Thermodynamics and Phase Transition of a Gauss-Bonnet Black Hole in a Cavity

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Abstract

Considering a canonical ensemble, in which the temperature and the charge on a wall of the cavity are fixed, we investigate the thermodynamics of a $D$-dimensional Gauss-Bonnet black hole in a finite spherical cavity. Moreover, it shows that the first law of thermodynamics is still satisfied. We then discuss the phase structure and transition in both five and six dimensions. Specifically, we show that there always exist two regions in the parameter space. In one region, the system possesses one single phase. However in the other region, there could coexist three phases and a van der Waals-like phase transition occurs. Finally, we find that there is a fairly close resemblance in thermodynamic properties and phase structure of a Gauss-Bonnet-Maxwell black hole, either in a cavity or in anti-de Sitter space.
1 Introduction

The thermodynamics of black holes has been a subject of intensive study for several decades since J. Bekenstein and S. Hawking discovered that the black hole entropy was proportional to the area of its event horizon \[1, 2\]. A Schwarzschild black hole in asymptotically flat space has a negative specific heat and hence become unstable as a thermodynamic system. That being said, the Schwarzschild black hole radiates more when it becomes smaller. To make a black hole thermally stable, it is necessary to put the black hole in a closed system. One of the most popular way to approach this aim is to place a black hole in anti-de Sitter (AdS) space with the negative cosmological constant, in which the timelike boundary can reflected radiation back into the bulk. Hawking and Page in \[3\] first studied the thermodynamic properties of a Schwarzschild black hole in AdS space and discovered the Hawking-Page phase transition. Since then, the thermodynamic properties and phase transition of various black holes in AdS space have been discussed in \[4–10\].

On the other hand, another popular choice is considering a cavity in an asymptotic flat space, where the Dirichlet boundary condition is imposed on the wall of the cavity. It was discovered by York in \[11\] that a Schwarzschild black hole in a cavity is thermally stable and a Hawking-Page-like transition can occur as the temperature decreases, which is quite similar to the behavior of a Schwarzschild-AdS black hole. Later, a Reissner-Nordstrom (RN) black hole in a cavity was studied in a grand canonical ensemble \[12\] and a canonical ensemble \[13\]. It showed that a Hawking-Page-like phase transition and a van der Waals-like one occur in the canonical ensemble and grand canonical ensemble, respectively, which is similar to the AdS case \[14\]. In the following papers \[15–20\], the phase structure of various black holes in the cavity is studied, where Hawking-Page-like or van der Waals-like phase transitions were found except for some special cases. Considering charged
scalars, boson stars and hairy black holes in a cavity in [21–24], it showed that the phase structure of the gravity system in a cavity is strikingly similar to that of holographic superconductors in the AdS gravity. In [25–32], the stabilities of solitons, stars and black holes in a cavity were also investigated. It is interesting to note that most studies in the literature have been consider in the framework of the Eisenstein-Maxwell theory. On the other hand, we recently found that the phase structure of a nonlinear electrodynamics black hole in a cavity can be different from that in AdS space [33, 34]. It naturally raises a question whether there exists other theory beyond the Eisenstein-Maxwell theory, in which thermodynamics of a black hole can be dependent on boundary conditions.

The Gauss-Bonnet gravity is the simplest case of Lovelock theories, which extends the general relativity theory through adding higher derivative terms into the Einstein-Hilbert action. In [35, 36], it showed that the Gauss-Bonnet term is naturally consistent with the first-order $\alpha'$ correction of closed string low energy effective action. The Gauss-Bonnet AdS black solution was first obtained in [8]. After that, the thermodynamic properties and phase structure of a Gauss-Bonnet black hole in AdS space are discussed in various scenarios [37–40]. It is worth noting that the Gauss-Bonnet term is a topological invariant in four dimensions, and hence thermodynamics of a Gauss-Bonnet black hole is always analyzed in higher dimensions.

In this paper, we study the thermodynamic properties and phase structure of a Gauss-Bonnet-Maxwell black hole in a cavity in a canonical ensemble. We find that thermodynamics and phase structure of a Gauss-Bonnet-Maxwell black hole in a cavity bear striking resemblance to that in AdS space. This paper is organized as follows. In section 2 we first review the Gauss-Bonnet-Maxwell black hole solution and obtain the Euclidean action of the Gauss-Bonnet-Maxwell black hole in a cavity. In section 3 we then discuss thermodynamics of the Gauss-Bonnet-Maxwell black hole in a cavity and show that the first law of thermodynamics is satisfied. In section 4 the phase structure of the black hole in cavity is discussed in five and six dimensions. We summarize our results in section 5. Finally, we discuss the phase structure of a Gauss-Bonnet-Maxwell AdS black hole in the appendix. We take $G = \hbar = c = k_B = 1$ for simplicity in this paper.

2 Gauss-Bonnet Black Hole in a Cavity

In this section, we briefly review the Gauss-Bonnet-Maxwell black hole solution and obtain the Euclidean action. It is worth noting that in this paper, we study the black hole thermodynamics in a canonical ensemble, in which the temperature and charge are fixed on the boundary.

Considering the Gauss-Bonnet gravity coupled to Maxwell theory on a $D$-dimensional spacetime manifold $\mathcal{M}$ with a time-like boundary $\partial \mathcal{M}$, we can write the action as

\[
S = S_{\text{bulk}} + S_{\text{surf}}.
\]

(2.1)

Here, the bulk action is given by

\[
S_{\text{bulk}} = \frac{1}{16\pi} \int_{\mathcal{M}} d^D x \sqrt{-g} \left[ R + \alpha \left( R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \right) - F_{\mu\nu}F^{\mu\nu} \right],
\]

(2.2)

where $\alpha$ is the Gauss-Bonnet coupling constant. Generally, $\alpha$ is positive since it is associated with string length’s square of string theory [33]. Furthermore, $F_{\mu\nu}$ is the electromagnetic field strength tensor, which is defined as
\[ F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \] in terms of the vector potential \( A_{\mu} \). On the boundary \( \partial \mathcal{M} \), the surface terms are

\[
S_{\text{surf}} = -\frac{1}{8\pi} \int_{\partial \mathcal{M}} d^{D-1}x \sqrt{-\gamma} \left[ K + 2\alpha \left( J - 2\tilde{G}^{\mu\nu} K_{\mu\nu} \right) - K_0 - 2\alpha \left( J_0 - 2\tilde{G}_0^{\mu\nu} (K_0)_{\mu\nu} \right) \right] - \frac{1}{16\pi} \int_{\partial \mathcal{M}} d^{D-1}x \sqrt{-\gamma} \partial_{\mu} F_{\mu\nu} A_{\nu},
\]

(2.3)

where \( \gamma \) is the determinant of the induced metric on \( \partial \mathcal{M} \), \( K_{\mu\nu} \) is the external curvature of \( \partial \mathcal{M} \), \( K \) is the trace of the external curvature, and \( J \) is the trace of the correlative quantities when boundary \( \partial \mathcal{M} \) is embedded in flat spacetime. Note that the Gauss-Bonnet term is a topological invariant in four dimensions, so we will consider \( D \geq 5 \) in what follows.

By varying the action (2.1), we find the equations of motion

\[
R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + H_{\mu\nu} = 8\pi T_{\mu\nu},
\]

\[
\nabla_{\mu} F^{\mu\nu} = 0,
\]

(2.5)

where

\[
H_{\mu\nu} = -\frac{1}{2}\alpha \left( R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right) g_{\mu\nu} + 2\alpha \left( RR_{\mu\nu} - 2R_{\mu\alpha} R^{\alpha\beta} g_{\beta\nu} - 2R_{\mu\lambda\sigma\rho} R^{\lambda\sigma} + g_{\beta\nu} R_{\gamma\sigma\alpha} R^{\beta\gamma\sigma\alpha} \right),
\]

(2.6)

\[
T_{\mu\nu} = \frac{1}{4\pi} \left( -\frac{1}{4} F^{\mu\nu} g_{\mu\nu} + F_{\mu}^{\lambda} F_{\nu\lambda} \right).
\]

We consider a static spherically symmetric black hole solution with the metric

\[
ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{D-2},
\]

\[ A = A_t(r) dt. \]

(2.7)

The equations of motion then reduce to

\[
0 = \left[ (D - 3) (1 - f(r)) - rf'(r) \right] r^{D-4} + 2\alpha f'(r) (f(r) - 1) r^{D-5} + (D - 5) \bar{\alpha} (f(r) - 1)^2 r^{D-6} + \frac{4}{D - 2} \left( -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + F^\tau r \partial_\tau A_t(r) \right) r^{D-2},
\]

(2.8)

\[
0 = \left[ (D - 4) (D - 3) (1 - f(r)) - 2 (D - 3) rf'(r) - r^2 f''(r) \right] r^{D-5} + 2\bar{\alpha} \left[ f'(r)^2 + (f(r) - 1) f''(r) \right] r^{D-5} + 4 (D - 5) \bar{\alpha} (f(r) - 1) f'(r) r^{D-6} + (D - 5) (D - 6) \bar{\alpha} (f(r) - 1)^2 r^{D-7} - F^{\mu\nu} F_{\mu\nu} r^{D-3},
\]

(2.9)

\[
0 = \left[ r^{D-2} F^{\tau\tau} \right] \]

(2.10)

where we denote \( \bar{\alpha} = \alpha (D - 3) (D - 4) \) for simplicity. Integrating eqns. (2.8) and (2.10), we obtain the solution

\[
f(r) = 1 + \frac{r^2}{2\bar{\alpha}} \left[ 1 - \sqrt{1 + 4\bar{\alpha} \left( \frac{16\pi M}{(D - 2) \omega_{D-2} r^{D-1}} - \frac{32\pi^2 Q^2}{(D - 2) (D - 3) r^{2D-4} \omega_{D-2}^2} \right)} \right],
\]

(2.11)
where $M$ is the ADM mass and $Q$ is the charge of the black hole, and $\omega_{D-2}$ is the volume of the unit $D-2$ sphere. The outer event horizon radius $r_+$ of the black hole satisfies $f(r_+) = 0$. Therefore, the metric function $f(r)$ can be rewritten in terms of $r_+$:

$$f(r) = 1 + \frac{r^2}{2\tilde{\alpha}} \left[ 1 - \sqrt{1 + 4\tilde{\alpha} \left[ \frac{r_{+}^{D-5}}{r_{+}^{D-5} - \tilde{\alpha}} + \frac{r_{+}^{D-3}}{r_{+}^{D-3} - \tilde{\alpha}} + \frac{32\pi^2 Q^2}{(D-2)(D-3)r_{+}^{D-2}} \left( \frac{1}{r_{+}^{D-3}} - \frac{1}{r_{+}^{D-3}} \right) \right]} \right]. \quad (2.12)$$

The Euclidean action $S^E$ can be related to the action $S$ [2.1]: $S^E = iS$. Using the analytic continuation $t = i\tau$ and $A_\tau d\tau = A_t dt$, we can obtain

$$A_\tau = iA_t, \quad (2.13)$$

which gives $F^r \tau = iF^r t$. Suppose that the black hole lives in a spherical cavity, where the boundary $\partial \mathcal{M}$ is at $r = r_B$. Since the temperature $T$ is fixed on the boundary of the cavity, we can impose the boundary condition at $r = r_B$ in terms of the reciprocal temperature:

$$\int \sqrt{f(r_B)} d\tau = T^{-1}, \quad (2.14)$$

which identifies the Euclidean time $\tau$ as $\tau \sim \frac{1}{T \sqrt{f(r_B)}}$, and hence the period of $\tau$ is $\frac{1}{T \sqrt{f(r_B)}}$. Integrating the Euclidean action and using eqn. (2.12), the Euclidean action is rewritten as

$$S^E = \frac{1}{8\pi} (D-2) \frac{\omega_{D-2} r_B^{D-3}}{T} \left( 1 - \sqrt{f(r_B)} \right) - S + \frac{\tilde{\alpha}}{12\pi} (D-2) \frac{\omega_{D-2} r_B^{D-5}}{T} \left( \sqrt{f(r_B)} f(r_B) - 3\sqrt{f(r_B)} + 2 \right), \quad (2.15)$$

where $S = \frac{1}{4} \omega_{D-2} r_+^{D-2} \left[ 1 + 2\tilde{\alpha} (D-2) / (D-4) r_+^2 \right]$ is the entropy of the black hole.

### 3 Thermodynamics

In the semi-classical approximation, the on-shell Euclidean action is related to the free energy $F$:

$$F = -T \ln Z = TS^E. \quad (3.16)$$

From eqn. (2.16), we can express the the free energy $F$ in terms of the temperature $T$, the charge $Q$, the Gauss-Bonnet parameter $\alpha$ (or $\tilde{\alpha}$), the cavity radius $r_B$ and the horizon radius $r_+$:

$$F = \frac{1}{8\pi} (D-2) \omega_{D-2} r_B^{D-3} \left( 1 - \sqrt{f(r_B)} \right) - \frac{1}{4} S_{D-2} r_+^{D-2} \left( 1 + \frac{D-2}{D-4} \frac{2\tilde{\alpha}}{r_+^2} \right) T + \frac{\tilde{\alpha}}{12\pi} (D-2) \omega_{D-2} r_B^{D-5} \left( \sqrt{f(r_B)} f(r_B) - 3\sqrt{f(r_B)} + 2 \right). \quad (3.17)$$

where $T$, $Q$, $\alpha$ (or $\tilde{\alpha}$) and $r_B$ are parameters of the canonical ensemble and the horizon radius $r_+$ is the only variable,

$$F = F(r_+; T, Q, \alpha, r_B). \quad (3.18)$$

By extremizing the free energy $F(r_+; T, Q, \alpha, r_B)$ with respect to $r_+$, we can determine the only variable $r_+$:

$$\frac{dF(r_+; T, Q, \alpha, r_B)}{dr_+} = 0$$
\[ f'(r_+) = 4\pi T \sqrt{f(r_B)}. \]  

(3.19)

The solution \( r_+ = r_+ (T, Q, \alpha, r_B) \) of eqn. (3.19) is in relevance to a locally stationary point of \( F(r_+; T, Q, \alpha, r_B) \). Since the Hawking temperature of the black hole is defined as \( T_h = f'(r_+) / 4\pi \), eqn. (3.19) can be written as

\[ T = \frac{T_h}{\sqrt{f(r_B)}}, \]

(3.20)

where the Hawking temperature is

\[ T_h = \frac{(D - 5) \alpha + (D - 3) r_+^2 - \frac{1}{D-2} r_+^2 32\pi^2 Q^2}{4\pi (1 + \frac{\alpha}{r_+^2}) r_+^3}. \]

(3.21)

So for the observer on the wall, the temperature \( T \) on the cavity is blueshifted from Hawking temperature \( T_h \).

At the locally stationary point \( r_+ = r_+ (T, Q, \alpha, r_B) \), the free energy \( F(r_+; T, Q, \alpha, r_B) \) can be express only in terms of \( T, Q, \alpha (\tilde{\alpha}) \) and \( r_B \):

\[ F(T, Q, \alpha, r_B) \equiv F(r_+ (T, Q, \alpha, r_B); T, Q, \alpha, r_B). \]

(3.22)

For later convenience, \( F(r_+; T, Q, \alpha, r_B) \) and \( F(T, Q, \alpha, r_B) \) can be abbreviated to \( F(r_+) \) and \( F \), respectively. Furthermore, the thermal energy of the black hole in the cavity is

\[
E = -T^2 \frac{\partial(F/T)}{\partial T} = \frac{1}{8\pi} (D-2) \omega_ {D-2} r_B^{D-3} \left[ (1 - \sqrt{f(r_B)}) + \frac{2\tilde{\alpha}}{3} r_B^2 \left( \sqrt{f(r_B)} f(r_B) - 3\sqrt{f(r_B)} + 2 \right) \right].
\]

(3.23)

where the thermal energy \( E \) is expressed in terms of the entropy \( S \), the charge \( Q \) and the cavity radius \( r_B \). Moreover, we can define an electric potential and thermodynamic surface pressure as

\[ \Phi \equiv \frac{A_\alpha (r_B) - A_\alpha (r_+)}{\sqrt{f(r_B)}}, \lambda \equiv -\frac{\partial E}{\partial (S_{D-2} r_B^{D-2})}. \]

(3.24)

It is easy to verify that the differential \( E \) with respect to \( S, Q \) and area \( A \) is satisfied:

\[ \frac{\partial E}{\partial S} = T, \frac{\partial E}{\partial Q} = \Phi, \frac{\partial E}{\partial A} = \lambda. \]

(3.25)

where \( A \equiv S_{D-2} r_B^{D-2} \) is the surface area of the cavity. Using eqns. (3.25) and (3.24), the first law of thermodynamics can be established as

\[ dE = TdS + \Phi dq - \lambda dA. \]

(3.26)

To discuss the thermodynamic stability of the Gauss-Bonnet-Maxwell black hole in the cavity, we consider the specific heat at constant electric charge

\[ C_Q = T \left( \frac{\partial S}{\partial T} \right)_Q = \frac{1}{4} \omega_ {D-2} r_+^{D-3} (D-2) \left( 1 + \frac{\alpha}{r_+^2} \right) T \frac{\partial r_+ (T, Q, r_B, \alpha)}{\partial T}. \]

(3.27)

Since the system is thermally stable with \( C_Q > 0 \), the black holes is thermally stable when \( \partial r_+ (T, Q, r_B) / \partial T > 0 \). From \( \partial^2 F / \partial^2 T = -C_Q \), the thermally stable/unstable phases have concave downward/upward \( F-T \) curves.
The black hole phase is thermally stable/unstable when 
\[ r_+ (T, Q, \alpha, r_B) \] 
is a local minimum/maximum of 
\[ F (r_+). \]
Note that the physical space of \( r_+ \) has boundaries, such as
\[ r_e \leq r_+ \leq r_B, \] (3.28)
where \( r_e \) is the horizon radius of the extremal black hole.

4 Phase Transition

In this section, we will discuss the phase transition of a Gauss-Bonnet-Maxwell black hole in a cavity for \( D = 5 \) and \( D = 6 \). For convenience, we express the variables in terms of \( r_B \):
\[
x \equiv \frac{r_+}{r_B}, \quad \bar{Q} \equiv \frac{Q}{r_B^{D-3}}, \quad \bar{\alpha} \equiv \frac{\alpha}{r_B^2}, \quad \bar{T} \equiv r_B T, \quad \bar{F}(x) \equiv \frac{8\pi F (r_+)}{(D-2) S_{D-2} r_B^{D-3}}.
\] (4.29)
From eqns. (3.17) and (2.12), we obtain the free energy as a function of \( x \):
\[
\bar{F}(x) = 1 - \sqrt{f(x)} - 2\pi x^{D-2} \left( 1 + 2(D-2)(D-3) \frac{\bar{\alpha}}{x^2} \right) \bar{T} + \frac{2(D-3)(D-4)}{3} \bar{\alpha} \left( \sqrt{f(x)} f(x) - 3\sqrt{f(x)} + 2 \right),
\] (4.30)
where the metric function becomes as
\[
f(x) = 1 + \frac{1}{2(D-3)(D-4)\bar{\alpha}} \left[ 1 - \left( \frac{1 + 8\bar{\alpha}}{3\pi x^2} \right)^\frac{3}{2} \left( \frac{32\pi^2 Q^2}{(D-2) (D-3) \omega_D^{2(D-2)}} \left( \frac{1}{x^{D-3}} - 1 \right) \right)^{\frac{3}{2}} \right].
\] (4.31)
Moreover, the Hawking temperature in eqn. (3.21) is rewritten as
\[
\bar{T}_h \equiv r_B T_h = \frac{(D-3)(D-4)(D-5) \bar{\alpha} + (D-3) x^2 \frac{32\pi^2 Q^2}{(D-2) x^{2(D-3)} \omega_D^{(D-2)}}}{4\pi \left( 1 + \frac{2(D-3)(D-4)\bar{\alpha}}{x^2} \right) x^3}.
\] (4.32)

4.1 Five dimensions

When \( D = 5 \), the thermodynamic expressions are rewritten as:
\[
\bar{F}(x) = 1 - \sqrt{f(x)} - \frac{2\pi}{3} x^3 \left( 1 + 12 \frac{\bar{\alpha}}{x^2} \right) \bar{T} + \frac{4\bar{\alpha}}{3} \left( \sqrt{f(x)} f(x) - 3\sqrt{f(x)} + 2 \right),
\] (4.33)
\[
\bar{T} = \frac{x^2 - \frac{4\bar{\alpha}^2}{3\pi x^2}}{2\pi \left( 1 + \frac{4\bar{\alpha}}{x^2} \right) x^3 \sqrt{f(x)}}.
\] (4.34)
where the metric function is simplified as
\[
f(x) = 1 + \frac{1}{4\bar{\alpha}} \left[ 1 - \sqrt{1 + 8\bar{\alpha} \left( 2\bar{\alpha} + x^2 + \frac{4Q^2}{3\pi^2 x^2} - \frac{4Q^2}{3\pi^2} \right)} \right].
\] (4.35)
When it comes to the phase structure, we need to consider the locally stationary points \( r_+ = r_+ (T, Q, \alpha, r_B) \), which can be multivalued and lead to more than one phase. The globally stable phase and phase transitions can be determined by calculating the free energy.
Figure 1: The two regions in the $\bar{\alpha}$-$\bar{Q}$ phase space of a $D = 5$ Gauss-Bonnet black hole in a cavity, each of which possesses distinct behavior of the phase structure and transition. Varying the temperature, there is only one phase in Regions I while a van der Waals-like LBH/SBH phase transition occurs in Regions II.

Figure 2: $\bar{\alpha} = 0.01$ and $\bar{Q} = 0.4$ in the Regions I of FIG. 1. There is no phase transition.

Figure 3: $\bar{\alpha} = 0.01$ and $\bar{Q} = 0.03$ in the Regions II of FIG. 1. There is first-order phase transition.

In FIG. 1 we show that there are two regions in the $\bar{\alpha}$-$\bar{Q}$ phase space of a Gauss-Bonnet-Maxwell black hole in a cavity, each of which possesses distinct behavior of the phase structure and transition. There is only one phase in Regions I while a van der Waals-like LBH/SBH phase transition occurs in Regions II. Moreover,
Figure 4: The two regions in the $\tilde{\alpha} - \tilde{Q}$ phase of a $D = 6$ Gauss-Bonnet black hole in a cavity, each of which possesses distinct behavior of the phase structure and transition. Varying the temperature, there is only one phase in Regions I, while a van der Waals-like phase transition occurs in Regions II.

FIG. 1 and the left panel of FIG. 7 show that the phase structure of a Gauss-Bonnet-Maxwell black hole in a cavity is quite similar to that of a Gauss-Bonnet-Maxwell AdS black hole. Specifically for a black hole with $\tilde{\alpha} = 0.01$ and charge $\tilde{Q} = 0.4$ in Region I, we plot the radius of the black hole horizon radius and the free energy against the temperature in FIG. 2, which shows that the system has a single phase structure and no phase transition. In FIG. 3 we consider a black hole with $\tilde{\alpha} = 0.01$ and $\tilde{Q} = 0.03$ in Region II and plot the horizon radius and the free energy against the temperature. The left panel of FIG. 3 shows that, in some ranges of the temperature, there exists more than one horizon radius of the black hole for a fixed value of temperature. This means that the system can possess a multi-phase structure, which consists of the small, intermediate and large black hole phases. From (3.27), the small and large black holes are thermally stable while the intermediate one is unstable. From the right panel of FIG. 2 we find that, as the temperature increases, the system undergoes a first-order van der Waals-like phase transition from a small black hole and a large one.

### 4.2 Six dimensions

In six dimensions, the thermodynamic expressions are simplified as follow:

$$\tilde{F}(x) = 1 - \sqrt{f(x)} - \frac{\pi}{2} x^4 \left( 1 + 24 \frac{\tilde{\alpha}}{x^2} \right) \tilde{T} + 4 \tilde{\alpha} \left( \sqrt{f(x)} f(x) - 3 \sqrt{f(x)} + 2 \right),$$

$$\tilde{T} = \frac{6 \tilde{\alpha} + 3 x^2 - \frac{9 \tilde{Q}^2}{x^2}}{4 \pi \left( 1 + \frac{12 \tilde{\alpha}}{x^2} \right) x^3 \sqrt{f(x)}},$$

where

$$f(x) = 1 + \frac{1}{12 \tilde{\alpha}} \left[ 1 - \sqrt{1 + 24 \tilde{\alpha} \left( 6 \tilde{\alpha} x + x^3 + \frac{9 \tilde{Q}^2}{24 \pi^2 x^3} - \frac{9 \tilde{Q}^2}{24 \pi^2} \right)} \right].$$

It is worth noting that, unlike the $D = 5$ case, the extremal temperature is dependent on $\tilde{\alpha}$ in the $D = 6$ case. The two regions in the $\tilde{\alpha}-\tilde{Q}$ phase space are plotted in FIG. 4. As shown in FIG. 5 there is only one phase.
for the black holes in Region I. On the other hand, FIG. 6 shows that, for the black holes in Region II, there exists a band of temperatures where three phases coexist, and a first-order van der Waals-like phase transition occurs. Note that the phase transition structure of a $D = 6$ Gauss-Bonnet-Maxwell black hole in a cavity is quite similar to that of a $D = 6$ Gauss-Bonnet-Maxwell AdS black hole, which is shown in the right panel of FIG. 7.

5 Conclusion

In this paper, we first calculated the Euclidean action of a Gauss-Bonnet-Maxwell black hole in a finite spherical cavity and obtained the corresponding free energy in a canonical ensemble by semi-classical approximation. Moreover, the first law of thermodynamics was found to be satisfied. In the rest of this paper, we mainly discussed the phase structure and transition of a Gauss-Bonnet-Maxwell black hole in a cavity for $D = 5$ and $D = 6$. In five dimensions, there are two regions in the $\bar{\alpha}-\bar{Q}$ phase space in FIG. 1. In Region I, there is a one-to-one correspondence between temperature and free energy, so no phase transition occurs. In Region II, there exists a three phase coexistence, in which the small and large black holes are both stable with a positive specific heat while the intermediate black hole is unstable. As the temperature of the system increases, the system starts from a small black hole, undergoes a van der Waals-like phase transition and ends in a large black hole. In six dimensions, the two regions are presented in the $\bar{\alpha}-\bar{Q}$ phase space in FIG. 4. Similarly, no phase transition and a van der Waals-like phase transition occur in Regions I and II, respectively. Finally, we
found that the phase structure of a Gauss-Bonnet-Maxwell black hole in cavity is almost the same as that of a Gauss-Bonnet-Maxwell in AdS space, which is discussed in the appendix.

Acknowledgements We are grateful to Qingyu Gan, Guangzhou Guo and Houwen Wu for useful discussions and valuable comments. This work is supported in part by the NSFC (Grant No. 11875196, 11375121 and 11005016).

A Gauss-Bonnet black hole in AdS space

In this appendix, we briefly discuss phase structure of a Gauss-Bonnet-Maxwell black hole in AdS space. In [8], the metric function was obtained as

$$f(r) = 1 + \frac{\rho^2}{2\bar{\alpha}} \left[ 1 - \sqrt{1 + 4\bar{\alpha} \left( \frac{1}{l^2} + \frac{16\pi M}{(D - 2) \omega_{D-2}r^{D-1}D-1} - \frac{32\pi^2 Q^2}{(D - 2)(D - 3) r^{2D-4} \omega_{D-2}^2} \right)} \right], \quad (A.39)$$

where $l$ is the radius of the AdS space. Using eqn. (A.39), we can express ADM mass $M$ of the black hole in terms of the horizon radius $r_+$,

$$M = \left( \frac{\bar{\alpha}}{r_+^2} + 1 + \frac{r_+^2}{l^2} + \frac{32\pi^2 Q^2}{(D - 2)(D - 3) r_+^{2D-6} \omega_{D-2}^2} \right) \frac{r_+^{D-3} (D - 2) \omega_{D-2}}{16\pi G}. \quad (A.40)$$

Furthermore, the black hole temperature and entropy are

$$T = \frac{f'(r_+)}{4\pi} = \frac{(D - 1) \frac{\rho^2}{l^2} + (D - 5) \frac{\bar{\alpha}}{r_+^2} + (D - 3) - \frac{32\pi^2 Q^2}{(D - 2) \omega_{D-2}^2 r_+^{D-6}}}{4\pi \left( 1 + \frac{2\bar{\alpha}}{l^2} \right) r_+}, \quad (A.41)$$

$$S = \int \frac{1}{T} \frac{\partial M}{\partial r_+} dr_+ = \frac{1}{4} \omega_{D-2} r_+^{D-2} \left( 1 + \frac{D - 2}{D - 4} \frac{2\bar{\alpha}}{r_+^2} \right), \quad (A.42)$$

respectively. The free energy of the black hole is defined as $F = M - TS$, which can be expressed in terms of horizon radius $r_+$, AdS radius $l$, black hole charge $Q$ and Gauss-Bonnet parameter $\bar{\alpha}$:

$$F = \left( \frac{\bar{\alpha}}{r_+^2} + 1 + \frac{r_+^2}{l^2} + \frac{32\pi^2 Q^2}{(D - 2)(D - 3) r_+^{2D-6} \omega_{D-2}^2} \right) \frac{r_+^{D-1} (D - 2) \omega_{D-2}}{16\pi} - T \frac{1}{4} \omega_{D-2} r_+^{D-2} \left( 1 + \frac{D - 2}{D - 4} \frac{2\bar{\alpha}}{r_+^2} \right). \quad (A.43)$$

We can also define

$$\bar{r}_+ = \frac{r_+}{T}, \quad \bar{Q} = \frac{Q}{l^{D-3}}, \quad \bar{\alpha} = \frac{\bar{\alpha}}{l^2}, \quad \bar{T} = Tl, \quad \bar{S} = \frac{S}{\omega_{D-2} l^{D-2}}, \quad \bar{F} = \frac{F}{\omega_{D-2} l^{D-3}}. \quad (A.44)$$

It should be noted that the Gauss-Bonnet parameter is constrained as

$$0 \leq \bar{\alpha} \leq \frac{1}{4 (D - 3) (D - 4)}, \quad (A.45)$$

since the square root of eqn. (A.39) should be greater than zero when $M = Q = 0$. Solving eqn. (A.41) for $r_+$ in terms of $T$, we can write the free energy $F$ as a function of the temperature $T$, the charge $Q$, the Gauss-Bonnet
Figure 7: The two regions in the $\bar{\alpha}$-$\bar{Q}$ represent the different phase structure of a Gauss-Bonnet black hole in AdS space. The left panel is for the black holes in five dimensions, while the right panel is in six dimensions. In both cases, the yellow regions (Region I) have only one phase, while a van der Waals-like phase transition occurs in the cyan regions (Region II).

parameter $\bar{\alpha}$, and the horizon radius $\bar{r}_+$. In the both $D = 5$ and $D = 6$, FIG. 7 shows that there are two regions in the $\bar{\alpha}$-$\bar{Q}$ phase space. In Region I, there is a single phase and no phase transition. However in Region II, three phases can coexist for some range of temperature, and a first-order van der Waals-like phase transition occurs.

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