Fixed-Time Flocking and Collision Avoidance Problem of a Cucker–Smale Model

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Multiagents systems are used in artificial intelligence, control theory, and social sciences. In this article, we studied a Cucker–Smale model with a continuous non-Lipschitz protocol. The methodology presented in the current paper is based on the explicit construction of a Lyapunov functional. By using the fixed-time control technology, we show that the flocking can occur in fixed-time and collision avoiding when a singular communication function with a weighted sum of sign functions of the relative velocities among agents, and we can obtain the estimation of the converging time which is independent of the initial states of agents. Theoretical results are supported by numerical simulations.

1. Introduction

In recent years, multiagent systems have various applications in widely fields such as biology, robotics, and control theory as well as sociology and economics [1–6]. Each agent typically has only a local, limited impact on other agents via the connections with its neighbor individuals in the given system, and the coherent behavior of the agents caused the emerging phenomenon in terms of flocking, in which the agents reach final agreement on controlled variables of interest. Multiagent dynamical systems are typically fragment for modelling of birds and fishes in nature world, and more and more scientists realized the importance of these models. Among others, the celebrated Cucker–Smale model [7, 8] provides a framework to explain the self-organizing behavior in various complex systems, and the Cucker–Smale model is given by the following ODE system:

\[
\begin{align*}
\frac{dx_i}{dt} &= v_i, \quad t > 0, i = 1, 2, \ldots, N, \\
\frac{dv_i}{dt} &= \sum_{j=1}^{N} a_{ij}(\|x_j - x_i\|)(v_j - v_i).
\end{align*}
\]

(1)

Subjecting to the initial configuration,

\[
(x_i(0), v_i(0)) = (x_{i0}, v_{i0}).
\]

(2)

Here, \(N\) is the number of particles, \(x_i = (x_{i1}, x_{i2}, \ldots, x_{id}) \in \mathbb{R}^d\) and \(v_i = (v_{i1}, v_{i2}, \ldots, v_{id}) \in \mathbb{R}^d\) denote the position and velocity of the \(i\)th particle at the time \(t\), and \(a_{ij}\) quantify the way the birds influence each other:

\[
a_{ij}^{CS}(\|x_j - x_i\|) = \frac{\psi(\|x_j - x_i\|)}{N}.
\]

(3)

\(\psi(r)\) called influence function quantifies the pairwise influence of the agent \(j\) on the alignment of the agent \(i\), as a function of the distance: that is,

\[
\psi_{ij}(\|x_j - x_i\|) = \frac{1}{(1 + \|x_j - x_i\|^2)^{\beta}}.
\]

(4)

In [7], it is shown that the unconditional flocking occurs when \(\beta < (1/2)\), while the conditional flocking occurs under some restricted conditions on the initial positions and velocities when \(\beta \geq (1/2)\). In [9], the authors introduced the Cucker–Smale model with a singular communication weight influence function:
\[ \psi(s) = s^{-\alpha}. \]  

(5)

The authors extended the conclusions of unconditional flocking to \( \bar{\beta} \leq (1/2) \) by the energy method. In [10], the authors introduced a nonsymmetric influence function and took into account relative distance between agents instead of the distance between agents:

\[ a_{ij}^{MT}(\|x_k - x_i\|) = \frac{\psi(\|x_k - x_i\|)}{\sum_{1 \leq k \leq N} \psi(\|x_k - x_i\|)}. \]  

(6)

Based on the notion of active sets, a sufficient condition for flocking was derived. The Cucker–Smale model was extensively studied, see, for example, time delay [11–15], pattern formation [16, 17], and hierarchical structure [14, 18–21]. However, the flocking phenomenon described in the most previous works is an asymptotic behavior without considered to collision avoidance behavior. In fact, the issue of collision avoiding plays a significant role in our real life, there are terrible consequences if collision avoidance is not considered. Meanwhile, under some occasional perturbations, individuals in bird flocks or fish schools can return back to ordered group motion after adjusting their states in a short time. In [22, 23], the authors introduced a modified Cucker–Smale mode by adding a term which prevents collisions. In [24], the authors used singular value influence function (5), through the Lyapunov method, when the initial value lies in the set of the admissible initial configurations can avoid collision. In [25], the authors improve the result of [24], when \( \alpha > 1 \), as long as the initial value satisfies with \( i \neq j \) and \( x_{ji} \neq x_{ij} \), and there will be no collision in the process of flocking; when \( \alpha \geq 2 \), the authors consider the singular communication weight of the form:

\[ \psi(s) = (s - \delta)^{-\alpha}. \]  

(7)

Expand the set of singular points of \( \psi \) from 0 to [0, \( \delta \)], if the initial data satisfy with condition that the difference between any two agents is greater than the control parameter \( \delta \), there will also be no collision in the process of flocking, and the minimum distance between agents is greater than the control parameter \( \delta \). In [26], the authors introduced a Cucker–Smale model with a continuous non-Lipschitz protocol, and then the flocking can occur in finite time when the communication rate function is locally Lipschitz continuous and has a lower bound. In [27], the authors introduced a Cucker–Smale model with a continuous non-Lipschitz protocol, by constructing a Lyapunov functional to obtain the finite-time flocking, which is the convergence time depending on the initial values. Although finite-time flocking has favourable properties, the estimation of convergence time depends on initial states of networked agents, and this restricts the applications in practice if the knowledge of initial conditions is unavailable in advance. There are lots of works in the field of fix-time consensus [28–30], and we know little about considering the fixed-time flocking and collision avoidance of the Cucker–Smale model.

The main purpose of this article is to investigate the fixed-time flocking and collision avoidance of a Cucker–Smale model. The remaining of this paper is organized as follows. In Section 2, a Cucker–Smale model with continuous non-Lipschitz protocol is presented and some useful preliminaries are also given in this section. The existence of a global smooth solution is obtained in Section 3, Section 4 states the definition of fixed-time flocking and proved the fixed-time flocking and collision avoidance results, and some numerical results are given in Section 5. Section 6 concludes this paper.

2. Problem Statement and Preliminaries

Consider a model consisting of \( N \) autonomous agents. \( x_i = (x_{i1}, x_{i2}, \ldots, x_{ip}) \in \mathbb{R}^p \) and \( v_i = (v_{i1}, v_{i2}, \ldots, v_{ip}) \in \mathbb{R}^p \) denote the position and velocity of the \( i \)th particle at the time \( t \), respectively, and the modified Cucker–Smale model in this paper can be described by the following equations:

\[
\begin{aligned}
\frac{dx_i}{dt} &= v_i, \quad t > 0, i = 1, 2, \ldots, N, \\
\frac{dv_i}{dt} &= k_1 \sum_{j=1}^{N} \psi(r_{ij} - \delta)(v_j - v_i) + k_2 \sum_{j=1}^{N} \frac{1}{r_{ij} + 1} \text{sig}(v_j - v_i)^p + k_3 \sum_{j=1}^{N} \frac{1}{r_{ij} + 1} \text{sig}(v_j - v_i)^q.
\end{aligned}
\]  

(8)

where \( 0 < p < 1 < q \) are two control parameters, \( k_1, k_2, \) and \( k_3 \) measure the interaction strength, and \( \psi \) was defined in (7); that is,

\[
\psi(r_{ij} - \delta) = (r_{ij} - \delta)^{-\alpha},
\]  

(10)

where \( r_{ij} = |x_i - x_j| \) is the distance of any two agents:

\[
\text{sgn}(s) = \begin{cases} 
1, & s > 0, \\
0, & s = 0, \\
-1, & s < 0.
\end{cases}
\]  

(12)

\( \delta > 0, R > 0 \) are two control parameters to make particles mutually exclude and attract with each other.
We have the following lemmas, which play an important role in the proof of the main results.

**Lemma 1** (see [31]). Consider the following equation:
\[ \dot{x} = f(t, x), x(0) = x_0, \]  
where \( x \in \mathbb{R}^n \) and \( f: \mathbb{R}_+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a nonlinear continuous function. Assume the origin is an equilibrium point of (13). If there exists a continuous radially unbounded function \( H: \mathbb{R}^n \rightarrow \mathbb{R}_+ \cup \{0\} \) such that
(i) \( H(z) = 0 \Leftrightarrow z = 0, \)
(ii) for some positive numbers \( \theta, \kappa > 0, 0 < a < 1 < b, \) any solution \( z(t) \) satisfies the inequality:
\[ H(z(t)) \leq -\theta H^a(z(t)) - \kappa H^b(z(t)). \]  
(14)

Then, the origin is globally fixed-time stable and \( H(t) \equiv 0 \) if
\[ t \geq \frac{1}{b(1-a)} + \frac{1}{\kappa(b-1)}. \]  
(15)

**Lemma 2** (see [32]). Let \( y \in \mathbb{R}^n \) and \( 0 < r < s. \) Then, the following norm equivalence property holds:
\[ \left( \sum_{i=1}^{n} |y_i|^s \right)^{(1/s)} \leq \left( \sum_{i=1}^{n} |y_i|^r \right)^{(1/r)}, \]  
(16)
\[ \left( \frac{1}{n} \sum_{i=1}^{n} |y_i|^{s} \right)^{(1/s)} \geq \left( \frac{1}{n} \sum_{i=1}^{n} |y_i|^{r} \right)^{(1/r)}. \]  
(17)

### 3. A Global Existence Theory

Throughout the paper, we use \( x=(x_1, x_2, \ldots, x_N), \) where \( x_i=(x_i^1, x_i^2, \ldots, x_i^d) \in \mathbb{R}^d \) denote the position of the particles while \( v=x \) is the velocity. For vectors \( x_i, v_i \in \mathbb{R}^d, \) their Euclidean norms and the inner products are defined as follows:
\[ \|x_i\| = \left( \sum_{k=1}^{d} (x_i^k)^2 \right)^{1/2}, \]  
\[ \|v_i\| = \left( \sum_{k=1}^{d} (v_i^k)^2 \right)^{1/2}, \]  
(18)
where \( x_i^k \) and \( v_i^k \) are the \( k \)th components of \( x_i \) and \( v_i. \) Respectively, we consider macroscopic variables:
\[ x_c = \frac{1}{N} \sum_{j=1}^{N} x_j, \]  
\[ v_c = \frac{1}{N} \sum_{j=1}^{N} v_j. \]  
(19)

By the symmetry of the indices, we have
\[ \sum_{i=1}^{N} \frac{dvi}{dt} = \frac{k_1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \psi(r_{ij} - \delta)(v_i - v_j) \]  
\[ + \frac{k_2}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{r_{ij} + R} \text{sign}(v_i - v_j)^p \]  
\[ + \frac{k_3}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{1}{r_{ij} + R} \text{sign}(v_i - v_j)^q = 0. \]  
(20)
Hence, we have explicit dynamics for the macroscopic variables:
\[ \frac{dx_c}{dt} = v_c, \]  
\[ \frac{dv_i}{dt} = 0, \]  
(21)
which yields that
\[ v_c(t) = v(0), \]  
\[ x_c(t) = x_c(0) + tv_c(0), \quad t \geq 0. \]  
(22)

We consider the fluctuations \( (\bar{x}_i, \bar{v}_i): \)
\[ \bar{x}_i = x_i - x_c, \]  
\[ \bar{v}_i = v_i - v_c. \]  
(23)
Then, systems (8)-(9) can be written as
\[
\begin{align*}
\frac{d\bar{x}_i}{dt} = \bar{v}_i, & \quad t > 0, i = 1, 2, \ldots, N, \\
\frac{d\bar{v}_i}{dt} = \frac{k_1}{N} \sum_{j=1}^{N} \psi(r_{ij} - \delta)(\bar{v}_i - \bar{v}_j) + \frac{k_2}{N} \sum_{j=1}^{N} \frac{1}{r_{ij} + R} \text{sign}(\bar{v}_i - \bar{v}_j)^p + \frac{k_3}{N} \sum_{j=1}^{N} \frac{1}{r_{ij} + R} \text{sign}(\bar{v}_i - \bar{v}_j)^q.
\end{align*}
\]  
(24)
With the initial value,
\[ (\hat{x}_i(0), \hat{v}_i(0)) = (\check{x}_i(0), \check{v}_i(0)) \]  
(25)

Through simple calculation, we can get
\[ \hat{x}_c = \frac{1}{N} \sum_{i=1}^{N} \hat{x}_i = 0, \]
\[ \hat{v}_c = \frac{1}{N} \sum_{i=1}^{N} \hat{v}_i = 0. \]
(26)

For convenience, we take the hat out of the variables and use \((x_i, v_i)\) instead of \((\hat{x}_i, \hat{v}_i)\). It is easy to see that
\[ \sum_{i=1}^{N} x_i = 0, \]
\[ \sum_{i=1}^{N} v_i = 0, \quad t \in [0, T). \]
(27)

The main goal of this section is to consider the existence of a global smooth solution of systems (8)-(9). For this purpose, we introduce the maximal life span \(T_0\) for the initial data \(x_0\) as follows:
\[ T_0 = \sup \{ x \in \mathbb{R} : \exists \text{solution } (x(t), v(t)) \text{ for systems in a time interval } [0, s] \}. \]
(28)

Let
\[ \mathcal{L}^{\alpha-2}(t) = \frac{1}{N^2} \sum_{i,j=1}^{N} \left( |x_i(t) - x_j(t)| - \delta \right)^{-(\alpha-2)}. \]
(29)

Now, we are in the position to present the main result of this section.

**Theorem 1.** Assume that \(\alpha \geq 2\) and the initial datum satisfies
\[ |x_i(0) - x_j(0)| > \delta \quad \text{for any } 1 \leq i \neq j \leq N. \]
(30)

Then, there exists a global smooth solution \((x_i, v_i)\) to systems (8)-(9). Moreover, for \(t \geq 0\), when \(\alpha = 2\), we have
\[ \left| \frac{1}{N^2} \sum_{i,j=1}^{N} \log \left( \left| x_i(t) - x_j(t) \right| - \delta \right) \right| \leq \left| \frac{1}{N^2} \sum_{i,j=1}^{N} \log \left( \left| x_i(0) - x_j(0) \right| - \delta \right) \right| + \frac{T_0}{2} + \frac{1}{2N} \sum_{i=1}^{N} |v_i(0)|^2, \]
when \(\alpha > 2\), we have
\[ \mathcal{L}^{\alpha-2}(t) \leq \mathcal{L}^{\alpha-2}(0) e^{Ct} \leq C e^{Ct} \frac{1}{N} \sum_{i=1}^{N} |v_i(0)|^2, \]
where \(C\) is a constant depending only on \(\alpha\).

**Proof.** Throughout the proof, we will denote for simplicity \(T_0 = T(x_0)\). Clearly, if the distances between agents are bigger than \(\delta\) for any finite time, then the solution can be prolonged indefinitely and \(T_0 = \infty\). Moreover, if (31) or (32) holds, then \(\inf_{t \in [0, \infty)} \min_{i \neq j \leq N} |x_i(t) - x_j(t)| > \delta\) and \(T_0 = \infty\). Therefore, in order to finish the proof it suffices to show (19) or (20), first we provide the energy dissipation of system (8), by (8)2 and a symmetry argument, and we have
\[ \frac{d}{dt} \left( \frac{1}{N} \sum_{i=1}^{N} |v_i(t)|^2 \right) = \frac{2}{N} \sum_{i=1}^{N} \langle v_i(t), \frac{d}{dt} v_i(t) \rangle = A_1 + A_2 + A_3, \]
(33)
where
\[ A_1 = \frac{2k_1}{N} \sum_{i=1}^{N} \langle v_i(t), \sum_{j=1}^{N} \psi(r_{ij} - \delta)(v_j(t) - v_i(t)) \rangle, \]
\[ A_2 = \frac{2k_2}{N} \sum_{i=1}^{N} \langle v_i(t), \sum_{j=1}^{N} \frac{1}{r_{ij} + R} \text{sig}(v_j(t) - v_i(t))^3 \rangle, \]
\[ A_3 = \frac{2k_3}{N} \sum_{i=1}^{N} \langle v_i(t), \sum_{j=1}^{N} \frac{1}{r_{ij} + R} \text{sig}(v_j(t) - v_i(t))^3 \rangle. \]
(34)

By the result of Theorem 1 in [27], we have
\[ A_1 \leq - \frac{k_1}{N^2} \sum_{i,j=1}^{N} \frac{|v_j - v_i|^2}{|x_j - x_i - \delta|^q}, \]
\[ A_2 \leq - \frac{k_2}{N^2} \sum_{i,j=1}^{N} \frac{1}{r_{ij} + R} |v_j - v_i|^{q+1}, \]
\[ A_3 \leq - \frac{k_3}{N^2} \sum_{i,j=1}^{N} \frac{1}{r_{ij} + R} |v_j - v_i|^{q+1}. \]
(35)

Thus, substituting (35)-(37) into (33), integrating the differential inequality (33) from 0 to \(t\) yields
\[ \frac{1}{N} \sum_{i=1}^{N} |v_i(t)|^2 + \int_0^t \frac{k_1}{N^2} \sum_{i,j=1}^{N} \frac{|v_j - v_i|^2}{|x_j - x_i - \delta|^q} \] \[ + \int_0^t \frac{k_2}{N^2} \sum_{i,j=1}^{N} \frac{1}{r_{ij} + R} |v_j - v_i|^{q+1} ds \]
\[ + \int_0^t \frac{k_3}{N^2} \sum_{i,j=1}^{N} \frac{1}{r_{ij} + R} |v_j - v_i|^{q+1} ds = \frac{1}{N} \sum_{i=1}^{N} |v_i(0)|^2, \quad \text{for } t \in [0, T_0]. \]
(38)

And this yields
\[ \int_0^t \frac{k_1}{N^2} \sum_{j=1}^{N} \frac{|v_j - v_i|^2}{|x_j - x_i - \delta|^q} ds = \frac{1}{N} \sum_{i=1}^{N} |v_i(0)|^2, \quad \text{for } t \in [0, T_0]. \]
(39)

By the same processes of the proof of Theorem 1 in [27], we have that (31) and (32) are established and there is no collision between agents for all time and \(T_0 = \infty\).
## 4. Fixed-Time Flocking

In this section, we shall show that systems (8)-(9) with continuous non-Lipschitz protocol have a fixed-time flocking. First, we first introduce the definition of the fixed-time flocking.

**Definition 1.** Systems (8)-(9) are said to reach a fixed-time flocking if and only if the system satisfies the following two conditions:

(i) Velocity alignment: the velocity fluctuations go to zero in the fixed-time $T$, and the time function $T$ is called the convergence time independent of the initial values:

$$
\|v_i - v_j\| = 0, \quad \forall t \geq T, \quad \text{for } i, j = 1, 2, \ldots, N. \quad (40)
$$

(ii) Forming a group: the position fluctuations are uniformly bounded in time $t$:

$$
\sup_{0 \leq t \leq T} \|x_i - x_j\|^2 < \infty, \quad \text{for } i, j = 1, 2, \ldots, N. \quad (41)
$$

**Theorem 2.** Assume that $\alpha \geq 2, 0 < p < 1 < q$, and the initial datum satisfies (30). Then, systems (8)-(9) can reach a fixed-time flocking. Moreover, the convergence time is independent of the initial states of agents which is estimated by

$$
T \leq T^* \leq \frac{2}{\delta (1 - p)} + \frac{2}{\kappa (q - 1)}, \quad (42)
$$

where

$$
\delta = (k_3 (2N)^{(p+1)/2} / N (M + R)), \quad \kappa = (k_3 (M + R) 2^{(q+1)/2}) d^{(1-q)/2} \text{ and } M = \max_{i \neq j} \|x_j - x_i\|.
$$

**Proof.** Let $x(t) = (x_1(t), x_2(t), \ldots, x_N(t)) \in \mathbb{R}^{dN}$ and $v(t) = (v_1(t), v_2(t), \ldots, v_N(t)) \in \mathbb{R}^{dN}$. Take the candidate Lyapunov function:

$$
V(t) \equiv \sum_{i=1}^{N} \|v_i\|^2. \quad (43)
$$

let

$$
X(t) \equiv \sum_{i=1}^{N} \|x_i\|^2. \quad (44)
$$

By using (27), we have

$$
\sum_{1 \leq i \neq j \leq N} \|v_j - v_i\|^2 = 2N \sum_{i=1}^{N} \|v_i\|^2 = 2NV, \quad (45)
$$

$$
\sum_{1 \leq i \neq j \leq N} \|x_j - x_i\|^2 = 2N \sum_{i=1}^{N} \|x_i\|^2 = 2NX. \quad (46)
$$

It is easy to see that the difference of all individuals' velocities will tend to zero in fixed-time if the function $V(t)$ tends to 0 in fixed-time, and the diameter of a group is bounded if the function $X(t)$ is bounded:

$$
\frac{d}{dt} V(t) = 2 \sum_{i=1}^{N} \langle v_i(t), \frac{d}{dt} v_j(t) \rangle = N A_1
$$

$$
+ \frac{2k_2}{N} \sum_{j=1}^{N} \langle v_j(t), \frac{1}{r_{ij} + R} \text{sig}(v_j - v_i)^p \rangle
$$

$$
+ \frac{2k_3}{N} \sum_{j=1}^{N} \langle v_j(t), \frac{1}{r_{ij} + R} \text{sig}(v_j - v_i)^q \rangle \quad (47)
$$

$$
\leq N A_1 - \frac{k_2}{N} \sum_{i,j=1}^{N} \frac{1}{r_{ij} + R} \|v_j - v_i\|^{p+1}
$$

$$
- \frac{k_3}{N} \sum_{i,j=1}^{N} \frac{1}{r_{ij} + R} \|v_j - v_i\|^{q+1}. \quad (48)
$$

Note that $0 < p < 1 < q$; then, employing Lemma 2, one can easily get that

$$
\left( \sum_{k=1}^{d} \|v_j - v_k\|^{p+1} \right)^{1/(p+1)} \geq \left( \sum_{k=1}^{d} \|v_j - v_k\|^2 \right)^{1/2} = \|v_j(t) - v_k(t)\|,
$$

$$
\left( \sum_{k=1}^{d} \|v_j - v_k\|^{q+1} \right)^{1/(q+1)} \geq d^{(1-q)/2} \left( \sum_{k=1}^{d} \|v_j - v_k\|^2 \right)^{1/2}
$$

$$
= d^{\frac{q}{p+q}} \|v_j(t) - v_k(t)\| \quad (49)
$$

Which means

$$
\sum_{k=1}^{d} \|v_j - v_k\|^{p+1} \geq \|v_j(t) - v_k(t)\|^{p+1}, \quad (49)
$$

$$
\sum_{k=1}^{d} \|v_j - v_k\|^{q+1} \geq d^{(1-q)/2} \|v_j(t) - v_k(t)\|^{q+1}. \quad (50)
$$

Therefore, combining (47)-(50) gives that

$$
\frac{d}{dt} V(t) \leq N A_1 - \frac{k_2}{N} \sum_{i,j=1}^{N} \frac{1}{r_{ij} + R} \|v_j(t) - v_k(t)\|^{p+1}
$$

$$
- \frac{k_3}{N} d^{(1-q)/2} \sum_{i,j=1}^{N} \frac{1}{r_{ij} + R} \|v_j(t) - v_k(t)\|^{q+1}. \quad (51)
$$

It follows from (31) and (32) that there exists a positive constant $M$, such that $|x_i - x_j| < M$. Noting that $A_1 \leq 0$, then we have from (51) that
\[ \frac{d}{dt} V(t) \leq -k_2 \frac{N}{N(M+R)} \sum_{i,j=1}^{N} \|v_j(t) - v_i(t)\|^{p+1} \]

\[ - \frac{k_3}{N(M+R)} d^{(1-q)/2} \sum_{i,j=1}^{N} \|v_j(t) - v_i(t)\|^{q+1} \]

\[ = - \frac{k_2}{N(M+R)} \sum_{i,j=1}^{N} \left( \|v_j(t) - v_i(t)\|^2 \right)^{(p+1)/2} \]

\[ - \frac{k_3}{N(M+R)} d^{(1-q)/2} \sum_{i,j=1}^{N} \left( \|v_j(t) - v_i(t)\|^2 \right)^{(q+1)/2}. \]

Choosing \( s = 1 \) and \( r = ((p + 1)/2) \) and applying (16) to the processing inequality show that

\[ \sum_{i,j=1}^{N} \left( \|v_j(t) - v_i(t)\|^2 \right)^{(p+1)/2} \geq \left( \sum_{i,j=1}^{N} \|v_j(t) - v_i(t)\|^2 \right)^{(p+1)/2}. \]

Choosing \( s = (q + 1)/2 \) and \( r = 1 \) and applying (17), we get

\[ \sum_{i,j=1}^{N} \left( \|v_j(t) - v_i(t)\|^2 \right)^{(q+1)/2} \geq N \left( \sum_{i,j=1}^{N} \|v_j(t) - v_i(t)\|^2 \right)^{(q+1)/2}. \]

It follows from (45) that

\[ \frac{d}{dt} V(t) \leq -k_2 \frac{2N}{N(M+R)} V(t)^{(p+1)/2} \]

\[ - \frac{k_3}{(M+R)} d^{(1-q)/2} V(t)^{(q+1)/2}. \]

It is easy to see that \( V(t) \) is a continuous and positive definite function. So, by Lemma 1, when \( a = (p + 1)/2 \) and \( b = (q + 1)/2 \), we have

\[ V(t) \equiv 0, \quad t \geq T, \quad \text{(56)} \]

and the convergence time independent of the initial values is estimated by

\[ T \leq T^* \geq \frac{1}{\beta(1 - ((p + 1)/2))} + \frac{1}{\kappa(\|(q + 1)/2\| - 1)} \]

\[ = \frac{2}{\beta(1 - p)} + \frac{2}{\kappa(q - 1)}, \quad \text{(57)} \]

where \( \beta = k_2 \frac{2N}{(p+1)/2}(N(M+R)) \) and \( \kappa = (k_3/ (M+R)) 2^{(q+1)/2}d^{(1-q)/2} \), which implies that condition (i) of the definition of fixed-time flocking holds.

Now, we prove condition (ii) of the definition of fixed-time flocking that the function \( X(t) \) is bounded.

It follows from (55) that \( V(t) \) is a nonincreasing function with respect to \( t \). That is, when \( t > 0, V(0) \geq V(t) \geq 0 \). By the triangle inequality and Cauchy–Schwarz inequality, we have

\[ \frac{dX(t)}{dt} = 2 \sum_{i=1}^{N} \langle x_i, v_j \rangle \leq 2 \sum_{i=1}^{N} \|x_i\|\|v_j\| \leq 2X^{1/2}V^{1/2}. \]

Integrating differential inequality (58) from 0 to \( t \) yields

\[ X^{1/2}(t) \leq X^{1/2}(0) + \int_{0}^{t} V^{1/2}(s)ds. \]

If \( t < T \), then it deduces from (59) that

\[ X^{1/2}(t) \leq X^{1/2}(0) + \int_{0}^{T} V^{1/2}(s)ds \leq X^{1/2}(0) + V^{1/2}(0)T < \infty. \]

If \( t > T \), then it deduces from (55) and (59) that

\[ X^{1/2}(t) \leq X^{1/2}(0) + \int_{0}^{T} V^{1/2}(s)ds + \int_{T}^{t} V^{1/2}(s)ds \leq X^{1/2}(0) + \int_{T}^{t} V^{1/2}(s)ds < \infty. \]

Thus, combining the processing inequality with (46) implies that

\[ \sup_{0\leq t\leq \infty} \|x_i - x_j\| < \infty, \quad \text{for } i, j = 1, 2, \ldots, N. \]

This completes the proof. \( \square \)

**Remark 1.** Compared to [27], the convergence time is estimated by

\[ T \leq T^* = \frac{2N}{k_2(1 - \beta)(2N)} \left( \sum_{i=1}^{N} \|v_0\|^2 \right)^{(1 - \beta)/2}, \]

which is formulated by the initial states of the all agents. We added the term

\[ \frac{k_3}{N} \sum_{j=1}^{N} \frac{1}{r_{ij}} + \frac{1}{\sqrt{d}} \left\| \sum_{i=1}^{N} v_j \right\|, \quad \text{(64)} \]

to the control protocol, the advantage of Theorem 2 is that the convergence time is independent of the initial states of agents which is estimated by

\[ T \leq T^* = \frac{2N}{k_2(2N)} \left( \left( \frac{p+1}{2} \right) \left( \frac{q+1}{2} \right) \right)^{1/2} + \frac{2}{k_3 d^{(1-q)/2}} (N^{q/2}) (q-1) \]

5. **Simulations**

**Example 1.** Choose \( N = 30, 0 < p = 0.2 < 1 < q = 2, d = 2, k_1 = 1, k_2 = k_3 = 2, \delta = 0.8, \alpha = 2, \) and \( R = 2, \) using the Euler algorithm, step length \( h = 0.01, \) for the random initial position generated on \([0, 20]\) and random velocity on \([0, 2]\), and the initial minimum distance between agents is satisfying (30); then, the following simulation results (Figures 1–3) are obtained. In Figure 1, the \( x, y \) direction velocity of the all agents is presented, and the velocity of all
agents converges to the same value after about $t = 7$. In order to shorten the convergence time of the system, according to formula (42), under the condition that the parameter $N, p, q$ was determined and increasing the parameter value of $k_2, k_3$ was used to shorten the convergence time of the system. In Figure 2, take $k_2 = k_3 = 4$, other parameters are the same as those in Figure 1 and the velocity of all agents converges to the same value about $t = 3$, which is half shorter than that of Figure 1. Moreover, Figure 3, shows that the minimum distance between all agents is stable after $t = 0.5$ and the maximum distance between all agents is stable after $t = 6$. Meanwhile, Figure 3 explains that the minimum distance among all agents is about $0.842 > \delta$. 

6. Conclusion

In this paper, we have investigated the flocking problem of a modified Cucker–Smale model with a continuous non-Lipschitz protocol. By using a Lyapunov functional, we show that the flocking can occur and collision avoidance when a singular communication function with a weighted sum of sign functions of the relative velocities among agents. The main results demonstrate that the flocking converging time is independent of the initial states of agents.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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