Automatically Discovering Relaxed Lyapunov Functions for Polynomial Dynamical Systems

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Abstract— The notion of Lyapunov function plays a key role in design and verification of dynamical systems, as well as hybrid and cyber-physical systems. In this paper, to analyze the asymptotic stability of a dynamical system, we generalize standard Lyapunov functions to relaxed Lyapunov functions (RLFs), by considering higher order Lie derivatives of certain functions along the system’s vector field. Furthermore, we present a complete method to automatically discovering polynomial RLFs for polynomial dynamical systems (PDSs). Our method is complete in the sense that it is able to discover all polynomial RLFs by enumerating all polynomial templates for any PDS.

I. INTRODUCTION

The notion of Lyapunov function plays a very important role in design and verification of dynamical systems, in particular, in performance analysis, stability analysis and controller synthesis of complex dynamical and controlled systems [1], [2], [3]. In recent years, people realized that the notion is quite helpful to safety verification of hybrid and cyber-physical systems as well [4].

However, the following two issues hinder the application of Lyapunov functions in practice. Firstly, it is actually not necessary for the first-order Lie derivative of a Lyapunov function to be strictly negative to guarantee asymptotic stability, which is shown by LaSalle’s Invariance Principle [2]. Such a condition could limit to scale up the method. Secondly, in general there is no effective way so far to find Lyapunov functions, although many methods have been proposed by different experts using their field expertise.

To address the above two issues, in this paper, we first generalize the standard concept of Lyapunov function to relaxed Lyapunov function (RLF) for asymptotic stability analysis. Compared with the conventional definition of Lyapunov function, the first non-zero higher order Lie derivative of RLF is required to be negative, rather than its first-order Lie derivative. Such a relaxation extends the set of admissible functions that can be used to prove asymptotic stability.

Another contribution of this paper is that we present a complete method to automatically discovering polynomial RLFs for polynomial dynamical systems (PDSs). The basic idea of our method is to predefine a parametric polynomial as a template of RLF first, and then utilize the Lie derivatives of the template at different orders to generate constraint on the parameters, and finally solve the resulting constraint. Our method is complete in the sense that it is able to generate all polynomial RLFs by enumerating all polynomial templates for any PDS.

Related Work. In [5], the same terminology “relaxed Lyapunov function” is used, with a different definition.

The idea of applying higher order Lie derivatives to analyze asymptotic stability is not new. For example, in [6], [7] the authors resorted to certain linear combinations of higher order Lie derivatives with non-negative coefficients such that the combination is always negative. This method could be included in the framework of vector Lyapunov functions method [8], [9]. Our method is essentially different from theirs because an RLF only requires its first non-zero higher order Lie derivative to be negative.

In the literature, there is a lot of work on constructing Lyapunov functions. For instance, in [10], [11], [12] methods for constructing common quadratic Lyapunov functions for linear systems were proposed, which were generalized in [13] and [14] for nonlinear systems wherein the generated Lyapunov functions are not necessarily quadratic. Another useful technique is the linear matrix inequality (LMI) method introduced in [15] and [16], which enables us to utilize the results of numerical optimization for discovering piecewise quadratic Lyapunov functions. Based on sums-of-squares (SOS) decomposition and semi-definite programming (SDP) [17], a method for constructing piecewise high-degree polynomial and piecewise non-polynomial Lyapunov functions was proposed in [18] and [19]. The SOS and SDP based method was also used in [20] to search for control Lyapunov functions for polynomial systems. In [21], the authors proposed a new method for computing Lyapunov functions for polynomial systems by solving semi-algebraic constraint using their tool DISCOVERER [22]. Approaches to constructing Lyapunov functions beyond polynomials using radial basis functions were proposed in [23], [24].

Our method has the following features compared to the related work. Firstly, it generates relaxed Lyapunov functions rather than conventional Lyapunov functions. Secondly, it is able to discover all polynomial RLFs by enumerating all polynomial templates for any PDS, whereas the Krasovskii’s method [25] and Zubov’s method [26] can only produce Lyapunov functions of special forms. Thirdly, the LMI method and SOS method are numerical, while our method...
is symbolic, which means it could provide a mathematically rigorous framework for the stability analysis of polynomial dynamical systems.

**Structure:** The rest of this paper is organized as follows. In Section III the theoretical foundations are presented. Section III shows a new criterion for asymptotic stability using the notion of relaxed Lyapunov functions. In Section IV we present a sound and complete method and a corresponding algorithm for automatically discovering polynomial RLFs on polynomial dynamical systems. The method is illustrated by an example in Section V. Finally, we conclude this paper and discuss possible future work in Section VI.

II. THEORETICAL FOUNDATIONS

In this section, we present the fundamental materials based on which we develop our method.

### A. Polynomial Ideal Theory

Let \( K \) be an algebraic field, and \( K[x_1, x_2, \ldots, x_n] \) denote the polynomial ring over \( K \). Customarily, let \( x \) denote the \( n \)-tuple \((x_1, \ldots, x_n)\). Then \( K[x_1, x_2, \ldots, x_n] \) can be written as \( K[x] \) for short, and a polynomial in \( K[x_1, x_2, \ldots, x_n] \) can simply be written as \( p(x) \) or \( p \). Particularly, \( K \) will be taken as the real field \( \mathbb{R} \) in this paper, and \( x \) takes value from the \( n \)-dimensional Euclidean space \( \mathbb{R}^n \).

In our method we will use polynomials with undetermined coefficients, called parametric polynomials or templates. Such polynomials are denoted by \( p(u, x) \), where \( u = (u_1, u_2, \ldots, u_t) \) is a \( t \)-tuple of parameters. A parametric polynomial \( p(u, x) \) in \( \mathbb{R}[x_1, x_2, \ldots, x_n] \) with real parameters can be seen equivalently as a regular polynomial in \( \mathbb{R}[u_1, u_2, \ldots, u_t, x_1, x_2, \ldots, x_n] \). Given \( u_0 \in \mathbb{R}^t \), we call the polynomial \( p_{u_0}(x) \) resulted by substituting \( u_0 \) for \( u \) in \( p(u, x) \) an instantiation of \( p(u, x) \).

The following are some fundamental results relative to polynomial ideals, which can be found in [27].

**Definition 1:** A subset \( I \subseteq K[x] \) is called an ideal iff

1. (a) \( 0 \in I \); 
2. (b) If \( p(x), g(x) \in I \), then \( p(x) + g(x) \in I \); 
3. (c) If \( p(x) \in I \), then \( p(x)h(x) \in I \) for any \( h(x) \in K[x] \).

It is easy to check that if \( p_1, \ldots, p_m \in K[x] \), then

\[
\langle p_1, \ldots, p_m \rangle = \{ \sum_{i=1}^{m} p_i h_i \mid \forall i \in [1, m], h_i \in K[x] \}
\]

is an ideal. In general, we say an ideal \( I \) is generated by polynomials \( g_1, g_2, \ldots, g_k \in K[x] \) if \( I = \langle g_1, g_2, \ldots, g_k \rangle \), where all \( g_i \) for \( i \in [1, k] \) are called generators of \( I \). In fact, we have

**Theorem 2 (Hilbert Basis Theorem):** Every ideal \( I \subseteq K[x] \) has a finite generating set. That is, \( I = \langle g_1, g_2, \ldots, g_k \rangle \) for some \( g_1, g_2, \ldots, g_k \in K[x] \).

From this result, it is easy to see that

**Theorem 3 (Ascending Chain Condition):** For any ascending chain

\[
I_1 \subseteq I_2 \subseteq \cdots \subseteq I_m \subseteq \cdots
\]

of ideals in polynomial ring \( K[x] \), there must be an \( N \) such that for all \( m \geq N \), \( I_m = I_N \).

### B. Dynamical Systems and Stability

We summarize some fundamental theories of dynamical systems here. For details please refer to [1], [2], [3].

1) **Dynamical Systems:** We consider autonomous dynamical systems modeled by first-order ordinary differential equations

\[
x = f(x),
\]

where \( x \in \mathbb{R}^n \) and \( f \) is a vector function from \( \mathbb{R}^n \) to \( \mathbb{R}^n \), which is also called a vector field in \( \mathbb{R}^n \).

In this paper, we focus on special nonlinear dynamical systems whose vector fields are defined by polynomials.

**Definition 4 (Polynomial Dynamical System):** Suppose \( f = (f_1, f_2, \ldots, f_n) \) in \( \mathbb{R}^n \). Then \( f \) is called a polynomial dynamical system (PDS for short) if for every \( 1 \leq i \leq n \), \( f_i \) is a polynomial in \( \mathbb{R}[x] \).

If \( f \) satisfies the local Lipschitz condition, then given \( x_0 \in \mathbb{R}^n \), there exists a unique solution \( x(t) \) of \( f \) defined on \((a, b)\) with \( a < 0 < b \) s.t.

\[
\forall t \in (a, b), \quad \frac{dx(t)}{dt} = f(x(t)) \quad \text{and} \quad x(0) = x_0.
\]

We call \( x(t) \) on \([0, b)\) the trajectory of \( f \) starting from initial point \( x_0 \).

Let \( \sigma(x) \) be a function from \( \mathbb{R}^n \) to \( \mathbb{R} \). Suppose both \( \sigma \) and \( f \) are differentiable in \( x \) at any order \( n \in \mathbb{N} \). Then we can inductively define the Lie derivatives of \( \sigma \) along \( f \), i.e.

\[
L_k f : \mathbb{R}^n \rightarrow \mathbb{R} \quad \text{for} \quad k \in \mathbb{N},
\]

as follows:

- \( L_0 \sigma(x) = \sigma(x) \),
- \( L_k \sigma(x) = \left( \frac{\partial}{\partial x} L_{k-1} \sigma(x), f(x) \right) \), for \( k > 0 \),

where \( (\cdot, \cdot) \) is the inner product of two vectors, i.e. \((a_1, \ldots, a_n), (b_1, \ldots, b_n)\) = \( \sum_{i=1}^{n} a_i b_i \).

Polynomial functions are sufficiently smooth, so given a PDS \( P \) and a polynomial \( p \), the vector field \( f \) of \( P \) satisfies the local Lipschitz condition, and the higher order Lie derivatives of \( p \) along \( f \) are well defined and are all polynomials. For a paramterized polynomial \( p(u, x) \), we can define \( L_k f p(u, x) : \mathbb{R}^t \rightarrow \mathbb{R} \) by seeing \( u \) as undetermined constants rather than variables. In the sequel we will implicitly employ these facts.

**Example 5:** Suppose \( f = (-x, y) \) and \( p(x, y) = x + y^2 \). Then \( L_1 f p = -x + 2y^2 \) and \( L_2 f p = -x + 4y^2 \).

2) **Stability:** The following are classic results of stability of dynamical systems in the sense of Lyapunov.

**Definition 6:** A point \( x_e \in \mathbb{R}^n \) is called an equilibrium or critical point of \( f \) if \( f(x_e) = 0 \).

We assume \( x_e = 0 \) w.l.o.g from now on.

**Definition 7:** Suppose \( 0 \) is an equilibrium of \( f \). Then

- \( 0 \) is called Lyapunov stable if for any \( \epsilon > 0 \), there exists a \( \delta > 0 \) such that if \( \|x_0\| < \delta \), then the corresponding solution \( x(t) \) of \( f \) satisfies \( \|x(t)\| < \epsilon \) for all \( t \geq 0 \).
- \( 0 \) is called asymptotically stable if it is Lyapunov stable and there exists a \( \delta > 0 \) such that for any \( \|x_0\| < \delta \), the corresponding solution \( x(t) \) of \( f \) can be extended to infinity and \( \lim_{t \to \infty} x(t) = 0 \).
Lyapunov first provided a sufficient condition, using so-called \textit{Lyapunov function}, for the Lyapunov stability as follows.

\textbf{Theorem 8 (Lyapunov Stability Theorem):} Suppose 0 is an equilibrium point of $f$. If there is an open set $U \subset \mathbb{R}^n$ with $0 \in U$ and a continuous differentiable function $V : U \rightarrow \mathbb{R}$ such that

(a) $V(0) = 0$,
(b) $V(x) > 0$ for all $x \in U \setminus \{0\}$ and
(c) $L_0^1 V(x) \leq 0$ for all $x \in U$,
then 0 is a stable equilibrium of $f$. Moreover, if condition (c) is replaced by
(c') $L_0^1 V(x) < 0$ for all $x \in U \setminus \{0\}$,
then 0 is an asymptotically stable equilibrium of $f$. Such $V$ is called a Lyapunov function.

For asymptotic stability, we have Barbashin-Krasovskii-LaSalle (BKLS) Principle which relaxes condition (c') in Theorem 8.

\textbf{Theorem 9 (BKLS Principle):} Suppose there exists $V$ satisfying the conditions (a), (b) and (c') in Theorem 8. If the set $M \triangleq \{ x \in \mathbb{R}^n \mid L_0^1 V(x) = 0 \} \cap U$ does not contain any trajectory of the system besides the trivial trajectory $x(t) \equiv 0$, then 0 is asymptotically stable.

Inspired by Theorem 9, we will define the \textit{relaxed Lyapunov function} (RLF) for short in the subsequent section, which guarantees the asymptotic stability of an equilibrium of a dynamical system.

\section*{III. RELAXED LYAPUNOV FUNCTION}

Intuitively, a Lyapunov function requires that any trajectory starting from $x_0 \in U$ cannot leave the region $\{ x \in \mathbb{R}^n \mid V(x) \leq V(x_0) \}$. While, in the asymptotic stability case, the corresponding $V$ forces any trajectory starting from $x_0 \in U$ to transit the boundary $\{ x \in \mathbb{R}^n \mid V(x) = V(x_0) \}$ towards the set $\{ x \in \mathbb{R}^n \mid V(x) < V(x_0) \}$. It is clear that $L_0^1 V(x) < 0$ is only a sufficient condition to guarantee asymptotic stability. When a point $x$ satisfies $L_0^1 V(x) = 0$, the transaction requirement may still be met if the first non-zero higher order Lie derivative of $V$ at $x$ is negative. To formalize this idea, we give the following definition.

\textbf{Definition 10 (Pointwise Rank):} Let $N^+$ be the set of positive natural numbers. Given sufficiently smooth function $\sigma$ and vector field $f$, the \textit{pointwise rank} of $\sigma$ w.r.t. $f$ is defined as the function $\gamma_{\sigma,f} : \mathbb{R}^n \rightarrow \mathbb{N} \cup \{ \infty \}$ given by

$$\gamma_{\sigma,f}(x) = \begin{cases} \infty, & \text{if } \forall k \in N^+, L_k^f \sigma(x) = 0, \\ \min \{ k \in N^+ \mid L_k^f \sigma(x) \neq 0 \}, & \text{otherwise.} \end{cases}$$

\textbf{Example 11:} For $f = (-x, y)$ and $p(x, y) = x + y^2$, by Example 5 we have $\gamma_{p,f}(0,0) = \infty$, $\gamma_{p,f}(1,1) = 1$, $\gamma_{p,f}(2,2) = 2$.

\textbf{Definition 12 (Transverse Set):} Given sufficiently smooth function $\sigma$ and vector field $f$, the \textit{transverse set} of $\sigma$ w.r.t. $f$ is defined as

$$\text{Trans}_{\sigma,f} \triangleq \{ x \in \mathbb{R}^n \mid \gamma_{\sigma,f}(x) < \infty \land L_f^{\gamma_{\sigma,f}(x)} \sigma(x) < 0 \}.$$ 

Intuitively, $\text{Trans}_{\sigma,f}$ consists of those points at which the first non-zero high order Lie derivative of $\sigma$ along $f$ is negative. Now we can relax condition (c') in Theorem 8 and get a stronger result for asymptotic stability.

\textbf{Theorem 13:} Suppose 0 is an equilibrium point of $f$. If there is an open set $U \subset \mathbb{R}^n$ with $0 \in U$ and a sufficiently smooth function $V : U \rightarrow \mathbb{R}$ s.t.

(a) $V(0) = 0$,
(b) $V(x) > 0$ for all $x \in U \setminus \{0\}$ and
(c) $x \in \text{Trans}_{V,f}$ for all $x \in U \setminus \{0\}$,
then 0 is an asymptotically stable equilibrium point of $f$.

\textbf{Proof:} First notice that condition (c) implies $L_0^1 V(x) \leq 0$ for all $x \in U \setminus \{0\}$. In order to show the asymptotic stability of 0, according to Theorem 9, it is sufficient to show that $M \triangleq \{ x \in \mathbb{R}^n \mid L_0^1 V(x) = 0 \} \cap U$ contains no nontrivial trajectory of the dynamical system.

If not, let $x(t), t \geq 0$ be such a trajectory contained in $M$ other than $x(t) \equiv 0$. Then $L_0^1 V(x(t)) = 0$ for all $t \geq 0$. Noting that $x_0 = x(0) \in \text{Trans}_{V,f}$, we can get the Taylor expansion of $L_0^1 V(x(t))$ at $t = 0$:

$$L_0^1 V(x(t)) = L_0^1 V(x_0) + L_2^1 V(x_0) \cdot t + L_3^1 V(x_0) \cdot \frac{t^2}{2!} + \cdots = L_0^1 V(x_0) \cdot \frac{t^1}{1!} + \cdots .$$

By Definition 12 there exists an $\epsilon > 0$ s.t. $\forall t \in (0, \epsilon), L_0^1 p(x(t)) < 0$, which contradicts the assumption.

\textbf{Definition 14 (Relaxed Lyapunov Function):} We refer to the function $V$ in Theorem 13 as a \textit{relaxed Lyapunov function}, denoted by RLF.

In the next section, we will explore how to discover polynomial RLFSs automatically for PDSs.

\section*{IV. AUTOMATICALLY DISCOVERING RLFSs FOR PDSs}

Given a PDS, the process of automatically discovering polynomial RLFSs is as follows:

- a template, i.e. a parametric polynomial $p(u, x)$, is predefined as a potential RLF;
- the conditions for $p(u, x)$ to be an RLF are translated into an equivalent formula $\Phi$ of the decidable \textit{first-order theory of reals} [28];
- constraint $\Phi'$ on parameters $u$, or equivalently a set $S_u$ of all $t$-tuples subject to $\Phi'$, is obtained by applying quantifier elimination (QE for short. See [29], [30]) to $\Phi$, and any instantiation of $u$ by $u_0 \in S_u$ yields an RLF $p_{u_0}(x)$.

\subsection*{A. Computation of Transverse Set}

Correct translation of the three conditions in Theorem 13 is crucial to our method. In particular, we have to show that for any polynomial $p(x)$ and polynomial vector field $f$, the transverse set $\text{Trans}_{p,f}$ can be represented by first order polynomial formulas. To this end, we first give several theorems by exploring the properties of Lie derivatives and polynomial ideas.

In what follows, given a parameterized polynomial $p(u, x)$, all Lie derivatives $L_k^f p$ are seen as polynomials in $[u, x]$. Besides, we will use the convention that $\bigvee_{i \in \mathbb{N}} \eta_i = \text{false}$ and $\bigwedge_{i \in \mathbb{N}} \eta_i = \text{true}$, where $\eta_i$ is logical formula.
Theorem 15 (Fixed Point Theorem): Given \( p \equiv p(u, x) \), if \( L^k p \in \langle L^1 p, \ldots, L^{k-1} p \rangle \), then for all \( m > i \), \( L^m p \in \langle L^1 p, \ldots, L^{k-1} p \rangle \).

Proof: We prove this fact by induction. Assume \( L^k p \in \langle L^1 p, \ldots, L^{k-1} p \rangle \) for \( k \geq i \). Then there are \( g_j \in \mathbb{R}[u, x] \) such that \( L^k p = \sum_{j=1}^{i-1} g_j L^j p \). By the definition of Lie derivative, it follows that

\[
L^{k+1} p = (\frac{\partial}{\partial x} L^k p, f) = (\frac{\partial}{\partial x} \sum_{j=1}^{i-1} g_j L^j p, f) = \sum_{j=1}^{i-1} \left( \frac{\partial}{\partial x} g_j, f \right) L^j p + \sum_{j=1}^{i-1} g_j L^j p^j+1 p = \sum_{j=1}^{i-1} \left( \frac{\partial}{\partial x} g_j, f \right) L^j p + \sum_{j=2}^{i-1} g_j L^j p + g_{i-1} L^k p.
\]

By induction hypothesis, \( L^k p \in \langle L^1 p, \ldots, L^{k-1} p \rangle \), so \( L^{k+1} p \in \langle L^1 p, \ldots, L^{k-1} p \rangle \). By induction, the fact follows immediately.

Theorem 16: Given \( p \equiv p(u, x) \), the number

\[ N_{p, f} = \min \{ i \in \mathbb{N}^+ \mid L^{i+1} p \in \langle L^1 p, \ldots, L^i p \rangle \} \]

is well-defined and computable.

Proof: First it is easy to show that \( N_{p, f} \) has an equivalent expression \( N_{p, f} = \min \{ i \in \mathbb{N}^+ \mid I_{i+1} = I_i \} \), where \( I_i = \langle L^i p, \ldots, L^1 p \rangle \subseteq \mathbb{R}[u, x] \). Notice that

\[ I_1 \subseteq I_2 \subseteq \cdots \subseteq I_k \cdots \]

forms an ascending chain of ideals. By Theorem 3 \( N_{p, f} \) is well-defined. Computation of \( N_{p, f} \) is actually an ideal membership problem, which can be solved by computation of Gröbner basis [27].

Example 17: For \( f = (-x, y) \) and \( p(x, y) = x + y^2 \), by Example 5 we have \( L^1 p \not\in \langle L^1 p \rangle \) and \( L^2 p \in \langle L^1 p, L^2 p \rangle \), so \( N_{p, f} = 2 \).

Theorem 18 (Rank Theorem): Suppose that \( p \equiv p(u, x) \). Then for all \( x \in \mathbb{R}^n \) and all \( u_0 \in \mathbb{R}^l \), \( \gamma_{p_{u_0}, f}(x) < \infty \) implies \( \gamma_{p_{u_0}, f}(x) < N_{p, f} \).

Proof: If the conclusion is not true, then there exist \( x_0 \in \mathbb{R}^n \) and \( u_0 \in \mathbb{R}^l \) such that \( N_{p, f} < \gamma_{p_{u_0}, f}(x_0) < \infty \).

By Definition 10 \( x_0 \) satisfies

\[
L^1 u_0 = 0 \land \cdots \land L^{N_{p, f}} u_0 = 0 \land L^{\gamma_{p_{u_0}, f}(x_0)} u_0 \neq 0.
\]

Then by Theorem 16 and 15 for all \( m > N_{p, f} \), we have \( L^m \gamma_{p_{u_0}, f}(x_0) = 0 \). In particular, \( L^{\gamma_{p_{u_0}, f}(x_0)} u_0(x_0) = 0 \), which contradicts \( L^{\gamma_{p_{u_0}, f}(x_0)} u_0(x_0) \neq 0 \).

Now we are able to show the computability of \( Trans_{p, f} \).

Theorem 19: Given a parameterized polynomial \( p \equiv p(u, x) \) and polynomial vector field \( f \), for any \( u_0 \in \mathbb{R}^l \) and any \( x \in \mathbb{R}^n \), \( x \in Trans_{p_{u_0}, f} \) if and only if \( u_0 \) and \( x \) satisfy \( \varphi_{p, f} \), where

\[
\varphi_{p, f} \equiv \bigcup_{i=1}^{N_{p, f}} \varphi_{p, f}^i, \quad \text{and} \quad \varphi_{p, f}^i \equiv \left( \bigwedge_{j=1}^{i-1} L^j f(p(u, x)) = 0 \right) \land L^i f(p(u, x)) < 0.
\]

Proof: \((\Rightarrow)\) Suppose \( x \in Trans_{p_{u_0}, f} \). By Definition 12 \( x \) satisfies

\[ L^1 f p_{u_0} = 0 \land \cdots \land L^{N_{p, f}} f \gamma_{p_{u_0}, f}(x) - 1 p_{u_0} = 0 \land L^{N_{p, f}} f \gamma_{p_{u_0}, f}(x) p_{u_0} < 0. \]

By Theorem 18 \( \gamma_{p_{u_0}, f}(x) \leq N_{p, f} \). Then it is easy to check that (3) implies (2) when \( u = u_0 \).

\((\Leftarrow)\) If \( u_0 \) and \( x \) satisfy \( \varphi_{p, f} \), then from Definition 12 we can see that \( x \in Trans_{p_{u_0}, f} \) holds trivially.

B. A Sound and Complete Method for Generating RLFs

Based on the results established in Section IV-A, we can give a sound and complete method for automatically generating polynomial RLFs on PDSs.

Given \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \), let \( \|x\| = \sqrt{\sum_{i=1}^{n} x_i^2} \) denote the Euclidean norm of \( x \). Let \( B(x, d) = \{ y \in \mathbb{R}^n \mid \|y - x\| < d \} \) for any \( d > 0 \). Then our main result can be stated as follows.

Theorem 20 (Main Result): Given a PDS \( \tilde{x} = f(x) \) with \( f(0) = 0 \) and a parametric polynomial \( p \equiv p(u, x) \). Let \( r_0 \in \mathbb{R} \) and \( u_0 = (u_{t_0}, u_{t_0}, \ldots, u_{t_0}) \in \mathbb{R}^l \). Then \( p_{u_0} \) is an RLF in \( B(0, r_0) \) if and only if

\[ (u_{t_0}, u_{t_0}, \ldots, u_{t_0}, r_0) \in QE(\varphi_{p, f}) , \]

where

\[
\varphi_{p, f} \equiv \varphi_{p, f}^1 \land \varphi_{p, f}^2 \land \varphi_{p, f}^3 , \quad (5)
\]

\[
\varphi_{p, f}^1 \equiv p(u, 0) = 0 , \quad (6)
\]

\[
\varphi_{p, f}^2 \equiv \forall x. (\|x\|^2 > 0 \land \|x\|^2 < r^2 \rightarrow p(u, x) > 0) , \quad (7)
\]

\[
\varphi_{p, f}^3 \equiv \forall x. (\|x\|^2 > 0 \land \|x\|^2 < r^2 \rightarrow \varphi_{p, f}) . \quad (8)
\]

Proof: First, in Theorem 13 the existence of an open set \( U \) is equivalent to the existence of an open set \( B(0, r_0) \). Then according to Theorem 19 it is easy to check that (6), (7) and (8) are direct translations of conditions (a), (b) and (c) in Theorem 13.

According to Theorem 20 we can follow the three steps at the beginning of Section IV to discover polynomial RLFs on PDSs. This method is “complete” because we can discover all possible polynomial RLFs by enumerating all polynomial templates.
C. Implementation

To construct $\phi_{p,f}$ in Theorem 20 we need to compute $N_{p,f}$ in advance, which is time-consuming. What is worse, when $N_{p,f}$ is a large number the resulting $\phi_{p,f}$ can be a huge formula, for which QE is difficult. For analysis of asymptotic stability, one RLF is enough. Therefore if an RLF can be obtained by solving constraint involving merely lower order Lie derivatives, there’s no need to resort to higher order ones. Regarding this, we give an incomplete but more efficient implementation of Theorem 20 by constructing $\phi_{p,f}$ and searching for RLFs in a stepwise manner.

Let

$$\psi_{p,f} = \bigcap_{j=1}^{i-1} L_i^j p(u,x) = 0,$$

and

$$\theta_{p,f} = \forall x.(\|x\|^2 > 0 \land \|x\|^2 < r^2 \land \psi_{p,f} \rightarrow L_i^j p(u,x) < 0)$$

and

$$\tilde{\theta}_{p,f} = \forall x.(\|x\|^2 > 0 \land \|x\|^2 < r^2 \land \psi_{p,f} \rightarrow L_i^j p(u,x) \leq 0).$$

Intuitively, for $x$ satisfying $\psi_{p,f}$, we have to impose constraints $\theta_{p,f}$ or $\tilde{\theta}_{p,f}$ on the $i$-th higher order Lie derivative of $p$ along $f$. Now the RLF generation algorithm (RLFG for short) can be formally stated as follows.

**Algorithm 1: Relaxed Lyapunov Function Generation**

1. Input: $f \in \mathbb{R}[x_1, \ldots, x_n]$ with $f(0) = 0$.
2. $p \in \mathbb{R}[u_1, \ldots, u_t, x_1, \ldots, x_n]$.
3. Output: $\text{Res} \subseteq \mathbb{R}^{t+1}$.
4. $i := 1$; $\text{temp} := \emptyset$; $L_i^j p := \left( \frac{\partial}{\partial x}\right)^i p(f)$.
5. $\text{Res}^0 := \text{QE}(\theta_{p,f}) \land \phi_{p,f}$.
6. If $\text{Res}^0 = \emptyset$ then return $\emptyset$.
7. Else repeat
8. \hspace{1em} $\text{temp} := \text{Res}^{i-1} \cap \text{QE}(\theta_{p,f})$.
9. \hspace{1em} If $\text{temp} \neq \emptyset$ then
10. \hspace{2em} return $\text{temp}$.
11. \hspace{1em} Else $\text{Res}^i := \text{Res}^{i-1} \cap \text{QE}(\tilde{\theta}_{p,f})$.
12. \hspace{1em} If $\text{Res}^i = \emptyset$ then
13. \hspace{2em} return $\emptyset$.
14. \hspace{1em} Else $i := i + 1$.
15. \hspace{1em} $L_i^j p := \left( \frac{\partial}{\partial x}\right)^i L_i^j p(f)$.
16. Until $L_i^j p \notin \langle L_i^j p, L_i^{j-1} p, \ldots, L_i^1 p \rangle$.
17. Return $\emptyset$.

**Remark** Formula $\phi_{p,f}$ and $\phi_{p,f}$ in line 5 are defined in [6] and [7]; QE in line 5, 10 and 14 is done in a computer algebra tool like REDLOG [29] or QEPACK [30]; in line 20 the loop test can be done by calling the IdealMembership command in Maple [31].

The idea of Algorithm 1 is: at the $i$-th step, we search for an RLF using constraint constructed from Lie derivatives with order no larger than $i$. If this fails to produce a solution, then we add the $(i+1)$-th order Lie derivative to the constraint. This process continues until either we succeed in finding a solution, or we can conclude that there is no RLF with the predefined template, or we get to the $N_{p,f}$-th iteration, which means no solution exists at all.

Correctness of the algorithm RLFG is guaranteed by the following theorem.

**Theorem 21**: For Algorithm 1 we have

1. Termination. RLFG terminates for any valid input.
2. Soundness. If $(u,r) = (u_1, u_2, \ldots, u_t, r) \in \text{Res}$, then $p_u(x)$ is an RLF in $\mathcal{B}(0,r)$.
3. Weak Completeness. If $\text{Res} = \emptyset$ then there does not exist an RLF in the form of $p(u,x)$.

**Proof**:

1. (1) The loop condition is $L_i^j p \notin \langle L_i^j p, L_i^{j-1} p, \ldots, L_i^1 p \rangle$. By Theorem 16 RLFG can run at most $N_{p,f}$ many iterations.
2. (2) Suppose $\text{Res}^0, \text{Res}^1, \ldots, \text{Res}^k$ is the longest sequence generated by RLFG when it terminates. We can inductively prove that this sequence satisfies the following properties.
3. (P1) $0 \leq k \leq N_{p,f}$.
4. (P2) $\text{Res}^k = \text{QE}(\phi_{p,f}^{i,k} \land \phi_{p,f}^{i,k} \land \tilde{\phi}_{p,f}^i)$, for $0 \leq i \leq k$, where
   \[
   \tilde{\phi}_{p,f}^i \equiv \forall x.(\|x\|^2 > 0 \land \|x\|^2 < r^2) \rightarrow \\
   ((\bigvee_{j=1}^i \phi_{p,f}^j) \lor \phi_{p,f}^{i+1}).
   \]
5. (P3) $\text{QE}(\phi_{p,f}^{i,k} \land \phi_{p,f}^{i,k} \land \tilde{\phi}_{p,f}^i) = \emptyset$, for $1 \leq i \leq k$, where
   \[
   \phi_{p,f}^i \equiv \forall x.(\|x\|^2 > 0 \land \|x\|^2 < r^2) \rightarrow \bigvee_{j=1}^i \phi_{p,f}^j(t).
   \]
6. (P4) $\text{Res} = \emptyset$ if and only if either $\text{Res}^k = \emptyset$ or $k = N_{p,f}$; otherwise $\text{Res} = \text{QE}(\phi_{p,f}^{i,k} \land \phi_{p,f}^{i,k} \land \tilde{\phi}_{p,f}^i)$.

Suppose $(u,r) \in \text{Res}$, then by (P1), (P4) and (5) we can get $\text{Res} \subseteq \text{QE}(\phi_{p,f})$. Thus $(u,r) \in \text{QE}(\phi_{p,f})$ and $p_u(x)$ is an RLF according to Theorem 20.

3. (3) Suppose $\text{Res} = \emptyset$, then by (P4) we have either $k = N_{p,f}$ or $\text{Res}^k = \emptyset$. If $k = N_{p,f}$ (i.e. 1), then by (P3) and (5) we get $\text{QE}(\phi_{p,f}) = \emptyset$; if $\text{Res}^k = \emptyset$, from (P1), (P2), (3) as well as the validity of
   \[
   \text{QE}(\phi_{p,f}) \subseteq \text{Res}^k = \emptyset.
   \]

So far we have proved $\text{Res} = \emptyset$ implies $\text{QE}(\phi_{p,f}) = \emptyset$. Again by applying Theorem 20 we get the final conclusion.

V. Example

We illustrate our method for RLF generation using the following example.
Example 22: Consider the nonlinear dynamical system
\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{pmatrix}
-x + y^2 \\
-xy
\end{pmatrix}
\tag{9}
\]
with a unique equilibrium point \(O(0,0)\). We want to establish the asymptotic stability of \(O\). First, the linearization of (9) at \(O\) has the coefficient matrix
\[
A = \begin{pmatrix}
-1 & 0 \\
0 & 0
\end{pmatrix}
\]
with eigenvalues \(-1\) and \(0\), so none of the principles of stability for linear systems apply. Besides, a homogeneous quadratic Lyapunov function \(x^2 + axy + by^2\) for verifying asymptotic stability of (9) does not exist in \(\mathbb{R}^2\), because
\[
\text{QE} \left( \forall x \forall y . \left( x^2 + y^2 > 0 \rightarrow x^2 + axy + by^2 > 0 \right) \land (2x \dot{x} + ay \dot{x} + ax \dot{y} + 2b \dot{y} < 0) \right)
\]
is false. However, if we try to find an RLF in \(\mathbb{R}^2\) using the simple template \(p = x^2 + ay^2\), then Algorithm 1 returns \(a = 1\) at the third iteration. This means (9) has an RLF \(x^2 + y^2\), and \(O\) is asymptotically stable.

From this example, we can see that RLFs really extend the class of functions that can be used for asymptotic stability analysis, and our method for automatically discovering RLFs can save us a lot of effort in finding conventional Lyapunov functions in some cases.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, we first generalize the notion of Lyapunov functions to relaxed Lyapunov functions by considering the higher order Lie derivatives of a smooth function along a vector field. The main advantage of RLF is that it provides us more probability of certifying asymptotic stability. We also propose a method for automatically discovering polynomial RLFs for polynomial dynamical systems. Our method is complete in the sense that we can enumerate all potential polynomial RLFs by enumerating all polynomial templates for a given PDS. We believe that our methodology could serve as a mathematically rigorous framework for the asymptotic stability analysis.

The main disadvantage of our approach is the high computational complexity: the complexity of the first-order quantifier elimination over the closed fields of reals is doubly exponential [32]. Currently we are considering improving the efficiency QE on first order polynomial formulas in special forms, and it will be the main focus of our future work.

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