**w-DIVISORIAL DOMAINS**

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**Abstract.** We study the class of domains in which each **w**-ideal is divisorial, extending several properties of divisorial and totally divisorial domains to a much wider class of domains. In particular we consider **PvMDs** and Mori domains.

**Introduction**

The class of domains in which each nonzero ideal is divisorial has been studied, independently and with different methods, by H. Bass [2], E. Matlis [25] and W. Heinzer [17] in the sixties. Following S. Bazzoni and L. Salce [3, 4], these domains are now called divisorial domains. Among other results, Heinzer proved that an integrally closed domain is divisorial if and only if it is a Prüfer domain with certain finiteness properties [17, Theorem 5.1].

Twenty years later E. Houston and M. Zafrullah introduced in [20] the class of domains in which each **t**-ideal is divisorial, which they called TV-domains, and characterized **PvMDs** with this property [20, Theorem 3.1]. However they observed that an integrally closed TV-domain need not be a **PvMD** [20, Remark 3.2]; thus in some sense the class of TV-domains is not the right setting for extending to **PvMDs** the properties of divisorial Prüfer domains.

The purpose of this paper is to investigate **w**-divisorial domains, that is domains in which each **w**-ideal is divisorial. This class of domains proves to be the most suitable **t**-analogue of divisorial domains. In fact, by using this concept we are able to improve and generalize several results proved for Noetherian and Prüfer divisorial domains in [3, 17, 28, 31].

The main result of Section 1 is Theorem 1.5. It states that **R** is a **w**-divisorial domain if and only if **R** is a weakly Matlis domain (that is a domain with **t**-finite character such that each **t**-prime ideal is contained in a unique **t**-maximal ideal) and **R**

**M** is a divisorial domain, for each **t**-maximal ideal **M**. In this way we recover the characterization of divisorial domains given in [3, Proposition 5.4].

In Section 2, we study the transfer of the properties of **w**-divisoriality and divisoriality to certain (generalized) rings of fractions, such as localizations at (**t**-)prime ideals, (**t**-)flat overrings and (**t**-)subintersections.

In Section 3 we consider **w**-divisorial **PvMDs**. We prove that **R** is an integrally closed **w**-divisorial domain if and only if **R** is a weakly Matlis **PvMD** and each **t**-maximal ideal is **t**-invertible (Theorem 3.3). This is the **t**-analogue of [17, Theorem 5.1]. We also prove that when **R** is integrally closed, each **t**-linked overring of **R** is **w**-divisorial if and only if **R** is a generalized Krull domain and each **t**-prime ideal is contained in a unique **t**-maximal ideal (Theorem 3.5). Since in the Prüfer case generalized Krull domains coincide with generalized Dedekind domains [4], we obtain that an integrally closed domain is totally divisorial if and only if it is a divisorial generalized Dedekind domain [28, Section 4].

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The last section is devoted to Mori \( w \)-divisorial domains. A Mori \( w \)-divisorial domain is necessarily of \( t \)-dimension one and each of its localizations at a height-one prime is Noetherian (Corollary \[4.3\]). Noetherian divisorial and totally divisorial domains were intensely studied in \[3\] \[2\] \[23\] \[31\]. It turns out that several of the results proved there can be extended to the Mori case by using different technical tools. In Theorem \[4.2\] we characterize \( w \)-divisorial Mori domains and in Theorems \[4.5\] and \[4.11\] we study \( w \)-divisoriality of their overrings. In particular, we show that generalized rings of fractions of \( w \)-divisorial Mori domains are \( w \)-divisorial and we prove that a domain whose \( t \)-linked overrings are all \( w \)-divisorial is Mori if and only if it has \( t \)-dimension one.

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Throughout this paper \( R \) will denote an integral domain with quotient field \( K \) and we will assume that \( R \neq K \).

We shall use the language of star-operations. A \textit{star operation} is a map \( I \to I^* \) from the set \( F(R) \) of nonzero fractional ideals of \( R \) to itself such that:

1. \( R^* = R \) and \((aI)^* = aI^*\), for all \( a \in K \setminus \{0\} \);
2. \( I \subseteq I^* \) and \( I \subseteq J \Rightarrow I^* \subseteq J^* \);
3. \( I^{**} = I^* \).

General references for systems of ideals and star operations are \[13\] \[15\] \[16\] \[21\].

A star operation \( * \) is of \textit{finite type} if \( I^* = \bigcup \{J^*: J \subseteq I \text{ and } J \text{ is finitely generated}\} \), for each \( I \in F(R) \). To any star operation \( * \), we can associate a star operation \( f \) of finite type by defining \( I^{*f} = \bigcup J^* \), with the union taken over all finitely generated ideals \( J \) contained in \( I \). Clearly \( I^{*f} \subseteq I^* \). A nonzero ideal \( I \) is \( * \)-\textit{finite} if \( I^* = J^* \) for some finitely generated ideal \( J \).

The identity is a star operation, called the \textit{d-operation}. The \( v \)- and the \( t \)-operations are the best known nontrivial star operations and are defined in the following way. For a pair of nonzero ideals \( I \) and \( J \) of a domain \( R \) we let \( (J: I) \) denote the set \( \{x \in K : xI \subseteq J\} \). We set \( I_v = (R: (R: I)) \) and \( I_t = \bigcup J_v \) with the union taken over all finitely generated ideals \( J \) contained in \( I \). Thus the \( t \)-operation is the finite type star operation associated to the \( v \)-operation.

A nonzero fractional ideal \( I \) is called a \( * \)-\textit{ideal} if \( I = I^* \). If \( I = I_v \), we say that \( I \) is \textit{divisorial}. For each star operation \( * \), we have \( I^* \subseteq I_v \), thus each divisorial ideal is a \( * \)-ideal.

The set \( F_*(R) \) of \( * \)-ideals of \( R \) is a semigroup with respect to the \( * \)-\textit{multiplication}, defined by \( (I, J) \to (IJ)^* \), with unity \( R \). We say that an ideal \( I \in F(R) \) is \( * \)-\textit{invertible} if \( I^* \) is a unit in the semigroup \( F_*(R) \). In this case the \( * \)-\textit{inverse} of \( I \) is \( (R : I) \). Thus \( I \) is \( * \)-invertible if and only if \((I(R : I))^* = R \). Invertible ideals are \( (\ast \text{-invertible}) \) \( * \)-ideals.

A prime \( * \)-ideal is also called a \( * \)-\textit{prime}. A \( * \)-\textit{maximal} ideal is an ideal that is maximal in the set of the proper \( * \)-ideals. A \( * \)-maximal ideal (if it exists) is a prime ideal. If \( * \) is a star operation of finite type, an easy application of Zorn’s Lemma shows that the set \( \ast\text{-Max}(R) \) of the \( * \)-maximal ideals of \( R \) is not empty. Moreover, for each \( I \in F(R) \), \( I^* = \cap_{M \in \ast\text{-Max}(R)} I^*R_M \); in particular \( R = \cap_{M \in \ast\text{-Max}(R)} R_M \) \[15\].

The \( w \)-operation is the star operation defined by setting \( I_w = \cap_{M \in \ast\text{-Max}(R)} IR_M \). An equivalent definition is obtained by setting \( I_w = \cup \{I : J \subseteq J \text{ is finitely generated} \} \). By using the latter definition, one can see that the notion of \( w \)-ideal coincides with the notion of \textit{semi-divisorial} ideal introduced by S. Glaz and W. Vasconcelos in 1977 \[13\]. As a star-operation, the \( w \)-operation was first considered by E. Hedstrom and E. Houston in 1980 under the name of \( F_{\infty} \)-operation \[18\]. Since 1997 this star operation was intensely studied by Wang Fanggui and R.
McCasland in a more general context. In particular they showed that the notion of $w$-closure is a very useful tool in the study of Strong Mori domains [32, 33].

The $w$-operation is of finite type. We have $w$-$\text{Max}(R) = t$-$\text{Max}(R)$ and $IR_M = I_w R_M \subseteq I_t R_M$, for each $I \in F(R)$ and $M \in t$-$\text{Max}(R)$. Thus $I_w \subseteq I_t \subseteq I_v$.

We denote by $t$-$\text{Spec}(R)$ the set of $t$-prime ideals of $R$. Each height one prime is a $t$-prime and each prime minimal over a $t$-ideal is a $t$-prime. We say that $R$ has $t$-dimension one if each $t$-prime ideal has height one.

1. $w$-Divisorial Domains

A divisorial domain is a domain such that each ideal is divisorial [8] and we say that a domain $R$ is $w$-divisorial if each $w$-ideal is divisorial, that is $w = v$. Since $I_w \subseteq I_t \subseteq I_v$, for each nonzero fractional ideal $I$, then $R$ is $w$-divisorial if and only if $w = t = v$. A domain with the property that $t = v$ is called in [20] a $TV$-domain. Mori domains (i.e., domains satisfying the ascending chain condition on proper divisorial ideals) are $TV$-domains. A domain such that $w = t$ is called a $TW$-domain [27]. An important class of $TW$-domains is the class of $PrvMD$s; in fact a $PrvMD$ is precisely an integrally closed $TW$-domain [22, Theorem 3.1].

(Recall that a domain $R$ is a Prüfer $v$-multiplication domain, for short a $PrvMD$, if $R_{\text{max}}$ is a valuation domain for each $t$-maximal ideal $M$ of $R$.) Since a Krull domain is a Mori $PrvMD$, a Krull domain is a $w$-divisorial domain. An example due to M. Zafrullah shows that in general $w \neq t \neq v$ [27, Proposition 1.2]. Also there exist $TV$-domains and $TW$-domains that are not $w$-divisorial [27, Example 2.7].

If $R$ is a Prüfer domain, in particular a valuation domain, then $w$-divisoriality coincides with divisoriality, because each ideal of a Prüfer domain is a $t$-ideal.

**Proposition 1.1.** A $w$-divisorial domain $R$ is divisorial if and only if each maximal ideal of $R$ is a $t$-ideal. Hence a one-dimensional $w$-divisorial domain is divisorial.

**Proof.** If each maximal ideal of $R$ is a $t$-ideal, then each ideal of $R$ is a $w$-ideal by [27, Proposition 1.3]. Hence, if $R$ is $w$-divisorial it is also divisorial. The converse is clear.

Following [11], we say that a nonempty family $\Lambda$ of nonzero prime ideals of $R$ is of finite character if each nonzero element of $R$ belongs to at most finitely many members of $\Lambda$ and we say that $\Lambda$ is independent if no two members of $\Lambda$ contain a common nonzero prime ideal. We observe that a family of primes is independent if and only if no two members of $\Lambda$ contain a common $t$-prime ideal. In fact a minimal prime of a nonzero principal ideal is a $t$-ideal.

The domain $R$ has finite character (resp., $t$-finite character) if $\text{Max}(R)$ (resp., $t$-$\text{Max}(R)$) is of finite character. If the set $\text{Max}(R)$ is independent of finite character, the domain $R$ is called by E. Matlis an $h$-local domain [28]; thus $R$ is $h$-local if it has finite character and each nonzero prime ideal is contained in a unique maximal ideal. A domain $R$ such that $t$-$\text{Max}(R)$ is independent of finite character is called in [11] a weakly Matlis domain; hence $R$ is a weakly Matlis domain if it has $t$-finite character and each $t$-prime ideal is contained in a unique $t$-maximal ideal.

Clearly, a domain of $t$-dimension one is a weakly Matlis domain if and only if it has $t$-finite character. A one-dimensional domain is a weakly Matlis domain if and only if it is $h$-local; if and only if it has finite character.

We recall that any $TV$-domain, hence any $w$-divisorial domain, has $t$-finite character by [20, Theorem 1.3]. The main result of this section shows that $w$-divisorial domains form a distinguished class of weakly Matlis domains.

We start by proving some technical properties of weakly Matlis domains.

**Lemma 1.2.** Let $R$ be an integral domain. The following conditions are equivalent:
(1) \( R \) is a weakly Matlis domain;

(2) For each \( t \)-maximal ideal \( M \) of \( R \) and a collection \( \{I_\alpha\} \) of \( w \)-ideals of \( R \) such that \( \cap_\alpha I_\alpha \neq 0 \), if \( \cap_\alpha I_\alpha \subseteq M \), then \( I_\alpha \subseteq M \) for some \( \alpha \).

Proof. (1) \( \Rightarrow \) (2). First, we show that each \( t \)-prime ideal is contained in a unique \( t \)-maximal ideal. We adapt the proof of [17 Theorem 2.4]. Let \( P \) be a \( t \)-prime which is contained in two distinct \( t \)-maximal ideals \( M_1 \) and \( M_2 \). Let \( \{I_\alpha\} \) be the set of all \( w \)-ideals of \( R \) which contain \( P \) but are not contained in \( M_1 \). Such a collection is nonempty since \( M_2 \) is in it. Let \( I = \cap I_\alpha \). Then \( I \nsubseteq M_1 \) and \( I \subseteq M_2 \). Take \( x \in I \setminus M_1 \). Since \( x^2 \notin M_1 \), then \( (P + x^2R)_w \in \{I_\alpha\} \) and so \( x \in (P + x^2R)_w \). Thus \( x \in (P + x^2R)R_{M_2} \notin R_{M_2} \), and \( sx = p + x^2r \) for some \( s \in R \setminus M_2 \), \( p \in P \) and \( r \in R \). Whence \( (s - rx)x = p \in P \subseteq M_1 \cap M_2 \). Now \( s - rx \notin P \) because \( s \notin M_2 \) and \( rx \in I \subseteq M_2 \). But also \( x \notin P \), since \( x \notin M_1 \); a contradiction because \( P \) is prime.

Next we show that \( R \) has \( t \)-finite character. Let \( 0 \neq x \in R \) and \( \{M_\beta\} \) be the set of all \( t \)-maximal ideals of \( R \) which contain \( x \). For a fixed \( \beta \), let \( A_\beta \) be the intersection of all \( w \)-ideals of \( R \) which contain \( x \) but are not contained in \( M_\beta \). By assumption \( A_\beta \nsubseteq M_\beta \). Set \( A = \sum_\beta A_\beta \). Then \( x \in A \) and \( A \) is contained in no \( M_\beta \). Hence \( A_\beta = R \). Let \( F = (a_{\beta_1}, a_{\beta_2}, \ldots, a_{\beta_k}) \), where \( a_{\beta_i} \in A_\beta \), be a finitely generated ideal of \( R \) such that \( F_1 = R \). Now, if \( M_\beta \notin \{M_{\beta_1}, M_{\beta_2}, \ldots, M_{\beta_k}\} \), necessarily \( M_\beta \subseteq F \), which is impossible because \( M_\beta \) is a proper \( t \)-ideal and \( F_1 = R \). We conclude that \( \{M_\beta\} = \{M_{\beta_1}, M_{\beta_2}, \ldots, M_{\beta_k}\} \) is finite. \( \square \)

**Lemma 1.3.** Let \( R \) be a \( w \)-divisorial domain, \( M \) a \( t \)-maximal ideal of \( R \) and \( \{I_\alpha\} \) a collection of \( w \)-ideals of \( R \) such that \( \cap_\alpha I_\alpha \neq 0 \). If \( \cap_\alpha I_\alpha \subseteq M \), then \( I_\alpha \subseteq M \) for some \( \alpha \).

Proof. Set \( A = \cap_\alpha I_\alpha \). Since \( R \) is a \( TW \)-domain, then the \( I_\alpha \)'s and \( A \) are \( t \)-ideals. Since \( R \) is also a \( TV \)-domain, by [20 Lemma 1.2], if \( I_\alpha \nsubseteq M \), for each \( \alpha \), then \( A \nsubseteq M \). \( \square \)

**Lemma 1.4.** If \( R \) is a weakly Matlis domain, then \( I_vR_M = (IR_M)_v \), for each nonzero fractional ideal \( I \) and each \( t \)-maximal ideal \( M \).

Proof. Apply [11 Corollary 5.3] for \( F = t\text{-Max}(R) \). \( \square \)

We are now ready to prove the \( t \)-analogue of [3 Proposition 5.4], which states that a domain \( R \) is divisorial if and only if it is \( h \)-local and \( R_M \) is a divisorial domain, for each maximal ideal \( M \). Local divisorial domains have been studied in [3 Section 5] and completely characterized in [11 Section 2].

**Theorem 1.5.** Let \( R \) be an integral domain. The following conditions are equivalent:

1. \( R \) is a \( w \)-divisorial domain;
2. \( R \) is a weakly Matlis domain and \( R_M \) is a divisorial domain, for each \( t \)-maximal ideal \( M \);
3. \( R \) is a \( TV \)-domain and \( R_M \) is a divisorial domain, for each \( t \)-maximal ideal \( M \);
4. \( IR_M = (IR_M)_v = I_vR_M \), for each nonzero fractional ideal \( I \) and each \( t \)-maximal ideal \( M \).

Proof. (1) \( \Rightarrow \) (2). That \( R \) is a weakly Matlis domain follows from Lemmas [13] and [12]. Now let \( M \) be a \( t \)-maximal ideal of \( R \) and \( I = JR_M \) a nonzero ideal of \( R_M \), where \( J \) is an ideal of \( R \). By Lemma [14] we have \( I_v = (JR_M)_v = J_vR_M \). Since \( J_v = J_w \), then \( I_v = J_wR_M = JR_M = I \). Hence \( R_M \) is a divisorial domain.
(2) \implies (4) follows from Lemma \[14\].

(4) \implies (1). Let \( t \) be a nonzero fractional ideal of \( R \). Then \( I_w = \cap_{M \in \text{t-Max}(R)} IR_M = \cap_{M \in \text{t-Max}(R)} I_tR_M = I_v \). Whence \( R \) is a \( w \)-divisorial domain.

(1) \implies (3) via (2).

(3) \implies (4). Since \( t = v \) in \( R \) and \( d = t = v \) in \( R_M \), for each nonzero fractional ideal \( I \) and each \( t \)-maximal ideal \( M \) of \( R \), we have
\[
IR_M = (IR_M)_v = (IR_M)_t = (I_tR_M)_t = I_tR_M = I_tR_M.
\]

Any almost Dedekind domain that is not Dedekind provides an example of a locally divisorial domain that is not \( w \)-divisorial, because it is not of finite character \[13\] Theorem 37.2).

**Corollary 1.6.** Let \( R \) be a domain of \( t \)-dimension one. Then \( R \) is \( w \)-divisorial if and only if \( R \) has \( t \)-finite character and \( R_P \) is divisorial, for each height one prime \( P \).

2. **Localizations of \( w \)-divisorial domains**

A domain whose overrings are all divisorial is called totally divisorial \[3\]. Not all divisorial domains are totally divisorial \[17\] Remark 5.4]; in fact a valuation domain \( R \) is divisorial if and only if its maximal ideal is principal \[17\] Lemma 5.2], but it is totally divisorial if and only if it is strongly discrete \[3\] Proposition 7.6], equivalently \( PR_P \) is a principal ideal for each prime ideal \( P \) of \( R \) \[3\] Proposition 5.3.8]. Since for valuation domains divisoriality coincides with \( w \)-divisoriality and each overring of a valuation domain is a localization at a certain \((t-)\)prime, we see that \( w \)-divisoriality is not stable under localization at \((t-)\)primes.

We say that an integral domain \( R \) is a strongly \( w \)-divisorial domain (resp., a strongly divisorial domain) if \( R \) is \( w \)-divisorial (resp., divisorial) and \( R_P \) is a divisorial domain for each \( P \in t-\text{Spec}(R) \) (resp., \( P \in \text{Spec}(R) \)). Note that if \( R \) is strongly \( w \)-divisorial (resp., strongly divisorial), then \( R_P \) is strongly divisorial for each \( P \in t-\text{Spec}(R) \) (resp., for each \( P \in \text{Spec}(R) \)).

By Theorem \[15\] (resp., \[3\] Proposition 5.4]), \( R \) is a strongly \( w \)-divisorial domain (resp., a strongly divisorial domain) if and only if \( R \) is a weakly Matlis domain (resp., an \( h \)-local domain) and \( R_P \) is a divisorial domain for each \( P \in t-\text{Spec}(R) \) (resp., \( P \in \text{Spec}(R) \)).

If \( R \) has \( t \)-dimension one, then \( R \) is \( w \)-divisorial if and only if it is strongly \( w \)-divisorial.

In this section we shall study the extension of \( w \)-divisoriality and divisoriality to distinguished classes of generalized rings of fractions such as localizations at \((t-)\)prime ideals, \((t-)\)flat overrings and \((t-)\)subintersections.

We recall the requisite definitions. A nonempty family \( F \) of nonzero ideals of a domain \( R \) is said to be a multiplicative system of ideals if \( IJ \in F \), for each \( I, J \in F \). If \( F \) is a multiplicative system, the set of ideals of \( R \) containing some ideal of \( F \) is still a multiplicative system, which is called the saturation of \( F \) and is denoted by \( \text{Sat}(F) \). A multiplicative system \( F \) is said to be saturated if \( F = \text{Sat}(F) \).

If \( F \) is a multiplicative system of ideals, the overring \( R_F := \cup \{ (R : J); J \in F \} \) of \( R \) is called the generalized ring of fractions of \( R \) with respect to \( F \). For any fractional ideal \( I \) of \( R \), \( I_F := \cup \{ (I : J); J \in F \} \) is a fractional ideal of \( R_F \) and \( IR_F \subseteq I_F \). Clearly \( I_F = I_{\text{Sat}(F)} \).

The map \( P \mapsto P_F \) is an order-preserving bijection between the set of prime ideals \( P \) of \( R \) such that \( P \notin \text{Sat}(F) \) and the set of prime ideals \( Q \) of \( R_F \) such that \( JR_F \nsubseteq Q \) for any \( J \in F \), with inverse map \( Q \mapsto Q \cap R \). In addition, \( R_P = (R_F)_{P_F} \) for each prime ideal \( P \notin \text{Sat}(F) \). If \( Q \) is a \( t \)-prime ideal of \( R_F \), then \( Q \cap R \) is a \( t \)-prime ideal of \( R \) \[10\] Proposition 1.3].
If Λ is a nonempty family of nonzero prime ideals of R, the set \( \mathcal{F}(\Lambda) = \{ J : J \subseteq R \text{ is an ideal and } J \not\subseteq P \text{ for each } P \in \Lambda \} \) is a saturated multiplicative system of ideals and \( I_{\mathcal{F}(\Lambda)} = \cap \{ IR_P : P \in \Lambda \} \), for each fractional ideal I of R; in particular \( R_{\mathcal{F}(\Lambda)} = \cap \{ R_P : P \in \Lambda \} \). A generalized ring of fractions of type \( R_{\mathcal{F}(\Lambda)} \) is called a subintersection of R; when \( \Lambda \subseteq t\text{-Spec}(R) \), we say that \( R_{\mathcal{F}(\Lambda)} \) is a \( t \)-subintersection of R.

A multiplicative system of ideals \( \mathcal{F} \) of R is finitely generated if each ideal \( I \in \mathcal{F} \) contains a finitely generated ideal \( J \) which is still in \( \mathcal{F} \). As in \([10]\), we say that \( \mathcal{F} \) is a \( v \)-finite multiplicative system if each \( t \)-ideal \( I \in \text{Sat}(\mathcal{F}) \) contains a finitely generated ideal \( J_v \in \text{Sat}(\mathcal{F}) \). A finitely generated multiplicative system is \( v \)-finite. If \( \mathcal{F} \) is \( v \)-finite, the set \( \Lambda \) of \( t \)-ideals which are maximal with respect to the property of not being in \( \text{Sat}(\mathcal{F}) \) is not empty, \( \Lambda \subseteq t\text{-Spec}(R) \), \( \mathcal{F}(\Lambda) \) is \( v \)-finite and \( T = R_{\mathcal{F}(\Lambda)} \) \([10]\) Proposition 1.9 (a) and (b)].

An overring \( T \) of R is said to be \( t \)-flat over R if \( T_M = R_{M \cap R} \), for each \( t \)-maximal ideal \( M \) of \( T \) \([23]\), equivalently \( T_Q = R_{Q \cap R} \), for each \( t \)-prime ideal \( Q \) of \( T \) \([7]\) Proposition 2.6]. Flatness implies \( t \)-flatness, but the converse is not true \([23]\) Rem. 2.12]. By \([7]\) Thm. 2.6], \( T \) is \( t \)-flat over R if and only if there exists a \( t \)-finite multiplicative system \( \mathcal{F} \) of R such that \( T = R_{\mathcal{F}} \). Thus \( T \) is \( t \)-flat if and only if \( T = R_{\mathcal{F}(\Lambda)} \), where \( \Lambda \) is a family of pairwise incomparable \( t \)-primes of \( R \) and \( \mathcal{F}(\Lambda) \) is \( v \)-finite. It follows that a \( t \)-flat overring of R is a \( t \)-subintersection of R.

In turn, any generalized ring of fractions is a \( t \)-linked overring; but the converse does not hold in general \([5]\) Proposition 2.2]. We recall that an overring \( T \) of an integral domain \( R \) is \( t \)-linked over \( R \) if, for each nonzero finitely generated ideal \( J \) of \( R \) such that \( (R : J) = R \), we have \((T : JT) = T \) \([5]\). This is equivalent to say that \( T = \cap T_{R/P} \), where \( P \) ranges over the \( t \)-primes of \( R \) \([5]\) Proposition 2.13(a)].

It is well known that if \( P \) is a \( t \)-prime ideal of R, then \( PR_P \) need not be a \( t \)-ideal of \( R_P \). When \( PR_P \) is a \( t \)-prime ideal, \( P \) is called by M. Zafrullah a \textit{well behaved} \( t \)-prime \([23]\) page 436]. We prefer to say that \( P \) \textit{t-localizes} or that it is a \textit{\( t \)-localizing prime}. Height-one prime ideals and divisorial \( t \)-maximal primes, e. g. \( t \)-invertible \( t \)-primes, are examples of \( t \)-localizing primes.

A large class of domains with the property that each \( t \)-prime ideal \( t \)-localizes is the class of \( v \)-coherent domains. We recall that a domain R is called \( v \)-\textit{coherent} if the ideal \( (R : J) \) is \( v \)-finite whenever \( J \) is finitely generated. This class of domains properly includes PrMD’s, Mori domains and coherent domains \([24]\) [11].

If \( R \) is a \( w \)-divisorial (resp., strongly \( w \)-divisorial) domain, then each \( t \)-maximal (resp., \( t \)-prime) ideal \( t \)-localizes.

**Lemma 2.1.** Let \( \Lambda \) be a set of \( t \)-localizing \( t \)-primes of R. Then:

1. \( P_{\mathcal{F}(\Lambda)} \in t\text{-Spec}(R_{\mathcal{F}(\Lambda)}) \), for each \( P \in \Lambda \).
2. \( \mathcal{F}(\Lambda) \) is \( v \)-finite, \( t\text{-Max}(R_{\mathcal{F}(\Lambda)}) = \{ P_{\mathcal{F}(\Lambda)} : P \text{ maximal in } \Lambda \} \).

**Proof.** Set \( \mathcal{F} = \mathcal{F}(\Lambda) \) and \( T = R_{\mathcal{F}} \).

1. Let \( P \in \Lambda \). Since \( R_P = P_{\mathcal{F}} \) and by hypothesis \( PR_P = P_{\mathcal{F}} T_{\mathcal{F}} \) is a \( t \)-ideal, then \( P_{\mathcal{F}} = P_{\mathcal{F}} T_{\mathcal{F}} \cap T \) is a \( t \)-ideal of \( T \).
2. Since \( P_{\mathcal{F}} \) is a \( t \)-ideal by part (1), we can apply \([10]\) Proposition 1.9 (c)]. \(\square\)

**Proposition 2.2.** Let \( \Lambda \) be a set of pairwise incomparable \( t \)-localizing \( t \)-primes of R. Then:

1. \( \Lambda \) is independent of finite character if and only if \( \mathcal{F}(\Lambda) \) is \( v \)-finite and \( R_{\mathcal{F}(\Lambda)} \) is a weakly Mal'tis domain.
2. If \( R_{\mathcal{F}(\Lambda)} \) is \( u \)-divisorial, then \( \Lambda \) is independent of finite character.

**Proof.** Set \( \mathcal{F} = \mathcal{F}(\Lambda) \) and \( T = R_{\mathcal{F}} \).
(1). If \( \mathcal{F} \) is \( v \)-finite, by Lemma 2.4(2) we have \( t\text{-Max}(T) = \{ P_T \mid P \in \Lambda \} \). It follows that \( \Lambda \) is independent of finite character if and only if \( t\text{-Max}(T) = \{ P_T \mid P \in \Lambda \} \) is independent of finite character, that is \( T \) is a weakly Matlis domain. On the other hand, if \( \Lambda \) is of finite character, then \( \mathcal{F} \) is \( v \)-finite by [10, Lemma 1.16].

(2). Since \( T \) is a weakly Matlis domain, by part (1) it suffices to show that \( \Lambda \) is of finite character.

By Lemma 2.4(1), \( P_T \) is a \( t \)-prime of \( T \), for each \( P \in \Lambda \). We show that each proper divisorial ideal of \( T \) is contained in some \( P_T \). We have \( T = \cap_{P \in \Lambda} R_P = \cap_{P \in \Lambda} T_{P_T} \). If \( I \) is a proper divisorial ideal of \( T \), there is \( x \in K \setminus T \) (where \( K \) is the quotient field of \( R \)) such that \( I \subseteq x^{-1} T \cap T \). Since \( x \notin T \), there exists \( P \in \Lambda \) such that \( x \notin T_P \), equivalently \( x^{-1} T \cap T \subseteq P_T \).

Since \( t = v \) on \( T \), we conclude that \( t\text{-Max}(T) = \{ P_T \mid P \in \Lambda \} \). Since \( T \) has \( t \)-finite character, it follows that \( \Lambda \) is of finite character.

**Theorem 2.3.** Let \( R \) be a \( w \)-divisorial domain. If \( \Lambda \subseteq t\text{-Max}(R) \), then \( R_{\mathcal{F}(\Lambda)} \) is a \( t \)-flat \( w \)-divisorial overring of \( R \).

**Proof.** Since \( R \) is a weakly Matlis domain (Theorem 1.5), \( t\text{-Max}(R) \) is independent of finite character; thus \( \Lambda \) has the same properties. In addition, each \( t \)-maximal ideal is a \( t \)-localizing prime ideal. It follows that \( \mathcal{F}(\Lambda) \) is \( v \)-finite and \( T := R_{\mathcal{F}(\Lambda)} \) is a \( t \)-flat weakly Matlis domain (Proposition 2.2(1)). By Lemma 2.4(2), for each \( N \in t\text{-Max}(T) \), there exists \( M \in \Lambda \) such that \( N = M_{\mathcal{F}(\Lambda)} \). It follows that \( T_N = R_M \) is divisorial and so \( T \) is \( w \)-divisorial by Theorem 1.5. \( \square \)

As we have mentioned above, the localization of a \( w \)-divisorial domain at a \( t \)-prime need not be a \( (w) \)-divisorial domain. Thus Theorem 2.4 does not hold for an arbitrary \( \Lambda \subseteq t\text{-Spec}(R) \). However, under the hypothesis that \( R \) is strongly \( w \)-divisorial, we have a satisfying result.

**Theorem 2.4.** Let \( R \) be a strongly \( w \)-divisorial domain and \( \Lambda \) a set of pairwise incomparable \( t \)-primes of \( R \). The following conditions are equivalent:

1. \( R_{\mathcal{F}(\Lambda)} \) is \( w \)-divisorial;
2. \( R_{\mathcal{F}(\Lambda)} \) is strongly \( w \)-divisorial;
3. \( R_{\mathcal{F}(\Lambda)} \) is a \( t \)-flat weakly Matlis domain;
4. \( R_{\mathcal{F}(\Lambda)} \) is a \( t \)-flat TV-domain;
5. \( \Lambda \) is independent of finite character.

**Proof.** Set \( \mathcal{F} = \mathcal{F}(\Lambda) \) and \( T = R_{\mathcal{F}} \). Since \( R \) is strongly \( w \)-divisorial, each \( P \in \Lambda \) \( t \)-localizes.

\( (1) \Rightarrow (5) \) by Proposition 2.2(2).

\( (5) \Rightarrow (3) \). By Proposition 2.2(1).

\( (3) \Rightarrow (2) \). If \( Q \) is a \( t \)-prime of \( T \), then \( P = Q \cap R \in t\text{-Spec}(R) \) and \( T_Q = R_P \) is divisorial. Whence \( T \) is strongly \( w \)-divisorial.

\( (3) \Leftrightarrow (4) \) By \( t \)-flatness, \( T_M \) is divisorial for each \( t \)-maximal ideal \( M \). Thus we can apply Theorem 1.5.

\( (2) \Rightarrow (1) \) is obvious. \( \square \)

Divisorial flat overrings of a strongly divisorial domain have a similar characterization. Recall that an overring \( T \) of \( R \) is flat if \( T_M = R_M \cap R \), for each maximal ideal \( M \) of \( T \); in this case \( T = R_{\mathcal{F}(\Lambda)} \), where \( \Lambda \) is a set of pairwise incomparable prime ideals of \( R \).

**Corollary 2.5.** Let \( R \) be a strongly divisorial domain and \( T = R_{\mathcal{F}(\Lambda)} \) a flat overring, where \( \Lambda \) is a set of pairwise incomparable prime ideals of \( R \). The following conditions are equivalent:

1. \( T \) is divisorial;
(2) $T$ is strongly divisorial;
(3) $T$ is $h$-local;
(4) $\Lambda$ is independent of finite character.

Proof. (1) $\Leftrightarrow$ (3). By [3, Proposition 5.4], $T$ is divisorial if and only if it is $h$-local and locally divisorial. But, since $T$ is flat and $R$ is strongly divisorial, for each maximal ideal $M$ of $T$, $T_M = R_{M\cap R}$ is divisorial, for each prime ideal $Q$ of $T$.

(1) $\Rightarrow$ (2). Since $T$ is flat and $R$ is strongly divisorial, then $T_Q = R_{Q\cap R}$ is divisorial, for each prime ideal $Q$ of $T$.

(2) $\Rightarrow$ (4). Since $R$ and $T$ are divisorial, then $d = w = t = v$ in $R$ and $T$. Thus we can apply Theorem 2.4 ((2) $\Rightarrow$ (5)).

(4) $\Rightarrow$ (1). Since $d = w = t = v$ in $R$, by Theorem 2.4 ((5) $\Rightarrow$ (1)), $T$ is $w$-divisorial. To prove that $T$ is divisorial, we show that each maximal ideal of $T$ is a $t$-ideal (Proposition 1.1). If $M$ is a maximal ideal of $T$, by flatness we have $T_M = R_{M\cap R}$. Since $R$ is strongly divisorial, $MT_M$ is a $t$-ideal and so $M = MT_M \cap T$ is a $t$-ideal.

Corollary 2.6. Let $R$ be an integral domain. The following conditions are equivalent:

(1) Each $t$-flat overring of $R$ is strongly $w$-divisorial;
(2) $R$ is strongly $w$-divisorial and each $t$-flat overring is a weakly Matlis domain;
(3) $R$ is strongly $w$-divisorial and each $t$-flat overring is a $TV$-domain;
(4) $R$ is strongly $w$-divisorial and each family $\Lambda$ of pairwise incomparable $t$-primes of $R$ such that $F(\Lambda)$ is $v$-finite is independent of finite character.

Proof. By Theorem 2.4 recalling that an overring $T$ is $t$-flat over $R$ if and only if $T = R_{F(\Lambda)}$, where $\Lambda$ is a family of pairwise incomparable $t$-primes of $R$ and $F(\Lambda)$ is $v$-finite.

In order to study $t$-subintersections, we need the following technical lemma.

Lemma 2.7. Let $R$ be an integral domain and $\mathcal{C}$ an ascending chain of $t$-localizing $t$-primes of $R$. If $R_{F(\mathcal{C})}$ is a $TV$-domain, then $\mathcal{C}$ is stationary.

Proof. Let $\mathcal{C} = \{P_\alpha\}$ and set $\mathcal{F} = F(\mathcal{C})$ and $T = R_\mathcal{F}$. By Lemma 2.1(1), $(P_\alpha)_T$ is a $t$-prime ideal of $T$, for each $\alpha$. It follows that $M = \cup_\alpha (P_\alpha)_T$ is a proper $t$-prime ideal of $T$ (since it is an ascending union of $t$-primes) and so $M$ is divisorial (because $T$ is a $TV$-domain). We have $T = \cap_\alpha T_{R\setminus P_\alpha}$; thus the map $I \mapsto I^* = \cap_\alpha JT_{R\setminus P_\alpha}$ defines a star operation on $T$. Since $M$ is divisorial, we have $M^* \subseteq M$; so that $M^*$ is a proper ideal. It follows that there exists $\alpha$ such that $M \cap R \subseteq P_\alpha$. Hence $M \cap R = P_\alpha$ and so $P_\beta = P_\alpha$ for $\beta \geq \alpha$.

Theorem 2.8. Let $R$ be an integral domain. The following conditions are equivalent:

(1) Each $t$-subintersection of $R$ is strongly $w$-divisorial;
(2) $R$ is a strongly $w$-divisorial domain which satisfies the ascending chain condition on $t$-prime ideals and each family $\Lambda$ of pairwise incomparable $t$-primes of $R$ is independent of finite character.

Proof. (1) $\Rightarrow$ (2). Clearly $R$ is a strongly $w$-divisorial domain. If $\Lambda$ is a set of pairwise incomparable $t$-prime ideals, then by assumption $R_{F(\Lambda)}$ is strongly $w$-divisorial. Hence $\Lambda$ is independent of finite character, by Theorem 2.4. It remains to show that $R$ has the ascending chain condition on $t$-prime ideals. This follows from Lemma 2.7. In fact, if $\mathcal{C}$ is an ascending chain of $t$-prime ideals of $R$, $R_{F(\mathcal{C})}$ is strongly $w$-divisorial. Hence each $t$-prime in $\mathcal{C}$ $t$-localizes and it follows that $\mathcal{C}$ is stationary.
(2) ⇒ (1). Let $R_{F(\Lambda)}$ be a $t$-subintersection of $R$. By the ascending chain condition on $t$-prime ideals, $\Lambda$ has maximal elements; thus we can assume that $\Lambda$ is a set of pairwise incomparable $t$-primes. The conclusion follows from Theorem 2.4. □

**Corollary 2.9.** Let $R$ be a domain. If each $t$-subintersection of $R$ is strongly $w$-divisorial, then each $t$-subintersection of $R$ is $t$-flat.

*Proof.* If each $t$-subintersection of $R$ is strongly $w$-divisorial, then $R$ satisfies the ascending chain condition on $t$-primes (Theorem 2.8). Thus each $t$-subintersection is of type $R_{F(\Lambda)}$, where $\Lambda$ is a family of pairwise incomparable $t$-primes. By Theorem 2.4, $R_{F(\Lambda)}$ is $t$-flat. □

**Remark 2.10.** If each subintersection of the domain $R$ is strongly divisorial, then clearly $R$ is strongly divisorial. In addition, since $d = w = t = v$ on $R$, then $R$ satisfies the ascending chain condition on prime ideals and each family $\Lambda$ of pairwise incomparable prime ideals of $R$ is independent of finite character (Theorem 2.8).

Conversely, assume that $R$ is a strongly divisorial domain satisfying the ascending chain condition on prime ideals and each family $\Lambda$ of pairwise incomparable prime ideals of $R$ is independent of finite character.

Then each subintersection $T$ of $R$ is of type $R_{F(\Lambda)}$, where $\Lambda$ is a family of pairwise incomparable prime ideals independent of finite character. Thus $F(\Lambda)$ is finitely generated $[10$, Lemma 1.16$]$ and $T$ is strongly $w$-divisorial and $t$-flat by Theorem 2.4. We conclude that $T$ is (strongly) divisorial if and only if each maximal ideal of $T$ is a $t$-ideal (Proposition 1.1) if and only if $T$ is flat.

We observe that in general, if $F$ is a finitely generated multiplicative system of ideals, then $R_F$ need not be a flat extension of $R$ $[11$, pag. 32$]$. On the other hand, we do not know any example of a strongly divisorial domain $R$ with a finitely generated multiplicative system $F$ such that $R_F$ is not flat.

If $R$ is any domain, we say that $\text{Spec}(R)$ (resp., $t\text{-Spec}(R)$) is treed (under inclusion) if any maximal (resp., $t$-maximal) ideal of $R$ cannot contain two incomparable primes (resp., $t$-primes). The Spectrum of a Prüfer domain and the $t$-Spectrum of a $PvMD$ are treed. If $\text{Spec}(R)$ is treed, then $\text{Spec}(R) = t\text{-Spec}(R)$ $[23$, Proposition 2.6$]$; in particular each maximal ideal is a $t$-ideal and so $w$-divisoriality coincides with divisoriality by Proposition 1.1.

If $t\text{-Spec}(R)$ is treed and $t\text{-Max}(R)$ is independent of finite character, then each family $\Lambda$ of pairwise incomparable $t$-prime ideals of $R$ is independent of finite character. Hence the next results are easy consequences of Theorem 2.4 and Theorem 2.8, respectively.

**Corollary 2.11.** Let $R$ be an integral domain such that $t\text{-Spec}(R)$ is treed. The following conditions are equivalent:

1. $R$ is strongly $w$-divisorial;
2. $R_{F(\Lambda)}$ is a $t$-flat $w$-divisorial domain, for each set $\Lambda$ of pairwise incomparable $t$-primes;
3. $R_{F(\Lambda)}$ is a $t$-flat strongly $w$-divisorial domain, for each set $\Lambda$ of pairwise incomparable $t$-primes.

If $R$ has $t$-dimension one, then clearly $t\text{-Spec}(R)$ is treed. In this case, The conditions stated in Corollary 2.11 are all satisfied if $R$ is $w$-divisorial (cf. Theorem 2.8).

**Corollary 2.12.** Let $R$ be an integral domain such that $t\text{-Spec}(R)$ is treed. The following conditions are equivalent:
(1) $R$ is a strongly $w$-divisorial domain which satisfies the ascending chain conditions on $t$-prime ideals;
(2) Each $t$-subintersection of $R$ is $t$-flat and strongly $w$-divisorial.

3. INTEGRLY CLOSED $w$-DIVISORIAL DOMAINS

W. Heinzer proved in [17] that an integrally closed domain is divisorial if and only if it is an $h$-local Prüfer domain with invertible maximal ideals. We start this section by showing that integrally closed $w$-divisorial domains have a similar characterization among $PvMD$s. Note that a divisorial $PvMD$ is a Prüfer domain.

**Lemma 3.1.** Let $R$ be a $w$-divisorial domain and $M \in t\text{-Max}(R)$. The following conditions are equivalent:

(1) $M$ is $t$-invertible;
(2) $MR_M$ is a principal ideal;
(3) $R_M$ is a valuation domain.

Proof. (1) $\iff$ (2). Since $t\text{-Max}(R)$ has $t$-finite character (Theorem 1.5), we can apply [34, Theorem 2.2 and Proposition 3.1].

(2) $\implies$ (3) follows from [31, Lemme 1, Section 4], because $R_M$ is a divisorial domain (Theorem 1.5), and (3) $\implies$ (2) follows from [17, Lemma 5.2].

**Proposition 3.2.** Let $R$ be a $w$-divisorial domain. Then $R$ is a $PvMD$ if and only if each $t$-maximal ideal of $R$ is $t$-invertible.

**Theorem 3.3.** Let $R$ be an integral domain. The following conditions are equivalent:

(1) $R$ is an integrally closed $w$-divisorial domain;
(2) $R$ is a weakly Matlis $PvMD$ and each $t$-maximal ideal of $R$ is $t$-invertible.

Proof. (1) $\implies$ (2). A domain $R$ is a $PvMD$ if and only if $R$ is an integrally closed $TW$-domain [22, Theorem 3.5]. Hence an integrally closed $w$-divisorial domain is a $PvMD$. By Theorem 1.5, $R$ is a weakly Matlis domain and by Proposition 3.2 each $t$-maximal ideal is $t$-invertible.

(2) $\implies$ (1). A $t$-maximal ideal $M$ of a $PvMD$ is $t$-invertible if and only if $MR_M$ is a principal ideal [19]. Since $R_M$ is a valuation domain, this means that $R_M$ is divisorial [17, Lemma 5.2]. Now we can apply Theorem 1.5.

The previous theorem can be proved also by using the fact that a domain $R$ is a $PvMD$ if and only if $R$ is an integrally closed $TW$-domain [22, Theorem 3.5] and the characterization of $PvMD$s which are $TV$-domains given in [20, Theorem 3.1].

Recall that a Prüfer domain $R$ is strongly discrete if $P^2 \neq P$ for each nonzero prime ideal $P$ of $R$ [8, Section 5.3] and that a generalized Dedekind domain is a strongly discrete Prüfer domain with the property that each ideal has finitely many minimal primes [30]. We say that a $PvMD$ $R$ is strongly discrete if $(P^2)_t \neq P$, for each $P \in t\text{-Spec}(R)$ [7, Remark 3.10]. If $R$ is a strongly discrete $PvMD$ and each $t$-ideal of $R$ has only finitely many minimal primes, then $R$ is called a generalized Krull domain [7].

The next theorem shows that the class of strongly $w$-divisorial domains and the class of strongly discrete $PvMD$s are strictly related to each other.

**Lemma 3.4.** Let $R$ be a domain. The following conditions are equivalent:

(1) $R$ is a strongly discrete $PvMD$;
(2) $R_M$ is a strongly discrete valuation domain, for each $M \in t\text{-Max}(R)$;
(3) $R_P$ is a strongly discrete valuation domain, for each $P \in t\text{-Spec}(R)$;
(4) $R_P$ is a valuation domain and $PR_P$ is a principal ideal, for each $P \in t\text{-Spec}(R)$;
Proof. (1) ⇔ (4). For each t-prime ideal \(P\) of \(R\), we have \((P^2)_t = P^2R_P \cap R\) [19 Proposition 1.3]. Hence \((P^2)_t \neq P\) if and only if \(P^2R_P \neq PR_P\). Now recall that a maximal ideal of a valuation domain is not idempotent if and only if it is principal.

(2) ⇔ (3) because each overring of a strongly discrete valuation domain is a strongly discrete valuation domain [8 Proposition 5.3.1(3)].

(3) ⇔ (4) by [17 Lemma 5.2].

□

Theorem 3.5. Let \(R\) be an integral domain. The following conditions are equivalent:

1. \(R\) is a strongly discrete \(P\) vMD and a weakly Matlis domain;
2. \(R\) is an integrally closed strongly \(w\)-divisorial domain;
3. \(R\) is integrally closed and each \(t\)-flat overring of \(R\) is \(w\)-divisorial;
4. \(R\) is integrally closed and each \(t\)-linked overring of \(R\) is \(w\)-divisorial;
5. \(R\) is a \(w\)-divisorial generalized Krull domain;
6. \(R\) is a generalized Krull domain and each \(t\)-prime ideal of \(R\) is contained in a unique \(t\)-maximal ideal.

Proof. (1) ⇒ (2). Clearly \(R\) is integrally closed. In addition, by Lemma 3.4, \(R_P\) is a divisorial domain, for each \(P \in t\)-Spec\((R)\). Hence \(R\) is a strongly \(w\)-divisorial domain.

(2) ⇒ (3) By Theorem 3.3, \(R\) is a \(P\) vMD; in particular \(t\)-Spec\((R)\) is treed. Thus we can apply Corollary 2.11.

(3) ⇒ (1). By Theorem 3.3, \(R\) is a weakly Matlis \(P\) vMD. Now, given \(P \in t\)-Spec\((R)\), \(R_P\) is a divisorial valuation domain. Hence \(R\) is a strongly discrete \(P\) vMD by Lemma 3.4.

(3) ⇔ (4). By Theorem 3.3 statements (3) and (4) imply that \(R\) is a \(P\) vMD. The conclusion now follows from the fact that each \(t\)-linked overring of a \(P\) vMD \(R\) is \(t\)-flat [23 Proposition 2.10].

(1) ⇒ (5). By (1)⇒(2), \(R\) is a \(w\)-divisorial domain. To show that \(R\) is a generalized Krull domain, let \(I\) be a \(t\)-ideal of \(R\). Since \(R\) has \(t\)-finite character, then \(I\) is contained in only finitely many \(t\)-maximal ideals. Furthermore, each \(t\)-prime ideal is contained in a unique \(t\)-maximal ideal. Thus \(I\) has just finitely many minimal \((t)\)-prime ideals. We conclude by using [7 Theorem 3.9].

(5) ⇒ (6) is clear.

(6) ⇒ (1). It is enough to show that \(R\) has \(t\)-finite character. This follows from the fact that each nonzero principal ideal has finitely many minimal \((t)\)-primes. □

As a consequence of Theorem 3.5, we obtain the following characterization of integrally closed totally divisorial domains (see also [25]).

Corollary 3.6. Let \(R\) be an integral domain. The following conditions are equivalent:

1. \(R\) is an integrally closed totally divisorial domain;
2. \(R\) is integrally closed and each flat overring of \(R\) is divisorial;
3. \(R\) is an integrally closed strongly divisorial domain;
4. \(R\) is an \(h\)-local strongly discrete Prüfer domain;
5. \(R\) is a divisorial generalized Dedekind domain;
6. \(R\) is a generalized Dedekind domain and each nonzero prime ideal is contained in a unique maximal ideal.

Proof. This follows from the fact that in a Prüfer domain the \(d\)- and \(t\)-operation coincide, that each overring of a Prüfer domain is a flat Prüfer domain, and that
a Prüfer domain is a generalized Krull domain if and only if it is a generalized Dedekind domain \[7\]. □

Recall that the complete integral closure of \( R \) is the overring \( \tilde{R} := \cup\{(I: I) ; I \text{ nonzero ideal of } R\} \). If \( R = \tilde{R} \), we say that \( R \) is completely integrally closed.

**Proposition 3.7.** Let \( R \) be an integral domain. The following conditions are equivalent:

1. \( R \) is an integrally closed \( w \)-divisorial domain of \( t \)-dimension one;
2. \( R \) is an integrally closed domain of \( t \)-dimension one and each \( t \)-linked overring of \( R \) is \( w \)-divisorial;
3. \( R \) is a completely integrally closed \( w \)-divisorial domain;
4. \( R \) is a Krull domain.

**Proof.** (1) \( \Leftrightarrow \) (2) \( \Leftrightarrow \) (4). Clearly a \( w \)-divisorial domain of \( t \)-dimension one is strongly \( w \)-divisorial. Since a generalized Krull domain of \( t \)-dimension one is a Krull domain [7, Theorem 3.11], we can conclude by applying Theorem 3.5.

(3) \( \Leftrightarrow \) (4) because a completely integrally closed \( TV \)-domain is Krull [20, Theorem 2.3]. □

It is well-known that a divisorial Krull domain is a Dedekind domain; hence by the previous proposition we recover that a completely integrally closed divisorial domain is a Dedekind domain [17, Proposition 5.5].

**Remark 3.8.** Recall that, for any domain \( R \), \( \tilde{R} \) is integrally closed and \( t \)-linked over \( R \) [5, Corollary 2.3]. Since each localization of a \( t \)-linked overring of \( R \) is still \( t \)-linked over \( R \), if each \( t \)-linked overring of \( R \) is \( w \)-divisorial, we have that \( \tilde{R} \) is an integrally closed strongly \( w \)-divisorial domain. In this case, by Theorem 3.5 \( \tilde{R} \) is a weakly Matlis strongly discrete \( P vMD \). If in addition \( \tilde{R} \) is completely integrally closed, for example if \((R: \tilde{R}) \neq 0\), by Proposition 3.7 \( \tilde{R} \) is a Krull domain.

In a similar way, by using Corollary 3.6 we see that if \( R \) is totally divisorial, the integral closure of \( R \) is an \( h \)-local strongly divisorial Prüfer domain.

### 4. Mori \( w \)-divisorial domains

We start by recalling some properties of Noetherian divisorial domains proved in [17, 31].

**Proposition 4.1.** Let \( R \) be a domain. The following conditions are equivalent:

1. \( R \) is a one-dimensional \( w \)-divisorial Mori domain;
2. \( R \) is a divisorial Mori domain;
3. \( R \) is a divisorial Noetherian domain;
4. \( R \) is a Mori domain and each two generated ideal of \( R \) is divisorial;
5. \( R \) is a one-dimensional Mori domain and \((R: M)\) is a two generated ideal, for each \( M \in \text{Max}(R)\);
6. \( R \) is a one-dimensional Noetherian domain and \((R: M)\) is a two generated ideal, for each \( M \in \text{Max}(R)\).

**Proof.** (1) \( \Rightarrow \) (2) by Proposition 1.1.

(2) \( \Rightarrow \) (3) because each \( v \)-ideal of a Mori domain is \( v \)-finite.

(3) \( \Rightarrow \) (1) because Noetherian divisorial domains are one-dimensional [17, Corollary 4.3].

(3) \( \Leftrightarrow \) (6) and (2) \( \Leftrightarrow \) (4) \( \Leftrightarrow \) (5) by [31, Theorem 3, Section 2]. □

An integrally closed \( w \)-divisorial Mori domain is a Krull domain. In fact it has to be a \( P vMD \) (Theorem 4.1). By Proposition 4.1 any Noetherian integrally closed
domain of dimension greater than one is a \( w \)-divisorial Noetherian domain that is not divisorial.

We say that a nonzero fractional ideal \( I \) of \( R \) is a \( w \)-divisorial ideal if \( I_w = I_w \).

With this notation, a \( w \)-divisorial domain is a domain in which each nonzero ideal is \( w \)-divisorial. We also say that, for \( n \geq 1 \), \( I \) is \( n \) \( w \)-generated if \( I_w = (a_1 R + \cdots + a_n R)_w \), for some \( a_1, \ldots, a_n \) in the quotient field of \( R \).

**Theorem 4.2.** Let \( R \) be a Mori domain. The following conditions are equivalent:

1. \( R \) is a \( w \)-divisorial domain;
2. Each two generated nonzero ideal is \( w \)-divisorial;
3. \( R \) has \( t \)-dimension one and \((R: M)\) is a two \( w \)-generated ideal, for each \( M \in t \- \text{Max}(R) \).

**Proof.** (1) \( \Rightarrow \) (2) is clear.

(2) \( \Rightarrow \) (3). Let \( M \in t \- \text{Max}(R) \). Since \( R \) is a Mori domain, then \( M \) is a divisorial ideal. Let \( x \in (R: M) \setminus R \), then \((R: M) = (R + Rx)_w \). So that by assumption \((R: M) = (R + Rx)_w \). To conclude, we show that \( R_M \) is one-dimensional. Let \( I \) be a nonzero two generated ideal of \( R_M \). Then, we can assume that \( I = (a, b) R_M \) for some \( a, b \in I \cap R \). Since \( R \) is a Mori domain, then \( I_v = ((a, b) R_M)_v = (a, b)_v R_M \). Hence \( I_v = (a, b)_w R_M = (a, b) R_M = I \). Thus each two generated ideal of \( R_M \) is divisorial. It follows from Proposition 1.1 that \( R_M \) is one-dimensional.

(3) \( \Rightarrow \) (1). Since \( R \) is a TV-domain, by Theorem 1.5, it is enough to show that \( R_M \) is a divisorial domain for each \( M \in t \- \text{Max}(R) \). This follows again from Proposition 1.1. In fact, by assumption \( R_M \) is a Mori domain of dimension one. Let \((R: M) = (a, b)_w \) for some \( a, b \in (R: M) \). Then \((R_M:MR_M) = (R: M)R_M = (a, b)_w R_M = (a, b) R_M \) is two generated (the first equality holds because \( M \) is \( v \)-finite). \( \square \)

Recall that a Strong Mori domain is a domain satisfying the ascending chain condition on \( w \)-ideals. A domain \( R \) is a Strong Mori domain if and only if it has \( t \)-finite character and \( R_M \) is Noetherian, for each \( t \) maximal ideal \( M \) [33, Theorem 1.9]. Thus a Mori domain is Strong Mori if and only if \( R_M \) is Noetherian, for each \( t \)-maximal ideal \( M \).

**Corollary 4.3.** [27, Corollary 2.5] A \( w \)-divisorial Mori domain is a Strong Mori domain of \( t \)-dimension one.

**Proof.** A \( w \)-divisorial Mori domain is Strong Mori (because \( w = v \)) and has \( t \)-dimension one by Theorem 4.2. \( \square \)

We next investigate \( w \)-divisoriality of overrings of Mori domains. Our first result in this direction shows that, if \( R \) is Mori, \( w \)-divisoriality is inherited by generalized ring of fractions. This improves [27, Theorem 2.4].

We observe that a Mori domain is a \( v \)-coherent TV-domain, because each \( t \)-ideal of a Mori domain is \( v \)-finite. We also recall that if \( R \) is \( v \)-coherent, we have \( I_t R_S = (IR_S)_t \), for each nonzero fractional ideal \( I \) and each multiplicative set \( S \).

**Proposition 4.4.** Let \( R \) be a \( v \)-coherent domain. The following conditions are equivalent:

1. \( R \) is a \( TV \)-domain;
2. All the nonzero ideals of \( R_M \) are \( t \)-ideals, for each \( M \in t \- \text{Max}(R) \);
3. All the nonzero ideals of \( R_P \) are \( t \)-ideals, for each \( P \in t \- \text{Spec}(R) \);
4. Each \( t \)-flat overring of \( R \) is a \( TV \)-domain.

**Proof.** (1) \( \Leftrightarrow \) (2). Let \( I \) be a nonzero ideal and \( M \) a \( t \)-maximal ideal of \( R \). If \( t = w \) on \( R \), then \( IR_M = I_w R_M = I_t R_M = (IR_M)_t \).
Conversely, we have $IR_M = (IR_M)_t = I_IR_M$. Thus

$$I_t = \cap_{M \in t\text{-Max}(R)} IR_M = \cap_{M \in t\text{-Max}(R)} I_IR_M = I_t.$$  

(2) $\Rightarrow$ (3). Let $I$ be a nonzero ideal of $R$, $P$ a $t$-prime of $R$ and $M$ a $t$-maximal ideal containing $P$. Then

$$IR_P = (IR_M)R_P = (IR_M)_tR_P = (I_IR_M)_tR_P = I_tR_P = (IR_P)_t.$$  

(3) $\Rightarrow$ (4). Let $T$ be a $t$-flat overring of $R$. Then $T$ is a $v$-coherent domain [10, Proposition 3.1]. If $N$ is a $t$-maximal ideal of $T$, then $P = N \cap R$ is a $t$-prime of $R$ and $T_N = R_P$. Hence, if (3) holds, each nonzero ideal of $T_N$ is a $t$-ideal and $T$ is a TW-domain by (2) $\Rightarrow$ (1).

(4) $\Rightarrow$ (1) is clear. 

\[\Box\]

**Theorem 4.5.** Let $R$ be a Mori domain. The following conditions are equivalent:

1. $R$ is $w$-divisorial;
2. $R$ is strongly $w$-divisorial;
3. Each $t$-flat overring of $R$ is $w$-divisorial;
4. Each generalized ring of fractions of $R$ is $w$-divisorial;
5. $R_M$ is a divisorial domain, for each $M \in t\text{-Max}(R)$.

**Proof.** Each generalized ring of fractions of a Mori domain is Mori [31, Corollaire 1, Section 3]; thus it is a TV-domain. In addition, each generalized ring of fractions of a Mori domain is $t$-flat, because each $t$-ideal is $v$-finite and so each multiplicative system of ideals is $v$-finite. Hence we can apply Proposition 4.4.

$t$-linked overrings of Mori domains do not behave as well as generalized rings of fractions. In fact a Mori non-Krull domain has $t$-linked overrings which are not $t$-flat [6, Corollary 2.10]. Also, if each $t$-linked overring of a Mori domain $R$ is Mori, then $R$ has $t$-dimension one [5, Proposition 2.20]. The converse holds if $R$ is a Strong Mori domain; precisely, we have the following result.

**Proposition 4.6.** Each $t$-linked overring of a Strong Mori domain of $t$-dimension one is either a field or a Strong Mori domain of $t$-dimension one.

**Proof.** It follows from [31, Theorem 3.4] recalling that an overring of a domain is a $w$-module if and only if it is $t$-linked [5, Proposition 2.13 (a)].

\[\Box\]

**Corollary 4.7.** If $R$ is a $w$-divisorial Mori domain, then each $t$-linked overring of $R$ is either a field or a Strong Mori domain of $t$-dimension one.

**Proof.** It follows from Corollary 4.3 and Proposition 4.6.

Our next purpose is to improve and generalize to Mori domains some results proved in [3] for Noetherian totally divisorial domains.

**Proposition 4.8.** Let $R$ be a domain. The following conditions are equivalent:

1. $R$ is a one-dimensional domain and each $t$-linked overring of $R$ is $w$-divisorial;
2. $R$ is a one-dimensional totally divisorial domain;
3. $R$ is a Noetherian totally divisorial domain;
4. Each ideal of $R$ is two generated.

**Proof.** (1) $\Rightarrow$ (2). Since $\dim(R) = 1$, each overring of $R$ is $t$-linked over $R$ [5, Corollary 2.7 (b)]. Hence each overring $T$ of $R$ is $w$-divisorial. Assume that $T$ is not a field. To prove that $T$ is divisorial it suffices to check that $\dim(T) = 1$ (Proposition 4.3). Let $R'$ be the integral closure of $R$ and $T'$ that of $T$. Since $R'$ is one-dimensional and $w$-divisorial, then $R'$ is divisorial. Thus $R'$, being integrally closed, is a Prüfer domain [17, Theorem 5.1]. It follows that the extension $R' \subseteq T'$
Lemma 4.10. Let there exists \( S \) nonzero ideal of \( R \) Then is divisorial if and only if it is totally reflexive, because in the Noetherian case a domain is totally divisorial if and only if it is reflexive. □

Lemma 4.9. Let be an integral domain, \( I \) an ideal of \( R \), \( P_1, \ldots, P_n \) a set of pairwise incomparable prime ideals and \( S = R \setminus (P_1 \cup \cdots \cup P_n) \). If \( x_1, \ldots, x_n \in I \), there exists \( x \in IR_S \) such that \( x \equiv x_i \pmod{IP_iR_{P_i}} \), for each \( i = 1, \ldots, n \).

Lemma 4.10. Let be an integral domain which has \( t \)-finite character and \( I \) a nonzero ideal of \( R \). Let \( n \) be a positive integer and assume that, for each \( M \in t\text{-Max}(R) \), a minimal set of generators of \( IR_M \) has at most \( n \) elements. Then \( I \) is \( w \)-generated by a number of generators \( m \leq \max(2, n) \).

Proof. If \( I \) is not contained in any \( t \)-maximal ideal, then \( I_w = R \). Otherwise, let \( M_1, \ldots, M_r \) be the \( t \)-maximal ideals of \( R \) which contain \( I \). For \( i = 1, \ldots, r \), let \( a_{i1}, \ldots, a_{in} \in I \) be such that \( IR_{M_i} = (a_{i1}, \ldots, a_{in})R_{M_i} \). By Lemma 4.10 if \( S = R \setminus (M_1 \cup \cdots \cup M_r) \), for each \( j = 1, \ldots, n \), there exists \( a_j \in IR_S \subseteq IR_{M_i} \) such that \( a_j = a_{ij} \pmod{IM_jR_{M_j}} \), for each \( i = 1, \ldots, r \). By going modulo \( IM_jR_{M_j} \) and using Nakayama’s Lemma, we get \( IR_{M_j} = (a_{1j}, \ldots, a_{nj})R_{M_j} \) for each \( i = 1, \ldots, r \). We can assume that the \( a_i \)'s are in \( I \) and \( a_1 \neq 0 \). Let \( N_1, \ldots, N_s \) be the set of \( t \)-maximal ideals which contain \( a_1 \), with \( N_1 = M_1, \ldots, N_r = M_r \). Let \( b_1 = a_1, b_2, \ldots, b_n \in I \) such that \( IR_{N_j} = (b_{1j}, \ldots, b_{nj})R_{N_j} \) for each \( j = 1, \ldots, s \). We claim that \( I_w = (b_{1j}, \ldots, b_{nj}) \). Let \( M \) be a \( t \)-maximal ideal of \( R \). If \( M = N_j \) for some \( j \), then \( IR_M = (b_{1j}, \ldots, b_{nj})R_{M_j} \). If \( M \neq N_j \) for \( j = 1, \ldots, s \), then \( IR_M = R_M = (b_{1j}, \ldots, b_{nj})R_{M_j} \), since \( b_1 = a_1 \notin M \).

Theorem 4.11. Let be a domain. The following conditions are equivalent:

(1) \( R \) has \( t \)-dimension one and each \( t \)-linked overring of \( R \) is \( w \)-divisorial;
(2) \( R \) is a Mori domain and each \( t \)-linked overring of \( R \) is \( w \)-divisorial;
(3) \( R \) is a Mori domain and \( R_M \) is totally divisorial, for each \( M \in t\text{-Max}(R) \);
(4) Each nonzero ideal of \( R \) is a \( w \)-generated \( w \)-divisorial ideal;
(5) Each nonzero ideal of \( R \) is \( w \)-generated.

Proof. (1) ⇒ (2). \( R \) has \( t \)-finite character, because it is \( w \)-divisorial. We now show that, for each \( M \in t\text{-Max}(R) \), \( R_M \) is Noetherian. Since \( R_M \) is a one-dimensional \( t \)-linked overring of \( R \), then \( R_M \) is divisorial. In addition, each overring \( T \) of \( R_M \) is \( t \)-linked over \( R_M \) and so it is \( t \)-linked over \( R \). Thus \( T \) is a \( w \)-divisorial domain. By Proposition 4.8, \( R_M \) is Noetherian. We conclude that \( R \) is a (Strong) Mori domain.

(2) ⇒ (3). \( R \) is clearly \( w \)-divisorial. Hence \( R_M \) is a one-dimensional Noetherian domain. Let \( T \) be a \( t \)-linked overring of \( R_M \). Hence \( T \) is \( t \)-linked over \( R \) and so by assumption it is \( w \)-divisorial. By Proposition 4.8, \( R_M \) is totally divisorial.

(3) ⇒ (4). \( R \) is \( w \)-divisorial by Theorem 4.5. Hence \( R_M \) is one-dimensional and Noetherian by Corollary 1.3. Let \( T \) be a \( t \)-linked overring of \( R_M \). Hence \( T \) is \( t \)-linked over \( R \) and so by assumption it is \( w \)-divisorial. By Proposition 4.8, \( R_M \) is totally divisorial.

(4) ⇒ (5). \( R \) is \( w \)-divisorial by Theorem 4.5. Hence \( R_M \) is one-dimensional and Noetherian by Corollary 1.3. Thus, for each \( M \in t\text{-Max}(R) \), each ideal of \( R_M \) is two generated by Proposition 4.8. By using Lemma 4.10, we conclude that every nonzero ideal of \( R \) is a \( w \)-generated \( w \)-divisorial ideal.
(4) ⇒ (5) is clear.
(5) ⇒ (3). If (5) holds, $R$ is a Strong Mori domain and so $R_M$ is a Noetherian domain, for each $M \in \text{t-Max}(R)$. Let $IR_M$ be a nonzero ideal of $R_M$, where $I$ is an ideal of $R$. By assumption, $I_w = (a,b)_w$ for some $a,b \in R$. Thus $IR_M = (a,b)_wR_M = (a,b)R_M$ is a two generated ideal. It follows from Proposition 4.8 that $R_M$ is a totally divisorial domain.

(3) ⇒ (2). $R$ is $w$-divisorial by Theorem 4.8. Let $T$ be a $t$-linked overring of $R$, $T \neq K$. By Corollary 4.7, $T$ is a Mori domain. To show that $T$ is $w$-divisorial, by Theorem 4.8 we have to prove that $T_N$ is a divisorial domain, for each $N \in t$-$\text{Max}(T)$. Since $R \subseteq T$ is $t$-linked, then $Q = (N \cap R)_t \neq R$ [25 Proposition 2.1]; but as $R$ has $t$-dimension one (Corollary 4.8), then $Q$ is a $t$-maximal ideal of $R$. Since $R_Q$ is totally divisorial and $R_Q \subseteq T_N$, then $T_N$ is a divisorial domain.

(2) ⇒ (1) by Corollary 4.8.

Corollary 4.12. Let $R$ be a domain and assume that each $t$-linked overring of $R$ is $w$-divisorial. Then $R$ is a Mori domain if and only if it has $t$-dimension one.

Example 4.13. Mori non-Krull and non-Noetherian domains satisfying the equivalent conditions of Theorem 4.11 can be constructed by using pullbacks, as the following example shows.

Let $T$ be a Krull domain having a maximal ideal $M$ of height one and assume that the residue field $K = T/M$ has a subfield $k$ such that $|K : k| = 2$. Let $R = \varphi^{-1}(k)$ be the pullback of $k$ with respect to the canonical projection $\varphi : T \rightarrow K$.

The domain $R$ is Mori and it is Noetherian if and only if $T$ is Noetherian [11, Theorems 4.12 and 4.18]. $M$ is a maximal ideal of $R$ that is divisorial; thus $M \in \text{t-Max}(R)$. Since $R_M$ is the pullback of $k$ with respect to the natural projection $T_M \rightarrow K$, $R_M$ is divisorial by [27 Corollary 3.5]. In addition $T_M$ is the only overring of $R_M$. In fact each overring of $R_M$ is comparable with $T_M$ under inclusion; but $T_M$ is a DVR and $[K : k] = 2$. Thus $R_M$ is totally divisorial.

If $N$ is a $t$-maximal ideal of $R$ and $N \neq M$, there is a unique $t$-maximal ideal $N' \in T$ such that $N' \cap R = N$ [12 Theorem 2.6(1)] and for this prime $T_N = R_N$. Thus $R_N$ is a DVR. It follows that $R_N$ is totally divisorial, for each $N \in \text{t-Max}(R)$.

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