Diffeomorphism Group Representations
in Nonrelativistic and Relativistic Quantum Theory:
Some Future Directions

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Abstract. Diffeomorphism groups and their unitary representations unify the description
of a wide variety of diverse nonrelativistic (Galilean) quantum systems. In recent work, we
have suggested an approach to relativistic quantum field theory that begins with hierarchies of
diffeomorphism group representations and the corresponding current densities. We introduce
(noncovariant) creation and annihilation fields that intertwine the representations, and restore
the (relativistic) spacetime symmetry by redefining local quantum fields and the Hamiltonian
operator in terms of the intertwining fields and current densities. Here we outline the main
ideas, and point to next steps and possible future directions of this line of inquiry.

This contribution is dedicated to the memory of Irving Segal, who inspired a generation
of mathematical physicists. The first author thanks the organizers of Group32, particularly
Vladimir Dobrev and Patrick Moylan, for the opportunity to present these ideas on the occasion
the 100th anniversary of Segal’s birth.

1. Introduction

The distinct irreducible, unitary representations of the group of compactly-supported $C^\infty$
diffeomorphisms of physical space, and its semidirect product with the additive group of
compactly-supported, real-valued $C^\infty$ functions, describe the quantum kinematics of a wide
variety of physical systems. In such representations, the infinitesimal generators of appropriate
1-parameter subgroups are the self-adjoint operators describing mass density (indexed by scalar
functions) and momentum flux density (indexed by vector fields). These form a local current
algebra. Particular representations describe configuration spaces and particle statistics for $N$-
particle systems and infinite systems [1][2][3].

This perspective leads naturally to an understanding of how and why exotic statistics come
about. Examples beyond Bose and Fermi statistics include parastatistics in physical spaces of
dimension greater than one, and the "braid group" statistics of anyons and nonabelian anyons
in two-dimensional spaces.

In earlier work, we found that we could introduce creation and annihilation fields naturally
intertwining hierarchies of diffeomorphism group representations describing
\( N \) identical particles (for \( N = 1, 2, 3, \ldots \)). We then recover as a consequence not only canonical commutation
and anticommutation relations (for bosonic and fermionic hierarchies respectively), but also
\( q \)-commutations relations in two-space (for anyonic hierarchies) where \( q \) is the anyonic phase
change under a single clockwise exchange of particles [4].

More recently we have proposed to study relativistic quantum fields making use of such
hierarchies of diffeomorphism group representations. We restore the (relativistic) spacetime
symmetry by redefining local quantum fields and the Hamiltonian operator in terms of the
intertwining fields and current densities [5]. After outlining the main ideas, we point here to
next steps and possible future directions for such an approach.

2. Diffeomorphism groups and Galilean quantum theory

To establish notation in a general framework, let \( \Sigma \) be the manifold of physical space. Of course
\( \Sigma \) can also be regarded as a submanifold of spacetime, e.g., with \( t = 0 \). In Galilean theory, \( \Sigma \) is
independent of the observer’s inertial frame of reference.

Let \( D = C^\infty_0(\Sigma) \) be the group of compactly-supported smooth real-valued functions on
\( \Sigma \), under pointwise addition. Let \( K = \text{Diff}_0(\Sigma) \) be the group of compactly supported \( C^\infty \)
diffeomorphisms of \( \Sigma \), under composition. Let \( G = D \times K \) be the natural semidirect product of
these groups. For \((f, \phi) \in D \times K\), write a continuous unitary representation (CUR) of \( G \) in a
Hilbert space \( \mathcal{H} \) as \( U(f)V(\phi) \).

Under very general conditions, we may realize the Hilbert space as \( \mathcal{H} = L^2(\Gamma, \mathcal{M}) \), where:
\( \Gamma \) is a configuration space whose elements (denoted \( \gamma \)) are continuous linear functionals on \( D \);
\( \mathcal{M} \) is a complex inner product space (accommodating vector-valued wave functions); elements
of \( \mathcal{H} \) are square-integrable functions \( \Psi \) on \( \Gamma \) taking values in \( \mathcal{M} \); and \( \mu \) is a measure on \( \Gamma \)
quasiinvariant under diffeomorphisms of \( \Sigma \) (which act naturally on \( \Gamma \)). The representation then
takes the form

\[
[U(f)](\gamma) = \exp i\langle \gamma, f \rangle \Psi(\gamma), \quad [V(\phi)](\gamma) = \chi_\phi(\gamma)\Psi(\phi\gamma)\sqrt{\frac{d\mu_\phi}{d\mu}(\gamma)},
\]

(1)

where \( \langle \gamma, f \rangle \) denotes the value of \( \gamma \) at \( f \); \( \phi\gamma \) refers to the group action of \( K \) on \( \Gamma \); \( d\mu_\phi/d\mu \) is
the Radon-Nikodym derivative of the transformed measure with respect to the original one; and
\( \chi_\phi(\gamma) \) is a unitary 1-cocycle acting in \( \mathcal{M} \).

Each of these can be given a fairly direct physical interpretation in quantum mechanics.

In particular, different possible choices of \((\Gamma, \mu)\) describe: \( N \)-particle systems, for distinct
values of \( N \); tightly-bound composite particles (quantum dipoles, quadrupoles, etc.); locally
finite systems having infinitely many particles, carrying Poisson measures, Gibbs measures,
or other measures; systems of infinitely many particles with accumulation points; and some
systems of extended configurations such as vortex patches, filaments, and tubes. In general, for
infinite-dimensional configuration spaces, inequivalent quasiinvariant measures describe distinct
dynamical possibilities.

Furthermore, different (noncohomologous) choices of the cocycle \( \chi_\phi(\gamma) \) describe: indistin-
guishable particles satisfying bosonic, fermionic, and in 2-dimensional space, anyonic (braid
group) statistics; particles obeying parastatistics, and in 2-dimensional space, nonabelian braid
statistics; and a certain class of nonlinear time-evolution possibilities in quantum mechanics and
associated nonlinear gauge transformations [5].
We thus have a rather beautiful unifying description of a wide variety of distinct nonrelativistic quantum systems as representations of a local symmetry group.

The physical interpretation of a representation is obtained via the associated local current algebra of self-adjoint operators in $\mathcal{H}$. Define

$$\rho(f) = m \lim_{s \to 0} \frac{U(sf) - I}{is}, \quad J(g) = \hbar \lim_{s \to 0} \frac{V(\phi^s g) - I}{is},$$

where $f \in \mathcal{D}$, $g \in \text{vect}_0^\infty(\Sigma)$, and $\phi^s$ is the flow generated by the vector field $g$ under the real parameter $s$; $m$ is a unit mass, and $\hbar$ is Planck’s constant (over $2\pi$). Then $\rho$ describes the (space-averaged) mass density, and $J$ the (space-averaged) momentum flux density. The Lie algebra of local currents is given by:

$$[\rho(f_1), \rho(f_2)] = 0, \quad [\rho(f), J(g)] = i\hbar \rho(L_g f),$$

$$[J(g_1), J(g_2)] = -i\hbar J([g_1, g_2]),$$

where $(L_g f)$ is the Lie derivative of $f$ in the direction of $g$, and $[g_1, g_2]$ is the Lie bracket of vector fields.

Why do we call this Galilean quantum mechanics, when we have not made explicit use of the Galilei group? It is because we have the ability to introduce the Galilean symmetry without changing the original spatial manifold $\Sigma$ on which the diffeomorphism group is defined, and so that general diffeomorphisms of $\Sigma$ do not violate Galilean causality. A Lorentz boost, on the other hand, maps $\Sigma$ into a different spacelike surface $\Sigma'$, and general diffeomorphisms of $\Sigma$ do not respect the Minkowskian causal structure of spacetime.

More specifically, consider how a diffeomorphism $\phi$ of an $s$-dimensional spatial manifold $\Sigma$ in spacetime could be extended to a diffeomorphism $\Phi$ of the $(s + 1)$-dimensional spacetime. Define a diffeomorphism $\Phi$ of Minkowski spacetime $M^{(s+1)}$ to be causal if for any pair of points $x, y \in M^{(s+1)}$, it preserves the sign $(+, 0, -)$ of $(x - y)^2 = (x - y)_\mu(x - y)^\mu$ (using the Minkowskian metric). Now in $(1 + 1)$-dimensional Minkowski spacetime, an infinite-dimensional group of causal diffeomorphisms does, in fact, exist: we can permit $\Phi$ to act independently on each of the two light cone coordinates defining a point in the spacetime. But in Minkowski spacetimes with $s > 1$, the causal diffeomorphisms are limited to the finite-dimensional group of Poincaré transformations together with uniform dilations. Thus with the exception of $(1 + 1)$-dimensional spacetime, general diffeomorphisms disrupt the Minkowskian causal structure. In contrast, in Galilean spacetime, diffeomorphisms that act on spatial coordinates only, even possibly in a time-dependent way, respect the causal structure.

3. Intertwining fields and a relativistic quantum field

3.1. Intertwining creation and annihilation fields

We have mentioned that irreducible, unitary diffeomorphism group representations fall naturally into hierarchies intertwined by creation and annihilation fields. For example, consider a family of $N$-particle representations, where $N = 0, 1, 2, ...$. For fixed $N$, call the Hilbert space $\mathcal{H}_N$, and the unitary representation $U_N(f)V_N(\phi)$. We call the family $\{U_N(f)V_N(\phi)\}$ a hierarchy if for each $N$ there exists an operator-valued distribution $\psi_N^\ast : \mathcal{H}_N \to \mathcal{H}_{N+1}$ (the creation field), such that $(\forall h \in \mathcal{D}, f \in \mathcal{D}, \phi \in \mathcal{K})$,

$$U_{N+1}(f)\psi_N^\ast(h) = \psi_N^\ast(U_{N=1}(f)h)U_N(f), \quad V_{N+1}(\phi)\psi_N^\ast(h) = \psi_N^\ast(V_{N=1}(\phi)h)V_N(\phi).$$

That is, beginning with an $N$-particle state, if we create a new particle in state $h$ and then transform the resulting $(N + 1)$-particle state by the unitary group representation $U_{N+1}$...
(respectively, $V_{N+1}$), the result is the same as if we first transform the $N$-particle state by $U_N$ (resp. $V_N$), and then create a new particle in the state obtained by transforming the 1-particle state $h$ by $U_{N-1}$ (resp. $V_{N-1}$). This is the most natural possible way in which a creation field could act. The field $\psi^*$ then acts in the Fock space $H = \bigoplus H_N$ by $\psi^* = \bigoplus \psi_N^*$. The adjoint operator $\psi$ is the annihilation field, mapping $H_N$ to $H_{N-\infty}$ for each $N$. These intertwining operators create and annihilate objects (in this case, point particles) of the same kind. The construction is nontrivial because creation and annihilation within a hierarchy must respect the particle statistics (bosonic, fermionic, or anyonic).

When $H_1 = L^2(\mathbb{R}^3, \mathbf{C})$, as in the 1-particle representation of $D \times K$, we write $\psi^*(h) = \int \psi^*(x) h(x) d^3x$. In effect, $\psi^*(x)$ creates a particle at $x \in \Sigma$, and its adjoint $\psi(x)$ annihilates a particle at $x$. In consequence of Eqs. (4), the creation and annihilation fields also obey natural commutator brackets with the density and current operators. Moreover, $\psi^*$ and $\psi$ satisfy a bracket with each other: canonical commutation relations when they intertwine $N$-particle Bose representations, anticommutation relations when they intertwine $N$-particle Fermi representations, and $q$-commutation relations when they intertwine $N$-anyon representations (in 2-space). These ideas were introduced in [4], where they were used to obtain the anyonic $q$-commutation relations.

3.2. The free relativistic neutral scalar Bose field

Let $F$ be fixed inertial frame of reference in Minkowski space, and let $\Sigma_F$ be the spacelike surface defined by $t = 0$ in $F$. Consider now the $N$-particle unitary representations of $G(\Sigma_F)$ satisfying bosonic exchange symmetry. These form a hierarchy as just discussed, and the intertwining operators are the usual creation and annihilation fields intertwining $N$-particle subspaces of Fock space. But these are not the components of the relativistic neutral scalar field describing free bosons; rather, they correspond to 2nd-quantized, “non-relativistic” fields satisfying canonical commutation relations. These intertwining fields were already obtained by Schweber [6] in the context of the free relativistic neutral scalar Bose field, in order to consider particle measurement operators in that context. The approach we are taking here is basically to reverse the direction of the construction of Wightman and Schweber [7] in the free field case.

Considering the free neutral scalar relativistic field in the Fock representation, we use the following notational conventions. In Minkowski space-time, $x^\mu = (x^0, \mathbf{x})$, with $\mu = 0, 1, 2, 3$ and $x^0 = ct$; the metric tensor $g_{\mu\nu} = \text{diag}[1, -1, -1, -1]$. The covariant momentum 4-vector is $p_\mu = (p_0, \mathbf{p})$, where $p_0 = E/c$; $E$ is the energy. The wave number 4-vector is $k = (k_0, \mathbf{k})$, where with $E = \hbar \omega$, we have $k_0 = \omega/c = E/\hbar c$, and $\mathbf{p} = \hbar \mathbf{k}$. Since $E^2 = p^2 c^2 + m^2 c^4$, we have $\omega^2 = k^2 c^2 + (m^2 c^4/\hbar^2)$. For a given value of $\mathbf{k}$, write also $k_0 = \omega_k/c$, where $\omega_k = \sqrt{k^2 c^2 + (m^2 c^4/\hbar^2)}$. With $a_k^*$ and $a_k$ respectively the creation and annihilation operators for the free relativistic neutral scalar field in the Fock representation, they satisfy the (relativistic) commutator bracket $[a_k, a_k^*] = \omega_k \delta^{(3)}(\mathbf{k} - \ell)$.

Following Schweber, particle measurement operators are obtained from the (nonrelativistic) fields $\phi_1$ and $\phi_1^*$, with

$$\phi_1(x) = \int_{k_0 > 0} \frac{d^3k}{k_0} \frac{1}{(2\pi)^{3/2}} (k_0)^{1/2} e^{-ikx} a_k^*;$$

setting $t = 0$ and $k_0 = \omega_k$, we write

$$\phi_1(x, 0) = \int \frac{d^3k}{\omega_k^{3/2}} \frac{1}{(2\pi)^{3/2}} e^{-ikx} a_k.$$

Then $\phi_1$ and $\phi_1^*$ satisfy the same equal-time canonical commutation relations as the nonrelativistic bosonic intertwining fields $\psi$ and $\psi^*$ discussed above.
This means that we can reverse the direction of argument! That is, we can obtained $\psi$ and $\psi^*$ by intertwining diffeomorphism group representations at $t = 0$ is a specific inertial reference frame. Then we can construct the relativistic field from relativistic creation and annihilation operators obtained from the inverse transform of the equations above. In particular,

$$\frac{1}{(2\pi)^{3/2}} \int d^3 x \psi(x) e^{ik \cdot x} = a_k/\omega_k^{1/2},$$

and likewise for the adjoints. Finally, the relativistic field is

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int_{k_0 > 0} \frac{d^3 k}{k_0} (a_k e^{-ik \cdot x} + a_k^* e^{ik \cdot x}).$$

Note that the entire construction takes place in the fixed inertial frame of reference $F$. In this frame, the “nonrelativistic” and the “relativistic” fields (for any value of $c$) actually coexist quite harmoniously. Once the relativistic fields have been constructed, we may bring in Lorentz symmetry and obtain a fully covariant quantum field theory. And the resulting theory does not depend on the inertial frame of reference with which we started, in which we selected a spacelike surface, introduced the diffeomorphism group, and intertwined the hierarchy of representations.

In our recent work [8] we also write the relativistic free Hamiltonian in the neutral scalar Bose theory explicitly in terms of the self-adjoint, “nonrelativistic” particle density and current density operators. Modeled in a way on this example, we propose a new approach to relativistic quantum field theory that begins with diffeomorphism group representations and potentially accommodates a considerable variety of possibilities.

### 4. General framework and future directions

#### 4.1. Main steps in a general framework

It is the spacetime symmetry group that describes how to relate inertial frames of reference to each other. Relativistic quantum field theory often begins with fields assumed to be covariant with respect to a unitary representations of the Poincaré group; special relativity thus constrains the theory from the outset. But with the exception of (1 + 1)-dimensional spacetime, general diffeomorphisms disrupt the Minkowskian causal structure.

In the approach we propose, we specify a fixed frame of reference and a spatial manifold, in which actual quantum measurements are to take place. This specification precedes any of the constructions that lead to configuration spaces, and does not yet require specifying the spacetime symmetry. Thus we retain the possibility of considering compactified space dimensions, nontrivial spatial topology, and symmetries other than Poincaré symmetry. We also anticipate the idea that the group of diffeomorphisms of a spacelike surface can serve as an invariance group in some approaches to canonical quantization of general relativity.

The main steps we envision are as follows:

1. Choose an inertial frame of reference $F$ (the frame of the observer).
2. Specify the spacetime manifold $M_F$, as observed from $F$.
3. Introduce coordinates in $M_F$, letting $x$ refers to “space” and $t$ to “time.” A “spacelike” surface $\Sigma_F$, coordinatized by $x$, is obtained by setting $t = 0$ in $M_F$. Note that $\Sigma_F$ may be a manifold or a manifold with boundary and/or singularities, and may have nontrivial homotopy.
4. Define the group $G = D \times K$ with respect to $\Sigma_F$ and consider its continuous unitary representations.
5. Identify a hierarchy of representations for consideration – there will be many to choose from – describing configurations consisting of entities of the same type (to be interpreted as quanta of the same field). Introduce fields as operators intertwining the unitary representations in the hierarchy.
6. Construct the associated Lie algebra of self-adjoint mass density and momentum flux density operators, with which measurements in $F$ can be described. At this stage in the general development we have a full description of the field quanta. We do not yet have the spacetime symmetry, the field respecting that symmetry, do not yet have the field – nor do we have a description of the dynamics, which is to be provided as usual by a Hamiltonian operator.

7. Specify the spacetime symmetry group, whose elements relate different inertial frames of reference; i.e., given a different frame $F'$, they transform $M_F$ to $M_{F'}$.

8. Define a field that is “relativistic” with respect to the specified spacetime symmetry, making use of the intertwining fields. Now the configuration-space entities in the hierarchy of diffeomorphism group representations are interpreted as quanta of this field as observable in the reference frame $F$.

9. Write the Hamiltonian $H$ describing the (relativistic) dynamics, in terms of the relativistic quantum field. It may also be expressed explicitly in terms of the local currents together with the original (nonrelativistic) intertwining fields.

The framework (i.e., the field operators, the Hamiltonian) should not ultimately depend on the particular reference frame $F$.

4.2. Comments and future directions of study

Everything that we do in the fixed frame of reference $F$ is unconstrained by relativistic covariance or invariance. That is, it should all be compatible with different possible choices of the spacetime symmetry group. In $F$ the measurement operators can be (in fact, will generally need to be) noncovariant and nonlocal in the spacetime, although local in space at a fixed time.

The shapes of spacetime regions (or of spacelike surfaces such as hyperplanes in Minkowskian spacetime, and of regions within those surfaces) must change with the frame of reference of the observer. Therefore, to describe quantum measurement as it takes place in a particular frame of reference, we are forced at some stage to deal with noncovariant objects. Thus it makes sense to begin with them.

One research direction is thus to explore alternate choices of spacetime symmetry. Of course, the corresponding operators for measurements taken in a different inertial reference frame will be different (depending on the spacetime symmetry). They are not obtainable directly from the first set of operators until after covariant fields (or some other way of encoding the spacetime symmetry) have been introduced.

We think it will be natural to regard the different spacetimes $M_F$ as fibers in a bundle over a base space of reference frames. On each fiber, one builds a copy of the theory: hierarchies of diffeomorphism group representations, intertwining fields, etc. The spacetime symmetry will eventually establish the isomorphism of the theories in each fiber. Another direction of research is to define and elaborate on this perspective.

We have available (again, in a specific frame $F$) the extensive development in the literature including configuration spaces (of particles or extended objects), measures on configuration spaces, and statistics based on the topologies of configuration spaces. Among other results, we have shown how particles with and without spin, point particles as well as filaments and other kinds of extended objects, can all be described in a natural way within the framework of diffeomorphism group representations. So another general research direction is to study fields whose quanta are entities that are extended objects. Intertwining fields can create vortex loops, closed and open strings, or more general embedded manifolds. They can also create linked or knotted configurations. Fock-like representations of relativistic fields whose quanta have spatial extent are thus of great interest from this point of view. And the approach is general enough to include higher-dimensional spacetimes (e.g., 10- or 26-dimensional) with some of the spatial dimensions compactified.
It would be interesting to see what could be learned about the dynamics of extended quantum objects, employing different approximations to the Hamiltonian written in terms of the local currents used to describe such objects.

As we have discussed the free relativistic neutral boson field with Lorentz symmetry, it is natural to address the relativistic fermion case. This is more difficult, as we must take account of particle spin. Furthermore, it is not just a matter of intertwining \( N \)-particle diffeomorphism group representations in the Fock space of antisymmetric wave functions. The hierarchy of fermionic \( N \)-particle representations of the group (i.e., representations with totally antisymmetric exchange statistics) leads to intertwining fields satisfying equal-time canonical anticommutation relations acting in the Fock space. But we must take account of particle spin; in the spinless case, after introducing the relativistic Hamiltonian, one cannot satisfy the important property of local causality. Thus the challenging problems that remain to be addressed include the application of our approach to relativistic fields describing particles with spin, the manner in which the spin-statistics theorem is manifested, and the treatment of particle production. We note that ruling out spinless fermions does not mean that antisymmetric \( N \)-particle representations of diffeomorphism groups are irrelevant. When we consider spin 1 bosons, for example, we need to include both symmetric and antisymmetric spatial wave functions in order to allow for all of the possible spinor symmetries under particle exchange.

We anticipate the value of studying interacting relativistic quantum fields constructed from diffeomorphism group representations.

Finally, another natural direction in which to extend these ideas is to non-Fock representations of \( G \) describing infinitely many particles.

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