Abstract

In this paper we analyse the stability of black hole Cauchy horizons arising in a class of 2d dilaton gravity models.

It is shown that due to the characteristic asymptotic Rindler form of the metric of these models, time dependent gravitational perturbations generated in the external region do not necessarily blow up when propagated along the Cauchy horizon. There exists, in fact, a region of nonzero measure in the space of the parameters characterizing the solutions such that both instability and mass inflation are avoided. This is a new result concerning asymptotically flat space-times, not shared by the well-known solutions of General Relativity.

Despite this fact, however, quantum backreaction seems to produce a scalar curvature singularity there.
1. Introduction

In the recent years there has been a long debate on the question of whether Einstein field equations alone are able to completely determine the evolution of the gravitational field, once initial data are given on a Cauchy hypersurface $\Sigma$.

There exist, in fact, spacetimes possessing null surfaces called Cauchy horizons - boundaries beyond which the future evolution of physical fields is no longer uniquely determinable from the data prescribed on $\Sigma$. Extra boundary conditions are then needed, sometimes, at spacetime singularities and of course these are completely arbitrary in the context of the classical theory. It seems, therefore, that predictability is lost at the Cauchy horizons in Einstein gravity (see for example Ref. [1]).

We will divide the geometries of our interest endowed with Cauchy horizons in two classes depending on their asymptotics; they are either

i) asymptotically flat

or

ii) non asymptotically flat.

The exact solutions of General Relativity included in i) are all those of Kerr-Newmann type, with nonvanishing electric charge $Q$ and/or angular momentum. A good representative of this class, that keeps all the essential features we wish to discuss, is the Reissner-Nordström solution, representing the spacetime of a spherically symmetric black hole of mass $m_0$ carrying a conserved abelian charge $Q$.

The simplest way to have a non flat asymptotic region is to add in the Einstein action a cosmological constant $\Lambda$. As $\Lambda > 0$, this results in the well known De Sitter solution, that is commonly used in cosmology to describe the inflationary era of our Universe (see for example Ref. [2]). We will take as a representative of class ii) the Reissner-Nordström-De Sitter geometry, which incorporates the features of a charged black hole immersed in an expanding universe.

The question of the classical predictability of the field equations can also be reformulated in this way: are Cauchy horizons traversible in physically realistic spacetimes?

It was noted long ago by Penrose [3] that the Cauchy horizon of the Reissner-Nordström geometry (for $m_0 > |Q|$) is an infinite blueshift surface. This means that a small perturbation in the external region is seen infinitely (exponentially) blueshifted by free falling observers who cross this surface.

For macroscopic black holes originated from the gravitational collapse of massive stars, these perturbations are caused by the radiative tail determined by Price [4]. Therefore, this tail is likely to modify the geometry of the spacetime close to the Cauchy horizon. Poisson, Israel and Ori (see [5]) have constructed a simple model that mimics the realistic scenario of the gravitational collapse and shown that the local mass function and the scalar curvature $R$ diverge at the Cauchy horizon: this is the mass inflation phenomenon. In this
way they save the predictive power of the theory: a singularity forms at the Cauchy horizon and any extension of the geometry beyond it is meaningless!

The peculiarity of the Reissner-Nordström-De Sitter geometry is the presence, besides the black hole horizons, of a cosmological horizon, which divides the black hole region from the cosmological one. Perturbations, regular at the cosmological horizon, have been constructed for this spacetime and simple calculations show that there’s a region of nonzero measure in parameter space \((m_0, Q, \Lambda)\) for which the Cauchy horizon is stable and the mass inflation doesn’t occur ([6] and [7]).

In this paper we check the stability issue for Cauchy horizons in the context of a one-parameter class of simple 2d dilaton-gravity models introduced in Ref. [8]. They are described by the action

\[
S = S_n + S_{EM},
\]

and the explicit form of \(S_{EM}\) will be shown in section 3. \(R\) is the scalar curvature associated to the two-dimensional metric tensor \(g_{ab}\), \(\phi\) is the dilaton field. In the case \(n = 1\), eq. (1.1) is the usual CGHS action [9].

The motivation for studying such models is essentially that they are exactly solvable at the semiclassical level and contain vacuum as well as asymptotically flat solutions.

In Ref. [10] it has been considered the global causal structure of the corresponding static black hole solutions. It is shown that for \(n < 1\) they are asymptotically flat, and therefore of the type i). However, they are different from the black holes of General Relativity for one main reason: the spacetime is asymptotically flat in Rindler coordinates and not, as usual, in Minkowski ones.

When \(n > 1\) the region where the spacetime becomes flat is instead an horizon, an acceleration horizon, where the line element takes again the Rindler form. The metric can be analytically extended across this horizon and a true asymptotic region doesn’t seem to exist.

The analysis of the stability of the Cauchy horizons in these spacetimes leads to the following results:

- for \(n > 1\) the behaviour of the Cauchy horizon doesn’t differ significantly from the Reissner-Nordström-De Sitter case;
- in the case \(n < 1\), surprisingly, there remains a region of nonzero measure in parameter space \((m_0, Q, \lambda, n)\) for which the Cauchy horizon is stable and the mass inflation doesn’t take place.

There is a simple physical reason behind this result: at the Cauchy horizon the (infinite) blueshift factor is no more exponential but power-law in the external inertial time coordinate. We will clarify on this point in section 4.

However, quantum backreaction seems to alter these conclusions. Such an analysis has already been carried out in the RST model in [11] and similar results are obtained here, suggesting that a scalar curvature singularity always forms at the Cauchy horizon, thus forbidding any further extension of the geometry above it.
2. The Cauchy horizons in General Relativity

2.1. The Reissner-Nordström spacetime

Let us consider the Reissner-Nordström spacetime. The metric reads

\[ ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega^2, \]  

(2.1)

where

\[ f = 1 - \frac{2m_0}{r} + \frac{Q^2}{r^2} \]  

(2.2)

and \( d\Omega^2 \) is the metric on the unit two-sphere.

The zeros of \( f \), namely \( r_\pm = m_0 \pm \sqrt{m_0^2 - Q^2} \) for \( m_0 > |Q| \), represent the Killing horizons. \( r_+ \) is the event horizon and \( r_- \) the inner, and Cauchy, horizon.

We define, for future use, the surface gravity at both horizons

\[ k_\pm = \frac{1}{2} \left| \partial_r f \right|_{r_\pm} = \frac{\sqrt{m^2 - Q^2}}{r_\pm^2}. \]  

(2.3)

\( r = 0 \) is the location of the singularity and \( r = \infty \) defines the flat asymptotic region, where the spacetime becomes minkowskian. The Penrose diagram of this spacetime is well known and is represented in Fig. 1. The null directions are represented by the coordinates \( u \) and \( v \), defined by \( dv = dt + \frac{d\sigma}{f} \), \( du = dt - \frac{d\sigma}{f} \).

We follow Ref. [4] and perturb this geometry introducing an ingoing flow of radiation from the exterior region, which represents the radiative tail of the gravitational collapse. It is described by a massless scalar field \( f(v) \) with energy-momentum tensor

\[ T_{vv} = \frac{L(v)}{4\pi r^2}, \quad \text{where} \quad L(v) \sim v^{-p}, \]  

(2.4)

where \( L(v) \) is a luminosity function and \( p \geq 12 \). We note that the null coordinate \( v \) is the natural advanced time coordinate used by inertial observers at infinity and its value is \( \infty \) both on future null infinity and on the Cauchy horizon.

An exact solution of the Einstein field equations taking into account the source given in eq. (2.4) is known and is called the Vaidya spacetime. The line element is written in the light-cone gauge

\[ ds^2 = -f(v, r)dv^2 + 2dvdr + r^2d\Omega^2, \]  

(2.5)

where

\[ f = 1 - \frac{2m(v)}{r} + \frac{Q^2}{r^2} \]  

(2.6)
and the mass function \( m(v) \) is given by the equation

\[
\frac{dm}{dv} = 4\pi r^2 T_{vv} = L(v) \sim v^{-p}.
\] (2.7)

The Vaidya spacetime so defined is regular at the Cauchy horizon. However, let us consider a geodesic observer crossing the surface \( v = \infty \) at the Cauchy horizon. From the geodesic equations it is easy to see that close to the horizon its velocity is such that

\[
\dot{r} \simeq -c, \quad \dot{v} \simeq ce^{k_-v},
\] (2.8)

where \( c \) is a positive constant and \( k_- \) is given in eq. (2.3). Therefore he measures an energy influx given by

\[
\rho_{obs} \sim T_{vv} \dot{v}^2 \sim v^{-p} e^{2k_-v}.
\] (2.9)

This quantity clearly diverges as \( v = \infty \) due to the presence of the exponential blueshift factor \( e^{2k_-v} \).\(^1\)

It is reasonable to expect that in even more realistic scenarios of gravitational collapse a singularity will appear at the Cauchy horizon. The key observation has been made by Poisson and Israel \([5]\): a star will in general emit also outgoing gravitational waves during its collapsing phase. It is striking that no matter how \( T_{uu} \) is, as long as it is nonvanishing, the combination of both fluxes will cause the mass function to diverge at the Cauchy horizon as

\[
m(v) \sim v^{-p} e^{k_-v}.
\] (2.10)

We note that the divergence in eq. (2.10) is in some sense ‘milder’ than in eq. (2.9). This behaviour is called the mass inflation phenomenon. It is responsible for the appearance of a scalar curvature singularity at the Cauchy horizon (\( R \) has the same divergence as in eq. (2.10)) and therefore predictability of the Einstein field equations is saved in this model.

A different picture arises when one considers perturbations of the extremal Reissner-Nordström black holes \([12]\). These are defined by the condition \( m_0 = |Q| \) and, from eqs. (2.2) and (2.3), \( f = (1 - \frac{m_0}{r})^2, \ r_+ = r_- = m_0 \) and \( k_\pm = 0 \).

An appropriate perturbation for this spacetime is of the type in eq. (2.4) (with \( p \) numerically smaller than for the non-extreme case, but still \( p \geq 8 \)). The geodesic equations can be easily solved in the vicinity of the degenerate horizon and give

\[
\dot{v} \sim v^2, \quad \dot{r} \sim -c.
\] (2.11)

\(^1\) This is the so called proper-time compression effect: the free falling observer sees in a finite amount of proper time an infinite number of ingoing waves accumulating at the Cauchy horizon.
The energy density measured by the free falling observer is then

\[ \rho_{\text{obs}} \sim v^{-p+4}, \]  

vanishing in the limit \( v \rightarrow \infty \), i.e. at the Cauchy horizon.

In this case the Cauchy horizon is stable and the backreaction of the influx eq. (2.12) together with a generic outflux does not affect the regularity of the geometry there.

To summarize, in the space \((m_0, Q)\) of the parameters characterizing the Reissner-Nordstrøm geometries the perturbation analysis has shown that the Cauchy horizons are generically unstable, except for the region of zero measure \( m_0 = |Q| \).

It is worthwhile to stress that the unperturbed Reissner-Nordstrøm are, by virtue of Birkhoff’s theorem, the only static, spherically symmetric and asymptotically flat solutions of Einstein-Maxwell equations.

### 2.2. The Reissner-Nordstrøm-De Sitter spacetime

The line element of the Reissner-Nordstrøm-De Sitter geometry can be written as in eq. (2.1), where now

\[ f = 1 - \frac{2m_0}{r} + \frac{Q^2}{r^2} - \frac{\Lambda}{3} r^2 \]  

and \( \Lambda \) is the cosmological constant.

The equation \( f = 0 \), which defines the horizons, is a quartic. We take three of its real roots to be physical and we call them \( r_i, r_h \) and \( r_c \), where \( r_i < r_h < r_c \). \( r = r_i \) denotes the location of the inner, and Cauchy, horizon; \( r_h \) is the outer horizon and \( r_c \) is the cosmological horizon. The surface gravity at the horizons is defined, as usual, by

\[ k = \frac{1}{2} |\partial_r f| \text{ evaluated at } r = r_i, \ r = r_h \text{ and } r = r_c. \]

Null directions are again \( u \) and \( v \), where

\[ dv = dt + \frac{d\sigma}{f} \quad \text{and} \quad du = dt - \frac{d\sigma}{f}. \]

The causal structure of this spacetime is represented in Fig. 2. As opposed to the diagram of Fig. 1, now the causal past of the Cauchy horizon does not contain all of the external region, its boundary being represented by the cosmological horizon.

To construct a perturbation appropriate for this geometry we have to take into account the fact that the coordinate \( v \) is not inertial at \( r = r_c \). A finite perturbation in the local inertial frame there once transformed to \( v \) coordinate reads

\[ T_{vv} = \frac{Ke^{-2k_c v}}{4\pi r^2}, \]  

where \( K \) is a constant and \( k_c \) is the surface gravity at the cosmological horizon.

An exact solution of the Einstein field equations taking into account the contribution of eq. (2.14) is the Vaidya-Reissner-Nordstrøm-De Sitter spacetime. Its line element is of the form in eq. (2.5), \( f(r, v) \) is given by an equation like (2.13) with \( m = m(v) \).
Despite the regularity of the geometry at the Cauchy horizon, a free falling observer in its vicinity will measure an influx of energy given by
\[ \rho_{\text{obs}} \sim e^{2(k_i - k_c)v}, \]  

(2.15)

where \( k_i \) is the surface gravity associated with the Cauchy horizon. It diverges provided \( k_i > k_c \).

We can proceed as in the case of the Reissner-Nordström spacetime, i.e. we couple the influx of matter with generic outflowing null radiation crossing the Cauchy horizon. The behaviour of the mass function in the limit \( v = \infty \) has been computed in Ref. \[7\] and the results show that
\[ m(v) \sim e^{(k_i - 2k_c)v}. \]  

(2.16)

From these results we can identify three different regions in parameter space \((m_0, Q, \Lambda)\):

i) \( k_i \leq k_c \), the Cauchy horizon is completely stable;

ii) \( k_c < k_i \leq 2k_c \), divergent influx of radiation at \( r_i \), but no mass inflation;

iii) \( k_i > 2k_c \), both divergent influx and mass inflation.

Therefore, this simple model of crossflowing streams of null matter shows that in general when we have non asymptotically flat spacetimes the traversibility of the Cauchy horizon can be physically possible. To analytically extend the metric beyond it we need to impose boundary conditions on the timelike singularities represented in Fig. 2. In particular, the diagram of Fig. 2 was obtained from the assumption that ‘nothing escapes from the singularity’.

We note finally that in obtaining the results in eqs. (2.15) and (2.16) the only features we used of the spacetime in Fig. 2 are the surface gravities at the Cauchy and cosmological horizons. We did not take into any account the structure of the geometry beyond the cosmological horizon.

### 3. Two-dimensional black holes in accelerated frames

In this section we simply recall the form of the classical black hole solutions of the models introduced in Ref. \[8\] and consider briefly, for the purposes of the present paper, their causal structure.

We consider the action \( S_{cl} = S_n + S_M + S_{EM} \), \( S_n \) given in eq. (1.1),
\[ S_M = -\frac{1}{4\pi} \sum_{i=1}^{N} \int d^2x \sqrt{-g} (\nabla f_i)^2 \]  

(3.1)

\[2\] Although the mass function at the Cauchy horizon is finite, in \[7\] it is shown that the 4d metric has the Kretschmann invariant \( R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} \) divergent there.
and (see Ref. [13])

\[ S_{EM} = \frac{1}{2\pi} \int d^2 x \sqrt{-g} e^{\frac{2n-4}{n}\phi} (-2F_{\mu\nu}F^{\mu\nu}), \]  
(3.2)

where \( F_{\mu\nu} \) is the e.m. field tensor.

The equations of motion derived from varying \( S_{cl} \) with respect to \( \phi \) and \( g_{\mu\nu} \) are

\[ \frac{R}{n} - \frac{4}{n^2}(\nabla\phi)^2 + \frac{4}{n} \nabla^2 \phi + 4\lambda^2 e^{\frac{2n}{n-2}\phi} - \frac{2n-4}{n} e^{\frac{2n-2}{n}\phi} F_{\mu\nu}F^{\mu\nu} = 0, \]  
(3.3)

\[ g_{\mu\nu}[\frac{4}{n}(-\frac{1}{2} + \frac{1}{n})(\nabla\phi)^2 - \frac{2}{n} \nabla^2 \phi + e^{\frac{2n}{n-2}\phi} F_{\mu\nu}F^{\mu\nu} - 2\lambda^2 e^{\frac{2n-2}{n}\phi}] + \frac{4}{n}(1 - \frac{1}{n}) \partial_\mu \phi \partial_\nu \phi - 4 e^{\frac{2n-2}{n}\phi} F_{\mu\alpha}F^{\alpha}_{\nu} = 0. \]  
(3.4)

The equations of motion of \( F_{\mu\nu} \) are easily derived from \( S_{EM} \):

\[ \nabla_\nu(e^{\frac{2n-4}{n}\phi} F^{\mu\nu}) = 0. \]  
(3.5)

The e.m. field tensor is totally antisymmetric and so can be written as \( F_{\mu\nu} = Fe_{\mu\nu} \), where \( e_{\mu\nu} = e_{[\mu\nu]} \) and \( e_{01} = \sqrt{-g} \). Inserting this into (3.4) we get the solution \( F_{\mu\nu} = Qe^{\frac{2n}{n-2}\phi} e_{\mu\nu} \), where \( Q \) is a constant representing the charge on the black hole.

The static black hole solutions can be easily derived in the ‘Schwarzschild-Rindler’ gauge (for the details see Ref. [13]) \((\sigma, t)\), where

\[ ds^2 = e^{2(1-n)\lambda\sigma} [-dt^2 + \frac{1}{f}d\sigma^2], \quad \phi = -n\lambda\sigma, \]  
(3.6)

and \( f \) is defined by

\[ f = 1 - \frac{2m_0}{\lambda} e^{\frac{2n}{n-2}\phi} + \frac{Q^2}{\lambda^2} e^{\frac{4}{n}\phi}. \]  
(3.7)

Consider first the case \( n < 1 \). For \( m_0 > |Q| \) we have two Killing horizons located at \( \sigma = \sigma_{\pm} \), where

\[ e^{2\lambda\sigma_{\pm}} = e^{-\frac{2}{n}\phi_{\pm}} = (\frac{m_0 \pm \sqrt{m_0^2 - Q^2}}{\lambda}). \]  
(3.8)

\( \sigma_{\pm} \) is the location of the event horizon and \( \sigma_{-} \) is the inner, and Cauchy, horizon. The causal diagram is the same as that in Fig. 1. The asymptotic region is defined by \( \sigma = \infty \) and note that there the metric is not Minkowskian, but Rindlerian, due to the presence of the conformal factor \( e^{2(1-n)\lambda\sigma} \).

We turn now to the case \( n > 1 \). Concerning the global causal structure of these spacetimes, we still have Cauchy and black hole horizons at \( \sigma = \sigma_{\pm} \) as in eq. (3.8). However, the region \( \sigma = \infty \) doesn’t represent anymore the asymptotic region, but is the location of another horizon, the acceleration horizon, where the line element takes the Rindler form. The metric can be analytically extended across this horizon and the results
of this analysis are presented in Ref. [10], where it is shown that a true asymptotic region doesn’t exist in these spacetimes.

However, as noted at the end of section 2.2, all we need to discuss the classical stability and the mass inflation phenomenon at the Cauchy horizon is the causal structure of the spacetime in its causal past. Therefore for our purposes the geometries with \( n < 1 \) have the same features of standard Reissner-Nordström spacetime and those for \( n > 1 \) are similar to the Reissner-Nordström-De Sitter geometry.

A useful quantity is the local surface gravity \( k \), defined by (see for example [14])

\[
\kappa = \frac{1}{2} \left| \frac{\partial_{\sigma} g_{tt}}{\sqrt{-g_{\sigma\sigma} g_{tt}}} \right| = \left| \lambda (1 - n) f + \frac{1}{2} f_{,\sigma} \right|. \tag{3.9}
\]

One finds that at \( \sigma_{\pm} \) it is

\[
\kappa_{\pm} = \frac{1}{2} \left| f_{,\sigma_{\pm}} \right| = \lambda e^{-2\lambda\sigma_{\pm}} \left( e^{2\lambda\sigma_{+}} - e^{2\lambda\sigma_{-}} \right). \tag{3.10}
\]

Note that, provided \( m_0 > |Q| \), it is always \( k_- > k_+ \). In the case \( n > 1 \) at the acceleration horizon

\[
\kappa_{ah} \equiv \kappa (\sigma = \infty) = \lambda (n - 1). \tag{3.11}
\]

Since in the following we will be interested also in solutions arising from purely incoming massless matter we write down the metric and the dilaton in the chiral gauge

\[
ds^2 = e^{2(1-n)\lambda\sigma} (-f dv^2 + 2vdv d\sigma), \quad \phi = -n\lambda\sigma, \tag{3.12}
\]

where

\[
f = 1 - \frac{2m(v)}{\lambda} e^{\frac{2}{4}\phi} + \frac{Q^2}{\lambda^2} e^{\frac{4}{4}\phi}. \tag{3.13}
\]

Allowing for purely ingoing \( f_i \)-wave \( f_i = f_i(v) \) the \( vv \) component of eq. (3.4) gives the additional condition

\[
\frac{dm}{dv} = \frac{1}{4} \sum (\partial_{\nu} f_i)^2. \tag{3.14}
\]

For \( f_i \neq 0 \) the solution resembles, apart from the conformal factor \( e^{2(1-n)\lambda\sigma} \), the Vaidya solution of General Relativity. In general \( m \) is a function of \( v \) and we will refer to the scalar \( m \) in

\[
(\nabla^2 \phi)^2 = n^2 \lambda^2 e^{-2(1-n)\lambda\sigma} \left( 1 - \frac{2m}{\lambda} e^{-2\lambda\sigma} + \frac{Q^2}{\lambda^2} e^{-4\lambda\sigma} \right) \tag{3.15}
\]

as the mass function of the black hole.
4. Cauchy horizon stability and mass inflation in the case $n < 1$ (asymptotically flat space-times)

The interesting and crucial thing to note about the solution in eq. (3.6) is that it is static as referred to the asymptotic Rindler observers. An inertial observer at infinity will, in fact, detect a time dependent gravitational field. The inertial frame there is defined by the coordinates

$$\lambda y^+ = \frac{e^{\lambda(1-n)v}}{(1-n)},$$

$$-\lambda y^- = \frac{e^{-\lambda(1-n)u}}{(1-n)}.$$  \hspace{1cm} (4.1)

Neglecting the term proportional to $Q$, which is subleading at infinity as compared to that proportional to $m_0$, in this frame the metric takes the form

$$ds^2 = -f^n(y^+, y^-)dy^+dy^- = -\frac{dy^+dy^-}{\left[1 + \frac{2m_0}{\lambda}(1-n)^2y^+y^-\right]^n}. \hspace{1cm} (4.3)$$

Note that in the external region $y^+ \in [0, +\infty[$ and $y^- \in ]-\infty, 0]$. In the following we will use the ‘natural’ frame of the accelerated observers, comoving with the black hole.

4.1. Stability of the Cauchy horizon

In order to analyse the stability of the Cauchy horizon, we must consider gravitational perturbations generated in the external (asymptotically flat) region. Following the discussion of section I, we model them in the form of ingoing null matter flowing into the black hole. Such perturbations decay with an inverse power law in the advanced time of the asymptotic inertial observer and are described by the energy-momentum tensor

$$T_{++} = \frac{1}{2}(\frac{df}{dy^+})^2 = 2\gamma(\lambda y^+)^{-p}, \hspace{1cm} (4.4)$$

$\gamma$ being a constant and $p \geq 12$. Transforming to $v$ coordinate we get

$$T_{vv} = 2\gamma(1-n)^pe^{\lambda(1-n)(-p+2)v}. \hspace{1cm} (4.5)$$

Provided that the influx is turned on at a finite $v_0$ the mass function from eq. (3.14) is given by

$$m(v) = m_0 - \frac{\gamma(1-n)^{p-1}}{\lambda(p-2)}e^{\lambda(1-n)(-p+2)v}. \hspace{1cm} (4.6)$$

This simply tells us that in the external region the black hole settles down to the final mass $m_0$. 

9
Let us now consider a free falling observer close to the Cauchy horizon, defined again by \( v = \infty \). Solving the geodesic equations we get, as in section 2, \( \dot{v} \sim e^{k_-v} \) and \( \dot{\sigma} = \text{const.} \), where a dot means derivative with respect to the proper time. Thus the energy density he measures is

\[
\rho_{\text{obs}} \sim T_{vv} \dot{v}^2 \sim e^{\lambda(1-n)(-p+2)v+2k_-v}.
\]

This quantity remains finite whenever

\[
(1-n)(-p+2)\lambda + 2k_- \leq 0.
\]

We can use eqs. (3.8) and (3.10) to rewrite eq. (4.8) in the form

\[
m_0^2 \leq Q^2 \frac{[4 + (1-n)(p-2)]^2}{16 + 8(1-n)(p-2)}
\]

(together with \( m_0^2 \geq Q^2 \)).

Physically, this result is due to the fact that the blueshift factor, which is exponential in the null time coordinate \( v \) in the Reissner-Nordström spacetime, now has a power law behaviour in \( y^+ \) (this is easily seen writing eq. (4.7) in terms of \( y^+ \)). The same thing happens for the extreme Reissner-Nordström black holes, see eqs. (2.11) and (2.12), where \( k_- = 0 \).

4.2. Mass inflation

Our aim is to construct an approximate solution to the equations of motion in the presence both of incoming and outgoing fluxes of null radiation close to the Cauchy horizon. It is useful to write down eqs. (3.3) and (3.4) in the conformal gauge defined by

\[
ds^2 = -e^{2\rho} dx^+ dx^- \equiv -F e^{2\phi} dx^+ dx^-,
\]

where we have introduced the quantity \( F \equiv e^{2(\rho-\phi)} \). It is

\[
\partial_+ \partial_- (\ln F) = -2Q^2 F e^{\frac{\phi}{\lambda}} \phi,
\]

\[
\partial_+ \partial_- (e^{-\frac{2\phi}{\lambda}}) = (-\lambda^2 + Q^2 e^{\frac{4\phi}{\lambda}}) F,
\]

---

3 Therefore, what makes the Cauchy horizon unstable is not just the infinite accumulation of ingoing waves in its vicinity, but also the ‘strength’ of this accumulation, i.e. the behaviour of the blueshift factor.

4 The stability of other 2d dilaton black holes with \( k_- = 0 \) at the Cauchy horizon is considered in Ref. [13].
while the constraints are
\[
\partial_\pm^2 (e^{-\frac{2}{n} \phi}) - \partial_\pm (e^{-\frac{2}{n} \phi}) \partial_\pm (\ln F) = -\frac{1}{2} \sum_{i=1}^{N} (\partial_\pm f)^2. \tag{4.13}
\]

For simplicity let us imagine that the inflow is turned on at a finite advanced time \(x^+ = x_0^+\) and that the outflow, crossing the Cauchy horizon, starts at \(x^- = x_0^-\) (see Fig. 3).

The spacetime is then divided in different zones: a pure inflow and a pure outflow regions, whose geometries are correctly described by the Vaidya type metric in eq. (3.12) (in the outflow case \(v\) is replaced by \(u\) and (3.12) remains the same apart from a minus sign in \(dud\sigma\)), static sectors where the metric takes the form eq. (3.6) and finally the most interesting one where both fluxes are present.

We concentrate our discussion to this last region, close to \(x_0^-\), and choose \(x^+\) such that \(x^+ = 0\) at the Cauchy horizon.

Note that for \(x^- < x_0^-\) and close to the Cauchy horizon the solution is almost static (see eq. (4.6) in the limit \(v \to \infty\), the dilaton reaching the finite value \(\phi_-\). We fix the \(x^+\) coordinate in such a way that \(x^+ = 0\) at the Cauchy horizon, i.e. \(\sim -e^{-k-v}\) (see for example Ref. [11]).

The form of \(T_{++}^M\) then follows from eq. (4.5) after a coordinate change, that is
\[
T_{++}^M \sim (x^+)^{-\frac{\Lambda(1-n)(-p+2)}{e_-^2}} - 2, \tag{4.14}
\]
while the exact relation between \(u\) and \(x^-\) along with the shape of \(T_{--}^M \neq 0\) will not concern us here, since the conclusions that we’ll draw will be independent on them.\(^5\)

An approximate solution to \(F\) close to \(x_0^-\) and to the Cauchy horizon following from eq. (4.11) and with the above initial conditions is
\[
F \approx e^{-2\phi_-} \exp \left( -2Q^2 e^{\frac{4}{n} - 2}\phi_- (x^- - x_0^-)(x^+ - x_0^+) \right). \tag{4.15}
\]

This permits to solve, in the same limit, the constraints eqs. (4.13) along with eq. (4.12) giving
\[
e^{-\frac{2}{n} \phi} \approx e^{-\frac{2}{n} \phi}(x_0^+, x_0^-) - \int_{x_0^+}^{x^+} d\xi F(\xi) \int_{x_0^+}^{\xi} \frac{d\xi'}{F(\xi')} T_{++}^M(\xi') - \int_{x_0^-}^{x^-} d\xi F(\xi) \int_{x_0^-}^{\xi} \frac{d\xi'}{F(\xi')} T_{--}^M(\xi') + x^+(x^+ - x_0^-) + Q^2 e^{\frac{4}{n} - 2}\phi_- (x^- - x_0^-). \tag{4.16}
\]

\(^5\) Note that \(T_{++}^M\) in eq. (4.14) has exactly the same divergence as \(\rho_{obs}\) in eq. (4.7).
Note that both $F$ and the metric stay bounded as $x^+ \to 0$. However when one calculates the Ricci scalar $R = 4F^{-1}e^{-2\phi}\partial_+\partial_-(2\phi + \ln F)$ we have a sum of different terms of which only one is potentially divergent, namely that proportional to

$$\int_{x_0^+}^{x^+} \frac{d\xi}{F(\xi)} T_{++}^M(\xi) \int_{x_0^-}^{x^-} \frac{d\xi'}{F(\xi')} T_{--}^M(\xi').$$

(4.17)

Under the assumption that $\int_{x_0^-}^{x^-} \frac{d\xi'}{F(\xi')} T_{--}^M(\xi')$ is finite, we can easily transform the other integral to $v$ coordinate and see that as $v \to \infty$ it behaves as

$$e^{\lambda(1-n)(-p+2)+k_-v}.$$  

(4.18)

We stress that the dependence on $k_-$ of (4.18) is the same as in eq. (2.10). This is typical of mass inflation.

We get therefore the result that if

$$\lambda(1-n)(-p+2)+k_- \leq 0$$

(4.19)

then $R$ is finite at the Cauchy horizon, but when this inequality is not satisfied the scalar curvature diverges. Eq. (4.19) can also be rewritten as

$$m_0^2 \leq Q^2 \left[ \frac{2 + (1-n)(p-2)}{4 + 4(1-n)(p-2)} \right]^2.$$  

(4.20)

Therefore, also at this level the possibility of having a regular Cauchy horizon is not completely ruled out.

According to the results in eqs (4.8) and (4.19) in these two-dimensional theories three different regimes can be identified:

i) $k_- \leq \frac{\lambda(1-n)(p-2)}{2}$, complete stability;

ii) $\frac{\lambda(1-n)(p-2)}{2} < k_- \leq \lambda(1-n)(p-2)$, $\rho_{obs}$ diverges but the spacetime remains regular;

iii) $k_- > \lambda(1-n)(p-2)$, both $\rho_{obs}$ and $R$ are unbounded at the Cauchy horizon.

A comparison with the analysis of perturbations carried out in the Reissner-Nordström geometry (section 2.1) shows that now complete stability of the Cauchy horizon is achieved in a region of nonzero measure in the space of the parameters $(m_0, Q, \lambda, n)$ (in the plane $(m_0^2, Q^2)$ it is the dotted region of Fig. 4).

This is a new result concerning asymptotically flat spacetimes.
5. Stability analysis for $n > 1$

We have already noted that this case is qualitatively similar to that considered in section 2.2, concerning the Reissner-Nordström-De Sitter spacetime. As considered there, we introduce a null perturbation in the spacetime which is regular at the acceleration horizon and write

$$T_{vv} = Ke^{-2k_{ah}v},$$  \tag{5.1}

where $K$ is a constant and $k_{ah}$ is given by eq. (3.11). Note that now the coordinates in eqs. (4.1) and (4.2) define the Kruskal frame, regular at the acceleration horizon.

Simple stability arguments at the Cauchy horizon, as in the previous section, show that geodesic observers there will measure an energy density given by

$$\rho_{obs} \sim e^{2(k_- - k_{ah})v}. \quad \tag{5.2}$$

As $v \to \infty$ the stability is guaranteed when the inequality

$$k_- \leq k_{ah} \quad \tag{5.3}$$

is satisfied. By making use of the eqs. (3.10) and (3.11) we can rewrite (5.3) as

$$m_0^2 \leq \frac{Q^2 (1 + n)^2}{4n}. \quad \tag{5.4}$$

The insertion of a nonvanishing outflux of null radiation modifies the picture more or less as in section 4.2. Here we don’t write down the explicit calculations, but just limit ourselves to note that as a consequence the curvature scalar $R$ at the Cauchy horizon goes as

$$R \sim e^{(k_- - 2k_{ah})v}. \quad \tag{5.5}$$

This implies that the resulting spacetime is regular at the Cauchy horizon only when

$$k_- \leq 2k_{ah}, \quad \tag{5.6}$$

which can be rewritten more elegantly as

$$m_0^2 \leq \frac{Q^2 n^2}{2n - 1}. \quad \tag{5.7}$$

The formulas in eqs. (5.3) and (5.6) are very similar to those derived in section 2.2, provided we replace $k_{ah}$ with $k_c$. Therefore all the conclusions presented there on the stability of the Cauchy horizon are valid here too.
6. Quantum effects, backreaction and conclusions

In this paper we have shown that the instability of the Cauchy horizon, peculiar to the asymptotically flat solutions of General Relativity, is not generic in two-dimensional theories of gravity.

Indeed, we presented a class of dilaton-gravity models for which, due to the asymptotic Rindlerian form of the metric, the Cauchy horizon can be stable under generic small perturbations generated in the external universe and, also, the scalar curvature stays regular there.

This suggests the idea that, in principle, an observer falling into the black hole is not destroyed at the spacetime singularity but, due to the timelike ‘repulsive’ character of the singularity itself, can safely travel through the black hole tunnel of Fig. 1 and then emerge into another asymptotically flat region.

It is interesting to investigate, at this point, whether quantum effects can modify the classical picture emerged in this paper. In Ref. [13] we have computed the v.e.v. of the energy-momentum tensor of \( N \) massless conformally coupled scalar fields in these spacetimes. Now we wish to determine their backreaction on the geometry close to the Cauchy horizon.

The semiclassical theory that was considered in [8], which keeps into account the effects of the Hawking radiation, is described by the action

\[
S = S_n + S_M + S_{EM} + \frac{\kappa}{2\pi} \int d^2x \sqrt{-g} \left[ \frac{1-2n}{2n} \phi R + \frac{n-1}{n} (\nabla \phi)^2 - \frac{1}{4} R(\nabla^2)^{-1} R \right], \tag{6.1}
\]

where \( \kappa = \frac{N}{12} \) and \( S_n, S_M \) and \( S_{EM} \) are considered in (1.1), (3.1) and (3.2).

Let’s perform the field redefinitions

\[
\chi = \kappa \rho + \left( \frac{1}{2n} - 1 \right) \kappa \phi + e^{-\frac{2}{n}\phi}, \tag{6.2}
\]

\[
\Omega = \frac{\kappa}{2n} \phi + e^{-\frac{2}{n}\phi} \tag{6.3}
\]

and work in the conformal gauge. If \( Q = 0 \) the action (6.1) can be cast in the ‘free field’ form

\[
S = \frac{1}{\pi} \int d^2x \left[ \frac{1}{\kappa} \left( -\partial_+ \chi \partial_- \chi + \partial_+ \Omega \partial_- \Omega \right) + \lambda^2 e^{\frac{2}{n}(\chi-\Omega)} \right] + \frac{1}{2} \sum_{i=1}^{N} \partial_+ f_i \partial_- f_i \tag{6.4}
\]

and the models become exactly solvable. For \( Q \neq 0 \) it is no more possible to write down explicitly the action, not knowing the inverse transformation \( \phi = \phi(\chi, \Omega) \). However we can still derive the equations of motion, i.e.

\[
\partial_+ \partial_- \chi = (-\lambda^2 + Q^2 e^{\frac{4}{n}\phi}) e^{\frac{2}{n}(\chi-\Omega)}, \tag{6.5}
\]
\[ \partial_+ \partial_-(\chi - \Omega) = \frac{2k}{n} Q^2 e^{\frac{4n}{\Omega}} e^{\frac{2n}{e}} (\chi - \Omega) \]  \hspace{1cm} (6.6) 

and the constraint equations
\[ \kappa t_\pm = \frac{1}{\kappa} (-\partial_\pm \chi \partial_\pm \chi + \partial_\pm \Omega \partial_\pm \Omega) + \partial^2_{\pm} \chi + \frac{1}{2} \sum_{i=1}^{N} \partial_\pm f_i \partial_\pm f_i. \]  \hspace{1cm} (6.7) 

Here the choice of \( t_+ \) becomes crucial. Physically, it measures how the quantum state of the \( f_i \) fields is related to the vacuum defined in terms of the modes \( x^\pm \), regular at the Cauchy horizon.

We could require, as in \[11\], physical conditions such that the black hole is in thermal equilibrium with a heat bath in order to isolate effects on the interior from those of the evaporation. Or, as we did in \[8\], we could demand that the state of the \( f_i \) fields is the one naturally associated to Rindler modes \( e^{-iwu}, e^{-iwv} \).

In both cases we find that
\[ t_+(x^+) \sim \frac{\alpha}{x^+}, \]  \hspace{1cm} (6.8) 

where \( \alpha \) is a positive constant (this is easily seen by performing the Schwarzian derivative between \( e^{k_+ v} = (-x^+)^{-\frac{k_+}{\kappa}} \) (in the first case) or \( v = -\frac{1}{k_-} \ln(-k_- x^+) \) (in the second case) and \( x^+ \)). The explicit form of both \( t_-(x^-) \) and \( T_{\pm}^- \) is again not important for our discussion.

Much care is needed in order to determine whether or not the r.h.s. of eq. (6.6) diverges before the inner horizon is encountered. Classically, there’s a range of values of \( m_0, Q \) and \( \lambda \) such that it is always \( e^{-\frac{2n}{\Omega}} > e^{-\frac{2n}{\Omega - \kappa}} = \frac{4}{\kappa} \) (and so \( \Omega' \neq 0 \)) as \( \sigma \geq \sigma_- \). Moreover at the quantum level the \( t_+ \) considered in eq. (6.8) is equivalent to introduce a negative energy flux along the Cauchy horizon, thus providing a ‘defocussing’ effect that is likely to guarantee that \( \Omega' \neq 0 \) during the entire evolution up to \( x^+ \to 0 \). In fact it is easy to show (similarly to \[11\]) that, as \( x^+ \to 0 \), we are again in a weak coupling region and (only finite terms are written)
\[ \chi \sim \Omega \sim -\alpha \kappa \ln(-k_- x^+) + \int_{x^-}^{x^+} d\xi \int_{\xi}^{\xi'} d\xi' [kt_- (\xi') - T_{\pm}^-(\xi')] \]  \hspace{1cm} (6.9) 

(\( T_{\pm}^+ \) of eq. (4.14) does not contribute in this limit). The Ricci scalar in the same limit diverges and behaves as
\[ R \sim ne^{(\frac{4n}{\Omega} - 2)\phi} \frac{1}{x^+} \int_{x^-}^{x^+} d\xi [T_{\pm}^-(\xi) - \kappa t_-(\xi)]. \]  \hspace{1cm} (6.10) 

Therefore while at the classical level we could always choose our parameters \( n \) and \( k_- \) such that the spacetime is regular at the Cauchy horizon, quantum-mechanically this is no more possible and a singularity always forms independently on the value of \( n \).

We can say that in this way quantum mechanics has helped us to restore the full predictive power of the gravitational field equations that had been lost at the classical level.

Acknowledgements: We thank D. Amati and R. Balbinot for help and useful suggestions.
References

[1] S.W. Hawking and G.F.R. Ellis, *The large scale structure of space-time* (Cambridge University Press, Cambridge, England, 1973).

[2] S.K. Blau and A.H. Guth: *Inflationary Cosmology* in ‘300 years of Gravitation’ pag. 524, ed. by S.W. Hawking and W. Israel (Cambridge University Press, 1987).

[3] R. Penrose in *Battelle Rencontres*, ed. by C.M. De Witt and J.A. Wheeler (Benjamin, New York, 1968).

[4] R.H. Price, Phys. Rev. D5 (1972) 2419.

[5] E. Poisson and W. Israel, Phys. Rev. D41 (1990) 1796; A. Ori, Phys. Rev. Lett. 67 (1991) 789.

[6] P.R. Brady and E. Poisson, Class. Quant. Grav. 9 (1992) 121.

[7] P.R. Brady, D. Nunez and S. Sinha, Phys. Rev. D47 (1993) 4239.

[8] A. Fabbri and J.G. Russo, Phys. Rev. D53 (1996) 6995.

[9] C. Callan, S. Giddings, J. Harvey and A. Strominger, Phys. Rev. D45 (1992) R1005.

[10] R. Balbinot and A. Fabbri, gr-qc/9602047 (to appear in Class. Quant. Grav.).

[11] R. Balbinot and P.R. Brady, Class. Quant. Grav. 11 (1994) 1763.

[12] E. Poisson, private communication.

[13] R. Balbinot and A. Fabbri, *Two-dimensional black holes in accelerated frames: quantum aspects*, to appear.

[14] R. M. Wald, *General Relativity* (University of Chicago Press, Chicago, 1984).

[15] J.S.F. Chan and R.B. Mann, Phys. Rev. D50 (1994) 7376.
Fig. 1: Penrose diagram of the Reissner-Nordström spacetime for $m_0 > |Q|$. Double lines represent the singularity, dashed lines the curves $r = \text{const.}$, regular lines the horizons and thick lines the asymptotic region.

Fig. 2: Causal structure of the Reissner-Nordström-De Sitter geometry.
Fig. 3: Perturbed model: the inflow starts at $x_0^+$ and the outflow at $x_0^-$. $CH$ stands for the Cauchy horizon and $EH$ is the event horizon.

Fig. 4: The dotted region represents the part of the $(m_0^2, Q^2)$ plane where the Cauchy horizon is completely stable (here we chose $n = \frac{1}{2}$, $p = 12$).