On the Action, Topology and Geometric

Invariants in Quantum Gravity

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Abstract

The action in general relativity (GR), which is an integral over the manifold plus an integral over the boundary, is a global object and is only well defined when the topology is fixed. Therefore, to use the action in GR and in most approaches to quantum gravity (QG) based on a covariant Lorentzian action, there needs to exist a preferred (global) timelike vector, and hence a global topology $R \times S^3$, for it to make sense. This is especially true in the Hamiltonian formulation of QG. Therefore, in order to do canonical quantization, we need to know the topology, appropriate boundary conditions and (in an open manifold) the conditions at infinity, which affects the fundamental geometrical scalar invariants of the spacetime (and especially those which may occur in the QG action).
**Introduction:** In Einstein’s general relativity (GR) matter and geometry are intimately linked, replacing the previous Newtonian paradigm of matter dynamics occurring on a fixed background. However, unlike in GR, time and space are not on an equal footing in quantum mechanics (time is treated classically whereas space is associated with a quantum description). It is questionable as to whether modern theories of quantum gravity (QG) respect Einstein’s revolutionary way of interpreting gravitational physics geometrically [32].

Most approaches to QG are based on a covariant Lorentzian action. Unlike most field theories, in GR the situation is complicated by the fact that the Einstein-Hilbert action includes a surface term. In most derivations of the gravitational Hamiltonian, the surface term is ignored. This results in a Hamiltonian which is just a multiple of a constraint. One must then add to this constraint appropriate surface terms so that its variation is well defined [30], and the action yields the correct equations of motion subject only to the condition that the induced three metric and matter fields on the boundary are held fixed. The action is well defined for spatially compact geometries, but diverges for noncompact ones.

**Action:** The dynamical content of GR is fully expressed by Einstein’s FE (EFE). Nevertheless, even in a purely classical (i.e., non-quantum) context, it is convenient and useful for many purposes to have Lagrangian and Hamiltonian formulations of GR. The interpretation that EFE describes the evolution of a ‘spatial metric’, with “time” is perhaps best expressed via the Hamiltonian formulation. Moreover, most prescriptions for formulating a quantum field theory of gravitation require that the associated classical theory be expressed in a Lagrangian or Hamiltonian form.

For GR, the field variable is the spacetime metric, $g_{ab}$, defined on a four-dimensional (4D) manifold, $M$. In this case, the natural volume element in the integrals is the volume element which itself depends on the field variables $g_{ab}$, and hence its variation must be taken into account when calculating functional derivatives. In the canonical approach [26] a family of spacelike surfaces is introduced and used to construct a Hamiltonian and canonical equal-time commutation relations (which is appropriate for strong gravitational fields and is supposed to ensure unitarity). However, the split into three spatial dimensions and one time dimension seems to be contrary to the whole spirit of GR [25]. Moreover, it restricts the topology of spacetime to be the product of the real line with some 3D manifold, whereas one would expect that QG would allow all possible topologies of spacetime including those which are not products. It is precisely these other topologies that seem to give the most interesting effects. It is usually implicitly assumed either that the 3-geometry is compact (a ‘closed’ universe) or that the gravitational and matter fields die off in some suitable way at spatial infinity (the asymptotically flat space) [3]. Under variations of the metric which vanish and whose normal derivatives also vanish on $\partial M$, the
boundary of a compact region $M$, this action is stationary if and only if the metric satisfies the EFE.

However this action is not an extremum if one allows variations of the metric which vanish on the boundary but whose normal derivatives do not vanish there [25]. The reason is that the Ricci curvature scalar $R$ contains terms which are linear in the second derivatives of the metric. By integration by parts, the variation in these terms can be converted into an integral over the boundary which involves the normal derivatives of the variation on the boundary. In order to cancel out this surface integral, and so obtain an action which is stationary for solutions of the EFE under all variations of the metric that vanish on the boundary, one has to add to the action a term of the form [25]:

$$I = \frac{1}{16\pi G} \int_M (R - 2\Lambda)(-g)^{1/2} \, d^4x + \int_M L_m(-g)^{1/2} \, d^4x + \frac{1}{8\pi G} \int_{\partial M} K(\pm h)^{1/2} \, d^3x + C,$$

where $K$ is the trace of the second fundamental form of the boundary, $h$ is the induced metric on the boundary, the plus or minus signs are chosen according to whether the boundary is spacelike or timelike, and $C$ is a term which depends only on the boundary metric $h$ and not on the values of $g$ at the interior points (which must be considered in open universes, including asymptotically flat space-times, but may be set to zero in closed universes).

The action above is a global object and is well-defined only if the global topology is fixed as $R \times S^3$. In a sense the Lagrangian formulation of a field theory is “spacetime covariant”. On the other hand, a Hamiltonian formulation of a field theory necessarily requires a global breakup of the spacetime into space and time.

For closed universes we only have the two volume integrals in the above, and these terms would be finite even if $M$ were chosen to be the entire spacetime (for suitable choices of the matter Lagrangian $L_m$). In the open universe case the boundary terms enter the theory in a fundamental way, and there is no good way to decide what these terms should be for arbitrary open universes. In addition, the surface integral above is also more complicated in open universes, since in this case it must contain timelike portions [4].

We note that the boundary term $\partial M$ is added precisely to cancel the surface terms and exactly produce the EFE of GR. Therefore, in GR, the EFE are more fundamental than the action. This is of importance mathematically, since the EFE and the EL equations derived from action can differ, different actions can give rise to the same FE, and boundary terms can affects geodesics [5, 6].
Topology: Therefore, to know the local evolution (Euler-Lagrange (EL) field equations (FE)), we need to know the topology, appropriate boundary conditions and (in an open manifold) the conditions at infinity. A global topology $R \times S^3$ is considered, where $S^3$ is usually assumed compact. Regarding the boundary conditions, the action certainly makes more sense in a closed universe. The surface integral is more complicated in open universes, in which boundary terms enter in a more fundamental way (and it is not known in general what these terms should be). Therefore, there are problems with boundary conditions at infinity for an open manifold: we need to know where infinity is (definition), and conditions at infinity (which might be timelike or null). In an open or closed universe we need to add surface terms on a case-by-case basis (e.g. different for each type of spacetime). None of this is really acceptable in GR. In particular, these issues lead to philosophical problems in cosmology. 2

Clearly, in order to do canonical quantization additional spacetime structure is needed. In GR, there is no background geometry. The space-time metric itself is the fundamental dynamical variable. The canonical and the covariant approaches have adopted dramatically different attitudes to face these problems. In many covariant approaches to QG (with fields evolving on a fixed background 1), such as string theory 27, the spacetime metric is split into a kinematical background and dynamical fluctuations. The fundamental degrees of freedom and the short scale dynamics in the final quantum theory are quite different to those of GR and classical spacetimes and gravitons only emerge in a suitable limit (although the resulting quantum GR turns out to be non-renormalizable). However, background independence is important in modern approaches to QG 29, in which the emphasis is on preserving the geometrical character of GR.

In the canonical approach the Hamiltonian formulation of GR is used and a fixed (‘spatial’) three-manifold (usually assumed compact). The very first step of the canonical quantization program requires a splitting of space-time into space and time thereby, as noted earlier, doing grave injustice to space-time covariance that underlies general GR. [Therefore, we only attempt to quantize a subset of spacetimes (e.g., with a global topology $R \times S^3$)]. This is a valid concern, but successful background independent approaches to quantum GR (such as LQG and causal dynamical triangulation) accepts this price.

We also note that analytic continuation (a generalization of a Wick rotation) is often used in computations in quantum theories. There is clearly a restriction on the class of Lorentzian metrics in QG, and hence on the (real) gravitational degrees of freedom, by assuming the existence of such a Wick-rotation. It is expected that a Lorentzian spacetime that allows for such an analytic continuation is necessarily globally 1+3 (and hence I-non-degenerate) 7. In spacetimes with a topology $R \times S^3$ there is an absolute time function, since the spacetime admits a foliation by globally space-like hypersurfaces, which restricts the possible existence of closed time-like curves; this is not necessarily the case in the
supersymmetric Godel solutions in string theory [31].

Geometric Invariants: Therefore, in the canonical approach to QG there exists a unique time (and space and time are essentially treated independently; therefore the structure of the Lorentzian manifold is not fully utilized). A Lorentzian spacetime with global topology $R \times S^3$ is $I$-non-degenerate and thus completely classified by its set of scalar polynomial curvature invariants [22]. In this case all gravitational degrees of freedom are curvature invariants. For example, in many theories of fundamental physics there are geometric classical corrections to GR. Different polynomial curvature invariants (constructed from the Riemann tensor and its covariant derivatives) are required to compute different loop-orders of renormalization of the Einstein-Hilbert action. In specific quantum models such as supergravity there are particular allowed local counterterms [27].

It was proven (in 4D, and it is also likely true in arbitrary dimensions) [22] that a spacetime metric is either $I$-non-degenerate or a degenerate Kundt metric. A Lorentzian degenerate Kundt spacetime [21] is not completely classified by its set of scalar polynomial curvature invariants [22]. The higher-dimensional Kundt class of spacetimes [21] are genuinely Lorentzian and have many mathematical properties quite different from their Riemannian counterparts. For example, they have important geometrical information that is not contained in the scalar invariants and, in principle, the Einstein-Hilbert action may require geometric corrections that are not scalar invariants. In particular, the physical fundamental properties that do not depend only on scalar invariants may lead to interesting and novel physics in models of QG and particularly in string theory.

A Lorentzian manifold admitting an indecomposable but non-irreducible holonomy representation (i.e., with a one-dimensional invariant lightlike subspace) is a degenerate Kundt (degenerately reducible) spacetime, which contains the VSI and (non locally homogeneous) CSI subclasses (in which all of the scalar invariants are zero or constant, respectively) as special cases. It is perhaps within string theory that the full richness of Lorentzian geometry is realised, where the Kundt spacetimes (which include, for example, VSI generalized pp-wave spacetimes and CSI generalized $AdS_5 \times S^5$ spacetimes, which are not globally $R \times M$) may play a fundamental role. Solutions of the classical FE for which the counter terms required to regularize quantum fluctuations vanish are of importance because they offer insights into the behaviour of the full quantum theory of gravity (regardless of what the exact form of this theory might be). A classical metric is called universal if the quantum correction is a multiple of the metric [23], and consequently such metrics can be interpreted as having vanishing quantum corrections to all loop orders and are automatically solutions to the quantum theory. In particular, VSI and CSI spacetimes are exact solutions in string theory to all perturbative orders in the string tension scale [24].
Finally, it should be noted that many non-fundamental theories are not derived from an action. Hamiltonian systems are non-dissipative. For example, the Navier-Stokes fluid equations and statistical mechanics are obtained by coarse graining. An action principle may not be necessary (or even possible) for QG or at least a low energy effective version of the theory. The results of averaging the geometry, however, are expected to be far from trivial, since the EFE are highly non-linear. The averaging problem in GR has been studied in [8]. So in the same sense that (linear) QED is averaged to obtain Maxwell eqns, QG should be averaged to get GR [9]. This leads to the problem of averaging at the level of the action [10].
References

[1] In the covariant approach the emphasis is on field-theoretic techniques. [In the context of QG, the term ‘covariant’ is somewhat misleading because the introduction of a background metric violates diffeomorphism covariance. It is used mainly to emphasize that this approach does not necessarily involve a 1+3 decomposition of space-time]. The first step is to split the space-time metric \( g_{\mu\nu} \) in two parts, \( g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \), where \( \eta_{\mu\nu} \) is the background metric, often chosen to be flat, and it is only \( h_{\mu\nu} \); that is quantized. [There are assumptions on spacetime in order that this decomposition be possible.] That is, it is assumed that the underlying spacetime can be taken to be a continuum, endowed with a smooth background geometry, and the quantum gravitational field can be treated as any other quantum field on this background. But string theory is certainly not restricted to perturbing about flat spacetime. However, the role of spacetime has changed dramatically (from the familiar dynamical spacetime of GR). The vanishing of the conformal anomaly imposes an equation on the space-time metric, which becomes the field equation for \( g_{\mu\nu} \). If space and time are not fundamental, what replaces them in QG? At a perturbative level, the spacetime metric appears as “coupling constants” in a 2D quantum field theory. Nonperturbatively, spacetime is not fundamental but must be reconstructed from a holographic, dual theory [32]. [However, the resulting QG is non-renormalizable and hence the strategy fails by its own criteria.]

[2] In particular, these issues lead to philosophical problems in cosmology. In an open universe it is necessary to specify condition at infinity (outside of our particle horizon), which is absurd. [In addition, how do we choose when and where to set boundary conditions and initial conditions? If inflation occurs, the Universe is approximately spatially homogeneous at large distance and we then face additional problems [12]].

[3] Teitelbaum [19] treated the boundary conditions when \( M \) is not compact. The boundary integrals are required to vanish, but the necessary additional surface terms may only cancel for asymptotically flat spacetimes. In particular, the appropriate fall off rate is not fast enough to allow integration by parts for cosmological models.

[4] Hawking and Horowitz [30] give a general derivation of the gravitational Hamiltonian starting from the Einstein-Hilbert action, keeping track of all surface terms. The discussion applies to spacetimes that can be foliated by complete, nonintersecting spacelike surfaces. Thus, there are no inner boundaries, and horizons play no special role at this point [there is a global timelike vector field and a family of spacelike surfaces.] Traditionally, the gravitational Hamiltonian has been studied in the context of either spatially closed universes or asymptotically flat spacetimes. However in recent years, there has been interest in more general boundary
conditions; e.g., the possibility of a negative cosmological constant, resulting in spacetimes which asymptotically approach anti-de Sitter space.

[To define the the action for noncompact geometries, one must choose a static reference background \( g_0, \phi_0 \). The physical action is then the difference \( I_P(g, \phi) \equiv I(g, \phi) - I(g_0, \phi_0) \), which is finite for a class of fields \( g, \phi \) which asymptotically approach \( g_0, \phi_0 \). For asymptotically flat spacetimes, the appropriate background is flat space with zero matter fields, and this reduces to the familiar form of the gravitational action. However, when matter (or a cosmological constant) is included, one may wish to consider spacetimes which are not asymptotically flat. In this case one cannot use flat space as the background. The 4D scalar curvature can be related to the 3D one and the extrinsic curvature of the surfaces. When substituted into the action, the two total derivative terms give rise to boundary contributions. The first depends only on the initial and final boundary and it completely cancels the \( \oint K \) term on these surfaces. The second term only contributes to the surface integral near infinity.]

[5] The necessary and sufficient conditions for the validity of the assertion (PSC) that for local gravitational theories built from a metric, given a group action, for any group-invariant Lagrangian the equations obtained by restriction of the EL equations to group-invariant fields are equivalent to the EL equations of a canonically defined, symmetry-reduced Lagrangian, was investigated in \[16\]. One of these conditions, obtained previously in the context of transverse symmetry group actions, provides a generalization of the result that the standard Einstein-Hilbert action functional (or corresponding ADM Hamiltonian) give the correct EFE in spatially homogeneous cosmological models under symmetry reduction if and only if the space belongs to Bianchi Class A \[12\]. As Hawking \[15\] pointed out, the difficulty which arises with the Bianchi class B models is due to the presence of a non-trivial boundary terms in the restricted variational principle; spatially homogeneous variations which vanish at such a boundary would vanish everywhere, and so are not of the allowed kind.

[6] In particular, the FE of a theory and the action principle approach are not mathematically equivalent. The action principle is usually only used as a computational tool to infer the evolution FE. Historically physicists (who developed action principles in mechanics and general physics) were motivated by teleology (since the motion of the physical system is determined in the action principle formulation by the knowledge of both the initial and final states of the system) \[11\].

[7] C Helleland and S Hervik, 2015, preprint, show that a metric of [arbitrary dimension and signature] which allows for a Wick rotation to a Riemannian metric necessarily has a purely global electric Riemann and Weyl tensor. A purely electric Lorentzian spacetime is of type G, Li, D or O (in higher dimensions \[20\]: type I/D in 4D). Thus spacetimes not of these types provide examples of spaces where such a Wick rotation is disallowed. Non-Wick-
rotatable metrics include the classes of I-degenerate (Kundt metrics) in Lorentzian geometry.

[8] In cosmology it is assumed that the real complex lumpy universe with a discrete matter distribution can be adequately approximated by an “effective averaged out” stress-energy tensor. The averaging problem in cosmology is of considerable importance for the correct interpretation of cosmological data. A rigorous mathematical definition of averaging in a cosmological model is necessary. In general, a cosmological spacetime is completely characterized by its scalar curvature invariants, and a particular spacetime averaging scheme based entirely in terms of scalar curvature invariants was proposed in [14].

[9] The idea of macroscopic gravity can be considered as an extension of Lorentz’s idea, formulated first for electrodynamics [13], regarding the existence of two levels, microscopic and macroscopic, of understanding classical physical phenomena. Lorentz formulated a microscopic theory of electromagnetism and showed Maxwell’s theory to be its macroscopic version. A space averaging is always necessary and unavoidable in all macroscopic settings [13]. However, in electrodynamics the field operator is linear in the fields and it can be easily averaged, and models of continuous electromagnetic media which relates to the structure of averaged (macroscopic) fields can be constructed.

[10] An averaged Lagrangian approach (variational principle) for the derivation of Isaacson’s equation (for metrics describing (weak) gravitational radiation, by treating the gravitational perturbations using the WKB version of the high-frequency approximation), by applying the averaged Lagrangian method of Whitham, was formulated in [17]. It was shown that an averaged high-frequency Lagrangian of a similar form can be derived by applying the macroscopic gravity averaging scheme [18]. That is, by applying the space-time averaging procedure and the rules of splitting out the products of geometric objects developed within macroscopic gravity theory [13] to the (microscopic) action of GR, an averaged (macroscopic) action which upon the variation of the macroscopic metric yields the FE of macroscopic gravity was obtained.

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