WEIGHTED TANGO BUNDLES ON $\mathbb{P}^n$ AND THEIR MODULI SPACES

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Abstract. We define a new class of algebraic $(n-1)$-bundles on $\mathbb{P}^n$, that contains the bundles introduced by Tango [14] and their stable generalized pull-backs; we show that these bundles are invariant under small deformations and that they correspond to smooth points of moduli spaces.

It is a very difficult problem to find examples of non-splitting algebraic vector bundles on the complex projective space $\mathbb{P}^n$ whose rank is less than $n$. In particular for $n \geq 6$ the only known examples are essentially the mathematical instantons $\mathbb{B}$ (for odd $n$) and the bundles introduced by Tango $[14]$; all of them have rank $n-1$. Of course, pulling back the Tango bundles by a finite morphism $\mathbb{P}^n \rightarrow \mathbb{P}^n$ gives other examples of rank $n-1$ bundles.

In [9], Horrocks introduced a new technique of constructing new bundles from old ones, which generalizes the pull-back. This method, that we can call generalized pull-back, has been extensively studied in [1] and [2] and it applies only to bundles whose symmetry group contains a copy of $\mathbb{C}^*$. In this paper we show that, for any $n \geq 3$, there exists a Tango bundle that is $\text{SL}(2)$-invariant: hence the generalized pull-back allows us to define a new class of $(n-1)$-bundles on $\mathbb{P}^n$.

More precisely, let $\alpha, \gamma$ be integer numbers such that $\gamma > n\alpha \geq 0$ and let $Q_{\alpha,\gamma}$ be the bundles on $\mathbb{P}^n$ described by the exact sequence:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-\gamma) \rightarrow \bigoplus_{k=0}^n \mathcal{O}_{\mathbb{P}^n}((n-2k)\alpha) \rightarrow Q_{\alpha,\gamma} \rightarrow 0.$$ 

$Q_{\alpha,\gamma}$ can also be defined as the generalized pull-back of the quotient bundle on $\mathbb{P}^n$ and, in particular, $Q_{0,1}$ is the quotient bundle. Let us define the rank $2n-1$ vector bundle:

$$\mathcal{V} = S^{2(n-1)}(\mathcal{O}_{\mathbb{P}^n}(\alpha) \oplus \mathcal{O}_{\mathbb{P}^n}(-\alpha)) = \bigoplus_{k=0}^{2(n-1)} \mathcal{O}_{\mathbb{P}^n}((2n-1-2k)\alpha).$$

It will be proven that there exists an exact sequence of algebraic vector bundles over $\mathbb{P}^n$:

$$0 \rightarrow Q_{\alpha,\gamma}(-\gamma) \rightarrow \mathcal{V} \rightarrow F_{\alpha,\gamma}(\gamma) \rightarrow 0. \tag{1}$$

The $(n-1)$-bundles $F_{\alpha,\gamma}$ are called weighted Tango bundles of weights $\alpha$ and $\gamma$ and they are stable if and only if $\gamma > 2(n-1)\alpha$. The bundles $F_{0,1}$ are the classical Tango bundles.

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bundles, moreover the generalized pull-backs of the Tango bundles are contained in the sequence (I). The main result of this paper is the following:

**Theorem 0.1.** Let \( F_{\alpha,\gamma}^o \) be a stable weighted Tango bundle on \( \mathbb{P}^n \) of weights \( \alpha \) and \( \gamma \) and let \( c_i \) be the \( i \)-th Chern class of \( F_{\alpha,\gamma}^o \) (in particular \( c_1 = 0 \)). There exists a smooth neighborhood of the point of the moduli space \( \mathcal{M}_{\mathbb{P}^n}(0, c_2, \ldots, c_{n-1}) \) corresponding to \( F_{\alpha,\gamma}^o \) entirely consisting of weighted Tango bundles of weights \( \alpha \) and \( \gamma \).

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1. **Introduction.**

Let \( V \) be a \((n+1)\)-dimensional vector space over \( \mathbb{C} \), and let \( \mathbb{P}^n = \mathbb{P}(V) \): it is possible to show (cf. [11]) that a Tango bundle \( F \) on \( \mathbb{P}^n \) is contained in the following exact sequence:

\[
0 \to Q(-1) \to \frac{\Lambda^2 V}{W} \otimes \mathcal{O}_{\mathbb{P}^n} \to F(1) \to 0;
\]

here \( Q \) is the quotient bundle (cf. [13]) on \( \mathbb{P}^n \) and \( W \subseteq \Lambda^2 V \) is a linear subspace such that:

\[
\begin{aligned}
\text{dim}_\mathbb{C} \mathbb{P}(W) &= m - 1 \\
\mathbb{P}(W) \cap \mathbb{G}(1, n) &= \emptyset
\end{aligned}
\]

where \( m = \frac{(n-2)(n-1)}{2} \) and \( \mathbb{G}(1, n) \) is the Grassmannian of the lines in \( \mathbb{P}^n = \mathbb{P}(V) \); hence \( W \) does not contain any decomposable bivectors. Moore [12] has shown that \( F \) is uniquely determined by the subspace \( W \subseteq \Lambda^2 V \) and so by a point of the variety \( \mathbb{G}(m - 1, N - 1) \), with \( N = \frac{n(n+1)}{2} \); furthermore if \( W \) is invariant under the action of a group \( G \subseteq \text{GL}(n+1) \) then the Tango bundle, associated to \( W \), is \( G \)-invariant too, i.e. \( G \subseteq \text{Sym} F \).

2. **Action of SL(2).**

Let \( U \) be a 2-dimensional vector space over \( \mathbb{C} \) and let us consider the complex projective space \( \mathbb{P}^n = \mathbb{P}(S^nU) \): in this way, we have a natural action of \( \text{SL}(2) = \text{SL}(U) \) over \( \mathbb{P}^n \).

We want to find a subspace \( W \subseteq \Lambda^2 S^nU \), \( \text{SL}(2) \)-invariant and that satisfies (2). For this purpose we prove the following:

**Proposition 2.1.** The decomposition of \( \Lambda^2 S^nU \) into irreducible representations is given by \( \text{S}^{2(n-1)}U \oplus \text{S}^{2(n-3)}U \oplus \text{S}^{2(n-5)}U \oplus \ldots \); moreover if \( W = \text{S}^{2(n-3)}U \oplus \text{S}^{2(n-5)}U \oplus \ldots \), then \( W \) satisfies (3).

This proposition immediately implies that for any \( n \in \mathbb{N} \), such a subspace \( W \) defines a \( \text{SL}(2) \)-invariant Tango bundle \( F \) on \( \mathbb{P}^n \), which is described by the exact sequence:

\[
0 \to Q(-1) \to \text{S}^{2(n-1)}U \otimes \mathcal{O}_{\mathbb{P}^n} \to F(1) \to 0.
\]

Before proceeding with the proof of the proposition, we prove the following lemma:
Lemma 2.2. Let \( \{v_0, \ldots, v_n\} \) be a basis of \( V \) and \( \omega \in G(1, n) \subseteq \wedge^2 V \) a non-vanishing decomposable bivector, then:

\[
\omega = x_{i_0,j_0}(v_{i_0} \wedge v_{j_0}) + \sum_{i+j>i_0+j_0} x_{i,j}(v_i \wedge v_j)
\]

where \( x_{i,j} \in \mathbb{C} \) and \( x_{i_0,j_0} \neq 0 \).

**Remark.** In order to simplify the notations, we will often write \( v_{i,j} \) instead of \( v_i \wedge v_j \).

**Proof.** We proceed by induction on \( n \). For \( n = 1 \), there is nothing to prove.

Let us suppose now \( n > 1 \) and let \( \omega = v \wedge v' \) where \( v = \sum x_i v_i \) and \( v' = \sum y_i v_i \).

Let \( z_{i,j} = x_i y_j - x_j y_i \) then

\[
\omega = \sum_{i<j} z_{i,j} v_{i,j},
\]

where \( k_0 = \min \{ k | z_{i,j} = 0 \text{ if } i + j = k \} \).

If there exist \( i_0, j_0 \neq 0 \) such that \( i_0 + j_0 = k \), and \( z_{i_0,j_0} \neq 0 \) then, since \( z_{0,0} = 0 \) it easily follows \( x_0 = y_0 = 0 \) : thus the lemma is true by induction.

Otherwise, if such \( i_0, j_0 \) do not exist, then:

\[
\omega = z_{0,k} v_{0,k} + \sum_{i+j > k_0} z_{i,j} v_{i,j}.
\]

\[\square\]

**Proof of proposition 2.4.**

Let \( V = S^n U \) and let \( \{x, y\} \) be a basis of \( V \): if \( v_0 = x^n, \ldots, v_n = y^n \), then \( \{v_0, \ldots, v_n\} \) is a basis of \( V \). The weights of \( S^n U \) are \( \{n, n-2, \ldots, -n\} \) (cf. II. pag. 146–153) and since the weights of \( \wedge^2 S^n U \) are given by the sums of couples of different weights of \( S^n U \), it easily follows:

\[
\wedge^2 S^n U = S^{2(n-1)} U \oplus S^{2(n-3)} U \oplus S^{2(n-5)} U \oplus \ldots
\]

Indeed if \( W = S^{2(n-3)} U \oplus S^{2(n-5)} U \oplus \ldots \), then \( \dim_{\mathbb{C}} W = m \).

Let us prove now that \( W \) does not contain any decomposable bivector, as required. We suppose that there exists \( \omega \in W \cap G(1, n) \), such that \( \omega \neq 0 \); by the previous lemma, we get:

\[
\omega = x_{i_0,j_0} v_{i_0,j_0} + \sum_{i+j > i_0+j_0} x_{i,j} v_{i,j}
\]

where \( x_{i_0,j_0} \neq 0 \). We want to show that, in this case, there exists a vector of weight \( 2(n-1) \) in \( W \): this contradicts with the fact that \( S^{2(n-1)} U \cap W = \{0\} \).

Let \( Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \), \( H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) \( \in \mathcal{S}(2) \), and let \( \tilde{Y}, \tilde{H} \) be the corresponding endomorphisms of \( \wedge^2 S^n U \). If we suppose \( v_{n+1} = 0 \), we have:

\[
\tilde{Y}(v_{i,j}) = (n-i) v_{i+1,j} + (n-j) v_{i,j+1} \quad \text{for any } i, j = 0, \ldots, n.
\]

Hence if \( k = (2n-1) - i_0 - j_0 \), then \( x_{i_0,j_0} \tilde{Y}(v_{i_0} \wedge v_{j_0}) = \tilde{Y}(k)(\omega) \in W \). On the other hand it results that \( \tilde{Y}(k)(v_{i_0,j_0}) = m v_{n,n-1} \), where \( m \) is a positive integer: this implies that \( v_{n,n-1} \in W \) and since \( \tilde{H}(v_{n,n-1}) = -2(n-1)v_{n,n-1} \), we see that \( W \) contains a vector of weight \( 2(n-1) \).

\[\square\]
Remark. Moore [12] has shown that the Tango bundles on $\mathbb{P}^4$ have all symmetry groups isomorphic to $\mathbb{P}O(3)$ and that $\mathbb{P} GL(5)$ acts transitively on the moduli space of the Tango bundles $\mathcal{M}_{\mathbb{P}^4}(0,2,2)$. In higher dimensions the situation is different: in fact, with the help of the software Macaulay 2 [7], it has been possible to prove that on $\mathbb{P}^5$ the generic Tango bundle has a discrete symmetry group and that there exist Tango bundles with the symmetry group isomorphic to $\mathbb{C}^*$ (for instance the one defined by $W = \langle v_{0,5} + 5v_{2,3}, v_{1,4} + 3v_{2,3}, v_{0,4} - 2v_{1,3}, v_{2,5} + v_{3,4}, v_{0,3} + 3v_{1,2}, 2v_{2,5} - 3v_{3,4} \rangle$).

The algorithm needed to calculate the dimension of the orbit of a subspace $W_0 \subseteq \wedge^2 V$ (where $n = 5$) under the action of $\mathbb{P} GL(6)$ was communicated to the author by G. Ottaviani. We describe the fundamental steps of it:

1. Let us choose $m$ as a $(6 \times 15)$-matrix whose rows represent the generators of the subspace $W_0 \subseteq \wedge^2 V$.
2. We denote by $g = \{g_{i,j}\}$ a generic $(6 \times 6)$-matrix and let’s define $m’ = m \ast \wedge^2 g$: $m’$ represents the image $gW_0$ of the matrix $g \in \mathbb{P} GL(6)$ by the map $\eta : \mathbb{P} GL(6) \to \mathbb{G}(6, \wedge^2 V)$; By the Plucker embedding $\phi : \mathbb{G}(m, \wedge^2 V) \to \mathbb{P}^{5004}$, the dimension of the orbit of $W_0$ is equal to the dimension of the ideal generated by the minors $6 \times 6$ of $m’$, but its calculation is, computationally, too difficult. Therefore in order to make the computation easier, we first calculate the derivative $d(\phi \circ \eta)$ at the identity matrix and then we compute the dimension of its image: this number is exactly the dimension of the orbit. We proceed as follows:

3. Let $v_1(g), \ldots, v_6(g)$ be the rows of $m’$, and let $v_i(g)_{g_{i,j}} = \frac{\partial v_i(g)}{\partial g_{i,j}}$.

In order to compute the derivative $d(\phi \circ \eta)$, we remind that, for any $I \subseteq \{1, \ldots, 15\}$ such that $\# I = 6$, we have:

$$\frac{\partial}{\partial g_{i,j}} \det \begin{pmatrix} v^I_1(g) \\ \vdots \\ v^I_6(g) \end{pmatrix} = \det \begin{pmatrix} v^I_1(g)_{g_{i,j}} \\ \vdots \\ v^I_6(g)_{g_{i,j}} \end{pmatrix} + \cdots + \det \begin{pmatrix} v^I_1(g) \\ \vdots \\ v^I_6(g)_{g_{i,j}} \end{pmatrix}$$

where $v^I_i(g)$ denotes the vector composed by the components of $v_i(g)$ with index in $I$.

4. Let’s define $M^k_{i,j} = \begin{pmatrix} v_1(Id_6) \\ \vdots \\ v_k(Id_6)_{g_{i,j}} \\ \vdots \\ v_6(Id_6) \end{pmatrix}$; let $p_{i,j}$ be the sum of the vectors in $\mathbb{P}^{5004}$ defined by the minors of $M^k_{i,j}$ with $k = 1, \ldots, 6$;

5. The rank of the matrix $\begin{pmatrix} p_{1,1} \\ p_{1,2} \\ \vdots \\ p_{6,6} \end{pmatrix}$ is the dimension of the orbit of $W_0$.  

\[ \begin{pmatrix} v_1(Id_6) \\ \vdots \\ v_k(Id_6)_{g_{i,j}} \\ \vdots \\ v_6(Id_6) \end{pmatrix} \]
3. Weighted Tango Bundles.

We have shown that for any $n$, there exists a Tango bundle $F$ on $\mathbb{P}(S^nU)$ that is invariant under the $\mathbb{C}^*$-action defined by:

$$(t^n \quad t^{n-2} \quad \cdots \quad t^{-n}) \in \mathbb{P} \text{GL}(n + 1) \quad \text{for any } t \in \mathbb{C}^*$$

This map induces an embedding of $\mathbb{C}^*$ in $\text{Sym} F$ and so it is possible to study the pull-backs over $\mathbb{C}^{n+1} \setminus 0$ of such bundles (cf. [1, 2]).

Let us fix $\alpha, \gamma \in \mathbb{N}$ such that $\gamma > n\alpha$ and let $f_0, \ldots, f_n \in \mathbb{C}[x_0, \ldots, x_n]$ homogeneous polynomial of degree:

$$\deg f_k = \gamma + (n - 2k) \alpha \quad \text{for each } k = 0, \ldots, n$$

and without common roots.

Let $\phi = (f_0, \ldots, f_n)$ and let us take into account the following diagram:

$$\begin{array}{ccc}
\mathbb{C}^{n+1} \setminus 0 & \xrightarrow{\phi} & S^nU \setminus 0 \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
\mathbb{P}^n & & \mathbb{P}^n
\end{array}$$

According to [1, 3], there exists an algebraic vector bundle $F_{\alpha,\gamma}$ on $\mathbb{P}^n$ such that $\pi_1^* F_{\alpha,\gamma} = \phi^* \pi_2^* F$. Furthermore, since $Q$ is an homogeneous bundle [13], there exists $Q_{\alpha,\gamma}$ such that $\pi_1^* Q_{\alpha,\gamma} = \phi^* \pi_2^* Q$. Such a bundle is contained in the weighted Euler sequence:

$$(3) \quad 0 \to O_{\mathbb{P}^n}(-\gamma) \to S^nU \to Q_{\alpha,\gamma} \to 0$$

where $\mathcal{U} = O_{\mathbb{P}^n}(-\alpha) \oplus O_{\mathbb{P}^n}(\alpha)$. In general, we will call weighted quotient bundle of weights $\alpha$ and $\gamma$ any bundles $Q_{\alpha,\gamma}$ contained in a sequence (3).

On the other hand $F_{\alpha,\gamma}$ is contained in the exact sequence:

$$(4) \quad 0 \to Q_{\alpha,\gamma}(-\gamma) \to \mathcal{V} \to F_{\alpha,\gamma}(\gamma) \to 0$$

where $\mathcal{V} = S^{2(n-1)}\mathcal{U}$ and $Q_{\alpha,\gamma}$ is the pull-back over $\mathbb{C}^{n+1} \setminus 0$ of the quotient bundle $Q$ defined by the map $\phi$. Also in this case, we will call weighted Tango bundle of weights $\alpha$ and $\gamma$ any bundles $F_{\alpha,\gamma}$ contained in the sequence (4), where $Q_{\alpha,\gamma}$ is any weighted quotient bundle of weights $\alpha$ and $\gamma$.

By these sequences, it immediately follows that $c_1(F_{\alpha,\gamma}) = 0$ and that $c_i(F_{\alpha,\gamma}) = c_i(\alpha, \gamma)$ for any $i = 2, \ldots, n - 1$ (i.e. the Chern classes do not depend on the map $\phi$).

**Proposition 3.1.** A weighted Tango bundle $F_{\alpha,\gamma}$ is stable if and only if $\gamma > 2(n - 1)\alpha$.

**Proof.** Let $\gamma > 2(n - 1)\alpha$. By the Hoppe criterion [3], it suffices to show that $H^0(\wedge^q F_{\alpha,\gamma}) = 0$ for any $q = 1, \ldots, n - 2$. By the sequence:

$$0 \to S^{k-1}S^nU(-\gamma) \to S^kS^nU \to S^kQ_{\alpha,\gamma} \to 0$$

obtained raising the sequence (3) to the $k$-th symmetric power, we see that: $H^i(S^kQ_{\alpha,\gamma}(t)) = 0$ for any $i = 1, \ldots, n - 2$ and $t \in \mathbb{Z}$. 
On the other hand by (4), we have the long exact sequence:

\[
0 \rightarrow S^qQ_{\alpha,\gamma}(-q\gamma) \rightarrow \cdots \rightarrow S^kQ_{\alpha,\gamma}(-k\gamma) \otimes \wedge^{q-k}V \rightarrow \cdots
\]
\[
\cdots \rightarrow Q_{\alpha,\gamma}(-\gamma) \otimes \wedge^{q-1}V \rightarrow \wedge^qV \rightarrow \wedge^qF_{\alpha,\gamma}(q\gamma) \rightarrow 0
\]

This sequence immediately implies that \(H^0(\wedge^qF_{\alpha,\gamma}) \subseteq H^0(\wedge^qV(-q\gamma))\), and since

\[
\max\{t \in \mathbb{Z} | \mathcal{O}_{\mathbb{P}^n}(t) \subseteq \wedge^qV(-q\gamma)\} = q((2n-q-1)\alpha - \gamma) < 0
\]

we have that \(H^0(\wedge^qF_{\alpha,\gamma}) = 0\) for any \(q = 1, \ldots, n-2\), and so \(F_{\alpha,\gamma}\) is stable.

Let us prove now that the condition is necessary. By the sequence s:

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-3\gamma) \rightarrow S^nU(-2\gamma) \rightarrow Q_{\alpha,\gamma}(-2\gamma) \rightarrow 0
\]
\[
0 \rightarrow Q_{\alpha,\gamma}(-2\gamma) \rightarrow V(-\gamma) \rightarrow F_{\alpha,\gamma} \rightarrow 0
\]

it follows that if \(\gamma \leq 2(n-1)\alpha\), then \(H^0(F_{\alpha,\gamma}) \neq 0\) and so \(F_{\alpha,\gamma}\) cannot be stable. \(\square\)

4. Small deformations of \(F_{\alpha,\gamma}\).

Let \(E\) be a vector bundle on \(\mathbb{P}^n\): we will indicate with \((\text{Kur}E, e)\) the Kuranishi space of \(E\) (cf. [5]), where \(e \in \text{Kur}E\) is the point corresponding to the bundle \(E\).

We are finally ready to introduce the main result of this paper:

**Proposition 4.1.** Let \(F_{\alpha,\gamma}^0\) be a weighted Tango bundle of weights \(\alpha\) and \(\gamma\). Every small deformation of \(F_{\alpha,\gamma}^0\) is still a weighted Tango bundle and its Kuranishi space is smooth at the point corresponding to \(F_{\alpha,\gamma}^0\).

Before proceeding with the proof of the proposition, let us look at some preliminaries:

**Lemma 4.2.** Let \(Q_{\alpha,\gamma}^0\) be a weighted quotient bundle. Every small deformation of \(Q_{\alpha,\gamma}^0\) is still a weighted quotient bundle and the Kuranishi space of \(Q_{\alpha,\gamma}^0\) is smooth at the point corresponding to its isomorphism class.

**Proof.** The proof of this lemma is very similar to the proof of prop. 3.1 of [4]. \(\square\)

**Lemma 4.3.** Let \(F_{\alpha,\gamma}\) and \(F'_{\alpha,\gamma}\) be two isomorphic weighted Tango bundles, defined by the sequences:

\[
0 \rightarrow Q_{\alpha,\gamma}(-\gamma) \rightarrow V \rightarrow F_{\alpha,\gamma}(\gamma) \rightarrow 0
\]
\[
0 \rightarrow Q'_{\alpha,\gamma}(-\gamma) \rightarrow V \rightarrow F'_{\alpha,\gamma}(\gamma) \rightarrow 0
\]

where \(Q_{\alpha,\gamma}\) and \(Q'_{\alpha,\gamma}\) are weighted quotient bundles. Then \(Q_{\alpha,\gamma}\) and \(Q'_{\alpha,\gamma}\) are isomorphic.

**Proof.** By joining together the sequences (3) and (4), we get:

\[
0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-2\gamma) \xrightarrow{\phi} S^nU(-\gamma) \rightarrow V \rightarrow F_{\alpha,\gamma}(\gamma) \rightarrow 0.
\]

By proposition 1.4 of [5] and by the fact that \(-2\gamma < -\gamma - n\alpha\), the last sequence is the minimal resolution of \(F_{\alpha,\gamma}(\gamma)\): hence \(Q_{\alpha,\gamma}(-2\gamma) = \text{Coker} \phi\) is directly defined by this resolution. \(\square\)
Lemma 4.4. Every isomorphism between two weighted Tango bundles \( F_{\alpha,\gamma} \rightarrow F'_{\alpha,\gamma} \) is induced by an isomorphism of sequences:

\[
\begin{array}{cccc}
0 & \rightarrow & Q_{\alpha,\gamma}(-\gamma) & \rightarrow \ V & \rightarrow & F_{\alpha,\gamma}(\gamma) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & Q'_{\alpha,\gamma}(-\gamma) & \rightarrow \ V & \rightarrow & F'_{\alpha,\gamma}(\gamma) & \rightarrow & 0
\end{array}
\]

Proof. By the sequence

\[
0 \rightarrow O_{\mathbb{P}^n}(-2\gamma) \otimes V \rightarrow S^nU(-\gamma) \otimes V \rightarrow Q_{\alpha,\gamma}(-\gamma) \otimes V \rightarrow 0,
\]

and since

\[h^1(S^nU(-\gamma) \otimes V) = h^2(O_{\mathbb{P}^n}(-2\gamma) \otimes V) = 0,\]

we get \(h^1(Q_{\alpha,\gamma}(-\gamma) \otimes V) = 0\); hence the lemma is proven.

Lemma 4.5. Two morphisms \( f, f' \in \text{Hom}(Q_{\alpha,\gamma}(-\gamma), V) \) give the same element of \( \text{Quot}_{\mathbb{P}^n} \) if and only if there exists an invertible \( h \in \text{End}(Q_{\alpha,\gamma}(-\gamma)) \) such that

\[f = f' \circ h.\]

Proof. It follows from the definition of \( \text{Quot}_{\mathbb{P}^n} \), (cf. [10]).

Proof of proposition 4.2.

For brevity’s sake, we will write \( \tilde{F}_0 \) instead of \( F^o_{\alpha,\gamma} \) and \( \tilde{Q}_o \) for \( Q^o_{\alpha,\gamma} \). Let also \( \sigma_0 \in \text{Hom}(\tilde{Q}_o(\gamma), V) \) be such that \( \tilde{F}_0 = \text{Coker} \sigma_0 \).

Let \( Q \) be the sub-variety of the irreducible component of \( \text{Quot}_{\mathbb{P}^n} \) composed by all the quotients of the maps \( 0 \rightarrow Q_{\alpha,\gamma}(-\gamma) \overset{\sigma}{\rightarrow} V \) for some weighted bundle \( Q_{\alpha,\gamma} \) and containing the point \( \sigma_0 \) corresponding to \( \tilde{F}_0 \): the morphisms \( \Phi : (Q, \sigma_0) \rightarrow (\text{Kur} \tilde{Q}_o, q_0) \) and \( \Psi : (Q, \sigma_0) \rightarrow (\text{Kur} \tilde{F}_0, f_0) \) are canonically defined.

A generic fiber of \( \Phi \) is given by all the cokernels of the morphisms \( Q_{\alpha,\gamma}(-\gamma) \rightarrow V \) with a fixed \( Q_{\alpha,\gamma} \), and so, by lemma 4.3, its dimension is constantly equal (\( \alpha \) and \( \gamma \) are fixed) to \( h^0(Q_{\alpha,\gamma}^o(\gamma) \otimes V) \rightarrow h^0(\text{End} Q_{\alpha,\gamma}) \). Hence, since lemma 4.2 implies that \( \dim_{\sigma_0}(\text{Kur} \tilde{Q}_o) = h^1(\text{End} \tilde{Q}_o) \), we get:

\[
\dim_{\sigma_0} Q = h^0(\tilde{Q}_o^*(\gamma) \otimes V) - h^0(\text{End} \tilde{Q}_o) + h^1(\text{End} \tilde{Q}_o)
\]

Let us study now the morphism \( \Psi : Q \rightarrow \text{Kur} \tilde{F}_0 \): if \( \Sigma = \{ \sigma \in \text{Quot}_{\mathbb{P}^n}[F_{\sigma} \simeq \tilde{F}_0] \} \), then it results \( \Psi^{-1}(f_0) \subseteq \Sigma \) and by lemma 4.3, 4.4 and 4.5, it follows:

\[
\dim_{\sigma_0} \Sigma = h^0(\text{End} V) - \dim\{ \varphi \in \text{End} V| \varphi \cdot \sigma_0 = \sigma_0 \} - h^0(\text{End} \tilde{Q}_o).
\]

By the sequence:

\[
0 \rightarrow \tilde{F}_0^*(\gamma) \otimes V \rightarrow \text{End} V \rightarrow \tilde{Q}_o^*(\gamma) \otimes V \rightarrow 0
\]

obtained tensoring the dual sequence of \( [4] \) with \( V \), we have that:

\[
\dim\{ \varphi \in \text{End} V| \varphi \cdot \sigma_0 = \sigma_0 \} = h^0(\tilde{F}_0^*(\gamma) \otimes V)
\]

and so:

\[
\dim_{\sigma_0} \Psi^{-1}(f_0) \leq \dim_{\sigma_0} \Sigma = h^0(\text{End} V) - h^0(\tilde{F}_0^*(\gamma) \otimes V) - h^0(\text{End} \tilde{Q}_o).
\]

Hence:

\[
h^1(\text{End} \tilde{F}_0) \geq \dim_{f_0}(\text{Kur} \tilde{F}_0) \geq h^1(\text{End} \tilde{Q}_0) + h^1(\tilde{F}_0^*(\gamma) \otimes V).
\]
To prove the proposition it suffices to show that
\[ h^1(\text{End} \, \widetilde{F}_o) \leq h^1(\text{End} \, \tilde{Q}_o) + h^1(\tilde{F}_o^*(-\gamma) \otimes \mathcal{V}). \]
In fact this implies that \( h^1(\text{End} \, \widetilde{F}_o) = \dim_{f_0}(\text{Kur} \, \tilde{F}_o), \) i.e. \( \text{Kur} \, \tilde{F}_o \) is smooth at the point \( f_0, \) and that \( \dim_{f_0}(\text{Kur} \, \tilde{F}_o) = \dim_{c_0} \mathcal{Q} - \dim \Psi^{-1}(f_0), \) i.e. \( \Psi \) is surjective.

By the exact sequence:
\[ 0 \to \tilde{Q}_o(-2\gamma) \otimes \tilde{F}_o^* \to \tilde{Q}_o(-\gamma) \otimes \mathcal{V} \to \text{End} \, \tilde{Q}_o \to 0 \]
and by the vanishing of \( H^1(\tilde{Q}_o(-\gamma) \otimes \mathcal{V}) \) and \( H^2(\tilde{Q}_o(-\gamma) \otimes \mathcal{V}), \) we have that \( H^1(\text{End} \, \tilde{Q}_o) = H^2(\tilde{Q}_o(-2\gamma) \otimes \tilde{F}_o^*). \) Hence by the sequence:
\[ 0 \to \tilde{Q}_o(-2\gamma) \otimes \tilde{F}_o^* \to \tilde{F}_o^*(-\gamma) \otimes \mathcal{V} \to \text{End} \, \tilde{F}_o \to 0 \]
and for what we have seen, we get the sequence of cohomology groups:
\[ \cdots \to H^1(\tilde{F}_o^*(-\gamma) \otimes \mathcal{V}) \to H^1(\text{End} \, \tilde{F}_o) \to H^1(\text{End} \, \tilde{Q}_o) \to \cdots \]
In particular \( h^1(\text{End} \, \tilde{F}_o) \leq h^1(\text{End} \, \tilde{Q}_o) + h^1(\tilde{F}_o^*(-\gamma) \otimes \mathcal{V}), \) as required. \( \square \)

Theorem 0.1 easily follows from the previous proposition. In fact if \( \gamma \geq 2(n-1)\alpha, \) we can consider the canonical algebraic map \( \mathcal{Q} \to \mathcal{M}(0, c_2, \ldots, c_{n-1}). \) The image of this map is a smooth quasi projective set composed uniquely by weighed Tango bundles and it is an open neighborhood of \( F_{\alpha, \gamma}^o \) in \( \mathcal{M}(0, c_2, \ldots, c_{n-1}). \)

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