Power series and analyticity over the quaternions

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Abstract We study power series and analyticity in the quaternionic setting. We first consider a function $f$ defined as the sum of a power series $\sum_{n \in \mathbb{N}} q^n a_n$ in its domain of convergence, which is a ball $B(0, R)$ centered at 0. At each $p \in B(0, R)$, $f$ admits expansions in terms of appropriately defined power series centered at $p$, namely $\sum_{n \in \mathbb{N}} (q - p)^n b_n$. The expansion holds in a ball $\Sigma(p, R - |p|)$ defined with respect to a (non-Euclidean) distance $\sigma$. We thus say that $f$ is $\sigma$-analytic in $B(0, R)$. Furthermore, we remark that $\Sigma(p, R - |p|)$ is not always a Euclidean neighborhood of $p$; when it is, we say that $f$ is strongly analytic at $p$. It turns out that $f$ is strongly analytic in a neighborhood of $B(0, R) \cap \mathbb{R}$ that can be strictly contained in $B(0, R)$. We then relate these notions of analyticity to the class of quaternionic functions introduced in Gentili and Struppa (Adv. Math. 216(1):279–301, 2007), and recently extended in Colombo et al. (Adv. Math. 222(5):1793–1808, 2009) under the name of slice regular functions. Indeed, $\sigma$-analyticity proves equivalent to slice regularity, in the same way as complex analyticity is equivalent to holomorphy. Hence the theory of slice regular quaternionic functions, which is quickly developing, reveals a new feature that reminds the nice properties of holomorphic complex functions.

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1 Introduction

Denote by \( \mathbb{H} \) the skew field of quaternions, obtained by endowing \( \mathbb{R}^4 \) with the following multiplicative operation: if \( 1, i, j, k \) denotes the standard basis, define

\[
i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j,
\]

let 1 be the neutral element and extend the operation by distributivity to all quaternions \( q = x_0 + x_1i + x_2j + x_3k \). We define the conjugate of such a \( q \) as \( \bar{q} = x_0 - x_1i - x_2j - x_3k \), its real and imaginary part as \( Re(q) = x_0 \) and \( Im(q) = x_1i + x_2j + x_3k \) and its modulus as \( |q| = \sqrt{Re(q)^2 + |Im(q)|^2} \), then the multiplicative inverse of each \( q \neq 0 \) is computed as

\[
q^{-1} = \frac{\bar{q}}{|q|^2}.
\]

Polynomials and power series with coefficients in the skew field \( \mathbb{H} \) are harder to handle than in the usual commutative setting. Indeed, if \( q = x_0 + x_1i + x_2j + x_3k \), then it is easy to verify that

\[
x_0 = \frac{1}{4} (q - iq - jq - kq), \quad x_1 = \frac{1}{4i} (q - iqi + jqj + kqk),
\]

\[
ex_2 = \frac{1}{4j} (q + iq - jqj + kqk), \quad x_3 = \frac{1}{4k} (q + iqi + jqj - kqk).
\]

Hence the set of polynomial functions of the quaternionic variable \( q \), built as sums of monomials of type \( a_0q + a_1q^2 + \cdots + a_nq^n \), can be identified with that of 4-tuples of polynomials in the four real variables \( x_0, x_1, x_2, x_3 \). This deprives such polynomial functions of the properties we are familiar with. For instance, some of them fail to have roots (see [15] or consider \( iq - qi - 1 \)). The classical approach of non commutative algebra (see e.g. [13]), which defines the ring of polynomials of \( \mathbb{H} \) as the set of the \( a_0 + a_1q + \cdots + a_nq^n \) for \( a_i \in \mathbb{H} \), leads instead to the quaternionic analog of the fundamental theorem of algebra (proven in [14] and extended in [8]) and to other nice properties. We adopt an analogous point of view and consider (polynomials and) power series of the following type.

**Definition 1** Let \( \{a_n\}_{n \in \mathbb{N}} \) be a sequence in \( \mathbb{H} \). We define

\[
f(q) = \sum_{n \in \mathbb{N}} q^na_n,
\]

as the power series associated to \( \{a_n\}_{n \in \mathbb{N}} \).

As stated in [11], the analog of Abel’s theorem holds.

**Theorem 1** For every power series \( f(q) = \sum_{n \in \mathbb{N}} q^na_n \) there exists an \( R \in [0, +\infty) \), called the radius of convergence of \( f(q) \), such that the series converges absolutely
and uniformly on compact sets in $B(0, R) = \{q \in \mathbb{H} : |q| < R\}$ and diverges in $\{q \in \mathbb{H} : |q| > R\}$.

As in the complex case, $R$ is defined by the formula

$$\limsup_{n \to \infty} |a_n|^{1/n} = 1/R.$$  

Thanks to Theorem 1, a power series having radius of convergence $R > 0$ defines a $C^\infty$ function $f : B(0, R) \to \mathbb{H}$. Such a function $f$ has very peculiar properties (see [9]): for instance, its zero set consists of isolated points or isolated 2-spheres. As the ring of polynomials (see [13]), the set of power series converging in $B(0, R)$ is endowed with the usual addition operation $+$ and the multiplicative operation $\ast$ defined by

$$\left(\sum_{n \in \mathbb{N}} q^n a_n\right) \ast \left(\sum_{n \in \mathbb{N}} q^n b_n\right) = \sum_{n \in \mathbb{N}} q^n \sum_{k=0}^n a_k b_{n-k}.$$  \hspace{1cm} (3)

Notice that these operations define a ring structure, as well as a structure of real algebra (see [9] and references therein).

Defining an appropriate notion of analyticity over the quaternions is a delicate issue. A first attempt could be to consider expansions (at $p$) in terms of series of monomials of the type $a_0(q-p)\alpha_1(q-p)\ldots\alpha_{n-1}(q-p)a_n$, as done in [12]. However, due to Eq. 1, this reduces the problem to that of analyticity in the four real variables $x_0, x_1, x_2, x_3$ (for details, see [17]). A second possibility is to consider series of monomials of the type $(q-p)^n\alpha$, but we are immediately discouraged by the following example.

Example 1 The simple monomial $q^2$ does not admit an expansion of the type

$$q^2 = (q-p)^2c_2 + (q-p)c_1 + c_0$$

at any point $p \in \mathbb{H}\backslash \mathbb{R}$. Indeed, such an equality would imply $q^2 = q^2 - pq + p^2 + qc_1 - pc_1 + c_0$ and $q(c_1 - p) = pq$ for all $q \in \mathbb{H}$. This would yield $c_1 - p = p$ and $qp = pq$ for all $q \in \mathbb{H}$, which is impossible. The aforementioned expansion can only be performed at points $x \in \mathbb{R} \subset \mathbb{H}$, where we get the usual formula $q^2 = (q-x)^2 + (q-x)2x + x^2$ for all $q \in \mathbb{H}$.

The difficulty arises in the previous example because $(q-p)^2$ is not a polynomial when $p \not\in \mathbb{R}$. We can solve this problem using $\ast$-powers of $q-p$ instead. Let $(q-p)^{\ast n} = (q-p)\ast\ldots\ast(q-p)$ denote the $n$-th power of the binomial $q-p$ with respect to $\ast$-multiplication. Remark that $(q-x)^{\ast n} = (q-x)^n$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$, while $(q-p)^{\ast n}$ and $(q-p)^n$ differ for $p \in \mathbb{H}\backslash \mathbb{R}$ and $n \geq 2$. A direct computation leads to the following version of the binomial theorem.

Proposition 1 Fix $p \in \mathbb{H}$ and $m \in \mathbb{N}$. Then

$$q^m = \sum_{k=0}^m (q-p)^{\ast k} p^{m-k}\binom{m}{k}.$$  \hspace{1cm} (4)
We notice that Proposition 1 provides all polynomials with expansions at an arbitrary point \( p \). It also hints that power series \( \sum_{n \in \mathbb{N}} q^n a_n \) converging in \( B(0, R) \) might expand at \( p \in B(0, R) \) in terms of series of the following type.

**Definition 2** Let \( p \in \mathbb{H} \) and let \( \{a_n\}_{n \in \mathbb{N}} \) be a sequence in \( \mathbb{H} \). We define
\[
\sum_{n \in \mathbb{N}} (q - p)^n a_n
\]
as the **power series centered at** \( p \) **associated to** \( \{a_n\}_{n \in \mathbb{N}} \).

We thus propose the notion of strong analyticity (the motivation for the adjective “strong” will become clear in the sequel).

**Definition 3** Let \( \Omega \) be an open subset of \( \mathbb{H} \). A function \( f : \Omega \to \mathbb{H} \) is strongly analytic at \( p \in \Omega \) if there exists a power series \( \sum_{n \in \mathbb{N}} (q - p)^n a_n \) converging in a neighborhood \( U \) of \( p \) in \( \Omega \) such that \( f(q) = \sum_{n \in \mathbb{N}} (q - p)^n a_n \) for all \( q \in U \). We say that \( f \) is strongly analytic if it is strongly analytic at all \( p \in \Omega \).

The first question we have to answer is: where does a series of type (5) converge? If the radius of convergence of the quaternionic power series defined by (2) is \( R > 0 \), one might expect the series (5) to converge for all \( q \) in the open Euclidean ball \( B(p, R) \) of center \( p \) and radius \( R \) in \( \mathbb{H} \). Surprisingly, this is not the case. We begin in Sect. 2 with an estimate of \( |(q - p)^n| \) in terms of the function \( \sigma \) defined as follows. If, for all \( I \in \mathbb{S} = \{q \in \mathbb{H} : q^2 = -1\} \), we denote \( L_I = \mathbb{R} + I\mathbb{R} \) the plane through 0, 1 and \( I \) (which can be identified with the complex plane \( \mathbb{C} \)) then
\[
\sigma(q, p) = \begin{cases} 
|q - p| & \text{if } p, q \text{ lie on the same complex plane } L_I \\
\omega(q, p) & \text{otherwise}
\end{cases}
\]
where
\[
\omega(q, p) = \sqrt{[Re(q) - Re(p)]^2 + [|Im(q)| + |Im(p)|]^2}.
\]
The map \( \sigma \) proves to be a distance that is not topologically equivalent to the Euclidean distance (see Sect. 3). Actually, the topology induced by \( \sigma \) in \( \mathbb{H} \) is finer than the Euclidean topology. The sets of convergence of series of type (5) turn out to be \( \sigma \)-balls centered at \( p \) (see Sect. 4). The shape of such sets, portrayed in Fig. 2, is quite curious but coherent with Theorem 4.2 in [4]. We then study the strong analyticity of the sum \( f \) of a series \( f(q) = \sum_{n \in \mathbb{N}} q^n a_n \) in Sect. 5. The main result of this section is the following.

**Theorem 2** Any function defined by a power series \( \sum_{n \in \mathbb{N}} q^n a_n \) which converges in \( B = B(0, R) \) is strongly analytic in the open set
\[
\mathcal{A}(B) = \{p \in \mathbb{H} : 2|Im(p)| < R - |p|\}.
\]
Since \( A(\mathbb{H}) = \mathbb{H} \), all quaternionic entire functions are strongly analytic in \( \mathbb{H} \). If \( B \neq \mathbb{H} \), the set \( A(B) \) is strictly contained in \( B \). Furthermore, the previous theorem is sharp: we exhibit an example of power series which converges in \( B \) and is strongly analytic in \( A(B) \) only. This suggests the definition of a weaker notion of analyticity.

**Definition 4** Let \( \Omega \) be a \( \sigma \)-open subset of \( \mathbb{H} \). A function \( f : \Omega \to \mathbb{H} \) is \( \sigma \)-analytic at \( p \in \Omega \) if there exists a power series \( \sum_{n \in \mathbb{N}}(q - p)^{\sigma n}a_n \) converging in a \( \sigma \)-neighborhood \( U \) of \( p \) in \( \Omega \) such that \( f(q) = \sum_{n \in \mathbb{N}}(q - p)^{\sigma n}a_n \) for all \( q \in U \). We say that \( f \) is \( \sigma \)-analytic if it is \( \sigma \)-analytic at all \( p \in \Omega \).

The notion of \( \sigma \)-analyticity produces the following result (to be compared to the complex case).

**Theorem 3** A power series \( \sum_{n \in \mathbb{N}}q^\sigma a_n \) having radius of convergence \( R \) defines a \( \sigma \)-analytic function on \( B(0, R) \).

Finally, in Sect. 6 we relate our study to that of slice regular functions, quaternionic analogs of holomorphic functions that have been recently introduced (see [4,11]).

**Theorem 4** A quaternionic function is slice regular in a domain if, and only if, it is \( \sigma \)-analytic in the same domain.

It is important to point out that, in view of Theorem 4, the notion of \( \sigma \)-analyticity, which naturally appears when studying the set of convergence of power series centered at \( p \in \mathbb{H} \), turns out not to be a new notion. It is in fact equivalent to slice regularity. This equivalence urges a comparison with the classical case of complex holomorphic functions and gives a deeper insight into the theory of slice regular functions. This theory is developing well in different directions and it has already been extensively studied in a series of papers (see for example [9] or [10]). Slice regular functions have already proven their interest: for instance, they have applications to the theory of quaternionic linear operators (see [1–3]). The achievements of this paper give a new viewpoint, and hence enrich, the theory of slice regular functions.

We conclude the paper by also investigating the notion of strong analyticity. Indeed, we are able to prove that

**Theorem 5** All slice regular functions on a domain \( \Omega \subseteq \mathbb{H} \) are strongly analytic in \( A(\Omega) = \{ p \in \Omega : 2|\text{Im}(p)| < \sigma(p, \partial_\sigma \Omega) \} \), where \( \partial_\sigma \Omega \) denotes the boundary of \( \Omega \) with respect to the \( \sigma \)-topology. Moreover, if \( \Omega \) intersects the real axis, then \( A(\Omega) \) contains a neighborhood of \( \Omega \cap \mathbb{R} \).

2 Estimate of the modulus of \((q - p)^{\sigma n}\)

In order to study the convergence of a power series \( \sum_{n \in \mathbb{N}}(q - p)^{\sigma n}a_n \) centered at \( p \in \mathbb{H} \), we first estimate the growth of \(|(q - p)^{\sigma n}|\) as \( n \to \infty \) approaches infinity. Define \( P(q) = q - p \) for all \( q \in \mathbb{H} \) and recall a well-known algebraic property of \( \mathbb{H} \): for all \( I \in \mathbb{S} = \{ q \in \mathbb{H} : q^2 = -1 \} \), the subalgebra \( L_I = \mathbb{R} + IR \) of \( \mathbb{H} \) is isomorphic to \( \mathbb{C} \).
C. In particular, if we choose \( I \) such that \( p \in L_I \) then \( p \) commutes with any \( z \in L_I \). As a consequence,

\[
P^{\infty}(z) = (z - p)^n
\]

and \(|P^{\infty}(z)| = |z - p|^n\) for all \( z \in L_I \). In order to estimate \(|P^{\infty}(q)|\) at a generic \( q \in \mathbb{H} \), we begin with what follows.

**Definition 5** For all \( p, q \in \mathbb{H} \), we define

\[
\omega(q, p) = \sqrt{[Re(q) - Re(p)]^2 + [|Im(q)| + |Im(p)|]^2}
\]

and

\[
\sigma(q, p) = \begin{cases} 
|q - p| & \text{if } p, q \text{ lie on the same complex plane } L_I \\
\omega(q, p) & \text{otherwise}
\end{cases}
\]

The function \( \sigma \) can also be expressed in the following manner.

**Lemma 1** Let \( p \in L_I \), choose any \( q \in \mathbb{H} \setminus L_I \) and let \( z, \bar{z} \) be the points of \( L_I \) such that \( Re(z) = Re(\bar{z}) = Re(q) \) and \( |Im(z)| = |Im(\bar{z})| = |Im(q)| \). Then

\[
\sigma(q, p) = \max\{|z - p|, |\bar{z} - p|\}.
\]

**Proof** The thesis follows by direct computation:

\[
\begin{align*}
\max\{|z - p|, |\bar{z} - p|\}^2 &= (Re(z) - Re(p))^2 + \max\{|Im(z) - Im(p)|^2, |Im(z) - Im(p)|^2\} \\
&= (Re(z) - Re(p))^2 + (|Im(z)| + |Im(p)|)^2 \\
&= (Re(q) - Re(p))^2 + (|Im(q)| + |Im(p)|)^2 = \sigma(q, p)^2.
\end{align*}
\]

In other words, if \( p \in L_I \setminus \mathbb{R} \) and if \( q \in \mathbb{H} \setminus L_I \) then we consider the rotations of \( \mathbb{H} \) fixing all points of \( \mathbb{R} \) and mapping \( q \) to \( L_I \). The point \( q \) can be mapped to a point \( w \) of the half plane bounded by \( \mathbb{R} \) and containing \( \bar{p} \), or to \( \tilde{w} \) (a point of the half plane bounded by \( \mathbb{R} \) and containing \( p \)). In this situation, \( \sigma(q, p) \) equals the Euclidean distance \(|w - p| = \max\{|w - p|, |\bar{w} - p|\}\). Furthermore, if \( r \) is the point of intersection between \( \mathbb{R} \) and the line through \( w \) and \( p \), then

\[
\sigma(q, p) = |w - p| = |w - r| + |r - p| = |q - r| + |r - p|
\]

as shown in Fig. 1.

We now estimate \(|(q - p)^n|\) in terms of the map \( \sigma \) defined above.

**Theorem 6** Fix \( p \in \mathbb{H} \). Then

\[
|(q - p)^n| \leq 2\sigma(q, p)^n
\]
for all \( n \in \mathbb{N} \). Moreover,

\[
\lim_{n \to \infty} |(q - p)^n|^{1/n} = \sigma(q, p). \tag{10}
\]

We will prove Theorem 6 by means of the following result, presented in [10]. For all \( x, y \in \mathbb{R} \), denote \( x + S y = \{ x + Iy : I \in \mathbb{S} \} \) (a 2-sphere if \( y \neq 0 \), a real singleton \( \{ x \} \) if \( y = 0 \).

**Theorem 7** Let \( f(q) = \sum_{n \in \mathbb{N}} q^n a_n \) have radius of convergence \( R > 0 \) and let \( x, y \in \mathbb{R} \) be such that \( x + S y \subset B(0, R) \). There exist \( b(x, y), c(x, y) \in \mathbb{H} \) such that

\[
f(x + Iy) = b(x, y) + Ic(x, y) \tag{11}
\]

for all \( I \in \mathbb{S} \).

Namely, \( b(x, y) = \sum_{n \in \mathbb{N}} \beta_n a_n \) and \( c(x, y) = \sum_{n \in \mathbb{N}} \gamma_n a_n \) with \( \{ \beta_n \}_{n \in \mathbb{N}} \) and \( \{ \gamma_n \}_{n \in \mathbb{N}} \) real sequences such that \( (x + Iy)^n = \beta_n + I\gamma_n \) for all \( I \in \mathbb{S} \) and all \( n \in \mathbb{N} \). The next Corollary can be immediately derived (see [5]).

**Corollary 1** If the hypotheses of Theorem 7 hold, then

\[
f(x + Jy) = \frac{f(x + Iy) + f(x - Iy) + Jf(x - Iy) - f(x + Iy)}{2}
= \frac{1 - JJ}{2} f(x + Iy) + \frac{1 + JJ}{2} f(x - Iy) \tag{12}
\]

for all \( I, J \in \mathbb{S} \).

We are now ready to prove the announced result.

**Proof (Proof of Theorem 6)** Let \( p \in L_I \) and \( P(q) = q - p \), consider \( P^n(q) \) for \( n \in \mathbb{N} \). As we already mentioned, for all \( z = x + Iy \in L_I \), \( z \) and \( p \) commute so...
that \( P^{\ast n}(z) = (z - p)^n \) and \(|P^{\ast n}(z)| = |z - p|^n = \sigma(z, p)^n \). For \( q = x + Jy \) with \( J \in \mathbb{S}\setminus\{\pm i\} \) we compute:

\[
P^{\ast n}(q) = \frac{1 - JI}{2} P^{\ast n}(z) + \frac{1 + JI}{2} P^{\ast n}(\bar{z}) = \frac{1 - JI}{2} (z - p)^n + \frac{1 + JI}{2} (\bar{z} - p)^n.
\]

If \(|z - p| < |\bar{z} - p|\) then \( Q = \frac{z - p}{\bar{z} - p} \) has modulus \(|Q| < 1\) and

\[
|P^{\ast n}(q)| = \left| \frac{1 - JI}{2} Q^n + \frac{1 + JI}{2} |\bar{z} - p|^n \right| \leq 2|z - p|^n.
\]

Moreover, we observe that \( Q^n \to 0 \) as \( n \to \infty \) and \( \frac{1 + JI}{2} \neq 0 \), so that

\[
\lim_{n \to \infty} \left| P^{\ast n}(q) \right|^{1/n} = \lim_{n \to \infty} \left| \frac{1 - JI}{2} Q^n + \frac{1 + JI}{2} |\bar{z} - p|^n \right|^{1/n} = |z - p|.
\]

If \(|z - p| \geq |\bar{z} - p|\) then similar computations show that

\[
|P^{\ast n}(q)| \leq 2|z - p|^n,
\]

\[
\lim_{n \to \infty} |P^{\ast n}(q)|^{1/n} = |z - p|.
\]

We conclude, since Lemma 1 implies \( \max(|z - p|, |\bar{z} - p|) = \sigma(q, p) \).

3 The map \( \sigma \) is a distance

Before going back to power series, let us study the properties of \( \sigma \) and \( \omega \). Notice that \( |q - p| \leq \sigma(q, p) \leq \omega(q, p) \) for all \( p, q \in \mathbb{H} \). Moreover,

**Proposition 2** The function \( \sigma : \mathbb{H} \times \mathbb{H} \to \mathbb{R} \) is a distance.

**Proof** Definition 5 yields that \( \sigma(q, p) \geq 0 \) for all \( p, q \in \mathbb{H} \) and that \( \sigma(q, p) = 0 \) if and only if \( q = p \). It also immediately implies that \( \sigma(q, p) = \sigma(p, q) \). The triangle inequality can be proven as follows. Let \( L_I \) be a complex plane through \( p \). If \( q \in L_I \) then for all \( o \in \mathbb{H} \)

\[
\sigma(q, o) + \sigma(o, p) \geq |q - o| + |o - p| \geq |q - p| = \sigma(q, p)
\]

as wanted. Now suppose \( q \in \mathbb{H}\setminus L_I \), i.e. \( q = x + Jy \in L_I \) with \( y > 0 \), \( J \in \mathbb{S}\setminus\{\pm i\} \).

(a) For all \( o \in L_I \), Lemma 1 implies:

\[
\sigma(q, o) + \sigma(o, p) = \max(|x + Iy - o|, |x - Iy - o|) + |o - p| \geq \max(|x + Iy - p|, |x - Iy - p|) = \sigma(q, p).
\]
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(b) For all \( o \in L_I \), the triangle inequality follows from (a), reversing the roles of \( q \) and \( p \).

(c) Finally, for \( o \in \mathbb{H} \setminus (L_I \cup L_J) \) we prove it as follows. Let \( z, \bar{z} \) be the two points of \( L_I \) having \( Re(z) = Re(o) = Re(\bar{z}) \) and \( |Im(z)| = |Im(o)| = |Im(\bar{z})| \). Without loss of generality, max\([|z - p|, |\bar{z} - p|] = |z - p| \) and Lemma 1 implies:

\[
\sigma(o, p) = \max\{|z - p|, |\bar{z} - p|\} = |z - p| = \sigma(z, p)
\]

where we used the fact that \( o \) and \( p \) do not lie in the same complex plane, while \( z \) and \( p \) do. Furthermore, taking into account that \( q \) does not lie in the same complex plane as \( o \) nor in the same as \( z \), we compute:

\[
\sigma(q, o) = \omega(q, o) = \sqrt{[Re(q) - Re(o)]^2 + (|Im(q)| + |Im(o)|)^2}
\]

\[
= \sqrt{[Re(q) - Re(z)]^2 + (|Im(q)| + |Im(z)|)^2} = \omega(q, z) = \sigma(q, z).
\]

Hence

\[
\sigma(q, o) + \sigma(o, p) = \sigma(q, z) + \sigma(z, p) \geq \sigma(q, p)
\]

where the last inequality follows from (a) since \( z, p \) lie in the same complex plane \( L_I \).

Let us conclude this section studying the \( \sigma \)-balls

\[
\Sigma(p, R) = \{ q \in \mathbb{H} : \sigma(q, p) < R \}.
\]

From \( |q - p| \leq \sigma(q, p) \leq \omega(q, p) \), we derive

\[
\Omega(p, R) = \{ q \in \mathbb{H} : \omega(q, p) < R \} \subseteq \Sigma(p, R) \subseteq B(p, R).
\]

More precisely:

Remark 1 If \( p \in L_I \subset \mathbb{H} \) and \( B_I(p, R) = \{ z \in L_I : |z - p| < R \} \) then

\[
\Sigma(p, R) = \Omega(p, R) \cup B_I(p, R). \tag{13}
\]

We are thus left with studying \( \Omega(p, R) \). In [10] we called a set \( C \subseteq \mathbb{H} \) axially symmetric if all the spheres \( x + S_y \) (with \( x, y \in \mathbb{R} \)) that intersect \( C \) are entirely contained in \( C \). The axially symmetric sets had been previously introduced, with a different approach, under the name of intrinsic sets (see [6]). Since an axially symmetric set is completely determined by its intersection with a complex plane \( L_I \), we can describe \( \Omega(p, R) \) as follows.

Remark 2 \( \Omega(p, R) \) is the axially symmetric open set whose intersection with the complex plane \( L_I \) through \( p \) is \( B_I(p, R) \cap B_I(p, R) \). In particular \( \Omega(p, R) \) is empty when \( R \leq |Im(p)| \), it intersects the real axis when \( R > |Im(p)| \) and it includes \( p \) if and only if \( R > 2|Im(p)| \).
Fig. 2 A view in $\mathbb{R} + i\mathbb{R} + j\mathbb{R}$ of $\sigma$-balls $\Sigma(p, R)$ centered at points $p \in L_i = \mathbb{R} + i\mathbb{R}$ and having $|\text{Im}(p)| \geq R, 0 < |\text{Im}(p)| < R$ and $\text{Im}(p) = 0$, respectively.

The fact that $\Omega(p, R)$ is axially symmetric is proven observing that, when $p$ is fixed, $\omega(x + Jy, p)$ only depends on $x$ and $y$. Moreover, $\Omega(p, R) \cap L_I = B_I(p, R) \cap B_I(\bar{p}, R)$ because $\omega(z, p) = \max(|z - p|, |\bar{z} - p|) = \max(|z - p|, |z - \bar{p}|)$ for all $z \in L_I$.

We conclude this section remarking that

**Remark 3** $\Omega(p, R)$ is the interior of $\Sigma(p, R)$ and $\Sigma(p, R) = \{q \in \mathbb{H} : \sigma(q, p) \leq R\}$ (with respect to the Euclidean topology).

### 4 Convergence of power series centered at $p$

The estimate given in Sect. 2 and the study of the $\sigma$-balls conducted in Sect. 3 allow the following result, which will be an important tool in the study of strong analyticity (Fig. 2).

**Theorem 8** Choose a sequence $\{a_n\}_{n \in \mathbb{N}} \subset \mathbb{H}$ and let $R \in [0, +\infty]$ be such that $1/R = \limsup_{n \to \infty} |a_n|^{1/n}$. For all $p \in \mathbb{H}$ the series

$$f(q) = \sum_{n \in \mathbb{N}} (q - p)^*a_n$$

converges absolutely and uniformly on the compact subsets of $\Sigma(p, R)$ and it does not converge at any point of $\mathbb{H} \setminus \Sigma(p, R)$. We call $R$ the $\sigma$-radius of convergence of $f(q)$.

**Proof** In each set $\Sigma(p, r)$ with $r < R$, the function series $\sum_{n \in \mathbb{N}} (q - p)^*a_n$ is dominated by the convergent number series $2 \sum_{n \in \mathbb{N}} r^n |a_n|$ thanks to Theorem 6. By the same theorem $\lim_{n \to \infty} |(q - p)^*a_n|^{1/n} = \sigma(q, p)$, so that

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\[
\limsup_{n \to \infty} |(q - p)^{\ast n}a_n|^{1/n} = \frac{\sigma(q, p)}{R}.
\]

Hence the series cannot converge at any point \( q \) such that \( \sigma(q, p) > R \).

5 Analyticity of power series

The algebra of power series centered at 0 can be endowed with the following derivation.

**Definition 6** Let \( f(q) = \sum_{n \in \mathbb{N}} q^n a_n \) be a power series. Its (formal) derivative is defined as the power series \( f'(q) = \sum_{n \in \mathbb{N}} q^n a_{n+1} (n + 1) \).

Remark that \( f'(q) \) (which we sometimes denote as \( f(q)' \)) has the same radius of convergence as \( f(q) \). A long computation proves Leibniz’s rule:

**Proposition 3** For all power series \( f(q) \) and \( g(q) \) centered at 0,

\[
(f(q) \ast g(q))' = f'(q) \ast g(q) + f(q) \ast g'(q).
\]

(15)

As a consequence, the \( n \)-th derivative of \( f(q) = \sum_{n \in \mathbb{N}} q^n a_n \), which we denote \( f^{(n)} \), can be expressed as:

\[
f^{(n)}(q) = \sum_{m \in \mathbb{N}} q^m a_{m+n} \frac{(m + n)!}{m!}.
\]

(16)

In particular, \( f^{(n)}(0) = a_n n! \) for all \( n \in \mathbb{N} \), so that

\[
f(q) = \sum_{n \in \mathbb{N}} q^n f^{(n)}(0) \frac{1}{n!}.
\]

To proceed, we will need the following estimate, which can be proven as in the complex case.

**Lemma 2** If \( f(q) \) has radius of convergence \( R > 0 \) and \( p \in B(0, R) \) then

\[
\limsup_{n \to \infty} \left| \frac{f^{(n)}(p)}{n!} \right|^{1/n} \leq \frac{1}{R - |p|}.
\]

(17)

We are now ready to exhibit expansions of \( f \) at any point \( p \in B(0, R) \).

**Theorem 9** Let \( f(q) = \sum_{n \in \mathbb{N}} q^n a_n \) have radius of convergence \( R > 0 \) and let \( p \in B(0, R) \). Then

\[
f(q) = \sum_{n \in \mathbb{N}} (q - p)^{\ast n} f^{(n)}(p) \frac{1}{n!}
\]

(18)

for all \( q \in \Sigma(p, R - |p|) \).
Proof Thanks to Theorem 8 and to Lemma 2, the series on the right hand side of Eq. 18 converges absolutely in $\Sigma(p, R - |p|)$. Applying formula (16) and Lemma 1, we compute

$$
\sum_{k \in \mathbb{N}} (q - p)^k f^{(k)}(p) \frac{1}{k!} = \sum_{k,n \in \mathbb{N}} (q - p)^k p^n a_{n+k} \frac{(n+k)!}{n!} \frac{1}{k!}
$$

$$
= \sum_{m \in \mathbb{N}} \left[ \sum_{k=0}^m (q - p)^k p^{m-k} \binom{m}{k} \right] a_m = \sum_{m \in \mathbb{N}} q^m a_m = f(q)
$$

for all $q \in \Sigma(p, R - |p|)$.

As a consequence, according to Definition 3, the function $f$ is strongly analytic at each $p \in B(0, R)$ such that $p$ lies in the interior of $\Sigma(p, R - |p|)$. Recalling that the interior of $\Sigma(p, R - |p|)$ is the set which we called $\Omega(p, R - |p|)$ and that the latter includes $p$ if and only if $R - |p| > 2|\text{Im}(p)|$, we get the following result.

Theorem 10 Any function defined by a power series $\sum_{n \in \mathbb{N}} q^n a_n$ which converges in $B = B(0, R)$ is strongly analytic in the open set

$$
\mathcal{A}(B) = \{ p \in \mathbb{H} : 2|\text{Im}(p)| < R - |p| \}.
$$

We remark that $\mathcal{A}(B) \subseteq B$ is the (open) region bounded by the hypersurface consisting of all points $p \in B$ such that $x = \text{Re}(p)$ and $y = |\text{Im}(p)|$ verify

$$
x^2 - 3\left( y - \frac{2}{3}R \right)^2 + \frac{R^2}{3} = 0.
$$

In other words, if for any $I \in \mathbb{S}$ we consider the arc of hyperbola $\mathcal{H}(I) = \{ x + iy \in L_I : 0 \leq y < R, \ x, y \text{ verify } (20) \}$, then the boundary of $\mathcal{A}(B)$ is the hypersurface of revolution generated rotating $\mathcal{H}(I)$ around the real axis as follows:

$$
\partial \mathcal{A}(B) = \bigcup_{I \in \mathbb{S}} \mathcal{H}(I).
$$

The previous result is quite surprising, if compared to the analogous result for complex power series. Even more surprisingly, it is sharp: the following example proves that some power series with radius of convergence $R$ are strongly analytic in $\mathcal{A}(B)$ only. Recall that, as explained in [16], the complex series $\sum_{n \in \mathbb{N}} z^n$ converges in the open unit disc of $\mathbb{C}$ and it does not extend to a holomorphic function near any point of the boundary.

Example 2 The sum of the quaternionic series $f(q) = \sum_{n \in \mathbb{N}} q^n$, which converges in $B = B(0, 1)$, is strongly analytic in $\mathcal{A}(B)$ only. Indeed, suppose $f$ were strongly analytic at $p \in B \setminus \mathcal{A}(B)$ (i.e. at $p \in B$ with $2|\text{Im}(p)| \geq 1 - |p|$). There would exist a $\sigma$-ball $\Sigma(p, r)$ of radius $r > 2|\text{Im}(p)|$ and a power series $\sum_{n \in \mathbb{N}} (q - p)^n a_n$ converging in $\Sigma(p, r)$ and coinciding with $f(q)$ for all $q \in B(0, 1) \cap \Sigma(p, r)$. The coefficients

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a_n would have to lie in the complex plane \( L_I \) through \( p \), because \( f(B_I(0, 1)) \subseteq L_I \) (using a slight variation of corollary 2.8 in [11]). Restricting to \( L_I = \mathbb{R} + I \mathbb{R} \simeq \mathbb{C} \), we would have \( f(z) = \sum_{n \in \mathbb{N}}(z - p)^n a_n \) for all \( z \in B_I(0, 1) \cap B_I(p, r) \), with \( B_I(p, r) \) not contained in the unit disc \( B_I(0, 1) \) because \( r > 2|\text{Im}(p)| \geq 1 - |p| \). The sum of the series \( \sum_{n \in \mathbb{N}}(z - p)^n a_n \) in \( B_I(p, R) \) would thus extend \( \sum_{n \in \mathbb{N}} z^n \) near some point of the boundary of \( B_I(0, 1) \), which is impossible.

The only case when we immediately get strong analyticity on the whole domain is the following.

**Theorem 11** A quaternionic entire function, i.e. the sum of a power series \( \sum_{n \in \mathbb{N}} q^n a_n \) converging in \( \mathbb{H} \), is always strongly analytic in \( \mathbb{H} \).

The peculiar results we have proven depend on the fact that the sets of convergence of power series are \( \sigma \)-balls, which are not open in the Euclidean topology. We are thus encouraged to define a weaker notion of analyticity in terms of the topology induced in \( \mathbb{H} \) by the distance \( \sigma \), a topology that is finer than the Euclidean.

**Definition 7** Let \( \Omega \) be a \( \sigma \)-open subset of \( \mathbb{H} \). A function \( f : \Omega \to \mathbb{H} \) is \( \sigma \)-analytic at \( p \in \Omega \) if there exist \( \sum_{n \in \mathbb{N}}(q - p)^n a_n \) and \( R > 0 \) such that \( f(q) = \sum_{n \in \mathbb{N}}(q - p)^n a_n \) in \( \Sigma(p, R) \cap \Omega \). We say that \( f \) is \( \sigma \)-analytic if it is \( \sigma \)-analytic at all \( p \in \Omega \).

We remark that in the previous definition (and in the rest of the paper) the series we consider are still meant to converge with respect to the usual Euclidean norm. We only make use of the \( \sigma \)-topology when dealing with subsets of \( \mathbb{H} \) (typically, with the domains of definition of slice regular functions) and only when explicitly stated.

We immediately deduce from Theorem 9 the following result. We notice the complete analogy with the complex case.

**Theorem 12** A power series \( \sum_{n \in \mathbb{N}} q^n a_n \) having radius of convergence \( R \) defines a \( \sigma \)-analytic function on \( B(0, R) \).

At this point, it would be natural to inquire about the strong analyticity and \( \sigma \)-analyticity of sums of power series \( \sum_{n \in \mathbb{N}}(q - p)^n a_n \) centered at an arbitrary point \( p \in \mathbb{H} \) and converging in a \( \sigma \)-ball \( \Sigma(p, R) \). Instead of employing the same techniques we applied in this section, we postpone this problem to the next section where such series are studied as part of a larger class of quaternionic functions.

### 6 Analyticity of slice regular functions

Besides its independent interest, the study we conducted in the previous sections is motivated by the theory of quaternion-valued functions of one quaternionic variable introduced in [11] and developed in subsequent papers (e.g. [4,9,10] and references therein). The theory, which is also the basis for a new functional calculus in a non commutative setting (see [1–3]), relies upon a definition inspired by Cullen [6] (see also [7]).
Definition 8 Let $\Omega$ be a domain in $\mathbb{H}$ and let $f : \Omega \to \mathbb{H}$ be a function. For all $I \in \mathbb{S}$, we denote $L_I = \mathbb{R} + I\mathbb{R}$, $\Omega_I = \Omega \cap L_I$ and $f_I = f|_{\Omega_I}$. The function $f$ is called slice regular if, for all $I \in \mathbb{S}$, the restriction $f_I$ is holomorphic, i.e. the function $\bar{\partial}_I f : \Omega_I \to \mathbb{H}$ defined by

$$\bar{\partial}_I f(x + Iy) = \frac{1}{2} \left( \frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x + Iy)$$  \hspace{1cm} (21)$$

vanishes identically.

As proven in [11], the class of slice regular functions includes the sums of the series $\sum_{n \in \mathbb{N}} q^n a_n$ in their balls of convergence $B(0, R)$. Moreover, the following can be proven by direct computation.

Theorem 13 Let $p \in \mathbb{H}$ and let $f(q) = \sum_{n \in \mathbb{N}} (q - p)^n a_n$ have $\sigma$-radius of convergence $R$. If $\Omega(p, R) \neq \emptyset$ then $f : \Omega(p, R) \to \mathbb{H}$ is a slice regular function.

Slice regular functions are not always as nice as the aforementioned. Indeed, if the domain $\Omega$ is not carefully chosen then a slice regular function $f : \Omega \to \mathbb{H}$ does not even need to be continuous.

Example 3 Choose $I \in \mathbb{S}$ and define $f : \mathbb{H} \setminus \mathbb{R} \to \mathbb{H}$ as follows:

$$f(q) = \begin{cases} 0 & \text{if } q \in \mathbb{H} \setminus L_I \\ 1 & \text{if } q \in L_I \setminus \mathbb{R} \end{cases}$$

This function is clearly slice regular, but not continuous.

Properties of the domains of definition of slice regular functions which prevent such pathologies have been identified in [4].

Definition 9 Let $\Omega$ be a domain in $\mathbb{H}$, intersecting the real axis. If $\Omega_I = \Omega \cap L_I$ is a domain in $L_I \simeq \mathbb{C}$ for all $I \in \mathbb{S}$ then we say that $\Omega$ is a slice domain.

The following result holds for slice regular functions on slice domains.

Theorem 14 (Identity principle) Let $\Omega$ be a slice domain and let $f, g : \Omega \to \mathbb{H}$ be slice regular. Suppose that $f$ and $g$ coincide on a subset $C$ of $\Omega_I$, for some $I \in \mathbb{S}$. If $C$ has an accumulation point in $\Omega_I$, then $f \equiv g$ in $\Omega$.

Analogs of many other classical results in complex analysis are proven in [4] for slice regular functions on slice domains: the integral representation formula, the mean value property, the maximum modulus principle, Cauchy’s estimates, Liouville’s theorem, Morera’s theorem, Schwarz’s lemma.

Adding the condition of axial symmetry (defined in Sect. 3) leads to even finer properties. Indeed, Theorem 7 extends to all slice regular functions on an axially symmetric slice domain $\Omega$, yielding that such functions are $C^\infty(\Omega)$. The condition of symmetry is not restrictive when studying slice regular functions on slice domains.
because of the following (surprising) theorem. Define the symmetric completion of each set \( T \subseteq \mathbb{H} \) as the axially symmetric set
\[
\tilde{T} = \bigcup_{x+Iy \in T} (x + S y).
\]

**Theorem 15 (Extension theorem)** Let \( \Omega \subseteq \mathbb{H} \) be a slice domain and let \( f : \Omega \to \mathbb{H} \) be a slice regular function. Then \( \tilde{\Omega} \) is an axially symmetric slice domain and \( f \) extends to a unique slice regular function \( \tilde{f} : \tilde{\Omega} \to \mathbb{H} \).

In addition to its intrinsic meaning, this result (proven in [4]) allows the definition of a multiplicative operation \( * \) on the set \( \mathcal{R}(\Omega) \) of slice regular functions on an (axially symmetric) slice domain \( \Omega \). It turns out that \( (\mathcal{R}(\Omega), +, *) \) is a (non-commutative) associative real algebra and this algebraic structure is strictly related to the distribution of the zeros of slice regular functions. Furthermore, the multiplicative inverse \( f^{-*} \) of any \( f \neq 0 \) in \( \mathcal{R}(\Omega) \) is computed in [4], extending results proven in [10]. Finally, the minimum modulus principle and the open mapping theorem presented in [10] extend to all slice regular functions on (axially symmetric) slice domains. We warn the reader: several results that we will prove in this section are stated for slice regular functions defined on generic domains \( \Omega \) in \( \mathbb{H} \), but they only acquire full significance when \( \Omega \) is a slice domain.

The study conducted in the previous sections allows us to provide slice regular functions on axially symmetric slice domains with power series expansions.

**Definition 10** Let \( \Omega \) be a domain in \( \mathbb{H} \) and let \( f : \Omega \to \mathbb{H} \) be a slice regular function. Its slice derivative is the slice regular function \( f' \) which equals
\[
\frac{\partial_I f}{\partial_I} (x + Iy) = \frac{1}{2} \left( \frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) f_I (x + Iy)
\] (22)
on \( \Omega_I \), for all \( I \in \mathbb{S} \).

We may denote the slice derivative of \( f \) by \( f' \) and its \( n \)-th slice derivative by \( f^{(n)} \) without causing any confusion, since the slice derivative of \( \sum_{n \in \mathbb{N}} q^n a_n \) coincides with the formal derivative \( \sum_{n \in \mathbb{N}} q^n a_{n+1} (n + 1) \). With this notation, we rephrase a result proven in [4] as follows.

**Theorem 16** Let \( f \) be a slice regular function on a domain \( \Omega \subseteq \mathbb{H} \) and let \( x \in \Omega \cap \mathbb{R} \). In each Euclidean ball \( B(x, R) \) contained in \( \Omega \) the function \( f \) expands as
\[
f(q) = \sum_{n \in \mathbb{N}} (q - x)^n f^{(n)}(x) \frac{1}{n!}.
\] (23)

We are now able to extend the previous result as follows.
Theorem 17 Let $f$ be a slice regular function on a domain $\Omega \subseteq \mathbb{H}$ and let $p \in \Omega$. In each $\sigma$-ball $\Sigma(p, R)$ contained in $\Omega$ the function $f$ expands as

$$f(q) = \sum_{n \in \mathbb{N}} (q - p)^n f^{(n)}(p) \frac{1}{n!}.$$  \hspace{1cm} (24)

Proof By construction $\Omega_f \supseteq \Sigma(p, R) \cap L_f = B_f(p, R)$. By the properties of holomorphic functions of one complex variable, $f_1(z)$ expands as $\sum_{n \in \mathbb{N}} (z - p)^n f^{(n)}(p) \frac{1}{n!}$ in $B_f(p, R)$. We conclude that the series in Eq. 24 converges in $\Sigma(p, R)$. If $\Omega(p, R)$ is empty then the assertion is proved. Otherwise, the series in Eq. 24 defines a slice regular function $g : \Omega(p, R) \to \mathbb{H}$. Since $f_1 \equiv g_1$ in $B_f(p, R)$, the identity principle 14 allows us to conclude that $f$ and $g$ coincide in $\Omega(p, R)$ (hence in $\Sigma(p, R)$, as desired).

Corollary 2 A quaternionic function is slice regular in a domain if, and only if, it is $\sigma$-analytic in the same domain.

The previous corollary recalls the complex theory, although its significance is quite different because of the properties of the topology involved in this case. Indeed, Example 3 proves that if the domain of definition is ill-chosen then a $\sigma$-analytic function need not be continuous. On the other hand, we might inquire about the strong analyticity of a slice regular function (which implies $C^\infty$ continuity). It turns out that the set of strong analyticity is far more complicated, as we are about to see. For all $T \subseteq \mathbb{H}$, we denote $\partial_T$ its $\sigma$-boundary, we define

$$\mathcal{A}(T) = \{ p \in T : 2|Im(p)| < \sigma(p, \partial_T) \}$$

and obtain the following direct consequence of Remarks 2, 3 and Theorems 8, 17.

Corollary 3 All slice regular functions on a domain $\Omega \subseteq \mathbb{H}$ are strongly analytic in $\mathcal{A}(\Omega)$.

Remark that if $\Omega$ does not intersect the real axis $\mathbb{R}$ then $\mathcal{A}(\Omega)$ is empty.

We now consider the case of a power series $f(q)$ centered at a point $p \in \mathbb{H}$ and converging in $\Sigma(p, R) = \Omega(p, R) \cup B_f(p, R)$. By the previous corollary, its sum $f$ is strongly analytic in $\mathcal{A}(\Omega(p, R))$. Suppose $p = x_0 + Iy_0$ with $I \in \mathbb{S}$ and $x_0, y_0, R \in \mathbb{R}$ such that $0 < y_0 < R$. By direct computation, $\mathcal{A}(\Omega(p, R)) \subset \Omega(p, R)$ is the (open) region bounded by the hypersurface of points $s \in \Omega(p, R)$ such that $x = Re(s)$ and $y = |Im(s)|$ solve the equation

$$\left(x - x_0\right)^2 - 3 \left(y - \frac{y_0 + 2R}{3}\right)^2 + \left(\frac{2y_0}{3} + R\right)^2 = 0, \hspace{1cm} (25)$$

i.e. the hypersurface of revolution $\bigcup_{J \in \mathbb{S}} \mathcal{H}(J)$ where $\mathcal{H}(J)$ is the arc of hyperbola

$$\mathcal{H}(J) = \{ x + Jy : 0 \leq y \leq R - y_0, \ x, y \ \text{verify} \ (25) \}$$

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Fig. 3 A view in \( \mathbb{R} + i \mathbb{R} + j \mathbb{R} \) of \( A(\Sigma(p, R)) = A(\Omega(p, R)) \cup A(B_I(p, R)) \), supposing \( p \in L_I = \mathbb{R} + i \mathbb{R} \)

for all \( J \in \mathbb{S} \). Furthermore, the following theorem proves that \( f \) is also strongly analytic in \( A(B_I(p, R)) \), which is the plane region in \( L_I \) lying between \( \mathcal{H}(-I) \) and the arc of hyperbola \( \mathcal{K}(I) \) consisting of all points \( x + iy \) with \( 0 \leq y \leq R - y_0 \) and

\[
(x - x_0)^2 - \frac{2}{3} \left( y + \frac{y_0 - R}{3} \right)^2 + \frac{(y - y_0)^2}{3} = 0.
\]

We notice that \( \mathcal{H}(I) \) always lies between \( \mathcal{H}(-I) \) and \( \mathcal{K}(I) \) in \( L_I \), so that the set \( A(B_I(p, R)) \) always includes \( A(\Omega(p, R)) \cap L_I \) (which is the plane region lying between \( \mathcal{H}(-I) \) and \( \mathcal{H}(I) \)). We also notice that \( \mathcal{K}(I) \) always lies in \( B_I(p, R) \cap B_I(\bar{p}, R) \), so that \( A(B_I(p, R)) \) is always included in \( \Omega(p, R) \) (Fig. 3).

**Theorem 18** Choose \( p \in \mathbb{H} \), let \( L_I \) be a complex plane through \( p \) and consider a power series \( f(q) = \sum_{n \in \mathbb{N}} (q - p)^n a_n \) having \( \sigma \)-radius of convergence \( R \). If \( \Omega(p, R) \) is not empty then \( f \) is strongly analytic at each point of

\[
A(\Sigma(p, R)) = A(\Omega(p, R)) \cup A(B_I(p, R)).
\]

The latter is a subset of \( \Omega(p, R) \) and it is not open, unless \( p \in \mathbb{R} \).

**Proof** We first prove equality (27). Let \( \Sigma = \Sigma(p, R), \Omega = \Omega(p, R) \) and \( B_I = B_I(p, R) \). For all \( s \in \Sigma \), we remark that \( \sigma(s, \partial_\Sigma) = R - \sigma(s, p) \). For all \( s \in \Sigma \setminus L_I = \Omega \setminus L_I \) we get \( \sigma(s, \partial_\Omega) = R - \omega(s, p) \) and the latter coincides with \( \sigma(s, \partial_\Sigma) \). This proves that

\[
A(\Omega) \setminus L_I = A(\Sigma) \setminus L_I.
\]

As for \( s \in L_I \), we get \( \sigma(s, \partial_\Sigma) = R - |s - p| = \sigma(s, \partial_{B_I}) \) so that

\[
A(\Sigma) \cap L_I = A(B_I).
\]
We conclude this first part of the proof recalling that $\mathcal{A}(B_I)$ always includes $\mathcal{A}(\Omega) \cap L_I$, so that

$$ (\mathcal{A}(\Omega) \setminus L_I) \cup \mathcal{A}(B_I) = \mathcal{A}(\Omega) \cup \mathcal{A}(B_I). $$

Now let us prove that $f$ is strongly analytic at all $s \in \mathcal{A}(\Sigma)$. We already noticed that it is strongly analytic at all $s \in \mathcal{A}(\Omega)$, due to Corollary 3. Finally, we prove the strong analyticity at all $s \in \mathcal{A}(B_I) \subset \Omega$ as follows. By the properties of holomorphic complex functions, the restriction $f_I(z)$ expands as $\sum_{n \in \mathbb{N}} (z - s)^n f^{(n)}(s) \frac{1}{n!}$ for all $z \in B_I(s, R - |s - p|)$. Hence the expansion $f(q) = \sum_{n \in \mathbb{N}} (q - s)^n f^{(n)}(s) \frac{1}{n!}$ at $s$ is valid not only in $\Sigma(s, R - \omega(s, p))$ as guaranteed by Theorem 17, but in the whole $\Sigma(s, R - |s - p|)$. Since $s \in \mathcal{A}(B_I)$ implies $R - |s - p| > 2|Im(s)|$, the $\sigma$-ball $\Sigma(s, R - |s - p|)$ is a Euclidean neighborhood of $s$ as desired.

We conclude with an application of Theorem 18.

**Proposition 4** For any domain $\Omega \subseteq \mathbb{H}$ intersecting the real axis, the set $\mathcal{A}(\Omega)$ contains an axially symmetric neighborhood of $\Omega \cap \mathbb{R}$.

**Proof** For all $x \in \Omega \cap \mathbb{R}$, there exists an $\varepsilon_x > 0$ such that $B(x, \varepsilon_x) \subseteq \Omega$. Then

$$ \mathcal{A}(\Omega) = \{ p \in \Omega : 2|Im(p)| < \sigma(p, \partial_x \Omega) \} $$

$$ \supseteq \{ p \in B(x, \varepsilon_x) : 2|Im(p)| < \varepsilon_x - |p| \} = \mathcal{A}(B(x, \varepsilon_x)). $$

The union of the $\mathcal{A}(B(x, \varepsilon_x))$ for $x \in \Omega \cap \mathbb{R}$ is an axially symmetric slice domain containing $\Omega \cap \mathbb{R}$.

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