THE IGUSA MODULAR FORMS AND “THE SIMPLEST” LORENTZIAN KAC–MOODY ALGEBRAS

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Abstract. We find automorphic corrections for the Lorentzian Kac–Moody algebras with the simplest generalized Cartan matrices of rank 3

\[
A_{1,0} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix} \quad \text{and} \quad A_{1,I} = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.
\]

For \( A_{1,0} \) this correction, which is a generalized Kac-Moody Lie superalgebra, is given by the Igusa \( Sp_4(\mathbb{Z}) \)-modular form \( \chi_{35}(\mathbb{Z}) \) of weight 35, and for \( A_{1,I} \) by some Siegel modular form \( \widetilde{\Delta}_{30}(\mathbb{Z}) \) of weight 30 with respect to a 2-congruence subgroup of \( Sp_4(\mathbb{Z}) \). We find the infinite product expansions for \( \chi_{35}(\mathbb{Z}) \) and \( \widetilde{\Delta}_{30}(\mathbb{Z}) \) and calculate multiplicities of all roots of the corresponding generalized Lorentzian Kac–Moody superalgebras. These multiplicities are given by Fourier coefficients of some Jacobi forms of weight 0 and index 1.

Our method of construction of \( \chi_{35}(\mathbb{Z}) \) and \( \widetilde{\Delta}_{30}(\mathbb{Z}) \) naturally leads to the general and direct construction by infinite product or sum expansions of Siegel modular forms, whose divisors are the Humbert surfaces with fixed discriminants. Existence of these forms was proved by van der Geer in 1982 using some geometrical considerations.

To show the perspective for further study, we announce a list of all hyperbolic symmetric generalized Cartan matrices \( A \) of rank 3 such that \( A \) has elliptic or parabolic type, \( A \) has a lattice Weyl vector, and \( A \) contains the parabolic submatrix \( \tilde{A}_1 \).

0. Introduction

The main starting point of this paper is to find an automorphic correction of the Lorentzian Kac–Moody algebra \( g(A_{1,0}) \) with the generalized Cartan matrix

\[
A_{1,0} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix}.
\]

It is recognized that this matrix is the simplest hyperbolic (i.e., with exactly one negative square) generalized Cartan matrix of rank \( \geq 3 \). It has the smallest possible coefficients for this type of matrices. There are many publications where the Kac–Moody algebra \( g(A_{1,0}) \) was considered. We only mention the paper of Feingold and Frenkel [FF]. In that paper there was the first attempt to connect this algebra with the theory of Siegel modular forms.
The Weyl group of $\mathfrak{g}(A_{1,0})$ is an integral orthogonal group of signature (2,1), and it is isomorphic to the extended modular group $PGL_2(\mathbb{Z})$. The Weyl-Kac denominator formula for $\mathfrak{g}(A_{1,0})$ is

$$\sum_{g \in PGL_2(\mathbb{Z})} \det(g) \exp(2\pi i \text{tr}((gPg^t - P)Z)) = \sum_{\alpha > 0} (1 - \exp(2\pi i \text{tr}((\alpha Z))))^{\text{mult}(\alpha)},$$

(0.1)

where $P = \left( \begin{array}{ccc} 3 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 2 & 2 \\ \frac{1}{2} & 2 & 2 \end{array} \right)$, $Z = \left( \begin{array}{ccc} z_1 & z_2 & z_2 \\ z_2 & z_3 & z_2 \\ z_2 & z_2 & z_3 \end{array} \right)$ is a matrix from the Siegel upper half-plane $\mathbb{H}_2$, and the product is taken over all positive roots of the algebra $\mathfrak{g}(A_{1,0})$ (see [FF]). Unfortunately an exact formula for the multiplicities of roots $\text{mult}(\alpha)$ is not known.

Our point of view on the Lorentzian Kac–Moody algebras theory (see [N7], [N8], [GN1]–[GN3], where we also use some pioneer ideas due to R. Borcherds [B1]–[B6]) is that Lorentzian Kac–Moody algebras of the so called elliptic or parabolic type and with a lattice Weyl vector (one can consider them as “the most symmetric”) have an automorphic correction. It means that for a Kac–Moody algebra $\mathfrak{g}(A)$ of this type, there exists a generalized Kac–Moody superalgebra without odd real simple roots $\mathfrak{g}(A)'$ such that $\mathfrak{g}(A) \subset \mathfrak{g}(A)'$, these algebras have the same rank, have the same real roots system (in particular, any real root is even) and Weyl group, but $\mathfrak{g}(A)'$ has good automorphic properties for its Weyl–Kac–Borcherds denominator function $\Phi(z)$. The last means that $\Phi(z)$ is an automorphic form on the classical hermitian symmetric domain of type IV $\Omega$ which one should consider as the complexification of the hyperbolic space $\mathcal{L}$ where the Weyl group $W$ of $\mathfrak{g}(A)$ acts. (The Weyl group of a hyperbolic Kac–Moody algebra is a subgroup of an orthogonal group of signature $(n,1)$. ) The automorphic form $\Phi(z)$ is invariant with some weight with respect to an arithmetical orthogonal group of signature $(n+1,2)$, which contains the Weyl group $W$.

We remark that for a Lorentzian Kac–Moody algebra $\mathfrak{g}(A)$ its denominator function $\Phi(z)$ is invariant only with respect to a hyperbolic orthogonal group. (For example, the function (0.1) does not contain enough Fourier coefficients to be a Siegel modular form.) Therefore, one should consider corrected algebras as having strictly larger groups of symmetries than Kac–Moody algebras. This is the first advantage of the notion of automorphic correction.

From the point of view of the automorphic form theory, a form realizing an automorphic correction is an automorphic form with a very special Fourier expansion related with the generalized Cartan matrix $A$ (see (3.3), (3.4), and [N8], [GN1] for the general setting). Using automorphic properties of $\Phi(z)$, it is possible to find the infinite product expansion of $\Phi(z)$ and calculate multiplicities of the root spaces decomposition of the corrected algebra $\mathfrak{g}(A)'$. This is the second important advantage of these algebras.

Surprisingly, some well known classical Siegel modular forms enable us to construct automorphic corrections. In the theory of Siegel modular forms of genus 2 there is an important exceptional modular form $\Delta_5(Z)$ of weight 5 with non-trivial character $v : Sp_4(\mathbb{Z}) \to \{\pm 1\}$. This function is defined as the product of all even theta-constants of genus 2 (see (1.6) below) and it delivers an automorphic correction of the Kac–Moody algebra $\mathfrak{g}(A_{1,II})$ where

$$A_{1,II} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ 2 & -2 & 2 \end{pmatrix}$$
is another simple hyperbolic symmetric generalized Cartan matrix of rank 3. This result was proved in [GN1], [GN2], where the first examples of automorphic corrections of elliptic Lorentzian Kac–Moody algebras were found. We proved there that the following infinite product expansion is valid

$$\Delta_5(Z) = (qs)^{\frac{1}{2}} \prod_{n,l,m \in \mathbb{Z}} \left(1 - q^n r^l s^m \right)^{f(4nm - l^2)}.$$  \hspace{1cm} (0.2)

The integral exponents $f(4nm - l^2)$ are defined by Fourier coefficients of a canonical weak Jacobi form of weight 0 and index 1 (see (1.8)–(1.9) below). In (0.2) we use the variables $q = \exp(2\pi i z_1)$, $r = \exp(2\pi i z_2)$, $s = \exp(2\pi i z_3)$.

There exists a formula for the integral exponents $f(4nm - l^2)$ in terms of so-called H. Cohen numbers. It gives us an exact formula for multiplicities of root spaces decomposition of the corrected algebra $g(A_{1, 0})$.

Surprisingly, exactly the same story happens with the simplest Kac–Moody algebra $g(A_{1, 0})$. One of the main results of this paper is that the automorphic correction of $g(A_{1, 0})$ is defined by the first Siegel modular form $\chi_{35}(Z)$ of odd weight. This function of weight 35 was constructed by Igusa in [Ig1] as a linear combination of all “azygous” triplets of theta-constants. Igusa proved that any Siegel modular form (with the trivial character) of odd weight is divisible by the form $\chi_{35}$. We call $\chi_{35}(Z)$ the Igusa modular form.

To prove that $\chi_{35}(Z)$ gives an automorphic correction of the algebra $g(A_{1, 0})$ and to find the infinite product formula for the Igusa modular form, we use the representation of $\chi_{35}(Z)$ in terms of $\Delta_5(Z) = \Delta_5(z_1, z_2, z_3)$. We show that, up to a constant, $\chi_{35}(Z)$ is defined by the quotient

$$\chi_{35}(Z) = \frac{\chi_{75}(Z)}{\Delta_5(Z)^8}$$ \hspace{1cm} (0.3)

where the modular form $\chi_{75}(Z)$ of weight 75 is defined by the product

$$\chi_{75}(Z) := \prod_{a,b,c \equiv 2 \text{ mod } 2} \Delta_5\left(\frac{z_1+a}{2}, \frac{z_2+b}{2}, \frac{z_3+c}{2}\right) \prod_{a \equiv 2 \text{ mod } 2} \Delta_5\left(\frac{z_1+a}{2}, z_2, 2z_3\right) \Delta_5(2z_1, z_2, \frac{z_2+a}{2})$$

$$\times \Delta_5(2z_1, 2z_2, 2z_3) \prod_{b \equiv 2 \text{ mod } 2} \Delta_5(2z_1, -z_1 + z_2, \frac{z_1-2z_2+z_3+b}{2}).$$ \hspace{1cm} (0.4)

One can consider the last formula as a “multiplicative” Hecke operator $T(2)$ applied to $\Delta_5(Z)$. To prove (0.3), we compare divisors of $\chi_{75}(Z)$ and of $\chi_{35}(Z)$ and use the Koecher principle. At the same time, this gives the new construction of the form $\chi_{35}(Z)$ and proves that $\chi_{35}(Z)$ is the Siegel modular form of the smallest odd weight. Due to (0.2)–(0.4), we get the infinite product expansion of $\chi_{35}(Z)$ which is similar to (0.2). Using this expansion, it is not difficult to prove that $\chi_{35}(Z)$ gives an automorphic correction of $g(A_{1, 0})$.

The divisors of the modular forms $\Delta_5(Z)$ and $\chi_{35}(Z)$ are well known. The divisor of $\Delta_5(Z)$ is the “diagonal” $H_1 = \{Z = \begin{pmatrix} z_1 & 0 \\ 0 & z_3 \end{pmatrix} \in Sp_4(\mathbb{Z}) \backslash \mathbb{H}_2\}$ of the quotient $Sp_4(\mathbb{Z}) \backslash \mathbb{H}_2$. The divisor $H_2$, which is the so-called Humbert surface of the divisors of the modular forms $\Delta_5(Z)$ and $\chi_{35}(Z)$, is another simple hyperbolic symmetric generalized Cartan matrix of rank 3. This result was proved in [GN1], [GN2], where the first examples of automorphic corrections of elliptic Lorentzian Kac–Moody algebras were found.
discriminant 1, gives moduli of products of two elliptic curves, and its complement in $Sp_4(\mathbb{Z}) \setminus \mathbb{H}_2$ gives moduli of curves of genus 2. (This geometrical description explains applications of the modular form $\Delta_5(\mathbb{Z})$ in the string theory.) The divisor of the Igusa modular form $\chi_{35}(\mathbb{Z})$ is the sum with multiplicities one of the Humbert surface $H_1$ above and the Humbert surface $H_4$ of discriminant 4.

The consideration with divisors shows us that the quotient

$$\tilde{\Delta}_{30}(\mathbb{Z}) = \chi_{35}(\mathbb{Z})/\Delta_5(2\mathbb{Z})$$

is a holomorphic modular form with respect to the congruence subgroup

$$\Gamma_0(2) = \{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_4(\mathbb{Z}) \mid C \equiv 0 \mod 2 \}.$$ 

It turns out that this function gives an automorphic correction of the Kac–Moody algebra $g(A_1,I)$ with another very simple generalized Cartan matrix

$$A_{1,I} = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$ 

Using (0.2)–(0.4), we get the product expansion of the denominator function $\tilde{\Delta}_{30}(\mathbb{Z})$ of $g(A_{1,I})'$ and find multiplicities of its roots.

One can also consider the second Igusa modular form

$$\chi_{30}(\mathbb{Z}) = \chi_{35}(\mathbb{Z})/\Delta_5(\mathbb{Z})$$

which is a Siegel modular form of weight 30 with the non-trivial character $v : Sp_4(\mathbb{Z}) \to \{\pm 1\}$. Its divisor is $H_4$. Its infinite product expansion follows from (0.2)–(0.4). We conjecture that $\chi_{30}(\mathbb{Z})$ gives an automorphic correction of the Kac–Moody superalgebra with the generalized Cartan matrix $A_{1,0}$ and the set of odd indices $\{2\} \subset \{1, 2, 3\}$ (see [K2], [K3] and [R] for the theory of such superalgebras). We don’t consider automorphic corrections of Kac–Moody superalgebras in this paper, but this example shows that they appear naturally in the subject. Of course, these considerations show that the four Kac–Moody algebras which we have considered are very closely related.

Similarly to (0.4), applying the Hecke operator $T(p)$ ($p > 2$ is a prime) to $\Delta_5(\mathbb{Z})$, we get a Siegel modular form $F^{(p^2)}(\mathbb{Z})$ whose divisor is the Humbert surface $H_{p^2}$ of discriminant $p^2$, and we obtain the infinite product and sum expansion of this form (see Theorem 1.2, Theorem 1.7 and Appendix A). We remark that the modular form $F^{(p^2)}(\mathbb{Z})$ is equal to a finite product (or quotient) of known infinite series. This leads to satisfactory formulae for the Fourier coefficients of $F^{(p^2)}(\mathbb{Z})$.

It was proved by van der Geer (see [vdG1]–[vdG2]), that for any natural $D$, there exists a Siegel modular form $F^{(D)}(\mathbb{Z})$ whose divisor is the Humbert surface $H_D$. Thus Theorem 1.7 gives the exact construction of these modular forms with divisors $H_{p^2}$. Moreover, in Appendix B, using the infinite product formula of the type (0.2), we define Siegel modular forms with divisor $H_D$ for $D$ being not a perfect square.

In Appendix A the general results are obtained which relate multiplicative Hecke operators acting on Siegel modular forms, with Hecke-Jacobi operators acting on
Jacobi forms. This is important for the theory of liftings of Jacobi forms. In particular, we get that exponents of the product formulae for all modular forms $F(D)(Z)$ above are related with Fourier coefficients of Jacobi forms of weight 0 and index 1.

In [GN1] and [GN3] we have pointed out the interpretation of the function $\Delta_5(Z)$ from the point of view of mirror symmetry for K3 surfaces and as the discriminant of K3 surfaces moduli with the condition $S \subset L_{K3}$ on Picard lattice of K3 surfaces where $(S)_{L_{K3}}$ is isomorphic to the lattice $2U(4) \oplus (-2)$. Here $U$ denote the even unimodular lattice with signature $(1,1)$.

Similarly, the Igusa form $\chi_{35}(Z)$ serves for K3 surfaces moduli with condition $S = U \oplus E_8(-1) \oplus E_7(-1) \subset L_{K3}$ on the Picard lattice. Here $(S)_{L_{K3}} \sim 2U \oplus (-2)$.

Using automorphic forms $F(D)(Z)$ with the divisor $H_D$, one can define K3 surfaces submoduli of codimension one $M_{S_1} \subset L_{K3} \subset M_{S}$, where $S_1 \subset L_{K3}$ is isomorphic to the lattice $2U(4) \oplus (-2)$. Here $S \subset L_{K3}$ is one of two conditions on the Picard lattice of K3 surfaces which we considered above. This gives K3 surfaces moduli interpretation of automorphic forms $F(D)(Z)$.

It would be interesting to understand the product and sum formulae of $F(D)(Z)$ from the point of view of this algebraic-geometric construction. See [GN3] for some related considerations.

In Sect. 4 we give the complete list of 13 symmetric hyperbolic generalized Cartan matrices $A$ of elliptic or parabolic type and with a lattice Weyl vector such that $A$ contains a submatrix $\tilde{A}_1 = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. We hope to publish similar results, as in this paper, for these 13 generalized Cartan matrices later. We remark that the cases $A_{1,II}$ and $A_{2,II}$, from this list, had been considered in [GN1], [GN2], and cases $A_{1,0}$ and $A_{1,I}$ are considering in this paper.

1. The Hecke Operators Construction of Siegel Modular Forms with Rational Quadratic Divisors and Their Infinite Product Expansions

A Siegel modular form of weight $k$ with respect to $Sp_4(\mathbb{Z})$ is, by definition, a holomorphic function $F(Z)$ on the Siegel upper half-plane

$$\mathbb{H}_2 = \{Z = \begin{pmatrix} z_1 & z_2 \\ z_2 & z_3 \end{pmatrix} \in M_2(\mathbb{C}) \mid \text{Im}(Z) > 0\},$$

that satisfies the functional equation

$$(F|_k M)(Z) = F(Z)$$  \hspace{1cm} (1.1)

where

$$(F|_k M)(Z) := \det(CZ + D)^{-k} F((AZ + B)(CZ + D)^{-1})$$

for any $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_4(\mathbb{Z})$. By $\mathfrak{M}_k(Sp_4(\mathbb{Z}))$ we denote the finite dimensional space of all Siegel modular forms of weight $k$.

In this section we define a multiplicative analogue of Hecke operators which transforms a Siegel modular form of weight $k$ into another Siegel modular form of a different weight. Let us consider a decomposition of a double coset $V$ in the finite union of left cosets with respect to $Sp_4(\mathbb{Z})$

$$V = Sp_4(\mathbb{Z})MSp_4(\mathbb{Z}) = \sum Sp_4(\mathbb{Z})M_i$$  \hspace{1cm} (1.2)
where $M$ and $M_i$ are elements of the group of symplectic similitudes

$$GSp_4(\mathbb{Z}) = \{ M \in M_4(\mathbb{Z}) \mid \forall M J M = \mu(M) J, \quad \mu(M) \in \mathbb{N} \} \quad (J = \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}).$$

**Definition.** Let $F(Z)$ be a Siegel modular form of weight $k$ and $V$ be the double coset $(1.2)$. We set

$$[F(Z)]_V := \prod_i (F|_k M_i)(Z). \quad (1.3)$$

The next lemma is evident.

**Lemma 1.1.** The function $[F(Z)]_V$ on $\mathbb{H}_2$ is correctly defined and is a Siegel modular form of weight $k\nu$ where $\nu$ is the number of the left cosets in $V$.

Let $p$ be a prime. In accordance with the elementary divisors theorem, there exists only one double coset

$$T(p) = Sp_4(\mathbb{Z}) M Sp_4(\mathbb{Z}) = Sp_4(\mathbb{Z}) \text{diag}(1, 1, p, p) Sp_4(\mathbb{Z})$$

with $\mu(M) = p$. One can find a system of representatives from the distinct left cosets in $T(p)$ consisting of $\nu(p) = (p^2 + 1)(p + 1)$ elements:

$$T(p) = Sp_4(\mathbb{Z}) \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \sum_{a_1, a_2, a_3 \mod p} Sp_4(\mathbb{Z}) \begin{pmatrix} 1 & 0 & a_1 & a_2 \\ 0 & 1 & a_2 & a_3 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}$$

$$+ \sum_{a \mod p} Sp_4(\mathbb{Z}) \begin{pmatrix} 1 & 0 & a & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \sum_{b_1, b_2 \mod p} Sp_4(\mathbb{Z}) \begin{pmatrix} p & 0 & 0 & 0 \\ -b_1 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_1 \\ 0 & 0 & 0 & p \end{pmatrix}. \quad (1.4)$$

We denote the modular form $[F(Z)]_{T(p)}$ simply by $[F(Z)]_p$.

**Remark on the definition of $[F(Z)]_p$.** One can give another equivalent definition of the form $[F(Z)]_p$ using other terms. Let $M_p = \text{diag}(pE_2, E_2)$. Then

$$\Gamma_0(p) = Sp_4(\mathbb{Z}) \cap M_p^{-1} Sp_4(\mathbb{Z}) M_p = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_4(\mathbb{Z}) \mid C \equiv 0 \mod p \right\}$$

is a congruence subgroup of the Siegel modular group. Obviously, one has

$$T_p = \sum_{\gamma \in \Gamma_0(p) \setminus Sp_4(\mathbb{Z})} Sp_4(\mathbb{Z}) M_p \gamma.$$

Hence

$$[F(Z)]_p = \prod_{\gamma \in \Gamma_0(p) \setminus Sp_4(\mathbb{Z})} F(pZ)|_{k\gamma}.$$
\[ \Delta_5(Z) = \frac{1}{64} \prod_{(a,b)} \vartheta_{a,b}(Z), \quad \text{where} \quad \vartheta_{a,b}(Z) = \sum_{l \in \mathbb{Z}^2} \exp \left( \pi i (Z[l] + \frac{1}{2}a) + ^t b l \right) \quad (Z[l] = ^t l \mathbb{Z} l) \]

and the product is taken over all vectors \( a, b \in (\mathbb{Z}/2\mathbb{Z})^2 \) such that \(^t a b \equiv 0 \mod 2\). (There are exactly ten different \((a, b)\).) Remark that \( \Delta_5(Z) \) has integral Fourier coefficients.

We shall apply the operator \([\ldots]_p \) to \( \Delta_5(Z) \). To avoid a problem with the formal definition, we fix the system of representatives (1.4) in \( T(p) \) taking \( x = 0, \ldots, p - 1 \) as a system modulo \( p \). Due to (1.5) the function \([\Delta_5(Z)]_p \) is a Siegel modular form with character \( v \) or with trivial character. The following result explains our interest to the operator \([\ldots]_p \).

**Theorem 1.2.** For \( p = 2 \) the quotient

\[ [\Delta_5(Z)] / \Delta_5(Z)^8 \]
is a Siegel modular form of weight 35 with trivial character. For a prime \( p > 2 \) the function

\[
[\Delta_5(Z)]_p / \Delta_5(Z)^{(p+1)^2}
\]

is a Siegel modular form of weight \( 5p(p^2 - 1) \) with trivial character.

Proof. We have mentioned that there exists the only non-trivial character \( v \) of \( Sp_4(\mathbb{Z}) \) which has values \( \pm 1 \). It is easy to see from (1.6) that

\[
v(\nabla) = -1 \quad \text{where} \quad \nabla = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

Let us apply \( \nabla \) to \( [\Delta_5(Z)]_p \). We remind that we fixed the system of residues \( \{0, 1, \ldots, p - 1\} \) modulo \( p \). Using (1.4), we obtain that

\[
[\Delta_5(\nabla < Z >)]_p = v(\nabla)^{p^2 + p}v(\nabla^p)^{p+1}[\Delta_5(Z)]_p
\]

where \( M < Z > = (AZ + B)(CZ + D)^{-1} \) for \( M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \). The factor \( v(\nabla)^{p^2 + p} \) equals 1 for any prime \( p \). Thus \( [\Delta_5(Z)]_p \) is a Siegel modular form of weight \( 5(p^2 + 1)(p + 1) \) with the trivial character. To prove the theorem, we find the divisor of \( [\Delta_5(Z)]_p \).

A rational quadratic divisor of \( \mathbb{H}_2 \) is, by definition, the set

\[
\mathcal{H}_\ell = \{ \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \in \mathbb{H}_2 \mid (z_2^2 - z_1z_3)d + cz_2 + bz_2 + az_1 + e = 0 \},
\]

where \( \ell = (e, a, b, c, d) \in \mathbb{Z}^5 \) is a primitive (i.e. with the greatest common divisor equals 1) integral vector. The number \( D(\ell) = b^2 - 4de - 4ac \) is called the discriminant of \( \mathcal{H}_\ell \). This divisor determines the Humbert surface \( H_D \) in the Siegel threefold \( \mathcal{A}_1 = Sp_4(\mathbb{Z}) \setminus \mathbb{H}_2 \) (the moduli space of abelian surfaces with principal polarization)

\[
H_D = \pi \left( \bigcup_{v \in \mathbb{Z}^5 \text{ primitive} \atop D(v) = D} \mathcal{H}_v \right) = \pi(\mathcal{H}_\ell)
\]

where \( \pi \) is the natural projection \( \pi : \mathbb{H}_2 \to \mathcal{A}_1 \). The following properties of Humbert surfaces in \( \mathcal{A}_1 \) are well-known and almost evident if one considers them from the point of view of the orthogonal group \( SO(3, 2) = PSp_4 \) (see for example [vdG1], [GH] where the case of non-principal polarizations have also been considered):

- \( \mathcal{H}_\ell \) is not empty \( \Leftrightarrow D(\ell) > 0 \),
- \( \forall \gamma \in Sp_4(\mathbb{Z}) \) the set \( \gamma(\mathcal{H}_\ell) \) is a rational quadratic divisor with the same discriminant \( D(\ell) \),
- \( H_D \) is irreducible,
- \( H_D \) can be represented by an equation \( az_1 + bz_2 + z_3 = 0 \) with \( b = 0 \) or \( b = 1 \).

The next fact is of the great importance for different subjects: the divisor of \( \Delta_5(Z) \) on \( Sp_4(\mathbb{Z}) \setminus \mathbb{H}_2 \) is equal to the Humbert surface \( H_1 = \pi(\{Z \in \mathbb{H}_2 \mid z_2 = 0\}) \) of discriminant 1.
Let $M_p = \text{diag}(p, p, 1, 1)$, then

$$M_p(\mathcal{H}_F) = \{ Z \in \mathbb{H}_2 \mid p^2(z_2^2 - z_1 z_3)d + pcz_3 + pbz_2 + paz_1 + e = 0 \}.$$ 

Thus $D(M_p(\mathcal{H}_F)) = p^2 D(\ell)$ if $(e, p) = 1$ and $D(M_p(\mathcal{H}_F)) = D(\ell)$ if $(e, p) = p$. From the properties mentioned above, it follows that for any element $M \in T(p) = Sp_4(Z)\text{diag}(p, p, 1, 1)Sp_4(Z)$ the divisor $M(\mathcal{H}_F)$ is a rational quadratic divisor of discriminant $D(\ell)$ or $p^2 D(\ell)$. Therefore the divisor of the modular form $[\Delta_5(Z)]_p$ is a sum of the Humbert surfaces $H_1$ and $H_{p^2}$ with some multiplicities $\alpha$ and $\beta$

$$\text{Div}([\Delta_5(Z)]_p) = T(p)^*(H_1) = \pi \left( \bigcup_{v \in \mathbb{Z}^5 \text{ primitive } M_i \in T(p)} \sum_{D(v) = 1} M_i^{-1}(\mathcal{H}_v) \right) = \alpha H_1 + \beta H_{p^2}$$

where the last sum is taken over representatives $M_i$ from the distinct cosets in $T(p)$. To define the integers $\alpha$ (resp. $\beta$), we have to find the number of the elements $M_i$ from the system (1.4) such that $M_i(\mathcal{H}^{(1)}) = \mathcal{H}_F$ (resp. $M_i(\mathcal{H}^{(p)}) = \mathcal{H}_F$) with $D(\ell) = 1$ for two quadratic divisors

$$\mathcal{H}^{(1)} = \{ Z \in \mathbb{H}_2 \mid z_2 = 0 \} \quad \text{and} \quad \mathcal{H}^{(p)} = \{ Z \in \mathbb{H}_2 \mid pz_2 = 1 \}.$$ 

It is easy to see that $\beta = 1$ (there is the only $M_i = \text{diag}(p, p, 1, 1)$) and $\alpha = (p + 1)^2$ (the first and third summands and the summands with $a_2 = 0$ and $b_1 = 0$ in (1.4)). Hence we proved that

$$\text{Div}([\Delta_5(Z)]_p) = (p + 1)^2 H_1 + H_{p^2}.$$ 

Therefore the function $[\Delta_5(Z)]_p/\Delta_5(Z)^{(p+1)^2}$ has no poles on $\mathbb{H}_2$. According to the Koecher principle this function is holomorphic at infinity. For $p > 2$ the quotient has trivial $Sp_4(Z)$-character. For $p = 2$ we consider $[\Delta_5(Z)]_p/\Delta_5(Z)^{(p+1)^2-1}$. This finishes the proof of the theorem.

□

From the proof of Theorem 1.2, we get

**Corollary 1.3.** Let $p \geq 2$ be a prime. The divisor of the Siegel modular form $[\Delta_5(Z)]_p/\Delta_5(Z)^{(p+1)^2}$ of weight $5p(p^2 - 1)$ coincides with the Humbert surfaces $H_{p^2}$ of discriminant $p^2$ taken with multiplicity one.

Moreover we obtain the next results

**Corollary 1.4.** 1. Any Siegel modular form which has zero on $H_{p^2}$ is divisible by the modular form $[\Delta_5(Z)]_p/\Delta_5(Z)^{(p+1)^2}$ of weight $5p(p^2 - 1)$.

2. (Igusa, [Ig1]) Up to a constant there exists only one Siegel modular form $\chi_{35}(Z)$ of weight 35, and an arbitrary Siegel modular form of odd weight is divisible by $\chi_{35}(Z)$.

3. An arbitrary Siegel modular form with the non-trivial character $v : Sp_4(Z) \to \{ \pm 1 \}$ can be represented as

$$\Delta_5(Z) F(Z) \quad \text{or} \quad \frac{[\Delta_5(Z)]_2}{\Delta_5(Z)} F(Z).$$
where $F(Z)$ is a Siegel modular form with the trivial character.

Proof. The first statement follows from the Koecher principle and Corollary 1.3.

Let $F(Z)$ be a Siegel modular form of odd weight. Then $F(z_1,0,z_3) \equiv 0$ since there do not exist $SL_2(\mathbb{Z})$-modular forms of odd weight. $F(Z)$ is 0 on the Humbert surface

$$H_4 = \pi(\{Z = \begin{pmatrix} z & z_2 \\ z_2 & z \end{pmatrix} \in \mathbb{H}_2\}),$$

since $F(z_1,z_2,z_3) = -F(z_3,z_2,z_1)$. The divisor of the form $[\Delta_5(Z)]_2/\Delta_5(Z)^8$ equals $H_1 + H_4$. It proves the second statement.

To prove the third one, we mention that any $Sp_4(\mathbb{Z})$-modular form of odd (resp. even) weight with the character $v$ vanishes on $H_1$ (resp. $H_4$).

Remark. The existence of Siegel modular forms with the divisors $H_D$ for any $D$ was proved by van der Geer using some geometrical considerations (see [vdG1], [vdG2]). The forms $[\Delta_5(Z)]_p$ are examples of Siegel modular forms having the divisors $H_{p^2}$. In Appendix B we show how one can construct explicitly the modular forms with divisors $H_D$ where $D$ is not a perfect square.

In the next theorem we find an infinite product formula for $[\Delta_5(Z)]_p$. To define this formula, we recall some notions and results.

The Fourier-Jacobi expansion of a Siegel modular form $F(Z) = F(z_1,z_2,z_3)$ is its Fourier expansion with respect to $z_3$

$$F(z_1,z_2,z_3) = f_0(z_1) + \sum_{m \geq 1} f_m(z_1,z_2) \exp(2\pi i mz_3)$$

where $z_1 = x_1 + iy_1$ ($y_1 > 0$) belongs to the usual upper half-plane $\mathbb{H}_1$ and $z_2 \in \mathbb{C}$. The function

$$\tilde{f}_m(Z) := f_m(z_1,z_2) \exp(2\pi i mz_3)$$

satisfies (1.1) for all elements $M \in \Gamma_\infty$ of the parabolic subgroup

$$\Gamma_\infty = \left\{ \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & 0 & * \\ 0 & 0 & 0 & * \end{pmatrix} \in Sp_4(\mathbb{Z}) \right\}. \quad (1.7)$$

The Fourier-Jacobi coefficient $f_m(z_1,z_2)$ (resp. $\tilde{f}_m(Z)$) is a Jacobi form (see [EZ] and Appendix A) of weight $k$ and index $m$ (resp. a $\Gamma_\infty$-modular form of weight $k$).

The Fourier expansions of the cusp forms $\Delta_5(Z)$, $\chi_{10}(Z)$ and $\chi_{12}(Z)$ can be determined only by the Fourier coefficients of their first Fourier-Jacobi coefficients. In other words, they are elements of the so-called Maass subspaces. For example, the first Fourier-Jacobi coefficient of $\Delta_5(Z)$ is a Jacobi form of weight 5 and index 1/2

$$\varphi_{5,1/2}(z_1,z_2) = \eta^9(z_1)\vartheta_{11}(z_1,z_2) = -q^{1/2}r^{-1/2} \prod_{n \geq 1} (1 - q^{n-1}r)(1 - q^nr^{-1})(1 - q^n)^{10}$$

$$= \sum g(n,l) \exp(\pi i (nz_1 + lz_2)).$$
Here we denote \( q = \exp(2\pi iz_1) \) and \( r = \exp(2\pi iz_2) \). Forms \( \eta(z_1) \) and \( \vartheta_{11}(z_1, z_2) \) are the Dedekind eta-function and the Jacobi theta-series respectively. The Fourier expansion of \( \Delta_5(Z) \) has the form (see [M2])

\[
\Delta_5(Z) = \sum_{n, l, m \equiv 1 \mod 2 \atop n, m > 0, 4mn - l^2 > 0} \sum_{d | (n, l, m)} d^l g(\frac{nm}{d^2} \cdot \frac{l}{d}) \exp(\pi i (nz_1 + lz_2 + mz_3)).
\]

Up to a constant there exists only one Jacobi cusp form of weight 12 and index 1. We consider the Jacobi form of weight 12 and index 1 with integral Fourier coefficients

\[
\phi_{12,1}(z_1, z_2) = \frac{1}{144} (E_4^2(z_1)E_{4,1}(z_1, z_2) - E_6(z_1)E_{6,1}(z_1, z_2))
\]

\[
= (r^{-1} + 10 + r)q + (10r^{-2} - 88r^{-1} - 132 - 88r + 10r^{-2})q^2 + \ldots .
\]

Here

\[
E_4(z_1) = 1 + 240 \sum_{n \geq 1} \sigma_3(n)q^n,
\]

\[
E_6(z_1) = 1 - 504 \sum_{n \geq 1} \sigma_5(n)q^n
\]

are Eisenstein series for \( SL_2(\mathbb{Z}) \), and \( E_{k,1}(z_1, z_2) \) is the Jacobi–Eisenstein series with integral Fourier coefficients of weight \( k \) and index 1 (see [EZ, §3]).

Let us introduce a weak Jacobi form of weight 0 and index 1 with integral Fourier coefficients

\[
\phi_{0,1}(z_1, z_2) = \phi_{12,1}(z_1, z_2)/\Delta_12(z_1) = \sum_{n \geq 0, l \in \mathbb{Z}} f(n, l) \exp(2\pi i (nz_1 + lz_2))
\]

\[
= (r^{-1} + 10 + r) + q(10r^{-2} - 64r^{-1} + 108 - 64r + 10r^2) + \ldots
\]

(1.8)

where \( \Delta_12(z_1) = q \prod_{n \geq 1} (1 - q^n)^{24} \) is the \( SL_2(\mathbb{Z}) \)-cusp form of weight 12. The Jacobi form \( \phi_{0,1}(z_1, z_2) \) is one of the two canonical generators of the ring of weak Jacobi forms (see [EZ, §9]). \( \phi_{0,1}(z_1, z_2) \) satisfies the same functional equation as usual Jacobi forms (i.e. \( \phi_{0,1}(z_1, z_2) \exp(2\pi iz_3) \) satisfies (1.1) for all \( M \in \Gamma_\infty \)) and has nonzero Fourier coefficients only with indices \((n, l) \in \mathbb{Z} \) such that \( n \geq 0 \) (since \( \phi_{12,1}(z_1, z_2) \) is a cusp form) and \( 4n - l^2 \geq -1 \). Its weight is even, thus \( f(n, l) = f(n, -l) \) and \( f(n, l) \) depends only on \( 4n - l^2 \). Therefore we may define a function \( f(N) \) of integral argument such that

\[
f(N) = \begin{cases} f(n, l) & \text{if } N = 4n - l^2 \\ 0 & \text{otherwise.} \end{cases}
\]

(1.9)

In particular, we have \( f(0) = 10, f(-1) = 1 \) and \( f(n) = 0 \) if \( n < -1 \).

In [GN1, Theorem 4.1] we proved that the modular form \( \Delta_5(Z) \) is the denominator function of a generalized Lorentzian Kac–Moody superalgebra and the product formula

\[
\Delta_5(Z) = (qr)^{\frac{1}{2}} \prod_{n, l, m \in \mathbb{Z}} (1 - q^n r^l s^m)^{f(4nm - l^2)}
\]

(1.10)
is valid where \( q = \exp(2\pi iz_1), \ r = \exp(2\pi iz_2), \ s = \exp(2\pi iz_3). \) The condition \((n, l, m) > 0\) in the product means that \( n \geq 0, \ m \geq 0, \ l \) is an arbitrary integer if \( n + m > 0, \) and \( l < 0 \) if \( n = m = 0. \) (We remark that the infinite product absolutely converges for all \( \Im(Z) > C \) for a large \( C > 0. \))

Let us introduce a modular form of weight 35

\[ \Delta_{35}(Z) := 2^{110} \exp\left(\frac{\pi i}{4} \frac{[\Delta_5(Z)]_2}{\Delta_5(Z)^8}\right). \]

**Theorem 1.5.** The Siegel modular form \( \Delta_{35}(Z) \) has integral Fourier coefficients and can be represented as infinite product:

\[ \Delta_{35}(Z) = q^2r^2s^2(q-s) \prod_{n,l,m\in\mathbb{Z}} (1-q^n r^l s^m)^{f_2(4nm-l^2)}. \]

The integral exponents \( f_2(4nm-l^2) \) are defined by the formula

\[ f_2(N) = 8f(4N) + 2\left( \frac{-N}{2} \right) - 1)f(N) + f\left( \frac{N}{4} \right) \]

where \( f(N) \) is the function (1.9) and

\[ \left( \frac{D}{2} \right) = \begin{cases} 
1 & \text{if } D \equiv 1 \mod 8 \\
-1 & \text{if } D \equiv 5 \mod 8 \\
0 & \text{if } D \equiv 0 \mod 2.
\end{cases} \]

The condition \((n, l, m) > 0\) in the last product means that \( n \geq 0, \ m \geq 0, \ l \) is an arbitrary integer if \( n + m > 0, \) and \( l < 0 \) if \( n = m = 0. \)

**Remark.** The factor \( q^2rs^2(q-s) \) shows that \( \Delta_{35}(Z) \) coincides with \((4i)\chi_{35}(Z)\) where \( \chi_{35} \) is the Igusa modular form (see [Ig1, Theorem 3]).

Below we give the proof of Theorem 1.5 together with its variant for arbitrary prime \( p. \) Before doing this, we formulate the next corollary.

Let \( \Delta_{30}(Z) = \Delta_{35}(Z)/\Delta_5(Z) \) be a holomorphic cusp form of weight 30 with non-trivial character \( v \) (see the proof of Theorem 1.2). Using Theorem 1.5 and (1.10), we get

**Corollary 1.6.** The following identity is valid for \( Z \) with a large imaginary part

\[ \Delta_{30}(Z) = q^{\frac{3}{2}}r^{\frac{5}{2}}s^{\frac{3}{2}}(q-s) \prod_{n,l,m\in\mathbb{Z}} (1-q^n r^l s^m)^{f'_2(4nm-l^2)} \]

where the integers \( f'_2(4nm-l^2) \) are defined by the formula

\[ f'_2(N) = 8f(4N) + 2\left( \frac{-N}{2} \right) - 3)f(N) + f\left( \frac{N}{4} \right). \]
The form $\Delta_{30}(Z)$ has integral Fourier coefficients, and the divisor of $\Delta_{30}(Z)$ on $Sp_4(Z) \setminus \mathbb{H}_2$ is equal to $H_4$.

Remark. Using the product formulae for $\Delta_{35}(Z)$ and $\Delta_{30}(Z)$, we can calculate the first non-trivial Fourier-Jacobi coefficients of these functions having indices $2$ and $3/2$ respectively. We have

$$f_2(-4) = f_2'(-4) = 1, \quad f_2(-1) = 0, \quad f_2'(-1) = -1, \quad f_2(0) = 70, \quad f_2(0) = 60.$$ 

Therefore

$$\phi_{35,2}(z_1, z_2) = \eta^{69}(z_1) \vartheta_{11}(z_1, 2z_2) = -q^{2r-1} \prod_{n \geq 1} (1 - q^{n-1}r^2)(1 - q^n r^{-2})(1 - q^n)^{70}$$

and

$$\phi_{30,3/2}(z_1, z_2) = \eta^{59}(z_1) \frac{\vartheta_{11}(z_1, 2z_2)}{\vartheta_{11}(z_1, 2z_2)} = q^{2r-2} \prod_{n \geq 1} (1 + q^{n-1}r)(1 - q^{2n-1}r^2)(1 - q^{2n-1}r^{-2})(1 + q^{n}r^{-1})(1 - q^n)^{60}.$$ 

Now we formulate an analogue of Theorem 1.5 for a prime $p > 2$. Let us define a Siegel modular form

$$F_p(Z) := (-1)^{\frac{p-1}{2}} p^{5p(p^2 + p + 1)} \frac{[\Delta_5(Z)]_p}{\Delta_5(Z)(p+1)^2}$$

of weight $5p(p^2 - 1)$.

**Theorem 1.7.** Let $p$ be an odd prime. The form $F_p(Z)$ has the infinite product expansion

$$F_p(Z) = q^{\frac{5p(p^2 - 1)}{2}} r^{\frac{(p-1)}{2}} s^{\frac{(p^2 - 1)}{2}} \left( \frac{r^p - 1}{r - 1} \right) \prod_{n, l, m \in \mathbb{Z}} (1 - q^n r^l s^m)^{f_p(4nm - l^2)}.$$ 

The integers $f_p(4nm - l^2)$ are defined by the formula

$$f_p(N) = p^3 f(p^2 N) + (p \left( -\frac{N}{p} \right) - p - 1) f(N) + f\left( \frac{N}{p} \right)$$

where $\left( -\frac{N}{p} \right)$ is the Legendre symbol of the quadratic residue. The product is taken over all integral triplets $(n, l, m)$ such that $m \geq 0$, $l$ is an arbitrary integer and $n \neq 0$ or $m \neq 0$. In particular, $F_p(Z)$ has integral Fourier coefficients.

**Proof of Theorem 1.5 and Theorem 1.7.** Let us consider the factor

$$(1 - q^n r^l s^m) = (1 - \exp(2\pi i r(NZ))) = (1 - e(N, Z)) \quad (N = \left( \frac{n}{p}, \frac{l/2}{p}, \frac{m}{p} \right)).$$
where \( e(N, Z) := \exp (2\pi i \tr(NZ)) \). For any element

\[
M = \begin{pmatrix} U & V \\ 0 & p^tU^{-1} \end{pmatrix}
\]

with \( V = UX, X = tX \in M_2(\mathbb{Z}), UX \mod (p^tU^{-1}) \),

from the system of representatives (1.4), we have

\[
(1 - e(N, M < Z >)) = (1 - e(p^{-1}N[U], Z + V)).
\]

Here \( N[U] := tUNU \). Therefore the following formulae are valid:

\[
(1 - e(N, Z))|_5 \begin{pmatrix} pE_2 & 0 \\ 0 & E_2 \end{pmatrix} = (1 - e(pN, Z))
\]

and

\[
\prod_{X = tX \mod p} (1 - e(N, Z))|_5 \begin{pmatrix} E_2 & X \\ 0 & pE_2 \end{pmatrix}
\]

\[
= p^{-10p^3} \begin{cases} (1 - e(N, Z))^{p^2}, & \text{if } N \not\equiv 0 \mod p, \\ (1 - e(p^{-1}N, Z))^{p^3}, & \text{if } N \equiv 0 \mod p, \end{cases}
\]

for the action of the first and the second summands from (1.4), and

\[
\prod_{V \mod p^tU^{-1}} (1 - e(N, Z))|_5 \begin{pmatrix} U & V \\ 0 & p^tU^{-1} \end{pmatrix}
\]

\[
= p^{-5p} \begin{cases} (1 - e(N[U], Z)), & \text{if } N[U] \not\equiv 0 \mod p, \\ (1 - e(p^{-1}N[U], Z))^p, & \text{if } N[U] \equiv 0 \mod p, \end{cases}
\]

for the action of the third and the fourth summands.

Let us introduce the function \( \tilde{f} : M_2(\mathbb{Z}) \rightarrow \mathbb{Z} \) of the matrix argument \( N = \begin{pmatrix} n & l/2 \\ l/2 & m \end{pmatrix} \) by the formula

\[
\tilde{f}(N) = \begin{cases} f(4 \det N) & \text{if } (n, l, m) > 0 \\ 0 & \text{otherwise} \end{cases}
\]

where \( f(4 \det N) \) is the function (1.9). Collecting together the above formulae, we obtain that the factor \((1 - e(N, Z))\) appears in the product \([\Delta_5(Z)]_p\) with the exponent

\[
\tilde{f}_p(N) = p^3 \tilde{f}(pN) + p \sum_U \tilde{f}(pN[U^{-1}]) + \begin{cases} \sum_U \tilde{f}(N[U^{-1}]) + p^2 \tilde{f}(N), & \text{if } N \not\equiv 0 \mod p, \\ \tilde{f}(p^{-1}N), & \text{if } N \equiv 0 \mod p. \end{cases}
\]

The last sum is taken over all \( U \in \left\{ \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} p & 0 \\ -b & 1 \end{pmatrix} \right\}_{b=0, \ldots, p-1} \) (see (1.4)). To find the formula for \( f_p(N) \), one has to calculate the number (which depends only
on $M \text{ mod } p$ of the elements $U$, from the system given above, such that $M[U^{-1}]$ is an integral matrix. It provides us with the given formula for $f_p(N)$.

Due to (1.11) and the choice (1.4), all factors $1 - e(N, Z)$ with non-zero $f_p(N)$ have $m \geq 0$. In the case $p = 2$ one easily finds that the product contains the unique factor with $n < 0$, namely $(1 - q^{-1}s)$. This finishes the proof of Theorems 1.5 and 1.7. □

The formulae of Theorem 1.5 and Theorem 1.7 show us that $f_p(N)$ for $p > 2$ and $f'_2(N)$ for $p = 2$ are the Fourier coefficients of the Jacobi form of weight 0 and index 1

$$p^3(\phi_{0,1}T^J(p))(z_1, z_2) - (p + 1)\phi_{0,1}(z_1, z_2) \quad (1.12)$$

where $T^J(p)$ is the Hecke operator on the space of Jacobi forms (see [EZ, §4]). We explain this phenomenon in Appendix A where we give another “functorial” proof of Theorem 1.5 and Theorem 1.7.

2. The orthogonal interpretation of the construction of $\Delta_{35}$ and application to moduli of K3 surfaces

Here we give the simple orthogonal interpretation of the construction in Sect. 1 of Igusa automorphic forms using $\Delta_5(Z)$. Moreover, we give an application to K3 surfaces moduli.

We use notations from [N1] for lattices. For a lattice (i.e., an integral symmetric bilinear form) $S$ and elements $a, b$ of $S$ (i.e., elements of the module of $S$) we denote by $(a, b)$ the value of the form of $S$ on elements $a, b$. We denote by $S(q)$ a lattice which one gets multiplying the form of $S$ by $q \in \mathbb{Q}$. For an integral symmetric matrix $A$ we denote by $\langle A \rangle$ a lattice which is defined by $A$. Thus, $S = \langle A \rangle$ means that the lattice $S$ has a bases with the Gram matrix $A$. We denote by $\oplus$ the orthogonal sum of lattices.

We set $U = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. The lattice $U$ is unique (up to isomorphism) even unimodular lattice of signature $(1, 1)$.

Let $T$ be a lattice of signature $(t, 2)$. One can define a symmetric domain of type IV $\Omega^+(T)$ which is one of two connected components of

$$\Omega(T) = \{ Z \in \mathbb{P}(T \otimes \mathbb{C}) \mid (Z, Z) = 0, \, (Z, \overline{Z}) < 0 \}. $$

We also consider the corresponding homogeneous cones (without zeros) $\tilde{\Omega}(T) \subset T \otimes \mathbb{C}$ and $\tilde{\Omega}^+(T) \subset T \otimes \mathbb{C}$ such that $\Omega(T) = \tilde{\Omega}(T)/\mathbb{C}^*$ and $\Omega^+(T) = \tilde{\Omega}^+(T)/\mathbb{C}^*$. Let $O^+(T)$ be the subgroup of index two of $O(T)$ which keeps the component $\Omega^+(T)$. It is well-known that $O^+(T)$ is discrete in $\Omega^+(T)$ and has a fundamental domain of finite volume.

A function $\Phi$ on $\tilde{\Omega}^+(T)$ is called an automorphic form of weight $k$ if $\Phi$ is holomorphic on $\tilde{\Omega}^+(T)$, $\Phi(c\omega) = c^{-k}\Phi(\omega)$ for any $c \in \mathbb{C}^*$, $\omega \in \tilde{\Omega}^+(T)$, and $\Phi(\gamma(\omega)) = \chi(\gamma)\Phi(\omega)$ for any $\gamma \in G$, $\omega \in \tilde{\Omega}^+(T)$. Here $G \subset O^+(T)$ is a subgroup of finite index and $\chi : G \rightarrow \mathbb{C}^*$ some character with the kernel of finite index in $G$. Then $\Phi$ is called automorphic with respect to $G$ with the character $\chi$. If $\dim \Omega(T) = \text{rk } T - 2 \leq 2$, the function $\Phi$ additionally should be holomorphic at infinity of $\Omega(T)$. If $\dim \Omega(T) > 2$, this condition is automatically valid by the Koecher principle.
For a lattice $T$ we denote $\Delta^{(2)}(T) = \{ \delta \in T \mid (\delta, \delta) = 2 \}$. For $e \in T$ with $(e, e) > 0$ we denote $\mathcal{H}_e = \{ Z \in \Omega^+(T) \mid (Z, e) = 0 \}$. The $\mathcal{H}_e$ is called the quadratic divisor orthogonal to $e$. The quadratic divisor $\mathcal{H}_e$ does not change if one changes $e$ to $te$, $t \in \mathbb{Q}$.

The starting point of this paper is finding in some cases lattices $T$ of signature $(t, 2)$ and automorphic forms on $\Omega^+(T)$ (with respect to subgroups of finite index of $O(T)$) with divisor which is a sum with some multiplicities of quadratic divisors $\mathcal{H}_e$, $e \in \Delta^{(2)}(T)$. For a general discussion see [N9] where similar subject was considered.

Conjecturally, the set of these lattices $T$ is finite for $\text{rk } T \geq 5$.

Let

$$L_{1,0} = 2U \oplus \langle 2 \rangle \quad \text{and} \quad L_{1,II} = 2U(4) \oplus \langle 2 \rangle.$$ 

Obviously, $L_{1,II} \cong 2L^*_{1,0}$ and $L_{1,0} \cong 2L^*_{1,II}$. It follows that $O(L_{1,0}) \cong O(L_{1,II})$. Using the natural isomorphism $O^+(L_{1,0})/\{\pm E_5\} \cong Sp_4(\mathbb{Z})/\{\pm E_4\}$ (see [GN1], for example), one can identify Siegel modular forms (with respect to subgroups of finite index of $Sp_4(\mathbb{Z})$) with automorphic forms on IV type domains $\Omega^+(L_{1,0})$ or $\Omega^+(L_{1,II})$.

The function $\Delta_5(Z)$ is an automorphic form on $\Omega^+(L_{1,II})$ with the divisor which is equal to the sum with multiplicities one of all quadratic divisors $\mathcal{H}_\delta$, $\delta \in \Delta^{(2)}(L_{1,II})$ (see [GN1] where this interpretation of $\Delta_5(Z)$ is given). By the Koecher principle, this function is unique (up to multiplying by a constant) and it is automorphic with respect to $O^+(L_{1,II})$ possibly with some character. Thus, the function $\Delta_5(Z)$ is defined (up to a multiplicative constant) by the lattice $L_{1,II}$ and we denote it by $\Delta(L_{1,II})_5(Z)$.

Now consider the lattice $L_{1,0} = 2U \oplus \langle 2 \rangle$. We want to construct an automorphic form $\Delta(L_{1,0})$ on $\Omega^+(L_{1,0})$ such that $\Delta(L_{1,0})$ has the divisor which is equal to the sum with multiplicities one of all quadratic divisors $\mathcal{H}_\delta$, $\delta \in \Delta^{(2)}(L_{1,0})$. By the Koecher principle, $\Delta(L_{1,0})$ is unique up to a multiplicative constant if it exists.

We first remark that elements $\delta \in \Delta^{(2)}(L_{1,0})$ and corresponding quadratic divisors $\mathcal{H}_\delta$ are of two different types. An element $\delta \in \Delta^{(2)}(L_{1,0})$ has type II if $(\delta, x) \equiv 0 \mod 2$ for any $x \in L_{1,0}$. Otherwise, $\delta \in \Delta^{(2)}(L_{1,0})$ has type I; then there exists $x \in L_{1,0}$ such that $(x, \delta) \equiv 1 \mod 2$. Two elements of $\Delta^{(2)}(L_{1,0})$ are conjugate by $O^+(L_{1,0})$ if and only if they have the same type: I or II.

Obviously, $\delta \in L_{1,0}$ has type II if and only if $\delta \in 2L^*_{1,0} \cong 2U(4) \oplus \langle 2 \rangle$. Here $2L^*_{1,0} \subset L_{1,0}$ is a canonical sublattice of $L_{1,0}$ which is equal (or isomorphic) to $2U(4) \oplus \langle 2 \rangle$. Since the IV type domain of any sublattice of finite index of $L_{1,0}$ is naturally identified with $\Omega^+(L_{1,0})$, the automorphic form $\Delta(2L^*_{1,0})_5$ gives the automorphic form on $\Omega^+(L_{1,0})$ with the divisor which is equal to the sum with multiplicities one of all quadratic divisors $\mathcal{H}_\delta$, where $\delta \in \Delta^{(2)}(L_{1,0})$ and $\delta$ has type II. Obviously, $O(L_{1,0}) = O(2L^*_{1,0})$. Thus the form $\Delta(2L^*_{1,0})_5$ is automorphic with respect to $O^+(L_{1,0})$ with some character.

Idea of construction of the automorphic form $\Delta(L_{1,0})$ is using other sublattices $T \subset L$ such that $T \cong L_{1,II}$, and considering the product of automorphic forms $\Delta(T)_5$ of these sublattices.
Let us consider the bases \((f_1, f_{-1}, f_2, f_{-2}, f_3)\) of \(L\) with the Gram matrix

\[
\begin{pmatrix}
0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}.
\]

Obviously, the sublattice \(T_0 = \mathbb{Z}f_1 + \mathbb{Z}f_{-1} + \mathbb{Z}2f_2 + \mathbb{Z}2f_{-2} + \mathbb{Z}f_3 \subset L_{1,0}\) is isomorphic to \(L_{1,II}\). The set \(\Delta^{(2)}(T_0)\) contains some elements of \(\Delta^{(2)}(L_{1,0})\) of type I. For example, \(f_{-1} + f_3 \in \Delta^{(2)}(T_0)\) has type I in \(L_{1,0}\). Thus, the automorphic form \(\Delta(T_0)_5\) has the divisor which is equal to the sum with multiplicities one of all quadratic divisors \(H_\delta\) where \(\delta \in \Delta^{(2)}(L_{1,0}) \cap T_0\); and there exists such \(\delta \in \Delta^{(2)}(L_{1,0}) \cap T_0\) of type I.

Consider all sublattices \(T = g(T_0) \subset L_{1,0}\) where \(g \in O(L_{1,0})\). Since \(4L_{1,0} \subset T_0\), their number is finite. We denote by \(R\) the whole set of these sublattices. Obviously, then

\[
\Psi(L_{1,0}) = \prod_{\{T \subset L_{1,0}\} \in R} \Delta(T)_5
\]

is an automorphic form on \(\Omega^+(L_{1,0})\) with the divisor which is the sum of quadratic divisors \(H_\delta, \delta \in \Delta^{(2)}(L_{1,0})\), with the same multiplicity \(a > 0\) for all \(\delta\) of type I and with the same multiplicity \(b \geq 0\) for all \(\delta\) of type II. By the Koecher principle, the function \(\Phi(L_{1,0}) = \Psi(L_{1,0})\Delta(2L_{1,0}^*)_5^{-b}\) is the automorphic form on \(\Omega^+(L_{1,0})\) with respect to \(O^+(L_{1,0})\) possibly with some character such that the divisor of \(\Phi(L_{1,0})\) is the sum with the same multiplicity \(a\) for all quadratic divisors \(H_\delta, \delta \in \Delta^{(2)}(L_{1,0})\) if \(a \neq 0\). It follows that the function \(\Delta(L_{1,0}) = \Phi(L_{1,0})^{1/a}\) is the automorphic form we are looking for.

The orthogonal interpretation of calculations we have done in Sect. 1 for \(p = 2\) is that \(#R = 15\), thus, \(\Psi(L_{1,0})\) has weight 75. The multiplicities \(a = 1\) and \(b = 9\). Thus, \(\Delta(L_{1,0}) = \Phi(L_{1,0})\) has weight 35, and further we denote this automorphic form by \(\Delta(L_{1,0})_{35}\). It is not difficult to repeat these calculations using the orthogonal language. Discriminant forms technique (see [N1]) is very useful for these calculations. We leave these calculations to an interesting reader.

Now it is not difficult to identify \(\Delta(L_{1,0})_{35}\) with the Igusa modular form \(\chi_{35}\) since these forms have the same odd weight and \(\chi_{35}\) is automorphic with respect to \(O^+(L_{1,0})\), i.e., it has the trivial character on \(SO^+(L_{1,0})\); it follows it has the character \(\text{det}(g)\) for \(g \in O^+(L_{1,0})\) since the weight is odd.

Consider \(\delta \in \Delta^{(2)}(L_{1,0})\). This \(\delta\) defines the reflection \(s_\delta \in O^+(L_{1,0})\) where

\[
s_\delta(x) = x - (\delta, x)\delta, \quad x \in L_{1,0}.
\]

It follows that \(s_\delta(\delta) = -\delta\) and \(s_\delta(x) = x\) if \((\delta, x) = 0\). Thus, \(\text{det}(s_\delta) = -1\). We have \(\chi_{35}(\omega) = \chi_{35}(s_\delta(\omega)) = -\chi_{35}(\omega) = 0\) if \(\mathbb{C}\omega \in H_\delta\). Thus \(\chi_{35}\) is equal to zero on the quadratic divisor \(H_\delta\). By the Koecher principle, \(\chi_{35}/\Delta(L_{1,0})_{35}\) is a holomorphic automorphic form of weight 0. Thus, it is a constant and \(\Delta(L_{1,0})_{35} = c\chi_{35}\) where \(c \in \mathbb{C}^*\).

We mention the following application of these considerations to geometry of moduli of K3 surfaces (see [N9] for a general setting).
Theorem 2.1. Igusa modular form $\chi_{35} = c \Delta(L_{1,0})_{35}$, $c \in \mathbb{C}^*$, gives the discriminant of K3 surfaces moduli with condition $S = U \oplus E_8(-1) \oplus E_7(-1) \subset L_{K3}$ on Picard lattice where $E_8$ and $E_7$ are the standard positive definite lattices which are defined by the root systems of the same type.

Proof. The lattice $L_{K3} = 3U \oplus 2E_8(-1)$ is the even unimodular lattice of signature $(3,19)$. Using discriminant form technique (see [N1]), it is very easy to calculate that $T = (S)^{\perp}_{K3} \cong 2U \oplus \langle -2 \rangle = L_{1,0}(-1)$. The discriminant $\Delta(S \subset L_{K3})$ of K3 surfaces with condition $S \subset L_{K3}$ on Picard lattice is equal to union of all quadratic divisors $\mathcal{H}_\delta$, $\delta \in \Delta(-2)(T) = \Delta(2)(L_{1,0})$. Thus, Igusa modular form $\chi_{35}$ is equal to zero exactly on this discriminant. It follows the statement. □

Similarly, $\Delta_5(Z)$ gives the discriminant of K3 surfaces moduli with condition $S \subset L_{K3}$ on Picard lattice where $S$ is the hyperbolic lattice with signature $(1,16)$ and with the discriminant quadratic form $2q_1^{(1)}(2)$. For this condition, $(S)^{\perp}_{K3} \cong 2U(4) \oplus \langle -2 \rangle = L_{II}(-1)$. See [GN1] and [GN3] where this case was considered.

At the same time, $\Delta_5(Z) = \Delta(2L^*_1,5)(Z)$ is equal to zero on the type II part of the discriminant of K3 surfaces moduli with condition $U \oplus E_8(-1) \oplus E_7(-1) \subset L_{K3}$ on Picard lattice, which is union of all quadratic divisors $\mathcal{H}_\delta$, $\delta \in \Delta(2)(L_{1,0})$ and $\delta$ has type II.

The Igusa modular form $\chi_{30} = c_1 \chi_{35}/\Delta_5 = c_2 \Delta(L_{1,0})_{35}/\Delta(2L^*_1,5)$, $c_1, c_2 \in \mathbb{C}^*$, on $\Omega^+(L_{1,0})$ gives the type I part of this discriminant since its divisor is the sum (with multiplicities one) of all quadratic divisors $\mathcal{H}_\delta$, where $\delta \in \Delta(2)(L_{1,0})$ and $\delta$ has type I. Let $L_{1,I}$ be a sublattice of $L_{1,0}$ which is generated by all elements $\delta \in \Delta(2)(L_{1,0})$ of type I. One can easily find that $L_{1,I} = L_{1,0}$. It follows that $\chi_{30}$ is not equal to the discriminant (or the full discriminant if one likes) of K3 surfaces moduli with condition on Picard lattice.

Let us consider a condition $S = U \oplus E_8(-1) \oplus E_7(-1) \subset L_{K3}$ on Picard lattice of K3 which we have considered above. Any primitive intermediate hyperbolic sublattice $S \subset S_1 \subset L_{K3}$ defines a condition $S_1 \subset L_{K3}$ on Picard lattice of K3 surfaces. It defines the submoduli $\mathcal{M}_{S_1 \subset L_{K3}} \subset \mathcal{M}_{S \subset L_{K3}}$ of moduli $\mathcal{M}_{S \subset L_{K3}}$ of K3 surfaces with the condition $S \subset L_{K3}$ on Picard lattice. Consider a negative definite sublattice $Q = S_1 \cap L \subset L$ where $L = (S)^{\perp}_{K3}$. If $Q = Z\delta$ has the rank one and $\delta^2 = -2p^2$, the submoduli $\mathcal{M}_{S_1 \subset L_{K3}}$ are the set of zeros of the Siegel modular form $F_p(Z)$ which we have constructed in Sect. 1. Similarly, one can use modular forms constructed in Sect. 1 and Appendix B to define any codimension one submoduli $\mathcal{M}_{S_1 \subset L_{K3}} \subset \mathcal{M}_{S \subset L_{K3}}$ by one modular form (with respect to the full group $O(L^+)$ equation. It gives the K3 surfaces interpretation of these Siegel modular forms. It is interesting to understand the algebraic-geometric sense of the infinite product (and the sum as well) expansions which we have found in Sect. 1 and Appendix B, of the corresponding modular forms. Compare with some considerations in [GN3].

Similar results we have for the condition $S \subset L_{K3}$ when $S^{\perp}_{K3} = 2U(4) \oplus \langle -2 \rangle$.

For the automorphic correction of Lorentzian Kac–Moody algebras which we consider below, it is important to have automorphic forms $\Phi$ on $\Omega(T)$ with the divisor which is a sum of some quadratic divisors $\mathcal{H}_\delta$, $\delta \in \Delta(2)(T)$, with the multiplicity one. The automorphic forms $\Delta(L_{1,0})_0$ and $\Delta(L_{1,0})_1$ give such automorphic forms.
forms. Quotients \( \Phi(L_{1,0})_{30} = \Delta(L_{1,0})_{35}/\Delta(T)_{5} \), where \( T \subset L_{1,0} \) and \( T \cong L_{1,II} \) also give automorphic forms with these property by the Koecher principle. There are three different types of embeddings \( T \subset L_{1,0} \) with \( T \cong L_{1,II} \).

The first type is defined by the condition \( L_{1,0}/T \cong (\mathbb{Z}/2\mathbb{Z})^{4} \). It gives \( T = 2L_{1,0}^{*} \) and \( \Delta_{30} = \Delta(L_{1,0})_{30} = \Delta(L_{1,0})_{35}/\Delta(2L_{1,0}^{*})_{5} = c\chi_{30} \).

The second type \( \{T \subset L_{1,II}\} \in R \) we have considered above. For this type, \( L_{1,II}/T \cong (\mathbb{Z}/4\mathbb{Z}) + (\mathbb{Z}/2\mathbb{Z})^{2} \). It gives \( \tilde{\Delta}_{30} = \tilde{\Delta}(L_{1,0})_{30} = \Delta(L_{1,0})_{35}/\Delta(T)_{5} \).

There is the third type when \( T = \mathbb{Z}f_{1} + \mathbb{Z}f_{2} + \mathbb{Z}f_{3} \subset L_{1,0} \) for the bases of \( L_{1,0} \) above. For this type \( L_{1,0}/T \cong (\mathbb{Z}/4\mathbb{Z})^{2} \). It gives \( \tilde{\Delta}_{30} = \tilde{\Delta}(L_{1,0})_{30} = \Delta(L_{1,0})_{35}/\Delta(T)_{5} \).

In Sect. 3 we use Fourier expansions at the appropriate cusps of the IV type domain \( \Omega^{+}(L_{1,0}) \) (a cusp is defined by the primitive isotropic element of \( L_{1,0} \)) of the automorphic forms \( \Delta_{5} = \Delta(L_{1,II})_{5} \), \( \Delta_{35} = \Delta(L_{1,0})_{35} \) and \( \tilde{\Delta}_{30} = \tilde{\Delta}(L_{1,0})_{30} \) for the automorphic correction of some Lorentzian Kac–Moody Lie algebras. It seems that the Igusa form \( \Delta_{30} = \Delta(L_{1,0})_{30} = c\chi_{30} \) and the automorphic form \( \tilde{\Delta}_{30} = \tilde{\Delta}(L_{1,0})_{30} \) may be used for the automorphic correction of some Lorentzian Kac–Moody Lie superalgebras (see Remark 3.2 below), but we don’t consider automorphic correction for this more general class of Lorentzian Kac–Moody algebras in this paper.

### 3. Automorphic correction of “the simplest” Lorentzian Kac–Moody algebras or rank 3 using Igusa automorphic forms

There are three closely related elliptic hyperbolic (here we use terminology of [N8]) generalized Cartan matrices of the rank 3 (see [K1] for general definitions and results related with Kac–Moody algebras):

\[
A_{1,0} = \begin{pmatrix}
2 & 0 & -1 \\
0 & 2 & -2 \\
-1 & -2 & 2
\end{pmatrix}, \quad A_{1,I} = \begin{pmatrix}
2 & -2 & -1 \\
-2 & 2 & -1 \\
-1 & -1 & 2
\end{pmatrix}, \quad A_{1,II} = \begin{pmatrix}
2 & -2 & -2 \\
-2 & 2 & -2 \\
-2 & -2 & 2
\end{pmatrix}.
\]

They (especially the first one \( A_{1,0} \)) give the simplest generalized Cartan matrices of this type. It does not mean that the corresponding to these matrices Lorentzian Kac–Moody algebras are the simplest ones. It is why we use “ ” in the title of the paper and this Section.

We consider a hyperbolic lattice \( M_{1,0} = \mathbb{Z}f_{2} + \mathbb{Z}f_{-2} + \mathbb{Z}f_{3} \) with the symmetric bilinear form \( (f_{2}, f_{2}) = (f_{-2}, f_{-2}) = 0, (f_{2}, f_{-2}) = -1, (f_{3}, f_{3}) = 2, (f_{2}, f_{3}) = (f_{-2}, f_{3}) = 0 \). Thus, the Gram matrix of \( f_{2}, f_{-2}, f_{3} \) is equal to

\[
\begin{pmatrix}
0 & -1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 2
\end{pmatrix}.
\]

First, we show that the generalized Cartan matrices \( A_{1,0}, A_{1,I} \) and \( A_{1,II} \) are defined by the lattice \( M_{1,0} \) and some its natural sublattices which we introduce below (here, of course, we consider different embeddings of \( L_{1,0} \) into \( M_{1,0} \)).
we repeat some results of [N5] and [GN1]). The lattice $M_{1,0}$ is hyperbolic, i.e. it has signature $(2,1)$. Thus $M_{1,0}$ defines a cone

$$V(M_{1,0}) = \{ x \in M_{1,0} \otimes \mathbb{R} \mid (x, x) < 0 \}$$

which is union of its two half-cones. (All general definitions and notations below hold for an arbitrary hyperbolic (i.e. with signature $(n,1)$) lattice, and we shall use them later.) We choose one of this half-cones $V^+(M_{1,0})$. We denote by $O^+(M_{1,0})$ the subgroup of $O(M_{1,0})$ of index two which fixes the half-cone $V^+(M_{1,0})$. It is well-known that the group $O^+(M_{1,0})$ is discrete in the corresponding hyperbolic space $L^+(M_{1,0}) = V^+(M_{1,0})/\mathbb{R}_{++}$ and has a fundamental domain of finite volume (below indices $++$ and $+$ denote positive and non-negative numbers respectively).

Any reflection $s_{\delta} \in O(M_{1,0})$ with respect to an element $\delta \in M_{1,0}$ with $(\delta, \delta) > 0$ is a reflection in the hyperplane

$$\mathcal{H}_{\delta} = \{ \mathbb{R}_{++} x \in L^+(M_{1,0}) \mid (x, \delta) = 0 \}$$

of $L^+(M_{1,0})$. This maps the half-space

$$\mathcal{H}^+_{\delta} = \{ \mathbb{R}_{++} x \in L^+(M_{1,0}) \mid (x, \delta) \leq 0 \}$$

to the opposite half-space $\mathcal{H}^+_{-\delta}$ which are both bounded

by the hyperplane $\mathcal{H}_{\delta}$. Here $\delta \in M_{1,0}$ is called orthogonal to $\mathcal{H}_{\delta}$ and $\mathcal{H}^+_{\delta}$. All reflections of $M_{1,0}$ generate the reflection subgroup $W(M_{1,0}) \subset O^+(M_{1,0})$.

The hyperbolic lattice $M_{1,0}$ is very special, and its automorphism group is well-known (see [N5], for example). Firstly, $O^+(M_{1,0}) = W^{(2)}(M_{1,0})$ where index 2 denote the subgroup generated by reflections in all elements of $M_{1,0}$ with square 2. Thus, $O^+(M_{1,0})$ is generated by reflections in $\Delta^{(2)}(M_{1,0})$. An element $\delta \in \Delta^{(2)}(M_{1,0})$ and the corresponding reflection $s_{\delta}$ have one of two types:

Type I: $(\delta, M_{1,0}) = \mathbb{Z}$.
Type II: $(\delta, M_{1,0}) = 2\mathbb{Z}$.

We introduce sublattices $M_{1,I}$ and $M_{1,II}$ which are generated by all elements $\delta \in \Delta^{(2)}(M_{1,0})$ of the type I and II respectively. We have

$$M_{1,I} = \{ nf_2 + lf_3 + mf_{-2} \in M_{1,0} \mid n + l + m \equiv 0 \mod 2 \},$$

$$M_{1,II} = \{ nf_2 + lf_3 + mf_{-2} \in M_{1,0} \mid n \equiv m \equiv 0 \mod 2 \}.$$

The lattice $M_{1,I}$ is unique in its genus and is defined by its discriminant form $q_{M_{1,I}} \cong q_{5}^{(2)}(8)$ (we use notations from [N1]). There exists a unique overlattice $M_{1,I} \subset M$ of index 2, and it is equal to $M_{1,0}$. The lattice $M_{1,II} = 2M_{1,0} \cong U(4) \oplus < 2 >$. The lattice $M_{1,0} = 2M_{1,II}$. It follows that sublattices or overlattices $M_{1,0}$, $M_{1,I}$, $M_{1,II}$ are defined by one another, and their automorphism groups are naturally identified.

An element $\delta \in \Delta^{(2)}(M_{1,0})$ has the type I (resp. II) if and only if $\delta \in M_{1,I}$ (resp. $\delta \in M_{1,II}$). It follows that the subgroup of $O^+(M_{1,0}) = W^{(2)}(M_{1,0})$ generated by all reflections $s_{\delta}$ of the type I (resp. II) is equal to $W^{(2)}(M_{1,I})$ (resp. $W^{(2)}(M_{1,II})$). Obviously, three sublattices $M_{1,0}$, $M_{1,I}$, $M_{1,II}$ are $W^{(2)}(M_{1,0})$ invariant, and both
subgroups $W(2)(M_{1, I})$ and $W(2)(M_{1, II})$ are normal in $W(2)(M_{1, 0})$. The index $[W(2)(M_{1, 0}) : W(2)(M_{1, I})] = 2$ and $[W(2)(M_{1, 0}) : W(2)(M_{1, II})] = 6$. Fundamental polyhedra $\mathcal{M}_{1, 0}$, $\mathcal{M}_{1, I}$, and $\mathcal{M}_{1, II}$ for groups $W(2)(M_{1, 0})$, $W(2)(M_{1, I})$ and $W(2)(M_{1, II})$ respectively are equal to $\mathcal{M}_{1, i} = \bigcap_{\delta \in P(\mathcal{M}_{1, i})} H_\delta^+$ where $P(\mathcal{M}_{1, i})$ are minimal (with this equality) sets of elements of $\Delta(2)(M_{1, i})$, where $i = 0, I, II$. They are called orthogonal vectors to polyhedra $\mathcal{M}_{1, i}$ and are equal to

$$P(\mathcal{M}_{1, 0}) = \{-f_2 + f_-, f_3, f_2 - f_3\};$$
$$P(\mathcal{M}_{1, I}) = \{f_2 + f_3, f_2 - f_3, -f_2 + f_-\};$$
$$P(\mathcal{M}_{1, II}) = \{2f_2 - f_3; 2f_2 - f_3, f_3\}.$$

Here $\mathcal{M}_{1, 0}$ is a triangle with angles $\pi/2$, $0$, $\pi/3$; $\mathcal{M}_{1, I}$ is a triangle with angles $\pi/3$, $0$, $\pi/3$ and $\mathcal{M}_{1, II}$ is a triangle with zero angles (i.e., with its vertices at infinity). It follows that the groups $W(2)(M_{1, 0})$, $W(2)(M_{1, I})$, $W(2)(M_{1, II})$ are generated by reflections in elements of $P(\mathcal{M}_{1, 0})$, $P(\mathcal{M}_{1, I})$ and $P(\mathcal{M}_{1, II})$ respectively. We denote by

$$\text{Sym} \ (P(\mathcal{M}_{1, i})) = \{g \in O^+(\mathcal{M}_{1, i}) \mid g(P(\mathcal{M}_{1, i})) = P(\mathcal{M}_{1, i})\}$$

the group of symmetries of the fundamental polyhedron $\mathcal{M}_{1, i}$ and its set $P(\mathcal{M}_{1, i})$ of orthogonal vectors. The group $\text{Sym} \ (P(\mathcal{M}_{1, 0}))$ is trivial, $\text{Sym} \ (P(\mathcal{M}_{1, I}))$ has order two and is generated by $s_{f_3}$, the group $\text{Sym} \ (P(\mathcal{M}_{1, II}))$ is the group of symmetries of the right triangle (i.e. it is $S_3$) and is generated by $s_{f_2 - f_3}, s_{f_2 - f_2}$. We can write down the groups $O^+(M_{1, 0}) = O^+(M_{1, I}) = O^+(M_{1, II})$ as the semi-direct products:

$$O^+(M_{1, 0}) = O^+(M_{1, I}) = O^+(M_{1, II}) = W(2)(M_{1, 0})$$
$$= W(2)(M_{1, I}) \rtimes \text{Sym} \ (P(\mathcal{M}_{1, I})) = W(2)(M_{1, II}) \rtimes \text{Sym} \ (P(\mathcal{M}_{1, II})).$$

The Gram matrices of sets $P(\mathcal{M}_{1, 0})$, $P(\mathcal{M}_{1, I})$ and $P(\mathcal{M}_{1, II})$ are respectively equal to the generalized Cartan matrices $A_{1, 0}$, $A_{1, I}$ and $A_{1, II}$ above. It shows that these matrices are naturally related with the lattice $M_{1, 0}$ and its natural sublattices $M_{1, I}$ and $M_{1, II}$.

To unify notation, further $i = 0, I$ or II. The lattice $M$ denote $M_{1, i}$, and $\mathcal{M}$ denote the fundamental polygon $\mathcal{M}_{1, i}$ above for $W = W(2)(M)$, and $P(\mathcal{M}) = \{\delta_1, \delta_2, \delta_3\}$. The generalized Cartan matrices $A = A_{1, i}$ (equivalently the Weyl groups $W(2)(M)$), have so called elliptic type. This means that the fundamental polygon $\mathcal{M}$ has finite volume in $L^+(M)$. Since the cone $V^+(M)$ is self-dual, it is equivalent to any of the embeddings

$$\mathbb{R}_+ \mathcal{M} \subset V^+(M) \subset \mathbb{R}_+ \delta_1 + \mathbb{R}_+ \delta_2 + \mathbb{R}_+ \delta_3$$

(3.1)

where

$$\mathbb{R}_+ \mathcal{M} = (\mathbb{R}_+ \delta_1 + \mathbb{R}_+ \delta_2 + \mathbb{R}_+ \delta_3)^* = \{x \in M \otimes \mathbb{R} \mid (x, P(\mathcal{M})) \leq 0\}.$$  

Another very important property of $A_{1, i}$ is that $P(\mathcal{M}_{1, i})$ has a lattice Weyl vector $\rho = \rho_i$ which is, by definition, an element $\rho_i \in M_{1, i} \otimes \mathbb{Q}$ such that

$$\langle \xi, \delta \rangle = \langle \delta, \delta \rangle / 2 = -1 \quad \text{for any } \delta \in P(\mathcal{M}_{1, i}).$$
We have
\[ \rho_0 = 3f_2 + 2f_{-2} - f_3/2, \quad \rho_1 = 2f_2 + f_1, \quad \rho_{11} = f_2 + f_{-2} - f_3/2. \]

Consider the complexified cone \( \Omega(V^+(M)) = M \otimes \mathbb{R} + iV^+(M) \). Remark that for all our lattices \( M = M_{1,i} \) these cones are identified (by the isomorphism of lattices over \( \mathbb{Q} \)), and we always can use the same coordinates \((z_1, z_2, z_3)\) for the point \( z = z_3f_2 + z_2f_3 + z_1f_{-2} \) of these domains. Considering the lattice \( L_k = U(k) \oplus M \) where \( U(k) = \begin{pmatrix} 0 & -k \\ -k & 0 \end{pmatrix} \), \( k \in \mathbb{N} \), we identify \( \Omega(V^+(M)) \) with one (of two) connected component of the symmetric domain

\[
\Omega(L_k) = \{ Z \in \mathbb{P}(L_k \otimes \mathbb{C}) \mid (Z, Z) = 0, \ (Z, \overline{Z}) < 0 \}
\]

of type IV. If \( e_1, e_{-1} \) is the bases of \( U(k) \), one should correspond to \( z \in \Omega(M) \) the point \( \mathbb{C} \left( ( (z, z)/2)e_1 + (1/k)e_2 + z \right) \in \Omega(L_k) \). Using these embeddings, we can speak about automorphic forms on \( \Omega(V^+(M)) \) with respect to subgroups of finite index of \( O(L_k) \). Thus, an automorphic form on \( \Omega(M) \) means a holomorphic function on \( \Omega(V^+(M)) \) which is an automorphic form (with some non-negative weight) with respect to some subgroup of finite index of \( O(L_k) \) for some \( k \).

One can correspond to the matrix \( A = A_{1,i} \) the Kac–Moody algebra \( g = g(A) \) (see [K1]). An automorphic correction (see [B3], [B4] and [N8] and [GN1] for a general setting) of the Kac–Moody algebra \( g(A) \) is an automorphic form \( \Phi(z) \) on \( \Omega(V^+(M)), \ M = M_{1,i} \) which has the Fourier expansion

\[
\Phi(z) = \sum_{w \in W} \det(w) \left( \exp(-2\pi i (w(\rho), z)) - \sum_{\alpha \in M^* \cap \mathbb{R}^+ M} m(a) \exp(-2\pi i (w(\rho + a), z)) \right)
\]

(3.3)

where all \( m(a) \) are integers. (This is a very special and very restricted type of Fourier expansion.) Using this Fourier expansion of \( \Phi(z) \), one can construct a generalized Kac–Moody Lie superalgebra without odd real simple roots \( g(M, s\Delta) \) which has the Weyl–Kac–Borcherds denominator function \( \Phi(z) \). This algebra contains the Kac–Moody algebra \( g(A) \) and has better automorphic properties for its denominator function. Using these automorphic properties it is possible to calculate the product formula for \( \Phi(z) \) (the product converges for \( z \) with large \( \text{Im} z \in V^+(M) \)):

\[
\Phi(z) = \sum_{w \in W} \det(w) \left( \exp(-2\pi i (w(\rho), z)) - \sum_{\alpha \in M^* \cap \mathbb{R}^+ M} m(a) \exp(-2\pi i (w(\rho + a), z)) \right)
\]

\[ = \exp(-2\pi i (\rho, z)) \prod_{\alpha \in \Delta(M)^{re} + \Delta(M^*)^{im}} (1 - \exp(-2\pi i (\alpha, z)))^{\text{mult } \alpha}. \]

(3.4)

Here \( \Delta(M)^{re} = (W(P(M)) \cap Q^+) \) where \( Q^+ = \mathbb{Z}_+ \delta_1 + \mathbb{Z}_+ \delta_2 + \mathbb{Z}_+ \delta_3 \); equivalently, \( \Delta(M)^{re} = \{ \delta \in \Delta^2(M) | (\rho, \delta) < 0 \} \). For an intermediate sublattice \( M \subset T \subset M^* \), we denote \( \Delta(T)^{im} = \overline{V^+(M)} \cap T \). Elements of \( \Delta(M)^{re} \) are called positive real roots (they have the square 2). Elements of \( \Delta(T)^{im} \) are called positive imaginary roots (they have non-positive squares). The exponents \( \text{mult } \alpha \) are integers. They
are called *multiplicities* of roots $\alpha$ and are very important invariants of the Kac–Moody superalgebra $\mathfrak{g}(M, s\Delta)$. Always $\text{mult} \, \alpha = 1$ if $\alpha \in \Delta(M)^{\text{re}}_{+}$. We refer to [GN1] for details.

Using results of Sect. 1, we find automorphic corrections for Kac–Moody algebras $\mathfrak{g}(A), A = A_{1,0}, A_{1,1}, A_{1,II}$. For the algebra $A_{1,II}$ this have been done in [GN1], [GN2] using $\Delta_{5}(Z)$.

We identify a point $Z = \left( \begin{array}{ccc} z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1 \\ z_3 & z_1 & z_2 \end{array} \right) \in \mathbb{H}_2$ with the point $z = z_3 f_2 + z_2 f_3 + z_1 f_{-2} \in \Omega(V^{+}(M))$ with the coordinates $(z_1, z_2, z_3)$ which we have introduced above. Then Siegel automorphic forms on $\mathbb{H}_2$ will give some automorphic forms on $\Omega(V^{+}(M))$ because of the well-known isomorphism $\text{Sp}_{4}(\mathbb{Z})/\{\pm E_4\} \cong O^{+}_{1}(L_1)/\{\pm E_5\}$ (see [GN1], for example). Thus, we can consider the automorphic form $\Delta_{5}(Z)$ and the Igusa automorphic forms $\chi_{35}(Z)$ and $\chi_{30}(Z)$ which we have studied in Sect. 1, as automorphic forms on $\Omega(V^{+}(M))$. We can rewrite

$$\exp(\pi i(nz_1 + lz_2 + mz_3)) = \exp(-\pi i(a, z)), \quad a = nf_2 - lf_3/2 + mf_{-2}$$

where $z = z_3 f_2 + z_2 f_3 + z_1 f_{-2}$.

Using the lattice $M_{1,II}$, we can rewrite $\Delta_{5}(Z)$ as

$$\Delta_{5}(z) = \exp(-\pi i(\rho_{II}, z)) \prod_{\alpha \in \Delta(M_{1,II})_{\text{re}}^{+} \cup \Delta(M_{1,II})_{\text{im}}^{+}} (1 - \exp(-\pi i(\alpha, z)))^{\text{mult} \, \alpha},$$

where

$$\text{mult} \, \alpha = f(-\alpha, \alpha)/2$$

for the function $f(N)$ introduced in (1.9). It follows that

$$\Delta_{5}(2z) = \exp(-2\pi i(\rho_{II}, z)) \prod_{\alpha \in \Delta(M_{1,II})_{\text{re}}^{+} \cup \Delta(M_{1,II})_{\text{im}}^{+}} (1 - \exp(-2\pi i(\alpha, z)))^{\text{mult} \, \alpha},$$

$$\text{mult} \, \alpha = f(-\alpha, \alpha)/2,$$ has the form (3.4).

Using the lattice $M_{1,0}$, we can rewrite $\Delta_{35}(z)$ as

$$\Delta_{35}(z) = \exp(-2\pi i(\rho_{0}, z)) \prod_{\alpha \in \Delta(M_{1,0})_{\text{re}}^{+} \cup \Delta(M_{1,0})_{\text{im}}^{+}} (1 - \exp(-2\pi i(\alpha, z)))^{\text{mult} \, \alpha},$$

where

$$\text{mult} \, \alpha = f_{2}(-2(\alpha, \alpha))$$

for the function $f_{2}(N)$ introduced in Theorem 1.5.

We consider an automorphic form $\tilde{\Delta}_{30}(z) := \Delta_{35}(z)/\Delta_{5}(2z)$. In the orthogonal language of Sect. 2, this is the automorphic form $\tilde{\Delta}_{30} = \tilde{\Delta}(L_{1,0})_{30}$ introduced in Sect. 2. Using the lattice $M_{1,1}$, we can rewrite $\tilde{\Delta}_{30}(z)$ as follows:

$$\tilde{\Delta}_{30}(z) = \Delta_{35}(z)/\Delta_{5}(2z) = \exp(-2\pi i(\rho_{1}, z)) \prod_{\alpha \in \Delta_{+}(M_{1,1})_{\text{re}}^{+} \cup \Delta(M_{1,0})_{\text{im}}^{+}} (1 - \exp(-2\pi i(\alpha, z)))^{\text{mult} \, \alpha}$$

where

$$\text{mult} \, \alpha = f_{2}(-2(\alpha, \alpha)) - \left\{ \begin{array}{ll} f(-\alpha, \alpha)/2 & \text{if } \alpha \in M_{1,II} = 2M_{1,0}^{*} \\ 0 & \text{if } \alpha \notin M_{1,II} = 2M_{1,0}^{*} \end{array} \right. \quad (3.11)$$

Using these preliminary calculations, we get
Theorem 3.1. The automorphic form $\Delta_{35}(z)$ gives the automorphic correction of the Kac–Moody algebra $g(A_{1,0})$ with the product expansion (3.8), (3.9).

The automorphic form $\tilde{\Delta}_{30}(z) = \Delta_{35}(z)/\Delta_{5}(2z)$ gives the automorphic correction of the Kac–Moody algebra $g(A_{1,1})$ with the product expansion (3.10), (3.11).

The automorphic form $\Delta_{5}(2z)$ gives the automorphic correction of the Kac–Moody algebra $g(A_{1,11})$ with the product expansion (3.7), (3.6).

Proof. The proof is similar to the proof of Theorem 2.3 in [GN1]. For example, consider $g(A_{1,0})$. From the product (3.8), one has that $\Delta_{35}(s_{\alpha}(z)) = -\Delta_{35}(z)$ for any $\alpha \in \Delta(M_{1,0})^{*e}$ because $\text{mult} \alpha = 1$. It follows that $\Delta_{35}(w(z)) = \text{det}(w)\Delta_{35}(z)$, $w \in W = W^{(2)}(M_{1,0})$. Since all $\text{mult} \alpha$ are integral, it then follows from the product that

$$\Delta_{35}(z) = \sum_{w \in W} \text{det}(w) \left( - \sum_{\rho_{0} + a \in M_{1,0}^{*} \cap \mathbb{R}_{+}M_{1,0}} m(a) \exp(-2\pi i(w(\rho_{0} + a), z)) \right)$$

where all $m(a)$ are integral and $m(0) = -1$. Consider $\rho_{0} + a \in M_{1,0}^{*} \cap \mathbb{R}_{+}M_{1,0}$. We have $(\rho_{0} + a, \delta_{i}) \leq 0$, $i = 1, 2, 3$. If $(\rho_{0} + a, \delta_{i}) = 0$, then the corresponding Fourier coefficient $m(a) = 0$, since $\Delta_{35}(z)$ is anti-invariant with respect to $s_{\delta_{i}}$. Thus, we can suppose that $(\rho_{0} + a, \delta_{i}) < 0$, $i = 1, 2, 3$, considering only non-zero $m(a)$. By definition of $\rho_{0}$, we then get that $(a, \delta_{i}) \leq 0$ since $a \in M_{1,0}^{*}$. By (3.1), then $a \in M_{1,0}^{*} \cap \mathbb{R}_{+}M_{1,0}$. It follows that $a \in \mathbb{R}_{+}M_{1,0}$ if $a \neq 0$. If $a = 0$, we have $m(a) = -1$. Thus $\Delta_{5}(z)$ has the form (3.3). It follows the statement.

□

Remark 3.2. In Sects. 1 and 2 we have considered the Igusa modular form $\Delta_{30}(z) = \Delta_{35}(z)/\Delta_{5}(z) = c_{30}$. It seems $\Delta_{30}(2z)$ with the product expansion of Corollary 1.6 gives an automorphic correction of the Kac–Moody superalgebra with the generalized Cartan matrix $A_{1,0}$ and with the set of odd indexes $\tau = \{2\} \subset \Gamma = \{1, 2, 3\}$. See [K3] and [R] about these algebras. We don’t consider automorphic corrections of Kac–Moody superalgebras in this paper.

4. The perspective

Using methods and results of [N5], [N8] (see also general results in [N3], [N4]), we can prove the following classification results:

Theorem 4.1. There are exactly 12 elliptic hyperbolic symmetric generalized Cartan matrices of rank 3 which have the Weyl group with a non-compact (i.e. with an infinite vertex) fundamental polygon and have a lattice Weyl vector. They are matrices $A_{1,0} - A_{3,III}$ below:

$$A_{1,0} = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -2 \\ -1 & -2 & 2 \end{pmatrix}, \quad A_{1,I} = \begin{pmatrix} 2 & -2 & -1 \\ -2 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},$$

$$A_{1,II} = \begin{pmatrix} 2 & -2 & -2 \\ -2 & 2 & -2 \\ -2 & -2 & 2 \end{pmatrix}, \quad A_{1,III} = \begin{pmatrix} 2 & -2 & -6 & -6 & -2 \\ -2 & 2 & 0 & -6 & -7 \\ -6 & 0 & 2 & -2 & -6 \\ -6 & -6 & -2 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$
IGUSA MODULAR FORM AND KAC–MOODY ALGEBRAS

\[
A_{2,0} = \begin{pmatrix}
2 & -2 & -2 \\
-2 & 2 & 0 \\
-2 & 0 & 2
\end{pmatrix},
\]

\[
A_{2,I} = \begin{pmatrix}
2 & -2 & -4 & 0 \\
-2 & 2 & 0 & -4 \\
-4 & 0 & 2 & -2 \\
0 & -4 & 2 & 2
\end{pmatrix},
A_{2,II} = \begin{pmatrix}
2 & -2 & -6 & -2 \\
-2 & 2 & -2 & -6 \\
-6 & 2 & -2 & 6 \\
-2 & -6 & 2 & 2
\end{pmatrix},
\]

\[
A_{2,III} = \begin{pmatrix}
2 & -2 & -8 & -18 & -14 & -8 & 0 \\
-2 & 2 & 0 & -8 & -14 & -16 & -8 \\
-8 & 0 & 2 & -2 & -16 & -18 & -14 \\
-16 & -8 & -2 & 2 & 0 & -8 & -14 & -18 \\
-18 & -14 & -8 & 0 & 2 & -2 & -16 & -8 \\
-14 & -18 & -16 & -8 & 2 & 0 & -8 & -14 & -18 \\
-8 & -16 & -18 & -14 & -8 & 0 & 2 & -2 & -8 \\
0 & -8 & -14 & -18 & -16 & -8 & -2 & 2
\end{pmatrix},
\]

\[
A_{3,0} = \begin{pmatrix}
2 & -2 & -2 \\
-2 & 2 & -1 \\
-2 & -1 & 2
\end{pmatrix},
A_{3,I} = \begin{pmatrix}
2 & -2 & -5 & -1 \\
-2 & 2 & -1 & -5 \\
-5 & 1 & 2 & -2 \\
-1 & -5 & 2 & 2
\end{pmatrix},
\]

Theorem 4.2. Let \( \mathcal{M}_{4,0} = U(16) \oplus (2) \). Let \( \mathcal{M}_{4,0} \) be a fundamental polygon for \( W = W^{(2)}(\mathcal{M}_{4,0}) \) in \( L^+(\mathcal{M}_{4,0}) \) and \( P(\mathcal{M}_{4,0}) \subset \Delta^{(2)}(\mathcal{M}_{4,0}) \) the set of all orthogonal vectors to faces of \( \mathcal{M}_{4,0} \).

The Gram matrix \( A_{4,0} = G(P(\mathcal{M}_{4,0})) \) of elements of \( P(\mathcal{M}_{4,0}) \) (this matrix is infinite) is the only parabolic hyperbolic symmetric generalized Cartan matrix of

\[
A_{3,II} =
\begin{pmatrix}
2 & -2 & -10 & -14 & -10 & -2 \\
-2 & 2 & -2 & -10 & -14 & -10 \\
-10 & -2 & 2 & -2 & -10 & -14 \\
-14 & -10 & -2 & 2 & -2 & -10 \\
-10 & -14 & -10 & -2 & 2 & -2 \\
-2 & -10 & -14 & -10 & -2 & 2
\end{pmatrix},
\]

\[
A_{3,III} =
\begin{pmatrix}
2 & -2 & -11 & -25 & -37 & -47 & -47 & -37 & -23 & -11 & -1 \\
-2 & 2 & -1 & -11 & -23 & -37 & -37 & -23 & -11 & -1 & -1 \\
-11 & -1 & 2 & -2 & -11 & -23 & -37 & -37 & -23 & -11 & -1 \\
-25 & -11 & -2 & 2 & -1 & -11 & -23 & -37 & -37 & -23 & -11 \\
-37 & -23 & -11 & -1 & 2 & -2 & -11 & -25 & -37 & -47 & -37 & -47 \\
-47 & -37 & -25 & -11 & -2 & 2 & -1 & -11 & -23 & -37 & -47 & -47 \\
-50 & -46 & -37 & -23 & -11 & -1 & 2 & -2 & -11 & -25 & -37 & -47 \\
-46 & -50 & -47 & -37 & -25 & -11 & -2 & 2 & -1 & -11 & -23 & -37 \\
-37 & -47 & -50 & -46 & -37 & -23 & -11 & -1 & 2 & -2 & -11 & -25 \\
-23 & -37 & -46 & -50 & -47 & -37 & -25 & -11 & -2 & 2 & -1 & -11 \\
-11 & -25 & -37 & -47 & -50 & -46 & -37 & -23 & -11 & -1 & 2 & -2 \\
-1 & -11 & -23 & -37 & -46 & -50 & -47 & -37 & -25 & -11 & -2 & 2
\end{pmatrix}
\]
rank 3 which has the Weyl group with a fundamental polygon which has at least one infinite vertex different from the cusp and has a lattice Weyl vector.

Using methods developed in [GN1]–[GN3] and in this paper, we can find automorphic corrections for all Lorentzian Kac–Moody algebras \( g(A) \) where \( A \) is one of 13 generalized Cartan matrices of Theorems 4.1 and 4.2. There are analogues of automorphic forms \( \Delta_5, \Delta_{35}, \Delta_{30}, \Delta_{30}, \Delta_{30} \) for all matrices \( A_{i,II}, A_{i,0}, A_{i,I}, i = 1, \ldots, 4 \), respectively. The cases \( A_{1,II} \) and \( A_{2,II} \) had been considered in [GN1], and the cases \( A_{1,0}, A_{1,I} \) have been considered in this paper. The cases \( A_{i,III}, i = 1, 2, 3 \), are more delicate. Using multiplicative Hecke operators applied to these forms, one can find analogues of modular forms \( F_p(Z) \) which we consider in Sect. 1, and modular forms of Appendix B, and their product and sum expansions. We hope to publish these results later.

**Appendix A. Multiplicative Hecke operators and the lifting of Jacobi forms**

In Appendix we give a proof of a generalization of Theorems 1.5 and 1.7. In particular, this provides us with a new proof of these theorems and explains why the Hecke-Jacobi operator \( T^J(p) \) appears in the formula (1.12) for the exponents in the infinite product expansion of the modular form \([\Delta_5(Z)]_p\).

For an arbitrary positive integer \( t \), we denote by

\[
\Gamma_t = \left\{ \begin{pmatrix} * & t & * & * \\ * & * & * & t^{-1} * \\ * & t & * & * \\ * & t & t & * \end{pmatrix} \in Sp_2(\mathbb{Q}) \right\},
\]

the paramodular group of type \((1, t)\) where all * are integers. (The threefold \( \Gamma_t \setminus \mathbb{H}_2 \) is a coarse moduli space of abelian surfaces with a polarization of type \((1, t)\).) For \( t = 1 \), we have \( \Gamma_1 = Sp_4(\mathbb{Z}) \).

**Definition.** A holomorphic function \( \phi(z_1, z_2) : \mathbb{H}_1 \times \mathbb{C} \to \mathbb{C} \) is called a *Jacobi form* of index \( t \in \mathbb{N} \) and weight \( k \) if the function \( \tilde{\phi}(Z) = \phi(z_1, z_2) \exp(2\pi i tz_3) \) on the Siegel upper half-plane \( \mathbb{H}_2 \) is a modular form of weight \( k \) with respect to the parabolic subgroup \( \Gamma_\infty \) (see (1.7)). It means that \( \tilde{\phi}(Z) \) satisfies the functional equation (1.1) for any \( M \in \Gamma_\infty \) and that \( \tilde{\phi}(Z) \) is holomorphic at infinity, i.e. it has a Fourier expansion of the type

\[
\tilde{\phi}(Z) = \sum_{n, l \in \mathbb{Z}} f(n, l) \exp(2\pi i (nz_1 + lz_2 + tz_3)).
\]

We remark that a Jacobi form of index \( t \) satisfies the functional equation (1.1) for any \( M \in \Gamma(t) = \Gamma_t \cap \Gamma_\infty(\mathbb{Q}) \). We shall use the term Jacobi form for the function \( \tilde{\phi}(Z) \) as well.

We call the function \( \phi(z_1, z_2) \) (or \( \tilde{\phi}(Z) \)) a *Jacobi cusp form* if we have the strict inequality \( 4nt > l^2 \) in the Fourier expansion.

The function \( \phi(z_1, z_2) \) (or \( \tilde{\phi}(Z) \)) is called a *weak Jacobi form* if we have the condition \( n > 0 \) instead of \( 4nt > l^2 \) in the
Fourier expansion (see [EZ, §9]).

The space of all Jacobi forms (resp. Jacobi cusp forms) of index \( t \) and weight \( k \) is denoted by 
\[ M_{k,t}^J \] (resp. \( S_{k,t}^J \)).

The maximal parabolic subgroup \( \Gamma_\infty \) is the semi-direct product of \( SL_2(\mathbb{Z}) \) and the integral Heisenberg group

\[
\Gamma_\infty / \{ \pm E_4 \} \cong \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \times \left\{ \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & \kappa \\ 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}
\]

where \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \) and \( \mu, \lambda, \kappa \in \mathbb{Z} \). Thus, for Jacobi forms, the equation (1.1) is equivalent to two separate functional equations: one for elements of \( SL_2(\mathbb{Z}) \), and another for elements of the Heisenberg group.

The Hecke algebra \( \mathcal{H}(\Gamma_\infty) = \mathcal{H}_Q(\Gamma_\infty, \Gamma_\infty \Gamma_\infty) \) will be very useful in our consideration. As a linear space over \( \mathbb{Q} \), this algebra is generated by the double cosets

\[ U = \Gamma_\infty M \Gamma_\infty = \sum_i \Gamma_\infty M_i \in \mathcal{H}(\Gamma_\infty) \]

where \( M \) and \( M_i \) are elements of the group of integral symplectic similitudes \( G \Gamma_\infty \) of type \( \Gamma_\infty \) (see (1.2)). The algebra \( \mathcal{H}(\Gamma_\infty) \) is equipped with the usual multiplication (see [G3], [G5] for a general point of view on such algebras).

This Hecke algebra acts on the space of Jacobi forms of all indices. Let \( V = \sum_i a_i \Gamma_\infty M_i \in \mathcal{H}(\Gamma_\infty) \) be an arbitrary element and let \( F(Z) \) be any function which is invariant with respect to the \( |k \)-action of the parabolic subgroup \( \Gamma_\infty \), i.e. \( F|k \gamma = F \) for any \( \gamma \in \Gamma_\infty \) (see (1.1)). Then we put

\[
F(Z) \to (F|k V)(Z) = \begin{cases} \\
\sum_i \mu(M_i)^{2k-3}a_i (F|k M_i)(Z) & \text{if } k > 0 \\
\sum_i a_i (F|k M_i)(Z) & \text{if } k = 0 \end{cases}
\]

where \( \mu(M_i) \) denotes the degree of the symplectic similitudes (see (1.2)). (In the definition for, \( k > 0 \), we keep the same normalizing factor as for the Hecke operators for \( Sp_4(\mathbb{Z}) \).

If \( \phi(Z) = \phi(z_1, z_2) \exp(2\pi i tz_3) \) is a Jacobi form of weight \( k \) and index \( t \), then \( (\phi|k V)(Z) \) is a sum of Jacobi forms of possibly different indices. To describe this representation, we need to define the concept of signature of double cosets from \( \mathcal{H}(\Gamma_\infty) \).

Let

\[ U = \Gamma_\infty \begin{pmatrix} * & 0 & * & * \\ * & a & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & d \end{pmatrix} \Gamma_\infty \in \mathcal{H}(\Gamma_\infty) \quad (a, d \in \mathbb{N}). \]

The rational number \( \text{Sign}(U) = ad^{-1} \) is called the signature of \( U \). It is a multiplicative invariant of the double cosets since

\[
\text{Sign} \left( \Gamma_\infty M_1 \Gamma_\infty \cdot \Gamma_\infty M_2 \Gamma_\infty \right) = \text{Sign} \left( \Gamma_\infty M_1 \Gamma_\infty \right) \cdot \text{Sign} \left( \Gamma_\infty M_2 \Gamma_\infty \right).
\]

One can easily prove (see [G5, Proposition 5.2]).
Lemma A.1. Let $\tilde{\phi}(Z) = \phi(z_1, z_2) \exp(2\pi i tz_3) \in M^J_{k,t}$, and $U \in \mathcal{H}(\Gamma_{\infty})$ be a double coset of signature $ad^{-1}$. Then we have $(\tilde{\phi}|_k U)(Z) \in M^J_{k, tad^{-1}}$ if $tad^{-1} \in \mathbb{N}$ and $(\tilde{\phi}|_k U)(Z) \equiv 0$ otherwise.

The Hecke algebra $\mathcal{H}(\Gamma_{\infty})$ is not commutative and contains zero divisors. This algebra has two important properties:

a) $\mathcal{H}(\Gamma_{\infty})$ might be viewed as a non-commutative extension of the standard Hecke algebra $\mathcal{H}(\Gamma_1) = \mathcal{H}_\mathbb{Q}(Sp_4(\mathbb{Z}), GSp_4(\mathbb{Z}))$ of the Siegel modular group;

b) $\mathcal{H}(\Gamma_{\infty})$ contains two copies of the Hecke algebra of the special linear group $\mathcal{H}(SL_2(\mathbb{Z})) = \mathcal{H}_\mathbb{Q}(SL_2(\mathbb{Z}), M^J_{2}(\mathbb{Z}))$.

These two algebras are the tensor products of their local components

$$\mathcal{H}(SL_2(\mathbb{Z})) = \bigotimes_p \mathcal{H}_p(SL_2(\mathbb{Z})), \quad \mathcal{H}(\Gamma_1) = \bigotimes_p \mathcal{H}_p(\Gamma_1)$$

which are the polynomial rings with generators

$$T(p) = SL_2(\mathbb{Z}) \text{diag}(1, p)SL_2(\mathbb{Z}), \quad T(p, p) = SL_2(\mathbb{Z}) pE_2 SL_2(\mathbb{Z})$$

and

$$T_p = \Gamma_1 \text{diag}(1, 1, p, p)\Gamma_1, \quad T_{1,p} = \Gamma_1 \text{diag}(1, p, p^2, p)\Gamma_1, \quad \Delta_p = \Gamma_1(pE_4)\Gamma_1$$

respectively. We remind that

$$\mathcal{H}_p(\Gamma) = \{ \sum a_i \Gamma M_i \in \mathcal{H}_p(\Gamma) \mid \mu(M_i) = p^{m_i} \}$$

where $\Gamma$ is one of the groups $\Gamma_t$, $\Gamma_{\infty}$ or $\Gamma^{(t)}$.

There is a natural embedding of $\mathcal{H}(\Gamma_1)$ into $\mathcal{H}(\Gamma_{\infty})$. If $V = \sum_i a_i \Gamma_1 M_i \in \mathcal{H}(\Gamma_1)$, then according to the elementary divisors theorem one can represent $V$ in the form $V = \sum_i a_i \Gamma_1 M'_i$, where $M'_i \in \mathcal{G}\Gamma_{\infty}(\mathbb{Z})$. It is easy to see that the map

$$\text{Im} : U = \sum_i a_i \Gamma_1 M_i \to \sum_i a_i \Gamma_{\infty} M'_i$$

is an embedding of $\mathcal{H}(\Gamma_1)$ into $\mathcal{H}(\Gamma_{\infty})$ (see [G3], [G5] for more general constructions).

We shall identify $\mathcal{H}(\Gamma_1)$ with its image in $\mathcal{H}(\Gamma_{\infty})$.

The ring $\mathcal{H}(\Gamma_{\infty})$ contains two subrings isomorphic to $\mathcal{H}(SL_2(\mathbb{Z}))$:

$$\mathcal{H}(SL_2(\mathbb{Z})) \downarrow \left( \begin{array}{c} j \downarrow \ j+  \\ \mathcal{H}(\Gamma_{\infty}) \end{array} \right)$$
It is enough to define the embeddings $j_{\pm}$ for the generators $T(p)$ and $T(p, p)$ whose images we denote by $T_{\pm}(p) = j_{\pm}(T(p))$ and $\Lambda_{\pm}(p) = j_{\pm}(T(p, p))$. By definition, we have

$$
T_{-}(p) = \Gamma_{\infty} \text{diag}(1, p, p, 1) \Gamma_{\infty}, \quad \Lambda_{-}(p) = \Gamma_{\infty} \text{diag}(p, p^2, p, 1) \Gamma_{\infty},
$$

$$
T_{+}(p) = \Gamma_{\infty} \text{diag}(1, 1, p, p) \Gamma_{\infty}, \quad \Lambda_{+}(p) = \Gamma_{\infty} \text{diag}(p, 1, p, p^2) \Gamma_{\infty}.
$$

These elements are connected with the standard generators of the symplectic Hecke algebra $H_p(\Gamma_1)$

$$
T(p) = T_{-}(p) + T_{+}(p), \quad T_{1,p} = \Lambda_{-}(p) + \Lambda_{+}(p) + T_{0}(p) + \nabla_p - \Delta_p \quad (A.2)
$$

where

$$
T_{0}(p) = \Gamma_{\infty} \text{diag}(1, 1, p, p^2) \Gamma_{\infty}, \quad \nabla_p = \sum_{r \in \mathbb{Z}/p\mathbb{Z}} \Gamma_{\infty} \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & r \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}
$$

and $\Delta_p = \Gamma_{\infty}(pE_4) \Gamma_{\infty}$. (The last element is a central element of $H(\Gamma_{\infty})$.)

For any $V = \sum_i a_i \Gamma_{\infty}M_i \in H(\Gamma_{\infty})$, we denote by $\text{Sign}_s(V)$ or $V^{(s)}$ its homogeneous part of signature $s$

$$
V^{(s)} = \text{Sign}_s(V) = \sum_{i, \text{sign}(M_i) = s} a_i \Gamma_{\infty} M_i.
$$

The commutativity of the elements $T_{\pm}(p)$, $\Lambda_{\pm}(p)$, $T_{0}(p)$ and $\nabla_p$, for different primes, follows from the lemma (see [G2, Satz 5.1]).

**Lemma A.2.** Let $U = \sum_a U^{(a)}$ and $V = \sum_b V^{(b)}$ be the decompositions in $H(\Gamma_{\infty})$ of elements $U, V \in H(\Gamma_1)$ in the sum of homogeneous components with the same signature. Then the identity

$$
\sum_{ab = r} U^{(a)} \cdot V^{(b)} = \sum_{ab = r} V^{(b)} \cdot U^{(a)}
$$

is valid.

We shall see that the formula of Theorem 1.7 for $f_{p}(N)$ and the formula (1.12) follow from commutator relations in the non-commutative ring $H(\Gamma_{\infty})$, which reflect some properties of local $L$-functions of $Sp_4(\mathbb{Z})$ and $SL_2(\mathbb{Z})$.

We remind that the Hecke element $T(m) \in H(SL_2(\mathbb{Z}))$ is equal to the sum of all distinct double cosets with determinant $m$

$$
T(m) = \sum_{a|b, ab = m} SL_2(\mathbb{Z}) \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} SL_2(\mathbb{Z}).
$$

For a prime $p$, we introduce the $p$-local Hecke polynomial

$$
Q_p(X) = 1 - T_p(X) + pT_p(p)X^2
$$
which is the denominator of the local zeta-function of the group $SL_2(\mathbb{Z})$

$$\sum_{d \geq 0} T(p^d) X^d = Q_p(X)^{-1}.$$  \hfill (A.3)

A similar to $Q_p(X)$ polynomial for the symplectic group is the polynomial over $\mathcal{H}_p(\Gamma_1)$

$$S_p(X) = 1 - T_p X + p(T_1 + (p^2 + 1)\Delta_p) X^2 - p^3 \Delta_p T_p X^3 + p^6 \Delta_p^2 X^4$$

which is the denominator of the Hecke series of type (A.3) for $Sp_4(\mathbb{Z})$. This polynomial is the local factor of the so-called Spin-$L$-function of $Sp_4(\mathbb{Z})$.

The next identity (see [G5]) shows a connection between the polynomial $S_p(X)$ and the ±-embeddings of $Q_p(X)$ into the parabolic extension $\mathcal{H}(\Gamma_1)$ of $\mathcal{H}(\Gamma_1)$. It is

$$S_p(X) = Q_{p,-}(X)(1 - K_p X^2)Q_{p,+}(X)$$  \hfill (A.4)

where

$$Q_{p,\pm}(X) = 1 - T_\pm(p) X + p\Lambda_\pm(p) X^2, \quad K_p = p^2 \Delta_p - p\nabla_p.$$  

(The proof of (A.4) and some more general identities of this type see in [G3], [G5].)

The “minus”-embedding is used in the construction of the arithmetical lifting of the Jacobi forms. By definition of the “minus”-embedding, we have

$$T_-(m) = \sum_{ad = m \mod b} \Gamma_\infty \begin{pmatrix} a & 0 & b & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Hence by Lemma A.1

$$(\tilde{\phi}|_k T_-(m))(Z) = m^{2k-3} \sum_{ad = m \mod b} d^{-k} \phi \left( \frac{az_1 + b}{d}, az_2 \right) \exp(2\pi i mtz_3) \in M^J_{k,mt}$$

for any $\tilde{\phi}(Z) = \phi(z_1, z_2) \exp(2\pi i tz_3) \in M^J_{k,t} (k > 0)$. In [G1, Theorem 3] it was proved that for any Jacobi cusp form $\tilde{\phi}(Z) \in S^J_{k,t} (k > 0)$ the function

$$\text{Lift}(\tilde{\phi})(Z) = \sum_{m=1}^{\infty} m^{2-k} (\tilde{\phi}|_k T_-(m))(Z)$$  \hfill (A.5)

is a cusp form of weight $k$ with respect to the paramodular group $\Gamma_t$.

Another application of the same method was found in [GN1] where we constructed modular forms with respect to the paramodular group $\Gamma_t$ of type

$$G(Z) = \tilde{\psi}(Z) \exp \left( - \sum_{m=1}^{\infty} m^{-1} (\tilde{\phi}|_0 T_-(m))(Z) \right)$$  \hfill (A.6)

where $\tilde{\psi}(Z)$ is a Jacobi cusp form of integral or half integral index and $\tilde{\phi}(Z)$ is a weak Jacobi form of weight 0 and index $t$. The identity of type (A.6) implies the product formula (0.2) of the modular form $\Delta_5(Z)$ (see [GN1, Theorem 4.1]).
For $t=1$ the lifting (A.5) coincides with the Maass lifting and it commutes with action of Hecke operators from $\mathcal{H}(\Gamma_1)$. A proof of this fact in [EZ, §6] is based on the exact formulae for action of Hecke operators on Fourier coefficients of Siegel modular forms. Another proof using parabolic Hecke operators from $\mathcal{H}(\Gamma_\infty)$, one can find in [G4], where the commutativity of the lifting and Hecke operators was proved for the Hermitian modular group $SU(2, 2)$. In [G4] we used that the lifting is a modular form.

Bellow we prove the commutativity of the lifting of Jacobi forms with action of Hecke operator as a corollary of some commutator identities in the non-commutative extension $\mathcal{H}(\Gamma_\infty)$. Such relations provide a purely algebraic proof of the result mentioned above for any paramodular group $\Gamma_t$ and an exact formula for the action of arbitrary multiplicative Hecke operator of type (1.3) on modular forms of type (A.6).

The next proposition will play the crucial role in the proof.

**Proposition A.3.** 1. For any element $V \in \mathcal{H}(\Gamma_1)$, the formal power series

$$R_p(V, X) = Q_{p,-}(X)^{-1} V Q_{p,-}(X) = \left( \sum_{d \geq 0} T_-(p^d)X^d \right) V Q_{p,-}(X)$$

is a polynomial in $X$ over $\mathcal{H}(\Gamma_\infty)$.

2. Let $V \in \mathcal{H}_p(\Gamma_1)$ and let $F_t(Z) = \phi(z_1, z_2) \exp(2\pi i t z_3)$ be a $|_k$-invariant function with respect to $\Gamma_\infty$ where $(t, p) = 1$. Then

$$F_t|_k R_p(V, p^{2-k}) = \sum_{d \geq 0} F_t|_k \text{Sign}_1(T_-(p^d)V) p^{(2-k)d}$$

where $\text{Sign}_1(U)$ denote the part of $U \in \mathcal{H}_p(\Gamma_\infty)$ with signature 1 (see Lemma A.2).

**Proof.** 1. $\mathcal{H}(\Gamma_1)$ is commutative, hence we have

$$S_p(X)^{-1} V S_p(X) = V$$

for any $V \in \mathcal{H}(\Gamma_1)$. Using (A.4), we get

$$Q_{p,-}(X)^{-1} V Q_{p,-}(X) = (1 - K_p X^2) Q_{p,+}(X) V Q_{p,+}(X)^{-1} (1 - K_p X^2)^{-1}. \quad (A.7)$$

By definition, we have

$$\text{Sign}(T_-(p^d)) = p^d, \quad \text{Sign}(T_+(p^d)) = p^{-d}, \quad \text{Sign}(K_p) = 1.$$ 

Comparing the signature of the both sides of (A.7), we see that the formal power series $Q_{p,-}(X)^{-1} V Q_{p,-}(X)$ is a polynomial, because

$$\min(\text{Sign}(V)) \leq \text{Sign} \left( Q_{p,-}(X)^{-1} V Q_{p,-}(X) \right) \leq \max(\text{Sign}(V))$$

where $\min(\text{Sign}(V))$ (resp. $\max(\text{Sign}(V))$) is the minimal (resp. maximal) signature of the components of $V$.

2. According to Lemma A.1 and to the condition $(t, p) = 1$, we have $F_t|_k W = 0$ if $\text{Sign}(W) = p^{-n}$ with $n > 0$. Hence all elements with signature of type $p^n$
Proposition A.3 and Lemma A.2 we get the formula of Corollary □
the definition of the lifting using a formal infinite product (see [G1])
For any \( V \) and index \( \Gamma \)
\( \text{Sign}(\phi) = \text{Sign}(M) = 1 \).
\( \text{Sign} \) of the function \( F \)
\( \text{Sign} \) is the component of signature \( p^d \) of the Hecke element
\( T(p^d) \) is the component of signature \( p^d \) of the Hecke element
\( T(p^d) \in \mathcal{H}(\Gamma_1) \). Therefore, by Lemma A.2,
\( \text{Lift} (\tilde{\phi})|_k V = \text{Lift} (\tilde{\phi}|_k J_k(V)). \)
Proof. Without loss of generality we suppose that \( V \) is a double coset. Let
\( V = \Gamma_1 \text{diag}(a, b, c, d) \Gamma_1 = \prod_{p|\nu} V_p, \quad (v = ac = bd = \mu(V)) \)
be the decomposition of \( V \in \mathcal{H}(\Gamma_1) \) as the product of its local components \( V_p \in \mathcal{H}_p(\Gamma_1) \). The element \( T_-(p^d) \) is the component of signature \( p^d \) of the Hecke element
\( T(p^d) \in \mathcal{H}_p(\Gamma_1) \). Therefore, by Lemma A.2,
\( \mathcal{J}_k(V) = \left( \prod_{p|\mu(V)} Q_{p,-}(p^{2-k})^{-1} \right) V \left( \prod_{p|\mu(V)} Q_{p,-}(p^{2-k}) \right) = \prod_{p|\mu(V)} R_p(V_p, p^{2-k}). \)
Due to the rationality of the Hecke series of \( \mathcal{H}(SL_2(\mathbb{Z})) \) (see (A.3)), we can rewrite the definition of the lifting using a formal infinite product (see [G1])
\( \text{Lift} (\tilde{\phi}) = \tilde{\phi}|_k \prod_{m=1}^{\infty} m^{2-k} T_-(m) = \tilde{\phi}|_k \prod_{p} Q_{p,-}(p^{2-k})^{-1}. \)
In view of Proposition A.3 and Lemma A.2 we get the formula of Corollary
\( \text{Lift} (\tilde{\phi})(V) = \left( \tilde{\phi}|_k \left( \prod_{p|\mu(V)} Q_{p,-}(p^{2-k})^{-1} V \prod_{p|\mu(V)} Q_{p,-}(p^{2-k}) \right) \right)|_k \prod_{p} Q_{p,-}(p^{2-k})^{-1}. \)
**Example A.5.** (See [EZ, Theorem 6.3].) One can calculate the images of the generators of the Hecke algebra $\mathcal{H}(\Gamma_1)$ under $J_k$ using (A.2). One can easily check that

$$T_- (p) T_+ (p) = p T_0 (p) + (p^3 + p^2) \Delta_p, \quad (A.9)$$

$$T_- (p^2) \Lambda_+ (p) = p^3 T_0 (p) \Delta_p + p^4 \Delta_p^2, \quad (A.10)$$

Hence

$$\widetilde{\phi} \big| k J_k (T_p) = \widetilde{\phi} \big| k (p^{3-k} T_0 (p) + p^{k-1} + p^{k-2}),$$

$$\widetilde{\phi} \big| k J_k (T_{1,p}) = \widetilde{\phi} \big| k ((p+1) T_0 (p) + p^{2k-6} (p^2 - 1)).$$

The element $T_0 (p) \in \mathcal{H}(\Gamma_\infty)$ has the expansion in the sum of left cosets

$$\Gamma_\infty \text{diag}(1, p, p^2, p) \Gamma_\infty = \sum_{a \mod p} \Gamma_\infty \left( \begin{array}{cccc} p^2 & 0 & 0 & 0 \\ -p a & p & 0 & 0 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & p \end{array} \right) + \sum_{a \mod p} \Gamma_\infty \left( \begin{array}{cccc} 1 & 0 & c & a \\ 0 & p & p a & 0 \\ 0 & 0 & p^2 & 0 \\ 0 & 0 & 0 & p \end{array} \right)$$

$$+ \sum_{b \not\equiv 0 \mod p} \sum_{a \mod p} \Gamma_\infty \left( \begin{array}{cccc} p & 0 & b & a b \\ 0 & p & a b & a^2 b \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{array} \right).$$

Using this system of representatives, one can see that the operator $| k T_0 (p)$ is connected with Hecke-Jacobi operator $T^J (p)$ of Eichler-Zagier

$$(\phi \big| k, t T^J (p) ) (z_1, z_2) = p^{k-4} \sum_{M \in SL_2(\mathbb{Z}) \backslash M_2^+ (\mathbb{Z})} \sum_{X \in \mathbb{Z}^2 / p \mathbb{Z}^2} \phi \big| k, t M \mid X \det (M) = p^2 \gcd (M) = 1 \text{ or } p^2$$

(see [EZ, §3]) by the formula

$$\widetilde{\phi} \big| k T_0 (p) = p^{k-3} \widetilde{\phi} \big| k T^J (p).$$

For the action of $T_0 (p)$ on a Jacobi form $\widetilde{\phi} (Z) = \phi (z_1, z_2) \exp (2 \pi i z_3)$ of index 1, we have the formula

$$\widetilde{\phi} \big| k T_0 (p) (Z) = \sum_{n, l} f_p (4 n - l^2) \exp (2 \pi i (n z_1 + l z_2 + t z_3))$$

with

$$f_p (N) = \begin{cases} p^{k-3} (f (p^2 N) + p^{k-2} f (N)) + p^{2k-3} f (\frac{N}{p}) & \text{if } k > 0, \\ p^3 f (p^2 N) + p f (N) + f (\frac{N}{p}) & \text{if } k = 0 \end{cases} \quad (A.11)$$

where we use the notation (1.8)–(1.9). We remark that Fourier coefficients $f (n, l)$ of an arbitrary Jacobi form of index 1 depend only on the norm $4 n - l^2$. 

**IGUSA MODULAR FORM AND KAC–MOODY ALGEBRAS**

33
The restriction \( t = 1 \) in Corollary A.4 is not principal. The lifting \((A.5)\) is a modular form with respect to the paramodular group \( \Gamma_t \). The Jacobi form \( \tilde{\phi}_t(Z) = \tilde{\phi}(z_1, z_2) \exp(2\pi i tz_3) \) of weight \( k \) and index \( t \) is invariant with respect to the parabolic subgroup \( \Gamma_\infty^{(t)} = \Gamma_t \cap \Gamma_\infty(\mathbb{Q}) \). Let us consider the Hecke algebra \( \mathcal{H}(\Gamma_\infty^{(t)}) \) containing \( \pm \)-embeddings of \( \mathcal{H}(SL_2(\mathbb{Z})) \) (one may use the same definition as before) and the Hecke algebra of all elements having a good reduction for the paramodular group \( \Gamma_t \)

\[
\mathcal{H}_*(\Gamma_t) = \bigotimes_{(p, t) = 1} \mathcal{H}_p(\Gamma_t) \cong \bigotimes_{(p, t) = 1} \mathcal{H}_p(\Gamma_1).
\]

For a prime \( p \), which does not divide \( t \), the new Hecke algebras are isomorphic to the Hecke algebras connected with the symplectic group

\[
\mathcal{H}_p(\Gamma_t) \cong \mathcal{H}_p(\Gamma_1), \quad \mathcal{H}_p(\Gamma_\infty^{(t)}) \cong \mathcal{H}_p(\Gamma_\infty).
\]

Thus we can define the same morphism \( J \) for any element \( V = \sum_{a} V^{(a)} \in \mathcal{H}_*(\Gamma_t) \)

\[
J_k^{(t)}(V) = \sum_{m \geq 1} m^{2-k} T_-(m) V^{(m^{-1})} \in \mathcal{H}^{(0)}(\Gamma_\infty^{(t)}).
\]

(A.8-t)

Then we have

**Corollary A.4-t.** Let \( \tilde{\phi}_t(Z) = \tilde{\phi}(z_1, z_2) \exp(2\pi i tz_3) \in S_{k,t}^J \) be a Jacobi cusp form of weight \( k \) and index \( t \), and \( V \in \mathcal{H}_*(\Gamma_t) \) be a Hecke element with good reduction. Then

\[
(Lift(\tilde{\phi}_t))|_k V = Lift(\tilde{\phi}_t|_k J_k^{(t)}(V))
\]

(See [G2] where the local factors of the Spin-\( L \)-function of the lifting were calculated for prime \( p \) with an arbitrary reduction.)

Now we consider an application of Proposition A.3 to the modular forms of type (A.6). Firstly, we have a Jacobi analogues of Lemma 1.1.

**Lemma A.6.** Let \( \phi(z_1, z_2) \exp(2\pi i tz_3) \in M_{k,t}^J \) be a Jacobi form of weight \( k \) and index \( t \), and \( V = \Gamma_\infty^{(t)} M_{\infty}^{(t)} = \sum_i \Gamma_\infty^{(t)} M_i \in \mathcal{H}(\Gamma_\infty^{(t)}) \) be a double coset of the parabolic Hecke ring. Then the function

\[
[\phi](Z)_V := \prod_i (\phi|_k M_i)(Z)
\]

is a Jacobi form of weight \( kv \) and index \((t\nu)\text{Sign}(V)\) where \( \nu \) is the number of the left cosets in \( V \).

**Theorem A.7.** Let

\[
G(Z) = \bar{\psi}(Z) \exp \left( -\sum_{m=1}^{\infty} \bar{\phi}|_0 m^{-1} T_-(m)(Z) \right)
\]

be a modular form (see (A.6)) of weight \( k \) with respect to the paramodular group \( \Gamma_t \) where \( \bar{\psi}(Z) \) is a Jacobi form of weight \( k \) and \( \bar{\phi}(Z) \) is a Jacobi form of weight \( 0 \) and index \( t \). Let \( V \in \mathcal{H}_*(\Gamma_t) \) be a Hecke element. Then the formula

\[
[G(Z)]_V = [\bar{\psi}(Z)]_V \exp \left( -\sum_{m=1}^{\infty} (\phi_{0,t}|_{J_0^{(t)}(V)})|_0 m^{-1} T_-(m)(Z) \right)
\]
holds where
\[ J_0^{(t)}(V) = \sum_{m \geq 1} m^{-1} T_-(m)V^{(m-1)} \in \mathcal{H}^{(0)}(\Gamma^t) \]
and \( |G(Z)|_V \) is defined in (1.3).

Remark. In the case of Jacobi forms of weight 0, we do not have a normalizing factor in the definition of the representation \( \phi_{0,t}|_0 V \) (see (A.1)). This explains different definition of the operator \( J_0 \).

Proof. Let us denote the function under exponent in \( G(Z) \) by \( F(Z) \). Each multiplicative Hecke operator \( \ldots |_V \) acting on \( \exp(F(Z)) \) turns into a usual Hecke operator on the function \( F(Z) \) under the exponent:

\[ [\exp(F(Z))]_V = \exp(F|_0 V(Z)). \]

We remark that the element \( V \) to the right is considered as an element of the Hecke ring \( \mathcal{H}(\Gamma^t) \). The function \( F(Z) \) is of the form considered in Corollary A.4

\[ F(Z) = \tilde{\phi}|_0 \prod_p Q_{p,-}(p^{-1})^{-1}. \]

Using Proposition A.3, we get the formula of Theorem A.7.

\( \square \)

Example A.8. Let us calculate \( [\Delta_5(Z)]_V \) for \( V = T_p \) and \( V = T_{1,p} \). Using (A.9)–(A.10), we have

\[ \tilde{\phi}|_0 J_0(T_p) = \tilde{\phi}|_0 (T_0(p) + p^2 + p) \]
\[ \tilde{\phi}|_0 J_0(T_{1,p}) = \tilde{\phi}|_0 ((p + 1)T_0(p) + p^2 - 1). \]

In view of (A.11), we get the formula (1.12) for the exponents \( f_p(N) \) in the infinite product representation of \( [\Delta_5(Z)]_p \).

Appendix B. Siegel modular forms with divisors \( H_D \).

In the remark after Corollary 1.4 we have mentioned a result of van der Geer. He proved in [vdG2] that there exists a Siegel modular form of weight \( -60H(2,D) \) with the divisor

\[ G_D = \bigcup_{e^2|D} H_{D/e^2}, \]

where \( H(2,D) \) is the so-called H. Cohen number and \( H_{D/e^2} \) is the Humbert surface of discriminant \( D/e^2 \). The proof in [vdG2] is based on the calculation of degree of the divisor \( G_D \) in a smooth compactification of the Siegel modular threefold and does not provide an exact construction of Siegel modular forms with known divisors. Like in Sect. 1 and Appendix, repeating exponential Hecke operators and considering quotients, one can easily construct, starting from the form \( \Delta_5(Z) \), Siegel modular forms with divisors \( H_{D/e} \) for any \( D \), and their infinite product expansions. In this appendix we describe an infinite product construction of modular forms with the divisors \( H_D \) where \( D \) is not a perfect square.
An example of the Siegel modular form of weight 24 with the divisor \( H_5 \) was found in [vdG2] in terms of the Igusa’s generators of the graded ring of Siegel modular forms:

\[
G^{(5)} = (\chi_{12} - 2^{-1}2^{3-3}E_6^2 + E_4^3)^2 - E_4(2 \cdot 3^{-1}\chi_{10} - 2^{11}3^{-6}E_6E_4)^2.
\]

We find below an infinite product expansion for this function.

Let \( j(\tau) \) be the \( SL_2(\mathbb{Z}) \)-modular invariant function of weight 0

\[
j(\tau) = \frac{E_4(\tau)^3}{\Delta_{12}(\tau)} = \frac{(1 + 240q + 2160q^2 + \ldots)^3}{q - 24q^2 + 253q^3 + \ldots} = q^{-1} + 744 + 19684q + 21493760q^2 + \ldots
\]

where \( E_4(\tau) \) is the Eisenstein series of weight 4 and \( \Delta_{12}(\tau) \) is the cusp form of weight 12 for \( SL_2(\mathbb{Z}) \).

The product \( j(z_1)\phi_{0,1}(z_1, z_2) \) (see (1.8)) is a meromorphic Jacobi form of weight 0 and index 1. This function is not a weak Jacobi form, since it has the Fourier expansion

\[
j(z_1)\phi_{0,1}(z_1, z_2) = q^{-1}(r^{-1} + 10 + r) + (10r^{-2} + 680r^{-1} + 7548 + 680r + 10r^2) + q(\ldots).
\]

Let us define another Jacobi form of the same type

\[
\phi_{0,1}^{(5)}(z_1, z_2) = j(z_1)\phi_{0,1}(z_1, z_2) - 10(\phi_{0,1} | T_0(2))(z_1, z_2) - 680\phi_{0,1}(z_1, z_2)
\]

where \( T_0(2) \) is the Hecke-Jacobi operator (see (A.11)). This Jacobi form has the Fourier expansion of the type

\[
\phi_{0,1}^{(5)}(z_1, z_2) = \sum_{n \geq -1, l \in \mathbb{Z}} g(n, l) \exp(2\pi i (nz_1 + lz_2)) = q^{-1}(r^{-1} + r) + 48 + q(\ldots).
\]

Like in §1 we use the Fourier coefficients of the last Jacobi form to define an infinite product which is a Siegel modular form.

**Theorem B.1.** The infinite product

\[
F^{(5)}(Z) = r^{-1}(qr - s)(q - rs) \prod_{n,l,m \in \mathbb{Z}, (n,l,m) > 0} (1 - q^n r^l s^m)^{g(nm,l)},
\]

where \( (n,l,m) > 0 \) means that \( n \geq 0, m \geq 0, n + m > 0 \) and \( l \) is an arbitrary integer, defines a Siegel modular form of weight 24. The divisor of \( F^{(5)}(Z) \) is the Humbert surface \( H_5 \) of discriminant 5.

**Proof.** One can easily proof this theorem using the method we used in [GN1, Theorem 4.1] (see also [B5, Theorem 5.1]). We remark only the main steps of the proof.
i) The given infinite product absolutely converges for all \( \text{Im}(Z) > C \) for a large \( C > 0 \) and can be continued to a multi-valued analytic function on \( \mathbb{H}_2 \).

ii) It is invariant with respect to the changing of variables \( q \to s, s \to q \) (equivalently \( z_1 \to z_3, z_3 \to z_1 \)).

iii) The function \( F^{(5)}(Z) \) has a representation in terms of the Hecke operators \( T_{-}(m) \)

\[
F^{(5)}(Z) = \Delta_{12}(z_1)^2 \exp \left( - \sum_{m=1}^{\infty} m^{-1} \left( \phi_{0,1}^{(5)} \mid_0 T_{-}(m) \right)(Z) \right).
\]

Hence the product is a \( \Gamma_{\infty} \)-invariant function of weight 24. The parabolic subgroup \( \Gamma_{\infty} \) and the transformation \( z_1 \to z_3, z_3 \to z_1 \) generate the Siegel modular group.

iv) Only the factors with negative square \( l^2 - 4nm < 0 \) of the infinite product have zeros on \( \mathbb{H}_2 \). There factors exist only with square \( l^2 - 4nm = -5 \), and they define the Humbert surface \( H_5 \).

\[ \square \]

We remark that the construction given above is general. It gives us a modular form with the irreducible divisor \( H_D \) for any \( D \). For \( D = 4d + 1 \) one can consider the Jacobi form

\[
j(z_1)^d \cdot \phi_{0,1}(z_1, z_2) = q^{-d}(r^{-1} + 10 + r) + q^{-d-1}(\ldots).
\]

For \( D = 4d \) one can start with the Jacobi form of weight 0

\[
\left( \phi_{0,1} \mid_0 T_0(2) \right)(z_1, z_2) - 2\phi_{0,1}(z_1, z_2) = q^{-1} + (r^2 + r^{-2} + 70) + q(\ldots)
\]

which defines the infinite product expansion of \( \Delta_{35}(Z) \).

We hope to present this construction in more details somewhere.

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