CABLE LINKS AND L-SPACE SURGERIES

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Abstract. An L-space link is a link in $S^3$ on which all sufficiently large integral surgeries are L-spaces. We prove that a cable link is an L-space link if and if only each component is an L-space knot. We also compute HFL and $\hat{\text{HFL}}$ of an L-space cable link in terms of its Alexander polynomial. As an application, we confirm a conjecture of Licata [Lic12] regarding the structure of $\hat{\text{HFL}}$ for $(n,n)$ torus links.

1. Introduction

Heegaard Floer homology is a package of 3-manifold invariants defined by Ozsváth and Szabó [OS04a, OS04b]. In its simplest form, it associates to a closed 3-manifold $Y$ a graded vector space $\hat{\text{HF}}(Y)$. For a rational homology sphere $Y$, they show that $\dim \hat{\text{HF}}(Y) \geq |H_1(Y;\mathbb{Z})|$. If equality is achieved, then $Y$ is called an L-space.

A knot $K \subset S^3$ is an L-space knot if $K$ admits a positive L-space surgery. Let $S^{3}_{p/q}(K)$ denote $p/q$ Dehn surgery along $K$. If $K$ is an L-space knot, then $S^{3}_{p/q}(K)$ is an L-space for all $p/q \geq 2g(K) - 1$ where $g(K)$ denotes the Seifert genus of $K$ [OS11, Corollary 1.4]. A link $L \subset S^3$ is an L-space link if all sufficiently large integral surgeries on $L$ are L-spaces. In contrast to the knot case, if $L$ admits a positive L-space surgery, it does not necessarily follow that all sufficiently large surgeries are also L-spaces; see [Liu14, Example 2.3].

For relatively prime integers $m$ and $n$, let $K_{m,n}$ denote the $(m,n)$ cable of $K$, where $m$ denotes the longitudinal winding; without loss of generality, we will assume that $m > 0$. Work of Hedden [Hed09] ("if" direction) and the second author [Hom11] ("only if" direction) completely classifies L-space cable knots.

Theorem 1 ([Hed09, Hom11]). Let $K$ be a knot in $S^3$, $m > 1$ and $\gcd(m,n) = 1$. The cable knot $K_{m,n}$ is an L-space knot if and only if $K$ is an L-space knot and $n/m > 2g(K) - 1$.

We generalize this theorem to cable links with many components.

Theorem 2. Let $K$ be a knot in $S^3$, $m > 1$ and $\gcd(m,n) = 1$. The d-component cable link $K_{dn,dn}$ is an L-space link if and only if $K_{m,n}$ is an L-space knot. Furthermore, if $K$ is an L-space knot and $n \geq 2g(K) - 1$ then $K_{d,dn}$ is an L-space link.

Remark 1.1. If $m = 1$ then Theorem 1 is not needed since $K_{1,n} = K$ for all $n$. However, if $K$ is an L-space knot then it is not clear if $K_{d,dn}$ is an L-space link for $d > 1$ and $n < 2g(K) - 1$.

In [OS05], Ozsváth and Szabó show that if $K$ is an L-space knot, then $\hat{\text{HF}}(K)$ is completely determined by $\Delta_K(t)$, the Alexander polynomial of $K$. Consequently, the Alexander polynomials

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of L-space knots are quite constrained (the non-zero coefficients are all ±1 and alternate in sign) and the rank of \( \widehat{\text{HFK}}(K) \) is at most one in each Alexander grading. In [Liu14, Theorem 1.15], Liu generalizes this result to give bounds on the rank of \( \text{HFL}(L) \) in each Alexander multi-grading and on the coefficients of the multi-variable Alexander polynomial of an L-space link \( L \) in terms of the number of components of \( L \). For L-space cable links, we have the following stronger result.

**Definition 1.2.** Define the \( \mathbb{Z} \)-valued functions \( h(k) \) and \( \beta(k) \) by the equations:

\[
(1.1) \quad \sum_k h(k)t^k = \frac{t^{-1} \Delta_{m,n}(t)(tm^{nd/2} - t^{-mnd/2})}{(1-t^{-1})^2(tm^{nd/2} - t^{-mnd/2})}, \quad \beta(k) = h(k - 1) - h(k) - 1,
\]

where \( \Delta_{m,n}(t) \) is the Alexander polynomial of the cable knot \( K_{m,n} \).

Throughout, we work with \( F = \mathbb{Z}/2\mathbb{Z} \) coefficients. The following theorem gives a complete description of the homology groups \( \widehat{\text{HFL}} \) for cable links with \( n/m > 2g(K) - 1 \).

**Theorem 3.** Let \( K_{dm,dn} \) be a cable link with \( n/m > 2g(K) - 1 \).

(a) If \( \beta(k) + \beta(k + 1) \leq d - 2 \) then:

\[
\widehat{\text{HFL}}(K_{dm,dn}, k, \ldots, k) \simeq \bigoplus_{i=0}^{\beta(k)} \left( \binom{d-1}{i} \right) F_{-2h(k) - i} \oplus \bigoplus_{i=0}^{\beta(k+1)} \left( \binom{d-1}{i} \right) F_{-2h(k)+2-d+i}
\]

(b) If \( \beta(k) + \beta(k + 1) \geq d - 2 \) then:

\[
\widehat{\text{HFL}}(K_{dm,dn}, k, \ldots, k) \simeq \bigoplus_{i=0}^{d-2-\beta(k+1)} \left( \binom{d-1}{i} \right) F_{-2h(k) - i} \oplus \bigoplus_{i=0}^{d-2-\beta(k)} \left( \binom{d-1}{i} \right) F_{-2h(k)+2-d+i}
\]

(c) If \( v \) has \( j \) coordinates equal to \( k - 1 \) and \( d - j \) coordinates equal to \( k \) for some \( k \) and \( 1 \leq j \leq d - 1 \), then:

\[
\widehat{\text{HFL}}(K_{dm,dn}, (k-1)^j, k^{d-j}) \simeq \left( \binom{d-2}{\beta(k)} \right) F_{-2h(k)-\beta(k)-j}.
\]

(d) For all other Alexander gradings the groups \( \widehat{\text{HFL}} \) vanish.

We prove the parts of this theorem as separate Theorems 4.18, 4.20 and 4.21. We compute \( \widehat{\text{HFL}} \) explicitly for several examples in Section 5. In particular, we use Theorem 3 to confirm a conjecture of Joan Licata [Lic12, Conjecture 1] concerning \( \widehat{\text{HFL}} \) for \((n, n)\) torus links.

**Theorem 4.** Suppose that \( 0 \leq s \leq \frac{n-1}{2} \). Then

\[
\widehat{\text{HFL}} \left( T(n,n), \frac{n-1}{2} - s, \ldots, \frac{n-1}{2} - s \right) = \bigoplus_{i=0}^{s} \left( \binom{n-1}{i} \right) F_{-s^2 - s - i} \oplus \bigoplus_{i=0}^{s-1} \left( \binom{n-1}{i} \right) F_{-s^2 - s - n + 2 + i}.
\]

Combined with [Lic12, Theorem 2], this completes the description of \( \widehat{\text{HFL}}(T(n,n)) \).

The following theorem describes the homology groups \( \text{HFL}^- \) for cable links with \( n/m > 2g(K) - 1 \).

**Theorem 5.** Let \( K \) be an L-space knot and \( n/m > 2g(K) - 1 \). Consider an Alexander grading \( v = (v_1, \ldots, v_n) \). Suppose that among the coordinates \( v_i \) exactly \( \lambda \) are equal to \( k \) and all other coordinates are less than \( k \). Let \( |v| = v_1 + \ldots + v_n \). Then the Heegaard-Floer homology group \( \text{HFL}^-(K_{dm,dn}, v) \) can be described as follows:

(a) If \( \beta(k) < d - \lambda \) then \( \text{HFL}^-(K_{dm,dn}, v) = 0 \).
(b) If $\beta(k) \geq d - \lambda$ then

$$\text{HFL}^- (K_{dm, dn}, v) \simeq (\mathbb{F}(0) \oplus \mathbb{F}(-1))^{d - \lambda} \otimes \bigoplus_{i=0}^{\beta(k)-d+\lambda} \left( \frac{\lambda - 1}{i} \right) \mathbb{F}(-2h(v) - i),$$

where $h(v) = h(k) + kd - |v|$.

We prove this theorem in Section 4.2. The structure of the homology for $n/m = 2g(K) - 1$ (which is possible only if $m = 1$) is more subtle and is described in Theorem 4.22.

Finally, we describe $\text{HFL}^-$ as a $\mathbb{F}[U_1, \ldots, U_d]$-module. We define a collection of $\mathbb{F}[U_1, \ldots, U_d]$-modules $M_\beta$ for $0 \leq \beta \leq d - 2, M_{d-1,k}$ for $k \geq 0$ and $M_{d-1,\infty}$. These modules can be defined combinatorially and do not depend on a link.

**Theorem 6.** Let $R = \mathbb{F}[U_1, \ldots, U_d]$ and suppose that $n/m > 2g(K) - 1$. There exists a finite collection of diagonal lattice points $a_i = (a_i, \ldots, a_i)$ (determined by $m, n$ and the Alexander polynomial of $K$) such that $\text{HFL}^-$ admits the following direct sum decomposition:

$$\text{HFL}^- (K_{dm, dn}) = \bigoplus_i R \cdot \text{HFL}^-(K_{dm, dn}, a_i).$$

Furthermore, for $\beta(a_i) \leq d - 2$ one has $R \cdot \text{HFL}^-(K_{dm, dn}, a_i) \simeq M_{\beta(a_i)}$, and for $\beta(a_i) = d - 1$ one has either $R \cdot \text{HFL}^-(K_{dm, dn}, a_i) \simeq M_{d-1,k}$ for some $k$ or $R \cdot \text{HFL}^-(K_{dm, dn}, a_i) \simeq M_{d-1,\infty}$.

We compute $\text{HFL}^-$ explicitly for several examples in Section 5.

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## 2. Dehn surgery and cable links

In this section, we prove Theorem 2. We begin with a result about Dehn surgery on cable links (cf. [Hei74]).

**Proposition 2.1.** The manifold obtained by $(mn, c_2, \ldots, c_d)$-surgery on the $d$-component link $K_{dm, dn}$ is homeomorphic to $S_{n/m}^3(K) \# L(m, n) \# L(c_2 - mn, 1) \# \cdots \# L(c_d - mn, 1)$.

**Proof.** Recall (see, for example, [Hed09, Section 2.4]) that $mn$-surgery on $K_{m,n}$ gives the manifold $S_{n/m}^3(K) \# L(m, n)$. Viewing $K_{m,n}$ as the image of $T_{m,n}$ on $\partial N(K)$, we have that the reducing sphere is given by the annulus $\partial N(K) \setminus N(T_{m,n})$ union two parallel copies of the meridional disk of the surgery solid torus; we obtain a sphere since the surgery slope coincides with the surface framing.

The link $K_{dm, dn}$ consists of $d$ parallel copies of $K_{m,n}$ on $\partial N(K)$. Label these $d$ copies $K^1_{m,n}$ through $K^d_{m,n}$. We perform $mn$-surgery on $K^1_{m,n}$ and consider the image $\tilde{K}^i_{m,n}$ of $K^i_{m,n}$, $2 \leq i \leq d$, in $S_{n/m}^3(K) \# L(m, n)$. Each $\tilde{K}^i_{m,n}$ lies on $\partial N(K) \setminus N(T_{m,n})$ and thus on the reducing sphere. In particular, each $\tilde{K}^i_{m,n}$ bounds a disk $D^2_i$ in $S_{n/m}^3(K) \# L(m, n)$ such that the collection $\{D^2_2, \ldots, D^2_d\}$ is disjoint. It follows that performing surgery on $\bigcup_{i=2}^d \tilde{K}^i_{m,n}$ yields $d - 1$ lens space summands. To see which lens spaces we obtain, note that the $mn$-framed longitude on $K^i_{m,n} \subset S^3$ coincides with the 0-framed longitude on $\tilde{K}^i_{m,n} \subset S_{n/m}^3(K) \# L(m, n)$. Thus, $c_i$-surgery on $K^i_{m,n}$ corresponds to $(c_i - mn)$-surgery on $\tilde{K}^i_{m,n}$, and the result follows. \[\square\]
Let us recall that the linking number between each two components of $K_{dm, dn}$ equals $l := mn$. It is well-known that the cardinality of $H_1$ of the manifold obtained by $(c_1, c_2, \ldots, c_d)$-surgery on $K_{dm, dn}$ equals $|\det \Lambda(c_1, \ldots, c_d)|$, where

$$\Lambda_{ij} = \begin{cases} c_i, & \text{if } i = j, \\ l, & \text{if } i \neq j. \end{cases}$$

This cardinality can be computed using the following result.

**Proposition 2.2.** One has the following identity:

$$\det \Lambda(c_1, \ldots, c_d) = (c_1 - l) \cdots (c_d - l) + l \sum_{i=1}^{d} (c_1 - l) \cdots (c_i - l) \cdots (c_d - l). \quad (2.1)$$

**Proof.** One can easily check that $\det \Lambda(l, c_2, \ldots, c_d) = l(c_2 - l) \cdots (c_d - l)$. The expansion of the determinant in the first row yields a recursion relation

$$\det \Lambda(c_1, \ldots, c_d) = \det \Lambda(l, c_2, \ldots, c_d) + (c_1 - l) \det \Lambda(c_2, \ldots, c_d) =$$

$$= l(c_2 - l) \cdots (c_d - l) + (c_1 - l) \det \Lambda(c_2, \ldots, c_d).$$

Now (2.1) follows by induction in $d$.

**Corollary 2.3.** If $c_i \geq l$ for all $i$ then $\det \Lambda(c_1, \ldots, c_d) \geq 0$.

In order to prove Theorem 2, we will need the following:

**Theorem 2.4 ([Liu14, Proposition 1.11]).** A link $L$ is an $L$-space link if and only if there exists a surgery framing $\Lambda(c_1, \ldots, c_d)$, such that for all sublinks $L' \subseteq L$, $\det(\Lambda(c_1, \ldots, c_d)|_{L'}) > 0$ and $S^3_{\Lambda|_{L'}}(L')$ is an $L$-space.

**Proof of Theorem 2.** If $K_{dm, dn}$ is an L-space link, then by [Liu14, Lemma 1.10] all its components are L-space knots. On the other hand, its components are isotopic to $K_{m,n}$.

Conversely, suppose that $K_{m,n}$ is an L-space knot. Let us prove by induction on $d$ that $(c_1, \ldots, c_d)$-surgery on $K_{dm, dn}$ is an L-space if $c_i > l$ for all $i$. For $d = 1$ it is clear. By Proposition 2.1, the link $K_{dm, dn}$ admits an L-space surgery with parameters $l, c_2, \ldots, c_d$. Let us apply Theorem 2.4. Indeed, by Corollary 2.3, one has $\det(\Lambda(l, c_2, \ldots, c_d)|_{L'}) > 0$ and by the induction assumption $S^3_{\Lambda|(l, c_2, \ldots, c_d)}(L')$ is an L-space for all sublinks $L'$. By [Liu14, Lemma 2.5], $(c_1, \ldots, c_d)$-surgery on $K_{dm, dn}$ is also an L-space for all $c_1 > l$. Therefore $K_{dm, dn}$ is an L-space link.

3. A spectral sequence for L-space links

In this section we review some material from [GN14]. Given $u, v \in \mathbb{Z}^d$, we write $u \preceq v$ if $u_i \leq v_i$ for all $i$, and $u < v$ if $u \preceq v$ and $u \neq v$. Recall that we work with $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ coefficients.

**Definition 3.1.** Given a $d$-component oriented link $L$, we define an affine lattice over $\mathbb{Z}^d$:

$$\mathbb{H}(L) = \bigoplus_{i=1}^{d} \mathbb{H}_i(L), \quad \mathbb{H}_i(L) = \mathbb{Z} + \frac{1}{2} \text{lk}(L_i, L - L_i).$$

Let us recall that the Heegaard-Floer complex for a $d$-component link $L$ is naturally filtered by the subcomplexes $A^i_L(L; v)$ of $\mathbb{F}[U_1, \ldots, U_d]$-modules for $v \in \mathbb{H}(L)$. Such a subcomplex is spanned by the generators in the Heegaard-Floer complex of Alexander filtration less than or equal to $v$ in
the natural partial order on $\mathbb{H}(L)$. The group $\text{HFL}^-(L, v)$ can be defined as the homology of the associated graded complex:

$$
(3.1) \quad \text{HFL}^-(L, v) = H_* \left( A^- (L; v) / \sum_{u < v} A^- (L; u) \right).
$$

One can forget a component $L_d$ in $L$ and consider the $(d-1)$-component link $L - L_d$. There is a natural forgetful map $\pi_d : \mathbb{H}(L) \to \mathbb{H}(L - L_d)$ defined by the equation:

$$
\pi_d(v_1, \ldots, v_d) = (v_1 - \text{lk}(L_1, L_d)/2, \ldots, v_{d-1} - \text{lk}(L_{d-1}, L_d)/2).
$$

Similarly, one can define a map $\pi_{L'} : \mathbb{H}(L) \to \mathbb{H}(L')$ for every sublink $L' \subset L$. Furthermore, for large $v_d \gg 0$ the subcomplexes $A^- (L; v)$ stabilize, and by [OS08, Proposition 7.1] one has a natural homotopy equivalence $A^- (L; v) \sim A^- (L - L_d; \pi_d(v))$. More generally, for a sublink $L' = L_{i_1} \cup \ldots \cup L_{i_{d'}}$ one gets

$$
(3.2) \quad A^- (L'; \pi_{L'}(v)) \sim A^- (L; v), \text{ if } v_i \gg 0 \text{ for } i \notin \{i_1, \ldots, i_{d'}\}.
$$

We will use the “inversion theorem” of [GN14], expressing the $h$-function of a link in terms of the Alexander polynomials of its sublinks, or, equivalently, the Euler characteristics of their Heegaard-Floer homology. Define $\chi_{L,v} := \chi(\text{HFL}^- (L, v))$. Then by [OS08]

$$
\chi_L(t_1, \ldots, t_d) := \sum_{v \in \mathbb{H}(L)} \chi_{L,v} t_1^{e_1} \cdots t_d^{e_d} = \begin{cases} (t_1 \cdots t_d)^{1/2} \Delta(t_1, \ldots, t_d), & \text{if } d > 1, \\
\Delta(t)/(1 - t^{-1}), & \text{if } d = 1,
\end{cases}
$$

where $\Delta(t_1, \ldots, t_r)$ denotes the symmetrized Alexander polynomial.

**Remark 3.2.** We choose the factor $(t_1 \cdots t_d)^{1/2}$ to match more established conventions on the gradings for the hat-version of link Floer homology. For example, the Alexander polynomial of the Hopf link equals 1, and one can check [OS08] that $\text{HFL}$ is supported in Alexander degrees $(\pm 1, \pm 1)$. Since the maximal Alexander degrees in $\text{HFL}$ and $\text{HFL}^-$ coincide, one gets $\chi_{T(2,2)}(t_1, t_2) = t_1^{1/2} t_2^{1/2}$. The following “large surgery theorem” underlines the importance of $A^- (L, v)$.

**Theorem 3.3 ([MO10]).** The homology of $A^- (L, v)$ is isomorphic to the Heegaard-Floer homology of a large surgery on $L$ with spin$_c$-structure specified by $v$. In particular, if $L$ is an L-space link, then $H_* (A^- (L, v)) \simeq \mathbb{F}[U]$ for all $v$ and all $U_i$ are homotopic to each other.

One can show that for L-space links the inclusion $h_e : A^- (L, v) \hookrightarrow A^- (S^3)$ is injective on homology, so it is multiplication by $U^{h_L(v)}$. Therefore the generator of $H_* (A^- (L, v)) \simeq \mathbb{F}[U]$ has homological degree $-2 h_L(v)$. The function $h_L(v)$ will be called the $h$-function of a link $L$. In [GN14] it was called an “$HFL$-weight function”.

Furthermore, if $L$ is an L-space link, then for large $N \in \mathbb{H}(L)$ one has

$$
\chi \left( A^- (L; N) / A^- (L, v) \right) = h_L(v),
$$

hence by (3.1) and the inclusion-exclusion formula one can write:

$$
(3.3) \quad \chi_{L,v} = \sum_{B \subseteq \{1, \ldots, d\}} (-1)^{|B|-1} h_L(v - e_B),
$$

where $e_B$ denotes the characteristic vector of the subset $B \subset \{1, \ldots, d\}$. Furthermore, by (3.2) for a sublink $L' = L_{i_1} \cup \ldots \cup L_{i_{d'}}$ one gets

$$
(3.4) \quad h_{L'}(\pi_{L'}(v)) = h_L(v), \text{ if } v_i \gg 0 \text{ for } i \notin \{i_1, \ldots, i_{d'}\}.
$$
For \( d = 1 \) equation (3.3) has the form \( \chi_{L,v} = h(v - 1) - h(v) \), so \( h(v) \) can be easily reconstructed from the Alexander polynomial: \( h_L(v) = \sum_{u \geq v+1} \chi_{L,u} \). For \( d > 1 \), one can also show that equation (3.3) (together with the boundary conditions (3.4)) has a unique solution, which is given by the following:

**Theorem 3.4** ([GN14]). The \( h \)-function of an L-space link is determined by its Alexander polynomial as following:

\[
(3.5) \quad h_L(v_1, \ldots, v_d) = \sum_{L' \subseteq L} (-1)^{d' - 1} \sum_{u \geq \pi_{L'}(v+1)} \chi_{L',u},
\]

where the sublink \( L' \) has \( d' \) components and \( 1 = (1, \ldots, 1) \).

Given an L-space link, we construct a spectral sequence whose \( E_2 \) page can be computed from the multi-variable Alexander polynomial by an explicit combinatorial procedure, and whose \( E_\infty \) page coincides with the group \( \text{HFL}^- \). The complex (3.1) is quasi-isomorphic to the iterated cone:

\[
\mathcal{K}(v) = \bigoplus_{B \subset \{1, \ldots, d\}} A^-(L, v - e_B),
\]

where the differential consists of two parts: the first acts in each summand and the second acts by inclusion maps between summands. There is a spectral sequence naturally associated to this construction. Its \( E_1 \) term equals

\[
E_1(v) = \bigoplus_{B \subset \{1, \ldots, d\}} H_* (A^-(L, v - e_B)) = \bigoplus_{B \subset \{1, \ldots, d\}} \mathbb{F}[U] \langle z(v - e_B) \rangle,
\]

where \( z(u) \) is the generator of \( H_* (A^-(L, u)) \) of degree \(-2h_L(u)\). The next differential \( \partial_1 \) is induced by inclusions and reads as:

\[
(3.6) \quad \partial_1(z(v - e_B)) = \sum_{i \in B} U^{h(v - e_B) - h(v - e_B - i)} z(v - e_B + e_i).
\]

We obtain the following result.

**Theorem 3.5** ([GN14]). Let \( L \) be an L-space link with \( d \) components and let \( h_L(v) \) be the corresponding \( h \)-function. Then there is a spectral sequence with \( E_2(v) = H_* (E_1, \partial_1) \) and \( E_\infty \simeq \text{HFL}^- (L, v) \).

**Remark 3.6.** Let us write more precisely the bigrading on the \( E_2 \) page. The \( E_1 \) page is naturally bigraded as follows: a generator \( U^m z(v - e_B) \) has cube degree \(|B|\) and its homological degree in \( A^-(L, v - e_B) \) equals \(-2m - 2h(v - e_B)\). In short, we will write

\[
\text{bideg} (U^m z(v - e_B)) = (|B|, -2m - 2h(v - e_B)).
\]

The homological degree of the same generator in \( E_1(v) \) equals the sum of these two degrees. The differential \( \partial_1 \) has bidegree \((-1, 0)\), and, more generally, the differential \( \partial_k \) in the spectral sequence has bidegree \((-k, k - 1)\).

In the next section we will compute the \( E_2 \) page for cable L-space links and show that \( E_2 = E_\infty \). Let us discuss the action of the operators \( U_i \) on the \( E_2 \) page. Recall that \( U_i \) maps \( A^-(L, v) \) to \( A^-(L, v - e_i) \), and in homology one has:

\[
(3.7) \quad U_i z(v) = U^{1 - h(v - e_i) + h(v)} z(v - e_i).
\]

Since \( U_i \) commutes with the inclusions of various \( A^- \), we get the following result.
Proposition 3.7. Equation (3.7) defines a chain map from $K(v)$ to $K(v - e_i)$ commuting with the differential $\partial_1$, so we have a well-defined combinatorial map

$$U_i : H_*(E_1(v), \partial_1) \to H_*(E_1(v - e_i), \partial_1).$$

If $E_2 = E_\infty$ then one obtains $U_i : \text{HFL}^-(L, v) \to \text{HFL}^-(L, v - e_i)$.

Furthermore, by the definition of $\widehat{\text{HFL}}$ [OS08, Section 4] one gets:

$$\widehat{\text{HFL}}(L, v) = H_*(A^-(L, v)/\left[\sum_{i=1}^d A^-(v - e_i) \oplus \sum_{i=1}^d U_i A^-(v + e_i)\right]).$$

This implies the following result:

Proposition 3.8. There is a spectral sequence with $E_1$ page

$$\widehat{E}_1 = \bigoplus_{B \subset \{1, \ldots, d\}} \text{HFL}^-(L, v + e_B)$$

and converging to $\widehat{E}_\infty = \widehat{\text{HFL}}(L, v)$. The differential $\widehat{\partial}_1$ is given by the action of $U_i$ induced by (3.7).

4. Heegaard-Floer homology for cable links

4.1. The Alexander polynomial and $h$–function. The Alexander polynomial of cable knots and links is given by the following well-known formula:

$$\Delta_{K_{dm, dn}}(t_1, \ldots, t_d) = \Delta_K(t_1^m \cdots t_d^m) \cdot \Delta_{T(dm, dn)}(t_1, \ldots, t_d),$$

where $T(dm, dn)$ denotes the $(dm, dn)$ torus link. Throughout, let $t = t_1 \cdots t_d$ and $l = mn$.

Lemma 4.1. The generating functions for the Euler characteristics of $\text{HFL}^-$ for $K_{dm, dn}$ and $K_{m, n}$ are related by the following equation:

$$\chi_{K_{dm, dn}}(t_1, \ldots, t_d) = \chi_{K_{m, n}}(t) \cdot (t^{l/2} - t^{-l/2})^{d-1}.$$  

Proof. The statement follows from the identity (4.1) and the expression for the Alexander polynomials of torus links:

$$\chi_{T(dm, dn)}(t_1, \ldots, t_d) = \frac{(t^{mn/2} - t^{-mn/2})^d}{(t^{m/2} - t^{-m/2})(t^{n/2} - t^{-n/2})}.$$  

From now on we will assume that $K$ is an L-space knot and $n/m \geq 2g(K) - 1$, so $K_{dm, dn}$ is an L-space link for all $d$. To simplify notation, we define $h_{dm, dn}(v) = h_{K_{dm, dn}}(v)$ and $\chi_{dm, dn}(v) = \chi_{K_{dm, dn}, v}$. Let $c = l(d - 1)/2$.

Theorem 4.2. Suppose that $v_1 \leq v_2 \leq \ldots \leq v_d$. Then the following equation holds:

$$h_{dm, dn}(v_1, \ldots, v_d) = h_{m, n}(v_1 - c) + h_{m, n}(v_2 - c + l) + \ldots + h_{m, n}(v_d - c + (d - 1)l).$$

Proof. We will use Theorem 3.4 to compute $h(v)$. Let $L'$ be a sublink of $K_{dm, dn}$ with $d'$ components, i.e., $L' = K_{dm', d'n}$. By (4.2), one has

$$\chi_{K_{dm', d'n}}(t_1, \ldots, t_{d'}) = \chi_{K_{m, n}}(t) \cdot t^{(d'-1)/2} \sum_{j=0}^{d'-1} (-1)^j \binom{d'-1}{j} t^{-lj},$$
hence $\chi_{L',u}$ does not vanish only if $u = (s, \ldots, s)$, and
\[
\chi_{L', s, \ldots, s} = \sum_{j=0}^{d'-1} (-1)^j \binom{d'-1}{j} \chi_{m,n} (s-l(d'-1)/2 + lj).
\]
Therefore
\[
\sum_{u \geq \pi_{L'}(v+1)} \chi_{L', u} = \sum_{s > \max(\pi_{L'}(v))} \sum_{j=0}^{d'-1} (-1)^j \binom{d'-1}{j} \chi_{m,n} (s-l(d'-1)/2 + lj)
= \sum_{j=0}^{d'-1} (-1)^j \binom{d'-1}{j} \chi_{m,n} (\max(\pi_{L'}(v)) - l(d'-1)/2 + lj).
\]
Furthermore, if $L' = L_{i_1} \cup \ldots \cup L_{i_d'}$ then $\pi_{L'}(v) = (v_{i_1} - l(d-d')/2, \ldots, v_{i_{d'}} - l(d-d')/2)$, so 
\[
\max(\pi_{L'}(v)) = \max(v_{i_1}, \ldots, v_{i_d'}) - l(d-d')/2 = \max(v_{L'}) - l(d-d')/2.
\]
This means that (3.5) can be rewritten as follows:
\[
h_{m,n,d}(v_1, \ldots, v_d) = \sum_{L' \vdash j} (-1)^{d' - 1 + j} \binom{d'}{j} \chi_{m,n} (\max(v_{L'}) - l(d-1)/2 + lj)
= \sum_{i,j} \chi_{m,n} (v_i - l(d-1)/2 + lj) \sum_{L' \vdash v_i = \max(v_{L'})} (-1)^{d' - 1 + j} \binom{d'}{j}.
\]
One can check that the inner sum vanishes unless $j = i - 1$ (recall that $v_1 \leq v_2 \leq \ldots \leq v_d$), so one gets
\[
h_{m,n,d}(v_1, \ldots, v_d) = \sum_i \chi_{m,n} (v_i - l(d-1)/2 + l(i-1)).
\]

**Lemma 4.3.** The following identity holds:
\[
h_{m,n,d}(-v_1, \ldots, -v_d) = h_{m,n,d}(v_1, \ldots, v_d) + (v_1 + \ldots + v_d).
\]

**Proof.** Suppose that $v_1 \leq v_2 \leq \ldots \leq v_d$. Then $-v_1 \geq -v_2 \geq \ldots \geq -v_d$. Therefore
\[
h_{m,n,d}(-v_1, \ldots, -v_d) = \sum_{i=1}^{d} \chi_{m,n} (-v_i - l(d-1)/2 + l(d-i))
= \sum_{i=1}^{d} \chi_{m,n} (-v_i + l(d-1)/2 - l(i-1)).
\]
It is known (e.g., [HLZ13]) that for all $x$,
\[
\chi_{m,n}(-x) = \chi_{m,n}(x) + x,
\]
hence
\[
\chi_{m,n} (-v_i + l(d-1)/2 - l(i-1)) = \chi_{m,n} (v_i - l(d-1)/2 + l(i-1)) + (v_i - l(d-1)/2 + l(i-1)).
\]
Finally, $\sum_{i=1}^{d} (-l(d-1)/2 + l(i-1)) = 0$.

**Lemma 4.4.** One has $h_{m,n,d}(k, k, \ldots, k) = h(k)$, where $h(k)$ is defined by (1.1).
Proof. Indeed, by (4.3) we have
\[ h_{dm, dn}(k, \ldots, k) = h_{m,n}(k - (d - 1)/2) + h_{m,n}(k - (d - 1)/2 + l) + \ldots + h_{m,n}(k + (d - 1)/2), \]
so
\[ \sum_k h_{dm, dn}(k, \ldots, k) t^k = (t^{-l(d-1)/2} + \ldots + t^{l(d-1)/2}) \sum_k h_{m,n}(k) t^k = \frac{(d^l/2 - t^{-l/2})}{(t/2 - t^{-1/2})} \frac{t^{-1} \Delta_{m,n}(t)}{(1 - t^{-1})^2}. \]

For the rest of this section we will assume that \( n/m > 2g(K) - 1 \).

**Lemma 4.5.** If \( v \leq g(K_{m,n}) - l \) then \( \text{HFK}^- (K_{m,n}, v) \simeq \mathbb{F} \).

**Proof.** We have
\[ g(K_{m,n}) = mg(K) + \frac{(m-1)(n-1)}{2}, \]
so for \( n/m > 2g(K) - 1 \) we have
\[ 2g(K_{m,n}) = 2mg(K) + mn - m - n + 1 < mn + 1, \]
hence \( l = mn \geq 2g(K_{m,n}) \). On the other hand, it is well-known that for \( v \leq -g(K_{m,n}) \) one has \( \text{HFK}^- (K_{m,n}, v) \simeq \mathbb{F} \).

We will use the function \( \beta \) defined by (1.1).

**Lemma 4.6.** If \( \beta(k) = -1 \) then \( \text{HFK}^- (K_{m,n}, k - c) = 0 \). Otherwise
\[ \beta(k) = \max \{ j : 0 \leq j \leq d - 1, \ HFK^- (K_{m,n}, k - c + lj) \simeq \mathbb{F} \}. \]

**Proof.** By (1.1) and Lemma 4.4 we have
\[ \beta(k) + 1 = h_{dm, dn}(k-1, \ldots, k-1) - h_{dm, dn}(k, \ldots, k) = \sum_{j=0}^{d-1} (h_{m,n}(k - 1 + c + lj) - h_{m,n}(k - c + lj)). \]
If \( \text{HFK}^- (K_{m,n}, k - c + lj) = 0 \) then by Lemma 4.5 \( \text{HFK}^- (K_{m,n}, k - c + lj') = 0 \) for all \( j' > j \). Therefore if \( \text{HFK}^- (K_{m,n}, k - c) = 0 \) then \( \beta(k) = -1 \), otherwise \( \text{HFK}^- (K_{m,n}, k - c + lj) \simeq \mathbb{F} \) for all \( j \leq \beta(k) \).

Suppose that \( v_1 = \ldots = v_{\lambda_1} = u_1, v_{\lambda_1+1} = \ldots = v_{\lambda_1+\lambda_2} = u_2, \ldots, v_{\lambda_1+\ldots+\lambda_s+1} = \ldots = v_d = u_s \)
where \( u_1 < u_2 < \ldots < u_s \) and \( \lambda_1 + \ldots + \lambda_s = d \). We will abbreviate this as \( v = (u_1^{\lambda_1}, \ldots, u_s^{\lambda_s}) \).

**Lemma 4.7.** Suppose that \( \beta(u_s) < d - \lambda_s \). Then for any subset \( B \subset \{ 1, \ldots, d - 1 \} \) one has
\[ h_{dm, dn}(v - e_B) = h_{dm, dn}(v - e_B - e_d). \]

**Proof.** To apply (4.3), one needs to reorder the components of the vectors \( v - e_B \) and \( v - e_B - e_d \). Note that in both cases the last (largest) \( \lambda_s \) components are equal either to \( u_s \) or to \( u_s - 1 \), and the corresponding contributions to \( h_{dm, dn} \) are equal to \( h_{m,n}(u_s - c + l(d - \lambda_s) + lj) \) or to \( h_{m,n}(u_s - c + l(d - \lambda_s) + lj - l) \), respectively \( (j = 0, \ldots, \lambda_s - 1) \). On the other hand, by (4.4) one has
\[ \text{HFK}^- (K_{m,n}, u_s - c + l(d - \lambda_s) + lj) = 0 \]
and so
\[ h_{m,n}(u_s - c + l(d - \lambda_s) + lj - 1) = h_{m,n}(u_s - c + l(d - \lambda_s) + lj). \]

**Lemma 4.8.** If \( \beta(u_s) \geq d - \lambda_s \) then \( h_{dm, dn}(v) = h(u_s) + du_s - |v| \).
Proof. Since $\beta(u_s) \geq d - \lambda_s$, we have $\text{HFK}^-(K_{m,n}, u_s - c + l(d - \lambda_s)) \simeq \mathbb{F}$. For $i \leq d - \lambda_s$ we get $v_i - c + l(i - 1) < u_s - c + l(d - \lambda_s) - l$, so by Lemma 4.5,

$$h_{m,n}(v_i - c + l(i - 1)) = h_{m,n}(u_s - c + l(i - 1)) + (u_s - v_i).$$

Now the statement follows from Lemma 4.3. □

Lemma 4.9. Suppose that $\beta(u_s) \geq d - \lambda_s$. Then for any subsets $B' \subset \{1, \ldots, d - \lambda_s\}$ and $B'' \subset \{d - \lambda_s + 1, \ldots, d\}$ one has

$$h_{dm,dn}(v - e_{B'} - e_{B''}) = h_{dm,dn}(v) + |B'| + \min(|B''|, \beta(u_s) - d + \lambda_s + 1).$$

Proof. Since $\text{HFK}^-(K_{m,n}, u_s - c + l(d - \lambda_s)) \simeq \mathbb{F}$, we have $u_s - c + l(d - \lambda_s) \leq g(K_{m,n})$, so for all $i \leq d - \lambda_s$ one has $v_i - c + l(i - 1) < u_s - c + l(d - \lambda_s) - l \leq g(K_{m,n}) - l$, and by Lemma 4.5 $\text{HFK}^-(K_{m,n}, v_i - c + l(i - 1)) \simeq \mathbb{F}$, and $h_{m,n}(v_i - c + l(i - 1)) = h_{m,n}(u_s - c + l(i - 1)) + 1$. Therefore $h_{dm,dn}(v - e_{B'} - e_{B''}) = |B'| + h_{dm,dn}(v - e_{B''})$. Finally,

$$h_{dm,dn}(v - e_{B''}) - h_{dm,dn}(v) = \sum_{j=0}^{|B''|} (h_{m,n}(u_s - c + l(d - \lambda_s) + lj) - h_{m,n}(u_s - c + l(d - \lambda_s) + lj))$$

$$= \min(|B''|, \beta(u_s) - d + \lambda_s + 1).$$

□

4.2. Spectral sequence for HFL$^-$.

Definition 4.10. Let $E_d$ denote the exterior algebra over $\mathbb{F}$ with variables $z_1, \ldots, z_d$. Let us define the cube differential on $E_d$ by the equation

$$\partial(z_{a_1} \wedge \cdots \wedge z_{a_k}) = \sum_{j=1}^{k} (-1)^{j-1} z_{a_1} \wedge \cdots \wedge \hat{z_{a_j}} \wedge \cdots \wedge z_{a_k},$$

and the $b$-truncated differential on $E_d[U]$ by the equation

$$\partial^{(b)}(z_{a_1} \wedge \cdots \wedge z_{a_k}) = \begin{cases} U \partial(z_{a_1} \wedge \cdots \wedge z_{a_k}), & \text{if } k \leq b \\
\partial(z_{a_1} \wedge \cdots \wedge z_{a_k}), & \text{if } k > b. \end{cases}$$

More invariantly, one can define the weight of a monomial $z_\alpha = z_{a_1} \wedge \cdots \wedge z_{a_k}$ as $w(z_\alpha) = \min(|\alpha|, b)$, and the $b$-truncated differential is given by the equation:

$$\partial^{(b)}(z_\alpha) = \sum_{\alpha \in \alpha} U^{w(\alpha) - w(\alpha - a_i)} z_{\alpha - a_i}. \quad (4.5)$$

Indeed, $w(\alpha) - w(\alpha - a_i) = 1$ for $|\alpha| \leq b$ and $w(\alpha) - w(\alpha - a_i) = 0$ for $|\alpha| > b$.

Definition 4.11. Let $E_d^{red} \subset E_d$ be the subalgebra of $E_d$ generated by the differences $z_i - z_j$ for all $i \neq j$.

Lemma 4.12. The kernel of the cube differential $\partial$ on $E_d$ coincides with $E_d^{red}$.

Proof. It is clear that $\partial(z_i - z_j) = 0$, and Leibniz rule implies vanishing of $\partial$ on $E_d^{red}$. Let us prove that $\text{Ker} \partial \subset E_d^{red}$. Since $\partial$ is acyclic on $E_d$, it is sufficient to prove that the image of every monomial $z_{a_1} \wedge \cdots \wedge z_{a_k}$ is contained in $E_d^{red}$. Indeed, one can check that

$$\partial(z_{a_1} \wedge \cdots \wedge z_{a_k}) = (z_{a_2} - z_{a_1}) \wedge \cdots \wedge (z_{a_k} - z_{a_{k-1}}).$$

□
Lemma 4.13. The homology of \( \partial^{(b)} \) is given by the following equation:

\[
H_k(\mathcal{E}_d[U], \partial^{(b)}) = \begin{cases} 
\binom{d-1}{k}, & \text{if } k < b \\
0, & \text{if } k \geq b.
\end{cases}
\]

Proof. Since \( \partial \) is acyclic, one immediately gets \( H_k(\mathcal{E}_d[U], \partial^{(b)}) = 0 \) for \( k \geq b \). For \( k < b \), the homology is supported at the zeroth power of \( U \) and one has \( H_k(\mathcal{E}_d[U]) \simeq \ker(\partial|_{\wedge^k(z_1, \ldots, z_d)}) \). The dimension of the latter kernel equals

\[
\dim \ker(\partial|_{\wedge^k(z_1, \ldots, z_d)}) = \dim \wedge^k(z_1 - z_2, \ldots, z_1 - z_d) = \binom{d-1}{k}.
\]

\[ \square \]

Proof of Theorem 5. Let us compute \( \HFL^-(K_{dm,dn}, v) \) using the spectral sequence constructed in Theorem 3.5. By Lemma 4.7, in case (a) it is easy to see that the complex \((E_1, \partial_1)\) is contractible in the direction of \( e_d \) and \( E_2 = H_*(E_1, \partial_1) = 0 \).

In case (b) by Lemma 4.9 and (4.5) one can write \( E_1 = \mathcal{E}_{d-\lambda} \otimes \mathcal{E}_\lambda \otimes \mathbb{F}[U] \), and \( \partial_1 \) acts as \( U/\partial \) on the first factor and as \( \partial^{(\beta+1)} \) on the second one. By the Künneth formula and Lemma 4.13, the \( E_2 \) page agrees with the statement of the theorem, hence we need to prove that the spectral sequence degenerates at the \( E_2 \) page.

Indeed, the \( E_1 \) page is bigraded by the homological degree and \( |B| \) (see Remark 3.6). By Lemma 4.13 any surviving homology class on the \( E_2 \) page of cube degree \( x \) has bidegree \( (x, -2h_{dm,dn}(v) - 2x) \), so all bidegrees on the \( E_2 \) page belong to the same line. Therefore all higher differentials must vanish.

Finally, a simple formula for \( h_{dm,dn}(v) \) in case (b) follows from Lemma 4.8.

\[ \square \]

4.3. Action of \( U_i \). One can use Proposition 3.7 to compute the action of \( U_i \) on \( \HFL^-(K_{dm,dn}, v) \) for cable links. Recall that \( R = \mathbb{F}[U_1, \ldots, U_d] \). Throughout this section we assume \( m/n > 2g(K) - 1 \).

Definition 4.14. We define \( A_d = \mathcal{E}_d \otimes R \) and \( A_d^{red} = \mathcal{E}_d^{red} \otimes R \). Let \( I_{\beta} \) denote the ideal in \( A_d \) generated by the monomials \( z_{i_1} \cdots z_{i_{\beta+1}} U_{i_{\beta+1}} \cdots U_{i_{\beta+1}} \) for all \( s \leq \beta + 1 \) and all tuples of pairwise distinct \( i_1, \ldots, i_{\beta+1} \). Let \( I_{\beta} := I_{\beta} \cap A_d^{red} \) be the corresponding ideal in \( A_d^{red} \).

The algebras \( A_d \) and \( A_d^{red} \) are naturally \( \mathbb{Z}^{r+1} \)-graded: the generators \( z_i \) have Alexander grading 0 and homological grading \((-1)\), the generators \( U_i \) have Alexander grading \((-e_i)\) and homological grading \((-2)\).

Definition 4.15. We define \( \mathcal{H}(k) := \bigoplus_{\max(v) \leq k} \HFL^-(K_{dm,dn}, v) \). Since \( U_i \) decrease the Alexander grading, \( \mathcal{H}(k) \) is naturally an \( R \)-module.

The following theorem clarifies the algebraic structure of Theorem 5.

Theorem 4.16. The following graded \( R \)-modules are isomorphic:

\[ \mathcal{H}(k)/\mathcal{H}(k-1) \simeq A_d^{red}/I_{\beta(k)}[-2h(k)], \]

where \([\cdot]\) denotes the shift of the homological grading.

Proof. Let us first prove the isomorphism on the level of vector spaces for each Alexander grading. By definition \( \mathcal{H}(k)/\mathcal{H}(k-1) \) is supported on the set of Alexander gradings \( v \) such that \( \max(v) = k \).

Suppose that exactly \( \lambda \) components of \( v \) are equal to \( k \), without the loss of generality we can assume \( v_1, \ldots, v_{d-\lambda} < k \) and \( v_{d-\lambda+1} = \ldots = v_d = k \). It follows from Lemma 4.12 and the proof of
Theorem 5 that $\text{HFL}^{-}(K_{dm,dn}, v)$ is isomorphic to the quotient of $E_{d}^{\text{red}}$ by the ideal generated by degree $\beta - d + \lambda + 1$ monomials in $(z_{i} - z_{j})$ for $i, j > d - \lambda$.

On the other hand, the component of $A_{d}^{\text{red}}/I_{\beta(k)}$ of Alexander degree $(v_{1} - k, \ldots, v_{d} - k)$ is spanned by the products of $U_{1}^{k-v_{1}} \cdots U_{d}^{k-v_{d}-\lambda}$ and monomials in $(z_{i} - z_{j})$. Since the relations in $I_{\beta}$ imply that degree $\beta - d + \lambda + 1$ monomials in $(z_{i} - z_{j})$ for $i, j > d - \lambda$ annihilate $U_{1} \cdots U_{d-\lambda}$, we get the desired isomorphism.

Finally, the isomorphism of $R$–modules follows from Proposition 3.7 and the commutation of the above isomorphisms with the shifts of $v$ preserving $\max(v) = k$.

**Lemma 4.17.** Suppose that $\max(v) = k$ and $\max(v - e_{i}) = k - 1$, and the homology group $\text{HFL}^{-}(K_{dm,dn}, v)$ does not vanish. Then $\beta(k) = d - 1, \beta(k - 1) \geq d - 2$ and the map

$$U_{i} : \text{HFL}^{-}(K_{dm,dn}, v) \to \text{HFL}^{-}(K_{dm,dn}, v - e_{i})$$

is surjective.

**Proof.** Since $\max(v) = k$ and $\max(v - e_{i}) = k - 1$, the multiplicity of $k$ in $v$ equals 1, so by Theorem 5 $\beta(k) \geq d - 1$, hence $\beta(k) = d - 1$. Therefore $\text{HFL}^{-}(K_{dm,dn}, v) \simeq E_{d}^{\text{red}} \simeq (F(0) \oplus F(0))^{d-1}$, so $U_{i}$ is surjective. Finally, by (4.4) $\text{HFK}(K_{m,n}, k - c + l(d - 1)) \simeq F$, and by Lemma 4.5 $\text{HFK}(K_{m,n}, k - c + l(d - 2)) \simeq F$, so $\beta(k - 1) \geq d - 2$. \hfill $\square$

**Proof of Theorem 6.** Let us prove that the homology classes with diagonal Alexander gradings generate $\text{HFL}^{-}$ over $R$. Indeed, given $v = (v_{1} \leq \ldots \leq v_{d})$ with $\text{HFL}^{-}(K_{dm,dn}, v) \neq 0$, by Theorems 5 and 4.16 one can check that $\text{HFL}^{-}(K_{dm,dn}, v_{d}, \ldots, v_{d}) \neq 0$ and the map

$$U_{1}^{u_{1} - v_{1}} \cdots U_{d-1}^{u_{d-1} - v_{d-1}} : \text{HFL}^{-}(K_{dm,dn}, v_{d}, \ldots, v_{d}) \to \text{HFL}^{-}(K_{dm,dn}, v)$$

is surjective.

Let us describe the $R$–modules generated by the diagonal classes in degree $(k, \ldots, k)$. If $\beta(k) = -1$ then $\text{HFL}^{-}(K_{dm,dn}, k, \ldots, k) = 0$. If $0 \leq \beta(k) \leq d - 2$ then by Lemma 4.17 the submodule $R \cdot \text{HFL}^{-}(K_{dm,dn}, k, \ldots, k)$ does not contain any classes with maximal Alexander degree less than $k$, so by Theorem 4.16

$$R \cdot \text{HFL}^{-}(K_{dm,dn}, k, \ldots, k) \simeq A_{d}^{\text{red}}/I_{\beta(k)} =: M_{\beta(k)}$$

Suppose that $\beta(k) = d - 1$, and consider minimal $a$ and maximal $b$ such that $a \leq k \leq b$ and $\beta(i) = d - 1$ for $i \in [a, b]$. If there is no minimal $a$, we set $a = -\infty$. By Lemma 4.17, $\beta(a - 1) = d - 2$ and all the maps

$$\text{HFL}^{-}(K_{dm,dn}, b, \ldots, b) \xrightarrow{U_{1} \cdots U_{d}} \text{HFL}^{-}(K_{dm,dn}, b - 1, \ldots, b - 1) \to \ldots$$

$$\ldots \to \text{HFL}^{-}(K_{dm,dn}, a, \ldots, a) \xrightarrow{U_{1} \cdots U_{d}} \text{HFL}^{-}(K_{dm,dn}, a - 1, \ldots, a - 1)$$

are surjective. Therefore

$$R \cdot \text{HFL}^{-}(K_{dm,dn}, b, \ldots, b) \simeq A_{d}^{\text{red}}/(U_{1} \cdots U_{d})^{b-a}I_{d-2} =: M_{d-1,b-a+1}$$

is supported in all Alexander degrees with maximal coordinates in $[a, b]$ and in Alexander degrees with maximal coordinate $(a - 1)$ which appears with multiplicity at least 2.

Finally, we get the following decomposition of $\text{HFL}^{-}$ as an $R$–module:

$$\text{HFL}^{-}(K_{dm,dn}) = \bigoplus_{k: 0 \leq k \leq d-1} M_{\beta(k)} \oplus \bigoplus_{a,b: \beta(a-1) = d-2} M_{d-1,b-a+1} \oplus M_{d-1,\infty}.$$
Note that for $d = 1$ we get $M_{0,l} \simeq \mathbb{F}[U]/(U_1^l)$ and $M_{0,+\infty} \simeq \mathbb{F}[U]$.

4.4. Spectral sequence for $\widehat{\text{HFL}}$.

**Theorem 4.18.** If $\beta(k) + \beta(k+1) \leq d - 2$ then the spectral sequence for $\widehat{\text{HFL}}(K_{dm,dn}, k, \ldots, k)$ degenerates at the $E_2$ page and

$$\widehat{\text{HFL}}(K_{dm,dn}, k, \ldots, k) \simeq \bigoplus_{i=0}^{\beta(k)} \left( \frac{d-1}{i} \right) \mathbb{F}^{-2h(k)-i} \oplus \bigoplus_{i=0}^{\beta(k+1)} \left( \frac{d-1}{i} \right) \mathbb{F}^{-2h(k+2-d+i)}$$

**Proof.** By Proposition 3.8, for a given $v$ there is a spectral sequence with $E_1$ page

$$\widehat{E}_1 = \bigoplus_{B \subset \{1, \ldots, d\}} \text{HFL}^-(L, v + e_B)$$

and converging to $\widehat{E}_\infty = \widehat{\text{HFL}}(L, v)$. If $v = (k, \ldots, k)$ then (for $B \neq \emptyset$) the maximal coordinate of $v + e_B$ equals $k + 1$ and appears with multiplicity $\lambda = \lvert B \rvert$. Therefore by Theorem 5 HFL$^-(L, v + e_B)$ does not vanish if and only if either $B = \emptyset$ or $\lvert B \rvert \geq d - \beta(k+1)$, and it is given by Theorem 5. By (1.1) we have $h(k+1) = h(k) - \beta(k+1) - 1$.

The spectral sequence is bigraded by the homological (Maslov) grading at each vertex of the cube and the “cube grading” $\lvert B \rvert$. The differential $\widehat{\partial}_1$ acts along the edges of the cube, and decreases the Maslov grading by 2 and the cube grading by 1.

One can check using Theorem 4.16 that its homology $\widehat{E}_2$ does not vanish in cube degrees 0 and $d - \beta(k+1)$, so one can write $\widehat{E}_2 = \widehat{E}_2^0 \oplus \widehat{E}_2^{d-\beta(k+1)}$, and

$$\widehat{E}_2^0 \simeq \bigoplus_{i=0}^{\beta(k)} \left( \frac{d-1}{i} \right) \mathbb{F}^{-2h(k)-i}, \quad \widehat{E}_2^{d-\beta(k+1)} \simeq \bigoplus_{i=0}^{\beta(k+1)} \left( \frac{d-1}{i} \right) \mathbb{F}^{-2h(k+2-d+i)}.$$

By (1.1) we have $h(k+1) = h(k) - \beta(k+1) - 1$, so $-2h(k+1) - 3\beta(k+1) + i = -2h(k) + 2 - \beta(k+1) + i$.

A higher differential $\widehat{\partial}_s$ decreases the cube grading by $s$ and decreases the Maslov grading by $s + 1$. Therefore the only nontrivial higher differential is $\widehat{\partial}_{d-\beta(k+1)}$ which vanishes by degree reasons too. Indeed, the maximal Maslov grading in $\widehat{E}_2^{d-\beta(k+1)}$ equals $-2h(k) + 2$ while the minimal Maslov grading in $\widehat{E}_2^0$ equals $-2h(k) - \beta(k)$, so the differential can decrease the Maslov grading at most by $\beta(k) + 2$. On the other hand, $\widehat{\partial}_{d-\beta(k+1)}$ drops it by $d - \beta(k+1) + 1$, and for $\beta(k) + \beta(k+1) < d - 1$ one has $d - \beta(k+1) + 1 > \beta(k) + 2$. Therefore $\widehat{\partial}_{d-\beta(k+1)} = 0$ and the spectral sequence vanishes at the $E_2$ page.

We illustrate the proof of Theorem 4.18 by Examples 5.4 and 5.5

**Lemma 4.19.** The following identity holds:

$$\beta(1 - k) + \beta(k) = d - 2.$$

**Proof.** By (1.1) and Lemma 4.4 we have

$$\beta(k) = h(k - 1, \ldots, k - 1) - h(k, \ldots, k) - 1, \quad \beta(1 - k) = h(-k, \ldots, -k) - h(1 - k, \ldots, 1 - k) - 1.$$

By Lemma 4.3 we have

$$h(-k, \ldots, -k) = h(k, \ldots, k) + kd, \quad h(1 - k, \ldots, 1 - k) = h(k - 1, \ldots, k - 1) + d(k - 1).$$

These two identities imply the desired statement.
**Theorem 4.20.** If \( \beta(k) + \beta(k+1) \geq d - 2 \) then:

\[
\hat{\text{HFL}}(K_{dm,dn}, k, \ldots, k) \simeq \bigoplus_{i=0}^{d-2-\beta(k+1)} \binom{d-1}{i} \mathbb{F}_{-2h(k)-i} \oplus \bigoplus_{i=0}^{d-2-\beta(k)} \binom{d-1}{i} \mathbb{F}_{-2h(k)+2-d+i}
\]

**Proof.** By Lemma 4.19 we get \( \beta(-k) = d - 2 - \beta(k+1) \) and \( \beta(1-k) = d - 2 - \beta(k) \), so

\[
\beta(k) + \beta(k+1) + \beta(-k) + \beta(1-k) = 2(d - 2),
\]

so \( \beta(-k)+\beta(1-k) \leq d-2 \). By Theorem 4.18 the spectral sequence degenerates for \( \hat{\text{HFL}}(-k, \ldots, -k) \) and

\[
\hat{\text{HFL}}(K_{dm,dn}, -k, \ldots, -k) \simeq \bigoplus_{i=0}^{d-2-\beta(k+1)} \binom{d-1}{i} \mathbb{F}_{-2h(-k)-i} \oplus \bigoplus_{i=0}^{d-2-\beta(k)} \binom{d-1}{i} \mathbb{F}_{-2h(-k)+2-d+i}
\]

Finally, by [OS08, Proposition 8.2] we have

\[
\hat{\text{HFL}}_\ast(K_{dm,dn}, k, \ldots, k) = \hat{\text{HFL}}_{-2kd}(K_{dm,dn}, -k, \ldots, -k)
\]

and by Lemma 4.3 \( h(k) = h(-k) - kd \). \( \square \)

**Theorem 4.21.** Off-diagonal homology groups are supported on the union of the unit cubes along the diagonal. In such a cube with corners \((k, \ldots, k)\) and \((k+1, \ldots, k+1)\) one has

\[
\hat{\text{HFL}}(K_{dm,dn}, (k-1)j, l^{d-j}) \simeq \binom{d-2}{\beta(k)} \mathbb{F}_{-2h(k)-\beta(k)-j}.
\]

**Proof.** We use the spectral sequence from \( \text{HFL}^- \) to \( \hat{\text{HFL}} \). By Theorem 4.16, all the \( \hat{E}_2 \) homology outside the union of these cubes vanish (since some \( U_i \) would provide an isomorphism between \( \text{HFL}^-(K_{dm,dn}, v) \) and \( \text{HFL}^-(K_{dm,dn}, v - e_i) \)). Furthermore, if \( \beta(k) = d - 1 \) then the homology in the cube vanish too, so we can focus on the case \( \beta(k) \leq d - 2 \).

One can check that \( \hat{E}_2 \) does not vanish in cube degrees \( j - \beta(k), \ldots, j \) and

\[
\hat{E}_2^{j-c} \simeq \binom{j-1}{c} \binom{d-1-j}{\beta(k)-c} \mathbb{F}_{-2h(k)-\beta(k)-c}.
\]

Note that the total homological degree on \( \hat{E}_2^{j-c} \) equals \( -2h(k) - \beta(k) - j \) and does not depend on \( c \). Therefore all higher differentials in the spectral sequence must vanish and the rank of \( \hat{\text{HFL}} \) equals:

\[
\sum_{c=0}^{\beta} \binom{j-1}{c} \binom{d-1-j}{\beta(k)-c} = \binom{d-2}{\beta(k)}.
\]

\( \square \)

We illustrate this proof by Example 5.6.

4.5. **Special case:** \( m = 1, n = 2g(K) - 1 \). The case \( m = 1, n = 2g(K) - 1 \) is special since Lemma 4.5 is not always true. Indeed, \( K_{m,n} = K \) and \( l = n = 2g(K) - 1 \), but for \( v = g(K) - l = 1 - g(K) \) we have \( \text{HFL}^-(K, v) = 0 \). However, it is clear that in all other cases Lemma 4.5 is true, so for generic \( v \) Lemmas 4.7 and 4.9 hold true. This allows one to prove an analogue of Theorem 5.

**Theorem 4.22.** Assume that \( m = 1, n = 2g(K) - 1 \) (so \( l = 2g(K) - 1 \)) and suppose that \( v = (u_1^\lambda, u_2^\lambda, \ldots, u_s^\lambda) \) where \( u_1 < \ldots < u_s \). Then the Heegaard-Floer homology group \( \text{HFL}^-(K_{dm,dn}, v) \) can be described as following:
(a) Assume that \( u_s - c + l(d - \lambda_s) = g(K) - \nu l \) with \( 1 \leq \nu \leq \lambda_s \). Then

\[
HFL^-(K_{dm,dn}, v) \simeq (\mathbb{F}(0) \oplus \mathbb{F}(-1))^{d-\lambda_s} \otimes \left[ \bigoplus_{j=0}^{\nu-2} \binom{\lambda_s - 1}{j} \mathbb{F}(-2h(\nu-j)) \oplus \binom{\lambda_s - 1}{\nu} \mathbb{F}(-2h(\nu+2-\nu)) \right]
\]

(b) In all other cases, the homology is given by Theorem 5.

Proof. One can check that the proof of Lemma 4.7 fails if \( u_s - c + l(d - \lambda_s) = g(K) - l \), and remains true in all other cases. Similarly, the proof of Lemma 4.9 fails only if \( u_s - c + l(d - \lambda_s) + lj = g(K) - l \) for \( 1 \leq j \leq \lambda_s - 1 \), which is equivalent to \( u_s - c + l(d - \lambda_s) = g(K) - (j + 1) \). This proves (b).

Let us consider the special case (a). Note that

\[
h_{m,n}(u_s - c + l(d - \lambda_s) + lj - 1) - h_{m,n}(u_s - c + l(d - \lambda_s) + lj) =
\]

\[
\chi(HFK^-(K, g(K) + l(j - \nu))) = \begin{cases} 1, & \text{if } j < \nu - 1 \\ 0, & \text{if } j = \nu - 1 \\ 1, & \text{if } j = \nu \\ 0, & \text{if } j > \nu. \end{cases}
\]

Given a pair of subsets \( B' \subset \{1, \ldots, d - \lambda_s \} \) and \( B'' \subset \{d - \lambda_s + 1, \ldots, d \} \), one can write, analogously to Lemma 4.9:

\[
h_{dm,dn}(v - e_{B'} - e_{B''}) = h_{dm,dn}(v) + |B'| + w(B''),
\]

where

\[
w(B'') = \begin{cases} |B''|, & \text{if } |B''| \leq \nu - 1 \\ \nu - 1, & \text{if } |B''| = \nu \\ \nu, & \text{if } |B''| > \nu. \end{cases}
\]

By the Künneth formula, the \( E_2 \) page of the spectral sequence is determined by the “deformed cube homology” with the weight function \( w(B'') \), as in (4.5). If \( \partial \), as above, denotes the standard cube differential, then, similarly to Lemma 4.13, the homology of \( \partial''_{\nu} \) is isomorphic to the kernel of \( \partial \) in cube degrees \( 0, \ldots \nu - 2 \) and \( \nu \).

Finally, we need to prove that all higher differentials vanish. For a homology generator \( \alpha \) on the \( E_2 \) page of cube degree \( x \), its bidegree is equal either to \( (x, -2h(\nu) - 2x) \) or to \( (x, -2h(\nu) - 2x + 2) \). The differential \( \partial_k \) has bidegree \( (-k, k - 1) \) (see Remark 3.6), so the bidegree of \( \partial_k(\alpha) \) is equal either to \( (x - k, -2h(\nu) - 2x + k - 1) \) or to \( (x - k, -2h(\nu) - 2x + k + 1) \). Since \( -2x + k + 1 < -2(x - k) \) for \( k > 1 \), we have \( \partial_k(\alpha) = 0 \). \qed

The action of \( U_i \) in this special case can be described similarly to Theorem 4.16. However, it is not true that \( U_i \) is surjective whenever it does not obviously vanish. In particular, the following example shows that \( HFL^- \) may be not generated by diagonal classes, so Theorem 6 does not hold.

We leave the appropriate adjustment of Theorem 6 as an exercise to a reader.

**Example 4.23.** Consider \( T_{2,2} \), the \((2,2)\) cable of the trefoil. We have \( g(K) = l = 1 \) and \( c = 1/2 \), so by Theorem 4.22

\[
HFL^-(T_{2,2}, 1/2, 1/2) \simeq \mathbb{F}(-1), \quad HFL^-(T_{2,2}, -1/2, 1/2) \simeq \mathbb{F}(-2) \oplus \mathbb{F}(-3).
\]

Therefore \( U_1 \) is not surjective. Furthermore, the class in \( HFL^- (T_{2,2}, -1/2, 1/2) \) of homological degree \((-2)\) is not in the image of any diagonal class under the \( R \)-action.
5. Examples

5.1. \((n, n)\) torus links. The symmetrized multi-variable Alexander polynomial of the \((n, n)\) torus link equals (for \(n > 1\)):

\[
\Delta_{T_{n,n}}(t_1, \ldots, t_n) = ((t_1 \cdots t_n)^{1/2} - (t_1 \cdots t_n)^{-1/2})^{n-2}.
\]

Each pair of components has linking number 1, so \(c = (n-1)/2\). The homology groups \(HFL^-(T(n,n), v)\) are described by following theorem, which is a special case of Theorem 5.

**Theorem 5.1.** Consider the \((n, n)\) torus link, and an Alexander grading \(v = (v_1, \ldots, v_n)\). Suppose that among the coordinates \(v_i\) exactly \(\lambda\) are equal to \(k\) and all other coordinates are less than \(k\). Let \(|v| = v_1 + \ldots + v_n\). Then

\[
HFL^-(T(n,n), v) = \begin{cases} 0 & \text{if } k > \lambda - \frac{n+1}{2}, \\ \langle \mathbb{F}(0) \oplus \mathbb{F}(-1) \rangle^{n-\lambda} \otimes \bigoplus_{i=0}^{\lambda - \frac{n+1}{2}} \mathbb{F}_{k-i} & \text{if } k = \lambda - \frac{n+1}{2}, \\ \langle \mathbb{F}(0) \oplus \mathbb{F}(-1) \rangle^{n-\lambda} \otimes \bigoplus_{i=0}^{\lambda - \frac{n+1}{2}} \mathbb{F}_{k-i} & \text{if } -\frac{n-1}{2} \leq k \leq \lambda - \frac{n+1}{2}, \end{cases}
\]

where \(h(v) = \frac{1}{2}(\frac{n+1}{2} - k)(\frac{n+1}{2} - k + 1) + kn - |v|\) in the last case.

**Proof.** Indeed, \(\beta(k) = \frac{n-1}{2} - k\) for \(k > -\frac{n-1}{2}\) and \(\beta(k) = n-1\) for \(k \leq -\frac{n-1}{2}\). By Theorem 5, the homology group \(HFL^-(T(n,n), v)\) does not vanish if and only if

\[
(5.1) \quad k \leq \lambda - \frac{n+1}{2}.
\]

If \(k \geq -\frac{n+1}{2}\), equation (4.3) implies:

\[
h_{n,n}(v) = \frac{1}{2} \left( \frac{n-1}{2} - k \right) \left( \frac{n-1}{2} - k + 1 \right) + kn - |v|.
\]

If \(k \leq -\frac{n-1}{2}\), equation (4.3) implies \(h_{n,n}(v) = -|v|\). Furthermore, for all \(v\) satisfying (5.1) one has

\[
HFL^-(T(n,n), v) = \langle \mathbb{F}(0) \oplus \mathbb{F}(-1) \rangle^{n-\lambda} \otimes \bigoplus_{j=0}^{\lambda - n+1} \mathbb{F}_{-2h_{n,n}(v)-j}.
\]

Finally, if \(k = -\frac{n+1}{2}\), then (5.1) holds for all \(\lambda\) and \(\lambda - \frac{n+1}{2} - k > \lambda - 1\), hence

\[
HFL^-(T(n,n), v) = \langle \mathbb{F}(0) \oplus \mathbb{F}(-1) \rangle^{n-\lambda} \otimes \bigoplus_{j=0}^{\lambda - n+1} \mathbb{F}_{-2h_{n,n}(v)-j} = \langle \mathbb{F}(0) \oplus \mathbb{F}(-1) \rangle^{n-1} \otimes \mathbb{F}_{-2h_{n,n}(v)}.
\]

**Remark 5.2.** One can check that, in agreement with [GN14], the condition (5.1) defines the multi-dimensional semigroup of the plane curve singularity \(x^n = y^n\).

**Corollary 5.3.** We have the following decomposition of \(HFL^-\) as an \(R\)-module:

\[
HFL^-(T(n,n)) = M_0 \oplus M_1 \oplus M_2 \oplus \ldots \oplus M_{n-2} \oplus M_{n-1, +\infty}.
\]

To prove Theorem 4, we use Theorem 3.

**Proof of Theorem 4.** We have \(\beta\left(\frac{n-1}{2} - s\right) = s\) for \(s < n-1\), and

\[
\beta\left(\frac{n-1}{2} - s\right) + \beta\left(\frac{n-1}{2} - s + 1\right) = 2s - 1 \leq n - 2 \leq s \leq \frac{n-1}{2}.
\]
Therefore for \( s \leq \frac{n-1}{2} \) Theorem 4.18 implies the degeneration of the spectral sequence from \( \widehat{\text{HFL}}^{-} \) to \( \widehat{\text{HFL}} \), and
\[
\widehat{\text{HFL}} \left( T(n, n), \frac{n-1}{2} - s, \ldots, \frac{n-1}{2} - s \right) = \bigoplus_{i=0}^{s} \binom{n-1}{i} \mathbb{F}_{(-s^2-s-i)} \oplus \bigoplus_{i=0}^{s-1} \binom{n-1}{i} \mathbb{F}_{(-s^2-s-n+2+i)}.
\]

Let us illustrate the degeneration of the spectral sequence from \( \text{HFL}^{-} \) to \( \widehat{\text{HFL}} \) in some examples.

**Example 5.4.** For \( s = 0 \) we have \( \widehat{E}_{1} = \widehat{E}_{2} = \mathbb{F}_{(0)} \). For \( s = 1 \) the \( \widehat{E}_{1} \) page has nonzero entries in cube degree 0 where one gets
\[
\text{HFL}^{-} \left( T(n, n), \frac{n-1}{2} - 1, \ldots, \frac{n-1}{2} - 1 \right) \simeq \mathbb{F}_{(-2)} \oplus (n-1)\mathbb{F}_{(-3)},
\]
and in cube degree \( n \) where one gets \( \mathbb{F}_{(0)} \). Indeed, the differential \( \widehat{\partial}_{1} \) vanishes, so for \( n > 2 \)
\[
\widehat{\text{HFL}} \left( T(n, n), \frac{n-1}{2} - 1, \ldots, \frac{n-1}{2} - 1 \right) \simeq \mathbb{F}_{(-2)} \oplus (n-1)\mathbb{F}_{(-3)} \oplus \mathbb{F}_{(-n)}.
\]
Note that for \( n = 2 \) the differential \( \widehat{\partial}_{2} \) does not vanish, so the bound \( s \leq \frac{n-1}{2} \) is indeed necessary for the spectral sequence to collapse at \( \widehat{E}_{2} \) page.

**Example 5.5.** The case \( s = 2 \) is more interesting. The \( \widehat{E}_{1} \) page has nonzero entries in cube degree 0, \( n - 1 \) (where we have \( n \) vertices) and \( n \), where one has
\[
\widehat{E}_{1}^{0} = \mathbb{F}_{(-6)} \oplus (n-1)\mathbb{F}_{(-7)} \oplus \left( \frac{n-1}{2} \right)\mathbb{F}_{(-8)}, \quad \widehat{E}_{1}^{n-1} = n(\mathbb{F}_{(-4)} \oplus \mathbb{F}_{(-5)}) \quad \widehat{E}_{1}^{n} = \mathbb{F}_{(-2)} \oplus (n-1)\mathbb{F}_{(-3)}.
\]
The differential \( \widehat{\partial}_{1} \) cancels some summands in \( E_{1}^{n-1} \) and \( \widehat{E}_{1}^{n} \);
\[
\widehat{E}_{2}^{0} = \mathbb{F}_{(-6)} \oplus (n-1)\mathbb{F}_{(-7)} \oplus \left( \frac{n-1}{2} \right)\mathbb{F}_{(-8)}, \quad \widehat{E}_{2}^{n-1} = (n-1)\mathbb{F}_{(-4)} + \mathbb{F}_{(-5)}.
\]
For \( n > 4 \) all higher differentials vanish and
\[
(5.2) \quad \widehat{\text{HFL}} \left( T(n, n), \frac{n-1}{2} - 2, \ldots, \frac{n-1}{2} - 2 \right) \simeq \mathbb{F}_{(-6)} \oplus (n-1)\mathbb{F}_{(-7)} \oplus \left( \frac{n-1}{2} \right)\mathbb{F}_{(-8)} \oplus (n-1)\mathbb{F}_{(-3-n)} + \mathbb{F}_{(-4-n)}.
\]
The following example illustrates the computation of \( \widehat{\text{HFL}} \) for the off-diagonal Alexander gradings.

**Example 5.6.** Let us compute the homology \( \widehat{\text{HFL}}(T(n, n), v) \) for \( v = (\frac{n-1}{2} - 2)^{j}(\frac{n-1}{2} - 1)^{n-j} \) \( 1 \leq j \leq n - 1 \) using the spectral sequence from \( \text{HFL}^{-} \). In the \( n \) dimensional cube \( (v + e_{B}) \) almost all all vertices have vanishing \( \text{HFL}^{-} \), except for the vertex \( (\frac{n-1}{2} - 1, \ldots, \frac{n-1}{2} - 1) \)
\[
\text{HFL}^{-} \left( \frac{n-1}{2} - 1, \ldots, \frac{n-1}{2} - 1 \right) = \mathbb{F}_{(-2)} \oplus (n-1)\mathbb{F}_{(-3)}
\]
and \( j \) of its neighbors with homology \( \mathbb{F}_{(-4)} \oplus \mathbb{F}_{(-5)} \). Clearly, \( \widehat{E}_{2} \) is concentrated in degrees \( j \) (with homology \( (n-1-j)\mathbb{F}_{(-3-j)} \)) and \( (j-1) \) (with homology \( (j-1)\mathbb{F}_{(-4)} \)). Note that both parts contribute to the total degree \( (-3 - j) \), so
\[
\widehat{\text{HFL}}(T(n, n), v) = (n - 1 - j)\mathbb{F}_{(-3-j)} \oplus (j - 1)\mathbb{F}_{(-3-j)} = (n - 2)\mathbb{F}_{(-3-j)}.
\]
Finally, we draw all the homology groups $HFL^-$ for $(2,2)$ and $(3,3)$ torus links.

**Example 5.7.** For the Hopf link, one has two cases. If $v_1 < v_2$, then the condition (5.1) implies $v_2 \leq -1/2$. If $v_1 = v_2$, then (5.1) implies $v_2 \geq 1/2$.

The nonzero homology of the Hopf link is shown in Figure 1 and Table 1.

![Figure 1. HFL$^-$ for the (2,2) torus link: $F^2$ on thick lines and in the grey region.](image)

| Alexander grading | Homology                                      |
|-------------------|-----------------------------------------------|
| $(1/2, 1/2)$      | $F_{(0)}$                                    |
| $(a, b)$, $a, b \leq -1/2$ | $F_{(2a+2b)} \oplus F_{(2a+2b-1)}$ |

*Table 1. Maslov gradings for the (2,2) torus link*

**Example 5.8.** For the $(3,3)$ torus link, one has two cases. If $v_1 \leq v_2 < v_3$, then the condition (5.1) implies $v_3 \leq 1$. If $v_1 < v_2 = v_3$, then (5.1) implies $v_3 \leq 0$. Finally, if $v_1 = v_2 = v_3$, then (5.1) implies $v_3 \leq 1$. In other words, nonzero homology appears at the point $(1,1,1)$, at three lines $(0,0,k), (0,k,0), (k,0,0)$ ($k \leq 0$) and at the octant $max(v_1, v_2, v_3) \leq -1$.

This homology is shown in Figure 2 and Table 2.

| Alexander grading | Homology                                      |
|-------------------|-----------------------------------------------|
| $(1,1,1)$         | $F_{(0)}$                                    |
| $(0,0,0)$         | $F_{(-2)} \oplus 2F_{(-3)}$                  |
| $(0,0,k)$, $(0,k,0)$ and $(k,0,0)$ ($k < 0$) | $F_{(2k-2)} \oplus F_{(2k-3)}$ |
| $(a,b,c)$, $a, b, c \leq -1$ | $F_{(2a+2b+2c)} \oplus 2F_{(2a+2b+2c-1)} \oplus F_{(2a+2b+2c-2)}$ |

*Table 2. Maslov gradings for the (3,3) torus link*
5.2. More general torus links. The HFL\(^{-}\) homology of the \((4, 6)\) torus link is shown in Figure 3 and Table 3. Note that as an \(\mathbb{F}[U_1, U_2]\) module it can be decomposed into 5 copies of \(M_0 \simeq \mathbb{F}\), a copy of \(M_{1,1}\) and a copy of \(M_{1,+\infty}\). In particular, the map \(U_1 U_2 : \text{HFL}^{-}(-2, -2) \to \text{HFL}^{-}(-3, -3)\) is surjective with one-dimensional kernel.

5.3. Non-algebraic example. In this subsection we compute the Heegaard-Floer homology for the \((4, 6)\)-cable of the trefoil. Its components are \((2, 3)\)-cables of the trefoil, which are known to be
Figure 3. HFL$^-$ for the (4,6) torus link: $F^2$ on thick lines and in the grey region

| Alexander grading | Homology                  |
|-------------------|---------------------------|
| $(4, 4)$          | $F(0)$                    |
| $(2, 2)$          | $F(-2)$                   |
| $(1, 1)$          | $F(-4)$                   |
| $(0, 0)$          | $F(-6)$                   |
| $(-1, -1)$        | $F(-8)$                   |
| $(-2, k)$ and $(k, -2)$, $k \leq -2$ | $F_{(2k-6)} \oplus F_{(2k-7)}$ |
| $(-3, -3)$        | $F_{(-12)}$               |
| $(a, b)$, $a, b \leq -4$ | $F_{(2a+2b)} \oplus F_{(2a+2b-1)}$ |

Table 3. Maslov gradings for the (4, 6) torus link

L-space knots (cf. [Hed09]), but not algebraic knots. By Theorem 2, the (4, 6)-cable of the trefoil is an L-space link, but its homology is not covered by [GN14].
The Alexander polynomial of the (2, 3)-cable of the trefoil equals:
\[
\Delta_{T_{2,3}}(t) = \frac{(t^{6} - t^{-6})(t^{1/2} - t^{-1/2})}{(t^{3/2} - t^{-3/2})(t^{2} - t^{-2})},
\]
hence the Euler characteristic of its Heegaard-Floer homology equals
\[
\chi_{2,3}(t) = \frac{\Delta_{T_{2,3}}(t)}{1 - t^{-1}} = t^{3} + 1 + t^{-1} + t^{-3} + t^{-4} + \ldots
\]
By (4.1), the bivariate Alexander polynomial of the (4, 6)-cable equals:
\[
\chi_{4,6}(t, t_{2}) = \chi_{2,3}(t_{1} \cdot t_{2})((t_{1}t_{2})^{3} - (t_{1}t_{2})^{-3})
= (t_{1}t_{2})^{6} + (t_{1}t_{2})^{3} + (t_{1}t_{2})^{2} + (t_{1}t_{2})^{-1} + (t_{1}t_{2})^{-2} + (t_{1}t_{2})^{-5}.
\]
The nonzero Heegaard-Floer homology are shown in Figure 4 and the corresponding Maslov gradings are given in Table 4. Note that as \(\mathbb{F}[U_{1}, U_{2}]\) module it can be decomposed in the following way:
\[
\text{HFL}^{-} \simeq 4M_{0} \oplus M_{1,1} \oplus M_{1,2} \oplus M_{1,1+\infty}.
\]

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| Alexander grading | Homology |
|-------------------|----------|
| (6, 6)            | $F(0)$   |
| (3, 3)            | $F(-2)$  |
| (2, 2)            | $F(-4)$  |
| $(0, k)$ and $(k, 0)$, $k \geq 0$ | $F_{(2k-6)} \oplus F_{(2k-7)}$ |
| ($-1$, $-1$)      | $F(-10)$ |
| ($-2$, $-2$)      | $F(-12)$ |
| ($-3, k$) and $(k, -3)$, $k \geq -3$ | $F_{(2k-8)} \oplus F_{(2k-9)}$ |
| ($-4, k$) and $(k, -4)$, $k \geq 10$ | $F_{(2k-10)} \oplus F_{(2k-11)}$ |
| ($-5, -5$)        | $F(-22)$ |
| $(a, b)$, $a, b \leq -6$ | $F_{(2a+2b)} \oplus F_{(2a+2b-1)}$ |

Table 4. Maslov gradings for the (4,6) cable of the trefoil
Figure 4. HFL$^{-}$ for the (4,6) cable of the trefoil: $F^2$ on thick lines and in the grey region.