I. INTRODUCTION

It is known\(^1\) that in order to apply the equations of Einstein’s General Relativity at the large length scales of interest in cosmology, it is necessary to first perform a smoothing or averaging operation, which will generate nontrivial corrections in these nonlinear equations. There is a debate in the literature concerning two basic questions regarding this smoothing operation: (a) How does one obtain consistent and observationally relevant variables associated with the corrections due to averaging or “backreaction”, and (b) what is the magnitude of this backreaction?

This discussion has had a long history\(^2\)\(^3\)\(^4\), although in recent times attention has been focussed on two promising candidates for a consistent\(^5\) nonperturbative\(^6\) averaging procedure, namely the spatial averaging (on average homogeneous) metric of the universe, of inhomogeneities which only perturbatively affect the literature\(^7\). It has been argued\(^8\)\(^9\)\(^10\) that effects of inhomogeneities which only perturbatively affect the (on average homogeneous) metric of the universe, cannot lead to effects large enough to account for the inferred acceleration of the universe from, say supernovae observations\(^11\). This has been countered by the argument\(^12\)\(^13\)\(^14\) that perturbation theory may break down during the epoch of fully nonlinear structure formation. Recently it was shown\(^15\) in the context of a specific model of spherical collapse, that an explicit coordinate transformation could be found, which brought the metric to the perturbed Friedmann-Lemaître-Robertson-Walker (FLRW) form,

\[
ds^2 = -(1 + 2\varphi)dt^2 + a^2(\tau)(1 - 2\psi)d\vec{x}^2 ,
\]

satisfying all conditions required for a perturbation formalism in \(\varphi\) and \(\psi\) to hold, even in the regime of fully nonlinear collapse.

In such a context, it is important to have a consistent formalism, free from issues such as gauge artifacts, in which one has derived expressions for the backreaction which can be applied in a straightforward manner to any model in which the metric can be brought to the form\(^16\). Since the Buchert framework by construction is best adapted to coordinates comoving with the matter, most applications using perturbation theory in this framework have focussed on the synchronous and comoving gauge, although recently Behrend et al.\(^17\) have also performed calculations in the conformal Newtonian gauge. However, as pointed out elsewhere\(^18\), calculations using the Buchert framework in perturbation theory necessarily face the ambiguity of dealing with two scale factors: one is the scale factor with which the perturbed FLRW metric is defined, and the other is the volume averaged scale factor defined by Buchert\(^19\), and it is not clear which of these scale factors is the observationally relevant one.

In this paper we shall deal with the covariant framework of Zalaletdinov’s Macroscopic Gravity (MG), and its spatial averaging limit proposed in\(^20\). We will see that here, one has a well defined averaged metric with a corresponding uniquely defined scale factor. Further, we will argue that the structure of MG allows us to completely specify a consistent, observationally relevant aver-
The organisation of this paper is as follows: Sec. II presents a brief recap of the basic ideas underlying MG and some useful expressions from standard cosmological perturbation theory (PT), together with a proposal for practically estimating the effect of the backreaction on the time evolution of the FLRW scale factor. Sec. III presents details of the MG averaging procedure adapted to cosmological PT, including general expressions for the leading order backreaction terms, with a discussion of gauge related issues and the definition of the averaging operator. The heart of the paper is in Sec. IV, where gauge related issues and the definition of the averaging operation adapted for perturbations to the FLRW W scale factor. Sec. V contains example calculations in first order PT, which show that the magnitude of the backreaction is, as expected, negligible compared to the homogeneous energy density of matter in the radiation dominated era and for a significant part of the matter dominated era. We conclude in Sec. VI with some final comments. Throughout the paper, lower case Latin indices $a,b,c,...$ will refer to spacetime indices 0,1,2,3, and upper case Latin indices $A,B,C,...$ to spatial indices 1,2,3. The speed of light $c$ is set to unity, and a prime refers to a derivative with respect to conformal time unless stated otherwise.

II. COVARIANT MACROSCOPIC GRAVITY (MG) AND COSMOLOGY

A. Recap of MG formalism

In this section we present a rapid overview of the generally covariant averaging formalism of Macroscopic Gravity (MG), developed by Zalaletdinov and coworkers \cite{4,5,22,23}, and its spatial averaging limit proposed in \cite{21}. The reader is referred to these papers for more details.

The MG formalism uses a bilocal operator $W^{ab}_{i}(x', x)$, called the coordination bivector, to define a covariant spacetime averaging operation on tensors in a spacetime manifold $\mathcal{M}$. (The prime here is being used to distinguish two different spacetime points, and must not be confused with a derivative with respect to conformal time.) The various properties that this operator must satisfy, and a proof of its existence can be found in Refs. \cite{4,5,21}. The form of the coordination bivector used in all MG calculations is

$$W^{ab}_{i}(x', x) = \frac{\partial x^{a}}{\partial x^{j}_{VP}} \left|_{x'} \right| \frac{\partial x^{b}}{\partial x^{j}_{VP}} \left|_{x} \right| ,$$

(2)

where $x^{j}_{VP}$ refers to a coordinate system in which the metric determinant is constant. The class of such coordinate systems is called volume preserving (VP). Not surprisingly, the entire formalism of MG simplifies considerably in VP coordinate systems, in which the coordination bivector reduces to the Kronecker delta

$$W^{ab}_{i}(x', x)_{|VP} = \delta^{a}_{i} \delta^{b}_{j} .$$

(3)

We will see later that VPC systems play an important role in consistently setting up cosmological perturbation theory in the context of MG.

The average of a tensor $P^{a}_{b}$ over a finite spacetime domain $\Sigma$ is given by

$$\bar{P}^{a}_{b}(x) = \langle \bar{P}^{a}_{b}(x) \rangle = \frac{1}{V_{\Sigma}} \int_{\Sigma} d^{4}x' \sqrt{-g'} P^{a}_{b}(x', x) ;$$

$$V_{\Sigma} = \int_{\Sigma} d^{4}x' \sqrt{-g'} ,$$

(4)

where $\bar{P}^{a}_{b}(x', x)$ is the bilocal extension of $P^{a}_{b}$ defined as

$$\bar{P}^{a}_{b}(x', x) = W^{a}_{b}(x, x') P^{b}_{a}(x') W^{a}_{i}(x', x) .$$

(5)

The averaging operation when appropriately applied to the connection $\Gamma^{a}_{bc}$ on $\mathcal{M}$, gives an averaged connection $\tilde{\Gamma}^{a}_{bc}$ which is taken to be the connection on an averaged manifold $\mathcal{M}$. In other words, the connection on $\mathcal{M}$ satisfies the condition

$$\langle \tilde{\Gamma}^{a}_{bc} \rangle = \tilde{\Gamma}^{a}_{bc} .$$

(6)

The metric $G_{ab}$ associated with the averaged connection can be assumed to be the average of the inhomogeneous metric $g_{ab}$ on $\mathcal{M}$, i.e. $G_{ab} = \bar{g}_{ab}$. (We will see later that in the perturbative setting this amounts to a very natural condition on the perturbations.) Averaging the Einstein equations on $\mathcal{M}$ leads to the equations satisfied by the averaged metric, which can be written as

$$E^{a}_{b} = 8\pi G_{N} T^{a}_{b} + \langle grav \rangle T^{a}_{b} .$$

(7)

Here $E^{a}_{b}$ is the Einstein tensor constructed from the metric $G_{ab}$ and its inverse $G^{ab}$, $T^{a}_{b}$ is the averaged energy-momentum tensor, $G_{N}$ is Newton’s gravitational constant, and $\langle grav \rangle T^{a}_{b}$ is a tensorial correlation object which acts like an effective gravitational energy-momentum tensor. For brevity we shall omit its detailed definition, referring the reader to Refs. \cite{4,5}, and only note that its broad structure can be symbolically represented as

$$\langle grav \rangle T \sim \langle \tilde{\Gamma}^{2} \rangle - \langle \tilde{\Gamma} \rangle^{2} ,$$

(8)
where \( \tilde{\Gamma} \) symbolically denotes the bilocal extension of the Christoffel connection on \( \mathcal{M} \). In general the total energy-momentum tensor \((8\pi G_N T^a_b + (\text{grav}) T^a_b)\) is covariantly conserved,

\[
(8\pi G_N T^a_b + (\text{grav}) T^a_b)_{\alpha} = 0, \quad (9)
\]

with the semicolon denoting the covariant derivative with respect to the averaged geometry (\( \tilde{\Gamma}^a_{bc} \) and \( G_{ab} \)).

In Ref. [21] it was argued that in the cosmological context, it is essential to consider a spatial averaging limit of the covariant averaging used in MG. The simplest way to see this is to note that the homogeneous and isotropic FLR W spacetime must be left invariant under the averaging procedure more exactly.

Throughout this paper we will assume that the metric given by

\[
\frac{1}{a^2} \left[ a \frac{da}{d\tau} \right]^2 = \frac{8\pi G_N}{3} \rho - \frac{1}{6} \left[ P^{(1)} + S^{(1)} \right], \quad (13a)
\]

\[
\frac{1}{a^2} \frac{d^2a}{d\tau^2} = -\frac{4\pi G_N}{3} (\rho + 3p) + \frac{1}{3} \left[ P^{(1)} + P^{(2)} + S^{(2)} \right], \quad (13b)
\]

where the combinations \((P^{(1)} + S^{(1)})\) and \((P^{(1)} + P^{(2)} + S^{(2)})\) are generally covariant scalars defined by the relations (see Eqs. 88 of Ref. [21], with \( f = a \))

\[
P^{(1)} = \frac{1}{a^2} \left[ \langle \tilde{\Gamma}_{0A}^A \tilde{\Gamma}_{0B}^B \rangle - \langle \tilde{\Gamma}_{0B}^B \tilde{\Gamma}_{0A}^A \rangle - 6\mathcal{H}^2 \right], \quad (14a)
\]

\[
S^{(1)} = \langle \tilde{g}^{JK} \rangle \left( \langle \tilde{\Gamma}_{JA}^A K_B \rangle - \langle \tilde{\Gamma}_{JB}^A K_A \rangle \right), \quad (14b)
\]

\[
P^{(2)} + P^{(1)} = -\frac{1}{a^2} \langle \tilde{\Gamma}_{0A}^A \tilde{\Gamma}_{00}^0 \rangle - \langle \tilde{g}^{JK} \rangle \langle \tilde{\Gamma}_{JA}^A K_B \rangle + 6\mathcal{H}^2 \frac{a^2}{a^2}, \quad (14c)
\]

\[
S^{(2)} = \frac{1}{a^2} \langle \tilde{\Gamma}_{00}^A \tilde{\Gamma}_{0A}^A \rangle + \langle \tilde{g}^{JK} \rangle \langle \tilde{\Gamma}_{0J}^0 \tilde{\Gamma}_{0K}^A \rangle, \quad (14d)
\]

where we have defined

\[
\mathcal{H} = \frac{1}{a^2} \frac{da}{d\eta} \equiv \frac{a'}{a}, \quad (15)
\]

and accounted for the fact that in general, the average of the inverse inhomogeneous metric, need not equal the inverse metric \( G^{ab} \). However, we will soon see that in the perturbative setting we can in fact set these two tensors to be equal via a very natural condition on the perturbations. The index 0 in Eqs. (14) refers to the conformal time \( \eta \). The averaging in Eqs. (15) is assumed to be a spatial averaging in an unspecified spatial slicing in the inhomogeneous manifold \( \mathcal{M} \); in Sec. III we will specify the averaging procedure more exactly.

In addition, the following “cross-correlation” constraints must also be satisfied by the inhomogeneities (see
Eqns. 89 of Ref. [21]

\[
\frac{1}{a^2} \left[ \langle \tilde{\Gamma}_0^0 \tilde{\Gamma}_0^m \rangle - \langle \tilde{\Gamma}_B^0 \tilde{\Gamma}_B^m \rangle \right] + \langle \tilde{g}^{JK} \rangle \left[ \langle \tilde{\Gamma}_0^0 \tilde{\Gamma}_B^J \tilde{\Gamma}_B^K \rangle - \langle \tilde{\Gamma}_J^0 \tilde{\Gamma}_B^K \rangle \right] = 0 \quad (16a)
\]

\[
\frac{1}{a^2} \left[ \langle \tilde{\Gamma}_B^A \tilde{\Gamma}_B^m \rangle - \langle \tilde{\Gamma}_m^0 \tilde{\Gamma}_B^A \rangle \right] + \langle \tilde{g}^{JK} \rangle \left[ \langle \tilde{\Gamma}_B^A \tilde{\Gamma}_K^J \tilde{\Gamma}_K^m \rangle - \langle \tilde{\Gamma}_K^A \tilde{\Gamma}_B^J \tilde{\Gamma}_K^m \rangle \right] = 0 \quad (16b)
\]

\[
\frac{1}{a^2} \left[ \langle \tilde{\Gamma}_B^0 \tilde{\Gamma}_B^m \rangle - \langle \tilde{\Gamma}_m^0 \tilde{\Gamma}_B^A \rangle \right] + \langle \tilde{g}^{JK} \rangle \left[ \langle \tilde{\Gamma}_B^A \tilde{\Gamma}_K^J \tilde{\Gamma}_K^m \rangle - \langle \tilde{\Gamma}_K^A \tilde{\Gamma}_B^J \tilde{\Gamma}_K^m \rangle \right] = 0 \quad (16c)
\]

where the lower case index \( m \) is the last equation runs over all spacetime indices 0, 1, 2, 3.

B. Cosmological Perturbations and Gauge Transformations

For ready reference, in this subsection we present expressions for the metric, its inverse, and the Christoffel connection in first order cosmological PT, in an arbitrary, unified gauge. The notation we use is similar to that used in Ref. [25]. We will also give expressions for the first order gauge transformations of the perturbation functions (see e.g. Ref. [26]).

The first order perturbed FLRW metric in an arbitrary gauge can be written as

\[
ds^2 = a^2(\eta) \left[ -(1 + 2\varphi)d\eta^2 + 2\omega_A d\eta^A d\eta^B + ((1 - 2\psi)\gamma_{AB} + \chi_{AB}) d\eta^A d\eta^B \right]. \quad (17)
\]

The functions \( \varphi \) and \( \psi \) are scalars under spatial coordinate transformations. The functions \( \omega_A \) and \( \chi_{AB} \) can be decomposed as follows

\[
\omega_A = \partial_A \omega + \tilde{\omega}_A \quad ; \quad \chi_{AB} = D_{AB} \chi + 2\nabla_{(A} \tilde{\chi}_{B)} + \tilde{\chi}_{AB}, \quad (18)
\]

where the parentheses indicate symmetrization; \( D_{AB} \) is the tracefree second derivative defined by

\[
D_{AB} \equiv \nabla_A \nabla_B - (1/3) \gamma_{AB} \nabla^2 \quad \nabla^2 \equiv \gamma^{AB} \nabla_A \nabla_B, \quad (19)
\]

with \( \nabla_A \) the covariant spatial derivative compatible with \( \gamma_{AB} \); and \( \omega_A \), \( \tilde{\omega}_A \) and \( \chi_{AB} \), \( \tilde{\chi}_{AB} \) satisfy

\[
\nabla_A \omega^A = 0 = \nabla_A \tilde{\omega}^A \quad ; \quad \nabla_A \tilde{\chi}^A = 0 = \tilde{\chi}_A, \quad (20)
\]

where spatial indices are raised and lowered using \( \gamma_{AB} \) and its inverse \( \gamma^{AB} \). From their definitions it is clear that \( \varphi, \psi, \omega \) and \( \chi \) each correspond to one scalar degree of freedom, the transverse 3-vectors \( \tilde{\omega}_A \) and \( \tilde{\chi}_A \) each correspond to two functional degrees of freedom, and the transverse tracefree 3-tensor \( \tilde{\chi}_{AB} \) corresponds also to two functional degrees of freedom. This totals to 10 degrees of freedom, of which 4 are coordinate degrees of freedom which can be arbitrarily fixed, which is what one means by a gauge choice. For example, the conformal Newtonian or longitudinal or Poisson gauge \([20, 21, 22]\) is defined by the conditions

\[
\omega = 0 = \chi; \quad \tilde{\chi}^A = 0. \quad (21)
\]

For the metric (17) we have at first order,

\[
\sqrt{-\det g} = a^4(\eta) \left( 1 + \varphi - 3\psi \right). \quad (22)
\]

The inverse of metric (17), correct to first order, has the components

\[
g^{00} = -\frac{1}{a^2} (1 - 2\varphi); \quad g^{0A} = \frac{1}{a^2} \omega^A; \quad g^{AB} = \frac{1}{a^2} \left( (1 + 2\psi)\gamma^{AB} - \chi^{AB} \right). \quad (23)
\]

Denoting \( \mathcal{H} = (a'/a) \), the prime denoting a derivative with respect to conformal time \( \eta \), the first order accurate Christoffel symbols are

\[
\Gamma^0_{00} = \mathcal{H} \varphi'; \quad \Gamma^0_{0A} = \partial_A \varphi + \mathcal{H} \omega_A, \quad \Gamma^A_{00} = \partial_A \varphi + \mathcal{H} \omega^A, \quad \Gamma^0_{AB} = (\mathcal{H} - \psi - 2\mathcal{H}(\varphi + \psi)) \gamma_{AB} - \nabla_{(A} \omega_{B)} + \frac{1}{2} \mathcal{H} \chi_{AB}, \quad \Gamma^A_{0B} = (\mathcal{H} - \psi - 2\mathcal{H}(\varphi + \psi)) \gamma^A_B + \nabla_A \omega_B + \mathcal{H} \chi_{AB}, \quad \Gamma^A_{BC} = \frac{1}{2} \nabla_B \omega^A - \frac{1}{2} (\nabla_B \gamma^A_{C} + \nabla_C \chi^A_B - \nabla_C \chi^A_B), \quad (24)
\]

where \( \tilde{\Gamma}_{BC} \) denotes the Christoffel connection associated with the homogeneous 3-metric \( \gamma_{AB} \).

Gauge transformations : While the concept of gauge transformations can be described in a rather sophisticated language using pullback operators between manifolds [20], for our purposes it suffices to implement a gauge transformation using the simpler notion of an infinitesimal coordinate transformation (also known as the “passive” point of view) [29]. Hence, denoting the coordinates and perturbation functions in the new gauge with a tilde (i.e. \( \tilde{x}^a, \tilde{\varphi}, \tilde{\omega}_A \), and so on), we have

\[
\tilde{x}^a = x^a + \xi^a(x) \quad ; \quad x^a = \tilde{x}^a - \xi^a. \quad (25)
\]
where the infinitesimal 4-vector $\xi^a$ can be decomposed as

$$\xi^a = (\xi^0, \xi^A) = (\alpha, \partial^A \beta + d^A), \quad (26)$$

where $\alpha$ and $\beta$ are scalars and $d^A$ is a transverse 3-vector satisfying $\nabla_A d^A = 0$.

It is then easy to show that if this transformation is assumed to change the metric \[17\] by changing only the perturbation functions but leaving the background intact (a so-called “steady” coordinate transformation), then the old perturbations and the new are related by \[20\]

$$\varphi = \tilde{\varphi} + \alpha' + H\alpha, \quad \psi = \tilde{\psi} - \frac{1}{3} \nabla^2 \beta - H\alpha, \quad \omega = \tilde{\omega} - \alpha + \beta', \quad \hat{\omega}^A = \tilde{\omega}^A + d^{A'}, \quad \chi = \tilde{\chi} + 2\beta, \quad \hat{\chi}^A = \tilde{\chi}^A + d^A, \quad \hat{\chi}_{AB} = \tilde{\chi}_{AB}. \quad (27)$$

The last equality shows that the transverse tracefree tensor perturbations are gauge invariant. They correspond to gravitational waves.

**C. Time evolution of the background: An iterative approach**

Before we move on to deriving formulae for the correlation terms \[13\] in terms of perturbation functions in the metric, there is one issue which merits discussion. The cosmological perturbation setting, together with the paradigm of averaging, presents us with a rather peculiar situation. On the one hand, the time evolution of the scale factor is needed in order to solve the equations satisfied by the perturbations. Indeed, the standard practice is to fix the time evolution of the background once and for all, and to use this in solving for the evolution of the perturbations. On the other hand, the evolution of the perturbations (i.e. – the inhomogeneities) is needed to compute the correlation terms appearing in Eqns. \[13\].

Until these terms are known, the behaviour with time of the scale factor cannot be determined; and until we know the scale factor as a function of time, we cannot solve for the perturbations. Note that this is a generic feature independent of all details of the averaging procedure.

It would appear therefore, that we have reached an impasse. To clear this hurdle, one can try the following iterative approach: Symbolically denote the background as $a$, the inhomogeneities as $\varphi$, and the correlation objects as $C$. Note that $a$, $\varphi$ and $C$ all refer to functions of time. We start with a chosen background, say a standard flat FLRW background with radiation, baryons and cold dark matter (CDM), and solve for the perturbations in the usual way, without accounting for the correlation terms $C$. In other words, for this “zeroth iteration”, we artificially set $C$ to zero and obtain $a^{(0)}$ and $\varphi^{(0)}$ using the standard approach (see e.g. Ref. \[30\]). Clearly, since the “true” background (say $a_*$) satisfies Eqns. \[13\] with a nonzero $C$, we can calculate the zeroth iteration correlation objects $C^{(0)}$ by applying the prescription to be developed later in this paper. As a first correction to the solution $a^{(0)}$, we now solve for a new background $a^{(1)}$, with the known functions $C^{(0)}$ acting as sources in Eqns. \[13\]. This first iteration will then yield a solution $\varphi^{(1)}$ for the inhomogeneities, and hence a new set of correlation terms $C^{(1)}$, and this procedure can be repeatedly applied. [See however the first paper in Ref. \[14\] for an alternative approach exclusively using averaged quantities in solving the full MG equations.] Pictorially,

$$a^{(0)} \rightarrow \varphi^{(0)} \rightarrow C^{(0)} \rightarrow a^{(1)} \rightarrow \varphi^{(1)} \rightarrow \ldots \quad (28)$$

As for convergence, if perturbation theory is in fact a good approximation to the real universe, then one can expect that the correlation terms will tend to be small compared to other background objects, and will therefore not affect the background significantly at each iteration, leading to rapid convergence. On the other hand, if the correlation terms are large, this procedure may not converge and one might expect a breakdown of the perturbative picture itself \[31\]. We will see that in the linear regime of cosmological perturbation theory, the correlation terms do in fact remain negligibly small.

**III. THE AVERAGING OPERATION AND GAUGE RELATED ISSUES**

In this section, we will describe the details of the MG (spatial) averaging procedure adapted to the setting of cosmological PT.

**A. Volume Preserving (VP) Gauges and the Correlation Scalars**

It will greatly simplify the discussion if we start with symbolic calculations which allow us to see the broad structure of the objects we are after. Since the correlation objects in Eqns. \[13\] depend only on derivatives of the metric, we will primarily deal with metric fluctuations; matter perturbations will only come into play when solving for the actual dynamics of the system. Before dealing with the issue of which gauge to choose in order to set the condition \[6\], we will show that irrespective of this choice, the leading order contribution to the correlations requires knowledge of only first order perturbation functions.

We will use the following symbolic notation:

- Inhomogeneous connection: $\Gamma$
- FLRW connection: $\Gamma_F$
• Perturbation in the connection: \( \delta \Gamma \equiv \Gamma - \Gamma^F = \delta \Gamma^{(1)} + \delta \Gamma^{(2)} + \ldots \)

• Coordination bivector: \( W \equiv 1 + \delta W = 1 + \delta W^{(1)} + \delta W^{(2)} + \ldots \)

• Bilocal extension of the connection: \( \tilde{\Gamma} \)

• Inhomogeneous part of the bilocal extension of the connection: \( \delta \tilde{\Gamma} \equiv \tilde{\Gamma} - \Gamma = \delta \tilde{\Gamma}^{(1)} + \delta \tilde{\Gamma}^{(2)} + \ldots \)

• Correlation object: \( (\text{grav}) T \)

The integer superscripts denote the order of perturbation. The form of the coordination bivector arises from the fact that in perturbation theory, in the spatial averaging limit, a transformation from an arbitrary gauge to a VP one can be achieved by an infinitesimal coordinate transformation. By a VP gauge we mean a gauge in which the metric determinant is independent of the spatial coordinates to the relevant order in PT, but may be a function of time. It can be shown that such a function of time (which will typically be some power of the scale factor), is completely consistent with all definitions and requirements of MG in the spatial averaging limit. An easy way of seeing this is to note that in any averaged quantity, the metric determinant appears in two integrals, one in the numerator and the other in the denominator (which gives the normalising volume). In the “thin time slicing” approximation we are using to define the averaging, any overall time dependent factor in the denominator (which gives the normalising volume). In the integrals, one in the numerator and the other in the denominator. In the “thin time slicing” approximation, this gives the normalising volume. It is not hard to show that in the thin time slicing approximation, this gives the same coordination bivector \( W^\prime_x(x', x) \) as the VP gauge definition above.

To see that first order perturbations are sufficient to calculate \( (\text{grav}) T \) to leading order, we only have to note that the background connection \( \Gamma^F \) satisfies

\[
\langle \Gamma^F \rangle = \Gamma^F,
\]

and that the structure of \( (\text{grav}) T \) is given by Eqn. (29). \( (\text{grav}) T \) then reduces to

\[
(\text{grav}) T = \langle \tilde{\Gamma}^2 \rangle - \langle \tilde{\Gamma} \rangle^2,
\]

which is exact. Clearly, the correlation is quadratic in the perturbation as expected, and hence to leading order, \( \tilde{\Gamma} \) above can be replaced by \( \tilde{\Gamma}^{(1)} \).

Eqs. (29) and (30) treat the averaging operation at a conceptual level only. To make progress however, we also need to prescribe how to practically impose the averaging assumption

\[
\langle \tilde{\Gamma} \rangle = \Gamma^F \quad \text{i.e.} \quad \langle \tilde{\Gamma} \rangle = 0,
\]

in any given perturbative context. This requires some discussion since, for example, the bilocal extension of the connection \( \tilde{\Gamma} \) has the structure

\[
\tilde{\Gamma} = \Gamma W + W^{-1}(\partial + \partial')W,
\]

where \( \partial \) is a derivative at \( x \) and \( \partial' \) a derivative at \( x' \). [The reader is referred to Ref. [6, 7] for the detailed expression. Suffice it to note that this structure ensures that the averaged connection has the correct transformation properties.] The actual MG averaging operation in general is therefore a rather involved procedure. Additionally, it is also necessary to address certain gauge related issues.

To clarify the situation, let us start with a fictitious setting in which the geometry has exactly the flat FLRW form, with no physical perturbations. Clearly, if we work in the standard comoving coordinates in which the metric \( \gamma_{AB} \) of Eqn. (10) is simply \( \gamma_{AB} = \delta_{AB} \), then since these coordinates are volume preserving in the sense described above, the coordination bivector becomes trivial. The averaging involves a simple integration over 3-space, and we can easily see that Eqn. (29) is explicitly recovered.

Now suppose that we perform an infinitesimal coordinate transformation, after imposing Eqn. (29). Since the averaging operation is covariant, then from the point of view of a general coordinate transformation, both sides of Eqn. (29) will be affected in the same way. However, suppose that we had performed the transformation before imposing Eqn. (29). In the language of cosmological PT, we would then be dealing with some “pure gauge” perturbations around the fixed, spatially homogeneous background. If we did not know that these perturbations were pure gauge, we might naively construct the nontrivial coordination bivector for this metric, compute the bilocal extension of the connection according to Eqn. (32) and try to impose Eqn. (31). This would be incorrect since these perturbations were arbitrarily generated and need not average to zero (for example they could be positive definite functions). In order to maintain consistency, it is then necessary to ensure in practice that the averaging condition (31) is applied only to gauge invariant inhomogeneities present. Note from Eqn. (2) that the coordination bivector has the structure

\[
W = \frac{\partial x}{\partial x^\prime} \bigg|_{x^\prime} \frac{\partial x^\prime}{\partial x} \bigg|_x
\]

where \( x \) denotes the coordinates we are working in and \( x^\prime \) a set of VPCs. In perturbation theory (in the spatial averaging limit) we will have, at leading order,

\[
x = x^\prime - \xi \quad ; \quad x^\prime = x + \xi,
\]

where \( \xi \) symbolically denotes an infinitesimal 4-vector defining the transformation, and hence

\[
(\partial x^\prime)/(\partial x) = 1 + \partial \xi
\]
and so on, which gives us

\[ W = 1 - (\partial \xi)_x' + (\partial \xi)_x + \ldots = 1 + \delta W^{(1)} + \ldots \]  

(36)

Now when we compute a quantity such as \( \langle \Gamma F \delta W^{(1)} \rangle \) which appears in the expression for \( \langle \Gamma \rangle \), we will be left with a fluctuating (\( \vec{x} \)-dependent) term of the form 

\[ \Gamma_F \langle \partial \xi - \partial \xi \rangle, \]

where \( \vec{x} \) denotes the 3 spatial coordinates. Hence if we try to impose Eqn. (31) we will ultimately be left with a fluctuating (\( \vec{x} \)-dependent) term of the form

\[ \langle f \rangle (\vec{x}) - f(\vec{x}) = 0 \]  

(37)

for some functions derived from the inhomogeneities which we have collectively denoted \( f \). In other words, consistency would seem to demand that the inhomogeneities vanish in this coordinate system, which is of course not desirable.

It therefore appears that we are forced to impose Eqn. (31) in a volume preserving gauge, since by definition, only in such a gauge will we have \( W = 1 \) exactly. We emphasize that this is a purely practical aspect related to defining the averaging operation, and is completely decoupled from, e.g., the choice of gauge made when studying the time evolution of perturbations. We are in no way breaking the usual notion of gauge invariance by choosing an averaging operator. The conditions Eqn. (37) now reduce to the form

\[ \langle f_{\text{VPC}} \rangle (\vec{x}) = 0 \]  

(38)

which are far more natural than Eqn. (37). The averaging condition is now unambiguous, but depends on a choice of the VP gauge which defines the averaging operation, an issue we shall discuss in the next subsection. For now, all we can assert is that this VP gauge must be such that in the absence of gauge invariant fluctuations, it must reduce to the standard comoving (volume preserving) coordinates of the background geometry as in Eqn. (11). This of course is simply the statement that the VP gauge must be well defined and must not contain any residual degrees of freedom.

The averaging operation now takes on an almost trivial form – to leading order it is easy to show that for any quantity \( f(\eta, \vec{x}) \) (with or without indices), the average of \( f \) in a VP gauge in the spatial averaging limit, is given by

\[ \langle f \rangle (\eta, \vec{x}) = \frac{1}{V_L} \int_{V(\vec{x})} d^3y f(\eta, y), \]  

(39)

where the integral is over a spatial domain \( V(\vec{x}) \) with a constant volume \( V_L \). The spatial coordinates are the comoving coordinates of the background metric, and at leading order the boundaries of \( V(\vec{x}) \) can be specified in a straightforward manner as, e.g.,

\[ V(\vec{x}) = \{ \vec{y} | \ x^A - L/2 < y^A < x^A + L/2, A = 1, 2, 3 \} , \]  

(40)

where \( L \) is a comoving scale over which the averaging is performed (in which case \( V_L = L^3 \)). The averaging definition can be written more compactly in terms of a window function \( W_L(\vec{x}, \vec{y}) \) as

\[ \langle f \rangle (\eta, \vec{x}) = \int d^3y W_L(\vec{x}, \vec{y}) f(\eta, \vec{y}) , \]

\[ \int d^3y W_L(\vec{x}, \vec{y}) = 1 , \]  

(41)

where \( W_L(\vec{x}, \vec{y}) \) vanishes everywhere except in the region \( V(\vec{x}) \), with the integrals now being over all space. This expression will come in handy when working in Fourier space, as we shall do in later sections.

A couple of comments are in order at this stage. Firstly, we have not specified the magnitude of the averaging scale \( L \). The general philosophy is that this scale must be large enough that a single averaging domain encompasses several realisations of the random inhomogeneous fluctuations, and small enough that the observable universe contains a large number of averaging domains. However, as we will show later in Sec. IV if one is ultimately interested in quantities which are formally averaged over an ensemble of realisations of the universe (as is usually done in interpreting observations), then the actual value of the averaging scale becomes irrelevant.

This brings us to the second issue. The above discussion is valid only in the situation where there are no fluctuations at arbitrarily large length scales, since in the presence of such fluctuations the averaging condition (31) loses meaning (in such a situation it would be impossible to isolate the background from the perturbation by an averaging operation on any finite length scale). Indeed, we shall see a manifestation of this restriction in Sec. IV where the correlation scalars will be seen to diverge in the presence of a nonzero amplitude at arbitrarily large scales, of the power spectrum of metric fluctuations.

We will end this subsection by explicitly writing out the averaging condition in an “unfixed VP” gauge, to be defined below, and also writing the correlation terms appearing in Eqn. (13), in this gauge. As we can see from Eqn. (22), the basic condition to be satisfied by a VP gauge is

\[ \tilde{\psi} = 3\psi . \]  

(42)

Hereafter, all VP gauge quantities will be denoted using a tilde. This should not be confused with the similar notation that was used for the bilocal extension in Sec. IIA which will not be needed in the rest of the paper. \( \tilde{\psi} \) and \( \psi \) are the scalar potentials appearing in the perturbed FLRW metric (17). The single condition (42) leaves 3 degrees of freedom to be fixed, in order to completely specify the VP gauge one is working with. The MG formalism by itself does not prescribe a method to choose a particular VPC system; in fact this freedom of choice of VPCs is an inherent part of the formalism. We shall return to this issue in the next subsection. For now we define the “unfixed VP (uVP) gauge” by the single requirement (42), with 3 unfixed degrees of freedom, and
present the expressions for the averaging condition and the correlation scalars, with this choice.

It is straightforward to determine the consequences of requiring Eqn. (10) to hold, with the right hand side corresponding to the FLRW connection in conformal coordinates, and remembering that the coordination bivector (in the spatial averaging limit) is now trivial (see Eqn. (13)). Together with some additional reasonable requirements, namely

$$\langle \nabla^2 s \rangle = 0 = \langle \nabla^2 \partial_A s \rangle,$$  \hspace{1cm} (43)

for any scalar $s(\eta, \vec{x})$, the averaging condition in the uVP gauge reduces to

$$\langle \tilde{\psi} \rangle = 0; \quad \langle \tilde{\omega}_A \rangle = 0 = \langle \tilde{\psi}' \rangle; \quad \langle \tilde{\chi}_{AB} \rangle = 0,$$

$$\langle \nabla_C \tilde{\chi}^A_B \rangle + \langle \nabla_B \tilde{\chi}^C_A \rangle - \langle \nabla^A \tilde{\chi}^B_C \rangle = 0,$$

$$\langle \nabla_A \tilde{\omega}_B \rangle = \langle \nabla_B \tilde{\omega}_A \rangle = \mathcal{H}(\tilde{\chi}_{AB}),$$  \hspace{1cm} (44)

where we have used the expressions in Eqn. (24) with the uVP condition (12). We will also make the additional reasonable requirement that

$$\langle \tilde{\chi}_{AB} \rangle = 0,$$  \hspace{1cm} (45)

using which it is easy to see that the perturbed FLRW metric (17) and its inverse (23), in the uVP gauge, both on averaging reduce to their respective homogeneous counterparts, namely

$$\langle g_{ab} \rangle = g_{ab}^{(FLRW)}; \quad \langle g^{ab} \rangle = g^{ab}_{(FLRW)}.$$  \hspace{1cm} (46)

Using these results, the expressions (14) simplify to give, in the uVP gauge,

$$P^{(1)} = \frac{1}{a^2} \left[ 6\langle (\tilde{\psi}')^2 \rangle + \langle \nabla [A \tilde{\omega}_B] + \nabla [A \tilde{\omega}_B] \rangle \right.$$  \hspace{1cm} (47a)

$$- \frac{1}{4} \left( \tilde{\chi}_{AB} \tilde{\chi}^{AB} \right),$$

$$S^{(1)} = \frac{1}{a^2} \left[ -10 \langle \partial_A \tilde{\psi} \partial_A \tilde{\psi} \rangle - 2 \langle \partial_A \tilde{\psi} \nabla_B \tilde{\chi}^{AB} \rangle ight.$$  \hspace{1cm} (47b)

$$+ \frac{1}{4} \langle \nabla^2 \tilde{\chi}_{AC} + 2(\nabla_A \tilde{\chi}_{BC} - \nabla_B \tilde{\chi}_{AC} \rangle \right],$$

$$P^{(1)} + P^{(2)} = \frac{1}{a^2} \left[ 6\langle (\tilde{\psi}')^2 \rangle - 24\mathcal{H}(\tilde{\psi}' \tilde{\psi}) ight.$$  \hspace{1cm} (47c)

$$- \langle \tilde{\psi}' \nabla^2 \tilde{\omega} \rangle + \frac{1}{2} \langle \tilde{\chi}_{AB} \nabla^A \tilde{\omega}_B \rangle ight.$$  \hspace{1cm} (47c)

$$- \frac{1}{4} \left( \tilde{\chi}_{AB} (\tilde{\chi}^{AB} + 2\mathcal{H} \tilde{\chi}^{AB} \right),$$.  \hspace{1cm} (47c)

$$S^{(2)} = \frac{1}{a^2} \left[ 3\langle \tilde{\omega}^A \partial_A \tilde{\psi} \rangle + \mathcal{H}(\tilde{\omega}^A \tilde{\omega}'_A \rangle \right],$$  \hspace{1cm} (47d)

where square brackets denote antisymmetrization.

### B. Choice of VP Gauge

In this subsection we will prescribe a choice for the VP gauge which defines the averaging operation. In general, the class of volume preserving coordinate systems for any spacetime, is very large (see Ref. [22] for a detailed characterisation). We have so far managed to pare it down by requiring that the VP gauge we choose should reduce to the standard FLRW coordinates in the absence of fluctuations. It turns out to be somewhat difficult to go beyond this step, since there does not appear to be any unambiguously clear guiding principle governing this choice. We will therefore motivate a choice for the VP gauge based on certain details of cosmological PT which one knows from the standard treatments of the subject.

In particular, we shall make use of certain nice properties of the conformal Newtonian or longitudinal or Poisson gauge, which is defined by the conditions [21] [24] [27] (henceforth we shall refer to this gauge as the cN gauge for short). Since this gauge is well defined and has no residual degrees of freedom, all the nonzero perturbation functions in the cN gauge, namely $\varphi, \psi, \tilde{\omega}_A$ and $\tilde{\chi}_{AB}$ in the notation of Sec. [11] are equal to gauge invariant objects. This is trivially true for $\tilde{\chi}_{AB}$, as seen in the last equation in [27]. For the rest, note that in any arbitrary unfixed gauge, the following combinations are gauge invariant at first order

$$\Phi_B = \varphi + \frac{1}{a} \partial_0 \left[ \frac{1}{a} \left( \omega - \frac{1}{2} \chi' \right) \right],$$

$$\Psi_B = \psi - \mathcal{H} \left( \omega - \frac{1}{2} \chi' \right) + \frac{1}{6} \nabla^2 \chi,$$

$$\tilde{V}_A = \tilde{\omega}_A - \chi_A,$$  \hspace{1cm} (48)

which can be easily checked using Eqns. (27), and in the cN gauge, $\omega, \chi$ and $\tilde{\chi}_{AB}$ all vanish. Here $\Phi_B$ and $\Psi_B$ are the Bardeen potentials [32] (upto a sign), and $\Psi_B$ in particular has the physical interpretation of giving the gauge invariant curvature perturbation, which is the quantity on which initial conditions are imposed post inflation [33].

Additionally, it is also known that the cN gauge for the metric remains stable even during structure formation, when matter inhomogeneities have become completely nonlinear. (See Ref. [17] for an intuitive description of why this is so, and Ref. [20] for an explicit demonstration in a toy model of structure formation.) We believe that this is a strong argument in favour of using the cN gauge to define a VP gauge which will then define the averaging operation in the perturbative context. This will ensure that this “truncated” averaging operation, defined for first order PT, will remain valid at leading order even during the nonlinear epochs of structure formation.

To implement this in practice, consider a transformation from the cN gauge to the uVP gauge defined by Eqn.
The transformation equations \(27\) reduce to
\[
\alpha' + 4\mathcal{H}\alpha + \nabla^2\beta = \varphi - 3\psi,
\]
\[
\tilde{\psi} = \frac{1}{3}\varphi - \alpha' - \mathcal{H}\alpha,
\]
\[
\tilde{\omega} = \alpha' - \beta',
\]
\[
\tilde{\chi} = \omega^A - d^A \xi,
\]
\[
\tilde{\chi} = -2\beta,
\]
\[
\tilde{\chi}^A = -d^A,
\]
\[
\tilde{\chi}_{AB} = \hat{\chi}_{AB}.
\]
(49)

Recall that to completely specify a VP gauge, we need to fix 3 degrees of freedom in the uVP gauge. Our requirement regarding the “well defined”-ness of the VP gauge, forces us to set \(d^A = 0\), and to choose \(\alpha\) and \(\beta\) such that they vanish in the case where \(\varphi = 0 = \psi\).

This has fixed 2 degrees of freedom, in addition to the condition \(12\) which is just the definition of the uVP gauge, and has hence not yielded a uniquely specified VP gauge. To do this, we shall make the following additional requirement. Since we are dealing with a spatial averaging, it seems reasonable to require that the VP gauge being used to define the averaging, should be “as close as possible” to the cN gauge in terms of time slicing, and for this reason we shall set the function \(\alpha\) to zero.

To summarize, the VP gauge chosen is defined in terms of the gauge transformation functions \(\xi^a = (\alpha, \partial^A\beta + d^A)\) between the cN gauge and the VP gauge, by the following relations
\[
\alpha = 0 = d^A,
\]
and
\[
\bar{\varphi} = 3\tilde{\psi} = \varphi,
\]
\[
\nabla^2\beta = \varphi - 3\psi,
\]
\[
\bar{\omega} = -\beta' ; \quad \bar{\chi} = -2\beta,
\]
\[
\tilde{\omega} = \hat{\omega}_A = 0
\]
\[
\tilde{\chi}^A = \hat{\chi}^A = 0 ,
\]
\[
\tilde{\chi}_{AB} = \hat{\chi}_{AB}.
\]
(50)

where the function \(\beta\) is restricted not to contain any nontrivial solution of the homogeneous (Laplace) equation \(\nabla^2\beta = 0\).

Having made this choice for the VP gauge, we are now assured that all averaged quantities which we compute are gauge invariant: our choice ensures that the averaging procedure does not introduce any pure gauge modes, and the philosophy of “steady” coordinate transformations ensures that all background objects are, by assumption, unaffected by gauge transformations. In particular, the correlation objects in Eqs. \(17\) are all gauge invariant. This is different from the gauge invariance conditions derived in the first paper of Ref. \[4\], where the background was also taken to change under gauge transformations at second order in the perturbations. It is at present not clear how these results are related to ours.

Note that all these arguments are valid at first order in PT, which is sufficient for our present purposes. A consistent treatment at second order would require more work, although as long as one is interested only in the leading order effect, these arguments are expected to go through.

### IV. The Correlation Scalars

With the VP gauge choice defined by Eqns. \(51\), it is straightforward to rewrite the correlation objects in Eqs. \(17\) (which are in the uVP gauge) in terms of the perturbation functions in the cN gauge. We will restrict the subsequent calculations in this paper to the case where there are no transverse vector perturbations, i.e.,
\[
\tilde{\omega}_A = 0,
\]
in the cN gauge. This is a reasonable choice since such vector perturbations, even if they are excited in the initial conditions, decay rapidly and do not source the other perturbations at first order \[30\].

In addition, for simplicity (and to keep this paper concise) we will choose to ignore the gauge invariant tensor perturbations as well,
\[
\tilde{\chi}_{AB} = 0.
\]
(53)

It will be an interesting exercise to account for the effects of gravitational waves in the correlation scalars, however we will leave this to future work. Thus, the results presented here apply only to scalar perturbations.

In terms of the scalar perturbations in the cN gauge, for a flat FLRW background, the correlation objects \(17\) reduce to
\[
\mathcal{P}^{(1)} = \frac{1}{a^2} \left[ 2 \langle (\psi')^2 \rangle + \langle (\varphi' - \psi')^2 \rangle - \langle (\nabla_A\nabla_B\beta') (\nabla^A\nabla^B\beta') \rangle \right],
\]
(54a)
\[
\mathcal{S}^{(1)} = -\frac{1}{a^2} \left[ 6 \langle \partial_A\psi\partial^A\psi \rangle + \langle \partial_A(\varphi - \psi)\partial^A(\varphi - \psi) \rangle - \langle (\nabla_A\nabla_B\nabla_C\beta)(\nabla^A\nabla^B\nabla^C\beta) \rangle \right],
\]
(54b)
\[
\mathcal{P}^{(1)} + \mathcal{P}^{(2)} = \frac{1}{a^2} \left[ \langle \varphi'(\varphi' - \psi') \rangle - 2\mathcal{H} \langle \varphi'\varphi \rangle - \langle \psi'\psi \rangle + \langle \varphi'\varphi \rangle + \langle (\nabla_A\nabla_B\beta)(\nabla^A\nabla^B\beta) \rangle \right],
\]
(54c)
\[
\mathcal{S}^{(2)} = -\frac{1}{a^2} \left[ \langle \partial^A\beta'\partial_A\varphi - \mathcal{H}\partial_A\beta' \rangle \right],
\]
(54d)
where $\beta$ is defined in Eqn. 51b.

Since we are working with a flat FLRW background, it becomes convenient to transform our expressions in terms of Fourier space variables. This will also highlight the problem with large scale fluctuations which was mentioned in Sec. III. We will use the following Fourier transform conventions: For any scalar function $f(\eta, \vec{x})$, its Fourier transform $f_\vec{k}(\eta)$ satisfies

$$f(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} f_\vec{k}(\eta),$$

$$f_\vec{k}(\eta) = \int d^3xe^{-i\vec{k} \cdot \vec{x}} f(\eta, \vec{x}).$$

(55)

Consider an average of a generic quadratic product of two scalars $f^{(1)}(\vec{x})$ and $f^{(2)}(\vec{y})$ where we have suppressed the time dependence since it simply goes along for a ride. Using the definition (11), and keeping in mind that the scalars are real, it is easy to show that we have

$$\langle f^{(1)} f^{(2)} \rangle(\vec{x}) = \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} W_L(\vec{k}_1 - \vec{k}_2, \vec{x}) f^{(1)}_{\vec{k}_1} f^{(2)*}_{\vec{k}_2},$$

(56)

where $W_L(\vec{k}, \vec{x})$ is the Fourier transform of the window function $W_L(\vec{x}, \vec{y})$ on the variable $\vec{y}$, and the asterisk denotes a complex conjugate.

In the present context, the functions $f^{(1)}$ and $f^{(2)}$ will typically be derived in terms of the initial random fluctuations in the metric $\varphi_\vec{k}$, which are assumed to be drawn from a statistically homogeneous and isotropic Gaussian distribution with some given power spectrum. In order to ultimately make contact with observations, it seems necessary to perform a formal ensemble average over all possible realisations of this initial distribution of fluctuations. The statistical homogeneity and isotropy of the initial distribution implies that the functions $f^{(1)}$ and $f^{(2)}$ will satisfy a relation of the type

$$[ f^{(1)}_{\vec{k}_1} f^{(2)*}_{\vec{k}_2} ]_{ens} = (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{k}_2) P_{f^{(1)} f^{(2)}}(\vec{k}_1),$$

(57)

for some function $P_{f^{(1)} f^{(2)}}(k, \eta)$ which is derivable in terms of the initial power spectrum of metric fluctuations, and where $[ ... ]_{ens}$ denotes an ensemble average and $\delta^{(3)}(\vec{k})$ is the Dirac delta distribution.

Applying an ensemble average to Eqn. (50) introduces a Dirac delta which forces $\vec{k}_1 = \vec{k}_2$. Further, the normalisation condition on the window function in Eqn. (11) implies that we have

$$W_L(\vec{k} = 0, \vec{x}) = 1,$$

(58)

which means that all dependence on the averaging scale and domain drops out, and we are left with

$$[ \langle f^{(1)} f^{(2)} \rangle ]_{ens} = \int \frac{d^3k}{(2\pi)^3} P_{f^{(1)} f^{(2)}}(k).$$

(59)

Note however, that the right hand side of Eqn. (59) is precisely what we would have obtained, had we treated the spatial average $\langle ... \rangle$ to be the ensemble average $[ ... ]_{ens}$ to begin with. Therefore for all practical purposes, we are justified in replacing all the spatial averages in the expressions for the correlation scalars (53), by ensemble averages.

It is convenient to define the transfer function $\Phi_k(\eta)$ via the relation

$$\varphi_\vec{k}(\eta) = \varphi_\vec{k} \Phi_k(\eta).$$

(60)

For the calculations in this paper, we shall assume that the cN gauge scalars $\varphi(\eta, \vec{x})$ and $\psi(\eta, \vec{x})$ are equal

$$\varphi(\eta, \vec{x}) = \psi(\eta, \vec{x}),$$

(61)

a choice which is valid in first order PT when anisotropic stresses are negligible (see Ref. 50). This simplifies many of the expressions we are dealing with. The Fourier transform of $\beta$ can be written, using Eqs. (51b) and (61), as

$$\beta_\vec{k}(\eta) = \frac{2}{k^2} \varphi_\vec{k}(\eta).$$

(62)

Finally, in terms of the transfer function $\Phi_k(\eta)$ and the initial power spectrum of metric fluctuations defined by

$$[ \varphi_{\vec{k} i} \varphi^{*}_{\vec{k} i} ]_{ens} = (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{k}_2) P_{\varphi_{\vec{k} i} \varphi^{*}_{\vec{k} i}}(\vec{k}_1),$$

(63)

the correlation scalars (53) can be written as (compare Eqs (58)-61) of Ref. 10,

$$P^{(1)} = - \frac{2}{a^2} \int \frac{dk}{2\pi^2} k^2 P_{\varphi_{\vec{k} i} \varphi^{*}_{\vec{k} i}}(k) (\Phi_k')^2,$$

(64a)

$$S^{(1)} = - \frac{2}{a^2} \int \frac{dk}{2\pi^2} k^2 P_{\varphi_{\vec{k} i} \varphi^{*}_{\vec{k} i}}(k) (k^2 \Phi_k^2),$$

(64b)

$$P^{(1)} + P^{(2)} = - \frac{8\pi}{a^2} \int \frac{dk}{2\pi^2} k^2 P_{\varphi_{\vec{k} i} \varphi^{*}_{\vec{k} i}}(k) (\Phi_k \Phi_k'),$$

(64c)

$$S^{(2)} = - \frac{2}{a^2} \int \frac{dk}{2\pi^2} k^2 P_{\varphi_{\vec{k} i} \varphi^{*}_{\vec{k} i}}(k) (\Phi_k' \Phi_k' - \frac{2H}{k^2} \Phi_k^2).$$

(64d)

These expressions highlight the problem of having a finite amplitude for fluctuations at arbitrarily large length scales ($k \to 0$), which was mentioned in Sec. III. As a concrete example, consider the frequently discussed Harrison-Zel’dovich scale invariant spectrum $34$ which satisfies the condition

$$k^3 P_{\varphi_{\vec{k} i} \varphi^{*}_{\vec{k} i}}(k) = \text{constant}.$$  

(65)

Eqns. (54) now show that if the transfer function $\Phi_k(\eta)$ has a finite time derivative at large scales (as it does in the standard scenarios – see the next section), then the correlation objects $P^{(1)}$, $P^{(2)}$ and $S^{(2)}$ all diverge due to
contributions from the $k \to 0$ regime. This demonstrates the importance of having an initial power spectrum in which the amplitude dies down sufficiently rapidly on large length scales (which is a known issue, see Ref. [33]). Perturbation theory cannot adequately describe the behaviour of inhomogeneities with arbitrarily large length scales [34]. Keeping this in mind, we shall concentrate on initial power spectra which display a long wavelength cutoff [30]. Models of inflation leading to such power spectra have been discussed in the literature [37], and more encouragingly, analyses of WMAP data seem to indicate that such a cutoff in the initial power spectrum is in fact realised in the universe [38].

A final comment before proceeding to explicit calculations: In addition to picking up nontrivial correlation corrections in the cosmological equations, the averaging formalism also requires that the “cross-correlation” constraints in Eqs. (11) be satisfied. In the absence of vector and tensor modes, it is straightforward to show that the statistical homogeneity and isotropy of the scalar metric fluctuations corresponds correspondingly becomes negligible. In practice therefore, one can extend the linear calculation well into the matter dominated era, with the expectation that the order of magnitude of the various integrals will not change significantly due to nonlinear effects (see also the discussion in the last section).

The model we will use is the standard Cold Dark Matter (sCDM) model consisting of radiation and CDM [30]. We will neglect the contribution of baryons, and at the end we shall discuss the effects this may have on the final results. We shall also discuss, without explicit calculation, the effects which the introduction of a cosmological constant is likely to have. In the following, $\Omega_r$ and $\Omega_m$ denote the density parameters of radiation and CDM respectively at the present epoch $\tau_0$, with $\tau$ denoting cosmic time. $\Omega_r$ is assumed to contain contributions from photons and 3 species of massless, out-of-equilibrium neutrinos. At the “zeroth iteration” (see Sec. III.C) we have

$$\left(\frac{1}{a} \frac{da}{d\tau}\right)^2 = H^2(a) = H_0^2 \left[ \frac{\Omega_m}{a^4} + \frac{\Omega_r}{a^4} \right],$$

where $H_0$ is the standard Hubble constant, the scale factor is normalised so that $a(\tau_0) = 1$, and $H$ and $\mathcal{H}$ are related by

$$\mathcal{H}(a) = aH(a).$$

The comoving wavenumber corresponding to the scale which enters at the matter radiation equality epoch, is given by

$$k_{eq} = a_{eq}H(a_{eq}) = H_0 \left( \frac{2 \Omega_m^2}{\Omega_r} \right)^{1/2} \sim H_0 \cdot 10^{5/2},$$

where we have set (see Refs. [30, 39] for details)

$$\Omega_r = \Omega_{\text{photon}} + 3 \Omega_{\text{neutrino}} = \Omega_{\text{photon}} \left( 1 + 3 \cdot \frac{7}{8} \cdot \left( \frac{4}{49} \right)^{4/3} \right) = 4.15 \times 10^{-5} h^{-2},$$

where $h$ is the dimensionless Hubble parameter defined by $H_0 = 100h$ km/s/Mpc. For all calculations we shall set $h = 0.72$ [40].

V. WORKED OUT EXAMPLES

We will now turn to some explicit calculations of the correlation integrals (backreaction), which will show that the magnitude of the effect remains negligibly small for most of the evolution duration in which linear PT is valid. At early times, linear PT is valid at practically all scales including the smallest scales at which we wish to apply General Relativity. As matter fluctuations grow, the small length scales progressively approach nonlinearity, and linear PT breaks down at these scales. As we will see, however, by the time a particular length scale becomes nonlinear, its contribution to the amplitude of the metric fluctuations correspondingly becomes negligible. In practice therefore, one can extend the linear calculation well into the matter dominated era, with the expectation that the order of magnitude of the various integrals will not change significantly due to nonlinear effects (see also the discussion in the last section).

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where $H_0$ is the standard Hubble constant, the scale factor is normalised so that $a(\tau_0) = 1$, and $H$ and $\mathcal{H}$ are related by

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where we have set (see Refs. [30, 39] for details)

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where $h$ is the dimensionless Hubble parameter defined by $H_0 = 100h$ km/s/Mpc. For all calculations we shall set $h = 0.72$ [40].

A. EdS background and non-evolving potentials

Before dealing with the full model (which requires a numerical evolution) let us consider the simpler situation, described by an Einstein-deSitter (EdS) background, with negligible radiation and a non-evolving potential $\varphi = \varphi(\vec{x})$ (which is a consistent solution of the Einstein equations in the sCDM model at least at subhorizon scales at late times [30]). Although not fully accurate, this example requires some very simple integrals and will help to give us a feel for the structure and magnitude of the backreaction.

With a constant potential, the only correlation object which survives is $S^{(1)}$, which evolves like $\sim a^{-2}$, where the scale factor refers to the “zeroth iteration”. The constant of proportionality can be written in terms of the BBKS transfer function $T_{BBKS}(k/k_{eq})$ [30, 41], to give

$$S^{(1)} = -\frac{2}{a^2} \int \frac{dk}{2\pi^2} k^4 P_{\varphi\varphi}(k) T_{BBKS}(k/k_{eq}),$$

where we have [41]

$$T_{BBKS}(x) = \frac{\ln \left[ 1 + 0.171x \right]}{(0.171x)} \left[ 1 + 0.284x + (1.18x)^2 + (0.399x)^3 + (0.490x)^4 \right]^{-0.25},$$

(71)
where \( x \equiv (k/k_{eq}) \).

The integral in Eqn. (70) is well-behaved even in the presence of power at arbitrarily large scales, for a (nearly) scale invariant spectrum. Since we are only looking for an estimate, we shall evaluate the integral in the absence of a large scale cutoff, and leave a more accurate calculation for the next subsection. For the initial spectrum given by

\[
\frac{k^3 P_{\varphi}(k)}{2\pi^2} = A(k/H_0)^{n_*-1},
\]

(72)

where the scalar spectral index \( n_* \) is close to unity, the integral in Eqn. (70) can be easily performed numerically and has the order of magnitude

\[
\int \frac{dk}{2\pi^2} k^4 P_{\varphi}(k) T_{BBKS}^2(k/k_{eq}) \sim A (k_{eq})^2 \sim AH_0^2 \cdot 10^5,
\]

up to a numerical prefactor of order 1. Since the amplitude of the power spectrum is \( A \sim 10^{-9} \), the overall contribution of the backreaction is

\[
\frac{S^{(1)}}{H_0^2} \sim \frac{1}{a^2}(10^{-4}).
\]

(74)

Now, as long as the correlation objects give a negligible backreaction to the usual background quantities, when we proceed with the next iteration, the effect of the backreaction on the evolution of the perturbations will also remain negligible (at least at the leading order). Hence in practice there will be essentially no difference between the zeroth iteration and first iteration perturbation functions. This amounts to saying that when the backreaction is negligible, convergence to the “true” solution for the scale factor at the leading order, is essentially achieved in a single calculation.

B. Radiation and CDM without baryons

Let us now turn to the full sCDM model (without baryons). An analytical discussion of this model in various regions of \((k, \eta)\)-space, can be found e.g. in Ref. [30]. Since we are interested in integrals over \( k \) across a range of epochs \( \eta \), it is most convenient to solve this model numerically. It is further convenient to use \((\ln a)\) in place of \( \eta \), as the variable with which to advance the solution. Also, it is useful to introduce transfer functions like \( \Phi_k(\eta) \) for all the relevant perturbation functions in exactly the same manner (see Eqn. (69)), namely by pulling out a factor of \( \varphi_{k,0} \), since the initial conditions are completely specified by the initial metric perturbation. For a generic perturbation function \( s_k(\eta) \) (other than the metric fluctuation \( \varphi_k \)) the transfer function corresponding to \( s \) will be denoted by a caret, so that

\[
s_k(\eta) = \varphi_k^{'*} \hat{s}_k(\eta)
\]

(75)

The relevant Einstein equations can be brought to the following closed set of first order ordinary differential equations (adapted from Eqns. (7.11)-(7.15) of Ref. [30]),

\[
\frac{\partial \Phi_k}{\partial (\ln a)} = - \left[ \left( 1 + \frac{K^2}{3E^2} \right) \Phi_k + \frac{1}{2E^2a} \left( \Omega_m \delta_k + \frac{4}{a} \Omega_r \hat{\Theta}_{0k} \right) \right],
\]

(76a)

\[
\frac{\partial \delta_k}{\partial (\ln a)} = - \frac{K}{E} \hat{V}_k + 3 \frac{\partial \Phi_k}{\partial (\ln a)},
\]

(76b)

\[
\frac{\partial \hat{\Theta}_{0k}}{\partial (\ln a)} = - \frac{K}{E} \hat{\Theta}_{1k} + \frac{\partial \Phi_k}{\partial (\ln a)},
\]

(76c)

\[
\frac{\partial \hat{\Theta}_{1k}}{\partial (\ln a)} = \frac{K}{3E} \left( \hat{\Theta}_{0k} + \Phi_k \right),
\]

(76d)

\[
\frac{\partial \hat{V}_k}{\partial (\ln a)} = - \hat{V}_k + \frac{K}{E} \Phi_k.
\]

(76e)

Here we have introduced the dimensionless variables

\[
K \equiv \frac{k}{H_0} \quad ; \quad E(a) = \frac{\mathcal{H}(a)}{H_0} = \frac{\mathcal{H}(a)}{H_0},
\]

(77)

and the various perturbation functions are defined as follows: \( \delta_k \) is the k-space density contrast of CDM, \( \Theta_{0k} \) and \( \Theta_{1k} \) are the monopole and dipole moments respectively of the k-space temperature fluctuation of radiation, and \((-i\hat{V}_k)\) is the k-space peculiar velocity scalar potential of CDM (i.e., the real space peculiar velocity is \( v_A = \partial A_v \) where \( v \) is the Fourier transform of \((-i\hat{V}_k)\)).

Assuming adiabatic perturbations, the initial conditions satisfied by the transfer functions at \( a = a_i \) are (adapted from Ch. 6 of Ref. [30])

\[
\Phi_k(a_i) = 1 \quad ; \quad \delta_k(a_i) = -\frac{3}{2} \quad ; \quad \hat{\Theta}_{0k}(a_i) = -\frac{1}{2},
\]

\[
\hat{V}_k(a_i) = 3 \hat{\Theta}_{1k}(a_i) = \frac{1}{2} \frac{K}{E(a_i)}.
\]

(78)

We choose \( a_i = 10^{-16} \), which corresponds to an initial background radiation temperature of \( T \sim 10^{9} \text{GeV} \). While this is not as far back in the past as the energy scale of inflation (which is closer to \( \sim 10^{15} \text{GeV} \)), it is on the edge of the energy scale where known physics begins. This makes Eqn. (69) unrealistic since we have ignored all of Big Bang Nucleosynthesis and also the fact that neutrinos were in equilibrium with other species at temperatures higher than about 1 MeV. However the modifications due to these additional details are not expected to drastically change the final results, and these assumptions lead to some simplifications in the code used. The
goal here is only to demonstrate an application of the formalism; more realistic calculations accounting for the effects of baryons can also be performed (see, e.g. Behrend et al. [10] who incorporate these effects for the post-recombination era, albeit in the Buchert formalism).

In order to partially account for the fact that inflationary initial conditions are actually set much earlier than $a = 10^{-16}$, we impose an absolute small wavelength cutoff at the scale which enters the horizon at the initial epoch which we have chosen. In the above notation this corresponds to setting $K_{max} = E(a_i) \approx 10^{13}$. This makes sense since scales satisfying $K \gg K_{max}$ have already entered the horizon and decayed considerably by the epoch $a = 10^{-16}$. There is a source of error due to ignoring scales $K \gtrsim K_{max}$ which have not yet decayed significantly, but this error rapidly decreases with time as progressively larger length scales enter the horizon and decay. [In fact, in practice to compute the integrals at any given epoch $a = a_*$ one only needs to have followed the evolution of modes with $K < \sim 5000E(a_*)$: more on this in the next subsection.] More important is the cutoff at long wavelengths, which we set at $K_{min} = 1$ (corresponding to $k_{min} = H_0$), which is firstly a natural choice given that $H_0^{-1}$ is the only large scale in the system, and is secondly also guided by analyses of CMB data which have detected such a cutoff [38]. We will see that reducing $K_{min}$ even by a few orders of magnitude, does not affect the final qualitative results significantly.

\[ \Phi_k(y) = \frac{1}{10y^3} \left[ 16 \sqrt{1 + y + 9y^3 + 2y^2 - 8y - 16} \right], \]

(79)

where $y \equiv a/a_{eq}$, and this function is also shown. Clearly all the curves in Fig. 1 are practically identical.

Fig. 2 shows the function $\Phi_k$ normalised by its (constant) value at large scales, at the epoch $a = 500a_{eq} \approx 0.04$, (which is well into the matter dominated era). The dotted line is the BBKS fitting form given in Eqn. (71) with $k_{eq}$ given by Eqn. (68).

To numerically estimate the integrals in Eqns. (63), the values of $\Phi_k$ and its first and second derivatives with respect to $(\ln a)$ are needed across a range of $K$ values. For reference, note that the following relations hold for a generic function of time $w(\eta)$,

\[ \frac{dw}{d\eta} = aH \frac{dw}{da} = \mathcal{H} \frac{dw}{d(\ln a)}. \]

(80)

Based on the earlier discussion, the initial power spectrum $P_{\Phi_0}(k)$ is taken to satisfy

\[ \frac{k^3P_{\Phi_0}(k)}{2\pi^2} = A, \quad \text{for } H_0 < k < k_{max} = \mathcal{H}(a_1), \]

(81)
and zero otherwise, and we set

\[ A = 1.0 \times 10^{-9}, \]

\[ \] which, for the sCDM model follows from the convention (see Eqn. (6.100) of Ref. [30]) \[ A = (5\delta H/3)^2 \] with \( \delta H \approx 2 \times 10^{-3} \) (see, e.g. Ref. [42]).

Consider Figs. 3 and 4 which highlight two issues discussed earlier. Fig. 3 shows the integrand of \( S^{(1)} \) at three sample epochs, and we see that the integrand dies down rapidly at increasingly smaller \( k \) values for progressively later epochs. (The other integrands, not displayed here, also show this rapid decline for large \( k \).) We have not shown the integrand at the later two epochs for all values of \( k \) since this was computationally expensive, but the declining trend of the curves can be extrapolated to large \( k \), which is well understood analytically [30]. This justifies the statement in the beginning of this section, that at any epoch \( a \) is sufficient to have followed the evolution of scales satisfying \( K < 5000 E(a_*) \) for computing the integrals. Secondly, Fig. 4 shows the behaviour of \( k^{3/2} |\delta_k| = A^{1/2} |\delta_k| \) at the same three epochs, and comparing with Fig. 3 we see that at any epoch, the region of \( k \)-space where linear PT has broken down, does not contribute significantly to the integrals. This is in line with the conjecture in Ref. [43] that the effects of the backreaction should remain small since the mass contained in the nonlinear scales is subdominant.

Due to the structure of the integrals and the chosen initial power spectrum, it is convenient to compute the integrands in Eqs. (1) equally spaced in \( (\ln K) \), and then perform the integrals using the extended Simpson’s rule [42]. If \( 2^N + 1 \) points are used to evaluate a given integral, resulting in a value \( \mathcal{I}_N \) say, then the error can be estimated by computing the integral with \( 2^{N-1} + 1 \) points to get \( \mathcal{I}_{N-1} \), and estimating the relative error as

\[ \] |\mathcal{I}_{N-1}/\mathcal{I}_N| - 1. \] With \( N = 10 \), the estimated errors in all the integrals at all epochs were typically less than 0.1%. A bigger error is incurred in computing the integrand itself at any given epoch, leading to estimated errors of order \( \sim 1\% \) in \( S^{(1)}, P^{(1)} \) and \( P^{(1)} + P^{(2)} \), with a larger error in \( S^{(2)} \) as explained below.

The second derivative \( \partial^2 \Phi_k/\partial(\ln a)^2 \) proves to be difficult to track numerically. At early times, when most scales are superhorizon, the Kodama-Sasaki analytical solution (79) is a good approximation for most values of \( k \). Using this one can see that at early times the value of the derivative is numerically very small, and is difficult to reliably estimate due to roundoff errors. For this reason the integral \( S^{(2)} \) could not be accurately estimated at early times. However, the structure of the integrand of \( S^{(2)} \) (64a) shows that the largest contribution comes from large (superhorizon) scales (the small scales being subdominant due to the presence of \( \Phi_k \) and \( 1/k^2 \)). An analysis using the Kodama-Sasaki solution then shows in a fairly straightforward manner that the behaviour of the backreaction term is \( |S^{(2)}/H^2| \sim 10^{-6}(a/a_*) (H_0/k_{\text{min}})^2 \) for our choices of parameters, where \( (a/a_*) \ll 1 \). At intermediate times around \( a \sim a_\text{eq} \) and later, although it becomes computationally expensive to obtain converged values for the second derivative at all relevant scales (10), moderately good accuracy (1-5%) can be achieved.

The results are shown in Fig. 5 in which the magnitudes of the correlation integrals of Eqn. (64), normalised by the Hubble parameter squared \( H^2(a) = (\mathcal{H}/a)^2 \) are plotted as a function of the scale factor in a log-log plot. The values for \( S^{(2)} \) are shown only for epochs later than \( a \approx 0.01a_\text{eq} \sim 10^{-6} \). We see that at all epochs, the correlation terms remain negligible compared to the chosen zeroth iteration background. Also, in the radiation dominated epoch all the correlation scalars (except \( S^{(2)} \) whose evolution couldn’t be accurately obtained) track
the \( \sim a^{-4} \) behaviour of the background radiation density (see also the second paper in Ref. [4]). The discussion above shows however that the magnitude of \( S^{(2)} \) is far smaller than the other backreaction functions at early times, for a cutoff at \( k_{\text{min}} = H_0 \). On the other hand, in the matter dominated epoch \( S^{(1)} \) dominates the backreaction and settles into a curvature-like \( \sim a^{-2} \) behaviour (note that in the matter dominated epoch we have \( H^2 \sim a^{-3} \)). This can also be compared with the results of Ref. [3]. As for the signs of the correlations, \( S^{(1)}, S^{(2)} \), and \( P^{(1)} \) are negative throughout the evolution while \( P^{(1)} + P^{(2)} \) is positive throughout.

Finally, a few comments regarding the effects of ignoring baryons, nonlinear corrections, etc. Including baryons in the problem (with a background density parameter of \( \Omega_m \approx 0.05 \)) will lead to a significant suppression of small scale power (by introducing pressure terms which will tend to wipe out inhomogeneities) and also a small suppression of large scale power. This effect causes a (downward) change in the late time transfer function of roughly 15-20\% [30], and therefore cannot increase the contribution of the backreaction. Quasi-linear corrections can lead to significant changes in the transfer function, but do not cause shifts by several orders of magnitude (see Ref. [17] and references therein). Hence accounting for changes due to quasi-linear behaviour will also not increase the magnitude of the backreaction by a large amount (see also Ref. [10]). As for effects from fully nonlinear scales, we have seen that these can be expected to remain small, or at least not orders of magnitude larger than those from linear scales (see also the discussion in the last section, and Ref. [3]).

Adding a cosmological constant (and retaining a flat background geometry) will change the qualitative features of the correlation functions by shifting the scale \( k_{eq} \) (due to a reduced \( \Omega_m \), which will also increase the power spectrum amplitude [30], but again not by orders of magnitude). Also, the late time behaviour of the correlation scalars will be affected since the potential \( \Phi_k \) will decay at late times instead of remaining constant. Regardless, the backreaction is expected to remain small even in this case (which is also indicated by the calculations of Behrend et al. [10] in the Buchert framework [2]).

VI. DISCUSSION

This paper has presented an analysis of cosmological perturbation theory (PT) in the fully covariant averaging framework of Zalaletdinov’s Macroscopic Gravity (MG) [6, 7] and its restriction to spatial averaging [21]. While this framework is generally covariant, the issue of gauge dependence in perturbation theory introduces certain subtleties in the problem. We have shown that, provided one takes seriously the idea that the cosmological background must be defined by an averaging procedure [23], it is possible to attach a gauge invariant meaning to the averaging condition and the corresponding correlation objects which appear as corrections to the cosmological equations. While there remains considerable freedom in an explicit choice of the averaging operator (through a choice of the volume preserving gauge used in its definition), this freedom can be fixed by some additional requirements based on knowledge of cosmological PT in the standard implementations. In particular we have seen that properties of the conformal Newtonian or Poisson gauge can be used to motivate a fully specified choice of the averaging operation adapted to first order PT.

One prerequisite to the formulation of a consistent averaging framework in the context of perturbation theory, is the absence of perturbative fluctuations with arbitrarily large wavelengths, since such fluctuations would render meaningless the notion of recovering a homogeneous background on averaging. This problem also manifested itself in the correlation integrals [24], which diverge in the presence of a finite amplitude of fluctuations as the wavenumber \( k \to 0 \). Accordingly, all the calculations of this paper have assumed that the initial power spectrum of metric fluctuations has a sharp cutoff at the scale corresponding to \( k = H_0 \), a hypothesis which is in fact supported by analysis of CMB data [25].

The main purpose of this paper was to lay down the formalism of MG in a language most convenient from the point of view of cosmological PT. This was accomplished by writing Equations [17], [51] and the Fourier space version [52] (with certain simplifying assumptions regarding vector and tensor perturbations which can if needed be relaxed in a completely straightforward manner). This was supplemented by calculations in the sCDM model [30] (which is the flat FLRW model with radiation and Cold Dark Matter but no Dark Energy) with the additional simplification of ignoring the baryons. The ana-
lytical results of Sec. A as well as the more detailed numerical results of Sec. B show that the correlation objects or “backreaction” remain negligibly small up to epochs corresponding to a scale factor of $a \sim 0.01$. While the calculations ignored corrections from quasi-linear and non-linear scales, these are not expected to contribute dramatically to the correlations obtained here, an expectation which is justified by the work of Behrend et al. [10] and further by the calculation in Ref. [48] (see below).

We have seen that by using the framework of MG, we have completely bypassed the problem mentioned in the Introduction, which one faces when applying the Buchert framework to cosmological PT, namely of having to deal with two scale factors. Here, one has a single well-defined scale factor associated with the background metric, and its evolution can be obtained in an iterative fashion as described in Sec. II C. In practice, we saw that since the backreaction is small, convergence can be achieved by essentially a single calculation, at least in the context of first order PT.

This brings us to a final, and very important issue: What is the magnitude and behaviour of the backreaction in the fully nonlinear regime of structure formation? There are (at least) two possible avenues to approach this question. The first is to set up the problem in a manner which is suitable for $N$-body simulations. The iterative approach suggested earlier can presumably be adapted to full-fledged $N$-body codes as well. While this is a possibility worth pursuing, there is also a less involved (but correspondingly less realistic) way of determining the effect of nonlinearities on the backreaction, which is to study toy models of structure formation. As mentioned earlier, such a toy model of spherical collapse was recently presented in Ref. [20], and it was shown by an explicit coordinate transformation, that the metric can be brought to the conformal Newtonian form $\Box^2$. Now, it is worth noting that while all the calculations of this paper assumed that first order PT is valid, the actual expressions for the correlation integrals in Eqs. [54] only assume that the potentials $\varphi$ and $\psi$ satisfy $|\varphi|,|\psi| \ll 1$. The dynamics governing the potentials is irrelevant at this stage. This means that, as long as one is interested in leading order effects only, the expressions in Eqs. [54] can be directly applied to any model of structure formation where the metric can be brought to the conformal Newtonian form $\Box^2$, in particular they can be applied to the model of Ref. [20]. This has been done in Ref. [48], and one finds that even in the fully nonlinear regime, the effect of the backreaction remains negligible. In this context see also Ref. [3] for a third approach.

To conclude, all our calculations and arguments seem to indicate that the averaging of perturbative inhomogeneities in a consistent manner appears to lead only to very small effects. This does not, however, mean that the effects do not exist. It remains to be seen whether any observable consequences of the backreaction may be detectable by future experiments.

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Note that strictly speaking, “conformal Newtonian” refers to the first order PT gauge whereas “Poisson” refers to its higher order generalisation. We will not make this distinction here.