WELL-POSEDNESS OF FULLY NONLINEAR AND NONLOCAL CRITICAL PARABOLIC EQUATIONS

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Abstract. In this paper we prove the existence of smooth solutions to fully nonlinear and nonlocal parabolic equations with critical index. The proof relies on the apriori Hölder estimate for advection fractional-diffusion equation established by Silvestre [11].

1. Introduction and main result

In this paper we are interested in solving the following fully nonlinear and nonlocal parabolic equation:

$$\partial_t u = F(t, x, u, \nabla u, (-\Delta)^{\frac{\alpha}{2}} u), \quad u(0) = \varphi, \quad \alpha \in (0, 2),$$

(1.1)

where $F(t, x, u, w, q) : [0, 1] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$ is a measurable function, and $(-\Delta)^{\frac{\alpha}{2}}$ is the usual fractional Laplacian defined by

$$(-\Delta)^{\frac{\alpha}{2}} u = \mathcal{F}^{-1}(\cdot ^{\alpha} \mathcal{F} u), \quad u \in S(\mathbb{R}^d),$$

where $\mathcal{F}$ denotes the Fourier’s transform, $S(\mathbb{R}^d)$ is the Schwartz class of smooth real-valued rapidly decreasing functions.

Recently, in the sense of viscosity solutions, fully nonlinear and nonlocal elliptic and parabolic equations have been extensively studied (cf. [4], [10], [3], [9], etc.). In [4], Caffarelli and Silvestre studied the following type of nonlocal equation:

$$I_\alpha u(x) := \sup_i \inf_j \left( c^{ij} + b^{ij} \cdot \nabla u(x) + \int_{\mathbb{R}^d} [u(x + y) - u(x)] a^{ij}(y) |y|^{-d-\alpha} dy \right) = 0,$$

where $\alpha \in (0, 2)$, $i, j$ ranges in arbitrary sets, $c^{ij} \in \mathbb{R}$ and $b^{ij} \in \mathbb{R}^d$, the kernel $a^{ij}(y)$ satisfies

$$a^{ij}(y) = a^{ij}(-y), \quad a_0 \leq a^{ij}(y) \leq a_1.$$

This type of equation appears in the stochastic control problems. In [4], the extremal Pucci operators are used to characterize the ellipticity, and the ABP estimate, Harnack inequality and interior $C^{1,\beta}$-regularity were obtained. In [11], Silvestre studied the following nonlocal parabolic equation with critical index $\alpha = 1$:

$$\partial_t u = I_1 u, \quad u(0) = \varphi,$$

and established $C^{1,\beta}$-regularity of viscosity solutions. In particular, the following first order Hamilton-Jacobi equation is covered by the above equation when $H$ is Lipschitz continuous:

$$\partial_t u + H(\nabla u) + (-\Delta)^{\frac{\alpha}{2}} u = 0.$$

In [9], Lara and Davila extended Silvestre’s result to the more general case, and in particular, focused on the uniformity of regularity as $\alpha \to 2$.

However, it is not known how to solve the fully nonlinear and nonlocal equation (1.1) in Sobolev spaces. Let us fix the main idea of the present paper for solving (1.1). Assume that $F$ does not depend on $u$. Taking the gradient with respect to $x$ for equation (1.1), we have

$$\partial_x \nabla u = - (\partial_q F)(-\Delta)^{\frac{\alpha}{2}} \nabla u + (\nabla_w F) \nabla^2 u + \nabla_x F.$$
We make the following observation:

\[-(-\Delta)^{\frac{1}{2}}u = (-\Delta)^{\frac{1}{2}}\text{div}\nabla u = R^\circ \cdot \nabla u,\]

where \(R^\circ := (-\Delta)^{\frac{1}{2}}\text{div}\) is a bounded linear operator from Bessel potential space \(L^p_{\text{B}}\) to \(L^p\) provided \(p > 1\). If we set \(w := \nabla u\), then \(w\) satisfies the following quasi-linear parabolic system:

\[
\partial_t w = -\partial_q F(w, R^\circ w)(-\Delta)^{\frac{1}{2}}w + (\nabla_w F)(w, R^\circ w)\nabla w + (\nabla_x F)(w, R^\circ w). \tag{1.2}
\]

It is noticed that the classical quasi-geostrophic equation takes the same form (cf. [6, 5, 8], etc.):

\[
\partial_t \theta + (-\Delta)^{\frac{1}{2}}\theta + R \cdot \nabla \theta = 0, \quad R := \nabla^\perp(-\Delta)^{-\frac{1}{2}}.
\]

Assume now that one can solve equation (1.2), then it is natural to define

\[
u(t, x) := \varphi(x) + \int_0^t F(s, x, w(s, x), R^\circ w(s, x))ds.
\]

Thus, if one can show

\[
\nabla u = w, \tag{1.3}
\]

then it follows that

\[
u(t, x) = \varphi(x) + \int_0^t F(s, x, \nabla u(s, x), -(-\Delta)^{\frac{1}{2}}u(s, x))ds.
\]

For solving equation (1.2), we shall use Silvestre’s Hölder estimate [11] about the following linear parabolic equation:

\[
\partial_t u = -a(-\Delta)^{\frac{1}{2}}u + b \cdot \nabla u + f. \tag{1.4}
\]

For proving (1.5), we need to solve a linear equation like

\[
\partial_t u = a(-\Delta)^{\frac{1}{2}}\Box u + b \cdot (\nabla u - (\nabla u)^\perp), \tag{1.5}
\]

where \(\Box := \text{div}\nabla - \nabla\text{div}\) is a symmetric operator on \(L^2(\mathbb{R}^d; \mathbb{R}^d)\) and

\[
\langle \Box u, u \rangle_2 = -||\nabla u||^2_2 + ||\text{div}u||^2_2.
\]

Notice that in one dimensional case, \(\Box = 0\).

In this work, we mainly concentrate on the critical case \(\alpha = 1\) and prove the following result:

**Theorem 1.1.** Assume that \(\partial_q F \geq a_0 > 0\) and for some \(\kappa_0 > 0\),

\[
|F(t, x, u, 0, 0)| \leq \kappa_0(|u| + 1); \tag{1.6}
\]

and for any \(R > 0\),

\[
F \in L^\infty([0, 1]; C^\infty_b(\mathbb{R}^d \times B^1_R \times B^1_R \times B^1_R)), \tag{1.7}
\]

\[
\partial_q F, \nabla_w F \in L^\infty([0, 1]; C^1_b(\mathbb{R}^d \times B^1_R \times B^1_R \times \mathbb{R})), \tag{1.8}
\]

\[
\partial_q F \in L^\infty([0, 1] \times \mathbb{R}^d \times B^1_R \times \mathbb{R}^d \times \mathbb{R}), \tag{1.9}
\]

where \(B^1_R\) denotes the open ball in \(\mathbb{R}^d\) with radius \(R\) and center \(0\); and for any \(j \in \mathbb{N}\) and \(R > 0\), there exist \(C_{R,j} \geq 0\), \(\gamma_{R,j} \geq 0\) and \(h_{R,j} \in (L^1 \cap L^\infty)(\mathbb{R}^d)\) such that for all \((t, x, u, w, q) \in [0, 1] \times \mathbb{R}^d \times B^1_R \times \mathbb{R}^d \times \mathbb{R},\)

\[
|\nabla^j_F(t, x, u, w, q)| \leq C_{R,j}|w|(|w|^\gamma_{R,j} + 1) + h_{R,j}(x), \tag{1.10}
\]

where \(\gamma_{R,1} = 0\). Then for any initial value \(\varphi \in \mathbb{S}^\infty := \bigcap_k \mathbb{U}^k\), where \(\mathbb{U}^k\) is defined by (2.4) below, there exists a unique \(u \in C([0, 1]; \mathbb{U}^\infty)\) solving equation (1.1) with \(\alpha = 1\). Moreover,

\[
\sup_{t \in [0, 1]} ||u(t)||_{\text{loc}} \leq e^{\kappa_0}(|\varphi|_{\text{loc}} + \kappa_0).
\]
Remark 1.2. Let \( A(q) \in C^\infty(\mathbb{R}) \) have bounded derivatives of first and second orders and \( \partial_q A \) be bounded below by \( a_0 > 0 \). Let \( H \in C^\infty(\mathbb{R}^d) \) and \( f \in C^\infty(\mathbb{R}) \) satisfy \( |f(u)| \leq \kappa_0(|u| + 1) \). Then

\[
F(t, x, u, w, q) := A(q) + H(w) + f(u)
\]
satisfies all the conditions (1.6)-(1.10).

In the subcritical case \( \alpha \in (1, 2) \), when we adopt the same argument described above to solve the fully nonlinear equation (1.1), there are two difficulties occurring: on one hand, we need to prove a stronger apriori Hölder estimate for equation (1.4) (see Theorem 2.4 below)

\[
\sup_{t \in [0,1]} \sup_{x \in \mathbb{R}^d} \frac{|u(t, x) - u(t, y)|}{|x - y|^{\beta}} \leq C, \quad \exists \beta \in (\alpha - 1, 1),
\]

where \( C \) only depends on the bounds of \( a, b, f \) and \( u(0) \); on the other hand, for \( \alpha \in (1, 2) \), it is not known whether the uniqueness holds for equation (1.5) in the class of smooth solutions. In the case of \( \alpha \in (0, 1) \), this problem can be solved by observing \( \text{div} \Delta u = 0 \) (see Lemma 5.1).

In the supercritical case \( \alpha \in (0, 1) \), it is well-known that there exists an explosion solution for one-dimensional fractal Burger’s equation (cf. [7] [11]). Nevertheless, from the proof of Theorem 1.1 one can see that the approach also works for the following fully nonlinear equation:

\[
\partial_t u = F(t, x, u, u, u, \cdots), \quad u(0) = \varphi.
\]

The paper is organized as follows: In Section 2, we prepare some notations and recall some well-known facts for later use. In Section 3, we solve the linear equation in Sobolev spaces. In Section 4, we prove the existence of smooth solutions for the quasi-linear nonlocal parabolic system. In Section 5, we give the proof of Theorem 1.1.

2. Preliminaries

Let \( N_0 := \mathbb{N} \cup \{0\} \). For \( p \in (1, \infty) \) and \( \beta \in N_0 \), let \( Y^{\beta, p} \) be the completion of \( S(\mathbb{R}^d) \) with respect to the norm

\[
\|f\|_{\beta, p} := \sum_{k=0}^{\beta} \|\nabla^k f\|_p,
\]

where \( \nabla^k \) denotes the \( k \)-order gradient; and for \( 0 < \beta \neq \text{integer} \), let \( Y^{\beta, p} \) be the completion of \( S(\mathbb{R}^d) \) with respect to the norm

\[
\|f\|_{\beta, p} := \|f\|_p + \sum_{k=[\beta]}^{\beta} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{\|\nabla^k f(x) - \nabla^k f(y)\|_p}{|x - y|^{\beta + |\beta|}} \, dx \, dy \right)^{\frac{1}{\beta}}, \quad (2.1)
\]

where for a number \( \beta > 0 \), \( [\beta] \) denotes the integer part of \( \beta \) and \( \lfloor \beta \rfloor = \beta - [\beta] \). It is well-known that for \( k \leq m, \theta \in (0, 1) \) with \( (1 - \theta)k + m \theta \notin \mathbb{N} \) (cf. [14] p.185, (2)),

\[
(\mathcal{W}^{k, p}, \mathcal{W}^{m, p})_{\beta, p} = \mathcal{W}^{\lfloor (1-\theta)k + m \theta \rfloor, p}, \quad (2.2)
\]

where \( (\cdot, \cdot)_{\beta, p} \) stands for the real interpolation space. For \( t \in [0, 1] \), write \( \mathcal{W}^{k, p}_t := L^p([0, t]; \mathcal{W}^{k, p}) \) with the norm

\[
\|u\|_{\mathcal{W}^{k, p}_t} := \left( \int_0^t \|u(s)\|_{k, p}^p \, ds \right)^{\frac{1}{p}},
\]

and let \( \mathcal{X}^{k, p}_t \) be the completion of all functions \( u \in C^\infty([0, t]; S(\mathbb{R}^d)) \) with respect to the norm

\[
\|u\|_{\mathcal{X}^{k, p}_t} := \sup_{s \in [0, t]} \|u(s)\|_{k-1, p} + \|u\|_{\mathcal{W}^{k, p}_s} + \|\partial_t u\|_{\mathcal{W}^{k-1, p}}.
\]

It is well-known that (cf. [11] p.180, Theorem III 4.10.2)

\[
\mathcal{X}^{k, p}_t \hookrightarrow C([0, t]; \mathcal{W}^{k-\frac{1}{p}, p}). \quad (2.3)
\]
Let $U^{k,p}$ be the Banach space of the completion of $C_c^{\infty}(\mathbb{R}^d)$ with respect to the norm:

$$
\|f\|_{U^{k,p}} := \|f\|_\infty + \|\nabla f\|_{k,p}. \tag{2.4}
$$

For simplicity of notation, we also write

$$
\chi^{k,p}_\ast := \chi^{k,p}_1, \quad \chi^{k,p} := \chi^{k,p}_1
$$

and

$$
\mathcal{W}^{\infty} := \cap_{k,p} \mathcal{W}^{k,p}, \quad \mathcal{W}^{\ast} := \cap_{k,p} \mathcal{W}^{k,p}, \quad \mathcal{W}^{\infty} := \cap_{k,p} \mathcal{W}^{k,p}, \quad \mathcal{W}^{\infty} := \cap_{k,p} \mathcal{W}^{k,p}.
$$

Let $\Omega$ be an open domain of $\mathbb{R}^d$. For $k \in \mathbb{N}_0 \cup \{\infty\}$, we use $C_k^p = C^k(\Omega)$ to denote the space of all bounded and $k$-order continuous differentiable functions with all bounded derivatives up to $k$-order. For $\beta \in (0, 1)$, let $\mathcal{C}^\beta$ be the completion of $\mathcal{S}(\mathbb{R}^d)$ with respect to the norm

$$
\|f\|_{C^\beta} := \|f\|_\infty + \|f\|_{C^\beta},
$$

where $\| \cdot \|_\infty$ is the sup-norm and

$$
|f|_{C^\beta} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\beta}. \tag{2.5}
$$

Notice that $C^{\infty}_c(\mathbb{R}^d) \subset \mathcal{C}^\beta$. By the Sobolev embedding theorem, one has

$$
\mathcal{W}^{1-p} \subset C^{1-\frac{d}{p}}, \quad p > d.
$$

Let $\mathcal{R}$ be the class of all linear operators $\mathcal{R} : \mathcal{W}^{\infty} \to \mathcal{W}^{\infty}$ satisfying that for each $\beta \geq 0$ and $p > 1$,

$$
\mathcal{R} : \mathcal{W}^{\beta,p} \to \mathcal{W}^{\beta,p} \text{ is a bounded linear operator},
$$

and for each $\beta \in (0, 1)$,

$$
|\mathcal{R}f|_{C^\beta} \leq C_{d,\beta} |f|_{C^\beta}, \quad \forall f \in C^\beta. \tag{2.6}
$$

A typical example of such an operator is the Riesz transform:

$$
\mathcal{R}_j f := (-\Delta)^{\frac{d}{2}} \partial_j f = \lim_{\varepsilon \to 0} \int_{|y|<\varepsilon} f(x-y) \frac{y_j}{|y|} \frac{1}{|y|^{d+1}} \, dy.
$$

Indeed, it holds that for any $p > 1$ (cf. [12]),

$$
\|\nabla f\|_p \simeq \|(-\Delta)^{\frac{1}{2}} f\|_p. \tag{2.7}
$$

Recalling that for any $f \in \mathcal{S}(\mathbb{R}^d)$,

$$
(-\Delta)^{\frac{d}{2}} f(x) = c_d \int_{\mathbb{R}^d} \frac{f(x) - f(x+y)}{|y|} |y|^{d-1} \, dy, \tag{2.8}
$$

where $c_d > 0$ is a universal constant, we have

$$
(-\Delta)^\frac{d}{2} (fg) = g(-\Delta)^{\frac{d}{2}} f + f(-\Delta)^{\frac{d}{2}} g - \mathcal{E}(f, g), \tag{2.9}
$$

where

$$
\mathcal{E}(f, g)(x) := c_d \int_{\mathbb{R}^d} (f(x) - f(x+y))(g(x) - g(x+y)) |y|^{d-1} \, dy. \tag{2.10}
$$

From this formula, it is easy to derive that (see [16]),

**Lemma 2.1.** Let $\zeta \in \mathcal{S}(\mathbb{R}^d)$ and set $\zeta_*(x) := \zeta(x - z)$ for $z \in \mathbb{R}^d$. Then for any $p \in [1, \infty)$, there exists a constant $C = C(p, d) > 0$ such that for all $f \in \mathcal{W}^{1,p}$,

$$
\int_{\mathbb{R}^d} \|(-\Delta)^{\frac{d}{2}}(f \zeta) - (-\Delta)^{\frac{d}{2}} f \zeta\|_p^p \, dz \leq C \|\zeta\|_{2,p} \|f\|_{p/2}^p \|f\|_{1,p}^p. \tag{2.11}
$$
For given $\lambda_0 > 0$, $f \in L^{\infty}([0, 1]; \mathbb{W}^{\infty})$ and $\varphi \in \mathbb{W}^{\infty}$, let us consider the following heat equation:

$$\partial_t u + \lambda_0(-\Delta)^{\frac{d}{2}} u = f, \quad u(0) = \varphi.$$  

It is well-known that the unique solution can be represented by

$$u(t, x) = \mathcal{P}^{\lambda_0}_t \varphi(x) + \int_0^t \mathcal{P}^{\lambda_0}_{t-s} f(s, x) ds,$$

where $(\mathcal{P}^{\lambda_0}_t)_{t \geq 0}$ is the Cauchy semigroup associated with $\lambda_0(-\Delta)^{\frac{d}{2}}$ and given by

$$\mathcal{P}^{\lambda_0}_t \varphi(x) := c_d t \int_{\mathbb{R}^d} \frac{\varphi(\lambda_0 y + x)}{(|y|^2 + t^2)^{d+1/2}} dy,$$

where $c_d > 0$ is a universal constant. By the classical Littlewood-Paley-Stein theory, there exists a constant $C > 0$ only depending on $\lambda_0, d, p$ such that for any $f \in L^p([0, 1] \times \mathbb{R}^d)$ (cf. [12, 16]),

$$\int_0^1 \left\| \nabla \int_0^t \mathcal{P}^{\lambda_0}_{t-s} f(s) ds \right\|^p_p ds \leq C \int_0^1 \|f(s)\|^p_p ds. \tag{2.12}$$

We now use the probabilistic technique to extend the above estimate to the more general case. Let $(L_t)_{t \geq 0}$ be a $d$-dimensional Cauchy process with Lévy measure $\nu(dx) = dx/|x|^{d+1}$. It is well-known that (cf. [2])

$$\mathcal{P}^{\lambda_0}_t \varphi(x) = \mathbb{E} \varphi(x + \lambda_0 L_t).$$

Let $\theta : [0, 1] \to \mathbb{R}^d$ and $\lambda : [0, 1] \to [0, \infty)$ be two bounded measurable functions. Define

$$\mathcal{T}^{\lambda, \theta}_{t,s} \varphi(x) := \mathbb{E} \varphi \left( x - \int_s^t \theta(r) dr + \int_s^t \lambda(r) dL_r \right). \tag{2.13}$$

By the theory of stochastic differential equation (cf. [2, p.402, Theorem 6.7.4]), one knows that

$$\partial_t \mathcal{T}^{\lambda, \theta}_{t,s} \varphi + \lambda(-\Delta)^{\frac{d}{2}} \mathcal{T}^{\lambda, \theta}_{t,s} \varphi + \theta \cdot \nabla \mathcal{T}^{\lambda, \theta}_{t,s} \varphi = 0.$$  

Now if we define

$$u(t, x) := \int_0^t \mathcal{T}^{\lambda, \theta}_{t,s} f(s, x) ds,$$

then it is easy to see that

$$\partial_t u + \lambda(-\Delta)^{\frac{d}{2}} u + \theta \cdot \nabla u = f, \quad u(0) = 0.$$  

We have

**Theorem 2.2.** Let $\theta : [0, 1] \to \mathbb{R}^d$ and $\lambda : [0, 1] \to [\lambda_0, \infty)$ be two bounded measurable functions, where $\lambda_0 > 0$. For any $p \in (1, \infty)$, there exists a constant $C$ depending only on $\lambda_0, d, p$ such that for all $f \in L^p([0, 1] \times \mathbb{R}^d)$,

$$\int_0^1 \left\| \nabla \int_0^t \mathcal{T}^{\lambda, \theta}_{t,s} f(s) ds \right\|^p_p ds \leq C \int_0^1 \|f(s)\|^p_p ds.$$  

**Proof.** Let $(L^{(i)}_t)_{t \geq 0}, i = 1, 2$ be two independent copies of Cauchy process $(L_t)_{t \geq 0}$. By the theory of stochastic differential equation (cf. [2, 15]), one can write

$$\mathcal{T}^{\lambda, \theta}_{t,s} \varphi(x) = \mathbb{E} \varphi \left( x - \int_s^t \theta(r) dr + \int_s^t (\lambda(r) - \lambda_0) dL^{(1)}_r + \lambda_0 (L^{(2)}_t - L^{(2)}_s) \right) = \mathbb{E} \mathcal{P}^{\lambda_0}_{t-s} \varphi \left( x - X_t + X_s \right), \tag{2.14}$$

where $\mathcal{P}^{\lambda_0}_t \varphi(x) := \mathbb{E} \varphi(x + \lambda_0 L^{(2)}_t)$ is the semigroup associated with $\lambda_0(-\Delta)^{\frac{d}{2}}$, and

$$X_t := \int_0^t \theta(r) dr - \int_0^t (\lambda(r) - \lambda_0) dL^{(1)}_r.$$
Define
\[ u(t, x) := \int_0^t \mathcal{P}^t_{t-s} f(s, x + X_s) \, ds. \]

By (2.14) one has
\[ \int_0^t \mathcal{L}^t_{t-s} f(s, x) \, ds = \mathbb{E} u(t, x - X_t). \]

Hence, by Hölder’s inequality and Fubini’s theorem,
\[
\int_0^1 \left\| \nabla \int_0^t \mathcal{L}^t_{t-s} f(s, x) \, ds \right\|_p^p \, dt = \int_0^1 \left\| \mathbb{E} \nabla u(t, \cdot - X_t) \right\|_p^p \, dt \leq \mathbb{E} \int_0^1 \left\| \nabla u(t, \cdot - X_t) \right\|_p^p \, dt
\]
\[
= \mathbb{E} \int_0^1 \left\| u(t) \right\|_p^p \, dt = \mathbb{E} \int_0^1 \left\| \mathcal{P}^t_{t-s} f(s, \cdot + X_s) \, ds \right\|_p^p \, dt \leq C \mathbb{E} \int_0^1 \left\| f(s, \cdot + X_s) \right\|_p^p \, ds = C \int_0^1 \left\| f(s) \right\|_p^p \, ds.
\]

The proof is finished. \(\square\)

Below we prove a maximum principle for the fully nonlinear equation (1.1).

**Theorem 2.3. (Maximum principle)** Let \( F(t, x, w, q) : [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R} \) be a measurable function. Assume that for any \( R > 0 \) and all \( (t, x, w, q) \in [0, 1] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \) with \( |w|, |q| \leq R \),
\[
0 \leq \partial_t F(t, x, w, q) \leq a_{R,1}, \quad |\nabla w F|(t, x, w, q) \leq a_{R,2},
\]
where \( a_{R,1}, a_{R,2} > 0 \). Let \( u \in C([0, 1]; C^2_b(\mathbb{R}^d)) \) satisfy
\[ u(t, x) = u(0, x) + \int_0^t F(s, x, \nabla u(s, x), (-\Delta)^{1/2} u(s, x)) \, ds. \]

If \( F(s, x, 0, 0) \leq 0 \), then for all \( t \in [0, 1] \),
\[
\sup_{x \in \mathbb{R}^d} u(t, x) \leq \sup_{x \in \mathbb{R}^d} u(0, x). \tag{2.16}
\]

In particular,
\[
\|u(t)\|_{\infty} \leq \|u(0)\|_{\infty} + \int_0^t \|F(s, \cdot, 0, 0)\|_{\infty} \, ds. \tag{2.17}
\]

**Proof.** First of all, we assume
\[
F(s, x, 0, 0) \leq \delta < 0. \tag{2.18}
\]

Suppose that (2.16) does not hold, then there exists a time \( t_0 \in (0, 1] \) such that
\[
\sup_{x \in \mathbb{R}^d} u(t_0, x) \neq \sup_{x \in [0, 1] \times \mathbb{R}^d} u(t, x).
\]

Let \( x_n \in \mathbb{R}^d \) be such that
\[
\lim_{n \to \infty} u(t_0, x_n) = \sup_{x \in \mathbb{R}^d} u(t_0, x) \geq u(t, x), \quad \forall (t, x) \in [0, 1] \times \mathbb{R}^d.
\]

We have for any \( \varepsilon \in (0, t_0) \),
\[
0 \leq \frac{1}{\varepsilon} \left( \lim_{n \to \infty} u(t_0, x_n) - \lim_{n \to \infty} u(t_0 - \varepsilon, x_n) \right)
\]
\[
\leq \frac{1}{\varepsilon} \left( \lim_{n \to \infty} (u(t_0, x_n) - u(t_0 - \varepsilon, x_n)) \right)
\]
\[
= \frac{1}{\varepsilon} \lim_{n \to \infty} \int_{t_0 - \varepsilon}^{t_0} F(s, x_n, \nabla u(s, x_n), (-\Delta)^{1/2} u(s, x_n)) \, ds. \tag{2.19}
\]
and for any $h \in \mathbb{R}^d$,
\[
0 \leq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left( \lim_{n \to \infty} \frac{1}{E} \left( \lim_{n \to \infty} u(t_0, x_n) - \lim_{n \to \infty} u(t_0, x_n - \varepsilon h) \right) \right)
\leq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \lim_{n \to \infty} (u(t_0, x_n) - u(t_0, x_n - \varepsilon h))
= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \int_0^1 h \cdot \nabla u(t_0, x_n - \varepsilon h) \, ds
= \lim_{n \to \infty} (h \cdot \nabla u(t_0, x_n)).
\]
In particular, by the arbitrariness of $h$, we get
\[
\lim_{n \to \infty} \nabla u(t_0, x_n) = 0. \tag{2.21}
\]
On the other hand, since for any $y \in \mathbb{R}^d$,
\[
u(t_0, x_n + y) - u(t_0, x_n) \leq \sup_{x \in \mathbb{R}^d} u(t_0, x) - u(t_0, x_n) \to 0,
\]
by (2.8) we have
\[
\lim_{n \to \infty} -(-\Delta)^{\frac{1}{2}} u(t_0, x_n) \leq c_d \int \lim_{n \to \infty} [u(t_0, x_n + y) - u(t_0, x_n)] |y|^{-d-1} \, dy \leq 0. \tag{2.22}
\]
Moreover, since by $u \in C([0, 1]; C^2_0(\mathbb{R}^d))$,
\[
\lim_{s \to t_0} \|\nabla u(s) - u(t_0)\|_\infty = 0
\]
and
\[
\lim_{s \to t_0} \|(-\Delta)^{\frac{1}{2}} (u(s) - u(t_0))\|_\infty = 0,
\]
we have by (2.15),
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{t_0 - \varepsilon}^{t_0} \|F(s, \nabla u(s), -(-\Delta)^{\frac{1}{2}} u(s)) - F(s, \nabla u(t_0), -(-\Delta)^{\frac{1}{2}} u(t_0))\|_\infty \, ds = 0.
\]
Hence, by (2.20), (2.21) and (2.18),
\[
0 \leq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \lim_{n \to \infty} \int_{t_0 - \varepsilon}^{t_0} F(s, x_n, \nabla u(t_0, x_n), -(-\Delta)^{\frac{1}{2}} u(t_0, x_n)) \, ds
= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \lim_{n \to \infty} \int_{t_0 - \varepsilon}^{t_0} F(s, x_n, 0, -(-\Delta)^{\frac{1}{2}} u(t_0, x_n)) \, ds
\leq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \lim_{n \to \infty} \int_{t_0 - \varepsilon}^{t_0} [F(s, x_n, 0, -(-\Delta)^{\frac{1}{2}} u(t_0, x_n)) - F(s, x_n, 0, 0)] \, ds + \delta
= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \left[ a_{n, \varepsilon} \cdot \left(-(-\Delta)^{\frac{1}{2}} u(t_0, x_n)\right) \right] + \delta, \tag{2.23}
\]
where
\[
a_{n, \varepsilon} := \frac{1}{\varepsilon} \int_{t_0 - \varepsilon}^{t_0} \int_0^1 (\partial_q F)(s, x_n, 0, -r(-\Delta)^{\frac{1}{2}} u(t_0, x_n)) \, ds \, ds.
\]
Let $R := \|(-\Delta)^{\frac{1}{2}} u(t_0)\|_\infty$. Noticing that
\[
0 \leq a_{n, \varepsilon} \leq a_{R, 1},
\]
by (2.22), (2.23) and $\delta < 0$, we obtain a contradiction.
We now drop assumption (2.18). For $\delta < 0$, set
\[
u_\delta(t, x) = u(t, x) + \delta t.
Then
\[ u_0(t, x) = u_0(0, x) + \int_0^t \left[ \delta + F(s, x, \nabla u_\delta(s, x), -(\Delta)\frac{1}{2} u_\delta(s, x)) \right] ds. \]
So, for all \( t \in [0, 1] \),
\[ \sup_{x \in \mathbb{R}^d} u(t, x) \leq \sup_{x \in \mathbb{R}^d} u_\delta(t, x) - \delta t \leq \sup_{x \in \mathbb{R}^d} u(0, x) - \delta t. \]
Letting \( \delta \uparrow 0 \), we conclude the proof of (2.16).

As for (2.17), by considering
\[ \tilde{u}(t, x) = u(t, x) - \int_0^t \| F(s, \cdot, 0, 0) \|_\infty \, ds \]
and using (2.16) for \( \tilde{u}(t, x) \) and \(-\tilde{u}(t, x)\) respectively, we immediately obtain (2.17).

Next we recall Silvestre’s Hölder estimate about the linear advection fractional-diffusion equation. The following result is taken from [16, Corollary 6.2]. Although the proofs given in [11] and [16] are only for constant diffusion coefficient \( a(t, x) \), by slight modifications, they are also adapted to the general bounded measurable function \( a(t, x) \).

**Theorem 2.4.** (Silvestre’s Hölder estimate) Let \( a : [0, 1] \times \mathbb{R}^d \to \mathbb{R} \) and \( b : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d \) be two bounded measurable functions. Let \( u \in C([0, 1]; C_b^2(\mathbb{R}^d)) \) satisfy
\[ u(t, x) = u(0, x) - \int_0^t (a(\Delta)^{\frac{1}{2}} u)(s, x) \, ds + \int_0^t (b \cdot \nabla u)(s, x) \, ds + \int_0^t f(s, x) \, ds. \]
If \( a(t, x) \geq a_0 > 0 \), then for any \( \gamma \in (0, 1) \), there exist \( \beta \in (0, 1) \) and \( C > 0 \) depending only on \( d, a_0, \gamma \) and \( \| u \|_\infty, \| b \|_\infty \) such that
\[ \sup_{t \in [0,1]} |u(t)|_{\epsilon^{\beta}} \leq C(\| u \|_\infty + \| f \|_\infty + |u(0)|_{\epsilon^{\gamma}}), \quad (2.24) \]
where \( | \cdot |_{\epsilon^{\beta}} \) is defined by (2.25).

### 3. Linear Nonlocal Parabolic Equation

In this section, we consider the following linear scalar nonlocal equation:
\[ \partial_t u + a(\Delta)^{\frac{1}{2}} u + b \cdot \nabla u = f, \quad u(0) = \varphi, \quad (3.1) \]
where \( a : [0, 1] \times \mathbb{R}^d \to \mathbb{R} \) and \( b : [0, 1] \times \mathbb{R}^d \to \mathbb{R}^d \) are two bounded measurable functions.

An increasing function \( \omega : \mathbb{R}^+ \to \mathbb{R}^+ \) is called a modulus function if \( \lim_{s \to 0} \omega(s) = 0 \). We make the following assumptions on \( a \) and \( b \):

**\( \text{H}^{a,b}_x \)** Let \( k \in \mathbb{N}_0 \), \( a, b \in L^\infty([0, 1]; C_b^k) \), and there are two modulus functions \( \omega_a \) and \( \omega_b \) such that for all \( t \in [0, 1] \) and \( x, y \in \mathbb{R}^d \),
\[ |a(t, x) - a(t, y)| \leq \omega_a(|x - y|), \quad |b(t, x) - b(t, y)| \leq \omega_b(|x - y|). \quad (3.2) \]
Moreover, for some \( a_0, a_1 > 0 \) and all \( (t, x) \in [0, 1] \times \mathbb{R}^d \),
\[ a_0 \leq a(t, x) \leq a_1. \]

We first prove the following important apriori estimate.

**Lemma 3.1.** For given \( p \in (1, \infty) \) and \( k \in \mathbb{N} \), let \( f \in Y^{k-1,p} \) and \( u \in X^{k,p} \) satisfy that for almost all \( (t, x) \in [0, 1] \times \mathbb{R}^d \),
\[ \partial_t u(t, x) + a(t, x)(\Delta)^{\frac{1}{2}} u(t, x) + b(t, x) \cdot \nabla u(t, x) = f(t, x). \quad (3.3) \]
Then under \( \text{H}^{a,b}_{k-1} \), there exists a constant \( C_{k,p} > 0 \) such that for all \( t \in [0, 1] \),
\[ \| u \|_{Y^{k,p}} \leq C_{k,p} \left( \| u(0) \|_{\epsilon^p} + \| f \|_{Y^{k-1,p}} \right), \quad (3.4) \]
where $C_{1,p}$ depends only on $a_0, a_1, \|b\|_\infty, d, p$ and $\omega_a, \omega_b$. In particular, equation \(3.3\) admits at most one solution in $\mathbb{R}^{k,p}$.

**Proof.** Let $\rho \in C^\infty_0(\mathbb{R}^d)$ be a family of mollifiers in $\mathbb{R}^d$, i.e., $\rho_\varepsilon(x) = \varepsilon^{-d} \rho(\varepsilon^{-1} x)$, where $\rho \in C^\infty_0(\mathbb{R}^d)$ is nonnegative and has support in $B_1$ and $\int \rho = 1$. Define

$$u_\varepsilon(t) := u(t) \ast \rho_\varepsilon, \quad a_\varepsilon(t) := a(t) \ast \rho_\varepsilon, \quad b_\varepsilon(t) := b(t) \ast \rho_\varepsilon, \quad f_\varepsilon(t) := f(t) \ast \rho_\varepsilon.$$  

Taking convolutions for both sides of \(3.3\), we have

$$\partial_t u_\varepsilon(t, x) + a_\varepsilon(t, x)(-\Delta)^{\frac{1}{2}} u_\varepsilon(t, x) + b_\varepsilon(t, x) \cdot \nabla u_\varepsilon(t, x) = h_\varepsilon(t, x), \quad (3.5)$$

where

$$h_\varepsilon(t, x) := f_\varepsilon(t, x) + a_\varepsilon(t, x)(-\Delta)^{\frac{1}{2}} u_\varepsilon(t, x) - [(a(t)(-\Delta)^{\frac{1}{2}} u(t)) \ast \rho_\varepsilon](x) + b_\varepsilon(t, x) \cdot \nabla u_\varepsilon(t, x) - [(b(t) \cdot \nabla u(t)) \ast \rho_\varepsilon](x).$$

By \(3.2\), it is easy to see that for all $\varepsilon \in (0, 1)$ and $t \in [0, 1]$ and $x, y \in \mathbb{R}^d$,

$$|a_\varepsilon(t, x) - a_\varepsilon(t, y)| \leq \omega_a(|x - y|), \quad |b_\varepsilon(t, x) - b_\varepsilon(t, y)| \leq \omega_b(|x - y|),$$

and

$$|a_\varepsilon(t, x) - a(t, x)| \leq \omega_a(\varepsilon), \quad |b_\varepsilon(t, x) - b(t, x)| \leq \omega_b(\varepsilon).$$

Moreover, by the property of convolutions, we also have

$$\lim_{\varepsilon \to 0} \int_0^1 \|h_\varepsilon(t) - f(t)\|_p^p dt = 0.$$  

Below, we use the method of freezing the coefficients to prove that for all $t \in [0, 1]$,

$$\|u_\varepsilon\|_{\mathcal{C}^1_{t,p}} \leq C \left( \|u_\varepsilon(0)\|_{1-\frac{1}{p}, p} + \|h_\varepsilon\|_{\mathcal{C}^0_{t,p}} \right), \quad (3.7)$$

where the constant $C$ is independent of $\varepsilon$. After proving this estimate, \(3.4\) with $k = 1$ immediately follows by taking limits for \(3.7\).

For simplicity of notation, we drop the subscript $\varepsilon$ below. Fix $\delta > 0$ being small enough, whose value will be determined below. Let $\zeta$ be a smooth function with support in $B_\delta$ and $\|\zeta\|_p = 1$. For $z \in \mathbb{R}^d$, set

$$\zeta_z(x) := \zeta(x - z), \quad \lambda_z^a := a(t, z), \quad \theta_z^b(t) := b(t, z).$$

Multiplying both sides of \(3.5\) by $\zeta_z$, we have

$$\partial_t (u_\varepsilon \zeta_z) + \lambda_z^a(\Delta)^{\frac{1}{2}} (u_\varepsilon \zeta_z) + \theta_z^b \cdot \nabla (u_\varepsilon \zeta_z) = g_z^\varepsilon,$$

where

$$g_z^\varepsilon := \lambda_z^a - a(\Delta)^{\frac{1}{2}} u_\varepsilon \zeta_z + \lambda_z^a((\Delta)^{\frac{1}{2}} u_\varepsilon \zeta_z) - (-\Delta)^{\frac{1}{2}} u_\varepsilon \zeta_z + b \cdot \nabla u_\varepsilon \zeta_z + h_\varepsilon \zeta_z.$$  

Let $T_{t,s}^{\varepsilon,\zeta}$ be defined by \(2.13\). Then $u_\varepsilon \zeta_z$ can be written as

$$u_\varepsilon \zeta_z(t, x) = T_{t,0}^{\varepsilon,\zeta}(u(0) \zeta_z)(x) + \int_0^t T_{t,s}^{\varepsilon,\zeta} g_z^\varepsilon(s, x) ds,$$

and so that for any $T \in [0, 1]$,

$$\int_0^T \|\nabla (u_\varepsilon \zeta_z)(t)\|_p^p dt \leq 2^{p-1} \left( \int_0^T \|\nabla T_{t,0}^{\varepsilon,\zeta}(u(0) \zeta_z)\|_p^p dt + \int_0^T \|\nabla \int_0^T T_{t,s}^{\varepsilon,\zeta} g_z^\varepsilon(s) ds\|_p^p dt \right)$$

$$=: 2^{p-1}(I_1(T, z) + I_2(T, z)).$$
For $I_1(T, z)$, by (2.14) and (2.7), we have

$$
\int_0^T \|\nabla T_{t,0}(u(0)\zeta_2)\|_p^p \, dt = \int_0^T \|\nabla \mathcal{P}_t\zeta_0(u(0)\zeta_2)\|_p^p \, dt \\
\leq C \int_0^T \|(-\Delta)^{\frac{1}{2}} \mathcal{P}_t\zeta_0(u(0)\zeta_2)\|_p^p \, dt \\
\leq C \|u(0)\zeta_2\|_{1-\frac{1}{p},p},
$$

(3.8)

where the last step is due to [14, p.96 Theorem 1.14.5] and (2.2). Thus, by definition (2.1), it is easy to see that

$$
\int_{\mathbb{R}^d} I_1(T, z) \, dz \leq C \int_{\mathbb{R}^d} \|u(0)\zeta_2\|_{1-\frac{1}{p},p} \, dz \leq C \left( \|u(0)\|_{1-\frac{1}{p},p} \|\zeta\|_p + \|u(0)\|_{1-\frac{1}{p},p} \|\zeta\|_p \right).
$$

For $I_2(T, z)$, by Theorem 2.2 we have

$$
I_2(T, z) \leq C \int_0^T \|g^p_z(s)\|_{1-\frac{1}{p},p} \, ds \\
+ C \int_0^T \|\partial_z^p((-\Delta)^{\frac{1}{2}} u\zeta_2 - (-\Delta)^{\frac{1}{2}} u\zeta_2)\|_p \, ds \\
+ C \int_0^T \|\partial_z^p \cdot \nabla (u\zeta_2)\|_p \, ds \\
+ C \int_0^T \|u \zeta_2\|_{1-\frac{1}{p},p} \|\zeta\|_p \|u(0)\|_{1-\frac{1}{p},p} \|\zeta\|_p.
$$

For $I_{21}(T, z)$, by (3.6) and $\|\zeta\|_p = 1$, we have

$$
\int_{\mathbb{R}^d} I_{21}(T, z) \, dz \leq C \omega_0^p(\delta) \int_{\mathbb{R}^d} \int_0^T \|(-\Delta)^{\frac{1}{2}} u\zeta_2\|_p \, ds \, dz \\
= C \omega_0^p(\delta) \int_0^T \|\nabla u(s)\|_p \, ds \\
\leq C \omega_0^p(\delta) \int_0^T \|\nabla u(s)\|_p \, ds.
$$

For $I_{22}(T, z)$, by (2.11) and Young’s inequality, we have

$$
\int_{\mathbb{R}^d} I_{22}(T, z) \, dz \leq C a_1 \int_0^T \int_{\mathbb{R}^d} \|(-\Delta)^{\frac{1}{2}} u\zeta_2 - (-\Delta)^{\frac{1}{2}} u\zeta_2\|_p \, ds \, dz \\
\leq C \int_0^T \|u(s)\|_p \, ds + C \int_0^T \|u(s)\|^{p/2} \|\nabla u(s)\|_p^{p/2} \, ds \\
\leq C \int_0^T \|u(s)\|_p \, ds + \frac{1}{4p} \int_0^T \|\nabla u(s)\|_p \, ds.
$$

For $I_{23}(T, z)$, as above we have

$$
\int_{\mathbb{R}^d} I_{23}(T, z) \, dz \leq C \omega_0^p(\delta) \left( \int_0^T \|\nabla u(s)\|_p \, ds + \|\zeta\|_p \int_0^T \|u(s)\|_p \, ds \right).
$$

Moreover, it is easy to see that

$$
\int_{\mathbb{R}^d} I_{24}(T, z) \, dz \leq C \|b\|_{\infty} \|\nabla \zeta\|_p \int_0^T \|u(s)\|_p \, ds, \\
\int_{\mathbb{R}^d} I_{25}(T, z) \, dz \leq C \int_0^T \|h(s)\|_p \, ds.
$$
Combining the above calculations, we get

\[
\int_0^T \| \nabla u(s) \|^p_\rho \, ds = \int_0^T \int_{\mathbb{R}^d} \| \nabla u(s) \cdot \xi \|^p_\rho \, dz \, ds
\]

\[
\leq 2^{p-1} \int_0^T \int_{\mathbb{R}^d} \| \nabla(u\xi) \|^p_\rho \, dz \, ds + 2^{p-1} \| \nabla \xi \|^p_\rho \int_0^T \| u(s) \|^p_\rho \, ds
\]

\[
\leq C \| u(0) \|_{1-\frac{1}{p}, \rho}^p + \left( \frac{1}{4} + C(\omega^p(\delta) + \omega^p(\delta)) \right) \int_0^T \| \nabla u(s) \|^p_\rho \, ds
\]

\[
+ C \int_0^T \| u(s) \|^p_\rho \, ds + C \int_0^T \| h(s) \|^p_\rho \, ds.
\]

Choosing \( \delta_0 > 0 \) being small enough so that

\[C(\omega^p(\delta_0) + \omega^p(\delta_0)) \leq \frac{1}{4},\]

we obtain that for all \( T \in [0, 1], \)

\[
\int_0^T \| \nabla u(s) \|^p_\rho \, ds \leq C \| u(0) \|_{1-\frac{1}{p}, \rho}^p + C \int_0^T \| u(s) \|^p_\rho \, ds + C \int_0^T \| h(s) \|^p_\rho \, ds. \quad (3.9)
\]

On the other hand, by (3.5), it is easy to see that for all \( t \in [0, 1], \)

\[\| u(t) \|_\rho \leq C \| u(0) \|_{1-\frac{1}{p}, \rho}^p + C \int_0^t \| \nabla u(s) \|^p_\rho \, ds + C \int_0^t \| h(s) \|^p_\rho \, ds,
\]

which together with (3.9) and Gronwall’s inequality yields that for all \( t \in [0, 1], \)

\[
\sup_{s \in [0, t]} \| u(s) \|_\rho + \int_0^t \| \nabla u(s) \|^p_\rho \, ds \leq C \left( \| u(0) \|_{1-\frac{1}{p}, \rho}^p + \int_0^t \| h(s) \|^p_\rho \, ds \right). \quad (3.10)
\]

From equation (3.3), by (2.7) we also have

\[
\int_0^t \| \partial_t u(s) \|^p_\rho \, ds \leq C \left( \| a \|_\rho \int_0^t \| (-\Delta)^{\frac{1}{2}} u(s) \|^p_\rho \, ds + \| b \|_\rho \int_0^t \| \nabla u(s) \|^p_\rho \, ds + \int_0^t \| h(s) \|^p_\rho \, ds \right)
\]

\[
\leq C \left( \| a \|_\rho + \| b \|_\rho \int_0^t \| \nabla u(s) \|^p_\rho \, ds + \int_0^t \| h(s) \|^p_\rho \, ds \right),
\]

which together with (3.10) gives (3.7), and therefore (3.4) with \( k = 1. \)

Let us now estimate the higher order derivatives. For \( n = 1, 2, \ldots, k, \) let

\[w^{(n)}(t, x) := \nabla^n u(t, x).
\]

By the chain rule, we have

\[\partial_t w^{(n)} + a(-\Delta)^{\frac{1}{2}} w^{(n)} + b \cdot \nabla w^{(n)} = h^{(n)},\]

where

\[h^{(n)} := \nabla^n f - \sum_{j=1}^n \frac{n!}{(n-j)!j!} (\nabla^j a \cdot \nabla^{n-j} (-\Delta)^{\frac{1}{2}} u + \nabla^j b \cdot \nabla^{n-j+1} u).
\]

Thus, by (3.4) with \( k = 1 \) and the assumptions, we have

\[
\| \nabla^n u \|_{X^1_\rho} = \| w^{(n)} \|_{X^1_\rho} \leq C \left( \| w^{(n)}(0) \|_{1-\frac{1}{p}, \rho} + \| h^{(n)} \|_{X^1_\rho} \right)
\]

\[
\leq C \left( \| \nabla^n u(0) \|_{1-\frac{1}{p}, \rho} + \sum_{j=1}^n \| \nabla^{n-j+1} u \|_{X^1_\rho} + \| \nabla^n f \|_{X^0_\rho} \right),
\]

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which implies that
\[ \|u\|_{X^{k,p}} \leq C \left( \|u(0)\|_{a+1-rac{1}{p},p} + \|u\|_{Y^{k,p}} + \|f\|_{Y^{k,p}} \right). \]

By induction method, one obtains (3.7).

Now we prove the existence of solutions to equation (3.1).

**Theorem 3.2.** Let \( p > 1 \) and \( k \in \mathbb{N} \). Under (H\(_{k-1}^{a,b}\)), for any \( \varphi \in \mathcal{Y}^{k-1,p} \) and \( f \in \mathcal{Y}^{k-1,p} \), there exists a unique \( u \in \mathcal{X}^{k,p} \) with \( u(0) = \varphi \) solving equation (3.7).

**Proof.** We use the continuity method. For \( \lambda \in [0, 1] \), define an operator \( U_\lambda := \partial_t + \lambda a(-\Delta)^\frac{1}{2} + \lambda b \cdot \nabla + (1 - \lambda)(-\Delta)^\frac{1}{2} \).

By (2.7), it is easy to see that \( U_\lambda : \mathcal{X}^{k,p} \to \mathcal{Y}^{k-1,p} \).

For given \( \varphi \in \mathcal{Y}^{k-1,p} \), let \( \mathcal{X}^{k,p}_\varphi \) be the space of all functions \( u \in \mathcal{X}^{k,p} \) with \( u(0) = \varphi \). It is clear that \( \mathcal{X}^{k,p}_\varphi \) is a complete metric space with respect to the metric \( k \cdot \| \cdot \|_{2k,p} \). For \( \lambda = 0 \) and \( f \in \mathcal{Y}^{k-1,p} \), it is well-known that there is a unique \( u \in \mathcal{X}^{k,p}_\varphi \) such that
\[ U_0u = \partial_t u + (-\Delta)^\frac{1}{2} u = f. \]

In fact, Duhamel’s formula, the unique solution can be represented by
\[ u(t, x) = \mathcal{P}_t^1 \varphi(x) + \int_0^t \mathcal{P}_{t-s}^1 f(s, x)ds. \]

Suppose now that for some \( \lambda_0 \in [0, 1] \), and for any \( f \in \mathcal{Y}^{k-1,p} \), the equation
\[ U_{\lambda_0}u = f \]
admits a unique solution \( u \in \mathcal{X}^{k,p}_\varphi \). Then, for fixed \( f \in \mathcal{Y}^{k-1,p} \) and \( \lambda \in [\lambda_0, 1] \), and for any \( u \in \mathcal{X}^{k,p}_\varphi \), by (3.11), the equation
\[ U_{\lambda_0}w = f + (U_{\lambda_0} - U_{\lambda})u \]
admits a unique solution \( w \in \mathcal{X}^{k,p}_\varphi \). Introduce an operator
\[ w = Q_\lambda u. \]

We want to use Lemma 3.1 to show that there exists an \( \varepsilon > 0 \) independent of \( \lambda_0 \) such that for all \( \lambda \in [\lambda_0, \lambda_0 + \varepsilon] \),
\[ Q_\lambda : \mathcal{X}^{k,p}_\varphi \to \mathcal{X}^{k,p}_\varphi \]
is a contraction operator.

Let \( u_1, u_2 \in \mathcal{X}^{k,p}_\varphi \) and \( w_i = Q_\lambda u_i, i = 1, 2 \). By equation (3.12), we have
\[ U_{\lambda_0}(w_1 - w_2) = (U_{\lambda_0} - U_{\lambda})(u_1 - u_2) = (\lambda_0 - \lambda)((a - 1)(-\Delta)^\frac{1}{2} + b \cdot \nabla)(u_1 - u_2). \]

By (3.4) and (2.7), one sees that
\[ \|Q_\lambda u_1 - Q_\lambda u_2\|_{\mathcal{Y}^{k,p}} \leq C_{k,p}\lambda_0 - \lambda \cdot \|((a - 1)(-\Delta)^\frac{1}{2} + b \cdot \nabla)(u_1 - u_2)\|_{\mathcal{Y}^{k-1,p}} \]
\[ \leq C_0\lambda_0 - \lambda \cdot \|u_1 - u_2\|_{\mathcal{Y}^{k,p}} \leq C_0\lambda_0 - \lambda \cdot \|u_1 - u_2\|_{\mathcal{X}^{k,p}}, \]
where \( C_0 \) is independent of \( \lambda, \lambda_0 \) and \( u_1, u_2 \). Taking \( \varepsilon = 1/(2C_0) \), one sees that
\[ Q_\lambda : \mathcal{X}^{k,p}_\varphi \to \mathcal{X}^{k,p}_\varphi \]
is a 1/2-contraction operator. By the fixed point theorem, for each \( \lambda \in [\lambda_0, \lambda_0 + \varepsilon] \), there exists a unique \( u \in \mathcal{X}^{k,p}_\varphi \) such that
\[ Q_\lambda u = u, \]
which means that
\[ U_1 u = f. \]
Now starting from \( \lambda = 0 \), after repeating the above construction \( \lfloor \frac{1}{\epsilon} \rfloor + 1 \)-steps, one obtains that for any \( f \in \mathbb{X}^{k,p} \),
\[ U_1 u = f \]
admits a unique solution \( u \in \mathbb{X}^{k,p} \).

\[ \square \]

4. Quasi-linear nonlocal parabolic system

Consider the following quasi-linear nonlocal parabolic system:

\[ \partial_t u + a(u, R_u u)(-\Delta)^{\frac{1}{2}} u + b(u, R_u u) \cdot \nabla u = f(u, R_f u), \tag{4.1} \]

where \( u = (u^1, \ldots, u^n) \) and
\[
\begin{align*}
    a(t, x, u, r) & : [0, 1] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^k \rightarrow \mathbb{R}, \\
b(t, x, u, r) & : [0, 1] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^k \rightarrow \mathbb{R}^d, \\
f(t, x, u, r) & : [0, 1] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^k \rightarrow \mathbb{R}^m,
\end{align*}
\]
are measurable functions, and
\[ R_u = (R^{ij}_u), R_b = (R^{ij}_b), R_f = (R^{ij}_f) \in \mathbb{H}^{k,m}. \]

Here we have used that \( R_u = \sum_{i=1}^m R^{ij}_u u^j \), similarly for \( R_b u \) and \( R_f u \).

The main result of this section is:

**Theorem 4.1.** Suppose that \( a(t, x, u, r) \geq a_0 > 0 \), and for any \( R \geq 0 \),
\[
\begin{align*}
a, b, f & \in L^\infty([0, 1]; C^\infty_0(\mathbb{R}^d \times B^m_R \times B^k_R)), \tag{4.2} \\
a, b & \in L^\infty([0, 1]; C^\infty_0(\mathbb{R}^d \times B^m_R \times \mathbb{R}^k)), \tag{4.3}
\end{align*}
\]
where \( B^m_R \) denotes the ball in \( \mathbb{R}^m \) with radius \( R \), and for each \( j \in \mathbb{N}_0 \), there exist \( C_{f,j}, \gamma_j \geq 0 \) and \( h_j \in (L^1 \cap L^\infty)(\mathbb{R}^d) \) such that
\[
|\nabla_t f(t, x, u, r)| \leq C_{f,j} |u| |u|^{\gamma_j} + h_j(x), \tag{4.4}
\]
and for some \( C_f \geq 0 \),
\[
\langle u, f(t, x, u, r) \rangle_{\mathbb{R}^m} \leq C_f (|u|^2 + 1). \tag{4.5}
\]

Then for any \( \varphi \in \mathcal{W}^{k,p} \), there exists a unique \( u \in \mathbb{X}^{k,p} \) solving equation \( (4.7) \). Moreover,
\[
\sup_{t \in [0,1]} \|u(t)\|^2_{\mathbb{X}^{k,p}} \leq e^{C_f} (\|\varphi\|^2_{\mathbb{X}^{k,p}} + C_f).
\]

**Proof.** First of all, for any \( R \in \mathcal{R} \) and \( u \in \mathbb{X}^{k,p} \), by the boundedness of \( \mathcal{R} \) in Sobolev space \( \mathcal{W}^{k,p} \), one has
\[
(t, x) \mapsto \mathcal{R} u(t, x) \in \mathbb{X}^{k,p}.
\]
Thus, by \( (4.2) \) and the chain rules, one sees that for any \( u \in \mathbb{X}^{k,p} \),
\[
\begin{align*}
(t, x) & \mapsto a(t, x, u(t, x), \mathcal{R}_u u(t, x)) \in L^\infty([0, 1]; C^\infty_0), \\
(t, x) & \mapsto b(t, x, u(t, x), \mathcal{R}_b u(t, x)) \in L^\infty([0, 1]; C^\infty_0),
\end{align*}
\]
and by \( (4.4) \),
\[
(t, x) \mapsto f(t, x, u(t, x), \mathcal{R}_f u(t, x)) \in \mathcal{W}^{k,p}.
\]

Set \( u_0(t, x) \equiv 0 \). By Theorem 3.2 we can recursively define \( u_n \in \mathbb{X}^{k,p} \) by the following linear equation:
\[
\partial_t u_n + a(u_{n-1}, \mathcal{R}_u u_{n-1})(-\Delta)^{\frac{1}{2}} u_n + b(u_{n-1}, \mathcal{R}_b u_{n-1}) \cdot \nabla u_n = f(u_{n-1}, \mathcal{R}_f u_{n-1}) \tag{4.6}
\]
subject to the initial value \( u_n(0) = \varphi \in \mathcal{W}^p \).

We first assume that \( \gamma_0 = 0 \). For simplicity of notation, we set

\[
\begin{align*}
    a_n(t, x) &:= a(t, x, u_{n-1}(t, x), \mathcal{R}_a u_{n-1}(t, x)), \\
    b_n(t, x) &:= b(t, x, u_{n-1}(t, x), \mathcal{R}_b u_{n-1}(t, x)), \\
    f_n(t, x) &:= f(t, x, u_{n-1}(t, x), \mathcal{R}_f u_{n-1}(t, x)).
\end{align*}
\]

By the maximum principle (see Theorem 2.4) and in view of \( \gamma_0 = 1 \), it is easy to see that

\[
||u_n(t)||_\infty \leq ||\tilde{u}_n(t)||_\infty + \int_0^t ||f(s, \cdot, u_{n-1}(s, \cdot), \mathcal{R}_f u_{n-1}(s, \cdot))||_\infty ds,
\]

\[
\leq ||\tilde{u}_n(0)||_\infty + \int_0^t ||f_n(s)||_\infty ds + ||h_0||_\infty ds,
\]

which yields by Gronwall’s inequality that

\[
\sup_{t \in [0,1]} ||u_n(t)||_\infty \leq e^{Cf_0} (||\varphi||_\infty + ||h_0||_\infty) =: K_0.
\]  

(4.7)

By Theorem 2.4 and (2.6), there exist \( \beta \in (0, 1) \) and constant \( K_1 > 0 \) depending on \( K_0 \) such that for all \( n \in \mathbb{N} \),

\[
\sup_{t \in [0,1]} ||u_n(t)||_{\varphi, \beta} + \sup_{t \in [0,1]} ||\mathcal{R}_a u_n(t)||_{\varphi, \beta} + \sup_{t \in [0,1]} ||\mathcal{R}_b u_n(t)||_{\varphi, \beta} \leq K_1.
\]

Thus, by (4.3) we have

\[
|a_n(t, x) - a_n(t, y)| \leq ||\nabla_x a||_{L^\infty_{t_0}} |x - y| + K_1 \left( ||\nabla_x a||_{L^\infty_{t_0}} + ||\nabla_x a||_{L^\infty_{t_0}} \right) |x - y|^\beta,
\]

(4.8)

\[
|b_n(t, x) - b_n(t, y)| \leq ||\nabla_x b||_{L^\infty_{t_0}} |x - y| + K_1 \left( ||\nabla_x b||_{L^\infty_{t_0}} + ||\nabla_x b||_{L^\infty_{t_0}} \right) |x - y|^\beta,
\]

(4.9)

where \( ||\cdot||_{L^\infty_{t_0}} \) denotes the sup-norm in \( L^\infty([0, 1] \times \mathbb{R}^d \times B^p_{t_0} \times \mathbb{R}^k) \).

For \( k = 0, 1, 2, \cdots \), set

\[
w^{(k)}_n(t, x) := \nabla^k u_n(t, x).
\]

By the chain rule, we have

\[
\partial_t w^{(k)}_n + a_n(-\Delta)^{\frac{k}{2}} w^{(k)}_n + b_n \cdot \nabla w^{(k)}_n = g^{(k)}_n,
\]

where \( g^{(0)}_n = f_n \) and for \( k \geq 1 \),

\[
g^{(k)}_n := \nabla^k f_n - \sum_{j=1}^k \frac{k!}{(k-j)!j!} \left( \nabla^j a_n \cdot \nabla^{k-j}(-\Delta)^{\frac{j}{2}} u_n + \nabla^j b_n \cdot \nabla^{k-j} \nabla u_n \right).
\]

By (4.8), (4.9) and Lemma 3.1 we have for all \( p > 1 \) and \( t \in [0, 1] \),

\[
||w^{(k)}_n||_{\varphi, \beta} \leq C_{k,p} \left( ||\nabla^k \varphi||_{1-\frac{1}{p}} + ||g^{(k)}_n||_{\varphi, \beta} \right),
\]

(4.10)

where \( C_{k,p} \) is independent of \( n \).

For \( k = 0 \), by (4.10), (4.4) and (4.7), we have

\[
||u_n(t)||_p + \int_0^t ||u_n(s)||^p \, ds \leq C ||\varphi||^p_{1-\frac{1}{p}, \rho} + C \int_0^t ||f_n(s)||^p \, ds \leq C ||\varphi||^p_{1-\frac{1}{p}, \rho} + \int_0^t \left( C_{f,0}(K_0^\gamma + 1) ||u_{n-1}(s)||_p + ||h_0(s)||_p \right)^p \, ds \leq C ||\varphi||^p_{1-\frac{1}{p}, \rho} + C \int_0^t ||u_{n-1}(s)||^p \, ds + C \int_0^t ||h_0(s)||^p \, ds.
\]
By Gronwall’s inequality, one gets
\[ \sup_{n \in \mathbb{N}} \sup_{t \in [0,1]} \| u_n(t) \|_p^p \leq C_p, \]
and therefore, for all \( p > 1, \)
\[ \sup_{n \in \mathbb{N}} \| u_n \|_{\mathcal{X}^1,p} \leq C_p. \]

Now for any \( k = 1, 2, \cdots, \), since by the chain rules, \( g_n^{(k)} \) only contains the powers of all derivatives up to \( k \)-order of \( u_n, \mathcal{R}_a u_n, \mathcal{R}_b u_n \) and \( \mathcal{R}_f u_n, \) by induction method and using Hölder’s inequality, it is easy to see that for all \( k \in \mathbb{N} \) and \( p > 1, \)
\[ \sup_{n \in \mathbb{N}} \| u_n \|_{\mathcal{X}^k,p} \leq C_{k,p}. \] (4.11)

Below we write
\[ w_{n,m}(t, x) := u_n(t, x) - u_m(t, x). \]

Then
\[ \partial_t w_{n,m} + a_n(-\Delta)\frac{t}{2} w_{n,m} + b_n \cdot \nabla w_{n,m} = g_{n,m}, \]
where
\[ g_{n,m} := f_n - f_m + (a_m - a_n)(-\Delta)\frac{t}{2} u_m + (b_m - b_n) \cdot \nabla u_m. \]

By Lemma 3.1 again, we have for all \( p > 1 \) and \( t \in [0, 1], \)
\[ \| w_{n,m} \|_{L^1_t, \mathcal{X}_p} \leq C \| g_{n,m} \|_{L^1_t, \mathcal{X}_p}. \]

Here and below, \( C > 0 \) is independent of \( n, m. \) Using (4.11) and (4.3), we have
\[ \| g_{n,m} \|_{L^1_t, \mathcal{X}_p} \leq C \left( \| f_n - f_m \|_{L^1_t, \mathcal{X}_p} + \| a_n - a_m \|_{L^1_t, \mathcal{X}_p} + \| b_n - b_m \|_{L^1_t, \mathcal{X}_p} \right) \]
\[ \leq C \left( \| \nabla_a f \|_{L^\infty_{t,x}} + \| \nabla_r f \|_{L^\infty_{t,x}} + \| \nabla_a g \|_{L^\infty_{t,x}} + \| \nabla_r g \|_{L^\infty_{t,x}} \right) \]
\[ + \| \nabla_a b \|_{L^\infty_{t,x}} + \| \nabla_r b \|_{L^\infty_{t,x}} \) \| w_{n-1,m-1} \|_{L^1_t, \mathcal{X}_p}. \]

Hence,
\[ \sup_{s \in [0,t]} \| w_{n,m}(s) \|_p^p \leq C \int_0^t \| w_{n-1,m-1}(s) \|_p^p \, ds. \]

Taking sup-limits and by Fatou’s lemma, we obtain
\[ \lim_{n,m \to \infty} \sup_{s \in [0,t]} \| w_{n,m}(s) \|_p^p \leq C \int_0^t \lim_{n,m \to \infty} \sup_{s \in [0,t]} \| w_{n,m}(s) \|_p^p \, ds. \]

So,
\[ \lim_{n,m \to \infty} \sup_{s \in [0,1]} \| w_{n,m}(s) \|_p^p = 0, \]
which together with (4.11) and the interpolation inequality yields that for all \( k \in \mathbb{N} \) and \( p > 1, \)
\[ \lim_{n,m \to \infty} \sup_{s \in [0,1]} \| w_{n,m}(s) \|_{\mathcal{X}^k,p}^p = 0. \]

Thus, there exists a \( u \in \mathcal{X}^\infty \) such that for all \( k \in \mathbb{N} \) and \( p > 1, \)
\[ \lim_{n,m \to \infty} \sup_{s \in [0,1]} \| u_n(s) - u(s) \|_{\mathcal{X}^k,p}^p = 0. \]

Taking limits for (4.6), one sees that \( u \) solves equation (4.1).

Now we want to drop \( \gamma_0 = 0 \) and assume (4.5). For \( R > 0, \) let \( \chi_R \in C_0^\infty(\mathbb{R}^d) \) be a nonnegative cutoff function with \( \chi_R(u) = 1 \) for \( |u| \leq R \) and \( \chi_R(u) = 0 \) for \( |u| > R + 1. \) Set
\[ f_R(t, x, u, r) := f(t, x, u, r) \chi_R(u) \]
Let \( u_R \in X^\infty \) solve
\[
\partial_t u_R + a(u_R, R_u u_R)(-\Delta)^\frac{1}{2} u_R + b(u_R, R_u u_R) \cdot \nabla u_R = f_R(u_R, R_f u_R).
\]
Noticing that by (2.9),
\[
2\langle (-\Delta)^\frac{1}{2} u_R, u_R \rangle_{\mathbb{R}^n} = (-\Delta)^\frac{1}{2} |u_R|^2 + \langle u_R, u_R \rangle,
\]
we have
\[
2\partial_t |u_R|^2 + a(u_R, R_u u_R)(-\Delta)^\frac{1}{2} |u_R|^2 + b(u_R, R_u u_R) \cdot \nabla |u_R|^2
\]
\[
= 2\langle u_R, f_R(u_R, R_f u_R) \rangle_{\mathbb{R}^n} - a(u_R, R_u u_R) \langle u_R, u_R \rangle \leq 2C_f |u_R|^2 + 1.
\]
Thus, by the maximal principle, we have
\[
||u_R(t)||^2_{\infty} \leq ||\varphi||^2_{\infty} + C_f \int_0^t (||u_R(s)||^2_{\infty} + 1) \, ds,
\]
which implies that for all \( R > 0 \),
\[
\sup_{t \in [0,1]} ||u_R(t)||^2_{\infty} \leq e^{C_f}(||\varphi||^2_{\infty} + C_f).
\]
The proof is finished by taking \( R := [e^{C_f}(||\varphi||^2_{\infty} + C_f)]^{1/2} \).

The following lemma will play a key role in proving the existence.

**Lemma 5.1.** Let \( a \in L^\infty([0,1]; C^1_b(\mathbb{R}^d)) \) be bounded below by \( a_0 > 0 \) and \( b \in L^\infty([0,1]; C^1_b(\mathbb{R}^d)) \). Let \( u : [0,1] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \) belong to \( \mathcal{X}^{2,p} \) for some \( p > 1 \) and satisfy
\[
\partial_t u = a(-\Delta)^{-\frac{1}{2}} u + b \cdot (\nabla u - (\nabla u)') \tag{5.1}
\]
where \( \Box := \text{div} \nabla - \nabla \text{div} \). Then we have
\[
||u||_{\mathcal{X}^{1,p}} + ||U||_{\mathcal{X}^{1,p}} \leq C\left(||u(0)||_{1,p} + ||U(0)||_{1,p}\right),
\]
where \( U := \nabla u - (\nabla u)' \).

**Proof.** By equation (5.1), one sees that
\[
\partial_t u = -a(-\Delta)^{-\frac{1}{2}} u - a(-\Delta)^{-\frac{1}{2}} \nabla \text{div} u + b \cdot U,
\]
and
\[
\partial_t U = -a(-\Delta)^{-\frac{1}{2}} U + b \cdot \nabla U + (\nabla b) \cdot U - [(\nabla b) \cdot U]' + A,
\]
where
\[
A := (\nabla a)' \cdot (-\Delta)^{-\frac{1}{2}} \Box u - ((-\Delta)^{-\frac{1}{2}} \Box u)' \cdot \nabla a.
\]
By Lemma [3.1], there exists a constant \( C > 0 \) such that for all \( t \in [0,1] \),
\[
||u||_{\mathcal{X}^{1,p}} \leq C||u(0)||_{1,\frac{p}{p-1},p} + C||a(-\Delta)^{-\frac{1}{2}} \nabla \text{div} u||_{\mathcal{X}^{0,p}} + C||b \cdot U||_{\mathcal{X}^{0,p}}
\]
\[
\leq C||u(0)||_{1,\frac{p}{p-1},p} + C||a||_{\infty}||\nabla u||_{\mathcal{X}^{0,p}} + C||b||_{\infty}||U||_{\mathcal{X}^{0,p}},
\]
and
\[
||U||_{\mathcal{X}^{1,p}} \leq C||U(0)||_{1,\frac{p}{p-1},p} + C||\nabla b \cdot U + U \cdot (\nabla b)' + A||_{\mathcal{X}^{0,p}}
\]
\[
\leq C||U(0)||_{1,\frac{p}{p-1},p} + C||\nabla a||_{\infty} + ||\nabla b||_{\infty}||\nabla u||_{\mathcal{X}^{0,p}}.
\]
In particular, for all \( t \in [0,1] \),
\[
||u(t)||_p^p + \int_0^t ||u(s)||_p^p \, ds \leq C||u(0)||_p^p + C \int_0^t ||\text{div} u(s)||_p^p \, ds + C \int_0^t ||U(s)||_p^p \, ds
\]
Thus, there is at least one point
\[ \int \]
In particular,
\[ \int_0 \]
So, for some
\[ \int_0 \]
where
\[ \int \]
Hence,
\[ \int \]
Repeating the above proof, we obtain the desired estimate.
\[ \Box \]
As introduced in the introduction, let
\[ \int \]
we have
\[ \int \]
Hence,
\[ \int \]
\[ \int \]
we have
\[ \int \]
\[ \int \]
Now starting from
\[ \int \]
On the other hand, noticing that
\[ \int \]
we have
\[ \int \]
\[ \int \]
\[ \int \]
\[ \int \]
\[ \int \]
\[ \int \]
\[ \int \]
\[ \int \]
Now substituting (5.3) and (5.4) into (5.2), we obtain that for all \( t \in [0, 1] \),
\[ \int \]
where \( C_0, C_1 \) are independent of \( ||u(0)||_{1,p} \) and \( ||U(0)||_{1,p} \). Choosing \( t_0 := 1/(2C_1) \), we arrive at
\[ \int \]
So, for some \( C_2 > 0 \),
\[ \int \]
In particular,
\[ \int \]
Thus, there is at least one point \( s_0 \in [2t_0/3, t_0] \) such that
\[ \int \]
Now starting from \( s_0 \), as above, one can prove that for the same \( t_0 \),
\[ \int \]
Repeating the above proof, we obtain the desired estimate.
\[ \Box \]
We are now in a position to give
\[ \int \]
We divide the proof into three steps.
\[ \int \]
In this step we consider the following fully non-linear and nonlocal parabolic equation:
\[ \int \]
As introduced in the introduction, let
\[ \int \]
For any $\varphi \in \mathcal{U}^\infty = \cap_{k,p} \mathcal{U}^{k,p}$, where $\mathcal{U}^{k,p}$ is defined by (2.4), by Theorem 4.1 there exists a unique $w \in \mathbb{R}^\infty$ solving the following parabolic system:

$$
\partial_t w = - (\partial_q F)(w, \mathcal{R}w)(-\Delta)^{\frac{1}{2}} w + (\nabla_w F)(w, \mathcal{R}w) \nabla w + \nabla F(w, \mathcal{R}w)
$$

subject to $w(0) = \nabla \varphi$. Define

$$
u(t,x) := \varphi(x) + \int_{0}^{t} F(s,x,w(s,x),\mathcal{R}w(s,x)) \, ds
$$

and

$$h(t,x) := \nabla \nu(t,x) - w(t,x).
$$

Then we have

$$
\partial_t h = (\partial_q F)(w, \mathcal{R}w)(\nabla \mathcal{R}w + (-\Delta)^{\frac{1}{2}} w) + (\nabla_w F)(w, \mathcal{R}w)((\nabla w)^{\dagger} - \nabla w)
$$

subject to $h(0) = 0$, where $\Box := \text{div} \nabla - \text{div}$. By Lemma 5.1 we have

$$
h = 0 \Rightarrow w = \nabla \nu.
$$

Thus, by (5.5),

$$
\partial_t \nu(t,x) = F(t,x,\nabla \nu(t,x),\mathcal{R} \nabla \nu(t,x)) = F(t,x,\nabla \nu(t,x),-(-\Delta)^{\frac{1}{2}} u(t,x)).
$$

By the maximum principle (see Theorem 2.3), we have

$$
||u(t)||_\infty \leq ||\varphi||_\infty + \int_{0}^{t} ||F(s,\cdot,0,0)||_\infty \, ds. \tag{5.6}
$$

In particular, $u \in C([0,1] ; \mathcal{U}^\infty)$.

**Step 2.** Now we consider the general case. Set $u_0(t,x) = 0$. Let $u_n \in C([0,1] ; \mathcal{U}^\infty)$ be defined recursively by the following equation:

$$
\partial_t u_n = F(t,x,u_{n-1},\nabla u_n,-(-\Delta)^{\frac{1}{2}} u_n), \quad u_n(0) = \varphi. \tag{5.7}
$$

By (5.6) and (1.6), we have

$$
||u_n(t)||_\infty \leq ||\varphi||_\infty + \int_{0}^{t} ||F(s,\cdot,u_{n-1}(s,\cdot),0,0)||_\infty \, ds
$$

$$
\leq ||\varphi||_\infty + \kappa_0 \int_{0}^{t} (||u_{n-1}(s)||_\infty + 1) \, ds.
$$

By Gronwall’s inequality, we get

$$
||u_n(t)||_\infty \leq e^{\kappa_0} (||\varphi||_\infty + \kappa_0) =: K_0. \tag{5.8}
$$

On the other hand, by taking gradients with respect to $x$ for equation (5.7), we have

$$
\partial_t \nabla u_n = -\partial_q F(t,x,u_{n-1},\nabla u_n,-(-\Delta)^{\frac{1}{2}} u_n)(-\Delta)^{\frac{1}{2}} \nabla u_n + \nabla_w F(t,x,u_{n-1},\nabla u_n,-(-\Delta)^{\frac{1}{2}} u_n) \nabla^2 u_n
$$

$$
+ \partial_q F(t,x,u_{n-1},\nabla u_n,-(-\Delta)^{\frac{1}{2}} u_n) \nabla u_{n-1} + \nabla \nabla F(t,x,u_{n-1},\nabla u_n,-(-\Delta)^{\frac{1}{2}} u_n).
$$

By the maximum principle again and (1.9), (1.10) with $\gamma_{K_0,1} = 0$, we have

$$
||\nabla u_n(t)||_\infty \leq ||\nabla \varphi||_\infty + \int_{0}^{t} ||\partial_q F(s,x,u_{n-1},\nabla u_n,-(-\Delta)^{\frac{1}{2}} u_n) \nabla u_{n-1}||_\infty \, ds
$$

$$
+ \int_{0}^{t} ||\nabla \nabla F(s,x,u_{n-1},\nabla u_n,-(-\Delta)^{\frac{1}{2}} u_n)||_\infty \, ds
$$

$$
where \( C \) is independent of \( n \). By Gronwall’s inequality, we get

\[
\sup_n \sup_{t \in [0,1]} \|\nabla u_n(t)\|_\infty < +\infty. \tag{5.9}
\]

Moreover, by (1.8), (1.9), (1.10), (5.8), Theorem 2.4 and Lemma 2.1, as in the proof of Theorem 4.1, we have for all \( p > 1 \),

\[
\|\nabla u_n\|_{\dot{X}^1,p} \leq C \left( \|\nabla \varphi\|_{1-\frac{1}{p},p} + \|\nabla u_{n-1}\|_{\dot{X}^1,p} + \|\nabla u_n\|_{\dot{X}^1,p} + \|h_1\|_p \right),
\]

which implies by Gronwall’s inequality that

\[
\sup_n \|\nabla u_n\|_{\dot{X}^1,p} < +\infty, \tag{5.10}
\]

and furthermore, for all \( k \in \mathbb{N} \) and \( p > 1 \),

\[
\sup_n \|\nabla u_n\|_{\dot{X}^{k,p}} < +\infty. \tag{5.11}
\]

This together with (5.8) gives

\[
\sup_{n} \sup_{t \in [0,1]} \|u_n(t)\|_{\dot{X}^{k,p}} < +\infty. \tag{5.12}
\]

(Step 3). Next we want to show that \( u_n \) converges to some \( u \) in \( C([0,1]; \dot{U}^{k,p}) \). For \( n, m \in \mathbb{N} \), set

\[
v_{n,m}(t, x) := u_n(t, x) - u_m(t, x).
\]

Then

\[
\partial_t v_{n,m} = -a_{n,m}(-\Delta)^{\frac{1}{2}} v_{n,m} + b_{n,m} \cdot \nabla v_{n,m} + f_{n,m} v_{n-1,m-1},
\]

where

\[
a_{n,m} := \int_0^1 (\partial_q F)(u_{n-1}, \nabla u_n, -(-\Delta)^{\frac{1}{2}} (sv_{n,m} + u_m)) ds,
\]

\[
b_{n,m} := \int_0^1 (\nabla_w F)(u_{n-1}, \nabla (sv_{n,m} + u_m), -(-\Delta)^{\frac{1}{2}} u_m) ds,
\]

\[
f_{n,m} := \int_0^1 (\partial_u F)(sv_{n-1,m-1} + u_{n-1}, \nabla u_m, -(-\Delta)^{\frac{1}{2}} u_m) ds.
\]

By the maximum principle, we have

\[
\|v_{n,m}(t)\|_\infty \leq C \int_0^t \|v_{n-1,m-1}(s)\|_\infty ds,
\]

and by Gronwall’s inequality,

\[
\lim_{n,m \to \infty} \sup_{t \in [0,1]} \|v_{n,m}(t)\|_\infty = 0. \tag{5.13}
\]

On the other hand, by Lemma 2.1 and (5.12), we may derive that for all \( t \in [0,1] \),

\[
\|v_{n,m}\|_{\dot{X}^1,p} \leq C \|v_{n-1,m-1}\|_{\dot{X}^1,p},
\]

and so,

\[
\lim_{n,m \to \infty} \|v_{n,m}\|_{\dot{X}^1,p} = 0.
\]

This together with (5.11), the interpolation inequality and (5.13) yields that for all \( k \in \mathbb{N} \) and \( p > 1 \),

\[
\lim_{n,m \to \infty} \sup_{t \in [0,1]} \|v_{n,m}(t)\|_{\dot{X}^{k,p}} = 0.
\]
Thus, there is a $u \in C([0, 1]; U^\infty)$ such that for all $k \in \mathbb{N}$ and $p > 1$,

$$\lim_{n,m \to \infty} \sup_{t \in [0, 1]} \|u_n(t) - u(t)\|_{L^p} = 0.$$ 

The proof is complete by taking limits for approximation equation (5.7).

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