{18}\) for the distinction of these two kinds of rules.

\(^5\)In the sequel, we only consider deductive consequence defined in axiomatic systems, and will not consider other proof systems (like sequent calculus and tableau systems).

\(^6\)For example, Kleene’s three valued logic does not have any theorems, but it has valid inferences.
which considers only theorems and not consequences is gradually abandoned. A
bit surprisingly, a relatively new and popular textbook [19] also adopts omitted
definition.

The second way is called classical definition (as in [3] and [6]). This definition
follows the standard definition of deductive consequence for classical propositional
logic and transfers the treatment of the rule of necessitation to the deduction theo-
rem, which either has different contents (as in [3]) or is augmented with additional
constraints. The latter is also the treatment for the rule of universal generalization
in some early textbooks ([11]) in mathematical logic. There are two drawbacks of
classical definition. First, the deduction theorem is made too complicated, which is
not friendly to students. Second, there is no strong soundness under this definition,
which means the deductive consequence does not characterize valid inferences.

The third way is called ternary definition (as in [8]). In this definition, premises
are divided into two parts: a global part and a local part. The notation \( \Gamma \vdash \Lambda \Delta \Rightarrow \varphi \) means that \( \varphi \) can be deduced from global premises \( \Gamma \) and local premises \( \Delta \).\(^7\) The rule of necessitation can be applied to global premises but not to local premises. Thus,
the deduction becomes a ternary relation. Accordingly, the deduction theorem splits
into two for the two kinds of premises. As this definition is too distinctive, it has
not been widely adopted.

The fourth way is called reduced definition (as in [13], [4], [2], [9], and open text-
book [20]). In this definition, \( \Gamma \vdash \varphi \) if \( \varphi \) has a formal proof in \( \Gamma \) and local premises \( \Delta \).\(^8\) The essential idea is to reduce deduction with
premises to that without premises. Since \( \vdash \Lambda p \rightarrow \Box p \) usually does not hold, under
this definition \( p \vdash \Box p \) does not hold either, and hence the deductive consequence
accords to the semantics. Moreover, the deduction theorem holds almost trivially
without any constraints. The rule of uniform substitution can also be explicit without
hidden in axiom schemata. Under this definition, system \( \textbf{K} \) and its deductive
consequence is defined as follows.

\begin{definition}[reduced definition] The axiomatic system \( \textbf{K}^r \) consists of the fol-
lowing axioms and rules of inference.
\end{definition}

\begin{align*}
\text{PC1} & \quad p \rightarrow (q \rightarrow p) \\
\text{PC2} & \quad (p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r)) \\
\text{PC3} & \quad (\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p) \\
\text{K} & \quad \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow q) \\
\text{dual} & \quad \Diamond p \leftrightarrow \neg \Box \neg p \\
\text{MP} & \quad \varphi, \varphi \rightarrow \psi \quad \frac{}{\psi} \\
\text{RN} & \quad \varphi \quad \frac{\Box \varphi}{\Box \varphi}
\end{align*}

\(^7\)Fitting’s original notation is \( \Gamma \vdash_{\text{A}} \Delta \rightarrow \varphi \). To distinguish \( \rightarrow \) from material implication in
the object language, we use \( \Rightarrow \) instead.

\(^8\)If \( \Phi = \{\varphi_1, \ldots, \varphi_n\} \) then \( \bigwedge \Phi = \varphi_1 \land \cdots \land \varphi_n \); if \( \Phi = \emptyset \) then \( \bigwedge \Phi = \top \).
We say $\varphi$ is derivable from $\Gamma$ in $K^r$, denoted $\Gamma \vdash_{K^r} \varphi$, if there exists a finite subset $\Phi$ of $\Gamma$ such that $\bigwedge \Phi \rightarrow \varphi$ has a formal proof in $K^r$, i.e., there exists a finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that $\varphi_n = \varphi$, and for each $\varphi_i$, either $\varphi_i$ is an axiom of $K^r$, or is obtained from proceeding formulas in the sequence using the rules of inference of $K^r$. We abbreviate $\emptyset \vdash_{K^r} \varphi$ as $\vdash_{K^r} \varphi$.

Reduced definition is the most popular one in modal logic for deductive consequence. But it has three drawbacks. First, it is not as intuitive as classical definition, especially for students who learned classical definition in classical propositional logic before. Without explanation, they may be confused why in modal logic the definition has to be modified. Second, technically the definition relies on the logical constant $\rightarrow$. But not all logical languages contain this logical constant. Thereby, reduced definition is not general enough. Third, though the deduction theorem holds under this definition, it has no practical use at all. To prove $\varphi \rightarrow \psi$, we can not assume $\varphi$ and prove $\psi$ (so the proof may go easier) but have to prove $\varphi \rightarrow \psi$ directly, since $\varphi \vdash_{\Lambda} \psi$ is just defined by $\vdash_{\Lambda} \varphi \rightarrow \psi$.

The fifth way is called bounded definition (as in [10]). This definition restricts the use of the rule of necessitation, so that it can only be applied to conclusions derived from empty premises. Under this definition, the deduction theorem also holds without constraints. Compared to the above definitions, it has more advantages. But it can not be easily defined precisely in standard forms of axiomatic systems. Here we give a precise definition that is closer to standard axiomatic systems, taking system $K$ as an example.

**Definition 15** (bounded definition). The axiomatic system $K^b$ consists of the following axiom schemata and rules of inference.

- **sPC1** $\varphi \rightarrow (\psi \rightarrow \varphi)$
- **sPC2** $(\varphi \rightarrow (\psi \rightarrow \chi)) \rightarrow ((\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \chi))$
- **sPC3** $(\neg \varphi \rightarrow \psi) \rightarrow (\psi \rightarrow \varphi)$
- **sK** $\Box((\varphi \rightarrow \psi) \rightarrow (\Box \varphi \rightarrow \Box \psi))$
- **sdual** $\Diamond \varphi \rightarrow \neg \Box \neg \varphi$
- **MP** $\frac{\varphi, \psi \rightarrow \psi}{\varphi}$
- **RN** $\frac{\varphi}{\Box \varphi}$

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9For any $\sigma : PV \rightarrow L^b$, $\varphi^\sigma$ is inductively defined as follows. $p^\sigma = \sigma(p)$, $(\neg \psi)^\sigma = \neg (\psi)^\sigma$, $(\psi \rightarrow \chi)^\sigma = (\psi)^\sigma \rightarrow (\chi)^\sigma$, $\Diamond \varphi^\sigma = \Box (\varphi)^\sigma$.

10The other logical constant $\wedge$ can be eliminated by defining $\Gamma \vdash_{\Lambda} \varphi$ by: there exists $\{\varphi_1, \ldots, \varphi_n\} \subseteq \Gamma$ such that $\varphi_1 \rightarrow (\varphi_2 \rightarrow \cdots \rightarrow (\varphi_n \rightarrow \varphi) \cdots)$ has a formal proof in $\Lambda$.

11The definition in [10] is not given in standard forms of axiomatic systems but in a form more like sequent calculus.
A finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ is a formal proof of $\varphi$ in $K^b$ from empty premises, if $\varphi_n = \varphi$, and for each $\varphi_i$:

- either $\varphi_i$ is an instance of one of the axiom schema of $K^b$, or
- there exists $j, k < i$ such that $\varphi_k = \varphi_j \rightarrow \varphi_i$, i.e., $\varphi_i$ can be obtained from proceeding formulas in the sequence using the rule MP, or
- there exists $j < i$ such that $\varphi_i = \square \varphi_j$, i.e., $\varphi_i$ can be obtained from a proceeding formula in the sequence using RN.

We say $\varphi$ is derivable from $\Gamma$ in $K^b$, denoted $\Gamma \vdash_{K^b} \varphi$, if there exists a finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ such that $\varphi_n = \varphi$ and for each $\varphi_i$:

- either $\varphi_i \in \Gamma$, or
- $\varphi_i$ is an instance of one of the axiom schema of $K^b$, or
- there exists $j, k < i$ such that $\varphi_k = \varphi_j \rightarrow \varphi_i$, i.e., $\varphi_i$ can be obtained from proceeding formulas in the sequence using the rule MP, or
- there exists $j < i$ such that $\varphi_i = \square \varphi_j$, and there exists $\varphi_{j_1}, \ldots, \varphi_{j_k} \in \{\varphi_1, \ldots, \varphi_j\}$ such that $\varphi_{j_1}, \ldots, \varphi_{j_k}$ is a formal proof of $\varphi_j$ in $K^b$ from empty premises.

We abbreviate $\emptyset \vdash_{K^b} \varphi$ by $\vdash_{K^b} \varphi$.

We list some easily proved results for $\vdash_{K^b}$ without proofs.

**Lemma 16.** $\vdash_{K^b} \varphi \rightarrow \varphi$.

**Theorem 17** (Deduction Theorem). If $\Gamma, \psi \vdash_{K^b} \varphi$, then $\Gamma \vdash_{K^b} \psi \rightarrow \varphi$.

**Theorem 18** (Compactness). $\Gamma \vdash_{K^b} \varphi$ iff there exists a finite subset $\Phi$ of $\Gamma$ such that $\Phi \vdash_{K^b} \varphi$.

**Corollary 19** (Derivation Theorem). $\Gamma \vdash_{K^b} \varphi$ iff there exists a finite subset $\Phi$ of $\Gamma$ such that $\vdash_{K^b} \mathbf{\bigwedge} \Phi \rightarrow \varphi$.

The derivation theorem indicates that the deduction relation $\vdash_{K^b}$ can be characterized by the theorems of $K^b$. The next theorem is used for proving Theorem 27.

**Theorem 20** (Substitution Theorem). If $\vdash_{K^b} \varphi^\sigma$ then $\vdash_{K^b} \varphi^\sigma$, for any substitution $\sigma$.

The sixth way is called deflationary definition. This definition hides the rule of necessitation in axiom schemata, so that the set of axioms is closed under prefixing $\square$. Then the rule of necessitation is redundant. According to [10], this definition was suggested in [17] for provability logic. The idea may come from some textbooks in mathematical logic (as [5]) , in which the rule of universal generalization is hidden in axiom schemata so that the set of axioms is closed under taking universal quantification. The advantage of this definition is that we can follow the standard definition of deductive consequence in classical propositional logic without destroying the deduction theorem, and strong soundness is also maintained. Unfortunately, no textbooks in modal logic adopts this definition. That said, this definition also has a drawback. For logics that does not contain RN but only weaker rules like RE, this definition is not applicable any more.
In the sequel, we take $K$ as an example to give the precise deflationary definition and show its equivalence to reduced definition and bounded definition for reference of teaching.

**Definition 21** (deflationary definition). The set of axioms $\Xi_{K^d}$ of the axiomatic system $K^d$ includes

(PC1)  \[ p \to (q \to p) \]

(PC2)  \[ (p \to (q \to r)) \to ((p \to q) \to (p \to r)) \]

(PC3)  \[ (\neg p \to \neg q) \to (q \to p) \]

(K)  \[ \Box(p \to q) \to (\Box p \to \Box q) \]

(dual)  \[ \Diamond p \leftrightarrow \neg \Box \neg p \]

and satisfies the following conditions.

- It is closed under necessitation, i.e., if $\varphi \in \Xi_{K^d}$ then $\Box \varphi \in \Xi_{K^d}$.
- It is closed under substitution, i.e., if $\varphi \in \Xi_{K^d}$ then $\varphi^\sigma \in \Xi_{K^d}$, where $\sigma$ is any substitution.
- No other formulas are in $\Xi_{K^d}$.

The only rule of inference of $K^d$ is

\[(MP) \quad \varphi, \varphi \to \psi \quad \vdash \Box \psi. \]

A finite sequence of formulas $\varphi_1, \ldots, \varphi_n$ is a formal proof of $\varphi$ from $\Gamma$ in $K^d$, if $\varphi = \varphi_n$ and for each $\varphi_i$,

- either $\varphi_i \in \Gamma$, or
- $\varphi_i \in \Xi_{K^d}$, or
- there exist $j, k < i$ such that $\varphi_k = \varphi_j \to \varphi_i$, i.e., $\varphi_i$ is obtained from proceeding formulas in the sequence using modus ponens.

We say $\varphi$ is derivable from $\Gamma$ in $K^d$, denoted $\Gamma \vdash_{K^d} \varphi$, if there is a formal proof of $\varphi$ from $\Gamma$ in $K^d$. We abbreviate $\emptyset \vdash_{K^d} \varphi$ as $\vdash_{K^d} \varphi$.

**Theorem 22** (Deduction Theorem). If $\Gamma, \psi \vdash_{K^d} \varphi$, then $\Gamma \vdash_{K^d} \psi \to \varphi$.

**Theorem 23** (Compactness). $\Gamma \vdash_{K^d} \varphi$ iff there exists a finite subset $\Phi$ of $\Gamma$ such that $\Phi \vdash_{K^d} \varphi$.

**Corollary 24** (Derivation Theorem). $\Gamma \vdash_{K^d} \varphi$ iff there exists a finite subset $\Phi$ of $\Gamma$ such that $\vdash_{K^d} \varphi$. 

The derivation theorem indicates that the deduction relation $\vdash_{K^d}$ can be characterized by the theorems of $K^d$.

**Theorem 25** (Generalization Theorem). If $\Gamma \vdash_{K^d} \varphi$ then $\Box \Gamma \vdash_{K^d} \Box \varphi$.

The following results is easily obtained from the generalization theorem.

**Corollary 26.** $K^d$ has the following admissible rules.\(^\text{12}\)

\[\text{A rule } \Gamma \vdash_{S} \psi \text{ is admissible in } S, \text{ if for any substitution } \sigma, \vdash_{S} \Gamma^\sigma \text{ implies } \vdash_{S} \psi^\sigma, \text{ where } \vdash_{S} \Gamma^\sigma \text{ means } \vdash_{S} \psi^\sigma \text{ for all } \psi \in \Gamma.\]

\[12\text{A rule } \Gamma \vdash_{S} \psi \text{ is admissible in } S, \text{ if for any substitution } \sigma, \vdash_{S} \Gamma^\sigma \text{ implies } \vdash_{S} \psi^\sigma, \text{ where } \vdash_{S} \Gamma^\sigma \text{ means } \vdash_{S} \psi^\sigma \text{ for all } \psi \in \Gamma.\]
The first one is missing the axiom \( \square \varphi \iff \neg \square \neg \varphi \), when choosing \( \square \) as the primitive concept of deduction consequence in modal logic.

We can also adopt the most convenient one among them according to the demand in applications or proofs in different situations.

5 Conclusion

We discuss four common mistakes in the teaching and textbooks of modal logic. The first one is missing the axiom \( \square \varphi \iff \neg \square \neg \varphi \), when choosing \( \square \) as the primitive concept of deduction consequence in modal logic. We can also adopt the most convenient one among them according to the demand in applications or proofs in different situations.
modal operator, misunderstanding that □ and ◊ are symmetric. The second one is forgetting to make the set of formulas for filtration closed under subformulas, when proving the finite model property through filtration, neglecting that □φ and ◊φ may be abbreviations of formulas. The third one is giving wrong definitions of canonical relations in minimal canonical models that are unmatched with the primitive modal operators. These three mistakes are all related to the choice of primitive modal operators. The moral is that we can not take for granted that □ and ◊ are symmetric and can be chosen arbitrarily in certain definitions. They have to be carefully selected according to which modal operator is taken as primitive.

The fourth mistake is misunderstanding the rule of necessitation, without knowing its distinction from the rule of modus ponens. We summarizes six methods of defining deductive consequence in modal logic: omitted definition, classical definition, ternary definition, reduced definition, bounded definition, and deflationary definition, and show that the last three definitions are equivalent to each other.

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