On Isosceles Triangles and Related Problems in a Convex Polygon

Amol Aggarwal
Saratoga High School
Saratoga, California
June 19, 2010

Abstract

Given any convex $n$-gon, in this article, we: (i) prove that its vertices can form at most $n^2/2 + \Theta(n \log n)$ isosceles triangles with two sides of unit length and show that this bound is optimal in the first order, (ii) conjecture that its vertices can form at most $3n^2/4 + o(n^2)$ isosceles triangles and prove this conjecture for a special group of convex $n$-gons, (iii) prove that its vertices can form at most $\lfloor n/k \rfloor$ regular $k$-gons for any integer $k \geq 4$ and that this bound is optimal, and (iv) provide a short proof that the sum of all the distances between its vertices is at least $(n-1)/2$ and at most $\lfloor n/2 \rfloor \lceil n/2 \rceil (1/2)$ as long as the convex $n$-gon has unit perimeter.

1 Introduction

In 1959, Erdős and Moser asked the following question in [11]: What is the maximum number of unit distances that can be formed by vertices of a convex $n$-gon? They conjectured that this bound should be linear, and in [9], Edelsbrunner and Hajnal provided a lower bound of $2n - 7$. On the other hand, Füredi provided an upper bound of $2\pi n \log_2 n - \pi n$ in [12], and recently in [6], Brass and Pach gave an upper bound of $9.65n \log_2 n$ using induction and geometric constraints different from those provided by Füredi. These bounds were later improved to $n \log_2 n + 4n$ in [2].

In [3], Altman proved that the number of distinct distances among all of the vertices of any convex $n$-gon is at least $\lceil n/2 \rceil$, a bound that is achieved by a regular polygon. Moreover, in [4], Altman proved several useful properties about the lengths of the diagonals of convex $n$-gons. Dumitrescu showed in [7] that at most $(11n^2 - 18n)/12$ isosceles triangles can be created by the vertices of a convex $n$-gon and uses this upper bound to show that there are at least $\lceil (13n - 6)/36 \rceil$ distinct distances from some vertex, thereby making progress on Erdős’s conjecture in [10] that there is a vertex in a convex $n$-gon that is at distinct distances from at least $\lceil n/2 \rceil$ other vertices. In [17], Pach and Tardos showed that the number of isosceles triangles formed by a set of $n$ vertices in the plane is at most $O(n^{2.136})$. In [1], Ábrego and Fernández-Merchant showed that there are at most $n - 2$ equilateral triangles that can be created by the vertices of any convex $n$-gon. Furthermore, in [16], Pach and Pinchasi showed that the number of unit distance equilateral triangles is at most $2(n - 1)/3$, and they exhibit a convex $n$-gon for which this bound is achieved.
Before we discuss the results of this paper, we define a few terms with regard to a convex polygon. Call an edge of a unit edge if the length of the edge is one and call a triangle a unit isosceles triangle if it has at least two unit edges. We call vertex $v$ a centroid if there exist three vertices, $v_1, v_2, v_3$ such that $d(v, v_1) = d(v, v_2) = d(v, v_3)$, where $d(u, v)$ is the Euclidean distance between two points $u$ and $v$ in the plane. The circle with center $v$ and radius $d(v, v_1)$ is one of $v$’s centroid-circles. Note that $v$ can have multiple centroid-circles. We say that two centroid-circles intersect if they share a vertex of the polygon, and call a centroid-circle intersecting if it intersects at least one other centroid-circle. In this article, we prove the following results.

**Theorem 1:** There are at most $n^2/2 + \Theta(n \log n)$ unit isosceles triangles formed by vertices of any convex $n$-gon.

This bound is sharp in the first term because we exhibit a convex $n$-gon that forms $(n^2 - 3n + 2)/2 + [(n - 1)/3]$ unit isosceles triangles.

**Theorem 2:** Suppose that $\mathcal{P}$ is a $n$-gon that has no centroid-circles that intersect. Then, there are at most $3(n + 1)^2/4$ isosceles triangles formed by vertices of $\mathcal{P}$.

**Theorem 3:** Suppose that $\mathcal{P}$ is a convex $n$-gon that has $k$ intersecting centroid-circles with $k = o(n^{2/3})$. Then, there are at most $3n^2/4 + o(n^2)$ isosceles triangles formed by vertices of $\mathcal{P}$.

In Section 3, we show that the there exists a convex $n$-gon that creates $(3n^2 - 11n + 8 + 2\lfloor n/2 \rfloor)/4$ isosceles triangles, meaning that these bounds are sharp in the first order.

**Theorem 4:** Let $n$ and $k$ be integers greater than 3. The maximum number of regular $k$-gons that can be found in a convex $n$-gon is $\lfloor n/k \rfloor$ and this bound is sharp.

**Theorem 5:** For any convex $n$-gon with unit perimeter, the sum $S_n$ of distances between its vertices satisfies $(n - 1)/2 \leq S_n \leq (1/2)\lfloor n/2 \rfloor \lfloor n/2 \rfloor$.

In Section 5, we show that the results of Altman in [4] can be easily used to prove a conjecture given in [5] by Audet, Hansen, and Messine regarding the sum of distances between the vertices of a convex $n$-gon with unit perimeter. This result has also been proven by Larcher and Pillichshammer in [13], and Dumitrescu later extends their proof to work for concave polygons in [8].

## 2 Number of Unit Isosceles Triangles

**Proposition 1:** There exists a polygon that forms $\frac{n^2 - 3n + 2}{2} + \left\lceil \frac{n - 1}{3} \right\rceil$ unit isosceles triangles.
Proof: Consider vertices \( v, v_1, v_2, v_3, \ldots, v_{n-1} \) such that \( v_1v_2\cdots v_nv \) is convex, \( d(v, v_i) = 1 \) for all \( 1 \leq i \leq n-1 \), and \( d(v_i, v_{i+k}) = d(v_{i+k}, v_{i+2k}) = 1 \), where \( k = \lfloor n/3 \rfloor \) and \( 1 \leq i \leq \lfloor (n-1)/3 \rfloor \). Then, \( \triangle vv_j \) is isosceles for any \( 1 \leq i < j \leq n-1 \). Moreover, \( \triangle v_i v_{i+k}v_{i+2k} \) is isosceles for any \( 1 \leq i \leq \lfloor (n-1)/3 \rfloor \). Thus, we have a total of \( (n^2-3n+2)/2 + \lfloor (n-1)/3 \rfloor \) isosceles triangles. \( \blacksquare \)

Theorem 1: The number of unit isosceles triangles that can be formed by vertices of a convex \( n \)-gon is at most \( n^2/2 + 4n \log n + 20n + 8 \) for sufficiently large \( n \).

Proof: The idea of the proof is based on Dumitrescu’s paper [8] and Moser’s paper [14]. Let the convex \( n \)-gon be \( P \). Consider the smallest circle that covers all vertices of \( P \). At least two vertices of the polygon lie on this circle. We examine two cases: one in which there are precisely two vertices on this circle and one in which there are at least three vertices on this circle.

Case 1: Only two vertices of \( P \) lie on this circle. Then, these two vertices must form the diameter of the circle. Let the polygon be \( v_1v_2v_3\cdots v_n \) with \( v_1v_k \) as the diameter of the circle. Let the vertices \( v_1, v_2, v_3, \ldots, v_k \) form set \( S \) and let \( v_k, v_{k+1}, v_{k+2}, \ldots, v_n, v_1 \) form set \( S' \). Let \( |S| = a \) and \( |S'| = b \). Then, \( n+2 \geq a+b \). Consider any vertex \( v_j \) in \( S \). For any \( i, j \) so that \( 1 < i \leq j \leq k \), \( \angle v_j v_{i-1} v_i \geq \angle v_1 v_i v_k \geq \pi/2 \), so \( d(v_{i-1}, v_j) > d(v_j, v_i) \), implying that the distances from \( v_j \) to the vertices between \( v_1 \) and \( v_j \) in \( S \) are all distinct. By similar logic, the distances between \( v_j \) and vertices between \( v_{j+1} \) and \( v_k \) in \( S \) are also distinct. Similarly, if \( v_j \in S' \), the distances from \( v_j \) to vertices between \( v_1 \) and \( v_j \) in \( S' \) would be distinct and so would those from \( v_j \) to vertices between \( v_j \) and \( v_k \) in \( S' \).

Consider any vertex \( v_i \) in \( S \). From the discussion in the previous paragraph, the number of vertices in \( S \) that are of unit distance from \( v_i \) is at most two. Therefore, the number of unit isosceles triangles with apex vertex \( v_i \) that are completely within \( S \) is one, and hence there are at most \( |S| = a \) unit isosceles triangles in \( S \). A similar result holds for \( S' \). Now, consider the number of unit isosceles triangle with its base completely within \( S \). There are \((a^2-a)/2\) bases in \( S \), and for each one, its perpendicular bisector can intersect \( S' \) in at most one place (or else convexity would be contradicted). Hence, there are at most \((a^2-a)/2\) unit isosceles triangles with their bases completely in \( S \). A similar result holds for \( S' \).

Finally, consider unit isosceles triangles such that a vertex of the base and the apex vertex are either both in \( S \) or \( S' \). Suppose both are in \( S \). For any vertex \( v \in P \), define \( g_s(v) \) to be the number of vertices in a subset \( s \in P \) that are of unit distance from \( v \) and let \( g(v) = g(v) \). Then, by the arguments given above, for any \( v \in S \), \( g_s(v) \leq 2 \). Suppose that two such vertices exist, namely \( v_1 \) and \( v_2 \) such that \( d(v_1, v) = d(v_2, v) = 1 \). Then, \( \triangle v_1 vu \) is isosceles if and only if \( d(v, u) = 1 \), so there are at most \( 2g(v) \) unit isosceles triangles with apex vertex \( v \) such that the base is within \( S \). Summing over all \( v \in P \), we attain that the number of unit isosceles triangles of the above type is at most \( 2 \sum_{v \in P} g(v) \), which corresponds to four times the number of unit distances in a convex \( n \)-gon. In [2], Aggarwal proved that there are at most \( n \log_2 n + 4n \) unit distances in a convex \( n \)-gon, and hence, there are at most
Conjecture 1: Let $I(n)$ denote the maximum possible number of isosceles triangles formed by vertices of a convex $n$-gon, with $n \geq 3$. Then, $I(n) \leq \frac{3n^2}{4} + \Theta(n)$.

4 $n \log_2 n + 16n$ triangles of this form.

Upon summing, we attain that there are at most

$$\frac{a^2 + b^2 - a - b}{2} + a + b + 4n \log_2 n + 16n < \frac{(a + b + 1)^2}{2} + 4n \log_2 n + 16n$$

$$\leq \frac{(n + 3)^2}{2} + 4n \log_2 n + 16n < \frac{n^2}{2} + 4n \log_2 n + 19n + \frac{9}{2}$$

unit isosceles triangles for sufficiently large $n$.

Case 2: There are at least three vertices of $P = v_1v_2v_3 \cdots v_n$ on the circle. Three of these vertices must form an acute triangle, say $v_1, v_x, v_y$ with $1 \leq x \leq y \leq n$. Let the vertices $v_1, v_2, \cdots v_x$ form $S_1$, the vertices $v_xv_{x+1}v_{x+2} \cdots v_y$ form $S_2$, and the vertices $v_yv_{y+1}v_{y+2} \cdots v_nv_1$ form $S_3$. Let $|S_1| = a, |S_2| = b$, and $|S_3| = c$. Since all vertices of $P$ lie in the region defined by the union of $\triangle v_xv_yv_1$, and the semicircles with diameters $v_1v_x, v_xv_y,$ and $v_1v_y$, $a + b + c \leq n + 3$. We proceed in a similar manner as before. Again count the total number of isosceles triangles included only in $S_1$, in only $S_2$, and only in $S_3$. By using the same argument as applied in Case 1, this number is at most $a + b + c \leq n + 3$. Also, by using the same reasoning as provided in Case 1, the number of unit isosceles triangles with a vertex of the base and the apex vertex in the same set is at most $n \log_2 n + 4n$. Now, we consider the case in which each vertex of the base is in a different set from the set in which the apex vertex resides. Suppose both vertices of the base lie in $S_1$. Then, there are $(a^2 - a)/2$ possible bases and the perpendicular bisector can hit $P - S_1$ in at most one place, thereby yielding at most $(a^2 - a)/2$ possible unit isosceles triangles with the base exclusively in $S_1$. Similar results hold for $S_2$ and $S_3$. Next, the case in which a base has one vertex in $S_2$ and the other in $S_3$. There are at most $bc$ such bases, and since each of their perpendicular bisectors can only hit $S_1$ in one place, there are at most $bc$ unit isosceles triangles with a base partly in $S_2$ and partly in $S_3$. Similar results hold for the others cases.

Upon summing these four quantities, the number of unit isosceles triangles is at most

$$\frac{a^2 + b^2 + c^2}{2} + ab + bc + ac + \frac{a + b + c}{2} + 4n \log_2 n + 16n$$

$$< \frac{(a + b + c + 1)^2}{2} + 4n \log_2 n + 16n < \frac{(n + 4)^2}{2} + 4n \log_2 n + 16n < \frac{n^2}{2} + 4n \log_2 n + 20n + 8$$

for sufficiently large $n$.

Remark: If the number of unit distances in a convex polygon can be shown to be at most $\Theta(n)$, then the number of unit isosceles triangles can be proven to be at most $n^2/2 + cn$ for a suitable constant $c$.

3 Number of General Isosceles Triangles
3.1 Preliminary Observations

Proposition 2: \( I(n) \geq \frac{1}{4}(3n^2 - 11n + 8 + 2 \left\lfloor \frac{n}{2} \right\rfloor) \).

Proof: Suppose \( n \) is even (the logic is identical for odd \( n \)) and let \( n = 2x \). Consider vertices \( v_1, v_2, v_3, \cdots, v_{n-1} \) on a circle with center \( v \) such that the polygon \( vv_{n-1}v_{n-2} \cdots v_1 \) is convex and \( d(v_i, v_{i+1}) = d(v_j, v_{j+1}) \) for all \( 1 \leq i, j \leq n - 2 \). Then, \( \triangle vv_i v_j \) is an isosceles triangle for all \( 1 \leq i < j \leq n \), and hence we obtain \( (n^2 - 3n + 2)/2 \) such isosceles triangles. Moreover, \( \triangle v_{i-1}v_i v_{i+1}, \triangle v_{i-2}v_i v_{i+2}, \cdots, \triangle v_1v_i v_{2i-1} \) are all isosceles for any integer \( 1 \leq i \leq x \). Hence, \( v_i \) is the apex vertex of \( i - 1 \) isosceles triangles. Summing this over \( 1 \leq i \leq x \) gives \( \sum_{i=1}^{x}(i-1) = (x^2 - x)/2 \) isosceles triangles with apex vertex being among the set \( \{v_1, v_2, v_3, \cdots, v_k\} \). Moreover, \( v_{n-i} \) is the apex vertex of \( i - 1 \) isosceles triangles for \( n-x-1 \leq i \leq n \), forming another \( \sum_{i=1}^{n-x-1}(i-1) = (n-x-1)(n-x-2)/2 \) isosceles triangles. Summing all three quantities yields the number of isosceles triangles to be \((3n^2 - 10n + 8)/4\). When \( n \) is odd, we can repeat the same process and attain \((3n^2 - 10 + 7)/4\) isosceles triangles.  

Definitions: Define the apex vertex of an isosceles triangle to be the vertex common to both legs of the triangle and say that an isosceles triangle belongs to its apex vertex. Let \( T(P) \) denote the number of isosceles triangles in a convex polygon \( P \), and note that \( I(n) = \max(T(P)) \) over all convex \( n \)-gons \( P \).

Proposition 3: Suppose \( P \) does not have any centroids. Then \( T(P) \leq n \left\lfloor \frac{n-1}{2} \right\rfloor \).

Proof: We in fact show that no vertex can be the apex vertex of more than \( \lfloor (n - 1)/2 \rfloor \) isosceles triangles, which proves the desired bound. Suppose that some vertex \( v \) is the apex vertex of more than \( (n - 1)/2 \) isosceles triangles. There are two base vertices for each isosceles triangle, which entails that there are more than \( n - 1 \) base vertices total. However, there are \( n - 1 \) vertices other than \( v \) in \( P \), and hence one vertex \( u \) is used in two isosceles triangles. Suppose that \( \triangle vut \) and \( \triangle vus \) are isosceles. Then, \( d(v, s) = d(v, u) = d(v, t) \), so \( v \) is a centroid, which is impossible, thereby proving proposition 3. Note that equality holds when \( P \) is a regular polygon.

3.2 Proof for Non-Intersecting Centroid-Circles

Theorem 2: Let \( P \) be a convex \( n \)-gon composed of \( k \) non-intersecting centroid-circles; then, \( T(P) \leq 3(n + 1)^2/4 \).

Proof: We prove \( T(P) \leq 3n^2/4 \) when \( n \) is even, which implies that \( T(P) \leq 3(n + 1)^2/4 \) when \( n \) is odd. Suppose that the centroid-circles are \( C_1, C_2, C_3, \cdots, C_k \), suppose that \( C_i \) has \( a_i \) vertices on its circle, and without loss of generality, suppose that \( a_1 \geq a_2 \geq a_3 \geq \cdots \geq a_k \). Since the centroids do not intersect, \( \sum_{i=1}^{k} a_i \leq n \). Let \( v_{ij} \) be the \( j \)th vertex in counterclockwise order on the circle of \( C_i \). Consider two cases: one in which there is a centroid-circle with more than \( n/2 \) vertices and the other in which there is not.
Case 1: $a_1 > \frac{n}{2}$, so $a_1 > \sum_{i=2}^{k} a_i$.

Set $a_1 - n/2 = x$. Let $S_1$ consist of $v_{i1}$ for $1 \leq i \leq x$. Let $S_2$ consist of $v_{i1}$ for $x + 1 \leq j \leq n - x$, and let $S_3$ consist of $v_{i1}$ for $n - x + 1 \leq i \leq n$. Note that the number of isosceles triangles due to all centroids is at most $\sum_{i=1}^{k} (a_i^2 - a_i)/2$. Take some $v_{ij}$ in $S_1$ that is not a centroid. $v_{ij}$ cannot be on the perpendicular bisector of the segment formed by two vertices on the circle of $C_1$ between $v_{i1}$ and $v_{i(j-1)}$, or else since $C_1$ also lies on this perpendicular bisector, contradicting convexity. Moreover, notice that $v_{ij}$ cannot be part of two triangles with apex vertex $v_{ij}$, or else $v_{ij}$ is a centroid by the logic used in proposition 3. Hence, $v_{ij}$ can have at most $j - 1$ isosceles triangles having a vertex on $C_1$ between $v_{i1}$ and $v_{i(j-1)}$. Through similar reasoning, $v_{ij}$ cannot be the apex vertex of a triangle with the two base vertices in $S_3$, so at least one of the vertices in any isosceles triangle with apex vertex $v_{ij}$ that does not have a vertex between $v_{i1}$ and $v_{ij}$ has a vertex in $C_2, C_3, \ldots, C_k$, which has cardinality $n - a_1$. Again, no vertex among these can be in two isosceles triangles with apex vertex $v_{ij}$, implying that $v_{ij}$ is an apex vertex of at most $n - a_1$ triangles having a base not entirely within $S_1$; as a result, $v_{ij}$ is an apex vertex of at most $n - a_1 + j - 1$ triangles. Analagously, if we take $v_{i(n-j)}$, for $j \leq x - 1$, at most $n - a_1 + j$ isosceles triangles can be formed. Summing this over all vertices in $S_1$ and $S_3$ yields at most $2 \sum_{i=1}^{x} (n - a_1 + j) \leq 2x(n - a_1) + x^2$ isosceles triangles. Now, each of the vertices in $S_2$ or $P - S_1 - S_2 - S_3$ can be the apex vertex of at most $n/2$ isosceles triangles, totalling $n(n - a_1)$ isosceles triangles. Suppose that $n - a_1 = b$. Summing the four quantities yields a total of

$$(a_1 + b)b + \frac{(a_1 - b)^2}{4} + (a_1 - b)b + \sum_{i=1}^{k} a_i(a_i - 1) < \frac{a_1^2 + 6a_1b + b^2}{4} + \frac{a_1^2 + b^2}{2} = \frac{3}{4} \cdot n^2$$

isosceles triangles.

Case 2: $a_i \leq \frac{n}{2}$ for all $1 \leq i \leq k$.

The centroid vertices give at most $\sum_{i=1}^{k} (a_i^2 - a_i)/2$ isosceles triangles. Any non-centroid vertex of $P$ can form at most $[(n - 1)/2]$ isosceles triangles by proposition 3, so the total number of isosceles triangles formed by non-centroid vertices is at most $n[(n - 1)/2]$, and hence the total number of isosceles triangles is at most

$$n \left[ \frac{n - 1}{2} \right] + \sum_{i=1}^{k} \frac{a_i^2 - a_i}{2} < \frac{n^2}{2} + \sum_{i=1}^{k} \frac{a_i^2}{2}$$

Since the function $f(x) = x^2$ is convex and $a_i \leq n/2$ for $1 \leq i \leq k$, $\sum_{i=1}^{k} a_i^2$ is maximized when $a_1 = a_2 = n/2$, yielding the number of isosceles triangles to be less than $3n^2/4$. ■

3.3 Potential Progress Towards Intersecting Centroid-Circles

We omit the proof of the following partial result:
Theorem 3: Suppose that $\mathcal{P}$ is a convex $n$-gon that has $k$ intersecting centroid-circles with $k = o(n^{2/3})$. Then, there are at most $3n^2/4 + o(n^2)$ isosceles triangles formed by vertices of $\mathcal{P}$.

3.4 Number of Distinct Distances From a Vertex

Proposition 4: In a convex polygon $\mathcal{P} = v_1v_2v_3 \cdots v_n$, let $d(v_i)$ be the number of distinct lengths among $v_1v_i, v_2v_i, \ldots, v_nv_i$. Let $d(\mathcal{P}) = \max_{1 \leq i \leq n} d(v_i)$. If $I(n) \leq 3n^2/4 + o(n^2)$, then $d(\mathcal{P}) \geq 5n/12 + o(n)$.

Proof: The method is identical to that of Dumitrescu given in [7]. Let $I(v)$ be the number of triangles a vertex $v \in \mathcal{P}$ is an apex of. Then, $\sum_{v \in \mathcal{P}} I(v) = T(\mathcal{P}) \leq 3n^2/4 + o(n^2)$. Let $k$ be the maximum number of distinct lengths coming from a single vertex. As noted by Dumitrescu, $T(\mathcal{P})$ is minimized when, for each vertex $v \in \mathcal{P}$, the other $n-1$ vertices distributed evenly on concentric circles centered at $v$, i.e., each circle contains either 2 or 3 vertices. Let there be $x$ circles with 2 vertices and $y$ circles with 3 vertices about some vertex $v$. Then, $2x + 3y = n - 1$ and $x + y \leq k$. Therefore, $x \leq 3k - n + 1$, thus $I(v) = x + 3y \geq 2n - 2 - 3k$. Consequently, $3n^2/4 + o(n^2) \geq I(\mathcal{P}) \geq n(2n - 2 - 3k)$, and so $k \geq 5n/12 + o(n)$.

4 Number of Regular Polygons

Pach and Pinchasi proved in [16] that there are at most $\lfloor 2(n-1)/3 \rfloor$ unit equilateral triangles in a convex $n$-gon, whereas in [1], Abrego and Fernández-Merchant provided an upper bound of $n-2$ (not necessarily unit) equilateral triangles. However, the precise bound on the number of equilateral triangles remains open. Therefore, we believe

Conjecture 2: The maximum number of equilateral triangles in a convex $n$-gon is at most $\lfloor 2(n-1)/3 \rfloor$.

Remark: Notice that $\lfloor 2(n-1)/3 \rfloor$ equilateral triangles are formed in the following position: let vertices $v, v_1, v_2, v_3, \ldots, v_{n-1}$ be such that $v_1v_2 \cdots v_nv$ is convex, $d(v, v_i) = 1$ for all $1 \leq i \leq n-1$, and $d(v_i, v_{i+k}) = d(v_{i+k}, v_{i+2k}) = 1$, where $k = \lfloor n/3 \rfloor$ and $1 \leq i \leq \lfloor (n-1)/3 \rfloor$. Then, $\Delta vv_{i+k}v_{i+2k}$ and $\Delta vv_{i+k}$ are equilateral for all $1 \leq i \leq \lfloor (n-1)/3 \rfloor$, which gives a total of $\lfloor 2(n-1)/3 \rfloor$ equilateral triangles. This configuration has been mentioned by both Abrego and Fernández-Merchant in [1] and Pach and Pinchasi in [16].

While we are unable to prove conjecture 2, we are able to find precise bounds for the number of regular $k$-gons in a convex $n$-gon, for $k \geq 4$.

Theorem 4: Let $n$ and $k$ be integers greater than 3. The maximum number of regular $k$-gons that can be found in a convex $n$-gon is $\lfloor n/k \rfloor$ and this bound is sharp.
Proof: We first show that equality can be achieved. Let \( n = qk + r \), where \( 0 \leq r < k \). Consider a regular \( qk \)-gon \( v_1v_2v_3 \cdots v_{qk} \) and place the other \( r \) vertices on the circumcircle of the \( k \)-gon arbitrarily. The polygon formed is convex and for any \( 1 \leq i \leq q \), \( v_iv_{q+i}v_{2q+i} \cdots v_{kq-q+i} \) is a regular \( k \)-gon and hence there are \( q = \lceil n/k \rceil \) regular \( k \)-gons in this polygon.

We now prove the upper bound. Let the polygon be \( v_1v_2v_3 \cdots v_n \). Let the degree of a vertex denote the number of regular \( k \)-gons that pass through that vertex and call two polygons disjoint if the intersection between the two polygons has area zero. We show that the degree of any vertex is at most one. Suppose to the contrary that a vertex, say \( v_1 \), has degree at least two. Consider the largest regular \( k \)-gon passing through \( v_1 \). Let it be \( U = u_1u_2 \cdots u_{k-1}v_1 \). Let one of the other regular \( k \)-gons be \( t_1t_2t_3 \cdots t_{k-1}v_1 \). If these polygons are disjoint, then \( \angle t_1v_1t_{k-1} = \pi - (2\pi/k) = \angle u_1v_1u_{k-1} \), and thus there is an angle of at least \( 2\pi - (4\pi/k) \geq \pi \) since \( k \geq 4 \), which contradicts convexity. Consequently, these \( k \)-gons are not disjoint and so, for some \( i \), \( v_it_i \) passes through \( U \). Let \( v_it_i \) hit \( U \) again at \( t \) and suppose \( t \) lies on \( u_ju_{j+1} \) for some \( j \). Then, \( d(v_1, t) < d(v_1, t_i) \leq d(v_1, u_1) = d(v_1, u_{k-1}) \).

Now, if \( \angle v_1u_1u_j \geq \pi/2 \) and \( \angle v_1u_1u_j \geq \pi/2 \) (or they are both at most than \( \pi/2 \), in which case rather than considering \( u_1 \), consider \( u_{k-1} \)), then \( \angle v_1u_1t \geq \pi/2 \), so \( v_it > v_1u_1 \), which is a contradiction. Otherwise, \( j \) is the unique vertex such that \( \angle v_1u_1u_j \leq \pi/2 \) and \( \angle v_1u_1u_j \geq \pi/2 \), hence \( j = \lfloor k/2 \rfloor \). In this case, \( \angle v_1u_1t \geq \angle v_1u_1u_{j+1} \) and \( \angle u_1v_1t \geq \angle u_1v_1u_j \), thus \( \angle v_1u_1t + \angle u_1v_1u_j \geq \pi - (2\pi/k) \), so \( \angle u_1v_1t \leq 2\pi/k \), implying that if \( k - j - 1 \geq 2 \), \( \angle u_1v_1t \geq 2\pi/k \geq \angle u_1v_1t \). This implies that \( d(v_1, u_1) \leq d(v_1, t) \), which is a contradiction. Consequently, \( k = 4 \), so \( \angle v_1u_1t = \pi/2 \), entailing that \( d(v_1, t) > d(v_1, u_1) \), which is a contradiction.

Therefore, every vertex has degree at most one, and as a result, the sum of the degrees is at most \( n \). However, every \( k \)-gon has \( k \) vertices, each having degree one, so there are at most \( \lceil n/k \rceil \) regular \( k \)-gons. ■

5 Polygons With Unit Perimeters

In this section, we use theorems of Altman given in [4] to prove Audet, Hansen, and Messine’s conjecture given in [5]. In 2008, in [13], Larger and Pillichshammer also prove this conjecture. Here, we give a simpler proof.

Theorem 5: For any convex \( n \)-gon with unit perimeter, the sum \( S_n \) of distances between its vertices satisfies

\[
\frac{n - 1}{2} \leq S_n \leq \frac{1}{2} \cdot \left\lceil \frac{n}{2} \right\rceil \left\lfloor \frac{n}{2} \right\rfloor
\]

Proof: Let the polygon be \( v_1v_2v_3 \cdots v_n \) and let \( \sum_{i=1}^{n} d(v_i, v_{i+j}) = u_j \) (where indices are taken modulo \( n \)). In his first theorem in [4], Altman shows that \( u_i < u_j \) whenever \( 1 \leq i < j \leq \lfloor n/2 \rfloor \). Since \( u_1 \) is the perimeter of the polygon, \( u_j \geq u_1 = 1 \) for all \( 1 \leq j \leq \lfloor n/2 \rfloor \). Moreover,
notice that, for any \( i \) and any \( j, k \leq \lfloor n/2 \rfloor \), \( d(v_i, v_{i+j}) + d(v_{i+j}, v_{i+k+j}) > d(v_i, v_{i+j+k}) \) by the triangle inequality. Summing over all \( i \) yields \( u_j + u_k > u_{j+k} \). In particular, \( u_2 < 2u_1 = 2 \), and by induction, \( u_i < i \) for all \( 1 \leq i \leq \lfloor n/2 \rfloor \). Observe that \( S_n = \sum_{i=1}^{(n-1)/2} u_i \) when \( n \) is odd and \( S_n = \sum_{i=1}^{(n-2)/2} u_i + (u_{n/2})/2 \) when \( n \) is even.

Therefore, if \( n \) is odd, then the following two inequalities hold:

\[
S_n = \sum_{i=1}^{n-1} u_i \geq \sum_{i=1}^{n-1} 1 = \frac{n - 1}{2}; \quad S_n = \sum_{i=1}^{n-1} u_i \leq \sum_{i=1}^{n-1} i = \frac{(n-1)(n+1)}{8} = \frac{1}{2} \cdot \left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil
\]

thereby, proving the theorem. Analogously, if \( n \) is even,

\[
S_n = \sum_{i=1}^{n-1} u_i + \frac{u_{n/2}}{2} \geq \sum_{i=1}^{n-1} 1 + \frac{1}{2} = \frac{n - 1}{2}; \quad S_n = \sum_{i=1}^{n-1} u_i + \frac{u_{n/2}}{2} \leq \sum_{i=1}^{n-1} i + \frac{n}{4} = \frac{n^2}{8}
\]

thereby, proving the theorem. \( \blacksquare \)

**Remark:** Audet, Hansen, and Messine have already shown that the lower bound is approached with a segment \([0, 1/2]\) with \( v_1 \) at 0, and \( v_2, v_3, ..., v_n \) arbitrarily close to \( 1/2 \) and the upper bound is approached with \( v_1, v_2, ..., v_{[n/2]} \) arbitrarily close to 0, and \( v_{[n/2+1]}, ..., v_n \) arbitrarily close to \( 1/2 \).

**References**

[1] B. Ábrego and S. Fernández-Merchant, On the maximum number of equilateral triangles II, *DIMACS Technical Report*, 99-47 (1999).

[2] A. Aggarwal, On Unit Distances in a Convex Polygon, manuscript, 2010.

[3] E . Altman, On a problem of P. Erdős, *Amer. Math. Monthly*, 70 (1963), 148-154.

[4] E. Altman, Some theorems on convex polygons, *Canad. Math. Bull.*, 15 (1972), 329-340.

[5] C. Audet, P. Hansen, and F. Messine, Extremal Problems for Convex Polygons, *Journal of Global Optimization*, 38(2) (2007), 163-169.

[6] P. Brass and J. Pach, The maximum number of times the same distance can occur among the vertices of a convex \( n \)-gon is \( O(n \log n) \), *J. Combin. Theory Ser. A*, 94 (2001), 178-179.

[7] A. Dumitrescu, On distinct distances from a vertex of a convex polygon, *Discrete and Computational Geometry*, 36 (2006), 506-509.

[8] A. Dumitrescu, Metric inequalities for polygons, [http://arxiv.org](http://arxiv.org), manuscript, 2010.
[9] H. Edelsbrunner and P. Hajnal, A lower bound on the number of unit distances between points of a convex polygon, *J. Combin. Theory Ser. A*, **56** (1991), 312-316.

[10] P. Erdős, On sets of distances of $n$ points, *Amer. Math. Monthly*, **53** (1946), 249-250.

[11] P. Erdős and L. Moser, Problem 11, *Canadian Math. Bulletin*, **2** (1959), 53.

[12] Z. Füredi, The maximum number of unit distances in a convex $n$-gon, *J. Combin. Theory Ser. A*, **55** (1990), 316-320.

[13] G. Larcher and F. Pillichshammer, The sum of distances between vertices of a convex polygon with unit perimeter, *American Mathematical Monthly*, **115** (2008), 350-355.

[14] L. Moser, On different distances determined by $n$ points, *Amer. Math Monthly*, **59** (1952), 85-91.

[15] J. Pach and P. K. Agarwal, *Combinatorial Geometry*, John Wiley, New York, 1995.

[16] J. Pach and R. Pinchasi, How many unit equilateral triangles can be generated by $n$ points in convex position?, *Amer. Math. Monthly*, **110** (2003), 400-406.

[17] J. Pach and G. Tardos, Isosceles triangles determined by a planar point set, *Graphs and Combinatorics*, **18** (2002), 769-779.