Calculation and design of lattice cylindrical shells manufactured of unidirectional CFRPs

S Golushko\textsuperscript{1} and B Semisalov\textsuperscript{2}
\textsuperscript{1} Institute of Computational Technologies SB RAS, Novosibirsk, Russia
\textsuperscript{2} Novosibirsk State University, Novosibirsk, Russia
E-mail: s.k.golushko@gmail.com, ViBiS@ngs.ru

Abstract. The work is devoted to numerical simulation of the stationary stress-strain state of lattice cylindrical shells made of unidirectional carbon fiber reinforced plastics and to their optimal design. After averaging stiffness characteristics of lattice structure, the displacement and stress formulations for continuum anisotropic equivalents of lattice shells are given on the basis of classical relations of 3D theory of elasticity. While using a special fast pseudo-spectral algorithm, the solutions to problems of axial compression of anisogrid shells are obtained and compared one to another. The algorithm provides exponential decrease of error of approximation and requires small computational resources. Using this algorithm we have found the optimal geometrical parameters of structure ensuring its bearing capacity under given non-uniform loads.

1. Introduction
Anisogrid structures are grid shells manufactured of unidirectional carbon fiber reinforced plastics (CFRPs) by automated method of continuous filament winding. These structures demonstrate enhanced properties of strength and stiffness and have significant potential in the field of construction of rockets and spacecrafts, aircrafts and deep-diving vehicles, buildings and high-duty structures \cite{1}.

Anisogrid shells of zero Gaussian curvature including conical payload adapters of launch vehicles and cylindrical elements of their airframes (see Figure 1) are efficient technological solution for space industry. Tendency to minimize the weight of such structures leads to decrease of safety factors during their design that requires the description of stress-strain state (SSS) to be reliable and accurate, and the numerical simulation to have high time efficiency. This work aims at numerical simulation of stationary SSS of anisogrid cylindrical shells accounting for three mentioned principles.

To this end we applied the continuum approach verified in \cite{2}, used the classical equations of elasticity theory without any additional hypotheses \cite{3}, derived two formulations of problems and designed a special pseudo-spectral method to solve them based on Chebyshev and Fourier approximations of unknown functions \cite{4} that allows for estimation of approximation and round-off errors \cite{5}.

2. 3D formulations of problem describing deformation of anisogrid shells
Assume unidirectional CFRP to be orthotropic and quasi-homogeneous material with tension-compression ($E$, $E_T$) and shear ($G$, $G_T$) moduli describing stiffness along and across the direction
of its fibers and also with Poisson’s coefficients $\nu$, $\nu_T$, $\nu_L$, \cite{6}–\cite{8}. In this case the relation between stresses and strains in ribs of anisogrid shell can be described by generalized Hooke’s law. Further, the continuum model is used for averaging the stiffness characteristics of lattice shells and for passing to its solid anisotropic equivalent (see \cite{2}, \cite{9}).

Anisogrid shell consists of several sets of unidirectional ribs. Each set has its own number $j$ ($j = 1, \ldots, N_\delta$) and can be defined by angle of rib’s slope $\phi^j$ specified in coordinate system associated with shell, by thickness of ribs $\delta^j$ and by distance between ribs $a_j$. In accordance with the concept of ”smearing” \cite{2}, each set of ribs should be replaced with a solid surface with characteristics of stiffness obtained by averaging those of ribs by volume. After summing the characteristics over all sets of ribs in \cite{6}, \cite{7} a relation between stresses and strains occurring in the continuum equivalent of lattice shell was derived. It is as follows

\[
\mathbf{\sigma} = \mathbf{A} \mathbf{\varepsilon}, \quad (1)
\]

Relations (1) are written in Cartesian coordinate system $(x, y, z)$ (see Figure 1), associated with shell for components of stress and strain tensors $\mathbf{\sigma} = (\sigma_x, \sigma_y, \sigma_z, \tau_{yz}, \tau_{xz}, \tau_{xy})$, $\mathbf{\varepsilon} = (\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{yz}, \gamma_{xz}, \gamma_{xy})$. Matrix of stiffness coefficients $\mathbf{A}$ has block diagonal form: $A_{ij} \neq 0$ as $i, j = 1, \ldots, 4$ and $i, j = 5, 6$, their values are expressed by stiffness characteristics of CFRPs and geometrical parameters of sets of ribs. Other values $A_{ij} = 0$.

2.1. Displacement-based formulation of the problem

Now it is convenient to pass from $(x, y, z)$ to the cylindrical coordinate system $(r, \varphi, z)$. Let $l$ be height of shell, $h$ be its constant thickness, $R$ be radius of middle surface of shell (Figure 1). In this section problems of deformation of anisogrid cylindrical shells are posed in the domain

\[
\Omega = \{(r, \varphi, z) : 0 \leq z \leq l, \quad 0 \leq \varphi \leq 2\pi, \quad R - h/2 \leq r \leq R + h/2\}.
\]

Constitutive equations (1) in system $(r, \varphi, z)$ remain valid. Lets supplement them with
equilibrium equations

\[ L_r = \frac{\partial}{\partial r} (r \sigma_r) - \sigma_r + \frac{\partial \tau_{r\varphi}}{\partial \varphi} + \frac{\partial \tau_{r\varphi}}{\partial z} (r \tau_{r\varphi}) + F_r, r = 0, \]
\[ L_\varphi = \frac{\partial \sigma_\varphi}{\partial \varphi} + \frac{\partial}{\partial z} (r \tau_{r\varphi}) + \frac{\partial \tau_{r\varphi}}{\partial r} (r \tau_{r\varphi}) + \tau_{r\varphi} + F_\varphi, \]
\[ L_z = \frac{\partial}{\partial z} (r \sigma_z) + \frac{\partial \tau_{r\varphi}}{\partial \varphi} + \tau_{r\varphi} + F_z, r = 0, \]

and compatibility equations

\[ \varepsilon_r \approx \frac{\partial \sigma_r}{\partial r}, \quad \varepsilon_\varphi \approx \frac{1}{r} \frac{\partial \sigma_\varphi}{\partial \varphi} + \frac{u}{r}, \quad \varepsilon_z \approx \frac{\partial \sigma_z}{\partial z}, \]
\[ \gamma_{r\varphi} \approx \frac{1}{r} \frac{\partial \tau_{r\varphi}}{\partial r} + \frac{\partial}{\partial \varphi} \left( \frac{v}{r} \right), \quad \gamma_{r\varphi} \approx \frac{\partial \tau_{r\varphi}}{\partial z} + \frac{1}{r} \frac{\partial \tau_{r\varphi}}{\partial \varphi}, \quad \gamma_{rz} \approx \frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \tau_{rz}}{\partial z}, \]

where \( u, v, w \) are displacements of points of shell depending on \( r, \varphi, z; F_r, F_\varphi, F_z \) are components of field of strength in shell. Substituting (3) into (1) and (1) into (2) one obtains resolving system of differential equations with unknown displacements in \( \Omega \):

\[ \Delta_1 u = f_1(u, v, w, A_{ij}), \quad \Delta_2 v = f_2(u, v, w, A_{ij}), \quad \Delta_3 w = f_3(u, v, w, A_{ij}), \]

where \( \Delta_{1,2,3} \) are differential operators, \( f_{1,2,3} \) are linear but rather cumbersome functions of the displacements, of their first and second derivatives and of stiffness coefficients \( A_{ij} \). Solution to the system of equations (4) is assumed to be smooth enough and \( 2\pi \)-periodic with respect to \( \varphi \) coordinate. Two types of boundary conditions can be specified for the system. They are Dirichlet conditions (displacements are specified on border) and Neumann conditions (loads \( p = (p_x, p_y, p_z) \) and \( q = (q_x, q_y, q_z) \) are specified on border), see Figure 1. In last case the values of normal derivatives of \( u, v, w \) are specified

- on \( \Gamma = \{(r, \varphi, z) : z = 0, l, 0 \leq \varphi \leq 2\pi, R - h/2 \leq r \leq R + h/2 \} \) and
- on \( \Sigma = \{(r, \varphi, z) : 0 \leq \varphi \leq 2\pi, r = R - h/2, R + h/2 \} \).

The complete expressions for all of equations and boundary conditions are given in [6].

2.2. Stress-based formulation of the problem

Stress-based formulation of theory of elasticity should take into account Saint-Venant’s compatibility condition. This condition means that the medium remains solid while deforming. Discontinuities and voids are absent. It can be expressed in mathematical sense as continuity of functions \( u, v, w \) that is provided by vanishing of six independent components of Riemann-Christoffel tensor, [3]:

\[ R_{mnpq} = 0, \quad mnpq \in \{1212, 1313, 2323, 1213, 2123, 3132\}. \]

In case of infinitely small values of relative displacements in curvilinear coordinate system, these conditions have the following form, see [14]:

\[ \frac{\partial^2 \varepsilon_{mq}}{\partial x^m \partial x^n} + \frac{\partial^2 \varepsilon_{np}}{\partial x^m \partial x^n} - \frac{\partial^2 \varepsilon_{nq}}{\partial x^m \partial x^n} - \frac{\partial^2 \varepsilon_{mp}}{\partial x^m \partial x^n} - 2 \varepsilon_{rs} (\Gamma^r_{qn} \Gamma^s_{pn} - \Gamma^r_{mp} \Gamma^s_{qn}) + 2 \Gamma^r_{np} \varepsilon_{mpr} - 2 \Gamma^r_{pq} \varepsilon_{mrp} - 2 \Gamma^r_{mr} \varepsilon_{pmn} = 0, \]

where \( (x^1, x^2, x^3) \) are curvilinear coordinates, presence of index \( r \) implies summation over all values of \( r = 1, 2, 3 \), \( \varepsilon_{npr} = \frac{1}{2} \left( \frac{\partial \varepsilon_{pr}}{\partial x^n} + \frac{\partial \varepsilon_{rn}}{\partial x^p} - \frac{\partial \varepsilon_{pn}}{\partial x^r} \right) \), \( \Gamma^k_{ij} \) are Christoffel symbols of the second
were computed, where parameters, like relations of elastic moduli computations in Wolfram Mathematica environment. The derivation of resolving equations is computerized and implemented using symbolic functions of stresses, of their first and second derivatives and of coefficients denoted by $U$

To derive resolving system, the following transformations were performed.

1. System (1) is resolved with regard to $\varepsilon$: $\varepsilon = \mathfrak{B} \sigma$, the elements $B_{ij}$ of matrix $\mathfrak{B} = A^{-1}$ were computed, $i, j = 1, 2, 3$.

2. Substitution of components of $\varepsilon$ into (6) was performed. As a result, the expressions $U_{a}(\sigma)$ $U_{a,b}(\sigma)$ were obtained with $a, b \in \{r, \varphi, z\}$.

3. The relations for $\frac{\partial^2 \tau_{r\varepsilon}}{\partial r \partial \varphi}, \frac{\partial^2 \tau_{rz}}{\partial \varphi \partial z}$ were obtained using differentiation of linear transformations of (2). The results of its substitution into formulae for $U_{a}(\sigma)$ $U_{a,b}(\sigma)$ was denoted by $U_{a}^{*}(\sigma)$ $U_{a,b}^{*}(\sigma)$.

4. The resolving system of differential equations for stresses consists of six relations

$$
\begin{align*}
\frac{U_{r}^{*} + U_{z}^{*}}{B_{11}^{*}} &= 0 , \quad \frac{U_{r}^{*} + U_{z}^{*}}{B_{22}^{*}} = 0 , \quad \frac{U_{r}^{*}}{B_{33}^{*}} = 0 , \quad \frac{U_{r}^{*}}{B_{44}^{*}} = 0 , \quad \frac{U_{r}^{*}}{B_{55}^{*}} = 0 , \quad \frac{U_{r}^{*}}{B_{66}^{*}} = 0 .
\end{align*}
$$

Thus, a system of elliptic type equations for unknown components of stress tensor $\sigma$ was obtained in the form

$$
\begin{align*}
\lambda_{1} \sigma_{1} &= g_{1}(\mathfrak{B}, B_{ij}) , \quad \Delta \lambda_{2} = g_{2}(\mathfrak{B}, B_{ij}) , \quad \Delta \lambda_{3} = g_{3}(\mathfrak{B}, B_{ij} ,) , \\
\lambda_{4} \tau_{12} &= g_{12}(\mathfrak{B}, B_{ij}) , \lambda_{5} \tau_{13} = g_{13}(\mathfrak{B}, B_{ij}), \lambda_{6} \tau_{23} = g_{23}(\mathfrak{B}, B_{ij}),
\end{align*}
$$

where $\lambda_{1} - \lambda_{6}$ are differential operators, $g_{1,2,3}$, $g_{12,13,23}$ are linear but rather cumbersome functions of stresses, of their first and second derivatives and of coefficients $B_{ij}$.

The system should be supplemented with the values of stresses on the border of domain $\Omega$. The derivation of resolving equations is computerized and implemented using symbolic computations in Wolfram Mathematica environment.

3. Fast computational algorithm and numerical simulations

3.1. Pseudo-spectral method without saturation

The 3D boundary-value problems of displacement and stress formulations (4), (7) have a high computational complexity due to presence in them of small parameters. They are physical parameters, like relations of elastic moduli $E_{T}/E$, $G_{T}/G$, and geometric parameters, like the relation $a_{j}/b^{3}$ and $h$ (square of the relative shell’s thickness) that will appear as a scale factor at the highest order derivatives in (4), (7) after approximating them.
The solutions to these problems require application of fast and highly accurate algorithms providing the possibility to control the errors. For this purpose a new pseudo-spectral algorithm was developed on the basis of iterative stabilization method, collocation method and the so-called approximations without saturation. Particular, for approximation of unknown displacements or stresses the Kronecker products of interpolation polynomials with Chebyshev nodes (with respect to coordinates \( r, z \)) and with Dirichlet kernel (with respect to \( \phi \) coordinate) were used, see [6]–[8]. Following the works by K. I. Babenko [11] we can conclude, that the accuracy of such interpolations strictly corresponds to that of best polynomial approximations for any finite order of smoothness of sought-for function, while for infinitely differentiable functions an exponential (or geometrical) convergence can be achieved (this case was described in detail in [13]).

The resolving systems of equations (4), (7) have elliptic type and the applied load can be approximated using smooth function with steep gradients. Thus, the solutions are of high order of smoothness. This ensures the efficiency of applied approximations and the minimization of computational efforts.

Another way for reducing these efforts is a new approach for passage from differential problems to linear algebra. In this work we use the collocation method to derive the tensors approximations of derivatives that yields the Sylvester tensor equations. To solve these equations we use a special fast method proposed in [4]. The essential advantage of this method is the slow growth of condition numbers of matrices forming the tensors with grows of number of interpolation nodes. Detailed description of the algorithm together with estimates of its approximation and round-off errors can be found in [5].

### 3.2. Simulation of non-uniform axial compression of a cylindrical shell

Under assumption that the lower base of structure is rigidly fixed, the proposed algorithm was used for computing displacements of the upper one with respect to \( z \).

\[
\begin{align*}
    w^* &= \max_{j,k} w(r_j, \varphi_k, l), \\
    w^{**} &= \max_{j,k} \int_0^l \sigma_z(r_j, \varphi_k, z) dz,
\end{align*}
\]

where \( w \) is numerical solution obtained for displacement formulation of problem (4) (problem 1); \( \sigma_z \) is solution obtained for stress formulation of problem (see (7)) – problem 2; \( w^{**} \) is defined in accordance with Cesaro’s formulae. Let \( N, K, M \) be the numbers of interpolation nodes of algorithm with respect to \( r, \varphi, z \) axis correspondingly. Here we assume \( N = K = M \) and denote \( \lambda = |w^* - w^{**}|/w^* \). Further, \( T \) is running time of algorithm in seconds, values \( \varepsilon_N \) and \( \varepsilon_R \) are estimations of approximation and round-off errors of algorithm, they were computed while solving both problems for \( w \) and \( \sigma_z \). All these values were obtained using one core of Intel Core i3-3217U 1.8 Ghz CPU with DDR3 6Gb 1600 Mhz memory.

Here a solution to the problem of non-uniform compression is described. This problem may occur in many applications, for example, while describing the deformation of first stage adapter of Proton rocket. The first stage has six engines placed round a circle under the adapter (see https://en.wikipedia.org/wiki/Proton_(rocket_family)). Engine thrust is \( F_E = 10.02 \text{MN} \), load from the top in the moment of rocket’s start can be estimated as \( F_L = 2.94 \text{MN} \). Let us specify the following values of parameters: \( R = 2\text{m}, l = 2\text{m}, h = 16\text{mm}, a_h = 125.47\text{mm}, a_c = 139.66\text{mm}, \delta_h = 7.5\text{mm}, \delta_c = 4.2\text{mm} \). These values for lattice structure were taken from [9], [10]. The non-uniform load of six engines on the bottom of shell should be specified in accordance with the mechanical experiments with structure, but these data in open literature is absent. Here we
consider the test example with load of the form

\[ P(r, \varphi, 0) = \frac{F_E}{c_L} [\sin 3\varphi]^{10} \text{MN}, \quad c_l = \int_{R-h/2}^{R+h/2} \int_0^{2\pi} r(\sin 3\varphi)^{10} d\varphi dr = \frac{63h\pi r}{128}. \]

The results of solution to the problems (4), (7) with the described loads are given in Table 3.2 (here we take different values of \( N \) and \( K = 41, M = 7 \)).

### Table 1. Results obtained for non-uniform axial compression of cylindrical shell

| \( N \) | Problem | \( \varepsilon_{N,\%} \) | \( \varepsilon_{R,\%} \) | \( T, s \) | \( w^{+,\%}, \text{mm} \) | \( \lambda, \% \) |
|---|---|---|---|---|---|---|
| 7 | 1 | 0.1398 | 2.51e-3 | 526.1 | 13.12 | 1.8 |
| 7 | 2 | 0.4218 | 1.31e-5 | 19.3 | 13.36 | 1.8 |
| 11 | 1 | 0.222 | 3.08e-2 | 768 | 13.24 |
| 11 | 2 | 0.4344 | 1.35e-4 | 35.5 | 13.372 |
| 23 | 1 | 1.7458 | 1.1 | 1675.9 | 13.124 |
| 23 | 2 | 0.4346 | 97.6 | 280.8 | 13.372 |

Note that for a given non-uniform load the applied algorithm allows to obtain the solution of problem 1 only for small values of \( M \) \((M = 5 \sim 13)\) and it requires more time than the solution of problem 2. The reason is that the computational stability of boundary-value problem with Neumann conditions is much lower then that with Dirichlet conditions; that affects the values of round-off errors and the convergence of applied iteration process. But in any case, as it is seen from Table 3.2, the round-off error for this test does not exceed 1.1%.

Figure 2 contains the solutions of problem 2 with \( N = 21, K = 71, M = 21 \). They are values of \( \sigma_z \) in section \( r = 2 \) (Figure 2, a) and of \( u \) in section \( z = 0 \) (Figure 2, b). For this case we found that \( \varepsilon_R = 0.0342\% \) and \( \varepsilon_{N} = 1.94 \times 10^{-3}\% \). Thus, we may say, that the solutions have five accurate significant digits.

![Figure 2](image-url)  
*Figure 2. Numerical solutions to the problem of non-uniform axial compression.*

### 3.3. Design of a cylindrical shell bearing non-uniform axial loads

Here by varying the values of angles of spiral ribs slope \( \phi^2 \) \((\phi^2 = -\phi^3)\) and the thicknesses of ribs \( h \) we shall design the cylindrical shell of minimal mass bearing the axial loads described above. As the criterions of loss of bearing capacity the following expressions have been considered
Figure 3. Analysis of the bearing capacity of a shell – (a) values of $\phi^2$, $h$ ensuring the bearing capacity, (b) dependence of $M$ on $\phi^2$, $h$. The points on graphs show optimal values.

- Exceeding the CFRPs ultimate stress while tension or compression along ($s_{TL}$, $s_{CL}$) or across ($s_{TT}$, $s_{CT}$) the direction of fibers, or while applying share loads ($s_S$).
- The loss of stability of shell’s rib. For a rib of length $l$ with cross section $b \times h$ and with $E$ and $G$ elasticity and share moduli measured in tests with loads applied along the direction of fibers, the ultimate load is (see [12])

$$T_L = \frac{\pi^2 K D}{K^2 + \pi^2 D}, \quad K = bhG, \quad D = EJ, \quad J = \frac{bh^3}{12}. \quad (8)$$

Using (8) the ultimate loads on structure’s ribs were computed taking the following values from [9]: $s_{TL} = 2 \times 10^9$Pa, $s_{CL} = 1.2 \times 10^9$Pa, $s_{TT} = 5 \times 10^7$Pa, $s_{CT} = 1.7 \times 10^8$Pa, $s_S = 7 \times 10^7$Pa.

Performing the calculations using the described models and methods and also the bisection method we separated on the plane ($\phi^2$, $h$) the domain of parameters ensuring the bearing capacity of structure for a given load (see Fig 3, a). The mass of shell was computed as a sum of masses of all ribs assuming that the density of CFRP is $\rho \approx 1500$kg/m$^3$ (see Fig 3, b). Taking into account the obtained bounds on the values of $\phi^2$, $h$, the values ensuring the minimal weight of structure were computed. They are $\phi^2 = 12.78^\circ$, $h = 53.84$mm. The minimal mass of shell is $M = 249.176$kg. Let us note, that in the case of uniform load in [10], [12] it was recommended to take $\phi^2 = 26.6^\circ$ as optimal angle. In the case of considered non-uniform load it leads to increase of structure’s mass up to $M = 445.5$kg, that is 79 percent more than in optimal case.

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