Coherent Resonance in Trapped Bose Condensates
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Abstract

Coherent resonance is the effect of resonant excitation of nonlinear coherent modes in trapped Bose condensates. This novel effect is shown to be feasible for Bose-condensed trapped gases. Conditions for realizing this effect are derived. A method of stabilizing Bose condensates with attractive interactions is advanced. The origin of dynamic critical phenomena is elucidated. Interference effects are studied. The existence of atomic squeezing and multiparticle coherent entanglement is demonstrated. Coherent resonance is a generalization of atomic resonance, involving internal states of an individual atom, to collective states of a multiparticle system.
1 Introduction

Resonant interaction of an alternating electromagnetic field with an atom makes it possible to select a pair of atomic levels and to treat the atom as an effectively two-level system [1]. This resonant interaction is in the basis of one of the main directions of optics. The possibility of realizing atomic resonance is due to the existence of discrete energy levels of electrons in an atom, the energy spectrum being described by the Schrödinger equation. The Bose-condensed atoms are also described by the Schrödinger equation, though nonlinear, which is often termed the Gross-Pitaevskii equation [2–4]. Trapped atoms, because of a confining potential, also possess a discrete spectrum [5]. A specific feature is that this is a spectrum of collective nonlinear states of an ensemble of coherent atoms. The collective coherent and nonlinear nature of these states is what makes them principally different from single-particle linear states of an individual atom. To emphasize this difference, the collective states of Bose-condensed atomic gases are called nonlinear coherent modes [5]. Since the spectrum of these modes, pertaining to trapped atoms, is discrete, it has been suggested [5] that a resonant coupling of two such modes is feasible by means of an alternating field with a frequency tuned to the transition frequency between the chosen levels. This type of resonance for exciting nonlinear coherent modes can be named coherent resonance. This phenomenon, if realised, could open a whole new field of possible applications, analogous to those accomplished with atomic resonance.

In the present communication, we investigate the conditions required for realizing the coherent resonance. We propose a new method for stabilizing Bose-condensed gas with attractive interactions by transferring atoms to an excited coherent mode. We study dynamic critical phenomena, discovered numerically [6–8], and give their explanation. We also describe some new properties of Bose condensate subject to coherent resonance, such as interference effects, atomic squeezing, and multiparticle coherent entanglement.

2 Coherent Resonance

We consider a system of trapped Bose-condensed atoms at low temperature, when the Bose gas in a coherent state. The coherent-field wave function satisfies the Gross-Pitaevskii equation

\[ i \hbar \frac{\partial}{\partial t} \varphi(\mathbf{r}, t) = \left( \hat{H}[\varphi] + \hat{V} \right) \varphi(\mathbf{r}, t), \]  

in which the coherent field is normalized to unity, \( ||\varphi||^2 = 1. \) The nonlinear Hamiltonian is

\[ \hat{H}[\varphi] \equiv -\frac{\hbar^2 \nabla^2}{2m_0} + U(\mathbf{r}) + NA_s|\varphi(\mathbf{r}, t)|^2, \]  

where \( U(\mathbf{r}) \) is a trapping potential; \( m_0 \) is atomic mass; \( N \) is the number of atoms; \( A_s \equiv 4\pi\hbar^2a_s/m_0, \) with \( a_s \) being a scattering length. The alternating field is

\[ V(\mathbf{r}, t) = V_1(\mathbf{r}) \cos \omega t + V_2(\mathbf{r}) \sin \omega t. \]
Note that Eq. (1) is an exact equation for the coherent field \[9\], and \(\varphi(r,t)\) does not need to be interpreted as an average of a field operator, for which case Eq. (1) would be only a mean-field approximation.

The nonlinear coherent modes are the stationary solutions to the eigenproblem

\[ \hat{H}[\varphi_n] \varphi_n(r) = E_n \varphi_n(r) . \]  

(4)

Since the Hamiltonian (2) is nonlinear, the modes are not necessary orthogonal to each other, so that the scalar product

\[ (\varphi_m, \varphi_n) \equiv \int \varphi_m^* (r) \varphi_n (r) \, dr \]

is not compulsory a Kroneker delta. But each mode can always be normalized, \(||\varphi_n||^2 = 1\). Some properties of the nonlinear modes have been discussed \[5–8,10–13\] and dipole mode has been observed experimentally \[14\].

Let us select two spectrum levels, with the energies \(E_1\) and \(E_2\), such that \(E_1 < E_2\). The related transition frequency is

\[ \omega_{21} \equiv \frac{1}{\hbar} (E_2 - E_1) . \]

(5)

The frequency of the alternating field (3) is tuned close to this transition frequency (5), which implies the resonance condition

\[ \left| \frac{\Delta \omega}{\omega} \right| \ll 1 , \quad \Delta \omega \equiv \omega - \omega_{21} . \]

(6)

The solution to the evolution equation (1) can be presented in the form

\[ \varphi(r,t) = \sum_n c_n(t) \varphi_n(r) \exp \left( -i \frac{\hbar}{E_n} t \right) . \]

(7)

We assume that \(c_n(t)\) is slow as compared to the exponential, such that

\[ \frac{\hbar}{E_n} \left| \frac{dc_n}{dt} \right| \ll 1 . \]

(8)

There are two transition amplitudes in the system, one is the internal transition amplitude

\[ \alpha_{mn} \equiv A_s \frac{N}{\hbar} \int |\varphi_m(r)|^2 \left[ 2|\varphi_n(r)|^2 - |\varphi_m(r)|^2 \right] \, dr , \]

(9)

caused by atomic interactions, and another is the external transition amplitude

\[ \beta_{mn} \equiv \frac{1}{\hbar} \int \varphi_m^* (r) [V_1(r) - iV_2(r)] \varphi_n (r) \, dr , \]

(10)

due to the driving resonant field. Similarly to atomic resonance, in order to avoid power broadening, the transition amplitudes are to be smaller than the transition frequencies,

\[ \left| \frac{\alpha_{mn}}{\omega_{mn}} \right| \ll 1 , \quad \left| \frac{\beta_{mn}}{\omega_{mn}} \right| \ll 1 . \]

(11)
Under conditions (6), (8), and (11), the Gross-Pitaevskii equation (1) reduces to the set of equations
\[
i \frac{dc_1}{dt} = \alpha_{12} |c_2|^2 c_1 + \frac{1}{2} \beta_{12} c_2 e^{i \Delta \omega t}, \quad i \frac{dc_2}{dt} = \alpha_{21} |c_1|^2 c_2 + \frac{1}{2} \beta^*_{12} c_1 e^{-i \Delta \omega t}
\] (12)
for an effective two-mode system, where \(|c_1|^2 + |c_2|^2 = 1\). This reduction is valid provided all transition amplitudes, involved in Eq. (12), are small in the sense of condition (11), which yields the inequalities
\[
\left| \frac{\alpha_{12}}{\omega_{21}} \right| \ll 1, \quad \left| \frac{\alpha_{21}}{\omega_{21}} \right| \ll 1, \quad \left| \frac{\beta_{12}}{\omega_{21}} \right| \ll 1.
\] (13)

To analyse the above inequalities, we consider a cylindrical trap modelled by the harmonic trapping potential
\[
U(r) = \frac{m_0}{2} \left( \omega_r^2 r_x^2 + \omega_r^2 r_y^2 + \omega_z^2 r_z^2 \right),
\]
with the aspect ratio
\[
\nu \equiv \omega_z / \omega_r.
\] (14)
Introduce the dimensionless coupling parameter
\[
g \equiv 4\pi \frac{a_s}{l_r} N \left( l_r \equiv \sqrt{\frac{\hbar}{m_0 \omega_r}} \right).
\] (15)
The spectrum of nonlinear coherent modes is defined by the eigenproblem (4). The external transition amplitude (10) can always be made sufficiently small by regulating the amplitude of the resonant field. We need to calculate the internal transition amplitude (9) and to compare it with a chosen transition frequency (5). Calculations can be accomplished by means of the optimized perturbation theory [15–17] (for reviews see [18,19]). The first two inequalities (13) can be valid only outside the radius of convergence of the strong-coupling expansion. This imposes the restriction
\[
|g \nu| < \frac{[2p^2 + (q \nu)^2]^{5/4}}{14 p \sqrt{q} I_{nmj}}
\] (16)
on the coupling parameter (15). Here the integral \(I_{nmj} \sim (|\psi_{nmj}|^2, |\psi_{nmj}|^2)\), and \(p \equiv 2n + |m| + 1, q \equiv 2j + 1\) are the combinations of the radial, \(n\), azimuthal, \(m\), and axial, \(j\), quantum numbers. The restriction (16) can be rewritten as the limitation on the admissible number of particles \(N < N_0\), with the limiting number
\[
N_0 = \frac{[2p^2 + (q \nu)^2]^{5/4}}{56 \pi \nu p \sqrt{q} I_{nmj}} \left| \frac{l_r}{a_s} \right|.
\] (17)
For the ground-state level, this reduces to
\[
N_0 = \sqrt{\pi} \frac{(2 + \nu^2)^{5/4}}{14 \nu} \left| \frac{l_r}{a_s} \right|.
\] (18)
Thus, the number of particles, which can be resonantly transferred to an excited coherent mode, is limited by a number $N_0$, which depends on the quantum numbers of the coupled modes. The number $N_0$ also essentially depends on the trap shape through the aspect ratio $(14)$.

It turns out that the limiting numbers $(17)$ and $(18)$ are close to the critical values defining the maximal number of atoms with a negative scattering length, which can be condensed in a trap. Examples of atoms with attractive interactions are $^7$Li (see review [20]) and $^{85}$Rb (see Ref. [21]). Such atoms can form condensates only if their number does not exceed a critical value $[5,22–25]$. In that case, the condensate forms a long-lived metastable state, with a very slow decay, caused by quantum tunneling $[26–28]$, with the tunneling probability being negligible as compared to the probability of escaping from the trap because of depolarizing collisions $[23,29]$. But if the number of atoms is larger than critical, the condensate collapses. With a supply of atoms from an external source, say by a continuous loading from another trap $[30]$, the condensate grows again and, after surpassing the critical number, again collapses. Thus a series of growths and collapses take place, as was observed experimentally $[31,32]$ in Bose-condensed $^7$Li and described theoretically by means of the Gross-Pitaevskii equation supplemented by relaxation terms and by the related rate equations. Explosion of an attractive condensate of $^{85}$Rb atoms by manipulating the atomic interactions with an external magnetic field near a Feshbach resonance $[36]$ has also been observed $[21,37]$.

To increase the number of atoms in a condensate with attractive interactions, it was suggested $[38]$ to drive a quadrupole collective excitation. We may notice that transferring atoms to an excited nonlinear coherent mode may also stabilize such a condensate. This follows from the form $(17)$, which shows that the limiting number $N_0$ increases for higher excited modes as

$$N_0 \sim (2n + |m| + 1)^{3/2}, \quad N_0 \sim (2j + 1)^2.$$  

To estimate the number of atoms for highly excited modes, one could also use an optimized quasiclassical approximation $[39]$. As Eq. $(18)$ demonstrates, the limiting number $N_0$ is larger for essentially anisotropic traps, that is, for cigar-shape $(\nu \ll 1)$ and disk-shape $(\nu \gg 1)$ traps.

The temporal behaviour of the system under coherent resonance is described by the evolution equations $(12)$. Numerical solution of these equations, for the varying amplitude of the resonant field, displays dynamical critical phenomena $[6–8]$. Here we present a general picture of these phenomena and elucidate their origin.

Let us present the population amplitudes in the form

$$c_1 = \sqrt{1 - s \over 2} \exp \left\{ i \left( \pi_1 + {\Delta \omega \over 2} t \right) \right\}, \quad c_2 = \sqrt{1 + s \over 2} \exp \left\{ i \left( \pi_2 - {\Delta \omega \over 2} t \right) \right\}, \quad (19)$$

where $\pi_1 = \pi_1(t)$ and $\pi_2 = \pi_2(t)$ are the phases. The variable

$$s \equiv |c_2|^2 - |c_1|^2$$

is the population difference. Introduce the notation for the average internal transition amplitude

$$\alpha \equiv {1 \over 2} (\alpha_{12} + \alpha_{21}) \quad (21)$$
and let us take into account that the external transition amplitude (10) is, in general, complex valued,
\[ \beta_{12} = \beta e^{i\gamma}, \quad \beta \equiv |\beta_{12}|. \] (22)

Also, define the effective detuning
\[ \delta \equiv \Delta \omega + \frac{1}{2} (\alpha_{12} - \alpha_{21}). \] (23)

And introduce the phase difference
\[ x \equiv \pi_2 - \pi_1 + \gamma. \] (24)

Substituting Eqs. (19) into the evolution equations (12), we come to the set of equations for the population difference (20) and phase difference (24),
\[ \frac{ds}{dt} = -\beta \sqrt{1 - s^2} \sin x, \quad \frac{dx}{dt} = \alpha s + \frac{\beta s}{\sqrt{1 - s^2}} \cos x + \delta. \] (25)

Numerical solution of Eqs. (25) demonstrates the existence of dynamical critical phenomena, when a tiny variation of parameters results in sharp changes of temporal behaviour of \( s(t) \) and \( x(t) \). We illustrate this in the following figures, where time is measured in units of \( \alpha^{-1} \). There are two independent dimensionless parameters
\[ b \equiv \frac{\beta}{\alpha}, \quad \varepsilon \equiv \frac{\delta}{\alpha}, \] (26)

which can be varied. In what follows, we vary \( b \) in the interval \([-1, 1]\) and keep \( |\varepsilon| \ll 1 \). And there are also two initial conditions \( s_0 = s(0) \) and \( x_0 = x(0) \).

In Fig. 1, we set \( s_0 = -1, x_0 = 0, \) and \( \varepsilon = 0 \), while varying the parameter \( b \) corresponding to the amplitude of the pumping resonant field. In the range \( 0 < b < 0.5 \), the population difference oscillates between \(-1\) and zero and the phase difference monotonically decreases, as is shown in Fig. 1a and 1b. The critical point \( b_c = 0.5 \) separates the regions of two qualitatively different temporal behaviours. After \( b \) surpasses \( b_c \), the population difference starts oscillating between \(-1\) and \(+1\), while the phase difference becomes a periodic function, as is demonstrated in Figs. 1c to 1e.

As follows from the definition of the internal transition amplitude (9), its value may become negative, either for strongly energetically separated modes or for atoms with attractive interactions, when \( A_s < 0 \). In such a case, the parameter (21) is negative, \( \alpha < 0 \), and so is the parameter \( b < 0 \) from Eq. (26). The variation of \( b \) in the negative region is also accompanied by dynamic critical phenomena, as for positive \( b \), though the behaviour of the phase difference is not the same as for \( b > 0 \). This is illustrated by Fig. 2, where we set the same initial conditions \( s_0 = -1 \) and \( x_0 = 0 \), and zero detuning. For \( b \) in the interval \(-0.5 < b < 0 \), the population difference is in the range \(-1 \leq s \leq 0 \), as is shown in Figs. 2a and 2b. The critical point now is \( b_c = -0.5 \). After crossing the critical point, when \( b < b_c \), the amplitude of the population difference increases by a jump, so that \( s(t) \) oscillates now in the diapason \(-1 \leq s \leq 1 \). The phase difference is an oscillating function for both \( b > b_c \).
and $b < b_c$, but its shape changes drastically when crossing $b_c$, as can be seen in Figs. 2c and 2d.

In this way, there are two critical lines for each given pair of initial conditions $s_0$ and $x_0$. Changing initial conditions changes the location of the critical line on the manifold of the parameters $b$ and $\varepsilon$. Thus, in Fig. 3, we show the course of events for the initial conditions $s_0 = -0.8$ and $x_0 = 0$, with the detuning $\varepsilon = 0$ and for the parameter $b$ varying in the whole range $-1 \leq b \leq 1$. Under the fixed detuning $\varepsilon = 0$ and given initial conditions, there are two critical points $b_{c1} = -0.8$ and $b_{c2} = 0.2$.

For a nonzero detuning, the location of the critical line changes, but the overall picture remains the same. A finite detuning also makes the shape of the function $x(t)$ a little asymmetric, as is shown in Fig. 4.

To understand the origin of these critical phenomena, we studied the phase portrait on the plane $\{s, x\}$ for different parameters $b$ and $\varepsilon$. Examples are demonstrated in Fig. 5. We also accomplished the stability analysis of fixed points. This investigation clarify the origin of the dynamic critical phenomena. The latter are related to the existence of the saddle separatrix defined by the equation

$$\frac{s^2}{2} - b\sqrt{1 - s^2} \cos x + \varepsilon s = b.$$  \hfill (27)

The separatrices divide the phase plane $\{s, x\}$ into the regions with different dynamics. As is evident from Eq. (27), changing any of the parameters $b$ or $\varepsilon$ shifts the separatrices. When a separatrix crosses an initial point $\{s_0, x_0\}$, the system trajectory passes to another region of the phase plane, as a result of which the system dynamics changes qualitatively. The separatrix crossing of an initial point corresponds to the critical line

$$\frac{s_0^2}{2} - b\sqrt{1 - s_0^2} \cos x_0 + \varepsilon s_0 = |b|.$$  \hfill (28)

on the manifold of the parameters $b$ and $\varepsilon$. It is this separatrix crossing effect that causes the appearance of the dynamic critical phenomena. It can be shown [7,8] that a time-averaged system displays critical phenomena, typical of phase transitions in stationary statistical systems, on the same critical line.

There exist several more interesting effects occurring in the process of coherent resonance. Not to overload the present communication, we mention these effects only briefly. A more detailed consideration will be done in separate publications.

### 2.1 Interference effects

Coherent resonance couples two selected nonlinear modes, leaving other modes unpopulated. Therefore, the coherent field (7) is, effectively, a sum

$$\varphi(r, t) = \varphi_1(r, t) + \varphi_2(r, t)$$  \hfill (29)

of two terms

$$\varphi_j(r, t) = c_j(t)\varphi_j(r) \exp\left(-\frac{i}{\hbar} E_j t\right) \quad (j = 1, 2).$$
Using this, one can define the interference pattern

$$\rho_{\text{int}}(r, t) \equiv \rho(r, t) - \rho_1(r, t) - \rho_2(r, t) \, , \tag{30}$$

in which

$$\rho(r, t) = |\varphi(r, t)|^2 \, , \quad \rho_i(r, t) = |\varphi_i(r, t)|^2 \, ,$$

and the interference current

$$\mathbf{j}_{\text{int}}(r, t) \equiv \mathbf{j}(r, t) - \mathbf{j}_1(r, t) - \mathbf{j}_2(r, t) \, , \tag{31}$$

where

$$\mathbf{j}(r, t) = \frac{\hbar}{m_0} \text{Im} \varphi^*(r, t) \nabla \varphi(r, t) \, , \quad \mathbf{j}_i(r, t) = \frac{\hbar}{m_0} \text{Im} \varphi_i^*(r, t) \nabla \varphi_i(r, t) \, .$$

These quantities show that the total density of atoms in a trap is not simply a sum of the partial mode densities but it includes also interference fringes and that, because of the different mode topology, there arises a local interference current. The interference effects can be observed experimentally.

### 2.2 Atomic squeezing

The evolution equations for the effective two-mode system, obtained by means of the coherent resonance, can be interpreted as the equations for the statistical averages of quasispin operators

$$S_\alpha \equiv \sum_{i=1}^N S_i^\alpha \, , \quad S_\pm \equiv S_x \pm iS_y \, ,$$

where $\alpha = x, y, z$ and $S_i^\alpha$ is an $\alpha$-component of the spin-1/2 operator. For instance, the population difference (20) can be written as the average

$$s = \frac{2}{N} < S_z > \, .$$

The squeezing factor may be defined [40] as

$$Q_z \equiv \frac{2\Delta^2(S_z)}{\sqrt{< S_z >^2 + < S_y >^2}} = \frac{2\Delta^2(S_z)}{| < S_\pm > |} \, , \tag{32}$$

where $\Delta^2(S_z) \equiv < S_z^2 > - < S_z >^2$. One says that $S_z$ is squeezed with respect to $S_\pm$ if $Q_z < 1$. Calculating the factor (32), we find

$$Q_z = \sqrt{1 - s^2} \, . \tag{33}$$

Since $s^2 \leq 1$, one has $Q_z \leq 1$. The maximal squeezing is achieved when one of the modes is completely populated, i.e. $s = \pm 1$. Squeezed $S_z$ means that the population difference can be measured with a better accuracy than the phase difference.
### 2.3 Multiparticle entanglement

The system of interacting atoms possessing internal states may form multiparticle entangled states \([41,42]\). In our case, atoms have not internal but collective coherent modes. This makes the principal difference of the resonant Bose condensate having collective nonlinear modes from atoms with internal single-particle states. But the resonant Bose condensate also forms a multiparticle entangled state. This follows from the Schrödinger representation for the wave function of \(N\) atoms in the resonant condensate,

\[
\Psi_N(r_1, r_2, \ldots, r_N, t) = c_1(t) \prod_{i=1}^{N} \varphi_1(r_i) + c_2(t) \prod_{i=1}^{N} \varphi_2(r_i),
\]

where \(\varphi_j(r)\) is a coherent \(j\)-mode, with \(j = 1, 2\). The function (34) cannot be factorized into a product of single-particle functions, provided that \(c_1 \neq 0\) and \(c_2 \neq 0\). The maximal entanglement is reached for \(|c_1| = |c_2| = 1/\sqrt{2}\), that is, when \(s = 0\). Since the coefficients \(c_1(t)\) and \(c_2(t)\) are defined by the evolution equations (12) or (25), from where

\[
|c_1|^2 = \frac{1-s}{2}, \quad |c_2|^2 = \frac{1+s}{2},
\]

their values can be manipulated by applying the resonant field and switching it off at the appropriate moment. In this way, it is possible to create any degree of entanglement and then one can disentangle the created entangled state. The resonant field, thus, acts as a quantum disentanglement eraser \([43]\).

### 3 Conclusion

In trapped Bose condensates, the effect of coherent resonance is feasible. This is achieved by applying an alternating field with the frequency tuned to the transition frequency between two nonlinear modes. To preserve the resonance condition requires that the number of atoms be limited by a maximal value. Estimates show that this value is around \(10^3\) for a typical spherical trap and for alkali atoms. The limiting number is higher for cigar-shape and disk-shape traps, and can be about \(10^5\) atoms. This number also increases for higher excited nonlinear modes. Temporal behaviour of fractional populations exhibits dynamic critical phenomena occurring on a critical line in the parametric manifold. The origin of these critical phenomena is the saddle separatrix crossing by the starting point of a trajectory. Actually, there exists a whole bunch of critical lines, each of which corresponds to a different starting point.

As a result of coherent resonance in a system of Bose-condensed atoms, a new type of matter is formed, which can be called resonant Bose condensate. This is analogous to a resonant atom, with the principal difference that the resonant condensate is a coherent multiparticle system. Collective nature of the latter causes the appearance of interference patterns, interference current, atomic squeezing, and of multiparticle coherent entanglement.

We have considered here a single-component Bose gas. Possible extensions of the theory could be to binary mixtures of Bose condensates and, generally, to multicomponent condensates composed of different atomic species. Another possible generalization could be to
the systems of cold atoms in optical lattices [44] and to radiating atoms interacting with electromagnetic field [45]. We think that the effect of coherent resonance can find numerous applications, such as creating selected modes of atoms lasers, information processing, and quantum computing.

Acknowledgement

The work was accomplished in the Research Center for Optics and Photonics, University of São Paulo, São Carlos. Financial support from the São Paulo State Research Foundation is appreciated.
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**Figure Captions**

**Fig. 1.** The population difference $s(t)$ (dashed line) and phase difference $x(t)$ (solid line) as functions of time, measured in units of $\alpha^{-1}$, for the zero detuning $\varepsilon = 0$, initial conditions $s_0 = -1$, $x_0 = 0$, and varying amplitude of the resonant field: (a) $b = 0.470$; (b) $b = 0.495$; (c) $b = 0.501$; (d) $b = 0.7$; (e) $b = 1$.

**Fig. 2.** Temporal behaviour of the population difference $s(t)$ (dashed line) and phase difference $x(t)$ (solid line), under the same conditions as for Fig. 1, but for the negative parameter $b < 0$ taking the values: (a) $b = -0.40$; (b) $b = -0.49$; (c) $b = -0.6$; (d) $b = -1$.

**Fig. 3.** The population difference $s(t)$ (dashed line) and phase difference $x(t)$ (solid line) as functions of dimensionless time for $\varepsilon = 0$ and the initial conditions $s_0 = -0.8$, $x_0 = 0$, with varying the parameter $b$ as: (a) $b = -1$; (b) $b = -0.85$; (c) $b = -0.8001$ (slightly below the critical point $b_{c1}$); (d) $b = -0.7999$ (slightly above the critical point $b_{c1}$); (e) $b = -0.5$; (f) $b = 0.18$; (g) $b = 0.1999$ (slightly below the critical point $b_{c2}$); (h) $b = 0.2001$ (slightly above the critical point $b_{c2}$); (i) $b = 0.5$; (j) $b = 1$.

**Fig. 4.** Temporal behaviour of $s(t)$ (dashed line) and $x(t)$ (solid line) for the initial conditions $s_0 = -1$, $x_0 = 0$, fixed $b = -1$, and a nonzero detuning: (a) $\varepsilon = -0.05$; (b) $\varepsilon = -0.1$.

**Fig. 5.** Phase portrait on the plane $\{s, x\}$ for a finite detuning $\varepsilon = 0.1$ and different values of the pumping-field amplitude: (a) $b = 0.51$; (b) $b = 0.8$. 