The Ratios conjecture for real Dirichlet characters and multiple Dirichlet series

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Abstract

Conrey, Farmer and Zirnbauer introduced a recipe to find asymptotic formulas for the sum of ratios of products of shifted L-functions. These ratios conjectures are very powerful and can be used to determine many statistics of L-functions, including moments or statistics about the distribution of zeros.

We consider the family of real Dirichlet characters, and use multiple Dirichlet series to prove the ratios conjectures with one shift in the numerator and denominator in some range of the shifts. This range can be improved by extending the family to include non-primitive characters. All of the results are conditional under the Generalized Riemann hypothesis.

This extended range is good enough to enable us to compute an asymptotic formula for the sum of shifted logarithmic derivatives near the critical line. As an application, we compute the one-level density for test functions whose Fourier transform is supported in \((-2, 2)\), including lower-order terms.

1. Introduction

In [Mon73], Montgomery studied the pair correlation of zeros of the Riemann zeta function, and Dyson famously observed that the resulting density matches that of the pair correlation of eigenvalues in a GUE ensemble of random matrices. Since then, it is believed that random matrix theory can be used to model many statistics of L-functions, such as moments [CFKRS05], or the distribution of low-lying zeros [KaSa99a], [KaSa99b].

Farmer [Far93] conjectured that

$$\int_0^T \frac{\zeta(s + \alpha)\zeta(1 - s + \beta)}{\zeta(s + \gamma)\zeta(1 - s + \delta)} dt \sim T^{(\alpha + \delta)(\beta + \gamma)}(\alpha + \beta)(\gamma + \delta) - T^{-\alpha - \beta}(\delta - \beta)(\gamma - \alpha)(\alpha + \beta)(\gamma + \delta),$$

(1.1)

where \( s = 1/2 + it \), and \( \alpha, \beta, \gamma, \delta \sim \frac{1}{\log T} \), and noticed that it has many implications including Montgomery’s pair correlation conjecture.

In [CFZ08], [CFZ05], based on the conjecture above, Conrey, Farmer and Zirnbauer came up with the ratios conjectures, which give a general recipe to predict asymptotic formulas for the sum of ratios of products of shifted L-functions. These are very powerful as they are able to predict many other local or global statistics, which agree with the predictions coming from random matrix theory (see [CoSn07] for some applications). The conjectured asymptotics are expected to hold with a very strong error term, so, unlike random matrix theory, they are also

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We will study the family of quadratic Dirichlet L-functions with one shift in the numerator and in the denominator. For a fundamental discriminant $d$, we denote by $\chi_d$ the Kronecker symbol ($d$). In this case, the ratios conjecture has the form

$$\sum_{d \leq X}^{\ast} \frac{L(1/2 + \alpha, \chi_d)}{L(1/2 + \beta, \chi_d)} = \sum_{d \leq X}^{\ast} \left( \frac{\zeta(1 + 2\alpha)}{\zeta(1 + \alpha + \beta)} A_D(\alpha, \beta) \right) + O(X^{1/2+\varepsilon}),$$

(1.2)

where the star indicates that the sums run over fundamental discriminants, and

$$A_D(\alpha, \beta) = \prod_p \left( 1 - \frac{1}{p^{1+\alpha+\beta}} \right)^{-1} \left( 1 - \frac{1}{(p+1)p^{1+2\alpha}} - \frac{1}{(p+1)p^{\alpha+\beta}} \right).$$

(1.3)

The original conjecture asserts that this formula holds uniformly in $\alpha, \beta$ with $|\text{Re}(\alpha)| < 1/4, \frac{1}{\log X} \ll \text{Re}(\beta) < 1/4$, and $\text{Im}(\alpha), \text{Im}(\beta) \ll X^{-\varepsilon}$. We, conditionally under the generalized Riemann hypothesis (GRH), prove the conjecture in a smaller range of the shifts.

We are not aware of any other results towards the proof of this conjecture over number fields. See the recent work of Bui, Florea and Keating [BFK21] for similar results over function fields, using a different method.

We now briefly describe the recipe of Conrey, Farmer and Zirnbauer. The L-functions in the numerator are replaced by Dirichlet polynomials using the approximate functional equation, while those in the denominator are expanded into a full Dirichlet series. To obtain the main terms in the conjecture, the sums are completed, only the diagonal terms are retained, and they are replaced by their average over the family.

Our strategy is to use Mellin inversion (or Perron’s formula), which leads to an integral that involves a triple Dirichlet series $A(s, w, z)$. The main terms arise naturally from residues of $A(s, w, z)$, which gives another evidence for the validity of the heuristics of Conrey, Farmer and Zirnbauer. An advantage of our approach is that there are no non-diagonal terms to be bounded. The error term depends on how far we can meromorphically continue the triple Dirichlet series, which we obtain from showing that $A(s, w, z)$ satisfies some functional equations.

Let us now state our results, which are conditional under GRH. Theorem 1.1 is weaker than Theorem 1.2, its purpose is mainly to illustrate the method of the proof and to illustrate how the main terms arise from the residues of the multiple Dirichlet series. The difference in the results can be explained on the level of functional equations satisfied by the associated multiple Dirichlet series, which is explained in Section 3.

Let

$$R_D(X, \alpha; \beta; f) = \sum_{d \geq 1}^{\ast} \frac{L(1/2 + \alpha, \chi_d)}{L(1/2 + \beta, \chi_d)} f(d/X),$$

(1.4)

where the sum runs over positive fundamental discriminants. We also let $f(x)$ be a smooth, fast-decaying weight function.

To simplify some of the formulas, we denote by $\Gamma_e(s)$ and $\Gamma_o(s)$ the ratios of gamma factors
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that appear in the functional equation for even or odd characters, so that

\[ \Gamma_e(s) = \frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}, \quad \Gamma_o(s) = \frac{\Gamma(\frac{2-s}{2})}{\Gamma(\frac{s+1}{2})}. \]  

(1.5)

We denote by \( \mathcal{M}f(s) \) the Mellin transform of \( f(s) \), defined by (2.27).

**Theorem 1.1.** Assume GRH, and let \( 0 < |\text{Re}(\alpha)| < \text{Re}(\beta) < 1/2 \). Then

\[
R_D(X, \alpha, \beta; f) = \frac{X\mathcal{M}f(1)\zeta(1+2\alpha)}{2\zeta(2)\zeta(1+\alpha+\beta)} \left[P_D(1/2 + \beta, 1/2 + \alpha) + X^{1-\alpha}\mathcal{M}f(1-\alpha)\zeta(1-2\alpha)\pi^\alpha \Gamma_e(1/2 + \alpha) \right. \\
+ \left. \frac{1}{2\zeta(2)\zeta(1-\alpha+\beta)} P_D(1/2 + \beta, 1/2 - \alpha) + O_{\alpha,\beta}(X^{1-\text{Re}(\alpha)/2-\text{Re}(\beta)/2+\varepsilon}) \right],
\]

(1.6)

where

\[
P_D(z, w) = \prod_p \left(1 + \frac{1 - p^{-w}}{(p^z + w - 1)(p + 1)}\right).
\]

(1.7)

The error term in this result is not uniform in the shifts \( \alpha, \beta \). It is possible to obtain a uniform result using a similar estimate as in Section 6.5, which would introduce a factor of size \( |\alpha|^{c+\varepsilon} |\beta|^\varepsilon \) for some \( c > 0 \) into the error term, limiting the size of the imaginary parts. Note also that this theorem doesn’t hold in the range \( \alpha = \beta \), which is important for some applications. Both of these aspects can be improved by making the sum run over all quadratic characters, not only the primitive ones, as shown in Theorem 1.2.

The main terms from (1.6) agree with those in (1.2) (adjusted for the smooth weights). We remark that the computations leading to them are the same as those in the recipe of Conrey, Farmer and Zirnbauer, but they come from a different source. While in the heuristic, one obtains the main terms by discarding the non-diagonal terms assuming that they only contribute into the error, in our case they come from the residues of certain triple Dirichlet series.

It turns out that we can get a better range of the shifts if we extend our family to contain all characters, including non-primitive ones. The reason is that in this case, the associated triple Dirichlet series has an extra functional equation. We will change notation from Section 6 on and denote by \( \chi_n \) the Jacobi symbol \( (\frac{n}{2}) \). We denote by \( L_{(2)}(s, \chi_n) \) the L-function with the Euler factor at 2 removed. Then we get the following result:

**Theorem 1.2.** Assume GRH and let \( \varepsilon > 0 \). Then for \( \text{Re}(\alpha) > 0, \text{Re}(\beta) > \varepsilon, 1 + \text{Re}(\beta) > \text{Re}(\alpha) \), we have

\[
\sum_{n \text{ odd}} \frac{L_{(2)}(1/2 + \alpha, \chi_n)f(n/X)}{L_{(2)}(1/2 + \beta, \chi_n)} = X\mathcal{M}f(1) \zeta_{(2)}(1+2\alpha) \prod_{p>2} \left(1 + \frac{p^{\alpha-\beta} - 1}{p^{1+\alpha-\beta}(p^{1+\alpha+\beta} - 1)}\right) \\
+ X^{1-\alpha}\mathcal{M}f(1-\alpha)\pi^\alpha \left(\Gamma_o\left(\frac{1}{2} + \alpha\right) + \Gamma_e\left(\frac{1}{2} + \alpha\right)\right) \frac{\zeta(1-2\alpha)P\left(\frac{3}{2} - \alpha + \beta\right)}{\zeta(2)\zeta(1-\alpha+\beta)(6-2^{\alpha-\beta+1})} + O \left( (1 + |\alpha|)^\varepsilon |\beta|^\varepsilon X^{N(\alpha,\beta)+\varepsilon} \right),
\]

(1.8)
where

\[
P(z) = \prod_p \left( 1 + \frac{1}{(p^{z/2} - 1)(p + 1)} \right), \quad (1.9)
\]

and

\[
N(\alpha, \beta) = \max \left\{ 1 - 2\Re(\alpha), 1 - 2\Re(\beta), \frac{1}{2} - \Re(\alpha), \frac{1}{2} - \Re(\beta), -\frac{5}{2} \right\}. \quad (1.10)
\]

The condition \(1 + \Re(\beta) > \Re(\alpha)\) is to ensure convergence of the product \(P(3/2 - \alpha + \beta)\), and the \(-5/2\) in the error term comes from our definition of \(S_j\) in Section 6.5.

In this case, the error term is uniform in \(\alpha, \beta\). Let us emphasize that the imaginary parts may grow as fast as any power of \(X\), which is better than the original conjecture.

This result also allows us to differentiate with respect to \(\alpha\) and take \(\alpha = \beta = r\), thus obtain

**Theorem 1.3.** Assume GRH and let \(\Re(r) > \varepsilon\). Then

\[
\sum_{\substack{n \geq 1 \\text{n odd} \\text{odd}}} \frac{L'(1/2 + r, \chi_n)}{L(1/2 + r, \chi_n)} f(n/X) = \frac{XMf(1)}{2} \left( \frac{\zeta'(1 + 2r)}{\zeta(1 + 2r)} + \sum_{p > 2} \frac{\log p}{p(p^{1+2r} - 1)} \right) \\
- X^{1-r}Mf(1 - r)\pi^r \left( \Gamma_o \left( \frac{1}{2} + r \right) + \Gamma_e \left( \frac{1}{2} + r \right) \right) \frac{\zeta(1 - 2r)}{4} \\
+ O \left( 1 + |r|^\varepsilon X^{N(r)+\varepsilon} \right), \quad (1.11)
\]

where

\[
N(r) = \max \{ 1 - 2\Re(r), 1/2 - \Re(r), -5/2 \}. \quad (1.12)
\]

At this point, we can sieve out the non-primitive characters and obtain an asymptotic formula for the sum running over square-free integers:

**Theorem 1.4.** Assume GRH and let \(\varepsilon < \Re(r) < 1/4\). Then

\[
\sum_{\substack{n \geq 1 \\text{n odd} \\text{odd}}} \frac{\mu^2(n)L'(1/2 + r, \chi_n)}{L(1/2 + r, \chi_n)} f(n/X) = \frac{2XMf(1)}{3\zeta(2)} \left( \frac{\zeta'(1 + 2r)}{\zeta(1 + 2r)} + \sum_{p > 2} \frac{\log p}{p(p^{1+2r} - 1)} \right) \\
- X^{1-r}Mf(1 - r)\pi^r \left( \Gamma_o \left( \frac{1}{2} + r \right) + \Gamma_e \left( \frac{1}{2} + r \right) \right) \frac{\zeta(1 - 2r)}{4\zeta(2)(2 - 2r)} \\
+ O \left( |r|^\varepsilon X^{1-2\Re(r)+\varepsilon} \right), \quad (1.13)
\]

As opposed to Theorems 1.2 and 1.3, but similarly as Theorem 1.1, we now only have primitive characters, so we can compare the result with the prediction. This is done in Appendix B, where we show that the main terms in Theorem 1.4 agree with those coming from the recipe. However, in this case, the computations leading to Theorem 1.4 are different from those in the heuristic.

As an application of our results, we compute the one-level density in our family of quadratic Dirichlet characters. For an even Schwartz function \(h(x)\) whose Fourier transform \(\hat{h}\) is supported
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in \([-a, a]\) for some \(a > 0\), the one-level density is defined by

\[
D(X; h) = \frac{1}{F(X)} \sum_{n \geq 1, n \text{ odd}} \mu^2(n) f \left( \frac{n}{X} \right) \sum_{\gamma_n} h \left( \frac{\gamma_n \log X}{2\pi} \right),
\]

(1.14)

where \(\gamma_n\) runs over the imaginary parts of the non-trivial zeros of \(L(s, \chi_n)\), and

\[
F(X) = \sum_{n \geq 1, n \text{ odd}} \mu^2(n) f \left( \frac{n}{X} \right).
\]

(1.15)

By the conjectures of Katz and Sarnak [KaSa99a], [KaSa99b],

\[
D(X; h) \sim \int_{-\infty}^{\infty} h(u) \left( 1 - \frac{\sin(2\pi u)}{2\pi u} \right) du,
\]

(1.16)

where the kernel of the integral governs the distribution of eigenvalues close to 1 in a symplectic ensemble of random matrices. This has been proved for \(a < 1\) and for \(a < 2\) under GRH by Özlük and Snyder [ÖzSn99], and increasing the support further is a notoriously difficult problem. The ratios conjecture (1.2) implies that the above asymptotic holds for arbitrarily large \(a\), and it allows us to compute all lower order terms up to square-root error. In [Mil08], Miller showed unconditionally that for limited \(a\), the ratios conjecture prediction agrees with reality, including lower order terms. See also [Mil09], [MiMo11], [GJM+10], [HMM11] for similar results in other families, [BFK11] for related work over function fields, or [DHJ15], where the ratios conjecture is applied to compute the one-level density in two families of elliptic curves.

On the other hand, Fiorilli and Miller [FiMi15] computed the one-level density in the family of all Dirichlet characters modulo \(q\), including lower order terms beyond square-root. They discovered a term not predicted by the ratios conjecture, concluding that the conjectured error is essentially best possible.

As a consequence of Theorem 1.4, we compute the one-level density provided \(a < 2\), thus recovering the results of Özlük and Snyder, including lower order terms with a power-saving error.

**Corollary 1.5.** Assume GRH and let \(h\) be as above. Then

\[
\sum_{n \geq 1, n \text{ odd}} \mu^2(n) f \left( \frac{n}{X} \right) \sum_{\gamma_n} h \left( \frac{\gamma_n \log X}{2\pi} \right)
\]

\[
= \frac{2X}{\log X} \int_{-\infty}^{\infty} h(u) \left\{ 2Mf(1) \frac{\zeta'}{\zeta} \left( 1 + \frac{4\pi i u}{\log X} \right) + \sum_{p > 2} \frac{\log p}{(p + 1)(p^{1 + 4\pi i u/\log X} - 1)} \right\} du
\]

\[-e^{-2\pi i u} Mf \left( 1 - \frac{2\pi i u}{\log X} \right) \frac{2\pi i u}{\log X} \left( \frac{1}{2} + \frac{2\pi i u}{\log X} \right) + \Gamma \left( \frac{1}{2} + \frac{2\pi i u}{\log X} \right) \right) \zeta \left( 1 - \frac{4\pi i u}{\log X} \right) \right\} du
\]

\[-\frac{1}{\log X} \sum_{n \geq 1, n \text{ odd}} \mu^2(n) f \left( \frac{n}{X} \right) \log \left( \frac{\pi}{n} \right) \int_{-\infty}^{\infty} h(u) du
\]

\[-\frac{X Mf(1)}{3\zeta(2) \log X} \int_{-\infty}^{\infty} h(u) \left( \Gamma' \left( \frac{1}{2} + \frac{2\pi i u}{\log X} \right) + \frac{1}{\Gamma_0} \left( 1 - \frac{2\pi i u}{\log X} \right) \right) du
\]

\[+ O \left( X^{1/2+a/4+\varepsilon} \right).
\]

(1.17)
This implies that if \( a < 2 \),
\[
D(X; h) = \int_{-\infty}^{\infty} h(u) \left( 1 - \frac{\sin(2\pi u)}{2\pi u} \right) du + O \left( \frac{1}{\log X} \right).
\] (1.18)

The proof is given in Section 10, where we also explain that any improvement of the error term in Theorem 1.4, which is directly related to a meromorphic continuation of certain triple Dirichlet series, would allow us to increase the support.

Our strategy to prove Theorems 1.1 and 1.2 is to rewrite the sums as integrals using Mellin inversion (a smooth version of Perron’s formula), and then investigate the analytic properties of the relevant triple Dirichlet series. We show that they have two poles, whose residues give rise to the two main terms. The error then depends on how far our triple Dirichlet series can be meromorphically extended, which in turn depends on whether they satisfy certain functional equations.

A general theory of multiple Dirichlet series has been developed by Bump, Chinta, Diaconu, Friedberg, Hoffstein and others (see for example [BFG12], [Blo11], [DGH03], [GoHo85], or the expository papers [Bum], [BFH96], [CFH06] for an introduction to the theory). In [DGH03], the authors prove that if certain multiple Dirichlet series admit a meromorphic continuation beyond a certain point, then the moments of the Riemann zeta function and quadratic Dirichlet L-functions satisfy the asymptotics predicted by random matrix theory.

According to the heuristics developed in [BFH96] and [CFH06] that we present in Section 3, our triple Dirichlet series are expected to satisfy two functional equations. To prove these rigorously, one usually twists the L-functions in the coefficients of the triple Dirichlet series by certain weights, carefully chosen so that the equations are satisfied. For more details, see [CFH06] for an exposition of the general theory, or Blomer’s work [Blo11], where the situation is “simple” enough, so the author gives very explicit results and computations.

We use a different method in this paper. In the case of fundamental discriminants, we only have one functional equation, which is essentially due to the fact that
\[
L_D(s, \chi) = \sum_{d \geq 1}^* \frac{\chi(d)}{d^s} \tag{1.19}
\]
doesn’t satisfy any relation between \( L_D(s, \chi) \) and \( L_D(1 - s, \chi) \). This explains why we obtain a weaker result in this case.

When we include the non-primitive characters, we are summing over a nicer set, so both functional equations potentially hold. However, we have to deal with the L-functions of non-primitive characters – this is usually done by inserting the extra weights mentioned above. Instead, we introduce a functional equation that is valid for all Dirichlet characters, but where on the other side, one has a different Dirichlet series whose coefficients are twisted Gauss sums (see Proposition 2.3 and Appendix A). It then becomes straightforward to prove the functional equations for our triple Dirichlet series, but with a different triple Dirichlet series on the other side.

2. Preliminaries

Throughout the paper, \( \varepsilon \) will denote a small positive number, that may be different at each appearance. All implied constants can depend on \( \varepsilon \).
For an odd positive integer $n$, $\chi_n$ denotes the quadratic Dirichlet character given by the Jacobi symbol $\left(\frac{\cdot}{n}\right)$. The character $\chi_n$ is primitive for square-free $n$, and it is even for $n \equiv 1 \pmod{4}$ and odd for $n \equiv 3 \pmod{4}$. We will also work with the Kronecker symbol $\left(\frac{\cdot}{4}\right)$, which is periodic modulo $k$ only if $k \equiv 0, 1 \pmod{4}$. However, we often use the fact that if $n$ is odd, then $\left(\frac{4}{n}\right) = \left(\frac{4}{\overline{n}}\right)$, and $\left(\frac{4}{n}\right)$ is a Dirichlet character modulo $4k$ for any odd $k$. We also denote by $\psi_1, \psi_{-1}$ the principal and non-principal characters modulo $4$ respectively, and by $\psi_2, \psi_{-2}$ the two primitive quadratic characters modulo $8$ given by the Kronecker symbols $\psi_j(n) = \left(\frac{j}{n}\right)$, and by $\psi_0$ the primitive principal character, so that $\psi_0(n) = 1$ for all $n \in \mathbb{Z}$.

In sections 3, 4 and 5, $\chi_d$ denotes the Kronecker symbol $\left(\frac{d}{4}\right)$ for a fundamental discriminant $d$.

For a Dirichlet character $\chi$ modulo $n$, we define the shifted Gauss sums

$$\tau(\chi, q) = \sum_{j \, (\text{mod } n)} \chi(j)e(jq/n),$$

where we use the standard notation $e(x) = e^{2\pi i x}$.

If $\chi$ is a primitive character, then

$$\tau(\chi, q) = \bar{\chi}(q)\tau(\chi, 1).$$

Lemma 2.1. Let $\chi_1, \chi_2$ be two Dirichlet characters modulo $n_1$ and $n_2$, respectively, and assume that $(n_1, n_2) = 1$. Then for $\chi_1\chi_2$ considered as a Dirichlet character modulo $n_1n_2$, we have

$$\tau(\chi_1\chi_2, q) = \chi_1(n_2)\chi_2(n_1)\tau(\chi_1, q)\tau(\chi_2, q).$$

Proof. By the Chinese remainder theorem, we can write any $j \in \mathbb{Z}/(n_1n_2\mathbb{Z})$ uniquely as

$$j = j_1n_2\overline{n_2} + j_2n_1\overline{n_1},$$

where $n_1\overline{n_1} \equiv 1 \pmod{n_2}$, $n_2\overline{n_2} \equiv 1 \pmod{n_1}$, $j_1 \in \mathbb{Z}/n_1\mathbb{Z}$, and $j_2 \in \mathbb{Z}/n_2\mathbb{Z}$. Hence, we have

$$\tau(\chi_1\chi_2, q) = \sum_{j \, (\text{mod } n_1n_2)} \chi_1(j)\chi_2(j)e\left(\frac{jq}{n_1n_2}\right)
= \sum_{j_1 \, (\text{mod } n_1)} \sum_{j_2 \, (\text{mod } n_2)} \chi_1(j_1)\chi_2(j_2)e\left(\frac{qj_1n_2}{n_1}\right)e\left(\frac{qj_2n_1}{n_2}\right)
= \chi_1(n_2)\chi_2(n_1)\tau(\chi_1, q)\tau(\chi_2, q).$$

Lemma 2.2. (i) If $\ell \equiv 1 \pmod{4}$, then

$$\tau\left(\left(\frac{4\ell}{\cdot}\right), q\right) = \begin{cases} 0, & \text{if } q \text{ is odd,} \\
-2\tau\left(\left(\frac{\ell}{\chi}\right), q\right), & \text{if } q \equiv 2 \pmod{4}, \\
2\tau\left(\left(\frac{\ell}{\chi}\right), q\right), & \text{if } q \equiv 0 \pmod{4}. \end{cases}$$

(ii) If $\ell \equiv 3 \pmod{4}$, then

$$\tau\left(\left(\frac{4\ell}{\cdot}\right), q\right) = \begin{cases} 0, & \text{if } q \text{ is even,} \\
-2i\tau\left(\left(\frac{\ell}{\chi}\right), q\right), & \text{if } q \equiv 1 \pmod{4}, \\
2i\tau\left(\left(\frac{\ell}{\chi}\right), q\right), & \text{if } q \equiv 3 \pmod{4}. \end{cases}$$

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Proof. Quadratic reciprocity gives for $\ell \equiv 1 \pmod{4}$

$$\tau \left( \left( \frac{4\ell}{\cdot} \right), q \right) = \tau \left( \left( \frac{\cdot}{\ell} \right) \psi_1, q \right),$$

and for $\ell \equiv 3 \pmod{4}$

$$\tau \left( \left( \frac{4\ell}{\cdot} \right), q \right) = \tau \left( \left( \frac{\cdot}{\ell} \right) \psi_{-1}, q \right).$$

The result then follows from Lemma 2.1 with $n_1 = \ell$ and $n_2 = 4$. \hfill \Box

For quadratic characters, we also define $G(\chi_n, q)$ by

$$G(\chi_n, q) = \left( \frac{1 - i}{2} + \left( \frac{-1}{n} \right) \frac{1 + i}{2} \right) \tau(\chi_n, q) =
\begin{cases}
\tau(\chi_n, q), & \text{if } n \equiv 1 \pmod{4}, \\
\tau(\chi_n, q), & \text{if } n \equiv 3 \pmod{4},
\end{cases}$$

whose advantage is that they are multiplicative in $n$, so that for $(m, n) = 1$, we have

$$G(\chi_{mn}, q) = G(\chi_m, q)G(\chi_n, q).$$

Moreover, if $p$ is an odd prime and $p^a || q$, we have by [Sou00, Lemma 2.3]

$$G \left( \left( \frac{\cdot}{p^k} \right), q \right) =
\begin{cases}
\varphi(p^k), & \text{if } k \leq a, k \text{ even}, \\
0, & \text{if } k \leq a, k \text{ odd}, \\
-p^a, & \text{if } k = a + 1, k \text{ even}, \\
\left( \frac{q^{-a}}{p} \right) p^a \sqrt{p}, & \text{if } k = a + 1, k \text{ odd}, \\
0, & \text{if } k \geq a + 2,
\end{cases}$$

For primitive Dirichlet characters of conductor $d$, we have

$$L(s, \chi) = \varepsilon(\chi) \left( \frac{\pi}{4} \right)^{s-1/2} \Gamma_{e/o}(s)L(1 - s, \bar{\chi}),$$

where $\varepsilon(\chi) = \frac{\varphi(\chi_m)}{\sqrt{d}}$, and $a = 1$ and $\Gamma_{e/o} = \Gamma_e$ if $\chi$ is even, or $a = -i$ and $\Gamma_{e/o} = \Gamma_o$ if $\chi$ is odd.

We now introduce a functional equation valid for all Dirichlet characters $\chi$ modulo $n$.

**Proposition 2.3.** Let $\chi$ be any character modulo $n$. Then we have

$$L(s, \chi) = \varepsilon(\chi) \frac{\pi^{s-1/2}}{n^s} \Gamma_{e/o}(s)K(1 - s, \chi),$$

where

$$K(s, \chi) = \sum_{q=1}^{\infty} \frac{\tau(\chi, q)}{q^s},$$

$$\Gamma_{e/o}(s) =
\begin{cases}
\Gamma_e(s), & \text{if } \chi \text{ is even, or} \\
\Gamma_o(s), & \text{if } \chi \text{ is odd},
\end{cases}$$

and

$$\varepsilon(\chi) =
\begin{cases}
1, & \text{if } \chi \text{ is even, or} \\
-i, & \text{if } \chi \text{ is odd}.
\end{cases}$$
Note that if \( \chi \) is a primitive character, then we can use (2.2) and recover (2.13) from (2.14).

**Proof.** Follow one of the usual proofs of the functional equation that uses Poisson summation. The application of the Poisson summation leads to some Gauss sums, and for a primitive character \( \chi \), one uses (2.2) to change the Gauss sums back to characters. Skipping this last step and leaving the Gauss sums unchanged gives the proof.

We include the details in Appendix A.

We denote by \( L_k(s, \chi) \) the L-function with Euler factors of \( p \mid k \) removed, so

\[
L_k(s, \chi) = L(s, \chi) \prod_{p \mid k} \left( 1 - \frac{\chi(p)}{p^s} \right)
\]

We now record some useful estimates that hold under GRH. First is the Lindel"of bound: for \( \text{Re}(s) \geq 1/2 \),

\[
|L(s, \chi_n)| \ll |sn|^{\varepsilon}.
\]

Next, if \( n \) is squarefree so that \( \chi_n \) is primitive, and for \( \text{Re}(r) > \varepsilon \), we also have

\[
\left| \frac{L'(1/2 + r, \chi_n)}{L(1/2 + r, \chi_n)} \right| \ll \log^2 (|r|n).
\]

Finally, for any \( n \) and \( \text{Re}(s) > 1/2 + \varepsilon \), we have

\[
\frac{1}{|L(s, \chi_n)|} \ll |sn|^{\varepsilon}.
\]

Let

\[
L_D(s, \chi) = \sum_{d \geq 1}^{*} \frac{\chi(d)}{d^s}
\]

be the Dirichlet series where the sum runs over fundamental discriminants.

**Lemma 2.4.** For a Dirichlet character \( \chi \), we have

\[
L_D(s, \chi) = \left( 1 + \frac{\chi(4)}{2 \cdot 4^s} + \frac{\chi(8)}{8^s} \right) \frac{L(s, \chi\psi_1)}{L(2s, \chi^2\psi_1)} + \left( 1 - \frac{\chi(4)}{2 \cdot 4^s} \right) \frac{L(s, \chi\psi_1)}{L(2s, \chi^2\psi_1)}
\]

**Proof.** Any integer \( d \) is a fundamental discriminant if either

a) \( d \equiv 1 \pmod{4} \) and is squarefree, or

b) \( d = 4m \) with \( m \equiv 3 \pmod{4} \) and \( m \) is squarefree, or

c) \( d = 4m \) with \( m \equiv 2 \pmod{4} \) and \( m \) is squarefree.

We will compute the contribution of each of these three sets separately.

For part a), we have:

\[
\sum_{d \equiv 1 \pmod{4}} \frac{\chi(d)\mu^2(d)}{d^s} = \frac{1}{2} \sum_{d \geq 1} \frac{\chi(d)\mu^2(d)(\psi_1(d) + \psi_{-1}(d))}{d^s}
\]
Part b) gives:

\[
\sum_{d=4m, \ m \equiv 3 \ (\text{mod } 4)} \frac{\chi(d)\mu^2(m)}{d^s} = \frac{\chi(4)}{2 \cdot 4^s} \sum_{m \geq 1} \frac{\chi(m)\mu^2(m)(\psi_1(m) - \psi_{-1}(m))}{m^s} = \frac{\chi(4)}{2 \cdot 4^s} \left( \frac{L(s, \chi \psi_1)}{L(2s, \chi^2 \psi_1)} - \frac{L(s, \chi \psi_{-1})}{L(2s, \chi^2 \psi_1)} \right). \tag{2.25}
\]

The condition in part c) is equivalent to \(d = 8m\) with \(m\) odd and squarefree. Hence we obtain

\[
\sum_{d=8m, \ m \ odd} \frac{\chi(d)\mu^2(m)}{d^s} = \frac{\chi(8)}{8^s} \sum_{m \geq 1} \frac{\chi(m)\psi_1(m)\mu^2(m)}{m^s} = \frac{\chi(8)}{8^s} \frac{L(s, \chi \psi_1)}{L(2s, \chi^2 \psi_1)}. \tag{2.26}
\]

Adding the three parts gives the result.

\[
\square
\]

For a function \(f(x)\), we denote by \(\mathcal{M}f(s)\) its Mellin transform defined by

\[
\mathcal{M}f(s) = \int_0^\infty f(x)x^{s-1}dx, \tag{2.27}
\]

for these \(s\) where the integral converges. If \(f(x)\) is smooth, \(\mathcal{M}f(s)\) decays faster than any polynomial in vertical strips.

If \(\mathcal{M}f(s)\) is analytic for \(a < \text{Re}(s) < b\), then the inverse Mellin transform is given by

\[
f(x) = \frac{1}{2\pi i} \int_{(c)} x^{-s} \mathcal{M}f(s)ds, \tag{2.28}
\]

where the integral is taken over a vertical line \(\text{Re}(s) = c\) and \(a < c < b\) is arbitrary.

The following estimate is a consequence of Stirling’s formula: for a fixed \(\sigma \in \mathbb{R}\) and \(|t| \geq 1\), we have

\[
|\Gamma(\sigma + it)| \asymp e^{-|t|^{\frac{\sigma}{2}}} |t|^{\sigma - 1/2}. \tag{2.29}
\]

We have Legendre’s duplication formula

\[
\Gamma(s)\Gamma(s + 1/2) = 2^{1-2s}\sqrt{\pi}\Gamma(2s) \tag{2.30}
\]

and Euler’s reflection formula

\[
\Gamma(1 - s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}. \tag{2.31}
\]

We will also use the formula

\[
\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} = \frac{2^s \sin(\pi s/2) \Gamma(1-s)}{\sqrt{\pi}}. \tag{2.32}
\]

Recall that \(\Gamma_o(s)\) and \(\Gamma_e(s)\) denote the ratios of the gamma factors that appear in the functional equation for even or odd characters, so

\[
\Gamma_e(s) = \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}, \tag{2.33}
\]
and

\[ \Gamma_o(s) = \frac{\Gamma \left( \frac{2-s}{2} \right)}{\Gamma \left( \frac{s+1}{2} \right)}. \] (2.34)

**Lemma 2.5.** We have

\[ \Gamma_o(s) + \Gamma_e(s) = \frac{2^{s+1/2} \Gamma(1-s) \cos \left( \frac{\pi s}{2} - \frac{\pi}{4} \right)}{\sqrt{\pi}}. \] (2.35)

**Proof.** We have

\[ \Gamma_o(s) + \Gamma_e(s) = \frac{\Gamma \left( \frac{2-s}{2} \right) \Gamma \left( \frac{s}{2} \right) + \Gamma \left( \frac{1-s}{2} \right) \Gamma \left( \frac{s+1}{2} \right)}{\Gamma \left( \frac{s+1}{2} \right) \Gamma \left( \frac{s}{2} \right)}. \] (2.36)

Using the reflection formula in the numerator and the duplication formula in the denominator, this equals

\[ \frac{2^{s-1} \sqrt{\pi}}{\Gamma(s)} \left( \frac{1}{\sin(\pi s/2)} + \frac{1}{\cos(\pi s/2)} \right), \] (2.37)

and another application of the reflection formula in the denominator gives

\[ \frac{2^{s-1} \Gamma(1-s) \sin(\pi s)}{\sqrt{\pi}} \left( \frac{1}{\sin(\pi s/2)} + \frac{1}{\cos(\pi s/2)} \right). \] (2.38)

The lemma follows after using the identity

\[ \sin(\pi s) \left( \frac{1}{\sin(\pi s/2)} + \frac{1}{\cos(\pi s/2)} \right) = 2\sqrt{2} \cos \left( \frac{\pi s}{2} - \frac{\pi}{4} \right). \] (2.39)

A key tool from multivariable complex analysis that we use is Bochner’s Tube Theorem [Boc38]. For a set \( U \subset \mathbb{R}^n \), we define \( T(U) = U + i\mathbb{R}^n \subset \mathbb{C}^n \). Then we have the following theorem:

**Theorem 2.6** Bochner’s Tube Theorem. Let \( U \subset \mathbb{R}^n \) be a connected open set and \( f(z) \) be a function that is holomorphic on \( T(U) \). Then \( f(z) \) has a holomorphic continuation to the convex hull of \( T(U) \).

A more general version of the theory of domains of holomorphy allows us to show that in the situation of Bochner’s tube theorem, some of the properties of \( f(z) \) also hold for its holomorphic continuation. An example that we include in Appendix C is Theorem C.5, which we use to bound the continuation of some triple Dirichlet series in vertical strips. See also sections 4.3 and 4.4 of [DGH03].

**3. Overview of the proofs**

For simplicity, let us now assume our sums run over all positive integers. Our goal is to find an asymptotic formula for

\[ \sum_{n \leq X} \frac{L(1/2 + \alpha, \chi_n)}{L(1/2 + \beta, \chi_n)}. \] (3.1)
We insert a smooth weight into the sum and use Mellin inversion to rewrite it as an integral. Thus we obtain
\[
\sum_{n \geq 1} \frac{L(1/2 + \alpha, \chi_n)}{L(1/2 + \beta, \chi_n)} f(n/X) = \frac{1}{2\pi i} \int_{(c)} A^*(s, 1/2 + \alpha, 1/2 + \beta) X^s M f(s) ds,
\] (3.2)
where \(A^*(s, w, z)\) is the triple Dirichlet series
\[
A^*(s, w, z) = \sum_{n \geq 1} \frac{L(w, \chi_n)}{L(z, \chi_n)n^s} = \sum_{m,n,k \geq 1} \frac{\mu(k)\chi_n(k)\chi_n(m)}{n^s m^w k^z}. \tag{3.3}
\]
To be able to evaluate the integral, we need to investigate the analytic properties of \(A^*(s, w, z)\). Assuming GRH and using (2.19) and (2.21), the middle series of (3.3) series is absolutely convergent (up to a simple pole at \(w = 1\)) in the region
\[
\{(s, w, z) \in \mathbb{C}^3 : \text{Re}(s) > 1, \text{Re}(s + w) > 3/2, \text{Re}(z) > 1/2\}, \tag{3.4}
\]
where the second condition comes after using the functional equation for the L-function in the numerator if \(\text{Re}(w) < 1/2\) (neglecting for now that it only works for primitive characters). We remark that we get the same region without assuming GRH, as we only need the Lindelöf bound on average (see for example [Blo11] for details), which is provided by moments estimates.

Comparing the integral with the prediction of the ratios conjecture, we may expect that \(A^*(s, w, z)\) has a meromorphic continuation to \(\text{Re}(s) > 1/2\), with poles at \(s = 1\) and \(s = 1 - \alpha = 3/2 - w\).

We now present a heuristic of Bump, Friedberg and Hoffstein (see [BFH96]) suggesting that \(A^*(s, w, z)\) satisfies some functional equations that can be used to obtain a meromorphic continuation, and that it has the predicted poles. In these heuristics, we assume that all characters are primitive, and that the quadratic reciprocity holds in the form \(\chi_n(m) \approx \chi(m)\). We also don’t write the gamma factors in the functional equations, so they have the form
\[
L(s, \chi_n) \approx n^{1/2-s} L(1-s, \chi_n). \tag{3.5}
\]
Using this functional equation in \(w\), we obtain the first heuristic functional equation for \(A^*(s, w, z)\):
\[
A^*(s, w, z) \approx \sum_n \frac{L(w, \chi_n)}{L(z, \chi_n)n^s} \approx \sum_n \frac{L(1-w, \chi_n)}{L(z, \chi_n)n^{s+w-1/2}} \approx A^*(s + w - 1/2, 1 - w, z). \tag{3.6}
\]
On the other hand, we can expand the \(L\)-functions into Dirichlet series, first sum over the \(n\) variable, and use the functional equation in \(s\). Thus we obtain the second heuristic functional equation:
\[
A^*(s, w, z) \approx \sum_{m,n,k} \frac{\mu(k)\chi_n(k)\chi_n(m)}{n^s m^w k^z} \approx \sum_{m,k} \frac{\mu(k)L(s, \chi_{mk})}{m^w k^z} \approx \sum_{m,k} \frac{\mu(k)L(1-s, \chi_{mk})}{m^{s+w-1/2} k^{s+z-1/2}} \approx A^* \left(1 - s, s + w - \frac{1}{2}, s + z - \frac{1}{2}\right). \tag{3.7}
\]
We also see from this computation that there is a pole at \(s = 1\) coming from the terms with \(mk = \square\), and (3.6) then gives the pole at \(s = 3/2\). We can also use (3.6) to see a pole at \(w = 1\), which becomes a pole at \(s = 3/2 - w\) after (3.7).

An important aspect of our result is the admissible range of the parameters \(\alpha\) and \(\beta\), and the error term. This depends on the region of meromorphic continuation of \(A^*(s, w, z)\), which we obtain from the functional equations and a careful application of Bochner’s Tube Theorem.
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Let us now be more precise. For Theorem 1.1, we rewrite $R_D(X, \alpha, \beta; f)$ as an integral as

$$R_D(X, \alpha, \beta; f) = \frac{1}{2\pi i} \int_{(2)} A_D \left( s, \frac{1}{2} + \alpha, \frac{1}{2} + \beta \right) X^s \mathcal{M} f(s) ds,$$  

(3.8)

where $A_D(s, w, z)$ is the triple Dirichlet series

$$A_D(s, w, z) = \sum_{d \geq 1}^\star \frac{L(w, \chi_d)}{L(z, \chi_d) d^s} = \sum_{d \geq 1}^\star \sum_{m, k \geq 1} \frac{\mu(k) \chi_d(k) \chi_d(m)}{k^w m^z d^s}$$

$$= \sum_{m, k \geq 1} \frac{\mu(k) L_D(s, (\frac{4mk}{m^w k^z}))}{m^w k^z}.$$  

(3.9)

Since all characters are even and primitive, it is straightforward to get the functional equation in $w$. However, after exchanging summations, we obtain $L_D(s, \chi)$ instead of $L(s, \chi)$ in the heuristic in (3.7), so we don’t have the functional equation in $s$. This explains the weaker results for this family. The details are written in Section 4.

For Theorem 1.2, we similarly obtain

$$\sum_{n \geq 1, n \ odd} \frac{L(1/2 + \alpha, \chi_n)}{L(1/2 + \beta, \chi_n)} f\left(\frac{n}{X}\right) = \frac{1}{2\pi i} \int_{(2)} A \left( s, \frac{1}{2} + \alpha, \frac{1}{2} + \beta \right) X^s \mathcal{M} f(s) ds,$$  

(3.10)

where

$$A(s, w, z) = \sum_{n \geq 1, n \ odd} \frac{L(1/2 + \alpha, \chi_n)}{L(1/2 + \beta, \chi_n) n^s} = \sum_{k, m, n \ odd} \frac{\mu(k) \chi_n(k) \chi_n(m)}{k^w m^z n^s}$$

$$= \sum_{m, k \geq 1, m, k \ odd} \frac{L(s, (\frac{4mk}{m^w k^z}))}{m^w k^z}.$$  

(3.11)

In this case, we can potentially obtain both functional equations, but we have to deal with the presence of non-primitive characters. A key tool here is the functional equation in Proposition 2.3, which is valid for all Dirichlet characters. After applying it, we obtain a relation between $A$ and some different triple Dirichlet series, whose coefficients involve Gauss sums. We elaborate on this in Section 6.

4. The triple Dirichlet series for fundamental discriminants

As in (3.8) we can write $R_D(X, \alpha, \beta; f)$ as

$$R_D(X, \alpha, \beta; f) = \frac{1}{2\pi i} \int_{(c)} A_D(s, w, z) X^s \mathcal{M} f(s) ds,$$  

(4.1)

where

$$A_D(s, w, z) = \sum_{d \geq 1}^\star \frac{L(w, \chi_d)}{L(z, \chi_d) d^s},$$  

(4.2)

and $w = 1/2 + \alpha, z = 1/2 + \beta$. In this section, we investigate the analytic properties of the triple Dirichlet series $A_D(s, w, z)$ that allow us to shift the integral and compute the main terms coming from the two poles of $A_D(s, w, z).$
4.1 Region of absolute convergence
Using (2.19), (2.21) and the functional equation for $L(w, \chi_d)$ if $\text{Re}(w) < 1/2$, up to a simple pole at $w = 1$, the series (4.2) converges absolutely in the region

$$R_0 = \left\{ (s, w, z) : \text{Re}(s) > 1, \text{Re}(s + w) > \frac{3}{2}, \text{Re}(z) > \frac{1}{2} \right\},$$

and is polynomially bounded in vertical strips in this region, i.e., away from the possible poles, we have

$$|A_D(s, w, z)| \ll_{\text{Re}(s), \text{Re}(w), \text{Re}(z)} ((1 + |s|)(1 + |w|)(1 + |z|))^c$$

for some constant $c$. By exchanging summations and using Lemma 2.4, we have

$$A_D(s, w, z) = \sum_{m,k \geq 1} \sum_{d \geq 1} \mu(k) \chi_d(k) \chi_d(m) \frac{s^m w^k z}{d^s m^w k^z}$$

$$= \sum_{m,k \geq 1} \left( \frac{1}{2} + \left( \frac{4}{mk} \right) \frac{8}{8s} \right) \frac{\mu(k) \zeta(2) \prod_{p \mid 2mk} \left( \frac{p}{p^s} \right)}{m^w k^z \zeta(2s, \frac{mk}{2^s} \psi_1)}$$

This expression converges absolutely in the region

$$R_1 = \left\{ (s, w, z) : \text{Re}(s) > \frac{1}{4}, \text{Re}(w) > 1, \text{Re}(z) > 1, \text{Re}(s + w) > \frac{3}{2}, \text{Re}(s + z) > \frac{3}{2} \right\},$$

except there is a pole at $s = 1$ coming from the terms in the first sum when $mk = \Box$. Moreover, $(s - 1)A_D(s, w, z)$ is polynomially bounded in vertical strips in $R_1$. Bochner’s tube theorem allows us to meromorphically continue $A_D(s, w, z)$ to the convex hull of $R_0$ and $R_1$, which is

$$R_2 = \left\{ (s, w, z) : \text{Re}(s) > \frac{1}{4}, \text{Re}(z) > \frac{1}{2}, \text{Re}(s + w) > \frac{3}{2}, \text{Re}(s + z) > \frac{3}{2} \right\},$$

and by Proposition C.5, $(s - 1)A_D(s, w, z)$ is polynomially bounded in vertical strips in $R_2$.

4.2 Residue at $s = 1$
We see from expression (4.5) that $A_D(s, w, z)$ has a pole at $s = 1$ coming from the terms with $mk = \Box$. In this case, we have

$$\frac{L(s, \frac{\psi_1}{m})}{L(2s, \frac{\psi_1}{mk^2})} = \frac{\zeta(s)}{\zeta(2s)} \prod_{p \mid 2mk} \frac{p^s}{p^s + 1}. $$

If $mk$ is an odd square, then $mk \equiv 1 \pmod{8}$, so

$$\left( \frac{1}{2} + \left( \frac{4}{mk} \right) \frac{8}{8} \right) \prod_{p \mid 2mk} \frac{p}{p + 1} = \frac{3}{4} \prod_{p \mid mk} \frac{p}{p + 1} = \frac{1}{2} \prod_{p \mid mk} \frac{p}{p + 1},$$

and the same holds if $mk$ is an even square. Therefore, we have

$$\text{res}_{s=1} A_D(s, w, z) = \frac{1}{2\zeta(2)} \sum_{mk = \Box} \frac{\mu(k)}{m^w k^z} \prod_{p \mid mk} \frac{p}{p + 1}. $$
This is the same expression as in the heuristic computation (2.21) in [CoSn07], and can be written as

\[ \text{res}_{s=1} A_D(s, w, z) = \frac{\zeta(2w)}{2\zeta(2)} \zeta(z + w) P_D(z, w), \]  

where

\[ P_D(z, w) = \prod_p \left( 1 - \frac{1}{p^z + w} \right)^{-1} \left( 1 - \frac{1}{(p + 1)p^{2w}} - \frac{p}{(p + 1)p^z + w} \right) \]  

\[ = \prod_p \left( 1 + \frac{1 - p^{z-w}}{(p^z + w - 1)(p + 1)} \right). \]  

**4.3 Functional equation and meromorphic continuation**

We now use the functional equation of \( L(w, \chi_d) \) to obtain a functional equation for \( A_D(s, w, z) \), which gives us a meromorphic continuation beyond the region \( R_2 \). All of the characters \( \chi_d \) are even and primitive of conductor \( d \), so (2.13) gives

\[ A_D(s, w, z) = \sum_{d \geq 1}^\star \frac{L(w, \chi_d)}{L(z, \chi_d)d^s} = \pi^{w-1/2} \Gamma_e(w) \sum_{d \geq 1}^\star \frac{L(1 - w, \chi_d)}{L(z, \chi_d)d^{s+w-1/2}} \]

\[ = \pi^{w-1/2} \Gamma_e(w) A_D(s + w - 1/2, 1 - w, z). \]  

This functional equation provides a meromorphic continuation to the region

\[ R_3 = \{(s, w, z) : \text{Re}(s + w) > \frac{3}{4}, \text{Re}(z) > \frac{1}{2}, \text{Re}(s) > 1, \text{Re}(s + w + z) > 2\}, \]  

and gives rise to a new pole of \( A_D(s, w, z) \) at \( s = 3/2 - w \) with residue

\[ \text{res}_{s=3/2-w} A_D(s, w, z) = \pi^{w-1/2} \Gamma_e(w) \frac{\zeta(2-2w)}{2\zeta(2)} \zeta(1 + z - w) P_D(z, 1 - w). \]  

Bochner’s tube theorem allows us to meromorphically continue \( A_D(s, w, z) \) to the convex hull of \( R_2 \) and \( R_3 \), which is the region

\[ R_4 = \left\{ (s, w, z) : \text{Re}(s) > \frac{1}{4}, \text{Re}(z) > \frac{1}{2}, \text{Re}(s + w) > \frac{3}{4}, \text{Re}(2s + w) > \frac{7}{4}, \text{Re}(s + z) > \frac{3}{2}, \text{Re}(2s + w + z) > 3, \text{Re}(s + w + z) > 2 \right\}. \]  

Moreover, \((s - 1)(w - 1)(s + w - 3/2)A_D(s, w, z)\) is polynomially bounded in vertical strips in the region \( R_3 \), and by Proposition C.5 also in the region \( R_4 \).

**5. Proof of Theorem 1.1**

In this section, we prove Theorem 1.1. Assume that \(-1/2 < \text{Re}(\alpha) < 1/2, 0 < \text{Re}(\beta) < 1/2, \text{Re}(\beta) > |\text{Re}(\alpha)|\). We have

\[ R_D(X, \alpha, \beta; f) = \frac{1}{2\pi i} \int_{(2)} A_D(s, w, z)X^s \mathcal{M}(s)ds, \]  

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where \( w = 1/2 + \alpha, z = 1/2 + \beta \). Let

\[
M(\alpha, \beta) = 1 - \frac{\text{Re}(\alpha)}{2} - \frac{\text{Re}(\beta)}{2}.
\] (5.2)

By (4.16), we can shift the integral to \( \text{Re}(s) = M(\alpha, \beta) + \varepsilon \), and our assumptions about \( \text{Re}(\alpha), \text{Re}(\beta) \) ensure that we cross the poles at \( s = 1, s = 1 - \alpha \), and that the Euler products \( P_D\left(\frac{1}{2} + \beta, \frac{1}{2} \pm \alpha\right) \) converge absolutely.

\[
R_D(X, \alpha, \beta; f) = XMf(1)\text{res}_{s=1}A_D(s, 1/2 + \alpha, 1/2 + \beta) + X^{1-\alpha}Mf(1-\alpha)\text{res}_{s=1-\alpha}A_D(s, 1/2 + \alpha, 1/2 + \beta)
\]
\[
+ \frac{1}{2\pi i} \int_{(M(\alpha, \beta)+\varepsilon)} A_D(s, 1/2 + \alpha, 1/2 + \beta)Mf(s)X^s ds.
\] (5.3)

Since \( A_D(s, w, z) \) is polynomially bounded in vertical strips, we can bound the last integral and obtain

\[
R_D(X, \alpha, \beta; f) = \frac{XMf(1)\zeta(1+2\alpha)}{2\zeta(2)\zeta(1+\alpha+\beta)}P_D(1/2 + \beta, 1/2 + \alpha)
\]
\[
+ \frac{X^{1-\alpha}Mf(1-\alpha)\zeta(1-2\alpha)\pi^\alpha\Gamma_\varepsilon(1/2 + \alpha)}{2\zeta(2)\zeta(1-\alpha+\beta)}P_D(1/2 + \beta, 1/2 - \alpha)
\]
\[
+ O_{\alpha, \beta}(X^{M(\alpha, \beta)+\varepsilon}).
\] (5.4)

6. The triple Dirichlet series for all characters

We now proceed with proving the results for the family with non-primitive characters. In view of (3.10), we begin by studying the properties of the triple Dirichlet series \( A(s, w, z) \).

6.1 Region of absolute convergence

Writing \( n = n_0n_1^2 \) with \( n_0 \) squarefree, we have

\[
A(s, w, z) = \sum_{n \geq 1, n \text{ odd}} \frac{L(2, w, \chi_n)}{L(2, z, \chi_n)n^s} = \sum_{n_0, n_1 \text{ odd}} \frac{\mu^2(n_0)L(2, w, \chi_{n_0n_1^2})}{L(2, z, \chi_{n_0n_1^2})n_0^sn_1^2s}
\] (6.1)

\[
= \sum_{n_1 \text{ odd}} \frac{1}{n_1^s} \sum_{n_0 \text{ odd}} \frac{\mu^2(n_0)L(2, s, \chi_{n_0})}{L(2, z, \chi_{n_0n_1^2})n_0^s} \prod_{p|n_1} \left(1 - \frac{\chi_{n_0}(p)}{p^w}\right).
\]

Using

\[
\prod_{p|n_1} \left(1 - \frac{\chi_{n_0}(p)}{p^w}\right) \ll n_1^{\varepsilon + \max\{0, -\text{Re}(w)\}},
\] (6.2)

the above sum can be bounded by

\[
\sum_{n_1 \text{ odd}} n_1^{\varepsilon + \max\{0, -\text{Re}(w)\}} \sum_{n_0 \text{ odd}} \frac{\mu^2(n_0)|L(w, \chi_{n_0})|}{|L(z, \chi_{n_0n_1^2})|^n_0^{\text{Re}(s)}}
\] (6.3)

By (2.19) and (2.21), the double sum is absolutely convergent in the region

\[
\{(s, w, z) : \text{Re}(s) > 1, \text{Re}(w) \geq 1/2, \text{Re}(z) > 1/2\}.
\] (6.4)
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If \( \text{Re}(w) < 1/2 \), we may use the functional equation for \( L(w, \chi_{n_0}) \), obtaining the region of absolute convergence

\[
\{(s, w, z) : \text{Re}(w) < 1/2, \ \text{Re}(s + w) > 3/2, \ \text{Re}(z) > 1/2\}
\] (6.5)

(note that in the above region, we have \( \text{Re}(s) > 1/2, \ \text{Re}(2s + w) > 1 \)).

Combining (6.4) and (6.5), we see that \( A(s, w, z) \) can be initially defined in the region

\[
S_0 = \{(s, w, z) : \text{Re}(s) > 1, \ \text{Re}(s + w) > 3/2, \ \text{Re}(z) > 1/2\}. \] (6.6)

We also have

\[
A(s, w, z) = \sum_{m, n, k \geq 1, \ m, n, k \text{ odd}} \frac{\chi_n(m)\chi_n(k)\mu(k)}{n^s m^w k^z} = \sum_{m, k \geq 1, \ m, k \text{ odd}} \frac{\mu(k)L\left(s, \left(\frac{4mk}{m^w k^z}\right)\right)}{m^w k^z},
\] (6.7)

which is similarly defined in the region

\[
S_1 = \{(s, w, z) : \text{Re}(w) > 1, \ \text{Re}(z) > 1, \ \text{Re}(s + w) > \frac{3}{2}, \ \text{Re}(s + z) > \frac{3}{2}\}, \] (6.8)

except the pole at \( s = 1 \) coming from the summands with \( mk = \square \). The convex hull of \( S_0 \) and \( S_1 \) is

\[
S_2 = \{(s, w, z) : \text{Re}(w) > 1/2, \ \text{Re}(s + w) > 3/2, \ \text{Re}(s + z) > 3/2\}. \] (6.9)

6.2 Pole and residue at \( s = 1 \)

We see from (6.7) that \( A(s, w, z) \) has a pole at \( s = 1 \) coming from the summands where \( mk = \square \). In this case, we have

\[
L\left(s, \left(\frac{4mk}{m^w k^z}\right)\right) = \zeta(s) \prod_{p \mid 4mk} \left(1 - \frac{1}{p^s}\right),
\] (6.10)

so denoting by \( a(n) \) the multiplicative function with \( a(p^k) = 1 - \frac{1}{p} \), we have

\[
\text{res}_{s=1} A(s, w, z) = \sum_{m, k \text{ odd}} \frac{\mu(k)a(4mk)}{m^w k^z} = \frac{1}{2} \sum_{m, k \text{ odd}} \frac{\mu(k)a(mk)}{m^w k^z}.
\] (6.11)
We can write the last sum as an Euler product, slightly abusing notation by writing \( p^k \) for the prime factors of \( k \), and similarly for \( m \). We thus obtain
\[
\frac{1}{2} \prod_{p>2} \left( \sum_{m,k \geq 0, \text{ } m+k \text{ even}} \frac{\mu(p^k)a(p^{m+k})}{p^{mw+kz}} \right)
= \frac{1}{2} \prod_{p>2} \left( \sum_{m \geq 0, \text{ } m \text{ even}} \frac{a(p^m)}{p^{mw}} - \sum_{m \geq 0, \text{ } m \text{ odd}} \frac{a(p^{m+1})}{p^{z+mw}} \right)
= \frac{1}{2} \prod_{p>2} \left( 1 + \left( 1 - \frac{1}{p} \right) \cdot \frac{p^{-2w}}{1-p^{-2w}} - \left( 1 - \frac{1}{p} \right) \frac{p^{-z-w}}{1-p^{-2w}} \right)
\]
\[= \frac{\zeta(2)(2w)}{2\zeta(2)(z+w)} \prod_{p>2} \left( 1 - \frac{1}{p^{1+w-z}(p^{z+w}-1)} + \frac{1}{p(p^{z+w}-1)} \right) \]
\[= \frac{\zeta(2)(2w)}{2\zeta(2)(z+w)} \prod_{p>2} \left( 1 + \frac{p^{w-z}-1}{p^{1+w-z}(p^{z+w}-1)} \right). \tag{6.12} \]

Setting \( w = 1/2 + \alpha, \) \( z = 1/2 + \beta \) gives
\[
\text{res}_{s=1} A(s,1/2+\alpha,1/2+\beta) = \frac{\zeta(2)(1+2\alpha)}{2\zeta(2)(1+\alpha+\beta)} \prod_{p>2} \left( 1 + \frac{p^{\alpha-\beta}-1}{p^{1+\alpha-\beta}(p^{1+\alpha+\beta}-1)} \right). \tag{6.13} \]

### 6.3 Functional equation in \( s \)

To get a functional equation for \( A(s,w,z) \), we use expression (6.7) together with the functional equation from Proposition 2.3. Since \( \left( \frac{4mk}{\cdot} \right) \) is an even Dirichlet character modulo \( 4mk \) for any \( m,k \geq 1 \), we obtain
\[
A(s,w,z) = \sum_{m,k \geq 1, \text{ } m,k \text{ odd}} \frac{\mu(k)L\left(s,\left(\frac{4mk}{\cdot}\right)\right)}{m^wk^z}
= \pi^{s-1/2} \Gamma\left(\frac{1-s}{2}\right) \sum_{m,k \geq 1, \text{ } m,k \text{ odd}} \frac{\mu(k)K\left(1-s,\left(\frac{4mk}{\cdot}\right)\right)}{m^{s+w}k^{s+z}} \tag{6.14} \]
\[= \pi^{s-1/2} \Gamma\left(\frac{1-s}{2}\right) C(1-s,s+w,s+z), \]

where \( C(s,w,z) \) is the triple Dirichlet series
\[
C(s,w,z) = \sum_{m,k \geq 1, \text{ } m,k \text{ odd}} \frac{\mu(k)K\left(s,\left(\frac{4mk}{\cdot}\right)\right)}{m^wk^z} = \sum_{m,k,q \geq 1, \text{ } m,k \text{ odd}} \frac{\mu(k)\tau\left(\left(\frac{4mk}{\cdot},q\right)\right)}{q^sm^wk^z} \tag{6.15} \]

### 6.4 Region of convergence of \( C(s,w,z) \)

By (6.9) and the functional equation (6.14), \( C(s,w,z) \) is initially defined in the region
\[
P = \{(s,w,z) : \text{Re}(s+z) > 3/2, \text{ Re}(w) > 3/2, \text{ Re}(z) > 3/2\}. \tag{6.16} \]
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To extend this region, we exchange the summations in \( C(s, w, z) \) and get

\[
C(s, w, z) = \sum_{m,k,q \geq 1, \quad m,k \text{ odd}} \frac{\mu(k) \tau \left( \left( \frac{4mk}{q^s m^w k^z} \right), q \right)}{q^s m^w k^z} = \sum_{q=1}^{\infty} \frac{1}{q^s} \sum_{\ell \geq 1, \quad \ell \text{ odd}} \frac{\tau \left( \left( \frac{4\ell}{q^s} \right), q \right)}{\ell^w} \sum_{k \mid \ell} \mu(k) k^{z-w},
\]

(6.17)

where we also substituted \( mk = \ell \). Our goal now is to study the properties of the inner Dirichlet series, that is, the sum over \( \ell \).

We denote by \( a_t(\ell) \) the multiplicative function with \( a_t(p^k) = 1 - \frac{1}{p^t} \) and use Lemma 2.2, so we can rewrite (6.17) as

\[
C(s, w, z) = \sum_{\ell \equiv 1, q \equiv 1 \text{ (mod 4)}} -2\tau \left( \left( \frac{\ell}{q} \right), q \right) a_{z-w}(\ell) + \sum_{\ell \equiv 1, q \equiv 3 \text{ (mod 4)}} 2i\tau \left( \left( \frac{\ell}{q} \right), q \right) a_{z-w}(\ell)
\]

\[
+ \sum_{\ell \equiv 3, q \equiv 1 \text{ (mod 4)}} -2i\tau \left( \left( \frac{\ell}{q} \right), q \right) a_{z-w}(\ell) + \sum_{\ell \equiv 3, q \equiv 3 \text{ (mod 4)}} 2i\tau \left( \left( \frac{\ell}{q} \right), q \right) a_{z-w}(\ell)
\]

\[
= -\frac{2}{2^s} \sum_{\ell \equiv 1 \text{ (mod 4)}, \quad q \text{ odd}} \frac{\tau \left( \left( \frac{\ell}{q} \right), q \right)}{\ell^w} a_{z-w}(\ell) + \frac{2}{4^s} \sum_{\ell \equiv 1 \text{ (mod 4)}, \quad q \equiv 1 \text{ (mod 4)}} \tau \left( \left( \frac{\ell}{q} \right), q \right) a_{z-w}(\ell)
\]

(6.18)

All of the terms in (6.18) can be written as combinations of terms of the form

\[
C(s, w, z; \psi, \psi') := \sum_{\ell, q \geq 1} \frac{G \left( \left( \frac{\ell}{q} \right), \psi(\ell) \psi'(q) \right)}{\ell^w q^s} \sum_{k \mid \ell} \mu(k) k^{z-w},
\]

(6.19)

where \( \psi, \psi' \) are characters modulo \( b \mid 8 \), and \( \psi(2) = 0 \). In particular, we have

\[
C(s, w, z) = -2^{-8} \left( C(s, w, z; \psi_2, \psi_1) + C(s, w, z; \psi_{-2}, \psi_1) \right)
\]

\[
+ 4^{-8} \left( C(s, w, z; \psi_1, \psi_0) + C(s, w, z; \psi_{-1}, \psi_0) \right)
\]

\[
+ C(s, w, z; \psi_1, \psi_{-1}) - C(s, w, z; \psi_{-1}, \psi_1).
\]

(6.20)

In \( C(s, w, z; \psi, \psi') \), all coefficients are multiplicative in \( \ell \). We write

\[
C(s, w, z; \psi, \psi') = \sum_{q=1}^{\infty} \frac{\psi'(q)}{q^s} \cdot D(w, z - w, q; \psi),
\]

(6.21)

where

\[
D(w, t, q; \psi) = \sum_{\ell=1}^{\infty} \frac{G \left( \left( \frac{\ell}{q} \right), \psi(\ell) a_t(\ell) \right)}{\ell^w}.
\]

(6.22)

We have the following lemma:

**Lemma 6.1.** (i) \( D(w, t, q; \psi) \) has a meromorphic continuation to the region

\[
\{(w, t) : \Re(w) > 1, \Re(w + t) > 1\}.
\]

(6.23)

The only pole in this region is at \( w = \frac{3}{2} \), which occurs if \( q = 1 \), \( \psi \) is a principal character and \( t \neq 0 \).
(ii) For $\text{Re}(w) > 1 + \varepsilon$ and $\text{Re}(t + w) > 1 + \varepsilon$, away from the possible poles, we have
\[
|D(w, t, q, \psi)| \ll |w(t + w)|^\varepsilon q^{\max\{\varepsilon, \varepsilon - \text{Re}(t)\}}.
\] (6.24)

Proof. We can write $D(w, t, q, \psi)$ as an Euler product
\[
D(w, t, q; \psi) = \prod_p \left( \sum_{k=0}^{\infty} \frac{G \left(\frac{-\psi}{p^k}\right) \psi(p^k)a_t(p^k)}{p^{kw}} \right)
\] (6.25)
\[= P_{p|q}(w, t; q; \psi) P_{p|q}(w, t; q; \psi),
\]
where $P_{p|q}$ is the product over odd primes not dividing $q$, and $P_{p|q}$ is the rest. Since $\psi(2) = 0$, we can only consider odd primes. Also note that $P_{p|q}$ is a finite product, and each factor has only finitely many terms by (2.12).

For an odd $p \nmid q$, we have
\[
G \left(\frac{-\psi}{p^k}\right), q = \begin{cases} 1, & \text{if } k = 0, \\ \left(\frac{q}{p}\right) \sqrt{p}, & \text{if } k = 1, \\ 0, & \text{if } k \geq 2, \end{cases}
\] (6.26)
so for $P_{p|q}$, we have
\[
P_{p|q}(w, t, q; \psi) = \prod_{p|q} \left( 1 + \frac{\left(\frac{4q}{p}\right) \psi(p)}{p^{w-1/2}} \right) \prod_{p|q} \frac{1 + \left(\frac{4q}{p}\right) \psi(p) (1-p^{-t})}{1 + \left(\frac{4q}{p}\right) \psi(p)}
\] (6.27)
\[
= L \left(\frac{w-1/2}{\zeta(4q)(2w-1)}\right) E(w, t, q; \psi),
\]
where
\[
E(w, t, q; \psi) = \prod_p \left( 1 - \frac{\left(\frac{4q}{p}\right) \psi(p)}{p^{t+w-1/2}} \cdot \frac{1}{1 + \left(\frac{4q}{p}\right) \psi(p) \sqrt{p}} \right)
\] (6.28)
\[
= \frac{1}{L \left(\frac{t+w-1/2}{\left(\frac{4q}{p}\right) \psi}\right)} \prod_p \left( 1 + \frac{1 - \left(\frac{4q}{p}\right) \psi(p)}{1 + \left(\frac{4q}{p}\right) \psi(p) \sqrt{p}} \right)
\]
\[
= \frac{1}{L \left(\frac{t+w-1/2}{\left(\frac{4q}{p}\right) \psi}\right)} \prod_p \left( 1 + \frac{\left(\frac{4q}{p}\right)^2}{p^{t+w-1/2} - \left(\frac{4q}{p}\right) \psi(p) + \left(\frac{4q}{p}\right) \psi(p)} \right).
\]
For $\text{Re}(t+w) > 1 + \varepsilon$, $\text{Re}(w) > 1 + \varepsilon$, the last Euler product is absolutely convergent and $\ll 1$. This finishes the proof of part (i) of the Lemma.

To prove part (ii), we have to estimate the size of the remaining factors.

We have
\[
\zeta(4q)(2w-1)^{-1} = \zeta(2w-1)^{-1} \prod_{p\nmid 4q} \left( 1 - \frac{1}{p^{2w-1}} \right)^{-1},
\] (6.29)
and for $\text{Re}(w) > 1/2 + \varepsilon$, the product can be bounded by
\[
\prod_{p \mid q} \left(1 - \frac{1}{p^{2w-1}}\right)^{-1} = \prod_{p \mid q} \left(1 + \frac{1}{p^{2w-1} - 1}\right) \ll c\omega(4q) \ll q^\varepsilon, \tag{6.30}
\]
where $c$ is a suitable constant (depending on $\varepsilon$), and the last bound follows from the elementary estimate $\omega(n) \ll \frac{\log n}{\log \log n}$ (here $\omega(n)$ denotes the number of prime factors of $n$).

It remains to bound $P_{p \mid q}(w, t, q; \psi)$. By (2.12), we can write it as
\[
\prod_{p \mid q} \left(1 + \sum_{k=1}^{\lfloor \frac{q}{p} \rfloor} \frac{\varphi(p^{2k})\psi(p^{2k})}{p^{2kw}}(1 - p^{-t}) + G\left(\left(\frac{p^{2w}}{p^{(a+1)w}}\right), q, \psi(p^{a+1})\right)\right) \ll \prod_{p \mid q} \left(1 + (1 - p^{-t}) \sum_{k=1}^{\infty} p^{2k(1-w)}\right), \tag{6.31}
\]
and the geometric series is $\ll 1$ for $\text{Re}(w) > 1 + \varepsilon$. We have
\[
1 - p^{-t} = \begin{cases} 1, & \text{if } \text{Re}(t) \geq 0, \\ p^{-\text{Re}(t)}, & \text{if } \text{Re}(t) < 0, \end{cases} \tag{6.32}
\]
so the last expression in (6.31) is
\[
\ll \prod_{p \mid q} \left(1 + c_1 \left(1 + p^{-\text{Re}(t)}\right)\right) \ll \begin{cases} q^\varepsilon, & \text{if } \text{Re}(t) \geq 0, \\ q^{-\text{Re}(t)+\varepsilon}, & \text{if } \text{Re}(t) < 0, \end{cases} \tag{6.33}
\]
where the bounds were obtained similarly as in (6.30).

Using this Lemma, we can extend each of $C(s, w, z; \psi, \psi')$, and hence also $C(s, w, z)$ to the region
\[
\{(s, w, z) : \text{Re}(w) > 1, \text{Re}(z) > 1, \text{Re}(s) + \min\{0, \text{Re}(z - w)\} > 1\}. \tag{6.34}
\]
Using the functional equation (6.14) enables us to extend $A(s, w, z)$ to
\[
S_3 = \{(s, w, z) : \text{Re}(s + w) > 1, \text{Re}(s + z) > 1, \text{Re}(1 - s) + \min\{0, \text{Re}(z - w)\} > 1\}. \tag{6.35}
\]
The convex hull of $S_2$ and $S_3$ contains
\[
S_4 = \left\{ (s, w, z) : \text{Re}(s + 2w) > 2, \text{Re}(s + 2z) > 2, \text{Re}(s + z) > 1, \text{Re}(s + w) > 1, \text{Re}(z) > 1/2 \right\}. \tag{6.36}
\]

### 6.5 Bounding $A(s, w, z)$ in vertical strips

We now give bounds for $|A(s, w, z)|$ in vertical strips, which are necessary to bound the error term coming from the shifted integral. To get the desired result, we need a bound of the form $|wz|^{\varepsilon} |s|^R$ for some constant $K$, so we can be a little wasteful in the exponent of $s$.

For the earlier defined regions $S_j$, we define
\[
\tilde{S}_j = S_{j, \varepsilon} \cap \{(s, w, z) : \text{Re}(s) > -5/2, \text{Re}(w) > 1/2 - \varepsilon\}, \tag{6.37}
\]
where \( S_j \varepsilon = S_j + \varepsilon \bar{v} \), with \( \bar{v} = (1, 1, 1) \). Let also

\[
p(s, w) = (s - 1)(w - 1)(s + w - 3/2),
\]

so that \( A(s, w, z)p(s, w) \) is an analytic function in the considered regions. We also denote \( \tilde{p}(s, w) = 1 + |p(s, w)| \).

We first give bounds in the restricted regions \( \tilde{S}_0, \tilde{S}_1 \). We have

\[
|p(s, w)A(s, w, z)| = \left| p(s, w) \sum_{n \geq 1, n \text{ odd}} \frac{L(2)^{(w, \chi_n)}}{L(2)^{(z, \chi_n)}n^s} \right| \leq \tilde{p}(s, w)|wz|^\varepsilon, \tag{6.38}
\]

valid in \( \tilde{S}_0 \). Exchanging summations and using (2.19), we also get the bound

\[
|p(s, w)A(s, w, z)| = \left| p(s, w) \sum_{m, k \geq 1, m, k \text{ odd}} \frac{\mu(k)L(s, (4mk \cdot m^w m^k))}{m^w k^z} \right| \leq \tilde{p}(s, w)(1 + |s|)\max\{|\varepsilon, \frac{1}{2} - \Re(s) + \varepsilon\}|, \tag{6.39}
\]

which holds in \( \tilde{S}_1 \). Using Proposition C.5, we get the bound

\[
|p(s, w)A(s, w, z)| \ll \tilde{p}(s, w)|wz|^{\varepsilon}(1 + |s|)^{3+\varepsilon} \tag{6.40}
\]

in the convex hull of \( \tilde{S}_0, \tilde{S}_1 \), which is \( \tilde{S}_2 \).

By (6.20) and Lemma 6.1, we have

\[
|(w - 3/2)C(s, w, z)| \ll (1 + |w - 3/2|)|wz|^\varepsilon \tag{6.41}
\]

in the region

\[
\{(s, w, z) : \Re(w) > 1 + \varepsilon, \Re(z) > 1 + \varepsilon, \Re(s) + \min\{0, \Re(z - w) > 1 + \varepsilon\}\}, \tag{6.42}
\]

so the functional equation (6.14) gives the bound

\[
|p(s, w)A(s, w, z)| \ll \tilde{p}(s, w)|wz|^{\varepsilon}(1 + |s|)^{3+\varepsilon} \tag{6.43}
\]

in the region \( \tilde{S}_4 \). Using Proposition C.5 once more gives us the final bound

\[
|p(s, w)A(s, w, z)| \ll \tilde{p}(s, w)|wz|^{\varepsilon}(1 + |s|)^{3+\varepsilon} \tag{6.44}
\]

in the convex hull of \( \tilde{S}_2 \) and \( \tilde{S}_3 \), which is \( \tilde{S}_4 \).

Finally, dividing everything by \( p(s, w) \), we obtain the following bound valid in \( \tilde{S}_4 \) and away from the poles of \( A(s, w, z) \):

\[
|A(s, w, z)| \ll |wz|^{\varepsilon}(1 + |s|)^{3+\varepsilon}. \tag{6.45}
\]

6.6 Residue of \( A(s, w, z) \) at \( s = 3/2 - w \)

We have

\[
C(s, w, z; \psi, \psi') = \sum_{q \geq 1} \frac{\psi(q)}{q^s} \cdot D(w, z - w, q; \psi), \tag{6.46}
\]
and if \( \psi = \psi_1, q = \square \), and \( z - w \neq 0 \), \( D(w, z - w, q; \psi) \) has a pole at \( w = 3/2 \). Using the notation from the proof of Lemma 6.1, we have

\[
D(w, t, q; \psi_1) = \frac{L \left( w - 1/2, \left( \frac{4q}{p} \right) \psi_1 \right)}{\zeta(4q)(2w - 1)} E(w, t, q; \psi_1) P_{p|q}(w, t, q; \psi_1). \tag{6.48}
\]

We now compute the residue. The residue at \( w = 3/2 \) of \( \frac{L(w-1/2, (4q)\psi_1)}{\zeta(4q)(2w-1)} \) is

\[
\frac{1}{\zeta(2)} \prod_{p|4q} \frac{p}{p + 1}. \tag{6.49}
\]

If \( p \) is an odd prime and \( p^{2a} \| q \), then by (2.12)

\[
P_{p|q} \left( \frac{3}{2}, z - \frac{3}{2}, q; \psi_1 \right) = \prod_{p|q, \ p \ odd} \left[ \sum_{k=0}^{\infty} G \left( \left( \frac{4q}{p^k} \right), q \right) \psi(p^k) a_{z - \frac{3}{2}} (p^k) \right] \frac{p^{\frac{a}{2}}}{p^{\frac{a}{2}}}
\]

\[
= \prod_{p|q, \ p \ odd} \left( 1 + \sum_{k=1}^{a} G \left( \left( \frac{4q}{p^k} \right), q \right) a_{z - \frac{3}{2}} (p^k) \right) \frac{p^{a + 1/2}}{p^{3a + 1/2}}
\]

\[
= \prod_{p|q, \ p \ odd} \left( 1 + \left( 1 - p^{\frac{3}{2} - z} \right) \sum_{k=1}^{a} \frac{\varphi(p^{2k})}{p^{3k}} + \left( 1 - p^{\frac{3}{2} - z} \right) \frac{p^{2a} \sqrt{p}}{p^{3a + 1/2}} \right)
\]

\[
= \prod_{p|q, \ p \ odd} \left( 1 + \frac{1 - p^{\frac{3}{2} - z}}{p} \right). \tag{6.50}
\]

Finally, for \( q = \square \) and \( \psi = \psi_1 \), we have

\[
E \left( \frac{3}{2}, z - \frac{3}{2}, q; \psi \right)
\]

\[
= \frac{1}{L \left( z - \frac{1}{2}, \left( \frac{4q}{p} \right) \right)} \prod_{p} \left( 1 + \frac{\left( \frac{4q}{p} \right)^2}{p^{z - 1/2} - \left( \frac{4q}{p} \right) \left( p + \left( \frac{4q}{p} \right) \right)} \right)
\]

\[
= \frac{1}{\zeta \left( z - \frac{1}{2} \right)} \prod_{p|4q} \left( 1 - \frac{1}{p^{z - \frac{1}{2}}} \right)^{-1} \prod_{p|4q} \left( 1 + \frac{1}{p^{z - 1} - 1} \right) \left( p + 1 \right). \tag{6.51}
\]

If we now denote

\[
P(z) = \prod_{p} \left( 1 + \frac{1}{p^{z - \frac{1}{2} - 1} \left( p + 1 \right)} \right), \tag{6.52}
\]
implies that
\[ \frac{P(z)}{\zeta(z - \frac{1}{2})} \prod_{p|q} \left(1 - \frac{1}{p^{z - \frac{1}{2}}} \right)^{-1} \left(1 + \frac{1}{(p^{z - \frac{1}{2}} - 1)(p + 1)} \right)^{-1} = \frac{P(z)}{\zeta(z - \frac{1}{2})} \prod_{p|q} \frac{p^{z-1/2}(p + 1)}{p^{z-1/2} - 1}(p + 1) + 1, \]

Putting everything together, we find that
\[ \text{res}_{w=3/2} C(s, w, z; \psi_1, \psi') = \frac{P(z)}{\zeta(2)\zeta(z - \frac{1}{2})} \sum_{q>1} \frac{\psi'(q^2)}{q^{2s}} \times \prod_{p|q, \ p \ odd} \frac{p}{p + 1} \cdot \frac{p^{z-1/2}(p + 1)}{(p^{z-1/2} - 1)(p + 1) + 1} \prod_{p|q, \ p \ odd} \frac{p + 1 - p^{3/2-z}}{p}. \] (6.54)

We have
\[ \frac{p}{p + 1} \cdot \frac{p^{z-1/2}(p + 1)}{(p^{z-1/2} - 1)(p + 1) + 1} \cdot \frac{p + 1 - p^{3/2-z}}{p} = 1, \] (6.55)

so
\[ \text{res}_{w=3/2} C(s, w, z; \psi_1, \psi') = \frac{P(z)}{\zeta(2)\zeta(z - \frac{1}{2})} \cdot \frac{2^{z+1/2}}{3 \cdot 2^{2-s} - 2} \cdot L(2s, \psi^2). \] (6.56)

Using (6.20) gives
\[ \text{res}_{w=3/2} C(s, w, z) = 4^{-s} \text{res}_{w=3/2} C(s, w, z; \psi_1, \psi_0) + \text{res}_{w=3/2} C(s, w, z; \psi_1, \psi_1) \]
\[ = \frac{P(z)}{\zeta(2)\zeta(z - 1/2)} \cdot \frac{2^{z+1/2}}{3 \cdot 2^{2-s} - 2} \left(4^{-s}\zeta(2s) + L(2s, \psi_1)\right) \]
\[ = \frac{P(z)}{\zeta(2)\zeta(z - 1/2)} \cdot \frac{2^{z+1/2}}{3 \cdot 2^{2-s} - 2}. \] (6.57)

Note that this is also true in the case \( w = z \), when there is no pole and the residue is 0.

The functional equation
\[ A(s, w, z) = \frac{\pi^{s-1/2} \Gamma \left( \frac{1-s}{2} \right)}{4^{s} \Gamma \left( \frac{z}{2} \right)} C(1-s, s+w, s+z) \] (6.58)
implies that
\[ \text{res}_{s=3/2-w} A(s, w, z) = \frac{\pi^{1-w} \Gamma \left( \frac{w-1/2}{2} \right)}{\Gamma \left( \frac{3/2-w}{2} \right)} \cdot \frac{P(3/2 - w + z)\zeta(2w - 1)}{\zeta(2)\zeta(1-w+z)} \cdot \frac{2^{z+w-1}}{3 \cdot 2^{1-w} - 2}. \] (6.59)

Substituting \( w = 1/2 + \alpha, \) \( z = 1/2 + \beta \), we obtain
\[ \text{res}_{s=1-\alpha} A(s, 1/2 + \alpha, 1/2 + \beta) = \frac{\pi^{1/2-\alpha} \Gamma \left( \frac{\alpha}{2} \right)}{\Gamma \left( \frac{1-\alpha}{2} \right)} \cdot \frac{P(3/2 - \alpha + \beta)\zeta(2\alpha)}{\zeta(2)\zeta(1-\alpha+\beta)} \cdot \frac{2^{\alpha+\beta}}{3 \cdot 2^{1-\alpha+\beta} - 2}. \] (6.60)
The Ratios conjecture and multiple Dirichlet series

Using the functional equation

\[ \zeta(2\alpha) = \pi^{2\alpha-1/2} \frac{\Gamma\left(\frac{1-2\alpha}{2}\right)}{\Gamma(\alpha)} \zeta(1-2\alpha), \]  

(6.61)

the last expression becomes

\[ \frac{\pi^\alpha \Gamma\left(\frac{\alpha}{2}\right) \Gamma\left(\frac{1-2\alpha}{2}\right)}{\Gamma(\frac{1-\alpha}{2}) \Gamma(\alpha)} \cdot \frac{P(3/2 - \alpha + \beta)\zeta(1 - 2\alpha)}{\zeta(2)\zeta(1 - \alpha + \beta)} \cdot \frac{2^{\alpha+\beta}}{3 \cdot 2^{1-\alpha+\beta} - 1}. \]  

(6.62)

Now using the relation (2.32) gives

\[ \pi^{\alpha - \frac{1}{2}} \cos\left(\frac{\pi \alpha}{2}\right) \Gamma\left(\frac{1}{2} - \alpha\right) \frac{P\left(\frac{3}{2} - \alpha + \beta\right)\zeta(1 - 2\alpha)}{\zeta(2)\zeta(1 - \alpha + \beta)} \cdot \frac{2^\beta}{3 \cdot 2^{1-\alpha+\beta} - 1}, \]  

(6.63)

and an application of Lemma 2.5 yields

\[ \pi^\alpha \left( \Gamma_\sigma \left(\frac{1}{2} + \alpha\right) + \Gamma_\varepsilon \left(\frac{1}{2} + \alpha\right) \right) \frac{P\left(\frac{3}{2} - \alpha + \beta\right)\zeta(1 - 2\alpha)}{\zeta(2)\zeta(1 - \alpha + \beta)} \cdot \frac{1}{6 - 2^{1+\alpha-\beta}}. \]  

(6.64)

7. Proof of Theorem 1.2

We will now prove Theorem 1.2. We have

\[ \sum_{n \geq 1, \ n \text{ odd}} \frac{L(2, \frac{1}{2} + \alpha, \chi_n)}{L(2, \frac{1}{2} + \beta, \chi_n)} f\left(\frac{n}{X}\right) = \frac{1}{2\pi i} \int \mathcal{A} \left( s, \frac{1}{2} + \alpha, \frac{1}{2} + \beta \right) \mathcal{M} f(s) X^s ds. \]  

(7.1)

By (6.36) and the definition of \( \tilde{S}_4 \) in (6.37), we can shift the integral to \( \text{Re}(s) = N(\alpha, \beta) + \varepsilon \), where

\[ N(\alpha, \beta) = \max \left\{ 1 - 2\text{Re}(\alpha), 1 - 2\text{Re}(\beta), \frac{1}{2} - \text{Re}(\alpha), \frac{1}{2} - \text{Re}(\beta), -\frac{5}{2} \right\}. \]  

(7.2)
We capture the residues at \( s = 1 \) (6.13) and \( s = 1 - \alpha \) (6.64), and then use (6.46) to estimate the error term. Thus if \( \text{Re}(\alpha) > 0, \text{Re}(\beta) > \varepsilon \), we get

\[
\sum_{n \geq 1, n \text{ odd}} \frac{L(2)(1/2 + \alpha, \chi_n)}{L(2)(1/2 + \beta, \chi_n)} f(n/X) = X M f(1) \text{res}_{s=1} A \left( s, \frac{1}{2} + \alpha, \frac{1}{2} + \beta \right) + X^{1-\alpha} M f(1-\alpha) \text{res}_{s=1-\alpha} A \left( s, \frac{1}{2} + \alpha, \frac{1}{2} + \beta \right)
\]

\[
+ \frac{1}{2\pi i} \int_{(N(\alpha,\beta)+\varepsilon)} A(s, w, z) M f(s) X^s \, ds \quad \text{(7.3)}
\]

\[
= X M f(1) \frac{\zeta(2)(1 + 2\alpha)}{2\zeta(2)(1 + \alpha + \beta)} \prod_{p > 2} \left( 1 + \frac{p^{\alpha-\beta} - 1}{p^{1+\alpha-\beta}(p^{1+\alpha+\beta} - 1)} \right)
\]

\[
+ X^{1-\alpha} M f(1-\alpha) \pi^{\alpha} \left( \Gamma_{\alpha} \left( \frac{1}{2} + \alpha \right) + \Gamma_{\beta} \left( \frac{1}{2} + \alpha \right) \right) \times \frac{P \left( \frac{\beta}{\alpha} - \alpha + \beta \right) \zeta(1 - 2\alpha)}{\zeta(2) \zeta(1 - \alpha + \beta)(6 - 2^{1-\alpha + \beta + 1})}
\]

\[
+ O \left( (1 + |\alpha|)^{\varepsilon} |\beta|^{\varepsilon} X^{N(\alpha,\beta)+\varepsilon} \right).
\]

8. Proof of Theorem 1.3

Fix \( r \) with \( \text{Re}(r) > \varepsilon \). We let

\[
M_1(\alpha, \beta) = M f(1) \frac{\zeta(2)(1 + 2\alpha)}{2\zeta(2)(1 + \alpha + \beta)} \prod_{p > 2} \left( 1 + \frac{p^{\alpha-\beta} - 1}{p^{1+\alpha-\beta}(p^{1+\alpha+\beta} - 1)} \right), \quad \text{(8.1)}
\]

and

\[
M_2(\alpha, \beta) = M f(1-\alpha) \pi^{\alpha} \left( \Gamma_{\alpha} \left( \frac{1}{2} + \alpha \right) + \Gamma_{\beta} \left( \frac{1}{2} + \alpha \right) \right) \times \frac{P \left( \frac{\beta}{\alpha} - \alpha + \beta \right) \zeta(1 - 2\alpha)}{\zeta(2) \zeta(1 - \alpha + \beta)(6 - 2^{1-\alpha + \beta + 1})}, \quad \text{(8.2)}
\]

so that (1.8) is

\[
\sum_{n \geq 1, n \text{ odd}} \frac{L(2)(1/2 + \alpha, \chi_n)}{L(2)(1/2 + \beta, \chi_n)} f(n/X) = X M_1(\alpha, \beta) + X^{1-\alpha} M_2(\alpha, \beta) + E(X, \alpha, \beta), \quad \text{(8.3)}
\]

where \( E(X, \alpha, \beta) \ll X^{N(\alpha,\beta)+\varepsilon} \) is the error term. Note that the left-hand side and both \( M_1(\alpha, \beta) \) and \( M_2(\alpha, \beta) \) are analytic functions of \( \alpha, \beta \), so \( E(X, \alpha, \beta) \) is analytic too.

To deduce Theorem 1.3, we fix \( \beta = r \) with \( \text{Re}(\beta) > \varepsilon \), differentiate with respect to \( \alpha \), and set \( \alpha = \beta = r \).

For the first term, we get

\[
\frac{d}{d\alpha} X M_1(\alpha, \beta) \bigg|_{\alpha=\beta=r} = X M f(1) \frac{\zeta'(2)(1 + 2r)}{2 \zeta(2)(1 + 2r)} + \sum_{p > 2} \frac{\log p}{p^{1+2r} - 1} \left( \frac{p^{\alpha-\beta} - 1}{p^{1+\alpha-\beta}(p^{1+\alpha+\beta} - 1)} \right). \quad \text{(8.4)}
\]
For the second term, we notice that due to the factor \( \frac{1}{\zeta(1-\alpha+\beta)} \), only one term survives. We also note that \( P(3/2) = \zeta(2) \), so we get
\[
\frac{d}{d\alpha} X^{1-\alpha} M_2(\alpha, \beta) \bigg|_{\alpha=\beta=r} = -X^{1-r} M f(1-r) \pi^r \left( \Gamma_o \left( \frac{1}{2} + r \right) + \Gamma_e \left( \frac{1}{2} + r \right) \right) \frac{\zeta(1-2r)}{4}.
\]
(8.5)

Therefore we have
\[
\sum_{\substack{n \geq 1, \\ n \text{ odd}}} L'_2(1/2 + r, \chi_n) \frac{L(1/2 + r, \chi_n)}{L(1/2 + r, \chi_n)} f(n/X) =
\]
\[
= XM f(1) \left( \frac{\zeta(2)(1+2r)}{\zeta(2)(1+2r)} + \sum_{p > 2} \frac{\log p}{p^{p^{1/2+2r}}} \right) -
\]
\[
- X^{1-r} M f(1-r) \pi^r \left( \Gamma_o \left( \frac{1}{2} + r \right) + \Gamma_e \left( \frac{1}{2} + r \right) \right) \frac{\zeta(1-2r)}{4} +
\]
\[
+ \frac{d}{d\alpha} E(X, \alpha, \beta) \bigg|_{\alpha=\beta=r}.
\]
(8.6)

Since \( E(X, \alpha, \beta) \) is analytic in \( \alpha \), we can use Cauchy’s integral formula to compute its derivative. We have
\[
\frac{d}{d\alpha} E(X, \alpha, \beta) = \frac{1}{2\pi i} \int_{C_\alpha} \frac{E(X, z, \beta)}{(z - \alpha)^2} dz,
\]
(8.7)

where \( C_\alpha \) is a circle centered at \( \alpha \) of radius \( \rho \) with \( \varepsilon/2 < \rho < \varepsilon \). Then
\[
\left| \frac{d}{d\alpha} E(X, \alpha, \beta) \right| \ll \frac{1}{\rho} \max_{z \in C_\alpha} |E(X, z, \beta)| \ll (1 + |\alpha|)^\varepsilon |\beta|^{\varepsilon} X^{N(\alpha, \beta)+\varepsilon}.
\]
(8.8)

Taking \( \alpha = \beta = r \) and denoting \( N(r) := N(r, r) \) gives
\[
\left| \frac{d}{d\alpha} E(X, \alpha, \beta) \right| \ll |r|^\varepsilon X^{N(r)+\varepsilon}.
\]
(8.9)

Now we recover the Euler factors at 2 that we removed from \( A(s, w, z) \). For odd \( n \), we have
\[
L(1/2 + r, \chi_n) = L_2(1/2 + r, \chi_n) \left( 1 - \left( \frac{2}{n} \right) \right)^{-1},
\]
(8.10)

so
\[
\sum_{\substack{n \geq 1, \\ n \text{ odd}}} \frac{L'_2(1/2 + r, \chi_n)}{L(1/2 + r, \chi_n)} f(n/X) =
\]
\[
= \sum_{\substack{n \geq 1, \\ n \text{ odd}}} \frac{L'_2(1/2 + r, \chi_n)}{L(1/2 + r, \chi_n)} f(n/X) - \log 2 \sum_{\substack{n \geq 1, \\ n \text{ odd}}} \frac{\left( \frac{2}{n} \right)}{2^{1/2+2r} - \left( \frac{2}{n} \right)} f(n/X).
\]
(8.11)

Since for any \( a \in \mathbb{Z}/8\mathbb{Z} \), partial summation gives
\[
\sum_{n \equiv a \pmod{8}} f(n/X) = \frac{X}{8} M f(1) + O(1),
\]
(8.12)
we get
\[ \sum_{n \geq 1, \ n \ odd} \frac{L'(1/2 + r, \chi_n)}{L(1/2 + r, \chi_n)} f(n/X) = \]
\[ = \sum_{n \geq 1, \ n \ odd} \frac{L'(1/2 + r, \chi_n)}{L(1/2 + r, \chi_n)} f(n/X) - \frac{XMf(1)}{2} \cdot \frac{\log 2}{2^{1+2r} - 1} + O(1). \]  
(8.13)

We also have
\[ \frac{\zeta'(1 + 2r)}{\zeta(1 + 2r)} = \frac{\zeta'(2)(1 + 2r)}{\zeta(2)(1 + 2r)} - \frac{\log 2}{2^{2r+1} - 1}, \]  
(8.14)
so using (8.6) and (8.14) in (8.13), we obtain
\[ \sum_{n \geq 1, \ n \ odd} \frac{L'(1/2 + r, \chi_n)}{L(1/2 + r, \chi_n)} f(n/X) = \]
\[ = \frac{XMf(1)}{2} \left( \frac{\zeta'(1 + 2r)}{\zeta(1 + 2r)} + \sum_{p > 2} \frac{\log p}{p(p^{1+2r} - 1)} \right) - \]
\[ - X^{1-r}Mf(1 - r)\pi^r \left( \Gamma_o \left( \frac{1}{2} + r \right) + \Gamma_e \left( \frac{1}{2} + r \right) \right) \frac{\zeta(1 - 2r)}{4} + \]
\[ + O \left( 1 + |r|^{\frac{\varepsilon}{2}}X^{N(r)+\varepsilon} \right). \]  
(8.15)

9. Proof of Theorem 1.4

In this section, we assume that \( \varepsilon < \text{Re}(r) < 1/4 \). We write all odd integers \( n \) as \( n = n_0n_1^2 \) with \( n_0 \) square-free. Then we have
\[ \sum_{n \geq 1, \ n \ odd} \pi^2(n) \frac{L'(1/2 + r, \chi_n)}{L(1/2 + r, \chi_n)} f(n/X) = \]
\[ = \sum_{d \geq 1, \ d \ odd} \mu(d) \sum_{n \geq 1, \ n \ odd} \frac{L'(1/2 + r, \chi_{nd^2})}{L(1/2 + r, \chi_{nd^2})} f \left( \frac{nd^2}{X} \right). \]  
(9.1)

From
\[ L(s, \chi_{nd^2}) = L(s, \chi_n) \prod_{p|d} \left( 1 - \frac{\chi_n(p)}{p^s} \right), \]  
(9.2)
we obtain
\[ \frac{L'(1/2 + r, \chi_{nd^2})}{L(1/2 + r, \chi_{nd^2})} = \frac{L'(1/2 + r, \chi_n)}{L(1/2 + r, \chi_n)} + \sum_{p|d} \frac{\chi_n(p) \log p}{p^{1/2+r} - \chi_n(p)}, \]  
(9.3)
so (9.1) equals
\[
\sum_{d \geq 1, \ d \ odd} \mu(d) \sum_{n \geq 1, \ n \ odd} \frac{L'(1/2 + r, \chi_n)}{L(1/2 + r, \chi_n)} f \left( \frac{nd^2}{X} \right) + \sum_{d \geq 1, \ d \ odd} \mu(d) \sum_{n \geq 1, \ n \ odd} f \left( \frac{nd^2}{X} \right) \sum_{p \mid d} \frac{\chi_n(p) \log p}{p^{1/2 + r - \chi_n(p)}}.
\] (9.4)

Let us now consider the first sum. The terms with \( d > \sqrt{X} \) here contribute \( \ll |r|^\varepsilon X^{1/2} \) by (2.20). For the other terms, we use (1.11) (with \( X/d^2 \) instead of \( X \)). Writing the right-hand side of (1.11) as \( XM_1(r) - X^{1-r} M_2(r) + E(X, r) \), this part of (9.4) contributes
\[
XM_1(r) \sum_{d \leq \sqrt{X}, \ d \ odd} \frac{\mu(d)}{d^2} - X^{1-r} M_2(r) \sum_{d \leq \sqrt{X}, \ d \ odd} \frac{\mu(d)}{d^{2-2r}} + \sum_{d \leq \sqrt{X}, \ d \ odd} \mu(d) E \left( \frac{X}{d^2}, r \right)
\] (9.5)
\[
= \frac{4XM_1(r)}{3\zeta(2)} - \frac{X^{1-r} M_2(r)}{\zeta(2)(2 - 2r)} + O(|r|^\varepsilon X^{1-2r+\varepsilon}).
\]

Now we compute the second term in (9.4). Setting \( s = \frac{1}{2} + r \) and exchanging summations gives
\[
\sum_{p \ odd} \frac{\log p}{p^s} \sum_{d \geq 1, \ d \ odd} \mu(pd) \sum_{n \geq 1, \ n \ odd} \frac{\chi_n(p)}{1 - \chi_n(p)/p^s} f \left( \frac{nd^2 p^2}{X} \right) = \sum_{p \ odd} \frac{\log p}{p^s} \sum_{d \geq 1, \ d \ odd} \mu(pd) \sum_{n \geq 1, \ n \ odd} \chi_n(p) f \left( \frac{nd^2 p^2}{X} \right) \sum_{k=0}^\infty \frac{\chi_n(p)^k}{p^{ks}}.
\] (9.6)

We split the last sum depending on the parity of \( k \).

For an even \( k \), the sum equals
\[
\sum_{p \ odd} \frac{\log p}{p^{(k+1)s}} \sum_{d \geq 1, \ d \ odd} \mu(pd) \sum_{n \geq 1, \ n \ odd} \chi_n(p) f \left( \frac{nd^2 p^2}{X} \right).
\] (9.7)

If we denote
\[
\tilde{\chi}_p = \begin{cases} \chi_p & \text{if } p \equiv 1 \pmod{4}, \\ \chi_p \psi_1 & \text{if } p \equiv 3 \pmod{4}, \end{cases}
\] (9.8)

the inner sum is
\[
\frac{1}{2\pi i} \int_{(c)} \frac{L(2)(u, \tilde{\chi}_p)Mf(u)X^u}{d^{2u}p^{2u}} du \ll \frac{X^{1/2+\varepsilon}}{d^{1+2\varepsilon}p^{1+\varepsilon}}
\] (9.9)

where we shifted the integral to \( c = 1/2 + \varepsilon \) and used Lindelöf’s bound (2.19).

Using (9.9) and summing (9.7) over all even \( k \geq 0 \), we see that the contribution of even \( k \) in (9.6) is
\[
\ll X^{1/2+\varepsilon} \sum_{k=0}^\infty \sum_{p \ odd} \frac{\log p}{p^{(2k+1)s+1+\varepsilon}} \sum_{d \geq 1} \frac{1}{d^{1+2\varepsilon}} \ll X^{1/2+\varepsilon}.
\] (9.10)

Now we compute the contribution of odd values of \( k \) into (9.6). For an odd \( k \), the summand
equals
\[
\sum_{p > 2} \frac{\log p}{p^s} \sum_{d \geq 1, \ d \ odd} \mu(pd) \sum_{n \geq 1, \ n \ odd} \frac{\chi_n(p)^{k+1}}{p^{ks}} f \left( \frac{nd^2 p^2}{X} \right)
\]
\[= \sum_{p > 2} \frac{\log p}{p^{(k+1)s}} \sum_{d \geq 1, \ d \ odd} \mu(pd) \sum_{n \geq 1, \ n \ odd, \ p \nmid n} f \left( \frac{nd^2 p^2}{X} \right). \tag{9.11}
\]

We write the innermost sum as a Mellin integral:
\[
\sum_{n \geq 1, \ n \ odd, \ p \nmid n} f \left( \frac{nd^2 p^2}{X} \right) = \frac{1}{2\pi i} \int_{(c)} \frac{\zeta((2p))XuMf(u)}{d^{2u}p^{2u}} \ du
\]
\[= \frac{XMF(1)(1 - 1/p)}{2d^2p^2} + O \left( \frac{X^{1/2+\varepsilon}}{d^{1+2\varepsilon}p^{1+2\varepsilon}} \right), \tag{9.12}
\]
where the last equation holds after shifting the integral to \(c = 1/2 + \varepsilon\). Summing the error term over all odd \(k, d\) and \(p\) contributes \(\ll X^{1/2+\varepsilon}\), so it remains to compute the contribution of the main term. This gives
\[
\frac{XMF(1)}{3\zeta(2)} \sum_{p > 2} \left( \frac{1}{p} - \frac{1}{p^2} \right) \log p \sum_{d \geq 1, \ d \ odd} \mu(pd) \frac{d}{d^2} = \frac{XMF(1)}{3\zeta(2)} \sum_{p > 2} \frac{\log p}{(p+1)p^{1+(k+1)s}}. \tag{9.13}
\]

Summing this over odd values of \(k\) gives
\[
\frac{-2XMF(1)}{3\zeta(2)} \sum_{p > 2} \frac{\log p}{(p+1)p^{1+s}} \sum_{k=0}^{\infty} \frac{1}{p^{2(k+1)s}} = \frac{-2XMF(1)}{3\zeta(2)} \sum_{p > 2} \frac{\log p}{p(p+1)(p^{2s} - 1)} \tag{9.14}
\]

Putting all together, we obtain the final result
\[
\sum_{n \geq 1, \ n \ odd} \frac{\mu^2(n)L'(1/2 + r, \chi_n)}{L(1/2 + r, \chi_n)} f(n/X)
\]
\[= \frac{2XMF(1)}{3\zeta(2)} \left( \frac{\zeta'(1+2r)}{\zeta(1+2r)} + \sum_{p > 2} \frac{\log p}{(p+1)(p^{1+2r} - 1)} \right) - X^{1-r}MF(1-r)\pi^r \left( \Gamma_0 \left( \frac{1}{2} + r \right) + \Gamma_e \left( \frac{1}{2} + r \right) \right) \frac{\zeta(1-2r)}{4\zeta(2)(2-2r)}
\]
\[+ O \left( |r|^{\varepsilon}X^{1-2r+\varepsilon} \right). \tag{9.15}
\]

10. Proof of Corollary 1.5

In this section, we show how the one-level density can be computed using the ratios conjecture. We follow Section 3 of [CoSn07].

Let \(h(x)\) be an even Schwartz function whose Fourier transform \(\hat{h}\) is supported in the interval \([-a, a]\) for some \(a > 0\). It follows that \(h\) has an analytic continuation to the whole \(\mathbb{C}\) via Fourier inversion.
Recall that the one-level density is defined by
\[
D(X; h) = \frac{1}{F(X)} \sum_{n \geq 1, \ n \ odd} \mu^2(n) f \left( \frac{n}{X} \right) \sum_{\gamma_n} h \left( \frac{\gamma_n \log X}{2\pi} \right),
\]
where \( \gamma_n \) runs over the imaginary parts of the non-trivial zeros of \( L(s, \chi_n) \), and
\[
F(X) = \sum_{n \geq 1, \ n \ odd} \mu^2(n) f \left( \frac{n}{X} \right) = \frac{1}{2\pi i} \int_{(c)} \zeta(2) \zeta(2s) X^s \mathcal{M} f(s) ds.
\]
(10.1)
(10.2)
(10.3)
(10.4)
(10.5)

By the residue theorem, we have
\[
F(X) D(X; h) = \frac{1}{2\pi i} \left( \int_{(c)} - \int_{(1-c)} \right) h \left( \frac{\log X}{2\pi i} \left( s - \frac{1}{2} \right) \right) \sum_{n \geq 1, \ n \ odd} \mu^2(n) f(n/X) \frac{L'(s, \chi_n)}{L(s, \chi_n)} ds
\]
for any \( 1/2 < c < 1 \).

Using the functional equation (2.13) and the fact that \( h \) is even, the integral over \( (1-c) \) equals
\[
\frac{1}{2\pi i} \int_{(c)} h \left( \frac{\log X}{2\pi i} \left( s - \frac{1}{2} \right) \right) \times
\]
\[
\times \left( \sum_{n \geq 1, \ n \ odd} \mu^2(n) f \left( \frac{n}{X} \right) \left( \log \left( \frac{n}{X} \right) + \frac{\Gamma_n'(1-s)}{\Gamma_n(1-s)} \right) \right) ds,
\]
where \( \Gamma_n(s) \) is \( \Gamma_e(s) \) or \( \Gamma_o(s) \), depending on the parity of \( \chi_n \).

Hence
\[
F(X) D(X; h) = \frac{1}{\pi i} \int_{(c)} h \left( \frac{\log X}{2\pi i} (s - 1/2) \right) \sum_{n \geq 1, \ n \ odd} \mu^2(n) f(n/X) \frac{L'(s, \chi_n)}{L(s, \chi_n)} ds
\]
\[
- \frac{1}{2\pi i} \int_{(c)} h \left( \frac{\log X}{2\pi i} (s - 1/2) \right) \sum_{n \geq 1, \ n \ odd} \mu^2(n) f(n/X) \left( \log \left( \frac{n}{X} \right) + \frac{\Gamma_n'(1-s)}{\Gamma_n(1-s)} \right) ds.
\]
(10.5)

We now consider the first integral. We start with the following lemma:

**Lemma 10.1.** For \( h \) as above, and any integer \( A > 0 \), we have
\[
h \left( \frac{s \log X}{2\pi i} \right) \ll_A \frac{X^{\alpha \Re(s)}}{|s|^A \log X^A}.
\]
(10.6)
Proof. Using Fourier inversion and integrating by parts $A$-times, we obtain

$$h\left(\frac{s \log X}{2\pi i}\right) = \int_{-\infty}^{\infty} \hat{h}(t)e^{ts \log X} dt$$

$$= \frac{1}{s^A (\log X)^A} \int_{-\infty}^{\infty} \hat{h}^{(A)}(t) X^{ts} dt$$

$$\ll_A \frac{X^{\alpha - \Re(s)}}{|s|^A (\log X)^A}.$$ (10.7)

We now split the integral, writing

$$\frac{1}{\pi i} \int_{(c)} h\left(\frac{\log X}{2\pi i} (s - 1/2)\right) \sum_{n \geq 1, \ n \ odd} \mu^2(n) f(n/X) \frac{L'(s, \chi_n)}{L(s, \chi_n)} ds =: I_1 + I_2,$$ (10.8)

where $I_1$ is the part of the integral with $|\Im(s)| \leq X$, and $I_2$ the part with $|\Im(s)| > X$.

For $I_2$, we use Lemma 10.1 and (2.20), and obtain the bound

$$\sum_{n \geq 1, \ n \ odd} \mu^2(n) f\left(\frac{n}{X}\right) \log^2(n) \int_{\Re(s) = c, |\Im(s)| > X} \log^2(|s|) \left|h\left(\frac{\log X}{2\pi i} (s - 1/2)\right)\right| ds$$

$$\ll_A X^{1+\varepsilon + a(c-1/2)} \int_{|t| > X} \frac{\log^2(|t|)}{t^A} dt \ll X^\varepsilon,$$ (10.9)

where we took any $A > a(c - 1/2) + 1$.

For $I_1$, we substitute $s = 1/2 + r + it$ with $0 < r < 1/4$ and use Theorem 1.4. Thus $I_1$ is

$$\frac{X}{\pi} \int_{-X}^{X} h\left(\frac{\log X}{2\pi i} (r + it)\right) \left\{ \frac{2Mf(1)}{3\zeta(2)} \left(\frac{\zeta'(1 + 2r + 2it)}{\zeta(1 + 2r + 2it)} + \sum_{p > 2} \frac{\log p}{(p + 1)(p^{1+2r+2it} - 1)} \right) \right.$$ (10.10)

$$- \frac{Mf(1 - r - it) \pi^{r+it}}{X^{r+it} \zeta(2)} \left(\frac{\Gamma_0\left(\frac{1}{2} + r + it\right) + \Gamma_1\left(\frac{3}{2} + r + it\right)}{\Gamma_0\left(\frac{3}{2} + r + it\right) + \Gamma_1\left(\frac{5}{2} + r + it\right)} \right) \zeta(1 - 2r - 2it)$$

$$+ O(|t|^\varepsilon X^{1-2r+\varepsilon}) \right\} dt.$$

Using Lemma 10.1, we can bound the error term by

$$X^{1-2r+\varepsilon} \int_{-X}^{X} h\left(\frac{\log X}{2\pi i} (r + it)\right) |t|^\varepsilon dt \ll X^{1-2r+a r+\varepsilon},$$ (10.11)

and this is $o(X)$ as long as $a < 2$. Setting $r = 1/4 - \varepsilon$ gives the error term in Corollary 1.5.

We remark here that this is the only restriction on $a$, so any improvement of the error term in Theorem 1.4 would allow us to extend the support of the Fourier transform of our test functions.

Now we compute the main terms of (10.10). Since the integrand is regular at $r + it = 0$, we can shift the line of integration to $r = 0$, where the contribution of the horizontal integrals is by
(2.20) and Lemma 10.1

\[ \ll X^{1+\varepsilon} \int_0^{1/4} \left| h \left( \frac{\log X}{2\pi i} (r \pm iX) \right) \right| \ll_A X^{1+\varepsilon+a/4-A}. \quad (10.12) \]

We extend the range of integration from \(-\infty\) to \(\infty\) introducing an error of size \(X^\varepsilon\), and get

\[
\frac{X}{\pi} \int_{-\infty}^{\infty} h \left( \frac{\log X}{2\pi i} \right) \left\{ \frac{2Mf(1)}{3\zeta(2)} \left( \frac{\zeta'(1+2it)}{\zeta(1+2it)} \right) + \sum_{p>2} \frac{\log p}{(p+1)(p^{1+2it} - 1)} \right\} dt
- \frac{X^{-it}Mf(1-it)^{it} \left( \Gamma_o(1/2+it) + \Gamma_e(1/2+it) \right) \zeta(1-2it)}{4\zeta(2)(2-2it)}
= \frac{2X}{\log X} \int_{-\infty}^{\infty} h(u) \left\{ \frac{2Mf(1)}{3\zeta(2)} \left( \frac{\zeta'(1+\frac{2\pi iu}{\log X})}{\zeta(1+\frac{2\pi iu}{\log X})} \right) + \sum_{p>2} \frac{\log p}{(p+1)(p^{1+\frac{2\pi iu}{\log X}} - 1)} \right\}
- \frac{Mf \left( 1 - \frac{2\pi iu}{\log X} \right) \pi^{\frac{2\pi iu}{\log X}} \left( \Gamma_o \left( \frac{1}{2} + \frac{2\pi iu}{\log X} \right) + \Gamma_e \left( \frac{1}{2} + \frac{2\pi iu}{\log X} \right) \right) \zeta \left( 1 - \frac{4\pi iu}{\log X} \right)}{e^{2\pi iu}, 4\zeta(2)(2-\frac{4\pi iu}{\log X})} \right\} du. \quad (10.13)
\]

We now compute the second integral in (10.5). We shift the line of integration to \(c = 1/2\) and substitute \(\frac{\log X}{2\pi i} (s - \frac{1}{2}) = u\), getting

\[
\frac{1}{\log X} \int_{-\infty}^{\infty} h(u) \sum_{n \geq 1, \ n \ odd} \mu^2(n)f \left( \frac{n}{X} \right) \left( \log \left( \frac{\pi}{n} \right) - \frac{\Gamma'_n}{\Gamma_n} \left( \frac{1}{2} - \frac{2\pi iu}{\log X} \right) \right) du. \quad (10.14)
\]

The first part of Corollary 1.5 now follows from (10.11) with \(r = 1/4 - \varepsilon\), (10.13), (10.14) and noticing that a similar computation as in Lemma B.1 gives

\[
\sum_{n \geq 1, \ n \ odd} \frac{\Gamma'_n}{\Gamma_n} (1-s) = \frac{\Gamma'_n}{\Gamma_e(1-s)} \sum_{n \equiv 1 \ (mod \ 4)} \mu^2(n)f(n/X) + \frac{\Gamma'_o(1-s)}{\Gamma_o(1-s)} \sum_{n \equiv 3 \ (mod \ 4)} \mu^2(n)f(n/X) = \quad \text{(10.15)}
\]

\[
\left( \frac{\Gamma'_e(1-s)}{\Gamma_e(1-s)} + \frac{\Gamma'_o(1-s)}{\Gamma_o(1-s)} \right) \left( 3Mf(1) \frac{X}{3\zeta(2)} + O(\sqrt{X}) \right).
\]

To prove the second part, we use the Laurent expansions \(\zeta(s) = \frac{1}{s-1} + \cdots\), \(\zeta'(s) = \frac{-1}{(s-1)^2} + \cdots\), so (10.13) is

\[
\frac{2XMf(1)}{3\zeta(2)} \int_{-\infty}^{\infty} h(u) \left( e^{-\frac{2\pi iu}{\log X}} - 1 \right) du + O \left( \frac{X}{\log X} \right). \quad (10.16)
\]

We have

\[
\sum_{n \geq 1, \ n \ odd} \mu^2(n)f \left( \frac{n}{X} \right) \log \left( \frac{\pi}{n} \right) = -\frac{2Mf(1)X}{3\zeta(2)} + O(X), \quad (10.17)
\]

so by (10.15), (10.14) is

\[
\frac{-2XMf(1)}{3\zeta(2)} \int_{-\infty}^{\infty} h(u) du + O \left( \frac{X}{\log X} \right). \quad (10.18)
\]
Altogether, we obtain
\[ D(X; h) = \int_{-\infty}^{\infty} h(u) \left( 1 + \frac{e^{-2\pi i u} - 1}{2\pi i u} \right) du + O \left( \frac{1}{\log X} \right). \] (10.19)

Finally, to see that this result agrees with the density conjecture of Katz and Sarnak, we use the fact that \( h(u) \) is even. We can thus drop the last term in the integral and replace \( e^{-2\pi i u} \) by
\[ \frac{1}{2} \left( e^{-2\pi i u} - e^{2\pi i u} \right) = -\frac{\sin(2\pi u)}{2\pi u}, \] (10.20)
which completes the proof.

Appendix A. Proof of the functional equation from Proposition 2.3

We now give the proof of the functional equation in Proposition 2.3. We only give the details in the case of even characters, and then explain the usual modification for odd characters. For a detailed proof of the classical functional equation, see for example [Gar].

Let \( \chi \) be an even character modulo \( q \). We define two theta functions
\[ \theta_\chi(y) = \sum_{n \in \mathbb{Z}} \chi(n)e^{-\pi n^2 y}, \] (A.1)
and
\[ \theta_{\tau(\chi)}(y) = \sum_{n \in \mathbb{Z}} \tau(\chi, n)e^{-\pi n^2 y}. \] (A.2)

Then using the fact that \( \chi(n) \) is even, we have the integral representation of the L-function:
\[ \int_{0}^{\infty} \frac{\theta_\chi(y)}{y} dy = \sum_{n \geq 1} \chi(n) \int_{0}^{\infty} e^{-\pi n^2 y} dy = \pi^{-s} \Gamma \left( \frac{s}{2} \right) L(s, \chi), \] (A.3)
and similarly
\[ \int_{0}^{\infty} \frac{\theta_{\tau(\chi)}(y)}{y} dy = \pi^{-s} \Gamma \left( \frac{s}{2} \right) K(s, \chi). \] (A.4)

These integrals converge absolutely for \( s \) with \( \text{Re}(s) > 0 \).

The two theta functions are related by the following functional equation, which follows after an application of the Poisson summation.

**Lemma A.1.** Let \( \chi \) be a character modulo \( q \). Then
\[ \theta_\chi(y) = \frac{1}{q\sqrt{y}} \theta_{\tau(\chi)} \left( \frac{1}{yq^2} \right). \] (A.5)

**Proof.** We have
\[ \theta_\chi(y) = \sum_{n \in \mathbb{Z}} \chi(n)e^{-\pi n^2 y} = \sum_{j \pmod{q}} \chi(j) \sum_{n \in \mathbb{Z}} e^{-\pi (qn+j)^2 y}. \] (A.6)

The Fourier transform of the function in the inner sum is
\[ \int_{-\infty}^{\infty} e^{-\pi (qt+j)^2 y} e^{-2\pi i t n} dt = \frac{1}{q\sqrt{y}} \int_{-\infty}^{\infty} e^{-\pi u^2} e^{-2\pi i n \left( \frac{u}{q} - \frac{j}{q} \right)} du \]
\[ = \frac{e^{2\pi ijn/q}}{q\sqrt{y}} \cdot e^{-\frac{n^2}{q}}, \] (A.7)
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so Poisson summation gives

$$\theta \chi(y) = \frac{1}{q \sqrt{y}} \sum_{j \pmod{q}} \chi(j) \sum_{n \in \mathbb{Z}} e^{\frac{2\pi ij n}{q}} \cdot e^{-\frac{n^2}{2y}}$$

$$= \frac{1}{q \sqrt{y}} \sum_{n \in \mathbb{Z}} e^{-\frac{n^2}{2qy}} \sum_{j \pmod{q}} \chi(j) e^{\frac{2\pi ij n}{q}}$$

$$= \frac{1}{q \sqrt{y}} \sum_{n \in \mathbb{Z}} \tau(\chi, n) e^{-\frac{n^2}{2qy}}$$

$$= \frac{1}{q \sqrt{y}} \theta_{\tau(\chi)} \left( \frac{1}{yq^2} \right). \quad (A.8)$$

Substituting $y \mapsto \frac{1}{yq^2}$ and using relation $(A.5)$, we obtain

$$\pi^{-\frac{s}{2}} \Gamma \left( \frac{s}{2} \right) L(s, \chi) = \int_0^\infty y^{s/2} \frac{\theta_{\chi}(y)}{2} dy$$

$$= \int_0^\infty \left( \frac{1}{yq^2} \right)^{s/2} \theta_{\chi} \left( \frac{1}{yq^2} \right) dy$$

$$= q^{-s} \int_0^\infty y^{\frac{s+1}{2}} \frac{\theta_{\tau(\chi)}(y)}{2} dy$$

$$= \pi^{\frac{s+1}{2}} q^{-s} \Gamma \left( \frac{1-s}{2} \right) K(1-s, \chi), \quad (A.9)$$

which holds for $s$ with $0 < \text{Re}(s) < 1$. However, since the left-hand side has a meromorphic continuation to the whole complex plane, the functional equation holds for all $s \in \mathbb{C}$. This finishes the proof for even characters.

For odd characters, the functions $\theta_{\chi}(y), \theta_{\tau(\chi)}(y)$ are identically 0, so we work with the following functions instead:

$$\tilde{\theta}_{\chi}(y) = \sum_{n \in \mathbb{Z}} n \chi(n) e^{-\pi n^2 y}, \quad (A.10)$$

and

$$\tilde{\theta}_{\chi}(y) = \sum_{n \in \mathbb{Z}} n \tau(\chi, n) e^{-\pi n^2 y}. \quad (A.11)$$

The sign in the resulting functional equation comes from the fact that the function $f(x) = xe^{-\pi x^2}$ is an eigenfunction of the Fourier transform with eigenvalue $-i$.

Appendix B. Ratios conjecture predictions for the family $(\frac{\omega}{n})$ for odd square-free $n$

In this section, we follow the recipe of Conrey, Farmer and Zirnbauer for the family of Dirichlet characters $\chi_n = (\frac{\omega}{n})$ with $n$ odd and square-free, which is slightly different from the family of $(\frac{d}{2})$ for positive fundamental discriminants $d$ usually considered in the literature. In our case, the characters $\chi_n$ are primitive and even if $n \equiv 1 \pmod{4}$, or odd if $n \equiv 3 \pmod{4}$. We therefore
split the family according to the parity of the character as

\[
\sum_{\substack{n \leq X, \\ n \text{ odd}}} \frac{\mu^2(n) L(1/2 + \alpha, \chi_n)}{L(1/2 + \beta, \chi_n)} = \sum_{\substack{n \leq X, \\ n \equiv 1 \pmod{4}}} \frac{\mu^2(n) L(1/2 + \alpha, \chi_n)}{L(1/2 + \beta, \chi_n)} + \sum_{\substack{n \leq X, \\ n \equiv 3 \pmod{4}}} \frac{\mu^2(n) L(1/2 + \alpha, \chi_n)}{L(1/2 + \beta, \chi_n)}. \tag{B.1}
\]

The approximate functional equation is

\[
L(1/2 + \alpha, \chi_n) \approx \sum_{m \leq x} \frac{\chi_n(m)}{m^{1/2 + \alpha}} + \left( \frac{n}{\pi} \right)^{-\alpha} \Gamma_{e/o}(1/2 + \alpha) \sum_{m \leq y} \frac{\chi_n(m)}{m^{1/2 - \alpha}}, \tag{B.2}
\]

where \(xy \approx n/2\pi\), and \(G_{e/o}\) is \(\Gamma_e\) or \(\Gamma_o\), depending on the parity of \(\chi_n\).

We now consider the first sum in (B.1). We write the numerator using the approximate functional equation and the denominator as a Dirichlet series. The first part of the functional equation then gives

\[
\sum_{\substack{n \leq X, \\ n \equiv 1 \pmod{4}}} \mu^2(n) \sum_{m \leq x} \frac{\chi_n(m)}{m^{1/2 + \alpha}} \sum_{k \geq 1} \frac{\mu(k) \chi_n(k)}{k^{1/2 + \beta}}. \tag{B.3}
\]

We extend the sum over \(m\) all the way to infinity, and assume that the main contribution comes from the diagonal terms, which is computed in the following lemma:

**Lemma B.1.** For \(b = 1\) or \(3\), we have

\[
\sum_{\substack{n \leq X, \\ n \equiv b \pmod{4}}} \mu^2(n) \chi_n(\ell) = \begin{cases} \frac{X}{2^{s/2}} a(4\ell) + \text{small} & \text{if } \ell = \square, \\ \text{small} & \text{if } \ell \neq \square, \end{cases} \tag{B.4}
\]

where

\[
a(k) = \prod_{p|k} \frac{p}{p+1}. \tag{B.5}
\]

**Proof.** Since \(n\) runs over odd integers, we may replace \(\ell\) by \(4\ell\). Then we have

\[
\sum_{\substack{n \leq X, \\ n \equiv b \pmod{4}}} \mu^2(n) \chi_n(4\ell) = \sum_{n \leq X} \mu^2(n) \chi_n(4\ell) \psi_1(n) \pm \psi_{-1}(n) \tag{B.6}
\]

for an appropriate sign depending on \(b\). By Perron’s formula,

\[
\sum_{n \leq X} \mu^2(n) \chi_n(4\ell) \psi(n) = \frac{1}{2\pi i} \int_{(c)} A \left( s, \left( \frac{4\ell}{\cdot} \right) \psi \right) \frac{X^s}{s} ds, \tag{B.7}
\]

where

\[
A \left( s, \left( \frac{4\ell}{\cdot} \psi \right) \right) = \sum_{n=1}^{\infty} \frac{\mu^2(n) \left( \frac{4\ell}{n} \psi(n) \right)}{n^s} = \frac{L \left( s, \left( \frac{4\ell}{\cdot} \psi \right) \right)}{L \left( 2s, \left( \frac{4\ell}{\cdot} \psi \right)^2 \right)}. \tag{B.8}
\]

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$A(s, (4\ell) \psi)$ has a pole at $s = 1$ only if $\ell = \Box$ and $\psi = \psi_1$ is the principal character, in which case we have

$$A\left(s, \left(\frac{4\ell}{p}\right) \psi\right) = \frac{\zeta(s)}{\zeta(2s)} \prod_{p \nmid 4\ell} \left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{p^2}\right)^{-1} = \frac{\zeta(s)}{\zeta(2s)} a(4\ell). \quad (B.9)$$

Therefore the residue at $s = 1$ is $\frac{a(4\ell)}{\zeta(2)}$, which gives the main term

$$\frac{X}{2\zeta(2)} a(4\ell). \quad (B.10)$$

By Lemma B.1, the main term of (B.3) should be

$$\frac{X}{2\zeta(2)} \sum_{m_k = \Box} \frac{\mu(k) a(4mk)}{m^{1/2+\alpha} k^{1/2+\beta}}. \quad (B.11)$$

We split the sum depending on the parity of $mk$ and expand into an Euler product:

$$\sum_{m_k = \Box} \frac{\mu(k) a(4mk)}{m^{1/2+\alpha} k^{1/2+\beta}} = \frac{2}{3} \sum_{m_k = \Box, \mkod} \frac{\mu(k) a(4mk)}{m^{1/2+\alpha} k^{1/2+\beta}} + \sum_{m_k = \Box, \mkod} \frac{\mu(k) a(mk)}{m^{1/2+\alpha} k^{1/2+\beta}}$$

$$= \frac{2}{3} \prod_{p > 2} \sum_{m+k \text{ even}} \frac{\mu(p^k) a(p^{m+k})}{p^{m(1/2+\alpha)+k(1/2+\beta)}}$$

$$+ \left( \sum_{m+k \text{ even}, \mkgeq1} \frac{\mu(2^k) a(2^{m+k})}{2^{m(1/2+\alpha)+k(1/2+\beta)}} \right) \prod_{p > 2} \sum_{m+k \text{ even}} \frac{\mu(p^k) a(p^{m+k})}{p^{m(1/2+\alpha)+k(1/2+\beta)}}$$

$$= \frac{2}{3} \left( 1 + \sum_{m+k \text{ even}, \mkgeq1} \frac{\mu(2^k)}{2^{m(1/2+\alpha)+k(1/2+\beta)}} \right) \prod_{p > 2} \sum_{m+k \text{ even}} \frac{\mu(p^k) a(p^{m+k})}{p^{m(1/2+\alpha)+k(1/2+\beta)}}. \quad (B.12)$$

The product over $p > 2$ is the same as in the ratios conjecture for fundamental discriminants, so using (2.25) in [CoSn07], it equals

$$\frac{2}{3} \cdot \frac{\zeta(2)(1+2\alpha)}{\zeta(2)(1+\alpha+\beta)} \cdot P_{D,2}(\alpha, \beta), \quad (B.13)$$

where

$$P_{D,2}(\alpha, \beta) = \prod_{p > 2} \left( 1 + \frac{p^{\alpha-\beta} - 1}{p^{\alpha-\beta}(p+1)(p^{1+\alpha+\beta} - 1)} \right). \quad (B.14)$$

The remaining factor in (B.12) is

$$1 + \sum_{m \geq 1, \text{ m even}} \frac{1}{2^{m(1/2+\alpha)}} - \sum_{m \geq 0, \text{ m odd}} \frac{1}{2^{m(1/2+\alpha)+1/2+\beta}}$$

$$= \left( 1 - \frac{1}{2^{1+\alpha+\beta}} \right) \left( 1 - \frac{1}{2^{1+2\alpha}} \right)^{-1}, \quad (B.15)$$
so it recovers the missing factors in zeta’s in (B.13). Therefore the first main term for \( n \equiv 1 \pmod{4} \) is
\[
\frac{X}{3 \zeta(2)} \frac{\zeta(1 + 2\alpha)}{\zeta(1 + \alpha + \beta)} P_{D,2}(\alpha, \beta).
\] (B.16)

The second part of the functional equation contributes
\[
\Gamma_e(1/2 + \alpha) \sum_{n \leq X} \mu^2(n) \left( \frac{n}{\pi} \right)^{-\alpha} \sum_{m k = \square} \frac{\mu(k) a(4mk)}{m^{1/2-\alpha} k^{1/2+\beta}}.
\] (B.17)

The inner sum is similar as above with \( \alpha \) replaced by \( -\alpha \), so this term gives
\[
\frac{X^{1-\alpha}}{(1-\alpha)3\zeta(2)} \cdot \frac{\pi^\alpha \Gamma_e(1/2 + \alpha) \zeta(1 - 2\alpha)}{\zeta(1 - \alpha + \beta)} P_{D,2}(-\alpha, \beta).
\] (B.18)

Finally, the computation for \( n \equiv 3 \pmod{4} \) will be similar with \( \Gamma_e \) replaced by \( \Gamma_o \) in the second main term, so we obtain

**Conjecture B.2.** Let \(-1/4 < \text{Re}(\alpha) < 1/4\), \( \frac{1}{\log X} \ll \text{Re}(\beta) < 1/4\), and \( \text{Im}(\alpha), \text{Im}(\beta) \ll X^{1-\varepsilon}. \) Then
\[
\sum_{n \leq X, \ n \text{ odd}} \frac{\mu^2(n)L(1/2 + \alpha, \chi_n)}{L(1/2 + \beta, \chi_n)} = \frac{2X}{3 \zeta(2)} \frac{\zeta(1 + 2\alpha)}{\zeta(1 + \alpha + \beta)} P_{D,2}(\alpha, \beta)
\]
\[+ \frac{X^{1-\alpha} \pi^\alpha (\Gamma_e \left( \frac{1}{2} + \alpha \right) + \Gamma_o \left( \frac{1}{2} + \alpha \right)) \zeta(1 - 2\alpha)}{(1-\alpha)3\zeta(2)} \zeta(1 - \alpha + \beta) P_{D,2}(-\alpha, \beta)
\]
\[+ O(X^{1/2+\varepsilon}).
\] (B.19)

To obtain an asymptotic for the sum of logarithmic derivatives, we differentiate with respect to \( \alpha \), and set \( \alpha = \beta = r \).

For the first term, we have
\[
\frac{d}{d\alpha} \frac{\zeta(1 + 2\alpha)}{\zeta(1 + \alpha + \beta)} P_{D,2}(\alpha, \beta) \bigg|_{\alpha = \beta = r} = \frac{\zeta'(1 + 2r)}{\zeta(1 + 2r)} + P'_{D,2}(r, r),
\] (B.20)

where we noted that \( P_{D,2}(r, r) = 1 \), and
\[
P'_{D,2}(r, r) = \sum_{p > 2} \frac{\log p}{(p + 1)(p^{1+2r} - 1)},
\] (B.21)

so the contribution of the first term is
\[
\frac{2X}{3 \zeta(2)} \left( \frac{\zeta'(1 + 2r)}{\zeta(1 + 2r)} + \sum_{p > 2} \frac{\log p}{(p + 1)(p^{1+2r} - 1)} \right).
\] (B.22)

For the second term, we notice that only one term in the derivative survives due to the factor \( \frac{1}{\zeta(1-\alpha+\beta)} \), so it equals
\[
-\frac{X^{1-r} \pi^r (\Gamma_e \left( \frac{1}{2} + r \right) + \Gamma_o \left( \frac{1}{2} + r \right)) \zeta(1 - 2r)}{(1-r)3\zeta(2)} P_{D,2}(-r, r).
\] (B.23)

We are thus led to the following conjecture:
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Conjecture B.3. For \( \frac{1}{\log X} \ll \text{Re}(r) \ll \frac{1}{4} \), \( \text{Im}(r) \ll X^{1-\varepsilon} \), we have

\[
\sum_{n \leq X} \frac{\mu^2(n)L'(1/2 + r, \chi_n)}{L(1/2 + r, \chi_n)} = \frac{2X}{3\zeta(2)} \left( \frac{\zeta'(1 + 2r)}{\zeta(1 + 2r)} + \sum_{p > 2} \frac{\log p}{(p + 1)(p^{1+2r} - 1)} \right) - \frac{X^{1-\varepsilon}}{\pi r} \Gamma_e\left(\frac{1}{2} + r\right) + \Gamma_o\left(\frac{1}{2} + r\right) \zeta(1 - 2r) \cdot (1 - r)3\zeta(2) P_{D,2}(-r, r) + O\left(X^{1/2+\varepsilon}\right).
\]

(B.24)

The two main terms match our result in Theorem 1.4, after using \( Mf(s) \approx 1/s \) and noticing that

\[
\frac{P_{D,2}(-r, r)}{3\zeta(2)} = \frac{1}{3\zeta(2) 2 \left( 1 + \frac{1 - p^{2r}}{(p + 1)(p - 1)} \right)} = \frac{1}{3\zeta(2) 2 \left( 1 - 1/p^2 \right)^{-1} (1 - p^{2r-2})} = \frac{1}{4\zeta(2)(2-2r)}
\]

(B.25)

Appendix C. Multivariable complex analysis

A general reference for the theory of multivariable complex analysis is [Hör66].

Definition C.1. An open set \( R \subset \mathbb{C}^n \) is a domain of holomorphy if there are no open sets \( R_1, R_2 \subset \mathbb{C}^n \) such that \( \emptyset \neq R_1 \subset R \cap R_2 \), \( R_2 \) is connected and not contained in \( R \), and for any holomorphic function \( f \) on \( R \), there is a function \( f_2 \) holomorphic on \( R_2 \) such that \( f = f_2 \) on \( R_1 \).

Open balls \( B(c, r) \) centered in \( c \) of radius \( r \) are domains of holomorphy. The following is a generalization of vertical strips in \( \mathbb{C}^n \).

Definition C.2. An open set \( T \subset \mathbb{C}^n \) is a tube if there is an open set \( U \subset \mathbb{R}^n \) such that \( T = U + i\mathbb{R}^n = \{ z \in \mathbb{C}^n : \text{Re}(z) \in U \} \).

The following is (a generalization of) Bochner’s Tube Theorem [Boc38].

Theorem C.3. A tube domain is a domain of holomorphy if and only if it is convex.

We denote the convex hull of \( T \) by \( \hat{T} \). In particular, every holomorphic function on \( T \) has a holomorphic continuation to \( \hat{T} \).

The following is useful in showing that some properties of holomorphic functions extend to their analytic continuations.

Theorem C.4. Let \( R_1 \subset \mathbb{C}^m, R_2 \subset \mathbb{C}^n \) be domains of holomorphy, and \( f : R_1 \to \mathbb{C}^n \) a holomorphic map. Then

\[
R = f^{-1}(R_2) = \{ z \in R_1 : f(z) \in R_2 \}
\]

is a domain of holomorphy.
The following proposition is used to estimate the size of the meromorphic continuations of our triple Dirichlet series in vertical strips.

**Proposition C.5.** Assume that \( T \subset \mathbb{C}^n \) is a tube domain, \( g, h : T \to \mathbb{C} \) are holomorphic functions, and let \( \tilde{g}, \tilde{h} \) be their holomorphic continuation to \( \tilde{T} \). If \( |g(z)| \leq |h(z)| \) for all \( z \in T \), and \( h(z) \) is nonzero in \( T \), then also \( |\tilde{g}(z)| \leq |\tilde{h}(z)| \) for all \( z \in \tilde{T} \).

**Proof.** Since \( h(z) \) is nonzero, the function \( f(z) = g(z)/h(z) \) is holomorphic in \( T \), and hence has a holomorphic continuation \( \tilde{f} \) to \( \tilde{T} \). By our assumptions, \( |\tilde{f}(z)| \leq 1 \) for all \( z \in T \), so \( T \subset \tilde{f}^{-1}(B(0,1)) \). However, by Theorem C.4, \( \tilde{f}^{-1}(B(0,1)) \) is a domain of holomorphy, so it is the whole \( \tilde{T} \).

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**References**

- Blo11 V. Blomer, *Subconvexity for a double Dirichlet series*, Compositio Math. 147 (2011), 355-374.
- Boc38 S. Bochner, *A theorem on analytic continuation of functions in several variables*, Ann. of Math. (2) 39 (1938), 14–19.
- Bum D. Bump, *Multiple Dirichlet series*, available online at [http://sporadic.stanford.edu/bump/multiple.pdf](http://sporadic.stanford.edu/bump/multiple.pdf).
- BFG12 D. Bump, S. Friedberg, D. Goldfeld (editors), 2012, *Multiple Dirichlet Series, L-functions and Automorphic Forms*, vol. 300 of Progress in Mathematics. Birkhäuser/Springer, New York, 2012.
- BFH96 D. Bump, S. Friedberg, J. Hoffstein, *On some applications of automorphic forms to number theory*, Bull. A.M.S 33 (1996), 157–175.
- BFK11 H. Bui, A. Florea, J. Keating, *Type-I contributions to the one and two level densities of quadratic Dirichlet L-functions over function fields*, J. Number Theory 221 (2021), 389-423.
- BFK21 H. M. Bui, A. Florea, J. P. Keating, *The Ratios conjecture and upper bounds for negative moments of L-functions over function fields*, preprint (2021), arXiv:2109.10396.
- CFH06 G. Chinta, S. Friedberg, J. Hoffstein, *Multiple Dirichlet series and automorphic forms*. In Multiple Dirichlet series, automorphic forms, and analytic number theory, Proc. Sympos. Pure Math., vol. 75 (2006), 3-41, Amer. Math. Soc., Providence, RI.
- CFKRS05 J. B. Conrey, D. W. Farmer, P. Keating, M. Rubinstein, N. Snaith, *Integral moments of L-functions*, Proc. Londond Math. Soc. (3) 91 (2005), no. 1, 33-104.
- CFZ05 J. B. Conrey, D. W. Farmer, M. R. Zirnbauer, *Howe pairs, supersymmetry, and ratios of random characteristic polynomials for the unitary groups UN*, preprint, 2005. arXiv math-ph/0511024.
- CFZ08 J. B. Conrey, D. W. Farmer and M. R. Zirnbauer, *Autocorrelation of ratios of L-functions*, Commun. Number Theory Phys., 2 (3) (2008), 593-636.
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CoSa07 J. B. Conrey, N. Snaith, Applications of the L-functions Ratios conjecture, Proc. Lond. Math. Soc. 93 (3) (2007), 594–646.

DGH03 A. Diaconu, D. Goldfeld, J. Hoffstein, Multiple Dirichlet series and moments of zeta and L-functions, Compositio Math. 139 2003, 297-360.

DHJ15 C. David, D. K. Huynh, J. Parks, One-level density of families of elliptic curves and the Ratios Conjecture, Research in Number Theory 1 (2015), no. 1, 1-37.

Far93 D. W. Farmer, Long mollifiers of the Riemann zeta-function, Mathematika 40 (1) (1993), 71–87.

FiMi15 D. Fiorilli, S. J. Miller, Surpassing the ratios conjecture in the 1-level density of Dirichlet L-functions, Algebra Number Theory 9 (2015), no. 1, 13-52.

Flo21 A. Florea, Negative moments of L-functions with small shifts over function fields, preprint (2021), arXiv:2111.10477.

Gar P. Garrett, Analytic continuation, functional equation: examples, online notes available at https://www-users.cse.umn.edu/~garrett/m/mfms/notes_c/analytic_continuations.pdf

GoHo85 D. Goldfeld, J. Hoffstein, Eisenstein series of 1/2-integral weight and the mean value of real Dirichlet L-series, Invent. Math. 80 (1985), 185-208.

GJM+10 J. Goes, S. Jackson, S.J. Miller, D. Montague, K. Ninsuwan, R. Peckner, T. Pham, A unitary test of the L-functions Ratios Conjecture, J. Number Theory 130 (2010), 2238–2258.

Hör66 L. Hörmander, An introduction to complex analysis in several variables, Van Nostrand, Princeton, N.J., 1966.

HMM11 D. K. Huynh, S. J. Miller, R. Morrison, An elliptic curve test of the L-functions Ratios conjecture, J. Number Theory 131 (2011), no. 6 1117-1147.

KaSa99a N. Katz and P. Sarnak, Random Matrices, Frobenius Eigenvalues and Monodromy, AMS Colloquium Publications 45, AMS, Providence, 1999.

KaSa99b N. Katz and P. Sarnak, Zeros of zeta functions and symmetries, Bull. AMS 36, 1999, 1-26.

Mil08 S.J. Miller, A symplectic test of the L-functions Ratios Conjecture, Int. Math. Res. Not. 2008 (3) (2008), 36 pp.

Mil09 S.J. Miller, An orthogonal test of the L-Functions Ratios Conjecture, Proc. Lond. Math. Soc. (2009), doi:10.1112/plms/pdp009.

MiMo11 S. J. Miller, D. Montague, An orthogonal test of the L-functions Ratios Conjecture II, Acta Arith. 146 (2011) 53–90.

Mon73 H. Montgomery, The pair correlation of zeros of the zeta function, Analytic Number Theory, Proc. Sympos. Pure Math. 24, Amer. Math. Soc., Providence, 1973, 191-193.

ÖzSn99 A. E. Özlıük and C. Snyder, On the distribution of the nontrivial zeros of quadratic L-functions close to the real axis, Acta Arith. 91 (1999), 209–228.

Sou00 K. Soundararajan, Nonvanishing of Quadratic Dirichlet L-functions at s=1/2, Ann. of Math. (2) 152 (2000), 447–488.

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