Research Article

Eventually Periodic Solutions of a Max-Type Difference Equation

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We study the following max-type difference equation

\[ x_n = \max\{\frac{A_n}{x_{n-r}}, x_{n-k}\}, \quad n = 1, 2, \ldots \]

where \(\{A_n\}_{n=1}^{\infty}\) is a periodic sequence with period \(p\), and \(k, r \in \{1, 2, \ldots\}\) with \(\gcd(k, r) = 1\) and \(k \neq r\), and the initial conditions \(x_{-d}, x_{-d-1}, \ldots, x_0\) are real numbers with \(d = \max\{r, k\}\).

1. Introduction

The max operator arises naturally in certain models in automatic control theory (see [1]). In recent years, the discrete case involving difference equations with maximum has been receiving increasing attention, for some results in this area, see, for example, [2–4].

In this paper, we consider the following max-type equation:

\[ x_n = \max\left\{\frac{A_n}{x_{n-r}}, x_{n-k}\right\}, \quad n = 1, 2, \ldots \tag{1} \]

where \(\{A_n\}_{n=1}^{\infty}\) is a periodic sequence with period \(p\) and \(k, r \in \{1, 2, \ldots\}\) with \(\gcd(k, r) = 1\) and \(k \neq r\), and the initial conditions \(x_{-d}, x_{-d-1}, \ldots, x_0\) are real numbers with \(d = \max\{r, k\}\).

In [5], Iričanin and Elsayed showed that every well-defined solution of (1) is eventually periodic with period 4 when \(k = 4, r = 1, p = 1\). Elsayed and Stević [6] showed that every well-defined solution of (1) is eventually periodic with period 3 when \(k = 3, r = 1, p = 1\). In [7], Xiao and Shi showed that if \(k = 2, r = 1, p = 1\), then every well-defined solution of (1) is eventually periodic with period 2. Qin et al. [8] showed that every well-defined solution of (1) is eventually periodic with period \(k\) when \(r = 1\) and \(p = 1\).

In this paper, we will generalize the results of [5–8] to the general case.

2. Main Results and Example

In this section, we are ready to state and prove the main results.

Theorem 1. Let \(\{A_n\}_{n=1}^{\infty}\) be a periodic sequence with period \(p\), and \(k, r \in \{1, 2, \ldots\}\) with \(\gcd(k, r) = 1\) and \(k \neq r\).

(1) If \(p \geq 2\) and \(k\) is odd, then every well-defined solution of (1) is eventually periodic with period \(k\).

(2) If \(p = 1\), then every well-defined solution of (1) is eventually periodic with period \(k\).

Proof. Let \(\{x_n\}_{n=1}^{\infty}\) be a well-defined solution of (1). It follows from (1) that, for any \(n \geq 0\) and any \(i \geq 0\),

\[ x_{(n+1)k+i} = \max\left\{\frac{A_{(n+1)k+i}}{x_{(n+1)k+i-r}}, x_{nk+i}\right\} \geq x_{nk+i} \tag{2} \]
Then, for every \(i \geq 0\), \(\{x_{nk}\}_{n=0}^{\infty}\) is increasing, and \(x_{nk} < 0\) for all \(n \geq 0\) or there exists \(N_i > 0\) such that \(x_{nk+i} > 0\) for all \(n \geq N_i\).

We claim that \(\{x_{nk}\}_{n=0}^{\infty}\) is a constant sequence eventually. Indeed, if \(\{x_{nk}\}_{n=0}^{\infty}\) is not constant sequence eventually, then there exist \(kr < n_1 < n_2 < \cdots\) such that \(x_{nk} > x_{nk+i}\) and \(A_{nk}\) is a constant sequence for all \(i \geq 1\) since \(\{A_n\}_{n=1}^{\infty}\) is a periodic sequence. Thus we have

\[
\begin{align*}
x_{n,k} &= \max\left\{ \frac{A_{n,i+k} - x_{n,i-k}}{x_{n,i+k} - x_{n,k}} \right\} \\
&= \frac{A_{n,i+k}}{x_{n,i+k}} > \frac{x_{n,i-k}}{x_{n,i-k}} \quad (3)
\end{align*}
\]

From this we obtain that, for all \(i \geq 1\),

\[
\frac{A_{n,k} (x_{n,i+k} - x_{n,k})}{x_{n,i+k} - x_{n,k}} < 0. \quad (4)
\]

It follows that, for all \(i \geq 1\),

\[
x_{n,k} x_{n,k-i} = A_{n,k} < 0, \quad x_{n,i+k} > x_{n,k-r}. \quad (5)
\]

Therefore we have \(x_{nk} x_{nk-r} < 0\) eventually. By induction, we can show that \(x_{nk-i} x_{nk-kr} < 0\) eventually for all \(1 \leq i \leq k-1\) and every \(\{x_{nk-i}\}_{n=0}^{\infty} (1 \leq i \leq k-1)\) is not constant sequence eventually.

If \(p \geq 2\) and \(k = 2\), then we have \(x_{nk} x_{nk-2} x_{nk-4} \cdots x_{nk-2(k-1)} < 0\) eventually. This is a contradiction.

If \(p = 1\) and \(k = 2\), then we write \(A_{n} = A\) for all \(n \geq 1\) and choose \(m_0 \geq m_1 \geq \cdots \geq m_2k\) such that \(x_{m,j+i} < x_{(m+1),j+i} < x_{(m+2),j+i}\) and \(x_{(m+1),j+i} < x_{(m+2),j+i}\) for any \(j \in \{0, 1, 2k-1\}\). Thus

\[
x_{(m+1),k} = \max\left\{ \frac{A_{m,k}}{x_{m+1},k} \right\} \\
&= \frac{A}{x_{m+1},k} \\
&= \frac{A}{x_{m+1},k} \\
&= \max\left\{ \frac{A}{x_{m+1},k - 2r} x_{m+1},k - r \right\}
\]

which is a contradiction. This completes the proof of the claim.

By the above claim we may choose an \(N > 0\) such that \(x_{nk} = x_{Nk}\) for all \(n \geq N\). Since \(A_n\) is a periodic sequence, we can choose an \(N_1 > N\) such that \(A_{n,k+1} / x_{Nk} = \max\{A_{n,k+1} / x_{Nk} : n \geq N\}\). Then, for all \(n \geq N_1\),

\[
x_{nk} = \max\left\{ \frac{A_{nk+1}}{x_{Nk}} \right\}. \quad (7)
\]

Thus \(x_{nk} = x_{Nk}\) for all \(n \geq N_1\), and \(x_{nk} > x_{Nk}\) for some \(m \geq N_1\), then we have \(x_{nk+1} = A_{nk+1}/x_{Nk} > x_{(n-1)k} > x_{Nk+1}/x_{Nk}\) for all \(n \geq N_1\), which is a contradiction. By induction, we can show that \(x_{nk} = x_{Nk}\) for all \(n \geq N_1\) if \(x_{nk} > x_{Nk}\) is a constant sequence eventually for every \(1 \leq j \leq k\).

Note that \(j \mod k = \{0, 1, 2, \ldots, k-1\}\) since \(\gcd(k,r) = 1\). Then \(x_{nk} = x_{Nk}\) is constant eventually for every \(i \in \{0, 1, \ldots, k-1\}\), which implies that \(\{x_{nk}\}_{n=1-d}^{\infty}\) is eventually periodic with period \(k\).

From the proof of Theorem 1 we obtain the following corollary.

**Corollary 2.** Let \(k, r \in \{1, 2, \ldots\} \) with \(k \neq r\). If \(\{A_n\}_{n=1}^{\infty}\) is a periodic sequence, then every positive (or negative) solution of (1) is eventually periodic with period \(k\).

**Theorem 3.** Let \(\{A_n\}_{n=1}^{\infty}\) be a periodic sequence with period \(p \geq 2\), and \(k, r \in \{1, 2, \ldots\} \) with \(\gcd(k,r) = 1\) and \(k \neq r\). If \(A_1 \geq 0\) for some \(s \in \{1, 2, \ldots, p\}\), then every well-defined solution of (1) is eventually periodic with period \(k\).

**Proof.** Let \(\{x_{nk}\}_{n=0}^{\infty}\) be a well-defined solution of (1). Using arguments similar to the ones developed in the proof of Theorem 1, we know that, for every \(i \geq 0\), \(\{x_{nk+i}\}_{n=0}^{\infty}\) is increasing, and \(x_{nk+i} < 0\) for all \(n \geq 0\) if there exists \(N_i > 0\) such that \(x_{nk+i} > 0\) for all \(n \geq N_i\).

We may assume without loss of generality that \(A_1 \geq 0\). We claim that \(\{x_{nk}\}_{n=0}^{\infty}\) is a constant sequence eventually. Indeed, \(\{x_{nk}\}_{n=0}^{\infty}\) is not constant sequence eventually, then there exist \(kr < n_1 < n_2 < \cdots\) such that \(x_{nk} > x_{nk+i}\) with \(A_{nk}\) being a constant sequence for all \(i \geq 1\). Thus we have

\[
x_{n+1,k} = \max\left\{ \frac{A_{n+k}}{x_{n+1,k-1}} \right\} \\
&= \frac{A_{nk}}{x_{n+1,k-1}} \\
&= \frac{A_{nk}}{x_{n+1,k-1}} \\
&= \max\left\{ \frac{A_{n+k}}{x_{n+1,k-2}} x_{n+1,k-1} \right\}
\]
Consider the max-type equation 

\[ x_{n,k} = \max \left\{ A_{k}^{n}, x_{k,n-1} \right\} \]

\[ x_{n,k} = A_{n,k}, \quad x_{n,k-1} \geq A_{k} \geq 0. \] (8)

From this we obtain that, for all \( i \geq 1 \),

\[ x_{n,k} x_{n,k-1} \geq A_{k} \geq 0. \] (9)

Thus \( A_{n,k} \geq 0 \) and

\[ A_{n,k} \left( x_{n,k-1} - x_{n,k-2} \right) < 0. \] (10)

It follows that, for all \( i \geq 1 \),

\[ x_{n,i} - x_{n,i-1} < x_{n,i-1} - x_{n,i-2}. \] (11)

This is a contradiction.

Using arguments similar to the ones developed in the proof of Theorem 1, we can show that \( x_{n,k,i} \) is a constant sequence eventually for every \( 1 \leq j \leq k \). Note \( \{ j \mid 1 \leq j \leq k \} \mod k = \{ 0, 1, 2, \ldots, k-1 \} \) since \( \gcd(k, r) = 1 \). Then \( x_{n,i} \) is a constant sequence eventually for every \( i \in \{ 0, 1, \ldots, k-1 \} \), which implies that \( \{ x_{n,i} \}_{i=1}^{k-1} \) is eventually periodic with period \( k \).

Now we construct an example with \( p \geq 2 \) and \( k \) being even which has a well-defined solution that is not eventually periodic.

**Example 4.** Consider the max-type equation

\[ x_{n} = \max \left\{ A_{n}, x_{n-r}, x_{n-s} \right\}, \quad n = 1, 2, \ldots \] (12)

where \( k, r \in \{ 1, 2, \ldots \} \) and \( k \) is even with \( \gcd(k, r) = 1 \) and \( k \neq r \) and \( A_{n} \) is a periodic sequence with \( A_{2i} = A_{2i-1} = B < 0 \) for all \( i \geq 1 \). Choose the initial conditions \( x_{i} = B \) for odd \( i \in \{ 0, 1, \ldots, d \} \) and \( x_{i} = 1 \) for even \( i \in \{ 0, 1, \ldots, d \} \) with \( d = \max\{r, k\} \); we can obtain a solution \( \{ x_{n} \}_{n=1}^{\infty} \) of (12) such that

(1) If \( r < k \), then

\[ x_{n,i} = \begin{cases} B & \text{if } i \in \{ 1, 3, \ldots, r \}, \\ A & \text{if } i \in \{ r+2, \ldots, k-1 \}, \\ B & \text{if } i \in \{ 2, \ldots, k \}. \] (13)

It is easy to verify that \( \lim_{n \to \infty} x_{2n} = \infty \) and \( \lim_{n \to \infty} x_{2n-1} = 0. \)

(2) If \( r > k \), then

\[ x_{n,i} = \begin{cases} B & \text{if } i \in \{ 1, 3, \ldots, r \}, \\ A & \text{if } i \in \{ r+2, \ldots, 2r-1 \}, \\ B & \text{if } i \in \{ 2, \ldots, 2r \}. \] (14)

It is easy to verify that \( \lim_{n \to \infty} x_{2n} = \infty \) and \( \lim_{n \to \infty} x_{2n-1} = 0. \)

**Remark 5.** Consider the max-type equation

\[ x_{n} = \max \left\{ A_{n}, x_{n-r}, x_{n-s} \right\}, \quad n = 1, 2, \ldots \] (15)

where \( \{ A_{n} \}_{n=1}^{\infty} \) is a periodic sequence with period \( ps \) and \( s, k, r \in \{ 1, 2, \ldots \} \) with \( \gcd(k, r) = 1 \) and \( k \neq r \), and \( p = 1 \) (or \( p \geq 2 \)), and the initial conditions \( x_{1}, x_{2}, \ldots, x_{0} \) are real numbers with \( d = \max\{sr, sk\} \). Write \( y_{n} = x_{n+i} \) for every \( 1 \leq i \leq s \) and \( n = 0, 1, 2, \ldots \). Then (12) reduces to the equation

\[ y_{n} = \max \left\{ A_{n+i}, y_{n-r}, y_{n-s} \right\}, \quad 1 \leq i \leq s, \quad n = 0, 1, 2, \ldots \] (16)

(1) If \( p = 1 \) (or \( p \geq 2 \) and \( k \) is odd), then it follows from Theorem 1 that, for every \( 1 \leq i \leq s \), every well-defined solution of equation \( y_{n} = \max\{ A_{n+i}, y_{n-r}, y_{n-s} \} \) is eventually periodic with period \( k \). Thus every well-defined solution of (15) is eventually periodic with period \( sk \).

(2) If \( p \geq 2 \) and, for every \( 1 \leq i \leq s \), there exists some \( j_{i} \) such that \( A_{j_{i}} \geq 0 \), then it follows from Theorem 3 that for every \( 1 \leq i \leq s \), every well-defined solution of equation \( y_{n} = \max\{ A_{n+i}, y_{n-r}, y_{n-s} \} \) is eventually periodic with period \( k \). Thus every well-defined solution of (15) is eventually periodic with period \( sk \).

(3) If \( p \geq 2 \) and \( k \) is even, then it follows from Example 4 that, for every \( 1 \leq i \leq s \), we can construct an equation \( y_{n} = \max\{ A_{n+i}, y_{n-r}, y_{n-s} \} \) such that it has a well-defined solution which is not eventually periodic. Thus we can construct an equation

\[ x_{n} = \max \left\{ A_{n}, x_{n-r}, x_{n-s} \right\}, \quad n = 1, 2, \ldots \] (17)

such that it has a well-defined solution which is not eventually periodic.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.
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