A priori estimates for elliptic equations with reaction terms involving the function and its gradient

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Abstract
We study local and global properties of positive solutions of $-\Delta u = u^p + M |\nabla u|^q$ in a domain $\Omega$ of $\mathbb{R}^N$, in the range $\min\{p, q\} > 1$ and $M \in \mathbb{R}$. We prove a priori estimates and existence or non-existence of ground states for the same equation.

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1 Introduction

This article is concerned with local and global properties of positive solutions of the following type of equations

$$-\Delta u = M'|u|^{p-1}u + M|\nabla u|^q,$$

in $\Omega \setminus \{0\}$ where $\Omega$ is an open subset of $\mathbb{R}^N$ containing 0, $p$ and $q$ are exponents larger than 1 and $M, M'$ are real parameters. If $M' \leq 0$ the equation satisfies a comparison principle and a big part of the study can be carried via radial local supersolutions. This no longer the case when $M' > 0$ which will be assumed in all the article, and by homothety (1.1) becomes

$$-\Delta u = |u|^{p-1}u + M|\nabla u|^q.$$

If $M = 0$ (1.2) is called Lane–Emden equation

$$-\Delta u = |u|^{p-1}u.$$

It turns out that it plays an important role in modelling meteorological or astrophysical phenomena [13,15], this is the reason for which the first study, in the radial case, goes back to the end of nineteenth century and the beginning of the twentieth. A fairly complete presentation can be found in [18]. If $N \geq 3$, This equations exhibits two main critical exponents $p = \frac{N}{N-2}$ and $p = \frac{N+2}{N-2}$ which play a key role in the description of the set of positive solutions which can be summarized by the following overview:

1. If $1 < p \leq \frac{N}{N-2}$, there exists no positive solution if $\Omega$ is the complement of a compact set. Even in that case solution can be replaced by supersolution. This is easy to prove by studying the inequality satisfied by the spherical average of a solution of the equation.

2. If $1 < p < \frac{N+2}{N-2}$, there exists no ground state, i.e. positive solution in $\mathbb{R}^N$. Furthermore any positive solution $u$ in a ball $B_R = B_R(a)$ satisfies

$$u(x) \leq c(R - |x - a|)^{\frac{2}{p-1}},$$

where $c = c(N, p) > 0$, see [19].
3. If \( p = \frac{N+2}{N-2} \) all the positive solutions in \( \mathbb{R}^N \) are radial with respect to some point \( a \) and endow the following form

\[
u(x) := u_\lambda(x) = \frac{(N(N-2)\lambda)^{\frac{N-2}{4}}}{(\lambda + |x-a|^2)^\frac{N-2}{2}}.
\] (1.5)

All the positive solutions in \( \mathbb{R}^N \setminus \{0\} \) are radial, see [12].

4. If \( p > \frac{N+2}{N-2} \) there exist infinitely many positive ground states radial with respect to some points. They are obtained from one say \( v \), radial for example with respect to 0 by the scaling transformation \( T_k \) where \( k > 0 \) with

\[
T_k[v](x) = k^{\frac{2}{p-1}} v(kx).
\] (1.6)

Indeed, the first significant non-radial results deals with the case \( 1 < p \leq \frac{N}{N-2} \). They are based upon the Brezis–Lions lemma [11] which yields an estimate of solutions in the Lorentz space \( L^{\frac{N}{N-2},\infty} \), implying in turn the local integrability of \( u^q \). Then a bootstrapping method as in [21] leads easily to some a priori estimate. Note that this subcritical case can be interpreted using the famous Serrin’s results on quasilinear equations [24]. The first breakthrough in the study of Lane–Emden equation came in the treatment of the case \( 1 < p < \frac{N+2}{N-2} \), it is due to Gidas and Spruck [19]. Their analysis is based upon differentiating the equation and then obtaining sharp enough local integral estimates on the term \( u^q - 1 \) making possible the utilization of Harnack inequality as in [24]. The treatment of the critical case \( p = \frac{N+2}{N-2} \), due to Caffarelli, Gidas and Spruck [12], was made possible thanks to a completely new approach based upon a combination of moving plane analysis and geometric measure theory. As for the supercritical case, not much is known and the existence of radial ground states is a consequence of Pohozaev’s identity [22], using a shooting method.

The study of (1.2) when \( M \neq 0 \) presents some similarities with the one of Lane–Emden equation in the cases 1 and 2, except that the proof are much more involved. Actually the approach we develop in this article is much indebted to our recent paper [9] where we study local and global aspects of positive solutions of

\[- \Delta u = u^p |\nabla u|^q, \] (1.7)

where \( p \geq 0, 0 \leq q < 2 \), mostly in the superlinear case \( p + q - 1 > 0 \). Therein we prove the existence of a critical line of exponents

\[ (\mathcal{E}) := \{(p, q) \in \mathbb{R}_+ \times [0, 2) : (N-2)p + (N-1)q = N\}. \] (1.8)

The subcritical range corresponds to the fact that \( (p, q) \) is below \( (\mathcal{E}) \). In this region Serrin’s celebrated results [24] can be applied and we prove [9, Theorem A] that positive solutions of (1.7) in the punctured ball \( B_2 \setminus \{0\} \) satisfy, for some constant \( c > 0 \) depending on the solution,

\[
u(x) + |x| |\nabla u(x)| \leq c |x|^{2-N} \text{ for all } x \in B_1 \setminus \{0\}.
\] (1.9)
When \((p, q)\) is above \((\mathcal{L})\), i.e. in the supercritical range, we introduced two methods for obtaining a priori estimate of solutions: The pointwise Bernstein method and the integral Bernstein method. The first one is based upon the change of unknown \(u = v - \beta\), and then to show that \(|\nabla v|\) satisfies an inequality of Keller–Osserman type. When \((p, q)\) lies above \((\mathcal{L})\) and verifies

\[(i) \text{ either } 1 \leq p < \frac{N+3}{N-1} \text{ and } p + q - 1 < \frac{4}{N-1},
\]

\[(ii) \text{ or } 0 \leq p < 1 \text{ and } p + q - 1 < \frac{(p+1)^2}{p(N-1)},\]

we prove that any positive solution of \((1.7)\) in a domain \(\Omega \subset \mathbb{R}^N\) satisfies

\[|\nabla u^a(x)| \leq c^* (\text{dist}(x, \partial \Omega))^{-1-a - \frac{2-q}{p+q-1}} \quad \text{for all } x \in \Omega, \quad (1.10)\]

for some positive \(c^*\) and \(a\) depending on \(N, p\) and \(q\) [9, Theorem B]. As a consequence we prove that any positive solution of \((1.7)\) in \(\mathbb{R}^N\) is constant. With the second method we combine the change of unknown \(u = v - \beta\) with integration and cut-off functions. We show the existence of a quadratic polynomial \(G\) in two variables such that for any \((p, q) \in \mathbb{R}_+ \times [0, 2)\) satisfying \(G(p, q) < 0\) any positive solution of \((1.7)\) in \(\mathbb{R}^N\) is constant [9, Theorem C]. The polynomial \(G\) is not simple but it is worth noting that if \(0 \leq p < \frac{N+2}{N-2}\), there holds \(G(p, 0) < 0\), which recovers Gidas and Spruck result [19].

For Eq. \((1.2)\) we first observe that the equation is invariant under the scaling transformation \((1.6)\) for any \(k > 0\) if and only if \(q\) is critical with respect to \(p\), i.e.

\[q = \frac{2p}{p+1}.
\]

In general the transformation \(T_k\) exchanges \((1.2)\) with

\[\Delta v = v^p + Mk^{2p-2p(p+1)}|\nabla v|^q, \quad (1.11)\]

hence if \(q < \frac{2p}{p+1}\), the limit equation when \(k \to 0\) is \((1.3)\). We say that the exponent \(p\) is dominant. We can also consider the transformation

\[S_k[v](x) = k^{\frac{2-q}{q-1}} v(kx), \quad (1.12)\]

when \(q \neq 2\), which is the same as \(T_k\) if \(q = \frac{2p}{p+1}\), and more generally transforms \((1.2)\) into

\[\Delta v = k^{\frac{q-p(2-q)}{q-1}} v^p + M|\nabla v|^q. \quad (1.13)\]

Hence if \(q > \frac{2p}{p+1}\), the limit equation when \(k \to 0\) is the Riccati equation

\[\Delta v = M|\nabla v|^q. \quad (1.14)\]
It is also important to notice that the value of the coefficient $M$ (and not only its sign) plays a fundamental role, only if $q = \frac{2p}{p+1}$. If $q \neq \frac{2p}{p+1}$ the transformation

$$u(x) = a v(y) \quad \text{with} \quad a = |M|^{-\frac{2}{(p+1)(q-p)}} \quad \text{and} \quad y = a \frac{p-1}{2} x$$

allows to transform (1.2) into

$$-\Delta v = |v|^{p-1} v \pm |\nabla v|^q. \quad (1.16)$$

Equation (1.2) has been essentially studied in the radial case when $M < 0$ in connection with the parabolic equation

$$\partial_t u - \Delta u + M |\nabla u|^q = |u|^{p-1} u, \quad (1.17)$$

see [14,16,17,25,27,30,31]. The studies mainly deal with the case $q \neq \frac{2p}{p+1}$, although not complete when $q > \frac{2p}{p+1}$. When $q = \frac{2p}{p+1}$ the existence of a ground state is proved in dimension 1. Some partial results that we will improve, already exist in higher dimension. The case $M > 0$ attracted less attention.

In the nonradial case, any nonnegative nontrivial solution is positive since $p, q > 1$. We first observe, using a standard averaging method applied to positive supersolutions of (1.3), that if $M \geq 0$, $1 < p \leq \frac{N}{N-2}$ when $N \geq 3$, any $p > 1$ if $N = 1, 2$, then for any $q > 0$ there exists no positive solution in an exterior domain. When $0 < q < \frac{2p}{p+1}$ the equation endows some character of the pure Emden–Fowler equation (1.3) by the transformation $T_k$. In [23] it is proved that if $0 < q < \frac{2p}{p+1}$, $1 < p < \frac{N+2}{N-2}$ and $M \in \mathbb{R}$, any positive solution of (1.3) in an open domain satisfies

$$u(x) + |\nabla u(x)|^{\frac{2}{p+1}} \leq c_{N,p,q,M} \left( 1 + (\text{dist} (x, \partial\Omega))^{-\frac{2}{p-1}} \right) \quad \text{for all} \quad x \in \Omega. \quad (1.18)$$

Note that this does not imply the non-existence of ground state. In [1] Alarcón, García-Melián and Quass study the equation

$$-\Delta u = |\nabla u|^q + f(u), \quad (1.19)$$

in an exterior domain of $\mathbb{R}^N$ emphasizing the fact that positive solutions are super harmonic functions. They prove that if $1 < q \leq \frac{N}{N-1}$ and if $f$ is positive on $(0, \infty)$ and satisfies

$$\limsup_{s \to 0} s^{-p} f(s) > 0, \quad (1.20)$$

for some $p > \frac{N}{N-2}$, then (1.19) admits no positive supersolution. The same authors also study in [2] existence and non-existence of positive solutions of (1.19) in a bounded domain with Dirichlet condition.

The techniques we developed in this paper are based upon a delicate extension of the ones already introduced in [9]. Our first nonradial result dealing with the case $q > \frac{2p}{p+1}$ is the following:
**Theorem A** Let $N \geq 1$, $p > 1$ and $q > \frac{2p}{p+1}$. Then for any $M > 0$, any solution of (1.2) in a domain $\Omega \subset \mathbb{R}^N$ satisfies

$$|\nabla u(x)| \leq c_{N, p, q} \left( M^{\frac{p+1}{(p+1)q-2p}} + (M \text{dist}(x, \partial\Omega))^{-\frac{1}{q-1}} \right) \quad \text{for all } x \in \Omega.$$  
(1.21)

As a consequence, any ground state has at most a linear growth at infinity:

$$|\nabla u(x)| \leq c_{N, p, q} M^{\frac{2}{2(p-1)q}} \quad \text{for all } x \in \mathbb{R}^N.$$  
(1.22)

Our proof relies on a direct Bernstein method combined with Keller–Osserman’s estimate applied to $|\nabla u|^2$. It is important to notice that the result holds for any $p > 1$, showing that, in some sense, the presence of the gradient term has a regularizing effect.

In the case $q < \frac{2p}{p+1}$ we prove a non-existence result

**Theorem A’** Let $N \geq 1$, $p > 1$, $1 < q < \frac{2p}{p+1}$ and $M > 0$. Then there exists a constant $c_{N, p, q} > 0$ such that there is no positive solution of (1.2) in $\mathbb{R}^N$ satisfying

$$u(x) \leq c_{N, p, q} M^{\frac{2}{2(p-1)q}} \quad \text{for all } x \in \mathbb{R}^N.$$  
(1.23)

When $q$ is critical with respect to $p$ the situation is more delicate since the value of $M$ plays a fundamental role. Our first statement is a particular case of a more general result in [1], but with a simpler proof which allows us to introduce techniques that we use later on.

**Theorem B** Let $N \geq 2$, $p > 1$ if $N = 2$ or $1 < p \leq \frac{N}{N-2}$ if $N = 3$, $q = \frac{2p}{p+1}$ and $M > -\mu^*$ where

$$\mu^* := \mu^*(N) = (p + 1) \left( \frac{N - (N - 2)p}{2p} \right)^{\frac{p}{p+1}}.$$  
(1.24)

Then there exists no nontrivial nonnegative supersolution of (1.2) in an exterior domain.

In this range of values of $p$ this result is optimal since for $M \leq -\mu^*$ there exists positive singular solutions. The constant $\mu^*$ will play an important role in the description developed in [10] of radial solutions of (1.2). Using a variant of the method used in the proof of Theorem B we obtain results of existence and nonexistence of large solutions.

**Theorem B’** Let $N \geq 1$, $p > 1$ and $q = \frac{2p}{p+1}$.

1. If $\Omega$ is a domain with a compact boundary satisfying the Wiener criterion and $M \geq -\mu^*(2)$ there exists no positive supersolution of (1.2) in $\Omega$ satisfying

$$\lim_{\text{dist}(x, \partial\Omega) \to 0} u(x) = \infty.$$  
(1.25)
2. If $G$ is a bounded convex domain, $\Omega = \overline{G}$ and $M < -\mu^*(1)$ there exists a positive solution of (1.2) in $\Omega$ satisfying (1.25).

We show in [10] that the inequality $M < -\mu^*(1)$ is the necessary and sufficient condition for the existence of a radial large solution in the exterior of a ball.

Concerning ground states, we prove their nonexistence for any $p > 1$ provided $M > 0$ is large enough: indeed

**Theorem C** Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a domain, $p > 1$, $q = \frac{2p}{p+1}$. For any

$$M > M^\dagger := \left( \frac{p - 1}{p + 1} \right)^{\frac{p-1}{p+1}} \left( \frac{N(p+1)^2}{4p} \right)^{\frac{p}{p+1}} \quad (1.26)$$

and any $\nu > 0$ such that $(1 - \nu)M > M^\dagger$, there exists a positive constant $c_{N, p, \nu}$ such that any solution $u$ in $\Omega$ satisfies

$$\left| \nabla u(x) \right| \leq c_{N, p, \nu} \left( (1 - \nu)M - M^\dagger \right)^{-\frac{p+1}{p+1}} (\text{dist} \ (x, \partial \Omega))^{-\frac{p+1}{p+1}} \quad \text{for all } x \in \Omega.$$

Consequently there exists no nontrivial solution of (1.2) in $\mathbb{R}^N$.

The next result, based upon an elaborate Bernstein method, complements Theorem C under a less restrictive assumption on $M$ but a more restrictive assumption on $p$.

**Theorem D** Let $1 < p < \frac{N+3}{N-1}$, $N \geq 2$, $1 < q < \frac{N+2}{N}$ and $\Omega \subset \mathbb{R}^N$ be a domain. Then there exist $a > 0$ and $c_{N, p, q} > 0$ such that for any $M > 0$, any positive solution $u$ in $\Omega$ satisfies

$$\left| \nabla u^a(x) \right| \leq c_{N, p, q} (\text{dist} \ (x, \partial \Omega))^{-\frac{2a}{p-1}} \quad \text{for all } x \in \Omega.$$  

Consequently there exists no nontrivial nonnegative solution of (1.2) in $\mathbb{R}^N$.

It is remarkable that the constants $a$ and $c_{N, p, q}$ do not depend on $M > 0$, a fact which is clear when $q \neq \frac{2p}{p+1}$ by using the transformation $T_k$, but much more delicate to highlight when $q = \frac{2p}{p+1}$ since (1.2) is invariant. When $|M|$ is small, we use an integral method to obtain the following result which contains, as a particular case, the estimates in [19] and [10]. The key point of this method is to prove that the solutions in a punctured domain satisfy a local Harnack inequality.

**Theorem E** Let $N \geq 3$, $1 < p < \frac{N+2}{N-2}$, $q = \frac{2p}{p+1}$. Then there exists $\epsilon_0 > 0$ depending on $N$ and $p$ such that for any $M$ satisfying $|M| \leq \epsilon_0$, any positive solution $u$ in $B_R \setminus \{0\}$ satisfies

$$u(x) \leq c_{N, p} |x|^{-\frac{2}{p-1}} \quad \text{for all } x \in B_{\frac{r}{\epsilon}} \setminus \{0\}. \quad (1.29)$$

As a consequence there exists no positive solution of (1.2) in $\mathbb{R}^N$, and any positive solution $u$ in a domain $\Omega$ satisfies

$$u(x) + |\nabla u(x)|^{\frac{2}{p+1}} \leq c'_{N, p} (\text{dist} \ (x, \partial \Omega))^{-\frac{2}{p-1}} \quad \text{for all } x \in \Omega.$$  

$$\quad (1.30)$$
Note that under the assumptions of Theorem E, there exist ground states for $|M|$ large enough when $1 < p < \frac{N}{N-2}$, or any $p > 1$ if $N = 1, 2$.

If $u$ is a radial solutions of (1.2) in $\mathbb{R}^N$ it satisfies

$$-u'' - \frac{N-1}{r}u' = |u|^{p-1}u + M|u'|^q,$$

(1.31)
on (0, \infty). Using several type of Lyapounov type functions introduced by Leighton [20] and Anderson and Leighton [3], we prove some results dealing with the case $M > 0$ which complement the ones of [25] relative to the case $M < 0$.

**Theorem F** 1. Let $p > 1$ and $q > \frac{2p}{p+1}$. Then there exists no radial ground state $u$ satisfying $u(0) = 1$ when $M > 0$ is too large.

2. Let $1 < p < \frac{N+2}{N-2}$. If $1 < q \leq p$ there exists no radial ground state for any $M > 0$. If $q > p$ there exists no radial ground state for $M > 0$ small enough.

3. Let $N \geq 3$, $p > \frac{N+2}{N-2}$ and $q \geq \frac{2p}{p+1}$. Then there exist radial ground states for $M > 0$ small enough.

We end the article in proving the existence of non-radial positive singular solutions of (1.2) in $\mathbb{R}^N \setminus \{0\}$ obtained by bifurcation from radial explicit positive singular solutions. Our result shows that the situation is very contrasted according $M > 0$ where a bifurcation from $(M, X_M)$ occurs only if $p \geq \frac{N+1}{N-3}$ and $M \geq 0$ and $M < 0$ where there exists a countable set of bifurcations from $(M_k, X_{M_k}), k \geq 1$, when $1 < p < \frac{N+1}{N-3}$.

In a subsequent article [10] we present a fairly complete description of the positive radial solutions of (1.2) in $\mathbb{R}^N \setminus \{0\}$ in the scaling invariant case $q = \frac{2p}{p+1}$.

## 2 The direct Bernstein method

We begin with a simple property in the case $M \geq 0$ which is a consequence of the fact that the positive solutions of (1.2) are superharmonic.

**Proposition 2.1** 1. There exists no positive solution of (1.2) in $\mathbb{R}^N \setminus \overline{B}_R$, $R \geq 0$ if one of the two conditions is satisfied:

(i) $M \geq 0$, $q \geq 0$ and either $N = 1, 2$ and $p > 1$ or $N \geq 3$ and $1 < p \leq \frac{N}{N-2}$.

(ii) $M > 0$, $N \geq 3$, $p \geq 1$ and $1 < q \leq \frac{N}{N-1}$.

2. If $N \geq 3$, $q \geq 1$, $p > \frac{N}{N-2}$ and $u(x) = u(r, \sigma)$ is a positive solution of (1.2) in $\mathbb{R}^N \setminus \overline{B}_R$, $R \geq 0$. Then there exists $\rho \geq R$ such that

$$\frac{1}{N\omega_N} \int_{S^{N-1}} u(r, \sigma) dS := \overline{u}(r) \leq c_0 r^{-\frac{2}{p-1}} \quad \text{for all } r > \rho, \quad (2.1)$$

with $c_0 := \left(\frac{2N}{p-1}\right)^{\frac{1}{p-1}}$ and

$$\left| \frac{1}{N\omega_N} \int_{S^{N-1}} u_r(r, \sigma) dS \right| := |\overline{u}_r(r)| \leq (N-2)c_0 r^{-\frac{p+1}{p-1}} \quad \text{for all } r > \rho. \quad (2.2)$$

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3. If $M > 0$, $p \geq 0$, and $q > \frac{N}{N-1}$ there holds for

$$|\bar{u}_r(r)| \leq \left( \frac{(q-1)(N-1)-1}{(q-1)M} \right)^{1-q} r^{-\frac{1}{q-1}}$$

for all $r > \rho$, \hspace{1cm} (2.3)

and

$$\bar{u}(r) \leq \left( \frac{q-1}{2-q} \right) \left( \frac{(q-1)(N-1)-1}{(q-1)M} \right)^{1-q} r^{-\frac{q-2}{q-1}}$$

for all $r > \rho$, \hspace{1cm} (2.4)

Furthermore, if $R = 0$, inequalities (2.1), (2.2) and (2.3) hold with $\rho = 0$.

**Proof** Assertion 1-(i) is not difficult to obtain by integrating the inequality satisfied by the spherical average of the solution and using Jensen’s inequality. For the sake of completeness, we give a simple proof although the result is actually valid for much more general equations (see e.g. [5] and references therein). In this statement we denote by $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$ the spherical coordinates in $\mathbb{R}^N$, by $\omega_N$ the volume of the unit N-ball and thus $N\omega_N$ is the (N-1)-volume of the unit sphere $S^{N-1}$. Writing (1.2) in spherical coordinates and using Jensen formula, we get

$$- r^{1-N} \left( r^{N-1} \bar{u}_r \right)_r \geq \bar{u}^p + M |\bar{u}_r|^q.$$ \hspace{1cm} (2.5)

It implies that $r \mapsto w(r) := - r^{N-1} \bar{u}_r$ is increasing on $(R, \infty)$, thus it admits a limit $\ell \in (-\infty, \infty]$. If $\ell \leq 0$, then $\bar{u}_r(r) > 0$ on $(R, \infty)$. Hence $\bar{u}(r) \geq \bar{u}(\rho) := c > 0$ for $r \geq \rho > R$, then

$$\left( r^{N-1} \bar{u}_r \right)_r \leq - c^p r^{N-1} \implies \bar{u}_r(r) \leq \frac{\rho^{N-1}}{r^{N-1}} \bar{u}_r(\rho) - \frac{c^p}{N} \left( r - \frac{\rho^N}{r^{N-1}} \right),$$

which implies $\bar{u}_r(r) \to -\infty$, thus $\bar{u}(r) \to -\infty$ as $r \to -\infty$, contradiction. Therefore $\ell \in (0, \infty]$ and either $\bar{u}_r(r) < 0$ on $(R, \infty)$ or there exists $r_\ell > R$ such that $\bar{u}_r(r_\ell) = 0$, $\bar{u}$ is increasing on $(R, r_\ell)$ and decreasing on $(r_\ell, \infty)$. If $\bar{u}_r(r) < 0$ on $(R, \infty)$, then we have for $r > 2R$

$$- r^{N-1} \bar{u}_r(r) \geq \int_{r}^{r_\ell} t^{N-1} \bar{u}^p(t) dt \geq \frac{r^N \bar{u}^p(r)}{2N} \implies \left( \bar{u}^{1-p} \right)_r$$

$$\geq \frac{(p-1)r}{2N} \implies \bar{u}(r) \leq \left( \frac{2N}{(p-1)r^2} \right)^{1-p},$$

which yields (2.1). If we are in the second case with $r_\ell > R$, we apply the same inequality with $r > 2r_\ell$ and again (2.1) for $r > 2r_\ell$. Since $\bar{u}$ is superharmonic, the function $v(s) = \bar{u}(r)$ with $s = r^{2-N}$ is concave on $(0, R^{2-N})$ and it tends to 0 when $s \to 0$. Thus

$$v_s(s) \leq \frac{v}{s} \implies |\bar{u}_r(r)| \leq (N-2) \frac{\bar{u}(r)}{r} \leq (N-2)c_0 r^{-\frac{p+1}{p-1}}.$$
This implies (2.1) and (2.2). Note that the case \( r_\ell > R \) cannot happen if \( R = 0 \), so in any case, if \( R = 0 \) then \( \rho = 0 \).

If \( M > 0 \), we have with \( w(r) = -r^{N-1}u_r \)

\[
w_r \geq Mr^{(1-q)(N-1)} |w|^q.
\]

We have seen that \( w(r) > 0 \) at infinity with limit \( \ell \in (0, \infty) \), hence, on the maximal interval containing \( \infty \) where \( w > 0 \), we have

\[
(w^{1-q})_r \leq (1 - q)Mr^{(N-1)(1-q)}.
\]

We have for \( r > s > R \)

\[
w^{1-q}(r) - w^{1-q}(s) \leq M \ln \left( \frac{r}{s} \right),
\]

if \( q = \frac{N}{N-1} \) and

\[
w^{1-q}(r) - w^{1-q}(s) \leq \frac{M(q-1)}{(q-1)(N-1) - 1} \left( r^{1-(q-1)(N-1)} - s^{1-(q-1)(N-1)} \right)
\]

if \( q < \frac{N}{N-1} \), and both expressions which tend to \(-\infty\) when \( r \to \infty \), a contradiction.

This proves 1-(ii). If \( q > \frac{N}{N-1} \), the above expression yields, when \( r \to \infty \),

\[
\ell^{1-q} - w^{1-q}(s) \leq -\frac{(q-1)M}{(q-1)(N-1) - 1} s^{1-(q-1)(N-1)}.
\]

This implies

\[
w(s) \leq \left( \frac{(q-1)(N-1) - 1}{(q-1)M} \right)^{\frac{1}{q-1}} s^{N-1-\frac{1}{q-1}},
\]

and (2.3).

\( \square \)

**Remark** The previous is a particular case of a much more general one dealing with quasilinear operators proved in [5, Theorem 3.1].

### 2.1 Proof of Theorems A, A′ and C

The function \( u \) is at least \( C^{3+\alpha} \) for some \( \alpha \in (0, 1) \) since \( p, q > 1 \). Hence \( z = |\nabla u|^2 \) is \( C^{2+\alpha} \). Since there holds by Bochner’s identity and Schwarz’s inequality

\[
-\frac{1}{2} \Delta z + \frac{1}{N} (\Delta u)^2 + \langle \nabla \Delta u, \nabla u \rangle \leq 0,
\]

we obtain from (1.2),

\[
-\frac{1}{2} \Delta z + \frac{|u|^{2p}}{N} + 2M^2 |u|^{p-1}u_z z^q + M^2 z^q - p|u|^{p-1}z - Mq \frac{z^{q-1}}{2} \langle \nabla z, \nabla u \rangle \leq 0.
\]
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Since for $\delta > 0$,

$$z^{q-1} |\langle \nabla z, \nabla u \rangle| \leq \left| z^{-\frac{1}{2}} \nabla z \right| \left| \nabla u \right| = \left| z^{-\frac{1}{2}} \nabla z \right| z^{\frac{q}{2}} \leq \delta z^q + \frac{1}{4\delta} \frac{|\nabla z|^2}{z},$$

we obtain for any $\nu \in (0, 1)$, provided $\delta$ is small enough,

$$- \frac{1}{2} \Delta z + \frac{|u|^2 p}{N} + \frac{2M}{N} |u|^{p-1} u z^\gamma + \frac{M^2 (1 - \nu)^2}{N} z^q - p |u|^{p-1} z \leq c_1 \frac{|\nabla z|^2}{z}, \quad (2.7)$$

where $c_1 = c_1(M, N, \nu) > 0$.

### 2.1.1 Proof of Theorem A

We recall the following technical result proved in [9, Lemma 2.2] which will be used several times in the course of this article.

**Lemma 2.2** Let $S > 1$, $R > 0$ and $v$ be continuous and nonnegative in $\overline{B} R$ and $C^1$ on the set $U_+ = \{ x \in B R : v(x) > 0 \}$. If $v$ satisfies, for some real number $a$,

$$- \Delta v + v^S \leq a \frac{|\nabla v|^2}{v}$$

on each connected component of $U_+$, then

$$v(0) \leq c_{N, S, a} R^{-\frac{S}{S-1}}. \quad (2.9)$$

**Abridged proof** Assuming $a > 0$, we set $W = v^\alpha$ for $0 < \alpha \leq \frac{1}{a+1}$, this transforms (2.8) into

$$- \Delta W + \frac{1}{\alpha} W^{\alpha(S-1)+1} \leq 0, \quad (2.10)$$

and then we apply Keller–Osserman inequality. $\square$

**Proof of Theorem A** Suppose $\frac{2p}{p+1} < q$. We set $r = \frac{2p}{p-1}$, $r' = \frac{r}{r-1}$, then, for any $\epsilon > 0$

$$p |u|^{p-1} z \leq \frac{\epsilon r |u|^{(p-1)r}}{r} + \frac{z^r}{\epsilon r'} = \left( p - 1 \right) \frac{\epsilon r |u|^{2p}}{2} + (p + 1) \frac{z^{\frac{2p}{p+1}}}{2 \epsilon r'}. \quad (2.11)$$

We fix $\eta \in (0, 1)$ and $\epsilon$ so that $\epsilon r' = \frac{2(1-\eta)}{N(p-1)}$ and get

$$p |u|^{p-1} z \leq (1 - \eta) \frac{|u|^{2p}}{N} + c_2 z^{\frac{2p}{p+1}}, \quad (2.12)$$
where \( c_2 = \frac{\frac{p+1}{2}(N(p-1))}{2(1-\eta)} \). We perform the change of scale (1.6) in order to reduce (1.2) to the case \( M = 1 \) by setting \( u(x) = \alpha^{\frac{2}{p-1}} v(\alpha x) \) with \( \alpha = M^{-\frac{p-1}{(p+1)q-2p}} \). Then the equation for \( z = |\nabla v|^2 \) is considered in \( \Omega_\alpha = \alpha \Omega \). Choosing now \( \eta = \frac{1}{2} \) we obtain

\[
c_2 \frac{2p}{p+1} \leq \frac{1}{4N} \varepsilon^q + c_3,
\]

where \( c_3 = c_3(N, \ p, \ q) > 0 \), hence

\[
-\frac{1}{2} \Delta z + \frac{v^{2p}}{2N} + \frac{1}{4N} \varepsilon^q \leq c_3 + c_1 \frac{|\nabla z|^2}{z}.
\]

Put \( \tilde{z} = \left( z - (4Nc_3)^{\frac{1}{q}} \right)_+ \), then

\[
-\frac{1}{2} \Delta \tilde{z} + \frac{1}{4N} \tilde{z}^q \leq c_1 \frac{|\nabla \tilde{z}|^2}{\tilde{z}},
\]

hence, from Lemma 2.2, we derive

\[
\tilde{z}(y) \leq c_4 \left( \text{dist} \ (y, \partial \Omega_\alpha) \right)^{\frac{2}{q-1}}
\]

where \( c_4 = c_4(N, \ q, \ c_1) > 0 \) which implies

\[
|\nabla v(y)| \leq c_4' \left( 1 + \left( \text{dist} \ (y, \partial \Omega_\alpha) \right)^{-\frac{1}{q-1}} \right) \quad \forall \ y \in \Omega_\alpha.
\]

Then (1.21) and (1.22) follow.

Assume now that there exists a ground state \( u \). Fix \( y \in \mathbb{R}^N \) and consider \( \{y_n\} \subset \mathbb{R}^N \) such that \( |y_n| = 2n > |y| \). We apply (2.11) with \( \Omega_\alpha = B_n(y_n) \). Then

\[
|\nabla v(y)| \leq c_4' \left( 1 + 2n - |y||^{-\frac{1}{q-1}} \right),
\]

and letting \( n \to \infty \) we infer

\[
|\nabla v(y)| \leq c_4' \quad \forall \ y \in \mathbb{R}^N.
\]

Hence, by the definition of \( v \) and \( y \) we see that

\[
|\nabla u(x)| \leq c_4' M^{-\frac{p+1}{(p+1)q-2p}} \quad \forall \ x \in \mathbb{R}^N
\]

which is exactly (1.22). \( \square \)
2.1.2 Proof of Theorem A'

Suppose $1 < q < \frac{2p}{p+1}$. By scaling we reduce to the case $M = 1$ and we replace $u$ by $v$ defined by (1.6) as in the proof of Theorem A with $\alpha = M^{\frac{p-1}{2p-(p+1)q}}$. From (2.7) with $v = \frac{1}{4}$ the function $z = |\nabla v|^2$ satisfies

$$-\frac{1}{2} \Delta z + \frac{v^{2p}}{N} + \frac{1}{2N} z^q - pv^{p-1}z \leq c_1 \frac{|\nabla z|^2}{z}.$$  \hspace{1cm} (2.13)

By Hölder’s inequality,

$$pv^{p-1}z \leq \frac{1}{4N} z^q + p(4Np)q'v^{q'v}.$$  \hspace{1cm} (2.14)

Since $(p-1)q' = 2p + \frac{2p-(p+1)q}{q-1}$ we derive

$$-\frac{1}{2} \Delta z + \frac{v^{2p}}{N} \left( 1 - \frac{4q'v}{Nq'^2} v^{\frac{2p-(p+1)q}{q-1}} \right) + \frac{1}{4N} z^q \leq c_1 \frac{|\nabla z|^2}{z}.$$  \hspace{1cm} (2.15)

If max $v \leq c_{N,p,q} := (4q'v^{q'}Nq'^2)^{-\frac{q-1}{2p-(p+1)q}}$, we obtain

$$-\frac{1}{2} \Delta z + \frac{1}{4N} z^q \leq c_1 \frac{|\nabla z|^2}{z},$$

which implies that $z = 0$ by Lemma 2.2, hence $v$ is constant and thus $v = 0$ from the equation. \hspace{1cm} \Box

Remark If $u$ is a positive ground state of (1.2) radial with respect to 0, it satisfies $u_r(0) = 0$ and it is a decreasing function of $r$. The previous theorem asserts that it must satisfy

$$u(0) > c_{N,p,q} M^{\frac{2}{2p-(p+1)q}}.$$  \hspace{1cm} (2.16)

2.1.3 Proof of Theorem C

Suppose $\frac{2p}{p+1} = q$. For $A > 0$ we consider the expression

$$(u^p + A |\nabla u|^q)^2 - Npu^{p-1} |\nabla u|^2$$

$$= \left( u^p + A |\nabla u|^q - \sqrt{Np u^{p-1}} |\nabla u| \right) \left( u^p + A |\nabla u|^q + \sqrt{Np u^{p-1}} |\nabla u| \right).$$
Now the function $Z \mapsto \Phi_A(Z) = u^p + AZ^q - \sqrt{Np} u^{\frac{p-1}{2}} Z$ achieves its minimum at $Z_0 = \left( \frac{\sqrt{Np}}{qA} \right)^{\frac{p+1}{p+1}}$ and

$$\Phi_A(Z_0) = \left[ 1 - \frac{p-1}{p+1} \left( \frac{N(p+1)^2}{4p} \right)^{\frac{p}{p+1}} A^{-\frac{p+1}{p+1}} \right] u^p.$$ 

Thus setting

$$M_\dagger = \left( \frac{p-1}{p+1} \right) \left( \frac{N(p+1)^2}{4p} \right)^{\frac{p}{p+1}}, \quad (2.15)$$

we obtain that if $A \geq M_\dagger$, then $\Phi_A(Z) \geq 0$ for all $Z$. Put $M_\nu = (1 - \nu)M$ for $\nu \in (0, 1)$ such that $M_\dagger < M_\nu$, we derive from (2.7)

$$-\frac{1}{2} \Delta z + \frac{(u^p + M_\dagger z^q)^2}{N} - pu^{p-1}z + \frac{M_\nu^2 - M_\dagger^2}{N} z^q \leq c_1 \frac{|\nabla z|^2}{z}, \quad (2.16)$$

which yields

$$-\frac{1}{2} \Delta z + \frac{M_\nu^2 - M_\dagger^2}{N} z^q \leq c_1 \frac{|\nabla z|^2}{z}. \quad (2.17)$$

Using again Lemma 2.2 we obtain

$$|\nabla u(x)| \leq c_1' \left( (1 - \nu)M - M_\dagger \right)^{-\frac{1}{q-1}} (\text{dist} \,(x, \partial \Omega))^{-\frac{1}{q-1}}, \quad (2.18)$$

which is equivalent to (1.27).

\section*{2.2 Proof of Theorems B and B'}

\subsection*{2.2.1 Proof of Theorem B}

Since the result is known when $M \geq 0$ from Proposition 2.1, we can assume that $M = -m < 0$ and $N = 1, 2$ or $N \geq 3$ with $p < \frac{N}{N-2}$, $u$ is a nonnegative supersolution of (1.2) in $\overline{B_R^c}$ and we set $u = v^b$ with $b > 1$. Then

$$-\Delta v \geq (b-1) \frac{|\nabla v|^2}{v} + \frac{1}{b} v^{1+b(p-1)} - mb^{q-1} v^{(b-1)(q-1)} |\nabla v|^q. \quad (2.18)$$

Here again $q = \frac{2p}{p+1}$, setting $z = |\nabla v|^2$ we obtain

$$-\Delta v \geq \frac{\Phi(z)}{bv}.$$
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\[ \Phi(z) = b(b - 1)z - mb^{\frac{2p}{p+1}}v^{\frac{2+b(p-1)}{p+1}} - \frac{p}{p+1}z + v^{2+b(p-1)}. \]

Thus \( \Phi \) achieves its minimum for

\[ z_0 = \left( \frac{m p b^{p-1}}{(b - 1)(p + 1)} \right)^{p+1} b^{p-1} v^{2+b(p-1)} \]

and

\[ \Phi(z_0) = v^{2+b(p-1)} \left( 1 - \frac{p}{(p + 1)^{p+1}} \left( \frac{b}{b - 1} \right)^p m^{p+1} \right). \quad (2.19) \]

In order to ensure the optimal choice, when \( N \geq 3 \) we take \( 1 + b(p-1) = \frac{N}{N-2} \), hence \( b = \frac{2}{(N-2)(p-1)} \) which is larger than 1 because \( p < \frac{N}{N-2} \). Finally

\[ \Phi(z_0) = v^{\frac{N}{N-2}+1} \left( 1 - \frac{1}{(p + 1)^{p+1}} \left( \frac{2p}{N - p(N - 2)} \right)^p m^{p+1} \right). \]

Hence, if

\[ m < (p + 1) \left( \frac{N - p(N - 2)}{2p} \right)^{\frac{p}{p+1}} = \mu^*(N), \quad (2.20) \]

we have for some \( \delta > 0 \),

\[ -\Delta v \geq \delta v^{\frac{N}{N-2}}, \quad (2.21) \]

and by Proposition 2.1 that is no positive solution in an exterior domain of \( \mathbb{R}^N \).

If \( N = 2 \) for a given \( b > 1 \) we have from (2.19) that if

\[ m < (p + 1) \left( \frac{b - 1}{b p} \right)^{\frac{p}{p+1}}, \]

then, for some \( \delta > 0 \),

\[ -\Delta v \geq \delta v^{1+b(p-1)}. \quad (2.22) \]

The result follows from Proposition 2.1 by choosing \( b \) large enough. \( \Box \)

2.2.2 Proof of Theorem B’

1. We assume that such a supersolution \( u \) exists and we denote \( u = e^v \), then

\[ -\Delta v \geq F(|\nabla v|^2), \quad (2.23) \]

where

\[ F(X) = X + e^{(p-1)v} + Me^{\frac{p-1}{p+1}v} X^{\frac{p}{p+1}}. \]
Clearly, if $M \geq 0$, then $F(X) \geq 0$ for any $X \geq 0$. Next we assume $M < 0$, then

$$F(X) \geq F(X_0) = e^{(p-1)v} \left(1 - p^p \left(\frac{|M|}{p + 1}\right)^{p+1}\right) = e^{(p-1)v} \left(1 - \left(\frac{|M|}{\mu^*(2)}\right)^{p+1}\right).$$

Hence, if $|M| \leq \mu^*(2)$, $v$ is a positive superharmonic function in $\Omega$ which tends to infinity on the boundary. Such a function is larger than the harmonic function with boundary value $k > 0$ for any $k$ (and taking the value $\min_{|x| = R} v(x)$ for $R$ large enough if $\Omega$ is an exterior domain). Letting $k \to \infty$ we derive a contradiction.

2. Let $R > 0$ such that $\Omega^c \subset B_R$ and let $w$ be the solution of

$$-\Delta w - ae^{(p-1)w} = 0 \quad \text{in } B_R \cap \Omega$$
$$\lim_{\text{dist}(x, \partial B_R) \to 0} w(x) = -\infty$$
$$\lim_{\text{dist}(x, \partial \Omega) \to 0} w(x) = \infty,$$

with $a = 1 - \left(\frac{|M|}{\mu^*(2)}\right)^{p+1} < 0$, obtained by approximations. By the argument used in 1,

$$ae^{(p-1)w} \leq |\nabla w|^2 + e^{(p-1)w} - |M| e^{\frac{p-1}{p+1} w} |\nabla w|^{\frac{2p}{p+1}},$$

hence

$$-\Delta w \leq |\nabla w|^2 + e^{(p-1)w} - |M| e^{\frac{p-1}{p+1} w} |\nabla w|^{\frac{2p}{p+1}}.$$

Therefore $v = e^w$ is nonnegative and satisfies

$$-\Delta v - v^p + |M| |\nabla v|^{\frac{2p}{p+1}} \leq 0 \quad \text{in } B_R \cap \Omega$$
$$v = 0 \quad \text{on } \partial B_R$$
$$\lim_{\text{dist}(x, \partial \Omega) \to 0} v(x) = \infty.$$  \hfill (2.25)

Next we extend $v$ by zero in $B_R^c$ and denote by $\tilde{v}$ the new function. It is a nonnegative subsolution of (1.2) which tends to $\infty$ on $\partial \Omega$. For constructing a supersolution we recall that if $M \leq -\mu^*(1)$ there exist two types of explicit solutions of

$$-u'' = u^p + M |u'|^{\frac{2p}{p+1}}$$  \hfill (2.26)

defined on $\mathbb{R}$ by $U_{j,M}(t) = \infty$ for $t \leq 0$ and $U_{j,M}(t) = X_{j,M} t^{-\frac{2}{p+1}}$, $j=1,2$, for $t > 0$ where $X_{1,M}$ and $X_{2,M}$ are respectively the smaller and the larger positive root of

$$X^{p-1} - |M| \left(\frac{2}{p-1}\right)^{\frac{2}{p+1}} X^{\frac{p-1}{p+1}} + \frac{2(p+1)}{(p-1)^2} = 0. \hfill (2.27)$$
Since $\Omega^c$ is convex it is the intersection of all the closed half-spaces which contain it and we denote by $H_{\Omega}$ the family of such hyperplanes which are touching $\partial \Omega$. If $H \in H_{\Omega}$ let $n_H$ be the normal direction to $H$, inward with respect to $\Omega$, $H_+ = \{ x \in \mathbb{R}^N : \langle n_H, x - n_H \rangle > 0 \}$ and we define $U_H$ in the direction $n_H$ by putting

$$U_H(x) = U_{2,M}(\langle n_H, x - n_H \rangle) = X_{2,M}(\langle n_H, x - n_H \rangle)^{-\frac{2}{p-1}} \quad \text{for all } x \in H_+.$$ 

Hence and set, for $x \in \Omega := \cap_{H \in H_{\Omega}} H_+$,

$$u_{\Omega}(x) = \inf_{H \in H_{\Omega}} U_H(x). \quad (2.28)$$ 

Then $u_{\Omega}$ is a nonnegative supersolution of (1.2) in $\Omega$ and

$$u_{\Omega}(x) \geq X_{2,M}(\text{dist } x, \Omega))^{-\frac{2}{p-1}} \quad \forall x \in \Omega.$$

Next $v_{\Omega} = \ln u_{\Omega}$ blows up on $\partial \Omega$, is finite on $\partial B_R$ and satisfies

$$- \Delta v_{\Omega} - a e^{(p-1)v_{\Omega}} \geq 0 \quad \text{in } B_R \cap \Omega. \quad (2.29)$$

By comparison with $w$ since $a < 0$, $v_{\Omega} \geq w$. Hence $u_{\Omega} \geq v$ in $B_R \setminus \Omega^c$. Extending $v$ by zero as $\tilde{v}$ we obtain $u_{\Omega} \geq \tilde{v}$ in $\Omega^c$. Hence $u_{\Omega}$ is a supersolution in $\Omega^c$ where it dominates the subsolution $\tilde{v}$. It follows by [29, Theorem 1-4-6] that there exists a solution $u$ of (1.2) satisfying $\tilde{v} \leq u \leq u_{\Omega}$, which ends the proof. \hfill $\Box$

### 3 The refined Bernstein method

The method is a combination of the one used in the previous proofs. It is based upon the replacement of the unknown by setting first $u = v^{-\beta}$ as in [7] and [19] and the study of the equation satisfied by $|\nabla v|$. However we do not use integral techniques. Since $u$ is a positive solution of (1.2) in $B_R$, the function $v$ is well defined and satisfies

$$- \Delta v + (1 + \beta) \frac{|\nabla v|^2}{v} + \frac{1}{\beta} v^{1-\beta(p-1)} + M |\beta|^{q-2} \beta v^{(\beta+1)(1-q)} |\nabla v|^q = 0 \quad (3.1)$$

in $B_R$. We set

$$z = |\nabla v|^2, \quad s = 1 - q - \beta(q - 1) = (1 - q)(\beta + 1), \quad \sigma = 1 - \beta(p - 1),$$

\[ Springer\]
and derive
\[ \Delta v = (1 + \beta) \frac{z}{v} + \frac{1}{\beta} v^\sigma + M |\beta|^{q-2} \beta v^s z^\frac{q}{r}. \]  
(3.2)

Combining Bochner’s formula and Schwarz identity we have classically
\[ \frac{1}{2} \Delta z \geq \frac{1}{N} (\Delta v)^2 + \langle \nabla \Delta v, \nabla v \rangle. \]

We explicit the different terms
\[
(\Delta v)^2 = (1 + \beta) \frac{z^2}{v^2} + M^2 \beta^{2(q-1)} v^{2s} z^q + \frac{v^{2\sigma}}{\beta^2} + 2M (1 + \beta) |\beta|^{q-2} \beta v^s z^{1+\frac{q}{r}} \\
+ \frac{2(1 + \beta)}{\beta} v^{\sigma-1} z + 2M |\beta|^{q-2} \beta v^s z^{\frac{q}{r}},
\]
\[
\nabla \Delta v = (1 + \beta) \frac{\nabla z}{v} - (1 + \beta) z \nabla v + \frac{\sigma}{\beta} v^{\sigma-1} \nabla v + M s |\beta|^{q-2} \beta v^s z^{\frac{q}{r}} \nabla v \\
+ \frac{M q}{2} |\beta|^{q-2} \beta v^s z^{\frac{q}{r}-1} \nabla z,
\]
\[
\langle \nabla \Delta v, \nabla v \rangle = \left(\frac{1 + \beta}{v} + \frac{M q}{2} |\beta|^{q-2} \beta v^s z^{\frac{q}{r}-1} \right) \langle \nabla z, \nabla v \rangle - \left(\frac{1 + \beta}{v^2} + \frac{\sigma}{\beta} v^{\sigma-1} z \right)
\]
\[
+ M s |\beta|^{q-2} \beta v^s z^{\frac{q}{r}+1}.
\]

Hence
\[
- \frac{1}{2} \Delta z + \frac{1}{N} (\Delta v)^2 + \left(\frac{1 + \beta}{v} + \frac{M q}{2} |\beta|^{q-2} \beta v^s z^{\frac{q}{r}-1} \right) \langle \nabla z, \nabla v \rangle \\
- \left(\frac{1 + \beta}{v^2} + \frac{\sigma}{\beta} v^{\sigma-1} z \right) + M s |\beta|^{q-2} \beta v^s z^{\frac{q}{r}+1} \leq 0.
\]  
(3.3)

3.1 Proof of Theorem D

We develop the term \((\Delta v)^2\) in (3.3) and get
\[
- \frac{1}{2} \Delta z + \left(\frac{(1 + \beta)^2}{N} - (1 + \beta) \right) \frac{z^2}{v^2} + \frac{M^2 \beta^{2(q-1)}}{N} v^{2s} z^q \\
+ M \left( s + \frac{2(1 + \beta)}{N} \right) |\beta|^{q-2} \beta v^s z^{1+\frac{q}{r}} \\
+ \frac{v^{2\sigma}}{N \beta^2} + \left(\frac{1 + \beta}{v} + \frac{M q}{2} |\beta|^{q-2} \beta v^s z^{\frac{q}{r}-1} \right) \langle \nabla z, \nabla v \rangle \\
+ \frac{N \sigma + 2(1 + \beta)}{N \beta} v^{\sigma-1} z + \frac{2M |\beta|^{q-2}}{N} v^s z^{\frac{q}{r}} \\
\leq 0.
\]  
(3.4)
Next we set $z = v^{-k}Y$ where $k$ is a real parameter. Then $\nabla z = -kv^{-k-1}Y \nabla v + v^{-k} \nabla Y$,

$$
\langle \nabla z, \nabla v \rangle = -kv^{-k-1}Y z + v^{-k} \langle \nabla Y, \nabla v \rangle = -kv^{-2k-1}Y^2 + v^{-k} \langle \nabla Y, \nabla v \rangle,
$$

$$
\frac{\langle \nabla z, \nabla v \rangle}{v} = -kv^{-2k-2}Y^2 + v^{-k-1} \langle \nabla Y, \nabla v \rangle,
$$

$$
M v^s z^{\frac{q}{2}-1} \langle \nabla z, \nabla v \rangle = -k M v^s \frac{2k-1}{2} Y + M v^s \frac{q}{2} \frac{q}{2} - 1 (\nabla Y, \nabla v),
$$

$$
-\Delta z = \text{div} \left( kv^{-k-1}Y \nabla v - v^{-k} \nabla Y \right)
= kv^{-k-1}Y \Delta v - k(k + 1)v^{-k-2}Y z + 2kv^{-k-1} \langle \nabla Y, \nabla v \rangle - v^{-k} \Delta Y
= kv^{-k-1}Y \Delta v - k(k + 1)v^{-2k-2}Y^2 + 2kv^{-k-1} \langle \nabla Y, \nabla v \rangle - v^{-k} \Delta Y.
$$

From (3.2)

$$
\Delta v = (1 + \beta) v^{-k-1}Y + \frac{1}{\beta} v^\sigma + M |\beta|^{q-2} \beta v^{s-k} Y \frac{q}{2},
$$

therefore

$$
-\Delta z = k(\beta - k) v^{-2k-2}Y^2 + \frac{k}{\beta} \sigma v^{-k-1}Y + k M |\beta|^{q-2} \beta v^{s-k} Y \frac{q}{2} - 1
+ 2kv^{-k-1} \langle \nabla Y, \nabla v \rangle - v^{-k} \Delta Y.
$$

Replacing $\langle \nabla z, \nabla v \rangle$ and $\Delta z$ given by the above expressions in (3.4) and $z$ by $v^{-k}Y$, leads to

$$
-\Delta Y + \left( \frac{k(\beta - k)}{2} + \frac{(1 + \beta)^2}{N} - (k + 1)(\beta + 1) \right) v^{-k-2}Y^2 + \frac{v^{2\sigma + k}}{N \beta^2}
+ \frac{M^2 \beta^{2(q-1)}}{N} v^{2s+k-kq} Y^q
+ \left( \frac{k + \beta + 1}{v} + \frac{Mq |\beta|^{q-2} \beta v^{s+k-kq} Y \frac{q}{2} - 1}{2} \right) \langle \nabla Y, \nabla v \rangle
+ \frac{2M |\beta|^{q-2}}{N} v^{s+\sigma+k-kq} Y \frac{q}{2}
+ \left( s + \frac{2(1 + \beta)}{N} - \frac{k(q-1)}{2} \right) M |\beta|^{q-2} \beta v^{s-k} Y \frac{q}{2} + 1
+ \frac{1}{\beta} \left( \frac{k}{2} + \sigma + \frac{2(1 + \beta)}{N} \right) v^{\sigma-1} Y \leq 0.
$$
For $\epsilon_1, \epsilon_2 > 0$,

$$\frac{1}{v} |(\nabla Y, \nabla v)| \leq \epsilon_1 v^{-k-2} Y^2 + \frac{1}{4\epsilon_1} \frac{|\nabla Y|^2}{Y},$$

$$v^{s+k-k\frac{q}{2}} Y^{\frac{q}{2}-1} |(\nabla Y, \nabla v)| \leq \epsilon_2 v^{2s-kq+k} Y^q + \frac{1}{4\epsilon_2} \frac{|\nabla Y|^2}{Y}.$$ 

Hence

$$- \Delta Y + \frac{v^{2\sigma+k}}{N\beta^2} + \frac{2M |\beta|^{q-2}}{N} v^{s+\sigma+k-k\frac{q}{2}} Y^{\frac{q}{2}} + \left( \frac{M^2 \beta^{2(q-1)}}{N} - \frac{M q \epsilon_2 |\beta|^{q-1}}{2} \right) v^{s+k-kq} Y^q$$

$$+ \left( \frac{k(\beta-k)}{2} + \frac{(1+\beta)^2}{N} - (k+1)(\beta+1) - |k+\beta+1| \epsilon_1 \right) v^{-k-2} Y^2$$

$$+ \frac{1}{\beta} \left( \frac{k}{2} + \sigma + \frac{2(1+\beta)}{N} \right) v^{\sigma-1} Y + \left( s + \frac{2(1+\beta)}{N} - \frac{k(q-1)}{2} \right)$$

$$\times M |\beta|^{q-2} \beta v^{s-k\frac{q}{2}-1} Y^{1+\frac{q}{2}}$$

$$\leq \left( \frac{|k+\beta+1|}{\epsilon_1} + \frac{M q |\beta|^{q-1}}{2\epsilon_2} \right) \frac{|\nabla Y|^2}{4Y}. \quad (3.5)$$

We first choose $\epsilon_2 = \frac{M|\beta|^{q-1}}{qN}$, then

$$- \Delta Y + \frac{v^{2\sigma+k}}{N\beta^2} + \frac{k(\beta-k)}{2} + \frac{(1+\beta)^2}{N} - (k+1)(\beta+1) - |k+\beta+1| \epsilon_1 \right) v^{-k-2} Y^2$$

$$+ \frac{1}{\beta} \left( \frac{k}{2} + \sigma + \frac{2(1+\beta)}{N} \right) v^{\sigma-1} Y + \frac{M^2 \beta^{2(q-1)}}{2N} v^{2s+k-kq} Y^q + \frac{2M |\beta|^{q-2}}{N} v^{s+k-k\frac{q}{2}} Y^{\frac{q}{2}}$$

$$+ \left( s + \frac{2(1+\beta)}{N} - \frac{k(q-1)}{2} \right) M |\beta|^{q-2} \beta v^{s-k\frac{q}{2}-1} Y^{1+\frac{q}{2}}$$

$$\leq \left( \frac{|k+\beta+1|}{\epsilon_1} + \frac{N q^2}{2} \right) \frac{|\nabla Y|^2}{4Y}. \quad (3.6)$$

In order to show the sign of the terms on the left in (3.5), we separate the terms containing the coefficient $M$ from the ones which do not contain it. Indeed these last terms are associated to the mere Lane–Emden equation (1.3) which is treated, as a particular case, in [9, Theorem B] where the exponents therein are $q = 0$, and $p \in \left( 1, \frac{N+3}{N-1} \right)$. We set

$$H_{\epsilon_1,1} = \frac{v^{2\sigma+k}}{N\beta^2} + \frac{k(\beta-k)}{2} + \frac{(1+\beta)^2}{N} - (k+1)(\beta+1) - |k+\beta+1| \epsilon_1 \right) v^{-k-2} Y^2$$

$$+ \frac{1}{\beta} \left( \frac{k}{2} + \sigma + \frac{2(1+\beta)}{N} \right) v^{\sigma-1} Y$$

$$= v^{2\sigma+k} \tilde{H}_{\epsilon_1,1}(v^{-1-k-\sigma} Y), \quad (3.7)$$
where

\[
\tilde{H}_{\epsilon_1}(t) = \left( \frac{k(\beta - k) + (1 + \beta)^2}{2} + \frac{(k + 1)(\beta + 1) - |k + \beta + 1|\epsilon_1}{N} \right) t^2 \\
+ \frac{1}{\beta} \left( \frac{k}{2} + \sigma + \frac{2(1 + \beta)}{N} \right) t + \frac{1}{N\beta^2}.
\]  
(3.8)

and

\[
H_{M,2} = \frac{M^2\beta^{2(q-1)}}{2N}v^{2s+k-kqYq} + \frac{2M|\beta|^{q-2}}{N}v^{s+\sigma+k-kqYq} \\
+ \left( s + \frac{2(1 + \beta)}{N} - \frac{k(q - 1)}{2} \right) M|\beta|^{q-2}\beta v^{r-kqYq} + \tilde{L}.
\]  
(3.9)

Then

\[
-\Delta Y + v^{2\sigma+k}\tilde{H}_{\epsilon_1}(v^{-1-k-\sigma}Y) + H_{M,2} \leq \left( \frac{|k + \beta + 1|}{\epsilon_1} + \frac{Nq^2}{2} \right) \frac{|\nabla Y|^2}{4Y}.
\]

The sign of \( \tilde{H}_{\epsilon_1} \) depends on its discriminant \( D_{\epsilon_1} \) which is a polynomial in its coefficients. Then if for \( \epsilon_1 = 0 \) this discriminant is negative \( D_0 \) is negative, the discriminant \( D_{\epsilon_1} \) of \( \tilde{H}_{\epsilon_1} \) shares this property for \( \epsilon_1 > 0 \) small enough and therefore \( H_{\epsilon_1} \) is positive. The proof is similar as the one of [9, Theorem B] in case (i) but for the sake of completeness we recall the main steps. Firstly

\[
D'_0 := N^2\beta^2D_0 = \left( \frac{Nk}{2} + \sigma N + 2(1 + \beta) \right)^2 \\
- 4 \left( \frac{Nk(\beta - k)}{2} + (1 + \beta)^2 - N(k + 1)(\beta + 1) \right).
\]

Then

\[
D'_0 = \left( \frac{N(p - 1)}{4} - 1 \right) (2\sigma + k)^2 + 2(p - 1)(2\sigma + k) + \tilde{L}
\]

where \( \tilde{L} = (p - 1)k^2 + p(\lambda + 2)^2 > 0 \). Put

\[
S = \frac{2\sigma + k}{k + 2} = 1 - \frac{2\beta(p - 1)}{k + 2} \quad \text{and}
\]

\[
T(S) = \left( \frac{(N - 1)(p - 1)}{4} - 1 \right) S^2 + (p - 1)S + p.
\]

After some computations we get, if \( k \neq -2 \),

\[
D'_1 := \frac{D'_0}{(k + 2)^2} = (p - 1) \left( \frac{k}{k + 2} - \frac{S}{2} \right)^2 + T(S).
\]  
(3.10)
We choose \( S > 2 \) such that \( \frac{k}{k+2} - \frac{S}{2} = 0 \), hence \( \beta = \frac{2-k}{2(p-1)} \). If \( p < \frac{N+3}{N-1} \) the coefficient of \( S^2 \) in \( T(S) \) is negative. Hence \( T(S) < 0 \) provided \( S \) is large enough which is satisfied if \( k < -2 \) with \( |k+2| \) small enough. We infer from this that \( \beta > 0 \), \( D_0 < 0 \) and \( \tilde{H}_{1,1} > 0 \) if \( \epsilon_1 \) is small enough. In particular \( \tilde{H}_{1,1}(t) \geq c_6(t^2 + 1) \) for some \( c_6 = c_6(N, p, q) > 0 \), which means

\[
u^{2\sigma+k}\tilde{H}_{1,1}(v^{-1-k-\sigma}Y) \geq c_6\left(v^{-k-2}Y^2 + \nu^{2\sigma+k}\right). \tag{3.11}
\]

Secondly the positivity of \( H_{M,2} \) is ensured, as \( \beta \) and \( M \) are positive, by the positivity of \( A := s + \frac{2(1+\beta)}{N} - \frac{k(q-1)}{2} \).

Replacing \( s \) by its value, we obtain, since \( 1 < q < \frac{N+2}{N} \) and \( \beta + \frac{2+k}{2} > 0 \), which can be assume by taking \( |k+2| \) small enough,

\[
A = 2 - (q-1) \left( \beta + 1 + \frac{k}{2} \right) > -\frac{k}{N}
\]

Then we deduce that

\[-\Delta Y + c_6\left(v^{-k-2}Y^2 + \nu^{2\sigma+k}\right) \leq c_7 \frac{|\nabla Y|^2}{Y}. \tag{3.12}\]

and \( c_7 = c_7(N, p, q) > 0 \) is independent of \( M \). Since \( S = 1 - \frac{2\beta(p-1)}{k+2} = 1 - \frac{2-k}{k+2} = \frac{2k}{k+2} > 0 \), we have

\[
2Y^\frac{2S}{S+1} = 2 \left(\frac{Y^2}{v^{(k+2)}}\right)^\frac{S}{S+1} v^\frac{2(k+1)}{S+1} \leq \frac{Y^2}{v^{(k+2)}} + v^{(k+2)} = \frac{Y^2}{v^{(k+2)}} + \nu^{2\sigma+k}. \tag{3.13}
\]

From this we infer the inequality

\[-\Delta Y + 2c_6Y^\frac{2S}{S+1} \leq c_7 \frac{|\nabla Y|^2}{Y}. \tag{3.14}\]

Then we derive from Lemma 2.2 that in the ball \( B_R \) there holds

\[Y(0) \leq c_8R^{-\frac{2(S+1)}{S-1}} = c_8R^{-2+\frac{2(k+2)}{p(p-1)}}. \tag{3.15}\]

From this it follows

\[
|\nabla u|^{-\frac{2+k}{2p}}(0) \leq \frac{|k+2|}{\sqrt{c_8}} R^{-1+\frac{k+2}{p(p-1)}}. \tag{3.16}\]
Setting $a = -\frac{k+2}{2m} > 0$ we get that for any domain $\Omega \subset \mathbb{R}^N$ any positive solution in $\Omega$ satisfies

$$|\nabla u^a(x)| \leq \frac{|k+2|}{2} \sqrt{c_8} (\text{dist} (x, \partial \Omega))^{-1 - \frac{2m}{p-1}} \quad \text{for all } x \in \Omega. \quad (3.17)$$

The non existence of any positive of (1.2) solution in $\mathbb{R}^N$ follows classically. \(\Box\)

**Corollary 3.1** Let $\Omega$ be a smooth domain in $\mathbb{R}^N$, $N \geq 2$ with a bounded boundary, $1 < p < \frac{N+3}{N-1}$, $1 < q < \frac{N+2}{N}$ and $M > 0$. If $u$ is a positive solution of (1.2) in $\Omega$ there exists $d_0$ depending on $\Omega$ and $c_9 = c_9(N, p, q) > 0$ such that

$$u(x) \leq c_9 \left( (\text{dist} (x, \partial \Omega))^{-\frac{2}{p-1}} + \max_{\text{dist}(z, \partial \Omega) = d_0} u(z) \right) \quad \text{for all } x \in \Omega. \quad (3.18)$$

**Proof** It is similar to the one of [9, Corollary B-2]. \(\Box\)

### 4 The integral method

#### 4.1 Preliminary inequalities

We recall the next inequality [6, Lemma 3.1].

**Lemma 4.1** Let $\Omega \subset \mathbb{R}^N$ be a domain. Then for any positive $u \in C^2(\Omega)$, any nonnegative $\eta \in C_0^\infty (\Omega)$ and any real numbers $m$ and $d$ such that $d \neq m + 2$, the following inequality holds

$$A \int_\Omega \eta u^{m-2} |\nabla u|^4 \, dx - \frac{N-1}{N} \int_\Omega \eta u^m (\Delta u)^2 \, dx - B \int_\Omega \eta u^{m-1} |\nabla u|^2 \Delta u \, dx \leq R,$$

where

$$A = \frac{1}{4N} \left( 2(N - m)d - (N - 1)(m^2 + d^2) \right), \quad B = \frac{1}{2N} (2(N - 1)m + (N + 2)d),$$

and

$$R = \frac{m + d}{2} \int_\Omega u^{m-1} |\nabla u|^2 (\nabla u, \nabla \eta) \, dx + \int_\Omega u^m \Delta u (\nabla u, \nabla \eta) \, dx + \frac{1}{2} \int_\Omega u^m |\nabla u|^2 \Delta \eta \, dx.$$

It is noticeable that $d$ is a free parameter which plays a role only in the coefficients of the integral terms. The following technical result is useful to deal with the multi-parameter constraints problems which occur in our construction. It was first used in [7] under a simpler form and extended in [6, Lemma 3.4].
Lemma 4.2 For any $N \in \mathbb{N}$, $N \geq 3$ and $1 < p < \frac{N+2}{N-2}$ there exist real numbers $m$ and $d$ verifying

(i) $d \neq m + 2,$

(ii) $\frac{2(N-1)p}{N+2} < d,$

(iii) $\max\left\{-2, 1 - p, \frac{(N-4)p - N}{2}\right\} < m \leq 0,$

(iv) $2(N-m)d - (N-1)(m^2 + d^2) > 0.$  

(4.2)

4.2 Proof of Theorem E

Step 1: The integral estimates. Let $\eta \in C^\infty_0(\Omega)$, $\eta \geq 0.$ We apply Lemma 4.1 to a positive solution $u \in C^2(\Omega)$ of (1.2), firstly with $q > 1$ and then with $q = \frac{2p}{p+1}$.

$$A \int_\Omega \eta u^{m-2} |\nabla u|^4 \, dx - \frac{N-1}{N} \int_\Omega \eta \left(u^{m+2p} + 2Mu^{m+p} \, |\nabla u|^q + M^2u^m \, |\nabla u|^{2q}\right) \, dx$$

$$- B \int_\Omega \eta u^{m-1} |\nabla u|^2 \, u \, dx \leq R. \quad (4.3)$$

We multiply (1.2) by $\eta u^{m+p}$ and integrate over $\Omega$. Then

$$\int_\Omega \eta \left(u^{m+2p} + Mu^{m+p} \, |\nabla u|^q\right) \, dx = - \int_\Omega \eta u^{m+p} \, \Delta u \, dx$$

$$= \int_\Omega u^{m+p} \langle \nabla u, \nabla \eta \rangle \, dx + (m+p) \int_\Omega \eta u^{m+p-1} \, |\nabla u|^2 \, dx.$$

We set

$$F = \int_\Omega \eta u^{m-2} |\nabla u|^4 \, dx, \quad P = \int_\Omega \eta u^{m-1} |\nabla u|^{q+2} \, dx, \quad V = \int_\Omega \eta u^{m+2p} \, dx,$$

$$T = \int_\Omega \eta u^{m+p-1} \, |\nabla u|^2 \, dx, \quad W = \int_\Omega \eta u^{m+p} \, |\nabla u|^q \, dx, \quad U = \int_\Omega \eta u^m \, |\nabla u|^{2q} \, dx,$$

$$S = \int_\Omega u^{m+p} \langle \nabla u, \nabla \eta \rangle \, dx,$$

so that there holds

$$AF - \frac{N-1}{N} \left(V + 2MW + M^2U\right) + BT + BMP \leq R, \quad (4.4)$$

and

$$V + MW = (m+p)T + S. \quad (4.5)$$
Eliminating $V$ between (4.4) and (4.5), we get

$$AF + B_0 T + M \left( BP - \frac{N - 1}{N} W - \frac{N - 1}{N} MU \right) \leq R - \frac{N - 1}{N} S,$$

(4.6)

where

$$B_0 = B - \frac{N - 1}{N} (m + p) = \frac{N + 2}{2N} d - \frac{N - 1}{N} p.$$ 

Also

$$2P = 2 \int_{\Omega} \eta u^m \frac{\nabla u^2}{u} \left\| \nabla u \right\| \leq \int_{\Omega} \eta u^m \left( \frac{\left\| \nabla u \right\|^4}{u^2} + \left\| \nabla u \right\|^{2q} \right) dx = F + U.$$ 

We fix now $q = \frac{2p}{p+1}$, then

$$U = \int_{\Omega} \eta u^m \left\| \nabla u \right\|^{2q} dx = \int_{\Omega} \eta u^m \left( \frac{\left\| \nabla u \right\|^4}{u^2} \right)^{(q-1)} u^{2(q-1)} \left\| \nabla u \right\|^{4-2q} dx$$

$$\leq \frac{p - 1}{p + 1} \int_{\Omega} \eta u^{m-2} \left\| \nabla u \right\|^4 dx + \frac{2}{p + 1} \int_{\Omega} \eta u^{m+p-1} \left\| \nabla u \right\|^2 dx$$

$$\leq \frac{p - 1}{p + 1} F + \frac{2}{p + 1} T.$$ 

(4.7)

hence

$$P \leq \frac{1}{2} F + \frac{1}{2} U \leq \frac{p}{p + 1} F + \frac{1}{p + 1} T$$

(4.8)

and

$$2W = 2 \int_{\Omega} \eta u^{m+p} \left\| \nabla u \right\|^q dx \leq \int_{\Omega} \eta u^{m+2p} dx + \int_{\Omega} \eta u^m \left\| \nabla u \right\|^{2q} dx = V + U$$

$$\leq U + (m + p)T + S - MW.$$ 

(4.9)

Next we assume that $|M| \leq 1$. From (4.7), (4.9), it follows that

$$W \leq U + (m + p)T + S \leq F + (m + p + 1)T + S.$$ 

(4.10)

From now we fix $m$ and $d$ according Lemma 4.2. Therefore $A > 0$ by (4.2)-(iv) and $B > 0$ by combining (4.2)-(ii) and (4.2)-(iii). Furthermore $B_0 > 0$ by (4.2)-(ii). Hence, from (4.7), (4.8) and (4.10) we derive, since $\frac{N - 1}{N} < 1$ and $m \leq 0$ from (4.2)-(ii)

$$\left| BP - \frac{N - 1}{N} W - \frac{N - 1}{N} MU \right| \leq B (F + T) + F + (p + 1)T + S + F + T,$$

$$\leq (B + 2) F + (B + p + 2) T + S.$$
Plugging these estimates into (4.6) we infer

$$AF + B_0T - |M| ((B + 2) F + (B + p + 2) T + S) \leq R - \frac{N - 1}{N} S. \quad (4.11)$$

Since $A$ and $B_0$ are positive, there exists $\mu_1 \in (0, 1)$ such that for any $|M| < \mu_1$,

$$A_1 := A - |M| (B + 2) > \frac{A}{2} \quad \text{and} \quad B_1 := B_0 - |M| (B + p + 2) > \frac{B_0}{2}.$$ 

Set $A_2 = \min\{A_1, B_1\}$, then, and whatever is the sign of $S$,

$$A_2(F + T) \leq |R| + |S|. \quad (4.12)$$

Using (4.7) and (4.8) we have

$$A_2(U + P) \leq 2A_2(F + T) \leq 2(|R| + |S|). \quad (4.13)$$

In the sequel we denote by $c_j$ some positive constants depending on $N$ and $p$. Then

$$U + P + F + T + W \leq c_1(|R| + |S|).$$

On the other hand, we have

$$|R| \leq c_2 \int_{\Omega} \left( u^{m-1} |\nabla u|^3 |\nabla \eta| + u^{m+p} |\nabla u| |\nabla \eta| + u^m |\nabla u|^{q+1} |\nabla \eta| + u^m |\nabla u|^2 |\Delta \eta| \right) dx.$$ 

Since

$$|\nabla u|^q = \left( \frac{|\nabla u|}{\sqrt{u}} \right)^q u^{q-2} \leq \frac{|\nabla u|^2}{u} + u^{q-2} = \frac{|\nabla u|^2}{u} + u^p,$$

we deduce

$$\int_{\Omega} u^m |\nabla u|^{q+1} |\nabla \eta| dx \leq \int_{\Omega} u^{m-1} |\nabla u|^3 |\nabla \eta| dx + \int_{\Omega} u^{m+p} |\nabla u||\nabla \eta| dx.$$ 

Thus we derive from (4.13)

$$U + P + F + T + W \leq 2c_3 \left( \int_{\Omega} u^{m-1} |\nabla u|^3 |\nabla \eta| dx + \int_{\Omega} u^{m+p} |\nabla u||\nabla \eta| dx 
+ \int_{\Omega} u^m |\nabla u|^2 |\Delta \eta| dx \right). \quad (4.14)$$
From this point we can use the method developed in [7, p 599] for proving the Harnack inequality satisfied by positive solutions of (1.3) in \( \Omega \). We set \( \eta = \xi^h \) with \( \xi \in C^\infty_0(\Omega) \) with value in \([0, 1]\) and \( \lambda > 4 \). For \( \epsilon \in (0, 1) \) we have by the Hölder–Young inequality

\[
\int_\Omega u^{m-1} |\nabla u|^3 |\nabla \xi^\lambda| \, dx \leq \frac{\epsilon}{4c_3} \int_\Omega u^{m-2} |\nabla u|^4 \xi^\lambda \, dx + C(\epsilon, c_3) \int_\Omega u^{m+2} |\nabla \xi|^4 \xi^{\lambda-4} \, dx,
\]
(4.15)

and

\[
\int_\Omega u^{m+p} |\nabla u| |\nabla \xi^p| \, dx \leq \frac{\epsilon}{4c_3} \int_\Omega u^{m+p-1} |\nabla u|^2 \xi^p \, dx + C(\epsilon, c_3) \int_\Omega u^{m+p+1} |\nabla \xi|^2 \xi^{\lambda-2} \, dx,
\]
(4.16)

and

\[
\int_\Omega u^m |\nabla u|^2 |\Delta \xi^p| \, dx \leq \frac{\epsilon}{4c_3} \int_\Omega u^{m-2} |\nabla u|^4 \xi^p \, dx + C(\epsilon, c_3) \int_\Omega u^{m+2} (|\nabla \xi|^4 + |\Delta \xi|^2) \xi^{\lambda-4} \, dx.
\]
(4.17)

Hence

\[
U + P + F + T + W \leq c_4 \left( \int_\Omega u^{m+2} \left( |\nabla \xi|^4 + |\Delta \xi|^2 \xi^2 \right) \xi^{\lambda-4} \, dx 
+ \int_\Omega u^{m+p+1} |\nabla \xi|^2 \xi^{\lambda-2} \, dx \right).
\]
(4.18)

Let us denote by \( c_4 X \) the right-hand side of (4.18). Combining (4.5), (4.16) and (4.18) we also get

\[
S := \int_\Omega u^{m+p} |\nabla u| |\nabla \xi^p| \, dx \leq c_5 X \implies V := \int_\Omega u^{m+2p} \xi^p \, dx \leq c_6 X,
\]
(4.19)

and we finally obtain

\[
U + V + P + F + S + T + W \leq c_7 X.
\]
(4.20)

Finally we estimate the different terms in \( X \), using that \( m + p > 0 \) from (4.2)-(iii). For \( \epsilon > 0 \)

\[
\int_\Omega u^{m+2} \left( |\nabla \xi|^4 + |\Delta \xi|^2 \xi^2 \right) \xi^{\lambda-4} \, dx \leq \epsilon \int_\Omega u^{m+2p} \xi^\lambda \, dx 
+ C(\epsilon, c_7) \int \xi^{\lambda-2 \frac{m+2p}{p-1}} \left( |\nabla \xi|^4 + |\Delta \xi|^2 \right)^{\frac{m+2p}{2(p-1)}} \, dx,
\]
(4.21)
\[
\int_{\Omega} u^{m+p+1} |\nabla \xi|^2 \xi^{\lambda-2} \, dx \leq \epsilon \int_{\Omega} u^{m+2p} \xi^{\lambda} \, dx + C(\epsilon, c_7) \int_{\Omega} \xi^{\lambda-2} \frac{m+2p}{p-1} |\nabla \xi|^{\frac{2(m+2p)}{p-1}} \, dx. 
\]

(4.22)

At end we obtain
\[
U + V + P + F + S + T + W \leq c_8 \int_{\Omega} \xi^{\lambda-2} \frac{m+2p}{p-1} \left(|\nabla \xi|^4 + |\Delta \xi|^2\right)^{\frac{m+2p}{2(p-1)}} \, dx. 
\]

(4.23)

**Step 2: The Harnack inequality.** We suppose that \( \Omega = B_R \setminus \{0\} := B^*_R \) fix \( y \in B^*_R \), set \( r = |y| \), then \( B_r(y) \subset B^*_R \). Let \( \xi \in C_0^\infty(B_r(y)) \) such that \( 0 \leq \xi \leq 1 \), \( \xi = 1 \) in \( B^*_2(y) \), \( |\nabla \xi| \leq cr^{-1} \) and \( |\Delta \xi| \leq cr^{-2} \). We choose \( \lambda > \max \left\{ 4, \frac{m+2p}{p-1} \right\} \), then
\[
\int_{B_r(y)} \xi^{\lambda-2} \frac{m+2p}{p-1} \left(|\nabla \xi|^4 + |\Delta \xi|^2\right)^{\frac{m+2p}{2(p-1)}} \, dx \leq c_9 r^{N-\frac{2(m+2p)}{p-1}},
\]
and hence
\[
\int_{B^*_2(y)} u^{m+2p} \, dx \leq V \leq c_{10} r^{N-\frac{2(m+2p)}{p-1}}.
\]

(4.24)

We write (1.2) under the form
\[
\Delta u + D(x)u + M(G(x), \nabla u) = 0, 
\]

(4.25)

with
\[
D(x) = u^{p-1} \quad \text{and} \quad G(x) = |\nabla u|^{-\frac{2}{p+1}} \nabla u.
\]

Set \( \sigma = \frac{m+2p}{p-1} \), then \( \sigma > \frac{N}{2} \) by (4.2)-(iii) and
\[
\int_{B^*_2(y)} D^\sigma \, dx \leq V \leq c_{10} r^{N-\frac{2(m+2p)}{p-1}} = c_{10} r^{N-2\sigma}.
\]

(4.26)

Next we estimate \( G \). For \( \tau, \omega, \gamma > 0 \) and \( \theta > 1 \), we have with \( \theta' = \frac{\theta}{\theta - 1} \),
\[
|\nabla u|^{(q-1)\tau} = u^\omega |\nabla u|^\gamma u^{-\omega} |\nabla u|^{(q-1)\tau - \gamma} \leq u^{\omega \theta'} |\nabla u|^{\gamma \theta} + u^{-\omega \theta} |\nabla u|^{((q-1)\tau - \gamma)\theta'}.
\]

We fix
\[
\tau = 2 \frac{2p + m}{p - 1} = 2\sigma, \quad \omega = \frac{(2 - m)(p + m - 1)}{p + 1} \quad \text{and} \quad \theta = \frac{p + 1}{2 - m}.
\]
Then $\omega > 0$ and $\theta > 1$ from (4.2)-(iii), $\omega > 0$. Then $u^{\omega \theta} |\nabla u|^{\gamma \theta} = u^{p+m-1} |\nabla u|^2$ and $u^{-\omega \theta} |\nabla u|^{((q-1)\tau - \gamma)\theta} = u^{m-2} |\nabla u|^4$, thus

$$
\int_{B_{\frac{r}{2}} (y)} |\nabla u|^{(q-1)\tau} \, dx \leq F + T \leq c_{11} \int_{\Omega} \xi^{\lambda - 2m/(p-1)} \left( |\nabla \xi|^4 + |\Delta \xi|^2 \xi^2 \right)^{m+2p/(p-1)} \, dx.
$$

This implies

$$
\int_{B_{\frac{r}{2}} (y)} G^{\tau} (x) \, dx \leq c_{12} r^{N - \tau},
$$

(4.27)

with $\tau > N$. Using the results of [28, Section 5], we infer that a Harnack inequality, uniform with respect to $r$, is satisfied. Hence there exists $c_{13} > 0$ depending on $N$, $p$ such that for any $r \in (0, \frac{R}{2}]$ and $y$ such that $|y| = r$ there holds

$$
\max_{z \in B_{\frac{r}{2}} (y)} u(z) \leq c_{13} \min_{z \in B_{\frac{r}{2}} (y)} u(z) \, \forall 0 < r \leq \frac{R}{2} \, \forall y \text{ s.t. } |y| = r,
$$

(4.28)

which implies

$$
u(x) \leq c_{14} u(x') \, \forall x, x' \in \mathbb{R}^N \, \text{ s.t. } |x| = |x'| \leq \frac{R}{2},
$$

(4.29)

and actually $c_{14} = c_{13}^7$ by a simple geometric construction. By (4.24)

$$
\frac{r^N \omega_N r^N}{|z|^{2(m+2p)/(p-1)}} \left( \min_{z \in B_{\frac{r}{2}} (y)} u(z) \right)^{m+2p} \leq 4^N c_{10} r^{N - \frac{2(m+2p)}{p-1}},
$$

where $\omega_N$ is the volume of the unit N-ball. This implies

$$
u(x) \leq c_{14} |x|^{-\frac{2}{p-1}} \, \forall x \in B_{\frac{R}{2}}.
$$

(4.30)

The proof follows. $\square$

**Remark** Using standard rescaling techniques (see e.g. [29, Lemma 3.3.2]) the gradient estimate holds

$$
|\nabla u(x)| \leq c_{15} |x|^{-\frac{p+1}{p-1}} \, \forall x \in B_{\frac{R}{2}}.
$$

(4.31)

And the next estimate for a solution $u$ in a domain $\Omega$ satisfying the interior sphere condition with radius $R$ is valid

$$
u(x) \leq c_{14} \left( \text{dist} \, (x, \partial \Omega) \right)^{-\frac{2}{p-1}} \, \forall x \in \Omega \, \text{ s.t. } \text{dist} \, (x, \partial \Omega) \leq \frac{R}{2}.
$$

(4.32)
5 Radial ground states

We recall that if $q \neq \frac{2p}{p+1}$ and $M \neq 0$, (1.2) can be reduced to the case $M = \pm 1$ by using the transformation (1.15). Since any ground state $u$ of (1.2) radial with respect to 0 is decreasing (this is classical and straightforward), it achieves its maximum at 0 and the following equivalence holds if $v$ is defined by (1.15)

\[-u'' - \frac{N-1}{r}u' = |u|^{p-1}u + M |ur|^q \quad \text{s.t. max } u = u(0) = 1\]

\[\iff \quad -v'' - \frac{N-1}{r}v' = |v|^{p-1}v \pm |vr|^q \quad \text{s.t. max } v = v(0) = |M|^{\frac{2}{(p+1)q-2p}}.\]  

(5.1)

Hence large or small values of $M$ for $u$ are exchanged into large or small values of $v(0)$ for $v$ and in the sequel we will essentially express our results using the function $u$.

5.1 Energy functions

We consider first the energy function

\[r \mapsto H(r) = \frac{u^{p+1}}{p+1} + \frac{u^2}{2}.\]  

(5.2)

Then

\[H'(r) = M |u'|^{q+1} - \frac{N-1}{r}u'^2.\]

Hence, if $M \leq 0$, $H$ is decreasing, a property often used in [25]. This implies in particular that a radial ground state satisfies

\[|u'(r)| \leq \sqrt{\frac{2}{p+1}} (u(0))^{\frac{p+1}{2}}.\]  

(5.3)

A similar estimate holds in all the cases.

**Proposition 5.1** Let $M > 0$, $p, q > 1$. If $u$ is a radial ground state solution of (1.2), then the function $H$ defined in (5.2) is decreasing and in particular (5.3) holds.

**Proof** Let $u$ be such a radial ground state. By Proposition 2.1 we must have $q > \frac{N}{N-1}$ and

\[\frac{r}{u'^2} H' = Mr |u'|^{q-1} + 1 - N \leq \frac{(N-1)q - N}{q - 1} + 1 - N = -\frac{1}{q - 1},\]

this implies the claim. \(\square\)
5.1.1 Exponential perturbations

As we have seen it in the introduction, if $q < \frac{2p}{p+1}$ Eq. (1.2) can be seen as a perturbation of the Lane–Emden equation (1.3) while if $q > \frac{2p}{p+1}$ it can be seen as a perturbation of the Ricatti equation (1.14). Two types of transformations can emphasize these aspects.

(1) For $p > 1$ set

$$u(r) = r^{-\frac{2}{p-1}} x(t), \quad u'(r) = -r^{-\frac{p+1}{p-1}} y(t), \quad t = \ln r,$$

then

$$x_t = \frac{2}{p-1} x - y \quad (5.5)$$

$$y_t = -K y + x^p + M e^{-\omega t} y^q \quad (5.6)$$

with

$$K = \frac{(N - 2)p - N}{p - 1}, \quad (5.7)$$

and

$$\omega = \frac{(p + 1)q - 2p}{p + 1}. \quad (5.8)$$

If $q > \frac{2p}{p+1}$ (resp. $q < \frac{2p}{p+1}$), then $\omega > 0$ (resp. $\omega < 0$) system (5.7) is a perturbation of the Lane–Emden system

$$x_t = \frac{2}{p-1} x - y \quad (5.8)$$

$$y_t = -K y + x^p,$$

at $\infty$ (resp. $-\infty$). The following energy type function introduced in [20] is natural with (5.8)

$$N(t) = L(x(t), y(t)) = \frac{K}{p-1} x^2 - x^{\frac{p+1}{p+1}} - \left(\frac{2}{p-1}\right)^q M e^{-\omega t} \frac{x^{q+1}}{q+1} + 1 - \frac{2x}{p-1} \frac{1}{y^q} - \frac{2x}{p-1} - y^2, \quad (5.9)$$

and it satisfies

$$N'(t) = \left(\frac{2x}{p-1} - y\right) \left[ L \left(\frac{2x}{p-1} - y\right) - M e^{-\omega t} \left(\left(\frac{2x}{p-1}\right)^q - y^q\right)\right]$$

$$+ \omega \left(\frac{2}{p-1}\right)^q M e^{-\omega t} \frac{x^{q+1}}{q+1}, \quad (5.10)$$

where $L = N - 2 - \frac{4}{p-1} = K - \frac{2}{p-1}$. Relation (5.10) will be used later on.

(2) For $p, q > 1$ set

$$u(r) = r^{-\frac{2-q}{q-1}} \xi(t), \quad u'(r) = -r^{-\frac{1}{q-1}} \eta(t), \quad t = \ln r,$$

$$\omega = \frac{(p + 1)q - 2p}{p + 1}. \quad (5.11)$$
then
\[
\begin{align*}
\dot{\xi}_t &= \frac{2 - q}{q - 1} \xi - \eta \\
\dot{\eta}_t &= -\frac{(N - 1)q - N}{q - 1} \eta + \omega^\alpha \xi^p + M \eta^q
\end{align*}
\]  

(5.12)

where
\[
\omega = \frac{p - 1}{q - 1} \omega.
\]

(5.13)

Note that if \( q < \frac{2p}{p+1} \) this system at \( \infty \) endows the form
\[
\begin{align*}
\dot{\xi}_t &= \frac{2 - q}{q - 1} \xi - \eta \\
\dot{\eta}_t &= -\frac{(N - 1)q - N}{q - 1} \eta + M \eta^q.
\end{align*}
\]

(5.14)

It is therefore autonomous and much easier to study.

5.1.2 Pohozaev–Pucci–Serrin type functions

Let \( \alpha, \gamma, \theta, \kappa \) be real parameters with \( \alpha, \kappa > 0 \). Set
\[
\mathcal{Z}(r) = r^\kappa \left( \frac{u^2}{2} + \frac{u^{p+1}}{p+1} + \alpha \frac{uu'}{r} - \gamma u' |u'|^q \right).
\]

(5.15)

This type of function has been introduced in [25] in their study of Eq. (1.2) with \( M = 1 \) with specific parameters. We use it here to embrace all the values of \( M \). We define \( \mathcal{U} \) by the identity
\[
\mathcal{Z}' + \theta |u'|^{q-1} \mathcal{Z} = r^{\kappa-1} \mathcal{U}.
\]

(5.16)

Then
\[
\mathcal{U} = \left( \frac{\kappa}{2} + \alpha + 1 - N \right) u^2 + \left( \frac{\kappa}{p+1} - \alpha \right) u^{p+1} + \alpha(\kappa - N) \frac{uu'}{r} + \left( \frac{\theta}{p+1} - \gamma q \right) ru^{p+1} |u'|^{q-1}
\]
\[
+ \left( M + \gamma + \frac{\theta}{2} \right) r |u'|^{q+1} + (((N - 1)q - \kappa) \gamma
\]
\[
- \alpha(\theta + M)) u |u'|^q - \gamma (\theta + q M) ru |u'|^{2q-1}.
\]

(5.17)

5.2 Some known results in the case \( M < 0 \)

We recall the results of [14,25] and [23] relative to the case \( M < 0 \).
Theorem 5.2 1. Let $N \geq 3$ and $1 < p \leq \frac{N}{N-2}$.

1-(i) If $q > \frac{2p}{p+1}$, there is no ground state for any $M < 0$ [25, Theorem C].

1-(ii) If $1 < q < \frac{2p}{p+1}$ there exists a ground state when $|M|$ is large [14, Proposition 5.7] and there exists no ground state when $|M|$ is small [23].

2 Assume $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ and let $\overline{q}$ be the unique root in $(\frac{2p}{p+1}, p)$ of the quadratic equation
\[(N-1)(X - p)^2 - (N + 2 - (N-2)p)((p+1)X - 2p)X = 0.\]

2-(i) If $\overline{q} \leq q < p$ there exists no ground state for any $M < 0$ [25, Theorem C].

2-(ii) If $\frac{2p}{p+1} < q < \overline{q}$, there exists no ground state for $|M|$. It is an open question whether there could exist a finite number of $M$ for which there exists a ground state [25, Theorem 4].

2-(iii) If $1 < q < \frac{2p}{p+1}$, there exists a ground state for large $|M|$ [14, Proposition 5.7] and no ground state when $|M|$ is small [23].

3 Assume $p > \frac{N+2}{N-2}$ and $q > 1$ and let $Q_{N,p} = \frac{2(N-1)p}{2N+p+1} \in (\frac{2p}{p+1}, p)$.

3-(i) If $Q_{N,p} < q < p$ there exists a ground state for $|M|$ small.

3-(ii) If $1 < q \leq Q_{N,p}$ there exists a ground state for any $M < 0$ [25, Theorem A].

4 Assume $p = \frac{N+2}{N-2}$. There exists at least one $M < 0$ such that there exists a ground state if and only if $1 < q < p$. More precisely:

4-(i) If $\frac{2p}{p+1} < q < p$ there exists ground state if $|M|$ is small [25, Theorem B].

4-(ii) If $q \geq \frac{2p}{p+1}$ there exists a ground state for any $M < 0$ [25, Theorem A].

Remark It is interesting to quote that when $M < 0$ and $q \geq \frac{2p}{p+1}$, there holds [25, Theorem 3],
\[u(r) = O(r^{-\frac{2}{p-1}}) \quad \text{and} \quad u'(r) = O(r^{-\frac{p+1}{p-1}}) \quad \text{when} \quad r \to \infty.\]

5.3 The case $M > 0$

The next result is a consequence of Theorem A.

Theorem 5.3 Let $M > 0$, $p > 1$ and $q > \frac{2p}{p+1}$ then there exists no radial ground state satisfying $u(0) = 1$ when $M$ is large.

Proof Suppose that such a solution $u$ exists. From Theorem A and Proposition 2.1 there holds
\[
\sup_{r > 0} |u'(r)| \leq c_{N,p,q}|M|^{-\frac{p+1}{(p+1)(p-2q)}} \quad \text{and} \quad \sup_{r > 0} r^{\frac{p+1}{p-1}} |u'(r)| \leq c_{N,p}. \tag{5.18}
\]
As a consequence, if $r > R > 0$,

\[ 1 - u(r) = u(0) - u(r) = u(0) - u(R) + u(R) - u(r) \]

\[ \leq c_{N,p,q} |M|^{-\frac{p+1}{p+q-2p}} R + \int_R^\infty |u'(s)| \, ds \]

\[ \leq c_{N,p,q} |M|^{-\frac{p+1}{p+q-2p}} R + c'_{N,p} R^{-\frac{2}{p-1}}, \]

with $c'_{N,p} = \frac{p-1}{2} c_{N,p}$. Since $u(r) \to 0$ when $r \to \infty$, we take $R = |M|^{-\frac{p+1}{p+q-2p}}$ and derive

\[ 1 \leq \left( c_{N,p,q} + c'_{N,p} \right) |M|^{-\frac{2}{p+1(q-2p)}}, \]

and the conclusion follows. \hfill \square

**Remark** If we use Proposition 5.1 we can make estimate (5.19) more precise.

### 5.3.1 The case $M > 0$, $1 < p \leq \frac{N+2}{N-2}$

It is a consequence of our general results that there is no radial ground state for large $M$ or for small $M$ when $1 < q \leq \frac{2p}{p+1}$ and $1 < p < \frac{N+2}{N-2}$. Indeed, if $1 < q < \frac{2p}{p+1}$ is a consequence of the equivalence statement between a priori estimate and non-existence of ground state proved in [23], and if $q = \frac{2p}{p+1}$ it follows from Theorems C and E. Actually in the radial case, the result is more general.

**Theorem 5.4** Let $M > 0$ and $1 < p < \frac{N+2}{N-2}$. If $1 < q \leq p$, there exists no radial ground state for any $M$. If $q > p$ there exists no radial ground state for $M$ small enough.

**Proof** By Proposition 2.1, we may assume $N \geq 3$ and

\[ \frac{N}{N-2} < p \leq \frac{N+2}{N-2} \quad \text{and} \quad q > \frac{N}{N-1}. \]

(i) Assume first $q < \frac{2p}{p+1}$. We use the system (5.5). Then $\omega$, defined by (5.7) is negative. Hence the Leighton function $\mathcal{N}$ defined by (5.9) is nonincreasing since $L \leq 0$ when $p \leq \frac{N+2}{N-2}$. Furthermore since $(x(t), y(t)) \to (0, 0)$ when $t \to -\infty$ and $e^{-\omega t} \to 0$, we get $\mathcal{N}(-\infty) = 0$ it follows that $\mathcal{N}(t) < 0$ for $t \in \mathbb{R}$. Moreover, by Proposition 2.1,

\[ u(t) = O(r^{-\frac{2-q}{q-1}}) \quad \text{as} \quad r \to \infty \iff x(t) = O(e^{\frac{q(p+1)-2p}{q-1}t}) = o(1) \quad \text{as} \quad t \to \infty \]

This implies $e^{-\omega t} x^{q+1}(t) = O(e^{\frac{2q(p+1)-2p}{q-1}t}) = o(1)$ as $t \to \infty$ and $\mathcal{N}(\infty) = 0$, contradiction.

(ii) Assume next $\frac{2p}{p+1} \leq q \leq p$. We consider the function (5.15) with the parameters

\[ \kappa = \frac{2(p+1)(N-1)}{p+3} = (p+1)\alpha \quad \text{and} \quad \gamma = -\frac{2M}{q(p+1)+2} = \frac{\theta}{q(p+1)}, \]

\[ \square \]
already used by [25] when $M = -1$, and we get with $\mathcal{U}$ defined by (5.16),
\[
\mathcal{U} = \frac{2}{(p + 3)^2} \frac{u |u'|}{r} \left( A + BM \chi + CM \chi^2 \right) \quad \text{with} \quad \chi = \frac{p + 3}{2 + q(p + 1)} r |u'|^{q-1},
\]
where
\[
A = (N - 1)(N + 2 - (N - 2)p) , \quad B = 2(N - 1)(p - q) , \quad C = q(p + 1) - 2p.
\]
By our assumptions $A \geq 0$, $B \geq 0$ and $C > 0$. Hence $\mathcal{U} > 0$. This implies
\[
\mathcal{Z}(r) = \frac{1}{\mathcal{U}(s)} = \mathcal{Z}(0) + \int_0^r \frac{\mathcal{Z}(s)}{\mathcal{U}(s)} ds
\]
\[
= \int_0^r \frac{\mathcal{Z}(0)}{\mathcal{U}(s)} ds,
\]
since $\mathcal{Z}(0) = 0$. If $u$ is a ground state, then $u'(r) \to 0$ as $r \to \infty$, thus $u |u'|^q \leq u |u'|^{\frac{2p}{p+1}}$. Hence, from Proposition 2.1, $u'^2(r) = O(r^{-\frac{2p+1}{p+1}})$ as $r \to \infty$. The other terms $u^{p+1}(r), r^{-1}u(r)u'(r)$ and $u |u'|^{\frac{2p}{p+1}}$ satisfy the same bound, hence
\[
\mathcal{Z}(r) = O(r^{\frac{2(p+1)}{p+1}} - \frac{2(p+1)(N-1)}{p+1}) = O(r^{\frac{2(p+1)(N-2)p-(N+2)}{(p+3)(p-1)}}).
\]
Then $\mathcal{Z}(r) \to 0$ when $r \to \infty$, contradiction.

(iii) Suppose $q > p$ and $u$ is a ground state. By Proposition 5.1 and (5.18), there holds
\[
r |u'|^{q-1} = \frac{r}{\mathcal{U}} \left| u' \right|^{\frac{p+1}{p+1}} |u'|^{q-\frac{2p}{p+1}} \leq c_{N,p}.
\]
Then $\chi = \frac{p+3}{p+1} r |u'|^{q-1} \leq c_{N,p}$. Hence, if $M \leq M_{N,p}$ for some $M_{N,p} > 0$, $\mathcal{U}$ is positive as $A$ is. We conclude as above. \[\square\]

5.3.2 The case $M > 0, p > \frac{N+2}{N-2}$

We recall that in Theorem C if $q = \frac{2p}{p+1}$ and $p > 1$ there is no ground state whenever $M > M_{N,p}$, see (1.26). In Theorem A' if $1 < q < \frac{2p}{p+1}$ and $p > 1$ there is no ground state $u$ such that $u(0) = 1$ if $M$ is too large. In the next result we complement Theorem 5.3 for small value of $M$ in assuming $q > \frac{2p}{p+1}$.

Theorem 5.5 If $p > \frac{N+2}{N-2}$ and $q \geq \frac{2p}{p+1}$ then there exist radial ground states for $M > 0$ small enough.

Proof First we consider the function $\mathcal{Z}$ with $k = N$ and obtain
\[
\mathcal{Z}(r) = r^N \left( \frac{u'^2}{2} + \frac{u^{p+1}}{p + 1} + \alpha \frac{uu'}{r} - \gamma u |u'|^q \right).
\]
The function vanishes at the origin. We compute \( \mathcal{U} \) from the identity \( \mathcal{Z} + \theta |u'|^{q-1} \mathcal{Z} = r^{N-1} \mathcal{U} \) and get

\[
\mathcal{U} = \left( \alpha - \frac{N-2}{2} \right) u^2 + \left( \frac{N}{p+1} - \alpha \right) u^{p+1} + \left( \frac{\theta}{p+1} - \gamma q \right) ru^{p+1} |u'|^{q-1} + \left( M + \gamma + \frac{\theta}{2} \right) r |u'|^{q+1} + \left[ ((N-1)q - N) \gamma - \alpha (\theta + M) \right] u |u'|^q - \gamma (\theta + qM) ru |u'|^{2q-1}.
\]

If \( \gamma = 0 \) and \( \theta = -2M \), then

\[
\mathcal{U} = \left( \alpha - \frac{N-2}{2} \right) u^2 + \left( \frac{N}{p+1} - \alpha \right) u^{p+1} - \frac{2M}{p+1} ru^{p+1} |u'|^{q-1} + \alpha Mu |u'|^q.
\]

If \( u \) is a regular solution which vanishes at some \( r_0 > 0 \), then \( \mathcal{Z}(r_0) = 2^{-1} r_0^2 u'' N (r_0) > 0 \). As \( p > \frac{N+2}{N-2} \), by choosing \( \alpha = \frac{1}{2} \left( \frac{N}{p+1} + \frac{N-2}{2} \right) \) we have \( \frac{N}{p+1} < \alpha < \frac{N-2}{2} \). We define \( \ell > 0 \) by \( (N-2) p - (N+2) = 4(p+1) \ell \), then \( \frac{N-2}{2} - \alpha = \alpha - \frac{N}{p+1} = \ell \) and then

\[
\mathcal{U} \leq - \ell (u^2 + u^{p+1}) + M \alpha u |u'|^q.
\]

Assume first \( q < 2 \), we have from Hölder’s inequality and \( 0 < r \leq r_0 \) where \( u \) is positive

\[
u |u'|^q \leq \frac{q}{2} u^2 + \frac{2-q}{2} |u|^{\frac{2}{q-2}} \leq u^2 + |u|^{\frac{2}{q-2}},
\]

and

\[
\mathcal{U} + (\ell - M) u^2 \leq M \alpha u^{\frac{2}{q-2}} - \ell u^{p+1} = \ell u^{p+1} \left( \frac{M \alpha}{\ell} \frac{q^{p+1} - 2 p}{\ell - 1} - 1 \right)
\]

since \( q \geq \frac{2p}{p+1} \) and \( u \leq u(0) = 1 \). Taking \( M \leq \frac{\ell}{\alpha} = \frac{(N-2)p - N-2}{(N-2)p + 3N-2} \), \( \mathcal{U} \) is negative implying that \( r \mapsto e^{-2M \int_0^r |u'|^{q-1} ds} \mathcal{Z}(r) \) is nonincreasing. Since \( \mathcal{Z}(0) = 0 \), \( \mathcal{Z}(r) \leq 0 \), a contradiction.

If \( q = 2 \), then \( \mathcal{U} \leq - \ell (u^2 + u^{p+1}) + M \alpha u^2 \) since \( u \leq 1 \) on \( [0, r_0] \). We still infer that \( \mathcal{U} \leq 0 \) if we choose \( M \leq \frac{\ell}{\alpha} \).

Finally, if \( q > 2 \), we have from Theorem A, \( u' \leq C_{N,p,q} M^{-\frac{p+1}{(p+1)q - 2p}} u^2 \). Therefore, using again the decay of \( u \) from \( u(0) = 1 \),

\[
M \alpha u |u'|^q \leq M \alpha u |u'|^{q-2} u^2 \leq M \alpha C_{N,p,q}^{q-2} M^{-\frac{(p+1)(q-2)}{(p+1)q - 2p}} u^2 = \alpha C_{N,p,q}^{q-2} M^{\frac{2}{(p+1)q - 2p}} u^2.
\]
Hence $\mathcal{U} \leq -\left( \ell - \alpha C_{N,p,q}^{q-2} M^{\frac{2}{\sqrt{p+1}q-2p}} \right) u^2$. Taking

$$M^{\frac{2}{\sqrt{p+1}q-2p}} \leq C_{N,p,q}^{2-q} (N-2)p - N - 2,$$

we conclude that $\mathcal{U} < 0$ which ends the proof as in the previous cases. \hfill $\Box$

Theorem F is the combination of Theorems 5.3, 5.4 and 5.5.

### 6 Separable solutions

We denote by $(r, \sigma) \in \mathbb{R}_+ \times S^{N-1}$ the spherical coordinates in $\mathbb{R}^N$. Then Eq. (1.2) takes the form

$$-u_{rr} - \frac{N-1}{r} u_r - \frac{1}{r^2} \Delta' u = |u|^{p-1} + M \left( u_r^2 + \frac{1}{r^2} |\nabla' u|^2 \right)^{\frac{q}{2}},$$

where $\Delta'$ is the Laplace–Beltrami operator on $S^{N-1}$ and $\nabla'$ the tangential gradient. If we look for separable nonnegative solutions of (1.2) i.e. solutions under the form $u(r, \sigma) = \psi(r) \omega(\sigma)$, then

$$q = \frac{2p}{p+1}, \quad \psi(r) = r^{\frac{2}{p+1}}, \quad \text{and } \omega \text{ is a solution of}$$

$$-\Delta' \omega + \frac{2K}{p-1} \omega = \omega^p + M \left( \left( \frac{2}{p-1} \right)^2 \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p}{p+1}},$$

where $K$ is defined in (5.6). Throughout this section we assume

$$p > 1 \quad \text{and} \quad q = \frac{2p}{p+1}.$$  

### 6.1 Constant solutions

The constant function $\omega = X$ is a solution of (6.2) if

$$X^{p-1} + M \left( \frac{2}{p-1} \right)^{\frac{2p}{p+1}} X^{\frac{p-1}{p+1}} - \frac{2K}{p-1} = 0.$$  

(6.4)

For $N = 1, 2$ and $p > 1$ or $N \geq 3$ and $1 < p < \frac{N}{N-2}$, we recall that $\mu^* = \mu^*(N)$ has been defined in (1.24). The following result is easy to prove

**Proposition 6.1** 1. Let $M \geq 0$ then there exists a unique positive root $X_M$ to (6.4) if and only if $p > \frac{N}{N-2}$. Moreover the mapping $M \mapsto X_M$ is continuous and decreasing from $[0, \infty)$ onto $(0, \left( \frac{2K}{p-1} \right)^{\frac{1}{p+1}}]$.  

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2. Let \( M < 0, N \geq 3 \) and \( p \geq \frac{N}{N-2} \) then there exists a unique positive root \( X_M \) to (6.4) and the mapping \( M \mapsto X_M \) is continuous and decreasing from \((\infty, 0)\) onto \([\left( \frac{2K}{p-1} \right)^{\frac{1}{p-1}}, \infty)\).

3. Let \( M < 0, N = 1, 2 \) and \( p > 1 \) or \( N \geq 3 \) and \( 1 < p < \frac{N}{N-2} \) then there exists no positive root to (6.4) if \( -\mu^* < M \leq 0 \). If \( M = M^* := -\mu^* \) there exists a unique positive root \( X_{M^*} = \left( \frac{2|K|}{p(p-1)} \right)^{\frac{1}{p-1}} \). If \( M < -\mu^* \) there exists two positive roots \( X_{1,M} < X_{2,M} \). The mapping \( M \mapsto X_{1,M} \) is continuous and increasing from \((\infty, \mu^*)\) onto \((0, X_{M^*})\). The mapping \( M \mapsto X_{2,M} \) is continuous and decreasing from \((\infty, \mu^*)\) onto \([X_{M^*}, \infty)\).

**Abridged proof**

Set

\[
f_M(X) = X^{p-1} + M \left( \frac{2}{p-1} \right)^{\frac{2p}{p-1}} X^{\frac{p-1}{p+1}} - \frac{2K}{p-1},
\]

then

\[
f'_M(X) = (p - 1)X^{p-2} + M^{\frac{p-1}{p+1}} \left( \frac{2}{p-1} \right)^{\frac{2p}{p+1}} X^{-\frac{2}{p-1}}.
\]

1. If \( M \) is nonnegative, \( f_M \) is increasing from \(- \frac{2K}{p-1} = -\frac{2((N-2)p-N)}{(p-1)^2}\) to \( \infty \); hence, if \( p > \frac{N}{N-2} \) there exists a unique \( X_M > 0 \) such that \( f_M(X_M) = 0 \), while if \( 1 < p < \frac{N}{N-2} \), \( f_M \) admits no zero on \([0, \infty)\). Since \( f_M > f_M' \) for \( M > M' > 0 \), there holds \( X_M > X_{M'} \). By the implicit function theorem the mapping \( M \mapsto X_M \) is \( C^1 \) and decreasing from \([0, \infty)\) onto \((0, \left( \frac{2K}{p-1} \right)^{\frac{1}{p-1}})\). Actually it can be proved that (see [10, Proposition 2.2])

\[
X_M = \frac{p-1}{2} \left( \frac{K}{M} \right)^{\frac{p+1}{p-1}} (1 + o(1)) \text{ as } M \to \infty.
\]

2. If \( M \) is negative, \( f_M \) achieves its minimum on \([0, \infty)\) at \( X_0 = \left( \frac{2K}{p+1} \right)^{\frac{1}{p-1}} \)

\[
\left( \frac{2}{p-1} \right)^{\frac{2}{p-1}},
\]

and

\[
f_M(X_0) = -\frac{p}{(p+1)^{\frac{p+1}{p}}} \left( \frac{2}{p-1} \right)^{\frac{2}{p-1}} (-M)^{\frac{p+1}{p}} - \frac{2K}{p-1}
\]

\[
= -\left( \frac{2}{p-1} \right)^{\frac{2}{p-1}} \left( \frac{p}{(p+1)^{\frac{p+1}{p}}} (-M)^{\frac{p+1}{p}} + (N-2)p-N \right).
\]
Since $K > 0$, there exists a unique $X_M > 0$ such that $f_M(X_M) = 0$ and $X_M > X_0$. The mapping $M \mapsto X_M$ is $C^1$ and decreasing from $(-\infty, 0]$ onto $I\left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}, \infty)$. The following estimate holds

$$
\max \left\{ \left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}, \left(\frac{2}{p-1}\right)^{\frac{2}{p-1}} |M|^{\frac{p+1}{p(p-1)}} \right\} \leq X_M
$$

$$
\leq 2^{\frac{2}{p-1}} \left( \left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}} + \left(\frac{2}{p-1}\right)^{\frac{2}{p-1}} |M|^{\frac{p+1}{p(p-1)}} \right). \quad (6.7)
$$

3. If $N = 1, 2$ and $p > 1$ or $N \geq 3$ and $1 < p < \frac{N}{N-2}$, then $f_M(0) > 0$. Hence, if $f_M(X_0) > 0$ there exists no positive root to $f_M(X) = 0$. Equivalently, if $-\mu^* < M < 0$ if $f_M(X_0) = 0$, $X_0$ is a double root and this is possible only if $M = -\mu^*$, hence $X_{-\mu^*} = \left(\frac{2|K|}{p(p-1)}\right)^{\frac{1}{p-1}}$. If $f_M(X) < 0$, or equivalently, if $M < -\mu^*$, the equation $f_M(X) = 0$ admits two positive roots $X_{1,M} < X_0 < X_{2,M}$. The monotonicity of the $X_{j,M}$, $j=1,2$, and their range follows easily from the monotonicity of $M \mapsto f_M(X)$ for $M < 0$. Actually the following asymptotics hold when $M \to -\infty$,

$$
X_{1,M} = \frac{p-1}{2} \left(\frac{K}{M}\right)^{\frac{p+1}{p-1}} (1+o(1)) \text{ and } X_{2,M} = \left(\frac{2}{p-1}\right)^{\frac{2}{p-1}} (-M)^{\frac{p+1}{p(p-1)}} (1+o(1)). \quad (6.8)
$$

6.2 Bifurcations

We set

$$
A(\omega) = -\Delta' \omega + \frac{2K}{p-1} \omega - \omega^p - M \left(\frac{2}{p-1}\right)^{2} \omega^2 + |\nabla' \omega|^2 \right)^{\frac{p}{p+1}}, \quad (6.9)
$$

If $\eta \in C^\infty(S^{N-1})$ and if there exists a constant positive solution $X$ to $A(X) = 0$ we have

$$
\frac{d}{d\tau} A(X + \tau \eta)_{|\tau=0} = -\Delta' \eta + \left(\frac{2K}{p-1} - p X^{p-1} - M \left(\frac{2}{p-1}\right)^{2} \right)^{\frac{2p}{p+1}} X^{\frac{p-1}{p+1}} \eta.
$$

Hence the problem is singular if

$$
-\frac{2K}{p-1} + p X^{p-1} + M \left(\frac{2}{p-1}\right)^{\frac{2p}{p+1}} X^{\frac{p-1}{p+1}} = \lambda_k, \quad (6.10)
$$
where $\lambda_k = k(k + N - 2)$ is the k-th nonzero eigenvalue of $-\Delta'$ in $H^1(S^{N-1})$. The following result follows classically from the standard bifurcation theorem from a simple eigenvalue (which can always be assumed if we consider functions depending only on the azimuthal angle on $S^{N-1}$ reducing the eigenvalue problem to a simple Legendre type ordinary differential equation) see e.g. [26, Chapter 13] and identity (6.4).

**Theorem 6.2** Let $M_0 \in \mathbb{R}$ and $X_{M_0}$ be a constant solution of (6.2). If $X_{M_0}$ satisfies for some $k \in \mathbb{N}^*$,

$$M_0 \left( \frac{2}{p - 1} \right)^{\frac{2p}{p + 1}} X_{M_0}^{\frac{p - 1}{p + 1}} = \frac{p + 1}{p(p - 1)} (2K - \lambda_k), \quad (6.11)$$

there exists a continuous branch of nonconstant positive solutions $(M, \omega_M)$ of (6.2) bifurcating from the $(M_0, X_{M_0})$.

Since $M \left( \frac{2}{p - 1} \right)^{\frac{2p}{p + 1}} X_{M}^{\frac{p - 1}{p + 1}} = \frac{2K}{p - 1} - X_{M}^{\frac{p - 1}{p + 1}}$ by (6.4) the following statements follow immediately from Proposition 6.1.

**Lemma 6.3** Set $\Phi(M) = M \left( \frac{2}{p - 1} \right)^{\frac{2p}{p + 1}} X_{M}^{\frac{p - 1}{p + 1}}$ when $X_M$ is uniquely determined, and $\Phi_j(M) = M \left( \frac{2}{p - 1} \right)^{\frac{2p}{p + 1}} X_{M}^{\frac{p - 1}{p + 1}} j = 1, 2$, if there exist two equilibria. Then

1. If $N \geq 3$ and $p > \frac{N}{N - 2}$, the mapping $M \mapsto \Phi(M)$ is continuous and increasing from $[0, \infty)$ onto $[0, \frac{2K}{p - 1})$.
2. If $N \geq 3$ and $p \geq \frac{N}{N - 2}$, the mapping $M \mapsto \Phi(M)$ is continuous and increasing from $(-\infty, 0]$ onto $(-\infty, 0]$.
3. If $N = 1, 2$ and $p > 3$ or $N \geq 3$ and $1 < p < \frac{N}{N - 2}$, the mapping $M \mapsto \Phi_1(M)$ (resp $M \mapsto \Phi_2(M)$) is continuous and decreasing (resp. increasing) from $(-\infty, -\mu^*)$ onto $[\frac{2K}{p - 1}, 0)$ (resp. $(-\infty, \frac{2K}{p - 1})$).

If we analyse the range $R[\Phi]$ of $\Phi$ or $R[\Phi_j]$ of $\Phi_j$, we prove the following result.

**Theorem 6.4** 1 Let $N \geq 3$ and $p \geq \frac{N}{N - 2}$.

1-(i) There exists a continuous curve of bifurcation $(M, \omega_M)$ issued from $(M_0, X_{M_0})$ for some $M_0 = M_0(p) \geq 0$ if and only if $p \geq \frac{N + 1}{N - 3}$ and $k = 1$. Furthermore $M_0(\frac{N + 1}{N - 3}) = 0$.

1-(ii) The bifurcation curve $s \mapsto (M(s), \omega_M(s))$, is defined on $(-\epsilon_0, \epsilon_0)$ for some $\epsilon_0 > 0$ and verifies $(M(0), \omega_M(0)) = (M_0, X_{M_0})$.

2. Let $N \geq 3$ and $p \geq \frac{N}{N - 2}$.

2-(i) For any $k \geq 1$ there exist $M_k < 0$ and a continuous branch of bifurcation $(M, \omega_M)$ issued from $(M_k, X_{M_k})$, with the restriction that $p < \frac{N + 1}{N - 3}$ if $k = 1$. 

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2-(ii) The bifurcation curve $s \mapsto (M(s), \omega_{M(s)})$, is defined on $(-\epsilon_0, \epsilon_0)$ for some \(\epsilon_0 > 0\) and verifies $(M(0), \omega_{M(0)}) = (M_0, X_{M_0})$. Finally $M_k \to -\infty$ when $k \to \infty$.

3. let $N = 1, 2$ and $p > 1$, or $N \geq 3$ and $1 < p < \frac{N}{N-2}$.

3-(i) There exists no $M < 0$ such that \(\frac{2K}{p-1} < \Phi_1(M) < 0\), and a countable set of $M_k < 0$, $k \geq 1$, such that $\Phi_2(M_k) = \frac{p+1}{p(p-1)}(2K - \lambda_k)$.

3-(ii) There exist a countable branches of bifurcation of solutions $(M_k(s), \omega_{M_k(s)})$ issued from $(M_k, X_{2,M_k})$.

**Proof** Assertion 1. Since from Lemma 6.3, $R(\Phi) = [0, \frac{2K}{p-1})$ for $M \geq 0$, we have to see under what condition on $p \geq \frac{N}{N-2}$ one can find $k \geq 1$ such that

$$0 \leq \frac{p+1}{p(p-1)}(2K - \lambda_k) < \frac{2K}{p-1} \iff \frac{2K}{p+1} < \lambda_k \leq 2K.$$ 

Since $K < N$ and $\lambda_k \geq 2N$ for $k \geq 2$, the only possibility for this last inequality to hold is $k = 1$. The inequality $\frac{2K}{p+1} < N - 1$ always holds since $p > 1$, while the inequality $N - 1 = \lambda_1 \leq 2K$ is equivalent to $p \geq \frac{N+1}{N-3}$. Therefore $M_0 = 0$ and $X_{M_0} = \left(\frac{2K}{p-1}\right)^{\frac{1}{p-1}}$. If we consider only functions on the sphere $S^{N-1}$ which depend uniquely on the azimuthal angle $\theta = \tan^{-1}(x_N|\cdot|S^{N-1})$, the function $\psi_1(\sigma) = x_N|\cdot|S^{N-1}$ is an eigenfunction of $-\Delta'$ in $H^1(S^{N-1})$ with multiplicity one. Hence the bifurcation branch is locally a regular curve $s \mapsto (M(s), \omega_{M(s)})$ with $0 \leq s < \epsilon'_0$ and we construct the bifurcating solution on $S^{N-1}$ using the classical tangency condition [26, Theorem 13.5],

$$\omega_{M(s)} = X_{M_0} + s(\psi_1 + \zeta_s) \tag{6.12}$$

where $\zeta_s \in H^1(S^{N-1})$, is orthogonal to $\psi_1$ in $H^1(S^{N-1})$ and satisfies $\|\zeta_s\|_{C^1} = o(1)$ when $s \to 0$. This implies the claim.

**Assertion 2.** Since $R(\Phi) = (-\infty, 0)$ for $M < 0$, we have to find $k \geq 1$ such that

$$\frac{p+1}{p(p-1)}(2K - \lambda_k) < 0 \iff 2K < \lambda_k.$$ 

As in Case 1, $K < 2N$, then inequality $2K \leq \lambda_k$ holds for all $k \geq 2$, and if $k = 1$ this is possible only if $p < \frac{N+1}{N-3}$. The construction of the bifurcating curve is the same as in Case 1.

**Assertion 3.** We have $R(\Phi_1) = [\frac{2K}{p-1}, 0)$ for $M \leq -\mu^*$. If we look for the existence of some $k \geq 1$ such that

$$\frac{2K}{p-1} \leq \frac{p+1}{p(p-1)}(2K - \lambda_k) < 0 \iff 2K < \lambda_k < \frac{2K}{p+1};$$

we get an impossibility since $K < 0$. Hence there exists no $M_0 < 0$ such that $(M_0, X_{1,M_0})$ is a bifurcation point. We have also $R(\Phi_2) = (-\infty, \frac{2K}{p-1}]$ for $M \leq -\mu^*$. 

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Now the condition for the existence of a bifurcation branch issued from \((M_0, X_2, M_0)\) for some \(M_0 \leq -\mu^*\) is
\[
\frac{p+1}{p(p-1)} (2K - \lambda_k) \leq \frac{2K}{p-1} \iff \lambda_k \geq \frac{2K}{p+1},
\]
which is always true for any \(k \geq 1\) and \(1 < p < \frac{N}{N-2}\). \(\Box\)

**Remark** The exponent \(p = \frac{N+1}{N-3}\) is the Sobolev critical exponent on \(S^{N-1}\). If one consider the Lane–Emden equation with a Leray potential
\[
-\Delta u + \lambda |x|^{-2}u = u^{\frac{N+1}{N-3}},
\]
with \(\lambda \in \mathbb{R}\), then the separable solutions \(u(r, \sigma) = r^{-\frac{N-3}{2}}\omega(\sigma)\) verify
\[
-\Delta' \omega + \left(\frac{(N-1)(N-3)}{4} - \lambda\right) \omega - \omega^{\frac{N+1}{N-3}} = 0 \text{ on } S^{N-1}.
\]

It was observed in [7] that there exists a branch of bifurcation \((\lambda, \omega_{\lambda})\) with \(\lambda > 0\) issued from \((0, \omega_0)\), where \(\omega_0\) is the constant explicit solution of (6.14).

**Remark** In Theorem 6.4-1 and the above remark, we conjectured that on the bifurcating curve there holds locally \(M(s) < M_0\), and that for any \(p \geq \frac{N+1}{N-3}\) there exists \(M_0 := M_0(p)\) such that for \(M > M_0\) all the positive solutions to (6.2) are constant, furthermore \(M_0\) is defined by (6.11). When \(p = \frac{N+1}{N-3}\), then \(M = 0\) and there exists infinitely many positive solutions to (6.2) [7, Proposition 5.1]. When \(\frac{N}{N-2} < p < \frac{N+1}{N-3}\), it is unclear if the branches of bifurcation \((M(s), \omega_{M(s)})\) issued from \((M_0, \omega_{M_0})\) with \(M_0 < 0\) are such that \(M(s)\) keeps a constant sign. If it is the case one could expect that if \(M \geq 0\) and \(\frac{N}{N-2} < p < \frac{N+1}{N-3}\), all the positive solutions to (6.2) are constant.

The following statement is an immediate consequence of Theorem 6.4.

**Corollary 6.5**
1. If \(p > 1\) and \(q = \frac{2p}{p+1}\) there always exist nonradial positive singular solutions of (1.2) in \(\mathbb{R}^N \setminus \{0\}\) under the form \(u(r, \sigma) = r^{-\frac{2}{p-1}}\omega(\sigma)\).
2. If \(N \geq 4\) and \(p > \frac{N+1}{N-3}\), the functions are obtained by bifurcation from \(X_M\) with \(M > 0\).
3. If \(N \geq 3\) and \(\frac{N}{N-2} \leq p < \frac{N+1}{N-3}\), the functions are obtained by bifurcation from \(X_M\) with \(M < 0\).
4. If \(N = 1, 2\) and \(p > 1\) or \(N \geq 3\) and \(1 < p < \frac{N}{N-2}\), the functions are obtained by bifurcation from \((M_k, X_2, M_k)\) with \(M_k < -\mu^*\) and \(k \geq 1\).

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