KHOVANOV HOMOLOGY FROM FLOER COHOMOLOGY

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Abstract. This paper realises the Khovanov homology of a link in $S^3$ as a Lagrangian Floer cohomology group, establishing a conjecture of Seidel and the second author. The starting point is the previously established formality theorem for the symplectic arc algebra over a field $k$ of characteristic zero. Here we prove the symplectic cup and cap bimodules which relate different symplectic arc algebras are themselves formal over $k$, and construct a long exact triangle for symplectic Khovanov cohomology. We then prove the symplectic and combinatorial arc algebras are isomorphic over $\mathbb{Z}$ in a manner compatible with the cup bimodules. It follows that Khovanov cohomology and symplectic Khovanov cohomology co-incide in characteristic zero.

1. Introduction

Let $\mathcal{F}(\gamma)$ be a smooth fibre of the restriction of the adjoint quotient $\chi: \mathfrak{sl}_2(\mathbb{C}) \to \mathbb{C}^{2n-1}$ to a transverse slice $S \subset \mathfrak{sl}_2(\mathbb{C})$ at a nilpotent matrix with two equal Jordan blocks. Let $\mathcal{F}(\gamma_n)$ denote the Fukaya category of closed exact Lagrangian branes in $\mathcal{F}(\gamma)$, as constructed in [45]. The paper [50] defines a symplectic structure $\omega$ on $\gamma_n$ which is exact and has contact type at infinity, and an action of the braid group $Br_{2n}$ (by parallel transport varying $t$) on objects of $\mathcal{F}(\gamma_n)$. Let $\kappa$ be a link in $S^3$ realised as the closure of a braid $\beta_\kappa \times id \in Br_{2n}$, with $\beta_\kappa \in Br_n$. There is a distinguished Lagrangian submanifold $L_{\varphi_n} \subset \gamma_n$, and a relatively $\mathbb{Z}$-graded Floer cohomology group $Kh_{symp}(\kappa) = HF^*(L_{\varphi_n}, (\beta_\kappa \times id)(L_{\varphi_n}))$ called the symplectic Khovanov cohomology of $\kappa$. The main theorem of [50] proved that this is indeed a link invariant (independent of the choice of $\beta_\kappa$, and in particular of $n$, up to shifts; the relative grading can be refined to an absolute grading if one orients $\kappa$), and conjectured that it co-incided with a singly graded version of Khovanov’s combinatorial / representation-theoretic invariant $Kh(\kappa)$ from [24]. This paper proves that conjecture in characteristic zero.

The categories $\mathcal{F}(\gamma_n)$ for different $n$ are related by various canonical bimodules $\cup_i$ and $\cap_i$, $1 \leq i \leq 2n-1$, defined by symplectic analogues of the cup and cap bimodules of [25], cf. Section 4.5. Such bimodules play an implicit role in the construction of the link invariant $Kh_{symp}(\kappa)$, and were further considered in the work of Rezazadegan [37]. Here we prove that the Floer cohomology algebra generated by the Lagrangian iterated vanishing cycles associated to upper half-plane crossingless matchings (the “symplectic arc algebra”) is isomorphic over $\mathbb{Z}$ to Khovanov’s arc algebra [25], and we prove that the bimodules $\cup_i$ and $\cap_i$ are formal over any field $k$ of characteristic zero. We also construct a long exact triangle for symplectic Khovanov cohomology, analogous to the skein triangle obeyed (tautologically) by its combinatorial sibling. It follows that combinatorial and symplectic Khovanov cohomologies are isomorphic in characteristic zero, in particular have the same total rank over $\mathbb{Q}$.

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Outline of the paper. The paper broadly divides into three Parts.

1. In [6] we introduced an abstract formality criterion, due to Paul Seidel, for formality of an $A_\infty$-algebra. In the first Part (Sections 2-4), we provide a complementary formality criterion for $A_\infty$-bimodules. We explain how one can sometimes implement this criterion for bimodules over Fukaya categories of Stein manifolds via appropriate counts of holomorphic discs in a partial compactification, and as an application prove that the cup bimodules are formal.

2. In the second Part (Sections 5-6), we prove that the symplectic arc algebra is isomorphic over $\mathbb{Z}$ to Khovanov’s combinatorial arc algebra $H_n$. The essential difficulty is one of signs: over $\mathbb{Z}$, Floer complexes are not intrinsically based by intersection points, but by orientations of abstract lines associated to $\partial$-operators. We construct bases of the symplectic arc algebra on $2n$ strands in which products of positive generators are positive linear combinations of positive generators by an argument inductive on $n$. The induction relies on reduction to special cases, on plumbing models, and on the existence of a natural map from the cohomology of a symplectic manifold to the center of its Fukaya category.

3. In the third part (Section 7 and the Appendix) we construct the long exact triangle for symplectic Khovanov cohomology, and conclude the proof. The exact triangle adapts currently unpublished more general results due to the first author and Ganatra [3], partly based on forthcoming work of the first author and Seidel [5]. The argument uses a Fukaya category of a Morse-Bott Lefschetz fibration, similar in nature to the categories studied by Seidel in the Lefschetz case, and various canonical functors thereon. This formalism is reminiscent of the cobordism techniques developed by Biran and Cornea [11, 12], rather than the quilted Floer groups of Wehrheim and Woodward [56], and avoids technical issues stemming from non-compactness of the correspondences in our setting.

The last section of the paper draws the various pieces together to complete the proof. In short, the isomorphism of the 2nd part is compatible with cup functors, and hence (using formality) with their adjoint cap functors. Via the exact triangle, we infer that the Fukaya category $D\mathcal{F}(Y_n)$ generated by the Lagrangians associated to upper half-plane crossingless matchings is braid-group equivariantly equivalent to the derived category $D(mod - H_n)$ of modules over the arc algebra, which implies the main theorem. More precisely, we show that for any oriented link $\kappa$, and for the absolute grading on $\operatorname{Kh}_{\text{symp}}$ mentioned above, one has isomorphisms

$$\operatorname{Kh}_{\text{symp}}^k(\kappa) \cong \bigoplus_{i-j=k} \operatorname{Kh}^{i,j}(\kappa) \quad \forall k \in \mathbb{Z}$$

which establishes [50] Conjecture 2. Any Floer group $HF^*(L_{\varphi_0}, (\beta \times id)(L_{\varphi_0}))$ also inherits a generalised eigenspace decomposition from the endomorphism induced by the $nc$-vector field on $\mathcal{F}(Y_n)$ constructed in [6]. One can view this as an additional “weight” grading, $a priori$ by elements in the algebraic closure $\overline{K}$, and not $a priori$ invariant under Markov moves.

**Conjecture 1.1.** The relative weight grading on $HF^*(L_{\varphi_0}, (\beta \times id)(L_{\varphi_0}))$ is Markov invariant, and recovers the relative second grading on $\operatorname{Kh}_{\text{symp}}(\kappa)$.

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1 Even if one is primarily interested in closed Lagrangian submanifolds and their monodromy images, $a priori$ the quilt theory construction of the exact triangle requires compactness for spaces of curves whose only Lagrangian boundary condition is the correspondence itself.
this project. We would also like to thank Paul Biran and Octav Cornea for sharing a preliminary version of their manuscript \cite{12} with us.

2. Algebra

In this section we work over a coefficient field $k$, specialising where necessary to the case in which $k$ has characteristic zero.

2.1. Background. Let $A$ be a $\mathbb{Z}$-graded cohomologically unital $A_\infty$-algebra over $k$, equipped with $A_\infty$-products

\begin{equation}
\mu^d_A: A^\otimes d \to A, \ 1 \leq d
\end{equation}

of degree $2 - d$. The first two operations satisfy the Leibniz equation\footnote{Our sign conventions follow those of Seidel in \cite{45}: elements of $A$ are equipped with the reduced degree $||a|| = |a| - 1$, and operators act on the right.}

\begin{equation}
\mu^1_A(\mu^2_A(a_2, a_1)) + \mu^2_A(a_2, \mu^1_A(a_1)) + (-1)^{|a_1|-1}\mu^2_A(\mu^1_A(a_2), a_1) = 0.
\end{equation}

The cohomology groups with respect to $\mu^1_A$, denoted $A = H(A)$, naturally form an $A_\infty$ algebra for which all operations vanish except the product, which is induced by $\mu^2_A$.

**Definition 2.1.** $A$ is formal if it is quasi-isomorphic to $A$.

The prequel formulated and proved a necessary and sufficient condition for the formality of an $A_\infty$-algebra, due to Paul Seidel, in terms of the existence of a particular kind of degree one Hochschild cohomology class. We recall that the Hochschild cochain complex $CC^*(A, A)$ has chain groups

\[ CC^d(A, A) = \prod_{s \geq 0} \text{Hom}_d(A[1]^\otimes s, A) \]

where $[1]$ denotes downward shift by 1 and $\text{Hom}_d$ denotes $k$-linear maps of degree $d$, equipped with a differential

\begin{equation}
\delta: CC^{d-1}(A, A) \to CC^d(A, A)
\end{equation}

\begin{equation}
(\delta \sigma)^d(a_d, \ldots, a_1) = \sum_{i,j} (-1)^{|\sigma|-1} \mu^d_{A}(a_d, \ldots, \mu^j_A(a_{i+j}, \ldots, a_{i+1}), \ldots, a_1)
\end{equation}

\begin{equation}
+ \sum_{i,j} (-1)^{|\sigma|+1} \sigma^{d-j+1}(a_d, \ldots, \mu^j_A(a_{i+j}, \ldots, a_{i+1}), \ldots, a_1).
\end{equation}

Here $\dagger_i = \sum_{k=1}^i (|a_k| - 1)$.

**Definition 2.2.** An nc-vector field is a cocycle $b \in CC^1(A, A)$.

In the definition, nc stands for non-commutative. On a graded algebra, we have a canonical nc-vector field called the Euler vector field, which multiplies the graded piece $A^i \subset A$ of $A$ by $i$:

\begin{equation}
e: A^i \to A^i, \ a \mapsto i \cdot a.
\end{equation}
The fact that multiplication preserves the grading
\[(2.6) \quad |a_2a_1| = |a_2| + |a_1|\]
implies that \(e \in CC^1(A, A)\) is a cocycle, hence defines a class in \(HH^1(A, A)\) (which has no constant or higher order terms). Note that there is a natural projection of chain complexes
\[(2.7) \quad CC^*(A, A) \rightarrow A\]
\[(2.8) \quad b \mapsto b^0\]
induced by taking the order-0 part of a Hochschild cochain. Given an element of the kernel of this map, the first order part
\[(2.9) \quad b^1: A \rightarrow A\]
is a chain map, and hence defines an endomorphism of \(A\).

**Definition 2.3.** An nc-vector field \(b \in CC^1(A, A)\) is pure if \(b^0 = 0\), and the induced endomorphism of \(A\) agrees with the Euler vector field.

If \(A\) admits a pure vector field, in a minor abuse of notation we say that \(A\) itself is pure. The prequel paper proved:

**Theorem 2.4 (Seidel).** Suppose \(k\) has characteristic zero. If \(b\) is a pure vector field on \(A\), then \(A\) is formal. Indeed, there is an equivalence
\[(2.10) \quad \psi: A \rightarrow A\]
with left inverse \(\psi^{-1}\) so that the pullback of \(b\) to \(CC^*(A, A)\) agrees with the Euler vector field. \(\square\)

### 2.2. Formality for bimodules.

To shorten formulae, in this section we use the abbreviation
\[
\mathcal{C}(X_0, X_1, \ldots, X_r) = \text{home}(X_{r-1}, X_r) \otimes \text{home}(X_{r-2}, X_{r-1}) \otimes \cdots \otimes \text{home}(X_0, X_1)
\]
for any \(A_\infty\)-category \(\mathcal{C}\), where \(r \geq 1\) (for \(r = 0\) define the corresponding expression to equal the ground field).

Take two \(A_\infty\)-categories \(A, B\). Let \(P\) be an \((A, B)\)-bimodule. This comprises a collection of graded vector spaces \(P(Y, X)\) for any objects \(X \in A, Y \in B\), together with multi-linear maps
\[
\mu^{[r|s]}_P: A(X_0, \ldots, X_r) \otimes B(Y_0, \ldots, Y_s) \rightarrow P(Y_s, X_r)
\]
for \(r, s \geq 0\), and any objects \(X_i, Y_j\) of \(A\) respectively \(B\). (Our ordering convention is such that \(\text{hom}_A(X, X') = A(X, X')\) and if \(P\) is an \((A, B)\)-bimodule there are maps
\[
A(X, X') \otimes P(Y, X) \rightarrow P(Y, X')
\]
which fits with the ordering conventions for Floer theory multiplication from [45].) Assuming that the generators of the domain of \(\mu^{[r|s]}_P\) are written \(x_r \otimes \cdots \otimes x_1 \otimes \underline{m} \otimes y_1 \otimes \cdots \otimes y_s\), where we underline the bimodule element, we write the \(A_\infty\) relations as:
\[
(2.11) \quad 0 = \sum (-1)^r \mu^{[r-R+S|1]}_P (x_r, \ldots, x_{R+1}, \underline{m}^R, y_s, x_{S-1}, \ldots, x_1, \underline{y}, y_1, \ldots, y_s) + \\
\quad + \sum (-1)^s \mu^{[r-R+S-S]}_P (x_r, \ldots, x_{R+1}, y_s, \underline{x}, y_{S-1}, \ldots, y_1, \underline{m}, y_1, \ldots, y_s) + \\
\quad + \sum (-1)^o \mu^{[r|s-R+S]}_P (x_r, \ldots, x_1, \underline{m}, y_1, \ldots, y_{R-1}, \underline{y}^R, y_{S+1}, \ldots, y_s)
\]
summing over all \(R, S, o\), and where the signs are
\[
\circ = \sigma(y)^S_{S+1} \quad \ast = \sigma(y)^S_1 + \deg(m) + \sigma(x)^{S-1}_1
\]
with $\sigma^t_\ell = \sigma(x)^t_\ell = \sum_{t=1}^i |x| - 1$. The simplest equation in particular implies that the map $\mu^{0|10}_P$ squares to zero, hence defines the structure of a chain complex on each group $P(Y,X)$. Hence $P$ descends to a bimodule $HP$ of the corresponding cohomological categories, and we require that multiplication by the homological units of $HA$ and $HB$ agree with the identity map of $HP$.

Bimodules form a differential graded category, with morphisms from $P$ to $Q$ (sometimes called pre-morphisms) given by a collection of multi-linear maps

$$f^{r|s}_{r,s}: A(X_0, \ldots, X_r) \otimes P(Y_0, X_0) \otimes B(Y_s, \ldots, Y_0) \to Q(Y_s, X_r)$$

for $r, s \geq 0$. The differential assigns to such a sequence $f$ the linear maps

$$(x_r, \ldots, x_1, m, y_1, \ldots, y_s) \mapsto \sum(-1)^{s + |x|} f^{r|s}_{r,s}(x_r, \ldots, x_{R+1}, m_{A^{R+S-1}(x_{R}, \ldots, x_s)}, x_{S-1}, \ldots, x_1, m, y_1, \ldots, y_s) +$$

$$+ \sum(-1)^{s + |x|} f^{r|s}_{r,s}(x_r, \ldots, x_{R+1}, m_{A^{R+S-1}(x_{R}, \ldots, x_s)}, y_{S+1}, \ldots, y_s) +$$

$$+ \sum(-1)^{\deg(f)} \mu^{r|s}_{r,s}(x_r, \ldots, x_{R+1}, f^{r|s}_{r,s}(x_{R}, \ldots, x_1, m, y_1, \ldots, y_s), y_{S+1}, \ldots, y_s) +$$

summing over $R, S$, and where the signs are as before. The product is given by

$$(2.12) \quad \mu^2(f, g)(x_r, \ldots, x_1, m, y_1, \ldots, y_s) = \sum \sum(-1)^{g + |x|} f^{r|s}_{r,s}(x_r, \ldots, x_{R+1}, g_{A^{R+S-1}(x_{R}, \ldots, x_s)}, y_{S+1}, \ldots, y_s),$$

and all higher order operations vanish.

In particular, the bimodule endomorphisms of a fixed bimodule $P$ form a cochain complex we denote $(\mathcal{E}(P), \partial)$ with cohomology $\text{End}_{A,B}(P)$. There are maps

$$\nu_A : HH^*(A,A) \to \text{End}_{A,B}(P)$$

$$\nu_B : HH^*(B,B) \to \text{End}_{A,B}(P)$$

induced by a chain map which we formally denote by $\sigma \mapsto \mu^{r|s}_{r,s} \circ \sigma$, under which $\sigma$ eats some collection of the $A$-inputs respectively the $B$-inputs to $P$.

**Definition 2.5.** Let $b_A$ and $b_B$ be nc-vector fields on $A$ and $B$. A bimodule $P$ is equivariant if the induced cohomology classes agree:

$$\nu_A(b_A) = \nu_B(b_B) \in \text{End}_{A,B}^1(P).$$

A choice of equivariant structure is an endomorphism $c_P \in \mathcal{E}^0(P)$ for which

$$d c_P = \nu_A(b_A) - \nu_B(b_B).$$

Now suppose that $b_A$ and $b_B$ induce pure structures on $A$ and $B$. This means that all objects $X \in \text{Ob}A$ and $Y \in \text{Ob}B$ come with chosen equivariant structures $c_X, c_Y$. The elements $c_P, c_X, c_Y$ induce an endomorphism of the cohomology $H^*(P(X,Y))$ (taking cohomology with respect to $\mu^{0|10}_P$). More precisely

$$(2.15) \quad C^*(P(Y,X)) \ni \phi \mapsto c_P(\phi) - c_X \circ \phi + \phi \circ c_Y \in C^*(P(Y,X))$$

is a chain map, hence induces an endomorphism of $H^*(P(Y,X))$. We say that $(P, c_P)$ is pure if this endomorphism agrees with the Euler field, i.e. acts in degree $i$ by multiplication by $i$.

**Lemma 2.6.** If $c_P$ is pure, then $P$ is formal; more precisely, there are $A_{\infty}$-equivalences $H(A) \to A$ and $H(B) \to B$, and inverse equivalences of $H(A)$$-$$H(B)$ bimodules $\psi : H(P) \to P$ and $\phi : P \to$
we say that a

The essential image of the embedding of functors into bimodules can be determined as follows:

logically fully faithful embedding from the category of functors to that of

µ

with structure maps

Lemma

Proposition 5.22]:

of

B

We shall be particularly interested in the case of the diagonal: recall that the diagonal bimodule

F

When either

(2.20)

[20, Section 2.8]). The structure maps are

F

equation for functors given, e.g. as Equation (1.6) of [45]. The first such equation implies that

A

The

the structure maps of the functors

F

by pre-natural transformations, i.e. collections of maps

F

Given a

C

−

D

bimodule

P

and functors

F

: A → C and

G

: B → D, we obtain a 2-sided pullback

P

which is an

A

−

B

-bimodule which assigns

P(Y,FX)
to a pair of objects

X

and

Y

(see [28 Section 2.8]). The structure maps are

(2.20)

When either

F

or

G

are the identity, we obtain 1-sided pullbacks which we denote

P

and

P

.

we shall be particularly interested in the case of the diagonal: recall that the diagonal bimodule of

B

(which we denote

Δ

B

) assigns to a pair of objects

(Y,Y')

the morphism space

B(Y,Y')

, with structure maps

µ

with

(−1)

. The following standard result is proved e.g. in [28 Proposition 5.22]:

Lemma 2.7 (Yoneda Lemma). The assignment

F

→

Δ

B

(resp.

F

→

Δ

A

) extends to a cohomologically fully faithful embedding from the category of functors to that of

A

−

B

bimodules (resp.

B

−

A

-bimodules).

The essential image of the embedding of functors into bimodules can be determined as follows: we say that a

A

−

B

bimodule

P

is representable as a

B

-module if, for every object

X

of

A

, the

ϕ

ψ

H(P) and so that

(2.16)

ψ ◦ cP ◦ ϕ = Euler_{H(P)} ∈ \text{End}_{H(A),H(B)}(HP)

Sketch. Consider the category

A \coprod_P B

, whose set of objects is the union of the objects of

A

and

B

, such that morphisms from objects of

A

to those of

B

are given by

P

, and morphisms in the other direction vanish. The Hochschild complex of this category is the direct sum of the Hochschild complexes of

A

and

B

with the endomorphism complex of

P

, so the data

(b_A,c_P,b_B)

induces an

nc

-vector field on

A \coprod_P B

. By assumption, this vector field is pure, so we conclude from Theorem 2.4 that there are inverse equivalences

(2.17)

A \coprod_P B \xrightarrow{\phi} H(A) \coprod_{H(P)} H(B)

which map the

nc

-vector field to the Euler vector field. Translating back from the category

A \coprod_P B
to the bimodule, we obtain the desired result. □
Lemma 2.8. If $P$ is representable as a $\mathcal{B}$-module, then there is a functor $\mathcal{F} : \mathcal{A} \to \mathcal{B}$, such that
\begin{equation}
P \cong \mathcal{F} \Delta^\mathcal{B}.
\end{equation}

By the Yoneda Lemma, the representing functor is in fact unique up to quasi-isomorphism. We shall need an additional result comparing composition of functors (see [45, Sections (1b) and (1e)]) to the tensor product of bimodules. Recall the definition of the tensor product of $\mathcal{A} - \mathcal{B}$ and $\mathcal{B} - \mathcal{C}$ bimodules $P$ and $Q$. This is the $\mathcal{A} - \mathcal{C}$ bimodule, denoted $P \otimes_\mathcal{B} Q$, which is given by
\begin{equation}
(P \otimes_\mathcal{B} Q)(Z, X) = \bigoplus_{(Y_0, \ldots, Y_d) \in \text{Ob} \mathcal{B}} P(Y_d, X) \otimes \mathcal{B}(Y_0, \cdots, Y_d) \otimes Q(Z, Y_0),
\end{equation}
with differential induced by the structure maps of $P$ and $Q$ as $\mathcal{B}$ modules (together with the differential $\mu^{\mathcal{B}}_1$), and higher operations induced by the remaining structure maps of $P$ and $Q$ as bimodules (see [44, Equation (2.14)]).

Lemma 2.9. Let $\mathcal{F} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{C}$ be $A_\infty$ functors. There is a natural quasi-isomorphism of bimodules
\begin{equation}
\mathcal{F} \Delta^\mathcal{B} \otimes_\mathcal{B} \mathcal{G} \Delta^\mathcal{C} \to \mathcal{G} \circ \mathcal{F} \Delta^\mathcal{C}.
\end{equation}

Sketch of proof: The map from $\mathcal{B}(X_0, \ldots, X_r) \otimes \mathcal{B}(Y_d, \mathcal{F}X_0) \otimes \mathcal{B}(Y_0, \cdots, Y_d) \otimes \mathcal{C}(Z_0, \mathcal{G}Y_0) \otimes \mathcal{C}(Z_0, \ldots, Z_0)$ to $\mathcal{C}(Z_0, (\mathcal{G} \circ \mathcal{F})X_r)$ is:
\begin{equation}
(x_r, \ldots, x_1, p_1 y_0, \ldots, y_1, y_1, \ldots, y_s) \mapsto \sum (-1)^{\sigma} \mu^{\mathcal{C}}_{k+e+s} (\mathcal{C}^j(x_r, \ldots, x_{r-j+1}), \ldots, \mathcal{C}^{k+1+j}(x_{j_1}, \ldots, x_1, p_1 y_0, \cdots, y_{d-1} y_0+2), \ldots, \mathcal{C}^1(y_1, \cdots, y_1, q_1 z_1, \ldots, z_s)).
\end{equation}
At the linear level, we may fix an object $Z$ of $\mathcal{C}$. Letting $\mathcal{G} \mathcal{Y}_Z$ denote the pullback under $\mathcal{G}$ of the left Yoneda module of $Z$, the above map is exactly the linear term in the natural map of left $\mathcal{B}$-modules
\begin{equation}
\Delta^\mathcal{B} \otimes_\mathcal{B} \mathcal{G} \mathcal{Y}_Z \to \mathcal{G} \mathcal{Y}_Z.
\end{equation}
The result therefore follows from the fact that tensoring with the diagonal bimodule induces a quasi-isomorphism of bimodules by the acyclicity of the bar complex (see for instance [20, Proposition 2.2]).

2.4. Formality for functors. If $\mathcal{A}$ and $\mathcal{B}$ are equipped with nc-vector fields $b_\mathcal{A}$ and $b_\mathcal{B}$, we say that $\mathcal{F}$ is pure if either $\Delta^\mathcal{A}_\mathcal{F}$ or $\mathcal{F} \Delta^\mathcal{B}$ is pure. This implicitly means that $\mathcal{A}$ and $\mathcal{B}$ are equipped with pure equivariant structures; we write $\psi_\mathcal{A}$ and $\psi_\mathcal{B}$ for the induced equivalences from $H \mathcal{A}$ to $\mathcal{A}$ and $H \mathcal{B}$ to $\mathcal{B}$. In Corollary [24,22] below, we prove that purity as a left or right bimodule are equivalent conditions.
Remark 2.10. By considering only the pure case, we shall avoid having to appeal to the following result whose proof uses duality of bimodules: an equivariant structure on $\tau \Delta^B$ induces an equivariant structure on $\Delta^A_B$, and hence on $\tau \Delta^B_{\tau}$ via the natural equivalence
\begin{equation}
\tau \Delta^B \otimes_{\Psi} \Delta^B_{\Psi} \to \tau \Delta^B_{\tau}.
\end{equation}
Moreover, equipping the two-sided pullback with this equivariant structure, the natural map
\begin{equation}
\Delta^A \to \tau \Delta^B_{\tau}
\end{equation}
of $A - A$-bimodules is equivariant. If $\mathcal{F}$ has an adjoint, cf. Section 2.5, this map gives rise to the unit of the adjunction.

By Theorem 2.4, the pure equivariant structures on $A$ and $B$ induce quasi-equivalences of these categories with their cohomological categories. We say that $\mathcal{F}$ is formal if there is an equivalence of functors between $H\mathcal{F}$ and the composition
\begin{equation}
HA \xrightarrow{\psi_A} A \xrightarrow{\mathcal{F}} B \xrightarrow{\psi_B^{-1}} HB.
\end{equation}
Note that $\mathcal{F}$ is formal if and only if $\Delta^B_{\tau}$ (or $\tau \Delta^B$) is formal. To see this, note that the $HB-HA$ bimodule $\Delta^B_{HA}$ which represents $H\mathcal{F}$ is canonically isomorphic to $H(\Delta^B_{\tau})$. If $\mathcal{F}$ is formal, we conclude that the pullback of $\Delta^B_{\tau}$ to an $HB-HA$ bimodule is quasi-isomorphic to its cohomological module, hence is formal. In the other direction, if $\Delta^B_{\tau}$ is formal, we conclude that the bimodules representing $H\mathcal{F}$ and the composition of $\mathcal{F}$ with the equivalences from $A$ and $B$ to their cohomologies are quasi-isomorphic, hence the corresponding functors are equivalent by the Yoneda Lemma 2.7.

Proposition 2.11. $\mathcal{F}$ is pure if and only if it is formal.

Proof. We consider the case in which $\tau \Delta^B$ is pure: by Lemma 2.6, the pullback of $\tau \Delta^B$ to an $HA - HB$-bimodule is formal, hence equivalent to $H\mathcal{F} \Delta^B_{HA}$. The result follows from the fact that the left-sided pullback defines a (cohomologically) fully faithful embedding from the category of functors to the category of bimodules.

One can formulate the above Lemma more precisely, by noting that a quasi-isomorphism from $H\mathcal{F}$ to the composition in Equation (2.28) equips $\mathcal{F}$ with a pure structure, and that a pure structure induces a quasi-isomorphism between these two functors.

Corollary 2.12. If $\tau \Delta^B$ is pure, so are $\Delta^B_{\tau}$ and $\tau \Delta^B_{\tau}$, and vice-versa replacing left by right sided pullbacks.

Proof. Purity implies formality of $\mathcal{F}$, which implies formality of $\Delta^B_{\tau}$ and $\tau \Delta^B_{\tau}$; hence the existence of a pure equivariant structures on these bimodules.

2.5. Adjunctions. Let $\mathcal{F}: A \to B$ and $\mathcal{G}: B \to A$ be $A_{\infty}$ functors. As before, denote by $\tau \Delta^B$ the $A - B$ bimodule which is the pullback of the diagonal bimodule of $B$ on the left by $\mathcal{F}$, and by $\Delta^B_{\tau}$ the $A - B$ bimodule which is the pullback of the diagonal bimodule of $A$ on the right by $\mathcal{G}$.
**Definition 2.13.** \( F \) is left adjoint to \( G \) (equivalently \( G \) is right adjoint to \( F \)) if \( F \Delta^B \cong \Delta^A \Delta^B \) and \( \Delta^A \) are quasi-equivalent as \( A - B \) bimodules.

The equivalence between \( F \Delta^B \) and \( \Delta^A \) induces equivalences of \( A_\infty \)-bimodules
\[
\begin{align*}
F \Delta^B & \cong \Delta^A, \\
\Delta^A \otimes_F \Delta^B & \cong \Delta^A \otimes_F \Delta^B \cong \Delta^B.
\end{align*}
\]

In particular, the natural maps
\[
\begin{align*}
\Delta^A & \to \Delta^B, \\
\Delta^A \otimes F \Delta^B & \to \Delta^B
\end{align*}
\]
induce maps called the unit and counit of the adjunction:
\[
\begin{align*}
\Delta^A & \to \Delta^B, \\
\Delta^B & \to \Delta^B
\end{align*}
\]

Given that the category of functors embeds fully faithfully into the category of bimodules, we conclude that there are \( A_\infty \)-natural transformations
\[
\begin{align*}
\text{Id}_A & \to G \circ F, \\
F \circ G & \to \text{Id}_B,
\end{align*}
\]
again called the unit and counit of the adjunction. Finally, repeating the construction at the cohomological level, we obtain adjunctions:
\[
\begin{align*}
\text{Id}_{HA} & \to H \circ G \circ H \circ F, \\
H \circ G \circ H \circ F & \to \text{Id}_{HB}.
\end{align*}
\]

Adjunctions are natural with respect to natural transformations. Assume that we have a natural transformation \( T : \mathcal{F} \to \mathcal{F}' \) of \( A_\infty \)-functors which is a quasi-equivalence. Assuming furthermore that \( \mathcal{F} \) is left adjoint to \( \mathcal{G} \), this equivalence induces a quasi-isomorphism of \( A - B \) bimodules \( \mathcal{F} \Delta^B \cong \mathcal{F}' \Delta^B \), which implies that \( \mathcal{F}' \) is also left adjoint to \( \mathcal{G} \). In this case, the triangles
\[
\begin{align*}
\text{Id}_A & \to \mathcal{G} \circ \mathcal{F}, \\
\mathcal{F} \circ \mathcal{G} & \to \text{Id}_B
\end{align*}
\]
commute up to homotopy: this follows from the Yoneda Lemma and the commutativity of the corresponding diagrams at the level of bimodules.

Adjunctions are also natural with respect to composition of functors. Suppose that \( \phi : A \to A' \) and \( \psi : A' \to A \) are quasi-inverse equivalences. If \( \mathcal{F} \) is left adjoint to \( \mathcal{G} \), \( \mathcal{F} \circ \psi \) is left adjoint to \( \phi \circ \mathcal{G} \), and we have the following squares which commute up to homotopy:
\[
\begin{align*}
\phi \circ \text{Id}_A \circ \psi & \to \phi \circ \mathcal{G} \circ \mathcal{F} \circ \psi, \\
\mathcal{F} \circ \psi \circ \phi \circ \mathcal{G} & \to \text{Id}_B
\end{align*}
\]
Proposition 2.14. If $\mathcal{F}$ is formal and is either left or right adjoint to $\mathcal{G}$, then $\mathcal{G}$ is formal. Moreover, the units and counits are formal; i.e. the unit and counit of the adjuction on $HA$ and $HB$ are cohomologous (as $A_\infty$-natural transformations) to the pullback of the unit and counit on $A$ and $B$.

Proof. If $\mathcal{F}$ is formal, it is pure, hence $\Delta_f^B$ and $\mathcal{G}_\Delta^B$ are pure bimodules. If $\mathcal{G}$ is adjoint to $\mathcal{F}$, we conclude that $\Delta_A^G$ or $G_\Delta^A$ is pure, hence that $\mathcal{G}$ is formal. Formality of the units and counits follows from the naturality of the corresponding morphisms. □

3. Bimodules over Fukaya categories

This section collects geometric generalities in the spirit of [6, Section 3], but now concerning bimodules. We recall the standing hypotheses of the prequel [6], whose notation and conventions we adopt. We construct the Fukaya category in a slight generalisation of the framework introduced by Seidel [45], that allows for clean intersections of Lagrangians by combining Floer theory with Morse theory of auxiliary functions on intersections, as in Biran and Cornea’s “pearly trajectories” [10]. This approach was implemented by Seidel in [46] and Sheridan in [52]. We achieve compactness for moduli spaces of Floer holomorphic discs either by exactness or by using the existence of an effective compactification divisor at infinity which is nef on rational curves.

This section is a direct extension of the results of [6, Section 3] from categories to bimodules, and the reader will find a more leisurely exposition in the prequel.

3.1. Geometric set-up. The set-up introduced in [6] as a general setting for proving formality of a Floer $A_\infty$-algebra had the following ingredients. We begin with a smooth projective variety $\bar{M}$ of complex dimension $n$, equipped with a triple of reduced (not necessarily smooth or irreducible) effective divisors $D_0, D_\infty, D_r$. We denote by $\bar{M}$ the symplectic manifold obtained by removing $D_\infty$ from $\bar{M}$, and by $M$ the symplectic manifold obtained by removing the three divisors from $\bar{M}$. When the meaning is clear from context, we shall sometimes write $D_0$ for $D_0 \cap \bar{M}$, and $D_r$ for both $D_r \cap \bar{M}$ and $D_r \cap M$. We assume:

Hypothesis 3.1.

(3.1) the union $D_0 \cup D_\infty \cup D_r$ supports an ample divisor $D$ with strictly positive coefficients of each of $D_0, D_\infty, D_r$.

(3.2) $D_\infty$ is nef (or, at least, non-negative on rational curves).

(3.3) $\bar{M}$ admits a meromorphic volume form $\eta$ which is non-vanishing in $\bar{M}$, holomorphic on $D_r \cap \bar{M}$ and with simple poles along $D_0 \cap \bar{M}$.

(3.4) Each irreducible component of the divisor $D_0 \cap \bar{M}$ moves in $\bar{M}$, with base locus containing no rational curves.

Let $D_0'$ be a divisor linearly equivalent to and sharing no irreducible component with $D_0$, and $B_0 = D_0 \cap D_0'$, which is then a subvariety of $\bar{M}$ of complex codimension 2.

Fix a Kähler form $\omega_M$ in the cohomology class Poincaré dual to $D$. Ampleness implies that $\bar{M}$ is an affine variety, in particular an exact symplectic manifold which can be completed to a Weinstein manifold of finite type, modelled on the symplectization of a contact manifold near infinity. We will denote by $\lambda$ a primitive of the symplectic form $\omega_M$ given by restricting $\omega_M$ to
Let $A \in H_2(\bar{M}; \mathbb{Z})$ be a 2-dimensional homology class, with the property that
\begin{equation}
\langle D_r, A \rangle = 0 \text{ and } \langle D_0, A \rangle = 1.
\end{equation}
Consider the moduli space of stable rational curves in $\bar{M}$ with one marked point
\begin{equation}
\mathcal{M}_1(\bar{M} | 1) = \bigsqcup_{A \in H_2(\bar{M}; \mathbb{Z}) \text{ Condition (3.5) holds}} \mathcal{M}_{1; A}(\bar{M})
\end{equation}
which can be decomposed according to the homology class $A \in H_2(\bar{M}; \mathbb{Z})$ represented by each element. Via evaluation, we obtain a well-defined associated Gromov-Witten invariant $GW_1 \in H^2(M; \mathbb{Z})$ counting such curves (see [6, Lemma 3.6]).

**Hypothesis 3.2.**
\begin{equation}
GW_1 = \sum_{A} GW_{1; A}|_M = 0 \in H^{1f}_{2n-2}(M; \mathbb{Z}) \cong H^2(M; \mathbb{Z});
\end{equation}

$B_0$ is homologous to a cycle supported on the union $(D_0 \cap D_r) \cup D_0^{\text{sing}}$, where $D_0^{\text{sing}}$ denotes the singular locus of $D_0$.

Appealing to Hypothesis 3.2 we fix cochains
\begin{equation}
gw_1 \in C^1(M; \mathbb{Z})
\end{equation}
\begin{equation}
\beta_0 \in C^{1f}_{2n-3}(D_0, (D_0 \cap D_r) \cup D_0^{\text{sing}}; \mathbb{Z})
\end{equation}
satisfying
\begin{equation}
\partial(gw_1) = GW_1
\end{equation}
\begin{equation}
\partial(\beta_0) = [B_0].
\end{equation}

Let $\Delta$ denote the closed unit disc, $\mathcal{R}^1_{(0,1)}(L)$ denote the moduli space of maps from $(\Delta, \partial \Delta)$ to $(M, L)$, with the origin being the unique point mapping to $D_0$, and a point on the interval $(0, 1) \subset \Delta$ mapping to $D_0'$ (the last condition imposes an additional real codimension one constraint). By evaluation at 1, we obtain a cochain
\begin{equation}
\tilde{b}_D^0 = \text{ev}_* [\mathcal{R}^1_{(0,1)}(L)] \in C^1(L; \mathbb{Z}).
\end{equation}
Consider also the moduli space
\begin{equation}
\mathcal{R}^1_1(\bar{M}; (1, 0)|L)
\end{equation}
of discs in $\bar{M}$ with boundary on $L$ and one interior and one boundary marked point, with intersection numbers $(1, 0)$ with $(D_0, D_r)$; this has a natural map $\mathcal{R}^1_1(\bar{M}; (1, 0)|L) \to \mathcal{M}$ via evaluation at the interior marked point. We set
\begin{equation}
\co^0(\beta_0) = \text{ev}_*[\beta_0 \times_M \mathcal{R}^1_1(\bar{M}; (1, 0)|L)] \in C^1(L; \mathbb{Z}).
\end{equation}
We recall from [6] Lemma 3.11 that the sum of the restriction of $gw_1$ with $\tilde{b}_D^0$ and $\co^0(\beta_0)$ defines a cycle
\begin{equation}
\bar{b}_D^0 = \tilde{b}_D^0 + gw_1|_L + \co^0(\beta_0) \in C^1(L; \mathbb{Z}).
\end{equation}
A Lagrangian brane $L$ is \textit{(infinitesimally) equivariant} if the cycle in Equation (3.15) is nullhomologous. An \textit{(infinitesimally) equivariant} structure on $L$, over $k$, is a choice of bounding cochain in $C^0(L; k)$ for this cycle.

3.2. Bimodules from product Lagrangians. Consider two quasiprojective Kähler manifolds $M, N$, with (partial) compactifications $\bar{M} \subset M$ and $\bar{N} \subset N$ by divisors $D, D^N$ each satisfying Hypotheses 3.1 and 3.2 in particular $D^N$ is supported on the union of three divisors $D^0, D^1, D^{\infty}$. Let $M'$ denote the product $M \times N$, which is obtained by removing the divisor
\begin{equation}
D' = D \times \bar{N} \cup \bar{M} \times D^N
\end{equation}
from the compactification $\bar{M} \times \bar{N}$. We write $\pi_N$ for the projection from $M'$ to $N$.

We shall assign a \((\mathcal{F}(M')-\mathcal{F}(M))\) bimodule $\mathcal{X}$ to each Lagrangian brane $K \subset N$. By construction, this bimodule is representable, and hence corresponds to a functor on Fukaya categories. The theory of pseudo-holomorphic quilts due to Man-Wehrheim-Woodward [55, 30] could be used to give a more general construction than the one we provide here, provided the relevant moduli spaces of holomorphic curves with boundary on the correspondence behave well. We have avoided using quilt theory because the correspondence we consider later will not be compact in $M \times M'$, and we do not know that spaces of holomorphic curves with boundary on that correspondence are compact.

Given objects $L$ and $L'$ of $\mathcal{F}(M)$ and $\mathcal{F}(M')$, we chose a Hamiltonian
\begin{equation}
H_{L, L'}: [0, 1] \times M' \to \mathbb{R}
\end{equation}
whose flow maps $L \times K$ to a Lagrangian transverse to $L'$ and define
\begin{equation}
\mathcal{X}(L, L') \equiv CF^+(L \times K, L').
\end{equation}
We shall write $\mathcal{X}(L \times K, L')$ for the set of intersection points of the time-1 image of $L \times K$ and $L'$, so the vector space underlying the Floer complex is by definition the sum of the orientation lines $o_x$ associated to points $x \in \mathcal{X}(L \times K, L')$. Note that, given our conventions for multiplication from the right in Floer complexes, together with the usual conventions for bimodules as set out in Section 2.2 this is indeed an $\mathcal{F}(M')$-left module and an $\mathcal{F}(M)$-right module. To define the structure maps of this $A_{\infty}$ bimodule, we introduce the moduli space
\begin{equation}
\mathcal{R}^{[1]|s} \equiv \mathcal{R}^{r+s+2}
\end{equation}
which is a copy of the interior of the Stasheff associahedron. Ordered counter-clockwise, the marked points on the boundary of an element of $\mathcal{R}^{[1]|s}$ are denoted
\begin{equation}
y_0, z|s, \ldots, z|1, y_1, z_1|, \ldots, z_r|.
\end{equation}
This space admits a natural compactification with top dimensional boundary strata:
\begin{equation}
\prod_{1 \leq S \leq r} \mathcal{R}^{r-R+S|1|s} \times \mathcal{R}^{R-S+2}
\end{equation}
\begin{equation}
\prod_{1 \leq S \leq r} \mathcal{R}^{r-R+1|s-S} \times \mathcal{R}^{R|1|s}
\end{equation}
\begin{equation}
\prod_{1 \leq S \leq r} \mathcal{R}^{1|s-S+R} \times \mathcal{R}^{S-R+2}.
\end{equation}
Here, the middle stratum corresponds to the case where the points $y_0$ and $y_1$ belong to different components of the stable curve. These three degenerations are depicted in Figure 1.
On a given surface, the Floer data defining the pseudo-holomorphic curve equation comprise a closed 1-form $\alpha_\Sigma$ vanishing on the boundary and a family of Hamiltonians $H_z$ on $\bar{M}$ parametrised by $z \in \Sigma$. If $X_z$ is the Hamiltonian flow of $H_z$, the structure maps for the bimodule $K$ are defined by solutions to the family of inhomogeneous equations

$$ (du - X_z \otimes \alpha_\Sigma)^{0,1} = 0 $$

which are now parametrised by the universal family of punctured discs over $\mathcal{R}^{r|1|s}$. We require all the $H_z$ to be compactly supported, and use the product complex structure on $M'$. Near $y_0$ and $y_1$, the inhomogeneous term is induced by the function appearing in (3.17), near $\{z_j|1 \leq j \leq r\}$ by the Hamiltonians used in $\mathcal{F}(M')$, and near $\{z_j|1 \leq j \leq s\}$ by the pullbacks of the Hamiltonians used in the definition of $\mathcal{F}(M)$ under the projection $M' \to M$.

To be more precise, we first choose strip like ends $\{\epsilon_j|j = 0, 1\}$, $\{\epsilon_j|1 \leq j \leq r\}$, and $\{\epsilon_j|1 \leq j \leq s\}$ near each of the punctures, which vary smoothly with respect to the modulus, and which are compatible near the boundary with the data obtained from the boundary strata by gluing. The strip-like ends are incoming at all punctures except $y_0$, and outgoing at $y_0$, i.e. modelled on $Z_+$ at $\{y_1, z_i, z_i\}$ (respective $Z_-$ at $y_0$), where

$$ Z_- = (-\infty, 0) \times [0, 1] \text{ and } Z_+ = [0, \infty) \times [0, 1] $$

The inhomogeneous data consist of a 1-form $\alpha$ on each punctured disc $\Sigma$ in $\mathcal{R}^{r|1|s}$, and a Hamiltonian $H_z$ for each point $z \in \Sigma$. We require the pullback of $\alpha$ under each strip-like end to agree with $dt$.

To state the conditions on the Hamiltonian, fix sequences $(L_0, \ldots, L_s)$ and $(L'_0, \ldots, L'_r)$ of Lagrangian branes in $M$ and $M'$, i.e. objects of $\mathcal{F}(M)$ and $\mathcal{F}(M')$. We then require that

$$ H_{\epsilon_1}(s,t) = H_{L_0, L'_0}(t) $$
$$ H_{\epsilon_0}(s,t) = H_{L_s, L'_s}(t) $$
$$ H_{\epsilon_j}(s,t) = H_{L_{j-1}, L'_j}(t) \quad 1 \leq j \leq r $$
$$ H_{\epsilon_j}(s,t) = H_{L_j, L_{j-1}}(t) \quad 1 \leq j \leq s. $$

The Hamiltonians $H_{L_{j-1}, L'_j}$ and $H_{L_j, L_{j-1}}$ are those which are respectively used in the definition of the Floer complexes $CF^*(L'_j, L'_j)$ and $CF^*(L_j, L_{j-1})$; in the second case, we omit the

\footnote{We hope the use of $s$ as an index for boundary marked points on disks and as a co-ordinate on the strip $\mathbb{R} \times [0, 1]$ and hence for the Hamiltonians will cause no confusion.}
projection to $\mathcal{M}$ from the notation. With these assumptions, finite energy solutions $u$ of Equation (3.24), with boundary conditions given by $L_j'$ and $L_j \times K$ have the following convergence properties:

\begin{align}
\lim_{s \to +\infty} u \circ \epsilon_1(s,t) &\in \mathcal{X}(L_0 \times K, L_0') \\
\lim_{s \to -\infty} u \circ \epsilon_0(s,t) &\in \mathcal{X}(L_s \times K, L'_0) \\
\lim_{s \to +\infty} u \circ \epsilon_j(s,t) &\in \mathcal{X}(L_j', L_{j-1}') \quad 1 \leq j \leq r \\
\lim_{s \to +\infty} u \circ \epsilon_j(s,t) &\in \mathcal{X}(L_j, L_{j-1}) \times K \quad 1 \leq j \leq s.
\end{align}

Note that the last of these properties is implied by the removal of singularities theorem \[33\].

Indeed, $u \circ \epsilon_j$ is the product of a solution to Floer’s equation on the half-strip with target $\mathcal{M}$ and boundary conditions $L_{j-1}$ and $L_j$, with an ordinary holomorphic equation with target $\mathcal{N}$ and boundary condition $K$. Oh’s removal of singularities \[33\] implies that the factor with target $\mathcal{N}$ converges in the limit $s \to +\infty$ to a point in $K$. (If there were no inhomogeneous perturbations, exactness of $K$ in $\mathcal{N}$ would imply that the second factor was actually a constant disk.)

\begin{figure}
\centering
\includegraphics[width=\textwidth]{boundary_conditions}
\caption{Boundary conditions for the bimodule count}
\end{figure}

With this in mind, given chords

\begin{align}
m_0 &\in \mathcal{X}(L_s \times K, L'_r) \\
m_1 &\in \mathcal{X}(L_0 \times K, L'_0) \\
x'_{j-1} &\in \mathcal{X}(L_{j-1}', L_j') \quad 1 \leq j \leq r \\
x_{j-1} &\in \mathcal{X}(L_j, L_{j-1}) \quad 1 \leq j \leq s
\end{align}

we define the moduli space

\begin{align}
\mathcal{R}^{[1]}(m_0; x'_{r}, \ldots, x'_{1}, m_1, x_{1}, \ldots, x_{s})
\end{align}

to be the space of maps

\begin{align}
(\Sigma \to \mathcal{M}')
\end{align}

for $\Sigma \in \mathcal{R}^{[1]}$ with boundary segments mapping counter clockwise to

\begin{align}
(L_s \times K, \ldots, L_0 \times K, L'_0, \ldots, L'_r)
\end{align}
and asymptotic conditions along the ends given by the chords in Equations (3.31), (3.37), cf. Figure 2. More precisely, near the puncture \( z_{ij} \), we require that

\[
\lim_{s \to +\infty} \pi_M \circ u \circ \epsilon_{ij}(s, t) = x_{ij}(t)
\]

where \( \pi \) is the projection from \( M' \) to \( M \). In particular, for each integer \( 1 \leq j \leq s \), we have natural evaluation maps

\[
ev_{ij}: \mathcal{R}^{[1]}(M' \mid m_0; x_{r_1}', \ldots, x_{1}', m_1, x_{1}, \ldots, x_S) \to K.
\]

For generic choices of Floer data, the domain of (3.42) is a smooth manifold of dimension

\[
\deg(m_0) + r + s - 1 - \deg(m_1) - \sum_{j=1}^{s} \deg(x_j) - \sum_{j=1}^{r} \deg(x_j')
\]

and, having fixed brane data on all Lagrangians, it is naturally oriented relative to the orientation lines of the input and output chords. If

\[
\deg(m_0) = \deg(m_1) + \sum_{j=1}^{s} \deg(x_j) + \sum_{j=1}^{r} \deg(x_j') - r - s + 1,
\]

one therefore naturally associates to \( u \in \mathcal{R}^{[1]}(M' \mid m_0; x_{r_1}', \ldots, x_{1}', m_1, x_{1}, \ldots, x_S) \) a map

\[
o_{x_{r_1}'} \otimes \cdots \otimes o_{x_1'} \otimes o_{m_1} \otimes o_{x_1} \otimes \cdots \otimes o_{x_S} \to o_{m_0}.
\]

Taking the sum over all such pseudo-holomorphic curves \( u \), for all collections of chords satisfying Equation (3.41), we obtain the structure maps of \( \mathcal{K} \) as an \( A_\infty \)-bimodule

\[
\mu^{r_1} \mathcal{X} : CF^*(L_{r-1}', L_r') \otimes \cdots \otimes CF^*(L_0', L_1') \otimes \mathcal{K}(L_0, L_0')
\]

\[
\otimes CF^*(L_1, L_0) \otimes \cdots \otimes CF^*(L_s, L_{s-1}) \to \mathcal{K}(L_2, L_0').
\]

To check that these operations satisfy the \( A_\infty \)-relation, we assume that the Floer data are compatible with the boundary decomposition in Equations (3.21), (3.23) of the moduli spaces of domains. For the first stratum, this means that the component that lies in \( \mathcal{R}^{R-S+2} \) carries the data used in the definition of \( \mathcal{F}(M') \), while for the third stratum, the component that lies in \( \mathcal{R}^{S-R+2} \) carries the product of the data used in the definition of \( \mathcal{F}(M) \) with the (homogeneous) holomorphic curve equation with target \( N \). The strata of the moduli space \( \mathcal{R}^{[1]}(M' \mid m_0; x_{r_1}', \ldots, x_{1}', m_1, x_{1}, \ldots, x_S) \) of virtual codimension 1 are

\[
\prod_{1 \leq S \leq R \leq r} \mathcal{R}^{r-R+S}[1](M' \mid m_0; x_{r_1}', \ldots, x_{R+1}', x_{S-1}', \ldots, x_1', m_1, x_1, \ldots, x_S)
\]

\[
\times \mathcal{R}^{R-S+2}(M' \mid x_{1}', x_{R}', x_S')
\]

\[
\prod_{1 \leq R \leq s} \prod_{1 \leq S \leq s} \prod_{m \in \mathcal{X}(L_S \times K, L'_S)} \mathcal{R}^{r-S}[1](M' \mid m_0; x_{r_1}', \ldots, x_{R+1}', m, x_{S+1}, \ldots, x_S)
\]

\[
\times \mathcal{R}^{R-S}(M' \mid m_0; x_{r_1}', \ldots, x_{R+1}', m, x_{S+1}, \ldots, x_S, x_{S}').
\]
Indeed, the space of maps from elements of $\mathcal{R}^{s+R}(M'|m_0; x'_1, \ldots, x'_j, m_1, x_1, \ldots, x_{|R|-1}, x, x_{|S|+1}, \ldots, x_{|S|})$ 
\begin{equation}
\prod_{1 \leq R \leq S \leq s} \mathcal{R}^{r_1|s-R+R}(M'|m_0; x'_1, \ldots, x'_j, m_1, x_1, \ldots, x_{|R|-1}, x, x_{|S|+1}, \ldots, x_{|S|}) \times \mathcal{R}^{s-R+2}(M|x; x_R, \ldots, x_S).
\end{equation}

The last stratum can be more formally described as the fibre product 
\begin{equation}
\mathcal{R}^{r_1|s-R+R}(M'|m_0; x'_1, \ldots, x'_j, m_1, x_1, \ldots, x_{|R|-1}, x, x_{|S|+1}, \ldots, x_{|S|})_{\text{ev}} \times \pi_N
\end{equation}

Indeed, the space of maps from elements of $\mathcal{R}^{s-R+2}$ to $M'$ that we are considering splits as the product of $\mathcal{R}^{s-R+2}(M|x; x_R, x', x_S)$ with holomorphic discs with boundary on $K$. Since $K$ is an exact Lagrangian, all such latter curves are constant. Since the fibre product in Equation (3.50) is taken over $K$, we can remove this factor from the right side, and obtain Equation (3.49).

Since constant discs are regular, and the data defining $\mathcal{R}^{s-R+2}(M|x; x_R, x, x_S)$ are regular (part of the hypotheses that they could be used to define the Fukaya category), the right factor in Equation (3.49) is also regular. Proceeding by induction on the value of the sum $r + s$, we conclude:

**Lemma 3.3.** For generic choice of Floer data, 
\begin{equation}
\mathcal{R}^{r_1|s}(M'|m_0; x'_1, \ldots, x'_j, m_1, x_1, \ldots, x_{|S|})
\end{equation}

is a compact 1-dimensional manifold with boundary whenever 
\begin{equation}
\deg(m_0) = \deg(m_1) + \sum_{j=1}^{s} \deg(x_j) \sum_{j=1}^{r} \deg(x'_{j}) - r - s + 2.
\end{equation}

Its boundary strata are given by Equations (3.47)-(3.49). \hfill □

We now note that the stratum (3.47) corresponds to the first term in Equation (2.11), the stratum (3.48) to the second term, and (3.49) to the last term. We omit the discussion of signs, which is essentially the same as those for the usual $A_\infty$-relations in the Fukaya category; the details appear for instance in [20]. It follows that $\mathcal{K}$ does indeed define an $A_\infty$-bimodule. We will refer to this bimodule $\mathcal{K}$ defined from a fixed Lagrangian brane $K$ in this way as an **elementary** bimodule.

Finally, we note that, fixing an object $L$ of $\mathcal{F}(M)$, the module over $\mathcal{F}(M')$ obtained by considering $\mathcal{K}(L \times K, -)$ is quasi-isomorphic to the Yoneda module of $L \times K$. Appealing to Lemma 2.8 we conclude that there is a functor 
\begin{equation}
\mathcal{F}_K: \mathcal{F}(M) \longrightarrow \mathcal{F}(M'),
\end{equation}

unique up to quasi-isomorphism which represents $\mathcal{K}$.

### 3.3. Equivariance for elementary bimodules.

In the setting of the previous section, suppose we have **nc**-vector fields $b \in CC^1(\mathcal{F}(M), \mathcal{F}(M))$ and $b^N \in CC^1(\mathcal{F}(N), \mathcal{F}(N))$, induced by choosing bounding cochains 
\begin{align}
gw_1 & \in C^1(M; \mathbb{Z}) & gw^N_1 & \in C^1(N; \mathbb{Z}) \\
\beta_0 & \in C^1(D_0 \setminus (D_0 \cap D_r) \cup D^0_r \setminus \text{sing}; \mathbb{Z}) & \beta^N_0 & \in C^1(D^N_0 \setminus (D^N_0 \cap D^N_r) \cup D^{N,\text{sing}}_0; \mathbb{Z})
\end{align}

\begin{align}
gw_1 & \in C^1(M; \mathbb{Z}) & gw^N_1 & \in C^1(N; \mathbb{Z}) \\
\beta_0 & \in C^1(D_0 \setminus (D_0 \cap D_r) \cup D^0_r \setminus \text{sing}; \mathbb{Z}) & \beta^N_0 & \in C^1(D^N_0 \setminus (D^N_0 \cap D^N_r) \cup D^{N,\text{sing}}_0; \mathbb{Z})
\end{align}
We set
\[
D'_0 = \left( D_0 \times \tilde{N} \right) \cup \left( \tilde{M} \times D_0^N \right),
\]
and similarly for \(D'_r\) and \(D'_s\). We obtain cochains
\[
gw'_0 = gw_1 \times [N] + [M] \times gw_1^N \in C^1(M \times N; \mathbb{Z})
\]
\[
\beta'_0 = \beta_0 \times [N] + [M] \times gw_1^N \in C^1(D'_0 \setminus (D'_0 \cap D'_r) \cup (D'_0)^{\text{sing}}; \mathbb{Z}),
\]
which define an nc-vector field \(b' \in \mathcal{C}^1(\mathcal{F}(M'), \mathcal{F}(M'))\). Let us now in addition assume that \(K\) is equivariant with respect to \(b^N\) over the field \(k\). In particular, we fix a cochain
\[
c_K \in C^0(K; k)
\]
whose boundary is
\[
b^0_{\partial D} + gw_1^N |K + cd^0(\beta_0^N) \in C^1(K; k).
\]
For \(k \geq 0\), let \(R^{k+1}_{(0,1)}\) denote the moduli space of domains comprising the disc \(\Delta\)

1. with two marked points \(z_0 = 0\) and \(z_1 \in (0, 1)\), and
2. with \(k+1\) boundary punctures at \(p_0 = 1 \in \partial \Delta\) and points \(\{p_1, \ldots, p_k\} \subset \partial \Delta \setminus \{1\}\) ordered counter-clockwise.

It is important to note that the point 1 will play the role of an output, while the points \(p_i\) will correspond to inputs. Introduce the moduli spaces
\[
R^{r|s}[1]_{(0,1)} \simeq R^{r+s+2}_{(0,1)},
\]
where the marked points on the boundary are labelled as in Equation (3.21). We can describe the boundary of these moduli spaces exactly as in Section 3.2 except that it is more important to keep track of the position of the marked point \(y_1\). In particular, the strata in Equations (3.31) split as
\[
\prod R^{r-R+S|1]s}_{(0,1)} \times R^{R-S+2}
\]
\[
\prod R^{r-R|1]s-S}_{(0,1)} \times R^{R|1]S}
\]
\[
\prod R^{r|1]s-S+R}_{(0,1)} \times R^{S-R+2}
\]
while the boundary strata in Equation (3.34) split as
\[
\prod R^{r-R+S|1]s}_{(0,1)} \times R^{R-S+2}
\]
\[
\prod R^{r-R|1]s-S}_{(0,1)} \times R^{R|1]S}_{(0,1)}
\]
\[
\prod R^{r|1]s-S+R}_{(0,1)} \times R^{S-R+2}_{(0,1)}
\]
Given sequences \((L_0, \ldots, L_s)\) and \((L'_0, \ldots, L'_s)\) of Lagrangian branes in \(M\) and \(M'\), and chords as in Equations (3.31)-(3.37), we obtain a moduli space
\[
R^{r|s}[m_0; x'_1, \ldots, x'_1, m_1, x_1, \ldots, x_s],
\]
with boundary conditions as in Figure 2 and such that
\[
u(z_0) \in D_0 \text{ and } u(z_1) \in D'_0.
\]
These moduli spaces define a cochain
\begin{equation}
\tilde{c}_K \in \mathcal{E}_{B,A}(K)
\end{equation}
whenever $A \subset \mathcal{F}(M)$ and $B \subset \mathcal{F}(M')$ are subcategories of the Fukaya categories of $M$ and $M'$ containing the branes $L_i$ respectively $L'_j$.

In order to produce a cycle, we must add the contributions $co\chi(\beta'_0)$ and $CO\chi(gw'_1)$ which are defined in analogy with the corresponding structures at the level of categories; here $CO\chi$ is the open-closed map
\[ CO\chi : C^*(M \times N; k) \rightarrow \mathcal{E}_{B,A}(K) \]
which is induced from the usual open-closed map of $M'$. In addition, we define an operator
\begin{equation}
\nu(c_K) \in \mathcal{E}_{B,A}(K)
\end{equation}
by counting the moduli spaces
\begin{equation}
R^{r+s+1}_{(0,1)}(m_0; x'_1, \ldots, x'_s, m_1, x_{|1}, \ldots, x_{|s})_{ev_R} \times_K c_K
\end{equation}
(where $R$ varies). Here, $c_K$ is represented by a cycle in $K$, e.g. in terms of descending manifolds of critical points of a Morse function.

With this in mind, we define
\begin{equation}
c_K \equiv \tilde{c}_K + co\chi(\beta'_0) + CO\chi(gw'_1) + \nu(c_K) \in \mathcal{E}_{B,A}(K).
\end{equation}

**Proposition 3.4.** The cochain $c_K$ defines an equivariant structure on $K$.

**Sketch of proof.** The reader may wish to compare to the corresponding assertion in \cite[Proposition 3.20]{ABO}. We must verify Equation (2.14), and identify the boundary strata of the moduli spaces with the terms in this equation. We discuss the essential part of the computation, neglecting the contributions of the maps $CO$ and $co$ which are straightforward adaptations of the situation for categories discussed in the prequel.

The left hand side of Equation (2.14) corresponds to the boundary stratum of $R^{r+s+1}_{(0,1)}$ given in Equations (3.62)-(3.64) and Equation (3.67). The two terms on the right hand side of Equation (2.14) respectively correspond to the boundary strata in Equations (3.65) and those in Equation (3.67) for which the map on the factor $R^{S-K+2}_{(0,1)}$ is constant in the factor $N$. If the map is not constant in this factor, then we obtain
\begin{equation}
R^{r+s+1}_{(0,1)}(m_0; x'_1, \ldots, x'_s, m_1, x_{|1}, \ldots, x_{|s})_{ev_R} \times_K R^{1}_{(0,1)}(K).
\end{equation}
In the absence of contributions from the maps $co$ and $CO$, this is precisely the boundary of the moduli space in Equation (3.72), and it is accounted for by the term $\nu(c_K)$ in Equation (3.73). \hfill $\Box$

### 3.4. Purity for elementary bimodules.

We now further suppose that:

**Hypothesis 3.5.** The nc-vector field $b_M$ is pure on $A \subset \mathcal{F}(M)$, and $K \in \mathcal{F}(N)$ is pure.

Denote by $B = A \times \{K\}$ the full subcategory of $\mathcal{F}(M')$ comprising objects of the form $L' = L \times K$, for $L \in A$. The brane $K$ defines an elementary $(B - A)$-bimodule $\mathcal{K}$, as in the previous section.

**Lemma 3.6.** Assuming Hypothesis 3.5, $c_K$ defines a pure equivariant structure on $\mathcal{K}$.
Proof. After unwinding definitions, this is a repackaging of the Künneth theorem in Floer cohomology. Let \( L_i \in \mathcal{A} \), for \( i = 1, 2 \), and let \( L'_i = L_i \times K \in \mathcal{B} \subset \mathcal{F}(N) \). By choosing the Hamiltonian perturbation in Equation (3.17) to be the sum of the Hamiltonians on \( M \) and \( N \), we obtain an isomorphism of cochain complexes
\[
\mathcal{X}(L_1, L'_2) \equiv CF^*(L_1, L_2) \otimes CF^*(K, K).
\]

The equivariant structure \( c_\mathcal{X} \), together with the given \( \mathfrak{g} \)-vector fields on \( \mathcal{A} \) and \( \mathcal{B} \), yields an endomorphism of the cochain complex \( \mathcal{X}(L_1, L'_2) \) via \((2.15)\). By forcing all equations to split, our choices of auxiliary data ensure that this endomorphism also splits as a sum
\[
id \otimes b_1^Y + b^1 \otimes id \in \text{Aut}(\mathcal{X}(L_1, L'_2)).
\]

Purity for \( c_\mathcal{X} \) therefore follows from purity for \( b_1^Y \) and \( b^1 \). \( \square \)

**Corollary 3.7.** Under the previous hypotheses, the Künneth functor \( \mathcal{F}_K : \mathcal{A} \to \mathcal{B} \) of \((3.53)\) is formal.

**Proof.** This follows from the discussion before Proposition 2.11 together with Lemma 3.6. \( \square \)

4. **Cup bimodules**

4.1. **The Milnor fibre and the Hilbert scheme.** Let \( A_{2n-1} \) denote the Milnor fibre
\[
\left\{ x^2 + y^2 + \prod_{i=1}^{2n}(z - i) = 0 \right\} \subset \mathbb{C}^3
\]
equipped with the restriction of the standard Kähler form from \( \mathbb{C}^3 \). This is the total space of a Lefschetz fibration (from projection to the \( z \)-plane) with nodal fibres over \( \{1, 2, \ldots, 2n\} \). The Hilbert scheme of two points \( \text{Hilb} \{2\}(A_{2n-1}) \) contains a distinguished divisor \( D_{rel} \) which is the relative Hilbert scheme of the projection to the \( z \)-plane, i.e. subschemes whose projection to \( \mathbb{C}_z \) has length \( < 2 \), or equivalently, the relative second symmetric product of the fibration over \( \mathbb{C}_z \).

Let \( y_n \subset \text{Hilb}^{[n]}(A_{2n-1}) \) denote the open subset of the Hilbert scheme given by removing the image of \( \text{Hilb}^{[n-2]}(A_{2n-1}) \times D_{rel} \) under the canonical map from \( \text{Hilb}^{[n-2]} \times \text{Hilb}^{[2]} \to \text{Hilb}^{[n]} \), i.e. the divisor of subschemes whose projection to \( \mathbb{C}_z \) has length \( < n \). By Manolescu’s work \([29]\), this is holomorphically isomorphic to the Springer fibre appearing in \([50]\) (see Section 5.1). We equip \( y_n \) with an exact Kähler form which is product-like (induced from the product form on \( A_{2n-1}^{n} \)) outside an arbitrarily small neighbourhood of the diagonal.

Let \( Z \) denote the compactification \( \tilde{A}_{2n-1} \) of the Milnor fibre which is the blow-up of \( \mathbb{P}^1 \times \mathbb{P}^1 \) at \( 2n \) points. \( Z \) contains divisors defined by sections \( s_0, s_\infty \) of its Lefschetz fibration structure over \( \mathbb{P}^1 \) (extending the usual Lefschetz fibration \( A_{2n-1} \to \mathbb{C} \)), and a fibre \( F_\infty \) at infinity. In the notation of Section 3.1

- The projective variety is \( \tilde{M} = \text{Hilb}^{[n]}(Z) \).
- \( D_0 \) is the divisor of subschemes whose support meets \( s_0 \cup s_\infty \) in \( Z \).
- \( D_\infty \) is the divisor of subschemes whose support meets \( F_\infty \).
- \( D_r \subset \tilde{M} \) is the relative Hilbert scheme of the projection \( Z \to \mathbb{P}^1 \).

The previous discussion implies that \( M = y_n \) and \( \tilde{M} = \tilde{M} \setminus D_\infty = \text{Hilb}^{[n]}(\tilde{A}_{2n-1}) \). By taking push-offs of the sections \( s_0, s_\infty \) in \( Z \setminus F_\infty \) one obtains the linearly equivalent divisor \( D'_0 \). That
Hypotheses 5.1 and 5.2 hold for these choices (in particular vanishing of the Gromov-Witten invariant $GW_1$ in this situation) is proved in [8, Section 6], which also gives an explicit choice of bounding cochain $gw_1$. When we consider bimodules over products $Y_n \times Y_{n-1}$ we compactify each factor as above, which brings us into the framework considered previously modulo Remark 4.8 below.

4.2. Crossingless matchings. Any path $\gamma$ between critical values of $A_{2n-1} \to \mathbb{C}$ defines a Lagrangian matching sphere, so an $n$-tuple of pairwise disjoint paths $\varphi$ which join the critical points in pairs defines a Lagrangian submanifold $L_\varphi \cong (S^2)^n \subset Y_n$. The “Markov I” move of [50, Lemma 48] implies that one can slide any arc of $\varphi$ over any other arc (keeping the end-points fixed) without changing the Hamiltonian isotopy class of the associated Lagrangian submanifold $L_\varphi$, even though the corresponding isotopy necessarily passes through the diagonal locus in the Hilbert scheme (where the Kähler form is not product-like).

**Definition 4.1.** The symplectic arc algebra $\mathcal{H}_{symp}^n$ is the algebra $\bigoplus_{\varphi,\varphi'} HF^*(L_\varphi,L_{\varphi'})$, where $\varphi,\varphi'$ run over the set of upper-half-plane crossingless matchings in $\mathbb{C}$. The cochain level $A_\infty$-algebra will be denoted $C\mathcal{H}_{symp}^n$.

Since $Y_n$ is exact, and the Lagrangians $L_\varphi$ are simply-connected (hence exact), Floer cohomology is defined unproblematically. The Lagrangians $L_\varphi$ are orientable and admit unique spin structures, so one can work in characteristic zero. In fact $c_1(Y_{n}) = 0$ so the algebra $\mathcal{H}_{symp}^n$ admits an absolute $\mathbb{Z}$-grading. There are $\frac{1}{n} \binom{2n}{n}$ upper half-plane matchings, so the underlying graded algebra of $\mathcal{H}_{symp}^n$ is a finite-dimensional $k$-algebra.

For an upper-half-plane crossingless matching $\varphi$, we will denote by $\tilde{\varphi}$ its reflection into the lower half-plane. An iterative application of the Markov I move shows that $L_\varphi$ is Hamiltonian isotopic to $L_{\tilde{\varphi}}$. For any upper half-plane matchings $\varphi,\varphi'$, the Lagrangians $L_\varphi$ and $L_{\tilde{\varphi}}$ meet transversely. As shown in [6, Section 4.5], the mod 2 degrees of all intersection points co-incide, so the Floer differential in $CF^*(L_\varphi,L_{\tilde{\varphi}})$ vanishes identically. (We will introduce a different orientation scheme below; obviously, the vanishing of the Floer differential does not depend on such a choice.) There is therefore a vector space isomorphism

$$\mathcal{H}_{symp}^n \cong \bigoplus_{\varphi,\varphi'} CF^*(L_\varphi,L_{\tilde{\varphi}}).$$

As a relatively graded group, $CF^*(L_\varphi,L_{\tilde{\varphi}}) \cong H^*(S^2)^{\oplus c(\varphi,\varphi')}$, where $c(\varphi,\varphi')$ is the number of components of the planar unlink $\varphi \cup \varphi'$, see [6, Proposition 5.12].

4.3. Reduction to a basic case. We will say that crossingless matchings $\varphi,\varphi'$ of $2n$ points meet in codimension $k$, or that the corresponding Lagrangian submanifolds have that property, if $HF^*(L_\varphi,L_{\varphi'}) \cong H^*(S^2)^{\oplus \binom{n-k}{k}}$ has real rank $2n - 2k$. (The terminology is inherited from the structure of the “compact core” of the quiver variety underlying $Y_n$, cf. Lemma 5.2.) Thus, in general, $\varphi$ and $\varphi'$ meet in codimension $n - c(\varphi,\varphi')$. The following is then manifest:

**Lemma 4.2.** The matchings $\varphi$ and $\varphi'$ meet in codimension zero if and only if $\varphi = \varphi'$, and meet in codimension one if and only if $\varphi$ and $\varphi'$ share exactly $n - 2$ arcs. $\square$
If \( \wp \) and \( \wp' \) meet in codimension 1, then, for any crossingless matching \( \wp'' \), the codimension of intersection changes by 1, i.e.

\[
(4.1) \quad c(\wp, \wp'') = c(\wp', \wp'') \pm 1.
\]

The following result asserts that there is a way of interpolating between any two crossingless matchings so that the codimensions form a monotone sequence.

**Lemma 4.3.** For any pair of crossingless matchings \( \wp_0, \wp_k \) which meet in codimension \( k \), there is an interpolating sequence

\[
\wp_0, \wp_1, \ldots, \wp_k
\]

such that \( \wp_i \) and \( \wp_{i+1} \) meet in codimension one, \( \wp_0 \) and \( \wp_i \) in codimension \( i \), and \( \wp_i \) and \( \wp_k \) in codimension \( k - i \).

**Proof.** The proof is by induction on the the codimension. In the inductive step, we pick any arc in \( \wp_k \) which does not lie in \( \wp_i \); we define \( \wp_{i+1} \) to be the unique matching containing this arc which meets \( \wp_i \) in codimension 1 (this process is illustrated in Figure 3).

![Interpolating crossingless matchings by ones which intersect in codimension two](image)

**Figure 3.** Interpolating crossingless matchings by ones which intersect in codimension two

Since \( CF^*(L_\wp, L_{\wp'}) \cong H^*(S^2) \otimes c(\wp, \wp') \) as a relatively graded group, there is a well-defined rank one subspace

\[
(4.2) \quad k_{\min}(\wp, \wp') \subset CF^*(L_\wp, L_{\wp'}) \subset H^*_{\text{sym}}
\]

which is spanned by any minimal degree generator. This subspace does not depend on the choice of graded structures on the Lagrangians.

**Lemma 4.4.** Let \( \wp, \wp' \) and \( \wp'' \) be crossingless matchings such that \( \wp \) and \( \wp' \) meet in codimension one. Suppose moreover that \( c(\wp, \wp'') = c(\wp', \wp'') - 1 \). Then the Floer product

\[
k_{\min}(\wp', \wp'') \otimes k_{\min}(\wp, \wp') \rightarrow k_{\min}(\wp, \wp'')
\]

is non-trivial.

**Proof.** A stronger result follow from Lemma 5.19 in conjunction with Lemma 5.18, by appealing to a plumbing model for the Floer product. More directly, placing \( \wp, \wp' \) in the upper half-plane and \( \wp'' \) in the lower, the constant triangle at the odd critical points is regular, hence contributes
non-trivially to the product. Since the Lagrangians are exact, the area of any contributing holomorphic triangle is determined by the actions of the isolated intersection points, which implies that the constant triangle is then the only contribution. Compare to [6, Figure 13 & Lemma 5.18]. □

4.4. Purity of the symplectic arc algebra. We summarise the main result of [6]. Certain of the Lagrangian submanifolds \( L_\varphi \) have a special role.

- The Lagrangian associated to the crossingless matching comprising a sequence of adjacent arcs joining the critical values in pairs \( \{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\} \) is denoted \( L_{\varphi_{\text{plait}}} \); this is the \textit{plait} matching.
- The Lagrangian associated to the crossingless matching comprising a sequence of nested arcs joining the critical values in pairs \( \{1, 2n\}, \{2, 3\}, \ldots, \{2n-2, 2n-1\} \) is denoted \( L_{\varphi_{\text{mix}}} \); this is the \textit{mixed} matching.

\[
\text{Figure 4. The plait (left) and mixed (right) matchings of } 2n \text{ points.}
\]

**Proposition 4.5.** [6] For any \( \varphi, \varphi' \), the Floer cohomology \( HF^*(L_\varphi, L_{\varphi'}) \) is a cyclic module, generated by \( k_{\text{min}}(\varphi, \varphi') \), over each of \( HF^*(L_\varphi, L_\varphi) \) and \( HF^*(L_{\varphi'}, L_{\varphi'}) \). Furthermore, the rank one subspace of \( HF^*(L_\varphi, L_{\varphi'}) \) of largest cohomological degree lies in the image of the product

\[
HF^*(L_{\varphi_{\text{plait}}}, L_{\varphi'}) \otimes HF^*(L_{\varphi_{\text{plait}}}, L_{\varphi_{\text{plait}}}) \rightarrow HF^*(L_\varphi, L_{\varphi'}).
\]

□

Given a grading on the plait Lagrangian \( L_{\varphi_{\text{plait}}} \), there is a unique grading on each matching Lagrangian \( L_\varphi \) so that \( HF^*(L_{\varphi_{\text{plait}}}, L_\varphi) \) is symmetrically graded in the sense that the groups \( HF^*(L_{\varphi_{\text{plait}}}, L_\varphi) \) are supported in the same range of degrees \( n - c(\varphi, \varphi_{\text{plait}}) \leq * \leq n + c(\varphi, \varphi_{\text{plait}}) \).

The Gromov-Witten invariant \( GW_1 \in H^*(\bar{M}) \) is proved to vanish in [6]; since the \( L_\varphi \) are simply-connected, it follows that they admit equivariant structures (relative to the \( nc \)-vector field constructed from counting discs in the compactification \( \bar{M} \)). Given a choice of brane structures on the Lagrangians (which fix absolute gradings on Floer cohomology groups), we wish to choose these equivariant structures so that the weight gradings enjoy the same symmetry as the cohomological gradings.

**Theorem 4.6.** [6] For any choice of equivariant structure on \( L_{\varphi_{\text{plait}}} \), there are unique equivariant structures on the Lagrangians \( L_\varphi \) such that \( C\mathcal{H}^{\text{symp}}_n \) is pure. □

By Theorem 2.4, we obtain a fixed quasi-equivalence between \( C\mathcal{H}^{\text{symp}}_n \) and its cohomological algebra, which is \( \mathcal{H}^{\text{symp}}_n \).
4.5. **Some elementary bimodules.** Consider $y_n \subset \text{Hilb}^n(A_{2n-1})$, with $z: A_{2n-1} \to \mathbb{C}$ having critical values at $\{1, 2, \ldots, 2n\} \subset \mathbb{C}$. For each $1 \leq i \leq 2n$, let $y^i_n$ correspond to the intersection with $y_n$ of the Hilbert scheme of $n$ points lying in a subdomain $A_{2n+1}^i$ of the $A_{2n+1}$-surface defined by drawing a disk which encircles all but the $(i, i+1)$-pair of critical values of $A_{2n+1}$.

There is an inclusion
\begin{equation}
U^i \times y^i_n \hookrightarrow y^i_{n+1}
\end{equation}
where $U^i \subset A_{2n+1}$ is the $z$-preimage of a 2-disk encircling the $(i, i+1)$-critical values, so $U^i$ is a neighbourhood of a Lagrangian 2-sphere $L_i \subset A_{2n+1}$. See Figure 5.

**Figure 5.** Subsets of the base of the $A_{2k+1}$-surface.

**Lemma 4.7.** There is a deformation of exact symplectic manifolds from $y^i_n$ to $y_n$.

**Sketch.** We move the $(i, i+1)$-critical point pair towards infinity and then deform the subdisk defining $A_{2n+1}^i$ to a standard disk in the base over which $A_{2n-1}$ and $\bar{A}_{2n-1}$ fibre. This deformation of (compactified) Milnor fibres induces a deformation of Hilbert schemes and hence of the subspaces given by the nilpotent slices. □

The deformations are through subdomains inside manifolds satisfying Hypotheses 3.1 and 3.2, in particular the manifolds admit holomorphic projection maps to a continuously varying family of Stein domains. There are induced quasi-isomorphisms
\begin{equation}
F(y_n) \simeq F(y^i_n).
\end{equation}

Via the inclusion $y^i_{n+1} \supset y^i_n \times U^i$, and the identification $y^i_n \cong y_n$ of Lemma 4.7, the Lagrangian sphere $L^i \subset U^i$ defines an elementary $(F(y^i_{n+1}), F(y^i_n))$-bimodule. By Lemma 2.8, there is a corresponding functor
\[ \cup_i: F(y_n) \to F(y^i_{n+1}) \]
which we call $\cup_i$.

**Remark 4.8.** One minor difference with the setting in Section 3.2 is that the inclusion $y^i_n \times U^i \subset y^i_{n+1}$ is not an equality, and $\bar{y}^i_{n+1}$ does not co-incide with the product compactification $\bar{y}^i_n \times \bar{U}^i$. However, there is an embedding of partial compactifications (complements of the divisors of subschemes supported on the fibres over $\infty \in \mathbb{P}^1$)
\begin{equation}
\bar{y}^i_n \times \bar{U}^i \hookrightarrow \bar{y}^i_{n+1}
\end{equation}
coming from the subset of the Hilbert scheme of $A_{2n+1}$ of subschemes with a length one subscheme supported over $U^i$ and a length $n$ subscheme over the larger disk of Figure 5. This embedding of partial compactifications is compatible with the divisors $D_0, D_r$ at infinity, which is sufficient for constructing the moduli spaces used in Section 4.2.
4.6. Cup functors are formal. The theory of Section 3.4 defines an equivariant structure on $\cup_i$. It remains to prove that it is in fact pure, and hence formal. We separate out cases depending on the parity of the index $i$ at which we include the new component of the matching when applying $\cup_i$.

**Lemma 4.9.** If $i$ is odd, the cup functor $\cup_i$ is formal.

**Proof.** Let $C\mathcal{H}^{\text{sym}}_{n+1,i}$ denote the subcategory of $C\mathcal{H}^{\text{sym}}_{n+1}$ corresponding to crossingless matchings containing the $i$th cup. By construction, $\cup_i$ factors through the inclusion $C\mathcal{H}^{\text{sym}}_{n+1,i} \subset C\mathcal{H}^{\text{sym}}_{n+1}$. By Corollary 3.7, the functor $\cup_i: C\mathcal{H}^{\text{sym}}_{n} \to C\mathcal{H}^{\text{sym}}_{n+1,i}$ is formal with respect to the product equivariant structure on $C\mathcal{H}^{\text{sym}}_{n+1,i}$. Since the quasi-isomorphism from the cochains to cohomology is determined by the equivariant structure, it remains to show that we can choose the equivariant structure on the vanishing cycle $S^2$ factor so that the product equivariant structure agrees with the restriction of the equivariant structure on $C\mathcal{H}^{\text{sym}}_{n+1,i}$ fixed in Theorem 4.6.

To this end, it is useful to recall that the set of graded structures on a Lagrangian $L$, if non-empty, is parametrised by $H^0(L)$; similarly, the set of equivariant structures on $L$, if non-empty, is parametrised by $H^0(L)$. Elements of $H^0(L)$ act respectively by shifting the homological gradings and the weights. The hypothesis that $i$ is odd implies that the plait matching is an object of $C\mathcal{H}^{\text{sym}}_{n+1,i}$. We can therefore choose the graded and equivariant structures on the vanishing cycle $L^i$ so that the product equivariant structure on $L^{n+1}_{\text{plait}} = \cup_i L^n_{\text{plait}}$ agrees with the one fixed in Theorem 4.6.

For each Lagrangian $L_\varphi \in C\mathcal{H}^{\text{sym}}_n$, the Künneth formula in Floer cohomology implies that, when equipped with product gradings, the Floer cohomology between $\cup_i L^n_{\text{plait}}$ and $\cup_i L^{n+1}$ is still symmetrically graded; hence the product grading on $\cup_i L_\varphi$ agrees with the one fixed for Lagrangians in $\mathcal{H}_{n+1}$. The product equivariant structure on $\cup_i L_\varphi$ has the property that weights and gradings agree (i.e. it is pure, again by Künneth); hence it also agrees with the equivariant structure fixed in Theorem 4.6.

We conclude that the functor $\cup_i$ is pure with respect to the fixed equivariant structures from Theorem 4.6 hence is formal.

**Lemma 4.10.** If $i$ is even, the cup functor $\cup_i$ is formal.

**Proof.** While the distinguished component $L^{n+1}_{\text{plait}}$ is not in the image of the functor $\cup_i$, the matchings $\varphi^{n+1}_{\text{plait}}$ and $\cup_i \varphi^n_{\text{plait}}$ differ by a single handle-slide. It follows that the corresponding Lagrangian submanifolds $L^{n+1}_{\text{plait}}$ and $L^{n}_{\cup_i \text{plait}}$ intersect in codimension one in the sense of Section 4.3. By Lemma 4.4 we therefore know that the Floer triangle product for the triple

$$(L^{n+1}_{\text{plait}}, \cup_i L^n_{\text{plait}}, \cup_i L^n_\varphi)$$

is non-trivial. As with the proof of purity of the symplectic arc algebra in [6, Proposition 6.10], in the presence of a non-trivial Floer product, symmetry of the weights for two of the three possible pairs of components implies symmetry of the weights for the third pair. Indeed, by Proposition 3.4, the top degree class lies in the image of the product

$$HF^*(L^{n+1}_{\text{plait}}, \cup_i L^n_\varphi) \otimes HF^*(\cup_i L^n_{\text{plait}}, L^{n+1}_{\text{plait}}) \to HF^*(\cup_i L^n_{\text{plait}}, \cup_i L^n_\varphi)$$
which determines its weight given the weights on the domain groups; weights of all other classes in the target are fixed by the cyclicity of the module action over $H^{*}(\cup_{i}L_{i}^{n})$ say. For the two components of the triple \((4.7)\) in the image of \(\cup_{i}\) we have symmetry of weights by construction, and for the first two components we have symmetry by the choice of equivariant structure on \(L^{1} = S^{2}\). Varying \(\varphi\), we see that the equivariant structure on the target category is obtained by restriction from \(\mathcal{H}_{n+1}^{symp}\) as required.

\[\Box\]

5. Cohomology bases and conormal models

In the combinatorial arc algebra \(H_{n}\) from [51], the module associated to a pair of matchings \(\varphi \cup \varphi'\) is \(V^{\otimes c(\varphi, \varphi')}\), where \(V = \mathbb{Z}[x]/(x^{2})\) has a distinguished basis \(\{1, x\}\). We will pin down explicit bases for the corresponding Floer groups appearing in \(\mathcal{H}_{n}^{symp}\) inductively in \(n\), by making systematic use of the constraints imposed by compatibility with module structures and cup-functors.

For this and the next section, all (Floer) cohomology groups are taken with \(\mathbb{Z}\) coefficients.

5.1. Spaces of flags and the compact core. Let \(S = S_{n} \subset \mathfrak{gl}_{2n}(\mathbb{C})\) be the affine subspace consisting of matrices of the form

\[
A = \begin{pmatrix}
A_{1} & I \\
A_{2} & I \\
\vdots & \vdots \\
A_{n} & 0
\end{pmatrix}
\]

with \(A_{k} \in \mathfrak{gl}_{2}(\mathbb{C})\), and where \(I \in \mathfrak{gl}_{2}(\mathbb{C})\) is the identity matrix. (In [50] we considered the codimension one subspace lying in \(\mathfrak{gl}_{2}(\mathbb{C})\) and the adjoint quotient on configurations of total mass zero, but the results carry over \textit{mutatis mutandis} without the trace zero condition.) Grothendieck [33] described a simultaneous resolution of the adjoint quotient map

\[
(5.2) \quad \chi : S \rightarrow \text{Conf}_{2n}(\mathbb{C})
\]

via the space of pairs \((A, F)\), where \(F\) is a flag and \(A \in S\) preserves the flag. The simultaneous resolution maps to \(\mathbb{C}^{2n}\), the space of ordered configurations, since the flag orders the eigenspaces.

The resolution of \(\chi^{-1}(0)\) contains a “compact core” which is the space of flags fixed by the distinguished nilpotent matrix where \(A_{j} = 0 \forall j\). We now summarise results of Cautis and Kamnitzer [14, 21, 15], which give a fibrewise compactification of \(\chi\).

Fix a vector space \(\mathbb{C}^{2}\) with basis \(\{e_{1}, e_{2}\}\) and consider \(\mathbb{C}[z]\)-submodules \(F_{i}\) of \(\mathbb{C}^{2} \otimes \mathbb{C}[z]\) which contain \(F_{0} = \mathbb{C}^{2} \otimes \mathbb{C}[z]\). More precisely, for a tuple \(w = (w_{1}, \ldots, w_{2n}) \in \mathbb{C}^{2n}\) we consider the space of flags of modules:

\[
(5.1) \quad Y_{n}^{w} = \{F_{0} \subset F_{1} \subset \cdots \subset F_{2n} \subset \mathbb{C}^{2} \otimes \mathbb{C}[z], \text{rk}(F_{i}/F_{i-1}) = 1, (z - w_{i})F_{i} \subset F_{i-1}\}.
\]

As we vary \(w\), these spaces fit into a family \(Y_{n} : \mathbb{C}^{2n} \rightarrow\), we think of the base as the space of ordered configurations of \(2n\) points in \(\mathbb{C}\). There is an open subset

\[
(5.3) \quad Y_{n}^{\text{open}} = \{(F_{0}, \ldots, F_{n}) \in Y_{n}^{w} \mid F_{2n}/F_{0} = (z^{-1}e_{1}, z^{-1}e_{2}, \ldots, z^{-n}e_{1}, z^{-n}e_{2})\}
\]

We say \(w\) is generic if each \(w_{i} \neq w_{j}\). At the other extreme, when \(w = (0, \ldots, 0)\), we have the resolution of the nilpotent cone in \(S\). The compact core

\[
Z = \{(F_{0}, \ldots, F_{2n}) \mid z^{n}F_{2n} = F_{0}\} \subset Y_{n}^{\text{open}}
\]

is the locus lying over the matrix given by \(A_{j} = 0\) for each \(j\) in \((5.1)\).
Proposition 5.1. The family $Y_n \to \mathbb{C}^{2n}$ has the following properties.

1. Each fibre $Y_n^w$ is an iterated $\mathbb{P}^1$-bundle, diffeomorphic to $(S^2)^{2n}$. If $w$ is generic, then $Y_n^w \cong (\mathbb{P}^1)^n$ is holomorphically a product.
2. $Y_n$ is a fibrewise compactification of the simultaneous resolution $[57]$. In particular, if $w$ is generic, then $Y_n^{w,\text{open}} \cong Y_n^w$ are holomorphically isomorphic.
3. The complement $Y_n^w \setminus Y_n^{w,\text{open}}$ is an irreducible divisor, of class $(1, \ldots, 1) \in H^2(\mathbb{P}^1)^{2n}$ if $w$ is generic.

Sketch. The first two statements are directly from [14] Section 2.2 (see also [21 15]). For the third, since irreducibility is an open condition and $\chi$ is equivariant for a $\mathbb{C}^*$-action rescaling all eigenvalues, it suffices to work on the zero-fibre, i.e. with $w = 0$. In that case, $Y_n^0 \setminus Y_n^{0,\text{open}}$ is the locus where the natural map $z^nF_{2n} \to F_0$ is not an isomorphism, which is a divisor of determinantal type defined by vanishing of a section of the tensor product of the bundles $\mathcal{L}_i$ with fibres $F_i/F_{i-1}$. These deform to $\mathcal{O}(1)$-bundles on the factors of $(\mathbb{P}^1)^{2n}$, which gives the statement on the degree. Finally, the divisor maps with degree one to the space of flags of length $2n - 1$, which implies irreducibility (an Euler characteristic computation shows that the divisor is singular).

For an upper-half-plane crossingless matching $\varphi$, let $\hat{L}_\varphi$ denote the Lagrangian multi-antidiagonal of $(\mathbb{P}^1)^{2n}$ in which points paired by $\varphi$ should take antipodal values, with the factor labelled by the odd co-ordinates positively oriented. For instance, when $n = 2$ and we embed in $(\mathbb{P}^1)^4$, the two core components are

$$\hat{L}_{\varphi,\text{plan}} = \{(z, -1/\bar{z}, w, -1/\bar{w})\} \text{ and } \hat{L}_{\varphi,\text{mix}} = \{(z, -1/\bar{w}, w, -1/\bar{z})\}.$$ 

Lemma 5.2. $[57]$ The compact core $Z$ is homeomorphic to $\bigcup_\varphi \hat{L}_\varphi$ by a map which is a diffeomorphism on each component. In particular, $Z$ is a union of copies of $(S^2)^{n}$, indexed by crossingless matchings, meeting pairwise cleanly, and the “small antidiagonal”

$$(5.4)\quad \{(z, -1/\bar{z}, z, -1/\bar{z}, \ldots, z, -1/\bar{z})\} \subset (\mathbb{P}^1)^{2n}$$

lies in the common intersection locus of all of the $\hat{L}_\varphi$.

Since the divisor at infinity in $Y_n^0$ is ample when $w$ is generic, one can define the Fukaya category of $Y_n^0$ with respect to a finite volume exact Kähler form which extends smoothly to the compactification, arguing via positivity of intersection at infinity to ensure compactness of moduli spaces of pseudoholomorphic discs. The map $Y_n^w \to \mathbb{C}^{2n}$ is a differentiably trivial fibration with fibre $(S^3)^{2n}$, which is symplectically trivial in the class of the divisor at infinity by fragility of the two-dimensional Dehn twist [39] (of course the monodromy is not fragile relative to the divisor at infinity).

Lemma 5.3. The iterated vanishing cycle $L_\varphi$ is smoothly isotopic to $\hat{L}_\varphi$.

Proof. This follows from the description of the Morse-Bott degeneration of the spaces $Y_n^{w,\text{open}} \subset Y_n^w$ as a pair of eigenvalues coalesce, given in [21] Section 3, which in turn relies on the factorisation property of the affine Grassmannian due to Beilinson and Drinfeld [9] Section 5.3.10. In short, if we restrict the fibrewise compactification of $S$ to a disc $D_\epsilon$ parametrising $w = (\mu + \epsilon, \mu - \epsilon, w_3, \ldots, w_{2n})$, Kamnitzer shows the singular fibre at $\epsilon = 0$ is globally a product,
with one factor the compactification of the fibre of the smaller slice $S_{n-1}$ over $w' = (w_3, \ldots, w_{2n})$, and the other a singular quadric cone, i.e. a Hirzebruch surface $F_2$ with the $(-2)$-sphere collapsed to a point, which arises as the singular fibre of the compactified slice $S_1$. (The global product structure is not compatible with the divisors at infinity, so is not inherited by the resolution of the open slice.) On the compactification, this product structure shows the co-isotropic vanishing cycle is isotopic to the anti-diagonal in $(S^2 \times S^2)$ stabilised by the submanifold $(S^2)^{2n-2}$ corresponding to $w'$. Since the vanishing cycles are iterated convolutions of the correspondences, the result follows.

Lemma 5.3 constructs an isotopy in $Y_n^w$ and not necessarily in $Y_n^w$ open, but that is sufficient to control the cohomological restriction maps $H^*(Y_n) \to H^*(L_\wp)$.

Fix as usual the eigenvalues $1 = \{1, \ldots, 2n\} \subset \mathbb{C}$ and consider the fibre $\mathcal{Y}_n = \mathcal{Y}_n^1$ of the adjoint quotient. The cohomology $H^*(\mathcal{Y}_n; \mathbb{Z})$ is generated as a ring by $H^2(\mathcal{Y}_n; \mathbb{Z})$, and is torsion-free, so we can identify $H^2(\mathcal{Y}_n)$ and $H_2(\mathcal{Y}_n)^\vee$ integrally. Via complex orientations, there is a distinguished basis $e_i$ for $H^2(\mathbb{P}^1)^{2n} = \mathbb{Z}^{2n}$, where $e_i$ has 1 in the $i$-th place and 0’s elsewhere. Since $H^2(\mathcal{Y}_n)$ is the quotient of $H^2(\mathbb{P}^1)^{2n}$ by the $(1, \ldots, 1)$ class, the elements $(-1)^{i+1}e_i$, with $1 \leq i \leq 2n - 1$, therefore define a basis of $H^2(\mathcal{Y}_n; \mathbb{Z})$. By restricting these elements to $L_\wp$, we obtain bases for $H^2(L_\wp)$ which make the following true tautologically:

**Corollary 5.4.** The elements $v_i = (-1)^{i+1}e_i$, with $1 \leq i \leq 2n - 1$, define a basis of $H^2(\mathcal{Y}_n; \mathbb{Z})$ with the following property: under the restriction map

$$H^2(\mathcal{Y}_n; \mathbb{Z}) \to H^2(L_\wp; \mathbb{Z})$$

each basis element maps either to zero or to a basis element.

The force of the construction is that we restrict to zero or a basis element, and not to the negative of a basis element. We immediately obtain monomial bases for the entire cohomology $H^*(L_\wp)$, which is generated in degree two. Henceforth, we will abuse notation and denote by $v_j$ a basis element of either of $H^2(\mathcal{Y}_n)$ of $H^2(L_\wp)$, depending on the context.

The choice of basis of $H^2(\mathcal{Y}_n)$ also orients the 2-spheres $V_i$ in view of the proof of Lemma 5.3, which identifies them cohomologically with antidiagonals in factors of $(S^2)^{2n}$ (the $i$th and $i+1$st basis elements map to the same generator of the top cohomology of $V_i$). In particular, $H^2(L_{\cup_i \wp})$ inherits a basis from the Künneth theorem, i.e. the decomposition

$$H^2(L_{\cup_i \wp}; \mathbb{Z}) \to H^0(L_\wp) \otimes H^2(V_i) \oplus H^2(L_\wp) \otimes H^0(V_i).$$

**Corollary 5.5.** The basis of $H^2(L_\wp; \mathbb{Z})$ is preserved by the cup-functors, in the sense that the choice of basis induced by restriction from $H^2(\mathcal{Y}_n; \mathbb{Z})$ agrees with the basis induced by applying the Künneth theorem to a presentation as the image of a cup functor.

**Proof.** This follows since the Lagrangians $L_\wp$ are iterated convolutions of the correspondences, by Lemma 5.3. The orientations on the $V_i$ define bases for $H_2(L_\wp)$, and hence bases for $H^2(L_\wp)$, which by construction agree with those coming from $H^2(\mathcal{Y}_n)$.

5.2. **Another orientation convention.** In [50], and for the discussion of purity in Section 4.4, we adopted a grading convention in which the symplectic arc algebra was “symmetrically” graded, so for distinct matchings $\wp, \wp'$ meeting in codimension $k$ the group $HF^*(L_\wp, L_{\wp'})$ was

$$HF^*(L_\wp, L_{\wp'}) = \bigoplus_{n=2}^{2n} \bigoplus_{i=1}^{2n-i} \mathbb{Z}[w_{2i}, \ldots, w_{2n}]$$

with an extra grading on $w_{2i}$.
supported in degrees \( k \leq * \leq 2n - k \). For the purpose of computing products in the symplectic arc algebra, a different convention is more convenient.

**Definition 5.6.** Let \( \wp \) be a crossingless matching in the upper half-plane \( \mathbb{H} \). The depth of an arc \( \gamma \subset \wp \) is the number of other arcs \( \gamma' \subset \wp \setminus \gamma \) which meet a vertical line from \( \gamma \) to \( \infty \in \mathbb{H} \).

Informally, depth is the number of arcs under which \( \gamma \) is nested in \( \wp \). We orient the arcs \( \gamma \) by the convention that arcs of even depth are oriented clockwise, and arcs of odd depth are oriented anticlockwise: see Figure 6 for two examples. Equivalently, if the critical points are numbered \( \{1, 2, \ldots, 2n\} \) then all arcs are oriented towards their even end-point.

**Figure 6.** Cohomology bases for components of the compact core

In the Lefschetz fibration \( z : A_{2n-1} \to \mathbb{C} \), the monodromy of the generic fibre \( T^*S^1 \) around a critical point is a Dehn twist in the \( S^1 \), which preserves the orientation on \( S^1 \). Fix once and for all the standard (anticlockwise) orientation on \( S^1 \subset T^*S^1 = \mathbb{C}^* \subset \mathbb{C} \). Then a choice of orientation for a matching path \( \gamma \) determines a well-defined orientation on the Lagrangian sphere \( L_\gamma \subset A_{2n-1} \), which means that Definition 5.6 fixes bases for \( H_2(L_\wp) \) for all the Lagrangians \( L_\wp \).

**Lemma 5.7.** The orientation convention of Definition 5.6 yields the same bases for \( H_2(L_\wp) \) as the bases of Corollary 5.5. Furthermore, with these orientations the symplectic arc algebra is graded in even degrees.

**Proof.** An isolated intersection point of \( L_\wp \) and \( L_{\wp'} \) is a tuple of intersections of matching paths in the \( A_{2n-1} \)-surface, and the sign of the intersection is given by the product of the corresponding signs on the surface. At any critical point of the fibration \( A_{2n-1} \to \mathbb{C} \), the two matchings paths of \( \wp \) and \( \wp' \) meeting at that critical point are coherently oriented, in the sense that the local isotopy of Lefschetz thimbles given by rotating one path to the other through thimbles is an orientation-preserving isotopy. Therefore the local intersection number of the two thimbles in \( A_{2n-1} \) is positive. (It may be helpful to compare to the case of two copies of the same matching sphere in \( A_1 \), which defines \( S^2 \subset T^*S^2 \), and to recall that Maslov indices agree with Morse indices in a cotangent bundle \( T^*Q \) if \( \langle \partial_{q_1}, \ldots, \partial_{q_n}, \partial_{p_1}, \ldots, \partial_{p_n} \rangle \) is an oriented basis; this differs from the symplectic orientation of \( T^*Q \) by a global sign \( (-1)^{n(n+1)/2} \).) We conclude that the symplectic arc algebra is graded in even degrees.

5.3. \( A_2 \)-fibrations and the plumbing model for \( n = 2 \). Recall that two submanifolds \( Y_0, Y_1 \) meet cleanly if they intersect along a submanifold, and

\[
T_p(Y_0 \cap Y_1) = T_pY_0 \cap T_pY_1
\]
for every \( p \in Y_0 \cap Y'_1 \). The Lagrangians \( \hat{L}_p \) meet pairwise cleanly. It is not known if \( \hat{L}_p \) and \( \hat{L}_0 \) are Lagrangian isotopic (in \( Y_n \) rather than its compactification) in general, but pairs of crossingless matching Lagrangians can be Hamiltonian isotoped into a clean intersection model. This follows from the description in [50] of the degeneration of the fibre \( Y_n \) when three eigenvalues coalesce.

Take a tuple \( \mu = (\mu_1, \ldots, \mu_{2n}) \) with \( \mu_1 = \mu_2 = \mu_3 \), and which are otherwise pairwise distinct. The adjoint fibre \( Y_n^\mu \) contains two orbits: the regular orbit \( O^{reg} \) has an indecomposable Jordan block of size 3 for the eigenvalue \( \mu_1 \), whilst the subregular orbit \( O^{sub} \) has two Jordan blocks of sizes 1, 2 (the minimal orbit in the adjoint fibre \( \chi^{-1}(\mu) \) consists of matrices with three independent \( \mu_1 \)-eigenvectors, but this orbit is disjoint from \( S \)). The singular set of the fibre \( Y_n^\mu \) is smooth, and can be canonically identified with the fibre \( Y_n^{red} \) where \( \mu_{red} = (\mu_1, \mu_4, \ldots, \mu_{2n}) \).

At a point \( y \in O^{reg} \cap S \) let \( E_y \) be the \( \mu_1 \)-eigenspace of its semisimple part \( y_p \). These spaces form the fibres of a line bundle \( F \to O^{reg} \cap S \). We consider the associated vector bundle

\[
(5.7) \quad (F \setminus 0) \times_{C^*} C^4 = C \oplus F^{-2} \oplus F^2 \oplus C.
\]

We also introduce the map (the versal deformation of the \( A_2 \)-singularity)

\[
(5.8) \quad p : C^4 \to C^2, \quad p(a,b,c,d) = (d, a^3 - ad + bc).
\]

The relevant statement from [50] is then:

**Lemma 5.8.** Let \( P \to \text{Conf}_{2n}(C) \) be a small bidisc parametrized by \((d, z)\), corresponding to the set of eigenvalues

\[
(\mu_1 + \{\text{all solutions of } \lambda^3 - d\lambda + z = 0\}, \mu_4, \ldots, \mu_{2n}).
\]

There is a neighbourhood of \( O^{reg} \cap S \) inside \( \chi^{-1}(P) \cap S \), and an isomorphism of that with a neighbourhood of the zero-section inside \((F \setminus 0) \times_{C^*} C^4\), which fits into a diagram

\[
\begin{array}{ccc}
\chi^{-1}(P) \cap S & \xrightarrow{\text{local, defined near } O^{reg} \cap S} & (F \setminus 0) \times_{C^*} C^4 \\
\downarrow & & \downarrow p \\
P & \xrightarrow{(d, z)} & C^2
\end{array}
\]

where \( p \) is given by (5.8) on each \( C^4 \) fibre.

In particular, there is an open subset of a generic fibre \( Y_n \), for a tuple of eigenvalues sufficiently close to \( \mu \), which is an \( A_2 \)-fibration over \( Y_{n-1} \). In the lowest non-trivial case, an open subset of \( Y_2 \) is a (non-trivial) \( A_2 \)-fibration over \( T^*S^2 \). For any arc \( \gamma \) in the \( A_2 \)-space, there is an associated Lagrangian submanifold of \( Y_2 \) given by taking the matching sphere \( L_\gamma \subset A_2 \) fibrewise over the zero-section of \( T^*S^2 \). We will refer to such a Lagrangian as lying in fibred position.

**Lemma 5.9** (See Section 4.3 of [50]). The Lagrangians \( L_{\text{plait}} \) and \( L_{\text{mix}} \) in \( Y_2 \) can be Hamiltonian isotoped to lie in fibred position, given by arcs in \( A_2 \) which meet transversely once. \( \square \)

**Lemma 5.10.** Let \( L, L' \subset X \) be cleanly intersecting Lagrangian submanifolds of a symplectic manifold \( X \). There is an open neighbourhood \( U = U(L \cap L') \subset X \) of \( L \cap L' \) in \( X \), and a symplectic embedding \( U \hookrightarrow T^*L \) taking \( U \cap L \) to the zero-section and \( U \cap L' \) to the conormal bundle of \( L \cap L' \).
Proof. See e.g. [36, Proposition 3.4.1]. Briefly, Weinstein’s theorem gives a symplectomorphism $\psi : U \to T^*L$ taking $U \cap L'$ into the conormal bundle $\nu^*_L$. The image $\psi(L \cap U)$ is tangent to the zero-section along $L \cap L'$, hence can locally be written as the graph of a 1-form $\phi$ which vanishes along $L \cap L'$, so which is exact $\phi = dk$ in a small neighbourhood of the intersection locus. Composing the given symplectic embedding $\psi$ with the time-one Hamiltonian flow of $-k$, which preserves $\nu^*_L$ and takes $\psi(L \cap U)$ into the zero-section, yields the desired map $U \hookrightarrow T^*L$. □

According to [2], there is a “plumbing model” for the Fukaya category of a pair of exact graded Lagrangians which meet pairwise cleanly. We only require the somewhat simpler cohomological result, which goes back to Pozniak and follows fairly straightforwardly from Lemma 5.10; see for instance [22]. It may help to recall that given a diagram of cleanly intersecting submanifolds

![Diagram](image)

The convolution product

$$H^*(Q_1 \cap Q_2) \otimes H^*(Q_2) \to H^*(Q_1)$$

is by definition given by cup-product composed with the transfer (obtained from Poincaré duality on $Q_1 \cap Q_2$, the push-forward on homology, and Poincaré duality on $Q_1$).

**Proposition 5.11.** Let $Q_1$ and $Q_2$ be exact Lagrangian submanifolds of $(M, \omega)$ which meet cleanly along $C = Q_1 \cap Q_2$ of codimension $d$. Grade the $Q_i$ so that the minimal degree generator of $HF^*(Q_1, Q_2)$ lies in degree 0. The Donaldson category with objects $Q_1, Q_2$ is equivalent to the (ordinary) category

$$H^*(Q_1) \xrightarrow{H^*(C)} Q_2 \xleftarrow{H^*\nu^*_L} H^*(Q_1)$$

and with the non-trivial compositions given by convolution.

Note that the grading convention breaks symmetry: it amounts to choosing an ordering of $\{Q_1, Q_2\}$.

**Proof.** In the dg-model from [2], the endomorphisms of objects are chain complexes underlying classical cohomology, and the morphisms between the objects are given by cochains on a neighbourhood $U \subset Q_1$ of the intersection locus $C$ in one direction, and cochains relative boundary $C^*(U, \partial U)$ on that neighbourhood in the other. The identification of the morphism groups as given follows on passing to cohomology, and replacing $H^*(U, \partial U) \simeq H^{*-d}(C)$, by the Thom isomorphism theorem. For the product structure, note that in the model provided by Lemma 5.10 all holomorphic triangles are constant; there is a real-valued action functional and all intersection points have the same value of the action, so standard Morse-Bott techniques apply. The result then follows easily. □

The first arc algebra $H_1 = \mathbb{Z}\langle 1, x \rangle$ is isomorphic to $\mathcal{H}^{symp}_1 = H^*(S^2)$, where the cohomology group is based by Corollary 5.5 which in this case just amounts to saying that the anti-diagonal
in \( \mathbb{P}^1 \times \mathbb{P}^1 \) inherits a unique orientation from the complex orientation of the first factor, and the opposite orientation of the second factor. This comes with three functors \( \cup_{\text{comb}}: H_1 \to H_2 \). We also have the symplectic cup functors \( \cup_i: \mathcal{H}^{\text{symp}} \to \mathcal{H}^{\text{symp}}_i \), which come from the associated elementary bimodules.

**Proposition 5.12.** There are bases of \( HF^*(L_{\varphi \text{plait}}, L_{\varphi \text{mix}}) \) and of \( HF^*(L_{\varphi \text{mix}}, L_{\varphi \text{plait}}) \) such that the algebra

\[
\mathcal{H}^{\text{symp}}_2 = H^*(L_{\varphi \text{plait}}) \oplus HF^*(L_{\varphi \text{plait}}, L_{\varphi \text{mix}}) \oplus HF^*(L_{\varphi \text{mix}}, L_{\varphi \text{plait}}) \oplus H^*(L_{\varphi \text{mix}})
\]

is isomorphic to the combinatorial arc algebra \( H_2 \) in a manner which is compatible with the three cup functors \( \cup_i: \mathcal{H}_1 \to \mathcal{H}_2 \) respectively \( \cup_{\text{comb}}: H_1 \to H_2, i \in \{1, 2, 3\} \).

**Proof.** Lemma 5.9 shows that \( L_{\varphi \text{plait}} \) and \( L_{\varphi \text{mix}} \) can be isotoped to meet cleanly in a two-sphere \( C \). Knowledge of \( H^*(\mathbb{P}^2) \), or the flag description given in Section 5.1, shows that \( C \) a priori represents a class of square \( \pm 2 \) in each factor; with our orientation conventions, the plumbing model is that of two copies of \( S^2 \times S^2 \) which meet along the diagonal submanifold, of square \( +2 \).

We take the given bases of the groups \( H^*(L_{\varphi \text{plait}}), H^*(L_{\varphi \text{mix}}) \) from Corollary 5.5 and the induced basis of \( H^*(C) \) coming from restriction: this gives a uniquely defined choice of generator of \( H^2(C) \). The gradings agree with those in Proposition 5.11 by Lemma 5.7. The Floer products in the plumbing model are then cohomological (convolution products), as in Proposition 5.11. Since the Euler class of the normal bundle of the intersection locus is the \((1, 1)\)-class, products of positive generators all have positive coefficients. Given this, it is straightforward to compare to the arc algebra \( H_2 \).

We spell out the final comparison in the two most interesting examples. Consider the products

\[
\text{Hom}_{H_2}(\varphi \text{plait}, \varphi \text{mix}) \otimes \text{Hom}_{H_2}(\varphi \text{mix}, \varphi \text{plait}) \to \text{Hom}_{H_2}(\varphi \text{mix}, \varphi \text{mix})
\]

\[
\text{Hom}_{H_2}(\varphi \text{mix}, \varphi \text{plait}) \otimes \text{Hom}_{H_2}(\varphi \text{plait}, \varphi \text{mix}) \to \text{Hom}_{H_2}(\varphi \text{plait}, \varphi \text{plait}).
\]

In both cases, the diagrammatic product involves first merging two circles, and then splitting again, hence is given by the composition \( \Delta \circ m \), where

\[
m: \mathbb{Z}(1, x)^{\otimes 2} \to \mathbb{Z}(1, x)
\]

and \( \Delta: \mathbb{Z}(1, x) \to \mathbb{Z}(1, x)^{\otimes 2} \)

denote the product respectively co-product in the Frobenius algebra \( H^*(S^2) = \mathbb{Z}(1, x) \). The composite \( \Delta \circ m \) therefore takes

\[
1 \otimes 1 \mapsto 1 \otimes x + x \otimes 1, \quad 1 \otimes x \mapsto x \otimes x, \quad x \otimes 1 \mapsto x \otimes x, \quad x \otimes x \mapsto 0.
\]

The cohomology convolution product has exactly the same effect, with \( 1 \otimes x + x \otimes 1 \) arising from the cohomology class of the diagonal \( C \subset S^2 \times S^2 \). The other cases, and compatibility with the cup-functors, follow similarly on unwinding the definitions. \( \Box \)

### 5.4. Iterated \( A_2 \)-fibrations and plumbing models for arbitrary pairs

Recall \( c(\varphi, \varphi') \) denotes the number of components of the planar unlink \( \varphi \cup \varphi' \), and that the two associated Lagrangians meet in codimension one if \( c(\varphi, \varphi') = n - 1 \). We shall say that two arcs in the unlink are **consecutive** if one belongs to \( \varphi \) and one to \( \varphi' \) and they share exactly one end-point.

**Lemma 5.13.** Any pair \( L_\varphi, L_{\varphi'} \) may be Hamiltonian isotoped to meet pairwise cleanly in a submanifold \( (S^2)^{c(\varphi, \varphi')} \). In particular, any pair admits a plumbing model.
Proof. We give an inductive argument. Consider an innermost component of the unlink \( \varphi \cup \varphi' \), which is formed of matchings \( \varphi_n, \varphi'_n \) on a subset \( 2m \leq 2n \) of the critical points. If \( \varphi_n = \varphi'_n \) then these reduced matchings are both composed of a single arc which joins adjacent critical points. There is then a Morse-Bott degeneration of \( \mathcal{Y}_n \) which brings these eigenvalues together along the common arc (which might as well lie on the real axis). In the corresponding open subset the unknot \( \varphi_n \cup \varphi'_n \) admits a pair of consecutive arcs. We consider a degeneration of \( \mathcal{Y}_n \) in which the three eigenvalues which are end-points of these arcs come together at one of the outermost of the three points, by moving the eigenvalues along the given paths. This yields an open subset of \( \mathcal{Y}_n \) which is an \( A_2 \)-fibration over \( \mathcal{Y}_{n-1} \), and in which the matchings \( \varphi_n, \varphi'_n \) are explicitly fibred over (not necessarily half-plane) crossingless matchings in the base, with fibres being the core arcs of the \( A_2 \)-space which meet transversely once. This procedure can be iterated, until the Lagrangians in the base have no consecutive arcs, which happens only when they co-incide up to isotopy (the last conclusion uses the innermost condition; otherwise the arcs might differ by Markov I moves). The upshot is that there is an open subset of \( \mathcal{Y}_n \) which is an iterated \( A_2 \)-fibration over \( \mathcal{Y}_{n-k} \), where \( n - k = c(\varphi, \varphi') \), in which the two Lagrangians are fibred over Hamiltonian isotopic Lagrangians in \( \mathcal{Y}_{n-k} \), with fibres which are themselves products of pairwise transverse vanishing cycles in the \( A_2 \)-fibres. The result follows.

**Lemma 5.14.** In the situation of Lemma [5.13] one may further assume that the intersection of the cleanly intersecting Hamiltonian images of \( L_\varphi, L_{\varphi'} \) is smoothly isotopic to the iterated antidiagonal \( \hat{L}_\varphi \cap \hat{L}_{\varphi'} \). In particular, we have an isomorphism \( \text{HF}^*(L_\varphi, L_{\varphi'}) \cong \text{HF}^*(\hat{L}_\varphi \cap \hat{L}_{\varphi'}) \) compatible with the natural module structures of both cohomologies over \( H^*(\mathcal{Y}_n) \).

Proof. In the inductive construction of the previous set-up, consider some innermost component of \( \varphi \cup \varphi' \), which involves a subset \( I \subset \{1, 2, \ldots, 2n\} \) of cardinality \( 2m \leq 2n \) of the critical points. The degenerations along successive arcs for this component do not affect any of the other components, by the innermost condition. Furthermore, they exhibit the space \( \mathcal{Y}_n \) as an iterated \( A_2 \)-fibration over \( T^*S^2 \), and the Lagrangians \( L_{\varphi_n} \) and \( L_{\varphi'_n} \) intersect along a section of this iterated fibration, i.e. they both fibre over the zero-section with fibres meeting transversely once. It suffices to show that this intersection sphere is naturally identified with the small antidiagonal in a copy \( (S^2)^{2m} \subset (S^2)^{2n} \) indexed by the subset \( I \). As in Lemma [5.3] this holds because the simultaneous resolution shows the vanishing cycles of an \( A_1 \)-respectively \( A_2 \)-degeneration depend up to smooth isotopy only on the corresponding pair respectively triple of critical points which are brought together.

**Corollary 5.15.** The Floer cohomology group \( \text{HF}^*(L_\varphi, L_{\varphi'}) \) is a cyclic module over \( H^*(\mathcal{Y}_n) \), generated by any minimal degree generator.

### 5.5. Conormal triples

Lemma [5.10] asserts that a pair of cleanly intersecting Lagrangians can always be modelled by taking one to be a conormal bundle in the cotangent bundle of the other. For three Lagrangians, the situation is slightly more subtle. We recall the linear situation.
Lemma 5.16. Let $(V, \omega)$ be a symplectic vector space. An ordered triple of Lagrangian subspaces $(\Lambda_0, \Lambda_1, \Lambda_2)$ is determined up to symplectomorphism by the quintuple of integers

$$b_{ij} = \dim(\Lambda_i \cap \Lambda_j), \quad b_{012} = \dim(\Lambda_0 \cap \Lambda_1 \cap \Lambda_2), \quad s(\Lambda_0, \Lambda_1, \Lambda_2)$$

where $s$ is the Maslov triple index, i.e. the signature of the quadratic form on $\Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2$ given by $\omega(u_0, u_1) + \omega(u_1, u_2) + \omega(u_2, u_0)$.

A routine exercise shows that if the dimensions in (5.9) satisfy $\dim(V) + 2b_{012} = b_{01} + b_{02} + b_{12}$, then the Maslov triple index $s = 0$.

Lemma 5.17. Three cleanly intersecting real $m$-dimensional Lagrangians $L_0, L_1, L_2$ are locally symplectomorphic near $L_0 \cap L_1 \cap L_2$ to two conormal bundles inside $T^*L_0$ if and only if the dimension constraint holds:

$$m + 2b_{012} = b_{01} + b_{12} + b_{02}.$$ 

In particular, this condition is symmetric under permuting indices.

Sketch. Given two submanifolds $B_1, B_1 \subset L_0$ intersecting cleanly in a submanifold of dimension $d$, their conormal bundles $\nu^*_p \subset T^*L_0$ meet cleanly in a submanifold of dimension $2d$. This implies that a pair of conormal bundles satisfies the desired equality. Conversely, suppose the dimension constraint holds, and work inside $T^*L_0$ with $L_1 = \nu^*_{L_0 \cap L_1}$. From the linear algebra classification of triples of Lagrangian subspaces, at a point $p$ in the triple intersection $B$ we can suppose that $T_pL_0 = T_p(\nu^*_{L_0 \cap L_1})$. One then constructs a local Hamiltonian isotopy, in an open neighbourhood $U$ of $B$, taking $L_2 \cap U$ to $\nu^*_{L_0 \cap L_2} \cap U$, via a flow tangent to $L_0 \cap U$ and $\nu^*_{L_0 \cap L_1} \cap U$, as in Lemma 5.10.

We point out that (5.10) never holds if the triple intersection $L_0 \cap L_1 \cap L_2$ coincides with the three pairwise intersections $L_i \cap L_j$ (and the Lagrangians are not all identical). Given that, it is easy to see that there are triples of components $\tilde{L}_{\varphi_i}$ of the compact core $Z$ from Lemma 5.2 which, even though meeting cleanly, do not admit a conormal model.

For triples of Lagrangians, $A_2$-degenerations are not sufficient to bring the Lagrangians into clean intersection position (consider the case where no triple of critical points contains the end-points of at least one arc in each matching). Nonetheless, we have:

Lemma 5.18. Any triple of Lagrangians $L_\varphi, L_{\varphi'}, L_{\varphi''}$, two of which meet in codimension one, may be Hamiltonian isotoped to simultaneously meet pairwise cleanly. Moreover, the analogue of Lemma 5.14 again holds, and the cleanly intersecting Hamiltonian images satisfy the conormal condition (5.10).

Proof. Suppose $L_\varphi$ and $L_{\varphi'}$ meet in codimension one, meaning that they share $n - 2$ arcs. We repeat the argument of Lemma 5.13 but considering consecutive arcs one of which is common to $\varphi$ and $\varphi'$ and the other of which lies in $\varphi''$. The corresponding $A_2$-degeneration bringing eigenvalues together along such a consecutive pair of arcs manifests all three Lagrangians as fibred simultaneously, with $L_{\varphi}$ and $L_{\varphi'}$ having identical fibres in the local $A_2$-bundle, and $L_{\varphi''}$ having fibre the other core sphere in the $A_2$-fibres. This can now be iterated as before, and the analogue of Lemma 5.14 also holds as before.
For the final statement, in the notation of \(\{5,10\}\), let \(L_0 = L_{\varphi'}, L_1 = L_{\varphi'}\) and \(L_2 = L_{\varphi''}\), and set \(m = 2n\) to be the real dimension of \(L_{\varphi}\). Then \(b_{01} = 2n - 2\). The dimensions \(b_{02}\) and \(b_{12}\) differ by exactly 2, which means that (relabelling as necessary) we can assume \(b_{02} = 2j\) and \(b_{12} = 2j + 2\). The triple intersection has codimension at most 2 in \(L_{\varphi'} \cap L_{\varphi''}\), hence has dimension at least \(2j\), but is also contained in the \(2j\)-dimensional submanifold \(L_{\varphi} \cap L_{\varphi''}\). Therefore \(b_{012} = 2j\), and \(2n + 2b_{012} = b_{01} + b_{02} + b_{12}\). □

The convolution model for Floer product for a pair of cleanly intersecting Lagrangians admits a generalisation to the case of two conormal bundles. Again from \([\mathbb{22}, 31]\):

**Lemma 5.19.** Let \(A_i \subset Q\) be closed submanifolds of an oriented spin manifold \(Q\), \(i = 1, 2\). Let \(\nu_i^* \subset T^*Q\) denote the conormal bundle of \(A_i\). The product

\[
\begin{equation}
HF^*(\nu_1, \nu_2) \otimes HF^*(\nu_1, \nu_2) \longrightarrow HF^*(\nu_1 + \nu_2)
\end{equation}
\]

is a convolution product, given up to sign by restriction and push-forward

\[
\begin{equation}
H^*(A_1 \cap A_2) \otimes H^*(A_1) \rightarrow H^*(A_1 \cap A_2) \rightarrow H^*(A_2)
\end{equation}
\]

□

5.6. **Some non-zero Floer products.** We collect together several non-vanishing results for Floer products which will be used in the construction of positive bases. For a pair of matchings \(\varphi, \varphi'\) we will denote by

\[
\alpha_{\varphi, \varphi'} \in \kappa_{\min}(\varphi, \varphi') \subset HF^*(L_{\varphi}, L_{\varphi'})
\]
a minimal degree generator of \(HF^*(L_{\varphi}, L_{\varphi'})\). There are two choices, differing by sign. By Corollary \(\{5,16\}\) the Floer product

\[
\begin{equation}
HF^*(L_{\varphi'}, L_{\varphi''}) \otimes HF^*(L_{\varphi}, L_{\varphi'}) \longrightarrow HF^*(L_{\varphi}, L_{\varphi''})
\end{equation}
\]
is completely determined by that of the minimal degree generators and the module structure.

Despite its seeming innocuity, the following result is the main reason why the arc algebra and its symplectic analogue agree, since it asserts that, for the simplest products, basis elements appear with coefficients that have the same sign.

We now consider a triple of matchings \(\varphi, \varphi', \varphi''\), and suppose \(\varphi\) and \(\varphi'\) meet in codimension one. As remarked after Lemma \(\{5,12\}\) necessarily \(c(\varphi, \varphi'') = c(\varphi', \varphi'') \pm 1\).

**Lemma 5.20.** (1) If \(c(\varphi, \varphi'') = c(\varphi', \varphi'') - 1\), then

\[
\alpha_{\varphi', \varphi''} \cdot \alpha_{\varphi, \varphi'} = \pm \alpha_{\varphi, \varphi''}.
\]

(2) If \(c(\varphi, \varphi'') = c(\varphi', \varphi'') + 1\), then

\[
\alpha_{\varphi', \varphi''} \cdot \alpha_{\varphi, \varphi'} = \pm (v_{i_j} + v_{i_k}) \alpha_{\varphi, \varphi''} \neq 0
\]

for some \(v_{i_j}, v_{i_k} \in H^*(Y_{\varphi})\) elements of the distinguished basis. Specifically, \([v_{i_j} + v_{i_k}] \in H^*(\hat{L}_{\varphi} \cap \hat{L}_{\varphi''}) = HF^*(L_{\varphi}, L_{\varphi''})\) is Poincaré dual to the homology class of the triple intersection \([\hat{L}_{\varphi} \cap \hat{L}_{\varphi'} \cap \hat{L}_{\varphi''}]\).

**Proof.** The non-vanishing of the product if \(c(\varphi, \varphi'') = c(\varphi', \varphi'') - 1\) follows immediately from Lemma \(\{5,18\}\) and Lemma \(\{5,19\}\). If \(c(\varphi, \varphi'') = c(\varphi', \varphi'') + 1\), the product \(\alpha_{\varphi', \varphi''} \cdot \alpha_{\varphi, \varphi'}\) lands in degree exactly two higher than the minimal degree. For the cleanly intersecting representatives
of the three Lagrangians, the triple intersection \( L_\varphi \cap L_\varphi' \cap L_\varphi'' \) co-incident with the intersection \( L_\varphi \cap L_\varphi'' \), cf. the proof of Lemma 5.18. This defines a class

\[ [L_\varphi \cap L_\varphi' \cap L_\varphi''] \in H^2(L_\varphi' \cap L_\varphi'') \cong H^2((S^2)^C(\varphi', \varphi'')) \]

which is the product of the minimal degree generators by the convolution model for Floer products in conormal plumbings. The cohomology class can be identified with the corresponding class in the model of Lemma 5.2 by Lemma 5.14. Geometrically, the triple intersection is homologous to an antidiagonal in two factors

\[ S_{i_1}^2 \times S_{i_k}^2 \subset S_{i_1}^2 \times \cdots \times S_{i_m}^2 \subset (S^2)^{2n}, \]

\( m = c(\varphi', \varphi'') \), and each such is canonically the restriction of a class \((v_{ij} + v_{ik}) \in H^2(Y_n)\). This yields the second statement.

\[ \square \]

**Lemma 5.21.** If \( \varphi, \varphi' \) meet in codimension \( k \), i.e. \( c(\varphi, \varphi') = n - k \), the minimal degree generator \( \alpha_{\varphi, \varphi'} \in HF^*(L_\varphi, L_\varphi') \) can be written as a product

\[ \alpha_{\varphi, \varphi'} = \pm \prod_{j=k-1}^{0} \alpha_{\varphi_i, \varphi_i+1} \]

where \( \varphi = \varphi_0, \ldots, \varphi_k = \varphi' \) is a codimension one interpolating sequence as in Lemma 4.3.

**Proof.** The fact that the interpolating sequence has minimal possible length implies that

\[ c(\varphi, \varphi_{i+1}) = c(\varphi, \varphi_i) - 1 \]

for each \( i \geq 1 \). Then use Lemma 5.20. \[ \square \]

**Lemma 5.22.** For any \( \varphi, \varphi', \varphi'' \), there is an identity

\[ (5.15) \alpha_{\varphi', \varphi''} \cdot \alpha_{\varphi, \varphi'} = \pm \prod_{j}(v_{j_1} + v_{j_2})\alpha_{\varphi, \varphi'} \]

where each \( (v_{j_1} + v_{j_2}) \in H^2(Y_n) \) is a sum of positive basis elements and is dual to an antidiagonal in \((S^2)^{2n}\) (the right side of (5.15) may vanish).

**Proof.** Choose a sequence \( \varphi = \varphi_0, \varphi_1, \ldots, \varphi_k = \varphi' \) with where \( k = n - c(\varphi, \varphi') \) as before. We now write

\[ \alpha_{\varphi', \varphi''} \cdot \alpha_{\varphi, \varphi'} = \pm \prod_{j} ((\alpha_{\varphi', \varphi''} \alpha_{\varphi_{k-2}, \varphi_{k-1}}) \cdots \alpha_{\varphi_0, \varphi_1}) \]

and again appeal to Lemma 5.20. \[ \square \]

**Lemma 5.23.** For any \( \varphi, \varphi' \), the products

\[ \alpha_{\varphi_{\text{plait}}, \varphi'} \cdot \alpha_{\varphi, \varphi_{\text{plait}}} \in HF^*(L_\varphi, L_{\varphi'}) \quad \text{and} \quad \alpha_{\varphi_{\text{mix}}, \varphi'} \cdot \alpha_{\varphi, \varphi_{\text{mix}}} \in HF^*(L_\varphi, L_{\varphi'}) \]

are both non-zero.

**Proof.** For \( \varphi_{\text{plait}} \), this is exactly [3, Corollary 5.19]. To argue for \( \varphi_{\text{mix}} \), consider for a moment the fibre \( Y_n^\mu \) of the adjoint quotient corresponding to placing the \( 2n \) critical values at the roots of unity. There are “cyclic” analogues \( \varphi_{\text{plait}}^{\text{cyc}}, \varphi_{\text{mix}}^{\text{cyc}} \) of \( \varphi_{\text{plait}}, \varphi_{\text{mix}} \) for this configuration, which define Lagrangian submanifolds of \( Y_n^\mu \) which are Hamiltonian isotopic to our usual crossingless matching Lagrangians via parallel transport in the family over configuration space \( \text{Conf}_{2n}(\mathbb{C}) \) joining \( \mu \) and
6. An inductive construction of positive bases

6.1. Positive pairs, triangles, and triples. Our aim is to construct a basis of the symplectic arc algebra which is positive and preserved by cup functors, in the following sense.

Definition 6.1. A basis of $\mathcal{H}_{\text{symp}}$ is positive if every product of minimal degree generators is zero or of the form given in (5.15) with a $+$ sign. We say a map of based vector spaces preserves bases if it takes every basis element to zero or to a basis element.

Fix monomial bases of $HF^*(L_\varphi, L_\psi) = H^*(L_\varphi)$ for each $\varphi$ as in Corollary 6.5. Such a basis determines a trace

\[ \text{tr}: H^{2n}(L_\varphi) \to \mathbb{Z} \]

which is dual to the cohomology class $[L_\varphi] = v_1 \cdot v_2 \cdots v_n$.

Lemma 6.2. An $n$-fold product $\prod_{j=1}^n (v_{j1} + v_{j2})$ of sums of positive generators of $H^2(L_\varphi)$ either vanishes, or is a positive multiple of $[L_\varphi]$. \hfill \square

The cohomological shadow of the fact that the Fukaya category is cyclic is the statement that the trace is symmetric, i.e. that we have

\[ \text{tr}(\alpha \cdot \beta) = \text{tr}(\beta \cdot \alpha) \]

whenever $\alpha \otimes \beta \in HF^*(L, L') \otimes HF^*(L', L)$, cf. [12, Section 12(e)]. Note that there is no sign in the above formula, because all morphisms have even degree according to the grading convention fixed in Definition 5.9.

For any $\{\varphi, \varphi'\}$, $HF^*(L_\varphi, L_{\varphi'})$ is a quotient of both $HF^*(L_\varphi, L_\psi)$ and $HF^*(L_\psi, L_{\varphi'})$ via the natural module structures. A choice of minimal degree generator $\alpha_{\varphi,\varphi'}$ yields a basis for $HF^*(L_\varphi, L_{\varphi'})$ by multiplication by elements of $H^*(L_\varphi)$ on the right or by elements of $H^*(L_{\varphi'})$ on the left; these bases agree, since in both cases we can re-interpret the multiplication as coming from an element of $H^*(\mathcal{Y}_n)$, which acts centrally. In line with Definition 6.1 we will more generally say that a product of minimal degree generators is “positive” (or is “positive with respect to the minimal degree generator”) if it has the form of (5.15) with a $+$ sign. We say that a pair $(\varphi, \varphi')$ is positive if we have chosen minimal degree generators $\alpha_{\varphi,\varphi'}$ and $\alpha_{\varphi',\varphi}$ of the Floer groups between them so that any of the conditions in the following result hold:

Lemma 6.3. The following are equivalent:

1. There are positive basis elements $v_{jk} \in H^2(\mathcal{Y}_n)$ such that $\prod_j (v_{j1} + v_{j2}) \alpha_{\varphi,\varphi'} \cdot \alpha_{\varphi',\varphi}$ is a positive multiple of the fundamental class $[L_\varphi]$.

2. There are positive basis elements $v_{jk} \in H^2(\mathcal{Y}_n)$ such that $\prod_j (v_{j1} + v_{j2}) \alpha_{\varphi,\varphi'} \cdot \alpha_{\varphi',\varphi}$ is a positive multiple of the fundamental class $[L_{\varphi'}]$. 

\{1, 2, \ldots, 2n\}. The matchings $\varphi_{\text{cyc}}$ and $\varphi_{\text{mix}}$ are exchanged by the symplectomorphism of $\mathcal{Y}_n$ induced by cyclic rotation by $\exp(i\pi/n)$ in the base of the complex surface $A_{2n-1}$. That, and Hamiltonian invariance of the statement of the Lemma, implies the corresponding non-vanishing for products involving $L_{\varphi_{\text{mix}}}$. \hfill \square
Lemma 6.4. The following conditions are equivalent:

1. The triple product \( \alpha_{\varphi_1, \varphi_2} \cdot \alpha_{\varphi_0, \varphi_1} \cdot \alpha_{\varphi_2, \varphi_0} \) is positive.
2. The three possible triple products involving \( \alpha_{\varphi_1, \varphi_2} \), \( \alpha_{\varphi_0, \varphi_1} \), and \( \alpha_{\varphi_2, \varphi_0} \) are positive.

Proof. Consider a product \( \prod_j (v_j + v'_j) \) of sums of positive generators so that

\[
\prod_j (v_j + v'_j) \alpha_{\varphi_1, \varphi_2} \cdot \alpha_{\varphi_0, \varphi_1} \cdot \alpha_{\varphi_2, \varphi_0}
\]

is a positive multiple of the top degree generator of the cohomology of \( \mathcal{L}_{\varphi_2} \). Associativity and cyclicity of the trace imply that the product in a different order (but with the same cyclic order) is a positive multiple of a top degree generator of the cohomology of \( \mathcal{L}_{\varphi_0} \) or \( \mathcal{L}_{\varphi_1} \).

We say that \( \{\varphi_i\}_{i=0,1,2} \) form a positive triangle if we have fixed minimal degree generators \( \alpha_{\varphi_i, \varphi_{i+1}} \) so that the conditions of Lemma 6.4 hold in either cyclic ordering. There is no obstruction to a triangle being positive: given two of the three morphisms, we can pick the third so that all cyclic products are positive, and the choices for the two cyclic orderings are independent.

Definition 6.5. The matchings \( \varphi_0, \varphi_1, \varphi_2 \) form a positive triple if the corresponding triangle as well as all pairs are positive.

Lemma 6.6. If \( \varphi_0, \varphi_1, \varphi_2 \) form a positive triangle, two of the pairs are positive, and there are three non-cyclically ordered generators, involving all three Lagrangians, whose product does not vanish, then \( \varphi_0, \varphi_1, \varphi_2 \) form a positive triple.

Proof. By relabelling the matchings and using cyclic symmetry, we may assume that the pairs \( \varphi_0, \varphi_1 \) and \( \varphi_0, \varphi_2 \) are positive, that the triangle is positive, and that either (1) \( \alpha_{\varphi_2, \varphi_0} \cdot \alpha_{\varphi_1, \varphi_2} \cdot \alpha_{\varphi_2, \varphi_1} \) or (2) \( \alpha_{\varphi_2, \varphi_1} \cdot \alpha_{\varphi_0, \varphi_2} \cdot \alpha_{\varphi_2, \varphi_0} \) is non-zero. We must show in either case that the pair \( \varphi_1, \varphi_2 \) is positive.

Case 1: By associativity, we have

\[
\alpha_{\varphi_2, \varphi_0} \cdot (\alpha_{\varphi_1, \varphi_2} \cdot \alpha_{\varphi_2, \varphi_1}) = (\alpha_{\varphi_2, \varphi_0} \cdot \alpha_{\varphi_1, \varphi_2}) \cdot \alpha_{\varphi_2, \varphi_1}.
\]

Since the pair \( \varphi_0, \varphi_1 \) is positive, positivity of the triangle implies that the product \( \alpha_{\varphi_2, \varphi_0} \cdot \alpha_{\varphi_1, \varphi_2} \) is positive with respect to \( \alpha_{\varphi_1, \varphi_2} \) (i.e. the product of a positive basis element of \( H^* (\mathcal{L}_{\varphi_2}) \) with this class). Since the pair \( \varphi_0, \varphi_2 \) is positive, positivity of the triangle further implies that the
product of this class with \( \alpha_{\varphi_2, \varphi_1} \) is positive with respect to \( \alpha_{\varphi_2, \varphi_0} \). Equating the left and right hand sides above, we conclude that the product \( \alpha_{\varphi_1, \varphi_2} \cdot \alpha_{\varphi_2, \varphi_1} \) is positive.

**Case 2:** By Poincaré duality, our assumption implies that we have a non-zero product

\[
(6.5) \quad (\alpha_{\varphi_1, \varphi_2} \cdot \alpha_{\varphi_2, \varphi_1}) \cdot (\alpha_{\varphi_0, \varphi_2} \cdot \alpha_{\varphi_2, \varphi_0}) = \alpha_{\varphi_1, \varphi_2} \cdot (\alpha_{\varphi_2, \varphi_1} \cdot \alpha_{\varphi_0, \varphi_2} \cdot \alpha_{\varphi_2, \varphi_0}).
\]

Using positivity of the triangle under both cyclic orderings, we see that the right hand side is positive as in the previous case. Since the pair \((\varphi_0, \varphi_2)\) is positive, the expression \(\alpha_{\varphi_2, \varphi_1} \cdot \alpha_{\varphi_0, \varphi_2} \cdot \alpha_{\varphi_2, \varphi_0}\) is positive with respect to \(\alpha_{\varphi_2, \varphi_1}\). We conclude that \(\alpha_{\varphi_1, \varphi_2} \cdot \alpha_{\varphi_2, \varphi_1}\) is positive.

**Corollary 6.7.** Assume that \((\varphi_0, \varphi_1, \varphi_2)\) form a positive triangle, and two of \((\varphi_0, \varphi_1, \varphi_2)\) meet in codimension one. Given positive bases for two pairs of matchings \((\varphi_i, \varphi_j)\), there are unique bases for the remaining pair which yields a positive triple.

**Proof.** This follows on combining Lemma 5.20 which yields a suitable non-vanishing product, with the previous result. □

To ease notation, in the next Lemma we write \(\alpha_{ij}\) for \(\alpha_{\varphi_i, \varphi_j}\).

**Lemma 6.8.** Let \(\varphi_0, \varphi_1, \varphi_2, \varphi_3\) be matchings. If the triples containing \((\varphi_0, \varphi_3)\) are positive and the triangle \((\varphi_0, \varphi_1, \varphi_2)\) is positive, then the remaining triangle \((\varphi_1, \varphi_2, \varphi_3)\) is positive whenever the cyclic products \(\alpha_{01} \cdot \alpha_{30} \cdot \alpha_{23} \cdot \alpha_{12}\) and \(\alpha_{03} \cdot \alpha_{10} \cdot \alpha_{21} \cdot \alpha_{32}\) do not vanish.

**Proof.** We consider one of the two possible orderings: since \((\varphi_0, \varphi_1, \varphi_3)\) is a positive triangle, the product \(\alpha_{01} \cdot \alpha_{30}\) is a positive multiple of \(\alpha_{31}\). So \(\alpha_{31} \cdot \alpha_{23} \cdot \alpha_{12}\) is positive if and only if the quadruple cyclic product is positive. On the other hand, \(\alpha_{30} \cdot \alpha_{23}\) is a positive multiple of \(\alpha_{20}\) (because the triple \((\varphi_0, \varphi_2, \varphi_3)\) is positive), hence the quadruple product is positive because \(\alpha_{01} \cdot \alpha_{30} \cdot \alpha_{12}\) is positive. □

We note that non-trivial iterated products as required for the second part of Lemma 6.8 arise naturally when combining Lemmas 5.20 and 5.21.

6.2. **Description of the basis.** We construct a basis of \(\mathcal{H}^{symp}\) by the following procedure. Denote by \(V_i\) both the Lagrangian 2-sphere in \(A_{n-1}\) joining the points \(\{i, i+1\} \subset \{1, 2, \ldots, 2n\}\), and also the Lagrangian 2-sphere fibre of the elementary correspondence associated to the Morse-Bott degeneration which brings \(i, i+1\) together. Note that all these spheres are oriented.

We say a matching **contains an odd cup** if it contains the arc joining \(2j + 1, 2j + 2\) for some \(0 \leq j \leq n - 1\), and **contains an even cup** if it contains the arc joining \(2j, 2j + 1\) for some \(1 \leq j \leq n - 1\). Every matching contains at least one cup. The matching \(\varphi_{\text{plain}}\) is singled out by containing all \(n\) odd cups, and \(\varphi_{\text{mix}}\) is singled out by containing all \((n - 1)\) even cups.

The basis is constructed inductively, so we assume that we already have bases for the algebra \(\mathcal{H}_{n-2}^{symp}\), with the properties that (i) they are compatible with all cup-functors \(\cup_j : \mathcal{H}_{n-2}^{symp} \to \mathcal{H}_{n-1}^{symp}\), where compatible means that any basis element is taken either to zero or to a basis element by any given \(\cup_j\), and (ii) the algebra is isomorphic in the given bases to the arc algebra \(H_{n-1}\), by an isomorphism entwining \(\cup_j^\text{reg}\) and \(\cup_j^\text{sym}\). The induction is based by Proposition 5.12, so we may assume throughout that \(n \geq 3\).
that these bases are preserved by cup functors
bases described above are well-defined (independent of the choice s made along the way); to show
way that matches the product in
these bases, products of positive generators are linear combinat ions of positive generators in a
Lemma 6.9
Proof. Well-definition of the basis.
6.3. At this stage, all groups \( HF^* \) as in Corollary 8.3
(2) Suppose \( \varphi \) contains an odd cup \( \cup_{2j+1} \). Then both \( \varphi \) and \( \varphi_{\text{plait}} \) lie in the image of some \( \cup_{\text{odd}} \), say \( \varphi = \cup_{2j+1}(\varphi') \), \( \varphi_{\text{plait}} = \cup_{2j+1}(\varphi'_{\text{plait}}) \) (\( r \) for reduced), and hence the Künneth theorem gives canonical isomorphisms
\[
HF^*(L_{\varphi_{\text{plait}}}, L_{\varphi}) = HF^*(L_{\varphi'_{\text{plait}}}, L_{\varphi'}) \otimes H^*(V_{2j+1})
\]
\[
HF^*(L_{\varphi}, L_{\varphi_{\text{plait}}}) = HF^*(L_{\varphi'}, L_{\varphi'_{\text{plait}}}) \otimes H^*(V_{2j+1}).
\]
By induction we have a basis for the first factor on the right, by the orientation convention we have a basis for the second, and we take the induced basis.
(3) Suppose \( \varphi \) contains an even cup \( \cup_{2j} \). Write \( \varphi = \cup_{2j}(\varphi') \), \( \varphi_{\text{mix}} = \cup_{2j}(\varphi'_{\text{mix}}) \), and use Künneth and induction to take the bases induced from
\[
HF^*(L_{\varphi_{\text{mix}}}, L_{\varphi}) = HF^*(L_{\varphi'_{\text{mix}}}, L_{\varphi'}) \otimes H^*(V_{2j})
\]
\[
HF^*(L_{\varphi}, L_{\varphi_{\text{mix}}}) = HF^*(L_{\varphi'}, L_{\varphi'_{\text{mix}}}) \otimes H^*(V_{2j}).
\]
(4) We fix a basis of \( HF^*(L_{\varphi_{\text{plait}}}, L_{\varphi_{\text{mix}}}) \) as follows. Since \( n \geq 3 \) there is at least one matching \( \varphi \) which contains both an odd and an even cup. Pick some such; we then have bases for the pairs \( (\varphi_{\text{plait}}, \varphi) \) and \( (\varphi_{\text{mix}}, \varphi) \) (by the previous steps), and the triangle \( \{ \varphi, \varphi_{\text{plait}}, \varphi_{\text{mix}} \} \) admits a non-trivial product by Lemma 5.23. We pick the generators \( \alpha_{\varphi_{\text{plait}}, \varphi_{\text{mix}}} \) and \( \alpha_{\varphi_{\text{mix}}, \varphi_{\text{plait}}} \) so as to make this a positive triangle.
(5) If \( \varphi \) contains no odd cup it necessarily contains an even cup so bases for the pair \( (\varphi_{\text{mix}}, \varphi) \) are fixed in step (3). The triangle \( (L_{\varphi}, L_{\varphi_{\text{plait}}}, L_{\varphi_{\text{mix}}}) \) again admits a non-trivial product, and we pick the unique minimal generators for \( (L_{\varphi'}, L_{\varphi'_{\text{plait}}}) \) making this a positive triangle.
(6) For any \( \varphi, \varphi' \) we now have bases for the pairs \( (L_{\varphi}, L_{\varphi_{\text{plait}}}) \) and \( (L_{\varphi'}, L_{\varphi'_{\text{plait}}}) \). Since any triangle involving \( L_{\varphi_{\text{plait}}} \) has non-zero products, we now choose the basis for \( (L_{\varphi'}, L_{\varphi'}) \) to make the triangle \( (L_{\varphi}, L_{\varphi'}, L_{\varphi_{\text{plait}}}) \) a positive triangle.

At this stage, all groups in \( \mathcal{H}_{n-2}^{\text{sympl}} \) have bases. The remaining task is threefold: to show that the bases described above are well-defined (independent of the choices made along the way); to show that these bases are preserved by cup functors \( \mathcal{H}_{n-1} \rightarrow \mathcal{H}_n \); and to show that with respect to these bases, products of positive generators are linear combinations of positive generators in a way that matches the product in \( H_n \).

6.3. Well-definition of the basis. We show independence of choices.

Lemma 6.9. In Steps (2) or (3) above, the choice of odd respectively even cup in \( \varphi \) does not affect the resulting minimal degree generator.

Proof. Immediate from the Künneth theorem, and the fact that the basis for \( \mathcal{H}_{n-1}^{\text{sympl}} \) is compatible with all the cup functors from \( \mathcal{H}_{n-2}^{\text{sympl}} \).

In step (4), we make an arbitrary choice of a matching which contains both an odd and an even cup. When \( n \geq 4 \) one can interpolate between different choices, as follows. We are defining the basis for the pair \( (L_{\varphi_{\text{plait}}}, L_{\varphi_{\text{mix}}}) \) by choosing a matching \( \varphi = \cup_{2j+1} \cup_{2j} \varphi' \) which contains both an odd and an even cup. Note that necessarily \( i \) and \( j \) are not adjacent, i.e. the four end-points of the cups \( \cup_{2j} \) and \( \cup_{2i+1} \) are distinct, since both belong to some matching. Let \( \varphi_{2j} \) denote the matching which differs minimally from \( \varphi_{\text{plait}} \) whilst containing the cup joining the points
\( \{2j, 2j + 1\} \), so it does contain one even cup. One considers a configuration

\[
(6.6)
\]

The matchings in the bottom triple \((\varphi_{\text{plait}}, \varphi_{2j}, \cup_{2i+1} \cup_{2j} \varphi')\) all contain \(\cup_{2j}\), so this is a positive triple by the Künneth theorem and the choices fixed at Step (2). Similarly the three matchings in the outer triple \((\varphi_{\text{mix}}, \varphi_{2j}, \cup_{2i+1} \cup_{2j} \varphi')\) all contain the odd cup \(\cup_{2i+1}\), hence this is a positive triple by the Künneth theorem. There is a non-trivial product involving all four matchings, since \((\varphi_{\text{plait}}, \varphi_{2j})\) form a codimension one pair by construction of \(\varphi_{2j}\), and there is a non-trivial product involving the three matchings in the top left triple by Lemma 5.23. Therefore, the two possible basis elements for \((\varphi_{\text{plait}}, \varphi_{\text{mix}})\) on the dotted arrow defined by either choosing \(\varphi\) or \(\varphi_{2j}\) co-incide by applying Lemma 6.8. Iterating, one can compare any two choices of \(\varphi\) in Step (4) consistently.

We record an important consequence of the above construction:

**Lemma 6.10.** The triple \((\varphi_{\text{mix}}, \varphi_{2j}, \varphi_{\text{plait}})\) is positive. In particular, so is the pair \((\varphi_{\text{mix}}, \varphi_{\text{plait}})\).

**Proof.** The triangle \((\varphi_{\text{mix}}, \varphi_{2j}, \varphi_{\text{plait}})\) is positive by the previous discussion, whilst the pairs \((\varphi_{\text{mix}}, \varphi_{2j})\) and \((\varphi_{\text{plait}}, \varphi_{2j})\) are positive by the Künneth theorem. The result then follows from Corollary 6.7. \(\square\)

There is no further choice at Steps (5) and (6), so the bases are consistently determined.

6.4. **The case** \(n = 3\). We now prove independence of choices when \(n = 3\). There are exactly two matchings containing an odd and an even cup: the plait matching joins pairs \(\{(12), (34), (56)\}\), the mixed matching joins pairs \(\{(16), (23), (45)\}\) and the two possible matchings containing both an odd and even cup are

\[
\varphi_{25} = \{(14), (23), (56)\} \quad \text{and} \quad \varphi_{14} = \{(12), (36), (45)\}
\]

(the subscripts indicate which cups the matchings contain). Consider the triangle with vertices \(\varphi_{\text{mix}}, \varphi_{25}, \varphi_{14}\) and with an interior vertex labelled \(\varphi_{\text{plait}}\), cf. Figure 7. The previous steps of the inductive construction have fixed bases for the pairs including \(\varphi_{\text{mix}}\) or \(\varphi_{\text{plait}}\) and one of the other two vertices, at Steps (2) and (3). Moreover, the definition of the basis for the pair labelled \((\varphi_{14}, \varphi_{25})\) at step (6) ensures that the triangle \((\varphi_{\text{plait}}, \varphi_{14}, \varphi_{25})\) is positive (this is independent of the choice in Step (4)). The basis for the pair \((\varphi_{\text{plait}}, \varphi_{\text{mix}})\) can be fixed by choosing either of the other two vertices, and making the corresponding triangle positive. That leaves two further triangles: the outer triple, and the remaining internal triangle. These have the same sign by Lemma 6.8 (since Lemmas 5.21 and 5.23 together imply that there is a non-trivial product involving all four Lagrangians) but that sign is not determined by consistency with choices already made, so we need to compute it directly.
Lemma 6.11. Suppose \( n = 3 \). The two minimal degree generators for the pair \((\varphi_{\text{plait}}, \varphi_{\text{mix}})\) obtained from Step (4) of the inductive strategy respectively using \( \varphi_{14} \) or \( \varphi_{25} \) agree.

Proof. We draw the four matchings in question in the diagram of Figure 8 where we have used the Markov I move to slide certain arcs into the lower half-plane to remove excess intersections. The Markov I move preserves the orientation of each arc, hence preserves the bases of \( H_2(L_{\varphi}) \) and the associated monomial bases of \( H^*(L_{\varphi}) \), so this Hamiltonian isotopy introduces no signs into the computation of Floer products.

We have drawn the six critical points grouped into two triples, and consider the degeneration which simultaneously collapses these triples, i.e. we work in an open subset of \( Y_3 \) which is an \((A_2 \times A_2)\)-fibration over \( T^*S^2 \). In this local model, the four matchings are all fibred over the zero-section, with fibres being products of the two basic real arcs in \( A_2 \), one in each factor of \( A_2 \times A_2 \). The key claim is then that the two products corresponding to \( \beta \circ \alpha \) and \( \delta \circ \gamma \) in Figure 7 agree. We see this by explicit computation in the \((A_2 \times A_2)\)-fibred plumbing model. Schematically, the arrows \( \alpha, \beta, \gamma, \delta \) are as follows:

\[
\begin{align*}
\alpha & \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\beta & \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\gamma & \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \\
\delta & \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\end{align*}
\]
The six critical points are grouped in the triples which define the local \((A_2 \times A_2)\)-fibration. A solid line indicates that the morphism is between two Lagrangians which share that arc in Figure 8 and represents the fundamental class of the corresponding \(S^2\)-factor of the \(A_2\)-fibre; an arrow between a pair of dotted lines denotes the morphism given by the transverse intersection point of the corresponding matching spheres in the \(A_2\)-fibre. Since the fundamental class is a cohomological unit, \(\beta \circ \alpha\) and \(\delta \circ \gamma\) define the tensor product of the curved arrows in respectively the first two and last two rows, hence represent the same Floer cycle. \(\square\)

This shows that there is no ambiguity in the construction of the bases when \(n = 3\). By construction, all triangles among the quadruple \((\wp_{\text{plait}}, \wp_{14}, \wp_{25}, \wp_{\text{mix}})\) are positive. In each case, two pairs are also positive by the Künneth theorem, so Corollary 6.7 implies that all four triples are positive.

There is only one other matching when \(n = 3\), namely the “horseshoe matching” \(\wp_o\) (joining pairs \{\((16), (25), (34)\)\}) which enters into the definition of \(K_{\text{sym}}\). Consider the diagram

\[
\begin{array}{c}
  L_{\wp_{\text{mix}}} \\
  \downarrow \quad \downarrow
  \\
  L_{\wp_{\text{plait}}} \\
  \downarrow \quad \downarrow
  \\
  L_{\wp_{14}}
\end{array}
\]

where the solid lines denote pairs for which Floer groups were chosen either in Step (2) or Step (4) of the inductive construction, and the dotted lines are the pairs for which a choice is made only thereafter. The choice of bases up to Step (4) ensures that all solid lines connect positive pairs, and the top right (solid) triple is positive. The two dotted arrows are then chosen to make the remaining two internal triangles positive. Note that the bottom triangle is in fact a positive triple because \((\wp_{\text{plait}}, \wp_{14})\) is a codimension 1 pair. There is again a non-zero product involving all four Lagrangians since two meet in codimension one, so Lemma 6.8 implies that all triangles are positive. Noting that the top left triangle contains a codimension 1 pair, we conclude that the top left triple is also positive by Corollary 6.7. We conclude that all triples are positive.

Finally, by symmetry between \(L_{\wp_{14}}\) and \(L_{\wp_{25}}\) one easily checks that all products are positive when \(n = 3\). An examination of the argument furthermore shows that the bases for \(H_{\text{sym}}^3\) are indeed compatible with the cup functors \(\cup_i : H_{\text{sym}}^2 \to H_{\text{sym}}^3\); compare to the formally identical Lemma 6.13 proved below.

6.5. Cup-compatibility and positivity. We now return to the task of proving positivity and compatibility with the cup functors for \(n \geq 4\).

In steps (2)-(3) of the construction of bases, new generators are constructed from old ones using the Künneth theorem, and the positivity of the relevant pairs is inherited. The morphism constructed in step (4) was shown to be positive in Lemma 6.10.

**Lemma 6.12.** For the choice of basis in step (5), \((L_{\wp_{\text{plait}}}, L_{\wp_o})\) is a positive pair.
Proof. Fix an integer \( j \) so that \( L_\varphi \) lies in the image of \( \cup_{2j} \). Given the choices of bases on the pairs \((L_{\varphi_{\text{plait}}},L_{\varphi_{2j}})\) and \((L_\varphi,L_{\varphi_{2j}})\), which are induced by the Künneth theorem, and noting that the pair \((L_{\varphi_{\text{plait}}},L_{\varphi_{2j}})\) meet in codimension 1, Corollary \([6.7]\) implies that there is a unique choice of bases for the pair \((L_{\varphi_{\text{plait}}},L_\varphi)\) so that the triple \((L_{\varphi_{2j}},L_{\varphi_{\text{plait}}},L_\varphi)\) is positive. To show that the choice of basis fixed in step (5) agrees with this new one, it suffices to show that the triangle \((L_{\varphi_{\text{mix}}},L_{\varphi_{\text{plait}}},L_\varphi)\) is positive for the new choice of basis for the pair \((L_{\varphi_{\text{plait}}},L_\varphi)\). This is immediate from Lemma \([6.8]\) applied to the quadruple \((L_{\varphi_{2j}},L_{\varphi_{\text{plait}}},L_\varphi,L_{\varphi_{\text{mix}}})\):

\[
(6.7)
\]

Lemma \([6.10]\) implies that the triple \((L_{\varphi_{2j}},L_{\varphi_{\text{plait}}},L_{\varphi_{\text{mix}}})\) is positive. On the other hand, the triple \((L_{\varphi_{2j}},L_{\varphi_{\text{plait}}},L_\varphi)\) is positive by construction, while \((L_{\varphi_{2j}},L_\varphi,L_{\varphi_{\text{mix}}})\) is positive by Künneth. \(\square\)

**Lemma 6.13.** All triples containing \(\varphi_{\text{plait}}\) are positive. In particular, all pairs are positive.

Proof. Recall that matchings \(\varphi\) and \(\varphi''\) meet in codimension \(n - c(\varphi,\varphi'')\), where \(c(\varphi,\varphi'')\) is the number of components of the planar unlink \(\varphi \cup \varphi''\). Assume, by decreasing induction on \(c(\varphi,\varphi_{\text{plait}})\), that the triple \((\varphi,\varphi',\varphi_{\text{plait}})\) is positive whenever \(c(\varphi,\varphi_{\text{plait}}) \geq c(\varphi',\varphi_{\text{plait}})\). The base case \(\varphi = \varphi_{\text{plait}}\) is a reformulation of the positivity of pairs containing \(\varphi_{\text{plait}}\).

In the inductive step, we consider a Lagrangian \(L_{\varphi''}\) so that \(c(\varphi,\varphi'') = n - 1\), and \(c(\varphi,\varphi_{\text{plait}}) = c(\varphi'',\varphi_{\text{plait}}) - 1\). By the inductive hypothesis, in the quadruple \((L_{\varphi_{\text{plait}}},L_\varphi,L_{\varphi'},L_{\varphi''})\), the triples containing \((L_{\varphi_{\text{plait}}},L_{\varphi''})\) are both positive. The codimension 1 condition for \(\varphi\) and \(\varphi''\) implies, via Lemma \([5.18]\) and Lemma \([5.20]\) that either

\[
(6.8) \quad \alpha_{\varphi',\varphi''} \cdot \alpha_{\varphi''} = \pm \alpha_{\varphi',\varphi''} \text{ or } \alpha_{\varphi',\varphi''} \cdot \alpha_{\varphi'} = \pm \alpha_{\varphi',\varphi''}.
\]

The two cases are similar; we consider the first. Reversing the roles of \(\varphi\) and \(\varphi''\) in Lemma \([5.20]\), we see that multiplication by \(\alpha_{\varphi',\varphi''}\) is also necessarily non-trivial, yielding

\[
(6.9) \quad \alpha_{\varphi',\varphi''} \cdot \alpha_{\varphi'} = \pm (\psi_1 + \psi_2) \alpha_{\varphi',\varphi''}.
\]

We therefore obtain a non-zero product among the Lagrangians \((L_{\varphi_{\text{plait}}},L_{\varphi'},L_\varphi,L_{\varphi''})\) in forward and backward ordering. Applying Lemma \([6.8]\), we conclude that the triangle \((L_{\varphi'},L_\varphi,L_{\varphi''})\) is positive. The inductive hypothesis yields that the pairs containing \(L_{\varphi''}\) are positive. Since \((L_\varphi,L_{\varphi''})\) is a codimension 1 pair, we conclude that the triple \((L_{\varphi'},L_\varphi,L_{\varphi''})\) is positive, hence the pair \((L_{\varphi'},L_\varphi)\) is positive, which completes the proof of the inductive step. \(\square\)

**Lemma 6.14.** For the bases constructed by the induction of Section \([6.2]\), all Floer products are positive in the sense of Definition \([6.4]\).

Proof. The statement is equivalent to all triples being positive, and Lemma \([5.21]\) implies it suffices to know positivity for triples in which one pair is of codimension 1. Given a codimension one
pair $(\varphi, \varphi')$ and another matching $\varphi_1$, consider:

By Lemma 6.13 the triples involving $\varphi_{\text{plait}}$ are positive. Lemma 6.8 then says that the remaining triple is positive. □

**Lemma 6.15.** The bases constructed by the inductive procedure of Section 6.2 are compatible with all cup functors $\cup_i : \mathcal{H}_{n-1}^{\text{symp}} \to \mathcal{H}_n^{\text{symp}}$.

**Proof.** Compatibility with the odd cup functors is immediate from Step (2) of the construction, and the fact that at Step (6) arbitrary bases are defined by comparison with $\varphi_{\text{plait}}$ which contains all the odd cups. (Compatibility with the even cup functors requires a consistency check, since we have broken symmetry between $\varphi_{\text{plait}}$ and $\varphi_{\text{mix}}$ in Step (6) of the induction.) So suppose we have two matchings which are both in the image of an even cup functor, $\cup_2 \varphi$ and $\cup_2 \varphi'$. The argument then follows the same strategy as in Lemma 6.12. We consider the configuration

which admits a non-trivial product involving all 3 Lagrangians, by Lemma 5.23. The Künneth theorem in Floer cohomology, and the fact that all products are positive by Lemma 6.14, implies that if we use Künneth bases for the three groups in the triple, all of which lie in the image of $\cup_2$, then the triple is positive. Lemma 6.13 implies that the triple is also positive with respect to the basis we constructed in the induction, hence the two bases agree. □

The combinatorial arc algebra underlying Khovanov homology is based on a 2d TQFT with underlying Frobenius algebra $V = \mathbb{Z}(1, x)$ with the property that the co-product

$$V \to V \otimes V, \quad 1 \mapsto 1 \otimes x + x \otimes 1$$

gives a positive linear combination of elements of the tensor product basis: the two terms in the final expression have the same sign (rather than $1 \otimes x - x \otimes 1$). The corresponding positivity in our setting was contained in Lemma 5.20.

**Corollary 6.16.** The positive basis for $\mathcal{H}_n^{\text{symp}}$ constructed previously defines an isomorphism $\mathcal{H}_n^{\text{symp}} \to H_n$ which entwines the $\cup_i$ and $\cup_i^{\text{comb}}$.

**Proof.** A helpful model of the combinatorial arc algebra for our purposes is that given by Stroppel and Webster [54]: they set

$$H_n = \oplus_{\varphi, \varphi'} H^*(\hat{L}_\varphi \cap \hat{L}_{\varphi'})$$
and define a modified convolution product on these groups, which (they prove) co-incides with Khovanov’s original TQFT product. There are natural maps from $H_\ast(Y_n)$ to the centre of both $H_n$ and $\mathcal{H}_n^{\text{symp}}$. The clean intersection models for pairs of Lagrangians give identifications of $HF_\ast(L, L')$ and $H_\ast(L \cap L')$ as modules over $H_\ast(Y_n)$. To see that these identifications yield an algebra isomorphism, it is sufficient to know that one can choose minimal degree generators for all groups such that all products between minimal degree generators agree. Furthermore, we know from Lemma 5.21 and the Stroppel-Webster algorithm that it suffices to check products of a minimal degree generator with a degree one generator arising from a pair $(\varphi, \varphi')$ which meet in codimension one.

The algebra isomorphism now follows from comparing Lemma 5.20 with [54, Section 4.2]. In particular, one should compare the “push-forward” case on p. 503 of that paper, in which a saddle cobordism leads to multiplication by an element $\pm(z_{j,i+1} + z_{\sigma(j),i+1})$ which is a sum of two basis elements in $H^2(Y_n)$, with the corresponding situation in Lemma 5.20. The essential point is that the two $H^2$-classes which appear have the same sign, rather than opposite signs, and in both cases represent the Poincaré dual in cohomology to the appropriate triple intersection submanifold. ([54] gives an explicit sign recipe for ensuring that all the terms $\pm(z_{j,i+1} + z_{\sigma(j),i+1})$ appear with a sign +; our construction of a positive basis gives an implicit proof that on the symplectic side one can also ensure that only + signs appear.) Compatibility of the isomorphism with cup functors follows from Lemma 6.15.

Remark 6.17. The isomorphism of Corollary 6.16 identifies the projective module over $\mathcal{H}_n^{\text{symp}}$ defined by the Lagrangian submanifolds $L_\varphi$ with the projective module over $H_n$ defined by the idempotent associated to $\varphi$. In particular, the Lagrangian $L_\varphi$ which enters into the definition of symplectic Khovanov cohomology is identified with the corresponding module $P_\varphi$ over $H_n$.

7. The isomorphism

We now combine the previous results with the exact sequence formulated and proved in Appendix A to establish the isomorphism between Khovanov and symplectic Khovanov cohomology.

7.1. Parallel transport. We begin with a technical discussion of the choice of symplectic structure on the Slodowy slice, to ensure that the results of Appendix A can be applied to our setting.

Recall the slice $S$ from (5.1) and the adjoint quotient map (5.2), which defines a holomorphic fibration $S \to \text{Conf}_{2n}(\mathbb{C})$. Let $w = \{1, 2, \ldots, 2n\}$ and $y_n = Y^w_n$ denote the corresponding fibre of $\chi|S$. Fix a compact set $B \subset \text{Conf}_{2n}(\mathbb{C}) \setminus \Delta$ which lies away from the discriminant locus of non-simple configurations (corresponding to repeated eigenvalues).

Proposition 7.1. There is a Kähler form $\omega'$ on $y_n$, and an extension of this to a vertically non-degenerate closed 2-form on $S$, for which parallel transport maps are globally defined along path in $B$. Writing $\text{Symp}_\infty(y_n)$ for the group of symplectomorphisms of $(y_n, \omega')$ which are modelled on contactomorphisms at infinity, there is a global monodromy representation

$Br_{2n} \to \pi_0 \text{Symp}_\infty(y_n).$

In particular, the monodromy $\tau_i : y_n \to y_n$ of $W_i$ is well-defined in $\pi_0 \text{Symp}_\infty(y_n)$. 
In [50] a Kähler form on $S$, and hence by restriction on its general fibre $Y_n$, is defined as follows. For each co-ordinate $z$ of the matrix $A_i$, we take the function $\xi_i(z) = |z|^{4n/i}$. The sum of these define a proper $C^1$-smooth function $\xi : S \to \mathbb{R}$, which can be smoothed near the co-ordinate hyperplanes by replacing $\xi_i \mapsto \xi_i + \eta_i$ for suitable compactly supported functions $\eta_i$. Let $\psi$ denote the function obtained by adding up all the $\xi_i + \eta_i$ over all entries of $A \in S$. It has the following features [50] Section 5.1:

1. the set of critical values of $\psi$ is proper over $\text{Conf}_{2n}(\mathbb{C})$;
2. outside a compact set, $\|\nabla \psi\|^2 \leq C \psi$ for a constant $C > 0$.

Let $\omega = -dd^c \psi$. This defines an exact Kähler form on each fibre $Y_n^w$, provided $w$ is generic so the fibre is smooth. The Liouville vector field $Z_w$ of $\psi_w = \psi|_{Y_n^w}$ satisfies

$$Z_w \cdot \psi_w \leq C \psi_w$$

and hence has globally defined flow, so $(Y_n^w, \omega)$ is convex and has an infinite contact conical end. [50] related distinct fibres of $Y$ by “rescaled symplectic parallel transport maps” which were defined on compact subsets of the fibres. Here we expand on [50] Remark 30, defining symplectic parallel transport globally for paths which stay away from the discriminant locus; analogous arguments appear in [26] Section 6 and [23] Section 9, which we follow closely.

**Proof of Proposition 7.1**. Because of the properness of the critical values of $\psi$ and the $C^*$-equivariance of the projection to configuration space, we can find $c > 0$ for which the truncated fibres $Y_n^w \cap (\psi^{-1}(0, c))$ form a family of smooth Stein domains as in [50] Lemma 47, and Gray’s theorem implies that the contact boundaries vary locally trivially. Following [26] Section 6, we trivialise a collar neighbourhood of the horizontal boundary $\partial S$ of the fibration $S|_B \cap \psi^{-1}[0, c]$ by the flow of the fibrewise Liouville field $Z_w$, defining a diffeomorphism onto its image

$$(7.1) \quad \partial S \times [-\epsilon, 0] \to W \subset S|_B \cap \psi^{-1}[0, c], \quad (v, t) \mapsto \phi_{Z_w(v)}^t(v).$$

If $\alpha = -d^c \psi|_{\partial S}$ is the contact 1-form on $\partial S$, we obtain a 1-form $\alpha' = \psi^*(\phi_Z)\ast \alpha \in \Omega^1(W)$ with the property that $d\alpha'$ is fibrewise symplectic on $W$ and its associated horizontal subspaces are $\phi_{Z_w}$-equivariant in a collar neighbourhood of $\partial S$.

Fix a cut-off function $\eta : [-\epsilon, 0] \to \mathbb{R}$ which equals 0 on $[-\epsilon, -3\epsilon/4]$ and equals 1 on $[-\epsilon/4, 0]$. Let $\tilde{\eta} : W \to \mathbb{R}$ denote the function obtained from $\eta$ via (7.1). The form $\omega' = \omega + d(\tilde{\eta} \cdot (\alpha' + d^c \psi))$ is a closed vertically non-degenerate 2-form with the following properties:

- in $S|_B \cap \psi^{-1}[0, c]$ it agrees with $\omega$ on the complement of the neighbourhood $W$ of the horizontal boundary;
- its restriction to any fibre $Y_n^w \cap \psi^{-1}[0, c]$ agrees with the restriction of $\omega$, for $w \in B$;
- the $\omega'$-parallel transport maps over $B$ commute with the Liouville flow in $W$, so are cones on contactomorphisms.

The final condition implies that the $\omega'$-parallel transport maps have globally defined flows on the symplectic completions of the fibres (which was not obvious for the $\omega$-parallel transport maps). If we fix a regular fibre $(Y_n, \omega'|_{y_n})$ and write $\text{Symp}_\infty(Y_n)$ for the group of symplectomorphisms of $Y_n$ which are modelled on contactomorphisms at infinity, there is a global monodromy representation

$$Br_{2n} \to \pi_0 \text{Symp}_\infty(Y_n)$$
which is obtained from the previous construction by taking $B$ to be the complement of a relatively compact open neighbourhood of the discriminant in configuration space. Similarly, parallel transport maps exist over compact subsets disjoint from the critical values of the Morse-Bott Lefschetz fibrations obtained by restricting $\chi|_S$ to a disc transverse to the discriminant in $\text{Conf}_{2n}(\mathbb{C})$. It follows that parallel transport maps are globally defined for $W_i$ along paths which do not go into the origin, which completes the proof. 

For each $1 \leq i \leq 2n - 1$ we consider a map $w_i : \mathbb{C} \to \text{Conf}_{2n}(\mathbb{C})$ with $w_i(0) = \{1,2, \ldots ,i - 1, i + 1/2, i + 1/2, i + 2, \ldots ,2n\}$ and with image $\text{im}(w_i) \cap \Delta$ transverse at 0 to the discriminant in configuration space and otherwise disjoint from the discriminant. Let $w_i = \{1,2, \ldots ,i - 1, i + 1, \ldots ,2n\}$. Choosing $B$ to be sufficiently large, we pull back $\chi|_S$ by $w_i$ we obtain a holomorphic map

$$W_i : E_i^B \to D^2$$

which is Morse-Bott-Lefschetz at 0, with zero-fibre having singular locus canonically isomorphic to the space $\mathbb{Y}^n_{w_i}$, and with generic fibre $\mathbb{Y}_n$, and which is a flat symplectic fibration in a neighbourhood of the boundary of the disc. We write

$$(7.2) \quad W_i : E_i \to \mathbb{C}$$

for the extension of this map to a fibration over $\mathbb{C}$, which, outside the unit disc, is modelled after the symplectic mapping torus of the monodromy. By construction $W_i$ is an exact LG model in the sense of the Appendix, whose restriction to a neighbourhood of the critical point agrees with the map constructed by Seidel and Smith in [50].

**Remark 7.2.** There are at least three natural symplectic structures on $\mathbb{Y}_n$:

1. the one just discussed, from [50], which has good parallel transport properties helpful for establishing the long exact triangle;
2. the one discussed in Section 6.4 which extends smoothly to the compactification $(S^2)^{2n}$ and is convenient for computing the Lagrangians $L_\psi$ to the anti-diagonals $\hat{L}_\psi$ in the topological model and computing the ring structure in the symplectic arc algebra;
3. and the forms from Section 4.4 induced from $\mathbb{Y}_n \hookrightarrow \text{Hilb}^{[n]}(\mathbb{A}_{2n-1})$ which are product-like away from the diagonal, with respect to which the Lagrangians $L_\psi$ are products of fibred Lagrangians in Milnor fibres, and which we used in constructing the equivariant structure on the Fukaya category in [5] and on the cup bimodules in this paper.

One can embed a large compact subset of the Hilbert scheme symplectically into the nilpotent slice with the form $-dd^c\psi$ in such a way that the Lagrangians obtained as products of matching spheres are Hamiltonian isotopic to the iterated vanishing cycles, by a result of Manolescu [29]. The Kähler form from section 5.4 was only used to simplify the study of cohomological properties of the restriction $H^\ast(\mathbb{Y}_n) \to H^\ast(L_\psi)$, by reduction to the smooth models $\hat{L}_\psi$; those properties then carry over to any other model. We may therefore use the formality results for cup and cap functors, the long exact triangle, and the vanishing cycle description of Lagrangians (and the corresponding clean intersection models arising from Lemma 5.7.5) simultaneously.

7.2. The cup and cap functors revisited. Let $\text{Tw} \mathcal{H}_{n-1}$ denote the category of twisted complexes over $\mathcal{H}_{n-1}$, and let $\Delta_{i-1}$ denote the $\text{Tw} \mathcal{H}_n$ bimodule associated to the inverse monodromy of $W_i$ (i.e. the monodromy around a clockwise oriented path encircling $0 \in \mathbb{C}$). Recall that we previously constructed a functor $\cup_i : \mathcal{H}_{n-1} \to \mathcal{H}_n$, which we proved to be formal in Section 4.6.
Proposition 7.3. There is a functor \( i_n : \mathcal{H}_n \to Tw \mathcal{H}_{n-1} \) which is left adjoint to \( \cup_1 \). Moreover, the cone of the unit

\[
(7.3) \quad \Delta^{Tw \mathcal{H}_n} \to \Delta^{Tw \mathcal{H}_{n-1}}
\]

is quasi-isomorphic to the graph bimodule \( \Delta_{r_n}^{-1} \).

Proof. Keeping in mind that \( \tau^{-1} \) corresponds to a clockwise path around the critical point, this is a direct application of Proposition 7.2, whose assumptions it therefore suffices to verify. In the notation of op. cit. we have \( \mathcal{A} = \mathcal{H}_{n-1} \), with \( M = \mathcal{Y}_{n-1} \), and \( \mathcal{A}' = \mathcal{H}_n \) with \( M' = \mathcal{Y}_n \); we set \( E = \mathbb{C}^3 \to \mathbb{C} \) to be the standard Lefschetz fibration, with fibre \( N = \mathbb{T}^*S^2 \); and we obtain the inclusion \( M \times E \to E' = E_i \), compatible with the map \( W_i \), from the existence of \( A_1 \) degenerations as pairs of eigenvalues come together, as in [50, Lemma 27]. (Strictly speaking, \( E \) and \( N \) should be defined as subdomains of \( \mathbb{C}^3 \) respectively \( \mathbb{T}^*S^2 \) with contact boundary, and similarly for the fibration \( E_{L'} \) introduced below, but we will keep to the simpler notation and hope no confusion will arise.) To fulfil the hypotheses of Proposition A.2 we require:

1. For each object \( L' \) of \( \mathcal{A}' \), there exists a Lefschetz fibration \( E_{L'} \to \mathbb{C} \), with fibre \( N_{L'} \), including \( E \) as a subfibration containing all critical points, and an inclusion \( M \times E_{L'} \subset E' \) compatible with the maps to \( \mathbb{C} \) such that \( L' \) is contained in \( M \times N_{L'} \);
2. For each object \( L' \) of \( \mathcal{A}' \), there is an object \( L \) of \( \mathcal{A} \) such that either (i) \( L' \) is quasi-equivalent to the product of \( L \times K \) or (ii) \( L' \) is quasi-equivalent to a Lagrangian which meets \( L \times K \) cleanly along a section of the projection to \( K \).

Recall from Lemma 5,29 that if one places the eigenvalues \( w_{cyc} \in \text{Conf}_{2n}(\mathbb{C}) \) defining the fibre \( \mathcal{Y}^{w_{cyc}}_n \) at the roots of unity, there is a cyclic symmetry which acts transitively on consecutive pairs, and hence relates the different Morse-Bott fibrations \( W_i \). Without loss of generality we therefore suppose \( i = 1 \).

If \( L' \) is in the image of \( \cup_1 \), say \( L' = \cup_1 L \), we simply take \( E_{L'} = E \). This corresponds to case (i) of Condition (2) above.

If \( L' \) is not in the image of \( \cup_1 \), say \( L' = L_\varphi \), then since \( i = 1 \), we can suppose that \( \varphi \) contains the arcs \( (2, p) \) and \( (1, m) \) for some \( p < m \). The hypothesis that \( i = 1 \) additionally implies that the arc \( (1, m) \subset \varphi \) is necessarily outermost. Embed \( W_i \) into the larger two-parameter degeneration in which the triple of eigenvalues \( \{1, 2, p\} \) come together along the arcs \( [1, 2] \subset \mathbb{R} \) and \( (2, p) \subset \varphi \subset \mathbb{C} \), as in [50] Lemma 29, which is reproduced as Lemma 5.28 above. Let \( E_{L'} \) be \( \mathbb{T}^*S^2 \times \mathbb{C} \), equipped with the Lefschetz fibration \( E_{L'} \to \mathbb{C} \) which has generic fibre \( N_{L'} \) the \( A_2 \)-Milnor fibre, and a unique critical value. One can view this as a subset, containing only one of the two critical values, of the stabilisation of the fibration \( E \to \mathbb{C} \), which gives a fibre-preserving inclusion \( E \subset E_{L'} \). Lemma 5.28 implies that there is an open subset of \( E' \) which is an \( A_2 \)-fibration over \( \mathcal{Y}_{n-1} \cong \text{Crit}(W_i) \), and this fulfils the first set of required conditions.

From the construction of the \( A_2 \)-degeneration, \( L_\varphi \) is fibred over the matching given by removing the arc \( (2, p) \subset \varphi \), replacing the critical points \( \{1, 2, p\} \) by a single point \( * \) viewed as lying at position \( \{1\} \), and considering the matching \( \varphi_0 \) comprising \( \varphi \setminus (2, p) \), which contains the arc \( (*, m) \) (which is uniquely defined to isotopy, and still an upper half-plane matching, since \( (*, m) \) is still outermost). We set \( L = L_{\varphi_0} \in \mathcal{A} \). Then \( \cup_1 (L_{\varphi_0}) \) and \( L_{\varphi_0} \) meet in codimension one, and are given locally by fibrations over a common base with fibre the two arcs of the compact core of the \( A_2 \)-fibre \( N_{L'} \). This fulfils the requirements of case (ii) of Condition (2), completing the proof. \( \square \)
If we consider the monodromy twist $\tau_i$ instead of $\tau_i^{-1}$, functoriality of the construction of the graph bimodule implies that $\Delta_{\tau_i}$ is the bimodule inverse (under tensor product) to $\Delta_{\tau_i^{-1}}$. Since inverse bimodules are uniquely determined up to quasi-isomorphism, we conclude that this bimodule agrees with the diagramatic inverse twist:

**Corollary 7.4.** The bimodule $\Delta_{\tau_i}$ is quasi-isomorphic to the cone of the counit:

$$\Delta_{\tau_i} \simeq \text{Cone}(\Delta_{\text{Tw}(\beta^n)} \to \Delta_{\text{Tw}(\beta^n)})$$

We therefore reach the main theorem:

**Theorem 7.5.** For any link $\kappa$, and characteristic zero field $k$, we have an isomorphism

$$\text{Kh}_{\text{symp}}(\kappa; k) \cong \text{Kh}(\kappa; k).$$

**Proof.** Khovanov homology is completely determined by the arc algebra, the cup functors, and the distinguished module $P_{\varphi_0}$ associated to the $H_n$-idempotent defined by $\varphi_0$. Indeed, the cap functors are adjoints to cups, the twist functors are cones on adjunctions, and the link invariants are obtained from the resulting braid group action as Ext-groups

$$\text{Kh}(\kappa_\beta) = \text{Ext}_{D(\text{mod} - H_n)}(P_{\varphi_0}, (\beta \times \text{id})(P_{\varphi_0}))$$

for $\beta \in \text{Br}_n$ and $\beta \times \text{id} \in \text{Br}_2$ having closure $\kappa_\beta$. From our description of the long exact triangle for the braid group half-twists, the symplectic link invariant $\text{Kh}_{\text{symp}}$ is obtained by exactly the same procedure, starting from the symplectic arc algebra and its cup functors, and the Lagrangian $L_{\varphi_0}$ which corresponds to $P_{\varphi_0}$, cf. Remark 6.17. Therefore symplectic and combinatorial Khovanov cohomologies co-incide over $k$.

Up to this point, we have not discussed gradings. Let $\beta \in \text{Br}_n$ be a braid. The Lagrangian submanifold $L_{\varphi_0}$ admits a grading; by parallel transport, the image $(\beta \times \text{id})(L_{\varphi_0})$ is also graded, which yields an absolute $\mathbb{Z}$-grading on the group $HF^*(L_{\varphi_0}, (\beta \times \text{id})(L_{\varphi_0}))$, which is independent of choices. Then

$$\text{Kh}_{\text{symp}}^*(\kappa_\beta) = \text{HF}^{*+n+w}(L_{\varphi_0}, (\beta \times \text{id})(L_{\varphi_0}))$$

defines a Markov-invariant absolute $\mathbb{Z}$-grading on symplectic Khovanov cohomology, where $n$ is the number of strands in the braid $\beta$, and $w$ is the writhe, and the braid closure $\kappa_\beta$ is canonically oriented as a link by orienting each of the strands of $\beta$ in the same direction. We also recall the unoriented skein relation for Khovanov homology, which reads (for a positive respectively negative crossing)

$$\cdots \to \text{Kh}^{i,j}(\times) \to \text{Kh}^{i,j-1}(\vee) \to \text{Kh}^{i,v,j-3v-2}(\prec) \to \cdots$$

and

$$\cdots \to \text{Kh}^{i,j}(\times) \to \text{Kh}^{i+v+1,j-3v+2}(\prec) \to \text{Kh}^{i+1,j+1}(\vee) \to \cdots$$

According to long-standing convention, a positive (fibred Dehn) twist in the braid group corresponds to the negative crossing in a link diagram, and vice-versa.
Here, in the complement of the crossing under consideration, one takes the arc which ends at the top left corner of the crossing, and sets \( v \) to be the signed number of crossings between this arc and the other connected components of the complement (this compensates for the non-local change of orientation that occurs in the given way of resolving the crossing). To extend the definition of (7.6) to a diagram of a link which is not a braid closure, such as the unoriented crossing resolution, one should interpret the shift \( n + w \) as the sum of the number of cups \( n \) in the diagram with the writhe \( w \) as before, compare to [14, Section 5] or [38, Equation 40]. Given this, considering for instance the first of the two skein triangles (for the positive crossing and hence negative twist monodromy), one finds that in the \( k = i - j \) grading the terms which occur in the first line have degrees \( k, k + 1, k + 2v + 2 \) respectively. Suppose the given link is presented as a braid closure for \( \beta \in \text{Br}_n \) with writhe \( w \); then the corresponding absolute Floer gradings are for a sequence (in an obvious schematic notation)

\[
\cdots \rightarrow HF^{k+n+w}(\zeta) \rightarrow HF^{(k+1)+n+(w-1)}(\zeta) \rightarrow HF^{(k+2v+2)+(n+1)+(w-2v-1)}(\zeta) \rightarrow HF^{(k+1)+n+w}(\zeta) \rightarrow \cdots
\]

Here we note that the writhe decreases by one in \( \zeta \rightarrow \zeta' \) whilst the number of cups increases by one in \( \zeta' \rightarrow \zeta \). Furthermore, if the original \( \beta \) has \( a \) positive crossings and \( b \) negative crossings, then the unoriented resolution has \( a - v - 1 \) positive and \( b + v \) negative crossings, so the writhe of the third term differs from \( w \) by \( 2v + 1 \). In short, the degrees in the exact triangle in \( i - j \) grading are precisely those associated with the mapping cone construction of the twist in Corollary 7.4, with the boundary map of the mapping cone having cohomological degree \( +1 \) and the other arrows degree \( 0 \) respectively \( 2 \) (the last being the dimension of the vanishing sphere, cf. the degrees in [40]). The same discussion applies to the skein triangle for the negative crossing, versus the mapping cone of Proposition 7.3. Note that if \( \zeta \) is the unknot, the absolute grading on \( \text{Kh}^k_{\text{symp}}(\zeta) \) concentrates that group in symmetric degrees \( \{\pm 1\} \). Given that the cohomological grading in the symplectic arc algebra agrees with the \( i - j \)-grading collapse in the combinatorial arc algebra (which is concentrated in a single homological degree, and is given by tensor products of \( H^*(S^2)[1] \) in the \( j \)-grading), one infers that, in the absolute grading, one has an isomorphism

\[
\text{Kh}^k_{\text{symp}} = \bigoplus_{i-j=k} \text{Kh}^{i,j}
\]

which completes the proof of [50, Conjecture 2].

**Appendix A. The long exact sequence of a twist**

**A.1. The result.** Let \((E, \omega = d\theta, J)\) be an exact symplectic manifold equipped with a compatible almost complex structure \( J \) such that \( E \) is geometrically bounded in the sense of Sikorav [43].

**Definition A.1.** An exact Landau-Ginzburg model is a smooth map \( W : E \to \mathbb{C} \) such that, outside a compact set, \( W \) is a flat symplectic fibration which is holomorphic.

For convenience, we specify that the compact set is the disc of radius \( 1/4 \) centered at \((-1/2, 0)\), and assume that the triple \((E, \theta, J)\) is trivial over the right half-plane. For the purpose of defining \( \mathbb{Z} \)-graded Fukaya categories, we will furthermore assume that \( c_1(E) = 0 \), and fix a trivialisation of the top exterior power of \( TE \) as a complex vector bundle.
We will find it useful to study inclusions of Landau-Ginzburg models, in the following set-up. Recall that a Lefschetz fibration comprises an exact symplectic manifold $E$ with contact boundary, equipped with a proper map $E \to \mathbb{C}$ which is a symplectic fibration outside a neighbourhood of its critical points. These neighbourhoods are required to be modelled after the quadratic map
\begin{equation}
\sum z_i^2 : \mathbb{C}^n \to \mathbb{C}.
\end{equation}
Consider the following specific geometric situation: (i) $M$ is a Liouville domain, (ii) $W : E \to \mathbb{C}$ is a Lefschetz fibration with a unique critical point, whose smooth fibre we denote $N$, and (iii) $W' : E' \to \mathbb{C}$ is an exact Landau-Ginzburg model with fibre $M'$, equipped with an exact inclusion
\begin{equation}
M \times E \subset E'
\end{equation}
which is compatible with the map to $\mathbb{C}$. In particular, we have an inclusion $M \times N \subset M'$.

Consider subcategories $A \subset \mathcal{F}(M)$ and $A' \subset \mathcal{F}(M')$, and assume that the following technical condition holds:
\begin{equation}
\text{For each object } L' \text{ of } A', \text{ there exists a Lefschetz fibration } E_{L'} \to \mathbb{C}, \text{ with fibre } N_{L'},
\end{equation}
including $E$ as a subfibration containing all critical points, and an inclusion $M \times E_{L'} \subset E'$ compatible with the maps to $\mathbb{C}$ such that $L'$ is contained in $M \times N_{L'} \subset M'$.

We denote by $\phi : M' \to M'$ the inverse monodromy of the fibration, i.e. the symplectomorphism obtained by parallel transport along a loop going clockwise once around a circle lying in the region where $W'$ is locally flat. We write $\Delta_{\phi}^{\text{Tw}A'}$ for the graph bimodule over the category of twisted complexes on $A'$ associated to the endo-functor of the Fukaya category induced by $\phi$.

Fix a closed exact Lagrangian $K \subset N$. As in Section 4.6, let $\cup : A \to \mathcal{F}_M$ denote the functor which assigns to a Lagrangian $L \in \text{Ob} A$ its product with $K$ (we constructed this indirectly as a representing functor for the bimodule $\mathcal{X}$). The following result is proved in Section A.13, at the very end of this Appendix, as a consequence of Theorem A.4; the latter in turn is a special case, sufficient for our purposes, of a more general result which will appear in forthcoming work of the first author and Sheel Ganatra [3].

**Proposition A.2.** Assume that for every object $L'$ of $A'$, there is an object $L$ of $A$ such that either (i) $L'$ is quasi-equivalent to the product of $L \times K$ or (ii) $L'$ is quasi-equivalent to a Lagrangian which meets $L \times K$ cleanly along a section of the projection to $K$. There is a functor $\cap : \text{Tw}A' \to \text{Tw}A$ which is left adjoint to $\cup$. Moreover, the graph bimodule $\Delta_{\phi}^{\text{Tw}A'}$ is quasi-isomorphic to the cone of the unit:
\begin{equation}
\Delta_{\phi}^{\text{Tw}A'} \simeq \text{Cone} \left( \Delta_{\text{Tw}A'}^{\text{Tw}A'} \to \Delta_{\text{Tw}(\cup \cap)}^{\text{Tw}A'} \right).
\end{equation}

The reader may compare with Proposition 7.3 to see how this result is applied. Its assumptions are neither optimal nor natural. Assumption A.3 would more naturally be replaced by the property of being a fibration with Morse-Bott critical locus and globally integrable parallel transport, and the hypothesis on objects of $A'$ should be dropped. The global properties of parallel transport are unfortunately not well understood in the desired application, because the Kähler form on the slice $S_\alpha$ is only known to have well-defined parallel transport maps out of bounded subsets of the critical locus of a generic fibre in the discriminant locus of the adjoint quotient $\chi$. In a different direction, omitting the conditions (i) or (ii) on objects of $A'$ in the statement of the Proposition would require additional algebraic machinery, which whilst natural in a general development is
not needed for our particular application (this however is undertaken in the forthcoming work of Abouzaid-Ganatra).

A.2. The setting. The proof of Proposition A.2 relies on a more abstract adjunction involving the Fukaya category $\mathcal{F}_W$ of a Landau Ginzburg potential $W: E \to \mathbb{C}$, which we construct in section A.4. The idea that there should be such a Fukaya category is due to Kontsevich. There are many implementations, some of which are not yet in the literature (see, e.g. [45, 1, 47, 48, 12, 5]). We shall adopt a viewpoint which is a mixture of Seidel’s approach in [47] and Abouzaid-Seidel’s approach in [5], and where the Lagrangians we consider are similar to those studied by Biran and Cornea in [11, 12]. In particular, the objects of this category satisfy the following geometric condition:

**Definition A.3.** An exact Lagrangian in $E$ is horizontally admissible if it is proper over $\mathbb{C}$, and its image under $W$ agrees, outside a compact set, with a finite union of half-lines parallel to the positive real axis.

Note that [45, Lemmas 16.2 & 16.3] show that, over any half-line $\delta: [0, \infty) \to \mathbb{C}$ near infinity with $W(E) \supset \text{im}(\delta)$, such an admissible Lagrangian $L$ is smoothly fibred over $\delta$, and the component $L \cap W^{-1}(\text{im}(\delta))$ is obtained by parallel transport along $\delta$ of a smooth Lagrangian submanifold of $W^{-1}(\delta(0))$.

The height $h(L)$ of an admissible Lagrangian $L$ is the collection of real numbers which appear as $y$-coordinates of the corresponding half-lines near infinity. A brane structure on such a Lagrangian consists of a choice of Spin structure and a real lift of the $S^1$ valued phase. If $W(L)$ comprises a single half-line near infinity, we say that $L$ has one end.

Let $M$ denote a fibre of $W$ over the right half-plane. We denote by $\phi: M \to M$ the monodromy symplectomorphism obtained by parallel transport along a loop going clockwise once around a circle which is sufficiently large that it lies entirely in the region where $W$ is locally flat. In Section A.5 we construct a particular model for the Fukaya category $\mathcal{F}_M$ of this fibre. We write $\Delta_\phi$ for the graph bimodule of the endo-functor of the Fukaya category induced by $\phi$.

In section A.8 we construct a functor $\mathcal{D}: \mathcal{F}_M \to \mathcal{F}_W$ which the first author first heard described by Dima Orlov. Let $\mathcal{D}\Delta^{\mathcal{F}_W}_D$ denote the 2-sided pullback of the diagonal bimodule $\Delta^{\mathcal{F}_W}$ of $\mathcal{F}_W$ by $\mathcal{D}$. Applying $\mathcal{D}$ induces a natural map

$$\Delta^{\mathcal{F}_M} \to \mathcal{D}\Delta^{\mathcal{F}_W}_D$$

of bimodules which we call the unit.

**Theorem A.4 (Abouzaid-Ganatra).** The cone over the unit $\Delta^{\mathcal{F}_M} \to \mathcal{D}\Delta^{\mathcal{F}_W}_D$ is quasi-isomorphic to the graph bimodule $\Delta_\phi$ of the clockwise monodromy.

Very schematically, the functor $\mathcal{D}$ involves taking the parallel transport of a Lagrangian $L \subset M$ along an arc which has two ends and encircles the origin, cf. Figure 12. There is a canonical two-step filtration of Floer complexes for objects in the image of $\mathcal{D}$ with objects in the fibre, coming from the two-ended structure, and the exact triangle arises from that filtration.
A.3. Floer cochains and operations. If \( h(L) \) is disjoint from \( h(L') \), we define the Floer complex \( CF^*(L, L') \) using the methods of \cite{45}; i.e. we pick a compactly supported pair \((H, K)\) consisting of a Hamiltonian on \( E \) and a perturbation of the complex structure \( J \), so that regularity is achieved for all moduli spaces of finite energy solutions to Floer’s equation with boundary on \( L \) and \( L' \). The Floer complex is generated by Hamiltonian chords starting at \( L \) and ending on \( L' \), and the Floer differential counts solutions to Floer’s equation with such ends. The moduli spaces of Floer trajectories are precompact because projection to the base is holomorphic outside a compact set, so that the maximum principle applies. By choosing a proper Morse function \( f_L: L \to [0, \infty) \), and defining

\[
CF^*(L, L) = CM^*(L; f_L),
\]

we extend this definition to the case \( L = L' \).

There is a special case in which this complex may be readily computed: we say that \( L_{-\epsilon} \) is a small negative perturbation of \( L \) if is obtained by a small Hamiltonian isotopy which decreases all heights by \(-\epsilon < 0\). For an appropriate almost complex structure, the Floer complex \( CF^*(L, L_{-\epsilon}) \) is isomorphic to the Morse complex of a function on \( L \) whose gradient flow points outwards at infinity (see \cite{18, 1}). In particular:

**Lemma A.5.** There is a canonical degree 0 generator of \( CF^0(L, L_{-\epsilon}) \) for \(-\epsilon \) sufficiently small. □

We shall need a technical result, a version of which forms the basis to all approaches to the Fukaya category of \( W \). Consider a half-plane \( H_a = [a, \infty) \times \mathbb{R} \), and two collections of functions \( \{f_i\}_{i \in A} \) and \( \{f'_j\}_{j \in A'} \) on \([a, +\infty)\), which are locally constant on a neighbourhood of \( a \) and \( \infty \). Let \( \Gamma \) and \( \Gamma' \) denote the unions of the graphs of these functions; assume that the curves in each of these sets are pairwise disjoint.

**Definition A.6.** A monotone pair \((\Gamma, \Gamma')\) is a pair which satisfies:

\[
\frac{df_i}{dx}(x) \geq \frac{df'_j}{dx}(x) \quad \text{for all } (i, j) \in A \times A', \text{ with strict inequality if } f_i(x) = f'_j(x).
\]
Consider Lagrangians \( L \) and \( L' \) which respectively project to the graphs of \( \Gamma \) and \( \Gamma' \), and assume that their intersections over \( H_a \) are transverse. Assuming that the heights of \( L \) and \( L' \) are disjoint, let \( CF^*(L, L') \) be defined with respect to a Hamiltonian supported away from \( H_a \), and a perturbation of almost complex structures for which \( W \) is holomorphic over \( H_a \). Consider \( CF^*_a(L, L') \subset CF^*(L, L') \) to be the submodule generated by intersections projecting away from \( H_a \). We claim this is a subcomplex:

**Lemma A.7.** The differential preserves \( CF^*_a(L, L') \), and all holomorphic curves which contribute to the differential on this subcomplex have image which project to \( H_a \).

**Proof.** Let \( u \) be a holomorphic strip whose input is a generator of \( CF^*_a(L, L') \). After possibly deforming \( a \), each component \( \Sigma \) of \( (W \circ u)^{-1}(H_a) \) is a surface with corners; we shall show by contradiction that all such components are constant and contained in the vertical line \( x = a \). Letting \( v \) denote the restriction of \( W \circ u \) to such a component we first consider the situation in which this component does not include the outgoing end. In this case, every boundary segment of \( \Sigma \) which maps under \( v \) to a curve in \( \Gamma \) or \( \Gamma' \) must have both endpoints at the same point of the vertical line \( x = a \). In particular, the intersection number of the boundary of \( v(\Sigma) \) with a horizontal half-ray starting at \( x = a \) must vanish. This implies invariance of the signed count of pre-images of any point in the plane, which then vanishes because of vanishing at infinity. By the open mapping theorem, we conclude that the image of this component is constant, and contained in the boundary.

In order to extend this argument to the case that \( \Sigma \) includes the outgoing end, let \( v \) denote the extension of \( W \circ u \) to the compactification \( \overline{\Sigma} \), and let \( z \) denote the image of the point at infinity. The two segments of \( \partial \overline{\Sigma} \) meeting at the point at infinity map to curves \( \gamma \) and \( \gamma' \) in \( \Gamma \) and \( \Gamma' \) which meet at \( z \). The situation is summarised in Figure 10, and the ordering convention for the output of a Floer differential is such that the degree of \( v \) over the unique embedded triangle in that figure must be negative in order for the degree to vanish for large values of the \( x \)-coordinate. By positivity of degree (and the open mapping theorem) we conclude that such a component cannot exist. \( \square \)

The same methods yield a higher product

\[
\mu_d: CF^*(L_d, L_{d-1}) \otimes \cdots \otimes CF^*(L_0, L_1) \to CF^*(L_0, L_d)[2-d]
\]

whenever all Floer complexes are defined. The case in which some of the Lagrangians are equal is handled by counting configurations of holomorphic discs and (perturbed) gradient flow lines, as in [10, 46, 52]. Chosen inductively, these products satisfy the \( A_\infty \) relation. The same argument as in Lemma A.7 shows:
Lemma A.8. If $L_0, \cdots, L_d$ project to collections of curves $\Gamma_0, \cdots, \Gamma_d$ such that each pair $(\Gamma_i, \Gamma_j)$ is monotone if $i < j$, then $\mu_d$ preserves interior Floer cochains. □

A.4. The Fukaya category of $W$. The collection of horizontally admissible branes forms a partially ordered set, with $L > K$ if and only if $h(L) > h(K)$ as subsets of $\mathbb{R}$. We define a category $\mathcal{O}_W$ with objects such branes and morphisms

$$\mathcal{O}_W(L, K) = \begin{cases} CF^*(L, K) & \text{if } L \geq K \\ 0 & \text{otherwise.} \end{cases}$$  

(A.9)

The $A_\infty$ operations are given by Equation (A.8).

We shall define the Fukaya category as a localisation of $\mathcal{O}_W$, following [5]. To this end, consider a pair $(L, L')$ such that there is a real number $2 < a$, and a small negative perturbation $L_a$ of $L$ which projects to straight lines on $H_a$ such that (i) $L'$ agrees with $L_a$ away from $W^{-1}(H_a)$, and (ii) the pair $(L, L')$ projects to a monotone pair of arcs outside a compact set. In that case, Lemma A.7 implies that we have a subcomplex

$$CF^*(L, L_a) = CF^*_in(L, L') \subset CF^*(L, L'),$$  

(A.10)

hence Lemma A.5 yields a class which we call a quasi-unit

$$\kappa \in H\mathcal{O}_W(L, L').$$  

(A.11)

Definition A.9 (Abouzaid-Seidel). The Fukaya category $\mathcal{F}_W$ is the localisation of $\mathcal{O}_W$ with respect to quasi-units.

The definition of localisation relies on the quotient construction of $A_\infty$ categories. Following Drinfeld [17] in the differential graded case, a convenient model for such quotients was provided by [27]. By the universal property of localisations, there is a functor $\mathcal{O}_W \to \mathcal{F}_W$. The main justification of our definition is the following result:

Proposition A.10. If $L > K$, the localisation map induces an isomorphism on cohomology

$$H\mathcal{O}_W(L, K) \cong H\mathcal{F}_W(L, K).$$  

(A.12)

Sketch. The proof is completely formal once one shows that multiplication with respect to quasi-units induces an isomorphism on homology among directed objects, i.e. if $L > K$, and $\kappa_+: L_+ \to L$ and $\kappa_-: K \to K_-$ are quasi-units, then the maps

$$\mu_2(\kappa_+): CF^*(L, K) \to CF^*(L_+, K)$$  

(A.13)

$$\mu_2(\kappa_-): CF^*(L, K) \to CF^*(L, K_-)$$  

(A.14)

induce isomorphisms on homology. The essential point is that, in the isotopy between $L$ and $L_+$, no intersections with $K$ at infinity are created or destroyed (and similarly for swapping the roles of $K$ and $L$). The proof then follows from invariance of Floer cohomology under continuation maps [17, 11]; compare to [49, Lemma 10.7] for a related localisation construction. □

Remark A.11. The category $\mathcal{F}_W$ is a “partially wrapped” category, in the terminology of e.g. [7]. In constructing any version of the Fukaya category of a Liouville manifold which involves non-compact Lagrangian submanifolds $L$ and perturbations by a Reeb-type flow $\phi_H$ at infinity
defined by a Hamiltonian function $H$, one must always contend with the fact that $A_{\infty}$-operations in Hamiltonian Floer cohomology are defined by maps

\begin{equation}
CF^* (L, \phi_{H_1}(L)) \otimes \cdots \otimes CF^* (L, \phi_{H_k}(L)) \to CF^* (L, \phi_{H_1 + \cdots + H_k}(L))
\end{equation}

which involve multiples $H_i = \lambda_i H$ of a given Hamiltonian function, and the fact that the Floer complex on the right of (A.15) is not isomorphic at chain level to the factors on the left. The “telescope construction” of [4] circumvents this for “fully wrapped” categories, but there are additional complications for compactness of spaces of holomorphic curves, and well-definition of the required continuation maps for direct systems of partially wrapped Floer groups, when the Hamiltonian flow is degenerate on a subset of the contact boundary (roughly stemming from non-properness of $H$ on the completion). The localisation construction of [5], borrowed above, is designed to circumvent these issues.

One-ended Lagrangians, which by definition project to a single arc outside a compact set, play a special role in the theory. In our intended applications, we shall need a more flexible notion: to this end, we say that a Lagrangian $L$ is weakly one-ended if there exists a positive real number $x_0 \in \mathbb{R}$ such that $W(L)$ agrees near the vertical line $x = x_0$ with a horizontal line (say $y = y_0$). Let $\Lambda$ denote the fibre of $L$ over $(x_0, y_0)$, and let $L_{in} \subset L$ be the submanifold of $L$ (with boundary $\Lambda$) defined by the inequality that the real part of $W$ is bounded by $x_0$. We define $T_L$ be the Lagrangian which agrees with $L_{in}$ to the left of $x = x_0$, and with the parallel transport of $\Lambda$ along $y = y_0$ to the right of this line.

**Lemma A.12.** The Lagrangians $L$ and $T_L$ are quasi-isomorphic in $\mathcal{F}_W$.

**Sketch of proof:** To show that $L$ is a summand of $T_L$ in $\mathcal{F}_W$, it suffices to construct Lagrangians $T_L^+$ and $T_L^-$ which are quasi-equivalent to $T_L$ in $\mathcal{F}_W$, together with maps $f \in HF^* (T_L^+, L)$ and $g \in HF^* (L, T_L^-)$ such that the product

\begin{equation}
\mu^2 (g, f) \in HF^* (T_L^+, T_L^-)
\end{equation}

is a quasi-unit.

Pick a $C^2$-small Morse function $h$ on $L$ whose restriction to a neighbourhood of $\Lambda$ agrees with the sum of a Morse function on $\Lambda$ with a small multiple of $(x - x_0)^2 + (y - y_0)^2$. We can then pick $T_L^+$ (respectively $T_L^-$) to agree with Hamiltonian pushoff of $L_{in}$ by $-h$ (respectively $h$) in the region $x \leq x_0$, and with the parallel transport of $\Lambda$ by a line above $y = y_0$ (respectively below $y = y_0$) which does not intersect $W(L)$. We have

\begin{equation}
HF^* (L, T_L^-) \cong HF^* (T_L^+, L) \cong HF^* (T_L^+, T_L^-) \cong H^* (L_{in}),
\end{equation}

since all intersection points project to the left of the line $x = x_0$. These isomorphisms are compatible with multiplication, hence the quasi-unit in $HF^* (T_L^+, T_L^-)$ can be written as a product as in Equation (A.10). This completes the proof that $L$ is a summand of $T_L$; the proof that they are quasi-isomorphic follows by reversing their roles in the above argument. \hfill \square

**A.5.** The Fukaya category of $M$. Let $M$ be the fibre of $W$ at a point in the upper half-plane; by our assumptions on $W$, parallel transport in the right half-plane yields an identification of any pair of such fibres which preserves the primitive $\theta|M$ and the complex structure $J|M$.

To ease comparison with $\mathcal{F}_W$, we define a version of the Fukaya category of $M$ by localisation: fix a finite collection of exact Lagrangian branes $L_M$ in $M$. The objects of $\mathcal{O}_M$ are pairs $(L, i)$,
with \( L \in \mathcal{L}_M \), and \( i \) a negative integer. Choose a sequence of Hamiltonian perturbations \( \{ L^i \} \) of \( L \) which are uniformly \( C^2 \)-small, so that

\[
L^i \text{ is transverse to } K^j \quad \text{whenever } j \neq k.
\]

We obtain a directed category with morphisms

\[
\mathcal{O}_M((L,i),(K,j)) = \begin{cases} 
CF^*(L^i,K^j) & \text{if } (L,i) \geq (K,j) \\
0 & \text{otherwise},
\end{cases}
\]

where we again choose an auxiliary Morse function on each Lagrangian to define self-Floer cohomology, and the partial order is \( (L,i) > (K,j) \) if and only if \( i > j \). The definition of the Floer cochains uses perturbed almost complex structures, but we require that the inhomogeneous terms vanish. Given that the Hamiltonian perturbations were assumed to be \( C^2 \)-small, there is a canonical element

\[
\kappa_{i,j} \in HF^*(L^i,L^j)
\]

under the identification of Floer cohomology with Morse cohomology. If \( i \geq j \), this represents a morphism in \( H\mathcal{O}_M((L,i),(L,j)) \).

**Definition A.13.** The Fukaya category \( \mathcal{F}_M \) of \( M \) is the localisation of \( \mathcal{O}_M \) at the continuation elements \( \kappa_{i,j} \).

The objects \((L,i)\) and \((L,j)\) are quasi-isomorphic in \( \mathcal{F}_M \), but not identical. If \( \mathcal{F}(M) \) denotes the “usual” Fukaya category, as constructed in [15] and used in the body of this paper, one can construct an \( A\infty \)-functor \( \mathcal{O}_M \to \mathcal{F}(M) \) which sends \( \kappa_{i,j} \) to a quasi-isomorphism for every \( i,j \). From that point, a formal argument based on the properties of localisation again shows that \( \mathcal{F}_M \) and \( \mathcal{F}(M) \) are quasi-equivalent; see [5] and [49, Remark 10.8].

**A.6. Vertical Lagrangians.** Pick a monotonically increasing function \( g \) on \( \mathbb{R} \) which agrees with \(-1\) for \( y \ll 0 \), with \( 1 \) for \( 0 \ll y \), and which vanishes with non-zero derivative at the origin. Let \( g_i = (1 + \frac{1}{i})g(y) \), as shown on Figure 11. For each negative integer \( i \), Lagrangian \( L \in \mathcal{L}_M \), and point \( p \) in the right half plane, let \( \mathcal{V}_p(L,i) \) denote the parallel transport of \( L^i \) along the curve \( p + (g_n(y), y) \). The Lagrangians \( \mathcal{V}_p(L,i) \) are not horizontally admissible in the sense of Definition A.3, but they are vertically admissible. Extending the choices of almost complex structure on \( M \) used to define \( CF^*(L^0_0,L^1_1) \) to \( E \), we obtain a canonical isomorphism

\[
CF^*(L^0_0,L^1_1) \equiv CF^*(\mathcal{V}_p(L_0,i_0),\mathcal{V}_p(L_1,i_1))
\]

define the free abelian groups given by the inclusion of intersection points. Moreover, given a sequence \((L^0_0,\ldots,L^d_d)\), the maximum principle implies that all holomorphic discs in \( E \) with boundary on the sequence obtained by applying \( \mathcal{V}_p \) are contained in the fibre over \( 0 \), i.e. we have an
identification between moduli spaces of holomorphic discs in $E$ and $M$. The following result was proved by Seidel in [15]:

**Proposition A.14.** If $i_d < \cdots < i_0$, regularity for holomorphic discs with boundary conditions $(L_0^0, \ldots, L_d^d)$ is equivalent to regularity of the corresponding disc with boundary conditions $(\mathcal{V}_p(L_0, i_0), \ldots, \mathcal{V}_p(L_d, i_d))$. \hfill $\square$

We can extend this discussion to Lagrangians which are equal by choosing the Morse function on $\mathcal{V}_p(L, i)$ to be the sum of the Morse function on $L'$ with $(y - p)^2$. With these choices, the $A_\infty$-structure on $\mathcal{O}_M$ can be equivalently defined as a subcategory of a category of vertically admissible Lagrangians in $E$.

### A.7. The restriction bimodule.

Given a horizontally admissible Lagrangian $T$ and a vertically admissible Lagrangian $V$, we define $CF^*(T, V)$ by choosing a compactly supported Hamiltonian on $E$ which maps $T$ to a Lagrangian transverse to $V$. More generally, given sequences $(T_0, \ldots, T_s)$ and $(V_r, \ldots, V_0)$ of horizontally and vertically admissible Lagrangians projecting to arcs which are disjoint outside a compact set, the count of discs with $r + s + 2$ boundary marked points defines an operation

$$\tag{A.22} CF^*(V_{r-1}, V_r) \otimes \cdots \otimes CF^*(V_0, V_1) \otimes CF^*(T_s, V_0) \otimes CF^*(T_{s-1}, T_s) \otimes \cdots \otimes CF^*(T_0, T_1) \rightarrow CF^*(T_0, V_r),$$

which satisfies the equation for an $A_\infty$-bimodule.

We now define an $\mathcal{O}_M$-$\mathcal{O}_W$ bimodule $\mathcal{R}$, called the restriction bimodule, as follows: given objects $(L, i)$ of $\mathcal{O}_M$ and $T$ of $\mathcal{O}_W$, we set

$$\tag{A.23} \mathcal{R}(T, (L, i)) = \begin{cases} CF^*(T, V(L, i)) & \text{if } 2i + 1 < h(T) \\ 0 & \text{otherwise.} \end{cases}$$

The structure maps are obtained from Equation (A.22) and the identifications of morphism spaces in $\mathcal{O}_M$ with Floer groups among vertical Lagrangians. The key point is that, given sequences $(T_0, \ldots, T_s)$ and $(L_r, i_r), (L_0, i_0)$, the condition $2i_0 + 1 < h(T_0)$ implies $2i_r + 1 < h(T_0)$ whenever $i_k < i$ for all $0 \leq k \leq s - 1$ and $h(T_k) \leq h(T_{k+1})$ for all $0 \leq k \leq r - 1$. In particular, given sequences such that $T_k \leq T_{k+1}$ and $(L_k, i_k) \leq (L_{k+1}, i_{k+1})$, the Floer complexes between horizontal and vertical Lagrangian appearing in Equation (A.22) are respectively isomorphic to $\mathcal{R}(T_k, (L_0, i_0))$ and $\mathcal{R}(T_0, (L_r, i_r))$, and all other complexes are morphism groups in $\mathcal{O}_M$ or $\mathcal{O}_W$. The $A_\infty$ equation for the bimodule $\mathcal{R}$ therefore follows from the $A_\infty$ equation satisfied by Equation (A.22).

### A.8. The Orlov functor.

Fix a point $q \in \mathbb{C}$, lying in the region where $W$ is a locally flat symplectic fibration. For concreteness, we also assume that $q$ lies to the left of the line $x = 2$, though the entire construction can do without this assumption by changing constants below. In this section, we build a functor from $\mathcal{O}_M$ to $\mathcal{O}_W$; the key input is a careful construction of a sequence of arcs in the plane. Let $\gamma_i$ be a sequence of arcs as in Figure (12) indexed by negative integers $i$. More precisely, we first list the properties which only involve one curve at a time:

1. $\gamma_i$ agrees in $[3, +\infty) \times \mathbb{R}$ with horizontal lines at heights $2i + 1$ and $2i + 2$.
2. $\gamma_i$ intersects $[2, 3] \times \mathbb{R}$ in two components, which are graphs of monotonically decreasing functions with values $1 - 1/i$ and $-2 - 1/i$ at $2$. 
(3) $\gamma_i$ is disjoint from $(-1, 2) \times (-1, 1)$.

The next conditions are required for all pairs $i > j$:

1. $\gamma_i$ is transverse to $\gamma_j$.
2. The intersection of the pair $(\gamma_i, \gamma_j)$ with $[2, +\infty) \times \mathbb{R}$ is monotone (c.f. Definition A.6).
3. $\gamma_i \cap \gamma_j \cap (-\infty, 2] \times \mathbb{R} = \{q\}$.

To define a functor, we set $D(L, i)$ to be the parallel transport of $L^i$ along the curve $\gamma_i$, parametrised monotonically so that $t = 0$ maps to the point $q$. If $f_{L^i}$ is the Morse function used to define the self-Floer cochains of $L^i$, the function $f_{L^i} + t^2$ is Morse on $D(L, i)$, so we obtain an identification

$$CF^*(L^i, L^j) = CF^*(D(L, i), D(L, i)),$$

of self-Floer cochains. For pairs, we note that $D(L, i)$ and $D(K, j)$ are transverse if $i \neq j$, so we can define all $A_\infty$ operations among such Lagrangians without using inhomogeneous terms. We have an inclusion

$$CF^*(L^i, K^j) \subset CF^*(D(L, i), D(K, j)),$$

corresponding to the intersection points lying over $q \in \mathbb{C}$. Lemma A.7 implies that this is an inclusion of subcomplexes. Since $j < i$ implies that $h(D(K, j)) < h(D(L, i))$, we obtain an inclusion of morphisms in the directed categories:

$$\mathcal{O}_{M_q}((L, i), (K, j)) \subset \mathcal{O}_W((L, i), (K, j)).$$

This inclusion yields a functor

$$D: \mathcal{O}_{M_q} \to \mathcal{O}_W$$

with trivial higher order terms by the generalisation of Lemma A.7 to multiple Lagrangian boundary conditions. As quasi-units are defined in the same way on $\mathcal{O}_{M_q}$ and $\mathcal{O}_W$ we obtain, by
the universal property of localisation, a functor

(A.28) \( \mathcal{D} : \mathcal{F}_M \to \mathcal{F}_W \).

A.9. **An equivalence of bimodules.** The Orlov functor and the restriction bimodule can be compared whenever \( p = q \) lies in the right half plane. For concreteness, we set \( q = (1,1) \). In this case, note that \( \mathcal{D}(L,i) \) and \( \mathcal{V}(L,i) \) meet cleanly along a copy of \( L^i \) over \( q \), as shown in Figure 13. Given any horizontally admissible Lagrangian \( T \), the count of holomorphic polygons with corners mapping to this clean intersection defines a map

(A.29) \[
\text{CF}^*(\mathcal{D}(L_{r-1}, i_{r-1}), \mathcal{D}(L_r, i_r)) \otimes \cdots \otimes \text{CF}^*(\mathcal{D}(L_0, i_0), \mathcal{D}(L_1, i_1)) \\
\otimes \text{CF}^*(T_s, \mathcal{D}(L_0, i_0)) \otimes \text{CF}^*(T_{s-1}, T_s) \otimes \cdots \otimes \text{CF}^*(T_0, T_1) \to \text{CF}^*(T_0, \mathcal{V}(L,i)).
\]

Using the inclusion of morphism spaces in \( \mathcal{O}_M \) as subcomplexes of morphism spaces among the images of these Lagrangians under \( \mathcal{D} \), we obtain a map of \( \mathcal{O}_M \)-\( \mathcal{O}_W \) bimodules

(A.30) \( \mathcal{D} \Delta^{\mathcal{O}_W} \to \mathcal{R} \).

**Lemma A.15.** Whenever \( 2i + 2 < h(T) \), the induced map

(A.31) \( \mathcal{O}_W(T, \mathcal{D}(L,i)) \to \mathcal{R}(T, (L,i)) \)

is a chain equivalence.

**Proof.** Up to equivalence, the map is invariant under isotopies of \( \mathcal{D}(L,i) \) among horizontally admissible Lagrangians of fixed height, and all isotopies of \( \mathcal{V}(L,i) \) among vertically admissible Lagrangians. By moving the intersection point \( q \) so that its \( x \) and \( y \) coordinate are both much larger than 0, we may assume that all intersections points of \( T \) with \( \mathcal{D}(L,i) \) and \( \mathcal{V}(L,i) \) occur
along the ends of $T$, and all these ends have height smaller than the $y$-coordinate of $q$ (see Figure 13). Each end of $T$ projects to an arc which intersects the arcs defining $\mathcal{V}(L, i)$ and $\mathcal{D}(L, i)$ once, in the second case along the path going from $q$ to the horizontal line of height $2i + 2$ by assumption. The result follows from a straightforward count of holomorphic triangles which are constant in the fibre. □

**Corollary A.16.** $\mathcal{D}$ represents the bimodule $\mathcal{R}$, i.e. there is a natural quasi-equivalence of $\mathcal{F}_M$-$\mathcal{F}_W$-bimodules

\[
\mathcal{D} \Delta \mathcal{F}_W \simeq \mathcal{R}.
\]

□

**A.10. A two-step filtration.** Consider the right-pullback $\mathcal{W}_q$ of $\mathcal{R}$ by the functor $\mathcal{D}$: this is a $\mathcal{O}_{M_p}$-$\mathcal{O}_{M_q}$-bimodule which assigns to $(L, i) \in \text{Ob} \mathcal{O}_{M_p}$ and $(K, j) \in \text{Ob} \mathcal{O}_{M_q}$ the complex

\[
\mathcal{W}_q((K, j), (L, i)) = \begin{cases} 
\text{CF}^*(\mathcal{D}(K, j), \mathcal{V}(L, i)) & \text{if } i < j \\
0 & \text{otherwise.}
\end{cases}
\]

(A.33)

The bimodule structure maps arise from the identification of morphisms in $\mathcal{O}_{M_p}$ with the Floer complexes among the Lagrangians $\mathcal{V}(L, i)$, and the inclusion of morphisms in $\mathcal{O}_{M_q}$ as the subcomplex of the Floer complexes among the Lagrangians $\mathcal{D}(K, j)$ corresponding to the intersection points lying over $q$.

The definition of $\mathcal{W}_q$ is summarised in Figure 13 since the generators of all these complexes, as well as the intersection points among the Lagrangians corresponding to $\mathcal{D}$ and $\mathcal{V}$, take place in the region $x < 2$, we have only illustrated this subset of the base.

The intersection points between $\mathcal{D}(K, j)$ and $\mathcal{V}(L, i)$ which occur over points in the lower half-plane generate a submodule which we denote $\mathcal{W}_q^+((K, j), (L, i))$. 
Lemma A.17. The differential preserves $W^+_q((K,j),(L,i))$.

Proof. We apply the same argument as in Lemma A.7 to the dashed line in Figure 14 which separates the intersection points in the lower half plane from the region where the fibration is not flat and which contains the intersection points in the upper half plane. □

Generalising the above argument to the case of holomorphic polygons, we find a sub-bimodule $W^+_q \subset W_q$, given on pairs by the subcomplex $W^+_q((K,j),(L,i))$. We introduce the quotient complex

\[(A.34)\quad W^+_q \rightarrow W_q \rightarrow W^-_q,\]

whose value on a pair is the Floer complex $W^-_q((K,j),(L,i))$ generated by all intersection points occurring in the lower half plane.

A.11. Parallel transport and the exact triangle. Let us return to the situation considered in the proof of Lemma A.15 where $p = q$ is a point in the upper right quadrant of the plane. In this case, all intersection points defining the bimodule $W^+_q$ lie over this point and hence so do all holomorphic discs contributing to the bimodule structure maps. Since the Lagrangians $D(K,j)$ and $V(L,i)$ intersect this fibre along $K^j$ and $L^i$, the bimodule $W^+_q$ therefore represents the diagonal bimodule of $\mathcal{O}_{M_q}$.

Lemma A.18. The inclusion $W^+_q \rightarrow W_q$ agrees with the unit $\mathcal{O}_{M_q} \rightarrow \mathcal{D} \Delta_{\mathcal{D}}^{\mathcal{O}_{W_D}}$, i.e. the following diagram commutes:

\[(A.35)\quad \begin{array}{ccc}
\mathcal{O}_{M_q} & \rightarrow & W^+_q \\
\downarrow & & \downarrow \\
\mathcal{D} \Delta_{\mathcal{D}}^{\mathcal{O}_{W_D}} & \rightarrow & W_q
\end{array}\]

Proof. The bottom arrow is induced (by pullback under $\mathcal{D}$) by the equivalence between $\mathcal{D} \Delta_{\mathcal{D}}^{\mathcal{O}_{W_D}} \cong \mathbb{R}$. Commutativity is implied by the fact that all holomorphic curve counts take place over $q$, as in Lemma A.15. □

In order to compute $W^-_q$, we move $q$ along a path from $(1,1)$ to $(1,-1)$ that moves counterclockwise (e.g. going through $(-2,0)$). Having fixed the isomorphism between the categories $\mathcal{O}_{M(1,1)}$ and $\mathcal{O}_{M(1,-1)}$ arising from parallel transport in the right half-plane, the above path induces the monodromy symplectomorphism between these fibres; and hence acts correspondingly on Fukaya categories. On the other hand, setting $p = q = (1,-1)$ we have that $W^-_{(1,-1)}$ is isomorphic to diagonal bimodule. Pulling back again to a point in the upper right quadrant, we conclude

Lemma A.19. The bimodule $W^-_q$ is quasi-isomorphic to the graph bimodule of the clockwise monodromy. □

Combining the triangle $W^+_q \rightarrow W_q \rightarrow W^-_q$ with Lemma A.18 we obtain Theorem A.4 namely:
The graph bimodule $\Delta_\phi$ is quasi-isomorphic to the cone $\Delta_{\mathcal{F}} \to \mathcal{D} \Delta_{\mathcal{D}}^{3W}$ of the unit. \hfill □

**A.12. Lefschetz fibrations.** We now implement some of the above ideas in the setting of a Lefschetz fibration $W: E \to \mathbb{C}$ on a total space of dimension $2n$, with a unique critical point (by convention, we can set the value to be $-1/2$ to remain consistent with the previous section). Fix the thimble $T$ which projects to the subset $[-1/2, +\infty)$ of the real axis. As an object of $\mathcal{F}_W$, a thimble $T$ has self-Floer cochains given by any proper Morse function on $T \cong \mathbb{R}^n$. We fix such a Morse function $f_{\mathbb{R}^n}$ with a unique minimum, which near the unit sphere is the sum of a Morse function on the sphere with $(||W|| - 1)^2$. Denoting by $k$ the category with one object whose endomorphism group is $k$, we obtain a functor

$$\mathcal{T}: k \to \mathcal{F}_W,$$

which is a fully faithful embedding since the class of the minimum maps to the identity, which generates the self-Floer cohomology of a thimble. We abuse notation and write $\mathcal{T}$ either for the functor, or for the corresponding object of $\mathcal{F}_W$.

Let $K \subset M = \pi^{-1}(1)$ denote the vanishing cycle. Consider the Yoneda module $\mathcal{K}$ of this object of $\mathcal{F}_M$; note this $\mathcal{F}_M$ module can be equivalently thought of as a $k - \mathcal{F}_M$ bimodule.

**Lemma A.21.** The pullback of $\mathcal{K}$ by $\mathcal{T}$ is quasi-isomorphic to $\mathcal{K}$.

**Proof.** The Yoneda module assigns to any Lagrangian $L$ the Floer complex with $K$, computed in the fibre $M$, whereas the module $\tau \mathcal{K}$ is obtained by taking the Floer complex of the vertically admissible Lagrangian associated to $L$ with the thimble $\mathcal{T}$. For $L \neq K$, these agree by the maximum principle, whereas for $L = K$ they agree because our choice of Morse function on $\mathcal{T}$ restricts to a Morse function on $K$, which we use to define the self-Floer complex of $K$. \hfill □

Lemma [A.21] explains the nomenclature *restriction bimodule*: there is a basic link between $\mathcal{K}$ and the geometric process of “restricting” a Lefschetz thimble to a generic fibre. We next consider the functor $\mathcal{D}$.

**Lemma A.22.** There is a quasi-isomorphism $\mathcal{D}K \cong \mathcal{T} \oplus \mathcal{T}[n - 1]$.

**Proof.** Let $\mathcal{D}(K)$ be a representative of the image of the Orlov functor with ends at $\pm 1$, and $\mathcal{T}_\pm$ be representatives of $\mathcal{T}$ in $\mathcal{O}_W$ with ends at $\pm 2$ (see Figure [15]). Note that the product

$$HF^*(\mathcal{D}(K), \mathcal{T}_-) \otimes HF^*(\mathcal{T}_+, \mathcal{D}(K)) \to HF^*(\mathcal{T}_+, \mathcal{T}_-) \cong H^*(K)$$

is in the correct order for computing morphisms in $\mathcal{F}_W$. The existence of an embedding of two shifted copies of $\mathcal{T}$ as summands in $\mathcal{D}(K)$ is equivalent to the existence of classes $p_0, p_1 \in HF^*(\mathcal{D}(K), \mathcal{T}_-)$ and $t_0, t_1 \in HF^*(\mathcal{T}_+, \mathcal{D}(K))$, whose products satisfy

$$\mu_2(p_j, t_i) = \delta_{i,j},$$

since, up to shifts, this yields (mutually orthogonal) projections and inclusions of $\mathcal{T}_+ \to \mathcal{D}(K)$. By deforming the Lefschetz fibration if necessary, it suffices to prove the result for the standard Lefschetz fibration $\pi: \mathbb{C}^n \to \mathbb{C}$, $\pi: (z_1, \ldots, z_n) \mapsto \sum z_j^2$. The Lagrangians $\mathcal{T}_\pm$ project to the arcs shown in Figure [15] hence in particular agree with small perturbations of $i\mathbb{R}^n$ in a neighbourhood of the origin in $\mathbb{C}^n$. We can therefore perform the Floer-theoretic computation in $T^*\mathbb{R}^n$, in which
case we have a natural isomorphism $HF^*(\mathcal{T}_+, \mathcal{T}_-) \cong H^*_c(\mathbb{R}^n)$. Using the fact that $DS^{n-1}$ meets $i\mathbb{R}^n$ cleanly along $S^{n-1}$, we have a commutative diagram

$$
\begin{array}{ccc}
HF^*(DS^{n-1}, \mathcal{T}_-) & \otimes & HF^*(\mathcal{T}_+, DS^{n-1}) \\
\downarrow & & \downarrow \\
H^*(S^{n-1}) \otimes H^*(\mathbb{R}^n, \mathbb{R}^n \setminus S^{n-1}) & \rightarrow & H^*_c(\mathbb{R}^n).
\end{array}
$$

(If $U$ is an open neighbourhood of $S^{n-1} \subset i\mathbb{R}^n$, the plumbing model for clean intersections is usually stated with morphism groups $C^*(U)$ and $C^*(U, \partial U)$, cf. Proposition 5.11; we have used excision to identify $H^*(U, \partial U) \cong H^*(\mathbb{R}^n, \mathbb{R}^n \setminus S^{n-1})$.) The classes $p_0$ and $p_1$ can now be chosen to be generators of the two non-zero graded components of $HF^*(DS^{n-1}, \mathcal{T}_-) \cong H^*(S^{n-1})$, with $\iota_0$ and $\iota_1$ their Alexander-Lefschetz duals.

Consider now a closed exact Lagrangian $L \subset M$:

**Lemma A.23.** If $L$ meets $K$ at a single point, there is a quasi-isomorphic $DL \cong \mathcal{T}$.

**Proof.** Polterovich proved that, if Lagrangian submanifolds $K, L \subset M$ meet transversely at a single point, there is a Lagrangian cobordism $\Gamma \subset M \times \mathbb{C}$ between $K \sqcup L$ and the Lagrange surgery $K \# L$, fibred over a tripod (figure $\mathcal{Y}$, thickened at the vertex) with the surgery lying over a small neighbourhood of the trivalent vertex and the Lagrangians $K, L, K \# L$ lying over the three ends (see [35, 11]). Supposing further that $K \cong S^n \subset M$ is a Lagrangian sphere, this cobordism can be extended inside the Lefschetz fibration $E$ with fibre $M$ and vanishing cycle $K$ by continuing the edge labelled by $K$ into the critical point (this was also used by Biran and Cornea [12]). This yields a Lagrangian $TL \subset E$, which after Hamiltonian isotopy is horizontally admissible with ends fibred by copies of $L$ and the monodromy image $L \# K = \tau_K(L)$. The Lemma will follow from the claim that $L_\gamma$ is Hamiltonian isotopic to $TL$, together with Lemma [A.12] which asserts the equivalence of $TL$ with the thimble $\mathcal{T}$.

The Hamiltonian isotopy is a consequence of the $\lambda$-Lemma. Given a hyperbolic critical point $(x, y) = (0, 0) \in \mathbb{R}^k \times \mathbb{R}^m$ of a flow $(\phi_t)$, with local unstable manifold $W^u = \{0\} \times \mathbb{R}^m$ and stable manifold $W^s = \mathbb{R}^k \times \{0\}$, the $\lambda$-Lemma (see e.g. [10]) asserts that if $\Delta$ is an $m$-disc transverse to $W^s$, then its image under $\phi_R$, for sufficiently large $R \gg 0$, is the graph of a function $\phi : W^u \rightarrow \mathbb{R}$ with values and derivatives bound by $e^{\lambda T}$. (Schematically, although the given flow takes exponential time into the origin, one can consider the flow associated to a system
whose critical point is shifted slightly, and then reparametrise the flow-lines to have uniformly bounded time.) In the Lefschetz fibration, apply the Hamiltonian flow of $\text{Im}(\pi)$, which is the gradient flow of $\text{Re}(\pi)$. This flows the complement of an open neighbourhood of $-1 \in \gamma$ into the right half-plane, and the $\lambda$-lemma implies that the resulting fibred Lagrangian with boundary can be completed to a piecewise-smooth Lagrangian submanifold which contains a compact subset of the Lefschetz thimble, and which after smoothing is Hamiltonian isotopic to $L_\gamma$. That smoothing yields the surgery $T_L$ presented in fibred position, cf. Figure 16.

\[\blacksquare\]

A.13. The fibred twist. It would be natural, next, to consider the generalisation of the discussion of the previous section to the Morse-Bott case; for exact Morse-Bott fibrations with globally defined parallel transport that is relatively straightforward. However, to obtain a theory applicable to the geometric setting relevant to symplectic Khovanov cohomology with minimal further technical development, we consider the following setting introduced at the beginning of Appendix A. Namely, let $M$ be a Liouville domain, $W: E \to \mathbb{C}$ an exact Lefschetz fibration with a unique critical point and fibre $N$, and $W': E' \to \mathbb{C}$ an exact Landau-Ginzburg model with fibre $M'$. Assume there is an exact inclusion

\[(A.40)\quad M \times E \subset E'\]

compatible with the maps to $\mathbb{C}$. We now generalise the construction of the previous section to this setting. First, associated to each Lagrangian $L \subset M$, we obtain a thimble $T_L$ which is an object of $\mathcal{F}_W'$, by taking the product with the thimble $\mathcal{F}$ of $E$ which projects to the real axis.

Lemma A.24. The assignment $L \to T_L$ extends to a fully faithful embedding

\[(A.41)\quad \mathcal{J}: \mathcal{F}_M \to \mathcal{F}_W'.\]

Proof. We fix the Morse $f_{R^n}$ on $\mathbb{R}^n$ used in the previous section, with a unique minimum on the unit sphere. This induces an inclusion

\[(A.42)\quad CF^*(L_0, L_1) \subset CF^*(L_0 \times K, L_1 \times K),\]

which defines the $A_\infty$-homomorphism $\mathcal{J}$. As in Proposition A.14 holomorphic curves in $M$ which are regular are regular as curves in $M \times E$, hence in $E'$.

\[\blacksquare\]

Let $K \subset N = W^{-1}(1)$ denote the vanishing cycle as before. As in Section 3.2 we assign to $K$ an $\mathcal{F}_{M'}$-$\mathcal{F}_M$-bimodule $\mathcal{K}$ by considering Floer theory in $M'$. The proof of the next result is a straightforward generalisation of the proofs of Lemmas A.21 and A.22 using the fact that products of regular holomorphic curves are regular.

Lemma A.25. In the setting and notation of Lemma A.24,

1. The pullback of the bimodule $\mathcal{R}$ by $\mathcal{J}$ is quasi-isomorphic to $\mathcal{K}$.
For each Lagrangian \( L \subset M \), there is a quasi-isomorphism
\[
\mathcal{D}(L \times K) \cong \mathcal{T}L \oplus \mathcal{T}[n - 1].
\]

Consider now a closed exact Lagrangian \( L' \subset M' \), and a Liouville domain \( N_{L'} \) equipped with a Liouville inclusion \( N_{L'} \times M \subset M' \), containing the image of \( L' \). Moreover, we assume that \( N \subset N_{L'} \), and that the inclusion \( E \times M \subset E' \) extends to an inclusion
\[
E_{L'} \times M \subset E'
\]
where \( E_{L'} \to \mathbb{C} \) is a Lefschetz fibration with fibre \( N_{L'} \) all of whose critical points are contained in \( E \).

**Lemma A.26.** If \( L' \) meets \( K \times L \) cleanly along a section of the projection to \( K \), there is a quasi-isomorphic \( \mathcal{D}L' \cong \mathcal{T}L \).

**Proof.** This is the Morse-Bott version of Lemma A.23; the construction of the surgery and the cobordism takes place in the product of the Lefschetz fibration \( E_{L'} \) with \( M \), in which the gradient flow of the real part of \( W \) is integrable. The result follows by the Morse-Bott case of the \( \lambda \)-Lemma (see e.g. the proof of [8, Theorem 25]).

To conclude, we prove the version of the exact sequence of a twist that we use in the main part of the paper:

**Proof of Proposition A.2.** By assumption, every object of \( A' \) either satisfies the hypothesis of Lemma A.23 or is a product of a Lagrangian in \( M \) with \( K \). We conclude that the functor \( \mathcal{D}: A' \to \mathcal{T}W \) (composed with the inclusion \( \mathcal{T}W \to \text{Tw} \mathcal{T}W \)) is equivalent to a functor which factors through the image of \( \mathcal{T} \). Since \( \mathcal{T} \) is a fully faithful embedding, we may therefore fix a functor \( \cap: \text{Tw}A' \to \text{Tw}A \) so that we have a diagram which commutes up to equivalence:

\[
\begin{array}{ccc}
\text{Tw}A' & \xrightarrow{\cap} & \text{Tw}A \\
\mathcal{D} \downarrow & & \downarrow \mathcal{T} \\
\text{Tw} \mathcal{T}W \\
\end{array}
\]

Theorem A.4 therefore implies that the graph bimodule \( \Delta^\text{Tw}A \) is quasi-isomorphic to the cone of the unit \( \Delta^\text{Tw}A \to \cap \Delta^\text{Tw}A \). It remains to show that we have an equivalence
\[
\cap \Delta^\text{Tw}A \cong \Delta^\text{Tw}A' 
\]
i.e. that \( \cap \) is adjoint to \( \cup \), where \( \cup \) is the Künneth-type functor representing \( \mathcal{K} \). Observe that the two-sided pullback of \( \Delta^\mathcal{T}W \) by \( \mathcal{T} \) is equivalent to \( \Delta^A \) because \( \mathcal{T} \) is a fully faithful embedding, so \( \cap \Delta^A \) is equivalent to \( \mathcal{D} \Delta^\mathcal{T}W \). Corollary A.16 implies that \( \mathcal{R}_\mathcal{T} \) is represented by \( \mathcal{D} \Delta^\mathcal{T}W \), but Lemma A.25 implies that \( \cup \) is represented by \( \mathcal{R}_\mathcal{T} \), which proves the existence of the adjunction, hence establishes the equivalence in Equation (A.45).
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