Mode interaction in multi-mode optical fibers with Kerr effect

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We generalize the projection to orthogonal function basis (including polarization modes) method for nonlinear (Kerr medium) fiber and use this method in a case of two-mode waveguide. We consider orthogonal Bessel functions basis that fit the choice of cylindrical geometry of a fiber. The coupled nonlinear Schrödinger equations (CNLS) are derived. Analytical expressions and numerical results for coupling coefficients are given; the fiber parameters dependence is illustrated by plots.

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I. INTRODUCTION

It is well known that Nonlinear Shrödinger (NS) equation, similar to celebrated KdV, presents the most universal physical embedding of soliton behavior. However, its experimental realizations are still rather poor and rather demonstrate the solitons than introduce a true physical tool for nonlinear phenomena investigations. The main reason for this is its uni-dimensionality and uni-directionality - the embedding needs some kind of projecting with exact formal and experimental specification.

The important example of the theory is the optical guide (fiber) propagation that isolates modes due to boundary conditions in a fiber cross-section and the direction of propagation is fixed by a way in which the waveguide is excited. So the model based on NS is essentially one dimensional, hence the fiber propagation is perhaps, the one for physical realization of soliton behavior of electromagnetic wave-train. Nevertheless the existing theory adopts the model of waveguide propagation, that looks rather "technical" than physical one.

The authors use the parameter of NS equation as empirical ones without explicit expressions dependent on a whole geometry and physics of a waveguide.

In the papers authors investigate polarization behavior of light in a Kerr fiber; additionally the authors of took into account power exchange between modes. The authors of these articles declare the complete basis expansion for the problem solution but in fact use simplifications choosing for the basis functions (for higher order modes) the Gaussian functions. This approximation is used for nonlinear coefficients (in our notation $Q_1$) evaluation. The formulas for the coupling constants ($c_{ij}$ in $\mathbf{E}$) exhibit the fiber radius dependence (proportional to $1/r_0^2$) that essentially differs from ours.

By the work we started the detailed investigations; we described and discussed a method to derive CNLS (Coupled Nonlinear Shrödinger) equations for a multi-mode fibers focusing on a mono-mode case (which leads to NS equations) and made numerical calculations for it. We elaborated an approach which do not apply potentials, each component of electromagnetic field in cylindrical coordinates is represented directly as a series in correspondent Bessel functions and exponentials $J_l(\alpha_{nl}r)e^{\pm il\phi}$. Such scheme introduces natural mode notion in a fiber cross-section which have obvious link with the standard one (TE,TM modes).

Here we derive CNLS equations for two modes (where $l = 0, \pm 1$ and $n = 0$ in both cases). Information about the parameters allows to derive the explicit dependence of soliton properties on geometry and material constant of a medium. We use the model of step index waveguide with different refraction indexes (without assumption of weakly guided fiber like in $\mathbf{E}$, $\mathbf{G}$, $\mathbf{H}$). The main result of this part of our study is the analytical expression for mode interaction coefficients that exhibit rather strong dependencies on waveguide radius.

We take the fundamental mode (which is known from linear a theory as HE11) considering it and the second mode (TE01) as vectors of our basis. Both modes have different group velocities, this difference influence on the coupling coefficients magnitude. This model could be used in other waveguides with different geometries (e.g. the elliptical one $\mathbf{F}$). Both single-mode fibers with two polarizations and multi-mode fibers are used in optical-switching devices $\mathbf{S}$.

Considering two modes with different propagation constants (this mean $\delta \neq 0$), we expected that interaction between modes occurs only at the beginning of propagation (we inquire the propagation to begin from excitation at one of waveguide’s ends). This can appear in the strong birefringent photonic crystal fibers (PCF). In a consequence this interaction can lead to change power intensity of modes. For example, such properties are very important in case of photonic crystal fibers (PCF) or dispersion managed solitons (DM).

It is necessary to point out that in the cylindrical waveguide even three modes can interact: HE11, TE01 and TM01. Modes TE01 and TM01 have the same cut-off frequency and propagation constant $\mathbf{K}$. This case is

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very similar to single mode with two polarizations case and we considered this in \(r\). In case of HE11 - TE01 mode interaction, it is needed to take into account two mode interaction with different propagation constants. In this work we investigate only this (HE11-TE01) interactions to simplify equations.

We start in the second section from basic equations of linear and some of nonlinear theory of electromagnetic fields. In third section analytical solution of a given problem is presented. In fourth section some of numerical calculations for nonlinear coefficients as a function of the guide radius are given. The last section is a comparison with other cases of analyzed equations and general conclusion is presented.

II. BASIC EQUATIONS

Consider the propagation of optical pulse at isotropic medium and choose a cylindrical dielectric waveguide with small Kerr nonlinearity. The electric field can be written as

\[ E_i = \frac{1}{2} A_i e^{i\omega t} + c.c., \]  

where \(i = x, y, z\). We introduce a linearly polarized field as \(E_{x,y} = \frac{1}{2} E_{x,y}^+ + \frac{1}{2} E_{x,y}^-\), and the polarization vector component as

\[ P_z = 3 \chi_{xxx} \varepsilon_0 \left[ |A_x|^2 + \frac{2}{3} (|A_x|^2 + |A_y|^2) \right] A_z e^{i\omega t} \]

\[ + \frac{1}{3} \alpha_z (A_x^2 + A_y^2) e^{i\omega t} + c.c., \]

where \(\alpha_z\) is the complex conjugate of \(A_z\).

The electric field component in Bessel function basis, which is standard for cylindrical waveguide can be written as

\[ E_{x,y}^{\pm}(x, y, z, t) = \]

\[ + \frac{1}{2} \sum_{l, n} \frac{1}{\alpha_{ln}} p_l A_{ln}^{\pm} J_{l+1/2}(\alpha_{ln} r) e^{i(l+1/2)z} e^{i\omega t - ikz} + c.c., \]

(4a)

\[ E_{x,y}^{\pm}(x, y, z, t) = \]

\[ + \frac{1}{2} \sum_{l, n} \frac{i}{\alpha_{ln}} p_l A_{ln}^{\pm} J_{l+1/2}(\alpha_{ln} r) e^{i(l+1/2)z} e^{i\omega t - ikz} + c.c., \]

(4b)

here \(\alpha_{ln}\) is eigenvalue for linear cylindrical waveguide problem

\[ \alpha^2 = \omega^2 \varepsilon_0 \mu_0 \varepsilon_1 - k^2, \quad r \leq r_0, \]

\[ \beta^2 = k^2 - \omega^2 \varepsilon_0 \mu_0 \varepsilon_2, \quad r > r_0, \]

where \(r_0\) is waveguide radius and the variable amplitude \(\mathcal{A}\) depends on propagation axis and time.

Using the multi-mode model \(\mathcal{R}\) (where the modes orthogonality over the fiber cross section is used) we obtain

\[ (\Box + \alpha_{01}^2) A_{01}^p = \frac{2 \varepsilon_0 \mu_0}{\pi \varepsilon_{01}} \int_0^{r_0} \int_0^{2\pi} r J_0(\alpha_{01} r) \frac{\partial^2}{\partial z^2} P_z d\phi dr, \]

(6)

\[ (\Box + \alpha_{11}^2) A_{11}^p = \frac{2 \varepsilon_0 \mu_0}{\pi \varepsilon_{11}} \int_0^{r_0} \int_0^{2\pi} r J_1(\alpha_{11} r) e^{-i\varphi} \frac{\partial^2}{\partial z^2} P_z d\phi dr, \]

(7)

where \(\Box\) is defined by

\[ \Box = \mu_0 \varepsilon_0 \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial r^2}. \]

The coefficient at the r.h.s. denominators has the following form

\[ N_{nl} = \frac{r_0^2}{2} \left[ J_p^2(\alpha_{nl} r_0) - J_{l-1}(\alpha_{nl} r_0) J_{l+1}(\alpha_{nl} r_0) \right]. \]

(9)

In the case of a single-mode fiber (with two polarization) we could omit the difference between group velocities, assuming that the fiber is isotropic and do not have belling \(\mathcal{R}\). However in a multi-mode fiber it has to be taken into account, because different modes have substantially different group velocities. We describe it introducing the co-ordinate system which moves with average group velocity

\[ \xi = \sigma z, \]

\[ \tau = (t - \beta' z) c, \]

\[ \beta' = \frac{k_0^2 + k_1^2}{2}. \]

Next step is introducing a slowly varying amplitude of the wave envelope \(\mathcal{P}\) in the form

\[ \sigma X(\tau, \xi) e^{-i\kappa z}, \]

(11)

and \(X\) have the same unit as electrical field \([V/m]\).

III. TWO MODES INTERACTION

First, due to the isotropic material assumption, we have \(k_{01}^2 = k_{01}^2\) and \(k_{11}^2 = k_{11}^2\). Plugging a solution for electromagnetic field \(\mathcal{P}\) \(\mathcal{R}\) to the left hand side of the equations \(\mathcal{R} - \mathcal{R}\) yields four equations (two modes and each mode have two polarizations):

\[ i \partial_{\xi} X_{01}^{\pm} - i \partial_{\tau} X_{01}^{\pm} + \frac{\varepsilon_{01} k_0'}{2\sigma} \partial_{\tau} X_{01}^{\pm} = \]

\[ P_{01} \left[ O_1 |X_{01}^{\pm}|^2 + O_2 |X_{01}^{\pm}|^2 + O_3 |X_{11}^{\pm}|^2 \right] X_{01}^{\pm} + P_{01} \left[ O_4 |X_{11}^{\pm}|^2 + O_5 X_{11}^{\pm} X_{11}^{+} + O_6 X_{01}^{\pm} X_{11}^{+} \right] X_{01}^{\pm}, \]

(12a)
\[ i \partial_t X^\pm_{11} + i \delta r \partial_r X^\pm_{11} + \frac{\epsilon^2 k''}{2 \sigma} \partial_r r X^\pm_{11} = \]

\[ P_{11} \left[ Q_1 |X^\pm_{11}|^2 + Q_2 |X^\mp_{11}|^2 + Q_3 |X^\pm_{01}|^2 \right] X^\pm_{11} + P_{11} \left[ Q_4 |X^\mp_{01}|^2 + Q_5 X^\pm_{01} X^\mp_{01} + Q_6 X^\pm_{01} X^\mp_{01} \right] X^\pm_{11}, \] 

(12b)

where

\[ \delta = \frac{1}{2} (k'' - k'_{11}). \] 

(13)

The coupling coefficients have the following form

\[ Q_1 = \int_0^r r J_1^4(\alpha \Omega r) \, dr \] 

(14)

\[ Q_2 = \int_0^r 2r J_1^4(\alpha \Omega r) \, dr \] 

(15)

\[ Q_3 = \int_0^r 2r J_1^2(\alpha \Omega r) J_0^2(\alpha \Omega r) \, dr \] 

(16)

\[ Q_4 = \int_0^r \left[ 2r J_1^2(\alpha \Omega r) J_0^2(\alpha \Omega r) + J_0^2(\alpha \Omega r) \right] \, dr \] 

(17)

\[ \mathcal{Q}_5,6 = \mathcal{Q}_{5,6} = \int_0^r 2r J_1^2(\alpha \Omega r) J_0^2(\alpha \Omega r) \, dr, \] 

(22)

and

\[ P_{11} = \frac{3 \omega^2 \chi_{xxx} \chi_{xxx} n_l^2}{32 N_{11} N_{11} \ell^2}, \] 

(23)

where \( \ell = 0, 1 \) and \( P_{11} \) have units \( \left[ V^2 m^{-1} \right] \) while \( \chi \) is in \( \left[ m^2 / V^2 \right] \).
The coefficients $O_1$ and $O_2$ are the same as coefficients for one mode fiber with $l = 0$ and $n = 1$. The $Q_1$ and $Q_2$ coefficients for the multi-mode fiber are the same as for one mode one with $l = 1$ and $n = 0$. This part of equations describe interaction between polarization mode with same number $l$ and have been analyzed in [6].

Rest of the coefficients describe the coupling between two different modes (with different $l$ number and theirs polarizations).

Notice that coefficient $O_5$, $O_6$, $Q_5$ and $Q_6$ have same values because they describe mixed interaction (between different modes and different polarization).

IV. NUMERICAL RESULTS

First numerically evaluate eigenvalues $\alpha_{01}$ and $\alpha_{11}$ from Hondros-Debye equation. In the next steep integrals with Bessel function are calculated numerically. Eigenvalues are evaluated without approximation of weakly guided fiber where we have $\varepsilon_1 \approx \varepsilon_2$.

The results for inter-mode influence are shown at the figure. Note that the mode 01 have cutoff frequency near $V \approx 2.4$ for that reason the coefficients have different behaviour. Coefficients $O_5$, $O_6$, $Q_5$ and $Q_6$ describe mixed interactions (different modes and different polarizations) and for isotropic medium have same value. Figure 2 shows difference between coefficients ($Q_5 = Q_6 = Q_5 = Q_6$ is smallest and have smallest increment). All of this coefficients are smaller than $O_1 - O_4$ coefficients.

In our calculation we defined normalized frequency as

$$ V = \frac{\omega}{c} r_0 \sqrt{\varepsilon_1 - \varepsilon_2}, \quad (24) $$

and in numerical calculations we used physical parameters with following values:

$$ \omega = 12.2 \times 10^{14} \text{ Hz} \quad (\lambda \approx 1.54\mu m), \quad (25a) $$

$$ \varepsilon_1 = 2.25 \quad \text{(ref. index 1.5)}, \quad (25b) $$

$$ \varepsilon_2 = 1.96 \quad \text{(ref. index 1.4)}, \quad (25c) $$

$$ r_0 \quad \text{from} \ 1.2 \times 10^{-6} \text{m to} \ 10 \times 10^{-6} \text{m}. \quad (25d) $$

V. CONCLUSION

In this paper we considered the influence of Kerr non-linearity on the mode coupling in the case of two-mode fiber. The results show how the modes influence each other and we expect higher influence when $V$ is bigger. The reason is that the propagation constants $k_{01}$ for higher $V$ is roughly $k_{11}$ (fig. 3).

We can reconsider equation (12) again but with simplification, that we do not take into the account polarization interaction. Choosing two modes with the same
polarization yield:

\[
i\partial_\xi X_{01} - i\delta \partial_\tau X_{01} + \frac{\varepsilon^{2}k''}{2\sigma} \partial_\tau \tau X_{01} = \nonumber
\]

\[
\mathcal{P}_{01} \left[ \mathcal{O}_{1}|X_{01}|^{2} + \mathcal{O}_{3}|X_{11}|^{2} \right] X_{01}, \quad (26a)
\]

\[
i\partial_\xi X_{11} + i\delta \partial_\tau X_{11} + \frac{\varepsilon^{2}k''}{2\sigma} \partial_\tau \tau X_{11} = \nonumber
\]

\[
\mathcal{P}_{11} \left[ \mathcal{Q}_{1}|X_{11}|^{2} + \mathcal{Q}_{3}|X_{01}|^{2} \right] X_{11}. \quad (26b)
\]

In this case we have only four parameters (two for self phase modulation and two for cross phase modulation), which describe only interaction between modes. In boundary when we have two modes with the same group velocity (\(\delta = 0\)), we get system of equations like this in case of polarization interaction [6].

Method which we used for a cylindrical waveguide is easily reformulated for a different waveguide shape where second-modes (and higher modes) are important, for example in the elliptical waveguide [8]. This method also can be used for a photonic crystal fiber, directly or within the approximations of the approach of [13]. Here we have delivered all calculations for isotropic medium but there is a possibility to make it for anisotropic medium.

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