ON SHARP GLOBAL WELL-POSEDNESS AND ILL-POSEDNESS FOR A FIFTH-ORDER KDV-BBM TYPE EQUATION

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Abstract. We consider the Cauchy problem associated to the recently derived higher order hamiltonian model for unidirectional water waves and prove global existence for given data in the Sobolev space $H^s$, $s \geq 1$. We also prove an ill-posedness result by showing that the flow-map is not $C^2$ if the given data has Sobolev regularity $s < 1$. The results obtained in this work are sharp.

1. Introduction

In this work, we consider the Cauchy problem associated to the recently introduced higher order KdV-BBM type model by Bona et al in \textsuperscript{5}

$$
\begin{align*}
\eta_t + \eta_x - \gamma_1 \eta_{xxx} + \gamma_2 \eta_{xxxx} + \delta_1 \beta^2 \eta_{xxxxx} + \delta_2 \beta \eta_x + \gamma \eta^2_{xxx} - \frac{7}{58}(\eta_x^2)_x - \frac{1}{8}(\eta^3)_x &= 0, \\
\eta(x, 0) &= \eta_0(x),
\end{align*}
$$

(1.1)

where

$$
\begin{align*}
\gamma_1 &= \frac{1}{2}(b + d - \rho), \\
\gamma_2 &= \frac{1}{2}(a + c + \rho), \\
\delta_1 &= \frac{1}{4} \left[ 2(b_1 + d_1) - (b - d + \rho) \left( \frac{1}{6} - a - d \right) - d(c - a + \rho) \right], \\
\delta_2 &= \frac{1}{4} \left[ 2(a_1 + c_1) - (c - a + \rho) \left( \frac{1}{6} - a \right) + \frac{3}{8} \rho \right], \\
\gamma &= \frac{1}{24} \left[ 5 - 9(b + d) + 9 \rho \right].
\end{align*}
$$

(1.2)

The parameters appeared in (1.1) satisfy $a + b + c + d = \frac{1}{3}$, $\gamma_1 + \gamma_2 = \frac{1}{6}$, $\gamma = \frac{1}{24}(5 - 18 \gamma_1)$ and $\delta_2 - \delta_1 = \frac{19}{360} - \frac{1}{6} \gamma_1$ with $\delta_1 > 0$.

The model in (1.1) describes the unidirectional propagation of water waves. The authors in \textsuperscript{5} used the second order approximation in the two-way model, the so-called $abcd$-system.
introduced in [7, 8] and obtained a fifth order KdV-BBM type model (1.1). Also we note that, the model (1.1) possesses an energy conservation law

$$E(\eta(\cdot, t)) := \frac{1}{2} \int_{\mathbb{R}} \eta^2 + \gamma_1(\eta_x)^2 + \delta_1(\eta_{xx})^2 \, dx = E(\eta_0),$$

(1.3)

when the parameter $\gamma = \frac{7}{48}$. In this particular case, the model (1.1) turns out to be hamiltonian.

There are other higher order models of KdV and BBM type in the literature, see for example [15, 16, 17, 18, 23, 24] and references therein. These models are derived either by using Hamiltonian perturbation method [23, 24] or by expanding Dirichlet–Neumann operator in the Zakharov–Craig–Sulem formulation [21]. Also, most of the higher order KdV-BBM type models existing in the literature are either ill-posed or don’t have hamiltonian structure, see for example [1, 2, 3] and references therein. The model (1.1) posed on half-line is also studied in [14].

Well-posedness issues for the Cauchy problem (1.1) with initial data in the Sobolev spaces $H^s$ are studied in [5]. More precisely, for given data in $\eta_0 \in H^s(\mathbb{R})$, $s \geq 1$ the authors in [5] proved the following local well-posedness result.

**Theorem A.** Assume $\gamma_1, \delta_1 > 0$. For any $s \geq 1$ and for given $\eta_0 \in H^s(\mathbb{R})$, there exist a time $T_{\eta} = \frac{c_s}{\|\eta_0\|_{H^s}(1 + \|\eta_0\|_{H^s})}$ and a unique function $\eta \in C([0, T_{\eta}]; H^s)$ which is a solution of the Cauchy problem (1.1), posed with initial data $\eta_0$. The solution $\eta$ varies continuously in $C([0, T_{\eta}]; H^s)$ as $\eta_0$ varies in $H^s$.

In the case, when the parameter $\gamma = \frac{7}{48}$, conserved quality (1.3) allows one to get an a priori estimate in $H^2$ which in turn yields global well-posedness in $H^s$, for $s \geq 2$. For given data with certain range of Sobolev regularity below $H^2$, the authors in [5] used splitting argument introduced in [12, 13] (see also [9]) to extend local solution to global in time. More specifically, the global well-posedness result proved in [5] is the following.

**Theorem B.** Assume $\gamma_1, \delta_1 > 0$. Let $s \geq \frac{3}{2}$ and $\gamma = \frac{7}{48}$. Then the solution to the Cauchy problem (1.1) given by Theorem A can be extended to arbitrarily large time intervals $[0, T]$. Hence the problem is globally well-posed in this case.

Now, a natural question is: whether the results obtained in Theorems A and B are optimal? In this work we will try to respond this question. The first main result deals with the global well-posedness and is stated as follows.
Theorem 1.1. Assume $\gamma_1, \delta_1 > 0$. Let $1 \leq s < 2$ and $\gamma = \frac{7}{48}$. Then for any given $T > 0$, the solution to the Cauchy problem (1.1) given by Theorem A can be extended to the time interval $[0, T]$. Hence the Cauchy problem (1.1) is globally well-posed in this case. In addition, one also has

$$\eta(t) - S(t)\eta_0 \in H^2, \quad \text{for all } t \in [0, T]$$

and

$$\sup_{t \in [0,T]} \|\eta(t) - S(t)\eta_0\|_{H^2} \lesssim (1 + T)^{2-s},$$

where $S(t)$ is as defined in (2.4) below.

The second main result of this work address the sharpness of the well-posedness issue by proving the following ill-posedness result.

Theorem 1.2. Assume $\gamma_1, \delta_1 > 0$. For any $s < 1$ and for given $\eta_0 \in H^s(\mathbb{R})$, there exist no time $T = T(\|\eta_0\|_{H^s})$ such that the solution-map that takes initial data $\eta_0$ to the solution $\eta \in C([0, T]; H^s)$ to the Cauchy problem (1.1) is $C^2$.

In view of the results obtained in Theorems 1.1 and 1.2, the global well-posedness of the Cauchy problem (1.1) for data $\eta_0 \in H^s$, $s \geq 1$ is sharp.

Before leaving this section, we record the notations used in this work along with structure. We use standard notations of the PDE and explain wherever necessary in their first appearance. The structure of the paper is as follows. In Section 2 we prove the global well-posedness result stated in Theorem 1.1 while Section 3 is devoted to prove the ill-posedness result stated in Theorem 1.2.

2. Global Well-posedness Results

In this section we will consider $\gamma = \frac{7}{48}$ and $1 \leq s < 2$. Let $T > 0$ be large. Our objective in this section is to extend the local solution to the Cauchy problem (1.1) given by Theorem A to a large time interval $[0, T]$, for any given $T > 0$.

We start by writing the Cauchy problem (1.1) in the following form

$$\begin{cases}
    i\eta_t = \phi(\partial_x)\eta + \tau(\partial_x)\eta^2 - \frac{1}{8}\psi(\partial_x)\eta^3 - \frac{7}{48}\psi(\partial_x)\eta_x^2, \\
    \eta(x, 0) = \eta_0(x),
\end{cases}$$

(2.1)

where $\phi(\partial_x), \psi(\partial_x)$ and $\tau(\partial_x)$ are Fourier multiplier operators defined by,

$$\hat{\phi(\partial_x)}f(\xi) := \phi(\xi)\hat{f}(\xi), \quad \hat{\psi(\partial_x)}f(\xi) := \psi(\xi)\hat{f}(\xi) \quad \text{and} \quad \hat{\tau(\partial_x)}f(\xi) := \tau(\xi)\hat{f}(\xi),$$

(2.2)
with symbols
\[ \phi(\xi) = \frac{\xi(1 - \gamma_2\xi^2 + \delta_2\xi^4)}{\varphi(\xi)}, \quad \psi(\xi) = \frac{\xi}{\varphi(\xi)} \quad \text{and} \quad \tau(\xi) = \frac{3\xi - 4\gamma\xi^3}{4\varphi(\xi)}. \]

The common denominator
\[ \varphi(\xi) := 1 + \frac{1}{2}\gamma_1\xi^2 + \delta_1\xi^4, \]
is strictly positive because the parameters \( \gamma_1 \) and \( \delta_1 \) are taken to be positive.

Consider first the linear Cauchy problem associated to (2.1)
\[ \begin{cases} 
  i\eta_t = \phi(\partial_x)\eta, \\
  \eta(x,0) = \eta_0(x),
\end{cases} \tag{2.3} \]
whose solution is given by \( \eta(t) = S(t)\eta_0 \), where \( S(t) \) is defined via its Fourier transform
\[ S(t)\hat{\eta}_0 = e^{-i\phi(\xi)t}\hat{\eta}_0. \tag{2.4} \]

Clearly, \( S(t) \) is a unitary operator on \( H^s \) for any \( s \in \mathbb{R} \), so that
\[ \|S(t)\eta_0\|_{H^s} = \|\eta_0\|_{H^s}, \tag{2.5} \]
for all \( t > 0 \). Duhamel’s formula allows us to write the Cauchy problem (2.1) in the equivalent integral equation form,
\[ \eta(x,t) = S(t)\eta_0 - i \int_0^t S(t-t')\left(\tau(\partial_x)\eta^2 - \frac{1}{8}\psi(\partial_x)\eta^3 - \frac{7}{48}\psi(\partial_x)\eta^2\right)(x,t')dt'. \tag{2.6} \]

Local well-posedness results for the Cauchy problem (1.1) is obtained in [5] via the contraction mapping principle in the space \( C([0,T]; H^s) \), \( s \geq 1 \) using the Duhamel’s formula (2.6). To complete the contraction principle argument, the following estimates were crucial. In what follows we record these estimates along with some improvements, because they will be needed in our argument.

**Proposition 2.1.** For \( s \geq 0 \), there is a constant \( C = C_s \) for which
\[ \|\omega(\partial_x)(uv)\|_{H^s} \leq C\|u\|_{H^s}\|v\|_{H^s}, \tag{2.7} \]
where \( \omega(\partial_x) \) is the Fourier multiplier operator with symbol
\[ \omega(\xi) = \frac{\xi}{1 + \xi^2}. \tag{2.8} \]

**Proof.** See Lemma 3.1 in [5]. \( \square \)
Proposition 2.2. For any $s \geq 0$, there is a constant $C = C_s$ such that the inequality
\[ \| \tau(\partial_x)(\eta_1 \eta_2) \|_{H^s} \leq C \| \eta_1 \|_{H^s} \| \eta_2 \|_{H^s} \]  \tag{2.9} \]
and
\[ \| \partial_x \tau(\partial_x)(\eta_1 \eta_2) \|_{H^1} \leq C \| \eta_1 \|_{H^1} \| \eta_2 \|_{H^1} \]  \tag{2.10} \]
holds, where the operator $\tau(\partial_x)$ is as defined in \(2.2\).

Proof. The proof of the inequality (2.9) is in Corollary 3.2 of [5]. In order to prove (2.10), from definition of operator $\tau(\partial_x)$, we have
\[ \| \partial_x \tau(\partial_x)(\eta_1 \eta_2) \|_{H^1} = \| \langle \xi \rangle \xi \tau(\hat{\eta_1 \eta_2})(\xi) \|_{L^2} \]  \tag{2.11} \]
and
\[ |\xi \tau(\xi)| = \left| \frac{3\xi^2 - 4\gamma \xi^4}{4(1 + \gamma \xi^2 + \delta \xi^4)} \right| \leq c. \]
Thus, since $H^1$ is an algebra, we get
\[ \| \partial_x \tau(\partial_x)(\eta_1 \eta_2) \|_{H^1} \leq c \| \langle \xi \rangle \hat{\eta_1 \eta_2}(\xi) \|_{L^2} = c \| \eta_1 \eta_2 \|_{H^1} \lesssim \| \eta_1 \|_{H^1} \| \eta_2 \|_{H^1}. \]

Proposition 2.3. For $s \geq \frac{1}{6}$, there is a constant $C = C_s$ such that
\[ \| \psi(\partial_x)(\eta_1 \eta_2 \eta_3) \|_{H^s} \leq C \| \eta_1 \|_{H^s} \| \eta_2 \|_{H^s} \| \eta_3 \|_{H^s} \]  \tag{2.12} \]
and
\[ \| \partial_x \psi(\partial_x)(\eta_1 \eta_2 \eta_3) \|_{H^1} \leq C \| \eta_1 \|_{H^1} \| \eta_2 \|_{H^1} \| \eta_3 \|_{H^1}. \]  \tag{2.13} \]

Proof. The proof of the inequality (2.12) is in Proposition 3.3 of [5]. In order to prove (2.10), from definition of operator $\psi(\partial_x)$, we have
\[ \| \partial_x \psi(\partial_x)(\eta_1 \eta_2 \eta_3) \|_{H^1} = \| \langle \xi \rangle \xi \psi(\hat{\eta_1 \eta_2 \eta_3})(\xi) \|_{L^2} \]  \tag{2.14} \]
and
\[ |\xi \psi(\xi)| = \left| \frac{\xi^2}{1 + \gamma \xi^2 + \delta \xi^4} \right| \leq c. \]
Thus, since $H^1$ is an algebra, we get
\[ \| \partial_x \psi(\partial_x)(\eta_1 \eta_2 \eta_3) \|_{H^1} \leq c \| \langle \xi \rangle \hat{\eta_1 \eta_2 \eta_3}(\xi) \|_{L^2} = c \| \eta_1 \eta_2 \eta_3 \|_{H^1} \lesssim \| \eta_1 \|_{H^1} \| \eta_2 \|_{H^1} \| \eta_3 \|_{H^1}. \]
Proposition 2.4. For \( s \geq 1 \), the inequality
\[
\| \psi(\partial_x)(\eta_1)_x(\eta_2)_x) \|_{H^s} \leq C \| \eta_1 \|_{H^s} \| \eta_2 \|_{H^s}
\] (2.15)
and
\[
\| \partial_x \psi(\partial_x)(\eta_1)_x(\eta_2)_x) \|_{H^1} \leq C \| \eta_1 \|_{H^1} \| \eta_2 \|_{H^1}
\] (2.16)
hold.

Proof. The proof of the inequality (2.15) is in Lemma 3.5 of [5]. In order to prove (2.16), from definition of operator \( \psi(\partial_x) \), we have
\[
\| \partial_x \psi(\partial_x)(\eta_1)_x(\eta_2)_x) \|_{H^1} = \| \langle \xi \rangle \xi \psi(\xi)(\eta_1)_x(\eta_2)_x(\xi) \|_{L^2}.
\] (2.17)

Thus, using Plancherel identity and Proposition 2.1, we get
\[
\| \partial_x \psi(\partial_x)(\eta_1)_x(\eta_2)_x) \|_{H^1} \leq c \| \omega(\partial_x)(\eta_1)_x(\eta_2)_x \|_{L^2} \lesssim \| (\eta_1)_x \|_{L^2} \| (\eta_2)_x \|_{L^2} \lesssim \| \eta_1 \|_{H^1} \| \eta_2 \|_{H^1}.
\]

For the global well-posedness with initial data in \( H^s, s \geq 2 \), the conserved quantity (1.3) was used. While for the range of \( \frac{3}{2} \leq s < 2 \), splitting argument was used. In this work, we want to further lower the regularity condition on the initial data to get global solution to match that of the local existence. Here too, we will use the splitting argument introduced in [12, 13] and way earlier in [9]. This argument is very powerful to get global solution of the Cauchy problem with low regularity data and is used by several authors, see for example [5, 11, 19, 20] and references therein. It is worth noting that in [12, 13], energy estimate was used to complete the iteration argument, while in [19] and [20] \( L^pL^q \) estimates were used. In our earlier work [5] we used energy estimate evolving high frequency part of the initial data according to the original equation and the low frequency part according the difference equation. In this work, we will perform the other-way around, i.e., evolve the low frequency part according to the original equation and the high frequency part according to the difference equation so that the sum of two will give solution to the original problem.

Let \( \eta_0 \in H^s, s \geq 1 \), we split the initial data \( \eta_0 = u_0 + v_0, \hat{\eta}_0 = \hat{\eta}_0 \chi_{\{ |\xi| \leq N \}} \), where \( N \) is a large number to be chosen later, it is easy to see that \( u_0 \in H^\delta \) for any \( \delta \geq s \) and \( v_0 \in H^s \). In fact we have
\[
\| u_0 \|_{L^2} \leq \| \eta_0 \|_{L^2},
\]
\[
\| u_0 \|_{H^s} \leq \| \eta_0 \|_{H^s} \| \eta_0 \|_{H^s} N^{\delta-s}, \quad \delta \geq s,
\] (2.18)
and
\[ \|v_0\|_{H^s} \leq \|\eta_0\|_{H^s} N^{(\rho-s)}, \quad 0 \leq \rho \leq s. \] (2.19)

For each parts \( u_0 \) and \( v_0 \) of \( \eta_0 \) we associate the Cauchy problems
\[
\begin{align*}
&iu_t = \phi(\partial_x)u + F(u), \\
&u(x,0) = u_0(x),
\end{align*}
\]
where \( F(u) = \tau(\partial_x)u^2 - \frac{1}{8}\psi(\partial_x)u^3 - \frac{7}{48}\psi(\partial_x)u_x^2 \)
and
\[
\begin{align*}
&iv_t = \phi(\partial_x)v + F(u + v) - F(v), \\
v(x,0) = v_0(x),
\end{align*}
\] (2.21)
respectively, so that we have \( \eta(x,t) = u(x,t) + v(x,t) \), solves the original Cauchy problem \((1.1)\) in the common time interval of existence of \( u \) and \( v \). In what follows, we prove that there is a time \( T_u \) such that the Cauchy problem \((2.20)\) is locally well-posed in \([0, T_u]\). Fixing the solution \( u \) of \((2.20)\), we prove that there exists \( T_v \) such that the Cauchy problem \((2.21)\) is locally well-posed in \([0, T_v]\). In this way, for \( t_0 \leq \min\{T_u, T_v\} \), \( \eta = u + v \) solves the Cauchy problem \((1.1)\) in the time interval \([0, t_0]\) for given data in \( H^s \), \( s \geq 1 \). Our idea is to iterate this process maintaining \( t_0 \) as the length of existence time in each iteration to cover any given time interval \([0, T]\).

By Theorem A the Cauchy problem \((2.20)\) is locally well-posed in \( H^s \), \( s \geq 1 \) with existence time given by \( T_u = \frac{c_a}{\|u_0\|_{H^s}(1 + \|u_0\|_{H^s})} \) and by Theorem B globally well-posed in \( H^s \), \( s \geq 2 \).

Regarding the well-posedness of the Cauchy problem \((2.21)\) with variable coefficients that depend on \( u \), we have the following result.

**Theorem 2.5.** Assume \( \gamma_1, \delta_1 > 0 \) and \( u \) the solution to the Cauchy problem \((2.20)\). For any \( s \geq 1 \) and for given \( v_0 \in H^s(\mathbb{R}) \), there exist a time \( T_v = \frac{c_a}{(\|v_0\|_{H^s} + \|u_0\|_{H^s})(1 + \|v_0\|_{H^s} + \|u_0\|_{H^s})} \) and a unique function \( v \in C([0, T_v]; H^s) \) which is a solution of the IVP \((2.21)\), posed with initial data \( v_0 \). The solution \( v \) varies continuously in \( C([0, T_v]; H^s) \) as \( v_0 \) varies in \( H^s \).

**Proof.** Using Duhamel’s formula, the equivalent integral equation to \((2.21)\) is
\[
v(x,t) = S(t)v_0 - i \int_0^t S(t-t') \left( F(u + v) - F(v) \right)(x,t') dt'
\]
\[= : S(t)v_0 + h(x,t), \quad (2.22)\]
where
\[
F(u + v) - F(v) = \tau(\partial_x)(v^2 + 2vu) - \frac{1}{8}\psi(\partial_x)(3u^2v + 3uv^2 + v^3) - \frac{7}{48}\psi(\partial_x)(2uv_x + v_x^2). \quad (2.23)
\]
Let \( u \in C([0, T_a]; H^s) \) be the solution of Cauchy problem (2.20), given by Theorem A and satisfying
\[
\sup_{t \in [0, T_a]} \|u(t)\|_{H^s} \lesssim \|u_0\|_{H^s}.
\]
(2.24)

Let
\[
X^a_T = \{ v \in C([0, T]; H^s) : \|v\| := \sup_{t \in [0, T]} \|v(t)\|_{H^s} \leq a \}
\]
where \( a := 2\|v_0\|_{H^s} \), and consider an application
\[
\Phi_u(v)(x, t) = S(t)v_0 - i \int_0^t S(t-t')(F(u + v) - F(v))(x, t')dt'.
\]

We will prove that the application \( \Phi_u(v) \) is a contraction on \( X^a_T \). By definition \( S(t) \) is a unitary group in \( H^s(\mathbb{R}) \). Then for \( T \leq T_a \), we have
\[
\|\Phi_u(v)\|_{H^s} \leq \|v_0\|_{H^s} + T\|\partial_x(v^2 + 2vu) - \frac{1}{8}\psi(\partial_x)(3u^2v + 3uv^2 + v^3)
- \frac{7}{48}\psi(\partial_x)(2uxv_x + v^2_x)||.
\]
The inequalities (2.21), (2.12), (2.15) and (2.24) yield
\[
\|\Phi_u(v)\|_{H^s} \leq \|v_0\|_{H^s} + T\|\partial_x(v^2 + 2vu) - \frac{1}{8}\psi(\partial_x)(3u^2v + 3uv^2 + v^3)
- \frac{7}{48}\psi(\partial_x)(2uxv_x + v^2_x)||
\]
\[
\leq \frac{a}{2} + cT\|v\|\|v\|\|v\| + cT\|v\|\|v\|\|v\| + \|u_0\|_{H^s} + \|u_0\|_{H^s} + \|u_0\|_{H^s} + \|u_0\|_{H^s} + \|u_0\|_{H^s} + \|u_0\|_{H^s}
\]
\[
\leq \frac{a}{2} + cT[a(a + \|u_0\|_{H^s})(1 + a + \|u_0\|_{H^s})]
\]
If we choose
\[
cT[(a + \|u_0\|_{H^s})(1 + a + \|u_0\|_{H^s})] = \frac{1}{2}
\]
then \( \|\Phi_u(v)\|_{H^s} \leq a \), showing that \( \Phi_u(v) \) maps the closed ball \( X^a_T \) in \( C([0, T]; H^s) \) onto itself.

With the same choice of \( a \) and \( T \) and the same sort of estimates, one can prove that the application \( \Phi_u(v) \) is a contraction on \( X^a_T \) with contraction constant equal to \( \frac{1}{2} \). The rest of the proof is standard.

In what follows, we record a lemma which will play a fundamental role in the proof of the global well-posedness result.

**Lemma 2.6.** Let \( u \) be the solution of the Cauchy problem (2.20) and \( v \) be the solution of the Cauchy problem (2.21), then \( h = h(u, v) \) as defined in (2.22) is in \( C([0, t_0]; H^2) \) and,
\[
\|u(t_0)\|_{H^2} \lesssim N^2 \quad \text{and} \quad \|h(t_0)\|_{H^2} \lesssim N^{s-3},
\]
(2.25)
where \( t_0 \sim N^{-2(2-s)} \).

**Proof.** Observe that the energy conservation law (1.3), gives

\[
\|u(t_0)\|_{H^2} \sim E(u(t_0)) = E(u_0) \sim \|u_0\|_{H^2} \lesssim N^{2-s}.
\]

On the other hand, from (2.22) and (2.23), we have for \( 1 \leq \delta \leq s \)

\[
\|h(t_0)\|_{H^s} = \left\| \int_0^{t_0} S(-t') \left( F(u + v) - F(v) \right)(x, t') dt' \right\|_{H^s}
\]

\[
\leq \int_0^{t_0} \|S(-t') \left( F(u + v) - F(v) \right)(x, t') dt'\|_{H^s}
\]

\[
\leq \int_0^{t_0} \left( \|\tau(\partial_x)(v^2 + 2vu)\|_{H^s} + \frac{1}{8}\|\psi(\partial_x)(3u^2v + 3uv^2 + v^3)\|_{H^s} \right)
\]

\[
+ \frac{7}{48}\|\psi(\partial_x)(2u_xv_x + v_x^2)\|_{H^s}\)dt'.
\]

Now, using the Propositions 2.2, 2.3 and 2.4 we arrive to

\[
\|h(t_0)\|_{H^s} \lesssim \int_0^{t_0} (\|v\|_{H^s}^2 + \|v\|_{H^s}\|u\|_{H^s} + \|u\|_{H^s}\|v\|_{H^s}^2 + \|v\|_{H^s}^3) dt'.
\] (2.26)

The local theory and the inequalities (2.13) and (2.19) imply \( \|v\|_{H^s} \lesssim N^{δ-s} \) and \( \|u\|_{H^s} \lesssim c. \) Thus, if \( δ = 1 \) and \( s \geq 1 \), we have

\[
\|h(t_0)\|_{H^1} \lesssim \int_0^{t_0} (N^{2(1-s)} + N^{(1-s)} + N^{3(1-s)}) dt'
\]

\[
\lesssim t_0(N^{2(1-s)} + N^{(1-s)} + N^{3(1-s)})
\]

\[
\lesssim N^{-2(2-s)}(N^{2(1-s)} + N^{(1-s)} + N^{3(1-s)})
\]

\[
\lesssim N^{s-3} + N^{-2} + N^{-s-1}
\]

\[
\lesssim N^{s-3}.
\]

Furthermore

\[
\|\partial_x h(t_0)\|_{H^1} \leq \int_0^{t_0} (\|\partial_x\tau(\partial_x)(v^2 + 2vu)\|_{H^1} + \frac{1}{8}\|\partial_x\psi(\partial_x)(3u^2v + 3uv^2 + v^3)\|_{H^1})
\]

\[
+ \frac{7}{48}\|\partial_x\psi(\partial_x)(2u_xv_x + v_x^2)\|_{H^1}\)dt'.
\]

(2.29)

Using the Propositions 2.2, 2.3 and 2.4 we obtain

\[
\|\partial_x h(t_0)\|_{H^1} \lesssim \int_0^{t_0} (\|v\|_{H^1} + \|v\|_{H^1}\|u\|_{H^1} + \|u\|_{H^1}\|v\|_{H^1} + \|v\|_{H^1}^2 + \|v\|_{H^1}^3) dt'.
\] (2.30)

Similarly, as in (2.28) one can prove

\[
\|\partial_x h(t_0)\|_{H^1} \lesssim N^{s-3}.
\] (2.31)
Combining (2.28) and (2.31), one gets
\[ \|h(t_0)\|_{H^2} \sim \|h(t_0)\|_{H^1} + \|\partial_x h(t_0)\|_{H^1} \lesssim N^{s-3}, \] (2.32)
which completes the proof of lemma. \( \square \)

Now we are in position to supply the proof of the first main result of this work.

**Proof of Theorem 1.1.** Let \( \eta_0 \in H^s \), \( 1 \leq s < 2 \) and \( T > 0 \) be any given number. As discussed above, we split the initial data \( \eta_0 = u_0 + v_0 \) so that \( u_0 \) and \( v_0 \) satisfy the growth conditions (2.18) and (2.19) respectively.

We evolve \( u_0 \) according to the Cauchy problem (2.20) and \( v_0 \) according to the Cauchy problem (2.21). Using Theorems A and 2.5 we respectively obtain solutions \( u \) and \( v \) so that the sum \( \eta = u + v \) solves the Cauchy problem (1.1) in the common time interval of existence of \( u \) and \( v \).

Observe that from (1.3) and (2.18), we have
\[ E(u(t)) = E(u_0) \sim \|u_0\|_{H^2}^2 \lesssim N^{2(2-s)} \] (2.33)
and the local existence time in \( H^2 \), given in Theorem A is estimated by
\[ T_u = \frac{c_s}{\|u_0\|_{H^2}(1 + \|u_0\|_{H^2})} \geq \frac{c_s}{N(2-s)(1 + N(2-s))} \geq \frac{c_s}{N^{2(2-s)}} =: t_0. \] (2.34)

We observe that \( (\|v_0\|_{H^s} + \|u_0\|_{H^s})(1 + \|v_0\|_{H^s} + \|u_0\|_{H^s}) \lesssim \|\eta_0\|_{H^s}(1 + \|\eta_0\|_{H^s}) = C_s \), therefore
\[ T_v = \frac{c_s}{(\|v_0\|_{H^s} + \|u_0\|_{H^s})(1 + \|v_0\|_{H^s} + \|u_0\|_{H^s})} \geq \frac{c_s}{C_s} \geq t_0. \] (2.35)

The inequalities (2.34) and (2.35) imply that the solutions \( u \) and \( v \) are both defined in the same time interval \([0, t_0]\).

The inequality (2.33) implies that
\[ t_0 \lesssim \frac{1}{E(u_0)}. \] (2.36)

In view of (2.22) the local solution \( v \in H^s \) is given by
\[ v(x, t) = S(t)v_0 + h(x, t). \] (2.37)

Therefore, in the time \( t_0 \sim N^{-2(2-s)} \), the solution \( \eta \) can be written as
\[ \eta(t) = u(t) + v(t) = u(t) + S(t)v_0 + h(t), \quad t \in [0, t_0]. \] (2.38)
At the time \( t = t_0 \), we have
\[
\eta(t_0) = u(t_0) + S(t_0)v_0 + h(t_0) =: u_1 + v_1, \tag{2.39}
\]
where
\[
u_1 = u(t_0) + h(t_0) \quad \text{and} \quad v_1 = S(t_0)v_0. \tag{2.40}
\]
In the time \( t_0 \) we consider the new initial data \( u_1, v_1 \) and evolve them according to the Cauchy problems (2.20) and (2.21) respectively, and continue iterating this process. In each iteration we consider the decomposition of the initial data as in (2.40). Therefore \( v_1, \ldots, v_k = S(kt_0)v_0 \) have the same \( H^s \)-norm of \( v_0 \) i.e. \( \|v_k\|_{H^s} = \|v_0\|_{H^s} \). We expect that \( u_1, \ldots, u_k \) also have the same properties of \( u_0 \), i.e., the same growth properties as that of \( u_0 \), in order to ensure the same existence time interval \( [0, t_0] \) in each iteration and glue them to cover the whole time interval \( [0, T] \), then extending the solution of the systems (2.20) and (2.21). This fact is proved by induction. Here we will prove only the case \( k = 1 \) and note that a similar argument works in the general case. In order to attain this goal we will use the energy conservation (1.3).

We have
\[
E(u_1) = E(u(t_0) + h(t_0)) = E(u(t_0)) + \left[ E(u(t_0) + h(t_0)) - E(u(t_0)) \right] =: E(u(t_0)) + X. \tag{2.41}
\]
Now,
\[
X = 2 \int_R u(t_0)h(t_0)dx + \int_R h(t_0)^2dx + 2\gamma_1 \int_R u_x(t_0)h_x(t_0)dx \\
+ \gamma_1 \int_R h_x(t_0)^2dx + 2\delta_1 \int_R u_{xx}(t_0)h_{xx}(t_0)dx + \delta_1 \int_R h_{xx}(t_0)^2dx \tag{2.42}
\]
\[
\leq 2\|u(t_0)\|_{L^2}\|h(t_0)\|_{L^2}^2 + \|h(t_0)\|_{L^2}^2 + \gamma_1(2\|u_x(t_0)\|_{L^2}\|h_x(t_0)\|_{L^2} + \|h_x(t_0)\|_{L^2}^2) \\
+ \delta_1(2\|u_{xx}(t_0)\|_{L^2}\|h_{xx}(t_0)\|_{L^2} + \|h_{xx}(t_0)\|_{L^2}^2).
\]
Using the Lemma 2.6, the estimate (2.42) yields
\[
X \lesssim N^{2-s}N^{s-3} + N^{2(s-3)} + (\gamma_1 + \delta_1)(N^{2-s}N^{s-3} + N^{2(s-3)}) \tag{2.43}
\]
\[
\lesssim N^{-1}.
\]
Combining (2.41), (2.42) and (2.43), we conclude that
\[
E(u_1) \leq E(u(t_0)) + cN^{-1}. \tag{2.44}
\]
The number of steps in the iteration to cover the given time interval \([0, T]\) is
\[
\frac{T}{t_0} \sim TN^{2(2-s)}.
\]
Thus, by (2.44), for this to happen we need that
\[
TN^{2(2-s)} N^{-1} \lesssim N^{2(2-s)},
\]
which is possible if \(1 \leq s < 2\) and \(N = N(T) = T\).

From the discussion above we see that in each iteration one has
\[
\|u_k\|_{L^2} \sim E(u_k) \lesssim N^{2(2-s)}, \text{ uniformly and } \|v_k\|_{L^2} = \|v_0\|_{L^2}.
\]

Finally, let \(t \in [0, T]\), then there exist \(k \geq 0\) an integer, such that \(t = kt_0 + \tau\), for some \(\tau \in [0, t_0]\). In the \(k\)th-iteration (see equality (2.38)), one gets
\[
\eta(t) = u(\tau) + S(\tau)v_k + h(\tau) = u(\tau) + S(\tau)S(kt_0)v_0 + h(\tau) = S(t)\eta_0 + u(\tau) - S(t)u_0 + h(\tau).
\]
Thus
\[
\eta(t) - S(t)\eta_0 = u(\tau) - S(t)u_0 + h(\tau),
\]
and this completes the proof of Theorem 1.1. \(\square\)

3. ILL-POSEDNESS RESULT

In this section, we will consider the ill-posedness issue for the Cauchy problem (1.1). We will prove that for the given data with Sobolev regularity less than 1, the flow-map cannot be \(C^2\). This negative result makes sense, because if one uses contraction mapping principle to prove local well-posedness, the flow-map turns out to be smooth. We start the following result which is the main ingredient to prove the Theorem 1.2.

**Proposition 3.1.** Let \(s < 1\) and \(T > 0\). Then there does not exist a space \(X^s_T\) continuously embedded in \(C([0, T]; H^s(\mathbb{R}))\) such that
\[
\|S(t)\eta_0\|_{X^s_T} \lesssim \|\eta_0\|_{H^s}, \quad (3.1)
\]
and
\[
\left\| \int_0^t S(t - t') \left( \tau(\partial_x)^2\eta^2 - \frac{7}{48} \psi(\partial_x)^2\eta^2 \right) dt' \right\|_{X^s_T} \lesssim \|\eta\|_{X^s_T}^3, \quad (3.2)
\]
hold true.
Proof. The proof follows a contradiction argument. If possible, suppose that there exists a space $X^s_t$ that is continuously embedded in $C([0, T]; H^s(\mathbb{R}))$ such that the estimates (3.1) and (3.2) hold true. If we consider $\eta = S(t)\eta_0$, then from (3.1) and (3.2), we get

$$\| \int_0^t S(t-t') \left( \tau(D_x) (S(t)\eta_0)^2 - \frac{7}{48} \psi(\partial_x) \partial_x (S(t)\eta_0)^2 \right) dt' \|_{H^s} \leq \| \eta_0 \|^2_{H^s},$$

(3.3)

The main idea to complete the proof is to find an appropriate initial data $\eta_0$ for which the estimate (3.3) fails to hold whenever $s < 1$.

Let $N \gg 1$, $\alpha = \alpha(N)$ to be chosen later such that $0 < \alpha \ll 1$, $I_N := [N, N + 2\alpha]$ and define an initial data via the Fourier transform

$$\hat{\eta}_0(\xi) := N^{-\alpha} \chi^{-\frac{1}{2}} \left[ \chi_{I_N}(\xi) + \chi_{\{I_N\}}(\xi) \right].$$

(3.4)

A simple calculation shows that $\| \eta_0 \|_{H^s} \sim 1$.

To simplify the notations, let us define

$$f(x, t) := \int_0^t S(t-t') \left( \tau(D_x) (S(t)\eta_0)^2 - \frac{7}{48} \psi(\partial_x) \partial_x (S(t)\eta_0)^2 \right) dt'$$

(3.5)

and calculate the $H^s$ norm of $f(x, t)$.

Taking the Fourier transform in the space variable $x$, we get

$$\widehat{f(t)}(\xi) = \int_0^t e^{-i(t-t')\phi(\xi)} \left( \tau(\xi) (S(t')\eta_0)^2(\xi) - \frac{7}{48} \psi(\xi) (S(t')\partial_x \eta_0)^2(\xi) \right) dt'$$

$$= \int_0^t e^{-i(t-t')\phi(\xi)} \left( \frac{3\xi - 4\gamma\xi^3}{4\varphi(\xi)} (S(t')\eta_0)^2(\xi) - \frac{7}{48} \varphi(\xi) (S(t')\partial_x \eta_0)^2(\xi) \right) dt'$$

$$= \int_0^t e^{-i(t-t')\phi(\xi)} \left( \frac{3\xi - 4\gamma\xi^3}{4\varphi(\xi)} \int_\mathbb{R} e^{-iu\phi(\xi)} \hat{\eta}_0(\xi - \xi_1) e^{-iu\phi(\xi_1)} \hat{\eta}_0(\xi_1) d\xi_1 \right)$$

$$- \frac{7}{48} \varphi(\xi) \int_\mathbb{R} e^{-iu\phi(\xi)} \hat{\eta}_0(\xi - \xi_1) \hat{\eta}_0(\xi - \xi_1) e^{-iu\phi(\xi_1)} \hat{\eta}_0(\xi_1) d\xi_1 \right) dt'$$

$$= \int_\mathbb{R} e^{-iu\phi(\xi)} \left( \frac{3\xi - 4\gamma\xi^3}{4\varphi(\xi)} - \frac{7}{48} \frac{\xi_1 (\xi_1 - \xi)}{\varphi(\xi)} \right) \hat{\eta}_0(\xi_1) \hat{\eta}_0(\xi - \xi_1) \int_0^t e^{iu[\phi(\xi) - \phi(\xi) - \phi(\xi)]} dt' d\xi_1.$$  

(3.6)

We have that

$$\int_0^t e^{iu[\phi(\xi) - \phi(\xi) - \phi(\xi)]} dt' d\xi_1 = \frac{e^{iu[\phi(\xi) - \phi(\xi) - \phi(\xi)]} - 1}{i[\phi(\xi) - \phi(\xi) - \phi(\xi)]}.$$  

(3.7)

Now, inserting (3.7) in (3.6), one obtains

$$\widehat{f(t)}(\xi) = -i \int_\mathbb{R} \frac{\xi}{4\varphi(\xi)} \left( 3 - 4\gamma\xi^2 - \frac{7}{12} \xi_1 (\xi - \xi_1) \right) \hat{\eta}_0(\xi - \xi_1) \hat{\eta}_0(\xi_1) e^{-iu\phi(\xi)} \frac{e^{iu\Theta(\xi, \xi_1)} - 1}{\Theta(\xi, \xi_1)} d\xi_1,$$

(3.8)
where $\Theta(\xi, \xi_1) := \phi(\xi) - \phi(\xi - \xi_1) - \phi(\xi_1)$.

Let us define a set $K$ by

$$K := \{\xi_1 : \xi - \xi_1 \in I_N, \xi_1 \in -I_N\} \cup \{\xi_1 : \xi_1 \in I_N, \xi - \xi_1 \in -I_N\}. \quad (3.9)$$

With this notation, one infers that

$$\widehat{\phi(t)}(\xi) = -iN^{-2s} \alpha^{-1} e^{-it\phi(\xi)} \int_K \frac{\xi}{4 \varphi(\xi)} \left(3 - 4\gamma \xi^2 - \frac{7}{12} \xi_1 (\xi - \xi_1)\right) e^{i\Theta(\xi, \xi_1)} \frac{1}{\Theta(\xi, \xi_1)} d\xi_1. \quad (3.10)$$

Therefore,

$$\|f\|^2_{H^s} \lesssim \int_{\alpha/2}^{\alpha} \langle \xi \rangle^{2s} N^{-4s} \left| \int_K \frac{\xi g(\xi, \xi_1)}{4 \varphi(\xi)} e^{i\Theta(\xi, \xi_1)} - 1 \right| d\xi_1 \right|^2 \, d\xi, \quad (3.11)$$

where $g(\xi, \xi_1) := 3 - 4\gamma \xi^2 - \frac{7}{12} \xi_1 (\xi - \xi_1)$.

We have that $|K| \gtrsim \alpha$. Now, we move to show that the magnitude of $\Theta(\xi, \xi_1)$ in the set $K$ is very very small.

First note that, the phase function $\phi(\xi)$ is odd and consider the parameters $\gamma_1, \gamma_2, \delta_1$ and $\delta_2$ all positive. With these considerations, one can write

$$\Theta(\xi, \xi_1) = \int_0^1 \frac{d}{dt} [\phi(t\xi) - \phi(t\xi - \xi_1)] dt \quad (3.12)$$

$$= \int_0^1 [\xi \phi'(t\xi) - \xi \phi'(t\xi - \xi_1)] dt. \quad (3.13)$$

Using triangle inequality, one easily obtains from (3.12) that

$$|\Theta(\xi, \xi_1)| \leq |\xi| \int_0^1 |\phi'(t\xi)| \, dt + |\xi| \int_0^1 |\phi'(t\xi - \xi_1)| \, dt. \quad (3.13)$$

Let

$$p(\xi) := \frac{1 - \gamma_2 \xi^2 + \gamma_1 \xi^4}{1 + \gamma_1 \xi^2 + \delta_1 \xi^4}, \quad \phi'(\xi) = p(\xi) + \xi p'(\xi). \quad (3.14)$$

so that, one has $\phi(\xi) = \xi p(\xi)$ and $\phi'(\xi) = p(\xi) + \xi p'(\xi)$. Observe that

$$p'(\xi) = \frac{-2(\gamma_1 + \gamma_2)\xi + 4(\delta_2 - \delta_1)\xi^3 + 2(\gamma_2 \delta_1 + \gamma_1 \delta_2)\xi^5}{(1 + \gamma_1 \xi^2 + \delta_1 \xi^4)^2}. \quad (3.15)$$

For any $x \in \mathbb{R}$, one can infer $|\phi'(x)| \leq c$. In the domain of integration in the RHS of (3.11), we have $|\xi| \leq \alpha/2$. Therefore, in the light of the definition of $p$ in (3.14) and the expression for $p'$ in (3.15), one can easily obtain from (3.13) that

$$|\Theta(\xi, \xi_1)| \leq C \alpha. \quad (3.16)$$

From the last inequality, we can conclude that, there is some $\epsilon > 0$ such that

$$|\Theta(\xi, \xi_1)| \leq C \alpha \sim CN^{-\epsilon}. \quad (3.17)$$
Hence, for some fixed \( t > 0 \), we can obtain
\[
\left| \frac{e^{it\Theta(\xi, \xi_1)} - 1}{\Theta(\xi, \xi_1)} \right| \geq Ct. \tag{3.18}
\]

Now, using mean value theorem for the integrals and the lower bound (3.18), one can infer that
\[
\left| \int_K e^{it\Theta(\xi, \xi_1)} - \frac{1}{\Theta(\xi, \xi_1)} \right| d\xi_1 \geq C|K|. \tag{3.19}
\]

In the set \( K \), we have that \( \frac{\xi g(\xi_1)}{4\varphi(\xi)} \sim \alpha N^2 \). Using this last information along with the lower bound (3.19) in (3.11), we obtain
\[
1 \sim \| f \|^2_{H^s} \gtrsim N^{-4}N^4|K|^2 \alpha t^2 \gtrsim \alpha^3 N^{4-4}t^2. \tag{3.20}
\]

If we choose \( \alpha = N^{-\epsilon} \) for sufficiently large \( N \), the estimate (3.20) fails to hold for \( s < 1 \). □

Now, we prove the ill-posedness result.

**Proof of Theorem 1.2.** For \( \eta_0 \in H^s(\mathbb{R}) \), consider the Cauchy problem
\[
\begin{cases}
\dot{\eta} = \phi(\partial_x)\eta + \tau(\partial_x)\eta^2 - \frac{1}{8} \psi(\partial_x)\eta^3 - \frac{7}{48} \psi(\partial_x)\eta_x^2, \\
\eta(x, 0) = \epsilon \eta_0(x),
\end{cases} \tag{3.21}
\]
where \( \phi(\partial_x), \psi(\partial_x) \) and \( \tau(\partial_x) \) as in (2.2), and \( \epsilon > 0 \) is a parameter. The solution \( \eta(x, t) := \eta^\epsilon(x, t) \) of the IVP (3.21) depends on the parameter \( \epsilon \). The equivalent integral equation can be written as
\[
\eta^\epsilon(x, t) = \epsilon S(t)\eta_0 - i \int_0^t S(t-t')\left( \tau(\partial_x)\eta^2 - \frac{1}{8} \psi(\partial_x)\eta^3 - \frac{7}{48} \psi(\partial_x)\eta_x^2 \right)(x, t')dt', \tag{3.22}
\]
where \( S(t) \) is the unitary group describing the solution of the linear part of the IVP (3.21).

We differentiates \( \eta^\epsilon(x, t) \) in (3.22) with respect \( \epsilon \) and evaluate at \( \epsilon = 0 \), to obtain
\[
\frac{\partial \eta^\epsilon(x, t)}{\partial \epsilon} \bigg|_{\epsilon=0} = S(t)\eta_0(x) =: \eta_1 \tag{3.23}
\]
and
\[
\frac{\partial^2 \eta^\epsilon(x, t)}{\partial \epsilon^2} \bigg|_{\epsilon=0} = -i \int_0^t S(t-t')\left[ 2\tau(\partial_x)\eta^2 - \frac{7}{24} \psi(\partial_x)(\partial_x\eta_1) \right]dt' =: \eta_2. \tag{3.24}
\]

If the flow-map is \( C^2 \) at the origin from \( H^s(\mathbb{R}) \) to \( C([-T, T]; H^s(\mathbb{R})) \), we must have
\[
\| \eta_2 \|_{L_T^\infty H^s(\mathbb{R})} \lesssim \| \eta_0 \|^2_{H^s(\mathbb{R})}. \tag{3.25}
\]

But from Proposition 3.1 we have seen that the estimate (3.25) fails to hold for \( s < 1 \) if we consider \( \eta_0 \) given by (3.3) and this completes the proof of the Theorem. □
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