General first-order mass ladder operators for Klein–Gordon fields

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Abstract
We study the ladder operator on scalar fields, mapping a solution of the Klein–Gordon equation onto another solution with a different mass, when the operator is at most first order in derivatives. Imposing the commutation relation between the $d'$Alembertian, we obtain the general condition for the ladder operator, which contains a non-trivial case which was not discussed in the previous work (Cardoso et al 2017 Phys. Rev. D 96 024044). We also discuss the relation with supersymmetric quantum mechanics.

Keywords: Klein–Gordon equation, ladder operator, conformal killing vector

1. Introduction

Recently, the present authors have developed a formulation for ladder operators of the Klein–Gordon equation (KGE), which map a solution of the massive KGE onto another solution with different mass [1]. We thus call them mass ladder operators. The key of the formulation is the mass-dependence of the operators; the commutation relation with the $d'$Alembertian $\Box := g^{\mu\nu} \nabla_\mu \nabla_\nu$ is given by

$$[\Box, D] = \delta m^2 D + 2 \mathcal{Q}(\Box - m^2),$$

where $D$ is the mass ladder operator, $m^2$ and $\delta m^2$ are constants, and $\mathcal{Q}$ is a differential operator\textsuperscript{4}. Note that this commutation relation acts on an arbitrary scalar field. Since equation (1) is written in the form

\textsuperscript{4} If there exists a ladder operator $D$ which shifts the mass squared of the KGE from $m^2$ to $m^2 + \delta m^2$, we can show the existence of another ladder operator $D^*$ which shifts the mass squared of the KGE from $m^2 + \delta m^2$ to $m^2$ (see appendix A).
\[(\Box - (m^2 + \delta m^2)) D = (D + 2Q) (\Box - m^2),\]

one can see that, when \(D\) acts on a solution \(\Phi\) of the massive KGE with mass squared \(m^2\), i.e. \((\Box - m^2)\Phi = 0\), \(D\Phi\) is a solution to the massive KGE with mass squared \(m^2 + \delta m^2\), i.e.

\[\Box - (m^2 + \delta m^2)) (D\Phi) = 0.\]

As applications, the relation between supersymmetric quantum mechanics\(^5\), and the construction of Aretakis’ constant [6–8] in an extremal black hole were also discussed [1].

It is remarkable that such ladder operators are the consequence of conformal symmetry of spacetime. It was assumed in [1] that an \(n\)-dimensional spacetime admits a closed conformal Killing vector \(\zeta^\mu\) being an eigenvector of the Ricci tensor \(R_{\mu\nu}C^\nu = (n-1)\chi\zeta_\mu\). This assumption is satisfied in a maximally symmetric spacetime\(^6\). Acting the ladder operator on a solution of KGE several times, the mass \(m^2\) can be shifted to the minimum (or maximum) value \(m_\Delta^2\), satisfying \((n-1)^2/4 \geq m_\Delta^2/\chi \geq n(n-2)/4\). This implies that from the solutions of KGE with masses in this range, one can construct solutions for other masses, outside of this interval. A natural question is whether the physical properties of the KGE with different masses in this interval are different or not. While the mass ladder operator defined in [1] cannot connect these two masses, one still needs to consider the possibility that there exists another ladder operator which does so. This is one of the motivation to consider the general condition for the mass ladder operator.

One may consider similar ladder operators in Riemannian geometry. In that case, the d’Alembertian is replaced with the Laplacian, and the KGE becomes the eigenvalue equation for the Laplacian,

\[(\Delta - \lambda) \Phi = 0,\]

where \(\Delta\) is the scalar-Laplacian and \(\lambda\) is its eigenvalue. In two-dimensional sphere, one can construct the ladder operators for the spherical harmonics \(Y_{\ell,m}\), which shift the quantum number \(\ell\). Although the ladder operators were already known [9, 10], our procedure highlights that their construction and existence stems from the conformal symmetry of sphere.

In this paper, we consider the inverse problem; we investigate the necessary conditions for a spacetime to admit mass ladder operators of the KGE satisfying the commutation relation (1). In particular, we focus on first-order operators for simplicity. As shown later, if the spacetime admits a closed conformal Killing vector and it is an eigenvector of the Ricci tensor, the ladder operator derived from the general condition coincides with that in [1]. This suggests that the KGEs with different masses in the interval \((n-1)^2/4 \geq m_\Delta^2/\chi \geq n(n-2)/4\) have different physical properties, because they cannot be connected with the first order ladder operator, and this interval for the mass characterizes the KGE in this case. Also, we show that the general condition is of wider applicability than previously anticipated [1].

This paper is organized as follows. In section 2, we discuss the condition for the first order mass ladder operator. In section 3, we show the relation with the supersymmetric

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\(^5\)The relation between the massive KGE and a quantum mechanical system for the case of the anti-de Sitter (AdS) spacetime was discussed in [2–4]. These references also highlighted that the selection rule for resonance modes, which is related to the non-linear instability of AdS spacetime [5], can be understood in terms of a hidden symmetry. We find that a sufficient condition for the existence of the resonance modes, i.e. that the quantity \(\Delta\) (defined in equation (2.7) in [2]) is an integer, coincides with the condition such that the corresponding massive KGE is connected with the massless KGE by the mass ladder operator. While this might be just a coincidence, it seems to suggest some relation between the existence of the resonance modes and the mass ladder operator.

\(^6\)For example, there exist \(n + 1\) closed conformal Killing vectors in an \(n\)-dimensional anti-de Sitter spacetime \((\text{AdS}_n)\). Using them, one can construct \(n + 1\) mass ladder operators for the KGE on \(\text{AdS}_n\).
quantum-mechanical system. In section 4, The case with symmetry operator is discussed. Section 5 is devoted to summary and discussion.

2. Mass ladder operators

Suppose $\mathcal{D}$ is a first-order differential operator on an $n$-dimensional spacetime $(\mathcal{M}, g_{\mu\nu})$. Without loss of generality, we write $\mathcal{D}$ as

$$\mathcal{D} = \zeta^\mu \nabla_\mu + K,$$

(5)

where $\zeta^\mu$ and $K$ are a vector field and a function on $\mathcal{M}$, respectively. The commutation relation with the d'Alembertian is given by

$$[\Box, \mathcal{D}] = 2(\nabla^\mu \zeta^\nu) \nabla_\mu \nabla_\nu + (\Box \zeta^\nu + \zeta^\mu R^{\mu\nu} + 2\nabla^\mu K) \nabla_\nu + (\Box K),$$

(6)

where the commutator is assumed as acting on a scalar field. Since $\mathcal{D}$ is first order, $Q$ in equation (1) is required to be a function $Q$ and thus the right-hand side of equation (1) is calculated as

$$\delta m^2 \mathcal{D} + 2Q(\Box - m^2) = 2Q\Box + \delta m^2 \zeta^\nu \nabla_\nu + (\delta m^2 K - 2m^2 Q).$$

(7)

Substituting equations (6) and (7) into (1), we obtain the conditions (see appendix B for the detailed discussion)

$$\nabla_{(\mu} \zeta_{\nu)} = Qg_{\mu\nu},$$

(8)

$$\Box \zeta^\mu + R^\mu_{\nu} \zeta^\nu + 2\nabla_\mu K = \delta m^2 \zeta^\mu,$$

(9)

$$\Box K = \delta m^2 K - 2m^2 Q.$$  

(10)

In what follows, we look into these conditions in detail. First of all, equation (8) means that $\zeta^\mu$ is a conformal Killing vector field (CKV), and the function $Q$ is given by

$$Q = \frac{1}{n} \nabla_\mu \zeta^\mu,$$

(11)

which is called the associated function of $\zeta^\mu$ (see also appendix C for the basic property of CKVs).

If $Q = 0$, $\zeta^\mu$ becomes a Killing vector field (KV). In such a case, since we have $\nabla_\mu \zeta^\mu = 0$ and $\Box \zeta^\mu + R^\mu_{\nu} \zeta^\nu = 0$ from the divergence of the Killing equation $\nabla_{(\mu} \zeta_{\nu)} = 0$, equation (9) becomes $2\nabla_\mu K = \delta m^2 \zeta^\mu$, and hence $\Box K = 0$. Substituting this into equation (10), we obtain that, if $\delta m^2 \neq 0$, then $K = 0$. Thus, equation (9) leads $\zeta^\mu = 0$, however, this is not the case we are interested in. We thus find that, if $Q = 0$, equations (8)–(10) have the only trivial solution $\delta m^2 = 0$ and $K = \text{const}$ for a KV $\zeta^\mu \neq 0$. This has the well-known consequence that, if $\zeta^\mu$ is a KV, $\mathcal{D} = \zeta^\mu \nabla_\mu$ does not shift the mass. Hereafter, we focus on the non-trivial case when $Q \neq 0$.

The divergence of equation (8) leads to

$$\Box \zeta^\mu + R^\mu_{\nu} \zeta^\nu = (2 - n) \nabla_\mu Q.$$  

(12)

Substituting equations (12) into (9), we have

$$\delta m^2 \zeta^\mu = \nabla_\mu ((2 - n) Q + 2K).$$  

(13)
Since we are interested in a ladder operator which shifts the mass of KGE, we consider the case \( \delta m^2 \neq 0 \). In this case, equation (13) shows that \( \zeta_\nu \) is closed, i.e. \( \nabla[\nu \zeta_\nu] = 0 \). Hence \( \zeta_\mu \) is a closed conformal Killing vector (CCKV) which satisfies the equation

\[
\nabla_\mu \zeta_\nu = Q_{\mu\nu}.
\]

Thus, we find that a spacetime admitting a first-order mass ladder operator of the massive KGE must admit a CCKV.

A CCKV \( \zeta_\mu \) with a nonzero associated function \( Q \) must be either timelike or spacelike. If \( \zeta_\mu \) is null, \( 0 = \nabla_\mu (\zeta_\nu \zeta_\nu) = 2Q_{\mu\nu} \) and hence \( Q = 0 \). Since \( \zeta_\mu \) is a closed 1-form, one can introduce a function \( \lambda \) such that \( d\lambda = \zeta_\mu d\vec{x}_\mu \) in local coordinates \( x^\mu \). Using this function \( \lambda \) as one of local coordinates, we obtain the local form of metrics admitting a CCKV (see [11]),

\[
g_{\mu\nu} dx^\mu dx^\nu = \frac{d\lambda^2}{f(\lambda)} + f(\lambda) \tilde{g}_{ij} dx^i dx^j,
\]

where \( f(\lambda) \) is a function depending only on \( \lambda \), \( \tilde{g}_{ij} \) is an \((n-1)\)-dimensional metric, and \( x^\mu = (\lambda, x^i) \) are the local coordinates. In these coordinates, we have \( \zeta^\mu (\partial/\partial x^\mu) = f(\partial/\partial \lambda) \) and \( Q = f'/2 \). With this local form of the metric, the KGE can be solved by separation of variables between the coordinate \( \lambda \) and the others.

Now that \( \zeta_\mu d\vec{x}_\mu = d\lambda \), equation (13) reads \( d(\delta m^2 \lambda) = d((2-n)Q + 2K) \) where \( \delta m^2 \neq 0 \).

Integrating it, we have

\[
\delta m^2 \lambda = (2-n)Q + 2K + c,
\]

with an integration constant \( c \). This means that, since \( Q \) is a function of \( \lambda \), \( K \) is also a function of \( \lambda \) and is given by

\[
K = \frac{(n-2)}{4} f' + \frac{\delta m^2}{2} \lambda - \frac{c}{2}.
\]

Hence, the mass ladder operator is provided in general by

\[
D = f \frac{\partial}{\partial \lambda} + \frac{(n-2)}{4} f' + \frac{\delta m^2}{2} \lambda - \frac{c}{2}.
\]

It is also shown from the divergence of equation (14) that

\[
\nabla_\mu Q = \frac{1}{1-n} R_{\mu\nu} \zeta_\nu,
\]

which is same as equation (12) in the case when \( \zeta_\mu \) is a CCKV. If we use the metric form (15), it becomes

\[
R_{\mu\nu} \zeta_\nu = -\frac{(n-1)}{2} f'' \zeta_\mu.
\]

The remaining condition to solve is equation (10). Since the d’Alembertian acting on \( K \) is calculated as

\[\text{In the case } \delta m^2 = 0, \text{ the operator } D \text{ is called a symmetry operator. While a symmetry operator does not shift the mass, it is still interesting since it can map a solution of KGE into another solution of KGE with the same mass. This case will be discussed in section 4.}\]

\[\text{If } \zeta_\mu \text{ is timelike, i.e. } f \text{ takes negative value, } (-\tilde{g}_{ij}) \text{ should be positive definite metric so that the whole metric has } [-, +, \ldots, +] \text{ signature.}\]

\[\text{Moreover, it is shown that if } f''' = 0, \text{ the equation separated with } \lambda \text{ results in Legendre’s differential equation, which is explicitly solved by the Legendre functions (see appendix D).}\]
\[ \Box K = f^{-(n-2)/2} \frac{d}{d\lambda} \left( f^{n/2} \frac{d}{d\lambda} K \right) = fK'' + \frac{n}{2} f'K', \]  

(21)

Equation (10) becomes

\[ (n-2)ff''' + \frac{n(n-2)}{2}f'f'' + \alpha_1 f' + \alpha_2 + \alpha_3 \lambda = 0, \]

(22)

where

\[ \alpha_1 = 2(\delta m^2 + 2m^2), \quad \alpha_2 = 2\delta m^2 c, \quad \alpha_3 = -2(\delta m^2)^2. \]

(23)

In the next subsections, we explicitly solve this nonlinear differential equation for \( f \) by dividing it into two cases: (A) \( f''' = 0 \) and (B) \( f''' \neq 0 \).

2.1. \( f''' = 0 \) case

If \( f''' = 0 \), \( f \) is quadratic in \( \lambda \). By setting

\[ f = c_0 + c_1 \lambda - \chi \lambda^2 \]

(24)

with constants \( c_0 \), \( c_1 \) and \( \chi \), equation (22) becomes

\[ \alpha_2 + c_1 (\alpha_1 - n(n-2) \chi) + \lambda \left( \alpha_3 - 2\chi (\alpha_1 - n(n-2) \chi) \right) = 0. \]

(25)

For this equation to be satisfied for any value of \( \lambda \), we obtain

\[ m^2 = -\chi k(k+n-1), \quad \delta m^2 = \chi(2k+n-2), \quad c = \frac{(2k+n-2)c_1}{2}, \]

(26)

with a parameter \( k \). Imposing the condition such that both \( m^2 \) and \( m^2 + \delta m^2 \) are real values, we find that the parameter \( k \) also takes a real value. Solving the first equation in equation (26) w.r.t. \( k \), we obtain

\[ k = \frac{1 - n \pm \sqrt{(n-1)^2 - 4m^2/\chi}}{2}. \]

(27)

From the positivity of the inside of the square root of this equation, we find the range of masses so that the ladder operator exists as

\[ \frac{\chi(n-1)^2}{4} \leq m^2, \quad (\chi < 0), \quad \frac{\chi(n-1)^2}{4} \geq m^2, \quad (\chi > 0). \]

(28)

We note that the lower bound for \( \chi < 0 \) case is a negative value. In \( AdS \) case, this coincides with the Breitenlohner-Freedman bound [12, 13] which is the lowest mass for avoiding unstable modes. To summarize, we have obtained the mass squared \( m^2 \) and its shift \( \delta m^2 \) as one- parameter families, which enables us to shift various masses with the mass ladder operator.

In the present case, we can confirm from equation (20) that \( \zeta^\mu \) is an eigenvector of the Ricci tensor,

\[ R^\mu_{\nu \lambda \kappa} \zeta^\nu = (n-1) \chi \zeta^\mu, \]

(29)

which is the condition assumed in [1]. Since we also find that \( K = -kQ \), the mass ladder operator \( D \) is given by

\[ D = \zeta^\mu \nabla_\mu - kQ, \]

(30)

which is the mass ladder operator \( D_k \) discussed in [1].
2.2. $f''' \neq 0$ case

Let us consider the case when $f''' \neq 0$. This case happens only when $n \geq 3$ since equation (22) with $n = 2$ implies $f''' = 0$. In this case, $f'$, $\lambda$ and a nonzero constant must be linearly independent; otherwise, we have $f' = b_1 \lambda + b_2$ with some constants $b_1$ and $b_2$ and hence $f''' = 0$. We can write equation (22) in the form

$$-(n-2) \left( f'''' + \frac{n}{2} f''' \right) = \alpha_1 f' + \alpha_2 + \alpha_3 \lambda. \quad (31)$$

For a function $f$ fixed, the coefficients $\alpha_1$, $\alpha_2$, and $\alpha_3$ must be determined uniquely because $f'$, $\lambda$ and a nonzero constant are linearly independent. Solving equation (23) w.r.t. $m^2, \Delta m^2$ and $c$, we obtain

$$m^2 = m^2_{\pm} := \frac{\alpha_1 \pm \sqrt{-2\alpha_3}}{4}, \quad \Delta m^2 := \mp \frac{\sqrt{-\alpha_3}}{\sqrt{2}}, \quad c = c_{\pm} := \mp \frac{\alpha_2}{\sqrt{-2\alpha_3}}. \quad (32)$$

This means that for a fixed spacetime, i.e. for a fixed function $f$, $m^2$ can take only two values. From equation (16), the mass ladder operators are given by\(^\text{10}\)

$$\mathcal{D} = \mathcal{D}_{\pm} := \zeta^i \nabla_i + \frac{(n-2)Q + \Delta m^2_{\pm} \lambda - c_{\pm}}{2}. \quad (33)$$

Thus, we obtain mass ladder operators of the massive KGE with only two fixed masses $m^2_{\pm}$ in contrast to the case $f''' = 0$. We can see that $\mathcal{D}_{\pm}$ maps a solution of KGE with $m^2_{\pm}$ to that with $m^2_{\mp}$ because we have a relation $m^2_{\pm} + \Delta m^2_{\mp} = m^2_{\mp}$.

If we assume that the function $f$ can be expanded around $\lambda = 0$,

$$f = \sum_{i=0}^{\infty} \beta_i \lambda^i, \quad (34)$$

Equation (22) leads to the equations

$$-\alpha_2 - \alpha_1 \beta_1 - (n-2)(n\beta_1 \beta_2 + 6\beta_0 \beta_3) = 0, \quad (35)$$

$$-\alpha_3 - 2\alpha_1 \beta_2 - (n-2) \left( 2n\beta_2^2 + 3(2+n)\beta_1 \beta_3 + 24\beta_0 \beta_4 \right) = 0, \quad (36)$$

$$\alpha_1 (k-2) \beta_{k-2} + \sum_{i=0}^{k} i(i-1) \left( i - 2 + \frac{n}{2}(k-i) \right) \beta_{k-i} \beta_i = 0. \quad (k \geq 5) \quad (37)$$

There are six constant degrees of freedoms in these equations. For example, if we fix the constants $\alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2$, then $\beta_{k \geq 3}$ will be determined. We can see that $\beta_{k \geq 4} \neq 0$ if $\beta_3 \neq 0$. If we assume that $\beta_{k \geq 3} = 0$, the above conditions reduce to the case of $f''' = 0$. Alternatively, one can interpret them as the equations that determine the values of $m^2$, $\Delta m^2$ and $c$. Actually, this can be done with equations (35) and (36) if the values of the six parameters $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4$ and $\beta_5$ are provided; the other constants $\beta_i$ ($i \geq 6$) are determined by equation (37). Thus, equation (37) provides the condition for the metric to admit the mass ladder operator (33).

While the solution $f$ to equation (31) had only to be described as a series expansion in a generic case, equation (31) in the case of $n = 6$ and $\alpha_1 = 0$ can be solved explicitly. Actually, in that case, equation (31) becomes

\(^{10}\) Note that $\mathcal{D}, \mathcal{D}$ is the conjugate ladder operator to $-\mathcal{D}_+, \mathcal{D}_-$ (see appendix A).
\[(f^2)''' = 4m^4\lambda + 2m^2c,\]  
where we have used the condition \(\alpha_1 = 0\), i.e. \(\delta m^2 = -2m^2\), and this is easily solved by

\[f = \pm \sqrt{e_0 + e_1\lambda + e_2\lambda^2 + \frac{m^2c}{3}\lambda^3 + \frac{m^4}{6}\lambda^4},\]  
with integration constants \(e_0, e_1\) and \(e_2\).

### 3. Relation with supersymmetric quantum mechanics

In [1] a relation of the KGE admitting a mass ladder operator to the Schrödinger equation in a supersymmetric quantum-mechanical system was discussed, in which the mass ladder operator is mapped into the supercharge. We repeat this discussion in the present general framework.

For the metric (15), we consider the conformal transformation 
\[\bar{g}_{\mu\nu} = \Omega^{-2}g_{\mu\nu}\]  
with the conformal factor \(\Omega = 1/\sqrt{f}\). Since the resulting metric is given by

\[\bar{g}_{\mu\nu}d\bar{x}_\mu d\bar{x}_\nu = d\bar{\lambda}^2 + \bar{g}_{ij}d\bar{x}_i d\bar{x}_j = d\bar{\lambda}^2 + \tilde{g}_{ij}d\lambda_i d\lambda_j,\]  
the CCKV \(\zeta^\mu\) on \((M, g_{\mu\nu})\), which is given by

\[\zeta^\mu = f\partial^\mu = \partial^\lambda\]  
becomes a KV on \((M, \bar{g}_{\mu\nu})\).

Under this conformal transformation the KGE on \(g_{\mu\nu}\),

\[\Box - m^2)\Phi = \Omega^{(n+2)/2}(\Box - U)\Phi,\]  
where \(\Box\) is the d’Alembertian for \(g_{\mu\nu}\), \(\Phi = \Omega^{(2-n)/2}\Phi\) and

\[U = \frac{1}{16}(16m^2f + (n - 2)^2(f')^2 + 4(n - 2)f''').\]  
This means that the KGE on \(g_{\mu\nu}\), \((\Box - m^2)\Phi = 0\) is mapped into the equation on \(\bar{g}_{\mu\nu}\),

\[\bar{\Box} - U)\bar{\Phi} = f\frac{\partial}{\partial\lambda}\left(f\frac{\partial}{\partial\lambda}\Phi\right) + \bar{\Box}\Phi - U\bar{\Phi} = 0,\]  
where \(\bar{\Box}\) is the d’Alembertian for \(\bar{g}_{\mu\nu}\). Equation (43) admits separation of variable for the ansatz

\[\bar{\Phi}(\lambda, x') = \psi(\lambda)\Theta(x'),\]  
and it then reduces to the ODE

\[-f\frac{d^2}{d\lambda^2}f\frac{d}{d\lambda}\psi + U\psi = \nu^2\psi,\]  
where \(\nu^2\) is the separation constant.

Recall that the general condition for the existence of mass ladder operators was given by equation (22). Integrating equation (22), we have

\[(n - 2)f'' + \frac{(n - 2)^2}{4}(f')^2 + 2(\delta m^2 + 2m^2)f + 2\delta m^2c\lambda - (\delta m^2)^2\lambda^2 + \sigma = 0,\]  
where \(\sigma\) is an integration constant. Substituting this equation into equation (42), we obtain

\[U = \frac{1}{4}\left(-2\delta m^2f + \delta m^4\lambda^2 - 2\delta m^2c\lambda - \sigma\right).\]
It is remarkable that for this potential, equation (45) is factorized\textsuperscript{11} as
\[
\left( -f \frac{d}{d\lambda} + W \right) \left( f \frac{d}{d\lambda} + W \right) \psi = E \psi, \tag{48}
\]
where
\[
W = \frac{1}{2} \delta m^2 \lambda - \frac{c}{2}, \tag{49}
\]
and \( E := \nu^2 + (\sigma + c^2)/4 \). This factorization is important to see a relation with a supersymmetric quantum-mechanical system. We introduce the supercharges\textsuperscript{13}
\[
A^\dagger = -f \frac{d}{d\lambda} + W, \quad A = f \frac{d}{d\lambda} + W, \tag{50}
\]
and set the first Hamiltonian \( H \) by
\[
H := A^\dagger A. \tag{51}
\]
with
\[
V := \frac{1}{4} \left( -2 \delta m^2 f + \delta m^4 \lambda^2 - 2 \delta m^2 c \lambda + c^2 \right). \tag{52}
\]
Equation (48) is written as
\[
H \psi = E \psi. \tag{53}
\]
Setting the second Hamiltonian \( \tilde{H} \) by \( \tilde{H} := AA^\dagger \), we have
\[
\tilde{H} := \left( -f \frac{d}{d\lambda} + W \right) \left( f \frac{d}{d\lambda} + W \right) + \tilde{V}, \tag{54}
\]
where
\[
\tilde{V} := \frac{1}{4} \left( 2 \delta m^2 f + \delta m^4 \lambda^2 - 2 \delta m^2 c \lambda + c^2 \right). \tag{55}
\]
We consider the eigenvalue equation for \( \tilde{H} \)
\[
\tilde{H} \tilde{\psi} = \tilde{E} \tilde{\psi}. \tag{56}
\]
Now, by construction, we have the so-called intertwining relations
\[
\tilde{H} A = \Lambda H, \quad H A^\dagger = A^\dagger \tilde{H}. \tag{57}
\]
\textsuperscript{11}The factorization of equation (45) can be made for the potential (47) with a generic \( f \); however, we suppose here that \( f \) is a solution to equation (46).
\textsuperscript{12}In terms of the coordinate \( \tilde{\lambda} \) defined by \( d\lambda = d\tilde{\lambda}/f \), equation (48) is written as \( (-d/d\tilde{\lambda} + W)(d/d\tilde{\lambda} + W)\psi = E\psi \). Then equation (46) is regarded as the condition which determines \( W \) as a function of \( \tilde{\lambda} \).
\textsuperscript{13}In this section, the dagger \( ^\dagger \) denotes the adjoint operator in \( (M, \tilde{g}_{\mu \nu}) \) defined in [14]. If we set appropriate Hilbert spaces \( H_1 \) and \( H_2 \), \( A^\dagger \) can be the adjoint operator of \( A \) in the usual sense, that is, \( A (A^\dagger) \) is a linear map from the Hilbert space \( H_1 \) to \( H_2 \) (\( H_1 \)) and satisfies the condition \( < \psi_1, A^\dagger \psi_2 >_1 = < A \psi_1, \psi_2 >_2 \), where \( \psi_i \in H_i \) and \( <, > \) is the inner product in the Hilbert space \( H_i \). In the present case, \( <, > \) should be defined as \( < \psi, \phi > := \int d\lambda f^{-1} \psi \phi \) where \( \psi, \phi \in H_i \).
which shows that if $\psi$ is an eigenfunction for $H$ ($H\psi = E\psi$) is an eigenfunction for $\tilde{H}$ ($\tilde{H}\psi = E\psi$), and $A\psi$ ($A^\dagger\tilde{\psi}$) is an eigenfunction for $\tilde{H}$ ($\tilde{H}\psi = E\psi$) with the same eigenvalue. Indeed, we can easily check that

$$\tilde{H}A\psi = A\tilde{H}\psi = A(E\psi) = E(A\psi),$$

$$A^\dagger\tilde{\psi} = \tilde{A}^\dagger\tilde{\psi} = A\tilde{E}\tilde{\psi} = \tilde{E}(A^\dagger\tilde{\psi}).$$

If $f$ is a solution to equation (46) with the set of parameters $(m^2, \delta m^2, c)$, it is also a solution to equation (46) with $(m^2 + \delta m^2, -\delta m^2, -c)$. This means that, for a spacetime fixed (i.e. a solution $f$ fixed), we have ladder operators for the KGE with mass squared $m^2$ and $m^2 + \delta m^2$, which map solutions with the mass squared $m^2$ and $m^2 + \delta m^2$ into solutions with $m^2 + \delta m^2$ and $m^2$, respectively. As already seen the KGE with mass squared $m^2$ became equation (53); whereas, since we have $A^\dagger \rightarrow -A$ and $A \rightarrow -A^\dagger$ under the transformation $(m^2, \delta m^2, c) \rightarrow (m^2 + \delta m^2, -\delta m^2, -c)$, $H \rightarrow \tilde{H}$, and $E \rightarrow \tilde{E}$ under this transformation, and hence the KGE with mass squared $m^2 + \delta m^2$ becomes equation (56) with $\tilde{E} = E$. Thus, we see that the supercharges mapping eigenfunctions between two Hamiltonians $H$ and $\tilde{H}$ correspond to the mass ladder operators mapping solutions between two KGEs with mass squared $m^2$ and $m^2 + \delta m^2$. More explicitly, we can show the relation between the mass ladder operator and the supercharge. The mass ladder operator is given by (18), and the supercharge $A$ is given by (50). When we compare both expressions, we find

$$A\psi = D\Xi,$$

where $\Xi = \Omega^{(n-2)/2}\psi$ is the function of $\lambda$ separated in a solution $\Phi$ to the KGE as (D.2) (see appendix C for details).

Finally we remark that, in the $f''' = 0$ case, two Hamiltonians in the supersymmetric quantum mechanics admit shape invariance, so that the supercharges come to map wave functions to wave functions for the same Hamiltonian but between different energies (see [1] and see also appendix E). Due to this property we can construct all the energy spectrum for the system from a seed wave function by using the supercharge. This corresponds to shifting the masses of Klein–Gordon fields by acting the mass ladder operator many times. On the other hand, in the $f''' \neq 0$ case, the present system does not admit shape invariance. In that case, as was seen above, since the supercharges only map wave functions between two Hamiltonian $H$ and $\tilde{H}$ in the same energy level, the mass ladder operators also only map solutions between two KGEs with mass squared $m^2$ and $m^2 + \delta m^2$.

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14 We can also show this by an explicit calculation. We adapt the notation like $H^{(m', \delta m', \sigma)}$ which denotes an operator when the mass ladder operator exists for the KGE with mass squared $m^2$ and has parameters $\delta m^2$ and $c$. Here, we only consider a fixed function $f$ and a fixed value of $\sigma$. From the above discussion, there also exists $H^{(m', \delta m', -\delta m', -c)}$, and we can see the relation $H^{(m', \delta m', -\delta m', -c)} = H^{(m', \delta m', \sigma)}$. For a function $\psi_{m'}$ which satisfies

$$H^{(m', \delta m', \sigma)}\psi_{m'} = E^{(m', \delta m', \sigma)}\psi_{m'},$$

we have

$$H^{(m', \delta m', -\delta m', -c)}A^{(m', \delta m', \sigma)}\psi_{m'} = A^{(m', \delta m', \sigma)}H^{(m', \delta m', -\delta m', -c)}\psi_{m'} = A^{(m', \delta m', -\delta m', -c)}A^{(m', \delta m', \sigma)}\psi_{m'} = E^{(m', \delta m', -\delta m', -c)}A^{(m', \delta m', \sigma)}\psi_{m'},$$

where we used $E^{(m', \delta m', \sigma)} = E^{(m', \delta m', -\delta m', -c)}$ since it only depends on $c^2$. This shows that $\psi_{m'}$ is mapped into a solution of KGE with mass squared $m^2 + \delta m^2$ by $A^{(m', \delta m', \sigma)}$. 

We now specialize our discussion to the $\delta m^2 = 0$ case. The discussion is identical to the previous one, until equation (13). We should note that $\zeta^\mu$ is a CKV, but it is not necessarily closed, since equation (13) with $\delta m^2 = 0$ does not imply the closed condition for $\zeta^\mu$. Since we have $\delta m^2 = 0$, the commutation relation (1) is given by

$$[\Box, \mathcal{D}] = 2Q(\Box - m^2).$$

In other words, $\mathcal{D}$ is a symmetry operator of the KGE. When $\mathcal{D}$ acts on a solution $\Phi$ to the KGE with mass squared $m^2$, i.e. $(\Box - m^2)\Phi = 0$, one finds that $(\Box - m^2)(\mathcal{D}\Phi) = 0$.

It is shown from equation (13) with $\delta m^2 = 0$ that

$$2K + (2 - n)Q + c = 0,$$

where $c$ is an integration constant. This constant degree of freedom corresponds to constant shift symmetry for $K$ when $\mathcal{D}$ is a symmetry operator $\delta m^2 = 0$. We can see this from that equations (9) and (10) with $\delta m^2 = 0$ only contain derivative of $K$. So, it is enough to consider the case $c = 0$. Note that equation (63) seems to be a particular case of equation (16), but $\zeta^\mu$ is not necessarily closed now. From equations (10) and (63), we obtain

$$(n - 2)\Box Q + 4m^2 Q = 0.$$  

Thus, we find that if there exists a conformal Killing vector with the associated function which satisfies equation (64),

$$\mathcal{D} = \zeta^\mu \nabla_\mu + \frac{n-2}{2}Q,$$

becomes a symmetry operator of the KGE with mass squared $m^2$.

We consider several special cases as follows:

### 4.1. Killing vector case

When $\zeta^\mu$ is a KV, it satisfies the Killing equation $\nabla_\mu (\zeta_\nu) = 0$, which corresponds to the $Q = 0$ case in section 2. In this case, we obtain $K = \text{const}$ from equation (63). Since equation (64) holds for any $m^2$, $\mathcal{D}$ becomes a symmetry operator of the KGE with arbitrary mass squared. This is a well-known result. In this case, the commutation relation is given by $[\Box, \mathcal{D}] = 0$.

### 4.2. Homothetic vector case

When $\zeta^\mu$ is a homothetic vector (HV), it satisfies the conformal Killing equation $\nabla_\mu (\zeta_\nu) = Qg_{\mu\nu}$ with $Q$ constant. This case requires that $m^2 = 0$ and $K = \text{const}$ from equations (63) and (64). So, $\mathcal{D}$ becomes a symmetry operator of the massless KGE. In this case, the commutation relation is given by $[\Box, \mathcal{D}] = 2Q\Box$.

### 4.3. Closed and eigenvector of Ricci tensor case

If we consider that $\zeta^\mu$ is closed and an eigenvector of Ricci tensor $R_{\mu\nu}\zeta^\nu = \chi(n - 1)\zeta_\mu$, then we have $\Box Q + nQ = 0$. Compared with equation (64), we can see that $\mathcal{D}$ is a symmetry operator for KGE with $m^2 = n(n-2)\chi/4$. In this case, the ladder operator corresponds to $D_k$ with $k = -(n - 2)/2$, i.e. $\delta m^2 = \chi(2k + n - 2) = 0$ in [1].
4.4. Constant scalar curvature case

Using equation (C.8), equation (64) can be written as

$$L_c R = \frac{8 m^2 (n-1) - 2(n-2)R}{n-2} Q.$$  
(66)

This equation implies that if a spacetime has a constant scalar curvature $R = \chi n(n-1)$, $\mathcal{D}$ becomes a symmetry operator of the KGE with mass squared $m^2 = n(n-2)\chi/4$. In $n \geq 3$, it is known that all (non-trivial) CKVs are closed in a maximally symmetric spacetime, so in fact, this symmetry operator coincides with that constructed from CCKV in the previous subsection in maximally symmetric case.

4.5. Two dimensional case

From equation (64), we have $m^2 = 0$ for a CKV with $Q \neq 0$ if $n = 2$. Thus, the operator $\mathcal{D} = \zeta^\mu \nabla_\mu$ in equation (65) becomes a symmetry operator of the massless KGE for any CKV in two dimensions.

5. Summary and discussion

In this paper we have investigated mass ladder operators of the massive KGE. In particular, we focused on first-order mass ladder operators which satisfy the commutation relation (1). We found that if a spacetime admits such a ladder operator, the metric must admit a CCKV and hence the metric is written in the form (15). Moreover, if we assume that the mass ladder operator can act on scalar fields with continuous mass range for a fixed background spacetime, we can say $f''' = 0$ and obtain the same condition for the mass ladder operator in the previous paper [1]. We also obtain the $f''' \neq 0$ case which is not discussed in [1]. In that case, the ladder operator still exists but we can construct it for the KGE with two fixed masses in contrast to $f''' = 0$ case. We derived the general metric forms for both cases.

The meaning of the conditions for the mass ladder operator in the previous paper [1] are now clearer. As shown in appendix D, the presence of CCKV implies that the KGE has a solution with the form of the separation of variable. The additional condition $R_{\mu\nu} \zeta^\nu = (n-1)\chi\zeta_\mu$ in equation (29) (or $f''' = 0$) is the condition that the KGE can be solved by using the Legendre function. In that case, we can obtain the mass ladder operator from the differential recursion relations for the Legendre functions. This is the mathematical reason for the existence of the mass ladder operator in [1].

An interesting result we found concerns the existence condition for the general first-order mass ladder operator: the corresponding quantum-mechanical system becomes supersymmetric. The two fixed masses in $f''' \neq 0$ case correspond to the superpartners in this quantum-mechanical system. If the potential also has a shift symmetry, which corresponds to $f''' = 0$ case, the mass ladderoperator can act on scalar fields with various masses.

As a prospect, there are several possible generalizations of the present work. It might be interesting to consider the ladder operator for a system other than the KGE. As a simple example, we show a mass ladder operator for a scalar field which is coupled with a gauge field in appendix F. In [9] the symmetric tensor harmonics in $S^n$ were constructed by embedding $S^n$ into $S^{n+1}$ and using isometry of $S^{n+1}$, which seems to suggest the existence of ladder operators between tensor harmonics in a maximally symmetric space. This may be extended to the case of the Lorentz signature. Thus, it is interesting to see how far our formulation can be applied to vector, tensor and spinor fields. Another generalization is the extension to the operator which
contain higher derivative. In that case, the general conditions for the existence of such an operator will be more complicated; on the other hand, the general conditions will contain the Riemann curvature tensor, in contrast to the first-order case (see (8)–(10)), which will restrict the space-time admitting the ladder operator tightly. A naive expectation is that higher-order mass ladder operators are related to higher-rank conformal Killing tensors. In a maximally symmetric spacetime, any conformal Killing tensor is reducible, that is, decomposable as the symmetric tensor product of conformal Killing vector fields. Hence, it is expected that in a maximally symmetric spacetime, higher-order ladder operators are also decomposable as the multiple of first order mass ladder operators. It is an interesting question to ask whether or not there exists a spacetime admitting an irreducible higher-order ladder operator. In any case, such generalizations are challenging, because the conditions from the commutation relation contains the non-trivial case \( f''' \neq 0 \) even in the simplest case where the mass ladder operator on scalar contains at most first order derivative. It would be important to have a physical insight into them, while calculating the general conditions with power.

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Appendix A. Conjugate ladder operator

According to [14], in a spacetime \((M, g_{\mu\nu})\), for a differential operator \( \mathcal{L} \) which acts on scalar fields, we define the adjoint operator \( \mathcal{L}^\dagger \) as

\[
(\mathcal{L}^\dagger \phi_2)\phi_1 - \phi_2 \mathcal{L}\phi_1 = \nabla_\mu s^\mu, \tag{A.1}
\]

where \( \phi_1, \phi_2 \) are arbitrary scalar fields, and \( s^\mu \) is a vector field. From this definition, we can show \((\mathcal{L}_1 \mathcal{L}_2)^\dagger = \mathcal{L}_2^\dagger \mathcal{L}_1^\dagger\). We should note that the adjoint operator is defined locally in this definition, and it is different from the usual adjoint operator in the context of Hilbert spaces\(^{15}\).

Since we obtain by taking the adjoint of equation (2) that

\[
\mathcal{D}^\dagger (\Box - (m^2 + \delta m^2)) = (\Box - m^2) (\mathcal{D} + 2\mathcal{Q})^\dagger, \tag{A.2}
\]

where we have used the fact that \( \Box \) is self-adjoint, \((\mathcal{D} + 2\mathcal{Q})^\dagger \) rather than \( \mathcal{D}^\dagger \) is the operator which maps a solution to the massive KGE with mass squared \( m^2 + \delta m^2 \) to that with \( m^2 \).

In this paper,

\[
\mathcal{D}^\ast := (\mathcal{D} + 2\mathcal{Q})^\dagger \tag{A.3}
\]

will be said to be conjugate to \( \mathcal{D} \). Note that the existence of such a conjugate operator comes from the self-adjointness of the d’Alembertian \( \Box \), and this discussion holds even if \( \mathcal{D} \) contains higher derivative. We can see that \( \mathcal{D}^\ast \mathcal{D} \) becomes a symmetry operator for the KGE. When we consider the first-order mass ladder operator, \( \mathcal{D} \) is given by (5) with equations (8)–(10), and its conjugate operator is explicitly given by

\(^{15}\) See footnote 13.
\[ D^* = -\zeta^\mu \nabla_\mu + K - (n-2)Q. \]  

(A.4)

**Appendix B. Derivation of equations (8)–(10)**

First, we introduce the following proposition.

**Proposition 1.** Let \( A^{\mu\nu}, B^\mu, C \) be tensor, vector and scalar fields, respectively, and assume that the metric \( g_{\mu\nu} \) is C^2. If the equation

\[ A^{\mu\nu} \nabla_\mu \nabla_\nu \Phi + B^\mu \nabla_\mu \Phi + C \Phi = 0 \]  

(B.1)

holds for any scalar field \( \Phi \), the equations \( A^{\mu\nu} = B^\mu = C = 0 \) hold.

**Proof.** By taking the Riemann normal coordinate around a point \( p \) on a spacetime, the metric behaves \( g_{\mu\nu} = \eta_{\mu\nu} + O(x^2) \), where \( \eta_{\mu\nu} \) is a flat metric and we assumed that the point \( p \) is the origin of this coordinate. Choosing a scalar field as \( \Phi = c A^{\mu\nu} x^\mu x^\nu / 2 + c B^\mu x^\mu + c C \) with constants \( c A^{\mu\nu}, c B^\mu, c C \), we obtain an equation at \( p \) as

\[ A^{\mu\nu} |_p c A^{\mu\nu} + B^\mu |_p c B^\mu + C |_p c C = 0. \]  

(B.2)

Since \( \Phi \) is an arbitrary scalar, this equation holds for any \( c A^{\mu\nu}, c B^\mu, c C \), then \( A^{\mu\nu} |_p = B^\mu |_p = C |_p = 0 \). Since \( p \) is any point on the spacetime, we have \( A^{\mu\nu} = B^\mu = C = 0 \).

Let us consider the action of equation (1) on an arbitrary scalar field \( \Phi \),

\[ [\Box, D] \Phi = (\delta m^2 D + 2Q(\Box - m^2)) \Phi. \]  

(B.3)

Substituting equations (6) and (7) into this equation, we obtain

\[ (\nabla_\mu \zeta^\nu - Qg_{\mu\nu}) \nabla^\mu \nabla^\nu \Phi + (\Box \zeta^\mu + R^\mu \nabla^\mu \zeta^\nu + 2\nabla_\mu K - \delta m^2 \zeta^\mu) \nabla_\nu \Phi + (\Box K - \delta m^2 K - 2m^2 Q) \Phi = 0. \]  

(B.4)

Thus, we obtain equations (8)–(10) from the above proposition.

**Appendix C. Some formulas on conformal Killing vectors**

It is known that in \( n \geq 3 \) dimensions, the CKV equation \( \nabla_\mu \zeta^\nu = Qg_{\mu\nu} \) leads to

\[ \nabla_\mu \zeta^\nu = L_{\mu\nu} + Qg_{\mu\nu}. \]  

(C.1)

\[ \nabla_\mu L_{\nu\rho} = -R_{\mu\nu\rho}^\sigma \zeta^\sigma - 2g_{[\mu\nu} \eta_{\rho]}, \]  

(C.2)

\[ \nabla_\mu Q = \eta_\mu. \]  

(C.3)

\[ \nabla_\mu \eta_\nu = -(\nabla_\rho S_{\mu\rho}) \zeta^\nu - 2S_{\mu\nu} Q - \frac{2}{n-2} R^\rho (\mu L_{\nu})^\rho, \]  

(C.4)

where \( L_{\mu\nu} := \nabla_\mu \zeta^\nu, \eta_\mu := \partial_\mu Q \) and

\[ S_{\mu\nu} = \frac{1}{n-2} \left( R_{\mu\nu} - \frac{R}{2(n-1)} g_{\mu\nu} \right) \]  

(C.5)
is the Schouten tensor with the trace $S = (1/2(n-1))R$. These equations are sometimes called the prolonged equations, or, structure equations, which show that a CKV $\zeta^\mu$ is completely determined by the values of $\zeta^\mu$, $L_{\mu\nu}$, $Q$ and $\eta_{\mu}$ at a point on $M$ and hence the space of CKVs is a vector space of finite dimensions $(n+1)(n+2)/2$. In contrast, the space of CKVs in two dimensions is a vector space of infinite dimensions. Actually, since the Riemann tensor is given by $R_{\mu\nu\rho\sigma} = (R/2)(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})$ in two dimensions, equation (C.2) becomes

$$\nabla_\mu L_{\nu\rho} = -g_{\mu[\nu} \zeta_{\rho]} R - 2g_{\mu[\nu} \eta_{\rho]}, \quad \text{(C.6)}$$

and hence we do not have the counterpart to equation (C.4) in two dimensions.

From equations (C.1)–(C.4), we can derive useful relations

$$\nabla_\mu Q = \frac{1}{n} \nabla_\mu \nabla_\nu \zeta^\nu = -\frac{1}{n-2} \left( \Box \zeta^\mu + R_{\mu\nu} \zeta^\nu \right), \quad \text{(C.7)}$$

$$\Box Q = -\frac{1}{n-1} \left( \frac{1}{2} L_{\mu\nu} R + R Q \right), \quad \text{(C.8)}$$

$$[\nabla_\nu, \Box] \zeta^\nu = \nabla^\mu (R_{\mu\nu} \zeta^\nu). \quad \text{(C.9)}$$

Moreover, if we impose the closed condition $\nabla_{[\mu} \zeta_{\nu]} = 0$, we have $L_{\mu\nu} = 0$. Hence, equations (C.2) leads to (19).

**Appendix D. Separation of variables in the Klein–Gordon equation**

The existence of a CCKV restricts the metric form into (15), which leads to the separation of the variable $\lambda$ in the KGE. Actually, since the d’Alembertian for the metric form (15) acts on a scalar field $\Phi$ as

$$\Box \Phi = f^{-2} \partial_\lambda (f^{-2/2} \partial_\lambda \Phi) + f^{-1} \Box \Phi, \quad \text{(D.1)}$$

where $\Box := \tilde{g}^{ij} \nabla_i \nabla_j$ and $\nabla_i$ are the d’Alembertian and the covariant derivative of $\tilde{g}_{ij}$, the KGE $(\Box - m^2) \Phi = 0$ admits the separation of variable for the ansatz

$$\Phi(\lambda, x^i) = \Xi(\lambda) \Theta(x^i), \quad \text{(D.2)}$$

and reduces to the ODE

$$f^2 \Xi'' + \frac{n}{2} f f' \Xi' + \left( -m^2 f + \nu^2 \right) \Xi = 0, \quad \text{(D.3)}$$

where $\nu^2$ is the separation constant, satisfying the eigenvalue equation $\Box \Theta = \nu^2 \Theta$. Below, we discuss equation (D.3) in the respective case of $f''' = 0$ and $f''' \neq 0$. For both cases, the ladder operator $\mathcal{D}$ maps a solution $\Xi$ to the equation (D.3) into a solution $\Xi := \mathcal{D} \Xi$ to the equation

$$f^2 \Xi'' + \frac{n}{2} f f' \Xi' + \left( -(m^2 + \delta m^2) f + \nu^2 \right) \Xi = 0. \quad \text{(D.4)}$$

The difference between two cases is that, in the $f''' \neq 0$ case, $f$ is given independent of $m^2$, whereas, in the $f''' = 0$ case, $f$ depends on $m^2$.

In the $f''' = 0$ case, the function $f$ is given by the quadratic function (24). Hence, equation (D.3) becomes
\[(c_0 + c_1 \lambda - \chi \lambda^2)^2 \Xi'' + \frac{n}{2} (c_0 + c_1 \lambda - \chi \lambda^2) \Xi' \]
\[+ \left( -m^2 (c_0 + c_1 \lambda - \chi \lambda^2) + \nu^2 \right) \Xi = 0. \tag{D.5} \]

This contains 6 parameters; \(c_0, c_1, \chi, m^2, \nu^2 \) and \(n \). It is interesting that this second-order ODE can be solved by the Legendre functions. In fact, we perform the redefinition of the variable \( \Xi(\lambda) \) by \( \Xi(\lambda) = f(\lambda)^{(2-n)/4} P(\lambda) \). Moreover, after making the coordinate transformation \( \lambda \to z \)
\[\lambda = \frac{c_1 - z \sqrt{c_1^2 + 4 c_0 \chi}}{2 \sqrt{c_1^2 + 4 c_0 \chi}}, \tag{D.6} \]
with the reparametrization \( (\chi, m^2, \nu^2) \to (\kappa^2, \alpha, \beta) \)
\[\chi = \frac{\kappa^2 - c_1^2}{4 c_0}, \quad m^2 = \frac{(c_1^2 - \kappa^2)(-n(2-n) - 4 \alpha(1+\alpha))}{16 c_0}, \quad \nu^2 = \frac{\kappa^2(n - 2 - 2 \beta)(n - 2 + 2 \beta)}{16}, \tag{D.7} \]
Equation (D.5) reduces to the associated Legendre differential equation
\[(1 - z^2) \frac{d^2 P}{dz^2} - 2z \frac{dP}{dz} + \left\{ \alpha(\alpha + 1) - \frac{\beta^2}{1 - z^2} \right\} P = 0, \tag{D.8} \]
which is solved by the Legendre function \( P = P_{\alpha, \beta}(z) \). Hence, the solution to the equation (D.5) is given by
\[\Xi_{m^2, \nu^2}(\lambda) = C_{m^2, \nu^2} f(\lambda)^{(2-n)/4} P_{\alpha, \beta}(z(\lambda)), \tag{D.9} \]
where \( z(\lambda) = (c_1 - 2 \chi \lambda) \sqrt{c_1^2 + 4 c_0 \chi} \) and \( C_{m^2, \nu^2} \) is the normalization constant. Note that \( \alpha \) and \( \beta \), given by (D.7), are functions of \( m^2 \) and \( \nu^2 \), respectively.

The ladder operator in this case is given by
\[D = D_k = \int \frac{dP}{d\lambda} - \frac{k(c_1 - 2 \chi \lambda)}{2}, \tag{D.10} \]
and leads to the relation
\[D_k \Xi_{m^2, \nu^2}(\lambda, \chi') = \frac{C_{m^2, \nu^2}}{C_{m^2 + \delta m^2, \nu^2}} \Xi_{m^2 + \delta m^2, \nu^2}(\lambda, \chi'), \tag{D.11} \]
where \( \Xi_{m^2 + \delta m^2, \nu^2}(\lambda, \chi') = f(\lambda)^{(2-n)/4} P_{\alpha(\lambda + \delta m^2), \beta}(z(\lambda)), \) and \( k \) is chosen to satisfy the relations (26). This relation is essentially the same result as the well-known differential recursion relations for the Legendre functions.

In \( f''' \neq 0 \) case, equation (D.3) deals with a wider class of second-order ODEs than the associated Legendre differential equation. In this case, the masses we can shift are only two kinds, for a spacetime fixed.
Appendix E. Supersymmetric quantum-mechanical systems with a shape invariance

If the function $f$ is given by a quadratic function (24), $m^2, \delta m^2, c$ takes equation (26),

$$m^2 = m_k^2 = -\chi k(k + n - 1), \quad \delta m^2 = \delta m_k^2 = \chi(2k + n - 2), \quad c = c_k = \frac{(2k + n - 2)c_1}{2},$$

where we added the subscript $k$ for the convenience. From equation (46), $\sigma$ takes

$$\sigma = \sigma_k = -\frac{c_1^2(n - 2)^2}{4} + 4c_0k(k + n - 2)\chi.$$  \hfill (E.1)

In this case, $V$ and $\tilde{V}$ become

$$V = V_k = \frac{2k + n - 2}{16}\left(4(2k + n)\chi^2\lambda^2 - 4c_1(2k + n)\chi \lambda + c_{1}^2(2k + n - 2) - 8c_0\chi\right),$$

$$\tilde{V} = \tilde{V}_k = \frac{2k + n - 2}{16}\left(4(2k + n - 4)\chi^2\lambda^2 - 4c_1(2k + n - 4)\chi \lambda + c_{1}^2(2k + n - 2) + 8c_0\chi\right).$$

The quantum-mechanical systems for $H = H_k = A_k^2A_k$ and $\tilde{H} = \tilde{H}_k = A_k\lambda_k^2$ are

$$H_k\psi_k = E_k\psi_k,$$ \hfill (E.2)

$$\tilde{H}_k\tilde{\psi}_k = \tilde{E}_k\tilde{\psi}_k,$$ \hfill (E.3)

where $E_k$ is

$$E_k = \nu^2 + \frac{k(k + n - 2)(c_1^2 + 4c_0\chi)}{4}.\hfill (E.4)$$

and $\tilde{E}_k$ is a constant.

As shown in section 3, $H_k$ and $\tilde{H}_k$ are connected by the transformation $(m_k^2, \delta m_k^2, c_k, \sigma_k) \rightarrow (m_{k+1}^2, \delta m_{k+1}^2, -\delta m_k^2, -c_k, \sigma_k)$. Since $E_k$ is not changed under this transformation, the KGE with $m_k^2 + \delta m_k^2$ becomes equation (E.3) with $\tilde{E}_k = E_k$. So, $\tilde{H}_k\psi_k = E_k\lambda_k\lambda_k^2\psi_k$ implies that $A_k$ maps the solution of KGE with $m_k^2$ into that with $m_{k+1}^2 + \delta m_{k+1}^2 (= m_{k+1}^2)$.  \hfill (E.5)

We can also see that the quantum-mechanical systems have a shape invariance

$$\tilde{V}_{k+1} = V_k + \epsilon_k,\hfill (E.5)$$

with

$$\epsilon_k = \frac{(2k + n - 1)(c_1^2 + 4c_0\chi)}{4},\hfill (E.6)$$

then we have

$$\tilde{H}_{k+1} = H_k + \epsilon_k.\hfill (E.7)$$

Using this, we can show

\hfill (E.6) We can also see this explicitly as follows. The transformation $(m_k^2, \delta m_k^2, c_k, \sigma_k) \rightarrow (m_{k+1}^2, \delta m_{k+1}^2, -\delta m_k^2, -c_k, \sigma_k)$ corresponds to $k \rightarrow -k - n + 2$. Since we have $H_{-k-n+2} = H_k$, we can show $H_{-k-n+2}A_k\psi_k = H_k\psi_k = A_k\psi_k = E_k\lambda_k\lambda_k^2\psi_k$, where we used $H_k A_k = A_k H_k$ and $E_{-k-n+2} = E_k$. From the relation $m_{-k-n+2}^2 = m_{k+1}^2$, we can see that the solution of the KGE with $m_k^2$ is mapped into that with $m_{k+1}^2$ by $A_k$.  \hfill (E.7)
\[ H_{k+1} \psi_k = (H_k + \epsilon_k) \psi_k = (E_k + \epsilon_k) \psi_k = E_{k+1} \psi_k. \]  \hspace{1cm} (E.8)

From the relation \( A^+_k H_{k+1} = H_{k+1} A^+_k \) and the above equation, we have
\[ H_{k+1} A^+_k \psi_k = E_{k+1} A^+_k \psi_k. \]  \hspace{1cm} (E.9)

This corresponds to the map between solutions of two KGEs with \( m_k^2 \) and \( m_{k+1}^2 \).

Thus, for a given \( k \), we have two ladder operators, one shifts \( k \) to \( k - 1 \) by the supercharge, the other shifts \( k \) to \( k + 1 \) due to the shape invariance. Repeating this process, we can shift \( k \) to \( k \pm 1, k \pm 2, \cdots \). This is the explanation from the point of view of the supersymmetric quantum mechanics with a shape invariance potential.

**Appendix F. In the presence of a gauge field and a scalar potential**

When a Maxwell field \( A_{\mu} \), which satisfies \( \nabla^\mu F_{\mu\nu} = 0 \) with \( F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \), and a scalar potential \( U \) are present, the field equation is given by \((\mathcal{H} - m^2)f = 0\), where the operator \( \mathcal{H} \) is
\[ \mathcal{H} = g^{\mu\nu}(\nabla_\mu + ie A_\mu)(\nabla_\nu + ie A_\nu) + U, \]  \hspace{1cm} (F.1)
and \( e \) is an electric charge. In analogy with equation (30), we consider the operators
\[ D_k := \zeta^\mu(\nabla_\mu + ie A_\mu) - kQ, \]  \hspace{1cm} (F.2)
where \( \zeta^\mu \) is a CCKV satisfying \( \nabla_\mu \zeta_\nu = Qg_{\mu\nu} \), and \( k \) is a real parameter. We also assume that \( \zeta^\mu \) is non-null and an eigen vector of the Ricci tensor \( R_{\mu\nu}\zeta_\nu = (n-1)\chi \zeta_\mu \). Under these conditions, we discuss the conditions that the operators (F.2) become the ladder operators which satisfy the commutation relation (1). After a calculation, we obtain
\[ [\mathcal{H}, D_k] = \chi(2k + n - 2)D_k + 2Q(\mathcal{H} + \chi k(k + n - 1)) 
+ 2ie \zeta_\nu F^\mu\nu(\nabla_\mu + ie A_\mu) - \zeta^\mu \partial_\mu U - 2QU. \]  \hspace{1cm} (F.3)

Hence, imposing the additional conditions\(^\dagger\)
\[ F^\mu\nu \zeta_\nu = 0, \quad \zeta^\mu \partial_\mu U = -2QU, \]  \hspace{1cm} (F.4)
the commutation relation comes down to
\[ [\mathcal{H}, D_k] = \chi(2k + n - 2)D_k + 2Q(\mathcal{H} + \chi k(k + n - 1)). \]  \hspace{1cm} (F.5)
This is the commutation relation for the mass ladder operator with \( m^2 = -\chi k(k + n - 1) \) and \( \delta m^2 = \chi(2k + n - 2) \).

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\(^\dagger\) Note that if we impose the condition \( F^\mu\nu \zeta_\nu = i\sigma \zeta^\mu \) with a constant \( \sigma \), we can show \( \sigma = 0 \) by taking an inner product with this condition and \( \zeta_\mu \).
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