Branch points of area-minimizing projective planes

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Dedicated to the memory of Robert Osserman

Abstract

Minimal surfaces in a Riemannian manifold $M^n$ are surfaces which are stationary for area: the first variation of area vanishes. In this paper we focus on surfaces of the topological type of the real projective plane $\mathbb{R}P^2$. We show that a minimal surface $f: \mathbb{R}P^2 \to M^3$ which has the smallest area, among those mappings which are not homotopic to a constant mapping, is an immersion. That is, $f$ is free of branch points. As a major step toward treating minimal surfaces of the type of the projective plane, we extend the fundamental theorem of branched immersions to the nonorientable case. We also resolve a question on the directions of branch lines posed by Courant in 1950.

1 Introduction

Let $M$ be an $n$-dimensional Riemannian manifold, and let $\Sigma$ be a compact surface with boundary which carries a conformal structure; we do not assume $\Sigma$ is orientable. Many existence theorems for minimal surfaces in the literature [6], [17], [5] find solutions by minimizing the energy of a mapping $f: \Sigma \to M$, where both $f$ and the conformal structure of $\Sigma$ are allowed to vary. The energy may be written as

$$E(f) := \frac{1}{2} \int_{\Sigma} (|f_x|^2 + |f_y|^2) \, dx \, dy \quad (1.1)$$

where $(x, y)$ are local conformal coordinates for $\Sigma$, and subscripts are used to denote partial derivatives. Write $\frac{\partial}{\partial x}$, etc., for covariant partial derivatives in the Riemannian manifold $M$. If the mapping $f$ and the conformal structure on $\Sigma$ are stationary for $E$, then the resulting mapping is harmonic:

$$\Delta f := \frac{D}{\partial x} \frac{\partial f}{\partial x} + \frac{D}{\partial y} \frac{\partial f}{\partial y} = 0 \quad (1.2)$$

and conformal:

$$|f_x| \equiv |f_y|, \langle f_x, f_y \rangle \equiv 0. \quad (1.3)$$

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We shall refer to a conformally parameterized harmonic mapping as a *conformally parameterized minimal surface* (CMS). Observe that for any $W^{1,2}$ mapping, $E(f)$ is bounded below by the area
\[ A(f) := \int_{\Sigma} |f_x \wedge f_y| \, dx \, dy, \] (1.4)
with equality if and only if $f$ is a conformal mapping almost everywhere. In particular, a mapping which minimizes $E(f)$ in a geometrically defined class of mappings has minimum area among mappings of the admissible class (see [8], p. 232 or Remark 6 below). Moreover, a conformal mapping which is harmonic, that is, stationary for $E$, is minimal, that is, stationary for area.

A CMS is an immersion except at a discrete set of branch points. Let a point of $\Sigma$ be given by $(0,0)$ in some local conformal coordinates $(x,y)$ for $\Sigma$. Write $z = x + iy$. Then $(0,0)$ is a branch point of $f$ of order $m - 1$ if for some system of coordinates $u^1, \ldots, u^n$ for $M$ and for some complex vector $c$, $f(x,y)$ satisfies the asymptotic description
\[ f^1(x,y) + if^2(x,y) = cz^m + O(z^{m+1}) \]
and
\[ f^k(x,y) = O(z^{m+1}), \]
$k = 3, \ldots, n$, as $(x,y) \to (0,0)$. Here we have written $f^k(x,y)$ for the value of the $k^{th}$ coordinate $u^k$ at $f(x,y)$, $k = 1, \ldots, n$, and $O(z^{m+1})$ denotes any “remainder” function bounded by a constant times $|z^{m+1}|$. We shall refer to a mapping which is an immersion except at a discrete set of branch points as a branched immersion (see [12]).

Our main theorem is

**Theorem 1.** Suppose $\Sigma$ is of the topological type of the real projective plane $\mathbb{R}P^2$. Let a CMS $f : \Sigma^2 \to M^3$ have minimum area among all $h : \Sigma \to M$ which are not homotopic to a constant mapping. Then $f$ is an immersion.

In order to prove this theorem, we will distinguish between two types of branch points: see [15], [12], [1], [2], and [9]. A *false branch point* of a branched immersion $f : \Sigma \to M$ is a branch point $z_0$ such that the image set $f(U)$ is an embedded surface, under another parameterization, for some neighborhood $U$ of $z_0$ in $\Sigma$. Otherwise, we call it a *true branch point*. A branched immersion $f : \Sigma \to M$ is said to be *ramified* if there are two disjoint open sets $V, W \subset \Sigma$ with $f(V) = f(W)$. If $f$ is ramified in every neighborhood of a point $z_0$, we say that $z_0$ is a *ramified branch point*. Note that any false branch point of
a branched immersion must be ramified. Osserman showed that in codimension one, a branched immersion \( f : \Sigma^2 \to M^3 \) with a true branch point cannot minimize area, see [15] and Theorem 2 below, in contradiction to assertions of Douglas (p. 239 of [7]) and of Courant (footnote p. 46 of [4]). On the other hand, regarding false branch points, we shall extend to nonorientable surfaces the fundamental theorem of branched immersions in [10], and show that if a branched CMS \( f : \mathbb{R}P^2 \to M^n \) is ramified, with any codimension, then there is another CMS \( \tilde{f} : \mathbb{R}P^2 \to M \) with at most half the area of \( f \).

We would like to acknowledge the interest of Simon Brendle in this problem, whose questions, not used in [3], stimulated us to investigate this research topic. We are also indebted to the late Jim Serrin for pointing us toward Remark 6.

2 Analysis of branch points

This section reports on material that has appeared in the literature, see especially [9]. In this paper, we shall discuss certain steps in the interest of clarity and completeness.

Let \( \Sigma^2 \) be a compact surface with a conformal structure, \( M^n \) a Riemannian manifold, and let \( f : \Sigma \to M \) be a CMS. Consider a branch point \( z_0 \in \Sigma \) for \( f \).

Write \( D \) for the Riemannian connection on \( M \). Let local conformal coordinates \((x, y)\) for \( \Sigma \) and local coordinates \((q_1, \ldots, q_n)\) for \( M \) be introduced with \( z_0 = (0, 0) \) and \( f(z_0) = (0, \ldots, 0) \) in these coordinates. Then equation (1.2) may be rewritten

\[
\frac{D}{\partial \overline{z}} \frac{\partial f}{\partial z} = 0,
\]

where we write the complex coordinate \( z = x + iy \), \( \frac{\partial f}{\partial z} = \frac{1}{2} \left( \frac{\partial f}{\partial x} - i \frac{\partial f}{\partial y} \right) \), and \( \frac{D}{\overline{z}} = \frac{1}{2} \left( \frac{D}{\partial x} + i \frac{D}{\partial y} \right) \). In this form, we see that harmonicity implies that the complex tangent vector \( \frac{\partial f}{\partial z} \) is holomorphic to first order. It is readily shown that for some positive integer \( m \) and for some complex tangent vector \( c = a + ib \) to \( M \) at \( f(z_0) \),

\[
f(z) = \mathcal{R}\{cz^m\} + O_2(|z|^{m+1}).
\]

Here we write \( \mathcal{R}\{v\} \) for the real part of a complex vector \( v \), and we have used the big-O notation with the subscript 2, meaning that as \( z \to 0 \), the remainder term is bounded by a constant times \( |z|^{m+1} \), its first partial derivatives are bounded by a constant times \( |z|^m \) and its second partial derivatives are bounded by a constant times \( |z|^{m-1} \). It follows from the conformality condition (1.3) that the complex-bilinear inner product \( \langle c, c \rangle = |a|^2 - |b|^2 + 2i \langle a, b \rangle = 0 \). Choose a new system of coordinates \( p_1, \ldots, p_n \) for \( M \) near \( f(z_0) \) with \( \frac{\partial}{\partial p_1} = a \) and \( \frac{\partial}{\partial p_2} = b \); and
a new system of coordinates \((\tilde{x}, \tilde{y})\) for \(\Sigma\) with \(\tilde{z} = \tilde{x} + i\tilde{y} = |a|^{1/m}z\). Then along the mapping \(f\),

\[ p_1 + ip_2 = \tilde{z}^m + \sigma(\tilde{z}) \]

and

\[ p_\ell = \psi_\ell(\tilde{z}) , \]

\(\ell = 3, \ldots, n\), where \(\sigma(\tilde{z}), \psi_\ell(\tilde{z}) = O_2(\tilde{z}^{m+1})\). We now define a non-conformal complex parameter \(w = u_1 + iu_2\) on a neighborhood of the branch point in \(\Sigma\):

\[ w := \tilde{z} \left[ 1 + \tilde{z}^{-m} \sigma(\tilde{z}) \right]^{1/m} . \tag{2.2} \]

Then \(w\) is a \(C^{1,\alpha}\) coordinate on \(\Sigma\), for some Hölder exponent \(\alpha > 0\), in terms of which the coordinate representation of \(f\) is simplified:

\[ p_1 + ip_2 = w^m , \tag{2.3} \]

and

\[ p_\ell = \varphi_\ell(w) = O_2(w^{m+1}) , \]

\(\ell = 3, \ldots, n\).

We now turn our attention to the case \(n=3\) of codimension one. The self-intersection of the surface is determined by the single real-valued function \(\varphi(w) = \varphi_3(w)\). Define \(\overline{\varphi}(w) = \varphi(\zeta_m w)\), where \(\zeta_m = e^{2\pi i/m}\) is a primitive \(m^{th}\) root of unity, and let \(\Phi(w) = \varphi(w) - \overline{\varphi}(w)\). Then the zeroes of \(\Phi\) correspond to curves of intersection of the surface with itself. But both \(\varphi\) and \(\overline{\varphi}\) satisfy the same quasilinear minimal surface equation in \(M\), with the same coefficients. Therefore, their difference \(\Phi := \varphi - \overline{\varphi}\) satisfies a linear homogeneous PDE:

\[ \sum_{i,j=1}^{2} a_{ij} \Phi_{u_i u_j} + \sum_{i=1}^{2} a_i \Phi_{u_i} + a \Phi = 0 , \tag{2.4} \]

whose coefficients, as functions of \(w\), are obtained by integrating from the PDE satisfied by \(\varphi\) to the PDE satisfied by \(\overline{\varphi}\) along convex combinations. We have \(a_{ij}(0,0) = \delta_{ij}\).

It follows that \(\Phi\) satisfies an asymptotic formula

\[ \Phi(w) = \mathcal{R}\{Aw^N\} + O_2(w^{N+1}) , \tag{2.5} \]

for some integer \(N > m\) and some complex constant \(A \neq 0\) (see [13]). We shall call \(N - 1\) the proper index of the branch point. We may sketch the proof of Hartman and Wintner in [13]. We rewrite the PDE (2.4) in terms of the complex gradient \(\Phi_w = \frac{1}{2}(\Phi_{u_1} - i\Phi_{u_2})\). If \(\nabla \Phi(w) = o(w^{k-1})\), then we test
against the function \( g(w) = w^{-k}(w - \zeta)^{-1} \), where \( \zeta \neq 0 \) is small, to show that 
\[ \Phi_\zeta(\zeta) = a\zeta^k + o(\zeta^{k+1}) \]
for some \( a \in \mathbb{C} \). Proceeding by induction on \( k \), one finds formula (2.5) for some integer \( N \) and some \( A \neq 0 \), unless \( \nabla \Phi(w) \equiv 0 \). Details are as in [13], pp. 455-458.

Thus, there are two alternatives: \( \Phi \) is either identically zero or satisfies the asymptotic formula (2.5) where the integer \( N \) is \( \geq m+1 \) and the complex number \( A \neq 0 \). If \( \Phi \equiv 0 \), then \( \gamma_0 \) is a false branch point: see section 3 below.

If \( \Phi \) is not \( \equiv 0 \), then we have a true branch point.

**Theorem 2.** ([15], [9].) Suppose \( \Sigma \) is a surface with a conformal structure, \( M^3 \) a Riemannian manifold and \( f : \Sigma \rightarrow M \) a mapping which has smallest area in a \( C^0 \) neighborhood of \( f \). Then \( f \) has no true branch points.

**Proof:** As we have just seen, a true branch point \( \gamma_0 \) has an order \( m - 1 \geq 1 \) and a coordinate neighborhood in \( \Sigma \) with a \( C^1 \alpha \) complex coordinate \( w \), \( w = 0 \) at \( \gamma_0 \), such that \( f \) has the representation (2.3) near \( \gamma_0 \). Adjacent sheets \( p_3 = \varphi(w) \) and \( p_3 = \varphi(w) = \varphi(\zeta_m w) \) intersect when \( \Phi(w) = 0 \), which, according to formula (2.5), occurs along \( 2N \geq 6 \) arcs in \( \Sigma \) forming equal angles \( \pi/N \) when they leave \( w = 0 \). Let one of these arcs be parameterized as \( \gamma_1 : [0, \varepsilon] \rightarrow \Sigma \), and let the corresponding arc be \( \gamma_2 : [0, \varepsilon] \rightarrow \Sigma \), defined by \( w(\gamma_2(t)) = \zeta_m w(\gamma_1(t)) \). Then for all \( 0 \leq t \leq \varepsilon \), \( \varphi(\gamma_1(t)) = \varphi(\gamma_2(t)) \). Note from formula (2.3) that all three coordinates coincide: the mapping \( f(\gamma_1(t)) = f(\gamma_2(t)), 0 \leq t \leq \varepsilon \).

We may now construct a Lipschitz-continuous and piecewise smooth surface \( \tilde{f} \) which has the same area as \( f \), but has discontinuous tangent planes, following Osserman [15]. The idea of the following construction is that the parameter domain \( D \) may be cut along the arcs \( \gamma_1((0, \varepsilon)) \) and \( \gamma_2((0, \varepsilon)) \), opened up to form a lozenge, with two pairs of adjacent sides originally identified, and then closed up along the remaining two pairs of adjacent sides.

In detail: choose an open topological disk \( D \subset \mathbb{C} \) on which the coordinate \( w \) is defined, \( (0, 0) \in D \), and which is invariant under the rotation taking \( w \) to \( \zeta_m w \). Assume \( \gamma_1(\varepsilon) \) and \( \gamma_2(\varepsilon) \) are the first points along \( \gamma_1 \) resp. \( \gamma_2 \) which lie on the boundary of \( D \). We shall construct a discontinuous, piecewise \( C^1 \) mapping \( Q : B_1 \rightarrow D \), such that \( \tilde{f}(\zeta) := f(Q(\zeta)) \) is nonetheless continuous, and \( Q \) is one-to-one and onto except for sets of measure 0. Here, \( B_1 \) is the disk \( \{ z \in \mathbb{C} : |z| < 1 \} \). Choose points \( A_i = \gamma_1(\varepsilon/2), i = 1, 2 \). Then \( D \) is broken along \( \gamma_1 \) and \( \gamma_2 \) into two curvilinear pentagons with vertices \( \gamma_1(\varepsilon), A_1, (0, 0), A_2 \) and \( \gamma_2(\varepsilon) \). The edges of these pentagons are \( \gamma_1([\varepsilon/2, \varepsilon]), \gamma_1([0, \varepsilon/2]), \gamma_2([\varepsilon/2, \varepsilon]), \gamma_2([\varepsilon/2, \varepsilon]) \) and one of the two arcs of \( \partial D \) with endpoints \( \gamma_1(\varepsilon) \) and \( \gamma_2(\varepsilon) \). Similarly, break the unit disc \( B_1 \) along the interval \((-1, 1)\) of the \( x \)-axis and the interval \([-\frac{1}{2}, \frac{1}{2}]\) of the \( y \)-axis into two pentagons. Each pentagon in \( B_1 \) will be bounded by four line segments,
an interval along the $y$-axis being used twice, plus the upper or lower half-circle of $\partial B_1$. Denote the points $a = (0, \frac{1}{2})$, $e = (0, -\frac{1}{2})$, $c_1 = (1, 0)$, $c_2 = (-1, 0)$ and give the origin $(0, 0)$ four different names: $b_1$ when approached from the first quadrant $\{x > 0, y > 0\}$, $b_2$ when approached from the second quadrant $\{x < 0, y > 0\}$, $d_2$ when approached from the third quadrant $\{x < 0, y < 0\}$, and $d_1$ when approached from the fourth quadrant $\{x > 0, y < 0\}$. $Q$ will map the pentagon in $B_1$ in the upper half-plane $y > 0$ to the pentagon in $D$ lying counterclockwise from $\gamma_1$ and clockwise from $\gamma_2$, with $Q(c_1) = \gamma_1(\epsilon)$, $Q(b_1) = A_1$, $Q(a) = (0, 0)$, $Q(b_2) = A_2$, and $Q(c_2) = \gamma_2(\epsilon)$. This describes $Q$ on the boundary of one of the two pentagons; the other pentagon is similar. The interior of each pentagon may be made to correspond by a $C^1$ diffeomorphism. We require $Q$ to be continuous along the $x$-axis. Of course, $Q$ is discontinuous along the intervals $0 < y < \frac{1}{2}$ and $-\frac{1}{2} < y < 0$ of the $y$-axis. However, we can regain the continuity of $\tilde{f}$ by requiring that for $0 \leq y \leq \frac{1}{2}$, along the interval from $b_2$ to $a$, $Q(0, y) = \gamma_2(\epsilon y)$ and along the interval from $b_1$ to $a$, $Q(0, y) = \gamma_1(\epsilon y)$. Similarly, $Q$ will map the pentagon in the lower half-plane to the pentagon lying counterclockwise from $\gamma_2$ and clockwise from $\gamma_1$, taking care that for $-\frac{1}{2} \leq y \leq 0$, $Q(0, y) = \gamma_2(-\epsilon y)$ as approached from the third quadrant, and $Q(0, y) = \gamma_1(-\epsilon y)$ as approached from the fourth quadrant.

Then the continuous, piecewise $C^1$ image surfaces $\tilde{f}(B_1)$ and $f(D)$ consist of the same pieces of surface in $M$, and therefore have the same area. But the tangent planes of $\tilde{f}$ are discontinuous along the $y$-axis at each $y$ in the open intervals $-\frac{1}{2} < y < 0$ and $0 < y < \frac{1}{2}$, which implies that $\tilde{f}$, and therefore $f$, does not have minimum area: smoothing $\tilde{f}$ near these arcs reduces its area to first order.

Note that by choosing $\epsilon$ small, we may make $\tilde{f}$ arbitrarily close to $f$ in the uniform topology. □

Observation 3. R. Courant (see p. 123 of [5]) asked whether the curves of self-intersection near any true branch point of a minimal surface in Euclidean $\mathbb{R}^3$ meet at equal angles at the branch point. We may observe that this conjecture is partially correct, as seen above: the curves of intersection given by $\Phi(w) = 0$ make equal angles at the branch point. However, the totality of the curves of intersection may make a variety of angles at a branch point of order $m - 1 \geq 3$. Specifically, if $m = 4$ and the proper index $N - 1 = 5$, then the zeroes of $\Phi(w) = \varphi(w) - \varphi(\zeta_4 w)$ will occur along $2N = 12$ curves forming equal angles in the parameter plane, and therefore (since angles in the $w$-plane at 0 are multiplied by $m = 4$ in $\mathbb{R}^3$) equal angles in the tangent plane to $f(\Sigma)$ at the branch point. But their images
are only those curves of intersection coming from successive pairs of the four “sheets” of the surface. We also have the intersection of non-successive sheets, which are the images of zeroes of \( \Phi_2(w) := \varphi(w) - \varphi(-w) \), since \( \zeta_4^2 = i^2 = -1 \).

The leading term \( A w^6 \) of the asymptotic formula (2.5) cancels when we compute \( \Phi_2(w) = \varphi(w) - \varphi(-w) = \Phi(w) + \Phi(iw) \).

It follows from the PDE satisfied by \( \varphi(w) \) and by \( \varphi(-w) \) that \( \Phi_2 \) satisfies an elliptic PDE analogous to (2.4), and therefore an asymptotic relation analogous to (2.5) with leading term \( A_2 w^{N_2} \) for some integer \( N_2 > N \) and some \( A_2 \in \mathbb{C}\{0\} \).

That is, the curves of intersection of non-successive sheets form a family of equally spaced directions, which are presumably independent of the directions of the curves of intersection of successive sheets. This philosophy is justified by the following explicit example with \( N = 6 \) and \( N_2 = 7 \).

Choose \( a, b \in \mathbb{C}\{0\} \). Using the Weierstraß representation (see [16], p. 63) for a minimal surface \( f : \mathbb{C} \to \mathbb{R}^3 \) in Euclidean 3-space, based on the polynomials \( 4 z^3 \) and \( 2 a z^2 + 2 b z^3 \) (the latter representing the Gauß map in stereographic projection), we have the specific CMS with

\[
\begin{align*}
\varphi_1(z) &= \left[1 - (a z^2 + b z^3)^2\right] 2 z^3 \\
\varphi_2(z) &= -i \left[1 + (a z^2 + b z^3)^2\right] 2 z^3 \\
\varphi_3(z) &= 4 z^3 (a z^2 + b z^3),
\end{align*}
\]

which leads to

\[
\begin{align*}
w^4 := f^1 + i f^2 &= z^4 - \frac{a^2 z^8}{2} - \frac{8}{9} a b z^9 - \frac{2}{5} b^2 z^{10} \\
z &= w\left(1 + \frac{a^2 z^8}{8w^4} + \frac{2 a b z^9}{9w^4} + O_2(|w|^6)\right),
\end{align*}
\]

via an extensive, but straightforward, computation. Recall that each component \( f^k \) of \( f \) is real and harmonic as a function of \( z \). Rewriting

\[
\begin{align*}
f^3(z) &= 8 \Re\{\frac{a}{6} z^6 + \frac{b}{7} z^7\}
\end{align*}
\]

as a (non-harmonic) function of \( w \), we find

\[
\varphi(w) = 8 \Re\{\frac{a}{6} w^6 + \frac{b}{7} w^7 + \frac{|a|^2}{8} w^8 w^2 + O_2(|w|^{11})\}:
\]

the difference of \( \varphi \) on successive sheets is

\[
\Phi(w) := \varphi(w) - \varphi(iw) = 8 \Re\{\frac{a}{3} w^6 + \frac{b}{7} (1 + i) w^7\} + O_2(|w|^{10})
\]

(2.9)
and on non-successive sheets is

\[ \Phi_2(w) := \varphi(w) - \varphi(-w) = 8 R\{bw^7\} + O_2(|w|^{10}). \]  

(2.10)

From the formula (2.9), we see that \( N = 6 \), \( A = \frac{8a}{3} \) and the curves of intersection of successive sheets are curves in \( \mathbb{R}^3 \) leaving the branch point along the \((x_1, x_2)\)-plane, which is the tangent plane to \( \Sigma \) at the branch point, in the 12 directions \((\cos(4\theta), \sin(4\theta), 0)\), where \( 6\theta + \arg(a) \) is an integer multiple of \( \pi \). The 12 directions are paired off to form 6 curves in \( \mathbb{R}^3 \) leaving the branch point at equal angles \( \frac{2\pi}{3} \).

Similarly, from the formula (2.10), we see that \( N_2 = 7 \), \( A_2 = 8b \) and the curves of intersection of nonsuccessive sheets are seven curves leaving the branch point and making equal angles (images of 14 curves in the \( w \)-plane, paired). The arguments of \( a, b \in \mathbb{C} \setminus \{0\} \) may be given arbitrary values, so that the angle between a representative of the family of six curves of self-intersection and a representative of the family of seven curves of self-intersection may be chosen arbitrarily. For most choices, these 13 curves in \( \mathbb{R}^3 \) will not form equal angles at the branch point.

### 3 False branch points

The elimination of false branch points from an area-minimizing CMS \( f : \Sigma \to M \) is in general only possible by comparison with surfaces \( \Sigma_0 \) of reduced topological type (see [8], [11]): for orientable surfaces, \( \Sigma_0 \) has smaller genus or the same total genus and more connected components. As an oriented example, we may choose \( \Sigma \) to be a surface of genus 2 and \( \Sigma_0 \) to be a torus. Then there is a branched covering \( \pi : \Sigma \to \Sigma_0 \) with two branch points of order one. (Think of \( \Sigma \) as embedded in \( \mathbb{R}^3 \) so that it is invariant under a rotation by \( \pi \) about the \( z \)-axis, and meets the \( z \)-axis only at two points: the quotient under this rotation is a torus.) Now choose a minimizing CMS \( f_0 : \Sigma_0 \to M^n \), and let \( f : \Sigma \to M^n \) be \( f = f_0 \circ \pi \). Then \( f \) has two false branch points. In order to be sure that \( f \) minimizes area in its homotopy class, we may choose \( M^3 \) to be a flat 3-torus with two small periods and one large period. As one sees from this example, in order to show that false branch points do not occur, we must assume that \( f \) minimizes area among mappings from surfaces of the topological type of \( \Sigma \) and of lower topological type. This hypothesis was used by J. Douglas (see [8]), in a strict form, to find the existence of minimal surfaces \( f : \Sigma \to \mathbb{R}^3 \) with prescribed boundary. For \( \mathbb{R}P^2 \), however, there are no nonorientable surfaces of lower type.

Results in the literature for false branch points have until now assumed that \( \Sigma \) is oriented, see [9], [2], [12], [10], [11] and [18]. In order to treat false branch
points for nonorientable surfaces, we will need to extend certain known results. In particular, the following theorem appears in [10] for orientable surfaces, possibly with boundary, including surfaces of prescribed mean curvature vector not necessarily zero.

**Theorem 4.** *(Fundamental theorem of branched immersions)* Let $\Sigma^2$ be a compact surface with boundary endowed with a conformal structure, $\partial\Sigma$ possibly empty and $\Sigma$ not necessarily orientable. Let $M^n$ be a Riemannian manifold and $f : \Sigma \to M$ a CMS. Assume that the restriction of $f$ to $\partial\Sigma$ is injective. Then there exists a compact Riemann surface with boundary $\tilde{\Sigma}$, a branched covering $\pi : \Sigma \to \tilde{\Sigma}$ and a CMS $\tilde{f} : \tilde{\Sigma} \to M$ such that $f = \tilde{f} \circ \pi$. Moreover, the restriction of $\tilde{f}$ to $\partial\tilde{\Sigma}$ is injective. Further, $\tilde{\Sigma}$ is orientable if and only if $\Sigma$ is orientable.

*Proof:* If $\Sigma$ is orientable, then Theorem 4.5 of [10] provides an orientable quotient surface $\hat{\Sigma}$, a branched covering $\pi : \Sigma \to \hat{\Sigma}$ and an unramified CMS $\hat{f} : \hat{\Sigma} \to M$ such that $f = \hat{f} \circ \pi$.

There remains the case where $\Sigma$ is not orientable. Assume, without loss of generality, that $\Sigma$ is connected.

Let $p : \hat{\Sigma} \to \Sigma$ be the oriented double cover of $\Sigma$, with the induced conformal structure. Then $\hat{\Sigma}$ is connected and orientable, and $p$ is two-to-one. The composition $\hat{f} = f \circ p : \hat{\Sigma} \to M$ is a CMS, defined on an orientable surface, and we may apply Theorem 4.5 of [10] to find a compact orientable surface with boundary $\hat{\Sigma}$, an unramified CMS $\hat{f} : \hat{\Sigma} \to M$ and an orientation-preserving branched covering $\hat{\pi} : \hat{\Sigma} \to \hat{\Sigma}$ so that $\hat{f}$ factors as $\hat{f} \circ \hat{\pi}$.

Now let $\tilde{\Sigma}$ be the quotient surface of $\hat{\Sigma}$ under the identification of $\hat{\pi}(x^+) \in \hat{\Sigma}$ with $\hat{\pi}(x^-) \in \hat{\Sigma}$ whenever $x^+ \in \hat{\Sigma}$ and $p(x^+) = p(x^-)$ in $\Sigma$. Then for each $x \in \Sigma$, $p^{-1}(x)$ consists of two points $x^+, x^- \in \hat{\Sigma}$ and there are diffeomorphic neighborhoods of $\hat{\pi}(x^+)$ and of $\hat{\pi}(x^-)$ which are thereby identified in $\hat{\Sigma}$, with reversal of orientation. This implies that $\hat{\Sigma}$ is a differentiable 2-manifold. Write $\tilde{\pi} : \tilde{\Sigma} \to \hat{\Sigma}$ for the quotient mapping. Then $\tilde{f} : \tilde{\Sigma} \to M$ is well defined such that $\tilde{f} = \hat{f} \circ \tilde{\pi}$. Also, for $x \in \Sigma$, the two pre-images $x^+, x^- \in \tilde{\Sigma}$ have $\tilde{\pi} \circ \hat{\pi}(x^+) = \tilde{\pi} \circ \hat{\pi}(x^-)$, so that we may define $\pi : \Sigma \to \tilde{\Sigma}$ by $\pi(x) := \tilde{\pi} \circ \hat{\pi}(x^+)$.

Note that the mappings $p, \hat{\pi}$ and $\tilde{\pi}$ are surjective, and therefore also $\pi : \Sigma \to \tilde{\Sigma}$.

In the event that $\partial\Sigma$ is nonempty, since $f = \tilde{f} \circ \pi$ restricted to $\partial\Sigma$ is injective, it follows readily that the restriction of $\tilde{f}$ to $\partial\tilde{\Sigma}$ is injective.

Then in the above construction, for each $x \in \Sigma$, $f$ defines the same piece of surface, with opposite orientations, on neighborhoods of $x^+$ and of $x^-$. The branched covering $\hat{\pi} : \hat{\Sigma} \to \hat{\Sigma}$ preserves orientation, implying that $\hat{\pi}(x^+) \neq \hat{\pi}(x^-)$. Since $\hat{\Sigma}$ is connected, there is a path from $\hat{\pi}(x^+)$ to $\hat{\pi}(x^-)$ whose image in $\hat{\Sigma}$ re-
verses orientation. Therefore $\tilde{\Sigma}$ is not orientable. □

4 An immersion of $\mathbb{R}P^2$

We are now ready to give the proof of the main Theorem 1. Let $f : \mathbb{R}P^2 \to M^3$ be a CMS into a three-dimensional Riemannian manifold, which has minimum area among all mappings $\mathbb{R}P^2 \to M^3$ not homotopic to a constant. Write $\Sigma = \mathbb{R}P^2$. From Theorem 2, we see that $f$ has no true branch points. (For this conclusion, it would suffice that $f$ minimizes area in a $C^0$ neighborhood of each branch point.)

There remains the possibility of false branch points.

We first recall the computation of the Euler characteristic of a surface. For a compact, connected surface which is either orientable or nonorientable, the Euler characteristic

$$\chi(\Sigma) = 2 - r(\Sigma),$$

where $r(\Sigma)$ is the topological characteristic of $\Sigma$ [8], also known as the nonorientable genus; we shall adopt the term demigenus. If $\Sigma$ is orientable, then it has even demigenus and genus $\frac{1}{2}r(\Sigma)$. If it is non-orientable, then $\Sigma$ may be constructed by adding $r(\Sigma)$ cross-caps to the sphere. The demigenus of the sphere equals zero, of $\mathbb{R}P^2$ equals one, of the torus and the Klein bottle equals two. For other compact surfaces without boundary, the demigenus is $\geq 3$.

Now according to Theorem 4, there is a compact Riemann surface $\tilde{\Sigma}$, a branched covering $\pi : \Sigma \to \tilde{\Sigma}$ and an unramified CMS $\tilde{f} : \tilde{\Sigma} \to M$ such that $f = \tilde{f} \circ \pi$. We will apply the Riemann-Hurwitz formula to the branched covering $\pi$:

$$\chi(\Sigma) = d \chi(\tilde{\Sigma}) - O(\pi), \quad (4.1)$$

where $d$ is the degree of $\pi$, $O(\pi)$ is the total order of branching of $\pi$, and $\chi$ is the Euler number. Suppose that $f$ has a false branch point, or more generally, a ramified branch point. Then $O(\pi) \geq 1$, and the branched covering $\pi$ has degree $d \geq 2$.

Using the formula (4.1), we can determine the topological type of $\tilde{\Sigma}$. Since $\Sigma$ is homeomorphic to $\mathbb{R}P^2$, it has $\chi(\Sigma) = 1$. We also know that $d > 0$ and $O(\pi) \geq 1$. It follows that the demigenus $r(\tilde{\Sigma}) \leq 1$. Otherwise, the integer $r(\tilde{\Sigma})$ would be $\geq 2$, which implies $\chi(\tilde{\Sigma}) \leq 0$ and by the formula (4.1), $1 \leq -O(\pi) \leq -1$, a contradiction. That is, $\tilde{\Sigma}$ is either the sphere or the projective plane. But according to Theorem 4, since $\Sigma$ is not orientable, $\tilde{\Sigma}$ is not orientable; therefore, $\tilde{\Sigma}$ is homeomorphic to $\mathbb{R}P^2$. 

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Note that if $\tilde{f} : \tilde{\Sigma} \to M$ were homotopic to a constant mapping, then so would be $f = \tilde{f} \circ \pi$.

On the other hand, the area of $f$ equals the area of $\tilde{f}$ times the degree $d$ of $\pi$. But $d \geq 2$, so the area of $\tilde{f}$ is at most one-half the area of $f$. But this would mean that $f$ does not have minimum area among maps $\mathbb{RP}^2 \to M$ not homotopic to a constant mapping, contradicting our hypothesis. This implies that $f$ has no branch points, and is therefore an immersion. $\square$

Remark 5. We have treated conformally parameterized minimal surfaces in this paper. However, the proofs go through with only minor changes for projective planes of nonzero prescribed mean curvature $H : M^3 \to \mathbb{R}$, provided that $f(\Sigma)$ has a transverse orientation. This can occur only when $M$ is non-orientable.

It also appears plausible that a version of Theorem 4 can be extended to the more general case of mappings satisfying the unique continuation property, see [12].

Remark 6. An alternative approach to branch points of minimal surfaces of the type of the disk appears in the recent book [18] by Tromba. The second and higher variations of energy $E$ (see (1.1)) of a CMS $f$ are computed in a neighborhood of a branch point $z_0$, and the lowest nonvanishing variation is shown to be negative if the branch point is nonexceptional. This is defined in terms of the index $i > m$ of $z_0$, where $i + 1$ is the order of contact of the mapping with the tangent plane at the branch point. Note that the proper index $N$ is $\geq i$ (recall the definition (2.5)). If $i + 1$ is an integer multiple of $m$, where $m - 1$ is the order of the branch point, then the branch point is called exceptional. Tromba also shows that exceptional interior branch points will not occur, provided that the mapping has minimum area $A$ among surfaces with the same boundary curve.

In fact, minimizing area and minimizing energy, under such Plateau boundary conditions, are equivalent properties of a Lipschitz continuous mapping $f : B \to \mathbb{R}^n$, where $B$ is the unit disk in $\mathbb{R}^2$. Namely, if $E(f) \leq E(g)$ for all Lipschitz-continuous mappings $g : B \to \mathbb{R}^n$ defining the same boundary curve, then $A(f) \leq A(g)$ for all such $g$, as we now show.

Otherwise, for some $g : B \to \mathbb{R}^n$ with the same Plateau boundary conditions as $f$, $A(f) > A(g)$. Write $\eta = A(f) - A(g) > 0$. Approximate $g$ with $g^\delta(w) := (g(w), \delta w) \in \mathbb{R}^{n+2}$. Then $g^\delta$ is a Lipschitz immersion, so there are conformal coordinates $\tilde{w} =: F^{-1}(w)$ for some bi-Lipschitz homeomorphism $F : B \to B$ which preserves $\partial B$ (see [14]). Write $\tilde{g}^\delta(\tilde{w}) := g^\delta(F(\tilde{w}))$ for the conformal mapping with the same image as $g^\delta$. Define $\tilde{g} : B \to \mathbb{R}^n$ by composing $\tilde{g}^\delta$ with the projection.
from $\mathbb{R}^{n+2} \to \mathbb{R}^n$. Then the energy $E(\tilde{g}^\delta) = E(\tilde{g}) + \delta^2 E(F)$. Also, the area $A(\tilde{g}^\delta) \leq A(g) + C\delta$ for some constant $C$. It follows that

$$E(\tilde{g}) \leq E(\tilde{g}^\delta) = A(\tilde{g}^\delta) = A(g^\delta) \leq A(g) + C\delta < A(f) \leq E(f)$$

if $\delta$ is chosen small enough that $C\delta < \eta$. This implies that $f$ does not minimize energy, a contradiction.

The converse implication, that if $f$ minimizes area then it minimizes energy, follows similarly.

We may observe that with this remark, Tromba’s book [18] gives an independent proof that a mapping from the disk into $\mathbb{R}^3$ which minimizes energy for prescribed Plateau boundary conditions is an immersion in the interior.

In addition, the book contains new, partial results on boundary branch points.

References

[1] H. Wilhelm Alt, Verzweigungspunkte von H-Flächen I, Math. Z. 127 (1972), 333–362.

[2] H. Wilhelm Alt, Verzweigungspunkte von H-Flächen II, Math. Annalen 201 (1973), 33–56.

[3] Hugh Bray, Simon Brendle, Michael Eichmair and Andre Neves, Area-minimizing projective planes in 3-manifolds. Comm. Pure Appl. Math. 63 (2010), 12371247.

[4] Richard Courant, On a generalized form of Plateau’s problem, Trans. Amer. Math. Soca 50, 40–47(1941).

[5] Richard Courant, Dirichlet’s principle, Conformal Mapping and Minimal Surfaces. New York: Wiley Interscience 1950.

[6] Jesse Douglas, Solution of the problem of Plateau, Trans. Amer. Math. Soc. 33 (1931), 263–321.

[7] Jesse Douglas, One-sided minimal surfaces with a given boundary, Trans. Amer. Math. Soc. 34 (1932), 731-756.

[8] Jesse Douglas, Minimal Surfaces of Higher Topological Structure, Annals of Math. 40 (1939), 205–298.

[9] Robert Gulliver, Regularity of minimizing surfaces of prescribed mean curvature, Annals of Math. 97 (1973), 275–305.
[10] Robert Gulliver, Branched immersions of surfaces and reduction of topological type, I, Math. Z. **145** (1975), 267–288.

[11] Robert Gulliver, Branched immersions of surfaces and reduction of topological type, II, Math. Ann. **230** (1977), 25–48.

[12] Robert Gulliver, Robert Osserman and Halsey Royden, A theory of branched immersions of surfaces, Amer. J. Math **95** (1973), 750–812.

[13] Philip Hartman and Aurel Wintner, On the local behavior of solutions of non-parabolic partial differential equations, Amer. J. Math. **75** (1953), 258–287.

[14] Charles B. Morrey, On the solutions of quasi-linear elliptic partial differential equations. Trans. Amer. Math. Soc. **43** (1938), 126166.

[15] Robert Osserman, A proof of the regularity everywhere of the classical solution to Plateau's problem, Annals of Math. **91** (1970), 550–569.

[16] Robert Osserman, A Survey of Minimal Surfaces, New York: Van Nostrand Reinhold 1969.

[17] Tibor Radó, The problem of least area and the problem of Plateau, Math. Z. **32** (1930), 763–796.

[18] Anthony Tromba, A theory of branched minimal surfaces. Springer Verlag 2012.

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