GAME THEORETICAL METHODS IN PDES

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Nonlinear PDEs, mean value properties, and stochastic differential games are intrinsically connected. We will describe how the solutions to certain PDEs (of \( p \)-Laplacian type) can be interpreted as limits of values of a specific Tug-of-War game, when the step-size \( \epsilon \) determining the allowed length of move of a token, decreases to 0. This approach originated in [PSSW] and [PS]; for the case of deterministic games see the review SIAM News article [K] and [KS, KS2].

How the linear elliptic equations arise in probability.

Let us begin with the case governed by the discrete Brownian motion. Consider an open bounded set \( \Omega \subset \mathbb{R}^N \) and a non-empty portion of its boundary \( \Gamma_1 \subset \partial \Omega \). Place a token at a point \( x_0 \in \Omega \) and assume that at each step of the process, it is moved with equal probabilities \( \frac{1}{2^N} \), to one of the \( 2^N \) symmetric positions \( x_0 \pm \epsilon e_i, \ i : 1 \ldots N \). Denote by \( u_\epsilon(x_0) \) the probability that the first time the token exists \( \Omega \), it exits across \( \Gamma_1 \). By applying conditional probabilities, it is clear that \( u_\epsilon \) satisfy the mean value property:

\[
\frac{1}{2N} \sum_{i=1}^{N} \left( u_\epsilon(x + \epsilon e_i) + u_\epsilon(x - \epsilon e_i) \right) = u_\epsilon(x)
\]

Further, it follows that as \( \epsilon \to 0 \), the functions \( u_\epsilon \) converge uniformly in \( \Omega \) to a continuous \( u \in C(\Omega) \) which is a viscosity solution to the problem \( \Delta u = 0 \) in \( \Omega \), \( u = \chi_{\Gamma_1} \) on \( \partial \Omega \).

More precisely, this means that: (i) for each \( x_0 \in \Omega \) and each smooth test function \( \phi \) satisfying \( u(x) - \phi(x) > u(x_0) - \phi(x_0) = 0 \) for all \( x \neq x_0 \) in a small neighbourhood of \( x_0 \), one has: \( \Delta \phi(x_0) \leq 0 \), (ii) the same condition holds if we replace \( u \) by \( -u \). It is well known that the viscosity solutions to \( \Delta u = 0 \) coincide with the classical solutions. An advantage of working with the above, seemingly, more complex notion, is that the limiting properties of \( u_\epsilon \) follow quite naturally from the mean value property \([1]\). Namely, replacing the increments \( u_\epsilon(x \pm \epsilon e_i) - u_\epsilon(x) \) in the discontinuous \( u_\epsilon \) in \([1]\), by the same increments in the smooth \( \phi \), applying Taylor’s expansion and taking into account the assumed sign of \( u - \phi \), yields the sign of \( \Delta \phi \), whereas the first derivatives cancel out due to the symmetry in \([1]\).

Heuristically, this can be seen by writing the Taylor expansion of \( u \) at a given point \( x \in \Omega \) and averaging it on a ball \( B_\epsilon(x) \). One obtains:

\[
\int_{B_\epsilon(x)} u = u(x) + \frac{\epsilon^2}{2(N+2)} \Delta u(x) + o(\epsilon^2),
\]

which is a continuum version of \([1]\) when the second term in the right hand side vanishes. Consequently, the function \( u \) must be harmonic, i.e. \( \Delta u = 0 \).
The $p$-Laplacian and its mean value property.

To apply a similar reasoning to a nonlinear problem, consider the homogeneous $p$-Laplacian:
\[
\Delta^H_p u = \Delta u + (p - 2)\Delta_\infty u = |\nabla u|^{p-2} \text{div}(\nabla u), \quad 1 < p < \infty,
\]
where the infinity-Laplacian is given by: \(\Delta_\infty u = \langle \nabla^2 u : \nabla u \otimes \nabla u \rangle\). Parallel to (2) one gets the expansion:
\[
\frac{1}{2} \left( \sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) = u(x) + \frac{\epsilon^2}{2} \Delta_\infty u(x) + o(\epsilon^2).
\]
Forming a linear combination of (2) and (4) with coefficients \(\alpha = \frac{p-2}{p+N}\) and \(\beta = \frac{2+N}{p+N}\), yields:
\[
\frac{\alpha}{2} \left( \sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) + \beta \int_{B_\epsilon(x)} u = u(x) + \frac{\epsilon^2}{2(p+N)} \Delta^H_p u + o(\epsilon^2),
\]
and so the equation (5) suggests that a $p$-harmonic function $u$, i.e. a function satisfying $\Delta^H_p u = 0$, may be approximated by $p$-harmonious functions $u_\epsilon$, defined by the mean value property:
\[
\frac{\alpha}{2} \left( \sup_{B_\epsilon(x)} u_\epsilon + \inf_{B_\epsilon(x)} u_\epsilon \right) + \beta \int_{B_\epsilon(x)} u_\epsilon = u_\epsilon(x).
\]
As we shall see, the functions $u_\epsilon$ satisfying (6) have a probabilistic interpretation as values of Tug-of-War games with noise.

A Tug-of-War game with noise for $\Delta^H_p$.

A Tug-of-War is a two-person, zero-sum game, i.e. two players compete and the gain of Player I equals the loss of Player II. Initially, a token is placed at a point $x_0 \in \Omega$. At each step of the process (the game) one of the three actions takes place: (i) with probability $\frac{\alpha}{2}$, Player I is allowed to play, and she moves the token from its current position $x_n$ to her chosen position $x_{n+1} \in B_\epsilon(x_n)$, (ii) with probability $\frac{\alpha}{2}$, Player II moves the token to his chosen position in $B_\epsilon(x_n)$, (iii) with probability $\beta = 1 - \alpha \in [0, 1]$, the token is moved randomly in the ball $B_\epsilon(x_n)$. The game stops when the token leaves $\Omega$, whereas Player II pays to Player I the amount equal to the value of a given boundary pay-off function $F$ at the exit token position $x_\tau$ (see Figures 1 and 2).

Figure 1. Player I and Player II compete in a Tug-of-War with random noise [Ka].

Players I and II play according to strategies $\sigma_I$ and $\sigma_{II}$ respectively, which are (Borel measurable) functions assigning to each finite history of the game $x_n = (x_0, \ldots, x_n)$ the next position $x_{n+1}$ in the $\epsilon$-neighborhood $\Omega_\epsilon$ of $\Omega$, where the player will move the token once she/he is
allowed to play. These strategies determine a probability \( P_{\sigma_I, \sigma_{II}}^{x_0} \) on the space of all possible game runs in \((\Omega, \epsilon)^{\infty}\). Since \( \beta > 0 \), the game ends \( P_{\sigma_I, \sigma_{II}}^{x_0} \)-a.e., so that we can define the stopping time \( \tau(x_0, x_1, \ldots) = \inf\{n; x_n \notin \Omega\} \). The expected value of the game is then given by \( \mathbb{E}_{\sigma_I, \sigma_{II}}^{x_0}[F_{\tau}] = \int_{\Omega^\infty} F(x_\tau) \, dP_{\sigma_I, \sigma_{II}}^{x_0} \). Consequently, the minimum gain \( u_I \) that Player I can expect, and the maximum loss \( u_{II} \) of Player II in his best case scenario, are:

\[
(7) \quad u_I(x_0) = \sup_{\sigma_I} \inf_{\sigma_{II}} \mathbb{E}_{\sigma_I, \sigma_{II}}^{x_0}[F_{\tau}], \quad u_{II}(x_0) = \inf_{\sigma_{II}} \sup_{\sigma_I} \mathbb{E}_{\sigma_I, \sigma_{II}}^{x_0}[F_{\tau}].
\]

The following main results were achieved in [PSSW] for \( p = \infty \), and in [MPR] for \( p \in [2, \infty) \):

**Theorem A:** The two game values in (7) coincide: \( u_I = u_{II} \) and are equal to the \( p \)-harmonious function \( u_\epsilon \), which is the unique solution to the mean value law in (6) augmented by the boundary data: \( u_\epsilon = F \) on \( \mathbb{R}^n \setminus \Omega \).

**Theorem B:** As \( \epsilon \to 0 \), the game value \( u_\epsilon \) converges uniformly in \( \Omega \) to a function \( u \in C(\Omega) \), which is the unique viscosity solution to: \( \Delta^H u = 0 \) in \( \Omega \) and \( u = F \) on \( \partial \Omega \).

In brief, one obtains exactly the same as in the case of the discrete Brownian motion, whose value satisfied the averaging principle (1) and converged to a harmonic function. The equivalence of the notions of viscosity solution to \( \Delta^H u = 0 \) and weak solution to \( \text{div}(|\nabla u|^{p-2} \nabla u) = 0 \) has been proven in [JLM].

**The key martingale calculation.**

We now sketch the proof of \( u_{II} \leq u_\epsilon \); a symmetric argument yields \( u_\epsilon \leq u_I \), while \( u_I \leq u_{II} \) is trivially true. We take advantage of the cancellation encoded in the mean value property (6) by showing that certain quantities related to \( u_\epsilon \) are sub- and super-martingales.

Fix a small \( \eta > 0 \) and let \( \bar{\sigma}_{II} \) be an “almost optimal” strategy for Player II, so that:

\[
u_\epsilon(\bar{\sigma}_{II}(x_0, \ldots x_n)) \leq \inf_{B_{n}(x_n)} \nu_\epsilon + \frac{\eta}{2^{n+1}}.
\]

Player I plays according to an arbitrary strategy \( \sigma_I \). The key observation is that the sequence of random variables \( \{u_\epsilon(x_n) + \frac{\eta}{2^n} \mid (x_0, \ldots x_n)\}_{n \geq 1} \) is a supermartingale.

![Figure 2. Player I, Player II and random noise with their probabilities [Ka].](image)
To see this, compute the conditional expectation relative to the history $x_n = (x_0 \ldots x_n)$:

$$
\mathbb{E}_{\sigma_I, \sigma_{II}}^0 \{ u_\epsilon(x_{n+1}) + \frac{\eta}{2n+1} \}(x_n) = \frac{\alpha}{2} u_\epsilon(\sigma_I(x_n)) + \frac{\alpha}{2} u_\epsilon(\sigma_{II}(x_n)) + \beta \int_{B_\epsilon(x_n)} u_\epsilon + \frac{\eta}{2n+1} \leq \frac{\alpha}{2} \sup_{B_\epsilon(x_n)} u_\epsilon + \frac{\alpha}{2} \left( \inf_{B_\epsilon(x_n)} u_\epsilon + \frac{\eta}{2n+1} \right) + \beta \int_{B_\epsilon(x_n)} u_\epsilon + \frac{\eta}{2n+1}
$$

$$
= u_\epsilon(x_n) + \left( \frac{\alpha}{2} + 1 \right) \frac{\eta}{2n+1} \leq u_\epsilon(x_n) + \frac{\eta}{2n},
$$

where we first used the game’s rules, then the sub-optimality of $\sigma_{II}$, and further the formula (6) for $u_\epsilon$. Applying Doob’s optimal stopping time theorem we get the desired comparison result:

$$
u_{II}(x_0) \leq \sup_{\sigma_I} \mathbb{E}_{\sigma_I, \sigma_{II}}^0 [F_r] = \sup_{\sigma_I} \mathbb{E}_{\sigma_I, \sigma_{II}}^0 [u(x_\tau)] \leq \sup_{\sigma_I} \mathbb{E}_{\sigma_I, \sigma_{II}}^0 [u(x_\tau) + \frac{\eta}{2\tau}] \leq \sup_{\sigma_I} \mathbb{E}_{\sigma_I, \sigma_{II}}^0 [u(x_0) + \frac{\eta}{2n}] = u(x_0) + \eta, \quad \text{for all } \eta > 0.
$$

Strategies and inequalities.

We have seen how probability tools can be used to study nonlinear PDEs, where the key technical ingredient was assigning suitable strategies yielding the desired inequalities for game values. Below we sketch two further examples of this powerful technique.

The proof of uniform convergence in Theorem B relies on a variant of the Ascoli-Arzelá theorem valid for the discontinuous functions $u_\epsilon$. The verification [MPR] of the appropriate ‘equidiscontinuity’ property requires estimating quantities $|u_\epsilon(x_0) - u_\epsilon(y_0)|$, say for $x_0 \in \Omega$, $y_0 \in \partial \Omega$. If $F$ is Lipschitz, this reduces to estimating $|x_\tau - y_0|$, and the feasible strategy is that of Player II “pulling towards $y_0$", namely shifting the token by $\epsilon$ along the segment connecting its current position with $y_0$.

In [LPS2], the local Harnack inequality for $p$-harmonic functions for $p > 2$ is proven independent of the classical, yet technically challenging methods of De Giorgi or Moser. This is done via a uniform estimate on the oscillations of $u_\epsilon$. Let $x_0, y_0 \in \Omega$ and let $z$ be equidistant from both points by a multiple of $\epsilon$. Define strategies $\sigma_I^*$ in which Player $i$ cancels the earliest uncancelled move of her/his opponent, and otherwise “pulls towards $z$” as before, and let $\tau^*_I$ be the stopping time in which the game terminates when either Player $r$ has played sufficiently many turns to place the token at $z$ (modulo the random noise), or when the total amount of token’s shifts by her/his opponent and by the random noise, has passed an undesired large threshold $r$. Let now $\sigma_I, \sigma_{II}$ be two arbitrary strategies. By the symmetry of this construction, the bulk “nonlinear” parts in the two quantities: $\mathbb{E}_{\sigma_I, \sigma_{II}}^0 [u_\epsilon(x_{\tau^*_I})]$ and $\mathbb{E}_{\sigma_I, \sigma_{II}}^0 [u_\epsilon(x_{\tau^*_r})]$, corresponding to stopping the game due to the first condition, are equal. The remaining “linear” part in: $[\mathbb{E}_{\sigma_I, \sigma_{II}}^0 [u_\epsilon(x_{\tau^*_I})] - \mathbb{E}_{\sigma_I, \sigma_{II}}^0 [u_\epsilon(x_{\tau^*_r})]]$ can then be bounded by $\frac{|x_0 - y_0|}{r} \text{osc}(u_\epsilon, B_r(z))$, using a comparison with a cylinder walk. This concludes the proof, in view of: $|u_\epsilon(x_0) - u_\epsilon(y_0)| \leq \sup_{\sigma_I, \sigma_{II}} [\mathbb{E}_{\sigma_I, \sigma_{II}}^0 [u_\epsilon(x_{\tau^*_I})] - \mathbb{E}_{\sigma_I, \sigma_{II}}^0 [u_\epsilon(x_{\tau^*_r})]]$.

Further results.

Generalizations of Theorems A and B have been obtained in various contexts. For $p = \infty$, only the notion of a metric space is necessary to define the game, and indeed [PSSW] formulates its results for an arbitrary length space where the solutions to $\Delta_\infty u = 0$ are understood as Absolutely Minimizing Lipschitz Extensions. When $p \in [2, \infty)$, the game uses the notion of a metric and a measure, and it is amenable to the recent extension to Heisenberg groups in [FLM]. For $p \in (1, \infty)$ one needs the additional notion of perpendicularity [PS]. We see that as $p \to 1$, the
required complexity of structure increases. In the case $p = 1$ the game is naturally related to the mean curvature flow \cite{KS} and functions of least gradient.

Other extensions include the obstacle problems \cite{MRS}, finite difference schemes \cite{AS}, equations with right hand side $f \neq 0$, mixed boundary data \cite{APSS, CGR} and parabolic equations \cite{KS2, MPR2}.

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