General unit-disk representation for periodic multilayers

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We suggest a geometrical framework to discuss periodic layered structures in the unit disk. The band gaps appear when the point representing the system approaches the unit circle. We show that the trace of the matrix describing the basic period allows for a classification in three families of orbits with quite different properties. The laws of convergence of the iterates to the unit circle can be then considered as universal features of the reflection.

Photonic band gap structures can be dealt with by computing the band structure (using e.g. Bloch theory) or from the point of view of scattering. The essential difference is that a scattering experiment always involves a finite structure, while Bloch waves imply an endlessly repetition of the basic period.

In the context of electromagnetic optics, photonic crystals (that is, one-dimensional periodic layered structures) have attracted recently a lot of attention because the striking property of acting as omnidirectional reflectors: they can reflect light at any polarization, any incidence angle, and over a wide range of wavelengths.

The appearance of strong reflection (stop bands) depends on the properties of the basic period. Different theoretical approaches, involving equivalent medium theories, group velocity, and other quantities, have been developed to account for the detailed structure of these stop bands and their edges. Each one of these models emphasizes some aspects of the problem but, at the same time, has some drawbacks.

In the present Letter we introduce a geometrical setting that allows for a deeper understanding of periodic systems. Our treatment is quite general and only assumes linearity: it applies to any physical system whose transfer matrix belongs to the group SU(1, 1). The key point for our purposes is the fact that the multilayer transfer function induces a bilinear transformation in the unit disk. Since perfect mirrors are represented precisely by the unit circle, the route to a stop band can be understood as the convergence of the point representing the action of the system to the unit circle.

We start by examining the basic period of the system, which consists of a stack of plane-parallel layers sandwiched between two semi-infinite ambient (a) and substrate (s) media that we shall assume to be identical, since this is the common experimental case. Hereafter all the media are supposed to be lossless, homogeneous, and isotropic.

A monochromatic linearly polarized plane wave falls from the ambient making an angle \( \theta_0 \) with the normal to the first interface and with an amplitude \( E_a^{(+)} \). We consider as well another plane wave of the same frequency and polarization, and with amplitude \( E_a^{(-)} \), incident from the substrate at the same angle \( \theta_0 \). The output fields in the ambient and the substrate will be denoted \( E_a^{(-)} \) and \( E_s^{(+)} \), respectively.

The field amplitudes at each side of the multilayer are related by the linear relation

\[
\begin{pmatrix}
E_a^{(+)} \\
E_a^{(-)}
\end{pmatrix} = M_{as} \begin{pmatrix}
E_s^{(+)} \\
E_s^{(-)}
\end{pmatrix},
\]

where the multilayer transfer matrix \( M_{as} \) can be shown to be

\[
M_{as} = \begin{bmatrix}
1/T_{as} & R_{as}/T_{as} \\
R_{as}/T_{as} & 1/T_{as}
\end{bmatrix} \equiv \begin{bmatrix}
\alpha & \beta \\
\beta^* & \alpha^*
\end{bmatrix}.
\]

Here the complex numbers \( R_{as} \) and \( T_{as} \) are, respectively, the overall reflection and transmission coefficients for a wave incident from the ambient. Because \( |R_{as}|^2 + |T_{as}|^2 = 1 \), we have \( \det M_{as} = +1 \) and then the set of lossless multilayer matrices reduces to the group SU(1, 1).

We are often interested in the transformation properties of field quotients rather than the fields themselves. Therefore, we introduce the complex numbers

\[
z = \frac{E^{(-)}}{E^{(+)}},
\]

for both ambient and substrate. Equation defines then a transformation on the complex plane \( \mathcal{C} \), mapping the point \( z_a \) into the point \( z_a \), according to

\[
z_a = \Phi[M_{as}, z] = \frac{\beta^* + \alpha^* z}{\alpha + \beta z}.
\]

which is a bilinear (or M"obius) transformation. This action can be seen as a function \( z_a = f(z) \) that will be called the multilayer transfer function. One can check that the unit disk, the external region and the unit circle remain invariant under the multilayer action. Note that \( |z_a| = |z| = 1 \) for a perfect mirror with light incident from both the ambient and the substrate, so the transformation relates then points on the unit circle. When no light strikes from the substrate \( z_s = 0 \) and \( |z_s| = 1 \), so a mirror maps the origin into a point on the unit circle.

In what follows, the idea of fixed points of the transformation will prove to be essential. These points can be defined as the field configurations such that \( z_a = z_f \) in Eq. , i.e., \( z_f = \Phi[M_{as}, z] \), whose solutions are

\[
z_f = \frac{1}{2\beta} \left\{ -2i\, \text{Im}(\alpha) \pm \sqrt{(\text{Tr}(M_{as}))^2 - 4} \right\}.
\]
The trace of $M_{as}$ provides then a suitable tool for the classification of multilayers.

When $[\text{Tr}(M_{as})]^2 < 4$ the multilayer action is elliptic and it has only one fixed point inside the unit disk, while the other lies outside. When $[\text{Tr}(M_{as})]^2 > 4$ the action is hyperbolic and it has two fixed points on the unit circle. Finally, when $[\text{Tr}(M_{as})]^2 = 4$ the multilayer action is parabolic and it has only one (double) fixed point on the unit circle.

To proceed further let us note that by taking the conjugate of $M_{as}$ with any matrix $C \in \text{SU}(1, 1)$, that is

$$\hat{M}_{as} = C M_{as} C^{-1},$$

we obtain another matrix of the same type, since $\text{Tr}(\hat{M}_{as}) = \text{Tr}(M_{as})$. Conversely, if two multilayer matrices have the same trace, a matrix $C$ satisfying Eq. (6) can be always found.

The fixed points of $\hat{M}_{as}$ are then the image by $C$ of the fixed points of $M_{as}$. In consequence, given any matrix $M_{as}$, it can always be reduced to a unique $\hat{M}_{as}$ with one of the following canonical forms:

$$\hat{K}(\varphi) = \begin{bmatrix} \exp(i\varphi/2) & 0 \\ 0 & \exp(-i\varphi/2) \end{bmatrix},$$

$$\hat{A}(\chi) = \begin{bmatrix} \cosh(\chi/2) & i \sinh(\chi/2) \\ -i \sinh(\chi/2) & \cosh(\chi/2) \end{bmatrix},$$

$$\hat{N}(\eta) = \begin{bmatrix} 1 - i\eta/2 & \eta/2 \\ \eta/2 & 1 + i\eta/2 \end{bmatrix},$$

that have as fixed points the origin (elliptic), $+i$ and $-i$ (hyperbolic) and $+i$ (parabolic), and whose physical significance has been studied before.

The concept of orbit is especially appropriate for obtaining a picture of these actions. Given a point $z$, its orbit is the set of points $z'$ obtained from $z$ by the action of all the elements of the group. In Fig. 1.a we have plotted typical orbits for each one of the canonical forms. For matrices $K(\varphi)$ the orbits are circumferences centered at the origin. For $A(\chi)$, they are arcs of circumference going from the point $+i$ to the point $-i$ through $z$. Finally, for $N(\eta)$ the orbits are circumferences passing through the point $+i$ and joining the points $z$ and $-z^*$.

For an arbitrary matrix $M_{as}$ the orbits can be obtained by transforming the orbits described before with the appropriate matrix $C$. In Fig. 1.b we have plotted typical examples of such orbits for elliptic, hyperbolic, and parabolic actions. We stress that once the fixed points of the matrix $M_{as}$ are known, one can ensure that $z_a$ will lie in the orbit associated to $z_s$.

Assume now that we have a finite periodic structure obtained by repeating $N$ times the basic period represented by $M_{as}$. The overall transfer matrix for this system is $M_{as}^N$. In the unit-disk picture, the transformed field by the $N$-period structure is represented by the point $z_N = \Phi[M_{as}, z_{N-1}] = \Phi[M_{as}^N, z_0]$, (8)

where $z_0$ is the initial point $z_s$. The idea that maps iterates could be applied to this problem was recognized before, though with a somewhat different approach [16].

Henceforth, we shall take $z_0 = 0$, which is not a serious restriction and corresponds to the case in which no light incides from the substrate [$E_3^{(\infty)} = 0$], as it happens usually. Note also that all the points $z_N$ lie in the orbit associated to the initial point $z_0$ by the single period, which is determined by its fixed points: the character of these fixed points determine thus the behavior of the periodic structure.

To illustrate how this geometrical approach works in practice, we take the single period as a Fabry–Perot-like system formed by two identical plates separated by a spacer of phase thickness $\delta_2$. This is a symmetric system for which $R_{as}$ and $T_{as}$ can be easily computed [17]. By varying $\delta_2$ we can choose to work in the elliptic, the hyperbolic, or the parabolic case. In Fig. 2 we have plotted the sequence of successive iterates obtained numerically for these three regimes.

In the elliptic case, it is clear that the points $z_N$ revolve in the orbit centered at the fixed point and the system never reaches the unit circle.

On the contrary, for the hyperbolic and parabolic cases the iterates converge to one of the fixed points on the unit circle, although with different laws, which correspond to the band stop and band edges of the system, respectively [16].

To gain further insights into these behaviors, we compute explicitly the $N$th iterate. This can be easily done...
for the canonical forms in Eq. (7) and then, conjugating as in (6) we obtain, after some lengthy calculations, that for a hyperbolic action one has

$$z_N = \frac{1 - \xi^N}{1 - \xi^N(z_{f+}/z_{f-})} z_{f+},$$

(9)

where $\xi = (\alpha + \beta z_{f-})/(\alpha + \beta z_{f+})$ is a complex number satisfying $|\xi| < 1$. Analogously, for the parabolic case we have

$$z_N = \frac{N \beta z_{f}^2}{N \beta z_{f} - 1},$$

(10)

where $z_f$ is the (double) fixed point. It is quite obvious that in both cases $z_N$ converges to one of the fixed points on the unit circle, so $|z_N| \rightarrow 1$ when $N$ increases, a typical behavior of perfect mirror. In the mathematical literature this limit point is referred to as the Denjoy-Wolff point of the map [18].

To characterize the convergence of $z_N$ we note that, because $z_0 = z_s = 0$, this initial point is transformed by the single period into $z_{a s} = R_{as}$. Therefore, $z_N$ represents the reflection coefficient of the overall periodic structure $R_{as}^{(N)}$, which is obviously different from $(R_{as})^N$. One can then compute that for the hyperbolic case [19]

$$|z_N|^2 = \frac{|\beta|^2}{|\beta|^2 + \sinh(\chi)/\sinh(N\chi)^2},$$

(11)

that approaches the unit circle exponentially with $N$, as one could expect from a band stop, while for the parabolic action

$$|z_N|^2 = \frac{|\beta|^2}{|\beta|^2 + (1/N)^2},$$

(12)

that goes to unity with a typical behavior $O(N^{-2})$. This is universal in the physics of reflection, as put forward in a different framework by Yeh [20] and Lekner [21].

To conclude, we expect that the geometrical scenario presented here could provide an appropriate tool for analyzing and classifying the performance of periodic multilayers in an elegant and concise way that, additionally, is wider enough to accommodate other periodic systems appearing in physics.

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