TROPICALLY COMPACTIFY MODULI VIA GROMOV-HAUSDORFF COLLAPSE

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To the memory of Kentaro Nagao

Abstract. We compactify the moduli variety of compact Riemann surfaces (resp., of abelian varieties) by attaching moduli of graphs (resp., of tori) as boundary. The compactifications patch together to form a big connected moduli space in which \( \sqcup_g M_g \) is open dense (the same for \( A_g \)).

The “degenerations” as tropical varieties are obtained via Gromov-Hausdorff collapse by fixing diameters of Kähler-Einstein metrics. This phenomenon can be seen as an “archimedean” analogue of the tropicalisation of Berkovich analytification of \( M_g \) \([ACP]\). We also study topologies of the boundaries a little.

1. Introduction

The “canonical” geometric compactification of the moduli space of smooth projective curves usually means the one whose boundary parametrizes stable nodal curves (Deligne-Mumford \([DM]\)) and the constructions of similar compactifications of moduli spaces of abelian varieties are recently established (Alexeev \([Ale1]\), Nakamura \([Nak1]\, \[Nak2]\)).

Each of them has coarse projective moduli variety and the parametrized objects satisfy stability conditions, either GIT stability \((DM, [Nak1])\) or K-stability \((DM, [Ale1], [Nak2])\). The latter is all moduli spaces of K-(semi)stable varieties (“K-moduli” cf., \([Od3]\)). Indeed, the degenerations on the boundary are K-semistable due to \([Od1]\ Theorem 4.1(ii)] because they only have semi-log-canonical singularities \(([Ten])\) and have trivial canonical classes.

Now, forgetting about stability and those moduli for a while, we give very different compactification which are no longer varieties but compact Hausdorff (metric) topological space which is canonical in a differential geometric sense. Roughly speaking, each of them is a metric completion of certain Gromov-Hausdorff metric on \( M_g \) or \( A_g \) respectively, so that another way to express our work is to put tropical geometric structure on the boundary of the metric completions. We will
call our compactifications “tropical geometric compactifications” and denote them as $\bar{M}_g^T$ and $\bar{A}_g^T$. For Fano manifolds case, as we discussed in [OSS], [Od3], the two kinds of compactifications (algebro-geometric K-moduli / our Gromov-Hausdorff compactification with fixed diameters) should coincide because of the non-collapsing. However they are completely different in general due to collapse as we show in the present paper. Indeed, the author believes that Gromov-Hausdorff compactification while fixing volume (rather than diameter) if it exists in an appropriate sense, should be related to K-moduli. There are two inspirations for this work.

(i) Current extensive approach to the Strominger-Yau-Zaslow mirror symmetry conjecture ([SYZ]) being lead by Gross-Siebert (cf., e.g., a survey [Gross]) and Kontsevich-Soibelman (cf., e.g., [KS]) and other researchers.

(ii) Algebraicity of non-collapsed Gromov-Hausdorff limits of Kähler- Einstein manifolds ([DS]), its applications to moduli of Fano varieties ([Spo], [OSS], [Od3]).

There is a similarity between the above two as the first is in particular observing some sort of “algebraicity” of collapsed Gromov-Hausdorff limits while the second is that for non-collapsed case. This similarity made a motivation of this work.

In the present paper, first we start with classifications of all the possible Gromov-Hausdorff limits of compact Riemann surfaces (resp., of principally polarised abelian varieties) with Kähler-Einstein metrics of diameters 1. Then using the classifications, we construct the compactification and proceed to analysis of their structures and related discussions.

In particular, as mentioned before, the boundaries of the compactifications are moduli spaces of tropical varieties. Moduli spaces of tropical curves and tropical abelian varieties are introduced and discussed by several authors before (cf., [Cap], [BMV], [MZ]). I will show the relation between our boundary and their moduli spaces. Consequently, they are similar although not the same.

Our connection between classical algebro-geometric compactifications and tropical moduli spaces can be seen as an analogue of the tropicalization (skeleton) of Berkovich analytification of the moduli varieties which is recently studied in [ACP]. We explain this analogy later.

The author chose this terminology carefully to avoid overlap with J. Tevelev’s theory of “tropical compactification” although the context is very different, which is to compactify subvariety of tori nicely in toric varieties [Tev].
Another interesting point of our compactifications, say $\bar{M}_g^T$, is that they naturally patch together to form a big (infinite dimensional) connected moduli space of which $M_g$ are open subsets for all $g$. We will call them infinite join and denotes as $M_\infty^T$. We also have an analogue $A_\infty^T$ for abelian varieties case.

It would be interesting to pursue this line of research for moduli varieties of other polarized varieties. For instance, the author wonders about moduli scheme of smooth canonical models and how it relates to the recent Kollár-Shepherd-Barron-Alexeev (KSBA) compactification $[\text{Kol}]$ via semi-log-canonical models (note that it is a higher dimensional extension of $M_g$ of $[\text{DM}]$ and is a K-moduli scheme $[\text{Od2}]$). Another interesting case would be those of polarized K3 surfaces whose Gromov-Hausdorff collapse for maximally degenerate case are well studied recently ($[\text{GW}]$, $[\text{KS}]$, $[\text{GTZ}]$).

Throughout this article, we basically work over the complex number field $\mathbb{C}$, unless otherwise stated.

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This paper is dedicated to fifteen years memory of Kentaro Nagao. Looking back, I can never stop deeply thanking Nagao-san for all the inspirations from the beginning and the warm friendliness. Hope he would be delighted again.

2. Curves case

2.1. Precompactness of $M_g$. For each compact Riemann surface of genre $g$ parametrized in $M_g$, we put rescaled Kähler-Einstein metric of (fixed) diameter 1. The first point we should clarify is the precompactness of $M_g$ with the associated Gromov-Hausdorff metric on it. We denote it as $d_{\text{dGH}}$. During the process of degenerations i.e., going to boundary of $M_g$, the curvature tends to $-\infty$, so we can not apply the Gromov’s precompactness theorem $[\text{Grom}]$ in our situation. Instead we can apply the following theorem of Shiota $[\text{Shi}]$ and the Gauss-Bonnet theorem to prove it.

**Theorem 2.1 ([Shi Theorem 1.1])**. For two fixed positive real numbers $D > 0$ and $c > 0$, consider the set $S(D,c)$ of closed 2-dimensional Riemannian manifolds $(R,d)$ with

(i) the diameter $\text{diam}(d) < D$
\[ (ii) \text{ and the total absolute curvature } \int_R |K_{(R,d)}| \text{vol}(R) < c \text{ where } K_{(R,d)} \text{ and } \text{vol}(R) \text{ denotes the Gaussian curvature and the volume form with respect to the metric } d. \]

Then the set \( S(D,c) \) is precompact with respect to the associated Gromov-Hausdorff distance.

By applying the above theorem, we get the following desired precompactness.

**Corollary 2.2.** \( (M_g, d_{dGH}) \) is precompact.

**First proof.** It directly follows from the Shioya’s theorem above (2.1) since our total absolute curvature is constant due to the Gauss-Bonnet theorem.

We include another proof of Corollary 2.2 in the next subsection, which is direct enough to classify the Gromov-Hausdorff limits.

### 2.2. Gromov-Hausdorff collapse of Riemann surfaces

#### Before stating a theorem, we precisely fix some graph theoritic terminology we use in this paper.

**Definition 2.3.** In the present paper, a metrized (finite) graph means a finite connected non-directed graph with finite positive lengths attached to all edges. It is not necessarily simple, i.e., loops and several edges with the same ends are allowed. A contraction of a finite graph simply means a graph which can be obtained after a contraction of some edges.

The main result of this subsection is the following, which implies the precompactness of \( M_g \) and also classify all the possible Gromov-Hausdorff limits of the compact Riemann surfaces while fixing diameters.

**Theorem 2.4.** Let \( \{R_i\}_{i \in \mathbb{Z}_{>0}} \) be an arbitrary sequence of compact Riemann surfaces of fixed genus \( g \geq 2 \). Suppose \( \{(R_i, \frac{d_{KE}}{\text{diam}(R_i)})\}_{i} \) converges in the Gromov-Hausdorff sense. Here \( d_{KE} \) denotes the Kähler-Einstein metric on each \( R_i \) and its diameter is \( \text{diam}(R_i) \).

Then the limit is either

(i) a metrized (finite) graph of diameter 1 or
(ii) a compact Riemann surface of the same genus.

Moreover, in the former case, the combinatorial type of the graph is (possibly contractions of) the dual graph of the corresponding stable curve i.e., the limit of \([R_i]\) in the Deligne-Mumford compactification of
the moduli of curves $\bar{M}_g$, with non-negative metrics (possibly zero) on each edges.

Conversely, any metrized dual graph of the stable curve of genus $g$ with diameter 1 can occur in this way (i).

Proof. We fix a reference compact Riemann surface $S$ and regart the Teichmuller space $T_g$ as the set of marked compact Riemann surfaces $[\phi: S \rightarrow R]$ where we only care the isotopy type of $\phi$.

Consider a pants decomposition of $S$

$$S = \bigsqcup_{0 \leq a \leq g-2} P_a$$

with simple closed boundary geodesics $s_1, \cdots, s_{3g-3}$ with

$$\partial P_a = s_{3a+1} \sqcup s_{3a+2} \sqcup s_{3a+3}.$$ 

Then it naturally induces the corresponding pants decompositions

$$R = \bigsqcup_{0 \leq a \leq g-2} P_a(R)$$

for all elements $[\phi: S \rightarrow R]$ of $T_g$ as we can take simple closed boundary geodesics in the corresponding homology classes. The associated simple closed geodesics $\{s_j(R)\}_j$ gives the ($C^{\omega}$-) Fenchel-Nielsen coordinates on it

$$(l_1, \cdots, l_{3g-3}; \theta_1, \cdots, \theta_{3g-3}) : T_g \cong \mathbb{R}_{>0}^{3g-3} \times (S^1)^{3g-3},$$

where $l_j$ is the length of $s_j$ and $\theta_j$ is corresponding twist parameters (cf., [IT]).

We use the following well-known theorem due to L. Bers.

**Theorem 2.5 ([Bers]).** Fix a positive integer $g \geq 2$. Then there is a uniform constant $L_g$ satisfying the following.

For an arbitrary compact hyperbolic Riemann surface $R$, there is a pant decomposition whose corresponding lengths $l_j$ of simple closed geodesics are all $l_i < L_g$.

There are extensive studies to estimate the best value of $L_g$ (called the Bers constant) but for the purpose of our present paper, we do not need any effective value. On the other hand, note that the topological type of each pants decomposition corresponds to 3-valent (3-regular) graph with $2g - 2$ vertices. The number of edges of such a graph is $3g - 3$ so obviously their combinatorial types are finite. Therefore, for any given sequence of compact Riemann surfaces $\{R_i\}$, passing to an appropriate subsequence, we can assume that with a fixed combinatorial type of pants decomposition, we have $\lim_i l_j(R_i) = L_j$ with $L_j \in [0, \infty)$. 

The simple geodesics \( s_j \) of \( R_i \) with \( L_j = 0 \) shrink exactly to nodal singularities of the corresponding limit in Deligne-Mumford compactification \( \bar{M}_g^{DM} \). Note that if there is such \( j \) (i.e., with \( L_j = 0 \)), it means the diameter of non-rescaled hyperbolic metrics (i.e., with constant curvature \(-1\)) tends to infinity due to the collar theorem [Ke]. From now on, we assume these equivalent conditions are satisfied. Otherwise, the subsequence converges to a compact Riemann surface (i.e., “does not degenerate”), which corresponds to the case \((ii)\).

Let us denote the diameter of Kähler-Einstein (Poincaré) metric \( d_{KE} \) of \( R_i \) as \( d_i \). Then recall that what we are to analyse is the Gromov-Hausdorff limits of \((R_i, \frac{d_{KE}}{d_i})\). For that, it is enough to prove the existence of the Gromov-Hausdorff limit of \((P_a(R_i), \frac{d_{KE}}{d_i})\) and determine their structures. We do these simultaneously. Recall that the pant \( P_a \) can be cut and separate to two isometric hyperbolic hexagons \( Q_a \) and \( Q'_a \) canonically by geodesics which connect different boundaries among \( s_{3a+b} \) \((b = 1, 2, 3)\). Let us call the 3 boundaries of the hexagon which were originally part of the boundaries of the pant \( P_a \) as “boundary geodesics”. In any case, the important invariants are the lengths of the 3 boundary geodesics which are half of the boundary geodesics \( s_{3a+b} \) of the original pant \( P_a \). Indeed, it is a well-known fact that biholomorphic type of \( Q_a \) (so also for \( P_a \)) is determined by the lengths of the three boundary geodesics (cf., e.g., [IT]). Thus we reduce the problem to determination of the Gromov-Hausdorff limit of the “half pant” \( Q_a \) while fixing diameters.

Consider the hyperbolic hexagons \( Q_a \) inside a disc with center \( p \) and we denote the Poincaré metric \( d_{KE} \) on it. The collar theorem ([Ke]) says \( d_{KE}(p, s_{3a+b}(R_i)) \to +\infty \) for \( i \to \infty \) if and only if the corresponding boundary geodesic shrinks i.e., \( L_{3a+b} = 0 \).

Suppose we have \( \epsilon \) boundary geodesics of \( Q_a \) shrink \((0 \leq \epsilon \leq 3)\). We associate a tree \( \Gamma_a \) with

- the vertex set \( V(\Gamma_a) := \{v_a, w_j(s_j \text{ shrinks})\} \) and
- the edge set \( E(\Gamma_a) := \{v_aw_j \mid s_j \text{ shrinks}\} \).

Denote the diameter of the half pant \( Q_a(R_i) \) with respect to Poincaré metric as \( d_i(a) \). Then, it follows that depending on the ratios of diverging lengths of connecting geodesics, possible Gromov-Hausdorff limits of \((Q_a, \frac{d_{KE}}{d_i(a)})\) with \( i \to \infty \) are all metrized tree \( \Gamma_a \) of diameter 1 with possibly some contracted edges. We make this rigorous as follows. First we fix a constant \( 0 < \epsilon \ll 1 \) so that the sequence of the half pant \( \{Q_a(R_i)\}_i \) satisfies that the radius \((1-\epsilon)\) disc \( D(p, (1-\epsilon)) \) with center \( p \) contains all non-shrinking boundary geodesics of \( Q_a(R_i) \). Then as the diameter of \( \{(Q_a(R_i) \cap D(p, (1-\epsilon)), d_{KE})\}_i \) is bounded above
(by $2 \tanh^{-1}(1 - \epsilon)$) and the diameters of the collars from collapsing boundary geodesics tends to infinity by the collar theorem again \[Ke\]. Thus, considering the rescaled sequence of $(Q_a, \frac{d_{KE}}{d_i(a)})$, we get the assertion. And as $d_i$, which was defined as the diameter of whole $(R_i, d_{KE})$, tends to infinity as well, the Gromov-Hausdorff limit of $(Q_a, \frac{d_{KE}}{d_i})$ is either simply the Gromov-Hausdorff limit (the tree) of $(Q_a, \frac{d_{KE}}{d_i(a)})$ by a positive constant or a point.

Connecting all the (“local”) Gromov-Hausdorff limit of such $(Q_a, \frac{d_{KE}}{d_i})$, we see that the global Gromov-Hausdorff limit of $(R_i, \frac{d_{KE}}{d_{i(a)}})$ is a metrized graph which is obtained by gluing all $\Gamma_a$ at $w_j$s whose corresponding boundary geodesics $s_j$ are the same in $R_i$. The resulting graph is exactly the dual graph of the corresponding stable curve $R_\infty$. Note that depending on the limits of ratios of diverging lengths of connecting geodesics, we obtain all the metrized dual graph of $R_\infty$ of diameter 1 with possibly contracted edges.

This completes the proof of Theorem 2.4. □

For those who are unfamiliar, as a standard reference for the Teichmüller theory, we refer to Imayoshi-Taniguchi’s textbook \[IT\].

Remark 2.6. In the case $g = 1$, we have a similar phenomenon and compactification. As it can be regarded as a special case of the next section on the moduli spaces of principally polarized abelian varieties and also well-known to SYZ mirror symmetry experts, we give only brief description. Suppose there is a sequence of elliptic curve \{$C_i / \mathbb{Z} + \mathbb{Z} \tau_i$\} where $\tau_i$ locates in the standard fundamental domain $W$ of the upper half plane $\mathbb{H}$ modulo the modular group $SL(2, \mathbb{Z})$, that is

$$W := \{ \tau \in \mathbb{H} \mid |Re(\tau)| \leq 1, |\tau| \geq 1 \}.$$

If $Im(\tau_i)$ does not diverge, then after passing to a subsequence, they converge in the Gromov-Hausdorff sense to an elliptic curve. If $Im(\tau_i)$ diverges, then the Gromov-Hausdorff limit of a subsequence of $(R_i, d_{KE}/d_{i(a)})$ is $S^1(1/2\pi)$. On the other hand, the compactified Néron model (relative minimal model) after suitable base change is well-known to be $n$-gon with some $n \in \mathbb{Z}_{>0}$. Thus their dual graphs are topologically $S^1$, which is exactly the Gromov-Hausdorff limit.

Let us trace again the proof of our Theorem 2.4 to see analogy with the tropicalisation of the Berkovich analytification \[ACP\]. Starting with an arbitrary sequence of compact hyperbolic surfaces, we took a nice subsequence which converges to a singular stable curve in the Deligne-Mumford compactification and also converging in the Gromov-Hausdorff sense (with fixed diameters). Let us call such sequence of
compact hyperbolic surfaces of genus \( g(\geq 2) \) “strongly convergent sequence”. We denote the set of such strongly convergent sequences of compact hyperbolic Riemann surfaces as \( \mathcal{SM}_g \).

Then what we have constructed in the proof is the following two kinds of limiting maps

(1) \[ r: \mathcal{SM}_g \to \bar{M}_g \]

which maps \( \{R_i\} \) to the limit stable curve and

(2) \[ t: \mathcal{SM}_g \to S_g \]

which maps \( \{R_i\} \) to the Gromov-Hausdorff limit. Furthermore, we proved that \( r \) and \( t \) are compatible in the sense that underlying graph of \( t(\{R_i\}) \) is (possibly contraction of) the dual graph of the limit stable curve \( r(\{R_i\}) \).

On the other hand, in the recent paper [ACP] by Abramovich-Caporaso-Payne, the following is proved.

Fix an algebraically closed base field \( k \) with trivial valuation. If we consider the Berkovich analytification \( \bar{M}_g^{an} \) of the Deligne-Mumford compactification \( \bar{M}_g \), then the deformation retraction to the Berkovich skeleton \([Berk1]\) is “tropicalisation” which maps to the moduli of weighted tropical curves of genus \( g \).

Note that the Berkovich analytification parametrises stable curves over valuation fields which contains \( k \) (with trivial valuation) and the author vaguely sees (a subspace of) this as an “algebro-geometric” analogue of the set of strongly convergent sequence of compact Riemann surfaces \( \mathcal{SM}_g \). From this viewpoint, their tropicalisation (deformation retract) is an analogue of our \( t \) \([ACP]\) which maps to our tropical moduli space \( S_g \). The analogue of \( r \) in the Berkovich geometric setting \([ACP]\) is simply a reduction map \( \bar{M}_g^{an} \to \bar{M}_g \). The anti-continuity of the reduction map \([Berk1] (2.4)\) is related to the reverse of order of specialization / generisation while transferring to the tropical setting in our situation. The compatibility corresponding to those of our \( r \) and \( t \) is also proved in \([ACP]\).

2.3. The construction of \( \bar{M}_g^T \). Let us denote the metric completion of \( \bar{M}_g \) with respect to the Gromov-Hausdorff distance for our rescaled Kähler-Einstein metrics of (fixed) diameter 1. Using this, we define

\[^{2}\text{Here, } \mathcal{S} \text{ stands for a sequence.}\]
our tropical geometric compactification of moduli of curves first set-theoretically as
\[ \bar{M}_g^T := M_g \cup \partial \bar{M}_g^{dGH}. \]
Then we put a topology on it, whose open basis consists of those of \( M_g \) (the Euclidean topology) and metrics balls with center in \( \partial \bar{M}_g^{dGH} \). The metric ball which is centered at \([G]\) (\(G\) is a metric graph) and radius \( r \in \mathbb{R}_{>0} \) is simply defined as
\[ B([G], r) := \{ [C] \in \bar{M}_g^T | d_{dGH}([C], [G]) < r \}. \]
The obtained compactification \( \bar{M}_g^T \) is a compact Hausdorff topological space thanks to Theorem 2.4. Note also that the Gromov-Hausdorff metrics \( d_{dGH} \) on the boundary \( \partial \bar{M}_g^{dGH} \) partially extend via the Gromov-Hausdorff distance \( d_{dGH} \) on \( M_g/\text{Aut}(\mathbb{C}/\mathbb{R}) \) by defining a partial extension \( \tilde{d}_{dGH}(R, \iota R) \) as
\[ \inf \{ d_{dGH}(R, X) | X \in \partial \bar{M}_g^{dGH} \text{ or } X \text{ is a member of } M_g \text{ with diameter 1 rescaled hyperbolic metric s.t.,} \iota X \simeq X \}. \]
Note that the complex conjugate \( \iota \in \text{Aut}(\mathbb{C}/\mathbb{R}) \) simply makes the natural orientation of the corresponding Riemann surface reversed, which does not change it metric space structure.
A point here is we can not simply take the Gromov-Hausdorff metric completion of the set of compact Riemann surfaces of genus \( g \) by regarding the Riemann surfaces just as metric spaces. That is because it would discard the complex structures and ignore the effect of \( \iota \) above (cf., [Spo], [OSS]).
Let us set \( S_g := \partial \bar{M}_g^{dGH} \). For each finite (metrized) graph \( \Gamma \), let us denote the number of 1-valent vertices \( v_1(\Gamma) \) and denote the first betti number of \( \Gamma \) as \( b_1(\Gamma) \).

**Proposition 2.7.** \( S_g \) with the Gromov-Hausdorff metric topology is the moduli (metric) space of underlying metric spaces of finite metrized graphs of diameter 1 which satisfy \( v_1(\Gamma) + b_1(\Gamma) \leq g \).

Note there is a distinction between the metrized graph and the underlying metric space, which is simply a 1-dimensional CW complex. The point is that the underlying metric space does not see the 2-valent vertices. It is also not enough to consider metrized graphs without 2-valent vertices since a circle can not be obtained in that way.

**Proof.** From Theorem 2.4 we only need to specify the class of underlying (i.e., non-metrized) dual graphs of stable curves with genus \( g \).
Due to the genus formula, a stable curve $C$ of genus $g$ whose irreducible decomposition is $\bigcup_i C_i$ with dual graph $\Gamma$, we have

$$g = \sum_i g(C_i^\nu) + b_1(\Gamma),$$

where $\nu$ denotes the normalization and $b_1$ denotes the first Betti number. From the stability condition, for each component $C_i$ which corresponds to a 1-valent vertex of $\Gamma$, $g(C_i^\nu) \geq 1$. This is essentially the only numerical stability condition. Thus we have $g = \sum_i g(C_i^\nu) + b_1(\Gamma) \geq v_1(\Gamma) + b_1(\Gamma)$. Tracing back the above discussion, it is also easy to see that it is a sufficient condition as well. □

**Remark 2.8.** One useful remark is that in the above characterisation of metrized graphs which are parametrised in $S_g$, rather than putting “diameter 1” condition, it may be easier to impose a version that “the sum of lengths of edges is 1”. Note that the two tropical moduli spaces are naturally bijective and homeomorphic, simply by rescaling. This viewpoint will be used occasionally in this paper.

**2.4. Comparison with other tropical moduli spaces.** Recently Brannetti-Melo-Viviani [BMV] constructed moduli space $\bar{M}_g^{tr}$ of tropical curves and Caporaso [Cap] introduced its pointed versions $\bar{M}_g^{tr,p}$ (but note that the way of treating legs is slightly modified from [BMV] even in $n = 0$ case.) They are similar to our boundary $S_g$ but there is an essential difference as their definition of tropical curves even encode genus informations on the each component or not, which [BMV] and [Cap] called weights. As Caporaso’s moduli space [Cap] does not allow leaves with finite lengths, our moduli $S_g$ matches better with [BMV].

As in [CV], [BMV], [Cap] for the similar situations, the combinatorial types of underlying graphs of the metrized graphs gives a natural CW complex structure on $S_g$. A point of our moduli space $S_g$ is the following.

**Proposition 2.9.** With respect to the Gromov-Hausdorff topology on moduli of metrized finite graphs, the function $\Gamma \mapsto v_1(\Gamma) + b_1(\Gamma)$ is a lower semicontinuous function.

**Proof.** The assertion is essentially already proved by Theorem 2.4 combined with the precompactness (2.2) but let the author write more straightforward combinatorial proof.

It is enough to see that if we contract one edge $e$, $v_1 + b_1$ does not increase. If the edge $e$ is a loop, then the process decreases $b_1$ by 1 and $v_1$ increases at most 1. If the edge $e$ is not a loop, then the contraction does not change homotopy type so that it keeps $b_1$ unchanged neither, and $v_1$ does not increase (it may decrease by 1 (or 2)).
Note that via the modular interpretations, there is a sequence of canonical cellular closed embeddings
\[
S_g \hookrightarrow S_{g+1} \hookrightarrow \ldots
\]
unlike other compactifications of moduli of curves nor the moduli of weighted tropical curves by [BMV], [Cap].
Inside the moduli space $M^{tr}_g$ of (weighted) tropical curve in the sense of [BMV], let us consider the closed locus $S^w_g$ which parametrizes those which have diameters 1.

**Proposition 2.10.** We have natural morphisms as cell complexes as follows.

\[
\partial M^tr_g := M^tr_g \setminus \{ \text{a point with weight } g \} \cong S^w_g \times \mathbb{R}_{>0} \to S^w_g \to S_g.
\]

In addition, the last morphism is a proper map with generically finite fibers.

**Proof.** The (weighted) tropical curve in the sense of [BMV], [Cap] has finite non-zero diameter unless it is a point, so that we get the first isomorphism. Secondly, starting from their tropical curve which is not topologically a point, just by forgetting the weights, we get unweighted metrized graphs. It defines the next morphism $S^w_g \to S_g$. Hence, the non-finite locus of the morphism in $S^w_g$ should parametrizes those which have 2-valent vertices. Thus it maps to $S_{g-1} \subset S_g$ which proves the assertion. In particular, $S_g$ is purely $(3g - 4)$ dimensional for each $g(\geq 2)$. □

Note that a point with weight $g$ is a (weighted) dual graph of smooth projective curves of genus $g$, that is the open dense part $M^g$ in the Deligne-Mumford compactification $\bar{M}_g$. It reflects the fact that comparing usual algebro-geometric degeneration, partial smoothing and the tropical version reverses the order.

On the other hand, there is more classical theory of the outer space $X_n$ by Culler-Vogtmann [CV] in the 80s, which is an analogue of the Teichmuller space for metrized graphs. There, the corresponding analogous discrete group is the outer automorphism group $\text{Out}(F_n)$ of the free group $F_n$ with rank $n$, instead of the mapping class group. From now on, we use $g$ instead of their $n$ to adjust notation.

Recall that the quotient $X_g/\text{Out}(F_g)$ parametrizes graphs $\Gamma$ with $b_1(\Gamma) = g$ but $v_1(\Gamma) = 0$. 
We introduce another moduli space of graphs as a subset of $S_g$ (with induced topology) as

$$S_g^o := \{ \Gamma \in S_g \mid v_1(\Gamma) + b_1(\Gamma) = g \} \subset S_g.$$ 

The above is just a complement of $S_{g-1} \hookrightarrow S_g$ by the definition. These are related as follows.

**Proposition 2.11.** There is a canonical cellular open embedding $X_g/Out(F_g) \hookrightarrow S_g^o(\subset S_g)$. The image of $X_g/Out(F_g)$ is open dense in $S_g$ (thus so is $S_g^o$).

**Proof.** As $X_g/Out(F_g)$ parametrises metric graphs $\Gamma$ with $b_1(\Gamma) = g$ and $v_1(\Gamma) = 0$, what we want to show is that it forms an open dense subset of $S_g$. For each $\Gamma \in S_g^o$ with $v_1(\Gamma) + b_1(\Gamma) = g$ and $0 < \epsilon \ll 1$, we define graph(s) $\phi_\epsilon(\Gamma)$ as follows. For each leave $vw$ where $v$ is a 1-valent vertex, we put a small loop of length $\epsilon \ell(vw)$. Doing the same for all edges and rescale the metric on whole graph to make its diameter 1, then we call it $\phi_\epsilon(\Gamma)$. This naturally defines a perturbation of elements of $S_g^o$ to those of $X_g/Out(F_g)$.

The facts that all of these are unions of relative interiors of the cells with respect to that CW complex structure follow straightforward from the definitions.

We also need to prove $S_g^o$ is dense inside $S_g$. Although it follows from the fact that $S_g$ is a purely dimensional compact CW complex and $S_{g-1}$ is a proper closed subset, we provide an elementary proof for convenience. Let us analyse the neighborhood of $\Gamma \in S_{g-1} \subset S_g$. Starting from any such $\Gamma$ with a point $p \in \Gamma$, we can similarly consider $\Gamma$’s deformation $\psi_t(\Gamma) \in X_g/Out(F_g)$ for $t > 0$, for example, as follows. Set $v_1(\Gamma) + b_1(\Gamma) = g - d$. Taking a point $p$, we define $\psi_t(\Gamma)$ as a union of $\Gamma$ and a bouquet i.e., the union of $d$ length $t$ loops which passes through $p$. Thus in particular $X_g/Out(F_g)$ is open dense and hence so is $S_g^o$ as well. \hfill \Box

2.5. (Co)homology - curve case. We would like to make a first step of investigation of topologies of our compactification or its boundaries.

First, we recall that the moduli space of smooth projective curves has vanishing higher homology groups, due to J. Harer. His proof shows the existence of a deformation retract via the cell complex structure of the Teichmüller space (“arc complex”).

**Theorem 2.12 ([Har] Theorem 4.1).** For $g \geq 2$ and $i > 4g - 5$, we have

$$H_i(M_g; \mathbb{Q}) = 0 \text{ and } H^i(M_g; \mathbb{Q}) = 0.$$
So combined with the Poincaré-Lefschetz duality for orbifold, we get that for \( i \leq 2g - 2 \)

\[
H^i_c(M_g; \mathbb{Q}) = 0 \quad \text{and} \quad H^{BM}_i(M_g; \mathbb{Q}) = 0,
\]

where \( H^i_c \) denotes the cohomology group with compact supports and \( H^{BM}_i \) the Borel-Moore homology group.

We benefit from the above theorem as follows.

**Corollary 2.13.** For \( i \leq 2g - 2 \), we have

\[
H^i(\bar{M}_g^T; \mathbb{Q}) = H^i(S_g; \mathbb{Q}).
\]

**Proof.** It follows simply from the exact sequences of compactly supported cohomology groups or the Borel-Moore homology groups. \( \square \)

Thus the study of homology and cohomology of our tropical geometric compactification is at least partially reducible to that of the boundary for the specific range of the degrees. So let us study topology of our boundary \( S_g \).

First, we sketch the following case of very low \( g \) to grasp an intuition although these easiest cases are topologically trivial.

**Example 2.14.** \( S_1 \) is a point which stands for the circle of length 1. and \( S_2 \) is a two 2-simplices (triangles) patched together along one of their edges for each. The latter is also homeomorphic to a 2-simplex again so \( S_1 \) and \( S_2 \) are both contractable.

The open dense locus \( S^o_g \) of \( S_g \) is rationally a classifying space of \( \text{Out}(F_g) \), thus has in general highly nontrivial topology. Indeed its cohomology is those of \( \text{Out}(F_g) \) (cf., e.g., [EVHS] for non-vanishing cohomology for \( g = 5 \) case), we expect interesting topological structure on \( S_g \) for higher \( g \).

We put \( S_\infty := \varinjlim_g S_g = \bigcup_g S_g \) which is a limit with respect to the canonical embeddings \( S_{g-1} \hookrightarrow S_g \hookrightarrow S_{g+1} \cdots \) (cf., [3]). Then, while we expect that each \( S_g \) has highly nontrivial topologies in general, we observe the following asymptotic triviality.

**Theorem 2.15.** The (infinite dimensional) topological space \( S_\infty \) is contractible. In particular, for any \( k \geq 0 \), \( \varinjlim_g H_k(S_g; \mathbb{Q}) = 0 \).

**Proof.** Let us denote a cone of \( S_g \) as \( CS_g := (S_g \times [0,1])/\{(S_g \times 1)\} \).

It is enough to construct a continuous extension \( \phi_g : CS_g \to S_\infty \) with \( \phi_g(S_g \times \{1\}) = \{\text{interval of length 1}\} \) of the inclusion map \( S_g \hookrightarrow S_\infty \) and compatibility with other \( g \)s i.e., \( \phi_{g+1}|_{S_g} = \phi_g \).

We construct the map \( \phi_g \) by the following three steps.
Step 1. First we construct $\phi_g|_{S_g \times [0,\frac{1}{3}]}$. For any $(\Gamma, t) \in S_g \times [0,\frac{1}{3}]$, for each edge $e = vw \in E(\Gamma)$, we attach two new vertices $v', w'$ and new edges $v'w', wv'$ with the lengths $l(v'v') = l(ww') = 3tl(vw)$. Then rescale it to make it diameter 1. This is $\phi_g(\Gamma, t)$ with $t \leq \frac{1}{3}$. This $\phi_g$ is continuous as we multiplied 3$t$ at the definition of lengths. Note that the number of edges are tripled by this process.

Step 2. Next step is the construction of $\phi_g|_{S_g \times [\frac{1}{3}, \frac{2}{3}]}$. In this step of $t$ going from $\frac{1}{3}$ to $\frac{2}{3}$, we gradually contract the old edges. As the author thinks its definition is obvious by the expression, we omit the precise definition. Note that for each $t$, after the partial contraction, we need to rescale to make the diameter of each whole graph 1. Then the graphs finally obtained as $\phi_g|_{t=2/3}$ are trees which shapes like "*" i.e., precisely speaking, graphs whose

- vertices set is $\{v\} \cup \{w_i \mid 1 \leq i \leq m\}$ and
- edges set is $\{v'w_i \mid 1 \leq i \leq m\}$.

Let us call this type of tree "*type" with $m$ leaves.

Step 3. The final step is the construction of $\phi_g|_{S_g \times [\frac{2}{3}, 1]}$. The moduli space of *-type trees with $m$-leaves of diameter 1 is homeomorphic to the moduli space of those whose sum of lengths of edges is 1, simply by rescaling. And the latter is the simplex

$$\Delta_m := \{(x_1, \cdots, x_m) \mid 0 \leq x_1 \leq x_2 \leq \cdots x_m \leq 1, \sum x_i = 1\}.$$

Using the contractability of the simplex above, for each metrized tree of *-type $\Gamma$, we can cook up a segment connecting $\Gamma$ to the interval $[0,1]$ i.e., the metrized tree of * type with $m = 2$. This gives $\phi_g|_{S_g \times [\frac{2}{3}, 1]}$.

Combining all these steps to define $\phi_g$, we get a contractability of $S_\infty$. This completes the proof of Theorem 2.15.

Remark 2.16. We close the section by mentioning again on $\bar{M}_g$ of [DM]. It must be being assumed or at least conjectured by experts that pointed Gromov-Hausdorff limits of any convergent sequence of pointed compact Riemann surfaces of fixed genus $g \geq 2$ with volume constant (i.e., non-rescaled) Kähler-Einstein metrics are components of the stable curves with hyperbolic metrics on them. Asymptotic behaviour of hyperbolic metrics is precisely studied in [Wol].

Although I have not been able to find out any literatures claiming it so far.
2.6. **Finite join** $\overline{M_{\leq g}}^T$ and **infinite join** $\overline{M}_\infty^T$. An interesting phenomenon is that our tropical geometric compactification $\overline{M}_g^T$ naturally patches together for different $g$ thanks to the inductive structures (3) of $S_g$. Precisely speaking, we have the following joins of our compactifications.

**Definition 2.17.** The **finite join** of our tropical geometric compactifications is defined inductively as

\[
\overline{M}_{\leq 0}^T := \overline{M}_0^T = \{ \text{Riemann sphere } \mathbb{CP}^1 \},
\]

\[
\overline{M}_{\leq 1}^T := \overline{M}_1^T := M_1 \sqcup \{ S^1(1/2\pi) \} (= \overline{A}_1^T \text{ in the next section })
\]

(one point compactification)

and for $g \geq 2$ as

\[
\overline{M}_{\leq g}^T := \overline{M}_{\leq (g-1)}^T \cup_{S_{g-1}} \overline{M}_g^T.
\]

The last union is obtained via two inclusion maps $S_{g-1} \hookrightarrow S_g$ and $S_{g-1} \hookrightarrow \overline{M}_{\leq (g-1)}^T$. We call the above $\overline{M}_{\leq g}^T$ a **finite join** of our tropical geometric compactification.

From the definition, we have

\[
\cdots \overline{M}_{\leq (g-1)}^T \subset \overline{M}_{\leq g}^T \cdots
\]

Then we set

\[
\overline{M}_\infty^T := \lim_{g \to \infty} \overline{M}_{\leq g}^T = \cup_g \overline{M}_{\leq g}^T,
\]

and call it **infinite join** of our tropical geometric compactifications.

Note it is connected and all our tropical geometric compactification $\overline{M}_g^T$ is inside this infinite join. In particular, $M_g$ for all $g$ is inside this connected “big infinite dimensional moduli space”.

3. Abelian varieties case

3.1. **Gromov-Hausdorff collapse.** We similarly think about $g$-dimensional principally polarized abelian varieties with *rescaled* Kähler-Einstein metrics whose *diameters* are 1. Note that in this case, precompactness of the corresponding moduli variety $A_g$ with respect to the associated Gromov-Hausdorff distance follows from the famous Gromov’s precompactness theorem [Grom] (while it also directly follows from our discussion below. )

We proceed to classification of all the possible Gromov-Hausdorff limits. The author suspects it has been expected by experts and at
least partially known that those collapse should be (real) tori but unfortunately he could not find precise study nor results in literatures, so we present here a precise statement as well as its proof.

For simplicity and better presentation of ideas, let us first restrict our attention to maximally degenerating case.

Theorem 3.1. Consider an arbitrary sequence of $g$-dimensional principally polarized complex abelian varieties $\{V_i\}_{i \in \mathbb{Z}^>0}$ which is converging to the cusp $A_0$ of the boundary $\partial \bar{A}_g^\circ$ of the Satake compactification $\bar{A}_g^\circ$. We denote the flat (Kähler) metrics with respect to the polarization $d_{KE}(V_i)$ and their diameters $\text{diam}(d_{KE}(V_i))$. Then, after passing to an appropriate subsequence, we have a Gromov-Hausdorff limit of $\{(V_i, \frac{d_{KE}(V_i)}{\text{diam}(d_{KE}(V_i))})\}$, which is $(g-r)$-dimensional (flat) tori of diameter 1 with some $(0 \leq r < g)$.

Conversely, any such flat $(g-r)$-dimensional torus of diameter 1 with $0 \leq r < g$ can appear as a possible Gromov-Hausdorff limit of such sequence of $g$-dimensional principally polarised abelian varieties with fixed diameter 1.

Proof. Let us set our notations (mainly after [Chai]) on the Siegel upper half space and its compactification theory due to Satake [Sat] before the details of the proof. In our proof, we make essential use of the Siegel reduction theory.

For an element $Z = X + \sqrt{-1}Y$ of the Siegel upper half space $\mathfrak{H}_g$, we denote the Jacobi decomposition of $Y$ as $Y = 'BDB$, where

$$B = \begin{pmatrix}
1 & b_{1,2} & b_{1,3} & \cdots & b_{1,g} \\
& 1 & b_{2,3} & \cdots & b_{2,g} \\
& & 1 & \cdots & \vdots \\
& & & \ddots & \vdots \\
& & & & 1
\end{pmatrix},$$

$$D = \text{diag}(d_1, \cdots, d_g) = \begin{pmatrix}
d_1 & 0 & 0 & \cdots & 0 \\
0 & d_2 & 0 & \cdots & 0 \\
0 & 0 & d_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & d_g
\end{pmatrix}.$$

\[\text{\footnote{The character “サ” is Hiragana type character which we pronounce “SA”, the first syllable of Satake and the idea of using this character is after Namikawa’s book [Nam2] which used Katakana “サ” instead (but we japaneses rarely use katakana for writing japanese name). The corresponding Kanji character 佐 is more normal.}}\]
Equivalently, writing $Y = t\sqrt{Y} \sqrt{Y}$ with a matrix $\sqrt{Y} \in GL(g, \mathbb{R})$, 
\[ \sqrt{Y} = \text{diag}(\sqrt{d_1}, \ldots, \sqrt{d_g})B \]  
is the corresponding Iwasawa decomposition.

Using the above notation, recall that the Siegel subset $\mathcal{F}_g(u)$ of the Siegel upper half plane $\mathcal{H}_g$ is defined as
\[ \{X + \sqrt{-1} \in \mathcal{H}_g | |x_{ij}| < u, |1-b_{i,j}| < u, 1 < ud_i < ud_{i+1} \text{ for all } i, j\}. \]
It is known to satisfy
\[ Sp_{2g}(\mathbb{Z}) \cdot \mathcal{F}_g(u) = \mathcal{H}_g \quad \text{for } u \gg 0. \]
Let us set $\mathcal{H}_g^* := \mathcal{H}_g \sqcup \mathcal{H}_g - 1 \sqcup \mathcal{H}_g - 2 \sqcup \cdots \sqcup \mathcal{H}_0$.

Then the Satake compactification $\tilde{\mathcal{A}}_g^{\text{Sat}}$ can be defined as $\mathcal{F}_g^*(u) / \sim$ with some equivalent relation extending $Sp_{2g}(\mathbb{R})$-action on $\mathcal{H}_g$. Thanks to (5) we can suppose that, fixing sufficiently large $u_0 \gg 1$, we have $\tilde{\mathcal{A}}_g^*(u_0) / \sim$ with the same equivalent relation, where $\tilde{\mathcal{F}}_g^*(u_0) := \mathcal{F}_g(u_0) \sqcup \mathcal{F}_g - 1(u_0) \sqcup \cdots \sqcup \mathcal{F}_0(u_0) \subset \mathcal{H}_g^*$.

Hence we can assume that the whole sequence $\{V_i\}$ of principally polarize abelian varieties are parametrized by a sequence $Z_i = X_i + \sqrt{-1}Y_i$ in $\mathcal{F}_g(u_0)$ for the fixed $u_0$ which is sufficiently large. Thanks to such boundedness, appropriately passing to a subsequence, we can and do assume $X_i$ and $B_i$ converges.

As a metric space, our principarlly polarized variety which corresponds to $Z = X + \sqrt{-1}Y \in \mathcal{H}_g$ is
\[ \mathbb{C}^g / \left( \begin{array}{c} X \\ Y \end{array} \right) \mathbb{Z}^{2g} \]
are isometric to $\mathbb{R}^{2g} / \mathbb{Z}^{2g}$ with metric matrix
\[ \left( \begin{array}{c} 1 \\ X \\ Y \end{array} \right) \left( \begin{array}{cc} Y^{-1} & \cdot \\ \cdot & Y^{-1} \end{array} \right) \left( \begin{array}{c} 1 \\ X \\ Y \end{array} \right) = \left( \begin{array}{cc} Y^{-1} & Y^{-1}X \\ XY^{-1} & XY^{-1}X + Y \end{array} \right). \]
In particular, if $X = (0)$ (zero matrix), then the metric matrix of our torus $\mathbb{R}^{2g} / \mathbb{Z}^{2g}$ is
\[ \left( \begin{array}{c} Y^{-1} \\ \cdot \end{array} \right). \]

Thus what we would like to classify are possible Gromov-Hausdorff limits of
\[ \left( \begin{array}{cc} Y_i^{-1} & \cdot \\ \cdot & X_i Y_i^{-1} \end{array} \right) / \text{diam}(V_i). \]
Replacing \( \text{diam}(V_i) \) by \( d_g(V_i) \), let us first classify possible limits of

\[
\begin{pmatrix}
Y_i^{-1} & Y_i^{-1}X_i \\
X_iY_i^{-1} & X_iY_i^{-1}X_i + Y_i
\end{pmatrix} / d_g(V_i)
\]

instead. Our assumption that it converges to the cusp \( A_0 \in \partial \tilde{A}_g \) (“maximally degenerating”) is equivalent to that \( d_1(V_i) \to +\infty \) when \( i \to +\infty \) from the definition of Satake topology (cf., [Sat], [Chai]).

Now, let us set

\[ r := \max \{ (1 \leq j \leq g) \mid \liminf_{i \to +\infty} \frac{d_j(V_i)}{d_g(V_i)} = 0 \} \]

Then, after appropriately passing to a subsequence again, we can assume that \( \frac{d_r(V_i)}{d_g(V_i)} \to +0 \) so that \( \frac{d_j(V_i)}{d_g(V_i)} \to +0 \) for all \( j \leq r \).

Note that yet another process of replacing by a subsequence will let all the sequences \( \left\{ \frac{d_j+1(V_i)}{d_g(V_i)} \right\}_i \) converge for \( 1 \leq j \leq g - r \). We denote that convergence values as \( a_{r+j} \). We prove that then \( \{(V_i, \frac{d_{K\mathbb{F}}(V_i)}{d_g(V_i)})\}_i \) converges to a \((g - r)\)-dimensional torus.

As \( d_g(V_i) \to +\infty \), it follows that \( Y_i^{-1}/d_g(V_i) \to +0 \). On the other hand, from the definition of our \( r \) and the conditions which are certified due to the process of passing to a subsequence, the following holds

\[
\begin{pmatrix}
d_1(V_i) \\
d_2(V_i) & d_3(V_i) & 0 \\
0 & \ddots & \ddots \\
0 & \ddots & \ddots & d_g(V_i)
\end{pmatrix} / d_g(V_i)
\]

\[
\downarrow
\]

\[
\begin{pmatrix}
0 & \ddots & \ddots & 0 \\
\ddots & 0 & \ddots & \ddots \\
0 & \ddots & \ddots & a_{g+1} \\
0 & \ddots & \ddots & a_g = 1
\end{pmatrix}
\]

when \( i \to +\infty \). Please note that the downarrow between the above big matrices “\( \downarrow \)” means convergence. From the form of the second (limit) matrix above, it follows straightforward that \( \mathbb{R}^{2g}/\mathbb{Z}^{2g} \) with the normalized metric matrix

\[
\begin{pmatrix}
Y_i^{-1} & Y_i^{-1}X_i \\
X_iY_i^{-1} & X_iY_i^{-1}X_i + Y_i
\end{pmatrix} / d_g(V_i)
\]
converges to a \((g - r)\)-dimensional torus. From this result, we also observe the following.

**Claim 3.2.** In the above setting, we have \(d_g(V_i) \sim d_{KE}(V_i)\) i.e., the ratio of the left hand side and the right hand side is bounded when \(i\) (or \(V_i\) to all principally polarized abelian varieties) runs through.

Going back to proof of Theorem 3.1, now we would like to show the other direction i.e., to show that every \((g - r)\)-dimensional tori with \(0 \leq r \leq g\) of diameter 1 can appear as the above Gromov-Hausdorff limit. Indeed, we can construct a sequence in the following way. Fix \((a_{r+1}, \ldots, a_g) \in \mathbb{R}_{>0}^{g-r}\). Then set a sequence of diagonal matrices \(\{D_i\}_i\) as

\[d_j(V_i) := (u_0 + 1)^{j-1}\text{ for } j \leq r \text{ and } d_j(V_i) := (u_0 + 1)^r a_j\text{ for } j \geq r + 1.\]

If we fix \(X\) and \(B\), i.e., setting \(Z_i = X_i + \sqrt{-1} Y_i\) with constant \(X_i\) and \(B_i\), we get a \((g - r)\)-dimensional torus as a Gromov-Hausdorff limit. Letting \(B\) runs over all upper triangular matrices while keeping \(X\) to be constantly 0 (zero matrix), we get all \((g - r)\)-dimensional tori as such Gromov-Hausdorff limits. \(\square\)

Please note that \(r < g\) can really happen but please do not get confused with the conjectures [KS, Conjecture 1, p.19], [Gross, Conjecture 5.4] for the Strominger-Yau-Zaslow mirror symmetry on Calabi-Yau varieties as in that algebraic setting, the collapse only to just half dimensional affine manifolds (with singularities), i.e. \(i = g\) case are expected. This difference occurred since because we take an arbitrary sequence rather than dealing with proper algebraic family with maximal monodromy as they do.

For the general case, we prove the following.

**Theorem 3.3.** We use the same notation as Theorem 3.1. Suppose a sequence of \(g\)-dimensional principally polarized complex abelian varieties \(\{V_i\}_{i \geq 1}\) converges to a point of \(A_c \subset \partial \bar{A}_g\) with \(0 \leq c < g\) in the Satake compactification \(\bar{A}_g\). Then, after passing to a subsequence, \((V_i, \frac{d_{KE}(V_i)}{\text{diam}(d_{KE}(V_i))})\) converges to a \((g - r)\)-dimensional (flat) tori of diameter 1 with some \(c \leq r(< g)\), in the Gromov-Hausdorff sense. Conversely, any such flat \((g - r)\)-dimensional torus of diameter 1 with \(c \leq i \leq g\) can appear as a possible Gromov-Hausdorff limit of such sequence of \(g\)-dimensional principally polarized complex abelian varieties with diameter 1 rescaled Kähler-Einstein (flat) metrics.
Before going to the proof, let us analyse what the above particularly means. Note that the set of possible limits get smaller compared with the maximal degeneration case. This reflects the fact that “degeneration / deformation” order get reversed once we pass from algebro-geometric setting to its tropical analogue. Indeed, similar phenomenon happened in curve case \(2.4\) as the limit could be a contraction of the dual graph of the “corresponding” limit in the Deligne-Mumford compactification.

Another but related way of explaining the phenomenon is via the following quite simple observation that our Gromov-Hausdorff limit sees just “degenerating part” and ignores non-degenerating part.

**Proposition 3.4.** Suppose a sequence of compact metric spaces \(\{X^{(i)}\}_{i \in \mathbb{Z}_{>0}}\) decomposes as
\[
X^{(i)}_1 \times \cdots \times X^{(i)}_m
\]
as metric spaces with \(p\)-product metric for some \(p > 0\). If the last component \(X^{(i)}_m\) is “responsible of degeneration” in the sense that
\[
(i) \quad \text{diam}(X^{(i)}_m) \to +\infty \quad \text{and}
(ii) \quad \text{diam}(X^{(i)}_j) \leq \text{constant for all } j \neq m,
\]
then the Gromov-Hausdorff limit “only sees \(X^{(i)}_m\)” in the sense that
\[
\lim_{i \to +\infty} \left( X^{(i)}/\text{diam}(X^{(i)}) \right) = \lim_{i \to +\infty} \left( X^{(i)}_m/\text{diam}(X^{(i)}_m) \right).
\]
A trivial remark is that the statement of the above proposition is just equivalent to \(m = 2\) case but we stated as above just to get a better intuition for various applications.

**Proof.** The whole point is simply that there is a constant \(c\) which satisfies the inequality
\[
\text{diam}(X^{(i)}_m) \leq \text{diam}(X^{(i)}) \leq \text{diam}(X^{(i)}_m) + c
\]
for all \(i\). The assertion easily follows from the above. \(\Box\)

Thus indeed if a punctured family of abelian varieties with semiabelian reduction with torus rank \((g - r)\) of the central fiber, it follows that the torus part determines the Gromov-Hausdorff limit (with fixed diameters). Theorem 3.3 is reflecting that fact.

Let us now turn to the proof, i.e. the classification of our Gromov-Hausdorff limits of principally polarized abelian varieties.

**Proof of Theorem 3.3.** As the proof is very similar to the proof of maximal degeneration case \((3.1)\), we only sketch the proof with focus on the differences. As in \((3.1)\), thanks to the Siegel reduction theory, we
can and do fix sufficiently large \( u_0 \gg 0 \) so that our sequence can be parametrized by a sequence
\[
\{ Z_i = X_i + \sqrt{-1} Y_i \}_{i \geq 1}
\]
in the Siegel set \( \mathfrak{g}_g(u_0) \) (cf., the definition \( \mathfrak{g} \)). Again in the same manner, we can and do appropriately take a subsequence so that the following conditions hold.

(i) \( X_i \) converges when \( i \to +\infty \),
(ii) the upper triangle matrix part \( B(V_i) \) converges when \( i \to +\infty \),
(iii) \( d_j(V_i) \) for any \( (1 \leq j \leq c) \) converges when \( i \to +\infty \),
(iv) \( d_{c+j}(V_i) \to +\infty \) when \( i \to +\infty \) for any \( (1 \leq j \leq (g - c)) \).

Here, the notations are same as before. Let us set again
\[
r := \max \left\{ (1 \leq j \leq g) \mid \liminf_{i \to +\infty} \frac{d_j(V_i)}{d_g(V_i)} = 0 \right\}.
\]

Then in our general case, we have \( c \leq r \leq g \) from the definition of the Satake topology \[Sat\]. The rest of the proof that \( V_i \) converges to a \((g - r)\)-dimensional flat torus with diameter 1 is completely the same.

Conversely, for a given \( r \leq c \), let us prove that any \((g - r)\)-dimensional torus with diameter 1 can appear as the above limit. Consider \( a_1, \ldots, a_g \in \mathbb{R}_{>0} \) which satisfies \( a_1 \geq 1, a_{j+1} \leq u_0 a_j \) for all \( j \leq 1 \). Then if we define \( V_i(B, \{a_j\}) \) as
\[
d_j(V_i(B, \{a_j\}) := a_j \text{ for each } j \leq r
\]
\[
d_j(V_i(B, \{a_j\}) := ia_j \text{ for each } j \geq r + 1
\]
as before, it is straightforward to see that any \((g - r)\)-dimensional torus of diameter 1 can be obtained as a Gromov-Hausdorff limit of a sequence of the form \( \{ (V_i(B), \{a_j\}, \frac{d_{KE}(V_i(B, \{a_j\}))}{d_g(V_i(B), \{a_j\})} ) \}_{i} \).

The above proofs again show that, for an arbitrary sequence of principally polarized abelian varieties, passing to an appropriate nice subsequence, then we have a compatibility of the limit in the algebro-geometric compactifications by \[Ale1, Nak1, Nak2\] and the Gromov-Hausdorff limit (while fixing diameters). It is possible to formulate the analogues of limiting maps for \( M_g \) case \( r (1), t (2) \) and regard the compatibility of those two as that of these two maps and simultaneously indicates a possible extension of approach via Berkovich analytification \[ACP\] for abelian varieties case. We omit the details since we have not obtained more substantial result in this direction at this moment. The author wishes to come back to this problem of better formulation in future.
3.2. The construction of $\tilde{A}_g^T$ and comparison with other tropical moduli. Similarly as in the curves case, we define our tropical geometric compactification of the moduli space of principally polarized abelian varieties first set-theoretically as

$$\tilde{A}_g^T := A_g \sqcup \partial A_g^{dGH}.$$  

Then we put a topology on it whose open basis can be taken as those of $A_g$ and metric balls around point $[T]$ in $\partial A_g^{tr}$

$$B([T], r) := \{ [X] \in \tilde{A}_g^T \mid d_{GH}([X], [T]) < r \},$$

where $d_{GH}$ denotes, as in the previous section, the Gromov-Hausdorff distance with respect to the rescaled metric on each torus whose diameter is 1. Compactness and Hausdorff property follow from the Gromov’s precompactness theorem [Grom] and the definition of Gromov-Hausdorff convergence respectively.

Note that if we forget complex structures of principally polarized abelian varieties, it gives nontrivial morphism to a moduli (metrized) space of the same dimension.

The discreteness of the fibres of the forgetful morphism follows from the fact that, adding marking $[\pi_1(X) \xrightarrow{\approx} \mathbb{Z}^2]$], which is obviously discrete data, recovers the complex structure. It easily follows from the fact that the metric matrix (6) has enough information to recover $X$ and $Y$.

Now, let us discuss the relation with the moduli space $A_g^{tr}$ of tropical abelian varieties constructed in [BMV]. We follow their notations. From the above, it holds that the boundary of our tropical geometric compactification $\tilde{A}_g^T$ is

$$\partial \tilde{A}_g^T \cong A_g^{tr}/\mathbb{R}_{>0} = (\Omega^{tr} \setminus \{0\})/(GL(g, \mathbb{Z}) \cdot \mathbb{R}_{>0}),$$

where $A_g^{tr}$ is the moduli space of $g$-dimensional tropical (principally polarized) abelian varieties $\mathbb{R}^g/\Lambda$ in the sense of [BMV], $\Omega^{rt}$ (resp., $\Omega$) is the cone of positive semidefinite forms (resp., positive definite forms) on the universal covering $\mathbb{R}^g$ whose null space has basis inside the rational vector space $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$, following their notations. Note $\Omega \subset \Omega^{rt} \subset \Omega^t$.

Remark 3.5. We make a simple observation on the relation with the dual (intersection) complex (cf., [KS], [Gross]). In the conjectures of Kontsevich-Soibelman [KS] and Gross-Siebert (cf., [Gross]) for their approach to the Strominger-Yau-Zaslow conjecture [SYZ], they also predict with partial proofs indeed that given a maximal degeneration of general Calabi-Yau manifolds, the dual complex of the special fiber is

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5For example, at the locus which parametrizes products of $g$ elliptic curves, it gives generally $2^g$ to 1 morphism.
“close to” the Gromov-Hausdorff limit of Ricci-flat metrics with fixed diameters.

Suppose we have a semi-abelian reduction of abelian schemes and its relative compactification of Alexeev-Nakamura type ([Ale1], [AN], [Nak1], [Nak2]). Due to [AN, (3.17)], [Ale1], [Nak1, (4.9)], the dual complexes are the duals of Delaunay triangulations of \((g-r)\)-dimensional tori which are topologically of course tori of \((g-r)\)-dimension. This matches to our Theorem 3.3 except for a slight difference that the tori can get lower dimension as we take an arbitrary sequence.

3.3. (Co)homology - abelian varieties case. About the open dense locus \(A_g\), the following has been classically known as a result of A.Borel who proved via vector spaces of \(Sp_{2g}(\mathbb{R})\)-invariant different forms and the group cohomology interpretation that

\[ H^i(A_g; \mathbb{Q}) = H^i(Sp_{2g}(\mathbb{Z}); \mathbb{Q}). \]

**Theorem 3.6 ([Bor])**. \( H^i(A_g; \mathbb{Q}) = \mathbb{Q}[x_2, x_6, x_{10}, ...] |_{\text{degree}=i} \) for \(0 \leq i < g-1\), where the right hand side is a polynomial generated by \(x_{4a+2}\) whose weight is \(4a+2\). In particular, \( H^i(A_g; \mathbb{Q}) = 0 \) if \(i\) is odd, less than \(g-1\) and the stable cohomology is naturally \( \lim_{\rightarrow g} H^*(A_g) = \mathbb{Q}[x_2, x_6, x_{10}, ...] \).

There are also many studies on the homology of symplectic groups such as [Char], [MV] etc. Using such topological results on \(A_g\), at least partially the study of (co)homologies of the boundary \(\partial \bar{A}_g^{dGH}\) gives some informations on those of \(\bar{A}_g^T\). For instance, a simple observation is that \(\dim(\partial \bar{A}_g^{dGH}) = 3g - 4\) combined with the long exact sequence of the Borel-Moore homology groups gives that \(H_i(\bar{A}_g^T; \mathbb{Q}) = 0\) for if \(i\) is even and \(i > g^2\).

Motivated partially from the above discussion, from now on, let us study the boundary \(\partial \bar{A}_g^{dGH}\) which we denote as \(T_g\) for simplicity from now on. Note that \(T_g\) has the following orbifold as an open dense locus

\[ \Omega/(\mathbb{R}_{>0} \cdot GL(g, \mathbb{Z})), \]

which we will write \(T_g^o\). Then

\[ \partial \bar{A}_g^{dGH} = T_g^o \sqcup \partial A_{g-1}^{dGH}, \]

so that we can partially study the (co)homology of \(\partial \bar{A}_g^{dGH}\) inductively, once we know those of \(T_g = \Omega/(\mathbb{R}_{>0} \cdot GL(g, \mathbb{Z}))\). However, the author
does not know well how this cohomology behaves except for the asymptotic behaviour of the lower degree due to A. Borel [Bor], that is
\[ H^i(T_g; \mathbb{Q}) = H^i(GL(g; \mathbb{Z}); \mathbb{Q}) = \mathbb{Q}[x_3, x_5, x_7, \ldots] \mid \text{deg} = i, \]
for \( i \leq (g - 5)/4. \)

As in the discussion in the previous section for curve case, we have canonical closed embeddings

\[ T_g \hookrightarrow T_{g+1} \hookrightarrow \ldots \]

which is analogous to the boundary structure of the Satake compatification.

We have the following asymptotic triviality of the topologies, analogous to that of curves case (2.15).

**Proposition 3.7.** The (infinite dimensional) topological space \( T_\infty \) is contractible. \( \text{Im}(H_k(T_g; \mathbb{Q}) \to H_k(T_{g+1}; \mathbb{Q})) = 0 \) for any \( k \) and \( g \).

**Proof.** We imitate the idea of curve case (2.15) but in this abelian varieties case, it is easier. However, the whole point is still the same, that is to construct an extension \( \psi_g: CT_g \to T_\infty \) of the identity map of \( T_g \) where \( CT_g := (T_g \times [0, 1])/(T_g \times \{1\}) \), which is compatible with lower \( \psi \) i.e., \( \psi_g|_{T_{g-1}} = \psi_{g-1} \).

For \( ((X, d_X), t) \in T_g \times [0, 1] \) (\( d_X \) denotes the flat metric on \( X \)), we define
\[ \psi_g(X, t) := \text{rescale of } ((X, (1 - t)d_X) \times S^1(t)) \text{ with diameter } 1. \]

The continuity of the map is obvious. Here, the product means the 2-product metric (i.e., simply the square root of the sum of squares of direction-wise distances). It is straightforward to confirm the requirements of the map. \( \square \)

Intuitively speaking, the all \( g \)-dimensional tori continuously and simultaneously change to once \( (g + 1) \)-dimensional tori but later collapse to a circle of circumference 1.

On the other hand, we have the following exact sequence from which high nontriviality of the topologies of \( T_g \) follows.

**Proposition 3.8.** We have the following two long exact sequences.

\[ \cdots \to H_k(GL(g; \mathbb{Z}); \mathbb{Q})^* \to H^k(T_g, \mathbb{Q}) \to H^k(T_{g-1}; \mathbb{Q}) \to \cdots \]
\[ \cdots \to H_{k+1}(GL(g; \mathbb{Z}); \mathbb{Q})^* \to H^{k+1}(T_g, \mathbb{Q}) \to H^{k+1}(T_{g-1}; \mathbb{Q}) \to \cdots. \]
\( \cdots \to H_k(T_{g-1}; \mathbb{Q}) \to H_k(T_g; \mathbb{Q}) \to H^k(GL(g; \mathbb{Z}); \mathbb{Q})^* \to \cdots \)
\( \cdots \to H_{k-1}(T_{g-1}; \mathbb{Q}) \to H_{k-1}(T_g; \mathbb{Q}) \to H^{k-1}(GL(g; \mathbb{Z}); \mathbb{Q})^* \to \cdots \)

Proof. These are simply the long exact sequences of compactly supported cohomology groups and the Borel-Moore homology groups respectively, combined with Lefschetz duality for orbifold \( T_g \setminus T_{g-1} = \Omega/GL(g; \mathbb{Z}) \).

\( \square \)

3.4. Finite and infinite joins of \( A_g \). Completely similarly as for curve case (subsection 2.6), we can naturally construct joins of our tropical geometric compactifications \( \bar{A}_g^T \), thanks again to the inductive structure of the boundaries (7).

**Definition 3.9.** The finite join of our tropical geometric compactifications is defined inductively as

\[
\bar{A}_{\leq g}^T := \bar{A}_{\leq (g-1)}^T \cup_{T_{g-1}} \bar{A}_g^T.
\]

The union is obtained via two inclusion maps \( T_{g-1} \hookrightarrow T_g \) and \( T_{g-1} \hookrightarrow \bar{A}_{\leq (g-1)}^T \). We call \( \bar{A}_{\leq g}^T \) a finite join of our tropical geometric compactifications.

From the definition, we have

\[
\cdots \bar{A}_{\leq (g-1)}^T \subset \bar{A}_{\leq g}^T \cdots .
\]

Then we set

\[
\bar{A}_\infty^T := \lim_{\rightarrow g} \bar{A}_{\leq g}^T = \cup_g \bar{A}_{\leq g}^T,
\]

and call it the infinite join of our tropical geometric compactifications.

Note it is connected and all our tropical geometric compactification \( \bar{A}_g^T \) is inside this infinite join.

In particular, \( A_g \) for all \( g \) is inside this connected “big infinite dimensional moduli space”.

3.5. Torelli maps. It is natural to think how or whether the classical period map

\( t^{\text{alg}} : M_g \hookrightarrow A_g \)

extends between our two compactifications \( \bar{M}_g^T \) and \( \bar{A}_g^T \).

Let us briefly recall the recent study of Torelli problem in tropical setting by other mathematicians. After the introduction of Tropical Jacobian [MZ], [CV] and [BMV] established the existence of a continuous map

\( M_g^{tr} \to A_g^{tr} \)
and its compatibility with their “stacky fan” structure [BMV]. The Torelli property i.e., the injectivity of the above does not literally hold even in $g = 2$ case as pointed out in [MZ]. Indeed, the closure of only one of the 2-cells of $S_2$ which parametrizes those without connecting edge maps onto $T_2$. Nevertheless, they proved that it is “generically one to one” [CV], [BMV]. Let us follow this line.

Recall that for a unweighted (or weighted with zeroes) metrised graph $\Gamma$, the tropical Jacobian [MZ], [BMV] is simply $H_1(\Gamma, \mathbb{R}/\mathbb{Z})$ with the following positive definite quadratic form $Q$. It is defined as

$$Q(\sum_{e: \text{edge}} \alpha_e \cdot e) := \sum_{e} \alpha_e^2 \cdot l(e)$$

for each 1-cycle $\sum_{e: \text{edge}} \alpha_e \cdot e$ where $l(-)$ denotes the length function.

By rescaling the metric of their tropical Jacobian to make the diameter 1 and not introducing any 2-valent vertices, we can naturally associate a member of $T_g$ to each member $\Gamma$ of $S_g$, unless $\Gamma$ is a tree. If $\Gamma$ is a tree, then the tropical Jacobian of [MZ], [BMV] is just a point so that we cannot rescale to make the diameter 1. Let us denote the closed locus of $S_g$ which parametrizes trees simply by $S_g^{\text{tree}}$. Note that it is disjoint with $S_g^t$.

So we get the following continuous map

$$t^{\text{Trop}} : (S_g \setminus S_g^{\text{tree}}) \rightarrow T_g.$$

Of course, this is not injective as simply inheriting the failure of injectivity in [MZ] setting. Note that the (open) locus of $S_g$ where this map is defined parameterizes metrized dual graphs of stable curves whose generalized Jacobian is abstractly an abelian variety.

Now, it is natural to ask the following question of Namikawa-Mumford-Alexeev type (cf., [Nam1], [Ale2]) i.e., about the extension of $t^{\text{alg}} : \bar{M}_g \hookrightarrow \bar{A}_g$ to compactifications. It essentially asks the compactibility of Jacobians and tropical Jacobians. We do not know the answer yet.

**Question 3.10.** Do these two “period maps” $t^{\text{alg}}$ and $t^{\text{Trop}}$ glue to form a continuous map from $\bar{M}_g^{\mathcal{T}} \setminus S_g^{\text{tree}}$? If that would be the case, then analyze the behaviour of the map around the boundary $\partial \bar{M}_g^{\mathcal{T}}$.

### 3.6. Gromov-Hausdorff limits with other rescaling.

There are of course some other ways of rescaling the metrics of abelian varieties which could produce different (pointed) Gromov-Hausdorff limits. One of the nontrivial rescaling is via fixing the volumes while another is via fixing injectivity radius. We discuss the two rescales.
We keep using the previous notation of this section. Recall that for our sequence \( \{V_i\}_{i=1,2,\ldots} \) of principally polarized abelian varieties of \( g \)-dimension, the corresponding point in the Siegel set is denoted as \( Z_i = X_i + \sqrt{-1}Y_i \) with \( Y_i = iB_iD_iB_i \) (the Iwasawa decomposition of \( \sqrt{Y_i} \)).

Similarly as before, after passing to a subsequence, we can and do assume that for some \( 0 \leq r < g \),

(i) both \( X_i \) and \( B_i \) converge when \( i \) tends to infinity,
(ii) \( d_j(V_i) \) for all \( 1 \leq j \leq r \) converges to finite value while
(iii) \( d_j(V_i) \) for all \( j > r \) (strictly) diverges to infinity when \( i \) tends to infinity.

Here, we meant by the strict divergence, that all subsequences diverge. We assume the above three throughout the rest of present subsection.

Let us take as the simplest example we used in the proof of Theorems 3.1, 3.3.

(iv) \( X_i = 0, B_i = I_g \) (unit matrix),
(v) \( d_j(V_i) = a_j \) for all \( j \leq r \) and
(vi) \( d_j(V_i) = i \cdot a_j \) for all \( j > r \).

Here, \( a_1, \ldots, a_g \) are some real constant numbers with

\[ 1 < u_0a_0, a_i < u_0a_{i+1}. \]

Intuitively \( r \) is corresponding “torus rank” of limit. Then from the above assertions, it is easy to see that

**Proposition 3.11.** The pointed Gromov-Hausdorff limit of the rescaled Kähler-Einstein metrics on \( V_i(i \to \infty) \) with fixed injectivity radius 1 in the above notation is isometric to

\[
\prod_{1 \leq j \leq (g-r)} S^1\left(\frac{a_{g-j}}{2\pi a_1}\right) \times \mathbb{R}^{g+r},
\]

where \( S^1(a) \) denotes a circle with radius \( a \).

Note that “pointed” does not cause ambiguity in this situation, thanks to the homgeneity of abelian varieties.

As the limit above does not reflect any abelian part data (“\( a_{r+1}, \ldots, a_g \)” encoded in the boundary of the Satake compactification, we prefer the other Gromov-Hausdorff limits. Our main intention of this subsection is (still) to investigate relations with moduli compactifications.

We remove the assumptions (iv), (v), (vi) now while keep assuming (i), (ii), (iii) and analyse the corresponding volume fixed Gromov-Hausdorff limits in turn. Note that to fix the volume of \( \{V_i\} \), say as 1,
is simply resulting to the metric matrices
\[(8) \quad \begin{pmatrix} Y_i^{-1} & Y_i^{-1}X_i \\ X_iY_i^{-1} & X_iY_i^{-1}X_i + Y_i \end{pmatrix} \]
of \(\mathbb{R}^{2g}/\mathbb{Z}^{2g}\) without any normalization factor. Let \(B = \lim_{i \to \infty} B_i\) and let \(X = \lim_{i \to \infty} X_i\). We extract the \((r \times r)\) upper left part \(X'\) of \(X\) and \(Y'\) of \(Y\) as
\[
X' := \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,r} \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots & x_{2,r} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ x_{r,1} & x_{r,2} & x_{r,3} & \cdots & x_{r,r} \end{pmatrix},
\]
\[
Y' := \begin{pmatrix} y_{1,1} & y_{1,2} & y_{1,3} & \cdots & y_{1,r} \\ y_{2,1} & y_{2,2} & y_{2,3} & \cdots & y_{2,r} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ y_{r,1} & y_{r,2} & y_{r,3} & \cdots & y_{r,r} \end{pmatrix},
\]
and denote the \((r \times r)\) upper left part \(B'\) of \(B\) as
\[
B' := \begin{pmatrix} 1 & b_{1,2} & b_{1,3} & \cdots & b_{1,r} \\ 1 & b_{2,3} & \cdots & b_{2,r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.
\]
Then our metric matrices (8) converge to the following except for lower right i.e., \((*)\)-part of \((g-r) \times (g-r)\).
\[
\begin{pmatrix} F & 0 & \cdots & 0 & G & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ tG & 0 & \cdots & 0 & H & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}.
\]
Here, the submatrices \(F, G, H\) are those defined by \(X, X', Y, Y'\) as
- \((Y')^{-1} = F,\)
- \((Y')^{-1}X = G\) and
\[ X'(Y')^{-1}X' + Y' = H. \]

The corresponding \((\ast)\)-part of our metric matrix \(\mathbf{Y}\) is exactly the lower right part of \(\mathbf{Y}\) which is diverging due to the divergence of \(d_{r+j}(\mathbf{V}_i)\) \((i \to +\infty)\) for any \(j > 0\). More precisely that \((g - r) \times (g - r)\) part is positive definite with all eigenvalues strictly diverge to \(+\infty\).

The diverging part \(((g + r + j)\text{-th columns for } 1 \leq j \leq (g - r))\) yields \(\mathbb{R}^{g-r}\) and the rest of part converges to the \(2r\)-dimension real torus with the metric matrix as \((3.12)\).

Summarising up, from the discussions above, we have proved

**Proposition 3.12.** In the above setting, the pointed Gromov-Hausdorff limit of our \(\mathbf{V}_i\) with fixed volume 1 is isometric to

\[ (\mathbb{R}^{2r}/\mathbb{Z}^{2r}) \times \mathbb{R}^{g-r} \]

where the corresponding metric matrix of the first factor is

\[ \begin{pmatrix} F & G \\ iG & H \end{pmatrix}. \]

Note that the metric matrix corresponds exactly to the limit of \([\mathbf{V}_i]\) ∈ \(\mathcal{A}_g\) in the Satake compactification (cf., e.g., [Chai, 4.4]). In conclusion, we have proved that the Satake compactification \(\bar{\mathcal{A}}_g\) parametrizes the set of pointed Gromov-Hausdorff limits with fixed volumes of \(g\)-dimensional principally polarized abelian varieties. This means that the Satake compactification \(\text{Sat}\) can be differentially geometrically naturally reconstructed, i.e., in the spirit of Gromov-Hausdorff.

We conclude the article with one of the simplest examples to make the above proofs more intuitive.

**Example 3.13.** As in Ex 2.6, consider again a degenerating sequence of elliptic curves

\[ E_k := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}k(a\sqrt{-1})) \]

for \(k = 1, 2, \cdots\), while \(a > 1\) fixed. In this case, this is maximally degenerating so that the corresponding “torus rank” is \(r = 1 = g\).

The “diameter fixed” Gromov-Hausdorff limit is \(S^1(1/2\pi)\) as we observed. Instead if we fix the injectivity radius, then as the metric is standard metric of \(\mathbb{C}\) we get

\[ (\mathbb{R}/\mathbb{Z}) \times (\sqrt{-1}\mathbb{R}) \]

as the pointed Gromov-Hausdorff limit.

On the other hand, if we fix the volume of each \(E_k\), then we rescale the metric by multiplying the lengths by \(1/\sqrt{k}\). Then the pointed Gromov-Hausdorff limit is the imaginary axis

\[ (\sqrt{-1}\mathbb{R}) \subset \mathbb{C}. \]
In our Gromov-Hausdorff interpretation of the Satake compactification $\mathbb{C} \subset \mathbb{C} P^1$ discussed above [3.12], this line of infinite length is corresponding to the cusp $\{\infty\}$ while the open part $A_1 \cong \mathbb{C}$ parametrizes flat 2-dimensional tori of volume 1.

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