Spectral analysis of one class of the integro-differential operators

G V Garkavenko\textsuperscript{1}, A R Zgolich\textsuperscript{2} and N B Uskova\textsuperscript{3}

\textsuperscript{1} Voronezh State Pedagogical University, Physico-Mathematical Faculty, 86 Lenina St, Voronezh, Russia
\textsuperscript{2} Voronezh State University, Applied Mathematics, Informatics and Mechanics Faculty, 1 Universitetskaya Square, Voronezh, Russia
\textsuperscript{3} Voronezh State Technical University, Information Technologies and Computer Security Faculty, 14 Moskovsky prospekt, Voronezh, Russia

E-mail: g.garkavenko@mail.ru, arsenij112@mail.ru, nat-uskova@mail.ru

Abstract. Let $\mathcal{H} = L_2[0, \omega]$ be the Hilbert space of equivalence classes of square integrable Lebesgue measurable complex functions on the closed interval $[0, \omega]$. In this paper, we study the spectral properties of a second-order differential operator which perturbed by an integral operator. The integral operator belong to the two-sided ideal $S_2(\mathcal{H})$ of Hilbert-Schmidt operator in $\mathcal{H}$. We use similar operators method for studying this operator. The similar operators method was developed by A. G. Baskakov and his collaborators. This method allows us to reduce the study of the operator to the one with a diagonal and a block-diagonal matrix. Asymptotic estimates of eigenvalues, eigenvectors, and spectral projections of an integro-differential operator are obtained.

1. Introduction

Let $\mathcal{H} = L_2[0, \omega]$ be the Hilbert space of equivalence classes of square integrable Lebesgue measurable complex functions on closed interval $[0, \omega]$. The inner product on this space is given, as usually, by the formula

$$ (x, y) = \frac{1}{\omega} \int_0^\omega x(t)\overline{y(t)}dt, \quad x, y \in \mathcal{H}. $$

The norm in $\mathcal{H}$ is induced by this inner product. By $W_2^2[0, \omega]$ we denote the Sobolev space, $\{y \in L_2[0, \omega]: \text{y is absolutely continuous and y'' \in L}_2[0, \omega]\}$. $W_2^2[0, \omega] \subset \mathcal{H}$.

We consider an integro-differential operator $L : D(L) \subset \mathcal{H} \to \mathcal{H}$ generated by the integro-differential expression

$$ (lx)(t) = \frac{d^2x}{dt^2} - \int_0^\omega K(t, s)x(s)ds. \quad (1) $$

The domain $D(L)$ of the operator $L$ is defined by the periodic boundary conditions

$$ D(L) = \{x \in W_2^2[0, \omega]: x'(0) = x'(\omega) = 0\}. $$

Let $L = A - B$, where $A : D(A) = D(L) \subset \mathcal{H} \to \mathcal{H}$, and $B : D(B) = \mathcal{H} \to \mathcal{H}$.
(Ax)(t) = \frac{d^2x}{dt^2} \tag{2}

and

(Bx)(t) = \int_0^\omega K(t,s)x(s)ds. \tag{3}

We assume the integral operator \( B \) will have the following form

\[ \int_0^\omega \int_0^\omega |K(t,s)|^2dt ds < \infty. \]

The class of the operators under consideration arises in the problems of the chemical kinetics (see [1, 2]).

In this paper we study a spectral properties of the operator \( L \) defined by (1), such these the asymptotic estimates of eigenvalues, eigenvectors and spectral projectors.

To this end, we will use the similar operators method. The method has been extensively developed and used in the works of Anatoly Baskakov and his collaborators (see [3–12] and references therein). It provides a foundation for finding estimates of eigenvectors and eigenvalues of many classes of differential and difference operators [3–12].

We shall call the operator \( A \), which defined by (2), unperturbed of free. The operator \( A \) is self-ajoined and has a compact resolvent. Its spectrum \( \sigma(A) \) will have the following form

\[ \sigma(A) = \{-(\frac{\pi n}{\omega})^2, n \in \mathbb{Z}+ = \mathbb{N} \cup \{0\}\}, \]

where \( \lambda_n = -(\frac{\pi n}{\omega})^2 \) are simple isolated eigenvalues, \( E_n = \text{Span}\{e_n\}, n \in \mathbb{Z}+ \), is the eigenspace for an eigenvalue \( \lambda_n = -(\frac{\pi n}{\omega})^2 \), where \( e_n(t) = \sqrt{2}\cos \frac{\pi n}{\omega} t, n \in \mathbb{Z}+ \). The spectral projections \( P_n = P(\sigma_n, A), \sigma_n = \{\lambda_n\}, n \in \mathbb{Z}+ \) are given by \( P_n(x) = (x,e_n), e_n, x \in \mathcal{H} \).

We will use the notation \( \mathcal{H}_k = \text{Im} P_k, k \in \mathbb{Z}+ \), \( P_m = \sum_{i \leq m} P_i, \mathcal{H}(m) = \text{Im} P_m \).

Operator \( A - B \) is well-defined by condition \( D(A) \subset D(B) \). Operator \( B \) will play the role of the perturbation.

2. Materials and methods

In this section we introduce the necessary notation following the standard scheme of the similar operators method.

Given a function \( v \in L_2[0,\omega] \) its Fourier series are given by

\[ v(t) \sim \sum_{n \in \mathbb{N}} \hat{v}(n) \cos \frac{\pi n}{\omega} t + \frac{\hat{v}(0)}{2}, \]

where the Fourier coefficients are

\[ \hat{v}(n) = \frac{1}{\omega} \int_0^\omega v(t) \cos \frac{\pi n}{\omega} t dt = (v,e_n), n \in \mathbb{N}, \]

\[ \hat{v}(0) = \frac{1}{\omega} \int_0^\omega v(t)dt, \]

and \( P_n(x) = \hat{v}(n)e_n, n \in \mathbb{Z}+ \).

Throughout this section by \( \mathcal{H} \) we denote an abstract Hilbert space. For a Hilbert space \( \mathcal{H} \) by \( \text{End} \mathcal{H} \) we denote the Banach algebra of all bounded linear operators in \( \mathcal{H} \). We shall also use the ideal of Hilbert-Schmidt operator in \( \mathcal{H} \), denoted by \( \mathcal{S}_2(\mathcal{H}), \mathcal{S}_2(\mathcal{H}) \subset \text{End} \mathcal{H} \). The norm in
The operator $U$ is called the similarity transform of $A_1$ into $A_2$.

Directly from Definition 1, we have the following result about the spectral properties of similar operators.

**Lemma 1.** [4] Let $A_m : D(A_m) \subset \mathcal{H} \to \mathcal{H}$, $m = 1, 2$, be two similar operators with the operator $U$ being the similarity transform of $A_1$ into $A_2$. Then the following properties hold.

1) We have $\sigma(A_1) = \sigma(A_2)$, $\sigma_p(A_1) = \sigma_p(A_2)$, and $\sigma_c(A_1) = \sigma_c(A_2)$, where $\sigma_p$ denotes the point spectrum and $\sigma_c$ denotes the continuous spectrum;

2) If $\lambda$ is an eigenvalue of the operator $A_2$ and $x$ is a corresponding eigenvector, then $y = Ux$ is an eigenvector of the operator $A_1$ corresponding to the same eigenvalue $\lambda$.

3) If $P_\sigma$ is the spectral projection of $A_2$ that corresponds to the spectral component $\sigma \subseteq \sigma(A_2) = \sigma(A_1)$, then $P_\sigma = U P_\sigma U^{-1}$ is the spectral projection of $A_1$ that corresponds to the same spectral component $\sigma$.

4) If $A_2$ is a generator of $C_0\mathcal{R}$-semigroup (group) $T_2 : \mathcal{R} \to \text{End} \mathcal{H}$, $\mathcal{R} = \{ \mathbb{R}, \mathbb{R}_+ \}$, the operator $A_1$ generates the $C_0\mathcal{R}$-semigroup (group) $T_1 : \mathcal{R} \to \text{End} \mathcal{H}$, $\mathcal{R} = \{ \mathbb{R}, \mathbb{R}_+ \}$, $T_1(t) = U T_2(t) U^{-1}$.

The method of similar operators uses the commutator transform $ad_A : D(ad_A) \subset \text{End} \mathcal{H} \to \text{End} \mathcal{H}$ defined by

$$ad_A X = AX - XA, \quad X \in D(ad_A).$$

The domain $D(ad_A)$ consists of all $X \in \text{End} \mathcal{H}$ such that the following two properties hold:

1) $XD(A) \subset D(A)$;

2) The operator $ad_A X : D(A) \to \mathcal{H}$ admits a unique extension to a bounded operator $Y \in \text{End} \mathcal{H}$; we then let $ad_A X = Y$.

The key notion of the method of similar operators is that of an admissible triplet. Once such a triplet is constructed, achieving the goal of the method becomes a routine task.

**Definition 2.** [4, 7] Let $J,G \in \text{End} \mathfrak{S}_2(\mathcal{H})$. The collection $(\mathfrak{S}_2(\mathcal{H}), J,G)$ is an admissible triplet for the operator $A$, and the space $\mathfrak{S}_2(\mathcal{H})$ is the space of admissible perturbations, if the following five properties hold.

1. $J$ is idempotent.

2. $(GX)D(A) \subset D(A)$ and

$$(ad_A GX)x = (X - JX)x, \quad x \in D(A), \quad X \in \mathfrak{S}_2(\mathcal{H}),$$

moreover $Y = GX \in \text{End} \mathfrak{S}_2(\mathcal{H})$ is the unique solution of the equation

$$ad_A Y = AY - YA = X - JX,$$

that satisfies $JY = 0$.

3. $XGY, (GX)Y \in \mathfrak{S}_2(\mathcal{H})$ for all $X,Y \in \mathfrak{S}_2(\mathcal{H})$, and there is a constant $\gamma > 0$ such that

$$||G|| \leq \gamma, \quad \max\{||XGY||_2, ||(GX)Y||_2\} \leq \gamma ||X||_2 ||Y||_2.$$

4. $J((GX)JY) = 0$ for all $X,Y \in \mathfrak{S}_2(\mathcal{H})$. 

$3$. 

$\mathfrak{S}_2(\mathcal{H})$ is idempotent.
5. For every \( X \in \mathcal{S}_2(\mathcal{H}) \) and \( \varepsilon > 0 \) there exists a number \( \lambda_\varepsilon \in \rho(A) \), such that
\[
||X(A - \lambda_\varepsilon J^{-1})|| < \varepsilon.
\]
To formulate the main theorem of the similar operators method we will use the map
\[
\Phi : \mathcal{S}_2(\mathcal{H}) \rightarrow \mathcal{S}_2(\mathcal{H}),
\]
given by
\[
\Phi(X) = BGX - GXJX + B
\]  
(4)

Theorem 1. [9] Assume that \((\mathcal{S}_2(\mathcal{H}), J, G)\) is an admissible triplet for an unperturbed operator \( A \). Assume also that
\[
q = 6\gamma||B||_2 < 1,
\]
where \( \gamma \) comes from Property 3 of Definition 2 and \( ||J|| = 1 \). Then the map \( \Phi : \mathcal{S}_2(\mathcal{H}) \rightarrow \mathcal{S}_2(\mathcal{H}), \) given by (4) is a contraction and has a unique fixed point \( X^* \) in ball
\[
\mathcal{B} = \{ X \in \mathcal{S}_2(\mathcal{H}) : ||X - B||_2 \leq \frac{3}{2}||B||_2 \},
\]
which can be found as a limit of simple iterations: \( X_0 = 0, X_1 = \Phi(X_0) \), etc. The operator \( A - B, B \in \mathcal{S}_2(\mathcal{H}) \) is similar to the operator \( A - JX^*, X^* \in \mathcal{S}_2(\mathcal{H}) \) and the similarity transform of \( A - B \) into \( A - JX^* \) is given by \( I + GX^* \in \text{End}\mathcal{H}, GX^* \in \mathcal{S}_2(\mathcal{H}). \)

Proof. We use the Banach fixed-point theorem to prove existence of \( X^* \). We show that \( \Phi(\mathcal{B}) \subseteq \mathcal{B} \) and \( ||\Phi(X) - \Phi(Y)||_2 \leq q||X - Y||_2 \) for all \( X, Y \in \mathcal{B}. \) We have
\[
||X||_2 \leq \frac{3}{2}||B||_2,
\]
and
\[
||\Phi(X) - B||_2 \leq ||BGX - GXJX||_2 \leq 5\gamma||B||_2^2 < \frac{3}{2}||B||_2,
\]
and
\[
||\Phi(X) - \Phi(Y)||_2 \leq ||BGX - G(X - Y)JY - GXY(X - Y)||_2 \leq \\
\leq \gamma||B||_2 \cdot ||X - Y||_2 + \gamma||X||_2 \cdot ||X - Y||_2 + \gamma||Y||_2 ||X - Y||_2 \leq \\
\leq 6\gamma||B||_2 ||X - Y||_2 = q||X - Y||_2, \quad q < 1.
\]

Now the first part of the Theorem 1 is proved.

Next, we need to verify the following
\[
(A - B)(I + GX^*) = (I + GX^*)(A - JX). \quad (5)
\]
We have \( JX^* = J(BGX^*) - JB, \)
\[
(A - B)(I + GX^*) = A - B + AGX^* - BGX^* = \\
= A - B + GX^*A + X^* - JX^* - BGX^* = \\
= (I + GX^*)(A - JX^*) + X^* + GX^*JX^* - BGX^* - B.
\]

Then if \( X \) is the solution of the equation
\[
X = BGX - GXJX + B,
\]
then the formulae (5) takes place.

Next, we need to show that \( I + GX^* \) is continuously invertible if \( X \in \mathcal{B}. \)

We get
\[
||GX^*||_2 \leq \gamma||X^*||_2 \leq \gamma(||X^* - B||_2 + ||B||_2) \leq \frac{5}{2}\gamma||B||_2 < 1,
\]
yielding

\[(I + GX_*)^{-1} = \sum_{n=0}^{\infty} (-1)^n (GX_*)^n.\]

To complete the proof of this theorem we need to show that \((I + GX_*)D(A) = D(A)\) or \((I + GX_*)D(A) \subseteq D(A)\) and \((I + GX_*)^{-1}D(A) \subseteq D(A)\). The first formulae is Property 2 Definition 2. Using Property 2 once again, we get

\[GX_*(A - \lambda I)^{-1} = (A - \lambda I)^{-1}(A - \lambda I)GX_*(A - \lambda I)^{-1} =
\]

\[= (A - \lambda I)^{-1}(X_* - JX_*) + GX_*(A - \lambda I)^{-1} =
\]

\[= (A - \lambda I)^{-1}((X_* - JX_*)^{-1} + GX_*).\]

for any \(\lambda \in \rho(A)\). Using Property 5 we choose \(\lambda \in \rho(A)\) such that

\[||(X_* - JX_*)(A - \lambda I)^{-1} + GX_*|| < 1\]

and

\[(U + GX_*)^{-1}(A - \lambda I)^{-1} = (A - \lambda I)^{-1}(I + (X_* - JX_*) - I - GX_*)^{-1} + GX_*)^{-1}.\]

The theorem is proved.

We note, that usually used another version of the similar operator method. The main theorem of the similar operator method for example, in [4] is differs from the Theorem 1.

3. Results and discussions

Let \(H = L_2[0, \omega]\) and operators \(A\) and \(B\) be defined by (2), (3). For each operator \(X \in \mathfrak{S}_2(H)\) we define his operator matrix. The entries of the matrix will be the operators \(X_{nm} = P_nX_P, n, m \in \mathbb{Z}_+\), \(X = (X_{nm})\). The operator \(X_p = \sum_{n-m=p} X_{ij}\) is the p-th diagonal of the matrix of the operator \(X\) [15].

The transform \(J \in \text{End}\mathfrak{S}_2(H)\) is supposed to pick out the main diagonal of the matrix of operator \(X \in \mathfrak{S}_2(H), JX = \sum_{i \in \mathbb{Z}_+} P_iXP_i\), where the series converge unconditionally in \(\mathfrak{S}_2(H)\).

Observe that in this setting the matrix of the commutator \(ad_A X\) satisfies

\[(ad_A X)_{nm} = (\lambda_n - \lambda_m)X_{nm}, X \in D(ad_A).\]

Therefore, in view of Property 2 of Definition 2 it is natural to define the transform \(G \in \text{End}\mathfrak{S}_2(H)\) via

\[(GX)_{nm} = \begin{cases} \frac{X_{nm}}{\lambda_n - \lambda_m}, & n \neq m; \\ 0, & n = m, \end{cases}\]

here \(X \in \mathfrak{S}_2(H), GX = \sum_{n,m \in \mathbb{Z}_+} \frac{X_{nm}}{\lambda_n - \lambda_m}, \) where the series converge unconditionally in \(\mathfrak{S}_2(H)\).

We also define a family of transformers \(J_kX\) and \(G_kX, k \in \mathbb{Z}_+, X \in \mathfrak{S}_2(H)\) as follows:

\[J_kX = P(k)XP(k) + \sum_{i > k} P_iXP_i,\]

\[G_kX = G(X - J_kX)\]

\[J_0X = JX, \quad G_0X = GX.\]

We note, that the operators \(G_k, J_k \in \text{End}\mathfrak{S}_2(H)\) differ from the operators \(G, J \in \text{End}\mathfrak{S}_2(H)\) on operator of finite rank.
Theorem 2. For each $k \geq 0$ the triple $(\mathcal{S}_2(H), G_k, J_k)$ is admissible for operator $A$.

Proof. Since the projectors $P_n, n \in \mathbb{Z}_+$ are orthogonal, it follows that the norm in $\mathcal{S}_2(H)$ can be defined by the formula

$$
\|X\|_2^2 = \sum_{i,j \in \mathbb{Z}_+} \|P_iXP_j\|_2^2 = \sum_{p \in \mathbb{Z}} \|X_p\|_2^2.
$$

Obviously, $\|J_k\| = 1$, because for $X \in \mathcal{S}_2(H)$

$$
\|J_kX\|_2^2 = \|P(k)XP(k)\|_2^2 + \sum_{i > k} \|P_iXP_i\|_2^2 \leq \|X\|_2^2.
$$

If the matrix of operator $X$ is diagonal, i.e. $X = \sum_{i \in \mathbb{Z}_+} P_iXP_i$, then $J_kX = X$ and $\|J_kX\|_2 = \|X\|_2$.

Then,

$$
\|GX\|_2^2 = \|G_0X\|_2^2 = \sum_{i \neq j} \frac{\|X_{ij}\|_2^2}{|\lambda_i - \lambda_j|^2} \leq \left(\frac{\omega}{\pi}\right)^4 \sum_{i \neq j} \|X_{ij}\|_2^2 \leq \left(\frac{\omega}{\pi}\right)^4 \|X\|_2^2,
$$

$$
\|G_kX\|_2^2 = \sum_{i \neq j, \max\{i,j\} > k} \frac{\|X_{ij}\|_2^2}{|\lambda_i - \lambda_j|^2} \leq \left(\frac{\omega}{\pi}\right)^4 \frac{1}{(2k+1)^2} \sum_{i \neq j, \max\{i,j\} > k} \|X_{ij}\|_2^2 \leq \left(\frac{\omega}{\pi}\right)^4 \frac{1}{(2k+1)^2} \|X\|_2^2.
$$

It follows that

$$
\|G_kX\|_2 \leq \left(\frac{\omega}{\pi}\right)^2 \frac{1}{2k+1} \|X\|_2,
$$

and $\gamma = \gamma_k = \left(\frac{\omega}{\pi}\right)^2 \frac{1}{2k+1}$.

If operator $X,Y \in \mathcal{S}_2(H)$ then $X(G_kY), (G_kX)Y$ also belong to $\mathcal{S}_2(H)$ and $\|X(G_kY)\|_2 \leq \gamma_k \|X\|_2 \|Y\|_2, \|X(G_kX)Y\|_2 \leq \gamma_k \|X\|_2 \|Y\|_2$.

If follows that property 3 of Definition 2 is hold with

$$
\gamma = \gamma_k = \left(\frac{\omega}{\pi}\right)^2 \frac{1}{2k+1}.
$$

Operator $J$ is idempotent because

$$
J^2X = J(JX) = J \left( \sum_{i \in \mathbb{Z}_+} P_iXP_i \right) = \sum_{j \in \mathbb{Z}_+} P_j \left( \sum_{i \in \mathbb{Z}_+} P_iXP_i \right) P_j = \sum_{j \in \mathbb{Z}_+} P_jXP_j = JX.
$$

The fact that is also follows from the preceding argument is that $J_k^2X = J_kX, k \in \mathbb{Z}_+, X \in \mathcal{S}_2(H)$ and property 1 of Definition 2 holds true.

Property 2 is now verified by direct computation. In particular, one checks that for a given $X \in \mathcal{S}_2(H)$ the bounded operators $GX(A-\lambda I)^{-1} - (A-\lambda I)^{-1}GX$ and $(A-\lambda I)^{-1}(X-JX)(A-\lambda I)^{-1}$ have the same matrices. This follows from the fact that $GX(D(A)) = D(A)$.

Finally, property 5 is obvious since

$$
\|X(A-\lambda I)^{-1}\| \leq \|X\| \|(A-\lambda I)^{-1}\|.
$$
The first factor is finite and the other can be chosen arbitrarily small if \( \lambda_i = in \), where \( n \in \mathbb{N} \) is the larg natural number.

The proof is complete.

We note that
\[
\|B\|_2^2 = \frac{1}{\omega^2} \int_0^\omega \int_0^\omega |K(t, s)|^2 dt ds.
\]

**Theorem 3.** Assume that
\[
6\|B\|_2 \left( \frac{\omega}{\pi} \right)^2 < 1.
\]

Then the operator \( A - B \) is similar to the diagonal operator \( A - JX_s = A - \sum_{i \in \mathbb{Z}_+} P_i X_s P_i \), 
\( X_s \in \mathcal{S}_2(H) \), where \( X_s \) is limit of simple iteration: \( X_0 = 0, X_1 = \Phi(X_0) = B \), etc, with \( \Phi \) given by (4). The transform operator is \( U = I + GX_s \).

**Proof.** Follows immediately from Theorem 1 and Theorem 2.

**Theorem 4.** There exists a number \( k \in \mathbb{Z}_+ \) such that the operator \( A - B \) is similar to the operator \( A - P_k X_s P_k - \sum_{i > k} P_i X_s P_i \), where \( X_s \in \mathcal{S}_2(H) \). The identity
\[
(A - B)(I + G_k X_s) = (I + G_k X_s)(A - J_k X_s)
\]
holds true and subspaces \( H(k)_j = Im P_k \) and \( H_j = Im P_j \), \( j > k \) are invariant with respect to the operator \( A \) and \( J_k X_s \). The operator \( X_s \in \mathcal{S}_2(H) \) is the solution of the equation (4) and it can be found with the simple iteration method.

**Proof.** Follows immediately from Theorem 1, Theorem 2 and (6).

Assume that Theorem 4 holds true.

Then the similarity of operator \( L \) and \( A - J_k X_s \) yields \( \sigma(L) = \sigma(A - J_k X_s) = \sigma(A_k \bigcup (|i > k|A_i)) \), where \( A_k = (A - P_k X_s P_k)|_{H(k)} \), \( H(k) = Im P_k \), \( A_i = (A - P_i X_s P_i)|_{H_i} \), \( i > k \), \( H_i = Im P_i \). The operator \( X_s \in \mathcal{S}_2(H) \) is unknown and we know only approximations for it.

The first approximation is \( B \), then \( A_i = (P_i A - P_i BP_i - P_i X_s - B) P_i|_{H_i} \).

In what follows, by \( l_p, p \geq 1 \) we denote the Banach space of \( p \)-summable \( (p \geq 1) \) sequences of complex numbers with the norm \( \|y\|_p = (\sum_{n \in \mathbb{Z}_+} |y(n)|^p)^{1/p} \).

**Lemma 2.** Under the assumptions of Theorem 3 we have
\[
\|P_i(X_s - B)P_i\|_2 \leq \frac{2,5}{(2k + 1)^2} \int_0^\omega \int_0^\omega |K(t, s)|^2 dt ds,
\]

\[
\|P_i(X_s - B)P_i\|_2 \leq (2i - 1)^{-1} \alpha_i, \ i > k,
\]
where the sequence \( \{\alpha_i, i > k\} \) belong to \( l_1 \).

**Proof.** We have
\[
X_s - B = BG_k X_s - GX_s J X_s,
\]
\[
P_i(X_s - B)P_i = P_i(BG_k X_s)P_i,
\]
\[
\|P_i(X_s - B)P_i\|_2 \leq \|B\|_2 \|X_s\|_2 \gamma \leq 2,5\|B\|_2 \gamma \leq \frac{2,5}{(2k + 1)^2} \int_0^\omega \int_0^\omega |K(t, s)|^2 dt ds.
\]

We note that \( BG_k X_s \in \mathcal{S}_1(H) \) and \( \|P_i(X_s - B)P_i\|_2 = \|P_i BG_k X_s P_i\|_2 \leq (2i - 1)^{-1} \alpha_i \), where the sequence \( \{\alpha_i, i > k\} \) belong to \( l_1 \).

Theorem 5 follows from Lemma 1 and Lemma 2. This theorem describes the spectral properties of the operator \( L \).

**Theorem 5.** There is a number \( k \in \mathbb{Z}_+ \) such that the spectrum \( \sigma(L) \) of the operator \( L \) satisfies
\[
\sigma(L) = \tilde{\sigma}(k) \cup \bigcup_{i > k} \tilde{\sigma}_i,
\]
where \( \widetilde{\sigma}_{(k)} \) consists of no more than \( k + 1 \) eigenvalues. The sets \( \widetilde{\sigma}_i, i > k \) are one-point sets \( \widetilde{\sigma}_i = \{ \lambda_i \} \) and

\[
\widetilde{\lambda}_i = -\left(\frac{\pi i}{\omega}\right)^2 - \frac{1}{2\omega} \int_0^\omega \int_0^\omega K(t,s) \cos \frac{\pi it}{\omega} \cos \frac{\pi is}{\omega} \, dt \, ds + \alpha_i(2i - 1)^{-1}, i > k,
\]

\( \{\alpha_i, i > k\} \) belong to \( l_1 \).

The corresponding eigenvector \( \widetilde{e}_i, i \in \mathbb{Z}_+ \) form the Riesz basis in the space \( \mathcal{H} \). The following estimates hold for the eigenvector \( \widetilde{e}_i, i > k \):

\[
||e_i - \frac{1}{\sqrt{2}} \cos \frac{\pi n}{\omega} t||_2 \leq (2i - 1)^{-1} \beta_i,
\]

where the sequence \( \{\beta_i, i > k\} \) belong to \( l_2 \).

Let \( \tilde{P}_k, k \in \mathbb{Z}_+ \) and \( P_i, i > k \) be the spectral projections \( \tilde{P}_k = P(\widetilde{\sigma}_{(k)}, L), \tilde{P}_i = P(\widetilde{\sigma}_i, L) \) corresponding to the sets \( \widetilde{\sigma}_{(k)}, \widetilde{\sigma}_i, i > k \) described in Theorem 5.

From the Lemma 1 and Theorem 3 follows

**Theorem 6.** We have

\[
||\tilde{P}_i - P_i|| \leq M_1 i^{-1} \nu_i, i > k,
\]

where the sequence \( \{\nu_i, i > k\} \) belong to \( l_2 \).

\[
||\sum_{i \geq m} \tilde{P}_i - \sum_{i \geq m} P_i|| \leq M_2 m^{-1},
\]

where \( m > k, N > k, N > m, N \in \mathbb{Z}_+, M_1, M_2 > 0, k \in \mathbb{Z}_+ \).

The spectral projections \( \tilde{P}_k, \tilde{P}_i, i > k \) also satisfy the following estimates of uniform unconditional equiconvergence of spectral expansions:

\[
||P(\widetilde{\sigma}_{(k)}, L) + \sum_{i > k} \tilde{P}_i - \sum_{i = 0}^N P_i|| = O(N^{-1}),
\]

where \( N \in \mathbb{Z}_+, N > k \).

We need the following definition in the monograph [16].

**Definition 3.** [16] Let \( C : D(C) \subset \mathcal{H} \rightarrow \mathcal{H} \) be a linear operator in the space \( \mathcal{H} \) whose spectrum can be represented as a union

\[
\sigma(C) = \bigcup_{l \in J} \sigma_l
\]

of pairwise disjoint sets \( \sigma_l, l \in c \). Also let \( P_l, l \in \mathbb{Z}_+ \) be the Riesz projection corresponding to the spectral set \( \sigma_l, l \in \mathbb{Z}_+ \). An operator \( C \) is said to be spectral with respect to the expansion (7) (or generalized-spectral) if the series \( \sum_l P_l x \) is convergent for any vector \( x \in \mathcal{H} \).

If \( \sigma_l = \{ \lambda_l \}, l \in \mathbb{Z}_+ \), are one-point sets and \( CP_l = \lambda_l P_l \) for all \( l \in \mathbb{Z}_+ \), except for finitely many, then spectral with respect to the expansion (7) operator \( C \) is a spectral operator. Operator \( C \) is the spectral operator of scalar type if \( CP_l = \lambda_l P_l \) for all \( l \in \mathbb{Z}_+ \).

**Theorem 7.** The operator \( L \) is a spectral operator.

Let \( K_i = \frac{1}{\omega} \int_0^\omega \int_0^\omega K(t,s) \cos \frac{\pi it}{\omega} \cos \frac{\pi is}{\omega} \, dt \, ds, i > k, i \in \mathbb{Z}_+ \).

From the Lemma 1 and Theorem 4 follows
Theorem 8. The integro-differential operator $iL$ is a generator of a $C_0$-group $\tilde{T} : \mathbb{R} \to \text{End} \mathcal{H}$. This group is similar to the group $\hat{T} : \mathbb{R} \to \text{End} \mathcal{H}$, $T(t) = UT(t)U^{-1}, t \in \mathbb{R}$, $U = I + G_kX_\omega$, $X_\omega \in \mathcal{S}_2(\mathcal{H})$. The operator group $\tilde{T}$ can be written as

$$\tilde{T}(t) = e^{i(A(k)-X_\omega(k))t}P(k) + \left(\sum_{l>k} e^{i(-\frac{2\pi}{\omega}l^2-K_l+\alpha_l(2l-1)^{-1})t}P_l\right), t \in \mathbb{R}, k \in \mathbb{Z}_+.$$  

where sequence $\{\alpha_i, i > k\}$ belong to $l_2$.

The above results motivates the terminology in the following definition.

Definition 4. A strongly continuous operator (semi)group $T_0 : \mathbb{J} \to \text{End} \mathcal{H}$, $\mathbb{J} = \{\mathbb{R}, \mathbb{R}_+\}$, is called a basic (semi)group for a strongly continuous operator (semi)group $T : \mathbb{J} \to \text{End} \mathcal{H}$ if there exist a operator-valued function $V : \mathbb{J} \to \text{End} \mathcal{H}$ such that $T(t) = T_0(t)V(t)$ and $\lim_{t \to \infty} ||V(t)|| = 0$. If $||V(t)|| \leq e^{-\beta t}, t \geq 0$, for some $\beta > 0$, then we call $T_0$ an exponentially basic (semi)group.

We have $\tilde{T}(t) = T_0(t)V(t)$, where

$$T_0(t) = e^{i(A(k)-X_\omega(k))t}P(k) + \left(\sum_{l>k} e^{i(-\frac{2\pi}{\omega}l^2)t}P_l\right), t \in \mathbb{R},$$

$$V(t) = \sum_{l>k} e^{i(-K_l+\alpha_l(2l-1)^{-1})t}P_l, t \in \mathbb{R}.$$  

It follows that the group $T_0 : \mathbb{R} \to \text{End} \mathcal{H}$ is basic for the group $\tilde{T} : \mathbb{R} \to \text{End} \mathcal{H}$ if $\text{Im}K_l < 0, l > k, k \in \mathbb{Z}_+$.

The existence of the operator group $T$ guaranteed by Theorem 8 is important because it allows one to use the results, for example, from [17] on exponential dichotomy and the estimates for Green’s function.

4. Conclusion

In this paper we studied the spectral properties of a second order integro-differential operator in the close interval $[0, \omega]$ with the periodic boundary conditiones. The considered class of operators arises in problems of chemical kinetics. To this end, we used the similar operators method. This method allowed us to reduce the study of the operator to one with a block-diagonal matrix. This method was thoroughly described in the paper and its basic definitions, such as admissible triplet and commutator transform, were given. Main results of the paper were formulated in eight theorems and in two lemmas, proofs of some of the theorem and lemmas were also provided. The article is ten pages long and list of references consists of seventeen issues.

Acknowledgments

We thank I. Krishtal and A. Baskakov for stimulation discussion that helped us improve this manuscript. The research was supported by Russian Foundation for Basic Research (project no. 19–01–00732).

References

[1] Perov A, Glushko E and Tyulenevva I 1988 Differ. Uravn. 24 4
[2] Valko P and Matros Yu 1979 Dokl AH SSSR 248 4
[3] Baskakov A, Krishtal I and Romanova E 2017 J. Evol. Equat. 17 15
[4] Baskakov A and Polyakov D 2017 Sbornik Math. J. 208 43
[5] Baskakov A, Krishtal I and Uskova N 2018 J. Oper and Matr. 12 33
[6] Baskakov A, Derbushev A and Shcherbakov A 2011 Izv. Math. 75 24
[7] Baskakov A and Uskova N 2018 Ufa Math. J. 10 24
[8] Garkavenko G, Zgolich A and Uskova N 2018 J. of Physics: Conf. Series 973 10
[9] Baskakov A 1983 Siberian Math. J. 24 15
[10] Baskakov A and Krishtal I 2013 J. Anal. and Appl. 407 21
[11] Brauentigam I and Polyakov D 2018 Differ. Equat. 54 12
[12] Garkavenko G and Uskova N 2017 Siber. Electr. Math. Reports 14 16
[13] Dunford N and Schwartz J 1963 Linear operators. Part II. Spectral Theory. Spectral Theory Self Adjoint Operators in Hilbert Space (New York: Wiley-Interscience)
[14] Gohberg I and Krein M 1969 Introduction to the theory of linear nonselfadjoint operators in Hilbert space. Amer. Math.Soc., Providence, RI
[15] Baskakov A and Krishtal I 2014 J.Funct. Anal. 267 54
[16] Dunford N and Schwartz J 1971 Linear operators. Part III. Spectral operators (New York: Wiley-Interscience)
[17] Baskakov A 2015 Sbornik Math. J. 206 8