CONCENTRATION INEQUALITIES FOR RANDOM WALKS ON PROPER HYPERBOLIC SPACES

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Abstract. We prove Hoeffding-type concentration inequalities around the drift for the displacement of non-elementary random walks on proper hyperbolic spaces. The bounds depend on the size of support of the measure as in the classical case of sums of independent random variables, but also on the norm of the driving probability measure in the left regular representation of the group of isometries. We deduce uniform bounds in the case of hyperbolic groups and effective bounds for simple linear groups of rank-one. These results follow from a more general, but less explicit, concentration statement that we prove for cocycles which satisfy a certain cohomological equation. Finally, we also deduce subgaussian concentration bounds around the top Lyapunov exponent of random matrix products in arbitrary dimension.

1. Introduction

Let \((M, d)\) be a metric space and \(\text{Isom}(M)\) the group of isometries of \(M\). Consider a finitely supported probability measure \(\mu\) on \(\text{Isom}(M)\), let \((X_i)_{i \in \mathbb{N}}\) be a sequence of independent random variables with distribution \(\mu\) and denote by \(R_n\) the random variable given by the product \(X_1 \ldots X_n\). Fix a basepoint \(o \in M\) and consider the random walk \(R_n o\) on \(M\). A straightforward application of Kingman’s subadditive ergodic theorem shows that there exists a constant \(\ell(\mu) \geq 0\), called the drift of the random walk, such that

\[
\frac{1}{n} d(R_n o, o) \xrightarrow{a.s.} \ell(\mu), \quad (1.1)
\]

This can be seen as a generalization of the classical law of large numbers which corresponds to the case \(M = \mathbb{R}\) and \(\mu\) supported on the translations \(\mathbb{R} < \text{Isom}(\mathbb{R})\).

Understanding various aspects of the convergence (1.1) (e.g. central limit theorem (CLT), large deviation principles (LDP), Azuma-type concentration inequalities) in the aforementioned special case constitutes a fundamental part of classical probability theory. Various other cases have attracted considerable attention relatively more recently: starting in ’60s with the work of Furstenberg, Kesten, Oseledets, Kaimanovich [FK60, Fur63, Ose68, Kai89] for symmetric spaces of non-compact type and with Dynkin–Malyutov [DM61], Furstenberg [Fur73], Kaimanovich–Vershik [KV83] and others for random walks on countable groups. More recently, for general metric spaces with an assumption of coarse negative-curvature (namely Gromov hyperbolicity), a number of analogues of

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the classical results were proven including CLT’s [BQ16b, MS20], local limit theorems [Gou14], and closer to our considerations, LDP’s and exponential decay results [BMSS, Gou17]. Our goal in this paper is to establish Hoeffding-type concentration inequalities in the general setting of random walks on hyperbolic spaces. To the best of our knowledge, this aspect of the classical theory is far less developed in our setting.

Concentration inequalities around the mean \( \ell(\mu) \) have two distinctive features compared to asymptotic large deviations estimates: on the one hand, these are large deviation bounds for the fluctuations of the distance of the random walk that are valid \textit{uniformly over all times} as opposed to asymptotic estimates. On the other hand, the exponential decay rate is expressed \textit{as an explicit function of the normalized deviation distance} \( t \). As such, these inequalities have been useful in the classical case both from a pure mathematics and applied or computational perspectives. Accordingly, one of the main reasons that we focus our attention in this article to Gromov hyperbolic spaces is that, by following a geometric and harmonic analytic technique of Benoist–Quint [BQ16b], we are able to exploit their geometry and consequently pin down all involved quantities in a rather explicit way. Indeed, our approach also allows us, for example, to prove subgaussian concentration estimates for random matrix products, but with less explicit bounds. These results are also new and discussed later in the introduction.

Our general approach for proving concentration bounds is in line with Gordin’s method for proving the central limit theorem; it is based on relating the values of cocycles along random walks coming from group actions to martingales via a Poisson type equation. In particular, the explicit solutions by Benoist–Quint of associated cohomological equations for Busemann and norm cocycles, respectively on the boundary of hyperbolic spaces [BQ16b] and projective spaces [BQ16a], play a crucial role in the application of our general cocycle-concentration results to these settings. In the former case, we slightly extend this solution to adapt it to our purposes and get explicit bounds on the size. These bounds involve the norm \( \|\lambda_G(\mu)\|_2 \) of the regular representation \( \lambda_G \) of a probability measure \( \mu \) on the isometry group \( G = \text{Isom}(M) \). In a later part, we use various versions of uniform Tits’ alternatives to control the size of \( \|\lambda_G(\mu)\|_2 \) which in turn yields effective constants for example in the case of linear groups of rank one, thanks to the works of Breuillard [Bre08, Bre11a].

Let us now state our main result, some of its consequences and related remarks.

1.1. Subgaussian concentration estimates for random walks on hyperbolic spaces. We require some notation and definitions to state our main result.

Let \((M,d)\) be a proper metric space, we denote by \( G \) its group of isometries. It is a locally compact group and we denote by \( \mu_G \) a Haar measure on \( G \). For every \( r \in [0,1] \), we denote \( \mu_{r,\text{lazy}} = r\delta_{\text{id}} + (1-r)\mu \). Furthermore, we denote by \( \lambda_G(\mu) \) the operator given by the image of the probability measure \( \mu \) under the the left-regular representation of \( G \) on \( L^2(G) \). Finally, having fixed a basepoint \( o \in M \), for an element \( g \in G \), we set \( \kappa(g) := d(go,o) \) and for a set \( S \subset G \), \( \kappa_S := \sup\{\kappa(g); g \in S\} \). The set \( S \) is said to be bounded if \( \kappa_S < \infty \).
Given $\delta \geq 0$, by a $\delta$-hyperbolic metric space $M$, we understand a metric space $M$ such that for every $x, y, z, o \in M$, we have $(x|y)_o \geq (x|z)_o \wedge (z|y)_o - \delta$, where $(\cdot, \cdot)$ is the Gromov product given by $(x|y)_o = \frac{1}{2}(d(x, o) + d(y, o) - d(x, y))$. A probability measure $\mu$ is called non-elementary if its support $S$ generates a semigroup that contains two independent loxodromic elements (see §3.2). We can now state

**Theorem 1.2.** Let $(M, d)$ be a proper geodesic $\delta$-hyperbolic space and $o \in M$. Assume that the group $G = \text{Isom}(M)$ acts cocompactly on $M$. Then, there exist an explicit positive function $D(\cdot, \cdot) > 0$ and a constant $C_M > 0$ such that for every non-elementary probability measure $\mu$ on $G$ with bounded support $S$, for every $t \geq 0$ and $n \in \mathbb{N}$ we have

$$
P(|\kappa(R_n) - n\ell(\mu)| \geq nt) \leq 2 \exp\left(-\frac{nt^2}{\kappa_S D(\kappa_S, ||\lambda_G(\mu_{\text{r, lazy}})||_2)}\right)$$

for every $r \in [0, 1)$.

This statement will follow from a more general concentration result (Theorem 4.1) for the Busemann cocycle on the horofunction compactification of $M$.

To convey the dependence of this upper bound to the involved quantities and for practical use, in the following remark we provide a function that one can substitute for the function $D$ in the previous result.

**Remark 1.3 (On the upper bound).** One can take

$$D(\kappa, \lambda) \leq 32(16 \ln^+(\kappa) + 8A_0/3 + 33)^2 \frac{1}{(1 - \sqrt{\lambda})^4},$$

where $A_0 = \left(\frac{\mu_G(B_{2R(\delta)} + 2D_0)}{\mu_G(B_{R(\delta)} + D_0)}\right)^{1/2}$ with $R(\delta) = 14\delta + 4$, for $r \geq 0$, $B_r := \{g \in G; d(go, o) \leq r\}$ and $D_0 := 2\text{diam}(G \setminus M)$. We also set $D(\kappa, 1) = \infty$. Note that if $\mu$ is non-elementary, then for every $r \in (0, 1)$, we have $||\lambda_G(\mu)||_2 < 1$ (see Remark 4.4).

In the sequel, we will see that each of the two aspects of the upper bound in Theorem 1.2, namely its subgaussian form and its parameters of dependence, have implications and strengthenings. On the one hand, by combining this upper bound with versions of uniform Tits alternatives in various contexts (which entail uniform bounds for $||\lambda_G(\mu)||_2$, see Lemma 5.3), we will obtain uniform concentration estimates for a class of driving probability measures, see Corollaries 1.5 and 1.6. On the other hand, the subgaussian character allows us for instance to provide a global quadratic lower bound (see Corollary 1.9) for the rate function of large deviations, recently studied in this setting by [BMSS]. Let us now explain these consequences.

### 1.4. The cases of hyperbolic and rank-one linear groups.

Firstly, specifying Theorem 1.2 to hyperbolic groups, and using Koubi’s uniform Tits alternative [Kou98, Theorem 5.1], we obtain the following more precise concentration result for random walks on hyperbolic spaces.
Corollary 1.5. Let $(M, d)$ be a proper geodesic hyperbolic metric space and $o \in M$. Then there exist constants $C_M, A_M > 0$ such that for every non-elementary probability measure $\mu$ of finite support $S$ in $\text{Isom}(M)$ with the property that the group $\Gamma$ generated by $S$ acts properly and cocompactly on $M$, there exist constants $\alpha_\Gamma > 0$ and $N_\Gamma \in \mathbb{N}$ depending only on $\Gamma$ such that for every $t > 0$ and $n \in \mathbb{N}$, setting $m_\mu = \min_{g \in S} \mu(g)$, we have

$$
\mathbb{P} \left( |\kappa(R_n) - n\ell(\mu)| \geq nt \right) \leq 2 \exp\left( \frac{-nt^2}{m_\mu^{N_\Gamma} \alpha_\Gamma \kappa_S^2 (\ln^+ (\kappa_S) + A_M)} \right).
$$

Specifying Theorem 1.2 to rank one matrix groups and using the strong Tits alternative of Breuillard [Bre08, Bre11a], we obtain concentrations for random matrix products of discrete non-amenable subgroups of rank-one semisimple algebraic groups. A further aspect of the following corollary is that thanks to the work of Breuillard, the implied constants can be effectively calculated.

We need some notation to state the next corollary. Let $k$ be a local field (i.e. in characteristic zero $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$ for a prime number $p$ and in positive characteristic, a finite extension of $\mathbb{F}_p((T))$). We denote by $\| \cdot \|$ the canonical norm on $k^d$ for a fixed discrete valuation on $k$ and consider the associated operator norm on the space of $d \times d$-matrices. Moreover, if $S$ is a finite subset of $M_d(k)$, we denote by $\kappa_S := \sup \{\log \|g\| \mid g \in S\}$. Finally, if $\mu$ is a probability measure with finite first moment on $\text{GL}_d(k)$, we denote by $\ell(\mu)$ the top Lyapunov exponent, i.e. the almost sure limit of $\frac{1}{n} \log \|L_n\|$.

Corollary 1.6. Let $k$ be a local field and $\mathbb{H} \subseteq \text{SL}_d$ be a connected semisimple linear algebraic group of $k$ rank-one defined over $k$. For every $d \in \mathbb{N}$, there exists constants $\alpha_d, C_d > 0$, $N_d \in \mathbb{N}$ depending only on the dimension $d$ and constants $A = A(\mathbb{H}, k)$ such that for every finitely supported probability measure $\mu$ whose support generates a non-amenable discrete subgroup of $\mathbb{H}(k)$, for every $t > 0$ and $n \in \mathbb{N}$, the following holds:

$$
\mathbb{P} \left( \frac{1}{n} \log \|R_n\| - \ell(\mu) \geq t \right) \leq 2 \exp\left( \frac{-nt^2}{m_\mu^N \alpha_d \kappa_S^2 (\ln^+ (\kappa_S) + A_{\mathbb{H}, k})^2} \right), \quad (1.2)
$$

Remark 1.7 (About the discreteness assumption). 1. Both of the above corollaries are obtained from Theorem 1.2 in the following way: the respective versions of Tits’ alternatives allow us to deduce bounds on the norm $\|\lambda_\Gamma(\mu)\|$ of the regular representation on $\ell^2(\Gamma)$, which is equal to $\|\lambda_G(\mu)\|$ thanks to the discreteness assumption. In general, even though we have uniform upper bounds for $\|\lambda_\Gamma(\mu)\|$, we are not able to transfer this to a bound on $\|\lambda_G(\mu)\|$ without discreteness assumption. Indeed, by [Kur51, BG03], in any connected semisimple Lie group $G$, for any element $g \in G$, one can find pairs of elements $\{a_n, b_n\}$ that converge to $g$ and that generate a non-abelian free group, so that for the uniform probability measure $\mu_n$ supported on $\{a_n, b_n, a_n^{-1}, b_n^{-1}\}$, we have $\frac{\Delta^2}{2} = \|\lambda(\mu_n)\| < \|\lambda_G(\mu)\| \rightarrow 1$.

2. We also note that under the discreteness assumption, the fact that the support
$S$ generates a non-elementary group implies, thanks to various versions of Margulis Lemma, a positive lower bound for $\kappa_S$. This lower bound depends in Corollary 1.5 on some parameters of $M$ and the group generated by $S$ (see [BCGS17, Theorem 5.21]). In Corollary 1.6, it depends only on $H(k)$ (see e.g. [BGS85, Chapter 8]).

1.8. Rate function of LDP. We now mention a consequence of Theorem 1.2 concerning the rate function of large deviation principles of random walks on hyperbolic spaces recently studied by [BMSS]. The authors prove that the sequence of random variables $\frac{\kappa(R_n)}{n}$ satisfies a large deviation principle with proper convex rate function $I_{\mu} : [0, \infty) \to [0, +\infty]$ vanishing only at the drift $\ell(\mu)$. However, to the best of our knowledge, no explicit global estimate for the rate function exists in the literature. Theorem 1.2 allows us to give an explicit quadratic lower bound for the rate function $I_{\mu}$ in our setting, i.e. when $M$ is proper and the non-elementary probability measure $\mu$ has a bounded support.

Corollary 1.9 (Quadratic lower bound). Under the assumptions of Theorem 1.2, for every $t \in [0, \infty)$ the rate function $I_{\mu}$ of the sequence $\frac{1}{n}\kappa(R_n)$ satisfies

$$I_{\mu}(t) \geq \frac{(t - \ell(\mu))^2}{\kappa_S^2 D(\kappa_S, ||\lambda_G(\mu_r, \text{lazy})||)},$$

for every $r \in [0, 1)$.

1.10. Subgaussian concentrations for random matrix products. The concentration estimates that we obtain in Section 2 for general cocycles also allow us to deduce concentration estimates for random matrix products in arbitrary dimension, but these are less explicit compared to Theorem 1.2. Before stating the result we recall some known facts; we refer to §3.1 for more details.

Let $\mu$ be a probability measure on $\text{GL}_d(\mathbb{C})$ whose support generates a strongly irreducible and proximal subgroup, then there exists a unique $\mu$-stationary probability measure $\nu$ on the projective space of $\mathbb{C}^d$ ([Fur73, GR85]). The stationary measure $\nu$ enjoys some regularity properties. It is non-degenerate (i.e. does not charge any proper hyperplane) [Fur73], log-regular under a finite second order moment [BQ16a] and Hölder regular under a finite exponential moment assumption [Gui90]. Suppose now $\mu$ has bounded support and consider $c(\mu) := \sup_{x \in \mathbb{C}^d \setminus \{0\}} \int \log ||x|| ||y|| d\nu(Cy)$. It follows from the aforementioned regularity properties that this quantity is finite. Finally, we denote $\mu^*$ the pushforward of $\mu$ by the map $g \mapsto g^*$, where $g^*$ is the conjugate-transpose of $g$. With these at hand, we are now ready to state

Proposition 1.11. Let $\mu$ be a boundedly supported probability measure on $\text{GL}_d(\mathbb{C})$ such that the semigroup generated by the support $S$ of $\mu$ is strongly irreducible and proximal. Let $\kappa_S := \max\{\log ||g|| \vee \log ||g^{-1}|| : g \in S\}$ and $c = c(\mu^*)$. Then, for every $t > 0$ and $n \in \mathbb{N}$ such that $nt \geq \log d$, we have

$$P\left(\frac{1}{n}\kappa(R_n) - nt(\mu) \geq nt\right) \leq 2d \exp\left(-\frac{nt^2}{128(\kappa_S + c)}\right).$$
In this result, the fact that we have subgaussian estimates for every \( t > 0 \) small enough and with non-explicit constants can also be deduced from the spectral gap result of Le Page [LP82] using analytic perturbation methods. We also refer to [DK16, Ch. 5] for local concentrations that are uniform over small neighborhoods of irreducible cocycles.

**Remark 1.12.** Similarly to Corollary 1.9, the estimate in Proposition 1.11 allows one to obtain a quadratic lower bound (less explicit in its constants compared to the previous corollary) for the rate function of log-norms of random matrix products studied in [Ser19, XGL20]. We are also not aware of any such global estimate for the rate function in the literature of random matrix products (cf. [Ser19, Corollary 4.17].)

We end this introduction with a remark concerning the close relation between our concentration results and continuity of the drift.

**Remark 1.13 (On the consequence to continuity of the drift).**

1. A consequence of Theorem 1.2 is a uniform control, over different driving probability measures with controlled parameters \( \kappa_S \) and \( ||\lambda_G(\mu)||_2 \), of large deviations of the displacement around the drift. In turn, this allows one to deduce that the drift varies continuously when one perturbs \( \mu \) in such a way that that \( \kappa_S \) remains bounded and \( ||\lambda_G(\mu)||_2 \) remains away from 1, see Corollary 5.7. The idea of such a deduction, of continuity from uniform large deviations, already appears in the literature, see e.g. Duarte–Klein [DK16, Ch. 3].

2. However, this way of deducing the continuity is not optimal in our setting since Theorem 1.2 has the additional bounded support assumption. Indeed, as briefly shown in §5.6.1, a direct application of the method Hennion [Hen84] and Furstenberg–Kifer [FK83] based on the unique cocycle-average property, allows one to show such a continuity result under less restrictive assumptions. It is also worth noting that a crucial ingredient that we use in the proof of Theorem 1.2, namely the solution of an associated cohomological equation, already implies the unique cocycle-average property.

3. Finally, we mention that, in the case of a countable hyperbolic group, further regularity properties of the drift are known, see e.g. Erschler–Kaimanovich [EK13], Ledrappier [Led13] and Gouëzel [Gou17].

**Organization.** The article is organized as follows. In Section 2, we prove concentration estimates for a general cocycle that satisfies a certain cohomological equation. In Section 3, adapting the solutions by Benoist–Quint [BQ16a, BQ16b] of the associated cohomological equations, we deduce concentration estimates for random matrix products in arbitrary dimension and for random walks on hyperbolic spaces. In Section 4, exploiting a geometric and harmonic analytic approach of Benoist–Quint, we express the implied constants of the concentration estimates for random walks on hyperbolic spaces on which the isometry group acts cocompactly. Finally, in Section 5, using the works of Koubi and Breuillard, on uniform Tits’ alternatives, we deduce uniform/effective constants for concentrations of random walks on hyperbolic groups and linear semisimple groups of rank-one.
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2. Concentration inequalities for cocycles satisfying a Poisson equation

The goal of this section is to prove Proposition 2.1 yielding concentration inequalities for values of a cocycle for which the associated Poisson equation has a continuous solution. This result will provide the basis for the rest of the article where we will obtain more precise versions in the particular setups discussed in Introduction. We note that this section is inspired by the work of Furstenberg–Kifer [FK83] of which it can be seen as a quantitative analogue under an additional assumption (see Remark 2.2).

We start by recalling some standard terminology. Let $G$ be a group (endowed with the discrete $\sigma$-algebra) and $X$ a standard Borel space endowed with a measurable action of $G$. We shall refer to such a space as a $G$-space. A function $\sigma: G \times X \to \mathbb{R}$ is said to be an additive cocycle if it satisfies $\sigma(g_1g_2, x) = \sigma(g_1, g_2x) + \sigma(g_2, x)$ for every $g_1, g_2 \in G$ and $x \in X$. Given a probability measure $\mu$ on $G$, a probability measure $\nu$ on $X$ is said to be $\mu$-stationary if for every $\phi \in C(X)$, we have $\int \int \phi(gx)d\mu(g)d\nu(x) = \int \phi(x)d\nu(x)$. Moreover, we denote by $P_\mu$ the Markov operator acting on bounded measurable functions on $X$ by $P_\mu \phi(x) = \int \phi(gx)d\mu(g)$. Finally, denoting by $(X_i)_{i \in \mathbb{N}}$ a sequence of independent $G$-valued random variables with distribution $\mu$, we write $L_n$ for the left product $X_n \ldots X_1$. Although, $L_n$ and $R_n$ have the same distribution, it will be more convenient in this section to work with the left random walk $L_n$.

**Proposition 2.1.** Let $G$ be a group, $X$ a $G$-space and $\sigma: G \times X \to \mathbb{R}$ a bounded additive cocycle. Let $\mu$ be a probability measure on $G$ with support $S$. Denote by

$$\kappa_S := \max\{\sup_{x \in X} |\sigma(g, x)|; g \in S\}.$$

Let $\nu$ be a $\mu$-stationary probability measure on $X$ and

$$\ell(\mu) := \int_{G \times X} \sigma(g, x)d\mu(g)d\nu(x).$$

Assume that the set $E$ of bounded measurable solutions $\psi$ of the Poisson equation

$$\psi(x) - P_\mu(\psi)(x) = \int_G \sigma(g, x)d\mu(g) - \ell(\mu).$$

is non-empty and let $c := \inf\{||\psi||_\infty; \psi \in E\}$. Then, for every $t > 0$, $n \in \mathbb{N}$, and $x \in X$ we have

$$\mathbb{P}(|\sigma(L_n, x) - n\ell(\mu)| \ge nt) \le 2 \exp\left(-\frac{nt^2}{32(\kappa_S + c)^2}\right).$$

**Remark 2.2.** 1. Our assumption (2.1) implies that there is a unique cocycle average in the sense of [BQ16a, §3].

2. This result can be seen as an abstract quantitative refinement of [FK83, Theorem 2.1] under the assumption that the expected increase function is cohomologous to a constant.
The proof of the previous result is based on the following general probabilistic ingredient. We start by recalling some standard terminology on Markov chains. Let \( M \) be a standard Borel space, \( P \) a Markov operator on \( M \), i.e. a measurable map \( x \mapsto P_x \) from \( M \) to the space of probability measures on \( M \). This data naturally defines an operator on the space of bounded Borel functions on \( M \) by \( \phi \mapsto P\phi \), where \( P\phi(x) = \int \phi(y) dP_y(x) \). Given \( x \in M \), we denote by \( P_x \) the law of the Markov chain \((Z_n)\) on the space of trajectories, i.e. \( M^\mathbb{N} \) and \( \mathbb{E}_x \) the associated expectation operator. We say that a probability measure \( \pi \) is invariant (or stationary) under the Markov operator \( P \) if \( \int P\phi d\pi = \int \phi d\pi \) for every bounded measurable function \( \phi \) on \( M \).

**Proposition 2.3.** Let \((Z_n)\) be a Markov chain on a standard Borel space \( M \) associated to the Markov operator \( P \). Let \( \pi \) be a \( P \)-stationary probability measure on \( M \). Let \( f \) be a bounded measurable function on \( M \). We assume that \( f \) is cohomologous to \( \int_M f d\pi \), i.e. there exists a bounded measurable solution \( \phi \) of the equation:

\[
\phi - P\phi = f - \int_M f d\pi \tag{2.2}
\]

Then, for every \( t > 0 \), \( n \in \mathbb{N} \) and \( x \in M \), the following inequality holds

\[
\mathbb{P}_x \left( \left| \sum_{i=1}^n f(Z_i) - n \int_M f d\pi \right| \geq nt \right) \leq 2 \exp \left( -\frac{nt^2}{32\|\phi\|_\infty^2} \right).
\]

**Remark 2.4.** 1. Let \( M \) be a compact metric space and \( f : M \to \mathbb{R} \) a continuous function. Suppose, for simplicity, that the operator \( P \) is Markov-Feller and that \( f \) has a unique average \( \ell_f := \int f d\pi \) for all \( P \)-stationary probability measures \( \pi \). Even though \( f \) may not be cohomologous to the constant \( \ell_f \), Furstenberg–Kifer [FK83, Lemma 3.1] showed that for every \( t > 0 \), there exists a continuous function \( h_t \) on \( M \), cohomologous to \( f \) and such that \( \|h_t\|_\infty \leq \ell_f + t \). This can be used to show the exponential decay of \( \mathbb{P}(\left| \sum_{i=1}^n f(Z_i) - n\ell_f \right| > nt) \). This is a particular case of Benoist–Quint’s [BQ16a, Proposition 3.1]. In Proposition 2.3, thanks to the stronger assumption (2.2), one obtains subgaussian exponential decay with explicit constants.

2. In some particular cases, powerful concentration inequalities exist for sum of any function on a Markov chain [Gil98, DG15]. They are not applicable here since our Markov chains are not geometrically ergodic. On the other hand, the particular requirement (2.2) on the function \( f \) allows us to use the usual Hoeffding inequality for martingales and thereby deduce the previous concentration estimates in the generality of Markov chains that we consider.

**Proof of Proposition 2.1.** We start by defining the appropriate objects to which we will apply Proposition 2.3. We take the standard Borel space \( M \) to be \( F \times X \) and \( P \) the Markov operator defined by

\[
Pf((g,x)) = \int_G f(\gamma, gx) d\mu(\gamma)
\]
for every bounded measurable function \( f \) on \( X \). The associated Markov chain \((Z_n)_{n \in \mathbb{N}} \) on \( M \) starting from \( Z_0 = (e, x) \) is the process
\[
Z_0 = (e, x), \ Z_1 = (g_1, g_1 \cdot x), \ Z_2 = (g_2, g_2 g_1 \cdot x) \cdots Z_n = (g_n, L_n \cdot x), \cdots ,
\]
where the \( g_i \)’s are iid random variables on \( G \) with distribution \( \mu \). Let \( \pi \) be the probability measure on \( M \) defined by
\[
\int_M f d\pi := \int \int_{F \times X} f(g, g \cdot x) \, d\mu(g) \, d\nu(x)
\]
for every bounded measurable \( f \) on \( M \). Since \( \nu \) is a \( \mu \)-stationary, one readily checks that \( \pi \) is stationary for the Markov operator \( P \). Let now
\[
f : M \longrightarrow \mathbb{R}, (g, x) \longmapsto f(g, x) := \sigma(g, g^{-1} \cdot x).
\]

The following properties are immediate to check

1. Starting from \( Z_0 = (e, x) \), we have \( \sum_{i=1}^n f(Z_i) = \sigma(L_n, x) \),
2. \( \int_M f \, d\pi = \int \int f(g, g \cdot x) \, d\mu(g) \, d\nu(x) = \int \sigma(g, x) \, d\mu(g) \, d\nu(x) = \ell(\mu) \),
3. \(||f||_\infty \leq \kappa_S \).

Finally, we check that if (2.1) holds for some \( \psi \), then (2.2) holds. Indeed, let
\[
\phi : M \longrightarrow \mathbb{R}, (g, x) \longmapsto \phi(g, x) := \psi(x) + f(g, x).
\]

One readily checks that \( P_\psi = P_\phi \) and \( Pf(g, x) = \int_{Z'} \sigma(g, x) \, d\mu(g) \). Thus, by (2.1), \( \phi - P\phi = f - \int_M f \, d\pi \), and (2.2) is fulfilled. Since \(||\phi||_\infty \leq ||\psi||_\infty + \kappa_S \), Proposition 2.1 follows from Proposition 2.3.

**Proof of Proposition 2.3.** Let \( \alpha := \int_M f \, d\pi \) and \( \phi \) as in the statement so that \( f - \alpha = \phi - P\phi \). We write
\[
\sum_{i=1}^n f(Z_i) - n \alpha = \sum_{i=1}^n [\phi(Z_{i+1}) - P\phi(Z_i)] + [\phi(Z_1) - \phi(Z_{n+1})].
\]

On the one hand, the sequence \( D_i := \phi(Z_{i+1}) - P\phi(Z_i) \) is a martingale difference sequence with respect to the canonical filtration of \((Z_i)_i \). Moreover, \(|D_i| \leq 2||\phi||_\infty \). Thus \( M_n := \sum_{i=1}^n [\phi(Z_{i+1}) - P\phi(Z_i)] \) is a martingale with bounded differences. Applying Azuma–Hoeffding concentration inequality for martingales with bounded difference (see for instance [McDS9, Lemma 4.1]), we get that for every \( t > 0 \) and \( n \in \mathbb{N} \),
\[
\mathbb{P}(M_n \geq nt/2) \leq \exp\left(-\frac{nt^2}{32||\phi||^2}\right) \quad \text{and} \quad \mathbb{P}(M_n \leq -nt/2) \leq \exp\left(-\frac{nt^2}{32||\phi||^2}\right) \tag{2.4}
\]

On the other hand, the following crude upper bound holds for \( V_n := \phi(Z_1) - \phi(Z_{n+1}) \); for every \( n \in \mathbb{N} \), we have \(|V_n| \leq 2||\phi||_\infty \). Hence, \(|V_n| \leq nt/2 \) for every \( n \geq \frac{4||\phi||_\infty}{t} \). Combining this fact with (2.3) and (2.4), we get that for every \( t > 0 \) and every \( n \geq \frac{4||\phi||_\infty}{t} \),
\[
\mathbb{P}\left(\sum_{i=1}^n f(Z_i) - n \int f \, d\pi \geq nt\right) \leq \exp\left(-\frac{nt^2}{32||\phi||^2}\right).
\]


and
\[ \mathbb{P}\left( \sum_{i=1}^{n} f(Z_i) - n \int_M f \, d\pi \leq -nt \right) \leq \exp\left( -\frac{nt^2}{32||\phi||_\infty^2} \right). \]

Thus \( \mathbb{P}\left( |\sum_{i=1}^{n} f(Z_i) - n \int_M f \, d\pi| \leq nt \right) \leq 2 \exp\left( -\frac{nt^2}{32||\phi||_\infty^2} \right). \) This shows the desired inequality in the case \( n \gtrsim \frac{1}{t} ||\phi||_\infty. \) Suppose finally that \( n \lesssim \frac{1}{t^4} ||\phi||_\infty. \) In this case, \( nt^2 \lesssim 16||\phi||^2 \) and then \( \exp\left( -\frac{nt^2}{32||\phi||_\infty^2} \right) \gtrsim \exp\left( -1/2 \right) > \frac{1}{2}. \) The desired estimate holds trivially in this case. \( \square \)

3. Applications to random matrix products and random walks on hyperbolic spaces

The goal of this section is to obtain two consequences of Proposition 2.1 in the settings of random matrix products and random walks on hyperbolic spaces to get subgaussian concentration estimates, respectively, for the norm and Busemann cocycles. We then deduce concentration estimates for the norms of random matrix products and displacement of random walks on hyperbolic spaces.

3.1. Subgaussian concentrations for random matrix products. Let \( d \geq 1 \) be an integer, we consider \( \mathbb{C}^d \) endowed with the canonical Hermitian structure and \( M_d(\mathbb{C}) \) with the induced operator norm. For simplicity, we denote by \( ||\cdot|| \) both norms on \( \mathbb{C}^d \) and \( M_d(\mathbb{C}) \). We denote by \( X \) the projective space of \( \mathbb{C}^d \) and we endow it with the standard metric given by
\[ \delta([x],[y]) := \frac{||x \wedge y||}{||x|| ||y||}, \]
where the norm \( ||\cdot|| \) is the canonical norm on \( \wedge^2 \mathbb{C}^d \), \( [x] = \mathbb{C} x \) and \( [y] = \mathbb{C} y \).

A probability measure \( \mu \) on \( \text{GL}_d(\mathbb{C}) \) is said to be (strongly-)irreducible if the support \( S \) of \( \mu \) does not fix a (finite union of) non-trivial proper subspace(s) of \( \mathbb{C}^d \). An irreducible probability measure \( \mu \) is said to be proximal if the closure \( \overline{G\mu^+} \) in \( M_d(\mathbb{C}) \) of the semigroup \( G^+\mu \) generated by the support of \( \mu \) contains a rank-one linear transformation.

A probability measure \( \nu \) on \( X \) is said to be \( \mu \)-stationary if it is \( \mu \)-stationary for the Markov operator \( P_\mu \) associated to \( \mu \). We recall that for a strongly irreducible and proximal probability measure \( \mu \) on \( \text{GL}_d(\mathbb{C}) \), there exists a unique \( \mu \)-stationary probability measure \( \nu \) on \( X \). \( \text{[Fur73, GR85]} \). We denote by \( \mu^* \) the image of \( \mu \) under the map \( \text{g} \mapsto g^* \), where \( g^* \) denotes the conjugate-transpose of \( g \) and by \( \nu^* \) the unique-stationary measure of \( \mu^* \) (which is also proximal and strongly irreducible).

We denote by \( \sigma : \text{GL}_d(\mathbb{C}) \times X \to \mathbb{R} \) the (additive) norm-cocycle given by \( \sigma(g,[x]) = \log \frac{||gx||}{||x||} \). The solution of the Poisson equation (2.1) for the norm cocycle is closely related to regularity properties of the stationary measure \( \nu \) on \( X \). Indeed when \( \mu \) has an exponential moment, (2.1) can be solved using the result of Le Page \( \text{[LP82]} \) establishing a spectral gap for the Markov operator \( P_\mu \) acting on some Hölder functions of \( X \). As proved by Guivarc’h \( \text{[Gui90]} \) this spectral gap property implies the Hölder regularity of \( \nu \). When \( \mu \) has a
moment of order 2, Benoist–Quint [BQ16a] solved the same equation by using and proving the log-regularity of the stationary measure \( \nu \). We will rely on their results.

By [BQ16a], the following quantity
\[
\psi([x]) := -\int \log \frac{|x||y|}{|x,y|} \, d\nu^*([y])
\]  
(3.1)
is finite for every \( x \in X \) and defines a continuous function \( \psi \) on \( X \). Moreover, \( \psi \) satisfies the cohomological equation
\[
\psi - P_\mu \psi = \phi - \ell(\mu),
\]  
(3.2)
where \( \phi([v]) := \int \log \frac{|g||v|}{|v|} \, d\mu(g) \), the expected increase at \([v]\). This fact plays the key role in the proof of the following result:

**Proof of Proposition 1.11.** We will apply Proposition 2.1 with \( G = \text{GL}_d(\mathbb{C}) \), \( X = P(\mathbb{C}^d) \), and the norm-cocycle \( \sigma : G \times X \to \mathbb{R} \). Observe that for every \( g \in G \) and \( x \in X \),
\[
|\sigma(g, x)| \leq \max\{\log |g|, \log ||g^{-1}||\}.
\]
Furthermore, the equation (3.2) shows that the hypothesis (2.1) of Proposition 2.1 holds, and consequently, we deduce that for every \( t > 0, n \in \mathbb{N}, v \in \mathbb{C}^d \setminus \{0\} \)
we have
\[
\mathbb{P} \left( \left| \int \log \frac{|L_n v|}{||v||} \, d\mu(v) - n\ell(\mu) \right| \geq nt \right) \leq 2 \exp \left( -\frac{nt^2}{32(\kappa_S + c)^2} \right),
\]  
(3.3)
where \( c = c(\mu^*) = \sup_{[x] \in P(\mathbb{C}^d)} \psi([x]) \). Now, to get the concentration estimates for the matrix norm of \( L_n \), consider the canonical basis \( e_1, \ldots, e_n \) of \( \mathbb{C}^d \). For every \( g \in G \), we have
\[
||g|| \leq \sqrt{d} \max\{||ge_i||; i = 1, \ldots, d\}.
\]
Thus
\[
\mathbb{P} (\int |L_n|| - n\ell(\mu) \geq nt) \leq d \max_{i=1,\ldots,d} \mathbb{P} \left( \left| \int |L_n e_i|| - n\ell(\mu) \right| \geq nt - \frac{\log d}{2} \right).
\]
(3.4)
Suppose that \( nt \geq \log d \). Then \( nt - (\log d)/2 \geq \frac{nt}{2} \) and hence, by combining (3.3) and (3.4), we get that
\[
\mathbb{P} (\left| \int |L_n|| - n\ell(\mu) \right| \geq nt) \leq 2d \exp \left( -\frac{nt^2}{128(\kappa_S + c)^2} \right),
\]
as claimed. \( \square \)

### 3.2. Application to random walks on hyperbolic spaces.

We will apply Proposition 2.1 to the horofunction compactification \( \overline{M}^h \) and cocycle \( \sigma \) of a proper geodesic hyperbolic metric space. The key point in this application is to solve the cohomological equation (2.1) in this setting. This was previously done by Benoist–Quint [BQ16a] who gave a solution \( \psi \) on \( \partial_h M \); we will observe here that \( \psi \) extend to a solution on the larger space \( \overline{M}^h \) which will be more convenient for our purpose. Here we will content with a non-explicit bound on
the size of this solution. In the next Section 4, under an additional assumption on the space \( M \), we will make this bound explicit, which is be the key point that allows us to obtain explicit the estimates in our main result Theorem 1.2.

Let us start by recalling the definitions of these notions. Let \((M, d)\) be a proper geodesic metric space and denote by \( C(M) \) the set of continuous real valued functions on \( M \), endowed with the compact-open topology. Fixing \( o \in M \), for \( x \in M \), let the function \( h_x \in C(M) \), defined by \( h_x(m) = d(x, m) - d(x, o) \). The closure \( \overline{M} \) of \( \{h_x; x \in M\} \) is a compact subset of \( C(M) \), called the horofunction compactification of \( M \). Identifying \( M \) with its image under this embedding, \( M \) is open and dense in \( \overline{M} \) and the horofunction boundary of \( M \) is defined as \( \partial_h M := \overline{M} \setminus M \). The set \( \overline{M} \) consists, in particular, of Lipschitz functions that takes the value zero on the fixed point \( o \) and the group of isometries \( \text{Isom}(M) \) acts on \( \overline{M} \) by homeomorphisms given, for \( g \in \text{Isom}(M) \), \( h \in \overline{M} \) and \( m \in M \), by \( (g.h)(m) = h(g^{-1}m) - h(g^{-1}o) \). This extends equivariantly the isometric action of \( \text{Isom}(M) \) on \( M \) and the compact set \( \partial_h M \subset \overline{M} \) is invariant under \( \text{Isom}(M) \). The Busemann cocycle \( \sigma : \text{Isom}(M) \times \overline{M} \to \mathbb{R} \) is defined by

\[
\sigma(g, h) = h(g^{-1}o).
\]

Now, let \((M, d)\) be, moreover, a \( \delta \)-hyperbolic space. Since there are various equivalent definitions that cause the constant \( \delta \geq 0 \) to differ, we recall that, for us, this means for every \( x, y, z, o \in M \), we have

\[
(x|y)_o \geq (x|z)_o \wedge (z|y)_o - \delta,
\]

where \((.,.)\) is the Gromov product given by \((x|y)_o = \frac{1}{2}(d(x, o) + d(y, o) - d(x, y))\). For simplicity, we will often omit the basepoint \( o \) from the notation. We refer to [CDP90] for general properties of these spaces. An element \( \gamma \in \text{Isom}(M) \) is said to be loxodromic if for any \( x \in M \), the sequence \( (\gamma^n x)_{n \in \mathbb{Z}} \) constitutes a quasi-geodesic (see [CDP90, Ch. 3]). Equivalently, \( \gamma \) is loxodromic if and only if it fixes precisely two points \( x_+^\gamma, x_-^\gamma \) on the Gromov boundary \( \partial M \) of \( M \) [CDP90, Ch. 9 & 10]. Two loxodromic elements \( \gamma_1, \gamma_2 \) are said to be independent if the sets of fixed points \( \{x_+^{\gamma_i}, x_-^{\gamma_i}\} \) for \( i = 1, 2 \) are disjoint. Finally, a set \( S \), or equivalently a probability measure with support \( S \), is said to be non-elementary if the semigroup generated by \( S \) contains at least two independent loxodromic elements.

For \( h_1, h_2 \in \partial_h M \), we set

\[
(h_1|h_2)_o = -\frac{1}{2} \inf_{m \in M} (h_1(m) + h_2(m)).
\]

This is the continuous extension to \( \partial_h M \) of the usual Gromov product on \( M \) based at \( o \in M \). We note that \((h_1|h_2)_o = \infty \) if any only if \( h_1 \) and \( h_2 \) have the same projection to the Gromov boundary of \( M \).

Let now \( \mu \) be a non-elementary probability measure on \( G \) with bounded support. Similarly to the previous subsection, the next crucial ingredient of our concentration result is the solution by Benoist–Quint [BQ16b, Propositions 4.2, 4.3] of the Poisson equation (2.1) on \( \partial_h M \). Note that there might exist several \( \mu \)-stationary probability measures on \( \partial_h M \) but they all have the
same Busemann cocycle average which is given by the drift random walk on $M$ induced by $\mu$ [BQ16b, Proposition 3.3], i.e. for any $\mu$-stationary $\nu$ on $\mathcal{M}^h$,

$$\int_{G \times \mathcal{M}^h} \sigma(g, x) \, d\mu(g) \, d\nu(x) = \ell(\mu).$$

Namely, they showed that the function $\psi$ defined on $\partial_h \mathcal{M}$ as

$$\psi(x) := -2 \int_{\partial_h \mathcal{M}} (x|y)_o \, d\tilde{\nu}(y),$$

is bounded, measurable, and it satisfies the Poisson equation

$$\psi(x) - P_{\mu} \psi(x) = \int \sigma(g, x) \, d\mu(g) - \ell(\mu),$$

where $\tilde{\nu}$ is any stationary probability measure on $\partial_h \mathcal{M}$ for $\mu^{-1}$ and $\mu^{-1}$ is the non-elementary probability measure given by the image of $\mu$ by the map $g \mapsto g^{-1}$.

For our purposes in the sequel, it will be more convenient to consider the action of the Markov operator $P_{\mu}$ on the space of bounded measurable functions defined on the whole compactification $\mathcal{M}^h$. Accordingly, we will verify that the natural extension of the function $\psi$ given by (3.6) to the space $\mathcal{M}^h$ yields a solution to the equation (3.7). We summarize these in the next

**Lemma 3.3.** The function $\psi : \mathcal{M}^h \to \mathbb{R}$ defined by

$$\psi(x) = -2 \int_{\mathcal{M}^h} (x|y)_o \, d\tilde{\nu}(y)$$

is a bounded measurable function that satisfies the equation

$$\psi(x) - P_{\mu} \psi(x) = \int \sigma(g, x) \, d\mu(g) - \ell(\mu)$$

for every $x \in \mathcal{M}^h$.

**Proof.** The function $\psi$ is well-defined since it is well-defined on the boundary $\partial_h \mathcal{M}$ thanks to [BQ16b, Proposition 4.2] and since it is clearly finite on $M$ thanks to the trivial bound $0 \leq (x|y)_o \leq d(o, x)$ true for $x \in M$ and $y \in \partial_h \mathcal{M}$. The proof then goes similarly as Benoist–Quint’s proof [BQ16b, Propositions 4.2 & 4.6]; we indicate only the needed changes. First, any $\mu$-stationary probability measure on $\mathcal{M}^h$ is supported on the boundary $\partial_h \mathcal{M}$ since $\kappa(L_n) \to +\infty$ almost surely [BQ16b, Proposition 3.3] (see also [MT18, Theorem 1.2]). Since by [BQ16b, Proposition 3.3], the Busemann cocycle $\sigma : G \times \mathcal{M}^h \to \mathbb{R}$ has a unique cocycle average on the boundary $\partial_h \mathcal{M}$, it follows that it has a unique cocycle average on all of the compactification $\mathcal{M}^h$. Then, using Benoist–Quint’s large deviation result for cocycles [BQ16a, Prop. 3.2] applied to the Busemann cocycle on $\mathcal{M}^h$, we deduce that for every $t > 0$,

$$\sum_{n \geq 1} \sup_{\xi \in \mathcal{M}^h} \mathbb{P}(|\sigma(L_n, \xi) - n\ell(\mu)| > nt) < +\infty.$$  

Using now the same strategy as in the proof of [BQ16b, Lemma 4.5], this implies that $\psi$ is bounded on all $\mathcal{M}^h$. Finally, we remark that the following key identity

$$\sigma(g, x) = -2(x|g^{-1}y)_o + 2(gx|y)_o + \sigma(g^{-1}|y)$$
used by Benoist–Quint for every $g \in \text{Isom}(M)$ and $x, y$ on the boundary holds true for $x, y \in \overline{M}^h$. Equation (3.9) follows then by integrating both sides of the previous identity with respect to $d\tilde{\nu}(y)d\mu(g)$ and using the fact that $\ell(\mu) = \ell(\tilde{\mu})$.

**Remark 3.4.** In parallel with the remark concerning the results of Le Page and Guivarc’h mentioned in the paragraph preceding (3.1), in the current setup, one can substitute Maher’s result on Hölder regularity of the harmonic measure [Mah12, Lemma 2.10] (see also [BMSS, Proposition 2.16]) for large deviation results of Benoist–Quint to prove that the function $\psi$ is bounded.

**Proposition 3.5.** Let $(M, d)$ be a proper, geodesic, $\delta$-hyperbolic space, $o \in M$ and $\mu$ a non-elementary Borel probability measure on the group $G = \text{Isom}(M)$ with bounded support $S$. Denoting $\kappa_S = \sup_{g \in S} d(go, o)$ and $c = -\inf_{x \in X} \psi(x)$ (see Lemma 3.3), for every $t > 0$ and $n \in \mathbb{N}$, we have

$$\sup_{\xi \in \overline{M}^h} \mathbb{P}(|\sigma(L_n, \xi) - n\ell(\mu)| \geq nt) \leq 2 \exp\left(-\frac{nt^2}{32(\kappa_S + c)^2}\right).$$

(3.10)

In particular,

$$\mathbb{P}(|\kappa(L_n) - n\ell(\mu)| \geq nt) \leq 2 \exp\left(-\frac{nt^2}{32(\kappa_S + c)^2}\right).$$

(3.11)

**Proof.** Using (3.11) $|\sigma(g, \xi)| \leq \kappa(g)$ true for every $g \in G$ and $\xi \in \overline{M}^h$, the estimate (3.10) follows directly from Proposition 2.1 applied to $G = \text{Isom}(M)$, $X = \overline{M}^h$ and the Busemann cocycle $\sigma : G \times X \to \mathbb{R}$. Finally, (3.11) follows directly by specializing to $\xi = o$.

4. Explicit estimates for random walks on Gromov hyperbolic spaces

Here, we express the explicit dependence of concentration probabilities to some parameters of the measure $\mu$ driving the random walk in the setting of random walks on hyperbolic spaces. The following result is the first key step to establish effective estimates discussed in Introduction. It is heavily based on the harmonic analytic and geometric approach and results of Benoist–Quint [BQ16b, §5].

Let $(M, d)$ be a proper metric space, we denote by $G$ its group of isometries. It is a locally compact group [GK03, Theorem 6] and we denote by $\mu_G$ a Haar measure on $G$. For a probability measure $\mu$ on $G$, $S$ denotes the support of $\mu$ which is the smallest closed subset whose $\mu$-mass equals one. We recall that for every $r \in [0, 1)$, we denote by $\mu_{r, \text{lazy}} = r\delta_{\text{id}} + (1 - r)\mu$. Having fixed a basepoint $o \in M$, for an element $g \in \text{Isom}(M)$, we write $\kappa(g) = d(go, o)$ and for a bounded set $S$, we set $\kappa_S := \sup\{\kappa(g); g \in S\}$.

The main result of this section is the following result, which immediately implies Theorem 1.2 by specializing to $\xi = o$.

**Theorem 4.1.** Let $(M, d)$ be a proper geodesic hyperbolic space and $o \in M$. Assume that the group $\text{Isom}(M)$ acts cocompactly on $M$. Then, there exists an
explicit positive function \( D(.,.) > 0 \) and a constant \( C_M > 0 \) such that for every non-elementary probability measure \( \mu \) on \( G \) with bounded support \( S \), for every \( \xi \in M^h \), \( t > 0 \) and \( n \in \mathbb{N} \), we have

\[
\mathbb{P} (|\sigma(L_n, \xi) - n\ell(\mu)| \geq nt) \leq 4 \exp \left( \frac{-nt^2}{\kappa_S^2 D(\kappa_S, ||\lambda_G(\mu_{r,lazy})||_2)} \right)
\]

for every \( r \in [0,1) \).

With Proposition 3.5 at hand, the main ingredient for the proof of the Theorem 4.1 is the following

**Proposition 4.2.** Let \((M,d)\) be a proper geodesic \( \delta \)-hyperbolic space such that the group \( G \) of isometries of \( M \) acts cocompactly on \( M \). Then, there exists an explicit positive function \( C(.,.) \) such that for every boundedly supported non-elementary probability measure \( \mu \) on \( G \), we have

\[
\sup_{y \in M^h} \int_{\partial_h M} (x | y)_a \, d\nu(x) \leq C(\kappa_S, ||\lambda_G(\mu_{r,lazy})||_2).
\]

for every \( r \in [0,1) \).

**Remark 4.3.** The function \( C(\kappa, \lambda) \) is a function of \( \kappa > 0 \) and \( \lambda \in (0,1] \) and, for \( \lambda < 1 \), it can be given by

\[
\kappa \inf_{1 < c < \lambda^{-1/2}} \left[ 4 \left( \frac{\log \kappa}{\log e} \vee \frac{1}{(\log e)^2} \right) + \frac{A_0}{1 - e^2 - \lambda} \right],
\]

where \( A_0 = (\frac{\mu_G(B_R(\delta) + 2B_0)}{\mu_G(B_R(\delta) + B_0)})^{1/2} \) with \( R(\delta) = 14\delta + 4 \), for \( r \geq 0 \), \( B_r := \{ g \in G; d(go, o) \leq r \} \) and \( D_0 \) is the diameter \( 2 \sup_{x,y \in M} \inf_{g \in G} d(gx, y) \), which is finite by the cocompactness assumption. We set \( C(\kappa, 1) = \infty \). Finally, for concreteness, for \( \lambda < 1 \), specializing to \( c = (1 + \lambda^{-1/2})/2 \), one can get

\[
C(\kappa, \lambda) \leq \kappa (8 \ln^+(\kappa) + 4A_0/3 + 16) \frac{1}{(1 - \sqrt{\lambda})^2}.
\]

**Remark 4.4** (Why we also consider lazy random walks). Denoting by \( \rho(\mu) \) the spectral radius of \( \lambda_G(\mu) \), we have \( ||\lambda_G(\mu)||_2 = \sqrt{\rho(\mu * \mu^{-1})} \), where \( \mu^{-1} \) the image of \( \mu \) by the map \( g \mapsto g^{-1} \). When \( \mu \) is symmetric, \( ||\lambda_G(\mu)||_2 = \rho(\mu) \). Moreover, thanks to [BC74, Theorem 4], \( ||\lambda_G(\mu)||_2 < 1 \) as soon as \( \mu * \mu^{-1} \) is non-elementary and hence in this case, we may take \( r = 0 \) on the right-hand-side of the the inequality given by the previous result. Finally, for every non-elementary probability measure \( \mu \), for every \( r > 0 \), \( \mu_{r,lazy} \) satisfies the previous property and hence for \( r \in (0,1) \), \( ||\lambda_G(\mu_{r,lazy})||_2 < 1 \). We refer to [BC74, p177] for an example of a non-elementary probability measure \( \mu \) with \( ||\lambda_G(\mu)||_2 = 1 \).

For the proof, we require the following version of [BQ16b, Lemma 5.2] where we highlight the constants that appear in the aforementioned lemma for our purposes. This is the crucial harmonic analytic ingredient of the proof where the additional hypothesis (compared to Proposition 3.5) on the cocompactness action of \( G \) on the space \( M \) is used.
Lemma 4.5. Let \((M,d)\) be a proper metric space. Suppose that the group \(G\) of isometries of \((M,d)\) acts cocompactly on \(M\). Then, there exists a constant \(D_0 > 0\) depending only on \(M\) such that for every probability measure \(\mu\), for every \(R > 0\), \(n \geq 1\) and \(m, m' \in M\), we have

\[
\mathbb{P}(d(R_n m, m') \leq R - D_0) \leq A_0(R)\|\lambda_G(\mu)\|_2^n,
\]

where \(A_0(R) = \left(\frac{\mu_G(B_{2R})}{\mu_G(B_R)}\right)^{1/2}\).

Proof. The proof follows similarly as \([BQ16a, Lemma 5.2]\), we only indicate the necessary modifications.

- We replace the estimate \(\|\lambda_G(\mu)^n\| \leq C_0 a_0^n\) used in the proof of \([BQ16a, Lemma 5.2]\), by \(\|\lambda_G(\mu)^n\| \leq \|\lambda_G(\mu)\|^n\), and consequently, the constants \(C_0\) and \(a_0\) in \([BQ16a, Lemma 5.2]\) can, respectively, be taken to be 1 and \(\|\lambda_G(\mu)\|\).
- Regarding the constant \(A_0\) in \([BQ16a, Lemma 5.2]\), in their proof, Benoist–Quint assume that \(m\) and \(m'\) belongs to the same \(G\)-orbit. However, since the action of \(G\) on \(M\) is cocompact, there exists \(D_0\) such that for every \(x, y\), there exists \(g \in G\) with \(d(gx, y) \leq D_0/2\) and this allows us to take \(A_0 = \left(\frac{\mu_G(B_{2R})}{\mu_G(B_R)}\right)^{1/2}\). \(\square\)

We will need the following geometric lemma, which is an adaption of Inclusion (5.5) in the proof of \([BQ16b, Lemma 5.3]\) to the horofunction compactification. We point out that this is the key point where the geometric assumption of hyperbolicity is used.

Lemma 4.6. Let \((M,d)\) be a proper geodesic hyperbolic metric space and \(o \in M\). Then, there exists \(R(\delta) > 0\) such that for any \(\xi \in \partial_h M\), \(y \in \overline{M}^h\), and constant \(D > 0\), there exists a finite subset \(C \subset M \times M\) with at most \(D^2\) elements such that for any \(g \in \text{Isom}(M)\), there exists \((x', y') \in C\) such that

\[
(g\xi|y)_o \geq \kappa(g) \implies [\kappa(g) \geq D] \lor \{d(gx', y') \leq R(\delta)\}.
\]

Moreover, all elements constituting the tuples in \(C\) are contained in a ball of radius \(D\) around \(o\).

As it is will be shown, one can take the constant \(R(\delta) = 14\delta + 4\).

The proof will require some juggling between Gromov and horofunction boundaries to construct the set \(C\). We therefore start by recalling some standard facts on the relation between \(\partial_h M\) and \(\partial M\).

First, there exists a natural \(G\)-equivariant surjective map from \(\overline{M}^h\) to \(M \cup \partial M\). Namely, given \(h \in \partial_h M\), for any sequence \(x_n \in M\) such that \(h_{x_n} \to h\), the sequence \(x_n\) Gromov converges to infinity in the sense that \(\inf_{m,n \geq k} d(x_n, x_m) \to \infty\) as \(k \to \infty\). When we endow \(M \cup \partial M\) with the usual topology \([CDP90, Ch. 2]\), this projection is the unique map that continuously extends the identity map \(m \to m\) on \(M\). For \(y \in \overline{M}^h\), we denote by \(\pi_m\) its image in \(M \cup \partial M\).

Given two points \(\pi_x \neq \pi_y \in \partial M\), by using the defining inequality \((3.5)\) of a \(\delta\)-hyperbolic space, one checks that for any pair of pair of sequences \(x_n, x'_n\) and \(y_n, y'_n\), that Gromov converge to infinity and that are, respectively, in the
Suppose now that \((\xi, \eta)\) fix three rays \([\xi, \eta]\) oriented from \(i\) to \(j\) by \(\gamma_{ij}\), where \(i, j \in \{x, y, z, a, b, c\}\), we have for every \(t \geq 0\) in the respective interval of definition, all distances \(d(\gamma_{xb}(t), \gamma_{xa}(t)), d(\gamma_{ya}(t), \gamma_{yc}(t))\) and \(d(\gamma_{b2}(t), \gamma_{c3}(t))\) are \(\leq 6\delta\).

This lemma can be deduced from standard facts in hyperbolic geometry. We include a brief proof for reader’s convenience.

**Proof.** Consider the triangle \((x, y, z)\) whose edges are as given in the statement. Let \(\zeta_n\) be a sequence of points on the edge \((y, z)\) that converge to \(z\). Consider the segments \(\zeta_n\) from \(x\) to \(\zeta_n\). Since \(M\) is proper, by Arzelà–Ascoli Theorem, up to subsequence, they converge to a ray \(\zeta\) between \(x\) and \(z\).

For each triangle \((x, y, \zeta_n)\), fix points \(a_n, b_n, c_n\) respectively on the edges \([x, y]\), \([x, \zeta_n]\) and \([y, \zeta_n]\) that map to the junction point of the associated tripod \([\xi, \eta]\). Using the fact that \(M\) is proper and passing to a further subsequence of \(\zeta_n\), we may suppose that the sequences \(a_n, b_n, c_n\) converge, respectively, to the points, \(a \in [x, y], b' \in \zeta\) and \(c \in [y, z]\). Let \(a\) be the point on \([x, z]\) at distance \(d(x, a)\) from the \(x\).

Now note that by the tripod lemma \([\xi, \eta, \gamma]\), we have the required property within each triangle \((x, y, \zeta_n)\) with \(4\delta\). Since all points \(a_n, b_n, c_n\) converge to respectively \(a, b', c\), the same property is true at the limit triangle with \([x, z]\) replaced by \(\zeta\). Now since \([x, z]\) and \(\zeta\) are at parametrized-distance \(2\delta\)-apart, we get the required property with \(6\delta\). \(\square\)

**Lemma 4.8** (Fellow travellers). For every \(\xi, \eta \in \overline{M}^h\), let \(\gamma_\xi\) and \(\gamma_\eta\) be geodesic rays such that \(\gamma_\xi(0) = \gamma_\eta(0) = o\) and \(\gamma_\xi(t) \to \pi_\xi\) as \(t \to \infty\) for \(\zeta \in \{\xi, \eta\}\). Then for any \(r > 0\) such that \((\xi|\eta)_o \geq r\), we have \(d(\gamma_\xi(t), \gamma_\eta(t)) \leq 8\delta\) for every \(t \in [0, r]\).

**Proof.** We can suppose that \(\pi_\xi \neq \pi_\eta\) and \(r \geq 2\delta\). It follows by (4.2) that for every \(\epsilon > 0\) for every \(s > 0\) large enough, we have \((\gamma_\xi(s)|\gamma_\eta(s))_o \geq r - 2\delta - \epsilon\). The statement now follows by expanding the inequality \((\gamma_\xi(t)|\gamma_\eta(t))_o \geq (\gamma_\xi(t), \gamma_\xi(s)) \wedge (\gamma_\xi(s), \gamma_\eta(s)) \wedge (\gamma_\eta(s), \gamma_\eta(t)) - 2\delta\) for \(t \leq r\) and \(s \to \infty\). \(\square\)

We now give the

**Proof of Lemma 4.6.** We will prove the claim with \(R = 14\delta + 4\). Let such \(\xi \in \partial_h M\), \(y \in \overline{M}^h\) and \(D > 0\) be given. Let \(g \in \text{Isom}(M)\). To construct the set \(C\), fix three rays \([o, \pi_\xi], [o, \pi_\eta]\) and \([o, g\pi_\xi]\) and for \(i = 1, \ldots, [D] - 1\), let \(m_i, m'_i\) and \(m''_i\) be points on the respective rays satisfying \(d(z_i, o) = i \wedge d(o, \zeta)\) for every couple \((z_i, \zeta) \in \{(m_i, \pi_\xi), (m'_i, \pi_\eta), (m''_i, g\pi_\xi)\}\). We denote \(m_0 = m'_0 = m''_0 = o\). Suppose now that \((g\xi|y)_o \geq \kappa(g)\) and \(\kappa(g) < D\). Using the thin triangles Lemma
4.7 for \((o, go, g\xi)\), since \(D > \kappa(g)\), we deduce that there exists \(i_0, j_0 \leq \kappa(g)\) such that \(d(gm_{i_0}, m''_{j_0}) \leq 6\delta + 4\).

We now use the fellow-travellers Lemma 4.8 for \(g\xi, y \in \overline{M}^h\) with the geodesic rays \([o, g\pi]\) and \([o, \pi]\). Since \((g\xi y)_o \geq r := \kappa(g)\) and \(j_0 \leq \kappa(g)\), we find that there exists \(i_1 \leq \kappa(g)\) such that \(d(m''_{j_0}, m'_{i_1}) \leq 8\delta\).

We deduce that \(d(gm_{i_0}, m'_{i_1}) \leq 14\delta + 4\). Therefore, desired result holds with \(C := \{(m_i, m'_j)|0 \leq i, j \leq \lfloor D \rfloor - 1\}\). \(\square\)

**Proof of Proposition 4.2.** We fix \(r \in [0, 1)\) such that \(\|\lambda_G(\mu_{r, \text{lazy}})\| < 1\). The latter may be equal to 1 only for \(r = 0\) (see Remark 4.4) in which case the inequality holds trivially by setting \(C(., 1) \equiv \infty\). Note that the measure \(\nu\) is \(\mu_{r, \text{lazy}}\)-stationary and denoting by \(S_r\) the support of \(\mu_{r, \text{lazy}}\), we have \(\kappa_{S_r} = \kappa_S\). To ease the notation, in the proof, we write \(\mu\) for \(\mu_{r, \text{lazy}}\).

Let \(\nu\) be a \(\mu\)-stationary probability measure on \(\partial hM\). Let \((B = G^N, \beta = \mu \otimes \mathbb{N})\) be the Bernoulli space and \(T : B \to B\) the shift map. Since \(\partial hM\) is compact, metrizable (see e.g. [MT18, Proposition 3.1]) and \(\text{Isom}(M)\) acts continuously on \(\partial hM\), by a result of Furstenberg [Fur73], it follows that for \(\beta\)-almost every \(b \in B\), there exists a probability measure \(\nu_b\) on \(\partial hM\) such that the following weak convergence holds

\[
(b_1 \cdots b_n) \ast \nu \xrightarrow[n \to +\infty]{\text{weakly}} \nu_b. \tag{4.3}
\]

Moreover, for every \(n \in \mathbb{N}\), we have

\[
(b_1 \cdots b_n) \ast \nu T^n b = \nu_b \quad \text{and} \quad \nu = \int_B \nu_b d\beta(b). \tag{4.4}
\]

For every \(b \in B\) and \(n \in \mathbb{N}\), denote for simplicity \(R_n(b) := b_1 \cdots b_n\) and \(k_n(b) := \kappa(R_n(b))\). Let now \(\eta \in \overline{M}^h\). Using (4.4), the fact that \(\kappa_n(b) \to +\infty\) almost surely and multiple applications of Fubini–Tonelli theorem, we have

\[
\int_{\partial hM} (\eta|\xi)_o d\nu(\xi) = \int_B \int_{\partial hM} (\eta|y)_o d\nu_b(y) d\beta(b)
= \int_B \int_0^\infty \nu_b((\eta|y)_o \geq t) dt d\beta(b) = \int_0^\infty \int_B \nu_b((\eta|y)_o \geq t) dt d\beta(b) dt
\leq \int_0^\infty \int_B \sum_{n=0}^\infty 1_{\kappa_n(b) \leq t < \kappa_{n+1}(b)} \nu_b((\eta|y)_o \geq t) dt d\beta(b) dt
= \sum_{n=0}^\infty \int_B \int_{\kappa_n(b)}^{\kappa_{n+1}(b)} \nu_b((\eta|y)_o \geq t) 1_{\kappa_n(b) \leq t < \kappa_{n+1}(b)} dt d\beta(b)
\leq \sum_{n=0}^\infty \kappa_S \int_B \nu_b((\eta|y)_o \geq \kappa_n(b)) d\beta(b)
\]
Now using the first equality of (4.4), we have
\[
\int_{\partial_h M} (\eta|\xi)_o d\nu(\xi) \leq \sum_{n=0}^{\infty} \kappa_S \int_B \nu_{T^n b}(\eta|b_1 \ldots b_n y)_o \geq \kappa_n(b) d\beta(b)
\]
\[
= \sum_{n=0}^{\infty} \kappa_S \int_B \int_G \nu_0((\eta|gy)_o \geq \kappa(g)) d\mu^*(g) d\beta(b)
\]
\[
= \sum_{n=0}^{\infty} \kappa_S \int_B \int_{\partial_h M} \mu^*(\eta|gy)_o \geq \kappa(g)) d\nu_b(y) d\beta(b),
\]
where in the second line we used the fact that \(T^n \beta = \beta\) and in the last line we used Fubini-Tonelli’s theorem. We conclude that
\[
\int_X (\eta|\xi)_o d\nu(x) \leq \sum_{n=0}^{\infty} \kappa_S \sup_{y \in \partial_h M} \beta((\eta|R_n y)_o \geq \kappa(R_n)).
\]
Using now Lemma 4.6, we get that for every \(c > 1\), \(n \in \mathbb{N}\), \(\xi \in \partial_h M\), \(y \in \mathcal{M}^\mathcal{N}\),
\[
\mathbb{P} ((R_n \xi|y)_o \geq \kappa(R_n)) \leq \mathbb{P}(\kappa(R_n) \geq c^n) + c^{2n} \sup_{x', y' \in M} \mathbb{P}(d(R_n x', y') \leq R(\delta)).
\]
Thus
\[
\int_X (\eta|\xi)_o d\nu(x) \leq \kappa_S \left( \sum_{n=0}^{+\infty} a_n + \sum_{n=0}^{+\infty} b_n \right). \tag{4.5}
\]
On the one hand, since \(\kappa(R_n) \leq n\kappa_S\), we have \(\sum_{n=0}^{+\infty} a_n \leq \sum_{n=0}^{+\infty} 1_{c^n \leq n\kappa_S}.\)
Using this, it is not hard to deduce that
\[
\sum_{n=0}^{+\infty} a_n \leq \max\{ \frac{2 \ln \kappa_S}{\ln c}, \frac{4}{(\ln c)^2} \}. \tag{4.6}
\]
On the other hand, by Lemma 4.5, for every \(n \in \mathbb{N}\), we have for every \(x', y' \in M\),
\[
\mathbb{P}(d(R_n x', y') \leq R(\delta)) \leq A_0(R(\delta) + D_0) ||\lambda_G(\mu)||^n,
\]
where \(A_0(.)\) is the function defined in that lemma.
We deduce that for every \(1 < c < ||\lambda_G(\mu)||^{-1/2}\),
\[
\sum_{n=0}^{+\infty} b_n \leq A_0(R(\delta) + D_0) \sum_{n=0}^{+\infty} (c^2 ||\lambda_G(\mu)||)^n \leq \frac{A_0(R(\delta) + D_0)}{1 - c^2 ||\lambda_G(\mu)||}. \tag{4.7}
\]
The proof follows by combining (4.5), (4.6) and (4.7).

Our main result Theorem 4.1 now directly follows by putting together Propositions 3.5 and 4.2.

Proof of Theorem 4.1. Using the estimate (4.1) in combination with Lemma 3.3, one gets that in Proposition 3.5, the constant \(c\) is bounded above by \(2C(\kappa_S, ||\lambda_G(\mu_{r, lazy})||_2)\).
Since the right-hand-side of the inequality (3.11) is increasing in \(c\), we can substitute \(2C(\kappa_S, ||\lambda_G(\mu_{r,lazy})||_2)\) for \(c\), one gets that for every \(\xi \in \mathcal{M}^h\), \(n \in \mathbb{N}\) and \(t > 0\),

\[
\mathbb{P}(|\sigma(L_n, \xi) - n\ell(\mu)| \geq nt) \leq 2\exp(-\frac{nt^2}{32(\kappa_S + 2C(\kappa_S, ||\lambda_G(\mu_{r,lazy})||_2)^2)})
\]

for every \(r \in [0,1)\). This yields the desired estimate. \(\square\)

5. Concentration inequalities for random walks on Gromov hyperbolic groups and rank-one linear groups

The goal of this section is to prove Corollaries 1.5 and 1.6 using Theorem 4.1. An important ingredient that allows us to obtain concentration inequalities with implied constants that depends, in a minimal fashion, on the probability measure \(\mu\) is a version of uniform Tits alternative for group of isometries of hyperbolic spaces. For hyperbolic groups, we will use Koubi’s results [Kou98] and for linear groups the strong Tits alternative of Breuillard [Bre08].

We conclude this section by a discussion on the implication of our methods and results to continuity of the drift and the Hausdorff dimension of the harmonic measure on the Gromov boundary, and relating them to known results in the literature.

5.1. Concentration inequalities for random walks on Gromov hyperbolic groups. For the proof of Corollary 1.5, we will use the following result of Koubi:

**Theorem 5.2 ([Kou98]).** Let \(\Gamma\) be a finitely generated non-elementary hyperbolic group. There exists \(N_{\Gamma} \in \mathbb{N}\) such that for any finite subset \(S\) generating \(\Gamma\), there exists two elements \(a, b \in S^{N_{\Gamma}}\) that generate a free subgroup of rank two.

Here, by \(S\)-length of an element \(g \in \Gamma\), we mean the distance of \(g\) to the identity element in the word-metric induced by \(S\).

The previous result will be useful to us in combination with the following straightforward observation (see e.g. [Bre11b, §8])

**Lemma 5.3.** Let \(\Gamma\) be a countable group and \(S \subset \Gamma\) such that \(S^{N_0}\) contains a pair of elements that generates a free subgroup of rank two for some \(N_0 \in \mathbb{N}\). Let \(\mu\) be a probability measure with support \(S \cup \{\text{id}\}\) and set \(m_\mu = \min_{g \in S} \mu(g)\). Then,

\[
||\lambda_\Gamma(\mu)||_2 \leq \left(1 - \left(1 - \frac{\sqrt{3}}{2}\right) m_\mu^{2N_0}\right)^{\frac{1}{2N_0}}.
\]

**Proof.** Consider the probability measure \(\mu' = \mu \ast \tilde{\mu}\) and denote by \(S'\) its support. Since \(S\) contains identity, the set \(S'\) is symmetric and it contains \(S\). It follows that \((S')^{N_0}\) contains a set \(\{a, b, a^{-1}, b^{-1}\}\), where \(a, b\) are the generators of a free group of rank two.
Since $\mu'$ is symmetric, the operator $\lambda_\Gamma(\mu')$ on $L^2(\Gamma)$ is self-adjoint, and since $\lambda_\Gamma(\mu'^{N_0}) = \lambda_\Gamma(\mu')^{N_0}$, we have $||\lambda_\Gamma(\mu')||_2 = ||\lambda_\Gamma(\mu'^{N_0})||_{2/N_0}^{1/2}$. Therefore,

$$||\lambda_\Gamma(\mu)||_2 = ||\lambda_\Gamma(\mu')||_2^{1/2} = ||\lambda_\Gamma(\mu^{N_0})||_2^{1/2}$$

(5.1)

On the other hand, we write $\mu'^{N_0} = m_{\mu'}^{N_0} \eta + (1 - m_{\mu'}^{N_0})\zeta$, where $\eta$ is the uniform probability measure on $\{a, b, a^{-1}, b^{-1}\}$ and $\zeta$ some probability measure on $\Gamma$. Using the trivial bound $||\lambda_\Gamma(\zeta)||_2 \leq 1$, we deduce that

$$||\lambda_\Gamma(\mu'^{N_0})||_2 \leq 1 - \kappa m_{\mu'}^{N_0},$$

(5.2)

where $1 - \kappa = \sqrt{3}/2$ is the spectral radius of the uniform probability measure on the free group [Kes59, Theorem 3]. Combining (5.1) and (5.2), and using the fact that $m_{\mu'} \geq m_{\mu}^2$, we deduce that

$$||\lambda_\Gamma(\mu)||_2 \leq \left(1 - \kappa m_{\mu}^{2N_0}\right)^{1/2N_0}$$

(5.3)

□

Proof of Corollary 5.2. Note first that by [CCMT15, Proposition 2.6], the group $\Gamma$ is a non-elementary hyperbolic group. Therefore, the hypothesis of Lemma 5.3 is satisfied for every finite generating set of $\Gamma$ with a uniform constant $N_0 = N_1'$ thanks to Koubi’s Theorem 5.2. Applying Lemma 5.3 to $\mu_{1/2,\text{lazy}}$ and using the fact that $m_{\mu'} \geq m_{\mu}^2$, we deduce that

$$||\lambda_\Gamma(\mu_{1/2,\text{lazy}})||_2 \leq \left(1 - \kappa m_{\mu}^{2N_0}\right)^{1/2N_0}$$

(5.3)

□

5.4. Concentration inequalities for random walks on rank-one semisimple linear groups. For the proof of Corollary 1.6, we will use the following result of Breuillard [Bre08, Theorem 1.1] and [Bre11a] (see [Bre11b] for the particular case of $SL_2$).

Theorem 5.5 ([Bre08]). For every $d \in \mathbb{N}$ there is $N_d \in \mathbb{N}$ such that if $k$ is any field and $S$ is a finite symmetric subset of $GL_d(k)$ containing identity, either $S^{N_d}$ contains two elements which generate a non-abelian free group, or the group generated by $S$ contains a finite-index solvable subgroup.

We are now able to give the
Proof of Corollary 1.6. Let $d \in \mathbb{N}$ be given, and $k$ and $H \subseteq \mathbb{S}_{ld}$ be as in the statement. Let the natural number $N_d'$ (depending only on $d$) be as given by Theorem 5.5. Let $\mu$ be a probability measure whose support $S$ is a finite $\mathbb{H}(k)$ that generates a discrete non-amenable subgroup $\Gamma$ of $\mathbb{H}(k)$. Let $\mu' := \mu_{1/2, \text{lazy}}$, $\mu'' := \mu' \ast \mu^{-1}$, denote by $S'$ the support of $\mu'$ and $S''$ the support of $\mu''$. Notice that the finite set $S''$ is symmetric, contains the identity and it generates the group $\Gamma$. Therefore it follows by Theorem 5.5 that $S''N_d'$ contains two elements that generate a free group of rank two where the constant $\lambda$ where the constant $\lambda = 16$.

Since $\lambda > 0$ and $\mu''$ has bounded support, we deduce

$$||\lambda_{\Gamma}(\mu'')||_2 \leq 1 - m_{\mu}^2 \frac{2N_d'}{4N_d'^2}.$$ 

As in the proof of Corollary 1.5, by the discreteness assumption of $\Gamma$ it follows that $||\lambda_{\Gamma}(\mu_{1/2, \text{lazy}})||_2 = ||\lambda_{\Gamma}(\mu')||_2$, so that a direct application of Theorem 1.2 (with $r = 1/2$ on the right hand side of the theorem) concludes the proof with $N_d = 16N_d'$, $\alpha_d = 2^{25 + N_d^4} \frac{1}{(1 - \sqrt{3}/2)^2}$, and $A_{\mathbb{H}, k} = A_0/3 + 3$, where $A_0$ is the constant defined in Remark (4.3) applied for the isometry group $G$ of the symmetric space $M$ associated to the rank-one group $\mathbb{H}(k)$. $\square$

5.6. Applications to continuity of the drift.

5.6.1. Application to continuity of the drift. Here we briefly mention the connection between our concentration estimates and continuity of the drift. Subsequently, we include the proof of a general continuity statement that one can deduce directly from unique cocycle-average property for the Busemann cocycle (which was a key point in obtaining our concentration result).

Here is a direct consequence of Theorem 1.2 regarding the continuity of drift:

\textbf{Corollary 5.7.} Let $(M, d)$ be a proper geodesic hyperbolic metric space such that $\text{Isom}(M)$ acts cocompactly on $M$. Consider a sequence of non-elementary probability measures $(\mu_m)_{m \in \mathbb{N}}$ with bounded support $S_n$ in the group $\text{Isom}(M)$ such that $\lim \sup_{m \to \infty} \inf_{r \in [0, 1]} ||\lambda_G(\mu_{r, \text{lazy}})||_2 < 1$ and $\sup_{n \in \mathbb{N}} \kappa_n S_n < \infty$. Suppose that $\mu_n$ converges weakly to some probability measure $\mu_\infty$. Then, as $m \to \infty$

$$\ell(\mu_m) \to \ell(\mu_\infty).$$

\textbf{Proof.} Fix $t_0 > 0$ and let $\lambda > 1$ be a constant such that for every $m \in \mathbb{N}$, $\inf_{r \in [0, 1]} ||\lambda_G(\mu_{m, r, \text{lazy}})||_2 < \lambda$. Set $\kappa_0 = \sup_{n \in \mathbb{N}} \kappa_n S_n$. Choose $n_0$ large enough so that

$$\left| \frac{1}{n_0} \mathbb{E}_{\mu_\infty} [\kappa(L_{n_0})] - \ell(\mu_\infty) \right| < t_0,$$

and

$$2 \exp(-\frac{n_0 t_0^2}{\kappa_0^2 32 (16 \ln^+(\kappa_0) + 8 A_0/3 + 33)^2 (1 - \sqrt{\lambda})^4}) < \frac{t_0}{\kappa_0},$$

where the constant $A_0$ (depending only on $M$) is as in Remark 1.3.
The choice of $n_0$ satisfying (5.5) implies by using Theorem 1.2 and the bound on the function $D$ given in Remark 1.3 which is non-decreasing in $\kappa$ and in $\lambda$, that for every $m \in \mathbb{N}$ large enough, we have

$$\mathbb{P}_{\mu_n}(|\kappa(L_{n_0}) - n_0\ell(\mu_m)| > n_0t_0) \leq \frac{t_0}{\kappa_0}.$$

Since $\frac{1}{n_0}\kappa(L_{n_0}) - \ell(\mu_m) \leq \kappa_0$, this implies that for every $m$ large enough, we have

$$\frac{1}{n_0}\mathbb{E}_{\mu_n}[\kappa(L_{n_0})] - \ell(\mu_m) \leq 2t_0. \quad (5.6)$$

On the other hand, since $\mu_m \to \mu_\infty$ weakly, we have that as $m \to \infty$, $\mathbb{E}_{\mu_n}[\kappa(L_{n_0})] \to \mathbb{E}_{\mu_\infty}[\kappa(L_{n_0})]$. Therefore, combining (5.4) with (5.6), it follows that for every $n \in \mathbb{N}$ large enough, we have $|\ell(\mu_\infty) - \ell(\mu_m)| \leq 4t_0$ completing the proof.

A particular situation where the hypotheses of the previous result are satisfied is the following. Suppose that there exists a finite set $S \subset \text{Isom}(M)$ that contains the supports of all $\mu_n$ for $n \in \mathbb{N}$, $\mu_n \to \mu_\infty$ weakly and $\rho(\lambda_G(\mu_\infty)) < 1$ (see §5.6.1). As we will now prove a more general continuity statement, we leave the justification of this claim which can easily be deduced from the results of Berg–Christensen [BC74].

The following general continuity statement is the one can deduce from the unique cocycle-average property similarly to Hennion [Hen84] and Furstenberg–Kifer [FK83]. For a very similar proof closer to our setting and related remarks, see Gouëzel–Mathéus–Maucourant [GMM18, Proposition 2.3]. In the following statement, for a probability measure $\mu$ on $\text{Isom}(M)$, we denote $L_1(\mu) = \int \kappa(g)d\mu(g)$.

**Proposition 5.8.** Let $(M, d)$ be a proper geodesic hyperbolic metric space. Let $\mu_n$ be a sequence of non-elementary probability measures that converges weakly to a non-elementary probability measure $\mu_\infty$. Suppose furthermore that $L_1(\mu_n) \to L_1(\mu)$ as $n \to \infty$. Then,

$$\ell(\mu_n) \to \ell(\mu).$$

**Proof.** Let $\nu_n$ be a $\mu_n$-stationary probability measure on the horofunction boundary $X$ of $M$. By unique cocycle-average property [BQ16b, Proposition 3.3(c)], we have

$$\ell(\mu_n) = \int_{G \times X} \sigma(g, \xi)d\mu_n(g)d\nu_n(\xi). \quad (5.7)$$

Since $X$ is compact, up to passing to a subsequence of $\nu_n$, we can suppose that the sequence $\nu_n$ converges to a probability measure $\nu$ on $X$. Since $\mu_n \to \mu$ weakly, one deduces from the continuity of the action of $G$ on $X$ that $\nu$ is $\mu$-stationary. Using the hypothesis that $L_1(\mu_n) \to L_1(\mu)$ and the fact that $\kappa(g) \geq |\sigma(g, \xi)|$ for every $g \in G$ and $\xi \in X$, one gets by dominated convergence that the sequence of integrals in (5.7) converges to $\int_{G \times X} \sigma(g, \xi)d\mu(g)d\nu(\xi)$. But by unique cocycle-average property, the latter is equal to $\ell(\mu)$. This implies the claimed convergence. \qed
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