CLASSIFICATION OF OBSTRUCTED BUNDLES OVER A VERY GENERAL SEXTIC SURFACE AND MESTRANO-SIMPSON CONJECTURE

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Abstract. Let $S \subset \mathbb{P}^3$ be a very general sextic surface over complex numbers. Let $\mathcal{M}(H, c_2)$ be the moduli space of rank 2 stable bundles on $S$ with fixed first Chern class $H$ and second Chern class $c_2$. In this article we will classify the obstructed bundles in $\mathcal{M}(H, c_2)$ for small $c_2$. Using this classification we will make an attempt to prove Mestrano-Simpson conjecture on the number of irreducible components of $\mathcal{M}(H, 11)$ and prove the conjecture partially. We will also show that $\mathcal{M}(H, c_2)$ is irreducible for $c_2 \leq 10$.

1. Introduction

Let $S$ be a smooth projective irreducible surface over $\mathbb{C}$ and $H$ be an ample divisor on $S$. Let $r \geq 1$ be an integer, $L$ be a line bundle on $S$, and $c_2 \in H^4(S, \mathbb{Z}) \simeq \mathbb{Z}$. The moduli space of semistable torsion free sheaves on $S$ (w.r.t $H$) with fixed determinant $L$ and second Chern class $c_2$ was first constructed by Gieseker and Maruyama (see [12], [18]) using Mumford’s geometric invariant theory and proved that it’s a projective scheme (need not be reduced). After their construction many people have studied the geometry of this moduli space. The study has been done by fixing the underlying surface. For example when the surface is rational Barth [8], Costa and Rosa Maria [9], Le Potier [17] proved that the moduli space is reduced, irreducible and rational under certain conditions on rank and Chern classes. When the surface is K3 it has been studied by Mukai [19] and many others. When the surface is general, Jun Li [16] showed that for $c_2$ big enough, the moduli space is also of general type. The guiding general philosophy is that the geometry of the moduli space is reflected by the underlying geometry of the surface. The first result without fixing the underlying surface was given by O’Grady. In [20] O’Grady proved that for sufficiently large second Chern class $c_2$, the moduli space is reduced, generically smooth and irreducible. In fact O’Grady’s first step to prove irreducibility was to show each component is generically smooth of expected dimension. The generic smoothness result was also proved by Donaldson [11] in the rank 2 and trivial determinant case and Zuo [23] for arbitrary determinant.

After O’Grady’s result it was important to give an effective bound on $c_2$ for the irreducibility and generic smoothness of the moduli space. The moduli space of vector bundles over hypersurfaces is one of the important objects to study. When the underlying surface $S$ is a very general quintic hypersurface in $\mathbb{P}^3$ Simpson and Mestrano studied this question systematically and in [10], the current author with K. Dan, studied the question related to Brill-Noether loci. In a series of papers [1], [2], [3], Simpson and Mestrano proved that the moduli space of rank 2 $H$–stable torsion free sheaves with fixed determinant $H$ is generically smooth and irreducible, where $H := \mathcal{O}_S(1)$. This result was known before by an unpublished work by Nijsse for $c_2 \geq 16$ [22].

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Motivated by the results of Mestrano and Simpson we look at the next case i.e. the moduli space of rank 2 torsion free sheaves on a very general sextic surface $S$, that is, a very general hypersurface of degree 6 in $\mathbb{P}^3$.

In [2], Simpson and Mestrano showed that the moduli space of stable rank 2 bundles over sextic surface is not irreducible for $c_2 = 11$. In fact, they have shown that the moduli space in this case has at least two different irreducible components.

In fact, they constructed a 12-dimensional irreducible component $M_1$ consisting of vector bundles fitting in an exact sequence of the form

$$0 \to O_S \to E \to J_Z(1) \to 0,$$

where $Z$ is a zero-dimensional locally complete intersection subscheme contained in a rational cubic curve and a family of vector bundles $M_2$ of dimension at least 13 consisting of vector bundles fitting in an exact sequence of the form

$$0 \to O_S \to E \to J_Z(1) \to 0,$$

where $Z$ is a zero-dimensional locally complete intersection subscheme contained in a hyperplane section. Then they conjectured that $M_1$ and $M_2$ are the only two components. In other words, $M_2$ is an irreducible component of dimension at least 13 and $M_1$ and $M_2$ cover $\mathcal{M}(H,11)$, where $\mathcal{M}(H,c_2)$ denotes the moduli space of $H$-stable rank 2 locally free sheaves on $S$ with fixed determinant isomorphic to $H$ and second Chern class $c_2$.

Our main goal in this article is to classify the obstructed bundles of rank 2 over a very general sextic surface and apply it to give a partial proof of the above conjecture made by Mestrano and Simpson. Further more the natural questions one can ask are the following:

1) Is the moduli space irreducible for $c_2 \leq 10$ ?

2) Can one give an effective bound for $c_2$ such that the moduli space becomes irreducible ?

Let $S \subset \mathbb{P}^3$ be a very general sextic surface over $\mathbb{C}$ and $H$ denote the very ample line bundle $O_S(1)$. The Picard group of $S$ is generated by $H$. Let $\mathcal{M}(H,c_2)$ denote the moduli space of $H$-stable rank 2 locally free sheaves on $S$ with fixed determinant isomorphic to $H$ and second Chern class $c_2$ and $\overline{\mathcal{M}}(H,c_2)$ be the Gieseker-Maruyama moduli space of semistable torsion free sheaves.

It is known that $\overline{\mathcal{M}}(H,c_2)$ is projective and $\mathcal{M}(H,c_2)$ sits inside $\overline{\mathcal{M}}(H,c_2)$ as an open subset, whose complement is called the boundary.

The very first and main step towards a proof of the conjecture is to show that there is a bijection between the irreducible components of $\mathcal{M}(H,c_2)$ and $\Sigma_1(c_2) := \{E \in \mathcal{M}(H,c_2) : h^0(E) \neq 0\}$. Major part of this article is devoted to prove the above fact. The main idea to prove that fact is to investigate the obstructed bundles and the space of obstructions. Using it we also give a bound for the boundary strata of the moduli space. More precisely, we will prove the following Theorems.

**Theorem 1.1.** If $c_2 \leq 17$, then there is a bijection between the irreducible components of $\Sigma_1(c_2) := \{E \in \mathcal{M}(H,c_2) : h^0(E) \neq 0\}$ and the irreducible components of $\mathcal{M}(H,c_2)$.

**Theorem 1.2.** The moduli space $\mathcal{M}(H,c_2)$ is non-empty for $c_2 \geq 5, c_2 \neq 7$ and it is irreducible for $c_2 \leq 10$.

**Theorem 1.3.** Suppose $\mathcal{M}(H,c_2)$ is good for $c_2 \geq 27$. Then $\overline{\mathcal{M}}(H,c_2)$ is also good for $c_2 \geq 27$.

Finally as an application we will give a partial proof of the Mestrano-Simpson conjecture, more precisely we will prove that $M_1, M_2$ cover $\mathcal{M}(H,11)$ and $M_2$ is of dimension exactly 13.
containing an irreducible subset of dimension 13 and possibly one more irreducible component of dimension 12.

In particular, \( h^0(E) > 0 \) for any stable bundle \( E \in \mathcal{M}(H,11) \).

1.1. The organization of the paper: In section 2, we will recall some basic results which we will use in the subsequent sections. In section 3, we study the singularity of \( S \cap X \), where \( X \) is a hypersurface of degree 5. We show that there is no of degree 5 hypersurfaces \( X \) such that \( S \cap X \) is singular along a zero-dimensional subscheme \( P \) satisfying Cayley-Bacharach property for \( \mathcal{O}(5) \).

In section 4, we will classify the obstructed bundles. In fact, we will show that if \( c_2 \leq 18 \), then every vector bundle \( E \in \mathcal{M}(H,c_2) \) either admits a non-zero section or the space of obstructions \( H^2(\text{End}^0(E)) \) vanishes. In other words, \( E \in \mathcal{M}(H,c_2) \) is obstructed if and only if \( h^0(E) \neq 0 \).

In section 5, we will prove the Theorem 1.1 and as an application we will give a partial proof of the Mestrano-Simpson conjecture.

In section 6, we will show that the moduli space \( \mathcal{M}(H,c_2) \) is non-empty and irreducible for \( 5 \leq c_2 \leq 10 \), \( c_2 \neq 7 \) and for \( c_2 \leq 4 \) and \( c_2 = 7 \), it is empty.

Finally in section 7, we will give an upper bound for the boundary strata and using it we will show that if \( \mathcal{M}(H,c_2) \) is good for \( c_2 \geq 27 \) then \( \mathcal{M}(H,c_2) \) is also good for \( c_2 \geq 27 \). In other words, every component of \( \mathcal{M}(H,c_2) \) has the dimension equals to the expected dimension \( 4c_2 - 39 \).

Notation and convention

For the line bundle \( \mathcal{O}_{\mathbb{P}^3}(n) \) we write simply \( \mathcal{O}(n) \) and for a subscheme \( X \subset \mathbb{P}^3 \), we denote the pull back of \( \mathcal{O}(n) \) to \( X \) by \( \mathcal{O}_X(n) \).

If \( S \) is a very general hypersurface of degree 6 then, \( \text{Pic}(S) \simeq \mathbb{Z} \). It is not difficult to see that \( h^i(S,\mathcal{O}_S(n)) = h^i(\mathbb{P}^3,\mathcal{O}(n)) \). Thus a zero-dimensional subscheme \( P \subset S \) which satisfies Cayley-Bacharach property for \( \mathcal{O}_S(m) \) also satisfies Cayley-Bacharach property for \( \mathcal{O}(m) \) and vice-versa.

Thus if a zero-dimensional subscheme \( P \subset S \) satisfies Cayley-Bacharach property for \( \mathcal{O}_S(m) \) then with out loss of generality we can assume that \( P \) satisfies Cayley-Bacharach property for \( \mathcal{O}(m) \) in \( \mathbb{P}^3 \) and we write \( P \) satisfies CB(m).

We will always denote the length a zero dimensional scheme \( P \) by \( |P| \). Throughout the article, \( X_i \) denotes an irreducible hypersurface of degree \( i \) and by \( X = X_i \cap X_j \) we mean that \( X \) is decomposed into irreducible components of degree \( i \) and \( j \).

2. Preliminaries

In this section we will recall few results which we need in next sections.

**Theorem 2.1.** Let \( P \) be a set of points in \( \mathbb{P}^r \), and let \( d \geq 2 \) be an integer. If, for all \( k \geq 1 \), no \( dk + 2 \) points of \( P \) lie in a projective \( k \)-plane, then \( P \) impose independent conditions on forms of degree \( d \); in fact there is a multilinear form of degree \( d \) containing any subset consisting of all but one of the points, but missing the last.

**Proof.** See [6] Theorem 2.

**Theorem 2.2.** (Chasles) If a set \( \Gamma_1 \) of 8 points in \( \mathbb{P}^2 \) lies in the complete intersection \( \Gamma \) of two cubics, then any cubic vanishing on \( \Gamma_1 \) vanishes on \( \Gamma \).

**Proof.** See [7] Corollary 2.8.
Lemma 2.11. Note that any \( m \) points of a plane lying in an intersection of two cubics in \( \mathbb{P}^3 \), where \( m \geq 9 \), satisfies the Cayley-Bacharach property for \( \mathcal{O}(3) \) in \( \mathbb{P}^3 \).

Let \( L = L_d^{(r)}(−\sum_{i=1}^{n} 2p_i) \) be the linear system consisting of hypersurfaces of degree at most \( d \) in \( r+1 \) variables that are singular at \( \{p_i\} \). Then the expected dimension of \( L \) is \( \max\{-1, (r^2 + 4d − 1 − n(r + 1))\} \).

Definition 2.4. Then \( L \) is called special if the dimension of \( L \) is not of expected dimension.

Definition 2.5. In an \( n \) dimensional projective space \( \mathbb{P}^n \), we say that a subset of points are in general linear position if no \( k \) of them lie in a \( k − 2 \) plane in \( \mathbb{P}^n \).

Theorem 2.6. (Alexander-Hirschowitz Theorem): Fix \( r \geq 2 \) and \( d \geq 2 \), and consider the linear system \( L = L_d^{(r)}(−\sum_{i=1}^{n} 2p_i) \) consisting of hypersurfaces of degree at most \( d \) in \( r+1 \) variables that are singular at \( n \) general points \( \{p_i\} \). Then

(a) For \( d = 2 \), the linear system \( L \) is special if and only if \( 2 \leq n \leq r \).

(b) For \( d \geq 3 \), the linear system \( L \) is special if and only if the triple \( (r, d, n) \) is one of the following: \((2, 4, 5), (3, 4, 9), (4, 4, 14), (4, 3, 7)\).

Proof. See [4].

Remark 2.7. In the above theorem general points mean points in general linear position.

Remark 2.8. Note that, if \( P, P' \) are two sets of \( m \) and \( l \) points in \( \mathbb{P}^3 \) which together impose independent conditions on hypersurfaces of degree \( d \), then the dimension of degree \( d \) hypersurfaces which are singular along \( P \) and passes through \( P' \) is \( h^0(\mathcal{O}(d)) - (4m+l) \) provided \( (d, m) \neq (4, 9) \). In particular if \( 4m+l \geq h^0(\mathcal{O}(d)) \) then there is no such hypersurface of degree \( d \).

Definition 2.9. Let \( X \) and \( Y \) be closed subschemes of the \( n \) dimensional projective space \( \mathbb{P}^n_K \) over a fixed algebraically closed field \( K \), where \( n \) is a positive integer. The residual scheme \( \text{Res}_Y(X) \) of \( X \) with respect to \( Y \) is the closed subscheme of \( \mathbb{P}^n_K \) whose ideal sheaf is defined by the division(scolon) ideal sheaf \( \mathcal{J}_{\text{Res}_Y}(X) = (\mathcal{J}_X : \mathcal{J}_Y) \), where \( \mathcal{J}_X \) and \( \mathcal{J}_Y \) are the ideal sheaves of \( X \) and \( Y \) respectively.

Example 2.10. Let \( X = 2P_0 \subset \mathbb{P}^n \) be the two-fat point(i.e double point) defined by \( \mathbb{P}^2 = (x_1, ..., x_n)^2 \), and let \( H \) be the hyperplane \( \{x_n = 0\} \). Then, the residual scheme \( \text{Res}_H(Z) \subset \mathbb{P}^n \) is defined by: \( \text{Res}_H(Z) = (\mathbb{P}^2 : (x_n)) = (x_1, ..., x_n) = \mathbb{P} \), hence, it is a simple point of \( \mathbb{P}^n \).

2.1. Singularity of intersection. Let \( S \) be a smooth surface and \( P \) be a singular point of a curve \( C \subset S \), we use \( e(P, C) \) to denote the multiplicity of \( C \) at \( P \), that is, if \( \pi : W \to S \) is the blow-up of \( S \) at \( P \), and \( E \) is the exceptional divisor, then \( \pi^*C = C^* + e(P, C)E \). Here \( C^* \) is the proper transform of \( C \) by \( \pi \). If \( \{q_1, q_2, ..., q_s\} = C^* \cap E \), then the points \( q_i \) are said to be the infinitely near points of \( P \) on \( C \) of the first order. Inductively, infinitely near points of \( q_i(i=1, 2, ..., s) \) on \( C^* \) of \( j \)-th order are said to be the infinitely near points of \( P \) on \( C \) of the \( (j + 1) \)-th order. We define \( e(q_i, C) = e(q_i, C^*) \), and so on. If \( P_{ij}(j = 0, 1, ..., n_0) \) are all the singular points on \( C \), \( P_{ij}(j = 0, 1, ..., n_1) \) are all the infinitely near points on \( C \) of the \( i \)-th order, \( e_{ij} = e(P_{ij}, C) \), and \( E_{ij} \) is the exceptional divisor resulting from the blowing up at \( P_{ij} \), then \( C \) has a type \( \mu = (\mu_{ij}, P_{ij}, E_{ij})(i, j) \in \Gamma \) singularity with \( \Gamma = \{(i, j)|\mu_{ij} > 1\} \).

Lemma 2.11. Assume \( C = \{F = 0\} \cap \{G = 0\} \) is a reduced and irreducible curve on a smooth surface \( S = \{F = 0\} \) in \( \mathbb{P}^3 \), \( \deg(F) = d \), \( \deg(G) = k \), and \( C \) has a type \( \mu = (\mu_{ij}, P_{ij}, E_{ij})(i, j) \in \Gamma \) singularity. If \( Q \in H^0(\mathcal{O}(m)) \) is not in the homogeneous polynomial ideal \( (F, G) \) generated by \( F \) and \( G \), and the curve \( \{Q = 0\} \) on \( S \) has a weak type \( \mu - 1 = (\mu_{ij} - 1, P_{ij}, E_{ij})(i, j) \in \Gamma \)
singularity, then
\[ \sum_{(i,j) \in \Gamma} \mu_{ij}(\mu_{ij} - 1) \leq dkm. \]

Proof. See [24] Lemma 2.5] \hfill \Box

Remark 2.12. Note that if, $S$ is a very general hypersurface of degree $d \geq 5$ in $\mathbb{P}^3$ and $C$ is a curve in $S$ given by an irreducible hypersurface section of degree $< d$, then, $C$ satisfies the hypothesis of the above Lemma. Let $P_1, \ldots, P_m$ be $m$-nodes on $C$. In other words, $C$ has a type $\mu = (2, P_i, E_i | i \in \{1, 2, \ldots, n\})$ singularity. Then for a general hyperplane $H \in H^0(\mathcal{O}(1))$, $S \cap H$ has a weak type $\mu - 1$ singularity. Thus by above Lemma, we have $2m \leq dk$. In other words, $C$ can have at most $\frac{dk}{2}$ nodes.

3. Singularity of complete intersection curves along points satisfying Cayley
Bacharach property for line bundle on $\mathbb{P}^3$

Let $S$ be a very general irreducible hypersurface of degree 6 in $\mathbb{P}^3$. Let $P \subset S$ be a zero dimensional locally complete intersection subscheme satisfying $CB(5)$. In this section we shall show that if $P$ is not contained in a quadratic hypersurface, then $h^0(S, \mathcal{J}_P^2(5)) = 0$, where $\mathcal{J}_P$ denotes the ideal sheaf of $P$. More precisely,

Theorem 3.1. Let $S$ be a very general hypersurface of degree 6 in $\mathbb{P}^3$. Let $Z$ be a zero dimensional locally complete intersection subscheme of $S$ of length $l \geq 17$ satisfying $CB(5)$ and not contained in any quadratic hypersurface. Then $h^0(S, \mathcal{J}_Z^2(5)) = 0$.

To prove the Theorem we need the following easy Lemma.

Lemma 3.2. Fix an integer $d$. Let $P_1, P_2, P_3$ be three zero dimensional locally complete intersection subschemes in $\mathbb{P}^3$ satisfying the following property:
(a) $P_i, i = 1, 2$ is contained in a union of at most $k$ lines in a plane say $H_i, i = 1, 2$ with $k(d+1) \leq \left(\frac{d+2}{2}\right) - 1$.
(b) $P_3$ has length at most $d+1$ and contained in a line in another plane $H_3$ or $P_3$ has length at most $2d+1$ and contained in union of two lines in a plane $H_3$.
Then $P = P_1 \cup P_2 \cup P_3$ fails to satisfy $CB(d)$.

Proof. Since a line imposes $d+1$ independent conditions on sections of $\mathcal{O}(d)$, $P_i$ can impose at most $k(d+1)$ conditions on degree $d$ forms on $H_i$. Let $P_i$ imposes $m_i$ independent conditions on forms of degree $d$ in $H_i$. Then, since $k(d+1) < \left(\frac{d+2}{2}\right)$ any point on $H_i$ not lying in the union of those $k$ lines together with $P_i$ impose $m_i + 1$ independent conditions on forms of degree $d$ on $H_i$. In other words, given any point in $H_i$ not lying in the union of those $k$ lines, one can always find a degree $d$ form which contains those $k$ lines but not the given point. Since $P_3$ is contained in a union of at most two lines say, $l_1, l_2$ and they are not in the plane $H_1$ and $H_2$, the forms of degree $d$ on $H_1H_2$ have no base point in $l_1 \cup l_2$.

Assume $P_3$ has length at most $d+1$ and is contained in a line $l$. Since a general degree form intersects a line in $d$ points and the forms of degree of $d$ in $H_1H_2$ have no base point on $l$, it is always possible to find a degree $d$ form which contains all the points of $P_1 \cup P_2$ and all but one points of $P_3$ but not all of $P_3$. In other words, $P$ fails to satisfy $CB(d)$.

Let $P_3$ has length at most $2d+1$ and contained in a pair of lines $l_1, l_2$ in $H_3$. Since two lines in a plane together impose $2d+1$ independent conditions on forms of degree $d$, as earlier case one can find a degree $d$ form which contains $P_1 \cup P_2$ and all but one point of $P_3$ but not the whole $P_3$, which concludes the lemma. \hfill \Box
Remark 3.3. Note that if a line contains more than 3 points of \(\text{Sing}(S \cap X_i), i \leq 5\) then \(X_i\) has to be singular along the line itself. In fact, a line can not be tangent to \(S\) at more than 3 points as the multiplicity of each such point is 2 and a line intersects a sextic surface at 6 points (counted with multiplicity).

Proof of Theorem 3.1. Let \(Z\) be a zero dimensional locally complete intersection subscheme of length \(c_2 + 12\). If \(X \in H^0(\mathcal{H}_2^3(5))\) is irreducible then by remark 2.8, \(S \cap X\) have almost 15 nodes, which is a contradiction for \(c_2 \geq 5\). So \(X\) is reducible. Thus we have the following possibilities:

**Case(1):** \(X = X_1.X_4\).

**Case(2):** \(X = X_1.X_2.X'_2\).

**Case(3):** \(X = X_2.X_3\).

**Case(4):** \(X = X_1.X'_1.X_3\).

**Case(5):** \(X = X_1.X'_1.X''_1.X'''.1.X_1''\).

**Case(6):** \(X = X_1.X_1'.X_2'.X_2\).

We consider each case separately and show that they can not occur.

**Case(1):** \(X = X_1.X_4\)

Note that,

\[Z \subset \text{Sing}(S \cap X) = \text{Sing}(S \cap X_1) \cup \text{Sing}(S \cap X_4) \cup (S \cap X_1 \cap X_4)\]

where, \(\text{Sing}(S \cap X_i) := \) the singular locus of \(S \cap X_i\) which are not in the intersection.

If \(Z \cap \text{Sing}(S \cap X_4) = \emptyset\), then we have, \(Z \subset X_1 \subset X_1.H\), for some other plane \(H\), which means \(Z\) sits inside a quadric, a contradiction.

If \(Z \cap \text{Sing}(S \cap X_4) \neq \emptyset\), then by remark 2.11 we have, \(|Z \cap \text{Sing}(S \cap X_4)| \leq 12\).

Since \((Z \cap \text{Sing}(S \cap X_1)) \cup (Z \cap S \cap X_1 \cap X_4) \subset X_1\), \(\text{Res}_{X_1}(Z) \subset Z \cap \text{Sing}(S \cap X_4)\) satisfies CB(4), we have \(6 \leq |\text{Res}_{X_1}(Z)| \leq 12\). By \cite{3} Proposition 17.8, it follows that \(\text{Res}_{X_1}(Z)\) either lies in a plane (say \(H\)) or in a double line or in a pair of skew lines. If the last two possibilities occur, then by remark 3.3 \(X_4\) has to be singular along a union of two lines or along a double line. In that case one can see that \(X_4\) has to be reducible. Thus \(\text{Res}_{X_1}(Z)\) lies in a plane \(H\). Thus \(Z \subset X_1.H = \text{a quadric}, \) a contradiction.

**Case(2):** \(X = X_1.X_2.X'_2\).

Note the residual subscheme of \(Z\) with respect to \(X_1\) is \(Z \cap X_2 \cap X'_2\). If the length of the residual subscheme is \(\leq 13\) then they satisfy CB(4). In this situation one can easily see that the residual subscheme lie on a plane. Hence \(Z\) is contained in a quadric hypersurface. If the residual subscheme has length \(\geq 14\) then by remark 2.8 they are not in general position. Thus there is a plane containing at least 4 points.

(a) there is a plane \(H\) containing 5 points of the residual subscheme such that no 4 of them lie in a line then they imposes independent conditions on quadrics (by Theorem 2.11) on \(H\), a contradiction.

(b) If there is a plane \(H\) containing at least 8 points of the residual subscheme then also one can find five points in \(H\), no four of them lie on a line, a contradiction.

Let \(H\) be a plane containing at least 4 points not satisfying the conditions on (a) and (b). If the residual sub-scheme with respect to \(X_1H\) has length \(\leq 10\), then as they satisfies CB(3), one can easily see that they all lie on a plane. If the residual sub-scheme has length \(\geq 11\), then again we will get a plane \(G\) containing at least 4 points. If \(H\) and \(G\) do not cover the residual sub-scheme then there exist at least three skew lines contained in \(X_2\) and \(X'_2\), a contradiction. Thus \(X_1, G, H\) cover \(Z\) and have the configuration as in Lemma 3.2 a contradiction.
Case (3): $X = X_2X_3$.
Note that, the cubic $X_3$ contains all the points of $Z \setminus \text{Sing}(S \cap X_2)$. Thus by [2, 12] the residual of $Z$ with respect to $X_3$ has length at most 6 which satisfies CB(2). Hence if the residual is non-empty, it has length $\geq 4$ [3, Proposition 17.8]. If $Z \setminus \text{Sing}(S \cap X_2)$ has length $\leq 5$, then by remark [3, 3], $X_2$ has to be reducible. Thus $Z \setminus \text{Sing}(S \cap X_2)$ has length 6 and all lie on a plane $H$. In other words, they lie on $H \cap X_2$. In fact in this situation $H \cap X_2$ is pair of lines and the lines (in particular the plane $H$) have to be tangent to $S$ at those 6 points, a contradiction to the fact that no plane can be tangent to $S$ at more than 3 points. Thus $Z \setminus \text{Sing}(S \cap X_2)$ is empty.

Similarly, if $Z \setminus \text{Sing}(S \cap X_3)$ has cardinality $\geq 8$ then they all lie on a plane $H$ and $X_3$ is singular at these points. These only can happen if $H \cap X_3$ is union of a double line and a line (i.e., $H \cap X_3 = l^2_{l_1}$). The line $l$ has to contain at least 5 and $l_2$ contains at least 2 points. But in such position of at most 9 points fails to satisfy CB(3). Thus $Z \setminus \text{Sing}(S \cap X_3)$ has length $\leq 6$ and all lie on a line $l$.

In this situation one can choose a plane $H$ which contains only $l$. If the residue with respect to $l$ is $\leq 13$ then one can easily conclude that they all lie in a plane on contained in a pair of lines or in a double line. If the residue has length $\geq 14$, then as in the previous case one can show that either they all lie in 3 skew lines or in union of two planes. In the later case, each plane contains at most two lines which contain all the residual points. Thus $Z$ is in a configuration as in Lemma [3, 2]. Hence fails to satisfy CB(5), a contradiction.

Case (4): $X = X_1X_1'.X_3$.
Arguing similar to the case (2), one can get also a contradiction in this case.

Case (5): $X = X_1X_1'.X_1''.X_1'''$. 
In this case, if all the pairwise intersection $X_1 \cap X_1'$ etc., are non-empty, then they all have length at least 5. Thus the length of $Z$ is bigger than 30, a contradiction. Thus there exist at least one intersection say, $X_1 \cap X_1'$ is empty. Thus $Z$ is contained in $X_1'' \cup X_1''' \cup X_1'$ and have the configuration as in Lemma [3, 2] a contradiction.

Case (6): $X = X_1X_1'.X_2$.
As in case (2), $\text{Sing}(S \cap X_2)$ is empty. Thus $Z$ is contained in $X_1 \cup X_1' \cup X_1''$. Also note all the residual subschemes with respect to $X_1X_1', X_1X_1''$ and $X_1'X_1''$ are non-empty (otherwise $Z$ is itself contained in a quadratic hypersurface). If all such residual subschemes have length $\leq 7$ then they lie on lines and if the length is $\geq 8$ then one can easily see that they all lie on a union of at most 2 lines in a plane. Thus $Z$ lies in the union of 3 planes in the configuration as in Lemma [3, 2] Thus $Z$ fails to satisfy CB(5).

4. Obstructed bundles on a very general sextic surface in $\mathbb{P}^3$

Let $S$ be a smooth irreducible projective surface over the field of complex numbers. Let, $\mathcal{M}(r, H, c_2)$ be the moduli space of rank $r$, $H$-stable vector bundles with fixed determinant $H$ and second Chern class $c_2$, where $H := c_1(O_S(1))$. Consider a point $E \in \mathcal{M}(r, H, c_2)$. Then the obstruction theory is controlled by

$$\text{Obs}(E) := H^2(S, \text{End}^0(E)).$$

Here, $\text{End}^0(E) := \text{Ker}(\text{tr} : \text{End}(E) \rightarrow O_S)$.

By Serre duality,$$
H^2(S, \text{End}^0(E)) \cong H^0(S, \text{End}^0(E) \otimes K_S)^*.
$$
So, $\text{Obs}(E) \neq \{0\}$ if and only if there exists a non-zero element $\varphi \in H^0(S, \text{End}^0(E) \otimes K_S)$. In other words, there exists a twisted endomorphism
\[ \varphi : E \rightarrow E \otimes K_S, \text{with} \quad \text{tr}(\varphi) = 0. \]
A bundle $E$ is said to be obstructed if $\text{Obs}(E) \neq \{0\}$, otherwise we call $E$ to be unobstructed.

Here we only consider the case when $r = 2$.

Let $E \in \mathcal{M}(2, H, c_2)$ be a point which fits into an exact sequence
\[ 0 \rightarrow L \rightarrow E \rightarrow J_P \otimes L' \rightarrow 0, \tag{4.1} \]
where $P$ is a zero-dimensional subscheme of $S$ and $J_P$ denotes the ideal sheaf of $P$. Then we have $\det(E) \cong L \otimes L'$,
\[ E^* \otimes L' \cong E \otimes L^*, \]
and there is an exact sequence
\[ 0 \rightarrow E \otimes (L')^* \rightarrow \text{End}^0(E) \rightarrow J_P^2 \otimes L' \otimes L^* \rightarrow 0. \tag{4.2} \]

From now on we specialize to the case when $S$ is a very general sextic surface in $\mathbb{P}^3$. In this section we will classify the obstructed bundles on $S$.

Let $S \subset \mathbb{P}^3$ be a very general smooth surface of degree 6. Then the canonical line bundle $K_S \simeq \mathcal{O}_S(2)$. We denote by $\mathcal{M}(H, c_2)$ the moduli space of rank 2, $\mu-$stable vector bundles $E$ on $S$, with respect to $H := \mathcal{O}_S(1)$ and with $c_1(E) = H$ and $c_2(E) = c_2$. Note that in this case the stability and semistability are same. The expected dimension of $\mathcal{M}(H, c_2)$ is $4c_2 - c_2^2 - 3\chi(\mathcal{O}_S) = 4c_2 - 39$.

**Theorem 4.1.** If $5 \leq c_2 \leq 18$, then a point $E \in \mathcal{M}(H, c_2)$ is obstructed if and only if $E$ admits a non-zero section.

**Proof.** Note that, the Euler characteristic of a point $E \in \mathcal{M}(H, c_2)$ is equal to $19 - c_2$ which is $> 0$ for $c_2 \leq 18$. Thus $h^0(S, E) + h^2(S, E) = h^0(S, E) + h^0(S, E(1)) > 0$ for $c_2 \leq 18$. In particular, $H^0(S, E(1)) \neq 0$. Let us assume $E$ does not have any section. Then we have the following exact sequence:
\[ 0 \rightarrow \mathcal{O}_S(-1) \rightarrow E \rightarrow J_P(2) \rightarrow 0 \tag{4.3} \]
where, $J_P$ is the ideal sheaf of a zero-dimensional subscheme $P$ of length $12 + c_2$. Since $E$ is locally free, $P$ is locally complete intersection and satisfies CB(5).

Also taking $L = \mathcal{O}_S(-1)$ and $L' = \mathcal{O}_S(2)$ in (4.2) we have:
\[ 0 \rightarrow E \otimes \mathcal{O}_S(-2) \rightarrow \text{End}^0(E) \rightarrow J_P^2(3) \rightarrow 0. \tag{4.4} \]
Tensoring (4.4) by $\mathcal{O}_S(2)$ and considering the cohomology sequence, we have
\[ h^0(S, \text{End}^0(E) \otimes \mathcal{O}_S(2)) \leq h^0(S, E) + h^0(S, J_P^2(5)) = h^0(S, J_P^2(5)). \]
Thus the Theorem will follow if $h^0(S, J_P^2(5)) = 0$.

But by Theorem 5.1 if $P$ is not in any quadratic hypersurface, then $h^0(S, J_P^2(5)) = 0$. If $P$ lies in a quadratic hypersurface, then from the cohomology sequence of the exact sequence (4.3) we have $h^0(S, E) \neq 0$.

Conversely, let $E$ admits a section. Then we have the following exact sequence:
\[ 0 \rightarrow \mathcal{O}_S \rightarrow E \rightarrow J_P(1) \rightarrow 0 \tag{4.5} \]
where, \( J_P \) is the ideal sheaf of a zero-dimensional subscheme \( P \) of length \( c_2 \). Since \( E \) is locally free, \( P \) is locally complete intersection and satisfies CB(3).

Also taking \( L = \mathcal{O}_S \) and \( L' = \mathcal{O}_S(1) \) in (4.2) we have:

(4.6) \[
0 \to E \otimes \mathcal{O}_S \to \text{End}^0(E) \otimes \mathcal{O}_S(1) \to J_P^2(2) \to 0.
\]

Tensoring (4.4) by \( \mathcal{O}_S(1) \) and considering the cohomology sequence, we have

\[
h^0(S, \text{End}^0(E) \otimes \mathcal{O}_S(2)) \geq h^0(S, E(1)),
\]

which concludes the theorem. \( \square \)

5. Mestrano-Simpson conjecture

In this section we will prove the Mestrano-Simpson conjecture partially. Let us recall few results from [2] and the Mestrano-Simpson conjecture.

**Theorem 5.1.** The space of bundles \( E \) fitting into an exact sequence of the form

\[
0 \to \mathcal{O}_S \to E \to J_P(1) \to 0
\]

where, \( P \) is a length 11 subscheme of \( C \cap S \) for \( C \) a rational normal cubic in \( \mathbb{P}^3 \), consists of a single 12-—dimensional generically smooth irreducible component of the moduli space \( \mathcal{M}(H, 11) \) of stable bundles on \( S \).

**Proof.** See [2, Corollary 11.3]. \( \square \)

**Theorem 5.2.** The space of bundles \( E \) fitting into an exact sequence of the form

\[
0 \to \mathcal{O}_S \to E \to J_P(1) \to 0
\]

where, \( P \) is a length 11 subscheme of \( H \cap S \) for \( H \) a hyperplane, general with respect to \( S \) in \( \mathbb{P}^3 \), is contained in an irreducible component of dimension \( \geq 13 \) of the moduli space \( \mathcal{M}(H, 11) \) of stable bundles on \( S \).

**Proof.** See [2, Theorem 11.4]. \( \square \)

**Conjecture 5.1.** (Mestrano-Simpson Conjecture, [2, Conjecture 11.5])

The 13—dimensional family in Theorem 5.2 constitutes a full irreducible component of \( \mathcal{M}(H, 11) \); this component is nonreduced and obstructed. Together with the 12—dimensional generically smooth component in Theorem 5.1 these are the only irreducible components of \( \mathcal{M}(H, 11) \). In particular, \( h^0(E) > 0 \) for any stable bundle \( E \in \mathcal{M}(H, 11) \).

Let us consider the subsets \( \Sigma(11) := \{ E \in \mathcal{M}(H, 11) : h^0(E) \neq 0 \} \). Then any point \( E \in \Sigma(11) \) sits in an exact sequence of the form:

(5.1) \[
0 \to \mathcal{O} \to E \to J_P(1) \to 0,
\]

where \( P \) is a locally complete intersection zero-dimensional subscheme of length 11 satisfying CB(3).

Let us consider the subsets

\[
\mathcal{M}_1 := \{ E \in \Sigma(11) : P \text{ in 5.1 lies on a rational cubic curve } \}
\]

and

\[
\mathcal{M}_2 := \{ E \in \Sigma(11) : P \text{ in 5.1 lies on a plane } \}
\]

Clearly \( \mathcal{M}_1 \) and \( \mathcal{M}_2 \) are closed subset of \( \mathcal{M}(H, 11) \) and by Theorem 5.1 \( \mathcal{M}_1 \) is an irreducible component of dimension 12.
We will first show that there is a bijection between the irreducible components of \( \mathcal{M}(\mathcal{H}, c_2) \) and the irreducible components of \( \Sigma_1(c_2) := \{ E \in \mathcal{M}(\mathcal{H}, c_2) : h^0(E) \neq 0 \} \) for \( c_2 \leq 17 \). Then we will show that if, \( c_2 = 11 \), and \( P \) is not in any hyperplane then \( P \) lies in a rational normal cubic, which will tell us that the irreducible components \( \mathcal{M}(\mathcal{H}, 11) \) different from \( \mathcal{M}_1 \) is contained in \( \mathcal{M}_2 \). In particular \( h^0(E) \neq 0 \) for all \( E \in \mathcal{M}(\mathcal{H}, 11) \).

**Proof of Theorem 1.1**

**Proof.** Let \( c_2 \leq 17 \) and \( E \in \mathcal{M}(\mathcal{H}, c_2) \) such that, \( h^0(E) = 0 \). Then the Euler characteristic computation tells that \( h^0(E(1)) \neq 0 \). If \( E \) does not belong to any irreducible component of \( \Sigma_1(c_2) \), then \( E \) is contained in an irreducible component consisting of vector bundles which fit in the following exact sequence:

\[
0 \rightarrow \mathcal{O}_S(-1) \rightarrow E \rightarrow J_P(2) \rightarrow 0
\]

where, \( P \) is a zero dimensional locally complete intersection subscheme of \( S \) of length \( c_2 + 12 \) and satisfies CB(5). By Theorem 1.1 we have that for \( c_2 \leq 18 \), the space of obstructions at \( E \) vanishes. Thus if \( E \) is not contained in any component of \( \Sigma_1(c_2) \) then the component containing \( E \) has dimension \( 4c_2 - 39 \).

On the other hand, by [1] Corollary 3.1, every component of the space of bundles \( E \) which fits in the exact sequence of the form

\[
0 \rightarrow \mathcal{O}_S(-1) \rightarrow E \rightarrow J_P(2) \rightarrow 0
\]

has dimension at least \( 3c_2 + 36 - h^0(\mathcal{O}_S(5)) - 1 = 3c_2 - 21 \). If \( c_2 \leq 17 \) then \( 3c_2 - 21 > 4c_2 - 39 \), thus there does not exist any component of dimension \( 4c_2 - 39 \) containing \( E \), a contradiction. Therefore, \( E \) lies in an irreducible component of \( \Sigma_1(c_2) \), which concludes the Proposition. \( \Box \)

**Remark 5.3.** By theorem 1.1 to know the number of irreducible components of \( \mathcal{M}(\mathcal{H}, 11) \), it is enough to know the components of \( \Sigma_1(11) \). Let \( E \in \Sigma_1(11) \). Then \( E \) fits into an exact sequence of the form,

\[
(5.2) \quad 0 \rightarrow \mathcal{O}_S \rightarrow E \rightarrow J_P(1) \rightarrow 0
\]

where, \( P \) is a length 11 subscheme of \( S \) which satisfies CB(3).

**Proposition 5.4.** If there is no hyperplane passing through the zero-dimensional subscheme \( P \) in the exact sequence 5.2, then \( P \) lies in a rational normal cubic curve.

**Proof.** Since there is no hyperplane passing through \( P \), from the cohomology sequence of 5.2 we have \( h^0(E) = 1 \). On the other hand, the Euler characteristic computation says that \( h^0(E(1)) \geq 7 \). Again tensoring 5.2 by \( \mathcal{O}_S(1) \) and considering the cohomology sequence we have, \( h^0(J_P(2)) \geq 3 \). In other words, \( P \) lies in the intersection of at least 3 quadrics. If all the quadrics have a common component \( H \) where, \( H \) is a hyperplane, then one can show that \( P \) lies on \( H \), a contradiction. Choose two quadrics \( Q_1, Q_2 \) with out having a common component and passing through \( P \). Let \( Q_3 \) be another quadric which is not in the span of \( Q_1, Q_2 \). If \( Q_3 \) does not contain any component of \( Y =: Q_1 \cap Q_2 \), then \( Q_3 \) intersects \( Y \) at 8 points, a contradiction. Thus \( Q_3 \) contains a component of \( Y \).

**Case I:** \( Q_3 \) contains a component, say, \( C \) of degree 1.

Then the remaining components of \( Y \) intersect \( Q_3 \) at 6 points. Thus \( C \) contains at least 5 points. Thus there always exists a plane \( H \) containing at least 6 points and the residual of \( P \) with respect to \( H \) satisfies CB(2), hence the residual has length at least 4 and at most 5. Thus they all lie in a line, say \( l \). Clearly \( l \) can not intersect \( C \), otherwise we will get a plane containing 9 points hence all of \( P \). Thus \( C \) and \( l \) are skew lines. If \( C \) contains exactly 5 points of \( P \), then
10 points of $P$ lie in a pair of skew lines and one point out side the union of $C$ and $l$. In such a situation one can see that $P$ fails to satisfy CB(3), a contradiction. If $C$ contains 6 points then $l$ can contain exactly 4 points, again one can see that $P$ can not satisfy CB(3).

Case II: $Q_3$ contains a component $C$ of degree 2.

In this case the remaining components of $Y$ intersects $Q_3$ at 4 points. Thus $C$ contains at least 7 points. Since any degree 2 space curve lie in a plane, there is a plane containing 7 points of $P$. In this case arguing as case I, one can get a contradiction. Thus the only possibility is $Q_3$ contains a component $C$ of degree 3. Now any degree three space curve is either a plane curve or it is a rational normal cubic curve. In case of plane curve we can easily get a contradiction.

Thus $C$ is a rational normal cubic which contains at least 9 points of $P$. If the remaining points of $P$ do not lie on $C$, then $P$ imposes independent conditions on sections of $\mathcal{O}(3)$, hence fails to satisfy CB(3), a contradiction. Therefore, $P$ lies on $C$, which concludes the proposition.

\[ \square \]

**Proposition 5.5.** The subset $M_2$ is of 13 dimensional and it can have at most 2 irreducible components, one of dimension 13 and possibly one of dimension 12.

**Proof.** By Theorem 5.2, $M_2$ has dimension at least 13. Note that $M_2$ can be identified as a subspace of the space of triples,

$$\{(P, H, E) \in \text{Hilb}^{11}(S) \times \mathbb{P}(H^0(\mathcal{O}(1))) \times \mathbb{P}(H^1(\mathcal{J}_P(3))) : P \subset H\}$$

If $H$ is in general position with respect to $S$ and $P$ is a general set of points of length 11, then one can show (as in [2]), that such collection of triples constitute a 13 dimensional irreducible subset $M_2^1$.

Consider the subset $M_2^2 := \{(P, H, E) \in M_2$ such that $P$ is contained in an irreducible cubic curve in $H$. Then since an irreducible plane cubic curve intersects another cubic curve at 9 points, $h^0(\mathcal{J}_P(3)) = 11$ and such $P$ satisfies CB(3). Thus the space of extension $\text{Ext}^1(\mathcal{J}_P(1), \mathcal{O}_S)$ has dimension 2. Also note that dimension of such sub-schemes in a plane irreducible cubic curve $C$ is zero (there are only finitely many choices in $S \cap C$). Using monodromy argument one can show that $M_2^2$ is an irreducible subset of $M_2$.

On the other hand, we have $h^0(\mathcal{J}_P(1)) = 1$, since $P$ is contained in a plane, so $h^0(E) = 2$. This means that for a given bundle $E$, the space of choices of sections (modulo scaling) leading to the subscheme $P$, has dimension 1. Hence the dimension of the space of bundles obtained by this construction is one less than the dimension of the space of subschemes.

Count now the dimension of the space of choices of $P$: there is a three-dimensional space of choices of the plane $H$, and for each one we have an 9-dimensional space of choices of irreducible cubic curves in the plane $H$ and for each choice of $P$, one dimensional space of extensions upto scalars. This gives $\dim\{P\} = 3 + 9 + 1 - 1 = 12$.

Let us consider the other subsets, that is the sub-schemes $P$ such that the dimension of the space of extensions $\text{H}^1(\mathcal{J}_P(3))$ is high. So let us consider the set $\triangle_{H,i} := \{P \in \text{Hilb}^{11}(S \cap H) : h^0(H, \mathcal{J}_{P,H}(3)) = 1 + i\}$. Note that for $P \in \triangle_{H,i}, h^1(\mathcal{J}_P(3)) = i + 2$. Consider the incidence variety $T = \{(Z, C) : Z \subset C \} \subset \mathbb{P}(H^0(\mathcal{O}_H(3))) \times \text{Hilb}^{11}(S \cap H)$ and let $\pi_1, \pi_2$ be the projections. Since $Z$ is contained in at least two cubic plane curve, $C$ has to be reducible. In other words, the image of $\pi_2$ has dimension at most 8. Note that the dimension of $\pi_2^{-1}(C)$ is 0.
Thus we have, $8 \geq \dim T \geq \dim \pi_1^{-1}(\triangle_{H,i}) \geq \dim \triangle_{H,i} + i$. This implies that $\dim \triangle_{H,i}$ is bounded by $8 - i$. Hence as earlier the space of bundles $E$ obtained by this construction has dimension $\leq \dim(\cup_H(\triangle_{H,i}) + \dim$ of the extensions $-1 = 3 + 8 - i + i + 1 - 1 = 11$.

But by $[\Pi]$, Corollary 3.1, every component of the space of bundles $\mathcal{M}$ contains at most $2$ irreducible components of dimension 13 and 12 respectively.

Remark 5.7. If $\mathcal{M}_1$ contains $\mathcal{M}_2$, then $\mathcal{M}_2$ is irreducible and hence the the conjecture $[5,7]$ is true.

6. Dimension estimates of $\mathcal{M}(H,c_2)$ and its boundary for $c_2 \leq 19$

It is known $[3]$ Corollary 17.10] that the moduli space $\mathcal{M}(H,c_2)$ is empty for $c_2 \leq 4$.

Proposition 6.1. For $c_2 = 5, 6, \mathcal{M}(H,c_2)$ is irreducible of dimension 2 and 3 respectively and $\mathcal{M}(H,7)$ is empty.

Proof. By Proposition $[\Pi,$ it is enough to show that $\Sigma_1(c_2), 5 \leq c_2 \leq 7$ is irreducible. Let $E \in \Sigma_1(c_2), c_2 = 5, 6, 7$. Then we have the following exact sequence

$$0 \to \mathcal{O}_S \to E \to J_P(1) \to 0$$

where, $P$ is a zero-dimensional locally complete intersection subscheme of $S$ of length $c_2$ and satisfies CB(3). Thus by $[3]$ Proposition 17.8], $P$ lies in a line $l$. Since a line intersects $S$ at 6 points, $\Sigma_1(7)$ is empty. Hence $\mathcal{M}(H,7)$ is empty.

Note that a linear form which vanishes on $P$, also vanishes on $l$. Thus $h^0(S,J_P(1)) = h^0(S,J_1(1)) = 2$. Therefore, $h^0(E) = 3$.

Case 1: $c_2 = 5$.

Let $R$ be the subsheaf of $E$ generated by its global sections and $T$ be the co-kernel in the exact sequence

$$0 \to R \to E \to T \to 0.$$

We also have an exact sequence

$$0 \to J_{\pi^{-1}(S)}(1) \to J_P(1) \to T \to 0.$$

So $T$ has length 1. It is supported on a point $p \in S$. Since $h^0(E) = 3$, $R$ is generated by 3 sections. Thus we have the following exact sequence

$$0 \to \ker \to \mathcal{O}_S^3 \to R \to 0$$

with locally free kernel. From the degree computation, we have $\ker = \mathcal{O}_S(-1)$.

On the other hand, if $p \in \mathbb{P}^3$ is a point and $G := H^0(\mathbb{P}^3, J_p(1))$, then we have the canonical exact sequence

$$0 \to \mathcal{O}(-1) \to G^* \otimes \mathcal{O} \to R_p \to 0$$

where, $R_p$ is a reflexive sheaf with $c_2(R_p) = \text{ the class of the line }$. Thus $R_p|_S$ has $c_2 = 6$. 

Thus we can conclude that \( R = \mathcal{R}_p|_S \) and \( E \equiv \mathcal{R}_p|_* \). Thus we get a map \( \Sigma_1(5) \to S \) which takes \( E \to p \), the support of \( T \), with the inverse map, \( p \to \mathcal{R}_p|_* \), which gives an isomorphism of \( \Sigma_1(5) \) to \( S \).

**Case II:** \( c_2 = 6 \).

In this case \( P = l \cap S \). In other words, \( E \) itself is generated by its sections. Thus we have an exact sequence

\[
0 \to \mathcal{O}(-1) \to \mathcal{O}_S^3 \to E \to 0,
\]

which gives a subspace of dimension 3 of the space of linear forms in \( \mathbb{P}^3 \). Let \( p \) be the base locus of this 3–dimensional subspace. Then it is easy to see that \( E \) is isomorphic to \( \mathcal{R}_p \) which defines a map \( f : \Sigma_1(6) \to \mathbb{P}^3 \). On the other hand, the restriction of the co-kernel sheaf \( \mathcal{R}_p \) to \( S \) is locally free if and only if \( p \in \mathbb{P}^3 \setminus S \). Thus image of \( f \) is contained in \( \mathbb{P}^3 \setminus S \). In other words, we have a map \( f : \Sigma_1(6) \to \mathbb{P}^3 \setminus S \) with inverse \( p \to \mathcal{R}_p|_S \), which proves the Proposition.

6.1. **When** \( 8 \leq c_2 \leq 10 \). Let \( E \in \Sigma_1(c_2), 8 \leq c_2 \leq 10 \). Then \( E \) fits in the following exact sequence:

\[
(6.1) \quad 0 \to \mathcal{O}_S \to E \to \mathcal{J}_P(1) \to 0
\]

where, \( P \) is a zero-dimensional subscheme of \( S \) of length \( c_2 \) and \( P \) satisfies CB(3). Thus by [3, Proposition 17.8], \( P \) lies on a hyperplane or on a pair of skew lines or on a double line.

**Proposition 6.2.** \( \mathcal{M}(H, 8) \) is irreducible of dimension 7.

**Proof.** Again by Proposition [1.1], it is enough to show that \( \Sigma_1(8) \) is irreducible of dimension 7.

**Claim:** \( P \) in [6.1] lies in a plane curve of degree 2.

**Proof of the claim:**

By [3, Proposition 17.8], \( P \) in [6.1] lies on a hyperplane \( H \). Thus \( h^0(\mathcal{J}_P(1)) = 1 \). Note that \( h^2(\mathcal{J}_P(1)) = h^2(\mathcal{O}_S(1)) = 4 \). Thus from the long exact sequence of the canonical sequence

\[
0 \to \mathcal{J}_P(1) \to \mathcal{O}_S(1) \to \mathcal{O}_P(1) \to 0,
\]

we have \( h^1(\mathcal{J}_P(1)) = 5 \). On the other hand, from the long exact sequence of [6.1] we have the following exact sequence:

\[
0 \to H^1(E) \to H^1(\mathcal{J}_P(1)) \to H^2(\mathcal{O}_S) \to H^2(E) \to H^2(\mathcal{J}_P(1)) \to 0.
\]

Thus we have \( h^0(E(1)) = h^2(E) \geq 9 \). Tensoring [6.1] by \( \mathcal{O}_S(1) \) and considering its long exact sequence, one can see that \( h^0(\mathcal{J}_P(2)) \geq 5 \). Therefore, \( P \) lies in at least 5 quadratic hypersurface sections. But the quadratic hypersurface sections containing \( H \) has dimension 4. Thus there is a quadratic hypersurface \( Q \) containing \( P \) but not containing \( H \). Therefore, \( Q \cap H \) gives a plane curve of degree 2 containing \( P \), which proves our claim.

Since any two plane curves of degree 2 intersect at at most 4 points of \( S \), there exist a unique plane curve of degree 2 containing \( P \). On the other hand, given a plane curve of degree 2, it intersects \( S \) at 12 points and any 8 points of these 12 points satisfies CB(3), hence gives a vector bundle as an extension of the form:

\[
0 \to \mathcal{O} \to E \to \mathcal{J}_P(1) \to 0.
\]

Note that such an extension is unique up to isomorphism. Therefore, \( \Sigma_1(8) \) is isomorphic to the space of bundles \( E \) fitting into an exact sequence of the form [6.1], where \( P \) is a length 8 subscheme of \( C \cap S \) for \( C \) a rational plane curve of degree 2 in \( \mathbb{P}^3 \). Now using the monodromy argument on the set of choices of 8 points out of 12 points of \( C \cap S \), as \( C \) moves one can show that \( \Sigma_1(8) \) is irreducible. Now the dimension of plane curves in \( \mathbb{P}^3 \) is 8 and since \( h^0(E) = 2 \), the dimension of \( \Sigma_1(8) \) is \( 8 - 1 = 7 \), which concludes the Proposition.

\[\square\]
Lemma 6.3. Let $9 \leq c_2 \leq 10$. Then there is a bijection between the irreducible components of $\Sigma_1(c_2)$ and the irreducible components of the space of bundles $E$ fitting into the exact sequence of the form \((6.1)\) where $P$ is a length $c_2$ subscheme lying in a hyperplane.

Proof. Case I: $c_2 = 9$.
In this case $P$ lies on a hyperplane or on a pair of skew lines each of which contains at least 4 points of $P$ or $P$ lies on a double line.
If $P$ lies on a pair of skew lines say, $l_1$ and $l_2$. Then one of them say, $l_1$ contains exactly 4 points of $P$ and imposes 4 independent conditions on sections of $O(3)$, which implies that $P$ can not satisfy $CB(3)$. Therefore, $P$ can not lie on a pair of skew lines.
Since the dimension of lines in $\mathbb{P}^3$ is 4, it is easy to see that the space of bundles $E$ fitting into an exact sequence of the form \((6.1)\) where $P$ is a length 9 subscheme lying in a double line has dimension strictly smaller than 6. On the other hand, by \([1\text{. Corollary 3.1}]\), every irreducible component of $\Sigma_1(9)$ has dimension at least 6. Therefore there is a bijection between the irreducible components of $\Sigma_1(9)$ and the irreducible components of the space of bundles $E$ fitting into an exact sequence of the form \((6.1)\) where $P$ is a length 9 subscheme lying on a hyperplane.

Case II: $c_2 = 10$.
By \([1\text{. Corollary 3.1}]\), every irreducible component of $\Sigma_1(10)$ has dimension at least 9. On the other hand, if $P$ lies on a pair of skew lines or on a double line, then one can easily see that the dimension of the space of bundles fitting into an exact sequence of the form \((6.1)\) where $P$ is a length 10 subscheme lying in a pair of skew lines or in a double line has dimension strictly smaller than 9. Thus as earlier case there is a bijection between the the irreducible components of $\Sigma_1(10)$ and the irreducible components of the space of bundles $E$ fitting into an exact sequence of the form \((6.1)\) where $P$ is a length 10 subscheme lying on a hyperplane.

Proposition 6.4. $\mathcal{M}(H,9)$ is irreducible of dimension $\leq 10$.

Proof. By Proposition \([1.1]\) it is enough to show that $\Sigma_1(9)$ is irreducible of dimension 10. First of all we will show that $\dim(\Sigma_1(9)) \leq 10$.
By Lemma \([6.3]\), it is enough to consider the space of bundles $E$ fitting into an exact sequence of the form \((6.1)\) where $P$ is a length 9 subscheme lying on a hyperplane.

Let $H$ be a plane in general position with respect to $S$, and let $Y := S \cap H$. Let $P$ consists of a general collection of 9 points in $Y$. The map $H^0(O_H(3)) \to H^0(O_Y(3))$ is injective (since $Y$ is a curve of degree 6 in the plane $H$), so a general collection of 9 points in $Y$ imposes independent conditions on $H^0(O_H(3))$. Thus a general set of 9 points fails to satisfy $CB(3)$. Let $Z \subset \operatorname{Hilb}^0(Y)$ be the subset consisting of points which satisfy $CB(3)$. Clearly $\dim(Z) \leq 8$.
Also for a general point $P \in Z$, $h^0(J_{P/H}(3)) = 2$, where $J_{P/H}$ denotes the ideal sheaf of $P$ in $H$. Thus $h^1(J_{P}(3)) = 12$. In other words, $h^1(J_{P}(3)) = 1$. Thus $P$ determines a vector bundle $E$ uniquely up to isomorphism. Since $h^0(J_{P}(1)) = 1$, we have $h^0(E) = 2$. This means that for a given bundle $E$, the space of choices of sections $s$ (modulo scaling) leading to the subscheme $P$, has dimension 1. Hence the dimension of the space of bundles obtained by this construction is one less than the dimension of the space of subschemes.
Now there is a three-dimensional space of choices of the plane $H$, and for each one we have an 9-dimensional space of choices of the subscheme $P$. This gives the total dimension of the space of choices of $P = 3 + \dim(Z) - 1$. So total dimension of such bundles is $3 + \dim(Z) - 1$ which is $\leq 10$.

Irreducibility: Since $|P| = 9$ and it satisfies $CB(3)$, $h^0(J_{P}(3)) \geq 12$. Thus $P$ lies on a complete
intersection of two cubics. Now the space of complete intersections of two cubics is irreducible and by remark \[2.3\] any 9 points on such a complete intersection satisfies CB(3) and hence give a point in \(\Sigma_1(9)\). Again using monodromy argument one can conclude the proposition.

\[\square\]

**Proposition 6.5.** \(\mathcal{M}(H, 10)\) is irreducible of dimension 11.

*Proof.* By Proposition \[1.1\] it is enough to show that \(\Sigma_1(10)\) is irreducible of dimension 11. Again by Lemma \[6.3\] it is enough to consider the space of bundles \(E\) fitting into an exact sequence of the form \[6.1\] where, \(P\) is a length 10 subscheme lying on a hyperplane \(H\).

Since \(P\) satisfies CB(3), \(h^0(J_P(3)) \geq 11\). Thus there exists a cubic hypersurface containing \(P\) and not containing \(H\). By Bertini, a general such cubic is irreducible and hence intersects \(H\) in an irreducible cubic plane curve. Therefore, \(P\) lies in a plane cubic curve. Since any two plane cubic curves intersect at 9 points, there exists a unique plane cubic curve containing \(P\).

On the other hand, given a plane cubic curve intersects \(S\) at 18 points and a plane cubic curve imposes 9 independent conditions on the sections of \(O(3)\). Thus any 10 points of these 18 points satisfies CB(3) and hence gives a vector bundle as an extension of the form:

\[
0 \to O \to E \to J_P(1) \to 0.
\]

Note that such an extension is unique up to isomorphism. Therefore, \(\Sigma_1(10)\) is isomorphic to the space of bundles \(E\) fitting into an exact sequence of the form \[6.1\] where, \(P\) is a length 10 subscheme of \(C \cap S\) for \(C\) a plane cubic curve in \(\mathbb{P}^3\). Now using the monodromy argument on the set of choices of 10 points out of 18 points of \(C \cap S\), as \(C\) moves one can show that \(\Sigma_1(10)\) is irreducible.

Let \(Y\) be a plane cubic curve. Then for a general point \(P\) in \(\text{Hilb}^{c_2}(Y)\) one has \(h^0(J_P(3)) = 21 - c_2\), which gives \(h^1(J_P(3)) = 1\). In other words, \(P\) determines a vector bundle \(E\) uniquely up to isomorphism. Since \(h^0(J_P(1)) = 1\), we have \(h^0(E) = 2\). This means that for a given bundle \(E\), the space of choices of section \(s\) (modulo scaling) leading to the subscheme \(P\), has dimension 1. Hence the dimension of the space of bundles obtained by this construction is one less than the dimension of the space of subschemes.

Now there is a three-dimensional space of choices of the plane \(H\), and for each one we have a 10—dimensional space of choices of the cubic curves. This gives the total dimension of the space of choices of \(P = 3 + 9 = 12\). So total dimension of such bundles is \(12 - 1 = 11\).

\[\square\]

**Remark 6.6.** Note that \(\mathcal{M}(H, 11)\) has two components of dimension 12 and 13, respectively.

6.2. \(c_2 \geq 12\). Let \(E \in \Sigma_1(c_2)\). Then we have

\[
(6.2) \quad 0 \to O_S \to E \to J_P(1) \to 0
\]

where, \(P\) is a zero dimensional subscheme of \(S\) of length \(c_2\).

Let \(P\) lies in a hyperplane \(H\) and \(P' \subset P\) be a subset of 10 points. Then \(h^0(H, J_{P'/H}(3)) = 0\) and \(h^0(E) = 2\). So any cubic hypersurface which contains \(P\) will also contain \(H\). Therefore, \(h^0(J_P(3)) = 10\) and hence \(h^1(J_P(3)) = 9\). The total dimension of bundles which fit in the above exact sequence with \(P\) lying in a plane is equal to the dimension of such subschemes \(-1 + c_2 - 11\). Now the dimension of such subschemes is \(3 + c_2\). Thus Total dimension is \(2c_2 - 9\).

Also taking \(L = O_S\) and \(L' = O_S(1)\) in \[4.2\] we have:

\[
(6.3) \quad 0 \to E \otimes O_S(-1) \to \text{End}^0(E) \to J_P^2(1) \to 0.
\]

Tensoring the above exact sequence by \(O_S(2)\) and considering the cohomology sequence, we have

\[
h^0(S, \text{End}^0(E) \otimes O_S(2)) \leq h^0(S, E(1)) + h^0(S, J_P^2(3)).
\]
If $P$ is not in any hyperplane section, then $h^0(J_P(1)) = 0$. If $h^0(S, J^2_P(3)) \neq 0$, then there is a cubic hypersurface which is singular along $P$. But an irreducible cubic hypersurface can have at most 4 isolated singularities \cite{25}, thus any non-zero section of $J^2_P(3)$ is reducible, in this situation it has two components, a hyperplane and a quadratic hypersurface. On the other hand, any irreducible quadratic hypersurface has only one isolated singularity \cite{25}. Therefore, $h^0(S, J^2_P(3)) = 0$. Thus we have $h^0(S, \text{End}^0(E) \otimes \mathcal{O}_S(2)) \leq h^0(S, E(1)) = h^0(\mathcal{O}(1)) + h^0(J_P(2)) = 4 + h^0(J_P(2))$.

If $h^0(J_P(2)) \geq 3$, then $P$ lies in a complete intersection of two quadratic hypersurfaces with out having any common component, i.e. in a curve $C$ of degree 4. Thus for $c_2 \geq 13$, any cubic hypersurface which contains $P$ also contains $C$. Now a complete intersection curve of degree 4 imposes 12 independent conditions on sections of $\mathcal{O}(3)$. Thus we have, $h^0(J_P(3)) = 8$. Therefore, the dimension of the isomorphic classes of bundles determined by $P$, is $h^1(J_P(3)) - 1 = c_2 - 11$. Also we have $P$ lies in a curve $Y = Q \cap S$, where $Q$ is a quadratic hypersurface and $Q$ varies over a 9 dimensional variety of quadrics. Thus the dimension of such zero dimensional subschemes of length is at most $c_2 + 9$. Thus the total dimension of such bundles is at most $c_2 + 9 + c_2 - 11 = 2c_2 - 2$.

On the other hand, if $h^0(J_P(2)) \leq 2$, then $h^0(S, \text{End}^0(E) \otimes \mathcal{O}_S(2)) \leq 6$. Thus in this situation, every component of $\Sigma_1(c_2)$ has dimension at most $4c_2 - 39 + 6 = 4c_2 - 33$ (the expected dimension + the dimension of obstructions). Thus we have the following Proposition.

**Proposition 6.7.** Let $c_2 \geq 13$. Then every component of $\Sigma_1(c_2)$ has dimension at most $\max\{2c_2 - 2, 4c_2 - 33\}$ and for $c_2 = 12$, it has dimension $4c_2 - 39 + 10 = 4c_2 - 29$.

### 7. Boundary strata

The boundary $\partial \mathcal{M}(H, c_2) := \overline{\mathcal{M}(H, c_2)} - \mathcal{M}(H, c_2)$ is the set of points corresponding to torsion-free sheaves which are not locally free.

Let $\mathcal{M}(c_2, c'_2) := \{[F] \in \overline{\mathcal{M}(H, c_2)]} | F$ is not locally free with $c_2(F**) = c'_2\}$. Then the boundary has a decomposition into locally closed subsets

$$\partial \mathcal{M}(H, c_2) = \coprod_{c'_2 < c_2} \mathcal{M}(c_2, c'_2).$$

By the construction of $\mathcal{M}(c_2, c'_2)$, we have a well defined map,

$$\mathcal{M}(c_2, c'_2) \longrightarrow \mathcal{M}(H, c'_2).$$

The map takes $E \longrightarrow E**$. The fiber over $E \in \mathcal{M}(H, c'_2)$ is the Grothendieck Quot-scheme $\text{Quot}(E; d)$ of quotients of $E$ of length $d := c_2 - c'_2$. Thus $\dim(\mathcal{M}(c_2, c'_2)) = \dim(\mathcal{M}(H, c'_2)) + \dim(\text{Quot}(E; d))$. Now the dimension of $\text{Quot}(E; d)$ is $3(c_2 - c'_2)$. Therefore,

$$\dim(\mathcal{M}(c_2, c'_2)) = \dim(\mathcal{M}(H, c'_2)) + 3(c_2 - c'_2). \tag{7.1}$$

\text{7.1} allows us to fill in the dimensions of the strata $\mathcal{M}(c_2, c'_2)$ in the Tables \textbf{1} and \textbf{2} starting from the dimensions of the moduli spaces given by previous section. The entries in the second column are the expected dimensions $4c_2 - 39$; in the third column the upper bounds of the dimensions of $\mathcal{M}(H, c_2)$; and in the following columns, the upper bounds of the dimensions of $\mathcal{M}(c_2, c_2 - d)$, $d = 1, 2, ..., 19$.

**Definition 7.1.** A closed subset $X \subset \mathcal{M}(H, c_2)$ is called good if every irreducible component of $X$ contains a point $[E]$ with $H^2(S, \text{End}^0(E)) = 0$, where $\text{End}^0(E)$ denotes the traceless endomorphisms of $E$.

**Theorem 7.2.** Suppose $\mathcal{M}(H, c_2)$ is good for $c_2 \geq 27$. Then $\overline{\mathcal{M}(H, c_2)}$ is also good for $c_2 \geq 27$. 


Proof. Since we are given that \( \mathcal{M}(H, c_2) \) is good, every component has expected dimension \( 4c_2 - 39 \) for \( c_2 \geq 27 \). Thus every boundary strata \( \mathcal{M}(c_2, c'_2) \) for \( c_2 \geq 27 \) and \( 20 \leq c'_2 \leq 26 \) has smaller dimension. Also from the tables \[1\] and \[2\] we can see that other boundary stratas also have dimension smaller than the expected dimension. Thus every component of \( \mathcal{M}(H, c_2) \) for \( c_2 \geq 27 \) has expected dimension, which concludes the Theorem. \( \square \)

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