Comments on D-branes in $AdS_3$

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We study D-branes that preserve a diagonal $SL(2)$ affine Lie algebra in string theory on $AdS_3$. We find three classes of solutions, corresponding to the following representations of $SL(2)$: (1) degenerate, finite dimensional representations with half integer spin, (2) principal continuous series, (3) principal discrete series. We solve the bootstrap equations for the vacuum wave functions and discuss the corresponding open string spectrum. We argue that from the point of view of the AdS/CFT correspondence, the above D-branes introduce boundaries with conformal boundary conditions into the two dimensional spacetime. Open string vertex operators correspond to boundary perturbations. We also comment on the geometric interpretation of the branes.
1. Introduction

In this paper we continue our study [1] of string theory on $AdS_3$, focusing on the physics of D-branes in this background. There are a number of motivations for studying such D-branes (see also [1]):

1. The $SL(2, R)$ group manifold is one of the simplest non-compact curved backgrounds in string theory. Since D-branes are believed to be important for a microscopic understanding of string theory, it seems useful to develop a better understanding of D-brane dynamics in such backgrounds.

2. In the context of the AdS/CFT correspondence, the $AdS_3$ case is special in a number of ways: the relevant two dimensional conformal group is infinite dimensional, and the theory can be studied beyond the supergravity approximation, since it can be defined without turning on Ramond-Ramond backgrounds (see e.g. [2,3,4,5,6] for some recent discussions and additional references). It is thus interesting to study D-branes in $AdS_3$ to address various questions regarding the correspondence. For example, given a string background of the form $AdS_3 \times \mathcal{N}$, where $\mathcal{N}$ is a compact manifold [7], can one construct directly the dual “spacetime” two dimensional CFT, perhaps by studying D-branes in the bulk theory? Other (related) questions are what do D-branes in $AdS_3$ correspond to in the spacetime CFT? What excitations live on the branes and what is their role in the spacetime CFT?

3. As a special case of (1) above, there are some interesting backgrounds that are closely related to $SL(2, R)$; an understanding of D-branes in $AdS_3$ would be very useful for studying D-branes in these backgrounds. For example, Liouville theory and the Euclidean and Lorentzian two dimensional black hole are cosets of the form $SL(2)/U(1)$ [8,9,10]; Liouville corresponds to modding out by a Borel subgroup, while the Euclidean (Lorentzian) black hole is obtained by modding out by the timelike (spacelike) $U(1)$. The cosmological model of [11] corresponds to the coset $[SL(2) \times SU(2)]/U(1)^2$. All the above backgrounds are of interest in the context of holography. Liouville theory is central in two dimensional string theory. It is believed to be dual, at least to all orders in string perturbation theory, to a certain large $N$ matrix quantum mechanics (see e.g. [12] for a review). The two dimensional black hole appears in the near-horizon geometry of $NS5$-branes and Calabi-Yau singularities [13,14] and thus plays a role in Little String Theory (see e.g. [15] for a review). An outstanding problem

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1. See e.g. [7] for a construction of a large class of such backgrounds.
in two dimensional string theory and LST is to understand microscopically the high energy density of states, which was computed thermodynamically in [16,17]. Studying D-branes in these backgrounds might be useful for that.

(4) An important special case of (3) concerns the physics of D-branes located near the tip of the two dimensional semi-infinite cigar (or Euclidean two dimensional black hole). The near-horizon geometry describing two NS5-branes intersecting along $\mathbb{R}^{3,1}$ (or equivalently a resolved conifold) is $\mathbb{R}^{3,1} \times SL(2)/U(1)$. D4-branes stretched between the fivebranes correspond in this geometry to D-branes located near the tip of the cigar [18,19]. The low energy theory on a collection of such branes is $N = 1$ super-symmetric Yang-Mills theory [20], and one may hope that a better understanding of these D-branes will help understand holography for $N = 1$ SYM (see [21] for a recent discussion).

Unlike the case of D-branes in rational CFT’s, which is well studied (see e.g. [22] for a review), there has not been much work on non-compact interacting theories such as $SL(2, R)$, with some notable exceptions. The authors of [23,24] studied some aspects of the $SL(2, R)$ boundary states; [18] discussed some properties of D-branes on the cigar. The geometric, semi-classical interpretation of D-branes in $AdS_3$ was clarified in [23]. In particular, in this paper it was shown that there are several classes of D-branes that preserve a diagonal $SL(2)$ current algebra: pointlike D-instantons, Euclidean two dimensional branes with worldvolume $H_2$, D-strings with worldvolume $AdS_2$ stretched between two points on the boundary of $AdS_3$, and tachyonic D-strings with worldvolume $dS_2$. We will reproduce these results below from an algebraic analysis.

There was also some interesting work on D-branes in Liouville theory [26,27,28], where it was shown that there are two classes of branes in this case. The authors of [26,27] studied branes that are extended in the Liouville direction, while [28] constructed localized branes. The difference is reflected in the spectrum of open string excitations: the extended branes of [26,27] support open string excitations which carry arbitrary momentum in the Liouville direction, while the localized branes [28] have a finite number of open string Virasoro primaries. It was also shown in [26,27,28] that the boundary of the worldsheet is at a finite distance (in the dynamical worldsheet metric) from the bulk for the extended branes, while for localized branes this distance is infinite.

The above results on Liouville branes were obtained by solving the bootstrap equations for the one point functions of bulk operators on the upper half plane, and combining the resulting information with an analysis of the annulus amplitude in the open and closed
string channels ("modular bootstrap"). This seems to be a fruitful way of studying the theory. The main purpose of this paper is to repeat this analysis for the $AdS_3$ case. We will see that the analogs of the localized branes of [28] in this case are associated with D-instantons in $SL(2, R)$, while the analogs of the extended branes of [26,27] are the $H_2$, $dS_2$ and $AdS_2$ branes of [23]. In the process we will learn more about these branes and answer some of the questions mentioned above. In particular, we will see that branes in $AdS_3$ introduce boundaries into the base space on which the spacetime CFT is defined. Most of the detailed analysis will be done in the Euclidean version of $AdS_3$, $H_3$; we will discuss the continuation of the results to Lorentzian $AdS_3$.

The plan of the paper is as follows. In section 2 we start with a very brief summary of $AdS_3$ CFT on the plane. We mainly establish the notation and quote some results that are needed for the subsequent analysis. In section 3 we solve the bootstrap for the wavefunctions of localized D-branes satisfying "symmetric gluing conditions" $J^a(z) = \bar{J}^a(\bar{z})$ for $z = \bar{z}$ ($a = 3, \pm$). We find an infinite number of solutions labeled by an integer $r = 1, 2, 3, \cdots$. The basic solution, $r = 1$, does not contain any non-trivial primary boundary operators. The spectrum of excitations contains the current algebra block of the identity. This boundary state can be thought of as an analog of the identity Cardy state for rational CFT’s, or a pointlike instanton in $AdS_3$. The spacetime theory in the background of a $D$-instanton is equivalent to a two dimensional CFT on a manifold with a boundary. The $r > 1$ branes correspond to multi-instanton configurations, and contain a finite number of degenerate $SL(2)$ primaries, whose properties are very reminiscent of the $SU(2)$ case. This is analogous to the results of [28], who showed that the spectrum of excitations of the localized branes in Liouville theory contains a finite number of degenerate Virasoro representations, in a structure very reminiscent of minimal model branes.

In section 4 we discuss the annulus amplitude corresponding to the branes constructed in section 3, and show that the spectrum of boundary operators proposed in section 3 leads to sensible modular properties. This is used to determine the spectrum of open strings stretched between different branes.

In section 5 we discuss a second class of solutions to the bootstrap equations, describing two dimensional D-branes in $AdS_3$. We construct the operator corresponding to the worldvolume electric field on the D-brane and exhibit two classes of branes, one with a supercritical electric field, and the other with subcritical field. In section 6 we discuss the corresponding annulus amplitude and the spectrum of open strings living on these branes, and strings connecting them to D-instantons.
In section 7 we discuss the generalization of the formalism to “asymmetric” gluing conditions for the $SL(2)$ currents, obtained by twisting with automorphisms of the Lie algebra. We show that different gluing conditions correspond in the spacetime CFT to boundaries at different locations (related by conformal transformations). In Minkowski space one can describe both spacelike and timelike boundaries using different gluing conditions.

In section 8 we comment on the geometric interpretation of our results, and in particular their relation to [25]. We also discuss possible extensions. Some technical results appear in the appendices.

2. A brief review of conformal field theory on $AdS_3$

Lorentzian $AdS_3$ can be thought of as a pseudosphere in $\mathbb{R}^{2,2}$,

$$x_1^2 + x_2^2 - x_0^2 - x_3^2 = -l^2,$$  \hspace{1cm} (2.1)

where $l$ is the radius of curvature of $AdS_3$. A convenient parametrization of the space is via the coordinates $(r, t, \theta)$,

$$x_0 = \sqrt{l^2 + r^2} \cos t$$
$$x_3 = \sqrt{l^2 + r^2} \sin t$$
$$x_1 = r \cos \theta$$
$$x_2 = r \sin \theta.$$  \hspace{1cm} (2.2)

$r$ can be thought of as the radial coordinate on $AdS_3$, while $(t, \theta)$ parametrize the boundary. The metric on $AdS_3$ in these coordinates is

$$ds^2 = \left(1 + \frac{r^2}{l^2}\right)^{-1} dr^2 - l^2 \left(1 + \frac{r^2}{l^2}\right) dt^2 + r^2 d\theta^2.$$  \hspace{1cm} (2.3)

Another useful coordinate system is Poincare coordinates $(u, \gamma, \bar{\gamma})$, related to $(r, t, \theta)$ by the transformation

$$u = \frac{1}{l} \left(\sqrt{l^2 + r^2} \cos t + r \cos \theta\right)$$
$$\gamma = \frac{\sqrt{l^2 + r^2} \sin t + r \sin \theta}{\sqrt{l^2 + r^2} \cos t + r \cos \theta}$$
$$\bar{\gamma} = -\frac{\sqrt{l^2 + r^2} \sin t + r \sin \theta}{\sqrt{l^2 + r^2} \cos t + r \cos \theta}.$$  \hspace{1cm} (2.4)

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As usual, we will consider the infinite cover of $SL(2, R)$, where $t$ is not periodic.
The metric (2.3) is
\[ ds^2 = l^2 \left( \frac{du^2}{u^2} + u^2 d\gamma d\bar{\gamma} \right). \] (2.5)

The boundary of \( AdS_3 \) is at large \( u \). The coordinates (2.4) are obtained by parametrizing the \( SL(2, R) \) group manifold by the Gauss decomposition
\[ g = \begin{pmatrix} 1 & \bar{\gamma} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & u \end{pmatrix} \begin{pmatrix} \gamma & u^{-1} \\ \bar{\gamma} u & u \end{pmatrix} = \begin{pmatrix} \gamma \bar{\gamma} u + u^{-1} \bar{\gamma} u \\ \gamma u & u \end{pmatrix}. \] (2.6)

It is often convenient to analytically continue the geometry (2.3) to Euclidean space, e.g. by setting \( t = -i\tau \), or equivalently replacing \( x_3 \rightarrow ix_3 \) in (2.2). This gives rise to a three dimensional hyperbolic space \( H_3 \simeq SL(2, C)/SU(2) \). This is the model that will be mostly studied below. We will parametrize \( H_3 \) as in (2.6), with \( \gamma \) a complex variable whose complex conjugate is \( \bar{\gamma} \). It should be kept in mind that the relation to \( SL(2) \) is through the analytic continuation that maps the coordinates on \( H_3, (u, \gamma, \bar{\gamma}) \), to (2.4).

The WZNW model on \( AdS_3 \) is invariant under two copies of the \( SL(2, R) \) current algebra. The left moving symmetry is generated by the currents \( J^a(z) \), with \( a = 3, \pm \), satisfying the OPE algebra
\[ J^3(z)J^{\pm}(w) \sim \pm \frac{J^{\pm}(w)}{z-w}, \]
\[ J^3(z)J^3(w) \sim -\frac{k}{(z-w)^2} \]
\[ J^-(z)J^+(w) \sim \frac{k}{(z-w)^2} + \frac{2J^3(w)}{z-w}. \] (2.7)

A similar set of OPE’s holds for the right moving \( SL(2) \) current algebra. The level \( k \) of the current algebra (2.7) is a real number, related to \( l \) in (2.3) via the relation \( k = l^2 \) (in string units). It determines the central charge of the WZNW CFT via
\[ c = \frac{3k}{k-2}. \] (2.8)

One is typically interested in \( k > 2 \).

A natural set of observables is given by the eigenfunctions of the Laplacian on \( AdS_3 \),
\[ \Phi_h = \frac{1 - 2h}{\pi} \left( \frac{1}{|\gamma - x|^2 e^{\frac{Q\phi}{2}} + e^{-\frac{Q\phi}{2}}} \right)^{2h} = \]
\[ -e^{Q(h-1)\phi} \delta^2(\gamma - x) + O(e^{Q(h-2)\phi}) + \frac{(1 - 2h)e^{-Qh\phi}}{\pi|\gamma - x|^{4h}} + O(e^{-Q(h+1)\phi}), \] (2.9)
where we are using Poincare coordinates (2.5), with
\[ u = e^{Q \phi}. \] (2.10)

\( Q \) is related to \( k \) via
\[ Q^2 = \frac{2}{k-2} \equiv -\frac{2}{t}. \] (2.11)
The last equality defines
\[ t = -(k-2). \] (2.12)

\( x \) is an auxiliary complex variable whose role can be understood by expanding the operators \( \Phi_h \) near the boundary of \( AdS_3, \phi \rightarrow \infty \), as is done on the second line of (2.9). Note the difference between the behavior for \( h > 1/2 \) and \( h < 1/2 \) [3]. For \( h > 1/2 \), the operators \( \Phi_h \) are localized near the boundary at \( (\gamma, \bar{\gamma}) = (x, \bar{x}) \). For \( h < 1/2 \), the delta function is subleading, and the operators are smeared over the boundary. One can think of \( \Phi_h \) as the propagator of a particle with mass \( h(h-1) \) from a point \( (x, \bar{x}) \) on the boundary, to a point \( (\phi, \gamma, \bar{\gamma}) \) in the bulk of \( AdS_3 \). Thus, \( x \) labels the position on the boundary of \( AdS_3 \), which is the base space of the CFT dual to string theory on \( AdS_3 \) via the AdS/CFT correspondence.

The operators \( \Phi_h \) are primary under the \( \hat{SL}(2) \) current algebra (2.7); they satisfy
\[
\begin{align*}
J^3(z)\Phi_h(x, \bar{x}; w, \bar{w}) &\sim -\frac{(x\partial_x + h)\Phi_h(x, \bar{x})}{z-w} \\
J^+(z)\Phi_h(x, \bar{x}; w, \bar{w}) &\sim -\frac{(x^2\partial_x + 2hx)\Phi_h(x, \bar{x})}{z-w} \\
J^-(z)\Phi_h(x, \bar{x}; w, \bar{w}) &\sim -\frac{\partial_x \Phi_h(x, \bar{x})}{z-w} .
\end{align*}
\] (2.13)

Their worldsheet scaling dimensions are
\[ \Delta_h = -\frac{h(h-1)}{k-2} = \frac{h(h-1)}{t} . \] (2.14)

It is very convenient [29] to “Fourier transform” the \( SL(2) \) currents as well, and define
\[ J(x; z) \equiv -J^+(x; z) = 2xJ^3(z) - J^+(z) - x^2J^-(z) . \] (2.15)

\[^3\] Note that in (2.9) we have rescaled \( \phi \) and \( \Phi_h \) relative to equations such as (2.8) in [3].
Since $J_0^- = -\partial_x$ is the generator of translations in $x$ (see (2.13)) we can think of (2.13) as a result of “evolving” the currents $J^a(z)$ in $x$:

\[
J^+(x; z) = e^{-xJ_0^-} J^+(z) e^{xJ_0^-} = J^+(z) - 2xJ^3(z) + x^2J^-(z)
\]
\[
J^3(x; z) = e^{-xJ_0^-} J^3(z) e^{xJ_0^-} = J^3(z) - xJ^-(z) = -\frac{1}{2} \partial_x J^+(x; z)
\]
\[
J^-(x; z) = e^{-xJ_0^-} J^-(z) e^{xJ_0^-} = J^-(z) = \frac{1}{2} \partial_x^2 J^+(x; z).
\]  

(2.16)

The OPE algebras (2.7) and (2.13) can be written in terms of $J(x; z)$ as follows:

\[
J(x; z) J(y; w) \sim k \frac{(y - x)^2}{(z - w)^2} + \frac{1}{z - w} \left[ (y - x)^2 \partial_y - 2(y - x) \right] J(y; w)
\]

(2.17)

\[
J(x; z) \Phi_h(y, \bar{y}; w, \bar{w}) \sim \frac{1}{z - w} \left[ (y - x)^2 \partial_y + 2h(y - x) \right] \Phi_h(y, \bar{y}).
\]  

(2.18)

It is sometimes useful to expand the operators (2.9) in modes,

\[
\Phi_h(x, \bar{x}) = \sum_{m, \bar{m}} V_{h-1; m, \bar{m}} x^{-m-\hbar} \bar{x}^{-\bar{m}-\hbar}
\]

(2.19)

or

\[
V_{j; m, \bar{m}} = \int d^2 x x^j x^{\bar{m}} \Phi_{j+1}(x, \bar{x}).
\]  

(2.20)

Note that (2.13) implies that $V_{j; m, \bar{m}}$ transforms under $SL(2)$ as follows:

\[
J^3(z) V_{j; m, \bar{m}}(w) = \frac{m}{z - w} V_{j; m, \bar{m}}
\]

\[
J^\pm(z) V_{j; m, \bar{m}}(w) = \frac{(m \mp j)}{z - w} V_{j; m \pm 1, \bar{m}}.
\]  

(2.21)

As is clear from (2.14), the operators $\Phi_h$ and $\Phi_{1-h}$ are closely related. They are related by a reflection symmetry [30],

\[
\Phi_h(x, \bar{x}; z, \bar{z}) = \mathcal{R}(h) \frac{2h - 1}{\pi} \int d^2 x' |x - x'|^{-4h} \Phi_{1-h}(x', \bar{x}'; z, \bar{z})
\]

(2.22)

where the $x'$ integral runs over the complex plane. The reflection coefficient $\mathcal{R}(h)$ depends on the normalization of the operators (see [1]). In the normalization used in [1], which we will adopt here, it is equal to

\[
\mathcal{R}(h) = \frac{\Gamma(1 + \frac{2h-1}{t})}{\Gamma(1 - \frac{2h-1}{t})}.
\]  

(2.23)

Note that in the semiclassical limit $t \to -\infty$ the reflection coefficient goes to one; the resulting semiclassical relation (2.22) can be verified directly by using (2.9).
3. \( SL(2) \) conformal field theory on the upper half plane (I)

After reviewing \( AdS_3 \) CFT on the plane in the previous section, we turn next to the construction of D-branes in this background. We would like to analyze the theory on the upper half plane \( \text{Im} z \geq 0 \), with boundary conditions that preserve conformal symmetry. We will impose the standard requirement on the worldsheet stress tensor,

\[
T(z) = \bar{T}(\bar{z}); \quad \text{for } z = \bar{z}.
\]

We will furthermore require that the D-branes preserve a diagonal \( SL(2) \) current algebra. The latter is not necessary – one certainly expects to find D-branes that do not satisfy this requirement. Nevertheless, it is useful to analyze the most symmetric D-branes before moving on to less symmetric ones.

The simplest boundary conditions on the currents are:

\[
J^a(z) = \bar{J}^a(\bar{z}); \quad \text{for } z = \bar{z}, \quad a = 3, +, -. (3.2)
\]

In this section we will analyze localized D-branes that arise when we impose (3.1), (3.2). We will follow closely the discussion of [28].

Consider the one point function of the bulk observable (2.9) on the upper half plane,

\[
\langle \Phi_h(x, \bar{x}; z, \bar{z}) \rangle = \frac{U(h)}{(x - \bar{x})^{2h}(z - \bar{z})^{2\Delta_h}}. (3.3)
\]

The \( z \) dependence in (3.3) is fixed by the conformal symmetry which remains unbroken on the upper half plane (3.1). The \( x \) dependence follows from the unbroken \( SL(2) \) symmetry (3.2). As is standard in boundary conformal field theory (BCFT), one can think of the one point functions (3.3) as the wavefunctions of the corresponding boundary states [31]. We would like to determine them, and use that to characterize the boundary states.

Note that the one point function (3.3) exhibits a singularity as \( x \to \bar{x} \). We will see below that the meaning of this is the following. Recall that \( x \) labels the base space on which the CFT dual to string theory on \( AdS_3 \) lives [4,8]. Introducing a boundary with the boundary conditions (3.1), (3.2) into the worldsheet corresponds in the spacetime CFT to the appearance of a boundary at \( x = \bar{x} \) with conformal boundary conditions similar to

\[\text{Since } J^a + \bar{J}^a \text{ is a conserved charge, we see from (2.13) that the correlator (3.3) must satisfy } \partial_x + \partial_{\bar{x}} = 0, x\partial_x + \bar{x}\partial_{\bar{x}} + 2h = 0, x^2\partial_x + \bar{x}^2\partial_{\bar{x}} + 2h(x + \bar{x}) = 0, \text{ whose solution is const} \times (x - \bar{x})^{-2h}.\]
for the spacetime stress tensor constructed in [2,3]. The singularity as $x \to \bar{x}$ in (3.3) corresponds to the operator $\Phi_h$ approaching the boundary in spacetime.

It is also useful to note that $U(h)$ has a simple transformation under $h \to 1 - h$. To derive it we use the reflection symmetry (2.22). Take the expectation value of both sides of (2.22). This gives

$$\frac{U(h)}{(x - \bar{x})^{2h}} = \mathcal{R}(h) \frac{2h - 1}{\pi} U(1 - h) \int d^2x' \frac{|x - x'|^{-4h}}{(x' - \bar{x}')^{2(1 - h)}}$$  (3.4)

The $x'$ integral runs over the complex plane. Denote

$$x = x_1 + ix_2$$
$$x' = x'_1 + ix'_2$$  (3.5)

with $x_2 > 0$, and $x'_2$ running from $-\infty$ and $\infty$. By shifting $x'_1$ we can get rid of the $x_1$ dependence on the r.h.s. of (3.4), which is consistent with the fact that the l.h.s. is independent of $x_1$. We then find

$$\int d^2x' \frac{|x - x'|^{-4h}}{(x' - \bar{x}')^{2(1 - h)}} = \int_{-\infty}^{\infty} dx'_1 \int_0^{\infty} dx'_2 \left[ \frac{(x'_1)^2 + (x'_2 - x_2)^2}{(2e^{i\pi x'_2})^{2(1 - h)}} \right]^{-2h}$$

$$+ \int_{-\infty}^{\infty} dx'_1 \int_{-\infty}^{0} dx'_2 \left[ \frac{(x'_1)^2 + (x'_2 - x_2)^2}{(2e^{i\pi x'_2})^{2(1 - h)}} \right]^{-2h}$$

The $x'_1$ integral is performed by rescaling $x'_1 = z|x'_2 - x_2|$. One gets (apart from a power of $|x'_2 - x_2|$)

$$2A_h \equiv 2 \int_0^{\infty} dz (1 + z^2)^{-2h} = \frac{\Gamma(2h - \frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(2h)}$$  (3.7)

The remaining $x'_2$ integral looks like

$$\int_0^{\infty} dx'_2 \frac{|x'_2 - x_2|^{-4h}}{(2e^{i\pi x'_2})^{2(1 - h)}}$$  (3.8)

for the upper half plane, and

$$\int_{-\infty}^{0} dx'_2 \frac{|x'_2 - x_2|^{-4h}}{(2e^{i\pi x'_2})^{2(1 - h)}}$$  (3.9)

for the lower half plane. The contribution of the upper half plane vanishes. It is equal to

$$\frac{2A_h}{x_2^{2h} (2e^{i\pi x_2})^{2(1 - h)}} \int_0^{\infty} dx'_2 (x'_2)^{2(h - 1)} |1 - x'_2|^{-4h}$$  (3.10)
and by using the standard Euler integral
\[ \int_0^1 dx x^{a-1}(1-x)^{b-1} = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \tag{3.11} \]
one finds that this vanishes. The contribution from the lower half plane is non-zero; it is given by
\[ \frac{2A_h}{2e^{-\frac{i\pi}{2}}}(1-4h)\int_{-\infty}^0 dx'(-x')^2(1-x')^{1-4h}. \tag{3.12} \]

By using (3.11) again, and also
\[ \frac{1}{2}\Gamma(x)\Gamma(x+1), \tag{3.13} \]
one finds that
\[ U(h) = -R(h)U(1-h) = -\frac{\Gamma(1+\frac{2h-1}{t})}{\Gamma(1-\frac{2h-1}{t})}U(1-h). \tag{3.14} \]

Defining\footnote{The $(-)^{\Delta_h}$ is irrelevant for the present discussion since $\Delta_h = \Delta_{1-h}$, but it will be convenient later.}
\[ f(h) = \frac{(-)^{\Delta_h}U(h)}{\Gamma(1+\frac{2h-1}{t})} \tag{3.15} \]
we have
\[ f(h) = -f(1-h). \tag{3.16} \]

To determine $f(h)$ we apply a procedure that was used for Liouville theory in [28]. The $SL(2)$ CFT has degenerate operators of the form (2.9) with (see [30] for more details)
\[ h_{r,s} = \frac{1-r}{2} - \frac{1-s}{2}t \tag{3.17} \]
where $r, s = 1, 2, 3, \cdots$. For irrational $k$, the Fock module corresponding to $h_{r,s}$ contains a single null state at level $r(s-1)$. Consider, for example, the special case $s = 1$. The degenerate representations have $h_{r,1} = (1-r)/2$, and the null state is at level zero. Looking back at (2.9) this is natural: $\Phi_{h_{r,1}}$ is in this case a polynomial of degree $r-1$ in $x$ and $\bar{x}$, and the null state is
\[ \partial_x^r\Phi_{(1-r)/2} = \partial_{\bar{x}}^r\Phi_{(1-r)/2} = 0, \quad r = 1, 2, 3, \cdots. \tag{3.18} \]
The operators $\Phi_{(1-r)/2}$ correspond to finite $r$ dimensional representations of $SL(2)$. They are direct generalizations of the finite dimensional spin $(r - 1)/2$ representations of $SU(2)$ which are studied in the language used here in [29].

Equation (3.18) gives strong constraints on the OPE of the degenerate operators with generic operators $\Phi_h$. For example, as reviewed in [1], the first non-trivial operator $\Phi_{-\frac{1}{2}}$ satisfies

$$\Phi_{-\frac{1}{2}}(x)\Phi_h(y) = C_-(h)\Phi_{h-\frac{1}{2}}(y) + |x - y|^2 C_+(h)\Phi_{h+\frac{1}{2}}(y) + \cdots$$

(3.19)

where the “…” stand for current algebra descendants, and we have suppressed the dependence on worldsheet locations of the operators, which can be easily restored using (2.14). The structure constants $C_\pm$ are given by (in our normalizations; see [1] for a more detailed discussion)

$$C_+(h) = \frac{2}{\pi} R(-\frac{1}{2})$$

$$C_-(h) = \frac{2}{\pi} R(-\frac{1}{2}) \frac{\Gamma(-\frac{2(h-1)}{t})\Gamma(1+\frac{2h-1}{t})}{\Gamma(1+\frac{2h-1}{t})\Gamma(-\frac{2h-1}{t})}$$

(3.20)

In what follows, we would like to use the degenerate operators (3.17) to obtain constraints on the one point function of operators with generic $h$, (3.3).

To achieve that, consider the two point function on the upper half plane:

$$G_{-\frac{1}{2}}(h) = \langle \Phi_{-\frac{1}{2}}(x, \bar{x}; z, \bar{z})\Phi_h(y, \bar{y}; w, \bar{w}) \rangle$$

(3.21)

Using the unbroken worldsheet conformal symmetry (3.1), and $SL(2)$ symmetry (3.2), one can write the two point function as follows:

$$G_{-\frac{1}{2}}(h) = \frac{(w - \bar{w})^{-2\Delta_h}}{(z - \bar{z})^{2\Delta_{-\frac{1}{2}}}} \frac{x - \bar{x}}{(y - \bar{y})^{2h}} F(\eta_{ws}, \eta_{st})$$

(3.22)

where $\eta_{ws}$ and $\eta_{st}$ are the cross ratios on the worldsheet and in spacetime,

$$\eta_{ws} = \frac{|z - w|^2}{(z - \bar{z})(w - \bar{w})}$$

$$\eta_{st} = \frac{|x - y|^2}{(x - \bar{x})(y - \bar{y})}.$$ 

(3.23)

Consider the dependence of $G_{-\frac{1}{2}}$ on $x$. Since the operator $\Phi_{-\frac{1}{2}}$ satisfies the differential equation (3.18) with $r = 2$, the right hand side of (3.22) must be a linear function of $x$ and of $\bar{x}$. This means that the function $F$ is in fact linear in $\eta_{st}$:

$$F(\eta_{ws}, \eta_{st}) = a_0(\eta_{ws}) + \eta_{st}a_1(\eta_{ws})$$

(3.24)
As is well known, the two point function of bulk operators on the upper half plane is closely related to a four point function (of the operators and their mirror images) on the plane. In particular, the function of the cross ratio $F$ ([3.22], [3.24]) is given by a combination of the current algebra blocks on the sphere. Due to (3.19) there are in this case two blocks, and one has

$$F(\eta_{ws}, \eta_{st}) = a_-(F_0^{(-)} + \eta_{st}F_1^{(-)}) + a_+(F_0^{(+)}) + \eta_{st}F_1^{(+)},$$

where $F_0^{(\pm)}$ are given in [1]. We present them here for completeness:

$$F_0^{(-)} = x^{-a}(1-x)^{-a}F(-2a, 2b - 2a; b - 2a; x)$$
$$= x^{-a}(1-x)^{-a}F(b, -b; b - 2a; x)$$
$$F_1^{(-)} = \frac{2a}{b-2a}x^{-a}(1-x)^{-a}F(1 - 2a, 2b - 2a; b - 2a + 1; x)$$
$$= \frac{2a}{b-2a}x^{-a}(1-x)^{-a}F(b, 1 - b; b - 2a + 1; x)$$

$$F_0^{(+) = x^{a-b+1}(1-x)^{a-b}F(2a - 2b + 1, 2a + 1; 2a - b + 2; x)}$$
$$= x^{a-b+1}(1-x)^{-a}F(1 - b, 1 + b; 2a - b + 2; x)$$
$$F_1^{(+) = \frac{b-2a-1}{b}x^{a-b}(1-x)^{-a}F(1 - b, b; 2a - b + 1; x)}$$
$$= \frac{b-2a-1}{b}x^{a-b}(1-x)^{-a}F(2a - 2b + 1, 2a; 2a - b + 1; x),$$

where we used the notation

$$a \equiv \frac{h}{t}; \quad b \equiv \frac{1}{t}; \quad x \equiv \eta_{ws},$$

and $F$ is the hypergeometric function (see appendix A for some of its properties).

To determine the constants $a_{\pm}$ in [3.25], consider the two point function [3.21] in the limit $z \to w$ (i.e. $\eta_{ws} \to 0$). It is not difficult to see that $F^{(-)}$ corresponds to the contribution of the block of $\Phi_{h-\frac{1}{t}}$, while $F^{(+)}$ is the contribution of $\Phi_{h+\frac{1}{t}}$ and its descendants. Taking the limit $\eta_{ws} \to 0$ in (3.26), (3.27) and using the relations

$$\Delta_{h-\frac{1}{t}} - \Delta_{h} - \Delta_{h} = -\frac{h}{t} = -a$$
$$\Delta_{h+\frac{1}{t}} - \Delta_{h} - \Delta_{h} = \frac{h-1}{t} = a - b$$

one furthermore finds that the contribution of the primary $\Phi_{h-\frac{1}{t}}$ comes from the leading term in $F_0^{(-)}$, while that of $\Phi_{h+\frac{1}{t}}$ comes from $F_1^{(+)}$. The leading terms in $F_1^{(-)}$ and $F_0^{(+)}$ are the contributions of $\partial_y \Phi_{h-\frac{1}{t}}(y; w)$ and $J \Phi_{h+\frac{1}{t}}(y; w)$, respectively.
Using the OPE (3.19) and the behavior of $\mathcal{F}^{(\pm)}$ at small $\eta_{\text{ws}}$, we conclude that

$$a_- = C_-(h) U(h - \frac{1}{2})$$

$$a_+ = \frac{1}{1 - 2h - t} C_+(h) U(h + \frac{1}{2}).$$

(3.30)

Plugging this into (3.25) and comparing to (3.24) we see that

$$a_0(\eta_{\text{ws}}) = C_-(h) U(h - \frac{1}{2}) \mathcal{F}_0^{(-)} + \frac{1}{1 - 2h - t} C_+(h) U(h + \frac{1}{2}) \mathcal{F}_0^{(+)},$$

$$a_1(\eta_{\text{ws}}) = C_-(h) U(h - \frac{1}{2}) \mathcal{F}_1^{(-)} + \frac{1}{1 - 2h - t} C_+(h) U(h + \frac{1}{2}) \mathcal{F}_1^{(+)}.$$  

(3.31)

where $\mathcal{F}^{(\pm)}$ are the blocks of (3.26), (3.27). The resulting equation for the two point function $G_{-\frac{1}{2}}(h)$ involves known quantities such as the structure constants (3.20) and the conformal blocks (3.26), (3.27), and the unknown one point function $U(h)$, (3.3).

A non-trivial constraint on $U(h)$ comes from considering the two point function (3.21) in the limit $z \to \bar{z}$, $w \to \bar{w}$. In this limit the bulk operators $\Phi_{-\frac{1}{2}}$ and $\Phi_h$ approach the boundary of the worldsheet and we expect them to create boundary operators $\Psi_h$. It is natural to expect that these operators are restricted to the boundary in spacetime as well. This is an important general feature. In addition to the bulk observables (2.9), in the presence of D-branes one finds operators $\Psi_h(x; z)$ which live on the boundary of the worldsheet $\text{Im} z = 0$, and of spacetime $\text{Im} x = 0$. We will explain this further below. For now we note that the worldsheet boundary scaling dimension of $\Psi_h$ is

$$\Delta_h^{(b)} = \frac{h(h - 1)}{t}$$

(3.32)

while the spacetime boundary scaling dimension is $h$.

Returning to the two point function (3.21), clearly in the limit $z \to \bar{z}$ there should still be only two blocks. To describe them explicitly, consider the operator $\Phi_{-\frac{1}{2}}$. As mentioned above, the mode expansion (2.19) truncates:

$$\Phi_{-\frac{1}{2}}(x, \bar{x}; z, \bar{z}) = \sum_{m, \bar{m} = -\frac{1}{2}, \frac{1}{2}} V_{-\frac{1}{2}, m, \bar{m}}(z, \bar{z}) x^{\frac{1}{2} - m} \bar{x}^{\frac{1}{2} - \bar{m}}.$$  

(3.33)

Near the boundary of the worldsheet, $\Phi_{-\frac{1}{2}}$ (3.33) should be expanded in boundary operators, both on the worldsheet and in spacetime. The spacetime expansion is particularly simple; one can write

$$\Phi_{-\frac{1}{2}}(x, \bar{x}) = (x - \bar{x}) V_1 + [V_2 + (x + \bar{x}) V_3 + x \bar{x} V_4]$$

(3.34)
where $V_1, \ldots V_4$ are combinations of $V_{-\frac{3}{2}, m, \bar{m}}$ evaluated at $z \to \bar{z}$. In the limit $x \to \bar{x}$ one can think of (3.34) as follows:

$$\Phi_{-\frac{1}{2}}(x \simeq \bar{x}) \sim A_0(x - \bar{x}) + A_1 \Psi_{-1}(x; z) \quad (3.35)$$

The first term on the r.h.s. corresponds to the identity operator on the boundary, with $A_0$ the corresponding structure constant and $(x - \bar{x})$ taking care of the spacetime scaling dimension. In the second term, $\Psi_{-1}$ is a quadratic polynomial in $x$, corresponding to the finite dimensional spin one representation of $SL(2)$, and $A_1$ is the relevant structure constant. $\Psi_{-1}$ satisfies the boundary analog of (3.18). It belongs to an infinite set of boundary operators degenerate at level zero, $\Psi_{(1-r)/2}$, which satisfy

$$\partial_r^r \Psi_{(1-r)/2} = 0, \quad r = 1, 2, 3, \ldots \quad (3.36)$$

Note that group theoretically, (3.35) is simply the statement that multiplying two spin $\frac{1}{2}$ $SL(2)$ representations (corresponding to $\Phi_{-\frac{1}{2}}$ and its mirror image) gives representations with spins zero and one, $\Psi_0 = 1$ and $\Psi_{-1}$.

To reiterate, when $\Phi_{-\frac{1}{2}}$ approaches the boundary of the worldsheet in (3.21), it can be expanded in boundary operators (both on the worldsheet and in spacetime). There are two terms in the expansion, corresponding to the degenerate operators 1 and $\Psi_{-1}$. Similarly, when $\Phi_h$ approaches the boundary, it can be expanded in boundary operators; the two point function (3.21) is only sensitive to the contributions of the boundary operators 1 and $\Psi_{-1}$ in this expansion.

We will next obtain a constraint on the one point function $U(h)$ (3.3) by computing the contribution of the identity operator to the two point function (3.21) in two different ways. One is to use the explicit form (3.22) – (3.31) in the limit $z - \bar{z} \to i0^+$, $w - \bar{w} \to i0^+$, i.e. $\eta_{ws} \to -\infty$. The contribution of the identity operator in this limit is given by the constant term in $\mathcal{F}(\eta_{ws}, \eta_{st})$. Using equations (3.22) – (3.31) and the properties of hypergeometric functions reviewed in appendix A, we find

$$\mathcal{F}(\eta_{ws} \to -\infty) = (-1)^{-a} C_-(h) U(h) \frac{1}{2} \frac{\Gamma(b - 2a) \Gamma(2b)}{\Gamma(b) \Gamma(2b - 2a)} + \ldots \quad (3.37)$$

$$+ (-1)^{a-b} C_+(h) U(h) \frac{1}{2} \frac{\Gamma(2a - b + 1) \Gamma(2b)}{\Gamma(b) \Gamma(2a + 1)} + \ldots \quad (3.37)$$
where “...” stand for $\eta$ dependent terms, and $a, b$ are given in (3.28). On the other hand, the contribution of the identity operator to (3.21) factorizes as a product of one point functions,

$$
\langle \Phi_{-\frac{1}{2}}(x, \bar{x}; z, \bar{z}) \Phi_h(y, \bar{y}; w, \bar{w}) \rangle \sim \langle \Phi_{-\frac{1}{2}}(x, \bar{x}; z, \bar{z}) \rangle \langle \Phi_h(y, \bar{y}; w, \bar{w}) \rangle
$$

where in the last equality we used (3.3) twice. Some comments are in order here:

1. The factorization property (3.38) is the $SL(2)$ analog of eq. (2.12) in [28] for the Liouville case. It should be emphasized that it is important for its validity that the spectrum of states that live on the boundary is discrete (and as we will see, it is even finite). We will later encounter branes for which the open string spectrum is continuous, and there the relation (3.38) will not hold.

2. In the derivation of (3.38), the one point function $U(h)$ is defined by (3.3), where the correlator is normalized by dividing by the partition sum. Naively this means that $U(0) = 1/\pi$ (since $\Phi_0 = 1/\pi$; see (2.9)). However, as discussed above and in [1], we are using a normalization in which operators with $h < 1/2$ are renormalized by a factor of $R(h)$ (2.23) compared to their semiclassical form. Thus, we in fact expect

$$
U(0) = \frac{\langle \Phi_0 \rangle}{\langle 1 \rangle} = \frac{1}{\pi} R(0) = \frac{1}{\pi} \frac{\Gamma(1 - \frac{1}{t})}{\Gamma(1 + \frac{1}{t})}.
$$

We now have two different expressions for the contribution of the identity to the two point function (3.21) in the limit $\eta ws \to -\infty$. One is obtained by plugging (3.37) in (3.22); the other is (3.38). Equating the two, using the value of $C_\pm$ found in (3.20), and writing the relation in terms of $f(h)$ (3.15), we find

$$
\pi \Gamma(1 + \frac{1}{t}) f(-\frac{1}{2}) f(h) = f(h - \frac{1}{2}) + f(h + \frac{1}{2}),
$$

(3.40)

where $f(h) = -f(1 - h)$ (3.16).

Equation (3.40) has many solutions. We will next show that when one includes a similar constraint on the one point function coming from the degenerate operator $\Phi_{\frac{1}{2}}$ (corresponding to $r = 1, s = 2$ in (3.17)), $f(h)$ is fixed uniquely.

The degenerate operator $\Phi_{\frac{1}{2}}$ is discussed in [30,4]. As shown in these papers, the combination

$$
\theta(x) = \frac{1}{2} t(t + 1) \partial_x^2 J \Phi_{\frac{1}{2}} + (t + 1) \partial_x J \partial_x \Phi_{\frac{1}{2}} + J \partial_x^2 \Phi_{\frac{1}{2}},
$$

(3.41)
which is manifestly a current algebra descendant of $\Phi^\frac{1}{2}$, is also primary. In a unitary theory, this would imply that $\theta$ should be set to zero. $SL(2)$ CFT is not unitary, but it is believed that one should still set $\theta = 0$. In some sense, this is part of the definition of the theory \[30\].

The vanishing of $\theta$ imposes a constraint on the OPE’s of $\Phi^\frac{1}{2}$ with other observables. As described in \[30\], one has

$$
\Phi^\frac{1}{2}(x)\Phi(y) = C_1(h)\Phi_{h+\frac{1}{2}}(y) + \frac{C_2(h)}{|x-y|^{2t}}\Phi_{h-\frac{1}{2}}(y) + \frac{C_3(h)}{|x-y|^{2(t+2h-1)}}\Phi_{1-h-\frac{1}{2}} \tag{3.42}
$$

where the structure constants are:

$$
C_1(h) = -\frac{t^2(1-t)}{\pi(2h + t - 1)^2} \mathcal{R}(\frac{t}{2})
$$

$$
C_2(h) = \frac{1-t}{\pi} \mathcal{R}(\frac{t}{2})
$$

$$
C_3(h) = \frac{1}{(2h-1)(1-t)} \mathcal{R}(\frac{t}{2}) \Gamma(1 + \frac{2h-1}{t})\Gamma(1 - 2h)\Gamma(t)\Gamma(2h + t - 1) \frac{1}{\Gamma(1 - \frac{2h-1}{t})\Gamma(2h)\Gamma(2h + t - 1)\Gamma(2 - 2h - t)}.
$$

To derive a constraint on the one point function $U(h)$, we proceed in the same way as before. Consider the two point function on the upper half plane,

$$
G^\frac{1}{2}(h) = \langle \Phi^\frac{1}{2}(x, \bar{x}; z, \bar{z})\Phi_h(y, \bar{y}; w, \bar{w}) \rangle. \tag{3.44}
$$

The dependence on the worldsheet and spacetime positions is

$$
G^\frac{1}{2}(h) = \frac{(w - \bar{w})^{-2\Delta_h}}{(z - \bar{z})^{2\Delta^\frac{1}{2}}} \frac{(x - \bar{x})^{-t}}{(y - \bar{y})^{2h}} \mathcal{F}(\eta_{ws}, \eta_{st}) \tag{3.45}
$$

where $\eta_{ws}, \eta_{st}$ are given by \[3.23\]. Since the OPE \[3.42\] contains three $SL(2)$ representations, the function $\mathcal{F}$ is given in this case by

$$
\mathcal{F}(x; z) = a_AF_A + a_BF_B + a_CF_C, \tag{3.46}
$$

where $F_{A,B,C}$ are the current algebra blocks corresponding to $\Phi_{h+\frac{1}{2}}, \Phi_{h-\frac{1}{2}}$ and $\Phi_{1-h-\frac{1}{2}}$, respectively. They are given by (see \[30\])

$$
F_A(x; z) = z^h(1 - z)^hF_1(2h, t, 2h + t - 1; 2h + t; x, z)
$$

$$
F_B(x; z) = x^{-t}z^{1-h}(1 - z)^hF_1(t, t, 1 - t; 2 - 2h; \frac{z}{x}, z)
$$

$$
F_C(x; z) = z^h(1 - z)^h e^{-i\pi(1-t)} \frac{\Gamma^2(2h)}{\Gamma(2h + 1 - t)\Gamma(2h + t - 1)} \times \left[ Z_8 - \frac{\Gamma(2h + 1 - t)\Gamma(1 - 2h - t)}{\Gamma^2(1-t)} e^{2i\pi h} Z_1 \right] \tag{3.47}
$$
where
\[ Z_8 = x^{-2h} F_1(2h, 1 - t, t + 2h - 1; 2h + 1 - t; \frac{1}{x}, \frac{z}{x}) \]
\[ Z_1 = F_1(2h, t, t + 2h - 1; t + 2h; x, z). \]

In the last equations, \( F_1 \) is a hypergeometric function in two variables defined in appendix A. The structure constants \( a_A, a_B, a_C \) are determined as before by studying (3.44) in the limit \( z \to w, x \to y \) i.e. \( \eta_{ws}, \eta_{st} \to 0 \). The small \( x, z \) behavior of \( \mathcal{F} \) leads to
\[ a_A = C_1(h) U(h + \frac{t}{2}) \]
\[ a_B = C_2(h) U(h - \frac{t}{2}) \]
\[ a_C = C_3(h) U(1 - h - \frac{t}{2}). \]

If we now send the cross ratios to \(-\infty\), we find a non-linear equation for \( U(h) \) (3.3), as before.

In order to do that we need the behavior of the functions \( F_A, F_B, F_C \) at \( z \to -\infty, x \to -\infty \) (in this order of limits). Moreover, we are only interested in the constant term which gives the identity block in the t-channel. This term can be computed by using certain relations satisfied by hypergeometric functions in two variables, described in [32].

We outline the calculation in appendix B. We find
\[ F_A(x \to -\infty; z \to -\infty) \simeq (-h)^{2h + t - 1} \]
\[ F_B(x \to -\infty; z \to -\infty) \simeq (-h)^{2h - 1} \]
\[ F_C(x \to -\infty; z \to -\infty) \simeq (-h)^{\text{id}} \]

Using (3.50) in (3.44) and equating the result to the analog of (3.38) for this case, we get the relation:
\[ \pi \Gamma(1 + \frac{1}{t}) f(t) f(h) = f(h + \frac{t}{2}) + f(h - \frac{t}{2}) + f(1 - h - \frac{t}{2}). \]

The reflection symmetry (3.16) allows one to simplify (3.51) considerably:
\[ \pi \Gamma(1 + \frac{1}{t}) f(t) f(h) = f(h - \frac{t}{2}). \]
This relation\textsuperscript{6} together with (3.40) determines the function $f$ uniquely, at least for irrational $t$ (and then by analytic continuation). The solution is

$$f(h) = \rho(t) \frac{\sin \left[ \frac{\pi}{7}(2h - 1)(2h' - 1) \right]}{\sin \left[ \frac{\pi}{7}(2h' - 1) \right]}$$

(3.53)

where

$$\rho(t) = -\frac{1}{\pi \Gamma(1 + \frac{1}{t})}$$

(3.54)

and\textsuperscript{7}

$$0 \neq 2h' - 1 \in \mathbb{Z}.$$  

(3.55)

Note that this solution is normalized such that

$$U(h = 1) = -\frac{1}{\pi},$$

$$U(h = 0) = \frac{1}{\pi} \frac{\Gamma(1 - \frac{1}{t})}{\Gamma(1 + \frac{1}{t})}$$

(3.56)

in agreement with the expectation (3.39).

To summarize, the conclusion of the analysis is that boundary states are labeled by a positive integer $|2h' - 1|$ and have wavefunctions given by (3.53). These boundary states are in one to one correspondence with degenerate operators of the form $\Phi_{-j'}$ (3.18), with $-h' = j' = 0, \frac{1}{2}, 1, \cdots$.

Now that we understand the one point functions $U(h)$, we can compute the amplitude for creating $\Psi_{-1}$ on the boundary (see (3.35)). In analogy to \textsuperscript{28} we expect the following to happen: for $|2h' - 1| > 1$ we should get a non-zero amplitude for this, while for $|2h' - 1| = 1$ the amplitude should vanish. More generally, we expect that for $h' = -j$ the fundamental strings that connect the brane to itself contain only the degenerate operators $\Psi_{-j'}$, with $j'$ running from zero up to $2j$ in jumps of one (and of course their current algebra descendants).

\textsuperscript{6} which, together with (3.16) implies that $f(h + t) = f(h)$.

\textsuperscript{7} The denominator in (3.53) has singularities for rational $t$. These singularities are associated with the vanishing of the disk partition sum for the corresponding boundary states labeled by $h'$. One possible interpretation of this is that for rational $t$ one should restrict to a finite subset of the branes constructed here; \textit{e.g.} for $t \in \mathbb{Z}$ one would only keep branes with $|2h' - 1| < |t|$. It would be interesting to investigate this further.
To see that this general picture is indeed consistent with our results, consider again the two point function (3.21). As we see from (3.35), to study the appearance (or lack thereof) of $\Psi_{-1}$ in the limit $x \to \bar{x}$, we should look for a constant term (in $x - \bar{x}$) in $G_{-\frac{1}{2}}(h)$. This constant term will then measure the product of: (1) the amplitude to create $\Psi_{-1}$ from $\Phi_{-\frac{1}{2}} \to$ boundary; (2) the amplitude to create $\Psi_{-1}$ from $\Phi_h \to$ boundary; (3) the boundary two point function $\langle \Psi_{-1}\Psi_{-1} \rangle$.

As is clear from the form of the two point function (3.22), we are looking for a term that goes like $\eta_{ws}$ in $F$, i.e. we are interested in $a_1$ (see (3.24), (3.31)). Thus, we are interested in the behavior of $F_1(\pm)$ as $\eta_{ws} \to -\infty$. By using the explicit forms (3.26), (3.27), we find

$$F_1^(-) \simeq (-1)^{1 + \frac{h'}{2}} (-\eta_{ws})^{-\frac{1}{2}} \frac{\Gamma(1 - 2h) \Gamma(1 - \frac{2}{t})}{\Gamma(1 - \frac{1}{2}) \Gamma(-\frac{2h}{t})}$$
$$F_1^+ \simeq 2(-1)^{\frac{1 - h'}{2}} (-\eta_{ws})^{-\frac{1}{2}} \frac{\Gamma(2 + 2h - 1) \Gamma(-\frac{2}{t})}{\Gamma(1 - \frac{1}{2}) \Gamma(1 + 2h - 2)}$$

(3.57)

One can check that the power of $\eta_{ws}$ is precisely correct for describing the contribution of $\Psi_{-1}$. Plugging (3.57) into $a_1$ (3.31), we find that this contribution indeed vanishes for $|2h' - 1| = 1$ and is non-zero otherwise.

A similar check can be performed on the two point function $G_{-\frac{1}{2}}$ (3.43). The constraint (3.51) was obtained by analyzing the contribution of the identity operator in the limit $z \to \bar{z}$. We know from (3.42) that the other boundary operators that can in principle contribute to (3.45) in this limit are the degenerate boundary operators $\Psi_t$ and $\Psi_{1-t}$. However, an explicit calculation leads to the conclusion that these contributions vanish (the outline of the proof appears at the end of appendix B).

As we will see later, the above is the beginning of the emergence of a structure somewhat similar to that found for Liouville branes in [28]. The open string spectrum contains a finite number of degenerate operators with half integer spin (of the form (3.36)).

At this point, one can in principle continue developing the bootstrap approach by studying other correlation functions. For example, by analyzing the two point function on the upper half plane of the higher degenerate operators (3.18) with $\Phi_h$, $\langle \Phi_{(1-r)/2}\Phi_h \rangle$, one can compute the amplitude for creating a degenerate operator $\Psi_{-n}$ with $n > 1$ (and $n \in Z$) by taking a generic bulk operator $\Phi_h$ to the boundary. As we explained earlier, we expect to find this amplitude to be non-zero for the boundary state corresponding to a certain $h' \in Z/2$, when $n \leq 2|h'|$. It would be interesting to verify this.

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8 After projecting out the contribution of the identity sector.
Another interesting problem is to analyze the two point function of generic bulk operators on the upper half plane \( \langle \Phi \Phi \rangle \) and show that as the bulk operators approach the boundary, they only create degenerate operators (3.36). We will not proceed further in this direction here; instead we will turn next to the analysis of the annulus amplitude, which adds insight about the theory.

4. Modular bootstrap (I)

The problem that we would like to address in this section is the spectrum of open strings connecting two D-branes corresponding to, say, \(|2h' - 1| = n, m\) (or the spectrum of \((n, m)\) strings, in short). A priori, the most naive thing to expect is that these strings belong to the finite dimensional, degenerate representations of the diagonal \(SL(2)\) with half integer spin. More precisely, one might expect that if we write \(n = 2j_1 + 1\), \(m = 2j_2 + 1\), then the \((n, m)\) strings belong to finite dimensional representations of \(SL(2)\) which appear in the product

\[
 j_1 \otimes j_2 = |j_1 - j_2| \oplus \cdots \oplus j_1 + j_2 . \tag{4.1}
\]

Some of the reasons why this is a natural thing to expect are:
(1) Algebraically, since the boundary states were found to be labeled by an integer \(|2h' - 1|\) corresponding to a degenerate representation of \(SL(2)\) with half-integer spin, it is natural for the open strings to arrange themselves in the representations that one obtains by sending these degenerate operators to the boundary. Also, both the algebraic analysis and the resulting structure are very reminiscent of \(SU(2)\) branes, which makes the spectrum (4.1) seem plausible.
(2) Recall the situation for \(SU(2)\). There is a basic Cardy state, which algebraically corresponds to \(j = 0\) in the sense of (4.1), and geometrically describes a \(D0\)-brane at a point on the three-sphere. The open strings that live on this brane belong to the current algebra block of the identity (in agreement with (4.1)). Now consider the boundary state corresponding to \(n\) \(D0\)-branes on the sphere. Algebraically, it corresponds to a tensor product of the basic Cardy state describing a single \(D0\)-brane and \(n \times n\) Chan-Paton factors; geometrically it describes \(n\) coincident \(D0\)-branes on the sphere. This boundary state is marginally unstable: it contains a marginally relevant

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9 Recall that the finite dimensional spin \(j\) representation corresponds to the degenerate boundary operator \(\Psi_{-j}\) (3.36).
operator $J^a(z)I_a$, ($a = 3, +, -$), where $J^a(z)$ is the $SU(2)$ current which is conserved on the upper half plane (evaluated at the boundary), while $I_a$ is a constant $n \times n$ matrix representing $SU(2)$ in terms of the Chan-Paton degrees of freedom. In the presence of this interaction, the worldsheet CFT flows [33] to an interacting BCFT, described by the $SU(2)$ boundary state corresponding to spin $(n - 1)/2$. Geometrically, the $n$ D0-branes expand into a finite size two-sphere inside $S^3$. The situation is expected to be similar in the $SL(2)$ case. The boundary state with $|2h' - 1| = 1$ corresponds to a D-instanton in $AdS_3$ (we will motivate this further below), and should have only the current block of the identity living on it. The boundary state with $|2h' - 1| = n$ that we constructed above corresponds to $n$ D-instantons, presumably again in the presence of an interaction of the form $J^a(z)I_a$. Thus, $(n, m)$ strings connect an aggregate of $n$ instantons to $m$ instantons. One would expect to find $nm$ $SL(2)$ primaries in this sector. This is precisely what one finds if the operators belong to the representations (4.1).

With the above comments as motivation, we will analyze in this section the annulus partition sum corresponding to the degenerate representations with half-integer spin, and its properties under the modular transformation $\tau \rightarrow -1/\tau$.

We will be computing the character

$$Z(\tau, u) = \text{Tr} q^{L_0 - \frac{c}{24}} e^{2\pi i u J^0_3}, \quad q = e^{2\pi i \tau}. \quad (4.2)$$

We are really interested in the character with $u = 0$ but it might be convenient to compute with finite $u$ and take $u$ to be small at the end of the calculation.

For a degenerate representation with spin $j = \frac{r - 1}{2}$, $m$ takes values in the finite range $m = -j, -j + 1, \ldots, j - 1, j$. Performing the trace (4.2) one finds

$$Z_r(\tau, u) = \frac{\sin \pi ru}{\sin \pi u} \frac{q^{\pi^2}}{q^\frac{1}{8}} \prod_{n=1}^{\infty} \frac{1}{(1 - e^{2\pi i u q^n})(1 - q^n)(1 - e^{-2\pi i u q^n})}. \quad (4.3)$$

This can be concisely written in terms of a theta function,

$$\theta_1(u, \tau) = 2q^{1/8} \sin \pi u \prod_{n=1}^{\infty} (1 - e^{2\pi i u q^n})(1 - q^n)(1 - e^{-2\pi i u q^n}) \quad (4.4)$$

as

$$Z_r(\tau, u) = \frac{2q^{\pi^2} \sin \pi ru}{\theta_1(u, \tau)} \quad (4.5)$$
Now perform a modular transformation,

\[(\tau', u') = \left( -\frac{1}{\tau}, \frac{u}{\tau} \right). \tag{4.6} \]

The transformation property of the theta function is given e.g. in \[34\]

\[\theta_1\left( \frac{u}{\tau}, -\frac{1}{\tau} \right) = -i(-i\tau)^{1/2}\exp(\pi i u^2/\tau)\theta_1(u, \tau). \tag{4.7} \]

Thus, we have

\[Z_r(\tau', u') = \frac{2i(q')r^2/4t \sin \pi ru'}{(-i\tau)^{1/2}\exp(i\pi u^2/\tau)\theta_1(u, \tau)} . \tag{4.8} \]

Using the fact that

\[2i(q')r^2/4t e^{-i\pi u^2/\tau} \sin \pi ru' = \exp \left[ -\frac{i\pi}{\tau} \left( \frac{r^2}{2t} + u^2 - ru \right) \right] - \exp \left[ -\frac{i\pi}{\tau} \left( \frac{r^2}{2t} + u^2 + ru \right) \right] \tag{4.9} \]

and

\[\sqrt{\frac{i}{\tau}} e^{-\pi i(p')^2} = \int_{-\infty}^{\infty} dpe^{\pi i\tau p^2} e^{2\pi ipp'} \tag{4.10} \]

we conclude that:

\[Z_r(\tau', u') = \frac{1}{\theta_1(u, \tau)} \int_{-\infty}^{\infty} dpe^{\pi i\tau p^2} \left( e^{2\pi i\sqrt{\frac{r^2}{2t}+u^2-ru}} - e^{2\pi i\sqrt{\frac{r^2}{2t}+u^2+ru}} \right). \tag{4.11} \]

We can now send \(u \to 0\) and look at the resulting expressions.

The original degenerate character (4.3) becomes

\[Z_r(\tau, u \to 0) = \frac{rq^{r^2/4t}}{\eta^3(\tau)}. \tag{4.12} \]

The modular transformed character (4.11) goes to

\[Z_r(-\frac{1}{\tau}, u' \to 0) = \frac{1}{\eta^3(\tau)} \int_{-\infty}^{\infty} dpe^{\pi i\tau p^2} e^{2\pi i\frac{\sqrt{r^2}+u^2+ru}{\sqrt{2t}}} \tag{4.13} \]

We note in passing that (4.13) can be obtained directly from (4.12) by using

\[\eta(-\frac{1}{\tau}) = (-i\tau)^{1/2}\eta(\tau) \tag{4.14} \]

and (4.10):

\[p'(-i\tau)^{-3/2}e^{-\pi i(p')^2} = \frac{1}{2\pi i} \frac{d}{dp'} \sqrt{\frac{i}{\tau}} e^{-\pi i(p')^2} = \int_{-\infty}^{\infty} dp e^{\pi i\tau p^2} e^{2\pi ipp'}. \tag{4.15} \]
The modular transformation (4.6) takes us from the open string channel to the closed string channel. The original partition sum (4.12) can be thought of, for example, as the trace over open string states stretched between a brane with \(|2h' - 1| = r\) and the brane with \(|2h - 1| = 1\) (i.e. the partition sum of \((r, 1)\) strings). The transformed partition sum is a trace over closed strings that can be exchanged by the branes.

By comparing the closed string channel expression (4.13) to the contribution of closed strings with spin \(j\), one concludes that the states exchanged by the branes belong to the principal continuous series

\[ j = -\frac{1}{2} + i\lambda. \] (4.16)

The momentum in the radial direction, \(\lambda\), is related to \(p\) in (4.13) via the rescaling \(\frac{1}{2}p^2 = -\lambda^2/t\) or

\[ p = \lambda\sqrt{-\frac{2}{t}}. \] (4.17)

Using this rescaling, and remembering that \(\lambda\) and \(-\lambda\) (4.16) should be identified (this follows from (2.22)), one can rewrite the partition sum (4.13) as

\[ Z_r(-1/\tau) = -2\sqrt{-2/t} \int_0^\infty d\lambda \chi_\lambda(q) \sinh(\frac{2}{t}\pi\lambda r) \] (4.18)

where

\[ \chi_\lambda(q) = \frac{2\lambda q^{-\lambda^2/t}}{\eta^3(\tau)}. \] (4.19)

A potentially puzzling aspect of (4.19) is the apparent absence of a contribution from the sum over eigenvalues of \(J^3\), which naively gives an infinite multiplicative factor. We will not discuss this issue in detail here, but would like to make a few remarks about it.

The origin of the problem is the fact that we are considering here closed string exchange between coincident branes. In flat space, this is usually dealt with by separating the branes in a transverse direction, and computing the partition sum as a function of the separation. This could also be done here. For example, consider the annulus amplitude corresponding to \((r, 1)\) strings, with one of the branes located at \(\text{Im}x = 0\) and the other at \(\text{Im}x = x_0\). This can be implemented by imposing the symmetric boundary conditions (3.2) on one of the boundaries, and boundary conditions twisted by an automorphism (see section 7) on the other boundary. This would regularize the divergence in question.

When the separation between the branes goes to zero, one expects, as in flat space, to find that the partition sum in the closed string channel gets additional factors of \(\tau\), which are the remnant of the divergence. This is indeed what happens here, but as explained in
Eqs. (4.13) – (4.15), we chose to trade the powers of \( \tau \) for a power of \( p \) (or \( \lambda \)). Thus the factor of \( \lambda \) in (4.19) is the remnant of the sum over \( m \).

Returning to (4.18), general properties of BCFT imply that we should be able to write the partition sum of \((r,1)\) strings as

\[
Z_{r,1}(q') = \int d\lambda \chi_\lambda(q) \Psi_r(\lambda) \Psi_1(-\lambda)
\]

(4.20)

where the wavefunction \( \Psi_r(\lambda) \) is proportional to the one point function of the bulk primary with \( h = \frac{1}{2} + i\lambda \) in the state labeled by \( r \). The wavefunction in (4.18) is proportional to \( \sinh(2\pi \lambda r/t) \). Comparing to the one point function (3.53) taking into account the map between the variables \( h, h' \) in (3.53) and \( \lambda, r \) here,

\[
h = \frac{1}{2} + i\lambda; \quad r = |2h' - 1|
\]

we see that the \( h \) dependence is correct. Comparing (4.18) and (4.20) we furthermore find that

\[
\Psi_r(\lambda) \Psi_1(-\lambda) = -2 \left( -\frac{2}{t} \right)^{\frac{1}{2}} \sinh(\frac{2}{t} \pi \lambda r) .
\]

(4.21)

In particular,

\[
\Psi_1(\lambda) \Psi_1(-\lambda) = -2 \left( -\frac{2}{t} \right)^{\frac{1}{2}} \sinh(\frac{2}{t} \pi \lambda) .
\]

(4.22)

A solution to this is

\[
\Psi_1(\lambda) = \left( -\frac{t}{2} \right)^{\frac{1}{4}} \sqrt{2\pi/\lambda} \Gamma(-\frac{t}{2} i\lambda) .
\]

(4.23)

In general, we should be able to write the partition sum of \((r, r')\) strings as

\[
Z_{r,r'}(q') = \int d\lambda \chi_\lambda(q) \Psi_r(\lambda) \Psi_{r'}(-\lambda) .
\]

(4.24)

Plugging (4.21), (4.23) into (4.24) and using the identity

\[
\sum_{n=0}^{\min(r,r')-1} \sinh(\frac{2\pi \lambda}{t})(r + r' - 2n - 1) = \sinh(\frac{2\pi \lambda}{t}r) \sinh(\frac{2\pi \lambda}{t}r'), \quad r, r' \in Z
\]

(4.25)

we find that

\[
Z_{r,r'}(q) = \sum_{n=0}^{\min(r,r')-1} Z_{r+r'-2n-1}(q), \quad r = 2j + 1, \quad r' = 2j' + 1, \quad j, j' = 0, \frac{1}{2}, 1, \ldots
\]

(4.26)
This is precisely the spectrum proposed in the beginning of this section in (4.1).

To summarize, starting with the assumption that \((r, 1)\) strings transform in the degenerate spin \((r - 1)/2\) representation, we showed that the spectrum of \((r, r')\) strings is given by (4.1) and that the wavefunction of the boundary state labeled by \(r\) is proportional to the one point function \(f(h)\) (3.53), in agreement with expectations. This provides strong evidence for the validity of the overall picture.

5. \(SL(2)\) conformal field theory on the upper half plane (II)

In the previous sections we have described D-branes that correspond to instantons in \(AdS_3\). One certainly expects to find branes that correspond to real, physical objects in \(AdS_3\) as well. In this section we will outline the algebraic structure that underlies a class of branes that correspond (as we will see later) to Euclidean and Lorentzian two dimensional worldvolumes embedded in \(AdS_3\).

Algebraically, the branes that we will construct correspond to the principal discrete series \(\frac{1}{2} < h \in \mathbb{R}\), and the continuous series \(h = \frac{1}{2} + i\lambda, 0 < \lambda \in \mathbb{R}\) (recall that the instantonic branes of the previous sections correspond to finite dimensional degenerate representations of \(SL(2), (3.18)\)). Thus, they have a much richer spectrum of boundary \(SL(2)\) primaries \(\Psi_h(x; z)\) living on them than the branes described in sections 3,4. This is natural from the geometric point of view, since these branes are extended in two directions.

A fact that will play a role in the discussion is that one can turn on a constant electric field\(^\text{10}\) on these D-branes without breaking worldsheet conformal invariance (or going off-shell in spacetime). In flat spacetime, this is described by the Born-Infeld action

\[
\mathcal{L}_{BI} = \frac{1}{g_s} \sqrt{1 - E^2} \tag{5.1}
\]

and one expects to find a similar structure for branes in \(AdS_3\) [25].

In order to turn on an electric field on the worldvolume, one needs to add to the worldsheet Lagrangian the vertex operator of a zero momentum photon. The relevant vertex operator is

\[
I_b = \int dz J(x; z)\Psi_1(x; z). \tag{5.2}
\]

\(^{10}\) We use the terminology suitable for \(1 + 1\) Minkowski worldvolume, although some of the branes we will construct are Euclidean.
The integral over $z$ runs over the real line, the boundary of the worldsheet. The real variable $x$ parametrizes the boundary of the spacetime upper half plane. $\Psi_1$ is a primary of $SL(2)$ which lives on the boundary; it has worldsheet dimension zero \((3.32)\), and spacetime dimension one. $J(x;z)$ is the diagonal (conserved) $SL(2)$ current, restricted to the boundary.

Many of the properties of $I_b$ can be understood by noting its similarity to the vertex operator of the zero momentum dilaton,

$$ I = -\frac{1}{k^2} \int d^2z J(x;z)J(x;\bar{z})\Phi_1(x,\bar{x};z,\bar{z}), \quad (5.3) $$

introduced and discussed in \([3]\) (see also \([1]\)). $I_b$ is a marginal operator on the worldsheet; adding it to the action,

$$ S = S_0 - E \int dz J(x;z)\Psi_1(x;z), \quad (5.4) $$

corresponds to turning on an electric field on the brane (which, as mentioned above, is a modulus). While naively $I_b$ depends on $x$, one can show, along the lines of \([3]\), that in fact

$$ \frac{\partial I_b}{\partial x} = 0, \quad (5.5) $$

in agreement with the fact that its spacetime scaling dimension is zero \([3]\), and that it describes a constant electric field.

As discussed in \([1]\), $I$ is a dimension zero operator which is not proportional to the identity operator, despite the fact that it satisfies $\partial_x I = \partial_{\bar{x}} I = 0$ (i.e. it is constant in correlation functions, but not necessarily the same constant in different correlation functions). Similarly, $I_b$ is a dimension zero boundary operator in the spacetime CFT, which is not proportional to the identity (as we will see). Just like $I$ keeps track of sectors with different central charges \([1]\), $I_b$ keeps track of sectors with different boundary conditions on the upper half plane $\text{Im} x \geq 0$.

Since the operator $I$ corresponds to the zero momentum dilaton, adding it to the worldsheet action changes the dilaton expectation value and thus the closed string coupling \([3,1]\). Similarly, adding $I_b$ to the worldsheet action \((5.4)\) changes the electric field on the brane and thus the open string coupling. As discussed in \([2,3]\), the expectation value of $I$ is arbitrary classically but is quantized in the full quantum theory. Similarly, the expectation value of $I_b$ is classically arbitrary, but quantum mechanically it is quantized\([3]\).

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11 This quantization is associated with the fact that an electric field on a D-string is equivalent to fundamental string charge, and simply corresponds to bound states of one D-string and $n$ fundamental strings. The fact that $n$ is integer quantizes the electric field.
(this is standard for the gauge field on D-strings in flat space, and is explained in the context of \(AdS_3\) branes in [23]).

For some calculations it is useful to note that \(I_b\) is closely related to the "boundary Wakimoto screening operator." In the bulk case, this is shown in [1]. Similar arguments lead to the conclusion that in "Wakimoto variables" (\(\phi, \beta, \gamma\)), adding \(I_b\) to the action as in (5.4) is equivalent to perturbing by

\[
S = S_0 - E \int dz \beta e^{-\frac{1}{2}Q\phi(z)}. \tag{5.6}
\]

We would like next to repeat the analysis of section 3 for the extended branes, and derive the analog of eq. (3.40) for the one point function (3.3). Thus, consider the two point function \(G_{-\frac{1}{2}}(h)\) (3.21). We can again calculate the contribution of the identity in the limit \(z \to \bar{z}\) in two different ways, thereby obtaining a constraint on \(U(h)\). The discussion leading to eq. (3.37) is unchanged, since it does not depend on any detailed properties of the boundary state in which the calculation is done. However, the factorization property (3.38) is no longer valid here [26,28]. The reason is that there is now a continuum of boundary operators that can be created as \(\Phi_{-\frac{1}{2}} \to \text{boundary}\), and the "contribution of the identity" in (3.21) as \(z \to \bar{z}\) is a coherent effect of this continuum.

Nevertheless, we can proceed in a different way since, in analogy to [26,1], the physics associated with \(\Phi_{-\frac{1}{2}} \to \text{boundary}\) is expected to be perturbative in \(E\) (5.6) for branes which reach the boundary of \(AdS_3\). We can expand the action to first order in \(E\) and use free field OPE’s to deduce the structure constant in (3.35),

\[
\lim_{w \to \bar{w}} \Phi_{-\frac{1}{2}}(x, \bar{x}; w, \bar{w}) = A_0 \frac{x - \bar{x}}{(w - \bar{w})^{2\Delta - \frac{1}{2}}} + \cdots. \tag{5.7}
\]

Using (3.33) we find that

\[
\frac{A_0}{(w - \bar{w})^{2\Delta - \frac{1}{2}}} = E \int_{-\infty}^{\infty} dz \beta e^{-\frac{1}{2}Q\phi(z)} V_{-\frac{1}{2}, -\frac{1}{2}}(w) . \tag{5.8}
\]

As explained in [1], eq. (4.13), we have

\[
V_{-\frac{1}{2}, -\frac{1}{2}}(w) = -\frac{2}{\pi} R(-\frac{1}{2}) \gamma(w) e^{\frac{1}{2}Q\phi(z)}. \tag{5.9}
\]

Plugging (5.9) in (5.8) and performing the integral using free field contractions, we find

\[
A_0 = \tilde{C}(t) E, \tag{5.10}
\]

\[
27
\]
where \( \tilde{C}(t) \) is a calculable constant (function of \( t \)) that we will not write explicitly.

Returning to (3.38), we thus have in this case

\[
\langle \Phi_{-\frac{1}{2}}(x, \bar{x}; z, \bar{z})\Phi_{h}(y, \bar{y}; w, \bar{w}) \rangle \simeq \frac{(w - \bar{w})^{-2\Delta_h}}{(z - \bar{z})^{2\Delta - \frac{1}{2}}} \frac{x - \bar{x}}{(y - \bar{y})^{2h}} \tilde{C}(t) EU(h). \tag{5.11}
\]

We can now compare (5.11) to (3.37) (as in section 3) and conclude that the rescaled one point function \( f(h) \) (3.15) satisfies

\[
C(t)Ef(h) = f(h - \frac{1}{2}) + f(h + \frac{1}{2}) \tag{5.12}
\]

where \( C(t) \) is another calculable constant, whose precise form will not be needed here. In addition, \( f(h) \) still satisfies the reflection property (3.16). The solution of this equation that is relevant here has the form (compare to (3.53))

\[
f(h) = A(t) \sin\left[\frac{\pi}{t}(2h - 1)(2h' - 1)\right]. \tag{5.13}
\]

A few remarks are in order:

1. Unlike (3.53), in (5.13) we are computing \( \langle \Phi_{h} \rangle \) without dividing by the partition sum \( \langle 1 \rangle \) on the upper half plane.

2. In the next section we will verify (5.13) by an analysis of the annulus amplitude, but for now we will assume the form (5.13) and study some of its properties.

3. Note that the equation (5.12) does not determine the overall normalization of \( f(h) \). This ambiguity is absorbed in the factor \( A(t) \) in (5.13).

Like in section 3, one can think of \( h' \) as labeling the representation to which the boundary state in question corresponds. Equation (5.12) relates \( h' \) to the electric field by:

\[
\cos\frac{\pi}{t}(2h' - 1) = \frac{1}{2}EC(t). \tag{5.14}
\]

This is an interesting relation: for \( h' \) belonging to the principal discrete series \( (h' \in \mathbb{R}) \), the electric field is bounded from above,

\[
|E| \leq \frac{2}{C(t)}. \tag{5.15}
\]

For the continuous series, \( h' = \frac{1}{2} + i\lambda \), the electric field is instead bounded from below,

\[
|E| \geq \frac{2}{C(t)}. \tag{5.16}
\]
This is reasonable, since the principal discrete series is associated with massive particles in $AdS_3$, while the continuous series describes tachyons, and it is well known that above a certain critical field, D-branes become tachyonic (as is obvious e.g. from (5.1)). This interpretation suggests that the bound in (5.15), (5.16) is the critical electric field of open string theory.

To see that this is indeed the case, consider the partition sum on the disk with the boundary conditions corresponding to a certain boundary state labeled by $h'$. It is expected to depend on the electric field via the Born-Infeld form (5.1). Our formalism allows one to compute the partition sum as follows. First note that if we know the normalization factor $A(t)$ in (5.13), we can compute the partition sum by setting $h = 0$ in (5.13). Let us assume for now (we will return to this assumption shortly) that $A(t)$ does not depend on the boundary state (i.e. on $h'$), as implied by the notation. Then, ignoring a $t$ dependent overall factor, we have

$$Z_{\text{disk}} \simeq \sin \frac{\pi}{t} (2h' - 1). \tag{5.17}$$

Using (5.14) to write this in terms of $E$, we find

$$Z_{\text{disk}} \simeq \sqrt{1 - \left(\frac{1}{2}EC(t)\right)^2}. \tag{5.18}$$

After rescaling $E$, this is precisely of the Born-Infeld form (5.1), and the critical field is indeed the one we deduced from (5.15), (5.16) before.

Thus, the branes corresponding to $h'$ in the principal continuous series are tachyonic, while those in the principal discrete series are massive, in agreement with what one would expect.

We see that it is very natural to expect that $A(t)$ in (5.13) is independent of $h'$, but is this really the case? We believe the answer is yes, but will leave a detailed verification of this to future work. We would like to make some comments on this matter:

(1) $A(t)$ can be computed by using an idea from [26]. The basic point is the following: we know the $h$ dependence of $f(h)$ – it is given by (5.13). At the same time, there is an infinite number of values of $h$ for which (5.13) can be computed using free field methods, by perturbing in the Wakimoto coupling (the coefficient of $I$ (5.3) in the action – see [1]) and in $E$. This calculation can be used to determine $A(t)$.

(2) It is natural to expect that the only dependence on the electric field $E$ in (5.13) is via its relation to $h'$, (5.14); this was found to be the case in a similar calculation in boundary Liouville theory in [26].
Equation (5.13) was found above to be valid for the principal discrete and continuous series. It is interesting to note that it is valid for the finite dimensional representations discussed in section 3 as well. To show this, one computes the partition sum on the upper half plane, by setting \( h = 0 \) in (5.13) recalling the finite renormalizations (3.15), (3.54), (3.56). Dividing \( f(h) \) (5.13) by the resulting partition sum, one finds precisely the form (3.53).

The last comment also suggests that it might be possible to interpolate between the two dimensional branes of this section and the pointlike branes discussed in sections 3,4, in analogy to the situation in flat space where one can turn D2-branes into (collections of) D0-branes by turning on a large magnetic field on the worldvolume of the D2-brane.

6. Modular bootstrap (II)

As in the analysis of the pointlike branes before, we can supplement the bootstrap analysis with information from the annulus. Consider the annulus partition sum for open strings stretched between a brane labeled by \( h' \) (from the principal discrete or continuous series) and the D-instanton brane corresponding to \( h' = 0 \), or \( r = 1 \) in sections 3,4. It is natural to expect that these open strings belong to the \( SL(2) \) representation labeled by \( h' \). Thus, the partition sum is given by

\[
Z_{h',1} = \chi_{h'}(q),
\]

(6.1)

where \( \chi_{h'} \) is the open string character corresponding to the representation \( h' \),

\[
\chi_{h'} = \frac{2\lambda'q^{\frac{1}{2}(2h'-1)^2}}{\eta^{3}(\tau)},
\]

(6.2)

where, as before, \( \lambda' \) is related to \( h' \) via

\[
h' = \frac{1}{2} + i\lambda'.
\]

(6.3)

\( \lambda' \) is real for the continuous representations and imaginary for the discrete ones. We can now perform the modular transform \( \tau' = -1/\tau \), using (4.10), (4.11). This gives

\[
Z_{h',1}(q') = -2\sqrt{-2} \int_{0}^{\infty} d\lambda \chi_{\lambda}(q) \sin \frac{4\pi \lambda \lambda'}{t}.
\]

(6.4)
The same logic that led to (4.24) leads to the conclusion that the wave function $\Psi_{h'}(\lambda)$ corresponding to the boundary state labeled by $\lambda'$ is given by

$$\Psi_{h'}(\lambda)\Psi_1(-\lambda) = -2\sqrt{-\frac{2}{t}} \sin \frac{4\pi \lambda \lambda'}{t}$$  \hspace{1cm} (6.5)

Using $2i\lambda = 2h - 1$, we conclude that the one point function $f(h)$ in the boundary state labeled by $h'$ is proportional to

$$f(h) \propto \sin \frac{\pi}{t}(2h - 1)(2h' - 1)$$  \hspace{1cm} (6.6)

in agreement with the discussion of the previous section (compare (6.6) to (5.13)).

It is also possible to find the spectrum of open strings stretched between a brane labeled by $h'$ and an excited (multi-instanton) brane, corresponding to the degenerate representation with $|2h' - 1| = r$. By using the expression

$$Z_{h',r}(q') = \int d\lambda \chi_{h'}(\lambda)\Psi_{h'}(\lambda)\Psi_r(-\lambda)$$  \hspace{1cm} (6.7)

and the wavefunctions (4.21), (6.5), we find

$$Z_{h',r}(q) = \sum_{m=0}^{r-1} \chi_{h'-\frac{r-1}{2}+m}(q)$$  \hspace{1cm} (6.8)

from which we can read off the open string representations that appear in the sector $(h', r)$.

Finally, one can also study the spectrum of open strings that stretch between two branes labeled by $h'_1$ and $h'_2$. This spectrum is expected to be similar to the familiar one from closed string theory on $AdS_3$, with states belonging to the principal discrete and continuous series, and long string sectors corresponding to twisted representations, as in [5,7]. A detailed study of this is left for future work.

7. Asymmetric gluing

Up to this point, our discussion involved branes that are obtained by imposing symmetric gluing conditions on the currents, (3.2). As is well known, one can also study branes which preserve a different diagonal $SL(2)$, related to the original one by the chiral
application of an automorphism of the algebra (say, just on the left movers). Examples of automorphisms of the $SL(2)$ algebra \((2.7)\) are
\[
J^\pm \rightarrow -J^\mp; \quad J^3 \rightarrow -J^3 \\
J^\pm \rightarrow -J^\pm; \quad J^3 \rightarrow J^3 \\
J^\pm \rightarrow J^\mp; \quad J^3 \rightarrow -J^3. \tag{7.1}
\]
Twisting with the first line of \((7.1)\), one finds branes that correspond to the following boundary conditions on the $SL(2)$ currents at $z = \bar{z}$:
\[
J^\pm(z) = -J^\mp(\bar{z}); \quad J^3(z) = -J^3(\bar{z}). \tag{7.2}
\]
Thus, the $x$ dependence of the one point function \((3.3)\) changes. For example, in the particular case \((7.2)\), imposing the fact that the one point function should satisfy $J^3 - J^3 = 0$ and $J^\pm - J^\mp = 0$ (using \((2.13)\)), one finds that
\[
\langle \Phi_h(x, \bar{x}; z, \bar{z}) \rangle = \frac{U(h)}{(1 - x\bar{x})^{2h}(z - \bar{z})^{2\Delta_h}}. \tag{7.3}
\]
Replacing \((3.3)\) by \((7.3)\) one can now proceed with the bootstrap discussion of the previous sections, and find the same set of branes as before. The only difference is that this time the branes correspond to a boundary at $x\bar{x} = 1$, the boundary of the unit disk (in Euclidean space), which is conformally related to the upper half plane that appeared in the construction of the earlier sections. Similarly, the second line of \((7.1)\) gives rise to a boundary at $x + \bar{x} = 0$ in the spacetime CFT, while the third line corresponds to a boundary at $x\bar{x} + 1 = 0$. The latter has no solutions in Euclidean spacetime, but makes sense in Minkowski space.

To summarize, as one would expect, twisted boundary conditions on the currents, like \((7.2)\), give rise to the same set of branes that was discussed above. The branes give rise to a boundary in the spacetime CFT, and for different diagonal $SL(2)$ algebras one finds boundaries oriented in a different way in $x$ space. Of course, in Minkowski space, some of the constructions give rise to timelike boundaries while others produce spacelike ones, but from the point of view of the algebraic analysis performed here, this does not influence most of the results.

\footnote{For instance, branes corresponding to a boundary at $\bar{x} - x = c$, where $c$ is a constant, are obtained by imposing the gluing conditions $J^a = J^a$ with $J^a$ acting as the differential operators on the r.h.s. of \((2.13)\), with $x \rightarrow x + c$.}
8. Discussion

In this paper we have performed an algebraic analysis of D-branes in $AdS_3$. We solved the bootstrap equations for the one point function of bulk operators (3.3), which are the wavefunctions of the boundary states. We also studied the annulus partition sum, which provided additional information on the spectrum of open strings stretched between different branes. We found three classes of branes, labeled by representations of $SL(2, R)$:

1. Finite dimensional degenerate representations with

   \[ h' = -\frac{n}{2}, \quad n = 0, 1, 2, 3, \ldots \]  

   (8.1)

2. Principal continuous series,

   \[ h' = \frac{1}{2} + i \lambda, \quad \lambda \in \mathbb{R}. \]  

   (8.2)

3. Principal discrete series,

   \[ \frac{1}{2} < h' \in \mathbb{R}. \]  

   (8.3)

It is natural to ask what do the different branes correspond to geometrically. We argued that the first class of branes (8.1) are zero dimensional, while the last two (8.2), (8.3) are two dimensional. We showed that the continuous series branes (8.2) are tachyonic, since they have a supercritical electric field on the worldvolume, while the discrete series branes (8.3) have a subcritical electric field and positive mass squared.

The authors of [25] analyzed the semiclassical geometry of branes in $AdS_3$ by studying the corresponding (twined) conjugacy classes. The regular conjugacy classes are characterized by $Tr g$, where $g$ is the $SL(2)$ matrix (2.6):

\[ g = \frac{1}{l} \begin{pmatrix} x_0 - x_1 & x_2 - x_3 \\ x_2 + x_3 & x_0 + x_1 \end{pmatrix}, \]  

(8.4)

The conjugacy classes are:

1. $g = 1$, which is the point $x_1 = x_2 = x_3 = 0, x_0 = l$ on $AdS_3$.
2. Two dimensional de Sitter space ($dS_2$), corresponding to

   \[ Tr g = 2C, \]  

   (8.5)

   with $|C| > 1$.
3. Two dimensional hyperbolic plane ($H_2$), corresponding to (8.3) with $|C| < 1$. 

(4) Light-cone, corresponding to $|C| = 1$ in (8.5).

It was also shown in [25] that $dS_2$ branes have a supercritical electric field on their world-volume, and are thus tachyonic, while the Euclidean $H_2$ branes have a subcritical electric field. The $H_2$ branes also have a Minkowski counterpart ($AdS_2$ branes), obtained by considering twined conjugacy classes, i.e. twisting the gluing conditions as in section 7.

There is a natural map between the branes constructed here and those of [25]. The basic brane, corresponding to $h' = 0$ (8.1), is associated with the conjugacy class $g = 1$. The multi-instanton branes (8.1) with $n > 1$ correspond to $n$ of the $g = 1$ branes, in some sort of extended bound states. It is not quite clear to us what is the semiclassical description of these states.

The $dS_2$ branes of [25] correspond to the continuous series (8.2), while the $H_2$ and $AdS_2$ branes correspond to the principal discrete series (8.3). As explained in section 7, $H_2$ and $AdS_2$ branes give rise to equivalent boundary states, with the boundary oriented in $x$ space in a different way in the two cases.

The role of the light-cone conjugacy class, $|C| = 1$ in (8.5), is less clear. It is possible that the $n > 1$ branes (8.1) expand into the light-cone, but this requires a better understanding.

We finish with a few remarks about the above identification. Some of the comments below require much more work:

(1) Naively, it seems that there is a disagreement between the geometry of the branes in [25] and here. The conjugacy classes (8.5) correspond near the boundary of $AdS_3$ ($u \to \infty$ in Poincare coordinates (2.5), (2.6)) to

$$\gamma \bar{\gamma} + 1 = 0. \quad (8.6)$$

On the other hand, we saw in section 3 that symmetric gluing (3.3) leads to branes localized at $x = \bar{x}$. Since near the boundary of $AdS_3$, $\gamma \simeq x$, here one finds a boundary at

$$\gamma = \bar{\gamma}. \quad (8.7)$$

The discrepancy between (8.6) and (8.7) seems to be due to a different definition of “diagonal gluing” in the two cases. As we saw in section 7, one can get a boundary

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13 This is similar to the situation with excited branes in [28], which were constructed algebraically, but are not understood semiclassically.
at \( x\bar{x} + 1 = 0 \) by preserving a different \( SL(2) \) then (3.2). Presumably, this is what corresponds to the untwisted construction in [25].

(2) We argued above that string theory on \( AdS_3 \) in the presence of D-branes corresponds to studying the spacetime CFT on the upper half plane \( \text{Im} x \geq 0 \). This might seem puzzling, since the branes look more like defects stretched along the real line (8.7) and it is not clear why one should not think of the spacetime CFT as still living on the whole complex plane, in the presence of a line defect corresponding to the brane. Since the worldsheet CFT on \( AdS_3 \) is left-right symmetric, the structure that we find in the presence of the branes appears to be consistent with focusing on the theory on just the upper (or just the lower) half plane. For left-right asymmetric worldsheet theories one in general expects problems with defining D-branes.

(3) Another interesting question concerns the boundary conditions imposed on the spacetime CFT by the introduction of the branes. The D-instanton \( h' = 0 \) (8.1) seems to correspond to the basic Cardy state in the spacetime CFT. The fact that from the worldsheet point of view only the identity and its \( \hat{SL}(2) \) descendants live on the brane implies, using the results of [2,3], that in spacetime the (boundary) excitations of this brane include only the identity and its Virasoro descendants. It is less clear what boundary boundary CFT corresponds to the spectrum we found on the higher \( (|h'| > 0) \) multi-instanton branes, as well as the extended \( dS_2, AdS_2 \) and \( H_2 \) branes. It is an interesting remaining problem to describe the corresponding BCFT directly in spacetime. It seems that one must add boundary interactions of some sort; their exact nature is left for future work.

(4) One might be puzzled by the fact that the localized branes influence the physics near the boundary at all. For example, the D-instanton, which is associated with the conjugacy class \( g = 1 \), is classically located at \( r = 0 \) (or \( u = 1 \)) in the coordinates (2.3), (2.5), deep in the bulk of \( AdS_3 \). How can it introduce a boundary in \( x \) space, which naively parametrizes the large \( r \) part of space? The answer seems to be the following. In general in anti-de-Sitter space, observables in the spacetime CFT (local operators) correspond to wavefunctions that are supported near the boundary of \( AdS \) space. States in the spacetime CFT correspond to normalizables wavefunctions that are typically supported at \( u \simeq 1 \). Interactions take place in the bulk of the space, despite the fact that the observables are defined at large \( r \). The D-instanton, and more generally the zero dimensional branes (8.1), do not introduce any new observables into the theory. They do introduce an object, off which the non-normalizable observables
from the closed string sector can scatter, and therefore it is natural that they have a description of the sort proposed here.

(5) A related puzzling aspect of the pointlike D-instantons is the fact that they induce a finite size boundary in the spacetime CFT – a one dimensional boundary at, say, $x\bar{x} = -1$ in the construction of [25]. Naively one might expect that they should give rise to a pointlike defect (or local insertion in the spacetime CFT) at $\gamma = \bar{\gamma} = 0$. This “blow up” of the instantons seems to be related to a familiar phenomenon in the context of the $AdS_5/SYM_4$ correspondence. It is known [35,36] that the radial location of D-instantons in $AdS_5$ is related to the size of the corresponding instantons in super Yang-Mills. D-instantons near the boundary of $AdS_5$ correspond to small instantons, and as they approach the horizon of $AdS_5$, they grow in SYM. Similarly here, D-instantons located at $u = 1$ (far from both the horizon and the boundary of $AdS_3$) seem to correspond to objects of size “one” in the spacetime CFT. What is perhaps surprising is that closed string fields see the expanded instantons as sharply defined objects (as signaled by the singularities of the closed string correlators near the boundary in spacetime) – one might have expected an everywhere smooth behavior.

(6) There are many possible extensions of the work described here. It would be interesting to complete the bootstrap program and verify its full consistency. Also, we have restricted attention to the “short string” sector of the model. Perhaps additional insight could be obtained by applying our methods to sectors with long strings. It would also be interesting to extend the analysis to other related backgrounds, such as the Euclidean cigar, $SL(2)/U(1)$. In that case it is known that one should find both branes that are extended along the radial direction of the cigar, and branes localized near the tip [18,19]. It is natural to expect that branes localized near the tip (which are of interest for various application – see section 1) will arise from the localized branes in $AdS_3$, and the multi-instanton branes discussed here will correspond to small disks localized near the tip.

(7) It would also be interesting to understand the Liouville branes [26,27,28] from the perspective of $SL(2)$ branes, by dividing by the Borel subgroup. One way of making a direct connection is to consider string theory on $AdS_3$ in static gauge, $\gamma = z, \bar{\gamma} = \bar{z}$. This corresponds to studying the system in a vacuum containing a long string [2,37]. The transverse fluctuations of such a string are described by the Liouville Lagrangian for the radial coordinate $\phi$ (2.10) [37]. D-branes which reach the boundary of $AdS_3$ intersect the worldsheet of the long string, and thus it is natural that they give rise
to worldsheet boundaries at a finite distance from the bulk, as found for the extended branes in [26]. Thus, these branes correspond to the principal series solutions (8.2), (8.3), and the boundary cosmological constant \( \mu_B \) in [26] is a remnant of the electric field \( E \) on these branes. D-instantons do not reach the boundary of \( AdS_3 \), and thus do not intersect the worldsheet of the long string. They give rise in the Liouville description to a boundary at an infinite distance from the bulk, as found in [28]. It would be interesting to make this picture more precise.

(8) It would also be interesting to extend the present work to study the properties of D-branes in supersymmetric string theories on \( AdS_3 \times N \) [3,4]. For example, the type II superstring on \( AdS_3 \times S^3 \times T^4 \) has supersymmetric branes with an \( AdS_2 \times S^2 \) worldvolume (which were studied geometrically in [25,39]). Such supersymmetric \( AdS_d \) branes in \( AdS_{d+1} \) have some interesting properties, suggested recently in [40,41], and our techniques might be useful for studying them.

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**Appendix A. Some useful formulae**

The hypergeometric function is defined by the differential equation for a function \( u(x) \)

\[
x(1-x)u'' + [\gamma - (\alpha + \beta + 1)x] u' - \alpha \beta u = 0
\]

This equation has two solutions:

\[
u_1 = F(\alpha, \beta; \gamma; x) \\
u_2 = x^{1-\gamma} F(\alpha - \gamma + 1, \beta - \gamma + 1; 2 - \gamma; x)
\]

For small \( x \), \( F \) can be expanded as follows:

\[
F(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n! (\gamma)_n} x^n = \frac{\alpha \beta}{\gamma} x + \frac{\alpha(\alpha + 1)\beta(\beta + 1)}{2\gamma(\gamma + 1)} x^2 + \cdots ,
\]
where
\[(a)_n \equiv a(a + 1) \cdots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)}. \quad (A.4)\]

Two other identities that are sometimes useful are:
\[F(\alpha, \beta; \gamma; x) = (1 - x)^{\gamma - \alpha - \beta} F(\gamma - \alpha, \gamma - \beta; \gamma; x) \]
\[\frac{\partial F}{\partial x}(\alpha, \beta; \gamma; x) = \frac{\alpha \beta}{\gamma} F(\alpha + 1, \beta + 1; \gamma + 1; x) \quad (A.5)\]

Under \(x \to 1/x\):
\[F(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma) \Gamma(\beta - \alpha)}{\Gamma(\alpha) \Gamma(\gamma - \alpha)} \left( -\frac{1}{x} \right)^{\alpha} F(\alpha, \alpha + 1 - \gamma; \alpha + 1 - \beta; \frac{1}{x}) + \]
\[\frac{\Gamma(\gamma) \Gamma(\alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\beta - \gamma)} \left( -\frac{1}{x} \right)^{\beta} F(\beta, \beta + 1 - \gamma; \beta + 1 - \alpha; \frac{1}{x}) \quad (A.6)\]

Under \(x \to 1 - x\):
\[F(\alpha, \beta; \gamma; 1 - x) = \frac{\Gamma(\gamma) \Gamma(\gamma - \beta - \alpha)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)} F(\alpha, \beta + \alpha - 1 - \gamma; \alpha + 1 - \beta; 1 - x) + \]
\[x^{\gamma - \alpha - \beta} \frac{\Gamma(\gamma) \Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha) \Gamma(\beta)} F(\gamma - \alpha, \gamma - \beta; \gamma + 1 - \alpha - \beta; 1 - x) \quad (A.7)\]

The hypergeometric function in two variables \(F_1(x, y)\) can be defined as the analytic continuation of the small \(x, y (|x|, |y| < 1)\) expansion:
\[F_1(\alpha, \beta, \beta'; \gamma; x, y) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha)_m (\beta)_n (\beta')_n}{m! n! (\gamma)_{m+n}} x^m y^n, \quad (A.8)\]

where \((a)_n\) is defined in \((A.4)\). Some useful identities are:
\[F_1(\alpha, \beta, \beta'; \gamma; x, 1) = \frac{\Gamma(\gamma) \Gamma(\gamma - \beta' - \alpha)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta')} F(\alpha, \beta; \gamma - \beta'; x), \quad (A.9)\]
\[F_1(\alpha, \beta, \beta'; \gamma; x, x) = F(\alpha, \beta + \beta'; \gamma; x). \]

More useful identities:
\[\Gamma(x) \Gamma(1 - x) = \frac{\pi}{\sin(\pi x)} \quad (A.10)\]
\[\Gamma(1 + ix) \Gamma(1 - ix) = \frac{\pi x}{\sinh(\pi x)}, \quad x \in R \quad (A.11)\]
\[\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2} \quad (A.12)\]
\[\sinh x + \sinh y = 2 \sinh \frac{x + y}{2} \cosh \frac{x - y}{2} \quad (A.13)\]
Appendix B. The limit $z \to \infty$, $x \to \infty$ of $F_A$, $F_B$, $F_C$

In section 3 we used the behavior of certain combinations of hypergeometric functions in two variables, as their arguments $x, z$ go to infinity. In this appendix we outline the derivation of the result, eq. (3.50).

In order to study the limit $z \to \infty$, $x \to \infty$ of the hypergeometric function $F_1(\alpha, \beta, \beta'; \gamma; x, z)$ it is convenient to study its transformation under $z \to 1/z$, $x \to 1/x$. In [32] there is a function, denoted by $Z_3$, that implements precisely this transformation:

$$Z_3 = x^{-\beta}z^{-\beta'}F_1(\beta + \beta' + 1 - \gamma, \beta, \beta'; \beta + \beta' + 1 - \alpha; \frac{1}{x}, \frac{1}{z}).$$  \hspace{1cm} (B.1)

Thus to study the behavior of $F_A$ (3.47) as $z, x \to \infty$ it would be convenient to express $Z_3$ in terms of $F_A$, $F_B$ and $F_C$, or alternatively in terms of $Z_1$, $Z_5$, and $Z_8$ (see [32] for the general definition of $Z_5$; for our values of the parameters it is equal to $F_B$).

In fact, as explained in [32], precisely such a relation does indeed exist. One has

$$Z_3 = c_1Z_1 + c_5Z_5 + c_8Z_8$$  \hspace{1cm} (B.2)

where $c_i$, $i = 1, 5, 8$ are constants (independent of $x, z$). Similarly, in order to study the large $z, x$ behavior of $Z_5$ and $Z_8$ one is interested in the functions $Z_9$ and $Z_4$ in [32], and again, they can be expressed in terms of the basic functions as

$$Z_9 = d_1Z_1 + d_5Z_5 + d_8Z_8$$
$$Z_4 = e_1Z_1 + e_5Z_5 + e_8Z_8$$  \hspace{1cm} (B.3)

If one knows the coefficients $c_i$, $d_i$, $e_i$, one can analyze the behavior of the blocks $F_A$, $F_B$, $F_C$ in the limit $z, x \to \infty$.

Determining all the coefficients is somewhat tedious, but happily the part that is needed for deriving (3.50) is rather simple. First recall that we are only interested in the constant terms in $F_A$, $F_B$ and $F_C$ in the limit $z, x \to \infty$. These correspond to the contribution of the identity block in the limit studied in the text. Terms that depend on $x, z$ in this limit give the contributions of the boundary blocks $\Psi_t$ and $\Psi_{1-t}$ (which, as we mention in the text and discuss below, turn out to vanish in the end).

One can show by studying the small $x, z$ behavior of $Z_1$, $Z_5$, $Z_8$, that only the $Z_5$ terms on the r.h.s. of (B.2), (B.3) contribute to the identity block; the other terms contribute to the blocks corresponding to $\Psi_t$ and $\Psi_{1-t}$. Therefore, we need only to compute $c_5$, $d_5$ and $e_5$ to derive (3.50).
The second nice fact is that if we set $x = 1$ in (B.2), (B.3), the hypergeometric functions in two variables can be expressed in terms of standard hypergeometric functions; $Z_1$ and $Z_8$ give rise to the same hypergeometric function, while $Z_5$ produces a different one. Thus, by setting $x = 1$ one can compute precisely the coefficients that are needed for (3.50). By using standard relations between the hypergeometric functions $F$ and $F_1$, one finds

$$
c_5 = \frac{\Gamma(\beta + \beta' + 1 - \alpha)\Gamma(\gamma - \beta - 1)}{\Gamma(\beta')\Gamma(\gamma - \alpha)} (-)^{\beta' + 1 - \gamma}
$$
$$
d_5 = \frac{\Gamma(\alpha + 1 - \beta')\Gamma(\gamma - \beta - 1)}{\Gamma(\alpha)\Gamma(\gamma - \beta - \beta')} (-)^{\alpha + 1 - \gamma}
$$
$$
e_5 = \frac{\Gamma(\beta' + 2 - \gamma)\Gamma(\gamma - \beta - 1)}{\Gamma(\beta')\Gamma(1 - \beta)} (-)^{\beta' + 1 - \gamma}
$$

Substituting the values of $\alpha, \beta, \beta', \gamma$ relevant for our problem, which are

$$
\alpha = \beta = t, \ \beta' = 2h + t - 1, \ \gamma = 2t
$$

leads to the results (3.50).

In section 3 we argued that when the operator $\Phi_{z^2}$ approaches the boundary of the worldsheet ($z \to \bar{z}$) it only creates the identity operator (i.e. the coefficient of $\Psi_t$ and $\Psi_{1-t}$, which also seem to be allowed by the OPE (3.42), vanish). To prove this one notes that the function $F$ in (3.46) can be expanded in terms of the blocks relevant for the $z \to \bar{z}$ limit, $Z_3, Z_4, Z_9$ given in eqs. (B.1) – (B.3), as follows:

$$
F = a_3 Z_3 + a_4 Z_4 + a_9 Z_9
$$

The first two terms on the r.h.s. are linear combinations of the $\Psi_t$ and $\Psi_{1-t}$ blocks, while $a_9$ is the coefficient of the block of the identity. Plugging (3.49) into (3.46) one finds that $a_3 = a_4 = 0$. Therefore, the amplitude for $\Phi_{z^2}$ to create $\Psi_t$ and $\Psi_{1-t}$ on the boundary vanishes.
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