RECURSION CATEGORIES OF COALGEBRAS

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ABSTRACT. We construct recursion categories from categories of coalgebras. Let $F$ be a nontrivial endofunctor on the category of sets that weakly preserves pullbacks and such that the category $\text{Set}_F$ of $F$-coalgebras is complete. The category $\text{Set}_F$ may be embedded in the category $\text{Pfn}_F$ of $F$-coalgebras and partial morphisms, which is a $P$-category that is prodominical but not dominical in general. An existence theorem of A. Heller is applied to certain subcategories of $\text{Pfn}_F$ to obtain examples of recursion categories of coalgebras.

1. Introduction

The theory of universal coalgebras has recently undergone vigorous development as a descriptive study of automata, data structures, and the semantics of object oriented programming, among other applications [Rut00, Gum99]. Coalgebras as data structures include colorings, permutations, trees, partial self-mappings, the real numbers, streams and formal power series [Gum99, PP99, PE98]. Dominical categories [DPH87] were developed by DiPaola and Heller as an element-free generalization of classical recursion theory, in which one has categorical analogs of the construction of the universal Turing machine, the s-m-n theorem and the Kleene recursion theorem. Rosolini’s $P$-categories [Ros86] generalize the dominical categories and have gained acceptance as an appropriate setting for categorical developments of recursion theory and programming semantics [Hel90, Tay93]. We construct prodominical recursion categories of coalgebras; in particular, this shows that many categories of coalgebras possess a Turing morphism, which is a categorical generalization of the universal Turing machine.

Let $F$ be a nontrivial endofunctor on the category of sets such that $F$ weakly preserves pullbacks, and such that the category $\text{Set}_F$ of $F$-coalgebras is complete. For example, if the forgetful functor $\text{Set}_F \to \text{Set}$ has a right adjoint, then $\text{Set}_F$ is complete [Kur00]. Under various mild assumptions on the endofunctor $F$, such as boundedness or accessiblility, the right adjoint will exist [GS00c, Kur00, PW98]. The
category $\text{Set}_F$ may be embedded in the category $\text{Pfn}_F$ of $F$-coalgebras and partial morphisms, which is a $P$-category that is predominical but not dominical in general.

This paper is organized as follows. In the first section we recall definitions and results needed from the theory of coalgebras that we use to construct the category $\text{Pfn}_F$ of coalgebras and partial morphisms. Next, we review A. Heller’s theory of recursion categories [Hel90], and verify that $\text{Pfn}_F$ satisfies certain properties required for the application of Heller’s existence theorem. In the second section we state Heller’s existence theorem and apply it to obtain our existence result. In the final section we show that $\text{Pfn}_F$ contains many recursion categories. In conclusion, we mention related results beyond the scope of this paper.

1.1. Coalgebras. Let $F$ be an endofunctor on the category $\text{Set}$ of sets. An $F$-coalgebra $(A, \alpha)$ is a set $A$ together with a map $\alpha : A \to FA$ called its structure map. A morphism of $F$-coalgebras $\varphi : (A, \alpha) \to (B, \beta)$ is a map $\varphi : A \to B$ such that $\beta \circ \varphi = F\varphi \circ \alpha$. The $F$-coalgebras and their morphisms form a category, denoted by $\text{Set}_F$.

Coalgebras in this sense are more general than those associated with comonads [ML98]. The category $\text{Set}_F$ is cocomplete, co-wellpowered, and is closed under homomorphic images [Gum99].

Let $(A, \alpha)$ be a coalgebra. A subset $U$ of $A$ has at most one coalgebra structure such that the inclusion $U \hookrightarrow A$ induces a morphism of coalgebras. A subcoalgebra of a coalgebra $(A, \alpha)$ is a coalgebra $(U, \beta)$ such that the inclusion is a morphism. The set endofunctor $F$ is nontrivial if for any set $X$, $FX = \emptyset$ implies $X = \emptyset$. If $F$ is nontrivial, then the set of subcoalgebras of a given $F$-coalgebra $X$ forms a topology on $X$ called the coalgebra topology [GS00c].

A commutative square in a category $\mathbf{C}$ is a weak pullback if it satisfies all of the conditions of a pullback except for the uniqueness requirement for induced morphisms. Analogously, one may define a weak limit of a diagram in $\mathbf{C}$. A sink is a collection of morphisms of $\mathbf{C}$ with a common codomain. A (weak) generalized pullback is a (weak) limit of a sink. A functor preserves weak (generalized) pullbacks if it sends weak (generalized) pullbacks to weak (generalized) pullbacks. A functor weakly preserves (generalized) pullbacks if it sends (generalized) pullbacks to weak (generalized) pullbacks. If (generalized) pullbacks exist in $\mathbf{C}$, and if $F : \mathbf{C} \to \text{Set}$ is a set valued functor, then $F$ preserves weak (generalized) pullbacks if and only if $F$ weakly preserves (generalized) pullbacks [Gum99]. More generally, if $\mathbf{C}$ has all limits of certain diagrams $D$, then $F$ preserves weak limits of $D$-diagrams if and only if $F$ weakly preserves limits of $D$-diagrams.
The following \textbf{Set} endofunctors preserve weak limits of sinks: the identity functor, all constant functors, functors of the form $X \mapsto Y$ for a fixed set $Y$, functors of the form $X \mapsto X^Y$ for a fixed set $Y$, and the powerset functor. Moreover, sums, products and composites of \textbf{Set} endofunctors that preserve weak limits of sinks also preserve weak limits of sinks \cite{GS00a}.

Gumm and Schröder \cite{GS00a} prove that the \textbf{Set} endofunctor $F$ weakly preserves pullbacks along monomorphisms if and only for any morphism $\phi : A \to B$ of $F$-coalgebras and for any subcoalgebra $U$ of $B$, the preimage $\phi^{-1}[U]$ of $U$ under $\phi$ is a subcoalgebra of $A$.

A \textbf{Set} endofunctor $F$ is \textit{bounded by the cardinal} $\kappa$ if for each $F$-coalgebra $A$, and for each $a \in A$ there is a subcoalgebra $B$ of $A$ containing $a$ such that $|B| \leq \kappa$. Let $C$ and $M$ be two fixed, possibly empty sets. Any functor of the form $X \mapsto C \times X^M$ is bounded; in particular, the identity functor and the constant functors are bounded. Moreover, sums, products and compositions of bounded functors are bounded. A characterization of bounded functors is given in \cite{GS00a}. Examples of coalgebras for bounded functors include colorings, which are coalgebras for a constant \textbf{Set} endofunctor, self-maps of a set, which are coalgebras for the identity functor, and stream coalgebras, which are coalgebras for the \textbf{Set} endofunctor given by $X \mapsto A \times X$, where $A$ is a fixed nonempty set \cite{Gum99, PE98, PP99}. In each of these examples, the type endofunctor preserves weak limits of sinks.

Let $U : \text{Set}_F \to \text{Set}$ be the \textit{forgetful} functor, which sends an $F$-coalgebra $\alpha : A \to F(A)$ to the set $A$, and a morphism of coalgebras to itself as a morphism of \textbf{Set}. Kurz \cite{Kur00} has shown that if the forgetful functor $U$ has a right adjoint, then $\text{Set}_F$ is complete. The existence of the right adjoint implies that the cofree coalgebra on any set exists; limits are constructed as certain subcoalgebras of cofree coalgebras. Gumm and Schröder construct such a right adjoint to the forgetful functor $U$ in the case when the endofunctor $F$ is bounded; hence if $F$ is bounded, $\text{Set}_F$ is complete \cite{GS00c}.

Let $X_i, i = 0, 1$ be $F$-coalgebras. A \textit{bisimulation} between $X_0$ and $X_1$ is a relation $R \subseteq UX_0 \times UX_1$ equipped with an $F$-coalgebra structure $\rho : R \to F(R)$, called a \textit{bisimulation structure}, such that the projections $\pi_i : R \to X_i, i = 0, 1$ are morphisms of coalgebras \cite{Gum99, Rut00}. A bisimulation structure need not be unique if it exists. If $X$ is a coalgebra, and if $\varphi_i : X \to Y_i, i = 0, 1$ are coalgebra morphisms, then $(U \varphi_0 \times U \varphi_1) UX$ is a bisimulation between $Y_0$ and $Y_1$ \cite{Gum99}. Let $X,Y$ be coalgebras and let $\varphi : UX \to UY$ be a map in \textbf{Set}. Then $\varphi$ defines a coalgebra morphism $X \to Y$ if and only if its graph is a
bisimulation; it follows that for any coalgebra $X$, the diagonal $\Delta_X = \{(x, x) : x \in UX\}$ is a bisimulation \cite{Rut00}.

A functor between $\kappa$-accessible categories is $\kappa$-accessible if it preserves $\kappa$-filtered colimits. It follows from results of Power and Watanabe \cite{PW98} that if $F$ is an $\omega$-accessible endfunctor, then $\text{Set}_F$ is complete.

1.2. **Partial morphisms of coalgebras.** Let $F$ denote an endofunctor on $\text{Set}$. Under certain restrictions on $F$, the category $\text{Set}_F$ can be extended to the category of $F$-coalgebras and partial morphisms, denoted by $\text{Pfn}_F$, which we will obtain by following the procedure in Rosolini’s thesis \cite{Ros86} for constructing a category of partial maps.

Let $C$ be a category with finite limits. A dominion is a class $\mathcal{M}$ of monomorphisms of $C$ that is closed under identities, composition and pullbacks of morphisms of $C$ \cite{Ros86}. If $\mathcal{M}$ is a dominion of $C$ and if $X$ and $Y$ are objects of $C$, a partial map from $X$ to $Y$ defined in $\mathcal{M}$ is a pair $(m, \varphi)$, where $m : U \to X$ is in $\mathcal{M}$, and where $\varphi : U \to Y$ is a morphism of $C$. Two such pairs $(m, \varphi)$ and $(m', \varphi')$ are equivalent if there exists an isomorphism $\theta$ making the following obvious diagram commute.

\[ U \xrightarrow{\varphi} Y \\
\downarrow m \hspace{1cm} \downarrow \theta \\
X \xrightarrow{m'} U' \]

The equivalence class of the pair $(m, \varphi)$ is denoted by $\{m, \varphi\}$ and is called a partial morphism from $X$ to $Y$, also denoted by $X \rightarrow_{\text{Pfn}_F} Y$. Composition of partial morphisms $A \rightarrow_{\text{Pfn}_F} B$ and $B \rightarrow_{\text{Pfn}_F} C$ is defined by the following diagram in which the square is a pullback.

\[ W \xrightarrow{\varphi} V \xrightarrow{\varphi'} C \\
\downarrow \varphi' \downarrow \{m', \varphi'\} \downarrow \{m, \varphi\} \\
U \xrightarrow{m} B \\
\downarrow \varphi \\
A \]

The equivalence class $\{m'', \varphi''\}$ of the composite is determined by the monomorphism $m'' : W \to A$ obtained as the composite of the morphisms of the left vertical column, and the morphism $\varphi'' : W \to C$ obtained as the composite of the morphisms of the top horizontal row.
We can then express the composite by the equation
\[ \{m'', \varphi''\} = \{m', \varphi'\} \circ \{m, \varphi\}. \]

The composite is associative. For each object \( A \), the equivalence class \( \{1_A, 1_A\} \) acts as an identity for composition. Given a category \( C \) with finite products together with a dominion \( \mathcal{M} \), the category of partial maps \( P(C, \mathcal{M}) \) is the category whose objects are those of \( C \) and whose morphisms are the equivalence classes of partial maps defined in \( \mathcal{M} \).

We will define the category \( \mathbf{Pfn}_F \) as \( P(\mathbf{Set}_F, \mathcal{M}) \) by taking the dominion \( \mathcal{M} \) to be the class of monomorphisms of \( \mathbf{Set}_F \) subject to conditions on the endofunctor \( F \) that will make \( \mathbf{Set}_F \) complete and such that monomorphisms will correspond to subcoalgebras.

For the first requirement, we note that if the forgetful functor \( \mathbf{Set}_F \to \mathbf{Set} \) has a right adjoint, which happens, for example, if \( F \) is bounded or accessible, then \( \mathbf{Set}_F \) is complete \( [\text{GS}00, \text{Kur}00, \text{PW}98] \); in particular, \( \mathbf{Set}_F \) has products.

For the second requirement, we note that although a morphism in \( \mathbf{Set}_F \) that is injective in \( \mathbf{Set} \) is a monomorphism in \( \mathbf{Set}_F \), the converse does not hold in general \( [\text{Gum}99, \text{Rut}00] \), and so monomorphisms do not necessarily correspond to subcoalgebras in \( \mathbf{Set}_F \). To obtain a workable notion of a partial map, we will use the following result on factorization systems in \( \mathbf{Set}_F \) due to Kurz, who introduced in his thesis the technique of lifting factorization systems from the base category to obtain a characterization of subcoalgebras in general categories of coalgebras \( [\text{Kur}00]\).

Let \( C \) be a category. A monomorphism is regular if it is an equalizer. A monomorphism \( m \) in \( C \) is extremal if and only if \( m = fe \) for some \( f \) and for some epi \( e \) implies that \( e \) is an isomorphism. A monomorphism \( m \) is strong if and only if for all epis \( e \) and for all \( f, g \) such that the following square commutes, there is a unique \( d \) such that the triangles in the following diagram commute.

\[
\begin{array}{ccc}
A & \xrightarrow{e} & B \\
\downarrow{f} & & \downarrow{g} \\
C & \xrightarrow{m} & D
\end{array}
\]

Strong monomorphisms are closed under composition, intersection and left cancellation. The classes of monomorphisms and of regular, strong monomorphisms of \( \mathbf{Set}_F \) are precisely the injective morphisms; however, the use of factorization systems applies to more general categories of coalgebras.

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\[\text{GS}00, \text{Kur}00, \text{PW}98, \text{Gum}99, \text{Rut}00\]

\[\text{Kur}00\]

\[\text{It is sufficient for our application to cite Proposition 4.7 of [Rut00], which states that if } F \text{ preserves weak pullbacks, then monomorphisms of } \mathbf{Set}_F \text{ are precisely the injective morphisms; however, the use of factorization systems applies to more general categories of coalgebras.}\]
and extremal monomorphisms of \( C \) are denoted by \( \text{Mono} \), \( \text{RegMono} \), \( \text{StrongMono} \) and \( \text{ExtrMono} \), respectively. We have the inclusions

\[
\text{ExtrMono} \subseteq \text{StrongMono} \subseteq \text{RegMono} \subseteq \text{Mono}.
\]

**Theorem 1.1** (Theorem 1.3.9 [Kur00]). The following assertions hold in \( \text{Set}_F \).

1. \((\text{Epi}, \text{StrongMono})\) is a factorization system in \( \text{Set}_F \). Moreover, \( \text{Epi} \) contains the surjective coalgebra morphisms, \( \text{StrongMono} \) contains precisely the injective morphisms, and \((\text{Epi}, \text{StrongMono})\)-factorizations are calculated as \((\text{Epi}, \text{Mono})\)-factorizations in \( \text{Set} \).

2. The classes \( \text{ExtrMono} \), \( \text{StrongMono} \) and \( \text{RegMono} \) coincide.

3. If \( F \) preserves weak pullbacks, then \( \text{Mono} \) coincides with \( \text{StrongMono} \) and hence with \( \text{ExtrMono} \) and \( \text{RegMono} \).

In virtue of the preceding, unless stated otherwise we will assume henceforth that the \( \text{Set} \) endofunctor \( F \) is nontrivial, weakly preserves pullbacks and is either bounded or accessible. Fixing such an \( F \), we define the dominion \( \mathcal{M} \) in \( \text{Set}_F \) to be the class of monomorphisms of \( \text{Set}_F \) and we set \( \text{Pfn}_F = P(\text{Set}_F, \mathcal{M}) \). Next, we extend the product \( \times : \text{Set}_F^2 \to \text{Set}_F \) to \( \boxtimes : \text{Pfn}_F^2 \to \text{Pfn}_F \) by defining it on objects \( X,Y \) by \( X \boxtimes Y = X \times Y \) and on partial morphisms by the formula

\[
\{m, \varphi\} \boxtimes \{m', \varphi'\} = \{m \times m', \varphi \times \varphi'\}.
\]

This assignment makes sense since \( m \times m' \in \mathcal{M} \) and is independent of the choice of representatives. The reason for introducing the box product notation is to distinguish it among the four products in use, namely, the product of categories, the product in \( \text{Set} \), the product in \( \text{Set}_F \) and its extension to a near product, to be defined, on \( \text{Pfn}_F \). With these definitions, a coalgebraic version of Rosolini’s Proposition 2.1.1 is immediate [Ros86].

**Proposition 1.2.** There is a faithful embedding of \( \text{Set}_F \) into \( \text{Pfn}_F \) which sends each \( F \)-coalgebra to itself, and which sends the \( F \)-coalgebra morphism \( \varphi : A \to B \) to the equivalence class \( \{1_A, \varphi\} \). The product \( \times \) on \( \text{Set}_F \) extends to the bifunctor \( \boxtimes \) on \( \text{Pfn}_F \).

Since the forgetful functor \( U : \text{Set}_F \to \text{Set} \) creates colimits, it follows that the category \( \text{Pfn}_F \) is cocomplete.

1.3. \( P \)- and \( P\Sigma \)-categories of coalgebras. The category \( \text{Pfn}_F \) with the product \( \boxtimes \) has the structure of a \( P \)-category, a notion due to [Ros86] which generalizes the dominical categories of [DPHS87] and
the $B$-categories (categories with a binary product) of $[\text{Hel90}]$. We recall the relevant definitions from $[\text{Hel90}]$ with slight changes in notation and with some additional remarks.

Let $F, G : C \to D$ be functors. An infranatural transformation $\phi : F \to G$ is a family of morphisms $\phi_X : FX \to GX$, called components, for each object $X$ of $C$. The naturalizer of $\phi$, denoted by $\text{nat} \phi$, is the largest subcategory of $C$ containing all the objects of $C$ such that $\phi$ is a natural transformation $F|\text{nat} \phi \to G|\text{nat} \phi$.

Let $C$ be a category. The diagonal functor $\delta_C : C \to C \times C$ is given on objects by $A \mapsto (A, A)$ and on morphisms by $f \mapsto (f, f)$.

A $P$-category consists of a category $C$ together with a functor $\boxtimes : C \times C \to C$, called a near product, a natural transformation $\Delta : 1_C \to \boxtimes \circ \delta_C$, and infranatural transformations $p_i : \boxtimes \to \pi_i$, where $\pi_i : C^2 \to C$ is the projection onto the $i$-th factor for $i = 0, 1$. For morphisms $f : X \to Y, g : X \to Z$ of $C$, we set $\langle f, g \rangle = (f \boxtimes g)\Delta_X : X \to Y \boxtimes Z$.

These functors and transformations are subject to the following four conditions:

i) For objects $X$ in $C$ the following diagrams commute.

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta_X} & X \boxtimes X \\
\downarrow & & \downarrow \Delta_{\boxtimes X} \\
X \boxtimes X & \xrightarrow{p_{1X,X}} & X
\end{array}
\quad
\begin{array}{ccc}
X \boxtimes X & \xrightarrow{\Delta_{\boxtimes X}} & (X \boxtimes X) \boxtimes (X \boxtimes X) \\
\downarrow & & \downarrow 1 \\
X \boxtimes X & \xrightarrow{p_{0X,X} \boxtimes p_{1X,X}} & X \boxtimes X
\end{array}
\]

where $i = 0, 1$.

ii) If $P \subseteq C$ is the smallest subcategory closed under $\boxtimes$ containing all components of $p_i, i = 0, 1$, then $C \times P \subseteq \text{nat} p_0$ and $P \times C \subseteq \text{nat} p_1$. This implies that that projections satisfy certain identities; e.g., as in the following naturality square diagram, where we have used the property that the component $p_{0Y,Z} : Y \boxtimes Z \to Y$ is in $P$, and therefore $(1_X, p_{0Y,Z})$ is in $\text{nat} p_0$.

\[
\begin{array}{ccc}
(X, Y \boxtimes Z) & \xrightarrow{1_X \boxtimes p_{0Y,Z}} & X \boxtimes (Y \boxtimes Z) \\
\downarrow & & \downarrow \text{ass}_{\boxtimes} \\
(X, Y) & \xrightarrow{1_X} & X
\end{array}
\]

iii) There is a natural isomorphism

$\text{ass}_{\boxtimes} : ((- \boxtimes -) \boxtimes -) \to (- \boxtimes (- \boxtimes -))$

of functors $C^3 \to C$ whose component $\text{ass}_{\boxtimes X,Y,Z}$ is given by

$\langle p_{0X,Y}p_{0X,YZ}, \langle p_{1X,Y}p_{0X,YZ}, p_{1X,YZ} \rangle \rangle : (X \boxtimes Y) \boxtimes Z \to X \boxtimes (Y \boxtimes Z)$. 

iv) Let $\text{tr}_\times$ be the endofunctor on $C^2$ given by $(X,Y) \mapsto (Y,X)$. There is a natural isomorphism

$$\text{tr}_\times : \boxtimes \longrightarrow \boxtimes \circ \text{tr}_\times$$

of functors $C^2 \rightarrow C^2$ whose component $\text{tr}_\times \boxtimes_{X,Y}$ is given by

$$\langle p_1X,Y, p_0X,Y \rangle : X \boxtimes Y \longrightarrow Y \boxtimes X .$$

The natural isomorphisms $\boxtimes$ and $\text{tr}_\times \boxtimes$ must make $\boxtimes$ coherently associative and commutative; i.e., the natural isomorphism $\text{tr}_\times \boxtimes$ must satisfy the condition $\text{tr}_\times \boxtimes_{Y,X} \circ \text{tr}_\times \boxtimes_{X,Y} = 1_{X \boxtimes Y}$ and a hexagonal coherence condition; the natural isomorphism $\text{ass}_\boxtimes$ must satisfy a pentagonal coherence condition \cite{ML98, ML63}.

**Proposition 1.3.** The category $\text{Pfn}_F$ is a $P$-category.

**Proof.** This follows from Rosolini’s Theorem 2.1.9 \cite{Ros86} applied to $\text{Set}_F$. \hfill $\square$

1.4. **Prodominical categories.** A system of zero morphisms is a collection of morphisms $0_{X,Y} : X \rightarrow Y$ for each pair of objects $X$ and $Y$ of $C$ such that for objects $W,Z$ and morphisms $f : W \rightarrow X$ and $g : Y \rightarrow Z$ of $C$, one has $g0_{X,Y}f = 0_{W,Z}$. A system of zero morphisms is unique if it exists. For any coalgebra $A$, the empty map $0_{\emptyset, A} : \emptyset \rightarrow A$ is vacuously a morphism in $\text{Set}_F$. The equivalence classes $\{0_{\emptyset, A}, 0_{\emptyset, B}\}$ for coalgebras $A$ and $B$ form a system of zeros of $\text{Pfn}_F$.

A prodominical category $C$ is a $P$-category that is pointed; i.e., $C$ contains a system of zero morphisms and, for any $\phi : A \rightarrow B$, $\phi \boxtimes 0_{C,D} = 0_{A^C,B^C,D}$. A morphism $f : X \rightarrow Y$ of a pointed category is weakly total if for all $\phi : W \rightarrow X$, $f\phi = 0_{W,Y}$ implies that $\phi = 0_{W,X}$.

We take the following proposition from Heller \cite{He90} as the definition of a dominical category. A prodominical category is dominical if every weakly total morphism is total and if $\phi \boxtimes \psi = 0$ implies $\phi = 0$ or $\psi = 0$. This property fails in $\text{Pfn}_F$ in general, for two reasons.

The category $\text{Pfn}_F$ can fail to be dominical due to certain pathologies of the product in $\text{Set}_F$; Gumm and Schröder \cite{GS00b} give examples of three finite coalgebras $A, B, C$ for the finite powerset functor such that $A \times A \cong A$, $A \times B = \emptyset$ and such that $C \times C$ is infinite.

**Proposition 1.4.** There exists a nontrivial, bounded $\text{Set}$ endofunctor $F$ that weakly preserves pullbacks such that $\text{Pfn}_F$ is prodominical but not dominical.

**Proof.** The finite powerset functor $P_\omega$ is nontrivial, preserves weak pullbacks and is bounded \cite{GS00b, GS00a, PW98}, hence $\text{Pfn}_{P_\omega}$ is prodominical. Moreover there exist two nonempty finite coalgebras $A$ and
for the finite powerset functor such that $A \times B = \emptyset$. Let $i_A : A \to A \coprod B$ be the coproduct injection in $\textbf{Set}_{P\omega}$, and let $1_B$ be the identity on $B$. By previous remarks,

$$i_A \times 1_B : A \times B \to (A \coprod B) \times B = 0_{\emptyset \times B},$$

and therefore there exist nonzero $\phi, \psi \in \textbf{Pfn}_{P\omega}$ with $\phi \otimes \psi = 0$. \qed

Also, the category $\textbf{Pfn}_F$ can fail to be dominical if it contains a non-total weakly total morphism. An example of a dominical category that is not dominical because it has such a morphism is the syntactic $P$-category $T$, obtained from Peano arithmetic $\text{PA}$; this example is essentially due to Franco Montagna [Ros86]. The category $T$ has exactly one object. The morphisms of $T$ are the equivalence classes under provability of of the provably functional $\Sigma_1$-formula $F(x, y)$ of $\text{PA}$; i.e., those $\Sigma_1 F(x, y)$ for which

$$\vdash_{\text{PA}} \forall x, y, z (F(x, y) \land F(x, z) \Rightarrow y = z).$$

The identity morphism $1_T$ is defined to be the provable equivalence class of the formula $0 = 0$. The composition of morphisms $F(x, y)$ and $G(y, z)$ is defined by $(G \circ F)(x, z) \Leftrightarrow \exists y F(x, y) \land G(y, z)$. The domain of a morphism $F(x, y)$ is the morphism $D(x, y)$ defined by $\exists z F(x, z) \land y = z$. A morphism is total if $\vdash_{\text{PA}} \forall x D(x, x)$. A morphism $F(x, y)$ is a zero morphism (undefined) provided $\vdash_{\text{PA}} \forall x, y \neg F(x, y)$. A morphism $F(x, y)$ is weakly total if for all morphisms $G(w, x)$,

$$\vdash_{\text{PA}} \forall w, y \neg (F \circ G)(w, y) \text{ implies } \vdash_{\text{PA}} \forall w, x \neg (F \circ G)(w, x).$$

See Rosolini [Ros86] for the argument that Peano arithmetic contains a provably functional $\Sigma_1$ formula $F(x, y)$ that is weakly total but not total. Further examples of dominical categories that are not dominical are given in Montagna [Mon89].

However, it is even easier to produce a category of coalgebras and partial morphisms which possesses a non-total weakly total morphism. In $\textbf{Pfn}_F$, weak totality can be interpreted by means of the coalgebra topology. The topological criterion given below for a weakly total morphism to be total can be used to give an elementary proof that $\textbf{Pfn}_F$ is not dominical in general. The following statements are immediate.

**Proposition 1.5.** Let $\varphi : X \to Y = \{m : W \leftrightarrow X, \psi : W \to Y\}$ be a morphism in $\textbf{Pfn}_F$, with $m$ mono.

i) The morphism $\varphi$ is weakly total if and only if the image $m[W]$ of $m$ is dense in the coalgebra topology on $X$. 


ii) If the subcoalgebra $U$ of $X$ is dense in $X$, then the partial morphism $\{i, 1_U\}$ defined by

\[
\begin{array}{c}
U \\
i \\
\downarrow \\
\downarrow \\
\{i, 1_U\} \\
i \\
\downarrow \\
\downarrow \\
X
\end{array}
\]

is weakly total.

iii) The coalgebra $X$ is irreducible in the coalgebra topology if and only if every morphism $\varphi : X \to Y$ other than $0_{X,Y}$ is weakly total.

iv) Every weakly total morphism $\varphi : X \to Y$ is total if and only if $X$ is the only nonempty open dense subset of $X$.

v) If $X$ is Hausdorff, then every weakly total morphism $\varphi : X \to Y$ is total.

To obtain a non-total weakly total morphism in $\text{Pfn}_{F}$, it suffices to find a coalgebra with a proper nonempty subcoalgebra that is dense in the coalgebra topology.

**Proposition 1.6.** Let $1_{\text{Set}}$ be the identity functor on $\text{Set}$. Then $\text{Pfn}_{1_{\text{Set}}}$ is prodominical but not dominical.

**Proof.** A coalgebra for the identity functor is given by a self map of a set. Let $X = \{x, y\}$ be a set with $x \neq y$, and define $\alpha : X \to X$ by $\alpha(x) = y$ and $\alpha(y) = y$. Then $U = \{y\}$ is a proper subcoalgebra of $X$ that is open and dense in the coalgebra topology on $X$. It follows from Proposition 1.5, statement iv) that the partial map $X \to U$ of statement ii) is weakly total but not total. \qed

We record some non-pathological properties of the product in $\text{Set}_F$ that will be needed in the sequel. Fortunately, despite the example of two finite nonempty coalgebras $A, B$ for which $A \times B = \emptyset$, we have the following.

**Proposition 1.7.** If $\text{Set}_F$ has finite products, and if $A$ is a nonempty $F$-coalgebra, then for each $n \geq 1$, $A^n$ is nonempty.

**Proof.** Let $U : \text{Set}_F \to \text{Set}$ be the forgetful functor. If $A$ is nonempty, so is the diagonal $\Delta_A = \{(a, a) : a \in U(A)\}$, which is a bisimulation on $A$ \cite{Rut00}. Lemma 8.1 of \cite{GS00} states that if $A_0 \times A_1$ is a product in $\text{Set}_F$ with projections $\pi_i, i = 0, 1$, then $(U\pi_0 \times U\pi_1)U(A_0 \times A_1)$ is the largest bisimulation between $A_0$ and $A_1$; taking $A_0 = A_1 = A$, we have that $\emptyset \neq \Delta_A \subseteq (U\pi_0 \times U\pi_1)U(A \times A)$ so that $A \times A$ cannot be empty. It follows that $A^n \neq \emptyset$ for any $n \geq 1$. \qed
The product in $\text{Set}_F$ distributes over the coproduct. This follows essentially from theorem 6.4 of Worrel [Wor98], which states that if $F$ is a bounded endofunctor which preserves weak pullbacks, then $\text{Set}_F$ is a full reflective subcategory of a Grothendieck topos. Moreover, assuming only that $F$ is nontrivial and preserves weak pullbacks, Worrel’s proof implies that the following conditions of Giraud’s theorem hold. In $\text{Set}_F$, coproducts are disjoint and stable under pullback, every epimorphism is the coequalizer of its kernel pair, and epimorphisms are stable under pullback.

**Proposition 1.8.** Let $F$ be a nontrivial $\text{Set}$ endofunctor that weakly preserves pullbacks and such that $\text{Set}_F$ has finite limits. Let $X$ be an $F$-coalgebra, let $I$ be a set, and let $\{Y_\alpha\}_{\alpha \in I}$ be a family of $F$-coalgebras. There is a canonical isomorphism

$$X \times \coprod_I Y_\alpha \rightarrow \coprod_I (X \times Y_\alpha)$$

**Proof.** By a result of Gumm and Schröder [GS00a], the preimage of a subcoalgebra is a subcoalgebra if and only if $F$ weakly preserves pullbacks along monomorphisms; hence pullbacks of subcoalgebras are subcoalgebras in $\text{Set}_F$. Paraphrasing Worrel [Wor98], since the forgetful functor $U : \text{Set}_F \rightarrow \text{Set}$ creates colimits and since $U$ weakly preserves pullbacks if $F$ does, coproducts are disjoint and stable under pullback. Fix coalgebras $A$ and $B$; it follows by standard arguments [MLM92] that the pullback operation $- \times_A B$ distributes over coproducts; taking $A$ to be the terminal object, it follows that products distribute over coproducts. \hfill \Box

In a $P$-category $C$, the **domain** $\text{dom}\phi$ of a morphism $\phi : X \rightarrow Y$ is the composite $p_{0X,Y} \circ (1_X \phi) : X \rightarrow X$. For an object $X$ of $C$, let $\text{dom}(X)$ denote the set of domains $\text{dom}\phi$ for morphisms $\phi : X \rightarrow Y$. The set $\text{dom}(X)$ is a meet semilattice with meet defined by composition [DPH87, Hel90, Ros86]. Domains in $\text{Pfn}_F$ correspond precisely to subcoalgebras.

**Proposition 1.9.** Let $X$ be a coalgebra in $\text{Pfn}_F$. There is a meet semilattice isomorphism from $\text{dom}(X)$ to the lattice $\mathcal{L}_X$ of sub-coalgebras of $X$ (considered as a meet semilattice).

**Proof.** The map $\mathcal{L}_X \rightarrow \text{dom}(X)$ is defined by sending the sub-coalgebra $U$ of $X$ to the domain $p_{0X,Y} \circ (1_X \phi)$, where $\phi = \{U \hookrightarrow X, U \hookrightarrow 2N\}$.

---

2 Necessary and sufficient conditions for distributivity in categories of coalgebras were announced in joint work of H. Peter Gumm and Tobias Schröder of the University of Marburg, Germany, and Jesse Hughes of Carnegie Mellon University [GHS00].
The inverse map is defined as follows. Given a domain \( \phi \in \text{dom}(X) \), we may write \( \phi = p_{0,X,Y} \circ \langle 1_X \{ m, \varphi \} \rangle \), where \( m : V \to X \) is a monomorphism of the coalgebras, and where \( \varphi : V \to Y \) is a morphism in \( \text{Set}_F \). By definition of composition in \( \text{Pfn}_F \) one obtains the following diagram in which the squares are pullbacks. We adopt the convention of not showing the structure maps and induced morphisms of the coalgebras occurring in commutative diagrams of coalgebras.

\[
\begin{array}{c}
V 
\xrightarrow{1} 
X \times V 
\xrightarrow{1_X \times \varphi} 
X \times Y 
\xrightarrow{p_{0,X,Y}} 
X \\
\downarrow 1 
\downarrow 1 
\downarrow 1 
\downarrow p_{0,X,Y} \\
V 
\xrightarrow{1} 
X \times V 
\xrightarrow{1_X \times \varphi} 
X \times Y 
\xrightarrow{1_X \varphi} 
X \times Y \\
\downarrow \Delta_X 
\downarrow 1 
\downarrow \Delta_X 
\downarrow 1 \\
X 
\xrightarrow{} 
X \times X 
\xrightarrow{} 
X \times Y 
\xrightarrow{} 
X \\
\end{array}
\]

We use this diagram to define the coalgebra \( U \in L_X \) corresponding to the domain \( \phi \) as the image of \( V \) under the left vertical column of the diagram. Showing that the meets are preserved involves certain large diagrams such as those occurring in [DPH87].

Let \( \mathcal{C} \) be a \( P \)-category. A morphism \( \phi : X \to Y \mathcal{C} \) is total if \( \text{dom}\phi = 1_X \). The collection of total morphisms of \( \mathcal{C} \) generate its subcategory \( \mathcal{C}_T \) of total morphisms. The near-product and infranatural transformations of a \( P \)-category \( \mathcal{C} \) become a product and natural transformations, respectively, on its subcategory \( \mathcal{C}_T \), which has the structure of a \( B \)-category; for convenience we include the definition from Heller [Hel90]. A \( B \)-category is a category \( \mathcal{C} \) with a bifunctor \( \times : \mathcal{C}^2 \to \mathcal{C} \) and natural transformations \( p_0, p_1, \Delta, \text{ass}_x, \text{tr}_x \) satisfying the conditions i), iii) and iv) for \( P \)-categories above, with \( \times \) replacing \( \boxtimes \). It should be emphasized that the projections of a \( B \)-category are required to be natural and not merely infranatural transformations; we write \( \times \) for the product of a \( B \)-category and \( \boxtimes \) for the near product of a \( P \)-category. Dually, one may speak of a category with a binary coproduct, together with natural transformations \( i_0, i_1, \nabla, \text{ass}_{\nabla}, \text{tr}_{\nabla} \) satisfying the duals of the conditions 1), iii) and iv) in which the injections \( i_0, i_1 \) replace the projections \( p_0, p_1 \), the codiagonal \( \nabla \) replaces the diagonal \( \Delta \), where \( \text{ass}_{\nabla} \) and \( \text{tr}_{\nabla} \) replace \( \text{ass}_x \) and \( \text{tr}_x \), respectively, and where \( \boxplus \) replaces \( \boxtimes \). A \( B \)-category with a binary coproduct that has a natural
isomorphism called \( \text{dist} \) inverse to the natural transformation
\[
(X \times Y) \coprod (X \times Z) \to X \times (Y \coprod Z)
\]
is called a \( B^+ \)-category. A \( B^+ \)-category with a countable coproduct \( \coprod_n \) such that \( \times \) distributes over \( \coprod_n \) is called a \( B\Sigma \)-category.

By analogy with \( B^+ \) and \( B\Sigma \)-categories, one may define \( P^+ \) and \( P\Sigma \)-categories, in which the coproduct injections are required to be natural (and not merely infranatural) transformations. Under the embedding \( \text{Set}_F \to \text{Pfn}_F \), the canonical isomorphism of proposition 1.8 defines the natural isomorphism
\[
\text{dist}_{X,(Y_n)} : X \boxtimes \coprod_n Y_n \to \coprod_n X \boxtimes Y_n
\]
of functors \( \text{Pfn}_F \times \coprod_n \text{Pfn}_F \to \text{Pfn}_F \).

**Proposition 1.10.** The category \( \text{Pfn}_F \) is a \( P\Sigma \)-category.

**Proof.** Let \( f : X \to W \) in \( \text{Pfn}_F \) be given by \( f = \{ \mu, \phi \} \), where \( \mu : U \to X \) and \( \phi : U \to W \) are in \( \text{Set}_F \), with \( \mu \) mono, and for \( n \in \mathbb{N} \) let \( g_n : Y_n \to Z_n \) in \( \text{Pfn}_F \) be given by \( g_n = \{ \nu_n, \psi_n \} \), where \( \nu_n : V_n \to Y_n \) and \( \psi_n : V_n \to Z_n \) are in \( \text{Set}_F \), with \( \nu_n \) mono. Consider the following diagram.

We claim that the bottom parallelogram commutes in \( \text{Pfn}_F \). Observe that the top parallelogram is a pullback in \( \text{Set}_F \). It follows that the composite
\[
\coprod_n (f \boxtimes g_n) \circ \text{dist}_{X,(Y_n)}
\]
in \( \text{Pfn}_F \) is represented by the pair

\[
\left\{ \mu \times \coprod_n \nu_n, \coprod_n (\phi \times \psi_n) \circ \text{dist}_{U,(Y_n)} \right\}.
\]
On the other hand, the composite

\[ dist_{W,(Z_n)} \circ \left( f \boxtimes \prod_N g_n \right) \]

in \( \text{Pfn}_F \) is represented by the pair

\[ \left\{ \mu \times \prod_N \nu_n, dist_{W,(Z_n)} \circ \left( \phi \times \prod_N \psi_n \right) \right\}, \]

which equals (1) since the (back) rectangle commutes, as \( dist \) is natural in \( \text{Set}_F \).

In a \( P \)-category, if \( \phi : X \to Y \) is a morphism and if \( \varepsilon \in \text{Dom}Y \), we write \( \phi \prec \varepsilon \) if and only if \( \varepsilon \phi = \phi \); we say that \( \varepsilon \) receives \( \phi \). If \( \varepsilon \) receives \( \phi \), and in addition, \( \varepsilon \) satisfies for all appropriate \( \psi, \psi' \), \( \psi \phi = \psi' \phi \) implies \( \psi \varepsilon = \psi' \varepsilon \), then \( \varepsilon \) is the least domain in \( \text{Dom}Y \) receiving \( \phi \), since if \( \delta \in \text{dom}Y \) satisfies \( \delta \phi = \phi \), then \( \delta \phi = \varepsilon \phi \), which implies that \( \delta \varepsilon = \varepsilon \varepsilon = \varepsilon \) and therefore \( \varepsilon \prec \delta \). In this case we say that \( \phi \) has range \( \varepsilon \) and we write \( \text{ran} \phi = \varepsilon \).

If each morphism of the \( P \)-category \( C \) has a range, we say that \( C \) has ranges. Since \( \text{Set}_F \) has \( \text{(Epi,StrongMono)} \)-factorizations, the image of a coalgebra under a morphism is a subcoalgebra of the codomain; conversely every subcoalgebra is an image. If \( \phi = \{ m, \psi \} \) is a morphism of \( \text{Pfn}_F \), where \( m : U \to X \) and \( \psi : U \to Y \) are morphisms of \( \text{Set}_F \) with \( m \) mono, we define the image \( \text{im} \phi \) of \( \phi \) by \( \text{im} \phi = \psi[U] \). Under the semilattice isomorphism of Proposition 1.9, images of morphisms in \( \text{Pfn}_F \), which are coalgebras, correspond with ranges, which are morphisms.

**Proposition 1.11.** The category \( \text{Pfn}_F \) has ranges.

If every morphism of the \( P \)-category \( C \) has a range, and if for morphisms \( \phi, \psi \) of \( C \), \( \text{ran}(\phi \boxtimes \psi) = \text{ran}\phi \boxtimes \text{ran} \psi \), then one says [Hel90] that \( C \) is an \( rP \)-category; such a category has a calculus of ranges [DPH87]. In a \( B^+ \)-category, if \( f \) and \( g \) are morphisms with the same codomain \( Y \), then we define \( [f, g] = \nabla_Y(f \coprod g) \). In an \( rP^+ \)-\( (rP^\Sigma) \)-category, the meet semilattice \( \text{dom}(X) \) becomes a distributive lattice if one defines the join by \( \varepsilon \cup \delta = \text{ran}[\varepsilon, \delta] \) for \( \varepsilon, \delta \in \text{dom}(X) \) [DPH87, Hel90]. To show that \( \text{Pfn}_F \) is an \( rP^\Sigma \)-category, we first show that the image of a product of morphisms in \( \text{Set}_F \) is the product of the images; the proof is mostly folklore. The lack of a simple description of the product in \( \text{Set}_F \) and the possibility of pathologies seems to necessitate the use of categorical technique in the proof.

**Proposition 1.12.** For morphisms \( \phi, \psi \) of \( \text{Set}_F \),

\[ \text{im}(\phi \times \psi) = \text{im} \phi \times \text{im} \psi. \]
Proof. We first show that if \( f : A \to B \) is an epimorphism in \( \text{Set}_F \), then so is \( f \times 1 : A \times C \to B \times C \). The following diagram is a pullback in \( \text{Set}_F \).

\[
\begin{array}{ccc}
A 	imes C & \xrightarrow{f \times 1} & B \times C \\
p_A \downarrow & & \downarrow p_B \\
A & \xrightarrow{f} & B
\end{array}
\]

Apply the forgetful functor \( U : \text{Set}_F \to \text{Set} \) to this and take the pullback in \( \text{Set} \). Proposition 6.3 of [Wor98] states that if \( F \) preserves weak pullbacks, then the forgetful functor \( U \) preserves weak pullbacks; hence the square in the following diagram is a weak pullback, and therefore there exists an induced map as indicated.

\[
\begin{array}{ccc}
UA \times_{UB} U(B \times C) & \xrightarrow{U(f \times 1)} & U(B \times C) \\
U(A \times C) \downarrow u_{pA} & & \downarrow U_{pB} \\
UA & \xrightarrow{Uf} & UB
\end{array}
\]

Since \( U \) preserves epimorphisms, \( Uf \) is surjective. Since it is possible that \( U(B \times C) = \emptyset \), in that trivial case commutativity of the square forces \( U(A \times C) = \emptyset \), so that \( U(f \times 1) = 0_\emptyset, 0 = 1_\emptyset \). Otherwise, a diagram chase shows that \( U(f \times 1) \) is surjective and, since \( U \) reflects epimorphisms, \( f \times 1 \) is an epimorphism in \( \text{Set}_F \). Therefore, if \( f, g \) are epimorphisms in \( \text{Set}_F \), so is \( f \times g = (f \times 1) \circ (1 \times g) \).

Using the first pullback diagram alone, one has that if \( f \) is a monomorphism, then so is \( f \times 1 \), and therefore a product of monomorphisms in \( \text{Set}_F \) is a monomorphism. It follows that the epi-mono factorization of a product \( f \times g \) is the product of the epi-mono factorizations of \( f \) and of \( g \).

A theorem of Gumm and Schröder [GS00] states that if \( A_i \) is a subcoalgebra of \( B_i \) for \( i = 0, 1 \), and if the product \( B_0 \times B_1 \) exists, then so does \( A_0 \times A_1 \), which is a subcoalgebra of \( B_0 \times B_1 \). If \( \phi : X \to Y \) and \( \psi : W \to Z \) are coalgebra morphisms, then by previous remarks, one has two canonically isomorphic epi-mono factorizations of \( \phi \times \psi \) as
indicated in the following commutative diagram.

\[
\begin{array}{ccc}
X \times W & \rightarrow & \text{im}(\phi \times \psi) \\
\downarrow & & \downarrow \\
\text{im}\phi \times \text{im}\psi & \rightarrow & Y \times Z
\end{array}
\]

Since the morphisms into \(Y \times Z\) are inclusions, the induced map is the identity. \(\square\)

**Proposition 1.13.** The category \(\text{Pfn}_F\) is an \(rP\Sigma\)-category. Moreover, for any \(F\)-coalgebra \(X\), the meet semilattice isomorphism of Proposition 1.9 is a lattice isomorphism from \(\text{dom}(X)\) to the lattice \(L_X\) of sub-coalgebras of \(X\).

The following definitions are taken from Heller [Hel90]. In a \(P\)-category, a *section* of a morphism \(\phi : X \rightarrow Y\) is a morphism \(\sigma : Y \rightarrow X\) such that \(\phi \sigma = \text{dom}\sigma\) and \(\phi \sigma \phi = \phi\). A \(P\)-category satisfies the axiom of choice if every morphism has a section. A morphism \(\phi\) is a *partial monomorphism* if \(\phi\theta = \phi\theta'\) implies \((\text{dom}\phi)\theta = (\text{dom}\phi)\theta'\). A \(P\)-category satisfies the weak axiom of choice if each partial monomorphism has a section.

**Proposition 1.14.** The category \(\text{Pfn}_F\) satisfies the weak axiom of choice.

*Proof.* Let \(\phi : X \rightarrow Y\) be a partial monomorphism in \(\text{Pfn}_F\). Then \(\phi = \{\mu, \psi\}\), where \(\mu, \psi : U \rightarrow X\) are monomorphisms in \(\text{Set}_F\), and where \(U\) is a subcoalgebra of \(X\). The category \(\text{Set}_F\) has (Epi-StrongMono) factorizations, hence one has the following diagram in which \(\psi\) has the epi-mono factorization \(\psi = me\).

\[
\begin{array}{ccc}
Q & \xrightarrow{m} & Y \\
\downarrow^{e} & \downarrow^{f} & \downarrow^{1} \\
U & \xrightarrow{\psi} & Y \\
\downarrow^{\mu} & \downarrow^{\phi} & \downarrow^{=\{\mu, \psi\}} \\
X
\end{array}
\]

A morphism in \(\text{Set}_F\) is epi if and only if it is surjective in \(\text{Set}\). The morphism \(e\) is a monomorphism since \(m\) and \(\psi\) are. Therefore \(e\) is invertible in \(\text{Set}\) with inverse \(f\) which, since the image of \(m\) is contained in the image of \(\psi\), is a morphism in \(\text{Set}_F\) by a diagram lemma (e.g., Gumm [GS00c]).
The partial morphism \( \sigma \) defined by \( \sigma = \{ m, \mu f \} \) gives the required section. The required properties \( \phi \sigma = \text{dom} \sigma \) and \( \phi \sigma \phi = \phi \) can be verified in the following commutative diagram, in which the squares are pullbacks in \( \text{Set}_F \).

\[
\begin{array}{c}
U \xrightarrow{e} Q \xrightarrow{f} U \xrightarrow{\psi} Y \\
\downarrow 1 \downarrow 1 \downarrow \mu \\
U \xrightarrow{\mu} Q \xrightarrow{\mu f} X \\
\downarrow m \downarrow \sigma \\
U \xrightarrow{\psi} Y \\
\downarrow \mu \\
X
\end{array}
\]

In consequence, a generalization of what Gumm [Gum99] refers to as the first diagram lemma of \( \text{Set}_F \) holds in \( \text{Pfn}_F \). In Heller’s nomenclature, one says that \( \text{Set}_F \) and \( \text{Pfn}_F \) are factorial [Hel90].

**Lemma 1.15.** Let \( X, Y \) and \( Z \) be coalgebras, let \( \phi : X \to Y \) and \( \psi : X \to Z \) be partial morphisms in \( \text{Pfn}_F \) with \( \text{dom} \phi \subseteq \text{dom} \psi \), and suppose that for each \( x, y \in \text{dom} \phi \), \( \phi x = \phi y \) implies that \( \psi x = \psi y \) (in Heller’s terminology one says that \( \phi \) divides \( \psi \)). Then there is a unique partial map, denoted by \( \psi / \phi \), such that \( \text{dom} \psi / \phi = \text{ran} \phi \) and such that the following diagram commutes.

\[
\begin{array}{c}
X \xrightarrow{\phi} Y \\
\downarrow \psi \downarrow \psi / \phi \\
\downarrow \psi / \phi \\
Z
\end{array}
\]

1.5. **Free semigroups, \( B^\# \)- and iteration categories.** If \( C \) is a \( B \Sigma \)- or \( P \Sigma \)-category, one may construct the free semigroup \( X^\# = \coprod N X^{n+1} \) of an object \( X \) of \( C \). In view of the possibility of pathological products of coalgebras, we remark that Proposition 1.7 implies that the free semigroup of a nonempty coalgebra is nonempty. The associative multiplication \( m : X^\# \times X^\# \to X^\# \) of the free semigroup is the composite

\[
(\coprod N X^{n+1}) \times (\coprod N X^{n+1}) \longrightarrow \coprod N \times N X^{n+1} \times X^{m+1} \longrightarrow \coprod N X^{k+1}
\]

where the left map is obtained from two applications of the natural transformation \( \text{dist} \), and where the right map is, to within a natural
isomorphism, the coproduct of the codiagonal maps

\[ \nabla_k : \coprod_{n+m+1=k} X^{n+1} \times X^{m+1} \rightarrow X^{k+1} \]

for \( k \geq 1 \) (compare with the construction of the free monoid in [ML98], page 172). In particular, the categories \( \text{Set}_F \) and \( \text{Pfn}_F \) possess these canonical coalgebra morphisms.

More general than \( B\Sigma \)-categories are the \( B\# \)-categories, whose definition will be needed for the statement of Heller’s existence theorem. Though our results could be formulated in certain \( P\Sigma \)-categories without mention of \( B\# \)-categories and iteration categories (to be defined), technicalities involving products of coalgebras arise that compel us to consider the more general notion even when the free semigroup is available. A \( B\# \)-category has an associated monad, denoted by \( \# \), called the formally-free semigroup functor which, although lacking the universal property of the free semigroup, has enough of the structure of the free semigroup to define by categorical means operations corresponding to the manipulation of finite sequences. Such operations are used to define the iteration of a morphism in the sequel.

We remark on certain technicalities of products of coalgebras alluded to above. In the case of coalgebras for an endofunctor \( F \) which preserves products, the product of coalgebras may be constructed as it is constructed in \( \text{Set} \), and one may identify an element of a finite product of \( F \)-coalgebras with a finite sequence of elements of coalgebras. However, in most cases of interest in computer science, there is no simple description of products of coalgebras; products are not constructed from products of base sets. For the proof of the iteration lemma below, one must rely on the operations provided by the (formally) free semigroup in the general case.

The definition of a \( B\# \)-category, taken verbatim from Heller, follows. For this purpose some additional notation is needed. The formally-free semigroup functor \( \# \) comes with an associative multiplication \( m \); for morphisms \( f, g \) with common codomain \( X^\# \), one defines \( f \cdot g = m(f \times g) \). We define \( j_n : X^n \rightarrow X^\# \) by \( j_1 = j \) and \( j_{n+1} = j \cdot j_n \) for \( j > 1 \). When \( X^\# \) is the free semigroup, the \( j_n \) are (to within canonical isomorphism) the coproduct injections.
A $B^\#$-category is a $B^+$-category equipped with seven natural transformations

$$
m : X^\# \times X^\# \to X^\#
$$

$$
J : X \to X^\#
$$

$$
e : X^\# \to X
$$

$$
l : X^\# \to X \amalg (X \times X^\#)
$$

$$
r : X^\# \to (X^\# \times X) \amalg X
$$

$$
\text{par} : X \times Y^\# \to (X \times Y)^\#
$$

$$
wd : (X \amalg Y)^\# \to X^\# \amalg Y^\# \amalg (X^\# \times Y^\#)^\# \amalg (Y^\# \times X^\#)^\#
$$

subject to the following conditions.

i) $m$ is associative, hence $X^\#$ is a semigroup and, if $f$ is a morphism, then $f^\#$ is a semigroup homomorphism.

ii) $\#$ is a monad with unit $j$ and multiplication $e$, and each component of $e$ is a semigroup homomorphism.

iii) $l$ and $r$ are respectively the inverses of

$$
[j, j \cdot X^\#] : X \amalg (X \times X^\#) \to X^\#
$$

$$
[X^\# \cdot j, j] : (X^\# \times X) \amalg X \to X^\#.
$$

iv) The following diagram commutes.

$$
\begin{array}{ccc}
X \times Y^\# & \xrightarrow{\text{par}} & (X \times Y)^\#\\
\downarrow & & \downarrow^{p_{1,X,Y}^\#}\\
Y^\# & & \end{array}
$$

v) $wd$ is inverse to the morphism

$$
[i_0^\#, i_1^\#, e(i_0^\# \cdot i_1^\#)^\#, e(i_1^\# \cdot i_0^\#)^\#, i_0^\# \cdot e(i_0^\# \cdot i_1^\#)^\#, i_1^\# \cdot e(i_1^\# \cdot i_0^\#)^\#].
$$

A functor between $B^\#$-categories is called a $B^\#$-functor if it preserves the functors $\times, \amalg$ and $\#$ and the fourteen natural transformations $\Delta, p_0, p_1, \nabla, i_0, i_1, \text{dist}, m, j, e, l, r, \text{par}$ and $wd$ defined above. Let $C$ be a $B^\#$-category, and let $S$ be a set of morphisms of $C$. The $B^\#$-subcategory generated by $S$ is the smallest $B^\#$-category of $C$ containing $S$; this subcategory is denoted by $B^\#(S)$. If $C$ is a small $B^\#$-category, then
the set $C_0$ of objects of $C$ is an algebra with signature $(\times, \coprod, \#)$; such an algebra is called a $(\times, \coprod, \#)$-algebra. In the sequel, such algebras will be obtained from certain objects of $B^\#$-category called isotypical objects, to be defined. In particular, such an algebra will generate a $B^\#$-category, in the following sense.

**Proposition 1.16** (Hel90). If $C$ is a small $B^\#$-category, $D_0$ is a $(\times, \coprod, \#)$-algebra and $F_0 : D_0 \to C_0$ is a homomorphism, then there exists, uniquely to within canonical isomorphism, a $B^\#$-category $D$ with object algebra $D_0$, supplied with a full and faithful $B^\#$-functor $F : D \to C$ extending $F_0$.

**Proof.** Following the procedure in Hel90 for obtaining the smallest $B^\#$-category containing a given set of morphisms, we think of the elements of $D_0$ as the objects of an as yet unspecified $B^\#$-category $D$, and construct the set $D_1$ of morphisms accordingly. For each $A \in D_0$, we adjoin an identity morphism $1_A$ to $D_1$, subject to the functoriality relations $1_A \times B = 1_A \times 1_B$, and so on. Next, adjoin the values of the fourteen natural transformations above with arguments in $D_0$ to $D_1$, subject to the relations i) through v) above and subject to the relations that hold in a $B^+$-category. Finally, we close the set of morphisms that results under $\times$, $\coprod$, $\#$ and composition. This produces the category $D$. The homomorphism $F_0$ is extended to $D_1$ in the only way possible, following the three step construction of $D_1$. Identities in $D_1$ must be preserved by $F$. The image in $C$ under $F$ of a value of one of the fourteen natural transformations is completely determined by functoriality and by definition of $F_0$. For example, the value of $F$ on $\nabla_A : A \coprod A \to A$ is $\nabla_{FA} : FA \coprod FA \to FA$. Finally, we require that $F$ commute with $\times$, $\coprod$, $\#$ and composition; for example, $F(\nabla_X \times 1_{Y^\#}) = F\nabla_X \times 1_{F(Y^\#)} = \nabla_{FX} \times 1_{(FY)^\#}$. \hfill \square

Let $C$ be a $B^\#$-category. An object $X$ of $C$ generates by Proposition 1.13 a $B^\#$-category denoted by $C\langle X \rangle$, whose object algebra is free on the generator $X$, together with a $B^\#$-functor $F : C\langle X \rangle \to B^\#(1_X)$. In case the $(\times, \coprod, \#)$-algebra generated by $X$, namely $B^\#(1_X)_0$, is free, then $F$ gives an identification of $C\langle X \rangle$ with the full subcategory $B^\#(1_X)$ of $C$ and, following Heller, we use the notation

$$C = C\langle X \rangle$$

(2)

to say that $C_0$ is freely generated by $X$ as a $(\times, \coprod, \#)$-algebra.

The definition of an iteration category requires the notion of a stable union, a categorical notion which needs no special definition in $\text{Set}_F$. Let $C$ be a $P$-category, and let $X$ be an object of $C$, and let $\varepsilon_n \in \text{dom}(X)$ be a countable family of domains. A domain $\delta$ is a union of
\{\varepsilon_n\} if for each \(n\), \(\varepsilon_n \prec \delta\) and if, for any \(\phi\) and \(\phi'\), \(\phi\varepsilon_n = \phi'\varepsilon_n\) implies \(\phi\delta = \phi'\delta\). This determines \(\delta\) uniquely, for if \(\delta'\) is any other union of \(\{\varepsilon_n\}\), then \(\delta\varepsilon_n = \varepsilon_n = \delta'\varepsilon_n\), so \(\delta = \delta\delta = \delta'\delta = \delta'\delta'\).

Since composition of domains is commutative, \(\delta = \delta'\). In this case, we set \(\delta = \bigcup \varepsilon_n\). If for any object \(Y\) of \(C\),

\[
Y \boxtimes \bigcup \varepsilon_n = \bigcup (Y \boxtimes \varepsilon_n),
\]

the union is called stable \([\text{Hel90}]\). In a \(PS\)-category with ranges, every countable family of domains \(\{\varepsilon_n\}\) has the stable union \(\text{ran} (\bigvee \varepsilon_n)\).

An iteration category is a prodominical \(P^+\)-category with ranges that satisfies the weak axiom of choice, such that the \(B^+\)-category structure of \(C_T\) has been extended to a \(B^\#\)-category so that for each object \(X\), the formally free semigroup \(X^\#\) is the stable union of \(\{\text{ran} j_n\}\).

**Proposition 1.17.** \(Pfn_F\) is an iteration category.

**Proof.** Since \(Pfn_F\) is a \(PS\)-prodominical category with ranges that satisfies the weak axiom of choice, it is an iteration category. \(\square\)

We give a translation of the iteration lemma from Heller \([\text{Hel90}]\) into \(Pfn_F\). The iteration lemma and its application to Turing data in the sequel is an alternative to the use of fixed point semantics for iteration, as exemplified by Manes’ application of Kleene’s fixed point theorem to iterative specifications in \(\omega\)-complete categories \([\text{Man92}]\). The proof of the iteration lemma that we give is the transparent one available when the endofunctor \(F\) preserves products; in the general coalgebraic case we rely on the proof given in Heller, which makes full use of the structure of a \(B^\#\)-category, in lieu of our explicit manipulation of finite sequences below.

**Lemma 1.18** (Iteration Lemma \([\text{Hel90}]\)). Let \(X\) be a coalgebra, let \(i : U \hookrightarrow X\) be the inclusion of a sub-coalgebra \(U\) into \(X\), and let \(f : X \to X\) be a morphism in \(\text{Set}_F\) such that \(f \circ i = i\). Then there is a unique morphism \(\text{It}(f, U) : X \to X\) in \(Pfn_F\) with domain \(\bigvee_n f^{-n}[U]\) such that for each \(n \geq 0\), \(\text{It}(f, U)[f^{-n}[U]] = f^n[f^{-n}[U]]\). Moreover, \(\text{im} \, \text{It}(f, U) \subseteq U\) and, for appropriate \(g\) and \(V\),

\[
\text{It}(f \times g, U \times V) = \text{It}(f, U) \boxtimes \text{It}(g, V),
\]

\[
\text{It}(f \prod g, U \prod V) = \text{It}(f, U) \prod \text{It}(g, V).
\]

**Proof.** The proof we give in \(Pfn_F\) presupposes that the endofunctor \(F\) preserves products, since we assume that products are constructed as they are in \(\text{Set}\); the general case follows from \([\text{Hel90}]\).
Let \( X^\# = \Sigma_n X^{n+1} \) be the free semigroup generated by \( X \) and define the set
\[
W = \{(x, f x, \ldots, f^nx) \in X^\# : n \geq 1, x \in X, f^nx \in U\}.
\]
The set \( W \) is a sub-coalgebra of \( X^\# \) since, translating from Heller [Hel90] into this context,
\[
W = \Delta^{-1}_{X^\#} \left[ (j \circ \text{first} \cdot f^\#) \times (X^\# \cdot j \circ \text{last}) \right]^{-1} [\text{im}\Delta_{X^\#}] \cap \text{last}^{-1}[U],
\]
where \( \text{first} = \left[ X, p_0 \right] l_X \), \( \text{last} = \left[ p_1, X \right] r_X \), and where the dot \( \cdot \) denotes concatenation of sequences in \( X^\# \); i.e., \((x_1, \ldots, x_m) \cdot (y_1, \ldots, y_n) = (x_1, \ldots, x_m, y_1, \ldots, x_n)\).

Next we apply the diagram lemma in \( \text{Pfn}_F \). There exists a unique induced map in the following diagram
\[
\begin{array}{ccc}
X^\# & \xrightarrow{\text{first}|W} & X \\
\downarrow \text{last}|W & & \downarrow \text{last}|W \\
W & \xrightarrow{\text{It}\left([v, i_1], 0 \coprod Y\right)} & W \coprod Y \\
& \downarrow \text{first}|W & \\
Y & & Y
\end{array}
\]
We may suppose that \( n > m \). If
\[
\text{first}(x, f x, \ldots, f^mx) = \text{first}(y, f y, \ldots, f^ny),
\]
then \( x = y \), and as both of the indicated sequences are in \( W \), both \( f^mx \) and \( f^ny \) are in \( U \), and therefore \( f^{n-m} \circ f^mx = f^mx \). We take
\[
\text{It}(f, U) = \frac{\text{last}|W}{\text{first}|W}.
\]
The remaining assertions are immediate. \( \square \)

1.6. Turing developments and local connectedness. The iteration lemma [1.18] will be applied to maps \( f : X \to X \) that fix a sub-coalgebra \( U \) arising as a summand of a coproduct \( X = U \coprod V \).

Let \( C \) be an iteration category. A Turing datum in \( C \) is a diagram
\[
\begin{array}{ccc}
X & \xrightarrow{u} & W \\
\downarrow v & & \downarrow w \\
W \coprod Y & \xrightarrow{\text{It}(\left[v, i_1\right] \text{,} \emptyset \coprod Y)} & W \coprod Y \\
\end{array}
\]
in \( C_T \); in our case \( C = \text{Pfn}_F \) and \( C_T = \text{Set}_F \). The map \([v, i_1] : W \coprod Y \to W \coprod Y\) is total and satisfies \([v, i_1] (\emptyset \coprod Y) = \emptyset \coprod Y\), so by the iteration lemma [1.18] there is a map
\[
\text{It}(\left[v, i_1\right] \text{,} \emptyset \coprod Y) : W \coprod Y \to W \coprod Y.
\]
The Turing development \( \text{Tur}(u, v) : X \to Y \) of the given Turing datum is the composite
\[
\begin{array}{ccc}
X & \xrightarrow{i_{0u}} & W \coprod Y \\
\downarrow & & \downarrow \text{It}(\left[v, i_1\right] \text{,} \emptyset \coprod Y) \\
W \coprod Y & \xrightarrow{[0,1v]} & Y
\end{array}
\]
If $\mathcal{D}$ is a $B^+$-subcategory of $\mathcal{C}_T$, then the class of all Turing developments of Turing data in $\mathcal{D}$ is denoted by $\text{Tur}(\mathcal{D})$. It follows from Lemma 8.1 of Heller [Hel90] that $\mathcal{D} \subseteq \text{Tur}(\mathcal{D})$ and that $\text{Tur}(\mathcal{D})$ is closed under $\boxdot$ and $\bigcirc$. Under the additional assumption that $\mathcal{C}$ is “locally connected”, it follows that $\text{Tur}(\mathcal{D})$ is closed under composition [Hel90].

A coalgebra is connected if it is connected in the coalgebra topology; i.e., if it is not a nontrivial coproduct of subcoalgebras. If for each coalgebra $X$ of the iteration category $\text{Pfn}_F$, the coalgebra topology on $X$ is locally connected, we say that $\text{Pfn}_F$ is locally connected; in that case by Lemma 8.2 of Heller [Hel90], for any $B^+$ subcategory $\mathcal{D}$ of $\mathcal{C}$, $\text{Tur}(\mathcal{D})$ is closed under composition; moreover, it is a $+$-predominical subcategory of $\mathcal{C}$ containing $\mathcal{D}$.

**Proposition 1.19.** If the Set endofunctor $F$ weakly preserves generalized pullbacks of monomorphisms, then each $F$-coalgebra is locally connected in the coalgebra topology.

**Proof.** Under the hypothesis on $F$ it follows from Theorem 5.10 of Gumm and Schröder [GS00a] that for each $F$-coalgebra $X$ and for each $x \in X$, the subcoalgebra cogenerated by $x$ exists; hence the 1-cogenerated subcoalgebras of $X$ form a base for the coalgebra topology on $X$. Each 1-cogenerated subcoalgebra is connected in the coalgebra topology. \[ \square \]

1.7. **Isotypical objects and isotypes.** We paraphrase Heller [Hel90]. A category $\mathcal{C}$ is isotypical if any two of its objects are isomorphic. For example, if $\aleph$ is an infinite cardinal number, then the full subcategory $\text{Set}_{\aleph}$ of $\text{Set}$ containing the sets of cardinality $\aleph$ can be given the structure of a $B\Sigma$-category, hence of an isotypical $B\#$-category.

Let $\mathcal{C}$ be a $B\#$ category. An object $X$ of $\mathcal{C}$ is isotypical if it is isomorphic to each of $X \times X$, $X \bigsqcup X$, and $X^\#$. Such objects are used in the construction of recursion categories in the sequel. If $X$ is an isotypical object of a $B\#$-category $\mathcal{C}$, then $\mathcal{C}(X)$ (cf. equation 2) is an isotypical $B^\#$-category.

1.8. **Uniform generation.** The notion of the domain of a morphism carries over from $P$-categories to $B$-categories; in particular, if $\mathcal{C}$ is a $B$-category, then the notion of a total morphism is definable in $\mathcal{C}$. If $\mathcal{C}$ is a $B$-category and $t : W \times X \rightarrow X$ is a morphism in $\mathcal{C}$, then an index of $f : X \rightarrow X$ relative to the catalog $t$ is a total morphism $g : X \rightarrow W$.
such that the following diagram commutes.

\[
\begin{array}{c}
X \times X \xrightarrow{g \times 1} W \times X \\
p_1 \downarrow \quad \quad \downarrow t \\
X \quad \quad \quad F \quad \quad X
\end{array}
\]

(3)

If \( B \) is a subcategory of \( C \), the uniform list cataloged by \( t \) with indices in \( B \) is the set \( \mathcal{L}(B, t) \) of morphisms \( f \in C(X, X) \) for which there exists a total \( g \in B(X, W) \) such that the diagram (3) commutes.

If \( C \) is a \( B^\# \)-category and \( B = B(X) \) is an isotypical \( B^\# \)-subcategory, then \( B \) contains a frame \( b \) at \( X \), namely a collection of isomorphisms \( b_x : X \to X \times X, b_I : X \to X \bigsqcup X, b_\# : X \to X^\# \), along with their inverses. The subcategory \( B \) is called a uniformly generated \( B^\# \)-subcategory of \( C\langle X \rangle \) provided

\[
B = B^\# (b \cup \mathcal{L}(B, t))
\]

(4)

(cf. Proposition 1.16 and preceding remarks).

Uniformly generated isotypical categories have the following properties. If \( b \) and \( b' \) are frames at \( X \) contained in the uniformly generated category \( B \), then

\[
B = B^\# (b \cup \mathcal{L}(B, t)) = B^\# (b' \cup \mathcal{L}(B, t)).
\]

Any finitely generated isotypical \( B^\# \)-subcategory \( B = B\langle X \rangle \subset C\langle X \rangle \) is uniformly generated. For any \( t : W \times X \to X \) in \( C\langle X \rangle \) there is a uniformly generated \( B \subset C \) containing \( t \) and \( \mathcal{L}(B, t) \). For any frame \( b \) and for any catalogue \( t \) such that (4) holds, there exists a maximal uniformly generated category \( B \) satisfying (4).

2. Recursion categories

2.1. The existence theorem. A Turing morphism in a prodominical isotype is a morphism \( \tau : W \boxtimes X \to Y \) such that for any \( \phi : V \boxtimes X \to Y \) there exists a total \( g : V \to W \) such that the following diagram commutes.

\[
\begin{array}{c}
V \boxtimes X \xrightarrow{g \times 1_X} W \boxtimes X \\
\phi \downarrow \quad \quad \downarrow \tau \\
Y
\end{array}
\]

A recursion category is a prodominical isotype in which there is a Turing morphism.

The statement of Heller’s existence theorem follows.
Theorem 2.1 (Heller [Hel90]). Let $C$ be a locally connected iteration category. If $D = D(X)$ is a uniformly generated isotypical $B\#$-subcategory of $C(X)$, then $\text{Tur}(D)$ is a recursion category.

By preceding remarks, we have the following.

Theorem 2.2. Let $F$ be a nontrivial endofunctor on $\text{Set}$ that weakly preserves pullbacks. Suppose that $\text{Set}_F$ is locally connected and complete. Then for any uniformly generated isotypical subcategory $C$ of $\text{Pfn}_F$, $\text{Tur}(C)$ is a recursion category.

Corollary 2.2.1. Let $F$ be a nontrivial bounded or accessible endofunctor on $\text{Set}$ that weakly preserves generalized pullbacks. If $X$ is an isotypical coalgebra of $\text{Pfn}_F$ and if $b$ is a frame at $X$, then for any finite collection of morphisms $S \subset \text{Pfn}_F(X, X)$, $\text{Tur}(B\#(b \cup S))$ is a recursion category.

3. Examples and remarks

Taking the $\text{Set}$ endofunctor $F$ to be the identity functor and each coalgebra structure map $\alpha : X \to FX$ to be the identity morphism $1_X$, one obtains the category $\text{Pfn}$ of sets and partial functions as a subcategory of $\text{Pfn}_F$; hence $\text{Pfn}_F$ yields all the examples of recursion categories that come from sets and partial functions, as in [Hel90].

The set of functions computable in the Blum-Shub-Smale model of computation [BCSS98] over a ring can be obtained as morphisms of an appropriately defined recursion category. We omit the construction.

Since we suppose that the endofunctor $F$ is such that $\text{Set}_F$ is complete, e.g. if $F$ is bounded or $\omega$-accessible, $\text{Set}_F$ contains a terminal object $1$. For any $F$-coalgebra $X$, let $Y = \coprod N X$ be a countable copower, let $Z = \coprod N Y = \coprod Y \coprod Y^2 \coprod \cdots$ and let $W = Z\#$. By proposition 1.1 of [Hel90], $W$ is an isotypical object of $\text{Set}_F$ and hence of $\text{Pfn}_F$. If, in addition, $F$ preserves weak limits of sinks, then $\text{Pfn}_F$ is locally connected, and by the theorem its isotypical subcategory $\text{Pfn}_F\langle W \rangle$ contains many recursion categories.

We remark on further aspects of $P$-categories of coalgebras beyond the scope of this paper. On the assumption that $F$ be a nontrivial endofunctor on $\text{Set}$ that weakly preserves pullbacks and such that $\text{Set}_F$ is complete, $\text{Pfn}_F$ is a ranged Boolean category. Boolean categories were defined by Manes [Man92] as a categorical setting for predicate transformer semantics. Predicate transformers were introduced by Dijkstra and have been applied as a formal calculus of program derivation from specifications in logic [DS90, Kal90]. It follows from Manes’ theory that $\text{Pfn}_F$ admits a representation as a category of relations.
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