On $^1S_0$ pairing for neutrons in dense neutron matter induced by a soft pion

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(Dated: September 26, 2014)

Neutron pairing in the $^1S_0$ channel is shown to occur in dense neutron matter in a vicinity of the $\pi^0$ condensation point. The $^1S_0$ pairing gap $\Delta$ results from an exchange of a soft neutral pionic mode that contributes significantly to the $nn$ pairing force. A soft pion induced potential $V_S(p)$ is characterized by an attenuating oscillatory behavior in coordinate space, while in momentum space all $S$-wave matrix elements $V_S(p, p')$ are positive. The solution of the gap equation reveals strong momentum dependence.

PACS numbers: 21.65.-f, 26.60.-c, 71.10.Ay

I. INTRODUCTION

Pairing in neutron matter in the $^1S_0$ partial wave channel is sufficiently well studied. On the level of the BCS approach with the use of a bare $NN$ potential as a pairing interaction and free single-particle energies, it is well established that the $^1S_0$ pairing correlations exist in neutron matter in a range of densities $n \leq n_S \simeq 0.18$ fm$^{-3}$ [1]. The upper limit corresponds to the neutron Fermi momentum $p_F \simeq 1.75$ fm$^{-1}$. Various investigations of nucleon pairing beyond BCS have shown that account of in-medium corrections to the pairing interaction and renormalization of the single-particle spectrum both lead to rather strong reduction of the $^1S_0$ pairing gap whereas the critical value $n_S$ changes not significantly [2–8]. Therefore neutron superfluidity at densities $n \geq n_S$, which are relevant to a neutron star core region, $n \gtrsim 0.5 n_0$, where $n_0 \simeq 0.16$ fm$^{-3}$ is the nuclear saturation density, is usually connected with the $^3P_2^-$ $^3F_2$ coupled channel [1, 9, 10]. In this work we show that the $^1S_0$ pairing is possible to reappear in dense neutron matter in a vicinity of the $\pi^0$ condensation instability [11–13] where the collective $\pi^0$-like excitations become quasiquasifermions.

In homogeneous neutron matter, the $^1S_0$ propagator is given by the familiar expression,

$$D(\omega, q) = (\omega^2 - q^2 - m^2 - \Pi(\omega, q; n))^{-1},$$

where the density dependent polarization operator $\Pi(\omega, q; n)$ accounts for in-medium pion scattering processes including particle-hole ($NN$) and $\Delta$-isobar-hole ($\Delta N$) excitations. The dispersion law $\omega = \omega(q)$ of the in-medium pionic field is identified with poles of the propagator (1). It contains three branches which correspond to the $NN$, $\pi^0$ and $\Delta N$ degrees of freedom, which are strongly mixed.

In vicinity of the pion condensation point the static $\pi^0$ propagator takes the form

$$D(0, q) \simeq -\frac{1}{\alpha(q^2 - q_0^2)^2 + \beta(n_c - n)},$$

as follows from the Taylor expansion of the propagator (1) at the critical parameters $q_c$, $n_c$ and zero energy. The coefficients $\alpha, \beta$ are positive and can be calculated if a model for the pion polarization operator is suggested [13]. The realistic estimation of the critical density for the $\pi^0$ condensation is $n_c \simeq 0.2$ fm$^{-3}$, as follows from microscopic investigations of nuclear matter [14, 15]. The critical momentum $q_c$ is not known sufficiently well and is estimated to be in a range of (0.7 ± 1.0 fm$^{-1}$).

The presence of a soft collective mode in a Fermi system affects strongly the quasiparticle interaction [16]. In vicinity of the $\pi^0$ condensation point in neutron matter, the scalar Landau–Migdal interaction amplitude $F^{nn}(p_1, p_2)$ of two neutron quasi-particles with momenta $p_1, p_2$ is dominated by the contribution from an exchange of a soft pion $\delta F_q^{nn} \propto D(\omega = 0, |p_1 - p_2|)$ which has a strong momentum dependence [17], see Eq. (3). Several investigations have shown [17–21] that this strong momentum dependence of the interaction amplitude $F^{nn}(p_1, p_2)$ can trigger topological phase transitions in neutron matter from the Landau state to states with more than one sheet of the Fermi surface. Thus, the study of pairing effects near the $\pi^0$ condensation point should include consideration of a possible non Fermi-liquid topology of the underlying ground state where the pairing correlations are switched off. A general discussion of pairing aspects in a Fermi system in a state with two sheets of the Fermi surface may be found in [21] where the method developed in [22] was applied.

The investigation [18] of nuclear pairing in dense neutron matter showed that the spin-triplet $P$-wave neutron pairing is amplified by the soft pion exchange. In this article, we report on a possibility of spin-singlet $S$-pairing in the vicinity of the $\pi^0$ condensation point. We consider the pairing interaction induced by a soft pion and discuss specific features of the solution of the gap equation. This study is limited to the $^1S_0$ pairing in the Landau state.
II. SOFT PION INDUCED POTENTIAL

The static $nn$ potential produced by an exchange of one in-medium pion in neutron matter is given in momentum space in the standard notation \[ V_\pi(q) = \frac{\bar{f}^2}{m_\pi^2} (\sigma_1 q)(\sigma_2 q) D(\omega = 0, q), \quad (4) \]
where $\bar{f}$ is the in-medium $\pi n$ coupling constant. Projecting this potential onto the $nn$ spin-singlet state and taking into account Eq. (3) for the propagator of the soft $\pi^0$, one arrives at the formula
\[ V_\pi(q) = \frac{C_0 g_\pi}{(q^2/q_c^2 - 1)^2 + \eta^2}, \quad (5) \]
where $C_0 = \nu_F^{-1} = \pi^2/mp_F$ is the inverse unrenormalized density of states, $g_\pi > 0$ is an effective coupling constant and $\eta^2 \propto (n_c - n)/n_c$ is a dimensionless measure of proximity of the system to the $\pi^0$ instability. In the following we will adopt the values
\[ g_\pi = 2.8, \quad \eta = \sqrt{4.6(n_c - n)/n_c}, \quad (6) \]
obtained within a semi-microscopic model of the pion polarization operator considered in [19]. This model reproduces the realistic critical density $n_c \approx 0.2 \text{ fm}^{-3}$ and the model value $q_c = p_F$ for the wave vector of the soft $\pi^0$ mode.

The spatial behaviour of the Fourier transform $V_\pi(r)$ of the potential (5) in coordinate space is shown in Fig. 1.

It has both regions of repulsion and attraction and resembles Fridel oscillations while its asymptotic form at distances $r > 1/q_c$ is
\[ V_\pi(r) \approx \frac{C_0 g_\pi q_c^2}{\eta} \exp \left( -\frac{\eta q_c r}{2} \right) \left( \sin(q_c r) + \frac{\eta}{2} \cos(q_c r) \right). \quad (7) \]

The potential supports bound states in the $S$ channel if the bare two-particle problem is considered. The first two $S$-levels appear when the parameter $\eta$ successively reaches the values $\eta_1 \approx 0.55$ and $\eta_2 \approx 0.24$. The radial wave function $r \psi_\lambda(r)$ of the bound $S$-state of the potential $V_\pi(r)$ at $\eta = 0.3$ is displayed in the same Fig. 1. The energy $\varepsilon_\lambda$ of the lowest $S$-state eventually gets rather big values, of order the Fermi energy $\sqrt{2}p_F/2m$, increasing in magnitude for decreasing $\eta$. The role of the bare bound $S$-state for the pairing problem that we consider is discussed in the next sections, the energy $\varepsilon_\lambda$ and the average radius $\langle r_\lambda \rangle = \int r^2 |r \psi_\lambda|^2 dr / \int |r \psi_\lambda|^2 dr$ of a bare bound pair for several values of $\eta$ are presented in Table I.

The $S$-wave component of the potential $V_\pi(p - p')$ in momentum space is found by averaging of the expression
\[ V_\pi(p, p') = \frac{C_0 g_\pi q_c^2}{\eta} \left[ \arctan \frac{1}{\eta} \left( \left( \frac{p + p'}{q_c} \right)^2 - 1 \right) \right] - \arctan \frac{1}{\eta} \left( \left( \frac{p - p'}{q_c} \right)^2 - 1 \right). \quad (8) \]

Two specific properties of the matrix $V_\pi(p, p')$ are worth to be pointed out: i) all the matrix elements are positive; ii) the off-diagonal elements prevail over diagonal ones at small $\eta$. The first is obvious and the second follows from the comparison of the diagonal elements, $V_\pi(p, p) \propto \eta^{-1}$, with a representative off-diagonal one, $V_\pi(q_c, 0) \propto \eta^{-2}$. Some of the matrix elements as a function of momentum
are plotted in Fig. 2. The dominant off-diagonality of the matrix $V_{p,p'}$ is the mathematical reason why a non-trivial solution of the gap equation appears in our case. We note that the similar situation holds for the Reid soft core $NN$ potential [23] well known in nuclear physics.

III. PAIRING CORRELATIONS IN THE $^1S_0$ CHANNEL IN THE VICINITY OF PION CONDENSATION POINT

In order to consider the pairing phenomenon nearby the critical region of the $\pi^0$ condensation point in neutron matter it is necessary to construct the pairing interaction $V_{\text{pair}}$. A reasonable scheme is as follows:

$$V_{\text{pair}} = (V_{NN} - V_{\text{OPE}}) + V_\varepsilon,$$

where $V_{NN}$ is a bare $NN$ potential, $V_{\text{OPE}}$ is the one pion exchange potential in vacuum, and $V_\varepsilon$ is the in-medium $\pi^0$ exchange potential (5). The intuitive prediction could be that the conditions for the $^1S_0$ pairing become worse since the attractive in the $S$ channel force $V_{\text{OPE}}$ is subtracted and a positive in momentum space term $V_\varepsilon$ is added. Indeed, this is the case when the density of neutron matter is more than 5% lower than the critical one $n_c$. However, at higher densities the situation turns out to be different since the pairing interaction starts to be determined by the last term in the formula (9). For instance, the minimum of the bare $NN$ potential such as the Argonne $v_{14}(r)$ force [24] is about an order of magnitude shallower than the lowest minimum of the potential $V_\pi(r)$ for the values of $\eta \leq 0.47$. The corresponding constraint on the density is $(n_c - n)/n_c \lesssim 5\%$ according to the model (6). For this region of densities, we assume the pairing interaction to be $V_{\text{pair}} \simeq V_\varepsilon$.

To examine pairing correlations in the $^1S_0$ channel, we solve the gap equation with the interaction (8),

$$\Delta(p) = -\int V_\pi(p,p') \frac{\Delta(p')}{2E(p')} \frac{p'^2dp'}{2\pi^2},$$

where $E(p) = \sqrt{\langle \varepsilon(p) - \mu \rangle^2 + \Delta(p)^2}$ is the spectrum of Bogolubov quasiparticles. The superfluidity is regarded on the top of the Landau Fermi-liquid state with the quasiparticle momentum distribution $n_{FL}(p) = \theta(p_F - p)$ with one sheet of the Fermi surface at the Fermi momentum $p_F = (3\pi^2n)^{1/3}$. The spectrum $\varepsilon(p)$ of initial quasiparticles of the nonsuperfluid state is fixed to be $\varepsilon(p) = p^2/2m^*$ and the bare neutron mass $m^* = m$ is taken for simplicity. The particle number conservation condition is used to find the chemical potential $\mu$:

$$\int \mathcal{N}(p) \frac{p^2dp}{\pi^2} = n,$$

where $\mathcal{N}(p)$ is the momentum distribution of quasiparticles rearranged by pairing correlations:

$$\mathcal{N}(p) = \frac{1}{2} \left( 1 - \frac{\varepsilon(p) - \mu}{E(p)} \right).$$

The onset of superfluidity in neutron matter may be conveniently determined from the analysis of the linear equation

$$|2\varepsilon(p) - p_F^2/m| \kappa(p) = -\int V_\pi(p,p') \kappa(p') \frac{p'^2dp'}{2\pi^2},$$

that follows from Eq. (10) in the limit of $\Delta \to 0$. Here the anomalous density $\kappa(p) = \Delta(p)/2E(p)$ is introduced. The appearance of a non-trivial solution of this equation is the condition that the two-particle scattering amplitude acquires a pole at the total momentum $\mathbf{P} = 0$ and the energy $E = p_F^2/m$ of a pair, as it follows from the in-medium Bethe-Salpeter equation. We found that the pairing instability in the $^1S_0$ channel occurs at $\eta_\Delta \simeq 0.38$.

It is interesting to note that, since $\eta_\Delta < \eta_1 \simeq 0.55$, there is a range of values of the parameter $\eta$ where a pair of particles interacting by means of the potential $V_\pi$ is bound being hypothetically placed in vacuum while it is unbound in neutron matter. This situation is contrary to what one knows for the weak coupling attraction. The reasons why it takes place are the repulsion, $V_\pi(p_F,p_F') > 0$, of the pairing interaction at the Fermi surface and the overlap of such pairs in neutron matter at densities we deal with. A more detailed discussion is given in the next section.

The normalized solution of the gap equation (10) is
The gap function $\Delta(p)$ shows a strong momentum dependence. It gets maximum values inside the Fermi sphere and then changes its sign several times in order to satisfy the gap equation with the positive pairing interaction. The anomalous density $\kappa(p)$ corresponding to this gap function is plotted in panel (b) of the same figure. One may see that, besides the usual sharp maximum attained at the Fermi surface of the same figure. One may see that, besides the usual sharp maximum attained at the Fermi surface, the anomalous density also tends to this limit at the origin of the momentum axis as the parameter $\eta$ is decreased. This means that $\Delta(p)$ begins to prevail over the spectrum $\varepsilon(p) - \mu$ in the inner region of the Fermi sphere.

We present details of the solution of the gap equation at different values of the parameter $\eta$ in Table I. The quantities $\Delta_0$, $\Delta_F$ are the values of the gap function $\Delta(p)$ at the points $p = 0$, $p = p_F$, respectively. The variation of the chemical potential is defined as $\delta \mu = \mu - p_F^2/2m$. The correlation length $\xi$ is introduced below. The above mentioned characteristics $\varepsilon_\lambda$ and $(r_\lambda)$ of the bare two-particle bound $S$-state are presented in the last two columns.

The strong momentum dependence of the gap function has specific influence on the quasiparticle momentum distribution (12) which is plotted in Fig. 4. This figure shows a dip in the occupation numbers $\mathcal{N}(p)$ at low momenta and a permanent presence of a jump from one to zero in vicinity of the Fermi momentum for each value of the parameter $\eta$. At the same time, the chemical potential is almost unperturbed, $\mu \simeq p_F^2/2m$, as one may see from Table I. We note that the found solution of the gap equation has no relation to the strong coupling limit despite of the presence of a bound state in the pairing potential. In the latter case one has $\mu < 0$ and occupation numbers tend to be $\mathcal{N}(p) \ll 1$ opening a possibility to Bose-Einstein condensation [25].

In order to clarify the character of pairing regime we calculated the correlation length

$$\xi = \sqrt{\frac{\int \frac{\partial}{\partial p} \kappa(p)^2 \, p^2 \, dp}{\int |\kappa(p)|^2 \, p^2 \, dp}},$$

(14)

The dependence of this quantity on the parameter $\eta$ is given in Fig. 5, as well as in Table I. It is clearly seen that the correlation length exceeds the average interparticle distance $r_s = (9\pi/4)^{1/3}/p_F$ that implies a picture consistent with weak coupling. The figure also demonstrates a worsening of the usual estimate $\xi_F = p_F/m \sqrt{8\Delta_F}$ of the correlation length with decrease of the parameter $\eta$. Thus, we deal here with a weak coupling regime with an unusually strong momentum dependence of the gap function inside the Fermi sphere. The solution found by us may be related to a class of unconventional BSC solutions in the classification of the article [26] where an original point of view on pairing in neutron matter is presented.
IV. DISCUSSION

In this section we discuss in more detail several points concerning the solution of the gap equation with the interaction (8). The first point concerns the relation between the found weak coupling solution of the gap equation and the presence of a bound \( S \)-state in the pairing potential. In order to consider this problem it is useful to rewrite the gap equation in the following way: \[ 2(\varepsilon(p) - \mu) \kappa(p) = -(1 - 2\mathcal{N}(p)) \int \mathcal{V}_\pi(p, p') \kappa(p') \frac{p'^2 dp'}{2\pi^2} \], \[ \mathcal{N}(p) = \frac{1}{2} \left( 1 - \text{sgn}(\varepsilon(p) - \mu) \sqrt{1 - 4|\kappa(p)|^2} \right). \] (16)

The two-particle Schrödinger equation is then recovered for the negative chemical potential and \(|\kappa(p)| \ll 1\) when the approximate relation \(\mathcal{N}(p) \simeq |\kappa(p)|^2 \ll 1\) holds. In this limit the normalization condition (11) takes the form \(\int |\kappa(p)|^2 p^2 dp/\pi^2 = n\). Turning now to the pairing interaction (5) which has a characteristic radius \(1/\eta_{pF}\) one may conclude from the normalization condition that for a strongly bound two-particle state \(|\kappa(p)|^2 \sim 1/\eta^4 > 1\). Thus, the bound state solution cannot be properly normalized in order not to violate the constraint \(|\kappa(p)| \ll 1\). In other words the bound pairs of the Schrödinger equation overlap and the Pauli principle plays a significant role for the existence of the solution of the gap equation that we found.

The second point is the specific role of the Fermi surface in the formation of a pairing gap in the case of the pairing interaction positive in momentum space. Despite of the repulsion \(V_\pi(pF, \nu) > 0\) of the pairing interaction on the Fermi surface it is that manifold in momentum space where the anomalous density \(\kappa(p)\) appears from the outset. To understand this better, it is worth to consider the gap equation in the model space \(S_0 = (pF - \delta p, pF + \delta p)\) in terms of a renormalized pairing interaction \[ \Delta(p) = -\int_{S_0} V^\text{eff}_\pi(p, p') \frac{\Delta(p')}{2s_0(p')} \frac{p'^2 dp'}{2\pi^2}. \] (17)

The renormalized interaction obeys the equation

\[ V^\text{eff}_\pi(p, p') = \mathcal{V}_\pi(p, p') - \int_{S'} \frac{V_\pi(p, q)V^\text{eff}_\pi(q, p')}{2|\varepsilon(q) - \mu|} \frac{q^2 dq}{2\pi^2}. \] (18)
in which the integration is carried out in the subspace \(S'\) complementary to \(S_0\). The model space \(S_0\) have to be large enough for neglecting pairing effects \(|\Delta(p)| \ll |\varepsilon(p) - \mu|\) in the subspace \(S'\). However, if one approaches to the pairing instability from the superfluid side, \(\Delta(p) \rightarrow 0\), the model space may be chosen sufficiently small \(\delta p/pF \ll 1\). In this case the pairing gap in the model space can be obtained in the standard way: \(\Delta_F/\delta\varepsilon = \exp(2/\nu_F \nu_{F, F})\) where \(\nu_F = V^\text{eff}_\pi(pF, pF)\) and \(\delta\varepsilon = 2pF\delta p/m\). The transition to a nonsuperfluid state is associated with the vanishing of the negative effective pairing interaction \(\nu_{F, F} \rightarrow 0\). Thus, one may see that the anomalous density \(\kappa(p)\) concentrates in the model subspace of momentum space and shrinks with it to the Fermi surface as the superfluid phase transition is approached, see Fig. 3b. The effective interaction calculated from Eq. (18) for a narrow model space with \(\delta p/pF = 0.2\) is shown in Fig. 6 for several values of the parameter \(\eta\) near the critical one \(\eta\Delta \simeq 0.38\) mentioned above. We note that states away from the Fermi surface play an important role in the renormalization of the pairing interaction. Moreover, in our case all states inside the Fermi sphere must be included in the model space explicitly to obtain a correct solution for the strongly momentum dependent gap function at a distance beyond the superfluid transition.

V. CONCLUSION

We have examined a possibility of \(^1S_0\) neutron pairing in dense neutron matter in vicinity of the \(\pi^0\) condensation point. The investigation was performed in the framework of the BCS approach with the pairing interaction induced by an exchange of the soft neutral pionic mode. Superfluidity was searched on the top of the Landau state with one sheet of the Fermi surface. It is shown that a solution of the gap equation in the \(^1S_0\) channel appears in a domain of \((n_c - n)/n_c \lesssim 5\%\) near the critical density of the \(\pi^0\) condensation. The gap function reveals a strong momentum dependence as the off-diagonal \(S\)-wave matrix elements of the pairing interaction prevail over diagonal ones while all the matrix elements are positive in momentum space. We have also discussed the weak coupling nature of the superfluid phase that we found.
VI. ACKNOWLEDGMENTS

We are grateful to M.V. Zverev for useful discussions and remarks. Two of us, S.P. and E.S., thank INFN, Sezione di Catania, for hospitality during the stay in Catania when the major part of this work was done.

This research was partially supported by Grant No. NSh-932.2014.2 of the Russian Ministry for Science and Education and by the RFBR Grants 12-02-00955-a, 13-02-00085-a, 13-02-12106-ofi_m, 14-02-00107-a, 14-02-31353 mol_a.

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