NEARLY KÄHLER AND NEARLY PARALLEL G₂-STRUCTURES ON SPHERES

THOMAS FRIEDRICH

Abstract. In some other context, the question was raised how many nearly Kähler structures exist on the sphere $S^6$ equipped with the standard Riemannian metric. In this short note, we prove that, up to isometry, there exists only one. This is a consequence of the description of the eigenspace to the eigenvalue $\lambda = 12$ of the Laplacian acting on 2-forms. A similar result concerning nearly parallel G₂-structures on the round sphere $S^7$ holds, too. An alternative proof by Riemannian Killing spinors is also indicated.

Consider the 6-dimensional sphere $S^6 \subset \mathbb{R}^7$ equipped with its standard metric. Denote by $\Delta$ the Hodge-Laplace operator acting on 2-forms of $S^6$ and consider the space

$$E_{12} := \{ \omega^2 \in \Gamma(\Lambda^2(S^6)) : d^* \omega^2 = 0, \Delta(\omega^2) = 12 \cdot \omega^2 \}.$$ 

This space is an SO(7)-representation. Moreover, it coincides with the full eigenspace of the Laplace operator acting on 2-forms with eigenvalue $\lambda = 12$.

**Proposition 1.** The SO(7)-representation $E_{12}$ is isomorphic to $\Lambda^3(\mathbb{R}^7)$. More precisely, for any 2-form $\omega^2 \in E_{12}$, there exists a unique algebraic 3-form $A \in \Lambda^3(\mathbb{R}^7)$ such that

$$\omega^2_x(y, z) = A(x, y, z)$$

holds at any point $x \in S^6$ for any two tangent vectors $y, z \in T_x(S^6)$.

*Proof.* It is easy to check that any 2-form $\omega^2$ on $S^6$ defined by a 3-form $A \in \Lambda^3(\mathbb{R}^7)$ as indicated satisfies the differential equations $d^* \omega^2 = 0, \Delta(\omega^2) = 12 \cdot \omega^2$. Consequently, we obtain an SO(7)-equivariant map

$$\Lambda^3(\mathbb{R}^7) \rightarrow E_{12}.$$ 

Since $\Lambda^3(\mathbb{R}^7)$ is an irreducible SO(7)-representation, the map is injective. On the other hand, by Frobenius reciprocity, one computes the dimension of the eigenspace of the Laplace operator on 2-forms to the eigenvalue $\lambda = 12$. Its dimension equals 35. □

We recall some basic properties of nearly Kähler manifolds in dimension six (see the paper [1]). Let $(M^6, J, g)$ be a nearly Kähler 6-manifold. Then it is an Einstein space with positive scalar curvature $\text{Scal} > 0$. The Kähler form $\Omega$ satisfies the differential equations

$$d \ast \Omega = 0, \quad \Delta(\Omega) = \frac{2}{5} \cdot \text{Scal} \cdot \Omega.$$
In particular, the Kähler form $\Omega^J$ of any nearly Kähler structure $(\mathbb{S}^6, J, g_{\text{can}})$ on the standard sphere $\mathbb{S}^6$ is a 2-form on $\mathbb{S}^6$ satisfying the equations $d \ast \Omega^J = 0$ and $\Delta(\Omega^J) = 12 \cdot \Omega^J$. This observation yields the following result.

**Proposition 2.** The Kähler form $\Omega^J$ of any nearly Kähler structure $(\mathbb{S}^6, J, g_{\text{can}})$ on the standard sphere is given by an algebraic 3-form $A \in \Lambda^3(\mathbb{R}^7)$ via the formula

$$\Omega^J_x(y, z) = A(x, y, z)$$

where $x \in \mathbb{S}^6$ is a point in the sphere and $y, z \in T_x(\mathbb{S}^6)$ are tangent vectors.

Since the Kähler form $\Omega^J$ is a non-degenerate 2-form at any point of the sphere $\mathbb{S}^6$, the 3-form $A \in \Lambda^3(\mathbb{R}^7)$ is a non-degenerate vector cross product in the sense of Gray (see [2], [4], [5]). For purely algebraic reasons it follows that two forms of that type are equivalent under the action of the group $\text{SO}(7)$. Finally, we obtain the following

**Theorem 1.** Let $(\mathbb{S}^6, J, g_{\text{can}})$ be a nearly Kähler structure on the standard 6-sphere. Then the almost complex structure $J$ is conjugated – under the action of the isometry group $\text{SO}(7)$ – to the standard nearly Kähler structure of $\mathbb{S}^6$.

A similar argument applies in dimension seven, too.

**Theorem 2.** Let $(\mathbb{S}^7, \omega, g_{\text{can}})$ be a nearly parallel $G_2$-structure on the standard 7-sphere. Then it is conjugated – under the action of the isometry group $\text{SO}(8)$ – to the standard nearly parallel $G_2$-structure of $\mathbb{S}^7$.

**Remark.** Nearly Kähler structures in dimension six and nearly parallel structures in dimension seven correspond to Riemannian Killing spinors. It is well-known that the isometry group of the spheres $\mathbb{S}^6$ and $\mathbb{S}^7$ acts transitively on the set of Killing spinor of length one. This observation yields a second proof of the latter Theorems (see [3] and [6]).

**References**

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friedric@mathematik.hu-berlin.de

Institut für Mathematik
Humboldt-Universität zu Berlin
Sitz: WBC Adlershof
D-10099 Berlin, Germany