Restoring Gauge Invariance in Non-Abelian Second-class Theories

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Abstract
In this paper, we propose a generalization of the improved gauge unfixing formalism in order to generate gauge symmetries in the non-Abelian valued systems. This generalization displays a proper and formal reformulation of second-class systems within the phase space itself. Then, we present our formalism in a manifestly gauge invariant resolution of the SU(N) massive Yang-Mills and SU(2) Skyrme models, where gauge invariant variables are derived.

Keywords Improved gauge unfixing · Yang-Mills theory · Skyrme model

1 Introduction
Constrained dynamical systems have, among other important features, the existence of first and second-class constraints [1–6]. A constraint is said to be first-class when its Poisson brackets with all constraints, including itself, vanish weakly. A constraint that is not first-class is named a second-class one1. First-class constraints imply the existence of a gauge

1 Strictly speaking, constraints are second-class when the matrix of their Poisson brackets has maximum rank. However, for the simple case of two constraints, the definition above coincide with the general one [5].
approach to construct physical theories. Besides, it provides a natural geometric-topological set-up to describe a wide range of quantization aspects on pure algebraic grounds by using the homotopy or, (co-)homology, operators \([5, 9]\). Consequently, any fundamental theory can be better structured and understood if it is formulated as a genuine gauge theory, since it has several advantages.

A prototypical example of second-class constrained dynamical system is the massive Yang-Mills theory. At the original construction, the formation of a dynamical gluon mass \(m^2 A^a \partial_\mu A_\mu^a\) is forbidden due to the fact that it destroys the gauge invariance. However, there are powerful reasons to consider a mass term, since it might play an important role in the nonperturbative regime. This possibility was pointed out by the considerable amount of results obtained through theoretical and phenomenological studies, as in Gribov-Zwanziger, Schwinger-Dyson or QCD sum rules models, as well as from lattice simulations, for instance, refs. \([10–18]\) and references therein. Then, this is a clear example where it is desirable to reformulate the mass term to restore the original gauge symmetry. Nevertheless, at low energy QCD, standard perturbative techniques cannot be used due to the a strong coupling constant. Effective degrees of freedom in QCD are given by hadrons and another exotic mixed states, so the study of hadronic properties requires effective models, although numerical calculations in lattice QCD have technical problems to implement hadron properties also due to the chiral symmetry breaking.

A particular form to describe baryons and its interactions with a medium is given by the Skyrme model \([19]\), a nonlinear mesonic model with non-trivial soliton field (who acts as baryon). The Skyrme term is a chiral perturbative correction to the non-linear \(\sigma\)-model, which gives rise to the second-class constraints. It has a great versatility to model diverse problems of nuclear matter, among which we can mention spin-isospin correlations (axial coupling), charge radii, magnetic moments, many-bodies nuclear interaction, hadronic crystal lattice and medium modified meson properties \([20–23]\). For a complete review see \([24]\).

Gauge freedom can result in simplified algebras by choosing the adequate fixing term in each case. Several algebraic consequences of the first-class construction were explored in \([25, 26]\). The general idea is to rewrite a second-class system as a first-class one, and thus the complete dynamical system exhibits a gauge symmetry. Several methods in the Hamiltonian formalism are available to implement this process, among which we can mention the Batalin-Fradkin-Tyutin \([25–28]\) and the gauge unfixing (GU) formalism \([29–31]\). The last one accomplishes the conversion to equivalent gauge invariant theories without the extension of the original phase space variables. The original motivation of the GU formalism and its improved versions \([32, 33]\) is to use one of the second-class constraints, if the system has two second-class constraints, of course, as a symmetry generator. The other one will be discarded. Then, the invariant quantities obtained will be written as a powers series of the discarded constraint. The coefficients of this series will be given by successive transformations of the original variable under the new symmetry generator \([31, 34]\). In quantum field theory, this formalism has been successfully used in Abelian models such as Chern-Simons, Proca and Carroll-Field-Jackiw \([31, 34, 35]\). Then, the aim of the present work is to face the issue of the gauge invariance restitution of non-Abelian second-class theories. One of the attempts to attack a non-abelian model using the GU formalism was carried out in \([36]\) concerning a pure non-Abelian Chern-Simons theory (without Maxwell term).

In this work, we will provide a general analysis of the GU variables as a power series of the discarded constraint. In the mentioned papers related to the use of GU formalism, the correction terms in \(n \geq 2\) are null in the power series, including the non-Abelian Chern-Simons theory. This will not be the case for Yang-Mills theories, as we will show later. In fact, the issue of the gauge invariance for non-Abelian vector fields has been addressed in \([37]\), where the dressing functions written in terms of infinite sum of non-local terms are used to define invariant quantities. Here we can mention that, for the non-Abelian Stückelberg mechanism with a mass term, the invariance requirement implies a non-polynomial action \([38–42]\). Therefore, it is natural to be expected that, unlike the Abelian cases, Yang-Mills models, whose gauge transformations depend on the covariant derivative, will generate invariant variables without truncations in the power series.

The paper is organized as follows: In Sect. 2 we described an outline of the improved GU formalism and its non-Abelian extension. In Sect. 3 we applied our formalism to the \(SU(N)\) massive Yang-Mills model. Section 4 is devoted to the study of the \(SU(2)\) Skyrme model. Section 5 contains our conclusions.

### 2 The Improved Gauge Unfixing Formalism for Non-Abelian Case

Let us consider a constrained dynamical system possessing two second-class constraints, \(T^a_1\) and \(T^b_2\), with canonical variables valued in a Lie algebra with \(a = 1, 2, \ldots, N\), where \(N\) is the dimension of the representation of these fields with respect to the non-Abelian group. The Poisson bracket of these constraints is non-trivial and it is given by

\[
\{ T^a_1, T^b_2 \} = \delta^{ab} K,
\]

where \(K\) is a non-vanishing constant on the surface defined by the constraints \(T^a_1\) and \(T^b_2\). Following the gauge unfixing formalism \([30, 31]\), one of the constraints can be used as generator of infinitesimal transformations and the other one

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is disregarded, i.e., no longer considered a constraint after the reformulation. We can also set the generator of infinitesimal transformations as

$$\chi^a = K^{-1} T^a_1,$$

(2)

such that

$$\left\{ \chi^a, T^b_T \right\} = \left\{ K^{-1} T^a_1, T^b_T \right\} = K^{-1} \left\{ T^a_1, T^b_T \right\} + \left\{ K^{-1}, T^b_T \right\} T^a_1 = \delta^{ab} + \left\{ K^{-1}, T^b_T \right\} K \chi^a \approx \delta^{ab},$$

(3)

where the symbol “≈” means weak equality, namely, they hold only on the hyper-surface defined by the intersection of the constraints on the full phase space. We can see that $\chi^a$ and $T^a_T$ form an approximate canonically conjugate pair on $\chi^a \approx 0$. The constraint $T^a_T$ will be disregarded, that is, it will no longer be considered a constraint. Consider now a second-class function $F^a(q^a, p^a)$, where “$a, b, c \ldots$” are the color indices. In the GU formalism \cite{32, 33}, the invariant first-class function $\bar{F}^a$ is constructed by as a power series of the discarded constraint $T^a_T$

$$\bar{F}^a = F^a + c_{1} \partial^{ab} T^b_T + c_{2} \partial^{abc} T^b_T T^c_T \ldots$$

$$= F^a + \sum_{k=1}^{\infty} \frac{1}{k!} \varepsilon^k \prod_{i=1}^{k} T^{m_i}_T,$$

(4)

which has the following boundary condition

$$\bar{F}^a|_{T^a_T = 0} = F^a.$$

(5)

The coefficients $c_{a_{w_1 \ldots w_k}}$ in (4) are determined order by order in powers of $T^a_T$ if we employ the invariance condition:

$$\delta^m \bar{F}^a = \varepsilon \left\{ \bar{F}^a, \chi^m \right\} = 0$$

$$= \delta^m F^a + \sum_{k=1}^{\infty} \left( \delta^m c_k \prod_{i=1}^{k} T^{m_i}_T + c_k \partial^{a_{w_1 \ldots w_k}} \prod_{i=1}^{k} T^{m_i}_T \right) = 0,$$

(7)

where

$$\delta^m F^a = \varepsilon \left\{ F^a, \chi^m \right\}$$

(8)

$$\delta^m c_k = \varepsilon \left\{ c_k, \chi^m \right\}$$

(9)

$$\delta^m T^a_T = \varepsilon \left\{ T^a_T, \chi^m \right\} = -\delta^{am} \varepsilon,$$

(10)

where $\delta^m$ is the variational operator valued in the Lie algebra for the transformation induced by the generator $\chi^a$ and $\varepsilon$ is an arbitrary infinitesimal parameter. We will assume that there is no an a priori symmetry in the color indices of the coefficients $c_k$. Then, from Eq. (7), we can derive an equation for the zero order term in $T^a_T$

$$\delta^m F^a + c_1 \delta^{ab} T^b_T = 0 \Rightarrow c_{1} = \frac{\delta^m F^a}{\varepsilon}. $$

(11)

For the linear term in $T^a_T$, we obtain

$$\delta^m c_1 \partial^{ab} T^b_T + c_2 \delta^{abc} T^b_T T^c_T = 0 \Rightarrow c_{2} = \frac{\delta^m \delta^b F^a}{2 \varepsilon^2}. $$

(12)

From Eqs. (11) and (12), we can observe that, for $k \geq 3$, the general relation is

$$c_{k} = \frac{1}{k!} \prod_{i=1}^{k} \delta^{a_{w_1 \ldots w_k}} F^a,$$

(13)

This last expression is the non-Abelian analogue of the expression found in \cite{33, 35, 36}. Therefore, the expression for the invariant first-class function $\bar{F}^a$, Eq. (4), becomes

$$\bar{F}^a = F^a + T^b_T \frac{\delta^b F^a}{\varepsilon} + T^b_T T^c_T \frac{\delta^b \delta^c F^a}{2 \varepsilon^2} \ldots$$

$$= \left( 1 + \sum_{k=1}^{\infty} \frac{1}{k!} \prod_{i=1}^{k} T^{m_i}_T \varepsilon^k \right) F^a$$

(14)

where we have adopted an ordering prescription “$: \cdot :$”, which means that $T^a_T$ always comes before the variation operator

$$\frac{1}{\varepsilon} \delta^m f = \varepsilon \left\{ f, \chi^m \right\},$$

(15)

for any functional $f$ on the phase space. The last line in Eq. (14) shows us that, as pointed out by [31], any gauge invariant quantity can be generated by applying the projection operator $\mathcal{P} \equiv \varepsilon \left[ \cdot , \cdot \right]$ on a function in the phase space. In the next section, we will consider some specific models to show the use of the improved GU formalism in the non-Abelian theories.

## 3 Massive SU(N) Yang-Mills model

The massive $SU(N)$ Yang-Mills model, in $d = 4$ Euclidean dimensions, is represented by the following action

$$S = \int d^4 x \left( \frac{1}{4} F^a_{\mu \nu} F_{\mu \nu}^a + \frac{m^2}{2} A^a_{\mu} A^a_{\mu} \right),$$

(16)

where $F^a_{\mu \nu} = \partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu} + f^{abc} A^b_{\mu} A^c_{\nu}$ is the field strength. We are working in Euclidean space to avoid issues of the validity of the Wick rotation at the non-perturbative regime. In other words, the action in Eq. (16) is the non-Abelian version of the Proca action. Because of the presence mass
term, this Lagrangian is no longer invariant under gauge transformations. The canonical momenta are given by

\[ \pi^a_\mu = \frac{\partial L}{\partial \left( \partial_\mu A^a \right)} = F^a_\mu, \]  

from which we define the following fundamental Poisson brackets

\[ \left\{ A^a_\mu(x), A^b_\nu(y) \right\} = \left\{ \pi^a_\mu(x), \pi^b_\nu(y) \right\} = 0, \]  

\[ \left\{ A^a_\mu(x), \pi^b_\nu(y) \right\} = \delta^a_b \delta_{\mu\nu} \delta^{(3)}(x-y). \]  

From Eq. (17) we obtain the primary constraint of the model

\[ T^a_1 \equiv \pi^a_0. \]  

The canonical momenta \( \pi^a_\mu \) are given by

\[ \pi^a_\mu = F^a_\mu. \]  

For the complete investigation of the canonical structure of the model, we will now show the canonical Hamiltonian of the model, \( H_c \), which is obtained from a Legendre transformation that leads to

\[ H_c = \int d^3x \left\{ \frac{1}{2} \pi^a_\mu \pi^a_\mu + \pi^a_i \left( \partial_\mu A^a_0 - gf^{abc} A^b_0 A^c \right) \right. \]
\[ \left. - \frac{1}{4} F^a_\mu F^a_\mu - \frac{1}{2} m^2 (A^a_0 A^a_0 + A^a_i A^a_i) \right\}. \]  

From the time stability condition of the constraint \( T^a_1 \) [1–5],

\[ \dot{T}^a_1 = \left\{ T^a_1(x), H_p(y) \right\} = 0, \]  

we have the secondary constraint

\[ T^a_2 \equiv D^a_\mu \pi^\mu_0 + m^2 A^a_0. \]  

where \( H_p \) in Eq. (23) is the primary hamiltonian, namely

\[ H_p = H_c + \int d^3x \lambda^a_1(x)T^a_1(x), \]  

and

\[ D^a_\mu = \delta^a_{\mu\nu} \partial_\nu - gf^{abc} A^c_\mu \]  

is the covariant derivative in the adjoint representation. The time evolution of the constraint \( T^a_2 \) determines the Lagrange multiplier \( \lambda^a_2 \).

\[ \lambda^a_1 = D^a_\mu A^\mu_1. \]  

This indicates that no more constraints are generated via this iterative procedure. \( T^a_1 \) and \( T^a_2 \) are the total constraints of the theory. Therefore, they form the following second-class algebra

\[ \{ T^a_1(x), T^b_1(y) \} = 0, \]  

\[ \{ T^a_1(x), T^b_2(y) \} = -m^2 \delta^{ab} \delta^{(3)}(x-y), \]  

\[ \{ T^a_2(x), T^b_2(y) \} = gf^{abc} T^1_1 \delta^{(3)}(x-y). \]  

The total Hamiltonian is then given by

\[ H_T = H_c + \int d^3x (\lambda^a_1 T^a_1 + \lambda^a_2 T^a_2). \]  

Demanding again conservation of constraint \( T^a_1 \) in time, but using the total hamiltonian in Eq. (31), then the Lagrange multiplier \( \lambda^a_2 \) is fixed as

\[ \lambda^a_2 = \frac{1}{m^2} T^a_2 \approx 0. \]  

We are now able to apply the improved GU formalism in the massive Yang-Mills model. We have two available choices, as the system presents two second-class constraints. By choosing \( T^a_1 \) as the generator of infinitesimal transformations, the field variations are similar to those of the abelian case [34]. In contrast, the infinitesimal transformations generated by the constraint \( T^a_2 \) lead to new features in the non-Abelian case. Therefore, we will focus on the second case. We first redefine \( T^a_2 \) as

\[ \chi^a \equiv \frac{1}{m^2} T^a_2, \]  

such that

\[ \{ \chi^a(x), T^b_1(y) \} = \delta^{ab} \delta^{(3)}(x-y), \]  

and \( T^a_1 \) will be discarded as a constraint. The infinitesimal transformations generated by \( \chi^a \) are

\[ \delta^a A^b_0(x) = \int d^3y \varepsilon(y) (A^b_0(x), \chi^a(y)) = 0, \]  

\[ \delta^a A^b_0(x) = \int d^3y \varepsilon(y) (A^b_0(x), \chi^a(y)) = -\frac{1}{m^2} D^a_\mu \varepsilon(x). \]
\[ \delta^a \pi^i_a(x) = \delta^a T^i_\alpha(x) = \int d^3 y \epsilon(y) \{ \pi^i_a(x), \chi^a(y) \} = -\epsilon(x) \delta^a z. \]  

(37)

\[ \delta^a \pi^b_i(x) = \int d^3 y \epsilon(y) \{ \pi^b_i(x), \chi^a(y) \} = -g \delta^a \epsilon \pi^b_i(x). \]  

(38)

Hence, the invariant field \( \hat{A}_i^a \), which is constructed by a power series of the disregarded constraint \( T^i_\alpha \), Eq. (20), is given by

\[ \hat{A}_i^a(x) = A_i^a(x) + \sum_{k=1}^{\infty} \int \left( \prod_{r=1}^{k} d^3 x_r \right) c^{aw_{1}...w_{k}}_{ki}(x_1, ..., x_k, x) \left( \prod_{r=1}^{k} T^w_{r}(x_r) \right). \]  

(39)

The next step is to determine the coefficients \( c_{ki} \). For this goal, we impose the variational condition in Eq. (7)

\[ \delta^m \hat{A}_i^a(x) = \delta^m A_i^a(x) \]

\[ + \sum_{k=1}^{\infty} \int \left( \prod_{r=1}^{k} d^3 x_r \right) \delta^m c_{aw_{1}...w_{k}}(x_1, ..., x_k, x) \left( \prod_{r=1}^{k} T^w_{r}(x_r) \right) \]

\[ + \sum_{k=1}^{\infty} \int \left( \prod_{r=1}^{k} d^3 x_r \right) c^{aw_{1}...w_{k}}_{ki}(x_1, ..., x_k, x) \delta^m \left( \prod_{r=1}^{k} T^w_{r}(x_r) \right) = 0. \]  

(40)

Equation (40) generates an equation for zeroth-order term in \( T^i_\alpha \), from where we can obtain the first coefficient of Eq. (39)

\[ c_{1j}^{ab}(x, y) = \{ A_i^a(x), \chi^b(y) \} \]

\[ = \frac{1}{m^2} D^{ba}_{\alpha \beta}(x - y), \]  

(41)

where the subscript \( y \) in the covariant derivative indicates which space-time variable it is acting on. From the linear equation in \( T^i_\alpha \), we can find the second coefficient of Eq. (39)

\[ c_{2j}^{abc}(x, y, z) = \frac{1}{2} \left\{ c_{1j}^{bc}(x, y), \chi^a(z) \right\} \]

\[ = \frac{g \epsilon^{cde}}{2m^4} \delta^{(3)}(y - z)D^{ad}_{\alpha \beta}(x - y) \delta^{(3)}(y - z). \]  

(42)

Performing the same procedure for second-order terms, we arrive at the third coefficient of the series in Eq. (39)

\[ c_{3j}^{abcd}(x, y, z, w) = \frac{1}{3} \left\{ c_{2j}^{abcd}(x, y, z), \chi^d(z) \right\} \]

\[ = \frac{g \epsilon^{cde}}{3m^6} \delta^{(3)}(y - z)D^{ad}_{\alpha \beta}(x - y) \delta^{(3)}(y - z). \]  

(43)

From Eqs. (41), (42) and (43), we can conclude that, for \( k \geq 4 \), the coefficients \( c_{ki} \) have the following form

\[ F^{w_{1}...w_{k}}_{i} = f^{w_{1}...w_{k}}_{i} + \ldots + f^{w_{1}...w_{k}}_{i}, \ldots + f^{w_{1}...w_{k}}_{i}. \]  

(46)

Writing the series for \( \hat{A}_i^a \) explicitly, we have

\[ \hat{A}_i^a(x) = A_i^a(x) - \frac{1}{m^2} T^i_\alpha(x) \]

\[ + \sum_{k=1}^{\infty} \int \left( \prod_{r=1}^{k} d^3 x_r \right) c^{aw_{1}...w_{k}}_{ki}(x_1, ..., x_k, x) \left( \prod_{r=1}^{k} T^w_{r}(x_r) \right) \]

\[ = A_i^a(x) - \frac{1}{m^2} T^i_\alpha(x) \]

\[ - \sum_{k=1}^{\infty} \frac{g^{k-1}}{k! m^{2k}} F^{w_{1}...w_{k}}_{i} \left( \prod_{r=2}^{k} \pi^w_0(x) \left( D^{w}_{i} \pi^w_0(x) \right) \right), \]  

(45)

where

\[ F^{w_{1}...w_{k}}_{i} = f^{w_{1}...w_{k}}_{i} + f^{w_{1}...w_{k}}_{i} + \ldots + f^{w_{1}...w_{k}}_{i}, \ldots + f^{w_{1}...w_{k}}_{i}. \]  

(47)

(48)
From Eq. (49) it is possible to build up a general form of the coefficients for all \( k \geq 4 \)
\[
b_{k,j}^{m_{-1},m} (x, y, z, w) = \frac{g^k}{k! m^2 k} \int_0^w \cdots \int_0^{w_{m-1}} \int_0^{w_{m-2}} \cdots \int_0^{w_1} \cdots \int_0^{w_1} f_{w_{m-1}} f_{w_{m-2}} \cdots f_{w_1} f_{w_0}
\]
\[
\times \pi_i(x_i) \delta^3(x - x_i) \left( \prod_{j=2}^{k} \delta^3(x_j - x_{j+1}) \right).
\]

Hence, the results of Eqs. (49) and (50) allow us to obtain the expression for the GU corrected variable \( \tilde{\varphi}^a \)
\[
\tilde{\varphi}^a_i = \varphi^a_i \left[ \delta^{a0} + g^{a0b} \theta^b + \frac{g^2}{2!} f_{abc} f^{bhc} g^c \right.
\]
\[
+ \frac{g^3}{3!} f_{abc} f_{ced} f^{bhc} f^{edg} g^c g^d + \ldots \right].
\]

where \( \tilde{\varphi}^a \equiv \frac{\varphi^a}{m^2} \). We can verify that the new fields \( \tilde{\varphi}^a \) and \( \tilde{\varphi}^a \) (Eqs. (48) and (51)) respectively, satisfy the conditions \( \{ \tilde{\varphi}^a_i, \tilde{\varphi}^b_j \} = \{ \tilde{\varphi}^a_i, \gamma^b_j \} = 0 \). Therefore, \( \tilde{\varphi}^a \) and \( \tilde{\varphi}^a \) are the GU variables of the massive Yang-Mills model. Replacing Eqs. (48) and (51) in the second-class Hamiltonian, Eq. (22), then we can derive the gauge invariant Hamiltonian.

### 4 The SU(2) Skyrme Model

We now consider the SU(2) Skyrme model [19, 20, 25], which describes baryons and their interactions through soliton solution from the Lagrangian
\[
L = \int d^3 r \left[ -\frac{F_x^2}{12} \text{Tr} \left\{ \partial_t U \partial_t U^+ \right\} + \frac{1}{32e^2} \text{Tr} \left\{ \left[ U^+ \partial_x U, U^+ \partial_x U \right] \right\} \right],
\]

where \( F_x \) is the pion decay constant, \( e \) is a dimensionless parameter and \( U \) is an SU(2) matrix. Technical details can be found in [20]. The crucial point to be addressed here is that the SU(2) group has a Wess-Zumino term that vanishes, so it is triangle anomaly-free and the Hamiltonian admits a diagonal form. By inserting the collective coordinate \( x(t) \) into the Lagrangian in Eq. (52) and substituting \( U(r) \) by \( U(r) = \exp(i \vec{F}(r) \cdot \vec{r}) \), we obtain
\[
L = -M + \lambda \text{Tr} \left\{ \partial_0 A \partial_0 A^{-1} \right\},
\]

where \( M \) is the soliton mass which, in the hedgehog ansatz \( U(r) = \exp(i \vec{F}(r) \cdot \vec{r}) \), can be written as
\[
M = 4\pi \int_0^\infty dr r^2 \left[ \frac{1}{8} \frac{F_x^2}{r^2} \left( \frac{\partial F_x}{\partial x} \right)^2 + \frac{\sin^2 F_x}{r^2} \right]
\]
\[+ \frac{1}{2e^2} \frac{\sin^2 F_x}{r^2} \left( \frac{\sin^2 F_x}{r^2} + 2 \left( \frac{\partial F_x}{\partial r} \right)^2 \right) \right],
\]

and \( \lambda \) is called inertia moment, given by
\[
\lambda = \frac{2\pi}{3e^3 F_x} \Lambda,
\]

where
\[
\Lambda = \int_0^\infty dx x^2 \sin^2 F_x \left\{ 1 + 4 \left[ \left( \frac{\partial F_x}{\partial x} \right)^2 + \frac{\sin^2 F_x}{x^2} \right] \right\},
\]

in which \( x \) is a dimensionless variable defined by \( x \equiv eF_x r \).

The SU(2) matrix \( A \) can be written as \( A = a_0 + i a \cdot \tau \) with the constraint
\[
T_1 = a_0 a_i - 1 \approx 0, \quad i = 0, 1, 2, 3.
\]

Then, the Lagrangian in Eq. (52) can be rewritten as a function of the \( a_i \)’s as
\[
L = -M + 2\lambda \hat{a}_i \hat{a}_i.
\]

The canonical momenta are given by
\[
\pi_i \equiv \frac{\partial L}{\partial \hat{a}_i} = 4\lambda \hat{a}_i.
\]

Then, using the Legendre transformation we can derive the canonical Hamiltonian
\[
H_c = M + \frac{1}{8\lambda} \pi_i \pi_i.
\]

From the consistency condition that the constraint \( T_1 \) can not evolve in time, we obtain the secondary constraint
\[
T_2 = a_0 \pi_i.
\]

Then, \( T_1 \) and \( T_2 \), Eqs. (57) and (61), are the total second-class constraints of the theory, and the Poisson algebra is given by
\[
\{ T_a, T_b \} = 2\epsilon_{ab} a_i \pi_i, \quad a, b = 1, 2,
\]

where \( \epsilon_{ab} = 1 \). The second-class property shown by the constraints in Eq. (62) allows us to apply the improved GU formalism. Then we need to choose one of the two second-class constraints to be the generator of infinitesimal transformations. Two possibilities can be analysed separately. We will start by considering \( T_1 \) in Eq. (57) as being the gauge symmetry generator. We can redefine it so that the two constraints form an approximate canonically conjugate pair. Thus, we reclassify \( T_1 \) as
\[
\chi = \frac{T_1}{2a_i a_i} = \frac{a_i a_i - 1}{2a_i a_i}.
\]

The constraint \( T_2 \) in Eq. (61) can be discarded. The infinitesimal variations of the relevant quantities are given by

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\[ \delta a_i = \epsilon \{ a_i, \chi \} = 0. \]
\[ \delta \pi_i = \epsilon \{ \pi_i, \chi \} = -\frac{\epsilon a_i}{a_j a_j} + \frac{2 \epsilon a_i \chi}{a_j a_j} \]  \hspace{1cm} (64)
\[ \delta T_2 = \epsilon \{ T_2, \chi \} = -\epsilon - 2 \epsilon \chi. \]

As explained in Sect. 2, the GU variables are constructed as a power series of the disregarded constraint, namely
\[ \tilde{a}_i = a_i + \sum_{n=1} b_n T_2^n, \]
\[ \tilde{\pi}_i = \pi_i + \sum_{n=1} c_n T_2^n, \]  \hspace{1cm} (65)
where the coefficients \( b_n \) and \( c_n \) are calculated from Eq. (13).

By using the expression (64), we can see that, for the variable \( \tilde{a}_i \), all coefficients \( b_n \) are trivially zero, resulting in
\[ \tilde{a}_i = a_i. \]  \hspace{1cm} (66)

Then for \( \tilde{\pi}_i \), we obtain
\[ c_1 = \frac{(2 \chi - 1)a_i}{2a_j a_j - 1}. \]  \hspace{1cm} (67)
Because \( \delta c_1 = 0 \), all coefficients are null for \( n \geq 2 \). Therefore, the GU variable \( \tilde{\pi}_i \) is
\[ \tilde{\pi}_i = \pi_i + \frac{(2 \chi - 1)a_i}{2a_j a_j - 1} \pi_k. \]  \hspace{1cm} (68)

As the GU variables are gauge invariants, then the Hamiltonian constructed from these GU variables will also be gauge invariant. By the insertion of the Eq. (68) in the Hamiltonian in Eq. (60), we obtain the following gauge invariant Hamiltonian:
\[ \tilde{H}_c = M + \frac{1}{8 \lambda} \tilde{\pi}_i \tilde{\pi}_i \]
\[ = H_c + \frac{1}{4 \lambda} \left( \frac{(2 \chi - 1)(a_i \pi_j)^2}{2a_j a_j - 1} + \frac{1}{8 \lambda} \left( \frac{(2 \chi - 1)a_i a_j \pi_j}{2a_j a_k - 1} \right)^2 \right). \]  \hspace{1cm} (69)

We can verify that \( \{ \chi, \tilde{H}_c \} = 0 \), which means that \( \chi \) and \( \tilde{H}_c \) describe a gauge theory in this new system [5].

We will now consider the other choice for the gauge symmetry generator, which is \( T_2 \). This is the most optimal choice, as showed in [25], since this freedom of choice allows, from the redefined theories obtained, a gauge theory that presents good characteristics such as Lorentz invariance in a manifest way, locality, etc. We start by rescaling \( T_2 \) as
\[ \chi' = -\frac{T_2}{2a_i a_i} = -\frac{a_i \pi_i}{2a_j a_j}, \]  \hspace{1cm} (70)
and \( T_1 \), Eq. (57), will be ignored as a constraint. The infinitesimal transformations generated by \( \chi' \) are
\[ \delta a_i = -\epsilon \{ a_i, \chi' \} = -\frac{\epsilon a_i}{2a_j a_j}, \]
\[ \delta \pi_i = \epsilon \{ \pi_i, \chi' \} = \frac{\epsilon a_i}{2a_j a_j} + \frac{2 \epsilon \chi' a_i}{a_j a_k}, \]  \hspace{1cm} (71)
\[ \delta T_1 = \epsilon \{ T_1, \chi' \} = -\epsilon. \]

Hence, the GU variables \( \tilde{a}_i \) and \( \tilde{\pi}_i \) will be constructed as functions of the discarded constraint \( T_1 \), Eq. (57), as
\[ \tilde{a}_i = a_i + \sum b_n T_1^n, \]
\[ \tilde{\pi}_i = \pi_i + \sum c_n T_1^n. \]  \hspace{1cm} (72)

Considering both variables, we have an infinite number of coefficients because the series do not truncate. For \( \tilde{a}_i \), the first two coefficients are
\[ b_1 = -\frac{\pi_i}{2a_j a_j}, \]
\[ b_2 = -\frac{\pi_i}{8(a_j a_j)^2}. \]  \hspace{1cm} (73)

For \( n \geq 3 \), the general relation is
\[ b_n = -\frac{1}{2 \left( \frac{2}{3} - 1 \right) \cdots \left( \frac{2}{n+1} - 1 \right)} \frac{a_j}{(a_j a_k)^n}. \]  \hspace{1cm} (74)

Therefore, the GU variable \( \tilde{a}_i \) becomes
\[ \tilde{a}_i = a_i \left[ 1 - \frac{(a_j a_j - 1)}{2a_j a_k} \frac{(a_j a_j - 1)^2}{8(a_j a_k)^2} + \cdots \right], \]  \hspace{1cm} (75)
which can be put in a closed form as
\[ \tilde{a}_i = a_i \left[ 1 - \frac{(a_j a_j - 1)}{a_j a_k} \right]^{\frac{1}{2}} = \frac{a_i}{(a_j a_k)^{\frac{1}{2}}}. \]  \hspace{1cm} (76)

Now, for the GU variable \( \tilde{\pi}_i \), the first two coefficients are
\[ c_1 = \frac{\pi_i}{2a_j a_j} + \frac{a_j \chi'}{a_k a_k}, \]
\[ c_2 = \frac{\pi_i}{8(a_j a_j)^2} + \frac{a_j \chi'}{2(a_j a_k)^2}. \]  \hspace{1cm} (77)

Substituting the \( c_n \) coefficient in the second of the Eq. (72), we can derive the GU variable \( \tilde{\pi}_i \).
\[ \bar{\xi}_i = \xi_i + \left( \frac{\pi_i}{2a_ia_k} + \frac{2a_i\chi'}{a_ia_k} \right) (a_ia_j - 1) \]
\[ + \left( \frac{3\pi_i}{8(a_ia_k)} + \frac{a_i\chi'}{2(a_ia_k)^2} \right) (a_ia_j - 1)^2 \ldots \] (78)

Unlike \( \tilde{a} \) in Eq. (76), we can not obtain for \( \bar{\xi}_i \) in Eq. (78) a closed form. By the insertion of the Eq. (78) in the Hamiltonian (60), we obtain a new gauge invariant Hamiltonian
\[ \tilde{H}'_c = M + \frac{1}{8A} \bar{\xi}_i \bar{\xi}_i. \] (79)

It is possible to verify that \( \{ \chi, \tilde{H}'_c \} = 0 \). Furthermore, it is very easy to show that for the massive Yang-Mills model, as well as for the SU(2) Skyrme model, the Poisson brackets between the GU variables are equals to the Dirac brackets between the original second-class variables.

5 Conclusions

In this paper, we have extended the results of the improved GU formalism [32, 33] to take into account the presence of non-Abelian variables, i.e., variables with values in the Lie algebra of a non-commutative group. The main point of this formalism is the gauge invariance restitution of dynamical systems with second-class constraints. Following an iterative procedure given in the previous sections, the second-class variables can be converted to gauge invariant quantities and then we have a gauge invariant system. In the case of a system that has two second-class constraints, there are two ways to define the GU variables, and both depend on the constraint we have initially chosen to be the gauge symmetry generator.

In particular, we have investigated the gauge invariance restitution of the massive SU(N) Yang-Mills model and of the SU(2) Skyrme model. In the former, the proper choice of the gauge symmetry generator allows us to reproduce exactly the infinitesimal form of the gauge variations for the case of massless Yang-Mills. The expansions obtained for the GU variables, written in terms of a power series of the discarded constraint, reproduce exactly the (non-polynomial) Stueckelberg construction derived in previous work. On the other hand, we have obtained two different gauge theories for the SU(2) Skyrme model. For the first case (\( T_1 \) as a generator, Eq. (57)), the GU variables acquire a simpler form comparing with the second case (\( T_2 \) as a generator, Eq. (61)), where the respective GU variables have an infinite number of coefficients.

Both systems recover the original second-class property when we impose the initial boundary conditions. Finally, we can mention that for the massive Yang-Mills model and for the SU(2) Skyrme model, the Poisson brackets between the GU variables are equals to the Dirac brackets between the original second-class variables [33]. These results can indicate the validate of the consistency of our formalism.

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Data Availability All relevant data supporting the conclusions of this article are included within the article.

Declarations

Conflict of Interest There are non-financial interests that are directly or indirectly related to the work submitted for publication in Brazilian Journal of Physics.

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