An LMI Approach to Stability Analysis of PWM DC-DC Buck Converters

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Abstract

The paper considers a DC-DC buck power converter employing pulse width modulation and voltage feedback control. Global asymptotical stability of a periodical operating mode is examined. It is shown that the stability analysis can be reduced to study of a feasibility problem of semidefinite programming.

Keywords: Pulse width modulation, Switched power converters, Global asymptotical stability, Linear matrix inequalities.

1 Introduction

DC-DC buck (step down) power converters serve to convert a direct current (DC) voltage level to a lower DC voltage level [1]. One of the most popular types of power converters uses pulse width modulation (PWM). A pulse width modulator produces a train of square pulses with a constant frequency and with a variable duration. By controlling the duration one can control the output voltage. Last decade DC-DC PWM converters attracted much attention not only from engineers, but also from mathematicians and physicists. Their study lies in the mainstream of common interest to discontinuous and hybrid dynamical systems.

PWM converters may be described by nonlinear functional-differential equations, their operating modes correspond to periodic solutions. Many publications were devoted to the existence of periodic or quasi-periodic solutions, to the analysis of local stability of such solutions, to bifurcations and ways of chaotization (see, e.g., [2–6]). However, most of investigations do not concern global behavior of solutions, i.e., a behavior for all possible initial conditions.

There are two basic classes of mathematical models used for analysis and design of PWM power converters: discrete time models [7–10] and models based on averaging in time of the modulator’s output or of the state space vector [11–15]. The applicability of the averaging technique suggests that the switching frequency is high when compared to the frequencies of continuous-time signals.

In this paper we use another averaging approach, initiated by the work [16] and further developed in [17]. It is important to emphasize that our approach is not asymptotical or approximate, unlike the other averaging technique. However, it gives conservative estimates if the switching frequency is not high enough. Our method is based on ideas of the absolute stability theory [18] and leads to a system of linear matrix inequalities.
This work is a natural extension of the LMI technique proposed in [20]. Some other methods for stability study of switched power converters via LMIs can be found in [21, 22].

The main purpose of this paper is to provide a new computationally tractable procedure for stability investigations of PWM buck converters. It can be easily implemented with the help of a recently developed software for MATLAB and Scilab modelling systems, such as LMILab [23], LMITOOL [24] or SeDuMi solver [25] interfaced with YALMIP [26] or cvx [27].

2 Preliminaries

2.1 Power Stage

The part of a converter without a control loop is called a power stage. A conventional power stage of a buck converter is shown in Fig. 1 (see, e.g., [1]). Here $V_s$ is an input voltage, $S$ is an on-off switch, $L$ is an inductance, $C_0$ is a capacitance and $R$ is a load resistance. Let $U$ be a capacitor voltage (which is the output voltage at the same time) and $i_L$ be an inductor current. Then we get

$$\frac{dU}{dt} = \frac{1}{C_0} i_L - \frac{1}{RC_0} U,$$

$$\frac{di_L}{dt} = -\frac{1}{L} U + \frac{1}{L} V_s,$$

when the switch is closed and

$$\frac{dU}{dt} = \frac{1}{C_0} i_L - \frac{1}{RC_0} U,$$

$$\frac{di_L}{dt} = -\frac{1}{L} U,$$

when the switch is open.

The switch is on with some constant switching period $T$. Let $\tau$ be an on-time of the switch, i.e., a time when the switch is closed during the period. The value $D = \tau/T$ is called duty ratio.

The above described power stage consists of three parts — a voltage source, a low-pass filter and a load. The low-pass filter is designed to remove higher harmonics and pass only a constant component of the output voltage. However, in fact the output voltage has a small ripple. Neglecting this ripple, one obtains

$$U \approx \frac{T}{T} V_s = DV_s, \quad 0 \leq D \leq 1,$$

so the input voltage steps down.

In general, the power stage of a buck converter can be described with a system of equations

$$\frac{dx_p}{dt} = A_p x_p + B_p f, \quad U = C_p x_p$$

(3)

with

$$f(t) = \begin{cases} 1, & nT \leq t < nT + \tau, \\ 0, & nT + \tau \leq t < (n+1)T, \end{cases} \quad n = 0, 1, \ldots$$

(4)
The signal $f(t)$ is a train of square single-sign pulses of width $\tau$ and of period $T$. The matrix $A_p$ is square, $B_p$ is a column and $C_p$ is a row. The matrix $A_p$ is Hurwitz stable, i.e., all its eigenvalues have negative real parts.

E.g., for equations (1), (2) we have

$$A_p = \begin{bmatrix} 0 & -1/L \\ 1/C_0 & -1/RC_0 \end{bmatrix}, \quad B_p = \begin{bmatrix} V_s/L \\ 0 \end{bmatrix}, \quad C_p = [0 \ 1], \quad x_p = \begin{bmatrix} i_L \\ U \end{bmatrix}.$$  

Equations (3), (4) are the equations of an open loop system (without a control feedback). In practice, the pulse width $\tau$ is varied to control the output voltage $U$. Notice that equations (3), (4) apply to so called continuous conduction mode, which suggests that the inductance $L$ is sufficiently large.

### 2.2 Pulse-Width Modulation

In DC–DC converters a single sign modulation is commonly employed, i.e. a modulator’s output takes only two values, 0 and 1.

In this paper we will limit ourselves to a specific type of pulse-width modulation called natural or running modulation. The output of a modulator is

$$f(t) = \begin{cases} 1, & nT \leq t < nT + \tau_n, \\ 0, & nT + \tau_n \leq t < (n+1)T, \end{cases} \quad n = 0, 1, \ldots$$  

An input signal $\sigma(t)$ is compared with a sawtooth oscillating signal (a ramp)

$$\sigma_r(t) = \sigma_1 + \sigma_s(t - nT)/T, \quad nT \leq t < (n+1)T.$$  

Here $\sigma_s, \sigma_1$ are given positive parameters. A switching instant is determined from the relationship $\sigma(t) = \sigma_r(t)$ (see Fig. 2).

More accurately, the modulation law can be described as follows (see, e.g., [5]). If $\sigma(nT) \leq \sigma_1$, then $\tau_n = 0$. If $\sigma(nT + \tau) > \sigma_1 + \sigma_s \tau/T$, $0 \leq \tau \leq T$, then $\tau_n = T$. In all other cases $\tau_n$ is the smallest value $\tau \in [0, T]$ to satisfy the equation

$$\sigma(nT + \tau) = \sigma_1 + \sigma_s \tau/T.$$  

Figure 1: Equivalent circuit for a basic power stage
All the other roots of equation (7) are ignored. Such type of modulation is sometimes called modulation with a latch. In this case every switching interval contains at most one pulse, so such unpleasant effect as chattering cannot take place.

Besides the above scheme, where the trailing edge of a pulse is modulated, there are other modulation laws with modulation of the front edge or of both edges [28].

2.3 Control Loop

Here we consider a buck converter with a voltage feedback control. In most cases a closed loop system with a voltage mode control can be described by the equations

$$\frac{dx}{dt} = Ax + Bf + q, \quad \sigma = Cx + \psi, \quad f = M\sigma.$$  \hspace{1cm} (8)

Here $A$ is a constant square matrix, $B$ and $q$ are columns, $C$ is a row, and $\psi$ is a scalar. An operator $M$ describes a pulse modulator, so functions $\sigma(t)$ and $f(t)$ are the input and the output of the modulator, respectively. Suppose $A$ to be Hurwitz stable, i.e., all of its eigenvalues lie in the open left half-plane.

In the simplest case the control signal is defined by the formula

$$\sigma(t) = a(V_{ref} - U(t)).$$ \hspace{1cm} (9)

Here $V_{ref}$ is a constant reference signal and $a$ is a gain. Combine (3) and (9) to obtain (8) with

$$A = A_p, \quad B = B_p, \quad C = -aC_p, \quad \psi = aV_{ref}, \quad q = 0.$$  

In a more general case two additional linear circuits are added to the control scheme, as shown in Fig. 3 (see, e.g., [1]).
Let state space realizations of the compensator and of the sensor gain be

\[ \frac{dx_c}{dt} = A_c x_c + B_c \xi, \quad \sigma = C_c x_c + D_c \xi \]

(10) and

\[ \frac{dx_s}{dt} = A_s x_s + B_s U, \quad \eta = C_s x_s + D_s U, \]

(11) respectively, and

\[ \xi = V_{ref} - \eta. \]

(12)

Combining (10), (11) and (12) one obtains (8) with

\[
A = \begin{bmatrix}
A_p & 0 & 0 \\
B_s C_p & A_s & 0 \\
-B_c D_s C_p & -B_c C_s & A_c
\end{bmatrix},
\quad B = \begin{bmatrix}
B_p \\
0 \\
0
\end{bmatrix},
\quad q = \begin{bmatrix}
0 \\
0 \\
B_c V_{ref}
\end{bmatrix},
\]

\[
C = [-D_c D_s C_p, -D_c C_s, C_c],
\quad \psi = D_c V_{ref}.
\]

With the help of the change of variables \( \bar{x} = x + A^{-1}q \), equations (8) are transformed into the equations

\[ \frac{d\bar{x}}{dt} = A\bar{x} + Bf, \quad \sigma = C\bar{x} + \psi, \quad f = M\sigma \]

(13)

with \( \psi = \psi_0 - CA^{-1}q \).

### 3 Operating Modes

Operating modes of a converter correspond to periodical solutions of the closed loop system. As a rule, such solution is either \( T \)-periodical, or has a period multiple to \( T \).

The methods of finding periodical modes of a PWM system are well known. By straightforward calculations, one can obtain an explicit formula for the response of the linear part of a system to a square pulse signal. Taking into account periodicity one comes to a system of transcendental equations with respect to a pulse duration, which can be solved by numerical methods (see, e.g., [5, 17]).
Another way to find a periodical solution is to apply the harmonic balance method [12]. Certainly, in this case the linear part of the system needs to have good filtering properties.

In this work we will limit our considerations to $T$-periodical solutions, i.e., to solutions whose period coincides with the switching period. In addition, we require these solutions to be unsaturated, i.e., the cases $f(t) \equiv 0$ or $f(t) \equiv 1$ are excluded.

Let us formulate necessary and sufficient conditions for the existence of a $T$-periodical solution of (13). Define for any pair $(\tau, t)$, $0 \leq \tau \leq T$, $0 \leq t \leq T$, a function

$$\hat{\sigma}(\tau, t) = C e^{A(t-\tau)} \hat{x}(\tau) + \psi + C \left[ (e^{A(t-\tau)} - I) A^{-1} B \right]$$

(14)

with

$$\hat{x}(\tau) = - \left( I - e^{-A\tau} \right)^{-1} \left( I - e^{-A\tau} A^{-1} B \right).$$

(15)

Here $I$ is the identity matrix.

**Proposition 3.1** System (13) has an unsaturated $T$-periodical solution with a pulse duration $\tau$ if and only if the following conditions are satisfied:

$$0 < \tau < T; \quad \hat{\sigma}(\tau, \tau) = \Phi(\tau); \quad \hat{\sigma}(\tau, t) > \Phi(t) \quad \text{for all} \quad 0 \leq t < \tau.$$

Here $\Phi(t) = \sigma_1 + \sigma_* t/T$.

For this solution

$$\sigma(t) = \hat{\sigma}(\tau, t), \quad 0 \leq t \leq \tau,$n

$$\sigma(t) = \hat{\sigma}(\tau, t) + C \left[ I - e^{A(t-\tau)} \right] A^{-1} B, \quad \tau \leq t \leq T.$$

Proposition 3.1 is proved readily by a direct computation. The next theorem provides computationally tractable sufficient conditions for the existence of an unsaturated $T$-periodical solution.

**Theorem 3.1** Consider system (13). Suppose that the matrix $A$ is Hurwitz stable and the inequality

$$\sigma_1 < \psi < \sigma_1 + \sigma_* + CA^{-1}B$$

(16)

is valid. Let there exist a number $\varepsilon > 0$ and a symmetrical matrix $P$ such that the matrix inequalities

$$P(A^T + \varepsilon I) + (A + \varepsilon I)P \preceq \frac{1}{2\varepsilon} BB^T,$$

$$P \succ 0, \quad CAPA^T C^T \prec \gamma^2$$

(17)

are satisfied.\(^1\) Here $\gamma = \sigma_*/T - \min\{0, CB\}$. Then system (13) has an unsaturated $T$-periodical mode.

The first inequality (17) is not linear in the assembly of variables $P$ and $\varepsilon$ (it is bilinear in these variables). However, if $\varepsilon$ is fixed, we get an LMI in $P$. So, (17) can be checked by varying $\varepsilon$ in a loop. Obviously, the first inequality of (17) implies that the eigenvalues of the matrix $A + \varepsilon I$ have non-positive real parts. Thus $\varepsilon$ can be varied from zero to $-\text{Re} \lambda_1$, where $\lambda_1$ is an eigenvalue of $A$ with the minimal absolute value of a real part.

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\(^1\)The character $\prec (\succeq)$ denotes negative definiteness (negative semi-definiteness). The character $\succ (\preceq)$ denotes positive definiteness (positive semi-definiteness). The symbol $\top$ denotes transpose.
4 Stability Conditions

The following statement is the main result of this paper.

**Theorem 4.1** Assume that there exists a \( T \)-periodical solution \( x^0(t), \sigma^0(t) \) of (8) and an estimate

\[
\left| \frac{d\sigma^0(t)}{dt} \right| \leq L_1
\]

(18)

is valid, where \( L_1 \) is a positive constant. Let \( \tau^0 \) be a pulse duration for this solution.

Let there exist a symmetrical \( m \times m \) matrix \( H \) and scalars \( \varepsilon, \nu \) such that a system of linear matrix inequalities

\[
L(H) \prec R(\varepsilon, \nu), \quad H > 0, \quad \nu > 0, \quad \varepsilon > 0
\]

(19)

is satisfied. Here \( L, R \) are square symmetrical matrices,

\[
L(H) = \begin{bmatrix}
HA + A^\top H & HAB & HB & 0 \\
B^\top A^\top H & 0 & 0 & 0 \\
B^\top H & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix},
\]

\[
R(\varepsilon, \nu) = \begin{bmatrix}
0 & 0 & -\frac{1}{2} C^\top & \kappa A^\top C^\top \\
0 & 3\nu/T^2 & 0 & -\kappa_2 \kappa_1 \\
-\frac{1}{2} C & 0 & \sigma_* - TL_1 - \nu - \varepsilon - T|\kappa| & -\kappa_2 \kappa \\
\kappa_2 C A & -\kappa_2 \kappa_1 & -\kappa_2 \kappa & \varepsilon
\end{bmatrix}
\]

with \( \kappa = -CB, \kappa_1 = -CAB, \kappa_2 = T/\pi \).

Then for any solution \( x(t) \) of (8) the limit relationships

\[
\lim_{t \to +\infty} [x(t) - x^0(t)] = 0, \quad \lim_{n \to \infty} \tau_n = \tau^0
\]

(20)

are satisfied. Moreover, the solution \( x^0(t) \) is stable in the sense of Lyapunov.

Inequalities (19) can be easily verified with the help of standard software packages that are suitable for study an LMI feasibility problem [19].

5 Numerical Example

Consider a converter with a power stage described by (1), (2), where \( R = 22 \Omega, C_0 = 47 \mu F, L = 20 mH, T = 400 \mu s, V_s = 20 V \). Let a control be described by (9) with \( a = 1, V_{ref} = 13.5 V, \sigma_1 = 4 V \). The parameter \( \sigma_* \) will be chosen later. Then we have \( \psi = 13.5, q = 0 \).
Conditions of Theorem 3.1 are satisfied for

\[ \sigma_s \geq 12.83 \text{V}. \]  

(21)

Thus, under condition (21) the system has an unsaturated \( T \)-periodical mode.

Apply MATLAB to get a response of the power stage to the pulse signal with a fixed duty ratio \( \tau/T \). The direct modelling gives the following results:

| \( \tau/T \) | 0.1 | 0.3 | 0.5 | 0.7 | 0.9 |
|--------------|-----|-----|-----|-----|-----|
| \( TL_1 \)   | 0.15| 0.35| 0.44| 0.35| 0.15|

Thus \( TL_1 = 0.44 \) can be chosen for the worst case.

Applying Theorem 3.1 and the standard LMILab package for MATLAB gives

\[ \sigma_s \geq 17.78 \text{V}. \]  

(22)

Obviously, (21) follows from (22). In result, under condition (22) the system has a globally stable \( T \)-periodical mode.

If we neglect the ripple of the output voltage, the following approximate formula can be easily obtained:

\[ U^0 \approx V_s \frac{aV_{ref} - \sigma_1}{\sigma_s + aV_s}. \]

Choose \( \sigma_s = 18 \text{V} \), which satisfies (22). Then \( U^0 = 5 \text{V} \), so the converter steps the voltage down from 20 V to 5 V.

6 Proofs of the Theorems

A Useful Lemma. We precede the proof of Theorem 3.1 with a lemma, which is a version of a statement proved in [29].

**Lemma 6.1** Suppose that \( A \) is a square matrix, \( B \) is a column and \( \varepsilon \) is a positive scalar. Then the inequality

\[ H(A + \varepsilon I) + (A^\top + \varepsilon I)H \preceq -\frac{1}{2\varepsilon} HBB^\top H \]  

(23)

is equivalent to the relationship

\[ x^\top H(Ax + Bf) + \varepsilon(x^\top Hx - 1) \leq 0, \quad \forall x, \forall f, \quad |f| \leq 1. \]  

(24)

Since

\[ 2x^\top HBF - 2\varepsilon - \frac{1}{2\varepsilon}(x^\top HB)^2 f^2 = -\left( \frac{1}{\sqrt{2\varepsilon}} x^\top HBF - \sqrt{2\varepsilon} \right)^2, \quad \forall x, \forall f, \]

(24) follows from (23).
Let (24) be satisfied. Obviously, (24) preserves if one replaces $x$ for a vector $\alpha x$, where $\alpha$ is an arbitrary scalar. Let us fix arbitrary $x$ and $f$. Then (24) implies
\[
\alpha^2 x^\top H(A\varepsilon I) + \alpha x^\top H B f - \varepsilon \leq 0, \quad \forall \alpha.
\]
The latter relationship is satisfied for all $\alpha$ if and only if
\[
(x^\top HB f)^2 + 4\varepsilon x^\top H(A + \varepsilon I)x \leq 0.
\]
Since $|f| \leq 1$, we come to (23).

**Proof of Theorem 3.1:** The proof generally follows the proof of Theorem 6.2 [17]. Let us check that under condition (16) system (13) has no saturated periodical solutions (equilibria). Firstly, suppose that (13) has a solution with $f(t) \equiv 0$, that is $\tau = 0$. From the modulation law this means that $\sigma(0) \leq \sigma_1$. Hence, from (13) one concludes that $\psi \leq \sigma_1$. This contradicts the first inequality (16). Similarly, let $f(t) \equiv 1$, that is $\tau = T$. From (14), (15) it follows that $\tilde{\sigma}(t) = \tilde{\sigma}(T, t) = \psi - CA^{-1}B$. Then (6), (7) imply $\psi - CA^{-1}B \geq \sigma_1 + \sigma_*$, which contradicts the second inequality (16). Thus system (13) cannot have an equilibrium.

Let us prove the existence of a $T$-periodical solution under condition (17). Define $H = P^{-1}$. Then (17) imply $H \succ 0$, (23), and
\[
CAH^{-1}A^\top C^\top < \gamma^2. \tag{25}
\]
Firstly, consider the case, when $CB \leq 0$ and $\gamma = \sigma_*/T - CB$. Let us take an ellipsoid
\[
E = \{ x : x^\top Hx \leq 1 \}
\]
and define a Lyapunov function $V(x) = x^\top Hx$. From Lemma 6.1 it follows that
\[
\frac{dV}{dt} + 2\varepsilon (V - 1) \leq 0
\]
along the solutions of (13). Thus if a solution starts in $E$, it remains in $E$ for all subsequent times. Since
\[
\max_{x \in E} CAx = (CAH^{-1}A^\top C^\top)^{1/2},
\]
from inequality (25) it follows that
\[
\max_{x \in E} CAx < \sigma_*/T - CB. \tag{26}
\]

Consider the operator of translation along the trajectories of (13):
\[
S : \tilde{x}(0) \mapsto \tilde{x}(T).
\]
As it was shown above, $SE \subset E$. Let us prove that $S$ is continuous on $E$. It is easily seen that $S$ is continuous provided that the pulse width $\tau_0$ depends continuously on $\tilde{x}(0)$. Since $\tau_0$ is the minimal positive root of equation (7) with $n = 0$, it suffices to require
\[
\frac{d\sigma(\tau_0 - 0)}{d\tau} \neq \frac{\sigma_*}{T}.
\]
The last inequality can be rewritten as

\[ CA\tilde{x}(\tau_0) + CB \neq \sigma_\ast/T. \]

Thus continuity of \( S \) follows from (26).

We have found that the operator \( S \) is continuous on a closed, bounded and convex set. Hence, it has a fixed point, which corresponds to a \( T \)-periodical solution [30].

Let us turn to the case, when \( CB > 0 \) and \( \gamma = \sigma_\ast/T \). Consider an ellipsoid

\[ E_0 = \{ x : (x + A^{-1}B)^\top H(x + A^{-1}B) \leq 1 \} \]

and a Lyapunov function \( V_0(x) = (x + A^{-1}B)^\top H(x + A^{-1}B) \). Obviously,

\[ \frac{dV_0}{dt} = 2(x + A^{-1}B)^\top H[A(x + A^{-1}B) + B(f - 1)] \]

along the solutions of (13). Arguing as above, one obtains that the ellipsoid \( E_0 \) is invariant under the translation \( S \) and \( S \) is continuous on \( E_0 \). Hence, it has a fixed point on this ellipsoid.

**Proof of Theorem 4.1:** The proof is based on the proofs of Theorems 3.3, 4.7 [17] and follows the averaging scheme proposed by A. Kh. Gelig in [16]. The idea of the approach is that the state vector \( \tilde{x}(t) \) varies much slower than the modulator’s output \( f(t) \), so this output can be averaged in time [31].

Let \( x(t), \sigma(t) \) be an arbitrary solution of (8). Consider the deviations \( x_d(t) = x(t) - x^0(t), \sigma_d(t) = \sigma(t) - \sigma^0(t), f_d(t) = f(t) - f^0(t) \), where \( f^0 = M\sigma^0 \). Then we obtain

\[ \frac{dx_d}{dt} = Ax_d + Bf_d, \quad \sigma_d = Cx_d, \quad (27) \]

\[ f_d(t) = \begin{cases} 0, & nT \leq t < nT + \tau_{n_{\text{min}}} \\ \text{sgn}(\tau_n - \tau^0), & nT + \tau_{n_{\text{min}}} \leq t < nT + \tau_{n_{\text{max}}} \\ 0, & nT + \tau_{n_{\text{max}}} \leq t < nT + T. \end{cases} \quad (28) \]

Here \( \tau_{n_{\text{min}}} = \min\{\tau_n, \tau^0\}, \tau_{n_{\text{max}}} = \max\{\tau_n, \tau^0\} \). Recall that \( \tau^0 \) is the pulse duration for the given \( T \)-periodical mode.

Define a sequence of average values

\[ v_n = \frac{1}{T} \int_{nT}^{nT+T} f_d(t) \, dt = \frac{\tau_n - \tau^0}{T} \]

and consider a piecewise constant function

\[ v(t) = v_n, \quad nT \leq t < nT + T. \]

Unlike the signal \( f_d(t) \), which is modulated in width and position, the pulse signal \( v(t) \) is amplitude modulated. However, the average values of both signals coincide on each sampling interval. One has

\[ f_d(t) = v(t) + \frac{du(t)}{dt}, \quad \text{where} \quad u(t) = \int_0^t (f_d(s) - v(s)) \, ds. \]
Obviously, \( u(nT) = 0 \) for \( n = 0, 1, \ldots \). The function \( u(t) \) may be considered as an averaged error of the replacement of \( f_d(t) \) for \( v(t) \). In Chapter 3, [17] it was shown that \( u(t) \) is much less than \( v(t) \), provided that the switching period \( T \) is small. More precisely, 
\[
|u(t)| \leq T|v(t)| \text{ for all } t
\]
and
\[
\int_{nT}^{nT+T} u(t)^2 \, dt \leq \frac{T^2}{3} \int_{nT}^{nT+T} v(t)^2 \, dt, \quad n = 0, 1, \ldots.
\]

The following lemma plays the main role in the proof.

**Lemma 6.2** For any solution of (8) and for any \( n \) there exists a number \( \tilde{t}_n \) such that 
\[
0 \leq \frac{v_n}{\sigma_d(\tilde{t}_n)} \leq \frac{1}{\sigma_* - TL_1}.
\]  
(29)

Inequalities (29) present a version of Lur’e sectoral constraints, which are common in the absolute stability theory.

**Proof of Lemma 6.2:** Since the given \( T \)-periodic mode is unsaturated, the relationship
\[
\sigma^0(\tau^0) = \sigma^0(\tau^0 + nT) = \sigma_1 + \sigma_* \tau^0 / T
\]  
(30)
is valid. Consider the following three cases.

Case (i). Suppose that \( \tau_n \) is defined as a minimal positive root of equation (7). Subtract \( \sigma^0(nT + \tau_n) = \sigma^0(\tau_n) \) from both sides of equality (7) to obtain
\[
\sigma_d(nT + \tau_n) = \sigma^0(\tau^0) - \sigma^0(\tau_n) + \sigma_* v_n.
\]  
(31)

Since \( \tau_n = \tau^0 + T v_n \), one can write
\[
\sigma^0(\tau_n) - \sigma^0(\tau^0) = \sigma^0(\tau^0 + T v_n) - \sigma^0(\tau^0) = T v_n \frac{d\sigma^0}{dt}(t_s),
\]  
(32)
where \( t_s \) is some time instant lying between \( \tau^0 \) and \( \tau_n \). Then (31) and (32) imply
\[
\frac{v_n}{\sigma_d(nT + \tau_n)} = \frac{1}{\sigma_* - T \frac{d\sigma^0}{dt}(t_s)},
\]
so
\[
\frac{1}{\sigma_* + TL_1} \leq \frac{v_n}{\sigma_d(nT + \tau_n)} \leq \frac{1}{\sigma_* - TL_1}.
\]
Hence condition (29) is satisfied with \( \tilde{t}_n = nT + \tau_n \).

Case (ii). Suppose that
\[
\sigma(nT) < \sigma_1,
\]  
(33)
so \( \tau_n = 0 \) and \( v_n = -\tau^0 / T < 0 \). Let us estimate \( v_n \) from below. In view of (30), one has
\[
\sigma_* v_n = -\sigma_* \tau^0 / T = \sigma_1 - \sigma^0(\tau^0).
\]  
(34)
Then (33) and (34) imply
\[
\sigma_* v_n > \sigma(nT) - \sigma^0(\tau^0) = \sigma_d(nT) + \sigma^0(nT) - \sigma^0(\tau^0) .
\]
(35)
Write
\[
\sigma^0(0) - \sigma^0(\tau^0) = T v_n \frac{d \sigma^0}{d t}(t_s),
\]
where \( t_s \) is some time, \( 0 \leq t_s \leq \tau^0 \). Since \( \sigma^0(nT) = \sigma^0(0) \), (35) implies
\[
0 > v_n \left( \sigma_* - \frac{d \sigma^0}{d t}(t_s) \right) > \sigma_d(nT).
\]
Then we come to (29) with \( \tilde{t}_n = nT \).

Case (iii). Suppose that (6) is fulfilled, so \( \tau_n = T \) and \( v_n = 1 - \tau^0/T > 0 \). Set \( \tau = 0 \) and \( \tau = T \) in (6). Then
\[
\sigma(nT) > \sigma_1, \quad \sigma(nT + T) > \sigma_1 + \sigma_*.
\]
Firstly, consider the case when \( \sigma(nT) \geq \sigma_1 + \sigma_* \). Then
\[
\sigma_* v_n = \sigma_1 + \sigma_* - \sigma^0(\tau^0) \leq \sigma(nT) - \sigma^0(\tau^0) = \sigma_d(nT) + \sigma^0(0) - \sigma^0(\tau^0).
\]
Arguing as above, we get
\[
0 < v_n \left( \sigma_* - \frac{d \sigma^0}{d t}(t_s) \right) < \sigma_d(nT),
\]
so we come to (29) with \( \tilde{t}_n = nT \).

At last suppose that (6) is satisfied and \( \sigma(nT) < \sigma_1 + \sigma_* \). Since \( \sigma(nT + T) > \sigma_1 + \sigma_* \), there exists a number \( \tau_s \) such that \( 0 < \tau_s < T \) and \( \sigma(nT + \tau_s) = \sigma_1 + \sigma_* \). It is easily seen that (29) is valid with \( \tilde{t}_n = nT + \tau_s \). The proof of Lemma 6.2 is complete.

Now return to the proof of Theorem 3.1. Inequality (19) can be rewritten as
\[
L_0(H) < R_0(\varepsilon, \nu)
\]
(36)
with
\[
L_0(H) = \begin{bmatrix}
HA + A^\top H & HAB & HB \\
B^\top A^\top H & 0 & 0 \\
B^\top H & 0 & 0
\end{bmatrix},
\]
\[
R_0(\varepsilon, \nu) = \begin{bmatrix}
0 & 0 & -\frac{1}{2} C^\top \\
0 & 3\nu/T^2 & 0 \\
-\frac{1}{2} C & \sigma_* - TL_1 - \nu - \varepsilon - T|\kappa| & -\frac{\kappa^2}{\varepsilon} \begin{bmatrix}
A^\top C^\top \\
-\kappa_1 \\
-\kappa
\end{bmatrix}
\end{bmatrix} [CA - \kappa_1 - \kappa].
\]
Inequality (36) has less dimension than (19), however, it is not linear in \( \varepsilon \).
Let us make a change of variables \( y(t) = x_d(t) - Bu(t) \). Then (27) can be rewritten as

\[
\frac{dy}{dt} = Ay + Bv + ABu, \quad \sigma = Cy + CBu. \tag{37}
\]

Consider a Lyapunov function \( V(y) = y^\top H y \), where \( H \) satisfies (36). Define a quadratic form

\[
F(y, u, v) = \begin{bmatrix} y \\ u \\ v \end{bmatrix}^\top R_0(\varepsilon, \nu) \begin{bmatrix} y \\ u \\ v \end{bmatrix}.
\]

Then (36) implies

\[
\frac{dV(y(t))}{dt} \leq F(y, u, v) - \delta (\|y\|^2 + u^2 + v^2) \tag{38}
\]

along the solutions of (37). Here \( \delta \) is some small positive number. From Lemma 2 it follows that

\[
\int_{nT}^{nT+T} F(y(t), u(t), v(t)) \, dt \leq 0
\]

for all \( n \) (see Chapter 3 [17] for details). Then (38) implies \( y(t) \to 0, v(t) \to 0 \) as \( t \to +\infty \). Thus we come to (20). The Lyapunov stability can be proved as in [20].

## 7 Conclusion

We propose an LMI based approach to finding conditions for global stability of an operating mode of a PWM power converter. The LMI technique becomes a powerful tool for this type of problems when combined with well-known methods of calculation of periodical solutions for PWM systems. The LMI approach leads to computationally tractable criteria, which can be easily implemented with the help of standard modelling software.

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