THE CONTRAVARIANT FORM ON SINGULAR VECTORS OF A PROJECTIVE ARRANGEMENT

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ABSTRACT. We define the flag space and space of singular vectors for an arrangement $\mathcal{A}$ of hyperplanes in projective space equipped with a system of weights $\omega: \mathcal{A} \to \mathbb{C}$. We show that the contravariant bilinear form of the corresponding weighted central arrangement induces a well-defined form on the space of singular vectors of the projectivization. If $\sum_{H \in \mathcal{A}} a(H) = 0$, this form is naturally isomorphic to the restriction to the space of singular vectors of the contravariant form of any affine arrangement obtained from $\mathcal{A}$ by dehomogenizing with respect to one of its hyperplanes.

1. INTRODUCTION

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of affine hyperplanes in $\mathbb{C}^\ell$. Let $f_i: \mathbb{C}^\ell \to \mathbb{C}$ be an affine linear functional with zero locus $H_i$, for $1 \leq i \leq n$. Let $M = M(\mathcal{A}) = \mathbb{C}^\ell - \bigcup_{i=1}^n H_i$ be the complement to the arrangement. If $W$ is a $\mathbb{C}$-vector space, then $W^*$ denotes its dual space. Let $\mathbb{C}^x = \mathbb{C} - \{0\}$.

Let $\omega_i = d\log(f_i)$ for $1 \leq i \leq n$. Denote by $A$ the $\mathbb{C}$-subalgebra of the holomorphic De Rham complex of $M$ generated by the closed forms $1, \omega_1, \ldots, \omega_n$. The algebra $A$ is graded, $A = \oplus_{p=0}^n A^p$, and called the Arnol’d-Brieskorn-Orlik-Solomon algebra or the OS algebra of $\mathcal{A}$. The dual space $\mathcal{F} = \mathcal{F}(\mathcal{A}) := \oplus_{p \geq 0} \mathcal{F}^p$ of $A$ is called the flag space of $\mathcal{A}$, [SV91].

Let $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ be a vector of weights. The contravariant form of the weighted arrangement $(\mathcal{A}, a)$ is the symmetric bilinear form $S = \oplus S_p: \mathcal{F} \otimes \mathcal{F} \to \mathbb{C}$, where $S_p: \mathcal{F}^p \otimes \mathcal{F}^p \to \mathbb{C}$ is defined by

$$S_p(F, F') = \sum_J a_{J} F(\omega_J) F'(\omega_J).$$

The sum is over all sequences $J = (j_1, \ldots, j_p)$ with $1 \leq j_1 < \cdots < j_p \leq n$, $a_J = \prod_{j=1}^p a_{j_j}$, and $\omega_J = \omega_{j_1} \wedge \cdots \wedge \omega_{j_p}$, [SV91].

In particular, if $\{F_1, \ldots, F_n\} \subseteq \mathcal{F}^1$ is the basis dual to the basis $\{\omega_1, \ldots, \omega_n\}$ of $A^1 \cong \mathbb{C}^n$, then

$$S_1(F_i, F_j) = a_i \delta_{ij}.$$

The contravariant form has many remarkable properties, see [SV91] [V95] [V06] [V11]. It is a generalization of the Shapavalov form associated to a tensor product of highest weight representations of a simple Lie algebra – for this application $\mathcal{A}$ is a discriminantal arrangement and $a$ is determined by the representations.

The space $\mathcal{F}$ has a combinatorially defined differential $d: \mathcal{F}^p \to \mathcal{F}^{p+1}$. The space $A$ has a differential $\delta_a: A^p \to A^{p+1}$ defined by multiplication by $\omega_a := \sum_{i=1}^n a_i \omega_i$. The contravariant form $S$ induces a morphism of complexes $\psi: (\mathcal{F}, d) \to (A, \delta_a)$, see [SV91] and Section 2. The pair $(\mathcal{F}, d)$ is the flag complex of $\mathcal{A}$.

Let $\text{Sing}(\mathcal{F}^\ell) = \text{Sing}_a(\mathcal{F}^\ell) \subseteq \mathcal{F}^\ell$ be the annihilator of $\omega_a \wedge A^\ell-1$. It is called the subspace of singular vectors of $\mathcal{F}^\ell$, relative to $a$. This terminology is introduced in [V06] and motivated by

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In [SV91] the subspace $\text{Sing}(\mathcal{F}^\ell)$ for a discriminantal arrangement is interpreted as the subspace of singular vectors of a tensor product of Verma modules over a Kac-Moody algebra. The inclusion $\text{Sing}(\mathcal{F}^\ell) \hookrightarrow (A^\ell)^*$ induces an isomorphism
\[
\text{Sing}(\mathcal{F}^\ell) \rightarrow (H^\ell(A, \delta_a))^* = (A^\ell/(\omega_\alpha \wedge A^{\ell-1}))^*.
\]

Let $\Phi_a = \prod_i f_i^{-a_i}$ be the master function associated with $(A, a)$, and let $\mathcal{L}_a$ be the rank-one local system on $M$ whose local sections are the multiples of single-valued branches of $\Phi_a$. The inclusion of $(A, \delta_{ca})$ into the twisted algebraic de Rham complex of $\mathcal{L}_{ca}$ induces an isomorphism
\[
H^*(A, \delta_{ca}) \cong H^*(M, \mathcal{L}_{ca})
\]
for generic $c$ [SV91]. Since $\text{Sing}_{ca}(\mathcal{F}^\ell) = \text{Sing}_a(\mathcal{F}^\ell)$ for any nonzero $c$, this implies that $\text{Sing}_a(\mathcal{F}^\ell)$ is isomorphic to the local system homology $H_\ell(M, \mathcal{L}_{-ca})$ for generic $c$.

An important object is the restriction of the contravariant form $S_\ell$ to the subspace $\text{Sing}(\mathcal{F}^\ell)$. It relates linear and nonlinear characteristics of the weighted arrangement $(A, \delta)$ in projective hyperplanes $\mathcal{A}_\infty$. In [SV91] the subspace $\text{Sing}(\mathcal{F}^\ell)$ is defined, and the purpose of this note is to describe objects in such a way that they will not depend on the choice of the hyperplane at infinity of the flag space $\mathcal{F}$.

The orbit map $q: (\text{Ann}(\tilde{\omega}_a \wedge \tilde{A}) + \text{Ann}(q^*A^1))/\text{Ann}(q^*A^1), \tilde{S}_1|_{(\text{Ann}(\tilde{\omega}_a \wedge \tilde{A}) + \text{Ann}(q^*A^1))/\text{Ann}(q^*A^1)}$ described below.

Let $[u : v]$ be homogeneous coordinates on $\mathbb{P}^1$, with $x = \frac{-u}{v}$. The projectivization $\mathcal{A}_\infty$ of $\mathcal{A}$ is the arrangement in $\mathbb{P}^1$ of the points $p_1 = [1 : z_1], \ldots, p_n = [1 : z_n]$ and the point $p_0 = [1 : 0]$ at infinity. The weight of $p_i$ is $a_i$ for $1 \leq i \leq n$ and the weight of $p_0$ is $a_0 = -\sum_{i=1}^n a_i$.

In our construction we use the associated central arrangement in $\mathbb{C}^2$, the cone $\mathring{\mathcal{A}}$ of $\mathcal{A}_\infty$, consisting of the lines $v - z_i u = 0$ for $1 \leq i \leq n$ and the line $u = 0$. Introduce the following one-forms on $\mathbb{C}^2$: $\tilde{\omega}_i = d\log(v - z_i u)$ for $1 \leq i \leq n$, and $\tilde{\omega}_0 = d\log(u)$. The arrangement $\mathring{\mathcal{A}}$ is weighted with the weights $\tilde{a} = (a_0, \ldots, a_n)$. We will denote by $\tilde{M}, \tilde{A}, \tilde{F}, \tilde{\omega}, \tilde{S}$ the complement, Orlik-Solomon algebra, flag space of $\mathring{\mathcal{A}}$, special element, and the contravariant form of $(\mathring{\mathcal{A}}, \tilde{a})$, respectively.

The orbit map $q: \tilde{M} \to \tilde{M}/\mathbb{C}^\times = \mathbb{C} - \{z_1, \ldots, z_n\}$ induces an injection $q^*: A \to \tilde{A}$ whose image is the subalgebra generated by $\{\sum_{i=0}^n \lambda_i \tilde{\omega}_i \in \tilde{A}^1 \mid \sum_{i=0}^n \lambda_i = 0\} \subset \tilde{A}$. One computes
\[
q^*(\omega_i) = q^*(d\log(x - z_i)) = d\log(\frac{v}{u} - z_i) = \tilde{\omega}_i - \tilde{\omega}_0.
\]

Then the special element $\omega_a$ of $A^1$ is mapped by $q^*$ to the special element $\tilde{\omega}_a = \sum_{i=0}^n a_i \tilde{\omega}_i$ of $\tilde{A}^1$. Identifying $A^1$ with $q^*A^1$, the flag space $\mathcal{F}^1 = (A^1)^*$ is isomorphic to the quotient of $\tilde{F}$ by
the annihilator $\text{Ann}(q^*A^1) \subset \tilde{F}^1$ of $q^*A^1 \subset \tilde{A}^1$. The subspace $\text{Ann}(q^*A^1)$ is spanned by $\sum_{i=0}^n \tilde{F}_i$.

(Notice that in this consideration the index 0 does not play any special role.) The subspace $\text{Ann}(\tilde{w}_a \wedge \tilde{A}^0)$ of $\tilde{F}^1$ consists of flags $\sum_{i=0}^n c_i\tilde{F}_i$ such that $\sum_{i=0}^n c_i a_i = 0$. This subspace is orthogonal to the subspace $\text{Ann}(q^*A^1)$ relative to the contravariant form of $\tilde{A}$. Indeed we have $\tilde{S}_1(\sum_{i=0}^n \tilde{F}_i, \sum_{i=0}^n c_i\tilde{F}_i) = \sum_{i=0}^n c_i a_i = 0$. Thus, the contravariant form $\tilde{S}_1$ induces a well-defined form on the image of $\text{Ann}(\tilde{w}_a \wedge \tilde{A}^0)$ in $\tilde{F}^1/\text{Ann}(q^*A^1)$, namely, a form on

$$(\text{Ann}(\tilde{w}_a \wedge \tilde{A}^0) + \text{Ann}(q^*A^1))/\text{Ann}(q^*A^1) \cong \text{Ann}(\tilde{w}_a \wedge \tilde{A}^0)/(\text{Ann}(\tilde{w}_a \wedge \tilde{A}^0) \cap \text{Ann}(q^*A^1)).$$

The flags $\tilde{F}_1, \ldots, \tilde{F}_n$ induce a basis of $\tilde{F}^1/\text{Ann}(q^*A^1)$. Using this basis, we see that the form induced by $\tilde{S}_1$ on $(\text{Ann}(\tilde{w}_a \wedge \tilde{A}^0) + \text{Ann}(q^*A^1))/\text{Ann}(q^*A^1)$ corresponds to the restriction of the original form $S_1$ to the subspace $\text{Sing}(\tilde{F}^1)$ under the isomorphism of $\tilde{F}^1$ with $\tilde{F}^1/\text{Ann}(q^*A^1)$.

Notice that the form $\tilde{S}_1$ does not induce a well-defined form on $\tilde{F}^1 = \tilde{F}/\text{Ann}(q^*A^1)$ – the extension of $S_1|_{\text{Sing}(\tilde{F}^1)}$ defined by (1.2) depends on the choice of hyperplane at infinity.

In general, for any weighted affine arrangement $(A,a)$ in $\mathbb{C}^\ell$, we identify the pair $(\text{Sing}(\tilde{F}^\ell), S_1|_{\text{Sing}(\tilde{F}^\ell)})$ with the pair

$$\left((\text{Ann}(\tilde{w}_a \wedge \tilde{A}^{\ell-1}) + \text{Ann}(q^*A^\ell))/\text{Ann}(q^*A^\ell), \tilde{S}_\ell|_{(\text{Ann}(\tilde{w}_a \wedge \tilde{A}^{\ell-1}) + \text{Ann}(q^*A^\ell))/\text{Ann}(q^*A^\ell)}\right)$$

expressed in terms of the cone $\tilde{A}$ of the projectivization $A_\infty$ of $A$.

Our statement that the pair $(\text{Sing}(\tilde{F}^\ell), S_1|_{\text{Sing}(\tilde{F}^\ell)})$ can be constructed in terms of $A_\infty$, without choosing a particular hyperplane at infinity, is analogous to the following fact from representation theory. Let $V_{\Lambda_i}, i = 0, \ldots, n$, be irreducible finite dimensional highest weight representations of a simple Lie algebra. Here $\Lambda_i$ is the highest weight of $V_{\Lambda_i}$. Let $\Lambda_0^\vee$ be the highest weight of the representation dual to $V_{\Lambda_0}$. Let $S_1$ be the Shapovalov form on $V_{\Lambda_0}$. Let $\text{Sing}(\otimes_{i=0}^n V_{\Lambda_i})[0] \subset \otimes_{i=0}^n V_{\Lambda_i}$ be the subspace of singular vectors of weight zero and $\text{Sing}(\otimes_{i=1}^n V_{\Lambda_i})[\Lambda_0^\vee] \subset \otimes_{i=1}^n V_{\Lambda_i}$ the subspace of singular vectors of weight $\Lambda_0^\vee$. Then the pair $(\text{Sing}(\otimes_{i=1}^n V_{\Lambda_i})[\Lambda_0^\vee], (\otimes_{i=1}^n S_i)|_{\text{Sing}(\otimes_{i=1}^n V_{\Lambda_i})[\Lambda_0^\vee]})$ is isomorphic to the pair $(\text{Sing}(\otimes_{i=0}^n V_{\Lambda_i})[0], (\otimes_{i=0}^n S_i)|_{\text{Sing}(\otimes_{i=0}^n V_{\Lambda_i})[0]})$.

2. FLAG COMPLEX AND CONTRAVARIANT FORM OF A CENTRAL ARRANGEMENT

We recall in more detail some of the theory of flag complexes from [SV91]. The following notation, which differs from the notation of [OT92] will be used throughout the rest of the paper. For general background on arrangements see [OT92].

Suppose $A = \{H_0, \ldots, H_n\}$ is a central arrangement in $\mathbb{C}^{\ell+1}$. Let $f_0, \ldots, f_n \in (\mathbb{C}^{\ell+1})^*$ with $H_i = \text{ker}(f_i)$ for $0 \leq i \leq n$. Let $\omega_i = \frac{1}{f_i}$ for $0 \leq i \leq n$, and let $A$ be the OS algebra of $A$, as defined in [OT]. Let $E$ be the graded exterior algebra over $\mathbb{C}$ with generators $e_0, \ldots, e_n$ of degree one. Let $\partial: E^p \to E^{p-1}$ be defined by

$$\partial(e_{j_1} \wedge \cdots \wedge e_{j_p}) = \sum_{i=1}^p (-1)^{i-1} e_{j_1} \wedge \cdots \wedge \hat{e}_{j_i} \wedge \cdots \wedge e_{j_p},$$

where $\hat{\cdot}$ denotes deletion. If $J = (j_1, \ldots, j_p)$, denote the product $e_{j_1} \wedge \cdots \wedge e_{j_p}$ by $e_J$. Say $J$ is dependent if $\{f_i \mid i \in J\}$ is linearly dependent in $(\mathbb{C}^{\ell+1})^*$. Let $I$ be the ideal of $E$ generated by $\{\partial e_J \mid J$ is dependent\}. By [OS80], the surjection $E \to A$ sending $e_i$ to $\omega_i$ has kernel $I$. We tacitly identify $A$ with $E/I$. The map $\partial$ induces a well-defined map $\partial: A \to A$, a graded derivation of degree $-1$, and $(A, \partial)$ is a chain complex.

Let $L = L(A)$ be the intersection lattice of $A$, the set of intersections of subcollections of $A$, partially-ordered by reverse inclusion. Let $\text{Flag} = \bigoplus_{p=0}^{\ell+1} \text{Flag}^p$ be the graded $\mathbb{C}$-vector space with
Flag\(^p\) having basis consisting of chains \((X_0 \cdots < X_p)\) of \(L\) satisfying \(\text{codim}(X_i) = i\) for \(0 \leq i \leq p\). Such a chain will be called a flag. For each ordered subset \(J = (j_1, \ldots, j_p)\) of \(\{0, \ldots, n\}\), let \(\xi(J)\) be the chain \((X_0 < \cdots < X_p)\) of \(L\), where \(X_0 = \mathbb{C}^\ell + 1\) and \(X_i = \bigcap_{k=1}^i H_{j_k}\) for \(1 \leq i \leq p\). Note that \(\xi(J)\) is a flag if and only if \(\{H_i \mid i \in J\}\) is independent in \(\mathcal{A}\). If \(\pi\) is a permutation of \(\{1, \ldots, p\}\), let \(J^\pi = (j_{\pi(1)}, \ldots, j_{\pi(p)})\). For any flag \(F \in \text{Flag}^p\) and any ordered \(p\)-subset \(J\) of \(\{1, \ldots, n\}\), there is at most one permutation \(\pi\) such that \(F = \xi(J^\pi)\).

Define a bilinear pairing 
\[
\langle \cdot, \cdot \rangle: \text{Flag}^p \otimes E^p \to \mathbb{C}
\]
by
\[
\langle F, e_J \rangle = \begin{cases} 
\text{sgn}(\pi) & \text{if } \xi(J^\pi) = F \\
0 & \text{otherwise}
\end{cases}
\]
for every flag \(F\) in \(\text{Flag}^p\) and ordered \(p\)-subset \(J\) of \(\{0, \ldots, n\}\).

**Proposition 2.1** ([SV91]). \(\langle F, \partial e_J \rangle = 0\) for every \(F \in \text{Flag}^p\) and dependent \((p+1)\)-tuple \(J\). Moreover, if \((X_0 < \cdots < X_{i-1} < X_{i+1} \cdots X_p)\) is a chain in \(L\) with \(\text{codim}(X_j) = j\) then
\[
\left\langle \sum_{X_{i-1} < X < X_{i+1}} (X_0 < \cdots < X_{i-1} < X < X_{i+1} < \cdots X_p), e_J \right\rangle = 0,
\]
for every ordered \(p\)-subset \(J\) of \(\{0, \ldots, n\}\).

Let \(F = \oplus_{p=0}^{\ell+1} F^p\) be the quotient of \(\text{Flag}\) by the (homogeneous) subspace spanned by the sums
\[
\sum_{X_{i-1} < X < X_{i+1}} (X_0 < \cdots < X_{i-1} < X < X_{i+1} < \cdots X_p)
\]
as \((X_0 < \cdots < X_{i-1} < X_{i+1} < \cdots X_p)\) ranges over all chains in \(L\) with \(\text{codim}(X_j) = j\).

Denote the image of \((X_0 < \cdots < X_p)\) in \(\mathcal{F}^p\) by \([X_0 < \cdots < X_p]\). By Proposition 2.1, \(\langle \cdot, \cdot \rangle: \mathcal{F}^p \otimes A^p \to \mathbb{C}\)
induces a well-defined bilinear pairing \(\langle \cdot, \cdot \rangle: \mathcal{F}^p \otimes A^p \to \mathbb{C}\).

The pairing \(\langle \cdot, \cdot \rangle\) is a combinatorial model of the integration pairing of the ordinary homology and cohomology of the complement \(M\) with coefficients in \(\mathbb{C}\), see [SV91].

**Theorem 2.2** ([SV91]). The pairing \(\langle \cdot, \cdot \rangle: \mathcal{F}^p \otimes A^p \to \mathbb{C}\) is nondegenerate.

Let \(\varphi: A \to \mathcal{F}^*\) be defined by \(\varphi(x) = (-, x): \mathcal{F} \to \mathbb{C}\). By Theorem 2.2, \(\varphi\) is an isomorphism. The value of \(\varphi(\omega_J)\) in terms of the canonical basis of Flag is given in [SV91] (2.3.2). Similarly, \(\varphi^*: \mathcal{F} \to A^*\) is an isomorphism, with \(\varphi^*(F) = (F, -): A \to \mathbb{C}\). \(\mathcal{F}\) is called the flag space of \(A\).

Let \(d: \text{Flag}^p \to \text{Flag}^{p+1}\) be the linear map defined by
\[
d(X_0 < \cdots < X_p) = \sum_{\text{codim}(X) = p+1} (X_0 < \cdots < X_p < X).
\]

Clearly \(d\) induces a linear map \(d: \mathcal{F}^p \to \mathcal{F}^{p+1}\). Relations (2.2) imply \(d \circ d = 0\). The pair \((\mathcal{F}, d)\) is called the flag complex of \(A\). The following result is a reformulation of Lemma 2.3.4 of [SV91].

**Theorem 2.3.** For any \(F \in \mathcal{F}^p\) and \(x \in A^{p+1}\),
\[
\langle F, \partial x \rangle = \langle df, x \rangle.
\]

Let \(d^*: (\mathcal{F}^p)^* \to (\mathcal{F}^{p+1})^*\) be the adjoint of \(d: \mathcal{F}^{p+1} \to \mathcal{F}^p\).

**Corollary 2.4** ([SV91] Lemma 2.3.4). The map \(\varphi: (A, \partial) \to (\mathcal{F}^*, d^*)\) given by \(\varphi(x) = (-, x)\) is an isomorphism of chain complexes.
Similarly \( \varphi^* : (\mathcal{F}, d) \to (A^*, \partial^*) \) is an isomorphism of cochain complexes.

There is a decomposition of \( \mathcal{F} \) dual to the Brieskorn decomposition [OT92, Lemma 5.91] of \( A \). For \( X \in L \) let \( \mathcal{F}^p \) be the image in \( \mathcal{F} \) of the subspace of \( \text{Flag}^p \) spanned by flags that terminate at \( X \). Then by [SV91, (2.12)],

\[
\mathcal{F}^p = \bigoplus_{\text{codim}(X) = p} \mathcal{F}^p_X.
\]

Let \( a = (a_0, \ldots, a_n) \in \mathbb{C}^{n+1} \). Let \( \omega_a = \sum^n_{j=0} a_j \omega_j \) and \( \delta_a : A \to A \) with \( \delta_a(x) = \omega_a \wedge x \). Let

\[
S = \oplus S_p : \mathcal{F} \otimes \mathcal{F} \to \mathbb{C}
\]

be the contravariant form of the weighted arrangement \((\mathcal{A}, a)\), as defined in (1.1). \( S \) gives rise to the map \( \mathcal{F} \to \mathcal{F}^* \) that sends \( F \) to \( S(F, -) : \mathcal{F} \to \mathbb{C} \). By composing this map with the isomorphism \( \varphi^{-1} : \mathcal{F}^* \to A \), one obtains a map \( \psi : \mathcal{F} \to A \), characterized by the formula

\[
S_p(F, F') = \langle F, \psi(F') \rangle,
\]

for all \( F, F' \in \mathcal{F}^p \), for each \( p \). \( \psi \) is called the contravariant map.

**Theorem 2.5** ([SV91, Lemma 3.2.5]). The contravariant map \( \psi : (\mathcal{F}, d) \to (A, \delta_a) \) is a morphism of cochain complexes.

**Corollary 2.6.** For every \( F \in \mathcal{F}^p \) and \( F' \in \mathcal{F}^{p-1} \),

\[
S(F, dF') = \langle F, \omega_a \wedge \psi(F') \rangle.
\]

3. **Projective OS algebra and flag space**

Let \( \mathcal{A} \) be a central arrangement as in [2]. Let \( \tilde{\mathcal{A}} \) denote the projectivization of \( \mathcal{A} \), consisting of the projective hyperplanes

\[
\tilde{H}_i := (H_i - \{0\})/\mathbb{C}^\times
\]

in \( \mathbb{P}^\ell = (\mathbb{C}^{\ell+1} - \{0\})/\mathbb{C}^\times \), for \( 0 \leq i \leq n \). Let \( \tilde{M} = \mathbb{P}^\ell - \cup_{i=0}^\ell \tilde{H}_i \) be the complement to \( \tilde{\mathcal{A}} \) in \( \mathbb{P}^\ell \).

**Definition 3.1** ([CDFV10]). The OS algebra \( \tilde{A} = A(\tilde{\mathcal{A}}) \) of the projective arrangement \( \tilde{\mathcal{A}} \) is the kernel of \( \partial : A \to A \).

Denote by \( \iota : \tilde{\mathcal{A}} \to A \) the natural imbedding. Let \( (\mathcal{F}, d) \) be the flag complex of \( \mathcal{A} \).

**Definition 3.2.** The flag space \( \tilde{\mathcal{F}} = \mathcal{F}(\tilde{\mathcal{A}}) \) of the projective arrangement \( \tilde{\mathcal{A}} \) is the quotient \( \mathcal{F} / \text{im}(d) \).

Thus \( \tilde{\mathcal{F}} \) is obtained from \( \mathcal{F} \) by introducing the additional relations

\[
\sum_{X > X_p} (X_0 < \cdots < X_p < X) = 0,
\]

where \( (X_0 < \cdots < X_p) \) ranges over all flags of length \( p \) in \( L \), for \( 0 \leq p \leq \ell \).

Let \( \pi : \mathcal{F} \to \tilde{\mathcal{F}} \) be the canonical projection. For \( F \in \mathcal{F} \) we write \( \tilde{F} = \pi(F) \). Then, for instance, \( \sum_{i=0}^\ell \tilde{F}_i = 0 \), where \( \{F_0, \ldots, F_n\} \subseteq \tilde{F}^1 \) is the basis dual to \( \{\omega_0, \ldots, \omega_n\} \subseteq A^1 \).

**Theorem 3.3.** Let \( \rho : A^* \to A^* \) be given by restriction. Then the isomorphism \( \varphi^* : \mathcal{F} \to A^* \) induces an isomorphism \( \tilde{\varphi}^* : \tilde{\mathcal{F}} \to \tilde{A}^* \), given by the commutative diagram

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\varphi^*} & A^* \\
\pi \downarrow & & \downarrow \\
\tilde{\mathcal{F}} & \xrightarrow{\tilde{\varphi}^*} & \tilde{A}^*
\end{array}
\]
Theorem 3.3 is proved below.

Lemma 3.4 ([OT92 Lemma 3.13]). The complex
\[ 0 \to A^\ell \xrightarrow{\partial} A^{\ell-1} \to \cdots \to A^1 \xrightarrow{\partial} A^0 \to 0 \]
is exact.

Corollary 3.5. The complex
\[ 0 \to \mathcal{F}^0 \xrightarrow{d} \mathcal{F}^1 \to \cdots \to \mathcal{F}^{\ell-1} \xrightarrow{d} \mathcal{F}^\ell \to 0 \]
is exact.

Corollary 3.5 also follows from [SV91, Cor. 2.8].

Proposition 3.6. \(\text{Ann}(\bar{A}) = \text{im}(d; \mathcal{F} \to \mathcal{F})\).

Proof. By Lemma 3.4, \(\bar{A} = \text{im}(\partial)\). Then \(F \in \text{Ann}(\bar{A})\) if and only if \(\langle F, \partial x \rangle = 0\) for all \(x \in A\). By Theorem 2.2, this is equivalent to the statement \(\langle dF, x \rangle = 0\) for every \(x \in A\), or \(dF = 0\) by Theorem 2.2. Then \(\text{Ann}(\bar{A}) = \ker(d)\), which equals \(\text{im}(d)\) by Corollary 3.5. □

Proof of Theorem 3.3: The assertion now follows immediately from Proposition 3.6 and Definition 3.2.

Lemma 3.4 also has the following consequence.

Corollary 3.7. \(\bar{A}\) is the subalgebra of \(A\) generated by 1 and \(\bar{A}^1 = \{\sum_{i=0}^n c_i \omega_i \mid \sum_{i=0}^n c_i = 0\}\).

Proof. By Lemma 3.4 we have \(\bar{A} = \text{im}(\partial)\). One can show by induction that \(\partial \omega_J = (\omega_{j_2} - \omega_{j_1}) \wedge \cdots \wedge (\omega_{j_p} - \omega_{j_1})\), for any ordered subset \(J = (j_1, \ldots, j_p)\) of \(\{0, \ldots, n\}\) with \(p \geq 2\). Each factor on the right-hand side lies in \((\bar{A})^1\). Since such \(\omega_J\) (along with 1) span \(\bar{A}\), the result follows. □

Remark 3.8. The algebra \(\bar{A}\) is naturally isomorphic to \(H^*(M, \mathbb{C})\) and \(\iota: \bar{A} \to A\) is identified with the homomorphism \(q^*: H^*(M, \mathbb{C}) \to H^*(\bar{M}, \mathbb{C})\) induced by the orbit map \(q: M \to \bar{M}\). The space \(\mathcal{F}\) is naturally isomorphic to the homology space \(H_*(\bar{M}, \mathbb{C})\) and the projection \(\pi: \mathcal{F} \to \bar{F}\) is identified with the homomorphism \(q_*: H^*(M, \mathbb{C}) \to H^*(\bar{M}, \mathbb{C})\).

4. Singular subspace and contravariant form for projective arrangements

Let \(\mathcal{A} = \{H_0, \ldots, H_n\}\) be a central arrangement in \(\mathbb{C}^{\ell+1}\) as above. Let \(a = (a_0, \ldots, a_n) \in \mathbb{C}^{n+1}\) and \(\omega_a = \sum_{i=0}^n a_i \omega_i \in A^1\). We identify the flag space \(\mathcal{F} = \mathcal{F}(\mathcal{A})\) with \(A^*\) via the map \(\varphi^*\) of Section 2.

Definition 4.1. The singular subspace \(\text{Sing}(\mathcal{F}^\ell) \subset \mathcal{F}^\ell\) is
\[ \pi(\text{Ann}(\omega_a \wedge A^{\ell-1})) = (\text{Ann}(\omega_a \wedge A^{\ell-1}) + \text{im}(d)) / \text{im}(d) \subset \mathcal{F}^\ell / \text{im}(d) = \bar{\mathcal{F}}^\ell. \]

Let \(S: \mathcal{F} \otimes \mathcal{F} \to \mathbb{C}\) be the contravariant form of the central arrangement \(\mathcal{A}\), as defined in [1.1].

Theorem 4.2. The subspaces \(\text{Ann}(\omega_a \wedge A)\) and \(\text{im}(d)\) of \(\mathcal{F}\) are orthogonal with respect to \(S\).

Proof. In §2 we constructed the contravariant map \(\psi: \mathcal{F} \to A\) satisfying \(S_p(F, F') = \langle F, \psi(F') \rangle\) for every \(F, F' \in \mathcal{F}^p\). By Corollary 2.6, \(\psi\) satisfies \(S_p(F, dF') = \langle F, \omega_a \wedge \psi(F') \rangle\) for all \(F \in \mathcal{F}^p\) and \(F' \in \mathcal{F}^{p-1}\). Suppose \(F \in \text{Ann}(\omega_a \wedge A^{p-1}) \subset \mathcal{F}^p\). Then, for every \(F' \in \mathcal{F}^{p-1}\), \(\langle F, \omega_a \wedge \psi(F') \rangle = 0\). Then \(S_p(F, dF') = 0\) for every \(F' \in \mathcal{F}^{p-1}\). Thus \(\text{Ann}(\omega_a \wedge A)\) is orthogonal to \(\text{im}(d)\). □
Define the bilinear form
\[ \bar{S}_\ell : \text{Sing}(F^\ell) \otimes \text{Sing}(F^\ell) \to \mathbb{C} \]
by \( \bar{S}_\ell(F,F') = S_F(F,F') \).

**Corollary 4.3.** The form \( \bar{S}_\ell : \text{Sing}(F^\ell) \otimes \text{Sing}(F^\ell) \to \mathbb{C} \) is well-defined.

5. **Dehomogenization**

Throughout this section we assume \( a \in \mathbb{C}^{n+1} \) satisfies \( \sum_{i=0}^{n} a_i = 0 \). Then \( \omega_a \in \bar{A}^1 \) and \( \omega_a \wedge \bar{A} \subseteq \bar{A} \).

Fix a hyperplane \( H_j \in \mathcal{A} \). For simplicity of notation we assume \( j = 0 \), but the index 0 will play no special role. Choose coordinates \( (x_0, \ldots, x_\ell) \) on \( \mathbb{C}^{\ell+1} \) so that \( H_0 \) is defined by the equation \( x_0 = 0 \). The **decone** of \( \mathcal{A} \) relative to \( H_0 \) is an affine arrangement \( \mathcal{A} = \{dH_1, \ldots, dH_n\} \) in \( \mathbb{C}^\ell \). The affine hyperplane \( dH_i \) is defined by \( \hat{f}_i(x_1, \ldots, x_\ell) = 0 \), where \( \hat{f}_i(x_1, \ldots, x_\ell) = f_i(x_0, \ldots, x_\ell) \) and \( f_i : \mathbb{C}^{\ell+1} \to \mathbb{C} \) is a linear defining form for \( H_i \). Let \( \bar{F} = \mathbb{C}^\ell - \bigcup_{i=1}^n dH_i \), \( \hat{\omega}_i = d\log(\hat{f}_i) \), and let \( \bar{A} \) be the algebra of differential forms on \( \mathbb{C}^\ell \). We note for future reference that \( \hat{\omega}_0 \) and \( \hat{\omega}_i \) are well-defined.

**Lemma 5.1.** The map \( \varepsilon : \hat{A} \to \bar{A} \), \( \hat{\omega}_i \mapsto \hat{\omega}_i - \omega_0 \), is a well-defined isomorphism. Moreover, \( \varepsilon \) sends \( \hat{\omega}_a = \sum_{i=1}^n a_i \hat{\omega}_i \) to \( \omega_a \).

We note for future reference that
\[ (5.1) \quad e(\omega_j) = (\omega_{j_1} - \omega_0) \wedge \cdots \wedge (\omega_{j_p} - \omega_0) = \partial \omega_{(0,j)}, \]
for any ordered p-subset \( \hat{J} = (j_1, \ldots, j_p) \) of \( \{1, \ldots, n\} \), where \( (0, \hat{J}) = (0, j_1, \ldots, j_p) \).

As in \( \mathbb{2} \) the flag space \( \bar{F} = \mathcal{F}(\mathcal{A}) \) of the affine arrangement \( \mathcal{A} \) can be identified with \( \hat{A}^* \), and the singular subspace \( \text{Sing}(F^\ell) \subset F^\ell \) relative to \( \hat{a} \) is defined by \( \text{Sing}(F^\ell) = \text{Ann}(\hat{\omega}_0 \wedge \hat{A}^{\ell-1}) \). The contravariant form \( \bar{S} = \oplus \bar{S}_p \) of \( \mathcal{A} \) is given by
\[ \bar{S}_p(F, F') = \sum_{j} \hat{a}_j \bar{F}(\hat{\omega}_j) \bar{F}'(\hat{\omega}_j), \]
summing over increasing \( p \)-tuples \( \hat{J} = (j_1, \ldots, j_p) \) of elements of \( \{1, \ldots, n\} \). We identify \( \bar{F} \) with \( \hat{A}^* \) via the isomorphism \( \hat{\varphi}^* \) of Theorem 3.3.

In this section we prove the following theorem.

**Theorem 5.2.** The map \( \varepsilon^* : \bar{F} \to \hat{F} \) restricts to an isomorphism of inner-product spaces
\[ \varepsilon^* : (\text{Sing}(\hat{F}^\ell), \bar{S}_\ell|_{\text{Sing}(\hat{F}^\ell)}) \xrightarrow{\cong} (\text{Sing}(\bar{F}^\ell), \bar{S}_\ell|_{\text{Sing}(\bar{F}^\ell)}). \]

Recall that \( \bar{A} = \ker(\partial : A \to A) \). Define \( \sigma : A^{p-1} \to A^p \) by \( \sigma(x) = \omega_0 \wedge x \).

**Lemma 5.3.** For each \( p \), we have \( A^p = (\omega_0 \wedge \bar{A}^{p-1}) \oplus \bar{A}^p \).

**Proof.** By Lemma 3.4 the complex \( (A, \partial) \) is exact. Hence, for each \( p \), there is a short exact sequence
\[ (5.2) \quad 0 \longrightarrow \bar{A}^p \overset{i}{\longrightarrow} A^p \overset{\partial}{\longrightarrow} \bar{A}^{p-1} \longrightarrow 0. \]
This sequence splits: the map \( \sigma : \bar{A}^{p-1} \to A^p \) defined above is a section of \( \partial : A^p \to \bar{A}^{p-1} \). Indeed, \( \partial \circ \sigma(x) = \partial(\omega_0 \wedge x) = \partial \omega_0 \wedge x - \omega_0 \wedge \partial x = x \) for \( x \in \bar{A}^{p-1} \). Then \( A^p = \text{im}(\sigma) \oplus \ker(\partial) = (\omega_0 \wedge \bar{A}^{p-1}) \oplus A^p \) as claimed. \( \square \)

Recall that \( \ker(\pi) = \text{Ann}(\bar{A}) \). The map \( \pi : \bar{F}^p \to \bar{F}^{p-1} \) is the adjoint of the inclusion \( \bar{A} \to A \). Let \( \sigma^* : F^p \to F^{p-1} \) be the adjoint of \( \sigma : A^{p-1} \to A^p \).
Lemma 5.4. We have the following statements:

(i) \( \mathcal{F}^p = \ker(\sigma^*) \oplus \ker(\pi) \).

(ii) The restriction \( \pi|_{\text{Ann}(\omega_0 \wedge \bar{A}^{p-1})} : \text{Ann}(\omega_0 \wedge \bar{A}^{p-1}) \to \mathcal{F}^p \) is an isomorphism.

(iii) \( \text{Ann}(\omega_0 \wedge \bar{A}) = \text{Ann}(\omega_0 \wedge A) \).

Proof. Taking duals in (5.2), we obtain the exact sequence

\[
0 \to \mathcal{F}^{p-1} \to \mathcal{F}^p \to \pi \to 0.
\]

The map \( \sigma^* : \mathcal{F}^p \to \mathcal{F}^{p-1} \) satisfies \( \sigma^* \circ \partial^* = (\partial \circ \sigma)^* = \text{id}_{\mathcal{F}^{p-1}} \). Then \( \mathcal{F}^p = \ker(\sigma^*) \oplus \text{im}(\partial^*) \).

Statement (i) follows by exactness.

We have \( \sigma^*(F)(x) = \sigma(x) = F(\omega_0 \wedge x) \) for all \( x \in \bar{A} \). Then \( F \in \ker(\sigma^*) \) if and only if \( F \in \text{Ann}(\omega_0 \wedge \bar{A}^{p-1}) \), i.e., \( \ker(\sigma^*) = \text{Ann}(\omega_0 \wedge \bar{A}^{p-1}) \). Applying (i), we have \( \text{Ann}(\omega_0 \wedge \bar{A}^{p-1}) \cap \ker(\pi) = 0 \) and \( \mathcal{F}^p = \pi(\mathcal{F}^p) = \pi(\ker(\sigma^*)) = \pi(\text{Ann}(\omega_0 \wedge \bar{A}^{p-1})) \). This proves (ii).

For (iii), assume \( F \in \text{Ann}(\omega_0 \wedge \bar{A}) \) and \( x \in \omega_0 \wedge A \). Write \( x = \omega_0 \wedge y \) for \( y \in A \). By Lemma 5.3, we can write \( y = y_1 + y_2 \) with \( y_1 \in \omega_0 \wedge \bar{A} \) and \( y_2 \in \bar{A} \). Then \( \omega_0 \wedge y_1 = 0 \), so \( x = \omega_0 \wedge y_2 \in \omega_0 \wedge A \). Then \( F(x) = 0 \). Thus \( \text{Ann}(\omega_0 \wedge A) \subseteq \text{Ann}(\omega_0 \wedge A) \). The opposite inclusion holds because \( \omega_0 \wedge A \subseteq \omega_0 \wedge A \).

Recall the decomposition (2.5) of \( \mathcal{F} \). In this context Lemma 5.4 yields the following result, which we will consider in more detail in the next section.

Corollary 5.5. For \( 0 \leq p \leq \ell \),

\[
\mathcal{F}^p \cong \bigoplus_{\text{codim}(X) = p} \mathcal{F}^p_X
\]

Proof. Using (2.7) one can check easily that \( \{X_0 < \cdots < X_p\} \in \text{Ann}(\omega_0 \wedge \bar{A}^{p-1}) \) if and only if \( H_0 \not\subseteq X_p \). Then, for each \( X \in L \) of codimension \( p \),

\[
\text{Ann}(\omega_0 \wedge \bar{A}^{p-1}) \cap \mathcal{F}^p_X = \begin{cases} 0 & \text{if } H_0 \not\subseteq X \\ \mathcal{F}^p_X & \text{if } H_0 \subseteq X \end{cases}
\]

The claim then follows from parts (ii) and (iii) of Lemma 5.4.

Lemma 5.6. We have the following statements:

(i) \( \text{Ann}(\omega_0 \wedge \bar{A}) \cap \text{Ann}(\omega_0 \wedge A) \subseteq \text{Ann}(\omega_0 \wedge A) \).

(ii) \( \text{Ann}(\omega_0 \wedge \bar{A}) = \text{Ann}(\omega_0 \wedge A) + \text{Ann}(A) \).

(iii) \( \text{Sing}(\mathcal{F}^\ell) = \pi(\text{Ann}(\omega_0 \wedge A^{\ell-1})) \).

Proof. Let \( F \in \text{Ann}(\omega_0 \wedge \bar{A}) \cap \text{Ann}(\omega_0 \wedge A) \) and \( x \in \omega_0 \wedge A \). Write \( x = \omega_0 \wedge y \) with \( y \in A \).

By Lemma 5.3, we can write \( y = y_1 + y_2 \) with \( y_1 \in \omega_0 \wedge \bar{A} \) and \( y_2 \in \bar{A} \). Write \( y_1 = \omega_0 \wedge y'_1 \) with \( y'_1 \in \bar{A} \). Since \( \omega_0 \wedge \bar{A} \subseteq \bar{A} \), \( \omega_0 \wedge y'_1 = \omega_0 \wedge (\omega_0 \wedge y'_1) = \omega_0 \wedge (-y'_1) \in \omega_0 \wedge A \). Then \( x = \omega_0 \wedge y + \omega_0 \wedge y_2 \in \omega_0 \wedge A + \omega_0 \wedge A \). Then \( F(x) = 0 \). This proves (i).

Let \( F \in \text{Ann}(\omega_0 \wedge \bar{A}) \). By part (i) of Lemma 5.4, we can write \( F = F_1 + F_2 \) where \( F_1 \in \ker(\sigma^*) = \text{Ann}(\omega_0 \wedge \bar{A}) \) and \( F_2 \in \ker(\pi) = \text{Ann}(A) \). Since \( \omega_0 \wedge \bar{A} \subseteq \bar{A} \), \( F_2 \in \text{Ann}(\omega_0 \wedge \bar{A}) \). Then \( F_1 = F - F_2 \in \text{Ann}(\omega_0 \wedge A) \). Then \( F_1 \in \text{Ann}(\omega_0 \wedge A) \cap \text{Ann}(\omega_0 \wedge \bar{A}) \). Then \( F_1 \in \text{Ann}(\omega_0 \wedge A) \) by (i). Thus \( F = F_1 + F_2 \in \text{Ann}(\omega_0 \wedge \bar{A}) + \text{Ann}(A) \). This proves \( \text{Ann}(\omega_0 \wedge \bar{A}) \subseteq \text{Ann}(\omega_0 \wedge A) + \text{Ann}(A) \). The opposite inclusion follows easily from the fact that \( \omega_0 \wedge \bar{A} \subseteq (\omega_0 \wedge A) \cap \bar{A} \). This proves (ii).

Part (iii) follows immediately from (ii).
We note the following consequence of part (iii) of Lemma 5.6 for later use. As observed earlier, \((\bar{A}, \delta_a)\) is a subcomplex of \((A, \delta_a)\).

**Corollary 5.7.** The inclusion \(\text{Sing}(\mathcal{F}^\ell) \hookrightarrow \mathcal{F}^\ell = (\bar{A}^\ell)^*\) induces an isomorphism

\[
\text{Sing}(\mathcal{F}^\ell) \xrightarrow{\cong} (\mathcal{H}^\ell(\bar{A}, \delta_a))^*.
\]

**Proof.** Lemma 3.4 implies \(\bar{A}^{\ell+1} = 0\), so \(\mathcal{H}^\ell(\bar{A}, \delta_a) = \bar{A}^\ell/(\omega_a \wedge \bar{A}^{\ell-1})\). Then \((\mathcal{H}^\ell(\bar{A}, \delta_a))^*\) is isomorphic to the annihilator of \(\omega_a \wedge \bar{A}^{\ell-1}\) in \((\bar{A}^\ell)^*\). This annihilator is equal to

\[
(\text{Ann}(\omega_a \wedge \bar{A}^{\ell-1}) + \text{Ann}(\bar{A}^\ell))/\text{Ann}(\bar{A}^\ell).
\]

By Definition 4.1, Proposition 3.6, and Lemma 5.6(iii), this is equal to \(\text{Sing}(\mathcal{F}^\ell)\). \(\square\)

**Proof of Theorem 5.1.** Let \(F \in \text{Ann}(\omega_a \wedge \bar{A}^{\ell-1})\), and let \(\hat{x} \in \omega_a \wedge \bar{A}^{\ell-1}\). Then \(\epsilon^*(\mathcal{F})(\hat{x}) = F(\epsilon(\hat{x}))\). Since \(\epsilon(\omega_a) = \omega_a\), \(\epsilon(\hat{x}) \in \omega_a \wedge \bar{A}^{\ell-1}\), so \(F(\epsilon(\hat{x})) = 0\). Then \(\epsilon^*(\mathcal{F})(\hat{x}) = 0\). Thus \(\epsilon^*(\text{Sing}(\mathcal{F}^\ell)) \subseteq \text{Sing}(\mathcal{F}^\ell)\).

Conversely, suppose \(\hat{F} \in \text{Sing}(\mathcal{F}^\ell)\). Write \(\hat{F} = \epsilon^*(\mathcal{F})\) with \(F \in \mathcal{F}^\ell\). Let \(x \in \omega_a \wedge \bar{A}^{\ell-1}\). Then \(x \in \bar{A}^\ell\), so \(x = \epsilon(\hat{x})\) for some \(\hat{x} \in \bar{A}\). Since \(x \in \omega_a \wedge \bar{A}^{\ell-1}\), \(\hat{x} \in \omega_a \wedge \bar{A}^{\ell-1}\). Then \(\mathcal{F}(\hat{x}) = 0\) by definition of \(\text{Sing}(\mathcal{F}^\ell)\). Then \(F(x) = F(\epsilon(\hat{x})) = \epsilon^*(\mathcal{F})(\hat{x}) = \hat{F}(\hat{x}) = 0\). This shows that \(F \in \text{Ann}(\omega_a \wedge \bar{A}^{\ell-1})\). Then \(F \in \text{Ann}(\omega_a \wedge \bar{A}^{\ell-1})\) by part (iii) of Lemma 5.6. Then \(F \in \text{Sing}(\mathcal{F}^\ell)\) by definition of \(\text{Sing}(\mathcal{F}^\ell)\). Thus \(\text{Sing}(\mathcal{F}^\ell) \subseteq \epsilon^*(\text{Sing}(\mathcal{F}^\ell))\), and \(\epsilon^*\) restricts to an isomorphism \(\text{Sing}(\mathcal{F}^\ell) \rightarrow \text{Sing}(\mathcal{F}^\ell)\).

It remains to prove that \(\hat{S}_n(\epsilon^*(\mathcal{F}), \epsilon^*(\mathcal{F}')) = \hat{S}_n(\mathcal{F}, \mathcal{F}')\) for all \(\mathcal{F}, \mathcal{F}' \in \text{Sing}(\mathcal{F}^\ell)\). By (5.1), we have

\[
\hat{S}_n(\epsilon^*(\mathcal{F}), \epsilon^*(\mathcal{F}')) = \sum_j \hat{a}_j \epsilon^*(\mathcal{F})(\omega_j) \epsilon^*(\mathcal{F}')(\omega_j) = \sum_j a_j F(\epsilon(\omega_j)) F'(\epsilon(\omega_j)) = \sum_j a_j F(\partial(\omega_{(0,j)})) F'(\partial(\omega_{(0,j)})).
\]

The sum is over increasing \(p\)-tuples \(J\) of elements of \(\{1, \ldots, n\}\). By parts (ii) and (iii) of Lemma 5.4, we may assume that \(F, F' \in \text{Ann}(\omega_0 \wedge \bar{A}^{\ell-1})\). Since \(\partial(\omega_{(0,j)}) = \omega_j - \omega_0 \wedge \partial(\omega_j)\), this implies \(F(\partial(\omega_{(0,j)})) = F(\omega_j)\) and similarly for \(F'\). Then the last sum above is equal to \(\sum_j a_j F(\omega_j) F'(\omega_j)\). This sum is equal to \(\sum_j a_j F(\omega_j) F'(\omega_j)\), summing now over all increasing \(p\)-tuples \(J\) of elements of \(\{0, \ldots, n\}\), again because \(F, F' \in \text{Ann}(\omega_0 \wedge \bar{A}^{\ell-1})\). This equals \(\hat{S}_n(\mathcal{F}, \mathcal{F}')\) by definition. \(\square\)

We close this section with a topological remark. Consider the (multi-valued) master function \(\Phi_a = \prod_{i=0}^n f_i^{-a_i}\) on \(\mathbb{C}^{n+1}\). Since \(\sum_{i=0}^n a_i = 0\), \(\Phi_a\) is invariant under the action of \(\mathbb{C}^\times\), hence induces a (multi-valued) master function \(\Phi_a\) on \(\bar{M}\). We have \(\Phi_a = \Phi_{\bar{a}} \circ h\) where \(\Phi_{\bar{a}} = \prod_{i=1}^n \bar{f}_i^{-a_i}\) is the master function of \((d\mathcal{A}, \bar{a})\) on \(\bar{M}\), and \(h:\bar{M} \rightarrow M\) is the canonical diffeomorphism. The associated rank-one local systems \(\mathcal{L}_{\bar{a}}\) on \(\bar{M}\) and \(\mathcal{L}_{\bar{a}}\) on \(\bar{M}\) then satisfy \(h^*\mathcal{L}_{\bar{a}} = \mathcal{L}_{\bar{a}}\). The inclusion of \((\bar{A}, \delta_{ca})\) in the twisted algebraic de Rham complex of \(\mathcal{L}_{\bar{a}}\) induces an isomorphism of \(H^\ell(\bar{A}, \delta_{ca})\) with \(H^\ell(\bar{M}, \mathcal{L}_{\delta_{ca}})\) for generic \(c\). As before, \(\text{Sing}_{\bar{a}}(\mathcal{F}^\ell)\) is equal to \(\text{Sing}_{\delta_{ca}}(\mathcal{F}^\ell)\) for any nonzero scalar \(c\). Then, by Corollary 5.7, we have the following corollary.

**Corollary 5.8.** For generic \(c\), the inclusion \(\text{Sing}_{\bar{a}}(\mathcal{F}^\ell) \hookrightarrow (\bar{A}^\ell)^*\) induces an isomorphism

\[
\text{Sing}_{\bar{a}}(\mathcal{F}^\ell) \xrightarrow{\cong} H_\ell(\bar{M}, \bar{L}_{\delta_{ca}}).
\]
This isomorphism does not involve the choice of a hyperplane at infinity. Thus we have the following commutative diagram of isomorphisms, for generic $c$, in which the index 0 again plays no special role:

$$
\begin{array}{ccc}
\text{Sing}_0(\mathcal{F}) & \xrightarrow{\epsilon^*} & \text{Sing}_0(\mathcal{F}) \\
\downarrow & & \downarrow \\
H_\ell(M, \bar{L}_{-c_0}) & \xrightarrow{h^*} & H_\ell(M, \bar{L}_{-c_0})
\end{array}
$$

6. Transition functions

The right-hand side of the formula in Corollary 5.5 is the decomposition of the flag space $\hat{\mathcal{F}}^p$ of the decone $\mathcal{dA}$, see [SV91]. It can be considered to be the dehomogenization of the projective flag space $\mathcal{F}$ relative to $H_0$. The dehomogenizations relative to different hyperplanes form a set of

“affine charts” for $\hat{\mathcal{F}}$. We compute the transition functions.

For $0 \leq j \leq n$, let $\hat{A}_j, \mathcal{F}_j$, and $\hat{S}^{(j)}$ denote the OS algebra, flag complex, and contravariant form of the affine arrangement obtained by deconing $\mathcal{A}$ with respect to $H_j$. Let $\epsilon_j: \hat{A}_j \to \hat{A}$ be the isomorphism determined by $\epsilon(\hat{\omega}_k) = \omega_k - \omega_j$, for $0 \leq k \leq n$ and $k \neq j$, as in Lemma 5.1. Let $\epsilon_j^*: \hat{\mathcal{F}} \to \hat{\mathcal{F}}_j$ be the adjoint of $\epsilon_j$. For $0 \leq i < j \leq n$, set $\tau_{ij} = \epsilon_j^* \circ (\epsilon_i^*)^{-1}$. Then $\tau_{ij}: \mathcal{F}_i \to \mathcal{F}_j$ is an isomorphism. Theorem 5.2 has the following corollary.

**Corollary 6.1.** The restriction of $\tau_{ij}$ is an isomorphism of inner product spaces

$$
\tau_{ij}: (\text{Sing}(\hat{\mathcal{F}}^p), \hat{S}^{(i)}_{\mathcal{F}}|_{\text{Sing}(\hat{\mathcal{F}}^p)}) \xrightarrow{\approx} (\text{Sing}(\hat{\mathcal{F}}^p), \hat{S}^{(j)}_{\mathcal{F}}|_{\text{Sing}(\hat{\mathcal{F}}^p)}).
$$

According to Corollary 5.5, $\tau_{ij}$ can be considered to be an isomorphism

$$
\tau_{ij}: \bigoplus_{\text{codim}(X)=p \atop H_j \not\subset X} \mathcal{F}_X^p \longrightarrow \bigoplus_{\text{codim}(X)=p \atop H_j \not\subset X} \mathcal{F}_X^p.
$$

We describe this map explicitly.

In the special case $p = 1$ there is an easy formula for $\tau_{ij}$. Let $\{F_0, \ldots, F_n\}$ be the canonical basis of $\mathcal{F}^1$, and suppose $k \neq i$. Then

$$
\tau_{ij}(F_k) = \begin{cases} 
F_k & \text{if } k \neq j \\
- \sum_{r \neq j} F_r & \text{if } k = j.
\end{cases}
$$

To describe the general formula, we will use the following lemma.

**Lemma 6.2.** Let $X \in L$ with codim$(X) = p$, and let $H \in \mathcal{A}$. Then $\mathcal{F}_X^p$ is spanned by elements $[X_0 < \cdots < X_{p-1} < X]$ satisfying $H \not\subset X_{p-1}$.

**Proof.** We induct on $p$, the case $p = 0$ being trivial. Let $p > 0$ and $[X_0 < \cdots < X_{p-1} < X] \in \mathcal{F}_X^p$. By the inductive hypothesis, we may assume $H \not\subset X_{p-2}$. (Here we rely on the fact that the assignment $[X_0 < \cdots < X_{p-1}] \mapsto [X_0 < \cdots < X_{p-1} < X]$ determines a well-defined linear map $\mathcal{F}_{X_{p-1}}^{p-1} \to \mathcal{F}_{X_p}^p$.) If $H \not\subset X_{p-1}$ we are done. Otherwise, by (2.2), we have

$$
[X_0 < \cdots < X_{p-1} < X] = \sum_{X_{p-2} < X' < X \atop X' \neq X_{p-1}} -[X_0 < \cdots < X_{p-2} < X'] < X].
$$
Since \( H \not\leq X_{p-2} \) and \( H \not\leq X_{p-1} \), \( H \not\leq X' \) for any \( X' \neq X \) satisfying \( X_{p-2} < X' \) and \( \text{codim}(X') = p-1 \). Then every flag \( (X_0 < \cdots X_{p-2} < X' < X) \) that appears on the right-hand side satisfies the required condition. This completes the inductive step. \( \square \)

**Theorem 6.3.** Let \([X_0 < \cdots < X_p] \in \mathcal{F}^p \) with \( H_i \not\leq X_p \). If \( H_j \not\leq X_p \), then

\[
\tau_{ij}([X_0 < \cdots < X_p]) = [X_0 < \cdots < X_p].
\]

If \( H_j \leq X_p \) and \( H_j \not\leq X_{p-1} \), then

\[
\tau_{ij}([X_0 < \cdots < X_p]) = \sum_{X_{p-1} < X', X' \neq X_p} -[X_0 < \cdots < X_{p-1} < X'].
\]

**Proof.** By definition, \( \tau_{ij}([X_0 < \cdots < X_p]) \) is the unique element of \( \bigoplus_{\text{codim}(X) = p} \mathcal{F}^p_X \) that represents the same element of \( \bar{\mathcal{F}}^p \) as \([X_0 < \cdots < X_p] \). By (3.1), the right-hand side represents the same element of \( \mathcal{F}^p \) as \([X_0 < \cdots < X_p] \), in either case. An argument similar to the one used in the preceding lemma shows that the right-hand side lies in \( \bigoplus_{\text{codim}(X) = p} \mathcal{F}^p_X \) in either case. The claim follows. \( \square \)

By Lemma 6.2, this theorem is sufficient to determine \( \tau_{ij} \) uniquely. By Corollary 6.1, \( \tau_{ij} \) sends singular vectors of \( \hat{\mathcal{F}}^i \) to singular vectors of \( \hat{\mathcal{F}}^j \), and preserves the value of the contravariant form on such vectors.

Similarly, there is an algebra isomorphism \( \tau_{ji}^*: \hat{A}_i \to \hat{A}_j \) determined by

\[
\tau_{ji}^*(\hat{\omega}_k) = \begin{cases} 
\hat{\omega}_k - \hat{\omega}_i & \text{if } k \neq i \\
-\hat{\omega}_i & \text{if } k = i.
\end{cases}
\]

As in §2 there is an isomorphism \( \hat{\mathcal{F}}^* \to \hat{A}_i \) defined by the affine version of (2.1), and the contravariant map \( \psi_i: \hat{\mathcal{F}} \to \hat{A}_i \) characterized by the formula

\[
S(\hat{F}, \hat{F}') = (\hat{F}, \psi_i(\hat{F}')).
\]

The image of \( \psi_i \) is the complex of flag forms of \( \hat{A}_i \). (It is a subcomplex of \( (\hat{A}_i, \delta_{\hat{a}_i}) \).) Theorem 5.2 has the following consequence.

**Corollary 6.4.** The following diagram commutes:

\[
\begin{array}{ccc}
\text{Sing}(\hat{F}^i) & \xrightarrow{\psi_i} & \hat{A}_i^i \\
\tau_{ij} \downarrow & & \downarrow \tau_{ji}^* \\
\text{Sing}(\hat{F}_j) & \xrightarrow{\psi_j} & \hat{A}_j^j \\
\end{array}
\]

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