Abstract

Current studies of supersymmetric extensions of the Sachdev–Ye–Kitaev model stimulate a renewed interest in super–Schwarzian derivatives. While the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases are well understood, there is some controversy regarding the definition of an $\mathcal{N} = 4$ super–Schwarzian. In this work, we apply the method of nonlinear realizations to the finite–dimensional superconformal group $SU(1,1|2)$ and link its invariants to the $\mathcal{N} = 4$ super–Schwarzian derivative introduced by Matsuda and Uematsu.

Keywords: the method of nonlinear realizations, superconformal algebra, super–Schwarzian derivative
1. Introduction

Current studies of supersymmetric extensions of the Sachdev–Ye–Kitaev model\(^1\) stimulate a renewed interest in super–Schwarzian derivatives [2]–[5]. An \(\mathcal{N}\)–extended super–Schwarzian acts upon a fermionic superfield which specifies superconformal diffeomorphisms of the odd sector of \(\mathcal{R}^{1|\mathcal{N}}\) superspace. It enjoys a remarkable composition law which implies invariance of the \(\mathcal{N} = 1, 2, 3, 4\) super–Schwarzian under finite transformations forming \(OSp(1|2)\), \(SU(1, 1|1)\), \(OSp(3|2)\), and \(SU(1, 1|2)\) superconformal group, respectively.\(^2\)

A conventional way of introducing a super–Schwarzian derivative is to compute a (finite) superconformal transformation of the super stress–energy tensor underlying a 2\(d\), \(\mathcal{N}\)–extended conformal field theory, in which it shows up as the anomalous term [2]–[5]. Alternatively, one can study the cocycles describing central extensions of infinite dimensional Lie superalgebras (see, e.g., [6]). Because for \(\mathcal{N} \geq 5\) the construction of the central term operator is problematic [7], the \(\mathcal{N} = 1, 2, 3, 4\) instances mentioned above seem to exhaust all available options.

In a recent work [8], a third alternative was studied, which consists in applying the method of nonlinear realizations [9] to finite–dimensional superconformal groups. In this setting, a super–Schwarzian derivative is linked to the supergroup invariants. Within the method of nonlinear realizations, one usually starts with a coset space element \(\tilde{g}\), on which a (super)group representative \(g\) acts by the left multiplication \(\tilde{g}' = g \cdot \tilde{g}\), and then constructs the Maurer–Cartan one–forms \(\tilde{g}^{-1}d\tilde{g}\), where \(d\) is the (super)differential, which automatically hold invariant under the transformation. These invariants can then be used to impose constraints allowing one to express some of (super)fields parametrizing the coset element \(\tilde{g}\) in terms of the other [10]. If the algebra at hand is such that all but one (super)fields can be linked to a single unconstrained (super)field, then the last remaining Maurer–Cartan invariant describes a derivative of the latter, which holds invariant under the action of the (super)group one started with. In particular, in [8] the \(\mathcal{N} = 1\) and \(\mathcal{N} = 2\) super–Schwarzian derivatives were obtained by applying the procedure to \(OSp(1|2)\) and \(SU(1, 1|1)\) superconformal groups, respectively.\(^3\)

The goal of this paper is to provide a similar derivation of the \(\mathcal{N} = 4\) super–Schwarzian associated with \(SU(1, 1|2)\) superconformal group. The latter seems indispensable for building an \(\mathcal{N} = 4\) supersymmetric extension of the Sachdev–Ye–Kitaev model.

In the literature, there is some controversy regarding the definition of an \(\mathcal{N} = 4\) super–Schwarzian. A variant in [5] carries an external vector index and is linked to a superconformal field theory which involves non–primary fields. Mathematicians report an obstruction in constructing nontrivial cocycles for \(\mathcal{N} > 3\) (see, e.g., the discussion in [6]). A proposal in [12] seems to lack the invariance under finite \(SU(1, 1|2)\) transformations. One of our objectives in this work is to clarify the existing discrepancies. Our analysis below undoubtedly supports

\[^1\]There is an abundant literature on the subject. For a good recent account and further references see [1].

\[^2\]For the \(\mathcal{N} = 4\) case, one can define a derivative invariant under either \(SU(1, 1|2)\) or \(OSp(4|2)\) superconformal groups, these turn out to be particular instances of the most general superconformal group in one dimension \(D(2, 1; \alpha)\).

\[^3\]A non–supersymmetric case was previously studied in [11].
the result in [5].

The work is organized as follows.

In the next section, superconformal diffeomorphisms of $\mathcal{R}^{1|4}$ superspace are considered and conditions which follow from the requirement that the covariant derivatives transform homogeneously are analysed. For the fermionic superfield, which describes superconformal diffeomorphisms of the odd sector of $\mathcal{R}^{1|4}$, one reveals the chirality condition and an extra quadratic constraint [4]. The restrictions are solved explicitly and it is demonstrated that the latter is actually equivalent to a simpler linear equation (see Eq. (19) below).

In Sect. 3, the method of nonlinear realizations is applied to the finite-dimensional superconformal group $SU(1, 1|2)$ with the aim to link its invariants to an $\mathcal{N} = 4$ super–Schwarzian. First, each generator in the corresponding superalgebra is accompanied by a Goldstone superfield of the same Grassmann parity, which all together give rise to a group–theoretic element $\tilde{g}$. Then the Maurer–Cartan invariants $\tilde{g}^{-1}D^\alpha\tilde{g}$, where $D^\alpha$, $\alpha = 1, 2$, is the covariant derivative, are computed. After that, constraints are imposed, which enable one to link all the Goldstone superfields entering $\tilde{g}$ to a single fermionic superfield (a companion of the supersymmetry generator). Substituting the resulting relations back into the Maurer–Cartan invariants $\tilde{g}^{-1}D^\alpha\tilde{g}$, one unambiguously reproduces the $\mathcal{N} = 4$ super–Schwarzian derivative introduced in [5]. Finally, properties of the super–Schwarzian, including the finite $SU(1, 1|2)$ transformations which leave it invariant, are discussed.

In the concluding Sect. 4 we summarise our results and discuss possible further developments.

Our spinor conventions are gathered in Appendix.

Throughout the text summation over repeated indices is understood.

2. Superconformal diffeomorphisms of $\mathcal{R}^{1|4}$

Consider $\mathcal{R}^{1|4}$ superspace parametrized by a real bosonic coordinate $t$ and a pair of Hermitian conjugate anti–commuting $SU(2)$–spinors $(\theta_\alpha, \bar{\theta}^\alpha)$, $(\theta_\alpha)\dagger = \bar{\theta}^\alpha, \alpha = 1, 2$ (see Appendix for our spinor conventions). The $d = 1, \mathcal{N} = 4$ supersymmetry algebra

$$\{q_\alpha, q^\beta\} = 2\hbar\delta_\alpha^\beta$$

allows one to represent $\mathcal{R}^{1|4}$ as the supergroup manifold

$$\tilde{g} = e^{i\theta_\alpha q_\alpha},$$

while the left action of the supergroup on the superspace, $\tilde{g}' = e^{iah}e^{i\epsilon_\alpha q_\alpha + \bar{\epsilon}_\alpha \bar{q}^\alpha}\cdot \tilde{g}$, where $a$ and $(\epsilon^\alpha, \bar{\epsilon}_\alpha)$ are even and odd supernumbers, respectively, generates the $d = 1, \mathcal{N} = 4$ supersymmetry transformations

$$t' = t + a; \quad \theta'_\alpha = \theta_\alpha + \epsilon_\alpha, \quad \bar{\theta}'^\alpha = \bar{\theta}^\alpha + \bar{\epsilon}_\alpha,$$

$$t' = t - i \left(\epsilon_\alpha \bar{\theta}^\alpha + \bar{\epsilon}_\alpha \theta_\alpha\right).$$

Covariant derivatives, which anticommute with the supersymmetry generators, read

$$D^\alpha = \partial^\alpha + i\bar{\theta}^\alpha \partial_\gamma,$$

$$\bar{D}_\alpha = \bar{\partial}_\alpha + i\theta_\alpha \partial_\gamma,$$
where $\partial_t = \frac{\partial}{\partial t}$, $\partial^\alpha = \frac{\partial}{\partial x^\alpha}$, $\bar{\partial}_\alpha = \frac{\partial}{\partial \bar{x}_\alpha}$. They satisfy the relations

$$ \{D^\alpha, \bar{D}_\beta\} = 2i\delta^\alpha_\beta \partial_t, \quad D^\alpha D^\beta = -\frac{1}{2} \epsilon^{\alpha\beta} D^2, \quad \bar{D}_\alpha \bar{D}_\beta = -\frac{1}{2} \epsilon_{\alpha\beta} \bar{D}^2, $$

$$[D^2, \bar{D}_\alpha] = -4iD_\alpha \partial_t, \quad [D^2, D^\alpha] = -4iD^\alpha \partial_t, \quad [\bar{D}^2, D^\alpha] = -4i (D^\alpha \bar{D}_\alpha - \bar{D}_\alpha D^\alpha) \partial_t, $$

with $D^2 = D^\alpha D_\alpha$, $\bar{D}^2 = \bar{D}_\alpha \bar{D}^\alpha$.

In what follows, we will need a component decomposition of a chiral fermionic superfield $\psi_\beta$, which obeys the equation

$$ \bar{D}_\beta \psi_\gamma = 0. $$

Taking into account the identity $\bar{D}_\beta = e^{-i\theta \bar{\partial}_\beta} \partial_\beta e^{i\theta \partial_\beta}$, one gets

$$ \psi_\gamma(t, \theta, \bar{\theta}) = \alpha_\gamma(t) + \theta_\beta b_\gamma^\beta(t) - i\bar{\theta} \bar{\partial}_\gamma \bar{\alpha}(t) + \theta^2 \beta_\gamma(t) + \frac{i}{2} \bar{\bar{\theta}} \theta^2 b_\gamma^\beta(t) - \frac{1}{4} \theta^2 \bar{\bar{\theta}}^2 \bar{\alpha}_\gamma(t), $$

where $\alpha_\gamma(t)$, $\beta_\gamma(t)$ are complex fermionic components and $b_\gamma^\beta(t)$ is a complex bosonic matrix–valued function of $t$. The Hermitian conjugation rules

$$ (D^\alpha \rho)^\dagger = -\bar{D}_\alpha \rho, \quad (D^\alpha \psi_\beta)^\dagger = \bar{D}_\alpha \bar{\psi}_\beta, \quad (D^\alpha \bar{\psi}_\beta)^\dagger = -\bar{D}_\alpha \psi_\beta, $$

which involve a real bosonic superfield $\rho$, a complex fermionic superfield $\psi_\alpha$ and its Hermitian conjugate $\bar{\psi}_\alpha = (\psi_\alpha)^\dagger$, will be heavily used as well.

Similarly to the $\mathcal{N} = 1$ and $\mathcal{N} = 2$ cases [2, 3] (see also [13]), superconformal diffeomorphisms of $\mathcal{R}^{1|4}$ are introduced as the transformations

$$ t' = \rho(t, \theta, \bar{\theta}), \quad \theta_\alpha = \psi_\alpha(t, \theta, \bar{\theta}), \quad \bar{\theta}_\alpha = \bar{\psi}_\alpha(t, \theta, \bar{\theta}), $$

where $\rho$ is a real bosonic superfield and $\psi_\alpha$ is a complex fermionic superfield, under which the covariant derivatives transform homogeneously

$$ D^\alpha = (D^\alpha \psi_\beta) D^\beta, \quad \bar{D}_\alpha = (\bar{D}_\alpha \bar{\psi}_\beta) \bar{D}^\beta. $$

Eq. (10) yields the constraints

$$ D_\alpha \psi_\beta = 0, \quad D^\alpha \rho - i (D^\alpha \psi_\beta) \bar{\psi}_\beta = 0, \quad D^\alpha \bar{\psi}_\beta = 0, \quad \bar{D}_\alpha \rho - i (\bar{D}_\alpha \bar{\psi}_\beta) \psi_\beta = 0, $$

which also imply

$$ \partial_t \rho = -i \bar{\psi}_\alpha \partial_t \psi_\alpha + i \partial_t \bar{\psi}_\alpha \psi_\alpha + \frac{1}{2} (D \psi \bar{D} \psi), \quad (\psi_\alpha \bar{D}_\alpha \bar{\psi}_\beta = (D \psi \bar{D} \psi). $$

with $D^\alpha \psi_\beta \bar{D}_\alpha \bar{\psi}_\beta = (D \psi \bar{D} \psi)$. Thus, $\rho$ is fixed provided $\psi_\alpha$ is known.

The compatibility of (10) with the properties of the covariant derivatives (5) imposes further restrictions on $\psi_\alpha$. From $\{D^\alpha, \bar{D}_\beta\} = 2i\delta^\alpha_\beta \partial_t$ and the identity

$$ \partial_t = \partial_t \psi_\alpha D^\alpha + \partial_t \bar{\psi}_\alpha \bar{D}_\alpha + \frac{1}{2} (D \psi \bar{D} \psi) \partial_t, $$

(13)
one gets the quadratic constraint [4]

\[ \mathcal{D}^{\alpha} \psi_\lambda \bar{\mathcal{D}}^{\beta} \bar{\psi}^\lambda = \frac{1}{2} \delta^\alpha_\beta \left( \mathcal{D} \psi \bar{\mathcal{D}} \bar{\psi} \right). \]  

(14)

Thus, up to a factor, \( \mathcal{D}^{\alpha} \psi_\lambda \) is a unitary matrix, which also implies

\[ \mathcal{D}^{\lambda} \psi_\alpha \bar{\mathcal{D}}^{\lambda} \bar{\psi}^\beta = \frac{1}{2} \delta^\alpha_\beta \left( \mathcal{D} \psi \bar{\mathcal{D}} \bar{\psi} \right) \Rightarrow \mathcal{D}^{\alpha} \psi_\beta = -\frac{1}{2} \mathcal{D}^{\alpha} \bar{\psi}_\beta \left( \frac{\mathcal{D} \psi \bar{\mathcal{D}} \bar{\psi}}{\det(\mathcal{D} \psi)} \right), \]  

(15)

with \( \det(\bar{\mathcal{D}} \bar{\psi}) = -\frac{1}{2} \epsilon^\alpha_\beta \epsilon_{\gamma \delta} \bar{\mathcal{D}}^{\alpha} \bar{\psi}^\gamma \bar{\mathcal{D}}^{\beta} \bar{\psi}^\delta \). Computing the covariant derivatives of (14), one gets a chain of relations two of which

\[ \mathcal{D}^{\alpha} \left( \mathcal{D} \psi \bar{\mathcal{D}} \bar{\psi} \right) = 4i \partial_t \bar{\psi}_\beta \mathcal{D}^{\alpha} \psi_\beta, \quad \bar{\mathcal{D}}_\beta \left( \mathcal{D} \psi \bar{\mathcal{D}} \bar{\psi} \right) = 4i \partial_t \psi_\beta \bar{\mathcal{D}}^{\alpha} \bar{\psi}_\beta, \]  

(16)

will be important for what follows. Because the second line in (5) results from \( \{ \mathcal{D}^\alpha, \bar{\mathcal{D}}_\beta \} = 2i \delta^\beta_\alpha \partial_t \), it does not produce further restrictions on \( \psi_\alpha \).

Using the covariant projection method, in which components of a superfield are linked to its covariant derivatives evaluated at \( \theta_\alpha = 0, \bar{\theta}^\alpha = 0 \), one can verify that Eq. (14) relates the fermionic components of the chiral superfield (7) to each other and reduces the matrix–valued bosonic function \( b^\beta_\alpha(t) \) to a single unknown scalar

\[ \beta_\gamma(t) = i \dot{\alpha}_\gamma(t) e^{2iv}, \quad b^\beta_\alpha(t) = u(t) e^{iv} \left( \exp \left[ \frac{i}{2} \xi_c \sigma^c \right] \right)^\beta_\alpha. \]  

(17)

Here \( u(t) \) is an arbitrary real function of \( t \), \( v \) and \( \xi_a \), \( a = 1, 2, 3 \), are real bosonic constants, and \( \sigma_a \) are the Pauli matrices (see Appendix).

Substituting (17) into the covariant derivatives of \( \psi_\alpha \) and \( \bar{\psi}^\alpha \), one reveals the identity

\[ e^{-iv} \mathcal{D}^{\alpha} \psi_\beta + e^{iv} \bar{\mathcal{D}}^{\alpha} \bar{\psi}_\beta = 0. \]  

(18)

Because one can always redefine the fermionic superfield \( e^{-iv} \psi_\alpha \to \psi_\alpha \), in what follows we set the parameter \( v \) in (17) to vanish, thus reducing the quadratic constraint (14) to the linear restriction

\[ \mathcal{D}^{\alpha} \psi_\beta + \bar{\mathcal{D}}^{\alpha} \bar{\psi}_\beta = 0. \]  

(19)

The latter also implies \( \mathcal{D}^2 \psi_\alpha = -4i \partial_t \bar{\psi}_\alpha \). The fact that the quadratic equation (14) is equivalent to (19) seems to have escaped attention thus far.

The Taylor series expansion of \( u(t) \) and \( \alpha_\gamma(t) \) involves an infinite number of constant parameters, which represent the infinite–dimensional \( \mathcal{N} = 4 \) superconformal group, \( SU(1, 1|2) \) being its finite–dimensional subgroup.
3. $\mathcal{N} = 4$ super–Schwarzian via nonlinear realizations

As was mentioned in the Introduction, the primary goal of this work is to link an $\mathcal{N} = 4$ super–Schwarzian derivative to invariants of $SU(1,1|2)$ superconformal group. To this end, let us consider the structure relations of the superconformal algebra $su(1,1|2)$

$$[P, D] = iP, \quad [P, K] = 2iD,$$

$$[D, K] = iK, \quad [J_a, J_b] = i\epsilon_{abc}J_c,$$

$$\{Q_\alpha, \bar{Q}^\beta\} = 2P\delta^\beta_\alpha, \quad \{Q_\alpha, \bar{S}^\beta\} = 2i(\sigma_\alpha)^{\beta}_\beta J_a - 2D\delta^\beta_\alpha,$$

$$\{S_\alpha, \bar{S}^\beta\} = 2K\delta^\beta_\alpha, \quad \{\bar{Q}^\alpha, S_\beta\} = -2i(\sigma_\alpha)^{\alpha}_\beta J_a - 2D\delta^\alpha_\beta,$$

$$[D, Q_\alpha] = -\frac{i}{2}Q_\alpha, \quad [D, S_\alpha] = \frac{i}{2}S_\alpha,$$

$$[K, Q_\alpha] = iS_\alpha, \quad [P, S_\alpha] = -iQ_\alpha,$$

$$[J_a, Q_\alpha] = -\frac{1}{2}(\sigma_\alpha)^{\beta}_\beta Q_\beta, \quad [J_a, S_\alpha] = -\frac{1}{2}(\sigma_\alpha)^{\beta}_\beta S_\beta,$$

$$[D, \bar{Q}^\alpha] = -\frac{i}{2}\bar{Q}^\alpha, \quad [D, \bar{S}^\alpha] = \frac{i}{2}\bar{S}^\alpha,$$

$$[K, \bar{Q}^\alpha] = i\bar{S}^\alpha, \quad [P, \bar{S}^\alpha] = -i\bar{Q}^\alpha,$$

$$[J_a, \bar{Q}^\alpha] = \frac{1}{2}\bar{Q}^\beta(\sigma_\alpha)^{\beta}_\beta, \quad [J_a, \bar{S}^\alpha] = \frac{1}{2}\bar{S}^\beta(\sigma_\alpha)^{\alpha}_\beta. \quad (20)$$

Here $(P, D, K, J_a), a = 1, 2, 3,$ are (Hermitian) bosonic generators of translations, dilatations, special conformal transformations, and $su(2)$ rotations, respectively. $Q_\alpha$ and $S_\alpha$ are fermionic generators of supersymmetry transformations and superconformal boosts, $\bar{Q}^\alpha$ and $\bar{S}^\alpha$ being their Hermitian conjugates. $(\sigma_\alpha)^{\alpha}_\beta$ are the Pauli matrices (see Appendix).

Following the recipe in [9], each generator in the superalgebra is then accompanied by a Goldstone superfield of the same Grassmann parity and the group–theoretic element is introduced

$$\tilde{g} = e^{i\theta_\alpha q_\alpha + \bar{\theta}_\alpha \bar{q}^\alpha} e^{i\rho P} e^{i\phi_\alpha Q_\alpha + \bar{\phi}_\alpha \bar{Q}^\alpha} e^{i\delta_\alpha S_\alpha + \bar{\delta}_\alpha \bar{S}^\alpha} e^{i\mu K} e^{i\nu D} e^{i\lambda_a J_a}, \quad (21)$$

in which $(\rho, \mu, \nu, \lambda_a)$ are real bosonic superfields and $(\psi_\alpha, \phi_\alpha, \bar{\psi}^\alpha, \bar{\phi}^\alpha)$ are complex fermionic superfields. Here $\rho$ and $\psi_\alpha$ are identified with those in the preceding section and the constraints (11), (19) are assumed to hold. The choice of $\tilde{g}$ is prompted by the study of the $d = 1, \mathcal{N} = 4$ superconformal mechanics in [14].

The left multiplication by a group element

$$\tilde{g} \rightarrow g \cdot \tilde{g}, \quad g = e^{iaP} e^{i\alpha Q_\alpha + \bar{\kappa}_\alpha \bar{Q}^\alpha} e^{i\kappa_\alpha S_\alpha + \bar{\kappa}_\alpha \bar{S}^\alpha} e^{ibK} e^{ibD} e^{i\epsilon_a J_a}, \quad (22)$$
where \((a, b, c, \xi_\alpha)\) and \((\epsilon_\alpha, \kappa_\alpha)\) are bosonic and fermionic parameters, respectively, generates a finite \(SU(1,1|2)\) transformation. In most of the practical applications, it proves sufficient to focus on the infinitesimal transformations

\[
\rho' = \rho + a, \quad \psi'_\alpha = \psi_\alpha, \quad \bar{\psi}'_\alpha = \bar{\psi}_\alpha;
\]
\[
\rho' = \rho + b \rho, \quad \psi'_\alpha = \psi_\alpha + \frac{1}{2} b \psi_\alpha, \quad \bar{\psi}'_\alpha = \bar{\psi}_\alpha + \frac{1}{2} b \bar{\psi}_\alpha;
\]
\[
\rho' = \rho + c \rho^2 - \frac{1}{2} c \psi^2 \bar{\psi}^2, \quad \psi'_\alpha = \psi_\alpha + c \rho \psi_\alpha + \frac{i}{2} c \psi^2 \bar{\psi}_\alpha, \quad \bar{\psi}'_\alpha = \bar{\psi}_\alpha + c \rho \bar{\psi}_\alpha + \frac{i}{2} c \bar{\psi}^2 \psi_\alpha;
\]
\[
\rho' = \rho, \quad \psi'_\alpha = \psi_\alpha + i \frac{2}{\xi_\alpha (\sigma_\alpha)^\beta \psi_\beta}, \quad \bar{\psi}'_\alpha = \bar{\psi}_\alpha - i \frac{2}{\xi_\alpha \bar{\psi}^\beta (\sigma_\alpha)^\beta};
\]
\[
\rho' = \rho + i \left( \bar{\psi} \epsilon - \bar{\epsilon} \psi \right), \quad \psi'_\alpha = \psi_\alpha + \epsilon_\alpha, \quad \bar{\psi}'_\alpha = \bar{\psi}_\alpha + \epsilon^\alpha;
\]
\[
\rho' = \rho - i \rho \left( \bar{\psi} \kappa - \bar{\kappa} \psi \right), \quad \psi'_\alpha = \psi_\alpha - \rho \kappa_\alpha + i \bar{\psi} \psi \kappa_\alpha, \quad \bar{\psi}'_\alpha = \bar{\psi}_\alpha - \rho \bar{\kappa}_\alpha - i \psi \bar{\psi} \kappa_\alpha, \quad -i \psi^2 \bar{\kappa}_\alpha, \quad -i \bar{\psi}^2 \kappa_\alpha,
\]

which are obtained with the aid of the Baker–Campbell–Hausdorff formula

\[
e^{iA} T e^{-iA} = T + \sum_{n=1}^{\infty} \frac{i^n}{n!} [A, [A, \ldots [A, T] \ldots]], \tag{24}
\]

Note that both the original and transformed superfields depend on the same arguments \((t, \theta, \bar{\theta})\) such that the transformations affect the form of the superfields only, e.g. \(\delta \rho = \rho'(t, \theta, \bar{\theta}) - \rho(t, \theta, \bar{\theta})\). Computing the algebra of the infinitesimal transformations (23), one can verify that it does reproduce the structure relations (20).

As the next step, one builds the odd analogues of the Maurer–Cartan one–forms

\[
\tilde{g}^{-1} D^\alpha \bar{g} = i \omega_D^\alpha D + i \omega_K^\alpha K + \omega_Q^{\alpha \beta} Q_\beta + \omega_S^{\alpha \beta} S_\beta + \omega_S^{\alpha \beta} S_\beta + \omega_J^{\alpha a} J_a - q^a, \tag{25}
\]

where \(D^\alpha\) is the covariant derivative (4), which give rise to the \(SU(1,1|2)\) invariants\(^4\)

\[
\omega_D^\alpha = D^\alpha \nu + 2 i \bar{\phi} D^\alpha \psi_\beta, \\
\omega_K^\alpha = e^\nu \left( D^\alpha \mu - 2 i \mu \bar{\phi} D^\alpha \psi_\beta - i \bar{\phi} D^\alpha \phi_\beta - \bar{\phi}^2 \phi^\beta D^\alpha \psi_\beta - i \phi \bar{\phi} D^\alpha \bar{\phi}^\beta \right), \\
\omega_Q^{\alpha \gamma} = e^{-\frac{i}{2} D^\alpha \psi_\beta} \left( \exp \left[ \frac{i}{2} \lambda_c \sigma_c \right] \right)^\gamma, \\
\omega_S^{\alpha \gamma} = e^{\frac{i}{2} D^\alpha \phi_\beta} \left( \exp \left[ \frac{i}{2} \lambda_c \sigma_c \right] \right)^\gamma, \\
\omega_S^{\alpha \gamma} = e^{\frac{\nu}{2} \left( D^\alpha \phi_\beta + \mu D^\alpha \psi_\beta \right)} \left( \exp \left[ \frac{i}{2} \lambda_c \sigma_c \right] \right)^\gamma, \\
\omega_S^{\alpha \gamma} = e^{\frac{\nu}{2} \left( D^\alpha \phi_\beta + i \bar{\phi}^2 D^\alpha \psi_\beta \right)} \left( \exp \left[ \frac{-i}{2} \lambda_c \sigma_c \right] \right)^\gamma,
\]

\(^4\)The invariant \(\tilde{g}^{-1} D_a \bar{g}\) could be considered likewise, which would result in the Hermitian conjugates of (26).
\[ \omega_{\alpha}^{\alpha} \mathcal{J}_a = e^{i\lambda_c \mathcal{J}_c} \left( D^\alpha e^{-i\lambda_i \mathcal{J}_k} \right) + 2i \left( D^\alpha \psi^\beta (\sigma^\alpha)_{\beta} \phi^{\alpha} \right) \mathcal{J}_a. \] (26)

Because the \( \mathcal{N} = 4 \) super-Schwarzian derivative is expected to involve the fermionic superfield \( \psi_\alpha \) only, one is led to use the invariants (26), so as to eliminate \( (\nu, \mu, \lambda_\alpha, \phi_\alpha, \bar{\phi}^\alpha) \) from the consideration. Guided by a recent analysis of the \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) super-Schwarzian derivatives in a similar setting [8], let us impose the following constraints

\[ \omega^0_D = 0, \quad \omega^{\alpha}_D = 0, \quad \omega^{\alpha\gamma}_Q = \delta^{\alpha\gamma}, \quad \omega^{\alpha\gamma}_S = 0, \] (27)

where \( r^{\alpha\gamma} \) is a constant matrix with even supernumber elements (coupling constants).\(^5\)

Making use of the Hermitian conjugates, one can express \( \nu \) and \( \mu \) in terms of the fermionic superfields \( \psi_\alpha \) and \( \phi_\alpha \)

\[ e^{\nu} = \frac{(D\psi \bar{D}\psi)}{r \bar{r}}, \quad \mu (D\psi \bar{D}\psi) + (D\phi \bar{D}\psi) - i \bar{\phi} \bar{\phi} (D\psi \bar{D}\psi) = 0, \] (28)

where \( r \bar{r} = r^{\alpha\beta} \bar{r}_{\beta\alpha} \). Note that, computing the covariant derivative of the leftmost equation in (28) and taking into account \( \omega^0_D = 0 \), one gets

\[ D^\alpha (D\psi \bar{D}\psi) = 4i \partial_t \bar{\psi}^\beta D^\alpha \psi_\beta, \quad \bar{D}_\alpha (D\psi \bar{D}\psi) = 4i \partial_t \psi_\beta \bar{D}_\alpha \bar{\psi}^\beta. \] (29)

These equations are in agreement with (16) above.

In order to link \( (\phi_\alpha, \bar{\phi}^\alpha) \) to \( (\psi_\alpha, \bar{\psi}^\alpha) \), it suffices to contract \( \omega^0_D = 0 \) with \( \bar{D}_\alpha \bar{\psi}^\gamma \) and to make use of the quadratic constraints (15). The result reads

\[ \bar{\phi}_\alpha = -\frac{2\partial_t \psi_\alpha}{(D\psi \bar{D}\psi)}, \quad \bar{\phi}^\alpha = -\frac{2\partial_t \bar{\psi}^\alpha}{(D\psi \bar{D}\psi)}, \] (30)

which also simplifies the expression for \( \mu \)

\[ \mu = 2 \frac{(D^\alpha \partial_t \psi_\beta) (\bar{D}_\alpha \bar{\psi}^\beta)}{(D\psi \bar{D}\psi)^2}. \] (31)

Finally, substituting (28), (30), (31) back into the invariants (26), one reveals that \( \omega^0_D \) and \( \omega^{\alpha\gamma}_S \) vanish identically, \( \omega^{\alpha\gamma}_Q \) links \( \lambda_\alpha \) to \( \psi_\alpha \)

\[ \left( \exp \left[ i \frac{1}{2} \lambda_\alpha \sigma_a \right] \right)^\beta_\alpha = -\frac{2e^{\gamma} \tau^{\gamma\beta} \bar{D}_\gamma \bar{\psi}_\alpha}{D\psi \bar{D}\psi}, \] (32)

which proves to be consistent with \( \omega^{\alpha\beta}_\gamma \mathcal{J}_a = 0 \) evaluated at \( \mathcal{J}_a = \frac{1}{2} \sigma_a \). The left hand side of the equation \( \omega^{\alpha\gamma}_S = 0 \) gives rise to the second–rank tensor

\[ \mathcal{I}_\alpha^\beta = D_\alpha \bar{\psi}^\gamma D^\beta \left( \frac{\partial_t \psi_\gamma}{D\psi \bar{D}\psi} \right) - \frac{1}{2} \delta^\beta_\alpha \left( D_\mu \bar{\psi}^\gamma \bar{D}^\mu \left( \frac{\partial_t \psi_\nu}{D\psi \bar{D}\psi} \right) \right), \] (33)

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\(^5\)The consistency requires \( r^{\alpha\beta} \) and its Hermitian conjugate \( (r^{\alpha\beta})^\dagger = \bar{r}_{\beta\alpha} \) to obey \( \bar{r}_{\alpha\beta} r^{\beta\gamma} = \frac{1}{2} \delta^\gamma_\alpha (r \bar{r}), \) \( r^{\alpha\beta} \bar{r}_{\gamma\alpha} = \frac{1}{2} \delta^\alpha_\gamma (r \bar{r}) \) (see Eq. (32) below).
or, lowering the upper index,
\[ I_{\alpha\beta} = \frac{1}{8i} (D_\alpha \bar{D}_\beta + D_\beta \bar{D}_\alpha) \ln (D\psi \bar{D}\bar{\psi}), \]  
(34)

whereas \( \omega^\alpha_K \) reduces to \( D^\beta I_\alpha^\beta \). This proves to be the only \( SU(1,1|2) \) invariant combination involving solely \( \psi_\alpha \). Because \( I_{\alpha\beta} \) is traceless, its contains three independent components which can be extracted by contracting (33) with the (traceless) Pauli matrices
\[ D^\alpha(\sigma_a)_\alpha^\beta \bar{D}_\beta \ln (D\psi \bar{D}\bar{\psi}) := I_a[\psi(t,\theta,\bar{\theta}); t,\theta,\bar{\theta}], \]  
(35)

Eq. (35) coincides precisely with the \( N = 4 \) super–Schwarzian derivative introduced in [5]. Note that it transforms as a vector under \( SU(2) \) transformations acting in \( R^{1|4} \) superspace.

A few comments are in order. Firstly, focusing on the infinitesimal \( SU(1,1|2) \) transformations (23) and taking into account the constraints (11) and the equalities
\[ D\psi \bar{D}\bar{\psi}' = (1 + 2c\rho) D\psi \bar{D}\bar{\psi}, \quad D\psi' \bar{D}\bar{\psi}' = (1 + 2i(\bar{\kappa}\psi - \bar{\psi}\kappa)) D\psi \bar{D}\bar{\psi}, \\\nD\psi' \bar{D}\bar{\psi}' = (1 + b) D\psi \bar{D}\bar{\psi}, \]  
(36)

one can readily verify that (35) does hold invariant.

Secondly, considering the superconformal diffeomorphism described by Eqs. (9), (11), and (14) above, changing the argument \( \psi_\alpha(t,\theta,\bar{\theta}) \to \Omega_\alpha(\rho,\psi,\bar{\psi}) \) of the super–Schwarzian (35), and taking into account Eq. (10), which gives rise to the identity
\[ (D\Omega \bar{D}\Omega) = \frac{1}{2} (D\psi \bar{D}\bar{\psi}) (D'\Omega \bar{D}'\Omega), \]  
(37)

one finds the composition law [5]
\[ I_a[\Omega(t',\theta',\bar{\theta}'); t,\theta,\bar{\theta}] = I_a[\psi(t,\theta,\bar{\theta}); t,\theta,\bar{\theta}] + \frac{1}{2} M_{ab} I_b[\Omega(t',\theta',\bar{\theta}'); t',\theta',\bar{\theta}'], \]  
(38)

with \( M_{ab} = D^\alpha(\sigma_a)_\alpha^\beta D^\beta(\sigma_b)_\beta^\gamma \bar{D}_\gamma \). In deriving Eq. (38), the properties of the Pauli matrices exposed in Appendix have been used. In particular, the \( N = 4 \) super–Schwarzian holds invariant under the change of the argument \( \psi_\alpha(t,\theta,\bar{\theta}) \to \Omega_\alpha(\rho,\psi,\bar{\psi}) \), provided the last term in (38) vanishes.

Thirdly, assuming the constraints (11) and (19) to hold, which result in the restrictions\(^6\) (17) upon the components of the chiral superfield (7), and analysing the equation
\[ I_a[\psi(t,\theta,\bar{\theta}); t,\theta,\bar{\theta}] = 0, \]  
or equivalently \( I_{\alpha\beta} = 0 \), one can fix \( u(t) \) and \( \alpha_\gamma(t) \)
\[ u(t) = \frac{1}{ct + d}, \quad \alpha_\gamma(t) = \epsilon_\gamma + \frac{(\bar{\kappa}\kappa)\kappa_\gamma}{ct + d}, \]  
(39)

where \( (c,d) \) and \( (\epsilon_\gamma,\kappa_\gamma) \) are bosonic and fermionic parameters, respectively. The resulting superfield (7) determines a finite \( SU(1,1|2) \) transformation acting in the odd sector of \( R^{1|4} \).

\(^6\)Recall that \( v \) in (17) was set to zero.
superspace. In particular, it correctly reduces to Eqs. (23) in the infinitesimal limit.\textsuperscript{7} A finite $SU(1,1|2)$ transformation acting in the even sector of $\mathcal{R}^{1|4}$ can be found by integrating Eq. (12).

Finally, let us make a comment on the proposal in [12]. According to Ref. [12], an $\mathcal{N} = 4$ super–Schwarzian derivative reads

$$S[\psi(t, \theta, \bar{\theta}) ; t, \theta, \bar{\theta}] = \ln (\det [\mathcal{D}^\alpha \psi_\beta]) = \ln \left( \frac{1}{2} \mathcal{D} \psi \bar{\mathcal{D}} \bar{\psi} \right),$$

(40)

which coincides with the twisted variant of the $OSp(4|2)$ super–Schwarzian introduced in [4]. As compared to (35), the operator $\mathcal{D}^\alpha (\sigma_a)_\alpha^\beta \bar{\mathcal{D}}_\beta$ is missing. That (40) lacks some of the characteristic properties of a super–Schwarzian derivative is seen by considering a finite dilatation transformation with a real parameter $b$

$$\psi'_\alpha = b \psi_\alpha,$$

(41)

under which (40) does not hold invariant

$$S[\psi'; t, \theta, \bar{\theta}] = S[\psi; t, \theta, \bar{\theta}] + 2 \ln b.$$

(42)

Note that our consideration above relied upon invariants of the finite–dimensional supergroup $SU(1,1|2)$ alone. Neither infinite–dimensional extension of the supergroup nor the analysis of central charges/cocycles were needed. The method of nonlinear realizations thus undoubtedly supports the result in [5].

4. Discussion

To summarize, in this work we applied the method of nonlinear realizations to the superconformal group $SU(1,1|2)$ and linked its invariants to the $\mathcal{N} = 4$ super–Schwarzian derivative in [5]. As compared to other existing approaches in the literature, the advantage of the present consideration is that it is entirely focused on the finite–dimensional subgroup of the infinite–dimensional supergroup underlying an $\mathcal{N} = 4$ superconformal field theory. Despite the fact that (35) is not a scalar, it possesses all the attributes of a super–Schwarzian derivative and $\mathcal{L}_a \bar{\mathcal{L}}_a$ seems to be a reasonable candidate to be used when constructing an $\mathcal{N} = 4$ supersymmetric extension of the Sachdev–Ye–Kitaev model.

Apart from the $\mathcal{N} = 4$ super–Schwarzian associated with $SU(1,1|2)$, one can define a similar derivative which exhibit $OSp(4|2)$ superconformal invariance [4]. Both the supergroups are known to be particular instances of the most general superconformal group in one dimension $D(2, 1; \alpha)$. To the best of our knowledge, a $D(2, 1; \alpha)$ super–Schwarzian has not yet been constructed and we hope to report on its peculiarities soon.

\textsuperscript{7}In order to reproduce the infinitesimal form of the superconformal boosts entering (23), one sets $d = 1$, considers $c$ to be small, such that $\frac{1}{1 + ct} \approx 1 - ct$, and identifies $c(\bar{\kappa}\kappa)$ with the infinitesimal $\kappa_\gamma$ in (23). The resulting transformation is a superposition of the supersymmetry transformation, special conformal conformal transformation parametrized by $c$ and the superconformal boost associated with $c(\bar{\kappa}\kappa)\kappa_\gamma$. 

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Appendix

Throughout the text we use a lower Greek index to designate an $SU(2)$--doublet representation. Complex conjugation yields an equivalent representation to which one assigns an upper index

$$(\psi_\alpha)^\dagger = \bar{\psi}_\alpha, \quad \alpha = 1, 2.$$  

As usual, spinor indices are raised and lowered with the use of the $SU(2)$--invariant antisymmetric matrices

$$\psi^\alpha = \epsilon^{\alpha\beta} \psi_\beta, \quad \bar{\psi}_\alpha = \epsilon_{\alpha\beta} \bar{\psi}^\beta,$$

where $\epsilon_{12} = 1, \epsilon^{12} = -1$. For spinor bilinears we stick to the notation

$$\psi^2 = (\psi^\alpha \psi_\alpha), \quad \bar{\psi}^2 = (\bar{\psi}_\alpha \bar{\psi}^\alpha), \quad \bar{\psi} \psi = (\bar{\psi}^\alpha \psi_\alpha),$$

such that

$$\psi_\alpha \psi_\beta = \frac{1}{2} \epsilon_{\alpha\beta} \psi^2, \quad \bar{\psi}_\alpha \bar{\psi}^\beta = \frac{1}{2} \epsilon^{\alpha\beta} \bar{\psi}^2, \quad \psi_\alpha \bar{\psi}_\beta - \psi_\beta \bar{\psi}_\alpha = \epsilon_{\alpha\beta}(\bar{\psi} \psi),$$

$$\psi^\alpha \psi^\beta = -\frac{1}{2} \epsilon^{\alpha\beta} \psi^2, \quad \bar{\psi}_\alpha \bar{\psi}_\beta = -\frac{1}{2} \epsilon_{\alpha\beta} \bar{\psi}^2, \quad (\bar{\psi} \psi)^2 = \frac{1}{2} \psi^2 \bar{\psi}^2.$$  

(43)

The Pauli matrices $(\sigma_a)_\alpha^\beta$ are taken in the standard form

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which obey

$$(\sigma_a \sigma_b)_\alpha^\beta + (\sigma_b \sigma_a)_\alpha^\beta = 2 \delta_{ab} \delta_\alpha^\beta, \quad (\sigma_a \sigma_b)_\alpha^\beta - (\sigma_b \sigma_a)_\alpha^\beta = 2i \epsilon_{abc}(\sigma_c)_\alpha^\beta,$$

$$(\sigma_a \sigma_b)_\alpha^\beta = \delta_{ab} \delta_\alpha^\beta + i \epsilon_{abc}(\sigma_c)_\alpha^\beta, \quad (\sigma_a)_\alpha^\beta (\sigma_a)_\gamma^\rho = 2 \delta_\alpha^\rho \delta_\beta^\gamma - \delta_\alpha^\gamma \delta_\beta^\rho,$$

$$(\sigma_a)_\alpha^\beta \epsilon_{\beta\gamma} = (\sigma_a)_\gamma^\beta \epsilon_{\beta\alpha}, \quad \epsilon^{\alpha\beta}(\sigma_a)_\beta^\gamma = \epsilon^{\gamma\beta}(\sigma_a)_\alpha^\beta,$$

where $\epsilon_{abc}$ is the totally antisymmetric Levi-Civita tensor, $\epsilon_{123} = 1$. From the last line one also finds

$$\left( \exp \left[ -\frac{i}{2} \xi_a \sigma_a \right] \right)_\alpha^\gamma \epsilon_{\gamma\beta} = -\left( \exp \left[ \frac{i}{2} \xi_a \sigma_a \right] \right)_\beta^\gamma \epsilon_{\gamma\alpha},$$

$$\epsilon^{\alpha\gamma} \left( \exp \left[ -\frac{i}{2} \xi_a \sigma_a \right] \right)_\gamma^\beta = -\epsilon^{\beta\gamma} \left( \exp \left[ \frac{i}{2} \xi_a \sigma_a \right] \right)_\gamma^\alpha,$$
where $\xi_a$ is a real vector parameter.

Throughout the text we use the abbreviation $\bar{\psi}\sigma_a\psi = \bar{\psi}^\alpha(\sigma_a)^\alpha_\beta\psi^\beta$. Our convention for the Hermitian conjugation adopted above imply

$$(\bar{\psi}_a)^\dagger = -\psi^a, \quad (\psi^2)^\dagger = \bar{\psi}^2, \quad (\bar{\psi}\sigma_a\chi)^\dagger = \bar{\chi}\sigma_a\psi.$$ 

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