Coupled Multi-Physics Modelling in Continuum Mechanics

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Abstract. Contemporary models of solids of thermoelasticity requires to include multiphysics coupling and employ non classical e.g. elastic behavior. The permanent generalization of elastic model is the Cosserat micropolar model. Now this model is to be applied to growing solids, biomaterials, granular media, concrete. The basic concepts of the continuum mechanics is considered in connection with the difference in the dimensions of the continuum and external space. New field variables are introduced representing complex continuum properties. A generalization of the model of the micropolar continuum has been proposed. The action and the action density for the complex continuum model are discussed. A new field-theoretic model of a nonlinearly elastic continuum is developed assuming existence of an isometric immersing into an external plane space.

1. Introduction

Modern mathematical continuum models are based on the notion of differentiable manifold (or differentiable space) [1, 2]. Such models are best applicable to the mechanical behavior of solids, for example solids with microstructure. The continuum is assumed as of a differentiable manifold of a given dimension \( M ; M = 3 \) in applied problem.

It is often necessary to introduce additional structures on a differentiable manifold. For example, a Riemannian structure on a manifold allows us to define a differentiable spaces class [3], [4] playing an important role. The theory of finite deformations known from many discussions can be readily developed for a continuum with a Riemannian metric structure.

Historically, the theory of manifolds takes its origin from the geometry of curved surfaces in three-dimensional Euclidean space. Development of the theory of differentiable manifolds came to rejection of an outer (external) space notion. Internal geometry of manifolds as a special part of the theory has been developed. However, the external space with immersed solids is a permanent attribute of the continuum mechanics.

For a long time, such views do not cause any objections. Along with affine-metric structures of various complexity levels have been used to modeling continua with defects and damages. The best example of a complex continuum is a growing solid manufacturing by a specific technology or synthesized by a complex physicochemical process. The question about the possibility of immersion of a continuum in a 3D plane space has been never discussed previously. It is well known that continua with a complex affine-metric structure can not be immersed in a three-dimensional external plane space. Therefore it is impossible give a reason to deformation of such a continuum from the view point of three-dimensional representations and forms. Fundamentally new assumption of this study is the rejection of the external space concept as a 3D plane space with an immersed solid.
The present study deals with the foundations of the new mechanics of a complex continuum of an arbitrary dimension $M$ admitting an existence of a plane external space of dimension $N \geq M$. We are going to incorporate the formalism of the physical field theory [5, 6] into theory of differentiable manifold.

2. Mechanics of a complex continuum

A vicinity of each differentiable manifold point can be mapped to a domain of an affine space of dimension $M$. In this sense, the manifold admits local comparisons with domains of an affine space. The number of maps is not more than countable. Images of the manifold on intersecting maps are to be consistent. The concept of manifold (and consequently continuum) does not require any determination of the outer space into the continuum can be immersed. Individual points of the continuum are represented by a special variables denoted by $\xi$ identifying by the map coordinates $\xi^\alpha$. Thus, there is a correspondence (coordinate mapping)

$$
\xi \rightarrow \xi^\alpha,
$$

representing by $\xi^\alpha$ a continuum point $\xi$ on a map.

The idea of an outer space is explicitly or implicitly accepted in continuum mechanics. The solids are assumed to be immersed into an external space, providing an opportunity for the observer to discriminate the positions of material point in space, to measure distances and angles, and to register the changes of lengths and distortions of the angles of the linear material elements. From this point we able to introduce the concept of deformation. The external space is tacitly assumed to be 3D with the natural Euclidean metric. Another important assumption is that a continuum must always allow an embedding in the same external space during deforming. Since the continuum is modeled by some differentiable manifold, the question immediately arises of the possibility of its immersion into an external space. It is well known that such immersion in a 3D plane space is not always possible. For instance, a Riemannian manifold of dimension $M$ in the general case is immersed in a plane space of dimension $N = \frac{1}{2}M(M+1)$. In any case, the following inequality is fulfilled

$$
N \geq M.
$$

Let us elucidate the notion of a plane space. The plane space is usually defined as a special class of Riemannian spaces (see in details [3]). A Riemannian space is to be plane one if there is a coordinate system with constant components of the metric tensor. In a plane space there is always a coordinate system $y^j$ ($j = 1, \ldots, N$) with metrics determined by the differential form

$$
ds^2 = \sum_j c_j dy^j dy^j \quad (j = 1, \ldots, N),
$$

where constants $c_j = \pm 1$ depend on space type. The coordinate system $y^j$ is called Cartesian coordinate system. It is clear that for an external space the Cartesian coordinate system is preferable.

We can define embedding mapping (or emersion) for a continuum that allows immersion in an external space establishing a correspondence between the points of the continuum $\xi$ and positions (places) in external space. If the continuum admits an immersion into some external plane space of dimension $N$ then every point $\xi$ of the continuum takes some position $X$. The position $X$ in external space in the continuum mechanics is called referential one. By virtue of the embedding map, the referential position $X$ must be bijectively connected with the variable $\xi$:

$$
\xi \rightarrow X.
$$

It is possible taking into account the equations (1) and (3) to parameterize the referential position of
the continuum in the external space by the coordinates $\xi^\alpha$:

$$X \rightarrow \xi^\alpha.$$  \hfill (4)

The points of the continuum change their positions in the external space during deformation process. The spatial position of individual material point $\xi$ is denoted by $x$. In the general form, the deformation of the continuum can be expressed by mapping the individual points of the continuum into actual positions in the external space

$$\xi \rightarrow x.$$  \hfill (5)

Transformation (also called deformation) of the position variables in the external space according to equations (3) and (5) reads

$$X \rightarrow x.$$  \hfill (6)

Variables $X$ and $x$ can be represented by curvilinear coordinates $X^\alpha$ ($\alpha = 1, \ldots, M$), $x^j$ ($j = 1, \ldots, N$). The coordinates $\xi^\alpha$ can be equally used instead of the coordinates $X^\alpha$. The (Lagrangian) metrics $g_{\alpha\beta}$ is directly inherited from the external space. The spatial (Eulerian) metrics $g_{ij}$ is the one of the external space. The convective metrics is characterized by the metric tensor $g_{\alpha\beta}$ and is defined by deformation (6) contrary to the metrics $g_{\alpha\beta}$ and $g_{ij}$.

Modeling of the mechanical behavior of complex continua may require the introduction of additional degrees of freedom. Additional degrees of freedom (for example, rotational ones) are introduced into the theory in the form of tensor variables with contravariant indices, just as it does for translational freedom degrees related to the external space. These tensor variables are called $d$–variables and they are denoted by $d^k$. $d$–variables in the case of a micropolar continuum are a set of unit vectors (directors). The generalization of the micropolar continuum model immersed in a space of greater dimension is achieved due to the increase in the number of linearly independent directors associated with the infinitesimal element.

Furthermore, additional field variables can be used in the mechanics of a complex continuum. They are denoted also in the vector form by contravariant components $k^\alpha$. As an example, we note that in the hyperbolic thermomechanical continuum, the temperature displacement acts as an additional field variable.\(^1\)

Deformation gradient

$$\partial_\alpha x^j \quad (j = 1, \ldots, N; \; \alpha = 1, \ldots, M),$$  \hfill (7)

where $\partial_\alpha$ denotes the partial differentiation with respect to the referential variable characterizes the elementary affine deformation. The deformation gradient $\partial_\alpha x^j$ transforms the referential linear element $dX^\alpha$ to its actual position in the external space $dx^j$ according to

$$dx^j = (dX^\alpha) \partial_\alpha x^j.$$  \hfill (8)

The convective metrics $g_{\alpha\beta}$ is obtained by a deformation gradient and the Eulerian metrics $g_{ij}$

$$g_{\alpha\beta} = g_{ij} \left( \partial_\alpha x^i \right) \left( \partial_\beta x^j \right).$$  \hfill (9)

In the case $M = N$, a complete study of the finite deformation geometry can be given regardless of

\(^{1}\) The hyperbolic coupled thermomechanical continuum model is developed in monograph [6]. There thermal field variable is denoted by $\phi$.  

the mathematical dimension (see, for example, [7]). The study is based on the possibility of a polar decomposition of an arbitrary tensor of the second rank [8]. The polar decomposition of deformation gradient \( \partial_\alpha x^j \) (\( j = 1, \ldots, N \); \( \alpha = 1, \ldots, M \)) allows to present it in the form of a multiplicative composition of pure deformation (stretching relative to mutually orthogonal principal deformation axes in the referential position) and rotation in the external space of the referential polyhedron of the principal deformation axes to its actual position. Nothing similar can be said in the case when \( N > M \).

Further development of the theory is carried out within the frameworks of physical field theory. Therefore, we need to introduce the action integral and its density called also as Lagrangian. The functional arguments of the Lagrangian are field variables and their gradients. In the case of field theory on a \( P \)-dimension manifold of the action is a \( P \)-dimension integral variational functional:

\[
L = \int \left( \partial_\alpha \phi^k, \partial_\alpha \phi^k, \partial_\beta \phi^k, \ldots, X^\beta \right) d^P X,
\]

where the integration extends to an unspecified domain of the manifold \( L \) denotes natural Lagrangian density (action density); \( \phi^k \) are physical field variables; \( X^\beta (\beta = 1, \ldots, P) \) are space–time coordinates; \( d^P X \) are natural elemental volume (product of the coordinate differentials). In the case of a complex continuum immersed in an external space the coordinates of the external space \( x^j \) are treated as physical fields. The same is true for the system of \( d \)-variables and additional field variables. So the action density \( L \) can be given by the following functional form

\[
L = L \left( x^j, d^j, \partial_\alpha \phi^k, \partial_\alpha \phi^k, \partial_\alpha \phi^k, \partial_\alpha \phi^k, X^\beta \right). \tag{11}
\]

The functional form (11) plays a significant role in the mathematical model of a complex continuum. In addition to the differential field equations and the constitutive equations this form allow us to construct the objective deformation tensors. The first question in a study of complex continua is the strain tensors being an objective one and their complete systems sufficient for the mathematical representation of the deformed state. Objectivity is understood as the invariance of the strain tensors with respect to rotations of the preferred coordinate systems of the external space. The same can be said on the translations of coordinate systems. The action density \( L \) is an objective function. It must invariant under translations of the coordinates \( x^j \). As a result, we can conclude that the functional form (11) is clearly not dependent on external spatial coordinates (in this case, on the coordinates \( y^j \)). Therefore, it is reduced to the following form (we confine ourselves to static states):

\[
L = L \left( d^j, \partial_\alpha y^j, \partial_\alpha d^j \right). \tag{12}
\]

For the same reason, the action density \( L \) should not depend on rotations of the preferred coordinate system polyhedron in the external space. The invariance of the action density with respect to the rotations of the coordinate frame is a manifestation of the isotropy of the external space, i.e. absence of preferred directions in this space. For a functional form of the equation (12) this means that it actually functionally depends only on such algebraic combinations of contravariant vectors of the external space

\[
d^j, \partial_\alpha y^j, \partial_\alpha d^j, \tag{13}
\]

which invariant with respect to orthogonal transformations of the external coordinate system \( y^j \). The problem of constructing rational combinations of the contravariant vectors 13 that would not be sensitive to rotations of the preferred external coordinate system is solved by using the algebraic theory of rational invariants of tensors and tensor systems [9]. In particular, all irreducible rational invariant forms for the system (13) can be easily enumerated: they are inner and skew products of the
space vectors \((13)\). The mentioned predicts form a system of independent objective strain tensors. Consider a nonlinearly elastic continuum of dimension \(M\) whose metrics admits its isometric embedding into an external plane space of the same dimension. The functional form \((12)\) in this case reverts to

\[
L = L \left( \partial_{\alpha} y^\gamma \right),
\]

where the functional arguments are the \(M\) contravariant vectors of the external space \(y^\gamma\).

\[
\frac{\partial}{\partial y^\gamma}, \quad \det \left( \partial_{\alpha} y^\gamma \right).
\]

The form \((14)\) of irreducible invariant functional arguments of the Lagrangian \(L\) is furnished by

\[
\partial_{\alpha} y^\gamma \partial_{\beta} y^\gamma, \quad \det \left( \partial_{\alpha} y^\gamma \right).
\]

The first combination in \((16)\) is the convective metrics of the deformed continuum \(g_{\alpha\beta}\); the second one is the Jacobian of the deformation related to the convective metrics by the rational equation

\[
(\det \left( \partial_{\alpha} y^\gamma \right))^2 = \det \left( g_{\alpha\beta} \right).
\]

The latter equation is considered from the algebraic point of view as a syzygy for the system of invariants \((16)\), and from mechanical viewpoint it is the continuity equation in Lagrange variables. If we remain within the frameworks of such a scheme, which involves operating only with rational invariants, then the syzygy \((17)\) should be taken into account as a kinematic constraint, and the Lagrangian \((14)\) in accordance with the rule of Lagrange multipliers should be replaced by a new one

\[
L^\lambda \left( \partial_{\alpha} y^\gamma \partial_{\beta} y^\gamma, \det \left( \partial_{\alpha} y^\gamma \right) \right) = L \left( \partial_{\alpha} y^\gamma \partial_{\beta} y^\gamma, \det \left( \partial_{\alpha} y^\gamma \right) \right) - \lambda \left[ (\det \left( \partial_{\alpha} y^\gamma \right))^2 - \det \left( g_{\alpha\beta} \right) \right].
\]

where the functional arguments \((16)\) should be considered as kinematically independent, \(\lambda\) denotes indefinite Lagrange multiplier.

3. Conclusions
- The basic concepts of the continuum mechanics have been elucidated in connection with the difference in the dimensions of the continuum and external space.
- New field variables have been introduced representing complex continuum properties. A generalization of the model of the micropolar continuum has been proposed.
- The action and the action density for the complex continuum model have been discussed.
- A new field-theoretic model of a nonlinearly elastic continuum has been developed assuming existence of an isometric immersing into an external plane space.

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