Fidelity and entanglement entropy in the one-dimensional transverse-field quantum compass model

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Abstract
The one-dimensional extended quantum compass model in the presence of a transverse magnetic field is considered. Induced effects of the transverse magnetic field on the ground state of the system are studied from the viewpoint of fidelity. Using the numerical Lanczos method, the fidelity and susceptibility of fidelity are computed in finite chains. The critical exponent of the fidelity susceptibility is obtained in good agreement with the scaling behavior of the correlation length. In addition, the von Neumann entropy is calculated and its signature on the quantum phase transition is shown.

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(Some figures may appear in color only in the online journal)

1. Introduction

Recently, much attention has been paid to the role of orbital degrees of freedom in some materials such as transition metal compounds [1–3]. Mott insulators [4–6] such as Sr₂IrO₄, colossal magnetoresistances [7], high-temperature superconductivities [8] and so on demonstrate the significance of the role of orbital degrees of freedom. As a matter of fact, unremoved degeneracy of the d orbitals of transition metal ions determines how orbital degrees of freedom can interact selectively [9, 10].

A good candidate for explaining the low-temperature behavior of some Mott insulators is the quantum compass model. This model was first suggested in the context of the microscopic observation of Mott insulators possessing orbit degeneracy [4]. In this model, the orbital degrees of freedom play a major role in forming different bond directions and also have different types of interaction spin components. The one-dimensional (1D) quantum compass model [11–17], known as the quantum compass chain (QCC), is exactly soluble through its mapping to the quantum Ising model [11]. It is shown that this model exhibits a first-order phase transition between two disordered phases with opposite signs of certain local spin correlations. The model is also diagonalized exactly by a direct Jordan–Wigner transformation [13]. The obtained results by latter approach confirm the existence of the first-order phase transition in the ground-state phase diagram. In very accurate works it is found that the reported first-order phase transition, in fact occurs at a multicritical point where a line of the first-order transition meets a line of the second-order transition [15, 16]. In very recent work, by using the Jordan–Wigner transformation the diagonalized Hamiltonian at the multicritical point is obtained and analytical expressions for the spin-spin correlation functions are determined [18].

The effect of a transverse magnetic field on the 1D quantum compass model is also well studied recently [19–25]. On the first-order critical line, the energy gap is opened by applying the magnetic field [19]. In recent work [20], our group studied the QCC in a particular area of the ground-state magnetic phase diagram where the coupling on odd links is antiferromagnetic and larger than the coupling on even links. We showed that the QCC in a transverse magnetic field can be mapped to a 1D Ising model in a transverse magnetic field (ITF). Analytical investigation of the effective ITF Hamiltonian predicts the occurrence of two quantum phase
transitions (QPTs) by increasing the transverse magnetic field. In addition, we provided analytical and numerical results on the effect of a transverse magnetic field in the whole range of the ground-state phase diagram (GSPD) [21]. Simultaneity, the universality and scaling properties of the transverse susceptibility and nearest-neighbor correlation functions were studied by means of the Jordan–Wigner transformation [22]. By studying of quantum correlations in the presence of a transverse magnetic field, it is shown that the quantum discord discerns the orders of the phase transitions [23]. Recently, the thermodynamic properties of the model in the presence of a transverse magnetic field have also been studied. It is found that during an adiabatic demagnetization process, the temperature drops in the vicinity of the quantum critical fields [24].

In this work we continue our studies of a 1D quantum compass model in a transverse magnetic field using the numerical analysis. Within the framework of the Lanczos method, we compute the fidelity and the fidelity susceptibility of the ground state. We have to mention that, Ke-Wei Sun and Qing-Hu Chen [19] have also investigated the fidelity and fidelity susceptibility of QCC. Their results, however, remain solely restricted to considering the ground state of the model on the first- and the second-order phase transition lines [15, 16] and do not cover the whole range of the GSPD of QCC. Here, we carryout complete study of the fidelity in the whole range of the GSPD. We also investigated the von Neumann entropy in this system and showed that the entanglement entropy can be used to recognize different phases in the topic of quantum phase transitions.

This paper is organized as follows. In section 2, we introduce the QCC in the presence of a magnetic field and briefly discuss the application of fermionization technique on the model. In section 3, we present our numerical results, which include the fidelity and fidelity susceptibility functions within different regions of the GSPD. the von Neumann entropy is discussed in section 4. Finally, a summary is presented in section 5.

2. The model

The Hamiltonian of the QCC in the presence of a transverse magnetic field with periodic boundary conditions is given by [21, 22]

$$\mathcal{H} = \sum_{n=1}^{N/2} J_1 S_{2n-1}^z S_{2n}^z + J_2 S_{2n-1}^x S_{2n}^x + S_{2n-1}^y S_{2n}^y + L_1 S_{2n-1}^y S_{2n}^x + L_2 S_{2n-1}^x S_{2n}^y - \hbar \sum_{n=1}^{N} S_n^y,$$

where $S_n$ denotes the spin-1/2 operator on site $n$, $N$ is the number of spins and $J_1$, $J_2$, $L_1$ are the exchange couplings. It has been demonstrated that in the GSPD of the QCC (the case $\hbar = 0$), there are four regions based on exchange measures [15, 16]:

(I) $J_1/L_1 < 0, J_2/L_1 > 1$ with a hidden order where most of the two-site spin correlations vanish,

(II) $J_1/L_1 > 0, J_2/L_1 > 1$ with a hidden order,

(III) $J_1/L_1 < 0, J_2/L_1 < 1$ with the stripe antiferromagnetic (SAF) order and

(IV) $J_1/L_1 > 0, J_2/L_1 < 1$ with the Néel order.

In following, we briefly discuss the application of the fermionization technique on the model, which is described in detail in [21]. By Introducing two kinds of spinless fermion through the following Jordan–Wigner transformation,

$$S_{2n-1}^+ = a_n^e e^{i \sum_{m=1}^{n-1} (a_m b_m^+ + b_m^* a_m^+)}$$

$$S_{2n}^+ = b_n^e e^{i \sum_{m=1}^{n} (a_m b_m^* + b_m^* a_m^+)}$$

$$S_{2n-1}^+ = a_n^e a_n - \frac{i}{2}, \quad S_{2n}^+ = b_n^e b_n - \frac{i}{2},$$

the model is mapped to a 1D non-interacting spinless fermion system:

$$\mathcal{H}_f = \left( J_1 - \frac{J_2}{4} \right) \sum_{n=1}^{N/2} [a_n^e b_n^e - a_n^e b_n^e]$$

$$+ \left( J_1 + \frac{J_2}{4} \sum_{n=1}^{N/2} [a_n^e b_n^e - a_n^e b_n^e]$$

$$+ \frac{L_1}{4} \sum_{n=1}^{N/2} [b_n^e a_{n+1}^+ - b_n^e a_{n+1}^+ - b_n^e a_{n+1}^+ + b_n^e a_{n+1}^+]$$

$$- \hbar \sum_{n=1}^{N/2} \left[ \left( a_n^e a_{n+1} - \frac{1}{2} \right) + \left( b_n^e b_{n+1} - \frac{1}{2} \right) \right].$$

Then by means of Fourier transformations $a_n^e = \frac{1}{\sqrt{N/2}} \sum_k a_k^e \exp(i k n)$, the Hamiltonian is written as

$$\mathcal{H}_{MF} = \left( J_1 - \frac{J_2}{4} \right) \sum_{k} [a_k^e b_{-k}^e + a_k^e b_{-k}^e]$$

$$+ \sum_k \left( J_1 + \frac{J_2}{4} + L_1 \frac{e^{-ik}}{4} \right) [a_k^e b_k^e]$$

$$+ \sum_k \left( J_1 + \frac{J_2}{4} + L_1 \frac{e^{ik}}{4} \right) [b_{-k}^e a_k^e]$$

$$+ \left( \frac{L_1}{4} \right) \sum_k [e^{ik} a_k^e b_{-k}^e + e^{-ik} a_{-k}^e b_k^e]$$

$$- \hbar \sum_k (a_k^e a_k^e + b_k^e b_k^e).$$

Finally, by diagonalizing the Hamiltonian (equation (4)), critical transverse fields read as

$$h_{c_1} = \frac{1}{2} \sqrt{J_1 (L_1 + J_2)}$$

$$h_{c_2} = \frac{1}{4} \sqrt{J_1 (L_1 - J_2)}.$$
Fidelity and fidelity susceptibility

In this section we present results of our numerical experiment. We have used the Lanczos technique for simulation. One of the most accurate numerical methods for studying the zero-temperature behavior of low-dimensional spin systems is the Lanczos method, which is used to diagonalize exactly quantum compass chains with lengths up to $N = 24$ and periodic boundary conditions for different values of the exchanges and transverse fields.

3.1. Fidelity

Here, we are going to look at the fidelity function and try to establish a bridge between QPTs and fidelity in considerable detail through the QCC. As we know, a QPT identifies any point of non-analyticity in the ground state of an infinite lattice system [26]. Conventionally, local order parameters are needed to detect the non-analyticity in the ground-state properties as the system varies across the quantum critical point (QCP). However, the knowledge of the local order parameter is not easy to retrieve from a general many-body system. Equation (1) can be written as the Hamiltonian:

$$
\mathcal{H} = \mathcal{H}_0 + \lambda \mathcal{H}_\lambda,
$$

(6)

where $\lambda$ is a variable that typically parameterizes an interaction and exhibits a phase transition at some critical value, $\lambda_c$. In this form $\mathcal{H}_\lambda$ is then recognized as a term that drives the phase transition [27]. Recently, quantum fidelity has been widely used to study the QCP [28–35]. The fidelity is given by the modulus of the overlap [30] of normalized ground-state wave functions $|\psi(\lambda)\rangle$ with $|\psi(\lambda + \delta\lambda)\rangle$ for closely spaced Hamiltonian parameter $\lambda$ and $\lambda + \delta\lambda$, i.e.

$$
F(\lambda, \lambda + \delta\lambda) = \langle |\psi(\lambda)\rangle |\psi(\lambda + \delta\lambda)\rangle.
$$

(7)

Here we have selected the magnetic field ($h$) as a space Hamiltonian parameter. Figure 2 shows the ground-state fidelity of the QCC as a function of $h$. The presented numerical results in figure 2 cover all four regions and regard chains with lengths $N = 8, 12, 16, 20, 24$. As shown in figure 2, in region (I) of the GSPD, there is a size effect with the coefficients of the ground-state eigenvector in the absence of the transverse magnetic field. Increasing the magnetic field, this size effect vanishes in a way that overlapping of two different neighbor ground states will increase as the value of fidelity is close to one in large enough magnetic fields. In contrast, the results of fidelity’s calculation of region (II) shown in figure 2(b) indicate that in this region of the GSPD and in small magnetic fields ($h \to 0$) there is a no size effect. It seems that the sharp drops of fidelity can be described with a dramatic change in the structure of the ground state of the system during the QPTs. Away from these points, the fidelity almost equals unity; in other words, the ground states overlap each other completely. In figures 2(c) and (d) we have plotted the fidelity function related to regions (III) and (IV) respectively. In each of these figures, one can obviously recognize an abrupt drop for the 3rd and 4th regions of the GSPD correspondingly. As shown in the three last figures, the greater the number of particles, the sharper the drops at the critical fields. In fact, at the critical points since the ground-state eigenvector was replaced with another orthogonal quantum state eigenvector, the fidelity between two different ground states should be zero in the thermodynamic limit. By applying the extrapolation technique on our numerical results, we found that the minimum value of fidelity in the thermodynamic limit will be zero as expected. Therefore, it is concluded that the fidelity can be used to find the quantum critical points in the topic of quantum phase transition. However, how close to the really critical fields from the finite size scaling is assigned to the next section.

3.2. Fidelity susceptibility

As previously mentioned, fidelity has been confirmed to be so fruitful a device to study QPTs in condensed matter physics. In order to remove the artificial variation of external parameters, the concept of fidelity susceptibility is introduced [27]. Using equation (7), a series expansion of the ground-state fidelity can then be written as:

$$
F(\lambda) \equiv \chi_F = \frac{1}{2} \frac{\partial^2 F}{\partial \lambda^2}.
$$

(8)

where $\frac{\partial^2 F}{\partial \lambda^2} \equiv \chi_F$ is called the fidelity susceptibility. If the higher-order terms are taken to be negligibly small, then the fidelity susceptibility is defined as [27]

$$
\chi_F = \frac{2(1 - F(\lambda))}{\delta\lambda^2} \equiv \lim_{\delta\lambda \to 0} \frac{-2 \ln F}{(\delta\lambda)^2}.
$$

(9)

Figures 3(a), 4(a) and (b) show the numerical results of the per site fidelity susceptibility in the 2nd, 3rd and 4th regions of the GSPD, respectively. It can be seen now the averaged
Figure 2. Fidelity as a function of transverse magnetic field $h$, for different chain lengths $N = 8, 12, 16, 20, 24$ and exchanges $L_1 = 1.0$, (a) $J_1 = -3.0, J_2 = 3.0$, (b) $J_1 = 3.0, J_2 = 3.0$, (c) $J_1 = -3.0, J_2 = 0.5$ and (d) $J_1 = 3.0, J_2 = 0.5$.

Figure 3. (a) Fidelity susceptibility as a function of transverse magnetic field $h$ for chain lengths $N = 8, 12, 16, 20, 24$ and exchanges $L_1 = 1.0, J_1 = 3.0, J_2 = 3.0$ for the 2nd region of the GSPD ($h_{c1}^T$ and $h_{c2}^T$ are the theoretical results for critical fields obtained from exact solution [21]), (b) $\ln \chi_{max}^F$ versus $\ln N$ for both of its critical points. (c), (d) Best polynomial fitting of $h^*$ versus $1/N$ respectively for the first critical point and the second one.
fidelity susceptibility for different $N$ peaks at pseudo-critical field, $h^*$, in which $\chi_F/N$ becomes more pronounced. We can deduce that at the pseudo-critical point, $\chi_F/N$ is an extensive quantity. As the scaling relation [36–38] $\chi_F(h^*) \sim N^{\nu}$ governs at the pseudo-critical point, the maximum value of the fidelity susceptibility, $\chi_F^{\text{max}}$, depends on $N^{2/\nu}$ linearly. The critical exponent, $\nu$ is called the correlation length critical exponent. Figure 3(b), in which $\ln \chi_F^{\text{max}}$ is plotted versus $\ln N$, vouches for this linearly scaling relation at pseudo-critical points, and the slope of this plot corresponds to $2/\nu$. The results related to calculation of the correlation length exponent in various regions of the GSPD are shown in the table 1. Our numerical results are in complete agreement with the analytical results [22] and confirm that all quantum phase transitions in the QCM model take place in the ITF universality class. One should note that the small deviation from the exact value ‘one’ is related to the finite size effects.

On the other hand, Zanardi et al [39] claimed that the difference between the position of the pseudo-critical point ($h^*$) and the real critical point ($h_c$) is inversely related to the number of particles in a finite chain with a relation such as $|h^* - h_c| \sim N^{-1/\nu}$. By increasing the size of the chains, the position of the pseudo-critical point ($h^*$) tends to the real critical point. As shown in figures 3(c) and (d), related to the 2nd region of the GSPD of QCC, the peak positions of the pseudo critical field, $h^*$, are plotted versus $(1/N)^{1/\nu}$. Also, the insets of figures 4(a) and (b) are devoted to the best polynomial fitting of data for the 3rd and 4th regions of the GSPD. These fittings are listed in table 2.

Our results show that $[h^* - h_c] \sim N^{-1/\nu}$ should be substituted with $[h^* - h_c] \approx c_1 N^{-1/\nu} + c_2 N^{-2/\nu} + \cdots$ in small-sized systems, where $c_i$ are suitable coefficients. This result independently confirms the conclusion in [4]. A careful glance at the relations of $h^*$ shows us to augur mounts of

**Figure 4.** Fidelity susceptibility as a function of transverse magnetic field $h$ for chain lengths $N = 8, 12, 16, 20, 24$, and their insets indicating the best polynomial fitting of $h^* s$ versus $1/N$ for exchanges (a) $J_1 = -3.0, J_2 = 0.5$ related to the 3rd region of the GSPD, (b) $J_1 = 3.0, J_2 = 0.5$ related to the 4th region of the GSPD, (c) $\ln \chi_F^{\text{max}}$ versus $\ln N$ for the 3rd region and (d) $\ln \chi_F^{\text{max}}$ versus $\ln N$ for the 4th region.

**Table 1.** Critical exponent of the correlation length in different regions of the GSPD.

| Region of the GSPD | Correlation length exponent ($\nu$) |
|-------------------|-----------------------------------|
| II the 1st critical point | $1.11 \pm 0.01$ |
| the 2nd critical point | $1.01 \pm 0.01$ |
| III | $1.03 \pm 0.01$ |
| IV | $0.99 \pm 0.01$ |

**Table 2.** Functions of the pseudo-critical fields versus $N^{1/\nu}$ for the different regions of the GSPD.

| Region of the GSPD | Position of pseudo critical fields ($h^*$) |
|-------------------|---------------------------------------------|
| II the 1st critical point | $h_1^* \approx 1.202 + 0.241 N^{-1/\nu} + 5.029 N^{-2/\nu} - 8.473 N^{-3/\nu}$ |
| the 2nd critical point | $h_2^* \approx 1.714 - 0.587 N^{-1/\nu} - 11.827 N^{-2/\nu} + 35.482 N^{-3/\nu}$ |
| III | $h^* \approx 0.641 - 1.151 N^{-1/\nu} + 12.245 N^{-2/\nu} - 48.063 N^{-3/\nu}$ |
| IV | $h^* \approx 1.096 - 1.888 N^{-1/\nu} - 24.322 N^{-2/\nu} - 141.820 N^{-3/\nu}$ |
with the $\nu$ of the GSPD in the thermodynamic limit. Comparing the fidelity susceptibility $\chi$ for the second critical point $\nu_2$ for the second critical field in this area (figure 5(a)), the value of $\nu$ has been set to 1.11 and for the second critical field in this area (figure 5(b)), the value of $\nu$ is set to 1.01. On the other hand, to check the scaling function and to validate previous results for the 3rd and 4th regions, the data collapse is depicted in figures 6(a) and (b).

For the 3rd area the value of $\nu$ was set to 1.03, and for the 4th area the value of $\nu$ was set to 0.99.

4. Entanglement entropy

QPTs take place when controlling parameter changes across the critical point, and some properties of the many-body system will change dramatically [26]. Therefore, since the exotic magnetic behaviors at the QPTs being observed, many scientists are interested in researching phenomena that exhibit QPTs. During the past years, some important concepts in quantum information theory have been introduced to characterize QPTs [41]. For instance, entanglement, which is one of the main concepts in quantum information theory, can offer a useful signature for some QPTs [41]. Several measures for the entanglement have been used in the literature in order to investigate states of matter [41, 42]. Among them, the von Neumann entanglement entropy (EE) quantifies the bipartite entanglement between two parts of a quantum mechanical system [43].

Entanglement entropy quantifies the entanglement between a block of $L$ contiguous spins and the rest of the chain (ROC) and is defined as

$$E^{\nu \mathcal{N}} = -\langle \log \hat{\rho}_\Lambda \rangle = -\text{Tr}_\Lambda [\hat{\rho}_\Lambda \log \hat{\rho}_\Lambda],$$

where $\hat{\rho}_\Lambda = \text{Tr}_B [\hat{\rho}_\Lambda]$, and $\hat{\rho}$ represents the density matrix of the ground state. It is assumed that the system consists of subsystems A and B. The entanglement entropy (EE) quantifies the information describing the entanglement between the subsystems A and B. For QCC in which a unit cell consists of two particles, entanglement between various multi-spin blocks and the ROC can be studied. One-particle, two-particle and three-particle blocks with ROC. This plot demonstrates that in the absence of a magnetic field both selected blocks are entangled with the ROC. Despite the different amounts, EE has a similar descending behavior for different blocks. In addition, because of the existence of different exchange couplings in the odd and the even links, it is expected that various two-particle blocks in different links have different EE behaviors.

For the 1st region of the GSPD, figures 7(a) and (b) depict the behavior of EE between selected blocks and ROC. Figure 7(a) presents the EE treatment between one-particle and three-particle blocks with ROC. This plot demonstrates that in the absence of a magnetic field both selected blocks are entangled with the ROC. Despite the different amounts, EE has a similar descending behavior for different blocks. Furthermore, figure 7(b) depicts the EE behavior of a two-particle block placed at an even link is completely entangled with the ROC with respect to the two-particle block located on an odd link. In other words, a pair with a strong coupling does not perceive the presence of the rest of the system. This indicates that the value of entanglement of a two-particle block with the ROC is more dependent on the power of exchange couplings connecting the block with the ROC in good agreement with the area law

![Figure 5](image1.png)

**Figure 5.** Data collapse for the $\chi_{\nu}/L^{2/\nu}$ for different chain lengths $N = 8, 12, 16, 20, 24$ and exchanges $L_1 = 1.0$, $J_1 = 3.0$, $J_2 = 3.0$, 2nd region of the GSPD (a) for the first critical point $\nu_1 = 1.11$ and (b) for the second critical point $\nu_2 = 1.01$.

![Figure 6](image2.png)

**Figure 6.** Data collapse for $\chi_{\nu}/L^{2/\nu}$ for different chain lengths $N = 8, 12, 16, 20, 24$ and (a) exchanges $L_1 = 1.0$, $J_1 = 3.0$, $J_2 = 0.5$, the 3rd region of the GSPD $\nu_c = 1.03$ and (b) exchanges $L_1 = 1.0$, $J_1 = 3.0$, $J_2 = 0.5$, the 4th region of the GSPD $\nu_c = 0.99$.

exact critical fields. Comparing the $h^*$ in table 2 with the relation $|h^*-h| \approx c_1 N^{-1/d}\nu + c_2 N^{-2/d}\nu + \cdots$, we obtained $h_a = 1.202$ and $h_b = 1.714$ for the 2nd region of the GSPD, and $h_a = 0.641$ and $h_a = 1.096$ for the 3rd and 4th regions of the GSPD in the thermodynamic limit. Comparing the numerical values of critical points with equation (5) shows very good agreement with the exact analytical results.

In checking the validity of obtained critical exponents, one way is investigating the finite size scaling behavior of fidelity susceptibility. For this reason, we follow a scaling technique in which all graphs collapse on each other. The scaling technique based on the divergence of fidelity susceptibility close to the critical points (figure 3) determines that the behavior of $\chi_{\nu}$ in the vicinity of a quantum critical point for a finite system can be defined as $\chi_{\nu} = L^{2/\nu}/f_{\nu}(L^{1/\nu}|h-h^*)$, where $f_{\nu}$ is an unknown regular scaling function. The obtained data collapse for the 2nd area of GSPD is displayed in figure 5. For the first critical field in

![Figure 7](image3.png)

**Figure 7.** EE plots for the first (a) and second (b) regions of the GSPD.
Figure 7. Entanglement entropy as a function of transverse field $h$ for different chain lengths $N = 16, 20, 24$ and exchanges $L_1 = 1.0, J_1 = -3.0, J_2 = 3.0$. (a) Entanglement of one-particle block (1-P.B.) and a three-particle block (3-P.B.) with the rest of system. (b) Entanglement of two-particle blocks located in odd (2-P.B.O) and even links (2-P.B.E) with the rest of the system.

Figure 8. Entanglement entropy as a function of transverse field $h$ for different chain lengths $N = 16, 20, 24$ and exchanges $L_1 = 1.0, J_1 = 3.0, J_2 = 3.0$. (a) Entanglement of one-particle block (1-P.B.) and three-particle block (3-P.B.) with the rest of the system. (b) Entanglement of two-particle blocks located in odd (2-P.B.O) and even links (2-P.B.E) with the rest of the system.

Increasing a magnetic field cannot attain values for the entanglement entropy prior to the application of the field. In other words, for region (I) of the GSPD, every block shows a reduction treatment in the presence of a magnetic field. As was mentioned previously, in the 2nd region of the GSPD there are two critical fields. For this area of GSPD the EE between a single-particle and the rest of the system is shown in figure 8(a). Also evident from figure 8(a), the treatment of EE is related to the three-particle block. In the absence of a magnetic field, it is seen that a single particle irrespective of location in the chain is entangled with ROC. This holds true for increases in the magnetic field, up to the first critical field, $h = h_{c_1}$. At the first critical field, a severe reduction of the entanglement begins to set in. This decrement is continuous until the magnetic field reaches the 2nd critical point, $h = h_{c_2}$. This point has less reduction rate than the first critical point and for values of magnetic fields, $h > h_{c_2}$, the entanglement will vanish in good agreement with the saturated ferromagnetic phase. In addition, a the three-particle block is entangled with ROC in the fields less than the first critical field, while in the intermediate region, $h_{c_1} < h < h_{c_2}$, both parts of the system tend to be more entangled. In principle, by increasing the magnetic field, the entanglement of three-particle block increases and is maximized at the first critical field. By increasing the magnetic field, it decreases in the intermediate region $h_{c_1} < h < h_{c_2}$. However, in the saturated ferromagnetic phase ($h > h_{c_2}$) the mentioned entanglement will be zero.

It is also worth pointing out that the entanglement entropy behaves differently with the ROC for various two-particle blocks. As shown in figure 8(b), this behavior depends on the location of a two-particle block. In the absence of an applied field, if a block is located on an odd link with couplings $(J_1, J_2)$, it will almost be non-entangled with the ROC. On the other hand, a two-particle block in an even link, with coupling $(L_1)$, will be entangled with ROC. Also, the value of the entanglement of a two-particle block in an odd link is the same as that of an even link in the second critical field $h = h_{c_2}$.
Figure 9. Entanglement entropy as a function of transverse field $h$ for different chain lengths $N = 16, 20, 24$ and exchanges $L_1 = 1.0$, $J_1 = -3.0$, $J_2 = 0.5$. (a) Entanglement of a one-particle block (1-P.B.) and a three-particle block (3-P.B.) with the rest of the system. (b) Entanglement of two-particle blocks located in odd (2-P.B.O) and even (2-P.B.E) links with the rest of the system.

In the sequence of surveying the EE, we next arrive at the 3rd and 4th regions of GSPD. One- and three-particle blocks entanglements have qualitatively similar behavior for the 3rd and 4th regions. As presented in figures 9(a) and 10(a) one-particle block entanglement for these regions is recognizable to the three-block entanglement. This is because, for both regions, the one-particle block starts from a value close to 1 and reduces from their critical field values to zero in the larger magnetic field.

However, the three-particle block for both the 3rd and 4th regions exhibits a larger degree than the one-particle block. In addition, cusps near critical fields as presented in figures 9(a) and 10(a) specify three-particle block entanglement. Besides, for the 3rd and 4th regions odds- and even-block entanglements have almost identical values (figures 9(b) and 10(b)). This is in contrast with the 1st and 2nd regions, where the difference between the values is almost two orders of magnitude. The reason for this difference is that the value of $J_2$ in odd links for the 3rd and 4th regions is much less than that of the 1st and 2nd regions. Furthermore, in the 4th area of GSPD we can observe that at the critical field, $h_c$, the amount of entanglement for even and odd links is the same.

5. Conclusion

In this paper, we investigated the 1D quantum compass model with periodic boundary conditions in the presence of a transverse magnetic field. By using the exact diagonalization approach, we obtained the associated magnetic response functions at zero temperature. We first calculated the fidelity and fidelity susceptibility. Our computation of the fidelity function was shown to yield a maximum value ($\approx 1$) for the range of fields considered, and abrupt drops were observed as the magnetic field approached a quantum critical point. Using this method, we found that the critical fields in all regions of the ground-state phase diagram in complete agreement with our previous analytical and numerical results [20, 21]. It was seen that a drop in the fidelity raises the cusps in the
associated fidelity susceptibility. We showed that for a finite chain, the difference between the position of the sharp drop in the fidelity function, $h^*$, and the real critical field, $h_c$, is inversely proportional to the number of particles. It was also observed that fidelity susceptibility per site, i.e. the averaged fidelity susceptibility ($\chi_F/N$), is an intensive quantity in the off-critical fields and extensive in the critical fields.

We also studied the effect of a transverse magnetic field on the ground state of the QCC by looking at the von Neumann entropy for one-, two- or three-particle blocks separately. In the region (I), where a hidden magnetic order is suggested [15], one- and three-particle blocks are entangled with the ROC. But only two-particle blocks on even links are entangled with ROC. Applying the transverse magnetic field cannot destroy the von Neumann entropy in this gapped hidden order.

In the second region with a hidden order phase, in the absence of the magnetic field, one- and three-particle blocks are entangled with ROC. The big difference between this hidden order phase and another hidden order in region (I) is related to the blocks with two particles. In fact, in region (II) only two-particle blocks on ‘odd’ links are entangled with ROC in the absence of the magnetic field. Applying the magnetic field, entanglement of one- and two-particle blocks with the ROC remains almost constant up to the first critical field, $h_{c1}$. But, for three-particle blocks increases and is maximized at the first critical field. In the intermediate region, $h_{c1} < h < h_{c2}$, the entanglement of one, three and two-particle blocks on odd links with the ROC decreases by increasing field. But, for two-particle on even links first increases and passing of a maximum value and decreases by increasing the magnetic field. On the other hand, the value of the entanglement of a two-particle block in an odd link will be the same as that of an even link in the second critical field $h = h_{c2}$. Finally, all quantum correlations will be zero in the saturated ferromagnetic phase, $h > h_{c2}$.

Also, in the 3rd and 4th regions of the GSPD, one-, two-, and three-particle blocks are entangled with ROC at $h = 0$. The entanglement of one- and two-particle blocks on odd links decreases by increasing the magnetic field. But, the entanglement of three- and two-particle blocks on even links increases by adding the transverse magnetic field and is maximized in the critical field, $h = h_c$.

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