Quark Cluster Model Equations for $\beta\beta$ Decay

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Abstract

In a separate paper we have discussed the possibility that six quark clusters can affect the rate of double-beta decay. In this paper we present the notation and some of the formulae that are needed to carry through such calculations.

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1 Introduction

Multi-quark systems are Fermions containing more than three quarks and mesons containing more pairs than one quark one anti-quark. Such multi-quarks were later discovered experimentally, the penta-quark [1, 2] and the tetra-quark [3, 4]. On the latter see also the recent reviews [5],[6],[7],[8].

Since pentaquark and tetraquark systems, have already been found to exist one wonders why not multi quark configurations in he nucleus? So we will examine the possibility of the presence of six quark clusters in the nucleus, a much more complex problem. Such clusters, if present with a reasonable probability in the nucleus, may contribute to various processes, like neutrinoless double beta decay mediated by heavy neutrinos or other exotic particles. In conventional nuclear physics the relevant nuclear matrix elements are suppressed due to the presence of the nuclear hard core. In this presence of such clusters, however, the interacting quarks are in the same hadron. So one can have a contribution even in the case of of a $\delta$-function interaction [9]. Symmetries, of course, play a crucial role in reliably estimating the probability of finding such six quark clusters in the nucleus.

In this article we develop the formalism needed in the evaluation of the energy of all six-quark cluster configurations, which can arise in a harmonic oscillator basis up to $2\hbar\omega$ excitations. The symmetries that were found useful for this purpose were the combined spin color symmetry $SU_{cs}(6)$, the orbital symmetry $SU_{o}(6)$ and the isospin symmetry $SU_{I}(2)$.
2 Basis States

All states are products of four components: space, color, spin, and isospin; the color and spin will be combined in color-spin wave function described by $SU(6)_{cs}$. We assume the total color of the six quarks must be a color singlet. (Note that this is a restriction: the only requirement is that the entire nucleus be a color singlet which leaves open the possibility that the six quarks and the remaining $A - 2$ nuclear system could be non-singlet states while coupling to a color singlet state. However, shell model codes that would be used to described the $A - 2$ system currently assume the nucleons are in color singlet states so we make this assumption here.) The isospin will be described by $SU(2)$ and the spatial wave functions – being three-dimensional harmonic oscillator functions – will be labelled by $S(N)$, $[f]_r$, and $SU(3)_{r}$, $(\lambda \mu)_{r}$. The total wave function is necessarily antisymmetric. This imposes restrictions on the possible $SU(6)_{cs}$ and $SU(2)_{I}$ representations.

The $SU(3)$ representations, be they flavour, color, or spatial, are labelled by $(\lambda \mu)$. The dimension of a $(\lambda \mu)$ representation is $\frac{1}{2}(\lambda + 1)(\mu + 1)(\lambda + \mu + 2)$. The labels of the $SU(2)$ and $SU(1)$ subgroups are:

$$Y = p + q - \frac{1}{3}(2\lambda + \mu) \quad 0 \leq p \leq \lambda$$

$$I = \frac{1}{2}(\mu + p - q) \quad 0 \leq q \leq \mu$$

$$I_3 = r - I \quad 0 \leq r \leq \mu + p - q$$

Alternatively, the number of each of the three types of quarks for color/flavor/spatial $SU(3)$ is:

$$n_1/n_u/n_s = \frac{1}{3}(N - \lambda) + r - \frac{2}{3}\mu + q$$

$$n_2/n_d/n_x = \frac{1}{3}(N - \lambda) - r + \frac{1}{3}\mu + p$$

$$n_3/n_s/n_y = \frac{1}{3}(N + 2\lambda + \mu) - p - q$$

$$n_u + n_d = \frac{1}{3}(2N - 2\lambda - \mu) + p + q$$

where $N$ is the total number of quarks. Thus, for the $(0 0)$ representation, one has an equal number of each type of quark, as one should.

Below we will construct wavefunctions using $SU(2)_I$ rather than $SU(3)_f$. We now demonstrate that we can do this with no loss of generality if we assume there are no strange quarks in the wave function. From Eq. 2 above we see that the requirement for zero $s$ quarks is that

$$n_u + n_d = \frac{1}{3}(2N - 2\lambda - \mu) + p + q = N.$$
Table 1: The six-quark $SU(6)_{cs}$ representations that have color singlet states, $(00)_{c}$, the allowed spin and their concomitant isospin; these can be found from Ref. [12]. $SU(6)_{cs}$ representations with more than two columns are not allowed.

| $[f]_{cs}$ | S | I |
|-------|----|----|
| $[42]_{cs}$ | 1 | NA |
| $[411]_{cs}$ | 0 | NA |
| $[33]_{cs}$ | 0 | NA |
| $[321]_{cs}$ | 1, 2 | NA |
| $[3111]_{cs}$ | 1 | NA |
| $[222]_{cs}$ | 1, 3 | 0 |
| $[2211]_{cs}$ | 0, 2 | 1 |
| $[21^{4}]_{cs}$ | 1 | 2 |
| $[1^{6}]_{cs}$ | 0 | 3 |

Taking $p$ and $q$ to be their maximum values, we have

$$N = n_u + n_d = \frac{1}{3}(2N + \lambda + 2\mu)$$

or $N = \lambda + 2\mu$. Since the total number of quarks in a $SU(3)_f$ representation $[f_1f_2f_3]$ (which is contragradeint to $SU(6)_{cs}$) is

$$N_{tot} = f_1 + f_2 + f_3 = \lambda + 2\mu + 3f_3,$$

we have the requirement that $f_3 = 0$. Thus, we can label the representations by either $SU(3)_f$ or $SU(2)_I$.

2.1 0hω Excitations

The $SU(6)_{cs}$ representations that contain a color singlet and their spin content are shown in Table 1. All six quarks are assumed to be in the 0s state. The wave function is then

$$| (0s)^6[6]_r \ [f]_{cs}(00)_{c} \ S \ I = 1 > . \quad (2)$$

Since the spatial wave function is totally symmetric, the product of the color-spin and isospin wave functions must be antisymmetric. For $I = 1$, the isospin representation is necessarily$[42]_I$. The conjugate representation is $[2211]_{cs}$; from Table 1 one sees the spin can be either 0 or 2. The former is diproton-like but in the nucleus, one cannot exclude the S=2 possibility.

Note that the six quarks can also have isospin two or three as well as one or two. One may think of these states as having $\Delta$ admixtures. However, even if one decomposes the $I = 0$ six-quark wave function into two hadrons, there will also be $\Delta$ admixtures.

2.2 One Particle, 2hω Excitations

The wave function for positive parity is

$$| (0s)^5[5]_r \ [f]_{cs}(01)_{c} \ L_1 = 0 \ S_1 \ I_1 \times (1s0d)_r[1]_{cs}(10)_{c} \ S_2 = \frac{1}{2} \ I_2 = \frac{1}{2} ; \ (00)_{c} \ L_2 S J I = 1 > . \quad (3)$$
Two-Particle, $L, S, J, I$

A variety of total quark can be either 0 or 2. The possible states are listed in Table 3. These states can combine to form again, since the 0 states are not allowed because the contragradient representation must belong to $SU(2)_r$.

Table 2: $SU(6)_{cs}$ representations for five quarks that contain $(01)_c$, their spin, and isospin; these can be found from Ref. [12]. Note that those $SU(6)_{cs}$ representations that have more than two columns are not allowed because the contragradient representation must belong to $SU(2)_r$.

| $[f_{cs}]$ | $(\lambda\mu)_c$ | $S$ | $I$ |
|------------|-----------------|-----|-----|
| $[41]_{cs}$ | $(01)_c$ | $\frac{1}{2}$ | NA |
| $[32]_{cs}$ | $(01)_c$ | $\frac{1}{2}, \frac{3}{2}$ | NA |
| $[311]_{cs}$ | $(01)_c$ | $\frac{1}{2}, \frac{3}{2}$ | NA |
| $[221]_{cs}$ | $(01)_c$ | $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$ | $\frac{1}{2}$ |
| $[2111]_{cs}$ | $(01)_c$ | $\frac{1}{2}, \frac{3}{2}$ | $\frac{3}{2}$ |
| $[1^5]_{cs}$ | $(01)_c$ | $\frac{1}{2}$ | $\frac{5}{2}$ |

Table 3: Basis states for $0s^5(1s0d)$

Since the 0s quarks and the (1s0d) quarks are distinguishable, one need not antisymmetrize the entire wave function. The five-quark wave function then can have either $I_1 = \frac{1}{2}$ or $I_1 = \frac{3}{2}$ if the total isospin of the six quarks is 1. The five-quark states that have a $SU(3)_c$ representation of $(01)_c$ and their spin and isospin content are listed in Table 2.

If the isospin of the five quarks, $I_1$, is $\frac{1}{2}$, the SU(2) representation is $[32]_I$; the conjugate $SU(6)_{cs}$ is then $[221]_{cs}$ which allows $S_1 = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$. If the isospin, $I_1$, is $\frac{3}{2}$, the SU(2) representation is $[41]_I$; the conjugate $SU(6)_{cs}$ is then $[2111]_{cs}$ which allows $S_1 = \frac{1}{2}, \frac{3}{2}$. The angular momentum, $L_2$, of the (1s0d) quark can be either 0 or 2. The possible states are listed in Table 3. These states can combine to form a variety of total $L, S, J, I$.

2.3 Two-Particle, $2\hbar \omega$ Excitations

The wave function for positive parity states is

$$|(0s)^4[4]_r [f]_{cs} (\lambda_c \mu_c)_c L_1 = 0 \ S_1 \ I_1 \times (0p)^2_\gamma [f_\gamma] (\lambda_\gamma \mu_\gamma) \ [f_2]_{cs} (\mu_c \lambda_c)_c S_2 \ I_2; (00)_c \ L \ S \ J \ I = 1 >. \ (4)$$

Again, since the 0s quarks and the 0p quarks are distinguishable, one need not antisymmetrize the entire wave function. The 0p² wave function can have a $SU(6)_{cs}$ representation of either $[2]_{cs}$ or $[11]_{cs}$.
\begin{align*}
\begin{array}{|c|c|c|}
\hline
f_{cs} & (\lambda\mu)_c S & I \\
\hline
[22]_{cs} & (0\ 2)\ 0 & 0 \\
& (0\ 2)\ 2 & 0 \\
& (1\ 0)\ 1 & 0 \\
\hline
[211]_{cs} & (0\ 2)\ 1 & 1 \\
& (1\ 0)\ 0 & 1 \\
& (1\ 0)\ 1 & 1 \\
& (1\ 0)\ 2 & 1 \\
\hline
[11]_{cs} & (0\ 2)\ 0 & 2 \\
& (1\ 0)\ 1 & 2 \\
\hline
\end{array}
\end{align*}

Table 4: $SU(6)_{cs}$ representations for four quarks that contain $(02)_c$ or $(10)_c$, and the concomitant spin and isospin.

2.4 $0p^2$ States

We have an additional feature for the $0p^2$ wave functions, namely the spatial symmetry $[f_r]$ may be either $[2]$ or $[11]$. We must have

$$\text{space} \times SU(6)_{cs} \times \text{isospin} >$$

(5)
totally antisymmetric. For two quarks this is easily handled - if we had three of more quarks, one would need resort to more sophisticated results from the symmetric group. Using obvious notation one has the four possible states: $|SSA>$, $|SAS>$, $|ASS>$, and $|AAA>$. The spatial wave functions are $[2]_r(20)$ and $[11]_r(01)$ - since in our cases, the $SU(3)_r$ representations are determined by $[f]_r$, either the $[f]_r$ or $SU(3)_r$ label is redundant. The wave functions belonging to $[2]_r$ are spatially symmetric and $[11]_r$ antisymmetric. Note that with spatial symmetry included and which can be either symmetric or anti-symmetric, $[f]_{cs}$ no longer determines $[f]_f$ uniquely.

The $SU(6)$ color-spin representations are for $[2]_{cs}$ $(20)_c S = 1$ and $(01)_c S = 0$ (i.e., $|SS>$ or $|AA>$) and for $[11]_{cs}$, $(20)_c S = 0$ and $(01)_c S = 1$. If $(\lambda_c\mu_c)_c$ is the $SU(3)_c$ representation, then $(-)^{\mu_c+S}$ is -1 for the symmetric representations and 1 for the antisymmetric representations.

One is familiar with the rule for two nucleons that $L + S + I$ must be odd. We can generalize the rule to

$$(-)^{L+S+\mu_c+I}$$
must be even.

The possible two-quark states are thus $|[2]_r[2]_{cs}(01)_f I = 0>$, $|[2]_r[11]_{cs}(20)_f I = 1>$, $|[11]_r[2]_{cs}(01)_f I = 1>$, and $|[11]_r[11]_{cs}(20)_f I = 0>$. 

2.5 $0s^4$ States

Again, the spatial part is symmetric so we need deal only with color-spin and isospin. The color $SU(3)$ representation will be the contragradient of the color representation of the $(0p)^2$ particles and will be either $(02)_c$ or $(10)_c$. The possible $SU(2)$ isospin representations are $[22]I = 0$, $[31]I = 1$, and $[4]I = 2$. The corresponding $SU(6)_{cs}$ representations are $[22]_{cs}$, $[211]_{cs}$, and $[1111]_{cs}$, respectively. The color-spin content is shown in Table 4.
Table 5: Two-Quark States, Even Parity

| $q^N$ | $(\lambda\mu)_c$ | I | L |
|-------|-----------------|---|---|
| 1     | $0s^2$          | (2) | 0 | 0 |
| 2     | $0s^2$          | (0) | 0 | 0 |
| 3     | $0s^2$          | [11] | 1 | 0 |
| 4     | $0s^2$          | [11] | 1 | 0 |
| 5     | $0s1s$          | (2) | 0 | 0 |
| 6     | $0s0d$          | (2) | 0 | 0 |
| 7     | $0s1s$          | (0) | 0 | 0 |
| 8     | $0s0d$          | (0) | 0 | 0 |
| 9     | $0s1s$          | (2) | 0 | 0 |
| 10    | $0s0d$          | (2) | 0 | 0 |
| 11    | $0s1s$          | (0) | 0 | 0 |
| 12    | $0s0d$          | (0) | 0 | 0 |
| 13    | $0p^2$          | (0) | 0 | 0 |
| 14    | $0p^2$          | (2) | 0 | 0 |
| 15    | $0p^2$          | (0) | 0 | 0 |
| 16    | $0p^2$          | (2) | 0 | 0 |
| 17    | $0p^2$          | (0) | 0 | 0 |
| 18    | $0p^2$          | (2) | 0 | 0 |
| 19    | $0p^2$          | (0) | 0 | 0 |
| 20    | $0p^2$          | (2) | 0 | 0 |
| 21    | $0p^2$          | (0) | 0 | 0 |
| 22    | $0p^2$          | (2) | 0 | 0 |
| 23    | $0p^2$          | (0) | 0 | 0 |
| 24    | $0p^2$          | (2) | 0 | 0 |

Table 6: Two-Quark States, Odd Parity

| $q^N$ | $(\lambda\mu)_c$ | I | L | J |
|-------|-----------------|---|---|---|
| 1     | $0s0p$          | (2) | 0 | 1 |
| 2     | $0s0p$          | (0) | 0 | 1 |
| 3     | $0s0p$          | (2) | 0 | 1 |
| 4     | $0s0p$          | (0) | 0 | 1 |

3 Two-Quark Matrix Elements

3.1 Enumeration of Positive Parity Two-Quark States

For the $0\hbar\omega$ one has the states listed in Table 5.

3.2 Enumeration of Negative Parity Two-Quark States

For the $0\hbar\omega$ one has the states listed in Table 6.

4 Matrix Elements of the Two-Quark Interaction

Since the quark-quark interaction is two-body, we must separate off two quarks from the remaining four. We will use the same techniques as in the nuclear shell model[15] and in the MIT bag paper[13].
4.1 $<0s^6 \ | V \ | 0s^6>$

This is the simplest case. Two-body matrix elements for $0s^5$ and $0s^4$ will be similar except for the labels. Recall that $S$ in the basis state of Eq. (2) can be either 0 or 2. We take first the $J = S = 0$ case.

For the six-quark color-spin symmetry [2211]$_{cs}$ one may have [22]$_{cs} \times [11]_{cs}$, [211]$_{cs} \times [11]_{cs}$, [1111]$_{cs} \times [11]_{cs}$, and [211]$_{cs} \times [2]_{cs}$.

$$\langle (0s^6 \ [6]_r \ (00)_r \ [2211]_{cs} (00)_c \ S = 0 \ I = 1 \ J = 0 \rangle =$$

$$\sum_{f''(\lambda'',\mu''), I''(f_r), I_2} \frac{n f''}{n_{[2211]}} \langle \lambda'' \mu'' | c_2 S'' | \lambda_2 \mu_2 | S_2 \rangle \langle [2211]_{cs} (00)_c S \rangle \langle | [4]_r | [2]_r | [6]_r \rangle \langle [4]_r | (00)_r | (00)_r \rangle \langle [2]_r | (00)_r | (00)_r \rangle \langle [6]_r | (00)_r | (00)_r \rangle$$

$$\times \left[ | (0s^4 \ [4]_r \ (00)_r \ [f'']_c S'' I'' \rangle \times \left[ | (0s^2 \ [2]_r \ (00)_r \ [f_2]_c S'' I_2 \rangle \right] \right]^{(L,S,I,J)}.$$

The first factor is the ratio of the dimensions of the representation $[f'']$ and [2211] of the symmetric group. The next three factors on the right-hand side of Eq. (7) are the Clebsch-Gordan coefficients for the groups $SU(6)_{cs}$, $SU(3)_r$, and $SU(2)_I$, respectively. We also use the notation that $[ ]^{L,S,I,J}$ denotes the angular-momentum coupling of the relevant quantites to $L, S, I, J$. In this simple case, all orbital angular momentum is zero, so $L = 0$ and $S = J$. Also, the $\tilde{f}$ indicates the representation $[\tilde{f}]$ is contragradient to that of $[f]$.

The first of these factors can be obtained using the results of Ref. [13]; the relevant coefficients are shown in Table ???. Since all quarks are in a 0s state, the second factor, $\langle | [4]_r | (00)_r | (00)_r \rangle \langle [2]_r | (00)_r | (00)_r \rangle$, is necessarily of unit magnitude. We can without loss of generality assume it to be one. Finally, the coefficient $\langle | [\tilde{f}'']_r I'' \tilde{f}_2 I_2 | [42]_I \rangle$ describes how the $SU(2)_I$ representation [42]$_I$ with angular momentum $I$ can be decoupled into representations $[\tilde{f}'']_r$ and $[\tilde{f}_2]_r$. Since there is a one-to-one correspondence in $SU(2)$ between $[f]$ and the angular momentum $I$, these coefficients are all one; any other isospin dependence is in the Clebsch-Gordan coefficients summarized in $[\ldots]^I$.

With these simplifications we have

$$\langle (0s^6 \ [6]_r \ (00)_r \ [2211]_{cs} (00)_c \ S = 0 \ I = 1 \ J = 0 \rangle$$

$$= \sum_{f''(\lambda'',\mu''), I''(f_r), I_2} \frac{n f''}{n_{[2211]}} \langle \lambda'' \mu'' | c_2 S'' | \lambda_2 \mu_2 | S_2 \rangle \langle [2211]_{cs} (00)_c S \rangle$$
The allowed \([f'']_{cs}\) representations are from Table ??, [22], [211], and [1111]. The matrix elements of a two-body operator are then:

\[
\langle (0s)^4 [4]_r (00)_r [f'']_{cs}(\lambda'\nu'\mu'')_c S'' I'' \rangle \times \langle (0s)^2 [2]_r (00)_r [f_2]_{cs}(\lambda_2\mu_2)_c S_2 I_2 \rangle^{(L,S,I,J)}.
\]  

(8)

\[
\times \left[ (0s)^4 [4]_r (00)_r [f'']_{cs}(\lambda'\nu'\mu'')_c S'' I'' \right] \times \left[ (0s)^2 [2]_r (00)_r [f_2]_{cs}(\lambda_2\mu_2)_c S_2 I_2 \right]^{(L,S,I,J)}.
\]  

(8)
Table 9: Weight factors for two particles; ratios of the dimensions of representations of $S_n$.

\[
\begin{array}{|c|c|c|}
\hline
[22] & [2] \times [2] & \frac{1}{2} \\
[22] & [11] \times [11] & \frac{3}{2} \\
[211] & [2] \times [11] & \frac{3}{2} \\
[211] & [11] \times [2] & \frac{1}{2} \\
[211] & [11] \times [11] & \frac{3}{2} \\
[1111] & [11] \times [11] & 1 \\
[221] & [21] \times [2] & \frac{3}{2} \\
[221] & [21] \times [11] & \frac{3}{2} \\
[221] & [11] \times [11] & \frac{3}{2} \\
[211] & [22] \times [2] & \frac{3}{2} \\
[222] & [22] \times [11] & \frac{3}{2} \\
[221] & [211] \times [2] & \frac{3}{2} \\
[221] & [211] \times [11] & \frac{3}{2} \\
[211] & [2111] \times [11] & \frac{3}{2} \\
[221] & [2111] \times [11] & \frac{3}{2} \\
\hline
\end{array}
\]

Since all the orbital angular momentum is zero, the $9J$ symbol is one and $J'' = S''$, $J_2 = S_2$, and $S = J$. If the two-quark interaction does not change spin, $S'_2 = S_2$, then it follows that $(\lambda_2 \mu_2)_c = (\lambda'_2 \mu'_2)_c$. Hence,

\[
< (0s)^6 [2211]_{cs} (00)_c S = 0 \ I = 1 \ J = 0 \left| \sum_{i,j} V_{ij} \right| (0s)^6 [2211]_{cs} (00)_c S = 0 \ I = 1 \ J = 0 > = \frac{6 \cdot 5}{2 f''(\lambda''_{c} \mu''_{c}, I''_{f}, f_2)} \sum n_{f''} \left< \begin{array}{l}
[f'']_{cs} (\lambda''_{c} \mu''_{c}) S'' (\lambda_2 \mu_2)_c S_2 \\
[00]_c S
\end{array} \right> \left< \begin{array}{l}
[f'_2]_{cs} (\lambda'_2 \mu'_2)_c S'_2 \\
[00]_c S
\end{array} \right> \\
\]
These matrix elements will be used for $n = 4, 5, 6$ so we give the general formula here:

$$|(0s)^n [n]_r (00)_r [f]_{cs} (\lambda_c \mu_c)_c L_1 S_1 I_1 J_1 > =$$

$$\sum_{f''(\lambda_c'' \mu_c''), I''; [f_2], I_2} \sqrt{\frac{n_{f''}}{n_{fcs}}} \begin{vmatrix} [f'']_{cs} & [f_2]_{cs} & [2211]_{cs} \\ (\lambda_c'' \mu_c'')_c S'' & (\lambda_2 \mu_2)_c S_2 & (00)_c S_1 \\ \end{vmatrix}$$

$$\times \begin{vmatrix} (0s)^{n-2} [f_1]_{cs} (\lambda'' \mu'')_c S'' I'' \\ \sqrt{n_{fcs}} \sum_{L'S'} q^{N} [f']_{cs} (\lambda' \mu')_c S' J' \end{vmatrix} \times (L_1, S_1, I_1, J_1)$$

Using 2B CFPs for an n-quark system:

$$< q^N [f]_{cs} (\lambda \mu)_c L S J > = \frac{1}{2} N (N-1) \sum_{f''} \sqrt{n_{f''}} n_{f''}$$
\[
\times \sum_{f_{12},(\lambda'',\mu'')} \left[ \begin{array}{c} f'' \cr (\lambda'',\mu'') \end{array} \right]_{cs} \left[ \begin{array}{c} f_{12} \cr (\lambda_1 \mu_1) \end{array} \right]_{cs} S'' \left. \begin{array}{c} f \cr (\lambda \mu) \end{array} \right]_{cs} \left. \begin{array}{c} f' \cr (\lambda' \mu') \end{array} \right]_{cs} \left. \begin{array}{c} f'_{12} \cr (\lambda'_1 \mu'_1) \end{array} \right]_{cs} S'_{12} \right| \left. \begin{array}{c} f'' \cr (\lambda'',\mu'') \end{array} \right]_{cs} \left. \begin{array}{c} f_{12} \cr (\lambda_1 \mu_1) \end{array} \right]_{cs} S'' \left. \begin{array}{c} f \cr (\lambda \mu) \end{array} \right]_{cs} \left. \begin{array}{c} f' \cr (\lambda' \mu') \end{array} \right]_{cs} \left. \begin{array}{c} f'_{12} \cr (\lambda'_1 \mu'_1) \end{array} \right]_{cs} S'_{12} \right>
\times U \left( \begin{array}{ccc} L'' & L_{12} & L \\
S'' & S_{12} & S \\
J'' & J_{12} & J
\end{array} \right) \left. \begin{array}{ccc} L'' & L'_{12} & L' \\
S'' & S'_{12} & S' \\
J'' & J_{12'} & J
\end{array} \right) \left| q^{N-2} \left[ f'' \right]_{cs} \left( \lambda'',\mu'' \right)_{cs} L'' S'' J'' x q^2 \left[ f_{12} \right]_{cs} \left( \lambda_1 \mu_1 \right)_{cs} L_{12} S_{12}, J_{12} \right| V_{12} \right|
\left| q^{N-2} \left[ f'' \right]_{cs} \left( \lambda'',\mu'' \right)_{cs} L'' S'' J'' x q^2 \left[ f'_{12} \right]_{cs} \left( \lambda'_1 \mu'_1 \right)_{cs} L'_{12} S'_{12} J_{12} \right>
\]

The \( q^{N-2} \) part integrates to one and the remaining two-body matrix element is

\[
\langle q^2 \left[ f_{12} \right]_{cs} \left( \lambda_1 \mu_1 \right)_{cs} L_{12} S_{12} J_{12} \left| V_{12} \right| q^2 \left[ f'_{12} \right]_{cs} \left( \lambda'_1 \mu'_1 \right)_{cs} L'_{12} S'_{12} J_{12} \rangle \]

\[= (-)^{J_{12}-k} W(L_{12} L'_{12} S_{12} S'_{12}, k) \left\langle L_{12} \right| \left| V_{12}^{(k)} \right| L'_{12} \right\rangle \left\langle S_{12} \right| \left| V_{12}^{(k)} \right| S'_{12} \right\rangle \]

But all the orbital angular momenta for the 0s quarks are zero so

\[
U \left( \begin{array}{ccc} L'' & L_{12} & L \\
S'' & S_{12} & S \\
J'' & J_{12} & J
\end{array} \right) = U \left( \begin{array}{ccc} 0 & 0 & 0 \\
S'' & S_{12} & S \\
J'' & J_{12} & J
\end{array} \right) = 1,
\]

and the matrix element for \( q^N \) becomes

\[
< q^N \left[ f \right]_{cs}(\lambda \mu)_{cs} L S J \left| \sum_{i \leq j} V_{i,j} \right| q^N \left[ f' \right]_{cs}(\lambda' \mu')_{cs} L' S' J >= \frac{1}{2} N(N-1) \delta_{LL'} \delta_{SS'} \delta_{L0} \delta_{S0} \sum_{f_{cs}} \frac{n_{f_{cs}}}{\sqrt{n_{f_{cs}} n_{f_{cs}}}} \times
\]

\[
\times \sum_{f_{12},(\lambda'',\mu'')} \left[ \begin{array}{c} f'' \cr (\lambda'',\mu'') \end{array} \right]_{cs} \left[ \begin{array}{c} f_{12} \cr (\lambda_1 \mu_1) \end{array} \right]_{cs} S'' \left. \begin{array}{c} f \cr (\lambda \mu) \end{array} \right]_{cs} \left. \begin{array}{c} f' \cr (\lambda' \mu') \end{array} \right]_{cs} \left. \begin{array}{c} f'_{12} \cr (\lambda'_1 \mu'_1) \end{array} \right]_{cs} S'_{12} \right| \left. \begin{array}{c} f'' \cr (\lambda'',\mu'') \end{array} \right]_{cs} \left. \begin{array}{c} f_{12} \cr (\lambda_1 \mu_1) \end{array} \right]_{cs} S'' \left. \begin{array}{c} f \cr (\lambda \mu) \end{array} \right]_{cs} \left. \begin{array}{c} f' \cr (\lambda' \mu') \end{array} \right]_{cs} \left. \begin{array}{c} f'_{12} \cr (\lambda'_1 \mu'_1) \end{array} \right]_{cs} S'_{12} \right>
\times \frac{1}{\sqrt{S_{12}}} \left\langle L_{12} = 0 \right| \left| V_{12}^{(0)} \right| L'_{12} = 0 \right\rangle \left\langle S_{12} \right| \left| V_{12}^{(0)} \right| S_{12} \right\rangle \delta_{S_{12} S'_{12}} \delta_{S_J} \delta_{k0}.
\]

### 4.2.1 Examples

\[
< q^5 \left[ 221 \right]_{cs}(01)_{cs} L = 0 S J \left| \sum_{i \leq j} V_{i,j} \right| q^5 \left[ 2111 \right]_{cs}(01)_{cs} L' = 0 S' J >= 10 \delta_{SS'} \delta_{SJ} \sum_{f_{cs}} \frac{n_{f_{cs}}}{\sqrt{5} \cdot 4} \times
\]
\[
\sum_{f_{12}, (\lambda'', \mu''), \ldots} \left\langle \left[ f'' \right]_{cs} (\lambda'' \mu'') c S'' \left| (\lambda_2 \mu_2) c S_{12} \right\rangle \left| \left[ f'' \right]_{cs} (\lambda'' \mu'') c S'' \left| (\lambda_2' \mu_2') c S_{12} \right\rangle \right| 2111 \right|_{cs} \right|_{01} \left| S' \right\rangle \right|
\times \left( \left[ f_{12} \right] (\lambda'' \mu'') S'' \right| V \left| \left[ f_{12} \right] (\lambda'' \mu'') S'' \right\rangle .
\]

Note that \( S = J = S' \). Acceptable \([f'']\) are \([21]\) and \([111]\). If \([f_{12}] = \left[ f_{12} \right]\), from Table 9 we have as possible products: \([21] \times [11] \) and \([111] \times [11] \).

4.3 \(< 0 s^5 1 s 0 d \mid V \mid 0 s^5 1 s 0 d > \)

The \(0 s^5\) states for both the bra and ket are necessarily \([5]_r \) \((01), (01)_{sc}\); \(S_5\) can be \(\frac{1}{2}, \frac{3}{2}\) and \(I_5\) can be \(\frac{1}{2}, \frac{3}{2}\).

\[
< (0s)^5 [f]_{cs} \ S_5 \ I_5 J_5 \times 1s0d \ \ell \ ; (00)_{cs} L S J \left| \sum_{i,j} V_{ij} \right| (0s)^5 [f]_{cs} S_5 \ I_5 \times 1s0d \ \ell \ ; (00)_{cs} L S J >
= < (0s)^5 [f]_{cs} \ S_5 \ I_5 \left| \sum_{i,j} V_{ij} \right| (0s)^5 [f]_{cs} S_5 \ I_5 \rangle \delta_{\ell \ell} +

+ < (0s)^5 [f]_{cs} \ S_5 \ I_5 J_5 \times 1s0d \ \ell \ ; (00)_{cs} L S J \left| V_{sp} \right| (0s)^5 [f]_{cs} S_5 \ I_5 \times 1s0d \ \ell \ ; (00)_{cs} L = \ell S J >
\]

For \((0s)^5\) one can only have \(L_5 = 0\) so the 9J is one.

\[
< (0s)^5 (00)_{r} [f]_{cs} (01)_{cs} L_5 S_5 J_5 \left| V \right| (0s)^5 (00)_{r} [f]_{cs} (01)_{cs} L_5 = 0 \ S_5 \ I_5 J_5 >
= 10 \sum_{f'_{r}, (\lambda'_{r}, \mu'_{r}), l'_{r}, [f_2]_{cs}, L_5} \frac{n_{f_2}}{\sqrt{n_{cs} n_{f_2}}} \left\langle \left[ f' \right]_{cs} (\lambda' \mu') c S' \left| \left[ f_2 \right]_{cs} (\lambda_2 \mu_2) c S_2 \right\rangle \left| \left[ f \right]_{cs} (01)_{cs} S_5 \right\rangle \left. \left| \left[ f' \right]_{cs} (\lambda' \mu') c S' \left| \left[ f_2 \right]_{cs} (\lambda_2 \mu_2) c S_2 \right\rangle \left| \left[ f \right]_{cs} (01)_{cs} S_5 \right\rangle \right. \right| \left\langle (0s)^2 \right| [2]_{r} (00)_{r} [f]_{cs} (\lambda_2 \mu_2) c S_2 J_2 \left| V \right| (0s)^2 \right| [2]_{r} (00)_{r} [f]_{cs} (\lambda_2 \mu_2) c S_2 J_2 \right| \rangle
\]

The second term is more interesting

\[
\left| (0s)^5 \left[ f \right]_{cs} S_5 \ I_5 \times 1s0d ; (00)_{cs} L S J >
= \sum_{f'_{r}, (\lambda'_{r}, \mu'_{r}), l'_{r}, [f_2]_{cs}, L_5} \frac{n_{f_2}}{\sqrt{n_{f_2} n_{f_2} n_{f_2}}} \left\langle \left[ f' \right]_{cs} (\lambda' \mu') c S' \left| \left[ f_2 \right]_{cs} (\lambda_2 \mu_2) c S_2 \right\rangle \left| \left[ f \right]_{cs} (01)_{cs} S_5 \right\rangle \left| \left[ f' \right]_{cs} (\lambda' \mu') c S' \left| \left[ f_2 \right]_{cs} (\lambda_2 \mu_2) c S_2 \right\rangle \left| \left[ f \right]_{cs} (01)_{cs} S_5 \right\rangle \right. \right| \left\langle (0s)^2 \right| \left[ f' \right]_{cs} (\lambda' \mu') c S' \times 0s(10) \right| \left[ f \right]_{cs}(01)_{cs} S_5 \left. \left[ f_2 \right]_{cs} (\lambda_2 \mu_2) c S_2 \right\rangle \left| \left[ f_2 \right]_{cs} (01)_{cs} S_5 \right\rangle \right| \right| \left\langle (0s)^2 \right| (00)_{cs} L S J J >
\]
The ket on the right can be written as (dropping the bar for now)

\[
\sum_{(\lambda_1 \mu_1) S_{12} I_{12}} \left| (0s)^4 [f']_{cs} (\lambda' \mu') S' I' \times 0s \right|_{cs} (10) \frac{1}{2} \frac{1}{2} \times 1 s 0d \ (10) \frac{1}{2} \frac{1}{2} \left| S_{12} I_{12} (\lambda_1 \mu_1) \right> ; (00) c LSJ >
\]

\[
= \sum_{(\lambda_1 \mu_1) S_{12} I_{12} J_{12}} U \left( (\lambda' \mu') (10)(00) (10) ; (01) (\lambda_1 \mu_1) \right) U \left( S' \frac{1}{2} S \frac{1}{2} ; S_{12} \right) U \left( I' \frac{1}{2} I \frac{1}{2} ; I_{12} \right) U \left( 0 \ S' \ S \ S \right)
\]

\[
\times \left| (0s)^4 [f']_{cs} (\lambda' \mu') S' I' J' \times 0s \right|_{cs} (10) \frac{1}{2} \frac{1}{2} \times 1 s 0d \ (10) \frac{1}{2} \frac{1}{2} \left| S_{12} I_{12} (\lambda_1 \mu_1) \right> ; (00) c J >
\]

\[
= \sum_{(\lambda_1 \mu_1) S_{12} I_{12} J_{12} J'} \delta(\lambda' \mu')(10) \delta(01)(\lambda_1 \mu_1) U \left( S' \frac{1}{2} S \frac{1}{2} ; S_{12} \right) U \left( I' \frac{1}{2} I \frac{1}{2} ; I_{12} \right) \left( - \right) S_{12} + J' - S' - L_{12} \delta S_{12} \delta S_{12} W \left( S_{12} S J_{12} J ; S' L_{12} \right)
\]

where the results from Vergados (but we’ll just use the code) [16]

\[
U \left( (\lambda'' \mu'')(10)(00)(10) ; (01) (\lambda_1 \mu_1) \right) = \delta(\lambda'' \mu'')(10)
\]

which defines \( \Delta_{(\lambda_1 \mu_1)} \), and from ref. [15]

\[
U \left( 0 \ L_{12} \ S \ S \ right) = \left( - \right) S_{12} + J' - S' - L_{12} \delta S_{12} \delta S_{12} W \left( S_{12} S J_{12} J ; S' L_{12} \right)
\]

have been used.

The matrix element becomes

\[
\left< \left< 0s \right|^5 [f_5]_{cs} S_5 I_5 \times 1 s 0d \ell ; (01) c LSJ > \right| \sum_{ij} V_{ij} \left| 0s \right|^5 [\bar{f}_5]_{cs} S_5 \tilde{I}_5 \times 1 s 0d \ ; (01) c \ell \bar{L} \bar{S} J >
\]

\[
= 5 \sum_{f',(\lambda' \mu'),S',J',[f],I_2} \sqrt{\frac{n^2_{f'} n^2_{f}}{n^2_{f_5} n^2_{f_5}}} \left< f' \right|_{c} (\lambda' \mu')_{c} S' \left( 10 \right) \frac{1}{2} \cdot \left< f_5 \right|_{c} S_5 \left( 10 \right) \frac{1}{2} \cdot \left| [f']_{c} \right| (\lambda' \mu')_{c} S' \left( 10 \right) \frac{1}{2} \cdot \left| [f_5]_{c} \right| S_5 \left( 10 \right) \frac{1}{2}
\]

\[
\times U \left( S' \frac{1}{2} S \frac{1}{2} ; S_{12} \right) U \left( S' \frac{1}{2} S \frac{1}{2} ; S_{12} \right) U \left( I' \frac{1}{2} I \frac{1}{2} ; I_{12} \right) U \left( I' \frac{1}{2} I \frac{1}{2} ; I_{12} \right)
\]
\[ U((\lambda' \mu')(10)(00)(10); (01)(\lambda_{12}\mu_{12})) \times U((\lambda' \mu')(10)(00)(10); (01)(\tilde{\lambda}_{12}\tilde{\mu}_{12})) \]

\[ \times \left( \begin{array}{cc} 0 & \ell \\ S' & S' \\ J' & J' \\ \end{array} \right) \right) U \left( \begin{array}{cc} 0 & \ell \\ S & S \\ J & J \\ \end{array} \right) \]

\[ \times \left< 0s \times 1s0d l; (\lambda_{12}\mu_{12})c \ell S'_{12} J'_{12} \right| V \left| 0s \times 1s0d \bar{l}; (\tilde{\lambda}_{12}\tilde{\mu}_{12})c \bar{\ell} \bar{S}'_{12} \bar{J}'_{12} \right> \]  

(15)

Note that the SU(3) Racah coefficients lead to the requirement

\[ (\lambda' \mu') = (\mu_{23}\lambda_{23}) = (\mu'_{23}\lambda'_{23}) \]

However, there is no requirement that \( S_{12} = S'_{12} \).

4.3.1 Isospin

Consider the case in which the isospin of the five-quark state differs and consider only the interaction amongst the \( 0s^5 \) states:

\[ < s^5 \alpha I_L \times 1s0d \ell JJ | V | s^5 \beta I_R \times 1s0d, \ell' JJ > = \delta_{\ell\ell'} \sum < 0s^3 \alpha'' \times 0s^2 \alpha_{2L} | > s^5 \alpha > < 0s^3 \alpha'' \times 0s^2 \alpha_{2R} | > s^5 \beta > \]

\[ \times \left[ < 0s^2 \alpha_{2L} I_2 | V | 0s^2 \alpha_{2R} I_2 > < 0s^3 \alpha'' I_3 | 0s^3 \alpha'' I_3 > \right]_{I_M}^{I} \]

It is implicitly assumed the bra and ket states are each coupled to isospin \( I \):

\[ \left[ 0s^2 \alpha_{2R} I_2 M_2 > | 0s^3 \alpha'' I_3 > \right]_{I_M}^{I} = \sum_{M''_T, M_2} C_{I_2 M_2} I_3 M''_T I_M \left| 0s^2 \alpha_{2R} I_2 M_2 > | 0s^3 \alpha'' I_3 M''_T > \right> \]  

(16)

But the two-body matrix elements are independent of \( M_T \) and the three-body overlap is one so one can sum over the \( m_T \) components:

\[ \sum_{M''_T, M_2} C_{I_2 M_2} I_3 M''_T I_M \left| 0s^2 \alpha_{2R} I_2 M_2 > | 0s^3 \alpha'' I_3 M''_T > \right> \]

4.4 \( < 0s^4 0p^2 | V | 0s^4 0p^2 > \)

The matrix element can be written in obvious notation as

\[ V_{ss} + V_{pp} + V_{sp} \]
The first term is just $<0s^4 \mid V_{ss} \mid 0s^4>$ and the second is $<0p^2 \mid V_{pp} \mid 0p^2>$. Only the third term is new.

\[
\langle (0s)^4 [f_4]_c(\lambda_4\mu_4) S_4 I_4 \times (0p)^2 [f_2]_c(\lambda_2\mu_2) S_2 L_2 \mid (00)_c L S J I \mid \sum_{i,j} V_{ij} \rangle \\
\langle (0s)^4 \tilde{f}_4(\tilde{\lambda}_4\tilde{\mu}_4) \tilde{S}_4 \tilde{I}_4 \times (0p)^2 \tilde{f}_2(\tilde{\lambda}_2\tilde{\mu}_2) \tilde{L}_2 , \tilde{S}_2 \tilde{I}_2 ; (00)_c \tilde{L} \tilde{S} J I \rangle
\]
\[
= \langle (0s)^4 [f_4]_c(\lambda_4\mu_4) S_4 I_4 \mid \sum_{i,j} V_{ij} \rangle \langle (0s)^4 \tilde{f}_4(\tilde{\lambda}_4\tilde{\mu}_4) \tilde{S}_4 \tilde{I}_4 \rangle \delta_{I_1I_2} \delta_{S_2S_2} \delta_{22} + \\
+ \langle (0p)^2 [f_2]_c(\lambda_2\mu_2) S_2 L_2 \mid \sum_{i<j} V \rangle \langle (0p)^2 \tilde{f}_2(\tilde{\lambda}_2\tilde{\mu}_2) \tilde{L}_2 S_2 \rangle \delta_{11} + \\
+ \langle (0s)^4 [f_4]_c(\lambda_4\mu_4) S_4 I_4 \times (0p)^2 [f_2]_c(\lambda_2\mu_2) S_2 L_2 S_2 I_2 ; (00)_c L S J I \mid \sum_{i,j} V_{ij} \rangle \\
\langle (0s)^4 \tilde{f}_4(\tilde{\lambda}_4\tilde{\mu}_4) \tilde{S}_4 \tilde{I}_4 \times (0p)^2 \tilde{f}_2(\tilde{\lambda}_2\tilde{\mu}_2) \tilde{L}_2 , \tilde{S}_2 \tilde{I}_2 ; (00)_c \tilde{L} \tilde{S} J I \rangle
\]

This last matrix element can be evaluated using

\[
\langle (0s)^4 [f_4]_c(\lambda_4\mu_4) S_4 I_4 \times (0p)^2 [f_2]_c(\lambda_2\mu_2) S_2 L_2 S_2 I_2 ; (00)_c L S J I \mid \sum_{i,j} V_{ij} \rangle \\
\langle (0s)^4 \tilde{f}_4(\tilde{\lambda}_4\tilde{\mu}_4) \tilde{S}_4 \tilde{I}_4 \times (0p)^2 \tilde{f}_2(\tilde{\lambda}_2\tilde{\mu}_2) \tilde{L}_2 , \tilde{S}_2 \tilde{I}_2 ; (00)_c \tilde{L} \tilde{S} J I \rangle
\]
\[
= 8 \sum \left[ \frac{n_{f_4} n_{f_2}}{n_{f_2} n_{f_4}} \right] \begin{vmatrix} [f_4]_c & [1]_c \\ \left(\lambda_3\mu_3\right)_c & 3 \end{vmatrix} \begin{vmatrix} [f_2]_c & [1]_c \\ \left(\lambda_3\mu_3\right)_c & 3 \end{vmatrix} \begin{vmatrix} \tilde{f}_4 & [1]_c \\ \left(\lambda_4\mu_4\right)_c & 4 \end{vmatrix} \begin{vmatrix} \tilde{f}_2 & [1]_c \\ \left(\lambda_2\mu_2\right)_c & 2 \end{vmatrix} \begin{vmatrix} \tilde{S}_4 & \tilde{S}_2 \\ \tilde{S}_4 & \tilde{S}_2 \end{vmatrix} \begin{vmatrix} \tilde{S}_4 & \tilde{S}_2 \\ \tilde{S}_4 & \tilde{S}_2 \end{vmatrix} \\
\sum U \left( \begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & L_2 \\ 1 & 1 & L \end{array} \right) U \left( \begin{array}{ccc} 0 & 0 & 0 \\ 1 & 1 & L_2 \\ 1 & 1 & L \end{array} \right) U \left( \begin{array}{ccc} S_3' & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & S_2 \\ \frac{1}{2} & \frac{1}{2} & S_2 \end{array} \right) U \left( \begin{array}{ccc} S_3' & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & S_2 \\ \frac{1}{2} & \frac{1}{2} & S_2 \end{array} \right) \\
\times \sum_{J'J_{12}} U \left( \begin{array}{ccc} 1 & 1 & L \end{array} \right) U \left( \begin{array}{ccc} 1 & 1 & L \end{array} \right) U \left( \begin{array}{ccc} 1 & 1 & \tilde{L} \end{array} \right) U \left( \begin{array}{ccc} 1 & 1 & \tilde{L} \end{array} \right)
\times U \left( \begin{array}{ccc} \lambda_3\mu_3 & \lambda_4\mu_4 & \lambda_2\mu_2 \\ 10 & 10 & 10 \\ \lambda_2\mu_2 & \lambda_2\mu_2 & 00 \end{array} \right) U \left( \begin{array}{ccc} \lambda_3\mu_3 & \lambda_4\mu_4 & \lambda_2\mu_2 \\ 10 & 10 & 10 \\ \lambda_2\mu_2 & \lambda_2\mu_2 & 00 \end{array} \right)
\times U \left( \begin{array}{ccc} \lambda_3\mu_3 & \lambda_4\mu_4 & \lambda_2\mu_2 \\ 10 & 10 & 10 \\ \lambda_2\mu_2 & \lambda_2\mu_2 & 00 \end{array} \right) U \left( \begin{array}{ccc} \lambda_3\mu_3 & \lambda_4\mu_4 & \lambda_2\mu_2 \\ 10 & 10 & 10 \\ \lambda_2\mu_2 & \lambda_2\mu_2 & 00 \end{array} \right)
\times U \left( \begin{array}{ccc} \lambda_3\mu_3 & \lambda_4\mu_4 & \lambda_2\mu_2 \\ 10 & 10 & 10 \\ \lambda_2\mu_2 & \lambda_2\mu_2 & 00 \end{array} \right) U \left( \begin{array}{ccc} \lambda_3\mu_3 & \lambda_4\mu_4 & \lambda_2\mu_2 \\ 10 & 10 & 10 \\ \lambda_2\mu_2 & \lambda_2\mu_2 & 00 \end{array} \right)
\[ \times \left\langle 0s\,0p\, L_{12}s_{12}J_{12} \right| V \left| 0s\,0p\, L_{12}s_{12}J_{12} \right\rangle \]  

(17)

One can now simplify; the two orbital angular momentum nine-Js are unity. Also,

\[
U \left( \begin{array}{ccc}
(\lambda_1\mu_1)_c & (\lambda_2\mu_2)_c & (\lambda_{12}\mu_{12})_c \\
(\lambda_3\mu_3)_c & (\lambda_4\mu_4)_c & (\mu_{12}\lambda_{12})_c \\
(\lambda_{13}\mu_{13})_c & (\mu_{13}\lambda_{13})_c & (00)_c
\end{array} \right) = (-)^{\lambda_1+\mu_1+\lambda_{13}+\mu_{13}-\lambda_{12}-\mu_{12}-\lambda_4-\mu_4}U \left( (\lambda_{12}\mu_{12})_c(\lambda_1\mu_1)_c(\lambda_4\mu_4)_c(\lambda_3\mu_3)_c; (\lambda_{12}\mu_{12})_c(\lambda_{34}\mu_{34})_c \right) \]  

(18)

which in our case becomes

\[
(-)^{\lambda_3'+\mu_3'+\lambda_4-\mu_4^'}U \left( (10)_c(\lambda_{34}^\prime\mu_{34}^\prime)_c(10)_c; (\lambda_4\mu_4)_c(\lambda_4\mu_4)_c \right) \text{ check}
\]

4.5 \; < 0s^6 \left| V \right| 0s^4 \, 0p^2 >

\[ < (0s)^6 [f]_c (00)_c \, S \, I \left| \sum_{i,j} V_{ij} \left| (0s)^4 [f]_{cs}(\lambda_4\mu_4) \, S_4 \, I_4 \times (0p)^2 [f]_{cs}(\lambda_2\mu_2)_c \, L_2 \, S_2 \, I_2 \, ; (00)_c \, L \, S \, J_1 \right| > \]

\[ = \sum_{f^*,f_{12}s_{12}''} \sqrt{\frac{n_{fr}}{n_f}} \left\langle f'_{cs} \right| (\lambda''\mu'')_c \, S'' \mid f_{12}_{cs} \, (\mu''\lambda'')_c \, S_{12} \mid [f]_{cs} \, (00)_c \, S \right\rangle U \left( \begin{array}{ccc}
0 & 0 & 0 \\
S_4 & S_{12} & S \\
J_4 & J_2 & J
\end{array} \right)_L \left( \begin{array}{ccc}
0 & L_2 & L \\
S_4 & S_{12} & S \\
J_4 & J_2 & J
\end{array} \right)_R \]

\[ \times < (0s)^2 [f]_{cs} (\mu''\lambda'')_c \, S_2 \, I_2 \left| V_{12} \right| (0p)^2 [f]_{cs}(\lambda_2\mu_2)_c \, L_2 \, S_2 \, I_2 \right| > \]

\[ = (n_{\text{factor}}) \sum_{f_{12},s_{12},s_4} \sqrt{\frac{n_{fr}}{n_f}} \left\langle f_4_{cs} \right| (\lambda_4\mu_4)_c \, S_4 \mid f_1_{cs} \, (\lambda_4\mu_4)_c \, S_{12} \mid [f]_{cs} \, (00)_c \, S \right\rangle U \left( \begin{array}{ccc}
0 & L_2 & L \\
S_4 & S_{12} & S \\
J_4 & J_2 & J
\end{array} \right)_R \]

\[ \times < (0s)^2 [f]_{cs} (\mu_4\lambda_4)_c \, S_2 \, I_2 \left| V_{12} \right| (0p)^2 [f]_{cs}(\lambda_2\mu_2)_c \, L_2 \, S_2 \, I_2 \right| > \]  

(19)

4.6 \; < 0s^6 \left| V \right| 0s^5 \, 1s0d >

\[ < (0s)^6 [f]_{cs} (00)_c \, S_6 \, I \left| \sum_{i,j} V_{ij} \left| (0s)^5 [f]_{cs}(01) \, S_5 \, I_5 \times (1s0d) \mid [1]_{cs}(10)_c \, L_2 \, \frac{1}{2} J_2 \, ; (00)_c \, L \, S_R \, J_1 \right| > \]

\[ = \sum_{f^*,f_{12}s_{12}''} \left\langle f''_{cs} \right| (\lambda''\mu'')_c \, S'' \mid f_{12}_{cs} \, (\mu''\lambda'')_c \, S_{12} \mid [f]_{cs} \, (00)_c \, S_6 \right\rangle U \left( \begin{array}{ccc}
0 & 0 & 0 \\
S''_6 & S''_{12} & S''_{10} \\
J''_{12} & J''_{10} & J''_{10}
\end{array} \right) \frac{n_{fr}}{n_f} \]
\[\sum_{f''(\lambda'',\mu''),S''J'';[f_2,I_2]} \sqrt{\frac{n_{f''}}{n_{f_5}}} \left\langle f''_{\alpha\beta}\left(\lambda''\mu''\right) \left| S'' \right| \frac{1}{2} \right\rangle \left| f_5_{\alpha\beta}\right\rangle \times \left| V_{12} \right| \]

\[\left| \left(0s\right)^4 \left[f''_{\alpha\beta}\left(\lambda''\mu''\right) \left| S'' \right| I'' \times \left(0s\right)^2 \left[f_{12}_{\alpha\beta}\left(\mu''\lambda''\right)_{\alpha\beta} \left| S_{12}'' \right| \frac{1}{2}\right] \right]_{S_0J''} \right| \times 1s0d \left(10\right)_c \frac{1}{2} \frac{1}{2} \left(00\right)_c L \cdot S_R J > \]

\[= \sum_{f''(\lambda'',\mu''),S''J'';[f_2,I_2]} \left\langle f''_{\alpha\beta}\left(\lambda''\mu''\right) \left| S'' \right| \left| f_{12}_{\alpha\beta}\left(\mu''\lambda''\right)_{\alpha\beta} \left| S_{12}'' \right| \frac{1}{2} \right\rangle \times 1s0d \left(10\right)_c \frac{1}{2} \frac{1}{2} \left(00\right)_c L \cdot S_R J > \]

The last matrix element becomes

\[< \left(0s\right)^2 \left[f_{12}_{\alpha\beta}\left(\mu''\lambda''\right)_{\alpha\beta} \left| S'' \right| I'' \times \left(0s\right)^2 \left[f_{12}_{\alpha\beta}\left(\mu''\lambda''\right)_{\alpha\beta} \left| S_{12}'' \right| \frac{1}{2} \right] \right]_{S_0J''} \times 1s0d \left(10\right)_c \frac{1}{2} \frac{1}{2} \left(00\right)_c J > \]

Note that the two-body matrix element is diagonal in SU(3) by virtue that both the 0s^2 particles and the 0s0s1d pair must couple to the (\lambda''\mu'') states to form a color singlet. Finally, one has
\[
\times U\left((\lambda'',\mu'')(10)(00)(10); (01)(\mu''\lambda'')\right) U\left(S''\frac{1}{2}S_{R\frac{1}{2}}; S_5S_{12}\right) U\left(I''\frac{1}{2}I_{\frac{1}{2}}; I_5I_{12}\right) U\left(0\begin{array}{ccc}
S'' & \ell \\
S'' & S_{12} & S_R \\
\ell & J_2 & J \end{array}\right)
\]
\[
\times <(0s)^2 (\mu''\lambda')_{cs} S_{12}'I_2 I_1 \bigg| V_{12} \bigg| 0s \times 1s0d \ell : (\mu''\lambda'') L_{12} S_{12} J_{12} I_{12} >
\]

4.6.1 Example

\[
< (0s)^6 [2211]_{cs} (00)_c S_6 = 0 \mid I = 1 \mid \sum_{i,j} V_{ij} \bigg|
\]
\[
\bigg| (0s)^5 [221]_{cs} (01) S_5 = \frac{1}{2} I_5 = \frac{1}{2} \times (1s0d)\ [1]_{cs}(10)_c L_2 = 0 \frac{1}{2} I_2 = \frac{1}{2} ; (00)_c L = 0 S_R = 0 \ J = 0 I = 1 >
\]
\[
= \sum_{f_{12},(\lambda''\mu''),I''S_{12}J_{12}} f(N) \left[ \begin{array}{cc}
[f''_{cs}] & [f_{12}]_{cs} \\
(\lambda''\mu'')_{cs} S_{12}' & (\mu''\lambda'')_{cs} S_{12}' \\
(00)_{cs} S_6 = 0 & \end{array} \right] \sqrt{\frac{n_{f''}}{5 \cdot 9}} \left[ \begin{array}{cc}
[f''_{cs}] & [1]_{cs} \\
(\lambda''\mu'')_{cs} S_{12}' & (10)_{c\frac{1}{2}} S_5 = \frac{1}{2} \end{array} \right] \bigg]
\]
\[
\times U\left((\lambda''\mu'')(10)(00)(10); (01)(\mu''\lambda'')\right) U\left(S''\frac{1}{2}0\frac{1}{2}; \frac{1}{2}; S_{12}\right) U\left(I''\frac{1}{2}\frac{1}{2}; \frac{1}{2}; I_{12}\right) U\left(0\begin{array}{ccc}
S'' & 0 & 0 \\
S'' & S_{12} & 0 \\
0 & S_{12} & J = 0 \\
J & J_2 & J \end{array}\right)
\]
\[
\times < (0s)^2 (\mu''\lambda')_{cs} S_{12}'J_{12} I_2 \bigg| V_{12} \bigg| 0s \times 1s ; (\mu''\lambda'') L_{12} = 0 S_{12} J_{12} I_{12} >
\]

4.7  \[< 0s^4 (0p^2 |V| 0s^5 1s0d >
\]
One can use the results from above, namely Eq. (14) and Eq. (17), to obtain

\[
\left< (0s)^4 [f]_{cs}(\lambda_4\mu_4) S_4 I_4 \times (0p)^2 [f_2]_{cs}(\lambda_2\mu_2)_{cs} L_2 S_2 I_2 ; (00)_c L S J I \mid \sum_{i,j} V_{ij} \right|
\]
\[
\times \left| (0s)^5 [f]_{cs} S_5 I_5 \times 1s0d ; (01)_c L R S_R J \right>
\]
\[
= \sum_{(\lambda_4\mu_4)_{cs}S_4I_4J_12} U\left((\lambda_4\mu_4)(10)(00)(10); (01)(\lambda_1\mu_1)_{cs}\right) U\left(S_4\frac{1}{2} S_{R\frac{1}{2}}; S_5S_{12}\right) U\left(I_4\frac{1}{2} I_{R\frac{1}{2}}; I_5 I_{12}\right)
\]
\[
\times \sqrt{\frac{n_{f_4}}{n_{f_5}}} \left[ \begin{array}{cc}
[f]_{cs} & [1]_{cs} \\
(\lambda_4\mu_4)_{cs} S_4 & (10)_{c\frac{1}{2}} \end{array} \right] \left[ \begin{array}{c}
[f_5]_{cs} \\
(01)_{c\frac{1}{2}} S_5 \end{array} \right] U\left(0\begin{array}{ccc}
S_4 & L_2 & L \\
S_4 & S_2 & S \\
J_4 & J_{12} & J \end{array}\right) L\left(0\begin{array}{ccc}
S_4 & L_2 & L \\
S_4 & S_2 & S \\
J_4 & J_{12} & J \end{array}\right) R
\begin{align*}
&\times \left\langle (0s)^4 \left[ f_4 \right]_{cs} (\lambda_4 \mu_4) \ S_4 \ I_4 \times (0p)^2 \left[ f_2 \right]_{cs} (\lambda_2 \mu_2) \ c \ L_2 \ S_2 \ I_2 \ ; (00)_{c} \ L \ S \ J I \right\rangle \sum_{i,j} V_{ij}
&\times \left\langle (0s)^4 \left[ f_4 \right]_{cs} (\lambda_4 \mu_4) \ S_4 \ I_4 \ J_4 \times \left[ 0s \left[ 1 \right]_{cs} (10)_{c} 1 \frac{1}{2} \times 1s0d (10)_{c} l \frac{1}{2} \right] \right\rangle \ \left( \lambda_{12} \mu_{12} \right) \ S_{12} \ I_{12} \ J_{12} \ 
&= \sum_{S_{12} \ J_{12}} U \left( \left( \lambda_{4} \mu_{4} \right) (10) (00) ; (01) (\lambda_{12} \mu_{12}) \right) U \left( S_{4} \frac{1}{2} S_{R} \frac{1}{2} ; S_{5} S_{12} \right) U \left( I_{4} \frac{1}{2} I_{R} \frac{1}{2} ; I_{5} I_{12} \right) \ 
&\times \left\langle \left( 0s \right)^4 \left[ f_4 \right]_{cs} \left( \lambda_4 \mu_4 \right) S_4 \ J_4 \right\rangle \left\langle \left[ f_5 \right]_{cs} \left( \lambda_5 \mu_5 \right) S_5 \ J_5 \right\rangle \left\langle 0s \right| \left( 10 \right)_{c} \left| 1s0d \left( 10 \right)_{c} \left( \lambda_{12} \mu_{12} \right) \ S_{12} \ I_{12} \ J_{12} \right\rangle \ 
&\times \left\langle 0p \right| \left( 0s \right)^4 \left[ f_2 \right]_{cs} \left( \lambda_2 \mu_2 \right) L_2 S_2 I_2 J_{12} \right\rangle \sum_{i,j} V_{ij} \left( 0s \left( 10 \right)_{c} \times 1s0d \left( 10 \right)_{c} \right) \left( \lambda_{12} \mu_{12} \right) \ S_{12} \ I_{12} \ J_{12} \right) \ 
\end{align*}

One could simplify the 9j and the SU(3) Racah coefficient, but it is preferable to leave it in the above form and let the code fix the phases. There is no sum over (\lambda' \mu') because of the SU(3) Racah and the coupling to (00).
4.7.1 Example 1

Assume $J = I = 0$.

\[
< [211] (10)_c S_4 = 1 I_4 = 1 \times [11] (01)_c L_2 = 0 S_2 = 1 I_2 = 1; L = 0 S = 0 | V >
\]

\[
\times [221] (01)_c S_5 = \frac{1}{2} I_5 = \frac{1}{2} \times [1] \frac{11}{2} \ell = 0 L = 0 S = 0 >
\]

\[
= \sum_{S_{12}} U \left( (10)(10)(00)(10); (01)(01) \right) U \left( \frac{1}{2} \frac{10}{2}; \frac{1}{2} S_{12} \right) U \left( \frac{1}{2} \frac{01}{2}; \frac{1}{2} \right)
\]

\[
\times \sqrt{\frac{3}{5}} \left< \begin{array}{c|c} [211]_{cs} & [1]_{cs} \\ (10)_c 1 & (10)_c \frac{1}{2} \end{array} \right| U \left( \begin{array}{c c} 0 & \ell = 0 \\ 1 & S_{12} \end{array} \right) U \left( \begin{array}{c c} 0 \\ 1 \end{array} \right) S_{12} = 0
\]

\[
\times \left< (0p)^2 [11]_{cs} (01)_c L_2 = 0 S_2 = 1 I_2 = 1 J_2 = 1 \right| V_{ij} \left| 0s \times 1s0d \ell = 0 ; (01) S'_{12} I_{12} = 1 J_{12} \right>
\]

We have:

\[
\sqrt{\frac{n_f'}{n_f}} = \sqrt{\frac{3}{5}}
\]

\[
\left< \begin{array}{c|c} [211]_{cs} & [1]_{cs} \\ (10)_c 1 & (10)_c \frac{1}{2} \end{array} \right| \left< \begin{array}{c} [221]_{cs} \\ (01)_c \frac{1}{2} \end{array} \right> = -\sqrt{\frac{3}{2}}
\]

\[
U \left( \begin{array}{c c} 0 & \ell = 0 \\ 1 & S_{12} \end{array} \right) = 1
\]

\[
U \left( (10)(10)(00)(10); (01)(01) \right) = 1
\]

\[
U \left( \frac{1}{2} \frac{10}{2}; \frac{1}{2} S_{12} \right) = -1
\]

\[
U \left( \frac{1}{2} \frac{01}{2}; \frac{1}{2} \right) = -1
\]

\[
product = -\sqrt{\frac{2}{5}}
\]
4.7.2 Example 2

Assume $J = 0, I = 1$.

$$< [22] (02)_c S_4 = 0 I_4 = 0 \times [11] (20)_c L_2 = 0 S_2 = 0 I_2 = 1; L = 0 S = 0 | V >$$

$$\times [221] (01)_c S_5 = \frac{1}{2} I_5 = \frac{1}{2} \times [1] \frac{1}{2} \frac{1}{2} \ell = 0 L = 0 S = 0 >$$

$$= \sum_{S_{12}} U((02)(10)(00)(10); (01)(20)) U(\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} S_{12} = 0) U(\frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} 0)$$

$$\times \sqrt{\frac{3}{5}} \left[ [22]_{cs} \begin{array}{c} [1]_{cs} \\ (02)_c 1 \\ (10)_c \frac{1}{2} \\ (01)_c \frac{1}{2} \end{array} \right] \left[ [221]_{cs} \right] U(\begin{array}{c} 0 \\ 1 \\ S_{12} = 0 \\ S_{12} \end{array})$$

$$\times \left( (0p)^2 [11]_{cs} (01)_c L_2 = 0 S_2 = 1 I_2 = 1 J_2 = 1 | V_{ij} > \times 1 s o d \ell = 0; (01) S'_{12} I_{12} = 1 J_{12} \right)$$

one has:

$$\sqrt{\frac{n_p}{n_f}} = \sqrt{\frac{2}{5}}$$

$$\left[ [211]_{cs} \begin{array}{c} [1]_{cs} \\ (10)_c 1 \\ (10)_c \frac{1}{2} \\ (01)_c \frac{1}{2} \end{array} \right] \left[ [221]_{cs} \right] = \sqrt{\frac{3}{4}}$$

$$U(\begin{array}{c} 0 \\ 1 \\ S_{12} = 0 \\ S_{12} \end{array}) = 1$$

$$U((10)(10)(00)(10); (01)(01)) = 1$$

$$U(\begin{array}{c} 1 \frac{1}{2} \frac{1}{2} S_{12} \\ 0 \frac{1}{2} \frac{1}{2} \end{array}) = 1$$

$$U(\begin{array}{c} 1 \frac{1}{2} \frac{1}{2} 1 \\ 0 \frac{1}{2} \frac{1}{2} \end{array}) = -1$$

$$product = \sqrt{\frac{3}{10}}$$

This multiplies the 2BME

$$\langle 0p^2 [11]_{cs} (20)_c S_2 = 0 I_2 = 1 J_2 = 1 | V \rangle \times 1 s \times 1 s (20)_c S_{12} = 0 J_{12} = 1 I_{12} = 1 \rangle.$$
4.7.3 Example 3

Assume $J = 4, I = 1$.

\[
\langle [22] \ (02)_{c} S_{4} = 2 \ I_{4} = 0 \times [11] \ (20)_{c} L_{2} = 2 \ S_{2} = 0 \ I_{2} = 1; \ L = 2 \ S = 2 \ J = 4 | V | \\
\times [221] \ (01)_{c} S_{5} = 5 \ I_{5} = \frac{1}{2} \times \ [1] \ \frac{1}{2} \ \ell = 2 \ L = 2 \ S = 2 \ J = 4 \ I = 1 / >
\]

\[
= \sum_{S_{12}} U\left((02)(10)(00)(10); (01)(20)\right) U\left(2 \frac{1}{2} \frac{1}{2} ; \frac{5}{2} S_{12}\right) U\left(0 \frac{1}{2} \frac{1}{2} ; \frac{1}{2} I_{12}\right)
\times \ \sqrt{\frac{3}{5}} \ \langle [22]_{cs} \ (02)_{c} 1 \ (10)_{c} \frac{1}{2} \ | [221]_{cs} \ (01)_{c} \frac{1}{2} \rangle \ U\left(0 \ell = 0 \ 0 \ 1 \ S_{12} = 0 \ 0 \ 1 \ S_{12} = 0 \ 0 \right)
\times \ \langle (0p)^{2} [11]_{cs} \ (01)_{c} L_{2} = 0 \ S_{2} = 1 I_{2} = 1 J_{2} = 1 | V_{ij} \ 0 s \ \times 1 s 0 d \ \ell = 0 ; (01) S_{12} I_{12} = 1 J_{12} \rangle
\]
5 Appendix

5.1 SU(2) Conventions

Use the de-Shalit-Talimi [15] conventions for reduced matrix elements:

\[
\langle JM \mid T^{(k)} \mid J' M' \rangle = (-)^{J-M} \left( \frac{J}{M} \frac{k}{\kappa} \frac{J'}{M'} \right) \langle J \mid T^{(k)} \mid J' \rangle
\]

\[
\langle J \mid 1 \mid J' \rangle = \hat{J} \delta_{J,J'}
\]

\[
\langle J \mid J \mid J' \rangle = \hat{J} \delta_{J,J'}
\]

5.2 15j Symbols

\[
\left[ (L_4 \ell_5)_{J_5} (S_4 s_5)_{S_5} \right] \times \ell_6 s_6 \equiv LSJ = \sum U \left( \begin{array}{ccc} L_5 & \ell_6 & L \\ S_5 & s_6 & S \\ J_5 & j_6 & J \end{array} \right) \left[ (L_4 \ell_5)_{J_5} (S_4 s_5)_{S_5} \right]_{J_5} \times \ell_6 s_6 \partial_J \frac{J_5}{j_5} \frac{J}{J'} = \left( \frac{J_5}{J'} \right)
\]

\[
\left[ (L_4 \ell_5)_{J_5} (S_4 s_5)_{S_5} \right] \times \ell_6 s_6 \equiv LSJ = \sum U \left( \begin{array}{ccc} L_5 & \ell_6 & L \\ S_5 & s_6 & S \\ J_5 & j_6 & J \end{array} \right) \left[ (L_4 \ell_5)_{J_5} (S_4 s_5)_{S_5} \right]_{J_5} \times \ell_6 s_6 \partial_J \frac{J_5}{j_5} \frac{J}{J'} = \left( \frac{J_5}{J'} \right)
\]

\[
\left[ (L_4 \ell_5)_{J_5} (S_4 s_5)_{S_5} \right] \times \ell_6 s_6 \equiv LSJ = \sum U \left( \begin{array}{ccc} L_5 & \ell_6 & L \\ S_5 & s_6 & S \\ J_5 & j_6 & J \end{array} \right) \left[ (L_4 \ell_5)_{J_5} (S_4 s_5)_{S_5} \right]_{J_5} \times \ell_6 s_6 \partial_J \frac{J_5}{j_5} \frac{J}{J'} = \left( \frac{J_5}{J'} \right)
\]

Angular momenta involved: \( L_4, \ell_5, \ell_6, L_5, L, S_4, s_5, s_6, S_5, S, j_5, j_6, L_{56}, S_{56}, J \)

Alternatively,
\[
| (L_4 \ell_5^L (S_4 s_5)^S) \times \ell_6 s_6; LSJ \rangle = | (L_4 \ell_5^L \ell_6^L) \times (S_4 s_5)^S s_6 \rangle \]
\[
= \sum U \left( L_4 \ell_5 \ell_6; L_5 L_5^6 \right) U \left( S_4 s_5 S_6; S_5 S_5^6 \right) \left| L_4 (\ell_5 \ell_6)^L \times S_4 s_5 S_5^6 \rangle \right| J \right)
\[
= \sum U \left( L_4 \ell_5 \ell_6; L_5 L_5^6 \right) U \left( S_4 s_5 S_6; S_5 S_5^6 \right) U \left( \begin{array}{ccc}
L_4 & L_5 & L_5^6 \\
S_4 & S_6 & S_5 \\
J_4 & J_5^6 & J
\end{array} \right) \left| L_4 S_4 \right| (\ell_5 \ell_6)^L \times (S_5 S_5^6)^S ; J \right)
\]
\[
= U \left( \begin{array}{ccc}
L_4 & \ell_5 & L \\
S_4 & s_5 & S \\
J_4 & S_6 & J_5^6
\end{array} \right) \left| L_4 S_4 \right| (\ell_5 \ell_6)^L \times (S_5 S_5^6)^S ; J \right) \]

(22)

5.3 \( SU(3) \) \( \lambda \mu \) and \( \mu \lambda \) Special Cases

\[
U \left( \begin{array}{cccc}
\lambda_{1 \mu_1} & \lambda_{2 \mu_2} & \lambda_{12 \mu_{12}} \\
\lambda_{3 \mu_3} & \lambda_{4 \mu_4} & \mu_{12 \lambda_{12}} \\
\lambda_{13 \mu_{13}} & \mu_{13 \lambda_{13}} & (0 0)
\end{array} \right) = (-)^{\lambda_1 + \mu_1 + \lambda_{13} + \mu_{13} - \lambda_{12} - \mu_{12} - \lambda_4 - \mu_4} U \left[ (\lambda_2 \mu_2)(\lambda_1 \mu_1)(\lambda_4 \mu_4)(\lambda_3 \mu_3); (\lambda_{12} \mu_{12})(\lambda_{34} \mu_{34}) \right]
\]

(23)

\[
U \left[ (\lambda_2 \mu_2)(\lambda_1 \mu_1)(\mu_2 \lambda_2)(\mu_1 \lambda_1); (\lambda_{12} \mu_{12})(0 0) \right] = (-)^{\lambda_1 + \mu_1 + \lambda_2 + \mu_2 - \lambda_{12} - \mu_{12}} \frac{g(\lambda_{12} \mu_{12})}{g(\lambda_{1} \mu_{1}) g(\lambda_{2} \mu_{2})}
\]

(24)

\[
U \left( \begin{array}{ccc}
\lambda_{1 \mu_1} & \lambda_{2 \mu_2} & \lambda_{12 \mu_{12}} \\
\lambda_{3 \mu_3} & \lambda_{4 \mu_4} & \mu_{12 \lambda_{12}} \\
(0 0) & (0 0) & (0 0)
\end{array} \right) = \frac{g(\lambda_{12} \mu_{12})}{g(\lambda_{1} \mu_{1}) g(\lambda_{2} \mu_{2})}
\]

(25)

\[
U \left( (\lambda' \mu')(10)(00)(10); (01)(\lambda_{12} \mu_{12}) \right) = \delta(\lambda' \mu')(10) \delta(01)(\lambda_{12} \mu_{12})
\]

5.4 Matrix Elements

\[
\langle J_1(1) J_2(2), J \mid V_1 \mid J_3(1) J_4(2), J \rangle = \delta_{J_2 J_4} \langle J_1(1) \mid V_1 \mid J_3(1) \rangle
\]
5.5 Surface Delta Interaction

\[
\langle [J_1 J_2]^{J_{12}} J_3, J \rangle \left| T_{12}^{(k)} \right| \left[ \tilde{J}_1 \tilde{J}_2 \right]^{J_{12}} \tilde{J}_3, \tilde{J} \rangle = (-)^{2J_{12} + 2\tilde{J} + J_3 + J_{12} + k + J} \delta_{j_3 j_5} \tilde{J} \tilde{J}
\]

\[
\times W(J_{12} \tilde{J}_{12} J \tilde{J}; k J_3) \langle J_1 J_2, J_{12} \left| T_{12}^{(k)} \right| \tilde{J}_1 \tilde{J}_2, \tilde{J}_{12} \rangle
\]

(26)

Alternately,

\[
< \ell_2 j || C_k || \ell_2 j' > = (-)^{\ell + \ell' \sqrt{2j+1}} C \frac{j_1}{2} k j' \ell' \frac{1}{2} \left[ 1 + (-)^{\ell + k + \ell'} \right]
\]

(27)

\[
< j_1 j_2 J | (C_k \cdot C_k) | j_3 j_4 J > = (-)^{j_2 + J + j_3} \left\{ \begin{array}{c} j_1 \\ j_4 \\ j_3 \\ j \end{array} \right\} < j_1 || C_k || j_3 > < j_2 || C_k || j_4 >
\]

\[
= (-)^{j_1 + j_4} \left\{ \begin{array}{c} j_1 \\ j_4 \\ j_3 \\ j \end{array} \right\} C \frac{j_1}{2} k j_3 k j_4 k j_2 k j_4 k j_2
\]

(28)

Note that in Eq.(29) one could make use of an identity

\[
\left( \begin{array}{ccc} j_1 & j_2 & J \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{array} \right) = -\ell_1 \ell_2 \left( \begin{array}{ccc} \ell_1 & \ell_2 & J \\ 0 & 0 & 0 \end{array} \right) \left\{ \begin{array}{c} j_1 \\ j_2 \\ J \end{array} \right\}
\]

or

\[
(-)^{j_1 + j_2 + J} \left( \begin{array}{ccc} j_1 & J & j_2 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{array} \right) = (-)^{1 + \ell_1 + \ell_2 + J} \ell_1 \ell_2 \left( \begin{array}{ccc} \ell_1 & \ell_2 & J \\ 0 & 0 & 0 \end{array} \right) \left\{ \begin{array}{c} j_1 \\ j_2 \\ J \end{array} \right\}
\]
or

\[
\langle - \rangle^{j_1+j_2+J_1+\ell_1+\ell_J} \begin{pmatrix} j_1 & J & j_2 \\ \ell_1 & 0 & -\frac{1}{2} \end{pmatrix} = \langle - \rangle^{\ell_1} \hat{\ell}_1 \hat{\ell}_2 \begin{pmatrix} \ell_1 & J & \ell_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & J \\ \ell_2 & \ell_1 & \frac{1}{2} \end{pmatrix}
\]

(with \( \ell_1 + J + \ell_2 \) even) and Eq.(29) becomes

\[
\langle - \rangle^{\ell+J_1+J_2+K} \begin{pmatrix} \ell & \ell' & k \\ j' & j & \frac{1}{2} \end{pmatrix} \langle - \rangle^{\ell} \hat{\ell} \hat{\ell} \begin{pmatrix} \ell & k & \ell' \\ 0 & 0 & 0 \end{pmatrix} = \langle - \rangle^{\ell+J_1+J_2+K} \begin{pmatrix} j & k & j' \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix}
\]

\[
= \langle - \rangle^{\ell+J_1+J_2+K} \begin{pmatrix} j & k & j' \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \langle - \rangle^{J'-J+K} C \begin{pmatrix} j & k & j' \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}
\]

which agrees with Eq. (27) except for a minus...

In LS Coupling

\[
< \ell_1 \ell_2 L S J | (C_k \cdot C_k) | \ell_3 \ell_4 L S J > = \frac{1}{J} < LSJ || (C_k \cdot C_k) || LSJ >
\]

\[
= \langle - \rangle^{L+S+J} (2J + 1) \begin{pmatrix} j_1 & j_2 & J \\ \ell_1 & \ell_2 & \frac{1}{2} \end{pmatrix} < \ell_1 \ell_2 L || (C_k \cdot C_k) || \ell_3 \ell_4 L >
\]

\[
= \langle - \rangle^{L+S+J} \hat{\ell} \begin{pmatrix} j_1 & j_2 & J \\ \ell_1 & \ell_2 & \frac{1}{2} \end{pmatrix} \langle - \rangle^{L+S+J} \hat{\ell}_2 \hat{\ell}_3 \hat{\ell}_4 \begin{pmatrix} \ell_1 & k & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_2 & k & \ell_4 \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
= \langle - \rangle^{L+S+J+L_3+L_4+L_1+L_2+2K} \begin{pmatrix} j_1 & j_2 & J \\ \ell_1 & \ell_2 & \frac{1}{2} \end{pmatrix} \hat{\ell}_1 \hat{\ell}_2 \hat{\ell}_3 \hat{\ell}_4 \begin{pmatrix} \ell_1 & k & \ell_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell_2 & k & \ell_4 \\ 0 & 0 & 0 \end{pmatrix}
\]

In all cases the off-diagonal matrix elements are related to the corresponding diagonal elements:

\[
M_{ij} = \sqrt{M_{ii} M_{jj}}.
\]

5.6 SDI Matrix Elements - LS Coupling

5.6.1 Positive Parity, \( T=0 \)

\[
< L = 0 \ S = 1 \ J = 1^+ | V_{SDI} | LSJ = 1 >= \begin{pmatrix} s^2 & p^2 & d^2 \\ \frac{1}{\sqrt{3}} & \sqrt{3} & \sqrt{5} \\ \sqrt{5} & \sqrt{15} & 5 \end{pmatrix}
\]
5.6.2 Negative Parity, \( T=0 \)

5.6.3 Positive Parity, \( T=1 \)

\[ J = 0^+ \]

\[ < L = 0 \ S = 0 \ J = 0^+ \ |V_{SDI}| \ LSJ = 0 > = \begin{pmatrix} s^2 & p^2 & d^2 \\ \frac{1}{\sqrt{3}} & \sqrt{3} & \frac{\sqrt{5}}{\sqrt{15}} \\ \frac{\sqrt{5}}{\sqrt{15}} & \frac{3}{\sqrt{15}} & 5 \end{pmatrix} \] \( (31) \)

\[ \text{Trace} = 9 \]

\[ J = 2^+ \]

\[ < L = 1 \ S = 1 \ J = 2^+ \ |V_{SDI}| \ LSJ = 2 > = \begin{pmatrix} p^2 & d^2 \\ \frac{2}{15} & 0 \\ 0 & x \end{pmatrix} \] \( (32) \)

\[ < L = 0 \ S = 0 \ J = 2^+ \ |V_{SDI}| \ LSJ = 2 > = \begin{pmatrix} p^2 & sd & d^2 \\ \frac{2}{15} & \frac{19}{25} & \frac{818}{35} \\ \frac{12}{7} & -\sqrt{\frac{10}{7}} & \frac{818}{35} \end{pmatrix} \] \( (33) \)

\[ < sd \ L = 2 \ S = 1 \ J = 2^+ \ |V_{SDI}| \ LSJ = 2 > = \frac{6}{25} \] \( (34) \)

\[ < d^2 \ L = 2 \ S = 1 \ J = 2^+ \ |V_{SDI}| \ LSJ = 2 > = \] \( (35) \)

\[ \text{Trace} = \frac{127}{35} \]

\[ J = 4^+ \]

\[ < d^2 \ L = 3 \ S = 1 \ J = 4^+ \ |V_{SDI}| \ LSJ = 4 > = \frac{4}{35} \] \( (36) \)

\[ < d^2 \ L = 4 \ S = 0 \ J = 4^+ \ |V_{SDI}| \ LSJ = 4 > = \frac{26}{35} \] \( (37) \)

\[ \text{Trace} = \frac{6}{7} \]
5.6.4 Negative Parity, T=1

\[ < L = 1 S = 1 J = 0^- |V_{SDI}| LSJ = 0 > = \left( \begin{array}{cc} ps & pd \\ 1 & -\sqrt{2} \\ -\sqrt{2} & 2 \end{array} \right) \]  

(38)

\[ < L = 1 S = 1 J = 2^- |V_{SDI}| LSJ = 2 > = \left( \begin{array}{cc} ps & pd \\ \frac{2}{\sqrt{5}} & -\frac{\sqrt{8}}{5} \\ -\frac{\sqrt{8}}{5} & \frac{4}{5} \end{array} \right) \]  

(39)

\[ < L = 3 S = 1 J = 2^- |V_{SDI}| LSJ = 2 > = \frac{27}{35} \]  

(40)

\[ < L = 3 S = 1 J = 4^- |V_{SDI}| LSJ = 4 > = \frac{4}{7} \]  

(41)

5.7 SDI Matrix Elements - jj Coupling

5.8 Positive Parity, T=1

\[ < j_1j_2; J = 0^+ T = 1 |V_{SDI}| j_3j_4; J = 0^+ T = 1 > = \left( \begin{array}{cccccc} s_1s_1 & p_1p_1 & p_2p_2 & d_3d_3 & d_4d_4 \\ 1 & -1 & -\sqrt{2} & \sqrt{2} & \sqrt{3} \\ -1 & 1 & \sqrt{2} & -\sqrt{2} & -\sqrt{3} \\ -\sqrt{2} & \sqrt{2} & 2 & -2 & -\sqrt{6} \\ \sqrt{2} & -\sqrt{2} & -2 & 2 & \sqrt{6} \\ \sqrt{3} & -\sqrt{3} & -\sqrt{3} & \sqrt{6} & 3 \end{array} \right) \]  

(42)
\[ j_{1j2} : J = 2^+ \]

\[ < j_{1j2} : J = 2^+ | V_{SD1} | j_{3j4} : J = 2^+ > = \begin{pmatrix}
\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & \frac{2}{5} & -\frac{\sqrt{6}}{5} & -\frac{2\sqrt{21}}{35} & \frac{4\sqrt{21}}{35} \\
-\frac{2}{5} & \frac{2}{5} & \frac{2}{5} & -\frac{2}{5} & \frac{\sqrt{6}}{5} & \frac{2\sqrt{21}}{35} & -\frac{4\sqrt{21}}{35} \\
-\frac{2}{5} & \frac{2}{5} & \frac{2}{5} & -\frac{2}{5} & \frac{\sqrt{6}}{5} & \frac{2\sqrt{21}}{35} & -\frac{4\sqrt{21}}{35} \\
\frac{2}{5} & -\frac{2}{5} & -\frac{2}{5} & \frac{2}{5} & -\frac{\sqrt{6}}{5} & -\frac{2\sqrt{21}}{35} & \frac{4\sqrt{21}}{35} \\
-\frac{\sqrt{6}}{5} & \frac{\sqrt{6}}{5} & \frac{\sqrt{6}}{5} & -\frac{\sqrt{6}}{5} & \frac{3}{5} & \frac{3\sqrt{14}}{35} & -\frac{6\sqrt{14}}{35} \\
-\frac{2\sqrt{21}}{35} & \frac{2\sqrt{21}}{35} & \frac{2\sqrt{21}}{35} & -\frac{2\sqrt{21}}{35} & \frac{3\sqrt{14}}{35} & \frac{6}{35} & -\frac{12}{35} \\
\frac{4\sqrt{21}}{35} & -\frac{4\sqrt{21}}{35} & -\frac{4\sqrt{21}}{35} & \frac{4\sqrt{21}}{35} & -\frac{6\sqrt{14}}{35} & -\frac{12}{35} & \frac{24}{35} \\
\end{pmatrix} \tag{43} \]

Trace = \( \frac{107}{35} \)

\[ j_{1j2} : J = 4^+ \]

\[ < j_{1j2} : J = 4^+ T = 1 | V_{SD1} | j_{3j4} : J = 4^+ T = 1 > = \begin{pmatrix}
d_4 d_2 & d_2 d_5 \\
\frac{4}{7} & -\frac{\sqrt{2}}{7} \\
-\frac{\sqrt{2}}{7} & \frac{2}{7} \\
\end{pmatrix} \tag{44} \]

Trace = \( \frac{6}{7} \)

Negative Parity:
\[< j_1 j_2; J = 1^- | V_{S_{DI}} | j_3 j_4; J = 1^- > = \begin{pmatrix} s_1 p_{13} & s_1 p_{14} & p_1 d_{13} & p_1 d_{14} & p_3 d_{13} & p_3 d_{14} \\ 1/3 & \frac{\sqrt{2}}{5} & -\frac{\sqrt{2}}{3} & -\frac{\sqrt{10}}{15} & -\frac{2}{5} \\ \frac{\sqrt{2}}{5} & \frac{2}{3} & \frac{2}{3} & -\frac{2\sqrt{5}}{15} & -\frac{\sqrt{4}}{5} \\ -\frac{\sqrt{2}}{3} & \frac{-2}{3} & \frac{2}{3} & \frac{2\sqrt{5}}{15} & \sqrt{\frac{4}{5}} \\ -\frac{\sqrt{10}}{15} & -\frac{2\sqrt{5}}{15} & \frac{2\sqrt{5}}{15} & \frac{2}{5} & \frac{2}{5} \\ -\frac{\sqrt{2}}{5} & -\frac{\sqrt{1}}{5} & \sqrt{\frac{1}{5}} & \frac{2}{5} & \frac{6}{5} \end{pmatrix} \]

(45)

\[< j_1 j_2; J = 3^- | V_{S_{DI}} | j_3 j_4; J = 3^- >= \begin{pmatrix} p_1 d_{13} & p_1 d_{14} & p_3 d_{13} & p_3 d_{14} \\ \frac{18}{35} & -\frac{3}{7} \sqrt{\frac{6}{5}} & \frac{6\sqrt{6}}{35} \\ -\frac{3}{7} \sqrt{\frac{6}{5}} & \frac{3}{7} & -\frac{6\sqrt{5}}{35} \\ \frac{6\sqrt{6}}{35} & -\frac{6\sqrt{5}}{35} & \frac{12}{35} \end{pmatrix} \]

(46)
| index | $[f]_{cs}$ | $(\lambda\mu)_{cs} S$ |
|-------|-----------|-----------------|
| 1     | [222]     | (0 0) 1         |
| 2     | [222]     | (0 0) 3         |
| 3     | [221]     | (0 0) 0         |
| 4     | [221]     | (0 0) 2         |
| 5     | [221]     | (0 1) $\frac{1}{3}$ |
| 6     | [221]     | (0 1) $\frac{2}{3}$ |
| 7     | [221]     | (0 1) $\frac{4}{3}$ |
| 8     | [211]     | (0 1) $\frac{1}{2}$ |
| 9     | [211]     | (0 1) $\frac{3}{2}$ |
| 10    | [22]      | (0 2) 0         |
| 11    | [22]      | (0 2) 2         |
| 12    | [22]      | (1 0) 1         |
| 13    | [211]     | (0 2) 1         |
| 14    | [211]     | (1 0) 0         |
| 15    | [211]     | (1 0) 1         |
| 16    | [211]     | (1 0) 2         |
| 17    | [111]     | (0 2) 0         |
| 18    | [111]     | (1 0) 1         |
| 19    | [21]      | (1 1) $\frac{1}{3}$ |
| 20    | [21]      | (1 1) $\frac{2}{3}$ |
| 21    | [21]      | (0 0) $\frac{4}{3}$ |
| 22    | [111]     | (1 1) $\frac{1}{3}$ |
| 23    | [111]     | (0 0) $\frac{4}{3}$ |
| 24    | [2]       | (2 0) 1         |
| 25    | [2]       | (0 1) 0         |
| 26    | [1]       | (2 0) 0         |
| 27    | [1]       | (0 1) 1         |
| 28    | [1]       | (1 0) 0         |
| 29    | [1]       | (0 0) 1         |

Table 10: Index of $SU(6)_{cs}$ $(\lambda\mu)_{c} S$ representations.
Table 11: Cfps for four particles $SU(6)_{cs} (\lambda\mu)_{c} S$ representations $\rightarrow [21]$.

| [22] (02)$_c$ 2$S = 0$ $\rightarrow$ [21] | 1 | 1 | 1 |
| [22] (02)$_c$ 2$S = 4$ $\rightarrow$ [21] | 0.57735027 | -0.57735027 | -0.57735027 |
| [22] (10)$_c$ 2$S = 2$ $\rightarrow$ [21] | 0.70710678 | -0.70710678 | -0.70710678 |
| [211] (02)$_c$ 2$S = 2$ $\rightarrow$ [11] | 0.81649658 | 0.40824829 | 0.40824829 |
| [211] (10)$_c$ 2$S = 0$ $\rightarrow$ [21] | 0.70710678 | 1 |
| [211] (10)$_c$ 2$S = 2$ $\rightarrow$ [21] | 0.81649658 | 0.40824829 | 0.40824829 |
| [211] (10)$_c$ 2$S = 4$ $\rightarrow$ [21] | 1 |

Table 12: Cfps for four particles $SU(6)_{cs} (\lambda\mu)_{c} S$ representations $\rightarrow [111]$.

| [22] (01)$_c$ 2$S = 1$ $\rightarrow$ [22] | 0.86602540 | 0.61237244 | 0.79056942 |
| [22] (01)$_c$ 2$S = 3$ $\rightarrow$ [22] | -0.64549722 | -0.64549722 | 0.40824829 |
| [22] (01)$_c$ 2$S = 5$ $\rightarrow$ [22] | 1 |

Table 13: Cfps for five particles $SU(6)_{cs} (\lambda\mu)_{c} S$ representations $\rightarrow [22]$.

| [221] (01)$_c$ 2$S = 1$ $\rightarrow$ [221] | 0.40824829 | -0.40824829 | -0.81649658 |
| [221] (01)$_c$ 2$S = 3$ $\rightarrow$ [221] | -0.64549722 | 0.64549722 | 0.40824829 |
| [221] (01)$_c$ 2$S = 5$ $\rightarrow$ [221] | 1 |
| [2111] (01)$_c$ 2$S = 1$ $\rightarrow$ [211] | 0.57735027 | -0.57735027 | 0.57735027 |
| [2111] (01)$_c$ 2$S = 3$ $\rightarrow$ [211] | 0.73029674 | -0.36514837 | 0.57735027 |

Table 14: Cfps for five particles $SU(6)_{cs} (\lambda\mu)_{c} S$ representations $\rightarrow [211]$.

| [2111] (01)$_c$ 2$S = 1$ $\rightarrow$ [2111] | -0.77459667 | -0.63245553 | 1 |
| [2111] (01)$_c$ 2$S = 3$ $\rightarrow$ [2111] | 0.63245553 | 1 |

Table 15: Cfps for five particles $SU(6)_{cs} (\lambda\mu)_{c} S$ representations $\rightarrow [111]$.

| [222] (00)$_c$ 2$S = 2$ $\rightarrow$ [22] | 0.74535599 | 0.66666667 | 0.66666667 |
| [222] (00)$_c$ 2$S = 6$ $\rightarrow$ [22] | 1 |
| [2211] (00)$_c$ 2$S = 0$ $\rightarrow$ [221] | 1 |
| [2211] (00)$_c$ 2$S = 4$ $\rightarrow$ [221] | 0.8 | -0.6 |

Table 16: Cfps for six particles $SU(6)_{cs} (\lambda\mu)_{c} S$ representations $\rightarrow [2211]$.

| [2211] (00)$_c$ 2$S = 0$ $\rightarrow$ [2211] | -1 |
| [2211] (00)$_c$ 2$S = 4$ $\rightarrow$ [2211] | 1 |

Table 17: Cfps for six particles $SU(6)_{cs} (\lambda\mu)_{c} S$ representations $\rightarrow [2111]$. 
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