Abstract

Since the early work of Richard Stanley, it has been observed that several permutation statistics have a remarkable property with respect to shuffles of permutations. We formalize this notion of a shuffle-compatible permutation statistic and introduce the shuffle algebra of a shuffle-compatible permutation statistic, which encodes the distribution of the statistic over shuffles of permutations. This paper develops a theory of shuffle-compatibility for descent statistics—statistics that depend only on the descent set and length—which has close connections to the theory of $P$-partitions, quasisymmetric functions, and noncommutative symmetric functions. We use our framework to prove that many descent statistics are shuffle-compatible and to give explicit descriptions of their shuffle algebras, thus unifying past results of Stanley, Gessel, Stembridge, Aguiar–Bergeron–Nyman, and Petersen.

Keywords: permutations, shuffles, permutation statistics, $P$-partitions, quasisymmetric functions, noncommutative symmetric functions

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1. Introduction

We say that \( \pi = \pi_1 \pi_2 \cdots \pi_n \) is a permutation of length \( n \) (or an \( n \)-permutation) if it is a sequence of \( n \) distinct letters—not necessarily from 1 to \( n \)—in \( \mathbb{P} \), the set of positive integers. For example, \( \pi = 47381 \) is a permutation of length 5. Let \( |\pi| \) denote the length of a permutation \( \pi \) and let \( \mathcal{P}_n \) denote the set of all permutations of length \( n \).

A permutation statistic (or statistic) \( st \) is a function defined on permutations such that \( st(\pi) = st(\sigma) \) whenever \( \pi \) and \( \sigma \) are permutations with the same relative order.\(^2\) Three classical examples of permutation statistics are the descent set \( \text{Des} \), the descent number \( \text{des} \), and the major index \( \text{maj} \). We say that \( i \in [n-1] \) is a descent of \( \pi \in \mathcal{P}_n \) if \( \pi_i > \pi_{i+1} \). Then the descent set

\[
\text{Des}(\pi) := \{ i \in [n-1] : \pi_i > \pi_{i+1} \}
\]

\(^1\)In Section 2 we will in a few instances consider permutations with a letter 0. We note that, in these cases, every property of permutations that is used still holds when 0 is allowed to be a letter.

\(^2\)Define the standardization of an \( n \)-permutation \( \pi \) to be the permutation of \([n]\) obtained by replacing the \( i \)th smallest letter of \( \pi \) with \( i \) for \( i \) from 1 to \( n \). Then two permutations are said to have the same relative order if they have the same standardization.
of \( \pi \) is the set of its descents, the *descent number*

\[
\text{des}(\pi) := |\text{Des}(\pi)|
\]

its number of descents, and the *major index*

\[
\text{maj}(\pi) := \sum_{k \in \text{Des}(\pi)} k
\]

the sum of its descents.

Let \( \pi \in \mathfrak{S}_m \) and \( \sigma \in \mathfrak{S}_n \) be *disjoint* permutations, that is, permutations with no letters in common. We say that \( \tau \in \mathfrak{S}_{m+n} \) is a *shuffle* of \( \pi \) and \( \sigma \) if both \( \pi \) and \( \sigma \) are subsequences of \( \tau \). The set of shuffles of \( \pi \) and \( \sigma \) is denoted \( S(\pi, \sigma) \). For example, \( S(53, 16) = \{5316, 5136, 5163, 1653, 1536, 1563\} \). It is easy to see that the number of permutations in \( S(\pi, \sigma) \) is \( \binom{m+n}{m} \).

Richard Stanley’s theory of \( P \)-partitions \([25]\) implies that the descent set statistic has a remarkable property related to shuffles: for any disjoint permutations \( \pi \) and \( \sigma \), the multiset \( \{\text{Des}(\tau) : \tau \in S(\pi, \sigma)\} \)—which encodes the distribution of the descent set over shuffles of \( \pi \) and \( \sigma \)—depends only on \( \text{Des}(\pi), \text{Des}(\sigma) \), and the lengths of \( \pi \) and \( \sigma \) \([26, \text{Exercise 3.16}]\).

That is, if \( \pi \) and \( \pi' \) are permutations of the same length with the same descent set, and similarly with \( \sigma \) and \( \sigma' \), then the number of permutations in \( S(\pi, \sigma) \) with any given descent set is the same as the number of permutations in \( S(\pi', \sigma') \) with that descent set.

Stanley also proved a similar but more refined result for the joint statistic \((\text{des}, \text{maj})\), which is a special case of \([25, \text{Proposition 12.6 (ii)}]\). Bijective proofs were later found by Goulden \([10]\) and by Stadler \([24]\); they referred to this result as “Stanley’s shuffling theorem”.

Recall that the \( q \)-binomial coefficient \( \binom{n}{k}_q \) is defined by

\[
\binom{n}{k}_q := \frac{[n]_q!}{[k]_q! \, [n-k]_q!}
\]

where \( [n]_q! := (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}) \).

**Theorem 1.1** (Stanley’s shuffling theorem). Let \( \pi \in \mathfrak{S}_m \) and \( \sigma \in \mathfrak{S}_n \) be disjoint permutations, and let \( S_k(\pi, \sigma) \) be the set of shuffles of \( \pi \) and \( \sigma \) with exactly \( k \) descents. Then

\[
\sum_{\tau \in S_k(\pi, \sigma)} q^{\text{maj}(\tau)} = q^{\text{maj}(\pi) + \text{maj}(\sigma) + (k - \text{des}(\pi)) (k - \text{des}(\sigma))}
\]

\[
\times \binom{m - \text{des}(\pi) + \text{des}(\sigma)}{k - \text{des}(\pi)}_q \, \binom{n - \text{des}(\sigma) + \text{des}(\pi)}{k - \text{des}(\sigma)}_q.
\] (1.1)

A variant of the theorem gives the formula

\[
\sum_{\tau \in S(\pi, \sigma)} q^{\text{maj}(\tau)} = q^{\text{maj}(\pi) + \text{maj}(\sigma)} \binom{m + n}{m}_q.
\] (1.2)

see \([25, \text{p. 43}]\). These formulas show that the statistics \((\text{des}, \text{maj})\) and \(\text{maj}\) have the same property as \(\text{Des}\), and setting \( q = 1 \) in (1.1) shows that \(\text{des}\) has this property as well.
We call this property “shuffle-compatibility”. More precisely, we say that a permutation statistic \( st \) is \textit{shuffle-compatible} if for any disjoint permutations \( \pi \) and \( \sigma \), the multiset \( \{ st(\tau) : \tau \in S(\pi, \sigma) \} \) depends only on \( st(\pi), st(\sigma), |\pi|, \) and \( |\sigma| \). Hence \( \text{Des}, \text{des}, \text{maj}, \) and \( (\text{des}, \text{maj}) \) are examples of shuffle-compatible permutation statistics.

This paper serves as the first in-depth investigation of shuffle-compatibility, and we focus on the shuffle-compatibility of descent statistics, which are statistics that depend only on the descent set and length of a permutation. All of the statistics mentioned so far are descent statistics. In Section \( \text{2} \) we introduce some aspects of the general theory of descents in permutations and define some other descent statistics that we will be studying in this paper, including the peak set \( \text{Pk} \), the peak number \( \text{pk} \), the left peak set \( \text{Lpk} \), the left peak number \( \text{lpk} \), and the number of up-down runs \( \text{udr} \). There, we also give a bijective proof of the shuffle-compatibility of the descent set.

In Section \( \text{3} \) we define the “shuffle algebra” of a shuffle-compatible permutation statistic \( st \), which has a natural basis whose structure constants encode the distribution of \( st \) over shuffles of permutations (or more precisely, equivalence classes of permutations induced by the statistic \( st \)). Our first result is a characterization of the major index shuffle algebra using the variant \( (1.2) \) of Stanley’s shuffling theorem. We then prove several basic results that relate the shuffle algebras of permutation statistics that are related in various ways. Notably, if two statistics are related by a basic symmetry—reversion, complementation, or reverse complementation—and one of them is known to be shuffle-compatible, then both statistics are shuffle-compatible and have isomorphic shuffle algebras.

In Section \( \text{4} \) we introduce the algebra of quasisymmetric functions \( \text{QSym} \) (originally studied in \cite{7}) and observe that it is isomorphic to the descent set shuffle algebra. We establish a necessary and sufficient condition for the shuffle-compatibility of a descent statistic, which shows that the shuffle algebra of any shuffle-compatible descent statistic is isomorphic to a quotient algebra of \( \text{QSym} \). Using this condition, we give explicit descriptions for the shuffle algebras of \( \text{des} \) and \( (\text{des}, \text{maj}) \). We then observe that the peak set shuffle algebra is isomorphic to Stembridge’s “algebra of peaks” arising from his study of enriched \( \text{P} \)-partitions \cite{28}—thus showing that the peak set \( \text{Pk} \) is shuffle-compatible—and use Stembridge’s peak quasisymmetric functions to characterize the peak number shuffle algebra, thus showing that the peak number \( \text{pk} \) is shuffle-compatible. In the same vein, Petersen’s work \cite{19, 20} on left enriched \( \text{P} \)-partitions implies that the left peak set \( \text{Lpk} \) and left peak number \( \text{lpk} \) are shuffle-compatible.

In Section \( \text{5} \) we introduce the bialgebra of noncommutative symmetric functions \( \text{Sym} \) (originally studied in \cite{6}), whose coalgebra structure is dual to the algebra structure of \( \text{QSym} \). By exploiting this duality, we obtain a dual version of our shuffle-compatibility condition, which allows us to prove shuffle-compatibility of a descent statistic by constructing a suitable subcoalgebra of \( \text{Sym} \). We use this approach to describe the shuffle algebras of \( (\text{pk}, \text{des}) \), \( (\text{lpk}, \text{des}) \), \( \text{udr} \), and \( (\text{udr}, \text{des}) \), thus showing that these statistics are all shuffle-compatible.

Finally, in Section \( \text{6} \) we provide proofs for an alternate characterization of the \( \text{pk} \) and \( (\text{pk}, \text{des}) \) shuffle algebras, list some non-shuffle-compatible permutation statistics, and discuss some open questions and conjectures on the topic of shuffle-compatibility.

The appendix of this paper contains two tables. Table 1 lists all permutation statistics that we know to be shuffle-compatible, and Table 2 lists various equivalences (as defined in Section \( \text{3} \)) among the statistics that are studied in this paper.
We note that some permutation statistics, such as the number of inversions, satisfy a weak form of shuffle-compatibility: for disjoint permutations \( \pi \) and \( \sigma \), if every letter of \( \pi \) is less than every letter of \( \sigma \), then the multiset \( \{ \text{st}(\tau) : \tau \in S(\pi, \sigma) \} \) depends only on \( \text{st}(\pi) \), \( \text{st}(\sigma) \), \( |\pi| \), and \( |\sigma| \). Permutation statistics with this property are associated with quotients of the Malvenuto–Reutenauer algebra (also called the algebra of free quasisymmetric functions). Some of these statistics have been studied by Vong [29], but we do not consider them here.

Also, there is another class of algebras that are related to permutations and their descent sets, based on ordinary multiplication of permutations rather than shuffles. If \( \text{st} \) is a function defined on the \( n \)th symmetric group \( S_n \), we may consider the elements

\[
K_\alpha := \sum_{\pi \in S_n, \text{st}(\pi) = \alpha} \pi
\]

in the group algebra of \( S_n \), where \( \alpha \) ranges over the image of \( \text{st} \). Louis Solomon [23] proved that if \( \text{st} \) is the descent set, then the \( K_\alpha \) span a subalgebra of the group algebra of \( S_n \), called the descent algebra of \( S_n \). Several other descent statistics give subalgebras of the descent algebra, including the descent number [16]; the peak set [18, 22]; the peak number, and left peak number [1, 19, 20]; and the number of biruns and up-down runs [5, 15]. These descent statistics have the property that given values \( \alpha \) and \( \beta \) of \( \text{st} \), and \( \tau \in S_n \), the number of pairs \((\pi, \sigma)\) of permutations in \( S_n \) with \( \text{st}(\pi) = \alpha \), \( \text{st}(\sigma) = \beta \), and \( \pi \sigma = \tau \) depends only on \( \text{st}(\tau) \). In other words, these statistics are “compatible” under the ordinary product of permutations, and our work is an analogue of Solomon’s descent theory for statistics compatible under the shuffle product.

Although there is a significant overlap between shuffle-compatible permutation statistics and statistics corresponding to subalgebras of the descent algebra, neither class is contained in the other, as the number of biruns is not shuffle-compatible and the pair \((\text{pk}, \text{des})\) does not give a subalgebra of the descent algebra. The descent algebra and its subalgebras may also be studied through noncommutative symmetric functions (using the internal product of Sym [6] Section 5]) or quasisymmetric functions (using the internal coproduct of QSym [7]).

2. Permutations and descents

2.1. Increasing runs and descent compositions

We begin with a brief exposition on some basic material in permutation enumeration relating to descents.

Every permutation can be uniquely decomposed into a sequence of maximal increasing consecutive subsequences, which we call increasing runs (or simply runs). For example, the increasing runs of 21479536 are 2, 1479, 5, and 36. Equivalently, an increasing run of \( \pi \) is a maximal consecutive subsequence containing no descents. Let us call an increasing run short if it has length 1, and long if it has length at least 2. The initial run of a permutation refers to its first increasing run, whereas the final run refers to its last increasing run. For example, the initial run of 21479536 is 2 and its final run is 36. (If a permutation has only one increasing run, then it is considered to be both an initial run and a final run.)
The number of increasing runs of a nonempty permutation is one more than its number of descents; in fact, the lengths of the increasing runs determine the descents, and vice versa. Given a subset $A \subseteq [n-1]$ with elements $a_1 < a_2 < \cdots < a_j$, let $\text{Comp}(A)$ be the composition $(a_1, a_2 - a_1, \ldots, a_j - a_{j-1}, n - a_j)$ of $n$, and given a composition $L = (L_1, L_2, \ldots, L_k)$ of $n$, let $\text{Des}(L) := \{L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{k-1}\}$ be the corresponding subset of $[n-1]$. Then, $\text{Comp}$ and $\text{Des}$ are inverse bijections. If $\pi$ is an $n$-permutation with descent set $A \subseteq [n-1]$, then we call $\text{Comp}(A)$ the descent composition of $\pi$, which we also denote by $\text{Comp}(\pi)$. By convention, let us say that the empty permutation (i.e., permutation of length 0) has descent composition $\emptyset$. Note that the descent composition of $\pi$ gives the lengths of the increasing runs of $\pi$. Conversely, if $\pi$ has descent composition $L$, then its descent set $\text{Des}(\pi)$ is $\text{Des}(L)$.

A permutation statistic $st$ is called a descent statistic if it depends only on the descent composition, that is, if $\text{Comp}(\pi) = \text{Comp}(\sigma)$ implies $st(\pi) = st(\sigma)$ for any two permutations $\pi$ and $\sigma$. Equivalently, $st$ is a descent statistic if it depends only on the descent set and length of a permutation. Since two permutations with the same descent composition must have the same value of $st$ if $st$ is a descent statistic, we shall use the notation $st(L)$ to indicate the value of a descent statistic $st$ on any permutation with descent composition $L$.

We define several statistics based on increasing runs: the long run $lr$, long initial run $lir$, long final run $lfr$, short initial run $sir$, and short final run $sfr$ statistics. Let $lr(\pi)$ be the number of long runs of $\pi$, let $lir(\pi)$ be 1 if the initial run of $\pi$ is long and 0 otherwise, and let $lfr(\pi)$ be 1 if the final run of $\pi$ is long and 0 otherwise. Also, for nonempty $\pi$, let $sir(\pi) := 1 - lir(\pi)$ and $sfr(\pi) := 1 - lfr(\pi)$. By convention, if $\pi$ is empty, then all of these statistics are equal to zero. We will use these run statistics to give an alternative way of characterizing some of the descent statistics introduced in the next section.

2.2. Descent statistics

In the introduction to this paper, we saw four examples of descent statistics: the descent set $\text{Des}$, descent number $\text{des}$, major index $\text{maj}$, and the joint statistic $(\text{des}, \text{maj})$. The following are additional descent statistics that we will consider in our investigation of shuffle-compatibility:

- The comajor index $\text{comaj}$. The comajor index $\text{comaj}(\pi)$ of $\pi \in \mathfrak{S}_n$, a variant of the major index, is defined to be

  \[ \text{comaj}(\pi) := \sum_{k \in \text{Des}(\pi)} (n - k). \]

- The peak set $\text{Pk}$ and peak number $\text{pk}$. We say that $i$ (where $2 \leq i \leq n-1$) is a peak of $\pi \in \mathfrak{S}_n$ if $\pi_{i-1} < \pi_i > \pi_{i+1}$. The peak set $\text{Pk}(\pi)$ of $\pi$ is defined to be

  \[ \text{Pk}(\pi) := \{2 \leq i \leq n-1 : \pi_{i-1} < \pi_i > \pi_{i+1}\} \]

  and the peak number $\text{pk}(\pi)$ of $\pi$ to be

  \[ \text{pk}(\pi) := |\text{Pk}(\pi)|. \]
• The valley set Val and valley number val. We say that \(i\) (where \(2 \leq i \leq n - 1\)) is a *valley* of \(\pi \in \mathcal{P}_n\) if \(\pi_{i-1} > \pi_i < \pi_{i+1}\). Then \(\text{Val}(\pi)\) and \(\text{val}(\pi)\) are defined in the analogous way.

• The left peak set Lpk and left peak number lpk. We say that \(i \in [n - 1]\) is a *left peak* of \(\pi \in \mathcal{P}_n\) if \(i\) is a peak of \(\pi\) or if \(i = 1\) and is a descent of \(\pi\). Thus, left peaks of \(\pi\) are peaks of \(0\pi\) shifted by 1. The *left peak set* \(\text{Lpk}(\pi)\) is the set of left peaks of \(\pi\) and the *left peak number* \(\text{lpk}(\pi)\) is the number of left peaks of \(\pi\).

• The right peak set Rpk and right peak number rpk. These are defined in the same way as the corresponding left peak statistics, except that right peaks of \(\pi\) are peaks of \(\pi 0\).

• The exterior peak set Epk and exterior peak number epk. The *exterior peak set* \(\text{Epk}(\pi)\) of \(\pi\) is defined by

\[
\text{Epk}(\pi) := \begin{cases} 
\text{Lpk}(\pi) \cup \text{Rpk}(\pi), & \text{if } |\pi| \neq 1 \\
\{1\}, & \text{if } |\pi| = 1
\end{cases}
\]

and the *exterior peak number* \(\text{epk}(\pi)\) of \(\pi\) is defined by

\[
\text{epk}(\pi) := |\text{Epk}(\pi)|.
\]

• The number of biruns br and the number of up-down runs udr. A *birun* of a permutation is a maximal monotone consecutive subsequence, and the number of biruns of \(\pi\) is denoted \(\text{br}(\pi)\). An *up-down run* of a permutation \(\pi\) is either a birun or \(\pi_1 \pi_2\) when \(\pi_1 > \pi_2\), and the number of up-down runs of \(\pi\) is denoted \(\text{udr}(\pi)\). Thus the up-down runs of \(\pi\) are essentially the biruns of \(0\pi\). For example, the biruns of \(\pi = 871542\) are 871, 15, and 542, and the up-down runs of \(\pi\) are these biruns along with 8, so \(\text{br}(\pi) = 3\) and \(\text{udr}(\pi) = 4\).

• Ordered tuples of descent statistics, such as \((\text{pk, des})\), \((\text{lpk, des})\), and so on.

Before continuing, we give two lemmas that will help us understand some of the above statistics. The first lemma characterizes several statistics in terms of the run statistics introduced at the end of the previous section, and the second lemma reveals a close connection between the udr statistic and the lpk and val statistics.

**Lemma 2.1.** Let \(\pi \in \mathcal{P}_n\) with \(n \geq 1\). Then

(a) \(\text{pk}(\pi) = \text{lr}(\pi) - \text{lfr}(\pi)\)

(b) \(\text{val}(\pi) = \text{lr}(\pi) - \text{lir}(\pi)\)

(c) \(\text{lpk}(\pi) = \begin{cases} 
\text{lr}(\pi) + \text{sir}(\pi) - \text{lfr}(\pi), & \text{if } n \geq 2, \\
0, & \text{otherwise}.
\end{cases}\)

(d) \(\text{rpk}(\pi) = \text{lr}(\pi)\)

(e) \(\text{epk}(\pi) = \text{val}(\pi) + 1\)
Proof. Part (a) follows from the fact that every non-final long run ends in a peak, and every peak is at the end of a non-final long run. The same is true for valleys and non-initial long runs, and for right peaks and long runs, thus implying (b) and (d). Next,

$$\text{lpk}(\pi) = \begin{cases} \text{pk}(\pi) + \text{sir}(\pi), & \text{if } n \geq 2, \\ 0, & \text{otherwise,} \end{cases}$$

which together with (a) proves (c). Finally,

$$\text{epk}(\pi) = \text{rpk}(\pi) + \text{sir}(\pi)$$

$$= \text{lr}(\pi) + 1 - \text{lir}(\pi)$$

$$= \text{val}(\pi) + 1$$

proves (e).

Lemma 2.2. Let $\pi \in \mathcal{P}_n$ with $n \geq 1$. Then

(a) $\text{udr}(\pi) = \text{lpk}(\pi) + \text{val}(\pi) + 1$

(b) $\text{lpk}(\pi) = \lfloor \text{udr}(\pi)/2 \rfloor$

(c) $\text{val}(\pi) = \lfloor (\text{udr}(\pi) - 1)/2 \rfloor$

(d) If $n \geq 2$ and the final run of $\pi$ is short, then $\text{lpk}(\pi) = \text{val}(\pi) + 1$. Otherwise, $\text{lpk}(\pi) = \text{val}(\pi)$.

This is Lemma 2.1 of [31]; a proof can be found there. According to this result, not only do lpk and val determine udr, but udr determines both lpk and val. In other words, udr and (lpk, val) are equivalent permutation statistics in the sense that will be formally defined in Section 3.1.

We note that the definitions and properties of descents, increasing runs, descent compositions, and descent statistics extend naturally to words on any totally ordered alphabet such as $[n]$ or $\mathbb{P}$ if we replace the strict inequality $<$ with the weak inequality $\leq$, which reflects the fact that increasing runs are allowed to be weakly increasing in this setting. For example, $i$ is a peak of the word $w = w_1w_2 \cdots w_n$ if $w_{i-1} \leq w_i > w_{i+1}$.

2.3. Possible values of some descent statistics

In our study of shuffle-compatibility, it will be useful to determine all possible values that a descent statistic can achieve. It is clear that for $\pi \in \mathcal{P}_n$ and $n \geq 1$, we have $0 \leq \text{des}(\pi) \leq n-1$ and $\text{des}(\pi)$ can attain any value in this range for some $\pi \in \mathcal{P}_n$. It is also easy to check that the possible values of $\text{maj}(\pi)$ and $\text{comaj}(\pi)$ for $\pi \in \mathcal{P}_n$ range from 0 to $\binom{n}{2}$, and that all of these values are attainable. Finding such bounds for other descent statistics requires more work. Here, we determine all possible values for the $(\text{des}, \text{maj})$, $(\text{des}, \text{comaj})$, $(\text{pk}, \text{des})$, $(\text{lpk}, \text{des})$, and $(\text{udr}, \text{des})$ statistics.
Proposition 2.3 (Possible values of \((\text{des}, \text{maj})\)).

(a) For any permutation \(\pi \in \mathfrak{P}_n\) with \(n \geq 1\) and \(\text{des}(\pi) = j\), we have \((\binom{j+1}{2}) \leq \text{maj}(\pi) \leq nj - \binom{j+1}{2}\).

(b) If \(n \geq 1\), \(0 \leq j \leq n-1\), and \((\binom{j+1}{2}) \leq k \leq nj - \binom{j+1}{2}\), then there exists \(\pi \in \mathfrak{P}_n\) with \(\text{des}(\pi) = j\) and \(\text{maj}(\pi) = k\).

Proof. Among all \(n\)-permutations with \(j\) descents, it is clear that the smallest possible value of \(\text{maj}\) is attained when the descent set is \([1, 2, \ldots, j]\), in which case the major index is equal to \((\binom{j+1}{2})\). Similarly, the largest possible value of \(\text{maj}\) is attained when the descent set is \([n-j, n-j+1, \ldots, n-1]\), in which case the major index is equal to \(nj - \binom{j+1}{2}\). This proves (a).

Next we prove (b). The case \(j = 0\) is easy, so we assume that \(j \geq 1\). A permutation in \(\mathfrak{P}_n\) with descent set \([1, 2, \ldots, j]\) has major index \((\binom{j+1}{2})\). Now let \(\pi\) be a permutation in \(\mathfrak{P}_n\) with \(j\) descents, and suppose that for some \(i \in \text{Des}(\pi)\) we have \(i \neq n-1\) and \(i+1 \notin \text{Des}(\pi)\). Take \(\sigma \in \mathfrak{P}_n\) to have descent set \((\text{Des}(\pi) \setminus \{i\}) \cup \{i+1\}\). (This is possible because for any subset \(A\) of \([n-1]\), there exists an \(n\)-permutation with descent set \(A\).) Then \(\text{maj}(\sigma) = \text{maj}(\pi) + 1\).

We can repeat this process to increase the major index by 1 with every iteration until we reach a permutation with descent set \([n-j, n-j+1, \ldots, n-1]\), and thus major index \(nj - \binom{j+1}{2}\). This proves (b).

\(\square\)

Proposition 2.4 (Possible values of \((\text{des}, \text{comaj})\)).

(a) For any permutation \(\pi \in \mathfrak{P}_n\) with \(n \geq 1\) and \(\text{des}(\pi) = j\), we have \((\binom{j+1}{2}) \leq \text{comaj}(\pi) \leq nj - \binom{j+1}{2}\).

(b) If \(n \geq 1\), \(0 \leq j \leq n-1\), and \((\binom{j+1}{2}) \leq k \leq nj - \binom{j+1}{2}\), there exists \(\pi \in \mathfrak{P}_n\) with \(\text{des}(\pi) = j\) and \(\text{comaj}(\pi) = k\).

Proof. This follows from the previous proposition and the formula \(\text{comaj}(\pi) = n \text{des}(\pi) - \text{maj}(\pi)\).

\(\square\)

Proposition 2.5 (Possible values of \((\text{pk}, \text{des})\)).

(a) For any permutation \(\pi \in \mathfrak{P}_n\) with \(n \geq 1\), we have \(0 \leq \text{pk}(\pi) \leq \lfloor(n-1)/2\rfloor\). In addition, \(\text{pk}(\pi) \leq \text{des}(\pi) \leq n - \text{pk}(\pi) - 1\).

(b) If \(n \geq 1\), \(0 \leq j \leq \lfloor(n-1)/2\rfloor\), and \(j \leq k \leq n - j - 1\), then there exists \(\pi \in \mathfrak{P}_n\) with \(\text{pk}(\pi) = j\) and \(\text{des}(\pi) = k\).

Proof. Fix \(n \geq 1\). Recall from Lemma 2.1 (a) that \(\text{pk}(\pi)\) is equal to the number of non-final long runs of \(\pi\). It is clear that the number of non-final long runs of an \(n\)-permutation is between 0 and \(\lfloor(n-1)/2\rfloor\). Every peak is a descent, so \(\text{pk}(\pi) \leq \text{des}(\pi)\). For each peak \(i\), note that \(i-1 \in [n-1]\) is not a descent, so that \(\text{pk}(\pi) \leq n - 1 - \text{des}(\pi)\) and therefore \(\text{des}(\pi) \leq n - \text{pk}(\pi) - 1\). This proves (a).

To prove (b), it suffices to show that if \(n \geq 1\), \(0 \leq j \leq \lfloor(n-1)/2\rfloor\), and \(j \leq k \leq n - j - 1\) then there exists a composition of \(n\) with \(j\) non-final long parts (i.e., parts of size at least 2) and \(k+1\) total parts. Such a composition is \((2^j, 1^{k-j}, n - k - j)\). Hence, (b) is proved.

\(\square\)
Proposition 2.6 (Possible values of \((\text{lpk}, \text{des})\)).

(a) For any permutation \(\pi \in \mathfrak{P}_n\) with \(n \geq 1\), we have \(0 \leq \text{lpk}(\pi) \leq \lfloor n/2 \rfloor\). In addition, if \(\text{lpk}(\pi) = 0\), then \(\text{des}(\pi) = 0\); otherwise, \(\text{lpk}(\pi) \leq \text{des}(\pi) \leq n - \text{lpk}(\pi)\).

(b) If \(n \geq 1\), \(1 \leq j \leq \lfloor n/2 \rfloor\), and \(j \leq k \leq n - j\), then there exists \(\pi \in \mathfrak{P}_n\) with \(\text{lpk}(\pi) = j\) and \(\text{des}(\pi) = k\). In addition, for any \(n \geq 1\), there exists \(\pi \in \mathfrak{P}_n\) with \(\text{lpk}(\pi) = \text{des}(\pi) = 0\).

Proof. If \(\text{lpk}(\pi) = 0\), then \(\pi\) is an increasing permutation, so we also have \(\text{des}(\pi) = 0\). The other inequalities of part (a) follow from applying Proposition 2.5 (a) to the permutation \(0\pi\).

Now, fix \(n \geq 2\). (The case \(n = 1\) is obvious.) A permutation with descent composition \((n)\) has no left peaks and no descents. Suppose that \(1 \leq j \leq \lfloor n/2 \rfloor\) and \(j \leq k \leq n - j\). To complete the proof of (b), we show that there exists a composition \(L\) of \(n\) with exactly \(k + 1\) parts such that \(\text{lpk}(L) = \text{lr}(L) + \text{sr}(L) - \text{lfr}(L) = j\). Such a composition is \((1^{k-j+1}, 2^{j-1}, n - k - j + 1)\). This completes the proof of (b).

We say that \(i \in \lfloor n - 1 \rfloor\) is an ascent of an \(n\)-permutation \(\pi\) if \(\pi_i < \pi_{i+1}\). Let \(\text{asc}(\pi)\) denote the number of ascents of \(\pi\). It is clear that \(\text{des}(\pi) = n - 1 - \text{asc}(\pi)\).

Proposition 2.7 (Possible values of \((\text{udr}, \text{des})\)).

(a) For any permutation \(\pi \in \mathfrak{P}_n\) with \(n \geq 1\), we have \(1 \leq \text{udr}(\pi) \leq n\). In addition, if \(\text{udr}(\pi) = 1\), then \(\text{des}(\pi) = 0\); otherwise, \(\lfloor \text{udr}(\pi)/2 \rfloor \leq \text{des}(\pi) \leq n - \lceil \text{udr}(\pi)/2 \rceil\).

(b) If \(n \geq 1\), \(2 \leq j \leq n\), and \(\lfloor j/2 \rfloor \leq k \leq n - \lceil j/2 \rceil\), then there exists \(\pi \in \mathfrak{P}_n\) with \(\text{lpk}(\pi) = j\) and \(\text{des}(\pi) = k\). In addition, for any \(n \geq 1\), there exists \(\pi \in \mathfrak{P}_n\) with \(\text{udr}(\pi) = 1\) and \(\text{des}(\pi) = 0\).

Proof. It is clear that every nonempty permutation has at least one up-down run, and every up-down run of a permutation ends with a different letter, so \(1 \leq \text{udr}(\pi) \leq n\). The beginning of the \(2i\)th up-down run of \(\pi\) is always a descent of \(\pi\), so \(\text{des}(\pi) \geq \lfloor \text{udr}(\pi)/2 \rfloor\). The beginning of the \((2i - 1)\)th up-down run of \(\pi\) is an ascent of \(\pi\) for \(i \geq 2\), so the number of ascents of \(\pi\) is at least \(\lfloor (\text{udr}(\pi) - 1)/2 \rfloor = \lceil \text{udr}(\pi)/2 \rceil - 1\). Thus

\[\text{des}(\pi) = n - 1 - \text{asc}(\pi) \leq n - 1 - (\lfloor \text{udr}(\pi)/2 \rfloor - 1) = n - \lceil \text{udr}(\pi)/2 \rceil,\]

completing the proof of (a).

Now, fix \(n \geq 2\). (The case \(n = 1\) is obvious.) A permutation with descent composition \((n)\) has only one up-down run and no descents. Suppose that \(1 \leq j \leq n\) and \(\lfloor j/2 \rfloor \leq k \leq n - \lceil j/2 \rceil\). To complete the proof of (b), we show that there exists a composition \(L\) of \(n\) with exactly \(k + 1\) parts such that \(\text{udr}(L) = \text{lpk}(L) + \text{val}(L) + 1 = 2\text{sr}(L) + 2\text{lfr}(L) - \text{lfr}(L) = j\). For this, we consider three cases:

- If \(j = 2\), then we can take \((n - k, 1^k)\).
- If \(j > 2\) and \(j\) is even, then we can take \((1, n - j/2 - k + 2, 2^{j/2-2}, 1^{k-j/2+1})\).
- If \(j\) is odd, then we can take \((1, 1^{k-(j-1)/2}, 2^{(j-3)/2}, n - (j + 1)/2 - k + 2)\).

This completes the proof of (b). □
2.4. A bijective proof of the shuffle-compatibility of the descent set

Here we give a simple proof that the descent set is a shuffle-compatible permutation statistic. The idea of the proof is inspired by the theory of $P$-partitions [25].

Recall that in Section 2.1, we defined the inverse bijections Comp and Des between compositions of $n$ and subsets of $[n-1]$ in the following way: for a set $A = \{a_1, a_2, \ldots, a_j\} \subseteq [n-1]$ with $a_1 < a_2 < \cdots < a_j$, we let $\text{Comp}(A) := (a_1, a_2 - a_1, \ldots, a_j - a_{j-1}, n - a_j)$, and for a composition $L = (L_1, L_2, \ldots, L_k)$, we let $\text{Des}(L) := \{L_1, L_1 + L_2, \ldots, L_1 + \cdots + L_{k-1}\}$. Observe that these maps extend to inverse bijections between weak compositions of $n$ and multisubsets of $\{0\} \cup [n]$. (A weak composition allows 0 as a part.) For example, if $n = 7$ and $A = \{0, 2, 5\}$, then $\text{Comp}(A) = (0, 2, 0, 3, 2)$.

For two weak compositions $J = (J_1, J_2, \ldots, J_k)$ and $K = (K_1, K_2, \ldots, K_k)$ with the same number of parts, let $J + K$ denote the weak composition $(J_1 + K_1, J_2 + K_2, \ldots, J_k + K_k)$ obtained by summing the entries of $J$ and $K$ componentwise. Also, we define the refinement order on weak compositions of $n$ analogously to the refinement order on compositions of $n$; that is, $M$ covers $L$ if and only if $M$ can be obtained from $L$ by replacing two consecutive parts $L_i$ and $L_{i+1}$ with $L_i + L_{i+1}$. We say that $L$ is a refinement of $M$ if $L \leq M$ in the refinement order.

Lemma 2.8. Let $\pi \in \mathfrak{P}_m$ and $\sigma \in \mathfrak{P}_n$ be disjoint permutations, and let $A \subseteq [m + n - 1]$ and $L = \text{Comp}(A)$. Then the number of shuffles of $\pi$ and $\sigma$ with descent set contained in $A$ is equal to the number of weak compositions $J$ of $m$ and $K$ of $n$ such that $J$ is a refinement of $\text{Comp}(\pi)$, $K$ is a refinement of $\text{Comp}(\sigma)$, $J$ and $K$ have the same number of parts as $L$, and $J + K = L$.

Proof. Suppose that $L$ has $k$ parts, and let $J$ and $K$ satisfy the above conditions. For every $i \in \text{Des}(J)$, insert a bar immediately before the $(i+1)$th letter of $\pi$. Similarly, for every $i \in \text{Des}(K)$, insert a bar immediately before the $(i+1)$th letter of $\sigma$. This creates $k$ blocks of letters in each of the permutations $\pi$ and $\sigma$ such that the letters in each block are increasing. For example, take $\pi = 12879$, $\sigma = 4635$, $A = \{1, 5, 6\}$, $L = (1, 4, 1, 3)$, $J = (1, 2, 0, 2)$, and $K = (0, 2, 1, 1)$. Then this yields the “barred permutations” $1[28][79]$ and $[46][3][5]$.

For each $1 \leq i \leq k$, let $\tau^{(i)}$ denote the permutation obtained by merging the letters in the $i$th block of $\pi$ and the $i$th block of $\sigma$ in increasing order. Then let $\tau \in S(\pi, \sigma)$ be the concatenation $\tau^{(1)}\tau^{(2)}\cdots\tau^{(k)}$, which has descent set contained in $A$. For example, using the $\pi$ and $\sigma$ specified above, we have $\tau^{(1)} = 1$, $\tau^{(2)} = 2468$, $\tau^{(3)} = 3$, and $\tau^{(4)} = 579$, so $\tau = 124683579$. Since $J + K = L$, the descent set of $\tau$ is contained in $A$.

To show that this procedure gives a bijection between shuffles of $\pi$ and $\sigma$ with descent set contained in $A$ and pairs of weak compositions $(J, K)$ satisfying the stated conditions, we give an inverse procedure. Let $\tau \in S(\pi, \sigma)$ with $\text{Des}(\tau) \subseteq A$, and let $k = \lvert A \rvert + 1$. For every $i \in A$, insert a bar after the $i$th letter of $\tau$. Delete every letter in $\sigma$ from $\tau$ to obtain the permutation $\pi$ decorated with bars, which creates $k$ blocks of letters in $\pi$ such that the letters in each block are increasing. Similarly, by deleting every letter in $\pi$ from $\tau$, we obtain $k$ blocks of letters in $\sigma$ such that the letters in each block are increasing. Using the same example as above, we begin with $A = \{1, 5, 6\}$ and $\tau = 124683579$. Inserting bars, we have $1[2468][3][5]79$, from which we obtain $1[28][79]$ and $[46][3][5]$.

$^3$Since $\text{Des}(J)$ is a multiset, multiple bars may be inserted in any given position.
Now, for each \( 1 \leq i \leq k \), let \( J_i \) denote the size of the \( i \)th block in \( \pi \) and let \( K_i \) denote the size of the \( i \)th block in \( \sigma \). Then define the weak compositions \( J \) and \( K \) by \( J = (J_1, J_2, \ldots, J_k) \) and \( K = (K_1, K_2, \ldots, K_k) \). Continuing the example, we have \( J = (1, 2, 0, 2) \) and \( K = (0, 2, 1, 1) \). Since the letters in every block are weakly increasing, \( J \) is a refinement of \( \text{Comp}(\pi) \) and \( K \) is a refinement of \( \text{Comp}(\sigma) \). Moreover, it is clear that \( J \) and \( K \) have the same number of parts as \( L = \text{Comp}(A) \) and that \( J + K = L \).

Lemma 2.8 shows that the number of shuffles of \( \pi \) and \( \sigma \) with descent set contained in a specified set \( A \) depends only on \( \text{Des}(\pi), \text{Des}(\sigma), |\pi|, \) and \( |\sigma| \). By inclusion-exclusion, it follows that the number of shuffles of \( \pi \) and \( \sigma \) with descent set equal to \( A \) depends only on \( \text{Des}(\pi), \text{Des}(\sigma), |\pi|, \) and \( |\sigma| \). In other words, the descent set is shuffle-compatible.

We can use the shuffle-compatibility of the descent set to prove the shuffle-compatibility of a family of related statistics that we call “partial descent sets”. For non-negative integers \( i \) and \( j \), define the partial descent set \( \text{Des}_{i,j} \) by

\[
\text{Des}_{i,j}(\pi) := \text{Des}(\pi) \cap (\{1, 2, \ldots, i\} \cup \{n - 1, \ldots, n - j\}),
\]

where \( n = |\pi| \). In other words, \( \text{Des}_{i,j}(\pi) \) is the set of descents of \( \pi \) that occur in the first \( i \) or last \( j \) positions. For example, if \( i + j \geq |\pi| - 1 \) then \( \text{Des}_{i,j}(\pi) = \text{Des}(\pi) \), and for \( |\pi| \geq 2 \), \( |\text{Des}_{1,0}(\pi)| = \text{sfr}(\pi) \) and \( |\text{Des}_{0,1}(\pi)| = \text{str}(\pi) \).

**Theorem 2.9.** The partial descent sets \( \text{Des}_{i,j} \) for all \( i, j \geq 0 \) are shuffle-compatible.

**Proof.** We write \( \text{Des}_{i,j}(S(\pi, \sigma)) \) for the multiset \( \{ \text{Des}_{i,j}(\tau) : \tau \in S(\pi, \sigma) \} \). We define the equivalence relation \( \equiv_{i,j} \) on permutations of the same length by \( \pi \equiv_{i,j} \pi' \) if and only if \( \text{Des}_{i,j}(\pi) = \text{Des}_{i,j}(\pi') \) for all \( \sigma \) disjoint from both \( \pi \) and \( \pi' \). (It is immediate from the above definition that \( \equiv_{i,j} \) is reflexive and symmetric, and it is not hard to show that \( \equiv_{i,j} \) is also transitive.) For \( \pi \) and \( \pi' \) in \( \mathfrak{P}_m \), the following are sufficient conditions for \( \pi \equiv_{i,j} \pi' \):

(i) If \( \pi \) and \( \pi' \) have the same descent set, then \( \pi \equiv_{i,j} \pi' \).

(ii) If \( \pi_k = \pi'_k \) for all \( k \) with \( 1 \leq k \leq i + 1 \) or \( m - j \leq k \leq m \), then \( \pi \equiv_{i,j} \pi' \).

Condition (i) is a consequence of the shuffle-compatibility of the descent set. Condition (ii) follows from the fact that \( \text{Des}_{i,j}(\tau) \) for \( \tau \in S(\pi, \sigma) \) does not depend on the values of \( \pi_k \) with \( i + 1 < k < m - j \).

We claim that to prove the theorem it is sufficient to show that \( \text{Des}_{i,j}(\pi) = \text{Des}_{i,j}(\pi') \) implies \( \pi \equiv_{i,j} \pi' \). Indeed, let \( \pi \) and \( \pi' \) be two permutations of the same length with \( \text{Des}_{i,j}(\pi) = \text{Des}_{i,j}(\pi') \) and similarly with \( \sigma \) and \( \sigma' \), where \( \pi \) is disjoint from \( \sigma \) and \( \pi' \) is disjoint from \( \sigma' \). By (i), we can assume without loss of generality that \( \sigma \) is disjoint from \( \pi' \) as well, and thus if we have \( \pi \equiv_{i,j} \pi' \) and \( \sigma \equiv_{i,j} \sigma' \), then \( \text{Des}_{i,j}(S(\pi, \sigma)) = \text{Des}_{i,j}(S(\pi', \sigma)) = \text{Des}_{i,j}(S(\pi', \sigma')) \).

Now suppose that \( \pi \) and \( \pi' \) are in \( \mathfrak{P}_m \) with \( \text{Des}_{i,j}(\pi) = \text{Des}_{i,j}(\pi') \). We shall show that \( \pi \equiv_{i,j} \pi' \), considering three cases separately:

1. First, suppose that \( i + j \geq m - 1 \). Then \( \text{Des}(\pi) = \text{Des}(\pi') \), so \( \pi \equiv_{i,j} \pi' \) by (i).

This is because if \( i + 1 < k < m - j \), then upon shuffling \( \pi \) with any permutation \( \sigma \) disjoint from \( \pi \), the letter \( \pi_k \) cannot end up in the first \( i + 1 \) or last \( j + 1 \) positions of any element of \( S(\pi, \sigma) \).
2. Next, suppose that \( i + j \leq m - 3 \). It is enough to find permutations \( \bar{\pi} \) and \( \bar{\pi}' \) such that \( \bar{\pi} \equiv_{i,j} \pi, \bar{\pi}' \equiv_{i,j} \pi' \), and \( \text{Des}(\bar{\pi}) = \text{Des}(\bar{\pi}') \). To do this, we may choose some \( a \in \mathbb{P} \) greater than all the letters of \( \pi \) and \( \pi' \) and construct \( \bar{\pi} \) and \( \bar{\pi}' \) by replacing the letters in positions \( i + 2, i + 3, \ldots, m - j - 1 \) of both \( \pi \) and \( \pi' \) with the sequence \( a \ a + 1 \ \cdots \ a + (m - i - j - 3) \).

3. Finally, suppose that \( i + j = m - 2 \). In this case, \( \text{Des}_{i,j}(\pi) \) comprises all descents of \( \pi \) except in position \( i + 1 \), so that \( \text{Des}(\pi) \) and \( \text{Des}(\pi') \) are the same except that \( i + 1 \) may be in one but not the other. If \( \text{Des}(\pi) = \text{Des}(\pi') \) then \( \pi \equiv_{i,j} \pi' \), so let us suppose that \( i + 1 \) is a descent of exactly one of \( \pi \) and \( \pi' \). Let \( \sigma \in \mathbb{P}_n \) be disjoint from \( \pi \).

By (i), we may assume without loss of generality that no letter of \( \pi \) or \( \sigma \) has value strictly between \( \pi_{i+1} \) and \( \pi_{i+2} \). Let \( \pi^* \) be the result of switching \( \pi_{i+1} \) and \( \pi_{i+2} \) in \( \pi \). It is easy to see that switching \( \pi_{i+1} \) and \( \pi_{i+2} \) in an element \( \tau \) of \( S(\pi, \sigma) \) can change \( \text{Des}(\tau) \) only by adding or removing a single descent which is at least \( i + 1 \) and at most \( n + i + 1 = m + n - 1 - j \) and thus does not change \( \text{Des}_{i,j}(\tau) \). Thus, \( \pi^* \equiv_{i,j} \pi \). Since \( \text{Des}(\pi^*) = \text{Des}(\pi') \), we also have \( \pi^* \equiv_{i,j} \pi' \), so \( \pi \equiv_{i,j} \pi' \) as desired.

3. Shuffle algebras

3.1. Definition and basic results

Every permutation statistic \( st \) induces an equivalence relation on permutations; we say that permutations \( \pi \) and \( \sigma \) are \( st \)-equivalent if \( st(\pi) = st(\sigma) \) and \( |\pi| = |\sigma| \). We write the \( st \)-equivalence class of \( \pi \) as \( [\pi]_{st} \). For a shuffle-compatible statistic \( st \), we can then associate to \( st \) a \( \mathbb{Q} \)-algebra in the following way. First, associate to \( st \) a \( \mathbb{Q} \)-vector space by taking as a basis the \( st \)-equivalence classes of permutations. We give this vector space a multiplication by taking
\[
[\pi]_{st}[\sigma]_{st} = \sum_{\tau \in S(\pi, \sigma)} [\tau]_{st},
\]
which is well-defined (i.e., the choice of \( \pi \) and \( \sigma \) in an equivalence class does not matter) because \( st \) is shuffle-compatible. Conversely, if such a multiplication is well-defined, then \( st \) is shuffle-compatible. We denote the resulting algebra by \( A_{st} \) and call it the shuffle algebra of \( st \). Observe that \( A_{st} \) is graded, and \( [\pi]_{st} \) belongs to the \( n \)th homogeneous component of \( A_{st} \) if \( \pi \) has length \( n \).

As an example, we describe the shuffle algebra of the major index \( \text{maj} \).

**Theorem 3.1** (Shuffle-compatibility of the major index).

(a) The major index \( \text{maj} \) is shuffle-compatible.

(b) The linear map on \( A_{\text{maj}} \) defined by
\[
[\pi]_{\text{maj}} \mapsto \frac{[\text{maj}(\pi)]}{[|\pi|]_{q}} x^{|\pi|}
\]

The notion of \( st \)-equivalence should not be confused with that of “\( st \)-Wilf equivalence” [4].
is a \( \mathbb{Q} \)-algebra isomorphism from \( \mathcal{A}_{\text{maj}} \) to the span of

\[
\left\{ \frac{q^j}{[n]_q!} x^n \right\}_{n \geq 0, 0 \leq j \leq \binom{n}{2}},
\]

a subalgebra of \( \mathbb{Q}[[q]][x] \).

(c) The \( n \)th homogeneous component of \( \mathcal{A}_{\text{maj}} \) has dimension \( \binom{n}{2} + 1 \).

Proof. We know from (1.2) that \( \text{maj} \) is shuffle-compatible, so there is no need to prove (a). Let \( \phi: \mathcal{A}_{\text{maj}} \to \mathbb{Q}[[q]][x] \) denote the map given in the statement of (b). Then by (1.2), for \( \pi \in \mathfrak{S}_m \) and \( \sigma \in \mathfrak{S}_n \), we have

\[
\phi([\pi]_{\text{maj}}) \phi([\sigma]_{\text{maj}}) = \frac{q^{\text{maj}(\pi)}}{[m]_q!} \frac{q^{\text{maj}(\sigma)}}{[n]_q!} x^{m+n} = \frac{q^{\text{maj}(\pi) + \text{maj}(\sigma)}}{[m+n]_q!} x^{m+n} = \sum_{\tau \in S(\pi, \sigma)} \frac{q^{\text{maj}(\tau)}}{[m+n]_q!} x^{m+n} = \phi([\pi]_{\text{maj}}[\sigma]_{\text{maj}}),
\]

so \( \phi \) is an algebra homomorphism. The possible values for \( \text{maj}(\pi) \) for an \( n \)-permutation \( \pi \) range from 0 to \( \binom{n}{2} \), and since the elements \( q^j x^n/[n]_q! \) are linearly independent, \( \phi \) gives an isomorphism from \( \mathcal{A}_{\text{maj}} \) to the stated subalgebra, thus proving (b) and (c).

We say that two permutation statistics \( \text{st}_1 \) and \( \text{st}_2 \) are equivalent if \( [\pi]_{\text{st}_1} = [\pi]_{\text{st}_2} \) for every permutation \( \pi \). In other words, \( \text{st}_2(\pi) \) depends only on \( \text{st}_1(\pi) \) and \( |\pi| \) for every permutation \( \pi \), and vice versa. As shown in Lemma 2.2, \( \text{udr} \) and \( (\text{lpk}, \text{val}) \) are equivalent statistics. It also follows from the formula \( \text{comaj}(\pi) = n \text{des}(\pi) - \text{maj}(\pi) \) that \( (\text{des}, \text{maj}) \) and \( (\text{des}, \text{comaj}) \) are equivalent statistics.

**Theorem 3.2.** Suppose that \( \text{st}_1 \) and \( \text{st}_2 \) are equivalent statistics. If \( \text{st}_1 \) is shuffle-compatible with shuffle algebra \( \mathcal{A}_{\text{st}_1} \), then \( \text{st}_2 \) is also shuffle-compatible with shuffle algebra \( \mathcal{A}_{\text{st}_2} \) isomorphic to \( \mathcal{A}_{\text{st}_1} \).

Proof. Equivalent statistics have the same equivalence classes on permutations, so \( \mathcal{A}_{\text{st}_1} \) and \( \mathcal{A}_{\text{st}_2} \) (as vector spaces) have the same basis elements. If \( \text{st}_1 \) and \( \text{st}_2 \) are equivalent, then

\[
[\pi]_{\text{st}_2}[\sigma]_{\text{st}_2} = [\pi]_{\text{st}_1}[\sigma]_{\text{st}_1} = \sum_{\tau \in S(\pi, \sigma)} [\tau]_{\text{st}_1} = \sum_{\tau \in S(\pi, \sigma)} [\tau]_{\text{st}_2},
\]

which proves the result. \( \square \)
For example, it is easy to see that Des$_{1,0}$ is equivalent to sir and that Des$_{0,1}$ is equivalent to sfr. Thus, Theorem 2.9 implies that sir and sfr are shuffle-compatible as well.

We say that $st_1$ is a refinement of $st_2$ if for all permutations $\pi$ and $\sigma$ of the same length, $st_1(\pi) = st_1(\sigma)$ implies $st_2(\pi) = st_2(\sigma)$. For example, the statistics of which the descent set is a refinement are exactly what we call descent statistics.

**Theorem 3.3.** Suppose that $st_1$ is shuffle-compatible and is a refinement of $st_2$. Let $A$ be a $\mathbb{Q}$-algebra with basis $\{u_\alpha\}$ indexed by $st_2$-equivalence classes $\alpha$, and suppose that there exists a $\mathbb{Q}$-algebra homomorphism $\phi: A_{st_1} \to A$ such that for every $st_1$-equivalence class $\beta$, we have $\phi(\beta) = u_\alpha$ where $\alpha$ is the $st_2$-equivalence class containing $\beta$. Then $st_2$ is shuffle-compatible and the map $u_\alpha \mapsto \alpha$ extends by linearity to an isomorphism from $A$ to $A_{st_2}$.

**Proof.** It is sufficient to show that for any two disjoint permutations $\pi$ and $\sigma$, we have

$$u_{[\pi]_{st_2}} u_{[\sigma]_{st_2}} = \sum_{\tau \in S(\pi, \sigma)} u_{[\tau]_{st_2}}.$$

To see this, we have

$$u_{[\pi]_{st_2}} u_{[\sigma]_{st_2}} = \phi([\pi]_{st_1}) \phi([\sigma]_{st_1})$$

$$= \phi([\pi]_{st_1} [\sigma]_{st_1})$$

$$= \phi\left( \sum_{\tau \in S(\pi, \sigma)} [\tau]_{st_1} \right)$$

$$= \sum_{\tau \in S(\pi, \sigma)} u_{[\tau]_{st_2}}. \quad \square$$

### 3.2. Basic symmetries yield isomorphic shuffle algebras

Here we consider three involutions on permutations given by symmetries—reversion, complementation, and reverse-complementation—and their implications for the shuffle-compatibility of permutation statistics.

Given $\pi = \pi_1 \pi_2 \cdots \pi_n \in \mathcal{P}_n$, we define the reversal $\pi^r$ of $\pi$ to be $\pi^r := \pi_n \pi_{n-1} \cdots \pi_1$, the complement $\pi^c$ of $\pi$ to be the permutation obtained by (simultaneously) replacing the $i$th smallest letter in $\pi$ with the $i$th largest letter in $\pi$ for all $1 \leq i \leq n$, and the reverse-complement $\pi^{rc}$ of $\pi$ to be $\pi^{rc} := (\pi^r)^c = (\pi^c)^r$. For example, given $\pi = 139264$, we have $\pi^r = 462931$, $\pi^c = 941623$, and $\pi^{rc} = 326149$.

More generally, let $f$ be an involution on the set of permutations which preserves the length of a permutation. Then let $\pi^f$ denote $f(\pi)$. Given a set $X$ of permutations, let

$$X^f := \{ \pi^f : \pi \in X \},$$

so $f$ naturally induces an involution on sets of permutations as well.

We say that two permutation statistics $st_1$ and $st_2$ are $f$-equivalent if $st_1 \circ f$ is equivalent to $st_2$. Equivalently, $st_1$ and $st_2$ are $f$-equivalent if $([\pi^f]_{st_1})^f = [\pi]_{st_2}$ for all $\pi$. It is easy to verify that $st_1(\pi^f) = st_2(\pi)$ implies that $st_1$ and $st_2$ are $f$-equivalent (although this is not a necessary condition).
For example, Lpk and Rpk are r-equivalent, pk and val are c-equivalent, Pk and Val are c-equivalent, (pk, des) and (val, des) are rc-equivalent, and maj and comaj are rc-equivalent. It is less obvious that (lpk, val) and (lpk, pk) are rc-equivalent, so we provide a proof below.

**Proposition 3.4.** (lpk, val) and (lpk, pk) are rc-equivalent statistics.

**Proof.** Fix a permutation \( \pi \). We divide into four cases: (a) \( \pi \) has a short initial run and a long final run, (b) \( \pi \) has a short initial run and a short final run, (c) \( \pi \) has a long initial run and a long final run, and (d) \( \pi \) has a long initial run and short final run. In case (a), we know from Lemma 2.2 that \( \text{lpk}(\pi) = \text{val}(\pi) \). Then \( \text{pk}(\pi^{rc}) = \text{val}(\pi) \), and \( \pi^{rc} \) has a long initial run, so

\[
\text{lpk}(\pi^{rc}) = \text{pk}(\pi^{rc}) = \text{val}(\pi) = \text{lpk}(\pi).
\]

Thus, \( (\text{lpk, val})(\pi) = (\text{lpk, pk})(\pi^{rc}) \). The other three cases can be verified in the same way. \( \square \)

Let us say that \( f \) is **shuffle-compatibility-preserving** if for every pair of disjoint permutations \( \pi \) and \( \sigma \), there exist disjoint permutations \( \hat{\pi} \) and \( \hat{\sigma} \) with the same relative order as \( \pi \) and \( \sigma \), respectively, such that \( S(\hat{\pi}^f, \hat{\sigma}^f) = S(\pi, \sigma)^f \) and \( S(\pi^f, \sigma^f) = S(\hat{\pi}, \hat{\sigma})^f \).

We note that f-equivalences are not actually equivalence relations on statistics (although they are symmetric), but we shall show that if the statistics are shuffle-compatible and \( f \) is shuffle-compatibility-preserving, then \( f \)-equivalences induce isomorphisms on the corresponding shuffle algebras.

**Theorem 3.5.** Let \( f \) be shuffle-compatibility-preserving, and suppose that \( \text{st}_1 \) and \( \text{st}_2 \) are \( f \)-equivalent statistics. If \( \text{st}_1 \) is shuffle-compatible with shuffle algebra \( A_{\text{st}_1} \), then \( \text{st}_2 \) is also shuffle-compatible with shuffle algebra \( A_{\text{st}_2} \) isomorphic to \( A_{\text{st}_1} \).

**Proof.** Let \( \pi \) and \( \bar{\pi} \) be permutations in the same \( \text{st}_2 \)-equivalence class and similarly with \( \sigma \) and \( \bar{\sigma} \), such that \( \pi \) and \( \sigma \) are disjoint and \( \hat{\pi} \) and \( \hat{\sigma} \) are disjoint. Since \( \text{st}_1 \) and \( \text{st}_2 \) are \( f \)-equivalent, it follows that

\[
([\pi^f]_{\text{st}_1})^f = [\pi]_{\text{st}_2} = [\bar{\pi}]_{\text{st}_2} = ([\bar{\pi}^f]_{\text{st}_1})^f.
\]

Hence \( [\pi^f]_{\text{st}_1} = [\bar{\pi}^f]_{\text{st}_1} \) and similarly \( [\sigma^f]_{\text{st}_1} = [\bar{\sigma}^f]_{\text{st}_1} \).

Since \( f \) is shuffle-compatibility-preserving, there exist permutations \( \hat{\pi}, \hat{\sigma}, \tilde{\pi}, \) and \( \tilde{\sigma} \)—having the same relative order as \( \pi, \sigma, \bar{\pi}, \) and \( \bar{\sigma} \), respectively—satisfying \( S(\hat{\pi}^f, \hat{\sigma}^f) = S(\pi, \sigma)^f \), \( S(\pi^f, \sigma^f) = S(\hat{\pi}, \hat{\sigma})^f \), \( S(\tilde{\pi}^f, \tilde{\sigma}^f) = S(\tilde{\pi}, \tilde{\sigma})^f \), and \( S(\tilde{\pi}^f, \tilde{\sigma}^f) = S(\tilde{\pi}, \tilde{\sigma})^f \). By the “same relative order” property, we have

\[
[\hat{\pi}^f]_{\text{st}_1} = [\pi^f]_{\text{st}_4} = [\bar{\pi}^f]_{\text{st}_1} = [\hat{\bar{\pi}}^f]_{\text{st}_1}
\]

and

\[
[\hat{\sigma}^f]_{\text{st}_1} = [\sigma^f]_{\text{st}_4} = [\bar{\sigma}^f]_{\text{st}_1} = [\hat{\bar{\sigma}}^f]_{\text{st}_1}.
\]

Now, by shuffle-compatibility of \( \text{st}_1 \), we have the equality of multisets

\[
\{ \text{st}_1(\tau) : \tau \in S(\hat{\pi}^f, \hat{\sigma}^f) \} = \{ \text{st}_1(\tau) : \tau \in S(\hat{\bar{\pi}}^f, \hat{\bar{\sigma}}^f) \},
\]
which is equivalent to

$$\{ \text{st}_2(\tau) : \tau \in S(\hat{\pi}^f, \hat{\sigma}^f) \} = \{ \text{st}_2(\tau) : \tau \in S(\hat{\pi}^f, \hat{\sigma}^f) \}$$

by $f$-equivalence of $\text{st}_1$ and $\text{st}_2$, and from $S(\hat{\pi}^f, \hat{\sigma}^f) = S(\pi, \sigma)^f$ and $S(\hat{\pi}^f, \hat{\sigma}^f) = S(\pi, \sigma)^f$, we have

$$\{ \text{st}_2(\tau) : \tau \in S(\pi, \sigma) \} = \{ \text{st}_2(\tau) : \tau \in S(\pi, \sigma) \}.$$

Therefore, $\text{st}_2$ is shuffle-compatible.

It remains to prove that $A_{\text{st}_2}$ is isomorphic to $A_{\text{st}_1}$. Observe that

$$\sum_{\tau \in S(\hat{\pi}, \hat{\sigma})} [\tau]_{\text{st}_2} = \sum_{\tau \in S(\pi, \sigma)} [\tau]_{\text{st}_2},$$

since $\text{st}_2$ is shuffle-compatible. Define the linear map $\varphi_f : A_{\text{st}_2} \to A_{\text{st}_1}$ by $[\pi]_{\text{st}_2} \mapsto [\pi^f]_{\text{st}_1}$.

Then

$$\varphi_f([\pi]_{\text{st}_2} [\sigma]_{\text{st}_2}) = \varphi_f \left( \sum_{\tau \in S(\pi, \sigma)} [\tau]_{\text{st}_2} \right)$$

$$= \sum_{\tau \in S(\pi, \sigma)} \varphi_f([\tau]_{\text{st}_2})$$

$$= \sum_{\tau \in S(\pi, \sigma)} [\tau^f]_{\text{st}_1}$$

$$= \sum_{\tau \in S(\pi, \sigma)} [\tau^f]_{\text{st}_1}$$

$$= \sum_{\tau \in S(\pi, \sigma)^f} [\tau]_{\text{st}_1}$$

$$= [\pi^f]_{\text{st}_1} [\sigma^f]_{\text{st}_1}$$

$$= \varphi_f([\pi]_{\text{st}_2}) \varphi_f([\sigma]_{\text{st}_2}),$$

so $\varphi_f$ is an isomorphism from $A_{\text{st}_2}$ to $A_{\text{st}_1}$.

\[ \square \]

**Lemma 3.6.** Reversion, complementation, and reverse-complementation are shuffle-compatibility-preserving.

**Proof.** It is clear that $S(\pi^r, \sigma^r) = S(\pi, \sigma)^r$, so by taking $\hat{\pi} = \pi$ and $\hat{\sigma} = \sigma$, the equalities $S(\hat{\pi}^r, \hat{\sigma}^r) = S(\pi, \sigma)^r$ and $S(\pi^r, \sigma^r) = S(\pi, \sigma)^r$ come for free. Thus reversion is shuffle-compatibility-preserving.

Unlike with reversion, it is not true in general that $S(\pi^c, \sigma^c) = S(\pi, \sigma)^c$. For disjoint permutations $\pi = \pi_1 \pi_2 \cdots \pi_m$ and $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$, let $P = \{\pi_1, \ldots, \pi_m, \sigma_1, \ldots, \sigma_n\}$ be the set of letters appearing in $\pi$ and $\sigma$, and let $\rho : P \to P$ be the map sending the $i$th smallest letter of $P$ to the $i$th largest letter of $P$ for every $i$. By an abuse of notation, let $\rho(\pi)$ denote the permutation $\rho(\pi_1)\rho(\pi_2)\cdots \rho(\pi_m)$ obtained by applying $\rho$ to each letter in $\pi$. Then, let
\( \hat{\pi} = \rho(\pi^c) \) and \( \hat{\sigma} = \rho(\sigma^c) \). For example, let \( \pi = 413 \) and \( \sigma = 25 \). Then \( P = [5] \), \( \pi^c = 143 \), and \( \sigma^c = 52 \), and so \( \hat{\pi} = 523 \) and \( \hat{\sigma} = 14 \). Clearly, \( \pi \) has the same relative order as \( \hat{\pi} \), and similarly with \( \sigma \) and \( \hat{\sigma} \). It is also easy to see that \( \rho(\pi) = \hat{\pi}^c = \hat{\pi} \) and \( \rho(\sigma) = \hat{\sigma}^c = \hat{\sigma} \).

To see that \( S(\hat{\pi}^c, \hat{\sigma}^c) = S(\pi, \sigma)^c \), first let \( \tau \in S(\pi, \sigma) \). Then \( \tau \) contains both \( \pi \) and \( \sigma \) as subsequences, and to show that \( \tau^c \in S(\hat{\pi}^c, \hat{\sigma}^c) \), it suffices to show that \( \tau^c \) contains both \( \hat{\pi}^c = \rho(\pi) \) and \( \hat{\sigma}^c = \rho(\sigma) \) as subsequences. However, this follows from the fact that, when taking the complement of \( \tau \), the subsequence \( \pi \) appearing in \( \tau \) is transformed into \( \rho(\pi) \), and similarly \( \sigma \) turns into \( \rho(\sigma) \). The other inclusion follows by the same reasoning, and the equality \( S(\pi^r, \sigma^r) = S(\hat{\pi}, \hat{\sigma})^c \) follows directly from \( S(\hat{\pi}^c, \hat{\sigma}^c) = S(\pi, \sigma)^c \) and replacing \( \pi \) and \( \sigma \) with \( \pi^c \) and \( \sigma^c \), respectively. Hence complementation is shuffle-compatibility-preserving.

Finally, the equalities \( S(\pi^r, \sigma^r) = S(\pi, \sigma)^r \), \( S(\pi^c, \sigma^c) = S(\pi, \sigma)^c \), and \( S(\pi^c, \sigma^c) = S(\hat{\pi}, \hat{\sigma})^c \) imply \( S(\hat{\pi}^c, \hat{\sigma}^c) = S(\pi, \sigma)^rc \) and \( S(\pi^r, \sigma^r) = S(\hat{\pi}, \hat{\sigma})^rc \). Thus reverse-complementation is shuffle-compatibility-preserving.

**Corollary 3.7.** Suppose that \( st_1 \) and \( st_2 \) are \( r \)-equivalent, \( c \)-equivalent, or \( rc \)-equivalent statistics. If \( st_1 \) is shuffle-compatible with shuffle algebra \( A_{st_1} \), then \( st_2 \) is also shuffle-compatible with shuffle algebra \( A_{st_2} \) isomorphic to \( A_{st_1} \).

For example, since \( \text{maj} \) and \( \text{comaj} \) are \( rc \)-equivalent, it follows from Theorem 3.1 and Corollary 3.7 that \( \text{comaj} \) is shuffle-compatible and that its shuffle algebra \( A_{\text{comaj}} \) is isomorphic to \( A_{\text{maj}} \).

### 3.3. A note on Hadamard products

The operation of **Hadamard product** \( * \) on formal power series in \( t \) is given by

\[
\left( \sum_{n=0}^{\infty} a_n t^n \right) \ast \left( \sum_{n=0}^{\infty} b_n t^n \right) := \sum_{n=0}^{\infty} a_n b_n t^n.
\]

Many shuffle algebras that we study in this paper can be characterized as subalgebras of various algebras in which the multiplication is the Hadamard product in a variable \( t \). In the notation for these algebras, we write \( t^{*} \) to indicate that multiplication is the Hadamard product in \( t \). For example, \( \mathbb{Q}[[t^{*}, q]] [x] \) is the algebra of polynomials in \( x \) whose coefficients are formal power series in \( t \) and \( q \), where multiplication is ordinary multiplication in the variables \( x \) and \( q \) but is the Hadamard product in \( t \).

We note that the Hadamard product is only used in descriptions of shuffle algebras and in the proof of Lemma 5.4 where \( t^m \ast t^n \) denotes the Hadamard product of \( t^m \) and \( t^n \). (Here, \( t^m \) is the ordinary product of \( m \) copies of \( t \) and similarly with \( t^n \).) All other expressions should be interpreted as using ordinary multiplication. For instance, any expression with an exponent such as \( t^k \) or \( (1 + yt)^k \) is ordinary multiplication, and \( (1 - tf)^{-1} \) (as in Corollary 5.5) denotes \( \sum_{k=0}^{\infty} t^k f^k \).

### 4. Quasisymmetric functions and shuffle-compatibility

#### 4.1. The descent set shuffle algebra QSym

A formal power series \( f \in \mathbb{Q}[[x_1, x_2, \ldots]] \) of bounded degree in countably many commuting variables \( x_1, x_2, \ldots \) is called a **quasisymmetric function** if for any positive integers \( a_1, a_2, \ldots, a_k, \ldots \)
if \( i_1 < i_2 < \cdots < i_k \) and \( j_1 < j_2 < \cdots < j_k \), then

\[
[x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}] f = [x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}] f.
\]

It is clear that every symmetric function is quasisymmetric, but not every quasisymmetric function is symmetric. For example, \( \sum_{i<j<k} x_i^2 x_j x_k \) is quasisymmetric, but it is not symmetric because \( x_1^2 x_2 x_3 \) appears as a term yet \( x_1^2 x_2^2 x_3 \) does not.

Let \( L \models n \) indicate that \( L \) is a composition of \( n \), and let \( \text{QSym}_n \) be the set of quasisymmetric functions homogeneous of degree \( n \), which is clearly a vector space. For a composition \( L = (L_1, L_2, \ldots, L_k) \), the monomial quasisymmetric function \( M_L \) is defined by

\[
M_L := \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{L_1} x_{i_2}^{L_2} \cdots x_{i_k}^{L_k}
\]

It is clear that \( \{M_L\}_{L \models n} \) is a basis for \( \text{QSym}_n \), so for \( n \geq 1 \), \( \text{QSym}_n \) has dimension \( 2^n - 1 \), the number of compositions of \( n \).

Another important basis for \( \text{QSym}_n \) (and the most important basis for our purposes) is the basis of fundamental quasisymmetric functions \( \{F_L\}_{L \models n} \) given by

\[
F_L := \sum_{i_1 \leq i_2 \leq \cdots \leq i_n \atop i_j < i_{j+1} \text{ if } j \in \text{Des}(L)} x_{i_1} x_{i_2} \cdots x_{i_n}.
\]

It is easy to see that

\[
F_L = \sum_{\text{Des}(K) \supseteq \text{Des}(L) \atop |K| = |L|} M_K,
\]

so by inclusion-exclusion, \( M_K \) can be expressed as a linear combination of the \( F_L \). It follows that \( \{F_L\}_{L \models n} \) spans \( \text{QSym}_n \), so this set must be a basis for \( \text{QSym}_n \) since it has the correct number of elements.

The product of two quasisymmetric functions is quasisymmetric, with the product formula for the fundamental basis given by the following theorem, which may be proved using \( P \)-partitions; see [27, Exercise 7.93]. This theorem may also be derived from Lemma 2.8.

**Theorem 4.1.** Let \( c^J_{L,K} \) be the number of permutations with descent composition \( L \) among the shuffles of a permutation \( \pi \) with descent composition \( J \) and a permutation \( \sigma \) (disjoint from \( \pi \)) with descent composition \( K \). Then

\[
F_J F_K = \sum_L c^J_{L,K} F_L.
\]

If \( f \in \text{QSym}_m \) and \( g \in \text{QSym}_n \), then \( fg \in \text{QSym}_{m+n} \). Thus \( \text{QSym} := \bigoplus_{n=0}^\infty \text{QSym}_n \) is a graded \( \mathbb{Q} \)-algebra called the algebra of quasisymmetric functions with coefficients in \( \mathbb{Q} \), a subalgebra of \( \mathbb{Q}[[x_1, x_2, \ldots]] \). Motivated by Richard Stanley’s theory of \( P \)-partitions, the first author introduced quasisymmetric functions in [7] and developed the basic algebraic properties of \( \text{QSym} \). Further properties of \( \text{QSym} \) and connections with many topics of study in combinatorics and algebra were developed in the subsequent decades. Basic references include [27, Section 7.19], [14, Section 5], and [17].

Observe that Theorem 4.1 implies that \( \text{QSym} \) is isomorphic to the shuffle algebra for the descent set with the fundamental basis corresponding to the basis of Des-equivalence classes.
Corollary 4.2 (Shuffle-compatibility of the descent set).

(a) The descent set $\text{Des}$ is shuffle-compatible.

(b) The linear map on $A_{\text{Des}}$ defined by

$$[\pi]_{\text{Des}} \mapsto F_{\text{Comp}(\pi)}$$

is a $\mathbb{Q}$-algebra isomorphism from $A_{\text{Des}}$ to $\text{QSym}$.

Now, let $st$ be a descent statistic. Then not only does $st$ induce an equivalence relation on permutations, but it also induces an equivalence relation on compositions because permutations with the same descent composition are necessarily $st$-equivalent.

We establish a necessary and sufficient condition for the shuffle-compatibility of a descent statistic, which will also imply that the shuffle algebra of any shuffle-compatible descent statistic is isomorphic to a quotient of $\text{QSym}$.

Theorem 4.3. A descent statistic $st$ is shuffle-compatible if and only if there exists a $\mathbb{Q}$-algebra homomorphism $\phi_{st} : \text{QSym} \to A$, where $A$ is a $\mathbb{Q}$-algebra with basis $\{u_\alpha\}$ indexed by $st$-equivalence classes $\alpha$ of compositions, such that $\phi_{st}(F_L) = u_\alpha$ whenever $L \in \alpha$. In this case, the linear map on $A_{st}$ defined by

$$[\pi]_{st} \mapsto u_\alpha,$$

where $\text{Comp}(\pi) \in \alpha$, is a $\mathbb{Q}$-algebra isomorphism from $A_{st}$ to $A$.

Proof. Suppose that $st$ is a shuffle-compatible descent statistic. Let $A = A_{st}$ be the shuffle algebra of $st$, and let $u_\alpha = [\pi]_{st}$ for any $\pi$ satisfying $\text{Comp}(\pi) \in \alpha$, so that

$$u_\beta u_\gamma = \sum_\alpha c^\alpha_{\beta, \gamma} u_\alpha$$

where $c^\alpha_{\beta, \gamma}$ is the number of permutations with descent composition in $\alpha$ that are obtained as a shuffle of a permutation $\pi$ with descent composition in $\beta$ and a permutation $\sigma$ (disjoint from $\pi$) with descent composition in $\gamma$. Observe that $c^\alpha_{\beta, \gamma} = \sum_{L \in \alpha} c^L_{j,k}$ for any choice of $J \in \beta$ and $K \in \gamma$, where as before $c^L_{j,k}$ is the number of permutations with descent composition $L$ that are obtained as a shuffle of a permutation $\pi$ with descent composition $J$ and a permutation $\sigma$ (disjoint from $\pi$) with descent composition $K$.

Define the linear map $\phi_{st} : \text{QSym} \to A$ by $\phi_{st}(F_L) = u_\alpha$ for $L \in \alpha$. Then any $J \in \beta$ and $K \in \gamma$ satisfy

$$\phi_{st}(F_J F_K) = \phi_{st} \left( \sum_L c^L_{j,k} F_L \right)$$

$$= \sum_L c^L_{j,k} \phi_{st}(F_L)$$

$$= \sum_\alpha \sum_{L \in \alpha} c^L_{j,k} u_\alpha$$

$$= \sum_\alpha c^\alpha_{\beta, \gamma} u_\alpha$$

$$= u_\beta u_\gamma$$

$$= \phi_{st}(F_J) \phi_{st}(F_K),$$

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so $\phi_{st}$ is a $\mathbb{Q}$-algebra homomorphism, thus completing one direction of the proof. The converse follows directly from Theorem 3.3.

It is immediate from Theorem 4.3 that when $st$ is shuffle-compatible, its shuffle algebra is isomorphic to $\mathbb{Q}\text{Sym} / \ker(\phi_{st})$.

**Corollary 4.4.** The shuffle algebra of every shuffle-compatible descent statistic is isomorphic to a quotient algebra of $\mathbb{Q}\text{Sym}$.

### 4.2. Shuffle-compatibility of des and $(\text{des, maj})$

We now use Theorem 4.3 to characterize the shuffle algebras of the two other shuffle-compatible statistics mentioned in the introduction: the descent number $\text{des}$ and the pair $(\text{des, maj})$. For the latter, we will actually characterize the shuffle algebra of $(\text{des, comaj})$, but this is sufficient by Theorem 3.2 since $(\text{des, maj})$ and $(\text{des, comaj})$ are equivalent statistics. We note that these characterizations can be derived from Propositions 8.3 and 12.6 of Stanley [25] in a related way, though we emphasize the connection with quasisymmetric functions. We will first prove the result for $(\text{des, comaj})$ and then derive from it the result for $\text{des}$ using Theorem 3.3.

We denote the set of non-negative integers by $\mathbb{N}$.

**Theorem 4.5** (Shuffle-compatibility of $(\text{des, comaj})$).

(a) The ordered pair $(\text{des, comaj})$ is shuffle-compatible.

(b) The linear map on $A_{(\text{des, comaj})}$ defined by

$$
[\pi]_{(\text{des, comaj})} \mapsto q^{\text{comaj}(\pi)} \left( p - \text{des}(\pi) + |\pi| - 1 \right) x^{|\pi|} \mod q
$$

is a $\mathbb{Q}$-algebra isomorphism from $A_{(\text{des, comaj})}$ to the span of

$$
\{1\} \cup \left\{ q^{k} \left( p - j + n - 1 \right) x^{n} \right\}_{n \geq 1, 0 \leq j \leq n-1, \left(\frac{j+1}{2}\right) \leq k \leq n - \left(\frac{j+1}{2}\right)},
$$

a subalgebra of $\mathbb{Q}[q, x]^{\mathbb{N}}$, the algebra of functions $\mathbb{N} \to \mathbb{Q}[q, x]$ in the non-negative integer variable $p$.

(c) The linear map on $A_{(\text{des, comaj})}$ defined by

$$
[\pi]_{(\text{des, comaj})} \mapsto \begin{cases} 
q^{\text{comaj}(\pi)} t^{\text{des}(\pi) + 1} / (1 - t), & \text{if } |\pi| = 0, \\
\left(1 - t\right) (1 - qt) \cdots \left(1 - q^{|\pi|} t\right) x^{|\pi|}, & \text{if } |\pi| \geq 1,
\end{cases}
$$

is a $\mathbb{Q}$-algebra isomorphism from $A_{(\text{des, comaj})}$ to the span of

$$
\left\{ \frac{1}{1 - t} \right\} \cup \left\{ q^{k} t^{j+1} \left(1 - t\right) (1 - qt) \cdots \left(1 - q^{n} t\right) x^{n} \right\}_{n \geq 1, 0 \leq j \leq n-1, \left(\frac{j+1}{2}\right) \leq k \leq n - \left(\frac{j+1}{2}\right)},
$$

a subalgebra of $\mathbb{Q}[[t^*, q]][x]$. 

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(d) For $n \geq 1$, the $n$th homogeneous component of $A_{\text{des, comaj}}$ has dimension $\binom{n}{3} + n$.

Proof. We first prove parts (a) and (b). For $p$ a positive integer and $f$ a quasisymmetric function, let
\[
\phi^{(p)}_{\text{comaj, des}}(f) = f(x, qx, \ldots, q^{p-1}x)
\]
and let $\phi^{(0)}_{\text{comaj, des}}(f)$ be the constant term in $f$. It is clear that $\phi^{(p)}_{\text{comaj, des}}$ is a homomorphism from $\text{QSym}$ to $\mathbb{Q}[q, x]$, so the map that takes $f$ to the function $p \mapsto f(x, qx, \ldots, q^{p-1}x)$ is a homomorphism from $\text{QSym}$ to $\mathbb{Q}[q, x]^N$.

If $L$ is a composition of $n \geq 1$, then
\[
F_L(x, qx, \ldots, q^{p-1}x) = \sum_{0 \leq i_1 \leq \cdots \leq i_n \leq p-1 \atop i_j < i_{j+1} \text{ if } j \in \text{Des}(L)} q^{i_1 + \cdots + i_n - n} x^n
\]
where
\[
r_j = i_j - |\{ k : k \in \text{Des}(L) \text{ and } k < j \}|
\]
and
\[
e(L) = \sum_{j=1}^{n} |\{ k : k \in \text{Des}(L) \text{ and } k < j \}| = \text{comaj}(L).
\]
Since
\[
\sum_{0 \leq r_1 \leq \cdots \leq r_n \leq p-1 - \text{des}(L)} q^{r_1 + \cdots + r_n} = \binom{p - \text{des}(L) + n - 1}{n}_q
\]
[26, Proposition 1.7.3], it follows that
\[
\phi^{(p)}_{\text{comaj, des}}(F_L) = q^{\text{comaj}(L)} \left( p - \text{des}(L) + n - 1 \right)_q^{\text{comaj}(L)} x^n,
\]
and for $n = 0$ we have $\phi^{(p)}_{\text{comaj, des}}(F_{\varnothing}) = 1$.

Furthermore, it follows from the formula between equations (1.86) and (1.87) in [26] (a form of the $q$-binomial theorem) that
\[
\sum_{p=0}^{\infty} \left( \binom{p - \text{des}(L) + n - 1}{n}_q \right) t^p = \sum_{p=0}^{\infty} \left( \binom{p + n}{n}_q \right) t^{p + \text{des}(L) + 1} = \frac{t^{\text{des}(L) + 1}}{(1 - t)(1 - qt) \cdots (1 - q^n t)}.
\]
Equation (4.3) implies that the functions $q^k \binom{p - j + n - 1}{n}_q x^n$ are linearly independent as their generating functions are clearly linearly independent. Then parts (a) and (b) follow from Theorem 4.3 and Proposition 2.4.

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To prove (c), we define the map \( \psi : \mathbb{Q}[q, x]^N \to \mathbb{Q}[q, x][[t^*]] \) by the formula

\[
\psi(f) = \sum_{p=0}^{\infty} f(p)t^p.
\]

Then \( \psi \) is clearly an isomorphism and by (4.3), the images of the basis elements in (b) are those given in (c), which are in \( \mathbb{Q}[[t^*, q]][x] \). For \( n \geq 1 \), the number of \((\text{des}, \text{comaj})\)-equivalence classes for \( n \)-permutations is

\[
\sum_{j=0}^{n-1} \left( \left( nj - \frac{(j + 1)}{2} \right) \right) - \left( \frac{j + 1}{2} \right) + 1 = \sum_{j=0}^{n-1} \left( nj - 2 \left( \frac{j + 1}{2} \right) + 1 \right),
\]

which can be shown to be equal to \( \binom{n}{3} + n \) by a routine argument. This proves (d).

\[\square\]

**Theorem 4.6** (Shuffle-compatibility of the descent number).

(a) The descent number \( \text{des} \) is shuffle-compatible.

(b) The linear map on \( A_{\text{des}} \) defined by

\[
[\pi]_{\text{des}} \mapsto \left( p - \text{des}(\pi) + |\pi| - 1 \right) x^{|\pi|}
\]

is a \( \mathbb{Q} \)-algebra isomorphism from \( A_{\text{des}} \) to the span of

\[
\{1\} \cup \left\{ \left( p - j + n - 1 \right) x^n \right\}_{n \geq 1, 0 \leq j \leq n - 1},
\]

a subalgebra of \( \mathbb{Q}[p, x] \).

(c) \( A_{\text{des}} \) is isomorphic to the span of

\[
\{1\} \cup \{ p^j x^n \}_{n \geq 1, 1 \leq j \leq n},
\]

a subalgebra of \( \mathbb{Q}[p, x] \).

(d) The linear map on \( A_{\text{des}} \) defined by

\[
[\pi]_{\text{des}} \mapsto \begin{cases} 
\frac{t^{\text{des}(\pi) + 1}}{(1 - t)^{|\pi| + 1}} x^{|\pi|}, & \text{if } |\pi| \geq 1, \\
1/(1 - t), & \text{if } |\pi| = 0,
\end{cases}
\]

is a \( \mathbb{Q} \)-algebra isomorphism from \( A_{\text{des}} \) to the span of

\[
\left\{ \frac{1}{1 - t} \right\} \cup \left\{ \frac{t^{j + 1} x^n}{(1 - t)^{n + 1}} \right\}_{n \geq 1, 0 \leq j \leq n - 1},
\]

a subalgebra of \( \mathbb{Q}[[t^*]][x] \).

(e) For \( n \geq 1 \), the \( n \)th homogeneous component of \( A_{\text{des}} \) has dimension \( n \).

**Proof.** Applying Theorem 3.3 to Theorem 4.5 with the homomorphism that takes \( q \) to 1, together with the observation that polynomial functions in characteristic zero may be identified with polynomials, yields (a), (b), and (d). Parts (c) and (e) follow easily from (b). \[\square\]
4.3. Shuffle-compatibility of the peak set and peak number

In [28], Stembridge defined a subalgebra Π of QSym called the “algebra of peaks” using enriched $P$-partitions, a variant of Stanley’s $P$-partitions. Here, we observe that Stembridge’s algebra Π is isomorphic to the shuffle algebra $A_{P_k}$ of the peak set $P_k$, thus showing that $P_k$ is shuffle-compatible, and we use further results of Stembridge on enriched $P$-partitions to show that the peak number $p_k$ is shuffle-compatible and to characterize its shuffle algebra.

An enriched $P$-partition is a map defined for a poset $P$, but for our purposes, we only need to consider the case where $P$ is a chain. Then the notion of an enriched $P$-partition is equivalent to that of an “enriched $\pi$-partition” for a permutation $\pi$, which we define below.\(^6\)

Let $P'$ denote the set of nonzero integers with the following total ordering:

$$-1 \prec +1 \prec -2 \prec +2 \prec -3 \prec +3 \prec \cdots.$$  

For $\pi \in \mathfrak{P}_n$, an enriched $\pi$-partition is a map $f : [n] \rightarrow P'$ such that for all $i < j$ in $[n]$, the following hold:

1. $f(i) \leq f(j)$;
2. $f(i) = f(j) > 0$ implies $\pi(i) < \pi(j)$;
3. $f(i) = f(j) < 0$ implies $\pi(i) > \pi(j)$.

Let $\mathcal{E}(\pi)$ denote the set of enriched $\pi$-partitions, and let

$$\Gamma(\pi) := \sum_{f \in \mathcal{E}(\pi)} x_{|f(1)|} x_{|f(2)|} \cdots x_{|f(n)|}$$

be the generating function for enriched $\pi$-partitions in which both $k$ and $-k$ receive the same weight $x_k$. For example, let $\pi = 3125674$. Then the map $f$ given by $f(1) = -1, f(2) = -1, f(3) = -3, f(4) = 3, f(5) = 3, f(6) = -7, f(7) = 9$ is an enriched $\pi$-partition, which contributes $x_1^2 x_3^3 x_7 x_9$ to $\Gamma(\pi)$.

It is clear that $\Gamma(\pi)$ is a quasisymmetric function homogeneous of degree $n$ which depends only on the descent set of $\pi$, but a stronger statement is true: $\Gamma(\pi)$ depends only on the peak set of $\pi$ [28, Proposition 2.2]. Hence, it makes sense to define the quasisymmetric function

$$K_{n,\Lambda} := \Gamma(\pi)$$

where $\pi$ is any $n$-permutation with $P_k(\pi) = \Lambda$. These peak quasisymmetric functions $K_{n,\Lambda}$ are linearly independent over $\mathbb{Q}$ [28, Theorem 3.1(a)].

Let $F_n$ be the $n$th Fibonacci number defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. It is easy to see that, for $n \geq 1$, there are exactly $F_n$ peak sets among all $n$-permutations, so the $\mathbb{Q}$-vector space $\Pi_n$ spanned by the $K_{n,\Lambda}$ has dimension $F_n$ with basis elements corresponding to peak sets of $n$-permutations. The peak quasisymmetric functions $K_{n,\Lambda}$ multiply by the rule

$$K_{m,\text{P}_k(\pi)}K_{n,\text{P}_k(\sigma)} = \sum_{\tau \in S(\pi,\sigma)} K_{m+n,\text{P}_k(\tau)}$$

\(^6\)We note that, in the notation of [28], we are setting $A = \mathbb{P}$, $\gamma = \pi$, and $P = ([n], \prec)$. 

Equation (3.1)], so \( \Pi := \bigoplus_{n=0}^{\infty} \Pi_n \) is a \( \mathbb{Q} \)-algebra, the *algebra of peaks*. Then the shuffle-compatibility of \( P_k \) and our characterization of the shuffle algebra \( A_{P_k} \) is immediate from (4.4).

**Theorem 4.7** (Shuffle-compatibility of the peak set).

(a) *The peak set* \( P_k \) *is shuffle-compatible.*

(b) *The linear map on* \( A_{P_k} \) *defined by*

\[
[\pi]_{P_k} \mapsto \frac{2^{2\pi+1}t^{\pi+1}(1+t)^{|\pi|-2\pi-1}}{(1-t)^{|\pi|+1}}x^{|\pi|}, \quad \text{if } |\pi| \geq 1,
\]

\[
\frac{1}{1-t}, \quad \text{if } |\pi| = 0,
\]

*is a \( \mathbb{Q} \)-algebra isomorphism from* \( A_{P_k} \) *to* \( \Pi \).

By Corollary 3.7, the valley set \( V \) is also shuffle-compatible and \( A_{V} \) is isomorphic to \( \Pi \). Note that (4.4) implies that the map \( F_t \mapsto K_{n,P_k(L)} \) is a \( \mathbb{Q} \)-algebra homomorphism from \( \text{QSym} \) to itself, a fact that we shall use in the proof of the next theorem, which is the analogous result for the peak number (and by Lemma 2.1, Corollary 3.7 and Theorem 3.2, the valley number and exterior peak number as well).

**Theorem 4.8** (Shuffle-compatibility of the peak number).

(a) *The peak number* \( p_k \) *is shuffle-compatible.*

(b) *The linear map on* \( A_{p_k} \) *defined by*

\[
[\pi]_{p_k} \mapsto \left\{ \frac{1}{1-t} \right\} \bigcup \left\{ \frac{2^{2j+1}t^{j+1}(1+t)^{n-2j-1}}{(1-t)^{n+1}}x^n \right\}_{n \geq 1, 0 \leq j \leq \left\lfloor \frac{n}{2} \right\rfloor},
\]

*a subalgebra of* \( \mathbb{Q}[[t]] \).[x].

(c) The \( p_k \) shuffle algebra \( A_{p_k} \) is isomorphic to the span of

\[
\{1\} \cup \left\{ p^jx^n \right\}_{n \geq 1, 1 \leq j \leq n, j \equiv n \text{ (mod 2)}},
\]

*a subalgebra of* \( \mathbb{Q}[p, x] \).

(d) *For* \( n \geq 1 \), *the* \( n \)th *homogeneous component of* \( A_{p_k} \) *has dimension* \( \left\lfloor (n+1)/2 \right\rfloor \).

The proof below implies parts (a), (b), and (d). We postpone the proof of part (c) until Section 6.1.
Proof. For a quasisymmetric function \( f \), let \( f(1^k) \) denote \( f \) evaluated at \( x_i = 1 \) for \( 1 \leq i \leq k \) and \( x_i = 0 \) for \( i > k \). Define \( \phi_{pk} : \text{QSym} \to \mathbb{Q}[[t^*]][x] \) by the formula

\[
\phi_{pk}(F_L) = \sum_{k=0}^{\infty} K_{n, Pk(L)}(1^k) t^k x^n
\]

for \( L \models n > 0 \) and \( \phi_{pk}(F_{\emptyset}) = 1/(1-t) \). Then \( \phi_{pk} \) is the composition of the map \( F_L \mapsto K_{n, Pk(L)} \) with the map \( f \mapsto \sum_{k=0}^{\infty} f(1^k) t^k x^n \) (where \( f \) is homogeneous of degree \( n \)); since both of these maps are \( \mathbb{Q} \)-algebra homomorphisms, it follows that \( \phi_{pk} \) is a \( \mathbb{Q} \)-algebra homomorphism as well.

Stembridge \cite[Theorem 4.1]{stembridge} showed that

\[
\sum_{k=0}^{\infty} K_{n, Pk(L)}(1^k) t^k = \frac{2^{2P(L)+1} t Pk(L)+1 (1+t)^{n-2Pk(L)-1}}{(1-t)^{n+1}},
\]

so in fact

\[
\phi_{pk}(F_L) = \frac{2^{2P(L)+1} t Pk(L)+1 (1+t)^{n-2Pk(L)-1}}{(1-t)^{n+1}} x^n.
\]

We know from Proposition \ref{proposition:peak_range} that for an \( n \)-permutation \( \pi \), the possible values of \( Pk(\pi) \) range from 0 to \( \lfloor (n-1)/2 \rfloor \). Since the elements \( 2^{2j+1} t^{j+1} (1+t)^{n-2j-1} x^n / (1-t)^{n+1} \) are linearly independent, the result follows from Theorem \ref{theorem:linear_independence}.

An alternative proof of Theorem \ref{theorem:shuffle_compatibility} can be given using Theorems \ref{theorem:shuffling} and \ref{theorem:compatibility}.

4.4. Shuffle-compatibility of the left peak set and left peak number

Motivated by Stembridge’s theory of enriched \( P \)-partitions and the study of peak algebras \cite{petersen19, petersen20}, Petersen \cite{petersen19, petersen20} defined another variant of \( P \)-partitions called “left enriched \( P \)-partitions” that tells a parallel story for left peaks.

As before, we restrict our attention to when \( P \) is a chain. Let \( \mathcal{P}^{(\ell)} \) denote the set of integers with the following total ordering:

\[
0 < -1 < +1 < -2 < +2 < -3 < +3 < \cdots .
\]

Then for \( \pi \in \mathcal{P}_n \), a left enriched \( \pi \)-partition is a map \( f : [n] \to \mathcal{P}^{(\ell)} \) such that for all \( i < j \) in \([n]\), the following hold:

1. \( f(i) \preceq f(j) \);
2. \( f(i) = f(j) \geq 0 \) implies \( \pi(i) < \pi(j) \);
3. \( f(i) = f(j) < 0 \) implies \( \pi(i) > \pi(j) \).

Let \( \mathcal{E}^{(\ell)}(\pi) \) denote the set of left enriched \( \pi \)-partitions, and let

\[
\Gamma^{(\ell)}(\pi) := \sum_{f \in \mathcal{E}^{(\ell)}(\pi)} x_{|f(1)|} x_{|f(2)|} \cdots x_{|f(n)|}.
\]
Just as the generating function $\Gamma(\pi)$ for enriched $\pi$-partitions depends only on the peak set of $\pi$, Petersen proved that $\Gamma^{(\ell)}(\pi)$ depends only on the left peak set \[20\] Corollary 6.5, so we can define

$$K_{n,\Lambda}^{(\ell)} := \Gamma^{(\ell)}(\pi)$$

for any $\pi \in \mathfrak{P}_n$ with $\text{Lpk}(\pi) = \Lambda$. Unlike the peak functions $K_{n,\Lambda}$, the $K_{n,\Lambda}^{(\ell)}$ are not quasisymmetric functions but rather type B quasisymmetric functions$^7$.

Petersen briefly mentions that the span of the left peak functions $K_{n,\Lambda}^{(\ell)}$ forms a graded subalgebra $\Pi^{(\ell)}$ of the algebra of type B quasisymmetric functions, called the \textit{algebra of left peaks} \[20, \text{p. 604}\]. The $n$th homogeneous component of $\Pi^{(\ell)}$ has dimension $F_{n+1}$, which is easily seen to be the number of left peak sets among $n$-permutations. He does not explicitly state a multiplication rule for the $K_{n,\Lambda}^{(\ell)}$, but it follows from the fundamental lemma of left enriched $P$-partitions \[20, \text{Lemma 4.2}\] that the multiplication is given by

$$K_{m,\text{Lpk}(\pi)}^{(\ell)} K_{n,\text{Lpk}(\sigma)}^{(\ell)} = \sum_{\tau \in S(\pi,\sigma)} K_{m+n,\text{Lpk}(\tau)}^{(\ell)},$$

which implies the shuffle-compatibility of the left peak set (and by Corollary 3.7 the right peak set as well).

**Theorem 4.9** (Shuffle-compatibility of the left peak set).

(a) The left peak set $\text{Lpk}$ is shuffle-compatible.

(b) The linear map on $A_{\text{Lpk}}$ defined by

$$[\pi]_{\text{Lpk}} \mapsto K_{|\pi|,\text{Lpk}(\pi)}^{(\ell)}$$

is a $\mathbb{Q}$-algebra isomorphism from $A_{\text{Lpk}}$ to $\Pi^{(\ell)}$.

Although Petersen was the first to explicitly construct the algebra of left peaks, Theorem 4.9 also follows from the work of Aguiar, Bergeron, and Nyman, who constructed the coalgebra dual to the algebra of left peaks $[1, \text{Proposition 8.3 and Remark 8.7.3}]$. We will extensively study coalgebras dual to shuffle algebras in Section 5.

Petersen’s work can also be used (in conjunction with Proposition 2.6 and Theorem 4.3) to prove the shuffle-compatibility of the left peak number. The proof is similar to the proof of Theorem 4.8 but we use the identity

$$\sum_{p=0}^{\infty} K_{n,\text{Lpk}(L)}^{(\ell)} (1^p t^p) = \frac{2^{\text{Lpk}(L)} t^{\text{Lpk}(L)} (1 + t)^{n-2 \text{Lpk}(L)}}{(1 - t)^{n+1}}$$

\[20, \text{Theorem 4.6}\]. Alternatively, Theorems 3.3 and 5.10 can be used to produce a different proof.

$^7$We omit the definition of a type B quasisymmetric function, as they play no further role in this paper, but we refer the reader to $[3]$.

$^8$Petersen actually calls this algebra the “left algebra of peaks”, but the “algebra of left peaks” seems to us a more natural name.
Theorem 4.10 (Shuffle-compatibility of the left peak number).

(a) The left peak number $lpk$ is shuffle-compatible.

(b) The linear map on $A_{lpk}$ defined by

$$[\pi]_{lpk} \mapsto \begin{cases} \frac{2^{lpk(\pi)}(1 + t)^{|\pi|} - 2^{lpk(\pi)}}{(1 - t)^{|\pi| + 1}}x^{|\pi|}, & \text{if } |\pi| \geq 1, \\ 1/(1 - t), & \text{if } |\pi| = 0, \end{cases}$$

is a $\mathbb{Q}$-algebra isomorphism from $A_{lpk}$ to the span of

$$\left\{ \frac{1}{1 - t} \right\} \bigcup \left\{ \frac{2^{2j}(1 + t)^{n-2j}}{(1 - t)^{n+1}}x^n \right\}_{n \geq 1, 0 \leq j \leq \lfloor n/2 \rfloor},$$

a subalgebra of $\mathbb{Q}[[t^*]][x]$.

(c) The $n$th homogeneous component of $A_{lpk}$ has dimension $\lfloor n/2 \rfloor + 1$.

By Theorem 3.7, the right peak number (or, the number of long runs; see Lemma 2.1 (d)) is also shuffle-compatible.

5. Noncommutative symmetric functions and shuffle-compatibility

5.1. Algebras, coalgebras, and graded duals

In this section, we introduce another criterion for shuffle-compatibility that will be in a sense “dual” to the criterion in Theorem 4.3. For this, we shall need the notion dual to an algebra, which requires the following equivalent definition of an algebra.

Let $R$ be a commutative ring. An $R$-algebra $A$ is an $R$-module with an $R$-linear map $\mu: A \otimes A \to A$ such that the following diagram commutes:

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu \otimes \text{id}} & A \otimes A \\ \text{id} \otimes \mu \downarrow & & \downarrow \mu \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

The map $\mu$ is called a multiplication.$^9$

The notion dual to an algebra is a coalgebra, defined as follows. An $R$-coalgebra $C$ is an $R$-module with an $R$-linear map $\Delta: C \to C \otimes C$ such that the following diagram commutes:

$$\begin{array}{ccc} C \otimes C & \xleftarrow{\Delta \otimes \text{id}} & C \otimes C \\ \Delta \otimes \text{id} \uparrow & & \uparrow \Delta \\ C \otimes C & \xleftarrow{\Delta} & C \end{array}$$

$^9$The multiplication map $\mu$ satisfies $\mu(a \otimes b) = ab$ under the original definition of an algebra; from this, it is clear why $\mu$ is called “multiplication”.
Observe that this diagram is essentially the diagram in the definition of an algebra, but with arrows reversed. The map $\Delta$ is called a \textit{comultiplication}.\footnote{Typically, the definition of an algebra requires an additional linear map called a “unit” which satisfies a certain commutative diagram, and the definition of a coalgebra requires the dual concept of a “counit”, but these will not be necessary for our work.}

If an $R$-module $A$ is simultaneously an $R$-algebra and an $R$-coalgebra such that its comultiplication map is an $R$-algebra homomorphism, then we call $A$ an $R$-bialgebra.

Suppose now that $R$ is a field and that $V = \bigoplus_{n \geq 0} V_n$ is a graded $R$-vector space of finite type, that is, each component $V_n$ is finite-dimensional. Let $V^o$ denote the \textit{graded dual} $V^o := \bigoplus_{n \geq 0} V_n^*$, which is contained inside the dual space $V^*$ of $V$. We say that a linear map $\phi: V \to W$ is \textit{graded} if, for every $n \geq 0$, $\phi(V_n)$ is contained inside the $n$th homogeneous component of $W$. Every graded linear map $\phi: V \to W$ induces a graded linear map $\phi^o: W^o \to V^o$ given by

$$\phi^o(f)(v) = f(\phi(v))$$

for $f \in W^o$ and $v \in V$. In particular, if $A$ is a graded $R$-algebra—meaning that its vector space and multiplication are graded—and is of finite type, then by reversing the arrows in the commutative diagram, we see that $A^o$ has the structure of a graded $R$-coalgebra. In fact, if $A$ has basis $\{a_i\}$ with structure constants $\{c_{j,k}^i\}$, i.e.,

$$a_ja_k = \sum_i c_{j,k}^ia_i,$$

then the $\{c_{j,k}^i\}$ are also the structure constants for the comultiplication of the dual basis $\{f_i\}$ in $A^o$:

$$\Delta(f_i) = \sum_{j,k} c_{j,k}^if_j \otimes f_k.$$ 

Similarly, the graded dual of a graded $R$-coalgebra is a graded $R$-algebra, with the same correspondence of structure constants. If $\phi$ is an $R$-algebra homomorphism, then $\phi^o$ is an $R$-coalgebra homomorphism, and vice versa.

### 5.2. Noncommutative symmetric functions

The graded dual of QSym is the coalgebra of noncommutative symmetric functions, which also has an algebra structure. We begin by defining the algebra of noncommutative symmetric functions before introducing the comultiplication.

Let $\mathbb{Q}\langle\langle X_1, X_2, \ldots \rangle\rangle$ be the $\mathbb{Q}$-algebra of formal power series in countably many noncommuting variables $X_1, X_2, \ldots$. Consider the elements

$$h_n := \sum_{i_1 \leq \cdots \leq i_n} X_{i_1}X_{i_2}\cdots X_{i_n}$$

of $\mathbb{Q}\langle\langle X_1, X_2, \ldots \rangle\rangle$, with $h_0 = 1$, which are noncommutative versions of the complete symmetric functions $h_n$. Note that $h_n$ is the noncommutative generating function for weakly increasing words of length $n$ on the alphabet $\mathbb{P}$ of positive integers. For example, the weakly
increasing word 13449 is encoded by $X_1X_3X_4^2X_9$, which appears as a term in $h_5$. Given a composition $L = (L_1, \ldots, L_k)$, we let

$$h_L := h_{L_1} \cdots h_{L_k}. \quad (5.1)$$

Equivalently,

$$h_L = \sum_{i_1, \ldots, i_n} X_{i_1}X_{i_2} \cdots X_{i_n}$$

where the sum is over all $i_1, \ldots, i_n$ satisfying

$$i_1 \leq \cdots \leq i_{L_1}, i_{L_1+1} \leq \cdots \leq i_{L_1+L_2}, \ldots, i_{L_1+\cdots+L_{k-1}+1} \leq \cdots \leq i_n,$$

so $h_L$ is the noncommutative generating function for words in $\mathbb{P}$ whose descent set is contained in $\text{Des}(L)$.

Let $\text{Sym}_n$ denote the vector space spanned by $\{h_L\}_{L \vdash n}$, and let $\text{Sym} := \bigoplus_{n=0}^\infty \text{Sym}_n$. Then $\text{Sym}$ is a graded $\mathbb{Q}$-algebra called the *algebra of noncommutative symmetric functions* with coefficients in $\mathbb{Q}$, a subalgebra of $\mathbb{Q}\langle\langle X_1, X_2, \ldots \rangle\rangle$. The study of $\text{Sym}$ was initiated in [4], although noncommutative symmetric functions have appeared implicitly in earlier work, including the first author’s Ph.D. thesis [9]. Also see [8, 30, 31] for a series of recent papers by the present authors on the subject of permutation enumeration in which $\text{Sym}$ plays a role.

In the following sections, we will work with noncommutative symmetric functions with coefficients in either the ring $\mathbb{Q}[x, y]$ of polynomials in $x$ and $y$ with rational coefficients or the ring $\mathbb{Q}[[t^*]][x, y]$ of polynomials in $x$ and $y$ with coefficients in the ring of formal power series in $t$ in which multiplication is the Hadamard product in $t$ but ordinary multiplication in $x$ and $y$. We will also need to use formal sums of noncommutative symmetric functions of unbounded degree with these coefficient rings, for example, $\sum_{n=0}^\infty h_n x^n$. We will use the notation $\text{Sym}_{xy}$ for the algebra of noncommutative symmetric functions of unbounded degree with coefficients in $\mathbb{Q}[x, y]$ and $\text{Sym}_{txy}$ for noncommutative symmetric functions with coefficients in $\mathbb{Q}[[t^*]][x, y]$.

For a composition $L = (L_1, \ldots, L_k)$, we define

$$r_L := \sum_{i_1, \ldots, i_n} X_{i_1}X_{i_2} \cdots X_{i_n}$$

where the sum is over all $i_1, \ldots, i_n$ satisfying

$$i_1 \leq \cdots \leq i_{L_1}, i_{L_1+1} \leq \cdots \leq i_{L_1+L_2}, \ldots, i_{L_1+\cdots+L_{k-1}+1} \leq \cdots \leq i_n.$$

Then $r_L$ is the noncommutative generating function for words on the alphabet $\mathbb{P}$ with descent composition $L$.

Note that

$$h_L = \sum_{\text{Des}(K) \subseteq \text{Des}(L)} r_K, \quad (5.2)$$
so by inclusion-exclusion,

\[ r_L = \sum_{\text{Des}(K) \subseteq \text{Des}(L) \mid |K| = |L|} (-1)^{l(L)-l(K)} h_K \]  

(5.3)

where \( l(L) \) denotes the number of parts of the composition \( L \). Hence the \( r_L \) are noncommutative symmetric functions, and are in fact noncommutative versions of the ribbon skew Schur functions \( r_L \).

Since \( r_L \) and \( r_M \) have no terms in common for \( L \neq M \), it is clear that \( \{r_L\}_{L \sqsubseteq n} \) is linearly independent. From (5.2), we see that \( \{r_L\}_{L \sqsubseteq n} \) spans \( \text{Sym}_n \), so \( \{r_L\}_{L \sqsubseteq n} \) is a basis for \( \text{Sym}_n \). Because \( \{h_L\}_{L \sqsubseteq n} \) spans \( \text{Sym}_n \) and has the same cardinality as \( \{r_L\}_{L \sqsubseteq n} \), we conclude that \( \{h_L\}_{L \sqsubseteq n} \) is also a basis for \( \text{Sym}_n \).

Let us also consider the noncommutative generating function

\[ e_n := \sum_{i_1 > \cdots > i_n} X_{i_1} X_{i_2} \cdots X_{i_n} \]

for decreasing words of length \( n \) on the alphabet \( \mathbb{P} \). Then \( e_n \) is a noncommutative version of the elementary symmetric function \( e_n \), and \( e_n \in \text{Sym}_n \) since \( e_n = r_{(1^n)} \).

Let

\[ h(x) := \sum_{n=0}^{\infty} h_n x^n \]

be the generating function for the noncommutative complete symmetric functions \( h_n \), where \( x \) commutes with all the variables \( X_i \), and let

\[ e(x) := \sum_{n=0}^{\infty} e_n x^n \]

be the generating function for the \( e_n \). Then

\[ e(x) = h(-x)^{-1}, \]  

(5.4)

which is a consequence of the infinite product formulas

\[ h(x) = (1 - X_1 x)^{-1}(1 - X_2 x)^{-1} \cdots \text{ and } e(x) = \cdots (1 + X_2 x)(1 + X_1 x) \]

(cf. [9, p. 38] or [6, Section 7.3]).

The algebra \( \text{Sym} \) can be given a coalgebra structure by defining the comultiplication \( \Delta : \text{Sym} \to \text{Sym} \otimes \text{Sym} \) by

\[ \Delta h_n = \sum_{i=0}^{n} h_i \otimes h_{n-i} \]  

(5.5)

and extending by the rule

\[ \Delta(fg) = (\Delta f)(\Delta g). \]  

(5.6)
Since $\Delta$ is an algebra homomorphism, $\text{Sym}$ is a bialgebra. The comultiplication $\Delta$ extends naturally to $\text{Sym}_{xy}$ and $\text{Sym}_{txy}$ (but note that now tensor products are over the coefficient ring).

Next, we show that the graded dual of the algebra QSym is the coalgebra $\text{Sym}$; cf. [6, Theorem 6.1] or [14, Section 5.3]. We may extend the definition of $h_L$ to weak compositions $L$ by (5.1), so that if $L$ is a weak composition then $h_L = h_{L'}$, where $L'$ is the composition obtained from $L$ by removing all zero parts. Recall that, as defined in Section 2.4 weak compositions are added componentwise.

**Lemma 5.1.** Let $L$ be a composition. Then $\Delta h_L = \sum_{J,K} b_{J,K} h_J \otimes h_K$, where the sum is over all pairs of weak compositions $J$ and $K$ with the same number of parts such that $J + K = L$.

**Proof.** This follows easily from the fact that $\Delta h_{(L_1,\ldots,L_m)} = \Delta h_{L_1} \cdots \Delta h_{L_m}$ together with (5.5).

**Theorem 5.2.** The graded dual of the algebra QSym of quasisymmetric functions is isomorphic to the coalgebra $\text{Sym}$ of noncommutative symmetric functions. In particular, the monomial basis $\{M_L\}$ of QSym is dual to the complete basis $\{h_L\}$ of $\text{Sym}$ and the fundamental basis $\{F_L\}$ of QSym is dual to the ribbon basis $\{r_L\}$ of $\text{Sym}$.

**Proof.** We first consider the product of two monomial quasisymmetric functions. Define coefficients $b_{J,K}^L$ by

$$M_J M_K = \sum_L b_{J,K}^L M_L. \quad (5.7)$$

It is easy to see that $b_{J,K}^L$ is the number of pairs of weak compositions $(J', K')$ with the same number of parts such that $J'$ is obtained from $J$ by inserting zeros, $K'$ is obtained from $K$ by inserting zeros, and $J' + K' = L$.

Lemma 5.1 implies that

$$\Delta h_L = \sum_{J,K} b_{J,K}^L h_J \otimes h_K,$$

where the coefficients $b_{J,K}^L$ are the same as those in equation (5.7). Thus $\{M_L\}_{L=n}$ and $\{h_L\}_{L=n}$ are dual bases for $\text{QSym}_n$ and $\text{Sym}_n$.

We may define a pairing between QSym and Sym by

$$\langle M_K, h_L \rangle = \delta_{K,L} = \begin{cases} 1, & \text{if } K = L, \\ 0, & \text{otherwise}. \end{cases}$$

Then

$$\langle F_K, r_L \rangle = \left\langle \sum_{\text{Des}(I) \supseteq \text{Des}(K)} M_I, \sum_{\text{Des}(J) \subseteq \text{Des}(L)} (-1)^{l(L) - l(J)} h_J \right\rangle = \sum_{\text{Des}(J) \supseteq \text{Des}(K), \text{Des}(J) \subseteq \text{Des}(L)} (-1)^{l(L) - l(J)} = \delta_{K,L},$$

and this implies that $\{F_L\}$ and $\{r_L\}$ are dual bases. $\square$

---

$\text{Sym}$ and QSym are Hopf algebras (see [14] for a definition) and the duality between Sym and QSym given in the next theorem is in fact a Hopf algebra duality. However, we will not need the antipode in this paper, nor will we be concerned with the coalgebra structure of QSym.

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11In fact, both Sym and QSym are Hopf algebras (see [14] for a definition) and the duality between Sym and QSym given in the next theorem is in fact a Hopf algebra duality. However, we will not need the antipode in this paper, nor will we be concerned with the coalgebra structure of QSym.
5.3. Monoidlike elements

We call an element $f$ of a bialgebra monoidlike if $\Delta f = f \otimes f$. It is straightforward to show that the product of two monoidlike elements is monoidlike and that the inverse of a monoidlike element, if it exists, is monoidlike.

**Lemma 5.3.** $h(x)$, $e(x)$, and $e(xy)$ are monoidlike in $\text{Sym}_{xy}$.

*Proof.* We have

$$\Delta h(x) = \sum_{n=0}^{\infty} \Delta h_n x^n$$

$$= \sum_{n=0}^{\infty} \sum_{i+j=n} (h_i \otimes h_j) x^n$$

$$= \sum_{n=0}^{\infty} \sum_{i+j=n} h_i x^i \otimes h_j x^j$$

$$= \sum_{i,j=0}^{\infty} h_i x^i \otimes h_j x^j$$

$$= \left( \sum_{i=0}^{\infty} h_i x^i \right) \otimes \left( \sum_{j=0}^{\infty} h_j x^j \right),$$

so $h(x)$ is monoidlike. Since $e(x) = h(-x)^{-1}$, this implies that $e(x)$ and $e(xy)$ are monoidlike.

**Lemma 5.4.** Let $f = \sum_{n=0}^{\infty} a_n t^n$ be an element of $\text{Sym}_{txy}$ where each $a_n$ is an element of $\text{Sym}_{xy}$. Then $f$ is monoidlike in $\text{Sym}_{txy}$ if and only if each $a_n$ is monoidlike in $\text{Sym}_{xy}$.

*Proof.* We have

$$f \otimes f = \sum_{m,n=0}^{\infty} a_m t^m \otimes a_n t^n$$

$$= \sum_{m,n=0}^{\infty} (a_m \otimes a_n)(t^m \ast t^n)$$

$$= \sum_{n=0}^{\infty} (a_n \otimes a_n)t^n$$

and

$$\Delta f = \sum_{n=0}^{\infty} \Delta a_n t^n.$$ 

Thus $\Delta f = f \otimes f$ if and only if $\Delta a_n = a_n \otimes a_n$ for each $n$. 

---

12 A monoidlike element $f$ of a bialgebra is called grouplike if $\varepsilon(f)$ is the identity element of the coefficient ring, where $\varepsilon$ is the counit. In our bialgebras, the counit is the coefficient of $h_0$, the identity element of $Q$ or $Q[x, y]$ is 1, and the identity element of $Q[[t]][x, y]$ is $(1 - t)^{-1} = \sum_{k=0}^{\infty} t^k$. Nearly all of our monoidlike elements are actually grouplike, but exceptions occur in Corollary 5.5.
The next result follows immediately from Lemma 5.4.

**Corollary 5.5.** Suppose that \( f \) is monoidlike in \( \text{Sym}_{xy} \). Then \((1 - tf)^{-1}, (1 - t^2 f)^{-1}, \) and \( 1 + tf \) are monoidlike in \( \text{Sym}_{txy} \).

### 5.4. Implications of duality to shuffle-compatibility

Let \( st \) be a descent statistic. For each \( st \)-equivalence class \( \alpha \) of compositions, let

\[
    r^st_\alpha := \sum_{L \in \alpha} r_L.
\]

We call the noncommutative symmetric functions \( r^st_\alpha \) st-ribbons.

The following is the dual version of Theorem 4.3.

**Theorem 5.6.** A descent statistic \( st \) is shuffle-compatible if and only if for every \( st \)-equivalence class \( \alpha \) of compositions, there exist constants \( c^\alpha_{\beta,\gamma} \) for which

\[
    \Delta r^st_\alpha = \sum_{\beta,\gamma} c^\alpha_{\beta,\gamma} r^st_\beta \otimes r^st_\gamma;
\]

that is, the st-ribbons \( r^st_\alpha \) span a subcoalgebra of \( \text{Sym} \). In this case, the \( c^\alpha_{\beta,\gamma} \) are the structure constants for \( A_{st} \).

**Proof.** By Theorem 5.2, we have a pairing between quasisymmetric functions and noncommutative symmetric functions for which

\[
    \langle F_L, r_J \rangle = \begin{cases} 
        1, & \text{if } L = J, \\
        0, & \text{otherwise.}
    \end{cases}
\]

Suppose that the st-ribbons \( r^st_\alpha \) span a subcoalgebra of \( \text{Sym} \) with structure constants \( c^\alpha_{\beta,\gamma} \). Let \( D \) be the subcoalgebra spanned by the \( r^st_\alpha \) and let \( i: D \rightarrow \text{Sym} \) be the canonical inclusion map, a \( \mathbb{Q} \)-coalgebra homomorphism. Then \( i \) induces a \( \mathbb{Q} \)-algebra homomorphism \( i^0: \text{QSym} \rightarrow D^0 \) given by

\[
    i^0(F_L)(r^st_\alpha) = \langle F_L, i(r^st_\alpha) \rangle = \langle F_L, r^st_\alpha \rangle = \begin{cases} 
        1, & \text{if } L \in \alpha, \\
        0, & \text{otherwise.}
    \end{cases}
\]

Observe that \( i^0(F_L) = i^0(F_J) \) whenever \( L \) and \( J \) belong to the same \( st \)-equivalence class. Hence, we can define \( f_\alpha := i^0(F_L) \) for \( L \in \alpha \). Then \( \{f_\alpha\} \) is the basis of \( D^0 \) dual to \( \{r^st_\alpha\} \), so

\[
    f_\beta f_\gamma = \sum_{\alpha} c^\alpha_{\beta,\gamma} f_\alpha.
\]

By Theorem 4.3, \( st \) is shuffle-compatible with shuffle algebra isomorphic to \( D^0 \). We omit the proof of the reverse implication, as it is similar; we begin with a quotient algebra of \( \text{QSym} \) and then show that its basis elements are dual to the st-ribbons \( r^st_\alpha \). \( \square \)
While Theorem 4.3 tells us that we can prove the shuffle-compatibility of a descent statistic by constructing suitable quotients of QSym, Theorem 5.6 tells us that we could, alternatively, construct suitable subcoalgebras of Sym, and this is what we will do in Sections 5.5 to 5.7. Moreover, because it is straightforward to compute coproducts of noncommutative symmetric functions, Theorem 5.6 is useful for showing that a descent statistic is not shuffle-compatible and for conjecturing that a statistic is shuffle-compatible, which is not the case for Theorem 4.3.

Although Theorem 5.6 does not give us a way to describe the dual algebra \( A_{st} \), we can describe \( A_{st} \) explicitly using the following theorem. For an st-equivalence class \( \alpha \) of compositions, we let \( |\alpha| \) be the sum of the parts of any composition \( L \in \alpha \).

**Theorem 5.7.** Let \( st \) be a descent statistic and let \( u_{\alpha} \in \mathbb{Q}[\![t^*]\!][x,y] \) be linearly independent elements (over \( \mathbb{Q} \)) indexed by st-equivalence classes \( \alpha \) of compositions. Suppose that
\[
f = \sum_{\alpha} u_{\alpha} r_{\alpha}^st = \sum_{\beta} u_{\beta} r_{\beta}^st \otimes \sum_{\gamma} u_{\gamma} r_{\gamma}^st.
\]
and that there exist constants \( c_{\beta,\gamma}^\alpha \) such that
\[
u_{\beta} u_{\gamma} = \sum_{\alpha} c_{\beta,\gamma}^\alpha u_{\alpha} \quad \text{for all st-equivalence classes } \beta \text{ and } \gamma, \text{ where } c_{\beta,\gamma}^\alpha = 0 \text{ unless } |\alpha| = |\beta| + |\gamma|.
\]Then \( st \) is shuffle-compatible and the linear map defined by
\[
[p]_{st} \mapsto u_{\alpha},
\]
where \( \text{Comp}(\pi) \in \alpha \), is a \( \mathbb{Q} \)-algebra isomorphism from \( A_{st} \) to the subalgebra of \( \mathbb{Q}[\![t^*]\!][x,y] \) spanned by the \( u_{\alpha} \).

**Proof.** Since \( f \) is monoidlike, we have that
\[
\sum_{\alpha} u_{\alpha} \Delta r_{\alpha}^st = \Delta f = \left( \sum_{\beta} u_{\beta} r_{\beta}^st \right) \otimes \left( \sum_{\gamma} u_{\gamma} r_{\gamma}^st \right) = \sum_{\beta,\gamma} u_{\beta} u_{\gamma} r_{\beta}^st \otimes r_{\gamma}^st = \sum_{\alpha} u_{\alpha} \sum_{\beta,\gamma} c_{\beta,\gamma}^\alpha r_{\beta}^st \otimes r_{\gamma}^st.
\]
Extracting the linear combinations of elements of \( \text{Sym}_i \otimes \text{Sym}_j \), where \( i + j = n \), we obtain
\[
\sum_{|\alpha| = n} u_{\alpha} \Delta r_{\alpha}^st = \sum_{|\alpha| = n} u_{\alpha} \sum_{\beta,\gamma} c_{\beta,\gamma}^\alpha r_{\beta}^st \otimes r_{\gamma}^st.
\]
Since these are finite sums, linear independence of the \( u_{\alpha} \) implies
\[
\Delta r_{\alpha}^st = \sum_{\beta,\gamma} c_{\beta,\gamma}^\alpha r_{\beta}^st \otimes r_{\gamma}^st
\]
and it follows from Theorem 5.6 that \( st \) is shuffle-compatible and that the \( c_{\beta,\gamma}^\alpha \) are the structure constants for \( A_{st} \). Since
\[
u_{\beta} u_{\gamma} = \sum_{\alpha} c_{\beta,\gamma}^\alpha u_{\alpha}
\]
for all st-equivalence classes \( \beta \) and \( \gamma \), the map \([p]_{st} \mapsto u_{\alpha}\) is an algebra homomorphism from \( A_{st} \) to the subalgebra of \( \mathbb{Q}[\![t^*]\!][x,y] \) spanned by the \( u_{\alpha} \), and since the \( u_{\alpha} \) are linearly independent, this map is an isomorphism.
We note that Theorem 5.7 can be generalized to a statement about monoidlike elements of more general graded bialgebras; we stated it only in the special case that we will use.

Unfortunately, in our applications, it is difficult to show directly that the desired \( u_\alpha \) are closed under multiplication. The following variant of Theorem 5.7 uses a change of basis argument to deal with this problem.

**Theorem 5.8.** Let \( st \) be a descent statistic and let \( u_\alpha \in \mathbb{Q}[\{t^*\}][x, y] \) be linearly independent elements (over \( \mathbb{Q} \)) indexed by \( st \)-equivalence classes \( \alpha \) of compositions. Suppose that \( f = \sum_\alpha u_\alpha r_\alpha^{st} \) is monoidlike in \( \text{Sym}_{txy} \), where \( u_\alpha \) is \( x^{|\alpha|} \) times an element of \( \mathbb{Q}[\{t^*\}][y] \). Let \( s_{n,p,q} \) be the coefficient of \( x^n y^p t^q \) in \( \sum_\alpha u_\alpha r_\alpha^{st} \) and suppose that \( r_\alpha^{st} \in \text{Span}_\mathbb{Q}\{s_{n,p,q}\} \) for each \( \alpha \). Then \( st \) is shuffle-compatible and the linear map defined by

\[
[\pi]_{st} \mapsto u_\alpha,
\]

where \( \text{Comp}(\pi) \in \alpha \), is a \( \mathbb{Q} \)-algebra isomorphism from \( A_{st} \) to the subalgebra of \( \mathbb{Q}[\{t^*\}][x, y] \) spanned by the \( u_\alpha \).

**Proof.** Equating coefficients of \( x^n \) in

\[
f = \sum_\alpha u_\alpha r_\alpha^{st} = \sum_{n,p,q} x^n y^p t^q s_{n,p,q}
\]
gives

\[
\sum_{|\alpha|=n} u_\alpha r_\alpha^{st} = x^n \sum_{p,q} y^p t^q s_{n,p,q}.
\]

Since the sum on the left is finite, this shows that \( s_{n,p,q} \in \text{Span}_\mathbb{Q}\{r_\alpha^{st}\} \), so \( \text{Span}_\mathbb{Q}\{r_\alpha^{st}\} = \text{Span}_\mathbb{Q}\{s_{n,p,q}\} \).

Let \( f_q \) be the coefficient of \( t^q \) in \( f \). Then since \( f \) is monoidlike, \( f_q \) is monoidlike by Lemma 5.4 so

\[
\sum_{n,p} x^n y^p \Delta s_{n,p,q} = \Delta f_q = f_q \otimes f_q
\]

\[
= \left( \sum_{n_1,p_1} x^{n_1} y^{p_1} s_{n_1,p_1,q} \right) \otimes \left( \sum_{n_2,p_2} x^{n_2} y^{p_2} s_{n_2,p_2,q} \right)
\]

\[
= \sum_{n_1,p_1,n_2,p_2} x^{n_1+n_2} y^{p_1+p_2} s_{n_1,p_1,q} \otimes s_{n_2,p_2,q}
\]

Equating coefficients of \( x^n y^p \) shows that \( \text{Span}_\mathbb{Q}\{s_{n,p,q}\} \) is a subcoalgebra of \( \text{Sym} \) and thus so is \( \text{Span}_\mathbb{Q}\{r_\alpha^{st}\} \). As a result, there exist constants \( c^\alpha_{\beta,\gamma} \) such that

\[
\Delta r_\alpha^{st} = \sum_{\beta,\gamma} c^\alpha_{\beta,\gamma} r_\beta^{st} \otimes r_\gamma^{st},
\]

so it follows from Theorem 5.6 that \( st \) is shuffle-compatible and that the \( c^\alpha_{\beta,\gamma} \) are the structure constants for \( A_{st} \).
Moreover, since $\sum_{\alpha} u_{\alpha} r_{\alpha}^{st}$ is monoidlike, we have

$$\sum_{\beta, \gamma} \sum_{\alpha} u_{\alpha} c_{\beta, \gamma}^{\alpha} r_{\beta}^{st} \otimes r_{\gamma}^{st} = \sum_{\alpha} u_{\alpha} \sum_{\beta, \gamma} c_{\beta, \gamma}^{\alpha} r_{\beta}^{st} \otimes r_{\gamma}^{st}$$

$$= \sum_{\alpha} u_{\alpha} \Delta r_{\alpha}^{st}$$

$$= \Delta \left( \sum_{\alpha} u_{\alpha} r_{\alpha}^{st} \right)$$

$$= \left( \sum_{\beta} u_{\beta} r_{\beta}^{st} \right) \otimes \left( \sum_{\gamma} u_{\gamma} r_{\gamma}^{st} \right)$$

$$= \sum_{\beta, \gamma} u_{\beta} u_{\gamma} r_{\beta}^{st} \otimes r_{\gamma}^{st}.$$ 

Using the linear independence of the $r_{\beta}^{st} \otimes r_{\gamma}^{st}$ and the fact that for each $i$ and $j$, $r_{\beta}^{st} \otimes r_{\gamma}^{st} \in Sym_i \otimes Sym_j$ for only finitely many $\beta$ and $\gamma$, we may equate coefficients of $r_{\beta}^{st} \otimes r_{\gamma}^{st}$ to obtain $u_{\beta} u_{\gamma} = \sum_{\alpha} c_{\beta, \gamma}^{\alpha} u_{\alpha}$. Thus the map $[\pi]_{st} \mapsto u_{\alpha}$ is an algebra homomorphism from $A_{st}$ to the subalgebra of $Q[[t^*]][x, y]$ spanned by the $u_{\alpha}$, and since the $u_{\alpha}$ are linearly independent, this map is an isomorphism.

Before applying Theorem 5.8 to prove new results, let us see how it works in a simpler case, the shuffle-compatibility of the descent number (Theorem 4.6).

We start with the formula

$$(1 - t \mathbf{h}(x))^{-1} = \frac{1}{1 - t} + \sum_{n=1}^{\infty} \sum_{L \in n} \frac{t^{\text{des}(L)+1}}{(1 - t)^{n+1}} x^n r_L,$$ (5.8)

which is the case $y = 0$ of Equation (5.10) below, but is easily proved directly [9, p. 83, Equation (3)]. Let $r_{\alpha, j}^{\text{des}}$, for $n \geq 1$, denote the noncommutative symmetric function $r_{\alpha}^{\text{des}}$ where $\alpha$ is the des-equivalence class of compositions corresponding to $n$-permutations with $j - 1$ descents, and let $r_{0,j}^{\text{des}} = \delta_{0,j}$. Let

$$u_{n,j} = u_{\alpha} = \frac{t^j}{(1 - t)^{n+1}} x^n$$ (5.9)

for $n \geq 0$. Then $\sum_{\alpha} u_{\alpha} r_{\alpha}^{\text{des}}$ is equal to (5.8), which is monoidlike in $Q[[t^*]][x]$ by Lemma 5.3 and Corollary 5.5.

With the notation of Theorem 5.8, we have for fixed $n \geq 1$,

$$\sum_{q=0}^{\infty} t^q s_{n,0,q} = \sum_{j=1}^{n} \frac{t^j}{(1 - t)^{n+1}} r_{n,j}^{\text{des}},$$

Multiplying both sides by $(1 - t)^{n+1}$ and equating coefficients of powers of $t$ shows that $r_{n,j}^{\text{des}} \in \text{Span}_Q \{s_{n,0,q}\}$. So by Theorem 5.8 we obtain part (d) of Theorem 4.6.37
5.5. Shuffle-compatibility of (pk, des)

In the remainder of Section 5, we use Theorem 5.8 to establish the shuffle-compatibility and describe the shuffle algebras of the descent statistics (pk, des), (lpk, des), (udr, des), and udr. All computations are done in the algebra Sym_{txty} of noncommutative symmetric functions with coefficients in \( \mathbb{Q}[[t, x, y]] \). We start with the shuffle-compatibility of (pk, des).

**Theorem 5.9** (Shuffle-compatibility of (pk, des)).

(a) The pair (pk, des) is shuffle-compatible.

(b) The linear map on \( A_{(pk, des)} \) defined by

\[
[\pi]_{(pk, des)} \mapsto \begin{cases} 
\frac{t^{pk(\pi)+1}(y+t)^{des(\pi)-pk(\pi)}(1+yt)^{|\pi|-pk(\pi)-des(\pi)-1}(1+y)^{2pk(\pi)+1}}{(1-t)^{|\pi|+1}}x^{|\pi|}, & \text{if } |\pi| \geq 1, \\
1/(1-t), & \text{if } |\pi| = 0,
\end{cases}
\]

is a \( \mathbb{Q} \)-algebra isomorphism from \( A_{(pk, des)} \) to the span of

\[
\left\{ \frac{1}{1-t} \right\} \bigcup \left\{ \frac{t^{j+1}(y+t)^{k-j}(1+yt)^{n-j-k-1}(1+y)^{2j+1}}{(1-t)^{n+1}}x^n \right\}_{n \geq 1, \, 0 \leq j \leq [n-1]/2, \, 0 \leq k \leq n-j-1},
\]

a subalgebra of \( \mathbb{Q}[[t, x, y]] \).

(c) The (pk, des) shuffle algebra \( A_{(pk, des)} \) is isomorphic to the span of

\[
\{1\} \cup \{p^{n-j}(1+y)^{n}(1-y)^{n-2k}x^n\}_{n \geq 1, \, 0 \leq j \leq n-1, \, 0 \leq k \leq [j/2]},
\]

a subalgebra of \( \mathbb{Q}[p, x, y] \).

(d) For \( n \geq 1 \), the \( n \)th homogeneous component of \( A_{(pk, des)} \) has dimension \( \lfloor (n+1)^2/4 \rfloor \).

We prove here parts (a), (b), and (d). We postpone the proof of part (c) until Section 6.1.

**Proof.** By Lemma 4.1 of [31], we have the formula

\[
(1 - te(xy)h(x))^{-1} = \frac{1}{1-t} + \sum_{n=1}^{\infty} \sum_{L= \pi} t^{pk(L)+1}(y+t)^{des(L)-pk(L)(1+yt)^{n-pk(L)-des(L)-1}(1+y)^{2pk(L)+1}}\frac{x^n r_L}{(1-t)^{n+1}}.
\]

Let \( r_{n-j,k}^{(pk, des)} \) denote the noncommutative symmetric function \( r_{\alpha}^{(pk, des)} \) where \( \alpha \) is the (pk, des)-equivalence class of compositions corresponding to \( n \)-permutations with \( j-1 \) peaks and \( k-1 \)
descents. By (5.10) and Proposition 2.5, we have

\[(1 - te(xy)h(x))^{-1} = \frac{1}{1 - t} + \sum_{n=1}^{\infty} \sum_{j=0}^{\frac{(n-1)/2}{2}} \sum_{k=j}^{n-j+1} \frac{t^{j+1}(y + t)^{k-j}(1 + yt)^n(1 + y)^{2j+1}}{(1 - t)^{n+1}} x^n r_{n,j+1,k+1}^{(pk,des)} \]

and this is monoidlike by Lemma 5.3 and Corollary 5.5.

Now define \( s_{n,p,q} \) by

\[\sum_{n,p,q=0}^{\infty} x^n y^p t^q s_{n,p,q} = (1 - te(xy)h(x))^{-1}. \]

For fixed \( n \geq 1 \), we have

\[\sum_{p,q=0}^{\infty} y^p t^q s_{n,p,q} = \sum_{j=1}^{\frac{(n+1)/2}{2}} \sum_{k=j}^{n-j+1} \frac{t^{j}(y + t)^{k-j}(1 + yt)^n(1 + y)^{2j-1}}{(1 - t)^{n+1}} x^n r_{n,j,k}^{(pk,des)} .\]

This identity can be inverted to obtain

\[\sum_{j=1}^{\frac{(n+1)/2}{2}} \sum_{k=j}^{n-j+1} y^j t^k r_{n,j,k}^{(pk,des)} = (1 + u) \left( \frac{1 - v}{1 + uv} \right)^{n+1} \sum_{p,q=0}^{\infty} u^p v^q s_{n,p,q}, \]

where

\[ u = \frac{1 + t^2 - 2yt - (1 - t)\sqrt{(1 + t)^2 - 4yt}}{2(1 - y)t} \]

and

\[ v = \frac{(1 + t)^2 - 2yt - (1 + t)\sqrt{(1 + t)^2 - 4yt}}{2yt} \]

in the formal power series ring \( \mathbb{Q}[[t,y]] \). It is easily checked that \( u \) and \( v \) are both formal power series divisible by \( t \), so \( (1 - v)/(1 + uv) \) is a well-defined formal power series in \( t \) and \( y \).

Equating coefficients of \( y^p t^q \) shows that each \( r_{n,j,k}^{(pk,des)} \) is a linear combination of the \( s_{n,p,q} \). (Since \( u \) and \( v \) are divisible by \( t \), only finitely many terms on the right will contribute a term in \( t^q \).) Parts (a) and (b) then follow from Theorem 5.8.

By Proposition 2.5, we know that for \( n \geq 1 \), the number of \( (pk,des) \)-equivalence classes for \( n \)-permutations is

\[\sum_{j=0}^{\frac{(n-1)/2}{2}} ((n - j - 1) - j + 1) = \sum_{j=0}^{\frac{(n-1)/2}{2}} (n - 2j), \]

which is easily shown to be equal to \( \lfloor (n + 1)^2/4 \rfloor \). This proves (d). \( \square \)
Note that \((pk, des)\) and \((val, des)\) are \(rc\)-equivalent statistics, and that \((val, des)\) and \((epk, des)\) are equivalent statistics. Thus, by Corollary 3.7 and Theorem 3.2, we know that \((val, des)\) and \((epk, des)\) are also shuffle-compatible and have shuffle algebras isomorphic to \(A_{(pk, des)}\).

5.6. Shuffle-compatibility of \((lpk, des)\)

We now prove the shuffle-compatibility of \((lpk, des)\) and characterize its shuffle algebra.

**Theorem 5.10** (Shuffle-compatibility of \((lpk, des)\)).

(a) The pair \((lpk, des)\) is shuffle-compatible.

(b) The linear map on \(A_{(lpk, des)}\) defined by

\[
\pi \mapsto \begin{cases}  
\frac{t^{lpk(\pi)}(y + t)^{des(\pi)} - t^{lpk(\pi)}}{1 - t}|\pi| + 1} x^{|\pi|}, & \text{if } |\pi| \geq 1, \\
\frac{1}{1 - t}, & \text{if } |\pi| = 0,
\end{cases}
\]

is a \(Q\)-algebra isomorphism from \(A_{(lpk, des)}\) to the span of

\[
\left\{ \frac{1}{1 - t} \bigcup \left\{ \frac{(1 + yt)^n}{(1 - t)^n + 1} x^n \right\} \bigcup \left\{ \frac{t^j(y + t)^{k - j}(1 + yt)^{n - j - k}(1 + y)^2j}{(1 - t)^{n+1}} x^n \right\} \bigg| n \geq 2, 1 \leq j \leq \lfloor n/2 \rfloor, j \leq k \leq n - j \right\},
\]

a subalgebra of \(Q[[t^*]][x, y]\).

(c) The \(n\)th homogeneous component of \(A_{(lpk, des)}\) has dimension \(\lfloor n^2/4 \rfloor + 1\).

**Proof.** By Lemma 4.6 of [31], we have the formula

\[
h(x)(1 - te(xy)h(x))^{-1} = \frac{1}{1 - t} + \sum_{n=1}^{\infty} \sum_{L=L_n} \frac{t^{lpk(L)}(y + t)^{des(L)} - t^{lpk(L)}}{(1 - t)^n + 1} x^n r_{L}.
\]

Let \(r_{n,j,k}\) denote \(r_{\alpha}^{lpk, des}\) where \(\alpha\) is the \((lpk, des)\)-equivalence class of compositions corresponding to \(n\)-permutations with \(j\) left peaks and \(k\) descents. Define \(s_{n,p,q}\) by

\[
\sum_{n,p,q=0}^{\infty} x^n y^p t^q s_{n,p,q} = h(x)(1 - te(xy)h(x))^{-1}.
\]

Then the proofs for parts (a) and (b) follow in the same manner as for Theorem 5.9 using Proposition 2.6 and Corollary 5.5 along the way.

By Proposition 2.6, the number of \((lpk, des)\)-equivalence classes for \(n\)-permutations is

\[
1 + \sum_{j=1}^{\lfloor n/2 \rfloor} ((n - j) - j + 1) = 1 + \sum_{j=1}^{\lfloor n/2 \rfloor} (n - 2j + 1),
\]

which is easily shown to be equal to \(\lfloor n^2/4 \rfloor + 1\). This proves (c).
Although \((\text{lpk, des})\) and \((\text{rpk, des})\) are not equivalent, \(r\)-equivalent, \(c\)-equivalent, or \(rc\)-equivalent, this argument does show that \((\text{rpk, des})\) is shuffle-compatible and has shuffle algebra isomorphic to that of \((\text{lpk, des})\) because \((\text{lpk, des})\) is \(r\)-equivalent to \((\text{rpk, asc})\) — where \(\text{asc}\) is the number of ascents — and \((\text{rpk, asc})\) is equivalent to \((\text{rpk, des})\).

## 5.7. Shuffle-compatibility of \((\text{udr, des})\)

Finally, we prove our result for the pair \((\text{udr, des})\) and derive from it the analogous result for \(\text{udr}\), the number of up-down runs.

**Theorem 5.11** (Shuffle-compatibility of \((\text{udr, des})\)).

1. The pair \((\text{udr, des})\) is shuffle-compatible.
2. The linear map on \(\mathcal{A}_{(\text{udr, des})}\) defined by
   \[
   [\pi]_{(\text{udr, des})} \mapsto \begin{cases} 
   \frac{N_{\pi}}{(1-t)(1-t^2)^{\|\pi\|}} & \text{if } |\pi| \geq 1, \\
   1/(1-t) & \text{if } |\pi| = 0,
   \end{cases}
   \]
   where
   \[
   N_{\pi} = t^{\text{udr}(\pi)}(1+y)^{\text{udr}(\pi)-1}(1+yt^2)^{|\pi|-\text{des}(\pi)-\lfloor \text{udr}(\pi)/2 \rfloor}(y+yt^2)^{\text{des}(\pi)-\lfloor \text{udr}(\pi)/2 \rfloor}
   \]
   \[
   \times (1+yt)^{\lfloor \text{udr}(\pi)/2 \rfloor - \lfloor \text{udr}(\pi)/2 \rfloor - \lfloor \text{udr}(\pi)/2 \rfloor + \lfloor \text{udr}(\pi)/2 \rfloor},
   \]
   is a \(\mathbb{Q}\)-algebra isomorphism from \(\mathcal{A}_{(\text{udr, des})}\) to the span of
   \[
   \left\{ \frac{1}{1-t} \right\} \bigcup \left\{ \frac{t(1+yt)(1+yt^2)^{n-1}}{(1-t)(1-t^2)^n} x^n \right\}_{n \geq 1} \\
   \bigcup \left\{ \frac{t^j(1+y)^{j-1}(1+yt^2)^{n-k-\lfloor j/2 \rfloor}(y+yt^2)^{k-\lfloor j/2 \rfloor} S_j x^n}{(1-t)(1-t^2)^n} \right\}_{n \geq 1, j \geq 1, \lfloor j/2 \rfloor \leq n \leq \lfloor j/2 \rfloor},
   \]
   where \(S_j\) is \(1+yt\) if \(j\) is odd and is \(y+t\) if \(j\) is even, a subalgebra of \(\mathbb{Q}[[t^*]][x, y]\).
3. The \(n\)th homogeneous component of \(\mathcal{A}_{(\text{udr, des})}\) has dimension \(\binom{n}{2} + 1\).

**Proof.** By Lemma 4.11 of [31], together with Lemma 2.2 (b) and (c), we have
   \[
   (1-t^2 h(x) e(xy))^{-1}(1+th(x)) = \frac{1}{1-t} + \sum_{n=1}^{\infty} \sum_{L \in \mathcal{N}} \frac{N_L}{(1-t)(1-t^2)^n} x^n r_L \]
   where
   \[
   N_L = t^{\text{udr}(L)}(1+y)^{\text{udr}(L)-1}(1+yt^2)^{\text{des}(L)-\lfloor \text{udr}(L)/2 \rfloor}(y+yt^2)^{\text{des}(L)-\lfloor \text{udr}(L)/2 \rfloor}
   \]
   \[
   \times (1+yt)^{\lfloor \text{udr}(L)/2 \rfloor - \lfloor \text{udr}(L)/2 \rfloor - \lfloor \text{udr}(L)/2 \rfloor + \lfloor \text{udr}(L)/2 \rfloor}.
   \]
Note that \([u dr(L)/2] - [u dr(L)/2]\) is 1 if \(u dr(L)\) is odd and is 0 if \(u dr(L)\) is even. The left-hand side of (5.11) is monoidlike by Lemma 5.3 and Corollary 5.5.

Let \(r^n_{u dr, des}\) denote \(r^\alpha_{u dr, des}\) where \(\alpha\) is the \((u dr, des)\)-equivalence class of compositions corresponding to \(n\)-permutations with \(j\) up-down runs and \(k\) descents. Then by (5.11) and Proposition 2.7, we have

\[
(1 - t^2 h(x)e(xy))^{-1}(1 + th(x)) = \frac{1}{1 - t} + \sum_{n=1}^{\infty} \left( \frac{t(1 + yt)(1 + yt^2)^{n-1}}{(1 - t)(1 - t^2)^n} \right) r^n_{u dr, des} + \sum_{2 \leq j \leq n \atop \lfloor j/2 \rfloor \leq k \leq n - \lfloor j/2 \rfloor} t^j(1 + y)^{j-1}(1 + y^2)^{n-k - \lfloor j/2 \rfloor} (y + t)^{k - \lfloor j/2 \rfloor} S_j x^n r^n_{u dr, des} \tag{5.12}
\]

with \(S_j\) as in the statement of the theorem. Define \(s_{n,p,q}\) by

\[
\sum_{n,p,q=0}^{\infty} x^n y^p t^q s_{n,p,q} = (1 - t^2 h(x)e(xy))^{-1}(1 + th(x)). \tag{5.13}
\]

To prove (a) and (b), as in Theorems 5.9 and 5.10 it is sufficient to show that each \(r^n_{u dr, des}\) is in the span of the \(s_{n,p,q}\). Because of the floor and ceiling functions in (5.12), we are not able to use the generating function inversion method that we used in the proofs of Theorems 5.9 and 5.10 so we take a different approach.

Expanding the right side of (5.12) and comparing with (5.13) shows that, for fixed \(n\), each \(s_{n,p,q}\) is a linear combination (with integer coefficients) of the \(r^n_{u dr, des}\). We will show that these relations can be inverted to express each \(r^n_{u dr, des}\) as a linear combination of the \(s_{n,p,q}\).

We totally order \(\mathbb{N} \times \mathbb{N}\) colexicographically, so \((p_1, q_1) \leq (p_2, q_2)\) if and only if \(q_1 < q_2\) or \(q_1 = q_2\) and \(p_1 \leq p_2\). We shall show that for each \(j\) and \(k\), there exist \(p\) and \(q\) such that \(r^n_{u dr, des}\) appears with coefficient 1 in \(s_{n,p,q}\) and if \(r^n_{u dr, des}\) appears in \(s_{n,p,q}\) then \((k', j') \leq (k, j)\).

This will imply, by induction, that \(r^n_{u dr, des}\) is in \(\text{Span}_q\{s_{n,p,q}\}\).

With this total order, the monomial \(y^p t^q\) with minimal \((p, q)\) that appears in the coefficient of \(x^n r^n_{u dr, des}\) on the right side of (5.12) is easily seen to be \(y^{k_j} t^{j}\) (with coefficient 1), where \(k_j\) is \(k - \lfloor j/2 \rfloor + 1\) if \(j\) is even and is \(k - \lfloor j/2 \rfloor\) if \(j\) is odd. In other words, \(s_{n,p,q}\) does not contain any \(r^n_{u dr, des}\) for which \((p, q) < (k_j, j)\). Replacing \(p\) and \(q\) with \(k_j\) and \(j\), and replacing \(k\) and \(j\) with \(k'\) and \(j'\), we have that

\[
s_{n,k_j,j} = r^n_{u dr, des} + \sum_{j', k'} c_{j', k'} r^n_{u dr, des}
\]

where \(c_{j', k'} = 0\) unless \((k_j', j') < (k_j, j)\). It is easy to see that \((k_j', j') < (k_j, j)\) implies \((k', j') < (k, j)\), so we have

\[
s_{n,k_j,j} = r^n_{u dr, des} + \sum_{(k', j') < (k, j)} c_{j', k'} r^n_{u dr, des}
\]

and this completes the proof of (b).
By Proposition \[2.7\] the number of (udr, des)-equivalence classes for \(n\)-permutations is

\[
1 + \sum_{j=2}^{n} (n - \lfloor j/2 \rfloor - \lceil j/2 \rceil + 1) = 1 + \sum_{j=2}^{n} (n - j + 1) = 1 + \binom{n}{2}.
\]

This proves part (c).

We know from Lemma \[2.2\] that udr and (lpk, val) are equivalent statistics, from Lemma \[2.1\] (d) that val is equivalent to epk, and from Proposition \[3.4\] that (lpk, val) is \(rc\)-equivalent to (lpk, pk). It follows that (udr, des) is equivalent to (lpk, val, des) and (lpk, epk, des), and is \(rc\)-equivalent to (lpk, pk, des). Thus, by Theorem \[3.2\] and Corollary \[3.7\] the statistics (lpk, val, des), (lpk, epk, des), and (lpk, pk, des) are all shuffle-compatible and have shuffle algebras isomorphic to \(A_{\text{udr,des}}\).

**Theorem 5.12** (Shuffle-compatibility of the number of up-down runs).

(a) The number of up-down runs udr is shuffle-compatible.

(b) The linear map on \(A_{\text{udr}}\) defined by

\[
[\pi]_{\text{udr}} \mapsto \begin{cases} 
2^{\text{udr}(\pi)} - 1 - 2^{\text{udr}(\pi)}(1 + t^2)\lvert\pi\rvert - \text{udr}(\pi) \times \lvert\pi\rvert, & \text{if } \lvert\pi\rvert \geq 1, \\
(1 - t)^2(1 - t^2)^{\lvert\pi\rvert - 1}x^{\lvert\pi\rvert} / (1 - t), & \text{if } \lvert\pi\rvert = 0,
\end{cases}
\]

is a \(Q\)-algebra isomorphism from \(A_{\text{udr}}\) to the span of

\[
\left\{ \frac{1}{1 - t} \right\} \bigcup \left\{ \frac{2^{j-1}t^j(1 + t^2)^n - j}{(1 - t)^2(1 - t^2)^{n-1}} x^n \right\}_{n \geq 1, 1 \leq j \leq n},
\]

a subalgebra of \(Q[[t^\ast]][x]\).

(c) For \(n \geq 1\), the \(n\)th homogeneous component of \(A_{\text{udr}}\) has dimension \(n\).

**Proof.** Let \(\phi\) be the homomorphism from \(Q[[t^\ast]][x, y]\) to \(Q[[t^\ast]][x]\) obtained by setting \(y\) to 1. It is easy to check that \(\phi\) takes the image of \([\pi]_{\text{udr,des}}\) as described in Theorem \[5.11\] (b) to the image of \([\pi]_{\text{udr}}\) as given in (b). Then (a) and (b) follow from Theorem \[3.3\]. Part (c) follows from Proposition \[2.7\].
6. Miscellany

6.1. An alternate description of the pk and (pk, des) shuffle algebras

In Section 5.5, we showed that the (pk, des) shuffle algebra $A_{(pk,des)}$ is isomorphic to the span of

$$\left\{ \frac{1}{1-t} \right\} \bigcup \left\{ \frac{t^{j+1}(y+t)^{k-j}(1+yt)^{n-j-k-1}(1+y)^{2j+1}}{(1-t)^{n+1}} \right\}_{n \geq 1, 0 \leq j \leq \lfloor (n-1)/2 \rfloor, j \leq k \leq n-j-1}$$

where the multiplication is the Hadamard product in $t$. Let

$$P_{n,j,k}(y,t) := t^{j+1}(y+t)^{k-j}(1+yt)^{n-j-k-1}(1+y)^{2j+1}$$

for $n \geq 1, 0 \leq j \leq \lfloor (n-1)/2 \rfloor$, and $j \leq k \leq n-j-1$. Then by [26, Corollary 4.3.1], we can write

$$\frac{P_{n,j,k}(y,t)}{(1-t)^{n+1}} = \sum_{p=1}^{\infty} R_{n,j,k}(p,y) t^p$$

where $R_{n,j,k}(p,y)$ is a polynomial in $p$ of degree at most $n$, with coefficients that are polynomials in $y$. In this section, we give a simple description of the span of the polynomials $R_{n,j,k}(p,y)$, which yields an alternate characterization of the (pk, des) shuffle algebra that was stated in part (c) of Theorem 5.9. Similarly, a simple description of the span of the polynomials $R_{n,j,k}(p,1)$ yields an alternate characterization of the pk shuffle algebra, which is part (c) of Theorem 4.8.

It is simpler to work with the following transformations of the polynomials $R_{n,j,k}(p,y)$ and $P_{n,j,k}(y,t)$; let

$$Q_{n,j,k}(p,z) := (1-z)^n R_{n,j,k} \left( p, \frac{1+z}{1-z} \right)$$

and let

$$A_{n,j,k}(t,z) := (1-z)^n P_{n,j,k} \left( \frac{1+z}{1-z}, t \right)$$

$$= (1-z)^n t^{j+1} \left( \frac{1+z}{1-z} + t \right)^{k-j} \left( 1 + \frac{1+z}{1-z} t \right)^{n-j-k-1} \left( 1 + \frac{1+z}{1-z} \right)^{2j+1}$$

$$= 2^{2j+1} t^{j+1} (1+t+z(1-t))^{k-j} (1+z(1-t))^{n-j-k-1},$$

so that

$$\frac{A_{n,j,k}(t,z)}{(1-t)^{n+1}} = \sum_{p=1}^{\infty} Q_{n,j,k}(p,z) t^p. \quad (6.1)$$

Also, define $\bar{A}_{n,j,k}(t,z)$ by

$$\frac{\bar{A}_{n,j,k}(t,z)}{(1-t)^{n+1}} = \sum_{p=0}^{\infty} Q_{n,j,k}(-p,z) t^p. \quad (6.2)$$

Lemma 6.1. Each $Q_{n,j,k}(p,z)$, as a polynomial in $p$, has no constant term.
Proof. By [26, Proposition 4.2.3], from (6.1) and (6.2) follows the equality of rational functions
\[
\frac{A_{n,j,k}(t, z)}{(1 - t)^{n+1}} = -\frac{A_{n,j,k}(1/t, z)}{(1 - (1/t))^{n+1}},
\]
which implies
\[
A_{n,j,k}(t, z) = (-1)^{n+1} A_{n,j,k}(1/t, z)
= (-1)^{n+1} 2^{2j+1} t^{n+1} \left( \frac{1}{t} \right)^{j+1} \left( 1 + \frac{1}{t} + z \left( 1 - \frac{1}{t} \right) \right)^{k-j} \left( 1 + \frac{1}{t} - z \left( 1 - \frac{1}{t} \right) \right)^{n-j-k-1}
= (-1)^{n+1} 2^{2j+1} t^{n+1} (1 + t - z (1-t))^{k-j} (1 + t + z (1-t))^{n-j-k+1}.
\]
Evaluating at \( t = 0 \) yields \( A_{n,j,k}(0, z) = 0 \), so by (6.2), \( Q_{n,j,k}(0, z) = 0 \).

Lemma 6.2. Let \( n \geq 1 \). Then the polynomials \( Q_{n,j,k}(p, z) \) for \( 0 \leq j \leq \lfloor (n-1)/2 \rfloor \) and \( j \leq k \leq n - j - 1 \) are linearly independent.

Proof. It is easy to see that the polynomials \( P_{n,j,k}(y, t) \) are linearly independent, and that a linear dependence relation for the polynomials \( Q_{n,j,k}(p, z) \) would imply a linear dependence relation for the polynomials \( P_{n,j,k}(y, t) \).

Essentially the same argument can be used to show that the polynomials \( R_{n,j,k}(p, y) \) are also linearly independent.

Theorem 6.3. Let \( n \geq 1 \). Then
\[
\text{Span}_Q \{ Q_{n,j,k}(p, z) \}_{0 \leq j \leq \lfloor (n-1)/2 \rfloor, \quad 0 \leq k \leq n-j-1} = \text{Span}_Q \{ p^{n-a} z^{a-2b} \}_{0 \leq a \leq n-1, \quad 0 \leq b \leq \lfloor a/2 \rfloor}
\]

Proof. First, we show that each \( Q_{n,j,k}(p, z) \) can be written as a linear combination of the polynomials \( p^{n-a} z^{a-2b} \). Note that
\[
\sum_{p=1}^{\infty} Q_{n,j,k}(p, z) t^p = \frac{2^{2j+1} t^{j+1} (1 + t + z (1-t))^{k-j} (1 + t - z (1-t))^{n-j-k+1}}{(1 - t)^{n+1}}
\]
is a linear combination of terms of the form
\[
\frac{z^l t^q (1 - t)^l}{(1 - t)^{n+1}} = \frac{t^q z^l}{(1 - t)^{n-l+1}} = \sum_{p=0}^{\infty} z^l \binom{n - l + p - q}{n - l} t^p
\]
where \( 0 \leq l \leq n-2j-1 \) and \( j+1 \leq q \leq n-j-l \). Moreover, \( \binom{n-1+p-q}{n-l} \) is a polynomial in \( p \) of degree \( n-l \), so it is a linear combination of \( 1, p, p^2, \ldots, p^{n-l} \). This shows that each \( Q_{n,j,k}(p, z) \) is a linear combination of terms of the form \( p^{n-a} z^l \) with \( n-a \leq n-l \), or equivalently, \( l \leq a \), and \( a \leq n-1 \) by Lemma 6.1. We set \( c = a - l \), so that \( p^{n-a} z^l = p^{n-a} z^{a-c} \). It remains to show that \( c \) must be even.

Observe that \( (-p)^{n-a} (-z)^{a-c} = (-1)^a (-1)^c p^{n-a} z^{a-c} \). Thus, it suffices to show that \( Q_{n,j,k}(p, z) = (-1)^a Q_{n,j,k}(-p, -z) \). Recall that
\[
\bar{A}_{n,j,k}(t, z) = (-1)^{n+1} A_{n,j,k}(1/t, z),
\]
Therefore, we have
\[
\tilde{A}_{n,j,k}(t, -z) = (-1)^n t^{n+1} A_{n,j,k}(1/t, -z).
\]
Since
\[
A_{n,j,k}(t, z) = 2^{2j+1} t^{j+1} (t + 1 - z(t - 1))^{k-j}(t + 1 + z(t - 1))^{n-j-k+1}
\]
\[
= 2^{2j+1} t^{n+1} \left( \frac{1}{t} \right)^{j+1} \left( 1 + \frac{1}{t} - z \left( 1 - \frac{1}{t} \right) \right)^{k-j} \left( 1 + \frac{1}{t} + z \left( 1 - \frac{1}{t} \right) \right)^{n-j-k+1}
\]
\[
= t^{n+1} A_{n,j,k}(1/t, -z),
\]
we have
\[
\sum_{p=1}^{\infty} (-1)^n Q_{n,j,k}(p, z) t^p = \frac{(-1)^n A(t, z)}{(1-t)^{n+1}}
\]
\[
= \frac{(-1)^n t^{n+1} A(1/t, -z)}{(1-t)^{n+1}}
\]
\[
= \frac{\tilde{A}(t, -z)}{(1-t)^{n+1}}
\]
\[
= \sum_{p=1}^{\infty} Q_{n,j,k}(-p, -z) t^p.
\]

Therefore, \(Q_{n,j,k}(p, z) = (-1)^n Q_{n,j,k}(-p, -z)\), so each \(Q_{n,j,k}(p, z)\) is a linear combination of the polynomials \(p^{n-a}z^{a-2b}\).

Since we know that the polynomials \(Q_{n,j,k}(m, z)\) are linearly independent, it suffices to show that the two sets of polynomials have the same cardinality. The restrictions \(0 \leq a \leq n - 1\) and \(0 \leq b \leq \lfloor a/2 \rfloor\) can be reformulated as \(0 \leq b \leq \lfloor (n - 1)/2 \rfloor\) and \(2b \leq a \leq n - 1\); the restriction on \(b\) matches the condition on \(j\), and the number of possible values of \(a\) for a fixed \(b\) is equal to the number of possible values of \(k\) for a fixed \(j\). Hence, the two sets are equinumerous and thus their spans are equal.

We are now ready to prove our alternate characterization of \(A_{(pk,des)}\) and of \(A_{pk}\).

**Proof of Theorem 5.9 (c).** In this proof, we identify \(A_{(pk,des)}\) with its characterization given in part (b) of Theorem 5.9.

Let \(\psi: A_{(pk,des)} \to \mathbb{Q}[p, x, y]\) be the linear map defined by
\[
\psi \left( \sum_{p=1}^{\infty} R_{n,j,k}(p, y) t^p x^n \right) = R_{n,j,k}(p, y) x^n
\]
and \(\psi(1/(1-t)) = 1\). With the usual multiplication of \(\mathbb{Q}[p, x, y]\), it is easy to see that \(\psi\) is an algebra homomorphism, and thus restricts to an algebra isomorphism from \(A_{(pk,des)}\) to the subalgebra of \(\mathbb{Q}[p, x, y]\) spanned by the \(R_{n,j,k}(p, y) x^n\).

Observe that
\[
\text{Span}_\mathbb{Q} \{ R_{n,j,k}(p, y) \}_{0 \leq j \leq \lfloor (n-1)/2 \rfloor, \ 0 \leq k \leq n-j-1} = \text{Span}_\mathbb{Q} \{ p^{n-a}(1+y)^n(1-y)^{a-2b} \}_{0 \leq a \leq n-1, \ 0 \leq b \leq \lfloor a/2 \rfloor}
\]
this is immediate from the previous theorem and applying the inverse transformation: dividing by \((1-z)^n\) and setting \(z = (y-1)/(1+y)\). Then the result follows.
Proof of Theorem 4.8 (c). First note that setting $y = 1$ in the basis for $A_{(pk, \text{des})}$ given by part (b) of Theorem 5.9 gives the basis for $A_{pk}$ described in part (b) of Theorem 4.8. Thus setting $y = 1$ in the basis for $A_{(pk, \text{des})}$ given by part (c) of Theorem 5.9 will give a spanning set for $A_{pk}$.

The only polynomials $p^n - a(1 + y)^n(1 - y)^{a - 2b}x^n$ that are nonzero after setting $y = 1$ are those for which $a = 2b$, yielding the polynomials $2^n p^n - 2b x^n$ for $0 \leq 2b \leq n - 1$. The span of these polynomials is equal to the span of $p^j x^n$ for $1 \leq j \leq n$ with $j$ having the same parity as $n$.

We note that part (c) of Theorem 4.8 can also be proven using Stembridge’s self-reciprocity property for enriched order polynomials [28, Proposition 4.2].

Unfortunately, we were unable to use the approach in this section to give an alternate characterization of any of the shuffle algebras $A_{lpk}$, $A_{(lpk, \text{des})}$, $A_{udr}$, or $A_{(udr, \text{des})}$.

6.2. Non-shuffle-compatible permutation statistics

Although many well-known descent statistics have been shown to be shuffle-compatible, there are many descent statistics that are not shuffle-compatible. Here we list some of them.

**Theorem 6.4.** The set $P_k \cup \text{Val}$ and the tuples $(pk, \text{val})$, $(pk, \text{val}, \text{des})$, $(P_k, \text{des})$, $(P_k, \text{val})$, $(P_k, \text{val}, \text{des})$, $(P_k, \text{val})$, $(Lpk, \text{des})$, $(Lpk, \text{val}, \text{des})$, and $(Epk, \text{des})$ are not shuffle-compatible.

Recall that a birun of a permutation is a maximal monotone consecutive subsequence, and that $br(\pi)$ is the number of biruns of $\pi$. The number of biruns is not shuffle-compatible, and the only joint statistics involving $br$ that we have found that seem to be shuffle-compatible are $(Lpk, br)$ and $(Epk, br)$; however, these are easily shown to be equivalent to Epk, which is shuffle-compatible (see the discussion following Conjecture 6.7).

**Theorem 6.5.** The number of biruns $br$ and the tuples $(br, \text{des})$, $(br, \text{maj})$, $(br, \text{des}, \text{maj})$, $(br, pk)$, $(br, pk, \text{des})$, $(br, lpk)$, $(br, lpk, \text{des})$, and $(Pk, br)$ are not shuffle-compatible.

Although $(\text{des}, \text{maj})$ is shuffle-compatible, we have not found any other shuffle-compatible joint statistics involving the major index.

**Theorem 6.6.** The tuples $(pk, \text{maj})$, $(pk, \text{des}, \text{maj})$, $(lpk, \text{maj})$, $(lpk, \text{des}, \text{maj})$, $(Pk, \text{maj})$, $(Lpk, \text{maj})$, $(Lpk, \text{des}, \text{maj})$, $(udr, \text{maj})$, $(udr, \text{des}, \text{maj})$, and $(lir, \text{maj})$ are not shuffle-compatible.

In addition to the descent statistics examined in this paper, we mention that there are two additional families of descent statistics, one based on the classical notion of double descents and one based on the more recent notion of alternating descents. We say that $i$ (where $2 \leq i \leq n - 1$) is a double descent of $\pi \in P_n$ if $\pi_{i-1} > \pi_i > \pi_{i+1}$; then we can define the double descent set and double descent number—as well as variations of these such as the left double descent set and left double descent number—in the obvious way. We say that $i \in [n - 1]$ is an alternating descent if $i$ is an even ascent or an odd descent; then we can define the alternating descent set, alternating descent number, and alternating major index in the obvious way. Alternating descents were introduced by Chebikin [2] and have been more recently studied by Remmel [21] and by the present authors [8].
Aside from the alternating descent set—which is equivalent to the descent set—none of these statistics mentioned above are shuffle-compatible. Among joint statistics that involve one or more of these statistics, we have not found any that seem to be shuffle-compatible (other than a few that are equivalent to statistics that we know to be shuffle-compatible).

Lastly, among permutation statistics that are not descent statistics, we have not found any that seem to be shuffle-compatible.

6.3. Open problems and conjectures

To conclude this paper, we state a couple permutation statistics that we conjecture to be shuffle-compatible based on empirical evidence, and a few more general open problems and conjectures on the topic of shuffle-compatibility.

**Conjecture 6.7.** The tuples \((\text{udr, pk})\) and \((\text{udr, pk, des})\) are shuffle-compatible.

In a preliminary version of this paper, we included as part of Conjecture [6.7] the conjectured shuffle-compatibility of the exterior peak set \(\text{Epk}\) and the tuples \((\text{Pk, val, des}), (\text{Pk, udr}), (\text{Lpk, val}),\) and \((\text{Lpk, val, des})\). All of these have been addressed by Darij Grinberg. Specifically, Grinberg proved that \(\text{Epk}\) is shuffle-compatible using a \(P\)-partition argument [11], noted that \((\text{Pk, udr})\) and \((\text{Lpk, val})\) are both equivalent to \(\text{Epk}\), and found counterexamples showing that \((\text{Pk, val, des})\) and \((\text{Lpk, val, des})\) are not shuffle-compatible [13].

Prior to this, Grinberg had shown that \(Q\text{Sym}\) is a “dendriform algebra” [12], an algebra whose multiplication can be split into a “left multiplication” and a “right multiplication” satisfying certain nice axioms. Together with the shuffle-compatibility of \(\text{Epk}\), Grinberg proved that \(\mathcal{A}_{\text{Epk}}\) is a dendriform quotient of \(Q\text{Sym}\). More generally, he proved that a descent statistic is a dendriform quotient of \(Q\text{Sym}\) if and only if it is both “left-shuffle-compatible” and “right-shuffle-compatible”, which are combinatorial conditions that, together, refine the notion of shuffle-compatibility. Other descent statistics that Grinberg has shown to be both left- and right-shuffle-compatible include the descent number \(\text{des}\), the pair \((\text{des, maj})\), and the left peak set \(\text{Lpk}\). On the other hand, the major index \(\text{maj}\), the peak set \(\text{Pk}\), and the right peak set \(\text{Rpk}\) are neither left- nor right-shuffle-compatible.

From Theorem [6.4] we know that a pair of two shuffle-compatible statistics need not be shuffle-compatible. Hence, we pose the following question.

**Question 6.8.** Suppose that \(\text{st}_1\) and \(\text{st}_2\) are shuffle-compatible statistics. Are there simple conditions that imply that the pair \((\text{st}_1, \text{st}_2)\) is shuffle-compatible?

Similarly, if a pair is shuffle-compatible, then that does not imply that the individual statistics in the pair are both shuffle-compatible.

**Question 6.9.** Suppose that the pair \((\text{st}_1, \text{st}_2)\) is shuffle-compatible. Are there simple conditions that imply that \(\text{st}_1\) and \(\text{st}_2\) are both shuffle-compatible?

Recall that Goulden [10] and Stadler [24] gave combinatorial proofs for the shuffle-compatibility of \((\text{des, maj})\), and in Section [2.4] we provided combinatorial proofs for the shuffle-compatibility of the descent set \(\text{Des}\) and partial descent sets \(\text{Des}_{i,j}\).

**Question 6.10.** Can we find combinatorial proofs for the shuffle-compatibility of other statistics?
Finally, we present the following conjecture.

**Conjecture 6.11.** *Every shuffle-compatible permutation statistic is a descent statistic.*

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**A. Tables of permutation statistics**

The following table summarizes every permutation statistic st that we know to be shuffle-compatible, along with their shuffle algebra $A_{st}$ and the dimension of the $n$th homogeneous component of $A_{st}$.

| Permutation Statistic | Shuffle Algebra | Dimension of $n$th Homogeneous Component |
|-----------------------|-----------------|----------------------------------------|
| Des                   | QSym            | $2^{n-1}$                              |
| des                   | Theorem 4.6     | $n$                                    |
| maj, comaj            | Theorem 3.1     | $(\binom{n}{2}) + 1$                   |
| (des, maj), (des, comaj) | Theorem 4.5     | $(\binom{n}{3}) + n$                   |
| Pk, Val               | Algebra of peaks II | $F_n$                              |
| pk, val, epk          | Theorem 4.8     | $\lfloor (n + 1)/2 \rfloor$              |
| Lpk, Rpk              | Algebra of left peaks $\Pi^{(\ell)}$ | $F_{n+1}$                              |
| lpk, rpk, lr          | Theorem 4.10    | $\lfloor n/2 \rfloor + 1$              |
| Des$_{1,0}$, Des$_{0,1}$, sir, lir, sfr, lfr | | 2 |
| Des$_{i,j}$           |                 | $2^{i+j}$ (if $i + j \leq n - 1$)       |
| (pk, des), (val, des), (epk, des) | Theorem 5.9 | $\lfloor (n + 1)^2/4 \rfloor$              |
| (lpk, des), (rpk, des), (lr, des) | Theorem 5.10 | $\lceil n^2/4 \rceil + 1$                |
| udr, (lpk, val), (lpk, pk), (lpk, epk), (rpk, val), (rpk, pk), (rpk, epk), (lr, val), (lr, pk), (lr, epk) | Theorem 5.12 | $n$                                    |
| (udr, des), (lpk, val, des), (lpk, epk, des), (lpk, pk, des) | Theorem 5.11 | $(\binom{n}{2}) + 1$                   |
| Epk                   | $\mathbb{F}_{11}$ | $F_{n+2} - 1$                              |
The next table gives a partial list of equivalences, $r$-equivalences, $c$-equivalences, and $rc$-equivalences among permutation statistics that are studied in this paper. Not all of these are explicitly proven in this paper, but the proofs are very straightforward. We leave out some redundancies such as sir $\sim lir$—omitted since we include sir $\sim lir$—as well as equivalences like $\sim (Lpk, val, des) \sim (Lpk, br, des)$, which is an immediate consequence of $\sim (Lpk, val) \sim (Lpk, br)$.

Table 2: Equivalences among permutation statistics

| Equivalences | $r$-Equivalences | $c$-Equivalences | $rc$-Equivalences |
|--------------|------------------|------------------|------------------|
| Des $\sim Lpk \cup Val$ $\sim (Lpk, Val)$ | $Lpk \sim_r Rpk$ | $Ppk \sim_c Val$ | $(pk, des) \sim_{rc} (val, des)$ |
| val $\sim epk$ | $lpk \sim_r rpk$ | pk $\sim_c val$ | $(lpk, val) \sim_{rc} (lpk, pk)$ |
| rpk $\sim epk$ | sir $\sim_r lfr$ | | |
| rpk $\sim lr$ | sfr $\sim_r lir$ | | |
| udr $\sim (lpk, val)$ | | | |
| Epk $\sim (Epk, val)$ $\sim (Epk, udr)$ $\sim (Epk, br)$ $\sim (Lpk, val)$ $\sim (Lpk, udr)$ $\sim (Ppk, udr)$ | | | |
| sir $\sim lir \sim Des_{1,0}$ | | | |
| sfr $\sim lfr \sim Des_{0,1}$ | | | |
| $(Ppk, val)$ $\sim (Ppk, br)$ | | | |
| $(Lpk, val)$ $\sim (Lpk, br)$ | | | |
| $(pk, val) \sim (pk, br) \sim (val, br)$ | | | |

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