Time-uniform central limit theory with applications to anytime-valid causal inference

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Abstract

This work introduces time-uniform analogues of confidence intervals based on the central limit theorem (CLT). Our methods take the form of confidence sequences (CS) — sequences of confidence intervals that are uniformly valid over time. CSs provide valid inference at arbitrary stopping times, incurring no penalties for “peeking” at the data, unlike classical confidence intervals which require the sample size to be fixed in advance. Existing CSs in the literature are nonasymptotic, requiring strong assumptions on the data, while the classical (fixed-time) CLT is ubiquitous due to the weak assumptions it imposes. Our work bridges the gap by introducing time-uniform CSs that only require CLT-like assumptions. While the CLT approximates the distribution of a sample average by that of a Gaussian at a fixed sample size, we use strong invariance principles like the seminal work of Komlós, Major, and Tusnády to uniformly approximate the entire sample average process by an implicit Brownian motion. Applying Robbins’ normal mixture martingale method to this Brownian motion then yields closed-form time-uniform boundaries. We combine these boundaries with doubly robust estimators to derive nonparametric CSs for the average treatment effect (and other causal estimands). These allow randomized experiments and observational studies to be continuously monitored and adaptively stopped, all while controlling the type-I error.

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1 Introduction
The central limit theorem (CLT) is arguably the most widely-used result in applied statistical inference, namely for its ability to provide approximately valid confidence intervals (CI) and \(p\)-values in a broad range of problems, including but not limited to (a) nonparametric estimation of means, including population proportions, (b) M-estimation, including maximum likelihood estimation [48], and (c) modern semiparametric causal inference methodology such as inverse propensity score weighting and doubly robust estimation [33, 45, 19, 7].

While the CLT makes efficient statistical inference possible in a broad array of problems, the resulting CIs are only valid at a prespecified sample size \(n\), invalidating any inference that occurs in sequential environments or at data-dependent stopping times. The analogue of CIs that retain validity in sequential environments are known as confidence sequences (CS) [10, 31, 16] and can be used to make decisions at arbitrary stopping times (e.g. while adaptively sampling, continuously peeking at the data, etc.). Let us first briefly review some notation and key facts about CSs.

Time-uniform confidence sequences (CSs). Consider the problem of estimating the population mean \(\mu = \mathbb{E}(Y_t)\) of a sequence of iid data \((Y_t)_{t=1}^\infty = (Y_1, Y_2, \ldots)\) that are observed sequentially over time. A classical \((1 - \alpha)\)-CI\(^1\) for \(\mu\) is a set \(\hat{C}_n \equiv \hat{C}(Y_1, \ldots, Y_n)\) with the property that
\[
\forall n \in \mathbb{N}^+, \quad \mathbb{P}(\mu \in \hat{C}_n) \geq 1 - \alpha, \quad \text{or equivalently,} \quad \forall n \in \mathbb{N}^+, \quad \mathbb{P}(\mu \notin \hat{C}_n) \leq \alpha. \tag{1.1}
\]
The coverage guarantee (1.1) of a CI is only valid at some prespecified sample size \(n\), and thus we cannot make inferences after looking at the data (this is sometimes referred to as “peeking” or “\(p\)-hacking”). However, it seems unnecessarily restrictive to artificially fix the sample size \(n\) beforehand, and it is impossible to know \textit{a priori} whether \(n\) will be large enough (or too large) to efficiently detect some signal of interest. CSs provide precisely the flexibility to choose sample sizes data-adaptively while controlling the type-I error rate. Formally, a CS is a sequence of CIs \((\bar{C}_t)_{t=1}^\infty\) such that
\[
\mathbb{P}(\forall t \in \mathbb{N}^+, \ \mu \in \bar{C}_t) \geq 1 - \alpha, \quad \text{or equivalently,} \quad \mathbb{P}(\exists t \in \mathbb{N}^+: \mu \notin \bar{C}_t) \leq \alpha. \tag{1.2}
\]
The statements (1.1) and (1.2) look similar but are markedly different from the data analyst’s or experimenter’s perspective. In particular, employing a CS has the following implications:

(a) The CS can be (optionally) updated whenever new data become available;

(b) Experiments can be continuously monitored, adaptively stopped, or continued;

\(^1\)We use overhead dots \(\hat{C}_n\) to denote fixed-time (pointwise) CIs and overhead bars \(\bar{C}_t\) to denote time-uniform CSs.
(c) The type-I error is controlled at all stopping times, including data-dependent times. In fact, CSs may be equivalently defined as CIs that are valid at arbitrary stopping times, i.e.

\[ \mathbb{P}(\mu \in \mathcal{C}_\tau) \geq 1 - \alpha \text{ for any stopping time } \tau. \]  

(1.3)

A proof of the equivalence between (1.2) and (1.3) can be found in Howard et al. [16, Lemma 3]. While

![Confidence sets for the average treatment effect](image)

Figure 1: The left plot shows a time-uniform CS alongside a fixed-time CI for a parameter of interest (in this case, the average treatment effect (ATE), an example we expand on in Section 3). The true value of the ATE is covered by the CS simultaneously from time 30 to 10000. On the other hand, the CI fails to cover the ATE at several points in time. The right plot displays the cumulative probability of miscovering the ATE at any time up to \( t \). Notice that the CI error rate begins at \( \alpha = 0.1 \) and quickly grows, while the CS error rate never exceeds \( \alpha = 0.1 \) (and never will, even asymptotically).

CSs have been developed for several problems, they have thus far been nonasymptotic. Nonasymptotic CSs (as well as CIs) require strong assumptions on the distribution of the data (e.g. a parametric likelihood [54] or known bounds on the random variables themselves, [16, 55, 25], known bounds on their moments [53], or known bounds on their moment generating functions [16]). These added requirements are unlike the CLT which can be applied under the weak condition of a couple of finite moments (which need not be known). In particular, existing CSs cannot be applied to the estimation problems discussed previously (e.g. M-estimation, semiparametric causal inference, etc.). Even fixed-time (i.e., not time-uniform) inference for their parameters typically relies on asymptotics. Our work bridges this gap, bringing CLT-based inference to the sequential regime by making one simple modification to the usual CIs and requiring one additional weak moment assumption (requiring \( 2 + \delta \) moments, for \( \delta > 0 \), instead of 2 moments).

**Looking ahead: Applications to doubly robust causal inference.** Many scientific and policy questions of interest are inherently causal and cannot be answered by simply analyzing associations. For example, a pharmaceutical company may be interested in evaluating whether a new drug is more beneficial than the current standard treatment. State-of-the-art methods to answer such causal questions typically rely on the CLT to provide valid fixed-time CIs. We will pay particularly close attention to applications of our time-uniform analogues of the CLT to the problem of *sequential causal inference* both in randomized experiments and observational studies.

For the purposes of this paper, we will mainly be concerned with binary treatments (e.g. drug versus placebo), though we do address more complicated settings in Section 5. In particular, we will focus on the *average treatment effect* (ATE) which is defined in terms of counterfactuals. Specifically, let \( Y^a \) denote the counterfactual outcome which would have occurred if a randomly selected subject
received treatment level $a \in \{0, 1\}$. The ATE is defined as
\[ \text{ATE} := E(Y^1 - Y^0), \]
which may be interpreted as the average population difference in subject outcomes if everyone were assigned to treatment, versus everyone assigned to control. Suppose that we observe triplets $Z_1, Z_2, \ldots$ where $Z_i = (X_i, A_i, Y_i)$ such that $X_i \in \mathbb{R}^d$ is subject $i$’s measured baseline covariates, $A_i \in \{0, 1\}$ is their assigned treatment, and $Y_i \in \mathbb{R}$ is their outcome after receiving treatment $A_i$. Note that as currently written, the ATE cannot necessarily be estimated via the data $Z_1, Z_2, \ldots$ (even an infinite amount) since the ATE depends on the unobservable counterfactuals, $Y^1 - Y^0$. It is well known, however, that under the so-called “identifiability assumptions” (IA1) consistency: $A = a \implies Y = Y^a$, (IA2) no unmeasured confounding: $A \perp Y^a \mid X$, and (IA3) positivity: $\mathbb{P}(A = a \mid X) > 0$ almost surely, then the ATE can be written as
\[ \text{ATE} = E \{ E(Y \mid X, A = 1) \} - E \{ E(Y \mid X, A = 0) \}. \]
Assumptions (IA1)–(IA3) are well-known to be untestable, and must be reasoned about carefully in the context of the scientific problem of interest. Each assumption can be weakened in various ways, at the expense of losing point identification of the marginal counterfactual distribution [34, 1, 20, 4]. That said, even when these assumptions are violated, estimation of the adjusted contrast above still plays an important role. Therefore we will assume (IA1)–(IA3) throughout and focus on the purely statistical problem of estimating the ATE with as much precision as possible under nonparametric conditions.

Many estimators for the ATE have been proposed over the years, including doubly robust versions which are optimal in the sense that they attain the semiparametric efficiency bound [26, 3, 44, 42, 19]. Such doubly robust-style estimators — including targeted maximum likelihood estimation [46] and double machine learning [7] — permit the construction of CLT-based CIs for the ATE under nonparametric conditions, even when constructed via flexible machine learning algorithms. As mentioned before, however, CLT-based inference has thus far been confined the the fixed-time regime where the sample size must be specified in advance (with some exceptions in group-sequential trials [36] and experiments with bounded time horizons [2]). Section 3 brings doubly robust causal inference to the fully sequential regime, where sequential experiments or observational studies can be continuously monitored online, providing valid inference for the ATE at arbitrary stopping times.

Outline In Section 2.1, we introduce the notion of “asymptotic confidence sequences” as time-uniform analogues of asymptotic confidence intervals. We then go on to explicitly construct two asymptotic confidence sequences for the classical problem of nonparametric mean estimation from iid data with finite moments. Section 2.4 extends these asymptotic confidence sequences to the regime where distributions — and in particular means and variances — can change over time. Section 3 focuses on applying the time-uniform central limit theory of Section 2 to the problem of sequential causal inference. In Section 3.1, we introduce sequential sample splitting and cross-fitting so that nuisance functions can be estimated using flexible machine learning algorithms. Combining sequential cross-fitting with the asymptotic confidence sequences of Section 2.1, we derive confidence sequences for the ATE in randomized sequential experiments (Section 3.2) and observational studies with no unmeasured confounding (Section 3.3). In Section 4 we illustrate the use of the methods presented in Section 3.3 to a real dataset to estimate the effect of fluid intake on mortality in sepsis patients. Finally in Section 5, we discuss how all of the aforementioned techniques apply to functional estimation tasks more generally, and instantiate sequential estimation of the ATE as a special case.

2 Time-uniform central limit theory

We first define what it means for a sequence of intervals to form an asymptotic confidence sequence (Asymp-CS), especially since the term “confidence sequence” has always been used nonasymptotically
ever since its introduction by Robbins and collaborators [9]. Then, we derive a “universal” Asymp-CS in the sense that the Asymp-CS does not depend on any features of the distribution beyond its mean and variance. This universal Asymp-CS is fundamentally related to Gaussians — much like classical asymptotic confidence intervals based on the CLT — since it stems from Brownian motion approximations and so-called strong invariance principles. Finally, similar to the Lyapunov version of the CLT for non-i.i.d. sums, we derive a Lyapunov-type Asymp-CS.

### 2.1 What is an asymptotic confidence sequence?

As mentioned in Section 1, the literature on CSs has focused on the nonasymptotic regime where strong assumptions must be placed on the observed random variables, such as boundedness, a parametric likelihood, or upper bounds on their MGFs. On the other hand, in batch — i.e. fixed-time — statistical analyses, the CLT is routinely applied to obtain approximately valid CIs in large samples, as it requires weak finite moment assumptions and has a simple, universal closed-form expression. Definition 1 presents “asymptotic confidence sequences” as anytime-valid analogues of asymptotic CIs, making similarly weak moment assumptions and providing a universal closed-form boundary. Later, Theorem 1 provides an explicit construction of such a CS, and discusses its properties.

To motivate the definition, let us briefly review the CLT in the batch (non-sequential) setting. We use the term universal in the same way that the CLT and the law of large numbers are considered universality results [41], because they describe macroscopic behaviors that are independent of most microscopic details of the system.

#### Definition 1 (Asymptotic confidence sequences (Asymp-CS)).

We say that \((\hat{\mu}_n \pm \bar{B}_n)\) is a two-sided \((1 - \alpha)\)-asymptotic confidence sequence for a parameter \(\mu\) with approximation rate \(r_t\) if there exists a nonasymptotic two-sided \((1 - \alpha)\)-CS \((\hat{\mu}_t \pm \bar{B}_t^*)\) for \(\mu\) such that \(\bar{B}_t^* - \bar{B}_t = O_{a.s.}(r_t)\) where \(r_t = o_{a.s.}(\bar{B}_t)\).

In particular, this means that

\[
\frac{\bar{B}_t^*}{\bar{B}_t} \overset{a.s.}{\rightarrow} 1.
\]

We can similarly define one-sided upper (or lower) Asymp-CSs by replacing \(\pm\) by \(+\) (or \(-\)), respectively.

### 2.2 A universal asymptotic confidence sequence

We now construct an explicit Asymp-CS for the mean of iid random variables under weak moment assumptions by combining a variant of Robbins’ (nonasymptotic) Gaussian mixture boundary [31] with Komlós, Major, and Tusnády’s strong approximation theorems [22, 23]. This will serve as a time-uniform analogue of the CLT-based CI (2.1).

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2We use the term universal in the same way that the CLT and the law of large numbers are considered universality results [41], because they describe macroscopic behaviors that are independent of most microscopic details of the system.

3Technically, writing \(\sum Y_i = \sum Z_i + o_{a.s.}(1/\sqrt{n})\) may require enriching the probability space so that both \(Y\) and \(Z\) can be defined without changing their distributions and this may not always be possible (for example if \(Y_i\) is Bernoulli). See the proofs of Theorem 1 or [22, 23, 24] for precise statements.
**Theorem 1** (Gaussian mixture asymptotic confidence sequence). Suppose \((Y_t)_{t=1}^\infty \sim \mathbb{P}\) is an infinite sequence of iid observations from a distribution \(\mathbb{P}\) with mean \(\mu\) and \(q > 2\) finite absolute moments. Let \(\hat{\mu}_t := \frac{1}{t} \sum_{i=1}^t Y_i\) be the sample mean, and \(\hat{\sigma}_t^2 := \frac{1}{t} \sum_{i=1}^t Y_i^2 - (\hat{\mu}_t)^2\) the sample variance based on the first \(t\) observations. Then, for any prespecified constant \(\rho > 0\),

\[
\tilde{C}_t^\rho (\hat{\mu}_t \pm \mathbb{B}_t^\rho) := \left( \hat{\mu}_t \pm \hat{\sigma}_t \sqrt{\frac{2(t\rho^2 + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{t\rho^2 + 1}}{\alpha} \right)} \right)
\]

forms a \((1 - \alpha)\)-Asymp-CS for \(\mu\) with approximation rate \(r_t = o(\sqrt{\log \log t/t})\). Furthermore, if \(q \geq 4\), then \(r_t\) has a faster rate of \(o\left((\log \log t/t)^{3/4}\right)\). In either case, \(r_t = o_{a.s.}(\mathbb{B}_t^\rho) = o(\sqrt{\log \log t})\).

The proof in Appendix A.1 combines the strong approximation results due to Komlós et al. \([22, 23]\) with Ville’s inequality for nonnegative supermartingales \([52]\) applied to Gaussian mixture martingales. We can think of \(\rho > 0\) as a user-chosen tuning parameter which dictates the time at which (2.4) is tightest, and we discuss how to easily tune this value in Section C.1.

While (2.4) may look visually similar to Robbins’ (sub)-Gaussian mixture CS \([31]\) — written explicitly in Howard et al. \([16, \text{Eq. (14)}]\) — it is worth pausing to reflect on how they are markedly different. Firstly, Robbins’ CS is a nonasymptotic bound that is only valid for \(\sigma\)-sub-Gaussian random variables, meaning \(\mathbb{E}\{\exp(\lambda Y_1)\} \leq \exp(\sigma^2 \lambda^2/2)\) for some \textit{a priori} known \(\sigma > 0\), while Theorem 1 does not require the existence of a finite MGF at all (much less a known upper bound on it). Secondly, Robbins’ CS uses this known (possibly conservative) \(\sigma\) in place of \(\hat{\sigma}_t\) in (2.4), and thus it cannot adapt to an unknown variance, while (2.4) always scales with \(\sqrt{\text{var}(Y_t)}\). In simpler terms, Theorem 1 is a time-uniform analogue of the CLT in the same way that Robbins’ CS is a time-uniform analogue of a sub-Gaussian concentration inequality (e.g. Hoeffding’s inequality \([14]\)).

It is important not to confuse Theorem 1 with a martingale CLT as the latter still gives fixed-time CIs in the spirit of the usual CLT but under different assumptions on the martingale difference sequence. Furthermore, notice that we could not have arrived at Theorem 1 using Donsker’s (weak) invariance principle, because Theorem 1 provides the explicit rate of convergence \(r_t\), which relies on the (strong) invariance principle of Komlós et al. \([22, 23]\). Knowing this rate is crucial for deriving nonparametric Asymp-CSs for causal effects, a topic which we explore in Section 3.

### 2.3 An asymptotic confidence sequence with iterated logarithm rates

As a consequence of the law of the iterated logarithm, a confidence sequence for \(\mu\) cannot have an asymptotic width smaller than \(O(\sqrt{\log \log t/t})\). This is easy to see since

\[
\limsup_{t \to \infty} \frac{\left| \hat{\mu}_t - \mu \right|}{\sigma \sqrt{2t \log \log t}} = 1.
\]

This raises the question as to whether \(\tilde{C}_t^\rho\) can be improved so that the optimal asymptotic width of \(O(\sqrt{\log \log t/t})\) is achieved. Indeed, we can use the bound in Howard et al. \([16, \text{Equation (2)}]\) to derive such a confidence sequence, but as the authors discuss, mixture boundaries such as the one in Theorem 1 may be preferable in practice, because any bound that is tighter “later on” (asymptotically) must be looser “early on” (at practical sample sizes) because all such bounds have a cumulative miscoverage probability \(\leq \alpha\). Nevertheless, we present an Asymp-CS with an iterated logarithm rate here for completeness.

**Proposition 1** (Iterated logarithm asymptotic confidence sequences). Under the same conditions as Theorem 1,

\[
\tilde{C}_t^{\rho} (\hat{\mu}_t \pm \mathbb{B}_t^{\rho}) := \left( \hat{\mu}_t \pm \hat{\sigma}_t \cdot 1.7 \sqrt{\frac{\log \log(2t) + 0.72 \log(10.4/\alpha)}{t}} \right)
\]

forms a \((1 - \alpha)\)-Asymp-CS for \(\mu\) with the same rate \(r_t\) as in Theorem 1.
Both Theorem 1 and Proposition 1 form Asymp-CSs for the mean $\mu$ of iid random variables. In the following section, however, we extend Asymp-CSs to the non-iid regime where means and variances can change over time.

### 2.4 Lyapunov-type confidence sequences for time-varying means

All of the results thus far have focused on the situation where the observed random variables are independent and identically-distributed, as this is one of the most commonly-studied regimes in statistical inference. In practice, however, we may not wish to assume that means and variances remain constant over time. Nevertheless, an analogue of Theorem 1 still holds for independent random variables with time-varying means and variances under similar conditions to the Lyapunov CLT. In this case, rather than the CS covering some fixed $\mu^*$, it covers the average mean thus far: $\tilde{\mu}_t := \frac{1}{t} \sum_{i=1}^{t} \mu_i$.

Given the additional complexity introduced by considering time-varying distributions, we will first explicitly spell out the assumptions required to achieve a time-varying analogue of Theorem 1. Note however, that these assumptions are no more restrictive, meaning that they reduce to the assumptions of Theorem 1 in the iid regime. Suppose $(Y_t)_{t=1}^\infty$ is a sequence of independent random variables with individual means $\mathbb{E}(Y_t) = \mu_t$ and variance $\text{var}(Y_t) = \sigma_t^2$. First, we require that the average variance $\tilde{\sigma}_t^2 := \frac{1}{t} \sum_{i=1}^{t} \sigma_i^2$ either does not vanish, or does so superlinearly.

![Figure 2: A 90%-Asymp-CS for the time-varying mean $\tilde{\mu}_t$ where we have set $\mu_t := \frac{1}{2} (1 - \sin(2 \log(e + 10t))/\log(e + 0.01t))$ to produce the sinusoidal behavior of $\tilde{\mu}_t$. Notice that $\tilde{C}_t$ uniformly captures $\tilde{\mu}_t$, adapting to its non-stationarity.](image)

**Assumption 1 (Variance does not vanish).** Let $\tilde{\sigma}_t := \frac{1}{t} \sum_{i=1}^{t} \sigma_i^2$ be the average variance up until time $t$. Then,

$$\tilde{\sigma}_t^2 = \omega(1).$$

For example, (2.5) would hold if $\tilde{\sigma}_t^2 \to \sigma^2$ or in the familiar case where $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma^2$. Second, we require that $Y_t$ has $q > 2$ finite absolute moments that do not diverge too quickly.

---

4Throughout this section, we use the overhead tilde (e.g. $\tilde{\mu}_t$, $\tilde{\sigma}_t$, and $\tilde{C}_t$) to emphasize that these quantities can change over time. For example, Figure 2 explicitly displays means and CSs with sinusoidal behaviors resembling a tilde.
Assumption 2 (Restricted divergence of $q^{th}$ moments). Let $\delta > 0$ and $q > 2 + 2\delta$ such that

$$\sum_{t=1}^{\infty} \frac{\mathbb{E}|Y_t - \mu_t|^q}{t^{1+\delta}} < \infty. \quad (2.6)$$

Note that (2.6) would hold if all $q^{th}$ moments are bounded by a constant, i.e. $\forall t$, $\mathbb{E}|Y_t - \mu_t|^q < K$ for some $K < \infty$. Third and finally, we require a consistent variance estimator.

Assumption 3 (Consistent variance estimation). Let $\hat{\sigma}_t^2$ be an estimator of $\sigma_t^2$ constructed using $Y_1, \ldots, Y_t$ such that

$$\hat{\sigma}_t^2 - \sigma_t^2 = o(1). \quad (2.7)$$

Note that in the case where $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_n^2$, then (2.7) would hold for the sample variance by the strong law of large numbers. Given Assumptions 1, 2, and 3, we have the following Asymp-CS for the time-varying mean $\hat{\mu}_t := \frac{1}{t} \sum_{i=1}^{t} \mu_i$.

Theorem 2 (Lyapunov-type Asymp-CS). Suppose $(Y_t)_{t=1}^{\infty}$ is a sequence of independent random variables with individual means $\mathbb{E}(Y_t) = \mu_t$, and variances $\text{var}(Y_t) = \sigma_t^2$ satisfying Assumptions 1, 2, and 3. Define the running mean $\tilde{\mu}_t := \frac{1}{t} \sum_{i=1}^{t} \mu_i$. Then,

$$\tilde{C}_t \equiv (\hat{\mu}_t \pm \tilde{B}_t) := \left(\hat{\mu}_t \pm \sqrt{\frac{2(t\hat{\sigma}_t^2 + 1)}{t^2 \sigma_t^2} \log \left(\frac{\sqrt{t \sigma_t^2} \rho^2 + 1}{\alpha}\right)}\right) \quad (2.8)$$

forms a $(1 - \alpha)$-Asymp-CS with approximation rate $o(\sqrt{\log t/t})$ for the time-varying mean $\tilde{\mu}_t$.

The proof of Theorem 2 can be found in Section A.2 and uses Shao’s strong approximation theorems for non-identically distributed random variables [39]. Figure 2 exemplifies what $\tilde{C}_t$ may look like in practice. Note that when $\mu_1 = \mu_2 = \cdots = \mu_*$, and $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_*^2$, it is nevertheless the case that $\tilde{C}_t$ forms a $(1 - \alpha)$-Asymp-CS for $\mu_*$ under the same assumptions as Theorem 1. In this sense, we can view $(\tilde{C}_t)_{t=1}^{\infty}$ as “robust” to deviations from stationarity.\(^5\)

3 Confidence sequences for the average treatment effect

Given the groundwork laid in Section 2.1, we now focus on time-uniform inference for causal effects — namely the average treatment effect — via Asymp-CSs. This will enable researchers to quantify uncertainty for causal effects in fully sequential environments, where confidence sets from randomized experiments and observational studies can be continuously monitored as the data are being collected. However, obtaining Asymp-CSs for the average treatment effect is not as simple as applying Theorem 1 to some appropriately chosen random variable due to the presence of (potentially infinite-dimensional) nuisance parameters. Nevertheless, after introducing and carefully analyzing sequential sample-splitting and cross-fitting (Section 3.1), we will see that efficient time-uniform inference in these sequential settings is still possible.

To solidify the notation and problem setup, we expand on the prelude provided in Section 1. Suppose that we observe a (potentially infinite) sequence of independent and identically distributed (iid) variables $Z_1, Z_2, \ldots$ from a distribution $\mathbb{P}$, where $Z_t := (X_t, A_t, Y_t)$ denotes the $t^{th}$ subject’s triplet and

- $X_t \in \mathbb{R}^d$ is subject $t$’s measured baseline covariates,
- $A_t \in \{0, 1\}$ is the treatment that subject $t$ received, and

\(^5\)Here, the term “robust” should not be interpreted in the same spirit as “doubly robust”, where the latter is specific to the discussions surrounding functional estimation and causal inference in Section 3.
\begin{itemize}
  \item $Y_t \in \mathbb{R}$ is subject $t$'s measured outcome after treatment.
\end{itemize}

Our target estimand is the average treatment effect (ATE) $\psi$ defined as

$$\psi := \mathbb{E}(Y^1 - Y^0).$$

where $Y^a$ is the counterfactual outcome for a randomly selected subject had they received treatment $a \in \{0, 1\}$. The ATE can be interpreted as the average population outcome if everyone were treated $\mathbb{E}(Y^1)$ versus if no one were treated $\mathbb{E}(Y^0)$. However, without further identifying assumptions, we cannot hope to estimate this counterfactual quantity using the observed data $(Z_t)_{t=1}^n$. Consider the following standard causal identifying assumptions, which we require for $a \in \{0, 1\}$.

(IA1): Consistency: $A = a \implies Y = Y^a$,
(IA2): No unmeasured confounding: $A \perp Y^a \mid X$, and
(IA3): Positivity: $\mathbb{P}(A = a \mid X) > 0$ almost surely.

The consistency assumption (IA1) can be thought of as stating that there is no interference between subjects, so that an individual’s counterfactual does not depend on the treatment of others (which, e.g. could be violated in a vaccine efficacy trial where a subject is protected by the fact that their friends received a vaccine). (IA2) effectively states that the treatment is as good as randomized within levels of the observed covariates, and (IA3) simply ensures that all subjects have a nonzero probability of receiving treatment $a \in \{0, 1\}$. Throughout the remainder of the paper, we assume (IA1). In Section 3.2, we will consider an experimental setting in which (IA2) and (IA3) hold by design, while in the observational case which we consider in Section 3.3, (IA2) and (IA3) will need to be assumed. It is well-known that if identifying assumptions (IA1)–(IA3) hold, then the average treatment effect $\psi$ can be written as

$$\psi = \mathbb{E}\{\mathbb{E}(Y \mid X, A = 1) - \mathbb{E}(Y \mid X, A = 0)\},$$

a purely statistical quantity for which we aim to derive sharp Asympt-CSs under nonparametric assumptions. To this end, we briefly review efficient estimators for $\psi$ and the sense in which they are optimal.

A brief review of efficient estimators. For a detailed account of efficient estimation in semiparametric models, we refer readers to Bickel et al. [3], van der Vaart [49], van der Laan and Robins [44], Tsiatis [42] and Kennedy [19], but provide a brief overview of their fundamental relevance to estimation of the ATE here.

A central goal of semiparametric efficiency theory is to characterize the set of influence functions for a parameter (in our case, $\psi$). Of particular interest is finding the efficient influence function (EIF) as its variance acts as a semiparametric analogue of the Cramer-Rao lower bound, hence providing a benchmark for constructing optimal estimators (in an asymptotic local minimax sense). In the case of $\psi$, the (uncentered) EIF is given by

$$f(z) \equiv f(x, a, y) := \{\mu^1(x) - \mu^0(x)\} + \left(\frac{a}{\pi(x)} - \frac{1 - a}{1 - \pi(x)}\right)\{y - \mu^a(x)\}, \quad (3.1)$$

where $\mu^a(x) := \mathbb{E}(Y \mid X = x, A = a)$ is the regression function among those treated at level $a \in \{0, 1\}$ and $\pi(x) := \mathbb{P}(A = 1 \mid X = x)$ is the propensity score (i.e. probability of treatment) for an individual with covariates $x$. In particular, this means that no estimator of $\psi$ based on $t$ observations can have asymptotic mean squared error smaller than $\text{var}(f(Z))/t$ without imposing additional assumptions.

In a randomized experiment, the joint distribution of $(X, Y)$ is unknown but the conditional distribution of $A \mid X = x$ is known to be Bernoulli($\pi(x)$) by design. In this case, our statistical model
for $Z$ is a proper semiparametric model, and hence there are infinitely many influence functions, all of which take the form,

$$
\hat{f}(z) \equiv \hat{f}(x, a, y) := \left\{ \hat{\mu}^1(x) - \hat{\mu}^0(x) \right\} + \left( \frac{a}{\pi(x)} - \frac{1 - a}{1 - \pi(x)} \right) \left\{ y - \hat{\mu}^a(x) \right\},
$$

(3.2)

where $\hat{\mu}^a : \mathbb{R}^d \to \mathbb{R}$ is any function. However, when the joint distribution of $(X, A, Y)$ is left completely unspecified (such as in an observational study with unknown propensity scores), our statistical model for $\mathbb{P}$ is nonparametric, and hence there is only one influence function, the EIF given in (3.1).

Not only does the EIF $f(z)$ provide us with a benchmark against which to compare estimators, but it hints at the first step in deriving the most efficient estimator. Namely, $\frac{1}{T} \sum_{t=1}^{T} f(Z_t)$ is a consistent estimator for $\psi$ with asymptotic variance equal to the efficiency bound, $\text{var}(f)$ by construction. However, $f(Z)$ depends on possibly unknown nuisance functions $\eta := (\mu^1, \mu^0, \pi)$. A natural next step would be to simply estimate $\eta$ from the data ($Z_t$)$_{t=1}^{T}$. Crucially, it is possible to ensure that only a negligible amount of additional estimation error is incurred by replacing $\eta$ by a data-dependent estimate $\hat{\eta}_t$ — the essential technique here being sample splitting and cross-fitting [32, 56, 7]. In the following section, we introduce sequential sample-splitting and cross-fitting, allowing the same types of analyses of $\hat{\eta}_t$ to be carried out but in fully sequential settings.

3.1 Sequential sample-splitting and cross-fitting

Following Robins et al. [32], Zheng and van der Laan [56], and Chernozhukov et al. [7], we employ sample-splitting to derive an estimate $\hat{f}$ of the influence function $f$ on a “training” sample, and evaluate $\hat{f}$ on values of $Z_t$ in an independent “evaluation” sample. Sample-splitting sidesteps complications introduced from “double-dipping” (i.e. using $Z_t$ to both construct $\hat{f}$ and evaluate $\hat{f}(Z_t)$) and greatly simplifies the analysis of the downstream estimator. However, the aforementioned authors employed sample-splitting in the batch (non-sequential) regime where one can simply randomly split the data into two halves. Given our sequential setup where data are continually observed in an online stream over time, we modify the sample-splitting procedure as follows. We will denote $\mathcal{D}_{t}^{\text{trn}}$ and $\mathcal{D}_{t}^{\text{eval}}$ as the “training” and “evaluation” sets, respectively. At time $t$, we assign $Z_t$ to either group with equal probability:

$$
Z_t \in \begin{cases} 
\mathcal{D}_{t}^{\text{trn}} & \text{with probability } 1/2, \\
\mathcal{D}_{t}^{\text{eval}} & \text{otherwise.}
\end{cases}
$$

Note that at time $t + 1$, $Z_t$ is not re-randomized into either split — once $Z_t$ is randomly assigned to one of $\mathcal{D}_{t}^{\text{trn}}$ or $\mathcal{D}_{t}^{\text{eval}}$, they remain in that split for the remainder of the study. In this way, we can write $\mathcal{D}_{T}^{\text{trn}} = (Z_1^{\text{trn}}, Z_2^{\text{trn}}, \ldots)$ and $\mathcal{D}_{T}^{\text{eval}} = (Z_1^{\text{eval}}, Z_2^{\text{eval}}, \ldots)$ and think of these as independent, sequential observations from a common distribution $\mathbb{P}$. To keep track of how many subjects have been randomized to $\mathcal{D}_{T}^{\text{trn}}$ and $\mathcal{D}_{T}^{\text{eval}}$ at time $t$, define

$$
T := |\mathcal{D}_{T}^{\text{eval}}| \quad \text{and} \quad T' := |\mathcal{D}_{T}^{\text{trn}}| = t - T,
$$

(3.3)

where we have left the dependence on $t$ implicit.

**Remark 1.** Strictly speaking, under the iid assumption, we do not need to randomly assign subjects to training and evaluation groups for the forthcoming results to hold (e.g. we could simply assign even-numbered subjects to $\mathcal{D}_{t}^{\text{trn}}$ and odd-numbered subjects to $\mathcal{D}_{t}^{\text{eval}}$). However, the analysis is not further complicated by this randomization, and it can be used to combat bias in treatment assignments when the iid assumption is violated [12].

The sequential sample-split estimators $(\hat{\psi}_t^{\text{split}})_{t=1}^{T}$. After employing sequential sample-splitting, the sequence of sample-split estimators $(\hat{\psi}_t^{\text{split}})_{t=1}^{T}$ for $\psi$ are given by

$$
\hat{\psi}_t^{\text{split}} := \frac{1}{T} \sum_{i=1}^{T} \hat{f}_T(Z_i^{\text{eval}}),
$$

(3.4)
where $\hat{f}_T$ is given by (3.1) with $\eta \equiv (\mu^1, \mu^0, \pi)$ replaced by $\hat{\eta}_T \equiv (\hat{\mu}^1, \hat{\mu}^0, \hat{\pi})$ which is built solely from $D_{\text{trn}}^T$. The sample-splitting procedure for constructing $\hat{\psi}_{\text{split}}^t$ is summarized pictorially in Figure 3. In the batch setting for a fixed sample size, (3.4) is often referred to as the “doubly robust” or “augmented inverse probability weighted” (AIPW) estimator [35, 37] and we adopt similar nomenclature here.

The sequential cross-fit estimators $(\hat{\psi}_T^x)_{i=1}^T$. A commonly cited downside of sample-splitting is the loss in efficiency by using $T \approx t/2$ subjects instead of $t$ when evaluating the sample mean $\frac{1}{T} \sum_{i=1}^T \hat{f}_T(Z_i^\text{eval})$. An easy fix is to cross-fit: swap the two samples, using the evaluation set $D_{\text{eval}}^T$ for training and the training set $D_{\text{trn}}^T$ for evaluation to recover the full sample size of $t = T + T' \ [32, 56, 7]$. That is, construct $\hat{f}_T$ solely from $D_{\text{eval}}^T$ and define the cross-fit estimator $\hat{\psi}_T^x$ as

$$\hat{\psi}_T^x := \frac{\sum_{i=1}^T f_T(Z_i^\text{eval}) + \sum_{i=1}^{T'} f_T(Z_i^\text{trn})}{t},$$

and the associated cross-fit variance estimate

$$\hat{\text{var}}_T(f) := \frac{\hat{\text{var}}_T(\hat{f}_T) + \hat{\text{var}}_T(\hat{f}_T')}{2}.$$ (3.6)

Note that (3.6) is simply the average of sample variances of the cross-fit pseudo-outcomes $(\hat{f}_T(Z_i^\text{eval}))_{i=1}^T$ and $(\hat{f}_T(Z_i^\text{trn}))_{i=1}^{T'}$, respectively. All of the results that follow are stated in terms of the cross fit estimators $(\hat{\psi}_T^x)_{i=1}^T$ but they can be amended to use $(\hat{\psi}_{\text{split}}^x)_{i=1}^T$ instead. Using the assumptions laid out so far, we are ready to apply the confidence sequences of Section 2.1 to randomized sequential experiments.

### 3.2 Asymptotic confidence sequences in randomized experiments

Consider an experiment in which subjects are recruited sequentially and administered treatment in a randomized and controlled manner. In particular, suppose that samples are iid and a subject with covariates $x$ has a known propensity score given by

$$\pi(x) := \mathbb{P}(A = 1 \mid X = x).$$
Consider the doubly robust cross-fit estimator \( \hat{\psi}_t^* \) as given in (3.5) but with estimated propensity scores — \( \hat{\pi}_T(x) \) and \( \hat{\pi}_r(x) \) — replaced by their true values \( \pi(x) \), and with \( \hat{\mu}_t^a \) and \( \hat{\mu}_r^a \) being possibly misspecified estimators for \( \mu^a \). That is, we will assume that \( \hat{\mu}_t^a \) converges to some function \( \mu^a \), which need not coincide with \( \mu^a \). We are now ready to state the main result of this section.

**Theorem 3** (Confidence sequences for the ATE in randomized experiments). Let \( \hat{\psi}_t^* \) be the doubly robust cross-fit estimator as in (3.5). Suppose \( \| \hat{\mu}_t^a(X) - \mu^a(X) \|_{L_2(P)} = o(1) \) for each \( a \in \{0,1\} \) where \( \mu^a \) is some function (but need not be \( \mu^a \)), and hence \( \| \hat{f}_t - f \|_{L_2(P)} = o(1) \) for some influence function \( f \) of the form (3.2). Suppose that propensity scores are bounded away from 0 and 1, i.e. \( \pi(X) \in [\delta,1-\delta] \) for some \( \delta > 0 \), and suppose that the influence function \( f(Z) \) has at least four moments, \( \mathbb{E}|f(Z)|^4 < \infty \). Then for any prespecified constant \( \rho > 0 \),

\[
\hat{\psi}_t^* \pm \sqrt{\text{var}(\hat{\psi}_t)} \cdot \sqrt{\frac{2(t \rho^2 + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{t \rho^2 + 1}}{\alpha} \right)}
\]

forms a \( (1-\alpha) \)-Asymp-CS for \( \psi \) with approximation rate \( o(\sqrt{\log \log t} / t) \).

The proof in Appendix A.3 combines an analysis of the almost-sure convergence of \( (\hat{\psi}_t^* - \psi) \) with the Asymp-CS of Theorem 1. Notice that since \( \hat{\mu}_t^a \) is consistent for a function \( \mu^a \), we have that \( \hat{f}_t \) is asymptotically equivalent to an influence function \( f \) of the form (3.2). In practice, however, one must choose \( \hat{\mu}_t^a \). As alluded to at the beginning of Section 3, the best possible influence function is the EIF \( f(z) \) defined in (3.1), and thus it is natural to attempt to construct \( \hat{\mu}_t^a \) so that \( \| \hat{f}_t - f \|_{L_2(P)} = o(1) \). The resulting confidence sequences would inherit such optimality properties, a point which we discuss further in Section C.3.

![Figure 4](image_url)

Figure 4: Three 90%-Asymp-CSs for the average treatment effect in a simulated randomized experiment using different regression estimators. Notice that all three confidence sequences uniformly capture the average treatment effect \( \psi \), but increasingly sophisticated models do so more efficiently, with the Super Learner greatly outperforming an unadjusted estimator. For more details on this simulation, see Section B.1.

Since \( \mu^a \) is an unknown regression function with a potentially complex structure, we cannot in general expect to estimate it with a simple parametric model. Instead, we suggest building \( \hat{\mu}_t^a \) as a
Theorem 3 but with consistently estimated in Moreover, suppose that identifying assumptions (IA2) and (IA3) do not hold by design, we weaken to a product rate of 0. This advantage can be seen empirically in Figure 4 where the true regression functions \(\mu^0\) and \(\mu^1\) are non-smooth, nonlinear functions of covariates \(x \in \mathbb{R}^d\). See Section B.1 for more details on how this simulation was designed.

So far, flexible nonparametric regression techniques such as Super Learning have been used to build efficient estimators \(\hat{\mu}_t^a\) of \(\mu^a\) with \(a \in \{0, 1\}\), but were not required to derive valid confidence sequences for \(\psi\). In an observational setting where neither \(\mu^a(x)\) nor \(\pi(x)\) are known, flexible nuisance estimation will be an essential tool in ensuring that confidence sequences capture \(\psi\), as we will see in the following section.

### 3.3 Asymptotic confidence sequences in observational studies

We now consider a situation where identifying assumptions (IA2) and (IA3) do not hold by design, but must be assumed. This may occur in a purely observational sequential study, or in a randomized sequential experiment where subjects do not comply with their assigned treatments or have missing outcomes. In any case, it is well-known that (IA2) and (IA3) are untestable from the observed data, and we assume that they hold for the discussions that follow. As before, under (IA1)--(IA3), we have that

\[
\psi = \mathbb{E}(Y_1 - Y^0) = \mathbb{E}(\mathbb{E}(Y \mid X, A = 1) - \mathbb{E}(Y \mid X, A = 0)),
\]

which is the target parameter we aim to estimate. Since we no longer have knowledge of each subject’s propensity score \(\pi(x) := \mathbb{P}(A \mid X = x)\) we must instead estimate \(\pi(x)\) in addition to the regression functions \(\mu^a(x) := \mathbb{E}(Y \mid X = x, A = a)\) for each \(a \in \{0, 1\}\) under nonparametric conditions. Let \(\hat{\psi}_t^\ast\) denote the doubly robust cross-fit estimator (3.5) built using estimates of \(\pi\), \(\mu^1\), and \(\mu^0\). Then the following theorem provides the conditions under which we can construct Asympt-CSs for \(\psi\) in observational studies.

**Theorem 4** (Confidence sequence for the ATE in observational studies). Consider the same setup as Theorem 3 but with \(\pi(x)\) no longer known. Suppose that regression functions and propensity scores are consistently estimated in \(L_2(\mathbb{P})\) at a product rate of \(o(\sqrt{\log \log t/t})\), meaning that

\[
\left\| \hat{\pi}_t - \pi \right\|_{L_2(\mathbb{P})} \sum_{a=0}^{1} \left\| \hat{\mu}_t^a - \mu^a \right\|_{L_2(\mathbb{P})} = o \left( \sqrt{\log \log t/t} \right).
\]

Moreover, suppose that \(\|\hat{f}_t - f\|_{L_2(\mathbb{P})} = o(1)\) where \(f\) is the efficient influence function (3.1) and that \(f(Z)\) has at least four finite moments \(\mathbb{E}|f(Z)|^4 < \infty\). Then for any prespecified constant \(\rho > 0\),

\[
\hat{\psi}_t^\ast \pm \sqrt{\text{var}(\hat{f})} \cdot \sqrt{\frac{2(t\rho^2 + 1)}{t^2\rho^2} \log \left( \frac{\sqrt{t\rho^2 + 1}}{\alpha} \right)}
\]

forms a \((1 - \alpha)\)-Asympt-CS for \(\psi\) with approximation rate \(o(\sqrt{\log \log t/t})\).

The proof in Appendix A.3.2 proceeds similarly to the proof of Theorem 3 by combining Theorem 1 with an analysis of the almost-sure behavior of \((\hat{\psi}_t - \psi)\). Notice that the requirement that nuisance functions are estimated at a product rate of \(o(\sqrt{\log \log t/t})\) is weaker than the usual \(o(1/\sqrt{t})\) rate that appears in the fixed-time doubly robust estimation literature. In fact, this requirement can be weakened to a product rate of \(o(\sqrt{\log t/t})\) but we omit this derivation.
Figure 5: Three 90%-Asymp-CSs for the average treatment effect in an observational study using three different estimators. Unlike the randomized setup, only the nonparametric ensemble (Super Learner) is consistent, since parametric (and especially the unadjusted) estimators are misspecified. Not only is the doubly robust Super Learner confidence sequence converging to \( \psi \), but it is also the tightest of the three models at each time step. For more details on this simulation, see Section B.2.

Unlike the experimental setting of Section 3.2, Theorem 4 requires that \( \hat{\mu}^a_t \) and \( \hat{\pi}_t \) consistently estimate \( \mu^a \) and \( \pi \), respectively. As a consequence, \( \hat{f}_t \) converges to the efficient influence function \( f \) and thus \( \hat{\psi}_t \) not only consistently estimates \( \psi \) but also attains the nonparametric efficiency bound. Unadjusted estimators or those built using misspecified models may neither be efficient nor consistent (see Figure 5).

### 3.4 Time-varying treatment effects

The results in Sections 3.2 and 3.3 considered the classical regime where the ATE \( \psi \) is a fixed functional that does not change over time. Consider a strict generalization where distributions — and hence individual treatment effects in particular — may change over time. In other words,

\[
\psi_t := \mathbb{E} \{ Y_t^1 - Y_t^0 \} = \mathbb{E} \{ \mathbb{E}(Y_t | X_t, A_t = 1) - \mathbb{E}(Y_t | X_t, A_t = 0) \},
\]

where the equality (\( \ast \)) holds under the usual causal identification assumptions (IA1)–(IA3). Despite the non-stationary and non-iid structure, it is nevertheless possible to derive CSs for the time-varying average treatment effect \( \tilde{\psi}_t := \frac{1}{t} \sum_{i=1}^t \psi_i \) using the Lyapunov-type bounds of Theorem 2. However, given this more general and complex setup, the assumptions required are more subtle (but no more restrictive) than those for Theorems 3 and 4; as such, we explicitly describe their details here.

**Assumption 4** (Regression estimator behavior in \( L_2(\mathbb{P}) \)). We assume that regression estimators \( \hat{\mu}_t^a(Z_i) \) converge to any function \( \bar{\mu}^a \) in \( L_2(\mathbb{P}) \) regardless of the distribution of \( Z_i \), i.e.

\[
\sup_{1 \leq i \leq \infty} \| \hat{\mu}_t^a(X_i) - \bar{\mu}^a(X_i) \|_{L_2(\mathbb{P})} = o(1)
\]

for each \( a \in \{0, 1\} \).
Assumption 4 simply requires that the regression estimator $\hat{\mu}_t^a$ must converge to some function $\mu^a$, which need not coincide with true regression function $\mu^a$. In the iid setting where $X_1, X_2, \ldots$ all have the same distribution, we would simply drop the $\sup_{1 \leq i \leq n}$, recovering the conditions for Theorems 3 and 4.

**Assumption 5** (Iterated logarithm convergence of average nuisance errors). Let $\hat{\mu}_t^a$ be an estimator of the regression function $\mu^a$, $a \in \{0, 1\}$ and $\hat{\pi}_t$ an estimator of the propensity score $\pi$. We assume that the average bias shrinks at an LIL rate, i.e.,

$$
1 \sum_{i=1}^t \left\{ \frac{1}{n} \sum_{a=0}^1 \left( \| \hat{\pi}_t^a(X_i) - \pi(X_i) \|_{L_2(P)} \right) \right\} = o \left( \frac{\log \log t}{t} \right). \tag{3.9}
$$

Note that Assumption 5 would hold in two familiar scenarios. Firstly, in a randomized experiment (Theorem 5) where $\hat{\pi}_t = \pi$ is known by design, we have that (3.9) is always zero, satisfying Assumption 5 trivially. Second, in an observational study (Theorem 6) where the product of errors $\| \hat{\pi}_t(X_i) - \pi(X_i) \|_{L_2(P)} \| \hat{\mu}_t^a(X_i) - \mu^a(X_i) \|_{L_2(P)}$ vanishes at a rate of $\sqrt{\log \log t}/t$, for each $i$ and for both $a \in \{0, 1\}$, we also have that their average product errors vanish at the same rate (3.9). With these assumptions in mind, let us summarize how time-varying treatment effects can be captured in randomized experiments.

**Theorem 5** (Confidence sequences for time-varying effects in randomized experiments). Suppose $Z_1, Z_2, \ldots$ are independent triples $Z_i := (X_i, A_i, Y_i)$ and consider the individual treatment effects given by $\psi_i := \mathbb{E}(Y_i \mid X_i, A_i = 1) - \mathbb{E}(Y_i \mid X_i, A_i = 0)$. Suppose that regression estimators converge to some limit (Assumption 4). Assume that the treatment mechanism $\pi(x) := \mathbb{P}(A_i \mid X_i = x)$ is known, and hence we have that Assumption 5 holds by design. Finally, suppose that Assumptions 1, 2, and 3 hold but with $(Y_i)_{i=1}^\infty$ replaced by the influence functions $(\hat{f}(Z_i))_{i=1}^\infty$. Then,

$$
\hat{\psi}_t^{\hat{\pi}} \pm \sqrt{\frac{2(2p^2 \varpi(f) + 1)}{t^2 p^2}} \log \left( \frac{\sqrt{t} p^2 \varpi(f) + 1}{\alpha} \right) \tag{3.10}
$$

forms a $(1 - \alpha)$-Asymp-CS for $\hat{\psi}_t := \frac{1}{t} \sum_{i=1}^t \psi_i$ with approximation rate $o(\sqrt{\log \log t}/t)$.

The proof can be found in Section A.4. The important takeaway from Theorem 5 is that under some rather mild conditions on the variance of $(\hat{f}(Z_i))_{i=1}^\infty$, it is possible to derive an Asymp-CS for a time-varying treatment effect $\psi_t$. Nevertheless, under the commonly considered regime where the treatment effect is constant $\psi_1 = \psi_2 = \cdots = \psi$, we have that (3.10) forms a $(1 - \alpha)$-Asymp-CS for $\psi$. Let us now consider the extension to observational studies with unknown propensity scores.

**Theorem 6** (Confidence sequences for time-varying effects in observational studies). Consider the same setup as Theorem 5 but with $\pi(x)$ no longer known. Suppose we have estimators $(\hat{\pi}_t, \hat{\mu}_t^1, \hat{\mu}_t^0)$ such that Assumption 5 holds. Finally, suppose that Assumptions 1, 2, and 3 hold but with $(Y_i)_{i=1}^\infty$ replaced by the efficient influence functions $(f(Z_i))_{i=1}^\infty$. Then,

$$
\hat{\psi}_t^{\hat{\pi}} \pm \sqrt{\frac{2(2p^2 \varpi(f) + 1)}{t^2 p^2}} \log \left( \frac{\sqrt{t} p^2 \varpi(f) + 1}{\alpha} \right) \tag{3.11}
$$

forms a $(1 - \alpha)$-Asymp-CS for $\hat{\psi}_t := \frac{1}{t} \sum_{i=1}^t \psi_i$ with approximation rate $o(\sqrt{\log \log t}/t)$.

The proof can be found in Section A.4. Similar to Theorem 5, the important takeaway lies in the fact that it is possible to derive an Asymp-CS for the time-varying treatment effect $\psi_t$, but that nevertheless reduces to an Asymp-CS for $\psi$ in the familiar regime of a constant ATE $\psi_1 = \psi_2 = \cdots = \psi$. 

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Figure 6: Three 90\% Asymp-CSs for \( \tilde{\psi}_t \) constructed using various estimators via Theorem 5. Since this is a randomized experiment, all three CSs capture \( \tilde{\psi}_t \) uniformly over time with high probability. Similar to Figure 4, however, the doubly robust estimator constructed via Super Learning greatly outperforms those based on parametric or unadjusted estimators.

4 Application to the effects of IV fluid caps in sepsis patients

Let us now illustrate the use of doubly robust confidence sequences by sequentially estimating the effect of fluid-restrictive strategies on mortality in an observational study of real sepsis patients. We will use data from the Medical Information Mart for Intensive Care III (MIMIC-III), a freely available database consisting of health records associated with more than 45,000 critical care patients at the Beth Israel Deaconess Medical Center [18, 27]. The data are rich, containing demographics, vital signs, medications, and mortality, among other information collected over the span of 11 years.

Following Shahn et al. [38], we aim to estimate the effect of restricting intravenous (IV) fluids within 24 hours of intensive care unit (ICU) admission on 30-day mortality in sepsis patients. In particular, we considered patients at least 16 years of age satisfying the Sepsis-3 definition — i.e. those with a suspected infection and a Sequential Organ Failure Assessment (SOFA) score of at least 2 [40]. Sepsis-3 patients can be obtained from MIMIC-III using SQL scripts provided by Johnson and Pollard [17], but we provide detailed instructions for reproducing our data collection and analysis process on GitHub\(^6\). This resulted in a total of 5231 sepsis patients, each of whom received out-of-hospital followup of at least 90 days.

We considered IV fluid intake within 24 hours of ICU admission \( \mathcal{L}^{24h} \). To construct a binary treatment \( A \in \{0, 1\} \), we dichotomized \( \mathcal{L}^{24h} \) so that \( A_i = 1(\mathcal{L}^{24h}_i \leq 6L) \). The 30-day mortality \( Y \) was defined as 1 if the patient died within 30 days of hospital admission, and 0 otherwise. Baseline covariates \( X \) included for modelling consisted of the patients’ age and sex, whether they are diabetic, modified Elixhauser scores [51], and SOFA scores. We are interested in the causal estimand,

\[
\psi := \mathbb{P}\left( Y \mathcal{L}^{24h} \leq 6L = 1 \right) - \mathbb{P}\left( Y \mathcal{L}^{24h} > 6L = 1 \right),
\]

which is the difference in average 30-day mortality that would be observed if all sepsis patients were randomly assigned an IV fluid level according to the lower truncated distribution \( \mathbb{P}(\mathcal{L}^{24h} \leq l \mid l \leq 6L) \) versus the upper truncated distribution \( \mathbb{P}(\mathcal{L}^{24h} \leq l \mid l > 6L) \) [11]. While this is technically a stochastic

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\[^6\]github.com/WannabeSmith/drconfseq/tree/main/paper_plots/sepsis
intervention effect, we have that under causal identification assumptions (IA1)–(IA3), $\psi$ is identified as

$$\psi = E \{ E(Y | X, A = 1) - E(Y | X, A = 0) \},$$

which is the same functional considered in the previous sections. Therefore, we can estimate $\psi$ under the same assumptions and with the same techniques as Section 3.3. Similar to the simulations in Sections B.1 and B.2, we produced confidence sequences for $\psi$ using unadjusted, parametric, and Super Learner estimators to demonstrate the impacts of different modelling choices on estimation (see Figure 7).

Figure 7: Three 90%-Asymp-CSs for the effect of capped IV fluid intake (defined as $\leq 6$ litres) on 30-day mortality using the same three estimators as those outlined for Figure 5. Notice that an analysis using unadjusted estimators would conclude that the treatment effect is negative after observing fewer than 1500 patients.

**Remark 2.** These simple binary treatment and outcome variables were used so that the methods outlined in Section 3.3 are immediately applicable, but as we will discuss in Section 5, our confidence sequences may be used to sequentially estimate other causal functionals.

Our Super Learner-based confidence sequences cover the null treatment effect of 0 from the 1000th to the 5231st observed patient, and thus we cannot conclude with confidence whether 6L IV fluid caps have an effect on 30-day mortality in sepsis patients. Note that the Super Learner-based confidence sequences nearly drop below 0 after observing the 5231st patient’s outcome. If we were using fixed-time confidence intervals, we would need to resist the temptation to resume data collection (e.g. to see whether the null hypothesis $H_0 : \psi = 0$ could be rejected with a slightly larger sample size) as this would inflate type-I error rates (as seen in Figure 1). On the other hand, our confidence sequences permit exactly this form of continued sampling.

## 5 Extensions to general functional estimation

The discussion thus far has been focused on deriving confidence sequences for the ATE in the context of causal inference. However, the tools presented in this paper are more generally applicable to any pathwise differentiable functional with positive semiparametric information bound. Here we list some prominent examples in causal inference:
where $R$ is an instrumental variable, $M$ is a mediator, and the notation $\bar{a}_s$ is shorthand for the tuple $(a_1, a_2, \ldots, a_s)$. Some examples outside of causal inference include

- Expected density: $\mathbb{E}\{p(X)\}$;
- Entropy: $-\mathbb{E}\{\log p(X)\}$;
- Expected conditional variance: $\mathbb{E}\{\text{var}(Y \mid X)\}$,

where $p$ is the density of the random variable $X$.

All of the aforementioned problems, including estimation of the ATE in Section 3 can be written in the following general form. Suppose $Z_1, Z_2, \ldots \sim Q$ and let $\theta(Q)$ be some functional (such as those listed above) of the distribution $Q$. In the case of a finite sample size $n$, $\hat{\theta}_n$ is said to be an asymptotically linear estimator [42] for $\theta$ if

$$\hat{\theta}_n - \theta = \frac{1}{n} \sum_{i=1}^{n} \phi(Z_i) + o_Q \left( \frac{1}{\sqrt{n}} \right),$$

where $\phi$ is the influence function of $\hat{\theta}_n$. When the sample size is not fixed in advance, we may analogously say that $\hat{\theta}_t$ is an asymptotically linear time-uniform estimator if instead,

$$\hat{\theta}_t - \theta = \frac{1}{t} \sum_{i=1}^{t} \phi(Z_i) + o \left( \sqrt{\frac{\log \log t}{t}} \right),$$

(5.1)

with $\phi$ being the same influence function as before. For example, in the case of the ATE with $(Z_i)_{i=1}^{\infty} \sim P$, we presented an efficient estimator $\hat{\psi}_t$ which took the form,

$$\hat{\psi}_t - \psi = \frac{1}{t} \sum_{i=1}^{t} \left( f(Z_i) - \psi \right) + o \left( \sqrt{\frac{\log \log t}{t}} \right),$$

where $f$ is the uncentered efficient influence function (EIF) defined in (3.1). In order to justify that the remainder term is indeed $o \left( \sqrt{\log \log t/t} \right)$, we used sequential sample splitting and additional analysis in the randomized and observational settings (see the proofs in Sections A.3 and A.3.2 for more details).

In general, as long as an estimator $\hat{\theta}_t$ for $\theta$ has the form (5.1), we may derive Asymp-CSs for $\theta$ as a simple corollary of Theorem 1.

**Corollary 1.** Suppose $\hat{\theta}_t$ is an asymptotically linear time-uniform estimator of $\theta$ with influence function $\phi$, that is, satisfying (5.1). Additionally, suppose that $\mathbb{E}\{\phi(Z_i)^q\} < \infty$ for some $q > 2$. Then,

$$\hat{\theta}_t \pm \sqrt{\text{var}(\phi)} \cdot \sqrt{\frac{2(t\rho^2 + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{t\rho^2 + 1}}{\alpha} \right)}$$

forms an $(1 - \alpha)$-Asymp-CS for $\theta$. If desired, the iterated logarithm boundary of Proposition 1 can be used here in place of Theorem 1. If computing $\hat{\theta}_t$ additionally involves the estimation of a nuisance parameter $\eta$ such as in Theorems 3 and 4, this must be handled carefully on a case-by-case basis where sequential sample splitting and cross fitting (Section 3.1) may be helpful, and higher moments on $\phi(Z_i)$ may be needed.
6 Conclusion

This paper introduced the notion of an “asymptotic confidence sequence” as the time-uniform analogue of an asymptotic confidence interval based on the central limit theorem. We derived an explicit universal asymptotic confidence sequence for the mean from iid observations under weak moment assumptions by appealing to the strong invariance principles of Komlós et al. [22, 23] and Major [24]. These results were extended to the setting where observations’ distributions (including means and variances) can vary over time, such that our confidence sequences capture a moving parameter — the running average of the means so far. We then applied the aforementioned results to the problem of doubly robust sequential inference for the average treatment effect in both randomized experiments and observational studies under iid sampling. Finally, we showed how these causal applications remain valid in the non-iid setting where distributions change over time, in which case our confidence sequences capture a running average of individual treatment effects. The aforementioned results will enable researchers to continuously monitor sequential experiments — such as clinical trials and online A/B tests — as well as sequential observational studies even if treatment effects do not remain stationary over time.

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A Proofs of the main results

A.1 Proof of Theorem 1

We first introduce two lemmas which will later be used in the main proof of Theorem 1.

Lemma 1 (Almost-sure approximation of the standard deviation under four moments). Suppose \((Y_t)_{t=1}^\infty \sim \mathbb{P}\) and let \(\tilde{\mu}_t = \sum_{i=1}^t Y_i / t\). Consider the sample standard deviation estimator for all \(t \geq 2\),

\[
\hat{\sigma}_t := \sqrt{\frac{\sum_{i=1}^t (Y_i - \tilde{\mu}_t)^2}{t}}.
\]

If \(\mathbb{P}\) has a finite fourth moment, then

\[
\sigma = \hat{\sigma}_t + O \left( \left( \frac{\log \log t}{t} \right)^{1/4} \right).
\]

Proof. Define the partial sums,

\[
S_t := \sum_{i=1}^t (Y_i - \mu), \quad S'_t := \sum_{i=1}^t [Y_i^2 - (\mu^2 + \sigma^2)].
\]

Now, consider the quantity,

\[
\frac{1}{t} S'_t - \left( \frac{1}{t} S_t \right)^2 = \frac{1}{t} \sum_{i=1}^t [Y_i^2 - (\mu^2 + \sigma^2)] - \left( \frac{1}{t} \sum_{i=1}^t (Y_i - \mu) \right)^2
\]

\[
= \frac{1}{t} \sum_{i=1}^t Y_i^2 - \mu^2 - \sigma^2 - \bar{Y}_t \mu - \mu^2
\]

\[
= \tilde{\sigma}_t^2 + (- \mu^2 - \sigma^2 + 2\bar{Y}_t \mu - \mu^2)
\]

\[
= \tilde{\sigma}_t^2 - \sigma^2 + 2\mu (\bar{Y}_t - \mu)
\]

\[
= \tilde{\sigma}_t^2 - \sigma^2 + \frac{2\mu}{t} S_t.
\]

Therefore, we have by the law of the iterated logarithm (LIL),

\[
\sigma^2 - \tilde{\sigma}_t^2 = - \frac{1}{t} S'_t + \left( \frac{1}{t} S_t \right)^2 + \frac{2\mu}{t} S_t.
\]

\[
= O \left( \sqrt{\frac{\log \log t}{t}} \right) + O \left( \frac{\log \log t}{t} \right) + O \left( \sqrt{\frac{\log \log t}{t}} \right)
\]

\[
= O \left( \sqrt{\frac{\log \log t}{t}} \right).
\]

Finally, using the fact that \(|\sqrt{a} - \sqrt{b}| \leq \sqrt{|a - b|}\) for all \(a, b \geq 0\), we have that

\[
\sigma - \tilde{\sigma}_t = O \left( \left( \frac{\log \log t}{t} \right)^{1/4} \right),
\]

completing the proof. \(\blacksquare\)
Lemma 2 (Strong Gaussian approximation of the sample average). Let \((Y_t)_{t=1}^\infty\) be an iid sequence of random variables with mean \(\mu\), variance \(\sigma^2\), and \(q > 2\) finite absolute moments. Suppose \(\sigma_t\) is an almost-surely consistent estimator for \(\sigma\) (such as the sample standard deviation). Then (after sufficiently enriching the probability space), there exist iid Gaussian random variables \((G_t)_{t=1}^\infty\) such that

\[
\frac{1}{t} \sum_{i=1}^{t} (Y_i - \mu) = \frac{\sigma_t}{\sigma} \sum_{i=1}^{t} G_i + \varepsilon_t
\]

where \(\varepsilon_t = o(\sqrt{\log \log t/t})\) if \(q > 2\), and \(\varepsilon_t = O((\log \log t/t)^{3/4})\) if \(q \geq 4\).

Proof. First, by Komlós et al. and Major’s strong approximation theorems (KMT) \([22, 23, 24]\) we have that (after sufficiently enriching the probability space) there exist iid Gaussian random variables \((G_t)_{t=1}^\infty\) such that

\[
\frac{1}{t} \sum_{i=1}^{t} (Y_i - \mu) = \frac{\sigma}{\sigma_t} \sum_{i=1}^{t} G_i + \kappa_t \tag{A.1}
\]

with \(\kappa_t = O(\log t/t)\) if \(Y_1\) has a finite moment generating function, and \(\kappa_t = o(t^{1/q-1})\) if \(Y_1\) has \(q > 2\) finite absolute moments. Since \((G_t)_{t=1}^\infty\) have mean zero and unit variance, we have by the LIL that

\[
\frac{1}{t} \sum_{i=1}^{t} G_i = O \left( \sqrt{ \frac{\log \log t}{t} } \right). \tag{A.2}
\]

Case I: If \(Y_1\) has \(q > 2\) finite absolute moments, then by the strong law of large numbers, \(\hat{\sigma}_t \xrightarrow{a.s.} \sigma\). Combining this fact with (A.1) and, we have (A.2)

\[
\frac{1}{t} \sum_{i=1}^{t} (Y_i - \mu) = \frac{\sigma_t + o(1)}{\sigma} \sum_{i=1}^{t} G_i + \kappa_t
\]

\[
= \frac{\sigma_t}{\sigma} \sum_{i=1}^{t} G_i + \kappa_t + o \left( \sqrt{ \frac{\log \log t}{t} } \right)
\]

\[
= \frac{\sigma_t}{\sigma} \sum_{i=1}^{t} G_i + o \left( \sqrt{ \frac{\log \log t}{t} } \right).
\]

Case II: If \(Y_1\) has at least 4 moments, then by Lemma 1,

\[
\hat{\sigma}_t - \sigma = O \left( (\log \log t/t)^{1/4} \right).
\]

Combining the above with (A.1) and (A.2), we have

\[
\frac{1}{t} \sum_{i=1}^{t} (Y_i - \mu) = \frac{\sigma_t + O \left( (\log \log t/t)^{1/4} \right)}{\sigma} \sum_{i=1}^{t} G_i + \kappa_t
\]

\[
= \frac{\sigma_t}{\sigma} \sum_{i=1}^{t} G_i + \kappa_t + O \left( \left( \frac{\log \log t}{t} \right)^{3/4} \right)
\]

\[
= \frac{\sigma_t}{\sigma} \sum_{i=1}^{t} G_i + O \left( \left( \frac{\log \log t}{t} \right)^{3/4} \right),
\]

which completes the proof. □

Proof of the main theorem The proof proceeds in 3 steps. First, we use the fact that for any martingale \(M_t(\lambda)\), we have that \(\int \lambda \, M_t(\lambda) dF(\lambda)\) is also a martingale where \(F\) is any probability
distribution on \( \mathbb{R} [15, 16] \). We apply this fact to an exponential Gaussian martingale and use a Gaussian density \( f(\lambda; 0, \rho^2) \) as the mixing distribution. Second, we apply Ville’s inequality [52] to this mixture exponential Gaussian martingale to obtain Robbins’ normal mixture confidence sequence [31]. Third, we use Lemma 2 to approximate \( \sum_{i=1}^{n}(Y_i - \mu) \) by a cumulative sum of Gaussian random variables and apply the results from steps 1 and 2.

**Proof.** **Step 1.** Let \((G_i)_{i=1}^{\infty} \) be a sequence of iid standard Gaussian random variables and define their cumulative sum \( W_t := \sum_{i=1}^{t} G_i \). Write the exponential process for any \( \lambda \in \mathbb{R} 

\[ M_t(\lambda) := \exp \left\{ \lambda W_t - t \lambda^2 / 2 \right\}. \]

It is well-known that \( M_t(\lambda) \) is a nonnegative martingale starting at \( M_0 = 1 \) with respect to the canonical filtration \((\mathcal{F}_{i})_{i=0}^{\infty}\) where \( \mathcal{F}_i := \sigma(X_1, \ldots, X_i) \) is the sigma-field generated by \( X_1, \ldots, X_i \) and \( \mathcal{F}_0 \) is the trivial sigma-field [31]. In particular, consider the Gaussian probability distribution function \( f(\lambda; 0, \rho^2) \) with mean zero and variance \( \rho^2 > 0 \) as the mixing distribution. The resulting martingale can be written as

\[
M_t := \int_{\lambda \in \mathbb{R}} \exp \left\{ \lambda W_t - \frac{t \lambda^2}{2} \right\} f(\lambda; 0, \rho^2) d\lambda
\]

\[
= \frac{1}{\sqrt{2\pi\rho^2}} \int_{\lambda} \exp \left\{ \lambda W_t - \frac{t \lambda^2}{2} \right\} \exp \left\{ -\frac{\lambda^2}{2 \rho^2} \right\} d\lambda
\]

\[
= \frac{1}{\sqrt{2\pi\rho^2}} \int_{\lambda} \exp \left\{ \lambda W_t - \frac{\lambda^2(t \rho^2 + 1)}{2 \rho^2} \right\} d\lambda
\]

\[
= \frac{1}{\sqrt{2\pi\rho^2}} \int_{\lambda} \exp \left\{ -\frac{\lambda^2(t \rho^2 + 1) + 2 \lambda \rho^2 W_t}{2 \rho^2} \right\} d\lambda
\]

\[
= \frac{1}{\sqrt{2\pi\rho^2}} \int_{\lambda} \exp \left\{ -a(\lambda + b \lambda^2) \right\} d\lambda
\]

by setting \( a := t \rho^2 + 1 \) and \( b := \rho^2 W_t \). Focusing on the integrand and completing the square, we have

\[
\exp \left\{ -\frac{\lambda^2 + 2 \lambda \frac{b}{a} + \left( \frac{b}{a} \right)^2}{2 \rho^2 / a} \right\} = \exp \left\{ -\frac{(\lambda - b/a)^2}{2 \rho^2 / a} + \frac{a(b/a)^2}{2 \rho^2} \right\}
\]

\[
= \exp \left\{ -\frac{(\lambda - b/a)^2}{2 \rho^2 / a} \right\} \exp \left\{ \frac{b^2}{2 \rho^2 a} \right\}.
\]

Plugging this back into the integral and multiplying the entire quantity by \( \frac{e^{-1/2}}{a^{1/2}} \), we finally get the closed-form expression of the mixture exponential Wiener process,

\[
M_t := \exp \left\{ \frac{\rho^2 W_t^2}{2(G_{t+1})} \right\} \sqrt{t \rho^2 + 1}.
\]

(A.3)
Step 2. Since $M_t$ is a nonnegative martingale with initial value one, we have by Ville’s inequality \[ \mathbb{P}(\forall t \geq 1, \ M_t < 1/\alpha) \geq 1 - \alpha. \] Writing this out explicitly for $M_t$ and solving for $W_t$ algebraically, we have that
\[
\mathbb{P}\left(\forall t \geq 1, \ \frac{\rho^2 W_t^2}{2(t\rho^2 + 1)} < \log(1/\alpha) + \log\left(\frac{\sqrt{t\rho^2 + 1}}{\alpha}\right)\right)
= \mathbb{P}\left(\forall t \geq 1, \ \frac{1}{t} \sum_{i=1}^{t} G_i \right) \leq \sqrt{\frac{2(t\rho^2 + 1)}{t^2 \rho^2}} \log\left(\frac{\sqrt{t\rho^2 + 1}}{\alpha}\right) \geq 1 - \alpha.
\]

Step 3. First, note that by the triangle inequality,
\[
\left| \frac{1}{t} \sum_{i=1}^{t} Y_i - \mu \right| \leq \left| \frac{1}{t} \sum_{i=1}^{t} (Y_i - \mu) - \frac{\hat{\sigma}_t}{t} \sum_{i=1}^{t} G_i \right| + \frac{\hat{\sigma}_t}{t} \sum_{i=1}^{t} G_i,
\]
and thus by Lemma 2 and Step 2, we have with probability at least $(1 - \alpha)$,
\[
\forall t \geq 1, \ \left| \frac{1}{t} \sum_{i=1}^{t} Y_i - \mu \right| < \frac{\hat{\sigma}_t}{t} \sqrt{\frac{2(t\rho^2 + 1)}{t^2 \rho^2}} \log\left(\frac{\sqrt{t\rho^2 + 1}}{\alpha}\right) + \varepsilon_t
\]
where $\varepsilon_t$ is defined as in Lemma 2. This completes the proof. \hfill \Box

A.2 Proof of Theorem 2

Proof. The proof proceeds in three steps. First, we use a similar technique to that of Theorem 1 to obtain a nonnegative martingale for Gaussian observations with time-varying means and variances. Second, we combine Step 1 with Assumption 2 and a strong approximation theorem due to Shao \cite{shao1995approximation} to obtain an AsympCS in terms of the (unknown) time-varying variances $\hat{\sigma}_t^2$. Second and finally, we use Assumptions 1 and 3 to obtain the same AsympCS but with $\hat{\sigma}_t^2$ replaced by the empirical variance $\sigma_t^2$.

Step 1: A Gaussian martingale for time-varying means and variances. Let $(G_t)_{t=1}^{\infty}$ be a sequence of iid standard Gaussian random variables. Define $\sigma_t^2 := \text{var}(Y_t)$ and note that
\[
M_t^{\text{ind}}(\lambda) := \exp\left\{ \frac{t}{\lambda} \sum_{i=1}^{t} (\lambda \sigma_i G_i - \lambda^2 \sigma_i^2/2) \right\}
\]
is a nonnegative martingale starting at one. Mixing over $\lambda$ with the probability density $dF(\lambda)$ of a mean-zero Gaussian with variance $\rho^2$ as in the proof of Theorem 1, we have that
\[
\tilde{M}_t := \int_{\lambda \in \mathbb{R}} \tilde{M}_t(\lambda) dF(\lambda) = \exp\left\{ \frac{\rho^2 (\sum_{i=1}^{t} \sigma_i G_i)^2}{2(t\hat{\sigma}_t^2 \rho^2 + 1)} \right\} \cdot (t \hat{\sigma}_t^2 \rho^2 + 1)^{-1/2}
\]
is also a martingale. By Ville’s inequality for nonnegative (super)martingales, we have that $F(\exists t : \tilde{M}_t \geq 1/\alpha) \leq \alpha$ and hence with probability at least $(1 - \alpha)$,
\[
\forall t \geq 1, \ \left| \frac{1}{t} \sum_{i=1}^{t} \sigma_i G_i \right| < \sqrt{\frac{2(t\hat{\sigma}_t^2 \rho^2 + 1)}{t^2 \rho^2} \log\left(\frac{\sqrt{t\hat{\sigma}_t^2 \rho^2 + 1}}{\alpha}\right)}.
\]
Step 2: Strong approximation via Shao [39, Theorem 1.3]. Now, recall Shao’s strong approximation theorem [39, Theorem 1.3] which states that if \( \sum_{i=1}^{\infty} \mathbb{E}|Y_i|^q/H_i^q < \infty \) for some nondecreasing sequence \( (H_i)_{i=1}^{\infty} \), then there exists a sequence of iid standard Gaussians \( (G_i)_{i=1}^{\infty} \) such that

\[
\sum_{i=1}^{t} (Y_i - \mu_i) = \sum_{i=1}^{t} \sigma_i G_i + o(H_t)
\]

almost surely. By our Assumption 2, we have that \( \sum_{i=1}^{\infty} \mathbb{E}|Y_i|^q/t^{1+\delta} < \infty \), and hence we can apply Shao [39, Theorem 1.3] with \( H_t = t^{(1+\delta)/q} \) to obtain the following strong invariance principle,

\[
\sum_{i=1}^{t} (Y_i - \mu_i) = \sum_{i=1}^{t} \sigma_i G_i + o(t^{(1+\delta)/q}).
\]

Therefore, with probability at least \( (1 - \alpha) \),

\[
\forall t \geq 1, \quad \frac{1}{t} \sum_{i=1}^{t} (Y_i - \mu_i) \leq \sqrt{\frac{2(t\hat{\sigma}_t^2 \rho^2 + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{t\hat{\sigma}_t^2 \rho^2 + 1}}{\alpha} \right) + o \left( t^{(1+\delta)/q - 1} \right)}.
\]

In particular, since \( q > 2(1 + \delta) \) by assumption, we have that \( t^{(1+\delta)/q} < \sqrt{\log \log t} \), and hence

\[
\hat{\mu}_t \pm \sqrt{\frac{2(t\hat{\sigma}_t^2 \rho^2 + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{t\hat{\sigma}_t^2 \rho^2 + 1}}{\alpha} \right)}
\]

forms a \((1 - \alpha)\)-AsympCS for \( \hat{\mu}_t \).

Step 2: Deriving an AsympCS in terms of the empirical variance \( \hat{\sigma}_t^2 \). Writing out the margin of (A.5) combined with the assumption \( \hat{\sigma}_t^2 - \sigma_t^2 = o(1) \), we have that

\[
\sqrt{\frac{2(t\hat{\sigma}_t^2 \rho^2 + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{t\hat{\sigma}_t^2 \rho^2 + 1}}{\alpha} \right)} = \sqrt{\frac{2((t\hat{\sigma}_t^2 + o(1)) \rho^2 + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{(t\hat{\sigma}_t^2 + o(1)) \rho^2 + 1}}{\alpha} \right)}
\]

\[
= \sqrt{\frac{t(\hat{\sigma}_t^2 + o(1)) \rho^2 + 1}{t^2 \rho^2} \log \left( \frac{t(\hat{\sigma}_t^2 + o(1)) \rho^2 + 1}{\alpha^2} \right)}
\]

\[
= \sqrt{\frac{t^2 \rho^2 + o(t) + 1}{t^2 \rho^2} \log \left( \frac{t^2 \rho^2 + o(t) + 1}{\alpha^2} \right)}
\]

\[
= \sqrt{\frac{t^2 \rho^2 + 1}{t^2 \rho^2} + o(1/t)} \log \left( \frac{t^2 \rho^2 + o(1) + 1}{\alpha^2} \right). \tag{A.6}
\]
Focusing on the logarithmic factor, we have

\[
\log \left( \frac{t \hat{\sigma}_t^2 \rho^2 + o(t) + 1}{\alpha^2} \right) = \log \left( \frac{1 + t \hat{\sigma}_t^2 \rho^2}{\alpha^2} + o(t) \right)
\]

where the last two lines follow from the assumption that \( \hat{\sigma}_t^2 = \omega(1) \) and the Taylor expansion \( \log(1 + x) = x + o(1) \) for \( |x| < 1 \). Combining (A.6) and (A.7), we have that the margin of (A.5) can be written as

\[
\sqrt{\frac{2(t \hat{\sigma}_t^2 \rho^2 + 1)}{t^2 \rho^2}} \log \left( \frac{\sqrt{t \hat{\sigma}_t^2 \rho^2 + 1}}{\alpha} \right) = \sqrt{\frac{2(t \hat{\sigma}_t^2 \rho^2 + 1)}{t^2 \rho^2}} \log \left( \frac{1 + t \hat{\sigma}_t^2 \rho^2}{\alpha^2} + o(1) \right)
\]

where the last inequality follows from \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \) for \( a, b \geq 0 \). In particular, this means that

\[
(\hat{\mu}_t \pm \tilde{B}_t) := \left( \hat{\mu}_t \pm \sqrt{\frac{2(t \hat{\sigma}_t^2 \rho^2 + 1)}{t^2 \rho^2}} \log \left( \frac{\sqrt{t \hat{\sigma}_t^2 \rho^2 + 1}}{\alpha} \right) + o \left( \sqrt{\log \frac{t}{t}} \right) \right)
\]

forms a nonasymptotic \((1 - \alpha)\)-CS for \( \hat{\mu}_t \), meaning \( \mathbb{P} \left( \exists t \geq 1 : \hat{\mu}_t \notin (\hat{\mu}_t \pm \tilde{B}_t) \right) \leq \alpha \). Combined with Assumption 1, we have that

\[
(\hat{\mu}_t \pm \tilde{B}_t) := \left( \hat{\mu}_t \pm \sqrt{\frac{2(t \hat{\sigma}_t^2 \rho^2 + 1)}{t^2 \rho^2}} \log \left( \frac{\sqrt{t \hat{\sigma}_t^2 \rho^2 + 1}}{\alpha} \right) \right)
\]

forms a \((1 - \alpha)\)-AsympCS for \( \hat{\mu}_t \) since \( \tilde{B}_t = \omega(\sqrt{\log \frac{t}{t}}) \). This completes the proof.

\[\square\]

### A.3 Proof of Theorems 3 and 4

In the proofs that follow, we will make extensive use of some convenient notation, namely the sample average operator \( \mathbb{E}_t f(Z) = \frac{1}{t} \sum_{i=1}^{t} f(Z_i) \) and the conditional expectation operator \( \mathbb{E} f(Z) \equiv \mathbb{E}(f(Z) \mid Z_1', \ldots, Z_n') \) where \( Z_1', \ldots, Z_n' \) are the data used to construct \( \hat{f} \).
First, let us analyze the almost-sure behavior of the doubly robust estimator \( \hat{\psi}_t \) for the average treatment effect \( \psi \).

**Lemma 3** (Decomposition of \( \hat{\psi}_t - \psi \)). Let \( \hat{\psi}_t := \mathbb{P}_T(\hat{f}_T) = \frac{1}{T} \sum_{t=1}^{T} \hat{f}_T(Z_{\text{val}}) \) be a (possibly misspecified) estimator of \( \psi := \mathbb{P}(f) = \mathbb{E}(f(Z_{\text{val}})) \) based on \( (Z_{\text{val}}^1, \ldots, Z_{\text{val}}) \) where \( \hat{f}_T \) can be any estimator built from \( (Z_1^{\text{trn}}, \ldots, Z_T^{\text{trn}}) \) and \( f : \mathcal{Z} \rightarrow \mathbb{R} \) any function. Furthermore, assume that there exists \( \bar{f} \) such that \( \|\hat{f}_T - \bar{f}\|_{L_2(\mathbb{P})} \to 0 \). In other words, \( \hat{f}_T \) is an estimator of \( \bar{f} \) but may instead converge to \( \bar{f} \). Then we have the decomposition,

\[
\hat{\psi}_t - \psi = \Gamma^{SA}_t + \Gamma^{EP}_t + \Gamma^{B}_t
\]

where

\[
\Gamma^{SA}_t := (\mathbb{P}_T - \mathbb{P}) \hat{f} \quad \text{is the centered sample average term,}
\]

\[
\Gamma^{EP}_t := (\mathbb{P}_T - \mathbb{P})(\hat{f} - \bar{f}) \quad \text{is the empirical process term, and}
\]

\[
\Gamma^{B}_t := \mathbb{P}(\hat{f} - \bar{f}) \quad \text{is the bias term.}
\]

**Proof.** By definition of the quantities involved, we decompose

\[
\hat{\psi}_t - \psi = \mathbb{P}_T(\hat{f}_T) - \mathbb{P}(f)
= (\mathbb{P}_T - \mathbb{P})(\hat{f}_T) + \mathbb{P}(\hat{f}_T - \bar{f})
= (\mathbb{P}_T - \mathbb{P})(\hat{f}_T - \bar{f}) + (\mathbb{P}_T - \mathbb{P})\bar{f} + \mathbb{P}(\hat{f}_T - \bar{f}),
\]

which completes the proof. \( \square \)

Now, let us analyze the almost-sure behaviour of the empirical process term \( \Gamma^{EP}_t \) and the bias term \( \Gamma^{B}_t \) to show that they vanish asymptotically at sufficiently fast rates. First, let us examine \( \Gamma^{EP}_t \).

**Lemma 4** (Almost sure convergence of \( \Gamma^{EP}_t \)). Let \( \mathbb{P}_T \) denote the empirical measure over \( Z_1^{\text{val}} := (Z_1^{\text{val}}, \ldots, Z_T^{\text{val}}) \) and let \( \hat{f}_T(z) \) be any function estimated from a sample \( D_1^{\text{trn}} = (Z_1^{\text{trn}}, Z_2^{\text{trn}}, \ldots, Z_T^{\text{trn}}) \) which is independent of \( D_1^{\text{val}} \). If \( \hat{\pi}_i \in [\delta, 1 - \delta] \) almost surely, then,

\[
\Gamma^{EP}_t := (\mathbb{P}_T - \mathbb{P})(\hat{f}_T - \bar{f}) = O \left( \left\{ \sum_{a=0}^{1} \|\hat{\pi}_a^{\text{trn}} - \bar{\pi}^a\|_{L_2(\mathbb{P})} \right\} \sqrt{\frac{\log \log t}{t}} \right).
\]

In particular, if \( \|\hat{\pi}_a^{\text{trn}} - \bar{\pi}^a\|_{L_2(\mathbb{P})} = o(1) \) for each \( a \), then we have that \( \Gamma^{EP}_t \) almost-surely converges to 0 at a rate of \( o(\sqrt{\log \log t/t}) \), but possibly faster.

The proof proceeds in two steps. First, we use an argument from Kennedy et al. [21] and the law of the iterated logarithm to show \( \Gamma^{EP}_t = O \left( \|\hat{f}_t - \bar{f}\|\sqrt{\log \log t/t} \right) \). Second and finally, we upper bound \( \|\hat{f}_t - \bar{f}\| \) by \( O \left( \sum_{a=0}^{1} \|\hat{\pi}_a^{\text{trn}} - \bar{\pi}^a\| \right) \).

**Proof. Step 1.** Following the proof of Kennedy et al. [21, Lemma 2], we have that conditional on \( D_x^{\text{trn}} := (Z_1^{\text{trn}})_{i=1}^{T_x} \) and \( S_x^{\text{trn}} := (1(Z_i \in D_x^{\text{trn}}))_{i=1}^{T_x} \) the term of interest has mean zero:

\[
\mathbb{E} \left\{ \mathbb{P}_T(\hat{f}_T - \bar{f}) \mid D_x^{\text{trn}}, S_x^{\text{trn}} \right\} = \mathbb{E}(\hat{f}_T - \bar{f} \mid D_x^{\text{trn}}, S_x^{\text{trn}}) = \mathbb{P}(\hat{f}_T - \bar{f}).
\]

Now, we upper bound the conditional variance of a single summand,

\[
\text{var} \left\{ (1 - \mathbb{P})(\hat{f}_T - \bar{f}) \mid D_x^{\text{trn}}, S_x^{\text{trn}} \right\} = \text{var} \left\{ (\hat{f}_T - \bar{f}) \mid D_x^{\text{trn}}, S_x^{\text{trn}} \right\} \leq \|\hat{f}_T - \bar{f}\|^2.
\]
In particular, this means that
\[
\left( \frac{T(P_T - P)(\hat{f}_T - \hat{f})}{\|\hat{f}_T - \hat{f}\|} \right)_{\mathcal{D}_\infty^{\text{trn}}, \mathcal{S}_\infty^{\text{trn}}} \]
is a sum of iid random variables with conditional mean zero and conditional variance at most 1, and thus by the law of the iterated logarithm,
\[
P \left( \limsup_{t \to \infty} \frac{\pm \sqrt{T}(P_T - P)(\hat{f}_T - \hat{f})}{\sqrt{2 \log \log T \|\hat{f}_T - \hat{f}\|}} \leq 1 \mid \mathcal{D}_\infty^{\text{trn}}, \mathcal{S}_\infty^{\text{trn}} \right) = 1.
\]
Therefore, we have that
\[
P \left( \frac{(P_T - P)(\hat{f}_T - \hat{f})}{\|\hat{f}_T - f\|\sqrt{\log \log t/t}} = O(1) \mid \mathcal{D}_\infty^{\text{trn}}, \mathcal{S}_\infty^{\text{trn}} \right)
= \mathbb{P} \left( \frac{\|\hat{f}_T - \hat{f}\|}{\|\hat{f}_T - f\|} \frac{\sqrt{T}(P_T - P)(\hat{f}_T - \hat{f})}{\sqrt{\log \log t\|\hat{f}_T - \hat{f}\|}} = O(1) \mid \mathcal{D}_\infty^{\text{trn}}, \mathcal{S}_\infty^{\text{trn}} \right)
= \mathbb{P} \left( \lim_{t \to \infty} \frac{\|\hat{f}_T - \hat{f}\|}{\|\hat{f}_T - f\|} \frac{\sqrt{T}(P_T - P)(\hat{f}_T - \hat{f})}{\sqrt{\log \log t\|\hat{f}_T - \hat{f}\|}} = O(1) \mid \mathcal{D}_\infty^{\text{trn}}, \mathcal{S}_\infty^{\text{trn}} \right) = 1.
\]
Finally, by iterated expectation,
\[
P \left( \frac{(P_T - P)(\hat{f}_T - \hat{f})}{\|\hat{f}_T - f\|\sqrt{\log \log t/t}} = O(1) \right) = \mathbb{E} \left[ \mathbb{P} \left( \frac{(P_T - P)(\hat{f}_T - \hat{f})}{\|\hat{f}_T - f\|\sqrt{\log \log t/t}} = O(1) \mid \mathcal{D}_\infty^{\text{trn}}, \mathcal{S}_\infty^{\text{trn}} \right) \right] = \mathbb{E}1 = 1,
\]
which completes Step 1.

**Step 2.** Now, let us upper bound \( \|\hat{f}_T - f\| \) by \( O \left( \sum_{n=0}^{1} \|\hat{\mu}_n - \mu_n\| \right) \). To simplify the calculations which follow, define
\[
\hat{f}^1(Z) := \hat{\mu}^1(X_i) + \frac{A_i}{\pi(X_i)} \{ Y_i - \hat{\mu}^1 \} \quad \text{and} \quad f^1(Z) := \mu^1(X_i) + \frac{A_i}{\pi(X_i)} \{ Y_i - \mu^1 \}.
\]
Analogously define \( \hat{f}^0 \) and \( f^0 \) so that \( \hat{f} = \hat{f}^1 - \hat{f}^0 \) and \( f = f^1 - f^0 \). Writing out \( \|\hat{f}_T^1 - \hat{f}_T^1\| \),
\[
\|\hat{f}_T^1 - \hat{f}^1\| = \|\hat{\mu}^1_T + \frac{A}{\pi_T} \{ Y - \hat{\mu}^1_T \} - \frac{A}{\pi} \{ Y - \mu^1 \}\|
\leq \|\hat{\mu}^1_T - \mu^1\| \left( 1 - \frac{A}{\pi_T} \right) \|
\leq \|\hat{\mu}^1_T - \mu^1\| \left( 1 - \frac{A}{\pi_T} \right) \|
\leq \frac{1}{2} \||\hat{\mu}^1_T - \mu^1\| \cdot \|\hat{\pi}_T - A\| = O \left( \|\hat{\mu}^1_T - \mu^1\| \right),
\]

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Almost surely for some $\tilde{\mu}$, we recall this proof here as it is short and illustrative.

**Lemma 5** (Almost-surely bounding $\Gamma_B$ by $L_2(\mathbb{P})$ errors of nuisance functions). Suppose $\tilde{\pi} \in [\delta, 1 - \delta]$ almost surely for some $\delta > 0$. Then

$$
\Gamma_B(\mathbb{P}) = O \left( \|\tilde{\pi} - \pi\|_{L_2(\mathbb{P})} \left\{ \|\tilde{\mu}^1 - \mu^1\|_{L_2(\mathbb{P})} + \|\tilde{\mu}^0 - \mu^0\|_{L_2(\mathbb{P})} \right\} \right)
$$

This is an immediate consequence of the usual proof for $O_\pi$ combined with the fact that expectations are real numbers, and thus stochastic boundedness is equivalent to almost-sure boundedness. For completeness, we recall this proof here as it is short and illustrative.

**Proof.** To simplify the calculations which follow, define

$$
\tilde{f}(Z_i) := \tilde{\mu}^1(X_i) + \frac{A_i}{\pi(X_i)} \left\{ Y_i - \hat{\mu}^A(X_i) \right\} \text{ and } f^1(Z_i) := \mu^1(X_i) + \frac{A_i}{\pi(X_i)} \left\{ Y_i - \mu^1(X_i) \right\}.
$$

Analogously define $\tilde{f}$ and $f^0$ so that $\tilde{f} = \tilde{f}^1 - \tilde{f}^0$ and $f = f^1 - f^0$. Therefore,

$$
\mathbb{P} \left( \tilde{f} - f^1 \right) ^{(i)} = \mathbb{P} \left( \frac{A_i}{\pi} (Y - \hat{\mu}^A) + \tilde{\mu}^1 - \mu^1 \right)
$$

$$
\stackrel{(ii)}{=} \mathbb{P} \left[ \left( \frac{\pi}{\tilde{\pi}} - 1 \right) (\tilde{\mu}^1 - \mu^1) \right]
$$

$$
\stackrel{(iii)}{=} \frac{1}{\delta} \mathbb{P} (|\tilde{\pi} - \pi| |\tilde{\mu}^1 - \mu^1|)
$$

$$
\stackrel{(iv)}{=} \frac{1}{\delta} \|\tilde{\pi} - \pi\|_{L_2(\mathbb{P})} \|\tilde{\mu}^1 - \mu^1\|_{L_2(\mathbb{P})},
$$

where (i) and (ii) follow by iterated expectation, (iii) follows from the assumed bounds on $\tilde{\pi}$, and (iv) by Cauchy-Schwarz. Similarly, we have

$$
\mathbb{P}(\tilde{f} - f^0) \leq \frac{1}{1 - \delta} \|\tilde{\pi} - \pi\| \|\tilde{\mu}^0 - \mu^0\|.
$$

Finally by the triangle inequality,

$$
\mathbb{P}(\tilde{f} - f) = O \left( \|\tilde{\pi} - \pi\| \sum_{a=0}^1 \|\tilde{\mu}^a - \mu^a\| \right),
$$

which completes the proof.

**Lemma 6** (Almost-sure consistency of the influence function variance estimator). Suppose that $\|\tilde{f}_T - f\|_2 = o(1)$ and that $\tilde{f}(Z)$ has a finite fourth moment. Then,

$$
\text{var}_T(\tilde{f}_T) = \text{var}(f) + o(1).
$$
Proof. First, write
\[
\mathbb{V} \text{ar}_T(\hat{f}_T^r) - \text{var}(\bar{f}) = \mathbb{P}_T(\hat{f}_T^r) - \left( \mathbb{P}_T\hat{f}_T^r \right)^2 - \mathbb{P}^2 + (\mathbb{P}\bar{f})^2 \\
= \frac{\mathbb{P}_T(\hat{f}_T^r) - \mathbb{P}^2}{(i)} - \left( \frac{\mathbb{P}_T\hat{f}_T^r)^2 - (\mathbb{P}\bar{f})^2}{(ii)} \right).
\]
We will separately show that (i) and (ii) are $o(1)$.

Almost-sure convergence of (i). Decompose (i) into sample average, empirical process, and bias terms:
\[
\mathbb{P}_T(\hat{f}^2) - \mathbb{P}^2 = \mathbb{P}_T(\hat{f}^2 - f^2) + \mathbb{P}_T(f - \hat{f})^2 + \mathbb{P}(\hat{f}^2 - f^2).
\]
Since $\bar{f}(Z)$ has four finite moments, we have that $\bar{f}(Z)^2$ has a variance. In particular, $\Gamma^SA = o(1)$ by the strong law of large numbers, and $\Gamma^EP = O\left( \|\hat{f}^2 - f^2\|_2 \log \log t/\ell \right)$ by Step 1 of the proof of Lemma 4. By our assumption that $\|\hat{f} - f\|_2 = o(1)$, we have that $\Gamma^EP = o(\log \log t/\ell)$.

Now, let us upper-bound $\Gamma^B$ by $L^2_P$ norms:
\[
\Gamma^B = \mathbb{P}(\hat{f}^2 - f^2) \\
\leq \|\hat{f} - f\| \|\hat{f} + f\| \\
\leq o(1) \quad O(1) \\
= o(1),
\]
where the second inequality follows from Cauchy-Schwartz. Therefore, (i) = $o(1)$.

Almost-sure convergence of (ii) Using the same analysis as above, we have that
\[
\mathbb{P}_T\hat{f} - \mathbb{P}f = o(1),
\]
or equivalently,
\[
\mathbb{P}_T\hat{f} \xrightarrow{a.s.} \mathbb{P}f.
\]
By the continuous mapping theorem,
\[
(\mathbb{P}_T\hat{f})^2 \xrightarrow{a.s.} (\mathbb{P}f)^2,
\]
which completes the proof that (ii) = $o(1)$. Therefore, $\mathbb{V} \text{ar}_T(\hat{f}_T^r) - \text{var}(\bar{f}) = o(1)$.

Proposition 2 (General Asymptotic CEs under sequential cross-fitting). Consider the cross-fit estimator as defined in (3.5):
\[
\hat{\psi}_t^r := \sum_{i=1}^{T_t} f_T(z_{eval}^i) + \sum_{i=1}^{T_t} f_T(z_{trn}^i),
\]
and the cross-fit variance estimator as defined in (3.6):
\[
\mathbb{V} \text{ar}_t(f) := \frac{\mathbb{V} \text{ar}_T(\hat{f}_T^r) + \mathbb{V} \text{ar}_T(\bar{f}_T^r)}{2}.
\]

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Suppose that \( \Gamma^B_t \) and \( \Gamma^{EP}_t \) are both \( o(\sqrt{\log \log t/t}) \). Then,

\[
\hat{\psi}^x_t = \sqrt{\var_t(\hat{f})} \left( \frac{2(t \rho^2 + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{t \rho^2 + 1}}{\alpha} \right) \right)
\]

forms a \((1 - \alpha)\)-Asymp-CS for \( \psi \).

**Proof.** Writing out the centered cross-fit estimator \( \hat{\psi}^x_t - \psi \) using the decomposition of Lemma 3, we have

\[
\begin{aligned}
\hat{\psi}^x_t - \psi &= \frac{\sum_{i=1}^T \hat{f}_T(Z^\text{val}_i) + \sum_{i=1}^{t'} \hat{f}_T(Z^\text{trn}_i) - t\psi}{t} \\
&= \frac{\sum_{i=1}^T (f_T(Z^\text{val}_i) - \psi) + \sum_{i=1}^{t'} (\hat{f}_T(Z^\text{trn}_i) - \psi)}{t} \\
&= \frac{(T\Gamma^\text{SA}_{t,\text{eval}} + TT_{t,\text{eval}}^\text{SA}) + \Gamma^{EP}_t + T\Gamma^{EP}_{t,\text{eval}} + T\Gamma^B_{t,\text{eval}} + T\Gamma^B_{t,\text{trn}}}{t} \\
&= \frac{(\mathbb{P}_t - \mathbb{P})\hat{f}(Z) + \Gamma^{EP}_{t,\text{eval}} + T\Gamma^{EP}_{t,\text{trn}} + T\Gamma^B_{t,\text{eval}} + T\Gamma^B_{t,\text{trn}}}{t} + O(\Gamma^B_t + \Gamma^{EP}_t)
\end{aligned}
\]

(A.9)

where \( \Gamma^\text{EP}_{t,\text{eval}} := \frac{1}{T} \sum_{i=1}^T \hat{f}_T(Z^\text{val}_i) \) and \( \Gamma^\text{EP}_{t,\text{trn}} := \frac{1}{T} \sum_{i=1}^{t'} \hat{f}_T(Z^\text{trn}_i) \), and similarly for \( \Gamma^\text{SA}_{t,\text{eval}}, \Gamma^\text{SA}_{t,\text{trn}}, \Gamma^B_{t,\text{eval}}, \) and \( \Gamma^B_{t,\text{trn}} \). Applying the proof of Theorem 1 (but with variance consistency \( \var_t(\hat{f}) \xrightarrow{a.s.} \var(f) \) obtained via Lemma 6), we have that

\[
\begin{aligned}
\hat{\psi}^x_t &\pm \sqrt{\var_t(\hat{f})} \left( \frac{2(t \rho^2 + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{t \rho^2 + 1}}{\alpha} \right) \right) + o(\sqrt{\log \log t/t})
\end{aligned}
\]

forms a nonasymptotic \((1 - \alpha)\)-CS for \( \psi \). Consequently,

\[
\hat{\psi}^x_t \pm \sqrt{\var_t(\hat{f})} \left( \frac{2(t \rho^2 + 1)}{t^2 \rho^2} \log \left( \frac{\sqrt{t \rho^2 + 1}}{\alpha} \right) \right)
\]

forms a \((1 - \alpha)\)-Asymp-CS for \( \psi \) with rate \( o(\sqrt{\log \log t/t}) \) which completes the proof. \( \square \)

### A.3.1 Proof of Theorem 3

**Proof.** When propensity scores are known, we have that \( \Gamma^B_t = 0 \) by Lemma 5. By assumption, \( \mathbb{E}[\hat{\mu}_T(X) - \mu^a(X)] = o(1) \), and thus by Lemma 4, \( \Gamma^{EP}_t = o(\sqrt{\log \log t/t}) \). Combining these conditions on \( \Gamma^B_t \) and \( \Gamma^{EP}_t \) with Proposition 2, we obtain the desired result. This completes the proof of Theorem 3. \( \square \)

### A.3.2 Proof of Theorem 4

**Proof.** By Lemmas 4 and 5 we have that both \( \Gamma^B_t \) and \( \Gamma^{EP}_t \) are \( o(\sqrt{\log \log t/t}) \). Applying Proposition 2, we obtain the desired result, completing the proof of Theorem 4. \( \square \)
A.4 Proof of Theorems 5 and 6

**Lemma 7** (Decomposition of $t\hat{\psi}^\times - t\tilde{\psi}$). Let $\hat{\psi}^\times$ be as in (3.5). Furthermore, assume that there exists $\bar{f}$ such that $\|\hat{f}_s - \bar{f}\|_{L_1(p)} \to 0$. In other words, $\hat{f}_s$ is an estimator of $f$ but may instead converge to $\bar{f}$. Then we have the decomposition,

$$t\hat{\psi}^\times - t\tilde{\psi} = \bar{S}_t^{SA} + \bar{S}_t^{SE} + \bar{S}_t^{SB} + \bar{S}_t^{T \text{ eval}} + \bar{S}_t^{T \text{ trn}}$$  \hspace{1cm} (A.10)

where

$$\bar{S}_t^{SA} := \sum_{i=1}^t \left[ \bar{f}(Z_i) - \mathbb{P}(\bar{f}(Z_i)) \right],$$

$$\bar{S}_t^{EP} := \sum_{i=1}^T \left\{ \left[ \bar{f}_T(Z_i^{\text{eval}}) - \mathbb{P}(\bar{f}_T(Z_i^{\text{eval}})) \right] - \left[ \bar{f}(Z_i^{\text{eval}}) - \mathbb{P}(\bar{f}(Z_i^{\text{eval}})) \right] \right\},$$

$$\bar{S}_t^{EP} := \sum_{i=1}^{T'} \left\{ \left[ \bar{f}_T(Z_i^{\text{trn}}) - \mathbb{P}(\bar{f}_T(Z_i^{\text{trn}})) \right] - \left[ \bar{f}(Z_i^{\text{trn}}) - \mathbb{P}(\bar{f}(Z_i^{\text{trn}})) \right] \right\},$$

$$\bar{S}_t^{B} := \sum_{i=1}^T \mathbb{P}(\bar{f}_T(Z_i^{\text{eval}}) - f(Z_i^{\text{eval}})), \text{ and}$$

$$\bar{S}_t^{B} := \sum_{i=1}^{T'} \mathbb{P}(\bar{f}_T(Z_i^{\text{trn}}) - f(Z_i^{\text{trn}})).$$

**Proof.** First, note that $t\hat{\psi}^\times - t\tilde{\psi}$ can be written as

$$t\hat{\psi}^\times - t\tilde{\psi} = \sum_{i=1}^T \bar{f}_T(Z_i^{\text{eval}}) + \sum_{i=1}^{T'} \bar{f}_T(Z_i^{\text{trn}}) - \sum_{i=1}^T \mathbb{P}(\bar{f}(Z_i^{\text{eval}})) - \sum_{i=1}^{T'} \mathbb{P}(\bar{f}(Z_i^{\text{trn}}))

= \sum_{i=1}^T \left[ \bar{f}_T(Z_i^{\text{eval}}) - \mathbb{P}(\bar{f}(Z_i^{\text{eval}})) \right] + \sum_{i=1}^{T'} \left[ \bar{f}_T(Z_i^{\text{trn}}) - \mathbb{P}(\bar{f}(Z_i^{\text{trn}})) \right].$$

We will handle each sum separately and then combine them to arrive at the final decomposition (A.10). Taking a closer look at (i) first, we have

$$(i) = \sum_{i=1}^T \left\{ \left[ \bar{f}_T(Z_i^{\text{eval}}) - \mathbb{P}(\bar{f}_T(Z_i^{\text{eval}})) \right] - \left[ \bar{f}(Z_i^{\text{eval}}) - \mathbb{P}(\bar{f}(Z_i^{\text{eval}})) \right] \right\}

+ \sum_{i=1}^T \mathbb{P} \left\{ \bar{f}_T(Z_i^{\text{eval}}) - f(Z_i^{\text{eval}}) \right\} + \sum_{i=1}^T \left\{ \bar{f}(Z_i^{\text{eval}}) - \mathbb{P}(\bar{f}(Z_i^{\text{eval}})) \right\}.

Similarly for (ii), we have

$$(ii) = \sum_{i=1}^{T'} \left\{ \left[ \bar{f}_T(Z_i^{\text{trn}}) - \mathbb{P}(\bar{f}_T(Z_i^{\text{trn}})) \right] - \left[ \bar{f}(Z_i^{\text{trn}}) - \mathbb{P}(\bar{f}(Z_i^{\text{trn}})) \right] \right\}

+ \sum_{i=1}^{T'} \mathbb{P} \left\{ \bar{f}_T(Z_i^{\text{trn}}) - f(Z_i^{\text{trn}}) \right\} + \sum_{i=1}^{T'} \left\{ \bar{f}(Z_i^{\text{trn}}) - \mathbb{P}(\bar{f}(Z_i^{\text{trn}})) \right\}.$$
Putting (i) and (ii) together, we have
\[
\sum_{i=1}^{t} \left\{ f(Z_i - \hat{f}(Z_i)) \right\} + \sum_{i=1}^{T} \left\{ f(Z_i^\text{trn} - \hat{f}(Z_i^\text{trn})) \right\} + \hat{S}^\text{EP}_t + \hat{S}^B_t
\]
which completes the proof.

\[\hat{S}^\text{EP}_t = o \left( \left\{ \sum_{a=0}^{1} \sup_i \| \hat{\mu}_a^t(X_i) - \mu_a^0(X_i) \|_{L_2(P)} \right\} \sqrt{t \log \log t} \right). \tag{A.11}\]

**Proof.** We will show that the result (A.11) holds for each of \( \hat{S}^\text{EP}_{t,\text{eval}} \) and \( \hat{S}^\text{EP}_{t,\text{trn}} \), thereby yielding the same result for their sum \( \hat{S}^\text{EP}_t \). The proof proceeds in two steps. First, we use an argument from Kennedy et al. [21] and the law of the iterated logarithm to bound \( \hat{S}^\text{SA}_t \) in terms of \( \sup_i \| \hat{f}_T(Z_i) - \bar{f}(Z_i) \| \). Second and finally, we upper bound \( \sup_i \| \hat{f}_T(Z_i) - \bar{f}(Z_i) \| \) by \( O \left( \sum_{a=0}^{1} \sup_i \| \hat{\mu}_a^t(X_i) - \bar{\mu}(Z_i) \| \right) \).

**Step 1.** Let us first consider \( \hat{S}^\text{EP}_{t,\text{eval}} \). Following the proof of Kennedy et al. [21, Lemma 2] and of Lemma 4, note that conditional on \( D^\text{trn}_\infty := (Z_i^\text{trn})_{i=1}^{\infty} \) and \( S^\text{trn}_\infty := (I(Z_i \in D^\text{trn}_\infty))_{i=1}^{\infty} \) the summands of \( \hat{S}^\text{EP}_{t,\text{eval}} \) have mean zero:
\[
\mathbb{P} \left\{ \left[ \hat{f}_T(Z_i^\text{eval}) - \bar{f}(Z_i^\text{eval}) \right] - \left[ \bar{f}(Z_i^\text{eval}) - \mathbb{P}(f(Z_i^\text{eval})) \right] \mid D^\text{trn}_\infty, S^\text{trn}_\infty \right\} = 0.
\]
Similar to the proof of Lemma 4, we upper bound the conditional variance of a single summand,
\[
\text{var} \left\{ \left( 1 - \mathbb{P} \right)(\hat{f}_T(Z_i^\text{eval}) - \bar{f}(Z_i^\text{eval})) \mid D^\text{trn}_\infty, S^\text{trn}_\infty \right\} \leq \| \hat{f}_T(Z_i^\text{eval}) - \bar{f}(Z_i^\text{eval}) \|_{L_2(P)}^2.
\]
Denote the following process \( \nu_{t,\text{eval}} \) as the supremum of the above with respect to \( i \in \{1, 2, \ldots \} \):
\[
\nu_{t,\text{eval}} := \sup_{1 \leq i \leq \infty} \| \hat{f}_T(Z_i^\text{eval}) - \bar{f}(Z_i^\text{eval}) \|_{L_2(P)}.
\]
Then we can upper bound the following conditional probability
\[
\mathbb{P} \left( \limsup_{t \to \infty} \frac{\| \hat{S}^\text{EP}_{t,\text{eval}} \|_{L_2(P)}}{\nu_{t,\text{eval}} \sqrt{2t \log \log t}} \leq 1 \mid D^\text{trn}_\infty, S^\text{trn}_\infty \right)
\]

\[\leq \mathbb{P} \left( \sum_{i=1}^{T} \frac{\| \hat{f}_T(Z_i^\text{eval}) - \bar{f}(Z_i^\text{eval}) \|_{L_2(P)}}{\nu_{t,\text{eval}} \sqrt{2t \log \log t}} \leq 1 \mid D^\text{trn}_\infty, S^\text{trn}_\infty \right), \tag{A.12} \]
where $\zeta_i$ are independent mean-zero random variables with variance at most one (conditional on $\mathcal{D}_{\text{trn}}^{\text{trn}}, \mathcal{S}_{\text{trn}}^{\text{trn}}$). By the law of the iterated logarithm, we have that (A.12) = 1. In particular, since this event happens with probability one conditionally, it also happens with probability one marginally. It follows that

$$\tilde{S}_{t, \text{eval}}^{\text{EP}} = O \left( \sup_i \left\| \hat{f}_i(Z_i) - \bar{f}(Z_i) \right\|_{L_2(\mathbb{P})} \sqrt{t \log \log t} \right).$$

Applying the same technique to $\tilde{S}_{t, \text{trn}}^{\text{EP}}$, we have that $\tilde{S}_{t, \text{eval}}^{\text{EP}} = O \left( \sup_i \left\| \hat{f}_i(Z_i) - \bar{f}(Z_i) \right\|_{L_2(\mathbb{P})} \sqrt{t \log \log t} \right)$, and hence

$$\tilde{S}_{t, \text{eval}}^{\text{EP}} = O \left( \sup_i \left\| \hat{f}_i(Z_i) - \bar{f}(Z_i) \right\|_{L_2(\mathbb{P})} \sqrt{t \log \log t} \right).$$  \hspace{1cm} (A.13)

**Step 2.** Now, following the same technique as Step 2 in the proof of Lemma 4, we have that

$$\left\| \hat{f}_T(Z_i) - \bar{f}(Z_i) \right\| = O \left( \sum_{a=0}^{1} \left\| \hat{\mu}_T^a(X_i) - \bar{\mu}_T^a(X_i) \right\| \right).$$  \hspace{1cm} (A.14)

Combining (A.13) and (A.14), we have the desired result,

$$\tilde{S}_{t}^{\text{EP}} = O \left( \left\{ \sum_{a=0}^{1} \sup_{1 \leq t \leq \infty} \left\| \hat{\mu}_T^a(X_i) - \bar{\mu}_T^a(X_i) \right\| \right\} \sqrt{t \log \log t} \right),$$

which completes the proof.

Now, we examine the asymptotic almost-sure behaviour of the bias term, $\Gamma_t^B$ by upper-bounding this term by a product of $L_2(\mathbb{P})$ estimation errors of nuisance functions.

**Lemma 9 (Almost-sure behavior of $\tilde{S}_t^B$).** Suppose $\tilde{\pi}_t \in [\delta, 1 - \delta]$ for every $t$ almost surely for some $\delta > 0$. Then,

$$\tilde{S}_t^B = O \left( \sum_{i=1}^{t} \left\| \hat{\pi}_T(X_i) - \pi(X_i) \right\|_{L_2(\mathbb{P})} \sum_{a=0}^{1} \left\| \hat{\mu}_T^a(X_i) - \mu^a(X_i) \right\|_{L_2(\mathbb{P})} \right)$$

The proof proceeds similarly to that of Lemma 5 but with additional care given to the fact that observations are no longer iid.

**Proof.** Similar to the proof of Lemma 8, we will first prove the result for $\tilde{S}_{t, \text{eval}}^{B}$ and the proof proceeds similarly for $\tilde{S}_{t, \text{trn}}^{B}$, thereby yielding the desired result for $\tilde{S}_t^B \equiv \tilde{S}_{t, \text{eval}}^B + \tilde{S}_{t, \text{trn}}^B$. Following the same technique as Lemma 5, we have that

$$\mathbb{P}(\hat{f}_{T, i}^{\text{eval}} - f(Z_i^{\text{eval}})) = O \left( \left\| \hat{\pi}_{T, i}^{\text{eval}}(X_i^{\text{eval}}) - \pi(X_i^{\text{eval}}) \right\| \sum_{a=0}^{1} \left\| \hat{\mu}_{T, i}^a(X_i^{\text{eval}}) - \mu^a(X_i^{\text{eval}}) \right\| \right).$$

Putting the above term back into the sum $\tilde{S}_{t, \text{eval}}^{B}$, we have

$$\tilde{S}_{t, \text{eval}}^{B} := \sum_{i=1}^{T} \mathbb{P}(\hat{f}_{T, i}^{\text{eval}} - f(Z_i^{\text{eval}}))$$

$$= O \left( \sum_{i=1}^{T} \left\| \hat{\pi}_{T, i}^{\text{eval}}(X_i^{\text{eval}}) - \pi(X_i^{\text{eval}}) \right\| \sum_{a=0}^{1} \left\| \hat{\mu}_{T, i}^a(X_i^{\text{eval}}) - \mu^a(X_i^{\text{eval}}) \right\| \right).$$
Using a similar argument to bound $\tilde{S}_{t,\text{trn}}^B$ and putting these together, we have the following bound for $S_{t}^B = \tilde{S}_{t,\text{eval}}^B + \tilde{S}_{t,\text{trn}}^B$,

$$S_{t}^B = O\left(\sum_{i=1}^{t} \|\tilde{e}_i(X_i) - \pi(X_i)\| + \sum_{a=0}^{1} \|\tilde{\mu}_a^0(X_i) - \mu^a(X_i)\|\right), \quad (A.15)$$

which completes the proof.

\[ \square \]

**Proposition 3** (General Asymp-CSs for time-varying causal effects under sequential cross-fitting). Consider the cross-fit estimator as defined in (3.5):

$$\hat{\psi}_t^\propto := \sum_{i=1}^{T} f_T(Z_i^{\text{eval}}) + \sum_{i=1}^{T'} f_T(Z_i^{\text{trn}}),$$

and suppose we have access to a variance estimator $\hat{\text{var}}_t(\hat{f})$ such that

$$\hat{\text{var}}_t(\hat{f}) - \text{var}(\hat{f}) = o(1).$$

Suppose that $S_{B}^B$ and $S_{EP}^B$ are both $o(\sqrt{\log \log T})$, and that Assumptions 1 through 3 hold but with $(Y_i)_{i=1}^{\infty}$ replaced by $(\tilde{f}(Z_i))_{i=1}^{\infty}$. Then,

$$\hat{\psi}_t^\propto \pm \sqrt{\frac{2(t\rho^2\hat{\text{var}}_t(\hat{f}) + 1)}{t^2 \rho^2} \log \left(\frac{\sqrt{t\rho^2\hat{\text{var}}_t(\hat{f}) + 1}}{\alpha}\right)}$$

forms a $(1 - \alpha)$-Asymp-CS for $\hat{\psi}_t^\propto := \frac{1}{t} \sum_{i=1}^{t} \psi_i$ with rate $o(\sqrt{\log \log t/t})$.

**Proof.** Writing out the centered cross-fit estimator on the “sum scale” $t(\hat{\psi}_t^\propto - \tilde{\psi}_t)$ using the decomposition of Lemma 7, we have

$$t(\hat{\psi}_t^\propto - \tilde{\psi}_t) = \tilde{S}_{t,\text{trn}}^{\text{SA}} + \tilde{S}_{t,\text{eval}}^{\text{EP}} + \tilde{S}_{t,\text{trn}}^{\text{EP}} + \tilde{S}_{t,\text{eval}}^{B} + \tilde{S}_{t,\text{trn}}^{B}.$$

Therefore, we have that

$$\hat{\psi}_t - \tilde{\psi}_t = \frac{1}{t} \sum_{i=1}^{t} (\tilde{f}(Z_i) - \hat{\psi}_i) + o(\sqrt{\log \log t/t}).$$

Applying Theorem 2 to $(\tilde{f}(Z_i))_{i=1}^{\infty}$ above, we have that

$$C_t(\times \times) := \hat{\psi}_t^\propto \pm \sqrt{\frac{2(t\rho^2\hat{\text{var}}_t(\hat{f}) + 1)}{t^2 \rho^2} \log \left(\frac{\sqrt{t\rho^2\hat{\text{var}}_t(\hat{f}) + 1}}{\alpha}\right)} + o(\sqrt{\log \log t/t})$$

forms a nonasymptotic $(1 - \alpha)$-CS for $\tilde{\psi}_t^\propto := \frac{1}{t} \sum_{i=1}^{t} \psi_i$, meaning $P \left( \exists t : \tilde{\psi}_t^\propto \notin C_t(\times \times) \right) \leq \alpha$. Consequently,

$$\hat{\psi}_t^\propto \pm \sqrt{\frac{2(t\rho^2\hat{\text{var}}_t(\hat{f}) + 1)}{t^2 \rho^2} \log \left(\frac{\sqrt{t\rho^2\hat{\text{var}}_t(\hat{f}) + 1}}{\alpha}\right)}$$

forms a $(1 - \alpha)$-Asymp-CS for $\tilde{\psi}_t^\propto$, which completes the proof. \[ \square \]
A.4.1 Proof of Theorem 5

Proof. By Lemma 8 combined with Assumption 4, we have that \( \hat{S}^{EP}_t = o(\sqrt{t \log \log t}) \). In a randomized experiment, Assumption 5 holds by design, and thus by Lemma 9, we have that \( \hat{S}^{B}_t = o(\sqrt{t \log \log t}) \). Invoking Proposition 3, we obtain the desired result. \( \Box \)

A.4.2 Proof of Theorem 6

Proof. Similar to the proof of Theorem 5, by Lemma 8 combined with Assumption 4, we have that \( \hat{S}^{EP}_t = o(\sqrt{t \log \log t}) \). For observational studies, we assume that Assumption 5 holds, and thus by Lemma 9, we have that \( \hat{S}^{B}_t = o(\sqrt{t \log \log t}) \). Invoking Proposition 3, we obtain the desired result. \( \Box \)

B Simulation details

B.1 Simulated randomized experiment (Figure 4)

First, we describe the simulated randomized sequential experiment displayed in Figure 4.

Data-generating process Consider a randomized experiment with \( n = 10^4 \) subjects, each with 3 real-valued covariates. Generate \( 10^4 \) 3-tuples of said covariates \( X_1, X_2, \ldots, N_3(0,I_3) \) from a standard trivariate Gaussian. Randomly assign subjects to treatment or control groups with equal probability: \( A_1, \ldots, A_n \sim \text{Bernoulli}(1/2) \). Define the regression function,

\[
\mu^*(x_i) := 1 - x_{i,1}^2 - 2 \sin(x_{i,2}) + 3|x_{i,3}|,
\]

and the target parameter \( \psi := 1 \) (which we will ensure is the average treatment effect by design). Finally, generate outcomes \( Y_1, \ldots, Y_n \) as

\[
Y_i := \mu^*(X_{i,1}, X_{i,2}, X_{i,3}) + \psi \cdot A_i + \epsilon_i,
\]

where \( \epsilon_i \sim t_5 \) are drawn from a \( t \)-distribution with 5 degrees of freedom (we use this heavy-tailed distribution in an attempt to stress-test the finite fourth absolute moment condition of Theorems 3 and 4). We now describe the three models used to estimate \( \psi \) knowing the treatment assignment distribution of \( A_1, \ldots, A_n \) but without knowledge of \( \mu^* \) or the distribution of \( \epsilon_i \).

Estimators The unadjusted estimator \( \hat{\psi}^U_t \) used in this example is the simplest of the three and takes the form,

\[
\hat{\psi}^U_t := \frac{1}{t} \sum_{i=1}^{t} \left( \frac{A_i}{1/2} - \frac{1-A_i}{1/2} \right) Y_i.
\]

Since this estimator does not estimate the regression functions \( \mu^a \) for \( a = 0, 1 \), no sequential cross-fitting is needed. The other two estimators employ sequential cross-fitting as in Section 3.1 and take the form (3.5) but with various choices of \( \hat{\mu}^1_t \) and \( \hat{\mu}^0_t \). Specifically, the “Parametric” estimator uses linear regression to construct \( \hat{\mu}^1_t \) and \( \hat{\mu}^0_t \) which in this case is misspecified. The “Super Learner” estimator, uses a weighted ensemble of several machine learning algorithms. In this simulation, these consisted of adaptive regression splines, generalized additive models, generalized linear models with LASSO (\( \ell_1 \) regularization) and pairwise interactions, and random forests. The weights were chosen via cross-validation [47, 29]. We then applied Theorem 3 to obtain the confidence sequences displayed in Figure 4.
B.2 Simulated observational study (Figure 5)

Data-generating process  The simulation scenario used to produce Figure 5 is identical to the previous section but without complete Bernoulli randomization of treatments. Instead, each individual is assigned treatment with propensity score \( \pi(x_1, x_2, x_3) \), defined by

\[
\pi(x_1, x_2, x_3) := 0.2 + 0.6 \cdot \logit(\mu^*(x_1, x_2, x_3)),
\]

where \( \mu^* \) is the regression function defined in (B.1). A scale of 0.6 and a translation of 0.2 is applied to ensure that \( \pi(x_1, x_2, x_3) \in [0.2, 0.8] \) is bounded away from 0 and 1.

Estimators  As before, the unadjusted estimator does not make use of sequential cross-fitting and is defined in (B.2), but uses the cumulative fraction of treated subjects as an estimate of the propensity score, \( \pi \). On the other hand, the ‘Parametric’ and ‘Super Learner’ estimators invoke sequential cross-fitting and take the form (3.5) where \( \hat{\mu}_t^\pi \) is constructed in the same way as in the experimental setup of the previous section. Since \( \pi(x) \) is unknown, it must now be estimated. The ‘Parametric’ estimator uses logistic regression to accomplish this, while the ‘Super Learner’ uses the same ensemble as in the previous section (appropriately modified for classification rather than regression). Invoking Theorem 4 yields the confidence sequences of Figure 5.

C Additional discussions

C.1 Optimizing Robbins’ normal mixture for \((t, \alpha)\)

In this section, we outline how one can choose \( \rho \) to optimize the boundary \( \tilde{B}_t \) in Theorem 1 for a specific time \( t^* \) and type-I error level \( \alpha \in (0, 1) \). We will outline both the (computationally inexpensive) exact solution, and the closed-form approximate solution. Note that the derivations that follow are essentially the same as those in Howard et al. [16, Section 3.5] but we repeat them here to keep our results self-contained.

The exact solution.  If \( \alpha \leq \sqrt{W_0(1)} \approx 0.7531 \), where \( W_1 \) is the lower branch of the Lambert W function [8], we have that

\[
\arg\min_{\rho > 0} \tilde{B}_t^\star(\alpha) = \frac{-W_{-1}(-\alpha^2 \exp\{\alpha^2 - 1\}) - 1}{t^*}. \tag{C.1}
\]

Proof. Consider the boundary in Theorem 1 at time \( t \),

\[
\tilde{B}_t(\alpha) := \sqrt{\frac{2(t \rho^2 + 1)}{t^2 \rho^2}} \log\left( \frac{\sqrt{t \rho^2 + 1}}{\alpha} \right).
\]

Defining \( x := \rho^2 \) and after some simple algebra, notice that

\[
\arg\min_{\rho > 0} \tilde{B}_t(\alpha) = \sqrt{\arg\min_{x > 0} f(x)},
\]

where \( f(x) := \frac{tx + 1}{t^2 x^2} \log\left( \frac{tx + 1}{\alpha^2} \right) \).

Notice that \( \lim_{x \to 0} f(x) = \lim_{x \to \infty} f(x) = \infty \) and thus if we find that \( df/dx = 0 \) has exactly one positive solution, we know that it must be the minimizer of \( f \).
To that end, it is straightforward to show that
\[
\frac{df}{dx} = -\frac{1}{tx^2} \log \left( \frac{tx + 1}{\alpha^2} \right) + \frac{\alpha^2}{tx}.
\]
Setting the above to 0, we obtain
\[
-\alpha^2 \exp \{ \alpha^2 - 1 \} = -(tx + 1) \exp \{ -(tx + 1) \}.
\]
Notice that if we rewrite \( y := -(tx + 1) \), we have that \( y = W_{-1} \left( -\alpha^2 \exp \{ \alpha^2 - 1 \} \right) \) where \( W_{-1} \) is the lower branch of the Lambert \( W \) function. However, \( y = W_{-1}(z) \) only has a solution if \( z \geq -e^{-1} \), and thus we require that
\[
\alpha^2 \exp \{ \alpha^2 \} \leq 1,
\]
or equivalently, \( \alpha \leq \sqrt{\Omega} \) where \( \Omega = W_0(1) \approx 0.5671 \) and \( W_0 \) is the principal branch of the Lambert \( W \) function. In summary, as long as \( \alpha \leq \sqrt{\Omega} \approx 0.7531 \), we have that
\[
\arg\min_{\rho > 0} \mathbb{B}_t(\rho) = \sqrt{-W_{-1} \left( -\alpha^2 \exp \{ \alpha^2 - 1 \} \right) - 1}\frac{1}{t^*}.
\]
This completes the proof. \( \square \)

**An approximate solution.** We can derive a closed-form approximation to (C.1) by considering the Taylor series expansion to the Lambert \( W \) function [8],
\[
W_{-1}(z) = \log(-z) - \log(- \log(-z)) + o(1).
\]
Replacing \( W_{-1}(z) \) by \( \log(-z) - \log(- \log(-z)) \) in (C.1), we obtain the following approximate solution,
\[
\rho'(t^*) := \sqrt{-\alpha^2 - 2 \log \alpha + \log(-2 \log \alpha + 1 - \alpha^2)} \frac{1}{t^*}.
\]
In practice, we find that using (C.2) over (C.1) has negligible downstream effects on the resulting CSs, but both are inexpensive to compute. Moreover, notice that \( \rho'(t^*) \) is quite similar to \( \sqrt{2 \log(1/\alpha)/t^*} \), which is precisely what one would choose when sharpening a sub-Gaussian confidence interval based on the Cramér-Chernoff technique for a fixed sample size \( t^* \).

**C.2 Time-uniform convergence in probability is equivalent to almost sure convergence**

In Theorems 1, 3, and 4, we justified the asymptotic validity of our confidence sequences by showing that the approximation error
\[
\varepsilon_t \overset{a.s.}{\rightarrow} 0
\]
at a particular rate. At first glance, this may seem like a slightly stronger statement than required since we only need the approximation error \( \varepsilon_t \) to vanish time-uniformly in probability:
\[
\sup_{k \geq t} |\varepsilon_k| \overset{p}{\rightarrow} 0.
\]
As it turns out, however, (C.3) and (C.4) are equivalent. This is not a new result, but we present a proof here for completeness.

**Proposition 4.** Let \( (X_n)_{n=1}^\infty \) be a sequence of random variables. Then,
\[
X_n \overset{a.s.}{\rightarrow} 0 \iff \sup_{k \geq n} |X_n| \overset{p}{\rightarrow} 0.
\]
Proof. First, we prove (\(\Rightarrow\)). By the continuous mapping theorem, \(|X_n| \xrightarrow{a.s.} 0\). Therefore,

\[
1 = \mathbb{P}\left(\lim_{n} |X_n| = 0\right) \leq \mathbb{P}\left(\limsup_{n} |X_n| = 0\right) \leq 1.
\]

In other words, sup_{k \geq n} |X_k| \xrightarrow{a.s.} 0, which implies sup_{k \geq n} |X_k| \xrightarrow{p} 0.

Now, consider (\(\Leftarrow\)). Suppose for the sake of contradiction that \(\mathbb{P}\left(\lim_n |X_n| = 0\right) < 1\). Then with some probability \(\delta > 0\), we have that \(\lim_n |X_n| \neq 0\), meaning there exists some \(\epsilon > 0\) such that \(|X_k| > \epsilon\) for some \(k \geq n\) no matter how large \(n\) is. In other words,

\[
\delta < \mathbb{P}\left(\limsup_{n \to \infty} \sup_{k \geq n} |X_k| > \epsilon\right) \leq \mathbb{P}\left(\sup_{k \geq n} |X_k| > \epsilon\right) \text{ for any } n \geq 1.
\]

In particular, \(\mathbb{P}\left(\sup_{k \geq n} |X_k| > \epsilon\right) \to 0\), which is equivalent to saying sup_{k \geq n} |X_k| \xrightarrow{p} 0, a contradiction. This completes the proof. \(\square\)

### C.3 Unimprovability of Asymp-CSs using efficient influence functions

Consider the confidence sequence of Theorem 4,

\[
\hat{\psi}_i^\alpha \pm \sqrt{\text{var}_i(f)} \cdot \left[ \frac{2(t\rho^2 + 1)}{t^2\rho^2} \log \left( \frac{\sqrt{t\rho^2 + 1}}{\alpha} \right) \right] \text{ with rate } o\left( \sqrt{\frac{\log \log t}{t}} \right). \tag{C.5}
\]

It is natural to whether (C.5) can be tightened at all. In a certain sense, (C.5) inherits optimality from its three main components: (i) Robbins’ normal mixture boundary, (ii) the approximation error rate, and (iii) the estimated standard deviation \(\sqrt{\text{var}_i(f)}\) of the efficient influence function \(f\).

**Term (i)** Starting with the width, we have that in the case of iid Gaussian data \(G_1, G_2, \ldots \sim \mathcal{N}(\mu, \sigma^2)\), Robbins’ normal mixture confidence sequence [31] is obtained by first showing that

\[
M_i(\mu) := \exp \left\{ \frac{\rho^2 (\Sigma_{i=1}^d (G_i - \mu))^2}{2(t\rho^2 + 1)} \right\} (t\rho^2 + 1)^{-1/2}
\]

is a nonnegative martingale starting at one, and hence by Ville’s inequality [52],

\[
\mathbb{P}(\exists t \geq 1 : M_i(\mu) \geq 1/\alpha) \leq \alpha.
\]

The resulting confidence sequence \(\tilde{C}_t^N\) at each time \(t\) is defined as the set of \(m\) such that \(M_i(m) < 1/\alpha\), i.e. \(\tilde{C}_t^N := \{m \in \mathbb{R} : M_i(m) < 1/\alpha\}\) and consequently,

\[
\mathbb{P}(\exists t \geq 1 : \mu \notin \tilde{C}_t^N) = \mathbb{P}(\exists t \geq 1 : M_i(\mu) \geq 1/\alpha) \leq \alpha.
\]

This inequality is extremely tight, since Ville’s inequality almost holds with equality for nonnegative martingales. Technically, the paths of the martingale need to be continuous for equality to hold, which can only happen in continuous time (such as for a Wiener process). However, any deviation from equality only holds because of this “overshoot” and in practice, the error probability is almost exactly \(\alpha\). This means that the normal mixture confidence sequence \(\tilde{C}_t^N\) cannot be uniformly tightened: any improvement for some times will necessarily result in looser bounds for others. For a precise characterization of this optimality for the (sub-)Gaussian case, see Howard et al. [16, Section 3.6], or Ramdas et al. [30] for a more general discussion of admissible confidence sequences.
**Term (ii)** The error incurred from almost-surely approximating a sample average $\frac{1}{t} \sum_{i=1}^{t} f(Z_i)$ of influence functions by Gaussian random variables is a direct consequence of Komlós et al. [22, 23] and Major [24], and is unimprovable without additional assumptions. Further approximation errors result from using $\sqrt{\text{var}_t(\hat{f})}$ to estimate $\text{var}(f)$, where almost-sure law of the iterated logarithm rates appear, and are themselves unimprovable.

**Term (iii)** Using the approximations mentioned in (ii) permits the use of Robbins’ normal mixture confidence sequence in (i). However, a factor of $\sqrt{\text{var}_t(\hat{f})}$ necessarily appears in front of the width as an estimate of the standard deviation $\sqrt{\text{var}(f)}$ of the efficient influence function $f$ discussed in Section 3. Importantly, $\sqrt{\text{var}(f)}$ corresponds to the semiparametric efficiency bound, so that no estimator of $\psi$ can have asymptotic mean squared error smaller than $\text{var}(f(Z))/t$ without imposing additional assumptions [49].