Covariant Conservation Laws and the Spin Hall Effect in Dirac-Rashba Systems

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We present a theoretical analysis of two-dimensional Dirac-Rashba systems in the presence of disorder and external perturbations. We unveil a set of exact symmetry relations (Ward identities) that impose strong constraints on the spin dynamics of Dirac fermions subject to proximity-induced interactions. This allows us to demonstrate that an arbitrary dilute concentration of scalar impurities results in the total suppression of nonequilibrium spin Hall currents when only Rashba spin–orbit coupling is present. Remarkably, a finite spin Hall conductivity is restored when the minimal Dirac–Rashba model is supplemented with a spin–valley interaction. The Ward identities provide a systematic way to predict the emergence of the spin Hall effect in a wider class of Dirac-Rashba systems of experimental relevance and represent an important benchmark for testing the validity of numerical methodologies.

Systems exhibiting strong spin–orbit coupling (SOC) have received much attention because they host unique spin transport phenomena that can be harnessed for low-power spintronics [1, 2]. The spin Hall effect (SHE) [3, 4] is indubitably a landmark in this novel approach: combined with its reciprocal phenomenon (the inverse SHE), it allows all-electrical generation, detection and manipulation of nonequilibrium spin currents in nonmagnetic conductors [5–8]. The exploitation of the SHE has proved fruitful for manipulation of magnetic order via spin–orbit interactions preserving the inherent SU(2) spin structure, such as a spin–valley coupling. Such unique features make the Dirac–Rashba model an ideal testbed to re-examine the existence of a covariant conservation law for the spin current—stemming from SU(2) gauge invariance—allows us to obtain the analytic form of two-particle spin-current importance to understand whether the absence of SHE is a general property of nonmagnetic surfaces with broken inversion symmetry or, rather, a peculiarity of the 2DEG.

The interfacial enhancement of SOC in graphene has been recently demonstrated [37–42], making it a promising model system to explore the above issue. The departure from the standard Rashba effect in a 2DEG can be readily appreciated for a minimal model of graphene subject to z → −z asymmetric SOC. In the long-wavelength limit, the relevant spin–orbit interaction is obtained by replacing momentum with pseudospin operator \( p \to \sigma \) in \( H_{BR} \) [43, 44]. The Hamiltonian density \( \mathcal{H} = \mathcal{H}_0 + \mathcal{H}_{BR} \) for the \( \chi = \pm \) valley reads

\[
\mathcal{H} = \psi_\chi^\dagger \left\{ \chi \left[ -i \hbar v \sigma^i \partial_i + \lambda (\sigma \cdot s)z \right] - \epsilon \right\} \psi_\chi,
\]

where \( v \) is the bare Fermi velocity of massless Dirac electrons, \( \lambda \) is the Rashba coupling, \( \epsilon \) is the Fermi energy, and \( \sigma_i \) (\( i = 1, 2 \)) and \( s_j \) (\( j = 1, 2, 3 \)) are Pauli matrices acting on pseudospin and spin subspace, respectively. This model possess two noteworthy features. First, the band splitting occurs along the energy axis [Fig. (1)]. Secondly, the Dirac helical spin texture is momentum dependent, e.g., \( ||(s)\) is not conserved [44]. Moreover, Eq. (1) admits a straightforward generalization by adding further interactions preserving the inherent SU(2) spin structure, such as a spin–valley coupling. Such unique features make the Dirac–Rashba model an ideal testbed to re-examine the absence of SHE in interfaces with spin-split states.

In this Letter, we investigate Dirac–Rashba models in the presence of disorder and external perturbations. The existence of a covariant conservation law for the spin current—stemming from SU(2) gauge invariance—allows us to obtain the analytic form of two-particle spin-current...
vertex functions directly from the self energy of the Dirac fermions and show that the spin Hall conductivity in the minimal model [Eq. (1)] is zero for nonmagnetic disorder, irrepressibly of the Fermi level position. Furthermore, we show that when Eq. (1) is generalized to include additional interactions, the obtained Ward identity imposes strong constraints on the nonequilibrium spin responses. Remarkably, this allows us to predict what type of proximity spin–orbit interactions can lead to a robust SHE in Dirac–Rashba interfaces of experimental interest.

The suppression of SHE in 2DEGs subject to uniform Rashba interactions occurs in the presence of an arbitrary small concentration of scalar impurities. Formally, the disorder corrections resulting from the resummation of ladder diagrams exactly cancel the “clean” spin Hall (SH) conductivity [24–28]. In Ref. [27] it was shown that this puzzling cancellation has its origin in the existence of a covariant conservation law for the spin current. For example, the spin-y component satisfies

\[ \partial_t J^y_a(x,t) + \partial^i J^i_a(x,t) = -2\alpha m J^y_a(x,t), \quad (2) \]

where \( J^y_a \) \( (a = x, y, z) \) is the spin density, \( J^y_a \) is the pure spin current flowing in the \( i = x, y \) direction, \( m \) is the effective electron mass, and \( \alpha \) is the Rashba parameter. The main difference with respect to the charge continuity equation originates from the non Abelian nature of spin, which results in the additional contribution on the right hand side. Equation (2) suggests that in the steady state of a homogeneous system, \( J^y_a \) is zero irrespectively of the underlying relaxation mechanism. Below we show that, albeit the drastically different nature of electronic states in the Dirac–Rashba model [Fig. (1)], a similar covariant conservation law exists, and discuss its consequences.

**Conservation laws I.**—A peculiarity of Dirac theories is the possible existence of quantum anomalies due to the joint effect of an infinite Dirac sea of filled electron states and an external field [45, 46]. Let us consider the minimal coupling of Eq. (1) to a \( U(1) \) gauge field \( A_i \equiv (A_0, A_1) \) within a Minkowsky metric. To simplify notation, we take \( \chi = + \) and omit this index hereafter. We also use natural units (\( h \equiv 1 \equiv e \)) and the compact notation \( \partial_i \equiv (\partial_t, \partial_i) \) with summation over dummy indices. The Dirac spin and charge currents are, respectively, \( J^\mu_a(x) = \psi^\dagger(x) s^\mu \psi(x) \) and \( J^\mu_a(x) = \psi^\dagger(x) v_x \psi(x) \), where \( v^\mu = (1, v\sigma) \) and \( x \equiv (t, \mathbf{x}) \). The Heisenberg equation of motion for the spin density reads

\[ \partial^\nu J^\nu_a(x) = -\frac{2\lambda}{v} \epsilon^a_{bc} \epsilon^{bl} J^c_a(x) + i \int dy [\partial_t J^0_a(x), J^b_a(y)] A^a(y), \quad (3) \]

where \( \epsilon^{bl} \epsilon^a_{bc} \) is the Levi-Civita symbol of second (third) rank. The term on the left hand side and the first on the right result from the commutator of \( J^0_a \) respectively with the kinetic and the Rashba term and give a contribution identical to the one found in the 2DEG upon identification of \( m \rightarrow 1/v \), c.f. Eq. (2). Both terms can be combined as the covariant derivative \( D^\mu J^\mu_a = \partial^\mu J^\mu_a + 2 \epsilon^a_{bc} A^b_{\mu} \epsilon^{c}, \) where \( A^a_0 = 0, A^a_1 = -\lambda/v \epsilon^a \) is a SOC-induced, homogeneous gauge field. Hence, in the absence of an external field, Eq. (3) acquires the form of a covariant conservation law for the spin density \( D^\mu J^\mu_a = 0 \). The current commutator in the last term (Schwinger term) defines the anomaly. A careful analysis shows however that despite the Dirac nature of the theory, the commutator is identically zero – see supplemental material [47]; therefore, the argument of Ref. [27] implies a vanishing SHE in the Dirac–Rashba model. At first sight this result contradicts the claims of Ref. [49], where the SH conductivity was evaluated using linear response theory \( \sigma_{SH} = \lim_{q \to 0} \lim_{\omega \to 0} \Theta_{xx}(q, \omega)/\omega \), with the response function \( \Theta_{xx} \), taken in the disorder-free approximation. Using the Matsubara propagator given in [47] we find

\[ \sigma_{SH} = -\frac{e}{16\pi\lambda} \left[ \frac{2\lambda + \epsilon}{\epsilon + \lambda} + \theta(\epsilon - 2\lambda) \frac{2\lambda - \epsilon}{\epsilon - \lambda} \right], \quad (4) \]

in agreement with Ref. [49]. Here \( \theta(\cdot) \) is the Heaviside step function and we assumed \( \epsilon, \lambda > 0 \). The apparent contradiction is resolved by recalling that, without disorder, there is no true stationary state. In the following we show that Eq. (4) misses on important physics related to scattering-induced relaxation that leads to \( \sigma_{SH} = 0 \).

**Conservation laws II: disorder effects.**—Broadly speaking, the Fermi surface contribution to \( \sigma_{SH} \) is dominated by incoherent multiple scattering off impurities, which can be viewed as a series of skew scattering and side jump events [50–52]. To determine how such effects change the above picture, we add to the bare Hamiltonian (1) a random scalar potential \( V(x) \), which we will assume to be Gaussian distributed with zero mean: \( \langle V(x)V(x') \rangle = n_i \alpha_0^2 \delta(x-x') \), where \( n_i \) is the impurity areal density and \( \alpha_0 \) parametrizes the potential strength. This approximation is accurate in the limit of weak potential scattering provided cross sections are right–left
symmetric (see below). We note that short-range impurities lead to scattering potentials that are off-diagonal in both sublattice and valley spaces. The intervalley scattering produced by such matrix disorder affects the charge conductivity \( \sigma_{xx} \) [53], but it does not change the covariant conservation law for the spin current.

Disorder enters the evaluation of response functions both in the propagator (as a self-energy) and the interaction vertex [54]. These two quantities are not independent of each other but they are related by Ward identities (WIs); these relations are the key to establish gauge invariance in quantum electrodynamics at a non-perturbative level [46]. Remarkably, we find that the non-Abelian WI associated to the spin current vertex completely determines the spin current \( J^\gamma_y \) in the dc limit and therefore it can be used to directly evaluate the SH conductivity. To see this, consider the three-legged spin vertex function \( \Lambda^\gamma_{y}(x, x', x'') \) = \( \langle T \bar{c}(x) \bar{s}(x') \bar{s}(x'') \rangle \), where “\( T \)” stands for the imaginary time ordering operator. Moving to frequency-momentum space, we perform analytic continuation \( i \omega_n \to \omega + i\gamma \) (for real \( \omega_n \) ) and note that there are fermionic Matsubara frequencies. Vertex corrections appear perturbatively as a series of impurity lines ladder diagrams, where only combinations of Green’s functions having poles on opposite sides of the real axis contribute to the renormalization of the vertex [54]. In this way, by projecting the vertex function to the retarded (R) — advanced (A) sector, we find

\[
q^\mu \Lambda^\gamma_y = - \frac{2\lambda}{v} \Lambda^\gamma_y + \frac{1}{2} s_y G^R_{k+q}(\epsilon - G^A_{k} s_y),
\]

where \( k \) and \( q \) are three-vectors. The disorder-averaged Green’s function \( G^R_{\sigma}(\epsilon) = [k_0 - H - \Sigma^R(\epsilon)]^{-1} \), where \( H \) is given by the first quantization form of Eq. (1) and \( \Sigma^R(\epsilon) \) is the disorder induced self-energy (see SM [47] for an explicit form). Owing to the non-Abelian nature of the WI, taking the dc (\( q \to 0 \) ) limit in Eq. (5) completely determines the effective vertex. The final step consists in recasting \( \Lambda^\gamma_y \) in terms of the truncated vertex, \( \Lambda^\gamma_y = G^A \tilde{J}^\gamma_y G^R \) [55], as appearing in the Kubo formula. After algebraic manipulations, we arrive at the important intermediate result

\[
\tilde{J}^\gamma_y = -\frac{v}{4\lambda} \left\{ [s_y, \tilde{H}]_+ + i [s_y, \text{Im} \Sigma^R(\epsilon)]_+ \right\},
\]

where \( \pm \) stands for the (anti-)commutator and \( \tilde{H} = H + \text{Re} \Sigma \) is the Hamiltonian renormalized by the real part of the self energy. This result provides an exact relation between the truncated spin current vertex and the self energy, and as such, it is independent of the particular approximation scheme used to evaluate disorder effects. Within the Gaussian approximation, we find \( -\text{Im} \Sigma^R(\epsilon) = 1/(2\pi)[1 + \theta(2\lambda - \epsilon)\lambda/2\epsilon]s_0 s_y + \theta(2\lambda - \epsilon)\lambda/(2\pi) \) \( \sigma_0 s_3 - 1/8(\epsilon \sigma \times s)_z \), where \( 1/2\tau = n_s c^2_0/4\lambda^2 \) is the quasiparticle broadening. Using the expression of the self energy in Eq. (6), we arrive at

\[
\tilde{J}^\gamma_y = \frac{v}{2} \left\{ \frac{\sigma_y s_3 - \frac{1}{2\lambda} \sigma_0 s_y}{\lambda^2} \right\}(1 + \frac{1}{2\lambda^2} \lambda \sigma_0 s_y + \frac{1}{4\pi^2 \tau} \lambda \sigma_0 s_x), \quad \epsilon > 2\lambda
\]

The first term is just the bare spin current vertex \( j^\gamma_y = \sigma_y s_y \), while, for \( \epsilon > 2\lambda \), the second term, generated by the disorder, is the bare spin density vertex \( \sigma_0 s_y \), apart from the factor \( -v/2\lambda \). This shows that the parameter \( \lambda \tau \) plays a fundamental role in determining the importance of disorder. At first sight one could be tempted to think that when the weak disorder limit \( (\epsilon \tau \gg 1) \) and for strong SOC \((\lambda \tau \gg 1) \), all disorder corrections can be neglected. However, it turns out that the spin polarization response is of order \( \lambda \tau \) (see below), whereas the bare spin current response, due to the first term in Eq. (7), is of order \( (\lambda \tau)^0 \). Hence, the two terms are of the same order irrespective of the disorder strength. Similar considerations apply also for \( \epsilon < 2\lambda \).

**SHE evaluation using the WI.**—We start by computing the Fermi surface contribution

\[
\sigma_{\text{SH}}^I = \frac{1}{2\pi} \int \frac{dk}{(2\pi)^2} \text{tr} \left[ \tilde{J}^\gamma_y \bar{G}^R_{\epsilon} v_x G^A_{\epsilon} \right] = \sigma_{\text{SH}} + \sigma_{\text{SG}} + \sigma_{xx} + \sigma_{zz}
\]

where \( v_x = v \sigma_x s_0 \) is the bare charge current vertex. Moreover, \( \sigma_{\text{SH}}, \sigma_{\text{SG}}, \sigma_{xx}, \) and \( \sigma_{zz} \) are the conductivity “bubbles” corresponding to the various terms in Eq. (7), respectively, a spin Hall (\( \sigma_y s_z \)), spin galvanic (SG) (\( \sigma_0 s_y \)), longitudinal (\( \sigma_x s_0 \)) and “staggered” \( (\sigma_y s_x) \) conductivities. Outside the pseudo-gap, where the Fermi surface splits into two branches [Fig. (1)], we find \( \sigma_{xx} = \sigma_{zz} = 0 \) and \( \sigma_{\text{SH}} = -\sigma_{\text{SG}} \), where

\[
\sigma_{\text{SH}} = \frac{1}{8\pi} \left( \frac{\epsilon^2}{\epsilon^2 - \lambda^2} - \frac{1}{1 + 4\lambda^2 \tau^2} \right),
\]

and thus the type I contribution to the SH conductivity is zero, \( \sigma_{\text{SH}}^{\text{I}} = 0 \). This result deserves few comments: First, in the \( \lambda \tau \gg 1 \) limit, one recovers Eq. (4). Second, the “empty bubble” SH conductivity \( (\tilde{\sigma}_{\text{SH}}) \) is precisely counteracted by the corresponding “empty bubble” for spin density-charge current response function \( (\tilde{\sigma}_{\text{SG}}) \) [56]. This means that the absence of SHE is linked to the onset of a current-induced, in-plane spin polarization known as the inverse spin galvanic effect [17–19]. The remaining (type II) contribution

\[
\sigma_{\text{SH}}^{\text{II}} = \frac{-1}{2\pi} \int \frac{dk}{(2\pi)^2} \int_{-\infty}^{\infty} dk_0 \text{Re tr} \left[ \bar{G}^R_{\epsilon} j^\gamma_y \delta_{k_0} G^R_{\epsilon} v_x \right],
\]

accounts for processes away from the Fermi surface [57]. Explicit evaluation shows that \( \sigma_{\text{SH}}^{\text{II}} = 0 \) and thus \( \sigma_{\text{SH}} = \sigma_{\text{SH}}^{\text{I}} + \sigma_{\text{SH}}^{\text{II}} \) is zero, in agreement with our earlier argument viz., Eqs. (2)-(3). Interestingly, in the 2DEG–Rashba
model, the type II term is only zero in the formal limit \( \epsilon \tau \to \infty \) and can attain large values for \( \lambda \tau \approx 1 \) [58]. The exact vanishing of the off-Fermi surface contribution is a unique feature of the Dirac theory. We now move gears to the regime \( \epsilon < 2 \lambda \), where only one subband is occupied. We note that this regime has no analogue in the 2DEG model, for which the Fermi surface always consists of two disconnected rings [Fig. (1)]. The mechanism leading to the regime \( \lambda \tau \approx 1 \) is thus far from obvious. To investigate this issue, we evaluate the Fermi surface contribution making use of the WI [see Eq. (7)] and the type II contribution using Eq. (10). After a lengthy calculation, we find for both contributions

\[
\sigma^{I}_{\text{SH}} = \frac{1}{16\pi} \frac{\epsilon}{\lambda} , \quad \sigma^{II}_{\text{SH}} = -\sigma^{I}_{\text{SH}} \tag{11}
\]

so that \( \sigma_{\text{SH}} = 0 \). Note that since \( \sigma^{I}_{\text{SH}} \) is of order \( \tau \), we can evaluate the type II contribution [Eq. (10)] directly in the absence of disorder. The suppression of the SHE in the regime \( 0 < \epsilon < 2 \lambda \) therefore results from a compensation between scattering corrections to the “clean” SH conductivity and off-Fermi surface processes.

Diagrammatic evaluation.—We now show the consistency of our results with a standard diagrammatic evaluation. The renormalized charge current vertex satisfies the following Bethe-Salpeter coupled equations [see Fig. (2)]

\[
\tilde{v}_{x,\mu a} = v \delta_{\mu 1} \delta_{a 0} + T_{\mu a \rho d} v^{\mu \nu \lambda \epsilon} I_{\nu \rho \lambda \epsilon} v^{\rho d} \tag{12}
\]

\[
T_{\mu a \rho d} v^{\mu \nu \lambda \epsilon} = \text{tr}[\sigma_{\mu} s_{a} \sigma_{\nu} s_{b} \sigma_{\rho} s_{d} \sigma_{\lambda} s_{c}] \tag{13}
\]

\[
I_{\nu \rho \lambda \epsilon} = \frac{\eta_{3}}{4} \frac{\alpha_{5}^{2}}{(2\pi)^{2}} \mathcal{G}_{\nu \rho \lambda \epsilon}^{R} \mathcal{G}_{\nu \rho \lambda \epsilon}^{A} \tag{14}
\]

In principle, \( I \) spans the entire Clifford Algebra. However, not all matrix elements contribute to the renormalization of the charge vertex. It is convenient to consider the effect of a single impurity density insertion, for which the vertex has the structure: \( \tilde{v}_{x} = \delta v_{10} \sigma_{1} s_{0} + \delta v_{23} \sigma_{2} s_{3} + \delta v_{02} \sigma_{0} s_{2} + \delta v_{13} \sigma_{1} s_{3} \), with \( \delta v_{ij} \) some non-zero matrix elements. This result suggests the form of the ansatz for \( \tilde{v}_{x} \) to use in Eq. (12). Since no new matrix element is generated in this procedure, the ansatz closes the system. In addition to the renormalized charge vertex \( \tilde{v}_{x}^{10} \), we find that disorder induces an effective SHE \( (\tilde{v}_{x}^{10}) \), spin galvanic \( (\tilde{v}_{x}^{02}) \) and “staggered” \( (\tilde{v}_{x}^{31}) \) interaction. Their explicit form reads (for \( \epsilon > 2 \lambda \)): \( \tilde{v}_{x}^{10} = 2 v, \tilde{v}_{x}^{02} = -2 v (\lambda/\epsilon), \tilde{v}_{x}^{31} = 0 \) and \( \tilde{v}_{x}^{23} = 0 \). In order to evaluate the SH conductivity we use now Eq. (8), with the ladder series now included in the charge vertex (i.e. \( \tilde{j}_{x} \rightarrow \tilde{j}_{x}^{0} \) and \( v_{x} \rightarrow \tilde{v}_{x} \)). Using Eq. (12), it is now easy to relate the renormalized vertex directly to the SH and Drude conductivity

\[
\sigma_{\text{SH}} = \frac{1}{2\pi} \left( \frac{2 v}{\eta_{3} \alpha_{5}^{2}} \right) \tilde{v}_{x}^{23} = 0, \tag{15}
\]

\[
\sigma_{xx} = \frac{1}{2\pi} \left( \frac{4 v}{\eta_{3} \alpha_{5}^{2}} \right) (\tilde{v}_{x}^{10} - v) = \frac{2 \epsilon \tau}{\pi} \tag{16}
\]

Discussions.—We mentioned earlier that higher-order scattering contributions to the self energy (and ladder series) could generate important corrections. This happens when impurities in the system lead to skew scattering. In the 2DEG, it is well known that skew scattering is absent (unless other ingredients, such as spin–orbit active impurities are considered). The absence of skewness has in fact an intuitive explanation: the spin of Rashba eigenstates is locked in-plane, so that in a given scattering event quasiparticles cannot distinguish left and right. The same picture holds in the Dirac–Rashba model and so here too there should be no skewness. We verified this by means of the self-consistent diagrammatic approach introduced in Ref. [52] together with the WI [Eq. (6)].

The formalism developed in this Letter also allows to predict the behaviour of more complicated systems. For instance, it is easy to see that a non-zero SH conductivity emerges when adding suitable interactions to Eq. (1), altering the covariant conservation law expressed in Eq. (3) and hence the WI [Eq. (6)]. For example, let us consider a spin–valley interaction of the form \( A_{y} = \lambda' \delta_{z} \), with \( \lambda' \) a constant. This interaction generates in Eq. (3) a new term proportional to \( \langle s_{y}^{2} \rangle \), where \( \langle s_{y}^{2} \rangle \) is the nonequilibrium average of the \( \hat{x} \)-spin polarization at a given valley. Taking the steady state of a homogeneous system, we find an exact relation between the spin Hall current and the difference between the nonequilibrium spin density at the two inequivalent valleys, namely:

\[
\langle J_{y}^{2} \rangle = v \frac{\lambda'}{\lambda} \left( \langle s_{x}^{0} \rangle - \langle s_{x}^{0} \rangle \right) \tag{17}
\]

This suggests that SHE can emerge provided there is a mechanism to generate \( \langle s_{y}^{2} \rangle \neq 0 \) with opposite signs for \( \chi = \pm 1 \). A strong candidate is skew scattering. In principle, skewness is now allowed since the spin–orbit interaction of the form \( \delta v_{ij} \) is absent (unless other ingredients, such as spin–orbit active impurities are considered).

Figure 2: Feynman diagrams for (a) dressed SH conductivity. (b) Charge vertex renormalization. The empty dot represents the bare charge vertex while the red \( x \) and the black dots represent respectively impurity density and scattering potential insertions.
proximity SOC \cite{59, 60}. The possibility to have skew scattering exclusively driven by SOC in the band structure appears to be a unique feature of Dirac systems.

In this context, we note in passing that random spatial fluctuations in the Rashba coupling (e.g., due to corrugations) provide an alternative source of SHE \cite{61}. The skew scattering contribution discussed above is dominant in clean samples due to its characteristic scaling ($n_i^{-1}$ opposed to $n_i^0$ in the random mechanism) and the relatively small size of the fluctuations expected for atomically-flat interfaces.

Our work constitutes a major step towards a unified theory of spin and charge dynamics for Dirac-Rashba models in generic non-stationary conditions. Real-space methodologies for numerical evaluation of transverse conductivities have been recently proposed \cite{62, 63}, which can help tackling more complex scenarios. The exact symmetry relations presented here provide a stringent test for real-space numerical approaches, for which the achievable energy resolutions still represent a major limiting factor.

Data availability statement (EPSRC).—No new data were created during this study.

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In this supplemental material we provide explicit expressions for the propagators in the presence of Rashba spin orbit coupling (both in the clean and disordered case) used in the main text. We provide an explicit proof of the absence of anomalous commutators (Schwinger terms) in the continuity equation for the spin current and finally give some details on the evaluation of the SH conductivity.

I. SPINOR PROPAGATOR WITH RASHBA SOC

Due to the breaking of the spin rotational symmetry $SU_2(2)$ in the presence of the Rashba term, the fermionic propagator is not easily invertible. Here we show how to obtain the general expression of the propagator both in the clean and in the disordered case. The Dirac-Rashba Hamiltonian density in $\chi = \pm$ valley is given in Eq. (1) of the main text. In momentum space it reads

$$\mathcal{H}_\chi = \psi_\chi^\dagger \left\{ v \mathbf{\sigma} \cdot \mathbf{k} + \lambda (\sigma_1 s_2 - \sigma_2 s_1) \right\} \psi_\chi$$

with eigenvalues

$$E_{ab,\chi}(k) = \chi (a \lambda + b \sqrt{(v k)^2 + \lambda^2}),$$

(2)

where $b = \pm 1$ indexes the positive/negative energy bands and $a = \pm 1$ indexes the “helicity” sub-band [64]. It is convenient to introduce the angle $\theta_a = \arcsinh(-a \lambda/vk)$ in order to rewrite the eigenstates in compact form as

$$\Phi_{ab} = \frac{1}{2 \sqrt{\cosh \theta_a}} \begin{pmatrix} -i a b e^{-i \phi} e^{b \theta_a/2} \\ e^{b \theta_a/2} \\ -i a e^{-b \theta_a/2} \\ b e^{i \phi} e^{b \theta_a/2} \end{pmatrix},$$

(3)

where $\phi$ is the angle between $k_x$ and $k_y$. The projector over the basis of the energy eigenstates of the Hamiltonian (1) is then $P_{ab} = |\Phi_{ab}\rangle\langle\Phi_{ab}|$. For computational purpose, it is convenient to expand again the projector over the $SU_2(2) \times SU_2(2)$ spinor representation

$$P_{ab} = \frac{1}{4} \left\{ (\sigma_0 s_0) + (\sigma_3 s_3) b \tanh \theta_a + (\sigma^i s_0) \hat{k}_i + \frac{b}{\cosh \theta_a} \left[ (\sigma_1 s_2) a (e^{b \theta_a} \cos 2 \phi + e^{-b \theta_a}) + (\sigma_2 s_2) a (e^{b \theta_a} \sin 2 \phi) \right] \right\}$$

(4)

Note that we use the notation $k \equiv |\mathbf{k}|$ and $\hat{k}_i = k_i/k$. Using $P_{ab}$, we can rewrite the clean Matsubara propagator as

$$G(\mathbf{k}, \imath \nu_n) = \sum_{ab,\chi} \frac{P_{ab}}{\epsilon + \imath \nu_n - E_{ab,\chi}},$$

(5)
where \( \nu_n \) are fermionic Matsubara frequencies. The real time, disorder averaged propagator in the Retarded (R)/Advanced (A) sector reads instead

\[
G^{R/A}(k) = \sum_{ab,\chi} \frac{P_{ab}}{\epsilon - E_{ab,\chi} \pm i n_i \eta(E_{ab,\chi})},
\]

(6)

where \( n_i \) is the density of impurities, \( E_{ab,\chi} \) are the energy eigenvalues after disorder average. Finally, \( \eta(E_{ab}) \) is an energy dependent disorder broadening, whose value depends on whether the Fermi level is inside or outside the Rashba pseudogap (see main text). Below we provide the explicit expression for the matrix elements of these two propagators.

II. MATSUBARA PROPAGATOR

Here we list the matrix elements of the \( \chi \)-valley propagator. We expand the propagator over the spinor basis as

\[
G_\chi(k, \nu) = \sigma^\mu s^\nu G_{\chi,\mu\nu}(k, \nu),
\]

where summation over repeated indices is understood. The only non-zero matrix elements are

\[
G_{\chi,00}(k, \nu) = -\frac{1}{2} \left\{ (\epsilon + i \nu_n + \chi \lambda) L_{1,\chi}(k, \nu) + (\epsilon + i \nu_n - \chi \lambda) L_{2,\chi}(k, \nu) \right\}
\]

(7)

\[
G_{\chi,i0}(k, \nu) = -\frac{\chi^i v k_i}{2} \left\{ L_{1,\chi}(k, \nu) + L_{2,\chi}(k, \nu) \right\}
\]

(8)

\[
G_{\chi,12}(k, \nu) = \frac{\cos 2\phi}{4} \left\{ (\epsilon + i \nu_n + 2\chi \lambda) L_{1,\chi}(k, \nu) - (\epsilon + i \nu_n - 2\chi \lambda) L_{2,\chi}(k, \nu) \right\}
\]

(9)

\[
+ \frac{1}{4} (\epsilon + i \nu_n) \left\{ L_{1,\chi}(k, \nu) - L_{2,\chi}(k, \nu) \right\}
\]

\[
G_{\chi,21}(k, \nu) = \frac{\cos 2\phi}{4} \left\{ (\epsilon + i \nu_n + 2\chi \lambda) L_{1,\chi}(k, \nu) - (\epsilon + i \nu_n - 2\chi \lambda) L_{2,\chi}(k, \nu) \right\}
\]

(10)

\[
- \frac{1}{4} (\epsilon + i \nu_n) \left\{ L_{1,\chi}(k, \nu) - L_{2,\chi}(k, \nu) \right\}
\]

\[
G_{\chi,11}(k, \nu) = -\frac{\sin 2\phi}{4} \left\{ (\epsilon + i \nu_n + 2\chi \lambda) L_{1,\chi}(k, \nu) - (\epsilon + i \nu_n - 2\chi \lambda) L_{2,\chi}(k, \nu) \right\}
\]

(11)

\[
G_{\chi,22}(k, \nu) = -G_{\chi,11}(k, \nu)
\]

(12)

\[
G_{\chi,01}(k, \nu) = -\frac{\chi^i v k_i \sin \phi}{2} \left\{ L_{1,\chi}(k, \nu) - L_{2,\chi}(k, \nu) \right\}
\]

(13)

\[
G_{\chi,02}(k, \nu) = \frac{\chi^i v k_i \cos \phi}{2} \left\{ L_{1,\chi}(k, \nu) - L_{2,\chi}(k, \nu) \right\}
\]

(14)

\[
G_{\chi,33}(k, \nu) = -\frac{\chi^i v k_i}{2} \left\{ L_{1,\chi}(k, \nu) - L_{2,\chi}(k, \nu) \right\}
\]

(15)

We have defined the two “Kernels”

\[
L_{1,\chi}(k, \nu) = \frac{1}{v^2 k^2 - (\epsilon + i \nu_n)(\epsilon + i \nu_n + 2\chi \lambda)}
\]

(16)

\[
L_{2,\chi}(k, \nu) = \frac{1}{v^2 k^2 - (\epsilon + i \nu_n)(\epsilon + i \nu_n - 2\chi \lambda)}
\]

(17)

Note that under valley exchange, the two Kernels are also interchanged, i.e \( L_{1,3} = L_{2,1} \) and vice versa.

III. DISORDER AVERAGED PROPAGATOR

Here we list the matrix elements of the real time, disorder averaged propagator in the Retarded (R)/Advanced (A) sector. We use the notation \( G_\chi(k) = \sigma^\mu s^\nu G_{\chi,\mu\nu}(k) \), where the Green’s functions are now evaluated at zero real frequencies. We first define the following notation for the Heaviside step function: \( \theta_{i/2} = \theta(\epsilon \mp 2\lambda) \). In the gaussian
approximation, we define the following parameters
\[
m = \frac{\lambda}{4\pi\tau\epsilon} \log \left| \frac{\epsilon - 2\lambda}{\epsilon + 2\lambda} \right|,
\]
\[
\delta m = \frac{\lambda}{4\tau\epsilon}(\theta_1 - \theta_2),
\]
\[
\eta = \frac{1}{4\tau}\left(\theta_1 + \theta_2\right) + \frac{\lambda}{4\tau\epsilon}(\theta_1 - \theta_2),
\]
\[
\delta \lambda = \frac{1}{8\tau}(\theta_1 - \theta_2).
\]
Here \(m\) is a random mass term coming from the real part of the self energy while \(\delta m\) is the analogue term coming from the Imaginary part of the Self energy. Finally, \(\delta \lambda\) is the imaginary part of the Rashba coupling introduced by the disorder and \(\eta\) is the broadening. The quasi-particle lifetime is defined in the main text as \(1/2\tau = n_i \epsilon \alpha_0^2/4v^2\).

The components of the disorder averaged propagator now read
\[
G_{\chi,00}^{R/A}(k) = -\frac{1}{2}\left\{ \mathcal{L}_{1,\chi}[\epsilon + \chi (\lambda \pm i \delta \lambda) \pm i \eta] + \mathcal{L}_{2,\chi}[\epsilon - \chi (\lambda \pm i \delta \lambda) \pm i \eta] \right\}
\]
\[
G_{\chi,01}^{R/A}(k) = -\chi \frac{k \nu}{2} \sin(\phi) \{ \mathcal{L}_{1,\chi} - \mathcal{L}_{2,\chi} \}
\]
\[
G_{\chi,02}^{R/A}(k) = \chi \frac{k \nu}{2} \cos(\phi) \{ \mathcal{L}_{1,\chi} - \mathcal{L}_{2,\chi} \}
\]
\[
G_{\chi,10}^{R/A}(k) = -\chi \frac{k \nu}{2} \cos(\phi) \{ \mathcal{L}_{1,\chi} + \mathcal{L}_{2,\chi} \}
\]
\[
G_{\chi,20}^{R/A}(k) = -\chi \frac{k \nu}{2} \sin(\phi) \{ \mathcal{L}_{1,\chi} + \mathcal{L}_{2,\chi} \}
\]
\[
G_{\chi,33}^{R/A}(k) = \frac{\chi}{2}(m \pm i \delta m) \{ \mathcal{L}_{1,\chi} + \mathcal{L}_{2,\chi} \} - \frac{\chi}{2}\left(\lambda \pm i \delta \lambda\right)\{ \mathcal{L}_{1,\chi} - \mathcal{L}_{2,\chi} \}
\]
\[
G_{\chi,11}^{R/A}(k) = -\frac{\sin(2\phi)}{4}\left\{ \mathcal{L}_{1,\chi}[\epsilon - \chi (m \pm i \delta m) + 2\chi (\lambda \pm i \delta \lambda) \pm i \eta] \\
- \mathcal{L}_{2,\chi}[\epsilon - \chi (m \pm i \delta m) - 2\chi (\lambda \pm i \delta \lambda) \pm i \eta] \right\}
\]
\[
G_{\chi,22}^{R/A}(k) = -G_{\chi,11}^{R/A}(k)
\]
\[
G_{\chi,12}^{R/A}(k) = \frac{\cos(2\phi)}{4}\left\{ \mathcal{L}_{1,\chi}[\epsilon - \chi (m \pm i \delta m) + 2\chi (\lambda \pm i \delta \lambda) \pm i \eta] \\
- \mathcal{L}_{2,\chi}[\epsilon - \chi (m \pm i \delta m) - 2\chi (\lambda \pm i \delta \lambda) \pm i \eta] \right\} + \frac{1}{4}(\epsilon + \chi (m \pm i \delta m) \pm i \eta)\{ \mathcal{L}_{1,\chi} - \mathcal{L}_{2,\chi} \}
\]
\[
G_{\chi,21}^{R/A}(k) = \frac{\cos(2\phi)}{4}\left\{ \mathcal{L}_{1,\chi}[\epsilon - \chi (m \pm i \delta m) + 2\chi (\lambda \pm i \delta \lambda) \pm i \eta] \\
- \mathcal{L}_{2,\chi}[\epsilon - \chi (m \pm i \delta m) - 2\chi (\lambda \pm i \delta \lambda) \pm i \eta] \right\} - \frac{1}{4}(\epsilon + \chi (m \pm i \delta m) \pm i \eta)\{ \mathcal{L}_{1,\chi} - \mathcal{L}_{2,\chi} \}
\]
We have defined the two "Kernels"
\[
\mathcal{L}_{1,\chi} = \frac{1}{v^2 k^2 + (m \pm i \delta m)((m \pm i \delta m) - 2(\lambda \pm i \delta \lambda)) - (\epsilon \pm i \eta)(\epsilon \pm 2\chi (\lambda \pm i \delta \lambda) \pm i \eta)}
\]
\[
\mathcal{L}_{2,\chi} = \frac{1}{v^2 k^2 + (m \pm i \delta m)((m \pm i \delta m) + 2(\lambda \pm i \delta \lambda)) - (\epsilon \pm i \eta)(\epsilon - 2\chi (\lambda \pm i \delta \lambda) \pm i \eta)}
\]
As in the clean case, also in this case a symmetry under valley interchange exists. Note the form of the disorder-averaged propagators presented above can be generalised to the non-Gaussian treatment (T-matrix) considering that the matrix structure of the Self-energy is unaffected. Therefore it suffices to use the corresponding form in the T-matrix approximation of the parameters in Eqs. (18)-(21).

IV. ABSENCE OF SCHWINGER TERMS

In Eq. (3) of the main text we mentioned the possible existence of anomalous current commutators in Dirac theories. These commutators arise from the presence of the infinite Dirac sea and they are at the core of anomalies. For example,
The Matsubara frequency spin-current response function is defined as

\[ [J_0^a(x), J_\mu(y)] = \frac{1}{2} \{ \psi^\dagger(x) \gamma_0 s_a \psi(y), \psi^\dagger(y) \gamma_\mu \psi(x) \} \]

Using the definition of the spin and current densities: \( J_\mu^a(x) = \psi^\dagger(x) s^a \sigma_\mu v_\mu / 2 \psi(x) \) and \( J_\mu(x) = \psi^\dagger(x) \sigma_\mu v_\mu \psi(x) \), where \( v_\mu = (1, v) \) we can manipulate the commutator as

\[ [J_0^a(x), J_\mu(y)] = \frac{1}{2} \psi^\dagger(x) \gamma_0 s_a \psi(y), \psi^\dagger(y) \gamma_\mu \psi(x) \]

In order to be sure that this term is zero, we need first to regularize it and then evaluate it explicitly. As a regularization scheme we choose point splitting with two infinitesimal quantities \( \epsilon \) and \( \epsilon' \) [45] and use the normal ordering definition: \( AB = : AB : + \{ AB \} \)

\[ [J_0^a(x), J_\mu(y)] = \lim_{\epsilon, \epsilon' \to 0} \frac{1}{2} \left( : \psi^\dagger(x + \epsilon) s_a \gamma_0 \psi(y - \epsilon') : \delta(x - y - \epsilon - \epsilon') - : \psi^\dagger(y + \epsilon') \gamma_\mu s_a \psi(x - \epsilon) : \right. \]

\[ \times \delta(y - x - \epsilon - \epsilon') + \left( : \psi^\dagger(x + \epsilon) s_a \gamma_0 \psi(y - \epsilon') : \right. \]

\[ \times \delta(y - x - \epsilon - \epsilon') \]

At equal position, the above object is in principle singular as we are effectively subtracting two infinite quantities [45]. In order to be sure that this term is zero, we need first to regularize it and then evaluate it explicitly. As a regularization scheme we choose point splitting with two infinitesimal quantities \( \epsilon \) and \( \epsilon' \) [45] and use the normal ordering definition: \( AB = : AB : + \{ AB \} \)

\[ [J_0^a(x), J_\mu(y)] = \lim_{\epsilon, \epsilon' \to 0} \frac{1}{2} \left( : \psi^\dagger(x + \epsilon) s_a \gamma_0 \psi(y - \epsilon') : \delta(x - y - \epsilon - \epsilon') - : \psi^\dagger(y + \epsilon') \gamma_\mu s_a \psi(x - \epsilon) : \right. \]

\[ \times \delta(y - x - \epsilon - \epsilon') + \left( : \psi^\dagger(x + \epsilon) s_a \gamma_0 \psi(y - \epsilon') : \right. \]

\[ \times \delta(y - x - \epsilon - \epsilon') \]

where the expectation value is taken with respect to the filled Dirac sea and : : stands for normal ordering. Since the normal ordered terms are finite, we can now take their difference and so we are left with the expectation values only. Let us now fix \( a = 2 \) and choose the gauge such as \( E = \partial_t A_0 \) [65]. It is also convenient to move to momentum and imaginary frequency space, from which we arrive at

\[ \langle [J_0^2, J_0(-p)] \rangle = -\frac{1}{\beta} \sum_n \int \frac{d^2 p}{(2\pi)^2} \left( G_{02}(p + q, \nu_n + \omega_m) - G_{02}(p, \nu_n) \right) \]

that is the standard form of the Ward Identity. At this point we can safely shift the momentum in the first Green’s function as \( p + q \to p \) to obtain the cancellation. A longer but equivalent way consists in performing the integrals explicitly. In this case it is easy to see that the q-independent Green’s function is zero after angular average. As for the first Green’s function, one has to expand for small \( q \) and perform the integral explicitly to find again zero. This means that Eq. (3) of the main text reduces to a classical conservation law that completely determines the dynamics of the spin currents, and in particular the fact that \( J_{1/2}^a \to 0 \) as the system reaches a steady state.

V. DETAILS ON THE EVALUATION OF THE SPIN-HALL CONDUCTIVITY

A. Clean system

Here we show how to obtain the SH conductivity in the clean limit by a direct evaluation of the SH-response function. The starting point is the definition of the \( \sigma_{SH} \) in terms of its related correlation function

\[ \sigma_{SH} = \lim_{\omega \to 0} \lim_{q \to 0} \frac{\Theta_{21}^3(q, \omega)}{i \omega} \]

The Matsubara frequency spin-current response function is

\[ \Theta_{21}^3(q, i \omega_n) = -\frac{v^2}{2 \beta} \sum_n \int \frac{d^2 p}{(2\pi)^2} \text{tr} \left[ \{ \sigma_2 s_3 \} G(p + q, i \nu_n + i \omega_n) (\sigma_1 s_0) G(p, i \nu_n) \right] \]

\[ = -\frac{v^2}{2 \beta} \sum_n \int \frac{d^2 p}{(2\pi)^2} 4 i \{ G_{33}(p + q, i \nu_n + i \omega_n) G_{00}(p, i \nu_n) - G_{00}(p + q, i \nu_n + i \omega_n) G_{33}(p, i \nu_n) \} \]

At this point we can take \( q \to 0 \) and evaluate the summation over Matsubara fermionic frequencies \( \nu_n \)

\[ \Theta_{21}^3(0, i \omega_m) = -\frac{\lambda \omega_m}{2 \pi (4 \lambda^2 + \omega_m^2)} \int_{|\lambda|}^{+\lambda} dx \left[ \frac{2 \lambda^2 + 2 \lambda^2 + \omega_m^2}{4 \lambda^2 + \omega_m^2} \right] \{ f[x - (\epsilon - \lambda)] - f[x - (\epsilon + \lambda)] \} \]

where \( \lambda = 2 \lambda \omega_m / (4 \lambda^2 + \omega_m^2) \) and \( \omega_m = m \hbar c / 2 \).
where we have defined $x = \sqrt{v^2 p^2 + \lambda^2}$ and $f(x)$ are fermionic distribution functions. Next we consider the zero temperature limit of the above expression. There are clearly two different solutions of the above expression, corresponding to whether $\epsilon$ is greater or smaller than $\lambda$, i.e. if the Fermi energy intersects two or one energy bands. It is easy to find in these two regimes

$$
\sigma_{SH} = -\frac{1}{8\pi} \frac{\epsilon^2}{\epsilon^2 - \lambda^2} \quad \epsilon > 2\lambda \quad (40)
$$

$$
\sigma_{SH} = -\frac{1}{16\pi} \frac{\epsilon(\epsilon + 2\lambda)}{\lambda(\epsilon + \lambda)} \quad \epsilon < 2\lambda. \quad (41)
$$

In order to obtain the above results we have first performed analytic continuation to real frequencies ($i\omega_m \rightarrow \omega + i0^+$) and then expanded for $\omega < \lambda$. 

