ANALYSIS OF NANOFUID FLOW PAST A PERMEABLE STRETCHING/SHRINKING SHEET

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(Communicated by Tim Healey)

ABSTRACT. In this article we analyze a recently proposed model for boundary layer flow of a nanofluid past a permeable stretching/shrinking sheet. The boundary value problem (BVP) resulting from this model is governed by two physical parameters; \( \lambda \), which controls the stretching \((\lambda > 0)\) or shrinking \((\lambda < 0)\) of the sheet, and \( S \), which controls the suction \((S > 0)\) or injection \((S < 0)\) of fluid through the sheet. For \( \lambda \geq 0 \) and \( S \in \mathbb{R} \), we present a closed-form solution to the BVP and prove that this solution is unique. For \( \lambda < 0 \) and \( S < 2\sqrt{-\lambda} \) we prove no solution exists. For \( \lambda < 0 \) and \( S = 2\sqrt{-\lambda} \) we present a closed-form solution to the BVP and prove that it is unique. For \( \lambda < 0 \) and \( S > 2\sqrt{-\lambda} \) we present two closed-form solutions to the BVP and prove the existence of an infinite number of solutions in this parameter range. The analytical results proved here differ from the numerical results reported in the literature. We discuss the mathematical aspects of the problem that lead to the difficulty in obtaining accurate numerical approximations to the solutions.

1. Introduction. The novel properties of nanofluids make them useful in many different scientific and engineering processes [2]. As the number of uses grows, so does the need to understand the heat and mass transfer characteristics of nanofluids in various physical configurations [1], [6]. Recently, Jahan et. al. [4] considered one such configuration; flow of a nanofluid past a heated, permeable sheet that can either stretch or shrink. Using a similarity transformation, they reduce the governing partial differential equations to a system of ordinary differential equations. The equation governing fluid velocity is decoupled from those governing temperature and nanofluid particle volume and satisfies the following boundary value problem (BVP): Find \( f(\eta) \) such that

\[
 f''' + ff'' - f'^2 = 0, \quad 0 < \eta < \infty, \tag{1}
\]

subject to

\[
 f(0) = S, \quad f'(0) = \lambda, \tag{2}
\]

and

\[
 f'(\infty) = 0, \tag{3}
\]

2010 Mathematics Subject Classification. Primary: 34B15; Secondary: 76D10.

Key words and phrases. boundary value problem; closed-form solution; uniqueness; multiplicity.

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where the parameter $S$ measures the suction ($S > 0$) or injection ($S < 0$) of fluid through the sheet and $\lambda$ measures the stretching ($\lambda > 0$) or shrinking ($\lambda < 0$) rate of the sheet.

The stream function for the steady flow is given by $\psi = (b\nu)^{1/2} x f(\eta)$, where $\eta = (b\nu)^{1/2} y$, $\nu$ is viscosity and $b$ is related to the stretching/shrinking rate of the sheet. The horizontal and vertical spatial variables are given by $x$ and $y$, respectively. The velocities in the horizontal and vertical directions are then obtained through $u = \partial \psi / \partial y = b\nu x f'(\eta)$ and $v = -\partial \psi / \partial x = -(b\nu)^{1/2} f(\eta)$.

Jahan *et. al.* [4] studied this problem numerically for various values of $S$ and $\lambda$. For a range of positive values of $S$ (between 2.1 and 2.5) they find a value $\lambda_c < 0$ (dependent on $S$) such that no solution exists for $\lambda < \lambda_c$, one solution exists for $\lambda = \lambda_c$, and two solutions exist for $\lambda > \lambda_c$. In section 2, we prove that for $S > 0$ this critical value of $\lambda$ is given by $\lambda_c = -S^2/4$. Further, for $S > 0$ and $\lambda = \lambda_c$ we present a closed-form solution to the BVP. For $S > 0$ and $\lambda_c < \lambda < 0$ we present two closed-form solutions to the BVP. (One of these closed-form solutions differs markedly from the numerical calculations given in [4]. We explain this difference in section 6.) In section 3, we use an argument given by Troy *et. al.* [8] to show that no solution exists for $S > 0$ and $\lambda < \lambda_c$. In that paper, Troy *et. al.* [8] prove uniqueness of the solution $f(\eta) = 1 - e^{-\eta}$ to the BVP (1-4) for $\lambda = 1$ and $S = 0$. (See also [3], [5], [7].) We further show that no solution to the BVP exists in the range $S \leq 0$ and $\lambda < 0$. In section 4, we show that when $\lambda < 0$ and $S = 2\sqrt{-\lambda}$ the solution is unique and when $\lambda < 0$ and $S > 2\sqrt{-\lambda}$ we prove that an infinite number of solutions exist.

For $\lambda > 0$, the numerical calculations of Jahan *et. al.* [4] continue to find two solutions to the BVP. This conclusion is incorrect. In section 5, we present a closed-form solution to the BVP for the case $\lambda \geq 0$ and $S \in \mathbb{R}$ and prove that this solution is unique. In section 6, we examine the features of the problem that make numerical approximation difficult and explain the difference between the analytical results obtained here and the numerical results reported in [4].

2. Closed-form solutions for $\lambda < 0$ and $S \geq 2\sqrt{-\lambda}$. For $\lambda < 0$ and $S = 2\sqrt{-\lambda} > 0$ (i.e. $\lambda = -S^2/4$) a solution to the BVP is given by

$$f_0(\eta) = S \left(1 + e^{-S\eta / 2}\right) / 2.$$  \hfill (5)

For $\lambda < 0$ and $S > 2\sqrt{-\lambda}$, let $a_1 = (-S - \sqrt{S^2 + 4\lambda})/2 < 0$ and $a_2 = (-S + \sqrt{S^2 + 4\lambda})/2 < 0$ be the two roots of the quadratic equation

$$a^2 + Sa - \lambda = 0.$$ \hfill (6)

For this range of parameters two solutions to the BVP are given by

$$f_{a_1}(\eta) = S - \frac{\lambda}{a_1} + \frac{\lambda}{a_1} e^{a_1 \eta},$$ \hfill (7)

and

$$f_{a_2}(\eta) = S - \frac{\lambda}{a_2} + \frac{\lambda}{a_2} e^{a_2 \eta}.$$ \hfill (8)

For $\lambda = -1$ and $S > 2$, these solutions are given in equation (17) in [4].
3. Nonexistence for \( \lambda < 0 \) and \( S < 2\sqrt{-\lambda} \).

**Theorem 3.1.** For \( \lambda < 0 \) and \( S < 2\sqrt{-\lambda} \), the BVP (1-4) has no solution.

**Proof.** Suppose for contradiction a solution does exist. From the ODE (1) we see that \( f' \) cannot have a maximum. This along with \( f'(0) = \lambda < 0 \) and \( f'(\infty) = 0 \) implies that \( f'(\eta) < 0 \) for all \( \eta > 0 \) and \( f''(\eta) > 0 \) for \( \eta \) sufficiently large, say for \( \eta > \eta_0 \).

First consider the range \( \lambda < 0 \) and \( S \leq 0 \). Since \( f(0) = S \leq 0 \) and \( f'(\eta) < 0 \) for all \( \eta > 0 \), we have that \( f(\eta) < 0 \) for all \( \eta > 0 \). Thus for \( \eta > \eta_0 \) we have from (1) that

\[
 f''(\eta) = f'(\eta)^2 - f(\eta)f'''(\eta) > 0, \tag{9}
\]

which eventually forces \( f' \) positive, contradicting the fact the \( f'(\eta) < 0 \) for all \( \eta > 0 \). Thus no solution exists for the range \( \lambda < 0 \) and \( S \leq 0 \).

Next consider the range \( \lambda < 0 \) and \( 0 < S < 2\sqrt{-\lambda} \). Since \( f(0) = S > 0 \), \( f'(\eta) > 0 \) for small \( \eta > 0 \). If \( f \) were to vanish at some first \( \eta_1 \), then \( f(\eta) < 0 \) for all \( \eta > \eta_1 \) since \( f'(\eta) < 0 \) for all \( \eta > 0 \). The argument of the previous paragraph then implies that \( f' \) cannot satisfy \( f'(\infty) = 0 \). Thus for a solution to the BVP we must have \( 0 < f(\eta) < S \) for all \( \eta > 0 \).

Throughout the remainder of the paper we will make use of a result of Troy et. al. \[8\] which can be restated as follows:

**Lemma 3.2.** (see [8]) Suppose \( f(\eta) > 0 \) for all \( \eta > 0 \). Let \( r = f''/f' \). If there exists a point \( \hat{\eta} \) such that \( f'(\hat{\eta}) < 0 \), \( r(\hat{\eta}) \leq 0 \) and \( r'(\hat{\eta}) < 0 \), then there exists a point \( \hat{\eta} > \hat{\eta} \) such that \( f'(\hat{\eta}) = 0 \) and \( f''(\hat{\eta}) > 0 \).

The result of [8] assumes \( r(\hat{\eta}) = 0 \), but the result also holds if \( r(\hat{\eta}) \leq 0 \). Since \( f' \) cannot have a maximum, once \( f'(\hat{\eta}) \) increases through zero at \( \hat{\eta} \), it cannot satisfy the boundary condition \( f'(\infty) = 0 \), thus any solution to the ODE (1) that satisfies the conditions of this lemma cannot be a solution to the BVP.

Continuing the proof of nonexistence for the case \( \lambda < 0 \) and \( 0 < S < 2\sqrt{-\lambda} \) we next consider the value of \( f''(0) \). If \( f''(0) \geq 0 \) then \( f''(\eta) > 0 \) for all \( \eta > 0 \), and if \( f''(0) < 0 \), then \( f' \) must achieve a negative minimum, and thereafter monotonically increase toward zero. If \( f''(0) = 0 \), take \( \hat{\eta} = 0 \). If \( f''(0) = \alpha < 0 \), then take \( \hat{\eta} \) to be the value of \( \eta \) where the minimum of \( f' \) occurs. Then the Lemma of [8] cited above implies that \( f' \) must become positive at some point and therefore cannot be solution to the BVP. If \( f''(0) = \alpha > 0 \), then the argument of [8] can be adjusted to give nonexistence in this case as well. Following [8], let \( r = f''/f' \). Then \( r \) satisfies

\[
 r' + r^2 + fr - f' = 0, \tag{10}
\]

\[
 r(0) = \frac{\alpha}{\lambda} < 0, \tag{11}
\]

and

\[
 r'(0) = (\lambda^3 - S\lambda\alpha - \alpha^2)/\lambda^2. \tag{12}
\]

Recall that \( \lambda < 0 \) and \( 0 < S < 2\sqrt{-\lambda} \) and so \( r'(0) = (\lambda^3 - S\lambda\alpha - \alpha^2)/\lambda^2 < 0 \) for all \( \alpha \in \mathbb{R} \). Thus the argument given in [8] again implies that \( f' \) must become positive and therefore cannot satisfy \( f'(\infty) = 0 \) and the theorem is proved. \( \square \)
4. Multiplicity of solutions for $\lambda < 0$ and $S > 2\sqrt{-\lambda}$.

**Theorem 4.1.** For $\lambda < 0$ and $S = 2\sqrt{-\lambda}$, (i.e. $\lambda = -S^2/4$), $f_0(\eta) = S(1 + e^{-S\eta/2})/2$ is the unique solution to the BVP (1-4).

**Proof.** Note that $f''_0(0) = \frac{S^3}{8}$. For any other solution we would necessarily need $f''(0) = \alpha \neq \frac{S^3}{8}$. Let $r = f''/f'$ as before, then $r(0) = \alpha / \lambda = -4\alpha / S^2$ and

$$r'(0) = \frac{-\alpha^2 + S^3 \alpha / 4 - S^6 / 64}{S^4 / 16} = \frac{p(\alpha)}{S^4 / 16}. \tag{13}$$

If $r(0) = \alpha / \lambda = -4\alpha / S^2 \geq 0$, then $\alpha \leq 0$ and $\eta$ can be chosen as in the proof of Lemma 3.2. Thus, a value of $\alpha \leq 0$ cannot give a solution to the BVP. If $r(0) = \alpha / \lambda < 0$, then note that $p(\alpha) < 0$ in (13) for all $\alpha \neq S^3/8$. Thus $r'(0) < 0$ and again Lemma 3.2 above says such a value of $\alpha$ cannot give a solution to the BVP. Therefore $f''_0(0) = \frac{S^3}{8}$ gives the unique solution to the BVP. $\square$

**Theorem 4.2.** For $\lambda < 0$ and $S > 2\sqrt{-\lambda}$, the BVP (1-4) has infinitely many solutions corresponding to solutions of the of the ODE (1) satisfying the initial conditions $f(0) = S$, $f'(0) = \lambda$ and $f''(0) = \alpha$ equal to any value in the range $\lambda a_2 \leq f''(0) \leq \lambda a_1$ with $a_1$ and $a_2$ defined as in section 2. Further, values of $f''(0)$ outside this range do not give solutions to the BVP.

**Proof.** From (7) and (8) we have that the two closed form solutions satisfy $f''_0(0) = \lambda a_1$ and $f''_a(0) = \lambda a_2$, where $a_1$ and $a_2$ are the two roots of the quadratic (6). Next, again consider $r = f''/f'$ and let $f''(0) = \alpha$. Then $r(0) = \alpha / \lambda$ and

$$r'(0) = (\lambda^3 - S \lambda \alpha - \alpha^2) \lambda^2 = q(\alpha) / \lambda^2. \tag{14}$$

Note that $q(\alpha) = 0$ for $\alpha = \lambda a_1$ and $\alpha = \lambda a_2$. Further note that $q(\alpha) > 0$ if $\lambda a_2 < \alpha < \lambda a_1$ and $q(\alpha) < 0$ if $\alpha < \lambda a_2$ or $\alpha > \lambda a_1$.

If $\alpha < 0$, then $f'$ must have a negative minimum and the argument of Lemma 3.2 precludes such a value of $\alpha$ from giving a solution to the BVP. If $0 \leq \alpha < \lambda a_2$ or $\alpha > \lambda a_1$, then $r(0) = \alpha / \lambda \leq 0$ and $r'(0) = q(\alpha) / \lambda^2 < 0$ and Lemma 3.2 again precludes such a value of $\alpha$ from giving a solution to the BVP.

This leaves the range of $\alpha$ where $\lambda a_2 < \alpha < \lambda a_1$ as possible values of $f''(0)$ which could give further solutions to the BVP. In fact, all of these values give solutions to the BVP. To see this, we first show that for these values, $f'$ cannot increase through zero. If $f'$ were to increase through zero at some first $\eta_2$, then $f''(\eta_2) > 0$ (we could not have $f'(\eta_2) = 0$ and $f''(\eta_2) = 0$ simultaneously since this implies that $f' = 0$). Again referring to $r = f''/f'$ we see that $\lim_{\eta \to \eta_2} r(\eta) = -\infty$. As in [8], differentiating (10) we obtain

$$r'' + (2r + f)r' = 0, \tag{15}$$

from which we conclude that

$$r'(\eta) = r'(0) \exp \left( - \int_0^\eta (2r(s) + f(s)) \, ds \right) > 0, \tag{16}$$

since $r'(0) > 0$ for $\alpha$ in the range $\lambda a_2 < \alpha < \lambda a_1$. Thus $r(\eta)$ is increasing for as long as it exists and therefore we cannot have $\lim_{\eta \to \eta_2} r(\eta) = -\infty$ and $f'$ cannot increase through zero at $\eta_2$.

Next we show that if $f(0) = S$, $f'(0) = \lambda$ and $f''(0) = \alpha$ with $\alpha$ any value in the range $\lambda a_2 < \alpha < \lambda a_1$, the solution to the ODE (1) exists for all $\eta > 0$. Since $f''(0) = \alpha > 0$, we see from the ODE (1) that $f''(0) > 0$ for as long as the solution
exists. This, along with the fact that \( f'(\eta) \) cannot increase through zero implies that \( f'(\eta) \) is bounded on its maximal interval of existence. Thus, on integration, \( f(\eta) \) is bounded as well. An integration of (1) from 0 to \( \eta \) gives

\[
f'' = \alpha + S\lambda - f f' + 2 \int_0^\eta f'(t)^2 \, dt, \tag{17}\]

from which we conclude that \( f'' \) is bounded. Finally, using the ODE (1) we see that \( f''' \) is bounded as well, and the solution cannot cease to exist at a finite value of \( \eta \).

Since \( f' < 0 \) and \( f'' > 0 \) for all \( \eta > 0 \), \( f'(\infty) = L \leq 0 \) exists. If \( f' \to -\infty \) as \( \eta \to \infty \), then \( f' \to L < 0 \), and from (1) we have

\[
f''' = f'^2 - f f'' > 0 \tag{18}\]

for large \( \eta \). This would imply, since \( f'' > 0 \), that \( f' > 0 \) for large \( \eta \), contradicting \( f' \to L < 0 \). Therefore \( f'(\infty) = L = 0 \) and the Theorem is proved. \( \square \)

5. **Unique closed-form solution for** \( \lambda \geq 0 \) and \( S \in \mathbb{R} \).

**Theorem 5.1.** Let \( \lambda > 0 \) and \( S \in \mathbb{R} \), then the unique solution to the BVP (1-4) is \( f_\alpha(\eta) = S - \lambda(1 - e^{\alpha \eta})/\alpha \) where \( \alpha \) is the negative root of \( a^2 + Sa - \lambda = 0 \).

**Proof.** Direct substitution shows that \( f_\alpha(\eta) = S - \lambda(1 - e^{\alpha \eta})/\alpha \), where \( \alpha \) is the negative root of \( a^2 + Sa - \lambda = 0 \), gives a solution to the BVP. Note that \( f_\alpha(\eta) \) satisfies \( f_\alpha'(\eta) > 0 \) and \( f_\alpha''(\eta) < 0 \) for all \( \eta > 0 \).

We next show that there cannot be two solutions, both of which satisfy \( f' > 0 \) and \( f'' < 0 \) for all \( \eta > 0 \). To prove this we use the equation of first variation. Consider the initial value problem (IVP) given by (1-3) along with a third initial condition

\[
f''(0) = \alpha. \tag{19}\]

Let \( u = \partial f/\partial \alpha \). Then \( u \) satisfies the initial value problem:

\[
u''' + f u'' - 2 f' u' + f'' u = 0, \tag{20}\]

subject to

\[
u(0) = 0, \quad u'(0) = 0, \quad u''(0) = 1. \tag{21}\]

Thus \( u' \) is positive and increasing for small \( \eta > 0 \). We claim that \( u' \) cannot achieve a maximum. At such a maximum we would have \( u > 0, u' > 0, u'' = 0 \) and \( u''' \leq 0 \). The ODE (20) then implies that at such a maximum,

\[
u''' = 2 f' u' - f'' u > 0, \tag{22}\]

giving a contradiction. Thus \( u' \) will be positive and bounded away from zero for \( \eta \) large.

If we assume the existence of two solutions \( f(\eta; \alpha_1) \) and \( f(\eta; \alpha_2) \) with \( \alpha_2 > \alpha_1 \) that satisfy \( f'(\eta; \alpha_i) > 0 \) and \( f''(\eta; \alpha_i) < 0 \), \( i \in \{1, 2\} \), we then have from the Mean Value Theorem,

\[
f'(\eta; \alpha_2) - f'(\eta; \alpha_1) = \left( \frac{\partial f'(\eta; \alpha)}{\partial \alpha} \right)_{\alpha = \tilde{\alpha}} (\alpha_2 - \alpha_1) = u'(\eta; \tilde{\alpha})(\alpha_2 - \alpha_1), \tag{23}\]

for \( \tilde{\alpha} \in (\alpha_1, \alpha_2) \). Since \( u'(\eta; \tilde{\alpha}) \) is bounded away from zero for \( \eta \) large, there exists a constant \( M > 0 \) such that

\[
0 = f'(\infty; \alpha_2) - f'(\infty; \alpha_1) \geq \lim_{\eta \to \infty} u'(\eta; \tilde{\alpha})(\alpha_2 - \alpha_1) > M(\alpha_2 - \alpha_1) > 0, \tag{24}\]

giving a contradiction.
Thus if a second solution exists, it must violate one or both of the inequalities $f' > 0$ and $f'' < 0$. In fact, any second solution must violate both. For any solution, $f'$ cannot have a maximum, thus a solution that satisfied $f' > 0$ for all $\eta > 0$ would also have to satisfy $f'' < 0$ for all $\eta > 0$, and vice versa. We therefore conclude that any second solution must have a point with $f' = 0$ and $f'' < 0$ (as was seen earlier, we cannot have $f'' = 0$ and $f' = 0$ simultaneously since this implies that $f' \equiv 0$).

In order to satisfy $f''(\infty) = 0$, $f'$ must achieve a negative minimum and thereafter increase toward zero. This is exactly the scenario prohibited by Lemma 3.2. Thus for $\lambda > 0$ and $S \in \mathbb{R}$, the solution to the BVP (1-4) is unique.

Corollary 1. Let $\lambda = 0$ and $S \geq 0$, then the unique solution to the BVP (1-4) is $f(\eta) \equiv S$.

Proof. Direct substitution shows that $f(\eta) \equiv S$ is a solution to the BVP when $\lambda = 0$. For this solution we have $f''(0) = 0$. Any further solutions would necessarily require $f''(0) \neq 0$. We cannot have $f''(0) > 0$ since $f'$ cannot have a maximum. The case $f''(0) < 0$ is ruled out by the argument of Lemma 3.2.

6. Discussion. In this article we completely characterize the solution set for the BVP (1-4). The results proved here differ markedly from the computational results obtained in [4], and highlight the need for mathematical analysis to complement numerical analysis. The numerical results of [4] suggest the existence of dual solutions for all $\lambda > \lambda_c$. The results proved here show that this conclusion is incorrect, since for $\lambda \geq 0$ the solution is unique.

![Figure 1](image)

**Figure 1.** The value of $f''(0)$ as a function of $\lambda$ for various values of $S$, from far left, $S = 2.5$, $S = 2.3$ and $S = 2.1$.

Numerical integration does appear to show dual solutions for $\lambda \geq 0$, one of which has $f'$ become negative and approach zero from below. However, for this solution, if numerical integration is continued on a large enough interval of $\eta$, $f'$ eventually increases through zero. From the ODE (1) we see that $f'(\eta)$ cannot have a maximum, and thus once $f'(\eta)$ increases through zero, it cannot satisfy the boundary condition $f'(\infty) = 0$.

In the range $\lambda_c < \lambda < 0$ and $S > 0$, multiple solutions to the BVP do exist, all of which satisfy $f''(0) > 0$. However, the numerical calculations of Jahan et. al. [4] find a range of $\lambda < 0$ for which one of the solutions satisfies $f''(0) < 0$. (See Figure 1.)
2 in [4]. In section 4, we proved that this cannot happen. And again, if numerical integration is continued on a long enough interval of $\eta$, it is seen that for the solution with $f''(0) < 0$, $f'(\eta)$ will eventually become positive. The difference between the analytical results and the numerical results becomes clear by comparing Figure 1 of the current paper with Figure 2 of [4]. It should be noted that we initially obtained the exact same numerical results as [4]. It was only the need to reconcile our analytical results with the apparent numerical results that led us to continue integration on a sufficiently large interval of $\eta$ to confirm the actual behavior.

Several aspects of the BVP (1-4) explain the difficulty encountered when calculating solutions numerically. First, the problem is posed on an infinite interval, which makes it difficult to determine if any finite interval of numerical integration is large enough to capture all relevant behavior of the solution. Second, for a range of parameters with $\lambda < 0$ and $\lambda$ close to zero, the exponential decay rate of the solutions is quite small. For $\lambda_c < \lambda < 0$, Figure 2 displays the values of $a_1 = (-S - \sqrt{S^2 + 4\lambda})/2$ (solid curve) and $a_2 = (-S + \sqrt{S^2 + 4\lambda})/2$ (dashed curve) for various values of $S$. As $\lambda \to 0^-$, $a_2 \to 0^-$. Thus the decay of $f_{a_2}(\eta) = S - \lambda / a_2 + \lambda e^{a_2\eta} / a_2$ becomes slower and slower, taking longer and longer for $f'_{a_2}(\eta)$ to converge to zero as $\eta \to \infty$. Such solutions would likely be missed by numerical software packages integrating over any finite interval of $\eta$.

The results presented here also prove the existence of an infinite number of solutions when $\lambda < 0$ and $S > 2\sqrt{-\lambda}$. This continuum of solutions would not be detected by numerically shooting over a finite range of $\eta$, as this would lead to a continuum of (nonzero) values of $f'(\infty)$ detected at the right end point of the integration interval. Only by continually extending the length of the interval would it become clear that infinitely many of these solutions eventually satisfy $f'(\infty) = 0$.

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Received May 2019; revised December 2019.

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