1. Preliminaries

Motivated by prediction problems in a Poisson shot noise process, we consider two types of problems for random sums of iid random variables (r.v. or r.v.’s for short). Let $N$ be a non-negative integer-valued r.v. and denote an iid sequence of non-negative r.v.’s by $(X_i)_{i=1,2,...}$ so that $S_N = \sum_{i=1}^N X_i$ denotes the total sum. Our problem is how we could obtain the information of the number $N$ or each component $X_i$ when we only observe $S_N$. Although there are several methods for these quantities such as linear predictions $cS_N$ with $c$ some constant, our methods are those by conditional moments, which are minimizers of the mean square error. More precisely our focus is on the following two types of conditional moments:

$$E[N^k \mid S_N] \quad \text{and} \quad E[X_i^k \mid S_N], \quad \text{for } k \in \mathbb{N},$$

(1.1)

where $(X_i)$ may take both real and integer values and $\mathbb{N}$ denotes the set of natural numbers as usual.

This type of random sum $S_N$ has been studied for a long time and has applications in a variety of fields. One could find many examples in the book of Feller [5, XII] such as genetics, required service time, cosmic ray showers, and automobile accidents to name just a few. A large number of relevant researches have been conducted, including e.g. calculations for probability of $S_N$ (Sundt and Vernic [28]) or various limit theorems (see e.g. Gut [8] and consult a nice summary in Embrechts et al. [4, 2.5]). In recent years tail asymptotics have intensively studied, since accurate calculations of tail probabilities of $S_N$ are computationally quite expensive, while they are required in applications. See Jessen and Mikosch [9] for a survey with regularly varying tails and Goldie and Klüppelberg [6] for that with subexponential tails.

In this paper we do not go further into asymptotics but investigate precise calculations of quantities (1.1), which have not been studied yet except for some special cases. A motivating example is prediction in the Poisson shot noise process of the form

$$M(t) = \sum_{i=1}^{N(t)} L_i(t - T_i), \quad t > 0,$$

(1.2)
where \(0 < T_1 < T_2 < \cdots\) are points of a homogeneous Poisson \(N(t)\) with intensity \(\lambda > 0\) and \((L_i)\) is a sequence of iid Lévy processes independent of \((T_i)\) and such that \(L_{t_i} = 0\) a.s. \(t \leq 0\). The process of this type has many applications in rather different areas (see [2], [29], [21] and [15]). One of important research topics is the prediction of future increments \(M(t, t+s) := M(t+s) - M(t), s, t > 0\) based on the present observation \(M(t)\). For example, in non-life insurance \(M(t, t+s)\) is interpreted as the number or amount of future payment in the interval \((t, t+s)\) from an insurance company to the insured. Another interpretation is that \(M(t, t+s)\) may describe the workload to be managed by a large computer network for sources in the interval \((t, t+s)\). Due to the properties of both Lévy and Poisson processes, the prediction of future increments \(M(t, t+s)\) given \(M(t)\) reduces to

\[
E[M(t, t+s) \mid M(t)] = E[N(s)]E[L_1(t+s-U)] + E[L_1(s)]E[N(t) \mid M(t)],
\]

where \(U\) is a uniform r.v. on \((0, t)\) denoted by \(U(0, t)\) independent of \((L_i)\). The proof of (1.3) is given in Appendix [A] or [16] (2.1). Here computations of \(E[N(s)], E[L_1(s)]\) and \(E[L_1(t+s-U)]\) are trivial. Since points \((T_i)\) of Poisson have the order statistic property, we can regard the sequence \((T_i)\) in the quantity \(E[N(t) \mid M(t)]\) as that of iid \(U(0, t)\) r.v.'s. Accordingly, taking \(X_i := L_i(t-T_i)\) and \(N := N(t)\) of \(M(t)\), we obtain the form (1.1). Similarly, higher conditional moments \(E[M(t, t+s)^k \mid M(t)]\), \(k \in \mathbb{N}\) are obtained as functions of \(E[N^k(t) \mid M(t)]\).

A series of papers [20], [10] and [16] assumes particular marginal distributions for \((L_i)\) such as Poisson or negative binomial, and exploits their specific properties to obtain the conditional moments. Although asymptotic behaviors of \(E[N \mid S_N = k], k \to \infty\) have been studied in [10] and [25], [26], only limited distributions are treated. Our methods presented here require no particular assumptions on \(N\) and \((X_i)\) and therefore could be applicable under more general settings than those of previous papers.

In our paper, we rely on two numerical methods, i.e. the Panjer recursion and the Fourier method, which are useful tools for computing \(P(S_N = n)\) and which are competitive (1.1). The Panjer recursion scheme originated in Panjer [22] is known to be stable when \(N\) belongs to the Panjer class in most cases (1.1). Meanwhile, the Fourier method could be applicable to general \(N\), though it requires accurate numerical integrals. In the present paper, we show that these methods are also useful for computing quantities in (1.1).

We say that the probability mass function \(q_n = P(N = n)\) belongs to the Panjer \((a, b)\) class if it satisfies

\[
q_n = \left(a + \frac{b}{n}\right)q_{n-1}, \quad n \in \mathbb{N}.
\]

for some \(a, b \in \mathbb{R}\). Poisson, negative binomial and binomial distributions belong to this class. For later use, we present the Panjer recursion formula (see [24], [20] for details and the proof).

**Theorem 1.1.** Suppose that \(N\) belongs to the Panjer \((a, b)\) class and denote an iid sequence of non-negative integer-valued r.v.'s by \((X_i)\). Then

\[
P(S_N = 0) = E[P(X_1 = 0)^N], \quad n = 0,
\]

\[
P(S_N = n) = \frac{1}{1 - aP(X_1 = 0)} \sum_{j=1}^{n} \left(a + \frac{b j}{n}\right)P(X_1 = j)P(S_N = n - j), \quad n \geq 1.
\]

Here we let \(0^0 = 1\) conventionally.
For the latter convenience, we define the following notations related with generating functions. For a fixed r.v. $X$ and a non-negative function $f$ and $|u| \leq 1$,

$$G_X(u) := E[u^X], \quad G_f(u) := \sum_{k=0}^{\infty} u^k f(k), \quad G_{df}(u) := \int u^x df(x),$$

where the last one is defined as a Riemann-Stieltjes integral if exists. From these quantities we can obtain the Fourier (-Stieltjes) transforms $\varphi_{\{u\}}(u) = G_{\{u\}}(e^{iu})$. Moreover, we write $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$ and $\mathbb{R}_+ := [0, \infty)$ in the sequel.

In the present paper, random sums of discrete r.v.’s are treated in Section 2, where computations of $E[N^k \mid S_N]$, $k \in \mathbb{N}$ are investigated in Subsection 2.1 and those of $E[X_1^k \mid S_N]$ are studied in Subsection 2.2. Both the recursion method and the Fourier method are investigated. Since we have difficulty in applying the recursion for $E[X_1^k \mid S_N]$, we additionally consider $E[X_1^k \mid S_{N+1}]$ to which the recursion is applicable. In Section 3, we consider random sums of non-negative continuous r.v.’s. Although we derive integral equations for quantities $E[N^k \mid S_N \leq x]$ for the Panjer class $N$, a direct application seems intractable. Alternatively, we resort mainly to the Fourier approach for computations of both $E[N^k \mid S_N \leq x]$ and $E[X_1^k \mid S_N \leq x]$. Finally in Section 4, numerical examples are given, which show that proposed methods work reasonably. As applications, we consider predictors for both the Poisson shot noise process and the compound mixed Poisson process.

2. Random sums of discrete random variables

2.1. Estimation of number of iid components. In this section calculations for conditional moments $E[N^k \mid S_N]$, $k \in \mathbb{N}$ will be investigated, where iid random components $(X_i)$ are integer-valued. In case $N$ belongs to the Panjer class, we apply the recursion formula to the calculation of $E[N^k \mid S_N]$. For general $N$, we consider the generating function of $E[N^k \mid S_N = \cdot]$ and then apply the Fourier inversion.

Throughout we denote the expectation of a r.v. $X$ over a measurable subset $A \subset \Omega$ by $E[X; A] = E[X1_{(X \in A)}]$. Since we obtain $P(S_N)$ by Theorem 1.1, we mainly consider $E[N^k; S_N]$, $k \in \mathbb{N}$ which yields $E[N^k \mid S_N] = E[N^k; S_N]/P(S_N)$.

**Theorem 2.1.** Let $N$ belong to the Panjer $(a, b)$ class and iid r.v.’s $(X_i)$ take values in $\mathbb{N}_0$. Let $C_0 := aP(X_1 = 0)$. Then, the restricted moments $m_k(\ell) := E[N^k; S_N = \ell]$, $k \in \mathbb{N}$, $\ell \in \mathbb{N}_0$ satisfy the recursion,

$$m_k(0) = E[N^kP(X_1 = 0)^N],$$

$$m_k(\ell) = \frac{1}{1 - C_0} \left\{ C_0 \sum_{j=0}^{k-1} \binom{k}{j} m_j(\ell) + \sum_{j=1}^{\ell} \binom{a+b}{\ell} P(X_1 = j) \sum_{i=0}^{k} \binom{k}{i} m_i(\ell - j) \right\}, \quad \ell \geq 1. \quad (2.1)$$

Since for the calculation of $m_k(\ell)$, a combination of $m_i(\ell)$, $i \leq k - 1$ and $m_i(j)$, $i \leq k$, $j \leq \ell - 1$ is sufficient, we can recursively calculate the quantity.

**Proof.** The iidness of $(X_i)$ and independence between $N$ and $(X_i)$ yield

$$m_k(0) = E[N^k; S_N = 0] = E[N^k E[I(S_N = 0) \mid N]] = E[N^k P(X_1 = 0)^N].$$

Next we consider $m_k(\ell)$, $\ell \geq 1$. Conditioning argument and the Panjer $(a, b)$ class assumption yield

$$m_k(\ell) = \sum_{i=1}^\infty \ell^i P(S_i = \ell) q_i = \sum_{i=1}^\infty P(S_i = \ell) \ell^i \left( a + \frac{b}{\ell} \right) q_{i-1} \quad (2.2)$$
where \( q_n = P(N = n), \ n \in \mathbb{N}_0 \). Since \((X_j)\) are iid r.v.'s
\[
\left( a + \frac{b}{\ell} \right) = a + b \frac{1}{\ell} \sum_{j=1}^{\ell} E\left[ \frac{X_j}{S_i} \mid S_i = \ell \right] = a + bE\left[ \frac{X_1}{S_i} \mid S_i = \ell \right] = E\left[ a + b \frac{X_1}{\ell} \mid S_i = \ell \right].
\]
Moreover,
\[
E\left[ a + \frac{bX_1}{\ell} \mid S_i = \ell \right] = \sum_{j=0}^{\ell} \left( a + \frac{bj}{\ell} \right) P(X_1 = j \mid S_i = \ell) = \sum_{j=0}^{\ell} \frac{a + bj}{\ell} P(X_1 = j) P(S_{i-1} = \ell - j) P(S_{i} = \ell).
\]
Substitution of this into \((a + b/\ell)\) of (2.2) and multiple interchanges of the order of summations give
\[
m_k(\ell) = \sum_{i=1}^{\infty} \sum_{j=0}^{\ell} \left( a + \frac{bj}{\ell} \right) P(X_1 = j) \left( \sum_{i=1}^{\infty} P(S_{i-1} = \ell - j) i^k q_{i-1} \right)
= \sum_{j=0}^{\ell} \left( a + \frac{bj}{\ell} \right) P(X_1 = j) \left( \sum_{i=1}^{\ell} P(S_i = \ell - j) \sum_{h=0}^{k} \binom{k}{h} i^h q_i \right)
= \sum_{j=0}^{\ell} \left( a + \frac{bj}{\ell} \right) P(X_1 = j) \sum_{h=0}^{k} \binom{k}{h} m_h(\ell - j)
= aP(X_1 = 0)m_k(\ell) + aP(X_1 = 0) \sum_{h=0}^{k-1} \binom{k}{h} m_h(\ell) + \sum_{j=1}^{\ell} \left( a + \frac{bj}{\ell} \right) P(X_1 = j) \sum_{h=0}^{k} \binom{k}{h} m_h(\ell - j).
\]
Thus we obtain the desired result. \(\square\)

If we take \(k = 1\) with \(\ell \geq 1\) in Theorem 2.1, a rather simple expression is obtained
\[
m_1(\ell) = \frac{1}{1 - C_0} \left( C_0 P(S_N = 0) + \sum_{j=1}^{\ell} \left( a + \frac{bj}{\ell} \right) P(X_1 = j) P(S_N = \ell - j) + m_1(\ell - j) \right),
\]
which together with Theorem 1.1 yields the conditional expectation.

Next we consider the generating function for \(m_k(\ell)\) with \(N\) a general r.v.

**Proposition 2.2.** Let \(N\) be a r.v. on \(\mathbb{N}_0\) and let \((X_i)\) be an iid sequence of r.v.'s on \(\mathbb{N}_0\). Assume \(EN^k < \infty, k \in \mathbb{N}\). Then the generating function of the truncated k-th moment \(m_k(\ell) = E[N^k; S_N = \ell]\) has the form
\[
G_{m_k}(u) = \sum_{j=1}^{k} \binom{k}{j} G^j_{X_1}(u) G^j_{N}(G_{X_1}(u)), \quad |u| \leq 1,
\]
where the quantities by braces \(\{\}\) denote the Striling number of the second kind\(^1\) (p.824), and \(G^j_{Y}(u), j \in \mathbb{N}\) denotes the j-th derivative of \(G_Y(x)\) at \(x = u\).

\(^1\) In the sequel, we use this notation for the Striling number of the second kind in without mentioning them.
Proof. A direct calculation yields

\[ G_{N^k}(u) = \sum_{\ell=0}^{\infty} u^\ell \mathbb{E}[N^k; S_N = \ell] = \mathbb{E}N^k \sum_{\ell=0}^{\infty} u^\ell \mathbb{P}(S_N = \ell \mid N) = \mathbb{E}N^k G_{X_1}^N(u), \]

where \( \mathbb{E}N^k < \infty \) assures Fubini’s theorem since \( |G_{X_1}(u)| \leq 1 \). We use the relation of the falling factorial \((x)_k = x(x-1) \cdots (x-k+1) \) and \( x^k \),

\[
(2.4) \sum_{j=1}^{k} \left\{ \begin{array}{l} k \\ j \end{array} \right\} (x)_j = x^k, \quad \left\{ \begin{array}{l} k \\ 0 \end{array} \right\} = 0, \quad k > 0,
\]

namely,

\[
\mathbb{E}[N^k G_{X_1}^N(u)] = \sum_{j=1}^{k} \left\{ \begin{array}{l} k \\ j \end{array} \right\} \mathbb{E}[(N)^j] G_{X_1}^N(u)] = \sum_{j=1}^{k} \left\{ \begin{array}{l} k \\ j \end{array} \right\} G_{X_1}^j(u) G_{N}^{(j)}(G_{X_1}(u)),
\]

where we change the order of derivatives and the summation, which is valid from \( \mathbb{E}N^k < \infty \) and \( |G_{X_1}(u)| \leq 1 \). \( \square \)

In order to obtain \( m_k(\ell) = \mathbb{E}[N^k; S_N = \ell] \) from \( G_{m_k} \), two methods are considered. One requires numerical integrations and the other needs derivatives of \( G_{m_k} \) at the origin. Since \( |G_{m_k}(e^{i\nu})|^2 \leq (\mathbb{E}N^k)^2 < \infty \), we have \( \mathbb{G}_{m_k}(e^{i\nu}) \in L^2(-\pi, \pi) \). Then the Fourier expansion of \( \mathbb{G}_{m_k}(e^{i\nu}) \) is guaranteed and their coefficients satisfy formula

\[
(2.5) \quad m_k(\ell) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\ell \nu} \mathbb{G}_{m_k}(e^{i\nu}) d\nu, \quad \ell \in \mathbb{N}_0,
\]

which correspond to the inversion of the Fourier transform \( \mathbb{G}_{m_k}(e^{i\nu}) \). On the other hand, if we take derivatives of \( G_{m_k} \) at the origin, we obtain \( m_k(\ell) = \frac{1}{i} \mathbb{G}_{m_k}(0) \). In view of (2.3), however, the calculation of \( \mathbb{G}_{m_k}(\ell) \) would yield additional complexities, though we may possibly find some efficient recursion methods for a limited class of \( N \). The choice of the two methods depends on distributional assumptions on \( N \) and \( X_1 \) and we need numerical experiments to judge which is better.

2.2. Estimation of magnitude of each iid component. In this subsection we consider the expected magnitude of r.v. \( X_1^k, k \in \mathbb{N} \) under the observation of the total number \( S_N \). Since the conditional moments minimize mean squared errors, we will consider \( \chi_k = \mathbb{E}[X_1^k \mid S_N], k \in \mathbb{N} \). Since the direct application of the Panjer recursion seems difficult for \( \chi_k \) and easy for \( \chi_{k^+} \) := \( \mathbb{E}[X_1^k \mid S_{N+1}] \), we derive the recursion only for \( \chi_{k^+} \). Meanwhile, the Fourier approach is applied to both.

**Theorem 2.3.** Let \( N \) be a Panjer \((a, b)\) class distribution and let \((X_i)\) be a sequence of iid r.v.’s on \( \mathbb{N}_0 \). Assume \( \mathbb{E}X_i^k < \infty, k \in \mathbb{N} \), then the truncated k-th moment \( \chi_{k^+}(\ell) = \mathbb{E}[X_1^k ; S_{N+1} = \ell], \ell \in \mathbb{N} \) has the form

\[
\chi_{k^+}(1) = P(X_1 = 1) \mathbb{E}[P(X_1 = 0)^N], \quad \text{and} \quad \text{for } \ell \geq 2,
\]

\[
\chi_{k^+}(\ell) = \ell^k P(X_1 = \ell) P(S_N = 0) + \frac{1}{1 - aP(X_1 = 0)} \sum_{j=1}^{\ell-1} P(X_1 = j) \left\{ a\chi_{k^+}(\ell - j) + \frac{b^j}{\ell - j} \chi_{1^+}(\ell - j) \right\}.
\]

**Proof.** For \( \ell = 1 \), due to the iidness of \((X_i)\),

\[
\chi_{k^+}(1) = \mathbb{E}[X_1^k ; S_{N+1} = 1] = P(X_1 = 1) \mathbb{E}[P(X_1 = 0)^N].
\]
Let $C_1 := 1/(1-aP(X_1 = 0))$. For $\ell \geq 2$, the property of $N$ yields
\[
\chi_{k+}(\ell) = E[X_1^k; S_{N+1} = \ell] = \sum_{j=1}^{\ell} j^k P(X_1 = j) P(S_N = \ell - j) = \ell^k P(X_1 = \ell) P(S_N = 0) + \sum_{j=1}^{\ell-1} j^k P(X_1 = j) C_1 \sum_{m=1}^{\ell-j} (a + \frac{bm}{\ell-j}) P(X_1 = m) P(S_N = \ell - j - m)
\]
\[
= \ell^k P(X_1 = \ell) P(S_N = 0) + C_1 \left\{ a \sum_{m=1}^{\ell-1} j^k P(X_1 = j) P(X_1 = m) P(S_N = \ell - j - m) + b \sum_{j=1}^{\ell-1} \frac{j^k}{\ell-j} P(X_1 = j) \sum_{m=1}^{\ell-j} m P(X_1 = m) P(S_N = \ell - j - m) \right\}
\]
\[
= \ell^k P(X_1 = \ell) P(S_N = 0) + C_1 \left\{ a \sum_{m=1}^{\ell-1} P(X_1 = m) E[X_1^k; S_{N+1} = \ell - m] + b \sum_{j=1}^{\ell-1} \frac{j^k}{\ell-j} P(X_1 = j) E[X_1^k; S_{N+1} = \ell - j] \right\},
\]
where in the third step, we use the Panjer recursion for $P(S_N = \ell - j)$. Finally, we arrange two sums and obtain the result. 

For the calculation of $\chi_k = E[X_1^k | S_N]$ a direct application of the Panjer recursion seems difficult and alternatively we try the Fourier methods. For this we need the generating function of $\chi_k$. 

**Proposition 2.4.** Let $N$ be a r.v. on $\mathbb{N}_0$ and let $(X_i)$ be an iid sequence of r.v.'s on $\mathbb{N}_0$. Assume $EX_1^k < \infty$, $k \in \mathbb{N}$, then the generating function of the truncated $k$-th moment $\chi_k(\cdot) = E[X_1^k; S_N = \cdot]$ has the form

\[
G_{\chi_k}(u) = \sum_{j=1}^{k} \left\{ \begin{array}{c} k \\ j \end{array} \right\} u^j G_{X_1}^{(j)}(u) \frac{G_{S_N}(u)}{G_{X_1}(u)}, \quad |u| \leq 1.
\]

**Proof.** In view of
\[
\chi_k(\ell) = E[X_1^k; S_N = \ell] = \sum_{j=1}^{\ell} j^k P(X_1 = j) P(S_{N-1} = \ell - j),
\]
the function $\chi_k$ is the convolution of two non-negative functions $g_1(j) := j^k P(X_1 = j)$ and $g_2(j) := P(S_{N-1} = j)$. Since $G_{S_{N-1}}(u) = \sum_{j=1}^{\ell} u^j P(S_{N-1} = j) = E[G_{X_1}^{N-1}(u)] = G_{S_N}(u)/G_{X_1}(u)$, the generating function of $g_1(j)$ is enough. We use the relation of the falling factorial (2.4) and obtain
\[
E[X_1^k u^{X_1}] = \sum_{j=1}^{k} \left\{ \begin{array}{c} k \\ j \end{array} \right\} E[(X_1)_j u^{X_1}] = \sum_{j=1}^{k} \left\{ \begin{array}{c} k \\ j \end{array} \right\} u^j G_{X_1}^{(j)}(u),
\]
where we apply Fubini’s theorem, which is possible by $EX_1^k < \infty$. Now the product of $G_{S_{N-1}}$ and $E[X_1^k u^{X_1}]$ yields the result. 

\[\square\]
Similarly as before, two methods are considered to obtain $\chi_k(\cdot)$ from $G_{\chi_k}$. One is to use derivatives at the origin, $\chi_k(\ell) = G_{\chi_k}^{(\ell)}(0)/\ell!$, $\ell \geq 1$. The other is the inversion of generating function

$$
\chi_k(\ell) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\ell u} G_{\chi_k}(e^{iu}) \, du, \quad \ell \in N_0.
$$

In view of (2.6), the former method requires some efficient algorithm for calculating derivatives of $G_{\chi_k}$, whereas for the second one, accurate numerical integrations are inevitable.

3. Random sums of continuous random variables

In this section, we assume continuous distributions for an iid random sequence $(X_i)$ taking values on $\mathbb{R}_+$, while keeping $N$ to be r.v. on $\mathbb{N}_0$. Similarly as before we consider $E[N^k \mid S_N]$ and $E[X_1^k \mid S_N]$, $k \in \mathbb{N}$. Here the Fourier Stieltjes transform (FST for short) is our main tool.

3.1. Estimation of random number from random sum. We firstly consider $E[N^k \mid S_N \in [0, x]] = E[N^k \mid S_N \leq x]$ for $x \in \mathbb{R}_+$ and $k \in \mathbb{N}$. We are starting to observe the integral equation as in [24, Sec. 4.4.3], which corresponds to the recursion formula when $X_1$ is a discrete distribution.

**Theorem 3.1.** Let $N$ be a Panjer $(a, b)$ class distribution and assume iid r.v.’s $(X_i)$ take values on $\mathbb{R}_+$ with common distribution $F_{X_1}$. Then the restricted $k$-th moment to the Borel set by $[S_N \leq x]$, $m_k(x) = E[N^k; S_N \leq x]$, $k \in \mathbb{N}$ satisfies the integral equation,

$$
m_k(x) = a(m_k \ast F_{X_1})(x) + \sum_{j=0}^{k-1} \left\{ a\left(\frac{k}{j}\right) + b\left(\frac{k-1}{j}\right) \right\}(m_j \ast F_{X_1})(x), \quad x \geq 0,
$$

where the operation $\ast$ denotes the convolution as usual.

**Proof.** Since $N$ belongs to the Panjer $(a, b)$ class, we can write

$$
m_k(x) = \sum_{n=0}^{\infty} n^k q_n F_{X_1}^{*(n)}(x) = a \sum_{n=0}^{\infty} (n + 1)^k q_n F_{X_1}^{*(n+1)}(x) + b \sum_{n=0}^{\infty} (n + 1)^{k-1} q_n F_{X_1}^{*(n+1)}(x),
$$

where $F_{X_1}^{*(n)}(x)$ denotes the distribution of the $n$-th convolution of $X_1$. Using the binomial expansion and changing the order of summations, we obtain

$$
m_k(x) = a \sum_{n=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j} n^j q_n F_{X_1}^{*(n+1)}(x) + b \sum_{n=0}^{\infty} \sum_{j=0}^{k-1} \binom{k-1}{j} n^j q_n F_{X_1}^{*(n+1)}(x)
$$

$$
= a \sum_{n=0}^{\infty} n^k q_n F_{X_1}^{*(n+1)} + \sum_{n=0}^{\infty} \left\{ a \sum_{j=1}^{k-1} \binom{k}{j} n^j q_n F_{X_1}^{*(n+1)}(x) + b \sum_{j=0}^{k-1} \binom{k-1}{j} n^j q_n F_{X_1}^{*(n+1)}(x) \right\}
$$

$$
= a(m_k \ast F_{X_1})(x) + \sum_{j=0}^{k-1} \left\{ a\left(\frac{k}{j}\right) + b\left(\frac{k-1}{j}\right) \right\} \sum_{n=0}^{\infty} n^j q_n F_{X_1}^{*(n+1)}(x).
$$

}\)

In view of expression (3.1), the integral equation seems useless to obtain $m_k(x)$ and we need additional techniques such as discretization of the density function of $X_1$ as in [24, Example (p.123)]. However, it is helpful to obtain the generating function of $m_k$ by providing an efficient recursion.
Lemma 3.2. Assume that \( N \) belongs to the Panjer \((a, b)\) class and iid r.v.’s \((X_i)\) take values on \( \mathbb{R}_+ \) with common ch.f. \( \phi_{X_1} \). Then the FST of \( m_k(x) = E[N^k; S_N \leq x] \), \( k \in \mathbb{N} \) has the following form

\[
\phi_{m_k}(u) = \frac{1}{1 - a\phi_{X_1}(u)} \sum_{j=0}^{k-1} \left\{ a \binom{k}{j} + b \binom{k-1}{j} \right\} \phi_{m_j}(u) \phi_{X_1}(u), \quad u \in \mathbb{R}.
\]

The proof of Lemma is a straightforward calculation and we omit it. Notice that due to Lemma 3.2, \( \phi_{m_k}(u) \) can be presented by a combination of \( G_N(\phi_{X_1}(u)) \) and \( \phi_{X_1}(u) \) since \( \phi_{m_0}(u) = E[e^{iuS_N}] = G_N(\phi_{X_1}(u)) \). For a general \( N \), we directly calculate the FST of \( m_k \).

Proposition 3.3. Let \( N \) be a r.v. on \( \mathbb{N}_0 \) and let \((X_j)\) be an iid sequence of r.v.’s on \( \mathbb{R}_+ \). Assume \( EN^k < \infty \) for \( k \in \mathbb{N} \), then the FST of \( m_k(x) := E[N^k; S_N \leq x] \) has the form

\[
\int_0^\infty e^{iux} dm_k(x) = E[N^k \phi_{X_1}(u)] = \sum_{j=1}^k \left\{ k \binom{k}{j} \phi_{X_1}^{(j)}(u) \phi_{N}^{(j)}(\phi_{X_1}(u)),
\]

where the left integral exists in the sense of the improper Riemann-Stieltjes integral.

Proof. Observe that \( m_k(x) \) is a bounded non-decreasing function and \( e^{iux} \) is continuous for every \( u \in \mathbb{R} \), then a Riemann-Stieltjes integral \( \int_0^M e^{iux} dm_k(x) \) exists for all \( M > 0 \) \([\text{30}, (2.24) \text{Theorem}]\). Moreover, integration by parts (twice) and Fubini’s theorem yield

\[
\int_0^M e^{iux} dm_k(x) = [e^{iux} m_k(x)]_0^M - \int_0^M iue^{iux} m_k(x) dx
\]

\[
= [e^{iux} m_k(x)]_0^M - E \left[ N^k \int_0^M iue^{iux} P(S_N \leq x \mid N) dx \right]
\]

\[
= [e^{iux} m_k(x)]_0^M - E \left[ N^k \{ e^{iux} P(S_N \leq x \mid N) \}_0^M - \int_0^M e^{iux} dP(S_N \leq x \mid N) \right]
\]

\[
= E \left[ N^k \int_0^M e^{iux} dP(S_N \leq x \mid N) \right],
\]

where in the third step, we use \( m_k(x) := E[N^k P(S_N \leq x \mid N)] \) for all \( x \geq 0 \). To obtain the first equality of (3.2) take the limit \( M \to \infty \) on both side, where in the right-hand side, the limit and expectation are exchangeable due to \( EN^k < \infty \). Since the third equality follows similarly as in the proof of Proposition 2.2, we conclude the result. \( \square \)

The inversion of FST \( \mathcal{F}^{-1} \) is well known \([\text{12}, \text{Theorem 4.4.1}]\): Let \( \phi(u) \) be the FST of a bounded non-decreasing function \( F(x) \), then \( \mathcal{F}^{-1} \) is defined as

\[
F(x) = \mathcal{F}^{-1}[\phi(u)](x) = \lim_{T \to \infty} \frac{1}{2\pi} \int_{-T}^T [(e^{-iut} - 1) + it] \phi(t) dt, \quad x > 0.
\]

Hence, \( m_k(x) = \mathcal{F}^{-1}[\phi_{m_k}(\cdot)](x) \). When a r.v. \( N \) is a Poisson with parameter \( \lambda \), so that \( a = 0, b = \lambda \) in Theorem 3.1, we obtain \( m_1(x) = \lambda (m_0 * F_{X_1})(x) \) and \( m_2(x) = \lambda (m_1 * F_{X_1})(x) + m_1(x) \). Thus, it follows that

\[
m_1(x) = \lambda \mathcal{F}^{-1}[\phi_{X_1}(\cdot) e^{\lambda (\phi_{X_1}(\cdot) - 1)}](x), \quad \lambda \mathcal{F}^{-1}[\phi_{X_1}^2(\cdot) e^{\lambda (\phi_{X_1}(\cdot) - 1)}](x) + m_1(x).
\]
3.2. Estimation of magnitude of each iid component. A direct application of the Panjer recursion seems difficult for both $\chi_k$ and $\chi_{k+}$, and we alternatively invert FST of these functions. In order to obtain the FST, we represent $\chi_k$ and $\chi_{k+}$ in the form of a convolution.

Lemma 3.4. Let $N$ be a r.v. on $\mathbb{N}_0$ and $(X_i)$ is an iid sequence of r.v.'s on $\mathbb{R}_+$ with common ch.f. $\phi_{X_i}$ such that $EX_1^k < \infty$. Then $\chi_k(x) = E[X_1^k; S_N \leq x], k \in \mathbb{N}$ has the form,

$$\chi_k(x) = \frac{1}{i} k^{-1} [\phi_{S_N}(\cdot) \phi_{X_1}^{(k)}(\cdot) / \phi_{X_1}(\cdot)](x).$$

Proof. We exploit the expression

$$\chi_k(x) = E[X_1^k; S_N \leq x] = \int_0^x y^k P(S_{N-1} \leq x-y)dP_{X_1}(y),$$

which is the convolution of $P(S_{N-1} \leq \cdot)$ and $\int_0^y y^k dP_{X_1}(y)$. Since the ch.f. of $S_{N-1}$ is $\phi_{S_N}(u)/\phi_{X_1}(u)$, the conclusion is implied by the FST of $\int_0^\infty e^{\alpha x} x^k dP_{X_1}(x) = i^{-k} \phi_{X_1}^{(k)}(u)$, where $EX_1 < \infty$ assures the existence of $\phi_{X_1}^{(k)}(u)$. □

The corresponding result for $E[X_1^k; S_{N+1} \leq x]$ is obvious. Under the same condition of Lemma 3.4, we have $\chi_{k+}(x) = i^{-k} F^{-1} [\phi_{S_N}(\cdot) \phi_{X_1}^{(k)}(\cdot)](x)$.

4. Numerical Examples

We prepare notations of distributions used in examples. Denote a Poisson distribution with parameter $\lambda$ by $\text{Pois}(\lambda)$ and by $\text{Geo}(p)$, a geometric distribution with parameter $p$ of which probability is $P(X = k) = pq^k$, $q = 1-p$, $k \in \mathbb{N}_0$. As usual write $X \sim \cdot$ if r.v. $X$ follows the distribution after the tilde. All computations are done with Mathematica ver. 9 of Wolfram.

Firstly a simple example of $E[N | S_N]$ is presented by setting $N \sim \text{Pois}(\lambda)$ and $X_1 \sim \text{Pois}(\gamma)$. We examine two proposed methods for $m_1 = E[N; S_N]$, the recursion method and the Fourier inversion. For the probability of $S_N$, we use the ordinary recursion (Theorem 1.1), which yields

$$P(S_N = \ell) = \begin{cases} E[P(X_1 = 0)^N] = e^{\lambda(e^{-\gamma} - 1)}, & \ell = 0, \\ \sum_{j=1}^\ell \frac{\partial}{\partial \ell} P(X_1 = j)P(S_N = \ell - j), & \ell \geq 1. \end{cases}$$

We apply Theorem 2.1 to obtain the recursion,

$$m_1(\ell) = \begin{cases} E[NP(X_1 = 0)^N] = \lambda e^{-\gamma} e^{(e^{-\gamma} - 1)}, & \ell = 0, \\ \sum_{j=1}^\ell \frac{\partial}{\partial \ell} P(X_1 = j)P(S_N = \ell - j) + m_1(\ell - j), & \ell \geq 1. \end{cases}$$

Another method for $m_1(\ell)$ is to apply (2.5) to the Fourier transform (Proposition 2.2), which is

$$G_{m_1}(e^{i\mu}) = ENC_{X_1}(e^{i\mu}) = EN e^{\gamma(e^{i\mu} - 1)N} = \lambda e^{\gamma(e^{i\mu} - 1)} e^{i\ell(e^{i\mu} - 1)}. $$

In Figure 1, we plot $E[N | S_N = \ell] = m_1(\ell)/P(S_N = \ell), \ell \geq 0$ using both methods. Although they coincide when parameters are moderate, if either of parameters of Poisson for $N$ and $X_1$ is large, we observe instability for small $\ell$ in the Fourier approach (Figure 1. Right, squared dots), though for large $\ell$ there is no difference.

Next, we consider an example of $E[X_1 | S_{N+1}]$ for the recursion and that of $E[X_1 | S_N]$ with the Fourier transform, where $N \sim \text{Pois}(\lambda)$ and $X_1 \sim \text{Geo}(p)$. The Panjer recursion is applied
Figure 1. Left: the conditional moment $E[N \mid S_N = \ell]$, $\ell \in [0, 400]$ when Poisson parameters for $(N, X_1)$ are $(\lambda = 20, \gamma = 10)$. Right: $E[N \mid S_N = \ell]$, $\ell \in [0, 1000]$ under the setting $(\lambda = 100, \gamma = 5)$. Squared dots are values by the Fourier approach and round dots are those by the recursion method. In the former case values of both methods coincide. However, in the latter case instability is observed in the Fourier method for small $\ell$, though for large $\ell$ they coincide.

Figure 2. Left: the conditional moments $E[X_1 \mid S_N = \ell]$ (square dots) and $E[X_1 \mid S_{N+1} = \ell]$ (round dots) for $\ell \in [1, 300]$ when parameters of $\text{Pois}(\lambda)$ and $\text{Geo}(p)$ for $(N, X_1)$ are $(\lambda = 40, p = 0.25)$ respectively. Right: the same quantities of the left but with $\ell \in [1, 1000]$ and $(\lambda = 150, p = 0.2)$. In both graphs these quantities present quite similar curves for large $\ell$. However, in the right graph instability occurs in small $\ell$ of $E[X_1 \mid S_N = \ell]$ (the Fourier approach).

to both $P(S_N = \cdot)$ and $P(S_{N+1} = \cdot)$, for the latter of which we also use the convolution. For $\chi_{1+}(\ell) = E[X_1; S_{N+1} = \ell]$, we use the recursion by Theorem 2.3, i.e.

$$
\chi_{1+}(\ell) = \begin{cases} 
    pq e^{-\lambda q}, & \ell = 1, \\
    \ell pq e^{-\lambda q} + \sum_{j=1}^{\ell-1} p q^j \frac{\lambda}{\ell-j} \chi_{1+} (\ell - j), & \ell \geq 2.
\end{cases}
$$

For $\chi_1(\ell) = E[X_1; S_N = \ell]$, the inversion of the Fourier transform (2.5) is applied to

$$
G_m(e^{iu}) = \frac{q e^{iu}}{1 - q e^{iu}} \exp \left\{ i \frac{q(1 - e^{iu})}{q e^{iu} - 1} \right\}.
$$

In Figure 2 we plot $E[X_1 \mid S_N = \ell]$ and $E[X_1 \mid S_{N+1} = \ell]$ for $\ell \geq 1$. Since the graphs show very
similar curves for a moderate setting of parameters, we conclude that both methods work properly. However the instability is again observed in the Fourier approach (Figure 2). Right, squared dots) when the parameter \( \lambda \) is large and \( \ell \) is small.

4.1. Prediction in Poisson shot noise process. We pursue the prediction \( E[M(t, t+s) \mid M(t)] \), \( t, s > 0 \) of the model (1.2), i.e. calculate the quantity \( E[N(t) \mid M(t)] \) in (1.3). As mentioned, since the order of \( (T_j) \) in (1.2) does not change the distributional relation of \( N(t) \) and \( M(t) \), by the order statistics property of the Poisson, we may consider the model \( M(t) := \sum_{k=1}^{N(t)} U_k(t - U_k) \) with the iid \( U(0, t) \) sequence \( (U_i) \), and then study \( E[N(t) \mid M(t)] \). We assume that the processes \( L_k \)'s are iid compound Poisson processes such that the generic process \( L \) has the form \( L(t) = \sum_{j=1}^{N(t)} Y_j \), where \( N(t) \sim \text{Pois}(\lambda t) \), and \( (Y_j) \) denotes an iid sequence of non-negative jump sizes.

Now by setting \( N := N(t) \) and \( X_i := L_i(t - U_i) \), \( i \in \mathbb{N} \), the calculation of \( E[N(t) \mid M(t)] \) can be considered in the framework of \( E[N \mid S_N] \). For the probability of \( X_1 \), since \( N_0(t - U_1) \) does not belong to the Panjer class, we take the Fourier approach. For this we need ch.f. of \( X_1 := L(t - U_1) \), which is

\[
E[e^{i\alpha X_1}] = E[e^{i\alpha \sum_{j=1}^{N(t-U_1)} Y_j}] = \frac{e^{\alpha(t-1)(\phi Y_1(u)-1)} - e^{\gamma(\phi Y_1(u)-1)}}{e^{\gamma(1-\phi Y_1(u))}},
\]

where \( \phi Y_1(u) \) is the ch.f. of \( Y_1 \). Thus after putting \( Y_1 \sim \text{Pois}(\mu) \) so that \( \phi Y_1(u) = e^{\mu e^{\mu - 1}} \) we obtain the probability of \( X_1 \) by the Fourier inversion. For simplicity, we set \( t = 1 \), i.e. consider \( E[N(1) \mid M(1)] \), and apply Theorem 2.1 or equivalently apply the recursions (4.1) and (4.2) with initial values \( P(S_N = 0) = e^{\mu P(X_1 = 0)^{N-1}} \) and \( m_1(0) = E[NP(X_1 = 0)^N] = \lambda P(X_1 = 0)P(S_N = 0) \).

In Figure 3 (left), we plot the prediction \( E[N(1) \mid M(1) = \ell] \) for \( \ell \in [0, \infty) \) with \( \gamma = 100, \mu = 5 \) and \( \lambda = 30 \). In view of the graph, our computational method seems to work well, and one can see a non-linear curve which shows that the linear estimation of \( N(t) \) by \( M(t) \) is insufficient.

4.2. Prediction in compound mixed Poisson process. We consider an example of the compound mixed Poisson process mixed by a Gamma r.v. called compound Pólya process [7, Ex. 4.1]. Let \( \bar{N}(t) := \pi(\theta \Lambda(t)) \) denotes a mixed Poisson process where \( \pi(t) \) be a homogeneous Poisson process with intensity 1 on \([0, \infty)\), \( \Lambda(t) \) is an intensity measure and \( \theta \) is a Gamma \((\alpha, \beta)\) r.v. of which
density is \( f_\theta(x) = \frac{\beta^\theta}{1(\theta)} x^{\alpha-1} e^{-\beta x} \). Then the process has the form \( Z(t) = \sum_{j=1}^{\mathcal{N}(t)} X_j, t > 0 \), where \( X_j \)'s are iid r.v.'s on \( \mathbb{N}_0 \) or \( \mathbb{R}_+ \) such that \( \mathcal{N} \) and \( (X_j) \) are independent. Since the \( \sigma \)-fields \( \mathcal{G}_t \) by \( \{ \mathcal{N}(t), \mathcal{N}(t, t + s), Z(t) \}, t, s > 0 \) and \( \mathcal{H}_t \) by \( \{ \mathcal{N}(t), Z(t) \} \) are finer than that by \( \{ Z(t) \} \), the conditional expectation of increments \( Z(t, t + s) := Z(t + s) - Z(t) \) given \( Z(t) \) has

\[
E[Z(t, t + s) \mid Z(t)] = E[E[Z(t, t + s) \mid \mathcal{G}_t] \mid \mathcal{H}_t] \mid Z(t)
\]

(4.3)

\[
= E[X_1]E[E[\mathcal{N}(t, t + s) \mid \mathcal{H}_t] \mid Z(t)]
\]

\[
= E[X_1]E[E[\mathcal{N}(t, t + s) \mid \mathcal{N}(t)] \mid Z(t)]
\]

where in the third step we use the conditional independence of \( \mathcal{N}(t, t + s) \) and \( Z(t) \) given \( \mathcal{N}(t) \) (Prop. 6.6). Since

\[
E[\mathcal{N}(t, t + s) \mid \mathcal{N}(t) = m] = \sum_{k=0}^{\infty} \frac{\Lambda^k(t, t + s)}{k!} \frac{E[\theta^{k+m} e^{-\theta \Lambda(t+s)}]}{E[\theta^{m} e^{-\theta \Lambda(t)}]}
\]

\[=
\Lambda(t, t + s) \frac{E[\theta^{m+1} e^{-\theta \Lambda(t)}]}{E[\theta^{m} e^{-\theta \Lambda(t)}]}
\]

\[=
\Lambda(t, t + s) \frac{\alpha + m}{\Lambda(t) + \beta},
\]

where in the second step we exchange the infinite sum and the expectation operator (see also [7 (1.4)]), we proceed the calculation (4.3) to get

\[
E[Z(t, t + s) \mid Z(t)] = \frac{E[X_1] \Lambda(t, t + s)}{\Lambda(t) + \beta} (\alpha + E[\mathcal{N}(t) \mid Z(t)]).
\]

Now let \( \Lambda(t) := t, \beta := 1, \alpha := 7 \) and \( X_1 \sim \text{Geo}(1/4) \), we obtain

\[
E[Z(t, t + s) \mid Z(t)] = \frac{3s}{1 + t} (7 + E[\mathcal{N}(t) \mid Z(t)]).
\]

Since \( \mathcal{N}(t) \) does not belong to the Panjer class, we apply the Fourier approach. Due to Proposition 2.2 together with

\[
G_{S_N}(u) = E[G_{X_1}(u)] = \frac{1}{1 + t(1 - G_{X_1}(u))}, \quad \text{and} \quad G_{m_1}(u) = E[\mathcal{N} G_{X_1}(u)] = \frac{7G_{X_1}(u)}{1 + t(1 - G_{X_1}(u))^2},
\]

we obtain the quantity \( E[\mathcal{N}(t) \mid Z(t)] \) by the inversion formula (2.5). In Figure 3 (right), we plot \( E[Z(t, t + s) \mid Z(t) = \ell] \), \( \ell \in [0, 250] \) with \( s = t = 1 \), where one would again observe a non-linear curve.

**Appendix A. Calculation of (1.3)**

For the calculation of (1.3), we use the following properties.

(a) By definition, the \( \sigma \)-field by \( M(t) \) is included in the \( \sigma \)-filed by \( (L_k(t-T_k)) \) and \( (T_k) \).

(b) Since \( N(t) = \sum_{k=1}^{\infty} I_{(T_k \leq t)} \), the \( \sigma \)-field by \( N(t) \) is included in the \( \sigma \)-field by \( (T_k) \).

(c) By the order statistics property of a Poisson, given \( N(t) \) and \( N(t + s) \), the set of points \( (T_k) \in (0, t] \) and the set of points \( (T_k) \in (t, t + s] \) are independent.

(d) Given \( N(t, t + s) \), points \( (T_k) \in (t, t + s] \) are mutually independent.

(e) Stationary and independent increments of Lévy processes.
By a multiple use of iterated property of the conditional expectation [11, Theorem 6.1 (vii)], detailed the calculation of (1.3) is

\[
E[M(t, t + s) | M(t)]
\]

\[
= E[\sum_{j=N(t)+1}^{N(t+s)} L_j(t - T_j, t + s - T_j) | M(t)] + E[\sum_{j=1}^{N(t)} L_j(t - T_j, t + s - T_j) | M(t)]
\]

\[
= E[\sum_{j=N(t)+1}^{N(t+s)} E[L_j(t - T_j, t + s - T_j) | N(t), N(t + s), \{(T_k), (L_k(t - T_k))\}_{k:T_k \leq t}] | M(t)]
\]

\[
+ E[\sum_{j=1}^{N(t)} E[L_j(t - T_j, t + s - T_j) | \{(T_k), (L_k(t - T_k))\}_{k:T_k \leq t}] | M(t)]
\]

\[
= E[\sum_{j=N(t)+1}^{N(t+s)} E[L_j(t + s - T_j)I_{t < T_j \leq t + s}] | N(t), N(t + s)] | M(t)]
\]

\[
+ E[\sum_{j=1}^{N(t)} E[L_j(t - T_j, t + s - T_j) | T_j, L_j(t - T_j)] | M(t)]
\]

\[
= E[\sum_{j=N(t)+1}^{N(t+s)} E[L(t + s - U)] | M(t)] + E[\sum_{j=1}^{N(t)} E[L(s)] | M(t)]
\]

\[
= E[N(t, t + s) | M(t)]E[L(t + s - U)] + E[L(s)]E[N(t) | M(t)],
\]

where in the second step, the properties (a) and (b) are used, and in the third step, we exploit (c) and (e) so that the conditional independence of \((L_j(t - T_j, t + s - T_j))_{j:t < T_j \leq t + s}\) and \((T_i), (L_i(t - T_i))_{i:T_i \leq t}\)

In the fourth step we use (d) and (e). Finally since the quantity \(N(t, t + s)\) is independent of the \(\sigma\)-field \(\mathcal{F}_t\) constructed by all available set before \(t\), the conclusion holds.

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