A CATEGORICAL GENERALIZATION OF COUNTERPOINT

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ABSTRACT. We extend Mazzola’s counterpoint model in terms of category theory. One immediate outcome is the possibility of relaxing the “yes/no” character of the definitions of consonance, and stressing its dependence on context in general. A counterpoint model with sets instead of pure pitches is obtained.

1. Introduction

Our intention in generalizing Mazzola’s counterpoint theory [2] via category and topos theory is to relativize the notion of consonance and dissonance, and to be able to apply contrapuntal techniques to other musical objects besides pitch. In particular, the topologization of the counterpoint model via a Kuratowski closure can be put in perspective within the general investigation of closure operators (see [3, 4], for example).

The general plan of the article is to construct a categorical generalization of Mazzola’s original conception [8, Part VII], providing examples of certain gains we obtain from it, and thus refining our requirements on the ambient categories required to have a successful generalization; toposi appear as a satisfactory option. We close with a brief study of the Kuratowski operator introduced by Mazzola in order to “topologize” counterpoint in the generalized setting. We presuppose from the reader some familiarity with both counterpoint in general and Mazzola’s model in particular (see [2], [5], and [7] for general introductions on these topics).

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2. Quasipolarities

Let $\mathcal{E}$ be an appropriate\(^1\) category, $\mathcal{M}$ a category, and $F : \mathcal{M} \to \mathcal{E}$ a functor. Let $S$ be an object of $\mathcal{M}$. A quasipolarity is a morphism $p : S \to S$ of $\mathcal{M}$ satisfying the following conditions:

i) The identity $p \circ p = \text{id}_S$ holds.

ii) The unique morphism from the initial object $0$ of $\mathcal{E}$ to $F(S)$ is the equalizer (in $\mathcal{E}$) of the pair $F(p), \text{id}_{F(S)} : F(S) \to F(S)$.

Remark 2.1. If the election of the functor $F$ is obvious (particulary if it is the forgetful functor), then we will omit it.

This definition was done thinking of the following example.

Example 2.2. Take $\mathcal{E} = \text{Set}$, $\mathcal{M} = \text{ModAf}_{\mathbb{Z}_{12}}$ (where $\text{ModAf}_R$ denotes the category of modules over a commutative ring with affine transformations between them\(^2\)), $F$ the forgetful functor from modules to sets, $S = \mathbb{Z}_{12}$, then $p = e^25$ is a quasipolarity. \(\square\)

Example 2.3. The presence of the subcategory $\mathcal{M}$ in the definition of quasipolarity is crucial. For instance, if $\mathcal{E} = \text{Set}$ like in the previous example but now $\mathcal{M} = \text{FinSet}$ and $S = \{0, 1, \ldots, 11\}$ (which coincides with $\mathbb{Z}_{12}$), then the two permutations

$p = (0, 2)(1, 7)(3, 5)(4, 10)(6, 8)(9, 11)$,

$q = (0, 1)(2, 3)(4, 5)(6, 7)(8, 9)(10, 11)$

are quasipolarities; $p$ is exactly the same one as the one from the previous example as a function between sets, whereas $q$ does not come from an affine transformation.

3. Dichotomies

Let $p : S \to S$ be a quasipolarity. A dichotomy relative to $p$ is a pair of monomorphisms

$\kappa : K \to F(S)$ and $\delta : D \to F(S)$

in $\mathcal{E}$ such that

i) The canonical morphism from the coproduct $K \sqcup D$ (in $\mathcal{E}$) to $F(S)$ is an isomorphism.

ii) The monomorphisms $F(p) \circ \kappa$ and $\delta$ represent the same subobject of $F(S)$.

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\(^1\)We will be more precise on the meaning of “appropriate” later.

\(^2\)We prefer this notation instead of $\text{Mod}_R$ used by Mazzola, since the latter is the standard notation for the category of $R$-modules and $R$-homomorphisms.
We denote a dichotomy relative to \( p \) with \((\kappa/\delta)_p\) or simply \((\kappa/\delta)\) if the election of \( p \) is clear.

**Example 3.1.** The subsets

\[
K = \{0, 3, 4, 7, 8, 9\} \quad \text{and} \quad D = \{1, 2, 5, 6, 10, 11\}
\]

of \( \mathbb{Z}_{12} \) constitute a dichotomy \((K/D)_{e^{2.5}}\) for the polarity of the Example 2.2. □

It is not known in this setting if every quasipolarity has an associated dichotomy, but in the case of the category \( \textbf{FinSet} \) it is easy to see that this is true.

**4. Polarities**

Let \((\kappa/\delta)\) be a dichotomy relative to a quasipolarity \( p : S \to S \). We say that the dichotomy is *strong* if \( p \) is the unique quasipolarity such that \( F(p) \circ \kappa \) and \( \delta \) represent the same subobject; in that case, we say that \( p \) is a *polarity*.

**Example 4.1.** The dichotomy of the Example 3.1 is strong and its quasipolarity is a polarity. □

**Example 4.2.** The notion of polarity also stresses the importance of the choice of the category \( \mathcal{M} \) and the embedding functor\(^3\) \( \mathcal{M} \to \mathcal{E} \). For instance, if \( \mathcal{M}_1 = \mathbb{ModSAf}_{\mathbb{Z}_{12}} \), which is the subcategory of modules over a commutative ring with the morphisms restricted those of the form \( e^n. \pm 1 \), taking

\[
(\kappa/\delta) = (\{0, 2, 3, 4, 7, 8\}/\{1, 5, 9, 10, 11\}),
\]

we have that \( p = e^1. - 1 \) is a polarity, but in the supercategory \( \mathcal{M}_2 = \mathbb{ModAf}_{\mathbb{Z}_{12}} \) it is not since

\[
e^1. - 1(\kappa) = e^9.7(\kappa) = \delta.
\]

It is an open question whether every quasipolarity is a polarity.

**5. Counterpoint symmetries**

**Definition 5.1** (Consonances). Let \((\kappa/\delta)\) be a dichotomy. A *consonance* is a morphism \( \xi \) (of \( \mathcal{E} \)) from the terminal object \( 1 \) of \( \mathcal{E} \) to \( K \), that is, a point of \( K \). More generally, a *generalized consonance* can be defined as a morphism \( \xi \) of \( \mathcal{E} \) with codomain \( K \).

Let \((\kappa/\delta)\) be a dichotomy with quasipolarity \( p : S \to S \). A *counterpoint symmetry* for a consonance \( \xi : 1 \to K \) is an isomorphism \( g : S \to S \) of \( \mathcal{M} \) such that

\(^3\)A purely combinatorial relativization was explored in [1].
i) the morphism $\kappa \circ \xi$ is a point of $F(g) \circ \delta$, or, more precisely, $\kappa \circ \xi$ factors through $F(g) \circ \delta$.

ii) the identity $g \circ p = p \circ g$ holds, and

iii) if an isomorphism $g' : S \rightarrow S$ of $\mathcal{M}$ satisfies i and ii, then there is a monomorphism from the meet\(^4\) $(F(g') \circ \kappa) \wedge \kappa$ to the meet $(F(g) \circ \kappa) \wedge \kappa$ of $\mathcal{E}$.

**Remark 5.2.** The subobjects $F(g) \circ \kappa$ and $F(g) \circ \delta$ of $F(S)$ represent the $g$-deformed consonances and dissonances, respectively.

In the case when $\mathcal{E} = \text{Set}$ and $\mathcal{M} = \text{ModAf}_{\mathbb{Z}_{12}}$, the condition ii is exactly the same requirement stated by the classical model of Mazzola. The condition i captures the idea of revealing the tension between consonance and dissonances by making $\xi$ a $g$-deformed dissonance, while condition iii implies a maximum of artistic choices, for $(F(g) \circ \kappa) \wedge \kappa$ are the allowed steps.

More generally, a counterpoint symmetry for a generalized consonance $\xi : E \rightarrow K$ is an isomorphism $g : S \rightarrow S$ of $\mathcal{M}$ satisfying that $\kappa \circ \text{Im} (\xi)$ factors through $F(g) \circ \delta$, plus the conditions ii and iii above. In this case, we require the existence of the image $\text{Im} (\xi) \rightarrow K$ of $\xi$.

**Remark 5.3.** The notion of cantus firmus and discantus is obtained for general $R$-modules as follows. Let $R$ and $S$ be commutative rings. The restriction $r : R \rightarrow S$ gives rise to the functor

$$S \otimes_R - : \text{ModAf}_R \rightarrow \text{ModAf}_S$$

$$N \mapsto S \otimes_R N,$$

$$f = e^k.f_0 : N \rightarrow K \mapsto S \otimes_R f = e^{1 \otimes k}.S \otimes_R f_0.$$ 

Recall now that the dual numbers for a ring $R$ is the quotient

$$R[\varepsilon] := \frac{R[x]}{(x^2)} = \{a + \varepsilon.b : a, b \in R, \varepsilon^2 = 0\}$$

We have the restriction $i : R \rightarrow R[\varepsilon] : a \mapsto a + \varepsilon.0$, that allow us to construct, for an $R$-module $M$, the module $M[\varepsilon] = R[\varepsilon] \otimes_R M$, which is the algebra of dual numbers with respect to $M$. As $R$-modules, we have

$$M[\varepsilon] \cong M \oplus M,$$

\(^4\) We should keep in mind that a category can be seen as a generalization of the notion of partially ordered set; the meet of a pair of subobjects $m : S \rightarrow A$ and $m' : S' \rightarrow A$ can be obtained by means of the pullback of $m$ and $m'$, whenever the category has pullbacks.
and thus the first component of \((m_1, m_2) \in M[\epsilon]\) corresponds to the cantus firmus and \(m_2\) is the interval that separates it from the discantus. The composition of this functor with the forgetful functor \(F : \text{ModAf}_{\mathbb{Z}_2[\epsilon]} \rightarrow \text{Sets}\) recuperates the classical Mazzola model for counterpoint via counterpoint symmetries.

**Example 5.4.** Let \(S\) be finite set. We know that \(2^S\) can be seen as a \(\mathbb{Z}_2\)-module defining the product as intersection and addition as the symmetric difference

\[
A +_{2^S} B \equiv A \Delta B = (A \cup B) \setminus (A \cap B).
\]

In particular,

\[
p = e^S.1
\]

is a quasipolarity that coincides with the operation of calculating the complement with respect to \(S\), and it is the polarity of any family \(\kappa\) of \(2^{|S| - 1}\) sets such that \(T \in \kappa\) if, and only if,

\[
p(T) = S \Delta T = \mathbb{C}T \notin \kappa.
\]

Thus we can construct the algebra \(2^S[\epsilon]\) such that the cantus firmus and the interval are sets. The sets of consonances and dissonances in \(2^S[\epsilon]\) are, of course,

\[
\kappa[\epsilon] = 2^S + \epsilon.\kappa \quad \text{and} \quad \delta[\epsilon] = 2^S + \epsilon.\delta.
\]

As in the case of the classical counterpoint model, it can be proved that the calculation of counterpoint symmetries can be reduced to cantus firmus 0 and the quasipolarity

\[
p_0 = e^{e^S}.1.
\]

The affine morphism \(g = e^{e^U}.(1 + \epsilon.W)\) always commute with \(p_0\), and for a consonance \(\xi = \epsilon.K \in \kappa\) it will occur that it is a \(g\)-deformed dissonance if there is \(C + \epsilon.D \in \delta[\epsilon]\) such that

\[
\xi = \epsilon.K = g(C + \epsilon.D) = C + \epsilon.(U \Delta D \Delta(W \cap C)),
\]

thus \(C = 0\) and

\[
K = U \Delta D.
\]

From this point on it is trivial to generalize Hichert’s algorithm [2, Algorithm 2.1] to complete the calculations.
6. SOME COMMENTARIES

The nature of the original definitions from Mazzola’s counterpoint model are set-theoretical. And rightly so, since Renaissance counterpoint consolidated the notion of consonance as a Boolean one: an interval is consonant or dissonant, though not both. But in the framework of category theory, Heyting algebras are more natural, and could be used to reflect the historical development of counterpoint via the pseudocomplement. For instance: if we take consonances to define a fuzzy set, that is, a function $\kappa : S \to [0, 1]$, then the pseudocomplement

$$\neg \kappa(x) = \begin{cases} 0, & \kappa(x) > 0, \\ 1, & \kappa(x) = 0, \end{cases}$$

defines, in particular, a crisp set, even if we apply it to a general fuzzy set. Further applications of this pseudocomplement yield only crisp sets, so we get back to Boolean complements. This is relevant since the progressive definition of the nature of consonance and dissonance that ended up with a stark separation between them with good contrapuntal properties is a well known musicological problem (see [9]); we must point out to [8] for a discussion of how this could be understood as a mathematical fact discovered by musical means. Hence, the intuitionistic logic of topoi can be used to take the first steps in order to solve this problem.

This leads us to the issue of specifying an “appropriate” category from Section 2. We now recognize that we need:

i) Final objects, equalizers, subobject intersections (pullbacks). Hence finite limits would suffice.

ii) Initial object and coproducts. Thus we may require finite colimits.

iii) Images, such that they could be obtained as cokernel equalizers.

iv) Well-behaved subobjects, so we do not need something like regular subobjects.

v) The only morphism from the initial object $0$ in $S$ is a monomorphism.

A topos has all these features (for the last one, for example, see [6, p. 194]), plus some other desirable ones like the fact that canonical injections in coproducts are disjoint monomorphisms.

7. TOPOLOGY AND CLOSURE OPERATORS

7.1. The operator induced by an involutive automorphism. The Kuratowski operator introduced by Mazzola [2, Section 10.2.2] can be generalized with no difficulty to the categorical case. Let $f :
Let $E \rightarrow E$ be a morphism of $\mathcal{E}$ such that $f \circ f = \text{id}_E$. Define an operator on subobjects $M$ of $E$ by means of the equation
\[ \overline{M} := M \vee f(M). \]
We check this is indeed a Kuratowski closure.

1) If $m : M \rightarrow E$ is a subobject of $E$ in $\mathcal{E}$, then $M \leq M \vee f(M) = \overline{M}$.

2) If $m : M \rightarrow E$ is a subobject of $E$ in $\mathcal{E}$, then
\[
\overline{M} = M \vee f(M)
= M \vee f(M) \vee f(M \vee f(M))
= M \vee f(M) \vee f \circ f(M)
= M \vee f(M) = \overline{M}.
\]
As to the third equality above, note that isomorphisms preserve joins without extra assumptions.

3) If $m : M \rightarrow E$ and $m' : M' \rightarrow E$ are subobjects of $E$ in $\mathcal{E}$, then
\[
\overline{M \vee M'} = M \vee M' \vee f(M \vee M')
= M \vee M' \vee f(M) \vee f(M') = \overline{M} \vee \overline{M'}.
\]

### 7.2. The operator induced by an arbitrary endomorphism.

Suppose that $\mathcal{E}$ has images and pullbacks. Let $f : E \rightarrow E$ be an arbitrary endomorphism of $\mathcal{E}$. Define an operator on subobjects $M$ of $E$ by means of the equation $\overline{M} = M \vee f(M)$, where $f(M)$ denotes the image of $M$ under $f$. This operator satisfies the following properties:

i) If $m : M \rightarrow E$ is a subobject of $E$ in $\mathcal{E}$, then
\[ M \leq M \vee f(M) = \overline{M}. \]

ii) If $m : M \rightarrow E$ and $m' : M' \rightarrow E$ are subobjects of $E$ in $\mathcal{E}$, then
\[
\overline{M \vee M'} = M \vee M' \vee f(M \vee M')
= M \vee M' \vee f(M) \vee f(M')
= \overline{M} \vee \overline{M'}.
\]

In particular, this operator preserves the order on subobjects, though it needs not to be idempotent. However, by iterating this operator, we can obtain an idempotent operator satisfying the previous properties, that is, a Kuratowski operator. In fact, if the lattice of subobjects of $E$

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5This implies that, for each morphism of $\mathcal{E}$, the associated direct image functor is left adjoint to the associated inverse image functor, which in turn implies that the direct image functor preserves joins of subobjects.

6See [4, p. xiv] for another general discussion.
is complete (for example, if $\mathcal{E}$ is small-cocomplete), then the operator defined by the equation
\[
\overline{M} = \bigvee_{k \in \mathbb{N}} f^k(M) = M \lor f(M) \lor f^2(M) \lor \cdots
\]
satisfies the following properties:

i) If $m : M \rightarrow E$ is a subobject of $E$ in $\mathcal{E}$, then $M \leq \overline{M}$.

ii) If $m : M \rightarrow E$, then
\[
\overline{M} = \bigvee_{k \in \mathbb{N}} f^k(M) \\
= \bigvee_{k \in \mathbb{N}} f^k\left(\bigvee_{j \in \mathbb{N}} f^j(M)\right) \\
= \bigvee_{k \in \mathbb{N}} f^{k+j}(M) \\
= \bigvee_{k \in \mathbb{N}} f^k(M) = \overline{M}.
\]

iii) If $m : M \rightarrow E$ and $m' : M' \rightarrow E$ are subobjects of $E$ in $\mathcal{E}$, then
\[
\overline{M} \lor M' = \bigvee_{k \in \mathbb{N}} f^k(M \lor M') \\
= \bigvee_{k \in \mathbb{N}} f^k(M) \lor \bigvee_{k \in \mathbb{N}} f^k(M') \\
= \overline{M} \lor \overline{M'}.
\]

In the case when $f^n = \text{id}_E$ occurs for some $n \in \mathbb{N}$, this collapses to
\[
\overline{M} = M \lor f(M) \lor f^2(M) \lor \cdots \lor f^{n-1}(M).
\]

and of course, if $n = 2$, then we obtain Mazzola’s closure operator from the previous subsection. Furthermore, given an endomorphism $p : S \rightarrow S$ of $\mathcal{M}$, the definition above applies to the endomorphism $F(p) : F(S) \rightarrow F(S)$, and hence we have a Kuratowski operator on the subobjects of $F(S)$.

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