Josephson junctions of Weyl and multi-Weyl semimetals

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We study a Josephson junction involving a Weyl and a multi-Weyl semimetal separated by a barrier region of width $d$ created by putting a gate voltage $U_0$ over the Weyl semimetal. The topological winding number of such a junction changes across the barrier. We show that $I_c R_N$ for such junctions, where $I_c$ is the critical current and $R_N$ the normal state resistance, in the thin barrier limit, has a universal value independent of the barrier potential. We provide an analytical expression of the Andreev bound states and use it to demonstrate that the universal value of $I_c R_N$ is a consequence of change in topological winding number across the junction. We also study AC Josephson effect in such a junction in the presence of an external microwave radiation, chart out its current-voltage characteristics, and show that the change in the winding number across the junction shapes the properties of its Shapiro steps. We discuss the effect of increasing barrier thickness $d$ on the above-mentioned properties and chart out experiments which may test our theory.

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I. INTRODUCTION

The role of topology in shaping the low energy properties of condensed matter systems has received widespread attention in recent years. Specific examples of such materials include graphene topological insulators, and Dirac and Weyl semimetals. These are classes of materials whose low-energy quasiparticles obey Dirac or Weyl equations and host Dirac/Weyl nodes at specific points in their Brillouin zone. These nodes act as sources or sinks of topological winding numbers. Several unconventional low-energy properties of these materials arise from the presence of such nodes. For example, in graphene, the magneto-resistivity of electrons displays an additional contribution due to the non-zero Berry phase gathered by electrons whose semiclassical trajectories encompasses a Dirac node. For topological insulators, the surface quasiparticles obey spin-momentum locking as seen in spin- and angle-resolved photoemission experiment. In three-dimensional (3D) Weyl semimetals, the presence of such nodes leads to a host of unconventional phenomena such as Fermi arcs on their surface, negative magnetoresistance, chiral anomaly, and interaction induced phase transitions.

A typical Weyl semimetal hosts several Weyl nodes in its Brillouin zone. The effective low-energy of quasiparticles near any of such Weyl nodes is given by $H = \pm \hbar v_F \sum_i \vec{\sigma} \cdot \vec{k}$, where $\vec{k} = (k_x, k_y, k_z)$ denotes momentum measured from the Weyl node, $v_F$ is the Fermi velocity, the +(-) sign corresponds to nodes with positive(negative) chirality, and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are Pauli matrices in spin-space. Such Weyl nodes, occurring in pairs, are usually protected by either time-reversal or inversion symmetry. More recently, it was suggested that certain materials such as HgTe may host Weyl nodes with anisotropic dispersion along two transverse directions (denoted as $\vec{k}_i = (k_x, k_y)$ in this work): $E \sim \hbar v_F \sqrt{k_i^2 + \epsilon_0^2 \lvert k_i \rvert^{2n}}$, where $\epsilon_0$ is a constant whose value depends on the details of band dispersion in the material and $n \neq 1$ is an integer indicating the anisotropy. It was shown that such a dispersion arises out of symmetry protected (such as discrete three- or six-fold rotational symmetry) merger of $n \leq 3$ Weyl nodes. These materials are dubbed as multi-Weyl semimetals and are known to have several unconventional properties which are distinct from both conventional metals and Weyl semimetals. In particular, these multi-Weyl nodes have a topological winding number $3 \geq n > 1 (n \in \mathbb{Z})$ which is the number of Weyl nodes merged. Interestingly, an application of external strain, which lifts the rotational symmetry protecting the merger, leads to emergence of $n$ Weyl nodes of same chirality and unit winding number from a multi-Weyl node.

The study of ballistic transport in junctions of materials whose low-energy quasiparticles satisfy Dirac or Weyl equations provides a wealth of information regarding their topological properties. In addition, it also provides access to several unconventional transport behavior displayed by these materials. For example, in 2D materials with Dirac quasiparticles (such as graphene and surfaces of topological insulators), the tunneling conductance $G$ of normal-barrier-normal (NBN) junctions oscillates as a function of barrier potential, such a behavior is qualitatively distinct from the monotonic decay of $G$ with increasing barrier potential in conventional junctions. A similar oscillatory dependence has been reported for junctions of Weyl semimetals. In contrast, it was found that normal-barrier-normal (NBN) and normal-barrier-superconducting (NBS) junctions of Weyl and multi-Weyl semimetals display barrier independence of $G$ in the thin barrier limit.

For 2D Dirac materials an analogous oscillatory dependence of the Josephson current $I_J$ on the barrier potential has also been predicted for superconductor-barrier-superconductor (SBS) junctions. It is well known that for conventional SBS junctions, the product $I_c R_N$, where $I_c$ is the critical current and $R_N$ is the normal state resistance of the junction, is a monotonically decreasing function bar-
rrier strength. The value of $I_c R_N / [\pi \Delta_0/(2e)]$ (where $\Delta_0$ is the amplitude of the pair-potential $\Delta$ and $e$ is the electron charge) decreases from 2 to 1 in conventional Josephson junctions with increasing barrier strength for thin barriers. Its maximal value occurs in the Kulik–Omelyanchuk (KO) limit when the junction is transparent\(^{20}\) while the minimum value is reached for junctions with very large value of the barrier potential in the so-called Ambegaokar-Baratoff (AB) limit\(^{21}\). In contrast, SBS junctions whose quasiparticles obey Dirac-Bogoliubov-de Gennes (DBdG) equations never reach the AB limit\(^{22}\) the value of $I_c R_N / [\pi \Delta_0/(2e)]$ in these junctions oscillates as a function of barrier strength between 2 and 1.1.\(^{15}\)

A similar oscillatory behavior is observed in junctions of Weyl semimetals\(^{15}\). However, the properties of AC Josephson effect in the presence of microwave radiation has not been investigated for these junctions. Moreover, either DC or AC Josephson effect between junctions of Weyl and multi-Weyl semimetals has not been studied so far.

In this work, we study the DC and AC Josephson effect in SBS junctions between Weyl and multi-Weyl semimetals. The main results that we obtain are as follows. First, we find analytic expressions for Andreev bound states in these junctions in the thin barrier limit. In this limit, the barrier potential $V \to \infty$ and its thickness $d \to 0$ such that the dimensionless barrier strength $\chi = U_0 d/(\hbar v_F)$, where $v_F$ is the Fermi velocity of the DBdG quasiparticles and $\hbar$ is Planck’s constant, is finite. Using the analytic expression Andreev bound states, we also obtain expressions for both DC and AC Josephson current in such junctions. Second, we show that $I_c R_N$ in these junctions is a barrier independent constant in the thin barrier limit: $I_c R_N = c [\pi \Delta_0/(2e)]$ for any $\chi$. Here $c$ is a constant which depends weakly on $n_1$ and $n_2$; numerically we find $c = 1.56$ for $n_1 = 1$ and $n_2 = 2$ and $c = 1.62$ for $n_1 = 1$ and $n_2 = 3$. We demonstrate that this behavior, which is contrast to behavior of $I_c R_N$ in all Josephson junctions studied earlier, is a consequence of change in the topological winding number across the junction and estimate the deviation of $I_c R_N$ from its thin barrier limit as a function of $U_0$ for thicker barriers. Third, we study the AC Josephson effect in such junctions in the presence of a microwave radiation both with voltage and current biases. We obtain the current-voltage (I-V) characteristics of such junctions and demonstrate the presence of Shapiro steps in the I-V curves. We show that the width of these Shapiro steps for such junctions is independent of the barrier strength in the thin barrier limit. Finally, we suggest possible experiments which may test our theory.

The plan of the rest of this work is as follows. In Sec. II, we chart out the derivation of Andreev bound states and use it to show the barrier independence of $I_c R_N$. This is followed by Sec. III, where we discuss AC Josephson effect in such junctions. Finally, in Sec. IV, we summarize our main results, discuss experiments which may test our theory, and conclude.

II. ANDREEV BOUND STATES

In this section, we develop analytic expressions for Andreev bound states in SBS junctions involving a Weyl and a multi-Weyl semimetal and use them to obtain their Josephson current. In what follows, we consider a ballistic junction schematically shown in Fig. 1. The barrier region has width $d$ and is created by putting a barrier potential over a normal Weyl semimetal. In contrast, regions I and III have s-wave superconducting pair-potentials of amplitude $\Delta_0$: $\Delta = \Delta_0 \exp [i \phi]$, where $\phi$ is the global superconducting phase. In what follows we shall choose this phase to be zero in region I and $\phi_0$ in region III without loss of generality. We assume that superconductivity has been induced in these regions by proximate $s$-wave superconductors. The microscopic analysis leading to specific conditions for $s$-wave pair potentials for induced superconductivity in these materials has been carried out in Refs.\(^{15,22}\) here we shall assume that these conditions hold for our system. In what follows, we consider the situation, where there are two Weyl nodes of opposite chirality in regions I and III: the winding numbers corresponding to these nodes will be denoted as $n_1$ ($n_2$) in regions I (III). The DBdG quasiparticles of the superconductors arises from superposition of electrons in one of these nodes with holes in the other. We shall obtain the equations leading to expression of Andreev bound state for arbitrary $d$ and $U_0$ in these junctions; the analytic expression of these bound states will be presented in the thin-barrier limit as a function of $\chi$.

The low-energy effective Hamiltonian of a multi-Weyl semimetal with a topological winding number $n$ is given by $H = \sum_{\vec{k}, \sigma} \Psi_{\vec{k}\sigma}^{\dagger} H_1 \left[ n \right] \Psi_{\vec{k}\sigma}$, where $\Psi_{\vec{k}} = (c_{\vec{k} \uparrow}, c_{\vec{k} \downarrow})^T$ are two component fermions field, $c_{\vec{k}\sigma}$ denotes annihilation operator of a Weyl quasiparticle with momentum $\vec{k}$ and spin $\sigma$, and $H_1 \left[ n \right]$
is given by\[11\]

$$H_1[n] = \left[ \hbar v_F k_x \sigma_z + \epsilon_0 |\vec{k}|^n \cos(n\phi_{\vec{k}})\sigma_x \right. + i \sin(n\phi_{\vec{k}})\sigma_y - \mu_0 I \right] $$

(1)

Here $v_F$ is the Fermi velocity, $k_F = \mu_0/(\hbar v_F)$ is the Fermi momentum, $\mu_0$ is the chemical potential, $I$ denotes $2 \times 2$ identity matrix, $\epsilon_0 = \mu_0/k_F^2$ is a material dependent constant (chosen to be unity if $n = 1$) whose value is unimportant for our analysis, and $\phi_{\vec{k}} = \arctan(k_y/k_x)$ is the azimuthal angle in the transverse direction. In the presence of induced $s$-wave superconductivity, the low-energy Hamiltonian governing the DBdG quasiparticles are given by

$$H_s[n] = \sum_{\vec{k}} \Psi^\dagger_{\vec{k}} (H_1[n]\tau_z + \Delta\tau_+ + \Delta^\ast\tau_-) \Psi_{\vec{k}}^\dagger$$

(2)

where $\vec{\tau} = (\tau_x, \tau_y, \tau_z)$ denotes Pauli matrices valley space

$$\psi_{elq} = (\sin(\theta_{E_1}) \epsilon^{+\gamma_1}, \cos(\theta_{E_1}) \epsilon^{+\gamma_1}, \sin(\theta_{E_2}), \cos(\theta_{E_2})) e^{i(k_{\vec{k}}^0 z + \vec{k}_r \cdot \vec{r} - n_1 \phi_{\vec{k}} \sigma_z/2)}/\sqrt{2}$$

$$\psi_{hilq} = (\cos(\theta_{E_2}), \sin(\theta_{E_2}) \epsilon^{+\gamma_1} \cos(\theta_{E_2}), \epsilon^{+\gamma_1} \sin(\theta_{E_2})) e^{i(k_{\vec{k}}^0 z + \vec{k}_r \cdot \vec{r} - n_1 \phi_{\vec{k}} \sigma_z/2)}/\sqrt{2}$$

(4)

where $k_{\vec{k}}^{0(1)(2)} = +[-] \sqrt{(\mu_0 [-]+\gamma_0^2) - \epsilon_0^2 |\vec{k}|^{2n_1}/(\hbar v_F)$, $\zeta = \sqrt{\Delta_0^2 - \gamma^2}$, $\gamma = \arccos(E(\vec{k}^0)/\Delta_0)$, and $\tan(2\theta_{k_1}^{0(2)}) = \epsilon_0 |\vec{k}|^{n_1}/(\hbar v_F k_{\vec{k}}^{0(1)(2)})$. Here we have set the phase of the superconducting pair-potential in region I to zero without loss of generality and $\vec{r}_I = (x, y)$. Note that the dependence on $\phi_{\vec{k}}$ of these wavefunctions is equivalent to a rotation in spin-space by $n_1 \phi_{\vec{k}}$ about the $z-$axis.\[13\] The wavefunction in region I is thus given by

$$\psi_I = a_1 \psi_{elq} + b_1 \psi_{hilq}, \quad (5)$$

where $a_1$ and $b_1$ are arbitrary complex coefficients.

In region II, the Hamiltonian is given by $H$ with $n = n_1$. The wavefunction in region II is thus a superposition of left and right moving electrons and holes. The wavefunction of these quasiparticles in the four component notation is given by

$$\psi_{e+} = (\cos(\theta_{E_0}), \sin(\theta_{E_0}), 0, 0) e^{i(k_{\vec{k}} z + \vec{k}_r \cdot \vec{r} - n_1 \phi_{\vec{k}} \sigma_z/2)}/\sqrt{2}$$

$$\psi_{e-} = (\sin(\theta_{E_0}), \cos(\theta_{E_0}), 0, 0) e^{-i(k_{\vec{k}} z + \vec{k}_r \cdot \vec{r} - n_1 \phi_{\vec{k}} \sigma_z/2)}/\sqrt{2}$$

$$\psi_{h+} = (0, 0, -\sin(\theta_{k_0}^e), \cos(\theta_{k_0}^e)) e^{i(k_{\vec{k}}^0 z + \vec{k}_r \cdot \vec{r} - n_1 \phi_{\vec{k}} \sigma_z/2)}/\sqrt{2}$$

$$\psi_{h-} = (0, 0, \cos(\theta_{k_0}^e), -\sin(\theta_{k_0}^e)) e^{-i(k_{\vec{k}}^0 z + \vec{k}_r \cdot \vec{r} - n_1 \phi_{\vec{k}} \sigma_z/2)}/\sqrt{2}$$

(6)

and $\tau_{\pm} = (\tau_x \pm i\tau_y)/2$. Here the pair potential connects electrons and holes between two Weyl nodes of opposite chirality (two valleys) and $\Psi_{\vec{k}}^\dagger = (c_{\vec{k} \sigma}, c_{-\vec{k} \sigma}^\dagger)^T$ denotes the four-component fermionic fields. In our notation the electrons and holes of these four components wavefunctions belongs to two different nodes (valleys).

The Hamiltonian in region I (see Fig. 1) is given by $H_s[n = n_1]$. In what follows, we shall choose $z$ to be the longitudinal direction. Thus the Weyl equation whose solution yields left moving electron-like and hole-like DBdG quasiparticles in region I is given by

$$H_s[n_1; k_z \rightarrow -i\partial_z] \psi = E \psi$$

(3)

These wavefunctions are given by\[15\]

$$\psi = \psi_{elq} + q \psi_{e+} + r \psi_{e-} + s \psi_{h+}, \quad (7)$$

where $p, q, r, s$ are complex coefficients.

In region III, the Hamiltonian is given by $H_s[n = n_2]$ with $\mu_0 \rightarrow \mu_0'$. The change in chemical potential between the two regions is kept to point out the generality of the procedure; however, it is to be noted that in principle the chemical potentials in region I and III can always be made equal by applying a gate voltage between the two regions. The wavefunction in this region is again a superposition of electron- and hole-like DBdG quasiparticle wavefunctions. These wavefunctions are given by
where \( \phi_0 \), as defined earlier, is the relative phase between the superconductors in region I and III. Here the expressions for \( k_{z}^{(3)} \) and \( k_{z}^{(4)} \) may be obtained from those of \( k_{z}^{(1)} \) and \( k_{z}^{(2)} \) respectively by replacing \( \mu_0 \to \mu_0' \) and \( n_1 \to n_2 \). Similarly \( \theta_{E_3}^{(4)} \) can be read off from the expressions of \( \theta_{E_3}^{(2)} \) after making similar replacements. The wavefunction in region III can then be written as

\[
\psi'_{III} = a_2 \psi'_{E_3} + b_2 \psi'_{E_4},
\]

where \( a_2 \) and \( b_2 \) are complex coefficients.

To obtain the Andreev bound states, we now impose the usual current continuity condition along \( \bar{z} \). For Weyl or multi-Weyl electrons with linear longitudinal dispersion, it is well known that current continuity amounts to continuity of the wavefunction leading to the conditions

\[
\psi_I(z = 0) = \psi_{II}(z = 0), \quad \psi_{II}(z = d) = \psi_{III}(z = d)
\]

Eq. [10] leads to eight linear homogeneous equations. The energy of the Andreev bound states is to be found by demanding non-zero solutions of these equations. Here we concentrate on the regime \( \mu_0, \mu_0' \gg \Delta_0 \), for which \( \theta_{E_3}^{(2)}(\theta_{E_3}^{(4)}) = -\theta_{E_3}^{(1)}(\theta_{E_3}^{(3)}) \).

In this regime, it is possible to simplify these equations significantly. Eliminating \( p, q, r \) and \( s \) from these equations in this regime, one gets, after a straightforward calculation, a set of four linear homogeneous equations involving \( a_{1,2} \) and \( b_{1,2} \). These equations are given by

\[
\begin{align*}
& a_1 \left( \cos \theta_{E_2} \sin(\theta_{E_1} - \theta_{E_2})e^{i(\gamma - k_{z}d)} + \sin \theta_{E_2} \cos(\theta_{E_1} + \theta_{E_2})e^{i(\gamma - k_{z}d)} \right) + b_1 \left( \cos \theta_{E_2} \cos(\theta_{E_1} - \theta_{E_2})e^{ik_{z}d} \right. \\
& - \sin \theta_{E_2} \sin(\theta_{E_1} + \theta_{E_2})e^{-ik_{z}d} = \cos(2\theta_{E_2})e^{i(n_1 - n_2)\phi} \left( a_2 \cos \theta_{E_3} - a_1 \sin \theta_{E_5}e^{i(\phi_0 - k_{z}d)} \right) \\
& a_1 \left( \cos \theta_{E_2} \sin(\theta_{E_1} - \theta_{E_2})e^{i(\gamma - k_{z}d)} + \sin \theta_{E_2} \cos(\theta_{E_1} + \theta_{E_2})e^{i(\gamma - k_{z}d)} \right) + b_1 \left( \sin \theta_{E_2} \cos(\theta_{E_1} - \theta_{E_2})e^{ik_{z}d} \right. \\
& - \cos \theta_{E_2} \sin(\theta_{E_1} + \theta_{E_2})e^{-ik_{z}d} = \cos(2\theta_{E_2})e^{-i(n_1 - n_2)\phi} \left( a_2 \sin \theta_{E_3}e^{i(\gamma + k_{z}d)} - b_2 \cos \theta_{E_3}e^{i(\phi_0 - k_{z}d)} \right) \\
& a_1 \left( \cos \theta_{E_2} \sin(\theta_{E_1} + \theta_{E_2})e^{ik_{z}d} - \sin \theta_{E_2} \cos(\theta_{E_1} - \theta_{E_2})e^{-ik_{z}d} \right) + b_1 \left( \cos \theta_{E_2} \cos(\theta_{E_1} + \theta_{E_2})e^{i(\gamma - k_{z}d)} \right. \\
& + \sin \theta_{E_2} \sin(\theta_{E_1} - \theta_{E_2})e^{i(\gamma + k_{z}d)} = \cos(2\theta_{E_2})e^{i(n_1 - n_2)\phi} \left( a_2 \cos \theta_{E_3}e^{-i(\phi_0 - k_{z}d)} - b_2 \sin \theta_{E_3}e^{i(\gamma - k_{z}d)} \right) \\
& a_1 \left( \cos \theta_{E_2} \sin(\theta_{E_1} + \theta_{E_2})e^{ik_{z}d} - \sin \theta_{E_2} \cos(\theta_{E_1} - \theta_{E_2})e^{-ik_{z}d} \right) + b_1 \left( \sin \theta_{E_2} \cos(\theta_{E_1} + \theta_{E_2})e^{i(\gamma - k_{z}d)} \right. \\
& - \cos \theta_{E_2} \sin(\theta_{E_1} - \theta_{E_2})e^{-ik_{z}d} = \cos(2\theta_{E_2})e^{-i(n_1 - n_2)\phi} \left( a_2 \sin \theta_{E_3}e^{-i(\phi_0 - k_{z}d)} + b_2 \cos \theta_{E_3}e^{i(\gamma - k_{z}d)} \right)
\end{align*}
\]
\[
\mathcal{N} = \begin{pmatrix}
\sin(\theta_{k_1}) e^{i\gamma} & \cos(\theta_{k_1}) & -\cos(\theta_{k_3}) e^{i(\alpha_0 + \gamma)} & \sin(\theta_{k_3}) e^{i(\alpha_0 + \phi_0)} \\
\cos(\theta_{k_1}) e^{i\gamma} & -\sin(\theta_{k_1}) & -\sin(\theta_{k_3}) e^{i(-\alpha_0 + \gamma)} & -\cos(\theta_{k_3}) e^{i(-\alpha_0 + \phi_0)} \\
\sin(\theta_{k_1}) & \cos(\theta_{k_1}) & -\cos(\theta_{k_3}) e^{i(-\alpha_0 - \phi_0)} & \sin(\theta_{k_3}) e^{i(-\alpha_0 + \gamma)} \\
-\sin(\theta_{k_1}) e^{i\gamma} & \cos(\theta_{k_1}) & \sin(\theta_{k_3}) e^{-i(\alpha_0 + \phi_0)} & -\cos(\theta_{k_3}) e^{i(\gamma - \alpha_0)}
\end{pmatrix}
\]

(12)

Note that the dependence of the matrix elements of \( \mathcal{N} \) on \( \chi \) comes only through \( \alpha_0 \) in the thin barrier limit. To obtain the analytic expression of the Andreev bound states, we demand \( \text{Det}[\mathcal{N}] = 0 \), which yields the dispersions of these states in terms of the normal state transmission \( T_N(\tilde{k}_t) \equiv T_N \) of the junction as

\[
E_{\pm}(\phi_0) = \pm \Delta_0 \sqrt{1 - T_N \sin^2(\phi_0/2)}, \quad T_N = \frac{\cos(2\theta_{k_1}) \cos(2\theta_{k_3})}{(\cos \theta_{k_1} \cos \theta_{k_3} + \sin \theta_{k_1} \sin \theta_{k_3})^2 - \sin(2\theta_{k_1}) \sin(2\theta_{k_3}) \cos^2 \alpha_0}
\]

(13)

and is given by

\[
I_J = \frac{e \Delta_0}{2h} \sum_{k_t} \frac{T_N \sin(\phi_0)}{\sqrt{1 - T_N \sin^2(\phi_0/2)}} \tanh \left[ \frac{E_{\pm}(\phi_0)}{2k_B T_0} \right]
\]

(14)

Using Eq. 14, we now demonstrate the barrier independence of \( I_J R_N \). To this end, we note that the sum over transverse momenta in Eq. 14 can be replaced by integral over \( (k_x, k_y) \) for large enough sample. To carry out this integral, we use the parametrization

\[
k_x[k_y] = k_F [\sin 2\theta]^{1/n_1} \cos \phi[\sin \phi] \\
k_z = k_F \cos 2\theta
\]

(15)

One can equivalently parametrize \( \tilde{k}_t \) using \( n_2 = \mu_0 = h v_F k_F \), where \( k_F \) is the Fermi wave vector in region III. In what follows we shall always choose the parametrization with lower \( n \) (chosen to be \( n_1 \) without loss of generality in the rest of this work) for carrying out the integrals; this ensures that the integral over \( \theta \) has the range \(-\pi/4 \leq \theta \leq \pi/4\). Using this parametrization one gets

\[
I_J = \frac{I_0}{4n_1} \left( \frac{L k_F}{2 \pi} \right)^2 \int_0^{\pi/4} d\theta \int_0^{2\pi} d\phi \times (\sin 2\theta)^{2/n_1 - 1} \cos 2\theta \frac{T_N \sin \phi_0}{\sqrt{1 - T_N \sin^2(\phi_0/2)}}
\]

(16)

where \( L \) is the transverse dimension of the junction.

Next, we denote that value of \( \phi_0 \) for which \( I_J \) is maximum to be \( \phi_0^m \). This maximum is obtained by demanding that \( \phi_0^m \) is a solution of \( \partial_{\phi_0} I_J = 0 \) and is given by

\[
\phi_0^m = \arccos \left[ (T_N - 2) + 2 \sqrt{1 - T_N} \right] / T_N
\]

(17)
Note that $\phi_0^m$ depends on $k_z$ through $T_N$. Using Eqs. 15 and 17 and defining $I_0 = 2e\Delta_0/\hbar$ one finds, at $T_0 = 0$,

$$I_c = I_J[\phi_0 = \phi_0^m] = \frac{I_0}{n_1} \left( \frac{LK_F}{2\pi} \right)^2 I_1$$

$$I_1 = \frac{1}{4} \int_0^{\pi/4} d\theta \int_0^{2\pi} d\phi (\sin 2\theta)^{2/n_1-1} \cos 2\theta \times \frac{T_N \sin \phi_0^m}{\sqrt{1 - T_N \sin^2(\phi_0^m/2)}}$$

A plot of $I_J/I_c$ as a function of $\phi_0$ is shown in the left panel of Fig. 3 for two different values of $\chi$ and $n_1 = 1$ and $n_2 = 2$. We find that $I_J/I_c$ is independent of $\chi$; this feature is a consequence of integration over the azimuthal angle $\phi$, which eliminates the $\chi$ dependence, provided $n_1 \neq n_2$ (Eqs. 14 and 18). In contrast, for $n_1 = n_2$, varying $\chi$ leads to substantial change in $I_J/I_c$ as shown in the right panel of Fig. 3. Thus we find a qualitative difference between dependence of $I_J$ on $\chi$ for junctions, where the topological winding number changes across the junction and for those, where it does not.

For computing $R_N$, one finds first the normal state junction conductance $G_N = e^2/h \sum N \cdot T_N$. Using this, one finds the normal state resistance $R_N = G_N^{-1}$, with $G_N$ given by

$$G_N = \left( \frac{e^2}{\hbar n_1} \frac{LK_F}{2\pi} \right)^2 \int_0^{\pi/4} d\theta \int_0^{2\pi} d\phi \times (\sin 2\theta)^{2/n_1-1} \cos 2\theta T_N$$

$$= \frac{e^2}{2n_1 \pi \hbar} \left( \frac{LK_F}{2\pi} \right)^2 I_2$$

From the expression of $T_N$ in Eq. 13, it is evident that $I_N$ depends on $\phi$ and $\chi$ only through $\alpha_0$. Consequently, it is possible to carry out the integral over $\phi$ analytically; in particular, it is easy to show by expressing the integral over $\phi$ as a contour integral with the substitution $z = \exp(i\phi)$ [10], that $J_{\phi J} \phi T_N$ is independent of $\chi$ for $n_1 \neq n_2$. A similar $\chi$ independence can be shown for $I_1$ (Eq. 18) required to evaluate $I_c$. Note that for $n_1 = n_2$, $T_N$ becomes independent of $\phi$ and the $\chi$ dependence is retained. Thus we find that the change in topological winding number across the junction is crucial for both $I_c$ and $R_N$ to be independent of the barrier strength in the thin barrier limit. Using Eqs. 18 and 19, we find

$$I_c R_N = \frac{\pi \Delta_0}{2e} I_2 = \frac{\pi \Delta_0}{2e} c,$$

where the ratio $c = I_1/I_2$ depends on $n_1$ and $n_2$.

A plot of $I_c R_N$ as a function of $\chi$ is shown in Fig. 4. The left panel of Fig. 4 shows the barrier independent of $I_c R_N$ for $n_1 \neq n_2$ for several choice of $n_1$ and $n_2$. This allows us to compute $c$; we find $c = 1.56(1.62)$ for $n_1 = 1$ and $n_2 = 2(3)$. This demonstrates the weak dependence of $c$ on $n_1$ and $n_2$. In contrast, $c$ is an oscillatory function of $\chi$ for $n_1 = n_2$ as shown in right panel of Fig. 4. These plots therefore demonstrate that $I_c R_N$ in these junctions becomes a barrier independent universal constant for a fixed $n_1 \neq n_2$; the value of this constant depends weakly on $n_1$ and $n_2$. This behavior is in sharp contrast to all JJs made out of topological or conventional superconductors studied earlier [18, 20].

Next, we move away from the thin barrier limit and allow for arbitrary width $d$ of the barrier potential. To this end, we numerically solve Eq. 11 for $\mu_0 = \mu_0' \gg \Delta_0$ for a given transverse momentum $(k_x, k_y)$ and find the Andreev bound state energy as a function of $\phi_0$. Next, we follow Eqs. 16, 18 and 19 to compute $I_c R_N$ numerically. The result of this computation is shown in Fig. 5 for $2n_1 = n_2 = 2$. We find that with increasing barrier thickness, $I_c R_N$ starts to oscillate with the barrier strength $U_0/\Delta_0$. However, the magnitude of this oscillation decreases rapidly with decreasing $d$ as can be seen by comparing the plots for $k_0d = 0.1$ and $k_0d = 0.5$ in Fig. 6. Moreover, the amplitude of these oscillations, for $k_0d \approx 1$ is found to be small compared to the JJs involving Weyl semimetals with $n_1 = n_2 = 1$, particularly for $U_0 \gg \Delta_0$. These oscillations can thus be distinguished from the ones arising in JJs involving Weyl semimetals with $n_1 = n_2$. The amplitude of these oscillations in JJs with

FIG. 3: Plot of $I_J/I_c$ as a function of $\phi_0$ for $\chi = \pi$ (blue [green] solid lines in left [right] panels) and $\chi = \pi/4$ (red dots) with $n_1 = 1$ and $n_2 = 2$ (left panel) and $n_1 = n_2 = 1$ (right panel). Note that $I_J/I_c$ becomes independent of $\chi$ in the thin barrier limit for $n_1 \neq n_2$ (left panel) but depends substantially on $\chi$ if $n_1 = n_2$ (right panel). All other parameters are same as in Fig. 2. See text for details.

FIG. 4: Plot of $I_c R_N$ in units of $\pi \Delta_0/(2e)$ as a function of $\chi$. The left panel represents cases for $n_1 \neq n_2$: the brown dotted (violet solid) line represents $n_1 = 1(2)$ and $n_2 = 2(3)$. The right panel represents case for which $n_1 = n_2$: the red solid, blue dashed and the green dotted lines corresponds to $n_1 = n_2 = 1, 2$ and 3 respectively. All other parameters are same as in Fig. 2. See text for details.
n_1 \neq n_2$ for a fixed $d$ is a monotonically decaying function of $U_0/\Delta_0$, while those in JJs with $n_1 = n_2$ is almost independent of the barrier height for large $U_0/\Delta_0$.

### III. AC JOSEPHSON EFFECT

In this section we analyze the AC Josephson effect in topological junctions, where the topological winding number changes across the junction. To this end, we first consider the voltage biased junctions which are analytically more tractable compared to their current biased counterparts. For such junctions, the expressions of the Josephson current, in the presence of bias voltage and a microwave radiation, may be obtained by the usual substitution

$$
\phi_0 \rightarrow \phi_0 + \frac{2e}{\hbar} \int^t dt' [V_0 + V_1 \cos(\omega_D t')],
$$

where $V_0[V_1]$ are the amplitude of the DC voltage [microwave radiation] and $\omega_D$ is the frequency of radiation. Substituting this in Eq. [14] and using the identity $\exp[i a \sin(\omega_D t)] = \sum_{m=-\infty}^{\infty} J_m(a) \exp[i m \omega_D t]$, where $J_m$ denotes $m^{\text{th}}$ order Bessel function and $m$ takes integer values, we find

$$
I_J(t) = \frac{I_0}{4n_1} \left( \frac{Lk_F}{2\pi} \right)^2 \int_0^{\pi/4} d\theta \int_0^{2\pi} d\phi \frac{T_N \sum_{m=-\infty}^{\infty} (\sin 2\theta)^{2/n_1-1} \cos 2\theta J_m \left( \frac{2V_1}{\hbar \omega_D} \right) \sin(\phi_0 + (\omega_J - m \omega_D)t)}{\sqrt{(1 - T_N/2) + (T_N/2) \sum_{m=-\infty}^{\infty} J_m \left( \frac{2V_1}{\hbar \omega_D} \right) \cos(\phi_0 + (\omega_J - m \omega_D)t) \}},
$$

where $\omega_J = 2eV_0/\hbar$ is the Josephson frequency. We note that the $I_J$ develops a DC component whenever $\omega_J = m_0 \omega_D$ leading to the $m_0^{\text{th}}$ Shapiro step [20]. The width of the step is the difference between the maximal and minimal value of the DC component of the current. If these values are obtained for the values $\phi^m_0$ (obtained from Eq. [17] by substituting $T_N \rightarrow T_N J_{m_0} (2eV_1/(\hbar \omega_D))$) and $-\phi^m_0$ of the relative phase, the width of the $m_0^{\text{th}}$ step is obtained by

$$
(\Delta I)_{m_0} = \frac{I_0}{2n_1} \left( \frac{Lk_F}{2\pi} \right)^2 \int_0^{\pi/4} d\theta \int_0^{2\pi} d\phi \frac{T_N \sum_{m=-\infty}^{\infty} (\sin 2\theta)^{2/n_1-1} \cos 2\theta J_m \left( \frac{2V_1}{\hbar \omega_D} \right) \sin(\phi_0 + (\omega_J - m \omega_D)t)}{\sqrt{(1 - T_N/2) + (T_N/2) \sum_{m=-\infty}^{\infty} J_m \left( \frac{2V_1}{\hbar \omega_D} \right) \cos(\phi_0 + (\omega_J - m \omega_D)t) \}},
$$

Using the same argument elucidated in Sec. [II], we find that for $n_1 \neq n_2$, the width of the Shapiro steps are independent of the barrier strength $\chi$ for any $m_0$. From Eq. [23] we find that $(\Delta I)_{m_0}$ is a constant which depends on topological winding $n_1$ and $n_2$ and on the junction geometry. This behavior is to be contrasted with that found in conventional JJs, where the Shapiro step width is monotonically decreasing function of $\chi$ and with that for JJs involving 2D Dirac materials, where they oscillate with $\chi$. This is shown in Fig. [6]. The left panel shows that the Shapiro step width for $m_0 = 1$ is independent of $\chi$ for $n_1 \neq n_2$ while the right panel indicates that it has a clear oscillatory dependence on $\omega$ for $n_1 = n_2$.

Next, we consider a current biased junction in the presence of microwave radiation since such junctions are experimentally more relevant than their voltage-biased counterparts. Such junctions are typically characterized by an external resistor $R$ and capacitor $C$ connected to the JJ in parallel along with a current source. It is well known that the equation governing the phase dynamics in such junctions is given by [23,25]

$$
\frac{d^2}{dt^2} \phi_0 + \beta_0 \frac{d\phi_0}{dt} + I_J/I_c = \left[ I_0 + I_1 \cos(\omega_D t_0) \right]/I_c,
$$

where $I_0$ is the bias current, $I_1$ is the amplitude of the microwave radiation, $\omega_D$ is its frequency, $I_c$ is the critical current of the junction, $I_J$ is the Josephson current given by Eq. [14] with $\phi_0 \rightarrow \phi_0(t)$, and $\beta_0 = \sqrt{h/(2eL_c R C^2)}$ is the McCumber parameter of the junction. In Eq. [24] we have scaled $t \rightarrow \omega_D t$ and $\omega_D \rightarrow \omega_D/\omega_p$, where $\omega_p = \sqrt{2eI_c/(\hbar C)}$ is the Plasma frequency of the JJ. The junction is overdamped...
The solution of this equation is straightforward and yields steps occur at \( I_{1} \) first order equation in \( \gamma \), where \( I_{0} = \hbar \omega_{D}/(2e) \).

To obtain a semi-analytic understanding of the nature of \( I_{J} \) in current-biased Weyl junction, we provide an analytic, albeit perturbative, solution to Eq. 24 for \( \omega_{D}, I_{1} \gg \beta_{0} \). To this end, we note that for \( I_{0}, I_{1} \gg \beta_{0} \), Eq. 24 can be written as a first order equation in \( Y = d\phi_{0}/dt \) as

\[
\frac{dY}{dt} + \beta_{0}Y = \left( I_{0} + I_{1} \cos(\omega_{D}t) \right)/I_{c} \tag{25}
\]

The solution of this equation is straightforward and yields

\[
\phi_{0}(t) = \varphi_{0} + \frac{I_{0}t}{I_{c}\beta_{0}} + \frac{I_{1}\sin(\omega_{D}t + \alpha_{0})}{I_{c}\omega_{D}\gamma} \tag{26}
\]

where \( \gamma = \sqrt{\beta_{0}^{2} + \omega_{D}^{2}} \) and \( \alpha_{0} = \arccos(\omega_{D}/\gamma) \). Substituting Eq. 26 in Eq. 23 and taking note of the fact that the Shapiro steps occur at \( I_{0} = |m|\omega_{D}\beta_{0}I_{c} \), one finds

\[
I_{DC}[\phi_{0}]/I_{c} = \int d\theta d\phi (\sin 2\theta)^{2/m_{1}-1} \cos 2\theta \tag{27}
\]

The width of the steps can then be obtained as in the case of the voltage biased junction. The value of the phase \( \varphi_{0}^{\text{max}} + m\alpha_{0} \) for which the step-size is maximal is given by Eq. 17 with \( I_{N} \rightarrow TNJ_{m}[I_{1}/(I_{c}\gamma_{D})] \). The width of the \( m \)th Shapiro step is thus given by 2\( I_{DC}[\varphi_{0} = \varphi_{0}^{\text{max}}] \) (Eq. 27).

An exact numerical solution of Eq. 24 leads to the \( I-V \) characteristics shown in Fig. 7 for the underdamped (\( \beta_{0} = 0.2 \)) and in Fig. 8 for the overdamped (\( \beta_{0} = 2 \)) regimes. The left panels of both Figs. 7 and 8 correspond to \( n_{1} = n_{2} = 1 \) while the corresponding right panels present data for \( 2n_{1} = n_{2} = 2 \). We find clear existence of Shapiro steps in both the plots for small \( C \). The inset in Fig. 7 shows clear signature of the devil staircase structure which is consistent with the prediction of Ref. 19. We note that the step size in the overdamped region is much larger than that for a conventional junctions, where the quasiparticles obey Schrodinger equation. This can be seen by the comparing the plateaus indicated by the blue solid lines with those corresponding to red and green lines in Figs. 8. In contrast for the underdamped region, the step sizes are similar as can be seen from Fig. 7. These results are in accordance with standard expectation for such junctions. The Shapiro step widths can be discerned from these plots for both \( n_{1} = n_{2} = 1 \) and \( 2n_{1} = n_{2} = 2 \) (Figs. 7 and 8). A plot of these step width as a function of \( \chi \) for \( \omega_{D} = 3\omega_{J} = 6eV/h, \beta_{0} = 0.2 \) and \( I_{1} = I_{0} \) is shown in Fig. 9. The step width is found to be independent for \( n_{1} \neq n_{2} \) as per expectation. This confirms the barrier independence of the Shapiro step width for current biased JJs involving Weyl and multi-Weyl semimetals.

Finally we study the dependence of the Shapiro step width as a function of the dimensionless radiation amplitude \( A = I_{1}/I_{c} \). This behavior can be understood semi analytically using Eq. 27 for large drive amplitude. A plot of the step width, obtained from Eq. 27 is shown in the left panel of Fig. 10 for both conventional [in the AB limit] and Weyl JJs for several representative values of \( \chi \). We find that the step-width displays an oscillatory behavior as a function of \( A \); this behavior can be understood to be the consequence of the behavior of \( J_{m}[A/(\gamma\omega_{D})] \) as a function of \( A \). The behavior of these oscillations is therefore similar for junctions with same or different topological winding numbers. The right panel of Fig. 10 shows the comparison of results obtained from exact numerical solution of Eq. 24 with those obtained from perturbative analysis (Eqs. 26 and 27) for \( \chi = 0.01 \). We find that these results agree qualitatively even at small \( A \); however, as expected, a better quantitative agreement is achieved at large
we find that the Josephson current $I_J$ of the junction is independent of the dimensionless barrier strength $\chi$ in this limit. This also allows us to show that the product $I_c R_N$ in these junctions lies between the KO and the AB limits of conventional junctions and is independent of $\chi$: they depend only on the topological winding numbers $n_1$ and $n_2$ of the quasiparticles controlling the transport in these junctions. In this sense, for a given $n_1$ and $n_2$, $I_c R_N$ turns out to be a universal number. A deviation from the thin barrier limit leads to oscillation of $I_c R_N$ (and $I_J$) as a function of the barrier potential $U_0$. However, these oscillations differ in characteristics from their counterparts in junctions with $n_1 = n_2$: their amplitude decays with increasing $U_0$ for large $U_0/\Delta_0$ for a fixed junction width $d$. Thus we expect the barrier independence of $I_c R_N$ to be discernible from those in junctions of Weyl semimetals with $n_1 = n_2$.

For AC Josephson effect, a study of either current or voltage biased junctions in the presence of a microwave radiation leads to the expected Shapiro steps. For current biased junctions, we also find the devil staircase structure in the I-V characteristics in accordance with earlier predictions. The width of these Shapiro steps can be shown also to be independent of $\chi$ in the thin barrier limit. We have also studied the variation of the width of these steps for current biased junctions both using a semi-analytic perturbative approach and exact numerics; these approaches yield the expected oscillatory behavior of the step width with amplitude of the external radiation and lead to near identical results at large radiation amplitudes.

Our results can be verified by standard experiments used to detect Shapiro steps. Typically such experiments are carried out with a fixed external radiation frequency $\omega_D$; the amplitude of the microwave radiation, or equivalently $I_0$, is varied to detect the width of the step. Our proposition, for junctions between Weyl and multi-Weyl superconductors, is to carry out these experiments with different barrier potential $\chi$. We predict that in the thin barrier limit, the step width would be inde-

A.

IV. DISCUSSION

In this work, we have studied the DC and AC Josephson effects in a junction of Weyl-multi-Weyl semimetals. The key characteristic of such a junction lies in the fact that it consists of junction between two materials whose low-energy quasiparticles have different topological winding numbers. Our results indicate that the change in topological winding number across the junction leads to novel features in AC and DC Josephson effects which have no analog in conventional junctions. These features are qualitatively distinct from their counterparts found in junction between two topological materials with same winding number.

For DC Josephson effect, we derive an analytic expression for the Andreev bound state in the thin barrier limit. Using it, we find that the Josephson current $I_J$ of the junction is independent of the dimensionless barrier strength $\chi$ in this limit. This also allows us to show that the product $I_c R_N$ in these junctions lies between the KO and the AB limits of conventional junctions and is independent of $\chi$: they depend only on the topological winding numbers $n_1$ and $n_2$ of the quasiparticles controlling the transport in these junctions. In this sense, for a given $n_1$ and $n_2$, $I_c R_N$ turns out to be a universal number. A deviation from the thin barrier limit leads to oscillation of $I_c R_N$ (and $I_J$) as a function of the barrier potential $U_0$. However, these oscillations differ in characteristics from their counterparts in junctions with $n_1 = n_2$: their amplitude decays with increasing $U_0$ for large $U_0/\Delta_0$ for a fixed junction width $d$. Thus we expect the barrier independence of $I_c R_N$ to be discernible from those in junctions of Weyl semimetals with $n_1 = n_2$.

For AC Josephson effect, a study of either current or voltage biased junctions in the presence of a microwave radiation leads to the expected Shapiro steps. For current biased junctions, we also find the devil staircase structure in the I-V characteristics in accordance with earlier predictions. The width of these Shapiro steps can be shown also to be independent of $\chi$ in the thin barrier limit. We have also studied the variation of the width of these steps for current biased junctions both using a semi-analytic perturbative approach and exact numerics; these approaches yield the expected oscillatory behavior of the step width with amplitude of the external radiation and lead to near identical results at large radiation amplitudes.

Our results can be verified by standard experiments used to detect Shapiro steps. Typically such experiments are carried out with a fixed external radiation frequency $\omega_D$; the amplitude of the microwave radiation, or equivalently $I_0$, is varied to detect the width of the step. Our proposition, for junctions between Weyl and multi-Weyl superconductors, is to carry out these experiments with different barrier potential $\chi$. We predict that in the thin barrier limit, the step width would be inde-
dependent of $\chi$. A deviation from the thin barrier limit would lead to oscillations of the barrier width with $\chi$; however, the amplitude of these oscillations would be small at high barrier potentials which will make the behavior of these junctions distinct from their $r_{11} = r_{22}$ counterparts.

In conclusion, we have studied JJs between Weyl and multi-Weyl semimetals with induced $s$–$d_\pm$ wave superconductivity. We have shown that a change of topological winding number across these junctions ensures that their $I_cR_N$ will be independent of the bias potential $\chi$. We have also shown the independence of the Shapiro step widths on $\chi$ in both current and voltage biased junctions in this limit.

Both these properties have no analogs in JJs of either conventional or topological materials studied earlier and we have discussed experiments which can detect such behavior.

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