Abstract

The linear canonical transforms of position and momentum are used to construct the tomographic probability representation of quantum states where the fair probability distribution determines the quantum state instead of the wave function or density matrix. The example of Moshinsky shutter problem is considered.

Keywords: tomogram, canonical transform, probability distribution function.

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Introduction

Both in classical and quantum mechanics, the linear canonical transforms of position and momentum preserving the Poisson brackets and commutation relations, respectively, were studied in [1, 2]. The linear canonical transform of creation and annihilation operators in the form of Bogolubov transform [3] can be associated with squeezing phenomenon [4] and correlated squeezed states [5]. Canonical transforms are also important for considering linear integrals of motion for quadratic nonstationary systems [6, 7, 8] and light propagation within the framework of geometric optics [9, 10].

Recently [11, 12] the new formulation of quantum mechanics (the probability representation of quantum mechanics) was suggested, where the quantum states are described by fair probability distributions called symplectic tomograms. The tomograms are related to the Wigner function [13] of quantum states by means of integral Radon transform [14]. In [15], the problem of diffraction in time [16] connected with quantum shutter was solved within the framework of the probability representation of quantum mechanics.

The aim of this work is to point out the connection of linear canonical transform of position and momentum with quantum tomograms describing the quantum states and to discuss the quantumness and classicality of the system states.

The paper is organized as follows.

In Section 1, we review the tomographic probability representation. In Section 2, we consider the problem of Moshinsky shutter in the phase-space representation. In Section 3, the conclusions and prospectives are presented.

1 Symplectic tomography

The symplectic tomogram can be constructed using the star-product scheme suggested in [17, 18] with a pair of operators, which are the quantizer operator

\[ \hat{D}(X, \mu, \nu) = \frac{1}{2\pi} \exp \left( iX\hat{\mathbf{1}} - i\nu \hat{\mathbf{p}} - i\mu \hat{\mathbf{q}} \right) \]
and the dequantizer operator
\[ \hat{U}(X, \mu, \nu) = \delta(X \hat{1} - \mu \hat{q} - \nu \hat{p}), \]

where \( X, \mu, \) and \( \nu \) are real variables, and \( \hat{q} \) and \( \hat{p} \) are the position and momentum operators, respectively.

The symplectic tomogram of quantum state with the density operator \( \hat{\rho} \) is defined as
\[ w(X, \mu, \nu) = \langle \hat{U}(X, \mu, \nu) \rangle = \text{Tr}(\hat{\rho} \delta(X - \mu \hat{q} - \nu \hat{p})), \quad \hbar = 1. \]

The tomogram can be rewritten in terms of the Radon transform of the Wigner function \( W(q, p) \) as follows:
\[ w(X, \mu, \nu) = \frac{1}{2\pi} \int W(q, p) \delta(X - \mu q - \nu p) \, dq \, dp. \]

The inverse Radon transform reads
\[ W(q, p) = \frac{1}{2\pi} \int w(X, \mu, \nu) \exp[-i(\mu q + \nu p - X)] \, d\mu \, d\nu \, dX. \]

The tomogram is normalized, i.e.,
\[ \int w(X, \mu, \nu) \, dX = 1 \]
for all \( \mu \) and \( \nu \). Relation (4) corresponds to operator form of the inverse Radon transform for the density operator
\[ \hat{\rho} = \frac{1}{2\pi} \int w(X, \mu, \nu) e^{i(X - \mu q - \nu p)} \, dX \, d\mu \, d\nu. \]

The symplectic tomogram is the probability distribution of random position \( X \) measured in a specific reference frame in the phase space. The reference frame is determined by two real parameters \( \mu = s \cos \theta \) and \( \nu = s^{-1} \sin \theta \), where \( s \) is the scaling parameter and \( \theta \) is the rotation angle of the frame axes.

The tomographic symbol \( f_A(X, \mu, \nu) \) of any operator \( \hat{A} \) can be given as \( \text{Tr} \hat{A} \hat{U} = f_A(X, \mu, \nu) \).

We consider the operator \( |\psi_1\rangle\langle\psi_2| \), where \( |\psi_1\rangle = \exp(-i\hat{H}t_1)|\psi\rangle \) and \( |\psi_2\rangle = \exp(-i\hat{H}t_2)|\psi\rangle \).

Here the Hamiltonian is the sum of the kinetic and potential energies: \( \hat{H} = (\hat{p}^2/2) + U(\hat{q}) \). We use dimensionless units, \( \hbar = m = 1 \). The operator \( \hat{A} = |\psi_1\rangle\langle\psi_2| \) for \( t_1 = t_2 = t \) is the density operator
\[ \hat{\rho}(t) = \exp(-i\hat{H}t)|\psi\rangle\langle\psi| \exp(i\hat{H}t) \]

of the system state.

The Schrödinger equation in the position representation for the wave function \( \psi(x_1, t_1) = \langle x_1 | \psi_1 \rangle \) provides the equation for the function \( \langle x_1 | \hat{A} | x_2 \rangle = A(x_1, x_2, t_1, t_2) = \psi(x_1, t_1)\psi^*(x_2, t_2) \) in the form
\[ \left[ i \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) + \frac{1}{2} \left( \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} \right) + U(x_2) - U(x_1) \right] A(x_1, x_2, t_1, t_2) = 0. \]

At \( t_1 = t_2 = t \), this equation is the standard von Neumann equation for the density matrix in the position representation. This equation can be rewritten for Weyl symbol of the operator \( \hat{A} \),
\[ W_{\hat{A}}(q, p, t_1, t_2) = \text{Tr} \left( 2 \exp[2(\alpha \hat{a}^* - \alpha^* \hat{a})] \hat{P} \hat{A} \right), \]
where \( \alpha = (q + ip)/\sqrt{2} \), \( \hat{a} = (\hat{q} + i\hat{p})/\sqrt{2} \), and \( \hat{P} \) is the parity operator.

For \( t_1 = t_2 = t \), the symbol \( W_A(q, p, t_1 = t, t_2 = t) \) coincides with the Wigner function \( W(q, p, t) \) of the quantum state. The equation reads

\[
\left[ i \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) + ip \frac{\partial}{\partial q} - U \left( q + \frac{i}{2} \frac{\partial}{\partial p} \right) + U \left( q - \frac{i}{2} \frac{\partial}{\partial p} \right) \right] W_A(q, p, t_1, t_2) = 0. \tag{10}
\]

This equation provides the Moyal equation \[19\] for the Wigner function. Equation (8) can be also rewritten in the tomographic form using the symbol of operator \( \hat{A} \),

\[
\mathcal{W}(X, \mu, \nu, t_1, t_2) = \text{Tr} \left( \delta(X - \mu \hat{q} - \nu \hat{p}) |\psi_1\rangle \langle \psi_2| \right). \tag{11}
\]

This function satisfies the equation

\[
\left[ i \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) - i\mu \frac{\partial}{\partial \nu} - \left( U \left( \left( \frac{\partial}{\partial X} \right)^{-1} \frac{\partial}{\partial \mu} + \frac{i}{2} \nu \frac{\partial}{\partial X} \right) - \text{c.c.} \right) \right] \mathcal{W}(X, \mu, \nu, t_1, t_2) = 0. \tag{12}
\]

For \( t_1 = t_2 \), the symbol of operator \( \hat{A} \) becomes the quantum-state tomogram \( \mathcal{W}(X, \mu, \nu, t) \) which satisfies the evolution equation found in \[11\].

For harmonic oscillator, Eq. (12) reads

\[
\left[ \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) - \mu \frac{\partial}{\partial \nu} + \nu \frac{\partial}{\partial \mu} \right] \mathcal{W}(X, \mu, \nu, t_1, t_2) = 0. \tag{13}
\]

The quantumness and classicality of the system states can be formulated in terms of the tomograms as follows.

A given normalized nonnegative tomogram \( \mathcal{W}(X, \mu, \nu) \) satisfying the homogeneity condition \( \mathcal{W}(\lambda X, \lambda \mu, \lambda \nu) = |\lambda|^{-1} \mathcal{W}(X, \mu, \nu) \), which follows from the homogeneity property of the delta-function, corresponds to a quantum state iff the integral on the right-hand side of Eq. (6) is nonnegative operator. The tomogram satisfying the nonnegativity of the integral term in Eq. (5) corresponds to the classical state. The tomograms violating nonnegativity conditions of integrals in both Eqs. (5) and (6) correspond neither classical nor quantum states.

The quantumness condition can be expressed also in the form of entropic inequality \[20\]

\[
- \left[ \int \mathcal{W}(X, \cos \theta, \sin \theta) \ln \mathcal{W}(X, \cos \theta, \sin \theta) dX \\
+ \int \mathcal{W}(X, \sin \theta, - \cos \theta) \ln \mathcal{W}(X, \sin \theta, - \cos \theta) dX \right] \geq \ln \pi \epsilon \tag{14}
\]

for optical tomogram \( \mathcal{W}(X, \mu = \cos \theta, \nu = \sin \theta) \).

Inequality (14) can be violated in the classical domain but must be fulfilled in the quantum domain.

## 2 Moshinsky shutter and diffraction in time

In \[16\], Moshinsky considered the problem of diffraction in time. This problem corresponds to opening at time \( t = 0 \) completely absorbing shutter located at position \( x = 0 \), on which
a stream of particles of definite momentum $K$ was inpunged. The Schrödinger equation (in dimensionless units)

$$i\dot{\psi} = -\frac{1}{2} \frac{\partial^2}{\partial x^2} \psi$$

was solved in terms of error function which for the shutter problem is reduced to the Moshinsky function

$$M(x, k, t) = \frac{i}{2\pi} \int \frac{\exp[i(\kappa x - [\kappa^2 t/2])] d\kappa}{\kappa - k}.$$  \hspace{1cm} (15)

The expression $|M(x, k, t)|^2$ gives the probability density of finding the particle at point $x$ at time $t$, if initially it was on the left side of the shutter. Equation (15) gives (see, e.g., [15])

$$|M(x, k, t)|^2 = \frac{1}{2} \left\{ \left[ \frac{1}{2} - C(w) \right]^2 + \left[ \frac{1}{2} - S(w) \right]^2 \right\}.$$ \hspace{1cm} (16)

Here $C(\omega)$ and $S(\omega)$ are Fresnel integrals

$$C(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos^2 y dy, \quad S(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin^2 y dy.$$ \hspace{1cm} (17)

One can find also the solution to the Moyal equation for the shutter problem, and it provides the Wigner function of the problem [15] in the form

$$W(q, p, k, t) = \frac{1}{\pi (k - p)} \sin\{2(pt - q)(k - p)\} \theta(pt - q).$$ \hspace{1cm} (18)

The symplectic tomogram for the Moshinsky shutter problem can also be obtained as a solution to the tomographic evolution equation (12); for $t_1 = t_2 = t$, it reads [15]

$$w(X, \mu, \nu, t) = \frac{1}{2|\mu|} \left\{ \left[ \frac{1}{2} + C(\rho) \right]^2 + \left[ \frac{1}{2} + S(\rho) \right]^2 \right\},$$ \hspace{1cm} (19)

where

$$\rho = \frac{k(\mu t + \nu) - X}{\sqrt{2\mu(\mu t + \nu)}}.$$ \hspace{1cm} (20)

Thus, the shutter problem can be solved in the Schrödinger, Moyal, and tomographic-probability representations. One can check that the solutions of these equations contain information on the diffraction-in-time properties of the shutter. The solutions are related by the integral transforms. For example, tomogram (19) and Wigner function (18) satisfy (5).

3 Canonical transforms and tomography

In Eq. (11), the argument of delta-function provides the classical linear transform of the particle position $q \rightarrow \mu q + \nu p$. This transform together with the linear transform of the particle momentum $p \rightarrow \mu' q + \nu' p$ form canonical transform in the phase space preserving the Poisson brackets. The real $2 \times 2$ matrices

$$\Lambda = \begin{pmatrix} \mu & \nu \\ \mu' & \nu' \end{pmatrix},$$

with the determinant equal to unity, form the symplectic group Sp(2,R).
In the quantum phase space, there exists the transform of operators $\hat{q} \rightarrow \mu \hat{q} + \nu \hat{p}$, $\hat{p} \rightarrow \mu' \hat{q} + \nu' \hat{p}$ determined by the symplectic matrix $\Lambda$. This transform corresponds to the unitary irreducible representation of the symplectic group. It can be considered as a combination of the scaling transform $\hat{S} = \exp is (\hat{q}\hat{p} + \hat{p}\hat{q}) / 2$ and rotation $\hat{R} = \exp i\theta (\hat{p}^2 + \hat{q}^2) / 2$.

The representations of the classical linear canonical transforms by means of the quantum operators acting in the Hilbert space of the particle states and the kernels of such operators were discussed in [1]. One can see that the quantizer and dequantizer of the symplectic tomography star-product scheme are based on using the quantum observables which are associated with the representations of the classical canonical transforms, i.e., with the representations of the symplectic group. An analogous construction can be given for systems with many degrees of freedom.

**Conclusions**

We point out the main results of our study.

We reviewed the notion of quantum state in the symplectic tomographic probability representation.

We constructed the evolution equation for the tomographic symbol of the operator corresponding to the product of the wave function $\psi(x_1, t_1)$ and its complex conjugate $\psi^*(x_2, t_2)$.

We considered the problem of Moshinsky shutter in the probability representation of quantum mechanics.

We formulated the quantumness of the system state as the operator inequality and as the inequality for tomographic entropy.

An extension of this approach to multimode states will be presented in future publications.

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