A New Symmetric Expression of Weyl Ordering

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Abstract

For the creation operator $a^\dagger$ and the annihilation operator $a$ of a harmonic oscillator, we consider Weyl ordering expression of $(a^\dagger a)^n$ and obtain a new symmetric expression of Weyl ordering w.r.t. $a^\dagger a \equiv N$ and $aa^\dagger = N + 1$ where $N$ is the number operator. Moreover, we interpret intertwining formulas of various orderings in view of the difference theory. Then we find that the noncommutative parameter corresponds to the increment of the difference operator w.r.t. variable $N$. Therefore, quantum (noncommutative) calculations of harmonic oscillators are done by classical (commutative) ones of the number operator by using the difference theory. As a by-product, nontrivial relations including the Stirling number of the first kind are also obtained.

1 Introduction

The problem for ordering of operators in Quantum Physics has been investigated then and now, see, for example, [1], [2], [3], [4] and their references.

Quantization

$\{ f, g \}_{PB} \rightarrow -\frac{i}{\hbar} [\hat{f}, \hat{g}]$

is a Lie algebra homomorphism, but not a ring homomorphism, where $\{ , \}_{PB}$ denotes a Poisson bracket and $[ , ]$ a commutator of operators on a Fock space. That is

$fg = gf \quad$ but $\quad \hat{f}\hat{g} \neq \hat{g}\hat{f}.$

This describes the noncommutativity in Quantum Physics. Therefore there exist various operator orderings and we need to select a suitable one depending on a problem in question.

Let $a^\dagger$ and $a$ be the creation operator and the annihilation operator of a harmonic oscillator respectively. The relation is

$[a, a^\dagger] = 1.$

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We put the number operator \( N \equiv a^\dagger a \).

There are various useful orderings in Quantum Physics \([5]\). In this paper, we deal with normal ordering, anti-normal ordering, and Weyl ordering. We denote them as follows:

1. Normal Ordering : \((a^\dagger)^n a^m \equiv (a^\dagger)^n a^m\),

2. Anti-Normal Ordering : \((a^\dagger)^n a^m \equiv a^m (a^\dagger)^n\),

3. Weyl Ordering : \((a^\dagger)^n a^m \equiv (n + m)\binom{-1}{n/m} (\text{sum of all symmetric products of } na^\dagger \text{ and } ma)\),

For example, when \(m = n\),

\[
: a^\dagger a :_W = \frac{1}{2}(a^\dagger a + aa^\dagger), \quad (1.1)
\]

\[
: (a^\dagger a)^2 :_W = \frac{1}{6}(a^\dagger a a^\dagger a + a^\dagger a a a^\dagger + a a^\dagger a^\dagger a + a a^\dagger a a^\dagger + a a a^\dagger a^\dagger) \quad (1.2)
\]

\[
: (a^\dagger a)^3 :_W = \frac{1}{20}(a^\dagger a^\dagger a^\dagger a a a + a^\dagger a^\dagger a a a^\dagger a + a^\dagger a^\dagger a a a a^\dagger a^\dagger + a^\dagger a a^\dagger a a a^\dagger a^\dagger + a^\dagger a a a a^\dagger a^\dagger a + a a a a^\dagger a a a^\dagger a + a a^\dagger a a a a^\dagger a + a a a a a a^\dagger a + a a a a a a a a^\dagger a + a a a a a a a a a^\dagger a), \quad (1.3)
\]

here we write \((a^\dagger)^n a^m \equiv : (a^\dagger a)^n \equiv : (\text{for simplicity } * = N, AN, W)\).

We often make proper use of these orderings. Therefore it is important to investigate what are the relations among them. The intertwining formulas of these orderings have been known by \([6]\), \([7]\) as a formula of “\(s\)-ordered power-series expansions”;

\[
\{(a^\dagger)^n a^m\}_s = \sum_{k=0}^{\min \{n,m\}} k! \binom{n}{k} \binom{m}{k} \left( \frac{t - s}{2} \right)^k \{(a^\dagger)^{n-k} a^{m-k}\}_t, \quad (1.4)
\]

where

\[
\{(a^\dagger)^n a^m\}_s \equiv \frac{\partial^{n+m}}{\partial^s \alpha^t \partial (-\alpha^*)^m} e^{a^\dagger a - a^* a} |\alpha|^2/2 \bigg|_{\alpha=0},
\]

and the ordering specified \(s, t = +1, 0, -1\) are, respectively, normal, Weyl, and anti-normal. On the other hand, the intertwining formulas among three orderings are used in the Weyl calculus. Recently, Omori et al. \([8]\) have investigated strange phenomena of \(*\)-exponential functions of
quadratic forms in the Weyl algebra. They have written the intertwining formulas as a form of linear operators by using the product formulas of the \( * \)-exponential functions;

\[
e^{\frac{1}{2}\partial_{a^\dagger}\partial_a} : f(a^\dagger, a) :_{N} = : f(a^\dagger, a) :_{W},
\]

\[
e^{-\frac{1}{2}\partial_{a^\dagger}\partial_a} : f(a^\dagger, a) :_{AN} = : f(a^\dagger, a) :_{W},
\]

where \( f(a^\dagger, a) \) is a suitable 2-variable function \( f(\bar{z}, z) \) substituting \( a^\dagger, a \) for \( \bar{z}, z \) respectively.

Under these circumstances, we investigate expressions of Weyl ordering with the intertwining formulas of orderings. The motivation is as follows; The definition of Weyl ordering is clear, but it is hard to give an explicit expression of it, like (1.1), (1.2), (1.3), especially when \( n \) is large. Therefore it is important to search a simple expression of it. If we study by using the formula (1.4) or (1.5), (1.6), though (1.4) give some explicit expressions of Weyl ordering, it is not “symmetric” that Weyl ordering would be. Therefore, we study this problem in another point of view, namely in the combinatorial theory and in the difference theory. Then we find a symmetric expression of Weyl ordering by using the number operator \( N \), which is new and would be useful in Quantum Physics. Since \( a^\dagger a = N \) and \( aa^\dagger = N + 1 \), we shall show Weyl ordering : \((a^\dagger a)^n :_{W}\) as a symmetric polynomial with respect to \( N \) and \( N + 1 \).

For example, (1.1), (1.2), (1.3) are

\[
: a^\dagger a :_{W} = \frac{1}{2}\{N + (N + 1)\},
\]

\[
: (a^\dagger a)^2 :_{W} = \frac{1}{2}\{N^2 + (N + 1)^2\},
\]

\[
: (a^\dagger a)^3 :_{W} = \frac{1}{2}\{N^3 + (N + 1)^3\} + \frac{1}{4}\{N + (N + 1)\}.
\]

In fact, for all \( n = 1, 2, \cdots \), we can show : \((a^\dagger a)^n :_{W}\) as a symmetric form like these examples in section 6. It is a new and very clear-cut expression of Weyl ordering.

The contents of this paper are as follows. In section 2, we prepare the difference theory and the Stirling number of the first kind. In section 3, first we prepare some facts about normal and anti-normal ordering. Next, we give a simple proof for the intertwining formulas of orderings as above for a help of the following sections. In section 4, we interpret these expressions by the difference theory. Then we find that the noncommutativity of harmonic oscillators is nothing but the the Newton expansion w.r.t. variable \( N \). In section 5, we describe nontrivial relations including the Stirling number of the first kind. These relations are derived from the noncommutativity of harmonic oscillators and are needed in the following section. In section 6, we describe the main result, that is a new symmetric expression of Weyl ordering. In section 7, we summarize our results.
2 Mathematical Preliminaries

In this section, we prepare the difference theory and the Stirling number of the first kind.

First we define factorial of degree \( n \) with an increment \( \epsilon \)[1]

\[
x^n \equiv x(x - \epsilon) \cdots (x - (n - 1)\epsilon) \quad (2.1)
\]

and

\[
x^n \equiv x(x + \epsilon) \cdots (x + (n - 1)\epsilon). \quad (2.2)
\]

We remark

\[
(x + \epsilon)^n = (x + \epsilon) \cdots (x + n\epsilon) = (x + n\epsilon)^n. \quad (2.3)
\]

Let \( p_n(x) \) be a polynomial of degree \( n \) w.r.t. \( x \), and \( \Delta_{+x} \) the forward difference operator defined by \( \Delta_{+x}f(x) \equiv \epsilon^{-1}\{f(x + \epsilon) - f(x)\} \). Then we have the Newton expansion

\[
p_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} \Delta_{+x}^k p_n(0) = \sum_{k=0}^{n} \frac{x^{n-k}}{(n-k)!} \Delta_{+x}^{n-k} p_n(0). \quad (2.4)
\]

We also remark that

\[
\Delta_{+x}x^n = nx^{n-1}. \quad (2.5)
\]

Next we define the Stirling number of the first kind and calculate some of them explicitly.

**Definition 2.1.** The Stirling number of the first kind \( s(n, i) \) \( (i = 1, \cdots, n) \) is defined by the following equation [10];

\[
x^n = x(x - \epsilon) \cdots (x - (n - 1)\epsilon) = \sum_{i=1}^{n} s(n, i) \epsilon^{n-i} x^i
\]

and \( s(n, 0) \equiv 0 \quad (n \geq 1) \), \( s(j, i) \equiv 0 \quad (j < i) \).

We remark

\[
x^n = x(x + \epsilon) \cdots (x + (n - 1)\epsilon) = \sum_{i=1}^{n} (-1)^{n-i}s(n, i) \epsilon^{n-i} x^i. \quad (2.6)
\]

**Remark 2.2.** We can obtain the Stirling number of the first kind explicitly by the following recursion formula and initial conditions.

\[
s(n + 1, i) = s(n, i - 1) - n \cdot s(n, i) \quad (n \geq i \geq 1), \quad (2.7)
\]

\[
s(n, n) = 1 \quad (n \geq 1), \quad (2.8)
\]

\[
s(n, 0) = 0. \quad (2.9)
\]
For example, if we put \( i = n \) in (2.7), then
\[
s(n + 1, n) = s(n, n - 1) - n \cdot s(n, n) = s(n, n - 1) - n.
\]

Therefore
\[
s(n, n - 1) = - \sum_{k=1}^{n-1} k = -\frac{n(n - 1)}{2}. \quad (2.10)
\]

Next, since
\[
s(n + 1, n - 1) = s(n, n - 2) - n \cdot s(n, n - 1) = s(n, n - 2) + \frac{1}{2} n^2(n - 1),
\]
we have
\[
s(n, n - 2) = \sum_{k=1}^{n-1} \frac{1}{2} k^2(k - 1) = \frac{1}{24} n(n - 1)(n - 2)(3n - 1). \quad (2.11)
\]

3 A Simple Proof for the Intertwining Formulas of Orderings

First we prepare some facts about normal and anti-normal ordering. For a while, we put \([a, a^\dagger] = \epsilon\) for more general situation.

Lemma 3.1. We have following formulas using the number operator \( N = a^\dagger a \):
\[
: (a^\dagger a)^n : N = N(N - \epsilon) \cdots (N - (n - 1)\epsilon) = N^\underline{n}, \quad (3.1)
\]
\[
: (a^\dagger a)^n : AN = (N + \epsilon)(N + 2\epsilon) \cdots (N + n\epsilon) = (N + \epsilon)^\underline{n}. \quad (3.2)
\]

Proof. By using simple relations
\[
aN = (N + \epsilon)a, \ a^\dagger N = (N - \epsilon)a^\dagger \quad \text{or} \quad Na = a(N - \epsilon), \ Na^\dagger = a^\dagger(N + \epsilon),
\]
we have an induction formula
\[
(a^\dagger)^n a^n = a^\dagger \cdots a^\dagger a^\dagger a a a \cdots a
\]
\[
= a^\dagger \cdots a^\dagger a^\dagger N a a \cdots a
\]
\[
= a^\dagger \cdots a^\dagger (N - \epsilon) a^\dagger a a \cdots a
\]
\[
= \cdots
\]
\[
= (N - (n - 1)\epsilon)(a^\dagger)^{n-1} a^{n-1}.
\]

Therefore we obtain \((3.1)\) by the mathematical induction. We also obtain \((3.2)\) in a similar way.
As we mentioned in section 1, since Omori et al have proved (1.5), (1.6) by using the product formulas of the $\ast$-exponential functions, we give an another simple proof of them. We have only to prove the next proposition. This proof helps account for our theory to be described in the following sections.

**Proposition 3.2.**

\[ e^{\frac{1}{2}i\partial_{a^\dagger}\partial_a} : (a^\dagger)^n a^m :_N = : (a^\dagger)^n a^m :_W, \]  

\[ e^{-\frac{1}{2}i\partial_{a^\dagger}\partial_a} : (a^\dagger)^n a^m :_{AN} = : (a^\dagger)^n a^m :_W. \]  

**Proof.** By using the generating function of the Weyl ordering,

\[ e^{\alpha a^\dagger + \beta a} = \sum_{n,m=0}^{\infty} \frac{\alpha^n \beta^m}{n!m!} : (a^\dagger)^n a^m :_W \]

and the fundamental Baker-Campbell-Hausdorff formula [11]

\[ e^{A+B} = e^{-\frac{1}{2}[A,B]}e^Ae^B \text{ whenever } ([A, [A,B]] = [B, [A,B]] = 0), \]

we have [12]

\[ e^{\alpha a^\dagger + \beta a} = e^{\frac{1}{2}i\alpha\beta} e^{\alpha a^\dagger} e^{\beta a} \text{ (normal order)} \]

\[ = e^{\frac{1}{2}i\partial_{a^\dagger}\partial_a} e^{\alpha a^\dagger} e^{\beta a}. \]  

If we expand both sides of (3.6) and compare the coefficients of $\alpha^n \beta^m$, we obtain (3.3). Similarly, we have

\[ e^{\alpha a^\dagger + \beta a} = e^{-\frac{1}{2}i\alpha\beta} e^{\beta a} e^{\alpha a^\dagger} \text{ (anti-normal order)} \]

\[ = e^{-\frac{1}{2}i\partial_{a^\dagger}\partial_a} e^{\alpha a^\dagger} e^{\beta a}. \]  

If we expand both sides of (3.7) and compare the coefficients of $\alpha^n \beta^m$, we obtain (3.4). \hfill \square

By the proposition 3.2 and the lemma 3.1, we obtain the following corollary.

**Corollary 3.3.** We can show Weyl ordering : $(a^\dagger)^n a^m :_W$ in terms of normal ordering.

\[ : (a^\dagger)^n a^m :_W = \sum_{k=0}^{\min\{n,m\}} \frac{\epsilon^{k!}}{2k} \binom{n}{k} \binom{m}{k} (a^\dagger)^{n-k} a^{m-k}. \]  

Especially, in case of $m = n$, if we use (3.7), we obtain an expression of $: (a^\dagger a)^n :_W$ in terms of the number operator $N$,

\[ : (a^\dagger a)^n :_W = \sum_{k=0}^{n} \frac{\epsilon^{k!}}{2k} \binom{n}{k}^2 N^{n-k}. \]
By a similar calculation, we obtain the next expression using anti-normal ordering.

**Corollary 3.4.**

\[
: (a^\dagger)^n a^m : W = \sum_{k=0}^{\min\{n,m\}} (-\epsilon)^k \binom{n}{k} \binom{m}{k} \frac{k!}{2^k} a^{n-k} (a^\dagger)^{m-k}.
\] (3.10)

Especially, in case of \( m = n \), if we use (3.2), we obtain another expression of \( : (a^\dagger a)^n : W \) in terms of the number operator \( N \),

\[
: (a^\dagger a)^n : W = \sum_{k=0}^{n} (-\epsilon)^k \binom{n}{k} \frac{2}{k!} (N + \epsilon)^{n-k}.
\] (3.11)

**Remark 3.5.** In [6], [7], K.E.Cahill and R.J.Glauber introduced “s-ordered power-series expansions” as follows;

Define the \( s \)-ordered displacement operator \( D(\alpha, s) \) by

\[
D(\alpha, s) = e^{\alpha a^\dagger - \alpha^* a} e^{s|\alpha|^2/2}, \quad \text{where } \alpha \in \mathbb{C} \text{ and } \alpha^* \text{ is its complex conjugate.} \] (3.12)

By the fundamental Baker-Campbell-Hausdorff formula (3.5), for the three discrete values of \( s = +1, 0, -1 \), the operator \( D(\alpha, s) \) can be written as an exponential which is, respectively, normal order

\[
D(\alpha, 1) = e^{\alpha a^\dagger} e^{-\alpha^* a},
\]
Weyl order

\[
D(\alpha, 0) = e^{\alpha a^\dagger - \alpha^* a},
\]
and anti-normal order

\[
D(\alpha, -1) = e^{-\alpha^* a} e^{\alpha a^\dagger}.
\]

They defined the \( s \)-ordered product \( \{ (a^\dagger)^n a^m \}_s \) by means of the Taylor series

\[
D(\alpha, s) = \sum_{n,m=0}^{\infty} \{ (a^\dagger)^n a^m \}_s \frac{\alpha^n (-\alpha^*)^m}{n!m!}
\]

or, equivalently,

\[
\{ (a^\dagger)^n a^m \}_s \equiv \frac{\partial^{n+m} D(\alpha, s)}{\partial \alpha^n \partial (-\alpha^*)^m} \bigg|_{\alpha=0}.
\]

The intertwining formula between \( s \)-ordered product and \( t \)-ordered product is

\[
D(\alpha, s) = e^{(s-t)|\alpha|^2/2} D(\alpha, t)
\]
or

\[
\{ (a^\dagger)^n a^m \}_s = \sum_{k=0}^{\min\{n,m\}} k! \binom{n}{k} \binom{m}{k} \frac{1}{(t-s)/2} \{ (a^\dagger)^{n-k} a^{m-k} \}_t.
\] (3.13)

If we put \((s, t) = (0, 1), (0, -1)\), we recover (3.8), (3.10) in case of \( \epsilon = 1 \) respectively.

We remark that since they did not write in the form of (3.3) and (3.4) in [6], [7] explicitly, though the idea of the proof was almost the same, we gave the proof of them in this section.
4 A Relation Between the Noncommutativity of Harmonic Oscillators and the Difference Theory

In this section, we describe a relation between the noncommutativity of harmonic oscillators and the difference theory. The noncommutativity of harmonic oscillators is as follows;

\[ e^{t\alpha_\beta}e^{\alpha_\beta^\dagger} = e^{s\alpha_\beta}e^{\beta_\beta^\dagger}, \tag{4.1} \]

where \( t - s = \epsilon \). If we expand both sides of (4.1), then we have

\[ \sum_{k,l,m} \frac{t^k \alpha_k^\beta \alpha_l^\dagger (a_l^\dagger) l \beta^m a_m^m}{k! l! m!} = \sum_{k,l,m} \frac{s^k \alpha_k^\beta \beta^l a_l^\dagger a_m^m (a_l^\dagger)^m}{k! l! m!}. \tag{4.2} \]

We choose coefficients of \( \alpha^n \beta^m \) of (4.2), then the left hand side is

\[ \sum_{k=0}^n k! \frac{n^k}{k!} \frac{t^k (a_l^\dagger)^{n-k} a_l^m}{(n-k)!} = \frac{1}{(n!)^2} \sum_{k=0}^n k! \frac{n^k}{k!} \frac{t^k (a_l^\dagger)^{n-k} a_l^m}{(n-k)!}, \]

\[ = \frac{1}{(n!)^2} \sum_{k=0}^n k! \frac{n^k}{k!} (N_{n-k}^n)^2. \]

By a similar calculation of the right hand side, we obtain

\[ \sum_{k=0}^n k! \frac{n^k}{k!} \frac{s^k (a_l^\dagger)^{n-k} a_l^m}{(n-k)!} = \sum_{k=0}^n k! \frac{n^k}{k!} (N_{n-k}^n)^2. \tag{4.3} \]

Remark 4.1. In the case of \( t = \epsilon \) and \( s = -\epsilon \), (4.3) is the equation that (3.9) and (3.11) are equal. Moreover, since \( (a_l^\dagger) a_n = N^n \) and

\[ \partial_{a_l^\dagger} (a_l^\dagger)^n a_n = n(a_l^\dagger)^{n-1} a_n = n N^{n-1} a_l, \quad \Delta_+ N^n = n N^{n-1}, \quad (\Delta_+ = \Delta_+ N) \tag{4.4} \]

relations using various orderings of \( (a_l^\dagger)^{n-1} a_n \) are equivalent to the relation (4.3) acted by \( \Delta_+ \) with some \( t \). Similarly, for any \( n, m \), relations using various orderings of \( (a_l^\dagger)^n a_m \) are equivalent to the relation (4.3) acted by some power of \( \Delta_+ \) and some \( t \).

Remark 4.2. From (4.3),

\[ \frac{d}{dt} \sum_{k=0}^n k! \frac{n^k}{k!} \frac{t^k (a_l^\dagger)^{n-k} a_l^m}{(n-k)!} = \frac{d}{dt} \sum_{k=0}^n k! \frac{n^k}{k!} \frac{(t - \epsilon)^k (n-k)!}{(n-k)!} (N + \epsilon)^{n-k}, \]

\[ \sum_{k=0}^n k! \cdot k \cdot t^{k-1} \frac{n^k}{k!} \frac{(n-k)!}{(n-k)!} = \sum_{k=0}^n k! \cdot k \cdot (t - \epsilon)^{k-1} \frac{n^k}{k!} \frac{(n-k)!}{(n-k)!}. \]
Therefore, if we put $t = \frac{1}{2}$ and $\epsilon = 1$, we obtain the following equation

$$\sum_{k=0}^{n} \frac{k! \cdot k}{2^{k-1}} \binom{n}{k}^2 \left\{ N^{n-k} + (-1)^k (N + 1)^{n-k} \right\} = 0.$$ \hspace{1cm} (4.5)

Next we show the equation of noncommutativity of harmonic oscillators (4.3) by using the difference theory. We put $p_n(N) = \sum_{k=0}^{n} k! s^k \binom{n}{k}^2 (N + \epsilon)^{n-k} = \sum_{k=0}^{n} k! s^k \binom{n}{k}^2 (N + (n-k)\epsilon)^{n-k}$, \hspace{1cm} (4.6)

and we expand (4.6) using the forward difference operator $\Delta_+$. Since

$$\Delta_+^l p_n(0) = \sum_{k=0}^{n} k! s^k \binom{n}{k}^2 (n-k)(n-k-1) \cdots (l-k+1) (0 + (n-k)\epsilon)^{l-k}$$

$$= \sum_{k=0}^{n} k! s^k \binom{n}{k} \frac{n!}{k!(n-k)!} (n-k)(n-k-1) \cdots (l-k+1) (n-k)! \epsilon^{l-k}$$

$$= l! \binom{n}{l} \sum_{k=0}^{n} s^k \binom{n}{k} \left( \frac{d}{d\epsilon} \right)^{n-l} \epsilon^{n-k}$$

$$= l! \binom{n}{l} \left( \frac{d}{d\epsilon} \right)^{n-l} (s + \epsilon)^n$$

$$= l! \binom{n}{l} \frac{n!}{l!} \epsilon^l,$$

by (2.4), we have

$$p_n(N) = \sum_{l=0}^{n} \Delta_+^l p_n(0) \frac{N^{n-l}}{(n-l)!}$$

$$= \sum_{l=0}^{n} l! \binom{n}{l} \frac{n!}{l!} \epsilon^l \frac{N^{n-l}}{(n-l)!}$$

$$= \sum_{l=0}^{n} l! \epsilon^l \binom{n}{l}^2 N^{n-l}.$$

Therefore, the noncommutativity of harmonic oscillators is nothing but the Newton expansion w.r.t. variable $N$. More precisely, the noncommutative parameter $\epsilon$ in $[a, a^\dagger] = \epsilon$ corresponds to the increment of the difference operator $\Delta_+ = \Delta_+ N$. 

9
5 Relations Including the Stirling Number of the First Kind

We shall describe nontrivial relations including the Stirling number of the first kind. These relations are derived from the noncommutativity of harmonic oscillators. Hereafter, we put $\epsilon = 1$ again.

**Theorem 5.1.** We have

$$
\sum_{k=0}^{2j+1} \frac{k!}{2^k} \binom{n-1}{k} \binom{n}{k} s(n-k, n-2j-1) = 0 \quad (j = 0, 1, \ldots).
$$

(5.1)

For example,

$$
\sum_{k=0}^{1} \frac{k!}{2^k} \binom{n-1}{k} \binom{n}{k} s(n-k, n-1) = s(n, n-1) + \frac{(n-1)n}{2} s(n-1, n-1)
$$

$$
= -\frac{n(n-1)}{2} + \frac{(n-1)n}{2}
$$

$$
= 0.
$$

This theorem is used in a new symmetric expression of $: (a^\dagger a)^n : W$ with respect to $a^\dagger a = N$ and $aa^\dagger = N + 1$.

**Proof.** From (4.3),

$$
\sum_{k=0}^{n} k! \left( \frac{n}{k} \right)^2 \Delta_+ N^{n-k} = \sum_{k=0}^{n} k! \left( t - 1 \right)^k \left( \frac{n}{k} \right)^2 \Delta_+ (N + n - k)^{n-k}
$$

$$
\sum_{k=0}^{n} k! \left( \frac{n}{k} \right)^2 (n-k) N^{n-k-1} = \sum_{k=0}^{n} k! \left( t - 1 \right)^k \left( \frac{n}{k} \right)^2 (n-k)(N + n - k)^{n-k-1}.
$$

Therefore, if we put $t = \frac{1}{2}$, we obtain the following equation.

$$
\sum_{k=0}^{n} k! \cdot \frac{(n-k)}{2^k} \left( \frac{n}{k} \right)^2 \left\{ N^{n-k-1} - (-1)^k (N + n - k)^{n-k-1} \right\} = 0.
$$

(5.2)

If we multiply (5.2) by $(N+1)$, then $(N + n - k)^{n-k-1} \cdot (N+1) = (N + 1)^{n-k}$ and the left
hand side is
\[
\sum_{k \geq 0} \frac{k!}{2^k} n \binom{n-1}{k} \binom{n}{k} \left\{ (N + 1)^{n-k} + (-1)^{k+1} (N + 1)^{n-k} \right\}
\]
\[
= n \sum_{k \geq 0} \frac{k!}{2^k} \binom{n-1}{k} \binom{n}{k} \left\{ \sum_{i=1}^{n-k} s(n-k, i)(N+1)^i + (-1)^{k+1} \sum_{i=1}^{n-k} (-1)^{n-k-i} s(n-k, i)(N+1)^i \right\}
\]
\[
= n \sum_{i \geq 1} \sum_{k \geq 0} \frac{k!}{2^k} \binom{n-1}{k} \binom{n}{k} s(n-k, i) \left\{ 1 + (-1)^{n-i} \right\} (N+1)^i
\]
\[
= 2n \sum_{j \geq 0} \sum_{k \geq 0} \frac{k!}{2^k} \binom{n-1}{k} \binom{n}{k} s(n-k, n-2j-1)(N+1)^{n-2j-1}.
\]
Therefore we obtain the theorem \[5.1\]  

**Remark 5.2.** More generally, by the equation
\[
e^{-\frac{1}{2} \partial_a \partial_a} : (a^\dagger)^{n+m} a^n : = : (a^\dagger)^{n+m} a^n :_W = e^{-\frac{1}{2} \partial_a \partial_a} : (a^\dagger)^{n+m} a^n :_{AN},
\]
we obtain the next theorem.

**Theorem 5.3.** For \( n > 0 \) and \( m = -n+1, -n+2, \cdots, -1, 0 \), we have
\[
((-1)^{n+m-i} + 1) \sum_{k=0}^{n+m+1-i} \frac{k!}{2^k} \binom{n+m}{k} \binom{n}{k} s(n+m+1-k, i)
\]
\[
+ \sum_{l=i+1}^{n+m+1} (m+1)^{l-i} \binom{l-1}{i-1} \sum_{k=0}^{n+m+1-l} \frac{k!}{2^k} \binom{n+m}{k} \binom{n}{k} s(n+m+1-k, l) = 0 \quad (5.3)
\]
(for \( i = 1, 2, \cdots, n+m+1 \)),

and for \( n > 0 \) and \( m = 0, 1, 2, \cdots \), we have
\[
((-1)^{n-i} + 1) \sum_{k=0}^{n+1-i} \frac{k!}{2^k} \binom{n+m}{k} \binom{n}{k} s(n+1-k, i)
\]
\[
+ \sum_{l=i+1}^{n+1} (1-m)^{l-i} \binom{l-1}{i-1} \sum_{k=0}^{n+1-l} \frac{k!}{2^k} \binom{n+m}{k} \binom{n}{k} s(n+1-k, l) = 0 \quad (5.4)
\]
(for \( i = 1, 2, \cdots, n+1 \)).

### 6 A Symmetric Expression of Weyl Ordering

We have expressed \( : (a^\dagger a)^n :_W \) as a polynomial of the number operator \( N \) in two ways, namely normal ordering expression \[4.9\] and anti-normal one \[4.11\]. Moreover, we have a new symmetric expression of \( : (a^\dagger a)^n :_W \) with respect to \( a^\dagger a = N \) and \( aa^\dagger = N+1 \).
Theorem 6.1.

\[(a^\dagger a)^n : W = \sum_{j=0}^{[(n-1)/2]} \alpha(n, 2j)\{N^{n-2j} + (N + 1)^{n-2j}\}, \]  

(6.1)

where the constants \(\alpha(n, i)\) are defined by

\[
\alpha(n, i) \equiv \frac{1}{2} \sum_{k=0}^{i} \frac{k!}{2^k} \binom{n}{k} \binom{n-1}{k} s(n-k, n-i) \quad (i = 1, 2, \cdots),
\]

and we have \(\alpha(n, 2j + 1) = 0 \quad (j = 0, 1, \cdots)\).

For example,

\[
\alpha(n, 0) = \frac{1}{2},
\]

(6.2)

and by using (2.8), (2.10) and (2.11), we have easily

\[
\begin{align*}
\alpha(n, 2) &= \frac{1}{2} \sum_{k=0}^{2} \frac{k!}{2^k} \binom{n}{k} \binom{n-1}{k} s(n-k, n-2) \\
&= \frac{1}{2} \left\{s(n, n-2) + \frac{1}{2} n(n-1) \cdot s(n-1, n-2) + \frac{1}{8} n(n-1)^2 (n-2) \cdot s(n-2, n-2)\right\} \\
&= \frac{1}{24} n(n-1)(n-2).
\end{align*}
\]

(6.3)

We remark that if we put \(n = 1, 2, 3\) in (6.2), (6.3), we recover (1.7), (1.8) and (1.9).

By a similar calculation, we have

\[
\begin{align*}
\alpha(n, 4) &= \frac{1}{2880} n(n-1)(n-2)(n-3)(n-4)(5n-7), \\
\alpha(n, 6) &= \frac{1}{725760} n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)(35n^2 - 147n + 124),
\end{align*}
\]

where we used Mathematica.

Proof. First, by (4.5), we remark that we have

\[
\begin{align*}
\sum_{k=0}^{n} \frac{k!}{2^k} \binom{n}{k} \binom{n-1}{k-1} \left\{N^{n-k} + (-1)^k (N + 1)^{n-k}\right\} \\
&= \frac{1}{2n} \sum_{k=0}^{n} \frac{k! \cdot k}{2^{k-1}} \binom{n}{k}^2 \left\{N^{n-k} + (-1)^k (N + 1)^{n-k}\right\} \\
&= 0.
\end{align*}
\]
By the proposition 3.2 and the corollary 3.3, 3.4,

\[ : (a^\dagger a)^n :_W = \frac{1}{2} \left( e^{\frac{i}{2} \partial_{a^\dagger} \partial_a} : (a^\dagger a)^n :_N + e^{-\frac{i}{2} \partial_{a^\dagger} \partial_a} : (a^\dagger a)^n :_{AN} \right) \]

\[ = \frac{1}{2} \left\{ \sum_{k=0}^{n} \frac{k!}{2^k} \binom{n}{k} 2^{n-k} \left( \frac{N^{n-k}}{2^k} \right) \sum_{k=0}^{n} (-1)^k k! \left( \frac{n}{k} \right)^2 (N + 1)^{n-k} \right\} \]

\[ = \frac{1}{2} \sum_{k=0}^{n-1} \frac{k!}{2^k} \binom{n}{k} \left\{ \binom{n-1}{k} + \binom{n-1}{k-1} \right\} \left\{ N^{n-k} + (-1)^k (N + 1)^{n-k} \right\} \]

\[ = \frac{1}{2} \sum_{k=0}^{n-1} \frac{k!}{2^k} \binom{n}{k} \left( \frac{n-1}{k} \right) \left\{ \sum_{i \geq 1} s(n-k,i) N^i \right\} \]

\[ + (-1)^k \sum_{i \geq 1} (-1)^{n-k-i} s(n-k,i) (N + 1)^i \right\} \]

\[ = \sum_{i \geq 1} \left\{ \frac{1}{2} \sum_{k=0}^{n-i} \frac{k!}{2^k} \binom{n}{k} \left( \frac{n-1}{k} \right) s(n-k,i) N^i \right\} \]

\[ + (-1)^{n-i} \frac{1}{2} \sum_{k=0}^{n-i} \frac{k!}{2^k} \binom{n}{k} \left( \frac{n-1}{k} \right) s(n-k,i) (N + 1)^i \right\} \]

\[ = \sum_{i \geq 1} \alpha(n, n-i) \{ N^i + (-1)^{n-i} (N + 1)^i \}, \]

where we used \( s(n-k,i) = 0 \) \((n-k < i)\).

If \( i = n - 2j - 1 \) \((j = 0, 1, \cdots)\), \( \alpha(n, 2j + 1) \) vanish because of (5.1).

And if \( i = n - 2j \) \((j = 0, 1, \cdots)\), since \((-1)^{n-i} = 1\), we obtain the theorem 6.1.

\[ \square \]

### 7 Discussion

We obtained a new symmetric expression of : \((a^\dagger a)^n :_W\) w.r.t. \(a^\dagger a = N\) and \(aa^\dagger = N + 1\) by using the difference theory. We also found that the noncommutative parameter \(\epsilon\) corresponded to the increment of the difference operator w.r.t. variable \(N\). Therefore, quantum (noncommutative) calculations of harmonic oscillators were done by classical (commutative) ones of the number operator by using the difference theory. As a by-product, nontrivial relations including the Stirling number of the first kind (5.3), (5.4) were also obtained.
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