On Semiprime Rings with (α,α)-Symmetric Derivations

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Abstract

The main purpose of this paper is to study and investigate concerning a (α,α)-symmetric derivations D on semiprime rings and prime rings R, we give some results when R admits a (α,α)-symmetric derivations D to satisfy some conditions on R.(i) D([x,y]n+1) =0 for all x, y ∈ R. (ii) [D(xn+1),α(y)] = 0 for all x, y ∈R. (iii) [[D(x),α(x)],α(x)]= 0 for all x ∈R. Where α: R → R is an automorphism mapping.

Keywords

Semiprime Ring, Prime Ring, (α,α)-Derivations, (α,α)-Symmetric Derivation

1. Introduction

Several researchers always ask why derivation? derivations on rings help us to understand rings better and also derivations on rings can tell us about the structure of the rings. For instance a ring is commutative if and only if the only inner derivation on the ring is zero. Also derivations can be helpful for relating a ring with the set of matrices with entries in the ring (see, Pajoohesh, 2007). Derivations play a significant role in determining whether a ring is commutative, see (Herstein, 1978), (Bell, Daif, 1995) and (Andima, and Pajoohesh, 2010). Derivations can also be useful in other fields. For example, derivations play a role in the calculation of the eigenvalues of matrices (see, Baker, 1959) which is important in mathematics and other sciences, business and engineering. Derivations also are used in quantum physics (see, Da Providencia, 1994). A lot of work has been done in this field [see (Bell, and Martindale, 1987; Bresar, 1993)]. In (Lee, 2001), he was introduce the (α,α)-derivation and α-commuting mapping in the following way: If \([f(x),α(x)] = 0 \) for all x ∈R, then f is said to be α-commuting, where α is an automorphism. An additive map D:R→R is said to be an (α,α)-derivation and α-commuting mapping in the following way: If D(xy)=D(x)α(y)+α(x)D(y) is fulfilled for all x, y ∈R. (Bresar, 1992). (Park, 2009) he was proved, let n ≥ 2 be a fixed positive integer and let R be a non-commutative n!-torsion free prime ring. Suppose that there exists a symmetric n-derivation Δ: Rn→R such that the trace δ of Δ is commuting on R. Then we have Δ = 0. (Çeven and Öztürk ,2007), they proved let N be a 2-torsion free 3-prime near-ring, D a symmetric bi-(σ,τ)-derivation of N and d the trace of D. If xd(N) = 0 for all x ∈ N, then x = 0 or D = 0, where a near ring N is 3-prime if aNb = {0} implies that a= 0 or b= 0, and a mapping D:N×N→N is said to be symmetric if D(x,y) = D(y,x) for all x,y ∈ N. A mapping d:N→N denoted by d(x) = D(x,x) is called the trace of D where D:N×N→N is a symmetric mapping. It is obvious that, if D:N×N→N is a symmetric mapping which also bi-additive (i.e., additive in both arguments), then the trace of D satisfies the relation d(x+y) = d(x)+2D(x,y)+d(y) for all x, y ∈ N. A symmetric bi-additive mapping D:N×N→N is called a symmetric bi-derivation if D(xy,z) = D(x,y)z+xD(y,z) is fulfilled for all x, y, z ∈ N. In this paper we investigate concerning a (α,α)-symmetric derivations D on semiprime rings and prime rings R.

2. Preliminaries

Throughout, R is an associative semiprime ring. We shall write \([x,y] \) for \(xy-yx \). Then \([xy,z] = x[y,z] + [x,z]y; \) \([x,yz] = y[x,z] + [x,y]z \) for all x, y, z ∈ R. Recall that R is prime if aRb=(0) implies a= 0 or b= 0 and semiprime if aRa=(0)
implies $a = 0$. An additive mapping $D$ from $R$ into itself is called a derivation if $D(xy) = D(x)y + xD(y)$ for all $x, y \in R$. A mapping $f$ of $R$ into itself is called a commuting if $[f(x), x] = 0$ and centralizing if $[f(x), x] \in Z(R)$ for all $x \in R$. $Z(R)$ denotes the centre of $R$. An additive mapping $F$ from $R$ to $R$ is said to be a commuting (resp. centralizing) if $[F(x), x] = 0$ (resp. $[F(x), x] \in Z(R)$) holds for all $x \in R$, and $d$ is said to be central if $F(x) \in Z(R)$ holds for all $x \in R$. More generally, for a positive integer $n$, we define a mapping $F$ to be $n$-commuting if $[F(x), x^n] = 0$ for all $x \in R$. An additive $(\alpha, \alpha)$-derivation $D: R \rightarrow R$ is called an $(\alpha, \alpha)$-symmetric derivation on a ring $R$ if $D(xy) = D(yx)$ for all $x, y \in R$. (Breˇsar ,and Vukman,1989) have introduced the notion of a reverse derivation as an additive mapping $d$ from a ring $R$ into itself satisfying $d(xy) = d(y)x + yd(x)$ for all $x, y \in R$. Obviously, if $R$ is commutative, then both derivation and reverse derivation are the same. An additive map $D: R \rightarrow R$ is said to be an $(\alpha, \alpha)$-derivation if $D(xy) = D(x)\alpha(y) + \alpha(x)D(y)$ for all $x, y \in R$.

We shall need the following well-known and frequently used lemmas.

**Lemma 2.1.** (Shu-Hua, and Feng-Wen, 2002: Corollary 1) Let $R$ be a semiprime ring and $D$ derivation of $R$, $U = 0$. If $D$ is centralizing or derivation and $\alpha$ an automorphism such that $D(\alpha(x))^2 = 0$ this gives $D(\alpha(x)) = 0$ for all $x, y \in R$. Hence $D$ is a central mapping. Then according to Lemma 2.2., a mapping $D$ on a semiprime ring $R$ is a reverse derivation.

**Theorem 3.2.** Let $R$ is a ring and $D$ is an $(\alpha,\alpha)$-symmetric mapping on a ring $R$, then $D([x,y]^{m+1}) = 0$ for all $x, y \in R$, where $n$ is a positive integer.

**Proof:** We can prove the theorem with the help of mathematical induction.

(i) When $n=1$, we then have $D([x,y]^2) = 0$. As $D$ is an $(\alpha,\alpha)$-symmetric mapping on a ring $R$, so by $D$ is an $(\alpha,\alpha)$-symmetric mapping on a ring $R$, we have $D([x,y]) = D([y,x]) = 0$ for all $x, y \in R$. Then

$D([x,y]) = 0$ for all $x, y \in R$. Also

$D([x,y]) = D([x,y])\alpha([x,y]) + \alpha([x,y])D([x,y])$. Since, we have

$D([x,y]) = 0$ for all $x, y \in R$. Therefore, $D([x,y]) = 0$.

(ii) When $n=m$, then $D([x,y]^{m+1}) = 0$ for all $x \in R$.

(iii) Suppose that true when $n=m+1$, then

$D([x,y]^{m+1}) = 0$ for all $x \in R$.

$D([x,y]^{m+1})\alpha([x,y]) + \alpha([x,y])D([x,y]) = 0$. Accoring to Lemma 2.3.(iii), we get

$\alpha([x,y]^{m+1})D([x,y]) = 0$.

Again according $D$ is an $(\alpha,\alpha)$-symmetric mapping on a ring $R$, we have $D([x,y]) = 0$ for all $x, y \in R$. Then by right-multiplying by $\alpha([x,y])$,we get

$D([x,y])\alpha([x,y]) = 0$ for all $x, y \in R$. Then from them relations, we get

$D([x,y])\alpha([x,y]) + \alpha([x,y])D([x,y]) = 0$ for all $x, y \in R$.

$D([x,y]) = 0$ for all $x, y \in R$. Right-multiplying by $D([x,y])$ for all $x, y \in R$, leads to $D([x,y])^{m+1} = 0$ for all $x, y \in R$. This completes the proof.

**Theorem 3.3.** Let $R$ is a prime ring and $D$ be a $(\alpha,\alpha)$-symmetric derivation on $R$, then either $R$ is commutative or $D$ is zero.

**Proof:** By Lemma 2.3 (ii) $D([x,y]) = 0$ implies that $D(\alpha([x,y])) + \alpha([x,y])D([x,y]) = 0$ also in view of Lemma 2.3(i), we have $D(\alpha([x,y])) = 0$ for all $x, y \in R$. Further in view of Lemma 2.3(i), $D([x,y]) = 0$ gives

$\alpha([x,y])D(x) = 0$.

Replacing $y$ by $yz$ in $\alpha([x,y])D(x) = 0$, and using it again, we get

$\alpha([x,y])D(x) = 0$.

Pre multiplying (4) by $\alpha([x,y])D(x)v$ we get

$\alpha([x,y])D(x)v = 0$.

$\alpha([x,y])D(x)v = 0$.
Let $R$ be a semiprime ring and $D$ be an $(\alpha, \alpha)$-symmetric derivation on $R$, then $[D(x), \alpha(y)] = 0$ for all $x, y \in R$, where $n$ is a positive integer.

**Proof:** Since $D$ is an $(\alpha, \alpha)$-symmetric derivation, according to Lemma 2.4, replacing $z$ by $xz$ in $D(x[y,z]u) = 0$ for all $x, y, z \in R$, we can get $D(x[y,z]u) = 0$ for all $x, y, z \in R$. Again replacing $z$ by $xz$ in the equation obtained and using $D(x[y,z]u) = 0$ for all $x, y, z \in R$, we have $D(x[y, z]) = 0$.

Continuing this process, we get $D(x^{n+1}[y,z]) = 0$ for all $x, y, z \in R$.

As $D$ is an $(\alpha, \alpha)$-symmetric derivation, therefore

$$D((y,z)x^{n+1}) = 0$$

for all $x, y, z \in R$.

$$D((x^{n+1}[y,z]) = 0$$

In view of Lemma 2.3 (i),

$$D((x^{n+1})\alpha([y,z]) = 0.$$  \hspace{1cm} (6)

As $D$ be an $(\alpha, \alpha)$-symmetric derivation, therefore

$$D((y,z)x^{n+1}) = 0.$$  \hspace{1cm} (7)

Subtracting (7) from (6), we get $D((x^{n+1}), \alpha([y,z])) = 0$.

Replacing $z$ by $y$ and using again it, we get $D((x^{n+1}), \alpha(y)) \alpha([y,z]) = 0$. After that

**Theorem 3.6.**

Let $R$ be a 2-torsion free semiprime ring, $U$ a nonzero left ideal of $R$ such that $r_0(U) = 0$, and $\alpha : R \to R$ is an automorphism. If there exists an $(\alpha, \alpha)$-derivation $D : R \to R$ such that $[D(x), \alpha(x)] = 0$ for all $x \in U$, then $D(R) \subseteq Z(R)$ and the ideal generated by $D(R)$ is in the $Z(R)$.

**Proof:** At first, we need to define a mapping $H(\ldots) : R \times R \to R$

by $H(x,y) = [D(x), \alpha(y)] + [D(y), \alpha(x)]$ for all $x \in U$, $y \in R$.

Then it is easy to see that $H(x,y) = H(y,x)$ for all $x \in U$, $y, z \in R$, and additive in both arguments.

Using the Jacobian identities and definition of $H(x,y)$, we can conclude the following:

$$H(xy,z) = H(x,y) + H(x,z) + D(x)(H(y,z) + D(x)[\alpha(y), \alpha(z)]) + [\alpha(x), \alpha(z)]$$

$$D(y).$$  \hspace{1cm} (8)

Define the mapping $h : R \to R$ such that $h(x) = H(x,x)$, then

$$h(x) = 2D(x), \alpha(x).$$  \hspace{1cm} (9)

One can conclude $h(x+y) = h(x) + h(y) + 2H(x,y)$. By $[[D(x),\alpha(x)],\alpha(x)] = 0$, we have, $h(x, \alpha(x)) = 0$. Linearization of last equation gives

$$[h(x), \alpha(y)] + [h(y), \alpha(x)] + 2H(x,y, \alpha(x)) + 2H(x,y, \alpha(y)) = 0.$$  \hspace{1cm} (10)

Replacing $x$ by $−x$ in (10) and using the fact $h(−x) = h(x)$, we get

$$[h(x), \alpha(y)] + [h(y), \alpha(x)] + 2H(x,y, \alpha(x)) + 2H(x,y, \alpha(y)) = 0.$$  \hspace{1cm} (11)

Adding (10) and (11) and using the fact that $R$ is 2-torsion free, we get

$$[h(x), \alpha(y)] + [h(y), \alpha(x)] + 2H(x,y, \alpha(x)) + 2H(x,y, \alpha(y)) = 0.$$  \hspace{1cm} (12)

Replacing $z$ by $\alpha(x)D(x^{n+1})$ and again replacing $z$ by $\alpha(y)D(x^{n+1})$ it again in (13), we get

$$[h(x), \alpha(y)] + [h(y), \alpha(x)] + 2H(x,y, \alpha(x)) + 2H(x,y, \alpha(y)) = 0.$$  \hspace{1cm} (13)

Replacing $y$ by $\alpha(y)$ in (12) and using $H(x,y) = H(y,x)$, we get

$$[h(x), \alpha(y)] + [h(y), \alpha(x)] + 2H(x,y, \alpha(x)) + 2H(x,y, \alpha(y)) = 0.$$  \hspace{1cm} (14)

Replacing $y$ by $\alpha(y)$ in (13), we get

$$[h(x), \alpha(y)] + [h(y), \alpha(x)] + 2H(x,y, \alpha(x)) + 2H(x,y, \alpha(y)) = 0.$$  \hspace{1cm} (15)

Putting $y = \alpha(x)D(x^{n+1})$ and using (15), the last equation becomes

$$3h(x)[\alpha(y), \alpha(x)] + 2D(x)[\alpha(y), \alpha(x)] = 0.$$  \hspace{1cm} (16)

Pre multiplying (13) by $D(x)$ and subtracting from (16), we get
\begin{align}
(3h(x)D(x)−D(x)h(x))[a(y),a(x)]&=0. \quad (17) \\
\text{Post multiplying (17) by } a(z), \text{ we get} \\
(3h(x)D(x)−D(x)h(x))[a(y),a(x)]a(z)&=0. \quad (18) \\
\text{Replacing } y \text{ by } yz \text{ in (17), we get} \\
(3h(x)D(x)−D(x)h(x))\{a(y)\{a(z),a(x)\}+\{a(y),a(x)\}a(z)\}&=0. \quad (19) \\
\text{Subtracting (18) from (19), we get} \\
\{3h(x)D(x)−D(x)h(x)\}a(y)\{a(z),a(x)\}&=0. \quad (20) \\
\text{Replacing } z \text{ by } a^{-1}(2D(x)) \text{ and } y \text{ by } a^{-1}(t) \text{ in (20), we get} \\
\{3h(x)D(x)−D(x)h(x)\}t\{3h(x)D(x)−D(x)h(x)\}&=0. \quad (21) \\
\text{Semiprimeness of } R \text{ implies that } 3h(x)D(x)−D(x)h(x) = 0. \quad (22) \\
\text{That is} \\
3h(x)D(x) = D(x)h(x). \quad (23) \\
\text{Now replacing } y \text{ by } zy \text{ in (14) and using (14) again and then putting} \\
y = a^{-1}(D(x)) \text{ and } z = y, \text{ we get} \\
3[a(y),a(x)]D(x)h(x)+2[a(y),a(x)]h(x)D(x)+2\{[a(y),a(x)],a(x)\}(D(x))²&=0. \quad (24) \\
\text{Post multiplying (14) by } D(x) \text{ and using in (24), we get} \\
[a(y),a(x)]3D(x)h(x)−h(x)D(x))&=0. \quad (25) \\
\text{Replacing } y \text{ by } zy \text{ in (25) and using (25) again, we get} \\
[a(z),a(x)]3D(x)h(x)−h(x)D(x))&=0. \quad (26) \\
\text{Replacing } z \text{ by } a^{-1}(2D(x)) \text{ in the last equation and using (9), we get} \\
h(x)a(y)3D(x)h(x)−h(x)D(x))&=0. \quad (27) \\
\text{Pre multiplying (24) by } 3D(x) \text{ and replacing } y \text{ by } a^{-1}(t), \text{ we get} \\
3D(x)h(x)3D(x)h(x)−h(x)D(x))&=0. \quad (28) \\
\text{Replacing } y \text{ by } a^{-1}(D(x)t) \text{ in (26), we get} \\
h(x)D(x)t3D(x)h(x)−h(x)D(x))&=0. \quad (29) \\
\text{Subtracting (28) from (27) and using the fact that } R \text{ is semiprime, we get} \\
3D(x)h(x) = h(x)D(x). \quad (30) \\
\text{Using (29) in (23) and by 2-torsion freeness of } R, \text{ we get} \\
h(x)D(x) = 0. \quad (31) \\
\text{and also } D(x)h(x) = 0. \text{ Now take} \\
h(x)D(y)+2H(x,y)D(x)=h(x)D(y)+2([D(x),a(y)]+D(y),a(x))]D(x). \quad (32) \\
\text{Replacing } y \text{ by } x \text{ and using (9) and (30), we get} \\
h(x)D(y) + 2H(x,y)D(x) = h(x)D(y) = 0. \quad (33) \\
\text{Replacing } y \text{ by } xy \text{ in (33), we have} \\
h(x)a(y)a(x)D(x)+D(x)\{a(y),a(x)\}a(x)D(x)&=0. \quad (34) \\
\text{Post multiplying (33) by } a(x), \text{ we get} \\
h(x)a(y)a(x)\{a(y),a(x)\}D(x)\{a(x)D(x)&=0. \quad (35) \\
\text{Subtracting (34) from (35) and using (9), we get} \\
h(x)D(y)+2H(x,y)D(x)=h(x)D(y)+2D(x)\{a(y),\alpha(x)\}D(x)&=0. \quad (36) \\
\text{As } [h(x),a(x)]&=0 \text{ and 2-torsion freeness of } R \text{ the last equation becomes} \\
h(x)yD(x) = D(x)\{a(y),\alpha(x)\}D(x)&=0. \quad (37) \\
\text{Replacing } y \text{ by } xy \text{ in (37), we have} \\
h(x)a(y)a(x)D(x)+D(x)\{a(y),a(x)\}a(x)D(x)&=0. \quad (38) \\
\text{In view of (37), we get } h(x)D(x) = 0 \text{ and semiprimeness of } R \text{ implies that } h(x)=0 \text{ for all } x \in R. \text{ That is } [D(x),\alpha(x)]&=0 \text{ for all } x \in R, \text{ since } \alpha:R \rightarrow R \text{ is an automorphism, then we obtain} \\
[D(x),x] = 0 \text{ for all } x \in U. \text{ This lead to } [D(x),x] \in \mathbb{Z}(R), \text{ for all } x \in U. \text{ Thus, according to Lemma 2.1, we get } D(R) \subseteq \mathbb{Z}(R) \text{ and the ideal generated by } D(R) \text{ is in the } \mathbb{Z}(R). \text{ This completes the proof. } 
\end{align}
Theorem 3.8.
Let $R$ be a $2$-torsion free and $3$-torsion free semiprime ring, $U$ a nonzero left ideal of $R$ such that $r_{R}(U)=0$, and $\alpha : R \to R$ is an automorphism. If there exists an $(\alpha, \alpha)$-derivation $D : R \to R$ such that $[D(x), \alpha(x)], \alpha(x)] \in Z(R)$, then $D(R) \subseteq Z(R)$ and the ideal generated by $D(R)$ is in the $Z(R)$.

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