On pro-$p$ link groups of number fields

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Abstract. As an analogue of a link group, we consider the Galois group of the maximal pro-$p$-extension of a number field with restricted ramification which is cyclotomically ramified at $p$, i.e., tamely ramified over the intermediate cyclotomic $\mathbb{Z}_p$-extension of the number field. In some basic cases, such a pro-$p$ Galois group also has a Koch type presentation described by linking numbers and mod 2 Milnor numbers (Rédei symbols) of primes. Then the pro-$2$ Fox derivative yields a calculation of Iwasawa polynomials analogous to Alexander polynomials.

1 Introduction

Let $p$ be a fixed prime number. We often regard the ring $\mathbb{Z}_p$ of $p$-adic integers as the additive group. Let $k$ be a number field, i.e., an extension of finite degree over the rational number field $\mathbb{Q}$, and let $P$ be the set of all primes of $k$ lying over $p$. For an algebraic extension $K/k$ and a finite set $S$ of primes of $k$ none of which are complex archimedean, we denote by $K_S$ the maximal pro-$p$-extension of $K$ unramified outside primes lying over $v \in S$, and put $G_S(K) = \text{Gal}(K_S/K)$. Since only pro-$p$-extensions over $k$ are treated here, we assume that the absolute norm $|N_{k/\mathbb{Q}}v| \equiv 1 \pmod{p}$ if $v \in S \setminus P$ is a prime ideal, and that $S$ contains no (real) archimedean primes if $p \neq 2$. The finitely presented pro-$p$ group $G_S(k)$ has a Koch type presentation in some basic cases (cf. e.g. [22, 23, 54]), and has been studied with a viewpoint of arithmetic topology (cf. [1, 34, 35, 52] etc.).

Let $k^{\text{cyc}}$ be the cyclotomic $\mathbb{Z}_p$-extension of $k$. A main object of Iwasawa theory is the $S$-ramified Iwasawa module $G_S(k^{\text{cyc}})^{ab}$, which is the abelianization of $G_S(k^{\text{cyc}})$. If $P \subset S$, then $k^{\text{cyc}} \subset k_S$, and hence $G_S(k^{\text{cyc}})^{ab}$ can be studied as a subquotient of $G_S(k)$ with the action of $\text{Gal}(k^{\text{cyc}}/k)$ induced by inner automorphism. On the other hand, we assume that $S \cap P = \emptyset$ throughout the following. In a similar way, the tamely ramified Iwasawa module $G_S(k^{\text{cyc}})^{ab}$ can be also studied as a subquotient of the Galois group

$$\tilde{G}_S(k) = \text{Gal}((k^{\text{cyc}})_S/k) \simeq G_S(k^{\text{cyc}}) \rtimes \text{Gal}(k^{\text{cyc}}/k)$$

of the maximal pro-$p$-extension $(k^{\text{cyc}})_S/k$ which is unramified outside $S \cup P$ and ‘cyclotomically ramified’ at any $v \in P$. Since the pro-$p$ group $\tilde{G}_S(k)$ is also finitely presented (cf. [2, 44]), we can know the structure of any subquotient in principle if we

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obtain the explicit presentation of $\widetilde{G}_S(k)$ by generators and relations. In particular when $S = \emptyset$, $(k^{\text{cyc}})_\emptyset$ is the union of the $p$-class field towers of intermediate fields of $k^{\text{cyc}}/k$, and the closed subgroup $G_\emptyset(k^{\text{cyc}})$ of $\widetilde{G}_\emptyset(k)$ is a central object in nonabelian Iwasawa theory of $\mathbb{Z}_p$-extensions (Theorem $3.1$). The purpose of this paper is to study these subjects from a viewpoint of analogy between Iwasawa theory and Alexander-Fox theory (cf. $[15, 18, 26, 36, 51]$ etc.), regarding the pro-$p$ Galois group $\widetilde{G}_S(k)$ as an analogue of a link group $\pi_1(X)$ (cf. Remark $3.6$). A summary of results in each section is the following.

In Section $2$ we recall Salle’s Shafarevich type formula on the generator rank and the relation rank of $\widetilde{G}_S(k)$, by referring the results and arguments of $[2, 44]$ with a little refinement in the case where $p = 2$.

In Section $3$ we obtain a Koch type presentation of $\widetilde{G}_S(k)$ in some basic cases (Theorems $3.1$ and $3.2$), where the relations modulo the 3rd step of the lower central series are described by linking numbers of primes. If $k = \mathbb{Q}$ and the archimedean prime $\infty \in S$, the Rédei symbols appear in the relations modulo the 4th step of the Zassenhaus filtration as the mod $2$ Milnor numbers (Proposition $3.7$ and Corollary $3.9$). In particular, we obtain a triple of Borromean primes including $p = 2$ (Example $3.10$). Moreover, by the arguments as in $[2, 44]$ etc., $\widetilde{G}_S(\mathbb{Q})$ is a mild pro-$p$ group of deficiency zero if $\infty \notin S$ and $S \cup \{p\}$ forms a ‘circular set’ of primes (cf. Section $3.3$).

Then the cohomological dimension $cd(\widetilde{G}_S(\mathbb{Q})) = 2$.

In Section $4$ we focus on Iwasawa polynomials, which are defined as the characteristic polynomials of Iwasawa modules, and certainly analogous to Alexander polynomials (cf. Remark $4.3$). Then the Koch type presentation of $\widetilde{G}_\emptyset(k)$ induces another proof of Gold’s theorem $[10]$ (Theorem $4.2$). If $\mathbb{Q}_S$ contains a quadratic extension $K/\mathbb{Q}$ ramified at any $v \in S \setminus \{\infty\}$, the unramified Iwasawa module $X = G_{(\infty)}(K^{\text{cyc}})^{ab}$ (in the narrow sense) is a subquotient of $\widetilde{G}_S(\mathbb{Q})$. Following the arguments by Fröhlich and Koch (cf. $[22, 23, 52]$), we obtain an approximation of the initial Fitting ideal of $X$ from the Koch type presentation of $\widetilde{G}_S(\mathbb{Q})$ by the pro-$2$ Fox free differential calculus (Theorem $4.4$). In particular, for a certain family of imaginary quadratic fields $K$, we calculate an approximation of the quadratic Iwasawa polynomial of $X$, which is described by $4$th power residue symbols (Theorem $4.7$). An explicit presentation of $\widetilde{G}_{(\infty)}(K) = \text{Gal}(K^{\text{cyc}}_{(\infty)}/K)$ is also calculated for a certain family of real quadratic fields $K$ (Theorem $4.10$).

Notations. For a pro-$p$ group $G$, we denote by $[h, g] = h^{-1}g^{-1}hg$ the commutator of $g, h \in G$. For a closed subgroup $H$ of $G$, $[H, G]$ (resp. $H^p$) denotes the minimal closed subgroup containing $\{[h, g] | h \in H, g \in G\}$ (resp. $\{h^p | h \in H\}$). Then $G^{ab} = G/[G, G]$. The $i$th cohomology group with coefficients in $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is denoted by $H^i(G) = H^i(G, \mathbb{Z}/p\mathbb{Z})$. For a set $Y$, $|Y|$ denotes the cardinality. For objects $x$ and $y$, $\delta_{x,y} = 1$ if $x = y$, and $\delta_{x,y} = 0$ otherwise.
2 Generator and RANK relation

2.1 Pro-p class field theory. First, we recall the idèlic class field theory by [17]. For each prime \( v \) of \( k \), we put \( \hat{k}_v^\times = \lim \left( k_v^\times / (k_v^\times)^P \right) \), where \( k_v \) is the completion of \( k \) at \( v \). For a nonarchimedean prime \( v \), we put \( U_v = \text{Ker}(k_v^\times \to \mathbb{Z}) \) is the unit group of the local field \( k_v \). The local class field theory yields that \( \text{Gal}(k_v^{ab,p}/k_v) \cong \hat{k}_v^\times \) and \( \text{Gal}(k_v^{ur,p}/k_v) \cong \hat{k}_v^\times / U_v \), where \( k_v^{ab,p} \) denotes the maximal abelian pro-p-extension of \( k_v \), and \( k_v^{ur,p} \) denotes the unramified \( \mathbb{Z}_p \)-extension of \( k_v \). For the cyclotomic \( \mathbb{Z}_p \)-extension \( k_v^{\text{cyc}} \) of \( k_v \), put

\[ N_v = \text{Ker}(\hat{k}_v^\times \to \lim k_v^\times / N_{k_v^{\text{cyc}}/k_v}(k_v^{\text{cyc}}) \cong \text{Ker}(\text{Gal}(k_v^{ab,p}/k_v) \to \text{Gal}(k_v^{\text{cyc}}/k_v)), \]

where \( k_v^{\text{cyc}} \) denotes the subextension of \( k_v^{\text{cyc}} \) such that \( [k_v^{\text{cyc}} : k_v] = p^n \). Then we put \( \tilde{U}_v = U_v \cap N_v \). Note that \( \tilde{U}_v = U_v = N_v \) (i.e., \( k_v^{\text{cyc}} = k_v^{ur,p} \)) if \( v \notin P \), and that \( \hat{k}_v^\times / \tilde{U}_v \cong \text{Gal}(k_v^{ur,p}/k_v^\times / k_v^{\text{cyc}}/k_v) \cong \hat{k}_v^\times / U_v \) for each prime \( v \). On the other hand, we put \( \tilde{U}_v = U_v = \hat{k}_v^\times \) for archimedean \( v|\infty \), where we note that \( \hat{k}_v^\times \cong \mathbb{R}^\times / \mathbb{R}_{>0} \) if \( p = 2 \) and \( v \) is real, and \( \hat{k}_v^\times = 1 \) otherwise.

Example 2.1 (cf. e.g. [3, Appendix §3]). Suppose that \( k_v = \mathbb{Q}_p \). Then \( \hat{k}_v^\times = p^\mathbb{Z}_p \). If \( p \neq 2 \), we have \( U_v = (1 + p)^{\mathbb{Z}_p} \), \( N_v = p^{\mathbb{Z}_p} \), \( \tilde{U}_v = \{1\} \). If \( p = 2 \), we have \( U_v = (1 + p)^{\mathbb{Z}_p} \), \( N_v = (-1)^{2\mathbb{Z}_p} \), \( \tilde{U}_v = \{1\} \).

Let \( J_k = \text{Ker}(\prod_v \hat{k}_v^\times \to \prod_v (\hat{k}_v^\times / U_v) / \bigoplus_v (\hat{k}_v^\times / \tilde{U}_v)) \) be the idèles of \( k \) as defined in [17], and put \( k^\times = k^\times \otimes \mathbb{Z}_p \subset J_k \), where we identify \( k^\times \) with the image of the diagonal embedding \( k^\times \to \lim k^\times / (k^\times)^{P} \text{diag} J_k \) (cf. [17] Remarques 1 (iii), Théorème et définition 1.4). We also identify \( \hat{k}_v^\times \) with \( \hat{k}_v^\times \times \prod_{v \neq w} \{1\} \subset J_k \) for each prime \( w \) of \( k \). The reciprocity map in the pro-p version of global class field theory [17] is the isomorphism

\[ \text{rec} : J_k / k^\times \cong \text{Gal}(k_v^{ab,p}/k) \]

such that \( \text{rec}(U_v k^\times / k^\times) \) (resp. \( \text{rec}(k_v^{ab,p} k^\times / k^\times) \)) is the inertia group (resp. decomposition group) of each prime \( v \), where \( k_v^{ab,p} \) is the maximal abelian pro-p-extension of \( k \). In particular, we have \( J_k / \mathcal{U} k^\times \cong G_\emptyset(k) \), where \( \mathcal{U} = \prod_{v} U_v \). Recall that \( S \cap \emptyset = \emptyset \). Put

\[ \tilde{U}_S = \prod_{v \in S} \tilde{U}_v = \prod_{v \notin S \cup P} U_v \times \prod_{v \in P} \tilde{U}_v \times \prod_{v \in S} \{1\} \subset \mathcal{U}. \]

Then we have the following isomorphism.

Proposition 2.2 (cf. [44] Proposition 1.2). The reciprocity map (2.1) induces an isomorphism

\[ \text{rec}_S : J_k / \tilde{U}_S k^\times \cong G_S(k) \]

Proof. See the proof of [44] Proposition 1.2 with the consideration of the ramification of \( v|\infty \). The same arguments hold for \( S \) containing some \( v|\infty \).
2.2 Generator rank and Kummer Groups. The map \( \overline{k^\times} = k^\times \otimes \mathbb{Z}_p \rightarrow \lim \overline{k^\times}/(k^\times)^p \rightarrow \overline{k^\times}/(k^\times)^p \) induces an isomorphism

\[
(2.2) \quad k^\times/(k^\times)^p \simeq \overline{k^\times}/(k^\times)^p : a(k^\times)^p \mapsto (a \otimes 1)(\overline{k^\times})^p.
\]

Put \( V = U_\mathcal{J}_k \cap k^\times \) and \( \tilde{V}_S = \tilde{U}_S \mathcal{J}_k \cap k^\times \). Then \( k^\times/(k^\times)^p \simeq k^\times/(k^\times)^p \) and \( U_v/U_v^p \simeq U_v/U_v^p \) for each \( v \), and since \( k^\times/(k^\times)^p \simeq k^\times/(k^\times)^p \) for \( v \in P \), we have

\[
(2.3) \quad \overline{V}_S/(k^\times)^p \simeq \overline{B}_S/(k^\times)^p \subset B_S/(k^\times)^p \subset B_0/(k^\times)^p \simeq V/(k^\times)^p
\]

under the isomorphism \((2.2)\), where \( B_0 \) consists of \( a \in k^\times \) such that the principal ideal \((a) = a^p\) is a \( p \)-th power of an ideal \( a \) of \( k \), and \( B_S = \{ a \in B_0 \mid a \in (k^\times)^p \text{ for all } v \in S \}, \)

\[
\overline{B}_S = \{ a \in B_S \mid a \in (N_{k_{v,1}/k_v}(k_{v,1}^{\text{cyc}}) \cap U_v)(k^\times)^p \text{ for all } v \in P \}.
\]

There is also an exact sequence

\[
(2.4) \quad 0 \rightarrow E_k/E_k^p \rightarrow B_0/(k^\times)^p \rightarrow \overline{B}_S/(k^\times)^p \rightarrow C_l(k) \rightarrow \overline{C}_l(k),
\]

where \( E_k \) is the unit group of \( k \), and \( C_l(k) \) is the ideal class group of \( k \).

Remark 2.3. The following facts are induced from \((2.3)\) and \((2.4)\): If \( p \neq 2 \) and either \( k = \mathbb{Q} \) or \( k \) is an imaginary quadratic field with class number \( h_k = |C_l(k)| \) not divisible by \( p \), then \( \overline{B}_S = B_0 = (k^\times)^p \) except when \( p = 3 \) and \( k = \mathbb{Q}(\sqrt{-3}) \). If \( p = 2 \) and \( k = \mathbb{Q} \), and if \( S \) contains \( \infty \) or a prime number \( q \equiv 3 \pmod{4} \), then \( \overline{B}_S = B_0 = (k^\times)^p \) (cf. \[44\] Example 11.12).

Then we have the following formula.

Proposition 2.4 (cf. [44, Théorème 3.3]). Under the settings above, the minimal number \( \overline{d}_S \) of generators of \( \overline{G}_S(k) \) is

\[
\overline{d}_S = \dim_{\mathbb{F}_p} H^1(\overline{G}_S(k)) = |S| + |P| + \dim_{\mathbb{F}_p} \overline{B}_S/(k^\times)^p - \dim_{\mathbb{F}_p} E_k/E_k^p.
\]

Proof. See the proof of \[44\, Théorème 3.3\] (cf. also \[2\, Theorem 3.5 (i)]\) with the consideration of the ramification of \( v|\infty \). Since \( U_v/U_v^p \simeq \mathbb{Z}/p\mathbb{Z} \) also for archimedean \( v \in S \), the same arguments hold.

\[ \square \]

2.3 Relation rank and the Shafarevich kernel. Let \( k_v^{\text{pro-}p} \) (resp. \( k_v^{\text{pro-}p} \)) be the maximal pro-\( p \)-extension of \( k_v \) (resp. \( k \)). We fix an embedding \( k^{\text{pro-}p} \subset k_v^{\text{pro-}p} \) for each \( v \). Then

\[
\overline{G}_v = \text{Gal}(k_v^{\text{pro-}p}/k_v).
\]
is identified with the decomposition subgroup of \( \overline{G} = \text{Gal}(k^{\text{pro-p}}/k) \) for a fixed prime \( \wp|v \) of \( k^{\text{pro-p}} \) (cf. e.g. [15, II, §6.1]), and \( \overline{G}_{S,v} = \text{Gal}((k^{\text{cyc}})_S/k_v) \) is also identified with the decomposition subgroup of \( \overline{G}_S(k) = \text{Gal}((k^{\text{cyc}})/k) \) for the prime \( \wp \) of \( (k^{\text{cyc}})_S \) such that \( \wp v \). Since the maximal unramified pro-\( p \)-extension of \( k^{\text{cyc}}_v \) is \( k^{\text{ur-p}}k^{\text{cyc}}_v \), we have \( (k^{\text{cyc}})_S k_v \subset k^{\text{ur-p}}k^{\text{cyc}}_v \) if \( v \not\in S \) and \( v \nmid \infty \), particularly if \( v \in P \). For each \( v \in P \), we denote by

\[
G^\text{cr}_v = \text{Gal}(k^{\text{ur-p}}k^{\text{cyc}}_v/k_v) \cong \mathbb{Z}_p^2
\]

the maximal ‘cyclotomically ramified’ quotient of \( \overline{G}_v \). Put

\[
\Pi^2(\overline{G}_S(k)) = \text{Ker}(H^2(\overline{G}_S(k)) \xrightarrow{\text{loc}} \bigoplus_{v \in S^*} H^2(\overline{G}_v) + \bigoplus_{v \in P} H^2(G^\text{cr}_v))
\]

with the localization map loc induced from \( \{ \overline{G}_v \xrightarrow{\wp v} \overline{G}_{S,v} \subset \overline{G}_S(k) \}_{v \in S^*} \) and \( \{ G^\text{cr}_v \xrightarrow{\wp v} \overline{G}_{S,v} \subset \overline{G}_S(k) \}_{v \in P} \), where \( S^* = S \setminus \{ w \} \) with an arbitrary (but suitable) \( w \in S \) if \( S \neq \emptyset \) and \( k \) contains a primitive \( p \)th root \( \zeta_p \) of unity, and \( S^* = S \) otherwise.

By [44, Théorème 4.1] with a refinement in the case where \( p = 2 \), we obtain the following theorem.

**Theorem 2.5** (cf. [44 Théorème 4.1]). *Under the settings above, an inequality*

\[
\dim_{\mathbb{F}_p} \Pi^2(\overline{G}_S(k)) \leq \dim_{\mathbb{F}_p} \overline{B}_S/(k^\times)^p
\]

is satisfied, and the minimal number \( \tilde{r}_S \) of relations of \( \overline{G}_S(k) \) satisfies

\[
\tilde{r}_S = \dim_{\mathbb{F}_p} H^2(\overline{G}_S(k)) \leq \dim_{\mathbb{F}_p} \Pi^2(\overline{G}_S(k)) + |S| - (1 - \delta_{S,0})\theta + |P|,
\]

where \( \theta = 1 \) if \( \zeta_p \in k \), and \( \theta = 0 \) otherwise.

**Proof.** Following the proof of [22 Theorem 3.5] (cf. also [44 Théorème 4.1]), we give a summary proof valid also for \( p = 2 \). Let \( T_v \) be the inertia subgroup of \( \overline{G}_v \). Since \( \overline{G}_v \) is the decomposition subgroup of \( \overline{G} \) for \( \wp|v \), and since \( k^{\text{cyc}}_v = k^{\text{cyc}} k_v \), \( T_v \cap \text{Gal}(k^{\text{pro-p}}/k^{\text{cyc}}_v) \) is the inertia subgroup of \( \text{Gal}(k^{\text{pro-p}}/k^{\text{cyc}}_v) \) for \( \wp \). For each \( v \), we put \( H_v = \overline{G}_v/D_v \) with \( D_v \) defined as follows:

\[
\cdot \quad D_v = T_v \text{ if } v \not\in S \cup P. \text{ Then } H_v \cong \mathbb{Z}_p \text{ if } v \nmid \infty, \text{ and } H_v \cong 1 \text{ if } v|\infty.
\]

\[
\cdot \quad D_v = T_v \cap \text{Gal}(k^{\text{pro-p}}/k^{\text{cyc}}_v) \text{ if } v \in P. \text{ Then } H_v = G^\text{cr}_v \cong \mathbb{Z}_p^2.
\]

\[
\cdot \quad D_v = \{1\} \text{ if } v \in S. \text{ Then } H_v = \overline{G}_v.
\]

Then \( \overline{G}_S = \overline{G}_S(k) \cong \overline{G}/D_S \), where \( D_S = (\bigcup_{v \in S} D_v)\overline{\wp} \) is the minimal closed normal subgroup of \( \overline{G} \) containing \( D_v \) for all \( v \not\in S \). Put

\[
P_0 = \{ v \in P \mid 0 \to H^2(H_v) \xrightarrow{\text{inf}} H^2(\overline{G}_v) \text{ (exact)} \}
\]
Ker φ snake lemma for (2.7) yields an injective homomorphism (2.7), and since where we note the following: The injective map (2.8) Lemma 3.9. Since the continuous surjective homomorphism is well-defined. Since and hence is also well-defined. Since since (2.7) yields an injective homomorphism

\[ \text{III}^2(\widetilde{G}_S(k)) \hookrightarrow \text{Coker } \phi. \]

There is also a commutative diagram

\[ \begin{array}{cccccc}
\widetilde{U}_S/\widetilde{U}_S' & \xrightarrow{\xi} & J_k/k^\times J_k^p & \xrightarrow{\xi} & J_k/\widetilde{U}_S k^\times J_k^p & \rightarrow 0 \\
\xrightarrow{\phi'} & & \xrightarrow{\phi} & & & \\
0 & \xrightarrow{\phi'} & \widetilde{G}_S/\widetilde{G}_S'/[G, G] & \xrightarrow{\phi} & \widetilde{G}^\text{ab}/p & \rightarrow 0 \\
\end{array} \]

with exact rows, where we note that \( D_v / [D_v, G] = D_v/[G_v, G_v] \) if \( v \notin S \) by [2, Lemma 3.9]. Since the continuous surjective homomorphism \( \phi' \) is the dual of \( \phi \) in (2.7), and since \( k^\times \cap J_k^p = (k^\times)^p \) by Hasse principle (cf. e.g. [11, II, Theorem 6.3.3]) and (2.2), we have

\[ \text{Hom}(\text{Coker } \phi, \mathbb{F}_p) \simeq \text{Ker } \phi' \simeq \text{Ker } \xi' \simeq \text{Ker } \xi \]

\[ \simeq (\widetilde{U}_S \cap k^\times J_k^p) / J_k^p \simeq \widetilde{V}_S J_k^p / J_k^p \simeq \widetilde{V}_S / (k^\times)^p. \]
Therefore the injection $\mathbb{III}^2(\tilde{G}_S(k)) \hookrightarrow \text{Hom}(\tilde{B}_S/(k^\times)^p, \mathbb{F}_p)$ yields the inequality (2.5). If $v \in S \cup P$ and $v \nmid \infty$, then $H_v \simeq \mathbb{Z}_p \times \mathbb{Z}_p$ (cf. [23, Theorem 10.2]). Since $H^2(H_v) \simeq \mathbb{Z}/p\mathbb{Z}$ if $v \in S^* \cup P$, the inequality (2.6) also holds by the definition of $\mathbb{III}^2(\tilde{G}_S(k))$. 

Several consequences are also refined when $p = 2$ as follows.

**Corollary 2.6** (cf. [11, Corollaries 4.3]). The second partial Euler-Poincaré characteristic $\chi_2(\tilde{G}_S(k)) = 1 - \tilde{d}_S + \tilde{r}_S$ satisfies

$$\chi_2(\tilde{G}_S(k)) \leq 1 + \dim_{\mathbb{F}_p} E_k/E_k^p - (1 - \delta_{S,0})\theta.$$ 

In particular, $\chi_2(\tilde{G}_S(\mathbb{Q})) \leq 1$.

**Proof.** The inequalities are induced from Proposition 2.4 and Theorem 2.5 where we note that $\tilde{G}_0(\mathbb{Q}) \simeq G_{1/p}(\mathbb{Q}) \simeq \mathbb{Z}_p$. 

**Corollary 2.7** (cf. [2, Corollary 3.11]). If

$$|S| + |P| \geq \dim_{\mathbb{F}_p} E_k/E_k^p + 2 + 2\sqrt{1 + \dim_{\mathbb{F}_p} E_k/E_k^p - (1 - \delta_{S,0})\theta},$$

then $\tilde{G}_S(k)$ and $G_S(k^{\text{cyc}})$ are not $p$-adic analytic. In particular, $\tilde{G}_S(\mathbb{Q})$ and $G_S(\mathbb{Q}^{\text{cyc}})$ are not $p$-adic analytic if $|S| \geq 3 + \delta_{p,2}$. Moreover, $\tilde{G}_0(k)$ and $G_0(k^{\text{cyc}})$ are not $p$-adic analytic if $k/\mathbb{Q}$ is a totally imaginary extension of degree $[k: \mathbb{Q}] \geq 4(3 + \theta)$ in which $p$ splits completely.

**Proof.** Note that $\dim_{\mathbb{F}_p} E_k/E_k^p \geq 1$ if $\theta = 1$. The inequality $1 + \dim_{\mathbb{F}_p} E_k/E_k^p - (1 - \delta_{S,0})\theta \leq 0$ can not occur. Hence $\tilde{d}_S \geq |S| + |P| - \dim_{\mathbb{F}_p} E_k/E_k^p \geq 3$ by Proposition 2.4 and the assumption. Suppose that $\tilde{G}_S(k)$ is $p$-adic analytic. Then $\tilde{G}_S(k)$ has ‘finite rank’ (cf. [3, Theorem 8.36]), and hence the Golod-Shafarevich inequality $\tilde{r}_S > \frac{1}{4}(\tilde{d}_S)^2$ is satisfied (cf. [3, Theorem D1]). By Proposition 2.4 and Theorem 2.5, we have

$$(\tilde{d}_S)^2 < 4\tilde{r}_S \leq 4(\tilde{d}_S + \dim_{\mathbb{F}_p} E_k/E_k^p - (1 - \delta_{S,0})\theta),$$

and hence

$$|S| + |P| - \dim_{\mathbb{F}_p} E_k/E_k^p \leq \tilde{d}_S < 2 + 2\sqrt{1 + \dim_{\mathbb{F}_p} E_k/E_k^p - (1 - \delta_{S,0})\theta}.$$ 

This contradicts the assumption. Therefore $\tilde{G}_S(k)$ is not $p$-adic analytic. Since $\tilde{G}_S(k)/G_S(k^{\text{cyc}}) \simeq \mathbb{Z}_p$, $G_S(k^{\text{cyc}})$ is also not $p$-adic analytic. (cf. [3, Exercise 3.1, Corollary 8.33, Theorem 8.36]).

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Y. Mizusawa, On pro-p link groups of number fields
Remark 2.8. All sets $S$ with prometacyclic $G_S(\mathbb{Q}^{\text{cyc}})$ have been characterized arithmetically (cf. [16 31 33]). Then $\tilde{G}_S(\mathbb{Q})$ is $p$-adic analytic, and actually there exists such $S$ of cardinality $|S| = 2 + \delta_{p,2}$. There are other examples of prometacyclic $G_S(k^{\text{cyc}})$ for $p = 2$ and quadratic $k/\mathbb{Q}$ (cf. [28 30 15] and Theorem 4.10). Moreover, all imaginary quadratic fields $k$ with prometacyclic (or abelian) $G_{k}(k^{\text{cyc}})$ have been characterized (cf. [29 32 39]). There also exists $p$ such that $G_{\mathbb{Q}(\sqrt{p})^{\text{cyc}}}$ is abelian and not procyclic (cf. [49]).

Example 2.9 (cf. [31 Theorems 7.3 and 9.2]). Suppose that $p = 2$ and $k = \mathbb{Q}$. If $S = \{\ell_1, \ell_2\}$ ($\ell_1 \neq \ell_2$) with $\ell_1 \equiv \ell_2 \equiv 3 \pmod{8}$, then $\tilde{G}_S(\mathbb{Q}) \simeq \mathbb{Z}_2 \rtimes \mathbb{Z}_2$, $d_S = 2$, $\tilde{r}_S = 1$, and $\chi_2(\tilde{G}_S(\mathbb{Q})) = 0$. If $S = \{\ell, \infty\}$ with $\ell \equiv 5 \pmod{16}$, then $\tilde{G}_S(\mathbb{Q}) \simeq D_{2\infty} \rtimes \mathbb{Z}_2$, $d_S = \tilde{r}_S = 2$, and $\chi_2(\tilde{G}_S(\mathbb{Q})) = 1$, where $G_S(\mathbb{Q}^{\text{cyc}}) \simeq D_{2\infty} \simeq \mathbb{Z}_2 \rtimes (\mathbb{Z}/2\mathbb{Z})$ is a prodihedral pro-$2$ group.

Example 2.10 (cf. [33 Theorem 1]). Suppose that $p 
eq 2$, $k = \mathbb{Q}$ and $S = \{\ell_1, \ell_2\}$ with $\ell_1 \not\equiv 1 \pmod{p^2}$, $\ell_2 \not\equiv 1 \pmod{p^2}$ such that $G_{\mathbb{Q}}(K^{\text{cyc}}) \simeq \{1\}$ for the $p$-extension $K = \mathbb{Q}_{(\ell_1)} \mathbb{Q}_{(\ell_2)}$ of degree $p^2$. (All such $S$ have been characterized arithmetically in [55].) Then $\tilde{G}_S(\mathbb{Q}) \simeq ((\mathbb{Z}/p^2\mathbb{Z}) \rtimes \mathbb{Z}_p) \rtimes \mathbb{Z}_p$, $d_S = \tilde{r}_S = 3$, and $\chi_2(\tilde{G}_S(\mathbb{Q})) = 1$. Moreover, there is a minimal presentation

$$1 \longrightarrow R \longrightarrow F \longrightarrow \tilde{G}_S(\mathbb{Q}) \longrightarrow 1$$

with a free pro-$p$ group $F$ generated by $\{a, b, c\}$, and the normal subgroup $R$ normally generated by $\{a^{-p}[a, b], [a, c], a^{-p}b^{-p}[b, c]\}$. One can see that $a^{p^2} \in R$ from a calculation of $c^{-1}(a^{-p}[a, b])c \in R$. There is also a similar example for $p = 2$ (cf. [33 Theorem 2]).

3 Finite presentation

3.1 Koch type presentation. Suppose that $k = \mathbb{Q}$. For each prime number $\ell \in S$, we fix an integer $\alpha_\ell$ such that $\alpha_\ell + (\mathbb{Z}/\ell\mathbb{Z})$ is a generator of cyclic group $(\mathbb{Z}/\ell\mathbb{Z})^\times$. For a pair $(\ell, \ell')$ of prime numbers $\ell, \ell' \in S \cup \{p\}$, we define the linking number $\text{lk}(\ell, \ell')$ as follows: If $\ell \neq \ell' \equiv 1 \pmod{p}$, then $\text{lk}(\ell, \ell')$ is an integer such that

$$\ell^{-1} \equiv \alpha_\ell^{\text{lk}(\ell, \ell')} \pmod{\ell'}$$

and $0 \leq \text{lk}(\ell, \ell') < \ell'$. If $\ell \neq \ell' = p$, then $\text{lk}(\ell, p) \in \mathbb{Z}_p$ is a $p$-adic integer satisfying

$$\ell = \begin{cases} (1 + p)^{\text{lk}(\ell, p)} & \text{if } p \neq 2, \\ (-1)^{\frac{\text{lk}(\ell, p)}{2}} & \text{if } p = 2. \end{cases}$$

If $\ell = \ell'$, we put $\text{lk}(\ell, \ell') = 0$. 
Theorem 3.1. Assume that \( k = \mathbb{Q} \) and
\[
S = \begin{cases} 
\{\ell_1, \ldots, \ell_d\} & \text{if } p \neq 2, \\
\{\ell_1, \ldots, \ell_d, \infty\} \text{ or } \{\ell_1, \ldots, \ell_d, q\} & \text{if } p = 2,
\end{cases}
\]
where \( \ell_i \equiv 1 \pmod{p} \) for \( 1 \leq i \leq d \), and \( q \equiv 3 \pmod{4} \). Put \( \ell_0 = p \). Then \( \widetilde{G}_S(\mathbb{Q}) \) has a minimal presentation
\[
1 \longrightarrow R \longrightarrow F \xrightarrow{\pi} \widetilde{G}_S(\mathbb{Q}) \longrightarrow 1
\]
where \( F = \langle x_0, x_1, \ldots, x_d \rangle \) is a free pro-\( p \) group with \( d + 1 \) generators \( x_i \) such that \( \pi(x_i) \) generates the inertia group of a prime \( \ell_i \) of \( \mathbb{Q}^{\text{cyc}} \) lying over \( \ell_i \), and \( R = \langle r_0, r_1, \ldots, r_d \rangle_F \) is a normal subgroup of \( F \) normally generated by \( d + 1 \) relations \( r_i \) of the form
\[
r_i = \begin{cases} 
[x_0^{-1}, y_0^{-1}] & \text{if } i = 0, \\
x_i^{-1}[x_i^{-1}, y_i^{-1}] & \text{if } 1 \leq i \leq d
\end{cases}
\]
with \( y_i \in F \) such that \( \pi(y_i) \) is a Frobenius automorphism of \( \ell_i \) in \( \widetilde{G}_S(\mathbb{Q}) \), and
\[
y_i \equiv \begin{cases} 
\prod_{j=0}^{d} x_j^{\mathbb{lk}(\ell_i, \ell_j)} & \text{mod } [F,F] \text{ if } p \neq 2 \text{ or } \infty \in S, \\
\prod_{j=1}^{d} x_j^{\mathbb{lk}(\ell_i, \ell_j)+1} & \text{mod } [F,F] \text{ otherwise.}
\end{cases}
\]

On the other hand, we suppose that \( p \neq 2 \) and \( k \) is an imaginary quadratic field with class number \( h_k \neq 0 \pmod{p} \). Assume that \( k \neq \mathbb{Q}(\sqrt{-3}) \) if \( p = 3 \). For each prime ideal \( v \in S \cup P \), we fix a generator \( \beta_v \in \mathcal{O}_k \) of the principal ideal \( \beta_v \mathcal{O}_k = v^{h_k} \), where \( \mathcal{O}_k \) denotes the ring of algebraic integers in \( k \). For each \( v \in S \), we fix an element \( \alpha_v \in \mathcal{O}_k \cap \mathcal{U}_v \) such that \((\mathcal{O}_k/v)^{\times} = (\alpha_v \mod v)\). For each \( v \in S \), we also fix \( \alpha_v \in \mathcal{U}_v \) such that \( \mathcal{U}_v/\hat{\mathcal{U}}_v = \langle \tau_v(\alpha_v) \hat{\mathcal{U}}_v \rangle \simeq \mathbb{Z}_p \), where \( \tau_v : k_v^\times \rightarrow k_v^\times \) is the natural homomorphism. If \( k_v = \mathbb{Q}_p \), we choose \( \alpha_v = (1 + p)^{-1} \) (cf. Example 2.11). For a pair \((v, w)\) of prime ideals \( v, w \in S \cup P \), the linking number \( \mathbb{lk}(v, w) \) is defined as follows: If \( v \neq w \in S \), then \( \mathbb{lk}(v, w) \) is an integer such that
\[
\beta_v^{-1} \equiv \alpha_w^{\mathbb{lk}(v, w)} \pmod{w}
\]
and \( 0 \leq \mathbb{lk}(v, w) < |\mathcal{O}_k/w| \). If \( v \neq w \in P \), then \( \mathbb{lk}(v, w) \in \mathbb{Z}_p \) is a \( p \)-adic integer satisfying
\[
\tau_w(\beta_v)^{-\mathbb{lk}(v, w)} \equiv \tau_w(\alpha_w)^{\mathbb{lk}(v, w)} \pmod{\hat{\mathcal{U}}_w}.
\]
If \( v = w \), we put \( \mathbb{lk}(v, w) = 0 \).
Theorem 3.2. Assume that $p \neq 2$ and $k$ is an imaginary quadratic field with class number $h_k \neq 0 \pmod{p}$, and that $k \neq \mathbb{Q}(\sqrt{-3})$ if $p = 3$. Then $\tilde{G}_S(k)$ has a minimal presentation

$$1 \rightarrow R \rightarrow F \rightarrow \widetilde{G}_S(k) \rightarrow 1$$

where $F$ is a free pro-$p$ group with generators $\{x_v\}_{v \in S \cup P}$ such that $\pi(x_v)$ generates the inertia group of a prime $\tilde{v}$ of $(k^{\text{cycl}})_S$ lying over $v$, and $R$ is a normal subgroup of $F$ normally generated by $\{r_v\}_{v \in S \cup P}$ of the form

$$r_v = \begin{cases} x_v^{[N_{k/v}]1} & \text{if } v \in P, \\ x_v^{[N_{k/v}]1 - 1} & \text{if } v \in S \end{cases}$$

with $y_v \in F$ such that $\pi(y_v)$ is a Frobenius automorphism of $\tilde{v}$ and

$$y_v \equiv \prod_{w \in S \cup P} x_w^{\text{lk}(v,w)} \pmod{[F,F]}.$$ 

Remark 3.3. Suppose that $p$ splits in $k$ as $pO_k = p_1p_2$. Then $P = \{p_1, p_2\}$, $\tilde{U}_v = \{1\}$, and $\alpha_{p_i} = (1 + p)^{-1}$. Since $t_{p_i}$ is injective from $U_{p_i}^{p-1}$, the linking number $\text{lk}(v, p_i) \in \mathbb{Z}_{p_i}$ satisfies

$$\beta_{p_i}^{p-1} = (1 + p)(p-1)\text{lk}(v, p_i) \in k_{p_i}.$$ 

Hence $\text{lk}(v, p_i) \equiv 0 \pmod{p^n}$ for $1 \leq n \leq \mathbb{Z}$ if and only if $\beta_{p_i}^{p-1} \equiv 1 \pmod{p_i^{n+1}}$. Since an isomorphism $k_{p_1} \xrightarrow{\sim} k_{p_2}$ is induced from the nontrivial element of $\text{Gal}(k/\mathbb{Q})$, we have $\text{lk}(p_1, p_2) = \text{lk}(p_2, p_1)$.

3.2 Proof of Theorems 3.1 and 3.2 Put $S' = S \setminus \{\infty\}$ if $\infty \in S$, and $S'' = S$ otherwise. Put $S^* = S \setminus \{q\}$ if $p = 2$ and $\infty \not\in S$, and put $S^* = S''$ otherwise. If $k = \mathbb{Q}$, we put $\alpha_p = (1 + p)^{-1}$ or $\alpha_p = 5^{-1}$ according to $p \neq 2$ or $p = 2$, and put $\beta_k = \ell$ for each prime number $\ell \in S^* \setminus \{p\}$.

Let $\iota_v : k_v^x \rightarrow \tilde{k}_v^x$ be the natural homomorphism for each prime $v$. Note that $U_v/(U_v \cap \ker \iota_v)$ is the maximal pro-$p$ quotient of $U_v$ if $v \nmid \infty$. If $v \in S'$, the finite cyclic $p$-group $\tilde{U}_v$ is generated by $\iota_v(\alpha_v)$, and

$$\iota_v(\beta_v)^{-1} = \iota_v(\alpha_v)^{\text{lk}(w,v)}$$

for $w \in (S' \setminus \{v\}) \cup P$. If $v \in P$, we have $\widetilde{U}_v = (\iota_v(\alpha_v)\tilde{U}_v) \simeq \mathbb{Z}_{p}$ and

$$\iota_v(\beta_v)^{-1} = \iota_v(\alpha_v)^{\text{lk}(w,v)} \pmod{\tilde{U}_v}$$

for $w \in S' \cup (P \setminus \{v\})$ (cf. Example 2.1). Then we obtain the following congruences in $J_k \subset \prod_v \tilde{k}_v^x$,

$$\iota_w(\beta_w) = (\iota_w(\beta_w), (1)_{v \not\in \{w\}}) \equiv (1, (\iota_v(\beta_v)^{-1})_{v \not\in \{w\}})$$

$$\equiv \left(\left((1)_{v \in (S' \cup P) \setminus \{w\}}, (\iota_v(\beta_v)^{-1})_{v \in (S' \cup P) \setminus \{w\}}\right)\right) \equiv \prod_{\substack{v \in S' \cup P \setminus \{w\}}} \iota_v(\alpha_v)^{\text{lk}(w,v)} \pmod{\tilde{U}_S k^x},$$

(3.1)
where we note that $\beta_v \otimes 1 \in k^\times$, and that $\beta_v = \ell \in \mathbb{R}_{>0}$, i.e., $\nu_\ell(\beta_v) = 1 \in \mathbb{Q}$ if $v = \infty \in S$. If $p = 2$ and $\infty \notin S$, then $k = \mathbb{Q}$, and

\[ 1 \equiv (\nu_\ell(-1))_v \equiv \left((1)_v \in S, (\nu_\ell(\alpha_v))_v \in S\right) \equiv \nu_\ell(\alpha_v)^{-1} \prod_{v \in S^*} \nu_\ell(\alpha_v) \equiv \tilde{U}_S k^\times \mod U_S k^\times, \]

where we note that $-1 \in \tilde{U}_p$ (cf. Example 2.1), and that $\frac{q - 1}{2} \equiv -1 \mod 2$, i.e., $U_q = \mathbb{Z}/2\mathbb{Z}$. Note that $\beta_v \nu_\ell^{-1} \in U_v$ for arbitrary uniformizer $\nu_\ell$ of $k_v$, where $h_k = 1$ if $k = \mathbb{Q}$. Since $h_k \in \mathbb{Z}^\times$, we have $\tilde{U}_v / U_v = \langle \nu_\ell(\beta_v) \rangle \simeq \mathbb{Z}_p$ for $v \in S' \cup P$.

Recall that $\tilde{G}_{S,v}$ is identified with the decomposition subgroup of $\tilde{G}_S = \tilde{G}_S(k)$ for each fixed prime $\tilde{v}|v$ of $(k^{\text{ncyc}})_S$. Let $\tilde{I}_v \subset \tilde{G}_{S,v}$ be the inertia subgroup. If $v \notin P$ and $v \nmid \infty$, then the inertia subgroup of $\tilde{C}_v \simeq \mathbb{Z}_p \times \mathbb{Z}_p$ is isomorphic to $\mathbb{Z}_p$ (cf. e.g. [23, Theorem 10.2]), and hence $\tilde{I}_v$ is also procyclic. If $v \in P$, then $(k^{\text{ncyc}})_S k_v \subset k_v^{ur,p} k_v^{\text{ncyc}}$, and hence

\[ \tilde{I}_v = \text{Gal}(k^{\text{ncyc}})_S k_v / (k^{\text{ncyc}})_S k_v \cap k_v^{ur,p} \simeq \text{Gal}(k_v^{ur,p} k_v^{\text{ncyc}} / k_v^{ur,p}) \simeq \mathbb{Z}_p. \]

Proposition 2.2 implies the existence of a Frobenius element $\sigma_v \in \tilde{G}_{S,v}$ and a generator $\tau_v$ of procyclic $\tilde{I}_v$ for $v \in S' \cup P$ such that

\[ \text{rec}_S(\nu_\ell(\beta_v) \tilde{U}_S k^\times) = \sigma_v^{h_k} \tilde{G}_S, \tilde{G}_S, \text{ rec}_S(\nu_\ell(\alpha_v) \tilde{U}_S k^\times) = \tau_v^{h_k} \tilde{G}_S, \tilde{G}_S. \]

Since $k$ has no $p$-extensions unramified outside $S \setminus S^*$, the pro-$p$ group $\tilde{G}_S$ is generated by $\{\tau_v\}_{v \in S' \cup P}$. Since $\tilde{G}_S = \langle S^* \cup P \rangle$ by Remark 2.3 and Proposition 2.4, $\tilde{G}_S$ has a minimal presentation

\[ 1 \longrightarrow R \longrightarrow F \overset{\pi}{\longrightarrow} \tilde{G}_S \longrightarrow 1 \]

with a free pro-$p$ group $F$ generated by $\{x_v\}_{v \in S^* \cup P}$ such that $\pi(x_v) = \tau_v$. By (3.1), (3.2) and (3.3), there is an element $y_v \in F$ for each $w \in S^* \cup P$ such that $\pi(y_v) = \sigma_w$ and

\[ y_v = \left\{ \begin{array}{ll}
\prod_{v \in S^* \cup P} x_v^{\nu_\ell(w,v)} & \text{mod } [F, F] \text{ if } p \neq 2 \text{ or } \infty \in S, \\
\left( \prod_{v \in S^* \cup P} x_v^{\nu_\ell(w,v)} \right)^{\nu_\ell(w,q)} \prod_{v \in S^* \cup P} x_v^{\nu_\ell(w,v)} & \text{mod } [F, F] \text{ otherwise.}
\end{array} \right. \]

For $v \in S^* \cup P$, we put

\[ H_v = \left\{ \begin{array}{ll}
\tilde{G}_v \simeq \mathbb{Z}_p \times \mathbb{Z}_p & \text{if } v \in S^*, \\
\tilde{G}_v^{cr} \simeq \mathbb{Z}_p \times \mathbb{Z}_p & \text{if } v \in P,
\end{array} \right. \]
as in the proof of Theorem 2.5 and let $F_v$ be a free pro-$p$ group generated by $\{s_v, t_v\}$. Then there exists a commutative diagram

$$
\begin{array}{ccc}
1 & \longrightarrow & R \\
| & | & | \\
1 & \longrightarrow & F_v
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
\chi_v & \varphi_v & \pi_v \\
\downarrow & \downarrow & \downarrow \\
H_v & \longrightarrow & 1
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & \longrightarrow \\
\pi & \varphi & \chi \\
\downarrow & \downarrow & \downarrow \\
1 & \longrightarrow & R_v
\end{array}
$$

with minimal presentations of $H_v$, such that $\chi_v(t_v) = x_v, \chi_v(s_v) = y_v, \varphi_v(H_v) = \tilde{G}_S, v$, and $\pi_v(s_v)$ (resp. $\pi_v(t_v)$) is a Frobenius element (resp. a generator of the inertia subgroup) of $H_v$, where $R_v$ is a normal subgroup of $F_v$ normally generated by either $\ell_{2[k_q, q]}^{-1}[t_v^{-1}, s_v^{-1}]$ or $[t_v^{-1}, s_v^{-1}]$ according to $v \in S^*$ or $v \in P$ (cf. [23 Theorem 10.2]). Since $\text{III}^2(\tilde{G}_S) = \{\phi\}$ by Remark 2.3 and Theorem 2.5, the localization map

$$H^2(\tilde{G}_S) \to \bigoplus_{v \in S^* \cup P} H^2(H_v)$$

(which is induced from $\{\varphi_v\}_{v \in S^* \cup P}$) is injective, and hence $R$ is normally generated by

$$\{x_v^{[N_k, q]^{-1}}[x_v^{-1}, y_v^{-1}]\}_{v \in S} \text{ and } \{[x_v^{-1}, y_v^{-1}]\}_{v \in P}$$

as a normal subgroup of $F$ (cf. [23 Theorem 6.14]). Thus the proof of Theorems 3.1 and 3.2 is completed.

REMARK 3.4. Suppose that $p = 2, \infty \in S$ and $\ell_d \equiv 3 \pmod{4}$ in the case of Theorem 3.1. If we put $\alpha_\infty = -1$ and $H_\infty = \tilde{G}_\infty \simeq \mathbb{Z}/2\mathbb{Z}$, then

$$1 \equiv (\iota_v(-1))_v \equiv \iota_d(\alpha_\ell_d^{-1}) \iota_\infty(\alpha_\infty^{-1}) \prod_{v \in S, (\ell, \ell_d)} \iota_v(\alpha_v^{-1}) \equiv \tilde{U}_S k^\times,$$

and hence the same arguments using $S \setminus \{\ell_d\}$ instead of $S^*$ yield the existence of $x_\infty$ and $\{\eta_i\}_{0 \leq i < d}$ such that $1 \neq \pi(x_\infty) \in \tilde{I}_\infty = \tilde{G}_{S, \infty} \simeq \mathbb{Z}/2\mathbb{Z},$

$$x_\infty \equiv x_\infty \prod_{j=1}^{d-1} x_j^{\ell_j^{-1}} \pmod{[F : F]},$$

$$\eta_i \equiv x_0^{\ell_j(\ell_j, \ell_d)} x_\infty^{\ell_j(\ell_j, \ell_d)} \prod_{j=1}^{d-1} x_j^{\ell_j(\ell_j, \ell_d)} \equiv \prod_{j=1}^{d-1} x_j^{\ell_j(\ell_j, \ell_d)} \pmod{[F : F]},$$

and $R$ is normally generated by $\{x_i^{\ell^{-1}x_i^{-1}, \eta_i^{-1}}\}_{1 \leq i < d}, [x_0^{-1}, \eta_0^{-1}]$ and $x_\infty^2$.

REMARK 3.5. Let $\text{Gal}$ be the element of $\text{Gal}(k(\mu_\infty)/k)$ such that $\text{Gal}(\zeta) = \zeta^\kappa$ for any $\zeta \in \mu_\infty$, where $\kappa = 1 + p$ or $\kappa = 5$ according to $p \neq 2$ or $p = 2$, and $\mu_\infty$ denotes the group of $p$-power roots of unity. If $k_v = \mathbb{Q}_p$ for $v \in P$, there is a commutative diagram

$$
\begin{array}{ccccccc}
k_\infty^\times \overset{\iota_v \text{ mod } \tilde{U}_S k^\times}{\longrightarrow} & J_k/\tilde{U}_S k^\times & \overset{\text{rec}_{\tilde{G}_S}}{\longrightarrow} & \tilde{G}_S & \overset{\text{Gal}(k_\infty^\times)}{\longrightarrow} & \text{Gal}(k_\infty^\times / k) \\
\downarrow & \downarrow & | & \downarrow & | \\
\text{Gal}(\mathbb{Q}_p(\mu_\infty)/\mathbb{Q}_p) & \overset{\approx}{\longrightarrow} & \text{Gal}(k(\mu_\infty)/k)
\end{array}
$$
with the reciprocity map $\rho$ of local class field theory, which satisfies $\rho(\alpha_v)(\zeta) = \zeta^{\alpha_v^{-1}} = \zeta^\alpha$ for any $\zeta \in \mu_p = \mathbb{Z}_p^\times$ (cf. e.g. [11 II, Exercise 3.4.3]). Then we have $\pi(x_v)^{h_v}|_{\mathcal{Q}_{\text{cyc}}} = \bar{\gamma}|_{\mathcal{Q}_{\text{cyc}}}$. In particular, $\pi(x_0)|_{\mathcal{Q}_{\text{cyc}}} = \bar{\gamma}|_{\mathcal{Q}_{\text{cyc}}} f0 \in F \text{ of Theorem } 3.1$

**Remark 3.6.** A link group $\pi_1(X)$ is the fundamental group of the complement $X$ of the open tubular neighborhood $\bigsqcup_{i=1}^d V^2_{K_i}$ of a link $L = \bigsqcup_{i=1}^d K_i$ in a rational homology 3-sphere $M$. The image of $\pi_1(\partial V_{K_i}) \simeq \mathbb{Z} \times \mathbb{Z}$ in $\pi_1(X)$ is analogous to the decomposition group of a ramified prime $v \in S^* \cup P$ in $\hat{G}_S(k)$, which is a quotient of the local Galois group $H_v \simeq \mathbb{Z}_p \times \mathbb{Z}_p$ (cf. (3.3)). Hence the Koch type presentation of $\hat{G}_S(k)$ is analogous to a Milnor presentation of $\pi_1(X)$ (cf. e.g. [36]).

### 3.3 Preliminaries for consequences

Let $G$ be a pro-$p$ group. Put $G_1 = G$, and let $G_n = [G_{n-1}, G]$ for $2 \leq n \in \mathbb{Z}$ recursively. Then $\{G_n\}_{n \geq 1}$ is the lower central series of $G$. Put $G(n) = \{g \in G \mid g - 1 \in (I_G)^n\}$ for $1 \leq n \in \mathbb{Z}$, where $I_G = \text{Ker}(\mathbb{F}_p[[G]] \to \mathbb{F}_p)$ is the augmentation ideal of $\mathbb{F}_p[[G]]$. Then $G^{(1)} = G$ and $\{G(n)\}_{n \geq 1}$ is the Zassenhaus filtration of $G$. Put $[g_1, g_2, g_3] = [[g_1, g_2], g_3] \in G_3$ for $g_1, g_2, g_3 \in G$.

Let $F = \langle x_0, x_1, \ldots, x_d \rangle$ be a free pro-$p$ group generated by $\{x_i\}_{0 \leq i \leq d}$. Let $\varepsilon_{\mathbb{Z}_p[[F]]} : \mathbb{Z}_p[[F]] \to \mathbb{Z}_p$ be the augmentation map, and $\frac{\partial}{\partial x_i} : \mathbb{Z}_p[[F]] \to \mathbb{Z}_p[[F]]$ be the pro-$p$ Fox derivative. For a multi-index $I = (i_1 \cdots i_n)$ with $0 \leq i_1, \ldots, i_n \leq d$ and $y \in F$, we put

$$
\varepsilon_I(y) = \varepsilon_{\mathbb{Z}_p[[F]]}\left(\frac{\partial^n y}{\partial x_{i_1} \cdots \partial x_{i_n}}\right) \in \mathbb{Z}_p
$$

which is the coefficient of $X_{i_1} \cdots X_{i_n}$ in the expansion of $y$ by the pro-$p$ Magnus isomorphism

$$
\hat{M} : \mathbb{Z}_p[[F]] \simeq \mathbb{Z}_p\langle X_1, \ldots, X_d \rangle : x_j \mapsto 1 + X_j
$$

(cf. [36], Proposition 8.14). We also put

$$
\varepsilon_{I,p}(y) = \varepsilon_I(y) + px \quad \text{for } x \in \mathbb{F}_p
$$

the mod $p$ Magnus coefficient.

Suppose that $F$ is the free pro-$p$ group in Theorem 3.1 and let $y_i \in F$ be the element obtained in Theorem 3.1. Then $\pi(y_i)$ is an analogue of the longitude of a component of a link, and the mod $p$ Milnor number

$$
\mu_p(i_1 \cdots i_n) = \varepsilon_{I,p}(y_i)
$$

for a multi-index $(i_1 \cdots i_n)$ is defined.
3.4 RÉDEI SYMBOLS. We consider the case where \( p = 2 \) and \( S = \{ \ell_1, \ldots, \ell_d, \infty \} \). Put \( \ell_0 = \ell_0 = 2 \), and put \( \ell_i = (-1)^{\frac{\ell_i - 1}{2}} \ell_i \equiv 1 \pmod{4} \) for \( 1 \leq i \leq d \). For \( 0 \leq i \leq d \), the elements \( y_i \) in Theorem 3.1 can be written in the form

\[
y_i \equiv x_d^{\text{lk}(\ell, \ell_d)} \cdots x_1^{\text{lk}(\ell, \ell_1)} x_0^{\text{lk}(\ell, \ell_0)} \prod_{a < b} [x_a, x_b]^{c_{iab}} \pmod{F_3}
\]

with some \( c_{iab} = -c_{iba} \in \mathbb{Z}_2 \), where all pairs \((a, b)\) with \( 0 \leq a < b \leq d \) run in the product. Assume that there are distinct prime numbers \( \ell_a, \ell_b \in S \cup \{2\} \) satisfying

\[
\text{lk}(\ell, \ell_a) \equiv \text{lk}(\ell, \ell_b) \equiv 0 \pmod{2}.
\]

For such \( \ell_a \) and \( \ell_b \), the quadratic field \( \mathbb{Q}(\sqrt{\ell_a \ell_b}) \) has a unique cyclic extension \( K_{(a,b)} \) of degree 4 unramified outside \( \infty \) (cf. e.g. [54]). Then \( \mathbb{Q}(\sqrt{\ell_a}, \sqrt{\ell_b}) \subset K_{(a,b)} \), and \( K_{(a,b)}/\mathbb{Q} \) is the Rédei extension, which is a dihedral extension of degree 8 unramified outside \( \{\ell_a, \ell_b, \infty\} \). Moreover if a prime number \( \ell_i \) (\( 0 \leq i \leq d \)) satisfies

\[
\text{lk}(\ell, \ell_a) \equiv \text{lk}(\ell, \ell_b) \equiv 0 \pmod{2},
\]

then a prime ideal \( \mathfrak{l}_i \) of \( \mathbb{Q}(\sqrt{\ell_a \ell_b}) \) lying over \( \ell_i \) splits in \( \mathbb{Q}(\sqrt{\ell_a}, \sqrt{\ell_b}) \), and the Rédei symbol (cf. [43]) for such triple \((\ell_a, \ell_b, \ell_i)\) can be defined as

\[
[\ell_a, \ell_b, \ell_i] = [\ell_b, \ell_a, \ell_i] = \begin{cases} 1 & \text{if } \mathfrak{l}_i \text{ splits completely in } K_{(a,b)}, \\ -1 & \text{otherwise.} \end{cases}
\]

The following proposition relates \( c_{iab}, [\ell_a, \ell_b, \ell_i] \), and the mod 2 Milnor number \( \mu_2(abi) \).

**Proposition 3.7.** Under the settings above, we have

\[
[\ell_a, \ell_b, \ell_i] = (-1)^{c_{iab}} = (-1)^{\mu_2(abi)}
\]

if (3.6) and (3.7) are satisfied.

**Proof.** Let

\[
\begin{array}{ccc}
1 & \longrightarrow & R \\
& \longrightarrow & F \\
& \longrightarrow & \tilde{G}_S(\mathbb{Q}) \\
& \longrightarrow & 1
\end{array}
\]

be the presentation of \( \tilde{G}_S(\mathbb{Q}) \) obtained in Theorem 3.1. Let \( N_{(a,b)} \) be the minimal normal subgroup of \( F \) including \( \{x_j | j \notin \{a, b\}\} \cup \{x_a^2, x_b^2\} \). By Theorem 3.1 and (3.6), we have \( y_a, y_b \in F_2 N_{(a,b)} \), and hence \( r_a, r_b \in F_3 N_{(a,b)} \). Since \( r_j \in N_{(a,b)} \) for any \( j \notin \{a, b\} \), we have \( R \subset F_3 N_{(a,b)} \). Since

\[
(x_a x_b)^2 \equiv [x_a, x_b] \pmod{N_{(a,b)}},
\]

and
Thus the proof of Proposition 3.7 is completed.

By (3.5) and (3.7), we have an isomorphism hence we have

\[ F/3N(a,b) \simeq \text{Gal}(K_{(a,b)}/\mathbb{Q}) : xF_3N(a,b) \mapsto \pi(x)|_{K_{(a,b)}}. \]

By (3.5) and (3.7), we have

\[ y_i \equiv [x_a, x_b]^{c_{ab}} \mod F_3N(a,b). \]

If \( i \notin \{a, b\} \), \( \ell_i \) splits completely in \( \mathbb{Q}(\sqrt[3]{a}, \sqrt[3]{b}) \) by (3.7). Then \( \pi(y_i)|_{K_{(a,b)}} \) generates the decomposition group of any primes lying over \( \ell_i \) by Theorem 3.1, and hence we have \([\ell_a, \ell_b, \ell_i] = (-1)^{c_{ab}}\). Suppose that \( i \in \{a, b\} \). Then \( \ell_i \) splits in \( \mathbb{Q}(\sqrt[3]{j}) \) for \( i \neq j \in \{a, b\} \) by (3.6). By the definition of Rédei symbol, we have \([\ell_a, \ell_b, \ell_i] = -1\) if and only if \( \mathbb{Q}(\sqrt[3]{j}) \) is equal to the decomposition field of any primes of \( K_{(a,b)} \) lying over \( \ell_i \). By Theorem 3.1, the elements \( \pi(x_i)|_{K_{(a,b)}} \), \( \pi(y_i)|_{K_{(a,b)}} \) of order at most 2 generate the decomposition group of the prime of \( K_{(a,b)} \) which is the restriction of \( \ell_i \).

Hence we have \([\ell_a, \ell_b, \ell_i] = (-1)^{c_{ab}}\).

Put \( h_i = [x_a, x_b]^{-c_{ab}}y_i \in F_3N(a,b) \). By the basic properties of mod 2 Magnus coefficients (cf. e.g. [52, Proposition 2.18]), we have

\[ \varepsilon_{(ab),2}^2([x_a, x_b]) = 1, \]

\[ \varepsilon_{(ab),2}(y_i) = \varepsilon_{(ab),2}([x_a, x_b]^{c_{ab}}) + \varepsilon_{(a),2}([x_a, x_b]^{c_{ab}})\varepsilon_{(b),2}(h_i) + \varepsilon_{(ab),2}(h_i), \]

and \( \varepsilon_{I,2}(ghh'g) = 0 \) for any \( g \in F \) and \( I \in \{(a), (b), (ab)\} \) if \( \varepsilon_{I,2}(h) = \varepsilon_{I,2}(h') = 0 \) for all \( I \in \{(a), (b), (ab)\} \). Since \( \varepsilon_I(F_3) = 0 \) (cf. e.g. [36, Proposition 8.15]) and \( \varepsilon_{I,2}(x_{a}^{\pm 1}) = \varepsilon_{I,2}(x_{a}^{\pm 2}) = \varepsilon_{I,2}(x_{b}^{\pm 2}) = 0 \) for any \( I \in \{(a), (b), (ab)\} \) and \( j \notin \{a, b\} \), the continuity of \( \varepsilon_{I,2} : F \to \mathbb{F}_2 \) yields that \( \varepsilon_{I,2}(F_3N(a,b)) = 0 \). In particular, \( \varepsilon_{(ab),2}(h_i) = \varepsilon_{(ab),2}(h_i) = 0 \). Therefore

\[ \mu_2(ab) = \varepsilon_{(ab),2}(y_i) = c_{ab} + 2\mathbb{Z}_2. \]

Thus the proof of Proposition 3.7 is completed. \( \square \)

The following proposition yields that the Rédei symbols \([\ell_a, \ell_b, \ell_i] \) with \( i \in \{a, b\} \) are written by the quartic residue symbols \((\cdot)_4^\ell\) defined as follows; \((\cdot)_4^\ell \equiv \pm 1 \equiv z^{\ell-1}_4 \mod \ell\) for a prime number \( \ell \equiv 1 \) (mod \( 4 \)) and \( z \in \mathbb{Z}_\ell^\times \) such that \((\ell)_4 = 1\), and \((\cdot)_4^\ell = (-1)^{\frac{z^{\ell-1}_4}{4}}\) for \( z \in \mathbb{Z} \) such that \( z \equiv 1 \) (mod \( 8 \)).
Proposition 3.8 (cf. [43]). Under the settings above, we have the following equations for distinct prime numbers \( \ell_a, \ell_b \in S \cup \{2\} \) satisfying (3.10):

\[
[\ell_a, \ell_b, \ell_a] = \left(\frac{\ell_b}{\ell_a}\right)_4 \text{ if } \ell_a \equiv 3 \pmod{4} \text{ and } \ell_b \not\equiv 3 \pmod{4},
\]

\[
[\ell_a, \ell_b, \ell_a] = [\ell_a, \ell_b, \ell_b] = \left(\frac{-\ell_b}{\ell_a}\right)_4 \text{ if } \ell_a \not\equiv 3 \pmod{4} \text{ and } \ell_b \equiv 3 \pmod{4}.
\]

Proof. See [43] (53)–(53³). Alternatively, since \([\ell_a, \ell_b, \ell_a] = -1\) if and only if the narrow class number of \( \mathbb{Q}(\sqrt{\ell_a \ell_b}) \) is not divisible by 8 and the narrow ideal class of \( \ell_a \) is nontrivial, the statement is obtained as a translation of well known results on the narrow class groups (cf. e.g. [56] Propositions 3.3–3.6)).

As a special case of Theorem 3.1, we obtain the following result similar to [52, Theorem 3.12].

Corollary 3.9. Suppose that \( k = \mathbb{Q}, \ell_0 = p = 2 \) and \( S = \{\ell_1, \cdots, \ell_d, \infty\} \). Assume that \( \ell_i \equiv 1 \pmod{4} \) for all \( 1 \leq i \leq d \), and that

\[
\text{lk}(\ell_a, \ell_b) \equiv \text{lk}(\ell_a, \ell_b) \equiv 0 \pmod{2}
\]

for all pairs \((a, b)\) with \( 0 \leq a, b \leq d \). Then \( d + 1 \) relations \( r_i \) of the presentation of \( \widetilde{G}_S(\mathbb{Q}) \) in Theorem 3.7 are written in the form

\[
\begin{aligned}
 r_i \equiv & \prod_{a < b} [x_{a, x_b, x_i}]^{\mu_2(ab)}, \prod_{a=0}^{i-1} [x_{a, x_i, x_a}]^{\mu_2(aia)}, \prod_{b=i+1}^{d} [x_{i, x_b, x_b}]^{\mu_2(ibi)} \pmod{F(4)},
\end{aligned}
\]

for each \( 0 \leq i \leq d \), where \( F(4) \) denotes the 4th step of the Zassenhaus filtration of \( F \).

Proof. Note that \( F(2) = F^2 F_2, F(3) = F^4 F_2^2 F_3 \) and \( F(4) = F^4 F_2^2 F_4 \) (cf. [43] Theorems 11.2 and 12.9). By (3.5), we have

\[
y_i^{-1} \equiv x_0^{2e_{i0}} x_1^{2e_{i1}} \cdots x_d^{2e_{id}} \prod_{a < b} [x_a, x_b]^{c_{iab}} \pmod{F(3)}
\]

with some \( e_{ij} \in \{0, 1\} \) such that \( 2e_{ij} \equiv \text{lk}(\ell_i, \ell_j) \pmod{4} \). Then \( e_{ii} = 0 \) and

\[
(-1)^{e_{ij}} = \left(\frac{\ell_i}{\ell_j}\right)_4 = [\ell_i, \ell_j, \ell_j]
\]

for \( j \neq i \) by Proposition 3.8. Since \([F(n), F(4-n)] \subset F(4)\) for \( n \in \{1, 2\} \), a direct calculation shows that

\[
r_i \equiv [x_i^{-1}, y_i^{-1}] \equiv \prod_{a < b} [x_{a, x_b, x_i}]^{c_{iab}} \prod_{j=0}^{d} [x_{i, x_j, x_j}]^{e_{ij}} \pmod{F(4)}.
\]

By Proposition 3.7, we obtain the claim. \( \square \)
Example 3.10. Put $S = \{\ell_1, \ell_2, \infty\}$ with $\ell_1 = 113$ and $\ell_2 = 593$. Then $\text{lk}(\ell_a, \ell_b) \equiv 0 \pmod{4}$ for any $0 \leq a, b \leq 2$. By Proposition 3.7 and PARI/GP [42], one can see that $\mu_2(abi) = 1$ if $abi$ is a permutation of $012$, and $\mu_2(abi) = 0$ otherwise. Then
\[ r_0 \equiv [x_1, x_2, x_0], \quad r_1 \equiv [x_2, x_0, x_1], \quad r_2 \equiv [x_0, x_1, x_2] \mod F(4) \]
by Corollary 3.9 and hence $(2, 113, 593)$ is also a triple of (proper) Borromean primes modulo 2 in the sense of [36] [52].

Example 3.11. For $S = \{\ell_1, \ell_2, \infty\}$ with $\ell_1 = 337$ and $\ell_2 = 593$, one can see that $\text{lk}(\ell_a, \ell_b) \equiv 0 \pmod{4}$ and $\mu_2(abi) = 0$ for any $0 \leq a, b, i \leq 2$ by Proposition 3.7 and PARI/GP [42]. Then
\[ \tilde{G}_S(\mathbb{Q})/\tilde{G}_S(\mathbb{Q})_{(4)} \simeq F/F(4) \]
by Corollary 3.9.

3.5 Mild pro-$p$ Groups. A finitely presented pro-$p$ group $G$ is said to be ‘mild’ (with respect to the Zassenhaus filtration) when $G$ has a presentation $F/R \simeq G$ with a system of relations which make a ‘strongly free sequence’ in the graded Lie algebra $\text{gr} F = \bigoplus_{n \geq 1} F(n)/F(n+1)$ (cf. [24]). Labute [24] gave the first example of mild $G_S(\mathbb{Q})$, in particular having the cohomological dimension $\text{cd}(G_S(\mathbb{Q})) = 2$, by showing that $G_S(\mathbb{Q})$ is mild if $S$ is a ‘circular set’. Such criteria for mildness have been reformulated as a ‘cup-product criterion’ (cf. [47] Theorem 5.5 and [6] [9] [25]), which induces the existence of mild $\tilde{G}_S(k)$ in various situations (cf. [2]). We also obtain circular sets of primes including $\ell_0 = p = 2$ as follows.

Theorem 3.12. Suppose that $k = \mathbb{Q}$, $S^* = \{\ell_1, \cdots, \ell_d\}$,
\[ S = \begin{cases} S^* & \text{if } p \neq 2, \\ S^* \cup \{q\} & \text{if } p = 2, \end{cases} \]
and $d = |S^*| > 1$ is odd, where $q \equiv 3 \pmod{4}$ and $q, \infty \notin S^*$. Put $\ell_0 = p$. Assume that $\{p\} \cup S^*$ is a ‘circular set’, i.e., there is a bijection $\sigma : \mathbb{Z}/(d+1)\mathbb{Z} \to \{0, 1, \cdots, d\}$ satisfying the following conditions:

1. $\ell_{\sigma(i)} \not\equiv 3 \pmod{4}$ if $p = 2$ and $i$ is even,
2. $\text{lk}(\ell_{\sigma(i)}, \ell_{\sigma(j)}) \equiv 0 \pmod{p}$ if $i$ and $j$ are even,
3. $\prod_{i=0}^{d} \tilde{\text{lk}}(\ell_{\sigma(i)}, \ell_{\sigma(i+1)}) \neq \prod_{i=0}^{d} \tilde{\text{lk}}(\ell_{\sigma(i+1)}, \ell_{\sigma(i)}) \pmod{p}$, where
\[ \tilde{\text{lk}}(\ell_i, \ell_j) = \begin{cases} \text{lk}(\ell_i, \ell_j) & \text{if } p \neq 2, \\ \text{lk}(\ell_i, \ell_j) + \delta_{\ell_i + 42, \ell_j + 42} \text{lk}(\ell_i, q) & \text{if } p = 2. \end{cases} \]
Then $\tilde{G}_S(\mathbb{Q})$ is a mild pro-$p$ group (with respect to the Zassenhaus filtration) of deficiency zero. In particular, the cohomological dimension $\text{cd}(\tilde{G}_S(\mathbb{Q})) = 2$, the Euler-Poincaré characteristic $\chi(\tilde{G}_S(\mathbb{Q})) = 1$, and $\tilde{G}_S(\mathbb{Q})$ is not $p$-adic analytic.

Proof. The pro-$p$ group $\tilde{G}_S(\mathbb{Q})$ is mild if the $\mathbb{F}_p$-vector space $H^1(\tilde{G}_S(\mathbb{Q}))$ has a decomposition $H^1(\tilde{G}_S(\mathbb{Q})) = U \oplus V$ with the subspaces $U, V$ such that

$$U \cup V = H^2(\tilde{G}_S(\mathbb{Q})) \quad \text{and} \quad V \cup V = \{0\}$$

(cf. e.g. [9, Cup-product criterion]), where $U \cup V$ (resp. $V \cup V$) denotes the image of $U \otimes V$ (resp. $V \otimes V$) by the cup product

$$\cup : H^1(\tilde{G}_S(\mathbb{Q})) \otimes H^1(\tilde{G}_S(\mathbb{Q})) \rightarrow H^2(\tilde{G}_S(\mathbb{Q})).$$

Let

$$1 \xrightarrow{} R \xrightarrow{\pi} \tilde{G}_S(\mathbb{Q}) \xrightarrow{} 1$$

be the minimal presentation of $\tilde{G}_S(\mathbb{Q})$ obtained in Theorem 3.1. Put

$$b_{i,j} = \tilde{\iota}(\ell_i, \ell_j) + p\mathbb{Z}_p \in \mathbb{F}_p$$

for $0 \leq i, j \leq d$. Then the normal subgroup $R$ of $F = \langle x_0, \cdots, x_d \rangle$ is normally generated by $d + 1$ relations $r_i = x_i^{(t_i-1)(1-\delta_{i,0})}[x_i^{-1}, y_i]^{-1}$ ($0 \leq i \leq d$) with $y_i$ written in the form

$$y_i \equiv \prod_{j=0}^{d} x_j^{b_{i,j}} \mod (2).$$

Let $\chi_i \in H^1(F)$ be the dual element of $x_i$, i.e., $\chi_i(x_j) = \delta_{i,j}$ for $0 \leq j \leq d$. Then $\chi_0, \cdots, \chi_d$ form a basis of the $\mathbb{F}_p$-vector space $H^1(F) \simeq H^1(G_S(\mathbb{Q}))$. For each $r \in R$, the trace map

$$\text{tr}_r : H^2(\tilde{G}_S(\mathbb{Q})) \rightarrow \mathbb{F}_p : \varphi \mapsto \text{tg}^{-1}(\varphi)(r)$$

is defined as an element of the dual space $H^2(\tilde{G}_S(\mathbb{Q}))^\vee$, where $\text{tg} : H^1(R)^{F/R} \simeq \text{Hom}(R/R^p[F,R], \mathbb{F}_p) \xrightarrow{\sim} H^2(\tilde{G}_S(\mathbb{Q}))$ is the transgression isomorphism. Then

$$H^2(\tilde{G}_S(\mathbb{Q}))^\vee = \sum_{i=0}^{d} \mathbb{F}_p \text{tr}_r.$$ 

For $i \geq 0$ and a multi-index $I = (i_1, i_2)$ with $0 \leq i_1, i_2 \leq d$, we have

$$\text{tr}_{r_i}(\chi_{i_1} \cup \chi_{i_2}) = \varepsilon_{I,p}(r_i) = \delta_{p,2}\delta_{i_1} \delta_{i_2} \delta_{i_1,2} \delta_{i_1,3} + \delta_{i_1,2} b_{i_1,2} + \delta_{i_1,2} b_{i_1,1}$$

by the basic properties of mod $p$ Magnus coefficients (cf. e.g. [8, Theorem 2.4] and [52, Proposition 2.18]).
Under the assumptions, we put
\[ U = \bigoplus_{j: \text{odd}} \mathbb{F}_p \chi(j) \quad \text{and} \quad V = \bigoplus_{j: \text{even}} \mathbb{F}_p \chi(j) . \]

If \( i_1 \) and \( i_2 \) are even, we have \( \text{tr}_{r_i} (\chi_{\sigma(i_1)} \cup \chi_{\sigma(i_2)}) = 0 \) for any \( 0 \leq i \leq d \) by the assumption. Hence \( V \cup V = 0 \) by the nondegeneracy of the pairing \( H^2(\tilde{G}_S(\mathbb{Q})) \times H^2(\tilde{G}_S(\mathbb{Q}))^\vee \to \mathbb{F}_p \). Put
\[ \varphi_j = \chi(j) \cup \chi(j+1) = -\chi(j+1) \cup \chi(j) \in U \cup V , \]
and put
\[ a_{i,j} = \text{tr}_{r_{\sigma(i)}} \varphi_j = -\delta_{i,j} b_{\sigma(j),\sigma(j+1)} + \delta_{i,j+1} b_{\sigma(j+1),\sigma(j)} \in \mathbb{F}_p \]
for \( 0 \leq i,j \leq d \). Then the \((d+1) \times (d+1)\) matrix
\[ A = (a_{i,j})_{i,j} = \begin{pmatrix}
- b_{\sigma(0),\sigma(1)} & b_{\sigma(0),\sigma(2)} & \cdots & b_{\sigma(0),\sigma(d)} \\
 b_{\sigma(1),\sigma(0)} & - b_{\sigma(1),\sigma(2)} & \cdots & \vdots \\
 \vdots & \vdots & \ddots & \vdots \\
 b_{\sigma(d),\sigma(0)} & \cdots & \cdots & - b_{\sigma(d),\sigma(d)} \\
\end{pmatrix} \]
has the nonzero determinant \( \det A = \prod_{i=0}^d b_{\sigma(i),\sigma(i+1)} \) by the assumption. The linearity of trace maps \( \text{tr}_{r_{\sigma(i)}} \) yields that the elements \( \Phi_j \in U \cup V \) defined by
\[ (\Phi_0, \Phi_1, \cdots, \Phi_d) = (\varphi_0, \varphi_1, \cdots, \varphi_d) A^{-1} \]
satisfy \( \text{tr}_{r_{\sigma(i)}} \Phi_j = \delta_{i,j} \) for any \( 0 \leq i,j \leq d \). Hence the \( d + 1 \) elements \( \Phi_j \in U \cup V \) are linearly independent. This implies that \( U \cup V = H^2(\tilde{G}_S(\mathbb{Q})) \) and the deficiency \( \tilde{d}_S - \tilde{r}_S = 0 \). By the cup-product criterion, \( \tilde{G}_S(\mathbb{Q}) \) is a mild pro-\( p \) group. The latter statement also holds as the basic properties of a mild pro-\( p \) group (cf. e.g. [8, Theorem 2.4]). \( \square \)

**Example 3.13.** For \( \ell_0 = p = 3 \) and \( (\ell_1, \ell_2, \ell_3) = (13, 73, 61) \), one can easily see that \( \text{lk}(\ell_0, \ell_2) \equiv \text{lk}(\ell_2, \ell_0) \equiv 0 \pmod{3} \) and \( \text{lk}(\ell_0, \ell_1)\text{lk}(\ell_1, \ell_2)\text{lk}(\ell_2, \ell_3)\text{lk}(\ell_3, \ell_0) \neq 0 \equiv \text{lk}(\ell_0, \ell_3) \pmod{3} \). Then \( \{p\} \cup S = \{\ell_0, \ell_1, \ell_2, \ell_3\} \) is a circular set with \( \sigma \) such that \( \sigma(i) \equiv i \pmod{4} \) for all \( i \), and hence \( \tilde{G}_S(\mathbb{Q}) \) is a mild pro-3 group by Theorem 3.12.

**Example 3.14.** For \( \ell_0 = p = 2 \) and \( S = \{\ell_1, \ell_2, \ell_3, q\} \), suppose that \( \ell_1 \equiv 7 \pmod{8} \), \( \ell_2 \equiv 1 \pmod{8} \), \( \ell_3 \equiv 5 \pmod{8} \), \( \left(\frac{\ell_1}{\ell_2}\right) = \left(\frac{\ell_2}{\ell_3}\right) = \left(\frac{\ell_3}{\ell_0}\right) = -1 \), and \( q \equiv 3 \pmod{8} \). For example, \( (\ell_1, \ell_2, \ell_3, q) = (7, 17, 5, 3) \). Then the assumption of Theorem 3.12 is satisfied for \( \sigma \) such that \( \sigma(i) \equiv i \pmod{4} \) for all \( i \), and hence \( \tilde{G}_S(\mathbb{Q}) \) is a mild pro-2 group.
4 ALEXANDER INVARIANTS IN Iwasawa theory

4.1 Iwasawa polynomial. Let \( \Lambda = \mathbb{Z}_p[[T]] \) be the ring of formal power series in a variable \( T \) with coefficients in \( \mathbb{Z}_p \). Any finitely generated \( \Lambda \)-module \( X \) has a finite presentation

\[
\Lambda^{d_2} \xrightarrow{Q} \Lambda^{d_1} \xrightarrow{} X \xrightarrow{} 0
\]

with a \( d_1 \times d_2 \) presentation matrix \( Q \) such that \( d_2 \geq d_1 \geq 1 \). Independently on the choice of such presentation, the \( i \)th elementary ideal (\( i \)th Fitting ideal) \( E_i(X) \) is defined as an ideal of \( \Lambda \) generated by \((d_1 - i) \times (d_1 - i)\) minors of \( Q \) if \( 0 \leq i < d_1 \), and \( E_i(X) = \Lambda \) if \( i \geq d_1 \). The 'divisorial hull' \( \widetilde{E} \) of an ideal \( E \) is defined as the intersection of all principal ideals containing \( E \) (cf. \[12\]). Since \( \Lambda \) is a unique factorization domain, \( \widetilde{E} \) is a principal ideal generated by the greatest common divisor of generators of \( E \) if \( E \neq \{0\} \). Since \( \text{Ann}(X)^{d_1} \subset E_0(X) \subset \text{Ann}(X) \) for the annihilator ideal \( \text{Ann}(X) \) of \( X \) (cf. e.g. \[27\], Appendix), we have \( \text{Ann}(X)^{d_1} = \{0\} \) (i.e., \( E_0(X) = \{0\} \)) if and only if \( X \) is not \( \Lambda \)-torsion.

Iwasawa polynomials are defined analogous to Alexander polynomials as follows. For simplicity and convenience, we assume that \( k \cap \mathbb{Q}^{\text{cyc}} = \mathbb{Q} \) and \( k^{\text{cyc}}/k \) is totally ramified at any \( v \in P \). Let \( \overline{\tau} \) be the element of \( \text{Gal}(k(\mu_{p^\infty})/k) \) such that \( \overline{\tau}(\zeta) = \zeta^\kappa \) for any \( \zeta \in \mu_{p^\infty} \), where \( \kappa = 1 + p \) or \( \kappa = 5 \) according to \( p \neq 2 \) or \( p = 2 \). Then \( \overline{\tau}|_{k^{\text{cyc}}} \) is a generator of \( \text{Gal}(k^{\text{cyc}}/k) \), and there is \( \overline{\gamma} \in \text{Gal}(k^{\text{cyc}}S/kS) \) such that \( \overline{\gamma}|_{k^{\text{cyc}}} = \overline{\tau}|_{k^{\text{cyc}}} \). Recall that \( S \cap P = \emptyset \). Let \( K/k \) be a finite subextension of \( kS/k \). Then \( K \cap k^{\text{cyc}} = k \), and \( K^{\text{cyc}}/K \) is also totally ramified at any primes lying over \( p \). Put \( \gamma = \overline{\gamma}|_{k^{\text{cyc}}} \), which is the generator of \( \Gamma = \text{Gal}(K^{\text{cyc}}/K) \) such that \( \gamma|_{k^{\text{cyc}}} = \overline{\tau}|_{k^{\text{cyc}}} \). The left action of \( \Gamma \) on \( G_S(K^{\text{cyc}}) \) is defined by \( \gamma g = \overline{g}\overline{\gamma}^{-1} \) for \( g \in G_S(K^{\text{cyc}}) \). This induces the action of \( \Lambda \) on \( X = G_S(K^{\text{cyc}})_{ab} \) for any \( \Sigma \subset S \) via the isomorphism \( \Lambda \simeq \mathbb{Z}_p[[\Gamma]] : 1 + T \leftrightarrow \gamma \), where \( \mathbb{Z}_p[[\Gamma]] = \varprojlim \mathbb{Z}_p[[\Gamma/P^\nu]] \) is the complete group ring.

Then the Iwasawa module \( X \) is a finitely generated \( \Lambda \)-torsion \( \Lambda \)-module. The 'Iwasawa polynomial' \( \Delta(T) = p^\mu P(T) \in \mathbb{Z}_p[T] \) of \( X \) is defined as the generator of the divisorial hull \( \widetilde{E}_0(X) = \Delta(T)\Lambda \) of \( E_0(X) \) (cf. Remark 4.1 below) with \( 0 \leq \mu \in \mathbb{Z} \) and monic \( P(T) \equiv T^\lambda \pmod{p} \), where \( \lambda = \deg P(T) \). There exists \( \nu \in \mathbb{Z} \) satisfying Iwasawa's class number formula \( |G_S(K_n)|_{ab} = p^{\lambda n + \mu p^n + \nu} \) for all sufficiently large \( n \) (cf. e.g. \[53\]), where \( K_n \) is the fixed field of \( P^\nu \). The theorem of Ferrero-Washington \[5\] implies that \( \mu = 0 \) if \( K/Q \) is an abelian extension.

Remark 4.1. Suppose that there is a \( \Lambda \)-homomorphism \( f : X \to Y \) of finitely generated \( \Lambda \)-modules with finite cokernel. Then \( (p^n, T^n) \subset E_0(\text{Coker} f) \) for sufficiently large \( n \). By the basic properties of Fitting ideals (cf. e.g. \[27\], Appendix), we have

\[
(p^n, T^n)E_0(X) \subset (p^n, T^n)E_0(\text{Im} f) \subset E_0(\text{Coker} f)E_0(\text{Im} f) \subset E_0(Y),
\]
and hence \( \widetilde{E}_0(X) \subset \widetilde{E}_0(Y) \). Moreover if \( \text{Ker} f \) is also finite (i.e., \( f \) is a pseudo-isomorphism), and if \( X \) and \( Y \) are \( A \)-torsion, then there is a pseudo-isomorphism \( g : Y \to X \), and hence \( \widetilde{E}_0(X) = \widetilde{E}_0(Y) \). In particular when \( Y = \bigoplus_{i=1}^{m} \mathbb{A}/\varphi_i^{n_i} \) with some prime ideals \( \varphi_i \) of height 1 and \( n_i \geq 1 \), we have \( \widetilde{E}_0(X) = \prod_{i=1}^{m} \varphi_i^{n_i} \).

Based on the analogy between \( \widetilde{G}_0(k) \) and \( \pi_1(X) \), we obtain the following another proof of Gold’s theorem analogous to [19, Theorem 3.2].

**Theorem 4.2** (cf. [10], also [46]). Assume that \( p \neq 2 \) and \( k \) is an imaginary quadratic field with class number \( h_k \neq 0 \) (mod \( p \)), and that \( p \) splits in \( k \) as \( p \mathcal{O}_k = \mathfrak{p}_1 \mathfrak{p}_2 \). Let \( \Delta(T) \) be the Iwasawa polynomial of \( X = G_0(k)\)\textsubscript{cyc}\(ab \). Then \( \Delta(T) = T \) (i.e., \( \widetilde{G}_0(k) \simeq \mathbb{Z}_p^2 \)) if and only if \( \text{lk}(\mathfrak{p}_1, \mathfrak{p}_2) \neq 0 \) (mod \( p \)).

**Proof.** Suppose \( K = k \) and \( \Sigma = \emptyset \). By Theorem 3.2, \( \widetilde{G}_0(k) \) has a presentation

\[
1 \longrightarrow R \longrightarrow F \longrightarrow \widetilde{G}_0(k) \longrightarrow 1
\]

with a free pro-\( p \) group \( F = \langle x_1, x_2 \rangle \) and the normal subgroup \( R \) which is normally generated by \( r_1, r_2 \) of the form

\[
r_1 \equiv r_2^{-1} \equiv [x_1^{-1}, x_2^{-1}]^{\text{lk}(\mathfrak{p}_1, \mathfrak{p}_2)} \mod F_3
\]

(cf. also Remark 3.3). Then \( RF_3 = F_2^{\text{lk}(\mathfrak{p}_1, \mathfrak{p}_2)} F_3 \). In particular \( R \subset F_2 \), i.e., \( \widetilde{G}_0(k)\)\textsubscript{ab} \( \simeq \mathbb{Z}_p^2 \), and there is a surjective homomorphism \( X \to \mathbb{A}/\mathcal{T}A \). Hence \( \Delta(T) \in TA \). It is well known that \( X \simeq \mathbb{Z}_p^3 \) (cf. e.g. [53] Corollary 13.29). Therefore \( \widetilde{G}_0(k) \simeq \mathbb{Z}_p^2 \) if and only if \( X \simeq \mathbb{A}/\mathcal{T}A \) (i.e., \( \Delta(T) = T \)). Since \( F_2 = R \) if and only if \( F_2 = F_3 R \), we obtain the claim. \( \Box \)

**Remark 4.3.** An analogue of Iwasawa module is a finitely generated \( \mathbb{Z}[t^{\pm 1}] \)-module \( H_1(X^{\text{cyc}}, \mathbb{Z}) \simeq \pi_1(X^{\text{cyc}})\)\textsubscript{ab} where \( X^{\text{cyc}} \) is an infinite cyclic covering of \( X \) with \( \Gamma = \text{Gal}(X^{\text{cyc}}/X) = \langle t^2 \rangle \). Then the Alexander polynomial \( \Delta(t) \in \mathbb{Z}[t^{\pm 1}] \) is defined as a generator of \( \widetilde{E}_0(H_1(X^{\text{cyc}}, \mathbb{Z})) = \Delta(t)\mathbb{Z}[t^{\pm 1}] \). Regarding \( \Delta(1 + T) \in \Lambda \), Iwasawa invariants \( \lambda, \mu \) and \( \nu \) for \( X^{\text{cyc}}/X \) are analogously defined and studied (cf. [15] [18] [19] [20] [34] [50] [51] etc.). For \( \mathbb{L} = K_1 \cup K_2 \subset M = S^3 \) (and \( X^{\text{cyc}} \) such that the meridians of \( K_1 \) and \( K_2 \) are nontrivial in \( \Gamma \)), an analogue [19, Theorem 3.2] of Gold’s theorem states that \( \Delta(1 + T)\Lambda = \Lambda T \) if and only if the linking number \( \text{lk}(K_1, K_2) \neq 0 \) (mod \( p \)).

Alexander polynomials can be calculated from a presentation of \( \pi_1(X) \) by Fox derivative (cf. [7] [36] etc.). Analogously, in some special cases, one can calculate approximation of Iwasawa polynomials by pro-\( p \) Fox derivative as in the following sections.
4.2 Iwasawa Module as a Subquotient. Suppose that \( p = 2, k = \mathbb{Q}, S^* = \{\ell_1, \cdots, \ell_d\} \subset S \) with \( \ell_i \equiv 1 \pmod{2} \), and \( \Sigma = S \setminus S^* \) is either \( \{\infty\} \) or \( \{q\} \) with \( q \equiv 3 \pmod{4} \). Let \( K = \mathbb{Q}(\sqrt{D}) \subset \mathbb{Q}_S \) be the quadratic field of discriminant

\[
D = \begin{cases} 
\prod_{i=1}^{d} \ell_i^2 \quad &\text{if } \Sigma = \{\infty\}, \\
\prod_{i=1}^{d} \ell_i \quad &\text{if } \Sigma = \{q\}, 
\end{cases}
\]

where \( \ell_i^2 = (-1)^{\frac{q-1}{2}} \ell_i \). Let

\[
1 \longrightarrow R \longrightarrow F \longrightarrow \tilde{G}_S(Q) \longrightarrow 1
\]
be the presentation of \( \tilde{G}_S(Q) \) obtained in Theorem 3.1. Put the closed subgroup

\[
H = \langle x_0, x_d^{-1}x_0x_d, x_1x_d, \cdots, x_{d-1}x_d, x_1^2, \cdots, x_d^2 \rangle
\]

of \( F \) generated by such \( 2d + 1 \) elements, where we ignore \( x_1x_d, \cdots, x_{d-1}x_d \) if \( d = 1 \). Recall that \( \pi(x_i) \) generates the inertia group \( T_i \) of \( \tilde{\ell}_i \) in \( \tilde{G}_S(Q) \). Then \( \text{Gal}(K/\mathbb{Q}) = \langle \pi(x_i) \rangle \) for any \( 1 \leq i \leq d \), and \( \pi(x_0)|_K = 1 \). Hence \( H \) is contained in the kernel of the surjective homomorphism

\[
|K \circ \pi : F \longrightarrow \tilde{G}_S(Q) \longrightarrow \text{Gal}(K/\mathbb{Q})|.
\]

Since \( x_i^{-1}x_0x_i = (x_i^{-1}x_d)(x_d^{-1}x_0x_d)(x_d^{-1}x_i) \) and

\[
(x_i^{-1}x_j)x_j^2 = x_i^2(x_j^{-1}x_j) = x_i x_j = (x_i x_d)(x_j x_d)^{-1} x_j^2 \in H
\]

for \( 1 \leq i, j \leq d \), one can easily see that \( H \) is a normal subgroup of \( F \) such that \( F/H \simeq \langle x_i, H \rangle \simeq \mathbb{Z}/2\mathbb{Z} \) for any \( 1 \leq i \leq d \). Then the maximal subgroup \( H \) of \( F \) is a free pro-2 group freely generated by the \( 2d + 1 \) elements (cf. e.g. [37, Corollary (3.9.6)]), and \( R \subset F^2F_2 \subset H \). Since \( |K \circ \pi| \) induces the isomorphism \( (F/R)/(H/R) \simeq \text{Gal}(K/\mathbb{Q}) \), \( \pi \) induces a presentation

\[
1 \longrightarrow R \longrightarrow H \longrightarrow \tilde{G}_S(K) \longrightarrow 1
\]

of \( \tilde{G}_S(K) = \text{Gal}(\langle \mathbb{Q}^{\text{cyc}} \rangle_S/K) \). The inertia group

\[
T_i \cap \tilde{G}_S(K) = \text{Ker}(T_i \rightarrow \text{Gal}(K/\mathbb{Q}) : \pi(x_i) \mapsto \pi(x_i)|_K)
\]

of \( \tilde{\ell}_i \) in \( \tilde{G}_S(K) \) is generated by \( \pi(x_i^2) \). Hence \( \pi \) also induces a presentation

\[
1 \longrightarrow NR \longrightarrow H \longrightarrow \tilde{G}_\Sigma(K) \longrightarrow 1
\]

of \( \tilde{G}_\Sigma(K) = \text{Gal}(\langle K^{\text{cyc}} \rangle_S/K) \), where \( \varpi = |(K^{\text{cyc}}) \circ \pi \), and \( N = \langle x_1^2, \cdots, x_d^2 \rangle_H \) is the normal subgroup of \( H \) normally generated by \( x_1^2, \cdots, x_d^2 \). Note that \( g^{-1}x_i^2g = \ldots \)
\[(x_i^{-1}g)^{-1}x_i^2(x_i^{-1}g) \in N \text{ for any } g \in F = H \cup x_iH \text{ and } 1 \leq i \leq d. \] Then \( N \) is also a normal subgroup of \( F \), and hence a presentation

\[
1 \longrightarrow NR \longrightarrow F \xrightarrow{\varpi} \text{Gal}((K^{\text{cyc}})_{\Sigma}/\mathbb{Q}) \longrightarrow 1
\]

of \( \text{Gal}((K^{\text{cyc}})_{\Sigma}/\mathbb{Q}) \) is also induced. Recall that \( R \) is generated by \( r_0, r_1, \ldots, r_d \) as a normal subgroup of \( F \), and put

\[
\rho_i = [x_i^{-1}, y^{-1}_i] \equiv r_i \mod N
\]

for \( 0 \leq i \leq d \). Since

\[
g^{-1}\rho_ig = (x_i^{-1}g)^{-1}(y_i^2x_i^{-2}y_i^{-1})^{-1}\rho_i^{-1}x_i^2(x_i^{-1}g) \equiv (x_i^{-1}g)^{-1}\rho_i^{-1}(x_i^{-1}g) \mod N
\]

for any \( g \in F = H \cup x_iH \) and \( 1 \leq i \leq d \),

\[
NR = \langle x_1^2, \ldots, x_d^2, \rho_0, \rho_1, \ldots, \rho_d \rangle_H
\]

is normally generated by such \( 2d + 2 \) elements in \( H \). Put \( \overline{F} = F/N \), and put \( \overline{g} = gN \in \overline{F} \) for any \( g \in F \). The universality of the free pro-2 group implies that

\[
\overline{H} = H/N = \langle w_{01}, w_{02}, w_1, \ldots, w_{d-1} \rangle
\]

is a free pro-2 group of rank \( d + 1 \), where

\[
w_{01} = \overline{x_0}, \quad w_{02} = \overline{x_1^{-1}x_0x_1}, \quad w_i = \overline{x_i^{-1}x_d}
\]

for \( 1 \leq i \leq d - 1 \). We ignore \( w_i \) if \( d = 1 \). Then \( \overline{\varpi} \) induces a presentation

\[
1 \longrightarrow \overline{\mathcal{R}} \longrightarrow \overline{\mathcal{P}} \xrightarrow{\overline{\varpi}} \tilde{G}_\Sigma(K) \longrightarrow 1
\]

of \( \tilde{G}_\Sigma(K) \) and an exact sequence

\[
1 \longrightarrow \overline{\mathcal{R}} \longrightarrow \overline{\mathcal{P}} \xrightarrow{\overline{\varpi}} \text{Gal}((K^{\text{cyc}})_{\Sigma}/\mathbb{Q}) \longrightarrow 1,
\]

where

\[
\overline{\mathcal{R}} = NR/N = \langle \overline{r_0}, \overline{x_1^{-1}r_0x_1}, \overline{r_1}, \ldots, \overline{r_d} \rangle_{\overline{\mathcal{P}}}
\]

and \( \overline{\varpi}(\overline{g}) = \overline{\varpi}(g) = \pi(g)|_{(K^{\text{cyc}})_{\Sigma}} \) for \( g \in F \). Recall that we can put \( \tilde{\gamma} = \pi(x_0) \) (cf. Remark 3.3). Then \( \Gamma = \text{Gal}(K^{\text{cyc}}/K) = \langle \gamma \rangle \) with \( \gamma = \tilde{\gamma}|_{K^{\text{cyc}}} \). Let \( \tau \) be the generator of \( \text{Gal}(K^{\text{cyc}}/Q^{\text{cyc}}) \simeq \mathbb{Z}/2\mathbb{Z} \). Then \( \text{Gal}(K^{\text{cyc}}/Q) = \langle \tau, \gamma \rangle \simeq \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}_p \), and \( \pi(x_i)|_{K^{\text{cyc}}} = \tau \) for all \( 1 \leq i \leq d \). Since \( \overline{\varpi}(w_{01})|_{K^{\text{cyc}}} = \overline{\varpi}(w_{02})|_{K^{\text{cyc}}} = \gamma \) and \( \overline{\varpi}(w_i)|_{K^{\text{cyc}}} = 1 \), the kernel of

\[
|_{K^{\text{cyc}}} \circ \overline{\varpi} : \overline{H} \xrightarrow{\overline{\varpi}} \tilde{G}_\Sigma(K) \xrightarrow{|_{K^{\text{cyc}}}} \Gamma
\]
is
\[ \overline{M} = (w_{01}, w_1, \ldots, w_{d-1})_\mathbb{H}. \]
Then \( \overline{H}/\overline{M} \cong \mathbb{Z}_2 \) and \( \overline{G}(\overline{M}) = G_\Sigma(K^{\text{cyc}}) \). By defining the action of \( \gamma \) on \( m \in \overline{M} \) as
\[ \gamma m = w_{01} m w_{01}^{-1}, \]
\( \overline{M} \) is a free pro-2-\( \Gamma \) operator group of rank \( d \), and \( \overline{M}/\overline{M}_2 \cong \Lambda^d \) is a free \( \Lambda \)-module of rank \( d \). Since \( \overline{F}/\overline{H} = (\overline{x}_d \overline{M}) \) and
\[
\overline{x}_d (w_{01}w_{02}^{-1}) \overline{x}_d = (w_{01}w_{02}^{-1})^{-1} = [\overline{x}_d, w_{01}^{-1}],
\]
\[ \overline{x}_d w_i \overline{x}_d = w_i^{-1} \]
for all \( 1 \leq i \leq d - 1 \), \( \overline{M} \) is a normal subgroup of \( \overline{F} \), and \( \overline{G} \) induces the isomorphism
\[ \overline{F}/\overline{M} = (\overline{x}_d \overline{M}, w_{01} \overline{M}) \cong \text{Gal}(K^{\text{cyc}}/\mathbb{Q}) = \langle \tau, \gamma \rangle : \overline{x}_d \overline{M} \mapsto \tau, w_{01} \overline{M} \mapsto \gamma. \]
In particular, we have \( \overline{R} \subset \overline{F}_2 \subset \overline{M} \). Note that
\[
\overline{x}_d(\overline{h} \overline{m} \overline{h}^{-1}) \overline{x}_d^{-1} = [\overline{x}_d^{-1}, \overline{h}^{-1}[\overline{x}_d \overline{m} \overline{x}_d^{-1}] \overline{h}^{-1}, \overline{x}_d^{-1}] \equiv \overline{h}(\overline{x}_d \overline{m} \overline{x}_d^{-1}) \overline{h}^{-1} \mod \overline{M}_2
\]
for any \( m \in \overline{M} \) and \( h \in H \). By defining the action of \( \tau \) on \( \overline{M}/\overline{M}_2 \) as
\[ \tau (m \overline{M}_2) = \overline{x}_d \overline{m} \overline{x}_d^{-1} \overline{M}_2 = (m \overline{M}_2)^{-1} \]
for \( m \in \overline{M} \), we regard \( \overline{M}/\overline{M}_2 \) as a \( \mathbb{Z}_2[[\text{Gal}(K^{\text{cyc}}/\mathbb{Q})]] \)-module, particularly as a \( \Lambda \)-module. Since \( \overline{M}_2 \overline{R}/\overline{M}_2 \) is a \( \mathbb{Z}_2[[\text{Gal}(K^{\text{cyc}}/\mathbb{Q})]] \)-submodule of \( \overline{M}/\overline{M}_2 \), \( \overline{G} \) induces an isomorphism
\[ \overline{M}/\overline{M}_2 \overline{R} \cong G_\Sigma(K^{\text{cyc}})^{ab} \]
as \( \mathbb{Z}_2[[\text{Gal}(K^{\text{cyc}}/\mathbb{Q})]] \)-modules, particularly as \( \Lambda \)-modules. Since
\[ \overline{x}_d^{-1} \overline{\rho}_0 \overline{x}_d \overline{M}_2 = \tau (\overline{\rho}_0 \overline{M}_2) = (\overline{\rho}_0 \overline{M}_2)^{-1}, \]
this isomorphism induces a presentation
\[ \Lambda^{d+1} \xrightarrow{\cong} \Lambda^d \xrightarrow{\\equiv} G_\Sigma(K^{\text{cyc}})^{ab} \xrightarrow{0} \]
of the Iwasawa module \( X = G_\Sigma(K^{\text{cyc}})^{ab} \), which implies that the Iwasawa polynomial is computable in principle from the presentation of \( \tilde{G}_S(\mathbb{Q}) \). We obtain the following theorem concerning an approximate computation of the initial Fitting ideal.
Theorem 4.4. Under the settings above, let \( E_0(X) \) be the initial Fitting ideal of the \( \Lambda \)-module \( X = G_\Sigma(K^{cyc})^{ab} \), and put

\[
\varepsilon = \begin{cases} 
1 & \text{if } D \equiv 1 \pmod{8}, \\
0 & \text{if } D \equiv 5 \pmod{8}.
\end{cases}
\]

Suppose that \( \rho_i = \left[ x_i^{-1}, y_i^{-1} \right] \equiv \rho_{i,n} \pmod{\mathfrak{M}} \) with some \( \rho_{i,n} \in \mathcal{H} \) for each \( 0 \leq i \leq d \) and \( n \geq 3 \), where

\[
\mathfrak{M} = M^{2n-1-\varepsilon} M_2 \prod_{i=2}^{n} F_i^{2n-i}.
\]

Then the ideal \( E_0(X) + (2,T)^{n-1-\varepsilon} \) of \( \Lambda \) is generated by \( (2,T)^{n-1-\varepsilon} \) and the \( d \times d \) minors of the \( d \times (d+1) \) matrix

\[
Q_n = \left( \Phi \left( \frac{\partial \rho_{i,n}}{\partial w} \right) \right)_{w,i}
\]

with rows and columns indexed by \( w \in \{ w_{01}, w_1, \cdots, w_{d-1} \} \) and \( 0 \leq i \leq d \), where

\[
\Phi : \mathbb{Z}_2[[\mathcal{H}]] \xrightarrow{\sim} \mathbb{Z}_2[[\Gamma]] \xrightarrow{\sim} \Lambda
\]
is the \( \mathbb{Z}_2 \)-linear ring homomorphism naturally extended from \( |_{K^{cyc} \circ \varpi} \circ \mathcal{H} \to \Gamma \).

4.3 Proof of Theorem 4.4. Put \( M^{(1)} = \mathcal{M} \supset F_2 \), and put \( M^{(n)} = [M^{(n-1)}, F] \supset F_{n+1} \) for \( n \geq 2 \) recursively. Arbitrary \( \overline{\gamma} \in F \) can be written in the form \( \overline{\gamma} = w_{01}x_d m' \) with some \( z \in \mathbb{Z}_2 \), \( e \in \{0, 1\} \) and \( m' \in \mathcal{M} \). Then, since

\[
[m, \overline{\gamma}] \equiv m^{-2e} (\gamma^{-1} m^{-1} (-1)^{e}) \pmod{M_2}
\]

for any \( m \in \mathcal{M} \), we have

(4.2) \[
\overline{M}^{(n)} M_2 / \mathcal{M}_2 = (2,T)^{n-1}(\mathcal{M}/M_2)
\]

for any \( n \geq 1 \) by induction. Then \( F_i^{2n-i} \subset (\mathcal{M}^{(i-1)})^{2n-i} \subset \mathcal{M}^{(n-1)} \mathcal{M}_2 \) for any \( n \geq i \geq 2 \), and hence

(4.3) \[
\mathfrak{M}^{(n)} \subset \overline{M}^{(n-1)} \mathcal{M}_2 \subset \overline{M}^{(2)} \mathcal{M}_2
\]

for \( n \geq 3 \). In particular, we have

\[
\overline{M}^{(2)} \mathcal{M}_2 R = \overline{M}^{(2)} \mathcal{M}_2 R^{(n)},
\]

where we put

\[
R^{(n)} = \langle \rho_{0,n}, x_{d}^{-1}, \rho_{0,n} x_{d}, \rho_{1,n}, \cdots, \rho_{d,n} \rangle_{\mathcal{H}} \subset \mathcal{M}.
\]
Lemma 4.5. \( \overline{M}^{(n-\varepsilon)} \overline{M}^{(n-\varepsilon)}_2 \mathbb{R} = \overline{M}^{(n-\varepsilon)} \overline{M}^{(n-\varepsilon)}_2 \mathbb{R}^{(n)} \) for any \( n \geq 3 \).

Proof. If \( D \equiv 1 \pmod{4} \), we obtain the claim by (1.3). Suppose that \( D \equiv 5 \pmod{8} \). Let \( L \) be the fixed field of \( (2, T)X \). Then \( L/K^{cyc} \) is the maximal elementary abelian 2-extension unramified outside \( \Sigma \) which is abelian over \( K \). Since
\[
\overline{M}/\overline{M}^{(2)}_2 \mathbb{R} \simeq X/(2, T)X \simeq \text{Gal}(L/K^{cyc})
\]
as \( \mathbb{Z}_2[[\text{Gal}(K^{cyc})/\mathbb{Q}]] \)-modules by (4.2), we have
\[
\overline{F}/\overline{M}^{(2)}_2 \mathbb{R} \simeq \text{Gal}(L/\mathbb{Q}).
\]
Let \( L' \) be the inertia field in \( L/K \) of the unique prime of \( K \) lying over \( 2 \). Since \( L/L' \) and \( K^{cyc}/K \) are totally ramified, we have \( L = L'K^{cyc} \) and \( L' \cap K^{cyc} = K \). Hence \( L' \) is an elementary abelian 2-extension of \( K \) unramified outside \( \Sigma \). Since \( G_{\Sigma}(\mathbb{Q})^{ab} \simeq \{1\} \), \( 1 + \tau \) annihilates the Sylow 2-subgroup of the ray class group of \( K \) modulo \( v \in \Sigma \), i.e., \( \tau|_K \in \text{Gal}(K/\mathbb{Q}) \) acts as inverse on \( G_{\Sigma}(K)^{ab} \). The trivial action of \( \text{Gal}(K/\mathbb{Q}) \) on \( G_{\Sigma}(K)^{ab}/2 \) implies that the maximal elementary abelian 2-extension of \( K \) unramified outside \( \Sigma \) is an abelian extension over \( \mathbb{Q} \). Therefore \( L = L'\mathbb{Q}^{cyc} \) is abelian over \( \mathbb{Q} \), i.e.,
\[
\overline{F}_2 \subset \overline{M}^{(2)}_2 \mathbb{R} = \overline{M}^{(2)}_2 \mathbb{R}^{(n)}.
\]
Since \( \overline{M}^{(n)} \mathbb{R}^{(n)} \) is also a normal subgroup of \( \overline{F} \) by (4.1), we have
\[
\overline{F}_i \subset \overline{M}^{(i)} \mathbb{R} \mathbb{R} \cap \overline{M}^{(i)} \mathbb{R}^{(n)} \mathbb{R}
\]
for any \( i \geq 2 \). Since \( (\overline{M}^{(i)})^{2^{n-1}} \subset \overline{M}^{(n)} \mathbb{R} \) for any \( i \geq 1 \) by (4.2), we have
\[
\mathcal{M}^{(n)} \subset \overline{M}^{(n)} \mathbb{R} \mathbb{R} \cap \overline{M}^{(n)} \mathbb{R}^{(n)} \mathbb{R}
\]
and hence
\[
\overline{M}^{(n)} \overline{M} \mathbb{R}^{(n)} = \overline{M}^{(n)} \overline{M}^{(n)} \mathbb{R}^{(n)} = \overline{M}^{(n)} \overline{M}^{(n)} \mathbb{R}^{(n)} \mathbb{R} = \overline{M}^{(n)} \mathbb{R}^{(n)} \mathbb{R}.
\]
Thus Lemma 4.5 is proved. \( \square \)

Lemma 4.5 above and (4.2) yield that
\[
(4.4) \quad X/(2, T)^{n-1-\varepsilon} X \simeq \overline{M}/\overline{M}^{(n-\varepsilon)} \overline{M}^{(n-\varepsilon)}_2 \mathbb{R} \simeq X^{(n)}/(2, T)^{n-1-\varepsilon} X^{(n)},
\]
where
\[
X^{(n)} = \overline{M}/\overline{M}^{(n)}/2 \mathbb{R}^{(n)} \simeq \text{Ker}(H/\mathbb{R}^{(n)} \psi_n \Gamma)^{ab}.
\]
Since \( \pi_n : H \to H/\mathbb{R}^{(n)} \) satisfies \( \psi_n \circ \pi_n = |K^{cyc}| \mathbb{R} \), there is a presentation
\[
A^{d+2} \xrightarrow{\mathcal{Q}^n} A^{d+1} \xrightarrow{A^\psi_n} 0
\]
of the complete $\psi_n$-differential module $\mathfrak{A}_{\psi_n}$ (cf. [36, Corollary 9.15]) with the $(d+1) \times (d+2)$ matrix $\tilde{Q}_n$ obtained by adding to $Q_n$ a row with $w = w_{02}$ and a column

$$\left( \Phi \left( \frac{\partial x_1 x_2 \cdots x_d}{\partial w} \right) \right)_w = \left( \frac{\partial \rho_0}{\partial w} \right)_w$$

which is the inverse of the column of $i = 0$ by (4.1) and the following lemma.

**Lemma 4.6.** For any $w \in \{w_{01}, w_{02}, w_1, \cdots, w_{d-1}\}$, we have

$$\Phi \left( \frac{\partial m_1 m_2}{\partial w} \right) = z_1 \Phi \left( \frac{\partial m_1}{\partial w} \right) + z_2 \Phi \left( \frac{\partial m_2}{\partial w} \right)$$

for any $z_1, z_2 \in \mathbb{Z}_2$ and $m_1, m_2 \in \mathcal{M}$. In particular,

$$\Phi \left( \frac{\partial m}{\partial w} \right) = 0$$

if $m \in \mathcal{M}_2$.

**Proof.** Recall that $\mathcal{M}$ is the kernel of $|_{K_{cyc}} \circ \varpi$, i.e., $\varpi(m_i)|_{K_{cyc}} = 1$. Hence the claim holds by the basic properties (cf. e.g. [36, Proposition 8.13]) of the pro-$p$ Fox derivative $\frac{\partial}{\partial w}$ and the continuity of $\Phi \circ \frac{\partial}{\partial w}$. \(\square\)

The involution $\tau$ on $\mathcal{M}$ defined by $\tau(h) = x_d^{-1} h x_d$ satisfies $(w_{01})^\tau = w_{02}$ and $(w_i)^\tau = w_i^{-1}$, in particular, $\varpi(h)|_{K_{cyc}} = \varpi(h)|_{K_{cyc}}$ for any $h \in \mathcal{M}$. By the chain rule of the Fox derivative (cf. [7, (2.6)]), we have

$$\frac{\partial \rho_{i,n}}{\partial w_{02}} = \frac{\partial (\rho_{i,n})^\tau}{\partial w_{02}} = \sum_w \left( \frac{\partial \rho_{i,n}^\tau}{\partial w} \right)^\tau \frac{\partial w^\tau}{\partial w_{02}} = \left( \frac{\partial \rho_{i,n}^\tau}{\partial w_{01}} \right)^\tau,$$

and hence (4.1) and Lemma 4.6 yield that

$$\Phi \left( \frac{\partial \rho_{i,n}}{\partial w_{02}} \right) = -\Phi \left( \frac{\partial \rho_{i,n}}{\partial w_{01}} \right)$$

for any $0 \leq i \leq d$, i.e., the row of $w = w_{02}$ is the inverse of the row of $w = w_{01}$ in $\tilde{Q}_n$. Since the Crowell exact sequence (cf. [38] or [36, Theorem 9.17])

$$0 \longrightarrow X^{(n)} \longrightarrow \mathfrak{A}_{\psi_n} \longrightarrow \Lambda^{\varepsilon_{x_2|x_1}} \longrightarrow 0$$

yields that $\mathfrak{A}_{\psi_n} \simeq X^{(n)} \oplus \Lambda$, the Fitting ideal $E_0(X^{(n)}) = E_1(\mathfrak{A}_{\psi_n})$ of the $\Lambda$-module $X^{(n)}$ is generated by $d \times d$ minors of $Q_n$, i.e., of $Q_n$ (cf. [36, Example 9.18]). Since

$$E_0(X) + (2, T)^{n-1-\varepsilon} = E_0(X^{(n)}) + (2, T)^{n-1-\varepsilon}$$

by (4.3) (cf. [27, Appendix]), we obtain the claim of Theorem 4.4.
4.4 Application of Theorem 4.3. In the following, we assume that \( \Sigma = \{ \infty \} \). Recall that \( \rho_i = [x_i^{-1}, y_i^{-1}] \). By Theorem 3.1 and (3.5),

\[
y_i \equiv x_d^{c_{id}} \cdots x_1^{c_{i1}} x_0^{c_{i0}} \prod_{a<b} [x_a, x_b]^{c_{iab}} \mod F_3N
\]

with \( c_{i1}, \ldots, c_{id} \in \{0, 1\} \) and \( c_{iab} \in \mathbb{Z}_2 \) for \( 0 \leq i \leq d \), where

\[
c_{i0} = \operatorname{lk}(\ell_i, \ell_0), \\
c_{ij} \equiv \operatorname{lk}(\ell_i, \ell_j) \mod 2
\]

for \( 1 \leq j \leq d \). Then we have

\[
(4.5) \quad \rho_i \equiv \prod_{j=0}^d [x_i^{-1}, x_j^{-1}]^{c_{ij}} \cdot [x_i, x_0, x_0]^{c_{i0}(c_{i0}-1)/2} \cdot \prod_{a<b} [x_i, x_a, x_b]^{c_{iab}}[x_a, x_b, x_i]^{c_{iab}} \mod F_3^2F_4(F_2)_{2N}
\]

by \([37]\) Propositions (3.8.3) and (3.8.6)\)] and the multilinearity of brackets

\[
[ , , ] : F/F_2 \otimes F/F_2 \otimes F/F_2 \rightarrow F/F_3 \otimes F/F_2 \rightarrow F_3/F_4.
\]

For convenience, we put \( w_d = x_dx_d = 1 \). Then, for any \( j \neq 0 \) and \( b \neq 0 \),

\[
[x_i^{-1}, x_j^{-1}] = \begin{cases} 
  w_{01} w_j w_0^{-1} w_j^{-1} & \text{if } i = 0, \\
  (w_i w_j^{-1})^2 & \text{if } i \neq 0,
\end{cases}
\]

\[
[x_i, x_j, x_0] = \begin{cases} 
  [w_j w_0 w_j^{-1}, w_0] & \text{if } i = 0, \\
  [(w_i w_j^{-1})^2, w_0] & \text{if } i \neq 0,
\end{cases}
\]

and

\[
[x_i, x_j, x_b] \equiv \begin{cases} 
  (w_{01}^{-1} w_j^{-1} w_0 w_j)^2 & \text{mod } M_2 \text{ if } i = 0, \\
  (w_i w_j^{-1})^{-4} & \text{mod } M_2 \text{ if } i \neq 0.
\end{cases}
\]

By (4.5), we have

\[
(4.6) \quad \rho_i \equiv \rho_{i,4} \mod F_3^2F_4M_2 \subset M^{(4)}
\]

with

\[
\rho_{0,4} = \prod_{j=1}^d (w_{01} w_j w_0 w_j^{-1})^{c_{0j}} [w_j w_0 w_j^{-1}, w_0]^{c_{0j}} \\
\quad \cdot \prod_{0<a<b} (w_{02}^{-1} w_a w_0 w_a)^{2c_{0a}c_{0b}}[(w_a w_b^{-1})^2, w_0]^{c_{0ab}}
\]
and
\[
\frac{\partial \mu_i}{\partial w} = \prod_{j=1}^d (w_i w_j^{-1}) (w_i^{-1} w_j^{-1} w_0 w_i w_j) \cdot (w_i w_j^{-1} w_0 w_i^{-1} w_j^{-1} w_0) \cdot \prod_{0 < a < b} (w_a w_b^{-1})^{-c_{ab}}.
\]

for \(1 \leq i \leq d\). Since

\[
\Phi \left( \frac{\partial (w_i w_j^{-1} w_0)}{\partial w} \right) = \delta_{w,w_0} - \delta_{w,w_0} + T \delta_{w,w_1},
\]
\[
\Phi \left( \frac{\partial (w_i w_j^{-1} w_0)}{\partial w} \right) = T \delta_{w,w_0} - T \delta_{w,w_0} + T^2 \delta_{w,w_1},
\]
\[
\Phi \left( \frac{\partial (w_i w_j^{-1} w_0)}{\partial w} \right) = 2(T + 1) \delta_{w,w_0} - 2(T + 1) \delta_{w,w_0} + 2T \delta_{w,w_1},
\]
\[
\Phi \left( \frac{\partial (w_i w_j^{-1} w_0)}{\partial w} \right) = 2T \delta_{w,w_0} + 2T \delta_{w,w_1}
\]

modulo \((2, T)^3\) for \(1 \leq i, a, b \leq d\), a routine calculation using Lemma 4.6 shows that

\[
\Phi \left( \frac{\partial \mu_i}{\partial w} \right) \equiv q_{w,i} \mod (2, T)^3
\]

with \(q_{w,i} \in \Lambda\) such that

\[
q_{w,0} = \begin{cases} 
\sum_{j=1}^d (c_{0j} + c_{00j} T) + \sum_{0 < a < b} 2c_{ab}c_{0b}(T + 1) & \text{if } w = w_0, \\
c_{0m} T + c_{00m} T^2 + \sum_{b=m+1}^d 2(c_{0m} c_{0b} + c_{0mb}) T + \sum_{a=1}^{m-1} 2c_{0am} T & \text{if } w = w_m,
\end{cases}
\]

and

\[
q_{w,i} = \begin{cases} 
\sum_{j=1}^d 2(c_{0j} c_{ij} + c_{00j})(T + 1) - c_{0i} + \frac{c_{0i}^2(T + 1)}{2} & \text{if } w = w_0, \\
\sum_{j=1}^d 2c_{ij}(1 + c_{0j} T) + 2c_{0j} T - c_{0i} T + \frac{c_{0i}^2(T + 1)}{2} T^2 & \text{if } w = w_i, \\
+ \sum_{0 < a < b} 2c_{ai} c_{ab} + \sum_{b=i+1}^d 2c_{abi} + \sum_{a=1}^{i-1} 2c_{iai} & \text{if } w = w_i, \\
-2c_{im} + 2c_{0im} T + \sum_{b=m+1}^d 4(c_{im} c_{ib} + c_{imb}) + \sum_{a=1}^{m-1} 4c_{iam} & \text{if } w = w_m \neq w_i,
\end{cases}
\]

for \(1 \leq i \leq d\) and \(1 \leq m \leq d - 1\). Then (1.9), (1.17) and Theorem 4.4 for \(n = 4\) yield that \(E_0(X) + (2, T)^{3-\varepsilon}\) is generated by \((2, T)^{3-\varepsilon}\) and \(d \times d\) minors of the \(d \times (d + 1)\) matrix \((q_{w,i})_{w,i}\). In particular when \(d = 2\), the \(2 \times 2\) minors

\[
\Delta_0(T) = \begin{vmatrix} q_{w_0,0} & q_{w_0,1} \\
q_{w_0,1} & q_{w_1,1} \end{vmatrix}, \quad \Delta_1(T) = \begin{vmatrix} q_{w_0,1} & q_{w_0,2} \\
q_{w_1,1} & q_{w_1,2} \end{vmatrix}, \quad \Delta_2(T) = \begin{vmatrix} q_{w_0,2} & q_{w_0,0} \\
q_{w_1,2} & q_{w_1,0} \end{vmatrix}
\]

29
of the $2 \times 3$ matrix $(q_{u,v})_{u,v}$ satisfy the following congruences modulo $(2, T)^3$;

\[
\Delta_0(T) \equiv (c_{10}c_{02} + c_{02} \frac{c_{01}(c_{20} - 1)}{2})T^2 - c_{10}c_{02}T + 2c_{12}(c_{01} + c_{02})
+ 2(c_{12}(c_{10}c_{02} + c_{001} + c_{002}) + c_{01}c_{102} + c_{02}c_{101} + c_{10}c_{012})T
+ 4(c_{12}c_{01}c_{02} + c_{112}(c_{01} + c_{02}))
\]

\[
\Delta_1(T) \equiv (c_{10} \frac{c_{20}(c_{20} - 1)}{2} + c_{20} \frac{c_{01}(c_{20} - 1)}{2})T^2 - c_{10}c_{20}T + 2(c_{12}c_{20} + c_{21}c_{10})
+ 2(c_{12}c_{20}(c_{10} + c_{20}) + c_{21}(c_{01} + c_{012}) + c_{12}(c_{201} + c_{202}) + c_{10}c_{212} + c_{20}c_{112})T
+ 4(c_{12}c_{21}(c_{10} + c_{20}) + c_{21}(c_{101} + c_{102}) + c_{12}(c_{201} + c_{202}) + c_{10}c_{212} + c_{20}c_{112}),
\]

\[
\Delta_2(T) \equiv (c_{20}c_{001} + c_{01} \frac{c_{01}(c_{20} - 1)}{2})T^2 - c_{01}c_{20}T + 2c_{21}(c_{01} + c_{02})
+ 2(c_{21}(c_{01}c_{20} + c_{001} + c_{002}) + c_{01}c_{202} + c_{02}c_{201} + c_{20}c_{012} + c_{01}c_{020})T
+ 4(c_{21}c_{01}c_{02} + c_{212}(c_{01} + c_{02})).
\]

Then we obtain the following theorems.

**Theorem 4.7.** Assume that $k = \mathbb{Q}$, $p = 2$ and $S = \{\ell_1, \ell_2, \infty\}$. Let $\Delta(T)$ be the Iwasawa polynomial of $X = G_0(K^{\text{cy}})^{ab}$ for an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-\ell_1\ell_2})$, and let $c_{012}, c_{201}$ be $2$-adic integers defined by (3.3).

1. If $\ell_1 \equiv 9$ (mod 16), $\ell_2 \equiv 3$ (mod 8) and $\left(\frac{\ell_1}{\ell_2}\right) = 1$, then

\[
\Delta(T) \equiv T^2 + (1 + \left(\frac{2}{\ell_2}\right)_4)T + 2(1 - \left(\frac{2}{\ell_1}\right)_4) \mod 4\mathbb{Z}_2 + 8\mathbb{Z}_2
\]

and $c_{012} + c_{201} \equiv \frac{1}{2}(1 + \left(\frac{2}{\ell_2}\right)_4) \mod 2$.

2. If $\ell_1 \equiv 7$ (mod 16), $\ell_2 \equiv 5$ (mod 8) and $\left(\frac{\ell_1}{\ell_2}\right) = 1$, then

\[
\Delta(T) \equiv T^2 + 2(1 - \left(\frac{\ell_1}{\ell_2}\right)_4) \mod 4\mathbb{Z}_2 + 8\mathbb{Z}_2
\]

and $c_{012} \equiv c_{201} \mod 2$.

**Proof.** Assume that $\left(\frac{\ell_1}{\ell_2}\right) = 1$ and either $\ell_1 \equiv 9$ (mod 16) and $\ell_2 \equiv 3$ (mod 8) or $\ell_1 \equiv 7$ (mod 16) and $\ell_2 \equiv 5$ (mod 8). Then, since $X \simeq \mathbb{Z}_2^2$ as a $\mathbb{Z}_2$-module (cf. [4, 21]), $E_0(X)$ is a principal ideal of $\mathcal{A}$ generated by $\Delta(T) \equiv T^2$ (mod 2) (cf. [53] p.299, Example (3)). By (4.6), (4.7) and Theorem 4.4, the ideal $E_0(X) + (2, T)^3$ is generated by $(2, T)^3$ and $\Delta_0(T), \Delta_1(T), \Delta_2(T)$. Hence

\[
\Delta_i(T) \equiv u_i(T) \Delta(T) \mod (2, T)^3
\]

with some $u_i(T) \in \Lambda^*$ for each $i \in \{0, 1, 2\}$. By the assumption, we have $c_{01} = 0, c_{02} = 1, c_{12} = c_{21} = 0, c_{10} \equiv 2$ (mod 4), $c_{20} \equiv 1$ (mod 2). Then $c_{20}c_{01}(c_{20} - 1) \equiv 1$ (mod 2). Moreover, $(-1)^{c_{010}} = \left(\frac{\ell_1}{\ell_2}\right)_4 = -1$ by Propositions 3.7 and 3.8. Hence

\[
\Delta_0(T) \equiv \Delta_1(T) \equiv T^2 + 2(c_{101} + 1)T + 4c_{112} \mod (2, T)^3,
\]

\[
\Delta_2(T) \equiv T^2 + 2(c_{012} + c_{201})T + 4c_{212} \mod (2, T)^3.
\]
In particular, $\Delta_i(T) \in (2, T)^2$ for all $i \in \{0, 1, 2\}$. Since

$$T^2 \equiv \Delta_i(T) \equiv u_i(T)\Delta(T) \equiv u_i(0)T^2 \mod (2, T^3),$$

we have $u_i(T) \in \Lambda^\times$, and hence

$$\Delta(T) \equiv u_i(T)^{-1}\Delta_i(T) \equiv \Delta_i(T) \mod (2, T)^3$$

for all $i \in \{0, 1, 2\}$. In particular, $c_{101} + 1 \equiv c_{012} + c_{201} \pmod{2}$. Since

$$(−1)^{c_{101}} = \begin{cases} \left(\frac{2}{\ell_1}\right)_4 & \text{if } \ell_1 \equiv 9 \pmod{16}, \\ \left(\frac{2}{\ell_1}\right)_4 = -1 & \text{if } \ell_1 \equiv 7 \pmod{16} \end{cases}$$

and

$$(−1)^{c_{112}} = (−1)^{c_{212}} = \begin{cases} \left(\frac{2}{\ell_2}\right)_4 = \left(\frac{2}{\ell_1}\right)_4 & \text{if } \ell_1 \equiv 9 \pmod{16}, \ell_2 \equiv 3 \pmod{8}, \\ \left(\frac{2}{\ell_2}\right)_4 = \left(\frac{2}{\ell_1}\right)_4 & \text{if } \ell_1 \equiv 7 \pmod{16}, \ell_2 \equiv 5 \pmod{8} \end{cases}$$

by Propositions 3.7 and 3.8 we obtain the statement of Theorem 4.7.

**Remark 4.8.** Although the Rédei symbols $[\ell_1, \ell_2, \ell_0]$ and $[\ell_0, \ell_1, \ell_2]$ are not defined in the situation of Theorem 4.7, the congruences for $c_{012}$ and $c_{201}$ imply a certain decomposition law of primes in the 2-extension $K_{(0,1)}K_{(1,2)}/\mathbb{Q}$ of degree 32.

**Remark 4.9.** In the situation of Theorem 4.7 the theorems of Mazur and Wiles (Iwasawa main conjecture, cf. [13, 27] etc.) yields the equality $\frac{1}{2}f(T)\Lambda = \Delta(T)\Lambda$ for $f(T) \in 2\Lambda$ such that $L_2(s, \chi) = f(K^{\times} - 1)$ is the Kubota-Leopoldt 2-adic $L$-function for a quadratic character $\chi$ associated to $\mathbb{Q}(\sqrt{\ell_1\ell_2})$. The construction of $f(T)$ via Stickelberger elements induces an algorithm of approximate computation of $\Delta(T)$.

On the other hand, the proof of Theorem 4.7 does not use these results.

**Theorem 4.10.** Suppose $p = 2$, and let $K = \mathbb{Q}(\sqrt{\ell_1\ell_2})$ be a real quadratic field with prime numbers $\ell_1 \equiv 7 \pmod{16}$, $\ell_2 \equiv 3 \pmod{8}$. Then $\breve{G}_{(\infty)}(K)$ has a minimal presentation

$$1 \longrightarrow \tilde{R} \longrightarrow \tilde{H} \longrightarrow \breve{G}_{(\infty)}(K) \longrightarrow 1$$

where $\tilde{H} = \langle w_{01}, w_1 \rangle$ is a free pro-2 group with two generators $w_{01}, w_1$, and $\tilde{R}$ is a normal subgroup of $\tilde{H}$ normally generated by two relations $w_{01}^2$, $[w_{01}, w_1, w_{01}]$.

**Proof.** By the assumption, $c_{01} = 0$, $c_{10} \equiv 2 \pmod{4}$, $c_{20} \equiv c_{02} = 1 \pmod{2}$ and $c_{21} = 1 - c_{12}$. Put $i = 2c_{21}$, $j = 2c_{12} \in \{0, 2\}$. Then $\Delta_i(T) \equiv 2 \pmod{(2, T)^2}$, and hence

$$2\Delta_i(T) \equiv 4 \mod (2, T)^3, \quad T\Delta_i(T) \equiv 2T \mod (2, T)^3.$$
Since \( \frac{c_0(c_0-1)}{2} \equiv 1 \pmod{2} \), and \((-1)^{c_001} = \left( \frac{-1}{2} \right)_4 = -1 \) by Propositions 3.7 and 3.8, we have \( \Delta_j(T) \equiv T^2 \pmod{(4,2T) + (2,T)^3} \), and hence \( E_0(X) + (2,T)^3 = (2,T^2) \)

by (4.8), (4.7) and Theorem 4.3. Since

\[
(2,T)^n \subset (2,T)^{n-2}(2,T^2) \subset E_0(X) + (2,T)^{n+1},
\]

one can see that \( (2,T^2) = E_0(X) + (2,T)^n \) for any \( n \geq 3 \) by induction, and hence \( E_0(X) = (2,T^2) \). Since \( X/TX \cong G_{(\infty)}(K)^{ab} \) is cyclic, \( X \) is a cyclic \( \Lambda \)-module, and hence

\[
X = G_{(\infty)}(K^{cyc})^{ab} \cong \Lambda/E_0(X) = \Lambda/(2,T^2).
\]

In particular, the commutator subgroup of \( \widetilde{G}_{(\infty)}(K) \) is \( G_{\emptyset}(K^{cyc}(\sqrt{-\ell_1})) \), and \( G_{(\infty)}(K^{cyc})^{ab} \) is an abelian group of type \([2,2]\). Since

\[
G_{\emptyset}(Q^{cyc}(\sqrt{-\ell_1}))^{ab} \cong \Lambda/T \cong \mathbb{Z}_2
\]
(cf. [3][21]), the maximal subgroup \( G_{\emptyset}(K^{cyc}(\sqrt{-\ell_1})) \) of \( G_{(\infty)}(K^{cyc}) \) has infinite abelian quotient. Therefore \( G_{(\infty)}(K^{cyc}) \) is a prodihedral pro-2 group. Then

\[
\text{Gal}((K^{cyc})_{(\infty)}/K^{cyc}(\sqrt{-\ell_1})) = G_{\emptyset}(K^{cyc}(\sqrt{-\ell_1})) \cong \mathbb{Z}_2,
\]

and hence \((K^{cyc})_{(\infty)} = KQ^{cyc}(\sqrt{-\ell_1})_\emptyset\), which is a \( \mathbb{Z}_2 \)-extension of \( K(\sqrt{-\ell_1}) \) by (4.8). Recall that the homomorphism \( \varpi \) gives isomorphisms \( \widetilde{G}_{(\infty)}(K) \cong \mathbb{H}/\mathbb{R} \) and \( \text{Gal}((K^{cyc})_{(\infty)}/\mathbb{Q}) \cong \mathbb{F}/\mathbb{R} \). Since \( w_{01}^{-1}w_{02} = [x_0, x_d] \in \mathbb{F}_2 \), we have \( \varpi(w_{01}^{-1}w_{02}) \in G_{\emptyset}(K^{cyc}(\sqrt{-\ell_1})) \), and hence \( \varpi \) induces a minimal presentation

\[
1 \longrightarrow \mathbb{H} \cap \mathbb{R} \longrightarrow \mathbb{H} \overset{\varpi}{\longrightarrow} \widetilde{G}_{(\infty)}(K) \longrightarrow 1,
\]

where \( \mathbb{H} = \langle w_{01}, w_1 \rangle \subset \mathbb{H} \). Then \( G_{(\infty)}(K^{cyc}) \) is a prodihedral pro-2 group generated by \( \varpi(w_1) \) and \( \varpi([w_{01}, w_1]) \), which has the procyclic maximal subgroup \( G_{\emptyset}(K^{cyc}(\sqrt{-\ell_1})) \) generated by \( \varpi([w_{01}, w_1]) \). This implies that

\[
w_{01}^{-1}w_{01}^{-1} [w_{01}, w_1] \equiv [w_{01}, w_1]^{-1} \pmod{\mathbb{H} \cap \mathbb{R}}.
\]

Then \( (w_{01}^{-1}w_1)^2 \equiv w_{01}^{-2} \pmod{\mathbb{H} \cap \mathbb{R}} \) for any \( z \in \mathbb{Z}_2 \), and hence \( w_{01}^{-1} \equiv 1 \pmod{\mathbb{H} \cap \mathbb{R}} \). Moreover, since the maximal subgroup \( \widehat{G}_{\emptyset}(K(\sqrt{-\ell_1})) \) of \( \widetilde{G}_{(\infty)}(K) \) is an abelian pro-2-group generated by \( \varpi([w_{01}, w_1]) \), we have \( [w_{01}, w_1, w_{01}] \in \mathbb{H} \cap \mathbb{R} \). Therefore \( \mathbb{R} \subset \mathbb{H} \cap \mathbb{R} \), i.e., there is a surjective homomorphism \( \mathbb{H}/\mathbb{R} \rightarrow \widehat{G}_{(\infty)}(K) \).

Since \( w_{01}^{-1} \in \mathbb{R} \) and

\[
[w_{01}, w_1]w_{01}^{-1}[w_{01}, w_1]w_1 = [w_01, w_1^2] \in \mathbb{R},
\]

there are surjective homomorphisms \( \mathbb{Z}_2 \oplus \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{H}/\mathbb{H}_2\mathbb{R} \) and \( \mathbb{Z}_2 \rightarrow \mathbb{H}_2\mathbb{R}/\mathbb{R} \). This implies that \( \mathbb{H}/\mathbb{R} \cong \widehat{G}_{(\infty)}(K) \), i.e., \( \mathbb{R} = \mathbb{H} \cap \mathbb{R} \). Thus the proof of Theorem 4.10 is completed. \( \square \)
Remark 4.11. In Theorem 4.10 the finiteness of $G_0(K^{cyc})^{ab}$ (Greenberg’s conjecture, cf. [12]) is certainly verified, using the same description of $\Delta(T) \mod (2, T)^3$ by $c_{ij}$ and $c_{iab}$ as in the case of Theorem 4.7. In fact, $G_0(K^{cyc}) = \{1\}$, i.e., $\tilde{G}_0(K) \simeq \mathbb{Z}_2$ in this case (cf. e.g. [11]).

Remark 4.12. In Theorem 4.7 we calculated Iwasawa polynomials from a Koch type presentation of $\tilde{G}_S(\mathbb{Q})$. Conversely, there is a case where an explicit presentation of $\tilde{G}_0(k)$ is obtained from the Iwasawa polynomial: If $p = 2$ and $k = \mathbb{Q}(-\ell_1\ell_2)$ with prime numbers $\ell_1 \equiv 7 \pmod{16}$, $\ell_2 \equiv 3 \pmod{8}$, then $\tilde{G}_0(k)$ has a minimal presentation

$$1 \longrightarrow R \longrightarrow F \longrightarrow \tilde{G}_0(k) \longrightarrow 1$$

with a free pro-2 group $F$ generated by $\{a, b, c\}$ and the normal subgroup $R$ normally generated by

$$a^2[a, b], \quad a^2[b, c, b], \quad [b, c, a], \quad [a, c], \quad [b, c]^{-C_1}[c, b, c]a^{C_1}b^{-C_0},$$

where $C_1, C_0 \in 2\mathbb{Z}_2$ are the coefficients of the Iwasawa polynomial $\Delta(T) = T^2 + C_1T + C_0$ of $X = G_0(k^{cyc})^{ab}$ (cf. [29] Theorem 2.2)).

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33
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