Feedbacks between strategies and the environment are common in social-ecological, evolutionary ecological and even psychological-economic systems. Using common resources is always a dilemma for community members, like the tragedy of the commons. Here, we consider replicator dynamics with feedback-evolving games, where the pay-offs switch between two different matrices. Although each pay-off matrix on its own represents an environment where cooperators and defectors cannot coexist stably, we show that it is possible to design appropriate switching control laws and achieve persistent oscillations of strategy abundance. This result should help guide the widespread problem of population state control in microbiological experiments and other social problems with eco-evolutionary feedback loops.

1. Introduction

Game theory is based on the principle that individuals make rational decisions regarding their choice of actions given suitable incentives [1]. In practice, the incentives are represented as strategy-dependent pay-offs. Evolutionary game theory extends game-theoretic principles to model dynamic changes in the frequency of strategies [2,3]. Replicator dynamics is one commonly used framework for such models [4]. In replicator dynamics, the frequency of strategists changes over time as a function of the social makeup of the community. The frequency of a strategy increases when the fitness of those who adopt it is greater than the average fitness of the population.

But individual actions do not only modify the social makeup, they very often modify the environment as well [5]. The strategies that individuals employ impact
the environment through time, and the state of the environment in turn influences the pay-offs of the game. This environment-dependent feedback occurs across scales from microbes [6–9] to humans [10–16] in the public goods game and the tragedy of the commons [17]. The tragedy of the commons is a fundamental problem that is intensively studied [18–21]. A well-known example is the danger of overfishing. Fishermen are motivated to catch the maximum amount of fish because restraint could only work if all others were behaving similarly. Otherwise, fish are driven to extinction, which is the worst scenario for everyone. Similarly, in overgrazing of common pasture lands, an individual’s short-term benefit seems to be in conflict with the long-term interest of a larger population. The common feature of these cases is that human activity influences the actual state of resources, which has a negative feedback for not only those who degrade the environment but also for the whole community.

The feedback loop between the environment and individual behavioural strategies also appears in microbial populations [22], including microbes, bacteria and viruses. Among microbes, feedback may arise due to fixation of inorganic nutrients given depleted organic nutrient availability, the production of extracellular nutrient-scavenging enzymes like siderophores or enzymes like invertase that hydrolyse diffusible products, and the release of extracellular antibiotic compounds. The incentive for public goods production changes as the production influences the environmental state.

Such joint influence occurs in human societies [23], for example, when individuals decide to vaccinate or not [24]. Decisions not to vaccinate have been linked most recently to outbreaks of otherwise-preventable childhood infectious diseases in northern California. These outbreaks modify the subsequent incentives for vaccination. Such coupled feedback also arises in public goods dilemmas involving water [25] or other resource use such as antibiotic use [26]. In a period of replete resources, there is less incentive for restraint. However, overuse in times of replete resource availability can lead to depletion of the resource and changes in incentives.

A new framework of replicator dynamics with feedback-evolving games has been proposed [10] to characterize the phenomenon that the environment and individual behaviour coevolve in many social-ecological and psychological-economic systems [27]. The environmental feedback can result in oscillating dynamics for both the environment quality and strategy states, for example, multiple waves of infections, as seen in the COVID-19 pandemic [28]. This unified approach to analyse and understand feedback-evolving games is called eco-evolutionary game theory. It denotes the coupled evolution of strategies and the environment. The cumulative feedback of decisions can subsequently alter environment-dependent incentives, thereby leading to new dynamical phenomena and new challenges for control.

Using common resources always imposes a dilemma for community members. While cooperators restrain themselves and help maintain the proper state of resources, defectors demand more than their supposed share for a higher pay-off. The nature of the feedback and the rate of ecological changes can relax or aggravate social dilemmas and promote persistent periodic oscillations of strategy abundance and environmental quality. To alleviate the tragedy of the commons, questions that inspire our present work include: how to manage and conserve public resources, which processes drive human cooperation, and how institutions, norms and other feedback mechanisms can be used to reinforce positive behaviours.

Strategies and the environment are coupled. In human societies, the institutions structuring social interactions can be seen as part of the environment that coevolve with strategic behaviours. Many problems need to be solved like pollution control and climate change mitigation, and the proper management of food webs, soil weathering and earth system processes; the psychology of decision-making, species interaction and climate change action.

In this paper, we show some simple examples where the original system has unstable equilibria under two distinct environments, and we can trap the system between the unstable equilibria to achieve a dynamic steady state for the system. We hope this can provide some intuition for policy-making in social dilemmas. Controlled decision-making is likely to be costly but will allow individuals to choose optimal behaviour.
2. Model

Consider a two-player asymmetrical game of normal form. Player 1 has the strategy set \( S = \{s_1, s_2\} \), and player 2 has the strategy set \( T = \{t_1, t_2\} \). When the pair of strategies \((s_i, t_j)\) is chosen, the pay-off to player 1 is \( a_{ij} = u_1(s_i, t_j) \) and the pay-off to player 2 is \( b_{ij} = u_2(s_i, t_j) \). Then the values of the pay-off functions can be given by two matrices [29], respectively, for the two types of players 1 and 2,

\[
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.
\]

We put these two matrices together in the following canonical way for a bimatrix game.

\[
M = A \oplus B = \begin{bmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) \end{bmatrix}.
\]

Now we consider a bimatrix game where the pay-off bimatrix is not constant and depends on the environment.

\[
M(t) = \begin{bmatrix} (a_{11}(t), b_{11}(t)) & (a_{12}(t), b_{12}(t)) \\ (a_{21}(t), b_{21}(t)) & (a_{22}(t), b_{22}(t)) \end{bmatrix}.
\]

For simplicity, we start with the combination of two constant bimatrices,

\[
M(t) = \eta(t)M_1 + (1 - \eta(t))M_{II},
\]

where

\[
M_1 = \begin{bmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) \\ (a_{21}, b_{21}) & (a_{22}, b_{22}) \end{bmatrix} \quad \text{and} \quad M_{II} = \begin{bmatrix} (a'_{11}, b'_{11}) & (a'_{12}, b'_{12}) \\ (a'_{21}, b'_{21}) & (a'_{22}, b'_{22}) \end{bmatrix},
\]

and \( \eta(t) \) is the environment evolving indicator as the environment switches between two states I and II. There are many possibilities for the function \( \eta(t) \). One of them is the following periodic function (figure 1) [30].

\[
q(t) = \begin{cases} 0, & kT \leq t < kT + \frac{T}{2}, \quad k = 0, 1, 2, \ldots \\ 1, & kT + \frac{T}{2} \leq t < (k + 1)T, \quad k = 0, 1, 2, \ldots \end{cases}
\]

Now we introduce mixed strategies in our game and assume \( X = (x_1, x_2) \), where \( x_1 \) represents the probability of player 1 choosing strategy \( s_1 = l \) and \( x_2 \) represents the probability of player 1
choosing strategy \( s_2 = r \). We have \( x_1 + x_2 = 1 \).

Similarly we have \( Y = (y_1, y_2) \), where \( y_1, y_2 \) represent the probabilities of player 2 choosing strategies \( t_1 = L \) and \( t_2 = R \). Then
\[
y_1 + y_2 = 1.
\]

We note that identical notations and equations can be used to describe eco-evolutionary dynamics in subpopulations consisting of type 1 and type 2 players.

The pay-off of player 1 when choosing strategy \( s_1 = l \) is
\[
\pi_l(X, Y) = a_{11}(t)y_1 + a_{12}(t)y_2.
\]

The pay-off of player 1 when choosing strategy \( s_2 = r \) is
\[
\pi_r(X, Y) = a_{21}(t)y_1 + a_{22}(t)y_2.
\]

Let \( E(\pi) = x_1\pi_l + x_2\pi_r \) be the expected pay-off for player 1. The replicator equation for \( x_1 \) is
\[
\dot{x}_1 = x_1(1 - x_1)[(a_{11}(t)y_1 + a_{12}(t)y_2) - (a_{21}(t)y_1 + a_{22}(t)y_2)].
\]

Plug in the values of \( \pi_l, \pi_r \), we get
\[
\dot{x}_1 = x_1(1 - x_1)[(a_{11}(t)y_1 + a_{12}(t)y_2) - (a_{21}(t)y_1 + a_{22}(t)y_2)].
\] (2.1)

And the replicator equation for \( x_2 \) is
\[
\dot{x}_2 = -\dot{x}_1,
\]

since \( x_1 + x_2 = 1 \).

Similarly we can write down the pay-offs of strategies \( t_1 = L \) and \( t_2 = R \).
\[
\pi_L(X, Y) = b_{11}(t)x_1 + b_{21}(t)x_2
\]
and
\[
\pi_R(X, Y) = b_{12}(t)x_1 + b_{22}(t)x_2.
\]

The replicator equations for \( y_1 \) and \( y_2 \) are
\[
\dot{y}_1 = y_1(1 - y_1)(\pi_L - \pi_R)
\]
and
\[
\dot{y}_2 = -\dot{y}_1.
\]

Plug in the values of \( \pi_L \) and \( \pi_R \), we get
\[
\dot{y}_1 = y_1(1 - y_1)[(b_{11}(t)x_1 + b_{21}(t)x_2) - (b_{12}(t)x_1 + b_{22}(t)x_2)].
\] (2.2)

Equations (2.1) and (2.2) combined with
\[
\eta(t) = q(t),
\] (2.3)
governs the dynamics of our system.

3. Simple one-dimensional model

Before we try to solve the general two-player bimatrix game \( A \neq B^T \), let us start with the simplified one-dimensional model obtained from assuming \( A = B^T \).

In this case, we have a system governed by the replicator equation of the following form:
\[
\dot{x} = x(1 - x)[a(t)x - b(t)], \quad 0 < x < 1,
\] (3.1)

where \( a(t) = a_{11}(t) - a_{12}(t) - a_{21}(t) + a_{22}(t) \) and \( b(t) = b_{22}(t) - a_{12}(t) \).

Let \( a^* = b(t)/a(t) \). As one can see, this equation has three equilibrium points, 0, 1, \( a^* \). For our purposes, we only consider cases where \( a^* \in (0, 1) \) by assuming the existence of an interior equilibrium.
Based on the signs of $a(t)$ and $b(t)$, we have two possible situations for the phase line of the system.

As figure 2 shows, the dynamics of the system is quite clear if $a(t)$ and $b(t)$ are fixed constants. However, if $a(t)$ and $b(t)$ are changing over time, the dynamics becomes much more complicated.

Let us consider a simple case where $a(t)$ and $b(t)$ switch between two pairs of constants periodically. More specifically, we are interested in the case where the two separate states both have unstable fixed points, $a^*_1$ and $a^*_2$, yet if we choose the period carefully, we can trap the system between $a^*_1$ and $a^*_2$, therefore achieving a dynamic equilibrium by switching between two different unstable systems.

Below we give two explicit examples with specific numbers for all the parameters to gain some intuition.

(a) Symmetric one-dimensional example

Consider two states governed by the following equations. By symmetry, we mean that the term $a(t) = a_{11}(t) - a_{12}(t) - a_{21}(t) + a_{22}(t)$ is identical in both environments.

$$\dot{x} = x(1 - x)(3x - 1)$$

and

$$\dot{x} = x(1 - x)(3x - 2).$$

Both equations are separable and we can solve them explicitly. In particular, we care about the solution between the two unstable fixed points $1/3$ and $2/3$.

$$\frac{dx}{x(1 - x)(3x - 1)} = \left( -\frac{1}{x} + \frac{1}{2} \frac{1}{1 - x} + \frac{9}{2} \frac{1}{3x - 1} \right) dx = dt,$$

$$-\ln x - \frac{1}{2} \ln(1 - x) + \frac{3}{2} \ln(3x - 1) = t + c,$$

$$\frac{(3x - 1)^3}{(1 - x)x^2} = c_1 e^{2t}, \quad \frac{1}{3} < x < 1.$$

This is the solution to equation (3.2) and similarly we can solve (3.3).

$$\frac{dx}{x(1 - x)(3x - 2)} = \left( -\frac{1}{x} + \frac{1}{2} \frac{1}{1 - x} + \frac{9}{2} \frac{1}{3x - 2} \right) dx = dt,$$

$$-\frac{1}{2} \ln x - \ln(1 - x) + \frac{3}{2} \ln(2 - 3x) = t + c,$$

$$\frac{(2 - 3x)^3}{(1 - x)^2 x} = c_2 e^{2t}, \quad 0 < x < \frac{2}{3}.$$

Assume at time $t = 0$, $x$ starts at position $1/3 + \epsilon$, $0 < \epsilon < 1/3$, and the system obeys equation (3.2). Then as time goes on $x$ will move right towards 2/3, if we manage to switch the governing equation to (3.3) before it reaches 2/3, then $x$ will move back left.
Plug in the initial condition, we get
\[ c_1 = \frac{(3\epsilon)^3}{(2/3 - \epsilon)(1/3 + \epsilon)^2}. \]

Therefore, \( x \) will reach 2/3 at time \( t \) that satisfies
\[ c_1 e^{2t} = \frac{1}{1/3 \times (2/3)^2} = \frac{27}{4} \]
and
\[ t = \frac{1}{2} \left[ \ln \left( \frac{27}{4} \right) - \ln c_1 \right] = \frac{1}{2} \left[ \ln \left( \frac{2}{3} - \epsilon \right) + 2 \ln \left( \frac{1}{3} + \epsilon \right) - 3 \ln \epsilon - 2 \ln 2 \right]. \]

Take the derivative of \( t \) on \( \epsilon \), we get
\[ t' = \frac{1}{2} \left[ \frac{-1}{2/3 - \epsilon} + \frac{2}{1/3 + \epsilon} - \frac{3}{\epsilon} \right] < \frac{1}{2} \left[ \frac{2}{1/3 + \epsilon} - \frac{3}{\epsilon} \right] < 0, \]
\[ 0 < \epsilon < \frac{1}{3}. \]

Therefore, \( t \) decreases as \( \epsilon \) increases between 0 and 1/3. The smallest time it takes for \( x \) to go from its initial position to 2/3 happens when \( \epsilon \rightarrow 1/3 \).

In fact when \( \epsilon = 1/3 \), we have
\[ t = \frac{1}{2} \left[ \ln \left( \frac{1}{3} \right) + 2 \ln \left( \frac{2}{3} \right) + 3 \ln 3 - 2 \ln 2 \right] = 0. \]

And as \( \epsilon \rightarrow 0 \), we can see \( t \rightarrow \infty \). This means if we put \( x \) initially at 1/3, it takes forever for \( x \) to move over to 2/3. And if we put \( x \) initially at 2/3, well, it is already there and if we do not want \( x \) to move further right past 2/3, we have to immediately switch the system to satisfy equation (3.3).

If we can control \( \epsilon \), the upper bound of the time we want to switch over to equation (3.3) is given by the equation
\[ t_l = \frac{1}{2} \left[ \ln \left( \frac{2}{3} - \epsilon \right) + 2 \ln \left( \frac{1}{3} + \epsilon \right) - 3 \ln \epsilon - 2 \ln 2 \right]. \] (3.4)

Now let us look at the other end of our trapping zone, close to 2/3. Assume \( x \) is positioned at \( 2/3 - \delta, 0 < \delta < 1/3 \) at time \( t = 0 \) and the system satisfies equation (3.3).

Plug in the initial condition, we get
\[ c_2 = \frac{(3\delta)^3}{(1/3 + \delta)^2(2/3 - \delta)}. \]

\( x \) will move left and reach 1/3 at time \( t \).
\[ c_2 e^{2t} = \frac{1}{(2/3)^2 \times 1/3} = \frac{27}{4}. \]

Unsurprisingly this is totally symmetric with the analysis we did before with \( \epsilon \). Therefore we know here \( t \) also decreases as \( \delta \) increases between 0 and 1/3. In fact, we can see the two differential equations are symmetric as well. If we can find the time it takes for \( x \) to move from \( 1/3 + \epsilon \) to \( 2/3 - \epsilon, 0 < \epsilon < 1/6 \), we can trap \( x \) in \((1/3 + \epsilon, 2/3 - \epsilon)\) forever, just by switching between the two
equations periodically. Now we compute critical value for the period \( T \).

\[
c_1 e^{2T} = \frac{(1 - 3\epsilon)^3}{(1/3 + \epsilon)(2/3 - \epsilon)^2},
\]

\[
e^{2T} = \left( \frac{1 - 3\epsilon}{3\epsilon} \right)^3 \frac{1}{2/3 - \epsilon},
\]

\[
T = \frac{1}{2} \left\{ 3[\ln(1 - 3\epsilon) - \ln(3\epsilon)] + \ln \left( \frac{1}{3} + \epsilon \right) - \ln \left( \frac{2}{3} - \epsilon \right) \right\}.
\]

Again we can take a derivative of \( T \) on \( \epsilon \),

\[
T' = \frac{1}{2} \left\{ 3 \left[ \frac{-3}{1 - 3\epsilon} - \frac{3}{3\epsilon} \right] + \frac{1}{1/3 + \epsilon} - \frac{-1}{2/3 - \epsilon} \right\} < 0,
\]

\[
0 < \epsilon < \frac{1}{6}.
\]

Again \( T \) decreases as \( \epsilon \) increases. When \( \epsilon \to 0 \), \( T \to \infty \). When \( \epsilon = 1/6 \), \( T = 0 \). For all values \( \epsilon \in (0, 1/6) \), we can achieve trapping between \((1/3 + \epsilon, 2/3 - \epsilon)\) by setting the initial condition to be either \( x(0) = 1/3 + \epsilon \) with equation (3.2) or \( x(0) = 2/3 - \epsilon \) with equation (3.3), and switching between the two equations with period

\[
T = \frac{1}{2} \left\{ 3[\ln(1 - 3\epsilon) - \ln(3\epsilon)] + \ln \left( \frac{1}{3} + \epsilon \right) - \ln \left( \frac{2}{3} - \epsilon \right) \right\}.
\] (3.5)

(b) Asymmetric one-dimensional example

The above example is symmetric and therefore special. We would like to explore the more general scenario of asymmetric cases, where \( a(t) = a_{11}(t) - a_{12}(t) - a_{21}(t) + a_{22}(t) \) is of different values under each environment.

Let us replace equation (3.2) with the following equation:

\[
\dot{x} = x(1 - x)(4x - 1). \quad (3.6)
\]

The unstable equilibrium of equation (3.6) happens at 1/4. We want to study the behaviour of the solution between 1/4 and 2/3.

\[
\frac{dx}{x(1-x)(4x-1)} = \left( -\frac{1}{x} + \frac{1}{3} \frac{1}{1-x} + \frac{16}{3} \frac{1}{4x-1} \right) dx = dt,
\]

\[
- \ln x - \frac{1}{3} \ln(1-x) + \frac{4}{3} \ln(4x-1) = t + c,
\]

\[
\frac{(4x-1)^4}{x^3(1-x)} = c_3 e^{3t}, \quad \frac{1}{4} < x < 1.
\]

Again assume we start the system at \( x(0) = 1/4 + \epsilon, 0 < \epsilon < 5/12 \) with equation (3.6), then we get

\[
c_3 = \frac{(4\epsilon)^4}{(1/4 + \epsilon)^3(3/4 - \epsilon)}.
\]

And \( x \) arrives at 2/3 at time \( t \), then

\[
c_3 e^{3t} = \frac{(5/3)^4}{(2/3)^3(1/3)} = \frac{5^4}{2^3}.
\]
\[ t = \frac{1}{3} [4 \ln 5 - 3 \ln 2 - \ln c_3] \]
\[ = \frac{1}{3} \{ 4[\ln 5 - \ln(4\epsilon)] + 3 \left[ \ln \left( \frac{1}{4} + \epsilon \right) - \ln 2 \right] + \ln \left( \frac{3}{4} - \epsilon \right) \}. \]

Again we can take a derivative,
\[ t' = \frac{1}{3} \left( -\frac{4}{\epsilon} + \frac{3}{1/4 + \epsilon} - \frac{1}{3/4 - \epsilon} \right) < 0, \]
\[ 0 < \epsilon < 5/12. \]

As \( \epsilon \to 0, t \to \infty \) since \( 1/4 \) is an equilibrium point. And \( t = 0 \) if \( \epsilon = 5/12 \). The time it takes for \( x \) to move from \( 1/4 + \epsilon \) to \( 2/3 - \delta \), \( 0 < \delta < 5/12 - \epsilon < 5/12 \) is
\[ t_l = \frac{1}{3} \left\{ 4 \left[ \ln \left( \frac{5}{3} - 4\delta \right) - \ln(4\epsilon) \right] + 3 \left[ \ln \left( \frac{1}{4} + \epsilon \right) - \ln \left( \frac{2}{3} - \delta \right) \right] \right. \]
\[ \left. + \ln \left( \frac{3}{4} - \epsilon \right) - \ln \left( \frac{1}{3} + \delta \right) \right\}. \]

(3.7)

Now if we switch the system to equation (3.3), the time it takes for \( x \) to move from \( 2/3 - \delta \) back to \( 1/4 + \epsilon \) is
\[ t_r = \frac{1}{2} \left\{ 3 \left[ \ln \left( \frac{5}{3} - 3\epsilon \right) - \ln(3\delta) \right] + 2 \left[ \ln \left( \frac{1}{3} + \delta \right) - \ln \left( \frac{3}{4} - \epsilon \right) \right] \right. \]
\[ \left. + \ln \left( \frac{2}{3} - \delta \right) - \ln \left( \frac{1}{4} + \epsilon \right) \right\}. \]

(3.8)

There is no obvious reason to think \( t_l \) and \( t_r \) are equal. In fact, if we take \( \epsilon = 1/12, \delta = 1/6 \), we can compute \( t_l = 5/3 \ln 2 \neq 1/2 \ln(27/4) = t_r \).

So under this asymmetric system, we could start with equation (3.6) at \( x(0) = 1/4 + \epsilon \), switch to equation (3.3) after time \( t_l \), then switch back to equation (3.6) after time \( t_r \) and repeat. Or we could start the system with equation (3.3) at \( x(0) = 2/3 - \delta \), switch to equation (3.6) after time \( t_r \), then switch back to equation (3.3) after time \( t_l \) and repeat. In either case, we manage to trap our system between \( 1/4 + \epsilon \) and \( 2/3 - \delta \).

With this asymmetric example, the time periods we stay on each state is different, but we can still achieve trapping by repeating a fixed control pattern of staying on one system for a fixed period and then the other on a different fixed period.

(c) General asymmetric one-dimensional model

In general, let us say we have the system switch between the following two equations:
\[ \dot{x} = x(1 - x)(a_1 x - b_1), \quad a_1 > b_1 > 0. \]

(3.9)

and
\[ \dot{x} = x(1 - x)(a_2 x - b_2), \quad a_2 > b_2 > 0. \]

(3.10)

Without loss of generality, we assume \( a_1^* = b_1 / a_1 < a_2^* = b_2 / a_2 \). Again we want to trap the system between \( (a_1^*, a_2^*) \subseteq (0, 1) \).

Assume we start the system on equation (3.9) at \( x(0) = a_1^* + \epsilon \), and we switch the system to equation (3.10) when \( x \) reaches \( a_2^* - \delta \), then we switch back to equation (3.9) after \( x \) reaches \( a_1^* + \epsilon \) again and repeat this process. Here, \( \epsilon, \delta > 0, \epsilon + \delta < a_2^* - a_1^* \). To find the period the system stays on...
equation (3.9), we need to solve the equation.

\[
\frac{dx}{x(1-x)(a_1 x - b_1)} = \left[ -\frac{1}{b_1} \frac{1}{1} + \frac{1}{a_1 - b_1} + \frac{1}{a_2 (1-b_1) a_1 x - b_1} \right] \, dx = dt,
\]

\[
- \frac{1}{b_1} \ln x - \frac{1}{a_1 - b_1} \ln(1-x) + \frac{a_1}{b_1(a_1 - b_1)} \ln(a_1 x - b_1) = t + c,
\]

\[
\frac{(a_1 x - b_1)^{a_1}}{x^{a_1-b_1}(1-x)^{b_1}} = c e^{b_1(a_1-b_1)} t, \quad a_1^* < x < 1.
\]

Plug in the initial condition, we get

\[
c_l = \frac{(a_1 e)^{a_l}}{(a_1^* + e)^{a_1-b_1}(1-a_1^* - e)^{b_1}}.
\]

The time it takes for \(x\) to move to \(a_2^* - \delta\) is

\[
t_l = \frac{1}{b_1(a_1-b_1)} \{ a_1 \ln(a_1 a_2^* - a_1 \delta - b_1) - \ln(a_1 e) \}
\]

\[+ (a_1 - b_1) \{ \ln(a_2^* + e) - \ln(a_2^* - \delta) \} + b_1 \{ \ln(1-a_1^* - \epsilon) - \ln(1-a_2^* + \delta) \} \}.
\]

One can check formulae (3.4), (3.5) and (3.7) from the symmetric and asymmetric examples are special cases of (3.11).

Similarly we can solve (3.10).

\[
\frac{dx}{x(1-x)(a_2 x - b_2)} = \left[ -\frac{1}{b_2} \frac{1}{1} + \frac{1}{a_2 - b_2} + \frac{1}{a_2^2 (a_2 - b_2) a_2 x - b_2} \right] \, dx = dt,
\]

\[
- \frac{1}{b_2} \ln x - \frac{1}{a_2 - b_2} \ln(1-x) + \frac{a_2}{b_2(a_2 - b_2)} \ln(b_2 - a_2 x) = t + c,
\]

\[
\frac{(b_2 - a_2 x)^{a_2}}{x^{a_2-b_2}(1-x)^{b_2}} = c e^{b_2(a_2-b_2)} t, \quad 0 < x < a_2^*.
\]

Plug in the initial condition \(x(0) = a_2^* - \delta\), we get

\[
c_r = \frac{(a_2 \delta)^{a_2}}{(a_2^* - \delta)^{a_2-b_2}(1-a_2^* + \delta)^{b_2}}.
\]

The time it takes \(x\) to move from \(a_2^* - \delta\) to \(a_1^* + \epsilon\) is

\[
t_r = \frac{1}{b_2(a_2 - b_2)} \{ a_2 \ln(b_2 - a_2 a_1^* - a_2 \epsilon) - \ln(a_2 \delta) \}
\]

\[+ (a_2 - b_2) \{ \ln(a_2^* - \delta) - \ln(a_2^* + \epsilon) \} + b_2 \{ \ln(1-a_2^* + \delta) - \ln(1-a_2^* - \epsilon) \} \}.
\]

Again we can check formulae (3.5) and (3.8) are special cases of (3.12). We can trap our system between \((a_1^*, a_2^*)\) by starting from equation (3.9) at \(x(0) = a_1^* + \epsilon\), switch the equation to (3.10) after time \(t_l\), then switch back to (3.9) after time \(t_r\), and repeat this process. Or we can start from equation (3.10) with \(x(0) = a_2^* - \delta\), switch the equation after time \(t_r\) to (3.9), and switch again back to (3.10) after time \(t_l\) and repeat the process. Here, \(t_r\) and \(t_l\) are given by formulae (3.11) and (3.12).

(d) Continuous variation

Both the symmetric and asymmetric examples above switch between two different equations.

What if the equation parameters vary continuously?

Still assuming \(a(t) > b(t) > 0\), let \(a^* = b(t)/a(t)\) be a periodic function that ranges between \(p_0\) and \(p_1\), \(0 < p_0 < p_1 < 1\). Again we can analyse the behaviour of the system with figure 2. One way to ensure \(x(t)\) is trapped between \(p_0\) and \(p_1\), at any moment in time, is if \(a^*\) is increasing, we want \(x(t) > a^*\) so that \(x(t)\) does not come to the left of \(a^*\) and converge to 0. Similarly if \(a^*\) is decreasing, we want \(x(t) < a^*\) so that \(x(t)\) does not converge to 1.
To summarize, we need
\[ [x(t) - a^*]a^* \geq 0, p_0 < x(t) < p_1, \quad \forall t \]  
(3.13)
for the system to be trapped in between \( p_0 \) and \( p_1 \). Plug \( a^* = b(t)/a(t) \) into (3.13), we get
\[ [a(t)x(t) - b(t)]|[b'(t)a(t) - a'(t)b(t)] \geq 0, p_0 < x(t) < p_1, \quad \forall t. \]  
(3.14)

4. Dynamics of the mixed strategy bimatrix game

Considering \( x_1 + x_2 = 1, y_1 + y_2 = 1 \), our system of mixed strategy bimatrix game has the following replicator equations:
\[
\begin{align*}
\dot{x}_1 &= x_1(1 - x_1)\{(a_{11}(t) + a_{22}(t) - a_{12}(t) - a_{21}(t))y_1 - [a_{22}(t) - a_{12}(t)]\}, \\
\dot{y}_1 &= y_1(1 - y_1)\{(b_{11}(t) + b_{22}(t) - b_{12}(t) - b_{21}(t))x_1 - [b_{22}(t) - b_{12}(t)]\}.
\end{align*}
\]

Here,
\[
M(t) = \\
= \begin{cases} 
M_{II}, t \in T_2, \\
M_I, t \in T_1,
\end{cases} \\
T_1 \cup T_2 = [0, \infty), T_1 \cap T_2 = \emptyset, \sup(T_1) = \sup(T_2) = \infty
\]

\[
M_I = \begin{bmatrix} (a_{11}, b_{11}) & (a_{12}, b_{12}) \\
(a_{21}, b_{21}) & (a_{22}, b_{22}) \end{bmatrix}, M_{II} = \begin{bmatrix} (a'_{11}, b'_{11}) & (a'_{12}, b'_{12}) \\
(a'_{21}, b'_{21}) & (a'_{22}, b'_{22}) \end{bmatrix}.
\]

Although we started with a two-player bimatrix game with mixed strategies, where \( x_1 \) and \( y_1 \) represent the probabilities of each player choosing the corresponding strategies, this model also works for two groups of players. There \( x_1 \) and \( y_1 \) represent the proportions or percentages of each player group that adopt the corresponding strategies.

Let
\[
a^* = \frac{a_{22}(t) - a_{12}(t)}{a_{11}(t) + a_{22}(t) - a_{12}(t) - a_{21}(t)}
\]
and
\[
b^* = \frac{b_{22}(t) - b_{21}(t)}{b_{11}(t) + b_{22}(t) - b_{12}(t) - b_{21}(t)}.
\]

Then \((b^*, a^*)\) is a fixed point of our system, in addition to the points \((0, 0)\), \((1, 1)\), \((0, 1)\) and \((1, 0)\). Since \( x_1 \) and \( y_1 \) are probabilities or proportions, we only consider their values in \([0, 1]\). The reasonable range for \( a^* \) and \( b^* \) is \((0, 1)\). We restrict our analysis for the case where the interior fixed point is unstable for each game environment; stable coexistence is impossible in any given single-game environment but can be sustained within a ‘trapping’ region under feedback-evolving games that lead to dynamic switching between the two game environments. For example, for times \( t \in T_2 \), if \( A' = \begin{bmatrix} a'_{11} & a'_{12} \\
\end{bmatrix} \) and \( B' = \begin{bmatrix} b'_{11} & b'_{12} \\
\end{bmatrix} \) are diagonal matrices and their diagonal entries are positive, we have
\[
a^* = \frac{a'_{22}}{a'_{11} + a'_{22}} \in (0, 1) \quad \text{and} \quad b^* = \frac{b'_{22}}{b'_{11} + b'_{22}} \in (0, 1).
\]

To analyse the dynamics of the system, we plot the nullclines and phase plot below (figure 3). Here, the blue nullclines are for \( x_1 \) and the red ones are for \( y_1 \). The intersection of a blue line and a red line is a fixed point, or equilibrium point. There are five equilibrium points in total. The blue arrows indicate whether \( x_1 \) is increasing or decreasing in the region, and the red arrows indicate the behaviour of the \( y_1 \) value over time. The purple arrow indicates qualitatively the direction of movements for points in that region. According to this qualitative phase portrait, we can see...
Figure 3. Phase plot for positive diagonal pay-off matrices.

Figure 4. Phase plot for oscillation around coexistence.

The equilibria (0, 0), (1, 1) are locally stable, and equilibria (0, 1), (1, 0) are unstable. The coexistence equilibrium \((b^*, a^*)\) is unstable. To be more specific, it is a saddle point.

If the matrices \(A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\) and \(B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\) are also positive diagonal matrices, then for times \(t \in T_1\), we have the same qualitative phase portrait as figure 3.

If we want the probabilities/proportions to oscillate around the coexistence equilibrium \((b^*, a^*)\), the qualitative phase portrait should look like figure 4.

For such a phase portrait, we need \(0 < a^*, b^* < 1\) and also

\[a_{11}(t) + a_{22}(t) - a_{12}(t) - a_{21}(t) < 0,\]

and

\[b_{11}(t) + b_{22}(t) - b_{12}(t) - b_{21}(t) > 0.\]

Solve all four inequalities, we get

\[a_{11}(t) < a_{21}(t), a_{22}(t) < a_{12}(t)\]

and

\[b_{11}(t) > b_{12}(t), b_{22}(t) > b_{21}(t).\]
These are the conditions for an oscillatory system around the coexistence equilibrium. In this case, the coexistence equilibrium is a centre.

5. Trapping regions

Assume we have two different saddle points under environment matrices $M_I$ and $M_{II}$. We wish to do what we did with the one-dimensional examples, achieve trapping between the two unstable fixed points by switching between the two environments.

Let us draw a qualitative phase portrait. Assume $(0, 0)$ and $(1, 1)$ are always stable, we have $E_1$ as the saddle point for environment $M_I$ and $E_2$ as the saddle point for environment $M_{II}$.

Here, the blue dense dotted lines represent the stable manifold for the saddle points and the red sparse dotted lines represent the unstable manifolds. Then as long as the blue lines and red lines form a region bounded by alternating coloured edges, we have a trapping region, shaded in green in figure 5. This is because, qualitatively, under environment I, the movements of points inside the region are marked by the two lower arrows, and when we switch to environment II, the movements of points inside the region are marked by the two upper arrows. Together they form a closed circular motion. As long as we make sure to switch the environment before the points move out of the boundary of the region, they should be trapped in the region forever.

Now let us look at the blue shaded region instead. Unlike the green region, along the boundaries of the blue region, the three orange arrows do not form a consecutive circular path. Indeed, when under environment I, the movements of points inside the blue region should follow the two orange arrows to the left, and if we do not ever switch the environment, the points will escape the blue region from the right. Now, if we switch the environment before the points escape, they will now follow the orange arrow on the right. If we stay on environment II for long enough the points will move on into the green region and escape the blue region. If we manage to switch back to environment I before the points move into the green region, well, they will keep moving right until they escape the blue region on the right.
This analysis can be applied to the purple, pink and yellow regions as well. We can have a trapping region if and only if the arrows along the boundary form a closed circular path.

6. Linearization of the system

Although we do not have explicit solutions for our nonlinear system, we can still linearize it near the fixed point \((b^*, a^*)\).

\[
\begin{bmatrix}
    x_1' \\
    y_1'
\end{bmatrix} = \begin{bmatrix}
    0 & a \\
    b & 0
\end{bmatrix} \begin{bmatrix}
    x_1 \\
    y_1
\end{bmatrix} - \begin{bmatrix}
    b^* \\
    a^*
\end{bmatrix},
\]

(6.1)

where

\[
\alpha = b^*(1 - b^*)[a_{11}(t) + a_{22}(t) - a_{12}(t) - a_{21}(t)]
\]

and

\[
\beta = a^*(1 - a^*)[b_{11}(t) + b_{22}(t) - b_{12}(t) - b_{21}(t)].
\]

Still assuming we have a system that switches between environments \(M_I = A \oplus B \) and \(M_{II} = A' \oplus B' \). Here \(A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, A' = \begin{bmatrix} a'_{11} & a'_{12} \\ a'_{21} & a'_{22} \end{bmatrix}, B' = \begin{bmatrix} b'_{11} & b'_{12} \\ b'_{21} & b'_{22} \end{bmatrix} \) are all constant matrices. Now assume

\[
a^* = \frac{a_{22} - a_{12}}{a_{11} + a_{22} - a_{12} - a_{21}},
\]

\[
b^* = \frac{b_{22} - b_{21}}{b_{11} + b_{22} - b_{12} - b_{21}},
\]

\[
c^* = \frac{a'_{22} - a'_{12}}{a'_{11} + a'_{22} - a'_{12} - a'_{21}},
\]

\[
d^* = \frac{b'_{22} - b'_{21}}{b'_{11} + b'_{22} - b'_{12} - b'_{21}},
\]

\[
\alpha = b^*(1 - b^*)[a_{11} + a_{22} - a_{12} - a_{21}],
\]

\[
\beta = a^*(1 - a^*)[b_{11} + b_{22} - b_{12} - b_{21}],
\]

\[
\alpha' = d^*(1 - d^*)[a'_{11} + a'_{22} - a'_{12} - a'_{21}],
\]

\[
\beta' = c^*(1 - c^*)[b'_{11} + b'_{22} - b'_{12} - b'_{21}].
\]

Under environment \(M_I\), the linear system of ODEs has solutions

\[
\begin{bmatrix}
    x_1 \\
    y_1
\end{bmatrix} = \begin{bmatrix}
    b^* \\
    a^*
\end{bmatrix} + c_1 e^{\sqrt{\alpha}t} \begin{bmatrix}
    \sqrt{\alpha} \\
    \sqrt{\beta}
\end{bmatrix} + c_2 e^{-\sqrt{\alpha}t} \begin{bmatrix}
    -\sqrt{\alpha} \\
    -\sqrt{\beta}
\end{bmatrix}.
\]

And under environment \(M_{II}\), this linear system of ODEs has solutions

\[
\begin{bmatrix}
    x_1 \\
    y_1
\end{bmatrix} = \begin{bmatrix}
    d^* \\
    c^*
\end{bmatrix} + c_1 e^{\sqrt{\alpha'}t} \begin{bmatrix}
    \sqrt{\alpha'} \\
    \sqrt{\beta'}
\end{bmatrix} + c_2 e^{-\sqrt{\alpha'}t} \begin{bmatrix}
    -\sqrt{\alpha'} \\
    -\sqrt{\beta'}
\end{bmatrix}.
\]

The fixed points \(E_1 = (b^*, a^*), E_2 = (d^*, c^*) \) are saddle points. Note that their two eigenvector directions are symmetric by the \(y\)-axis. If the two fixed points are positioned as in figure 6a, then the green shaded region is a trapping region. Here, the blue and light blue straight lines represent the contracting directions, or stable manifolds and the red and pink lines represent expanding directions, or unstable manifolds.

To have this left-right configuration, we need

\[
\left| \frac{a^* - c^*}{b^* - d^*} \right| < \min \left\{ \frac{\sqrt{\beta'}}{\sqrt{\alpha'}}, \frac{\sqrt{\beta}}{\sqrt{\alpha}} \right\}.
\]

(6.2)

The contracting and expanding directions of \(E_1\) divides our phase space into four parts. We should have four configurations with a convex quadrilateral trapping region. Only two of these
are distinct considering symmetry between $E_1$ and $E_2$, we can always rename or relabel them. The other distinct configuration, the up-down configuration in figure 6(b) requires

$$\left| \frac{a^* - c^*}{b^* - d^*} \right| > \max \left\{ \sqrt{\frac{\beta}{\alpha}}, \sqrt{\frac{\beta'}{\alpha'}} \right\}. \quad (6.3)$$

To show that this linear approximation actually is close enough to the nonlinear system, we study a specific nonlinear system here as well. The trapping region in figure 7a is not a convex quadrilateral any more, we plot a specific trapping trajectory in magenta and the corresponding frequencies of strategies in figure 7b. The equations for this nonlinear system are

$$\begin{align*}
x' &= x(1-x)(2y-1) \\
y' &= y(1-y)(4x-3)
\end{align*} \quad (6.4)$$

and

$$\begin{align*}
x' &= x(1-x)(2y-1), \\
y' &= y(1-y)(4x-1).
\end{align*} \quad (6.5)$$

At time $t_0 = 0$, we start with (6.4), and initial conditions $x_0 = 0.51$, $y_0 = 0.8$. We switch over to (6.5) at $t_1 = 6.15$. Then after $t_2 = 8.35$, we switch again, and the trajectory in magenta happens when we switch again after $t_3 = 4$ and $t_4 = 2$. The corresponding solutions $x(t)$ and $y(t)$ are plotted here as well (figure 7). We can see the curve for $x(t)$ is rather smooth while the curve for $y(t)$ has sharp tips. This is because the equation for $x(t)$ actually stays the same between the two environments.

This example is rather simple but it is nonlinear, and it corresponds very well to the left-right linear trapping region (figure 6a). Given this resemblance, we have a good reason to think the linear trapping regions, although not exactly what we will see in realistic nonlinear systems, still give us reasonable approximation in nonlinear trapping.

In addition to the left-right and up-down configurations in figure 6, if $E_2$ lies on the stable or unstable manifold of $E_1$, or the other way around, we will also have a trapping region. These cases, co-require

$$\left| \frac{a^* - c^*}{b^* - d^*} \right| = \sqrt{\frac{\beta}{\alpha}} \quad \text{or} \quad \sqrt{\frac{\beta'}{\alpha'}}. \quad (6.6)$$

In figure 8, we have the very special case where one fixed point is on the stable, or unstable manifold of the other fixed point, and their other manifolds are parallel to each other. When one
fixed point is on the stable manifold of the other, the trapping region degenerates to a line segment between the two fixed points. While when one fixed point is on the unstable manifold of the other, the trapping region becomes much larger and takes up a big part of our phase space.

Still considering one fixed point on the stable or unstable manifold of the other fixed point, we now focus on the cases where their other manifolds are not parallel to each other. In figure 9, one fixed point is on the unstable manifold of the other fixed point, depending on whether the manifolds of the fixed point are less steep or steeper, we have two slightly different triangular trapping regions. And in figure 10, one fixed point is on the stable manifold of the other fixed point. We end up with similar triangular trapping regions.

One might ask, what happens when

$$\min \left\{ \sqrt{\frac{\beta}{\alpha}}, \sqrt{\frac{\beta'}{\alpha'}} \right\} < \left| \frac{a^n - c^n}{b^m - d^m} \right| < \max \left\{ \sqrt{\frac{\beta}{\alpha}}, \sqrt{\frac{\beta'}{\alpha'}} \right\}? \quad (6.7)$$
Note that (6.2), (6.3), (6.6) and (6.7) include all possible configurations of the two distinct fixed points $E_1$ and $E_2$. In the case of (6.7), we still have trapping regions but they are a little more complicated than the convex trapping regions we have shown so far.

Whenever we have the slope of the segment $E_1E_2$ smaller than the eigenvector slope of $E_1$, that means $E_2$ is either on the left or the right of the four parts of the phase space divided by the contracting and expanding directions of $E_1$. This is why (6.2) corresponds to the case when $E_1$ and $E_2$ are in the ‘left-right’ configuration for each other. Similarly (6.3) corresponds to when the two fixed points are in the ‘up-down’ configuration for each other. Intuitively (6.7) means $E_1$ is in the ‘left-right’ configuration w.r.t. $E_2$ while $E_2$ is in the ‘up-down’ configuration w.r.t. $E_1$, or the other way around. We draw two different examples in figure 11.

Here, we have a slightly more complicated mode of trapping. For example, in figure 11a, each of the green triangular regions on their own is a trapping region. But the red shaded region combined with the two green triangles is also a trapping region. When we consider the combined
shape, which is not convex any more, we could end up with a trapped loop that forms a butterfly shape or ‘8’ shape. The trajectory starts in the red region, shoots into one of the green ears and then returns to the red region. If we switch the governing equations in time, then the trajectory is going to shoot into the other green ear and return to the red region. Although figure 11a,b has different locations for the trapping regions, the dynamics is the same. The phase portraits for cases where the red region is located at the upper-right corner and lower-left corner can also be drawn but we leave that up to the readers.

For the sake of completeness, we also put a simplified sketch of all possible linear trapping regions in figure 12. We only draw the upper-left location trapping for the mixed configuration.

7. Constant of motion for the system

Our system is governed by these equations

\[
\begin{align*}
\dot{x}_1 &= x_1(1-x_1)[a_{11}(t) + a_{22}(t) - a_{12}(t) - a_{21}(t)]y_1 - [a_{22}(t) - a_{12}(t)], \\
\dot{y}_1 &= y_1(1-y_1)[b_{11}(t) + b_{22}(t) - b_{12}(t) - b_{21}(t)x_1 - [b_{22}(t) - b_{21}(t)].
\end{align*}
\]

For convenience, let us assume \(a_{ij}, b_{ij}'s\) are constants here, for \(i, j = 1, 2\). Let \(p = a_{11} + a_{22} - a_{12} - a_{21}, q = a_{22} - a_{12}, u = b_{11} + b_{22} - b_{12} - b_{21}, v = b_{22} - b_{21}\). We will write \(x\) for \(x_1\), \(y\) for \(y_1\) in the following solving process to make the symbols easier to read and write.

Our system becomes

\[
\dot{x} = x(1-x)(py - q)
\]

and

\[
\dot{y} = y(1-y)(ux - v).
\]

Divide the second equation by the first, we get

\[
\frac{dy}{dx} = \frac{y(1-y)(ux - v)}{x(1-x)(py - q)}.
\]
Figure 12. Sketches of all possible cases for linear trapping. (Online version in colour.)

This is a separable equation.

\[
\frac{(py - q) \, dy}{y(1 - y)} = \left( \frac{p - q}{1 - y} - \frac{q}{y} \right) \, dy = \left( \frac{u - v}{x(1 - x)} - \frac{v}{x} \right) \, dx,
\]

\[
(q - p) \ln(1 - y) - q \ln y = (v - u) \ln(1 - x) - v \ln x + c,
\]

\[
\frac{(1 - y)^{q-p}}{y^q} = c \left( \frac{1 - x}{x^u} \right)^{v-u}, x^v(1 - y)^{q-p} = cy^q(1 - x)^{v-u},
\]

\[
V(x, y) = x^v(1 - x)^{u-v}y^{-q}(1 - y)^{q-p} = c.
\]

\[V(x, y) = c\] is a constant of motion [31] for our system. It gives the analytic implicit solution to our system. Without an explicit solution, we still do not have the exact behaviour of \(x(t)\) and \(y(t)\).

8. Discussion

Understanding how cooperative traits are maintained in populations is an essential problem of deep importance not only in evolutionary biology, but also in microbial ecology and systems biology. Ecological factors, such as population density, disturbance frequency, population
dispersal, resource supply, spatial structuring of populations, the presence of mutualisms or the presence of a competing species in the environment, often play an important role in the evolution of cooperation. The effect that these and other ecological factors have on the evolution of cooperation is in general well understood. However, the reverse process, i.e. the effect that the evolution of cooperative traits may have on the ecological properties of populations, is worthy of investigation using the framework of eco-evolutionary dynamics.

In this paper, we considered eco-evolutionary games that incorporate the impact of switching environment on game dynamics. We consider a changing environment, which is represented as a switched system between two different pay-off matrices. Motivated by the study of the tragedy of the commons in evolutionary biology, we demonstrate how alternative dynamics can arise. By switching between two environments, we can achieve an oscillating dynamic equilibrium and maintain coexistence, which is impossible otherwise. Cooperators and defectors can coexist if the system state does not exceed the trapping regions.

Our model uses simplifying assumptions and is just a first step towards a more sophisticated eco-evolutionary model. Still, we strongly believe that our results provide valuable insights in designing control mechanisms. We may consider relaxing or extending these simplifying assumptions for future research. For example, instead of a switch between two different pay-off matrices, we could consider a linear combination of the two. The possibilities under a general nonlinear pay-off matrix are likely even broader. We could also consider incorporating multiplayer games into our current model towards a more realistic description of population dynamics.

Other possible extensions of our work could be to consider feedback between strategies and patch quality, where not all players experience the same environment. Negative environmental feedback gives immediate benefits in rich environments but results in net costs in poor environments.

In the analysis of the trapping regions, we find multiple segments of stable and unstable manifolds in phase graphs, which is a new dynamical phenomenon that is different from the interior fixed equilibrium situation obtained in previous works. We provide constructive specific control examples where the system states evolve contained inside the trapping regions. In some very special cases, we even get an equilibrium manifold, depending on the initial condition, our system could be trapped on the equilibrium manifold. This combination of eco-evolutionary dynamics and control theory result could be used to help steer the population to desired states in social dilemma games. Although our present study is focused on mixed strategy bimatrix games, system properties are unchanged in a small neighbourhood, our results on control can be generalized to many other important situations.

Data accessibility. This article has no additional data.

Authors’ contributions. L.S.: formal analysis, investigation, validation, visualization, writing—original draft; F.F.: conceptualization, supervision, writing—review and editing.

Both authors gave final approval for publication and agreed to be held accountable for the work performed therein.

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