Adaptive CVaR Optimization for Dynamical Systems with Path Space Stochastic Search

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Abstract

We present a general framework for optimizing the Conditional Value-at-Risk for dynamical systems using stochastic search. The framework is capable of handling the uncertainty from the initial condition, stochastic dynamics, and uncertain parameters in the model. The algorithm is compared against a risk-sensitive distributional reinforcement learning framework and demonstrates outperformance on a pendulum and cartpole with stochastic dynamics. We also showcase the applicability of the framework to robotics as an adaptive risk-sensitive controller by optimizing with respect to the fully nonlinear belief provided by a particle filter on a pendulum, cartpole, and quadcopter in simulation.

1 Introduction

Stochastic Optimal Control (SOC) solves the optimization problem of a cost function subject to stochasticity in the dynamics. Additionally, uncertainty in the problem can arise in the form of uncertain parameters or initial conditions. In most cases, such uncertainties are handled by optimizing the expectation of the cost function. Despite its success in many SOC and Reinforcement Learning (RL) applications, expectation optimization has limited effect in controlling the distribution of trajectories. Common ways of incorporating the concept of risk are the mean-and-variance and expected exponential utility. These risk measures do not allow for precise control of the risk distribution. Conditional Value-at-Risk (CVaR) measures the level of risk by looking at the tail distribution or worst-case scenario. It has been used as a risk metric extensively in the field of finance, power utility, supply chain management, etc. In recent years, it is also seeing a rise in popularity in robotics research. However, popular techniques in SOC and RL that are based on dynamic programming cannot be directly applied to the CVaR optimization problem due to its time inconsistency. Several methods have been proposed to overcome this problem by lifting the state space of the problem. In [9] and [10], the CVaR decomposition theorem is introduced to obtain a dual representation of CVaR and optimizes the associated Bellman equation over a space of probability densities. Alternatively, the convex extremal formulation of CVaR can be used to alleviate the time-inconsistency problem [11]. Both approaches, however, require some form of state space augmentation and require solving an additional optimization problem. On the other hand, algorithms that directly optimize the policy are not affected by the time-inconsistency problem since they do not rely on the dynamic programming principles.

An effective and promising sampling-based approach for solving general nonlinear optimization problems is stochastic search. Stochastic search is a general class of optimization methods that optimizes the objective function by randomly sampling and updating candidate solutions. Many well-known algorithms, such as Cross Entropy Method (CEM), genetic algorithm, and simulated annealing fall into this category. Recently, a Gradient-based Adaptive Stochastic Search (GASS) algorithm was proposed by [15]. GASS updates the candidate solution by taking

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the gradient with respect to the sampling distribution parameters of solutions and approximating the gradient with Monte Carlo sampling. [16] extended GASS to constrained dynamic optimization problems, and [17] showed that dynamic GASS results in the same update law as the Information Theoretic Model Predictive Path Integral control [18] in the case of Gaussian and Poisson sampling distribution.

In this paper, we extend the risk sensitive formulation of GASS [19] and present Risk Sensitive Stochastic Search (RS3), a general framework to solve CVaR optimization for dynamical systems. The resulting algorithm bypasses the problem of time-inconsistency by directly performing stochastic gradient descent on the sampling distribution parameters. The framework is capable of handling uncertain initial states, parameters, and system stochasticity. We show that our framework outperforms in terms of final cost in average and under other risk measures compared to a risk-sensitive distributional RL algorithm, the Sample-based Distributional Policy Gradients (SDPG) [20]. The comparison was done on a pendulum and cartpole system in simulation. In addition, we demonstrate the applicability of our framework to robotics by combining it with a particle filter to perform risk-sensitive belief space control on a pendulum, cartpole and quadcopter in simulation.

The rest of this paper is organized as follows: in Section 2 we discuss related works. In Section 3 we formulate the problem and introduce the concept of CVaR. In Section 4 we derive the stochastic search framework. We present the algorithm and its application to belief space optimization in Section 5. The simulation results are included in Section 6. Finally, we conclude the paper in Section 7.

2 Related Works

In the majority of SOC and RL literature, the uncertainty in dynamical optimization, problems are handled by simply optimizing with respect to the expectation. However, in many cases, having the capability of reducing the risk of the resulting policy is important. Most risk-sensitive approaches consider a variance related measure [21, 22] or the expected exponential utility [23, 24]. These risk measures work well when the underlying distribution is Gaussian, as they optimize with respect to the first two moments of the cost function. To better minimize the tail end of the risk distribution, percentile risk [25], and more recently Value-at-Risk (VaR), have been used. Due to its extra capacity in tail minimization and computational advantage over VaR, CVaR has attracted significant attention over the past decade. Our work is similar in spirit to [26], where the gradient of CVaR with respect to the policy is taken directly. The main difference is that our work samples candidate policy parameterizations and approximates the gradient through averaging while a single copy of the policy is updated in their work.

Under stochasticity in the dynamics and the observation model, the true state is not available for decision-making. To deal with these uncertainties under risk-sensitive control, the belief-space control framework is introduced. Optimization is performed over a probability distribution over states provided by a state estimation framework such as the Extended Kalman Filter (EKF) [27, 29] or belief trees [30]. In our work, we use a particle filter (PF) to represent the belief because PF naturally fits our sampling-based approach.

In the belief space, planning and control algorithms including iterative Linear Quadratic Regulator (iLQR) [28], iterative Linear Quadratic Gaussian (iLQG) [31], stochastic Differential Dynamic Programming (sDDP) [29], and various type of Markov Decision Processes (MDPs) perform optimization with respect to the state distribution. However, multiple issues arise from the use of dynamic programming approaches such as time inconsistency and the curse of dimensionality. To bypass these issues, we use Stochastic Search (SS).

3 Problem Formulation

We consider the problem of minimizing the CVaR of some cost

$$U^* = \arg\min_{U \in \mathcal{U}} \text{CVaR}^\gamma[J(X, U)],$$

subject to nonlinear stochastic dynamics

$$x_{t+1} \sim p(x_{t+1} | x_t, u_t; \phi).$$
Here we have $U = \{u_0, \cdots, u_{T-1}\} \in \mathcal{U}$ as the control path and $X = \{x_0, \cdots, x_T\} \in \mathbb{R}^{n_x \times T}$ as the state path where $T \in [0, \infty)$ is the optimization horizon. $\mathcal{U} \subset \mathbb{R}^{n_u \times T-1}$ is the set of admissible control sequences, and $\phi$ is the system parameters. This formulation is capable of handling stochasticity in dynamics, $p(x_{t+1}|x_t, u_t)$, uncertain parameters, $p(\phi)$, and uncertain initial condition, $p(x_0)$. The CVaR is computed with respect to the uncertainty distributions and is defined as

$$\text{CVaR}^\gamma(X) = \frac{1}{1 - \gamma} \int_0^1 \text{VaR}^\gamma(X) dr,$$

where $\gamma \in [0, 1]$ is the risk level, and the VaR is defined as

$$\text{VaR}^\gamma(X) = \inf\{t : \mathbb{P}(X \leq t) \geq \gamma\}.$$

If $X$ has a continuous distribution, CVaR can be written as $\text{CVaR}^\gamma = \mathbb{E}[X | X \geq \text{VaR}^\gamma(X)]$.

### 4 Stochastic Search

Assuming that $p(x_{t+1}|x_t, u_t; \phi), p(x_0), p(\phi)$ are independent continuous density functions and $J$ is continuous, the minimization problem (1) can be rewritten as

$$U^* = \arg\min_{U \in \mathcal{U}} \mathbb{E}_{p(\chi)}[J(X, U)|J \geq \text{VaR}^\gamma(J)].$$

where $p(\chi) = p(x_0) p(\phi) \prod_{t=0}^{T-1} p(x_{t+1}|x_t, u_t)$ is the joint pdf of all uncertainty distributions. We parameterize the control $u$ with a policy $\pi_\eta$ characterized by its parameters $\eta$. The policy can be of any functional form, i.e. open-loop ($u_t = \nu_t, \eta_t = \nu_t$), linear feedback ($u_t = k_t x_t + \nu_t, \eta_t = \{k_t, \nu_t\}$) or even a neural network. We then define a sampling distribution for the policy parameters from the exponential family with a pdf of the form

$$p(\eta; \theta_t) = h(\eta_t) \exp\left(\theta_t^T T(\eta_t) - A(\theta_t)\right),$$

where $\theta$ is the natural parameters of the distribution and $T(\eta)$ is the sufficient statistics of $\eta$. The minimization is now performed with respect to the natural parameters

$$\theta^* = \arg\min_{\theta \in \Theta} \mathbb{E}_{p(\chi, \theta)}[J(X, U)|J \geq \text{VaR}^\gamma(J)].$$

The expectation is now taken with respect to the joint distribution of uncertainty and the sampling distribution. It is easy to show that the expected cost in (7) is an upper bound on the optimal cost in (5). For notational simplicity the joint distribution is dropped from the expectation.

For different algorithmic developments later, we turn the minimization problem into a maximization one by optimizing with respect to $-J$ and introduce a shape function $S : \mathbb{R} \rightarrow \mathbb{R}^+$. The problem is then transformed into

$$\theta^* = \arg\max_{\theta \in \Theta} \mathbb{E}[S(-J(X, U)) | J \geq \text{VaR}^\gamma(J)].$$

The shape function needs to satisfy the following conditions:

i) $S(y)$ is nondecreasing in $y$ and bounded from above and below for bounded $y$, with the lower bound being away from zero.

ii) The set of optimal solutions $\{\arg\max_{y \in \mathcal{Y}} S(H(y))\}$ after the transform is a non-empty subset of the solutions $\{\arg\max_{y \in \mathcal{Y}} H(y)\}$ of the original problem.

Common shape functions include: 1. the exponential function, $S(y; \kappa) = \exp(\kappa y)$, which leads to an update law similar to (13); 2. the sigmoid function, $S(y; \kappa, \varphi) = (y - y_h) \frac{1}{1 + \exp(-\kappa(y - \varphi))}$, where $y_h$ is a lower bound for the cost and $\varphi$ is the $(1 - \rho)$-quantile, which results in an update law similar to the CEM with elite threshold $\rho$.

Finally, we apply another log transformation to obtain scale-free gradient and the optimization problem becomes

$$\theta^* = \arg\max_{\theta \in \Theta} \ln \mathbb{E}[S(-J(X, U)) | J \geq \text{VaR}^\gamma(J)] = \arg\max_{\theta \in \Theta} l(\theta).$$
Since \( \ln : \mathbb{R}^+ \to \mathbb{R} \) is a strictly increasing function, it does not change the maximization objective. We can now take its gradient with respect to the parameters. Writing the expectation as an integral with respect to the path probability \( p(X, U; \theta) \) we get
\[
\mathbb{E}[S(-J(X, U)) | J \geq \text{VaR}^\gamma(J)] = \int_{\Omega_\chi \times \Omega_\eta} S(-J(X, U)) p(X, U; \theta) d\chi d\eta,
\]
where \( \Omega_\chi \) is defined such that \( J \geq \text{VaR}^\gamma(J) \) if and only if \( \chi \in \Omega_\chi \), and \( \Omega_\eta \) is defined such that \( \pi_{\eta_t}(x_t) \in \mathcal{U}_t, \forall t \). The path probability distribution can be decomposed as
\[
p(X, U; \theta) = p(x_T | x_{T-1}, \pi_{\eta_{T-1}}(x_{T-1}); \phi)p(\eta_{T-1}; \theta_{T-1}) \cdots p(x_0)p(\phi)
\]
where
\[
p(x_T | x_{T-1}, \pi_{\eta_{T-1}}(x_{T-1}); \phi) = \prod_{t=0}^{T-1} p(x_{t+1} | x_t, \pi_{\eta_t}(x_t); \phi)
\]
Note that since the uncertainty and sampling distribution are independent, their joint distribution can be broken into the product of the two. The gradient of the objective function with respect to the parameters can be taken as
\[
\nabla_\theta l(\theta) = \frac{\mathbb{E}_{p(\chi)}[\mathbb{E}_{p(\eta)}[S(-J(X, U)) | J \geq \text{VaR}^\gamma(J)] \nabla_\eta \left( \sum_{t=0}^{T-1} \ln p(\eta_t; \theta_t) \right)]}{\mathbb{E}[S(-J(X, U)) | J \geq \text{VaR}^\gamma(J)]}.
\]
The detailed derivation can be found in Appendix A. The gradient of the log parameter distribution at each time step can be calculated as
\[
\nabla_{\theta_t} \ln p(\eta_t; \theta_t) = \nabla_{\theta_t} \ln \left( h(\eta_t) \exp \left( \theta_t^T T(\eta_t) - A(\theta_t) \right) \right)
\]
\[
= \nabla_{\theta_t} (\theta_t^T T(\eta_t) - A(\theta_t))
\]
\[
= T(\eta_t) - \nabla_{\theta_t} A(\theta_t).
\]
Plugging it back into the gradient of the cost function we get
\[
\nabla_{\theta_t} l(\theta) = \frac{\mathbb{E}_{p(\chi)}[\mathbb{E}_{p(\eta)}[S(-J(X, U)) | J \geq \text{VaR}^\gamma(J)] (T(\eta_t) - \nabla_{\theta_t} A(\theta_t))]}{\mathbb{E}[S(-J(X, U)) | J \geq \text{VaR}^\gamma(J)]}.
\]
With this we have a gradient ascent update law for the parameters as
\[
\theta_{t+1} = \theta_t + \alpha k \nabla_{\theta_t} l(\theta^k).
\]
where the step size sequence \( \alpha^k \) satisfies the typical assumptions in Stochastic Approximation (SA):
\[
\alpha^k > 0 \quad \forall k, \quad \lim_{k \to \infty} \alpha^k = 0, \quad \sum_{k=0}^{\infty} \alpha_k = \infty
\]
The conditional expectation \( \mathbb{E}_{p(\chi)}[S(-J(X, U)) | J \geq \text{VaR}^\gamma(J)] = \text{CVaR}^\gamma(J) \) can be approximated with
\[
\hat{C}^\gamma_n = \hat{V}^\gamma_n + \frac{1}{M(1 - \gamma)} \sum_{m=1}^{M} (J_{n,m} - \hat{V}^\gamma_n)^+\]
\[
\hat{V}^\gamma_n = \inf\{x : \frac{1}{M} \sum_{m=1}^{M} \mathbb{1}_{\{J_{n,m} \leq x\}} \geq \gamma\}.
\]
The outer expectation can simply be approximated as \( \mathbb{E}_{p(\theta)}[\text{CVaR}^\gamma(J)] = \frac{1}{N} \sum_{n=1}^{N} \hat{C}^\gamma_n \). Note that the conditional expectations in (17) are computed as averages over costs defined on entire trajectory samples.

**Model Predictive Control Formulation**: The parameter update in Equation (18) can be used for trajectory optimization as well as in a receding horizon or Model Predictive Control (MPC) fashion. MPC is a powerful algorithmic approach to nonlinear feedback control which is essential in tasks...
that involve risk measures or high order statistical characteristics of cost functions. In this paper we will leverage parallelization using GPUs to implement Equation (18) in MPC fashion.

Adaptive Stochastic Search: The MPC formulation allows online interaction with the stochastic system dynamics. Data from this online interaction can be used to feed adaptive or state estimation schemes that update the probability distribution \( p(\chi) \) in an online fashion. In this work, we make use of a nonlinear state estimator, namely a particle filter, to propagate and update distribution \( p(\chi) \) over time. The resulting control architecture is a sampling-based risk-sensitive adaptive MPC scheme that optimizes CVaR while adapting the probability distribution over parametric uncertainties. The details of this approach are further explained in the next section.

5 Algorithm

In this section we present the RS3 algorithm implemented in MPC fashion, as shown in the algorithms below. At initial time, the policy parameter distribution’s natural parameters are initialized. Given an initial state distribution provided by a state estimator, \( M \) i.i.d. samples of initial states can be obtained. In the presence of uncertain parameters in the model, each sample is also associated with an i.i.d. sample of the model parameter distribution. \( N \) policies are sampled from the policy parameter distribution and each copy of the policy is applied to all \( M \) samples of the initial states. In the case of stochastic dynamics, the states of each of the \( M \) samples are propagated with an independent realization of the stochastic dynamics. A cost is then calculated for each of the total \( N \times M \) trajectories. For each policy sample, its associated CVaR cost is approximated with the \( M \) cost samples using (20) and (21). Using the CVaR values, the policy parameter distributions’ natural parameters can be updated using (17) and (18). In our simulations, we use Gaussian distributions with fixed variance to sample policy parameters, for which the sufficient statistics \( T(\eta) = \eta \) and \( \nabla p(\theta) = E[T(\eta)] \). The parameter update step is detailed in algorithm 2. As is common in SA

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**Algorithm 1 Stochastic Search for CVaR Optimization**

Given:
- \( F \): Transition Model; \( K \): Optimization Iterations; \( N \): Number of Control Samples; \( M \): Number of CVaR Optimization Samples; \( T \): Number of timesteps; \( \phi \): System parameters vector; \( c_0 \): Initial step size; \( \alpha \): Step size decay rate; \( \eta \): Policy; \( \gamma \): Risk level;

Initialize:
- \( \theta^0 \): natural parameters for policy; \( S \): Shape function;

while task not completed do
- \( \{x_0^m\}_{m=1,..,M} \leftarrow \text{StateEstimator}() \)
- \( \{\phi^m\}_{m=1,..,M} \leftarrow \text{SampleSystemParameters}() \)
for \( k \leftarrow 1 \) to \( K \) do
  \( \alpha_0 \leftarrow \frac{1}{k} \)
  \( \{\eta_{n,k}\}_{n=1,..,N} \sim p(\eta; \theta^k) \)
for \( n \leftarrow 1 \) to \( N \) in parallel do
  for \( m \leftarrow 1 \) to \( M \) in parallel do
    for \( t \leftarrow 0 \) to \( T-1 \) do
      \( u_{k,t}^m \sim \pi_{\eta^k}(x_{k,t}^m) \)
      \( x_{t+1}^m \leftarrow F(x_{t}^m, u_{k,t}^m, \epsilon^m, \phi^m) \)
    for \( m \leftarrow 1 \) to \( M \) do
      Compute cost: \( J_{n,m}(X_{n,m}, U_k^n) \)
    end for
    \( C_{n}^{k} \leftarrow \text{ComputeCVar}(S(-J_{n,m}), \gamma) \)
  for \( n \leftarrow 1 \) to \( N \) do
    \( \omega^{n} \leftarrow \frac{1}{N} \exp(-\tilde{C}^{n} - \beta) \)
  end for
  \( \eta^{k+1} = \Pi(\eta^{k} + \alpha^k \sum_{n=1}^{N} \nabla_{\eta}^{n}) \)
end for
end for
end for
Perform Polak averaging: \( \bar{\theta} \leftarrow \frac{1}{K} \sum_{k=1}^{K} \theta^k \)
Sample policy parameters \( \tilde{\eta} \sim p(\eta; \bar{\theta}) \)
Apply policy: \( \pi_{\tilde{\eta}} \) for \( \tau \) timesteps
Recede horizon: \( \theta^0 = \omega(\theta^K, \tau) \)
end while

**Algorithm 2 Parameter Update**

Given:
- \( N \): Number of Control Samples; \( \alpha^k \);
- Stepsize; \( \{\eta^k\} \): Policy samples; \( \{\tilde{C}\} \): CVaR values; \( \Pi \): Projection operator for control constraints;
- \( \mu \leftarrow u \)
- \( \beta \leftarrow \min(\tilde{C}) \)
- \( \eta \leftarrow \sum_{n=1}^{N} \exp\left(-\tilde{C}^{n} - \beta\right) \)
for \( n \leftarrow 1 \) to \( N \) do
  \( \omega^{n} \leftarrow \frac{1}{N} \exp\left(-\tilde{C}^{n} - \beta\right) \)
  \( \nabla_{\eta}^{n} = \eta^{n} - \frac{1}{N} \sum_{n=1}^{N} \eta^{n} \)
end for
\( \eta^{k+1} = \Pi(\eta^{k} + \alpha^k \sum_{n=1}^{N} \nabla_{\eta}^{n}) \)
Table 1: RS3 vs. risk-sensitive SDPG [20]

| System | Pendulum | Cartpole |
|--------|----------|----------|
|        | Mean     | Var      | CVaR     |
| RS3    | 169.2    | 170.3    | 170.7    |
|        | 172.0    | 175.9    | 177.6    |
|        | 177.0    | 185.1    | 189.3    |
|        | 183.1    | 196.1    | 203.7    |
|        | 291.0    | 291.9    | 292.3    |
|        | 299.3    | 302.5    | 304.2    |
|        | 311.9    | 320.1    | 326.9    |
| SDPG   | 171.1    | 172.4    | 172.9    |
|        | 173.4    | 177.9    | 179.5    |
|        | 178.4    | 188.3    | 191.6    |
|        | 185.2    | 201.1    | 206.8    |
|        | 302.1    | 302.6    | 302.8    |
|        | 430.4    | 607.9    | 617.8    |
|        | 591.3    | 650.8    | 659.6    |

algorithms, Polyak averaging is performed on the natural parameters to improve the convergence rate [33]. With the Polyak averaged natural parameters \( \bar{\eta} \), an optimal policy can be sampled and applied to the system for \( \tau \) timesteps. Finally, we apply a shift operator \( \omega(\theta, \tau) \) that recedes the optimization horizon and outputs \( \bar{\theta}_t = \theta_{t+\tau} \). The last \( \tau \) timesteps of the natural parameters are re-initialized.

Note that the RS3 algorithm can handle any or all uncertainties from initial state distribution, uncertain parameters and stochastic dynamics provided that i.i.d. samples can be generated from the uncertain distribution. To turn off a source of uncertainty, one can simply set the distribution as a Dirac delta function centered at the true value of initial state or parameter (or 0 in the case of stochastic dynamics). In our simulation examples of belief space control, we use a particle filter to provide the initial state distribution. To handle uncertain model parameters, we augment the states by the uncertain parameters and use a particle filter to learn its distribution (detailed in Appendix B). In both cases, i.i.d. particles from non-Gaussian distributions can be used directly used in the RS3 algorithm. However, we want to stress that any filter can be used together with the RS3 algorithm.

6 Simulation Results

In this section, we showcase the general applicability of RS3 in dealing various types of uncertainty. 1) External noise: Comparing its performance against the SDPG algorithm for CVaR optimization. 2) Uncertain system parameters: combining it with a particle filter to perform risk sensitive control in belief space. 3) Uncertain initial condition: This is included in the Appendix due to space constraints.

All simulations were performed with a risk level of 0.9. The open loop policy \( \pi_\eta(x) = \eta \) is used for all simulations, where \( \eta \) directly maps to the controls. The multivariate normal distribution is chosen as the sampling distribution, and we use the method proposed in [16] to handle the box control constraints in the simulation tasks by sampling from a truncated multivariate normal distribution. The tuning parameters of all simulations are included in the Appendix.

6.1 Comparison Against Sample-based Distributional Policy Gradients

In the literature, Sample-based Distributional Policy Gradients algorithm [34, 20] is one of the most recent work on optimizing the CVaR for dynamical systems. SDPG [34] and the risk-sensitive version of SDPG [20] are actor-critic type policy gradient algorithms in the distributional RL [35] setting. The actor network parameterizes the policy and the critic network learns the return distribution by reparameterizing simple Gaussian noise samples. The risk-sensitive version of SDPG [20] is an extension of the naive SDPG [34] algorithm by using CVaR as a loss function to train the actor network to learn a risk-sensitive policy. We compared RS3 and the risk-sensitive SDPG on two classic control systems in OpenAI Gym [36], a pendulum and a cartpole.

Typical RL algorithms always receive some state feedback either fully or partially from environments. Thus, to fairly compare against SDPG, we exploited an MPC scheme in RS3 to implicitly receive the state feedback and perform a receding-horizon optimization. In addition, as SDPG is unable to handle uncertainty in the initial states and controls, we consider deterministic initial states and system dynamics with additive noise h the control channels. In the SDPG framework [34, 20], control noise was only added during training for state space exploration, but to clearly assess the risk-sensitive optimization performance, it was also injected to both algorithms at test time. To match the RS3 framework, the Gym environment’s controls were modified to be continuous and use a quadratic cost function instead of the typical RL reward function -1, 0, or +1 implemented in Gym. The cost function used in the simulation can be found in Appendix C1.1. The discount rate (typically
Under the aforementioned conditions, RS3 is shown to outperform SDPG overall, as it was able to converge to a lower CVaR value, especially for larger noise levels. The mean, VaR, and CVaR values of the final costs obtained from both algorithms for the pendulum and cartpole simulation are shown in Table 1 and the estimated p.d.f. plots for the final costs are shown in Figure 1. The state, control, and cost histograms for all the simulation in Table 1 can be found in Appendix C.1.

It is clearly shown in Figure 1 that the distribution of the final cost has sharper tail on the high cost region in RS3’s results compared to SDPG’s. As a result, the Mean, VaR, and CVaR of RS3’s final costs are smaller than SDPG’s.

The reason why our method outperforms the RL framework is that we perform online update of our policy whereas the RL policy is fixed after training. This disadvantage of RL algorithms comes from the nature of RL. Once a model is trained on a specific dataset or with a specific noise profile, the model fails to output correct predictions under a new environment or given unseen inputs or noise. Our online optimization scheme solves this issue and fits better in risk-sensitive control.

6.2 Belief Space Optimization

We next show results for the uncertain parameter case from the pendulum, cartpole and quadcopter systems. In each trajectory plot, the dotted lines represent estimates from the particle filter with the error bars showing the ±3σ uncertainties of the nonlinear belief. The solid line represents the ground truth states.

Pendulum: We first apply RS3 to a pendulum for a swingup task with unknown pendulum mass. We assume deterministic initial condition and state transition model. The pendulum’s true mass is set to 2 kg. The prior for pendulum mass is set to be $\mathcal{N}(5.0, 4.0)$. The initial states $x = [\theta, \dot{\theta}]$ are drawn from a normal distribution with mean $[\pi, 0]$ and covariance matrix diag($[0.1, 0.1]$). We assume full-state observability with additive measurement noise $\xi \sim \mathcal{N}(0, 1)$. From Figure 2 we can observe that RS3 is able to correctly estimate the mass of the pendulum in the parameter estimation.
case. Without parameter estimation, RS3 overestimates the control effort required and overshoots the target angle.

**Cartpole:**

We apply the proposed algorithm to the task of cartpole swingup with unknown pole mass. The prior over the mass of the pole is a normal distribution $N(5.0, 5.0)$ and the true value is 0.1 kg. Our algorithm is able to learn the true mass of the pole and successfully perform a swing up (Figure 3). We compare this with the case of not estimating the mass of the pole. Without performing parameter estimation, the algorithm does not sample from the correct dynamics and is unable to correctly optimize for a trajectory that does swing up.

**Quadcopter:** Finally, we apply our algorithm to the Quadcopter system, where the task is to fly a Quadcopter with states $[x, y, z, x, y, z, p, y, p, y, p, y]$ from $[0, 0, 0]$ to $[2, 2, 2]$. The drag coefficient of the system is unknown, the prior over the drag is a normal distribution $N(0.5, 0.5)$ and the true value is 0.1. The algorithm is once again able to learn the correct drag coefficient and manages to pilot the quadcopter to the target position without significant overshoot (Figure 4). For the case where we do not perform parameter estimation, since the prior of the drag coefficient is greater than the actual value, the control policy found by the RS3 framework results in overshooting behavior before convergence, although it still manages to converge to the target state.
7 Conclusion

In this paper we introduced a general framework for CVaR optimization for dynamical systems. The resulting algorithm, RS3, is capable of handling uncertainties arising from uncertain initial conditions, unknown model parameters and system stochasticity. The algorithm can be readily combined with any filter for belief space risk sensitive control. We compared RS3 against SDPG on the systems of a pendulum and cartpole and demonstrated outperformance in terms of final CVaR cost. In addition, we combined RS3 with a particle filter for adaptive risk-sensitive control on non-Gaussian belief under different sources of uncertainty.

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A Stochastic Search Update Law Derivation

The stochastic search update law \([13]\) can be derived as

\[
\nabla_{\theta} l(\theta) = \frac{\nabla \mathbb{E}[S(-J(X,U)) | J \geq \text{VaR}^\alpha(J)]}{\mathbb{E}[S(-J(X,U)) | J \geq \text{VaR}^\alpha(J)]} = \frac{\nabla \int_{\Omega_\chi \times \Omega_\eta} S(-J(X,U)) p(X,U;\theta) d\chi d\eta}{\mathbb{E}[S(-J(X,U)) | J \geq \text{VaR}^\alpha(J)]} = \frac{\int_{\Omega_\chi \times \Omega_\eta} S(-J(X,U)) \nabla_{\theta} p(X,U;\theta) d\chi d\eta}{\mathbb{E}[S(-J(X,U)) | J \geq \text{VaR}^\alpha(J)]} = \frac{\int_{\Omega_\chi} S(-J(X,U)) p(X,U;\theta) \nabla_{\theta} \ln p(X,U;\theta) d\chi d\eta}{\mathbb{E}[S(-J(X,U)) | J \geq \text{VaR}^\alpha(J)]} = \frac{\mathbb{E}_{p(\eta)}[\mathbb{E}_{p(\chi)}[S(-J(X,U)) | J \geq \text{VaR}^\alpha(J)] \nabla_{\theta} (\sum_{t=0}^{T-1} \ln p(\eta_t; \theta_t))] / \mathbb{E}[S(-J(X,U)) | J \geq \text{VaR}^\alpha(J)]]}{\mathbb{E}[S(-J(X,U)) | J \geq \text{VaR}^\alpha(J)]}.
\]

B Particle Filter and RS3 for Uncertain Model Parameters

In the case of unknown parameters \(\phi\) in the model, we use the particle filter to learn both the states and uncertain model distribution. We first augment our state space with those parameters as \(y = [x, \phi]^T\) and formulate the new dynamics as

\[
y_{t+1} = \begin{bmatrix} f(x_t, u_t, \phi_t) \\ 0 \\ \end{bmatrix} + \begin{bmatrix} 0 \\ w_t \\ \end{bmatrix}, \quad \phi_0 \sim p(\phi_0)
\]

where \(w_t\) is the process noise added to mitigate the sample impoverishment problem of the particle filter. Starting with deterministic initial states \(x_0\) and a prior distribution on the parameters \(p(\phi_0)\). The steps of this adaptive risk sensitive control framework at each timestep are

1) Sample from initial distribution: \(\{\hat{y}_i\}_{i=1,...,M} \sim p(y_0)\)
2) Compute policy: \(\eta_t = \text{RS3}(\{\hat{y}_i\}_{i=1,...,M})\)
3) Apply control: \(x_{t+1} \leftarrow F(x_t, \pi_{\eta_t}(x_t), 0; \phi)\)
4) Obtain observation: \(z_{t+1} \leftarrow h(x_{t+1}, \xi_{t+1})\)
5) Propagate particles: \(\{\hat{x}_{t+1}^i\}_{i=1,...,M} \leftarrow \hat{F}(u_t, \{\hat{x}_t^i\}_{i=1,...,M}, 0, \{\hat{\phi}_t^i\}_{i=1,...,M})\)
6) Update likelihood: \(\{q_i\}_{i=1,...,M} = L(z_{t+1}, \{\hat{x}_{t+1}^i\}_{i=1,...,M})\)

where \(F\) is the true system dynamics with real parameters \(\phi\) and \(\hat{F}\) is the model for particle filter and RS3, and both do not contain stochasticities in the dynamics. \(\xi\) is the observation noise.

C Additional Simulation Results

C.1 Comparison against SDPG

C.1.1 Cost function

We define the cost function \(J(X,U)\) as

\[
J(X,U) = g(x_T) + \sum_{t=T_0}^{T-1} l(x_t, u_t), \quad (22)
\]
where \( x_t \in \mathbb{R}^{n_x} \) is the state at time \( t \), \( u_t \in \mathbb{R}^{n_u} \) is the control at time \( t \), \( g \) is the state cost at the final time \( T \), and \( l \) is the running cost at time \( t \). In the simulations, we set \( g = l \) and the running cost is composed of the quadratic state cost \( (x_t - x_{\text{target}})^T Q (x_t - x_{\text{target}}) \) and the quadratic control cost \( u_t^T R u_t \).

The system dynamics are from OpenAI Gym and the pendulum state is defined as \([\theta, \dot{\theta}]\), the pendulum angle and the angular velocity. The cartpole state is defined as \([x, \dot{x}, \theta, \dot{\theta}]\), the cart position, position velocity, pole angle, and the pole angular velocity. The state and control cost used in pendulum dynamics are \( Q = \text{diag}(\begin{bmatrix} 3 \ 0.01 \end{bmatrix}) \), where \( \text{diag}() \) represents a diagonal matrix, and \( R = 0.01 \). In cartpole, the costs are \( Q = \text{diag}(\begin{bmatrix} 0.01 \ 0.1 \ 1 \ 0.1 \end{bmatrix}) \) and \( R = 0.001 \).

For Pendulum, the initial state is \([-\pi, 0]\) and target state is \([0, 0]\) and for the Cartpole, the initial state is \([0, 0, -\pi, 0]\) and target state is \([0, 0, 0, 0]\).

C.1.2 Pendulum

Four different levels of control noise are tested in the pendulum simulations comparing RS3 against SDPG: \( \mathcal{N}(0, 0.3^2), \mathcal{N}(0, 1.0^2), \mathcal{N}(0, 2.0^2), \) and \( \mathcal{N}(0, 3.0^2) \). The mass of the pendulum is set to 1 kg and its length is set to 1.0 m. The controls are constrained by the box constraints \(|u_t| < 10\). The results can be found in the figures below. As discussed in the main paper and can be observed from Figure 5 to Figure 8, RS3 outperforms SDPG in terms of the Mean, VaR, and CVaR of the final costs in all cases.
Figure 5: Top: Control and state trajectories from the Pendulum problem with noise in the control channel. The control noise is $N(0, 0.3^2)$. Bottom: Cost histogram and the estimated p.d.f. of it.

Figure 6: Top: Control and state trajectories from the Pendulum problem with noise in the control channel. The control noise is $N(0, 1.0^2)$. Bottom: Cost histogram and the estimated p.d.f. of it.
Figure 7: *Top:* Control and state trajectories from the Pendulum problem with noise in the control channel. The control noise is $\mathcal{N}(0,2.0^2)$. *Bottom:* Cost histogram and the estimated p.d.f. of it.

Figure 8: *Top:* Control and state trajectories from the Pendulum problem with noise in the control channel. The control noise is $\mathcal{N}(0,3.0^2)$. *Bottom:* Cost histogram and the estimated p.d.f. of it.

C.1.3 Cartpole

Similarly, 3 different levels of additive control noise were used to compare RS3 and SDPG in the cartpole problem: $\mathcal{N}(0,0.3^2), \mathcal{N}(0,1.0^2)$, and $\mathcal{N}(0,2.0^2)$. The mass of the cart are set to 1 kg, the
mass of the pole 0.1 kg and its length 0.5 m. The controls are constrained by the box constraints $|u_t| < 15$. The results can be found in Figure 9 to Figure 11.

Although the control becomes spiky with larger noise, RS3 is able to accomplish the task, whereas SDPG fails the task most of the time, as seen in Figure 10 and Figure 11. These results show that RS3 does a better job of risk-sensitive control on higher order systems compared to SDPG.
Figure 9: Top: Control and state trajectories from the Cartpole problem with noise in the control channel. The control noise is \( N(0, 0.3^2) \). Bottom: Cost histogram and the estimated p.d.f. of it.

Figure 10: Top: Control and state trajectories from the Cartpole problem with noise in the control channel. The control noise is \( N(0, 1.0^2) \). Bottom: Cost histogram and the estimated p.d.f. of it.
Figure 11: Top: Control and state trajectories from the Cartpole problem with noise in the control channel. The control noise is $N(0, 2.0^2)$. Bottom: Cost histogram and the estimated p.d.f. of it.

D Belief Space Risk Sensitive Control

D.1 Pendulum

We test RS3 on Pendulum in the case of uncertainty in the initial conditions and in the case of stochastic dynamics where the true state of the system is fully observable. The mass of the pendulum is set to 1 kg and its length is set to 1.0 m. The controls are constrained by the box constraints $|u_t| < 10$. Measurements of the state $x$ are corrupted with additive noise $\xi \sim N(0, \text{diag}([0.7, 0.3]))$. In the case of uncertain initial states, the initial states $x_0 = [\theta, \dot{\theta}]$ are drawn from a normal distribution with mean $[\pi, 0]$ and covariance matrix $\text{diag}([0.5, 0.5])$. In the case of stochastic dynamics, the controls are corrupted by noise with distribution $N(0, 3.0^2)$.

The results for uncertain initial conditions and stochastic dynamics can be found in Figure 12 and Figure 13 respectively.
Figure 12: Trajectories from the Pendulum problem with uncertainty in the initial conditions. 
*Top:* A comparison of the posterior state from the particle filter and the ground truth from a single trajectory. 
*Bottom:* Different trajectory realizations of the initial condition.

Figure 13: Trajectories from the Pendulum problem with stochastic dynamics. 
*Top:* A comparison of the posterior state from the particle filter and the ground truth from a single trajectory. 
*Bottom:* Different trajectory realizations of the stochastic dynamics.
D.2 Cartpole

Similarly, we test RS3 on Cartpole in the case of uncertainty in the initial conditions and in the case of stochastic dynamics. The mass of the cart are set to 1 kg, the mass of the pole 0.1 kg and its length 0.5 m. The controls are constrained by the box constraints $|u_t| < 15$. Measurements of the state $x$ are corrupted with additive noise $\xi \sim N(0, \text{diag}([1, 1, 0.25, 0.25]))$.

The initial states $x_0 = [x, \dot{x}, \theta, \dot{\theta}]$ are drawn from a normal distribution with mean $[0, 0, \pi, 0]$ and covariance matrix $\text{diag}(0.5, 0.5, 0.08, 0.05))$

The results for uncertain initial conditions and stochastic dynamics can be found in Figure 14 and Figure 15 respectively.

Figure 14: Trajectories from the Cartpole problem with is uncertainty in the initial conditions. Top: A comparison of the posterior state from the particle filter and the ground truth from a single trajectory. Bottom: Different trajectory realizations of the initial condition.
Figure 15: Trajectories from the Cartpole problem with stochastic dynamics. Top: A comparison of the posterior state from the particle filter and the ground truth from a single trajectory. Bottom: Different trajectory realizations of the stochastic dynamics.

D.3 Quadcopter

Finally, we test RS3 on the Quadcopter in the case of uncertainty in the initial conditions and stochastic dynamics. Measurements of the state $x$ are corrupted with additive noise $\xi \sim \mathcal{N}(0, \text{diag}(\{0.1, 0.1, 0.01, 0.01, 0.01, 0.08, 0.08, 0.08, 0.01, 0.1, 0.01\}))$. The controls are constrained by the box constraints $[0, -10, -10, -1] < u_t < [20, 10, 10, 1]$.

For the case of uncertain initial states, the states $x_0 = [x, y, z, \alpha, \beta, \gamma, \dot{x}, \dot{y}, \dot{z}, \dot{\alpha}, \dot{\beta}, \dot{\gamma}]$ are drawn from a normal distribution with zero mean and covariance matrix diag($\{0.3, 0.3, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2, 0.2\}$). In the case of stochastic dynamics, the controls are corrupted with noise drawn from $\mathcal{N}(0, \text{diag}(\{1, 1, 1, 1\}))$.

The results can be found in Figure 16 and Figure 17 respectively.
Figure 16: Trajectories from the Quadcopter problem with uncertainty in the initial conditions. Top: A comparison of the posterior state from the particle filter and the ground truth from a single trajectory. Bottom: Different trajectory realizations of the initial condition.
Figure 17: Trajectories from the Quadcopter problem with stochastic dynamics. Top: A comparison of the posterior state from the particle filter and the ground truth from a single trajectory. Bottom: Different trajectory realizations of the stochastic dynamics.
D.4 Parameters

To alleviate the particle degeneracy problem from particle filters, artificial process noise is added to each sample. The artificial process noises added are zero mean, and their covariances are

\[
\Sigma_{\text{Pendulum_est}} = \text{diag}([1 \times 10^{-5}, 1 \times 10^{-5}, 1 \times 10^{-9}])
\]

\[
\Sigma_{\text{Pendulum_no_est}} = \text{diag}([0.2, 0.2, 0])
\]

\[
\Sigma_{\text{Cartpole_est}} = \text{diag}([0.001, 0.001, 0.001, 1 \times 10^{-6}])
\]

\[
\Sigma_{\text{Cartpole_no_est}} = \text{diag}([0.1, 0.3, 0.2, 1 \times 10^{-6}, 0])
\]

\[
\Sigma_{\text{Quadcopter_est}} = \text{diag}([0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.04, 0.04, 0.04, 0.04, 0.04, 0.04, 0.04, 1 \times 10^{-9}])
\]

\[
\Sigma_{\text{Quadcopter_no_est}} = \text{diag}([0.05, 0.05, 0.05, 0.03, 0.03, 0.003, 0.04, 0.04, 0.04, 0.04, 0.04, 0.04, 1 \times 10^{-9}])
\]

where the est subscript denotes the case of uncertain parameters and no_est denotes the cases of uncertain initial condition and stochastic dynamics. Note that the required artificial process noise is much higher for simulation runs that do not estimate the parameters. This is because the incorrect priors for the parameters result in dynamics that are very different from ground truth, leading to the particle filter diverging very quickly.