Derivation of an explicit expression for mutually unbiased bases in even and odd prime power dimensions.

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Abstract: Mutually unbiased bases generalize the $X$, $Y$ and $Z$ qubit bases. They possess numerous applications in quantum information science. It is well-known that in prime power dimensions $N = p^m$ (with $p$ prime and $m$ a positive integer) there exists a maximal set of $N+1$ mutually unbiased bases. In the present paper, we derive an explicit expression for those bases, in terms of the (operations of the) associated finite field (Galois division ring) of $N$ elements. This expression is shown to be equivalent to the expressions previously obtained by Ivanovic in odd prime dimensions (J. Phys. A, 14, 3241 (1981)), and Wootters and Fields (Ann. Phys. 191, 363 (1989)) in odd prime power dimensions. In even prime power dimensions, we derive a new explicit expression for the mutually unbiased bases. The new ingredients of our approach are, basically, the following: we provide a simple expression of the generalised Pauli group in terms of the additive characters of the field and we derive an exact groupal composition law inside the elements of the commuting subsets of the generalised Pauli group, renormalised by a well-chosen phase-factor.

Introduction

Two orthonormal bases of a $N$ dimensional Hilbert space are said to be mutually unbiased if whenever we choose one state in the first basis, and a second state in the second basis, the modulus squared of their in-product is equal to $1/N$. It is
well-known that, when the dimension of the Hilbert space is a prime power, there exists a set of \( N + 1 \) mutually unbiased bases. This set is maximal because it is not possible to find more than \( N + 1 \) mutually unbiased bases in a \( N \) dimensional Hilbert space \([1, 2, 3]\). It is also a complete set because when we know all the probabilities of transition of a given quantum state towards the states of the bases of this set (there are \( N^2 - 1 \) of them), we can reconstruct all the coefficients of the density matrix that characterizes this state; in other words we can perform full tomography or complete quantum state determination \([1, 2, 4, 5]\). A crucial element of the construction is the existence of a finite commutative division ring (or field\(^2\)) of \( N \) elements. As it is well known, finite fields with \( N \) elements exist if and only if the dimension \( N \) is a power of a prime, and a derivation of a set of mutually unbiased bases is already known in such cases. Note that nobody managed until now to generalize this construction in the absence of finite field so that it is still an open question whether such sets exist when the dimension is not a prime power \([6, 7]\). In the present paper, we obtain, in a synthetic formulation, the expressions for the mutually unbiased bases that were derived in the past. In odd prime power dimensions, we recover by a slightly different approach the expressions already obtained in the past by Ivanovic \([1]\) and in odd prime power dimension \( p^m \) by Wootters and Fields \([2]\). We provide a synthetic expression that is also valid in even prime powers dimensions \( (2^m) \). The (discrete) Heisenberg-Weyl group \([3, 4, 8]\) (sometimes also called generalized Pauli group), a finite group of unitary transformations, plays a central role in our approach.

1 Preliminary concepts

In what follows, we shall systematically assume that we work in a Hilbert space of prime power dimension \( N = p^m \) with \( p \) a prime number, and \( m \) a positive integer. Then, as is well known, it is possible to find a finite field with \( N \) elements. We shall label these elements by an integer number \( i, 0 \leq i \leq N - 1 \), or, equivalently, by a \( m \)uple of integer numbers \((i_0, i_1, ..., i_{m-1})\) running from 0 to \( p - 1 \) that we get from the \( p \)-ary expansion of \( i: i = \sum_{k=0}^{m-1} i_np^n \). This field is characterized by two operations, a multiplication and an addition, that we shall denote \( \circ_G \) and \( \oplus_G \) respectively. It is always possible to label the elements of the field in such a way that the addition is equivalent with the addition modulo \( p \) componentwise. As all the

\(^2\)A field is a set with a multiplication and an addition operation which satisfy the usual rules, associativity and commutativity of both operations, the distributive law, existence of an additive identity 0 and a multiplicative identity 1, additive inverses, and multiplicative inverses for every element, 0 excepted.
fields are equivalent, up to a relabelling, there is no strict obligation to do so, but it is more natural and convenient. This property is a direct consequence of the fact that for all the finite fields the characteristics of the field, which is the smallest number of times that we must add the element 1 (neutral for the multiplication) with itself before we obtain 0 (neutral for the addition), is always equal to a prime number \( p \) when \( N = p^m \). The index \( G \) refers to Evariste Galois and is introduced in order not to confuse these operations with the usual (complex) multiplication and addition for which no index is written.

Let us denote \( \gamma_G \) the \( p \)th root of unity: \( \gamma_G = e^{i.2\pi/p} \). Exponentiating \( \gamma_G \) with elements \( g \) of the field (with the usual rules for exponentiation), we obtain complex phasors of the type \( \gamma_G^g \) \( (0 \leq g \leq N) \). Such phasors can take \( p \) different values. They can be considered as a \( p \)-uple generalisation of the (binary) parity operation \( e^{i.(2\pi/2).g} \) that corresponds to the qubit case in the sense that the phasor \( \gamma_G^g \) \( (0 \leq g \leq N) \) only depends on the value of the first component \( g_0 \) of the \( p \)-ary expansion of \( g \) which is nothing else than the remainder of \( g \) after division by \( p \), when the division by \( p \) is taken in the usual sense.

The following identity appears to play a fundamental role in our approach:

\[
\sum_{j=0}^{N-1} \gamma_G^{(j \odot G^i)} = N \delta_{i,0} \tag{1}
\]

Indeed, if \( i = 0 \), then \( \sum_{j=0}^{N-1} \gamma_G^{(j \odot G^i)} = N.1 = N \). Otherwise, \( \sum_{j=0}^{N-1} \gamma_G^{(j \odot G^i)} = \sum_{j'=0}^{N-1} \gamma_G^{j'} \) in virtue of the inversibility of the multiplication. Now the exponentiation of gamma by elements of the field does only depend on the remainder after division by \( p \), so that \( \sum_{j'=0}^{N-1} \gamma_G^{j'} = p^{m-1}. \sum_{j'=0}^{p-1} \gamma_G^{j'} = p^{m-1}. \frac{(1-\gamma_G^m)}{(1-\gamma_G)} = 0 \).

In virtue of the fact that the addition is the addition modulo \( p \), componentwise, we can derive the following identity which is also very useful:

\[
\gamma_G^i \cdot \gamma_G^j = \gamma_G^{(i \oplus G^j)} \tag{2}
\]

Indeed, \( \gamma_G^i \cdot \gamma_G^j = \gamma_G^{(i+j)} = \gamma_G^{(i_0+j_0)} = \gamma_G^{(i \oplus G^j)_0} = \gamma_G^{(i \oplus G^j)} \) (in the previous expression, we represented by the symbol \( x_0 \) the remainder of \( x \) after division by \( p \), where \( x \) is an element of the field, comprised between 0 and \( p^m - 1 = N - 1 \), and the division by \( p \) is taken in the usual sense.) This relation is well-known and expresses, in the language of mathematicians, that \( p \)th roots of unity are additive characters of the Galois field [32].
It is important to note, in order to avoid confusions, that different types of operations are present at this level: the internal field operations are labelled by the lower index $G$. They must not be confused with the modulo $N$ operations. In order to emphasise the difference between these operations, we give in example the corresponding tables in the case $N = 4 = 2^2$ in appendix. One can check that the field and modulo 4 multiplications are distributive relatively to the associated addition, but that there are no dividers of 0, 0 excepted, only in the case of the field multiplication. As a consequence, the field multiplication table, amputed from the first line and column exhibits an invertible (group) structure. All operations are commutative as can be seen from the symmetry of the tables 1 to 4 under transposition.

Remark that if we express quartits as products of two qubits: $|0\rangle_4 = |0\rangle_2 \otimes |0\rangle_2$, $|1\rangle_4 = |0\rangle_2 \otimes |1\rangle_2$, $|2\rangle_4 = |1\rangle_2 \otimes |0\rangle_2$, $|3\rangle_4 = |1\rangle_2 \otimes |1\rangle_2$. It is then easy to check the following property: If $|i\rangle_4 = |i\rangle_2 \otimes |i\rangle_2$, and $|j\rangle_4 = |j\rangle_2 \otimes |j\rangle_2$, then $|i \oplus_G j\rangle_4 = |i \oplus_{mod2} j_1 \rangle_2 \otimes |i \oplus_{mod2} j_2 \rangle_2$. This is an illustration of the fact that the field addition is equivalent with the addition modulo $p$ componentwise. It is also worth noting that the property $\sum_{N - 1}^{N - 1} \gamma^{(p \odot q)} = N \delta_{q,0}$ is true for the modulo $N$ multiplication as well, but $\gamma$ must be taken to be equal to the $N$th root of unity in this case. In prime dimensions $\gamma_G$ is the $N$th root of unity and the Galois and modulo $N$ operations coincide. In prime power but non-prime dimensions, this is no longer true.

2 Construction of the dual basis

Let us now consider the unitary transformations $V^0_l$, that shift each label of the states of the computational basis ($\{|0\rangle, |1\rangle, \ldots, |i\rangle, \ldots, |N - 1\rangle\}$) by a distance $l$ ($|i\rangle \rightarrow |i \oplus_G l\rangle$) (the reason for our choice of notation will be made obvious soon). The transformations $V^0_l$ form a commutative group with $N$ elements that is isomorphic to the Galois addition. Generalizing the procedure outlined in [9], we define the dual basis as follows:

$$|	ilde{j}\rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \gamma^{\oplus_G (k \oplus_G j)} |k\rangle$$

where the symbol $\oplus_G$ represents the inverse of the Galois addition $\oplus_G$. It is easy to check that the dual states are invariant, up to a global phase, under the transfor-
mations $V_l^0$. Indeed, we have:

\[
V_l^0 \langle \tilde{j} | = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \gamma_G^{(k \odot_G l)} |k \oplus_G l\rangle |k\rangle \tag{4}
\]

\[
\frac{1}{\sqrt{N}} \sum_{k'=0}^{N-1} \gamma_G^{(k' \odot_G l \odot_G j)} |k'\rangle = \gamma_G^{(l \odot_G j)} |\tilde{j}\rangle \tag{5}
\]

Obviously, the dual basis and the computational basis are mutually unbiased. When the dimension is prime ($N = p$), the dual basis is the discrete Fourier transform of the computational basis, when it is a power of 2, it is a Hadamard transform [9].

Let us denote $V_l^0$ the unitary transformations that shift each label of the states of the dual basis ($\{|\tilde{0}\rangle, |\tilde{1}\rangle, ..., |\tilde{i}\rangle, ..., |\tilde{N} - \tilde{1}\rangle\}$) by a distance $\odot_G l$ ($\tilde{i} \rightarrow |\tilde{i} \odot_G l\rangle$). The transformations $V_l^0$ form a commutative group with $N$ elements that is isomorphic to the Galois addition. It is easy to check that these operators are diagonal in the computational basis:

\[
V_l^0 = \sum_{k=0}^{N-1} |\tilde{k} \odot_G l\rangle \langle \tilde{k}| = \sum_{k=0}^{N-1} \gamma_G^{(k \odot_G l)} |k\rangle \langle k|. \tag{6}
\]

This is the dual counterpart of a similar expression for the shifts in the computational basis:

\[
V_l^0 = \sum_{k=0}^{N-1} |k \oplus_G l\rangle \langle k| = \sum_{k=0}^{N-1} \gamma_G^{(k \odot_G l)} |\tilde{k}\rangle \langle \tilde{k}| \tag{7}
\]

3 Construction of the remaining $N-1$ mutually unbiased bases

In the previous section we derived a set of two bases, the computational basis and the dual basis, that are mutually unbiased. In this section, we shall generalize this derivation in order to obtain $N-1$ other mutually unbiased bases (between each other, and also relatively to the computational and dual bases).

Let us denote $V_i^j$ the compositions of the shifts in the computational and the dual basis:
\[ V^j_i = V^j_0 . V^i_0 = \sum_{k=0}^{N-1} \gamma_G^{((k \oplus G i) \circ G j)} |k \oplus_G i \rangle \langle k|; i, j : 0 \ldots N - 1 \] (8)

Here, the product \( . \) expresses the matricial product (the usual composition law of two unitary transformations).

As \( V^0_0 \) is the identity, there is no disagreement with the previous definitions. Note that \( V^j_0 \) and \( V^0_j \) do not commute:

\[ V^j_i = V^0_j . V^i_0 = \sum_{l=0}^{N-1} \gamma^{l \circ G j} |l \rangle \langle l| . \sum_{k=0}^{N-1} |k \oplus_G i \rangle \langle k| \]
\[ = \sum_{k=0}^{N-1} \gamma^{((k \oplus G i) \circ G j)} |k \oplus_G G j \rangle \langle k| \] (9)

Although this expression appeared, to our knowledge, in ref. [19] for the first time, we show in the last section that the set of operators so-defined coincides with the generalised Pauli group studied in ref. [3].

\[ V^i_0 . V^j_0 = \sum_{k=0}^{N-1} |k \oplus_G i \rangle \langle k| . \sum_{l=0}^{N-1} \gamma^{l \circ G j} |l \rangle \langle l| \]
\[ = \sum_{k=0}^{N-1} \gamma^{((k \oplus G i) \circ G j)} |k \oplus_G i \rangle \langle k| \]
\[ = \gamma^{G (i \circ G j)} V^j_0 . V^i_0 = \gamma^{G (i \circ G j)} V^j_i \] (10)

The commutator is thus given by the following expression:

\[ V^j_i . V^i_0 - V^i_j . V^j_0 = (1 - \gamma^{G (i \circ G j)}) V^j_i . V^i_0 \] (11)

We recognize here a commutation rule that is known as the Weyl commutation rule, and was already studied a long time ago [3]. This is not astonishing because the set of unitary transformations \( V^j_i \) that we consider here is a discrete version of the so-called Heisenberg-Weyl group (compositions of translations in position and in impulsion). In dimension 2, it coincides with the Pauli group. When the dimension is a prime number, the field operations are the addition and multiplication modulo \( p \), and the properties of mutually unbiased bases are already well-known in that case [1], as well as their relation with the “Heisenberg-Weyl-Pauli” group [10]. In the present
approach, we consider, instead of the usual (modulo $N$) operations, the Galois addition and multiplication, also for non-prime but prime power dimensions. The connection with previous works related to the Pauli group approach \cite{2, 3, 11, 12, 13} is established in the last section.

By a straightforward computation, we can now derive the law of composition of these $N^2$ unitary transformations:

$$V_i^j \cdot V_l^k = V_0^j \cdot V_i^k \cdot V_l^0$$

$$= \gamma \oplus G (i \oplus j) \cdot V_0^j \cdot V_i^0 \cdot V_l^0$$

$$= \gamma \oplus G (i \oplus j \oplus k) \cdot V_0^j \oplus G (i \oplus l)$$

(12)

Up to a global phase, this looks like a groupal composition law. We shall now show that (up to phases) the $N^2$ unitary transformations $V_i^j$ form $N + 1$ commuting subgroups of $N$ elements that have only the identity in common. Moreover, each of these subgroups admits a diagonal representation in a basis that is mutually unbiased relatively to the the $N$ bases in which the other subgroups are diagonal. Note that the last property can be shown, following an alternative approach developed in ref. \cite{3} to be a consequence of the fact that the $V$ operators, up to phases, form what is called a maximally commuting basis of orthogonal unitary matrices (see also last section). The new ingredients in our approach are (1) the expression 9 for the generalized Pauli operators, and (2) the recognition of the fact that these operators exhibit exact groupal composition laws, provided (a) they commute and (2) they are multiplied by a convenient phase factor.

In order to derive all the results, we shall take them for granted in a first time, and check afterwards that our hypothesis was correct. It is convenient to introduce new notations and definitions before we pursue. We shall denote $U_i^l$ the elements of these subgroups, where $i$ labels the subgroup and runs from 0 to $N$ (there are $N + 1$ of them), while $l$ labels the elements of the subgroup and runs from 0 to $N - 1$ (each subgroup contains $N$ elements). We know already the two first subgroups, that admit a diagonal representation in the computational and dual bases: the first one ($i = 0$) contains the elements $V_0^l (l : 0...N - 1)$, so by definition $U_0^l = V_0^l (l : 0...N - 1)$. The second one contains the elements $U_1^l = V_0^l (l : 0...N - 1)$.

In virtue of the equalities 6 and 7 we can also write $U_0^l = \sum_{k=0}^{N-1} \gamma_G (k \circ l) |k \rangle \langle k|$ and $U_1^l = \sum_{k=0}^{N-1} \gamma_G (k \circ l) |\tilde{k} \rangle \langle \tilde{k}|$. A similar expression can be found for each of the $N - 1$ remaining subgroups as we shall now show. It is convenient at this level to parametrize the basis states that diagonalize these subgroups as follows: the $k$th
basis state that diagonalizes the $i$th subgroup will be denoted as $|e_k^i\rangle$. Our ultimate goal is to prove that there exist $N+1$ bases $|e_k^i\rangle$ and $N^2$ operators $U^i_1$ that are in one to one correspondence with the $V$ operators and differ from them by an appropriate phase factor, such that the following constraints are fulfilled:

$$U^i_1 = \sum_{k=0}^{N-1} \gamma_G^{(k\circ_G l)} |e_k^i\rangle \langle e_k^i| \quad (l : 0...N - 1; i : 0...N)$$  \hspace{1cm} (13)

$$\langle e_k^i|e_j^l\rangle . \langle e_j^l|e_k^i\rangle = \delta_{k,l} \delta_{i,j} + (1/N). (1 - \delta_{k,l}) \quad (k, l : 0...N, i, j : 0...N - 1)$$  \hspace{1cm} (14)

In virtue of the commutativity of the Galois multiplication, of the identity 2 and of the definition 13 the $U$ operations that are labelled by a same value $i$ form a commutative subgroup and obey the (exact) group composition law $U^i_1U^i_1 = U^i_1$. We can guess that they correspond to families of operators $V^k_1$ such that the (Galois) ratio $k/G l$ is constant, because the commutation of $V^k_1$ and $V^k_1'$ implies that $k'\circ_G l = k\circ_G l'$. It is thus natural to try the identification $U^i_1 = V^i_1^{(l-1)\circ_G l}$, up to a phase, when $i$ differs from 0 and $U^0_1 = V^0_1$ which is consistent with our previous conventions. There are in general several ways to fix the phases but in any case certain constraints must be satisfied:

- the phase $U^i_1/V_1^{(l-1)\circ_G l}$ is equal to 0 when $l = 0$, because $V^0_0 = 1$, and the identity is present in all subgroups

- as we mentioned already, the $U$ operators must obey the following composition law: $U^i_1 U^i_{1'} = U^i_{1'}$, but the composition law $V^i_1 V^k_1 = \gamma^{\circ_G (k\circ_G k)} V^j_{1'\circ_G k}$ must be guaranteed at the same time, which restricts seriously the arbitrariness in the choice of the phase.

Let us now assume that the phase ratio between $U^i_1$ and $V^i_1^{(l-1)\circ_G l}$ is fixed for all powers of $p$ between 0 and $m - 1$ ($l = p^n, 0 \leq n \leq m - 1$).

We shall firstly treat the odd dimensional case. Then, iterating $l$ times the composition law 12 (with $2 \leq l \leq m - 1$), we obtain the following constraints on the ratio between $U^i_{p^n\circ_G l}$ and $V^r_{p^n\circ_G l}$, $0 \leq l \leq m - 1, 0 \leq n \leq m - 1$:

$$\left( U^i_{p^n\circ_G l} \right) = \sum_{k=0}^{N-1} \gamma_G^{p^n\circ_G k\circ_G l} |e_k^i\rangle \langle e_k^i| = \left( U^i_{p^n} \right)^l$$

$$= \left( U^i_{p^n}/V^r_{p^n} \right)^l \cdot \gamma^{\circ_G (l-1)\circ_G l\circ_G (l\circ_G 1)\circ_G p^n\circ_G p^n\circ_G l} \cdot V^r_{p^n\circ_G l}$$  \hspace{1cm} (15)
Here, the symbols $/$ and $\circ_G$ indicate the multiplication by the multiplicative inverse for the usual (complex) and field multiplications respectively. In order to fix the phase \( (U_{p^n}^{(i-1)\circ_G p^n}) / V_{p^n}^{(i-1)\circ_G p^n} \) we can make use of the fact that the characteristics of the field is \( p \) (so to say \( 1 \oplus_G 1 \oplus_G \ldots \oplus_G 1 \) (\( p \) times) = 0), which implies that \( (U_{p^n})^p = 1 \), so that we obtain the following constraint:

\[
(U_{p^n}^i / V_{p^n}^{(i-1)\circ_G p^n})^p = \gamma_G^{(i-1)\circ_G p^n \circ_G (p \circ_G 1) \circ_G p^n \circ_G p^n / \sigma^2} = 1
\]

(16)

The phase \( (U_{p^n}^i / V_{p^n}^{(i-1)\circ_G p^n} ) \) is thus determined up to an integer power of \( \gamma_G = e^{i2\pi/p} \). This is true for each integer value of \( n \) between 0 and \( m-1 \) so that there are \( p^n = N \) different possible ways to "fix" the phases. Let us denote \( \gamma_n \), the \( p \)th root of unity that we choose to be equal to \( U_{p^n}^i / V_{p^n}^{(i-1)\circ_G p^n} \). Once this value is chosen, all the other phases are determined, as shows the following development:

\[
U_{p^n}^i = \Pi_{n=0}^{m-1} U_{l_n \circ_G p^n} \Pi_{n=0}^{m-1} (U_{p^n})^{l_n} = \Pi_{n=0}^{m-1} \gamma_G^{(i-1)\circ_G l_n \circ_G (l_n \circ_G 1) \circ_G p^n \circ_G p^n / \sigma^2} \gamma_n^{l_n} V_{p^n \circ_G l_n}^{(i-1)\circ_G p^n \circ_G G l_n},
\]

(17)

where the coefficients \( l_n \) are unambiguously defined by the \( p \)-ary expansion of \( l \): \( l = \sum_{k=0}^{m-1} l_n p^n \). Moreover, we can check by direct computation that the \( U \) operators so-defined obey an exact groupal composition law, independently on the choice that we could decide to perform, among the \( p^n = N \) different possible ways to "fix" the phases \( \gamma_n \):

\[
\begin{align*}
U_{l_1}^i U_{l_2}^i &= \Pi_{n=0}^{m-1} U_{l_1 \circ_G p^n} U_{l_2 \circ_G p^n} \\
&= \Pi_{n=0}^{m-1} \gamma_G^{(i-1)\circ_G l_1 \circ_G (l_1 \circ_G 1) \circ_G p^n \circ_G p^n / \sigma^2} \gamma_G^{(i-1)\circ_G l_2 \circ_G (l_2 \circ_G 1) \circ_G p^n \circ_G p^n / \sigma^2} \gamma_n^{l_1 + l_2} V_{l_1 \circ_G p^n \circ_G l_2 \circ_G p^n}^{(i-1)\circ_G p^n \circ_G G l_1 \circ_G G l_2 \circ_G G l_2} \\
&= \Pi_{n=0}^{m-1} \gamma_G^{(i-1)\circ_G p^n \circ_G l_1 \circ_G (l_1 \circ_G 1) \circ_G l_2 \circ_G (l_2 \circ_G 1) \circ_G p^n \circ_G p^n / \sigma^2} \gamma_G^{(i-1)\circ_G p^n \circ_G G l_1 \circ_G G l_2 \circ_G G l_2} \\
&= \Pi_{n=0}^{m-1} \gamma_G^{(i-1)\circ_G p^n \circ_G p^n \circ_G G ((l_1 \circ_G l_2 \circ_G l_2) \circ_G (l_1 \circ_G l_2 \circ_G l_2)) / \sigma^2} \gamma_n^{l_1 + l_2 + modp l_2} V_{p^n \circ_G (l_1 \circ_G l_2 \circ_G l_2) \circ_G G l_1 \circ_G G l_2 \circ_G G l_2}^{(i-1)\circ_G p^n \circ_G G l_1 \circ_G G l_2 \circ_G G l_2} \\
&= \Pi_{n=0}^{m-1} \gamma_G^{(i-1)\circ_G p^n \circ_G p^n \circ_G G ((l_1 \circ_G l_2 \circ_G l_2) \circ_G (l_1 \circ_G l_2 \circ_G l_2)) / \sigma^2} \gamma_n^{l_1 + l_2 + modp l_2} V_{p^n \circ_G (l_1 \circ_G l_2 \circ_G l_2) \circ_G G l_1 \circ_G G l_2 \circ_G G l_2}^{(i-1)\circ_G p^n \circ_G G l_1 \circ_G G l_2 \circ_G G l_2} \\
&= \Pi_{n=0}^{m-1} (U_{p^n}^i)^{l_1 + l_2 + modp l_2} = \Pi_{n=0}^{m-1} (U_{l_1 \circ_G G l_2}^{(i-1)\circ_G p^n \circ_G G l_2 \circ_G G l_2}) = U_{l_1 \circ_G G l_2}^i
\end{align*}
\]

(18)

In even prime power dimensions, the treatment is similar, although we may not divide by 2 in this case. Combining the constraints \( (V_{2n}^{(i-1)\circ_G 2^n})^2 = (\gamma_G^{(i-1)\circ_G 2^n \circ_G 2^n} V_0^n) = \)
\( \gamma_G^{(j-1) \odot G 2^n \odot G 2^n} \) and \( U_2^n U_2^n = U_2^n \oplus G 2^n = U_0^n = 1 \), we obtain the following decomposition law for the \( U \) operators, which expresses their factorisation in terms of qubit operators:

\[
U^j = \Pi_{n=0}^{m-1} U_{l_n \odot G 2^n} = \Pi_{n=0}^{m-1} (U_{2^n})^l_n \\
= \Pi_{n=0}^{m-1} \left( \gamma_G^{(j-1) \odot G 2^n \odot G 2^n} \right)^{\frac{1}{2}} V_2^n \gamma_G^{(j-1) \odot G 2^n \odot G l_n},
\]

(19)

where the coefficients \( l_n \) are unambiguously defined by the binary expansion of \( l \):
\( l = \sum_{k=0}^{m-1} l_n 2^n \); \( l_n = 0 \) or \( l_n = 1 \). Similarly to what happens in the odd dimensional case, the phase factors \( \gamma_G^{(j-1) \odot G 2^n \odot G 2^n} \) can be fixed with some arbitrariness, actually up to a sign in this case. Let us now check that the \( U \) operators so defined obey an exact group composition law:

\[
U^{j_1}_1 U^{j_2}_2 = \Pi_{n=0}^{m-1} \left( \gamma_G^{(j-1) \odot G 2^n \odot G 2^n} \right)^\frac{l_{1n} l_{2n}}{2} \left( V_2^n \right)^{l_{1n} + l_{2n} \mod 2^{2n}} = \Pi_{n=0}^{m-1} \left( \gamma_G^{(j-1) \odot G 2^n \odot G 2^n} \right)^\frac{l_{1n} + l_{2n} \mod 2^{2n}}{2} \left( V_2^n \right)^{l_{1n} + l_{2n} \mod 2^{2n}} = U^{j_1}_{1} \odot G l_2.
\]

(20)

We made use of the fact that \( (V_2^n)^2 = \gamma_G^{(j-1) \odot G 2^n \odot G 2^n} \) and \( \gamma_G = -1 \).

Let us now derive an explicit expression for the \( N \) phases \( U^{j_1}_1 / V_1^{(j-1) \odot G l} \). We shall treat separately even and odd prime power dimensions.

### 3.1 Odd prime power dimensions

In odd prime power dimensions, all possible consistent choices for determining the phases (there are \( p^m = N \) such choices) can be expressed as follows:

\[
U^{i_1}_1 / V_1^{(i-1) \odot G l} = \left( \gamma_G^{\odot (i-1) \odot G l \odot G l / G^2} \right) \gamma_G^{k \odot l},
\]

(21)

where \( k \) is an arbitrary element of the field. Each choice for \( k \) (there are \( N \) of them) leads to another consistent determination of the phase ratio between the \( U \) and \( V \) operators. This is due to the fact that if \( \gamma_G^{k \odot p^n} = \gamma_G^{k' \odot p^n} \), \( \forall n : 0 \leq n \leq m - 1 \), then \( k = k' \) in virtue of the identity \( \Box \).

Note that, when \( k = 0 \), which is the simplest determination of the phases, we obtain the following relation:

\[
U^{j}_1 = \left( \gamma_G^{\odot (i-1) \odot G l \odot G l / G^2} \right) V_1^{(i-1) \odot G l}
\]

(22)

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This corresponds to the choice of phases $\gamma_n = \gamma_G^{\oplus((i-1)\ominus_G p^q \ominus_G p^r)/g^2}$. It is worth noting that the relation (22) is also valid for $i = 1$, which corresponds to the dual basis derived in the previous section.

For $i = 0$, $U_0^i = V_0^i$, in agreement with the previous definitions.

In order to check the consistency of the expression (Eqn.21), it is sufficient, making use of the composition law for the $V$ operators (Eqn.12), to check by direct computation that the $U$ operators obey an exact (so to say not up to a phase) group composition law. Remark that if $\tilde{U}_i^i = \gamma_G^{k \ominus_i U_1^i}$, and that $U_i^i U_i^i = U_{l \ominus_G k'}^i$, then $\tilde{U}_i^i \tilde{U}_i^i = \tilde{U}_i^{i \ominus_G k'}$. Therefore, it is sufficient to establish the groupal composition law when the expression (Eqn.22) is valid, in order to establish it when the expression (Eqn.21) is valid, for any value of $k$.

\[
U_i^i U_i^i = (\gamma_G^{\ominus((i-1)\ominus_G l\ominus_G l)/g^2}) (\gamma_G^{\ominus((i-1)\ominus_G l'\ominus_G l')/g^2}) V_{l'}^{(i-1)\ominus_G l \ominus_G l'} V_{l'}^{(i-1)\ominus_G l' \ominus_G l'} = (\gamma_G^{\ominus((i-1)\ominus_G l\ominus_G l')\ominus_G (l' \ominus_G l')/g^2}) V_{l'}^{(i-1)\ominus_G (l' \ominus_G l')} = U_{l \ominus_G k'}^i, \quad i : 1...N, l, l' : 0...N - 1
\] (23)

Now that we have at our disposal an exact expression for the operators $U$, we can also derive an explicit expression for the $N - 1$ dual bases associated to the subgroups that correspond to the operators $U_i^i$; $i = 1...N - 1$. This can be realised thanks to the following identity, a direct consequence of Eqns.13 and 1:

\[
|e_k^i \rangle \langle e_k^i| = \frac{1}{N} \sum_{l=0}^{N-1} \gamma_l^{\ominus_G k \ominus_G l} U_l^i
\] (24)

Obviously, if we choose another determination of the phases, so to say if we replace $U_i^i$ by $\tilde{U}_i^i = \gamma_G^{k' \ominus_i U_1^i}$, we obtain the same basis states, with their labels shifted by $k'$. It is thus more convenient to choose in what follows the simplest phase determination (22).

By a straightforward but lengthy computation that we do not reproduce here, we obtain then the expression, in the computational basis, of the states of $N - 1$ bases that correspond to the non-null values of the label $i - 1$.

\[
|e_k^i \rangle = \frac{1}{\sqrt{N}} \sum_{q=0}^{N-1} \gamma_G^{\ominus q \ominus G k} (\gamma_G^{((i-1)\ominus_G q \ominus_G q)/g^2}) |e_q^0 \rangle,
\] (25)
Remark that the previous expression is also valid when \( i = 1 \), which corresponds to the dual basis \([\tilde{j}]\).

Let us now check by direct computation that the \( N \) bases \( (N - 1 \text{ plus the dual basis}) \) obtained so are orthonormal and mutually unbiased between each other (it is easy to check that the computational basis also fulfills these requirements). Before we do so, we shall rewrite the factors \((\gamma_G^\otimes((i-1)\otimes G\otimes Gl))/a^2\) as follows:

\[
(\gamma_G^\otimes((i-1)\otimes G\otimes Gl))/a^2 = U_1^i/V_l^{(i-1)\otimes Gl} = (\gamma_G^\otimes((i-1)\otimes G\otimes Gl))^{\frac{1}{2}}
\] (26)

This redefinition is less precise than the previous one, because there exist two determinations of the square root of a complex number, nevertheless we adopt it, having in mind that in odd prime power dimensions, the previous expression fixes the sign of the square root of \((\gamma_G^\otimes((i-1)\otimes G\otimes Gl))\) without ambiguity. We shall show in the next section that a similar expression is also valid in the even dimensional case. Let us now prove that the expression (24) is valid.

\[
\langle e^j_i | e^k_i \rangle = \frac{1}{N} \sum_{q=0}^{N-1} \gamma_G^\otimes((i\otimes j)\otimes G\otimes Gq) (\gamma_G^\otimes((k-1)\otimes G\otimes Gq))^\frac{1}{2}
\]

(27)

\[
\langle e^j_i | e^k_i \rangle, \langle e^k_i | e^j_i \rangle,
\]

\[
= \frac{1}{N^2} \sum_{q=0}^{N-1} \gamma_G^\otimes((i\otimes j)\otimes G\otimes Gq) (\gamma_G^\otimes((l-1)\otimes G\otimes Gq))^\frac{1}{2} \sum_{q'=0}^{N-1} \gamma_G^\otimes((l\otimes j)\otimes G\otimes Gq') (\gamma_G^\otimes((k-1)\otimes G\otimes Gq')^\frac{1}{2})
\]

\[
= \frac{1}{N^2} \sum_{q,t=0}^{N-1} \gamma_G^\otimes((t\otimes t)\otimes G\otimes Gq) (\gamma_G^\otimes((l-1)\otimes G\otimes Gq))^\frac{1}{2} \sum_{q',t'=0}^{N-1} \gamma_G^\otimes((l\otimes t)\otimes G\otimes Gq') (\gamma_G^\otimes((k-1)\otimes G\otimes Gq')^\frac{1}{2})
\]

\[
= \frac{1}{N^2} \sum_{q,t=0}^{N-1} \gamma_G^\otimes((t\otimes t)\otimes G\otimes Gq) (\gamma_G^\otimes((l-1)\otimes G\otimes Gq))^\frac{1}{2} \sum_{q',t'=0}^{N-1} \gamma_G^\otimes((l\otimes t)\otimes G\otimes Gq') (\gamma_G^\otimes((k-1)\otimes G\otimes Gq')^\frac{1}{2})
\]

\[
= \frac{1}{N} \sum_{t=0}^{N-1} \delta((l-1)\otimes G\otimes G(k-1))\otimes Gt,0) \gamma_G^\otimes((t\otimes t)\otimes G\otimes Gq) (\gamma_G^\otimes((l-1)\otimes G\otimes Gq))^\frac{1}{2}
\]

\[
= \delta((l-1)\otimes G\otimes G(k-1),0) \delta_i,j + (1 - \delta((l-1)\otimes G\otimes G(k-1),0)) \frac{1}{N} \sum_{t=0}^{N-1} \delta_t,0 \gamma_G^\otimes((t\otimes t)\otimes G\otimes Gq) (\gamma_G^\otimes((l-1)\otimes G\otimes Gq))^\frac{1}{2}
\]

\[
= \delta_i,j \delta_k,j + (1/N). (1 - \delta_k,j)(28)
\]
We made use of the fact that there is no divider of 0 excepted 0 itself (the multiplication \( \odot_G \) forms a division ring). Henceforth, the following identity is valid: 
\[
\delta_{a \odot_G b, 0} = \delta_{a, 0} + (1 - \delta_{a, 0}) \cdot \delta_{b, 0}.
\]

Finally, let us control the validity of the postulated expression 13:
\[
\sum_{k=0}^{N-1} \gamma_G^{(k \odot_G l)} |e_k^i \rangle \langle e_k^i| = \sum_{q=0}^{N-1} \gamma_G^{q \odot_G k} \left( \gamma_G^{(i-1) \odot_G q \odot_G q'} \right)^{\frac{1}{2}} |e_q^0 \rangle \langle e_q^0| = \sum_{q'=0}^{N-1} \delta_{q,q'} \left( \gamma_G^{((i-1) \odot_G q' \odot_G q')} \right)^{\frac{1}{2}} |e_q^0 \rangle \langle e_q^0|
\]
\[
= \left( \gamma_G^{((i-1) \odot_G l \odot_G l')} \right)^{\frac{1}{2}} \sum_{q'=0}^{N-1} \gamma_G^{((i-1) \odot_G q' \odot_G q')} |e_{q'}^0 \rangle \langle e_{q'}^0| = U_l^i(l : 0...N - 1; i : 1...N)(29)
\]

### 3.2 Even prime power dimensions

In this case the explicit expressions for the mutually unbiased bases are less easy to manipulate. Once again, there are \( p^m \) (2\( m \)) in this case) possible ways to determine the phases \( U_l^i / V_l^{(j-1) \odot_G l} \), but they are equivalent, up to a relabelling of the basis states.

In the next development, we shall implicitly choose a certain determination of the square root of \( \gamma_G^{(j-1) \odot_G 2^n \odot_G 2^n} \) that is equal to \( \gamma_G^{(j-1) \odot_G 2^n \odot_G 2^n} \).
\[
U_l^i = \Pi_{n=0}^{m-1} U_{l_n \odot_G 2^n} = \Pi_{n=0}^{m-1} (\Pi_{l_n \neq 0} U_{l_n}^{j})_n
\]
\[
= \Pi_{n=0}^{m-1} (\gamma_G^{(j-1) \odot_G 2^n \odot_G 2^n})^{\frac{1}{2}} (V_{l_n}^{(j-1) \odot_G 2^n})_n
\]
\[
= \Pi_{n=0}^{m-1} \gamma_G^{(j-1) \odot_G 2^n \odot_G 2^n} (V_{l_n}^{(j-1) \odot_G 2^n})_n
\]
\[
= (\Pi_{n=0}^{m-1} \gamma_G^{(j-1) \odot_G 2^n \odot_G 2^n} (V_{l_n}^{(j-1) \odot_G 2^n})_n)_{l_n \neq 0} (j-1) \odot_G 2^n \odot_G 2^{n'} \gamma_G^{(j-1) \odot_G 2^n \odot_G 2^{n'}} V_{l}^{(j-1) \odot_G l} \quad (30)
\]
where the coefficients \( l_n \) are unambiguously defined by the \( p \)-ary (here binary) expansion of \( l_\ell : l = \sum_{k=0}^{m-1} l_n \cdot 2^n \), while \( n' \) is the smallest integer strictly larger than \( n \) such that \( l_{n'} \neq 0 \), if it exists, 0 otherwise.
This result is a generalisation of the identity\cite{22} because in both cases the phases are square roots of integer powers of gamma:

\[
(U_l^j / V_l^{(j-1) \odot G_l})^2 = \gamma_G \otimes ((j-1) \odot G_l \odot G_l).
\]

Nevertheless, in the present case we obtain the following determination of the square root of \((\gamma_G \otimes ((j-1) \odot G_l \odot G_l))\):

\[
(\gamma_G \otimes ((j-1) \odot G_l \odot G_l))^{\frac{1}{2}} = (\gamma_G \otimes ((j-1) \odot G_l \odot G_l))^{\frac{1}{2}} = \prod_{n=0, l_n \neq 0}^{m-1} (j-1) \odot G_2^n \odot G_2^{n'} \gamma_G^{(j-1) \odot G_2^n \odot G_2^{n'}}
\]

where \(n'\) and \(l_n\) were defined previously. What is particular with even prime powers is that the square root of an integer power of \(\gamma_G\) is not an integer power of \(\gamma_G\) as was the case in odd prime power dimensions. We are forced to introduce \(+i\) and \(-i\).

What is interesting is that this minimal extension is sufficient in order to diagonalize the operators of the generalized Pauli group in even prime power dimensions, a fact that was already recognized in previous references on the subject\cite{2,15}.

Combining the equations\cite{25} and\cite{26} we obtain the following synthetic expression:

\[
|e_k^j\rangle = \frac{1}{\sqrt{N}} \sum_{q=0}^{N-1} \gamma_G^{\odot G_q \odot G_k} (\gamma_G^{((j-1) \odot G_q \odot G_q)})^{\frac{1}{2}} |e_q^0\rangle
\]

(32)

Taking account of the Eqn\cite{31} we get an explicit expression for the mutually unbiased bases:

\[
|e_k\rangle = \frac{1}{\sqrt{N}} \sum_{q=0}^{N-1} \gamma_G^{\odot G_q \odot G_k} \prod_{n=0, q_n \neq 0}^{m-1} (j-1) \odot G_2^n \odot G_2^{n'} \gamma_G^{(j-1) \odot G_2^n \odot G_2^{n'}} |e_q^0\rangle,
\]

(33)

where \(q = \sum_{k=0}^{n-1} q_n 2^n\), while \(n'\) is the smallest integer strictly larger than \(n\) such that \(q_{n'} \neq 0\), if it exists, 0 otherwise.

As a consequence of the composition laws\cite{12} and\cite{20}:

\[
U_{l_1}^j \cdot U_{l_2}^j = (\gamma_G^{(j-1) \odot G_{l_1} \odot G_{l_2}})^{\frac{1}{2}} \cdot (\gamma_G^{(j-1) \odot G_{l_2} \odot G_{l_2}})^{\frac{1}{2}} \cdot (\gamma^{(j-1) \odot G_{l_1} \odot G_{l_2}}) V_{l_1 \odot G_{l_2}}^{(j-1) \odot G_{l_1} \odot G_{l_2}} V_{l_2 \odot G_{l_2}}^{(j-1) \odot G_{l_2} \odot G_{l_1}} = U_{l_1 \odot G_{l_2}}^j
\]

(34)

Therefore,

\[
(\gamma_G^{(j-1) \odot G_{l_1} \odot G_{l_1}})^{\frac{1}{2}} \cdot (\gamma_G^{(j-1) \odot G_{l_2} \odot G_{l_2}})^{\frac{1}{2}} \cdot (\gamma^{(j-1) \odot G_{l_1} \odot G_{l_2}}) = (\gamma_G^{(j-1) \odot G_{l_1} \odot G_{l_2} \odot G_{l_2}})^{\frac{1}{2}}.
\]

(35)
Formally we can rewrite the previous equation as follows:

\[ \left( \gamma_G^{(j-1) \circ_G (a \circ_G b) \circ_G (a \circ_G b)} \right)^{\frac{1}{2}} = \left( \gamma_G^{(j-1) \circ_G (a \circ_G a)} \right)^{\frac{1}{2}}, \]

which is reminiscent of the equation (2), although we are dealing here with half integer powers of \( \gamma_G \) instead of integer powers. Thanks to this property, it is possible to reproduce nearly literally the proofs given in odd prime power dimensions of the validity of the identities (13) and (14) because the automatisms of computation are nearly equivalent. It is important to note however that in even prime power dimensions the expressions of the type \( \left( \gamma_G^{(a \circ_G a)} \right)^{\frac{1}{2}} \) do well represent square roots of \( \gamma_G^{(a \circ_G a)} \) but must be considered as functions that depend on the \( 2^m \) variables \( a \) instead of only two variables, as would be the case if we considered literally square roots of integer powers of \( \gamma_G \) (with \( \gamma_G = -1 \)). When \( a \) is specified, the sign of the square root is also specified, according to the explicit expression (31). The even and odd dimensional cases are covered by the synthetic expression (32).

4 Open questions, comments and conclusions.

4.1 Other symmetries

At first sight, the computational basis plays a special role in our approach, but one can show that, to some extent, all the mutually unbiased bases can be treated on the same footing. This can be seen as follows. Now that we have at our disposal an explicit expression (Eqns (25, 33)) for all the mutually unbiased bases, we can “reevaluate the situation from the point of view of one of them”, say the \( i \)th basis (with \( i \) different from zero). In order to do so, we can express the action of the operator \( V_n^m \) in terms of its basis states. After a straightforward computation, we get that

\[ V_n^m(0) = \text{phase}V_n^m\circ_G\delta_G(i-1)\circ_Gm(i), \]

where \( V_n^m(0) = \sum_{k=0}^{N-1} \gamma_G^{((k \circ_G m) \circ_G n)} |e_0^{k \circ_G m}\rangle\langle e_0^k| \) and \( V_n^m(i) = \sum_{k=0}^{N-1} \gamma_G^{((k \circ_G m) \circ_G n)} |e_i^{k \circ_G m}\rangle\langle e_i^k| : i : 1...N. \) These relations (that we give without proof but are easy to derive from Eqns (25, 33)) are bijective. So the whole discrete Heisenberg-Weyl group is invariant (up to permutations and phase shifts) when we reexpress it in any of the \( N + 1 \) mutually unbiased bases. We shall not develop this question here, but this property has important implications in the theory of cloning machines, in relation with error operators and optimal cloning (9, 14). In prime dimensions, the invariance of the Heisenberg-Weyl group under conjugation by any unitary matrix that maps the computational basis onto a mutually unbiased basis is a basic property of a larger group that is known as the Jacobi or Clifford
group and possesses many applications in number theory and quantum computing [7].

Beside, there exists a one to one correspondence between generalised Bell states [19] and the Heisenberg-Weyl group. The properties of invariance of the Bell states in mutually unbiased bases appeared to be very useful in the resolution of the so-called mean king problem [10, 16, 17], where it also led to a compact and elegant expression valid in all prime power dimensions [18].

4.2 Connection with previous works.

The expression of the states of the mutually unbiased bases that we derived in the present paper \(|e_k^i\rangle = \frac{1}{\sqrt{N}} \sum_{q=0}^{N-1} \gamma_G^{\oplus \circ q \circ \gamma k} (\gamma_G^{((i-1)\circ q \circ \gamma q)})^{-\frac{1}{2}} |e_0^0\rangle\) is actually equivalent to the solution derived by Ivanovic [1] when the dimension is an odd prime (which can be shown, when rewritten according to our conventions, to be equivalent to the expression \(|e_k^i\rangle_{Ivan.} = \frac{1}{\sqrt{N}} \sum_{q=0}^{N-1} \gamma_G^{\oplus \circ q \circ \gamma k} (\gamma_G^{((i-1)\circ q \circ \gamma q)}) |e_0^0\rangle\). Our expression differs by a factor \(1/G^2\) in one of the exponents of \(\gamma_G\). When the dimension is prime and odd, it is easy to compensate the difference by a relabelling of the basis states, because the division by 2 is a permutation of the finite fields with \(p\) elements when \(p\) is a prime odd number, but contrary to Ivanovic’s expression, our expression is easily generalized in even prime dimension 2 (the qubit case), in which case we rederive the eigen bases of the Pauli operators, and in prime power dimensions.

As it was shown by Wootters and Fields [2], the generalisation in prime power dimensions of Ivanovic’s expression is the following:

\[ |e_k^i\rangle = \frac{1}{\sqrt{N}} \sum_{q=0}^{N-1} \gamma_G^{Tr.} (\gamma_G^{(\circ q \circ \gamma k)}) (\gamma_G^{Tr.} (\circ q \circ \gamma q)) |e_0^0\rangle, \]  

(36)

where \(Tr.\) represents the field theoretical trace. Our expression seems to be a bit simpler, but both expressions require to know the addition and multiplication tables of the field, so that the apparent gain in simplicity is relative. Moreover, both expressions are equivalent up to a relabelling in odd prime power dimensions as we shall now show. We could establish the equivalence at once but we prefer to base our derivation on the results of the reference [3] where the interrelation between the Pauli group approach and the expression of Wootters and Fields with the trace factor is made (section 4.3., [3]). Actually, there exist also different groups that present properties similar to those of the generalized Pauli group [12] but do not
obey the definition that we gave here. Nevertheless, the generators of the two subgroups corresponding to the computational and the dual basis that are given in the reference \[3\] coincide with our choice (the subgroups $V_0^l$ and $V_l^0$ correspond to the classes $C_0$ and $C_1$ studied in \[2\]) so that we are talking exactly about the same group, up to phases. As, in the same paper, a relation was established with the solution of Wootters and Fields \[2\], our expression for mutually unbiased bases must necessarily coincide with the expression derived by Wootters and Fields. In the reference \[3\], it is shown that when there exists a maximal commuting basis of orthogonal unitary matrices, the $N + 1$ bases that diagonalize these classes are unambiguously defined and, moreover, are mutually unbiased. A maximal commuting basis of orthogonal unitary matrices is a set of $N + 1$ sets of $N − 1$ commuting unitary operators (or classes) plus the identity such that these $N^2$ operators are orthogonal regarding the in-product induced by the (usual operator) trace denoted $tr$. It is easy to show that the $V$ operators defined in Eqn.8 are unitary with $(V_l^j)^+ = (V_l^j)^-1 = \gamma_G^{(i\otimes_G j)/2}V_{\otimes_G l}^{i\otimes_G l}V_{\otimes_G l}^{i\otimes_G l}$. Making use of the composition law \[12\] we obtain the relation $tr.(V_l^j)^+V_l^k = N.\delta_{l,0}.\delta_{j,0}$. This theorem suggests another way to derive an expression for the mutually unbiased bases: it is sufficient to find the common eigenstates of the classes of operators $V_l^{(i-1)\otimes_G l}$ (where $l$ varies from 0 to $N − 1$) in order to determine the value of the states of the $i$th mutually unbiased basis. When the dimension is an odd (even) prime power, one can check by direct substitution of the expression \[25\] that the states $|e_k^l\rangle$ are common eigenstates of the $i$th class:

\[
V_l^{(i-1)\otimes_G l}|e_k^l\rangle = \sum_{k'=0}^{N-1} \gamma_G^{((k'\otimes_G l)\otimes_G (i-1)\otimes_G l)}|k'\otimes_G l\rangle\langle k| \frac{1}{\sqrt{N}} \sum_{q=0}^{N-1} \gamma_G^{q\otimes_G c_k}\gamma_G^{((i-1)\otimes_G q\otimes_G q)/2}|e_q^l\rangle
\]

\[
= \frac{1}{\sqrt{N}} \sum_{q=0}^{N-1} \gamma_G^{((q\otimes_G l)\otimes_G (i-1)\otimes_G l) q\otimes_G c_k}\gamma_G^{((i-1)\otimes_G q\otimes_G q)/2}|e_q^l\rangle
\]

\[= \gamma_G^{((i\otimes_G c_k)\otimes_G (i-1)\otimes_G l)/2} \frac{1}{\sqrt{N}} \sum_{q=0}^{N-1} \gamma_G^{(q\otimes_G l)\otimes_G c_k}\gamma_G^{((i-1)\otimes_G q\otimes_G l)/2}|e_q^l\rangle(37)\]

Thanks to the product law \[8\] the proof is entirely similar in even prime power dimensions.

In order to establish explicitly and once for all the equivalence between the expression of Wootters and Fields \[36\] and ours \[25\], some work remains to be done because in the reference \[3\] no proof is given of the fact that the expression \[36\] represents eigenstates of the generalised Pauli operators. In order to prove this
result, it is useful to introduce two (field theoretical) dual bases: the first one, which is dual relatively to the trace contains \( m \) elements \( \tilde{p} \) of the field such that 
\[
Tr. p^i \odot_G \tilde{p}^j = \delta_{i,j} \text{;} \quad \text{the second one which is dual relatively to the rest after division by } p \text{ contains } m \text{ elements } \tilde{p}^j \text{ of the field such that } (p^i \odot_G \tilde{p}^j)_0 = \delta_{i,j}, \text{ where } (q)_0 \text{ represents (we work in dimension } p^m) \text{ the rest after division of } q \text{ by } p.\]

These bases can be shown to exist and to be unique, in virtue of the fact that the bilinear forms \( Tr.(x \odot_G y) \) and \( (x \odot_G y)_0 \) are non-singular \[32\], a direct consequence of the identity \[1\]. Beside, the \( m \) generators of the \( k \text{th class} \) considered in \[3\] are equal to \( X^i \Pi_{l=0}^{m-1}(\Pi_{j=0}^{m-1}(Z^j)^{b_{ij}})^{k_l} \), with \( j, l : 0...m-1 \text{.} \quad \text{In the previous expression, the coefficients } k_l \text{ are unambiguously defined by the } p\text{-ary expansion of } k: k = \sum_{i=0}^{m-1} k_ip^i \text{ while the multiplication matrix } b \text{ is defined as follows: } \gamma_i \odot_G \gamma_j = \sum_{l=0}^{m-1} b_{ij} \gamma_l, \text{ where the } \gamma \text{'s are a basis of the field (here we shall consider without loss of generality that } \gamma_0 = p_0.\) The operators \( X^i \) are “local” operators that shift the \( i \text{th component} \) of the label of the \( k \text{th class} \) state \( |e_k^0 \rangle \) (with \( k = \sum_{i=0}^{m-1} k_ip^i \)) by unity (modulo \( p \)):
\[
X^i|e_k^0 \rangle = |e_{k^l}^0 \rangle \text{ with } k^l_i = k_i + \text{mod}_p 1, k^l_i = k_i \text{ when } l \neq i. \text{ The operator } Z^j \text{ multiplies the state } |e_k^0 \rangle \text{ by a global phase equal to } \gamma_{G}^{k_j}. \text{ In our approach, the generators of the } k' \text{th class can be shown to be the same, provided the coefficients of } k' \text{ expressed in the double-tilded dual basis } \tilde{p}^j \text{ defined here above are the same as those of } k \text{ in the direct basis } p: k = \sum_{l=0}^{m-1} k_l p^l \text{ and } k' = \sum_{j=0}^{m-1} k_j \tilde{p}^j. \text{ Beside, the expressions with and without Trace \[28\] and \[30\] are equivalent, up to a bijective relabelling, in virtue of the following identity: } \text{Tr.}(r \odot_G k) = ((r'/G2) \odot_G k)_0 \text{ with } k \text{ and } r \text{ arbitrary elements of the Galois field with } N = p^m \text{ elements, } r = \sum_{i=0}^{m-1} r_i p^i \text{ and } r' = \sum_{i=0}^{m-1} r_i \tilde{p}^i \odot_G 2.\]

This comparison emphasises the difference between our approach and previous approaches: our expression \[8\] of the generalised Pauli operators is global and non-local, although they can be decomposed as products of local operators. It also shows that the field theoretical trace is replaced in our approach by another non-singular bilinear form, the rest after division by \( p.\)

Although it is out of the scope of the present paper, it would be interesting to understand the relation between our results in even prime power dimensions and the results presented in references \[2\] \[15\].

### 4.3 Other dimensions.

It is still an open question to know whether maximal sets of mutually unbiased bases exist in arbitrary dimensions. For instance in dimension 6 which is the smallest dimension that is not a power of a prime, nobody knows whether or not such
a maximal set exists [6, 7]. It is not possible to apply our treatment in this case because no finite field with 6 elements exists. We could try to repeat the procedure with operations that do not form a field; for instance we could try to find a distributive ring with 6 elements (such a ring obeys the same definition as a field (the definition that was given at the beginning of the paper), expected that the multiplication needs not be invertible-dividers of zero different from zero are allowed). One can show that there is only one distributive ring with 6 elements, that corresponds to the usual operations (multiplication and addition modulo 6). If we study the structure of the \( N^2 = 36 \) Heisenberg-Weyl unitary transformations in that case, we find that there are more than \( N + 1 = 7 \) subgroups of 6 elements (5+the identity). This is because, as a consequence of the non-invertibility of the multiplication modulo 6 (3 and 2 divide zero), certain operators present degeneracies and belong simultaneously to different subgroups (a treatment of similar type is given in detail in the reference [19] for the case \( N = 4 \)). The bases that diagonalize these operators are not mutually unbiased in general and the construction that was successfully applied in prime power dimensions does not provide a maximal set of mutually unbiased bases. Therefore the question of the existence of 7 mutually unbiased bases in a 6 dimensional Hilbert space is still open, and our approach does unfortunately not contribute to the elucidation of that problem.

### 4.4 Conclusions

As we already mentioned, there is a one to one correspondence between (generalised) Bell states and (generalised) Pauli operators [19] (see also [33] for a different approach based on additive and multiplicative characters of the Galois field). It can also be proven [18, 19] that the Bell states are invariant when we pass from one of the mutually unbiased bases to another one, an important result in the theory of cloning machines that was only conjectured until now [14]. Actually, the present results were largely inspired by results that we obtained in the framework of quantum cryptography [9] where the interest of mutually unbiased has been recognised several years ago, for what concerns encryption [20, 21, 22] and cloning as well [9, 14, 23].

It is worth noting that, beside quantum cryptography and quantum cloning, the Bell states found also many applications in quantum teleportation and dense coding and the connections between mutually unbiased bases, complete orthogonal families of unitary matrices, and teleportation, were already emphasised in the past [24, 25]. There exists also an impressive literature about the interrelation between finite fields and discrete Wigner representation [27, 28, 29]. It is worth noting that
if we perform a tomographic development of the density matrix in the basis of the
$V$ operators, we obtain \[19\] the following identity: $\rho = \frac{1}{N} \sum_{k,l} V^k_l \text{Tr}((V^k_l)^+ \rho)$.

It is very instructive to compare the amplitudes of the decomposition of a density
matrix into the $V$ basis with the Wigner function:

$$
Tr.(V^k_l)^+ \rho = \text{Tr}.(V^k_l)^+ \sum_{i,j=0}^{N-1} \rho_{i \oplus j, i} |e^0_i \rangle \langle e^0_i| = \sum_{i,j=0}^{N-1} \rho_{i \oplus j, i} \gamma^{\oplus(i \oplus l) \oplus k} \delta_{i,j} = 
$$

$$
\gamma^{\oplus(l \oplus k) \sum_{j=0}^{N-1} \rho_{j \oplus l, j} \gamma^{\oplus j \oplus k}}.
$$

The Wigner function can be written as follows \[26\] in terms of the conjugate
continuous variables $q$ and $p$:

$$
W(q, p) = C. \int d^3 r \rho(q-r, q+r)e^{2i.p.r/h} = C'. \int d^3 r' \rho(r', 2q-r')e^{-i.2p.r'/h} \text{ where } C \text{ is a normalisation constant, while } i \text{ is the square root of -1.}
$$

The analogy of both
expressions is striking and, as we can see, the tomography of a quantum state that we
realize in the Pauli group approach provides a discrete counterpart of the Wigner
representation. In general the coefficients $Tr.(V^k_l)^+ \rho$ are complex, which does
not meet the requirements of a properly discretized Wigner function, but in even
prime power dimensions this difficulty can be overcomen because the $V$ operators
are Hermitian up to a global phase that was defined in Eqn.31 (the $U$ operators
defined by Eqn.30 are Hermitian and unitary and also provide an orthogonal basis).

It is out of the scope of the present paper but it would be interesting to study the
connection with other proposals for discrete phase-space representation \[27, 28, 29\].

Note that as the $V$ operators are diagonal in the $N + 1$ mutually unbiased bases,
full tomography can be obtained by performing $N + 1$ von Neumann measurements,
as was already shown by Ivanovic in prime dimensions and Wootters and Fields in
prime power dimensions.

Finally, the properties of Bell states are also directly related to the error opera-
tors \[30, 31, 29\], and it would be worth investigating to which extent our formalism
contributes to a simplification of the theory of error correcting codes, in prime power
dimensions.

To conclude, we note that, despite of the fact that the problem (and its solutions)
seem to be regularly rediscovered by different generations of physicists, which means
also a lack of time and energy, our results about the Mean King’s problem \[18\]
confirm that it is important to explore alternative approaches in the treatment of
the question of mutually unbiased bases.

We wish that the present paper contributes to a deeper understanding of the
old problem of mutually unbiased bases.
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Appendix: Field and modulo $N$ operations for $N = 4$. 

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| ⊗₇ | 0 | 1 | 2 | 3 |
|-----|---|---|---|---|
| 0   | 0 | 0 | 0 | 0 |
| 1   | 0 | 1 | 2 | 3 |
| 2   | 0 | 2 | 3 | 1 |
| 3   | 0 | 3 | 1 | 2 |

Table 1: The field multiplication in dimension 4.

| ⊕₇ | 0 | 1 | 2 | 3 |
|-----|---|---|---|---|
| 0   | 0 | 1 | 2 | 3 |
| 1   | 1 | 0 | 3 | 2 |
| 2   | 2 | 3 | 0 | 1 |
| 3   | 3 | 2 | 1 | 0 |

Table 2: The field addition in dimension 4.

| .mod₄ | 0 | 1 | 2 | 3 |
|-------|---|---|---|---|
| 0     | 0 | 0 | 0 | 0 |
| 1     | 0 | 1 | 2 | 3 |
| 2     | 0 | 2 | 0 | 2 |
| 3     | 0 | 3 | 2 | 1 |

Table 3: The multiplication modulo 4.

| +mod₄ | 0 | 1 | 2 | 3 |
|-------|---|---|---|---|
| 0     | 0 | 1 | 2 | 3 |
| 1     | 1 | 2 | 3 | 0 |
| 2     | 2 | 3 | 0 | 1 |
| 3     | 3 | 0 | 1 | 2 |

Table 4: The addition modulo 4.