Energy Radiation of Charged Particles in
Conformally Flat Spacetimes

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Abstract

Original abstract: Consider the worldline of a charged particle in a static spacetime. Contraction of the time-translation Killing field with the retarded electromagnetic energy-momentum tensor gives a conserved electromagnetic energy vector which can be used to define the radiated electromagnetic energy. This note points out that for a conformally flat spacetime, the radiated energy is the same as for a flat spacetime (i.e. Minkowski space). This appears to be inconsistent with an equation of motion for such particles derived by DeWitt and Brehme [7] and later corrected by Hobbs [2]. [End of original abstract]

New abstract: Same as old abstract with last sentence deleted. The body of the paper is the same as previously. A new Appendix 2 has been added discussing implications to the previous arguments of recent work of Sonego (J. Math. Phys. 40 (1999), 3381-3394) and of Quinn and Wald (Phys. Rev. D 60 (1999), http://gr-qc/9610053).

1 Energy radiation in conformally flat spacetimes is the same as in Minkowski space.

Consider a spacetime which admits a Killing vector field \( K = K^i \). If \( T = T^{ij} \) is any symmetric tensor with vanishing covariant divergence, such as an energy-momentum tensor, then the vector \( E^j := K_\alpha T^{\alpha j} \) has vanishing divergence, and so defines a “conserved quantity” ([1], p. 96). In Minkowski space one obtains conservation of energy-momentum in this way from the Killing fields corresponding to space-time translations.

Now consider a static spacetime, which means that there is a “static” coordinate system \( x^i \) in which the metric coefficients \( g_{ij} = g_{ij}(x^1, x^2, x^3) \) are independent of the “time” coordinate \( x^0 \) and that \( g_{0j} = 0 \) for spatial indices \( I = 1, 2, 3 \). The infinitesimal generator \( \partial_0 \) of time translation is then a Killing generator.

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1 We follow the standard convention of using upper indices for contravariant vectors and lower indices for covariant vectors, with repeated upper and lower indices assumed summed unless otherwise indicated. Local coordinates are denoted by upper indices. The symbol “:=” means “equals by definition”.

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vector field, and this defines as above a conserved quantity $E^j := g_{00} T^{0j}$ which is usually interpreted as electromagnetic “energy” when $T$ is the usual energy-momentum tensor of the electromagnetic field. There are special cases in which this interpretation may not be appropriate (cf. [3]), but in this paper, we’ll follow tradition and call this conserved quantity “energy”. In this paper, “radiation” always means energy radiation unless otherwise specified. We want to study the energy radiation of a charged particle with a prescribed worldline, such as a freely falling particle.

It is “well known” that a stationary charged particle in a static spacetime does not radiate.\(^2\) A stationary particle is accelerated relative to local inertial frames, but is apparently unaccelerated with respect to the static coordinate frame. On the other hand, a freely falling particle is (by definition) unaccelerated with respect to local inertial frames but is, in general, apparently accelerated with respect to the static coordinate frame. One might guess that if a stationary (but accelerated) particle does not radiate, then a freely falling particle should radiate, the idea being that it may be acceleration relative to the static frame that determines radiation rather than acceleration relative to local inertial frames. This is not necessarily at variance with ideas of general covariance because singling out the time-like Killing vector field $\partial_0$ in a static spacetime destroys general covariance.

This problem has, in our opinion, never been satisfactorily treated in the literature. Unfortunately, this paper does not furnish a general analysis, but we do obtain an answer for any spacetime whose metric is of the special form:

$$g_{ij} dx^i dx^j = k(x^1, x^2, x^3)^2 \left[ (dx^0)^2 - \sum_{J=1}^{3} (dx^J)^2 \right],$$

(1)

that is, for any static, conformally flat metric. The essence of the answer is that a charged particle will radiate when and only when it is accelerated with respect to the “static” coordinate system $x^i$ with respect to which the static metric (1) is written. In particular, a freely falling particle can radiate.

For certain conformal factors $k$ in (1), this is at variance with the main results of [7], [2], [3]. They derive an equation of motion from considerations of energy-momentum balance, in effect calculating energy-momentum radiation without explicitly calling it that. After correction by [2] of an error in [7], this equation of motion (our equation (8) below) looks like the Lorentz-Dirac equation with an additional term involving the Ricci tensor. The method of [3] leads to the conclusion that if a metric $g$ of form (1) is such that the this additional term vanishes, then a charged particle can follow a geodesic (i.e., the same “freely falling” motion that an uncharged particle would have). This implies that there is no “radiation reaction” force and presumably, no radiation.

\(^2\)The phrase “widely believed” in place of “well known” would be more nearly accurate, but would also suggest a possibility of doubt which probably does not exist. That stationary particles do not radiate has never been rigorously proved in reasonable generality, but seems to be universally believed. Further discussion and a proof under certain auxiliary hypotheses can be found in [3], Appendix 2.
Thus the definition above of “energy radiation” is in this situation in conflict with the definition implicit in [7] and [2]. Both definitions are commonly used. Neither definition is unreasonable, but neither is beyond question. We discuss possible objections to both definitions but advocate neither. The purpose of this paper is simply to point out that the two definitions are sometimes (and probably usually) inconsistent.

Let $F = F_{ij}$ denote the electromagnetic field tensor, and $T = T^{ij} = F^i_\alpha F^{\alpha j} - (1/4)F^{\alpha\beta}F_{\alpha\beta}g^{ij}$ its corresponding energy-momentum tensor. The tensor $T$ implicitly depends on the conformal factor $k^2$; denote by $\tilde{T}$ the corresponding tensor for $k \equiv 1$; i.e., for Minkowski space. More generally, when we compare various tensors in the spacetime with conformal factor $k^2$ with the corresponding tensors in Minkowski space, we’ll consistently denote the Minkowski space versions by tildes as above. Index raising and lowering of tensors will be defined relative to the relevant metric; for example, denoting the Minkowski metric by $(\tilde{g}_{ij}) := \text{diag}(1, -1, -1, -1)$,

\begin{equation}
(\tilde{g}_{ij}) := \text{diag}(1, -1, -1, -1),
\end{equation}

we define $\tilde{T}_{ij} := \tilde{T}^{\alpha\beta}g_{\alpha\beta}$.

We want to compare $T$ and $\tilde{T}$. Because of the conformal invariance of Maxwell’s equations, the 2-covariant form $F_{ij}$ of $F$ is independent of the conformal factor $k^2$, but each index-raising with respect to $g$ introduces a factor of $k^{-2}$ relative to the same index-raising in Minkowski space, and hence $T_{ij}$ differs from the corresponding tensor $\tilde{T}_{ij}$ for Minkowski space by a factor $k^{-6}$: $T_{ij} = k^{-6}\tilde{T}_{ij}$.

Let $K = K^i$ denote the Killing vector corresponding to time translation; in differential-geometric notation, $K = \partial_0$. For future reference, note that $K^i$ is independent of the conformal factor $k^2$, but $\tilde{K}_i := g_{\alpha\beta}K^\alpha = k^2\tilde{K}_i$. To any such Killing vector field $K$ is associated a “conserved” (i.e. zero-divergence) vector field $E^i := K_\alpha T^{\alpha i}$. For our Killing field $K := \partial_0$, the integral of the normal component of $E$ over any spacelike submanifold is interpreted as the energy in the submanifold. A similar integral over the three-dimensional timelike submanifold $S$ obtained by letting a two-dimensional surface (such as a sphere) evolve through time is interpreted as the energy radiated through the surface over the time period in question. Such integrals will be denoted $\int_S E^\alpha dS_\alpha$. A fuller discussion is given in [5], and precise mathematical definitions can be found in [3], Section 2.8.

The relation between the above integral in the spacetime and in Minkowski space is:

\begin{equation}
\int_S E^\alpha dS_\alpha = \int_S k^{-4}\tilde{E}^\alpha k^4 d\tilde{S}_\alpha = \int_S \tilde{E}^\alpha d\tilde{S}_\alpha.
\end{equation}

For a timelike manifold $S$ consisting of a spacelike surface evolving through time, this says that the energy radiation through the surface is independent of the conformal factor $k^2$.

Since the cancellation of the factors of $k$ in (3) greatly simplifies our considerations, we would like to understand this with as little explicit calculation as
possible. To understand the symbolic substitution \( dS_\alpha = k^4 \tilde{dS}_\alpha \), think of an integral like \( \int_S E^\alpha \, dS_\alpha \) as obtained by the following process. Imagine decomposing the three-dimensional submanifold \( S \) into a large number of 3-dimensional “cubes”, each spanned by three tangent vectors to the submanifold. The factor \( k^4 \) in \( dS_\alpha = k^4 \tilde{dS}_\alpha \), as obtained by the following process. Imagine decomposing the three-dimensional submanifold \( S \) into a large number of 3-dimensional “cubes”, each spanned by three tangent vectors to the submanifold. The factor \( k^4 \) in \( dS_\alpha = k^4 \tilde{dS}_\alpha \), can be seen by looking at \( E^i := T^i_\alpha K_\alpha \). As noted above, passing from \( T \) to \( \tilde{T} \) involves three index-raisings of \( F_{ij} \), which introduces a factor of \( k^{-6} \), whereas \( K_\alpha = k^2 \tilde{K}_\alpha \).

In short, the energy radiation of a particle with an arbitrary worldline is the same as in Minkowski space, which is well-known ([4], p. 160) to be essentially given by the integral of the square of the proper acceleration over the worldline. More precisely, if the particle has charge \( q \) and is unaccelerated (in Minkowski space, not with respect to the metric (1)) in the distant past and future, then for a particle with Minkowski space four-velocity \( \tilde{u}(\tilde{\tau}) \) at Minkowski proper time \( \tilde{\tau} \) and Minkowski proper acceleration \( \tilde{a} := d\tilde{u}/d\tilde{\tau} \), the total energy radiation over all time is:

\[
\int_S E^\alpha \, dS_\alpha = \int_S \tilde{E}^\alpha \, \tilde{dS}_\alpha = -\frac{2}{3} q^2 \int_{-\infty}^{\infty} \tilde{\alpha}^2 \tilde{u}^0 \, d\tilde{\tau} ,
\]

where \( \tilde{\alpha}^2 := \tilde{\alpha}^\alpha \tilde{\alpha}_\alpha := \tilde{\alpha}^\alpha \tilde{g}_{\alpha\beta} \tilde{\alpha}^\beta \).

The reason for assuming that the particle be unaccelerated in distant past and future is that only under this hypothesis do all commonly used calculational methods give identical results. (Further explanation will be given in Section 2, and a complete treatment can be found in [4], Chapter 4.) Under this assumption we could also write the equivalent expression

\[
\int_S E^\alpha \, dS_\alpha = \int_S \tilde{E}^\alpha \, \tilde{dS}_\alpha = -\frac{2}{3} q^2 \int_{-\infty}^{\infty} \left[ \frac{d\tilde{u}^0}{d\tilde{\tau}} + \tilde{\alpha}^2 \tilde{u}^0 \right] \, d\tilde{\tau} ,
\]

and this will be convenient for later comparison with the results of [3]. It is customary to interpret the integrand

\[
-\frac{2}{3} q^2 \left[ \frac{d\tilde{u}^0}{d\tilde{\tau}} + \tilde{\alpha}^2 \tilde{u}^0 \right] ,
\]

as the proper-time energy radiation rate. (Interpreting \( (2q^2/3)\tilde{\alpha}^2 \tilde{u}^0 \) as the proper-time energy radiation rate leads to an inconsistency: cf. [4], p. 140.)

The interesting feature of (4) or (5) is that the acceleration which appears in the expression for the radiated energy is the Minkowski space acceleration \( \tilde{a} \).
rather than the acceleration computed relative to the metric $g$.\footnote{The “equivalence principle” might lead one to expect the opposite. However, this principle seems of dubious application to charged particles.} Put another way, the energy radiation is determined by the \textit{apparent} acceleration relative to the static coordinate frame rather than the physical acceleration experienced by the particle (i.e. the acceleration calculated using the semi-Riemannian connection induced by $g$).

This should not be surprising. For example, as previously noted, it seems universally believed that a stationary particle does not radiate energy even though a stationary particle is accelerated relative to $g$. (Here and elsewhere we use terms like “accelerated relative to $g$” as shorthand for “accelerated as measured by the unique connection compatible with $g$.”)

Now consider a charged particle which is stationary in the distant past, falls freely for a while, and then is brought to rest and remains stationary thereafter. By “falls freely” we mean that it is in a state of zero proper acceleration, so that its position $x^k(\tau)$ at proper time $\tau$ is governed by the geodesic equation,

$$\frac{d^2x^k}{d\tau^2} = -\Gamma^k_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau}$$  \hspace{1cm} (7)$$

where $\Gamma^k_{ij}$ is the connection induced by $g$.

Suppose that the conformal factor $k$ in (1) depends only on $x^1$: $k = k(x^1)$ . Then the nonzero connection coefficients are:

$$\Gamma^1_{00} \quad = \quad \Gamma^1_{11} = \frac{d \log k}{dx_1} = -\Gamma^1_{22} = -\Gamma^1_{33}$$

$$\Gamma^0_{01} \quad = \quad \Gamma^2_{12} = \Gamma^3_{13} = \frac{d \log k}{dx_1}$$

From this we see that a freely falling particle whose velocity is initially in the $x_1$-direction will maintain constant $x_2, x_3$ coordinates forever, and the $x_1$-component of (4) specializes to

$$\frac{d^2x^1}{d\tau^2} = -\frac{d \log k}{dx_1} \left[ \left( \frac{dx^0}{d\tau} \right)^2 + \left( \frac{dx^1}{d\tau} \right)^2 \right]$$

This implies that when $dk/dx^1 \neq 0$, which we assume in this section, the quantity $d^2x^1/d\tau^2$ does not vanish during the period of free fall.

The relationship between this quantity and the Minkowski space proper acceleration $\ddot{a}$ is messy, but one can see without detailed calculation that $\ddot{a}$ cannot vanish identically unless $d^2x^1/d\tau^2$ vanishes identically. One way to see this is to note that at the start of the free fall when the coordinate velocity $dx^1/d\tau$ vanishes,

$$\frac{d^2x^1}{d\tau^2} \quad = \quad \frac{d}{d\tau} \left( \frac{dx^1}{dx_0} \frac{dx^0}{d\tau} \right) = \frac{d^2x^1}{dx_0^2} \left( \frac{dx^0}{d\tau} \right)^2 + \frac{dx^1}{dx_0} \frac{d^2x^0}{d\tau^2} = \frac{1}{k^2} \frac{d^2x^0}{dx_0^2}$$
where the last equality is obtained by manipulating the identity $1 = k^2[(dx^0/d\tau)^2 - (dx^1/d\tau)^2]$ obtained from the definition (1) of the metric. Similarly, when the coordinate velocity vanishes, the Minkowski space proper acceleration $\ddot{a}$ satisfies
\[
\ddot{a}^2 = -(d^2 x^1/d\tau^2)^2 = -k^4\left(\frac{d^2 x^1}{d\tau^2}\right)^2
\]
so that $\ddot{a}$ cannot vanish at this time.

Given that $\ddot{a}$ does not vanish identically, the radiation given by (1) is nonzero because $\ddot{a}^2 \leq 0$. To summarize, in a spacetime with metric (1) and $k = k(x^1)$ satisfying $dk/dx_1 \neq 0$, there will be energy radiation (as defined above) from a charged particle initially at rest whose worldline thereafter is the same as that of a freely falling uncharged particle.

If energy is conserved, this implies that external forces would have to be applied to drive a charged particle along such a worldline, with the radiated energy supplied by these forces. Put another way, if energy as defined above is to be conserved, a charged particle acted on by no external forces could not fall freely, contrary to the main result of [3]. Section 4 explores this situation in more detail.

References [7] [2] [3] obtain their different results using, in effect, a different definition of energy-momentum radiation which we question in Section 2. They obtain, in effect, expressions for energy-momentum radiation in arbitrary spacetimes (not necessarily static), but they express these in terms of equations of motion obtained by setting up an equation of energy-momentum balance. When specialized to the case of the metric (1), [3] obtains the following equation of motion for a particle of mass $m$ and charge $q$ in an external field $F$:

\[
m\frac{du^i}{d\tau} = qF_{\alpha}^i u^\alpha + \frac{2}{3}q^2 \left[\frac{da^i}{d\tau} + a^2 u^i\right] - \frac{1}{3}q^2 \left[-R_{\beta}^i u^\beta + u^i R_{\alpha\beta} u^\alpha u^\beta\right]. \tag{8}
\]

Here $\tau$ is proper time, $\tau \mapsto z(\tau)$ the particle’s worldline, $u := dz/d\tau$ its four-velocity, $a := du/d\tau$ its proper acceleration, and $R$ the Ricci tensor. The difference in certain signs between (8) and equation (5.28) of [2] is because [2] uses a metric of signature $(-1, 1, 1, 1)$ opposite to ours.

The left side $m(du^i/d\tau)$ represents the rate of change of mechanical energy-momentum of the particle. The first term on the right, $qF_{\alpha}^i u^\alpha$, is the rate at which the external field furnishes energy-momentum. The remaining terms on the right represent the negative of the energy-momentum radiation rate.

This differs from our expression (4) for energy radiation in a several essential ways. First of all, the bracketed last term in (8) involving the Ricci tensor has no counterpart in our (4). However, [3] notes that the term involving the Ricci tensor vanishes for certain special conformal factors $k$. One such is

\[
k(x^0, x^1, x^2, x^3) = \frac{1}{x^1}, \tag{9}
\]
and we shall use this as a test case to compare the two approaches.
It is not entirely clear what is the proper way to compare (8) with our results (4) or (6), but it does not seem likely that the conclusions of [3] can be easily reconciled with ours. Since for certain conformal factors the term involving the Ricci tensor vanishes, the important radiation term in (8) would seem to be

\[ \frac{2}{3} q^2 \left[ \frac{da^i}{d\tau} + a^2 u^i \right]. \]  

(10)

In Minkowski space, this is just the radiation term in the Lorentz-Dirac equation.

If we interpret (10) as the proper-time energy-momentum radiation rate, then it would seem plausible to interpret the inner product of (10) with the unit vector \( k^{-1} \partial_0 \) as the proper-time energy radiation rate, the energy being calculated relative to the coordinate frame. Under this interpretation, the Hobbs method gives an energy radiation rate of

\[ \frac{2}{3} q^2 k \left[ \frac{da^0}{d\tau} + a^2 u^0 \right] = \frac{2}{3} q^2 k \left[ \frac{da^0}{d\tau} + a^2 k^{-1} u^0 \right], \]  

(11)

for the above case (9) in which the Ricci term vanishes. The factors of \( k \) are of no significance here. We are concerned with the proper-time radiation rate at a particular point on the worldline, and \( k \) can be normalized to unity at this point. The really significant difference between (11) and (6) is the replacement of the apparent (or Minkowski) acceleration \( \tilde{a} \) in (6) by the proper acceleration \( a \) in (11). Even if one questions the above interpretation, it is clear that this fundamental difference between our expressions and those of [3] remains. In the face of such inconsistency, it is appropriate to examine both methods for possible sources of error.

2 Objections to the DeWitt/Brehme/Hobbs method.

It seems to us that the most questionable feature of the method of [7] and [2] is that its derivation of the equation of motion of a charged particle in arbitrary spacetimes employs a physically unjustified identification of all tangent spaces in the neighborhood of a point on the particle’s worldline in order to calculate (in effect) the radiated energy-momentum.

In outline, their method is as follows. Surround the particle with a small two-dimensional sphere \( S_\tau \) associated with a given proper time \( \tau \) on the worldline. There are several reasonable ways to construct such spheres, and there is no reason to think that the final equation of motion will be independent of the method of construction (see below). However, let us pass over this point for the moment.

As this two-dimensional sphere evolves through time, it generates a three-dimensional “tube” \( \Sigma \) surrounding the worldline. In Minkowski space, the integral

\[ P^i := \int_{\Sigma} T^{i\alpha} d\Sigma_{\alpha}, \]  

(12)
of the electromagnetic energy-momentum tensor $T$ over this tube yields a vector quantity $P^i$ which is physically interpreted as the energy-momentum radiated by the particle. The equation of motion is then obtained as as an equation of energy-momentum balance. This is the method by which Dirac obtained the Lorentz-Dirac equation [9] for a charged particle in Minkowski space.

A fundamental difficulty in attempting to use the same method in an arbitrary spacetime, is that the integral (12) is not well-defined, because in effect it attempts the illegitimate mathematical operation of summing vectors in different tangent spaces. In Minkowski space this difficulty does not arise because the linear structure of the space gives natural identifications of all tangent spaces. To generalize (12) to arbitrary spacetimes, one needs some replacement for this natural identification.

Both [7] and [2] do recognize this difficulty, but the identifications they use are introduced without physical motivation. They seem arbitrary, and there seems to be no general principle guaranteeing that other, equally reasonable identifications would yield the same equation of motion. To illustrate, consider the following three methods.

**Method 1:** Choose a “base” point $z(\tau_0)$ on the worldline $\tau \mapsto z(\tau)$. Any point $x$ in a sufficiently small neighborhood of $z(\tau_0)$ can be joined to $z(\tau_0)$ by a unique geodesic lying in this neighborhood. Parallel translation along this geodesic yields an identification of the tangent space at $x$ with the tangent space at $z(\tau_0)$. In this way, all tangent spaces in a sufficiently small neighborhood of $z(\tau_0)$ are identified.

This identification would be expected to depend on the base point when the curvature does not vanish, since the difference between the identifications defined by two base points is parallel translation around a geodesic quadrilateral. Nevertheless, one could in principle use the method of [7] to obtain an equation of motion using this identification.

The dependence of the identification on the base point does expose the method to certain criticisms. Similar criticisms given below apply to the identifications actually used in [7] and [2].

**Method 2:** Again choose a base point $z(\tau_0)$ on the worldline. From an arbitrary point $x$ near $z(\tau_0)$, find a geodesic $\gamma$ connecting $x$ to a point $z(\tau_1)$ on the worldline with the property that at $z(\tau_1)$, $\gamma$ is orthogonal to the worldline. (In Minkowski space, $\gamma$ would be a straight line orthogonal to the worldline, as pictured in Figure 1) Identify the tangent space at $x$ with the tangent space at $z(\tau_0)$ by using parallel translation along $\gamma$ from $x$ to $z(\tau_1)$ and then Fermi-Walker transport along the worldline from $z(\tau_1)$ to $z(\tau_0)$.

This identification does not depend on the base point $z(\tau_0)$, but does have the more subtle defect of depending on the worldline. When one computes the integral (12) using this identification, it is not clear what is the physical meaning of the “vector” $P^i$ obtained. For instance, if we imagine performing the integration for two different worldlines through the same
base point $z(\tau_0)$, we obtain two different $P^i$’s, and there would seem to be no sensible way to compare them.

Such a comparison is not necessary for the derivations of [7] and [2], but if [12] is computing some real physical quantity, such a comparison should be possible. For instance, the rate of change of mechanical energy-momentum $m(du/d\tau)$ on the left of [5] can be sensibly compared for two worldlines, since these are tangent vectors at the same point. If the left sides of [5] can be sensibly compared, then one would think that it should be possible to compare the $P^i$’s on the right.

The difficulty can be seen more clearly by imagining a collision process in which the worldlines of two particles intersect at one point. For such a situation, it would seem reasonable to account for the interchange of mechanical energy-momentum by simply summing energy-momentum vectors of the incoming and outgoing worldlines, but the method of [7] and [2] would not be expected to correctly account for radiation in this context.

This is a situation which they do not consider, and the inapplicability of their method is not in itself a compelling objection. It could be that point collision processes are inherently unrealistic and that no method could properly account for them. However, it does seem a reason to closely scrutinize the method.

**Method 3:** This is a variant of Method 2. Instead of choosing $\gamma$ orthogonal to the worldline, choose $\gamma$ to be a lightlike geodesic from $x$ to the worldline.

An advantage of Methods 2 and 3 is that one needs to consider only spacelike or only lightlike geodesics, which sometimes leads to calculational simplifications.
Hobbs ([2], p. 145) uses Method 2, though his calculational method effectively bypasses the Fermi-Walker part of the translation.

Methods 2 and 3 also give more or less “natural” ways to construct the spheres \( S_\tau \). For example, with Method 2, \( S_{\tau_1} \) could be taken as the set of all \( x \) whose connecting geodesic to \( z(\tau_1) \) has a fixed length \( r \). For Method 3, one could replace the geodesic length by the geodesic parameter, under the normalization condition that the inner product of the geodesic tangent \( \gamma' \) with the worldline tangent \( z'(\tau_1) \) be unity.

In Minkowski space, these last two constructions yield tubes called Dirac and Bhabha tubes, respectively, and both calculations can be done exactly for the limit of a tube of vanishing radius. The integrals (12) over the two tubes between finite times \( \tau_1 \) and \( \tau_2 \) do not coincide ([4], p. 160), though they are close enough that a plausible argument can be made for the Lorentz-Dirac equation. The situation in arbitrary spacetimes seems much more obscure. We know of no good reason to think that any two of the three methods (or others equally “natural”) will yield equivalent results.

Another objection to the method of [7] and [2] is that certain (probably divergent) integrals are discarded with little discussion or physical justification. These discarded integrals are the integrals over the spacelike 3-volumes \( \Sigma_1 \) and \( \Sigma_2 \) and the integral over the 4-volume \( V \) in equation (5.2) of [2] and the similar integrals in equation (5.1) of [7]. The integrals over the spacelike 3-volumes (which may be roughly visualized as constant-time hypersurfaces) would yield mass renormalization terms in Minkowski space, assuming that the particle is unaccelerated in a neighborhood of the hypersurfaces. When the particle is accelerated at the hypersurfaces (as it would be in general in the formulations of [7] and [2]), the corresponding integrals have never been computed even in Minkowski space, and we know of no reason to believe that they will evaluate to mass renormalizations.

3 Critical discussion of our method.

Now let us look for possible sources of error in our expression (4) for the energy radiation. It is customary in the relativity literature to identify as “energy” the conserved quantity corresponding to \( \partial_0 \) in a static spacetime. Nevertheless, we have pointed out in [5] that such an identification is occasionally physically incorrect. For example, for the metric

\[
g^{ij}dx^idx^j = (x^1)^2(dx^0)^2 - \sum_{i=1}^{3}(dx^i)^2,
\]

the Riemann tensor vanishes, which implies that this spacetime may be metrically identified with a subset of Minkowski space, whose metric is

\[
ds^2 = dt^2 - dx^2 - dy^2 - dz^2.
\]
In Minkowski space, the energy is universally accepted as the conserved quantity associated with the Killing vector $\partial_t$, and this is not the same as the conserved quantity associated with $\partial_0$.

If there are timelike Killing vectors other than $\partial_0$ for our metric (11), it is conceivable that in analogy to the situation just presented, one of these might give the physically relevant energy as conserved quantity rather than $\partial_0$. However, it is rather rare that such hidden symmetries exist, and we show in the Appendix that for the metric (1) with $k$ given by (9), the only Killing vector fields which are invariant under translations and rotations in the $x^2$-$x^3$ plane are constant multiples of $\partial_0$. Thus apart from a trivial multiplicative constant, $\partial_0$ seems the only natural choice for the Killing field associated with “energy”.

### 4 A physical argument suggesting that the equation of motion (8) may be incorrect.

This section argues that if we do identify $\partial_0$ as the “energy” Killing vector, then it appears unlikely that Hobbs’ equation of motion (8) can be correct for a conformally flat spacetime (1) with $k$ given by (9).

It is well known that on any geodesic with tangent vector $u$, the inner product $\langle u, \partial_0 \rangle$ is constant ([8], p. 651). In the present situation, this says that the four-velocity $u$ of a freely falling particle satisfies

$$\langle u, \partial_0 \rangle = \frac{k}{\sqrt{1 - v^2}} = \text{constant},$$

where $v^2 := \sum_{j=1}^{3}(dx^j/dx^0)^2$ is the square of the coordinate velocity. To see this, note that $u = (dx^0/d\tau, dx/d\tau) = (dx^0/d\tau)(1, dx/dx^0)$, where $\tau$ is proper time. It follows that $1 = u^2 = k^2(dx^0/d\tau)^2(1 - v^2)$, so that $\langle u, \partial_0 \rangle = k^2(dx^0/d\tau) = k/\sqrt{1 - v^2}$.

The logarithm of equation (15) may be regarded as a sort of law of conservation of kinetic plus potential energy: log $k$ may be regarded as the potential energy. For example, if $k$ is decreasing as the particle moves along a geodesic, then $v$ is increasing. In particular, a freely falling particle which is released at a “height” $k = k_0$ with a particular velocity will arrive at a lower “height” $k = k_1 < k_0$ with a greater velocity.

If we had the potential energy given by a function $k = k(x^1)$ of the form sketched in Figure 2 then a freely falling particle released at rest at point $A$ would fall to arrive with nonzero velocity at the point $B$ of minimum potential energy. By symmetry, it would then climb to arrive at $C$ with zero velocity, after which it would fall back toward $B$. The oscillations $A$-$B$-$C$-$B$-$A$ would continue forever.

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5Since the motion is on the $x^1$-axis, the particle actually moves not from $A$ to $B$, but rather from the point on the horizontal axis whose $x^1$ coordinate is that of $A$ to one whose $x^1$ coordinate is $B$. However, speaking of “falling” from $A$ to $B$ allows us to describe the motion in physically suggestive language and should cause no confusion.
If a charged particle is radiating energy during this motion, we would have an infinite energy source. We are saved from a direct paradox by the fact that the graph of $k(x^1) = 1/x^1$ does not have a local minimum, but we can still obtain a result which is physically hard to believe by putting two similar $k$’s together as sketched in Figure 3. The sharp corner at the minimum at $x^1 = 1/2$ can be rounded off, if desired, leaving a small transition region near $B$ where the bracketed term involving the Ricci tensor in (8) does not vanish.

If a particle is released at rest at point $A$, the DeWitt/Brehme/Hobbs equation (8) implies that it will fall freely until it reaches the transition region near point $B$. Our energy calculation indicates that it is radiating energy during this period. The energy radiated at a point $D$ intermediate between $A$ and $B$ can presumably be collected by antennas located near $D$. If the particle oscillated indefinitely between $A$ and $C$, we would have an infinite energy source, so conservation of energy requires that the particle lose velocity (relative to the freely falling motion of an otherwise identical uncharged particle) in the transition region near $B$.

It is conceivable that the energy collected by antennas near points like $D$ where the particle is freely falling might be “borrowed” until the particle reaches the transition region near $B$, the decrease in velocity over the transition region being precisely that required to pay for the radiated energy. However, since there is nothing in the structure of the DeWitt/Brehme/Hobbs equation to guarantee this, it seems unlikely. Even if the equation did guarantee it, such “borrowing” of energy, though not a mathematical contradiction, seems physically very strange. We think it suggests that the DeWitt/Brehme/Hobbs equation should not be uncritically accepted.
Figure 3: The “potential” function $k$ is defined by $k(x^1) := 1/x^1$ for $0 \leq x^1 \leq 1/2$ and $k(x^1) := 1/(1-x^1)$ for $1/2 \leq x^1 \leq 1$. The sharp corner at $B$ can be rounded off, if desired, leaving a small transition region near $B$ where the bracketed term involving the Ricci tensor in (8) does not vanish.

5 Appendix

This appendix classifies the Killing vector fields $K = K^i \partial_i$ for a metric of the form (1) with $k$ given by (9):

$$g_{ij}dx^i dx^j = (1/x^1)^2[(dx^0)^2 - \sum_{J=1}^{3} (dx^J)^2] .$$

(16)

The result is that any such Killing field is of the form

$$K = K^0(x^2, x^3)\partial_0 + K^2(x^0, x^3)\partial_2 + K^3(x^0, x^2)\partial_3 ,$$

(17)

In other words, any Killing field for (16) is actually an $x^1$-independent Killing field for the three-dimensional Minkowski metric $(dx^0)^2 - (dx^2)^2 - (dx^3)^2$. Moreover, it will be apparent from the Killing equation (18) below that the space part $K^2(x^0, x^3)\partial_2 + K^3(x^0, x^2)\partial_3$ of $K$ is an $x^0$-dependent Killing vector for the Euclidean $x^2$-$x^3$ plane. The Killing fields for a Euclidean plane are well-known to be linear combinations (with constant coefficients) of constant vector fields and the infinitesimal generator of the rotation group; it follows that the only Killing fields for (16) which are invariant under translations and rotations in the $x^2$-$x^3$ plane are constant scalar multiples of $\partial_0$.

The condition that $K$ be a Killing field may be written as (II, p. 81, Problem 3.6.3(b)):

$$g_{\alpha \beta} \partial^\alpha K^{\beta} + g_{j \beta} \partial_\alpha K^{\beta} + K(g_{ij}) = 0 .$$

(18)

6The latter Killing fields are classified in (II), Chapter 13, under the additional hypothesis that the $K^i$ be analytic.
Here we use the differential-geometric convention of identifying tangent vectors with directional derivatives, so that $K_{ij} := K^a \partial_a g_{ij}$.

For a diagonal metric like (16), this simplifies to

$$g_{ii} \partial_j K^i + g_{jj} \partial_i K^j + K(g_{ij}) = 0 \quad \text{(NO SUMS)},$$

where the repeated indices $i$ and $j$ are not summed. We shall show that (19) can hold only if $K^1 = 0$. Assuming this, the fact that $g_{ij}$ depends only on $x^1$ implies that

$$K(g_{ij}) = K^1 \partial_1 g_{ij} = 0 \quad \text{(20)}.$$

From this and (19), it follows that $K^1$ is independent of both $x^1$ and $x^i$, establishing (17).

To show that $K^1 = 0$, we consider (1) for the special case in which $i$ and $j$ are both drawn from the set $\{0, 1\}$. While considering this special case, we simplify the notation by suppressing the $x^2, x^3$ dependence of $K^i$. Let $k(x^1) := 1/x^1$.

Our argument actually works for a large class of $k$'s.)

From (19) for $i = j = 1$, we obtain

$$0 = -2k^2 \partial_1 K^1 - K(k^2) = -2k^2 \partial_1 K^1 - K^1 \partial_1 (k^2).$$

If $K^1 \neq 0$ at some point, we may divide by $K^1$, obtaining near that point

$$\partial_1 \log K^1 = -\partial_1 \log k.$$

(21)

We'll show that this leads to a contradiction, so that the only alternative is the promised relation $K^1 = 0$, from which the rest of the conclusions follow routinely as outlined above.

From (21), it follows that

$$K^1(x^0, x^1) = \phi(x^0)/k(x^1) \quad \text{(22)}$$

for some function $\phi$ of $x^0$ (and the suppressed $x^2$ and $x^3$) alone. Next write (19) for $i = j = 0$ and again for $i = j = 1$ to conclude that

$$2k^2 \partial_0 K^0 = -K(k^2) = 2k^2 \partial_1 K^1,$$

whence

$$\partial_0 K^0 = \partial_1 K^1 \quad \text{(23)}.$$

Finally, write (19) for $i = 0, j = 1$ to see that

$$\partial_1 K^0 = \partial_0 K^1 \quad \text{(24)}.$$

Applying $\partial_0$ to (21) and using (23) shows that $K^1$ satisfies the wave equation

$$\partial_0^2 K^1 = \partial_1^2 K^1.$$

But a function of form (22) can satisfy the wave equation only if $1/k$ satisfies an equation of the form $(1/k)'' + \lambda (1/k) = 0$ for some constant $\lambda$; i.e. $1/k(x^1)$ must be a linear combination of complex exponentials $e^{\sqrt{\lambda}x^1}$. Since our $k(x^1) := 1/x^1$ is not of this form, we conclude that (19) can hold only for $K^1 = 0$.

Finally, from (18) and (20), it follows that $K^2(x^0, x^3) \partial_2 + K^3(x^0, x^2) \partial_3$ is an $x^0$-dependent Killing field on the $x^2$-$x^3$ plane. As noted above, the only such field invariant under translations and rotations is the zero field.
Appendix 2.

This appendix describes significant developments which have occurred since this paper was written in late 1995 and which affect some of its arguments. The first is a theorem of Sonego [12] showing the conformal invariance of the DeWitt/Brehme/Hobbs (abbreviated DBH below) radiation reaction term in equation (8),

\[
\frac{2}{3} q^2 \left[ d a^i \over d \tau + a^2 u^i \right] - \frac{1}{3} q^2 \left[ -R^i_{\beta} u^\beta + u^i R_{\alpha \beta} u^\alpha u^\beta \right].
\]  

(25)

The second is a long paper of Quinn and Wald [13] relating the “Killing vector” definition of energy radiation (3) to the DBH equation. These imply that the plausibility argument of Section 4 is basically a disguised version of a known physical objection to the Lorentz-Dirac equation discussed in detail in [5] and [6] and outlined below.

6.1 Implications of conformal invariance of the DBH radiation reaction

Sonego proved that the radiation-reaction term \( \phi^i \) in the DBH equation (8),

\[
\phi^i := \frac{2}{3} q^2 \left[ d a^i \over d \tau + a^2 u^i \right] - \frac{1}{3} q^2 \left[ -R^i_{\beta} u^\beta + u^i R_{\alpha \beta} u^\alpha u^\beta \right],
\]

(26)

is conformally invariant in the sense that if metrics \( g \) and \( \tilde{g} \) are related by \( g = k^2 \tilde{g} \) for some \( C^\infty \) function \( k \), then

\[
\phi^i = k \tilde{\phi}^i,
\]

(27)

where \( \tilde{\phi}^i \) is defined by the right side of (26) with all quantities computed relative to \( \tilde{g} \) instead of \( g \). This does not require that \( \tilde{g} \) be the Minkowski metric of the present work, but this is the only case we consider here.

For the \( g \)-geodesics considered in Section 4, by definition \( a = 0 \). Also, the term in (26) involving the Ricci tensor vanishes. Hence Sonego’s result implies that

\[
\frac{d \tilde{a}^i}{d \tilde{\tau}} + \tilde{a}^2 \tilde{a}^i = 0.
\]

(28)

This implies that the \( g \)-geodesics are uniformly accelerated relative to the Minkowski metric \( \tilde{g} \).

Indeed, this is one way to formulate the definition of “uniform acceleration”; after observing that \( \tilde{a}^2 = -(d \tilde{a}/d \tilde{\tau}, \tilde{a}) \), we see that it states that \( \tilde{a} \) is invariant under Fermi-Walker transport along the worldline. Alternatively, the identification of (28) with uniform acceleration for the present situation of one-dimensional motion can be established more directly by writing \( \tilde{a} = \tilde{A} \tilde{w} \) with \( \tilde{w} \) a unit vector orthogonal to \( \tilde{u} \) and \( \tilde{A} \) the scalar proper acceleration. Noting
that $d\tilde{w}/d\tilde{\tau}$ is orthogonal to $\tilde{w}$ (because $\tilde{w}$ is a unit vector) and extracting the $\tilde{w}$ component of

$$\frac{d\tilde{A}}{d\tilde{\tau}} = \frac{d\tilde{A}}{d\tilde{\tau}} \tilde{w} + \tilde{A} \frac{d\tilde{w}}{d\tilde{\tau}}$$

shows that implies (actually, is equivalent to) $d\tilde{A}/d\tilde{\tau} = 0$, i.e., constant scalar proper acceleration.

This establishes a connection between geodesic motion in the $g$-spacetime and uniformly accelerated motion in Minkowski space. We will say more about this connection later. But first we review some strange consequences of the assumption that the Lorentz-Dirac radiation reaction term correctly describes radiation reaction for a uniformly accelerated particle in Minkowski space. A more extensive discussion with full mathematical details can be found in [5].

Suppose a rocket ship contains a charged particle as payload. The ship also contains some additional mass which serves as fuel, by converting mass to energy.

Suppose the ship is unaccelerated in Minkowski space for all time up to some initial time $\tilde{\tau}_i$. At that time, it starts its engines and makes a smooth transition to a uniformly accelerated state at a slightly later time $\tilde{\tau}_i + \epsilon$. (All motion is in one spatial dimension.) It continues its uniform acceleration for a long time, finally smoothly removing the uniform acceleration during a transition interval $[\tilde{\tau}_f - \epsilon, \tilde{\tau}_f]$, after which it is unaccelerated again. In summary, it has a smooth worldline which is locally uniformly accelerated (with zero acceleration before the initial time $\tilde{\tau}_i$ and after the final time $\tilde{\tau}_f$) except during the transition intervals $[\tilde{\tau}_i, \tilde{\tau}_i + \epsilon]$ and $[\tilde{\tau}_f, \tilde{\tau}_f - \epsilon, \tilde{\tau}_f]$. It is assumed that the worldline is $C^\infty$; the function of the transition intervals is to permit a smooth transition from inertial motion (zero acceleration) to constant and nonzero uniform acceleration.

We will refer to this situation as “uniform acceleration for a finite time”. We emphasize that essential features of “uniform acceleration for a finite time” are absence of acceleration in distant past and future and smoothness of the worldline (at least $C^3$).

Despite the great confusion and controversy in the literature over the presumed behavior of uniformly accelerated charged particles, all authors seem to agree that a particle of charge $q$ accelerated (not necessarily uniformly) for a finite time interval $[\tau_i, \tau_f]$ in Minkowski space does radiate energy, given quantitatively by

$$\text{energy radiation} = \frac{2}{3} q^2 \int_{\tau_i}^{\tau_f} -\tilde{a}^2 \, d\tilde{\tau} .$$

(29)

The argument just given applies more generally to correspond to any uniformly accelerated worldline in the $g$-spacetime (of which geodesic motion is an instance), a uniformly accelerated worldline in Minkowski space. However, we do not need this.

Asymptotically vanishing acceleration in distant past and future might be good enough for some applications, but the additional generality thus obtained usually is insufficient compensation for the difficulty of carrying out the arguments with mathematical rigor. The problem is that one can easily obtain demonstrably incorrect results from plausible manipulations if one ignores conditions at the initial and final times $\tilde{\tau}_i$ and $\tilde{\tau}_f$. Both zero acceleration and smoothness of the worldline at these times are usually necessary to justify rigorously the manipulations customary in treating problems of radiation.
Note that the integrand is positive whenever $\tilde{a} \neq 0$, and the radiation can be made arbitrarily large for a given uniform acceleration by making $\tau_f - \tau_i$ sufficiently large.

The Lorentz-Dirac equation is equation (8) for Minkowski space (implying zero Ricci tensor). It states that the four-force on the particle is the external Lorentz force $qF^i{\alpha}u{\alpha}$ plus the Lorentz-Dirac radiation reaction term

$$\frac{2}{3}q^2\left[ \frac{d\tilde{a}^i}{d\tau} + \tilde{a}^2\tilde{u}^i \right].$$

(30)

But (28) states that this radiation reaction term vanishes outside the transition intervals $[\tilde{\tau}_i, \tilde{\tau}_i + \epsilon]$ and $[\tilde{\tau}_f - \epsilon, \tilde{\tau}_f]$. Hence those who believe that the Lorentz-Dirac term (30) correctly describes the radiation reaction must admit the strange consequence that the radiation reaction occurs only in the transition intervals at the beginning and ending of the trip. No matter how long the trip (i.e., no matter how large $\tilde{\tau}_f - \tilde{\tau}_i$), the radiation reaction is confined to two short time intervals of length $\epsilon$.

Note that radiation reaction is not a hypothetical quantity; in principle, it can be physically measured as the rate of fuel consumption of the rocket. Thus belief in the correctness of the Lorentz-Dirac radiation reaction requires the belief that the pilot of a charged rocket uniformly accelerated for a finite time observes no fuel consumption during the uniform acceleration. All the radiated energy is paid for by fuel consumed in the beginning and ending transition intervals.

Since the energy radiated during the beginning transition interval $[\tilde{\tau}_i, \tilde{\tau}_i + \epsilon]$ is presumably independent of the length of the uniform acceleration, for a very long uniform acceleration, most of the radiated energy is “borrowed”, to be inexorably paid for at the end of the trip.

What if the rocket doesn’t carry enough fuel to pay for the borrowed radiated energy (which can be made arbitrarily large by extending the period of uniform acceleration)? This problem is solved in detail in [5], and it turns out the rocket mass goes negative. The borrowed energy is paid for by negative mass at the end of the trip!

If we disallow negative mass as unphysical, then the Lorentz-Dirac equation of motion implies that the trip is possible only if the initial mass (fuel) is sufficient to pay for the radiated energy. After a small down payment for the initial transition period from inertial to uniformly accelerated motion, the radiated energy is “borrowed” until the end of the trip, at which time the debt is paid. If the initial mass is insufficient to pay the debt, the trip is presumably impossible, but under the assumption that the Lorentz-Dirac radiation reaction term identifies with the rate of fuel consumption, we don’t find this out until the end of the trip!

Since this sounds so unlikely, it should be emphasized that it is a mathematically rigorous consequence of identification of the energy radiation rate given by the Lorentz-Dirac radiation reaction term with fuel consumption. There are no approximations whatever in the argument leading to it. To my knowledge,
it has never been questioned.\footnote{Indeed, the paper \url{http://gr-qc/9303025} was rejected by several journals on the grounds that it is too trivial. No substantive objections have been raised to its mathematics. Nor was it considered one of those papers too vague to be judged correct or incorrect; several referees praised it as clearly written, though not sufficiently novel or mathematically complicated for their journals.}

There is no mathematical contradiction, but it is hard to believe that this behavior would be seen in nature. This is one of several reasons that many are skeptical about the Lorentz-Dirac equation (along with its generalization, the DBH equation).

We have the following situation:

1. To the best of my knowledge, all authors agree that a charged particle uniformly accelerated for a finite time in Minkowski space does radiate in accordance with (29).

2. Some authors (e.g., Singal \cite{15}) believe that nevertheless, a charged particle in Minkowski space uniformly accelerated for \textit{all} time would \textit{not} radiate. Indeed, Quinn and Wald \cite{14} take this as an \textit{axiom}, from which they obtain the DBH equation.

3. My opinion is that the question of whether radiation would be observed from a charged particle uniformly accelerated for all time is, according to taste, either meaningless or a matter of arbitrary definition. The reasons are given briefly in \cite{6}, and more fully in \cite{5}. See also \cite{4}, Chapters 4 and 5, for background.

The identification of geodesic motion in the $g$-spacetime with uniform acceleration in Minkowski space, and the fact that the corresponding radiation reaction terms vanish in both contexts, makes it unsurprising that there should exist in $g$-spacetime analogs of the crazy consequences of the Lorentz-Dirac equation for uniform acceleration in Minkowski space. The example of Section 4 is one such analog. There are other analogs which fit more smoothly into the framework of the energy conservation analysis of Quinn and Wald \cite{13}, but these require too much detailed knowledge of the Quinn/Wald setup to be worth presenting here.

However, it is not clear that there is any detailed correspondence between a charged particle uniformly accelerated for a finite time and satisfying the Lorentz-Dirac equation in Minkowski space and a particle undergoing geodesic motion for a finite time and satisfying the DBH equation in the $g$-spacetime of Section 4. The reason is that although the DBH radiation reaction term \footnote{Indeed, the paper \url{http://gr-qc/9303025} was rejected by several journals on the grounds that it is too trivial. No substantive objections have been raised to its mathematics. Nor was it considered one of those papers too vague to be judged correct or incorrect; several referees praised it as clearly written, though not sufficiently novel or mathematically complicated for their journals.} is conformally invariant, the DBH equation itself is not (because the left side is not conformally invariant). Sonego \cite{12} discusses noninvariance of the DBH equation in more detail.

Finally, I want to note some deficiencies in the present work. It implicitly assumes that the fields in a conformally flat spacetime are the same as in Minkowski space, and this assumption should have been stated explicitly. Due to the conformal invariance of the distributional Maxwell equations, the
Minkowski space fields do satisfy Maxwell’s equations in any conformally flat spacetime, but the uniqueness of such solutions seems not to have been rigorously established, though intuitively it is expected. Other authors (e.g., Hobbs [2]), appear to make the same assumption, so perhaps it can be justified in some way unknown to me.

The argument on pages 5 and 6 leading to the conclusion that $\dot{a}^2$ cannot vanish identically (for a freely falling particle starting at rest in the described conformally flat space time) is correct as stated, but its conclusion is not as strong as needed for the rest of the paper. The problem is that in order to unambiguously identify the energy radiation as given by (4), one needs to assume that the particle was unaccelerated in distant past and future, and that its worldline is smooth. If the particle is at rest up to the time the free fall starts, then its worldline fails to be differentiable at the starting time. This deficiency can be repaired by inserting a smoothing transition interval and invoking Sonego’s result [12] to conclude that in fact, $\dot{a}^2$ is a nonzero constant during the free fall.

6.2 Discussion of the proofs of [12] and [13]

The literature of relativistic electrodynamics is notoriously unreliable. The problems are physically subtle, and the mathematics tends to be complicated, with much tedious algebra. Errors in the literature are common and rarely corrected. I have found that the only way to be sure of a result is to check it oneself in detail, including the tedious algebra.

The main purpose of this appendix is to put in proper context the plausibility argument of the original. Although the original analysis still seems basically valid, the subsequent work of Sonego shows that it did not tell the whole story. A secondary purpose is to share with those interested in such problems my opinions concerning the mathematics of the proofs of Sonego and of Quinn and Wald.

The DBH radiation reaction term (26) applies only to conformally flat spacetimes. For general spacetimes, there is an additional term called the “tail term”. Sonego calls the term (26) the “local term”. The full DBH radiation reaction is the sum of the local term and the tail term. Sonego concludes that the local term and the tail term are separately conformally invariant.

I have checked his proof of conformal invariance of the local term. I regard this as a rigorously proved theorem.

I would hesitate to characterize conformal invariance of the tail term as a mathematically rigorous theorem. Sonego’s argument seems to require auxiliary assumptions for which I haven’t been able to find proofs. However, these assumptions are plausible, and I would expect some version of conformal invariance of the tail term to be rigorously provable, possibly under auxiliary technical

10I assume $C^\infty$ for mathematical simplicity, and this should have been explicitly stated in the original. With carefully chosen definitions, $C^2$ would be enough.

11Despite its appearance, this is not routine. One indication is the fact that it has escaped notice for the thirty years since Hobbs corrected the original DeWitt/Brehme equation.
hypotheses.

The present work (a copy of which was sent to Quinn and Wald in 1996) raised the question of whether the DBH equation conserves “energy” as defined by the usual construction in spacetimes with a timelike Killing vector. The 1999 paper of Quinn and Wald [13] answers this question by presenting a proof that indeed it does conserve energy. This is not in contradiction to the example of Section 4 for reasons explained above. Even if the DBH equation of motion conserves energy, for the geodesic motion for a finite time considered in Section 4, all the radiated energy is furnished by radiation reaction at the beginning and ending of the trip. For example, in the trip from $A$ to $B$ of Figure 3, all the radiated energy is furnished in the transition regions near $A$ and $B$ where the Ricci tensor does not vanish.\(^{12}\)

There is a major gap in the Quinn/Wald proof around their equation (42). In private correspondence, the authors have convinced me that it can probably be filled. However, the details of the repair are likely to be complicated, and I do not know if they have been written out.

It looks to me as if their equations (39) and (18) may be in error. If so, the errors are potentially serious enough to invalidate the paper’s main result. The authors have not answered a letter of October, 1999, enquiring how to justify these equations, nor was a followup letter of December (1999) answered. (This is being written in July, 2000.)

So, I cannot vouch for the correctness of the Quinn/Wald proof. However, after a careful study of much of the paper, I can vouch for its overall interest. It is clearly written and contains many potentially useful new ideas. I have learned much from it, and it will probably repay study for anyone seriously interested in fundamental problems of electrodynamics in curved spacetimes.

6.3 Reexamination of the original conclusions

The body of the paper is essentially the same as that posted in 1993. It was rejected by several journals on the grounds that it is mathematically too trivial, a judgment which I am not in a position to dispute. The only referee who saw any problem with its mathematics or conclusions was one who questioned whether Maxwell’s equations were conformally invariant! (That was his only objection.)

However, in the light of Sonego’s result, it is clear that its focus was to some degree misdirected. I found striking the fact that although radiation in a conformally flat spacetime could be computed as if it were Minkowski space, the DBH radiation term appeared to depend on the conformal factor. At the time, given known deficiencies in derivations of the DBH equation, this seemed presumptive evidence that “the DeWitt/Brehme/Hobbs equation should not be uncritically accepted”, evidence which seemed to go beyond the usual objections to the Lorentz-Dirac radiation reaction term. Sonego’s proof that the DBH

\(^{12}\)The original example should have included a transition region near $A$ in order to make the worldline smooth as mentioned above. Also, the curve depicted in Figure 3 should have started at $A$ (rather than being extended to the left of $A$) and ended at $C$. 
radiation reaction term is conformally invariant shows that this presumptive evidence was only illusory.\textsuperscript{13} (However, other objections to the DBH equation are unaffected.)

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\textsuperscript{13}In retrospect, it seems that an opportunity to conjecture Sonego’s result was missed here. Whatever the other questionable aspects of the DBH derivation, it was carried out in a covariant way starting from the conformally invariant Maxwell equations, so it would seem reasonable to conjecture that the result would be conformally invariant.
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