Skyrmions induced by dissipationless drag in $U(1) \times U(1)$ superconductors

Julien Garaud,1,2 Karl A. H. Sellin,2 Juha Jäykkä,3 and Egor Babaev2,1

1Department of Physics, University of Massachusetts Amherst, MA 01003 USA
2Department of Theoretical Physics, Royal Institute of Technology, Stockholm, SE-10691 Sweden
3Nordita, KTH Royal Institute of Technology and Stockholm University, Stockholm, SE-10691 Sweden

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Rather generically, multicomponent superconductors and superfluids have intercomponent current-current interaction. We show that in superconductors with substantially strong intercomponent drag interaction, the topological defects which form in external field are characterized by a skyrmionic topological charge. We then demonstrate that they can be distinguished from ordinary vortex matter by a very characteristic magnetization process due to the dipolar nature of inter-skyrmion forces. The results provide an experimental signature to confirm or rule out the formation of fractional vortices (i.e., fractional skyrmions) in $U(1)$ superconductors.

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In multicomponent superconductors and superfluids, the intercomponent current-current interaction is rather generic. It usually assumes the form of the scalar product of supercurrents in the two components $F_{a} \propto \mathbf{J}_{1} \cdot \mathbf{J}_{2}$. This kind of interaction between components can have various microscopic origins. It was discussed in connection with $^3$He-$^4$He mixtures,1 components of order parameters of spin-triplet superfluids and superconductors,2–5 hadronic superfluids in neutron stars,6–10 in metallic hydrogen and deuterium,11,12 in ultracold atomic mixtures13,14 and strongly correlated atomic mixtures in optical lattices.15 In the later case it was shown that it could be tuned to have arbitrary strength (in relative units).15 This kind of interaction for example affects rotational response of neutron stars12 and phase transitions, phase diagrams and rotational response of superfluid mixtures.12,16–21 Despite the generic character of such interaction, much less is known about its effect on the properties of topological excitations and magnetic response, beyond the simplest London approximation. Especially, little is known about collective properties of such defects. Here, we address this problem. We show that beyond a certain interaction threshold, the topological defects in the system acquire a skyrmionic topological charge. This results in long-range inter-skyrmion forces which alter dramatically the collective behaviour of vortex matter.

Note that current-current interaction is fourth order in the order parameters densities and second order in their derivatives. Importantly, it is not positively defined. Because the total free energy is positively defined, the drag term should come with other high-powers terms consistent with the $U(1) \times U(1)$ symmetry. Details of the model and how it relates to usual London models are discussed in Appendix A. The precise form of these terms is not principally important for the purpose of this work, so we investigate a minimal Ginzburg-Landau model (GL), which is positively defined and has the correct London limit

$$F = \frac{B^2}{2} + \sum_{a=1,2} \left( \frac{1}{2} |\mathbf{D}_a|^2 + \alpha_a |\psi_a|^2 + \frac{1}{2} |\beta_a \psi_a|^2 \right)$$

(1a)

$$+ \frac{\nu}{2} \left[ \text{Im}(\psi_a^\dagger \mathbf{D}_a \psi_a) + \text{Im}(\psi_\alpha^\dagger \mathbf{D}_\alpha \psi_\alpha) \right]^2.$$  

(1b)

Here, $\psi_a = |\psi_a| e^{i\varphi_a}$ are complex fields representing the independently conserved superconducting condensates denoted by indices $a = 1, 2$. The term (1b) contains the intercomponent current interaction, as well as higher order terms which makes the free energy bounded from below. Besides the drag interaction, the condensates are coupled by electromagnetic interactions in the kinetic terms $\mathbf{D} = \nabla + ie \mathbf{A}$. We set the Cooper pair charge as twice the electronic charge, then in these units the coupling constant $e$ parametrizes the London penetration length of the magnetic field $B = \nabla \times \mathbf{A}$, and the supercurrent reads as $\mathbf{J} \equiv \sum_a \mathbf{J}_a = \left[ |\psi_a|^2 \right] \mathbf{J}_a = \sum_a \text{Im}(\psi_a^\dagger \mathbf{D}_a \psi_a)$. In connection with spin-triplet systems such models are discussed in the situations where the variations of the relative phase $\varphi_2 - \varphi_1$ of the condensates variations are associated with spin degrees of freedom. The drag interaction is then associated with the spin stiffness.2,3

In this work we consider a two-dimensional model. The discussions thus also apply to three-dimensional systems invariant along the direction normal to the plane. The elementary topological excitations of the model are fractional vortices. These are field configurations with a $2\pi$ phase winding only in one phase (e.g. $\varphi_1$ has $\oint \nabla \varphi_1 = 2\pi$ winding while $\oint \nabla \varphi_2 = 0$). A fractional vortex in the $a$ condensate, carries a fraction of flux quantum $\Phi_a = \oint A_d \ell = |\psi_a|^2 \frac{\ell}{2\pi}$ with the flux quantum $\Phi_0 = 2\pi/e$ and the total superfluid density $\varrho^2 = \sum_a |\psi_a|^2$. Note that this flux quantization is the same as in two-component superconductors without drag.22 Fractional vortices have logarithmically divergent energy. However, a composite vortex being the bound state of fractional vortices in both condensates (each phase $\varphi_a$ winds $2\pi$) has finite energy and carries an inte-
The interacting energies $E_{ab}$, between vortices in the $a$ and $b$ condensates, are expressed in units of $2\pi|\psi_1|^2|\psi_2|^2/\theta^2$. $K_0$ is the modified Bessel of second kind and $R$ denotes the system size while the parameters $m$ and $w$ are $m = |\psi_1|^2/\theta^2$ and $w = 1 + \nu\theta^2$. $\lambda = \frac{1}{e\sqrt{w\theta^2}}$ is the penetration length of the magnetic field. For vanishing drag ($w = 1$) the minimum energy corresponds to an axially symmetric state of two co-centred fractional vortices. The Coulomb and Yukawa contributions in $E_{12}$ interaction compensate at $x = 0$.\textsuperscript{25} The drag term (1b) (i.e. when $w > 1$) penalizes co-directed currents so the Coulomb and Yukawa contributions of the interacting energy $E_{12}$ no longer cancel at $x = 0$ but at some finite separation. In the case of half-quantum vortices this process was studied in detail in London model.\textsuperscript{4}

Here we investigate the structure of single- and multi-vortex states, beyond the London limit. To this end we numerically minimize the free energy (1) within a finite element framework provided by the Freefem++ library.\textsuperscript{24} See technical details in Appendix C. We find that in contrast to the London limit, weak drag does not produce numerically detectable splitting of vortex cores. This is connected with the existence of finite cores where the current is modulated by a density suppression. Larger drag splits a composite vortex into a bound state of well separated fractional vortices. This is shown on Fig. 1. Note that a single fractional vortex has non trivial structure. In particular its magnetic field is not exponentially localized and can exhibit flux inversion.\textsuperscript{25} Fig. 1 shows that some of the features of isolated fractional vortices, reported in\textsuperscript{25} such as w-shaped modulation of densities, are preserved in the split composite vortex.

In general in multicomponent superconductors there could be terms which break the U(1) symmetry explicitly. A typical example is $-\eta|\psi_1||\psi_2|\cos\varphi_{12}$. Such terms result in asymptotically linear confinement of fractional vortices. We find that when such terms are not very strong, the splitting of cores is still present as shown on (see Fig. 1-G). In such a case dipolar forces are still present, but suppressed at the Josephson length.

The bound state of well separated fractional vortices is a skyrmion.\textsuperscript{23} This follows from mapping the two-component model (1) to an easy-plane non-linear $\sigma$-model.\textsuperscript{11,26} There, the pseudo-spin unit vector $\mathbf{n}$ is the projection of superconducting condensates on spin-1/2 Pauli matrices $\sigma$: $\mathbf{n} = \frac{\mathbf{\Psi} \times \mathbf{\nabla}\mathbf{\Psi}}{\sqrt{\mathbf{\nabla}\mathbf{\Psi} \cdot \mathbf{\nabla}\mathbf{\Psi}}}$ where $\mathbf{\Psi} = (\psi_1^*, \psi_2^*)$. When there is non-zero drag, the free energy (1) can be written in $\mathbf{n}$ representation as

$$\mathcal{F} = \frac{1}{2}(\nabla\theta)^2 + \frac{\theta^2}{8}\delta_{ij}\rho_{ij}\partial_i\rho_{ij} + \frac{\mathbf{J}^2}{2\epsilon w\theta^2} + V(\theta, n_z)$$

$$+ \frac{1}{2e^2} \left[ \epsilon_{ijk} \left( \partial_i \left( \frac{J_j}{e\theta w} \right) - \frac{1}{4} \epsilon_{abc}\rho_{ij}\partial_a\rho_{bc} \right) \right]^2,$$  \textsuperscript{(3)}

where $\epsilon$ is the Levi-Civita symbol and $V$ stands for the potential terms in (1a) (see Appendix A, for details of this derivation). The pseudo-spin is a map $\mathbf{n}: S^2 \rightarrow S^2$, classified by the homotopy class $\pi_2(S^2) \in \mathbb{Z}$, thus defining the integer valued topological (skyrmionic) charge $Q(\mathbf{n}) = \frac{1}{4\pi} \int_{S^2} \mathbf{\partial}_i \mathbf{n} \times \mathbf{\partial}_j \mathbf{n} \, dz dy$. Ordinary (compos-
ite) vortices with a single core $\Psi = 0$, have $Q = 0$. Here the core-split vortices have non-trivial skyrmionic charge $Q = N$, the number of flux quanta. The quantization of $Q$ follows from the flux quantization, and $\Psi = Q\Phi_0$ as long as cores are split ($\Psi \neq 0$).

The calculated pseudo-spin texture of $\mathbf{n}$ is shown on panel (H) in Fig. 1. Numerically calculated topological charge was found to be integer (with a negligible error of order $10^{-4}$). Note that these skyrmions are quite different from the skyrmions or non-axially symmetric vortices considered in superconducting states with different number of components and symmetries. In particular the structural differences in these skyrmions dictate different inter-skyrmion forces. This warrants investigation of a state of such a superconductor in external field, which we address in the following.

The mapping of fractional vortices to Coulomb charges (2) suggests that there will be asymptotically power-law inter-Skyrmion dipolar interaction forces (attractive for certain orientations and repulsive for other). Indeed the long-range Coulomb interaction originates in the phase difference mode $\varphi_{12} \equiv \varphi_2 - \varphi_1$. For the pair of fractional vortices it has a clear dipole-like structure shown on Fig. 1-(D). The total interaction forces, beyond the London limit do not reduce to Coulomb and Yukawa forces and are especially complicated at shorter distances due to the presence of density modes and Skyrme terms in (3). To investigate multi-quantum states we compute configurations carrying several flux quanta by energy minimization. First, as displayed in the first line in Fig. 2, they can form compact ‘checkerboard’ cluster. Unlike type-1.5 vortex clusters, where (composite) vortices can form cluster with inner triangular ordering,\textsuperscript{38–40} the dipolar-attraction driven structures have compact lattices with two interlaced square lattices. Other kind of structures which we found for few vortex states are loop- and stripe- like structures. These are shown on Fig. 3 and details about these configurations are included Appendix B. Some of these configurations are metastable local minima. The trend which we observed is that with increasing the drag coupling, multiple quanta configurations become more compact. Remarkably some of the vortex structures which we obtain are quite similar to those appearing in the easy-plane baby-Skyrme model consisting of the pseudo-spin $\mathbf{n}$ alone.\textsuperscript{42} This similarity in structures is an interesting fact which could not be a priori expected because $\mathbf{n}$ represents only a part of the degrees of freedom of GL theory (3), and does not account for all intervortex interaction forces. Moreover, at short length-scales, the GL model is certainly differently different from Skyrme model.\textsuperscript{11} Our observations demonstrate that at least in two dimensions there is a very close relationship between structure formation of topological defects in multicomponent superconductors and in pure baby-Skyrme models. Besides that we find that structure formation exhibit also complicated octagonal loop-like periodic structure as in the first line in Fig. 4. Their elementary cell carries $Q = 4$ flux quanta, and assumes octagonal geometry as a result of rotated underlying square fractional vortex structures.

Since the dipolar interactions are long-range they should dominate the tail of inter-skyrmion interactions. We therefore examine how much of the structure formation can be reproduced in the toy model of interacting point charges (2). To this end we perform Monte Carlo (MC) simulations using the Metropolis algorithm with parallel tempering.\textsuperscript{43} Although the point-charge model does not perfectly capture all the underlying physics, it reproduces some aspects of the structures obtained beyond the London limit (see Fig. 2 and Fig. 4). Moreover, the MC approach allows to investigate how the ordering depends on temperature. As shown on Fig. 2-(D) and Fig. 4-(D), thermal fluctuations can cause unbinding of the crystalline multi-quantum skyrmionic bound states held by dipolar forces. However fractional vortices are still paired and constitute well-defined skyrmions in higher temperature phases where there is no lattice structure.

Finally we address the magnetization process of the skyrmionic state. To this end we simulate the Gibbs free energy $\mathcal{G} = \mathcal{F} - \mathbf{B} \cdot \mathbf{H}$ of the system (1), on a finite domain in an increasing external field $\mathbf{H} = H\mathbf{e}_z$. Here, finite differences are used instead of finite elements, and a quasi-Newton (BFGS) method instead of conjugate gradients. For details, see\textsuperscript{44} and Appendix C. The magneti-
zation process of the skyrmionic states is quite specific. It can be easily distinguished from other unconventional magnetization processes such as those of chiral $p$-wave superconductors with multidomains,\textsuperscript{45} entropically stabilized square lattices,\textsuperscript{46} and type-1.5 superconductors.\textsuperscript{38,40} As shown on Fig. 5, it is heavily influenced by the existence of dipolar forces. In these simulations we typically observed that multi-skyrmion domains bound by dipolar forces are formed near boundaries. These domains are attracted to boundaries by long range dipolar interaction with image charges. This crucially modifies Bean-Livingston barrier physics because dipolar attraction to the image “anti-skyrmions” has longer range than the repulsion from the boundary due to surface Meissner current. These domains gradually fill the system until the repulsion from the boundary due to surface Meissner current. These domains gradually fill the system until merging to form a (checkerboard) square lattice of fractional vortices. When the field is increased further the density of skyrmions in the square lattice grows. Importantly, during the magnetization process, the skyrmionic charge does not change in integer steps. When the condensates are not equivalent there is a layer of one kind of fractional vortices (or half-skyrmions) near boundaries as can be seen in Fig. 5. This is in agreement with the thermodynamical stability of fractional vortices near boundaries demonstrated by Silaev, in the London limit without drag.\textsuperscript{36}

In conclusion we investigated topological defects and magnetic response of $U(1) \times U(1)$ superconductors with dissipationless drag, beyond the commonly used London approximation. In contrast to the London limit, it requires a critical strength of dissipationless drag to form unconventional split vortex solutions. We demonstrated that split fractional vortices in this model have a well defined skyrmionic charge. We established that, when the model is $U(1) \times U(1)$ or softly-broken $U(1) \times U(1)$, the vortex lattice structure is dominated by the long-range dipolar inter-Skyrmion forces. This results in unconventional magnetic response in low fields which features lack of hexagonal vortex lattice and formation of a layer of square lattice growing inward from boundaries of the sample. This magnetization process can be easily identified for example in scanning SQUID measurements and discriminated from other models for $p$-wave superconductivity which by contrast predict hexagonal vortex lattices in low fields and square lattice in high fields. It can also be straightforwardly distinguished from that of ordinary single-component type-II superconductors, or multicomponent type-1.5 superconductors or chiral $p$-wave multi-domain superconductors. For example the magnetic behavior of the putative triplet superconductor Sr$_2$RuO$_4$ is nontrivial, featuring phase separation.\textsuperscript{47–51} However since square vortex lattices were observed only at elevated fields and no boundary vortex states were reported, it is inconsistent with models which have long-range skyrmionic forces.

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Figure 3. (Color online)– Profile of the magnetic field for various bound states of vortices in the model (1), carrying $N = 8, 10, 16, 8$ and 4 flux quanta respectively. The corresponding potential parameters and details of the other physical quantities are given in Appendix B. Note that some regimes have extra bi-quadratic density potential term ($\sim |\psi_1|^2 |\psi_2|^2$) which is not essential but enriches the observed structures.

Figure 4. (Color online) A structure carrying $Q = 16$ flux quanta. The elementary cell here is a $Q = 4$ skyrmion. The parameters are $(\alpha_\epsilon, \beta_\epsilon) = (-5.0, 5.0)$ with $\epsilon = 0.6$ and the drag coupling $\nu = 2.0$. Displayed quantities are the same as in Fig. 2. Lower panel shows (D) shows a Monte Carlo simulation of sixteen particles of each kind for 0.036 particles per surface area. In the low temperature phase the fractional vortices are paired and ordered in a lattice, and for higher temperature the lattice melts but the vortices are still paired.
Figure 5. (Color online) – Sequences of the Skyrmionic states in the magnetization process of a finite sample in slowly increased magnetic flux. Corresponding values of the applied flux are respectively 96, 129, 201, 258 and 381 (in the unit of the flux quantum). Parameters of the Ginzburg-Landau free energy are the same as in Fig. 2. First line shows $|\psi_1|^2$, second line $|\psi_2|^2$ while the third displays the magnetic field $B$. The peaks of different intensities in the magnetic field, correspond to vortices carrying different fractions of flux quantum. Note that there is a layer of half-Skyrmions near boundary. This is consistent with the thermodynamic stability of fractional vortices near boundaries as discussed in.\(^3^6\) Animations of the magnetization process are available as online Supplemental Material.\(^3^7\)

at National Supercomputer Center at Linkoping, Sweden.

Appendix A: Details of theoretical framework

In two-component superconductors, the elementary topological excitations are fractional vortices. These are field configurations having a $2\pi$ phase winding only in one phase (e.g. $\varphi_1$ has $\oint \nabla \varphi_1 = 2\pi$ winding while $\oint \nabla \varphi_2 = 0$). The physics of fractional vortices, as well as the role of the intercomponent dissipationless drag can be enlightened by rewriting the theory in terms of charged and neutral modes. Here we derive the interaction between fractional vortices, for a two-component system. In particular this shows how, in the London limit, fractional vortices can be treated as point particles with Coulomb and Yukawa interactions. The Ginzburg-Landau free energy functional reads

$$\mathcal{F} = \frac{1}{2}(\nabla \times A)^2 + \sum_a \frac{1}{2}|D\psi_a|^2$$ \hspace{1cm} (A.1a)

$$\quad + \sum_a \alpha_a |\psi_a|^2 + \frac{1}{2} \beta_a |\psi_a|^4$$ \hspace{1cm} (A.1b)

$$\quad + \frac{1}{2} \gamma |\psi_1|^2 |\psi_2|^2$$ \hspace{1cm} (A.1c)

$$\quad + \frac{\nu}{2} |\text{Im}(\psi_1^* D\psi_1) + \text{Im}(\psi_2^* D\psi_2)|^2.$$ \hspace{1cm} (A.1d)

Note that for completeness we added bi-quadratic density coupling (A.1c) to the potential energy. Obviously, its effect is to enforce core splitting of fractional (when $\gamma > 0$) vortices. Since we mostly focus on the role of the drag interaction, the bi-quadratic density term is introduced here for sake of completeness rather than an essential ingredient of the physics we discuss.
1. Parametrization of the intercomponent drag and its London limit

Traditionally, the intercomponent current-current interaction is parametrized as the scalar product of supercurrents of two components $F_d \propto J_1 \cdot J_2$. Beyond the London limit, such a term reads explicitly $F_d \propto \text{Im}(\psi_1^* D\psi_1) \cdot \text{Im}(\psi_2^* D\psi_2)$. This term is fourth order in the order parameters densities and second order in their derivatives, moreover it is not positively defined. This leads to an unphysical instability: by creating strong counter-directed currents and increasing density, in a minimal GL model with such a term, makes free energy negative and unbounded from below. Thus this term should come with other high-power terms consistent with the symmetry, which make the total free energy positively defined. The precise form of these terms is not principally important for the purpose of this work, so we choose to use (A.1d), which is obviously positive. However one should also make sure that this term has the proper London limit. There, the free energy functional (A.1) reads as

$$F = \frac{1}{2} (\nabla \times A)^2 + \sum_{a=1}^2 \frac{1}{2} D\psi_a^2 + \nu \left[ \text{Im}(\psi_1^* D\psi_1) + \text{Im}(\psi_2^* D\psi_2) \right]^2.$$  

(A.2a)

Since the densities are constant, the covariant derivative reads as $D\psi_a = i\psi_a (\nabla \varphi_a + eA)$, expanding the drag term (A.2b) and collecting various orders, the free energy assumes the form typically used for discussing the problem in the London limit

$$F = \frac{1}{2} (\nabla \times A)^2 + \sum_{a=1}^2 \frac{1}{2} \rho_{aa} (\nabla \varphi_a + eA)^2 + \rho_d (\nabla \varphi_1 + eA) \cdot (\nabla \varphi_2 + eA).$$  

(A.3b)

Where the prefactors are

$$\rho_{aa} = |\psi_a|^2 (1 + \nu |\psi_a|^2)$$

$$\rho_d = \nu |\psi_1|^2 |\psi_2|^2.$$  

(A.4)

The term (A.3b) is the scalar product of the supercurrents of two components. Thus our parametrization (A.1d) of intercomponent current-current interaction has the conventional London limit.

2. Derivation of neutral and charged modes

To understand the role of the fundamental excitations (i.e. fractional vortices), the Ginzburg-Landau free energy (A.1) can be rewritten into charged and neutral modes by expanding the kinetic term (A.1a) and the drag term (A.1d)

$$F = \frac{1}{2} (\nabla \times A)^2 + \frac{J^2}{2\epsilon^2 w^2}$$

$$+ \sum_{a=1}^2 \frac{1}{2} (\nabla |\psi_a|^2)^2 + \alpha_a |\psi_a|^2 + \frac{\beta_a}{2} |\psi_a|^4$$

(A.5b)

$$+ \frac{\gamma |\psi_1|^2 |\psi_2|^2}{2\epsilon^2 w^2}$$

(A.5c)

$$+ \frac{|\psi_1|^2 |\psi_2|^2 (\nabla \varphi_{12})^2}{2\epsilon^2 w^2}.$$  

(A.5d)

Here $\varphi_{12} = \varphi_2 - \varphi_1$ is the phase difference and

$$w = 1 + \nu \varphi^2$$

and $\varphi^2 = \sum_a |\psi_a|^2$. (A.6)

The supercurrent defined from the Ampère’s equation $\nabla \times B = J = 0$, reads as

$$J_a = \epsilon w \varphi^2 A + \sum_{\alpha=1}^2 \left[ |\psi_\alpha|^2 \nabla \varphi_\alpha \right.$$  

(A.7a)

with the band index $\beta \neq \alpha$. The term on the second line is the current of the component $\alpha$ induced (dragged) by the component $\beta$. Assuming phase winding in all components and since far away from a vortex $J$ decays exponentially, the magnetic flux reads as

$$\Phi = \oint B \text{d}S = \oint A \text{d}\ell$$

$$= \frac{1}{\epsilon^2 w^2 \varphi^2} \oint \left( J - \nu \epsilon \sum_a |\psi_a|^2 \nabla \varphi_a \right) \text{d}\ell$$

(A.9)

where $\Phi_0 = 2\pi/\epsilon$ is the flux quantum and the closed path integration is done so that the flux is positive. The fraction of flux $|\psi_a|^2 \Phi_0/\varphi^2$ carried is the same as that of two-component superconductors without drag.22 The London limit, assumes that $|\psi_a| = \text{const}$ everywhere in space except small vortex core sharp cut-off. The expression (A.5) thus further simplifies

$$F = \frac{1}{2} \left( B^2 + \frac{J^2}{c^2 w^2 \nabla \times B} \right)^2$$

(A.10a)

$$+ \frac{|\psi_1|^2 |\psi_2|^2 (\nabla \varphi_{12})^2}{2\epsilon^2 w^2}.$$  

(A.10b)
where the Ampère’s law has been used to replace the current in (A.10a). The interaction energy of two non-overlapping fractional vortices can be approximated in this London limit by considering charged (A.10a) and neutral modes (A.10b), separately. With the identity
\[
|\nabla \times \mathbf{B}|^2 = \mathbf{B} \cdot \nabla \times \nabla \times \mathbf{B} - \nabla \cdot (\mathbf{B} \times \nabla \times \mathbf{B}),
\] (A.11)
the energy of the charged sector (A.10a) finally reads
\[
F_{\text{charged}} = \int \frac{B^2}{2} \left( \frac{1}{e^2 w^2} |\nabla \times \nabla \times \mathbf{B}| \right) .
\] (A.12)
The London equation for a (point-like) vortex placed at \(x_a\) and carrying a flux \(\Phi_a\) is
\[
\frac{1}{e^2 w^2} \nabla \times \nabla \times \mathbf{B} + B = \Phi_a \delta(x - x_a),
\] (A.13)
and its solution is
\[
\mathbf{B}(x) = \frac{\Phi_a e^2 w^2 K_0 \left( \frac{|x - x_a|}{\lambda} \right)}{2\pi}.
\] (A.14)
Here the London penetration length is \(\lambda = \frac{1}{\sqrt{|\psi|^2}}\) and \(K_0\) is the modified Bessel function of the second kind.

For two vortices located at \(x_a\) and \(x_b\), and carrying fluxes \(\Phi_a\) and \(\Phi_b\), the source term in London equation reads \(\Phi_a \delta(x - x_a) + \Phi_b \delta(x - x_b)\) and the magnetic field is the superposition of two contributions \(\mathbf{B}(x) = \mathbf{B}_a(x) + \mathbf{B}_b(x)\). Thus
\[
F_{\text{charged}} = \int \frac{1}{2} (\mathbf{B}_a + \mathbf{B}_b)(\Phi_a \delta(x - x_a) + \Phi_b \delta(x - x_b))
= \Phi_a \Phi_b e^2 w^2 K_0 \left( \frac{|x_2 - x_1|}{\lambda} \right) + E_{ab} + E_{ba}.
\] (A.15)
and \(E_{ab} \equiv \int \mathbf{B}_a(x_a) \Phi_a / 2\) denotes the (self-)energy of the vortex \(a\). Finally, the interaction energy of two vortices in components \(a, b\) reads
\[
E_{ab}^{\text{(int),charged}} = \frac{2\pi |\psi_a|^2 |\psi_b|^2}{\rho^2} K_0 \left( \frac{|x_a - x_b|}{\lambda} \right) .
\] (A.16)
The interaction of the charged sector is thus a Yukawa-like interaction given by the modified Bessel function. If we do not consider anti-vortices it is always positive (for any \(a, b\)), then it gives repulsive interaction between any kind of fractional vortices. On the other hand, the interaction through the neutral sector is logarithmic. It is attractive (resp. repulsive) for fractional vortices of the different (resp. same) kind. The energy associated with the neutral mode (A.10b) reads
\[
F_{\text{neutral}} = \frac{|\psi_1|^2 |\psi_2|^2}{2\rho^2} \int (\nabla \phi_{12})^2 .
\] (A.17)
A phase winding around some singularity located at the point \(x_a\), is (at sufficiently large distance) well approximated by \(\phi_a = \theta\). Thus
\[
\nabla \phi_a = \frac{e_a}{|x - x_a|} = z \times \nabla \ln |x - x_a| .
\] (A.18)
To evaluate the interaction between fractional vortices in different condensates and respectively located at \(x_a\) and \(x_b\), the neutral sector is expanded
\[
F_{\text{neutral}} = \frac{|\psi_1|^2 |\psi_2|^2}{2\rho^2} \int (\nabla \phi_a)^2 + (\nabla \phi_b)^2
- 2 \nabla \phi_a \cdot \nabla \phi_b .
\] (A.19)
Thus the interacting part reads
\[
E_{ab}^{\text{int,neutral}} = -\frac{|\psi_1|^2 |\psi_2|^2}{\rho^2} \int \nabla \phi_a \cdot \nabla \phi_b
= \frac{|\psi_1|^2 |\psi_2|^2}{\rho^2} \int \phi \Delta \phi
= 2\pi \frac{|\psi_1|^2 |\psi_2|^2}{\rho^2} \ln |x_b - x_a| .
\] (A.20)
Similarly, the interaction between two vortices in the same condensate \(a\) is computed by requiring that the phase is the sum of the individual phases \(\phi_a = \phi_a^{(1)} + \phi_a^{(2)}\), while \(\phi_b = 0\). Then the interaction reads
\[
E_{ab}^{\text{int,neutral}} = -2\pi \frac{|\psi_1|^2 |\psi_2|^2}{\rho^2} \ln |x_b^{(a)} - x_a^{(1)}| .
\] (A.21)
To summarize, the interaction of vortices in different condensates is then
\[
E_{ab}^{\text{int}} = \frac{|\psi_1|^2 |\psi_2|^2}{\rho^2} \left( \ln \frac{R}{R} + w K_0 \left( \frac{r}{\lambda} \right) \right) ,
\] (A.22)
while interactions of vortices of similar condensates are
\[
E_{ab}^{\text{int}} = -\frac{|\psi_1|^2 |\psi_2|^2}{\rho^2} \ln \frac{r}{R} + \frac{w |\psi_a|^4}{\rho^2} K_0 \left( \frac{r}{\lambda} \right) ,
\] (A.23)
with \(r \equiv |x_a - x_b|\) and \(R\) the sample size. Equations (A.22) and (A.23) give the different interactions between fractional vortices. Finally, choosing the energy scale to be \(2\pi |\psi_1|^2 |\psi_2|^2 / \rho^2\) and defining the parameters \(m\) and \(w\) as
\[
w = 1 + n \rho^2 = 1 + n(|\psi_1|^2 + |\psi_2|^2) ,
\] (A.24)
The interaction between fractional vortices in the various condensates reads
\[
E_{11} = \ln \frac{R}{R} + \frac{w m K_0 \left( \frac{r}{\lambda} \right) }{w m K_0 \left( \frac{r}{\lambda} \right) } ,
E_{22} = \ln \frac{R}{R} + \frac{w m K_0 \left( \frac{r}{\lambda} \right) }{w m K_0 \left( \frac{r}{\lambda} \right) } ,
E_{12} = -\ln \frac{R}{R} + \frac{w m K_0 \left( \frac{r}{\lambda} \right) }{w m K_0 \left( \frac{r}{\lambda} \right) } .
\] (A.25)
Thus vortex matter in the London limit of a two-component superconductor with intercomponent drag interaction is described by a 3-parameter family \((m, w, R)\).
gauge invariant current $J_n (A.1)$ in terms of the pseudo-spin and the free energy (A.5) can be written as

$$\varepsilon_n = \rho, \kappa.$$ 

where $\sigma$ is a model. Pauli matrices $\sigma$ the target sphere the plane, while colors give the magnitude of $n$. $J_n$ vanish $\alpha$ on the first line. On the second line, $J_1, J_2$ and $x \times \nabla \varphi_{12}$. The rightmost panel shows the normalized projection of $n$ onto the plane, while colors give the magnitude of $n_z$. Blue corresponds to the south pole (-1) while red is the north pole (+1) of the target sphere $S^2$.

3. Mapping to an easy-plane non-linear $\sigma$-model

The bound state of well separated fractional vortices is a Skyrmion. This follows from mapping the two-component model (A.1) to an easy-plane non-linear $\sigma$-model.\cite{11, 26} There, the pseudo-spin unit vector $n$ is the projection of superconducting condensates on spin-1/2 Pauli matrices $\sigma$:

$$n \equiv (n_x, n_y, n_z) = \frac{\Psi^* \sigma \Psi}{|\Psi|^2}, \text{ where } \Psi^* = (\psi_1^*, \psi_2^*).$$ 

(A.26)

The following identity is useful to rewrite the free energy (A.1) in terms of the pseudo-spin $n$, total density $\rho$ and gauge invariant current $J$

$$\frac{\partial^2}{4} \partial_k \partial_k n_a + (\nabla \delta)^2 \Psi^2 (A.27)$$

where summation on repeated indices is implied. Using the definition of the current (A.7) and noting that

$$4 \varepsilon_{ijk} \partial_k \left( \sum_a \frac{\psi_a^2}{\varepsilon} \partial_j \varphi_a \right) = \varepsilon_{ijk} \varepsilon_{abc} n_a \partial_k n_b \partial_j n_c, \text{ (A.28)}$$

where $\varepsilon$ is the Levi-Civita symbol, the magnetic field reads

$$B_k = \frac{1}{e} \varepsilon_{ijk} \partial_i \left( \frac{J_j}{\varepsilon \omega g^2} \right) = -\frac{1}{4} \varepsilon_{abc} n_a \partial_k n_b \partial_j n_c, \text{ (A.29)}$$

and the free energy (A.5) can be written as

$$\mathcal{F} = \frac{1}{2} (\nabla \delta)^2 + \frac{\varepsilon^2}{8} \partial_k n_a \partial_k n_a + \frac{J^2}{2 e^2 \varepsilon g^2} + V(\rho, n_z)$$

$$+ \frac{1}{2 e^2} \left[ \varepsilon_{ijk} \left( \frac{J_j}{\varepsilon \omega g^2} - \varepsilon_{abc} n_a \partial_k n_b \partial_j n_c \right) \right]^2, \text{ (A.30)}$$

where $V(\rho, n_z)$ stands for the potential terms (A.1b) and (A.1c). The easy plane potential explicitly reads

$$V(\rho, n_z) = \frac{\rho^2}{2} (a_1 + a_2 n_z) + \frac{\rho^4}{4} (b_1 + 2 b_2 n_z + b_3 n_z^2), \text{ (A.31)}$$

with the coefficients

$$b_1 = \frac{\beta_1 + \beta_2 + \gamma}{2}, \quad b_2 = \frac{\beta_1 - \beta_2}{2}, \quad b_3 = \frac{\beta_1 + \beta_2 - \gamma}{2},$$ 

$$a_1 = \alpha_1 + \alpha_2, \quad a_2 = \alpha_1 - \alpha_2. \quad \text{ (A.32)}$$

The pseudo-spin is a map from the one-point compactification of the plane ($\mathbb{R}^2 \approx S^2$) to the two-sphere target space spanned by $n$. That is $n : S^2 \to S^2$, classified by the homotopy class $\pi_2(S^2) \in \mathbb{Z}$, thus defining the integer valued topological (skyrminionic) charge

$$Q(n) = \frac{1}{4\pi} \int_{\mathbb{R}^2} n \cdot \partial_x n \times \partial_y n \ dx \ dy. \quad \text{ (A.33)}$$

Ordinary (composite) vortices with a single core $\Psi = 0$, have $Q = 0$. Core split vortices, on the other hand, have non-trivial skyrmionic charge $Q \in \mathbb{N}$ (with $N$ coincides with the number of carried flux quanta). The calculated pseudo-spin texture of $n$ is shown on the rightmost panel in Fig. 6. Numerically calculated topological charge was found to be integer (with a negligible error of order $10^{-5}$). It is worth emphasizing that the topological charge (A.33) is an integer, when integrated over the infinite plane $\mathbb{R}^2$, or at least an large enough domain $\Omega \subset \mathbb{R}^2$. By large enough, we understand that the fields should have recovered their ground state values at the boundary. Then the skyrmions shall not interact with the boundary. When the Skyrmion’s size is comparable with the size of the integration domain, truncation error appear and $Q$ is no more integer. Moreover when simulating a finite sample in applied field, in general the skyrmionic topological charge $Q$ will not be integer. This is because in general there are states where only a part of the Skyrmion texture enters the sample.
Figure 7. (Color online) – 8 vortex configuration. Parameters are $(\alpha_1, \beta_1) = (-3.6, 1.0)$ and $(\alpha_2, \beta_2) = (-3.0, 1.0)$ and $\gamma = 0.6$ with $\epsilon = 0.6$. There is no Andreev–Bashkin coupling $\nu = 0.0$ but fractional vortices are split by bi-quadratic density coupling only. Displayed quantities are respectively the magnetic field $B$, $|\psi_1|^2$, $|\psi_2|^2$ and the phase difference $\varphi_{12} \equiv \varphi_2 - \varphi_1$, on the first line. On the second line, $J$, $J_1$, $J_2$ and $z \times \nabla \varphi_{12}$. 

Appendix B: Additional material

The bi-quadratic density interaction (A.1c), in (A.1) also induces core splitting of the fractional vortices, for positive couplings $\gamma$. Unlike the drag term which induces splitting by energetically penalizing co-flowing currents, bi-quadratic density coupling (with $\gamma > 0$) penalizes core overlap directly. Indeed, it is energetically preferable to have singularities in each component sitting in different positions. Such a term is in general possible in multi-component systems. Note that when the coupling are strong, it is no more favourable to have coexisting condensates and the superfluid density of a given condensate is completely suppressed (i.e. phase separation).

Unlike the current drag interactions, the physics of the core splitting induced by bi-quadratic densities cannot be captured within the London limit (since it involves only densities). In general combining both dissipationless drag and bi-quadratic density interaction widely enriches the spectrum of various Skyrmionic structures which can be obtained. Figs. 7-12 show detail of multiskyrmion solutions from the main body of the paper.

Appendix C: Numerical Methods

1. Finite element energy minimization

We consider the two-dimensional problem (A.1) defined on the bounded domain $\Omega \subset \mathbb{R}^2$ with $\partial \Omega$ its boundary. In practice we choose $\Omega$ to be a disk. The problem is supplemented by the boundary condition $\mathbf{n} \cdot D\psi_a = 0$ with $\mathbf{n}$ the normal vector to $\partial \Omega$. Physically this condition implies there is no current flowing through the boundary. Since this boundary condition is gauge invariant, additional constraint can be chosen on the boundary to fix the gauge. Our choice is to impose the radial gauge on the boundary $e_\rho \cdot \mathbf{A} = 0$ (note that with our choice of domain, this is equivalent to $\mathbf{n} \cdot \mathbf{A} = 0$). With this choice, (most of) the gauge degrees of freedom are eliminated and the ‘no current flow’ condition separates in two parts

\[ \mathbf{n} \cdot \nabla \psi_a = 0 \quad \text{and} \quad \mathbf{n} \cdot \mathbf{A} = 0. \] (C.1)

Note that these boundary conditions allow a topological defect to escape from the domain. To prevent this in simulations of individual skyrmions or skyrmion groups without applied field, the numerical grid is chosen to be large enough so that the attractive interaction with the boundaries is negligible for a given numerical accuracy. Thus in this method one has to use large numerical grids, which is computationally demanding. The advantage is that it is guaranteed that obtained solutions are not boundary pressure artifacts.

The variational problem is defined for numerical computation using a finite element formulation provided by the Freefem++ library.\textsuperscript{24} Discretization within finite element formulation is done via a (homogeneous) triangulation over $\Omega$, based on Delaunay-Voronoi algorithm. Functions are decomposed on a continuous piecewise quadratic basis on each triangle. The accuracy of such method is controlled through the number of triangles, (we typically used $3 \sim 6 \times 10^4$), the order of expansion of the basis on each triangle (2nd order polynomial basis on each triangle), and also the order of the quadrature formula for the integral on the triangles.

Once the problem is mathematically well defined, a numerical optimization algorithm is used to solve the variational nonlinear problem (i.e. to find the minima of $\mathcal{F}$). We used here a nonlinear conjugate gradient method. The algorithm is iterated until relative variation of the norm of the gradient of the functional $\mathcal{F}$ with respect to all degrees of freedom is less than $10^{-6}$. 


Figure 8. (Color online) – Multiskyrmion carrying $Q = 8$ flux quanta, for identical components $(\alpha, \beta) = (-3.0, 1.0)$ and $\gamma = 0.6$ with $e = 0.8$. The Andreev–Bashkin coupling is $\nu = 1.0$. Displayed quantities are the same as in Fig. 7.

Figure 9. (Color online) – A 8 flux quanta configuration. Displayed quantities and the parameters are the same as in Fig. 7 except for the coupling $\nu = 1$.

Figure 10. (Color online) – A checkerboard cluster with $Q = 10$. Parameters are the same as in Fig. 8 except the gauge coupling $e = 0.6$. 
The initial field configuration carrying $N$ flux quanta is prepared by using an ansatz which imposes phase windings around spatially separated $N$ vortex cores in each condensate.

\[ \psi_a = |\psi_a| e^{i\theta_a}, \]
\[ |\psi_a| = u_a \prod_{k=1}^{N} \left( 1 + \tanh \left( \frac{4}{\xi_a} (R^a_k(x,y) - \xi_a) \right) \right), \]

(C.2)

where $a = 1, 2$ and $u_a$ is the ground state value of each condensate density. The parameters $\xi_a$ parametrize the core size while

\[ \Theta_a(x,y) = \frac{\sum_{k=1}^{N} \tan^{-1} \left( \frac{y - y_k^a}{x - x_k^a} \right)}{\xi_a}, \]
\[ R^a_k(x,y) = \sqrt{(x - x_k^a)^2 + (y - y_k^a)^2}. \]

(C.3)

$(x_k^a, y_k^a)$ determines the position of the core of $k$-th vortex of the $a$-condensate. The starting configuration of the vector potential is determined by solving Ampère's law on the background of the superconducting condensates specified by (C.2)-(C.3). Being a linear equation in $A$, this is an easy operation.

Once the initial configuration defined, all degrees of freedom are relaxed simultaneously, within the ‘no current flow’ boundary conditions discussed previously, to obtain highly accurate solutions of the Ginzburg-Landau equations.

2. Finite difference simulations

In our simulations using finite differences, the energy functional (A.1) is discretized in a gauge-invariance preserving manner using forward differences. For details of the discretization scheme, see. The constant applied external magnetic field $H = He_z$, is fixed by taking advantage of Stokes’s theorem and specifying that $A$ on the boundary satisfies

\[ \nabla \times A = H. \]

Stokes’s theorem then ensures the flux through the system is equal to $\int_{\partial \Omega} H \cdot dS$, but allowing $A$ and hence $B$ to vary arbitrarily inside the system. Note that this leaves gauge degrees of freedom in the system. However, in an energy minimization problem the algorithm only considers the energy which is a gauge-invariant quantity. Thus the possibility of evolving simply by a gauge transformation is eliminated since it does not lower the energy. The boundary condition is the discrete equivalent of $n \cdot D\psi_a = 0$ and ensures that no supercurrent escapes the sample. This boundary is located several lattice points inside the computational lattice. This is the boundary of the sample and outside it, $\psi_i$ are not solved for.

The lattice parameters, $h_i$, control the accuracy of the lattice approximation and the minimization algorithm is considered to be converged whenever the largest discrete gradient in the system is below $10^{-5} \Pi h_i$ or the sup-norm of the discrete gradients is below $10^{-7}$. Some control calculations with a more restrictive convergence criterion were made but with no appreciable change to the solutions.

We typically used domains of 401 × 403 lattices points with lattice spacing of $h_i = 0.1$. As an initial configuration, we set $\psi_a = 0$ outside the superconductor (these values are not part of the minimization process), $A = 0$ everywhere, and $\psi_a = \sqrt{\frac{2}{\beta_a}} \exp i\varphi_a(x,y)$, where phases $\varphi_a(x,y) \in [-\pi, \pi]$ are randomly chosen. At the beginning, therefore, we have $B = 0$ and this corresponds to a zero-field-cooled sample. When we have found a solution at a given external field, the boundary condition for $A$ is updated to reflect the new field and the old solution is used as an initial guess for the next solution. A quasi-
Newton algorithm with BFGS Hessian updates is used to simultaneously solve for all degrees of freedom subject to the boundary conditions at the two different boundaries (one for $A$ and one for $\Psi$). The program itself is an extension of the one used in\textsuperscript{44} (for further details, see\textsuperscript{44} and the relevant references therein).

### 3. Monte-Carlo simulations

In the Monte Carlo simulations, vortices are treated as a system of $N$ point particles of two different colors, interacting with potentials (A.25). The point particles live in a two-dimensional box $L \times L$ so that the number of particles per surface area is $N/L^2$. Periodic boundary conditions are imposed and the interaction is cut at half the box width. Tests with open boundary conditions without a cut-off have been performed and no structural differences are noted as compared to low-density simulations with periodic boundary conditions. Data are acquired during at least $10^4$ sweeps (a sweep constitutes a number of trial moves equal to the number of particles in the box), after an equilibration from a random initial configuration. The Monte Carlo trial moves consists of a single particle displacement, a pairwise displacement of a nearest-neighbours bound pair, or rotation of such a pair. The number of particles remains unchanged during the simulation. Furthermore, the maximal step length of a displacement is controlled such that approximately $10\%$ of the displacement trial moves are accepted. Parallel tempering is used in order for the low-temperature simulations to quickly converge into ordered low-energy states, as a low temperature simulation of these systems can easily be trapped in a metastable state.

The square lattice order parameter is defined as

$$
\Psi_4 = \frac{1}{4N} \sum_{i=1}^{N} \sum_{j=1}^{4} \exp(4i\phi_{ij})
$$

where the sum in $j$ runs over the four nearest neighbors of particle $i$, and $\phi_{ij}$ is the angle of the line joining particles $i, j$ with some arbitrary axis.

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Note that the topological charge is integer only for when a skyrmion is sufficiently far from boundaries. Since when simulating a finite sample in applied field, there are states where only part of the skyrmion texture enters the sample, in general the topological charge $Q$ will not be integer.

Note that formation of checkerboard square lattices for $p$-wave superconductors near $H_{c2}$ were found in. Here we consider a different situation of vortex cluster formed due to attractive dipolar interactions.

See online Supplemental Material http://people.umass.edu/garaud/Webpage/vortex-dipoles.html, for animations of the magnetization process.