Isotropy and Stability of the Brane

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We reexamine Wald’s no-hair theorem for global anisotropy in the brane world scenarios. We derive a set of sufficient conditions which must be satisfied by the brane matter and bulk metric so that a homogeneous and anisotropic brane asymptotically evolves to a de Sitter spacetime in the presence of a positive cosmological constant on the brane. We discuss the violations of these sufficient conditions and we show that a negative nonlocal energy density or the presence of strong anisotropic stress (i.e., a magnetic field) may lead the brane to collapse. We discuss the generality of these conditions.

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It has been proposed that we may live confined to a four dimensional brane in a higher dimensional spacetime [1]; in this scenario the fundamental high dimensional Planck mass could be of the same magnitude as the electroweak scale, resolving one of the hierarchy problems in the current standard model of high-energy physics. This brane world scenario generally predicts deviations from standard model processes at high energies, a direct consequence of the possibility that gravitons propagate from standard model processes at high energies, a direct consequence of the possibility that gravitons propagate in the bulk [2]. Nevertheless it remains viable given that there are no constraints on the gravitational force at very high energies (or very small distances); indeed current constraints are only beginning to probe sub-mm scales. One should also expect deviations from four dimensional Einstein gravity at high densities (or correspondingly strong gravitational fields). A first manifestation of this was seen when deriving the evolution equations for the cosmological scale factor, $a$, in the early universe: it was shown that $a$ is driven by an additional term which is proportional to $\rho^2$, where $\rho$ is the energy density of the cosmological fluid on the brane [2].

The two standard test situations for studying gravity in the strongly nonlinear regime are the formation and stability of black holes and the evolution of globally anisotropic models in the early universe. There have been attempts at constructing black hole solutions on the brane by extending the well-known four dimensional Schwarzschild solution [3]. Some work has also been done on the effect of global shear on the brane, in particular on its effect on the dynamics of the scale factor during inflation [2,4].

One of the only unambiguous predictions of the inflationary cosmology was put forward by Wald in 1983 [3]: it was shown that initially expanding homogeneous cosmological models isotropize in the presence of a positive cosmological constant. This leads to a firm prediction: if we detect any form of global anisotropy, then the universe cannot have undergone a de Sitter stage. As yet there is no detection of global anisotropy, and an upper bound on global shear was derived from the COBE four year data [9]. In this paper we wish to reexamine Wald’s result in the brane world scenario. The conditions we derive are more complicated than for the four dimensional case considered by Wald, but we can still assess their generality within the class of cosmological fluids currently being considered in cosmology.

We will attack this problem locally, i.e., in terms of dynamical quantities on the brane: in doing this we are ignoring possible constraints which we may eventually set on the bulk due to our assumption on the brane (we discuss the consequences of this approach at the end of this paper). Our starting point are the modified Einstein’s equations derived by Shiromizu, Maeda, and Sasaki [10] under the assumption of $Z_2$ symmetry in the extra dimension (for a five dimensional bulk). Following the notation of Maartens [11], these can be written as

$$G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu} + \tilde{\kappa}^2 S_{\mu\nu} - \mathcal{E}_{\mu\nu},$$

where $G_{\mu\nu}$, $g_{\mu\nu}$, and $T_{\mu\nu}$ are the four dimensional Einstein, metric, and energy-momentum tensors, respectively; $\Lambda$ is the effective cosmological constant on the brane, and $S_{\mu\nu}$ is a term quadratic in the energy-momentum tensor, defined by

$$4S_{\mu\nu} = \frac{1}{3} TT_{\mu\nu} - T_{\mu\nu} T^\rho_{\nu} - \frac{1}{2} g_{\mu\nu} \left( \frac{1}{3} T^2 - T_{\rho\sigma} T^{\rho\sigma} \right).$$

The quantities $\kappa^2/8\pi$ and $\tilde{\kappa}^2/8\pi$ are the effective Newton constant on the brane and the fundamental Newton constant in the bulk, respectively. The quadratic term $S_{\mu\nu}$ represents the matter corrections of the effective four dimensional gravity, and is significantly important at high energy, i.e., when $|T_{\mu\nu}| \gtrsim \lambda = 6\kappa^2/\tilde{\kappa}^2$, where $\lambda$ is the brane tension. Finally, $\mathcal{E}_{\mu\nu}$, which represents the projection of the five dimensional bulk Weyl tensor on the brane, is symmetric and traceless, and transmits nonlocal gravitational degrees of freedom from the bulk to the brane. We do not make any speculation about the origin of the cosmological constant $\Lambda$ which we assume to be positive.
Let us now consider two components of Eq. (1), the “initial-value” constraint equation,
\[
G_{\mu\nu}u^\mu u^\nu = \Lambda + \kappa^2 T_{\mu\nu}u^\mu u^\nu + \kappa^4 S_{\mu\nu}u^\mu u^\nu - \mathcal{E}_{\mu\nu}u^\mu u^\nu,
\]
where \( u_\mu \) is the unit normal to the spatial homogeneous hypersurfaces, and the Raychaudhuri equation,
\[
R_{\mu\nu}u^\mu u^\nu = -\Lambda + \kappa^2 (T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T)u^\mu u^\nu + \kappa^4 (S_{\mu\nu} - \frac{1}{2}g_{\mu\nu}S)u^\mu u^\nu - \mathcal{E}_{\mu\nu}u^\mu u^\nu.
\]
Both \( G_{\mu\nu}u^\mu u^\nu \) and \( R_{\mu\nu}u^\mu u^\nu \) can be expressed in terms of the three-curvature of the homogeneous hypersurfaces and the extrinsic curvature \( \Theta_{\mu\nu} = \nabla_\nu u_\mu \) of these surfaces using the Gauss-Codazzi equations. For convenience we decompose \( \Theta_{\mu\nu} \) into its trace \( \Theta \) and trace-free part \( \sigma_{\mu\nu} \),
\[
\Theta_{\mu\nu} = \frac{1}{3}\Theta h_{\mu\nu} + \sigma_{\mu\nu},
\]
where \( h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu \) projects orthogonal to \( u_\mu \), and we rewrite Eqs. (3) and (4) as
\[
\Theta^2 - 3\Lambda = \frac{3}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} - \frac{3}{2}(3\kappa^2)R + 3\kappa^4 S_{\mu\nu}u^\mu u^\nu - 3\mathcal{E}_{\mu\nu}u^\mu u^\nu + \frac{3}{2}\kappa^4 S_{\mu\nu}u^\mu u^\nu - 3\mathcal{E}_{\mu\nu}u^\mu u^\nu.
\]
and
\[
\dot{\Theta} - \Lambda + \frac{1}{3}\Theta^2 = -\sigma_{\mu\nu}\sigma^{\mu\nu} - \kappa^2 \left( T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \right)u^\mu u^\nu - \kappa^4 \left( S_{\mu\nu} - \frac{1}{2}g_{\mu\nu}S \right)u^\mu u^\nu + \mathcal{E}_{\mu\nu}u^\mu u^\nu,
\]
where the dot denotes \( u^\mu \nabla_\mu \). Eq. (1) is thus the evolution equation for the expansion rate \( \Theta \) and \( \sigma_{\mu\nu} \) is the shear of the timelike geodesic congruence orthogonal to the homogeneous hypersurfaces. The scalar curvature \((3\kappa^2)R\) is given in terms of the structure-constant tensor of the Lie algebra of the spatial symmetry group (see [12,13] for Bianchi models).

We now quickly review Wald’s argument which proceeds using the well known energy conditions on the energy-momentum tensor \( T_{\mu\nu} \). These state that for any non-spacelike vector \( b^\mu \) the following inequalities hold:
\[
T_{\mu\nu}b^\mu b^\nu \geq 0 \quad \text{and} \quad \left( T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T \right) b^\mu b^\nu \geq 0,
\]
known as the weak and strong energy conditions, respectively.

On the other hand, in standard gravity, the local quadratic term and the nonlocal term in Eqs. (3) and (4) are absent, and it is also well known that for all Bianchi models, except Bianchi type-IX, \((3\kappa^2)R \leq 0\). Therefore, Eqs. (3) and (4) together with the energy conditions (3), yield the inequalities
\[
\dot{\Theta} - \Lambda + \frac{1}{3}\Theta^2 \leq 0.
\]

The second of these two inequalities tells us that \( \Theta \) never passes through zero and if the universe is expanding at some time it will do it forever. Indeed, we have that \( \Theta \geq (3\Lambda)^{1/2} \) at all time. Moreover, the first inequality can be integrated and yields
\[
\Theta \leq \frac{(3\Lambda)^{1/2}}{\tanh(t\sqrt{\Lambda/3})}.
\]

Thus \( \Theta \) is “squeezed” between the lower limit \((3\Lambda)^{1/2}\) and the upper limit in Eq. (10), which exponentially approaches \((3\Lambda)^{1/2}\) on a time scale \((3/\Lambda)^{1/2}\). Again using Eq. (3) for standard gravity (without quadratic and nonlocal terms), as a final result one obtains that
\[
\sigma_{\mu\nu}\sigma^{\mu\nu} \leq \frac{2}{3}(\Theta^2 - 3\Lambda) \to 0,
\]
so that the shear of the homogeneous hypersurfaces rapidly approaches zero. One can generalize this result to Bianchi type-IX models provided that \( \Lambda \) is sufficiently large compared to spatial-curvature terms (3).

If one is to extend Wald’s result to the brane world scenario, one must consider two new (additional) constraints that, taken together, can be seen as sufficient conditions for its validity. These new conditions come from the presence of the local quadratic term and the nonlocal term which, in the early universe, when the isotropization is supposed to take place, may play an important role. Indeed, when
\[
\mathcal{E}_{\mu\nu}u^\mu u^\nu \leq 0
\]
and
\[
S_{\mu\nu}u^\mu u^\nu \geq 0 \quad \text{and} \quad \left( S_{\mu\nu} - \frac{1}{2}g_{\mu\nu}S \right) u^\mu u^\nu \geq 0,
\]
the r.h.s. of Eq. (4) is positive and the r.h.s. of Eq. (7) is negative, and Wald’s theorem remains valid.

Let us now separately discuss these two conditions and the consequence of their possible violations. The first sufficient condition imposes a constraint on the projection of the bulk Weyl tensor. Using the symmetry properties of \( \mathcal{E}_{\mu\nu} \), we can decompose it irreducibly as
\[
\mathcal{E}_{\mu\nu} = -\left( \frac{\kappa}{\mathcal{R}} \right)^4 \left[ \mathcal{U} \left( u_\mu u_\nu + \frac{1}{3}h_{\mu\nu} \right) + \mathcal{P}_{\mu\nu} + 2\mathcal{Q}_{\mu\nu} u_\nu \right],
\]
where \( \mathcal{U}, \mathcal{Q}_{\mu\nu}, \) and \( \mathcal{P}_{\mu\nu} \) are the effective nonlocal energy density, flux, and anisotropic stress arising from the free gravitational field in the bulk, respectively. The condition therefore becomes that

\[
\dot{\Theta} \leq \Lambda - \frac{1}{3}\Theta^2 \leq 0.
\]
Note that this imposes restrictions on the dynamics of the metric off the brane. As pointed out a number of times, there are no local constraints which can fix the sign of $\mathcal{U}$, and this remains a crucial problem for general issues. However it was recently shown from a global five-dimensional point of view that for all possible homogeneous and isotropic cosmological solutions on the brane, the bulk spacetime is Schwarzschild-AdS with the mass parameter of the black hole proportional to $|\mathcal{U}|^{1/3}$. In this setting one has some constraints on $\mathcal{U}$ (e.g. $\mathcal{U}$ is positive for closed S-AdS spacetime [15,16]). For anisotropic models such constraints do not exist.

We therefore choose to consider the possibility that $\mathcal{U}$ assumes whatever value and sign and we explore all possible brane solutions as a function of its value. As a first approach we neglect the effect of matter on the brane: as we will show, it is straightforward to generalize and include a cosmological fluid on the brane, but this would obscure the simple analysis we undertake here. We then consider coupled equations for $\sigma_{\mu\nu}$, $\Theta$, and $\mathcal{U}$ in order to study the importance of the bulk term. If we restrict ourselves to Bianchi class A models we can perform a relatively general analysis without concerning ourselves with off diagonal terms in the Ricci tensor. For this class of models we thus have $Q_{\mu} = 0$, which simplifies the set of the equations to solve.

From the nonlocal energy conservation we have

$$\dot{\mathcal{U}} + \frac{4}{3} \Theta \mathcal{U} + \sigma_{\mu\nu} \mathcal{P}_{\mu\nu} = 0,$$

and from the Gauss-Codazzi equation we have an evolution equation for the shear,

$$\dot{\sigma}_{\mu\nu} + \Theta \sigma_{\mu\nu} + (3)R_{\mu\nu} - \frac{1}{3} (3)R h_{\mu\nu} = \left(\frac{\kappa}{\lambda}\right)^4 \mathcal{P}_{\mu\nu},$$

where $(3)R_{\mu\nu}$ is the Ricci tensor of the homogeneous hypersurfaces. We first nevertheless restrict ourselves to models with $(3)R_{\mu\nu} = 0$, e.g. Bianchi type-I models.

The system of equation obtained by combining Eqs. (3), (10), and (11) is closed. This is a particular example of a more general result: due to the lack of an evolution equation for the nonlocal anisotropic stress $\mathcal{P}_{\mu\nu}$, the evolution of physical quantity on the brane cannot be predicted without making further assumptions on the bulk configuration [10,11]. We therefore choose to set $\sigma_{\mu\nu} \mathcal{P}_{\mu\nu} = 0$. As already noted in (3), this condition implies an evolution equation for $\mathcal{P}_{\mu\nu}$, i.e.,

$$\dot{\mathcal{P}}_{\mu\nu} + (\kappa/\lambda)^4 \mathcal{P}_{\mu\nu} \mathcal{P}^{\mu\nu} = 0,$$

which can be derived from Eq. (17) and is consistent on the brane, allowing us to close the system. We do not know whether this corresponds to a plausible physical form of the bulk metric, but this assumption is completely consistent from what concerns the brane point of view. Other assumptions under which one can still solve the system are nevertheless possible and we will mention some of them at the end of the paper.

Solving for the nonlocal energy density $\mathcal{U}$ and the expansion rate $\Theta$, we can perform a stability analysis of the dynamics by defining a new set of dimensionless variables:

$$\tau = \frac{t}{(3)\Lambda}, \quad X = \frac{\sigma_{\mu\nu} \sigma_{\mu\nu}}{2\Lambda},$$

$$\mathcal{U} = \frac{\Theta}{(3)\Lambda^{1/2}}, \quad U = \frac{\mathcal{U}}{\kappa^2 \lambda \Lambda}.$$  \hspace{1cm} (17)

This yields a second order autonomous system of nonlinear differential equations (with $'=d/d\tau$),

$$X' + 6XY = 0,$$

$$Y' + Y^2 + U - 1 + 2X = 0,$$

$$U' + 4UY = 0,$$

plus a constraint equation,

$$Y^2 - X - U = 1,$$  \hspace{1cm} (20)

which can be solved to find the asymptotic evolution of the $U$-term, the shear, and the expansion. Note that, if one considers the constraint equation with $U = 0$, $Y^2 - X - 1 = 0$, one recovers the evolution equations of standard gravity for the shear and expansion rate.

This system of equations is quite simple to analyze, and reveals the novel behavior of the asymptotic evolution of the brane. One can identify three critical points: $(X,Y,U) = (0,1,0)$, which is an attractor and is the same fixed asymptotically stable point we find in standard four dimensional cosmology ($\Theta \rightarrow (3\Lambda)^{1/2}$, $\sigma_{\mu\nu} \sigma_{\mu\nu} \rightarrow 0$); $(X,Y,U) = (0,-1,0)$, which is a repeller, an unstable point in the contracting phase; and $(X,Y,U) = (2,0,-3)$, which is a saddle point, and therefore unstable. The latter is the most interesting point of the system since it corresponds to $\Theta = 0$ and around this point the brane “decides” whether to enter the de Sitter phase or to collapse.

The orbits for this system of equations are given by

$$X = C|U|^{3/2},$$

$$Y^2 = 1 + U + C|U|^{3/2},$$  \hspace{1cm} (21)

where $C$ is an integration constant which depends on the initial conditions. The direction field is shown in Fig. 1. The thick red line represents the condition $X = 0$. Our physical phase space resides where the shear is non-negative, i.e., $Y^2 \geq 1+U$. The black line passing through the saddle point represents, instead, the separatrix between the region of asymptotic stability (above) and the one corresponding to instability, and is given by

$$Y = \begin{cases} - (1 + U + C_0 |U|^{3/2})^{1/2}, & U < -3 \\ (1 + U + C_0 |U|^{3/2})^{1/2}, & U > -3 \end{cases}.$$  \hspace{1cm} (22)


Let us go back to the second of the sufficient conditions, Eq. (13). We can covariantly decompose the brane energy momentum tensor $T_{\mu\nu}$ into its irreducible decomposition,

$$T_{\mu\nu} = \rho u_\mu u_\nu + p h_{\mu\nu} + \pi_{\mu\nu} + 2q_{(\mu} u_{\nu)},$$

(23)

where $\rho$ and $p$ are the energy density and isotropic pressure, $q_\mu$ is the energy flux, and $\pi_{\mu\nu}$ is the anisotropic stress of the matter on the brane.

The restrictions for the quadratic term therefore becomes

$$2\rho^2 - 3\pi_{\mu\nu} \pi^{\mu\nu} \geq 0 \quad \text{and} \quad 2\rho^2 + 3pp - 3q_\mu q^\mu \geq 0.$$

(24)

These conditions are trivially satisfied for the case of a perfect fluid (which satisfies the energy conditions) and for a free homogeneous scalar field with a velocity flow coincident with $u^\mu$. However they may not be satisfied in general.

An unambiguous example of violation of one of the above conditions is given by a cosmological pure magnetic field $B^\mu$, such that $B_\mu u^\mu = 0$ (for an extensive list of anisotropic fluids and sources see [17]); in this case we have that

$$2\rho_B^2 - 3\pi_{\mu\nu} \pi^{\mu\nu} = -\frac{3}{2} B^4 < 0,$$

$$2\rho_B^2 + 3B_B B_B - q_\mu q^\mu = \frac{3}{2} B^4 > 0,$$

where

$$\rho_B = 3p_B = B^2/2$$

(25)

and

$$\pi_{\mu\nu} = -B_\mu B_\nu + (1/3) h_{\mu\nu} B^2.$$

(26)

For simplicity we restrict ourselves to the case of Bianchi type-I cosmology, which can naturally accommodate a spatially homogeneous ordered magnetic field, and we neglect any other form of matter. The presence of the magnetic field introduces a preferred direction and we can consider the possibility that the background anisotropy is due to this presence. This corresponds to the typical situation in which the magnetic field is a shear eigenvector,

$$\sigma_{\mu\nu} B^\mu = \sigma_\mu B_\mu,$$

(27)

which does not represent the most general case; however it considerably simplifies the equations [18]. We consider

where $C_0 = 2/3^{3/2}$. The dynamics is therefore such that, for any initial condition above this line, the system asymptotically converges to $(X = 0, Y = 1, U = 0)$, which corresponds to $\sigma_{\mu\nu} \sigma^{\mu\nu} = 0, \Theta = \sqrt{3\Lambda}$, and $U = 0$; however, for initial conditions below this separatrix, the brane universe collapses and the anisotropy grows to infinity.

Isotropy is reached for any initial condition corresponding to $U \geq 0$. We therefore recover the sufficient condition for the validity of Wald’s theorem mentioned already above. When $U < 0$ the brane may not isotropize and can instead collapse. The conventional four dimensional cosmology resides on the subspace (the $Y$ axis) given by $U = 0$. As a side result, we can see that when the brane enters the de Sitter phase, the $U$ term approaches zero asymptotically. Furthermore, there is a region of initial conditions corresponding to small $U$ where it is possible to start with the brane contracting and still reach the stable point. This reflects the new properties of the modified gravity and the possibility of avoiding the cosmological singularity in the presence of the nonlocal term $U$, which was already noted in [3].

What we have found is not really surprising: the projected bulk Weyl tensor plays a crucial role in the isotropization of the brane. This means that there are additional constraints one must impose in the bulk if one is to find a complete generalisation of Wald’s theorem to the brane; in some sense this defeats the purpose of such a no-hair theorem. However, we have been able to present a local analysis in which we have closed the system by simplifying the effect of the bulk and still obtained non-trivial dynamics. Note that we can obtain trajectories with $U < 0$; however, this is not the crucial diagnostic for whether the brane isotropizes or not. Of course it is of paramount importance to see what global constraints the bulk imposes on $U$ in the same way as has been done for homogeneous and isotropic models.
the back-reaction from the bulk but, to be able to close the system, we assume \( \sigma^{\mu\nu} P_{\mu\nu} = 0 \) and \( P_{\mu\nu} B^\mu = 0 \). In this case, together with the Raychauduri equation, the full set of equations to solve becomes

\[
\dot{\rho}_B + \Theta (\rho_B + p_B) + \sigma^{\mu\nu} \pi_{\mu\nu} = 0
\]

from the local energy conservation,

\[
\dot{U} + \frac{4}{3} \Theta U = \frac{k^4}{24} (6 \rho_B \dot{\pi}_{\mu\nu} + 6 (\rho_B + p_B) \sigma^{\mu\nu} \pi_{\mu\nu} + 2 \Theta \pi^{\mu\nu} \pi_{\mu\nu} - 2 \sigma^{\mu\nu} \pi_{\mu\alpha} \pi_{\nu}^\alpha]
\]

from the nonlocal conservation equation, and

\[
\sigma_{\mu\nu} \dot{\sigma}^{\mu\nu} + \Theta \sigma_{\mu\nu} \sigma^{\mu\nu} = \kappa^2 \pi_{\mu\nu} \sigma^{\mu\nu} - \frac{k^4}{12} (\rho_B + 3 p_B) \pi_{\mu\nu} - \pi_{\mu\alpha} \pi_{\nu}^\alpha \]

from the Gauss-Codazzi equation.

Defining the additional new variables

\[
W = \sigma x (3 \frac{1}{\lambda})^{1/2} \quad \text{and} \quad Z = \frac{\rho_B}{\lambda}
\]

we obtain an autonomous nonlinear system,

\[
\begin{align*}
X' + 6 Y X + 4 x W [Z - (4/3) Z^2] &= 0, \\
Y' + Y^2 - 1 + 2 X + 2 x [Z + (3/2) Z^2] + U &= 0, \\
Z' + 4 Z Y - 2 W Z &= 0, \\
U' + 4 U Y - 4 x (5/3) W - 6 Y &= 0, \\
W' + 3 Y W + 8 x [Z - (4/3) Z^2] &= 0,
\end{align*}
\]

with the constraint equation

\[
Y^2 - 2 x [Z - (3/2) Z^2] - U - X = 1.
\]

Here \( \chi = \kappa^2 \lambda/2 \Lambda \).

This system has at least three critical points which correspond to those obtained analyzing the case without magnetic field: \((X = 0, Y = 1, Z = 0, U = 0, W = 0)\) is the de Sitter attractor, \((X = 0, Y = -1, Z = 0, U = 0, W = 0)\) is the repeller, and \((X = 2, Y = 0, Z = 0, U = -3)\) is a saddle point for any value of \( W \). They are all at \( Z = 0 \) and, indeed, \( \rho_B = 0 \) is an invariant submanifold of the system, i.e., if an orbit starts with \( \rho_B = 0 \) it will maintain this condition. When the magnetic field is much weaker than the brane tension \( Z < 1 \), we can neglect all the \( Z^2 \) contributions. In this case we still find a region of initial conditions corresponding to collapse of the brane, but only for \( U < -3 \); this means that the negativity of the \( U \)-term is responsible for the collapse and not the magnetic field.

When neglecting the \( Z^2 \) contribution we can eventually set \( U = 0 \) and recover standard gravity. Note that this is not possible when the quadratic term is included. Indeed, by analyzing the full system of differential equations one finds that \( U = 0 \) is not a consistent solution. By setting \( U \) and its derivative to zero we have a conservation equation involving quadratic terms of the energy momentum tensor and implying \( W = (18/5)Y \), which is not consistent with the evolution equations for \( Y \) and \( W \). We therefore find an interesting result: in this setup we cannot set the nonlocal energy density to zero unless we neglect quadratic terms. It is conceivable that this is a consequence of discarding the effect of the nonlocal anisotropic stress, and we are aware that without a full analysis of the bulk to completely solve the evolution equations this must remain an open problem in this work. This example, nevertheless, underline the importance of the back-reaction of nonlocal bulk effects in the presence of an anisotropic stress on the brane. (As a perturbation, the effect of anisotropic stresses was considered and discussed in \([13]\).)

When \( \chi = 0 \) the evolutions of \( X, Y, \) and \( U \) are decoupled from \( Z \) and \( W \), and the phase space is given by the product \((Z, W) \times (X, Y, U)_{Z = 0}\). The existence of other critical points depends on the value of the parameter \( \chi \). For

\[
0 \leq \chi < \frac{16}{27}
\]

there are three more saddle points, one at \((X = 2 - (27/8) \chi, Y = 0, Z = 3/4, U = -3 + (57/16) \chi, W = 0)\), and other two at \( Y \neq 0 \). These saddle points degenerate to one when \( \chi = 16/27 \) at the bifurcation point \((X = 0, Y = 0, Z = 3/4, U = -3 + (57/27), W = 0)\). For \( \chi > 16/27 \) there is no other saddle point.

The novelty in the behavior of this system is the fact that there are regions of phase space where the brane is unstable even when the \( U \)-term is positive. These regions can be found by numerically integrating system \([23]\) and they typically appear for large value of the magnetic field \( \rho_B \gtrsim \lambda \). Therefore the introduction of the anisotropic stress present in the quadratic term of the projected Einstein’s equations makes the brane unstable even when the \( U \)-term is positive. We conclude that the violation of the second sufficient condition for the validity of Wald’s theorem is enough to cause an expanding brane to collapse, provided that the source of violation is sufficiently strong.

In order to visualize the five-dimensional phase space and have an idea of the orbits of the system we use the fact that \( W^2 = 4X \) is another invariant submanifold. It corresponds to the axial symmetry of the brane universe around the \( z \) direction and is realized when \( \sigma_{\mu\nu} \sim \pi_{\mu\nu} \). This condition allows us to get rid of the evolution equation for \( X \) and use the constraint equation to eliminate \( U \). We are thus left with a closed system of three differential equations in \( Z, W, \) and \( Y \). In Fig. 2 we plot some typical trajectories of this system for \( \chi = 1/2 \). As one can see, for large value of the energy density of the magnetic field, \( Z \gtrsim 1 \), the brane can collapse.

Let us discuss some possible generalizations of our results. In the analysis undertaken so far, we have assumed
stability is enlarged by the introduction of the curvature ties, we can easily conclude that the region of asymptotic

By again considering Fig. 1 and applying these inequalities, one obtains the following behavior of the "new" or-

\[ R' = \frac{(3)R}{\Lambda}, \quad (35) \]

the relevant equations become

\[ Y' + Y^2 + U - 1 + 2X = 0, \quad (36) \]

\[ U' + 4UY = 0, \]

with the constraint

\[ Y^2 - X - U - 1 = -R/2 \geq 0, \quad (37) \]

where we have used that \((3)R \leq 0\). This, as already stated, is true for all Bianchi models except type-IX.

Let us define with \(Y_{R=0}(U)\) the "old" orbits of the system obtained by setting the curvature term to zero, and given by Eq. (21). By carefully integrating the evolution equations for \(Y\) and \(U\) and using the inequality (27), one obtains the following behavior of the "new" orbits \(Y(U)\): if at some point of the phase space of Fig. 1 \(Y(U(0)) = Y_{R=0}(U(0))\) (a new orbit crosses an old one at \(\tau = 0\)), then

\[ Y(U(\tau)) \geq Y_{R=0}(U(\tau)), \quad \tau > 0, \]

\[ Y(U(\tau)) \leq Y_{R=0}(U(\tau)), \quad \tau < 0. \]

By again considering Fig. 1 and applying these inequalities, we can easily conclude that the region of asymptotic stability is enlarged by the introduction of the curvature term. Note that the orbits never cross the \(Y\) axis since there \(U' = 0\). Indeed we still find the attractor and the repeller at the usual positions \((X, Y, U) = (0, \pm 1, 0)\), corresponding to \(R = 0\). However, it is impossible to show the existence of a saddle point at \(Y = 0\) unless one can show that \(R\) is stationary at this point. This is true for models in which the surfaces of transitivity are isotropic (see (32)),

\[ (3)R_{\mu\nu} = \frac{1}{3} (3)Rh_{\mu\nu}, \quad (38) \]

but is not generally the case. Looking at Fig. 1, in these models the saddle point is displaced towards the left, at \(U = -3 + R\).

According to these inequalities it is still possible that the universe recollapses. Other types of non de Sitter behavior are of course nevertheless possible, in particular, an asymptotic approach to an Einstein static brane. However, even with the introduction of a curvature term, it seems reasonable to conjecture that the only types of stable asymptotic behaviors are an asymptotic approach to de Sitter spacetime and recollapse, as in the case of \((3)R_{\mu\nu} = 0\). It is well known that the Kasner (vacuum Bianchi type-I model) collapsing solution is an attractor of Bianchi class A models with ordinary matter content. It would of course be important to understand if this is still the case in the presence of a nonlocal bulk effect. We reserve this study for future work.

The effect of matter and other possible cosmological fluids has been discarded in our analysis, but it can be included quite straightforwardly by adding a perfect fluid moving orthogonally to the homogeneous hypersurfaces with state equation \(p = \omega \rho\) to our system of equations. The presence of a new variable \(\rho\) introduces an extra dimension in the phase space and a new parameter \(\omega\).

When the magnetic field is not taken into account we can integrate the orbits of the system using the dimensionless variable \(Z_F = \rho / \lambda\). This yields

\[ X = C|U|^{3/2}, \]

\[ Z_F = D|U|^{3/(\omega + 1)/4}, \]

\[ Y^2 = 1 + U + C|U|^{3/2} \]

\[ + \chi(2D|U|^{3/(\omega + 1)/4} + D^2|U|^{3/(\omega + 1)/4}), \]

where \(D\) is a new integration constant. The attractor and the repeller are at the usual positions \((X, Y, U) = (0, \pm 1, 0)\) and \(Z_F = 0\). However, now the saddle point at \(Y = 0\) becomes a line which depends on \(Z_F:\)

\[ U = -3(1 - \chi(\omega - 1)Z_F - \chi \omega Z_F^2). \quad (39) \]

Note, however, that for \(\omega > -1/3\) the allowed values of \(Z_F\) of this line are limited by the constraint \(X \geq 0\). The separatrix between the region of asymptotic stability and instability is the two dimensional surface which forms the saddle line at the intersection with \(Y = 0\). The region of
instability always lies below this separatrix. The general characteristics of the phase space of the system, as illustrated in Fig. 1, remain unchanged: the only two stable asymptotic behaviors are an approach to the de Sitter spacetime and recollapse; recollapse can only occur when $U$ is negative. On the $U = 0$ plane, we recover the results of [2].

In order to close the system of equations discussed above, we have always chosen to neglect the nonlocal anisotropic stress $P_{\mu\nu}$. As already discussed, this always corresponds to a consistent solution in the bulk, provided that the equations are consistent on the brane, although it may not correspond to some plausible physical bulk configuration. For future investigations we mention three other possibilities which allow one to consistently close the system on the brane. If you have some sort of anisotropic stress on the brane a very simple possibility is of course to assume that $\dot{\pi}_{\mu\nu}$ is of course to assume that $\dot{\pi}_{\mu\nu}$ is of course $\pi_{\mu\nu}$ is negative. On the $\pi_{\mu\nu} = 0$ plane, we recover the results [11]. Given that we do not have an evolution equation for the nonlocal anisotropic stress $P_{\mu\nu}$, we have only considered a subspace of possible solutions such that we could close the system. We should point out, however that the complementary approach, i.e., constructing complete bulk solutions and assessing their effect on the brane, may also be restrictive although the two approaches are clearly necessary for a complete understanding of the asymptotic stability of the brane.

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We briefly summarize the present work. We have derived a set of sufficient conditions to be satisfied by the local brane matter and the nonlocal energy density, so that a homogeneous and anisotropic brane with tension $\lambda$, satisfying modified Einstein’s equations with a positive effective cosmological constant, asymptotically evolves to de Sitter spacetime. We have discussed the violation of these conditions, showing that, in the presence of a negative nonlocal energy density $U < 0$ and/or strong anisotropic stress $|\pi_{\mu\nu}| \gtrsim \lambda$, a Bianchi type-I brane, even if initially expanding, is unstable and may collapse. This is our main result. We have also given an explicit example of source of anisotropic stress by considering the case of a cosmological magnetic field on the brane, and we have shown that the back-reaction of nonlocal effects cannot be neglected. We have discussed possible generalisations.

Finally, let us reemphasise a key aspect of our analysis. We have studied the asymptotic behavior of the brane from a local point of view. On the one hand, this approach is incomplete and this is evident in what we have found: the bulk back-reaction plays a crucial role in the evolution of the brane. This back-reaction has been discarded in previous local analyses where isotropic fluids were considered [this is in fact consistent with our analysis, merely corresponding to a choice of integration constant when solving Eqs. (20)].

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