Complex unit gain bicyclic graphs with rank 2, 3 or 4 *

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Abstract

A $\mathbb{T}$-gain graph is a triple $\Phi = (G, \mathbb{T}, \varphi)$ consisting of a graph $G = (V, E)$, the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and a gain function $\varphi : \overrightarrow{E} \to \mathbb{T}$ such that $\varphi(e_{ij}) = \varphi(e_{ji})^{-1} = \varphi(e_{ji})$. The rank of $\mathbb{T}$-gain graph $\Phi$, denoted by $r(\Phi)$, is the rank of the adjacency matrix of $\Phi$. In 2015, Yu, Qu and Tu [G. H. Yu, H. Qu, J. H. Tu, Inertia of complex unit gain graphs, Appl. Math. Comput. 265(2015) 619–629] obtained some properties of inertia of a $\mathbb{T}$-gain graph. They characterized the $\mathbb{T}$-gain unicyclic graphs with small positive or negative index. Motivated by above, in this paper, we characterize the complex unit gain bicyclic graphs with rank 2, 3 or 4.

Key Words: $\mathbb{T}$-gain graph; Rank; Bicyclic graph; Complex unit gain graph.

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1 Introduction

All graphs considered in this article are simple graphs. Let $G = (V, E)$ be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. A gain graph is a graph whose edges are labeled orientably by elements of a group $M$. That is, if an edge $e$ in one direction has label a group element $m$ in $M$, then in the other direction it has label $m^{-1}$ (the invertible element of $m$ in $M$). We call the group $M$ to be the gain group. A gain graph is a generalization of a signed graph, where the gain group $M$ has only two elements 1 and $-1$, see Zaslavsky [10].

A $\mathbb{T}$-gain graph (or complex unit gain graph) is a graph with the additional structure that each orientation of an edge is given a complex unit, called a gain, which is the inverse of the complex unit assigned to the opposite orientation. For a simple graph $G = (V, E)$ of order $n$, let $\overrightarrow{E}$ be the set of oriented edges, it is obvious that this set contains two copies of each edge with opposite directions. We write $e_{ij}$ for the oriented edge from $v_i$ to $v_j$. The circle group, which is denoted by $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, is a subgroup of the multiplicative group of all nonzero complex numbers $\mathbb{C}^\times$. A $\mathbb{T}$-gain graph is a triple $\Phi = (G, \mathbb{T}, \varphi)$ consisting of a graph $G = (V, E)$, the circle group $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and a gain function $\varphi : \overrightarrow{E} \to \mathbb{T}$ such that $\varphi(e_{ij}) = \varphi(e_{ji})^{-1} = \varphi(e_{ji})$, where $G$ is the underlying graph of the $\mathbb{T}$-gain graph. We often

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write $\Phi = (G, \varphi)$ or $G^\varphi$ for a $\mathbb{T}$-gain graph. The adjacency matrix associated to the $\mathbb{T}$-gain graph $\Phi$ is the $n \times n$ complex matrix $A(\Phi) = (a_{ij})$, where $a_{ij} = \varphi(e_{ij})$ if $v_i$ is adjacent to $v_j$, otherwise $a_{ij} = 0$. It is obvious to see that $A(\Phi)$ is Hermitian and its eigenvalues are real.

If the gain of every edge is 1 in $\Phi$, then the adjacency matrix $A(\Phi)$ is exactly the adjacency matrix $A(G)$ of the underlying graph $G$. It is obvious that a simple graph is assumed as a $\mathbb{T}$-gain graph with all positive gain 1’s.

The positive inertia index $i_+(\Phi)$, the negative inertia index $i_- (\Phi)$ and the nullity $\eta(\Phi)$ of $\Phi$ are defined to be the number of positive eigenvalues, negative eigenvalues and zero eigenvalues of $A(\Phi)$ including multiplicities, respectively. The rank $r(\Phi)$ of an $n$-vertex $\mathbb{T}$-gain graph is defined as the rank of $A(\Phi)$. Obviously, $r(\Phi) = i_+ (\Phi) + i_- (\Phi) = n - \eta (\Phi)$.

An induced subgraph of $\Phi$ is an subgraph of $\Phi$ and each edge preserves the original gain in $\Phi$. For a vertex $v \in V(\Phi)$, we write $\Phi - v$ for the gain graph obtained from $\Phi$ by removing the vertex $v$ and all edges incident with $v$. For an induced subgraph $H$ of $\Phi$, let $\Phi - H$ be the subgraph obtained from $\Phi$ by deleting all vertices of $H$ and all incident edges. The degree of a vertex $v$ for a gain graph $\Phi$ is the number of the vertices incident to $v$ in its underlying graph $G$. A vertex of a gain graph $\Phi = (G, \varphi)$ is called pendant if its degree is 1 in $\Phi$, and is called quasi-pendant vertex if it is adjacent to a pendant vertex in $\Phi$. Denoted by $S_n$, $K_n$, $P_n$, $C_n$ a star, a complete graph, a path and a cycle all of order $n$, respectively. A graph is called trivial if it has one vertex and no edges, it is sometimes denoted by $K_1$ or $P_1$.

A bicyclic graph is a graph in which the number of edges equals the number of vertices plus one. Let $G$ be a bicyclic graph, the base of $G$ is the unique bicyclic subgraph of $G$ containing no pendant vertices.

Let $C_p$ and $C_q$ be two vertex-disjoint cycles and $v \in V(C_p)$, $u \in V(C_q)$, $P_l = v_1v_2 \ldots v_l$ ($l \geq 1$) be a path of length $l-1$. Let $\infty(p,l,q)$ (as shown in Fig. 1) be the graph obtained from $C_p$, $C_q$ and $P_l$ by identifying $v$ with $v_1$ and $u$ with $v_l$, respectively. The bicyclic containing $\infty(p,l,q)$ as its base is called an $\infty$-graph. We denote $\infty^\varphi(p,l,q)$ be the $\mathbb{T}$-gain $\infty(p,l,q)$ graph, and the $\mathbb{T}$-gain bicyclic graph containing $\infty^\varphi(p,l,q)$ as its base is called a $\mathbb{T}$-gain $\infty$-graph.

Let $P_{p+2}, P_{l+2}, P_{q+2}$ be three paths, where $\min\{p,l,q\} \geq 0$ and at most one of $p, l, q$ is 0. Let $\theta(p,l,q)$ (as shown in Fig. 1) be the graph obtained from $P_{p+2}$, $P_{l+2}$ and $P_{q+2}$ by identifying the three initial vertices and terminal vertices. The bicyclic graph containing $\theta(p,l,q)$ as its base is called a $\theta$-graph. We denote $\theta^\varphi(p,l,q)$ be the $\mathbb{T}$-gain $\theta(p,l,q)$ graph, and the $\mathbb{T}$-gain bicyclic graph containing $\theta^\varphi(p,l,q)$ as its base is called a $\mathbb{T}$-gain $\theta$-graph.

Recently there are many widely investigated research results about spectral-based graph invariants, such as inertia index of a graph [7, 9], skew energy [1, 2] and skew-rank [3, 4, 5] of oriented graphs. Nathan Reff [6] defined the adjacency, incidence and Laplacian matrices of a complex unit gain graph. Some eigenvalue bounds for the adjacency and Laplacian matrices were present. Yu, Qu and Tu [8] obtained some properties of inertia of a $\mathbb{T}$-gain graph. They characterized the $\mathbb{T}$-gain unicyclic graphs with small positive or negative index. Motivated by above, in this paper, we will investigate some characterizations about $\mathbb{T}$-gain bicyclic graphs.

The rest of this paper is organized as follows: in Section 2, we list some elementary lemmas and known results. In Section 3, we characterize the rank of $\mathbb{T}$-gain bicyclic graphs. In Section 4, we characterize the $\mathbb{T}$-gain bicyclic graphs with rank 2, 3 or 4.

2 Preliminaries
Lemma 2.1. \[\Phi = \Phi_1 \cup \Phi_2 \cup \ldots \cup \Phi_t, \text{ where } \Phi_1, \Phi_2, \ldots, \Phi_t \text{ are connected components of a } \overline{T}\text{-gain graph } \Phi. \text{ Then } i_+(\Phi) = \sum_{i=1}^{t} i_+(\Phi_i).\]

(b) Let \(\Phi\) be a \(\overline{T}\)-gain graph on \(n\) vertices. Then \(i_+(\Phi) = 0\) if and only if \(\Phi\) is a graph without edges.

(c) Let \(H^\varphi\) be an induced subgraph of a \(\overline{T}\)-gain graph \(G^\varphi\). Then \(i_+(H^\varphi) \leq i_+(G^\varphi), i_-(H^\varphi) \leq i_-(G^\varphi)\).

Lemma 2.2. \[\Phi = (G, \varphi) \text{ be a } \overline{T}\text{-gain graph containing a pendant vertex } v \text{ with the unique neighbor } u. \text{ Then } i_+(\Phi) = i_+(\Phi - u - v) + 1, i_-(\Phi) = i_-(\Phi - u - v) + 1, i_0(\Phi) = i_0(\Phi - u - v). \text{ Moreover, } r(\Phi) = r(\Phi - u - v) + 2.\]

Let \(C_n^\varphi\) be a weighted cycle with vertex set \(\{v_1, v_2, \ldots, v_n\}\) such that \(v_i v_{i+1} \in E(C_n^\varphi) (1 \leq i \leq n - 1)\), \(v_1 v_n \in E(C_n^\varphi)\). Let \(w_i = \varphi(v_i v_{i+1})\) and \(w_n = \varphi(v_n v_1)\).

In [8], Yu gave Definition 1 as follows about a \(\overline{T}\)-gain cycle. In fact, Definition 1 should be Definition 2.

**Definition 1.** \[\text{A } \overline{T}\text{-gain even cycle } C_n^\varphi \text{ is said to be of Type A if } (-1)^{\frac{n}{2}} w_n = w_1 w_2 w_3 \ldots w_{n-2} w_{n-1}; \]

\(C_n^\varphi\) is said to be of Type B if \((-1)^{\frac{n}{2}} w_n \neq w_1 w_2 w_3 \ldots w_{n-2} w_{n-1}.\)

A \(\overline{T}\text{-gain odd cycle } C_n^\varphi \text{ is said to be of Type C if }\]

\[\text{Re} \left((-1)^{\frac{n-1}{2}} w_1 w_2 w_3 \ldots w_{n-2} w_{n-1} w_n\right) > 0; \]

\(C_n^\varphi\) is said to be of Type D if \[\text{Re} \left((-1)^{\frac{n-1}{2}} w_1 w_2 w_3 \ldots w_{n-2} w_{n-1} w_n\right) < 0;\]

\(C_n^\varphi\) is said to be of Type E if \[\text{Re} \left((-1)^{\frac{n-1}{2}} w_1 w_2 w_3 \ldots w_{n-2} w_{n-1} w_n\right) = 0, \]

where \(\text{Re} (\cdot)\) is the real part of a complex number.

In the proof of Theorem 7 in [8], let \(w_i = \varphi(v_i v_{i+1}) (1 \leq i \leq n - 1)\), \(w_n = \varphi(v_n v_1)\). The authors of [8] gave the adjacency matrix \(A(C_n^\varphi)\) of \(C_n^\varphi\) as follows:

\[
A(C_n^\varphi) = \begin{pmatrix}
0 & w_1 & 0 & 0 & \cdots & 0 & w_n \\
\overline{w_1} & 0 & w_2 & 0 & \cdots & 0 & 0 \\
0 & \overline{w_2} & 0 & w_3 & \cdots & 0 & 0 \\
0 & 0 & \overline{w_3} & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & w_{n-1} & w_n \\
\overline{w_n} & 0 & 0 & \cdots & w_{n-1} & 0 & 0
\end{pmatrix}.
\]
From $A(C_n^\varphi)$, we can see that $a_{1n} = w_n$ should be $a_{1n} = \overline{w_n}$ and $a_{n1} = w_n$ should be $a_{n1} = \overline{w_n}$. So, they made a mistake. In the following, we will give the right definition.

**Definition 2.** A $T$-gain even cycle $C_n^\varphi$ is said to be of Type A if

\[
(-1)^\frac{n}{2} w_n = w_1 w_2 w_3 \cdots w_{n-2} w_{n-1}
\]

(i.e., $\varphi(e_{n-1,n}) + (-1)^{\frac{n}{2}} \frac{\varphi(e_{1,n})\varphi(e_{32})\varphi(e_{34})...\varphi(e_{n-1,n-2})}{\varphi(e_{12})\varphi(e_{34})...\varphi(e_{n-3,n-2})} = 0$);

$C_n^\varphi$ is said to be of Type B if

\[
(-1)^\frac{n}{2} w_n \neq w_1 w_2 w_3 \cdots w_{n-2} w_{n-1}
\]

(i.e., $\varphi(e_{n-1,n}) + (-1)^{\frac{n}{2}} \frac{\varphi(e_{1,n})\varphi(e_{32})\varphi(e_{34})...\varphi(e_{n-1,n-2})}{\varphi(e_{12})\varphi(e_{34})...\varphi(e_{n-3,n-2})} \neq 0$).

A $T$-gain odd cycle $C_n^\varphi$ is said to be of Type C if

\[
\Re \left( (-1)^{\frac{n-1}{2}} w_1 w_2 w_3 \cdots w_{n-2} w_{n-1} w_n \right) > 0
\]

(i.e., $\Re \left( (-1)^{\frac{n-1}{2}} \frac{\varphi(e_{32})\varphi(e_{34})...\varphi(e_{n-1,n-1})\varphi(e_{1,n})}{\varphi(e_{12})\varphi(e_{34})...\varphi(e_{n-2,n-1})} \right) > 0$);

$C_n^\varphi$ is said to be of Type D if

\[
\Re \left( (-1)^{\frac{n-1}{2}} w_1 w_2 w_3 \cdots w_{n-2} w_{n-1} w_n \right) < 0
\]

(i.e., $\Re \left( (-1)^{\frac{n-1}{2}} \frac{\varphi(e_{32})\varphi(e_{34})...\varphi(e_{n-1,n-1})\varphi(e_{1,n})}{\varphi(e_{12})\varphi(e_{34})...\varphi(e_{n-2,n-1})} \right) < 0$);

$C_n^\varphi$ is said to be of Type E if

\[
\Re \left( (-1)^{\frac{n-1}{2}} w_1 w_2 w_3 \cdots w_{n-2} w_{n-1} w_n \right) = 0
\]

(i.e., $\Re \left( (-1)^{\frac{n-1}{2}} \frac{\varphi(e_{32})\varphi(e_{34})...\varphi(e_{n-1,n-1})\varphi(e_{1,n})}{\varphi(e_{12})\varphi(e_{34})...\varphi(e_{n-2,n-1})} \right) = 0$),

where $\Re(\cdot)$ is the real part of a complex number.

**Lemma 2.3.** [8] Let $C_n^\varphi$ be a $T$-gain cycle of order $n$. Then

\[
(i_+(C_n^\varphi), i_-(C_n^\varphi), i_0(C_n^\varphi)) = \begin{cases} 
(n-2, n-2, 2), & \text{if } C_n^\varphi \text{ is of Type A}, \\
(n, n, 0), & \text{if } C_n^\varphi \text{ is of Type B}, \\
(n+1, n-1, 0), & \text{if } C_n^\varphi \text{ is of Type C}, \\
(n-1, n+1, 0), & \text{if } C_n^\varphi \text{ is of Type D}, \\
(n-2, n+1, 1), & \text{if } C_n^\varphi \text{ is of Type E}.
\end{cases}
\]

Two pendant vertices are called pendant twins in a $T$-gain graph $\Phi$ if they have the same neighbor in $\Phi$.

**Lemma 2.4.** Let $u, v$ be pendant twins of a $T$-gain graph $\Phi$. Then $r(\Phi) = r(\Phi-u) = r(\Phi-v)$.

**Proof.** Assume that all vertices in $\Phi$ are indexed by $\{v_1, v_2, \ldots, v_n\}$ with $v_1 = v, v_2 = u$. Without loss of generality, we assume $v, u$ are incident with $w = v_3$. Then it follows that

\[
A(\Phi) = \begin{pmatrix} 
0 & 0 & \varphi(e_{13}) & 0 & \cdots & 0 \\
0 & 0 & \varphi(e_{23}) & 0 & \cdots & 0 \\
\varphi(e_{31}) & \varphi(e_{32}) & 0 & \varphi(e_{34}) & \cdots & \varphi(e_{3n}) \\
0 & 0 & \varphi(e_{43}) & 0 & \cdots & \varphi(e_{4n}) \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \varphi(e_{n3}) & \varphi(e_{n4}) & \cdots & 0
\end{pmatrix}.
\]

By elementary row and column transformations on $A(\Phi)$, we have
\[ r(A(\Phi)) = r \begin{pmatrix} 0 & \varphi(e_{13}) & 0 & \cdots & 0 \\ \varphi(e_{31}) & 0 & \varphi(e_{34}) & \cdots & \varphi(e_{3n}) \\ 0 & \varphi(e_{43}) & 0 & \cdots & \varphi(e_{4n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \varphi(e_{n3}) & \varphi(e_{n4}) & \cdots & 0 \end{pmatrix}. \]

So, \( r(\Phi) = r(\Phi - u) \). Similarly, \( r(\Phi) = r(\Phi - v) \). \( \square \)

From Lemmas 2.1, 2.2 and 2.4, we can get the following two lemmas.

**Lemma 2.5.** Let \( \Phi_0 = (G_0, \varphi_0) \) be a \( T \)-gain graph of order \( n - p \) such that \( u \in V(\Phi_0) \). Let \( \Phi_1 = (G_1, \varphi_1) \) be the \( T \)-gain graph obtained from \( \Phi_0 \) and \( S^e_t \) by inserting an edge between \( u \) and the center \( v_1 \) of \( S^e_t \). Let \( \Phi_2 = (G_2, \varphi_2) = \Phi_1 - \{v_1v_2, v_1v_3, \ldots, v_1v_p\} + \{uv_2, uv_3, \ldots, uv_p\} \). Then \( r(\Phi_1) \geq r(\Phi_2) \).

**Lemma 2.6.** Let \( \Phi_0 = (G_0, \varphi_0) \) be a \( T \)-gain graph of order \( n - l - t \) and \( v_1, v_2 \in V(\Phi_0) \). Assume that \( \Phi_1 = (G_1, \varphi_1) \) be the \( T \)-gain graph obtained from \( \Phi_0 \), \( S^e_t \) and \( S^e_t \) by identifying \( v_1 \) with the center of \( S^e_t \), \( v_2 \) with the center of \( S^e_t \), respectively. Let \( \Phi_2 = (G_2, \varphi_2) \) be the \( T \)-gain graph obtained from \( \Phi_0 \), \( S^e_t \) by identifying \( v_1 \) with the center of \( S^e_t \). Then \( r(\Phi_1) \geq r(\Phi_2) \).

**Lemma 2.7.** Let \( \Phi_1 = (G_1, \varphi_1) \) and \( \Phi_2 = (G_2, \varphi_2) \) be two \( T \)-gain graphs and \( v \in V(\Phi_1) \), \( u \in V(\Phi_2) \). Let \( P^e_t(l \geq 3) \) be a \( T \)-gain path with two end-vertices \( v_1 \) and \( v_2 \). Let \( \Phi = (G, \varphi) \) be the \( T \)-gain graph obtained from \( \Phi_1 \cup \Phi_2 \cup P^e_t \) by identifying \( v \) with \( v_1 \), \( u \) with \( v_1 \). Let \( \Phi' = (G', \varphi') \) be the \( T \)-gain graph obtained from \( \Phi_1 \cup \Phi_2 \) by identifying \( v \) with \( u \) and adding \( l - 1 \) pendant vertices to \( v \). Then \( r(\Phi) \geq r(\Phi') \).

**Proof.** By Lemma 2.2, we have \( r(\Phi') = 2 + r(\Phi_1 - v) + r(\Phi_2 - u) \).

If \( l = 3 \), by Lemmas 2.1(c) and 2.2 then we have \( r(\Phi) \geq 2 + r(\Phi_1 - v) + r(\Phi_2 - u) \).

If \( l \geq 4 \), by Lemmas 2.1(c) and 2.2 then we have \( r(\Phi) \geq 2 + r(\Phi_1 - v) + r(\Phi_2) \). Note that \( r(\Phi_2) \geq r(\Phi_2 - u) \), therefore \( r(\Phi) \geq r(\Phi') \). \( \square \)

**Theorem 2.8.** Let \( C^e_n \) be a \( T \)-gain cycle of order \( n (n \geq 3) \) and \( H = (G_1, \varphi_1) \) be a \( T \)-gain graph of order \( m \geq 1 \). Assume that \( \Phi = (G, \varphi) \) is the \( T \)-gain graph obtained by identifying a vertex of \( C^e_n \) with a vertex of \( H \) (i.e., \( V(C^e_n) \cap V(H) = v \)). Let \( F = (G_2, \varphi_2) \) be the induced subgraph obtained from \( H \) by deleting the vertex \( v \) and its incident edges. Then

\[
\begin{align*}
    r(\Phi) &= n - 2 + r(H), & \text{if } C^e_n \text{ is of Type A,} \\
    r(\Phi) &= n + r(F), & \text{if } C^e_n \text{ is of Type B,} \\
    r(\Phi) &= n - 1 + r(H), & \text{if } C^e_n \text{ is of Type E,} \\
    n - 1 + r(F) &\leq r(\Phi) \leq n + r(H), & \text{if } C^e_n \text{ is of Type C or D.}
\end{align*}
\]

**Proof.** Case 1. When \( n \) is even, without loss of generality, we assume that \( n = 2p \) (\( p \geq 2 \)) and \( V(C^e_n) \cap V(H) = v_{2p} \). Then the adjacency matrix \( A(\Phi) \) of \( \Phi \) can be expressed as:
So, we can get the following results

\[ A(\Phi) = \begin{pmatrix}
0 & \varphi(e_{1,2}) & \varphi(1,2p) \\
\varphi(e_{21}) & 0 & \varphi(e_{21}) \\
\varphi(e_{22}) & 0 & \varphi(e_{22}) \\
\vdots & \ddots & \ddots \\
\varphi(1) & \ldots & 0 \\
0 & \varphi(0_{m-1}) & M
\end{pmatrix}, \]

where \( v_iv_{i+1} \in E(C_{n}^{\varphi_0})(1 \leq i \leq 2p-1), \) \( v_1v_{2p} \in E(C_{n}^{\varphi_0}), \) \( \alpha_j \in E(H), \) \( j = 1, 2, \ldots, m-1, \) \( M = A(F). \)

By elementary row and column transformations, it is obvious to show that \( A(\Phi) \) is congruent to

\[ A(\Phi_1) = \begin{pmatrix}
A_1 & \cdots & A_{p-1} \\
\downarrow & \ddots & \downarrow \\
0 & a & \varphi(1) & \varphi(\alpha_{m-1}) \\
0 & \varphi(\alpha_1) & \ddots & \ddots \\
\vdots & \ddots & \ddots & M \\
0 & \varphi(\alpha_{m-1}) & \ddots & \ddots
\end{pmatrix}, \]

where

\[ A_i = \begin{pmatrix}
0 & \varphi(e_{2i-1,2i}) \\
\varphi(e_{2i,2i-1}) & 0
\end{pmatrix}, \quad i = 1, 2, \ldots, p-1, \]

\[ a = \varphi(e_{2p-1,2p}) + (-1)^{2p-2} \frac{\varphi(e_{1,2p})\varphi(e_{32}) \cdots \varphi(e_{2p-1,2p-2})}{\varphi(e_{12})\varphi(e_{34}) \cdots \varphi(e_{2p-3,2p-2})}. \]

So, we can get the following results

(a) If \( a = \varphi(e_{2p-1,2p}) + (-1)^{2p-2} \frac{\varphi(e_{1,2p})\varphi(e_{32}) \cdots \varphi(e_{2p-1,2p-2})}{\varphi(e_{12})\varphi(e_{34}) \cdots \varphi(e_{2p-3,2p-2})} = 0, \) i.e., \( C_{n}^{\varphi_0} \) is of Type A, then we have \( r(\Phi) = 2p - 2 + r(H) = n - 2 + r(H). \)

(b) If \( a = \varphi(e_{2p-1,2p}) + (-1)^{2p-2} \frac{\varphi(e_{1,2p})\varphi(e_{32}) \cdots \varphi(e_{2p-1,2p-2})}{\varphi(e_{12})\varphi(e_{34}) \cdots \varphi(e_{2p-3,2p-2})} \neq 0, \) i.e., \( C_{n}^{\varphi_0} \) is of Type B, then we have \( r(\Phi) = 2p + r(F) = n + r(F). \)

**Case 2.** When \( n \) is odd, without loss of generality, we assume that \( n = 2p - 1 \) \( (p \geq 2) \) and \( V(C_{n}^{\varphi_0}) \cap V(H) = v_{2p-1}. \) By elementary row and column transformations, we can show \( A(\Phi) \) is congruent to

\[ A(\Phi_1) = \begin{pmatrix}
A_1 & A_2 & \cdots & A_{p-1} \\
A_2 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & C \\
A_{p-1} & \cdots & \cdots & \cdots
\end{pmatrix}, \]
where

$$A_i = \begin{pmatrix} 0 & \varphi(e_{2i-1,2i}) \\ \varphi(e_{2i-1,2i-1}) & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} a & \varphi(\alpha_1) & \cdots & \varphi(\alpha_{m-1}) \\ \varphi(\alpha_1) & M \\ \vdots \\ \varphi(\alpha_{m-1}) \end{pmatrix},$$

$$a = 2\Re \left( (-1)^{2p-2} \varphi(e_{1,3p-1}) \varphi(e_{1,3p}) \cdots \varphi(e_{p+1,2p+2}) \right),$$

for $i = 1, 2, \ldots, p - 1$. So, we can get the following results

(c) If $a = 0$, i.e., $C_n^\varphi$ is of Type E, then we have $r(\Phi) = 2p - 2 + r(H) = n - 1 + r(H)$.

(d) If $a \neq 0$, i.e., $C_n^\varphi$ is of Type C or D, by Lemmas 2.5, 2.6 and 2.7, we will distinguish the following two cases.

3 The rank of $T$-gain bicyclic graphs

In this section, we shall characterize the rank of $T$-gain bicyclic graphs. First, we will give two theorems.

**Theorem 3.1.** Let $\Phi = (G, \varphi)$ be a $T$-gain bicyclic graph of order $n$ ($n \geq p+q$) with pendant vertices containing two edge disjoint cycles $C_{p}^{\varphi_1}$ and $C_{q}^{\varphi_2}$. Then

$$r(\Phi) \geq \begin{cases} p + q, & \text{if } p, q \text{ are odd,} \\ p + q - 2, & \text{if } p, q \text{ are even,} \\ p + q - 1, & \text{otherwise.} \end{cases}$$

This bound is sharp.

**Proof.** Let $\Phi^* = (G^*, \varphi^*)$, where $G^*$ (as shown in Fig. 2) is the underlying graph of $\Phi^*$. By Lemma 2.2, we have $r(\Phi^*) = 2 + r(P_{p+1}^{\varphi_1}) + r(P_{q-1}^{\varphi_2})$.

Let $\Phi = (G, \varphi)$ be a $T$-gain bicyclic graph of order $n$ ($n \geq p+q$) with pendant vertices containing two edge disjoint cycles $C_{p}^{\varphi_1}$ and $C_{q}^{\varphi_2}$. In the following we shall prove that $r(\Phi) \geq r(\Phi^*)$. By Lemmas 2.5, 2.6 and 2.7, we will distinguish the following two cases.

**Case 1.** $\Phi$ is a $T$-gain bicyclic graph obtained from $\infty^\varphi(p, 1, q)$ by attaching $n - p - q + 1$ ($n \geq p + q$) pendant vertices to a vertex, different from $x$, of $C_{p}^{\varphi_1}$ or $C_{q}^{\varphi_2}$, where $x$ is the common vertex of $C_{p}^{\varphi_1}$ and $C_{q}^{\varphi_2}$.

Without loss of generality, we suppose all the pendant vertices are attached at $C_{p}^{\varphi_1}$. Then we have

$$r(\Phi) = 2 + r(P_{p-1}^{\varphi_1}) + \begin{cases} r(P_{q-1}^{\varphi_2}), & \text{if } p \text{ is odd,} \\ r(C_{q-1}^{\varphi_2}), & \text{if } p \text{ is even.} \end{cases}$$
From Lemma 2.1(c) and Lemma 2.3 we have \( r(C_q^{p_2}) \geq r(P_{q-1}^{p_2}) \). So we have \( r(\Phi) \geq r(\Phi^*) \).

**Case 2.** \( \Phi \) is a \( T \)-gain bicyclic graph obtained from \( \infty \bar{\varphi}(p, 2, q) \) by attaching \( n-p-q \) \( (n \geq p+q+1) \) pendant vertices to a vertex \( w \) of \( C_{p}^{q+1} \) or \( C_{q}^{p+2} \).

Without loss of generality, we suppose all the pendant vertices are attached at \( C_{p}^{q+1} \). If \( w = v \) (where \( v \) is shown in Fig. 2), then we have \( r(\Phi) = 2 + r(P_{q-1}^{p_2}) + r(C_{q}^{p_2}) \geq r(\Phi^*) \). If \( w \neq v \), then we have

\[
r(\Phi) = 2 + \begin{cases} 
    r(P_{q-1}^{p_2}) + r(C_{q}^{p_2}), & \text{if } p \text{ is odd,} \\
    p + r(P_{q-1}^{p_2}) \text{ or } r(P_{q-1}^{p_2}) + r(C_{q}^{p_2}), & \text{if } p \text{ is even.}
\end{cases}
\]

So we have \( r(\Phi) \geq r(\Phi^*) \). □

Let \( u, v \) be two vertices in \( \theta^{p_2}(p, l, q) \) (as shown in Fig. 1). Let \( \Phi^{**} = (G^{**}, \varphi) \) be the \( T \)-gain bicyclic graph with \( n-p-q-l-2 \) \( (n \geq p+q+l+3) \) pendant vertices attached to \( v \) in \( \theta^{p_2}(p, l, q) \) (as shown in Fig. 2).

**Theorem 3.2.** Let \( \Phi \) be a \( T \)-gain bicyclic graph of order \( n \) \( (n \geq p+q+l+3) \) with pendant vertices containing \( \theta^{p_2}(p, l, q) \) as its base. If \( plq \neq 0 \), then

\[
r(\Phi) \geq \begin{cases} 
    p + q + l + 2, & \text{if } p + q + l \text{ is even,} \\
    p + q + l + 1, & \text{if } p, q, l \text{ are odd, or } p \text{ is odd, } C_{q+l+2}^{p} \text{ is} \\
    p + q + l + 3, & \text{otherwise.}
\end{cases}
\]

This bound is sharp.

**Proof.** Let \( \Phi^{**} = \Phi_1 \) be a \( T \)-gain graph with \( G^{**} \) as the underlying graph. By Lemma 2.2 we have

\[
r(\Phi_1) = \begin{cases} 
    2 + p + r(P_{q+l+1}^{p_2}), & \text{if } p \text{ is even,} \\
    3 + p + r(P_{q}^{p_2}) + r(P_{q}^{p_2}), & \text{if } p \text{ is odd.}
\end{cases}
\]

Let \( \Phi_2 \) be a \( T \)-gain bicyclic graph of order \( n \) \( (n \geq p+q+l+3) \) with pendant vertices containing \( \theta^{p_2}(p, l, q) \) as its base. Consider the graph \( \Phi_2 \) in which all \( n-p-q-l-2 \) pendant vertices are attached to a vertex, different from \( u \) and \( v \) of \( \theta^{p_2}(p, l, q) \). Without loss of generality, assume that \( n-p-q-l-2 \) pendant vertices are attached to a vertex of \( P_{q+l+2}^{p_2} \) in \( \Phi_2 \). By Lemma 2.2 we have

\[
r(\Phi_2) = \begin{cases} 
    2 + p + r(P_{q+l+1}^{p_2}), & \text{if } p \text{ is even,} \\
    3 + p + r(P_{q}^{p_2}) + r(P_{q}^{p_2}) + 1 + p + r(C_{q+l+2}^{p_2}), & \text{if } p \text{ is odd.}
\end{cases}
\]

Combining the ranks of \( \Phi_1, \Phi_2 \) and Lemmas 2.2 and 2.3, we can get the results. □

Next we consider the special case that one of \( p, q, l \) is zero. Without loss of generality, we may assume that \( l = 0 \). By a similar discussion as in the proof of Theorem 3.2, we can get the following result.

**Theorem 3.3.** Let \( \Phi \) be a \( T \)-gain bicyclic graph of order \( n \) with pendant vertices containing \( \theta^{p_2}(p, 0, q) \) as its base. Then

\[
r(\Phi) \geq \begin{cases} 
    2 + p + q, & \text{if } p + q \text{ is even,} \\
    1 + p + q, & \text{otherwise.}
\end{cases}
\]

This bound is sharp.
Combining Theorems 3.1, 3.2 and 3.3, we can get the following theorem.

**Theorem 3.4.** Let $\Phi = (G, \varphi)$ be a $T$-gain bicyclic graph of order $n$ with pendant vertices. Then

1. If $\Phi$ is a $T$-gain $\infty$-graph, then $r(\Phi) \geq 6$.
2. If $\Phi$ is a $T$-gain $\theta$-graph, then $r(\Phi) \geq 4$.

**Table 1**

The gain conditions for each gain graph in Theorem 4.1 with rank 2, 3 or 4.

| Rank of $G^r$ | Gain graph $G^r$ and its gain conditions                                                                 |
|---------------|--------------------------------------------------------------------------------------------------------|
| $r(G^r) = 2$  | $G^r_1$, the subgraph induced on vertices 1, 2, 3 is of Type E and the subgraph induced on vertices 1, 2, 4, 3 is of Type A. $G^r_2$, the subgraphs induced on vertices 1, 2, 3, 4 and 1, 2, 5, 4 are of Type A. |
| $r(G^r) = 3$  | $G^r_1$, the subgraph induced on vertices 1, 2, 3 is of Type C or D, and the subgraph induced on vertices 1, 2, 4, 3 is of Type A. |
| $r(G^r) = 4$  | $G^r_1$, $Re \left(-\varphi(e_{15})\frac{e_{12}}{e_{15}}\right) + Re \left(-\varphi(e_{12})\frac{e_{15}}{e_{12}}\right) = 0$. $G^r_2$, the subgraph induced on vertices 3, 4, 5, 6 is of Type A and the subgraph induced on vertices 1, 2, 3 is of Type E. $G^r_3$, the subgraphs induced on vertices 1, 2, 4, 3 and 4, 5, 6, 7 are of Type A. $G^r_4$, the subgraph induced on vertices 1, 2, 3 is of Type C, or D, or E and the subgraph induced on vertices 1, 2, 4, 3 is of Type B. $G^r_5$, $Re \left(-\varphi(e_{15})\frac{e_{12}}{e_{15}}\right) + Re \left(-\varphi(e_{12})\frac{e_{15}}{e_{12}}\right) = 0$. $G^r_6$, the subgraph induced on vertices 1, 2, 6 is of Type E and the subgraph induced on vertices 1, 2, 3, 4, 5, 6 is of Type A. $G^r_7$, $\varphi(e_{16})\varphi(e_{12})\varphi(e_{14}) - \varphi(e_{16})\varphi(e_{34})\varphi(e_{52}) + \varphi(e_{12})\varphi(e_{34})\varphi(e_{56}) = 0$. $G^r_8$, the subgraph induced on vertices 1, 2, 3, 4, 5, 6 is of Type E and the subgraph induced on vertices 1, 2, 5, 4 is of Type B. $G^r_9$, the subgraph induced on vertices 1, 2, 3, 4, 5 is of Type B and the subgraph induced on vertices 1, 2, 5, 4 is of Type A or B. $G^r_{10}$, the subgraph induced on vertices 1, 2, 3, 4, 5 is of Type E and the subgraph induced on vertices 1, 2, 6, 5 is of Type A. $G^r_{11}$, the subgraphs induced on vertices 1, 2, 3, 4, 5, 6 and 1, 2, 7, 6 are of Type A. |

4 The $T$-gain bicyclic graphs with rank 2, 3 or 4

In this section, we shall characterize the $T$-gain bicyclic graphs with rank 2, 3 or 4.

**Theorem 4.1.** Let $\Phi = (G, \varphi)$ be a $T$-gain bicyclic graph without pendant vertex. Then

1. $r(\Phi) = 2$ if and only if $\Phi$ with one of $G_i$’s ($i = 5, 9$) (as shown in Fig. 3) as its underlying graph, and the corresponding gain conditions are as shown in Table 1.
2. $r(\Phi) = 3$ if and only if $\Phi$ with $G_5$ (as shown in Fig. 3) as its underlying graph, and the corresponding gain conditions are as shown in Table 1.
By simple calculations, we have

\[ r(\Phi) = 4 \text{ if and only if } \Phi \text{ with one of } G_i \text{'s } (i = 1 - 3, \ 5 - 11) \text{ (as shown in Fig. 3) as its underlying graph, and the corresponding gain conditions are as shown in Table 1.} \]

**Proof.** Let \( \Phi = (G, \varphi) \) be a \( \mathbb{T} \)-gain bicyclic graph without pendant vertices. Then \( G \) is a underlying graph of \( \Phi \), and \( G^p \) is a base.

**Case 1.** \( G \) is an \( \infty \)-graph \( \infty(p, l, q) \).

Without loss of generality, we suppose that \( p \leq q \). If \( (p, l, q) \in \{(3, 1, 3), \ (3, 1, 4), \ (4, 1, 4), \ (3, 2, 3)\} \), then \( G \) is one of the graphs \( G_1 - G_4 \) in Fig. 3.

\[
A(G_1^p) = \begin{pmatrix}
0 & \varphi(e_{12}) & 0 & 0 & \varphi(e_{15}) \\
\varphi(e_{21}) & 0 & 0 & 0 & \varphi(e_{25}) \\
0 & 0 & 0 & \varphi(e_{34}) & \varphi(e_{35}) \\
0 & 0 & \varphi(e_{43}) & 0 & \varphi(e_{45}) \\
\varphi(e_{51}) & \varphi(e_{52}) & \varphi(e_{53}) & \varphi(e_{54}) & 0
\end{pmatrix}.
\]

By simple calculations, we have

\[
r(G_1^p) = r \begin{pmatrix}
0 & \varphi(e_{12}) \\
\varphi(e_{21}) & 0
\end{pmatrix} + r \begin{pmatrix}
0 & \varphi(e_{34}) \\
\varphi(e_{43}) & 0
\end{pmatrix} + r(a + \overline{a} + b + \overline{b}),
\]

where \( a = -\varphi(e_{15})\varphi(e_{12})/\varphi(e_{12}) \), \( b = -\varphi(e_{35})\varphi(e_{34})/\varphi(e_{34}) \). So \( r(G_1^p) = 4 \) if and only if \( a + \overline{a} + b + \overline{b} = 0 \), i.e., \( Re \left(-\varphi(e_{15})\varphi(e_{12})/\varphi(e_{12}) \right) + Re \left(-\varphi(e_{35})\varphi(e_{34})/\varphi(e_{34}) \right) = 0 \).

By Lemma 2.3 and Theorem 2.8 we have \( r(G_2^p) = 4 \) if and only if the subgraph induced on vertices 3, 4, 5, 6 is of Type A and the subgraph induced on vertices 1, 2, 3 is of Type E.

By Lemma 2.3 and Theorem 2.8 we have \( r(G_3^p) = 4 \) if and only if all the subgraphs induced on vertices 1, 2, 3 and 4, 5, 6, 7 are of Type A.

\[
A(G_4^p) = \begin{pmatrix}
0 & \varphi(e_{12}) & 0 & 0 & \varphi(e_{15}) & 0 \\
\varphi(e_{21}) & 0 & 0 & 0 & \varphi(e_{25}) & 0 \\
0 & 0 & \varphi(e_{34}) & 0 & \varphi(e_{36}) & 0 \\
0 & 0 & \varphi(e_{43}) & 0 & \varphi(e_{46}) & 0 \\
\varphi(e_{51}) & \varphi(e_{52}) & 0 & 0 & \varphi(e_{56}) & 0 \\
0 & 0 & \varphi(e_{63}) & \varphi(e_{64}) & \varphi(e_{65}) & 0
\end{pmatrix}.
\]

By simple calculations, we have

\[
r(G_4^p) = r \begin{pmatrix}
0 & \varphi(e_{12}) \\
\varphi(e_{21}) & 0
\end{pmatrix} + r \begin{pmatrix}
0 & \varphi(e_{34}) \\
\varphi(e_{43}) & 0
\end{pmatrix} + r \begin{pmatrix}
a + \overline{a} & \varphi(e_{56}) \\
\varphi(e_{65}) & b + \overline{b}
\end{pmatrix},
\]
where $a = -\varphi(e_{15})\frac{\varphi(e_{52})}{\varphi(e_{12})}$, $b = -\varphi(e_{36})\frac{\varphi(e_{64})}{\varphi(e_{34})}$.

By simple calculations, we have $1 \leq r(G_5^{\varphi}) \leq 6$ since $\varphi(e_{56}) \neq 0$ and $\varphi(e_{65}) \neq 0$.

If $p \geq 3$, $l \geq 1$, $q \geq 5$ or $p \geq 3$, $l \geq 2$, $q \geq 4$, by Lemmas 2.1, 2.2 and 2.3 then we have $r(\Phi) \geq 6$.

**Case 2.** $G$ is a $\theta$-graph $\theta(p, l, q)$.

Without loss of generality, we suppose that $p \leq l \leq q$. If $(p, l, q) \in \{(0, 1, 1), (0, 1, 2), (0, 1, 3), (0, 2, 2), (1, 1, 1), (1, 1, 2), (1, 1, 3)\}$, then $G$ is one of the graphs $G_5 - G_{11}$ in Fig. 3.

$$A(G_5^{\varphi}) = \begin{pmatrix} 0 & \varphi(e_{12}) & \varphi(e_{13}) & 0 \\ \varphi(e_{21}) & 0 & \varphi(e_{23}) & \varphi(e_{24}) \\ \varphi(e_{31}) & \varphi(e_{32}) & 0 & \varphi(e_{34}) \\ 0 & \varphi(e_{42}) & \varphi(e_{43}) & 0 \end{pmatrix}.$$  

By simple calculations, we have

$$r(G_5^{\varphi}) = r\left( \begin{pmatrix} 0 & \varphi(e_{12}) \\ \varphi(e_{21}) & 0 \end{pmatrix} \right) + r\left( \begin{pmatrix} a + \bar{a} \\ b + \bar{b} \end{pmatrix} \right),$$

where $a = -\varphi(e_{13})\frac{\varphi(e_{32})}{\varphi(e_{12})}$, $b = \varphi(e_{43}) - \varphi(e_{13})\frac{\varphi(e_{42})}{\varphi(e_{12})}$.

So, we have

$$r(G_5^{\varphi}) = \begin{cases} 2, & \text{if } \Re(a) = 0, \quad b = 0, \\ 3, & \text{if } \Re(a) \neq 0, \quad b = 0, \\ 4, & \text{if } \Re(a) = 0, \quad b \neq 0 \text{ or } \Re(a) \neq 0, \quad b \neq 0. \end{cases}$$

i.e., $r(G_5^{\varphi}) = 2$ if and only if the subgraph induced on vertices 1, 2, 3 is of Type E and the subgraph induced on vertices 1, 2, 4, 3 is of Type A.

$r(G_5^{\varphi}) = 3$ if and only if the subgraph induced on vertices 1, 2, 3 is of Type C or D and the subgraph induced on vertices 1, 2, 4, 3 is of Type A.

$r(G_5^{\varphi}) = 4$ if and only if the subgraph induced on vertices 1, 2, 3 is of Type E and the subgraph induced on vertices 1, 2, 4, 3 is of Type B, or the subgraph induced on vertices 1, 2, 3 is of Type C or D and the subgraph induced on vertices 1, 2, 4, 3 is of Type B.

$$A(G_6^{\varphi}) = \begin{pmatrix} 0 & \varphi(e_{12}) & 0 & 0 & \varphi(e_{15}) \\ \varphi(e_{21}) & 0 & \varphi(e_{23}) & 0 & \varphi(e_{25}) \\ 0 & \varphi(e_{32}) & 0 & \varphi(e_{34}) & 0 \\ 0 & 0 & \varphi(e_{43}) & 0 & \varphi(e_{45}) \\ \varphi(e_{51}) & \varphi(e_{52}) & 0 & \varphi(e_{54}) & 0 \end{pmatrix}.$$  

By simple calculations, we have

$$r(G_6^{\varphi}) = r\left( \begin{pmatrix} 0 & \varphi(e_{12}) \\ \varphi(e_{21}) & 0 \end{pmatrix} \right) + r\left( \begin{pmatrix} 0 & \varphi(e_{34}) \\ \varphi(e_{43}) & 0 \end{pmatrix} \right) + r(a + \bar{a} + b + \bar{b}),$$

where $a = -\varphi(e_{15})\frac{\varphi(e_{52})}{\varphi(e_{12})}$, $b = \frac{\varphi(e_{15})\varphi(e_{52})\varphi(e_{54})}{\varphi(e_{12})\varphi(e_{34})}$.

So, $r(G_6^{\varphi}) = 4$ if and only if $\Re\left( -\varphi(e_{15})\frac{\varphi(e_{52})}{\varphi(e_{12})} \right) + \Re\left( \frac{\varphi(e_{15})\varphi(e_{52})\varphi(e_{54})}{\varphi(e_{12})\varphi(e_{34})} \right) = 0.$
By simple calculations, we have

$$r(G_7^\varphi) = r \left( \begin{array}{cc} 0 & \varphi(e_{12}) \\ \varphi(e_{21}) & 0 \end{array} \right) + r \left( \begin{array}{cc} 0 & \varphi(e_{34}) \\ \varphi(e_{43}) & 0 \end{array} \right) + r \left( \begin{array}{cc} 0 & b \\ b & a + \pi \end{array} \right),$$

where \( a = -\varphi(e_{16})/\varphi(e_{12}) \), \( b = \varphi(e_{56}) + \varphi(e_{16})\varphi(e_{32})\varphi(e_{54})/\varphi(e_{12})\varphi(e_{34}) \).

So, \( r(G_7^\varphi) = 4 \) if and only if the subgraph induced on vertices 1, 2, 6 is of Type E and the subgraph induced on vertices 1, 2, 3, 4, 5, 6 is of Type A.

\[
A(G_7^\varphi) = \begin{pmatrix}
0 & \varphi(e_{12}) & 0 & 0 & 0 & \varphi(e_{16}) \\
\varphi(e_{21}) & 0 & \varphi(e_{23}) & 0 & 0 & \varphi(e_{26}) \\
0 & \varphi(e_{32}) & 0 & \varphi(e_{34}) & 0 & 0 \\
0 & 0 & \varphi(e_{43}) & 0 & \varphi(e_{45}) & 0 \\
0 & 0 & 0 & \varphi(e_{54}) & 0 & \varphi(e_{56}) \\
\varphi(e_{61}) & \varphi(e_{62}) & 0 & 0 & \varphi(e_{65}) & 0
\end{pmatrix}.
\]

By simple calculations, we have

\[
r(G_8^\varphi) = r \left( \begin{array}{cc} 0 & \varphi(e_{12}) \\ \varphi(e_{21}) & 0 \end{array} \right) + r \left( \begin{array}{cc} 0 & \varphi(e_{34}) \\ \varphi(e_{43}) & 0 \end{array} \right) + r \left( \begin{array}{cc} 0 & a \\ a & 0 \end{array} \right),
\]

where \( a = \varphi(e_{16})\varphi(e_{32})\varphi(e_{54}) - \varphi(e_{16})\varphi(e_{34})\varphi(e_{52}) + \varphi(e_{12})\varphi(e_{34})\varphi(e_{56}) \).

So, \( r(G_8^\varphi) = 4 \) if and only if \( a = 0 \), i.e.,

\[
\varphi(e_{16})\varphi(e_{32})\varphi(e_{54}) - \varphi(e_{16})\varphi(e_{34})\varphi(e_{52}) + \varphi(e_{12})\varphi(e_{34})\varphi(e_{56}) = 0.
\]

\[
A(G_8^\varphi) = \begin{pmatrix}
0 & \varphi(e_{12}) & 0 & 0 & \varphi(e_{14}) & 0 \\
\varphi(e_{21}) & 0 & \varphi(e_{23}) & 0 & 0 & \varphi(e_{25}) \\
0 & \varphi(e_{32}) & 0 & \varphi(e_{34}) & 0 & 0 \\
\varphi(e_{41}) & 0 & \varphi(e_{43}) & 0 & \varphi(e_{45}) & 0 \\
0 & \varphi(e_{52}) & 0 & 0 & \varphi(e_{54}) & 0 \\
\varphi(e_{61}) & \varphi(e_{62}) & 0 & 0 & 0 & \varphi(e_{65})
\end{pmatrix}.
\]

By simple calculations, we have

\[
r(G_9^\varphi) = r \left( \begin{array}{cc} 0 & \varphi(e_{12}) \\ \varphi(e_{21}) & 0 \end{array} \right) + r \left( \begin{array}{cc} 0 & \varphi(e_{14}) \\ \varphi(e_{34}) & 0 \end{array} \right) + r \left( \begin{array}{cc} 0 & a \\ a & 0 \end{array} \right),
\]

where \( a = \varphi(e_{34}) - \varphi(e_{14})\varphi(e_{12})/\varphi(e_{16}) \), \( b = \varphi(e_{54}) - \varphi(e_{14})\varphi(e_{52})/\varphi(e_{12}) \).

So, we have

\[
r(G_9^\varphi) = \begin{cases} 2, & \text{if } a = 0, b = 0, \\
4, & \text{if } a = 0, b \neq 0, \text{ or } a \neq 0, b = 0, \text{ or } a \neq 0, b \neq 0.
\end{cases}
\]
i.e., \( r(G^\varphi_9) = 2 \) if and only if the subgraphs induced on vertices 1, 2, 3, 4 and 1, 2, 5, 4 are of Type A.

\( r(G^\varphi_9) = 4 \) if and only if the subgraph induced on vertices 1, 2, 3, 4 is of Type A and the subgraph induced on vertices 1, 2, 5, 4 is of Type B, or the subgraph induced on vertices 1, 2, 3, 4 is of Type B and the subgraph induced on vertices 1, 2, 5, 4 is of Type A or B.

\[
A(G^\varphi_{10}) = \begin{pmatrix}
0 & \varphi(e_{12}) & 0 & 0 & 0 & \varphi(e_{15}) & 0 \\
\varphi(e_{21}) & 0 & \varphi(e_{23}) & 0 & 0 & \varphi(e_{26}) & 0 \\
0 & \varphi(e_{32}) & 0 & \varphi(e_{34}) & 0 & 0 & 0 \\
0 & 0 & \varphi(e_{43}) & 0 & \varphi(e_{45}) & 0 & 0 \\
\varphi(e_{51}) & 0 & 0 & \varphi(e_{54}) & 0 & \varphi(e_{56}) & 0 \\
0 & \varphi(e_{62}) & 0 & 0 & \varphi(e_{65}) & 0 & 0
\end{pmatrix}.
\]

By simple calculations, we have

\[
r(G^\varphi_{10}) = r \begin{pmatrix}
0 & \varphi(e_{12}) \\
\varphi(e_{21}) & 0
\end{pmatrix} + r \begin{pmatrix}
0 & \varphi(e_{34}) \\
\varphi(e_{43}) & 0
\end{pmatrix} + r \begin{pmatrix}
a + \pi & b \\
b & 0
\end{pmatrix},
\]

where \( a = \frac{\varphi(e_{15})\varphi(e_{32})\varphi(e_{54})}{\varphi(e_{12})\varphi(e_{34})}, \quad b = \varphi(e_{65}) - \frac{\varphi(e_{15})\varphi(e_{62})}{\varphi(e_{12})}.\)

So, \( r(G^\varphi_{10}) = 4 \) if and only if \( Re(a) = 0 \) and \( b = 0 \), i.e., the subgraph induced on vertices 1, 2, 3, 4, 5 is of Type E and the subgraph induced on vertices 1, 2, 6, 5 is of Type A.

\[
A(G^\varphi_{11}) = \begin{pmatrix}
0 & \varphi(e_{12}) & 0 & 0 & 0 & \varphi(e_{16}) & 0 \\
\varphi(e_{21}) & 0 & \varphi(e_{23}) & 0 & 0 & 0 & \varphi(e_{27}) \\
0 & \varphi(e_{32}) & 0 & \varphi(e_{34}) & 0 & 0 & 0 \\
0 & 0 & \varphi(e_{43}) & 0 & \varphi(e_{45}) & 0 & 0 \\
0 & 0 & 0 & \varphi(e_{54}) & 0 & \varphi(e_{56}) & 0 \\
\varphi(e_{61}) & 0 & 0 & 0 & \varphi(e_{65}) & 0 & \varphi(e_{67}) \\
0 & \varphi(e_{72}) & 0 & 0 & 0 & \varphi(e_{76}) & 0
\end{pmatrix}.
\]

By simple calculations, we have

\[
r(G^\varphi_{11}) = r \begin{pmatrix}
0 & \varphi(e_{12}) \\
\varphi(e_{21}) & 0
\end{pmatrix} + r \begin{pmatrix}
0 & \varphi(e_{34}) \\
\varphi(e_{43}) & 0
\end{pmatrix} + r \begin{pmatrix}
a & 0 & 0 \\
\pi & 0 & b \\
b & 0 & 0
\end{pmatrix},
\]

where \( a = \varphi(e_{56}) + \frac{\varphi(e_{16})\varphi(e_{32})\varphi(e_{54})}{\varphi(e_{12})\varphi(e_{34})}, \quad b = \varphi(e_{76}) - \frac{\varphi(e_{16})\varphi(e_{72})}{\varphi(e_{12})}.\)

So, we have \( r(G^\varphi_{11}) = 4 \) if and only if \( a = 0, \quad b = 0.\)

i.e., \( r(G^\varphi_{11}) = 4 \) if and only if the subgraphs induced on vertices 1, 2, 3, 4, 5, 6 and 1, 2, 7, 6 are of Type A.

If \( p \geq 0, \ l \geq 1, \ q \geq 4 \) or \( p \geq 0, \ l \geq 2, \ q \geq 3 \) or \( p \geq 1, \ l \geq 1, \ q \geq 4, \) by Lemmas 2.1, 2.2 and 2.3 then we have \( r(\Phi) \geq 6.\)

This proof is complete. \( \square \)

**Theorem 4.2.** Let \( \Phi = (G, \varphi) \) be a \( T \)-gain bicyclic graph with pendant vertices but no pendant twins. Then \( r(\Phi) = 4 \) if and only if \( \Phi \) with one of \( G_i \)'s \( (i = 12 - 22) \) (as shown in Fig. 4) as its underlying graph, and the corresponding gain conditions are as shown in Table 2.

**Proof.** By Theorems 3.2, 3.3 and 3.4, we need only characterize all \( T \)-gain bicyclic graphs with \( \theta(0,1,1), \theta(0,1,2), \) and \( \theta(1,1,1) \) as bases.
Table 2

The gain conditions for each gain graph in Theorem 4.2 satisfying \( r(G^\varphi) = 4 \).

| Gain graphs \( G^\varphi \) | Gain conditions of \( G^\varphi \) |
|---------------------------|------------------------------------------|
| \( G_{12}^\varphi \)     | the subgraph induced on vertices 1, 2, 3 is of Type E. |
| \( G_{13}^\varphi, G_{14}^\varphi, G_{19}^\varphi, G_{20}^\varphi \) | any gain. |
| \( G_{15}^\varphi, G_{16}^\varphi \) | the subgraph induced on vertices 1, 2, 3 is of Type E and the subgraph induced on vertices 1, 2, 4, 3 is of Type A. |
| \( G_{17}^\varphi, G_{18}^\varphi \) | the subgraph induced on vertices 1, 2, 3, 4 is of Type A. |
| \( G_{21}^\varphi, G_{22}^\varphi \) | the subgraphs induced on vertices 1, 2, 3, 4 and 1, 2, 5, 4 are of Type A. |

![Diagram of gain graphs](image)

Figure 4: The eleven graphs in Theorem 4.2.

**Case 1.** \( G \) is a bicyclic graph with \( \theta(0,1,1) \) as a base.

**Subcase 1.1** \( G - \theta(0,1,1) \) is a collection of isolated vertices.

If \( |G - \theta(0,1,1)| = 1 \), then \( G \) is \( G_{12} \) or \( G_{13} \) (as shown in Fig. 4). For \( G_{12}^\varphi \), by Lemmas 2.2 and 2.3 we have \( r(G_{12}^\varphi) = 4 \) if and only if the subgraph induced on vertices 1, 2, 3 in \( G_{12}^\varphi \) is of Type E. For \( G_{13}^\varphi \), by Lemma 2.2 we have \( r(G_{13}^\varphi) = 4 \), where each edge in \( G_{13}^\varphi \) has any gain.

If \( |G - \theta(0,1,1)| = 2 \), then \( G \) is \( G_{14} \) (as shown in Fig. 4). By Lemma 2.2 we have \( r(G_{14}^\varphi) = 4 \), where each edge in \( G_{14}^\varphi \) has any gain.

**Subcase 1.2** If \( G - \theta(0,1,1) = P_3 \), then \( G \) is \( G_{15} \) or \( G_{16} \) (as shown in Fig. 4). By Lemma 2.2 and Theorem 4.1 we have \( r(\Phi) = 4 \) if and only if the subgraph induced on vertices 1, 2, 3 is of Type E and the subgraph induced on vertices 1, 2, 4, 3 is of Type A in \( G_{15}^\varphi \) or \( G_{16}^\varphi \).

**Subcase 1.3** If \( G - \theta(0,1,1) \) contains the union of \( P_2 \) and isolated vertices or contains \( P_3 \) as an induced subgraph, by Lemmas 2.2 and 2.3 then we have \( r(\Phi) \geq 6 \).

**Case 2.** \( G \) is a bicyclic graph with \( \theta(0,1,2) \) as a base.

**Subcase 2.1** \( G - \theta(0,1,2) \) is a collection of isolated vertices.

If \( |G - \theta(0,1,2)| = 1 \), then \( G \) is \( G_{17} \) (as shown in Fig. 4). By Lemmas 2.2 and 2.3 we have \( r(G_{17}^\varphi) = 4 \) if and only if the subgraph induced on vertices 1, 2, 3, 4 in \( G_{17}^\varphi \) is of Type A.

If \( |G - \theta(0,1,2)| \geq 2 \), by Lemma 2.2 then we have \( r(\Phi) \geq 6 \).

**Subcase 2.2** \( G - \theta(0,1,2) \) contains \( P_2 \) as an induced subgraph.

In this case, by Lemma 2.2 and Theorem 4.1 we have \( r(\Phi) \geq 6 \).

**Case 3.** \( G \) is a bicyclic graph with \( \theta(1,1,1) \) as a base.

**Subcase 3.1** \( G - \theta(1,1,1) \) is a collection of isolated vertices.

If \( |G - \theta(1,1,1)| = 1 \), then \( G \) is \( G_{18} \) or \( G_{19} \) (as shown in Fig. 4). For \( G_{18}^\varphi \), by Lemmas 2.2 and 2.3 we have \( r(G_{18}^\varphi) = 4 \) if and only if the subgraph induced on vertices 1, 2, 3, 4 in \( G_{18}^\varphi \) is of Type A. For \( G_{19}^\varphi \), by Lemma 2.2 we have \( r(G_{19}^\varphi) = 4 \), where each edge in \( G_{19}^\varphi \) has any gain.

If \( |G - \theta(1,1,1)| = 2 \), then \( G \) is \( G_{20} \) (as shown in Fig. 4). By Lemma 2.2 we have \( r(G_{20}^\varphi) = 4 \), where each edge in \( G_{20}^\varphi \) has any gain.

If \( |G - \theta(1,1,1)| \geq 3 \), by Lemma 2.2 then we have \( r(\Phi) \geq 6 \).
**Subcase 3.2** If $G - \theta(1, 1, 1) = P_2$, then $G$ is $G_{21}$ or $G_{22}$ (as shown in Fig. 4). By Lemma 2.2 and Theorem 4.1, we have $r(\Phi) = 4$ if and only if the subgraphs induced on vertices 1, 2, 3, 4 and 1, 2, 5, 4 are of Type A in $G_{21}^x$ or $G_{22}^x$.

**Subcase 3.3** If $G - \theta(1, 1, 1)$ contains the union of $P_2$ and isolated vertices or contains $P_3$ as an induced subgraph, by Lemmas 2.2 and 2.3 then we have $r(\Phi) \geq 6$.

This proof is complete. □

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