FINITENESS AND VANISHING RESULTS ON THE WEIGHTED POINCARÉ INEQUALITY OF COMPLETE MANIFOLDS

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Abstract. We study manifolds satisfying a weighted Poincaré inequality, which was first introduced by Li-Wang [8]. We generalized one of their results by relaxing the Ricci curvature bound condition only being satisfied outside a compact set and established a finitely many ends result. We also generalized a result of [7] to the weighted Poincaré inequality case and established a vanishing result for $L^2$ harmonic 1-form provided that the weight function $\rho$ is of sub-quadratic growth of the distance function.

Introduction

In a work of Witten-Yau [11], they proved that if $M^n$ is a conformally compact, Einstein, $n \geq 3$ dimensional manifold whose boundary has positive Yamabe constant, then $M$ must have only one end. Shortly after that, Cai-Galloway [1] relaxed the assumption to allow the boundary of $M$ has non-negative Yamabe constant. In [10], X. Wang proved the following:

Theorem. (X. Wang) Let $M^n$ be an $n$-dimensional ($n \geq 3$), conformally compact manifold with Ricci curvature bounded from below by

$$\text{Ric}_M \geq -(n-1).$$

Let $\lambda_1(M)$ be the lower bound of the spectrum of the Laplacian on $M$. If

$$\lambda_1(M) \geq n-2,$$

then either

(a) $H^1(L^2(M)) = 0$; or

(b) $M = \mathbb{R} \times N$ with the warped product metric $ds^2 = dt^2 + \cosh^2 t \, ds^2_N$, where $N$ is a compact manifold with $\text{Ric}_N \geq n-2$. In particular, $M$ either has only one end or it must be a warped product given as above.

The above result generalized the results of Witten-Yau and Cai-Galloway since a theorem of Mazzeo [9] identifies the $L^2$ cohomology group $H^1(L^2(M))$ with the relative cohomology group $H^1(M, \partial M)$ for conformally compact manifolds and a theorem of Lee [3] asserts that $\lambda_1(M) = \frac{(n-1)^2}{4}$ for a $n$-dimensional Einstein, conformally compact manifold with non-negative Yamabe constant for its boundary. In [7], Li-Wang considered a complete, $n$-dimensional, Riemannian manifold $M^n$ whose Ricci curvature is bounded from below by

$$\text{Ric}_M \geq -\frac{n-1}{n-2} \lambda_1(M),$$

where $\lambda_1(M)$, the greatest lower bound of the spectrum of the Laplacian acting on $L^2$ functions, is assumed to be positive. They proved the following
Theorem. (Li-Wang) Let $M^n$ be a complete Riemannian manifold of dimension $n \geq 3$. Suppose $\lambda_1(M) > 0$ and
\[ \text{Ric}_M \geq -\frac{n-1}{n-2} \lambda_1(M). \]
Then either
1. $M$ has only one end with infinite volume; or
2. $M = \mathbb{R} \times N$ with the warped product metric
\[ ds_M^2 = dt^2 + \cosh^2 \left( \sqrt{\frac{\lambda_1(M)}{n-2}} t \right) ds_N^2, \]
where $N$ is a compact manifold with Ricci curvature bounded from below by
\[ \text{Ric}_N \geq -\lambda_1(M). \]

Since all the ends of a conformally compact manifold must have infinite volume, the above theorem thus generalized the work of [1], [10] and [11] to complete manifolds with positive spectrum. Since $\lambda_1(M) > 0$, the variational principle for $\lambda_1(M)$ implies the following Poincaré inequality
\[ \lambda_1(M) \int_M \phi^2 \leq \int_M |\nabla \phi|^2; \]
for any compactly supported smooth function, $\phi \in C_\infty^\infty(M)$. In [8], the authors considered manifolds satisfying a weighted Poincaré inequality and generalized most of their results in [7] for manifolds with positive spectrum to manifolds satisfying a weighted Poincaré inequality. A manifold $M^n$ is said to be satisfied a weighted Poincaré inequality with a non-negative weight function $\rho(x)$ if
\[ \int_M \rho(x) \phi^2(x) dV \leq \int_M |\nabla \phi|^2 dV, \]
for any compactly supported smooth function $\phi \in C_\infty^\infty(M)$. In particular, when $\rho(x) = \lambda_1(M)$ is a positive constant, $M$ is a manifold with positive spectrum. We say that a manifold $M$ has property ($P_\rho$) if a weighted Poincaré inequality is valid on $M$ with some non-negative weight function $\rho(x)$ and the $\rho$-metric, defined by
\[ ds_\rho^2 = \rho \ ds_M^2 \]
is complete. Define
\[ S(R) = \sup_{B_\rho(R)} \sqrt{\rho}, \]
where $B_\rho(R)$ is the ball of radius $R$ (with respect to some fixed point $p$) under the $\rho$-metric. In [8], the authors proved the following:

Theorem 1. (Li-Wang) Let $M^n$ be a complete manifold with dimension $n \geq 3$. Assume that $M$ satisfies property ($P_\rho$) for some nonzero weight function $\rho \geq 0$. Suppose
\[ \text{Ric}_M(x) \geq -\frac{n-1}{n-2} \rho(x) \]
for all $x \in M$. If $\rho$ satisfies the growth estimate
\[ \liminf_{R \to \infty} \frac{S(R)}{F(R)} = 0, \]
where
\[
F(R) = \begin{cases} 
\exp\left(\frac{n-3}{n-2}R\right) & \text{for } n \geq 4 \\
\frac{R}{n-2} & \text{for } n = 3 
\end{cases}
\]
then either

1. \(M\) has only one nonparabolic end; or
2. \(M\) has two nonparabolic ends and is given by \(M = \mathbb{R} \times N\) with the warped product metric
   \[
ds^2_M = dt^2 + \eta^2(t)ds^2_N,
\]
   for some positive function \(\eta(t)\), and some compact manifold \(N\). Moreover, \(\rho(t)\) is a function of \(t\) alone satisfying
   \[
   \eta'' \eta^{-1} = \rho
\]
   and
   \[
   \liminf_{x \to \infty} \rho(x) > 0;
\]
or
3. \(M\) has one parabolic end and one nonparabolic end and is given by \(M = \mathbb{R} \times N\) with the warped product metric
   \[
ds^2_M = dt^2 + \eta^2(t)ds^2_N,
\]
   for some positive function \(\eta(t)\), and some compact manifold \(N\). Moreover, \(\rho(t)\) is a function of \(t\) alone satisfying
   \[
   \eta'' \eta^{-1} = \rho
\]
   and
   \[
   \liminf_{x \to \infty} \rho(x) > 0 \text{ on the nonparabolic end.}
\]

It is interesting to see if a similar theorem holds by relaxing the above assumptions to be only satisfied outside a compact set of \(M\). In this article, the following theorem has been established:

**Theorem 2.** Let \(M^n\) be a complete manifold with dimension \(n \geq 3\). Assume that \(M\) has property \((P_{\rho})\). Also assume that the Ricci curvature of \(M\) satisfies the lower bound

\[
\text{Ric}_{M \setminus K}(x) \geq -\frac{n-1}{n-2}\rho(x) + \tilde{\varepsilon}
\]

for some \(\tilde{\varepsilon} > 0\), compact set \(K \subseteq M\). If \(\rho\) satisfies the growth estimate

\[
\liminf_{R \to \infty} \frac{S(R)}{F(R)} = 0,
\]

where
\[
F(R) = \begin{cases} 
\exp\left(\frac{n-3}{n-2}R\right) & \text{for } n \geq 4 \\
\frac{R}{n-2} & \text{for } n = 3 
\end{cases}
\]
then \(M\) has only finitely many non-parabolic ends. If in addition \(\liminf_{x \to \infty} \rho(x) > 0\), then \(M\) has only finitely many infinite volume ends.

On the other hand, Li-Wang also proved a vanishing theorem for \(L^2\) integrable harmonic 1-forms on \(M\) in [7]:
Theorem 3. (Li-Wang) Let $M$ be a $n$-dimensional complete Riemannian manifold with $\lambda_1(M) > 0$ and

$$\text{Ric}_M \geq -\frac{n}{n-1} \lambda_1(M) + \varepsilon,$$

for some $\varepsilon > 0$. Then $H^1(L^2(M)) = 0$.

In this article, we generalized above theorem to manifolds satisfying a weighted Poincaré inequality:

Theorem 4. Let $M$ be a $n$-dimensional complete Riemannian manifold satisfying a weighted Poincaré inequality with a non-negative weight function $\rho(x)$. Assume the Ricci curvature satisfies

$$\text{Ric}_M(x) \geq -\frac{n}{n-1} \rho(x) + \varepsilon,$$

for some $\varepsilon > 0$. Let $r(x,p)$ be the distance function from $x$ to some fixed point $p$. If $\rho(x) = O(r^{2-\alpha}(x,p))$, for some $0 < \alpha < 2$. Then $H^1(L^2(M)) = 0$.

Some lemmas

The following lemma is modified from [8] to suit our situation.

Lemma 5. Assume that

$$\text{Ric}_M(x) \geq -\frac{n-1}{n-2} l(x)$$

for some function $l(x)$. If $f$ is a positive harmonic function on $M$, then

$$|\nabla f|(x) \leq \left( (n-1) \sup_{B_r(x,1)} \sqrt{\rho(y)} + C_1 \sup_{B_r(x,1)} \sqrt{\rho(y)} \right) f(x),$$

where $C_1$ is a constant only depending on $n$. In particular, if the lower bound of the Ricci curvature of $M$ satisfies

$$\text{Ric}_{M \setminus K}(x) \geq -\frac{n-1}{n-2} \rho(x) + \tilde{\varepsilon}$$

where $K$ is a compact sub-domain of $M$ and for some $\tilde{\varepsilon} > 0$. Then

(1) $$|\nabla f|(x) \leq C \left( \sup_{B_r(x,1)} \rho \right) f(x),$$

where $C = C(n)$, provided that $B_r(x,1) \cap K = \emptyset$.

Proof. Cheng-Yau’s [2] (see also [7]) local gradient estimate for positive harmonic functions implies that for any $R > 0$,

(2) $$|\nabla f|(x) \leq \left( (n-1) \sup_{B(x,R)} \sqrt{\rho} + CR^{-1} \right) f(x),$$

where $C = C(n)$. Consider the function $g(r) = r - (\sup_{B(x,r)} \sqrt{\rho})^{-1}$. Since $g$ is negative when $r \to 0$ and $g \to +\infty$ as $r \to +\infty$. Hence we can choose $R_0 > 0$ such
that \( g(R_0) = 0 \), that is, \( R_0 = \left( \sup_{B(x,R_0)} \sqrt{\rho(\gamma)} \right)^{-1} \). For any point \( y \in B(x, R_0) \), let \( \gamma \) be a minimizing geodesic (with respect to \( ds^2_M \)) joining \( x, y \), then

\[
\begin{align*}
    r_{\rho}(x, y) &\leq \int_{\gamma} \sqrt{\rho(\gamma(t))} dt \\
&\leq \left( \sup_{B(x,R_0)} \sqrt{\rho(y)} \right) R_0 \\
&\leq 1,
\end{align*}
\]

hence \( B(x, R_0) \subseteq B_{\rho}(x, 1) \). Combining with (2) by choosing \( R = R_0 \), the result follows. \( \quad \square \)

The following version of Bochner formula is well-known and was first used by Yau [12].

**Lemma 6.** [12] (see also [8]) Let \( M^n \) be a complete Riemannian manifold of dimension \( n \geq 2 \). Assume that the Ricci curvature of \( M \) satisfies the lower bound

\[
\text{Ric}_M(x) \geq -(n-1)\tau(x).
\]

Assume \( f \) is a nonconstant harmonic function on \( M \). Then the function \( h = |\nabla f| \) satisfies

\[
\Delta h \geq -(n-1)\tau h + \frac{|\nabla h|^2}{(n-1)h}.
\]

In addition, if we write \( g = h^{\frac{n-2}{n-1}} \), the above inequality becomes

\[
\Delta g \geq -(n-2)\tau g.
\]

**Proof.** For the sake of completeness, we outline the proof here. We choose a local orthonormal frame \( \{e_1, e_2, \cdots, e_n\} \) such that \( e_1 f = |\nabla f| \) and \( e_{\alpha} f = 0 \), for \( \alpha = 2, \cdots, n \) at a point \( x \).

\[
\begin{align*}
    h^2 &= |\nabla f|^2 \\
    hh_j &= \sum_{i=1}^n f_{ij} f_i.
\end{align*}
\]

Hence

\[
\begin{align*}
    h_j &= f_{1j}, \\
|\nabla h|^2 &= \sum_{j=1}^n h_j^2 = \sum_{j=1}^n f_{1j}^2, \quad \text{(3)}
\end{align*}
\]
when evaluated at $x$. Using the fact that $f$ is harmonic and the Ricci formula, we compute

$$|\nabla h|^2 + h \triangle h = \sum_{j=1}^{n} (h_j^2 + hh_{jj})$$

$$= \sum_{i,j=1}^{n} (f_{ij}^2 + f_{ijj} f_i)$$

$$= \sum_{i,j=1}^{n} (f_{ij}^2 + f_{jji} f_i + R_{ij} f_i f_j)$$

$$= \sum_{i,j=1}^{n} (f_{ij}^2 + R_{ij} f_i f_j)$$

$$\geq -(n-1)\tau h^2 + \sum_{i,j=1}^{n} f_{ij}^2,$$

and

$$\sum_{i,j=1}^{n} f_{ij}^2 = 2 \sum_{\alpha=2}^{n} f_{i\alpha}^2 + f_{11}^2 + \sum_{\beta=2}^{n} f_{\beta\beta}^2$$

$$\geq 2 \sum_{\alpha=2}^{n} f_{i\alpha}^2 + f_{11}^2 + \frac{1}{n-1} \left( \sum_{\beta=2}^{n} f_{\beta\beta} \right)^2$$

$$= 2 \sum_{\alpha=2}^{n} f_{i\alpha}^2 + \frac{n}{n-1} f_{11}^2$$

$$\geq \frac{n}{n-1} \sum_{j=1}^{n} f_{ij}^2.$$

Combining the above inequality with (3), (4) and evaluated at $x$, the result follows. □

A theory of Li-Tam [6] allows us to count the number of non-parabolic ends of $M$ by counting the dimension of $K^0(M)$, a subspace of the space of all harmonic functions on $M$. We outline the construction of Li-Tam here. Assume that $M$ has at least two non-parabolic ends, $E_1, E_2$, for $R > 0$, we solve the following equation

$$\begin{cases}
\triangle f_R = 0 & \text{in } B(R) \\
f_R = 1 & \text{on } \partial B(R) \cap E_1 \\
f_R = 0 & \text{on } \partial B(R) \setminus E_1.
\end{cases}$$

By passing to a convergent subsequence, the sequence $\{f_R\}$ converges to a nonconstant harmonic function $f_1$ with finite Dirichlet integral, satisfying $0 \leq f_1 \leq 1$. Clearly for each non-parabolic end $E_i$, we can construct a corresponding $f_i$ by the above process. Let $K^0(M)$ be the linear space containing all the $f_i$’s constructed as above. By the construction, the number of nonparabolic ends of $M$ is given by the dimension of $K^0(M)$. The following lemma of Li is useful in proving finiteness type theorems.
**Lemma 7.** Let $\mathcal{H}$ be a finite dimensional subspace of $L^2$ $p$-forms defined over a set $D \subseteq M^n$. If $V(D)$ denotes the volume of the set $D$, then there exists $\omega_0 \in \mathcal{H}$ such that

$$\dim \mathcal{H} \int_D |\omega_0|^2 \leq V(D) \cdot \sup_D |\omega_0|^2 \cdot \min \left\{ \left( \frac{n}{p} \right), \dim \mathcal{H} \right\}.$$ 

**Result of finitely many ends**

**Theorem 8.** Let $M^n$ be a complete manifold with dimension $n \geq 3$. Assume that $M$ has property $(P_\rho)$. Also assume that the Ricci curvature of $M$ satisfies the lower bound

$$\text{Ric}_{M \setminus K}(x) \geq -\frac{n-1}{n-2} \rho(x) + \tilde{\varepsilon}$$

for some $\tilde{\varepsilon} > 0$, compact set $K \subseteq M$. If $\rho$ satisfies the growth estimate

$$\liminf_{R \to \infty} \frac{S(R)}{F(R)} = 0,$$

where

$$F(R) = \begin{cases} \exp\left( \frac{n-3}{n-2} R \right) & \text{for } n \geq 4 \\ R & \text{for } n = 3 \end{cases},$$

then $M$ has only finitely many nonparabolic ends. If in addition $\liminf_{x \to \infty} \rho(x) > 0$, then $M$ has only finitely many infinite volume ends.

**Proof.** By the discussion above lemma 7 it is sufficient to estimate $\dim K^0(M)$. We may assume that $M$ has at least two nonparabolic ends and hence there exists a nonconstant bounded harmonic function $f \in K^0(M)$ with finite Dirichlet integral. By maximal principle, we may assume that $\inf f = 0$ and $\sup f = 1$ (see [6]). Bochner formula and the assumption on the Ricci curvature give us

$$\triangle |\nabla f| \geq -\frac{n-1}{n-2} |\nabla f|(\rho - \varepsilon) + \frac{|\nabla \nabla f|^2}{(n-1)|\nabla f|},$$

where $\varepsilon = \frac{n-2}{n-1} \tilde{\varepsilon}$. Applying lemma [4] the above equation becomes

$$\triangle g + \rho g \geq \varepsilon g$$

in $M \setminus K$, where $g = |\nabla f|^{\frac{n-2}{n-1}}$. Let $\phi \in C^\infty_c(M \setminus K)$ be a non-negative smooth function with compact support in $M \setminus K$. Using the property $(P_\rho)$ of $M$ and integration by parts, we have

\begin{align*}
\int_M \phi^2 \rho g^2 &\leq \int_M |\nabla (\phi g)|^2 \\
&= \int_M |\nabla \phi|^2 g^2 + \int_M \phi^2 |\nabla g|^2 + 2 \int_M \phi g \langle \nabla \phi, \nabla g \rangle \\
&= \int_M |\nabla \phi|^2 g^2 + \int_M \phi^2 |\nabla g|^2 + \frac{1}{2} \int_M \langle \nabla (\phi)^2, \nabla g \rangle^2 \\
&= \int_M |\nabla \phi|^2 g^2 - \int_M \phi^2 g \triangle g.
\end{align*}
Combining (5) and (6), we have
\[ \varepsilon \int_M \phi^2 g^2 \leq \int_M |\nabla \phi|^2 g^2, \forall \phi \in C_c^\infty (M \setminus K). \]

Since \( K \) is compact, we may choose \( R_0 > 0 \) such that
\[ K \subseteq \bigcup_{x \in K} B_p(x, 1) \subseteq B(R_0 - 1). \]

Let \( R > 0 \) be such that \( B(R_0) \subseteq B_p(R - 1) \). The above inequality implies
\[ \varepsilon \int_{B_p(R) \setminus B(R_0 - 1)} \phi^2 g^2 \leq \int_{B_p(R) \setminus B(R_0 - 1)} |\nabla \phi|^2 g^2, \]
for any \( \phi \in C_c^\infty (B_p(R) \setminus B(R_0 - 1)) \). If we write \( \phi = \psi \cdot \chi \), the right hand side of (7) becomes
\[ \int_{B_p(R) \setminus B(R_0 - 1)} |\nabla \phi|^2 g^2 \leq 2 \int_{B_p(R) \setminus B(R_0 - 1)} |\nabla \psi|^2 \chi^2 g^2 \]
\[ + 2 \int_{B_p(R) \setminus B(R_0 - 1)} |\nabla \chi|^2 \psi^2 g^2. \]

Now we choose \( \chi, \psi \) as in (8):
\[ \chi(x) = \begin{cases} 0 & \text{on } \mathcal{L}(0, \delta \bar{\varepsilon}) \cup \mathcal{L}(1 - \delta \bar{\varepsilon}, 1) \\ (-\log \delta)^{-1} (\log f - \log(\delta \bar{\varepsilon})) & \text{on } \mathcal{L}(\delta \bar{\varepsilon}, \bar{\varepsilon}) \cap (M \setminus E_1) \\ (-\log \delta)^{-1} (\log(1 - f) - \log(\delta \bar{\varepsilon})) & \text{on } \mathcal{L}(1 - \bar{\varepsilon}, 1 - \delta \bar{\varepsilon}) \cap E_1 \\ 1 & \text{otherwise} \end{cases}, \]
for some \( 0 < \delta < 1 \) and \( 0 < \bar{\varepsilon} < \frac{1}{2} \) to be determined later, where
\[ \mathcal{L}(a, b) = \{ x \in M : a < f(x) < b \}. \]
\[ \psi(x) = \begin{cases} 0 & \text{on } B(R_0 - 1) \\ 1 & \text{on } B_p(R - 1) \setminus B(R_0) \\ R - r_p & \text{on } B_p(R) \setminus B_p(R - 1) \\ 0 & \text{on } M \setminus B_p(R) \end{cases}. \]

The first term on the right hand side of (8) can be estimated by
\[ 2 \int_{B_p(R) \setminus B(R_0 - 1)} |\nabla \psi|^2 \chi^2 g^2 \leq 2 \int_{B_p(R) \setminus B_p(R - 1)} |\nabla \psi|^2 \chi^2 g^2 \]
\[ + 2 \int_{B(R_0) \setminus B(R_0 - 1)} |\nabla \psi|^2 \chi^2 g^2. \]
Now consider
\[
\int_{E_1} \left| \nabla \chi \right|^2 \psi^2 g^2 \leq (\log \delta)^{-2} \int_{L(1-\tau, 1-\delta \varepsilon) \cap E_1 \cap (B_p(R) \setminus B(R_0-1))} |\nabla f|^{2+ \frac{2(n-1)}{n}} (1-f)^{2} \\
\leq C S \frac{2(n-2)}{n(n-1)} (R+1)(\log \delta)^{-2} \int_{L(1-\tau, 1-\delta \varepsilon) \cap E_1 \cap (B_p(R) \setminus B(R_0-1))} |\nabla f|^2 (1-f)^{\frac{2(n-2)}{n-1}},
\]
where the second inequality follows from (1) by replacing \(f\) by \(1-f\) and the choice of \(R_0\) so that \(B_p(x, 1)\) does not intersect \(K\) for any \(x \in B_p(R) \setminus B(R_0-1)\). Now, we are exactly the same situation as of [8]. We follow the choices of \(\delta, \tau\) as in [8], \(\tau \to 0\) as \(R \to +\infty\) and we have the followings:
\[
\int_M |\nabla \chi|^2 \psi^2 g^2 \to 0 \quad \text{as} \quad R \to +\infty, \quad \text{and} \quad \int_{B_p(R) \setminus B_p(R-1)} |\nabla \psi|^2 \chi^2 g^2 \to 0 \quad \text{as} \quad R \to +\infty.
\]
Now if we let \(R \to +\infty\) and combining the above results with (8) and (9), (7) becomes
\[
\varepsilon \int_{M \setminus B(R_0)} g^2 \leq \tilde{C} \int_{B(R_0) \setminus B(R_0-1)} g^2,
\]
where \(\tilde{C}\) is a constant depending only on \(n\), which implies
\[
\int_{B(2R_0)} g^2 \leq C \int_{B(R_0)} g^2,
\]
where \(C = C(\varepsilon, n)\). Since the function \(g\) satisfies the differential inequality
\[
\Delta g \geq -\alpha g
\]
on \(B(p, 2R_0)\), where \(\alpha = \inf_{B(p, 2R_0)} \text{Ric}_M\), the mean value inequality of Li-Tam [5] gives
\[
g^2(x) \leq C_1 \int_{B(x, R_0)} g^2 \leq C_1 \int_{B(p, 2R_0)} g^2
\]
for any \(x \in B(p, R_0)\), where \(\nu = \inf_{x \in B(p, R_0)} V_x(R_0)\) and \(C_1 = C_1(n, \alpha, \nu)\). Combining with (11), we have
\[
\sup_{B(p, R_0)} g^2 \geq C_2 \int_{B(p, R_0)} g^2,
\]
where \(C_2 = C_2(\varepsilon, n, \alpha, \nu)\). On the other hand, the Schwarz’s inequality implies that
\[
\int_{B(R_0)} g^2 \leq \left( \int_{B(R_0)} |\nabla f|^2 \right) \rightarrow \frac{\varepsilon}{V_p(R_0)} \varepsilon.
\]
Therefore we have

\[
\sup_{B(R_0)} |\nabla f|^2 \leq C_3 \int_{B(R_0)} |\nabla f|^2,
\]

where \( C_3 = C_3(\varepsilon, n, \alpha, \nu, R_0) \) is a constant independent of \( f \in K^0(M) \). By unique continuation,

\[
\int_{B(R_0)} |\nabla f|^2 \neq 0,
\]

provided that \( f \) is not a constant function. Therefore,

\[
\int_{B(R_0)} \langle \nabla f, \nabla g \rangle
\]

defines a non-degenerate bilinear form on the space of 1-forms \( K = \{ df : f \in K^0(M) \} \).

Lemma 7 asserts that there exists \( f_0 \in K^0(M) \) such that

\[
\dim K \int_{B(R_0)} |df_0|^2 \leq nV_p(R_0) \sup_{B(R_0)} |df_0|^2.
\]

Combining the above with (12) implies

\[
\dim K^0(M) = \dim K + 1 \leq C_4
\]

for some fixed constant \( C_4 = C_4(C_3, V_p(R_0)) \), which completes the proof. The second part of the theorem follows from [8], an end is nonparabolic if and only if it has infinite volume, provided that \( \lim_{x \to \infty} \rho > 0 \).

\[\square\]

**Remark 9.** We would like to point out that the growth condition is not too restrictive. It is satisfied if the weight function \( \rho \) does not grow too fast, for instance, if \( \rho \) has only polynomial growth (\( n \geq 4 \)).

**Vanishing theorems on \( L^2 \) harmonic forms**

In this section we study \( H^1(L^2(M)) \), the space of \( L^2 \) integrable harmonic 1-forms. If \( f \) is a harmonic function with finite Dirichlet integral, then the exterior derivative of \( f \), \( df \) is a \( L^2 \) integrable harmonic 1-form. By the theory of Li-Tam [6] (see also [7]), we have

\[
\dim H^1(L^2(M)) + 1 \geq \dim K^0(M) \geq \text{number of non-parabolic ends of } M.
\]

If we further assume that \( \lambda_1(M) > 0 \), then

\[
\dim H^1(L^2(M)) + 1 \geq \text{number of infinite volume ends of } M.
\]

Therefore, an estimate on \( \dim H^1(L^2(M)) \) is, in general, a stronger estimate than an estimate on the number of nonparabolic ends (infinite volume ends if \( \lambda_1(M) > 0 \)). It is known that if \( \omega \) is a \( L^2 \) harmonic 1-form, then it is both closed and co-closed. In particular, \( h = |\omega| \) satisfies a Bochner type formula

\[
\Delta h \geq \frac{\text{Ric}_M(\omega, \omega)}{h} + \frac{|\nabla h|^2}{(n-1)h}.
\]
We start by proving an estimate for functions that satisfy the above Bochner type formula. We believe the estimate is of independent interest and will be useful in many other situations.

**Lemma 10.** Let \( b > -1 \). Assume \( h \) satisfies differential inequality

\[
\triangle h \geq -a h + b |\nabla h|^2,
\]

for some constant \( a \). For any \( \varepsilon > 0 \), we have the following estimate

\[
(b(1 - \varepsilon) + 1) \int_M |\nabla (\phi h)|^2 \leq \left( b \left( \frac{1}{\varepsilon} - 1 \right) + 1 \right) \int_M h^2 |\nabla \phi|^2 + a \int_M \phi^2 h^2,
\]

for any compactly supported smooth function \( \phi \in C^\infty_c(M) \). In addition, if

\[
\int_{B_\rho(R)} h^2 = o(R^2),
\]

then

\[
\int_M |\nabla h|^2 \leq \frac{a}{b + 1} \int_M h^2.
\]

In particular, \( h \) has finite Dirichlet integral if \( h \in L^2(M) \).

**Proof.** Let \( \phi \in C^\infty_c(M) \) be a smooth function with compact support. Integration by parts implies

\[
(13) \quad \int_M \phi^2 h \triangle h = -\int_M (\nabla (\phi^2 h), \nabla h)
\]

\[
= \int_M (h \nabla \phi + \nabla (\phi h), h \nabla \phi - \nabla (\phi h))
\]

\[
= \int_M h^2 |\nabla \phi|^2 - \int_M |\nabla (\phi h)|^2.
\]

By the differential inequality satisfied by \( h \), we have

\[
\int_M \phi^2 h \triangle h \geq -a \int_M \phi^2 h^2 + b \int_M \phi^2 |\nabla h|^2.
\]

Combining the above inequality with (13) gives

\[
(14) \quad \int_M h^2 |\nabla \phi|^2 + a \int_M \phi^2 h^2 \geq b \int_M \phi^2 |\nabla h|^2 + \int_M |\nabla (\phi h)|^2.
\]

On the other hand, Schwarz inequality implies

\[
\int_M \phi^2 |\nabla h|^2 = \int_M |\nabla (\phi h) - h \nabla \phi, \nabla (\phi h) - h \nabla \phi)
\]

\[
= \int_M (|\nabla (\phi h)|^2 - 2 \langle \nabla (\phi h), h \nabla \phi \rangle + h^2 |\nabla \phi|^2)
\]

\[
\geq (1 - \varepsilon) \int_M |\nabla (\phi h)|^2 + \left(1 - \frac{1}{\varepsilon}\right) \int_M h^2 |\nabla \phi|^2,
\]

for any \( \varepsilon > 0 \). Combining the above with (14) gives the first result of the lemma. For the second part, we choose

\[
\phi = \begin{cases} 
1 & \text{on } B(R) \\
0 & \text{on } M \setminus B(2R)
\end{cases}
\]
such that $|\nabla \phi| \leq C/R$ on $B(2R) \setminus B(R)$. The estimate of the lemma implies
\[
(b(1 - \varepsilon) + 1) \int_{B(R)} |h|^2 \leq \left( b\left(\frac{1}{\varepsilon} - 1\right) + 1 \right) R^{-2} \int_{B(2R) \setminus B(R)} h^2 + \int_M h^2.
\]
Using $\int_{B_p(R)} h^2 = o(R^2)$, and let $R \to +\infty, \varepsilon \to 0$, the second part of the lemma is achieved.

\[ \square \]

Remark 11. By the proof of the above lemma, the above conclusions are still valid if we assume $a = a(x)$ is a function of $x$, in the following forms
\[
(b(1 - \varepsilon) + 1) \int_M |\nabla (\phi h)|^2 \leq \left( b\left(\frac{1}{\varepsilon} - 1\right) + 1 \right) \int_M h^2 |\nabla \phi|^2 + \int_M a(x) \phi^2 h^2;
\]
and
\[
\int_M |\nabla h|^2 \leq \frac{1}{b + 1} \int_M a(x) h^2.
\]

The following theorem is an immediate application of the above lemma.

Corollary 12. With all the assumptions in lemma 10 if $\lambda_1(M) > 0$ and the Ricci curvature satisfies
\[
\text{Ric}_M \geq -(b + 1)\lambda_1(M) + \delta,
\]
for some $\delta > 0$. Then $H^1(L^2(M)) = 0$. In particular, we recover a theorem of Li-Wang in [7]: Let $M^n$ be a complete noncompact $n$-dimensional Riemannian manifold with $\lambda_1(M) > 0$, and the Ricci curvature satisfies
\[
\text{Ric}_M \geq -\frac{n}{n - 1} \lambda_1(M) + \delta,
\]
for some $\delta > 0$. Then $H^1(L^2(M)) = 0$.

Proof. Let $\omega \in H^1(L^2(M))$ and $h = |\omega| \in L^2(M)$. Bochner formula implies $h$ satisfies the differential inequality
\[
\triangle h \geq -ah + b \frac{|\nabla h|^2}{h},
\]
with $a = (b + 1)\lambda_1(M) - \delta$. Combining lemma 10 with the variational principle of $\lambda_1(M)$, we have
\[
(b(1 - \varepsilon) + 1)\lambda_1(M) \int_M \phi^2 h^2 \leq (b(1 - \varepsilon) + 1) \int_M |(\nabla \phi h)|^2 \leq a \int_M \phi^2 h^2 + \left( b\left(\frac{1}{\varepsilon} - 1\right) + 1 \right) \int_M h^2 |\nabla \phi|^2,
\]
for any $\varepsilon > 0$ and any compactly supported smooth function $\phi \in C^\infty_c(M)$. The above inequality implies
\[
\delta \int_M \phi^2 h^2 \leq b\varepsilon \lambda_1(M) \int_M \phi^2 h^2 + \left( b\left(\frac{1}{\varepsilon} - 1\right) + 1 \right) \int_M h^2 |\nabla \phi|^2.
\]
Let
\[
\phi = \begin{cases} 
1 & \text{on } B(R) \\
0 & \text{on } M \setminus B(2R)
\end{cases}
\]
such that $|\nabla \phi|^2 \leq C/R^2$ on $B(2R) \setminus B(R)$. The above inequality becomes
\[
\delta \int_{B(R)} h^2 \leq b \varepsilon \lambda_1(M) \int_{B(2R)} h^2 + R^{-2} \left( b \left( \frac{1}{\varepsilon} - 1 \right) + 1 \right) \int_{B(2R) \setminus B(R)} h^2.
\]
Combining the above inequality with the assumption $h \in L^2(M)$ and letting $R \to +\infty$ and then taking $\varepsilon \to 0$, we conclude that $\int_M h^2 \leq 0$ and thus $h \equiv 0$. For the second part, we just need to notice that in general, for a Riemannian manifold $M^n$ of dimension $n$, Bochner formula for $h = |\nabla \omega|$ is valid with $b = 1/n$. □

**Theorem 13.** Let $M^n$ be a complete noncompact manifold of dimension $n$ satisfying the weighted Poincaré inequality with a non-negative weight function $\rho(x)$. Assume the Ricci curvature satisfies
\[
\text{Ric}_M(x) \geq -\frac{n}{n-1} \rho(x) + \delta,
\]
for some $\delta > 0$. If $\rho(x) = O(r_p^{2-\alpha}(x))$, where $r_p(x)$ is the distance function from $x$ to some fixed point $p$, for some $0 < \alpha < 2$. Then $H^1(L^2(M)) = 0$.

**Proof.** Let $\omega \in H^1(L^2(M))$ and $h = |\omega| \in L^2(M)$. Applying Lemma 10 with $b = \frac{1}{n-1}$, $a = (b+1)\rho - \delta$ and using the weighted Poincaré inequality we have
\[
(b(1-\varepsilon) + 1) \int_M \rho \phi^2 h^2 \leq (b(1-\varepsilon) + 1) \int_M |(\nabla \phi h)|^2
\leq \int_M a(x) \phi^2 h^2 + \left( b \left( \frac{1}{\varepsilon} - 1 \right) + 1 \right) \int_M h^2 |\nabla \phi|^2.
\]
Thus
\[
\delta \int_M \phi^2 h^2 \leq ((1-b) + b\varepsilon^{-1}) \int_M h^2 |\nabla \phi|^2 + b\varepsilon \int_M \rho \phi^2 h^2,
\]
for any $\varepsilon > 0$ and any compactly supported smooth function $\phi \in C_c^\infty(M)$. Let $\varepsilon = R^{2-2/\alpha}$ and
\[
\phi = \begin{cases} 1 & \text{on } B(R) \\ 0 & \text{on } M \setminus B(2R) \end{cases}
\]
such that $|\nabla \phi|^2 \leq C/R^2$ on $B(2R) \setminus B(R)$. The above inequality becomes
\[
\delta \int_{B(R)} h^2 \leq bR^{-\alpha/2} \int_{B(2R)} h^2 + (1-b)R^{-2} + bR^{-\alpha/2} \int_{B(2R)} h^2.
\]
Combining the above inequality with the assumption $h \in L^2(M)$ and letting $R \to +\infty$, we conclude that $\int_M h^2 \leq 0$ and thus $h \equiv 0$. □

**Theorem 14.** Let $M^n$ be a complete noncompact manifold of dimension $n$ satisfying the weighted Poincaré inequality with weight function $\rho > 0$. Assume the Ricci curvature satisfies $\text{Ric}_M \geq -\frac{n}{n-1} - \delta \rho$, for some $\delta > 0$, and $\rho = O(r_p^{2-\alpha}(x))$ for some $0 < \alpha < 2$. Then $H^1(L^2(M)) = 0$. 


Proof. Let $\omega \in H^1(L^2(M))$ and $h = |\omega| \in L^2(M)$. Applying Lemma 10 with $b = \frac{1}{n-1}$, $a = (b+1-\delta)\rho$, and weighted Poincaré inequality we have

$$\begin{align*}
(b(1-\varepsilon) + 1) \int_M \rho \phi^2 h^2 &\leq (b(1-\varepsilon) + 1) \int_M |(\nabla \phi h)|^2 \\
&\leq a \int_M \phi^2 h^2 + \left(b(\frac{1}{\varepsilon} - 1) + 1\right) \int_M h^2 |\nabla \phi|^2.
\end{align*}$$

Thus

$$\delta \int_M \rho \phi^2 h^2 \leq be \int_M \rho \phi^2 h^2 + (be^{-1} - b + 1) \int_M h^2 |\nabla \phi|^2.$$

If we choose

$$\phi = \begin{cases} 1 & \text{on } B(R) \\ 0 & \text{on } M \setminus B(2R) \end{cases}$$

Using $\rho(x) = O(r^{2-\alpha}(x,p))$ and let $\varepsilon = R^{\alpha/2-2}$, the above inequality implies

$$\delta \int_{B(R)} \rho h^2 \leq C R^{-\alpha/2} \int_{B(2R)} h^2 + (bR^{-\alpha/2} + (1-b)R^{-2}) \int_{B(2R)} h^2,$$

for some constant $C$. Using $h \in L^2(M)$ and letting $R \to +\infty$, we conclude that $\int_M \rho h^2 = 0$. By Lemma 10 we have

$$\int_M |\nabla h|^2 \leq \left(1 - \frac{\delta}{b+1}\right) \int_M \rho h^2 = 0.$$

Therefore $|\nabla h| \equiv 0$. Hence $h \equiv c \in L^2(M)$ for some constant $c$. Since $M$ is non-parabolic, it must have infinite volume and thus $h \equiv 0$. □

References

[1] M. Cai and G. J. Galloway, Boundaries of zero scalar curvature in the ADS/CFT correspondence, Adv. Theor. Math. Phys. 3 (1999) 1769–1783.

[2] S. Y. Cheng and S. T. Yau, Differential equations on Riemannian manifolds and their geometric applications, Comm. Pure App. Math. 28 (1975) 333–354.

[3] J. Lee, The spectrum of an asymptotic hyperbolic Einstein Manifold, Comm. Anal. Geom. 3 (1995) 253–271.

[4] P. Li, On the Sobolev constant and the p-spectrum of a compact Riemannian manifold, Ann. Scient. Éc. Norm. Sup. 4, T 13 (1980) 451–469.

[5] P. Li and L. F. Tam, The heat equation and harmonic maps of complete manifolds, Invent. Math. 105 (1991) 1–46.

[6] ———, Harmonic functions and the structure of complete manifolds, J. Diff. Geom. 35 (1992) 359–383.

[7] P. Li and J. Wang, Complete manifolds with positive spectrum, J. Diff. Geom. 58 (2001) 501–534.

[8] ———, Weighted Poincaré inequality and rigidity of complete manifolds, Ann. Scient. Éc. Norm. Sup., 4e série, t. 39 (2006) 921–982.

[9] R. Mazzeo, The Hodge cohomology of a conformally compact metric, J. Diff. Geom. 28 (1988) 309–339.

[10] X. Wang, On conformally compact Einstein manifolds, Math. Res. Lett. 8 (2001) 671–688.

[11] E. Witten and S. T. Yau, Connectedness of the boundary in the AdS/CFT correspondence, Adv. Theor. Math. Phys. 3 (1999) 1635–1655.

[12] S. Y. Yau, Harmonic functions on complete Riemannian manifolds, Comm. Pure Appl. Math. 28 (1975) 201–228.
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