RATIONAL CONNECTEDNESS
OF LOG $\mathbb{Q}$-FANO VARIETIES

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Abstract. In this paper, we give an affirmative answer to a conjecture in the Minimal Model Program. We prove that log $\mathbb{Q}$-Fano varieties are rationally connected. We also study the behavior of the canonical bundles under projective morphisms.

§1. Log $\mathbb{Q}$-Fano varieties are rationally connected

Let $X$ be a log $\mathbb{Q}$-Fano variety, i.e., if there exists an effective $\mathbb{Q}$-divisor $D$ such that the pair $(X, D)$ is Kawamata log terminal (klt) and $-(K_X + D)$ is nef and big. By a result of Miyaoka-Mori [15], $X$ is uniruled. A standard conjecture ([10], [12], [13], [16]) predicts that $X$ is actually rationally connected. In this paper we apply the theory of weak (semi) positivity of the direct images of (log) relative dualizing sheaves $f_*(K_{X/Y} + \Delta)$ (which has been developed by Fujita, Kawamata, Kollár, Viehweg and others) to show that a log $\mathbb{Q}$-Fano variety is indeed rationally connected.

Remark: The rational connectedness of smooth Fano varieties was established by Campana [1] and Kollár-Miyaoka-Mori [12]. However their approach relies heavily on the (relative) deformation theory which seems quite difficult to extend to the singular case.

Theorem 1. Let $X$ be a log $\mathbb{Q}$-Fano variety. Then $X$ is rationally connected, i.e., for any two closed points $x, y \in X$ there exists a rational curve $C$ which contains $x$ and $y$.

Remark: When $\dim(X) \leq 3$ and $D = 0$, this was proved by Kollár-Miyaoka- Mori in [13]. On the other hand, the result is false for log canonical singularities (see 2.2 in [13]).

As a corollary, we can show the following result which was obtained by S. Takayama [16].
Corollary 1. Let $X$ be a log $\mathbb{Q}$-Fano variety. Then $X$ is simply connected.

Proof. $\pi_1(X)$ is finite by Theorem 1 and a result of F. Campana [2]. On the other hand, $h^i(X, \mathcal{O}_X) = 0$ for $i \geq 1$ by Kawamata-Viehweg vanishing ([4], [9]). Thus we have $\chi(X, \mathcal{O}_X) = 1$ and hence $X$ must be simply connected.

Our Theorem 1 is a consequence of the following proposition.

Proposition 1. Let $X$ be a log $\mathbb{Q}$-Fano variety and let $f : X \dasharrow Y$ be a dominant rational map, where $Y$ is a projective variety. Then $Y$ is uniruled if $\dim Y > 0$.

Let us assume Proposition 1. Let $X'$ be a resolution of $X$, then $X'$ is uniruled. There exists a nontrivial maximal rationally connected fibration $f : X' \dasharrow Y$ ([1], [12]). By a result of Graber-Harris-Starr [5], $Y$ is not uniruled. However Proposition 1 tells us that $Y$ must be a point and hence $X$ is rationally connected.

The general strategy for proving Proposition 1 is as follows. By a result of Miyaoka-Mori [15], it suffices to construct a covering family of curves $C_t$ on $Y$ with $C_t \cdot K_Y < 0$ for every $t$. To this end, we apply the positivity theorem of the direct images of (log) relative dualizing sheaves to $f : X \dasharrow Y$. We show that there exist an ample $\mathbb{Q}$-divisor $H$ on $Y$ and an effective $\mathbb{Q}$-divisor $D$ on $X$ such that $-K_Y = D + f^*H$ (modulo some exceptional divisors). Now let $C_t = f(F_t)$, where $F_t$ are the general complete intersection curves on $X$. We have a covering family of curves $C_t$ on $Y$ with $C_t \cdot K_Y < 0$ for every $t$.

Before we start to prove Proposition 1, let us first give some related definitions. Also the proof of Proposition 1 depends heavily on Kawamata’s paper [8] and Viehweg’s paper [17].

We work over the complex number field $\mathbb{C}$ in this paper.

Definition 1 [9]. Let $X$ be a normal projective variety of dimension $n$ and $K_X$ the canonical divisor on $X$. Let $D = \sum a_i D_i$ be an effective $\mathbb{Q}$-divisor on $X$, where $D_i$ are distinct irreducible divisors and $a_i \geq 0$. The pair $(X, D)$ is said to be Kawamata log terminal (klt) (resp. log canonical) if $K_X + D$ is a $\mathbb{Q}$-Cartier divisor and if there exists a desingularization (log resolution) $f : Z \to X$ such that the union $F$ of the exceptional locus of $f$ and the inverse image of the support of $D$ is a divisor with normal crossing and

$$K_Z = f^*(K_X + D) + \sum_i e_j F_j,$$

with $e_j > -1$ (resp. $e_j \geq -1$). $X$ is said to be Kawamata log terminal (resp. log canonical) if so is $(X, 0)$. 

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**Definition 2.** Let $X$ be a normal projective variety of dimension $n$ and $K_X$ the canonical divisor on $X$. We say $X$ is a $Q$-Gorenstein variety if there exists some integer $m > 0$ such that $mK_X$ is a Cartier divisor. A $Q$-Cartier divisor $D$ is said to be nef if the intersection number $D \cdot C \geq 0$ for any curve $C$ on $X$. $D$ is said to be big if the Kodaira-Iitaka dimension $\kappa(D)$ attains the maximum $\dim X$.

The following lemma due to Raynaud [19] is quite useful:

**Lemma 1.** Let $g : T \to W$ be surjective morphism of smooth varieties. Then there exists a birational morphism of smooth variety $\tau : W' \to W$ and a desingularization $T' \to T \times_W W'$, such that the induced morphism $g' : T' \to W'$ has the following property: Let $B'$ be any divisor of $T'$ such that $\text{codim}(g'(B')) \geq 2$. Then $B'$ lies in the exceptional locus of $\tau : T' \to T$.

**Proof of Proposition 1.** By the Stein factorization and desingularizations, we may assume that $Y$ is smooth. Resolving the indeterminacy of $f$ and taking a log resolution, we have a smooth projective variety $Z$ and the surjective morphisms $g : Z \to Y$ and $\pi : Z \to X$,

$$
\begin{array}{c}
Z \\
\downarrow g \\
Y
\end{array}
\quad
\begin{array}{c}
\pi \\
\downarrow \\
X
\end{array}
$$

such that

$$K_Z = \pi^*(K_X + D) + \sum_i e_i E_i$$

with $e_i > -1$, where $\sum E_i$ is a divisor with normal crossing.

Since $-(K_X + D)$ is nef and big, by Kawamata base-point free theorem, there exists an effective $Q$-divisor $A$ on $X$ such that $-(K_X + D) - A$ is an ample $Q$-divisor. Thus we can choose another ample $Q$-divisor $H$ on $Y$ and an effective $Q$-divisor $\Delta = \sum_i \delta_i E_i$ on $Z$ (with small $\delta \geq 0$ and $E_i$ are $\pi$-exceptional if $\delta_i > 0$) such that $-\pi^*(K_X + D + A) - \Delta - g^*(H) = L$ is again an ample $Q$-divisor on $Z$. We may also assume that $\text{Supp}(\Delta + \cup_i E_i + A)$ is a divisor with simple normal crossing and the pair $(X, L + \Delta + \sum_i (-e_i) E_i + A)$ is klt. Thus we have

$$K_{Z/Y} + \sum_i \epsilon_i E_i \sim_Q \sum_i m_i E_i - g^*(K_Y + H)$$

where $m_i = \lceil e_i \rceil$ are non-negative integers ($\{, \}$ is the fractional part and $\lceil, \rceil$ is the round up). Also $E_i$ on the left side contain components of $L$, $A$ and $\Delta$ with $0 \leq \epsilon_i < 1$.  

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We follow closely from Kawamata’s paper [8].
After further blowing-ups if necessary, and by Lemma 1, we may assume that:

(1) There exists a normal crossing divisor $Q = \sum l Q_l$ on $Y$ such that $g^{-1}(Q) \subset \sum_i E_i$ and $g$ is smooth over $Y \setminus Q$.

(2) If a divisor $W$ on $Z$ with $\operatorname{codim}(g(W)) \geq 2$, then $W$ is $\pi$-exceptional.

Let $D = \sum_i (\epsilon_i - m_i)E_i = \sum_i d_iE_i = D^h + D^v$, where

(1) $g : D^h \to Y$ is surjective and smooth over $Y \setminus Q$ (we say $D^h$ is $g$-horizontal)

(2) $g(D^v) \subset Q$ (we say $D^v$ is $g$-vertical [8]).

Notice that here besides those $E_i$ from the log resolution $\pi$, $D$ also contains the components of $\operatorname{Supp}(L + \Delta + A)$. Nevertheless $d_i < 1$ for all $i$.

We have two cases:

(1): Every $\pi$-exceptional divisor $E_i$ with $d_i < 0$ is $g$-vertical. In this case, the natural homomorphism $O_Y \to g_*(\lfloor -D \rfloor)$ is surjective at the generic point of $Y$.

Let $g^*Q_l = \sum j w_{lj}E_j$

and

$a_j = \frac{d_j + w_{lj} - 1}{w_{lj}}$ if $g(E_j) = Q_l$

and

$b_l = \{ \max \{ a_j \} : g(E_j) = Q_l \}$.

Let $N = \sum b_l Q_l$ and $M = -H - K_Y - N$. Then by a result of Kawamata [8, Theorem 2], $M$ is nef.

On the other hand, $g^*N = F + G$, where

(1) $\operatorname{Supp}(F)$ is $\pi$-exceptional (from those $Q_l$ with $g^*Q_l = \sum_j w_{lj}E_j$ and each $E_j$ is $\pi$-exceptional).

(2) $G$ is effective (from those $Q_l$ with $g^*Q_l = \sum_j w_{lj}E_j$ and at least one $E_j$ has coefficient $d_j \geq 0$).

Now let $C$ be a general complete intersection curve on $Z$ such that $C$ does not intersect with $\operatorname{Supp}(F)$ (e.g., pull-back a general complete intersection curve from $X$). Then

$g_*(C) \cdot (-K_Y - H) = C \cdot g^*(-K_Y - H) = C \cdot g^*(M + N) \geq 0$

and hence $g_*(C) \cdot K_Y \leq g_*(C) \cdot (-H) < 0$. Since the family of the curves $g(C)$ covers a Zariski open set of $Y$, by [15] $Y$ is uniruled.
(2): Some $\pi$-exceptional divisor is $g$-horizontal, in particular, $D^h$ is not zero.

We have $K_{Z/Y} + D \sim_Q -g^*(K_Y + H)$. By the stable reduction theorem and the covering trick ([8],[17]), there exists a finite morphism $p: Y' \to Y$ such that $Q' = \text{Supp}(p^*Q)$ is a normal crossing divisor and the induced morphism $g': Z' \to Y'$ from a desingularization $Z' \to Z \times_Y Y'$ is semistable over $Y' \setminus B$ with $\text{codim}(B) \geq 2$. Let $Z' \to Z$ be the induced morphism.

We can write

$$K_{Z'/Y'} + D' \sim_Q g'*p^*(-K_Y - H),$$

where $D' = \sum_j d'_j E'_j$.

The coefficients $d'_j$ can be calculated as follows [8]:

(1) If $E'_j$ is $g'$-horizontal and $q(E'_j) = E_j$, then $d'_j = d_j$.

(2) If $E'_j$ is $g'$-vertical with $q(E'_j) = E_j$ and $g'(E'_j) = Q'_l$, then $d'_j = e_j(d_j + w_l - 1)$, where $e_l$ is the ramification index of $q$ at the generic point of $E'_j \to E_j$.

(3) We are not concerned with those $E'_j$ such that $g'(E'_j) \subset B$.

Note: In Kawamata’s paper [8], he replaced $D$ by $D - g^*N$. Thus all the coefficients there are $\leq 0$. However, here we do not make such replacement and in our case some coefficients $d_i = \epsilon_i > 0$

Therefore, by the standard trick (keep the fractional part on the left side and the integral part on the right side, also blow-up $Z'$ if necessary). We have

$$K_{Z'/Y'} + \sum_{j'} \epsilon'_{j'} E'_{j'} \sim_Q \sum_{k'} n_{k'} E'_{k'} - g'^*p^*(K_Y + H) - V + G$$

such that

(1) $\sum_{k'} n_{k'} E'_{k'}$ is Cartier, $\text{Supp}(\sum_{k'} E'_{k'})$ is $q\circ\pi$-exceptional and $g'(\sum_{k'} n_{k'} E'_{k'})$ is not contained in $B$.

(2) $V$ is an effective Cartier divisor which is $g'$-vertical (from those $g$-vertical $E_i$ with $d_i = \epsilon_i \geq 0$ and $d'_{j'} = e_{j'}(d_j + w_l - 1) \geq 1$ for some $l$.)
(3) $G$ is also Cartier and $q \circ \pi$-exceptional (from those $E'_j$ with $g'(E'_j) \subset B$).

(4) $\sum j' d_{j'} E'_{j'}$ remains on the left side, where $E'_{j'}$ are $g'$-horizontal with $0 < d_{j'} = d_j = \epsilon_j < 1$.

(5) If $E_{j'}'$ is $g'$-horizontal, then $n_{k'} \geq 0$.

(6) $(Z', \sum_{j'} \epsilon'_{j'} E'_{j'})$ is klt.

There also exists a cyclic cover $[9] p' : Y'' \rightarrow Y'$ such that $Y''$ is smooth and $p'^* p^*(H) = 2H'$, where $H'$ is an ample Cartier divisor. Since $H$ is an ample $Q$-divisor, we can choose the covering $p'$ in such way that the ramification locus $R_{p'}$ of $p'$ intersects $\text{Supp} \, Q$ and $B$ transversely. Let $g'' : Z'' \rightarrow Y''$ be the induced morphism from a desingularization $Z'' \rightarrow Z' \times_Y Y''$.

\[
\begin{array}{ccc}
Z'' & \xrightarrow{q'} & Z' & \xrightarrow{q} & Z & \xrightarrow{\pi} & X \\
\downarrow g'' & & \downarrow g' & & \downarrow g & & \\
Y'' & \xrightarrow{p'} & Y' & \xrightarrow{p} & Y & & \\
\end{array}
\]

Since $g'$ is semistable over $Y' \setminus B$, we have $q'^* K_{Z'/Y'} = K_{Z''/Y''}$ over $Y'' \setminus p'^{-1}(B)$. Thus again we can write

\[
K_{Z''/Y''} + \sum_{j''} \epsilon''_{j''} E''_{j''} \sim Q \sum_{k''} n_{k''} E''_{k''} - g''^* p'^* p^* K_Y - 2g''^* H' - V' + G'
\]

where

(1) $\sum_{k''} n_{k''} E''_{k''}$ is Cartier and $\text{Supp}(\sum_{k''} E''_{k''})$ is $q' \circ q \circ \pi$-exceptional.

(2) $V'$ is an effective Cartier divisor which is $g''$-vertical.

(3) $G'$ is also Cartier and $q' \circ q \circ \pi$-exceptional (since codimg''$(G') \geq 2$).

(4) If $E''_{k''}$ is $g''$-horizontal, then $n_{k''} \geq 0$.

(5) $(Z'', \sum_{j''} \epsilon''_{j''} E''_{j''})$ is klt and $\sum_{j''} \epsilon''_{j''} E''_{j''}$ is $Q$-linearly equivalent to a Cartier divisor.

Since all the $g''$-horizontal divisors $E''_{k''}$ have non-negative coefficients,

\[g''_* \left( \sum_{k''} n_{k''} E''_{k''} - g''^* p'^* p^* K_Y - 2g''^* H' - V' + G' \right)\]

is not a zero sheaf. Let $\omega = \sum_{k''} n_{k''} E''_{k''} - g''^* p'^* p^* K_Y - V' + G'$.

Applying the results of Kollár [11], Viehweg [17, Lemma 5.1] and Kawamata [7, Theorem 1.2], we may assume that

\[g''_* (\omega - 2g''^* H') = g''_* (\omega) \otimes \mathcal{O}_{Y''}(-2H')\]
is torsion free and weakly positive over $Y''$.

*Note:* In [7], Kawamata proved that in fact (after blow-up $Y''$ further) we may assume that $g''_*(\omega)$ is locally free and semipositive. However the weak positivity is sufficient in our case.

By the weak positivity of $g''_*(\omega) \otimes \mathcal{O}_{Y''}(-2H')$, we have

$$\hat{S}^n g''_*(\omega) \otimes \mathcal{O}_{Y''}(-2nH' + nH')$$

is generically generated by its global sections over $Y''$ for some $n > 0$, where $\hat{S}^n$ denotes the reflexive hull of $S^n$.

We have a natural homomorphism

$$S^n g''_*(\omega) \otimes g''_* \mathcal{O}_{Y''}(-nH') \to \omega^n \otimes g''_* \mathcal{O}_{Y''}(-nH')$$

By the torsion freeness of $g''_*(\omega)$, there exists an open set $U \subset Y''$ with codim$(Y'' \setminus U) \geq 2$ such that

$$\hat{S}^n g''_*(\omega) \otimes \mathcal{O}_{Y''}(-nH')|_U = S^n g''_*(\omega) \otimes \mathcal{O}_{Y''}(-nH')|_U$$

and hence

$$g''_* \hat{S}^n g''_*(\omega) \otimes g''_* \mathcal{O}_{Y''}(-nH')|_W = S^n g''_*(\omega) \otimes g''_* \mathcal{O}_{Y''}(-nH')|_W$$

where $W = g''^{-1}(U)$. If $B' = Z'' \setminus W$, then $B'$ is $g''$-exceptional. Since $g''_* \hat{S}^n g''_*(\omega) \otimes g''_* \mathcal{O}_{Y''}(-nH')$ is also generically generated by its global sections over $Z''$, there is a non trivial morphism

$$\bigoplus \mathcal{O}_{Z''}|_W \to \omega^n \otimes g''_* \mathcal{O}_{Y''}(-nH')|_W$$

i.e., $\omega^n \otimes g''_* \mathcal{O}_{Y''}(-nH')$ admits a meromorphic section which has poles only along $B'$. Thus we may choose some large integer $k$ such that $\omega^n \otimes g''_* \mathcal{O}_{Y''}(-nH') + kB'$ has a holomorphic section, i.e.,

$$n(\sum_{k''} n_{k''} E''_{k''} - g''_* p^* p^* K_Y - V' + G') - ng''_* H' + kB'$$

is effective.

Again as before, we can choose a family of general complete intersection curves $C$ on $Z''$ such that $C$ does not intersect with the exceptional locus of $Z'' \to X$ (such as $E''_{k''}$, $B'$ and $G'$). Thus $g''_*(C) \cdot p^* p^*(K_Y) \leq g''_*(C) \cdot (-H') < 0$ and hence $Y$ is uniruled [15]. q.e.d.
The behavior of the canonical bundles under projective morphisms

Let $X$ and $Y$ be two projective varieties and $f : X \to Y$ be a surjective morphism. Assume that the Kodaira dimension $\kappa(X) \leq 0$. In general, it is almost impossible to predict the Kodaira dimension of $Y$. The following example shows that even when $\dim Y = 1$, we have no control of the genus of $Y$:

Example. ([6], [13]): Let $C$ be a smooth curve of arbitrary genus $g$ and $A$ be an ample line bundle on $C$ such that $\deg A > 2\deg K_C$. Let $S = \text{Proj}_C(\mathcal{O}(A) \oplus \mathcal{O}_C)$ be the projective space bundle associated to the vector bundle $\mathcal{O}(A) \oplus \mathcal{O}_C$. Then $K_S = \pi^*(K_C + A) - 2L$, where $L$ is the tautological bundle and $\pi : S \to C$ is the projection. An easy computation show that $-K_S$ is big (i.e, $h^0(S, -mK_S) \approx c \cdot m^2$ for $m \gg 0$ and some $c > 0$). In particular, the Kodaira dimension $\kappa(S) = -\infty$. However, the genus of $C$ could be large.

On the other hand, it is not hard to find the following facts:

1. Let $D = -K_S = 2(L - \pi^*A) + \pi^*(A - K_C)$, then $D$ is effective and the pair $(S, D)$ is not log canonical [9].
2. $-K_S$ is not nef, i.e, there exists some curve $B$ (e.g., choose $B$ to be the section corresponding to $\mathcal{O}(A) \oplus \mathcal{O}_C \to \mathcal{O}_C \to 0$) on $S$ such that $(-K_S) \cdot B < 0$.
3. For any integer $m > 0$, the linear system $-mK_S$ contains some fixed component (e.g., $m(L - \pi^*A)$) which dominates $C$.

In view of the above example, we give a few sufficient conditions which guarantees nice behavior of the Kodaira dimension (and the canonical bundle).

Theorem 2. Let $f : X \to Y$ be a surjective morphism. Assume that $D \equiv -K_X$ is an effective $Q$-divisor and the pair $(X, D)$ is log canonical. Moreover assume that $Y$ is normal and $Q$-Gorenstein. Then either $Y$ is uniruled or $K_Y$ is numerically trivial. (In particular, $\kappa(Y) \leq 0$.)

Corollary 2. Let $f : X \to Y$ be a surjective morphism. Assume that $X$ is log canonical and $K_X$ is numerically trivial (e.g., a Calabi-Yau manifold). Moreover assume that $Y$ is normal and $Q$-Gorenstein. Then either $Y$ is uniruled or $K_Y$ is numerically trivial.

Theorem 3. Let $f : X \to Y$ be a surjective morphism. Assume that $-K_X$ is nef and $X$ is smooth. Moreover assume that $Y$ is normal and $Q$-Gorenstein. Then either $Y$ is uniruled or $K_Y$ is numerically trivial.
Remark: Theorem 3 was proved in [18] (in particular, the result solved a conjecture proposed by Demailly, Peternell and Schneider [3]). However, the proof given there was incomplete (as pointed out to me by Y. Kawamata. I wish to thank him). The problem lies in Proposition 1 in [18], the point is that the nefness in general is not preserved under the deformations (mod $p$ reductions in our case). We shall present a new proof of this proposition (see Proposition 2).

**Theorem 4.** Let $f : X \rightarrow Y$ be a surjective morphism. Assume that there exists some integer $m > 0$ such that $-mK_X$ is effective and has no fixed locus which dominates $Y$. Moreover assume that $X$ is log canonical and $Y$ is normal and $\mathbb{Q}$-Gorenstein. Then either $Y$ is uniruled or $K_Y$ is numerically trivial.

As an immediate consequence, we can show the following result about the Albanese maps.

**Corollary 3.** Let $X$ be a smooth projective variety. Then the Albanese map $\text{Alb}_X : X \rightarrow \text{Alb}(X)$ of $X$ is surjective and has connected fibers if $X$ satisfies one of the following conditions:

1. $D \equiv -K_X$ is an effective $\mathbb{Q}$-divisor and the pair $(X, D)$ is log canonical ("\equiv" means numerically equivalent).
2. $-K_X$ is nef.
3. There exists some integer $m > 0$ such that $-mK_X$ is effective and has no fixed component which dominates $\text{Alb}(X)$.

The main ingredients of the proofs are the Minimal Model Program (in particular, a vanishing theorem of Esnault-Viehweg, Kawamata and Kollár plays an essential role), and the deformation theory. It is interesting to notice that the proofs of Theorem 2 and Theorem 3 are completely different in nature.

The following vanishing theorem of Esnault-Viehweg, Kawamata and Kollár is important to us:

**Vanishing Theorem** [4], [22]. Let $f : X \rightarrow Y$ be a surjective morphism from a smooth projective variety $X$ to a normal variety $Y$. Let $L$ be a line bundle on $X$ such that $L \equiv f^*M + D$, where $M$ is a $\mathbb{Q}$-divisor on $Y$ and $(X, D)$ is Kawamata log terminal. Let $C$ be a reduced divisor without common component with $D$ and $D + C$ is a normal crossing divisor. Then

1. $f_*(K_X + L + C)$ is torsion free [20].
2. Assume in addition that $M$ is nef and big. Then $H^i(Y, R^j f_*(K_X + L + C)) = 0$ for $i > 0$ and $j \geq 0$. 

Remark: The $C = 0$ case was done by Esnault-Viehweg, Kawamata and Kollár [11]. The generalized version given here essentially was proved by Esnault-Viehweg in [4] and by Fujino in [22]. I thank Professor Viehweg for informing on the matter. C. Hacon pointed out an inaccuracy and informed me of the reference [22]. I would like to thank him. Below, we give an outline of the proof which was provided to me by E. Viehweg.

Sketch of the proof. By [4, 5.1 and 5.12], we have an injective morphism

$$0 \to H^j(X, K_X + L + C) \to H^j(X, K_X + L + C + B)$$

for any $j$, where $B = f^*(F)$ for some divisor $F$ on $Y$. If we choose $F$ to be a very ample divisor, we have the exact sequence:

$$0 \to \mathcal{R}^j f_*(K_X + L + C) \to \mathcal{R}^j f_*(K_X + L + C + B) \to \mathcal{R}^j f_*(K_B + L + C) \to 0.$$

By induction on dim $Y$ and the Leray-spectral sequence associated with $\mathcal{R}^j f_*$, we can prove the result (see [4] for details). q.e.d.

Proof of Theorem 2: Let $g : Z \to X$ be a log resolution and let $\pi = f \circ g$. Then

$$K_Z = g^*(K_X + D) + \sum a_i E_i,$$

where each $a_i \geq -1$.

We can rewrite $\sum a_i E_i = \sum b_j E_j + \sum c_k E_k + \sum d_l E_l$ where $b_j \geq 0$, $0 > c_k > -1$ and $d_l = -1$.

Let $C$ be a general complete intersection curve on $Y$ and $W = \pi^{-1}(C)$. We have

$$K_Z = K_{Z/Y} + \pi^* K_Y$$

and $K_{Z/Y}|_W = K_{W/C}$.

Thus

$$K_W + \pi^* (K_Y|_C) + \sum -c_k E_k|_W + \sum \{-b_j\} E_j|_W + \sum E_l|_W \equiv \pi^* K_C + \sum [b_j] E_j|_W.$$

Let us assume that $K_Y \cdot C > 0$. Since $(K_W, \sum -c_k E_k|_W + \sum \{-b_j\} E_j)$ is Kawamata log terminal and $\sum [b_j] E_j|_W$ is exceptional, the Vanishing Theorem yields

$$H^1(C, \pi_* (\pi^* K_C + \sum [b_j] E_j|_W)) = H^0(C, \mathcal{O}_C) = 0,$$

a contradiction. So we must have $K_Y \cdot C \leq 0$. If $K_Y \cdot C < 0$, $Y$ is uniruled by [15]. If $K_Y \cdot C = 0$, $K_Y$ is numerically trivial by Hodge index theorem. q.e.d.

Proof of Theorem 3. Let us first establish the following proposition (Proposition 1 in [18]).
Proposition 2. Let $\pi : X \rightarrow Y$ be a surjective morphism between smooth projective varieties over $\mathbb{C}$. Then for any ample divisor $A$ on $Z$, $-K_{X/Y} - \delta \pi^* A$ is not nef for any $\delta > 0$.

Proof of Proposition 2. We shall give a new proof of this proposition by modifying the arguments we used before [18]. Again, the main idea and method comes from [14].

Let $C \subset X$ be a general smooth curve of genus $g(C)$ such that $C \not\subseteq \text{Sing}(\pi)$. Let $p \in C$ be a general point and $B = \{p\}$ be the base scheme. Denoting by $\nu : C \rightarrow X$ the embedding of $C$ to $X$. Let $D_Y(\nu, B)$ be the Hilbert scheme representing the functor of the relative deformation over $Y$ of $\nu$. Then by [14], we have

$$\dim_{\nu} D_Y(\nu, B) \geq -\nu_*(C) \cdot K_{X/Y} - g(C) \cdot \dim X$$

Suppose now that $-K_{X/Y} - \delta \pi^* A$ is nef for some $\delta > 0$. Let $H$ be an ample divisor on $X$ and $\epsilon > 0$ be a small number, we may assume that

$$\nu_*(C) \cdot (\delta \pi^* A - \epsilon H) > 0.$$ 

Then

$$\dim_{\nu} D_Y(\nu, B) \geq \nu_*(C) \cdot (\delta \pi^* A - \epsilon H) - g(C) \cdot \dim X$$

Since $-K_{X/Y} - \delta \pi^* A + \epsilon H$ is an ample divisor on $X$ and the ampleness is indeed an open property in nature. By the method of modulo $p$ reductions [14], after composing $\nu$ with suitable Frobenius morphism if necessary, we can assume that there exists another morphism [14] $\nu' : C \rightarrow X$ such that

1. $\deg \nu'_*(C) < \deg \nu_*(C)$, where $\deg \nu_*(C) = \nu_*(C) \cdot H$.
2. $\pi \circ \nu' = \pi \circ \nu$.

However, we have

$$\nu'_*(C) \cdot (\delta \pi^* A - \epsilon H) > \nu_*(C) \cdot (\delta \pi^* A - \epsilon H)$$

by (1) and (2). This guarantees the existence of a non-trivial relative deformation of $\nu'$. Since $\deg \nu'_*(C) < \deg \nu_*(C)$, this process must terminate, which is absurd. q.e.d.

Proof of Theorem 3 continued: We keep the same notations as in Theorem 2. Let $-K_{W/C} = -K_X|_W + f^*(K_Y|_C)$, where $C$ is a general complete intersection curve on $Y$ and $W = f^{-1}(C)$. Applying Proposition 2, we deduce that $K_Y \cdot C \leq 0$ and we are done. q.e.d.

Proof of Theorem 4. Replacing $X$ by a suitable resolution if necessary, we may assume that $-pK_X = L + N$ for some positive integer $p$, where $|L|$ is
base-point free and $N$ is the fixed part. Multiplying both sides by some large integer $m$, we can write $-K_X = \epsilon L_m + N_m$ as $Q$-divisors, where $\epsilon > 0$ is a small rational number. We may again assume that the linear system $L_m$ is base-point free and $N_m$ is the fixed part. The point is that $(X, \epsilon L_m)$ is log canonical. If the fixed locus does not dominate $Y$, we can choose a general complete intersection curve $C$ on $Y$ such that $C$ only intersects $f(\text{Supp}(N_m))$ at some isolated points. Using the same notations as before, we have

$$K_W + f^*(K_Y|_{C}) + \epsilon L_m|_W \equiv f^*K_C + E - N_m|_W$$

where $E$ is exceptional. If $C \cdot K_Y > 0$, then by the Vanishing Theorem

$$H^1(C, f_*(K_W + f^*(K_Y|_{C}) + \epsilon L_m|_W + \{N_m\})) = H^0(C, -f_*(-[N_m|_W])) = 0$$

Since $\text{Supp}(N_m)$ is contained in some fibers of $f$, we reach a contradiction. The remaining arguments are exactly the same as in the proof of Theorem 2. q.e.d.

**Proof of Corollary 3.** Let $X \xrightarrow{f} Y \xrightarrow{\pi} \text{Alb}(X)$ be the Stein factorization of $\text{Alb}_X$. Then by Theorem 2-4, we conclude that $\kappa(Y) = 0$. We may assume that $Y$ is smooth, otherwise we can take a desingularization $Y'$ of $Y$. This will not affect our choices for the general curve $C$ (since $Y$ is smooth in codimension 1). Therefore $\text{Alb}(X)$ must be an abelian variety and hence $\text{Alb}_X$ is surjective and has connected fibers (see [18] for details). q.e.d.

**Remark:** The notion of special varieties was introduced and studied by F. Campana in [20]. He also conjectured that compact Kähler manifolds with $-K_X$ nef are special. S. Lu [21] proved the conjecture for projective varieties. In particular, if $X$ is a projective variety with $-K_X$ nef and if there is a surjective map $X \rightarrow Y$. Then $\kappa(Y) \leq 0$. Our focus however, is on the uniruledness of $Y$.

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