LOCAL MULTIPLICITY OF CONTINUOUS MAPS BETWEEN MANIFOLDS

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Abstract. Let $M$ and $N$ be smooth (real or complex) manifolds, and let $M$ be equipped with some Riemannian metric. A continuous map $f: M \to N$ admits a local $k$-multiplicity if, for every real number $\omega > 0$, there exist $k$ pairwise distinct points $x_1, \ldots, x_k$ in $M$ such that $f(x_1) = \cdots = f(x_k)$ and $\text{diam}\{x_1, \ldots, x_k\} < \omega$. In this paper we systematically study the existence of local $k$-multiplicities and derive criteria for the existence of local $k$-multiplicity in terms of Stiefel–Whitney classes and Chern classes of the vector bundle $f^*\tau N \oplus (-\tau M)$. For example, as a corollary of one criterion we deduce that for $k \geq 2$ a power of 2, $M$ a compact smooth manifold with the integer $s := \max\{\ell : \overline{w}_\ell(M) \neq 0\}$, and $N$ a parallelizable smooth manifold, if $s \geq \dim N - \dim M + 1$ and $\overline{w}_s(M)k^{s-1} \neq 0$, any continuous map $M \to N$ admits a local $k$-multiplicity. Furthermore, as a special case of this corollary we recover, when $k = 2$, the classical criterion for the non-existence of an immersion $M \ni N$ between manifolds $M$ and $N$.

1. Introduction

Let $M$ and $N$ be smooth manifolds, and let $k \geq 2$ be an integer. A continuous map $f: M \to N$ admits a $k$-multiplicity if there exist $k$ pairwise distinct points $x_1, \ldots, x_k$ on $M$ such that $f(x_1) = \cdots = f(x_k)$. For example, a continuous map $f$ that admits a 2-multiplicity is not a (topological) embedding. An interesting result of Gromov on $k > 2$ multiplicities [12, p. 447] shows that for every $m$-dimensional manifold $M$ there exists a smooth map $M \to \mathbb{R}^m$ that does not admit $k$-multiplicity for $k \geq 4m + 1$.

Let us in addition assume that the manifold $M$ is equipped with some Riemannian metric. A continuous map $f: M \to N$ admits a local $k$-multiplicity if, for every real number $\omega > 0$, there exist $k$ pairwise distinct points $x_1, \ldots, x_k$ in $M$ such that $f(x_1) = \cdots = f(x_k)$ and $\text{diam}\{x_1, \ldots, x_k\} < \omega$. For example, the map $f: \mathbb{C} \to \mathbb{C}$ given by $f(z) = z^k$ admits a local $k$-multiplicity. Existence of a local $k$-multiplicity for the map $f$ implies the existence of a $k$-multiplicity for the same map. In the case $k = 2$ if a smooth map $f: M \to N$ admits a local 2-multiplicity, then $f$ is not an immersion. The property of being an immersion is of course stronger than the property of having no local 2-multiplicity. Although the notion of the local multiplicity for continuous maps is a natural extension of the non-immersibility property for smooth maps, it has not been not studied systematically before. In this paper we develop, and apply, a topological

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framework to study the existence of local multiplicity of continuous maps between real or complex manifolds.

1.1. The statements of the main results. The central results of this paper are the following two theorems that, for a given continuous map, give a cohomological criterion for the existence of local multiplicity.

**Theorem 1.1.** Let $k \geq 2$ be a power of 2, let $M$ be a compact smooth manifold, let $N$ be a smooth manifold, and let $f: M \to N$ be a continuous map. Denote by $w_i := w_i(f^*\tau N \oplus (-\tau M))$ the $i$-th Stiefel–Whitney class of the vector bundle $f^*\tau N \oplus (-\tau M)$ for $i \geq 0$, and $w_i = 0$ for $i < 0$.

If there exists an integer $s \geq 0$ such that the characteristic class

$$u_s(f^*\tau N \oplus (-\tau M)) := \det(w_{\dim N - \dim M + 1 + i + j})_{i,j \leq k-1}$$

does not vanish, then the continuous map $f$ admits a local $k$-multiplicity.

In the case when both manifolds $M$ and $N$ allow almost complex structure an additional criterion can be used.

**Theorem 1.2.** Let $k \geq 2$ be an odd prime, let $M$ be a compact smooth almost complex manifold, and let $f: M \to N$ be a continuous map. Furthermore, let us denote by $c_i := c_i(f^*\tau N \oplus (-\tau M))$ the $i$-th Chern class mod $k$ of the complex vector bundle $f^*\tau N \oplus (-\tau M)$ for $i \geq 0$, and $c_i = 0$ for $i < 0$. If there exists an integer $s \geq 0$ such that the characteristic class

$$v_s(f^*\tau N \oplus (-\tau M)) := \det(c_{\dim N - \dim M + 1 + i + j})_{i,j \leq k-1}$$

does not vanish, then the continuous map $f$ admits a local $k$-multiplicity.

In both theorems $-\tau M$ denotes the inverse (real or complex) vector bundle of the tangent vector bundle $\tau M$.

A special case of Theorem 1.1 is the following result.

**Theorem 1.3.** Let $k \geq 2$ be a power of 2, let $M$ be a compact smooth manifold with $s := \max\{\ell : \tilde{w}_\ell(M) \neq 0\}$, and let $N$ be a parallelizable smooth manifold. If $s \geq \dim N - \dim M + 1$ and $\tilde{w}_s(M)^{k-1} \neq 0$, then any continuous map $M \to N$ admits a local $k$-multiplicity.

Here $\tilde{w}_\ell(M)$ denotes the dual $i$-th Stiefel–Whitney class of the tangent vector bundle $\tau M$. An assumption that the manifold $N$ is parallelizable in Theorem 1.3 can be weakened and we can assume that $w(f^*\tau N) = 1$ instead. Moreover, in the case when $k = 2$ the statement of Theorem 1.3 yields the classical obstruction for the non-existence of an immersion $M \to N$ between manifolds $M$ and $N$, see for example [17, Cor. 3.5]

A similar consequence of Theorem 1.2 can be derived in the case, when $k$ is an odd prime.

**Theorem 1.4.** Let $k \geq 2$ be an odd prime, let $M$ be a compact smooth almost complex manifold with $s := \max\{\ell : \tilde{c}_\ell(M) \neq 0\}$, let $N$ be a parallelizable smooth complex manifold, and let $f: M \to N$ be a continuous map. If $s \geq \dim N - \dim M + 1$ and $\tilde{c}_s(M)^{k-1} \neq 0$, then any continuous map $M \to N$ admits a local $k$-multiplicity.

Here $\tilde{c}_\ell(M)$ denotes the $i$-th Chern class mod $k$ of the inverse of the complex tangent vector bundle $\tau M$, and not the $i$-th Chern class of the dual complex vector bundle.

Using well known facts about the Stiefel–Whitney classes of tangent bundles of projective spaces we derive following corollaries of Theorem 1.3.
Corollary 1.5. Let \( a \geq 1 \) and \( \ell \geq 1 \) be integers, let \( k \geq 2 \) be a power of 2, and let \( k(a + 1) \leq 2^\ell - 1 \). Then any continuous map
\[
\mathbb{R}P^{2^\ell - 2 - a} \rightarrow \mathbb{R}^{2^\ell - 2}
\]
admits a local \( k \)-multiplicity.

For \( k = 2 \) this corollary recovers, and slightly extends, the result of Milnor [16] from 1957 and implies that there is no immersion \( \mathbb{R}P^{2^\ell - 1} \rightarrow \mathbb{R}^{2^\ell - 2} \), or more precisely that there cannot exist a continuous map \( \mathbb{R}P^{2^\ell - 1} \rightarrow \mathbb{R}^{2^\ell - 2} \) that does not admit a local 2-multiplicity.

The next corollary is a consequence of Theorem 1.1 rather than Theorem 1.3 even the proof is almost identical to the proof of Corollary 1.5.

Corollary 1.6. Let \( a \geq 1 \) and \( \ell \geq 1 \) be integers, let \( k \geq 2 \) be a power of 2, and let \( k(a + 1) \leq 2^\ell - 1 \). Then any continuous map
\[
\mathbb{R}P^{2^\ell - 2 - a} \rightarrow S^{2^\ell - 2}
\]
admits a local \( k \)-multiplicity.

The next consequence is obtained via direct application of Theorem 1.3.

Corollary 1.7. Let \( a \geq 1 \) and \( \ell \geq 1 \) be integers, let \( k \geq 2 \) be a power of 2, and let \( k(a - 1) \leq 2^\ell - 1 \). Then any continuous map
\[
\mathbb{C}P^{2^\ell - a} \rightarrow \mathbb{R}^{2^{\ell+1} - 3}
\]
admits a local \( k \)-multiplicity.

Using the knowledge on Chern classes of tangent bundles of the complex projective spaces we get the following corollary of Theorem 1.4.

Corollary 1.8. Let \( a \geq 1 \) and \( \ell \geq 1 \) be integers, let \( k \) be an odd prime, and let \( 2 \leq a \leq \frac{k^\ell + 1}{k^\ell - 1} \). If \( k(a - 1) \leq k^\ell - 1 \) then any continuous map
\[
\mathbb{C}P^{k^\ell - a} \rightarrow \mathbb{C}^{k^\ell - 2}
\]
admits a local \( k \)-multiplicity.

The previous results motivates many natural questions. We state the first and most obvious one that, in the case of \( k = 2 \), extends the well known problem of the existence of an immersion between smooth manifolds.

Question 1.9. Let \( M \) be a smooth compact manifold of dimension \( m \), and let \( k \geq 2 \) be an integer. What is the minimal dimension \( n \) such that there exists a continuous map \( f : M \rightarrow \mathbb{R}^n \) such that \( f \) does not admit a local \( k \)-multiplicity?

It is our belief that in the process of answering this or similar questions about local multiplicities many new fascinating ideas will come to life as was the case when the immersion conjecture was studied by Brown and Peterson [8] and resolved by Ralph Cohen [10] during the 1970s and 1980s.

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1.2. Geometric applications. The concept of $k$-multiplicity, as also the concept of local $k$-multiplicity, are natural topological properties of a continuous map. After studying these properties from a topological point of view an immediate question arises: Can the existence of $k$-multiplicity or local $k$-multiplicity of a continuous map be used in solving problems outside the obvious realm of topology? In the following, using Corollary 1.6, we demonstrate how to obtain lower bounds for several questions in convex geometry.

In 1963 Branko Grünbaum [13, Sec. 6.5] posed many interesting problems. We consider the following two problems.

Let $K \subset \mathbb{R}^m$ be a convex body, that is a convex compact subset of $\mathbb{R}^m$ with non-empty interior. An affine diameter of a convex body $K$ in direction $\ell \in \mathbb{R}P^{m-1}$ is any affine line $\ell + v$ for $v \in \mathbb{R}^m$ with the property that

$$\text{length}(K \cap (\ell + v)) = \max \{\text{length}(K \cap (\ell + u)) : y \in \mathbb{R}^m\}.$$ 

Here length denotes the length of an interval. In general an affine diameter in direction $\ell \in \mathbb{R}P^{m-1}$ is not unique. For example, if $K$ is a square in $\mathbb{R}^2$, then in any direction parallel to one of the edges there are infinitely many affine diameters. On the other hand, if $K$ is a strictly convex body then in any direction there is a unique affine diameter.

The first question of Grünbaum we consider asks for the number of affine diameters of a convex body intersecting in a single point.

**Question 1.10.** Let $K \subset \mathbb{R}^m$ be a convex body. Is there at least $m+1$ pairwise distinct affine diameters $\ell_1 + v_1, \ldots, \ell_{m+1} + v_{m+1}$ of $K$ such that

$$\emptyset \neq (\ell_1 + v_1) \cap \cdots \cap (\ell_{m+1} + v_{m+1}) \cap \text{int } K?$$

This problem was studied intensively by many authors that employed diverse methods in addressing this question, see for example the work of Bárany et al., [2], [1], and for survey of known result [18]. Here we relate the number of pairwise distinct affine diameters intersecting in a single point inside a strictly convex body with a multiplicity of a continuous map $\mathbb{R}P^m \to S^m$.

**Theorem 1.11.** Let $K \subset \mathbb{R}^m$ be a strictly convex body. There exists a continuous map $f_K: \mathbb{R}P^m \to S^m$ with the property that if $f_K$ admits a $k$-multiplicity, then there exist $k$ pairwise distinct affine diameters $\ell_1 + v_1, \ldots, \ell_k + v_k$ of $K$ such that

$$\emptyset \neq (\ell_1 + v_1) \cap \cdots \cap (\ell_k + v_k) \cap \text{int } K.$$

**Proof.** The space of all affine lines in $\mathbb{R}^m$ can be identified with the total space $E(\gamma_m^{m-1})$ of the tautological bundle $\gamma_m^{m-1}$ over the Grassmann manifold $G_{m-1}(\mathbb{R}^m)$ of all $(m-1)$-dimensional vector subspaces of $\mathbb{R}^m$. Indeed, let $w \in S^{m-1}$ be a unit vector and let $\ell_w = \text{span}\{w\} \in \mathbb{R}P^{m-1}$ be the corresponding 1-dimensional vector.
subspace of $\mathbb{R}^m$. Let $v \in \mathbb{R}^m$. Then the correspondence between affine lines and points in $E(\gamma_m^{-1})$ is given by

$$\ell + v \longleftrightarrow (\ell^\perp, v - (w_\ell \cdot v) w_\ell).$$

Here "\perp" denotes the standard inner product in $\mathbb{R}^m$, while $\ell^\perp$ the orthogonal complement of $\ell$. Furthermore, recall that the projective space $\mathbb{RP}^m$ can be identified with the Grassmann manifold $G_{m-1}(\mathbb{R}^m)$ via the homeomorphism

$$\mathbb{RP}^m \ni \ell \longleftrightarrow \ell^\perp \in G_{m-1}(\mathbb{R}^m). \quad (1)$$

The convex body $K$ is strictly convex and therefore for every $\ell \in \mathbb{RP}^m$ there exists a unique affine diameter in direction $\ell$. The choice of an affine diameter of the strictly convex body $K$ defines a continuous map $s_K : \mathbb{RP}^m \rightarrow E(\gamma_m^{-1})$ from the space of all directions to the space of all affine lines in $\mathbb{R}^m$. If the projective space $\mathbb{RP}^m$ is identified with the Grassmann manifold $G_{m-1}(\mathbb{R}^m)$ via (1), then the function $s_K$ becomes a section of the tautological bundle $\gamma_m^{-1}$.

Next we consider the vector bundle $\gamma_m^{-1} \oplus (\gamma_m^{-1})^\perp$ over $G_{m-1}(\mathbb{R}^m)$, i.e., over $\mathbb{RP}^m$ via the identification (1). The subspace of the total space

$$X := \{(\ell, s_K(\ell) \oplus u) \in E(\gamma_m^{-1} \oplus (\gamma_m^{-1})^\perp) : \ell \in \mathbb{RP}^m, s_K(\ell) \in \ell^\perp, u \in \ell, u + s_K(\ell) \in \text{int}(K)\}$$

is homeomorphic to the total space of a disc bundle of the vector bundle $\gamma_m^1$ over $\mathbb{RP}^m$.

Let $g_K : X \rightarrow \text{int}(K)$ be a continuous map defined by

$$(\ell, s_K(\ell) \oplus u) \mapsto s_K(\ell) + u.$$ 

The map $g_K$ is a proper map and thus it induces a continuous map between one point compactifications $f_K : \tilde{X} \rightarrow \text{int}(\tilde{K})$. Since $\text{int}(K)$ is homeomorphic to an $m$-dimensional disc, $\text{int}(\tilde{K}) \approx S^m$. The one point compactification $\tilde{X}$ is homeomorphic to the Thom space of the vector bundle $\gamma_m^1$, and therefore it is homeomorphic to $\mathbb{RP}^m$, see [19, Prop. p.68]. The map $f_K$ has the desired property: If there exist $k$ pairwise distinct points $x_1, \ldots, x_k \in \tilde{X} \approx \mathbb{RP}^m$ such that $f_K(x_1) = \cdots = f_K(x_k)$ then

$$x_1 = (\ell_1, s_K(\ell_1) \oplus u_1) \in X, \ldots, x_k = (\ell_k, s_K(\ell_k) \oplus u_k) \in X,$$

and

$$s_K(\ell_1) + u_1 = \cdots = s_K(\ell_k) + u_k \in (\ell_1 + s_K(\ell_1)) \cap \cdots \cap (\ell_k + s_K(\ell_k)) \cap \text{int} K.$$ 

This concludes the proof of the theorem.

In the case when $m = 2^k - 2$ for $\ell \geq 2$ from Corollary 1.6 we get that: For every strictly convex body $K \subset \mathbb{R}^m$ there exist at least $k = 2^\ell - 1$ pairwise distinct affine diameters $\ell_1 + v_1, \ldots, \ell_k + v_k$ of $K$ such that

$$\emptyset \neq (\ell_1 + v_1) \cap \cdots \cap (\ell_k + v_k) \cap \text{int} K.$$ 

The second question indicated by Grünbaum in the same publication [13] is the following one. Let $K \subset \mathbb{R}^m$ be a convex body. For every affine hyperplane $H$ let, for example,

- $m(H)$ denote the center of mass of the convex body $K \cap H$ in $H$, and let
- $s(H)$ denote the Steiner point of $K \cap H$ in $H$, that is the center of mass of the Gaussian curvature of $\partial K \cap H$.

whenever $H \cap K$ is non-empty.
Question 1.12. What is the maximal number $k$ such that, for any strictly convex $K$, there exist $k$ pairwise distinct affine hyperplanes $H_1, \ldots, H_k$ whose centers of mass, or Steiner points coincide, that is

$$m(H_1) = \cdots = m(H_k), \quad \text{or} \quad s(H_1) = \cdots = s(H_k).$$

Relationship of a $k$-multiplicity and a coincidence of centers of mass or Steiner points is explained in the following theorem.

Theorem 1.13. Let $K \subset \mathbb{R}^m$ be a convex body. There exists a continuous map $f_K : \mathbb{R}P^m \rightarrow S^m$ such that if $f_K$ admits a $k$-multiplicity, then there exist $k$ pairwise distinct affine hyperplanes $H_1, \ldots, H_k$ such that

$$m(H_1) = \cdots = m(H_k) \quad \text{or} \quad s(H_1) = \cdots = s(H_k).$$

Proof. The space of all affine hyperplanes in $\mathbb{R}^m$ can be identified with the total space $E(\gamma_1^m)$ of the tautological bundle $\gamma_1^m$ over $\mathbb{R}P^{m-1}$. Indeed, the affine hyperplane $H$ corresponds to the point $(\ell, \ell \cap H) \in E(\gamma_1^m)$ where $\ell$ is the line through the origin perpendicular to $H$. Let us consider the space

$$Y := \{ H \in E(\gamma_1^m) : H \cap \text{int} \ K \neq \emptyset \}.$$

Similarly to the proof of Theorem 1.11, this is an open segment subbundle of $E(\gamma_1^m)$ and it can be identified with the whole $E(\gamma_1^m)$. The assignment of $m(H)$ to every $H \in Y$ determines a continuous proper map

$$g_K : Y \rightarrow \text{int} \ K,$$

which has an extension to the one-point compactifications to give

$$f_K : \mathbb{R}P^m \rightarrow S^m.$$  

The $k$-multiplicity of $f_K$ gives $k$ pairwise distinct hyperplanes with coinciding centers. \qed

For example, Corollary 1.6 implies that for $m = 2^\ell - 2$ and a convex body $K \in \mathbb{R}^m$ there exist $k = 2^\ell - 1$ pairwise distinct affine hyperplanes whose centers of mass, or Steiner points, coincide.

By restricting to hyperplanes passing through the origin the above argument works well to prove:

Theorem 1.14. Let $K \subset \mathbb{R}^m$ be a convex body such that $0 \in \text{int} \ K$. There exists a continuous map $f_K : \mathbb{R}P^{m-1} \rightarrow \mathbb{R}^m$ such that if $f_K$ admits a $k$-multiplicity, then there exist $k$ pairwise distinct linear hyperplanes $H_1, \ldots, H_k$ such that

$$m(H_1) = \cdots = m(H_k),$$
or the same with $s(H_i)$.

An example of an explicit bound follows from Corollary 1.6 for $m = 2^k - 2$ and $k = 2^{l-2}$. Then for any convex body $K \subset \mathbb{R}^m$ there exist $k$ pairwise distinct linear hyperplanes $H_1, \ldots, H_k$ whose centers of mass coincide.

2. Multiplicity of fiberwise maps between vector bundles

In this section we introduce and study the notion of $k$-multiplicity of a fiberwise map between vector bundles. Then we derive a criterion, which guarantees that, for an integer $k \geq 2$ and any two vector bundles over the same base space, any continuous fiberwise map between them admits a $k$-multiplicity.

Let $\xi$ be a vector bundle over $X$. Then $E(\xi)$ denotes the total space of $\xi$, $p_\xi : E(\xi) \to X$ denotes the corresponding projection map, and $F_\xi(\xi)$ stands for the fiber of $\xi$ over the point $x \in X$.

Let $X$ be a topological space with the homotopy type of a finite CW-complex, and let $k \geq 2$ be an integer. Consider vector bundles $\xi$ and $\eta$ over $X$, and a continuous fiberwise map $\Phi : \xi \to \eta$ between them. The map $\Phi$ admits a $k$-multiplicity if there exists a point $x \in X$ in the base space and $k$ pairwise distinct vectors $v_1, \ldots, v_k$ in $F_\xi(\xi)$, the fiber of $\xi$ over $x$, such that

$$\Phi(x, v_1) = \cdots = \Phi(x, v_k).$$

Here we abuse notation and instead of writing $\Phi(e)$ where $e \in E(\xi)$ we keep track of the fiber to which $e$ belongs to.

The fiberwise configuration space of the continuous map $p_\xi$ is a subspace of the configuration space $\text{Conf}(E(\xi), k)$ defined by

$$\text{Conf}_{p_\xi}(E(\xi), k) = \{(e_1, \ldots, e_k) \in \text{Conf}(E(\xi), k) : p_\xi(e_i) = p_\xi(e_j) \text{ for all } i, j\} = \{(x; v_1, \ldots, v_k) : x \in X \text{ and } (v_1, \ldots, v_k) \in \text{Conf}(F_\xi(\xi), k)\}.$$  

Here $\text{Conf}(E(\xi), k)$ denotes the classical configuration space of $k$ ordered pairwise distinct points in $E(\xi)$. The projection map $p_\xi$ of the vector bundle $\xi$ induces the following bundle

$$\text{Conf}_{p_\xi}(E(\xi), k) \to X, \quad (x; v_1, \ldots, v_k) \mapsto x.$$  

Any continuous fiberwise map $\Phi : \xi \to \eta$ gives rise of the following continuous fiberwise map

$$\text{Conf}_{p_\xi}(E(\xi), k) \xrightarrow{\Phi_{\oplus k}} E(\eta^{\oplus k})$$  

defined by

$$(x, v_1 \oplus \cdots \oplus v_k) \mapsto \Phi(x, v_1) \oplus \cdots \oplus \Phi(x, v_k).$$

This is a restriction of the continuous fiberwise map $\Phi^{\oplus k} : \xi^{\oplus k} \to \eta^{\oplus k}$. The fiberwise configuration space $\text{Conf}_{p_\xi}(E(\xi), k)$ as well as the total space $E(\eta^{\oplus k})$ are equipped with the obvious fiberwise $\mathfrak{S}_k$-actions in such a way that the fiberwise map $\Phi^{\oplus k}$ becomes an $\mathfrak{S}_k$-equivariant map. The action on $\text{Conf}_{p_\xi}(E(\xi), k)$ permutes $k$ pairwise distinct vectors in each fiber and therefore is free. Let $\mathbb{R}^{\oplus k}$ denote the trivial bundle over $X$ with fiber $\mathbb{R}^k$ equipped with the action of $\mathfrak{S}_k$ that permutes summands. Then $\eta^{\oplus k} \cong \eta \otimes_{\mathbb{R}} \mathbb{R}^{\oplus k}$, where the action on the tensor product is the diagonal action, assuming trivial action on $\eta$. 


The first step in the finding a useful criterion for the existence of a $k$-multiplicity of a continuous fiberwise map is the following stabilization lemma.

**Lemma 2.1.** Let $\xi$, $\zeta$ and $\eta$ be vector bundles over the space $X$. The continuous fiberwise map $\Phi: \xi \longrightarrow \eta$ admits a $k$-multiplicity if and only if the continuous fiberwise map $\Phi \oplus \text{Id}: \xi \oplus \zeta \longrightarrow \eta \oplus \zeta$ also admits a $k$-multiplicity.

Here $\text{Id}: \zeta \longrightarrow \zeta$ denotes the identity map that is also a fiberwise map.

**Proof.** (1) Let the fiberwise map $\Phi: \xi \longrightarrow \eta$ admit a $k$-multiplicity. Consequently, there exists $x_0 \in X$ and $(v_1, \ldots, v_k) \in \text{Conf}(F_{x_0}(\xi), k)$ such that

$$\Phi(x_0, v_1) = \cdots = \Phi(x_0, v_k).$$

Since, $(\Phi \oplus \text{Id})(x, v \oplus u) = (x, \Phi(x, v) \oplus u)$ for every $(x, v \oplus u) \in F_x(\xi \oplus \eta) \cong F_x(\xi) \oplus F_x(\eta)$, we get the following $k$-multiplicity of $\Phi \oplus \text{Id}$:

$$(\Phi \oplus \text{Id})(x_0, v_1 \oplus 0) = \cdots = (\Phi \oplus \text{Id})(x_0, v_\ell \oplus 0).$$

Thus, if $\Phi$ admits a $k$-multiplicity, then $\Phi \oplus \text{Id}$ also admits a $k$-multiplicity. (2) Now, let the fiberwise map $\Phi \oplus \text{Id}: \xi \oplus \zeta \longrightarrow \eta \oplus \zeta$ admit a $k$-multiplicity. Thus, there exists $x_0 \in X$ and $(v_1 \oplus u_1, \ldots, v_k \oplus u_k) \in \text{Conf}(F_{x_0}(\xi \oplus \zeta), k)$ such that

$$(\Phi \oplus \text{Id})(x_0, v_1 \oplus u_1) = \cdots = (\Phi \oplus \text{Id})(x_0, v_\ell \oplus u_\ell),$$

that is

$$(x_0, \Phi(x_0, v_1) \oplus u_1) = \cdots = (x_0, \Phi(x_0, v_\ell) \oplus u_\ell) \in F_{x_0}(\xi \oplus \zeta) \cong F_{x_0}(\xi) \oplus F_{x_0}(\zeta).$$

Consequently, $u_1 = \cdots = u_k$ and $\Phi(x_0, v_1) = \cdots = \Phi(x_0, v_\ell)$. Hence, if $\Phi \oplus \text{Id}$ admits a $k$-multiplicity, then $\Phi$ also admits a $k$-multiplicity. □

Instead of studying $k$-multiplicity of $\Phi: \xi \longrightarrow \eta$ directly we are going to use Lemma 2.1 and consider $k$-multiplicity of $\Phi \oplus \text{Id}: \xi \oplus \zeta \longrightarrow \eta \oplus \zeta$ where $\zeta$ is an inverse bundle of $\xi$, that means $\xi \oplus \zeta$ is a trivial vector bundle. When convenient we denote the bundle $\zeta$ by $-\xi$. In that case the fiberwise configuration space associated to the projection map $p_{\xi \oplus \zeta}: E(\xi \oplus \zeta) \longrightarrow X$ becomes a trivial bundle and decomposes as follows

$$\text{Conf}_{p_{\xi \oplus \zeta}}(E(\xi \oplus \zeta), k) \cong X \times \text{Conf}(\mathbb{R}^N, k),$$

where $N = \text{rank } \xi + \text{rank } \zeta$, and the projection map coincides with the projection on the first coordinate. The continuous fiberwise map (2) induced now by $\Phi \oplus \text{Id}$ has form

$$X \times \text{Conf}(\mathbb{R}^N, k) \cong \text{Conf}_{p_{\xi \oplus \zeta}}(E(\xi \oplus \zeta), k) \xrightarrow{(\Phi \oplus \text{Id})^\oplus k} E((\eta \oplus \zeta)^\oplus k) \quad (3)$$

where $p_1$ denotes the projection on the first factor. Typical fiber of the bundle $\text{Conf}_{p_{\xi \oplus \zeta}}(E(\xi \oplus \zeta), k)$ is homeomorphic to the configuration space $\text{Conf}(\mathbb{R}^N, k)$ and is equipped with the free action of the symmetric group $\mathfrak{S}_k$. If $T = \text{rank } \eta + \text{rank } \zeta$, then the fiber of $(\eta \oplus \zeta)^\oplus k$ is a real $\mathfrak{S}_k$-representation $(\mathbb{R}^T)^{\oplus k}$ where the action is given by permutation of factors in the $k$-fold direct sum. The actions on the fibers induce $\mathfrak{S}_k$-actions on $\text{Conf}_{p_{\xi \oplus \zeta}}(E(\xi \oplus \zeta), k)$ and $E((\eta \oplus \zeta)^\oplus k)$. Again, the fiberwise map $(\Phi \oplus \text{Id})^\oplus k$, as well as its fiberwise restrictions, are $\mathfrak{S}_k$-equivariant maps.

Next, consider vector bundle monomorphism $\Delta: \eta \oplus \zeta \longrightarrow (\eta \oplus \zeta)^{\oplus k}$, that is a continuous fiberwise map linear on each fiber, given by the diagonal embedding. It is an $\mathfrak{S}_k$-equivariant map, and its image $\alpha := \text{im } \Delta$ is an $\mathfrak{S}_k$-invariant vector subbundle of $(\eta \oplus \zeta)^{\oplus k}$. Let $\beta$ be the orthogonal complement of $\alpha$ in $(\eta \oplus \zeta)^{\oplus k}$. Hence, $(\eta \oplus \zeta)^{\oplus k} \cong \alpha \oplus \beta$, and $\beta$ is an $\mathfrak{S}_k$-invariant vector subbundle of $(\eta \oplus \zeta)^{\oplus k}$. 
On the level of the fibers this decomposition becomes \( (\mathbb{R}^T)^{\oplus k} \cong W_k^T \oplus \mathbb{R}^T \), where \( \mathbb{R}^T \) is a trivial \( \mathcal{S}_k \)-representation and \( W_k = \{(y_1, \ldots, y_k) \in \mathbb{R}^k : \sum y_i = 0\} \) is a subrepresentation of \( \mathbb{R}^k \) where the action is given by coordinate permutation. Observe that there exists an \( \mathcal{S}_k \)-equivariant isomorphism \( \beta \cong (\eta \oplus \zeta) \oplus \mathbb{R} W_k \) where, as before, \( W_k \) denotes a trivial vector bundle over \( X \) with fiber \( W_k \).

Furthermore, let \( \Pi \): \( (\eta \oplus \zeta)^{\oplus k} \cong \alpha \oplus \beta \longrightarrow \beta \) denote the vector bundle morphism given by the projection. The projection \( \Pi \) is an \( \mathcal{S}_k \)-equivariant map. Now consider the following composition \( \Pi \circ (\Phi \oplus \text{Id})^{\oplus k} \) of continuous fiberwise maps:

\[
\begin{array}{ccc}
X \times \text{Conf}(\mathbb{R}^N, k) \cong \text{Conf}_{p_{\mathbb{R}^N}}(E(\xi \oplus \zeta), k) & \xrightarrow{(\Phi \oplus \text{Id})^{\oplus k}} & E((\eta \oplus \zeta)^{\oplus k}) \\
\downarrow \quad \Pi & & \downarrow \\
& X & \\
\end{array}
\]

The composition of fiberwise maps \( \Pi \circ (\Phi \oplus \text{Id})^{\oplus k} \) is an \( \mathcal{S}_k \)-equivariant map with respect to the already defined actions. Moreover, it has the following important property: If there exists a point \( x_0 \in X \) such that the image of the fiber of the bundle \( \text{Conf}_{p_{\mathbb{R}^N}}(E(\xi \oplus \zeta), k) \) over \( x_0 \) along the fiberwise map \( \Pi \circ (\Phi \oplus \text{Id})^{\oplus k} \) contains zero of the fiber over \( x_0 \) of \( \beta \), then the continuous fiberwise map \( \Phi \oplus \text{Id} \), and consequently \( \Phi \), admits a \( k \)-multiplicity. Thus, if the image of every continuous \( \mathcal{S}_k \)-equivariant fiberwise map \( \text{Conf}_{p_{\mathbb{R}^N}}(E(\xi \oplus \zeta), k) \longrightarrow E(\beta) \) does not avoid the zero section of the vector bundle \( \beta \), then a \( k \)-multiplicity of any continuous fiberwise map \( \xi \longrightarrow \eta \) is guaranteed.

In order to translate this property in a more convenient language we consider the following pullback vector bundle:

\[
\begin{array}{ccc}
E(p_1^* \beta) & \xrightarrow{\Psi} & E(\beta) \\
\downarrow \quad \Pi \circ (\Phi \oplus \text{Id})^{\oplus k} & & \downarrow \\
X \times \text{Conf}(\mathbb{R}^N, k) \cong \text{Conf}_{p_{\mathbb{R}^N}}(E(\xi \oplus \zeta), k) & \xrightarrow{p_1} & X,
\end{array}
\]

where

\[
E(p_1^* \beta) = \{(x, (v_1, \ldots, v_k), u) \in X \times \text{Conf}(\mathbb{R}^N, k) \times E(\beta) : p_1(x, (v_1, \ldots, v_k)) = x = p_\beta(u)\},
\]

and the bundle morphism \( \Phi \) is given by \((x, (v_1, \ldots, v_k), u) \mapsto u\). The action of the symmetric group \( \mathcal{S}_k \) on \( E(p_1^* \beta) \) is given by the diagonal action on the product \( X \times \text{Conf}(\mathbb{R}^N, k) \times E(\beta) \), having in mind that the actions on \( X \) and \( E(\beta) \) are trivial.

The bundle morphism \( \Pi \circ (\Phi \oplus \text{Id})^{\oplus k} \) induces the \( \mathcal{S}_k \)-equivariant section of the pullback bundle \( s : \text{Conf}_{p_{\mathbb{R}^N}}(E(\xi \oplus \zeta), k) \longrightarrow E(p_1^* \beta) \) defined by

\[
(x, (v_1, \ldots, v_k)) \mapsto (x, (v_1, \ldots, v_k), (\Pi \circ (\Phi \oplus \text{Id})^{\oplus k})(x, (v_1, \ldots, v_k))).
\]

Now, the continuous \( \mathcal{S}_k \)-equivariant fiberwise map \( \Pi \circ (\Phi \oplus \text{Id})^{\oplus k} \) hits the zero section of the vector bundle \( \beta \) if and only if the \( \mathcal{S}_k \)-equivariant section \( s \) defined in (5) hits the corresponding zero section. Thus, the aim is to prove that every \( \mathcal{S}_k \)-equivariant section of the pullback bundle hits the zero section. Indeed, this would in particular imply that the section \( s \) hits the zero section, consequently \( \Pi \circ (\Phi \oplus \text{Id})^{\oplus k} \) hits the zero section and so the continuous fiberwise map \( \Phi \) admits a \( k \)-multiplicity.
The vector bundle $\beta$ is isomorphic to the tensor product $(\eta \oplus \zeta) \otimes_{\mathbb{R}} W_k$. Since the tensor product and the Whitney sum are compatible with the pullback we have
\[ E(p_1^*\beta) \cong E(p_1^*\eta \oplus p_1^*\zeta) \otimes_{\mathbb{R}} E(p_1^*W_k). \tag{6} \]

The action of $S_k$ on the total and on the base space of the pullback vector bundle is free. Moreover, the projection map of $p_1^*\beta$ is an $S_k$-equivariant map. Therefore, when dividing out the $S_k$ action we obtain the vector bundle
\[ E(p_1^*\beta)/S_k \rightarrow \text{Conf}_{p_1\otimes k}(E(\xi \oplus \zeta), k)/S_k. \tag{7} \]

Every $S_k$-equivariant section of the pullback bundle
\[ E(p_1^*\beta) \rightarrow \text{Conf}_{p_1\otimes k}(E(\xi \oplus \zeta), k) \]
induces a section of the quotient bundle
\[ E(p_1^*\beta)/S_k \rightarrow \text{Conf}_{p_1\otimes k}(E(\xi \oplus \zeta), k)/S_k. \]

Furthermore, presentation (6) implies the following presentation of the vector bundle (7) as a tensor product
\[ E(p_1^*\beta)/S_k \cong E(p_1^*\eta \oplus p_1^*\zeta)/S_k \otimes_{\mathbb{R}} E(p_1^*W_k)/S_k. \tag{8} \]

Note that the action of the symmetric group $S_k$ on $E(p_1^*\eta \oplus p_1^*\zeta) \cong E(p_1^*(\eta \oplus \zeta))$ is induced by the trivial action on $E(\eta \oplus \zeta)$ and by the diagonal action on the base space $X \times \text{Conf}(\mathbb{R}^N, k)$, assuming trivial action on $X$.

We have derived a criterion for the existence of a $k$-multiplicity for a fiberwise map between two vector bundles over the same base space. Assuming already introduced notation we formulate the following criterion.

**Theorem 2.2.** Let $X$ be a topological space with the homotopy type of a finite CW-complex, and let $k \geq 2$ be an integer. Let $\xi$ and $\eta$ be vector bundles over $X$. If the vector bundle
\[ E(p_1^*(\eta \oplus (-\xi)) \otimes_{\mathbb{R}} W_k)/S_k \cong E(p_1^*\eta \oplus p_1^*(-\xi))/S_k \otimes_{\mathbb{R}} E(p_1^*W_k)/S_k \rightarrow \text{Conf}_{p_1\otimes k}(E(\xi \oplus (-\xi)), k)/S_k \cong X \times \text{Conf}(\mathbb{R}^N, k)/S_k \]
do not admit a nowhere zero section, then any continuous fiberwise map $\xi \rightarrow \eta$ admits a $k$-multiplicity.

In the case of a fiberwise map between complex vector bundles the following analogous criterion for the existence of a $k$-multiplicity can be derived in the footsteps of the construction presented in this section. The only difference occurs in the place of the vector bundle $W_k$, which will be substituted by its complexification $W_k \otimes_{\mathbb{R}} \mathbb{C}$, which as a real vector bundle is isomorphic to $W_k \oplus 2$.

**Theorem 2.3.** Let $X$ be a topological space with the homotopy type of a finite CW-complex, and let $k \geq 2$ be an integer. Let $\xi$ and $\eta$ be complex vector bundles over $X$. If the complex vector bundle
\[ E(p_1^*(\eta \oplus (-\xi)) \otimes (W_k \otimes_{\mathbb{R}} \mathbb{C}))/S_k \cong E(p_1^*\eta \oplus p_1^*(-\xi))/S_k \otimes_{\mathbb{C}} E(p_1^*(W_k \otimes_{\mathbb{R}} \mathbb{C}))/S_k \rightarrow \text{Conf}_{p_1\otimes k}(E(\xi \oplus (-\xi)), k)/S_k \cong X \times \text{Conf}(\mathbb{C}^N, k)/S_k \]
do not admit a nowhere zero section, then any continuous fiberwise map $\xi \rightarrow \eta$ admits a $k$-multiplicity.
3. FROM A CONTINUOUS MAP TO A MULTIPlicity OF FIBERWISE MAP

In this section we relate the existence of a local $k$-multiplicity for continuous maps between two Riemannian manifolds with the existence of a $k$-multiplicity for a continuous fiberwise map between appropriately constructed vector bundles.

Let $M$ be an $m$-dimensional smooth closed manifold, let $N$ be an $n$-dimensional smooth manifold, and let $k \geq 2$ be an integer. A continuous map $f: M \rightarrow N$ admits a $k$-multiplicity if there exist $k$ pairwise distinct points $x_1, \ldots, x_k$ on $M$ such that $f(x_1) = \cdots = f(x_k)$. For example, the existence of a 2-multiplicity for any continuous map $M \rightarrow N$ between manifolds $M$ and $N$ means that the manifold $M$ cannot be embedded into the manifold $N$.

Let us assume that in addition manifold $M$ is equipped with some Riemannian metric. A continuous map $f: M \rightarrow N$ admits a local $k$-multiplicity if, for every real number $\omega > 0$, there exist $k$ pairwise distinct points $x_1, \ldots, x_k$ in $M$ such that

$$f(x_1) = \cdots = f(x_k) \quad \text{and} \quad \text{diam}\{x_1, \ldots, x_k\} < \omega.$$  

Note that in the case when $M$ is compact this definition does not depend on the choice of a Riemannian metric.

For our methods to work smoothly we need to make further assumptions on manifolds $M$ and $N$. Let us assume that $M$ is a compact Riemannian manifold and that $N$ is also a Riemannian manifold with positive injectivity radius, denoted by $\rho(N) > 0$. The compactness of $M$ implies that injectivity radius $\rho(M)$ of $M$ is also positive. Furthermore, let us fix a real number $\omega > 0$. The method we present can be in principle also applied when manifold $M$ is open, but additional assumptions need to be imposed.

Now, let $D_\delta(\tau M)$ and $D_\varepsilon(\tau N)$ denote open disc subbundles (of disc radius $\delta$ and $\varepsilon$, respectively) of the tangent bundles $\tau M$ and $\tau N$. The exponential maps associated to $M$ and $N$ are denoted by

$$\exp_M: E(\tau M) \rightarrow M \times M \quad \text{and} \quad \exp_N: E(\tau N) \rightarrow N \times N.$$  

Then, for a continuous map $f: M \rightarrow N$ and for every $0 < \varepsilon < \rho(N)$, there exists $0 < \delta < \min\{\rho(M), \frac{\varepsilon}{2}\}$ such that

$$(f \times f) \circ \exp_M(E(D_\delta(\tau M))) \subseteq \exp_N(E(D_\varepsilon(\tau N))).$$

Since the exponential map is injective inside the injectivity radius the following composition is well defined

$$\Phi': = \exp_N^{-1} \circ (f \times f) \circ \exp_M: E(D_\delta(\tau M)) \rightarrow E(D_\varepsilon(\tau N)), \quad (9)$$

and illustrated in the diagram below:

\[
\begin{array}{c}
\text{E}(D_\delta(\tau M)) \xrightarrow{\exp_M} M \times M \xrightarrow{f \times f} N \times N \\
\Phi' \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \exp_N \\
\text{E}(D_\varepsilon(\tau N)).
\end{array}
\]

The map $\Phi'$ is a continuous fiberwise map covering the continuous map $f: M \rightarrow N$. It plays a role of a differential map $df$ that cannot be considered in this case since $f$ is not assumed to be smooth. Furthermore, the continuous fiberwise map $\Phi'$ induces a continuous fiberwise map $\Phi: E(D_\delta(\tau M)) \rightarrow f^*E(D_\varepsilon(\tau N))$ between the bundle $E(D_\delta(\tau M))$ and the pullback bundle $f^*E(D_\varepsilon(\tau N))$ by

$$\Phi(x, v) = (x, \Phi'(x, v)), \quad (10)$$
and the following diagram commutes

\[
\begin{array}{ccc}
E(D_b(\tau M)) & \xrightarrow{f} E(D_\varepsilon(\tau N)) & \xrightarrow{f^*} E(D_\varepsilon(\tau N)) \\
\downarrow \Phi' & & \downarrow \Phi \\
M & \xrightarrow{f} & N.
\end{array}
\]

Now we prove a theorem that relates the existence of a \( k \)-multiplicity of an appropriately defined continuous fiberwise map with the existence of \( k \)-multiplicity for a continuous map \( f: M \to N \). In the following we use already introduced notations.

**Theorem 3.1.** Let \( k \geq 2 \) be an integer, let \( M \) be a compact Riemannian manifold, and let \( N \) be a Riemannian manifolds with positive injectivity radius. If, for a continuous map \( f: M \to N \), every continuous fiberwise map \( E(\tau M) \to f^*E(\tau N) \) admits a \( k \)-multiplicity, then the map \( f \) admits a local \( k \)-multiplicity.

**Proof.** Let us fix \( \omega > 0 \) and \( 0 < \varepsilon < \rho(N) \). There exists \( 0 < \delta < \min\{\rho(M), \frac{\varepsilon}{2}\} \) such that

\[(f \times f) \circ \exp_M(E(D_b(\tau M))) \subseteq \exp_N(E(D_\varepsilon(\tau N))).\]

Now we can construct the fiberwise map \( \Phi: E(D_b(\tau M)) \to f^*E(D_\varepsilon(\tau N)) \), as defined in (10). There are fiberwise homeomorphisms

\[E(\tau M) \cong E(D_b(\tau M)) \text{ and } E(\tau N) \cong E(D_\varepsilon(\tau N)).\]

Thus the assumption that every fiberwise map \( E(\tau M) \to E(f^*\tau N) \) admits a \( k \)-multiplicity implies that every continuous fiberwise map

\[D_b(\tau M) \to E(f^*D_\varepsilon(\tau N)),\]

also admits a \( k \)-multiplicity, for appropriate choice of \( \delta \) and \( \varepsilon \). Consequently, the same is true for the map \( \Phi \). Hence, there exists a point \( x \in M \) and \( k \) pairwise distinct vectors \( v_1, \ldots, v_k \in T_xM \) of the norm less than \( \delta \), such that

\[\Phi(x, v_1) = \cdots = \Phi(x, v_k).\]

By the definition \( \Phi(x, v) = (x, \Phi'(x, v)) \) and consequently

\[\Phi'(x, v_1) = \cdots = \Phi'(x, v_k).\]

Furthermore, in (9), we have defined that \( \Phi' = \exp^{-1}_N \circ (f \times f) \circ \exp_M \) and so \( \exp_N \circ \Phi' = (f \times f) \circ \exp_M \) implying

\[(f \times f) \circ \exp_M(x, v_1) = \cdots = (f \times f) \circ \exp_M(x, v_k).\]

Let \( y_i \) denote the point on the geodesic, that starts at \( x \) in direction \( v_i \), on length \( \|v_i\| \) from \( x \), \( i \in \{1, \ldots, k\} \). Then the previous equalities imply that

\[(f(x), f(y_1)) = \cdots = (f(x), f(y_k)) \implies f(y_1) = \cdots = f(y_k).\]

Since \( v_1, \ldots, v_k \) are pairwise distinct with the norm less then injectivity radius of \( M \) we have that \( y_1, \ldots, y_k \) are also pairwise distinct. Moreover, since \( \delta < \frac{\varepsilon}{2} \) we have that \( \text{diam}\{y_1, \ldots, y_k\} < \omega \). Therefore, \( f \) admits a local \( k \)-multiplicity. \( \square \)

Combining Theorem 3.1 we just proved with Theorem 2.2 we get the following criterion for the existence of local \( k \)-multiplicity of the given continuous map \( f: M \to N \).
Theorem 3.2. Let \( k \geq 2 \) be an integer, let \( M \) be a compact Riemannian manifold, and let \( N \) be a Riemannian manifold with positive injectivity radius. If, for a continuous map \( f: M \longrightarrow N \), the vector bundle
\[
E(p_1^*(f^*\tau N \oplus (-\tau M)) \otimes_{\mathbb{R}} W_k)/\mathcal{G}_k \cong E(p_1^*(f^*\tau N \oplus (-\tau M)))/\mathcal{G}_k \otimes_{\mathbb{R}} E(p_1^*W_k)/\mathcal{G}_k \longrightarrow M \times \text{Conf}(\mathbb{R}^{\dim TM + \dim(-TM)}, k)/\mathcal{G}_k
\]
does not admit a nowhere zero section, then the map \( f \) admits a local \( k \)-multiplicity.

Again, like in the case of Theorem 2.3, we can derive a criterion for the existence of local \( k \)-multiplicity for a given continuous map \( f: M \longrightarrow N \) between, now, complex manifolds \( M \) and \( N \). As already explained, the only difference is in the place of the vector bundle \( W_k \), which is substituted by its complexification \( W_k \otimes_{\mathbb{R}} \mathbb{C} \).

Theorem 3.3. Let \( k \geq 2 \) be an integer, let \( M \) be a compact complex Riemannian manifold, and let \( N \) be a complex Riemannian manifold with positive injectivity radius. If, for a continuous map \( f: M \longrightarrow N \), the vector bundle
\[
E(p_1^*(f^*\tau N \oplus (-\tau M)) \otimes_{\mathbb{C}} (W_k \otimes_{\mathbb{R}} \mathbb{C}))/\mathcal{G}_k \cong E(p_1^*(f^*\tau N \oplus (-\tau M)))/\mathcal{G}_k \otimes_{\mathbb{C}} E(p_1^*(W_k \otimes_{\mathbb{R}} \mathbb{C}))/\mathcal{G}_k \longrightarrow M \times \text{Conf}(\mathbb{C}^{\dim TM + \dim(-TM)}, k)/\mathcal{G}_k
\]
does not admit a nowhere zero section, then the map \( f \) admits a local \( k \)-multiplicity.

4. PROOFS OF THE MAIN RESULTS

4.1. Proof of Theorem 1.1. Let us fix a continuous map \( f: M \longrightarrow N \) and assume that:
\begin{itemize}
  \item \( k \geq 2 \) is a power of two,
  \item \( M \) is a compact \( m \)-dimensional smooth manifold,
  \item \( N \) is an \( n \)-dimensional smooth manifold,
  \item \( r := 2m - 1 - s \leq 2m - 1 \), and
  \item \( v_*(f^*\tau N \oplus (-\tau M)) = \det(w_{m+n-r-i+j}(f^*\tau N \oplus (-\tau M)))_{i,j,k-1} \neq 0 \).
\end{itemize}

Since \( M \) and \( N \) are smooth manifolds according to the work of Green [11] they can be equipped with a Riemannian metric in such a way that corresponding injectivity radii are positive.

Now, according to Theorem 3.2 a continuous map \( f \) admits a local \( k \)-multiplicity if the vector bundle
\[
E(p_1^*(f^*\tau N \oplus (-\tau M)))/\mathcal{G}_k \otimes_{\mathbb{R}} E(p_1^*W_k)/\mathcal{G}_k \longrightarrow M \times \text{Conf}(\mathbb{R}^{\dim TM + \dim(-TM)}, k)/\mathcal{G}_k
\]
does not admit a nowhere zero section. Thus, it suffices to prove that the Euler class, or the top Stiefel–Whitney class of the vector bundle (11) does not vanish.

The Whitney embedding theorem applied on the Riemannian manifold \( M \) implies that \( M \) can be embedded into \( \mathbb{R}^{2m} \). Consequently, we can assume that the inverse bundle \(-\tau M\) is the normal bundle of the existing embedding of \( M \) into \( \mathbb{R}^{2m} \), and therefore an \( m \)-dimensional vector bundle. Thus, the bundle (11) becomes
\[
E(p_1^*(f^*\tau N \oplus (-\tau M)))/\mathcal{G}_k \otimes_{\mathbb{R}} E(p_1^*W_k)/\mathcal{G}_k \longrightarrow M \times \text{Conf}(\mathbb{R}^{2m}, k)/\mathcal{G}_k.
\]

In order to prove the theorem we will show that the mod 2 Euler class, which is the \(((m+n)(k-1))\)-st Stiefel–Whitney class of the vector bundle (12) does not vanish.
To simplify notation let us denote by $\xi$ the vector bundle $E(p_1^*W_k)/\mathcal{S}_k$ and by $\eta$ the bundle $E(p_1^*(f^*\tau N \oplus (-\tau M)))/\mathcal{S}_k$. With the notation just introduced we will compute the $((m+n)(k-1))$-st Stiefel–Whitney class

$$w := w_{(m+n)(k-1)}(\eta \otimes \xi)$$

of the vector bundle $\eta \otimes \xi$. The cohomology class $w$ lives in the following cohomology group that decomposes into the direct sum by the Künneth formula [7, Thm. VI.1.6]:

$$H^{(m+n)(k-1)}(M \times \text{Conf}(\mathbb{R}^{2m}, k)/\mathcal{S}_k; \mathbb{F}_2) \cong \bigoplus_{i=0}^{(m+n)(k-1)} H^i(M; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^{(m+n)(k-1)-i}(\text{Conf}(\mathbb{R}^{2m}, k)/\mathcal{S}_k; \mathbb{F}_2).$$

Now we prove that the Stiefel–Whitney class $w$ does not vanish is several steps.

4.1.1. We first analyze the vector bundle $\xi = E(p_1^*W_k)/\mathcal{S}_k$. Consider the vector bundle

$$\zeta: W_k \to \text{Conf}(\mathbb{R}^{2m}, k) \times_{\mathcal{S}_k} W_k \to \text{Conf}(\mathbb{R}^{2m}, k)/\mathcal{S}_k,$$

and the projection map

$$p_2: M \times \text{Conf}(\mathbb{R}^{2m}, k)/\mathcal{S}_k \to \text{Conf}(\mathbb{R}^{2m}, k)/\mathcal{S}_k.$$

Then there is an isomorphism of vector bundles $\xi \cong p_2^*\zeta$. Consequently, by the naturality property of the Stiefel–Whitney classes [17, Ax. 2, p. 37] and the fact that $p_2$ is a projection onto the second factor we have that

$$w_i(\xi) = p_2^*(w_i(\zeta)) = 1 \otimes_{\mathbb{F}_2} w_i(\zeta) \in H^{0i}(M; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^i(\text{Conf}(\mathbb{R}^{2m}, k)/\mathcal{S}_k; \mathbb{F}_2), \quad (13)$$

for any integer $i \geq 0$. In particular, we have that $w_{k-1}(\xi) = 1 \otimes_{\mathbb{F}_2} w_{k-1}^*(\zeta)$. According to [4, Lem. 8.14], and as in [5, eq. (2), p. 7], the following equivalence holds

$$w_{k-1}(\xi)^j \neq 0 \quad \text{if and only if} \quad 0 \leq j \leq 2m - 1. \quad (14)$$

More information about characteristic classes of the vector bundle $\xi$ is given in the following lemma [5, Cor. 2.16].

**Lemma 4.1.** Consider a matrix $[j_{r,s}]_1 \leq r \leq t, 1 \leq s \leq k-1$ of non-negative integers with pairwise distinct rows. Assume that, for some $0 \leq j \leq 2m - 1$ and each $1 \leq r \leq t$,

$$\sum_{s=1}^{k-1} s \cdot j_{r,s} = (k-1)j.$$

Then, for some $\lambda_1, \ldots, \lambda_t \in \mathbb{F}_2$,

$$\sum_{r=1}^{t} \lambda_r \cdot w_1(\xi)^{j_{r,1}} \cdots w_{k-2}(\xi)^{j_{r,k-2}} w_{k-1}(\xi)^{j_{r,k-1}} = w_{k-1}(\xi)^j$$

if and only if there exists a (unique) $r_0 \in \{1, 2, \ldots, t\}$ such that

- $\lambda_r = 0$ if and only if $r \neq r_0$, and
- $j_{r_0,1} = \cdots = j_{r_0,k-2} = 0, j_{r_0,k-1} = j$. 
Thus, projection map by: $s$ The Schur function tensor product of vector bundles known formula \[20, \text{Thm. 1}], [17, \text{Pr. 7-C}] for the total Stiefel–Whitney class of the $\eta$

4.1.2. Second, consider the vector bundle $\xi$ and $\sigma$ part of the polynomial $b$

In order to compute the class $w$ we will identify the $(19)$ that belongs to the ring of symmetric polynomials

Here $P$ denotes the polynomial

that belongs to the ring of symmetric polynomials

Here $\sigma_1, \ldots, \sigma_{m+n}$ stand for the elementary symmetric polynomials in variables $a_1, \ldots, a_{m+n}$, and $\sigma'_1, \ldots, \sigma'_{k-1}$ are the elementary symmetric polynomials in variables $b_1, \ldots, b_{k-1}$.

In order to compute the class $w$ we will identify the $(m+n)(k-1)$-homogenous part of the polynomial $P$ expressed in terms of elementary symmetric polynomials $\sigma_1, \ldots, \sigma_{m+n}$ and $\sigma'_1, \ldots, \sigma'_{k-1}$ that correspond to the Stiefel–Whitney classes of $\eta$ and $\xi$.

4.1.4. The $(m+n)(k-1)$-homogenous part $P_{(m+n)(k-1)}$ of the polynomial $P$, that computes the Stiefel–Whitney class $w_{(m+n)(k-1)}(\eta \otimes \xi)$, can be expressed in the following form

According to a dual version of the Cauchy identity [14, Eq. (0.11')]
On the other hand, the Nägelsbach–Kosta formula [14, Eq. (0.3)] gives a presentation of the Schur function $s_\lambda(a)$ in terms of elementary symmetric functions $\sigma_1, \ldots, \sigma_{m+n}$ as follows:

$$s_\lambda(a_1, \ldots, a_{m+n}) = \det(\sigma_{\lambda_i-i+j})_{1 \leq i, j \leq t} = \begin{vmatrix} \sigma_{\lambda_1} & \sigma_{\lambda_1+1} & \cdots & \cdots & \sigma_{\lambda_1+t} \\ \sigma_{\lambda_2} & \sigma_{\lambda_2+1} & \cdots & \cdots & \sigma_{\lambda_2+t} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{\lambda_t} & \sigma_{\lambda_t+1} & \cdots & \cdots & \sigma_{\lambda_t+t} \end{vmatrix}$$

(20)

where $t$ is the length of the conjugate partition $\lambda'$. Here we assume that $\sigma_0 = 1$, and $\sigma_i = 0$ for $i < 0$ or $i > m + n$.

4.1.5. In the next step, having in mind Lemma 4.1, we want to identify all the Schur functions $s_{\tilde{\lambda}}(b)$ in the formula (19) that have a power of the elementary symmetric polynomial $\sigma_{k-1}'$ in their presentation. Recall that $\sigma_{k-1}'$ corresponds to the Stiefel–Whitney class $w_{k-1}(\xi)$.

From the Nägelsbach–Kosta formula (20) the Schur function $s_{\tilde{\lambda}}(b)$ has a power of $(\sigma_{k-1}')^t$ in its presentation if and only if

$$\tilde{\lambda} = (k-1, \ldots, k-1) \iff \lambda = (k-1, \ldots, k-1)_{m+n-t} \iff \lambda' = (m+n-t, \ldots, m+n-t, k-1).$$

In this case $s_{\tilde{\lambda}}(b) = (\sigma_{k-1}')^t$, and

$$s_\lambda(a_1, \ldots, a_{m+n}) = \begin{vmatrix} \sigma_{m+n-t} & \sigma_{m+n-t+1} & \cdots & \cdots & \sigma_{m+n-t+k-2} \\ \sigma_{m+n-t-1} & \sigma_{m+n-t} & \cdots & \cdots & \sigma_{m+n-t+k-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{m+n-t+k+2} & \sigma_{m+n-t+k+3} & \cdots & \cdots & \sigma_{m+n-t} \end{vmatrix} (\sigma_{k-1}')^t + \sum_{j \in J} \alpha_j \beta_j,$$  

(21)

Thus, we have the following presentation:

$$P_{(m+n)(k-1)} = \prod_{i=1}^{m+n} \prod_{j=1}^{k-1} (a_i + b_j) = \sum_\lambda s_\lambda(a)s_{\tilde{\lambda}}(b)$$

$$= \sum_{t=0}^{m+n} \begin{vmatrix} \sigma_{m+n-t} & \sigma_{m+n-t+1} & \cdots & \cdots & \sigma_{m+n-t+k-2} \\ \sigma_{m+n-t-1} & \sigma_{m+n-t} & \cdots & \cdots & \sigma_{m+n-t+k-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \sigma_{m+n-t+k+2} & \sigma_{m+n-t+k+3} & \cdots & \cdots & \sigma_{m+n-t} \end{vmatrix} (\sigma_{k-1}')^t + \sum_{j \in J} \alpha_j \beta_j,$$  

(21)

where $\alpha_j \in \mathbb{F}_2[\sigma_1, \ldots, \sigma_m]$ and $\beta_j \in \mathbb{F}_2[\sigma_1', \ldots, \sigma_{k-1}']$ are monomials, and no $\beta_j$ is a power of $\sigma_{k-1}'$. Now combining (17) and (21) we get that

$$w_{(m+n)(k-1)}(\eta \otimes \xi) = \begin{vmatrix} \sum_{t=0}^{m+n} w_{m+n-t}(\eta) & w_{m+n-t+1}(\eta) & \cdots & w_{m+n-t+k-2}(\eta) \\ w_{m+n-t-1}(\eta) & w_{m+n-t}(\eta) & \cdots & w_{m+n-t+k-3}(\eta) \\ \cdots & \cdots & \cdots & \cdots \\ w_{m+n-t+k+2}(\eta) & w_{m+n-t+k+3}(\eta) & \cdots & w_{m+n-t}(\eta) \end{vmatrix} w_{k-1}(\xi)^t + \sum_{j \in J} \alpha_j \beta_j,$$  

(22)

where $\alpha_j \in \mathbb{F}_2[w_1(\eta), \ldots, w_m(\eta)]$ and $\beta_j \in \mathbb{F}_2[w_1(\xi), \ldots, w_{k-1}(\xi)]$ are non-constant monomials, and no $\beta_j$ is a power of $w_{k-1}(\xi)$. 
Let us introduce notation

\[ u_t(\alpha) := \begin{pmatrix}
  w_{m+n-t}(\alpha) & w_{m+n-t+1}(\alpha) & \cdots & w_{m+n-t+k-2}(\alpha) \\
  w_{m+n-t-1}(\alpha) & w_{m+n-t}(\alpha) & \cdots & w_{m+n-t+k-3}(\alpha) \\
  \vdots & \vdots & \ddots & \vdots \\
  w_{m+n-t-k+2}(\alpha) & w_{m+n-t-k+3}(\alpha) & \cdots & w_{m+n-t}(\alpha)
\end{pmatrix}, \]

where \( \alpha \) is an \((m + n)\)-dimensional vector bundle. It is important to keep in mind that \( \deg(u_t) = (m + n - t)(k - 1) \). Then the formula (22) becomes:

\[ w_{(m+n)(k-1)}(\eta \otimes \xi) = \sum_{t=0}^{m+n} u_t(\eta) w_{k-1}(\xi)^t + \sum_{j \in J} \alpha_j \beta_j, \quad (23) \]

where \( \alpha_j \in \mathbb{F}_2[w_1(\eta), \ldots, w_m(\eta)] \) and \( \beta_j \in \mathbb{F}_2[w_1(\xi), \ldots, w_{k-1}(\xi)] \) are non-constant monomials, and \( \textbf{no} \beta_j \) is a power of \( w_{k-1}(\xi) \).

4.1.6. Now, having in mind (13) and (15) we transform expression (23) as follows:

\[ w_{(m+n)(k-1)}(\eta \otimes \xi) = \sum_{t=0}^{m+n} u_t(f^* \tau N \oplus (-\tau M)) \otimes_{\mathbb{F}_2} w_{k-1}(\xi)^t + \sum_{j \in J} \alpha_j' \otimes_{\mathbb{F}_2} \beta_j', \quad (24) \]

where

\[ \alpha_j = \alpha_j' \otimes_{\mathbb{F}_2} 1 \quad \text{and} \quad \beta_j = 1 \otimes_{\mathbb{F}_2} \beta_j', \]

for some \( \alpha_j' \in H^{21}(M; \mathbb{F}_2) \), and some \( \beta_j' \in H^{21}(\text{Conf}(\mathbb{R}^{2m}, k)/\mathbb{S}_k; \mathbb{F}_2) \) not a power of \( w_{k-1}(\xi) \).

Next, recall that by an assumption of the theorem there exists \( r \leq 2m - 1 \) with the property

\[ u_r(f^* \tau N \oplus (-\tau M)) = \det(w_{\dim M + \dim N - r-i+j}(f^* \tau N \oplus (-\tau M)))_{1 \leq i,j \leq k-1} \neq 0. \]

Hence, consider the projection induced by the Künneth formula decomposition

\[ \Lambda: H^{(m+n)(k-1)}(M \times \text{Conf}(\mathbb{R}^{2m}, k)/\mathbb{S}_k; \mathbb{F}_2) \longrightarrow H^{(m+n-r)(k-1)}(M; \mathbb{F}_2) \otimes_{\mathbb{F}_2} H^{r(k-1)}(\text{Conf}(\mathbb{R}^{2m}, k)/\mathbb{S}_k; \mathbb{F}_2). \]

Then from (24) we get that

\[ \Lambda(w) = u_r(f^* \tau N \oplus (-\tau M)) \otimes_{\mathbb{F}_2} w_{k-1}(\xi)^r + \sum_{j \in J''} \alpha_j'' \otimes_{\mathbb{F}_2} \beta_j'', \quad (25) \]

where \( \deg \alpha''_j = (m + n - r)(k - 1) \), \( \deg \beta''_j = r(k - 1) \) for \( r \geq 1 \), and \( \beta''_j = 0 \) for \( r = 0 \). Moreover, \( \textbf{no} \beta''_j \) is equal to \( w_{k-1}(\xi)^r \).

Since \( u_r(f^* \tau N \oplus (-\tau M)) \neq 0 \) and by (14) we have \( w_{k-1}(\xi)^r \neq 0 \) (because \( r \leq 2m - 1 \)), we have that the first summand in the formula (25) does not vanish. On the other hand by [5, Cor. 2.16] we get that first and second summand in (25) do not coincide, or since we are working with coefficients in \( \mathbb{F}_2 \), they do no cancel. Thus, \( \Lambda(w) \neq 0 \) and consequently the mod 2 Euler class \( w \) of the vector bundle \( \eta \otimes \xi \) does not vanish. This concludes the proof of the theorem.

4.2. **Proof of Theorem 1.2.** Let us fix a continuous map \( f: M \longrightarrow N \) and assume that:

- \( k \geq 2 \) is an odd prime,
- \( M \) is a compact \( m \)-dimensional smooth almost complex manifold,
- \( -\tau M \) is an \( m' \)-dimensional complex vector bundle,
- \( N \) is an \( n \)-dimensional smooth complex manifold,
- \( r \leq m + m' - 1 \), and
- \( u_r(f^* \tau N \oplus (-\tau M)) = \det(c_{m+n-r-1+j}(f^* \tau N \oplus (-\tau M)))_{1 \leq i,j \leq k-1} \neq 0. \)
Here we put \( r = m + m' - 1 - s \). Furthermore, the Chern classes we work with in this proof are considered mod \( k \). Since \( M \) and \( N \) are smooth manifolds they can be equipped with a Riemannian metric in such a way that corresponding injectivity radii are positive, see [11].

From Theorem 3.3 we have that the continuous map \( f \) admits a local \( k \)-multiplicity if the complex vector bundle

\[
E(p_1^*(f^*\tau N \oplus (-\tau M)) \otimes_{\mathbb{C}} (W_k \otimes_{\mathbb{R}} \mathbb{C})) / \mathcal{S}_k \cong E(p_1^*(f^*\tau N \oplus (-\tau M)) / \mathcal{S}_k \otimes_{\mathbb{C}} E(p_1^*(W_k \otimes_{\mathbb{R}} \mathbb{C})) / \mathcal{S}_k \longrightarrow M \times \text{Conf}(\mathbb{C}^{\dim M + \dim (-\tau M)}, k) / \mathcal{S}_k
\]

(26)
does not admit a nowhere zero section. It suffices to prove that the Euler class, or the top Chern class of the complex vector bundle (26) does not vanish.

Since the inverse bundle \(-\tau M\) is an \( m'\)-dimensional complex vector bundle the bundle (26) becomes

\[
E(p_1^*(f^*\tau N \oplus (-\tau M)) / \mathcal{S}_k \otimes_{\mathbb{C}} E(p_1^*(W_k \otimes_{\mathbb{R}} \mathbb{C})) / \mathcal{S}_k \longrightarrow M \times \text{Conf}(\mathbb{C}^{m+m'}, k) / \mathcal{S}_k.
\]

(27)

To prove the theorem we will demonstrate that the mod \( k \) Euler class, which in this case coincides with the \((m'+n)(k-1))\)-st Chern class of the complex vector bundle (27) does not vanish.

We simplify notation again by denoting:

\[
\xi = E(p_1^*(W_k \otimes_{\mathbb{R}} \mathbb{C})) / \mathcal{S}_k \text{ and } \eta = E(p_1^*(f^*\tau N \oplus (-\tau M)) / \mathcal{S}_k.
\]

Thus we need to compute the following the mod \( k \) Chern class

\[
c := c_{(m+n)(k-1)}(\eta \otimes_{\mathbb{C}} \xi)
\]

of the complex vector bundle \( \eta \otimes_{\mathbb{C}} \xi \). The class \( c \) belongs to the following cohomology group that decomposes into the direct sum by the Künneth formula [7, Thm. VI.1.6]:

\[
H^{2(m'+n)(k-1)}(M \times \text{Conf}(\mathbb{C}^{m+m'}, k) / \mathcal{S}_k; \mathbb{F}_k) \cong \bigoplus_{i=0}^{2(m'+n)(k-1)} H^i(M; \mathbb{F}_k) \otimes_{\mathbb{F}_k} H^{2(m'+n)(k-1)-i}(\text{Conf}(\mathbb{C}^{m+m'}, k) / \mathcal{S}_k; \mathbb{F}_k).
\]

(28)

The computation of the Chern class \( c \) will be done in steps.

4.2.1. First we analyze the complex vector bundle \( \xi = E(p_1^*(W_k \otimes_{\mathbb{R}} \mathbb{C})) / \mathcal{S}_k \). Consider the complex vector bundle

\[
\zeta : W_k \otimes_{\mathbb{R}} \mathbb{C} \longrightarrow \text{Conf}(\mathbb{C}^{m+m'}, k) \times \mathcal{S}_k (W_k \otimes_{\mathbb{R}} \mathbb{C}) \longrightarrow \text{Conf}(\mathbb{C}^{m+m'}, k) / \mathcal{S}_k,
\]

and the projection map

\[
p_2 : M \times \text{Conf}(\mathbb{C}^{m+m'}, k) / \mathcal{S}_k \longrightarrow \text{Conf}(\mathbb{C}^{m+m'}, k) / \mathcal{S}_k.
\]

There is an isomorphism of complex vector bundles \( \xi \cong p_2^*\zeta \). The naturality property of the Chern classes [17, Lem. 14.2] and the fact that \( p_2 \) is a projection onto the second factor imply that

\[
c_i(\xi) = p_2^*c_i(\zeta) = 1 \otimes_{\mathbb{F}_k} c_i(\zeta) \in H^i(M; \mathbb{F}_k) \otimes_{\mathbb{F}_k} H^i(\text{Conf}(\mathbb{C}^{m+m'}, k) / \mathcal{S}_k; \mathbb{F}_k),
\]

(29)

for any integer \( i \geq 0 \).

Next we recall some know fact about the cohomology of the unordered configuration space \( \text{Conf}(\mathbb{C}^{m+m'}, k) \) with \( \mathbb{F}_k \) coefficients, consult for example [9, Prop. 5.1(iii)] and Thm. 5.2

**Lemma 4.2.** Let \( k \) be an odd prime, and let \( m \geq 1 \) be an integer...
where \( \text{class} \) homogenous part and \( \sigma \) part of the polynomial \( P \) Here For this we use the following known formula for the total Chern class of the tensor

\[
\eta = \frac{1}{m!} \pi_1 \left( \sum_{\lambda} \text{Conf}(\lambda) \right) \left( \sum_{\lambda} \text{Conf}(\lambda) \right) \left( \sum_{\lambda} \text{Conf}(\lambda) \right)
\]

Furthermore, by \([3, \text{Thm. 4.1}]\) we have that \( c_{k-1}(\zeta) = 0 \) and only if \( 0 \leq j \leq m + m' - 1 \).

4.2.2. Now we consider the complex vector bundle \( \eta = E(p_1((f^* \tau N \oplus (-\tau M))))/\mathfrak{S}_k \). The projection map \( p_1 : M \times \text{Conf}(\mathbb{C}^{m+m'}, k) \to M \) factors as follows

\[
M \times \text{Conf}(\mathbb{C}^{m+m'}, k) \xrightarrow{\pi_2} M \times \text{Conf}(\mathbb{C}^{m+m'}, k)/\mathfrak{S}_k \xrightarrow{\pi_1} M,
\]

and therefore \( \eta \equiv \pi_1^{(f^* \tau N \oplus (-\tau M))} \). The naturality property for the Chern classes yields

\[
c_i(\eta) = \pi_1^{(c_i(f^* \tau N \oplus (-\tau M)))} = c_i(f^* \tau N \oplus (-\tau M)) \otimes \mathfrak{S}_k 1,
\]

every integer \( i \geq 0 \). According to one of theorem's assumptions

\[
v_i(f^* \tau N \oplus (-\tau M)) = \det(c_{m+n-r-i+j}(f^* \tau N \oplus (-\tau M)))_{1 \leq i, j \leq m' - 1} \neq 0.
\]

4.2.3. Next we make the first step in computation of \( c = c((m'+n)(k-1))(\eta \otimes \xi) \). For this we use the following known formula for the total Chern class of the tensor product of vector bundles \([15, \text{p. 67}], [6, \text{Eq. (21.9)}] \) we get

\[
c(\eta \otimes \xi) = P(c_1(\eta), \ldots, c_{m+n}(\eta), c_1(\xi), \ldots, c_{k-1}(\xi)).
\]

Here \( P \) is the same polynomial as in Section 4.1.4:

\[
P(\sigma_1, \ldots, \sigma_{m+n}, \sigma'_1, \ldots, \sigma'_{k-1}) = \prod_{i=1}^{m+n} \prod_{j=1}^{k-1} (1 + a_i + b_j)
\]

that belongs to the ring of symmetric polynomials

\[
\mathbb{F}_2[\sigma_1, \ldots, \sigma_{m+n}, \sigma'_1, \ldots, \sigma'_{k-1}] = \mathbb{F}_2[a_1, \ldots, a_{m+n}, b_1, \ldots, b_{k-1}] \otimes \mathfrak{S}_{m+n} \times \mathfrak{S}_{k-1}.
\]

In order to compute the class \( c \) we need to identify the relevant homogenous part of the polynomial \( P \). The elementary symmetric polynomials \( \sigma_1, \ldots, \sigma_{m+n} \) and \( \sigma'_1, \ldots, \sigma'_{k-1} \) correspond to the Chern classes of \( \eta \) and \( \xi \). The \((m'+n)(k-1)\)-homogenous part \( P_{(m'+n)(k-1)} \) of the polynomial \( P \), that computes the Chern class class \( c \), can be expressed as in the proof of Theorem 1.1 as follows

\[
P_{(m'+n)(k-1)} = \prod_{i=1}^{m+n} \prod_{j=1}^{k-1} (a_i + b_j) = \sum_{\lambda} s_{\lambda}(a) s_{\lambda}(b),
\]

where

\[
\begin{align*}
\lambda &= (\lambda_1, \ldots, \lambda_{m'+n}) \text{ is a partition, that means } \lambda_1 & \geq \cdots & \lambda_{m'+n} \geq 0, \\
\lambda_1 &\leq k-1, \\
\lambda &\equiv (k-1 - \lambda_{m'+n}, k-1 - \lambda_{m'+n-1}, \ldots, k-1 - \lambda_1),
\end{align*}
\]
The Nægelsback–Kosta formula [14, Eq. (0.3)] gives a presentation of the Schur function $s_\lambda$ in terms of elementary symmetric functions $\sigma_1, \ldots, \sigma_{m+n}$ as follows:

$$
s_\lambda(a_1, \ldots, a_{m+n}) = \det (\sigma_{\lambda'-i-j})_{1 \leq i, j \leq t}
\begin{array}{cccc}
\sigma_{\lambda'_1} & \sigma_{\lambda'_1+1} & \cdots & \sigma_{\lambda'_1+t-1} \\
\sigma_{\lambda'_1+1} & \sigma_{\lambda'_2} & \cdots & \sigma_{\lambda'_2+t-1} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{\lambda'_t} & \sigma_{\lambda'_t+1} & \cdots & \sigma_{\lambda'_t+t-1} \\
\sigma_{\lambda'_t+1} & \sigma_{\lambda'_t+2} & \cdots & \sigma_{\lambda'_t+t} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{\lambda'_t+t-1} & \sigma_{\lambda'_t+t+1} & \cdots & \sigma_{\lambda'_t+2t-2} \\
\sigma_{\lambda'_t+t} & \cdots & \sigma_{\lambda'_t+2t-1} & \sigma_{\lambda'_t+2t}
\end{array}
$$

where $t$ is the length of the conjugate partition $\lambda'$. Here we assume that $\sigma_0 = 1$, and $\sigma_i = 0$ for $i < 0$ or $i > m' + n$.

4.2.4. In the next step, having in (29), (30) and (31), we are going to identify all the Schur functions $s_\lambda(b)$ in the formula (35) that have a power of the elementary symmetric function $\sigma_{k-1}$ in their presentation. In this case, the symmetric polynomial $\sigma_{k-1}'$ corresponds to the only non-zero Chern class of positive degree $c_{k-1}(\xi)$.

As we have already seen in the proof of Theorem 1.1 according to the Nægelsback–Kosta formula (36) the Schur function $s_\lambda(b)$ has a power of $(\sigma_{k-1}')^t$ in its presentation if and only if

\[
\tilde{\lambda} = (k-1, \ldots, k-1) \quad \iff \quad \lambda = (k-1, \ldots, k-1) \quad \text{with} \quad m' + n - t
\]

Thus

\[
s_\lambda(a_1, \ldots, a_{m+n}) = \begin{array}{cccc}
\sigma_{m'+n-t} & \sigma_{m'+n-t+1} & \cdots & \sigma_{m'+n-t+k-2} \\
\sigma_{m'+n-t+1} & \sigma_{m'+n-t} & \cdots & \sigma_{m'+n-t+k-3} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{m'+n-t+k-2} & \sigma_{m'+n-t+k-3} & \cdots & \sigma_{m'+n-t}
\end{array}
\]

Since $c_i(\xi) = 0$ for $i \notin \{0, k-1\}$, unlike in the proof of Theorem 1.1, we have that all Schur functions $s_\lambda(a_1, \ldots, a_{m+n})$ vanish when

\[
\lambda \neq \underbrace{(k-1, \ldots, k-1)}_{m'+n-t}
\]

for some $t$. Thus

\[
P_{(m'+n)(k-1)} = \prod_{i=1}^{m'+n} a_i \prod_{j=1}^{k-1} b_j = \sum_{\lambda} s_\lambda(a)s_{\lambda'}(b)
\]

\[
= \sum_{t=0}^{m'+n} \begin{array}{cccc}
\sigma_{m'+n-t} & \sigma_{m'+n-t+1} & \cdots & \sigma_{m'+n-t+k-2} \\
\sigma_{m'+n-t+1} & \sigma_{m'+n-t} & \cdots & \sigma_{m'+n-t+k-3} \\
\vdots & \vdots & \ddots & \vdots \\
\sigma_{m'+n-t+k-2} & \sigma_{m'+n-t+k-3} & \cdots & \sigma_{m'+n-t}
\end{array} \quad (\sigma_{k-1}')^t
\]  \quad (37)
Finally, by collecting previous computation we have that
\[ c = c_{(m' + n)(k - 1)}(\eta \otimes c \xi) = \left\{ \begin{array}{cccc}
\sum_{t=0}^{m' + n} c_{m' + n - t}(\eta) & c_{m' + n - t + 1}(\eta) & \cdots & c_{m' + n - t + k - 2}(\eta) \\
\cdots & c_{m' + n - t - 1}(\eta) & \cdots & c_{m' + n - t + k - 3}(\eta) \\
\cdots & \cdots & \cdots & c_{m' + n - t}(\eta) \\
c_{k - 1}(\xi)^t 
\end{array} \right. \]
\[ = \sum_{t=0}^{m' + n} v_t(f^*\tau N \oplus (\tau M)) \otimes F_k c_{k - 1}(\xi)^t. \quad (38)\]

Observe that each summand in (38) belongs to a different summand in the Künneth decomposition (28) of the ambient group. More precisely
\[ v_t(f^*\tau N \oplus (\tau M)) \otimes F_k c_{k - 1}(\xi)^t \in H^{2(\ell + n - t)(k - 1)}(M; \mathbb{F}_k) \otimes \mathbb{F}_k H^{2(k - 1)}(\text{Conf}(C^m + m', k) / \mathbb{S}_k \mathbb{F}_k). \]

Hence, if one of the summands \( v_t(f^*\tau N \oplus (\tau M)) \otimes F_k c_{k - 1}(\xi)^t \) does not vanish then the Chern class \( c \) will also not vanish. Since, by the theorem assumption, \( v_t(f^*\tau N \oplus (\tau M)) \neq 0 \) and \( r \leq m + m' - 1 \), then according to (31)
\[ v_r(f^*\tau N \oplus (\tau M)) \otimes F_k c_{k - 1}(\xi)^t \neq 0, \]
and consequently \( c \neq 0 \). Thus, we completed the proof of the theorem.

4.3. Proof of Theorem 1.3. The proof of the theorem is obtained by applying Theorem 1.1 to an arbitrary continuous map \( f: M \to N \). We verify that all assumptions necessary for application of Theorem 1.1 are met. Let us denote by \( m := \dim M \) and by \( n := \dim N \).

First, we simplify the vector bundle \( f^*\tau N \oplus (\tau M) \). The assumption that \( N \) is parallelizable implies that the tangent bundle \( \tau N \) is trivial, meaning \( \tau N \cong \mathbb{R}^n \) as a vector bundle over \( N \). Consequently, the pullback bundle \( f^*\tau N \) is also a trivial vector bundle, denoted again by \( \mathbb{R}^n \), but now over \( M \). Thus,
\[ w(f^*\tau N \oplus (\tau M)) = w(\mathbb{R}^n \oplus (\tau M)) = w(\tau M) = \bar{w}(M), \]
where \( \bar{w}(M) \) denotes the total dual Stiefel–Whitney class of the tangent vector bundle \( \tau M \).

Since \( r := m - s = \min \{ \ell : \bar{w}_{m - \ell}(M) \neq 0 \} \) and moreover \( \bar{w}_{m - r}(M)^{k - 1} \neq 0 \) we have that
\[ u_{n + r}(f^*\tau N \oplus (\tau M)) = \left\{ \begin{array}{cccc}
\bar{w}_{m - r}(M) & \bar{w}_{m - r + 1}(M) & \cdots & \bar{w}_{m - r + k - 2}(M) \\
\bar{w}_{m - r - 1}(M) & \bar{w}_{m - r}(M) & \cdots & \bar{w}_{m - r + k - 3}(M) \\
\cdots & \cdots & \cdots & \bar{w}_{m - r}(M) \\
\bar{w}_{m - r - (k - 2)}(M) & \bar{w}_{m - r - (k - 1)}(M) & \cdots & \bar{w}_{m - r}(M) \\
\bar{w}_{m - r}(M) & 0 & \cdots & 0 \\
\bar{w}_{m - r - 1}(M) & \bar{w}_{m - r}(M) & \cdots & 0 \\
\cdots & \cdots & \cdots & \bar{w}_{m - r}(M) \\
\bar{w}_{m - r - (k - 2)}(M) & \bar{w}_{m - r - (k - 1)}(M) & \cdots & \bar{w}_{m - r}(M) \\
\bar{w}_{m - r}(M)^{k - 1} & \neq 0.
\end{array} \right. \]

Finally, assumption that \( r \leq 2m - n - 1 \) implies that
\[ n + r \leq n + 2m - n - 1 = 2m - 1. \]
Now, Theorem 1.1 implies that the continuous map \( f: M \to N \), that was chosen arbitrary, admits a \( k \)-multiplicity. This concludes the proof of the theorem.
4.4. Proof of Theorem 1.4. In order to prove the theorem we apply Theorem 1.2 to an arbitrary continuous map \( f : M \to N \). We just verify that all assumptions necessary for application of Theorem 1.1 are satisfied. Let \( m := \dim M, m' = \dim(−τM) \), and \( n := \dim N \).

In this case the vector bundle \( f'^*τN \oplus (−τM) \) can be simplified. Since \( N \) is a parallelizable then the tangent bundle \( τN \) is trivial, meaning \( τN \cong \mathbb{C}^n \) as a vector bundle over \( N \). Consequently, \( f'^*τN \) is also a trivial vector bundle, denoted also by \( \mathbb{C}^n \), but now as a complex vector bundle over \( M \). Thus, \[ c(f'^*τN \oplus (−τM)) = c(\mathbb{C}^n \oplus (−τM)) = c(−τM) = c(M), \] where \( c(M) \) denotes the \( i \)-th Chern class of the inverse of the tangent complex vector bundle \( τM \).

Since by the theorem assumption \( r := m' - s = \min\{ℓ : c_{m'−ℓ}(M) \neq 0\} \), and moreover \( c_{m'−r}(M)k^{−1} \neq 0 \), we can evaluate the following class \[ v_{n+r}(f'^*τN \oplus (−τM)) = \begin{vmatrix} \bar{c}_{m'−r}(M) & \bar{c}_{m'−r+1}(M) & \cdots & \bar{c}_{m'−r+k−2}(M) \\ \bar{c}_{m'−r−1}(M) & \bar{c}_{m'−r}(M) & \cdots & \bar{c}_{m'−r+k−3}(M) \\ \cdots & \cdots & \cdots & \cdots \\ \bar{c}_{m'−r−k+2}(M) & \bar{c}_{m'−r−k+3}(M) & \cdots & \bar{c}_{m'−r}(M) \end{vmatrix} = \begin{vmatrix} \bar{c}_{m'−r}(M) & 0 & \cdots & 0 \\ \bar{c}_{m'−r−1}(M) & \bar{c}_{m'−r}(M) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \bar{c}_{m'−r−k+2}(M) & \bar{c}_{m'−r−k+3}(M) & \cdots & \bar{c}_{m'−r}(M) \end{vmatrix} = \bar{c}_{m'−r}(M)k^{−1} \neq 0. \]

In the last step we have that the assumption \( r \leq m + m' − n − 1 \) implies \( n + r \leq n + m + m' − n − 1 = m + m' − 1 \). Hence, Theorem 1.2 implies that the continuous map \( f : M \to N \), that was chosen arbitrary, admits a \( k \)-multiplicity.

4.5. Proof of Corollary 1.5. Let \( a \geq 1 \) and \( ℓ \geq 1 \) be integers, let \( k \geq 2 \) be a power of 2, and let \( k(a+1) ≤ 2^ℓ − 1 \). Furthermore, set \( m = 2^ℓ − 2 − a \) and \( n = 2^ℓ − 2 \).

It order to apply Theorem 1.3 we need first to find the integer \( r = m − s = \min\{ℓ : \bar{w}_{m−ℓ}(\mathbb{RP}^m) \neq 0\} \).

It is well known that the total Stiefel–Whitney class \( w(\mathbb{RP}^m) = (1 + t)^{m+1} \), where \( H^∗(\mathbb{RP}^m; \mathbb{F}_2) = \mathbb{F}_2[t]/(t^{m+1}) \) and \( \deg(t) = 1 \). Consult for example [17, Thm. 4.5].

Then \[ \bar{w}(\mathbb{RP}^m) = (1 + t)^{2^m−m−1} = (1 + t)^a+1 = \sum_{i=0}^{a+1} \binom{a+1}{i} t^i = 1 + (a+1)t + \cdots + t^{a+1}. \]

Since \( a + 1 \leq \frac{1}{2}(2^ℓ − 1) \) and \( k \geq 2 \) we have that \( a + 1 \leq 2^ℓ − 1 \). Consequently, \( m = 2^ℓ − 2 − a \geq 2^ℓ − 1 \), and so \( a + 1 \leq m \). Thus, \( r = m − a − 1 \). Furthermore, \( \bar{w}_{a+1}(\mathbb{RP}^m)^k = t^{(a+1)(k−1)} \neq 0 \) because \( (a+1)(k−1) = k(a+1) − a − 1 \leq 2^ℓ − 1 − a − 1 = m \).

It remains to confirm that \( r \leq 2m − n − 1 \). Indeed, the following inequality holds:

\( m − a − 1 = r \leq 2m − n − 1 = m + 2^ℓ − 2 − a − (2^ℓ − 2) − 1 = m − a − 1 \).

Now, Theorem 1.3 applied to the manifolds \( M = \mathbb{RP}^m \) and \( N = \mathbb{R}^n \) concludes the proof of the corollary, meaning that every continuous map \( \mathbb{RP}^{2^ℓ−2−a} \to \mathbb{R}^{2^ℓ−2} \) admits a local \( k \)-multiplicity.
4.6. Proof of Corollary 1.6. Let $a \geq 1$ and $\ell \geq 1$ be integers, let $k \geq 2$ be a power of 2, and let $k(a+1) \leq 2^\ell - 1$. Furthermore, set $m = 2^\ell - 2 - a$ and $n = 2^\ell - 2$, and fix a continuous map $f: \mathbb{R}P^m \to S^n$.

It order to apply Theorem 1.1 we need to find an integer $r = 2m-1-s \leq 2m-1$ such that

$$u_r(f^*\tau S^n \oplus (-\tau \mathbb{R}P^m)) := \det(w_{m+n-r-i+j})_{1 \leq i,j \leq k-1}$$

does not vanish. Here $w_i := w_i(f^*\tau S^n \oplus (-\tau \mathbb{R}P^m))$ is the $i$-th Stiefel–Whitney class of the vector bundle $f^*\tau S^n \oplus (-\tau \mathbb{R}P^m)$ for $i \geq 0$, and $w_i = 0$ for $i < 0$. Since $w(\tau S^n) = 1$ we have that $w(f^*\tau S^n) = 1$. Consequently,

$$w(f^*\tau S^n \oplus (-\tau \mathbb{R}P^m)) = w(f^*\tau S^n)w(-\tau \mathbb{R}P^m) = w(-\tau \mathbb{R}P^m) = \bar{w}(\mathbb{R}P^m).$$

As we have seen in the proof of Corollary 1.5 the dual Stiefel–Whitney class of $\mathbb{R}P^m$ is

$$\bar{w}(\mathbb{R}P^m) = (1+t)^{2^\ell-m-1} = (1+t)^{a+1} = \sum_{i=0}^{a+1} \binom{a+1}{i}t^i = 1 + (a+1)t + \cdots + t^{a+1}.$$

Again, $a+1 \leq \frac{1}{2}(2^\ell - 1)$ and $k \geq 2$ imply that $a+1 \leq 2^{k-1} - 1$. Hence, $m = 2^\ell - 2 - a \geq 2^{k-1}$, and so $a+1 \leq m$. Thus, $\bar{w}_{a+1}(\mathbb{R}P^m) = t^{a+1} \neq 0$, and $\bar{w}_i(\mathbb{R}P^m) = 0$ for $i > a+1$.

Now, if we take $r = 2^\ell+1 - 2a - 5$ then $r = 2m-1$ and

$$u_r(f^*\tau S^n \oplus (-\tau \mathbb{R}P^m)) := \det(w_{a+1+i+j}(\mathbb{R}P^m))_{1 \leq i,j \leq k-1} = \det(\bar{w}_{a+1+i+j}(\mathbb{R}P^m))_{1 \leq i,j \leq k-1} = \bar{w}_{a+1}(\mathbb{R}P^m)^{k-1} = t^{(a+1)(k-1)}.$$

Since $(a+1)(k-1) = k(a+1) - a - 1 \leq 2^\ell - 1 - a - 1 = m$ we have that $t^{(a+1)(k-1)} \neq 0$, and consequently $u_r(f^*\tau S^n \oplus (-\tau \mathbb{R}P^m)) \neq 0$.

Thus, Theorem 1.1 applied to the manifolds $M = \mathbb{R}P^m$ and $N = S^n$ concludes the proof of the corollary.

4.7. Proof of Corollary 1.7. Let $a \geq 1$ and $\ell \geq 1$ be integers, let $k \geq 2$ be a power of 2, and let $k(a-1) \leq 2^\ell - 1$. Now set $m = 2^\ell - a$ and $n = 2^\ell+1 - 3$.

Again we apply Theorem 1.3. First, we need to find the integer $r = m-s$. A know fact is that the total Stiefel–Whitney class $w(\mathbb{C}P^m) = (1+x)^{m+1}$, where $H^*(\mathbb{C}P^m; \mathbb{F}_2) = \mathbb{F}_2[x]/(x^{m+1})$ and $\deg(x) = 2$, consult [17, Thm. 14.10]. Therefore, the total dual Stiefel–Whitney class can be computed as follows

$$\bar{w}(\mathbb{C}P^m) = (1+x)^{2^\ell-m-1} = (1+x)^{a-1} = \sum_{i=0}^{a-1} \binom{a-1}{i}x^i = 1 + (a-1)x + \cdots + x^{a-1}.$$

From the assumption $k(a-1) \leq 2^\ell - 1 \leftrightarrow (k-1)(c-m-1) \leq m$ we derive that $a-1 = 2^\ell - m - 1 \leq \frac{m}{2^\ell} \leq m$. Thus, $2m-r = 2a-2$ and so $r = 2m-2a+2$.

Next, $\bar{w}_{2m-2a+2}(\mathbb{C}P^m)^{k-1} = x^{(a-1)(k-1)} \neq 0$ because $(k-1)(a-1) \leq m$. Finally, we verify that $r \leq 4m-n-1$. Indeed,

$$r = 2m-2a+2 = 2m-2(2^\ell-m) + 2 = 4m - 2^\ell+1 + 2 = 4m-n-1.$$

Again, Theorem 1.3 applied to the manifolds $M = \mathbb{C}P^m$ and $N = \mathbb{R}^n$ yields the statement of the corollary, that is every continuous map $\mathbb{C}P^{2^\ell-a} \to \mathbb{R}^{2^\ell+1-3}$ admits a local $k$-multiplicity.
4.8. Proof of Corollary 1.8. Let $a \geq 1$ and $\ell \geq 1$ be integers, let $k$ be an odd prime, and let $2 \leq a \leq \frac{k+1}{2}$. Furthermore, let $m := k^\ell - a$, $m' = \dim(-\tau \mathbb{C}P^n)$, and $n := k^\ell - 2$. We will apply Theorem 1.4 and therefore we verify that its assumptions are satisfied.

The total Chern class of the projective space is $c(\mathbb{C}P^n) = (1 + x)^m$ where $H^* (\mathbb{C}P^n; \mathbb{F}_k) = \mathbb{F}_k[x]/(x^{m+1})$ and $\deg(x) = 2$, see [17, Thm.14.10]. Since $2 \leq a \leq \frac{k+1}{2}$ we have that

$$\frac{1}{2} (k^\ell - 1) < m + 1 < k^\ell \implies 0 < k^\ell - m - 1 < \frac{k^\ell + 1}{2} \leq m + 1.$$ 

Therefore, we have

$$c(\mathbb{C}P^n) = (1 + x)^{k^\ell - m - 1} = (1 + x)^{a-1} = \sum_{i=0}^{a-1} \binom{a-1}{i} x^i = 1 + (a-1)x + \cdots + x^{a-1},$$

where binomial coefficients are considered mod $k$. Thus $\bar{c}_{a-1}(\mathbb{C}P^n) \neq 0$ and $\bar{c}_i(\mathbb{C}P^n) = 0$ for all $i > a - 1$. Consequently $m' \geq a - 1$ and

$$r = m' - s = \min\{\ell : \bar{c}_{a-\ell}(\mathbb{C}P^n) \neq 0\} = m' - a + 1.$$

Since $k(a-1) \leq k\ell - 1 \iff (k-1)(a-1) \leq k\ell - a$ we have that $\bar{c}_{a-1}(\mathbb{C}P^n)^{k-1} \neq 0$.

It remains to verify that $r \leq m + m' - n - 1$. Indeed,

$$r = m' - a + 1 = m' + m - k^\ell + 1 = m' + m - n - 2 + 1 = m' + m - n - 1.$$

Hence, Theorem 1.4 applied to the manifolds $M = \mathbb{C}P^n$ and $N = \mathbb{C}^n$ yields the statement of the corollary, that is every continuous map $\mathbb{C}P^{k^\ell - a} \to \mathbb{C}P^{k^\ell - 2}$ admits a local $k$-multiplicity.

References

[1] Imre Bárány, Daniel Hug, and Rolf Schneider, Affine diameters of convex bodies, Proc. Amer. Math. Soc. 144 (2016), no. 2, 797–812.
[2] Imre Bárány and Tudor Zamfirescu, Diameters in typical convex bodies, Canad. J. Math. 42 (1990), no. 1, 50–61.
[3] Pavle V. M. Blagojević, Frederick R. Cohen, Wolfgang Lück, and Günter M. Ziegler, On complex highly regular embeddings and the extended Vassiliev conjecture, Int. Math. Research Notes (IMRN), Published online December 17, 2015, doi:10.1093/imrn/rnv341; arXiv:1410.6052.
[4] Pavle V. M. Blagojević, Wolfgang Lück, and Günter M. Ziegler, Equivariant topology of configuration spaces, J. Topol. 8 (2015), no. 2, 414–456.
[5] , On highly regular embeddings, Trans. Amer. Math. Soc. 368 (2016), no. 4, 2891–2912.
[6] Raoul Bott and Loring W. Tu, Differential forms in algebraic topology, Graduate Texts in Mathematics, vol. 82, Springer-Verlag, New York-Berlin, 1982.
[7] Glen E. Bredon, Topology and geometry, Graduate Texts in Mathematics, vol. 139, Springer-Verlag, New York, 1993.
[8] Edgar H. jun. Brown and Franklin P. Peterson, On immersions of $n$-manifolds, Adv. Math. 24 (1977), 74–77.
[9] Frederick R. Cohen, The homology of $C_{n+1}$-spaces, $n \geq 0$, “The Homology of Iterated Loop Spaces”, Lecture Notes in Math. 533, Springer, 1976, pp. 207–353.
[10] Ralph L. Cohen, The immersion conjecture for differentiable manifolds, Ann. Math. (2) 122 (1985), 237–328.
[11] R. E. Greene, Complete metrics of bounded curvature on noncompact manifolds, Arch. Math. (Basel) 31 (1978/79), no. 1, 89–95.
[12] Mikhail Gromov, Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry, Geom. Funct. Anal. 20 (2010), no. 2, 416–526.
[13] Branko Grünbaum, Measures of symmetry for convex sets, Proc. Sympos. Pure Math., Vol. VII, Amer. Math. Soc., Providence, R.I., 1963, pp. 233–270.
[14] I. G. Macdonald, Schur functions: theme and variations, Séminaire Lotharingien de Combinatoire (Saint-Nabor, 1992), Publ. Inst. Rech. Math. Av., vol. 498, Univ. Louis Pasteur, Strasbourg, 1992, pp. 5–39.
[15] _____. Symmetric functions and Hall polynomials, second ed., Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications.

[16] John W. Milnor, Lectures on characteristic classes, Published in 197 as Annals of Math. Studies 76, with Stasheff, 1957.

[17] John W. Milnor and James D. Stasheff, Characteristic classes, Princeton University Press, Princeton, 1974, Annals of Mathematics Studies, No. 76.

[18] V. Soltan, Affine diameters of convex-bodies—a survey, Expo. Math. 23 (2005), no. 1, 47–63.

[19] Robert E. Stong, Notes on cobordism theory, Mathematical notes, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1968.

[20] Emery Thomas, On tensor products of n-plane bundles, Arch. Math. 10 (1959), 174–179.