Convergence analysis of approximation formulas for analytic functions via duality for potential energy minimization

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Abstract

We investigate the approximation formulas that were proposed by Tanaka & Sugihara (2019), in weighted Hardy spaces, which are analytic function spaces with certain asymptotic decay. Under the criterion of minimum worst error of n-point approximation formulas, we demonstrate that the formulas are nearly optimal. We also obtain the upper bounds of the approximation errors that coincide with the existing heuristic bounds in asymptotic order by duality theorem for the minimization problem of potential energy.

Keywords—approximation, weighted Hardy space, convex optimization, duality, discrete energy minimization

1 Introduction

By taking over the arguments of Tanaka et al. (2017), Tanaka & Sugihara (2019) proposed an algorithm to design accurate approximation formulas in function spaces called weighted Hardy spaces defined by

\[ H_\infty(D_d, w) := \left\{ f : D_d \to \mathbb{C} \left| \text{f is analytic on } D_d, \sup_{z \in D_d} \left| f(z) \right| w(z) < \infty \right\} \right. , \]

where \( d > 0 \), \( D_d := \{ z \in \mathbb{C} \mid |\text{Im} z| < d \} \), and \( w \) is a weight function characterized later in Section 2.1. The spaces \( H_\infty(D_d, w) \) are often considered as spaces of transformed functions for well-used sinc approximation formulas shown later in (1.2). The objective of Tanaka et al. (2017) and Tanaka & Sugihara (2019) was to provide formulas outperforming the sinc formulas. However, their studies only provided heuristic analyses on the proposed formulas without any theoretical guarantees, although their methods have shown superiority to the sinc approximation formulas. In this study, we mathematically

1) prove near optimality of the formulas, and

2) provide a general upper bound of the errors of the proposed formulas and show that the bound coincides in asymptotic order with the heuristic bound derived by Tanaka et al. (2017).

Below we describe the background of this study more precisely. The spaces \( H_\infty(D_d, w) \) appear in literature as spaces of variable-transformed functions (Stenger 1993, Stenger 2011, Sugihara 2003, Tanaka et al. 2009). For example, the double exponential (DE) transform, which is well-used in numerical analysis (Takahasi & Mori 1974), has the form

\[ f(x) = g \left( \tanh \left( \frac{x}{2} \sinh(x) \right) \right) \]
and shows a double-exponential decay. Also, TANH transform $g(tanh(x/2))$ is commonly used (Schwartz 1969, Haber 1977). These variable transformations are employed for the accurate approximation of functions by yielding functions with rapid decay on $D_d$, which enables us to neglect the values of the functions for large $|x|$. This motivates us to analyze the approximation possibility over weighted Hardy spaces with general weight functions $w$. After Sugihara (2003) demonstrated near optimality of sinc approximation formulas
\[
    f(x) \approx \sum_{k=-N}^{k=N} f(kh) \text{sinc} \left( \frac{x}{h} - k \right)
\]
for several weight functions $w$, attempts to construct an optimal formula for general weight functions was started in the literature.

For this purpose, Tanaka et al. (2017) employed potential theoretical arguments to generate sampling points for the approximation of functions. Furthermore, Tanaka & Sugihara (2019) simplified the arguments and proposed accurate formulas $L_n[a^*; f](x)$ given later by (2.3) with special sets $a^*$ of sampling points. The formulas $L_n[a^*; f](x)$ outperform the sinc methods for functions $f \in H^\infty(D_d, w)$. The authors showed that
\[
    \sup_{\|f\| \leq 1, x \in \mathbb{R}} |f(x) - L_n[a^*; f](x)| \leq \exp \left( -\frac{F_{K,Q}^D(n)}{n - 1} \right),
\]
where $\|f\|$ is a norm of $f \in H^\infty(D_d, w)$ and $F_{K,Q}^D(n)$ is determined later in (2.11) by a “discrete” energy minimization problem. Furthermore, they considered the minimum worst error $E_{n}^{\text{min}}(H^\infty(D_d, w))$ in (2.2) of $n$-point approximation formulas in $H^\infty(D_d, w)$ and evaluated it as
\[
    \exp \left( -\frac{F_{K,Q}^C(n)}{n} \right) \leq E_{n}^{\text{min}}(H^\infty(D_d, w)),
\]
where $F_{K,Q}^C(n)$ is determined later in (2.3) by a “continuous counterpart” of the above energy minimization problem. The following problems about the formula $L_n[a^*; f](x)$ were unsolved in Tanaka & Sugihara (2019).

(i) Since (the RHS of (1.3)) \leq (the LHS of (1.3)), the formula $L_n[a^*; f](x)$ is assured of “near optimality” if $F_{K,Q}^C(n)$ and $F_{K,Q}^D(n)$ are close. However, their difference was not estimated.

(ii) To estimate the convergence rate of the error in the LHS of (1.3), we need to know how $F_{K,Q}^D(n)$ depends on $n$. However, it was not known.

In this paper, we provide solutions to these problems. Our contributions (1) and (2) mentioned in the first paragraph of this section correspond to the solutions to problems (i) and (ii), respectively. More precisely, we show the following statements.

(1) We show an evaluation like
\[
    F_{K,Q}^D(n) \lesssim F_{K,Q}^C(n) \lesssim 2F_{K,Q}^D(n).
\]
Its rigorous version is given by Theorem 2.3 in Section 2.3. The quantities $F_{K,Q}^D(n)$ and $F_{K,Q}^C(n)$ were obtained from the optimal solutions of the “discrete” energy minimization problem and its “continuous counterpart”, respectively. Therefore we construct a feasible solution for the latter using the optimal solution of the former to show this theorem.

(2) We show an inequality
\[
    \frac{F_{K,Q}^C(n)}{n} \geq \frac{Q(\alpha_n)}{2},
\]
where $Q(x) = -\log w(x)$ and $\alpha_n$ is determined by a tractable inequality. Its details are given by Theorem 2.4 in Section 2.3. By combining this inequality, the above statement (1), and Inequality (1.3), we obtain explicit convergence rates of the proposed formulas. To show this theorem, we consider the dual problem of the “continuous” energy minimization problem and provide its feasible solution. For preparation, we present a primal-dual theory of the energy minimization problem in Section 4.
As a result, we explicitly obtain lower bounds of $F_{K,Q}^C(n)$ and demonstrate that the rates of lower bounds coincide with those of heuristic bounds in Tanaka et al. (2017).

The rest of this paper is organized as follows. In Section 2, we present a mathematical overview of the existing studies and describe our main results as mathematical statements. Section 3 contains general arguments, which introduce the concept of “positive semi-definite in measure”. Then, we show that the problem under our interest is a special case of that concept and derive the duality theorem. The evaluations for the second result, described by Theorem 2.3, are given in Section 4. We compare the bounds with those in Tanaka et al. (2017) in Section 5. Finally, we describe the concluding remarks in Section 6.

2 Mathematical preliminaries and main results

2.1 General settings

We first give some definitions and formulate the problem mathematically. Let $D := \{ z \in \mathbb{C} \mid \text{Im} z < d \}$. Throughout this paper, a weight function $w : D \rightarrow \mathbb{C}$ is supposed to satisfy the following conditions:

1. $w$ is analytic and does not vanish over the domain $D$ and takes values in $(0, 1]$ on $\mathbb{R}$;
2. $w$ satisfies $\lim_{x \rightarrow \pm \infty} \int_{-d}^{d} |w(x+iy)| \, dy = 0$ and $\lim_{y \rightarrow \pm d} \int_{-\infty}^{\infty} (|w(x+iy)| + |w(x-iy)|) \, dx < \infty$;
3. $\log w$ is strictly concave on $\mathbb{R}$.

For a weight function with the above conditions, we define the weighted Hardy space $H^\infty(D, w)$ on $D$ as follows. Recall that it is defined by

$$H^\infty(D, w) := \left\{ f : D \rightarrow \mathbb{C} \mid f \text{ is analytic on } D, \sup_{z \in D} \left| \frac{f(z)}{w(z)} \right| < \infty \right\}. \quad (2.1)$$

We define

$$\|f\| := \sup_{z \in D} \left| \frac{f(z)}{w(z)} \right|$$

for $f \in H^\infty(D, w)$, and the expression $\|f\| < \infty$ shall also imply $f \in H^\infty$ in the following.

Based on Sugihara (2003) and Tanaka & Sugihara (2019), we adopt the minimum worst-case error

$$E_n^\min(\|H^\infty(D, w)\|) := \inf \left\{ \sup_{\|f\| \leq 1, \ x \in R} \left| f(x) - \sum_{j=1}^{n} \sum_{k=0}^{n_j-1} f^{(k)}(a_j) \phi_{jk}(x) \right| \mid 1 \leq l \leq n, \ m_1 + \cdots + m_l = n, a_j \in D_d \text{ are distinct}, \phi_{jk} : D \rightarrow \mathbb{C} \text{ are analytic} \right\} \quad (2.2)$$

as the optimal performance over all possible $n$-point interpolation formulas on $\mathbb{R}$, which is applicable to any $f \in H^\infty(D, w)$.

2.2 Properties of approximation formulas to be analyzed

Let us introduce some functions dependent on an $n$-sequence $a = \{a_j\}_{j=1}^{n} \subset \mathbb{R}$ as follows.

$$T_d(x) := \tanh \left( \frac{\pi}{4d} x \right),$$

$$B_n(x; a, D_d) := \prod_{j=1}^{n} \frac{T_d(x) - T_d(a_j)}{1 - T_d(a_j)T_d(x)},$$

$$B_{n,k}(x; a, D_d) := \prod_{1 \leq j \leq n, \ j \neq k} \frac{T_d(x) - T_d(a_j)}{1 - T_d(a_j)T_d(x)}.$$
Using these functions, we can give an n-point interpolation formula

\[ L_n[a; f](x) := \sum_{k=1}^{n} f(a_k) \frac{B_{n,k}(x; a, D_d)w(x)}{B_{n,k}(a_k; a, D_d)w(a_k) \cdot T_d(x - a_k)} \cdot T_d(0), \tag{2.3} \]

which is known to characterize the value \( E_n^{\text{min}}(\|D_d, w\|) \) as follows.

**Proposition 2.1.** (Sugihara 2003, Tanaka & Sugihara 2019) We have an upper bound of the error of (2.3) as

\[ \sup_{\|f\| \leq 1} \sup_{x \in \mathbb{R}} |f(x) - L_n[a; f](x)| \leq \sup_{x \in \mathbb{R}} |B_n(x; a, D_d)w(x)| \]

for any fixed sequence \( a = \{a_j\}_{j=1}^{n} \subset \mathbb{R} \) (of distinct points). Moreover, by taking infimum of the above expression over all n-sequences, it holds that

\[ E_n^{\text{min}}(\|D_d, w\|) = \inf_{a_j \in \mathbb{R}} \sup_{\|f\| \leq 1, x \in \mathbb{R}} |f(x) - L_n[a; f](x)| = \inf_{a_j \in \mathbb{R}} \sup_{x \in \mathbb{R}} |B_n(x; a, D_d)w(x)|. \]

By this assertion, it is enough to consider interpolation formulas of the form (2.3). Additionally, this motivates us to analyze the value \( \sup_{x \in \mathbb{R}} |B_n(x; a, D_d)w(x)| \), which is simpler than the worst-case error of (2.3). In Tanaka et al. (2017) and Tanaka & Sugihara (2019),

\[-\log \left( \inf_{a_j \in \mathbb{R}} \sup_{x \in \mathbb{R}} |B_n(x; a, D_d)w(x)| \right)\]

is treated as an optimal value of an optimization problem (justifiable by the addition rule of \( \tanh \))

\[ \text{(DC)} \quad \max \inf_{x \in \mathbb{R}} \left( \sum_{i=1}^{n} K(x - a_i) + Q(x) \right) \quad \text{subject to } a_1 < \cdots < a_n, \tag{2.4} \]

where \( K \) and \( Q \) are defined by

\[ K(x) := -\log |T_d(x)| = -\log \left| \tanh \left( \frac{\pi}{4d} \right) \right|, \tag{2.5} \]

\[ Q(x) := -\log w(x). \]

They considered a continuous relaxation of (DC) as

\[ \text{(CT)} \quad \max \inf_{x \in \mathbb{R}} \left( \int_{\mathbb{R}} K(x - y) \, d\mu(y) + Q(x) \right) \quad \text{subject to } \mu \in \mathcal{M}_c(\mathbb{R}, n), \]

where, we define \( \mathcal{M}(\mathbb{R}, n) \) as the set of all (positive) Borel measures \( \mu \) over \( \mathbb{R} \) with \( \mu(\mathbb{R}) = n \) and

\[ \mathcal{M}_c(\mathbb{R}, n) := \{ \mu \in \mathcal{M}(\mathbb{R}, n) \mid \text{supp } \mu \text{ is compact} \}. \]

Because each feasible solution of (DC) can be interpreted as a combination of \( \delta \)-measures being a feasible solution of (CT),

\[ \text{(the optimal value of (DC))} \leq \text{(the optimal value of (CT))} \tag{2.6} \]

Potential theoretical arguments (Saff & Totik 1997, Levin & Lubinsky 2001, Tanaka & Sugihara 2019) lead to the following proposition.

**Proposition 2.2.** (Tanaka & Sugihara 2019, Theorem 2.4, 2.5) The energy of \( \mu \in \mathcal{M}(\mathbb{R}, n) \) is defined as

\[ I_n^{\text{C}}(\mu) := \int_{\mathbb{R}} \int_{\mathbb{R}} K(x - y) \, d\mu(x) \, d\mu(y) + 2 \int_{\mathbb{R}} Q(x) \, d\mu(x). \tag{2.7} \]
Then, there exists a unique minimizer $\mu_n^*$ over $\mathcal{M}(\mathbb{R}, n)$ of $I_n^C(\mu)$ with a compact support and $\mu_n^*$ is also an optimal solution of (CT). Furthermore, if we define

$$F_{K,Q}^C(n) := I_n^C(\mu_n^*) - \int_{\mathbb{R}} Q(x) \, d\mu_n^*(x)$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} K(x - y) \, d\mu_n^*(x) \, d\mu_n^*(y) + \int_{\mathbb{R}} Q(x) \, d\mu_n^*(x),$$

(2.8)

the optimal value of (CT) coincides with $\frac{F_{K,Q}^C(n)}{n}$.

Following this proposition, Tanaka & Sugihara (2019) considered a discrete counterpart of $I_n^C(\mu)$ and $F_{K,Q}^C$, which are defined for $a = \{a_i\}_{i=1}^n$ ($a_1 < \cdots < a_n$) as

$$F_{K,Q}^D(n) := \sum_{i \neq j} K(a_i - a_j) + \frac{2(n - 1)}{n} \sum_{i=1}^n Q(a_i),$$

(2.9)

$$F_{K,Q}^D(n) := I_n^D(\mu^*) - \frac{n - 1}{n} \sum_{i=1}^n Q(a_i^*),$$

(2.10)

where $a^* = \{a_i^*\}_{i=1}^n$ is the unique minimizer of $F_{K,Q}^D(a)$, which certainly exists according to Theorem 3.3 in Tanaka & Sugihara (2019). We can easily obtain $a^*$ numerically as it is a solution of the convex programming and it is known to satisfy (Tanaka & Sugihara 2019, Theorem 4.1)

$$\sup_{\|f\| \leq 1, \, x \in \mathbb{R}} |f(x) - L_n[a^*; f](x)| \leq \exp \left( \frac{F_{K,Q}^D(n)}{n - 1} \right).$$

(2.11)

Then $E_n^{\min}(\mathbb{R}^\infty(D_d, w))$ is evaluated as (Tanaka & Sugihara 2019, Remark 4.2)

$$\exp \left( - \frac{F_{K,Q}^C(n)}{n} \right) \leq E_n^{\min}(\mathbb{R}^\infty(D_d, w)) \leq \exp \left( - \frac{F_{K,Q}^D(n)}{n - 1} \right).$$

Indeed, the left inequality holds true by (2.10) and Proposition 2.2 and the right inequality follows from (2.11). By this evaluation, we can consider $L_n[a^*; f](x)$ as a nearly optimal approximation formula if $F_{K,Q}^C(n)/n$ and $F_{K,Q}^D(n)/(n - 1)$ are sufficiently close.

### 2.3 Main results

In this paper, we demonstrate the following two theorems. The first and second theorems, respectively, correspond to (1) and (2) in Section 1.

**Theorem 2.3.** For $n \geq 2$, the following holds true

$$\frac{F_{K,Q}^D(n)}{n - 1} \leq \frac{F_{K,Q}^C(n)}{n} \leq \frac{n}{n - 1} \left( \frac{2F_{K,Q}^D(n)}{n - 1} + (3 + \log 2) \right).$$

Theorem 2.4. Suppose $w$ is even on $\mathbb{R}$. For $\alpha_n > 0$ that satisfies

$$\frac{2\alpha_n}{\pi \tanh(d)} \frac{Q(\alpha_n)^2 + Q'(\alpha_n)^2}{Q(\alpha_n)} \leq n,$$

we have

$$\frac{F_{K,Q}^C(n)}{n} \geq \frac{Q(\alpha_n)}{2}.$$

Theorem 2.3 shows the near optimality of the approximation formula $L_n[a^*; f](x)$. By the assertion of the theorem, we have, for arbitrary $\varepsilon > 0$,

$$\sup_{\|f\| \leq 1, \, x \in \mathbb{R}} |f(x) - L_n[a^*; f](x)| \leq \sqrt{2e^\varepsilon E_n^{\min}(\mathbb{R}^\infty(D_d, w))} \frac{1}{\sqrt{n}}$$

for each sufficiently large $n$. In addition, Theorem 2.4 (combined with Theorem 2.3) gives an explicit upper bound of $E_n^{\min}(\mathbb{R}^\infty(D_d, w))$ as

$$E_n^{\min}(\mathbb{R}^\infty(D_d, w)) \leq \sqrt{2e^\varepsilon \exp \left( - \frac{n - 1}{4n} Q(\alpha_n) \right)}.$$
2.4 Basic ideas to show the main results

The left inequality of Theorem 2.3 is from Theorems 3.4 and 3.5 in Tanaka & Sugihara (2019). To prove the right inequality of Theorem 2.3, we consider the optimization problem

\[
(P) \quad \text{minimize} \quad \int_{\mathbb{R}} \int_{\mathbb{R}} K(x - y) \, d\mu(x) \, d\mu(y) + 2 \int_{\mathbb{R}} Q(x) \, d\mu(x) \quad \mu \in \mathcal{M}(\mathbb{R}, n),
\]

whose solution provides \( F_{K,Q}^C(n) \) as shown in Proposition 2.2. The quantity \( F_{K,Q}^D(n) \) is obtained from the optimal solution of a discrete counterpart of \( P \) given by (2.9). Then, we construct a feasible solution of \( P \) given later by (3.1) from the optimal solution of the discrete counterpart. By using the feasible solution, we bound \( F_{K,Q}^D(n) \) from above by using \( F_{K,Q}^D(n) \).

To prove Theorem 2.3, we need a lower bound of the optimal value of \( P \). However, because \( P \) is a minimization problem, any concrete feasible solution does not help us. Therefore, we prove that \( P \) can be regarded as an infinite-dimensional convex quadratic programming, as \( K \) is positive semi-definite in measure (Definition 3.1), and take the dual problem (Dorn 1960, Luenberger 1997). We also show that the dual problem

\[
(D) \quad \text{maximize} \quad -\int_{\mathbb{R}} \int_{\mathbb{R}} K(x - y) \, d\nu(x) \, d\nu(y) + 2ns \quad \nu \text{ is a signed Borel measure}
\]

subject to \( s - \int_{\mathbb{R}} K(\cdot - y) \, d\nu(y) \leq Q \) satisfies the weak and strong duality (Theorem 3.3), i.e., the optimal value of \( D \) coincides with that of \( P \). By this, we can obtain a lower bound for the optimal value of \( P \), taking concrete \( \nu \) and \( s \). The practical advantage of taking \( D \) is that \( \nu \) can be a signed measure (though we indeed deal with a little wider class in Section 4), which means that we can define \( \nu \) as some Fourier transform of the symmetric function, without confirming the non-negativity. This solves one of the improper points of the evaluation in Tanaka et al. (2017).

Remark 2.5. Problem \( D \) in (2.12) needs to be more rigorous to realize a primal-dual theory for \( P \) and \( D \). In Section 3, we provide a rigorous form of \( D \) by introducing a set \( S_K \) for \( \nu \).

3 Proof of Theorem 2.3

To prove Theorem 2.3, we prepare the following lemmas.

Lemma 3.1. For arbitrary \( t > 0 \), the following holds true.

\[
\int_0^1 K(tx) \, dx \leq K(t) + 1.
\]

Proof. Consider the function \( g(x) := K(x) + \log \left( \frac{1}{2\pi x} \right) \) defined for \( x > 0 \). We first prove that \( g(x) \) is strictly increasing and satisfies \( \lim_{x \to 0} g(x) = 0 \). Let \( h(x) := \exp \left( g \left( \frac{2x}{t} \right) \right) \). Then, we have

\[
h(x) = \frac{x}{2 \tanh \frac{x}{2}} = \frac{2(e^x + 1)}{2(e^x - 1)}
\]

and

\[
h'(x) = \frac{(xe^x + x^2 + 1)(e^x - 1) - x(e^x + 1)e^x}{2(e^x - 1)^2} = \frac{e^{2x} - 2xe^x - 1}{2(e^x - 1)^2}.
\]

Because \( e^{2x} - 2xe^x - 1 = 2(e^{2x} - e^x - xe^x) = 2e^x(e^x - 1 - x) \) is valid, we have \( h'(x) > 0 \) for \( x > 0 \). Evidently, we also have \( \lim_{x \to 0} h(x) = 1 \). Thus, \( g \) satisfies the above properties.

Because \( g \) is positive and increasing, \( \int_0^1 g(tx) \, dx \leq g(t) \) is valid. Therefore, we have

\[
\int_0^1 K(tx) \, dx = \int_0^1 g(tx) \, dx - \int_0^1 \log \left( \frac{\pi}{4d} tx \right) \, dx \\
\leq g(t) - \log \left( \frac{\pi}{4d} t \right) + 1 \\
= K(t) + 1.
\]
as desired.

Lemma 3.2. For arbitrary \( x > 0 \), the following hold true.

\[
K \left( \frac{x}{2} \right) \leq K(x) + \log 2.
\]

Proof. By the definition of \( K \), it suffices to show that \( \tanh x \leq 2 \tanh \frac{x}{2} \). Indeed, we have

\[
\frac{2 \tanh \frac{x}{2}}{\tanh x} = \frac{2(e^x - 1)}{e^x + 1} \cdot \frac{e^{2x} + 1}{e^{2x} - 1} = \frac{2(e^{2x} + 1)}{(e^x + 1)^2} \geq \frac{(e^{2x} + 1) + 2e^x}{(e^x + 1)^2} = 1,
\]

where we have used \( e^{2x} + 1 \geq 2e^x \) (AM-GM inequality).

We can now prove the first theorem.

Proof of Theorem 2.3. The left inequality is from Theorem 3.4 and 3.5 in Tanaka & Sugihara (2019).

Let us prove the right inequality. Let \( a = (a_1, \ldots, a_n) \) (with \( a_1 < \cdots < a_n \)) be the minimizer of the discrete energy, satisfying

\[
F^D_{K,Q}(n) = \sum_{i \neq j} K(a_i - a_j) + \frac{n-1}{n} \sum_{i=1}^{n} Q(a_i).
\]

Let \( \mu \) be a measure with a density function \( p \) defined by

\[
p(x) = \begin{cases} \frac{n}{(n-1)(a_{i+1} - a_i)} & (x \in [a_i, a_{i+1}), \ i = 1, \ldots, n-1), \\ 0 & \text{(otherwise)} \end{cases}
\]

(3.1)

Then, we have

\[
F^C_{K,Q}(n) \leq I^C_n(\mu^*_n) \leq I^C_n(\mu).
\]

(3.2)

In the following, we obtain an upper bound of \( I^C_n(\mu) \). First, we evaluate \( \int_{\mathbb{R}} \int_{\mathbb{R}} K(x-y) \, d\mu(x) \, d\mu(y) \). For \( 1 \leq k \leq n-1 \) and \( y \in [a_k, a_{k+1}) \), we have

\[
\int_{\mathbb{R}} K(x-y) \, d\mu(x) = \int_{\mathbb{R}} K(x-y)p(x) \, dx
\]

\[
= \sum_{i=1}^{n-1} \frac{n}{(n-1)(a_{i+1} - a_i)} \int_{a_i}^{a_{i+1}} K(x-y) \, dx
\]

\[
= \frac{n}{n-1} \sum_{i=1}^{n-1} \int_{0}^{1} K(a_i + (a_{i+1} - a_i)z-y) \, dz.
\]

Here, because \( y \in [a_k, a_{k+1}), \) for \( i \notin \{k-1, k, k+1\} \), the convexity and monotonicity of \( K \) over \((-\infty, 0)\) or \((0, \infty)\) shows that

\[
\int_{0}^{1} K(a_i + (a_{i+1} - a_i)z-y) \, dz \leq \begin{cases} \frac{1}{2} (K(a_i - a_k) + K(a_{i+1} - a_k)) & (i \leq k-2), \\ \frac{1}{2} (K(a_i - a_{k+1}) + K(a_{i+1} - a_{k+1})) & (i \geq k+2). 
\end{cases}
\]

Therefore, by considering that \( K \) is non-negative, we have

\[
\sum_{i \neq k-1, k,k+1} \int_{0}^{1} K(a_i + (a_{i+1} - a_i)z-y) \, dz \leq \sum_{j \leq k-2} K(a_j - a_k) + \sum_{j \geq k+3} K(a_j - a_{k+1})
\]

\[
+ \frac{1}{2} (K(a_{k-1} - a_k) + K(a_{k+2} - a_{k+1}))
\]

(3.3)
Here, the terms that include an index of $a$ outside the domain $\{1, \ldots, n\}$ are void. Next, we consider the cases $i = k \pm 1$. If $k - 1 \geq 1$ is valid, we have
\[
\int_0^1 K(a_{k-1} + (a_k - a_{k-1})z - y) \, dz \leq \int_0^1 K(a_{k-1} + (a_k - a_{k-1})z - a_k) \, dz \\
= \int_0^1 K((a_k - a_{k-1})w) \, dw \\
\leq K(a_k - a_{k-1}) + 1 = K(a_{k-1} - a_k) + 1. \tag{3.4}
\]

Similarly, if $k + 2 \leq n$ is valid, we have, by Lemma 3.1,
\[
\int_0^1 K(a_{k+1} + (a_k - a_{k+1})z - y) \, dz \leq K(a_{k+1} - a_k) + 1. \tag{3.5}
\]

Finally, we deal with the case $i = k$. We show that the integral
\[
L_k(y) := \int_0^1 K(a_k + (a_{k-1} - a_k)z - y) \, dz
\]
is maximized at $y = \frac{a_k + a_{k+1}}{2}$ (over $y \in [a_k, a_{k+1}]$). If we define $t := \frac{y - a_k}{a_{k+1} - a_k}$ ($t \in [0, 1]$), the following holds true.
\[
L_k(y) = \int_0^t K((a_{k+1} - a_k)w) \, dw + \int_0^{1-t} K((a_{k+1} - a_k)w) \, dw.
\]

For $t < \frac{1}{2}$, we have
\[
\begin{align*}
L_k \left( \frac{a_k + a_{k+1}}{2} \right) - L_k(y) &= \int_0^{\frac{1}{2}} K((a_{k+1} - a_k)w) \, dw - \int_0^{1-t} K((a_{k+1} - a_k)w) \, dw \\
&= \int_0^{\frac{1}{2} - t} \left( K((a_{k+1} - a_k)(t + w)) - K \left( (a_{k+1} - a_k) \left( \frac{1}{2} + w \right) \right) \right) \, dw > 0.
\end{align*}
\]

By symmetry, $L_k(y) < L_k \left( \frac{a_k + a_{k+1}}{2} \right)$ is valid for $t > \frac{1}{2}$. Therefore, by Lemma 3.1 and 3.2,
\[
\int_0^1 K(a_k + (a_{k-1} - a_k)z - y) \, dz \leq L_k \left( \frac{1}{2} \right) \\
= 2 \int_0^{\frac{1}{2}} K((a_{k+1} - a_k)w) \, dw \\
= \int_0^1 K \left( \frac{a_{k+1} - a_k}{2} \right) \, dw \\
\leq K \left( \frac{a_{k+1} - a_k}{2} \right) + 1 \\
\leq K(a_{k+1} - a_k) + 1 + \log 2 \tag{3.6}
\]

By (3.3)–(3.6), we have the bound
\[
\left( \frac{n - 1}{n} \right)^2 \int_{a_k}^{a_{k+1}} \int_{\mathbb{R}} K(x - y) \, d\mu(x) \, d\mu(y) \\
\leq \frac{n - 1}{n} \sup_{y \in [a_k, a_{k+1}]} \int_{\mathbb{R}} K(x - y) \, d\mu(x) \\
\leq \sum_{j \leq k-2} K(a_j - a_k) + \sum_{j \geq k+3} K(a_j - a_{k+1}) + 3 + \log 2 \\
+ \frac{3}{2} K(a_{k-1} - a_k) + \frac{1}{2} K(a_k - a_{k+1}) + \frac{1}{2} K(a_{k+1} - a_k) + \frac{3}{2} K(a_{k+2} - a_{k+1}).
\]

8
Considering the sum of the right-hand side with respect to \( k = 1, \ldots, n - 1 \), the coefficient of each \( K(a_i - a_j) \) with \(|i - j| \geq 2\) is at most 1, and that of \( K(a_i - a_j) \) with \(|i - j| = 1\) is at most 2 \((= \frac{1}{2} + \frac{1}{2})\), where we have distinguished \( K(a_i - a_j) \) from \( K(a_j - a_i) \). Therefore, we have

\[
\left( \frac{n - 1}{n} \right)^2 \int_{\mathbb{R}} \int_{\mathbb{R}} K(x - y) \, d\mu(x) \, d\mu(y) \leq 2 \sum_{i \neq j} K(a_i - a_j) + (n - 1)(3 + \log 2). \tag{3.7}
\]

Let us now evaluate the second term of \( I_n^C(\mu) \), i.e., \( \int_{\mathbb{R}} Q(x) \, d\mu(x) \). By the convexity of \( Q \), we have

\[
\int_{\mathbb{R}} Q(x) \, d\mu(x) = \frac{n}{n - 1} \sum_{i=1}^{n-1} \int_{0}^{1} Q(a_i + (a_{i+1} - a_i)z) \, dz
\leq \frac{n}{n - 1} \sum_{i=1}^{n-1} \max\{Q(a_i), Q(a_{i+1})\}.
\]

It should be noted here that there are no duplicates for \( \max\{Q(a_i), Q(a_{i+1})\} \), i.e., it is impossible for \( Q(a_{i+1}) \) to be \( \max\{Q(a_i), Q(a_{i+1}), Q(a_{i+2})\} \), by the strong convexity. Therefore, the following holds true.

\[
\int_{\mathbb{R}} Q(x) \, d\mu(x) \leq \frac{n}{n - 1} \sum_{i=1}^{n} Q(a_i). \tag{3.8}
\]

Combining (3.7) and (3.8), we obtain

\[
I_n^C(\mu) = \int_{\mathbb{R}} \int_{\mathbb{R}} K(x - y) \, d\mu(x) \, d\mu(y) + 2 \int_{\mathbb{R}} Q(x) \, d\mu(x)
\leq 2 \left( \frac{n}{n - 1} \right)^2 \sum_{i \neq j} K(a_i - a_j) + \frac{2n}{n - 1} \sum_{i=1}^{n} Q(a_i) + \frac{n^2}{n - 1}(3 + \log 2)
\leq 2 \left( \frac{n}{n - 1} \right)^2 \left( \sum_{i \neq j} K(a_i - a_j) + \frac{n}{n - 1} \sum_{i=1}^{n} Q(a_i) \right) + \frac{n^2}{n - 1}(3 + \log 2)
\leq 2 \left( \frac{n}{n - 1} \right)^2 F_{K, Q}(n) + \frac{n^2}{n - 1}(3 + \log 2).
\]

Now, using (3.2), we reach the conclusion. \( \square \)

4 Duality theorem for convex programming of measures

The following definition is a variant of the existing definitions of positive definite kernel (Stewart 1976, Janing et al. 2009, Sriperumbudur et al. 2010).

**Definition 4.1.** Let \( X \) be a topological space. A non-negative measurable function \( k : X \times X \to \mathbb{R}_{\geq 0} \cup \{\infty\} \) is called **positive semi-definite in measure** if it satisfies

\[
\int_X \int_X k(x, y) \, d\mu(x) \, d\mu(y) + \int_X \int_X k(x, y) \, d\nu(x) \, d\nu(y) 
\geq \int_X \int_X k(x, y) \, d\mu(x) \, d\nu(y) + \int_X \int_X k(x, y) \, d\nu(x) \, d\mu(y) \tag{4.1}
\]

for arbitrary (positive) \( \sigma \)-finite Borel measures \( \mu, \nu \) on \( X \).

**Remark 4.2.** Let \( k \) be positive semi-definite in measure. Considering the Hahn-Jordan decomposition of a signed measure, we have

\[
\int_X \int_X k(x, y) \, d|\mu|(x) \, d|\mu|(y) < \infty \implies \int_X \int_X k(x, y) \, d\mu(x) \, d\mu(y) \geq 0
\]
for an arbitrary signed Borel measure \( \mu \) on \( X \) with \( |\mu| \) being \( \sigma \)-finite, where \( |\mu| \) denotes the total variation of \( \mu \). This is the generalization of the ordinary positive semi-definiteness. Notice that this non-negativity holds for a wider class of “measure”. Indeed, if we define

\[
S_k := \left\{ (\mu_+, \mu_-) \mid \int_X \int_X k(x,y) \, d\mu_+(x) \, d\mu_+(y), \int_X \int_X k(x,y) \, d\mu_-(x) \, d\mu_-(y) < \infty \right\}
\]

and for each \( \nu = (\nu_+, \nu_-) \in S_k \) define

\[
\int_X \int_X k(x,y) \, d\nu_+(x) \, d\nu_+(y) := \int_X \int_X k(x,y) \, d\nu_+(x) \, d\nu_+(y) + \int_X \int_X k(x,y) \, d\nu_-(x) \, d\nu_-(y)
\]

\[
- \int_X \int_X k(x,y) \, d\nu_+(x) \, d\nu_-(y) - \int_X \int_X k(x,y) \, d\nu_-(x) \, d\nu_+(y),
\]

then this integral is well-defined and the generalization of quadratic forms for ordinary signed measures. We formally write \( \nu = \nu_+ - \nu_- \) in such a situation, and call it also the Hahn-Jordan decomposition of \( \nu \).

**Lemma 4.3.** Let \( K : \mathbb{R} \to [0, \infty) \) be an even function. If \( K \in L^1(\mathbb{R}) \) and \( K \) is convex on \([0, \infty)\), and satisfies \( \lim_{x \to 0} K(x) = K(0) \), then \( K(x) \) is positive semi-definite in measure.

**Proof.** Because \( K \) is integrable and convex, \( K \) is continuous over \((0, \infty)\) and \( \lim_{x \to \infty} K(x) = 0 \) holds true. If \( K(0) < \infty \), \( K \) becomes continuous and this type of function is called \( \text{Pólya-type} \). Pólya-type functions are known to be a characteristic function of a positive bounded Borel measure, i.e., there exists a positive bounded measure \( \alpha \) on \( \mathbb{R} \) such that

\[
K(x) = \int_{\mathbb{R}} e^{-i\omega x} \, d\alpha(\omega)
\]

is valid (Jaming et al. 2009, Pólya 1949). Let \( \mu \) be a signed Borel measure with \( \int_\mathbb{R} \int_\mathbb{R} K(x-y) \, d\mu(x) \, d\mu(y) \) being finite and \( |\mu| \) being \( \sigma \)-finite. Then, we can take a sequence of increasing Borel sets \( A_1 \subset A_2 \subset \cdots \to \mathbb{R} \) satisfying \( |\mu|(A_k) < \infty \) for all \( k \). Let \( \mu = \mu_+ - \mu_- \) be the Hahn-Jordan decomposition and \( \mu^k_+ := \mu_+(A_k \cap \cdot) \), \( \mu^k_- := \mu_-(A_k \cap \cdot) \). For each \( k \), by Fubini’s theorem and (4.2), we have

\[
\int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} \int_\mathbb{R} K(x-y) \, d\mu^k_+(x) \, d\mu^k_+(y) \, d\mu^k_-(x) \, d\mu^k_-(y) = \int_\mathbb{R} \left| \int_\mathbb{R} e^{-ikx} \, d\mu^k_+(x) - \mu^k_-(x) \right|^2 \, d\alpha(\omega) \geq 0.
\]

This can be rewritten as

\[
\int_\mathbb{R} \int_\mathbb{R} K(x-y) \, d\mu^k_+(x) \, d\mu^k_+(y) + \int_\mathbb{R} \int_\mathbb{R} K(x-y) \, d\mu^k_-(x) \, d\mu^k_-(y)
\]

\[
\geq 2 \int_\mathbb{R} \int_\mathbb{R} K(x-y) \, d\mu^k_+(x) \, d\mu^k_-(y)
\]

and the monotone convergence theorem leads to the desired inequality, as, for example,

\[
\int_\mathbb{R} \int_\mathbb{R} K(x-y) \, d\mu^k_+(x) \, d\mu^k_-(y) = \int_\mathbb{R} \int_{A_k \times A_k} (x,y) K(x-y) \, d\mu^k_+(x) \, d\mu^k_-(y)
\]

is valid.

Let us consider the case \( K(0) = \infty \). In this case, \( K \) is continuous on \((0, \infty)\) and has a limit \( \lim_{x \to 0} K(x) \) for any \( \varepsilon > 0 \), define

\[
K_\varepsilon(x) := \frac{1}{\varepsilon} \int_0^\varepsilon K(|x| + z) \, dz, \quad x \in \mathbb{R}.
\]

Then, by \( K \in L^1(\mathbb{R}) \), \( K \) is bounded everywhere by \( \varepsilon^{-1} ||K||_{L^1} \). Moreover, \( K_\varepsilon \) is still convex, such that \( K_\varepsilon(x-y) \) is positive semi-definite in measure. Now, the continuity of \( K \) leads to

\[
K_\varepsilon(x) = \int_0^1 K(|x| + \varepsilon z) \, dz \to K(|x|) \quad (\varepsilon \searrow 0)
\]

by the monotone convergence theorem. Applying the monotone convergence theorem to both sides of (4.11) with \( K = K_\varepsilon \), we obtain the conclusion. \( \square \)
The function $K = -\log |\tanh \left( \frac{x}{t} \right) |$ satisfies the condition of Lemma 4.3. Thus, we can observe the optimization problem

$$\begin{align*}
(P) \quad \text{minimize} & \quad \int_\mathbb{R} \int_\mathbb{R} K(x-y) \, d\mu(x) \, d\mu(y) + 2 \int_\mathbb{R} Q(x) \, d\mu(x) \\
\text{subject to} & \quad \mu \in \mathcal{M}(\mathbb{R}, n)
\end{align*}$$

as convex quadratic programming. We can analogously make the dual problem to the finite-dimensional case in Dorn (1960), as

$$\begin{align*}
(D) \quad \text{maximize} & \quad -\int_\mathbb{R} \int_\mathbb{R} K(x-y) \, d\nu(x) \, d\nu(y) + 2ns \\
\text{subject to} & \quad \nu \in \mathcal{S}_K, \quad s - \int_\mathbb{R} K(\cdot - y) \, d\nu(y) \leq Q.
\end{align*}$$

Note that this is a rigorous version of problem (D) in (2.12). It should be noted here that we have not justified (D) as a formal (topologically) dual problem. There are arguments limited to the optimization of Radon measure over compact space (Ohtsuka 1966a, Ohtsuka 1966b, Wu 2001). While they are on quadratic programming problems, there exist more general theories on duality, such as Isii (1964), von Neumann’s minimax theorem (von Neumann 1928, Sion 1958) and Fenchel-Rockafellar duality theorem (Rockafellar 1966, Villani 2003). However, as it is essential that our duality can treat infinite measure $\nu$ with unbounded support (we indeed later use such a measure as a dual feasible solution), it is difficult to just apply existing studies and check all the conditions for (D) to be a topologically dual problem. Therefore, we here do not go deeper in this aspect, but just prove the assertion of Theorem 4.4. This assertion is sufficient to derive a lower bound of the optimal value of (P), which is our objective.

In the following, we demonstrate that the weak duality and strong duality are still valid in this infinite-dimensional primal-dual pair. It should be noted that $s = 0, \nu \equiv 0$ is a trivial feasible solution of (D) such that there exists an optimal value of (D).

**Theorem 4.4.** The optimal value of (D) is equal to the optimal value of (P).

**Proof.** First, we present the weak duality. Let $\mu$ and $(\nu, s)$ be feasible solutions of (P) and (D), respectively, and $\nu = \nu_+ - \nu_-$ be the Hahn-Jordan decomposition. If we write $\langle \alpha, \beta \rangle_K := \int_\mathbb{R} \int_\mathbb{R} K(x-y) \, d\alpha(x) \, d\beta(y)$ for measures $\alpha$ and $\beta$,

$$\langle \nu, \nu \rangle_K = \langle \nu_+, \nu_+ \rangle_K + \langle \nu_-, \nu_- \rangle_K - 2\langle \nu_+, \nu_- \rangle_K$$

holds true. Because $\langle \mu, \mu \rangle_K, \langle \nu_+, \nu_+ \rangle_K, \langle \nu_-, \nu_- \rangle_K < \infty$, we have $\langle \mu, \nu_+ \rangle_K, \langle \mu, \nu_- \rangle_K, \langle \nu_+ , \nu_- \rangle_K < \infty$ by $K$’s positive semi-definiteness in measure. Therefore, we have

$$\begin{align*}
& \left( \int_\mathbb{R} \int_\mathbb{R} K(x-y) \, d\mu(x) \, d\mu(y) + 2 \int_\mathbb{R} Q(x) \, d\mu(x) \right) - \left( - \int_\mathbb{R} \int_\mathbb{R} K(x-y) \, d\nu(x) \, d\nu(y) + 2ns \right) \\
& = \langle \mu, \mu \rangle_K + \langle \nu, \nu \rangle_K + 2 \int_\mathbb{R} (Q(x) - s) \, d\mu(x) \\
& \geq \langle \mu, \mu \rangle_K + \langle \nu_+, \nu_+ \rangle_K + \langle \nu_-, \nu_- \rangle_K - 2\langle \nu_+, \nu_- \rangle_K + 2 \int_\mathbb{R} \left( - \int_\mathbb{R} K(x-y) \, d\nu(y) \right) \, d\mu(x) \\
& = \langle \mu, \mu \rangle_K + \langle \nu_+, \nu_+ \rangle_K + \langle \nu_-, \nu_- \rangle_K - 2\langle \nu_+, \nu_- \rangle_K \geq 0
\end{align*}$$

by the positive semi-definiteness in measure. This indicates the weak duality.

To prove the strong duality, we construct the optimal solution of (D) using that of (P). By Theorem 2.4 in Tanaka & Sugihara (2019), $\mu^*$, the optimal solution of (P), satisfies

$$\int_\mathbb{R} K(x-y) \, d\mu^*(y) + Q(x) \geq \frac{F_{K,Q}(n)}{n} \tag{4.3}$$
for all $x \in \mathbb{R}$. Now, $\mu^*$ and $n^{-1}F^C_{K,Q}(n)$ is a feasible solution for (D). Moreover, the equality of (3.3) is valid on the support of $\mu^*$, such that we have

$$
- \int_{\mathbb{R}} \int_{\mathbb{R}} K(x - y) \, d\mu^*(x) \, d\mu^*(y) + 2n \frac{F^C_{K,Q}(n)}{n} = - \int_{\mathbb{R}} \int_{\mathbb{R}} K(x - y) \, d\mu^*(x) \, d\mu^*(y) + 2 \int_{\mathbb{R}} \left( Q(x) + \int_{\mathbb{R}} K(x - y) \, d\mu^*(y) \right) \, d\mu^*(x) \\
= \int_{\mathbb{R}} \int_{\mathbb{R}} K(x - y) \, d\mu^*(x) \, d\mu^*(y) + 2 \int_{\mathbb{R}} Q(x) \, d\mu^*(x).
$$

This shows the strong duality.

\[\square\]

5 Proof of Theorem 2.4

We can now give a lower bound of $F^C_{K,Q}(n)$ by using the dual problem (D) and prove Theorem 2.4. Let $\alpha > 0$ be a constant and $f$ be the inverse Fourier transform of $F\left|\omega\right| = \frac{\omega}{\pi \tanh(d\omega)} \int_{-\alpha}^{\alpha} (Q(\alpha) - Q(x)) e^{-i\omega x} \, dx$

Along with this, $f$ is $L^2(\mathbb{R})$-integrable by Theorem 4.4 in Tanaka et al. (2017). Here, the Fourier transform of a function $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is defined by

$$
\mathcal{F}[g](\omega) := \int_{\mathbb{R}} g(x) e^{-i\omega x} \, dx
$$

and for the whole space $L^2(\mathbb{R})$, $\mathcal{F}[\cdot]$ is defined as the continuous extension of $\mathcal{F}[\cdot]|_{L^1 \cap L^2}$. Because $Q(x)$ is even by the assumption, $f$ is an inverse Fourier transform of an even real function, so that $f$ itself is an even real function. Then, the formula (p.43, 7.112 in Oberhettinger 1990)

$$
\mathcal{F} \left[ \log \left| \tanh \left( \frac{x}{4d} \right) \right| \right](\omega) = -\frac{\pi}{\omega} \tanh(d\omega)
$$

leads to the (almost everywhere) equation

$$
\mathcal{F} \left[ \int_{\mathbb{R}} K(x - y) f(y) \, dy \right](\omega) = \mathcal{F}[K](\omega) \cdot \mathcal{F}[f](\omega) = \int_{-\alpha}^{\alpha} (Q(\alpha) - Q(x)) e^{-i\omega x} \, dx, \quad (5.1)
$$

where $K \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ (see Appendix A) and $f \in L^2(\mathbb{R})$ are used for the justification of the first equality. The integrability of $K(x - \cdot)f(\cdot)$ comes from $K, f \in L^2(\mathbb{R})$ and by Minkowski’s integral inequality (see, e.g., Hardy et al. 1952, Theorem 202), we have

$$
\left\| \int_{\mathbb{R}} K(\cdot - y) f(y) \, dy \right\|_{L^2} = \left\| \int_{\mathbb{R}} K(y) f(\cdot - y) \, dy \right\|_{L^2} \leq \int_{\mathbb{R}} \|K(y)\|_{L^2} \|f(\cdot - y)\|_{L^2} \, dy \leq \|K\|_{L^1} \|f\|_{L^2} < \infty.
$$

Considering the inverse Fourier transform of (5.1), we also have

$$
\int_{\mathbb{R}} K(x - y) f(y) \, dy = 1_{[-\alpha, \alpha]}(x)(Q(\alpha) - Q(x)).
$$

It should be noted that $f(x) \, dx \in S_K$ follows from the inequality

$$
\int_{\mathbb{R}} \int_{\mathbb{R}} K(x - y) |f(x)f(y)| \, dx \, dy \leq \|K \ast f\|_{L^2} \|f\|_{L^2} \leq \|K\|_{L^1} \|f\|_{L^2}^2 < \infty.
$$
These two relations imply that \((f(x) \, dx, Q(\alpha))\) is a feasible solution of (D). We can now evaluate the value of the objective function of (D). Let us define
\[
F(\alpha) := -\int_{\mathbb{R}} \int_{\mathbb{R}} K(x-y)f(x)f(y) \, dx \, dy + 2nQ(\alpha).
\] (5.2)
Because the first term can be considered as the inner product of \(K \ast f\) and \(f\) in \(L^2(\mathbb{R})\), it can be computed through the Fourier transform as
\[
\begin{align*}
\int_{\mathbb{R}} \int_{\mathbb{R}} K(x-y)f(x)f(y) \, dx \, dy &= \frac{1}{2\pi} \int_{\mathbb{R}} \left( \frac{\omega}{\pi \tanh(d\omega)} \int_{-\alpha}^{\alpha} (Q(\alpha) - Q(x)) e^{-i\omega x} \, dx \right) \left( \int_{-\alpha}^{\alpha} (Q(\alpha) - Q(x)) e^{-i\omega x} \, dx \right) \, d\omega \\
&= \frac{1}{2\pi^2} \int_{\mathbb{R}} \frac{\omega}{\pi \tanh(d\omega)} \left| \int_{-\alpha}^{\alpha} (Q(\alpha) - Q(x)) e^{-i\omega x} \, dx \right|^2 \, d\omega.
\end{align*}
\] (5.3)
Let \(G(\alpha)\) be the value of the right-hand side. \(G(\alpha)\) can be decomposed into two parts, which are defined as
\[
G_1(\alpha) := \frac{1}{2\pi^2} \int_{-1}^{1} \frac{\omega}{\pi \tanh(d\omega)} \left| \int_{-\alpha}^{\alpha} (Q(\alpha) - Q(x)) e^{-i\omega x} \, dx \right|^2 \, d\omega
\]
and
\[
G_2(\alpha) := \frac{1}{2\pi^2} \int_{[-1,1]^c} \frac{\omega}{\pi \tanh(d\omega)} \left| \int_{-\alpha}^{\alpha} (Q(\alpha) - Q(x)) e^{-i\omega x} \, dx \right|^2 \, d\omega.
\]
We first evaluate \(G_1\). Because the function \(\omega/\tanh(d\omega)\) is monotonically increasing in \([0, \infty)\) (see the proof of Lemma 3.1), we have
\[
\begin{align*}
G_1(\alpha) &\leq \frac{1}{\pi \tanh(d)} \cdot \frac{1}{2\pi} \int_{\mathbb{R}} \left| \int_{-\alpha}^{\alpha} (Q(\alpha) - Q(x)) e^{-i\omega x} \, dx \right|^2 \, d\omega \\
&= \frac{1}{\pi \tanh(d)} \|1_{[-\alpha, \alpha]}(x)(Q(\alpha) - Q(x))\|_{L^2}^2 \\
&\leq \frac{2}{\pi \tanh(d)} \alpha Q(\alpha)^2.
\end{align*}
\] (5.4)
Next, we similarly evaluate \(G_2\). By integration by parts, we get
\[
\omega \int_{-\alpha}^{\alpha} (Q(\alpha) - Q(x)) e^{-i\omega x} \, dx = -\frac{1}{i} \int_{-\alpha}^{\alpha} Q'(x) e^{-i\omega x} \, dx.
\]
Thus, we have
\[
\begin{align*}
G_2(\alpha) &\leq \frac{1}{2\pi^2} \int_{[-1,1]^c} \frac{1}{\pi \tanh(d\omega)} \left| \int_{-\alpha}^{\alpha} Q'(x) e^{-i\omega x} \, dx \right|^2 \, d\omega \\
&\leq \frac{1}{\pi \tanh(d)} \|1_{[-\alpha, \alpha]}(x)Q'(x)\|_{L^2}^2 \\
&\leq \frac{2}{\pi \tanh(d)} \alpha Q'(\alpha)^2.
\end{align*}
\] (5.5)
Finally, we reach the evaluation
\[
G(\alpha) \leq \frac{2\alpha}{\pi \tanh(d)} (Q(\alpha)^2 + Q'(\alpha)^2), \quad F(\alpha) \geq 2nQ(\alpha) - \frac{2\alpha}{\pi \tanh(d)} (Q(\alpha)^2 + Q'(\alpha)^2).
\]
By letting \(\alpha_n\) satisfy
\[
\frac{2\alpha_n}{\pi \tanh(d)} \frac{Q(\alpha_n)^2 + Q'(\alpha_n)^2}{Q(\alpha_n)} \leq n,
\]
we get \(nQ(\alpha_n)\) as a lower bound for the optimal value of (P). For such \(\alpha_n\), we finally have
\[
nQ(\alpha_n) \leq J_{K,Q}(\mu^*) \leq 2F_{K,Q}(n)
\]
and this is equivalent to the assertion of Theorem 2.4
6 Examples of convergence rates for several $Q(x)$'s

Although the asymptotic rates given in Tanaka et al. (2017, Section 4.3) are derived through mathematically informal arguments, we here demonstrate that those rates roughly coincide with the bound in Theorem 2.3.

Example 6.1. (The case $w$ is a single exponential) Consider the case

$$w(x) = \exp\left(-(\beta|x|)^\rho\right), \quad Q(x) = (\beta|x|)^\rho,$$

for $\beta > 0$ and $\rho \geq 1$. In this case, for a sufficiently large $\alpha$ (satisfying $\alpha \geq \rho$), we have

$$\frac{2\alpha}{\pi \tanh(d)} \frac{Q(\alpha)^2 + Q'(\alpha)^2}{Q(\alpha)} = \frac{2\alpha}{\pi \tanh(d)} \frac{(\beta\alpha)^{2\rho} + (\beta\rho)^2(\beta\alpha)^{2(\rho-1)}}{(\beta\alpha)^\rho} \leq \frac{4\beta\rho\alpha^{\rho+1}}{\pi \tanh(d)}$$

and $\alpha_n$ can be taken as

$$\alpha_n = \left(\frac{\pi \tanh(d)}{4\beta\rho}\right)^{\frac{1}{\rho-1}},$$

and

$$\frac{Q(\alpha_n)}{2} = \frac{1}{2} \beta^{\rho} \left(\frac{\pi \tanh(d)}{4\beta\rho}\right)^{\frac{1}{\rho-1}} \left(\beta\pi^\frac{\rho}{\rho-1}n\right), \quad (6.1)$$

for sufficiently large $n$. This rate roughly coincides with (4.37) in Tanaka et al. (2017).

Example 6.2. (The case $w$ is a double exponential) Consider the case

$$w(x) = \exp\left(-\beta\exp(\gamma|x|)\right), \quad Q(x) = \beta\exp(\gamma|x|),$$

for $\beta, \gamma > 0$. In this case,

$$\frac{2\alpha}{\pi \tanh(d)} \frac{Q(\alpha)^2 + Q'(\alpha)^2}{Q(\alpha)} = \frac{2\alpha\beta(1 + \gamma^2)\exp(\gamma\alpha)}{\pi \tanh(d)}$$

is valid. Let $\alpha_n > 0$ satisfy that the right-hand side is equal to $n$. Then, we have

$$\gamma\alpha_n = W\left(\frac{\pi \tanh(d)\gamma}{2\beta(1 + \gamma^2)n}\right) \left(\sim \log\left(\frac{\gamma}{\beta(1 + \gamma^2)n}\right)\right),$$

where $W$ is Lambert’s W function, i.e., the inverse of $x \mapsto xe^x$. Using this, we get

$$\frac{Q(\alpha_n)}{2} = \frac{\beta}{2\gamma\alpha_n} \cdot \gamma\alpha_n \exp(\gamma\alpha_n) = \frac{\beta}{2\gamma\alpha_n} \frac{\pi \tanh(d)\gamma}{2\beta(1 + \gamma^2)n} = \frac{\pi \tanh(d)n}{4(1 + \gamma^2)\alpha_n}$$

$$= \frac{\pi \tanh(d)\gamma}{4(1 + \gamma^2)} W\left(\frac{\pi \tanh(d)\gamma}{2\beta(1 + \gamma^2)n}\right) \left(\sim \frac{\pi \tanh(d)\gamma}{4(1 + \gamma^2)} \frac{n}{\log\left(\frac{\gamma}{\beta(1 + \gamma^2)n}\right)}\right). \quad (6.2)$$

This rate roughly coincides with the asymptotic order (4.44) in Tanaka et al. (2017) for each fixed constant $\gamma$.

Remark 6.1. We choose the weight functions in Examples 6.1 and 6.2 for simplicity although they are not (necessarily) analytic in the strip region $D_d$ for any $d > 0$. This is because we just need their asymptotic properties for finding $\alpha_n$.

7 Conclusion

In this study, we analyzed the approximation method proposed by Tanaka & Sugihara (2019) over weighted Hardy spaces $\mathbb{H}^\infty(D_d, w)$. We provided (1) proof of the fact that the approximation formulas are nearly optimal from the viewpoint of minimum worst-case error $E_n^{\min}(\mathbb{H}^\infty(D_d, w))$; and (2) upper bounds of $E_n^{\min}(\mathbb{H}^\infty(D_d))$ to evaluate the convergence rates of approximation errors.
with \( n \to \infty \). To obtain (2), we introduced the concept “positive semi-definite in measure” and by using this, provided a lower bound for \( F_{K,Q}^n \). We also compared the given bounds with those mentioned in the study by Tanaka et al. (2017), and demonstrated that they have the same convergence rate with \( n \to \infty \).

The new bounds do not indicate that the approximation formulas in Tanaka & Sugihara (2019) are optimal. Another method to bound the error is recently considered by van Meurs & Tanaka (2022), although their bound do not show the optimality, either. We need tighter bounds to show the optimality, which may require more sophisticated analysis. We leave such analysis to future work.

**Acknowledgements**

The authors are grateful to Ryunosuke Oshiro for his comment on signed measures. This study was supported by the Japan Society for the Promotion of Science with KAKENHI (17K14241 to K.T.).

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**Appendix A  Proof of $K \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$**

It suffices to consider the case $d = \pi/4$, i.e., $K(x) = -\log |\tanh(x)|$, and prove

$$\int_0^\infty (-\log \tanh(x)) \, dx < \infty \quad \text{and} \quad \int_0^\infty (-\log \tanh(x))^2 \, dx < \infty$$

as $K$ is even. By variable transformation $y = \tanh(x) \left( \Leftrightarrow x = \frac{1}{2} \log \frac{1 + y}{1 - y} \right)$, we have

$$\int_0^\infty (-\log \tanh(x)) \, dx = \int_0^1 (-\log y) \frac{1}{1 - y^2} \, dy.$$

Additionally, by setting $z = -\log y$, we get

$$\int_0^1 (-\log y) \frac{1}{1 - y^2} \, dy = \int_0^\infty \frac{ze^{-z}}{1 - e^{-2z}} \, dz.$$

Performing the same variable transformations, we have

$$\int_0^\infty (-\log \tanh(x))^2 \, dx = \int_0^\infty \frac{z^2 e^{-z}}{1 - e^{-2z}} \, dz.$$
Because $z \geq z^2$ over $(0, 1]$ and $z \leq z^2$ over $[1, \infty)$, it suffices to show that
\[
\int_0^1 \frac{ze^{-z}}{1 - e^{-2z}} \, dz < \infty \quad \text{and} \quad \int_1^\infty \frac{z^2 e^{-z}}{1 - e^{-2z}} \, dz < \infty.
\]
For the former, because $e^{2z} - 1 \geq 2z$ is valid, we have
\[
\int_0^1 \frac{ze^{-z}}{1 - e^{-2z}} \, dz = \int_0^1 \frac{ze^z}{e^{2z} - 1} \, dz \leq \int_0^1 \frac{e^z}{2} \, dz < \infty.
\]
For the latter, we have
\[
\int_1^\infty \frac{z^2 e^{-z}}{1 - e^{-2z}} \, dz \leq \frac{1}{1 - e^{-2}} \int_1^\infty z^2 e^{-z} \, dz \leq \frac{1}{1 - e^{-2}} \Gamma(3) < \infty.
\]
Therefore, we finally get the result $K \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. 