On n-Weakly Regular Rings

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ABSTRACT

As a generalization of right weakly regular rings, we introduce the notion of right n-weakly regular rings, i.e. for all $a \in N(R)$, $a \in aRaR$. In this paper, first give various properties of right n-weakly regular rings. Also, we study the relation between such rings and reduced rings by adding some types of rings, such as NCI, MC2 and SNF rings.

Keywords: Weakly Regular Rings, Reduced Rings.

Introduction:

Throughout this paper a ring $R$ denotes an associative ring with identity and all modules are unitary. For a subset $X$ of $R$, the left(right) annihilator of $X$ in $R$ is denoted by $r(X)(l(X))$. If $X=\{a\}$, we usually abbreviate it to $r(a)(l(a))$. We write $J(R)$, and $N(R)$, for the Jacobson radical and the set of nilpotent elements respectively.

The center of the ring $R$ is denoted by $Cent(R)$ and it is $Cent(R) = \{a \in R / ar = ra \ \forall \ r \in R\}$. A ring $R$ is called n-regular if for all $a \in N(R)$, $a \in aRa$ [7]. A right $R$-module $M$ is called $N$ flat if for any $a \in N(R)$, the mapping $1_M \otimes i : M \otimes_R Ra \to M \otimes_R R$ is monic, where $i : Ra \to R$ is the inclusion mapping [8]. A ring $R$ is called right (left) SNF if every simple right(left) $R$-module is $N$ flat [8].
A ring $R$ is called \textit{semiprime} if it has no nilpotent ideals \cite{6}. A ring $R$ is called \textit{reduced} if $N(R) = 0$ \cite{6}, or equivalently, $a^2 = 0$ implies $a = 0$ in $R$ for all $a \in R$. Recall that a ring $R$ is \textit{MERT} (resp. \textit{MELT}), if every maximal essential right (resp. left) ideal of $R$ is an ideal \cite{9}.

\section*{2. $n$-Weakly Regular Ring}

This section is devoted to give the definition of $n$-weakly regular rings with some of its characterizations and basic properties.

A ring $R$ is \textit{right (left) weakly regular} \cite{6}, if $a \in aRa$ $(RaRa)$ for every $a \in R$. We called $R$ is weakly regular if it is both right and left weakly regular.

\begin{definition}
A ring $R$ is to be \textit{right (left) $n$-weakly regular} if $a \in aRa$ $(aRaRa)$ for all $a \in N(R)$. We say that $R$ is $n$-weakly regular if it is right and left $n$-weakly regular ring.
\end{definition}

\textbf{Examples:}

1- Every reduced ring is $n$-weakly regular.
2- Every $n$-regular ring is $n$-weakly regular ring.
3- The ring $\mathbb{Z}$ of integers modulo 6, is reduced, $n$-regular, weakly regular ring, so it is $n$-weakly regular.
4- Let $R = \begin{bmatrix} Z_2 & Z_2 \\ Z_2 & Z_2 \end{bmatrix}$, $N(R) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 \end{bmatrix}$. $R$ is $n$-regular, weakly regular ring, so it is $n$-weakly regular but $R$ not reduced ring.
5- The ring $\mathbb{Z}$ of integer number is reduced, $n$-regular, so it is $n$-weakly regular but $\mathbb{Z}$ is not weakly regular ring.

\begin{proposition}
$R$ is a right $n$-weakly regular ring if and only if $aR$ is idempotent right ideal for all $a \in N(R)$.
\end{proposition}

\textbf{Proof:}

Let $R$ is a right $n$-weakly regular ring and $I$ is a principal right ideal of $R$ generated by a nilpotent element, then there exists $a \in N(R)$ such that $I = aR$, clearly $I^2 \subseteq I$.

On the other hand, since $R$ is right $n$-weakly, then there exists $y,z \in R$ such that $a = ayaz$. Now let $x \in I$, then there exists $r \in R$ such that $x = ar = ayazr \in I^2$. Therefore $I \subseteq I^2$. Hence $I^2 = I$.

Conversely, Let $a \in N(R)$, since $aR$ is idempotent right ideal of $R$, so $a \in aR = aRaR$. Therefore $R$ is right $n$-weakly regular ring. \hfill $\blacksquare$

\begin{proposition}
Let $R$ be a right $n$-weakly regular ring. If $aR \subseteq I$, for $a \in N(R)$ and $I$ is right or left ideal. Then $aRI = aR$.
\end{proposition}

\textbf{Proof:}

It is clearly that $bRI \subseteq bR$ for any $b \in R$. Now let $a \in N(R)$ and $x \in aR$, then there exists $r \in R$ such that $x = ar$. Since $R$ is right $n$-weakly regular ring then there exists $y,z \in R$ such that $a = ayaz$, $x = ayazr$, hence $azr \in aR \subseteq I$. So $aR \subseteq aRI$. Therefore $aRI = aR$. \hfill $\blacksquare$
Corollary 2.4

Let $R$ be a ring for all $a \in N(R)$ and any I right or left ideal of $R$ such that $aR \subseteq I$. Then the following condition are equivalent:

1- $R$ is right $n$-weakly regular.
2- For all $a \in N(R)$, $aRI=aR$.

Proof:
1 $\rightarrow$ 2 by Proposition 2.3.
2 $\rightarrow$ 1 Let $I=aR$ and by Proposition 2.2.

Proposition 2.5

Let $R$ be a right $n$-weakly regular ring. Then $N(R) \cap Cent(R)=0$

Proof:
If $N(R) \cap Cent(R) \neq 0$, then there exists $0 \neq a \in N(R) \cap Cent(R)$ such that $a^2=0$.

Then $R$ is right $n$-weakly regular then there exists $y,z \in R$ such that $a=ayaz=a^2yz=0yz=0$ ($a \in Cent(R)$). Therefore $a=0$. This shows that $N(R) \cap Cent(R)=0$.

Corollary 2.6

Let $R$ be a commutative ring. Then $R$ is reduced if and only if $R$ is right $n$-weakly regular.

Lemma 2.7 [2]

1- Every one sided or two sided nil ideal of $R$ is contained in $J(R)$.
2- In any ring $R$, $a \in J(R)$ if and only if $1-ar$ is invertible for all $r \in R$.

Now we have the following proposition

Theorem 2.8

Let $R$ be a right $n$-weakly regular ring. Then $N(R) \cap J(R)=0$.

Proof:
If $a \in N(R) \cap J(R)$, then there exists $y,z \in R$ such that $a=ayaz$. Hence $a(1-yaz)=0$.

Since $a \in J$, $yaz \in J$, then by Lemma 2.7(2), there exists invertible element $v \in R$ such that $(1-yaz)v=1$. So $(a-ayaz)v=a$, yield $a=0$. Therefore $N(R) \cap J(R)=0$.

Let $R$ be a ring we denoted to the upper nil radical for a ring $R$ by $Nil^+(R)$ and it is the sum of nil ideal in the ring $R$.

Corollary 2.9

Let $R$ be a right $n$-weakly regular ring. Then $Nil^+(R)=0$.

Proof:
Let $I$ be a nilpotent right ideal, by Lemma 2.7(1), we have that $I \subseteq J(R)$, $I \subseteq N(R) \cap J(R)=0$, Theorem 2.8, $I=0$, which is a contradiction. So $R$ not contain any nil ideal. Therefore $N^+(R)=0$.

Corollary 2.10

Let $R$ be a right $n$-weakly regular ring. Then $R$ is semiprime ring.

Proof:
Let $I$ be a nilpotent right ideal then $I \subseteq N(R) \cap J(R)=0$ (Theorem 2.8) $I=0$. Therefore $R$ is semiprime ring.

3. The Connection between $n$-Weakly Regular Rings and Other Rings.
In this section we gives the connection between n-weakly regular rings and reduced rings, SNF rings.

**Proposition 3.1**

The following conditions are equivalent for a ring R.
1. R is reduced.
2. R is right n-weakly regular and N(R) forms a right ideal of R.
3. R is right n-weakly regular and N(R) forms a left ideal of R.
4. R is right n-weakly regular and NI ring.
5. R is right n-weakly regular and N(R) ⊆ J(R).

**Proof:**

1 → 4 → 3 → 5, 1 → 2 → 5 it is trivial.

Suppose that R is right n-weakly regular ring. So N(R) ∩ J(R)=0, (Theorem 2.8). Since N(R) ⊆ J(R), then N(R) ∩ J(R)= N(R)=0. Therefore R is reduced. ■

**Theorem 3.2**

Let R be a ring with aR=Ra, for all a ∈ N(R). Then the following conditions are equivalent:
1. R is right n-weakly regular.
2. R is n-regular.
3. R is reduced.

**Proof:**

1 → 2
Let 0 ≠ a ∈ R, such that a^2=0. Since R is a right n-weakly regular, then a ∈ aR=RaR (Proposition 2.1)
= aRRa (Ra=aR)
= aRa.
so a ∈ aRa, hence R is n-regular.

2 → 3
Let 0 ≠ a ∈ R, such that a^2=0. Since R is a right n-regular, then there exists b ∈ R such that a=aba since aR=Ra there exists x ∈ R such that ab=xa, so a=aba=xa^2=x0=0.
Therefore R is reduced.

3 → 1 It is trivial. ■

A ring R is called NCI provided that N(R) contains a non zero ideal of R whenever N(R) ≠ 0 [1].

**Lemma 3.3 [1]**

Let R be a ring with N(R) ≠ 0. Then R is NCI if and only if N*(R) ≠ 0.

**Proposition 3.4**

Let R be a NCI ring, then R is right n-weakly regular if and only if R is reduced.

**Proof:**

Let N(R) ≠ 0, sinc R is NCI ring from Lemma 3.3, we get that N*(R) ≠ 0 but R is right n-weakly regular, N*(R)=0 (Corollary 2.9), which is contradiction. So N(R)=0. Therefore R is reduced. ■

A ring R is called weakly reversible if and only if for all a,b,r ∈ R such that ab=0, Rbra is a nil left ideal of R (equivalently braR is nil right ideal of R). Clearly ZI ring is weakly reversible [4].
Proposition 3.5
A ring R be right n-weakly regular ring and weakly reversible if and only if R is reduced.

Proof:
Let \( a \in R \) with \( a^2 = 0 \). Then \( a = ayaz \) for some \( y, z \in R \) because R is right n-weakly regular ring. Since R is weakly reversible then \( ayaR \) is nil right ideal of R so \( ayaR \subseteq J(R) \cap N(R) = 0 \) (Lemma 2.7(1) & Theorem 2.8) we get \( ayaR = 0 \), in particular \( a = ayaz = 0 \). Therefore R is reduced.

Converse, it is trivial. ■

Recall that a ring R is right MC2 if \( K \cong eR \) is simple, \( e^2 = e \), then \( K = gR \) for some \( g^2 = g \) [5].

Lemma 3.6 [9]
Let R be a left MC2, right SNF ring and MELT ring. Then R is a semiprime ring and right non singular.

Theorem 3.7 [9]
Let I be a right ideal of a ring R. Then \( R/I \) is N flat if and only if \( Ia = I \cap Ma \) for all \( a \in N(R) \).

Theorem 3.8
Let R be right SNF, left MC2 and MELT ring. Then R is left n-weakly regular ring.

Proof:
From Lemma 3.6, we get that R is a semiprime ring and \( a \in N(R) \). If \( RaR + I(a) \neq R \), then there exists a maximal left ideal \( M \) of R containing \( RaR + I(a) \), if \( M \) is not essential then we can write \( M = l(e) \), where \( e^2 = e \in R \) and \( e \neq 0 \), since \( RaRe = 0 \) because \( RaR \subseteq M \), \( (ReRa)^2 = 0 \) implies \( ReRa = 0 \) (since R is semiprime) \( ReRa = 0 \) in particular \( ea = 0 \) and \( e \in l(a) \subseteq M = l(e) \), then \( e^2 = 0 \), which is a contradiction. Therefore M is an essential, since R is MELT ring, then M is a two sided ideal then there exists a maximal right ideal \( L \) in R containing M, since R is right SNF ring then \( R/L \) is N flat right R-module, \( a = ma \) for some \( m \in M \) (Theorem 3.7), \( (1 - m)a = 0 \), \( 1 - m \in l(a) \subseteq M \subseteq L \) therefore \( 1 - m \in L \) implies \( 1 \in L \) which is a contradiction, therefore \( RaR + I(a) = R \) for all \( a \in N(R) \). Thus R is left n-weakly regular ring. ■

Theorem 3.9
Let R be MELT and right SNF ring with \( RaR \) is essential for all \( a \in N(R) \), then R is left n-weakly regular ring.

Proof:
Let \( a \in N(R) \). If \( RaR + I(a) \neq R \) then there exists a maximal left ideal \( M \) of R containing \( RaR + I(a) \), since \( RaR \) is left annihilator of a nilpotent element by the hypothesis \( RaR \) is essential left ideal in R, M is an essential ideal of R (MELT ring), there exists a maximal right ideal \( L \) in R such that \( M \subseteq L \). Since R is right SNF ring, then \( R/L \) is N flat right R-module, \( a = ma \) for some \( m \in M \) (Theorem 3.7), \( 1 - m \in l(a) \subseteq M \subseteq L \) therefore \( 1 - m \in L \) implies \( 1 \in L \), which is a contradiction. Therefore \( RaR + I(a) = R \) for all \( a \in N(R) \), and so R is left n-weakly regular ring. ■

Definition 3.10 [8]
A right $R$-module $M$ is said to be nil-injective, if for any $a \in N(R)$, any right $R$-homomorphism $f:aR \to M$ can be extended to $R \to M$, or equivalently $f=m$, where $m \in M$.

The ring $R$ is called right nil-injective if $R_R$ is right nil-injective. Clearly a reduced ring is a right nil-injective and $n$-regular ring is a right nil-injective [8].

**Theorem 3.11**

Let $R$ be a semiprime ring whose simple singular right $R$-module are nil-injective. Then $R$ is right $n$-weakly regular ring.

**Proof:**

Let $a \in N(R)$. We claim that $RaR+r(a)=R$ if not, there exists a maximal right ideal $M$ of $R$ containing $RaR+r(a)$. If $M$ is not essential in $R$ then $M=r(e)$, $e^2=e \in R$. Since $Rae \subseteq RaR \subseteq M=r(e)$, $eRae=0$, $(ae)^2=0$ but $R$ is semiprime, $aeR=0$, so $ae=0$. Thus $e \in r(a) \subseteq M=r(e)$, which is a contradiction. Hence $M$ is essential right ideal in $R$ and so $R/M$ is nil-injective. Define a mapping $f : aR \to R/M$ such that $f(ar)=r+M$, let $x,y \in R$ such that $ax=ay$, $a(x-y)=0$, $x+y \in M$, $x+M=y+M$, $f(ax)=x+M=y+M=f(ay)$, $f$ is well define. $1+M=f(a)=(b+M)(a+M)=ba+M$, $1-ba \in M$ because $ba \in RaR \subseteq M$ then $1 \in M$ which is a contradiction, so $RaR+r(a)=R$, in particular there is $y,z \in R$ and $v \in r(a)$ such that $yaz+v=1$, $ayaz+av=a$, $a=ayaz$. Therefore $R$ is right $n$-weakly regular ring. ■

**Proposition 3.12**

Let $R$ be a ring whose simple right $R$-module are nil-injective. Then $R$ is right $n$-weakly regular ring.

**Proof:**

Assume that $a \in R$ such that $aRa=0$. Then $RaR \subseteq r(a)$. If $a \neq 0$ then there exists a maximal right ideal $M$ containing $r(a)$. By hypothesis $R/M$ is nil-injective. We define a mapping $f : aR \to R/M$ such that $f(ar)=r+M$, $f$ is well define similar to Theorem 3.11, so there exists $b \in R$ such that $1+M=f(a)=ba+M$, $1-ba \in M$ because $ba \in RaR \subseteq M$ then $1 \in M$ which is a contradiction, so $a=0$. Therefore $R$ is a semiprime ring, by Theorem 3.11 we get that $R$ is right $n$-weakly regular ring. ■
REFERENCES

[1] Hwang, S. U., Jeon, Y. Ch. and Park, K. S. (2007); “On NCI rings”, Bull. Korean Math. Soc., Vol. 44, No. 2, pp. 215-223.

[2] Kasch, F. (1982) “Modules and Rings” Academic Press Inc. (Londeon) Ltd.

[3] Kim, N. K., Nam, S. B. and Kim, J. K. (1999); “On simple singular GP-injective modules” Comm. In Algebra, Vol. 27, No.5., pp. 2087-2096.

[4] Liang, Z. and Gang, Y. (2007); “On weakly reversible rings” Acta Math. Univ. Comentanae ,Vol. LXXVI, No. 2, pp. 189-192.

[5] Nicholson, W.K. and Yousif, M. F. (1997) “Mininjective ring” Journal of algebra, Vol. 187, pp. 548 -578.

[6] Ramamurthi, V. S. (1973) “Weakly regular ring” Canda. Math . Bull., Vol. 16, No. 3, pp.

[7] Stenström, B. (1977); “Ring of Quotient” Springer-Verlag, Berlin Heidelberg, New York.

[8] Wei, J. and Chen, J. (2007); “Nil-injective rings”, International Electronic Journal of Algebra, Vol. 2, No. , pp. 1-21.

[9] Wei, J. and Chen, J. (2008); “NPP rings, reduced rings and SNF rings”, International Electronic Journal of Algebra, Vol. 4, No. , pp. 9-26.

[10] Yue Chi Ming, R. (1980) “On V-rings and prime rings” Journal of algebra, Vol. 62, pp. 13-20.

[11] Yue Chi Ming, R. (1983) “On quasi-injectivity and Von Neumann regularity” Mh. Math., Vol. 95, pp. 25-32.