LOCAL AUTOMORPHISMS OF THE UNITARY GROUP AND THE GENERAL LINEAR GROUP ON A HILBERT SPACE

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Abstract

We prove that every 2-local automorphism of the unitary group or the general linear group on a complex infinite-dimensional separable Hilbert space is an automorphism. Thus these types of transformations are completely determined by their local actions on the two-points subsets of the groups in question.

1 Introduction

The study of automorphism groups of algebraic structures is of great importance in every field of mathematics. In a series of papers (see [1], [3], [4] and the references therein) we investigated these groups from the point of view of how they are determined by their local actions. Our investigations were motivated by the paper [2] of Kadison on local derivations and by a problem of Larson in [3] initiating the study of local automorphisms of Banach algebras. The structures that we treated so far were mainly $C^*$-algebras and we considered the following question: When is it true that any local automorphism, that is, any linear transformation which pointwise equals an automorphism (this automorphism may, of course, differ from point to point) is an automorphism?

It is easy to see that if we drop the assumption of the linearity of the transformations in question, then the corresponding statements are no longer true. However, if instead of linearity plus locality we assume the so-called 2-locality, then we can obtain positive results (for the first such result see [12]). 2-locality means that our transformation (linearity is not assumed any more) is supposed to be equal to an automorphism at every pair of points. Notice that in this way we arrive at a question that can be raised in any algebraic structure. For example, this observation motivated us to consider the problem for the orthomodular
A mapping \( \phi : GL(H) \to GL(H) \) is called a 2-local automorphism of the general linear group if for every \( X, Y \in GL(H) \) there is an automorphism \( \phi_{X,Y} \) of the group \( GL(H) \), depending on \( X \) and \( Y \), such that \( \phi_{X,Y}(X) = \phi(X) \) and \( \phi_{X,Y}(Y) = \phi(Y) \). In the case of the unitary group the situation is somewhat different. Because of certain reasons (see, for example, Theorem 2.1), \( U(H) \) is considered here as a topological group and by an automorphism of \( U(H) \) we mean a uniformly continuous group-automorphism. Now, not surprisingly, a mapping \( \phi : U(H) \to U(H) \) is called a 2-local automorphism of the unitary group if for every \( X, Y \in U(H) \) there is an automorphism \( \phi_{X,Y} \) of \( U(H) \) (in the above sense), such that \( \phi_{X,Y}(X) = \phi(X) \) and \( \phi_{X,Y}(Y) = \phi(Y) \).

### 2 Local automorphisms of the unitary group

It was proved in \cite{10} that any uniformly continuous group isomorphism between the unitary groups of two \( AW^\ast \)-factors is implemented by a linear or conjugate-linear \( \ast \)-isomorphism of the factors themselves. As a particular case, concerning \( B(H) \) we have the following result.

**Theorem 2.1** (Sakai) Let \( \phi : U(H) \to U(H) \) be a uniformly continuous automorphism. Then there exists either a unitary or an antiunitary operator \( U \) on \( H \) such that

\[
\phi(V) = UVU^\ast \quad (V \in U(H)).
\]  

(1)

As for the 2-local automorphisms of the unitary group \( U(H) \) we have the following statement. In the proof we use the notation \( x \otimes y \) which denotes the operator on \( H \) defined by

\[
(x \otimes y)(z) = \langle z, y \rangle x \quad (z \in H)
\]

for any \( x, y \in H \).

**Theorem 2.2** Every 2-local automorphism of \( U(H) \) is an automorphism.
Proof. Let $\phi : U(H) \to U(H)$ be a 2-local automorphism. For every projection $P \in B(H)$, the operator $I - 2P$ is unitary. Since $\phi$ is locally of the form (4), we obtain that $\phi(I - 2P) = I - 2P'$ for some projection $P'$. Consider the transformation

$$P \mapsto (I - \phi(I - 2P))/2.$$  

This is a 2-local automorphism of the orthomodular poset of all projections on $H$. By [3, Proposition], it is an automorphism and there is either a unitary or an antiunitary operator $U$ on $H$ such that our transformation is of the form

$$P \mapsto (I - \phi(I - 2P))/2 = UPU^*.$$  

We have $\phi(I - 2P) = U(I - 2P)U^*$ for every projection $P$. Transforming the original map $\phi$ by this operator $U$, we can suppose without loss of generality that $\phi(I - 2P) = I - 2P$ for every projection $P$. Let $V \in U(H)$ and pick an arbitrary unit vector $x \in H$. Let $P$ be the orthogonal projection onto the subspace spanned by $x$, that is, let $P = x \otimes x$. By the local property of $\phi$ we have an either unitary or antiunitary operator $U_{V,P}$ such that $\phi(V) = U_{V,P}VU_{V,P}^*$ and $\phi(I - 2P) = U_{V,P}(I - 2P)U_{V,P}^*$. Since $\phi(I - 2P) = I - 2P$, it follows that $P = U_{V,P}PU_{V,P}^*$. We compute

$$\langle \phi(V)x, x \rangle x \otimes x = x \otimes x \cdot \phi(V) \cdot x \otimes x = U_{V,P}PU_{V,P}^*U_{V,P}PVU_{V,P}^*P = U_{V,P}PVPU_{V,P}^* = U_{V,P} \cdot \langle Vx, x \rangle x \otimes x \cdot U_{V,P}^*.$$  

Since $U_{V,P}$ is either linear or conjugate-linear, we have either

$$\langle \phi(V)x, x \rangle x \otimes x = \langle Vx, x \rangle U_{V,P}x \otimes U_{V,P}x$$  

or

$$\langle \phi(V)x, x \rangle x \otimes x = \overline{\langle Vx, x \rangle} U_{V,P}x \otimes U_{V,P}x.$$  

We deduce that for every $x \in H$ either

$$\langle \phi(V)x, x \rangle = \langle Vx, x \rangle$$  

or

$$\langle \phi(V)x, x \rangle = \overline{\langle Vx, x \rangle}$$  

holds true. It is rather elementary to verify (see [3, Lemma]) that this implies that either $\phi(V) = V$ or $\phi(V) = V^*$. We show that either $\phi(V) = V$ for every $V \in U(H)$ or $\phi(V) = V^*$ for every $V \in U(H)$. To see this, observe that $\phi(iI)$ is either $iI$ or $-iI$. First assume that $\phi(iI) = iI$. We assert that in this case we have $\phi(V) = V$ ($V \in U(H)$). Suppose on the contrary that there is a non-selfadjoint unitary operator $V$ for which $\phi(V) = V^*$. Let $\lambda \in \sigma(V)$. By the spectral theorem of normal operators we can choose a sequence $(x_n)$ of pairwise orthogonal unit vectors in $H$ such that $\langle Vx_n, x_n \rangle \to \lambda$. We extend $(x_n)$ to a complete orthonormal sequence $(x_n')$ in $H$. Pick pairwise different
complex numbers $\lambda_n$ of modulus 1 from the open upper half-plane and consider the unitary operator $W = \sum_n \lambda_n x_n' \otimes x_n'$. By the local property of $\phi$ we have an either unitary or antiunitary operator $U_{i,W}$ such that $\phi(iI) = U_{i,W} i I U_{i,W}^*$ and $\phi(W) = U_{i,W} W U_{i,W}^*$. Since we have supposed that $\phi(iI) = iI$, it follows that $U_{i,W}$ is unitary. So, $\phi(W) = \sum_n \lambda_n U_{i,W} x_n' \otimes U_{i,W} x_n'$ and, on the other hand, we know that $\phi(W) = W$ or $\phi(W) = W^*$.

These result in $\phi(W) = W$. Once again, by the local property of $\phi$ we have an either unitary or antiunitary operator $U_{W,V}$ such that $\phi(W) = U_{W,V} W U_{W,V}^*$ and $\phi(V) = U_{W,V} V U_{W,V}^*$. Since $\phi(W) = W$, it follows that $U_{W,V}$ is necessarily linear and from the equalities

$$\sum_n \lambda_n x_n' \otimes x_n' = W = \phi(W) = U_{W,V} W U_{W,V}^* = \sum_n \lambda_n U_{W,V} x_n' \otimes U_{W,V} x_n'$$

we conclude that $U_{W,V}$ is diagonalizable with respect to $(x_n')$. Therefore, we can compute

$$\langle x_n, V x_n \rangle = \langle V^* x_n, x_n \rangle = \langle \phi(V) x_n, x_n \rangle = \langle U_{W,V} V U_{W,V}^* x_n, x_n \rangle = \langle V U_{W,V}^* x_n, U_{W,V} x_n \rangle = \langle V x_n, x_n \rangle.$$ 

If $n$ goes to infinity, we obtain that $\lambda = \overline{\lambda}$. Since $\lambda$ was an arbitrary element of $\sigma(V)$, we infer that $V = V^*$ which is a contradiction. So, we have $\phi(V) = V$ for every $V \in U(H)$ and this shows that $\phi$ is an automorphism of $U(H)$. We now consider the case when $\phi(iI) = -iI$. Similarly as above one can verify that then we have $\phi(V) = V^*$ ($V \in U(H)$).

By the local property of $\phi$ it follows that for every $V, V' \in U(H)$ there exists an either unitary or antiunitary operator $U$ such that $V^* = UVU^*$ and $V'^* = UV'U^*$. If $V$ is diagonal with respect to an orthonormal basis defined in the same way as $W$ and $V'$ permutes the same basis, then one can easily arrive at a contradiction.

The proof of the theorem is now complete.

3 Local automorphisms of the general linear group

We first remark that we were unable to find the description of the general form of automorphisms of $GL(H)$ in the literature. However, applying a result of Radjavi \[7\] on a factorization of invertible operators into a product of involutions and some automatic continuity techniques it is possible to obtain the general form of the automorphisms of $GL(H)$ as a consequence of results of Rickart on the isomorphisms of some analogues of the classical groups \[8, 9\]. So, we begin with a statement on the form of the automorphisms of the general linear group $GL(H)$.

**Theorem 3.1** Let $\phi : GL(H) \to GL(H)$ be an automorphism of the general linear group. Then $\phi$ is of one of the following forms:
(i) there exists a bounded linear invertible operator $T : H \to H$ such that
\[ \phi(X) = TXT^{-1} \quad (X \in GL(H)), \]

(ii) there exists a bounded conjugate-linear invertible operator $T : H \to H$ such that
\[ \phi(X) = TXT^{-1} \quad (X \in GL(H)), \]

(iii) there exists a bounded linear invertible operator $T : H \to H$ such that
\[ \phi(X) = (TX^{-1}T^{-1})^* \quad (X \in GL(H)), \]

(iv) there exists a bounded conjugate-linear invertible operator $T : H \to H$ such that
\[ \phi(X) = (TX^{-1}T^{-1})^* \quad (X \in GL(H)). \]

**Remark.** If $\phi$ is of type (i), (ii), (iii), (iv), then for every $X \in GL(H)$ we have $\sigma(\phi(X)) = \sigma(X)$, $\sigma(\phi(X)) = \sigma(X)$, $\sigma(\phi(X)) = (\sigma(X))^{-1}$, $\sigma(\phi(X)) = \sigma(X)^{-1}$, respectively.

**Proof.** Let $\phi : GL(H) \to GL(H)$ be an automorphism. By a result of Rickart [8, Theorem 5.1], [9, Theorem 1], $\phi$ must be either of the form $\phi(X) = \tau(X)TXT^{-1}$ or of the form $\phi(X) = \tau(X)(TX^{-1}T^{-1})^*$, where $\tau : GL(H) \to C$ is a multiplicative map and $T : H \to H$ is a bijective additive map satisfying $T(\lambda x) = f(\lambda)Tx$, $\lambda \in C$, $x \in H$, for some ring automorphism $f$ of $C$ (one has to be careful when applying the result of Rickart since $S^*$ in his papers denotes the adjoint of an operator $S$ defined as for Banach space operators, while here, of course, $S^*$ denotes the adjoint in the Hilbert space sense). Let us first show that $\tau(X) \equiv 1$. Since $\tau(I) = 1$ we have $\tau(S) \in \{-1, 1\}$ for any involution $S \in GL(H)$, that is, for any $S$ satisfying $S^2 = I$. According to [7] every element of $GL(H)$ can be written as a product of involutions, and consequently, the range of $\tau$ is contained in $\{-1, 1\}$. An arbitrary involution $S$ can be expressed as $S = (I - P) - P$ where $P$ is an idempotent. For $R = (I - P) + iP$ we have $S = R^2$, and therefore, $\tau(S) = 1$. Applying [7] once again we conclude that $\tau$ is identically equal to 1.

Therefore, either $\phi(X) = TXT^{-1}$ or $\phi(X) = (TX^{-1}T^{-1})^*$. In the second case we can compose $\phi$ by $X \mapsto (X^*)^{-1}$ to conclude that in both cases $X \mapsto TXT^{-1}$ is an automorphism of $GL(H)$. We have to prove that $f : C \to C$ is either the identity or the complex conjugation and that $T$ is bounded. To prove this one can apply automatic continuity techniques (note that for this part of the proof the assumption that $H$ is infinite-dimensional is indispensable). However, we shall use a shorter way of reducing the problem to a known result.

So, assume that $\phi(X) = TXT^{-1}$ is an automorphism of $GL(H)$. Clearly, $\phi^{-1}(X) = T^{-1}XT$. Define additive mappings $\psi, \varphi : B(H) \to A(H)$ by $\psi(X) = TXT^{-1}$ and
\[ \varphi(X) = T^{-1}XT, \ X \in B(H). \] If \(|\lambda| > ||X||\), then \(\psi(X) = \psi(X - \lambda I) + \psi(\lambda I) \in GL(H) + GL(H) \subset B(H).\) So, \(\psi\) is a multiplicative map from \(B(H)\) into \(B(H)\) which is also bijective since \(\varphi : B(H) \to B(H)\) is its inverse. By \(\Box\) there exists a bijective bounded linear or conjugate-linear map \(S : H \to H\) such that \(TXT^{-1} = SXS^{-1}\) for every \(X \in B(H)\), or equivalently, the additive map \(S^{-1}T\) commutes with every \(X \in B(H)\). It follows that \(T = \lambda S\) for some nonzero scalar \(\lambda\). This completes the proof of the statement that every automorphism \(\phi\) of \(GL(H)\) has one of the forms (i), (ii), (iii) or (iv).

In the proof of the main result of this section we shall need the following lemma. Let \(K\) be a nonempty subset of the complex plane and let \(\lambda\) be a complex number. We use the following notation: \(K - \lambda = \{\mu - \lambda : \mu \in K\}\), \(\overline{K} = \{\overline{\mu} : \mu \in K\}\), and \(r(K) = \sup\{|\mu| : \mu \in K\}\) is a bijective since \(\phi \in L\). Assume that \(\phi \in L\) and \(|\lambda| \geq r(K)\), which further yields that \(K - \lambda \neq (K - \lambda)^{-1}\) and \(K - \lambda \neq (K - \lambda)^{-1}\). Furthermore, our assumption implies that \(K - \lambda\) belongs to the open lower half-plane, and consequently, \(K - \lambda \neq (K - \lambda)^{-1}\). This completes the proof.

**Lemma.** Let \(K\) be a nonempty compact subset of \(C\) and \(\lambda\) a complex number such that \(\text{Im} \lambda > r(K) + 1\). Then \(0 \notin K - \lambda, \ K - \lambda \neq (K - \lambda)^{-1}, \ K - \lambda \neq \overline{K - \lambda}, \text{and} \ K - \lambda \neq (K - \lambda)^{-1}\).

**Proof.** Clearly, \(0 \notin K - \lambda\) and \(|\lambda| > r(K) + 1\). It follows that \(r(K - \lambda) \geq |\lambda| - r(K) > 1\).

On the other hand, \(r((K - \lambda)^{-1}) = \frac{1}{|\lambda - \lambda_0|} \) for some \(\lambda_0 \in K\).

\[
\frac{1}{|\lambda - \lambda_0|} \leq \frac{1}{|\lambda| - |\lambda_0|} \leq \frac{1}{|\lambda| - r(K)} < 1;
\]

we have \(r(K - \lambda) > r((K - \lambda)^{-1}) = r((K - \lambda)^{-1})\), which further yields that \(K - \lambda \neq (K - \lambda)^{-1}\) and \(K - \lambda \neq (K - \lambda)^{-1}\). Furthermore, our assumption implies that \(K - \lambda\) belongs to the open lower half-plane, and consequently, \(K - \lambda \neq (K - \lambda)^{-1}\). This completes the proof.

**Theorem 3.2** Every 2-local automorphism of \(GL(H)\) is an automorphism.

**Proof.** Assume that \(\phi\) is a 2-local automorphism of \(GL(H)\). Composing it with an appropriate automorphism of \(GL(H)\) we can assume with no loss of generality that \(\varphi(2iI) = 2iI\). It follows from the Remark that \(\phi_{X,2i}\) has to be of type (i) for every \(X \in GL(H)\). In particular, we have \(\sigma(\phi(X)) = \sigma(X)\) \((X \in GL(H))\), and \(\phi(\lambda I) = \lambda I\) \((\lambda \in C)\). Denote by \(S\) the set of all operators \(X \in GL(H)\) satisfying \(\sigma(X) \neq \sigma(X)\) and \(\sigma(X) \neq (\sigma(X))^{-1}\). If \(X \in S\) and \(Y\) is an arbitrary element of \(GL(H)\) then \(\phi_{X,Y}\) has to be of type (i). In particular, \(\phi_{X,Y}(\lambda I) = \lambda I\) \((\lambda \in C)\).

Now, let \(X \in B(H)\) be any bounded linear operator on \(H\). Denote by \(L_X\) the set of all complex numbers \(\lambda\) such that \(X - \lambda I\) is an invertible operator contained in \(S\). By Lemma, this set is always nonempty. We define \(\psi : B(H) \to B(H)\) by \(\psi(X) = \phi(X - \lambda I) + \lambda I\) where \(\lambda \in L_X\). First we have to show that \(\psi\) is well-defined. So, assume that \(\mu\) also belongs to \(L_X\). Then we already know that \(\phi_{X - \lambda I,X - \mu I}\) is of type (i), and consequently,

\[
\phi(X - \lambda I) + \lambda I = \phi_{X - \lambda I,X - \mu I}(X - \lambda I) + \lambda I =
\]
\[ \phi_{X - \lambda, X - \mu I}(X) = \phi(X - \mu I) + \mu I. \]

Next, we show that the restriction of \( \psi \) to \( GL(H) \) coincides with \( \phi \). In order to do this we first observe that if \( \varphi : GL(H) \to GL(H) \) is an automorphism of type (i) and if \( X, Y \) are arbitrary elements of \( GL(H) \) such that \( X + Y \) is also invertible, then \( \varphi(X + Y) = \varphi(X) + \varphi(Y) \). So, for any \( X \in GL(H) \) and \( \lambda \in L_X \setminus \{0\} \) we have

\[ \psi(X) = \phi(X - \lambda I) + \lambda I = \phi_{X, X - \lambda I}(X - \lambda I) + \lambda I = \phi(X). \]

It is well-known that every algebra automorphism of \( B(H) \) is inner. We show that \( \psi \) is a 2-local automorphism of \( B(H) \), that is, for every pair \( X, Y \in B(H) \) there exists a bounded linear invertible \( S : H \to H \) such that \( \psi(X) = SXS^{-1} \) and \( \psi(Y) = SYS^{-1} \). By Lemma we know that there exists \( \lambda \in \mathbb{C} \) such that \( \lambda \in L_X \cap L_Y \). The automorphism \( \phi_{X - \lambda I, Y - \lambda I} \) is of type (i), and therefore spatially implemented by a bounded linear invertible operator, say \( S \). Then

\[ \psi(X) = \phi(X - \lambda I) + \lambda I = \phi_{X - \lambda I, Y - \lambda I}(X - \lambda I) + \lambda I = S(X - \lambda I)S^{-1} + \lambda I = SXS^{-1}, \]

and similarly,

\[ \psi(Y) = SYS^{-1}. \]

Applying the result of the second author \([12]\) on 2-local automorphisms of \( B(H) \) we conclude that there exists a bounded linear invertible operator \( T : H \to H \) such that \( \psi(X) = TXT^{-1} \) for every \( X \in B(H) \). Consequently, \( \phi(X) = TXT^{-1} \) for every \( X \in GL(H) \). This completes the proof.

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