UNICITY OF TYPES AND LOCAL JACQUET–LANGLANDS CORRESPONDENCE

YUKI YAMAMOTO

Abstract. Let $F$ be a non-archimedean local field. For any irreducible representation $\pi$ of an inner form $G' = \text{GL}_m(D)$ of $G = \text{GL}_N(F)$, there exists an irreducible representation of a maximal compact open subgroup in $G'$ which is also a type for $\pi$. Then we can consider the problem whether these types are unique or not in some sense. If such types for $\pi$ are unique, we say $\pi$ has the strong unicity property of types. On the other hand, there exists a correspondence connecting irreducible representations of $G'$ and $G$, called the Jacquet–Langland correspondence. In this paper, we study the relation between the strong unicity of types and the Jacquet–Langlands correspondence.

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1. Introduction

Let $F$ be a non-archimedean local field. Let $m \in \mathbb{Z}_{>0}$, and let $D$ be a central division $F$-algebra. In the following, we consider smooth representations of locally profinite groups over the field of complex numbers.

To study representation theory of $G' = \text{GL}_m(D)$, the Bernstein decomposition is important. We recall it. We denote by $\mathcal{M}(G')$ the category of $G'$-representations. Let $\text{Irr}(G')$ be the set of isomorphism classes of irreducible representations of $G'$ over $\mathbb{C}$. Then there exists an equivalent relation $\sim$ on $\text{Irr}(G')$, called “the same inertial support.” We put $\mathcal{B}(G') := \text{Irr}(G')/\sim$, and we denote by $\mathfrak{s}$ the quotient map $\text{Irr}(G') \to \mathcal{B}(G')$. Then, for $\mathfrak{s}_0 \in \mathcal{B}(G')$, we can consider the full subcategory $\mathcal{M}(G')_{\mathfrak{s}_0}$ of $\mathcal{M}(G')$, whose objects are $G'$-representations $\pi$ such that any irreducible subquotient of $\pi$ is an element in $\mathfrak{s}_0$. In the setting, it is known that we have the decomposition of the category

$$\mathcal{M}(G') = \prod_{\mathfrak{s}_0 \in \mathcal{B}(G')} \mathcal{M}(G')_{\mathfrak{s}_0}.$$


called the Bernstein decomposition. Moreover, this decomposition is indecomposable. In this sense, the Bernstein decomposition is a concept to classify irreducible representations of \( G' \).

As a concept connecting with the Bernstein decomposition, there exists a class of representations of some compact open subgroups in \( G' \), called types. Let \( s_0 \in B(G') \). We say a pair \((J, \lambda)\) consisting a compact open subgroup \( J \) in \( G \) and an irreducible \( J \)-representation \( \lambda \) is an \( s_0 \)-type if

\[
\pi \in s_0 \text{ if and only if } \lambda \subset \pi|_J
\]

for any \( \pi \in \text{Irr}(G') \). Then, types are useful to examine whether an irreducible representation \( \pi \) is an element in \( s_0 \) or not by studying the restriction of \( \pi \) to some compact open subgroup in \( G' \).

For \( G' = \text{GL}_m(D) \), for any \( \pi \in \mathcal{A}_m(D) \) there exists an \( s(\pi) \)-type. For example, Sécherre [8, 9, 10, 11] constructed types for \( G' \), called simple types. It is a generalization of the construction by Bushnell–Kutzko [2] for \( G = \text{GL}_N(F) \) case.

Here, we can also consider another question whether for an irreducible representation \( \pi' \) of \( G' \), types for \( \pi \) are unique or not. This question is first considered in [4 Appendix A] to formulate the Breuil–Mézard conjecture for \( G = \text{GL}_2(\mathbb{Q}_p) \) case. For any irreducible representation \( \pi \) of \( \text{GL}_2(F) \), Henniart calculated the number of irreducible representations \( \rho \) of \( K = \text{GL}_2(\mathcal{O}_F) \) such that \((K, \rho)\) is also an \( s(\pi) \)-type, where \( \mathcal{O}_F \) is the ring of integers in \( F \).

To generalize this problem to more general reductive group case, we introduce archetypes, defined by Latham [5 Definition 4.3]. We denote by \( \text{MT}(s_0) \) the set of \( s_0 \)-types \((K, \rho)\) such that \( K \) is a maximal compact open subgroup in \( G' \). The group \( G \) acts \( \text{MT}(s_0) \) by conjugation. We call these orbits \( s_0 \)-archetypes.

When \( G' = \text{GL}_m(D) \) and \( \pi \) is irreducible and supercuspidal, Paškūnas showed in [7] that there exists a unique \( s(\pi) \)-archetype. On the other hand, it is shown in [4 A.1.5(3)] that if \( F \) satisfies some assumption, there exists a non-supercuspidal representation \( \pi \in \mathcal{A}_2(F) \) such that \( s(\pi) \)-archetypes are not unique. Let \( G' = \text{GL}_m(D) \). If \( \pi \in \text{Irr}(G') \) is supercuspidal and ‘unramified’ in some sense, the author [12] showed that there exists a unique \( s(\pi) \)-archetype. On the other hand, the author also constructed a supercuspidal representation \( \pi \) of \( \text{GL}_m(D) \) for some \( m \) and \( D \) such that \( s(\pi) \)-archetypes are not unique.

**Definition 1.1 ([6 §1])**. Let \( \pi \) be an irreducible representation of \( G \). We say \( \pi \) has the strong unicity property of types (SUP) if there exists a unique \( s(\pi) \)-archetype.

Here, we denote by \( \mathcal{A}_m(D) \) the set of isomorphism classes of irreducible, essentially square-integrable representations of \( \text{GL}_m(D) \). We put \( N = m \cdot (\dim_F D)^{1/2} \) and \( G = \text{GL}_N(F) \). Then there exists a correspondence

\[
\text{JL}_{m,D} : \mathcal{A}_m(D) \xrightarrow{\sim} \mathcal{A}_N(F)
\]

connecting representations of \( G' \) and \( G \), called the Jacquet–Langlands correspondence. It is known that \( \text{JL}_{m,D} \) satisfies some ‘nice’ properties. We also put \( \text{JL}_{m,D}^{-1} := (\text{JL}_{m,D})^{-1} \).

Then we have a natural question: “Does the Jacquet–Langlands correspondence preserve the strong unicity of types?” The goal of this paper is to study the behavior of the strong unicity property of types by the Jacquet–Langlands correspondence.

Our main theorem is the following.
Theorem 1.2 (Theorem [3,3]).

1. For some $F$, $m$ and $D$, the map $JL_{m,D}$ does not preserve the strong unicity property of types.

2. For some $F$, $m'$ and $D'$, the map $LJ_{m',D'}$ does not preserve the strong unicity property of types.

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2. Preliminaries

2.1. Notation. For a division $F$-algebra $D$, we denote by $\mathfrak{o}_D$ the ring of integers in $D$.

Let $G$ be a locally profinite group, and let $H$ be a closed subgroup in $G$. For $g \in G$, we put $^gH = gHg^{-1}$. For a representation $\rho$ of $H$, we define a $^gH$-representation $^g\rho$ by $^g\rho(ghg^{-1}) = \rho(h)$ for $h \in H$.

2.2. Simple type. We recall simple types for inner forms of $GL_N$, which are types for essentially square-integrable representations, from [8], [9], [10], [11].

Simple types are constructed from information of a 4-tuple $[\mathfrak{A}, n, 0, \beta]$, called a simple stratum. Simple strata $[\mathfrak{A}, n, 0, \beta]$ are defined by compact open subgroups $J = J(\beta, \mathfrak{A})$, $J^1 = J^1(\beta, \mathfrak{A})$ and $H^1 = H^1(\beta, \mathfrak{A})$ in $G' = GL_m(D)$. To construct simple types, we need two constructions of irreducible $J$-representations:

1. We can define a finite set $C(\beta, \mathfrak{A})$ of ‘simple characters’ from $[\mathfrak{A}, n, 0, \beta]$. When we take $\eta \in C(\beta, \mathfrak{A})$, we can define a unique irreducible $J^1$-representation $\eta_\theta$ up to isomorphism. Moreover, we can take a ‘nice’ extension $\kappa$ of $\eta_\theta$ to $J$, called a $\beta$-extension.

2. Suppose $J/J^1$ is isomorphic to $GL_d(f)^{\times j}$, where $f$ is a finite field. When $\eta$ is an irreducible cuspidal representation of $GL_d(f)$, we can take the inflation $\sigma$ of $\eta^{\otimes j}$ to $J$.

When we take $\kappa$ and $\sigma$ as above, we put $\lambda = \kappa \otimes \sigma$, and then $(J, \lambda)$ is a simple type. If $(J, \lambda)$ is a type for some supercuspidal representation, we say $(J, \lambda)$ is a maximal simple type.

From two distinct simple strata $[\mathfrak{A}, n, 0, \beta]$ and $[\mathfrak{A}', n', 0, \beta']$, we may construct the same simple type. In this case, it is shown in [11] Theorem 9.4 that $[F[\beta] : F] = [F[\beta'] : F]$.

2.3. Parametric degree. For $\pi \in \mathcal{A}_m(D)$, we can define a positive integer $\delta(\pi)$ as in [3] §2, called the parametric degree of $\pi$.

We recall the definition of $\delta(\pi)$ for some supercuspidal representation $\pi$. Let $(J, \lambda)$ be a maximal simple type which is also an $s(\pi)$-type. Suppose $(J, \lambda)$ is associated with a simple stratum $[\mathfrak{A}, n, 0, \beta]$. Then there exist two irreducible representations $\kappa$ and $\sigma$ of $J$ such that

- $\kappa$ is constructed from a ‘simple character’ associated with $[\mathfrak{A}, n, 0, \beta]$,
• \( \sigma \) is the inflation of some irreducible representation of a product of general linear groups over some finite field, and

• \( \lambda = \kappa \otimes \sigma \).

In this case, we can define a positive integer \( \delta_0(\sigma) \) as in \[3, 2.4, 2.6\] and we put \( \delta_0(\lambda) := \delta_0(\sigma) [F[\beta] : F] \).

**Proposition 2.1** ([3 Proposition 2.7]). (1) The integer \( \delta_0(\lambda) \) is independent of the choice of simple strata and representations \( \sigma \) and \( \kappa \) such that \( \lambda \equiv \kappa \otimes \sigma \).

(2) If \((J, \lambda)\) and \((J', \lambda')\) are simple types which are also \( s(\pi) \)-types for some irreducible representation \( \pi \), then \( \delta_0(\lambda) = \delta_0(\lambda') \).

Then, for \( \pi' \in \mathcal{A}_m(D) \) which is supercuspidal, we can define a positive integer \( \delta(\pi') \) by \( \delta(\pi') = \delta_0(\lambda) \) for a maximal simple type \((J, \lambda)\) which is also an \( s(\pi) \)-type.

We can generalize the parametric degree \( \delta(\pi) \) to any \( \pi \in \mathcal{A}_m(D) \). The integer \( \delta(\pi) \) satisfies the following properties.

**Proposition 2.2** ([3 2.7.3, 2.8 Corollary 1]). (1) For any \( \pi \in \mathcal{A}_m(D) \), the integer \( \delta(\pi) \) divides \( N = m \cdot (\dim_F D)^{1/2} \).

(2) For \( \pi \in \mathcal{A}_N(F) \), the representation \( \pi \) is supercuspidal if and only if \( \delta(\pi) = N \).

(3) For any \( m \) and \( D \), the map \( \mathcal{J}_{m,D} \) preserves the parametric degree, that is, we have \( \delta(\mathcal{J}_{m,D}(\pi')) = \delta(\pi') \) for any \( \pi' \in \mathcal{A}_m(D) \).

## 3. Main Examples

**Example 3.1.** We consider \( N = 2 \) and \( G' = D^x \), where \( D \) is a quaternion algebra of \( F \). Suppose the order of the residual field of \( F \) is 2. Then, by [11 A 1.5(3)] there exists an essentially square-integrable representation \( \pi \) of \( G = \text{GL}_2(F) \) and irreducible representations \( \rho_1 \) and \( \rho_2 \) of \( K = \text{GL}_2(\mathfrak{o}_F) \) such that \( \rho_1 \not\cong \rho_2 \) and \( (K, \rho_i) \) is an \( s(\pi) \)-type for \( i = 1, 2 \).

To show \( \pi \) does not have SUP, suppose \( \rho_1 \) and \( \rho_2 \) are \( G \)-conjugate. Then there exists \( g \in G \) such that \( K = gKg^{-1} \) and \( \rho_2 \cong \delta \rho_1 \). Since the normalizer of \( K \) in \( F^xK \), there exist \( t \in F^x \) and \( k \in K \) such that \( g = tk \). Here, we have \( k^i \rho_1 \cong \rho_1 \) since \( \rho_1 \) is a \( K \)-representation and \( \rho_1 \) is in \( K \). Moreover, we also have \( t^i \rho_1 \cong \rho_1 \) since \( t \) is an element in the center of \( G \). Therefore, \( \rho_2 \cong \delta \rho_1 = tk^i \rho_1 \cong \rho_1 \), which is a contradiction.

On the other hand, we put \( \pi' := \mathcal{J}_{1,D}(\pi) \). Since \( \pi' \) is an irreducible representation of \( D^x \), which is the multiplicative group of a central division \( F \)-algebra, the representation \( \pi' \) has SUP. That is, although \( \pi' \) has SUP, the image \( \pi = \mathcal{J}_{1,D}(\pi') \) does not have SUP. Therefore, \( \mathcal{J}_{1,D} \) does not preserve the strong unicity of types.

**Example 3.2.** We consider \( N = 4 \) and \( G' = \text{GL}_2(D) \), where \( D \) is a quaternion algebra of \( F \). Let \( E_1/F \) be a ramified quadratic extension of fields, and let \( E/E_1 \) be an unramified quadratic extension of fields. We consider an embedding \( E \hookrightarrow M_2(D) \) as in [12 Example 7.1]. We can take an element \( \beta \in E \) such that \( E = F[\beta] \) and a 4-tuple \( [\mathfrak{A}, n, 0, \beta] \) is a simple stratum, where \( \mathfrak{A} = M_2(\mathfrak{o}_D) \) and \( n \in \mathbb{Z}_{>0} \) is defined from \( \mathfrak{A} \) and \( \beta \). Let \((J, \lambda)\) be a maximal simple type associated with the simple stratum \([\mathfrak{A}, n, 0, \beta] \), and let \((\tilde{J}, \tilde{\lambda})\) be a maximal extension of \((J, \lambda)\) in \( G \). We put \( \pi' := c \cdot \text{Ind}_{\tilde{J}}^{\mathcal{J}_{1,D}} \tilde{\lambda} \). Then \( \pi' \) is irreducible and supercuspidal, and \( \pi' \) does not have SUP by [12 §7].
We put $\pi := \text{LJ}_2(D(\pi'))$. We will show $\pi$ has SUP. Since any irreducible supercuspidal representation of $\text{GL}_N(F)$ has SUP by \cite{7}, it is enough to show $\pi$ is supercuspidal. Then we consider the parametric degree $\delta(\pi)$ of $\pi$. To show $\pi$ is supercuspidal, it suffices to show $\delta(\pi) = N = 4$ by Proposition \ref{prop:supercuspidal-parametric}. Since $\delta(\pi) = \delta(\text{LJ}(\pi')) = \delta(\pi')$ by Proposition \ref{prop:supercuspidal-parametric}, it is enough to show $\delta(\pi') = 4$. Here, we have $[F[\beta] : F] = [E : F] = 4$. Since $\delta(\pi') = \delta_0(\lambda)$ is a multiple of $[F[\beta] : F]$ by the definition of $\delta_0(\lambda)$, we have $\delta(\pi') \in 4\mathbb{Z}$. On the other hand, the integer $\delta(\pi')$ divides 4 by Proposition \ref{prop:supercuspidal-parametric}. Then $\delta(\pi') = 4$ and $\pi$ is supercuspidal. That is, although $\pi$ has SUP, the image $\pi' = \text{LJ}_2(D(\pi))$ does not have SUP. Therefore, $\text{LJ}_2(D)$ does not preserve the strong unicity of types.

As a consequence of these examples, we obtain the following theorem.

**Theorem 3.3.**

1. Suppose the order of the residual field of $F$ is 2. Let $D$ be a quaternion algebra of $F$. Then $\text{LJ}_1(D)$ does not preserve the strong unicity property of types, that is, there exists $\pi' \in A_1(D)$ such that $\pi'$ has SUP and $\text{LJ}_1(D)(\pi') \in A_2(F)$ does not have SUP.

2. Let $D$ be a quaternion algebra of $F$. Then $\text{LJ}_2(D)$ does not preserve the strong unicity property of types, that is, there exists $\pi \in A_4(F)$ such that $\pi$ has SUP and $\text{LJ}_2(D)(\pi) \in A_2(D)$ does not have SUP.

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