COUETTE FLOWS OF A VISCOUS FLUID WITH SLIP EFFECTS AND NON-INTEGER ORDER DERIVATIVE WITHOUT SINGULAR KERNEL

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Abstract. Couette flows of an incompressible viscous fluid with non-integer order derivative without singular kernel produced by the motion of a flat plate are analyzed under the slip condition at boundaries. An analytical transform approach is used to obtain the exact expressions for velocity and shear stress. Three particular cases from the general results with and without slip at the wall are obtained. These solutions, which are organized in simple forms in terms of exponential and trigonometric functions, can be conveniently engaged to obtain known solutions from the literature. The control of the new non-integer order derivative on the velocity of the fluid moreover a comparative study with an older model, is analyzed for some flows with practical applications. The non-integer order derivative with non-singular kernel is more appropriate for handling mathematical calculations of obtained solutions.

1. Introduction. The nonslip boundary condition is one of the primary principles that provides basis for linearly viscous fluid’s mechanics to built on. For a large class of flows, most of the experiments function favourably for the nonslip boundary condition. Interesting debates concerning the acceptance of on nonslip condition are traced out. [7]. The success rate for nonslip condition is nonstatic: in case of the great variety of flows, it has proved its efficacy while it falls short of its performance in case of problems involving flows in micro channels or in wavy tubes, multiple interfaces, flows of polymeric liquids or flows of rarefied fluids.

A few years back, Navier [15] suggested a slip boundary condition where relative velocity that is also known as the slip velocity relies linearly on the shear stress. For describing the slip that happens at solid boundaries, a great number of models have

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been suggested. A list of these modals is found in the reference [17]. Monney’s study is a pioneer work done on the slip at the boundary [14]. Many researchers published their work undertaken to discuss the flows of Newtonian or non-Newtonian fluids with slip at the boundary. Khalid and Vafai [10] undertook the research on the effects of the slip condition on Stoke’s and Coutte flows because of an oscillating wall. Stoke’s flows of a Maxwell fluid with wall slip condition is studied by Vieru and Rauf [20]. Abelman et al [1] analysed the Coutte flow of a third grade fluid with rotating frame and slip condition.

A frequent discussion on Fractional calculus is found just as the standard differential and integral calculus are the most studied topics and a list of the application of fractional calculus is huge to be included here. Hence, It is significant to note the fact that fractional derivative generalizations of one dimensional viscoelastic models is of huge usage in modeling the response linear regime [9] and they are in agreement with the second principle of thermodynamics. Furthermore, a suitable agreement of experimental work was attained as an outcome of the work of Makris et al. [13], by using the non-integer order Maxwell model instead of an ordinary one. The aforementioned researchers substantiated that the applied method i.e. fractional method has got a stronger memory of the past than the other method called, ordinary method. It is noticed that the fractional calculus has been frequently used during the last few years. The researchers solved so many motion problems by using this method [11].

Most frequently, by restoring the integer order time derivatives through formal left hand Liouville or Riemann-Liouville differential operators, the governing equation analogous to motions of ordinary fluid models are adapted [12, 16, 5]. Nonetheless, a few deficiencies are found in both of these operators as well as the Caputo operator. Their essential core is singular and the nearly all results that have been attained through these methods are displayed in a complex way, encompassing some generalized functions even for Newtonian fluids [18, 8].

A contemporary definition of non integer order derivative is given by Caputo and Fabrizio [4] recently with smooth kernel so as to perform a function both for spatial and temporal variables. Ensuing, this derivative has already being applied for the solution of real problems because of its benefit when the Laplace transform is used to do problems with initial conditions [2, 6]. Atangana and Baleanu proposed a latest fractional operator whose kernel is also non-singular [3]. It has its foundations liked to the Mittag-Leffer function and is helpful in material and thermal sciences. It is significant to mention that both fractional derivatives namely, Atangana-Baleanu and Caputo-Fabrizio, have all the advantages which are provided by of Riemann-Liouville and Caputo operators. Additionally, their kernel is non-singular as well.

The main objective is to rely on modern definition of non-integer order derivative to attain precise common solutions for an incompressible viscous fluid’s Couette flows without singular kernel created by the movement of infinite plate, are analyzed under the slip condition at boundaries; whereas the bottom plate is supposed to be translated in its plane with a given velocity. The flow of the fluid is studied under the assumption that the relative velocity between the fluid at the wall and the wall itself is proportional to the shear rate at the wall. An integral transform determines the accurate expressions for velocity and shear stress. Hence, the velocity fields for viscous fluid analogous to both slip and non slip conditions are derived. The gained results are displayed in the comprehendible way including both trigonometric and exponential functions and can be expediently used to retrieve
the connected solutions for ordinary fluids. Finally the results for viscous fluids are viewed in comparison Newtonian fluids under both, slip and non slip conditions. Some properties of the flow are also presented.

2. Statement of the problem. As shown in the Fig. 1 consider $Oxyz$ Cartesian coordinate system with $y > 0$ in the upward direction and an infinite solid plane wall situated in the $(x, z)$-plane. The second infinite solid plane wall occupies the plane $y = h > 0$. An incompressible viscous fluid fills the slab $y \in (0, h)$. The fluid and plates are at rest at $t = 0$. At the moment $t = 0^+$, the fluid is set into motion due to translation of the bottom plate in the $x$-axis direction moving with the velocity $u_w(t) = U_0 f(t)$, where $U_0 > 0$ (constant) and $f(t)$ is a piecewise continuous Laplace transformable function defined on $[0, \infty)$ with $f(0) = 0$. For the flow parallel to the $x$-axis, we are interested in a solution for the velocity profile $\mathbf{V} = (u(y, t), 0, 0)$. For such motions, the continuity equation is identically verified while the motion and constitutive equations lead to the relevant partial differential equations

$$
\tau(y, t) = \mu \frac{\partial u(y, t)}{\partial y}, \ (y, t) \in (0, h) \times (0, \infty), \quad (1)
$$

$$
\frac{\partial u(y, t)}{\partial t} = \nu \frac{\partial^2 u(y, t)}{\partial y^2}, \ (y, t) \in (0, h) \times (0, \infty), \quad (2)
$$

where $\tau(y, t) = S_{xy}(y, t)$ is one of the non zero component of $\mathbf{S}$ and $\rho$ is the constant density of the fluid. We assume the presence of slip at the walls and consider that the relative velocity between the velocity of the fluid at the wall and wall is proportional to the shear rate at the wall. The appropriate initial and boundary conditions under the assumption of slip at the wall are;

$$
u u(y, 0) = \frac{\partial u(y, t)}{\partial t} \bigg|_{t=0} = 0, \ \tau(y, 0) = 0, \ y \in [0, h], \quad (3)
$$

$$
u u(0, t) - \beta \frac{\partial u(y, t)}{\partial y} \bigg|_{y=0} = U_0 f(t), \ t > 0, \quad (4)
$$

$$
u u(h, t) + \beta \frac{\partial u(y, t)}{\partial y} \bigg|_{y=h} = 0, \ t > 0, \quad (5)
$$

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{geometry.png}
\caption{Geometry of flow}
\end{figure}
where $\beta$ is the slip coefficient. We introduce the following dimensionless quantities

$$ t^* = \frac{t}{T}, y^* = \frac{y}{h}, u^* = \frac{u}{U_0}, \tau^* = \frac{\tau}{(\rho h U_0 T)}, \beta^* = \frac{\beta}{h}, $$

(6)

where $T > 0$ is a characteristic time, in the above model for the non-dimensionalization of the physical parameters. The non-dimensional flow equations after dropping the asterisks are

$$ \tau(y, t) = \frac{1}{R} \frac{\partial u(y, t)}{\partial y}, (y, t) \in (0, 1) \times (0, \infty), $$

(7)

$$ \frac{\partial u(y, t)}{\partial t} = \frac{1}{R} \frac{\partial^2 u(y, t)}{\partial y^2}, (y, t) \in (0, 1) \times (0, \infty), $$

(8)

$$ u(y, 0) = \frac{\partial u(y, t)}{\partial t} |_{t=0} = 0, \quad y \in [0, 1], $$

(9)

$$ \tau(y, 0) = 0, \quad y \in [0, 1], $$

(10)

$$ u(0, t) - \beta \frac{\partial u(y, t)}{\partial y} |_{y=0} = g(t), \quad t > 0 $$

(11)

and

$$ u(1, t) + \beta \frac{\partial u(y, t)}{\partial y} |_{y=1} = 0, \quad t > 0, $$

(12)

where $R = \frac{h^2}{\nu T}$ is the Reynolds number and $g(t^*) = f(T t^*)$. Applying Caputo-Fabrizio time Fractional Derivative we get the governing equation

$$ D^\alpha_t u(y, t) = \frac{1}{R} \frac{\partial^2 u(y, t)}{\partial y^2}, \quad (y, t) \in (0, 1) \times (0, \infty), $$

(13)

Here, unlike the previous published papers, the Caputo-Fabrizio derivative operator of order $\alpha$ [4, 5, 6]

$$ D^\alpha_t [b(t)] = \frac{1}{1 - \alpha} \int_0^t b'(t) \exp \left( -\frac{\alpha(t - s)}{1 - \alpha} \right) ds \quad \text{for} \quad 0 \leq \alpha \leq 1, $$

(14)

will be used. We firstly solve the fractional differential (13) with the initial and boundary conditions (IBC’s), and use the obtained results to develop the solution corresponding to the shear stress.

3. Solution of the problem.

3.1. Velocity field. Applying the Laplace transform to (13) and using the initial conditions from (9) we obtain, the following set of equations

$$ \frac{\partial^2 \tilde{u}(y, q)}{\partial y^2} - A \tilde{u}(y, q) = 0, $$

(15)

where $A = \frac{R q}{(1-\alpha)q+\alpha}$, solution of above differential equation is

$$ \tilde{u}(y, q) = C_1 \cosh \sqrt{A} y + C_2 \sinh \sqrt{A} y, $$

(16)

For constant $C_1$ and $C_2$ use boundary conditions after using Laplace transform

$$ C_2 = \frac{-G(q)(\cosh \sqrt{A} + \beta \sqrt{A} \sinh \sqrt{A})}{2 \beta \sqrt{A} \cosh \sqrt{A} + (\beta^2 A + 1) \sinh \sqrt{A}}, $$

and

$$ C_1 = \frac{G(q)(\beta \sqrt{A} \cosh \sqrt{A} + \sinh \sqrt{A})}{2 \beta \sqrt{A} \cosh \sqrt{A} + (\beta^2 A + 1) \sinh \sqrt{A}}. $$
where Laplace transform of \( g(t) \) is \( G(q) \). Using these values in (16), we get
\[
\tilde{u}(y, q) = \frac{G(q)\left(\beta \sqrt{A} \cosh \sqrt{A} + \sinh \sqrt{A} \right)}{2\beta \sqrt{A} \cosh \sqrt{A} + (\beta^2 A + 1) \sinh \sqrt{A}} \cosh \sqrt{A} y
\]
\[
- \frac{G(q)\left(\cosh \sqrt{A} + \beta \sqrt{A} \sinh \sqrt{A} \right)}{2\beta \sqrt{A} \cosh \sqrt{A} + (\beta^2 A + 1) \sinh \sqrt{A}} \sinh \sqrt{A} y.
\]

or
\[
\tilde{u}(y, q) = G(q) F_1(q) - G(q) F_2(q),
\]
where \( F_1(q) = \frac{\beta \sqrt{A} \cosh \sqrt{A} + \sinh \sqrt{A}}{2\beta \sqrt{A} \cosh \sqrt{A} + (\beta^2 A + 1) \sinh \sqrt{A}} \cosh \sqrt{A} y \)
and \( F_2(q) = \frac{\cosh \sqrt{A} + \beta \sqrt{A} \sinh \sqrt{A}}{2\beta \sqrt{A} \cosh \sqrt{A} + (\beta^2 A + 1) \sinh \sqrt{A}} \sinh \sqrt{A} y \)

In order to obtain the inverse Laplace transform of function \( F_1(y, q) \), we consider
the auxiliary function
\[
F_1(q) = \frac{\beta \sqrt{A} \cosh \sqrt{A} + \sinh \sqrt{A}}{2\beta \sqrt{A} \cosh \sqrt{A} + (\beta^2 A + 1) \sinh \sqrt{A}} \cosh \sqrt{A} y
\]
whose singular points are simple poles located at
\[
q_n = -\frac{\alpha p_n^2}{R + p_n^2 (1 - \alpha)}, \quad n = 1, 2, ...
\]
where \( p_n \neq 0 \) are the real roots of the equation
\[
\tan(p_n) = \frac{2\beta p_n}{\beta^2 p_n^2 - 1}.
\]

By using the residue theorem and after some simplifications we get, the Laplace inverse transform of \( F_1(y, q) \) as follows
\[
f_1(y, t) = L^{-1}\{F_1(y, q)\}
\]
\[
= \sum_{n=1}^{\infty} \text{Res}[F_1(y, q)e^{qt}, q_n],
\]
\[
L^{-1}\{F_1(y, q)\} = \sum_{n=1}^{\infty} A_n(y)e^{\frac{-p_n^2}{R + p_n^2 (1 - \alpha)}}
\]

Similarly
\[
L^{-1}\{F_2(y, q)\} = \sum_{n=1}^{\infty} B_n(y)e^{\frac{-p_n^2}{R + p_n^2 (1 - \alpha)}}
\]

where
\[
A_n(y) = \frac{2R \alpha \left[\beta p_n \cos \theta_n + \sin \theta_n \right] \cos \nu_n y}{\left[(\beta^2 p_n^2 - 2\beta - 1) \cos \nu_n + 2\beta p_n (\beta + 1) \sin \nu_n \right] \left(R + p_n^2 (1 - \alpha)\right)^2}
\]

\[
B_n(y) = \frac{2R \alpha \left[\beta p_n \cos \theta_n + \sin \theta_n \right] \cos \nu_n y}{\left[(\beta^2 p_n^2 - 2\beta - 1) \cos \nu_n + 2\beta p_n (\beta + 1) \sin \nu_n \right] \left(R + p_n^2 (1 - \alpha)\right)^2}
\]
and

\[ B_n(y) = \frac{-2Rp_n\alpha [\cos p_n - \beta p_n\sin p_n] \sin p_n y}{[(\beta^2 p_n^2 - 2\beta - 1)\cos p_n + 2\beta p_n(\beta + 1)\sin p_n] (R + p_n^2(1 - \alpha))^2} \]

Thus

\[ u(y, t) = \sum_{n=1}^{\infty} \int_0^t A_n(y) e^{\frac{-p_n^2\alpha}{R + p_n^2(1 - \alpha)}} g(t - \tau) d\tau + \sum_{n=1}^{\infty} \int_0^t B_n(y) e^{\frac{-p_n^2\alpha}{R + p_n^2(1 - \alpha)}} g(t - \tau) d\tau, \]

(22)

we can also write in the following equivalent form

\[ u_{GNs}(y, t) = \sum_{n=1}^{\infty} \frac{2Rp_n\alpha}{(R + p_n^2(1 - \alpha))^2} C_n \int_0^t e^{\frac{-p_n^2\alpha}{R + p_n^2(1 - \alpha)}} g(t - \tau) d\tau, \]

(23)

where

\[ C_n = \frac{\beta p_n\cos(1 - y)p_n + \sin(1 - y)p_n}{(\beta^2 p_n^2 - 2\beta - 1)\cos p_n + 2\beta p_n(\beta + 1)\sin p_n} \]

• For flows of generalized Newtonian fluids with a no slip boundary condition, that is \( \beta = 0 \), the transform domain solution is

\[ u_{GN}(y, t) = \sum_{n=1}^{\infty} \frac{2\pi R\alpha}{(R + \pi^2 n^2(1 - \alpha))^2} n\sin(n\pi y) \int_0^t e^{\frac{-n^2\alpha}{R + \pi^2 n^2(1 - \alpha)}} g(t - \tau) d\tau \]

(24)

• For flows of viscous Newtonian fluids with a no slip boundary condition, that is \( \alpha \to 1 \) and \( \beta = 0 \), the transform domain solution is

\[ u_{N}(y, t) = \frac{2\pi}{R} \sum_{n=1}^{\infty} n \sin(n\pi y) \int_0^t g(t - s) e^{\frac{-n^2\alpha}{R + \pi^2 n^2(1 - \alpha)}} ds, \]

(25)

which is similar to \([19], \text{Eq. (37)}\)

• The relative velocity between the fluid at the bottom wall and wall itself for fractional order Couette flow fluids

\[ u_{GNrel}(y, t) = u_{GNs}(0^+, t) - g(t) \]

\[ = \sum_{n=1}^{\infty} \frac{2Rp_n\alpha}{(R + p_n^2(1 - \alpha))^2} C_n \times \]

\[ \times \int_0^t e^{\frac{-p_n^2\alpha}{R + p_n^2(1 - \alpha)}} g(t - \tau) d\tau - g(t), \]

(26)

and for a viscous Newtonian fluid is given by for \( \alpha \to 1 \)

\[ u_{Nrel}(t) = u_{Ns}(0^+, t) - g(t) \]
\[ \tau(y, t) = \frac{1}{R} \frac{\partial u(y, t)}{\partial y}, \quad (y, t) \in (0, 1) \times (0, \infty), \] (28)

using the value of velocity from (23), we have the final expression as follow

\[ \tau(y, t) = \sum_{n=1}^{\infty} \frac{2p_n^2 \alpha}{(R + p_n^2(1 - \alpha))^2} C_n \times \]
\[ \int_0^t e^{\frac{-p_n^2 \alpha R}{R + p_n^2(1 - \alpha)}} g(t - \tau) d\tau. \] (29)

4. Numerical results and discussion. With a view to get a little insight of the results that have been obtained, three special cases with engineering applications are considered along with different graphical representations and are presented and discussed, as well as for comparison, the graphical representations are prepared only for the velocity fields corresponding to the three kinds of motions.

4.1. Translation of the plate with a constant velocity. The motion of the bottom plate is given by the function \( g(t) = H(t) \) and the velocity \( u(y, t) \) is obtained from (23) and (24) with \( g(t - \tau) = 1 \) for \( s \in (0, t) \). The velocity field corresponding to this type of the motion has the following expressions:

- For fractional order Newtonian fluid with slip at the boundary:
  \[ u_{GNs}(y, t) = \sum_{n=1}^{\infty} \frac{2R}{p_n(R + p_n^2(1 - \alpha))} C_n \times \]
  \[ H(t)[1 - e^{\frac{-p_n^2 \alpha R}{R + p_n^2(1 - \alpha)}}]. \] (30)

- For generalized Newtonian fluids with no slip at the boundary:
  \[ u_{GN}(y, t) = \sum_{n=1}^{\infty} \frac{2R}{n\pi(R + \pi^2n^2(1 - \alpha))} \sin(n\pi y)[1 - e^{\frac{-p_n^2 \alpha}{R + p_n^2(1 - \alpha)}}]. \] (31)

- The relative velocity for fraction order is given by
  \[ u_{GNrel}(y, t) = \sum_{n=1}^{\infty} \frac{2R}{p_n(R + p_n^2(1 - \alpha))} C_n \times \]
  \[ H(t)[1 - e^{\frac{-p_n^2 \alpha}{R + p_n^2(1 - \alpha)}}] - H(t). \] (32)

4.2. Translation of the plate with a constant acceleration. The motion of the bottom plate is given by the function \( g(t) = H(t)t \) and the velocity \( u(y, t) \) is
obtained from (23) and (24) with \( g(t - \tau) = t - \tau \) for \( s \in (0, t) \). The velocity field corresponding to this type of the motion has the following expressions:

- **Solution with slip at the boundary:**

  \[
  u_{GN,s}(y, t) = \sum_{n=1}^{\infty} \frac{2R}{p_n(R + p_n^2(1 - \alpha))} C_n \left[ e^{-R\frac{2\alpha}{p_n^2(1 - \alpha)}} t - 1 \right].
  \]

- **Solution with no slip at the boundary:**

  \[
  u_{GN}(y, t) = \sum_{n=1}^{\infty} \frac{2R}{n\pi(R + \pi^2n^2(1 - \alpha))} \sin(n\pi y) H(t)t + \sum_{n=1}^{\infty} \frac{2R}{n^3\pi^4\alpha} \sin(n\pi y) \left[ e^{-R\frac{2\alpha}{\pi^2n^2(1 - \alpha)}} t - 1 \right].
  \]

- **The relative velocity is given by**

  \[
  u_{GNrel}(y, t) = \sum_{n=1}^{\infty} \frac{2R}{p_n(R + p_n^2(1 - \alpha))} H(t)C_n t + \sum_{n=1}^{\infty} \frac{2R}{p_n^2\alpha} C_n H(t) \left[ e^{-R\frac{2\alpha}{p_n^2(1 - \alpha)}} t - 1 \right] - H(t)t.
  \]

### 4.3. Solution for the sinusoidal oscillations of the bottom plate

In this section, we will consider the function \( g(t) = \sin t \) for the motion of the bottom plate. The velocity fields corresponding to this type of motion are obtained from (23) and (24) if \( g(t - s) \) is replaced by \( \sin(t - s) \). The velocity field corresponding to this type of motion is

- **The velocity field corresponding to the Couette flow of fractional order viscous fluid with slip**

  \[
  u_{GN,s}(y, t) = \sum_{n=1}^{\infty} \frac{2Rp_n\alpha}{(p_n^2\alpha)^2(R + p_n^2(1 - \alpha))} C_n \times \left[ e^{-R\frac{2\alpha}{p_n^2(1 - \alpha)}} t + \frac{p_n^2\alpha}{R + p_n^2(1 - \alpha)} \sin t - \cos t \right].
  \]

- **The Couette flow of a generalized viscous fluid with no slip boundary condition**

  \[
  u_{GN}(y, t) = \sum_{n=1}^{\infty} \frac{2Rn\pi\alpha}{(\pi^2n^2\alpha)^2 + (R + \pi^2n^2(1 - \alpha))^2} \sin(n\pi y) \times \left[ e^{-R\frac{2\alpha}{\pi^2n^2(1 - \alpha)}} t + \frac{\pi^2n^2\alpha}{R + \pi^2n^2(1 - \alpha)} \sin t - \cos t \right].
  \]

- **The relative velocity is given by**

  \[
  u_{GNrel}(y, t) = \sum_{n=1}^{\infty} \frac{2Rp_n\alpha}{(p_n^2\alpha)^2(R + p_n^2(1 - \alpha))} C_n \times \left[ e^{-R\frac{2\alpha}{p_n^2(1 - \alpha)}} t + \frac{p_n^2\alpha}{R + p_n^2(1 - \alpha)} \sin t - \cos t \right] - \sin t.
  \]
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Figure 2. Velocity profiles (VP) varus $y$, with $\beta = 0.4$, $t = 0.2$ for different values of $\alpha$ and when translation of the plate with a constant velocity ($g(t) = H(t)$).

Figure 3. VP varus $y$, with $\beta = 0.4$, $t = 0.4$ for different values of $\alpha$ and $g(t) = H(t)$.

Figure 4. VP varus $y$, with $\beta = 0.4$, $t = 0.8$ for different values of $\alpha$ and $g(t) = H(t)$.
Figure 5. VP varus $y$, with $\beta = 0.7$, $t = 0.2$ for different values of $\alpha$ and $g(t) = H(t)$.

Figure 6. VP varus $y$, with $\alpha = 0.3$, $t = 0.2$ for different values of $\beta$ and $g(t) = H(t)$.

Figure 7. VP varus $y$, with $\alpha = 0.6$, $t = 0.2$ for different values of $\beta$ and $g(t) = H(t)$.
Figure 8. VP varus $y$, with $\alpha = 0.9$, $t = 0.2$ for different values of $\beta$ and $g(t) = H(t)$.

Figure 9. VP varus $t$, with $\beta = 0.0$ (no-slip), $y = 0.2$ for different values of $\alpha$ and $g(t) = H(t)$.

Figure 10. VP varus $t$, with $\beta = 0.4$, $y = 0.2$ for different values of $\alpha$ and $g(t) = H(t)$. 
Figure 11. VP varus $t$, with $\beta = 0.7$, $y = 0.2$ for different values of $\alpha$ and $g(t) = H(t)$.

Figure 12. VP varus $y$, with $\beta = 0.4$, $t = 0.2$ for different values of $\alpha$ and when translation of the plate with a constant acceleration ($g(t) = t$).

Figure 13. VP varus $y$, with $\beta = 0.4$, $t = 0.4$ for different values of $\alpha$ and $g(t) = t$. 
Figure 14. VP varus $y$, with $\beta = 0.4$, $t = 0.8$ for different values of $\alpha$ and $g(t) = t$.

Figure 15. VP varus $y$, with $\beta = 0.7$, $t = 0.2$ for different values of $\alpha$ and $g(t) = t$.

Figure 16. VP varus $y$, with $\alpha = 0.3$, $t = 0.2$ for different values of $\beta$ and $g(t) = t$. 
Figure 17. VP varus $y$, with $\alpha = 0.6$, $t = 0.2$ for different values of $\beta$ and $g(t) = t$.

Figure 18. VP varus $y$, with $\alpha = 0.9$, $t = 0.2$ for different values of $\beta$ and $g(t) = t$.

Figure 19. VP varus $t$, with $\beta = 0.0$ (no-slip), $y = 0.2$ for different values of $\alpha$ and $g(t) = t$. 
Figure 20. VP varus $t$, with $\beta = 0.4$, $y = 0.2$ for different values of $\alpha$ and $g(t) = t$.

Figure 21. VP varus $t$, with $\beta = 0.0$, $y = 0.2$ for different values of $\alpha$ and $g(t) = t$.

Figure 22. VP varus $y$, with $\beta = 0.4$, $t = 0.2$ for different values of $\alpha$ and with the sinusoidal oscillations of the bottom plate ($g(t) = \sin t$).
Figure 23. VP varus $y$, with $\beta = 0.4$, $t = 0.4$ for different values of $\alpha$ and $g(t) = \sin t$.

Figure 24. VP varus $y$, with $\beta = 0.4$, $t = 0.8$ for different values of $\alpha$ and $g(t) = \sin t$.

Figure 25. VP varus $y$, with $\beta = 0.7$, $t = 0.2$ for different values of $\alpha$ and $g(t) = \sin t$. 
Figure 26. VP varus y, with $\alpha = 0.3$, $t = 0.2$ for different values of $\beta$ and $g(t) = \sin t$.

Figure 27. VP varus y, with $\alpha = 0.6$, $t = 0.2$ for different values of $\beta$ and $g(t) = \sin t$.

Figure 28. VP varus y, with $\alpha = 0.9$, $t = 0.2$ for different values of $\beta$ and $g(t) = \sin t$. 
Figure 29. VP varus $t$, with $\beta = 0.0$ (no-slip), $y = 0.2$ for different values of $\alpha$ and $g(t) = \sin t$.

Figure 30. VP varus $t$, with $\beta = 0.4$, $y = 0.2$ for different values of $\alpha$ and $g(t) = \sin t$.

Figure 31. VP varus $t$, with $\beta = 0.7$, $y = 0.2$ for different values of $\alpha$ and $g(t) = \sin t$. 
5. Conclusions. In this work we discuss the channel flows of a viscous fluid with modern definition of time fractional Caputo-Fabrizio derivatives with non-singular kernel. Some limiting cases are obtained from general results and results are plotted for velocity field and spatial variable $y$. All the parameters are treated as dimensionless. For the first case (Translation of the plate with a constant velocity) Fig. (2 – 5) Shows the effect of non-integer order fractional parameter and time. It is observed that by increasing value of fractional parameter $\alpha$, the fluid velocity increasing. This is the reason by the using fractional model flow can be enhanced, i.e the fractional fluid has greater velocity than ordinary fluid. Further, velocity is the increasing function of time. Fig. (6 – 8) depicted to see the effect of slip on the fluid velocity. To see the influence of slip and without slip. When there is no slip at the wall fluid velocity is decreasing as we increased the value of slip parameter and non-integer order $\alpha$, the fluid velocity increases. Fig. (9 – 11) plotted between fluid velocity and time to see the fluid velocity for larger value of $\alpha$ increases in the absence of slip and as we increased the slip the momentum of boundary thickness decreases and fluid velocity also decreases. For the second case (Translation of the plate with a constant acceleration) Fig. (12 – 15) Shows the effect of $\alpha$ and time. It is noted that the fluid velocity increasing function of $\alpha$ and time. Fig. (16 – 18) shows the influence of slip and without slip. When there is no slip at the wall fluid velocity is decreasing as we increased the value of slip parameter and non-integer order $\alpha$, the fluid velocity increases. Fig. (19 – 21) plotted to see the fluid velocity for larger value of $\alpha$ increases in the absence of slip and as we increased the slip the momentum of boundary thickness decreases and fluid velocity also decreases. For the third case (sinusoidal oscillations of the bottom plate) Fig. (22 – 31) similar trend is observed as for other two cases.

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