NONLINEAR POTENTIALS AND TWO WEIGHT TRACE INEQUALITIES FOR GENERAL DYADIC AND RADIAL KERNELS

CARMÉ CASCANTE, JOAQUÍN M. ORTEGA, AND IGOR E. VERBITSKY

Abstract. We study trace inequalities of the type
\[ \|T_k f\|_{L^q(d\mu)} \leq C \|f\|_{L^p(d\sigma)}, \quad f \in L^p(d\sigma), \]
in the “upper triangle case” \( 1 \leq q < p \) for integral operators \( T_k \) with positive kernels, where \( d\sigma \) and \( d\mu \) are positive Borel measures on \( \mathbb{R}^n \). Our main tool is a generalization of Th. Wolff’s inequality which gives two-sided estimates of the energy \( E_k, \sigma[\mu] = \int_{\mathbb{R}^n} (T_k[\mu])^p d\sigma \) through the \( L^1(d\mu) \)-norm of an appropriate nonlinear potential \( W_k, \sigma[\mu] \) associated with the kernel \( k \) and measures \( d\mu, d\sigma \). We initially work with a dyadic integral operator with kernel \( K_D(x,y) = \sum_{Q \in D} K(Q) \chi_Q(x) \chi_Q(y), \) where \( D = \{Q\} \) is the family of all dyadic cubes in \( \mathbb{R}^n \), and \( K : D \to \mathbb{R}^+ \). The corresponding continuous versions of Wolff’s inequality and trace inequalities are derived from their dyadic counterparts.

1. Introduction

In the present paper we are concerned with integral inequalities of the type
\[ \|T_k f\|_{L^q(d\mu)} \leq C \|f\|_{L^p(d\sigma)}, \quad f \in L^p(d\sigma), \]
in the “upper triangle case” \( 1 \leq q < p \). Here \( d\sigma \) and \( d\mu \) are locally finite positive Borel measures on \( \mathbb{R}^n \), and
\[ T_k f(x) = \int_{\mathbb{R}^n} K(x,y) f(y) d\sigma(y), \quad x \in \mathbb{R}^n, \]
is an integral operator with nonnegative kernel \( K(x,y) \geq 0 \).

An important part of this work is a related study of generalized nonlinear potentials used originally by Hedberg and Wolff \cite{HeWo} in the special case of Riesz kernels \( k_\alpha(x,y) = |x-y|^{\alpha-n} \) \( (0 < \alpha < n) \) and when \( d\sigma \) is Lebesgue measure on \( \mathbb{R}^n \). Then \( T_{k_\alpha} f(x) = k_\alpha * f \) is the usual (linear) Riesz potential of \( d\nu = f \, dx \). For a positive locally finite measure \( \nu \) on \( \mathbb{R}^n \), the corresponding nonlinear potential is defined by:
\[ W_{k_\alpha}[\nu](x) = \int_0^{+\infty} \left( \frac{\nu(B(x,r))}{r^{n-\alpha p}} \right)^{p'-1} \frac{dr}{r}, \]

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where \( B(x, r) \) is a ball of radius \( r \) centered at \( x \in \mathbb{R}^n \), and \( p' = p/(p - 1) \), \( 1 < p < +\infty \). It follows that the energy \( E_{k_\alpha}[\nu] = \| k_\alpha \ast \nu \|_{L^{p'}(dx)} \) satisfies the classical Wolff’s inequality \( \text{HeWo} \):

\[
C_1 \int_{\mathbb{R}^n} W_{k_\alpha}[\nu] \, dv \leq E_{k_\alpha}[\nu] \leq C_2 \int_{\mathbb{R}^n} W_{k_\alpha}[\nu] \, dv,
\]

where the constants \( C_1, C_2 \) do not depend on \( \nu \). Here only the upper estimate is nontrivial. (A thorough discussion of Wolff’s inequality and its applications is given in \( \text{AdHe} \).)

The corresponding trace inequality

\[
\| T_{k_\alpha} f \|_{L^q(d\mu)} \leq C \| f \|_{L^p(dx)}, \quad f \in L^p(dx),
\]

in the case \( q = 1 \) by duality is equivalent to \( W_{k_\alpha}[\mu] \in L^1(d\mu) \). As was shown by the authors in \( \text{CaOrVe1} \), for \( 1 < q < p \), the preceding inequality is characterized by the condition \( W_{k_\alpha}[\mu] \in L^{\frac{q(p-1)}{p-q}}(d\mu) \). (Some extensions of this result for more general kernels and weights can be found in \( \text{Ve2}, \text{CaOr} \) and \( \text{CaOrVe2} \).) Another characterization of this inequality for \( q < p \) stated in capacitary terms was obtained earlier by Maz’ya and Netrusov \( \text{Ma}, \text{MaNe} \).

We observe that in the well-studied case \( p \leq q < +\infty \), the above inequalities have been fully characterized in several different ways, for Riesz potentials and Lebesgue measure \( dx \) in place of \( d\sigma \), in \( \text{Ad}, \text{KeSa}, \text{Ma}, \text{MaVe}, \text{Ve2} \), and for a wide class of integral operators \( T_k \) in \( \text{NaTrVo}, \text{SaWh}, \text{VeWh} \).

Our main goal is to unify and extend earlier results on Wolff’s inequality and trace inequalities for more general two weight estimates and integral operators \( T_k \) with dyadic and radial nonincreasing kernels in the case \( q < p \).

We first concentrate on dyadic integral operators \( T_{K_D} \) introduced below, and the corresponding integral inequalities. Later on we will show how continuous versions follow from their dyadic analogues. Let \( \mathcal{D} = \{Q\} \) be the family of all dyadic cubes \( Q \) in \( \mathbb{R}^n \), and \( K : \mathcal{D} \rightarrow \mathbb{R}^+ \). The kernel \( K_D(x, y) \) on \( \mathbb{R}^n \times \mathbb{R}^n \) is defined by

\[
K_D(x, y) = \sum_{Q \in \mathcal{D}} K(Q) \chi_Q(x) \chi_Q(y),
\]

where \( \chi_Q \) is the characteristic function of \( Q \in \mathcal{D} \).

Let \( \nu \) be a locally finite positive Borel measure on \( \mathbb{R}^n \), and let \( f \in L^1_{\text{loc}}(d\nu) \). We define the dyadic integral operator:

\[
T_{K_D}[f \, d\nu](x) = \int_{\mathbb{R}^n} K_D(x, y) f(y) \, d\nu(y) = \sum_{Q \in \mathcal{D}} K(Q) \chi_Q(x) \int_Q f \, d\nu.
\]

In case \( f \equiv 1 \), we set

\[
T_{K_D}[^{\nu}](x) = \sum_{Q \in \mathcal{D}} K(Q) \nu(Q) \chi_Q(x).
\]
If $0 < q, p < +\infty$, and $\sigma$ and $\mu$ are locally finite Borel measures on $\mathbb{R}^n$, the corresponding dyadic trace inequality is given by:

$$
(1.1) \quad \int_{\mathbb{R}^n} |T_{K_D}[f \, d\sigma]|^q \, d\mu \leq C \|f\|_{L^p(d\sigma)}^q, \quad f \in L^p(d\sigma).
$$

Assume for a moment that $q, p > 1$. Duality then gives that (1.1) is equivalent to the inequality:

$$
(1.2) \quad \int_{\mathbb{R}^n} |T_{K_D}[gd\mu]|^p \, d\sigma \leq C \|g\|_{L^p(d\mu)}^p, \quad g \in L^p(d\mu).
$$

The quantity on the left-hand side of (1.2) is a generalized version of the discrete energy of $d\nu = gd\mu$. For positive locally finite Borel measures $\nu$ and $\sigma$ on $\mathbb{R}^n$, the discrete energy associated with $\nu$ and $\sigma$ is defined by (cf. [HeWo]):

$$
(1.3) \quad \mathcal{E}_{K,\sigma}^D[\nu] = \int_{\mathbb{R}^n} (T_{K_D}[\nu])^p \, d\sigma = \int_{\mathbb{R}^n} \left( \sum_{Q \in \mathcal{D}} K(Q) \nu(Q) \chi_Q(x) \right)^p \, d\sigma(x).
$$

Fubini’s theorem gives an alternative expression for $\mathcal{E}_{K,\sigma}^D$:

$$
\mathcal{E}_{K,\sigma}^D[\nu] = \int_{\mathbb{R}^n} T_{K_D}[(T_{K_D}[\nu])^{p^{-1}}] \, d\sigma \, d\nu,
$$

where $T_{K_D}[(T_{K_D}[\nu])^{p^{-1}}] \, d\sigma$ is a dyadic analogue of the nonlinear potential of Havin–Maz’ya originally defined for $d\sigma = dx$ and with $k_\alpha$ in place of $K_D$ (see [AdHe], [Ma]).

In the special case where $d\sigma$ is Lebesgue measure on $\mathbb{R}^n$, $K(Q) = \frac{1}{|Q|^{1-(\alpha/n)}}, \quad 0 < \alpha < n$ and $|Q|$ is the Lebesgue measure of $Q$, i.e., when $K_D$ is a discrete Riesz kernel on $\mathbb{R}^n$, Hedberg and Wolff introduced a dyadic nonlinear potential defined by:

$$
\mathcal{W}_{\alpha, dx}^D[\nu](x) = \sum_{Q \in \mathcal{D}} \left( \frac{\nu(Q)}{|Q|^{\alpha/n}} \right)^{p^{-1}} \chi_Q(x).
$$

(Here $r_Q$ denotes the side length of $Q$.) A dyadic version of Wolff’s inequality established in [HeWo] shows that, for $1 < p < +\infty$,

$$
C_1 \mathcal{E}_{\alpha, dx}^D[\nu] \leq \int_{\mathbb{R}^n} \mathcal{W}_{\alpha, dx}^D[\nu](x) \, d\nu(x) \leq C_2 \mathcal{E}_{\alpha, dx}^D[\nu].
$$

Consequently, the trace inequality (1.1) holds for $q = 1, 1 < p < +\infty$, and $d\sigma = dx$ if and only if $\mathcal{W}_{\alpha, dx}^D[\nu]$ is in $L^1(d\mu)$. For $1 < q < p < +\infty$, as was shown in [CaOrVe1], (1.1) holds if and only if $\mathcal{W}_{\alpha, dx}^D[\mu] \in L^{\frac{p(p-1)}{p-q}}(d\mu)$.

In this line of argument, we need to define a suitable nonlinear potential associated with a pair of measures $\nu, \sigma$ and the kernel $K_D$ so that it is applicable to characterization of the trace inequality (1.1) for general dyadic kernels.

Let $\nu$ and $\sigma$ be positive locally finite Borel measures on $\mathbb{R}^n$. We denote by $\mathcal{K}(Q)$ the function

$$
\mathcal{K}(Q)(x) = \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} K(Q') \sigma(Q') \chi_{Q'}(x).
$$
For \( x \in \mathbb{R}^n \), we set
\[
W_{K, \sigma}^D[v](x) = \sum_{Q \in \mathcal{D}} K(Q) \sigma(Q) \left( \int_Q K(y) d\nu(y) \right)^{p'-1} \chi_Q(x).
\]

In [CaOrVe2], the following Wolff-type inequality was established for an arbitrary positive measure \( \nu \) on \( \mathbb{R}^n \), and \( dx \) in place of \( d\sigma \):
\[
C_1 \mathcal{E}_{K, dx}^D[\nu] \leq \int_{\mathbb{R}^n} W_{K, dx}^D[v] d\nu \leq C_2 \mathcal{E}_{K, dx}^D[\nu],
\]
where \( C_1, C_2 \) are constants which do not depend on \( \nu \) and \( K \). We note that some statements in [CaOrVe2] (in particular, Lemma 2.6, and consequently Lemma 2.7, Propositions 4.2 and 4.3) required certain corrections outlined in the erratum to that paper. They are presented with full proofs in Section 2 below. Our main results extend the preceding estimates, as well as characterizations of the discrete trace inequality in [CaOrVe1], [CaOrVe2], to arbitrary measures \( \sigma, \mu \) on \( \mathbb{R}^n \). In particular, we will prove the following theorems.

**Theorem A.** Let \( K : \mathcal{D} \to \mathbb{R}^+ \). Let \( 1 < p < +\infty \), and let \( \nu \) and \( \sigma \) be locally finite positive Borel measures on \( \mathbb{R}^n \). If \( \mathcal{E}_{K, \sigma}^D[\nu] \) and \( W_{K, \sigma}^D[\nu] \) are defined respectively by (1.3) and (1.4), then
\[
C_1 \mathcal{E}_{K, \sigma}^D[\nu] \leq \int_{\mathbb{R}^n} W_{K, \sigma}^D[\nu] d\nu \leq C_2 \mathcal{E}_{K, \sigma}^D[\nu],
\]
where \( C_1, C_2 \) are constants which do not depend on \( \nu \) and \( \sigma \).

**Theorem B.** Let \( K : \mathcal{D} \to \mathbb{R}^+ \). Let \( 1 \leq q < p < +\infty \), and let \( \mu \) and \( \sigma \) be locally finite positive Borel measures on \( \mathbb{R}^n \).

(i) Suppose there exists a constant \( C > 0 \) such that the trace inequality
\[
\int_{\mathbb{R}^n} |T_{K, \sigma} f d\mu|^q d\nu(x) \leq C \| f \|_{L^p(d\sigma)}^q,
\]
holds. Then \( W_{K, \sigma}^D[\mu] \in L^{q(p-1)/(p-q)}(d\mu) \).

(ii) Conversely, if \( W_{K, \sigma}^D[\mu] \in L^{q(p-1)/(p-q)}(d\mu) \) then the preceding trace inequality holds provided the pair \((K, \sigma)\) satisfies the dyadic logarithmic bounded oscillation condition (DLBO):
\[
\sup_{x \in Q} \overline{K}(Q)(x) \leq A \inf_{x \in Q} \overline{K}(Q)(x),
\]
where \( A \) does not depend on \( Q \in \mathcal{D} \).

If \( q = 1 \) then statement (ii) holds without the restriction \((K, \sigma) \in \text{DLBO} \). (In this case Theorem B is, by duality, an immediate consequence of Theorem A.)

In Section 3, we obtain continuous analogues of Theorems A and B. Here we state only a continuous version of the trace inequality for convolution operators with radial kernels \( k(x) = k(|x|) \),
\[
T_k[f](x) = \int_{\mathbb{R}^n} k(|x - y|) f(y) d\sigma(y).
\]
Here \( k = k(r), r > 0 \), is an arbitrary lower semicontinuous nonincreasing positive function.

The corresponding nonlinear potential is defined by

\[
W_{k, \sigma}[\mu](x) = \int_0^{+\infty} k(r) \sigma(B(x, r)) \left( \int_{B(x, r)} \overline{k}(r)(y) \, d\mu(y) \right)^{p'-1} \frac{dr}{r},
\]

where

\[
\overline{k}(r)(x) = \frac{1}{\sigma(B(x, r))} \int_0^r k(s) \sigma(B(x, s)) \frac{ds}{s},
\]

for \( x \in \mathbb{R}^n, r > 0 \).

**Theorem C.** Let \( 1 \leq q < p < +\infty \), and let \( \mu \) and \( \sigma \) be locally finite positive Borel measures on \( \mathbb{R}^n \). Assume that \( \sigma \) satisfies a doubling condition, and the pair \((k, \sigma)\) has the logarithmic bounded oscillation property (LBO):

\[
\sup_{y \in B(x, r)} \overline{k}(r)(y) \leq A \inf_{y \in B(x, r)} \overline{k}(r)(y),
\]

where \( A \) does not depend on \( x \in \mathbb{R}^n, r > 0 \). Then the following conditions are equivalent:

(i) There exists a constant \( C > 0 \) such that the trace inequality

\[
\int_{\mathbb{R}^n} |T_k[f \sigma]|^q(x) \, d\mu(x) \leq C \|f\|^q_{L^p(\sigma)}, \quad f \in L^p(\sigma),
\]

holds.

(ii) \( W_{k, \sigma}[\mu] \in L^{\frac{(a-1)}{p'-q}}(d\mu) \).

**Remark 1.** We observe that \((k, \sigma) \in \text{LBO}\) if \( d\sigma = dx \) is Lebesgue measure, for arbitrary nonincreasing radial kernels \( k(r) \), or if \( k(r) = r^{\alpha-n} \) is a Riesz kernel and \( \sigma \) satisfies a reverse doubling condition of order \( \gamma > n - \alpha \). (See details in Section 3.)

**Remark 2.** An example given in Sec. 3 demonstrates that Theorem C is no longer true for the nonlinear potential defined by

\[
\overline{W}_{k, \sigma}[\mu](x) = \int_0^{+\infty} \overline{k}(r)(x) \sigma(B(x, r)) \left( \int_{B(x, r)} \overline{k}(r)(y) \, d\mu(y) \right)^{p'-1} \frac{dr}{r},
\]

in place of \( W_{k, \sigma}[\mu] \), even when \( d\sigma \) is Lebesgue measure and \( \overline{k}(r) = r^{-n} \int_0^r k(s) s^{n-1} \, ds \) depends only on \( r \).

A similar example in Sec. 2 shows that Theorems A and B fail if one replaces \( W_{k, \sigma}[\mu] \) by

\[
\overline{W}_{k, \sigma}^{\mathcal{D}}[\mu](x) = \sum_{Q \in \mathcal{D}} \overline{K}(Q)(x) \sigma(Q) \left( \int_Q \overline{K}(Q)(y) \, d\mu(y) \right)^{p'-1} \chi_Q(x),
\]

even if \( d\sigma \) is Lebesgue measure and \( \overline{K}(Q)(x) \) is constant on \( Q \).

We conclude the introduction with a remark on our notation: we will adopt the usual convention of using the same letter for various “absolute” constants (which may
depend on \( q, p \) and \( n \) whose values may change in each occurrence, and we will write \( A \preceq B \) if there exists an absolute constant \( M \) such that \( A \leq MB \). We will say that two quantities \( A \) and \( B \) are equivalent if both \( A \preceq B \) and \( B \preceq A \), and in that case we will write \( A \simeq B \).

2.Discrete Wolff-type and trace inequalities

Let \( K : D \to \mathbb{R}^+ \) where \( \mathbb{R}^+ = [0, +\infty) \). We consider the kernel \( K_D \) given by

\[
K_D(x,y) = \sum_{Q \in D} K(Q) \chi_Q(x) \chi_Q(y),
\]

for \( x, y \) in \( \mathbb{R}^n \). If \( \mu \) is a positive locally finite Borel measure on \( \mathbb{R}^n \), we define the operator \( T_{K_D} \) by

\[
T_{K_D}[\mu](x) = \sum_{Q \in D} K(Q) \mu(Q) \chi_Q(x).
\]

Suppose \( \sigma \) is a fixed positive locally finite measure on \( \mathbb{R}^n \). For \( d\mu = f \, d\sigma \), where \( f \) is a nonnegative Borel measurable function, we will simply write

\[
T_{K_D}[f](x) = \sum_{Q \in D} K(Q) \chi_Q(x) \int_Q f(y) \, d\sigma(y).
\]

If \( 1 < p < +\infty \), the discrete energy of \( \mu \) is given by:

\[
\mathcal{E}_{K_D,\sigma}[\mu] = \int_{\mathbb{R}^n} (T_{K_D}[\mu](x))^p \, d\sigma(x) = \int_{\mathbb{R}^n} \left( \sum_{Q \in D} K(Q) \mu(Q) \chi_Q(x) \right)^p \, d\sigma(x).
\]

We now define a suitable nonlinear potential which generalizes the classical Hedberg-Wolff potential. For \( 1 < p < +\infty \) and \( Q \in D \), we first define the function

\[
\overline{K}(Q)(x) = \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} K(Q') \sigma(Q') \chi_{Q'}(x).
\]

Note that \( \overline{K}(Q)(x) \) is supported on \( Q \). If \( d\sigma = dx \) is Lebesgue measure on \( \mathbb{R}^n \) and \( K(Q) \) depends only on the size of \( Q \), i.e. there exists a nonincreasing function \( k : (0, +\infty) \to \mathbb{R}^+ \) such that for any \( Q \in \mathcal{D} \) \( K(Q) = k(r_Q) \), it is easy to check that

\[
\overline{K}(Q)(x) \simeq \frac{1}{r_Q^n} \int_0^{r_Q} k(t)t^{n-1} \, dt.
\]

(See the proof of part (i) of Proposition 2.4 below.) Next, we set

\[
W_{K,\sigma}[\mu](x) = \sum_{Q \in D} K(Q) \sigma(Q) \left( \int_Q \overline{K}(Q)(y) \, d\mu(y) \right)^{p'-1} \chi_Q(x), \quad x \in \mathbb{R}^n.
\]

It is worthwhile to observe that several other natural alternatives to \( W_{K,\sigma}[\mu] \) discussed in [CaOrVe2] fail to satisfy the desired analogue of Wolff’s inequality. (See an example at the end of this section.)
We will also deal with dyadic “shifted” versions of the above potential defined by

\[(2.2) \quad W_{K,\sigma}^d[\mu](x) = \sum_{Q \in D_z} K(Q)\sigma(Q) \left(\int_Q K(Q) \chi_Q(x) d\mu(y)\right)^{p'-1} \chi_Q(x), \quad x \in \mathbb{R}^n;\]

where \(D_z\) denotes the shifted dyadic lattice \(D_z = D + z = \{Q + z\}_{Q \in D}\).

We also introduce a dyadic maximal function associated with \(K^D\): \(M_K^D[\mu]\) given by

\[M_K^D[\mu](x) = \sup_{x \in Q} \frac{1}{\sigma(Q)} \left(\sum_{Q' \subset Q} K(Q')\sigma(Q')\mu(Q')\right).\]

We recall that if \(d\sigma = dx\) is Lebesgue measure on \(\mathbb{R}^n\) and \(K(Q)\) depends only on the size of \(Q\), this maximal function can be rewritten as the dyadic maximal function considered in \([KeSa]\), namely

\[M_K^D[\mu](x) = \sup_{x \in Q} K(Q)(x)\mu(Q).\]

Indeed, if \(Q \in D\), and \(x \in Q\), then for any \(l \geq 0\), \(2^{-l}Q\) is the unique cube in \(D\) satisfying \(x \in 2^{-l}Q\) and \(r_{2^{-l}Q} = 2^{-l}r_Q\). We then have

\[(3.3) \quad \frac{1}{r_Q^n} \sum_{Q' \subset Q} k(r_Q)r_Q^n\mu(Q') = \frac{\mu(Q)}{r_Q^n} \sum_{l \geq 0} k\left(\frac{r_Q}{2^l}\right) \left(\frac{r_Q}{2^l}\right)^n \approx \mu(Q)K(Q)(x).\]

Our first result can be viewed as a discrete version of the Wolff inequality \([HeWo]\) and Kerman-Sawyer inequality \([KeSa]\) for general measures \(\sigma\) and dyadic kernels \(K_D(x, y)\).

**Theorem 2.1.** Let \(K : D \to \mathbb{R}^+\) and \(1 < p < +\infty\). Let \(\mu\) and \(\sigma\) be locally finite positive Borel measures on \(\mathbb{R}^n\). Then the following quantities are equivalent:

(a) \(E_{K,\sigma}[\mu] = \int_{\mathbb{R}^n} \left(\sum_{Q \in D} K(Q)\mu(Q)\chi_Q(x)\right)^{p'} d\sigma(x);\)

(b) \(\int_{\mathbb{R}^n} W_{K,\sigma}^d[\mu] d\mu = \sum_{Q \in D} K(Q)\sigma(Q)\mu(Q) \left(\int_Q K(Q) \chi_Q(x) d\mu(y)\right)^{p'-1};\)

(c) \(\int_{\mathbb{R}^n} M_K^D[\mu]^{p'} d\sigma.\)

**Proof of Theorem 2.1**

Let \(1 < s < +\infty\), \(\Lambda = (\lambda_Q)_{Q \in D}\), \(\lambda_Q \in \mathbb{R}^+\), and let \(\sigma\) be a positive locally finite Borel measure. Our standing assumption will be that \(\lambda_Q = 0\) if \(\sigma(Q) = 0\). We will also follow the convention that \(0 \cdot \infty = 0\).
We define
\[
A_1(\Lambda) = \int_{\mathbb{R}^n} \left( \sum_{Q \in D} \frac{\lambda_Q}{\sigma(Q)} \chi_Q(x) \right)^s d\sigma(x),
\]
\[
A_2(\Lambda) = \sum_{Q \in D} \lambda_Q \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \lambda_{Q'} \right)^{s-1},
\]
\[
A_3(\Lambda) = \int_{\mathbb{R}^n} \sup_{x \in Q} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \lambda_{Q'} \right)^s d\sigma(x).
\]

The proof of the theorem will be a consequence of the following proposition.

**Proposition 2.2.** Let \( \sigma \) be a positive locally finite Borel measure on \( \mathbb{R}^n \). Let \( 1 < s < \infty \). Then there exist constants \( C_i > 0, i = 1, 2, 3 \), which depend only on \( s \), such that, for any \( \Lambda = (\lambda_Q)_{Q \in D}, \lambda_Q \in \mathbb{R}^+ \),
\[
A_1(\Lambda) \leq C_1 A_2(\Lambda) \leq C_2 A_3(\Lambda) \leq C_3 A_1(\Lambda).
\]

**Proof of Proposition 2.2.**

We begin by observing that if \( s \geq 1 \), then for any \( \Lambda = (\lambda_Q)_{Q \in D}, \lambda_Q \in \mathbb{R}^+ \), and \( x \in \mathbb{R}^n \), we have
\[
(2.4) \quad \left( \sum_{Q \in D} \lambda_Q \chi_Q(x) \right)^s \leq s \sum_{Q \in D} \lambda_Q \chi_Q(x) \left( \sum_{Q' \subset Q} \lambda_{Q'} \chi_{Q'}(x) \right)^{s-1}.
\]

We first prove (2.4) under the assumption that
\[
\sum_{Q \in D} \lambda_Q \chi_Q(x) < +\infty.
\]

Note that, for a fixed \( x \in \mathbb{R}^n \), the dyadic cubes containing \( x \) form a nested family of cubes. Hence using the elementary inequality \( b^s - a^s \leq s(b - a)b^{s-1} \), \( 0 \leq a \leq b \), \( 1 \leq s < \infty \), we obtain:
\[
\left( \sum_{Q' \subset Q} \lambda_{Q'} \chi_{Q'}(x) \right)^s - \left( \sum_{Q' \subset Q} \lambda_{Q'} \chi_{Q'}(x) \right)^s \leq s \lambda_Q \chi_Q(x) \left( \sum_{Q' \subset Q} \lambda_{Q'} \chi_{Q'}(x) \right)^{s-1}.
\]

From this (2.4) follows by a telescoping sum argument, taking the sums of both sides over all dyadic cubes \( Q \) that contain \( x \).

If \( \sum_{Q \in D} \lambda_Q \chi_Q(x) = +\infty \), but \( \sum_{Q \subset Q_0} \lambda_Q \chi_Q(x) < +\infty \) for some (and hence every) dyadic cube \( Q_0 \) which contains \( x \) then (2.4) follows by the same argument as above taking the sums over all \( Q \subset Q_0 \) and then letting \( |Q_0| \to +\infty \). Finally, in the case where \( \sum_{Q \subset Q_0} \lambda_Q \chi_Q(x) = +\infty \) for some \( Q_0 \), both sides of (2.4) are obviously infinite. This completes the proof of (2.4).

We now prove \( A_1(\Lambda) \leq C_1 A_2(\Lambda) \) for \( 1 < s \leq 2 \) (this obviously holds for \( s = 1 \) and \( C_1 = 1 \) as well). We may assume without loss of generality that there are only a finite
number of $\lambda_Q \neq 0$. By (2.4),

$$A_1(\Lambda) = \int_{\mathbb{R}^n} \left( \sum_{Q \in \mathcal{D}} \frac{\lambda_Q}{\sigma(Q)} \chi_Q(x) \right)^s d\sigma(x)$$

$$\leq s \sum_{Q \in \mathcal{D}} \frac{\lambda_Q}{\sigma(Q)} \int_Q \left( \sum_{Q' \subset Q} \frac{\lambda_{Q'}}{\sigma(Q')} \chi_{Q'}(x) \right)^{s-1} d\sigma(x).$$

Hölder’s inequality with exponent $\frac{s}{s-1} > 1$ gives:

$$\frac{1}{\sigma(Q)} \int_Q \left( \sum_{Q' \subset Q} \frac{\lambda_{Q'}}{\sigma(Q')} \chi_{Q'}(x) \right)^{s-1} d\sigma(x)$$

$$\leq \left( \frac{1}{\sigma(Q)} \int_Q \sum_{Q' \subset Q} \lambda_{Q'} \chi_{Q'}(x) d\sigma(x) \right)^{s-1} = \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \lambda_{Q'} \right)^{s-1}.$$

Consequently,

$$A_1(\Lambda) \leq s \sum_{Q \in \mathcal{D}} \lambda_Q \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \lambda_{Q'} \right)^{s-1} = s A_2(\lambda),$$

which proves the desired inequality with $C_1 = s$ for $1 < s \leq 2$. A similar inequality in the case $s > 2$ is proved by induction. For integer $k \geq 2$ we assume that the inequality $A_1(\Lambda) \leq C_1(s) A_2(\Lambda)$ holds for any $k - 1 < s \leq k$, and have to show that it also holds for $k < s \leq k + 1$. By (2.4),

$$A_1(\Lambda) = \int_{\mathbb{R}^n} \left( \sum_{Q \in \mathcal{D}} \frac{\lambda_Q}{\sigma(Q)} \chi_Q(x) \right)^s d\sigma(x)$$

$$\leq s \sum_{Q \in \mathcal{D}} \frac{\lambda_Q}{\sigma(Q)} \int_Q \left( \sum_{Q' \subset Q} \frac{\lambda_{Q'}}{\sigma(Q')} \chi_{Q'}(x) \right)^{s-1} d\sigma(x).$$
Applying the induction hypothesis for \( k - 1 < s - 1 \leq k \), with the measure \( \chi_Q \sigma \), and the set \( (\lambda_{Q'})_{Q'} \), where \( \lambda_{Q'} = 0 \) for cubes \( Q' \not\subset Q \), we obtain:

\[
\sum_Q \frac{\lambda_Q}{\sigma(Q)} \int_Q \left( \sum_{Q' \subset Q} \frac{\lambda_{Q'}}{\sigma(Q')} \chi_{Q'}(x) \right)^{s-1} d\sigma(x)
\]

\[
\leq C_1(s - 1) \sum_Q \frac{\lambda_Q}{\sigma(Q)} \sum_{Q' \subset Q} \lambda_{Q'} \left( \frac{1}{\sigma(Q')} \sum_{Q'' \subset Q'} \lambda_{Q''} \right)^{s-2}
\]

\[
= C_1(s - 1) \frac{\lambda_Q}{\sigma(Q)} \left( \frac{1}{\sigma(Q')} \sum_{Q'' \subset Q'} \lambda_{Q''} \right)^{s-2} \sum_{Q' \subset Q} \frac{\lambda_Q}{\sigma(Q)}
\]

\[
\leq C_1(s - 1) \int_{\mathbb{R}^n} \sum_{Q'} \frac{\lambda_{Q'}}{\sigma(Q')} \chi_{Q'}(x) \left( \frac{1}{\sigma(Q')} \sum_{Q'' \subset Q'} \lambda_{Q''} \right)^{s-2} \left( \sum_{Q} \frac{\lambda_Q}{\sigma(Q)} \chi_Q(x) \right) d\sigma(x).
\]

By Hölder’s inequality for sums with exponents \( s - 1 \) and \( (s - 1)/(s - 2) \) (note that \( s - 1 > k - 1 \geq 1 \)), we have

\[
\sum_{Q'} \frac{\lambda_{Q'}}{\sigma(Q')} \chi_{Q'}(x) \left( \frac{1}{\sigma(Q')} \sum_{Q'' \subset Q'} \lambda_{Q''} \right)^{s-2}
\]

\[
\leq \left( \sum_{Q'} \frac{\lambda_{Q'}}{\sigma(Q')} \chi_{Q'}(x) \right)^{1/(s-1)} \left( \sum_{Q'} \frac{\lambda_{Q'}}{\sigma(Q')} \chi_{Q'} \left( \frac{1}{\sigma(Q')} \sum_{Q'' \subset Q'} \lambda_{Q''} \right)^{s-1} \right)^{(s-2)/(s-1)}
\]

Substituting this estimate into the right-hand side of the preceding inequality, we obtain:

\[
A_1(\Lambda) \leq s C_1(s - 1) \int_{\mathbb{R}^n} \left( \sum_{Q'} \frac{\lambda_{Q'}}{\sigma(Q')} \chi_{Q'}(x) \right)^{1/(s-1)+1}
\]

\[
\times \left( \sum_{Q'} \frac{\lambda_{Q'}}{\sigma(Q')} \chi_{Q'}(x) \left( \frac{1}{\sigma(Q')} \sum_{Q'' \subset Q'} \lambda_{Q''} \right)^{s-1} \right)^{(s-2)/(s-1)} d\sigma(x).
\]

Applying now Hölder’s inequality for integrals with exponents \( s - 1 \) and \( (s - 1)/(s - 2) \), we have:

\[
A_1(\Lambda) \leq s C_1(s - 1) \left( \int_{\mathbb{R}^n} \left( \sum_{Q'} \frac{\lambda_{Q'}}{\sigma(Q')} \chi_{Q'}(x) \right)^s d\sigma(x) \right)^{1/s}
\]

\[
\times \left( \int_{\mathbb{R}^n} \sum_{Q'} \frac{\lambda_{Q'}}{\sigma(Q')} \chi_{Q'}(x) \left( \frac{1}{\sigma(Q')} \sum_{Q'' \subset Q'} \lambda_{Q''} \right)^{s-1} d\sigma(x) \right)^{s/(s-2)}
\]

\[
= s C_1(s - 1) A_1(\Lambda)^{1/s} A_2(\Lambda)^{s/(s-2)}.
\]
From this it follows that $A_1(\Lambda) \leq C_1(s) A_2(\Lambda)$ where $C_1(s) = (sC_1(s-1))^{\frac{s}{s-1}}$, for $k < s \leq k + 1$, and hence for every $1 < s < \infty$.

Next, let $1 < s < +\infty$. Then

$$A_3(\Lambda) = \int_{\mathbb{R}^n} \sup_{x \in Q} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \lambda_{Q'} \right)^s d\sigma(x)$$

$$\leq \int_{\mathbb{R}^n} M_{\sigma, D}^{HL} \left( \sum_{Q \in D} \frac{\lambda_Q}{\sigma(Q)} \chi_Q(x) \right)^s d\sigma(x) \leq CA_1(\Lambda),$$

where the dyadic Hardy-Littlewood maximal function $M_{\sigma, D}^{HL}[\nu]$ is defined by

$$M_{\sigma, D}^{HL}[\nu](x) = \sup_{x \in Q} \nu(Q).$$

Here we have used the fact that, for $d\nu = |f| d\sigma$, the operator $M_{\sigma, D}^{HL} : f \rightarrow M_{\sigma, D}^{HL}[f]$ is bounded on $L^s(\sigma)$ for $1 < s \leq +\infty$. This proves the estimate $A_3(\Lambda) \leq CA_1(\Lambda)$ with $C = ||M_{\sigma, D}^{HL}||_{L^s(\sigma) \rightarrow L^s(\sigma)}$.

We now prove the inequality $A_1(\Lambda) \leq C A_3(\Lambda)$ for $1 < s < \infty$. Using the estimate $A_1(\Lambda) \leq C_1 A_2(\Lambda)$ established above, we have:

$$A_1(\Lambda) = \int_{\mathbb{R}^n} \left( \sum_{Q \in D} \frac{\lambda_Q}{\sigma(Q)} \chi_Q(x) \right)^s d\sigma(x) \leq C_1 \sum_{Q \in D} \lambda_Q \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \lambda_{Q'} \right)^{s-1}$$

$$\leq C_1 \sum_{Q \in D} \lambda_Q \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \lambda_{Q'} \right)^{s-1} d\sigma(x)$$

$$\leq C_1 \int_{\mathbb{R}^n} \left( \sum_{Q \in D} \frac{\lambda_Q}{\sigma(Q)} \chi_Q(x) \right) \left( \sup_{x \in Q} \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \lambda_{Q'} \right)^{s-1} d\sigma(x)$$

$$\leq C_1 A_1(\Lambda)^{\frac{s}{s'}} A_3(\Lambda)^{\frac{1}{s'}},$$

where in the last estimate we have used Hölder’s inequality with exponents $s$ and $s'$. This chain of inequalities yields

$$A_1(\Lambda) \leq C_1 A_1(\Lambda)^{\frac{s}{s'}} A_3(\Lambda)^{\frac{1}{s'}},$$

which implies that $A_1(\Lambda) \leq C_1' A_3(\Lambda)$. It remains to show that $A_2(\Lambda) \leq CA_3(\Lambda)$. Note that

$$A_2(\Lambda) = \sum_{Q \in D} \lambda_Q \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \lambda_{Q'} \right)^{s-1} = \sum_{Q \in D} \lambda_Q \mu_Q,$$
where \( \mu_Q = \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \lambda_{Q'} \right)^{s^{-1}} \). Then

\[
\sum_{Q \in \mathcal{D}} \lambda_Q \mu_Q = \int_{\mathbb{R}^n} \left( \sum_{Q \in \mathcal{D}} \frac{\lambda_Q \mu_Q}{\sigma(Q)} \chi_Q(x) \right) d\sigma(x) \\
\leq \int_{\mathbb{R}^n} \left( \sum_{Q \in \mathcal{D}} \frac{\lambda_Q}{\sigma(Q)} \chi_Q(x) \right)^s \sup_{x \in Q} \mu_Q d\sigma(x).
\]

Hölder’s inequality gives that the above is bounded by

\[
\left( \int_{\mathbb{R}^n} \left( \sum_{Q \in \mathcal{D}} \frac{\lambda_Q}{\sigma(Q)} \chi_Q(x) \right)^s d\sigma(x) \right)^{\frac{1}{s}} \left( \int_{\mathbb{R}^n} \sup_{x \in Q} \mu_Q^s d\sigma(x) \right)^{\frac{1}{s'}}.
\]

Hence

\[
A_2(\Lambda) = \sum_{Q \in \mathcal{D}} \lambda_Q \mu_Q \leq A_1(\Lambda)^{\frac{1}{2}} A_3(\Lambda)^{\frac{1}{2}}.
\]

Since \( A_1(\Lambda) \leq C A_3(\Lambda) \), we finally obtain \( A_2(\Lambda) \leq C A_3(\Lambda) \). The proof of the proposition is complete. \( \square \)

Theorem 2.1 follows immediately from Proposition 2.2 with \( \lambda_Q = K(Q) \mu(Q) \sigma(Q) \) and \( s = p' \).

We are now in a position to characterize the trace inequality in the case \( q = 1 \).

**Corollary 2.3.** Let \( K : \mathcal{D} \to \mathbb{R}^+ \). Let \( \sigma, \mu \) be locally finite positive Borel measures on \( \mathbb{R}^n \), and let \( 1 < p < +\infty \). Then the following statements are equivalent:

(i) There exists \( C > 0 \) such that for any \( f \in L^p(d\sigma) \), \( f \geq 0 \),

\[
\left( \int_{\mathbb{R}^n} \left( \sum_{Q \in \mathcal{D}} \frac{\lambda_Q}{\sigma(Q)} \chi_Q(x) \right)^s d\sigma(x) \right)^{\frac{1}{s}} \left( \int_{\mathbb{R}^n} \sup_{x \in Q} \mu_Q^s d\sigma(x) \right)^{\frac{1}{s'}} \leq C \|f\|_{L^p(d\sigma)}.
\]

(ii) \( \mathcal{W}_{K,\sigma}[\mu](x) = \sum_{Q \in \mathcal{D}} \sigma(Q) \, K(Q) \, \chi_Q(x) \left( \int_Q \overline{K}(Q)(y) d\mu(y) \right)^{p'-1} \in L^1(d\mu) \).

**Proof of Corollary 2.3**

The trace inequality (i) can be restated equivalently via Fubini’s theorem as

\[
\left( \int_{\mathbb{R}^n} T_{K_D}[f] d\mu \right)^{\frac{1}{s'}} = \left( \int_{\mathbb{R}^n} \int_{\mathcal{D}} \sum_{Q' \subset Q} \lambda_{Q'}(y) \chi_Q(x) d\mu(y) \right)^{\frac{1}{s'}} \leq C \|f\|_{L^p(d\sigma)},
\]

which by duality is equivalent to \( \mathcal{E}^p_{K,\sigma}[\mu] < +\infty \). Now Theorem 2.1 gives that the dyadic energy is finite if and only if \( \mathcal{W}_{K,\sigma}[\mu] \) belongs to \( L^1(d\mu) \). \( \square \)

In what follows we will restrict ourselves to functions \( K : \mathcal{D} \to \mathbb{R}^+ \) and measures \( \sigma \) satisfying an extra assumption, namely that for any \( Q \in \mathcal{D} \), there exists a constant \( C \) that we will denote by \( \overline{K}(Q) \) such that

\[
\frac{1}{C} \overline{K}(Q) \chi_Q(x) \leq \overline{K}(Q)(x) \leq C \overline{K}(Q) \chi_Q(x),
\]

called the \( K \)-constant. This assumption is not restrictive since it is satisfied by many well-known kernels, such as the Cauchy kernel or the Riesz kernel.
where $C$ does not depend on $Q$. In other words, the oscillation of the function $\ln K(Q)(x)$ on $Q$ is bounded by a constant which is independent of $Q \in \mathcal{D}$. In this case we will say that the pair $(K, \sigma)$ satisfies the dyadic logarithmic bounded oscillation property, or DLBO, and simply write $(K, \sigma) \in \text{DLBO}$. Observe that strictly speaking, $K(Q)$ is not unequivocally defined, but since it behaves like $K(Q)(x)$ up to a multiplicative constant, we find this notation appealing.

We first give some examples.

**Proposition 2.4.** (i) Suppose that $d\sigma = dx$ is Lebesgue measure on $\mathbb{R}^n$ and $K(Q)$ depends only on the size of $Q$, i.e., there exists a nonincreasing function $k : (0, +\infty) \to \mathbb{R}^+$ such that for any $Q \in \mathcal{D}$, $K(Q) = k(r_Q)$ where $r_Q$ is the side length of $Q$. Then the pair $(K, dx)$ satisfies property (2.5) where

$$K(Q) = \frac{1}{r_Q^n} \int_0^{r_Q} k(t) t^{n-1} dt.$$  

(ii) Let $K(Q) = \frac{1}{r_Q^{n-\alpha}}$, $0 < \alpha < n$, be a discrete Riesz kernel. Suppose that $\sigma$ is a dyadic reverse doubling measure: $\sigma \in \text{DRD}_\gamma$ for some $\gamma > n - \alpha$, i.e., there exists $C > 0$ such that for any $j \geq 0$, $Q \in \mathcal{D}$,

$$\sigma(2^j Q) \geq C 2^{j\gamma} \sigma(Q),$$

where $2^j Q$ is the unique dyadic cube in $\mathcal{D}$ such that $Q \subset 2^j Q$ and $r_{2^j Q} = 2^j r_Q$. We then have that $(K, \sigma)$ satisfies property DLBO with $K(Q) = K(Q)$. Conversely, if $K$ is a discrete Riesz kernel and the pair $(K, \sigma)$ satisfies property DLBO with $K(Q) = K(Q)$, then $\sigma$ is a $\text{DRD}_\gamma$ measure for some $\gamma > n - \alpha$.

**Proof of Proposition 2.4.**

We begin with (i). We observe that if $Q \in \mathcal{D}$ and $x \in Q$, for any $l \geq 0$ there exists a unique $Q_l \subset Q$ in $\mathcal{D}$ such that $x \in Q_l$ and $r_{Q_l} = 2^{-l} r_Q$. Thus

$$K(Q)(x) = \frac{1}{r_Q^n} \sum_{Q' \subset Q} k(r_{Q'}) r_{Q'}^n \chi_{Q'}(x)$$

$$= \frac{1}{r_Q^n} \sum_{l \geq 0} k(2^{-l} r_Q) (2^{-l} r_Q)^n$$

$$\leq C \frac{1}{r_Q^n} \int_0^{r_Q} k(t) t^{n-1} dt.$$ 

For the converse estimate, the fact that $k$ is nonincreasing gives

$$K(Q)(x) \geq \frac{1}{r_Q^n} \sum_{l \geq 1} k(2^{-l} r_Q) (2^{-l} r_Q)^n \geq C \frac{1}{r_Q^n} \int_0^{r_Q} k(t) t^{n-1} dt.$$
Now we prove (ii). If $\sigma$ satisfies (2.6), $Q \in \mathcal{D}$, $x \in Q$ and $r_Q = 2^{-k}$, and for any $l \geq 0$, $2^{-l}Q$ is the unique cube in $\mathcal{D}$ such that $x \in 2^{-l}Q$ and $r_{2^{-l}Q} = 2^{-l}r_Q$, then
\[
\mathcal{K}(Q)(x) = \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \frac{\sigma(Q')}{r_{Q'}^{\alpha}} \chi_{Q'}(x) \leq \frac{1}{\sigma(Q)} \sum_{l \geq 0} \sigma(2^{-l}Q)2^{(l+k)(n-\alpha)}
\leq C \sum_{l \geq 0} 2^{l(2+k)(n-\alpha)} 2^{-l} = C 2^{k(n-\alpha)} \sum_{l \geq 0} 2^{l(n-\alpha-\gamma)} \leq C \frac{1}{r_Q^{n-\alpha}},
\]
since $n-\alpha-\gamma < 0$. Obviously,
\[
\mathcal{K}(Q)(x) \geq \frac{1}{\sigma(Q)} \frac{\sigma(Q)}{r_Q^{n-\alpha}} \chi_Q(x) = \frac{1}{r_Q^{n-\alpha}}.
\]

Suppose now that (2.5) with $\mathcal{K}(Q) = K(Q)$ holds. Let $Q \in \mathcal{D}$, $x \in Q$ and $r_Q = 2^{-k}$, where, as before, for any $l \geq 0$, $2^{-l}Q$ is the unique cube in $\mathcal{D}$ satisfying $x \in 2^{-l}Q$ and $r_{2^{-l}Q} = 2^{-l}r_Q$. We then have:
\[
C \frac{\sigma(Q)}{r_Q^{n-\alpha}} \geq \sum_{l \geq 0} \frac{\sigma(2^{-l}Q)}{r_{2^{-l}Q}^{n-\alpha}} = \sum_{l \geq 0} \frac{\sigma(2^{-l}Q)}{r_{2^{-l}Q}^{n-\alpha}} \geq \ldots \geq \left( \frac{1}{C} + 1 \right) \sum_{l \geq m} \frac{\sigma(2^{-l}Q)}{r_{2^{-l}Q}^{n-\alpha}} \geq \left( \frac{1}{C} + 1 \right)^m \frac{\sigma(2^{-m}Q)}{r_{2^{-m}Q}^{n-\alpha}}.
\]
Since $r_{2^{-m}Q} = 2^{-m}r_Q$, it follows that
\[
\sigma(Q) \geq \frac{1}{C} \left( \frac{1}{C} + 1 \right) 2^{n-\alpha} \sigma(2^{-m}Q),
\]
and (2.6) holds with $\gamma = n-\alpha + \log_2 \left( \frac{1}{C} + 1 \right)$.

**Theorem 2.5.** Let $K : \mathcal{D} \to \mathbb{R}^+$. Let $\mu$ and $\sigma$ be locally finite positive Borel measures on $\mathbb{R}^n$, $1 < q < p < +\infty$. We then have:
(a) If there exists $C > 0$ such that for any $f \geq 0$,
\[
\left( \int_{\mathbb{R}^n} (T_{K_{\mathcal{D}}} f)(x)^q \, d\mu(x) \right)^{\frac{1}{q}} \leq C \| f \|_p,
\]
then $\mathcal{W}^p_{K,\sigma} [\mu] = \sum_{Q \in \mathcal{D}} \sigma(Q) K(Q) \chi_Q \left( \int_Q \mathcal{K}(Q)(y) \, d\mu(y) \right)^{p'-1} \in L^{\frac{q(p-1)}{p-q}} (d\mu)$.

(b) Conversely, suppose that in addition $(K, \sigma) \in \text{DLBO}$. If $\mathcal{W}^p_{K,\sigma} [\mu] \in L^{\frac{q(p-1)}{p-q}} (d\mu)$, then (2.7) holds.

**Proof of Theorem 2.5.**
Duality gives an equivalent reformulation of (2.7), namely,
\[
\| T_{K_{\mathcal{D}}} [g \mu] \|_{L^{p'}} \leq C \| g \|_{L^{p'}(d\mu)},
\]
for any \( g \in L^d(d\mu), \, g \geq 0. \)

Theorem 5.1 applied to the positive measures \( gd\mu \) and \( \sigma \), gives:

\[
||T_{K\sigma}[gd\mu]||_{L^p(d\sigma)}^{\prime} = C \sum_{Q \in D} K(Q) \sigma(Q) \int_Q g(x) \, d\mu(x) \left( \int_Q \mathcal{R}(Q)(x) \, d\mu(x) \right)^{p'-1}.
\]

Assume that (2.7) (or equivalently (2.8)) holds. We then have that \( f \in L^q(d\mu), \, f \geq 0, \)

\[
\sum_{Q \in D} K(Q) \sigma(Q) \int_Q g(x) \, d\mu(x) \left( \int_Q \mathcal{R}(Q)(x) \, d\mu(x) \right)^{p'-1} \leq C ||g||_{L^q(d\mu)}^{\prime}.
\]

Let \( c_Q = K(Q) \sigma(Q) \mu(Q) \left( \int_Q \mathcal{R}(Q)(x) \, d\mu(x) \right)^{p'-1}. \) For \( \psi \in L^{\frac{q}{p}}(d\mu), \psi \geq 0, \) let

\[
g(x) = \left( M_{\mu, D}^H[\psi] \right)^{1/p'}(x) := \left( \sup_{Q \in D} \frac{1}{\mu(Q)} \int_Q \psi(y) \, d\mu(y) \right)^{1/p'}.
\]

The above estimate together with the \( L^q(d\mu) \)-boundedness of the dyadic Hardy-Littlewood maximal function \( M_{\mu, D}^H \), gives

\[
\sum_{Q \in D} c_Q \int_Q \psi(x) \, d\mu(x) \leq ||g||_{L^q(d\mu)}^{\prime} \leq C \psi \in L^{\frac{q}{p}}(d\mu).
\]

Using duality again we get:

\[
\mathcal{W}_{K, \sigma}^D[\mu] = \sum_{Q \in D} c_Q \left( \frac{\mu(Q)}{\mu(Q)} \right) \chi_Q \in L^{\frac{q(p-1)}{p-q}}(d\mu),
\]

which is condition (b).

Next we show part (b). Theorem 5.1 and (2.8) give that (2.7) holds if for any \( g \in L^q(d\mu), \, g \geq 0, \)

\[
\int_{\mathbb{R}^n} \mathcal{W}_{K, \sigma}^D[g \, d\mu](x) \, g(x) \, d\mu(x) \simeq \int_{\mathbb{R}^n} T_{K\sigma}[gd\mu]^{p'}(x) \, d\sigma(x) \leq C ||g||_{L^q(d\mu)}^{p'}.
\]

Since \((K, \sigma) \in DLBO, \) we have:

\[
\mathcal{W}_{K, \sigma}^D[g \, d\mu](x) \leq \left( M_{\mu, D}^{H, d}[g](x) \right)^{p'-1} \mathcal{W}_{K, \sigma}^D[\mu](x).
\]

Hölder’s inequality with exponent \( r = \frac{q'}{p'-1}, \) gives:

\[
\int_{\mathbb{R}^n} \mathcal{W}_{K, \sigma}^D[g \, d\mu](x) \, g(x) \, d\mu(x) \leq C \left( \int_{\mathbb{R}^n} \left( M_{\mu, D}^{H, d}[g](x) \right)^{q'} \, dx \right)^{r} \times \left( \int_{\mathbb{R}^n} (g(x) \mathcal{W}_{K, \sigma}^D[\mu](x))^{\prime'} \, d\mu(x) \right)^{1/r'}.
\]
Using now the boundedness of \( M^{H_{\mu,D}} \) on \( L^q(d\mu) \) and Hölder’s inequality with \( \lambda = \frac{q}{p} > 1 \) for the last integral, we see that the above integral is bounded by
\[
C \|g\|_{L^q(d\mu)}^{p'} \left( \int_{\mathbb{R}^n} W_{K,\sigma}[\mu](x)^{1/q'} \, d\mu(x) \right)^{1/q'}.
\]

Since \( r'\lambda' = \frac{a(p-1)}{p-q} \), the preceding estimate gives (2.11).

**Corollary 2.6.** Let \( K : \mathcal{D} \to \mathbb{R}^+ \). Let \( \mu \) and \( \sigma \) be locally finite positive Borel measures on \( \mathbb{R}^n \), \( 1 < q < p < +\infty \). Suppose that \((K,\sigma) \in \text{DLBO}\). Then the following statements are equivalent:

(a) There exists \( C > 0 \) such that for any \( f \geq 0 \),
\[
\left( \int_{\mathbb{R}^n} (T_{K,p}[f](x))^q \, d\mu(x) \right)^{1/q} \leq C \|f\|_p.
\]

(b) \( W_{K,\sigma}[\mu] = \sum_{Q \in \mathcal{D}} \sigma(Q)K(Q)\chi_Q \left( \int_Q K(Q) \, d\mu(y) \right)^{p' - 1} \in L^\frac{a(p-1)}{p-q}(d\mu). \)

In the general situation without assuming that \((K,\sigma) \in \text{DLBO}\), we can give some sufficient conditions in order that the trace inequality hold. We need to introduce another Wolff-type potential,
\[
(2.9) \quad \overline{W}_{K,\sigma}[\mu](x) = \sum_{Q \in \mathcal{D}} \sigma(Q)\overline{K}(Q) \left( \int_Q \overline{K}(Q) \, d\mu(y) \right)^{p' - 1}.
\]

Obviously \( K(Q)\chi_Q(x) \leq \overline{K}(Q)(x) \), so that for any \( x \in \mathbb{R}^n \), we have \( W_{K,\sigma}[\mu](x) \leq \overline{W}_{K,\sigma}[\mu](x) \). We also observe that when \( d\sigma = dx \) is Lebesgue measure on \( \mathbb{R}^n \) and \( \overline{K}(Q) = \frac{1}{r^\alpha}Q^{-\alpha} \) is a discrete Riesz kernel, then \( W_{K,\sigma}[\mu] \simeq \overline{W}_{K,\sigma}[\mu] \).

**Theorem 2.7.** Let \( 1 \leq q < p < +\infty \), and let \( \mu \) and \( \sigma \) be locally finite positive Borel measures on \( \mathbb{R}^n \). If \( \overline{W}_{K,\sigma}[\mu] \in L^\frac{a(p-1)}{p-q}(d\mu) \), then there exists \( C > 0 \) such that for any \( f \in L^p(d\sigma) \), \( f \geq 0 \),
\[
\int_{\mathbb{R}^n} (T_{K,p}[f \, d\sigma])^q \, d\mu \leq C \|f\|_{L^p(d\sigma)}^q.
\]

**Proof of Theorem 2.7**

We can assume that \( q > 1 \) since in Corollary 2.3 it was proved that for \( q = 1 \) the condition \( W_{K,\sigma}[\mu] \in L^1(d\mu) \) is necessary and sufficient for the trace inequality to hold, and as we have already observed \( W_{K,\sigma}[\mu] \leq \overline{W}_{K,\sigma}[\mu] \). Then \( \overline{W}_{K,\sigma}[\mu] \in L^\frac{a(p-1)}{p-q}(d\mu) \) is equivalent by duality to the fact that there exists \( C > 0 \) such that for any \( g \in L^\frac{q'}{p'}(d\mu) \), \( g \geq 0 \),
\[
(2.10) \quad \sum_{Q \in \mathcal{D}} \sigma(Q) \left( \int_Q \overline{K}(Q) \, d\mu(y) \right)^{p' - 1} \int_Q \overline{K}(Q)(y) g(y) \, d\mu(y) \leq C \|g\|_{L^\frac{q'}{p'}(d\mu)}^q.
\]
Next, let \( \varphi \in L^d(d\mu), \varphi \geq 0. \) Theorem 2.4 and the estimate \( W^D_{K,\sigma}[^\mu] \leq W^D_{K,\sigma}[^\mu] \) give that

\[
\int_{\mathbb{R}^n} (T_{K,\sigma}[\varphi d\mu](x))^{p'} d\sigma(x) \leq C \int_{\mathbb{R}^n} \mathcal{W}^D_{K,\sigma}[\varphi d\mu](x) \varphi(x) d\mu(x) \\
= C \sum_{Q \in D} \sigma(Q) \left( \int_Q \mathcal{K}(Q)(y) \varphi(y) d\mu(y) \right)^{p'}.
\]

Applying Hölder’s inequality and (2.10), we obtain:

\[
\int_{\mathbb{R}^n} (T_{K,\sigma}[\varphi d\mu](x))^{p'} d\sigma(x) \\
\leq C \sum_{Q \in D} \sigma(Q) \left( \int_Q \mathcal{K}(Q)(y) d\mu(y) \right)^{\frac{p'}{p}} \int_Q \mathcal{K}(Q)(y) \varphi(y)^{p'} d\mu(y) \\
\leq C \|\varphi\|^p_{L_{p'}(d\mu)} = C \|\varphi\|^p_{L_{p'}(d\mu)}.
\]

Duality again gives that there exists \( C > 0 \) such that for any \( f \in L^p(d\sigma), \ f \geq 0, \)

\[
\int_{\mathbb{R}^n} (T_{K,\sigma}[fd\sigma](x))^q d\mu(x) \leq C \|f\|^q_{L^p(d\sigma)}, \quad \square
\]

**Remark.** The condition \( W^D_{K,\sigma}[^\mu] \) is not necessary in general, as the following example shows for the case \( q = 1 \) and \( p = 2. \) Let \( k(r) = \frac{1}{r^{n-1} \log^\beta(C/r)} \) for \( 0 < r \leq 1, \) and \( k(r) = 0 \) for \( r > 1, \) where \( 1 < \beta \leq \frac{3}{2} \) and \( C > 0 \) is big enough so that \( k \) is nonincreasing (more precisely, we need \( C \geq e^\beta \)). Let \( Q_0 \) be the unit cube in \( \mathbb{R}^n, \) let \( \mu \) be Lebesgue measure restricted to \( Q_0, \) and let \( \sigma \) be Lebesgue measure on \( \mathbb{R}^n. \) Then \( \sum_Q k(r_Q) \mu(Q) \chi_Q(x) \) is zero unless \( x \in Q_0, \) and if \( x \in Q_0, \)

\[
\sum_Q k(r_Q) \mu(Q) \chi_Q(x) = \sum_{l \geq 0} k\left(\frac{1}{2^l}\right) 2^{-ln} = \sum_{l \geq 0} \frac{1}{2^l \log^\beta \frac{C}{2^l}}
\]

which is convergent, since \( \beta > 1. \) Consequently

\[
\mathcal{E}_{K,\sigma,\mu}[^\mu] = \int_{\mathbb{R}^n} \left( \sum_Q k(r_Q) \mu(Q) \chi_Q(x) \right)^2 dx \\
= \int_{Q_0} \left( \sum_Q k(r_Q) \mu(Q) \chi_Q(x) \right)^2 dx < +\infty.
\]

Corollary 2.3 gives then that the trace inequality for \( q = 1 \) and \( p = 2 \) holds.

On the other hand, if \( Q \subset Q_0, \)

\[
\mathcal{K}(Q)(x) \simeq \frac{1}{r_Q^n} \int_0^{r_Q} k(t) t^{n-1} dt \simeq \frac{1}{r_Q^n \log^\beta(C/r_Q)}.
\]
for any \( x \in Q \). Hence for \( x \in Q_0 \),

\[
\overline{W}^D_{K, \sigma}[\mu](x) = \sum_{Q \in D} \sigma(Q)K(Q)(x) \left( \int_Q K(Q)(y) d\mu(y) \right) \geq \\
\sum_{x \in Q \subset Q_0} r^n_Q \left( \frac{1}{r^n_Q \log^{\beta-1} \frac{C}{r_Q}} \right)^2 \mu(Q) = \\
\sum_{l \geq 1} \frac{1}{2ln} \left( \frac{2^{ln}}{\log^{\beta-1} (C2^l)} \right)^2 \frac{1}{2ln} \geq C \sum_{l \geq 1} \frac{1}{l^{2\beta-2}} = +\infty,
\]

since \( \beta \leq \frac{3}{2} \). Consequently, \( \overline{W}^D_{K, \sigma}[\mu](x) \notin L^1(d\mu) \). \( \square \)

3. CONTINUOUS WOLFF-TYPE INEQUALITIES AND APPLICATION TO CONTINUOUS TRACE INEQUALITIES

One of our main goals in this section is to derive the continuous version of Wolff’s inequality from its discrete counterpart. We start with some definitions.

Let \( k : (0, +\infty) \rightarrow \mathbb{R}^+ \) be a nonincreasing lower semicontinuous function, and let \( \sigma \) be a positive locally finite Borel measure on \( \mathbb{R}^n \). We set

\[
k(r)(x) = \frac{1}{\sigma(B(x, r))} \int_0^r k(l) \sigma(B(x, l)) \frac{dl}{l},
\]

for \( x \in \mathbb{R}^n \), \( r > 0 \).

Our first observation is that if \( \sigma \) is a doubling measure then \( k(\cdot)(x) \) satisfies a doubling condition.

**Lemma 3.1.** If \( \sigma \) is a doubling measure then there exists \( C > 0 \) such that, for any \( x \in \mathbb{R}^n \), \( r > 0 \),

\[
\frac{1}{C} k(2r)(x) \leq k(r)(x) \leq C k(2r)(x).
\]

**Proof of Lemma 3.1.**

Since \( \sigma \) is doubling, \( \sigma(B(x, r)) \simeq \sigma(B(x, 2r)) \). Then there exists \( C > 0 \) such that for any \( x \in \mathbb{R}^n \), \( r > 0 \), \( \overline{k}(r)(x) \leq C \overline{k}(2r)(x) \). On the other hand, the change of variables \( l = 2s \), together with the fact that \( k \) is nonincreasing, yields:

\[
\overline{k}(2r)(x) = \frac{1}{\sigma(B(x, 2r))} \int_0^{2r} \frac{k(l) \sigma(B(x, l)) dl}{l} \simeq \frac{1}{\sigma(B(x, r))} \int_0^r \frac{k(2s) \sigma(B(x, 2s)) ds}{s} \leq C \overline{k}(r)(x).
\]

The following lemma gives another equivalent reformulation of the function \( \overline{k}(r)(x) \).

**Lemma 3.2.** If \( \sigma \) is a doubling measure then there exists \( C > 0 \) such that, for any \( x \in \mathbb{R}^n \), \( r > 0 \),

\[
(3.1) \quad \frac{1}{C} \overline{k}(r)(x) \leq \frac{1}{\sigma(B(x, r))} \int_{B(x,r)} k(|x-y|) d\sigma(y) \leq C \overline{k}(r)(x).
\]
Proof of Lemma 3.2:
We begin by proving the second inequality. Since \(k\) is nonincreasing and \(\sigma\) is a doubling measure, we have:

\[
\int_{B(x,r)} k(|x - y|) d\sigma(y) = \sum_{l \geq 0} \int_{B(x, \frac{r}{2^l}) \setminus B(x, \frac{r}{2^{l+1}})} k(|x - y|) d\sigma(y)
\]

\[
\leq \sum_{l \geq 0} k(\frac{r}{2^{l+1}}) \left( \sigma(B(x, \frac{r}{2^l})) - \sigma(B(x, \frac{r}{2^{l+1}})) \right) \leq \sum_{l \geq 0} k(\frac{r}{2^{l+1}}) \sigma(B(x, \frac{r}{2^l}))
\]

\[
\leq C \sum_{l \geq 0} \int_{\frac{r}{2^{l+1}}}^{\frac{r}{2^l}} k(\frac{r}{2^{l+1}}) \sigma(B(x, \frac{r}{2^l})) \frac{ds}{s}
\]

\[
\leq C \sum_{l \geq 0} \int_{\frac{r}{2^{l+1}}}^{\frac{r}{2^l}} k(s) \sigma(B(x, s)) \frac{ds}{s} \leq C \int_0^{\frac{r}{2^l}} k(s) \sigma(B(x, s)) \frac{ds}{s} \leq C \mathcal{K}(r)(x) \sigma(B(x, r)).
\]

To prove the first inequality in (3.1), we recall that since \(\sigma\) is a doubling measure, it follows that there exists \(C > 0\) such that for any \(x \in \mathbb{R}^n, \ t > 0, \ l \in \mathbb{Z},\)

\[
\sigma(B(x, \frac{r}{2^l})) \leq C \sigma \left( B(x, \frac{r}{2^l}) \setminus B(x, \frac{r}{2^{l+1}}) \right) \leq C \sigma(B(x, \frac{r}{2^l})).
\]

Consequently,

\[
\int_0^{\frac{r}{2^l}} k(s) \sigma(B(x, s)) \frac{ds}{s} = \sum_{l \geq 0} \int_{\frac{r}{2^{l+1}}}^{\frac{r}{2^l}} k(s) \sigma(B(x, s)) \frac{ds}{s}
\]

\[
\leq \sum_{l \geq 0} \int_{\frac{r}{2^{l+1}}}^{\frac{r}{2^l}} k(\frac{r}{2^{l+1}}) \sigma(B(x, \frac{r}{2^l})) \frac{ds}{s}
\]

\[
\leq C \sum_{l \geq 0} k(\frac{r}{2^{l+1}}) \sigma \left( B(x, \frac{r}{2^l}) \setminus B(x, \frac{r}{2^{l+1}}) \right) \leq C \int_{B(x, \frac{r}{2^l})} k(|x - y|) d\sigma(y). \quad \Box
\]

Lemma 3.3. Let \(k : (0, +\infty) \to \mathbb{R}^+\) be a nonincreasing lower semicontinuous function, and let \(\sigma\) be a locally finite positive Borel measure on \(\mathbb{R}^n\) satisfying a doubling condition. There exists \(C > 0\) such that if \(Q \in \mathcal{D}\) and \(x \in Q\), then

\[
\frac{1}{C} \mathcal{K}(Q)(x) \leq \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} k(r_{Q'}) \sigma(Q') \chi_{Q'}(x) \leq C \mathcal{K}(r_Q)(x).
\]

In other words, if \(\mathcal{K}(Q)(x)\) is the function associated with \(K(Q) = k(\sigma)\) and \(\sigma\) is doubling, then \(\mathcal{K}(Q)(x) \simeq \mathcal{K}(r_Q)(x)\), for \(x \in Q\).

Proof of Lemma 3.3:
Observe that \(\sigma\) satisfies a doubling condition. Hence, if \(Q \in \mathcal{D}\) and \(x \in Q\), then \(\sigma(B(x, r_Q)) \simeq \sigma(Q)\).

For any \(Q \in \mathcal{D}, \ x \in Q\) and \(l \geq 0\) there exists a unique cube \(Q_l\) in \(\mathcal{D}\) such that \(x \in Q_l, \ Q_l \subset Q\) and \(r_{Q_l} = \frac{r_Q}{2^l}\). Hence \(Q_l \subset B(x, c \frac{r_Q}{2^l})\), where \(c > 0\) is a fixed constant which depends only on \(n\).
Since \( \sigma(Q_t) \simeq \sigma(B(x, c\frac{t}{2^n})) \), we have
\[
\frac{1}{\sigma(Q)} \sum_{Q' \subset Q} k(r_{Q'}) \sigma(Q') \chi_{Q'}(x) = \\
\frac{1}{\sigma(Q)} \sum_{x \in Q' \subset Q} k(r_{Q'}) \sigma(Q') \simeq \frac{1}{\sigma(Q)} \sum_{l \geq 0} k\left(\frac{r_Q}{2^l}\right) \sigma(B(x, c\frac{r_Q}{2^l})).
\]

But since \( \sigma \) satisfies a doubling condition, \( \sigma(B(x, c\frac{r_Q}{2^l})) \simeq \sigma(B(x, \frac{r_Q}{2^l})) \) for any \( l \geq 0 \). Thus the last sum is bounded above by
\[
C \frac{1}{\sigma(B(x, r_Q))} \int_0^{r_Q} k(t) \sigma(B(x, t)) \frac{dt}{t} = Ck(r_Q)(x).
\]

On the other hand,
\[
\frac{1}{\sigma(Q)} \sum_{l \geq 0} k\left(\frac{r_Q}{2^l}\right) \sigma(B(x, c\frac{r_Q}{2^l})) \simeq k(2r_Q)(x).
\]

Since by Lemma 3.1 \( k(2r_Q)(x) \simeq k(r_Q)(x) \), we obtain the lower estimate.

As for the discrete version, we will restrict ourselves to functions \( k \) and measures \( \sigma \) satisfying an extra assumption analogous to property DLBO, namely that there exists \( C > 0 \) such that for any \( x \in \mathbb{R}^n, r > 0, \) and \( z \in B(x, r) \),
\[
(3.2) \quad \frac{1}{C} k(r)(x) \leq k(r)(z) \leq Ck(r)(x).
\]

In this case we will say that the pair \((k, \sigma)\) satisfies the property of the logarithmic bounded oscillation, or simply write \((k, \sigma) \in \text{LBO}\). The following lemma shows the relationship between the LBO and DLBO properties. The proof is an immediate consequence of Lemma 3.3.

**Lemma 3.4.** Let \( k : (0, +\infty) \to \mathbb{R}^+ \) be a nonincreasing lower semicontinuous function, and let \( \sigma \) be a locally finite positive Borel measure on \( \mathbb{R}^n \) satisfying a doubling condition. Assume that \((k, \sigma) \in \text{LBO}\). If we set \( K(Q) = k(r_Q) \), then \((K, \sigma) \in \text{DLBO}\).  

We observe that the above lemma can be refined in the following sense: if for any \( z \in \mathbb{R}^n \) and \( Q \in \mathcal{D}_z \), where \( \mathcal{D}_z \) denotes the shifted dyadic lattice \( \mathcal{D}_z = \mathcal{D} + z \), we set \( K^z(Q) = k(r_Q) \), then \((K^z, \sigma) \in \text{DLBO}\), with constants that do not depend on \( z \in \mathbb{R}^n \). We will use this observation later on. We first check that the examples that we considered in Proposition 2.4 have continuous analogues satisfying property LBO.

**Proposition 3.5.** (i) Suppose that \( d\sigma = dx \) is Lebesgue measure on \( \mathbb{R}^n \), and suppose that \( k : (0, +\infty) \to \mathbb{R}^+ \) is a nonincreasing lower semicontinuous function. Then \((k, dx) \in \text{LBO}\).

(ii) Suppose that \( 0 < \alpha < n \), and that for any \( r > 0 \), \( k(r) = \frac{1}{r^{n-\alpha}} \), i.e., \( k(|x-y|) \) is the Riesz kernel on \( \mathbb{R}^n \). Suppose \( \sigma \) is a positive doubling measure on \( \mathbb{R}^n \), and \( \sigma \in RD_\gamma \) for some \( \gamma > n - \alpha \), i.e., there exists \( C > 0 \) such that for any \( x \in \mathbb{R}^n, A > 0, r > 0 \),
\[
(3.3) \quad \sigma(B(x, Ar)) \geq C A^\gamma \sigma(B(x, r)).
\]

Then \((k, \sigma) \in \text{LBO}\).
Proof of Proposition 3.5
Statement (i) is immediate since \( \sigma(B(x, t)) \approx t^n \) if \( d\sigma = dx \) is Lebesgue measure on \( \mathbb{R}^n \). Hence
\[
\overline{k}(r)(x) = \frac{1}{r^{n}} \int_{0}^{r} k(t) t^{n-1} dt
\]
is a radial function which obviously satisfies property LBO.

Let us show (ii). If \( k(r) = \frac{1}{r^{\gamma}} \) and \( \sigma \) is a doubling measure on \( \mathbb{R}^n \) such that \( \sigma \in RD_\gamma \), with \( \gamma > n - \alpha \), then for any \( x \in \mathbb{R}^n, r > 0, \)
\[
\int_{0}^{r} \frac{1}{t^{n-\alpha}} \sigma(B(x, t)) \frac{dt}{t} = \sum_{l \geq 0} \int_{\frac{r}{2^{l+1}}}^{\frac{r}{2^{l}}} \frac{1}{t^{n-\alpha}} \sigma(B(x, t)) \frac{dt}{t} \leq C \sum_{l \geq 0} l^{\gamma(n-\alpha)} \sigma(B(x, \frac{r}{2^l})).
\]
The fact that \( \sigma \) satisfies a reverse doubling condition gives \( \sigma(B(x, \frac{r}{2^l})) \leq C \frac{C}{2^l} \sigma(B(x, r)) \), and consequently that the above sum is bounded by
\[
\left( \sum_{l \geq 0} \frac{1}{2^{l(\gamma-(n-\alpha))}} \right) \frac{\sigma(B(x, r))}{r^{n-\alpha}}.
\]
Since \( \gamma - (n - \alpha) > 0 \), we have that \( \overline{k}(r)(x) \leq C \frac{1}{r^{\gamma(n-\alpha)}} \). The fact that \( \sigma \) is a doubling measure, gives that
\[
\int_{0}^{r} \frac{1}{t^{n-\alpha}} \sigma(B(x, t)) \frac{dt}{t} \geq \int_{\frac{r}{2^{l+1}}}^{\frac{r}{2^{l}}} \frac{1}{t^{n-\alpha}} \sigma(B(x, t)) \frac{dt}{t} \geq C \frac{\sigma(B(x, r))}{r^{n-\alpha}},
\]
and consequently that \( k(r)(x) \geq C \frac{1}{r^{\gamma(n-\alpha)}} \).

Remark. Observe that the two examples considered in the above proposition, satisfy a stronger property, namely that for any \( x, z \in \mathbb{R}^n, r > 0, \overline{k}(r)(x) \approx \overline{k}(r)(z). \)

We next define a continuous Wolff-type potential. If \( x \in \mathbb{R}^n \), we consider
\[
W_{k, \sigma}[\mu](x) = \int_{0}^{+\infty} k(r) \sigma(B(x, r)) \left( \int_{B(x, r)} \overline{k}(r)(y) d\mu(y) \right)^{p'-1} \frac{dr}{r}.
\]
Observe that if \( (k, \sigma) \in \text{LBO} \), then the above expression is equivalent to
\[
W_{k, \sigma}[\mu](x) = \int_{0}^{+\infty} k(r) \sigma(B(x, r)) \overline{\kappa}(r)(x)^{p'-1} \mu(B(x, r))^{p'-1} \frac{dr}{r}.
\]
For technical reasons we will also introduce the truncated Wolff-type potentials. If \( x \in \mathbb{R}^n, R > 0, \)
\[
W_{k, \sigma}^R[\mu](x) = \int_{0}^{R} k(r) \sigma(B(x, r)) \left( \int_{B(x, r)} \overline{k}(r)(y) d\mu(y) \right)^{p'-1} \frac{dr}{r}.
\]
If \( \mu \) and \( \sigma \) are positive locally finite measures on \( \mathbb{R}^n \), and \( 1 < p < +\infty \), the energy associated with \( k \) and \( \sigma \) is given by
\[
E_{k, \sigma}[\mu] = \int_{\mathbb{R}^n} (T_k[\mu](x))^p d\sigma(x).
\]
The following proposition gives a pointwise relationship between the dyadic Wolff potential and its continuous version.

**Proposition 3.6.** Let \( k : (0, +\infty) \to \mathbb{R}^+ \) be a nonincreasing lower semicontinuous function. Let \( 1 < p < +\infty \), and let \( \sigma \) be a positive locally finite Borel measure on \( \mathbb{R}^n \). Suppose also that \( \sigma \) satisfies a doubling condition. Let \( \overline{K}(Q)(x) \) be the function associated with \( K(Q) = k(r_Q) \). Then there exist constants \( c, C > 0 \) such that for any positive locally finite Borel measure \( \mu \) on \( \mathbb{R}^n \), and \( x \in \mathbb{R}^n \),

\[
\sum_{Q \in \mathcal{D}} k(c r_Q) \sigma(Q) \chi_Q(x) \left( \int_Q \overline{K}(Q)(y) d\mu(y) \right)^{p'-1} \leq C W_{k,\sigma}[\mu](x).
\]

**Proof of Proposition 3.6.**

If \( x \in \mathbb{R}^n \) and \( l \in \mathbb{Z} \), there exists a unique cube \( Q_l \) in \( \mathcal{D} \) such that \( x \in Q_l \) and \( r_{Q_l} = 2^l \). Hence \( Q_l \subset B(x, \frac{c}{2} 2^l) \) where \( c > 0 \) is a fixed constant which depends only on \( n \). Applying Lemma 3.3 we obtain:

\[
\sum_{Q \in \mathcal{D}} k(c r_Q) \sigma(Q) \chi_Q(x) \left( \int_Q \overline{K}(r_Q)(y) d\mu(y) \right)^{p'-1} = \sum_{l \in \mathbb{Z}} k(c 2^l) \sigma(Q_l) \left( \int_{Q_l} \overline{K}(r_{Q_l})(y) d\mu(y) \right)^{p'-1} \leq C \sum_{l \in \mathbb{Z}} k(c 2^l) \sigma(B(x, \frac{c}{2} 2^l)) \left( \int_{B(x, \frac{c}{2} 2^l)} \overline{K}(2^l)(y) d\mu(y) \right)^{p'-1} \leq C \sum_{l \in \mathbb{Z}} \int_{\frac{c}{2} 2^l}^{c 2^l} k(t) \sigma(B(x, t)) \left( \int_{B(x, t)} \overline{K}(t)(y) d\mu(y) \right)^{p'-1} \frac{dt}{t} \leq C W_{k,\sigma}[\mu](x). \quad \Box
\]

We now state a continuous version of Wolff’s theorem.

**Theorem 3.7.** Let \( k : (0, +\infty) \to \mathbb{R}^+ \) be a nonincreasing lower semicontinuous function. Let \( 1 < p < +\infty \), and let \( \sigma \) be a positive locally finite Borel measure on \( \mathbb{R}^n \). Suppose that \( \sigma \) satisfies a doubling condition and that \( (k, \sigma) \in \text{LBO} \). Then for any positive Borel measure \( \mu \) on \( \mathbb{R}^n \),

\[
\mathcal{E}_{k,\sigma}[\mu] \simeq \int_{\mathbb{R}^n} W_{k,\sigma}[\mu] d\mu,
\]

with constants that may depend on \( k \) and \( \sigma \), but not on \( \mu \).

**Proof of Theorem 3.7.**

For \( R > 0 \), we define the truncated operator \( T_k^R \) by

\[
T_k^R[\mu](x) = \int_{|x-y| \leq R} k(|x - y|) d\mu(y),
\]
where $\mu$ is a positive locally finite Borel measure on $\mathbb{R}^n$. The usual Fefferman-Stein argument (see [Sa] and also Lemma 2.2 in [SaWh]) shows, using the fact that $k$ is nonincreasing, that $T_k^R[\mu](x)$ is pointwise bounded by the average of the shifted dyadic potentials $T_{K_{D_j}}[\mu](x)$ associated with $\tilde{k}(r) = k(\frac{r}{4})$ and $\tilde{K}(Q) = \tilde{k}(r_Q)$, defined by $T_{K_{D_j}}[\mu](x) = \sum_{Q \in D_k} \tilde{k}(\frac{r_Q}{2}) \mu(Q + z) \chi_{Q^+(x)}$. That is, we have that there exists $j_0 \in \mathbb{Z}^+$, $C > 0$ such that for any $j \in \mathbb{Z}$,

\begin{equation}
T_k^{2^j}[\mu](x) \leq \frac{C}{2^{jn}} \int_{|z| \leq 2^{j+j_0}} T_{K_{D_j}}[\mu](x) dz.
\end{equation}

Indeed, fix $j_0$ such that $2^{j_0} > 2\sqrt{n} + 1$. Then for $x \in B_j = B(0, 2^j)$, and $l \leq j$, we denote by $\Omega$ the set of points $z \in B_{j+j_0}$, for which there exists $Q \in D$, $r_Q = 2^{l+1}$, and $B(x, 2^l) \subset Q + z$. It is geometrically evident that

\begin{equation}
|\Omega_l| \geq C |B_{j+j_0}| \sim 2^{jn}.
\end{equation}

Next the fact that $k$ is nonincreasing gives

\begin{align*}
T_k^{2^j}[\mu](x) &= \int_{|x-y| \leq 2^j} k(|x-y|) d\mu(y) \\
&\leq \sum_{l \leq j} k(2^{l-1}) \mu(B(x, 2^l)).
\end{align*}

Applying (3.9) to $l \leq j$ and $x \in B_j$, we obtain:

\begin{align*}
\mu(B(x, 2^l)) &\leq \frac{1}{|\Omega_l|} \int_{\Omega_l} \sum_{r_Q+z=2^{l+1}} \mu(Q+z) \chi_{Q^+(x)} dz \\
&\leq \frac{C}{2^{jn}} \int_{B_{j+j_0}} \sum_{r_Q+z=2^{l+1}} \mu(Q+z) \chi_{Q^+(x)} dz.
\end{align*}

Altogether, we deduce that

\begin{align*}
T_k^{2^j}[\mu](x) &\leq \frac{C}{2^{jn}} \int_{B_{j+j_0}} \sum_{r_Q+z=2^{l+1}} k(T_Q/4) \mu(Q+z) \chi_{Q^+(x)} dz \\
&\leq \frac{C}{2^{jn}} \int_{|z| \leq 2^{j+j_0}} T_{K_{D_j}}[\mu](x) dz,
\end{align*}

which gives (3.8). Now Hölder’s inequality together with (3.8) gives that for any $R > 0$,

\begin{align*}
\int_{\mathbb{R}^n} T_k^R[\mu](x)^p \ d\sigma(x) &\leq \frac{C}{R^n} \int_{|z| \leq c R} \int_{\mathbb{R}^n} T_{K_{D_z}}[\mu]^p(x) \ d\sigma(x) \ dz \\
&\leq C \sup_z \int_{\mathbb{R}^n} T_{K_{D_z}}[\mu]^p(x) \ d\sigma(x).
\end{align*}
Applying the dyadic Wolff inequality proved in Theorem 2.1 and Lemma 3.3 we obtain that the above expression is bounded by

\[ C \sup_z \sum_{Q \in D_z} \tilde{k}(r_Q)\sigma(Q)\mu(Q) \left( \int_Q \tilde{k}(r_Q)(y)d\mu(y) \right)^{p'-1}. \]

Next, Lemma 3.1 gives that \( \tilde{k} \simeq k \) so the preceding quantity is bounded by

\[ (3.10) \quad C \sup_z \sum_{Q \in D_z} k\left(\frac{r_Q}{4}\right)\sigma(Q)\mu(Q) \left( \int_Q k(r_Q)(y)d\mu(y) \right)^{p'-1}. \]

We now have to replace \( k\left(\frac{r_Q}{4}\right) \) in the last sum by \( k\left(c r_Q\right) \), where \( c > 0 \) is the constant given in Proposition 3.6. This is justified in the following lemma.

**Lemma 3.8.** Let \( k : (0, +\infty) \to \mathbb{R}^+ \) be a nonincreasing lower semicontinuous function. Let \( \sigma \) be a locally finite positive Borel measure on \( \mathbb{R}^n \), \( 1 < p < +\infty \) and \( 0 < c < +\infty \). Suppose that \( \sigma \) satisfies a doubling condition and that \( (k, \sigma) \in LBO \). Then for any positive Borel measure \( \mu \) on \( \mathbb{R}^n \),

\[ \sum_{Q \in D} k\left(c r_Q\right)\sigma(Q)\overline{k}(r_Q)^{p'-1}\mu(Q)^{p'} \simeq \sum_{Q \in D} k(r_Q)\sigma(Q)\overline{k}(r_Q)^{p'-1}\mu(Q)^{p'}, \]

with constants that do not depend on \( \mu \).

**Proof of Lemma 3.8.**

We first observe that since \( k \) is nonincreasing it follows that if \( Q \in D \) and \( 2Q \) is the unique cube in \( D \) such that \( Q \subset 2Q \) and \( r_{2Q} = 2r_Q \), then for any \( x \in Q \),

\[ \overline{K}(2Q)(x) \simeq \overline{K}(Q)(x). \]

Indeed, if \( x \in Q \)

\[ \overline{K}(2Q)(x) = \frac{\sigma(Q)}{\sigma(2Q)} \overline{K}(Q)(x) + k(2r_Q)\chi_{2Q}(x) \leq \overline{K}(Q)(x) + k(r_Q)\chi_Q(x) \leq 2\overline{K}(Q)(x). \]

We also observe that since \( \sigma \) satisfies a doubling condition, we have that for any \( Q \in D, \sigma(Q) \simeq \sigma(2Q) \), and consequently for any \( Q \in D, x \in Q \),

\[ \overline{K}(Q)(x) = \frac{\sigma(2Q)}{\sigma(Q)} \left( \overline{K}(2Q)(x) - k(2r_Q) \right) \leq C \overline{K}(2Q)(x). \]

Next, the fact that \( (k, \sigma) \in LBO \) implies by Lemma 3.1 that \( (K, \sigma) \in DLBO \), and consequently that \( \overline{K}(2Q) \simeq \overline{K}(Q) \). Since Lemma 3.3 shows that \( \overline{K}(Q) \simeq \overline{K}(r_Q) \), we begin showing that

\[ \sum_{Q \in D} k\left(c r_Q\right)\sigma(Q)\overline{K}(Q)^{p'-1}\mu(Q)^{p'} \leq C \sum_{Q \in D} k(r_Q)\sigma(Q)\overline{K}(Q)^{p'-1}\mu(Q)^{p'}. \]

Since \( k \) is a nonincreasing function, we can assume without loss of generality that \( c = \frac{1}{2^l}, l \geq 0 \). We have

\[ \sum_{Q \in D} \sigma(Q)k\left(\frac{1}{2^l} r_Q\right)\overline{K}(Q)^{p'-1}\mu(Q)^{p'} \simeq \sum_{Q \in D} \sigma(2^l Q)k(r_Q)\overline{K}(2^l Q)^{p'-1}\mu(2^l Q)^{p'}. \]
But \( \mu(2^lQ) = \sum_{Q' \subset 2^lQ, r_{Q'} = r_Q} \mu(Q') \), where the sum is taken over all cubes \( Q' \) in \( \mathcal{D} \) that are contained in \( 2^lQ \) and such that \( r_{Q'} = r_Q \). The doubling condition imposed on \( \sigma \) gives \( \sigma(2^lQ) \simeq \sigma(Q) \simeq \sigma(Q') \), and (3.11) implies \( \overline{K}(2^lQ) \simeq \overline{K}(Q) \simeq \overline{K}(Q') \). Thus the left-hand side of (3.14) is bounded above by

\[
C \sum_{Q \in \mathcal{D}} \sigma(Q) k(r_Q) \overline{K}(Q)^{p'-1} \mu(Q)^{p'},
\]

and we have (3.13) for \( c = \frac{1}{2^l}, l \geq 0 \). The converse estimate is obvious because \( k \) is a nonincreasing function.

We now complete the proof of the theorem. Lemma 3.8 implies that we can replace \( \frac{1}{2} \) in (3.10) by any positive constant, e.g., by the constant \( c > 0 \) given in Proposition 3.6. Therefore, (3.10) is bounded by

\[
C \sup_z \sum_{Q \in \mathcal{D}_z} k(c r_Q) \sigma(Q) k(r_Q)^{p'-1} \mu(Q)^{p'}.
\]

Finally, the pointwise inequality obtained in Proposition 3.6 gives that the above expression is bounded by \( C \int_{\mathbb{R}^n} \mathcal{W}_{k, \sigma}[\mu](x) d\mu(x) \), which shows that

\[
\mathcal{E}_{k, \sigma}[\mu] \leq C \int_{\mathbb{R}^n} \mathcal{W}_{k, \sigma}[\mu](x) d\mu(x).
\]

The converse estimate is proved in the following lemma.

**Lemma 3.9.** Let \( k : (0, +\infty) \rightarrow \mathbb{R}^+ \) be a nonincreasing lower semicontinuous function. Let \( 1 < p < +\infty \), and let \( \sigma \) be positive locally finite Borel measure on \( \mathbb{R}^n \). Suppose that \( (k, \sigma) \in \text{LBO} \). Then there exists \( C > 0 \) such that for any positive Borel measure \( \mu \) on \( \mathbb{R}^n \),

\[
(3.15) \quad \mathcal{E}_{k, \sigma}[\mu] \geq C \int_{\mathbb{R}^n} \mathcal{W}_{k, \sigma}[\mu](x) d\mu(x).
\]

**Proof of Lemma 3.9:**

Let \( M_{\sigma}^{\text{HL}} \) denote the centered Hardy-Littlewood maximal function with respect to the measure \( \sigma \). We have

\[
(3.16) \quad M_{\sigma}^{\text{HL}}[T_k[\mu]](x) \geq C M_k[\mu](x),
\]

where \( M_k \) is the following maximal function associated with the kernel \( \bar{k} \):

\[
M_k[\mu](x) = \sup_{r > 0} \bar{k}(r)(x) \mu(B(x, r)).
\]
Indeed, Fubini’s theorem gives
\[
M_{\sigma}^{\text{HL}}[T_k[\mu]](x) \geq \frac{1}{\sigma(B(x, 2r))} \int_{B(x, 2r)} T_k[\mu](y) \, d\sigma(y)
\]
\[
\geq \frac{C}{\sigma(B(x, r))} \int_{B(x, r)} \left( \int_{B(x, r)} k(|y - z|) \, d\mu(z) \right) \, d\sigma(y)
\]
\[
\geq \frac{C}{\sigma(B(x, r))} \int_{B(x, r)} k(|y - z|) \, d\mu(z)
\]
\[
\geq \frac{C}{\sigma(B(x, r))} \int_{B(x, r)} \overline{k}(r)(z) \, \sigma(B(z, r)) \, d\mu(z).
\]

Since \((k, \sigma) \in \text{LBO}\), we can replace \(\overline{k}(r)(z)\) in the last integral by \(\overline{k}(r)(x)\). Next, the fact that \(\sigma\) is a doubling measure gives that the above integral is bounded from below by \(\overline{k}(r)(x) \mu(B(x, r))\), and consequently that \(M_{\sigma}^{\text{HL}}[T_k[\mu]](x) \geq C M_k[\mu](x)\).

Since \(M_{\sigma}^{\text{HL}}\) is a bounded operator on \(L^{p'}(d\sigma)\), and \(M_{\sigma}^{\text{HL}}[T_k[\mu]] \geq T_k[\mu]\) a.e. (with respect to \(d\sigma\)), Fubini’s theorem gives
\[
||T_k[\mu]||^{p'}_{L^{p'}(d\sigma)} \geq C ||M_{\sigma}^{\text{HL}}[T_k[\mu]]||^{p'}_{L^{p'}(d\sigma)}
\]
\[
= C \int_{\mathbb{R}^n} M_{\sigma}^{\text{HL}}[T_k[\mu]](x) \left( M_{\sigma}^{\text{HL}}[T_k[\mu]] \right)^{p'-1}(x) \, d\sigma(x)
\]
\[
\geq C \int_{\mathbb{R}^n} T_k[\left( M_{\sigma}^{\text{HL}}[T_k[\mu]] \right)^{p'-1}](y) \, d\mu(y).
\]

But
\[
T_k[\left( M_{\sigma}^{\text{HL}}[T_k[\mu]] \right)^{p'-1}](y) = \int_{\mathbb{R}^n} k(|x - y|) \left( M_{\sigma}^{\text{HL}}[T_k[\mu]] \right)^{p'-1}(x) \, d\sigma(x)
\]
\[
= \sum_{l \in \mathbb{Z}} \int_{\frac{1}{2^{l+1}} \leq |x - y| < \frac{1}{2^l}} k(|x - y|) \left( M_{\sigma}^{\text{HL}}[T_k[\mu]] \right)^{p'-1}(x) \, d\sigma(x)
\]
\[
\geq \sum_{l \in \mathbb{Z}} \int_{\frac{1}{2^{l+1}} \leq |x - y| < \frac{1}{2^l}} \left( \frac{1}{2^{l}} \right) \left( M_{\sigma}^{\text{HL}}[T_k[\mu]] \right)^{p'-1}(x) \, d\sigma(x).
\]

Next (5.10) shows that \(M_{\sigma}^{\text{HL}}[T_k[\mu]](x) \geq C \overline{k}(\frac{1}{2^{l+1}})(x) \mu(B(x, \frac{1}{2^l}))\). Since for any \(y \in \mathbb{R}^n\) such that \(\frac{1}{2^{l+1}} \leq |x - y| < \frac{1}{2^l}\) we have \(B(y, \frac{1}{2^{l+2}}) \subset B(x, \frac{1}{2^l})\), and by (5.2) \(\overline{k}(\frac{1}{2^{l+1}})(x) \simeq \overline{k}(\frac{1}{2^{l+1}})(y)\) it follows that the above sum is bounded from below by
\[
C \sum_{l \in \mathbb{Z}} \left( \frac{1}{2^{l}} \right) \overline{k}(\frac{1}{2^{l+2}})(y)^{p'-1} \mu(B(y, \frac{1}{2^{l+2}}))^{p'-1} \sigma\left( \{ x \in \mathbb{R}^n : \frac{1}{2^{l+1}} \leq |x - y| < \frac{1}{2^l} \} \right).
\]

The fact that \(\sigma\) satisfies a doubling condition gives, as we have already pointed out earlier, that \(\sigma(\{ x \in \mathbb{R}^n : \frac{1}{2^{l+1}} \leq |x - y| < \frac{1}{2^l} \}) \simeq \sigma(B(y, \frac{1}{2^l}))\). Altogether we have that
the above sum is in its turn bounded from below by
\[
C \sum_{l \in \mathbb{Z}} k\left(\frac{1}{2}\right) \overline{k}\left(\frac{1}{2r-2}\right)(y)^{p'-1} \mu(B(y, \frac{1}{2r-1}))^{p'-1} \sigma(B(y, \frac{1}{2r}))
\]
\[
\geq C \sum_{l \in \mathbb{Z}} \int_{\frac{1}{2}}^{\frac{1}{2r-1}} k\left(\frac{1}{2}\right) \overline{k}\left(\frac{1}{2r-2}\right)(y)^{p'-1} \mu(B(y, \frac{1}{2r-1}))^{p'-1} \sigma(B(y, \frac{1}{2r})) \frac{dt}{t}
\]
\[
\geq C \sum_{l \in \mathbb{Z}} \int_{\frac{1}{2}}^{\frac{1}{2r-1}} k(t) \overline{k}(t)(y)^{p'-1} \mu(B(y, t))^{p'-1} \sigma(B(y, t)) \frac{dt}{t}
\]
Thus
\[
||T_k[\mu]||_{L^{p'}(d\sigma)} \geq C \int_{\mathbb{R}^n} \int_{0}^{+\infty} k(t) \overline{k}(t)(y)^{p'-1} \mu(B(y, t))^{p'-1} \sigma(B(y, t)) \frac{dt}{t} d\mu(y). \quad \square
\]

**Remark.** As in the dyadic case, Theorem 3.7 is no longer true with the nonlinear potential $\overline{\mathcal{W}}_{k, \sigma}[\mu]$ in place of $\mathcal{W}_{k, \sigma}[\mu]$ even in the case $d\sigma = dx$ when
\[
\overline{\mathcal{W}}_{k, dx}[\mu](x) = \int_{0}^{+\infty} r^n \overline{\mathcal{W}}(r)^{p'} \mu(B(x, r))^{p'-1} dr
\]
and $\overline{\mathcal{W}}(r) = r^{-n} \int_{0}^{r} k(s) s^{-n} ds$.

Indeed, let $q = 1$ and $p = 2$, and let $d\mu = \chi_{B_0} dx$ where $B_0 = B(0, 1)$ is the unit ball in $\mathbb{R}^n$. Then for $k(r) = r^{-n} \log\beta(C/r)$ if $0 < r < 1$, and $k(r) = 0$ if $r \geq 1$, where $1 < \beta \leq \frac{1}{2}$ and $C \geq e^{\beta/n}$, one has as in the example at the end of Sec. 2 that $\mathcal{W}_{k, dx}[\mu]$ is uniformly bounded and hence by Theorem 3.7 $E_{k, dx}[\mu] < +\infty$. On the other hand, $\overline{\mathcal{W}}_{k, dx}[\mu] \equiv +\infty$ on $B_0$ and so $\int_{\mathbb{R}^n} \overline{\mathcal{W}}_{k, dx}[\mu] d\mu = +\infty$. \quad \square

As in the discrete case, the continuous Wolff-type theorem that we have just proved yields a characterization of the corresponding trace inequality.

**Corollary 3.10.** Let $k : (0, +\infty) \to \mathbb{R}^+$ be a nonincreasing lower semicontinuous function. Let $1 < p < +\infty$, and let $\mu$ and $\sigma$ be positive locally finite Borel measures on $\mathbb{R}^n$. Suppose also that $\sigma$ satisfies a doubling condition and that $(k, \sigma) \in \text{LBO}$. Then the following assertions are equivalent:

(i) There exists $C > 0$ such that for any $f \in L^p(d\sigma)$, $f \geq 0$,
\[
\int_{\mathbb{R}^n} T_k[f] d\mu \leq C ||f||_p.
\]

(ii) If $\mathcal{W}^D_{k, \sigma}[\mu]$ are the dyadic shifted potentials defined in (2.2),
\[
\sup_{z \in \mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{W}^D_{k, \sigma}[\mu] d\mu < +\infty.
\]

(iii) $\mathcal{W}_{k, \sigma}[\mu] \in L^1(d\mu)$.

**Proof of Corollary 3.10**

Using Fubini’s theorem and duality it is easy to see that (i) is equivalent to the fact that $||T_k[\mu]||_{L^{p'}(d\sigma)} < +\infty$. By Theorem 3.7 this is in turn equivalent to $\mathcal{W}_{k, \sigma}[\mu] \in L^1(d\mu)$ which gives the equivalence of (i) and (iii).
Next, Proposition 3.6 together with Lemma 3.4 shows that there exists \( c > 0 \) such that for any \( z \in \mathbb{R}^n \),
\[
\sum_{Q \in \mathcal{D}_z} k(cr_Q)\sigma(Q)K(Q)^{p' - 1}\mu(Q)^{p'} \leq C \int_{\mathbb{R}^n} W_{k,\sigma}[\mu](x) d\mu(x).
\]

Lemma 3.8 gives that the constant \( c \) in the above sum on the left-hand side can be dropped, and consequently, we have that (iii) \( \Rightarrow \) (ii).

For the last implication, (ii) \( \Rightarrow \) (i), we proceed as in the proof of Theorem 3.7, using the fact that the truncated operator \( T_{K,\sigma} \) is pointwise bounded by the average of the shifted dyadic potentials \( T_{K,\sigma} \). Hölder’s inequality gives then that for any \( R > 0 \),
\[
\int_{\mathbb{R}^n} T_{K,\sigma}^R[\mu](x)^{p'} d\sigma(x) \leq C \sup_z \int_{\mathbb{R}^n} T_{K,\sigma}^R[\mu](x)^{p'} d\sigma(x).
\]

Applying the dyadic Wolff inequality established in Theorem 2.1 we obtain that the above expression can be bounded by
\[
C \sup_z \sum_{Q \in \mathcal{D}_z} \tilde{k}(r_Q)\sigma(Q)\mu(Q) \left( \int_{Q} \tilde{k}(r_Q)(y) d\mu(y) \right)^{p' - 1}.
\]

Since \( \tilde{k}(\cdot) \) satisfies a doubling condition, and by Lemma 3.8 we can replace \( \tilde{k}(r_Q) \) by \( k(r_Q) \), the fact that (ii) holds gives that \( \int_{\mathbb{R}^n} T_{K,\sigma}^R[\mu](x)^{p'} d\sigma(x) \leq C \) for any \( R > 0 \). Lebesgue’s monotone convergence theorem finally gives (i).

We now consider the trace inequality for \( q \geq 1 \).

**Theorem 3.11.** Let \( k : (0, +\infty) \rightarrow \mathbb{R}^+ \) be a nonincreasing lower semicontinuous function, \( 1 \leq q < p < +\infty \), and \( \mu \) and \( \sigma \) be positive locally finite Borel measures on \( \mathbb{R}^n \). Suppose also that \( \sigma \) satisfies a doubling condition and that \( (k,\sigma) \in \text{LBO} \). Then the following assertions are equivalent:

(i) There exists \( C > 0 \) such that for any \( f \in L^p(d\sigma) \), \( f \geq 0 \),
\[
\left( \int_{\mathbb{R}^n} T_k[f]^q d\mu \right)^{\frac{1}{q}} \leq C \|f\|_p.
\]

(ii) \( \sup_{z \in \mathbb{R}^n} \int_{\mathbb{R}^n} W_{k,\sigma}^{D_z}[\mu]^{\frac{2(p-1)}{p-q}} d\mu < +\infty \).

(iii) \( \int_{\mathbb{R}^n} W_{k,\sigma}[\mu]^{\frac{2(p-1)}{p-q}} d\mu < +\infty \).

**Proof of Theorem 3.11**

By the last corollary, we may assume that \( q > 1 \). We begin by showing that (i) implies (ii). We observe that Theorem 3.7 applied to the positive measure \( gd\mu \) gives that
\[
\|T_k[gd\mu]\|_{L^p(d\sigma)}^{p'} \simeq \int_{\mathbb{R}^n} W_{K,\sigma}[gd\mu](x)g(x) d\mu(x),
\]
with constants independent of \( g \) and \( \mu \). But Proposition 3.6 together with Lemma 3.8 applied to the shifted lattice \( \mathcal{D}_z \) show that the above integral is bounded from below.
by
\[ C \sup_{x \in \mathbb{R}^n} \sum_{Q \in D_k} k(r_Q)\sigma(Q)\mathcal{K}(Q)^{p'-1} \left( \int_Q g(x) \, d\mu(x) \right)^{p'} . \]

Now we can proceed as in the proof of (a)⇒(b) in Theorem 2.3 and obtain that (i)⇒(ii).

Before we present the rest of the proof, we need an estimate similar to (3.8) for the average of \( W_{K,\sigma}^p \) over the shifts of the dyadic lattice \( D \). Recall that, for \( R > 0, x \in \mathbb{R}^n \), we have
\[ W_{K,\sigma}^R[\mu](x) = \int_0^R k(r) \sigma(B(x, r)) \left( \int_{B(x,r)} \mathcal{K}(y) \, d\mu(y) \right)^{p'-1} \frac{dr}{r} . \]

Since \((k, \sigma) \in \text{LBO}\) it follows that the truncated Wolff type potential can be rewritten in the equivalent form:
\[ W_{K,\sigma}^R[\mu](x) = \int_0^R k(r) \sigma(B(x, r)) \mathcal{K}(r)^{p'-1}(x) \mu(B(x, r))^{p'-1} \frac{dr}{r} . \]

We then have that there exists \( j_0 \in \mathbb{Z}^+ \) and \( C > 0 \) such that for any \( j \in \mathbb{Z}, x \in B_{j} = B(0, 2^j) \),
\[ (3.17) \quad W_{K,\sigma}^{2^j}[\mu](x) \leq \frac{C}{|B_{j+j_0}|} \int_{B_{j+j_0}} W_{K,\sigma}^{2^j}[\mu](x) \, dz, \]
where \( \mathcal{K}(t) = k(\frac{t}{2^j}) \). The proof follows that of (3.8) in Theorem 3.7. With the notations used there, fix \( j_0 \) such that \( 2^{j_0} > 2\sqrt{n} + 1 \). Then for \( j \in \mathbb{Z}, x \in B_{j} = B(0, 2^j) \), and \( l \leq j \), \( \Omega_l \) is the set of points \( z \in B_{j+j_0} \), for which there exists \( Q \in D, r_Q = 2^{l+1} \), and \( I = B(x, 2^l) \subset Q + z \). We recall that by (3.9), \( |\Omega_l| \approx |B_{j+j_0}| \approx 2^{jn} \). The fact that \( \mathcal{K}(\cdot) \) satisfies a doubling condition gives
\[
W_{K,\sigma}^{2^j}[\mu](x) = \int_0^{2^j} k(r) \sigma(B(x, r)) \left( \int_{B(x,r)} \mathcal{K}(r)^{p'-1}(y) \, d\mu(y) \right)^{p'-1} \frac{dr}{r} \\
= \sum_{l \leq j} \int_{2^{l-1}}^{2^l} k(r) \sigma(B(x, r)) \left( \int_{B(x,r)} \mathcal{K}(r)^{p'-1}(y) \, d\mu(y) \right)^{p'-1} \frac{dr}{r} \\
\leq C \sum_{l \leq j} \sigma(B(x, 2^l)) k(2^{l-1}) \mathcal{K}(2^{l})(x)^{p'-1} \mu(B(x, 2^l))^{p'-1}.
\]

Applying (3.9) to \( l \leq j \) and \( x \in B_{j} \), we conclude that
\[
\mu(B(x, 2^j))^{p'-1} \leq \frac{1}{|\Omega_l|} \int_{\Omega_l} \sum_{r_Q + z = 2^{l+1}, B(x, 2^l) \subset Q + z} \mu(Q + z)^{p'-1} \chi_{Q+z}(x) \, dz \\
\leq \frac{C}{|B_{j+j_0}|} \int_{B_{j+j_0}} \sum_{r_Q + z = 2^{l+1}, B(x, 2^l) \subset Q + z} \mu(Q + z)^{p'-1} \chi_{Q+z}(x) \, dz.
\]
Hence, if \( x \in B_j \),
\[
\mathcal{W}_{K, \sigma}^{\partial z} [\mu](x) \leq \frac{C}{|B_{j+1}|} \int_{B_{j+1}} \sum_{l < j} \sum_{Q \in z = 2^{l+2}} \sigma(Q + z) k r Q (r Q)(x)^{p'-1} \times \mu(Q + z)^{p'-1} \chi_{Q+z}(x) dz \leq \frac{C}{|B_{j+1}|} \int_{B_{j+1}} \mathcal{W}_{K, \sigma}^{\partial z} [\mu](x) dz.
\]

In the last inequality we have used the estimate \( \tilde{k}(\cdot) \simeq \overline{k}(\cdot) \) which follows from the fact that \( k(\cdot) \) satisfies a doubling condition.

Now we can complete the proof of (ii)\( \Rightarrow \) (i). Duality and Theorem 3.7 gives that (i) holds if for any \( g \in L^{q'}(d\mu), \) \( g \geq 0, \)
\[
\int_{\mathbb{R}^n} \mathcal{W}_{k, \sigma}[gd\mu](x) g(x) d\mu(x) \simeq \int_{\mathbb{R}^n} T_k[gd\mu]^{p'}(x) d\sigma(x) \leq C ||g||_{L^{q'}(d\mu)}^{p'}.
\]

Now, we consider the translated dyadic Hardy-Littlewood maximal function with respect to \( \mu \) given by
\[
M^{\partial z}_{\mu, L} h(x) = \sup_{x \in Q + z \in \mathcal{D}} \frac{1}{\mu(Q + z)} \int_{Q+z} |h(y)| d\mu(y).
\]
We have that
\[
\mathcal{W}_{K, \sigma}^{\partial z} [gd\mu](x) \leq M^{\partial z}_{\mu, L} g(x)^{p'-1} \mathcal{W}_{K, \sigma}^{\partial z} [\mu](x).
\]
Hölder’s inequality with exponent \( r = \frac{q'}{p'-1} \), gives
\[
\int_{\mathbb{R}^n} \mathcal{W}_{K, \sigma}^{\partial z} [gd\mu](x) g(x) d\mu(x) \leq C \left( \int_{\mathbb{R}^n} M^{\partial z}_{\mu, L} g(x)^{q'} d\mu(x) \right)^{\frac{1}{q'}} \left( \int_{\mathbb{R}^n} \mathcal{W}_{K, \sigma}^{\partial z} [\mu](x) g(x)^{p'} d\mu(x) \right)^{\frac{1}{p'}}.
\]

Using the fact that \( M^{\partial z}_{\mu, L} \) is a bounded operator on \( L^{q'}(d\mu), \) and Hölder’s inequality with \( \lambda = \frac{q'}{p'} > 1, \) we have that the above integral is bounded by
\[
C ||g||_{L^{q'}(d\mu)}^{q'} \left( \int_{\mathbb{R}^n} \mathcal{W}_{K, \sigma}^{\partial z} [\mu](x)^{r' \lambda'} d\mu(x) \right)^{\frac{1}{r' \lambda'}}.
\]
Since \( r' \lambda' = \frac{q(p-1)}{p-q}, \) and we are assuming that (ii) holds, the last estimate and (3.17) easily give that
\[
\int_{B_j} \mathcal{W}_{k, \sigma}^{\partial z} [gd\mu](x) g(x) d\mu(x) \leq C ||g||_{L^{q'}(d\mu)}^{p'}
\]

once we show that in the expression
\[
\int_{\mathbb{R}^n} \mathcal{W}_{K, \sigma}^{\partial z} [\mu](x)^{r' \lambda'} d\mu(x),
\]
we can replace \( \tilde{K} \) by \( K \). This is proved in the following lemma.
Lemma 3.12. Let $k : (0, +\infty) \to \mathbb{R}^+$ be a nonincreasing lower semicontinuous function. Let $\sigma$ be locally finite positive Borel measure on $\mathbb{R}^n$, and let $1 < p, r < +\infty$. Assume that $\sigma$ satisfies a doubling condition and that $(k, \sigma) \in \operatorname{LBO}$. Then for any $c > 0$ there exists $C > 0$ such that for any positive Borel measure $\mu$ on $\mathbb{R}^n$,

$$
\frac{1}{C} \int_{\mathbb{R}^n} \left( \sum_{Q \in D} k(c r_Q) \sigma(Q) \overline{K}(Q)^{p'-1} \mu(Q)^{p'-1} \chi_Q(x) \right)^r d\mu(x)
$$

$$
\leq \int_{\mathbb{R}^n} \left( \sum_{Q \in D} k(r_Q) \sigma(Q) \overline{K}(Q)^{p'-1} \mu(Q)^{p'-1} \chi_Q(x) \right)^r d\mu(x)
$$

$$
\leq C \int_{\mathbb{R}^n} \left( \sum_{Q \in D} k(c r_Q) \sigma(Q) \overline{K}(Q)^{p'-1} \mu(Q)^{p'-1} \chi_Q(x) \right)^r d\mu(x).
$$

Proof of Lemma 3.12

Note that in the case $r = 1$ this lemma coincides with Lemma 3.8. Since $k$ is nonincreasing, we can assume without loss of generality that $c = \frac{1}{2^l}$, $l \geq 0$, and write $k_l(r)(x) = k\left(\frac{1}{2^l} r\right)(x)$, and $K_l(Q)(x) = k_l(r_Q)(x)$. The upper estimate is obvious because $l \geq 0$ and $k$ is nonincreasing. The lower estimate can be restated equivalently in the form

$$
(3.20) \quad \int_{\mathbb{R}^n} (\mathcal{W}^D_{K_l, \sigma}[\mu](x))^r d\mu(x) \leq C \int_{\mathbb{R}^n} (\mathcal{W}^D_{K, \sigma}[\mu](x))^r d\mu(x).
$$

We have that if $||g||_{L^r(\mu)} \leq 1$, then

$$
\left[ \int_{\mathbb{R}^n} (\mathcal{W}^D_{K_l, \sigma}[\mu](x))^r d\mu(x) \right]^\frac{1}{r} \leq \int_{\mathbb{R}^n} \mathcal{W}^D_{K_l, \sigma}[\mu](x) g(x) d\mu(x)
$$

$$
\leq C \sum_Q k\left(\frac{1}{2^l} r_Q\right) \overline{K}(r_Q)^{p'-1} \sigma(Q) \frac{1}{\mu(Q)} \int_Q g(x) d\mu(x).
$$

Similarly to the argument in Lemma 3.8 we estimate $\mu(Q)^{p'}$ by $C \sum_{Q'} \mu(Q')^{p'}$ where the sum is taken over the dyadic cubes $Q'$ contained in $Q$ such that $r_{Q'} = 2^{-l} r_Q$ (there are $2^{nl}$ such $Q'$). The doubling condition on $\sigma$ gives that $\sigma(Q') \simeq \sigma(Q)$ and $\overline{k}(r_{Q'})(x) \simeq \overline{k}(r_Q)(x)$. We finally obtain

$$
\left[ \int_{\mathbb{R}^n} (\mathcal{W}^D_{K, \sigma}[\mu](x))^r d\mu(x) \right]^\frac{1}{r} \leq C \int_{\mathbb{R}^n} \mathcal{W}^D_{K, \sigma}[\mu](x) M^D_{\mu}[g](x) d\mu(x).
$$

Applying Hölder’s inequality and the maximal inequality, we get (3.20).

To complete the proof of (ii)⇒(i) we let $j \to +\infty$ on the left-hand side of (3.18) and apply Lebesgue’s monotone convergence theorem.

The implication (iii)⇒(ii) follows from the pointwise estimate given by Proposition 3.6 using again the fact that by Lemma 3.12 we can replace $k(r_Q)$ by $k(c r_Q)$, for any $c > 0$, in the expression

$$
\int_{\mathbb{R}^n} (\mathcal{W}^D_{K, \sigma}[\mu](x))^r d\mu(x).
$$
It remains to prove that (ii) \(\Rightarrow\) (iii). Hölder’s inequality with exponent \(\frac{q(p-1)}{p-q} > 1\), together with (5.17), gives that

\[
\int_{B_k} \left( W^{2j}_{K,\sigma}[\mu](x) \right)^{\frac{q(p-1)}{p-q}} d\mu(x)
\leq \frac{C}{|B_{j+3}|^q} \int_{B_{j+3}} \int_{B_{j+3}} \left( W^{2j}_{K,\sigma}[\mu](x) \right)^{\frac{q(p-1)}{p-q}} dz d\mu(x)
= \frac{C}{|B_{j+3}|^q} \int_{B_{j+3}} \int_{B_{j+3}} \left( W^{2j}_{K,\sigma}[\mu](x) \right)^{\frac{q(p-1)}{p-q}} d\mu(x) dz.
\]

Now, Lemma 3.12 and (ii) easily give that

\[
\int_{B_k} \left( W^{2j}_{K,\sigma}[\mu](x) \right)^{\frac{q(p-1)}{p-q}} d\mu(x) \leq C,
\]
and letting \(j \to +\infty\), we obtain (iii).

\(\square\)

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Departament de Matemàtica Aplicada i Anàlisi, Facultat de Matemàtiques, Universitat de Barcelona, Gran Via 585, 08071 Barcelona, Spain

\textit{E-mail address}: cascante@mat.ub.es

Departament de Matemàtica Aplicada i Anàlisi, Facultat de Matemàtiques, Universitat de Barcelona, Gran Via 585, 08071 Barcelona, Spain

\textit{E-mail address}: ortega@mat.ub.es

Department of Mathematics, University of Missouri, Columbia, MO 65211, USA

\textit{E-mail address}: igor@math.missouri.edu