The Schrödinger Wave Functional
and Vacuum States in Curved Spacetime II
– Boundaries and Foliations

D.V. Long and G.M. Shore

Department of Physics
University of Wales Swansea
Singleton Park
Swansea, SA2 8PP, U.K.

Abstract

In a recent paper, general solutions for the vacuum wave functionals in the Schrödinger picture were given for a variety of classes of curved spacetimes. Here, we describe a number of simple examples which illustrate how the presence of spacetime boundaries influences the vacuum wave functional and how physical quantities are independent of the choice of spacetime foliation used in the Schrödinger approach despite the foliation dependence of the wave functionals themselves.

PACS numbers: 04.62+v, 04.70.Dy and 98.80.Cq

1D.V.Long@swansea.ac.uk
2G.M.Shore@swansea.ac.uk
1 Introduction

The Schrödinger wave functional provides a simple and intuitive description of vacuum states in quantum field theory in curved spacetimes. It is particularly useful in situations where the background metric is time-dependent or in the presence of boundaries.

This is the second paper in a series where we develop the Schrödinger picture formalism in curved spacetime. In the first paper [1], we reviewed and developed techniques for solving the Schrödinger wave functional equation for broad classes of spacetimes, viz. static (where the metric depends only on the spacelike coordinates), dynamic or Bianchi type I (where the metric depends only on the timelike coordinates) and a certain class of conformally static metrics including the Robertson-Walker spacetimes. Here, we continue this development by studying examples of spacetimes with boundaries, in particular regions described by coordinate patches which can be analytically extended to a larger spacetime. We describe how the presence of boundaries influences the choice of foliation in the Schrödinger formulation and determines the nature of possible vacuum states.

The main advantage of the Schrödinger picture over other ways to characterise vacuum states is that it describes states explicitly by a simple wave functional specified by a single, possibly time-dependent, kernel function satisfying a differential equation with the prescribed boundary conditions. This makes no reference to the assumed spectrum of excited states and so circumvents the difficulties of the conventional canonical description of a vacuum as a ‘no-particle’ state with respect to the creation and annihilation operators defined by a particular mode decomposition of the field, an approach which is not well suited to time-dependent problems. Unlike the alternative of specifying a vacuum state implicitly by giving a prescription for determining the Green functions, the Schrödinger wave functional is an explicit description, and this simplifies the interpretation of the nature of the states. In the end, of course, the same fundamental ambiguities appear in very similar guises in all these formalisms, but while the Green function approach is perhaps better suited to more elaborate issues such as renormalisation and higher-order perturbative calculations, the Schrödinger picture frequently gives the clearest insight into the nature of the vacuum state.

So far, we have spoken loosely about ‘the vacuum state.’ In fact, it is only for the very special class of static spacetimes that an essentially unique state exists which possesses most of the defining attributes of the Minkowski vacuum. In the general case, there may be no distinguished candidate at all for a vacuum state with the usual properties. For example, in a dynamic spacetime, there is a one-parameter family of ‘vacuum’ solutions to the Schrödinger wave functional equation and the selection of one of these requires a physically motivated initial condition on the first-order time-dependent equation for the kernel. Although
these states are stable, they are not stationary states with respect to the chosen time evolution.

Even in a static spacetime, the vacuum wave functional will depend on the foliation of spacetime chosen to define the Schrödinger equation. On the other hand, we expect physical observables to be independent of the choice of foliation, given the same spacetime and boundary conditions. The resolution of this potential paradox is illustrated here for a simple but non-trivial example.

Quantum field theories in spacetimes with boundaries have been extensively studied elsewhere [2, 3, 4]. In particular, questions of renormalisation and the Schrödinger picture have been addressed in considerable generality in [5]. In this paper, our approach is rather to illustrate general features in a number of simple and clear examples.

The content of this paper is as follows. In section 2, we review very briefly the solutions of the Schrödinger wave functional equation found in [1]. In section 3, we continue the development of [1] by looking at vacuum solutions in the Milne universe, an example of a dynamic spacetime of Robertson-Walker type which expands from an initial point but has no asymptotically static region. This is also of interest as an example of a spacetime which is just one coordinate patch of a larger spacetime, the covering spacetime in this case being simply Minkowski. It has also found a recent application in the dynamics of bubble nucleation in certain variants of the inflationary universe scenario [6].

In section 4, we consider the much-studied Rindler wedge, imposing vanishing boundary conditions on the field. The interpretation of the vacuum state defined with respect to a foliation respecting these boundary conditions is considered in some detail.

Taken together, two Rindler wedges and the Milne universe and its time-reversed counterpart comprise standard Minkowski spacetime. In section 5, we describe conventional Minkowski field theory using the foliation appropriate to the Rindler-Milne tiling and verify that, given the correct implementation of boundary conditions, the conventional Minkowski Green functions are recovered. This is strong evidence for the expected foliation independence of physical observables and an important consistency check on our interpretation of the Schrödinger picture formalism.

This example also serves as a technical warm-up for our eventual goal of determining the vacuum wave functional in the Kruskal black hole spacetime, which shares many features of the Rindler-Milne foliation of Minkowski spacetime (see, e.g. [1]).
2 Vacuum Wave Functionals

We begin by reviewing briefly the vacuum wave functional solutions described in [1] for different classes of spacetime. For notation and conventions, see ref.[1].

We consider globally hyperbolic spacetimes $\mathcal{M}$, with metric $g_{\mu\nu}$, which admit a foliation into a family of spacelike hypersurfaces $\Sigma$, with intrinsic coordinates $\xi^i$, labelled by a ‘time’ parameter $s$. The embeddings of $\Sigma$ in $\mathcal{M}$ are specified by the spacetime coordinates $x^\mu(s, \xi^i)$.

States are described by wave functionals $\Psi[\phi(\xi), s; N, N^i, h_{ij}]$, where the variables $\phi(\xi)$ are eigenvalues of the field operator on the equal-$s$ hypersurfaces $\Sigma$, and the Schrödinger equation describes their evolution along the integral curves of $s$. $N$, $N^i$ and $h_{ij}$ are respectively the lapse and shift functions characterising the embedding and the induced metric on $\Sigma$, The Schrödinger equation for a free massive scalar field theory is then

$$i \frac{\partial \Psi}{\partial s} = \int_\Sigma d^d \xi \left\{ \frac{1}{2} N \sqrt{-h} \left( \frac{\delta^2}{\delta \phi^2} - h^{ij} \partial_i \phi \partial_j \phi + (m^2 + \xi R) \phi^2 \right) - i \partial_s \phi \frac{\delta}{\delta \phi} \right\} \Psi$$ (1)

While this equation makes the dependence of the wave functional on the foliation explicit, it is much simpler in particular examples to choose spacetime coordinates which reflect the foliation. If we identify the spacetime coordinates $(t, \underline{x})$ with the embedding variables $(s, \underline{\xi})$, the lapse and shift functions reduce to $N = \sqrt{g_{00}}$ and $N^i = 0$ (so that $g_{0i} = 0$) while $h_{ij} = g_{ij}$. The Schrödinger equation then reduces to

$$i \frac{\partial \Psi}{\partial t} = \frac{1}{2} \int d^d \underline{x} \sqrt{-g} \left\{ \frac{g_{00}}{g} \frac{\delta^2}{\delta \phi^2} - g^{ij}(\partial_i \phi)(\partial_j \phi) + (m^2 + \xi R) \phi^2 \right\} \Psi$$ (2)

The ‘vacuum’ solutions to the Schrödinger equation are Gaussian functionals

$$\Psi_0[\phi(\underline{x}), t] = N_0(t) \psi_0[\phi, t]$$ (3)

with

$$\psi_0[\phi, t] = \exp \left\{ - \frac{1}{2} \int d^d \underline{x} \sqrt{-h} \int d^d \underline{y} \sqrt{-h} \phi(\underline{x}) G(\underline{x}, \underline{y}; t) \phi(\underline{y}) \right\}$$ (4)

and

$$\frac{d \ln N_0(t)}{dt} = - \frac{i}{2} \int d^d \underline{x} \sqrt{-h} \frac{\sqrt{g_{00}}}{\sqrt{h_{00}}} G(\underline{x}, \underline{x}; t)$$ (5)

where the kernel $G(\underline{x}, \underline{y}; t)$ satisfies

$$i \frac{\partial}{\partial t} \left( \sqrt{h_{00}} G(\underline{x}, \underline{y}; t) \right) = \int d^d \underline{z} \sqrt{-h} \frac{\sqrt{g_{00}}}{\sqrt{h_{00}}} \sqrt{h_{xy}} G(\underline{x}, \underline{z}; t) G(\underline{z}, \underline{y}; t) - \sqrt{h_{xy}} \delta^d(\underline{x}, \underline{y})$$ (6)

$^3$The $(d+1)$ dimensional spacetime Laplacian $\Box = \frac{1}{\sqrt{-\gamma}} \partial_{\mu}(\sqrt{-g} \partial_{\mu})$ can be split into a spatial part, $\Box_i$, and a time part, $\Box_0$. The delta function density is given by $\delta^d(\underline{x}, \underline{y}) = (\sqrt{-h_{00}})^{-1}\delta^d(\underline{x} - \underline{y})$ and satisfies $\int d^d \underline{x} \sqrt{-h} \delta^d(\underline{x}, \underline{y}) f(\underline{x}) = f(\underline{y})$. 

3
The kernel equation can be solved explicitly for special classes of spacetime. For ‘static’ spacetimes, where the metric depends only on the spacelike coordinates, the kernel (which in this case is time independent) is

$$G(x, y) = \sqrt{g^0_0} \int \frac{d\mu(\lambda)}{(2\pi)^d} \omega(\lambda) \tilde{\psi}(\lambda)(\omega, x) \tilde{\psi}^*(\lambda)(\omega, y)$$  \hspace{1cm} (7)

where \( \tilde{\psi}(\lambda, x) \) are a complete, orthonormal set of solutions to the eigenvalue equation

$$\left(\Box_i + m^2 + \xi R\right) \tilde{\psi}(\lambda, x) = g^{00}(\lambda) \tilde{\psi}(\lambda, x)$$  \hspace{1cm} (8)

and \( d\mu(\lambda) \) is the appropriate measure.

For ‘dynamic’ (Bianchi type I) spacetimes, where the metric depends only on the time coordinate, the kernel is

$$G(x, y; t) = -i \sqrt{g^0_0} \frac{1}{\sqrt{-h}} \int \frac{dk_0}{(2\pi)^d} e^{i k_0 (x-y)} \frac{\partial}{\partial t} \ln \tilde{\psi}^*(t, k)$$  \hspace{1cm} (9)

where \( \tilde{\psi}(t, k) \) satisfies the Fourier-transformed wave equation

$$\left(\Box_0 - g^{ij} k_i k_j + m^2 + \xi R\right) \tilde{\psi}(t, k) = 0$$  \hspace{1cm} (10)

The arbitrariness in the choice of solution is responsible for the one-parameter ambiguity (strictly, a one-function ambiguity, since the arbitrary coefficients in the general solution of eq.(10) may be functions of the momentum \( k \)) of the vacuum wave functional for dynamic spacetimes. It is important to notice that despite the time-dependence of the kernel, the vacuum states described by eq.(9) are stable and can allow time-independent expectation values for certain operators.

These solutions may be readily generalised to conformally static spacetimes where the conformal scale factor depends only on the time coordinate. This class includes the Robertson-Walker spacetimes with curved spatial sections.

Expectation values of operator products are given in the Schrödinger representation by

$$\langle 0 | O(\varphi, \pi) | 0 \rangle = \int D\phi \Psi_0^* O(\phi, -i \frac{\delta}{\delta \phi}) \Psi_0$$  \hspace{1cm} (11)

where \( \pi \) is the momentum conjugate to \( \varphi \) in the canonical formalism. Simple examples include

$$\langle 0 | \varphi(x) \varphi(y) | 0 \rangle = \frac{1}{2} \Delta_R(x, y; t)$$  \hspace{1cm} (12)

$$\langle 0 | \pi(x) \pi(y) | 0 \rangle = \frac{1}{2} \sqrt{h_x h_y} \left\{ G_R(x, y; t) + \int d^d u \int d^d v \sqrt{h_u h_v} G_I(u, v; t) \Delta_R(u, v; t) G_I(v, u; t) \right\}$$  \hspace{1cm} (13)
and
\[ \langle 0 | [\varphi(x), \pi(y)] | 0 \rangle = i\delta^d(x - y) \] (14)
\[ \langle 0 | \{\varphi(x), \pi(y)\} | 0 \rangle = -\int d^d u \sqrt{h_u h_u} G_T(x, u; t) \Delta_R(u, y; t) \] (15)
where \( \Delta_R \) is the inverse of the real part of the kernel \( G_R \).

The expectation value of the canonical energy-momentum tensor
\[ T_{\mu\nu}(x) = (\partial_\mu \phi)(\partial_\nu \phi) - \frac{1}{2} g_{\mu\nu} \left[ g^{\rho\sigma} (\partial_\rho \phi)(\partial_\sigma \phi) - (m^2 + \xi R) \phi^2 \right] \] (16)
can be written in terms of the kernel and its inverse if we point-split before calculating the expectation value, the coincidence limit being taken at the end of the calculation. In particular the expectation value of the ‘energy’ component is
\[ \langle 0 | T_{00}(x) | 0 \rangle = \lim_{x \to y} \langle 0 | T_{00}(x, y; t) | 0 \rangle \] (17)
\[ \langle 0 | T_{00}(x, y; t) | 0 \rangle = -\frac{g_{00}}{2} \left\{ \left( \frac{g_{00}}{g} \right) \langle 0 | \pi(x) \pi(y) | 0 \rangle + \left[ g^{ij} \frac{\partial^2}{\partial x^i \partial y^j} - m^2 \right] \langle 0 | \phi(x) \phi(y) | 0 \rangle \right\} 
= \frac{g_{00}}{4} \left\{ G_R(x, y; t) - \left[ g^{ij} \frac{\partial^2}{\partial x^i \partial y^j} - m^2 \right] \Delta_R(x, y; t) 
\quad + \int d^d u \int d^d v \sqrt{h_u h_v} G_T(x, u; t) \Delta_R(u, v; t) G_T(v, y; t) \right\} \] (18)

3 The Milne Universe

Our first example is a dynamic spacetime of Robertson-Walker type. The Milne universe is a two-dimensional spacetime which begins at an initial point and expands indefinitely. Quantum field theory in this spacetime has been previously studied in \[8, 9, 10\].

The metric is
\[ ds^2 = dz^2 - a^2(z) d\tau^2 \] (19)
where \( z \) is the time coordinate \( (z > 0) \) and \( \tau \) is the space coordinate \( (-\infty < \tau < \infty) \). The scale factor for the Milne universe is \( a(z) = z \).

With a rescaling of the time coordinate, it can be rewritten in manifestly conformally flat form:
\[ ds^2 = C(\eta)(d\eta^2 - d\tau^2) \] (20)
where \( \eta = \ln z \) and \( C(\eta) = e^{2\eta} \).

A further coordinate transformation, with \( t = z \cosh \tau \) and \( x = z \sinh \tau \), brings the metric to the form
\[ ds^2 = dt^2 - dx^2 \] (21)
where the coordinates are restricted to the range \(0 < t < \infty\) and \(-\infty < x < \infty\). In this form, it is clear that the Milne universe is simply the patch of Minkowski spacetime lying in the future light cone of the origin (see Fig (1)). This will be exploited in section 5.

The Milne universe is geodesically complete in the sense that it admits a foliation where each spacelike hypersurface is intersected exactly once by a semi-infinite timelike geodesic which does not intersect the boundary except at the special point at the origin. A suitable foliation in which to set up the Schrödinger formalism is shown in Fig (1) where we choose the Cauchy hypersurfaces \(\Sigma\) to be the lines of constant \(z\), and consider evolution in the time coordinate \(z\). The Schrödinger equation for a minimally coupled massive scalar field is

\[
i \frac{\partial \Psi}{\partial z} = \frac{1}{2} \int_{-\infty}^{\infty} d\tau \frac{1}{z} \left[ -\frac{\delta^2}{\delta \phi(\tau)^2} + \left[ \partial_\tau \phi(\tau) \right]^2 + z^2 m^2 \phi(\tau)^2 \right] \Psi \tag{22}
\]

where \(\Psi[\phi, z]\) is a functional of the field eigenvalues \(\phi(\tau)\) on the equal-\(z\) hypersurfaces. It may be solved as usual, giving

\[
\Psi[\phi, z] = N_0(z) \exp \left\{ -\frac{1}{2} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} d\tau' \ z^2 \phi(\tau) G(\tau, \tau'; z) \phi(\tau') \right\} \tag{23}
\]

where

\[
\frac{d}{dz} \ln N_0(z) = -\frac{i}{2} \int_{-\infty}^{\infty} d\tau \ z \ G(\tau, \tau; z) \tag{24}
\]

The kernel is

\[
G(\tau, \tau'; z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(\tau-\tau')} \tilde{G}(k; z) \tag{25}
\]

\[
\tilde{G}(k; z) = -\frac{i}{z} \frac{\partial}{\partial z} \ln \tilde{\psi}^*(z, k) \tag{26}
\]

\(\text{This is not true for null geodesics. In consequence, the conclusions of this section may not necessarily all be true for zero mass fields.}\)
where \( \tilde{\psi}(z, k) \) is a solution of the Fourier transformed wave equation
\[
\left( \frac{1}{z} \partial_z (z \partial_z) + m^2 + \frac{k^2}{z^2} \right) \tilde{\psi}(z, k) = 0
\] (27)
The general solution is a linear combination of Hankel functions of imaginary order (see [11, 12] for the required properties of Hankel and Bessel functions), i.e.
\[
\tilde{\psi}(z, k) = a(k) e^{-\frac{ik}{2} H_{ik}^1(mz)} + b(k) e^{\frac{ik}{2} H_{ik}^2(mz)}
\] (28)
Since the kernel depends only on the logarithm of \( \tilde{\psi}(z, k) \), only the ratio of the coefficient functions \( a(k) \) and \( b(k) \) survives as a one-parameter ambiguity in the vacuum wave functional. To fix this, we need to choose a suitable boundary condition.

In the cosmological models considered in [1], the spacetime had asymptotically Minkowski regions and the boundary condition was specified by choosing a vacuum wave functional that reproduced the standard Minkowski vacuum in the asymptotic limit. This is achieved by picking solutions of the wave equation which are positive frequency with respect to the usual Minkowski time coordinate. In the Milne universe, we have no analogous asymptotic region. However, we can still require that \( \tilde{\psi}(z, k) \) is a positive frequency solution (more precisely, a sum of positive frequency solutions) with respect to the proper time \( z \) of comoving observers in the expanding universe. Using a well-known integral representation of the Hankel functions we may rewrite eq.(28) as
\[
\tilde{\psi}(z, k) = -\frac{ia(k)}{\pi} \int_{-\infty}^{\infty} dte^{iz \cosh t - ikt} + \frac{ib(k)}{\pi} \int_{-\infty}^{\infty} dte^{-iz \cosh t - ikt}
\] (29)
So, remembering that for a comoving observer \((\tau = \text{const})\), \( z \) is simply proportional to \( t \), we restrict \( \tilde{\psi}(z, k) \) to be positive frequency in the above sense by choosing \( a = 0 \). The vacuum wave functional is therefore specified by the kernel (26) with \( \tilde{\psi}(z, k) = H_{ik}^2(mz) \).

To investigate the properties of this vacuum state, we evaluate first the two-point Wightman Green function then the vacuum expectation value of the energy-momentum tensor. The two-point function evaluated at equal \( z \)-time is simply
\[
\langle 0 | \varphi(\tau) \varphi(\tau') | 0 \rangle = \frac{1}{2} \Delta_R(\tau, \tau'; z)
\] (30)
where \( \Delta_R(\tau, \tau'; z) \) is the inverse of the real part of the kernel, \( G_R(\tau, \tau'; z) \). Defining the Fourier transform by
\[
\Delta_R(\tau, \tau'; z) = \int_{-\infty}^{\infty} dk \tilde{\Delta}_R(k; z) e^{ik(\tau - \tau')}
\] (31)
we find
\[
\tilde{\Delta}_R(k; z) = \frac{2i}{z} |\tilde{\psi}(z, k)|^2 W^{-1}[\tilde{\psi}^*(z, k), \tilde{\psi}(z, k)]
\] (32)
For the general solution (28), we have
\[
|\tilde{\psi}(z, k)|^2 = (|a|^2 + |b|^2)H_{ik}^1(mz)H_{ik}^2(mz) + a^* b e^{-\pi k} H_{ik}^2(mz) H_{ik}^2(mz) + a^* b e^{\pi k} H_{ik}^2(mz) H_{ik}^2(mz)
\]
while the Wronskian is
\[
W[\tilde{\psi}^*(z, k), \tilde{\psi}(z, k)] = (|a|^2 - |b|^2)W[H_{ik}^2(mz), H_{ik}^1(mz)] = \frac{4i}{\pi z}(|a|^2 - |b|^2)
\]
So, for the chosen comoving vacuum \((a = 0)\), we find
\[
\langle 0|\varphi(\tau)\varphi(\tau')|0\rangle_{\text{COM}} = \frac{\pi}{4} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(\tau - \tau')} H_{ik}^1(mz)H_{ik}^2(mz) = \frac{1}{2\pi} K_0(m\sigma)
\]
where \(\sigma\) is the geodesic interval along the equal-\(z\) hypersurface of the foliation, viz.
\[
\sigma = 2z \sinh\left(\frac{\tau - \tau'}{2}\right)
\]
Expressed in Minkowski coordinates \((t, x)\),
\[
\sigma^2 = (x - x')^2 - (t - t')^2
\]
The details of the calculation are given in section 5.

We see, therefore, that the two-point function in the comoving vacuum in the Milne universe is identical to the corresponding Green function in the complete Minkowski spacetime. This is not too surprising since we have used the same boundary condition in choosing the vacuum state, although it is less obvious that the Green function should be insensitive to the boundary, recalling that the Milne universe is simply the patch of Minkowski spacetime in the future light cone of the origin. This is assured by the property that the Milne patch admits a foliation for which the spacelike hypersurfaces are complete Cauchy surfaces for the full Minkowski manifold. This property is not shared by the other related example considered in this paper, the Rindler wedge (section 4).

As a second probe of the vacuum state, we may evaluate the expectation value of the energy-momentum tensor, eq.(16). The ‘energy’ component is expressed in terms of the expectation value with point-split argument, eq.(18)
\[
\langle 0|T_{zz}(\tau, \tau'; z)|0\rangle = \frac{1}{4} \left\{ G_R(\tau, \tau'; z) + \left[ \frac{1}{\zeta^2} \frac{\partial^2}{\partial \tau \partial \tau'} + m^2 \right] \Delta_R(\tau, \tau'; z) + 2z^2 \int_{-\infty}^{\infty} d\tau'' \int_{-\infty}^{\infty} d\tau''' G_I(\tau, \tau''; z) \Delta_R(\tau'', \tau'''; z)G_I(\tau''', \tau'; z) \right\}
\]
After some calculation (see appendix A for details), we find for the comoving vacuum,
\[
\langle 0|T_{zz}(\tau, \tau'; z)|0\rangle_{\text{COM}} = -\left( \frac{m}{2\pi \sigma} \right) K_1(m\sigma)
\]
where $\sigma$ is the geodesic interval along the equal-$z$ hypersurfaces of the foliation. Again, this agrees with the point-split energy-momentum tensor VEV for Minkowski spacetime, allowing for the coordinate transformation to the $(z, \tau)$ coordinates.

These two results confirm that physical quantities calculated in the Milne universe with the particular choice of state we have called the comoving vacuum are identical to those in Minkowski spacetime. In particular, they show no dependence on the boundary. However, other equally valid choices of vacuum state are possible corresponding to different choices of the arbitrary ratio $a/b$ in eq. (28). We now consider one of these, the so-called ‘conformal’ vacuum.

The conformal vacuum\footnote{Notice that, as in \cite{1}, we could equally well have formulated the Schrödinger equation for evolution in the conformal time, i.e. along the conformal Killing vector $\frac{\partial}{\partial \eta}$. However, since $\eta$ is a function of $z$ only (recall $\eta = \ln z$), the foliations into $z = \text{const}$ and $\eta = \text{const}$ surfaces are identical, so the Schrödinger equations are related by a trivial change of variable. In contrast, the choice of vacuum state is made at the level of imposing a boundary condition on the kernel equation. The comoving and ‘conformal’ vacua are distinguished by the choice of $\tilde{\psi}$ to be positive frequency with respect to the proper time $z$ of a comoving observer or (for massless fields) the conformal time $\eta$ respectively. This is a physical distinction unrelated to the foliation choice.} is selected by requiring that in the massless limit, where we are considering a conformal field theory on a conformally flat spacetime, the wave equation solutions determining the kernel should be positive frequency with respect to the conformal time $\eta$.

Yet another rewriting of the general solution (28) to the wave equation gives

$$\tilde{\psi}(z, k) = c(k)J_{\pm |k|}(mz) + d(k)J_{-|k|}(mz)$$ \hspace{1cm} (40)

where the $J_{\pm |k|}(mz)$ are Bessel functions of imaginary order. In terms of the $a$ and $b$ coefficients of eq. (28), we have for $k > 0$

$$c(k) = \alpha(k) a(k) + \beta(k) b(k)$$ \hspace{1cm} (41)

$$d(k) = \beta^*(k) a(k) + \alpha^*(k) b(k)$$ \hspace{1cm} (42)

where $\alpha(k)$ and $\beta(k)$ are Bogoliubov coefficients:

$$\alpha(k) = \frac{e^{\frac{\pi k}{2}}}{\sinh(\pi k)} \hspace{1cm} \beta(k) = -\frac{e^{-\frac{\pi k}{2}}}{\sinh(\pi k)}$$ \hspace{1cm} (43)

In the massless limit, $J_{-|k|}(mz) \sim z^{-|k|} e^{-|k|\eta}$, so is positive frequency with respect to the conformal time. The conformal vacuum is therefore specified by choosing $c = 0, d = 1$ in eq. (40). In terms of the original coefficients, it is specified by choosing the ratio $a/b = e^{-\pi k}$. Clearly, it is simply another of the one-parameter family of possible vacua characteristic of dynamic spacetimes. Of course, the conformal vacuum also corresponds to positive frequency behaviour.
with respect to the conformal time in the limit of early times (small $z$), as well as vanishing mass.

To show that this is indeed physically distinct from the comoving vacuum, we compare the expectation values of the energy-momentum tensor in the two states. Details of the calculations are given in appendix A. We find that the difference between the expectation values in the comoving and conformal vacua [13] is

$$\langle 0|T_{zz}(z)|0 \rangle_{\text{COM}} - \langle 0|T_{zz}(z)|0 \rangle_{\text{CONF}} = \frac{1}{\pi z^2} \int_0^\infty \frac{dk}{(e^{2\pi k} - 1)} + \frac{m^2}{8} \int_0^\infty \frac{dk}{\sinh^2(\pi k)} \left\{ 2e^{-\pi k} \left[ J_{ik}(mz) J_{-ik}(mz) + J_{ik+1}(mz) J_{-ik+1}(mz) \right] - J_{ik}(mz) J_{ik}(mz) - J_{-ik}(mz) J_{-ik}(mz) + J_{ik-1}(mz) J_{ik+1}(mz) + J_{-ik-1}(mz) J_{-ik+1}(mz) \right\} (44)$$

The first term dominates in the early time (small $z$) or small mass limits, since all the other terms are of $O(z^0)$. This term represents the energy density of radiation at a temperature $(2\pi z)^{-1}$, and shows that, in this limit, the comoving vacuum is an excited, thermal state with respect to the conformal vacuum.

4 Rindler Spacetime

Rindler spacetime [14] is the static spacetime described by the two-dimensional metric

$$ds^2 = z^2 d\tau^2 - dz^2$$ (45)

with $-\infty < \tau < \infty$ and $0 < z < \infty$. In coordinates which make the conformal flatness manifest,

$$ds^2 = C(\eta)(d\tau^2 - d\eta^2)$$ (46)

where $\eta = \ln z$ and $C(\eta) = e^{2\eta}$. Like the Milne universe, Rindler spacetime is simply a patch of Minkowski spacetime. To see this, make the coordinate transformation $t = z \sinh \tau, \ x = z \cosh \tau$. In these coordinates, the metric is simply

$$ds^2 = dt^2 - dx^2$$ (47)

where the range is restricted to $x > 0, |t| < x$. The spacetime is therefore just the R wedge in Fig (2).

Quantum field theory in this spacetime has been widely studied using many different formalisms (see for example [2, 15, 16] in the canonical formalism and [17, 18, 19] in the Schrödinger formalism). We have little to add to this discussion so the presentation here is very brief. It is intended mainly to illustrate the
importance of boundary conditions in specifying the vacuum state and to con-
trast with the results on foliation independence in the Rindler-Milne analysis of
Minkowski spacetime in section 5.

In order to apply the Schrödinger formalism, we need to choose a foliation into
a set of spacelike Cauchy hypersurfaces and consider evolution along a timelike
Killing vector field which is infinite in extent and in particular does not intersect
the boundary. The Rindler wedge is globally hyperbolic and thus geodesically
complete and so admits such a foliation.

A suitable foliation is given by choosing the spacelike hypersurfaces to be the
lines $\tau = \text{const}$ and considering evolution along the Killing vectors $\partial/\partial\tau$ as shown
in Fig (2). The Schrödinger equation is then

$$i \frac{\partial \Psi}{\partial \tau} = \frac{1}{2} \int_0^\infty dz \left\{ \frac{\delta^2}{\delta \phi(z)^2} + \left[ \frac{\partial_z \phi(z)^2}{2} + m^2 \phi(z)^2 \right] \right\} \Psi. \quad (48)$$

To solve this, we must impose boundary conditions on the field $\phi(z)$. A suitable
choice is the Dirichlet condition $\phi = 0$ at $z = 0$ (and, as usual, at spatial infinity,
$z \to \infty$). The vacuum wave functional is

$$\Psi[\phi, \tau] = N_0(\tau) \exp \left\{ -\frac{1}{2} \int_0^\infty dz \int_0^\infty dz' \phi(z)G(z, z')\phi(z') \right\} \quad (49)$$

where $N_0(t) = \exp \left\{ -\frac{i}{2} \int_0^\infty dz zG(z, z) \right\}$ and the kernel $G(z, z')$ is given by the

---

6The evolution path $z = \text{const}$ is the world line of a uniformly accelerating particle with
acceleration $1/z$ in Minkowski spacetime. This is the reason for the great interest in Rindler
spacetime [16] in modelling the behaviour of accelerated systems or observers.
general formula for static spacetimes, in this case

\[ G(z, z') = \frac{1}{zz'} \int_0^\infty \frac{d\omega}{2\pi} \omega \tilde{\psi}(\omega, z) \tilde{\psi}^*(\omega, z') \] (50)

The functions \( \tilde{\psi}(\omega, z) \) are Fourier transforms with respect to \( \tau \) of solutions of the wave equation, viz.

\[ \left( \frac{1}{z} \partial_z (z \partial_z) - m^2 + \frac{\omega^2}{z^2} \right) \tilde{\psi}(\omega, z) = 0 \] (51)

The boundary condition on \( \phi(z) \) is respected automatically if we choose \( \tilde{\psi}(\omega, z) \) such that \( \tilde{\psi} = 0 \) at \( z = 0 \). A suitable set, satisfying the orthonormality and completeness conditions

\[ \int_0^\infty \frac{dz}{2\pi} \frac{1}{z} \tilde{\psi}^*(\omega, z) \tilde{\psi}(\nu, z) = \delta(\omega - \nu) \] (52)

\[ \int_0^\infty \frac{d\omega}{2\pi} \tilde{\psi}^*(\omega, z) \tilde{\psi}(\omega, z') = z \delta(z - z') \] (53)

is

\[ \tilde{\psi}(\omega, z) = 2\sqrt{\frac{\omega \sinh(\pi \omega)}{\pi}} K_{i\omega}(mz). \] (54)

This specifies the vacuum state in Rindler spacetime subject to the given boundary condition. It is the ground state with respect to the energy associated with the chosen time evolution. It is unique in the same sense as is the usual vacuum in Minkowski spacetime. Of course, a different foliation satisfying the above criteria would yield a formally different expression for the vacuum wave functional, but all physical quantities derived from it would be identical. (The question of foliation independence is discussed in section 5.)

An alternative representation of the wave functional can be given in terms of the transforms \( \tilde{\phi}(\omega) \) of the field eigenvalues \( \phi(z) \),

\[ \phi(z) = \int_0^\infty \frac{d\omega}{2\pi} 2 \sqrt{\frac{\omega \sinh(\pi \omega)}{\pi}} K_{i\omega}(mz) \tilde{\phi}(\omega). \] (55)

As a functional of \( \tilde{\phi} \), the \( \tau \)-independent part of the vacuum wave functional is simply

\[ \Psi[\tilde{\phi}(\omega)] = \exp \left\{ -\frac{1}{2} \int_0^\infty \frac{d\omega}{(2\pi)} \omega \left| \tilde{\phi}(\omega) \right|^2 \right\}. \] (56)

\footnote{These results are immediate consequences of the Kontorovich-Lebedev transform}

\[ g(y) = \int_0^\infty dx f(x) K_{ix}(y) \]

\[ f(x) = 2\pi^{-2} x \sinh(\pi x) \int_0^\infty dy y^{-1} g(y) K_{ix}(y) \]
Excited states can be built as described in \[1\] by acting successively on \(\Psi[\tilde{\phi}]\) with the creation operators

\[
a^\dagger(\omega) = \int_0^\infty dz \tilde{\psi}(\omega, z) \left[ \frac{\omega}{z} \phi(z) - \frac{\delta}{\delta \phi(z)} \right]
\] (57)

To understand better the nature of this vacuum state, we again evaluate the Wightman function and the energy-momentum tensor. The Wightman function is simply the inverse kernel. Clearly, we have

\[
\Delta(z, z') = \int_0^\infty \frac{d\omega}{2\pi} \frac{1}{\omega} \tilde{\psi}(\omega, z) \tilde{\psi}^*(\omega, z')
\] (58)

and evaluating the integral over \(\omega\) we find

\[
\langle 0 | \varphi(z) \varphi(z') | 0 \rangle_{\text{RIND}} = \frac{1}{2\pi} K_0(m|z - z'|) - \frac{1}{\pi} \int_0^\infty \frac{dv}{\pi^2 + v^2} K_0(mZ)
\] (60)

where \(Z^2 = z^2 + z'^2 + 2zz' \cosh v\). The first term is simply the usual translation invariant Minkowski result. (Note that the geodesic interval \(\sigma^2 = (t - t')^2 - (x - x')^2\) for points with equal \(\tau\) is simply \(\sigma^2 = -(z - z')^2\). The second term shows a dependence on the absolute position and reflects the sensitivity to the boundary. This should be contrasted with the corresponding result in the Milne universe. The foliation hypersurfaces for the Rindler patch are not complete Cauchy hypersurfaces for the full Minkowski spacetime, so there is no reason to expect translation invariance in the Wightman function.

The energy-momentum tensor expectation values are computed as usual from the kernel and its inverse. After some calculation (see appendix B) we find

\[
\langle 0 | T_{\tau\tau}(z, z') | 0 \rangle_{\text{RIND}} = -z^2 \left( \frac{m}{2\pi|z - z'|} K_1(m|z - z'|) \right) + \int_0^\infty \frac{dv z^2}{2\pi(v^2 + \pi^2)} \times
\]

\[
\left( \left( \frac{P(v)}{z^2} - Q(v) m^2 \right) K_0(mZ) + \frac{m}{\pi^2} \left( 2 + \cosh v - 2Q(v) \right) K_1(mZ) \right)
\] (61)

The first term is exactly the same (up to a factor of \(g_{r\tau}\)) as the usual Minkowski result and depends only on the geodesic interval between the points. The second term, however, is not translation invariant and shows an explicit position dependence.

The energy density appropriate to evolution along the Killing vectors \(\partial/\partial \tau\) is therefore position dependent and sensitive to the boundary. However, if instead we calculate the expectation value of the corresponding Hamiltonian, given by

\[
\langle 0 | H_R | 0 \rangle_{\text{RIND}} = \int_0^\infty dz \sqrt{-g} g^{00} \langle 0 | T_{zz}(\tau, z) | 0 \rangle_{\text{RIND}}
\] (62)
we simply find the usual Minkowski-like sum of zero-point energies, viz.

\[ \langle 0|H_R|0 \rangle_{\text{RIND}} = \frac{1}{2} \int_0^\infty d\omega \ \omega \ \delta(0) \] (63)

The Rindler vacuum therefore shares most of the properties of the familiar Minkowski vacuum. It is the ground state with respect to the energy associated with the Hamiltonian generating the time evolution along the vector field \( \partial/\partial\tau \). A simple spectrum of excited states is generated by the creation operators \( a^\dagger(\omega) \). However, the lack of translation invariance in Rindler spacetime does affect the vacuum, showing up both in the Wightman function and in the explicit position dependence, or boundary sensitivity, of the local energy density.

Finally, we should make some remarks about observer dependence in the interpretation of this Rindler vacuum state.

In Minkowski spacetime, the Unruh effect implies that the vacuum state appears simple only to the class of inertial observers, whereas uniformly accelerated observers will experience a universal temperature effect \([15, 20]\). In Rindler spacetime, the rôle of preferred observers is taken by those following the timelike Killing vector fields \( \partial/\partial\tau \). These observers will be the analogues of the inertial observers in Minkowski spacetime and will perceive the Rindler vacuum to be a simple vacuum state. Other observers are accelerated relative to this class and will therefore experience an Unruh effect, perceiving the Rindler vacuum to be an excited state. For example, we expect observers following the Minkowski time evolution vectors \( \partial/\partial t \) to experience a universal, position-dependent temperature effect \( (T = 1/2\pi z) \), with the temperature increasing as the boundary is approached. This behaviour is in complete contrast to that of observers following the geodesically complete vector fields \( \partial/\partial\tau \), which are infinite in extent and never intersect the boundary.

5 Rindler-Milne Foliation of Minkowski Spacetime

This final example is designed to illustrate the foliation independence of physical quantities for quantum field theories in the same spacetime with the same boundary conditions.

In general, the foliation is specified by the deformation vector \( N^\mu(x) \) (which incorporates the lapse and shift functions \( N \) and \( N^i \)). The foliation determines the representation of operators in terms of the fields \( \varphi \) and conjugate momenta \( \pi \), so that both the operators and the wave functionals depend on \( N^\mu \). Foliation independence of physical quantities would then be expressed as a functional Ward
identity with respect to $N^\mu$. For example, for the physical VEV of a renormalisation group invariant operator $O(\pi, \varphi; N^\mu)$, we would have a Ward identity of the form

$$\frac{\delta}{\delta N^\mu} \int D\phi \, \Psi^*[\phi, s; N^\mu] \, O\left(-i \frac{\delta}{\delta \phi^0}, \phi; N^\mu\right) \, \Psi[\phi, s; N^\mu] = 0 \quad (64)$$

This encodes the invariance of the VEV under infinitesimal changes of the foliation hypersurfaces, although the wave functional itself is of course foliation dependent.

In this section, however, we consider ‘large’ changes of foliation. The example we choose is ordinary, $(d+1)$ dimensional Minkowski spacetime and we consider two foliations, first the standard one with hypersurfaces $t = \text{const}$ and second a ‘Rindler-Milne’ foliation where the spacetime is split into sections $P$, $L+R$, $F$ and the spacelike hypersurfaces are as shown in Fig (3).

![Figure 3: Rindler–Milne evolution surfaces in Minkowski spacetime.](image)

### 5.1 Minkowski foliation

The results of the standard Minkowski foliation \[1\] are well known and we simply quote them. The vacuum wave functional, which satisfies the Schrödinger equation

$$i \frac{\partial \Psi[\phi, t]}{\partial t} = \frac{1}{2} \int d^d x \left\{ - \frac{\delta^2}{\delta \phi^2} - \eta^{ij} (\partial_i \phi)(\partial_j \phi) + m^2 \phi^2 \right\} \Psi[\phi, t] \quad (65)$$

is

$$\Psi_0[\phi, t] = N_0(t) \, \exp \left\{ - \frac{1}{2} \int d^d x \int d^d y \, \phi(x) G(x, y) \phi(y) \right\} \quad (66)$$
where the kernel is
\[ G(x, y) = \int \frac{d^dk}{(2\pi)^d} \sqrt{k^2 + m^2} e^{ik(x-y)} \] (67)

The inverse kernel gives the Wightman function on a \( t = \text{const} \) hypersurface, viz.
\[ \langle 0 \mid \varphi(x) \varphi(y) \mid 0 \rangle_{\text{MINK}} = \frac{1}{2\pi} \left( \frac{m}{2\pi |x-y|} \right)^{d+1} K_{d+1}(m|x-y|) \] (68)

Notice that due to the manifest translation invariance, the Green function depends only on the distance \(|x-y|\) separating the points.

The (unrenormalised) VEV of the energy–momentum tensor is just the usual sum of zero-point energies,
\[ \langle 0 \mid T_{\mu\nu}(x) \mid 0 \rangle_{\text{MINK}} = g_{\mu\nu} \frac{1}{2} \int \frac{d^dk}{(2\pi)^d} \omega(k) \] (69)
where \( \omega^2(k) = k^2 + m^2 \). For later comparison the VEV of the energy component of the point–split energy–momentum tensor is
\[ \langle 0 \mid T_{00}(x,y) \mid 0 \rangle_{\text{MINK}} = -\left( \frac{m}{2\pi |x-y|} \right)^{d+1} K_{d+1}(m|x-y|) \] (70)

5.2 Rindler-Milne foliation

We now compare these results with those for the Rindler-Milne foliation. To set this up, we split Minkowski spacetime into the four wedges shown in Fig (3) and introduce coordinates \((\tau, z, x^a)\) in each wedge as follows:
\[
\begin{align*}
x^1 &= z \cosh \tau & x^1, t & \in R \\
x^1 &= -z \cosh \tau & x^1, t & \in L \\
x^1 &= z \sinh \tau & x^1, t & \in F \\
x^1 &= -z \sinh \tau & x^1, t & \in P.
\end{align*}
\] (71)

\( x^a (a = 2, \ldots, d) \) are retained as Minkowski coordinates.

**F and P patches**

In the F and P patches, the metric is \( ds^2 = dz^2 - z^2 d\tau^2 - (dx^a)^2 \) so appears dynamic in these coordinates. The spacelike hypersurfaces are chosen to be \( z = \text{const} \) and we consider evolution along \( \partial/\partial\tau \). These hypersurfaces are complete Cauchy surfaces for the whole of the Minkowski spacetime.

The analysis is precisely as in section 3, except that here we are working in \((d+1)\) dimensions. The Schrödinger equation is just the generalisation of eq.(22) and the vacuum wave functional is
\[ \Psi[\phi(\tau, x^a), z] = N_0(z) \exp\left\{ -\frac{1}{2} \int d^d x \int d^d y z^2 \phi(x)G(x,y,z)\phi(y) \right\} \] (72)
with kernel

\[ G(x, y; z) = \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)} \int \frac{d^{d-1}k_a}{(2\pi)^{d-1}} e^{i\omega(x-x')} e^{ik_a(x-x')} \tilde{G}(k; z) \]  

(73)

\[ \tilde{G}(k; z) = -\frac{i}{z} \frac{\partial}{\partial z} \ln \tilde{\psi}(z, \omega, k_a) \]  

(74)

Choosing boundary conditions on the wave equation solution \( \tilde{\psi}(z, \omega, k_a) \) as in section 3 so that it is a superposition of eigenfunctions which are positive frequency with respect to Minkowski time \( t \), we have

\[ \tilde{\psi}(\omega, k_a, z) = H_2^2(\omega q z) \]  

(75)

where \( q^2 = k^2 a + m^2 \). This resolves the one-parameter ambiguity of vacuum states in this foliation.

The two point function evaluated at equal \( z \) times in this vacuum state is given in term of the inverse kernel, which can be shown (as in section 3) to be

\[ \langle 0 | \varphi(z, \tau, x^a) \varphi(z, \tau', x^a) | 0 \rangle = \frac{\pi}{4} \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)} \int \frac{d^{d-1}k_a}{(2\pi)^{d-1}} e^{i\omega(\tau-\tau')} H_1^1(\omega q z) H_2^2(\omega q z). \]  

(76)

where we have only considered points separated in the \( x^1 \) direction. Rewriting the Hankel functions in terms of modified Bessel functions and performing the \( \omega \) integral gives

\[ \langle 0 | \varphi(z, \tau, x^a) \varphi(z, \tau', x^a) | 0 \rangle = \frac{2}{2^{d-1} \pi \Gamma(d/2)} \int_0^\infty dk k^{d-2} K_0(m \sigma) \]  

(77)

with \( \sigma = 2z \sinh(\frac{\tau-\tau'}{2}) \). The remaining integral can be performed by a Mellin transform to give the final expression for the two point function in this vacuum,

\[ \langle 0 | \varphi(z, \tau, x^a) \phi(z, \tau', x^a) | 0 \rangle = \frac{1}{2\pi} \left( \frac{m}{2\pi \sigma} \right)^{d/2} K_{d+1}(m \sigma) \]  

A similar calculation (see appendix A for details) gives the VEV of \( T_{zz} \) with point-split arguments,

\[ \langle 0 | T_{zz}(\tau, \tau'; z) | 0 \rangle = -\left( \frac{m}{2\pi \sigma} \right)^{d+1} K_{d+1}(m \sigma) \]  

(78)

Comparing with the equivalent Minkowski spacetime results, eqs.(68) and (70), and noticing that \( \sigma \) is simply the geodesic interval between the points \( (\tau, x^a, z) \) and \( (\tau', x^a, z) \) as explained in section 3, we see that as expected they are identical.

**L-R patch**
As already observed in section 4 where we studied the single Rindler wedge \( R \), the hypersurfaces \( \tau = \text{const} \) in this patch alone are not Cauchy complete in the extended Minkowski spacetime. To find such surfaces, which are necessary to have a correct foliation of the spacetime (i.e. respecting the global hyperbolicity and geodesic completeness), we have to treat the \( L \) and \( R \) wedges together. The metric for both patches is \( ds^2 = z^2 d\tau^2 - dz^2 - (dx^a)^2 \). The Cauchy hypersurfaces are then the surfaces \( \tau = \text{const} \) across both patches taken together, as shown in Fig(3), and we consider evolution in \( \partial / \partial \tau \) as shown.

The Schrödinger equation is then

\[
i \frac{\partial \Psi}{\partial \tau} = \frac{1}{2} \int \Sigma d^d x \left\{ -\frac{\delta^2}{\delta \phi(\mathbf{x})^2} + [\partial_x \phi(\mathbf{x})]^2 + [\partial_x \phi(\mathbf{x})]^2 + m^2 \phi(\mathbf{x})^2 \right\} \Psi
\]  

where the integral over the hypersurface is given by

\[
\int \Sigma d^d x = \int d^{d-1} x^a \left[ \int_0^\infty dz \Theta_x(L) + \int_0^\infty dz \Theta_x(R) \right]
\]

\( \Theta_x(L) \) is a theta function which is 0 when \( x \) is in the \( R \) region and is 1 when \( x \) is in the \( L \) region. The vacuum wave functional solution is

\[
\Psi[\phi(z, x^a), \tau] = N_0(\tau) \exp\left\{ -\frac{1}{2} \int \Sigma d^d x \int \Sigma d^d y \phi(\mathbf{x}) G(\mathbf{x}, \mathbf{y}) \phi(\mathbf{y}) \right\}
\]

with kernel\(^8\)

\[
G(\mathbf{x}, \mathbf{y}) = \frac{1}{z z'} \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)^{d-1}} \int \frac{d^{d-1}k_a}{(2\pi)^{d-1}} \omega \tilde{\psi}_k(\omega, \mathbf{x}) \tilde{\psi}_k(\omega, \mathbf{y})
\]

To construct \( G(\mathbf{x}, \mathbf{y}) \) we need a complete orthonormal set of eigenfunctions of the wave equation

\[
\left( \frac{1}{z} \partial_z (z \partial_z) + \partial_{x^a}^2 - m^2 + \frac{\omega^2}{z^2} \right) \tilde{\psi}(\omega, z, x^a) = 0
\]

The unique set\(^9\) consistent with the boundary condition that the field eigenvalues \( \phi(z, x^a) \) in the wave functional tend to zero at spatial infinity is found to be

\[
\tilde{\psi}_k(\omega, x, x^a) = \sqrt{\frac{2\omega}{\pi}} e^{i k_a x^a} K_{i \omega}(q z) \left[ e^{\frac{\pi}{2} \Theta_x(R)} + e^{-\frac{\pi}{2} \Theta_x(L)} \right]
\]

---

\(^8\) We use the notation \( \mathbf{x} = (z, x^a) \) and \( \mathbf{y} = (z', y^a) \) for space coordinates and \( \mathbf{k} = (\omega, k_a) \) and \( p = (\nu, p_a) \) for momenta.

\(^9\) We have already used the \( \tau \) independence of the metric to show that the kernel is a function of \( \mathbf{x}, \mathbf{y} \) only and Fourier transformed with respect to \( \tau \) to find solutions \( \tilde{\psi}(\omega, z, x^a) \) of the wave equation. As in Minkowski spacetime, this carries an implicit definition of the positive frequency convention.
with \( q^2 = (k_a^2 + m^2) \). These functions satisfy

\[
\int_{\Sigma} \frac{d^d x}{(2\pi)^d} \left( \frac{1}{z} \bar{\psi}_k(\omega, x) \psi_k(\omega, x) \right) = \delta(\omega - \nu)\delta^{d-1}(k_a - p_a) \tag{85}
\]

\[
\int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)} \int \frac{d^{d-1} k_a}{(2\pi)^{d-1}} \bar{\psi}_k(\omega, x) \psi_k(\omega, y) = \delta^{d-1}(x^a - y^a)\delta(z - z')
\]

\[
\times \left[ \Theta_x(R)\Theta_y(R) - \Theta_x(L)\Theta_y(L) \right]
\]

\[
\text{The two point function evaluated at equal } \tau \text{ times is given in terms of the inverse kernel}
\]

\[
\Delta(x, y) = \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)} \int \frac{d^{d-1} k_a}{(2\pi)^{d-1}} \frac{1}{\omega} \bar{\psi}_k^*(\omega, x) \psi_k(\omega, y).
\]

\[
\text{and is therefore}
\]

\[
\langle 0 | \phi(\tau, z, x^a) \phi(\tau, z', x^a) | 0 \rangle = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)} \int \frac{d^{d-1} k_a}{(2\pi)^{d-1}} \left[ e^{\frac{i\omega}{2}}\Theta_x(R) + e^{-\frac{i\omega}{2}}\Theta_x(L) \right]
\]

\[
\left[ e^{\frac{i\omega}{2}}\Theta_y(R) + e^{-\frac{i\omega}{2}}\Theta_y(L) \right] K_{i\omega}(qz) K_{i\omega}(qz') \tag{87}
\]

\[
\text{where again we have only considered points separated in the } x^1 \text{ direction. We can evaluate these integrals using the results of appendix C to give}
\]

\[
\langle 0 | \phi(x^1) \phi(y^1) | 0 \rangle = \frac{1}{2\pi} \left( \frac{m}{2\pi \Delta x} \right)^{\frac{d+1}{2}} K_{\frac{d+1}{2}}(m\Delta x) \tag{88}
\]

where

\[
\Delta x = \left\{ \begin{array}{ll}
|z - u| & x, y \in R, R \text{ or } x, y \in L, L \\
|z + u| & x, y \in R, L \text{ or } x, y \in L, R
\end{array} \right.
\]

\[
\Delta x = [z+u] \quad \Delta x = [z-u]
\]

Figure 4: Distance between two points on the \( t = \tau = 0 \) spacelike hypersurface.

As can be seen, Fig (4) this is equivalent to the Minkowski two point function. This is true on any \( \tau = \text{const} \) hypersurface because \( \Delta x \) is just the geodesic distance between the two points and exactly equals the geodesic distance between the same two points in Minkowski spacetime.

A similar calculation (see appendix B) gives the VEV of \( T_{\tau\tau} \) with point-split arguments,

\[
\langle 0 | T_{\tau\tau}(z, z') | 0 \rangle = -z^2 \left( \frac{m}{2\pi |z - z'|} \right)^{\frac{d+1}{2}} K_{\frac{d+1}{2}}(m|z - z'|) \tag{90}
\]
where the two points are in the same wedge. This again is identical to the Minkowski spacetime result, up to coordinate transformation factors.

In particular, notice that these results are quite different from those found for the single Rindler wedge $R$. A correct foliation of Minkowski spacetime must be based on spacelike hypersurfaces which are complete Cauchy surfaces for the whole spacetime.

We see, therefore, that despite the radically different choice of foliations, viz. equal-$t$ surfaces or Rindler-Milne, both the Wightman functions and energy-momentum tensor expectation values are identical. This provides impressive evidence that, in general, physical quantities will indeed be foliation independent, even though the vacuum wave functionals themselves necessarily depend on the foliation chosen to implement the Schrödinger picture.

**Acknowledgements**

One of us (GMS) would like to thank Prof. J-M. Leinaas for extensive discussions and hospitality at the University of Oslo and the Norwegian Academy of Sciences. We are both grateful to Dr. W. Perkins for many helpful discussions.
A Energy-momentum tensor in the Milne universe

In this appendix we calculate the expectation value of the ‘energy’ component of the energy-momentum tensor in the comoving and conformal vacua. The expectation value is the coincidence limit of (18), which in terms of the Fourier transformed kernel (74) is

\[
\langle 0| T_{zz}(\tau, \tau'; z)|0\rangle = \frac{1}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^{d-1}k_a}{(2\pi)^{d-1}} e^{i\omega(\tau-\tau')} \left\{ \overline{G(k^\omega_z z)} \overline{G^*(-k^\omega_z z)} + \frac{1}{\pi} \left[ \frac{\omega^2}{\pi^2} + q^2 \right] \right\}
\]

Here, we are working in \(d\) space dimensions, as needed in section 5, and have point-split in the \(x^1\) direction only.

The kernel for the comoving vacuum is specified by choosing the wave equation solution \(\tilde{\psi}(z, \omega) = H^2_{\omega}(qz)\). The expectation value in this vacuum is therefore

\[
\frac{\pi}{8} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^{d-1}k_a}{(2\pi)^{d-1}} e^{i\omega(\tau-\tau')} \left\{ \left[ \partial_z H^1_{\omega}(qz) \right] \left[ \partial_z H^2_{\omega}(qz) \right] + \left[ \frac{\omega^2}{\pi^2} + q^2 \right] H^1_{\omega}(qz) H^2_{\omega}(qz) \right\}
\]

where we have used the Wronskian \(\mathcal{W}[\psi, \tilde{\psi}^*] = \psi \partial_z \tilde{\psi}^* - [\partial_z \psi] \tilde{\psi}^* = -\frac{4i}{\pi^2}\). Using standard properties of derivatives of Hankel functions, this can be rewritten as

\[
\frac{\pi}{8} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int \frac{d^{d-1}k_a}{(2\pi)^{d-1}} e^{i\omega(\tau-\tau')} \left\{ q^2 \left[ H^1_{\omega}(qz) H^2_{\omega}(qz) - H^1_{\omega+1}(qz) H^2_{\omega-1}(qz) \right] + \frac{2\omega^2}{\pi^2} H^1_{\omega}(qz) H^2_{\omega}(qz) + i\omega q \left[ H^1_{\omega}(qz) H^2_{\omega-1}(qz) + H^1_{\omega+1}(qz) H^2_{\omega+1}(qz) \right] \right\}
\]

Expressing the Hankel functions as modified Bessel functions and performing the \(\omega\) integration (using eq. (105) and derivatives of it with respect to \(\rho\)) reduces this to

\[
-\frac{1}{2\pi} \int \frac{d^{d-1}k_a}{(2\pi)^{d-1}} \sqrt{k_a^2 + m^2} K_1(\sqrt{k_a^2 + m^2})
\]

Finally, we find

\[
\langle 0| T_{zz}(\tau, \tau'; z)|0\rangle_{\text{COM}} = -\left( \frac{m}{2\pi} \right)^{\frac{d+1}{2}} K_{\frac{d+1}{2}}(m\sigma)
\]

with \(\sigma\) is the geodesic interval between the two point-split points. The coincidence limit \((\tau \to \tau')\) is of course divergent.

An alternative representation of the expectation value is found by taking the coincidence limit before performing the \(\omega\) integration. Working in \((1+1)\) dimensions as in section 2 gives

\[
\langle 0| T_{zz}(z)|0\rangle_{\text{COM}} = \int_{0}^{\infty} \frac{d\omega}{2\pi} \left\{ \frac{\omega m^2}{4} \left[ H^1_{\omega}(mz) H^2_{\omega}(mz) - H^1_{\omega+1}(mz) H^2_{\omega-1}(mz) \right] \right\}
\]

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Reexpressing the Hankel functions in terms of Bessel functions it can be shown that

$$
\langle 0 | T_{zz}(z) | 0 \rangle_{\text{COM}} = \int_0^\infty \frac{d\omega}{2\pi} \frac{\omega \cosh(\pi\omega)}{z^2 \sinh(\pi\omega)} + \int_0^\infty \frac{d\omega}{2\pi} \frac{\pi m^2}{4 \sinh^2(\pi\omega) z^2} \times \left\{ 2 \cosh(\pi\omega) \left[ J_{i\omega}(mz) J_{-i\omega}(mz) + J_{i\omega+1}(mz) J_{-i\omega+1}(mz) \right] - [J_{-i\omega}(mz)]^2 \right. \\
\left. - [J_{i\omega}(mz)]^2 + J_{i\omega-1}(mz) J_{i\omega+1}(mz) + J_{-i\omega-1}(mz) J_{-i\omega+1}(mz) \right\}
$$

(93)

In the limit of small \( z \), only the first term is of \( O(z^{-2}) \), the others being of \( O(z^0) \). The first term is also \( m \) independent, while the others are of \( O(m^2) \).

The corresponding calculation of the expectation value of the \( T_{zz} \) component of the energy-momentum tensor in the conformal vacuum, defined by specifying \( \tilde{\psi}(z,\omega) = J_{-i\omega}(mz) \), gives

$$
\pi \int_0^\infty \frac{d\omega}{2\pi} \frac{1}{\sinh(\pi\omega)} \left\{ \left[ \partial_z J_{i\omega}(mz) \right] \left[ \partial_z J_{-i\omega}(mz) \right] + \left[ \frac{\omega^2}{z^2} + m^2 \right] J_{i\omega}(mz) J_{-i\omega}(mz) \right\}
$$

where \( W[\tilde{\psi},\tilde{\psi}^*] = \frac{2i}{\pi z} \sinh(\pi\omega) \). In this case, we find

$$
\langle 0 | T_{zz}(z) | 0 \rangle_{\text{CONF}} = \frac{z^2}{4} \left\{ G(x,y) + \left[ \frac{\partial^2}{\partial z \partial z'} + \frac{\partial^2}{\partial x^a \partial y^a} + m^2 \right] \Delta(x,y) \right\}
$$

(94)

Again, the first term dominates in the small \( z \) or small mass limits, being the only one of \( O(z^{-2}) \) or independent of \( m \).

**B Energy-momentum tensor in Rindler space-time**

In this appendix we calculate the VEV of the energy component of the energy–momentum tensor in the conformal vacuum, defined by specifying \( \tilde{\psi}(z,\omega) = J_{-i\omega}(mz) \), gives

$$
\langle 0 | T_{rr}(x,y) | 0 \rangle = \frac{z^2}{4} \left\{ G(x,y) + \left[ \frac{\partial^2}{\partial z \partial z'} + \frac{\partial^2}{\partial x^a \partial y^a} + m^2 \right] \Delta(x,y) \right\}
$$

Again we are working in \( d \) space dimensions and shall point–split in the \( x^1 \) direction only.

In the \( \text{R} \) Rindler wedge this expectation value reduces to

$$
\frac{z^2}{2\pi^2} \int_0^\infty d\omega \int \frac{d^{d-1}k_a}{(2\pi)^{d-1}} \left\{ \omega^2 \left[ \frac{\partial^2}{\partial z \partial z'} + q^2 \right] \sinh(\pi\omega) K_{i\omega}(qz) K_{i\omega}(qz') \right\}
$$
with \( q^2 = k^2_a + m^2 \) and where we have introduced the complete orthonormal set of solutions to the Fourier transformed wave equation, viz. (54). Using standard properties of modified Bessel functions and performing the \( \omega \) integration gives

\[
\frac{z^2}{2\pi} \int \frac{d^{d-1}k_a}{(2\pi)^{d-1}} \left\{ -qK_1(q|z - z'|) \right\} + \int_0^\infty \frac{dv}{(v^2 + \pi^2)} \left[ \frac{P(v)K_0(qZ)}{zz'} \right. \\
\left. + \frac{q}{Z}(2 + \cosh v)K_1(qZ) - Q(v)q^2K_2(qZ) \right\}
\]

where

\[
P(v) = 2(3v^2 - \pi^2)(v^2 + \pi^2)^{-2} \\
Q(v) = 1 + (z + z' \cosh v)(z' + z \cosh v)Z^{-2}
\]

Finally we find the expectation value in the \( R \) Rindler wedge is

\[
\langle 0 | T_{\tau \tau}(z, z') | 0 \rangle = -z^2 \left( \frac{m}{2\pi |z - z'|} \right)^{\frac{d+1}{2}} K_{\frac{d+1}{2}}(m|z - z'|) + \int_0^\infty \frac{dv}{2\pi(v^2 + \pi^2)} \times \\
\left[ 2\pi \frac{2 + \cosh v - (d + 1)Q(v)}{v^2 - (d + 1)Q(v)} \left( \frac{m}{2\pi Z} \right)^{\frac{d+1}{2}} K_{\frac{d+1}{2}}(mZ) \\
+ \left( \frac{P(v)}{zz'} - Q(v)m^2 \right) \left( \frac{m}{2\pi Z} \right)^{\frac{d+1}{2}} K_{\frac{d+1}{2}}(mZ) \right]
\]

(95)

In considering the expectation value in the \( L \) and \( R \) Rindler wedges together (as in section 5) we use the complete orthonormal set of solutions to the Fourier transformed wave equation, (54) which gives

\[
\frac{z^2}{4\pi^2} \int \frac{d\omega}{\omega^2} \int \frac{d^{d-1}k_a}{(2\pi)^{d-1}} \left\{ \frac{\partial^2}{\partial z \partial z'} + q^2 \right\} K_{i\omega}(qz)K_{i\omega}(qz') \times \\
\left[ e^{i\omega \Theta_x(R)\Theta_y(R) + \Theta_x(L)\Theta_y(L)} + e^{-i\omega \Theta_x(L)\Theta_y(L) + \Theta_x(R)\Theta_y(R)} \right]
\]

As we require the coincidence limit we shall only consider points separated in the same wedge. Implementing this and performing the \( \omega \) integration gives

\[
-\frac{z^2}{2\pi} \int \frac{d^{d-1}k_a}{(2\pi)^{d-1}} \frac{q}{|z - z'|} K_1(q|z - z'|)
\]

which results in

\[
\langle 0 | T_{\tau \tau}(z, z') | 0 \rangle = -z^2 \left( \frac{m}{2\pi |z - z'|} \right)^{\frac{d+1}{2}} K_{\frac{d+1}{2}}(m|z - z'|)
\]

(96)
C Integrals

The following integrals arise in the calculations of expectation values:

\[
\int_0^\infty dx K_{ix}(a)K_{ix}(b) = \frac{\pi}{2} K_0(a + b) \quad (97)
\]

\[
\int_0^\infty dx x^2 K_{ix}(a)K_{ix}(b) = \pi \frac{ab}{2(a + b)} K_1(a + b) \quad (98)
\]

\[
\int_0^\infty dx \cosh(\pi x)K_{ix}(a)K_{ix}(b) = \frac{\pi}{2} K_0(|a - b|) \quad (99)
\]

\[
\int_0^\infty dx x^2 \cosh(\pi x)K_{ix}(a)K_{ix}(b) = -\pi \frac{ab}{2|a - b|} K_1(|a - b|) \quad (100)
\]

\[
\int_0^\infty dx \sinh(\pi x)K_{ix}(a)K_{ix}(b) = \frac{\pi}{2} K_0(|a - b|) \quad (101)
\]

\[
-\pi \int_0^\infty \frac{dz}{\pi^2 + z^2} K_0(\sqrt{a^2 + b^2 + 2ab \cosh z})
\]

\[
\int_0^\infty dx x^2 \sinh(\pi x)K_{ix}(a)K_{ix}(b) = -\pi \frac{ab}{2|a - b|} K_1(|a - b|) \quad (102)
\]

\[
+2\pi \int_0^\infty \frac{dz (3z^2 - \pi^2)}{(\pi^2 + z^2)^3} K_0(\sqrt{a^2 + b^2 + 2ab \cosh z})
\]

valid for \(a, b > 0\) \[13, 19\].

The Mellin transform \[11\] of the Bessel function, with \(\text{Re} (s, \alpha, \beta) > 0\) is

\[
\int_0^\infty dx x^{s-1} (x^2 + \beta^2)^{-\nu} \frac{\Gamma(s)}{\Gamma(\nu)} K_{\frac{s}{2}}(\sqrt{a^2 + b^2 + 2ab \cosh z}) = a^{-\frac{s}{2}} 2^{\frac{\nu}{2}} 2^{\frac{\nu}{2}} - 1 \beta^{\frac{s}{2} - \nu} \Gamma(\frac{s}{2}) K_{\frac{s}{2} - \nu}(\alpha \beta). \quad (103)
\]

We also need \[12\],

\[
\int_{-\infty}^{\infty} dx e^{ix\rho} K_{i\nu}(a)K_{i\nu}(b) = \pi \left(\frac{a + b}{a + b}\right)^\nu K_{2\nu}(\sqrt{a^2 + b^2 + 2ab \cosh \rho}) \quad (104)
\]

\[
\int_{-\infty}^{\infty} dx e^{ix\rho} K_{i\nu}(ia)K_{i\nu}(-ia) = \pi e^{-\frac{\pi}{2}(\omega + \nu)} e^{i\frac{\pi}{2}(\nu - \omega)} K_{\nu}(2\alpha \sinh \frac{\rho}{2}) \quad (105)
\]

where \(|\text{arg}a| + |\text{arg}b| + |\text{Im}\rho| < \pi\).
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