Sufficient and Necessary Condition of Separability for Generalized Werner States

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We introduce a sufficient and necessary condition for the separability of a specific class of $N$-dimensional system (qudits) states, namely special generalized Werner state (SGWS): $W[d^N](v) = (1 - v)\psi_d^N + v|\psi_d^N⟩⟨\psi_d^N|$, where $|\psi_d^N⟩ = \sum_{i=0}^{d^N-1} \alpha_i|i⟩$ is an entangled pure state of $N$ qudits system and $\alpha_i$ satisfies two restrictions: (i) $\sum_{i=0}^{d^N-1} \alpha_i = 1$; (ii) Matrix $T = \text{Min}_{k \neq j} \{1/|\alpha_i\alpha_j|\}$, is a density matrix. Our condition gives quite a simple and efficiently computable way to judge whether a given SGWS is separable or not and previously known separable conditions are shown to be special cases of our approach.

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Entanglement and nonlocality are some of the most essential concepts embodied in quantum mechanics. A multiparticle system is entangled if the states of this system cannot be prepared locally by acting on the particles individually. Interest in entangled states has been heightened by proposed applications in quantum computation and information, such as quantum parallelism, quantum teleportation, dense coding, quantum cryptographic schemes, entanglement swapping and remote state preparation, etc. There are many publications that examined various aspects of entanglement, however, entanglement is not yet fully understood and many questions remain open.

One problem of great importance is to check whether a state, generally mixed, is entangled or separable. Historically, violation of Bell inequalities is the first sufficient condition of entanglement. If a state violates any Bell inequality, then it is certainly an entangled state. While the reverse is not valid, there are various entangled states that do not violate any Bell inequality. Another important problem concerning this problem was done by Peres, who proposed an necessary separability criterion based on partial transpose of the composite density operator. There have been some other necessary criteria for separability, such as reduction criterion, majorization criterion, entanglement witnesses, realignment and generalized realignment, as well as some necessary and sufficient operational criteria for low rank density matrices. More recently, some entanglement criteria are also presented for continuous bipartite states. Nevertheless, up to now there is no general necessary and sufficient condition for arbitrary states. Completely solving the separability problem is far away from us.

In this paper, we are interested in the separability properties of $N$ particles $d$-dimensional $(N$ qudits) systems. Their states are defined on finite dimensional Hilbert spaces $H^{(N)} = H_1 \otimes H_2 \cdots \otimes H_N$, where $H_i$ denotes the Hilbert space of the $i$th subsystem. A state in $H$ specified by a density matrix $\rho$ is said to be fully separable if it is a convex combination of tensor products:

$$\rho = \sum_{\lambda} p_\lambda \rho^{(1)}_\lambda \otimes \cdots \otimes \rho^{(N)}_\lambda,$$

where $0 \leq p_\lambda \leq 1$, $\sum_{\lambda} p_\lambda = 1$, and $\rho^{(k)}_\lambda$ is a density matrix on $H_k$. In the following, we will study the separability of a particular class of $N$ qudits states, namely special generalized Werner state (SGWS).

The Werner state was originally defined in for two qudits to show that some inseparable states admit the hidden variable interpretation. The special generalized Werner state (SGWS), $W[d^N](v)$ is defined as the convex combination of the completely random state $1/d^N I^{(N)}$ and an entangled pure state $|\psi_d^N⟩⟨\psi_d^N|$.}

$$W[d^N](v) = (1 - v)\frac{1}{d^N} I^{(N)} + v|\psi_d^N⟩⟨\psi_d^N|,$$

where $0 \leq v \leq 1$, $|\psi_d^N⟩ = \sum_{i=0}^{d^N-1} \alpha_i|i⟩$ (for simple and convenient, we let $i$ denote the repeated index $i \cdots i$ and define $|i⟩ = |i⟩ \otimes \cdots \otimes |i⟩$) is an entangled pure state of $N$ qudits system and $\alpha_i$ satisfies two restrictions: (i) $\sum_{i=0}^{d^N-1} \alpha_i \alpha_i^* = 1$; (ii) Matrix $T = \frac{1}{\rho^{(1)}_\lambda} + T \sum_{i \neq j} \alpha_i|\alpha_j⟩⟨\alpha_j|$, where $T = \text{Min}_{k \neq j} \{1/|\alpha_i\alpha_j|\}$, is a density matrix. The SGWS has been applied to study Bell’s inequality and local reality, and it has served as a test case of separability arguments in a number of studies. As a result, studies of the separability for SGWS itself is of great importance. In 24, 25, 26, Arthur O. Pittenger and Morton H. Rubin presented a necessary and sufficient condition of separability for a special case, they proved that, for $|\psi_d^N⟩ = \frac{1}{\sqrt{d}} \sum_{i=0}^{d^N-1} |i⟩$, the SGWS is separable if and only if $v \leq (1 + d^{n-1})^{-1}$. Nevertheless, for general $|\psi_d^N⟩$, no such result has been known.
In this letter, we shall provide a necessary and sufficient condition of separability for the SGWS. Our main result is as follows:

**Theorem 1:** The special generalized Werner state $W^{[d^N]}(v)$ defined in (2) is fully separable if and only if $v \leq \frac{T}{d^N + T}$.

**Proof.** We will take two steps to prove the above theorem. In the first step, we shall prove that the condition is a necessary condition, in the second step, we prove the sufficiency part. The method used here is similar to the method used in ref. [26].

Step 1: In this part, we will prove that the condition is a necessary condition. Typically, as mentioned above, the Peres partial transpose condition is a general necessary condition for separability, while it is not practical here to prove the necessity. In [21, 22] a weaker but useful condition for qudits was derived using the Cauchy-Schwarz inequality. We will reproduce here their construction for the sake of completeness. Let $n = n_1 \cdots n_N$ and $m = m_1 \cdots m_N$ differ in each component: $n_r \neq m_r$. If a density matrix $\rho$ in $H^{[N]}$ is separable, then the matrix elements of $\rho$ can be written as

$$\rho_{n,m} = \sum_\lambda p_\lambda \prod_{r=1}^N \rho_{n_r,m_r}^{(r)}(\lambda).$$

(3)

Since each $\rho_{n_r,m_r}^{(r)}(\lambda)$ is a density matrix, positivity requires that $\sqrt{\rho_{n_r,m_r}^{(r)}(\lambda)} \geq |\rho_{n_r,m_r}^{(r)}(\lambda)|$ for each $r$ and $\lambda$. Then using the Cauchy-Schwarz inequality we have:

$\sum_\lambda \left( \sum_{r=1}^N \sqrt{\rho_{n_r,m_r}^{(r)}(\lambda)} \right)^2 \geq \left( \sum_{r=1}^N \rho_{n_r,m_r}^{(r)}(\lambda) \right)^2$,

(4)

where, because of the Hermiticity of the density matrices, $(n_r, m_r)$ may be either $(n_r, m_r)$ or $(m_r, n_r)$ for each $r$. If the SGWS is fully separable, then use the inequality (4) and choose $n$ and $m$ appropriately, one can obtain the necessary condition:

$$1 - v \leq \frac{d^N}{d^N + T}.$$  

(5)

For all $i$ and $j$ the above equality is valid, then we have $1 - v \leq \frac{d^N}{d^N + T}$, i.e., $v \leq 1/(d^N \times \max_{i \neq j}|\alpha_i \alpha_j|) + 1$. Not that $T = \max_{i \neq j}|\alpha_i \alpha_j|$, it is easy to get $v \leq \frac{1}{d^N + T}$. Thus we have proved that the condition presented is a necessary condition of separability for the SGWS.

Step 2: In this part, we will show that the condition is a sufficient condition. We start with $v = \frac{T}{d^N + T}$, then we prove that for all $v < \frac{T}{d^N + T}$, the SGWS is also separable. For $v = \frac{T}{d^N + T}$, it is easy to rewrite $W^{[d^N]}(v)$ as:

$$W^{[d^N]}(v) = \frac{T}{d^N + T} \sum_{i=0}^{d-1} \alpha_i |i\rangle \langle i| \alpha_i^* +$$

(6)

$$\frac{d^N}{d^N + T} \left( I^{[N]} + T \sum_{i \neq j} \alpha_i |i\rangle \langle j| \alpha_j^* \right).$$

Since the first term in (6) is a sum of separable projections, we only need to show that the second term is also separable. It is convenient in what follows to introduce a set of fixed phase and to show that

$$\rho^{(N)} = \frac{1}{d \sum_{i} |\xi_i\rangle \langle \xi_i| \alpha_i \alpha_i^*}$$

(7)

where $\{ |\xi_i\rangle, i = 1, \ldots, d - 1 \}$, is separable. We proceed by induction. For $N = 1$, we have

$$\rho^{(1)} = \frac{1}{d} \left( I^{(1)} + T \sum_{i \neq j} \alpha_i |i\rangle \langle j| \alpha_j^* \right).$$

(8)

Since $T = \min_{i \neq j} |1/|\alpha_i \alpha_j||$, it follows that $|\rho_{i,j}| \leq 1/d$. From (8) and the definition of SGWS, it is obviously that $\rho^{(1)}$ is a density matrix for any choice of the parameters $\xi_i$.

Now assume that the density matrix of the form (7) is fully separable for $N \geq 2$, then we shall prove that it is fully separable for $N + 1$. Following the ideas in [26], for a fixed choice of parameters $\xi_i$ define

$$w^k = (\xi_0 \xi_d \cdots, \xi_{d-1} \xi_1),$$

(9)

where $z_i \in \{ \pm 1, \pm i \}$ and $\xi_i$ is independent of $k$. We have a total of $4^d$ different vectors. For each $w^k$ define the product state $\rho(k) = \rho(k)^{(N)}(w^k) \otimes \rho^{(1)}(z^k)$, where $z^k$ is equal to $w^k$ with all the $\xi_i$’s equal to 1. $\rho^{(N)}(w^k) = \frac{1}{d^N} \left( I^{[N]} + T \sum_{i \neq j} \alpha_i |\xi_i\rangle \langle \xi_i| \alpha_j^* \right)$ and $\rho^{(1)}(z^k) = \frac{1}{d} \left( I^{(1)} + T \sum_{z_i} z_i \langle r| |s\rangle \right)$. The states $\rho(k)$ are separable by the induction hypothesis, consequently, the convex combination $\rho^{(N+1)} = \frac{1}{d+N} \sum_k \rho(k)$ is also separable. Now we multiply out the terms:

$$\rho^{(N+1)} = \frac{1}{d+N} \left( I^{[N+1]} + I^{(N)} \otimes F^{(1)} + F^{(N)} \otimes I^{(1)} + G \right),$$

where $F^{(1)} = \sum_{z_i} \langle r| |s\rangle \frac{1}{d-N} \sum_k z_i \langle z_i| \langle j| \rangle \langle j| \alpha_j^* \right)$, $F^{(n)} = \sum_{i \neq j} \xi_i |\xi_i\rangle \langle i| \alpha_j^* \right) \frac{1}{d-N} \sum_k z_i \langle z_i| \langle j| \rangle \langle j| \alpha_j^* \right)$, and $G = \sum_{i \neq j} \xi_i |\xi_i\rangle \langle i| \alpha_j^* \right) \frac{1}{d-N} \sum_k z_i \langle z_i| \langle j| \rangle \langle j| \alpha_j^* \right)$.
\[
\sum_k z_i^{(k)} z_j^{(k)*} z_r^{(k)} z_s^{(k)*}. \text{ Since the components of } w^{(k)} \text{ are chosen independently of one another, } \sum_k z_i^{(k)} z_s^{(k)*} = 0 \text{ for } r \neq s; \text{ Consequently, } F^{(1)} \text{ and } F^{(n)} \text{ vanish and } \frac{1}{d} \sum_k z_i^{(k)} z_j^{(k)*} z_r^{(k)} z_s^{(k)*} = \delta(i, r) \delta(j, s), \text{ where } \delta(r, s) \text{ is the Kronecker delta function. Then}
\]
\[
\rho^{(N+1)} = \frac{1}{d^{N+1}} \left[ I^{(N+1)} + T \sum_{i \neq j} \alpha_i \alpha_j^* \xi_i \xi_j^* \right].
\]

\[
W^{[d^N]}(v) = (1 - v) \frac{I^{(N)}}{d^N} + v |\psi_d^N\rangle \langle \psi_d^N| = \left(1 - \frac{v}{d^N + T}\right) \frac{I^{(N)}}{d^N} + \frac{v}{d^N + T} \left[ \left(1 - \frac{T}{d^N + T}\right) \frac{I^{(N)}}{d^N} + \frac{T}{d^N + T} |\psi_d^N\rangle \langle \psi_d^N| \right].
\]

Obviously, the two parts of (11) are separable, thus the convex combination of them is also separable. This complete the proof of theorem.

Now returning to the definition of SGWS (see equation (2)), we see that the restriction (ii) for \(\alpha_i\) is needed in the proof of the theorem. However, rough numerical results show that, for some \(\alpha_i\) not satisfying this restriction, the theorem is also valid. Maybe we can rule out the restriction (ii) in the definition of SGWS, while we do not know how to prove analytically the theorem without it and we leave it as an open question.

As an example, we present a class of SGWS: Let \(\alpha_0 = \frac{1}{\sqrt{d}} \cos \theta\) and \(\alpha_i = \sqrt{\frac{d - \cos^2 \theta}{d(d - 1)}} \) \((i = 1, \cdots, d - 1)\), direct calculation shows that the matrix \(\varrho = \frac{1}{2} \left( I^{(1)} + T \sum_{i \neq j} \alpha_i \alpha_j^* \right) \) have \(d - 2\) zero eigenvalues and two nonzero eigenvalues: \(\frac{1}{2} \left(1 + \sqrt{\frac{(d - 2)^2 + (d - 4) \cos^2 \theta}{d(d - \cos^2 \theta)}}\right)\). It follows that the matrix \(\varrho\) is a density matrix and consequently the corresponding SGWS is separable if and only if \(v \leq \frac{1}{d^N + d(d - 1)} \). If we set \(\theta = 0\), then \(\alpha_i = \frac{1}{\sqrt{d}}\) and consequently the corresponding SGWS is separable if and only if \(v \leq \frac{1}{d^N + d^{N+1}}\), which is just the main result in [20]. Therefore, the necessary and sufficient separable conditions presented in [20] are shown to be special cases of our theorem.

Our theorem also accord with other previous results of low dimensional systems. Form the theorem, for \(d = 2\), \(|\psi_2^N\rangle = \sin \theta |0\rangle + \cos \theta |1\rangle\), we have \(T = 1/\sin \theta \cos \theta\) and \(W^{[2^N]}(v)\) is fully separable if and only if \(v \leq 1/(2^N \sin \theta \cos \theta + 1)\). On the other hand, for two qubits system, Wootters presented a simple formula for the calculation of the entanglement of formation [27]

\[
E(\rho) = E(C(\rho)) = \frac{1}{2} \left[1 + \sqrt{1 - C(\rho)^2}\right]
\]

where \(C(\rho) = \frac{\lambda_1 - \lambda_2}{\lambda_1 + \lambda_2} \) is the concurrence for two qubits density matrix, \(h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x)\), and the \(\lambda_i\)s are the eigenvalues, in decreasing order, of the Hermitian matrix \(R = \sqrt{\rho} \sqrt{\rho}^\dagger\). Since \(E(C(\rho))\) is monotonically increasing and ranges from 0 to 1 as \(C(\rho)\) goes from 0 to 1, one can take the concurrence as a measure of entanglement in its own right. For two qubits Werner states under consideration \(W^{[2^2]}\), direct calculation show that

\[
C(W^{[2^2]}) = \max \left\{0, \frac{1}{4} \left\{ \sqrt{1 + 2v + v^2 - 4v^2 \cos 4\theta + 4v \sin 2\theta \sqrt{(1 - v)^2 + 2v^2 \cos 4\theta}} \right. \right. \\
- \left. \sqrt{1 + 2v + v^2 - 4v^2 \cos 4\theta - 4v \sin 2\theta \sqrt{(1 - v)^2 + 2v^2 \cos 4\theta} - 2 + 2v} \right\}.
\]

It is easy to check that \(C(W^{[2^2]}) = 0\) if and only if \(v \leq 1/(2^2 \sin \theta \cos \theta + 1)\), which is also obtained form the theorem 1. What’s more, note that \(C(|\psi_2^2\rangle) = 2 \sin \theta \cos \theta\), one has that the critical value of \(v\) is monotonically decreasing form 1 to 1/3 as \(C(|\psi_2^2\rangle) = \sin \theta \cos \theta\) goes from 0 to 1. This implies that the critical value of \(v\) is related
to the entanglement degree of the entangled pure state $|\psi_3^N\rangle$, the greater the entanglement, the less the critical value of $v$. While, surprisingly this is not valid for $d \geq 3$, just as that the maximal entangled states do not violate the Bell inequality maximally for $d \geq 3$ [28]. From theorem 1, if one set two of $\alpha_i$s a little less than $1/\sqrt{2}$, then $T$ can be a value close to 2 and the critical value of $v$ is close to $2/(d^N + 2)$, which is smaller than $d/(d^N + d)$ for the maximal entangled pure states $|\psi_3^N\rangle = \frac{1}{\sqrt{3}} \sum_{i=0}^{d-1} |i\rangle$. For instance, when $d = 3$, $N = 2$, choose $|\psi_3^2\rangle = \frac{2}{3}|0\rangle + \frac{2}{3}|1\rangle + \frac{1}{3}|2\rangle$ which is not a maximal entangled states, one has that the corresponding SGWS is separable if and only if $v \leq 1/5$, while for $|\psi_3^2\rangle = \frac{1}{\sqrt{3}} \sum_{i=0}^{d-1} |i\rangle$, the corresponding SGWS is separable if and only if $v \leq 1/4$.

In summary, we have presented a sufficient and necessary condition of separability for SGWS. Our condition gives quite a simple and efficiently computable way to judge whether a given SGWS is separable or not and previously known separable conditions are shown to be special cases of our approach. Since the various use of SGWS in quantum information, our results may be very useful for the study of Bell inequalities, quantum entanglement measurement, distillation protocols, etc.

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