R-EQUIVALENCE AND $\mathbb{A}^1$-CONNECTEDNESS IN ANISOTROPIC GROUPS

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Abstract. We show that if $G$ is an anisotropic, semisimple, absolutely almost simple, simply connected group over a field $k$, then two elements of $G$ over any field extension of $k$ are $R$-equivalent if and only if they are $\mathbb{A}^1$-equivalent. As a consequence, we see that $\text{Sing}_* (G)$ cannot be $\mathbb{A}^1$-local for such groups. This implies that the $\mathbb{A}^1$-connected components of a semisimple, absolutely almost simple, simply connected group over a field $k$ form a sheaf of abelian groups.

1. Introduction

The notion of $R$-equivalence of rational points on a variety, introduced by Manin in 1970’s, has been extensively studied in the context of algebraic groups, where it provides a lot of information in the study of rationality properties. In this note, we explore a connection between the notions of $R$-equivalence in an algebraic group and the sheaf of $\mathbb{A}^1$-connected components, in the sense of Morel-Voevodsky.

Let $G$ be an algebraic group over a field $k$. If $G$ is an isotropic, semisimple, absolutely almost simple, simply connected group over $k$, classical results can be reinterpreted as saying that we have an isomorphism $G(k)/R \simeq \pi_{\mathbb{A}^1}^0 (G)(k)$, where $\pi_{\mathbb{A}^1}^0 (G)$ denotes the Nisnevich sheaf of $\mathbb{A}^1$-connected components of $G$ (see Theorem 3.4 below). In this note, we prove the following result:

Main Theorem. Let $G$ be an anisotropic, semisimple, absolutely almost simple, simply connected group over a field $k$ of characteristic $0$. Let $F$ be a field extension of $k$. Then the canonical morphism $G(F) \rightarrow \pi_{\mathbb{A}^1}^0 (G)(F)$ factors through the quotient morphism $G(F) \rightarrow G(F)/R$ and induces an isomorphism

$$G(F)/R \xrightarrow{\sim} \pi_{\mathbb{A}^1}^0 (G)(F).$$

Moreover, $\text{Sing}_* (G)$ is not $\mathbb{A}^1$-local. (Here $\text{Sing}_*$ denotes the Morel-Voevodsky singular complex construction in $\mathbb{A}^1$-homotopy theory.)

The conditions on $G$ in the statement of the Main Theorem are imposed only because our proof crucially depends on [7, Théorème 5.8], where they are required. It seems possible to lift the assumption on the characteristic of the base field (see Remark 4.3). It may be possible to generalize the Main Theorem to other classes of groups by proving a suitable generalization of [7, Théorème 5.8].

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This behaviour of anisotropic groups can be contrasted with the behaviour of isotropic groups. For instance, it has been shown that $\text{Sing}_G(G)$ is $\mathbb{A}^1$-local, when $G$ is smooth, split over a perfect field whose semisimple part has fundamental group of order prime to $\text{char } k$ (see [10, Proposition 5.11]), and when $G$ is an isotropic reductive group ([9, Proposition 4.1]). This allows one to study $\mathbb{A}^1$-connected components of $G$ in terms of naive $\mathbb{A}^1$-homotopies. Indeed, in this case, $\pi^1(G)(F)$ coincides with $S(G)(F)$ and with $G(F)/R$ and can be explicitly described as the quotient of $G(F)$ by its elementary subgroup $EG(F)$ (see [9] and Theorem 3.4 below).

A result of Chernousov-Merkurjev shows that the group of $R$-equivalence classes of a semisimple, absolutely almost simple, simply connected algebraic group over a field is abelian. This combined with classical results and our Main Theorem shows that for such groups, $\pi^1_0(\mathbb{A}^1)$ is a sheaf of abelian groups.

We now briefly outline the contents of this paper. In Section 2, we recollect preliminaries on $\mathbb{A}^1$-connectedness and a describe a geometric criterion for two points of an algebraic group to be $\mathbb{A}^1$-equivalent. In Section 3, we interpret known results about algebraic groups and $R$-equivalence in the setup of this paper. These facts are put together in Section 4 to give a proof of the Main Theorem.

2. Preliminaries on $\mathbb{A}^1$-connectedness

Let $k$ be a field and let $Sm/k$ denote the site of smooth schemes of finite type over $k$ along with the Nisnevich topology. We will work with the $\mathbb{A}^1$-homotopy category $\mathcal{H}(k)$ constructed in [8] by inverting all the projection maps of the form $X \times \mathbb{A}^1 \rightarrow X$ in the simplicial homotopy category $\mathcal{H}_s(k)$. We will follow the notation and terminology used in that paper. In this section, we will briefly recall some ideas from [2].

For any smooth scheme $U$ over $k$, we say that two morphisms $f, g : U \rightarrow X$ are $\mathbb{A}^1$-homotopic if there exists a morphism $h : U \times \mathbb{A}^1 \rightarrow X$ such that $h|_{U \times \{0\}} = f$ and $h|_{U \times \{1\}} = g$. We say that $h$ is an $\mathbb{A}^1$-homotopy and that it connects $f$ to $g$. We say that $f, g : U \rightarrow X$ are $\mathbb{A}^1$-chain homotopic if there exists a finite sequence $f_0 = f, \ldots , f_n = g$ such that $f_i$ is $\mathbb{A}^1$-homotopic to $f_{i+1}$, for all $i$. It is easy to see that $\mathbb{A}^1$-chain homotopy is an equivalence relation.

A simplicial sheaf $X$ is said to be $\mathbb{A}^1$-local if for any simplicial sheaf $\mathcal{Y}$, the projection map $\mathcal{Y} \times \mathbb{A}^1 \rightarrow \mathcal{Y}$ induces a bijection

$$\text{Hom}_{\mathcal{H}_s(k)}(\mathcal{Y}, X) \rightarrow \text{Hom}_{\mathcal{H}_s(k)}(\mathcal{Y} \times \mathbb{A}^1, X).$$

There exists an $\mathbb{A}^1$-localization endofunctor ([8, §2, Theorem 1.66 and p.107]) on the simplicial homotopy category $\mathcal{H}_s(k)$, denoted by $L_{\mathbb{A}^1}$, such that for every simplicial sheaf $X$, the simplicial sheaf $L_{\mathbb{A}^1}(X)$ is $\mathbb{A}^1$-local.

We next recall the Morel-Voevodsky singular complex construction $\text{Sing}_*$ in $\mathbb{A}^1$-homotopy theory (see [8, p.87]). For a simplicial sheaf $X$ on $Sm/k$, define $\text{Sing}_*(X)$ to be the simplicial sheaf given by

$$\text{Sing}_*(X)_n = \underline{\text{Hom}}(\Delta_n, X_n),$$
where $\Delta_•$ denotes the cosimplicial sheaf

$$\Delta_n = \text{Spec} \left( \frac{k[x_0, \ldots, x_n]}{(\sum_i x_i = 1)} \right)$$

with the natural coface and codegeneracy maps motivated from the ones on topological simplices.

**Definition 2.1.** Let $\mathcal{X}$ be a simplicial sheaf on $\text{Sm}/k$. The sheaf of $\mathbb{A}^1$-chain connected components of $\mathcal{X}$ is defined by

$$S(\mathcal{X}) := \pi^0_0(\text{Sing}_*(\mathcal{X})),$$

where $\pi^0_0$ of a simplicial sheaf denotes the sheaf of its simplicially connected components.

If $X$ is a scheme over $k$, then it is easy to see that $S(X)$ is the sheafification in Nisnevich topology of the presheaf on $\text{Sm}/k$ that associates with every smooth scheme $U$ over $k$ the set of equivalence classes in $X(U)$ under the relation of $\mathbb{A}^1$-chain homotopy.

**Definition 2.2.** Let $\mathcal{X}$ be a simplicial sheaf on $\text{Sm}/k$. The sheaf of $\mathbb{A}^1$-connected components of $\mathcal{X}$ is defined by

$$\pi^{\mathbb{A}^1}_0(\mathcal{X}) := \pi^0_0(L_{\mathbb{A}^1}(\mathcal{X})).$$

The main obstacle in the study of $\pi^{\mathbb{A}^1}_0$ of a simplicial sheaf is the explicit description of the $\mathbb{A}^1$-localization functor is cumbersome to handle. The following result, proved in [2], allows us to use geometric methods in the study of the $\mathbb{A}^1$-connected components sheaf of a smooth scheme over $k$.

**Theorem 2.3.** Let $\mathcal{F}$ be a sheaf of sets on $\text{Sm}/k$. Then the sheaf $\lim \rightarrow S^n(\mathcal{F})$ is $\mathbb{A}^1$-invariant. Moreover, if $\pi^{\mathbb{A}^1}_0(\mathcal{F})$ is $\mathbb{A}^1$-invariant, then the canonical map

$$\pi^{\mathbb{A}^1}_0(\mathcal{F}) \to \lim \rightarrow S^n(\mathcal{F})$$

is an isomorphism.

This suggests a method to verify when two sections of a sheaf map to the same element in its $\pi^{\mathbb{A}^1}_0$ (see Lemma 2.4 below). We will use a well-known characterization of Nisnevich sheaves, which we will recall here for the sake of convenience.

For any scheme $U$, an *elementary Nisnevich cover* of $U$ consists of two morphisms $p_1 : V_1 \to U$ and $p_2 : V_2 \to U$ such that:

(i) $p_1$ is an open immersion.

(ii) $p_2$ is an étale morphism and its restriction to $p_2^{-1}(U \setminus p_1(V_1))$ is an isomorphism onto $U \setminus p_1(V_1)$.

Then a presheaf of sets $\mathcal{F}$ on $\text{Sm}/k$ is a sheaf in Nisnevich topology if and only if the morphism

$$\mathcal{F}(U) \to \mathcal{F}(V_1) \times_{\mathcal{F}(V_1 \times_U V_2)} \mathcal{F}(V_2)$$

is an isomorphism, for all elementary Nisnevich covers $\{V_1, V_2\}$ of $U$. (See [8, §3, Proposition 1.4, p.96] for a proof.)
Lemma 2.4. Let $F$ be a sheaf of sets over $Sm/k$ such that the sheaf $\pi^A_0(F)$ is $A^1$-invariant. Let $U$ be a smooth scheme over $k$ and let $f, g : U \to F$ be two morphisms. Suppose that we are given data of the form

$$\{p_V : V \to A^1_U, p_W : W \to A^1_U\}, \{\sigma_0, \sigma_1\}, \{h_V, h_W\}, h$$

satisfying the following conditions:

- The two morphisms $\{p_V : V \to A^1_U, p_W : W \to A^1_U\}$ constitute an elementary Nisnevich cover.
- For $i \in \{0, 1\}$, $\sigma_i$ is a morphism $U \to V \coprod W$ such that $(p_V \coprod p_W) \circ \sigma_i : U \to U \times A^1$ is the closed embedding $U \times \{i\} \hookrightarrow U \times A^1$.
- $h_V$ and $h_W$ are morphisms from $V$ and $W$ respectively into $F$ such that $(h_V \coprod h_W) \circ \sigma_0 = f$ and $(h_V \coprod h_W) \circ \sigma_1 = g$.
- Let $\text{pr}_V : V \times A^1_U W \to V$ and $\text{pr}_W : V \times A^1_U W \to W$ denote the projection morphisms. Then $h = (h_1, \ldots, h_n)$ is an $A^1$-chain homotopy connecting the two morphisms $h_V \circ \text{pr}_V$ and $h_W \circ \text{pr}_W : V \times A^1_U W \to F$.

Then $f$ and $g$ map to the same element under the map $F(U) \to \pi^A_0(F)(U)$.

Proof. The data given above (which, in the terminology of [2], is a special case of an “$A^1$-ghost homotopy”), gives rise to a homotopy $H : A^1_U \to S(F)$. Indeed, since $\{p_V, p_W\}$ is an elementary Nisnevich cover and since $S(F)$ is a Nisnevich sheaf, the two compositions

$$U \xrightarrow{h_i} F \to S(F)$$

can be glued together to give a morphism $A^1_U \to S(F)$ which connects the images of $f$ and $g$ in $S(F)(U)$. Thus $f$ and $g$ map to the same element of $S^2(F)(U)$. We have the following commutative diagram:

$$\begin{array}{ccc}
S(F) & \longrightarrow & \pi^A_0(F) \\
\downarrow & & \downarrow \\
S^2(F) & \longrightarrow & \lim_{\longrightarrow} S^n(F)
\end{array}$$

Since $\pi^A_0(F)$ is $A^1$-invariant, by Theorem 2.3 we have $\pi^A_0(F) \sim \lim_{\longrightarrow} S^n(F)$. Therefore, $f$ and $g$ map to the same element of $\pi^A_0(F)(U)$. \qed

Remark 2.5. Using the arguments in [2, Section 4.1], one can see that Lemma 2.4 holds even without the hypothesis that $\pi^A_0(F)$ is $A^1$-invariant. However, we make this simplifying assumption since we only need to use it in a situation where $\pi^A_0(F)$ is known to be $A^1$-invariant.

3. Algebraic groups and $R$-equivalence

Definition 3.1. Let $G$ be an algebraic group over a field $k$. Two $k$-rational points $x, y$ of $G$ are said to be $R$-equivalent if there is a rational map $f : \mathbb{P}^1_k \to G$ defined at 0 and 1 such that $f(0) = x$ and $f(1) = y$.

The relation of $R$-equivalence generates a normal subgroup of $G(k)$ and one denotes the group of $R$-equivalence classes of the set $G(k)$ by $G(k)/R$. 
Proof. Note that the canonical quotient map through the map \( G_k \) almost simple group over an infinite field Theorem 3.4. Let which will play a crucial role in our proof of the Main Theorem.

is called the Whitehead group of \( G \) over \( F \).

We now state an interpretation of the known results in the isotropic case, which will play a crucial role in our proof of the Main Theorem.

Theorem 3.4. Let \( G \) be an isotropic, semisimple, simply connected, absolutely almost simple group over an infinite field \( k \). Then there is an isomorphism

\[
\pi_0^\mathbb{A}^1(G)(k) \simeq G(k)/R.
\]

Proof. Note that the canonical quotient map \( G(k) \to G(k)/R \) clearly factors through the map \( G(k) \to S(G)(k) \). By [7, Théorème 7.2], we identify \( G(k)/R \) with the Whitehead group \( W(k, G) \). Therefore, any two \( R \)-equivalent elements of \( G(k) \) differ by an element of \( G(k)^+ \), which gives an \( \mathbb{A}^1 \)-chain homotopy between the two elements. This shows that \( S(G)(k) = G(k)/R \).

A result of Völkel-Wendt [9, Corollary 3.4, Proposition 4.1] and Moser (unpublished) says that for an isotropic reductive group \( G \), \( \text{Sing}_a(G) \) is \( \mathbb{A}^1 \)-local. Therefore, the canonical map \( S(G) \to \pi_0^\mathbb{A}^1(G) \) is an isomorphism. \( \square \)

We next quote a straightforward consequence of [4, 8.2].

Theorem 3.5 (Borel-Tits). Let \( G \) be a smooth affine group scheme over a perfect field \( k \). Then the following are equivalent:

(1) \( G \) admits no \( k \)-subgroup isomorphic to \( \mathbb{G}_a \) or \( \mathbb{G}_m \).

(2) \( G \) admits a \( G \)-equivariant compactification \( \overline{G} \) such that \( G(k) = \overline{G}(k) \).

We end this section by noting down a few simple observations, which will be useful in the proof of the Main Theorem.

Lemma 3.6. Let \( G \) be an anisotropic group over a perfect field \( k \). Then any rational map \( h : \mathbb{P}^1_k \to G \) is defined at all the \( k \)-rational points of \( \mathbb{P}^1_k \).

Proof. By Theorem 3.5, there exists a compactification \( \overline{G} \) of \( G \) such that \( G(k) = \overline{G}(k) \). Clearly \( h \) can be extended to a morphism \( \overline{h} : \mathbb{P}^1_k \to \overline{G} \) and the lemma follows. \( \square \)

Lemma 3.7. Let \( G \) be an anisotropic group over a perfect field \( k \). Then there are no non-constant morphisms from \( \mathbb{A}^1_k \) into \( G \) and consequently,

\[
S(G)(k) = G(k).
\]

Proof. Again, obtain a compactification \( \overline{G} \) of \( G \) such that \( G(k) = \overline{G}(k) \) by applying Theorem 3.5. Any morphism \( h : \mathbb{A}^1_k \to G \) can be extended to a morphism \( \overline{h} : \mathbb{P}^1_k \to \overline{G} \). By Lemma 3.6, the morphism \( \overline{h} \) maps all the \( k \)-rational points of \( \mathbb{P}^1_k \) into \( G(k) \). Since \( h \) maps every point of \( \mathbb{P}^1_k \) other than \( \infty \) into \( G \).
anyway, we see that \( \overline{h} \) maps \( \mathbb{P}^1_k \) into \( G \) which is an affine scheme. Thus, \( \overline{h} \) is the constant map. This shows that \( S(G)(k) = G(k) \). \( \square \)

4. Proof of the main theorem

This section will be devoted to the proof of the Main Theorem stated in the introduction. We recall that according to [6, Theorem 4.18], for any algebraic group \( G \), the sheaf \( \pi^1_0(G) \) is \( \mathbb{A}^1 \)-invariant. This allows us to use Lemma 2.4 in the following proof.

**Conventions 4.1.** We will use the following conventions in this section:

1. For any scheme \( X \) over \( k \) and any field extension \( L/k \), \( X_L \) will denote the pullback \( X \times_{\text{Spec}(k)} \text{Spec}(L) \) over \( L \). Similarly, for any morphism \( f : X \to Y \) between schemes over \( k \), we will denote by \( f_L : X_L \to Y_L \) the pullback of \( f \) with respect to the projection \( Y_L \to Y \).
2. For any smooth scheme \( U \) over \( k \) and any sheaf \( F \) on \( \text{Sm}/k \), we will say that \( f, g \in F(U) \) are \( \mathbb{A}^1 \)-equivalent if they map to the same element of \( \pi^1_0(F)(U) \).

**Theorem 4.2.** Let \( G \) be an anisotropic, semisimple, absolutely almost simple, simply connected group over a field \( k \) of characteristic 0. Let \( F \) be a field extension of \( k \). Then two elements of \( G(F) \) are \( R \)-equivalent if and only if they are \( \mathbb{A}^1 \)-equivalent.

**Proof.** In view of Theorem 3.4, observe that it suffices to prove the theorem in the case \( F = k \).

**Proof of the “if” part:** By Theorem 3.5, there exists a compactification \( \overline{G} \) of \( G \) such that \( G(k) = \overline{G}(k) \). If two elements \( p \) and \( q \) of \( G(k) \) are \( \mathbb{A}^1 \)-equivalent, then \( p \) and \( q \) map to the same element in \( \pi^1_0(\overline{G})(k) \). Since \( \overline{G} \) is proper over \( k \), we can apply Theorem [1, Theorem 2.4.3] to conclude that \( p \) and \( q \) map to the same element in \( S(\overline{G})(k) \). Therefore, \( p \) and \( q \) are \( \mathbb{A}^1 \)-chain homotopic \( k \)-rational points of \( \overline{G} \). Since \( \overline{G}(k) \setminus G(k) = \emptyset \), it follows that \( p \) and \( q \) map to the same element in \( G(k)/R \).

**Proof of the “only if” part:** Let \( p \) and \( q \) be two elements of \( G(k) \), which are \( R \)-equivalent. Thus, there is a rational map \( h : \mathbb{P}^1_k \to G \) which is defined on 0 and 1 such that \( h(0) = p \) and \( h(1) = q \). Choose a compactification \( \overline{G} \) of \( G \) such that \( G(k) = \overline{G}(k) \). The rational map \( h \) can be uniquely extended to a morphism \( \overline{h} : \mathbb{P}^1_k \to \overline{G} \). By Lemma 3.6, \( \overline{h} \) maps all the \( k \)-rational points of \( \mathbb{P}^1_k \) into \( G \). Thus, we see that \( h \) is undefined only at points of \( \mathbb{A}^1_k \) having residue fields that are non-trivial finite extensions of \( k \). We define \( V := \overline{G}^{-1}(G) \cap \mathbb{A}^1_k \) which is a Zariski open subscheme of \( \mathbb{A}^1_k \). Let \( \mathbb{A}^1_k \setminus V = \{ p_1, \ldots, p_n \} \) and let the residue field at \( p_i \) be \( L_i \). We define \( h_V : V \to G \) by \( h_V := \overline{h}|_V \).

We claim that for each \( i \), \( G_{L_i} \) is an isotropic group. Indeed, the rational map \( h_{L_i} : \mathbb{P}^1_{L_i} \to G_{L_i} \) is not defined at an \( L_i \)-rational point. Hence, by Lemma 3.6, \( G_{L_i} \) cannot be anisotropic.

Since the group \( G_{L_i} \) is isotropic, we may apply [7, Théorème 5.8], which says that \( W(L_i, G) = W(L_i(t), G) \). Thus any element of \( G_{L_i}(L_i(t)) \) can be connected by an \( \mathbb{A}^1 \)-chain homotopy to an element in the image of the natural
Choose a preimage $p'_i$ of $p_i$ under the projection map $\mathbb{A}^1_{L_i} \to \mathbb{A}^1_k$ for each $i$ and denote by $V_i$ the open subscheme of $\mathbb{A}^1_{L_i}$ given by $V'_i \cup \{p'_i\}$. Let $q_i$ be the image of $q'_i$ under the projection $G_{L_i} \to G$. We define $h_i : V_i \to G$ to be the constant map taking $V_i$ to the point $q_i$. Let $W := \prod_i V_i$ and let $h_W : W \to G$ be the map $\prod_i h_i$.

We define $p_V : V \to \mathbb{A}^1_k$ to be the inclusion. For each $i$, we define $p_i : V_i \to \mathbb{A}^1_k$ to be the composition $V_i \to \mathbb{A}^1_{L_i} \to \mathbb{A}^1_k$. Let $p_W : W \to \mathbb{A}^1_k$ be the map $\prod_i p_i$. Since $p_W^{-1}(\mathbb{A}^1_k \setminus V) = \{p'_1, \ldots, p'_n\}$, it is easy to see that $\{p_V, p_W\}$ is an elementary Nisnevich cover of $\mathbb{A}^1_k$. In order to apply Lemma 2.4, we need to show that the morphisms $h_V \circ p_V$ and $h_W \circ p_W$ from $V \times_{\mathbb{A}^1_k} W$ to $G$ are $\mathbb{A}^1$-chain homotopic.

For every $1 \leq i \leq n$, we have $V \times_{\mathbb{A}^1_k} V_i = V'_i$. Thus $V \times_{\mathbb{A}^1_k} W = \prod_i V'_i$. The morphism $p_W|_{V'_i}$ is equal to the composition $V'_i \hookrightarrow V_{L_i} \to V$. Also, the morphism $p_W|_{V'_i}$ is equal to the composition of inclusions $V'_i \subset V_i \subset W$.

For each $i$, we have the commutative diagrams

$$
\begin{array}{ccc}
V'_i & \longrightarrow & V_{L_i} \\
\downarrow & & \downarrow \Phi \\
V & \rightarrow & G
\end{array}
$$

and

$$
\begin{array}{ccc}
V'_i & \longrightarrow & V_i \\
\downarrow & & \downarrow \hspace{1cm} h_W|_{V_i} \\
V & \rightarrow & G
\end{array}
$$

where $c_{q'_i}$ is the constant map taking the scheme $V_{L_i}$ to $q'_i$. By assumption, there exists an $\mathbb{A}^1$-chain homotopy connecting the maps $(h_V)_{L_i}|_{V'_i}$ to the map $c_{q'_i}|_{V'_i}$. On composing with the projection map $G_{L_i} \to G$, this gives an $\mathbb{A}^1$-chain homotopy connecting the morphism $h_V \circ p_V|_{V'_i}$ to the morphism $h_W \circ p_W|_{V'_i}$. Thus, there exists an $\mathbb{A}^1$-chain homotopy connecting the morphisms $h_V \circ p_V$ to the morphism $h_W \circ p_W$.

Thus, we may now apply Lemma 2.4 to conclude that $p$ and $q$ map to the same element in $\pi^\wedge_0(G)(k)$. This completes the proof of Theorem 4.2. □

Remark 4.3. An unpublished result of Gabber generalizes Theorem 3.5 to fields that are not perfect and to groups that are not necessarily smooth. This can be used to generalize Theorem 4.2 to fields that are not perfect by closely following the proof of Theorem 4.2. The only adjustment needed is in the proof of the “if” part, where one replaces the use of [1, Theorem 2.4.3] with the use of [2, Theorem 2 in the Introduction].
Corollary 4.4. Let $G$ be as in Theorem 4.2. Then $\text{Sing}_*(G)$ cannot be $\mathbb{A}^1$-local.

Proof. We simply note that there does exist a pair of distinct $R$-equivalent elements in $G(k)$. Indeed, this is an immediate consequence of the fact that $G$ is unirational over $k$ (see [3, Theorem 18.2]). Thus, the map $\mathcal{S}(G)(k) \rightarrow \pi^A_0(G)(k)$ is not a bijection. This shows that $\text{Sing}_*(G)$ cannot be $\mathbb{A}^1$-local.

□

This completes the proof of the Main Theorem.

Remark 4.5. A long-standing open question in the study of $R$-equivalence asks if the group of $R$-equivalence classes $G(k)/R$ of a reductive algebraic group is always abelian. This has been proved by Chernousov and Merkurjev (see [7, Théorème 7.7] and [5, 1.2]) in the case when $G$ is a semisimple, simply connected, absolutely almost simple and of classical type over $k$.

Thus, it is natural to conjecture the following.

Conjecture 4.6. Let $G$ be a reductive algebraic group over a field $k$. Then $\pi^A_0(G)(F) = G(F)/R$, for all field extensions $F$ of $k$.

We end with a question posed by Anastasia Stavrova, which is open:

Question 4.7. Let $G$ be a reductive algebraic group over a field $k$. Is $\pi^A_0(G)$ a sheaf of abelian groups?

Remark 4.8. We briefly explain how giving an affirmative answer to Question 4.7 is equivalent to giving an affirmative answer to the question of abelian-ness of the group of $R$-equivalence classes of a reductive algebraic group $G$ over a field $k$, if Conjecture 4.6 holds. One implication is obvious. For the other, observe that if $G(F)/R$ is abelian for any field extension $F/k$, to answer Question 4.7 affirmatively, it suffices to prove that $\pi^A_0(G)(\text{Spec } A)$ is an abelian group for regular henselian rings $A$ containing $k$. This follows from [6, Corollary 4.17], which implies that $\pi^A_0(G)(\text{Spec } A)$ injects into $\pi^A_0(G)(\text{Spec } Q(A))$, where $Q(A)$ denotes the quotient field of $A$. This proves the other implication. This gives an affirmative answer to Question 4.7 in the case when $G$ is a semisimple, simply connected automatically almost simple and of classical type over a field $k$ (see Remark 4.5 above).

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