RAMIFIED COVERING MAPS AND STABILITY OF PULLED BACK BUNDLES

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Abstract. Let \( f : C \rightarrow D \) be a nonconstant separable morphism between irreducible smooth projective curves defined over an algebraically closed field. We say that \( f \) is genuinely ramified if \( \mathcal{O}_D \) is the maximal semistable subbundle of \( f_*\mathcal{O}_C \) (equivalently, the homomorphism \( f_* : \pi^\text{et}_1(C) \rightarrow \pi^\text{et}_1(D) \) is surjective). We prove that the pullback \( f^*E \rightarrow C \) is stable for every stable vector bundle \( E \) on \( D \) if and only if \( f \) is genuinely ramified.

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1. Introduction

Let \( k \) be an algebraically closed field. Let \( f : C \rightarrow D \) be a nonconstant separable morphism between irreducible smooth projective curves defined over \( k \). For any semistable vector bundle \( E \) on \( D \), the pullback \( f^*E \) is also semistable. However, \( f^*E \) need not be stable for every stable vector bundle \( E \) on \( D \). Our aim here is to characterize all \( f \) such that \( f^*E \) remains stable for every stable vector bundle \( E \) on \( D \). It should be mentioned that \( E \) is stable (respectively, semistable) if \( f^*E \) is stable (respectively, semistable).

For any \( f \) as above the following five conditions are equivalent:

(1) The homomorphism between étale fundamental groups

\[ f_* : \pi^\text{et}_1(C) \rightarrow \pi^\text{et}_1(D) \]

induced by \( f \) is surjective.

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(2) The map $f$ does not factor through some nontrivial étale cover of $D$ (in particular, $f$ is not nontrivial étale).
(3) The fiber product $C \times_D C$ is connected.
(4) $\dim H^0(C, f^* f_* \mathcal{O}_C) = 1$.
(5) The maximal semistable subbundle of the direct image $f_* \mathcal{O}_C$ is $\mathcal{O}_D$.

The map $f$ is called genuinely ramified if any (hence all) of the above five conditions holds. Proposition 2.6 and Definition 2.5 show that the above statements (1), (2) and (5) are equivalent; in Lemma 3.1 it is shown that the statements (3), (4) and (5) are equivalent.

We prove the following (see Theorem 5.3):

**Theorem 1.1.** Let $f : C \to D$ be a nonconstant separable morphism between irreducible smooth projective curves defined over $k$. The map $f$ is genuinely ramified if and only if $f^* E \to C$ is stable for every stable vector bundle $E$ on $D$.

The key technical step in the proof of Theorem 1.1 is the following (see Proposition 3.5):

**Proposition 1.2.** Let $f : C \to D$ be a genuinely ramified Galois morphism, of degree $d$, between irreducible smooth projective curves defined over $k$. Then

$$f^*((f_* \mathcal{O}_C)/\mathcal{O}_D) \subset \bigoplus_{i=1}^{d-1} \mathcal{L}_i,$$

where each $\mathcal{L}_i$ is a line bundle on $D$ of negative degree.

When $k = \mathbb{C}$, a vector bundle $F$ on a smooth complex projective curve is stable if and only if $F$ admits an irreducible flat projective unitary connection [NS]. From this characterization of stable vector bundles it follows immediately that given a nonconstant map $f : C \to D$ between irreducible smooth complex projective curves, $f^* E$ is stable for every stable vector bundle $E$ on $D$ if the homomorphism of topological fundamental groups induced by $f$

$$f_* : \pi_1(C, x_0) \to \pi_1(D, f(x_0))$$

is surjective.

Theorem 4.4 was stated in [PS] (it is [PS, p. 524, Lemma 3.5(b)]) without a complete proof. In the proof of Lemma 3.5(b), which is given in three sentences in [PS, p. 524], it is claimed that the socle of a semistable bundle descends under any ramified covering map (the first sentence).

## 2. Genuinely ramified morphism

The base field $k$ is assumed to be algebraically closed; there is no restriction on the characteristic of $k$.

Let $V$ be a vector bundle on an irreducible smooth projective curve $X$ defined over $k$. If

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \subset V_n = V$$
is the Harder-Narasimhan filtration of $V$ \cite[p. 16, Theorem 1.3.4]{HL}, then define
\[
\mu_{\text{max}}(V) := \mu(V_1) := \frac{\deg(V_1)}{\text{rk}(V_1)} \quad \text{and} \quad \mu_{\text{min}}(V) := \mu(V/V_{n-1}).
\]
Furthermore, the above subbundle $V_1 \subset V$ is called the maximal semistable subbundle of $V$.

If $V$ and $W$ are vector bundles on $X$, and $\beta \in H^0(X, \text{Hom}(V, W)) \setminus \{0\}$, then it can be shown that
\[
\mu_{\text{max}}(W) \geq \mu_{\text{min}}(V). \tag{2.1}
\]
Indeed, we have
\[
\mu_{\text{min}}(V) \leq \mu(\beta(V)) \leq \mu_{\text{max}}(W).
\]

**Remark 2.1.** Let $f : X \rightarrow Y$ be a nonconstant separable morphism between irreducible smooth projective curves, and let $F$ be a semistable vector bundle on $Y$. Then it is known that $f^*F$ is also semistable. Indeed, fixing a nonconstant separable morphism $h : Z \rightarrow X$, where $Z$ is an irreducible smooth projective curve and $f \circ h$ is Galois, we see that $(f \circ h)^*F$ is semistable, because its maximal semistable subbundle, being $\text{Gal}(f \circ h)$ invariant, descends to a subbundle of $F$. The semistability of $(f \circ h)^*F = h^*f^*F$ immediately implies that $f^*F$ is semistable.

**Lemma 2.2.** Let $f : X \rightarrow Y$ be a nonconstant separable morphism of irreducible smooth projective curves. Then for any semistable vector bundle $E$ on $X$,
\[
\mu_{\text{max}}(f_*E) \leq \frac{\mu(E)}{\deg(f)}. \tag{2.2}
\]
More generally, for any vector bundle $E$ on $X$,
\[
\mu_{\text{max}}(f_*E) \leq \frac{\mu_{\text{max}}(E)}{\deg(f)}. \tag{2.3}
\]

**Proof.** Let $E$ be any vector bundle on $X$. The coherent sheaf $f_*E$ on $Y$ is locally free, because it is torsion-free. We have
\[
H^0(Y, \text{Hom}(F, f_*E)) \cong H^0(X, \text{Hom}(f^*F, E)) \tag{2.2}
\]
for any vector bundle $F$ on $Y$; see \cite[p. 110]{Ha}. Setting $F$ in (2.2) to be a semistable subbundle $V$ of $f_*E$ we see that $H^0(X, \text{Hom}(f^*V, E)) \neq 0$. The pullback $f^*V$ is semistable because $V$ is semistable and $f$ is separable; see Remark 2.1.

First take $E$ to be semistable. Hence for any nonzero homomorphism $\beta : f^*V \rightarrow E$,
\[
\deg(f) \cdot \mu(V) = \mu(f^*V) \leq \mu(E) \tag{2.3}
\]
(see (2.1)). Now setting $V$ in (2.3) to be the maximal semistable subbundle of $f_*E$ we conclude that
\[
\mu_{\text{max}}(f_*E) \leq \frac{\mu(E)}{\deg(f)}. \tag{2.4}
\]
To prove the general (second) statement, for any vector bundle $E$ on $X$, let
\[
0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E
\]
be the Harder–Narasimhan filtration of $E$ [HL, p. 16, Theorem 1.3.4]. Consider the filtration of subbundles

$$
0 = f_*E_0 \subset f_*E_1 \subset \cdots \subset f_*E_{n-1} \subset f_*E_n = f_*E. \tag{2.5}
$$

From (2.4) we know that

$$
\mu_{\max}((f_*E_i)/(f_*E_{i-1})) = \mu_{\max}(f_*(E_i/E_{i-1})) \leq \mu(E_i/E_{i-1})/\deg(f)
$$

$$
\leq \mu(E_1)/\deg(f) = \mu_{\max}(E)/\deg(f) \tag{2.6}
$$

for all $1 \leq i \leq n$. Observe that using (2.1) and the filtration in (2.5) it follows that

$$
\mu_{\max}(f_*E) \leq \max\{\mu_{\max}((f_*E_i)/(f_*E_{i-1}))\}_{i=1}^n,
$$

while from (2.6) we have

$$
\max\{\mu_{\max}((f_*E_i)/(f_*E_{i-1}))\}_{i=1}^n \leq \mu_{\max}(E)/\deg(f).
$$

Therefore, $\mu_{\max}(f_*E) \leq \mu_{\max}(E)/\deg(f)$, and this completes the proof. \hfill \Box

The following lemma characterizes the étale maps among the separable morphisms.

**Lemma 2.3.** Let $f : X \to Y$ be a nonconstant separable morphism between irreducible smooth projective curves. Then the following three conditions are equivalent:

1. The map $f$ is étale.
2. The degree of $f_*O_X$ is zero.
3. The vector bundle $f_*O_X$ is semistable.

**Proof.** We have $\mu_{\max}(f_*O_X) \geq 0$, because $O_Y \subset f_*O_X$ (see (2.2) and (2.1)). On the other hand, from Lemma 2.2 it follows that $\mu_{\max}(f_*O_X) \leq 0$, so

$$
\mu_{\max}(f_*O_X) = 0. \tag{2.7}
$$

From (2.7) it follows immediately that $f_*O_X$ is semistable if and only if $\deg(f_*O_X) = 0$. Therefore, statements (2) and (3) are equivalent.

Let $R$ be the ramification divisor on $X$ for $f$; define the effective divisor $B := f_*R$ on $Y$. We know that

$$
(\det f_*O_X)^{\otimes 2} = O_Y(-B)
$$

(see [Ha, p. 306, Ch. IV, Ex. 2.6(d)], [Sc]). Therefore, $f$ is étale (meaning $B = 0$) if and only if $\deg(f_*O_X) = 0$. So statements (1) and (2) are equivalent. \hfill \Box

Let $f : X \to Y$ be a nonconstant separable morphism between irreducible smooth projective curves. The algebra structure of $O_X$ produces an $O_Y$–algebra structure on the direct image $f_*O_X$.

**Lemma 2.4.** Let $V \subset f_*O_X$ be the maximal semistable subbundle. Then $V$ is a sheaf of $O_Y$–subalgebras of $f_*O_X$.
Proof. The action of $\mathcal{O}_Y$ on $f_*\mathcal{O}_X$ is the standard one. Let
\[ m : (f_*\mathcal{O}_X) \otimes (f_*\mathcal{O}_X) \rightarrow f_*\mathcal{O}_X \]
be the $\mathcal{O}_Y$–algebra structure on the direct image $f_*\mathcal{O}_X$ given by the algebra structure of the coherent sheaf $\mathcal{O}_X$. We need to show that
\[ m(V \otimes V) \subset V, \tag{2.8} \]
where $V$ is the maximal semistable subbundle of $f_*\mathcal{O}_X$.

Since $V$ is semistable of degree zero (see (2.7)), and $\mu_{\text{max}}((f_*\mathcal{O}_X)/V) < 0$, using (2.1) we conclude that in order to prove (2.8) it suffices to show that $V \otimes V$ is semistable of degree zero. Indeed, there is no nonzero homomorphism from $V \otimes V$ to $(f_*\mathcal{O}_X)/V$, if $V \otimes V$ is semistable of degree zero.

We have $\deg(V \otimes V) = 0$, because $\deg(V) = 0$. So $V \otimes V$ is semistable if it does not contain any coherent subsheaf of positive degree. As
\[ V \otimes V \subset (f_*\mathcal{O}_X) \otimes (f_*\mathcal{O}_X), \]
if $V \otimes V$ contains a subsheaf of positive degree, then $(f_*\mathcal{O}_X) \otimes (f_*\mathcal{O}_X)$ also contains a subsheaf of positive degree.

Therefore, to prove the lemma it is enough to show that $(f_*\mathcal{O}_X) \otimes (f_*\mathcal{O}_X)$ does not contain any subsheaf of positive degree.

The projection formula, [Ha, p. 124, Ch. II, Ex. 5.1(d)], [Se], says that
\[ (f_*\mathcal{O}_X) \otimes \mathcal{O}_Y (f_*\mathcal{O}_X) \cong f_*(f^*(f_*\mathcal{O}_X)). \tag{2.9} \]
Since $\mathcal{O}_Y \subset f_*\mathcal{O}_X$, we have
\[ \mathcal{O}_Y = \mathcal{O}_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_Y \subset (f_*\mathcal{O}_X) \otimes_{\mathcal{O}_Y} (f_*\mathcal{O}_X), \]
and hence $\mu_{\text{max}}((f_*\mathcal{O}_X) \otimes_{\mathcal{O}_Y} (f_*\mathcal{O}_X)) \geq 0$. Now from (2.9) it follows that
\[ \mu_{\text{max}}(f^*(f_*\mathcal{O}_X)) \geq 0. \tag{2.10} \]

Since $f$ is separable, the pullback, by $f$, of a semistable bundle on $Y$ is semistable (see Remark [2.1]), and consequently the Harder–Narasimhan filtration of $f^*F$ is the pullback, by $f$, of the Harder–Narasimhan filtration of $F$. Therefore, from (2.7) it follows that
\[ \mu_{\text{max}}(f^*(f_*\mathcal{O}_X)) = 0. \]
Now applying the second part of Lemma [2.2]
\[ 0 = \mu_{\text{max}}(f^*(f_*\mathcal{O}_X))/\deg(f) \geq \mu_{\text{max}}(f^*(f_*\mathcal{O}_X)). \]
This and (2.10) together imply that
\[ \mu_{\text{max}}(f^*(f_*\mathcal{O}_X)) = 0. \]
Therefore, using (2.9) it follows that
\[ \mu_{\text{max}}((f_*\mathcal{O}_X) \otimes (f_*\mathcal{O}_X)) = 0. \]
Hence $(f_*\mathcal{O}_X) \otimes (f_*\mathcal{O}_X)$ does not contain any subsheaf of positive degree. It was shown earlier that the lemma follows from the statement that $(f_*\mathcal{O}_X) \otimes (f_*\mathcal{O}_X)$ does not contain any subsheaf of positive degree. $\square$
**Definition 2.5.** A nonconstant separable morphism \( f : X \to Y \) between irreducible smooth projective curves is called *genuinely ramified* if \( \mathcal{O}_Y \) is the maximal semistable subbundle of \( f_*\mathcal{O}_X \).

**Proposition 2.6.** Let \( f : X \to Y \) be a nonconstant separable morphism between irreducible smooth projective curves. Then the following three conditions are equivalent:

1. The map \( f \) is genuinely ramified.
2. The map \( f \) does not factor through any nontrivial étale cover of \( Y \) (in particular, \( f \) is not nontrivial étale).
3. The homomorphism between étale fundamental groups induced by \( f \)

\[
\pi^\text{et}_1(X) \to \pi^\text{et}_1(Y)
\]

is surjective.

*Proof.* (1) \( \Rightarrow \) (2): If \( f \) factors through a nontrivial étale covering \( g : \tilde{Y} \to Y \), then \( g_*\mathcal{O}_{\tilde{Y}} \) is semistable of degree zero (see Lemma 2.3) and its rank coincides with the degree of \( g \). Since

\[
g_*\mathcal{O}_{\tilde{Y}} \subset f_*\mathcal{O}_X,
\]

this implies that \( f \) is not genuinely ramified.

(2) \( \Rightarrow \) (1): Lemma 2.4 says that the maximal semistable subbundle \( \mathcal{V} \subset f_*\mathcal{O}_X \) is a subalgebra. If \( f \) is not genuinely ramified, then by taking the spectrum of \( \mathcal{V} \) we obtain a separable, possibly ramified, covering map

\[
g : \tilde{Y} = \text{Spec} \mathcal{V} \to Y
\]

whose degree coincides with the rank of \( \mathcal{V} \). We have \( g_*\mathcal{O}_{\tilde{Y}} = \mathcal{V} \), and the inclusion map \( \mathcal{V} \hookrightarrow f_*\mathcal{O}_X \) defines a map

\[
h : X \to \tilde{Y}
\]

such that

\[
g \circ h = f.
\]

Since \( f \) is separable, from (2.12) it follows that \( g \) is also separable. It can be shown that \( g \) is étale. To prove this, first note that \( g_*\mathcal{O}_{\tilde{Y}} \) is semistable, because \( g_*\mathcal{O}_{\tilde{Y}} = \mathcal{V} \) and \( \mathcal{V} \) is semistable. Next, from (2.7) and the semistability of \( \mathcal{V} \) it follows that \( \mu(g_*\mathcal{O}_{\tilde{Y}}) = \mu_{\text{max}}(g_*\mathcal{O}_{\tilde{Y}}) = 0 \). Now Lemma 2.3 gives that the map \( g \) in (2.11) is étale.

Since \( g \) is étale, and (2.12) holds, we conclude that the statement (2) fails. Hence the statement (2) implies the statement (1).

The equivalence between the statements (2) and (3) follows from the definition of the étale fundamental group.

\( \square \)

Let \( f : X \to Y \) be a nonconstant separable morphism between irreducible smooth projective curves. Let

\[
g : \tilde{Y} := \text{Spec} \mathcal{V} \to Y
\]
be the étale covering corresponding to the maximal semistable subbundle $\mathcal{V} \subset f_*\mathcal{O}_X$ (see (2.11); it was shown that the map in (2.11) is étale). Let

$$h: X \to \tilde{Y}$$  \hspace{1cm} (2.13)

be the morphism given by the inclusion map $\mathcal{V} \hookrightarrow f_*\mathcal{O}_X$.

**Corollary 2.7.** The map $h$ in (2.13) is genuinely ramified.

**Proof.** Let $\beta: Z \to \tilde{Y}$ be an étale covering such that there is a map

$$\gamma: X \to Z$$

satisfying the condition $\beta \circ \gamma = h$. Since $(g \circ \beta) \circ \gamma = f$, we have

$$g_*\mathcal{O}_Y \subset (g \circ \beta)_*\mathcal{O}_Z \subset f_*\mathcal{O}_X;$$  \hspace{1cm} (2.14)

also, we have $\deg((g \circ \beta)_*\mathcal{O}_Y) = 0$, because $g \circ \beta$ is étale (see Lemma 2.3). But $\mathcal{V} = g_*\mathcal{O}_Y$ is the maximal semistable subsheaf of $f_*\mathcal{O}_X$. Hence from (2.14) it follows that $g_*\mathcal{O}_Y = (g \circ \beta)_*\mathcal{O}_Z$. This implies that $\deg(\beta) = 1$. Therefore, from Proposition 2.6 we conclude that the map $h$ in (2.13) is genuinely ramified. \hfill \Box

## 3. Properties of genuinely ramified morphisms

**Lemma 3.1.** Let $f: C \to D$ be a nonconstant separable morphism between irreducible smooth projective curves. Then the following three conditions are equivalent:

1. The map $f$ is genuinely ramified.
2. $\dim H^0(C, f^*f_*\mathcal{O}_C) = 1$.
3. The fiber product $C \times_D C$ is connected.

**Proof.** Let $\tilde{C} \times_D C$ be the normalization of the fiber product $C \times_D C$; it is a smooth projective curve, but it is not connected unless $f$ is an isomorphism. We have the commutative diagram

\[ \begin{array}{ccc} 
\tilde{C} \times_D C & \xrightarrow{\nu} & C \times_D C \\
\pi_2 & & \pi_2 \\
\pi_1 & & f \\
C & & D \\
\end{array} \]

By flat base change [Ha, p. 255, Proposition 9.3],

$$f^*(f_*\mathcal{O}_C) \cong \pi_1^*(\pi_2^*\mathcal{O}_C) = \pi_1^*\mathcal{O}_{C \times_D C}.$$  \hspace{1cm} (3.2)

(1) $\implies$ (2): Since $f$ is separable, $f^*F$ is semistable if $F$ is so (see Remark 2.1), and hence the maximal semistable subbundle of $f^*f_*\mathcal{O}_C$ is $f^*\mathcal{V}$, where $\mathcal{V} \subset f_*\mathcal{O}_C$ is the maximal semistable subbundle. If $f$ is genuinely ramified, then the maximal semistable subbundle of $f^*f_*\mathcal{O}_C$ is $f^*\mathcal{O}_D = \mathcal{O}_C$. On the other hand,

$$H^0(C, (f^*f_*\mathcal{O}_C)/(f^*\mathcal{V})) = 0,$$
because $\mu_{\max}((f^*f_*\mathcal{O}_C)/(f^*\mathcal{V})) < 0$ (see (2.1)). These together imply that
\[ \dim H^0(C, f^*f_*\mathcal{O}_C) = 1; \]
to see this consider the long exact sequence of cohomologies associated to the short exact sequence
\[ 0 \rightarrow f^*\mathcal{V} \rightarrow f^*f_*\mathcal{O}_C \rightarrow (f^*f_*\mathcal{O}_C)/(f^*\mathcal{V}) \rightarrow 0. \]

(2) $\iff$ (3): From (3.2) it follows that
\[ H^0(C, f^*f_*\mathcal{O}_C) = H^0(C, \pi_1^*\mathcal{O}_{C \times D_C}) = H^0(C \times D_C, \mathcal{O}_{C \times D_C}). \tag{3.3} \]
Consequently, $C \times_D C$ is connected if and only if $\dim H^0(C, f^*f_*\mathcal{O}_C) = 1$.

(3) $\implies$ (1): Assume that $f$ is not genuinely ramified. We will prove that $C \times_D C$ is not connected.
Let $g : \tilde{D} \rightarrow D$ be the étale cover of $D$ given by Spec $\mathcal{W}$, where $\mathcal{W} \subset f_*\mathcal{O}_C$ is the maximal semistable subbundle (as in (2.11)). The degree of this covering $g$ is at least two, because $f$ is not genuinely ramified. To prove that $C \times_D C$ is not connected it suffices to show that $\tilde{D} \times_D \tilde{D}$ is not connected.

The projection
\[ \gamma : \tilde{D} \times_D \tilde{D} \rightarrow \tilde{D} \]
to the first factor is evidently the base change of $g : \tilde{D} \rightarrow D$ to $\tilde{D}$, and hence the map $\gamma$ is étale. The diagonal $\tilde{D} \hookrightarrow \tilde{D} \times_D \tilde{D}$ is a connected component of $\tilde{D} \times_D \tilde{D}$. This implies that $\tilde{D} \times_D \tilde{D}$ is not connected, because the degree of $\gamma$ is at least two. $\square$

**Definition 3.2.** A nonconstant morphism $f : C \rightarrow D$ between irreducible smooth projective curves will be called a *separable Galois morphism* if $f$ is separable, and there is a reduced finite subgroup $\Gamma \subset \text{Aut}(C)$ such that $D = C/\Gamma$ and $f$ is the quotient map $C \rightarrow C/\Gamma$. Note that a separable Galois morphism need not be étale. A separable Galois morphism which is genuinely ramified will be called a *genuinely ramified Galois morphism*.

**Proposition 3.3.** Let $f : C \rightarrow D$ be a separable Galois morphism, of degree $d$, between irreducible smooth projective curves. Then $f^*((f_*\mathcal{O}_C)/\mathcal{O}_D)$ is a coherent subsheaf of $\mathcal{O}_C^\oplus(d-1)$.

**Proof.** The Galois group $\text{Gal}(f)$ of $f$ will be denoted by $\Gamma$. For any point $x \in C$, let
\[ \Gamma_x \subset \Gamma \]
be the isotropy subgroup that fixes $x$ for the action of $\Gamma$ on $C$. A point $(x, y) \in C \times_D C$ is singular if and only if $\Gamma_x$ is nontrivial. Note that for any $(x, y) \in C \times_D C$ the two isotropy subgroups $\Gamma_x$ and $\Gamma_y$ are conjugate, because $y$ lies in the orbit $\Gamma \cdot x$ of $x$. For any $\sigma \in \Gamma$, let
\[ C_{\sigma} \subset C \times_D C \tag{3.4} \]
be the irreducible component given by the image of the map
\[ \beta_{\sigma} : C \rightarrow C \times C, \quad x \mapsto (x, \sigma(x)); \]
clearly we have $\beta_{\sigma}(C) \subset C \times_D C$. In this way, the irreducible components of $C \times_D C$ are parametrized by the elements of the Galois group $\Gamma$. Note that there is a canonical identification

$$C \sim \rightarrow C_{\sigma}$$

(3.5)

for every $\sigma \in \Gamma$.

Let $\widehat{C \times_D C}$ be the normalization of $C \times_D C$. The maps $\beta_{\sigma}$, $\sigma \in \Gamma$, in (3.4) together produce an isomorphism

$$C \times \Gamma \sim \rightarrow \widehat{C \times_D C};$$

(3.6)

this map sends any $(y, \sigma) \in C \times \Gamma$ to $(y, \sigma(y))$ if $\Gamma y$ is trivial; if $\Gamma y$ is trivial, then $(y, \sigma(y))$ is a smooth point of $C \times_D C$ and hence $(y, \sigma(y))$ gives a unique point of $\widehat{C \times_D C}$. Consequently, we have

$$\tilde{\pi}_1^* O_{\widehat{C \times_D C}} = O_C \otimes_k k[\Gamma],$$

(3.7)

where $\tilde{\pi}_1$ is the projection in (3.1), and $k[\Gamma]$ is the group ring. The natural inclusion $O_{\widehat{C \times_D C}} \hookrightarrow \nu_* \widetilde{O}_{\widehat{C \times_D C}}$, where $\nu$ is the map in (3.1), induces an injective homomorphism

$$\varphi : \pi_1^* O_{C \times_D C} \hookrightarrow \pi_1^* \nu_* \widetilde{O}_{C \times_D C} = \tilde{\pi}_1^* \widetilde{O}_{\widehat{C \times_D C}};$$

(3.8)

where $\pi_1$ and $\tilde{\pi}_1$ are the maps in (3.1).

Let

$$\xi : O_C \longrightarrow O_C \otimes_k k[\Gamma]$$

(3.9)

be the composition of homomorphisms

$$O_C \longrightarrow \pi_1^* O_{C \times_D C} \overset{\varphi}{\longrightarrow} \tilde{\pi}_1^* \widetilde{O}_{\widehat{C \times_D C}} = O_C \otimes_k k[\Gamma]$$

(see (3.8) and (3.7)). Note that the image $\xi(O_C)$ in (3.9) is a subbundle of $O_C \otimes_k k[\Gamma]$, because the section

$$\xi(1_C) \subset \text{H}^0(C, \tilde{\pi}_1^* \widetilde{O}_{\widehat{C \times_D C}}) = k[\Gamma]$$

is nowhere vanishing, where $1_C$ is the constant function 1 on $C$. There is a trivial subbundle $\mathcal{E}$ of the trivial bundle $O_C \otimes_k k[\Gamma]$

$$O_C^{\oplus(d-1)} = \mathcal{E} \subset O_C \otimes_k k[\Gamma]$$

(3.10)

such that

$$\mathcal{E} \oplus \xi(O_C) = O_C \otimes_k k[\Gamma].$$

To see this, take any point $x \in C$, and choose a subspace

$$\mathcal{E}_x \subset (O_C \otimes_k k[\Gamma])_x = k[\Gamma]$$

such that $k[\Gamma] = \mathcal{E}_x \oplus \xi(O_C)_x$; then take

$$\mathcal{E} := O_C \otimes_k \mathcal{E}_x \subset O_C \otimes_k k[\Gamma].$$

This subbundle $\mathcal{E}$ clearly satisfies the condition in (3.10).

From the decomposition in (3.10) we conclude that $(O_C \otimes_k k[\Gamma]) / \xi(O_C) = \mathcal{E}$. Using the isomorphism in (3.7), the homomorphism $\varphi$ in (3.8) gives a homomorphism

$$\varphi' : (\pi_1^* O_{C \times_D C}) / O_C \longrightarrow (O_C \otimes_k k[\Gamma]) / (\varphi(O_C)) = (O_C \otimes_k k[\Gamma]) / (\xi(O_C)) = \mathcal{E}.$$
On the other hand, the isomorphism in (3.2) produces an isomorphism

\[(\pi_{1*}\mathcal{O}_{C \times_D C})/\mathcal{O}_C \cong f^*((f_*\mathcal{O}_C)/\mathcal{O}_D).\]

Combining this isomorphism with the homomorphism \(\varphi'\) in (3.11) we get a homomorphism

\[f^*((f_*\mathcal{O}_C)/\mathcal{O}_D) \rightarrow \mathcal{E} = \mathcal{O}_C^\oplus(d_{-1}).\]

This homomorphism is clearly an isomorphism over the nonempty open subset of \(C\) where \(f\) is étale. \(\square\)

Note that \(f^*((f_*\mathcal{O}_C)/\mathcal{O}_D) = (f^*f_*\mathcal{O}_C)/\mathcal{O}_C;\) but we use \(f^*((f_*\mathcal{O}_C)/\mathcal{O}_D)\) due to the relevance of \((f_*\mathcal{O}_C)/\mathcal{O}_D)\).

Let \(f : C \rightarrow D\) be a genuinely ramified Galois morphism, of degree \(d\), between irreducible smooth projective curves; see Definition 3.2. As before, the Galois group \(\text{Gal}(f)\) will be denoted by \(\Gamma\), so we have

\[\#\Gamma = d.\]

Assume that \(d > 1.\)

As in (3.4), the irreducible component of \(C \times_D C\) corresponding to \(\sigma \in \Gamma\) will be denoted by \(C_\sigma.\)

The following lemma formulated in the above set-up will be used in proving a variation of Proposition 3.3.

**Lemma 3.4.** There is an ordering of the elements of \(\Gamma\)

\[\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_d\}\]

and a self-map

\[\eta : \{1, 2, \cdots, d\} \rightarrow \{1, 2, \cdots, d\}\]

such that

1. \(\gamma_1 = e\) (the identity element of \(\Gamma\)),
2. \(\eta(1) = 1,\)
3. \(\eta(j) < j\) for all \(j \in \{2, \cdots, d\}\), and
4. \(C_\gamma \cap C_{\eta(j)} \neq \emptyset\) (see (3.4) for notation).

**Proof.** Set \(\Gamma_0 := \gamma_1\) to be the identity element \(e \in \Gamma\); also, set \(\eta(1) = 1\). Set \(N_0 = 1\).

Let

\[\Gamma_1 \subset \Gamma\]

be the subset consisting of all \(\gamma \neq e\) such that the action of \(\gamma\) on \(C\) has a fixed point. Therefore, \(\Gamma_1\) consists of all \(\gamma \neq e\) such that the irreducible component \(C_\gamma \subset C \times_D C\) intersects the component \(C_e = C_{\gamma_1}\). We note that \(\Gamma_1\) is nonempty, because otherwise \(C_{\gamma_1}\) would be a connected component of \(C \times_D C\), while from Lemma 3.1(3) we know that \(C \times_D C\) is connected; recall that \(\Gamma \neq \{e\}\) and \(f\) is genuinely ramified.
If \( \#\Gamma_1 = N_1 - 1 = N_1 - N_0 \), set \( \gamma_j \in \Gamma \), \( 2 \leq j \leq N_1 \), to be distinct elements of \( \Gamma_1 \) in an arbitrary order. Set

\[ \eta(j) = 1 \]

for all \( 2 \leq j \leq N_1 \).

If \( \Gamma_1 \cup \Gamma_0 \neq \Gamma \), let

\[ \Gamma_2 \subset \Gamma \setminus (\Gamma_1 \cup \Gamma_0) \]

be the subset consisting of all \( \gamma \in \Gamma \setminus (\Gamma_1 \cup \Gamma_0) \) such that the irreducible component

\[ C_\gamma \subset C \times_D C \]

intersects the component \( C_\sigma \) for some \( \sigma \in \Gamma \). Note that such a component \( C_\gamma \) does not intersect \( \bigcup_{i=0}^{n-1} \Gamma_i \), because in that case we would have \( \gamma \in \Gamma_1 \).

If \( \#\Gamma_2 = N_2 - N_1 \), set \( \gamma_j \in \Gamma \), \( N_1 + 1 \leq j \leq N_2 \), to be distinct elements of \( \Gamma_2 \) in an arbitrary order. For every \( N_1 + 1 \leq j \leq N_2 \), set

\[ \eta(j) \in \{2, \cdots, N_1\} \]

such that the component \( C_{\gamma_j} \subset C \times D C \) intersects the component \( C_{\gamma_{\eta(j)}} \); the above definition of \( \Gamma_2 \) ensures that such a \( \eta(j) \) exists. If there are more than one \( m \in \{2, \cdots, N_1\} \) such that \( C_{\gamma_j} \subset C \times D C \) intersects the component \( C_{\gamma_m} \), then choose \( \eta(j) \) arbitrarily from them.

Now inductively define

\[ \Gamma_n \subset \Gamma \setminus \bigcup_{i=0}^{n-1} \Gamma_i \],

if \( \Gamma_n \neq \emptyset \), to be the subset consisting of all \( \gamma \in \Gamma \setminus \bigcup_{i=0}^{n-1} \Gamma_i \) such that the irreducible component \( C_\gamma \subset C \times_D C \) intersects the component \( C_\sigma \) for some \( \sigma \in \Gamma_{n-1} \). Note that such a component \( C_\gamma \) does not intersect \( \bigcup_{i=0}^{n-2} \Gamma_i \), because in that case \( \gamma \in \bigcup_{i=0}^{n-1} \Gamma_i \).

If \( \#\Gamma_n = N_n - \sum_{i=0}^{n-1} \#\Gamma_i = N_n - N_{n-1} \), set \( \gamma_j \in \Gamma \), \( N_{n-1} + 1 \leq j \leq N_n \), to be distinct elements of \( \Gamma_n \) in an arbitrary order. For \( N_{n-1} + 1 \leq j \leq N_n \), set

\[ \eta(j) \in [N_{n-2} + 1, N_{n-1}] = \{1 + \sum_{i=0}^{n-2} \#\Gamma_i, \cdots, \sum_{i=0}^{n-1} \#\Gamma_i\} \]

such that the component \( C_{\gamma_j} \subset C \times_D C \) intersects the component \( C_{\gamma_{\eta(j)}} \). If \( C_{\gamma_j} \) intersects more than one such component, choose \( \eta(j) \) to be any one from them, as before.

Since \( \Gamma \) is a finite group, we have \( \Gamma_n = \emptyset \) for all \( n \) sufficiently large. Set

\[ S = \sum_{i=0}^{\infty} \#\Gamma_i = \max_{i \geq 0}\{N_i\} \]

Note that

\[ \bigcup_{i=1}^{S} C_{\gamma_i} \subset C \times D C \]
is the connected component of $C \times_D C$ containing $C_{\gamma_1}$. Hence from Lemma \[3.1(3)\] we know that

$$\bigcup_{i=1}^{s} C_{\gamma_i} = C \times_D C.$$ 

In other words, we have $S = d := \# \Gamma$. This completes the proof of the lemma. $\square$

**Proposition 3.5.** Let $f : C \longrightarrow D$ be a genuinely ramified Galois morphism, of degree $d$, between irreducible smooth projective curves. Then

$$f^*((f_* \mathcal{O}_C)/\mathcal{O}_D) \subset \bigoplus_{i=1}^{d-1} \mathcal{L}_i,$$

where each $\mathcal{L}_i$ is a line bundle on $D$ of negative degree.

**Proof.** As in Lemma \[3.4\] the Galois group $\text{Gal}(f)$ is denoted by $\Gamma$. The ordering in Lemma \[3.4\] of the elements of $\Gamma$ produces an isomorphism of $k[\Gamma]$ with $k^{\oplus d}$. Consequently, from \[3.7\] we have

$$\tilde{\pi}_1_* \mathcal{O}_{\tilde{C} \times_D \tilde{C}} = \mathcal{O}_C \otimes_k k[\Gamma] = \mathcal{O}_C^{\oplus d}. \quad (3.12)$$

Let

$$\Phi : \tilde{\pi}_1_* \mathcal{O}_{\tilde{C} \times_D \tilde{C}} = \mathcal{O}_C^{\oplus d} \longrightarrow \mathcal{O}_C^{\oplus d} = \tilde{\pi}_1_* \mathcal{O}_{\tilde{C} \times_D \tilde{C}}$$

be the homomorphism defined by

$$(f_1, f_2, \cdots, f_d) \mapsto (f_1 - f_{\eta(1)}, f_2 - f_{\eta(2)}, \cdots, f_d - f_{\eta(d)}), \quad (3.13)$$

where $\eta$ is the map in Lemma \[3.4\] more precisely, the $i$-th component of $\Phi(f_1, f_2, \cdots, f_d)$ is $f_i - f_{\eta(i)}$. It is straightforward to check that

$$\mathcal{F} := \Phi(\mathcal{O}_C^{\oplus d}) \subset \mathcal{O}_C^{\oplus d} = \tilde{\pi}_1_* \mathcal{O}_{\tilde{C} \times_D \tilde{C}}, \quad (3.14)$$

is a trivial subbundle of rank $d - 1$; the first component of $\Phi(f_1, f_2, \cdots, f_d)$ vanishes identically, because $\eta(1) = 1$. More precisely, we have

$$\mathcal{F} = \mathcal{O}_C^{\oplus(d-1)} \subset \mathcal{O}_C^{\oplus d} = \tilde{\pi}_1_* \mathcal{O}_{\tilde{C} \times_D \tilde{C}},$$

where $\mathcal{O}_C^{\oplus(d-1)}$ is the subbundle of $\mathcal{O}_C^{\oplus d}$ spanned by all $(f_1, f_2, \cdots, f_d)$ such that $f_1 = 0$.

From \[3.14\] it follows immediately that

$$\tilde{\pi}_1_* \mathcal{O}_{\tilde{C} \times_D \tilde{C}} = \mathcal{O}_C^{\oplus d} = \mathcal{F} \oplus \xi(\mathcal{O}_C), \quad (3.15)$$

where $\xi(\mathcal{O}_C)$ is the subbundle of $\mathcal{O}_C^{\oplus d} = \mathcal{O}_C \otimes_k k[\Gamma]$ in \[3.9\] (see \[3.12\]).

In \[3.1\] we have $\pi_1 = \pi_1 \circ \nu$, and hence, as in \[3.8\], there is a natural homomorphism

$$\varphi : \pi_1_* \mathcal{O}_{C \times D} \hookrightarrow \tilde{\pi}_1_* \mathcal{O}_{\tilde{C} \times D}, \quad (3.16)$$

which is an isomorphism over the open subset of $C$ where $f$ is étale. Therefore, from \[3.2\] and \[3.15\] we get an injective homomorphism of coherent sheaves

$$\Psi : f^*((f_* \mathcal{O}_C)/\mathcal{O}_D) \longrightarrow \mathcal{F} = \mathcal{O}_C^{\oplus(d-1)}; \quad (3.17)$$

it is similar to \[3.11\], except that now the direct summand $\mathcal{F}$ is chosen carefully (it was $\mathcal{E}$ in \[3.11\]). Note that since $\text{rk}(f^*((f_* \mathcal{O}_C)/\mathcal{O}_D)) = d - 1 = \text{rk}(\mathcal{O}_C^{\oplus(d-1)})$, the homomorphism
\(\Psi\) in (3.17) is generically an isomorphism, because it is an injective homomorphism of coherent sheaves. More precisely, \(\Psi\) is an isomorphism over the open subset of \(C\) where the map \(f\) is étale.

Consider the map \(\eta\) in Lemma 3.4. For every \(1 \leq i \leq d - 1\), choose a point
\[
z_i \in C_{\gamma_{i+1}} \cap C_{\tilde{\gamma}_{i}} ;
\]  
(3.18)
this is possible because the fourth property in Lemma 3.4 says that the intersection \(C_{\gamma_{i+1}} \cap C_{\tilde{\gamma}_{i}}\) is nonempty. Recall from (3.5) that \(C\) is identified with \(C_{\gamma_{i+1}}\). The point \(z_i \in C_{\gamma_{i+1}}\) in (3.18) will be considered as a point of \(C\) using this identification. Let
\[
L_i := \mathcal{O}_C(-z_i)
\]
be the line bundle corresponding to the point \(z_i \in C\).

For every \(1 \leq i \leq d - 1\), let
\[
P_i : \mathcal{O}_C^{(d-1)} \longrightarrow \mathcal{O}_C
\]  
(3.19)
be the natural projection to the \(i\)-th factor.

Consider the composition of homomorphisms \(P_i \circ \Psi\), where \(P_i\) and \(\Psi\) are constructed in (3.13) and (3.17) respectively. It can be shown that \(P_i \circ \Psi\) vanishes when restricted to the point \(z_i\) in (3.18). To see this, for any \(1 \leq j \leq d\), let
\[
\tilde{P}_j : \mathcal{O}_C^{(d-1)} \longrightarrow \mathcal{O}_C
\]
be the natural projection to the \(j\)-th factor. Recall the homomorphism \(\Phi\) constructed in (3.13). If \((f_1, f_2, \cdots, f_d)\) in (3.13) actually lies in the image of \(\pi_* \mathcal{O}_{C \times_D C}\) by the inclusion map \(\varphi\) in (3.16), then from (3.18) we have
\[
(\tilde{P}_{i+1} \circ \Phi)(f_1, f_2, \cdots, f_d)(z_i, \gamma_{i+1}) = f_{i+1}(z_i, \gamma) - f_{\tilde{\gamma}_{i+1}}(z_i, \tilde{\gamma}) = 0,
\]  
(3.20)
where \((z_i, \gamma_{i+1}) \in C \times \Gamma = C \times_D C\) (see (3.6)) and the same for \((z_i, \tilde{\gamma}_{i+1})\); note that from (3.18) it follows that the point in \(C\) corresponding to \(z_i \in C_{\gamma_{i+1}}\) (see (3.18)) by the identification \(C \sim C_{\gamma_{i+1}}\) in (3.5) coincides with the point corresponding to \(z_i \in C_{\gamma_{i+1}}\) (the element \(\gamma_{i+1}^{-1} \gamma_{i+1} \in \Gamma\) fixes this point of \(C\)). To clarify, there is a slight abuse of notation in (3.5) in the following sense: sections of \(\tilde{\pi}_* \mathcal{O}_{C \times_D C}\) over an open subset \(U \subset C\) are identified with function on \(\tilde{\pi}^{-1}(U)\). So \((f_1, f_2, \cdots, f_d)\) in (3.20) is considered as a function on \(\tilde{\pi}^{-1}(U)\); the above condition that \((f_1, f_2, \cdots, f_d)\) in (3.20) lies in the image of \(\pi_* \mathcal{O}_{C \times_D C}\) by the inclusion \(\varphi\) map in (3.16) means that \((f_1, f_2, \cdots, f_d)\) coincides with \(f \circ \nu\) for some function \(f\) on \(\pi_1^{-1}(U)\), where \(\nu\) is the map in (3.1). Now from (3.20) it follows that \(P_i \circ \Psi\) vanishes when restricted to the point \(z_i \in C\).

Since \(P_i \circ \Psi\) vanishes when restricted to the point \(z_i\), we have
\[
P_i \circ \Psi(f^*((f_* \mathcal{O}_C)/\mathcal{O}_D)) \subset L_i = \mathcal{O}_C(-z_i) \subset \mathcal{O}_C.
\]  
(3.21)
From (3.17) and (3.21) it follows immediately that
\[
f^*((f_* \mathcal{O}_C)/\mathcal{O}_D) \hookrightarrow \bigoplus_{i=1}^{d-1} L_i.
\]
Since \(\deg(L_i) = -1\), the proof of the proposition is complete. \(\square\)
4. Pullback of stable bundles and genuinely ramified maps

**Lemma 4.1.** Let \( f : C \to D \) be a genuinely ramified morphism between irreducible smooth projective curves. Let \( V \) be a semistable vector bundle on \( D \). Then

\[
\mu_{\text{max}}(V \otimes ((f_*\mathcal{O}_C)/\mathcal{O}_D)) < \mu(V).
\]

**Proof.** First assume that the map \( f \) is Galois. Take the line bundles \( \mathcal{L}_i, 1 \leq i \leq d - 1 \), in Proposition 3.5, where \( d = \deg(f) \). Then from Proposition 3.5 we have

\[
\mu_{\text{max}}(V \otimes ((f_*\mathcal{O}_C)/\mathcal{O}_D)) \leq \mu_{\text{max}}(V \otimes (\bigoplus_{i=1}^{d-1} \mathcal{L}_i)) \leq \max\{\mu(V \otimes \mathcal{L}_i)\}_i^{d-1},
\]

because \( V \otimes \mathcal{L}_i \) is semistable. On the other hand,

\[
\mu(V \otimes \mathcal{L}_i) < \mu(V),
\]

because \( \deg(\mathcal{L}_i) < 0 \). Combining these, we have

\[
\mu_{\text{max}}(V \otimes ((f_*\mathcal{O}_C)/\mathcal{O}_D)) \otimes \mathcal{L}_i < \mu(V),
\]

giving the statement of the proposition.

If the map \( f \) is not Galois, consider the smallest Galois extension

\( F : \hat{C} \to D \) (4.1)

such that there is a morphism \( \hat{f} : \hat{C} \to C \) for which

\( f \circ \hat{f} = F \). (4.2)

Note that \( \hat{C} \) is irreducible and smooth, and \( F \) is separable. From (4.2) it follows that

\( f_*\mathcal{O}_C \subset F_*\hat{\mathcal{O}}_{\hat{C}}. \) (4.3)

First assume that the map \( F \) in (4.1) is genuinely ramified. From (4.3) it follows that

\( (f_*\mathcal{O}_C)/\mathcal{O}_D \subset (F_*\hat{\mathcal{O}}_{\hat{C}})/\mathcal{O}_D. \) (4.4)

Since \( F \) is Galois, from Proposition 3.5 we know that \( (F_*\hat{\mathcal{O}}_{\hat{C}})/\mathcal{O}_D \) is contained in a direct sum of line bundles of negative degree. Hence the subsheaf \( (f_*\mathcal{O}_C)/\mathcal{O}_D \) in (4.4) is also contained in a direct sum of line bundles of negative degree. This implies that

\[
\mu_{\text{max}}(V \otimes ((f_*\mathcal{O}_C)/\mathcal{O}_D)) \otimes \mathcal{L}_i < \mu(V),
\]

giving the statement of the proposition.

Therefore, we now assume that \( F \) is not genuinely ramified. Let

\( (F_*\hat{\mathcal{O}}_{\hat{C}})_1 \subset F_*\hat{\mathcal{O}}_{\hat{C}} \)

be the maximal semistable subbundle. Let

\( g : \hat{\mathcal{D}} \to D \) (4.5)

be the étale cover defined by the spectrum of the bundle \( (F_*\hat{\mathcal{O}}_{\hat{C}})_1 \) of \( \mathcal{O}_{\mathcal{D}} \)-algebras (see Lemma 2.4); that the map \( g \) in (4.5) is étale follows from Lemma 2.3 and (2.7), because

\( g_*\mathcal{O}_{\hat{\mathcal{D}}} = (F_*\hat{\mathcal{O}}_{\hat{C}})_1 \) and \( (F_*\hat{\mathcal{O}}_{\hat{C}})_1 \) is semistable. We note that the Galois group \( \text{Gal}(F) \) for \( F \) acts naturally on \( F_*\hat{\mathcal{O}}_{\hat{C}} \), and this action of \( \text{Gal}(F) \) preserves the subbundle \( (F_*\hat{\mathcal{O}}_{\hat{C}})_1 \);
indeed, this follows from the uniqueness of the maximal semistable subbundle \((F_*\mathcal{O}_\hat{C})_1\). Therefore, \(\text{Gal}(F)\) acts on \(\hat{D}\), and the map \(g\) in (4.5) is \(\text{Gal}(F)\)–equivariant for the trivial action of \(\text{Gal}(F)\) on \(\hat{D}\). Consequently, the covering \(g\) in (4.5) is Galois.

Consider the following commutative diagram

\[
\begin{array}{ccc}
\hat{C} & \xrightarrow{\hat{g}} & \hat{D} \\
\downarrow{\hat{h}} & & \downarrow{\pi_2} \\
C \times_D \hat{D} & \xrightarrow{\pi_1} & \hat{D} \\
\downarrow{\pi_1} & & \downarrow{g} \\
C & \xrightarrow{f} & D
\end{array}
\]

(4.6)

The existence of the map \(h\) in (4.6) is evident. The map \(f\) being genuinely ramified, it follows from Lemma 2.6 that the homomorphism between étale fundamental groups

\[
f_* : \pi_1^{\text{et}}(C) \longrightarrow \pi_1^{\text{et}}(D)
\]

induced by \(f\) is surjective. This implies that the fiber product \(C \times_D \hat{D}\) is connected. The diagram in (4.6) should not be confused with the one in (3.1) — in (4.6), \(C \times_D \hat{D}\) is smooth as \(g\) is étale.

We will prove that the map \(\hat{g}\) in (4.6) is genuinely ramified and Galois. For this, first recall the earlier observation that \(\text{Gal}(F)\) acts on \(\hat{D}\). The map \(\hat{g}\) is evidently equivariant for the actions of \(\text{Gal}(F)\) on \(\hat{C}\) and \(\hat{D}\). This immediately implies that the map \(\hat{g}\) is Galois.

From Corollary 2.7 it follows that \(\hat{g}\) is genuinely ramified.

We will next prove that \(\hat{C} \times_D \hat{C}\) is a disjoint union of curves isomorphic to \(\hat{C} \times_D \hat{C}\). For this, first note that \(\hat{C} \times_D \hat{C}\) maps to \(\hat{D} \times_D \hat{D}\), and the curve \(\hat{D} \times_D \hat{D}\) is a disjoint union of copies of \(\hat{D}\) as \(g\) is étale Galois. The component of \(\hat{C} \times_D \hat{C}\) lying over any of these copies of \(\hat{D}\) is isomorphic to \(\hat{C} \times_D \hat{C}\), and therefore \(\hat{C} \times_D \hat{C}\) is a disjoint union of curves isomorphic to \(\hat{C} \times_D \hat{C}\).

From (4.3) we have

\[
F^*f_*\mathcal{O}_C \subset F^*F_*\mathcal{O}_{\hat{C}}.
\]

(4.7)

Since \(\hat{C} \times_D \hat{C}\) is a disjoint union of curves isomorphic to \(\hat{C} \times_D \hat{C}\), from (3.3) it follows that \(F^*F_*\mathcal{O}_{\hat{C}}\) is a direct sum of copies of \(\hat{g}^*\hat{g}_*\mathcal{O}_{\hat{C}}\). It was shown above that \(\hat{g}\) is genuinely ramified and Galois. So from Proposition 3.3 we know that \(\hat{g}^*((\hat{g}_*\mathcal{O}_{\hat{C}})/\mathcal{O}_{\hat{D}})\) is contained in a direct sum of line bundles of negative degree.

Since \(\hat{g}\) is genuinely ramified, we know from Lemma 3.1, Remark 2.1 and (2.7) that

\[
\mathcal{O}_{\hat{C}} = H^0(\hat{C}, \hat{g}^*\hat{g}_*\mathcal{O}_{\hat{D}}) \otimes \mathcal{O}_{\hat{C}}
\]

is the maximal semistable subbundle of \(\hat{g}^*\hat{g}_*\mathcal{O}_{\hat{C}}\). Since \(F^*F_*\mathcal{O}_{\hat{C}}\) is a direct sum of copies of \(\hat{g}^*\hat{g}_*\mathcal{O}_{\hat{C}}\), this implies that

\[
H^0(\hat{C}, F^*F_*\mathcal{O}_{\hat{C}}) \otimes \mathcal{O}_{\hat{C}} \subset F^*F_*\mathcal{O}_{\hat{C}}
\]

(4.8)
is the maximal semistable subbundle. On the other hand, we have $F^*f_*\mathcal{O}_C = \hat{f}^*f_*\mathcal{O}_C$. 
So from Lemma 3.1, Remark 2.1 and (2.7) we know that

$$\mathcal{O}_{\hat{\mathcal{C}}} = H^0(\hat{\mathcal{C}}, F^*f_*\mathcal{O}_C) \otimes \mathcal{O}_{\hat{\mathcal{C}}} \subset F^*f_*\mathcal{O}_C$$

(4.9)

is the maximal semistable subbundle.

Consider the inclusion homomorphism in (4.7). From (4.8) and (4.9) we conclude that

$$F^*F_*\mathcal{O}_{\hat{\mathcal{C}}}/(H^0(\hat{\mathcal{C}}, F^*F_*\mathcal{O}_{\hat{\mathcal{C}}}) \otimes \mathcal{O}_{\hat{\mathcal{C}}}).$$

(4.10)

Since $F^*F_*\mathcal{O}_{\hat{\mathcal{C}}}$ is a direct sum of copies of $\hat{g}^*\hat{g}_*\mathcal{O}_{\hat{\mathcal{C}}}$, the vector bundle in (4.10) is isomorphic to a direct sum of copies of $\hat{g}^*((\hat{g}_*\mathcal{O}_{\hat{\mathcal{C}}})/\mathcal{O}_{\hat{\mathcal{D}}})$.

It was shown above that $\hat{g}^*((\hat{g}_*\mathcal{O}_{\hat{\mathcal{C}}})/\mathcal{O}_{\hat{\mathcal{D}}})$ is contained in a direct sum of line bundles of negative degree. Therefore, the vector bundle in (4.10) is contained in a direct sum of line bundles of negative degree. Consequently, the subsheaf

$$F^*((f_*\mathcal{O}_C)/\mathcal{O}_D) \subset F^*F_*\mathcal{O}_{\hat{\mathcal{C}}}/(H^0(\hat{\mathcal{C}}, F^*F_*\mathcal{O}_{\hat{\mathcal{C}}}) \otimes \mathcal{O}_{\hat{\mathcal{C}}})$$

is also contained in a direct sum of line bundles of negative degree.

Since $F^*(f_*\mathcal{O}_C)/\mathcal{O}_D)$ is contained in a direct sum of line bundles of negative degree, we conclude that

$$\mu_{\text{max}}(F^*V \otimes (F^*((f_*\mathcal{O}_C)/\mathcal{O}_D))) < \mu(F^*V);$$

note that $F^*V$ is semistable by Remark 2.1 as $F$ is separable. From this it follows that

$$\mu_{\text{max}}(V \otimes ((f_*\mathcal{O}_C)/\mathcal{O}_D)) = \mu_{\text{max}}(F^*(V \otimes ((f_*\mathcal{O}_C)/\mathcal{O}_D)))/\deg(F)$$

$$< \mu(F^*V)/\deg(F) = \mu(V),$$

because $F^*V$ is semistable. This completes the proof.

\[ \square \]

**Remark 4.2.** When the characteristic of the base field $k$ is zero, the tensor product of two semistable bundles remains semistable [RR, p. 285, Theorem 3.18]. We note that Lemma 4.1 is a straightforward consequence of it, provided the characteristic of $k$ is zero.

**Lemma 4.3.** Let $f : C \to D$ be a genuinely ramified morphism between irreducible smooth projective curves. Let $V$ and $W$ be two semistable vector bundles on $D$ with

$$\mu(V) = \mu(W).$$

Then

$$H^0(D, \text{Hom}(V, W)) = H^0(C, \text{Hom}(f^*V, f^*W)).$$

\[ \text{Proof.} \] Using the projection formula, and the fact that $f$ is a finite map, we have

$$H^0(C, \text{Hom}(f^*V, f^*W)) \cong H^0(D, f_*\text{Hom}(f^*V, f^*W)) \cong H^0(D, f_*f^*\text{Hom}(V, W))$$

$$\cong H^0(D, \text{Hom}(V, W) \otimes f_*\mathcal{O}_C) \cong H^0(D, \text{Hom}(V, W \otimes f_*\mathcal{O}_C)).$$

(4.11)

Let

$$0 = B_0 \subset B_1 \subset \cdots \subset B_{m-1} \subset B_m = W \otimes ((f_*\mathcal{O}_C)/\mathcal{O}_D)$$
be the Harder–Narasimhan filtration of \( W \otimes ((f_*\mathcal{O}_C) / \mathcal{O}_D) \) \cite[p. 16, Theorem 1.3.4]{HL}.
Since \( W \) is semistable, and \( f \) is genuinely ramified, from Lemma \ref{Lemma 4.1} we know that
\[
\mu(B_i/B_{i-1}) \leq \mu(B_i) = \mu_{\text{max}}(W \otimes ((f_*\mathcal{O}_C) / \mathcal{O}_D)) < \mu(W)
\]
for all \( 1 \leq i \leq m \). In view this and the given condition that \( \mu(V) = \mu(W) \), from (2.1) we conclude that
\[
H^0(D, \text{Hom}(V, B_i/B_{i-1})) = 0
\]
for all \( 1 \leq i \leq m \); note that both \( V \) and \( B_i/B_{i-1} \) are semistable. This implies that
\[
H^0(D, \text{Hom}(V, W \otimes ((f_*\mathcal{O}_C) / \mathcal{O}_D))) = 0.
\]
Consequently, we have
\[
H^0(D, \text{Hom}(V, W \otimes f_*\mathcal{O}_C)) = H^0(D, \text{Hom}(V, W))
\]
by examining the exact sequence
\[
0 \rightarrow \text{Hom}(V, W) \rightarrow \text{Hom}(V, W \otimes f_*\mathcal{O}_C) \rightarrow \text{Hom}(V, W \otimes ((f_*\mathcal{O}_C) / \mathcal{O}_D)) \rightarrow 0.
\]
From this and (4.11) it follows that
\[
H^0(C, \text{Hom}(f^*V, f^*W)) = H^0(D, \text{Hom}(V, W)).
\]
This completes the proof. \( \square \)

**Theorem 4.4.** Let \( f : C \rightarrow D \) be a genuinely ramified morphism between irreducible smooth projective curves. Let \( V \) be a stable vector bundle on \( D \). Then the pulled back vector bundle \( f^*V \) is also stable.

**Proof.** Consider the Galois extension \( F : \hat{C} \rightarrow D \) and the diagram in (4.6). Since \( V \) is stable, from Lemma \ref{Lemma 4.3} it follows that \( f^*V \) is simple. As \( V \) is semistable, it follows that \( g^*V \) is also semistable, where \( g \) is the map in (4.6). Let
\[
E \subset g^*V
\]
be the unique maximal polystable subbundle with \( \mu(E) = \mu(g^*V) \) \cite[p. 23, Lemma 1.5.5]{HL}; this subbundle \( E \) is called the socle of \( g^*V \). Since \( g^*V \) is preserved by the action of the Galois group \( \text{Gal}(g) \) on \( g^*V \), there is a unique subbundle \( E' \subset V \) such that
\[
E = g^*E' \subset g^*V.
\]
As \( V \) is stable, we conclude that \( E' = V \), and hence \( g^*V \) is polystable. So we have a direct sum decomposition
\[
g^*V = \bigoplus_{j=1}^{m} V_j,
\]
where each \( V_j \) is stable with \( \mu(V_j) = \mu(g^*V) \).

Take any \( 1 \leq j \leq m \), where \( m \) is the integer in (4.12). Since \( V_j \) is stable, and \( \hat{g} \) in (4.6) is Galois (this was shown in the proof of Lemma 4.1), repeating the above argument involving the socle we conclude that \( \hat{g}^*V_j \) is also polystable. On the other hand, as \( \hat{g} \) is genuinely ramified (see the proof of Lemma 4.1), from Lemma \ref{Lemma 4.3} it follows that
\[
H^0(\hat{C}, \text{End}(\hat{g}^*V_j)) = H^0(\hat{D}, \text{End}(V_j)).
\]

(4.13)
But $H^0(\hat{D}, \text{End}(V_j)) = k$, because $V_j$ is stable. Hence from (4.13) we know that $H^0(\hat{C}, \text{End}(g^*V_j)) = k$. This implies that $g^*V_j$ is stable, because it is polystable.

Since $g^*V_j$ is stable, and $\pi_2 \circ h = \hat{g}$ (see (4.6)), we conclude that $\pi_2^*V_j$ is also stable with

$$\mu(\pi_2^*V_j) = \mu(\pi_2^*g^*V)$$

for all $1 \leq j \leq m$. This implies that

$$\pi_1^*f^*V = \pi_2^*g^*V = \bigoplus_{j=1}^{m} \pi_2^*V_j \quad (4.14)$$

is polystable.

The map $\pi_2$ is genuinely ramified because $\hat{g}$ is genuinely ramified (see the proof of Lemma 4.1) and $\hat{g} = h \circ \pi_2$. Indeed, if $\pi_2$ factors through an étale covering of $\hat{D}$, then the genuinely ramified map $\hat{g}$ factors through that étale covering of $\hat{D}$, and hence from Proposition 2.6 it follows that $\pi_2$ is genuinely ramified.

Since $\pi_2$ is genuinely ramified, and each $V_j$ in (4.14) is stable, from Lemma 4.3 it follows that

$$H^0(C \times_D \hat{D}, \text{Hom}(\pi_2^*V_i, \pi_2^*V_j)) = H^0(\hat{D}, \text{Hom}(V_i, V_j)) \quad (4.15)$$

for all $1 \leq i, j \leq m$. We know that $V_i$ and $\pi_2^*V_i$ are stable. So from (4.15) we conclude that $V_i$ is isomorphic to $V_j$ if and only if $\pi_2^*V_i$ is isomorphic to $\pi_2^*V_j$. From (4.15) it also follows that

$$H^0(C \times_D \hat{D}, \text{End}(\pi_2^*g^*V)) = H^0(\hat{D}, \text{End}(g^*V)) \quad (4.16)$$

we note that this also follows from Lemma 4.3.

The vector bundle $f^*V$ on $C$ is semistable, because $V$ is semistable and $f$ is separable. Let

$$0 \neq S \subset f^*V \quad (4.17)$$

be a stable subbundle with

$$\mu(S) = \mu(f^*V) \quad (4.18)$$

Since $S$ is stable with $\mu(S) = \mu(f^*V)$, and the map $\pi_1$ is Galois, using the earlier argument involving the socle we conclude that

$$\tilde{S} := \pi_1^*S \subset \pi_1^*f^*V = \pi_2^*g^*V = \bigoplus_{j=1}^{m} \pi_2^*V_j =: \tilde{V} \quad (4.19)$$

is a polystable subbundle with $\mu(\tilde{S}) = \mu(\tilde{V})$.

Consider the associative algebra $H^0(C \times_D \hat{D}, \text{End}(\tilde{V}))$, where $\tilde{V}$ is the vector bundle in (4.19). Define the right ideal

$$\Theta := \{ \gamma \in H^0(C \times_D \hat{D}, \text{End}(\tilde{V})) \mid \gamma(\tilde{V}) \subset \tilde{S} \} \subset H^0(C \times_D \hat{D}, \text{End}(\tilde{V})) \quad (4.20)$$

where $\tilde{S}$ is the subbundle in (4.19). The subbundle $\tilde{S} \subset \tilde{V}$ is a direct summand, because $\tilde{V}$ is polystable, and $\mu(\tilde{S}) = \mu(\tilde{V})$. Consequently, $\tilde{S}$ coincides with the subbundle generated by the images of endomorphisms lying in the right ideal $\Theta$. Since $\tilde{V}$ is semistable, the image of any endomorphism of it is a subbundle.
Consider $\tilde{V}$ in (4.19). The identification
$$H^0(C \times_D \hat{D}, \text{End}(\tilde{V})) = H^0(\hat{D}, \text{End}(g^*V))$$
in (4.16) preserves the associative algebra structures of
$$H^0(C \times_D \hat{D}, \text{End}(\tilde{V})) \quad \text{and} \quad H^0(\hat{D}, \text{End}(g^*V)),$$
because it sends any $\gamma \in H^0(\hat{D}, \text{End}(g^*V))$ to $\pi_2^*\gamma$. Let
$$\tilde{\Theta} \subset H^0(\hat{D}, \text{End}(g^*V))$$
be the right ideal that corresponds to $\Theta$ in (4.20) by the identification in (4.16). Let
$$\mathcal{S} \subset g^*V$$
be the subbundle generated by the images of endomorphisms lying in the right ideal $\tilde{\Theta}$ in (4.21). Since $g^*V$ is semistable, the image of any endomorphism of it is a subbundle. From the above construction of $\mathcal{S}$ it follows that
$$\tilde{\mathcal{S}} = \pi_2^*\mathcal{S},$$
where $\tilde{\mathcal{S}}$ is the subbundle in (4.20).

The isomorphism in (4.16) is equivariant for the actions of the Galois group $\text{Gal}(\pi_1) = \text{Gal}(g)$ on
$$H^0(C \times_D \hat{D}, \text{End}(\tilde{V})) = H^0(C \times_D \hat{D}, \text{End}(\pi_1^*f^*V))$$
and $H^0(\hat{D}, \text{End}(g^*V))$, because the isomorphism sends any $\gamma \in H^0(\hat{D}, \text{End}(g^*V))$ to $\pi_2^*\gamma$. Since $\tilde{\mathcal{S}} = \pi_1^*\mathcal{S}$ in (4.19) is preserved under the action of $\text{Gal}(\pi_1)$ on $\pi_1^*f^*V$, it follows that the action of $\text{Gal}(\pi_1)$ on $H^0(C \times_D \hat{D}, \text{End}(\pi_1^*f^*V))$ preserves the right ideal $\Theta$ in (4.20). These together imply that the action of $\text{Gal}(g)$ on $H^0(\hat{D}, \text{End}(g^*V))$ preserves the right ideal $\tilde{\Theta}$ in (4.21). Consequently, the subbundle
$$\mathcal{S} \subset g^*V$$
in (4.22) is preserved under the action of $\text{Gal}(g)$ on $g^*V$.

Since $\mathcal{S}$ is preserved under the action of $\text{Gal}(g)$ on $g^*V$, there is a unique subbundle
$$S_0 \subset V$$
such that $\mathcal{S} = g*S_0 \subset g^*V$. Given that $V$ is stable, and $\mu(S_0) = \mu(V)$ (this follows from (4.18)), we now conclude that $S_0 = V$. Hence the subbundle $S$ in (4.17) coincides with $f^*V$. Therefore, we conclude that $f^*V$ is stable. \qed

5. Characterizations of genuinely ramified maps

Let $D$ be an irreducible smooth projective curve, and let
$$\phi : X \longrightarrow D$$
be a nontrivial étale covering with $X$ irreducible. Let $L$ be a line bundle on $X$ of degree one.

**Proposition 5.1.**
(1) The direct image $\phi_*L$ is a stable vector bundle on $D$.
(2) The pulled back bundle $\phi^*\phi_*L$ is not stable.

Proof. Let $\delta$ be the degree of the map $\phi$; note that $\delta > 1$, because $\phi$ is nontrivial.

We have $\deg(\phi_*L) = \deg(L) = 1$ [Ha, p. 306, Ch. IV, Ex. 2.6(a) and 2.6(d)]. This implies that

$$\deg(\phi^*\phi_*L) = \delta \cdot \deg(L) = \delta.$$  

We have a natural homomorphism

$$H : \phi^*\phi_*L \longrightarrow L.$$  

(5.1)

This $H$ has the following property: For any coherent subsheaf $W \subset \phi_*L$, the restriction of $H$ to $\phi^*W \subset \phi^*\phi_*L$

$$H_W := H|_{\phi^*W} : \phi^*W \longrightarrow L$$  

(5.2)

is a nonzero homomorphism. Note that for any point $y \in D$, the fiber $(\phi^*\phi_*L)_y$ is $H^0(\phi^{-1}(y), L|_{\phi^{-1}(y)})$, and hence a nonzero element of $(\phi^*\phi_*L)_y$ must be nonzero at some point of $\phi^{-1}(y)$.

We will first show that $\phi_*L$ is semistable. To prove this by contradiction, let $V \subset \phi_*L$ be a semistable subbundle with

$$\mu(V) > \mu(\phi_*L) = \frac{1}{\delta}.$$  

(5.3)

Consider the nonzero homomorphism

$$H_V := H|_{\phi^*V} : \phi^*V \longrightarrow L$$  

in (5.2). We have $\mu(\phi^*V) = \delta \cdot \mu(V) > 1$ (see (5.3)), and also $\phi^*V$ is semistable because $V$ is so. Consequently, $H_V$ contradicts (2.1). As $\phi_*L$ does not contain any subbundle $V$ satisfying (5.3), we conclude that $\phi_*L$ is semistable.

Since $\text{rk}(\phi_*L)$ is coprime to $\deg(\phi_*L)$, the semistable vector bundle $\phi_*L$ is also stable. This proves statement (1).

The vector bundle $\phi^*\phi_*L$ is not stable, because the homomorphism $H$ in (5.1) is nonzero and $\mu(\phi^*\phi_*L) = \mu(L)$. \qed

**Proposition 5.2.** Let $f : C \longrightarrow D$ be a nonconstant separable morphism between irreducible smooth projective curves such that $f$ is not genuinely ramified. Then there is a stable vector bundle $E$ on $D$ such that $f^*E$ is not stable.

Proof. Since $f$ is not genuinely ramified, from Proposition 2.6 we know that there is a nontrivial étale covering

$$\phi : X \longrightarrow D$$

and a map $\beta : C \longrightarrow X$ such that $\phi \circ \beta = f$. As in Proposition 5.1, take a line bundle $L$ on $X$ of degree one. The vector bundle $\phi_*L$ is stable by Proposition 5.1(1).

The vector bundle $\phi^*\phi_*L$ is not stable by Proposition 5.1(2). Therefore,

$$f^*(\phi_*L) = \beta^*\phi^*(\phi_*L) = \beta^*(\phi^*\phi_*L)$$

is not stable. \qed
Theorem 4.4 and Proposition 5.2 together give the following:

**Theorem 5.3.** Let $f : C \to D$ be a nonconstant separable morphism between irreducible smooth projective curves. The map $f$ is genuinely ramified if and only if $f^*E$ is stable for every stable vector bundle $E$ on $D$.

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