Research Article

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Blow-up solutions with minimal mass for nonlinear Schrödinger equation with variable potential

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Abstract: This paper studies the mass-critical variable coefficient nonlinear Schrödinger equation. We first get the existence of the ground state by solving a minimization problem. Then we prove a compactness result by the variational characterization of the ground state solutions. In addition, we construct the blow-up solutions at the minimal mass threshold and further prove the uniqueness result on the minimal mass blow-up solutions which are pseudo-conformal transformation of the ground states.

Keywords: variable coefficient nonlinear Schrödinger equation; minimal mass blow-up solutions; variational characterization; ground state; compactness

MSC: 35Q55; 35B44

1 Introduction

In this paper, we study the behavior of blowup solutions for the variable coefficient nonlinear Schrödinger equation

\[ i\varphi_t + \Delta \varphi + c|x|^{-2}\varphi + |x|^{-b}\varphi^{p-1}\varphi = 0, \quad t \geq 0, \quad x \in \mathbb{R}^D, \]  

(1.1)

and

\[ \varphi(0, x) = \varphi_0, \quad x \in \mathbb{R}^D, \]  

(1.2)

in the case \( p = 1 + \frac{a-2b}{D} \). Here and hereafter, \( D \geq 3 \) is the space dimension, \( 0 < b < 2 \), \( \varphi = \varphi(t, x) : [0, T) \times \mathbb{R}^D \rightarrow \mathbb{C} \) is a complex value wave function with \( 0 < T \leq \infty \), \( i = \sqrt{-1} \), \( \Delta \) is the Laplace operator and \( c \in (0, c^*) \), where \( c^* = \frac{(D-2)^2}{4a} \) is the best constant in Hardy’s inequality:

\[ c^* \int_{\mathbb{R}^D} |x|^{-2} |u|^2 dx \leq \int_{\mathbb{R}^D} |\nabla u|^2 dx. \tag{1.3} \]

More precisely, we say that \( \varphi(\cdot) \) is a solution of the Cauchy problem (1.1)–(1.2) on \([0, T)\) if for all \( t \in [0, T) \),

\[ \varphi(t) = S(t)\varphi + i \int_0^t S(t-s)(|x|^{-b}|\varphi(s)|^{p-1}\varphi(s))ds, \]

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where $S(\cdot)$ is the group with infinitesimal generator $i(\Delta + c|x|^{-2})$.

From [2, 6, 15], we know that the Cauchy problem (1.1)-(1.2) is locally well-posed in $H^1(\mathbb{R}^D)$: there exists a unique solution $\varphi(t, x)$ of the Cauchy problem (1.1)-(1.2) in $C([0, T); H^1(\mathbb{R}^D))$ for some $T \in (0, \infty]$ (maximal existence time), and for all $t \in [0, T)$, we have the following alternatives:

$$T = +\infty,$$

or else

$$T < \infty \quad \text{and} \quad \lim_{t \to T^-} \|\varphi(t)\|_{H^1} = +\infty \text{(blowup)},$$

where $\| \cdot \|_{H^1}$ is the usual norm on $H^1$, and $H^1$ is $H^1(\mathbb{R}^D)$. Furthermore, the unique solution $\varphi(t, x)$ satisfies the following two conservation laws: for all $t \in [0, T)$,

$$\int_{\mathbb{R}^D} |\varphi(t, x)|^2 \, dx = \int_{\mathbb{R}^D} |\varphi_0(x)|^2 \, dx,$$

and

$$E(\varphi(t)) = \int_{\mathbb{R}^D} |\nabla \varphi(t)|^2 \, dx - \int_{\mathbb{R}^D} c|x|^2 |\varphi(t)|^2 \, dx - \frac{D}{D + 2 - b} \int_{\mathbb{R}^D} |x|^{-b} |\varphi(t)|^{2 + \frac{2b}{D - 2}} \, dx = E(\varphi_0).$$

Also, the $L^2$-norm of $\varphi(t, x)$ is invariant under the transformation

$$\varphi \to \varphi_\lambda(t, x) = \lambda^{D/2} \varphi(\lambda^2 t, \lambda x) \quad \text{for} \quad \lambda > 0.$$

In other words, $\|\varphi_\lambda(t)\|_{L^2} = \|\varphi(t)\|_{L^2}$ and the Cauchy problem (1.1)-(1.2) is called $L^2$-critical.

Eq.(1.1) is also called inhomogeneous nonlinear Schrödinger equation with inverse-square potential. Let us now recall some known results about the nonlinear Schrödinger equation. For the classical nonlinear Schrödinger equation, i.e. $c = b = 0$,

$$i\varphi_t + \Delta \varphi + |\varphi|^\frac{4}{D-2} \varphi = 0, \quad t \geq 0, \quad x \in \mathbb{R}^D.$$

(1.6)

Let $\varphi(t, x) = e^{it} u(x)$ be a standing wave solution of (1.6). Then $u(x)$ satisfies the following time-independent Schrödinger equation

$$\Delta u - u + |u|^\frac{4}{D-2} u = 0, \quad u \in H^1.$$

(1.7)

It is well known that (1.7) possesses a unique radial positive solution $Q(x)$ (see [2]). Moreover, the Cauchy problem (1.6)-(1.2) has no blow-up solution in the class $\{\varphi \in H^1 : \|\varphi\|_{L^2} < \|Q\|_{L^2}\}$ (see [19]), while in the class $\{\varphi \in H^1 : \|\varphi\|_{L^2} = \|Q\|_{L^2}\}$, there exists a unique blowup solution

$$\tilde{Q}(x, t) = \frac{1}{|t|^{D/2}} e^{i|x|^2/|t|-1} Q\left(\frac{x}{t}\right)$$

up to the invariances of the equation(see [10, 13, 14, 20]).

For the nonlinear Schrödinger equation with inverse-square potential, i.e. $b = 0, c \neq 0$,

$$i\varphi_t + \Delta \varphi + c|x|^{-2} \varphi + |\varphi|^\frac{4}{D-2} \varphi = 0, \quad t \geq 0, \quad x \in \mathbb{R}^D.$$

(1.8)

Let $\varphi(t, x) = e^{it} u(x)$ be a standing wave solution of (1.8). Then $u(x)$ satisfies the following time-independent Schrödinger equation

$$\Delta u + c|x|^{-2} u - u + |u|^\frac{4}{D-2} u = 0, \quad u \in H^1.$$

(1.9)

By solving a variational problem, Csobo and Genoud [5] proved that (1.9) has radial positive solutions to exist. Mukherjee, Nam and Nguyen [18] showed the uniqueness of radial positive solution of (1.9). In addition, Csobo and Genoud [5] proved that all $H^1$-solutions of Eq.(1.8) are global if $\|\varphi\|_{L^2} < \|U\|_{L^2}$ and by using the pesudo-conformal transformation, they constructed the minimal mass blow-up solutions defined as

$$\varphi(t, x) = e^{i\gamma_0} e^{i\frac{2\lambda x}{T-t}} e^{-i\frac{\lambda x^2}{2(T-t)}} \left( \frac{A_0}{T-t} \right)^{D/2} U \left( \frac{A_0 x}{T-t} \right),$$

where $A_0$ is the minimal mass blow-up solution de/f_ined as
such that $\|\varphi_0\|_{L^2} = \|U\|_{L^2}$, where $T \in \mathbb{R}, \lambda_0 > 0, \gamma_0 \in \mathbb{R}, U$ is the unique positive radial solution of (1.9).

For the inhomogeneous nonlinear Schrödinger equation, i.e. $c = 0, b \neq 0$,

$$i\varphi_t + \Delta \varphi + |x|^{-b} |\varphi|^{\frac{4+b}{2}} \varphi = 0, \ t \geq 0, \ x \in \mathbb{R}^D. \tag{1.10}$$

Let $\varphi(t, x) = e^{it} u(x)$ be a standing wave solution of (1.10). Then $u(x)$ satisfies the following time-independent Schrödinger equation

$$\Delta u - u + |x|^{-b} |u|^{\frac{4+b}{2}} u = 0, \ u \in H^1. \tag{1.11}$$

By solving a variational problem, Genoud [9] proved that (1.11) has a unique positive radial solution, which we will denote by $\varphi(x)$, we can assume that (1.12) has a unique positive radial solution, which we will denote by $\varphi(x)$. Moreover, assume that (1.12) has a unique positive radial solution, which we will denote by $\varphi(x)$, then (1.12) has a unique positive radial solution of (1.11). We also refer to Chen and Tang [3], Zhang and Ahmed [23] for related contributions on the nonlinear Schrödinger equation.

Motivated by the above studies, it is of interest to find the uniqueness of the minimal mass blow-up solutions for the Cauchy problem (1.1)-(1.2). In order to solve this problem, we need a notion of a ground state. So we consider the periodic solutions of equation of the form $\varphi(t, x) = e^{it} u(x)$, then $u(x)$ solves the nonlinear elliptic equation

$$\Delta u + c|x|^{-2} u - u + |x|^{-b} |u|^{\frac{4+b}{2}} u = 0, \ u \in H^1. \tag{1.12}$$

In section 2, we will prove the existence in $H^1$ of a positive radial solution of Eq.(1.12) by using Weinstein's argument [19] and that the set of solutions of (1.12) has a “minimal” element in $L^2$ called ground state. However, for $c > 0$, we are not aware of any uniqueness result of (1.12) on $\mathbb{R}^D$. According to [8], [18] and [21], we can assume that (1.12) has a unique positive radial solution, which we will denote by $W$ throughout the paper. In terms of the existence of ground state, we first show the global well-posedness condition for the Cauchy problem (1.1)-(1.2). Then, we shall exhibit blow-up solutions at the mass $\|\varphi_0\|_{L^2} = \|W\|_{L^2}$ by applying the pseudo-conformal transformation to the standing waves $e^{it} W(x)$, which is thus the minimal mass where blowup can occur.

Now we state our main result.

**Theorem 1.1.** (Determination of minimal blow-up solutions) Let initial data $\varphi_0 \in H^1$ such that the solution $\varphi(t, x)$ of the Cauchy problem (1.1)-(1.2) blows up in finite time $T > 0$. Moreover, assume that $\|\varphi_0\|_{L^2} = \|W\|_{L^2}$. Then there exist $\gamma_0 \in \mathbb{R}$ and $\lambda_0 > 0$ such that for all $t \in [0, T)$,

$$\varphi(t, x) := \varphi_{T, \lambda_0, \gamma_0} = \left(\frac{\lambda_0}{T-t}\right)^{D/2} e^{i\gamma_0} e^{i \frac{\lambda_0 x^2}{2(\lambda_0 - t)}} W \left(\frac{\lambda_0 x}{T-t}\right),$$

where $W(x)$ is the ground state of Eq.(1.12).

One can ask if there are other minimal mass blow-up solutions? That is, assume that

- $\varphi(t, x)$ blows up in finite time $T$, and
- $\|\varphi(t, x)\|_{L^2} = \|W\|_{L^2}$ for all $t$.

Is there $\gamma_0 \in \mathbb{R}$ and $\lambda_0 > 0$ such that

$$\tilde{\varphi}(t, x) = \varphi(t, x)?$$

The aim of this paper is to prove this uniqueness result on the minimal mass blow-up solutions. The method is mainly based on a compactness result.
The plan of this paper is as follows. In section 2, we prove the existence of ground state by solving a minimization problem. In section 3, we prove a compactness result, crucial in the proof of Theorem 1.1. In section 4, we construct the pseudo-conformal transformation and then give the proof of Theorem 1.1.

2 Ground State

In this section, we prove the existence of a positive radial solution $W \in H^1$ for (1.12). Such a solution is obtained by variational approach and called ground state. We first state the following result.

**Lemma 2.1.** For $D \geq 3$, $0 < c < c^* = (D - 2)^2$, $0 < b < 2$ and $u \in H^1 \setminus \{0\}$, define the functional

$$J(u) := \left( \int_{\mathbb{R}^D} (|\nabla u|^2 dx - c|x|^{-2} |u|^2 dx) \right) \left( \int_{\mathbb{R}^D} |u|^2 dx \right)^{\frac{2-b}{D}}, \tag{2.1}$$

and the variational problem

$$d := \inf_{u \in H^1 \setminus \{0\}} J(u). \tag{2.2}$$

Then one has the followings:

(i). There exists a positive radial function $v^*(x) \in H^1$ such that the variational problem (1.12) is attained at $v^*$.

(ii). Let $v^* = [(1 + \frac{2-b}{D})d]^{-\frac{b}{2D}} \lambda^\frac{b}{2} W(\lambda x)$ with $\lambda = \frac{2-b}{D}$. Then $W(x)$ is the positive radial solution of (1.12).

(iii). Assume that the positive radial solution $W(x)$ of the nonlinear elliptic equation (1.12) is unique. Then one has the sharp interpolation inequality for $u \in H^1(\mathbb{R}^D)$ with $D \geq 3$,

$$\int_{\mathbb{R}^D} |x|^{-b} |u|^{2 + \frac{2b}{D}} dx \leq \frac{D + 2 - b}{D} \left( \int_{\mathbb{R}^D} |W|^2 dx \right)^{-\frac{2-b}{D}} \left( \int_{\mathbb{R}^D} (|\nabla u|^2 dx - c|x|^{-2} |u|^2 dx) \right) \left( \int_{\mathbb{R}^D} |u|^2 dx \right)^{\frac{2-b}{D}}. \tag{2.3}$$

**Proof.** From Hardy inequality (1.3), one has that for $u \in H^1(\mathbb{R}^D)$,

$$\left(1 - \frac{c}{c^*}\right) \int_{\mathbb{R}^D} |\nabla u|^2 dx \leq \int_{\mathbb{R}^B} (|\nabla u|^2 - c|x|^{-2} |u|^2) dx \leq \int_{\mathbb{R}^B} |\nabla u|^2 dx. \tag{2.4A}$$

By (2.1), $J(u)$ is well-defined in $H^1 \setminus \{0\}$, and $J(u) \geq 0$. Furthermore, $J(u)$ is invariant under the scaling $u(x) \rightarrow u^{\mu, \lambda}(x) := \mu u(\lambda x)(\mu, \lambda > 0)$, that is

$$J(u^{\mu, \lambda}) = J(u) \quad \text{for} \quad \mu > 0, \lambda > 0. \tag{2.5}$$

Let $u^*$ be the Schwarz symmetrization of $u$. By [11, 12], one has that

$$\int_{\mathbb{R}^D} |x|^{-2} |u^*|^2 dx \geq \int_{\mathbb{R}^D} |x|^{-2} |u|^2 dx, \tag{2.6}$$

$$\int_{\mathbb{R}^D} |x|^{-b} |u^*|^{2 + \frac{2b}{D}} dx \geq \int_{\mathbb{R}^D} |x|^{-b} |u|^{2 + \frac{2b}{D}} dx, \tag{2.7}$$

$$\int_{\mathbb{R}^D} |\nabla u^*|^2 dx \leq \int_{\mathbb{R}^D} |\nabla u|^2 dx, \quad \int_{\mathbb{R}^D} |u^*|^2 dx = \int_{\mathbb{R}^D} |u|^2 dx. \tag{2.8}$$

By (2.2), we can choose a minimizing sequence $\{u_n\} \subset H^1 \setminus \{0\}$ such that $\lim_{n \rightarrow \infty} J(u_n) = d$. Then from (2.1), (2.2), (2.6), (2.7), (2.8), one has

$$\lim_{n \rightarrow \infty} J(u_n) = d. \tag{2.9}$$
where $u_n^*$ denotes the Schwarz symmetrization of $u_n$ for any $n \in \mathbb{N}$. Now we choose

$$
\lambda_n = \left[ \int_{\mathbb{R}^D} |u_n^*|^2 \, dx / \int_{\mathbb{R}^D} (|\nabla u_n^*|^2 - c|x|^{-2} |u_n^*|^2) \, dx \right]^{1/2},
$$

and take $v_n = |(u_n^*)^{\mu_n, \lambda_n}|$. By (2.5) and (2.9), we have that

$$
v_n \geq 0, \quad v_n(x) = v_n(|x|), \quad \int_{\mathbb{R}^D} |v_n|^2 \, dx = 1, \quad \int_{\mathbb{R}^D} (|\nabla v_n|^2 - c|x|^{-2} |v_n|^2) \, dx = 1.
$$

Thus we get the proof of (2.3).

It follows that

$$
\lim_{n \to \infty} \mu_n = \mu^* \in H^1.
$$

Thus we get the proof of (i). Now we prove (ii).

Since $v^*$ is the minimizer of (2.2), $v^*$ must satisfy the Euler-Lagrange equation

$$
\frac{d}{dx} \left. J(v^* + \varepsilon \eta) = 0, \text{ for all } \eta \in C_0^\infty(\mathbb{R}^D). \right)
$$

By (2.14), (2.15) it follows that

$$
\Delta v^* + c|x|^{-2} v^* - \frac{2 - b}{D} v^* + \left( 1 - \frac{2}{D} \right) d|x|^{-b} (v^*)^{1 + \frac{b}{2D}} = 0.
$$

Let $v^* = [(1 + \frac{2-b}{D}) d]^{-\frac{D}{2b}} \lambda \sqrt[2b]{D} W(\lambda x)$ with $\lambda = \sqrt{\frac{2-b}{D}}$. Then from (2.18), $W(x)$ is the positive radial solution of (1.12). This completes the proof of (ii). Then we prove (iii) in the following.

Since $v^* = [(1 + \frac{2-b}{D}) d]^{-\frac{D}{2b}} \lambda \sqrt[2b]{D} W(\lambda x)$ and $\lambda = \sqrt{\frac{2-b}{D}}$, by (2.15) it follows that

$$
d = \left( 1 + \frac{2-b}{D} \right)^{-1} \left( \int_{\mathbb{R}^D} |W|^2 \, dx \right)^{\frac{b}{2b}}.
$$

Thus if we assume that (1.12) has a unique positive radial solution, then for (ii) and (2.2), we get (2.3).

This completes the proof.
Remark 2.1. Let $\xi$ be the minimizer of $J(u)$, then we define a set $A$ as follows:

$$A = \left\{ V(x) : \xi(x) = \left( 1 + \left( \frac{2 - b}{D} \right) \right)^{-\frac{2}{b}} \lambda^\frac{b}{D} V(\lambda x) \text{ with } \lambda = \sqrt{\frac{2 - b}{D}} \right\}. \quad (2.20)$$

It is obvious that $W \in A$, then $\|W\|_{L^2}^2 = \left( 1 + \frac{2 - b}{D} \right)^{\frac{2}{b}}$. Moreover, $W$ satisfying the elliptic equation (1.12). Therefore, we call $W \in A$ ground state of Eq.(1.12).

In the following, we show that the ground state solutions play an important role in the global well-posedness for the Cauchy problem (1.1)–(1.2).

Corollary 2.1. Let $c \in (0, c^*)$, if $\varphi_0 \in H^1$ satisfies

$$\|\varphi_0\|_{L^2} < \|W\|_{L^2}, \quad (2.21)$$

then the corresponding solution of the Cauchy problem (1.1)–(1.2) is global, where $W \in A$ is the ground state of (1.12).

Proof. By the local well-posedness theory, we only need to show that $\int_{\mathbb{R}^D} (|\nabla \varphi|^2 - c|x|^{-2} |\varphi|^2) dx$ remains bounded. From the conservation laws of mass and energy,

$$E(\varphi_0) = \int_{\mathbb{R}^D} (|\nabla \varphi|^2 - c|x|^{-2} |\varphi|^2) dx - \frac{D}{D + 2 - b} \int_{\mathbb{R}^D} |x|^{-b} |\varphi|^{2 + \frac{4 - 2b}{D}} dx.$$  

Hence, by (2.3),

$$E(\varphi_0) \geq \int_{\mathbb{R}^D} (|\nabla \varphi|^2 - c|x|^{-2} |\varphi|^2) dx \left[ 1 - \left( \frac{\|\varphi_0\|_{L^2}}{\|W\|_{L^2}} \right)^{\frac{4 - 2b}{D}} \right].$$

If

$$\|\varphi_0\|_{L^2} < \|W\|_{L^2},$$

then $\int_{\mathbb{R}^D} (|\nabla \varphi|^2 - c|x|^{-2} |\varphi|^2) dx$ is bounded and so the solution is global. This completes the proof.

3 Compactness

In this section, we will establish a compactness result which is important in the proof of Theorem 1.1. For this aim, we introduce the notion of the variational characterization of ground state solutions.

Theorem 3.1. Let $W$ be the positive radial minimizer of $J(u)$ and $v \in H^1$ satisfy

$$\|v\|_{L^2} = \|W\|_{L^2} \text{ and } E(v) = 0. \quad (3.1)$$

Then there exist $\lambda_0 > 0$ and $\gamma_0 \in \mathbb{R}$ such that $v(x) = e^{i\gamma_0 \lambda_0^{D/2}} W(\lambda_0 x)$.

Proof. It follows from Lemma 2.1 and (3.1) that $v$ is a minimizer of $J(u)$. Since $\|\nabla (|v|)\|_{L^2} \leq \|\nabla v\|_{L^2}$, then $|v|$ is also a minimizer. Furthermore, any positive minimizer is radial thanks to a result of Hajaiej[11]. Indeed, suppose $v_0$ is a positive minimizer that is not radial, and consider its Schwarz symmetrization $v_0^*$. On the other hand,

$$\|\nabla v_0^*\|_{L^2} \leq \|\nabla v_0\|_{L^2}, \quad \|v_0^*\|_{L^2} \leq \|v_0\|_{L^2},$$

then we get $J(v_0^*) < J(v_0)$, a contradiction. We deduce that $|v|$ is radial. Furthermore, the Euler-Lagrange equation (2.17) expressing the fact that $|v|$ is a minimizer reads

$$\Delta(|v|) + c|x|^{-2}(|v|) - \frac{2 - b}{D} \frac{\int_{\mathbb{R}^D} (|\nabla v|^2 - c|x|^{-2} |v|^2) dx}{\int_{\mathbb{R}^D} |v|^2 dx} |v| + |x|^{-b} |v|^{1 + \frac{4 - 2b}{D}} = 0.$$
It now follows by the scaling properties of this elliptic equation, there exists $W \in A$ such that

$$|v(x)| = \lambda_0^{D/2} W(\lambda_0 x),$$

where $\lambda_0 = \sqrt{\frac{2 - b}{D}} \sqrt{\frac{\int_{\mathbb{R}^D} (|\nabla v|^2 - c|x|^{-2}|v|^2)dx}{\int_{\mathbb{R}^D} |v|^2 dx}}$.

It only remains to show that $w$ defined by $w(x) = \frac{\phi(x)}{|\phi(x)|}$ is constant on $\mathbb{R}^D$. Differentiating $|w|^2 = 1$ yields $Re(\bar{w}\nabla w) = 0$, thus

$$|\nabla v|^2 = |\nabla (|v|)|^2 + |v|^2 |\nabla w|^2 + 2|v|\nabla (|v|) \cdot Re(\bar{w}\nabla w),$$

and

$$\|\nabla v\|_{L^2}^2 = \|\nabla (|v|)\|_{L^2}^2 + \int_{\mathbb{R}^D} |v|^2 |\nabla w|^2 dx.$$

If $|\nabla w| \neq 0$, then $f(|v|) < f(v)$, which is a contradiction. Therefore, $w$ is constant, which completes the proof. Then we review the following result, for the proofs we refer readers to [7].

**Lemma 3.1.** Let $0 < b < 2, D \geq 3$ and $1 < p < 1 + \frac{4 - 2b}{D - 2}$. Then there exists a constant $C = C(D, b, p) > 0$ such that

$$\int_{\mathbb{R}^D} |x|^{-b} |\phi|^{p-1} - |v|^{p-1} |u| |\phi| dx \leq C (||\phi|^{p-1} - |v|^{p-1} ||_{L^p} ||u||_{L^p} ||\phi||_{L^p} + |||\phi|^{p-1} - |v|^{p-1} ||_{L^p} ||u||_{L^p} ||\phi||_{L^p})$$

for all $\phi, v, u, \phi \in H^1$, where

$$(p-1)\alpha = \beta \in \left(\frac{D(p+1)}{D-b}, 2^*\right)$$

and $(p-1)\eta = p + 1$.

In addition, we also need the following concentration-compactness Lemma which can be proved with a minor modification to the proof of Proposition 1.7.6 in [2].

**Lemma 3.2.** Let $\{v_n\}_{n \in \mathbb{N}} \in H^1$ satisfy

$$\lim_{n \to \infty} \|v_n\|_{L^2} = M < +\infty \text{ and } \sup_{n \in \mathbb{N}} H(v_n) < +\infty.$$ 

Then there exists a subsequence $\{v_{n_k}\}_{k \in \mathbb{N}}$ which satisfies one of the following properties:

(V) $\|v_{n_k}\|_{L^q} \to 0$ as $k \to \infty$ for all $q \in (2, 2^*)$.

(D) There are sequences $w_k, z_k \in H^1$ and a constant $\alpha \in (0, 1)$ such that:

1. $\text{dist}\text{(supp}(w_k), \text{supp}(z_k)) \to \infty$;
2. $\sup_{k \in \mathbb{N}} (\|w_k\|_{H^1} + |z_k|_{H^1}) < \infty$ for all $k \in \mathbb{N}$;
3. $\|w_k\|_{L^2} \to aM$ and $|z_k|_{L^2} \to (1-a)M$ as $k \to \infty$, for some $\alpha \in (0, 1)$;
4. $\lim_{k \to \infty} \int_{\mathbb{R}^D} |v_{n_k}|^q dx - \int_{\mathbb{R}^D} |w_k|^q dx - \int_{\mathbb{R}^D} |z_k|^q dx = 0$, for all $q \in [2, 2^*)$;
5. $\lim \inf_{k \to \infty} (H(v_{n_k}) - H(w_k) - H(z_k)) \geq 0$.

(C) There exist $\nu \in H^1$ and a sequence $\{\nu_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^D$ such that

$$v_{n_k} (\cdot - y_k) \to v \text{ in } L^q(\mathbb{R}^D), \forall q \in [2, 2^*).$$

Here and subsequently, for abbreviation, $H(\phi)$ stands for the Hardy functional defined by

$$\int (|\nabla v|^2 - c|x|^{-2}|v|^2)dx.$$ 

We are now show the main result of this section.

**Theorem 3.2.** Consider a sequence $\{v_n\}_{n \in \mathbb{N}} \subset H^1$ satisfying

$$\lim_{n \to \infty} \|v_n\|_{L^2} = \|W\|_{L^2}, 0 < \lim \sup_{n \in \mathbb{N}} H(v_n) < +\infty, \lim \sup_{n \to \infty} E(v_n) \leq 0. \tag{3.2}$$
Then there exists a subsequence \( \{v_{n_k}\}_{k \in \mathbb{N}} \) and \( \gamma_0 \in \mathbb{R} \) such that

\[
\lim_{k \to \infty} \|v_{n_k} - e^{i\gamma_0} W\|_{H^1} = 0,
\]

(3.3)

where \( W \in A \) is the ground state of (1.12).

**Proof.** The behavior of the sequence \( \{v_n\} \) is constrained by the concentration-compactness Lemma. The proof will be divided into three steps: we first show that property (C) holds in Lemma 3.2 by ruling out (V) and (D). And then prove the sequence \( \{y_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^D \) in (C) is bounded. We thus obtain the desired conclusion by using the variational characteristic of ground states.

Step 1: Compactness. Let \( \nu = 0 \) and \( \varphi = u = \phi = v_{n_k} \) in Lemma 3.1, there exists \( \beta \in (\frac{D(p+1)}{2b}, 2^*) \) such that

\[
\int_{\mathbb{R}^D} |x|^{-b}|v_{n_k}|^{p+1}dx \leq C(\|v_{n_k}\|_{L^{\beta}}^{p+1} + \|v_{n_k}\|_{L^{p+1}}).
\]

Since \( \beta, p+1 \in (2, 2^*) \), (V) would imply that \( \int_{\mathbb{R}^D} |x|^{-b}|v_{n_k}|^{p+1}dx \to 0 \) and so

\[
\lim_{k \to \infty} \sup_{k} E(v_{n_k}) = \limsup_{k \to \infty} \sup_{k} (H(v_{n_k}) - \frac{2}{p+1} \int_{\mathbb{R}^D} |v_{n_k}|^{p+1}dx) = \limsup_{k \to \infty} H(v_{n_k}) > 0,
\]

(3.4)

which contradicts (3.2).

Now suppose that (D) holds. It is worth pointing out that

\[
\lim_{k \to \infty} \int_{\mathbb{R}^D} |x|^{-b}|v_{n_k}|^{p+1}dx - \int_{\mathbb{R}^D} |x|^{-b}|w_k|^{p+1}dx - \int_{\mathbb{R}^D} |x|^{-b}|z_k|^{p+1}dx = 0.
\]

(3.5)

It then follows from property (D)(5) in Lemma 3.2 that

\[
\limsup_{k \to \infty} [E(w_k) + E(z_k)] \leq \liminf_{k \to \infty} H(v_{n_k}) - \frac{2}{p+1} \liminf_{k \to \infty} \int_{\mathbb{R}^D} |x|^{-b}|v_{n_k}|^{p+1}dx \leq \limsup_{k \to \infty} E(v_{n_k}) \leq 0.
\]

(3.6)

On the other hand, \( E(w_k) \geq 0 \), \( E(z_k) \geq 0 \) for sufficiently large \( k \) by using inequality (2.3) and property (D)(3) of Lemma 3.2, thus

\[
E(w_k) \to 0, \quad E(z_k) \to 0 \quad \text{as} \quad k \to \infty,
\]

which implies that

\[
H(w_k) \to 0, \quad H(z_k) \to 0 \quad \text{as} \quad k \to \infty.
\]

Therefore,

\[
\lim_{k \to \infty} \int_{\mathbb{R}^D} |x|^{-b}|v_{n_k}|^{p+1}dx = \lim_{k \to \infty} \left( \int_{\mathbb{R}^D} |x|^{-b}|w_k|^{p+1}dx + \int_{\mathbb{R}^D} |x|^{-b}|z_k|^{p+1}dx \right) = 0,
\]

again, it contradicts (3.4). Thus, to rule out (D), we just have to prove (3.5) holds. For this aim, defining \( \xi_k = v_{n_k} - w_k - z_k \), it follows from the construction of the sequence \( w_k \) and \( z_k \) in the proof of [2] that

\[
\|v_{n_k}\|_{L^{p+1}} - \|w_k\|_{L^{p+1}} - \|z_k\|_{L^{p+1}} \leq C\|v_{n_k}\| \|\xi_k\|
\]

and

\[
\|\xi_k\|_{L^2} \to 0 \quad \text{as} \quad k \to \infty.
\]

Since \( \|\xi_k\|_{L^q} \) is bounded by property (D)(5), applying the inequality (2.3), one can deduce that

\[
\|\xi_k\|_{L^q} \to 0 \quad \text{as} \quad k \to \infty, \quad \forall q \in (2, 2^*).
\]
Hence, it follows from Lemma 3.1 that

\[
\left| \frac{1}{\beta} \int_{|x-y_k|<R} |x-y|^{-\beta} |v_n(x-y_k)|^{p+1} \, dx \right| \leq C \int_{|x-y_k|<R} |x-y_k|^{-\beta} \left| v_n(x-y_k) \right| \, dx \leq C \left( \left\| v_n \right\|_{L^p}^{p+1} \right) \rightarrow 0 \text{ as } k \rightarrow \infty,
\]

which proves (3.5). Therefore, according to Lemma 3.2, there exist \( v \in H^1 \) and a sequence \( \{y_k\}_{k \in \mathbb{D}} \subset \mathbb{R}^D \) such that

\[ v_n (\cdot - y_k) \rightarrow v \text{ in } L^q(\mathbb{R}^D), \forall q \in [2, 2^*). \]  

(3.7)

Step 2: Localization. We will now show that \( \{y_k\} \) is bounded in \( \mathbb{R}^D \). Suppose that there exists a subsequence denoted still by \( \{y_k\} \) such that \( |y_k| \rightarrow \infty \) as \( k \rightarrow \infty \). Note that \( E(v_n) \) can be written as

\[ E(v_n) = H(v_n) - \frac{2}{p+1} \int_{\mathbb{R}^D} |x-y_k|^{-\beta} |v_n(x-y_k)|^{p+1} \, dx. \]  

(3.8)

Then we will show that the second term in the right hand side of (3.8) goes to zero as \( k \rightarrow \infty \), so that

\[ E(v_n) \rightarrow H(v_n) > 0 \text{ as } k \rightarrow \infty, \]  

(3.9)

which contradicts (3.2). We split the integral as

\[
\int_{|x-y_k|<R} |x-y_k|^{-\beta} |v_n(x-y_k)|^{p+1} \, dx + \int_{|x-y_k|>R} |x-y_k|^{-\beta} |v_n(x-y_k)|^{p+1} \, dx
\]

\[ = I + II, \]

for some \( R > 0 \). We first observe that, by Holder’s inequality,

\[ I \leq \left( \int_{|x-y_k|<R} |x-y_k|^{-\beta a} \, dx \right)^{1/a} \left( \int_{|x-y_k|<R} |v_n(x-y_k)|^{(p+1) \beta} \, dx \right)^{1/\beta}, \]  

(3.11)

where \( \alpha, \beta > 1 \) satisfy \( \frac{1}{\alpha} + \frac{1}{\beta} = 1 \). Then the first term in the right hand side of (3.11) is finite provided \( \beta > \frac{D}{D-\beta} \). Indeed, it is possible to choose \( \beta \) satisfies that \( \beta(p+1) \in (\frac{D(p+1)}{D-\beta}, 2^*) \) and it follows from (3.7) that

\[ \int_{|x-y_k|<R} |v_n(x-y_k)|^{(p+1) \beta} \, dx \rightarrow 0 \text{ as } k \rightarrow \infty. \]

On the other hand, by the Sobolev embedding Theorem and the boundedness of \( \{v_n(x-y_k)\} \) in \( H^1 \), the second term of (3.10) can be estimated as

\[ II \leq R^{-\beta} \int_{\mathbb{R}^D} |v_n(x-y_k)|^{p+1} \, dx \leq CR^{-\beta}. \]

Hence, \( II \) can be made arbitrarily small by choosing \( R \) large enough, uniformly in \( k \). This completes the proof of (3.9), and we conclude that the sequence \( \{y_k\} \subset \mathbb{R}^D \) is bounded.

Step 3: Conclusion. We shall suppose, without loss of generality, that \( y_k \rightarrow y^* \) as \( k \rightarrow \infty \), for some \( y^* \in \mathbb{R}^D \). Hence,

\[ v_n \rightarrow v^* = v(\cdot + y^*) \text{ in } L^q(\mathbb{R}^D), \forall q \in [2, 2^*). \]

Furthermore, we can also suppose that \( v_n \rightarrow v^* \) weakly in \( H^1 \). Since the Hardy functional \( H \) is weakly lower-continuous[17], \( v \mapsto \int_{\mathbb{R}^D} |x|^{-\beta} |v|^{p+1} \, dx \) is weakly sequentially continuous [9] and \( p + 1 \in (2, 2^*) \), it follows
that
\[ 0 \leq E(v^*) = H(v^*) - \frac{2}{p+1} \int |x|^{-b} |v^*|^{p+1} \, dx \leq \limsup_{k \to \infty} H(v_{n_k}) - \frac{2}{p+1} \int |x|^{-b} |v^*|^{p+1} \, dx = \limsup_{k \to \infty} E(v_{n_k}) \leq 0. \]

So from the inequality (2.3), one has that \( E(v^*) \geq 0 \), and so

\[ E(v^*) = 0 \quad \text{and} \quad H(v^*) = \lim_{k \to \infty} H(v_{n_k}). \] (3.12)

Together with \( \|v^*\|_{L^2} = \|W\|_{L^2} \) and the variational characteristic of the ground states, it implies that \( v^* = e^{i\gamma_0} W \) for some \( \gamma_0 \in \mathbb{R} \). Finally, we have \( \|v_{n_k}\|_{H^1} \to \|W\|_{H^1} = \|v^*\|_{H^1} \), \( \{v_{n_k}\} \) converges to \( v^* \) in \( H^1 \), which concludes the proof.

### 4 Classification

In this section, we construct finite time blow-up solutions with minimal mass \( \|W\|_{L^2} \) to the Cauchy problem (1.1)–(1.2). Furthermore, we prove the uniqueness of finite time blow-up solutions at the minimal mass threshold. We first show that the equation is invariant under the pseudo-conformal transformation, which is defined as follows.

**Lemma 4.1.** Let \( \varphi \) be a global solution of the Cauchy problem (1.1)–(1.2). Then, for all \( T \in \mathbb{R} \), the function

\[ \varphi_T(t, x) = e^{-i \frac{|x|^2}{4(T-t)^{D/2}}} \varphi \left( \frac{1}{T-t}, \frac{x}{T-t} \right) \]

is a solution of (1.1) and (1.2) on \( (-\infty, T) \), with \( \|\varphi_T\|_{L^2} = \|\varphi\|_{L^2} \).

Proof. By a direct calculation we have

\[ \partial_t \varphi_T(t, x) = e^{-i \frac{|x|^2}{4(T-t)^{D/2}}} \left[ -i \frac{|x|^2}{4} \varphi + \partial_t \varphi + \frac{D}{2} \varphi(T-t) + x \cdot \nabla \varphi \right] \left( \frac{1}{T-t}, \frac{x}{T-t} \right), \]

and

\[ \Delta \varphi_T = e^{-i \frac{|x|^2}{4(T-t)^{D/2}}} \left[ -i \frac{|x|^2}{4} \varphi - \frac{D}{2} \varphi(T-t) - ix \cdot \nabla \varphi + \Delta \varphi \right] \left( \frac{1}{T-t}, \frac{x}{T-t} \right) \]

for the derivatives. For the nonlinear term we find

\[ |x|^{-b} \varphi_T, \|x|^{-b} \varphi_T = e^{-i \frac{|x|^2}{4(T-t)^{D/2}}} \left[ \frac{x}{T-t} \right]^{-b} \left[ \varphi \left( \frac{1}{T-t}, \frac{x}{T-t} \right) \right], \]

and for the potential term

\[ |x|^{-2} \varphi_T = e^{-i \frac{|x|^2}{4(T-t)^{D/2}}} \left[ \frac{x}{T-t} \right]^{-2} \varphi \left( \frac{1}{T-t}, \frac{x}{T-t} \right). \]

Then it follows that

\[ i \partial_t \varphi_T + \Delta \varphi_T + c |x|^{-2} \varphi_T + |x|^{-b} \varphi_T \| \varphi_T \|^{\frac{4-b}{2}} \varphi_T \]

\[ = \frac{e^{-i \frac{|x|^2}{4(T-t)^{D/2}}} \left[ i \partial_t \varphi + \Delta \varphi + c |x|^{-2} \varphi + |x|^{-b} \varphi \| \varphi \|^{\frac{4-b}{2}} \varphi \right]}{\left( \frac{1}{T-t}, \frac{x}{T-t} \right)} = 0, \]

which proves the Lemma.
Remark 4.1. From Lemma 4.1, we claim that any solution \( u \) of (1.12) satisfies \( \| u \|_{L^2} \geq \| W \|_{L^2} \). Otherwise, applying Lemma 4.1 with \( u(t,x) = e^{it} \phi(x) \), one could construct a finite time blow-up solution below the minimal mass threshold, which would contradict the global well-posedness result Corollary 2.1.

We now construct a 3-parameter family solutions of the Cauchy problem (1.1)–(1.2) which blow-up in finite time by using the pseudo-conformal transformation and the symmetries of the equation.

**Proposition 4.1.** For all \( T \in \mathbb{R} \), \( \lambda_0 > 0 \) and \( \gamma_0 \in \mathbb{R} \), then the function \( \varphi_{T, \lambda_0, \gamma_0} \), defined by

\[
\varphi_{T, \lambda_0, \gamma_0}(t,x) = e^{i \gamma_0} e^{i \lambda_0 t} e^{-i \frac{|x|^2}{4T}} W\left( \lambda_0 x \right)
\]

(4.1)
is a minimal mass solution of the Cauchy problem (1.1)–(1.2) defined on \(( -\infty, T )\), and which blows up with speed

\[
\| \nabla \varphi_{T, \lambda_0, \gamma_0}(t, x) \|_{L^2} \sim \frac{C}{T-t}, \text{ as } t \to T, \text{ for } C > 0,
\]

where \( W \in \mathcal{A} \) is the ground state of (1.12).

**Proof.** The proof may be proved by applying Lemma 4.1 to the global solution

\[
\varphi_{\lambda_0, \gamma_0}(t,x) = e^{i \gamma_0} e^{i \lambda_0 t} W\lambda_0 x,
\]

which can be obtained under the scaling and phase symmetries of the standing wave \( \varphi(t,x) = e^{it} W(x) \).

**Remark 4.2.** Note that the blow-up solutions of the family showed in Proposition 4.1 can be retrieved from the solution

\[
\varphi_{0,1,0}(t,x) = e^{i \lambda_0 t} e^{-i \frac{1}{|t|^{D/2}}} W\left( \frac{x}{|t|^{D/2}} \right),
\]

defined on \(( -\infty, 0 )\) and which blows up at \( t = 0 \) with speed

\[
\| \nabla \varphi_{0,1,0}(0) \|_{L^2} \sim \frac{C}{|t|}, \text{ as } t \uparrow 0, \text{ for } C > 0.
\]

Indeed, all the solutions \( \varphi_{T, \lambda_0, \gamma_0} \) are equal to \( \varphi_{0,1,0} \), up to the symmetries. In other words, if we apply the changes \( \varphi(t,x) \to \lambda_0^{D/2} \varphi(\lambda_0^{-1} t, \lambda_0^{-1} x) \), \( \varphi(t,x) \to \varphi(t-T,x) \) and finally \( \varphi(t,x) \to e^{i \lambda_0 t} \varphi(t,x) \) to \( \varphi_{0,1,0} \), we obtain \( \varphi_{T, \lambda_0, \gamma_0} \).

In order to prove our main result, we also need to deduce information about \( \varphi(t) \) for \( t < T \) from the blow-up solution at \( t = T \). Using a result of Banica[1], we will show that \( \varphi(t) \in H^1 \) and \( x \varphi \in L^2(\mathbb{R}^D) \) for all \( t < T \). For \( \varphi \in H^1 \), \( \theta \in C_0^\infty(\mathbb{R}^D) \) and \( \eta \in \mathbb{R} \), we have that \( \nabla(\varphi e^{i\eta \theta}) = (\nabla \varphi + i \eta \nabla \theta) e^{i\eta \theta} \), and so

\[
|\nabla(\varphi e^{i\eta \theta})|^2 = |\nabla \varphi|^2 + 2 \eta \nabla \theta \cdot \text{Im}(\bar{\varphi} \nabla \varphi) + \eta^2 |\nabla \theta|^2 |\varphi|^2.
\]

By a direct calculation we have

\[
E(\varphi e^{i\eta \theta}) = E(\varphi) + 2 \eta \int_{\mathbb{R}^D} \nabla \theta \cdot \text{Im}(\bar{\varphi} \nabla \varphi) dx + \eta^2 \int_{\mathbb{R}^D} |\nabla \theta|^2 |\varphi|^2 dx.
\]

(4.2)

Then the refined Cauchy-Schwarz estimate for critical mass functions can be proved.

**Lemma 4.2.** Let \( \varphi \in H^1 \) be a function such that \( \| \varphi \|_{L^2} = \| W \|_{L^2} \). Then for all \( \theta \in C_0^\infty(\mathbb{R}^D) \), one has

\[
\left| \int_{\mathbb{R}^D} \nabla \theta \cdot \text{Im}(\bar{\varphi} \nabla \varphi) dx \right| \leq \sqrt{E(\varphi)} \left( \int_{\mathbb{R}^D} |\nabla \theta|^2 |\varphi|^2 dx \right)^{1/2}.
\]
Proof. For all $\eta \in \mathbb{R}$, we now have $\|\varphi e^{i\eta \theta}\|_{L^2} = \|\varphi\|_{L^2} = \|W\|_{L^2}$, so $E(\varphi e^{i\eta \theta}) \geq 0$ and $E(\varphi) \geq 0$ by (2.3). The result follows from the quadratic polynomial expression (4.2) in $\eta$ of $E(\varphi e^{i\eta \theta})$, which thus must have a non-positive discriminant. This completes the proof.

In the reminder of this section we prove the main result.

Proof of Theorem 1.1. Let $\varphi$ be a solution of the Cauchy problem (1.1)–(1.2) such that $\|\varphi\|_{L^2} = \|W\|_{L^2}$ and which blows up in finite time. The proof then falls into several steps.

Step 1. Let $\{t_n\}_{n \in \mathbb{N}} \subset \mathbb{R}$ be a sequence of times such that $t_n \to T$ as $n \to \infty$. We set

$$\varphi_n = \varphi(t_n), \quad \lambda_n = \sqrt{\frac{H(W)}{\mathcal{H}(\varphi, \varphi)}}.$$

We note that $\lambda_n \to \infty$ as $n \to \infty$, and

$$\|\varphi_n\|_{L^2} = \|\varphi\|_{L^2} = \|W\|_{L^2}, \quad H(\varphi_n) = \lambda_n^2 H(\varphi) = H(W).$$

On the other hand, by conservation of the energy,

$$E(\varphi_n) = H(\varphi_n) - \frac{D}{D + 2 - b} \int_{\mathbb{R}^d} |\varphi_n|^{-b} |\varphi_n|^{2+\frac{4}{d}} dx$$

$$= \lambda_n^2 H(\varphi) - \frac{D}{D + 2 - b} \lambda_n \int_{\mathbb{R}^d} |\varphi_n|^{-b} |\varphi_n|^{2+\frac{4}{d}} dx$$

$$= \lambda_n^2 E(\varphi) = \lambda_n^2 E(\varphi_0) \to 0 \text{ as } n \to \infty.$$

Hence, by Theorem 3.2, there exist $\gamma_0 \in \mathbb{R}$ such that, up to extracting a subsequence of $\{\varphi_n\}$, we have

$$\lim_{n \to +\infty} \|\varphi_n - e^{i\gamma_0} W\|_{H^1} = 0. \quad (4.4)$$

Step 2. Using the variational arguments, we prove that blowup’s profile is a Dirac function, i.e. in the sense of distributions,

$$|\varphi_n|^2 \to \|W\|_{L^2} \delta_0.$$

Indeed, by scaling (4.3), we observe that for $u \in C^\infty_0(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} |\varphi_n(x)|^2 u(\lambda_n x) dx = \int_{\mathbb{R}^d} |\varphi(x)|^2 u(x) dx,$$

and

$$\int_{\mathbb{R}^d} |\varphi_n(x)|^2 u(\lambda_n x) dx = \int_{\mathbb{R}^d} (|\varphi_n(x)|^2 - |W|^2) u(\lambda_n x) dx + \int_{\mathbb{R}^d} |W|^2 u(0) dx + \int_{\mathbb{R}^d} |W|^2 [u(\lambda_n x) - u(0)] dx.$$

Thus, we obtain

$$\left| \int_{\mathbb{R}^d} |\varphi_n(x)|^2 u(x) dx - \|W\|_{L^2}^2 u(0) \right| \leq \|u\|_{L^\infty} \int_{\mathbb{R}^d} (|\varphi_n(x)|^2 - |W|^2)^2 dx + \int_{\mathbb{R}^d} |W|^2 [u(\lambda_n x) - u(0)] dx.$$

(4.4) implies that $|\varphi_n|^2$ converges to $|W|^2$ strongly in $L^1(\mathbb{R}^d)$, so the first integral converges to 0 as $n \to +\infty$. Since $\lambda_n \to 0$, then the second integral also converges to 0 as $n \to +\infty$ by the dominated convergence theorem, thus

$$\int_{\mathbb{R}^d} |\varphi_n(x)|^2 u(x) dx \to \|W\|_{L^2} u(0). \quad (4.5)$$

Step 3. We claim that $\varphi(t) \in H^1$ and $x \varphi \in L^2(\mathbb{R}^d)$ for all $t \in [0, T)$. Let $\phi$ a nonnegative radial $C^\infty_0(\mathbb{R}^d)$ function such that $\phi(x) = |x|^2$ for $|x| \leq 1$ and $|\nabla \phi(x)|^2 \leq C \phi(x)$. For every $R > 0$, we define $\phi_R(x) = R^2 \phi(x/R)$, and for all $t \in [0, T),$

$$\phi_R(t) = R^2 \phi(x/R), \quad J_R(t) = \int_{\mathbb{R}^d} \phi_R(x) |\varphi(t, x)|^2 dx.$$
By a direct calculation (also see [2, 22]), we have
\[
J_R(t) = 2Re \int_{\mathbb{R}^D} \phi_R \overline{\psi} \, dx = 2 \int_{\mathbb{R}^D} \nabla \phi_R \cdot Im(\overline{\psi} \nabla \psi) \, dx.
\]
Since \( \|\phi\|_{L^2} = \|W\|_{L^2} \) and \( |\nabla \phi_R|^2 \leq C \phi_R \), we can apply Lemma 4.2 to get
\[
J_R(t) \leq 2 \sqrt{E(\phi)} \left( \int_{\mathbb{R}^D} |\nabla \phi_R|^2 |\psi|^2 \, dx \right)^{1/2} \leq C \sqrt{E(\phi_0)} \sqrt{J_R(t)}.
\]
By integration, we obtain that for \( t \in [0, T) \),
\[
|\sqrt{J_R(t)} - \sqrt{J_R(t_n)}| \leq C|t - t_n|.
\]
Since \( J_R(t_n) \to 0 \) by (4.5). Thus, letting \( t_n \to T \), we obtain that for all \( t \in [0, T) \) and all \( R > 0 \),
\[
J_R(t) \leq C(T-t)^2.
\]

Since the right-hand side of the last expression of \( R \), we obtain that by letting \( R \to \infty \), for all \( t \in [0, T) \),
\[
\varphi(t) \in H^1, \quad x \varphi \in L^2(\mathbb{R}^D) \quad \text{and} \quad 0 \leq J(t) \leq C(T-t)^2,
\]
where \( J(t) = \int_{\mathbb{R}^D} |x|^2 |\varphi|^2 \, dx \). From this estimate, we can extend by continuity \( J(t) \) at \( t = T \) by setting \( J(T) = 0 \), from which we also obtain \( J(T) = 0 \). Moreover, since \( \varphi(t) \in H^1, \quad x \varphi \in L^2(\mathbb{R}^D) \) and \( \varphi \) is a solution of the Cauchy problem (1.1)–(1.2), we obtain \( J(t) = 4E(\varphi_0)T^2 \), which finally gives, for all \( t \in [0, T) \),
\[
J(t) = 4E(\varphi_0)(T-t)^2.
\]

Letting \( t = 0 \), we find
\[
J(0) = \int_{\mathbb{R}^D} |x|^2 |\varphi_0|^2 \, dx = 4E(\varphi_0)T^2 \quad \text{and} \quad J'(0) = 4 \int_{\mathbb{R}^D} x \cdot Im(\overline{\psi_0} \nabla \varphi_0) \, dx = -8E(\varphi_0)T.
\]

Step 4. Determination of \( \varphi_0 \) and conclusion. We finally apply identity (4.2) to \( \varphi_0 \) and \( \eta = \frac{1}{2T} \), with \( \theta(x) = \frac{|x|^2}{2T} \).

Since \( \nabla \theta(x) = x \), we obtain
\[
E(\varphi_0 e^{i\frac{|x|^2}{2T}}) = E(\varphi_0) + \frac{1}{T} \int_{\mathbb{R}^D} x \cdot Im(\overline{\varphi_0} \nabla \varphi_0) \, dx = \frac{1}{4T^2} \int_{\mathbb{R}^D} |x|^2 |\varphi_0|^2 \, dx
\]
\[
= E(\varphi_0) + \frac{1}{T}(-2E(\varphi_0)T) + \frac{1}{4T^2}(4E(\varphi_0)T^2)
\]
\[
= 0.
\]

Note that this calculation justifies, a posteriori, the application of (4.2) with the function \( \theta(x) = \frac{|x|^2}{2T} \in C_0^\infty(\mathbb{R}^D) \).

Hence, we have \( \|\varphi_0 e^{i\frac{|x|^2}{2T}}\|_{L^2} = \|W\|_{L^2} \) and \( E(\varphi_0 e^{i\frac{|x|^2}{2T}}) = 0 \), and we deduce from the variational characteristic of ground state that there exist \( \lambda_0 > 0 \) and \( \gamma_0 \in \mathbb{R} \) such that
\[
\varphi_0 = e^{i\gamma_0} e^{-i\frac{|x|^2}{2T}} \lambda_0^\frac{T}{2} W(\lambda_0 x).
\]

Finally, we use the pseudo-conformal transformation. We define \( \widetilde{\lambda}_0 = \lambda_0 T > 0 \) and \( \tilde{\gamma}_0 = \gamma_0 - \lambda_0^\frac{T}{2} \in \mathbb{R} \), and write \( \varphi_0 \) as
\[
\varphi_0(x) = e^{i\gamma_0} e^{i\frac{|x|^2}{2T}} e^{-i\frac{|\lambda_0 T|^2}{2T}} \frac{\lambda_0^\frac{T}{2}}{\lambda_0} W \left( \frac{\lambda_0 x}{T} \right).
\]

Thus, \( \varphi_0(x) = \varphi_{T, \lambda_0, \gamma_0}(0) \), where \( \varphi_{T, \lambda_0, \gamma_0}(0) \) is defined by (4.1), which concludes the proof.

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