ON SOME FANO 4-FOLDS WITH LEFSCHETZ DEFECT 3

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Abstract. — We show that Fano 4-folds with Picard number 5 have Lefschetz defect 3 if and only if they are toric of combinatorial type $K$. We also find a characterization for such varieties in terms of Picard number of prime divisors. Moreover, we discuss classification results for 4-dimensional complex smooth projective varieties admitting some particular fiber type contractions.

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1. Introduction

Let us consider complex smooth and Fano varieties of dimension $n$. The classification of Fano manifolds has been achieved up to $n = 3$ and attracts a lot of attention also in higher dimensions, especially due to the MMP. Indeed we recall that Fano manifolds appear in the birational classification of varieties of negative Kodaira dimension: in this case the MMP is expected to end up with a fiber type morphism whose fibers are (mildly singular) Fano varieties.

In the early 80’s the classification of Fano 3-folds in [16] due to Mori and Mukai was the starting point to study Fano manifolds via their contractions. In fact, the Fano condition makes the situation special, because the Cone and the Contraction Theorems hold for the whole cone of effective curves.

Moreover, Birkar, Cascini, Hacon, and McKernan proved in [4] that Fano manifolds are Mori Dream Spaces. This allows to look more closely at the...
birational geometry of Fano varieties and to see a stronger behaviour of such manifolds with respect to Mori theory. See [14, 4] for details on Mori Dream Spaces. A modern different approach to classify Fano manifolds is done via mirror symmetry. We refer the reader to [7, 8] and reference therein for a recent account and results on this subject.

Apart from the techniques adopted, there is no complete classification of Fano varieties in dimension 4 and higher and we still lack a good understanding of the geometry of Fano 4-folds. See [6] for some recent results on Fano 4-folds with large Picard number, where these varieties are studied via birational geometry, throughout contractions and flips.

In this paper we focus on classification results of some Fano 4-folds, and to this end we follow the first and more classical approach together with some techniques and features of Mori Dream Spaces.

In order to introduce the main results of the paper, let us fix some notation. Given a manifold $X$, i.e. a complex smooth projective variety, we denote by $\mathcal{N}_1(X)$ the $\mathbb{R}$-vector space of one-cycles with real coefficients, modulo numerical equivalence, whose dimension is the Picard number $\rho_X$. Let $D \subset X$ be a prime divisor. The inclusion $i: D \hookrightarrow X$ induces a pushforward of one-cycles $i_*: \mathcal{N}_1(D) \to \mathcal{N}_1(X)$. We set $\mathcal{N}_1(D, X) := i_*(\mathcal{N}_1(D)) \subseteq \mathcal{N}_1(X)$, which is the linear subspace of $\mathcal{N}_1(X)$ spanned by numerical classes of curves contained in $D$. In [5] Casagrande introduced the following invariant, called Lefschetz defect:

$$\delta_X := \max\{\text{codim} \mathcal{N}_1(D, X) | D \subset X \text{ prime divisor}\}.$$  

In [5, Theorem 1.1] the author proved that if $X$ is a Fano manifold of arbitrary dimension $n$ with $\delta_X \geq 4$ then $X \cong S \times T$ with $S$ a del Pezzo surface, and $T$ a $(n-2)$-dimensional Fano manifold. As a consequence, all Fano 4-folds with $\delta_X \geq 4$ are well known, being product of two del Pezzo surfaces.

In this paper we deal with the case in which $X$ is a Fano 4-fold with $\delta_X = 3$. Under this assumption, by Theorem 2.5 and Proposition 2.4 (namely by [15, Theorem 1.1, Proposition 1.2]) we deduce that if $X$ is not a product of two del Pezzo surfaces, then $\rho_X \in \{5, 6\}$. Therefore in order to complete the classification of Fano 4-folds with $\delta_X = 3$ we are left to study the cases in which $\rho_X = 5$ and $\rho_X = 6$ (cf. Remark 4.2.2). Notice that by [15, Corollary 1.3] we already know that the varieties in which we are interested in are rational.

We complete the investigation when $\rho_X = 5$ by proving the following theorem which is the main result of this paper. We refer to [2, §4] for the terminology concerning toric varieties and their combinatorial type.

**Theorem 1.0.1.** — Let $X$ be a Fano 4-fold with $\rho_X = 5$. Then $\delta_X = 3$ if and only if $X$ is a toric variety of combinatorial type $K$.

Notice that on one hand Theorem 1.0.1 can be viewed as a classification result for Fano 4-folds with $\delta_X = 3$ and $\rho_X = 5$. On the other hand, it gives a characterization for toric varieties of combinatorial type $K$.

As a consequence, we also get the following characterization for such toric varieties in terms of Picard number of prime divisors.
Corollary 1.0.2. — Let $X$ be a Fano 4-fold with $\rho_X = 5$. Then $X$ is a toric variety of combinatorial type $K$ if and only if there exists a prime divisor $D \subset X$ such that $\rho_D = 2$.

Let us briefly discuss the strategy used to show Theorem 1.0.1. The fact that toric varieties of combinatorial type $K$ have Lefschetz defect 3 will easily follow from the Batyrev description in [2].

The most difficult part is to observe that if $X$ is a Fano 4-fold with $\delta_X = 3$ and $\rho_X = 5$ then $X$ is toric. To this end, we use the conic bundle structure $f: X \to Y$ with $\rho_X - \rho_Y = 3$ encoded by such a variety (cf. Theorem 2.5). Then the proof runs through such a fibration $f$, by taking a factorization of $f$ into extremal contractions. Indeed as we recall in Proposition 2.3, such a conic bundle $f$ admits a factorization

$$
\begin{array}{cccc}
X & \xrightarrow{f_1} & X_1 & \xrightarrow{f_2} & X_2 & \xrightarrow{g} & Y \\
\end{array}
$$

where $f_i$ are blow-ups along smooth surfaces of $X_i$ and $g$ is a smooth $\mathbb{P}^1$-fibration, that is called the elementary conic bundle in the factorization of $f$. At this point the crucial step is to classify the varieties which have such elementary conic bundle structure $g: X_2 \to Y$. Then analyzing the blow-ups $f_i$ we deduce that $X$ is toric and thanks to the classification of toric Fano 4-folds of [2] we are able to determine the variety $X$. The detailed study of $X_2$ arises from Theorem 3.0.1, which together with Proposition 3.0.2 is a classification result for 4-dimensional complex smooth projective varieties admitting some particular fiber type contractions.

Finally, we focus on Fano 4-folds with $\delta_X = 3$, and $\rho_X = 6$. One of the difference with the previous case is that in order to try to extend our methods we need to assume that in the above factorization for $f$, the variety $X_1$ is Fano. If this happens, we say that $X$ has a quasi-Fano bundle factorization (cf. Definition 4.2.1). Under this additional assumption, using Proposition 3.0.2 we deduce the possibilities for the variety $X_2$. This approach can be viewed as a first attempt to classify Fano 4-folds with $\delta_X = 3$ and $\rho_X = 6$, and as we point out in Remark 4.2.2 this is what we will need to complete the whole classification of Fano 4-folds with $\delta_X = 3$.

2. Notations and preliminaries

2.1. Notations and Conventions. — We work over the field of complex numbers. Let $X$ be a manifold, namely a smooth projective variety, of arbitrary dimension $n$. We call $X$ a Fano variety if $-K_X$ is an ample divisor.

$N_1(X)$ (respectively, $N^1(X)$) is the $\mathbb{R}$-vector space of one-cycles (respectively, divisors) with real coefficients, modulo numerical equivalence.

$\dim N_1(X) = \dim N^1(X) = \rho_X$ is the Picard number of $X$. 

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Let $C$ be a one-cycle of $X$, and $D$ a divisor of $X$. We denote by $[C]$ (respectively, $[D]$) the numerical equivalence class in $N_1(X)$ (respectively, $N^1(X)$). Moreover we denote by $\mathbb{R}[C]$ the linear span of $[C]$ in $N_1(X)$, and by $\mathbb{R}_{\geq 0}[C]$ the corresponding ray. The symbol $\equiv$ stands for numerical equivalence (for both one-cycles and divisors).

$\text{NE}(X) \subset N_1(X)$ is the convex cone generated by classes of effective curves.

A contraction of $X$ is a surjective morphism $\varphi: X \to Y$ with connected fibers, where $Y$ is normal and projective. If $\dim X > \dim Y$ we say that $\varphi$ is fiber type, otherwise it is birational.

We denote by $\text{Exc}(\varphi)$ the exceptional locus of $\varphi$, i.e. the locus where $\varphi$ is not an isomorphism. Given a subvariety $A \subset X$ we denote by $\mathcal{N}_{A/X}$ the normal bundle of $A$ in $X$.

The relative cone $\text{NE}(\varphi)$ of $\varphi$ is the convex subcone of $\text{NE}(X)$ generated by classes of curves contracted by $\varphi$.

A contraction of $X$ is called $K_X$-negative (or simply $K$-negative) if $-K_X \cdot C > 0$ for every curve $C$ contracted by $\varphi$.

A $\mathbb{P}^1$-bundle over a projective variety $Z$ is the projectivization of a rank 2 vector bundle on $Z$. A smooth $\mathbb{P}^1$-fibration is a smooth morphism such that every fiber is isomorphic to $\mathbb{P}^1$.

### 2.2. Lefschetz defect and Fano conic bundles

In this subsection we recall some crucial facts about the Lefschetz defect and Fano conic bundles which we need throughout the paper. We refer the reader to [18, 20, 17] for a deeper treatment concerning Fano conic bundles, and to [5, 9] for the properties of the Lefschetz defect.

Let $X$ be a smooth Fano variety and take $D \subset X$ a prime divisor. The inclusion $i: D \hookrightarrow X$ induces a pushforward of one-cycles $i_*: N_1(D) \to N_1(X)$, that does not need to be injective nor surjective.

We set $N_1(D, X) := i_*(N_1(D)) \subseteq N_1(X)$. Equivalently, $N_1(D, X)$ is the linear subspace of $N_1(X)$ spanned by classes of curves contained in $D$.

Working with $N_1(D, X)$ instead $N_1(D)$ means that we consider curves in $D$ modulo numerical equivalence in $X$, instead of numerical equivalence in $D$. Note that $\dim N_1(D, X) \leq \rho_D$.

In [5] Casagrande introduced the following invariant of $X$, called Lefschetz defect:

$$\delta_X := \max \{\text{codim} N_1(D, X) \mid D \text{ is a prime divisor of } X\}$$

We will relate $\delta_X$ with conic bundle structures of the variety, in the sense of the following:

**Definition 2.1.** — Let $X$ be a smooth, projective variety and let $Y$ be a normal, projective variety. A conic bundle $f: X \to Y$ is a fiber type $K$-negative contraction where every fiber is one-dimensional. A Fano conic bundle is a conic bundle $f: X \to Y$ where $X$ is a Fano variety.

**Definition 2.2.** — Let $f: X \to Y$ be a conic bundle. If $\rho_X - \rho_Y > 1$, $f$ is called non-elementary. Otherwise $f$ is called elementary.
Remark 2.2.1. — By [1, Theorem 3.1] it follows that given a conic bundle \( f: X \to Y \) as in Definition 2.2.1, the variety \( Y \) is also smooth. See also [19, Theorem 1.2] for a detailed proof concerning the flatness of \( f \), from which we get the smoothness of \( Y \) by [11, Proposition 17.3.3 (i)].

The following result holds in arbitrary dimension and in a more general situation but for the purposes of this paper we recall a weaker version when \( \dim X = 4 \) and \( \rho_X - \rho_Y = 3 \). The first three points are analyzed in [18, Proposition 3.5], the last item is proved in [18, Theorem 4.2 (2)].

Proposition 2.3. — Let \( f: X \to Y \) be a Fano conic bundle with \( \rho_X - \rho_Y = 3 \), and \( \dim X = 4 \). Then:

(a) There exists a factorization: \( X \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{g} Y \), where each \( f_i \) is a blow-up along a smooth surface of \( X_i \), and \( g \) is an elementary conic bundle;

(b) There exist prime divisors \( A_1, A_2 \) of \( Y \) such that \( f^*(A_i) = E_i + \hat{E}_i \), where \( E_i, \hat{E}_i \) are smooth prime divisors on \( X \), and \( E_i \to A_i \) and \( \hat{E}_i \to A_i \) are \( \mathbb{P}^1 \)-bundles for every \( i = 1, 2 \);

(c) Set \( \Delta_f := \{ y \in Y | f^{-1}(y) \text{ is singular} \} \) the discriminant divisor of \( f \). Then \( \Delta_f = A_1 \cup A_2 \), where \( A_i \) are smooth components of \( \Delta_f \) for \( i = 1, 2 \). Moreover, \( f \) has reduced fibers over \( A_1 \cup A_2 \).

(d) Let \( g: X_2 \to Y \) be as above. Then \( g \) is smooth, \( Y \) is also Fano and there exists a smooth \( \mathbb{P}^1 \)-fibration \( \xi: Y \to S \), where \( S \) is a del Pezzo surface.

In the next proposition we recall all the possible targets of Fano conic bundles \( f: X \to Y \) with \( \dim X = 4 \) and \( \rho_X - \rho_Y = 3 \).

Proposition 2.4. — [15, Proposition 1.2] Let \( f: X \to Y \) be a Fano conic bundle, where \( \dim X = 4 \), and \( \rho_X - \rho_Y = 3 \). Then \( 5 \leq \rho_X \leq 13 \). Moreover:

(a) If \( \rho_X = 5 \), then \( Y \) is one of the following Fano 3-folds: \( Y \cong \mathbb{P}^1 \times \mathbb{P}^2 \); \( Y \cong \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O}(1)) \); \( Y \cong \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O}(2)) \).

(b) If \( \rho_X = 6 \), then \( Y \) is one of the following Fano 3-folds: \( Y \cong \mathbb{P}^1 \times \mathbb{P}^1 \); \( Y \cong \mathbb{P}^1 \times \mathbb{P}^1(\mathcal{O}(-1, -1) \oplus \mathcal{O}) \); \( Y \cong \mathbb{P}^1 \times \mathbb{P}^1(\mathcal{O}(0, -1) \oplus \mathcal{O}(-1, 0)) \).

(c) If \( \rho_X \geq 7 \), then \( X \cong S_1 \times S_2 \), where \( S_1 \) is a del Pezzo surface with \( \rho_{S_1} = 4 \), \( Y \cong \mathbb{P}^1 \times S_1 \), and \( f \) is induced by a conic bundle \( S_1 \to \mathbb{P}^1 \).

Theorem 2.5. — [15, Theorem 1.1] Let \( X \) be a Fano 4-fold such that \( X \not\cong S_1 \times S_2 \), where each \( S_i \) is a del Pezzo surface. Then \( \delta_X = 3 \) if and only if there exists a conic bundle \( f: X \to Y \) such that \( \rho_X - \rho_Y = 3 \).

Let \( f: X \to Y \) be a Fano conic bundle, where \( X \) is toric with \( \rho_X = 5 \) and \( \rho_X - \rho_Y = 3 \). By Proposition 2.4 (a) we know that \( Y \cong \mathbb{P}^2(\mathcal{O} \oplus \mathcal{O}(a)) \) for some \( a \in \{0, 1, 2\} \). In particular \( X \) admits a locally trivial toric bundle over \( \mathbb{P}^2 \) whose fiber is a del Pezzo surface \( S \) with \( \rho_S = 4 \). Toric Fano 4-folds admitting such a fibration onto \( \mathbb{P}^2 \) were considered by Batyrev in [2, §3.2.9]. More precisely, we have the following:
Proposition 2.6. — Let $X$ be a toric Fano 4-fold with $\rho_X = 5$. Assume that there exists a conic bundle $f : X \to Y$ such that $Y \cong \mathbb{P}_2(\mathcal{O} \oplus \mathcal{O}(a))$ for some $a \in \{0, 1, 2\}$. Then $X \cong K_i$ for some $i \in \{1, 2, 3, 4\}$.

We recall by examples in [15, §5] that all the varieties $K_1, K_2, K_3, K_4$ admit a conic bundle $f : X \to Y$ onto $Y \cong \mathbb{P}_2(\mathcal{O} \oplus \mathcal{O}(a))$ for some $a \in \{0, 1, 2\}$.

3. Classification results

This section is devoted to discuss some classification results of 4-dimensional smooth projective varieties admitting particular fiber type contractions. The first result represents the key step to show Theorem 1.0.1. The listed assumptions arise in a natural way from our approach to study Fano 4-folds $X$ with $\delta_X = 3$ by looking at their conic bundle structure $f : X \to Y$ with $\rho_X - \rho_Y = 3$ (cf. Theorem 2.5). Indeed, taking a factorization for $f$ as in Proposition 2.3 (a), we will see that the variety which admits the elementary conic bundle structure in this factorization satisfies the hypothesis of the following theorem. Notice that Theorem 3.0.1 generalizes and improves [15, Proposition 4.1].

Theorem 3.0.1. — Let $\hat{X}$ be a 4-fold which admits a smooth $K$-negative $\mathbb{P}^1$-fibration $g : \hat{X} \to Y$, where $Y \cong \mathbb{P}_2(\mathcal{O} \oplus \mathcal{O}(a))$, with $a \in \{0, 1, 2\}$. Let us denote by $A_1, A_2$ two sections of $Y$, and let us consider the $\mathbb{P}^1$-bundles $g_{|\mathbb{P}^1_{x}^{-1}(A_i)} : D_i \to A_i$ for $i = 1, 2$. Assume that each $D_i$ has a section isomorphic to $A_i$ such the blow-up of $\hat{X}$ along this section is Fano. Then one of the following holds:

1. $\hat{X} \cong Y \times \mathbb{P}_2 \mathbb{P}_2(\mathcal{O} \oplus \mathcal{O}(b))$ with $b \in \{0, 1, 2\}$.
2. $\hat{X}$ is the blow-up of $\mathbb{P}_2(\mathcal{O} \oplus \mathcal{O}(c) \oplus \mathcal{O}(d))$ with $c \in \mathbb{Z}$, and $d \in \{0, 1, 2\}$ along a surface which is isomorphic to $\mathbb{P}^2$.

The strategy followed in some parts of the proof of Theorem 3.0.1 is similar to the one used in [15, Proposition 4.1] but here we are considering all the varieties $Y \cong \mathbb{P}_2(\mathcal{O} \oplus \mathcal{O}(a))$ with $a \in \{0, 1, 2\}$ and not only $Y \cong \mathbb{P}^1 \times \mathbb{P}^2$ as done in [15, Proposition 4.1]. Namely, by means of Proposition 2.4 (a) we are considering all the possible targets of Fano conic bundles $f : X \to Y$ where $\dim X = 4$ and $\rho_X - \rho_Y = 3$. This makes some parts of the proof more difficult and general, and it requires a more detailed analysis. We will refer to the proof of [15, Proposition 4.1] whenever we run the same arguments.

Proof of Theorem 3.0.1. — Let us consider the smooth $\mathbb{P}^1$-fibration $\xi : Y \to \mathbb{P}^2$. Take another extremal contraction $h$ of $\hat{X}$, different by $g$, and such that $\text{NE}(h) \subset \text{NE}(\xi \circ g)$. We get another factorization for $\psi : \xi \circ g : \hat{X} \to \mathbb{P}^2$, ...
given by $\tilde{X} \xrightarrow{h} Z \xrightarrow{\xi'} \mathbb{P}^2$, then the following commutative diagram:

$$
\begin{array}{ccc}
Y & \xrightarrow{\xi} & \mathbb{P}^2 \\
\downarrow{g} & & \downarrow{\xi} \\
\tilde{X} & \xrightarrow{\psi} & Z \\
\downarrow{h} & & \downarrow{\xi'} \\
\end{array}
$$

We show that $h$ is a $K$-negative contraction. For each $\mathbb{P}^1$-bundle $D_i \to A_i$ let us denote by $B_i$ the section which by our assumption is isomorphic to $A_i$. Assume by contradiction that there exists a curve $C \in \text{NE}(h)$ such that $K_{\tilde{X}} \cdot C \geq 0$. Since the blow-up of $\tilde{X}$ along $B_i$ is Fano, this means that $C \subset B_i$ for some $i = 1, 2$. Being $\rho_{B_i} = 1$, then $h$ should contract one of the $B_i$, which is impossible because by our assumption $B_i$ must dominate $\mathbb{P}^2$ through $\psi$.

We prove that all fibers of $h$ are one-dimensional \(^{(1)}\). By construction every fiber of $h$ is contained in a fiber of $\psi$, then it has dimension at most 2. Assume by contradiction that there exists a two-dimensional fiber of $h$. Then it is also a fiber of $\psi$ which is contracted by $h$. By the commutativity of the above diagram it follows that there is a fiber of $g$ which is contracted by $h$, and being $g$ and $h$ extremal this implies that $g = h$, a contradiction. Using [22, Corollary 1.4] there are only two possibilities for $h$:

\(^{(*)}\) $h$ is a conic bundle;

\(^{(**)}\) $h$ is a blow-up of $Z$ along a smooth surface.

Being $\psi$ a smooth $K$-negative contraction one has that for every $p \in \mathbb{P}^2$ the fiber $S_p := \psi^{-1}(p)$ is a smooth del Pezzo surface. As done in the proof of [15, Proposition 4.1] notice that in case \(^{(*)}\) all the fibers of $\psi$ are isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, while in the latter case \(^{(**)}\) they are isomorphic to $\mathbb{F}_1$. This is because of the deformation invariance of the Fano index (see for instance [10, Proposition 6.2]), then the fibers of $\psi$ must be all isomorphic each another.

Assume that we are in case \(^{(*)}\). Since $S_p \cong \mathbb{P}^1 \times \mathbb{P}^1$ for every $p \in \mathbb{P}^2$, and $h$ is an equidimensional morphism it follows that $Z$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$, and the fibers of $h$ are isomorphic to $\mathbb{P}^1$. In particular, the restrictions $\left. g \right|_{S_p^i} : S_p^i \to \mathbb{P}^1$, and $\left. h \right|_{S_p^i} : S_p^i \to \mathbb{P}^1$ corresponds to the two different natural projections. Let us consider the morphism given by the following composition

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{\tilde{f}} & X_1 \\
\downarrow{f_1} & & \downarrow{f_2} \\
\tilde{X} & \xrightarrow{h} & Z 
\end{array}
$$

where $f_i$ are blow-ups along $B_i$, respectively, and $\tilde{X}$ is Fano by our assumption. We observe that $\tilde{f}$ is a Fano conic bundle. To this end, we are left to prove

\(^{1}\) This is done as in the proof of [15, Proposition 4.1]. We recall the argument for the convenience of the reader.
that all the fibers of $\tilde{f}$ are one-dimensional. As we have already observed, each $B_i$ cannot be contracted by $h$. Take the $\mathbb{P}^1$-bundles $h_{|h^{-1}(h(B_i))}: \tilde{D}_i \to h(B_i)$ for $i = 1, 2$. Then $\tilde{D}_i$ are covered by fibers of $h$ which intersect transversally along $B_i$, because by our assumption $B_i$ are sections of $g$ and by the analysis on the fibers of $\psi$ it follows that the fibers of $g$ and $h$ have to intersect each other transversally at a point. Since $f_i$ are blow-ups along $B_i$, and all the fibers of $h$ are isomorphic to $\mathbb{P}^1$ it follows that $\tilde{f}$ has one-dimensional fibers, then it is a Fano conic bundle with discriminant divisor $\Delta_{\tilde{f}} = h(B_1) \cup h(B_2)$.

We observe that $\rho_{\tilde{X}} - \rho_Z = 3$, and applying Proposition 2.4 (a) to $\tilde{f}$, we obtain that $Z \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(b))$ with $b \in \{0, 1, 2\}$. Finally we get the commutative diagram:

$$
\begin{array}{ccc}
\tilde{X} & \xrightarrow{g} & Y \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(a)) \\
\downarrow h & & \downarrow \xi \\
Z \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(b)) & \xrightarrow{\xi'} & \mathbb{P}^2
\end{array}
$$

where we recall that $a, b \in \{0, 1, 2\}$, and with $\xi, \xi'$ smooth $\mathbb{P}^1$-fibrations. Then by the universal property of the fiber product, we get (1).

From now on suppose that (**) hold. Set $E := \text{Exc} (h)$. In the same spirit of the proof of [15, Proposition 4.1] we divide our argument in some steps.

**Step 1.** Take $C \cong \mathbb{P}^1$ a fiber of the $\mathbb{P}^1$-bundle $g_{|g^{-1}(A_1)}: D_1 \to A_1 \cong \mathbb{P}^2$. Then there exists a fiber $S_p \cong \mathbb{F}_1$ of $\xi \circ g: \tilde{X} \to \mathbb{P}^2$ such that $S_p \cap D_1 = C$. The same holds replacing $D_1$ with $D_2$.

**Proof of Step 1.** — We recall that $\psi := \xi \circ g: \tilde{X} \to \mathbb{P}^2$. Since $h$ is birational, as we have already observed all fibers of $\psi$ are isomorphic to $\mathbb{F}_1$. By the choice of $C$, one has $\{\xi(g(C))\} = \{p\}$, with $p \in \mathbb{P}^2$ a point. Take $S_p = \psi^{-1}(p) \cong \mathbb{F}_1$. We prove that $S_p$ cannot be contained in $D_1$. Assume by contradiction that $S_p \subset D_1$. Since $g_{|S_p}: S_p \to \mathbb{P}^1$ is a smooth $\mathbb{P}^1$-fibration, this implies that there would exist a fiber of $\xi$ contained in $g(D_1) = A_1$, which is impossible because $A_1$ dominates $\mathbb{P}^2$ thought $\psi$. It follows that $S_p \cap D_1$ is a curve and hence $S_p \cap D_1 = C$, since this intersection is reduced. Indeed, $C$ is contained in a fiber of $g_{|S_p}: S_p \to \mathbb{P}^1$ and all the fibers of $g_{|S_p}$ are isomorphic to $\mathbb{P}^1$. \hfill $\Box$

**Step 2.** For any point $p \in \mathbb{P}^2$, consider $S_p = \psi^{-1}(p) \cong \mathbb{F}_1$ and denote by $E_p$ the exceptional divisor of $h_{|S_p}: S_p \to \mathbb{P}^2$. Then $\{E_p\}_{p \in \mathbb{P}^2}$ is a family of $(-1)$-curves which covers $E$. Moreover, $E$ is a section of $g: \tilde{X} \to Y$.

**Proof of Step 2.** — Notice that each $E_p \subset E$, and being all fibers $S_p$ different, we obtain that $\{E_p\}_{p \in \mathbb{P}^2}$ is a family of $(-1)$-curves which covers $E$.

Take $\Gamma$ a fiber of $g$. We prove that $E \cdot \Gamma = 1$. Since all fiber of $g$ are algebraically equivalent as one-cycles, they are numerically equivalent, so that we can take a curve $C \subset D_1$ such that $g(C)$ is a point and $C \equiv \Gamma$. By Step
1, $S_p = \psi^{-1}(\xi(g(\Gamma)))$ is such that $D_i \cap S_p = C$. Since $\{E_p\}_{p \in \P^2}$ covers $E$, using the intersection theory of $\F_1 \cong S_p$ one has $E_p \cdot C = 1$. By our choice we get $E \cdot \Gamma = 1$.

Step 3. Let us denote by $A$ the center of the blow-up $h : \hat{X} \to Z$. Then $A$, $h(B_1)$, $h(B_2)$ are mutually disjoint sections of $\xi'$, and $Z \cong \P_2(\O \oplus \O(c) \oplus \O(d))$, with $c \in \Z$, and $d \in \{0, 1, 2\}$.

Proof of Step 3. — The fact that $Z$ is a $\P^2$-bundle over $\P^2$ is shown as in the proof of Step 2 of [15, Proposition 4.1]. We observe that $E \cap (B_1 \cup B_2) = \emptyset$.

To this end, let us recall by our assumptions that the blow-up of $\hat{X}$ along each $B_i$ is Fano. Using this information, as done in the proof of Step 1 of [15, Proposition 4.1], we deduce that if $B_i$ intersects a fiber of $h$ for some $i = 1, 2$, then this fiber has to be contained in such $B_i$, but this is impossible. Indeed, assume by contradiction there exists a fiber of $h$ contained in $B_i$ for some $i = 1, 2$. For simplicity, assume this happens for $i = 1$. Then $h(B_1)$ should be a point in $Z$ because $B_1 \cong A_1 \cong \P^2$, but $A_1$ is a section of $\xi' : Y' \to \P^2$, then we must have $h(B_1) \cong \P^2$. This implies that $E \cap (B_1 \cup B_2) = \emptyset$.

Then as in the proof of Step 2 of [15, Proposition 4.1] we deduce that $A$, $h(B_1)$, $h(B_2)$ are mutually disjoint sections of $\xi'$. This means that $Z = \P_2(\F)$, with $\F$ rank 3 decomposable vector bundle over $\P^2$.

Using Step 2 we know that $E \cong Y$, and by our assumption $Y \cong \P_2(\O \oplus \O(a))$, with $a \in \{0, 1, 2\}$. If $E \cong \P_2(\O \oplus \O(1))$, since $E \cong \P_2(\N'_{A_1/\Z})$, and that $A$ is a section of $\xi'$, then $Z \cong \P_2(\O \oplus \O(c) \oplus \O(1))$, with $c \in \Z$. If $E \cong \P_2(\O \oplus \O(2))$ then $Z \cong \P_2(\O \oplus \O(c) \oplus \O(2))$, with $c \in \Z$. If $Y \cong \P^1 \times \P^2$ then $Z \cong \P_2(\O \oplus \O \oplus \O(c))$, with $c \in \Z$. This implies that $E \cap (B_1 \cup B_2) = \emptyset$.

From this analysis it follows that when $h$ is birational we get part (2) of the statement, hence the claim.

As we will notice in Remark 5.0.1, the following proposition can be used as an intermediate step to classify Fano 4-folds with $\delta_X = 3$, and $\rho_X = 6$. The proof is gotten by running the same ideas of Theorem 3.0.1.

**Proposition 3.0.2.** — Let $\hat{X}$ be a 4-fold which admits a smooth $K$-negative $\P^1$-fibration $g : \hat{X} \to Y$, where $Y$ is one of the possible Fano 3-folds: $Y \cong \P^1 \times \P^1 \times \P^1$; $Y \cong \P_1 \times \P^1$; $Y \cong \P_{\P^1 \times \P^1}(\O(-1, -1) \oplus \O)$; $Y \cong \P_{\P^1 \times \P^1}(\O(0, -1) \oplus \O(-1, 0))$. Let us denote by $A_1$, $A_2$ two sections of $Y$, and consider the $\P^1$-bundles $g|_{g^{-1}(A_i)} : D_i \to A_i$, for $i = 1, 2$. Assume that each $D_i$ has a section isomorphic to $A_i$ such the blow-up of $\hat{X}$ along this section is Fano. Then one of the following holds:

1. $\hat{X} \cong \P^1 \times \P^1 \times \F_1$ or $\hat{X} = Y' \times_{\P^1 \times \P^1} Z$, where the possibilities for $Y'$ and $Z$ are the same listed above for $Y$ excluding $\F_1 \times \P^1$;
2. $\hat{X}$ is the blow-up along a surface isomorphic to $\P^1 \times \P^1$ of one of the following varieties: $\P_{\P^1 \times \P^1}(\O \oplus \O(0, -1) \oplus \O(-1, 0))$; $\P_{\P^1 \times \P^1}(\O \oplus \O(e, c, d) \oplus \O(e, f))$, with $(c, d) \in \{(0, 0), (-1, -1)\}$, and $(e, f) \in \Z \times \Z$. 

(3) $\hat{X}$ is the blow-up along a surface isomorphic to $F_1$ of $\mathbb{P}_{F_1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(c))$, with $c \in \mathbb{Z}$.

Proof. — Let us consider the smooth $\mathbb{P}^1$-fibration $\xi: Y \to S$ where depending on $Y$ one has $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $S \cong F_1$. As done in the proof of Theorem 3.0.1 we take another extremal contraction $h$ of $\hat{X}$, different by $g$, and such that $\text{NE}(h) \subset \text{NE}(\xi \circ g)$.

We get another factorization for $\psi := \xi \circ g: X \to S$, given by $\hat{X} \xrightarrow{h} Z \xrightarrow{\xi'} S$. It is easy to check that $h$ is a $K$-negative contraction with all fibers one-dimensional, so that again by [22, Corollary 1.4] there are only two possibilities for $h$:

(*) $h$ is a conic bundle;
(**) $h$ is a blow-up of $Z$ along a smooth surface.

If we are in case (*), as observed in the proof of Theorem 3.0.1 one has that $Z$ is a $\mathbb{P}^1$-bundle over $S$, and the fibers of $h$ are isomorphic to $\mathbb{P}^1$.

In the same way, we consider the morphism given by the following composition $\tilde{f}: \hat{X} \xrightarrow{f_1} X_1 \xrightarrow{f_2} \hat{X} \xrightarrow{h} Z$, where $f_i$ are blow-ups along surfaces $B_i \cong A_i$ for $i = 1, 2$, and we deduce that $\tilde{f}$ is a Fano conic bundle with $\rho_{\hat{X}} - \rho_Z = 3$.

Applying Proposition 2.4 (b) to $\tilde{f}$, we obtain all the possible $Z$. Then we have the following commutative diagram:

\[
\begin{array}{ccc}
\hat{X} & \xrightarrow{g} & Y \\
\downarrow h & & \downarrow \xi \\
Z & \xrightarrow{\xi'} & S \\
\end{array}
\]

with $S \cong F_1$ if $Y$ and $Z$ are both isomorphic to $F_1 \times \mathbb{P}^1$, otherwise $S \cong \mathbb{P}^1 \times \mathbb{P}^1$. In any case, as done in the proof of Theorem 3.0.1 we observe that $\hat{X}$ is the fiber product of $Y$ and $Z$ over $S$, so that we get part (1).

Assume now that (**) hold. We refer to the Steps used in the proof of Theorem 3.0.1, and we keep the same notation of that proof. Hence $E$ is the exceptional divisor of $h$, $A$ is the center of this blow-up, and we recall that $B_i$ are the two sections of $D_i$ which by our assumption are isomorphic to $A_i$. Let us prove that $B_i$ cannot intersect the fibers of $h$.

To this end, as done in the proof of Step 1 of [15, Proposition 4.1] we deduce that if a fiber of $h$ intersects $B_i$ for some $i = 1, 2$, then this fiber has to be contained in $B_i$. Assume by contradiction that this is the case, and for simplicity suppose $i = 1$.

If $B_1 \cong \mathbb{P}^1 \times \mathbb{P}^1$ then $h(B_1) = \mathbb{P}^1$ and this is impossible because $\xi'(h(B_1)) = \xi(g(B_1)) = \mathbb{P}^2$. Then $B_1 \cong F_1$, so that $S \cong F_1$ because by assumption $B_1 \cong A_1$ and $A_1$ is a section of $\xi$. Then $Y \cong F_1 \times \mathbb{P}^1$. Let us consider the following composition of contractions:

$\Psi: \hat{X} \xrightarrow{g} Y \cong F_1 \times \mathbb{P}^1 \xrightarrow{\pi} \mathbb{P}^1$
We show that $\Psi$ is finite on the fibers of $h$. Let $F \cong \mathbb{P}^1$ be a fiber of $h$.

Being $h$ and $g$ different extremal contractions of $\hat{X}$, it follows that $g(F)$ is a curve in $Y$, and by the commutativity of the above diagram we obtain that $\xi(g(F))$ is a point $p$ in $\mathbb{F}_1$. Being $\xi: \tilde{Y} \cong \mathbb{F}_1 \times \mathbb{P}^1 \to \mathbb{F}_1$ the projection, one has $g(F) = \{p\} \times \mathbb{P}^1$ which cannot be contracted by $\pi$. Since $\Psi$ contracts the divisors $D_i$ and $B_i \subset D_i$, one has that each $B_i$ cannot contain the fibers of $h$, and this is against our initial assumption. Therefore $E \cap (B_1 \cup B_2) = \emptyset$.

Steps 1 and 2 of the proof of Theorem 3.0.1 are shown in the same way, replacing the target of $\xi \circ g$ with $S$ being either $\mathbb{F}_1$ (if $Y \cong \mathbb{P}^1 \times \mathbb{F}_1$) or $\mathbb{P}^1 \times \mathbb{P}^1$ for the other possible $Y$. By repeating the same argument as in the proof of Step 2 of [15, Proposition 4.1] we observe $Z$ is a $\mathbb{P}^2$-bundle over $S$.

Using that $E \cap (B_1 \cup B_2) = \emptyset$, by the same proof it follows that $Z$ has three disjoint sections corresponding to $A$, $h(B_1)$, $h(B_2)$, hence $Z \cong \mathbb{P}_S(\mathcal{F})$ with $\mathcal{F}$ a rank 3 decomposable vector bundle over $S \cong \mathbb{F}_1$ if $Y \cong \mathbb{P}^1 \times \mathbb{F}_1$, over $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ in the other cases for $Y$.

We have already observed that $E \cong Y$. Now, if $E \cong \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, -1) \oplus \mathcal{O})$, using that $E \cong \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{N}_{A/Z})$, and that $A$ is a section of $\xi': Z \to \mathbb{P}^1 \times \mathbb{P}^1$ we get $Z \cong \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1, -1) \oplus \mathcal{O}(d, e))$ for $(d, e) \in \mathbb{Z} \times \mathbb{Z}$. Applying the same method for the other possibilities for $E \cong Y$ we obtain part (2) and (3) of the statement, hence the statement follows.

4. Discriminant divisors and conic bundle factorizations

The aim of this section is to built some intermediate steps which will be needed to show Theorem 1.0.1. In the first subsection, we give the complete description of the discriminant divisor of a Fano conic bundle $f: X \to Y$ with $\dim X = 4$ and $\rho_X - \rho_Y = 3$. In the second one, keeping in mind a factorization of $f$ as in Proposition 2.3 (a), we introduce the new notions of quasi-Fano and Fano type factorizations, and we prove existence results of such special factorizations.

4.1. Description of discriminant divisors. — Given a Fano conic bundle $f: X \to Y$, with $\dim X = 4$ and $\rho_X - \rho_Y = 3$, the goal of this subsection is to give an explicit description of the discriminant divisor $\Delta_f$ of $f$. This is a divisor of $Y$ that was introduced in [3] and studied further in [21, §1]. See also [17, §3.2] for a complete exposition about the main geometric properties related to $\Delta_f$. Let us recall its definition:

$$\Delta_f = \{ y \in Y \mid f^{-1}(y) \text{ is singular} \}.$$ 

More precisely, our aim is to find the components of $\Delta_f$ that in our case are two smooth disjoint divisors of $Y$, as we recalled in Proposition 2.3 (c). Notice that some partial results in this direction were gotten in [15, Corollary 3.4].

**Proposition 4.1.1.** Let $f: X \to Y$ be a Fano conic bundle, where $\dim X = 4$, and $\rho_X - \rho_Y = 3$. Let us take the smooth $\mathbb{P}^1$-fibration $\xi: Y \to S$ as in Proposition 2.3 (d), and $\Delta_f = A_1 \sqcup A_2$ the discriminant divisor of $f$, where
with $A_i$ its smooth components. Then $A_i$ are sections of $\xi: Y \to S$ for each $i = 1, 2$.

Proof. — By Proposition 2.4 we know that $5 \leq \rho_X \leq 13$, and if $\rho_X \geq 7$ then $X \cong S_1 \times S_2$ where each $S_i$ is a del Pezzo surface, $\rho_{S_i} = 4$, and $f$ is induced by a conic bundle $\tilde{f}: S_1 \to \mathbb{P}^1$. This means that $Y \cong \mathbb{P}^1 \times S_2$ with $f = (\tilde{f}, id_{S_2})$, being $id_{S_2}: S_2 \to S_2$ the identity morphism. We get the claim just by looking at the construction of $f$ (see [17, pag. 45-46] for details on induced conic bundles on del Pezzo surfaces and description of discriminant divisors).

Assume that $\rho_X = 5$. In this case the target of the smooth $\mathbb{P}^1$-fibration $\xi$ is $S \cong \mathbb{P}^2$. By Proposition 2.4 (a) one has $Y \cong \mathbb{P}_{p^2}(O \oplus O(a))$ with $a \in \{0, 1, 2\}$. By [15, Corollary 3.4] we are left to analyze what happens for $a \in \{1, 2\}$.

In the proof of [15, Proposition 1.2] it has been shown that this is the case in which only one of the divisor $A_i$ is nef, say $A_1$, while $A_2$ is the exceptional divisor of the birational contraction $\phi: Y \to Z$ given by $|mA_1|$, with $m \gg 0$, $m \in \mathbb{N}$, and $A_2$ is a section of $\xi: Y \to \mathbb{P}^2$. It remains to show that $A_1$ is also a section of $\xi$.

To this end, we use a factorization of $f$ as in Proposition 2.3 (a), and we keep the same notation. We denote by $B_i$ the blow-up centers of $f_i$. Let us consider the $\mathbb{P}^1$-bundles $g_{g^{-1}(A_i)}: D_i \to A_i$ for $i = 1, 2$. By the conic bundle structure, we know that $B_i$ is a section of $g_{g^{-1}(A_i)}: D_i \to A_i$ for $i = 1, 2$ (cf. [18, Proposition 3.5]). Moreover, let us take the prime divisors $E_i$, $\tilde{E}_i$ of $X$ such that $f^*(A_i) = E_i + \tilde{E}_i$, and $E_i \to A_i$, $\tilde{E}_i \to A_i$ are $\mathbb{P}^1$-bundles for $i = 1, 2$, as observed in Proposition 2.3 (b). We recall by Remark [18, Remark 3.7] that such a factorization of $f$ is not unique and depends on the choice of extremal rays spanned by $[e_1]$, $[\tilde{e}_1]$, with $e_1 \subset E_i$ and $\tilde{e}_1 \subset \tilde{E}_i$ the corresponding fibers for $i = 1, 2$. Moreover, each $E_i$ is either the exceptional divisor of $f_1$ or the strict transform in $X$ of the exceptional divisor of $f_2$, while each $\tilde{E}_i$ is the strict transform in $X$ of $D_i$.

Without loss of generality we can assume that $f_1$ corresponds to the contraction of the extremal ray $R_1 = \mathbb{R}_{\geq 0}[e_1]$, and $f_2$ is the contraction given by the extremal ray $R_2 = \mathbb{R}_{\geq 0}[\tilde{e}_2]$.

Let us take a section $\tilde{S}$ of $Y$, such that $\tilde{S} \cap A_2 = \emptyset$. We deduce that $\tilde{S} = A_1$. We first prove that $\tilde{S} \cap A_1 \neq \emptyset$. Let $C \subset \tilde{S}$ be an irreducible curve. Being $A_1$ a nef divisor of $Y$ we know that $A_1 \cdot C \geq 0$. We observe that $A_1 \cdot C > 0$ otherwise $\tilde{S}$ should be contracted by $\phi$, that is impossible because $\phi$ is an isomorphism outside $A_2$. Then $\tilde{S} \cap A_1 \neq \emptyset$, and in particular we can take an irreducible curve $\tilde{C} \subset A_1 \cap \tilde{S}$. Using Proposition 2.3 (d) we know that $g$ is a smooth $\mathbb{P}^1$-fibration, then $g_{g^{-1}(\tilde{S})}: g^{-1}(\tilde{S}) \to \tilde{S}$ is a $\mathbb{P}^1$-bundle.

Set $\tilde{S} := g^{-1}(\tilde{S})$, and continue to denote by $\tilde{S}$ its strict transform in $X$. We recall that $\tilde{E}_1$ is the strict transform of $D_1$ in $X$, and that $f^*(A_1) = E_1 + \tilde{E}_1$, thus the fibers over $A_1$ are reducible. In particular, for every point $p \in \tilde{C}$, one has that $f^{-1}(p)$ has an irreducible component contained in the exceptional divisor $E_1$ of $f_1$, and the other irreducible component is contained in $\tilde{E}_1 \cap \tilde{S}$. 
This implies that \( \hat{e}_1 \equiv \alpha \bar{s} \) where \( \bar{s} \subset \hat{S} \) is a fiber, and \( \alpha \in \mathbb{R} \). Thus if we change the factorization of \( f \) by replacing \( f_1 \) with the contraction of the extremal ray \( R = \mathbb{R}_{\geq 0}[\hat{e}_1] \), we deduce that \( \hat{E}_1 = \hat{S} \), so that \( f(\hat{E}_1) = A_1 = f(\hat{S}) = \hat{S} \).

Finally, suppose that \( \rho_X = 6 \). All possible targets of \( f \) are given by Proposition 2.4 (b). Looking at the smooth \( \mathbb{P}^1 \)-fibration \( \xi : Y \to S \) one has \( S \cong F_1 \) if \( Y \cong \mathbb{P}^1 \times F_1 \), while \( S \cong \mathbb{P}^1 \times \mathbb{P}^1 \) in the other possible cases for \( Y \).

If \( Y \cong \mathbb{P}^1 \times F_1 \) or \( Y \cong \mathbb{P}^1 \times \mathbb{P}^1 \), the statement has been proved in [15, Corollary 3.4]. By the proof of the same corollary, the claim follows also when \( Y \cong \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(0, -1) \oplus \mathcal{O}(-1, 0)) \). It remains to analyze the case in which \( Y \cong \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}) \). As observed in the proof of [15, Proposition 1.2] this happens when only one between \( A_1 \) and \( A_2 \) is a nef divisor. Then we run the above arguments to reach the statement.

Using Propositions 4.1.1 and 2.4, in the first column of the following table we write down all the possible targets of Fano conic bundles \( f : X \to Y \) with \( \dim X = 4 \), \( \rho_X - \rho_Y = 3 \) and \( \rho_X \in \{5, 6\} \). In the second column we give the complete description of the corresponding discriminant divisors.

| \( Y \) | \( \Delta_f \) |
|------|-----|
| \( \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(a)) \) with \( a \in \{0, 1, 2\} \) | \( \mathbb{P}^2 \sqcup \mathbb{P}^2 \) |
| \( \mathbb{P}^1 \times \mathbb{P}^1 \) | \( \mathbb{P}^1 \sqcup \mathbb{P}^1 \) |
| \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) | \( \mathbb{P}^1 \times \mathbb{P}^1 \sqcup \mathbb{P}^1 \times \mathbb{P}^1 \) |
| \( \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}) \) | \( \mathbb{P}^1 \times \mathbb{P}^1 \sqcup \mathbb{P}^1 \times \mathbb{P}^1 \) |
| \( \mathbb{P}_{\mathbb{P}^2 \times \mathbb{P}^2}(\mathcal{O}(0, -1) \oplus \mathcal{O}(-1, 0)) \) | \( \mathbb{P}^1 \times \mathbb{P}^1 \sqcup \mathbb{P}^1 \times \mathbb{P}^1 \) |

4.2. Conic bundle factorizations of quasi-Fano and Fano type. — For the purpose of this paper a crucial point is that given a non-elementary Fano conic bundle \( f : X \to Y \) we can consider a factorization as in Proposition 2.3 (a). Indeed, as we pointed out in the Introduction, in order to classify Fano 4-folds with \( \delta_X = 3 \) our strategy is to analyze the extremal contractions which factorize the non-elementary conic bundles encoded by such varieties (cf. Theorem 2.5). We recall by [18, Remark 3.7] that the factorization of \( f \) is not unique, hence it makes sense to introduce the new following

**Definition 4.2.1.** — Let \( f : X \to Y \) be a Fano conic bundle with \( \rho_X - \rho_Y = 3 \). We say that \( X \) has a conic bundle factorization of quasi-Fano type if there exists a factorization for \( f \) as described in Proposition 2.3 (a) where \( X_1 \) is Fano. If there exists such a factorization where both \( X_1 \) and \( X_2 \) are Fano, we say that \( X \) has a conic bundle factorization of Fano type. For short, we will also refer to a quasi-Fano type (or Fano-type) factorization.

**Lemma 4.1.** — Let \( f : X \to \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(a)) \) be a Fano conic bundle with \( \rho_X = 5 \), and \( a \in \{0, 1, 2\} \). Then \( X \) has a conic bundle factorization of quasi-Fano type. If \( a = 0 \) then \( X \) has a conic bundle factorization of Fano type.
**Proof of Lemma 4.1.** — Let us take a factorization for $f$ as in Proposition 2.3 (a), given by

\[ X \xrightarrow{f} \overline{X}_1 \xrightarrow{f_2} \overline{X}_2 \xrightarrow{g} Y. \]

We denote by $A_i$ the two components of the discriminant divisor of $f$ as in Proposition 2.3 (c). The statement has been proved in [15, Lemma 4.2] when $a = 0$, then we are left to analyze the cases $a \in \{1, 2\}$.

By Proposition 2.3 (b) we know that there exist two prime divisors $E_i$, $\overline{E}_i$ of $X$ such that $f^*(A_i) = E_i + \overline{E}_i$, and $E_i \to A_i$, $\overline{E}_i \to A_i$ are $\mathbb{P}^1$-bundles for $i = 1, 2$. Let us denote by $e_i \subset E_i$, $\hat{e}_i \subset \overline{E}_i$ the fibers for $i = 1, 2$.

By Proposition 2.3 (d) one has that $\overline{g}$ is a smooth $\mathbb{P}^1$-fibration so that $D_i := \overline{g}^*(A_i)$ are $\mathbb{P}^1$-bundles over $A_i$. By the conic bundle structure, each $D_i$ contains a section $B_i \cong A_i$ (cf. [18, Proposition 3.5]), such that $\overline{f}_i$ is the blow-up of $X_i$ along $B_i$ for $i = 1, 2$. Moreover, each $E_i$ is either the exceptional divisor of $\overline{f}_i$ or the strict transform in $X$ of the exceptional divisor of $f_2$, while each $\overline{E}_i$ is the strict transform in $X$ of $D_i$.

Applying [18, Remark 3.7] we know that the factorization of $f$ is not unique and depends on the choice of the extremal rays spanned by $[e_i]$, $[\hat{e}_i]$. To fix the notation, assume that $\overline{f}_1$ corresponds to the contraction of the extremal ray $R_1 = \mathbb{R}_{\geq 0}[e_1]$, and $\overline{f}_2$ is the contraction given by the extremal ray $R_2 = \mathbb{R}_{\geq 0}[e_2]$.

Using these facts, we prove that up to changing the factorization of $f$ we obtain that the target of the first blow-up of the factorization of $f$ is a Fano variety. Assume that $\overline{X}_1$ is not Fano. Then there exists an irreducible curve $C \subset \overline{X}_1$ such that $-K_{\overline{X}_1} \cdot C \leq 0$, and $C \subset B_1$. We recall by the proof of [15, Proposition 1.2] that one between $A_1$ and $A_2$ is a nef divisor of $Y$, and without loss of generality we can assume that it is $A_1$, so that $D_1 = \overline{g}^*(A_1)$ is a nef divisor of $\overline{X}_2$. Let us take the transform of $D_1$ in $\overline{X}_1$, which we continue to denote by $D_1$. We prove that $D_1$ is not Fano.

By the adjunction formula we get

\[ -K_{D_1} \cdot C = -K_{\overline{X}_1} \cdot C - D_1 \cdot C \leq 0. \]

Since $\overline{E}_1$ is the strict transform in $X$ of the divisor $D_1$, it follows that $\overline{E}_1$ is not a Fano variety.

By Proposition 4.1.1 (see also the table after this proposition) we deduce that $\rho_{A_1} = 1$, so that $\rho_{\overline{E}_1} = 2$. Now we replace $\overline{f}_1$ with the contraction of the extremal ray spanned by $[\hat{e}_1]$. In this way we get another blow-up $f_1: X \to X_1$ where $X_1$ is Fano by [22, Proposition 3.4]. \qed

**Remark 4.2.2.** — All Fano 4-folds $X$ with $\delta_X = 3$ and $\rho_X \geq 7$ are well known and correspond to product of del Pezzo surfaces $X \cong S_1 \times S_2$, with $\max\{\rho_{S_1} - 1, \rho_{S_2} - 1\} = \delta_X = 3$ (see [5, Example 3.1] for this equality). In fact, if $X$ is not a product of two del Pezzo surfaces, applying Theorem 2.5, and
Proposition 2.4 (c) we reach a contradiction. Then once we show Theorem 1.0.1, in order to conclude the classification of Fano 4-fold with $\delta_X = 3$ we are left to analyze the case $\rho_X = 6$. One of the problem to extend our methods to Fano 4-folds with $\delta_X = 3$ and $\rho_X = 6$ is that we still do not know if such Fano 4-folds have a conic bundle factorization of quasi-Fano type (cf. Definition 4.2.1). In fact, as will be shown in the next section, having a quasi-Fano type factorization is a key point to prove Theorem 1.0.1. We refer the reader to [15, §5] for examples of Fano conic bundles $f: X \to Y$ with $X$ toric 4-fold with $\rho_X = 6$ and $\rho_X - \rho_Y = 3$. We do not know examples of non toric Fano 4-folds with $\delta_X = 3$ and $\rho_X = 6$.

5. Proofs of the main results

The goal of this section is to prove Theorem 1.0.1 and Corollary 1.0.2. To this end, we collect the main results of the previous sections, especially Theorem 3.0.1, Proposition 4.1.1, and Lemma 4.1.

Proof of Theorem 1.0.1. — Suppose that $X$ is a Fano 4-fold with $\rho_X = 5$ and $\delta_X = 3$. We first observe that the condition $\delta_X = 3$ implies that $X$ admits a conic bundle $f: X \to Y$ with $\rho_X - \rho_Y = 3$. To this end, assume that $X \cong S_1 \times S_2$ with $S_i$ del Pezzo surfaces. By [5, Example 3.1] we know that $\max\{\rho_{S_1} - 1, \rho_{S_2} - 1\} = \delta_X = 3$ hence we get $\rho_{S_i} = 4$ for some $i = 1, 2$. Without loss of generality we can assume $\rho_{S_1} = 4$. Then there is a Fano conic bundle $f = (\tilde{f}, id_{S_2}): X \to Y \cong \mathbb{P}^1 \times S_2$ where $\tilde{f}: S_1 \to \mathbb{P}^1$ is a conic fibration, $id_{S_2}: S_2 \to S_2$ is the identity morphism, and $\rho_X - \rho_Y = 3$. If $X \not\cong S_1 \times S_2$, the claim follows by Theorem 2.5.

Applying Proposition 2.4 (a) to such a conic bundle $f: X \to Y$ we get $Y \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(a))$ with $a \in \{0, 1, 2\}$. Let us take a factorization for $f$ as in Proposition 2.3 (a), and let us keep the same notation.

By Lemma 4.1 we can assume that such a factorization is of quasi-Fano type. Moreover, using Proposition 2.3 (d) we know that the elementary conic bundle of this factorization $g: X_2 \to Y$ is a smooth $\mathbb{P}^1$-fibration. Applying Proposition 4.1.1 one has that the two smooth component $A_i$ of the discriminant divisor $\Delta_f$ of $f$ are sections of the smooth $\mathbb{P}^1$-fibration $\xi: Y \to \mathbb{P}^2$.

On the other hand, by the conic bundle structure we know that the $\mathbb{P}^1$-bundles $g_{y^{-1}(A_i)}: D_i \to A_i$ have sections $B_i \cong A_i$ such that $X$ is gotten by the blow-up along $B_i$ (see [18, Proposition 3.5]). To fix the notation, assume that each $f_i$ is the blow-up along $B_i$.

All the above observations imply that $X_2$ satisfies the assumptions of Theorem 3.0.1 from which we get the following possibilities:

(*) $X_2$ is the blow-up along a surface isomorphic to $\mathbb{P}^2$ of $\mathbb{P}^2_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(c) \oplus \mathcal{O}(d))$ with $c \in \mathbb{Z}$, and $d \in \{0, 1, 2\}$;

2. Note that more in general if $X$ is a Fano variety of dimension $n$ such that $X \cong S \times T$ with $S$ a del Pezzo surface and $T$ a $(n - 2)$-dimensional variety, the proof of [18, Theorem 4.2 (1)] shows that all conic bundle structure on $X$ are induced by a conic bundle $S \to \mathbb{P}^1$.目的地
(**) $X_2 \cong \mathbb{P}^2 (\mathcal{O} \oplus \mathcal{O}(a)) \times_{\mathbb{P}^2} \mathbb{P}^2 (\mathcal{O} \oplus \mathcal{O}(b))$ with $a, b \in \{0, 1, 2\}$.

We show that if case (*) holds, then $X$ is a toric variety. Set $\mathcal{E} := \mathcal{O} \oplus \mathcal{O}(c) \oplus \mathcal{O}(d)$ with $c, d \in \{0, 1, 2\}$, and $Z := \mathbb{P}^2 (\mathcal{E})$. Let us denote by $h : X_2 \rightarrow Z$ the blow-up along the surface $A \cong \mathbb{P}^2$. As observed in the proof of Step 3 of Theorem 3.0.1 (replacing $\mathcal{X}$ with $X_2$) one has that $h(B_1)$, $h(B_2)$ and $A$ are disjoint sections of $\mathbb{P}^2 (\mathcal{E}) \rightarrow \mathbb{P}^2$. Then looking at the composition of contractions

$$X \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{h} Z = \mathbb{P}^2 (\mathcal{E})$$

we deduce that $X$ is obtained as the blow-up of three disjoint sections isomorphic to $\mathbb{P}^2$ of the $\mathbb{P}^2$-bundle $\mathbb{P}^2 (\mathcal{E})$.

We show that the three centers of the blow-up correspond to the three possible quotients of the vector bundle $\mathcal{E}$ given respectively by the surjections $\mathcal{E} \rightarrow \mathcal{O}$, $\mathcal{E} \rightarrow \mathcal{O}(c)$, and $\mathcal{E} \rightarrow \mathcal{O}(d)$. By [13, Ex. 7.8] there is a correspondence between sections of the projective bundle $\pi : \mathbb{P}^2 (\mathcal{E}) \rightarrow \mathbb{P}^2$ and invertible quotient bundles of $\mathcal{E}$. Being $\mathcal{E}$ a decomposable bundle, its possible subbundles are its direct vector bundle summands. Let us take a section of $\pi$ corresponding to one of the possible direct line bundle summands of $\mathcal{E}$. We get an exact sequence:

$$0 \rightarrow \mathcal{O}(e) \oplus \mathcal{O}(f) \rightarrow \mathcal{E} \rightarrow \mathcal{L} \rightarrow 0$$

with $e, f \in \{0, c, d\}$ suitable taken so that $\mathcal{O}(e) \oplus \mathcal{O}(f)$ is a direct vector bundle summand of $\mathcal{E}$. Since $H^1 (\mathbb{P}^2, \mathcal{L}^\vee \otimes (\mathcal{O}(e) \oplus \mathcal{O}(f))) = 0$ then (1) splits, so that $\mathcal{L}$ corresponds to one of the possible direct line bundle summands of $\mathcal{E}$. This means that $X$ is gotten by blowing up invariant sections, then it is toric.

Being $X$ Fano with $\rho_X = 5$ by assumption, Proposition 2.6 allows to deduce that it has to be of combinatorial type $K$. In particular, looking at the Batyrev description in [2] we deduce the following possibilities for $X$: it is of type $K_1$ and $Z = \mathbb{P}^2 (\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(2))$ or $X$ of type $K_3$ and $Z = \mathbb{P}^2 (\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))$ or $X$ is of type $K_4$ and $Z = \mathbb{P}^2 \times \mathbb{P}^2$ (namely $c = d = 0$ in this latter case).

Now we analyze case (**). Set $\tilde{Z} := \mathbb{P}^2 (\mathcal{O} \oplus \mathcal{O}(b))$ with $b \in \{0, 1, 2\}$. By the proof of Theorem 3.0.1 we recall that case (**) occurs when $X_2$ has another smooth $K$-negative $\mathbb{P}^1$-fibration $h : X_2 \rightarrow \tilde{Z}$ different by $g$.

As done in the proof of case (*) of Theorem 3.0.1 we deduce that the morphism

$$f : X \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \xrightarrow{h} \tilde{Z}$$

is a Fano conic bundle (with $Z = \tilde{Z}$). Then we are in the situation in which there exist two conic bundles $f$ and $\tilde{f}$ admitting a factorization as in Proposition 2.3 (a) such that the two blow-ups $f_1$, and $f_2$ are the same and the variety $X_2$ has two extremal fiber type contractions $X_2 \xrightarrow{\delta} Y \cong \mathbb{P}^2 (\mathcal{O} \oplus \mathcal{O}(a))$ with $a \in \{0, 1, 2\}$, and $X_2 \xrightarrow{h} \tilde{Z}$ which are smooth $\mathbb{P}^1$-fibrations.
Namely we have the following diagram, where we also add the two smooth $\mathbb{P}^1$-fibrations $\xi$ and $\xi'$:

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X_1 \xrightarrow{f_1} X_2 \xrightarrow{g} Y \\
\Downarrow & & \Downarrow & & \Downarrow \\
\tilde{Z} & \xrightarrow{\xi} & \mathbb{P}^2 \\
\end{array}
\]

Claim. Let us denote by $\Delta_f = A_1 \cup A_2$ and $\Delta_{\tilde{f}} = \tilde{A}_1 \cup \tilde{A}_2$ respectively the discriminant divisor of $f$ and $\tilde{f}$. Take $D_i = g^*(A_i)$, and $\tilde{D}_i = h^*(\tilde{A}_i)$ for each $i = 1, 2$. We prove that $D_i \cap \tilde{D}_i = B_i$ with $B_i$ center of the blow-up $f_i$, and $B_i \cong \mathbb{P}^2$. Moreover $D_i \cong \tilde{Z}$, and $D_i \cong Y$ for $i = 1, 2$.

Proof of the claim. — As we have already observed we are in the situation in which the two Fano conic bundles $f: X \rightarrow Y$ and $\tilde{f}: X \rightarrow \tilde{Z}$ have a factorization as in Proposition 2.3 (a) such that the two blow-ups $f_i$ of these factorizations are the same contractions (see the diagram above). Then by the conic bundle structure, on one hand $\text{Exc} (f_i)$ is the exceptional divisor obtained blowing up a surface $B_i$ inside $D_i$, on the other hand it is gotten by blowing up a surface $\tilde{B}_i$ inside $\tilde{D}_i$. This means that $B_i = \tilde{B}_i$.

By the proof of [18, Proposition 3.5] we recall that the fibers of $g$ over $A_i$ intersects $D_i$ transversally at one point, then each $\mathbb{P}^1$-bundle $g_{|g^{-1}(A_i)}: D_i \rightarrow A_i$ has a section which is isomorphic to $A_i$ and this corresponds to the center $B_i$ of the blow-up $f_i$. Looking at the fibers of $h$ over $A_i$ we can run the same argument for the divisors $\tilde{D}_i$, then $D_i \cap \tilde{D}_i = B_i$. Moreover, using Proposition 4.1.1 it follows that $A_i$ are sections of $\xi: Y \rightarrow \mathbb{P}^2$ (respectively, $\tilde{A}_i$ are sections of $\xi': Y \rightarrow \mathbb{P}^2$) for $i = 1, 2$, then in particular $B_i \cong \mathbb{P}^2$ for every $i = 1, 2$. We know that $X_2$ is the fiber product of $Y$ and $Z$ over $\mathbb{P}^2$, and by the above arguments the divisors $D_i$ (respectively $\tilde{D}_i$) are sections of $h: X_2 \rightarrow \tilde{Z}$ (respectively of $g: X_2 \rightarrow Y$). Hence $D_i \cong \tilde{Z}$, and $\tilde{D}_i \cong Y$ for $i = 1, 2$. \hfill $\Box$

Being $X_2 = Y \times_{\mathbb{P}^2} Z$, with $Y$ and $Z$ toric varieties, $X_2$ is toric as well (see for instance [12, §2.6.5]). By the above claim it follows that the two surfaces $B_i$ that we blow-up from $X_2$ to get $X$ are invariant sections with respect to the action of the torus on $X_2$, then $X$ is toric. Since by our assumption $X$ is also Fano with $\rho_X = 5$, Proposition 2.6 again allows to deduce that $X$ is of combinatorial type $K$.

To get the statement we are left to prove that if $X$ is a toric Fano 4-fold of type $K$, then $\delta_X = 3$. To this end, we use Batyrev description of [2]. Suppose that $X$ is of type $K_4$, namely that $X \cong S \times \mathbb{P}^2$ with $S$ del Pezzo surface such that $\rho_S = 4$. Then [5, Example 3.1] gives $\delta_X = 3$. In the remaining other three cases, from Batyrev classification of toric Fano 4-folds we know that these varieties are obtained as blow-ups of surfaces isomorphic to $\mathbb{P}^2$ from

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X_1 \xrightarrow{f_1} X_2 \xrightarrow{g} Y \\
\Downarrow & & \Downarrow & & \Downarrow \\
\tilde{Z} & \xrightarrow{\xi} & \mathbb{P}^2 \\
\end{array}
\]
some toric Fano 4-folds (see [2] for details). Then in particular the exceptional divisor \( E \subset X \) obtained from the last blow-up is a prime divisor of \( X \) with \( \rho_E = 2 \). As we pointed out in \( \S 2.2 \) one has \( \dim \mathcal{N}_1(E, X) \leq \rho_D = 2 \), then by the very definition of the Lefschetz defect and being \( \rho_X = 5 \) it follows that \( \delta_X \geq 3 \). Using [5, Theorem 1.1] we get \( \delta_X = 3 \) because for \( i = 1, 2, 3 \) the varieties \( K_i \) are not product of two del Pezzo surfaces, then the claim.

**Remark 5.0.1.** — Assume that \( X \) is a Fano 4-fold with \( \delta_X = 3 \) and \( \rho_X = 6 \). As done in the first part of the proof of Theorem 1.0.1, we can consider a Fano conic bundle \( f: X \to Y \) with \( \rho_X - \rho_Y = 3 \), and Proposition 2.4 (b) allows to get all possible targets \( Y \). Take a factorization for \( f \) as in Proposition 2.3 (a) and keep the same notation. If we require that the factorization of \( f \) is of quasi-Fano type (cf. Definition 4.2.1), then arguing as done in the proof of Theorem 1.0.1 we deduce that the variety \( X_2 \) satisfies the assumption of Proposition 3.0.2, hence we deduce all possible cases for \( X_2 \). This approach can be viewed as the first attempt to classify Fano 4-folds with \( \delta_X = 3 \) and \( \rho_X = 6 \), then to complete the whole classification of Fano 4-folds with \( \delta_X = 3 \) (see also Remark 4.2.2).

We conclude this paper by proving Corollary 1.0.2 which gives a characterization for the toric varieties of combinatorial type \( K \), in terms of Picard number of prime divisors of such varieties.

**Proof of Corollary 1.0.2.** — Assume that \( X \) is one of the varieties of combinatorial type \( K \). As we have already observed in the final part of the proof of Theorem 1.0.1 we can find a prime divisor of \( X \) with Picard number 2. Conversely, suppose that \( X \) is a Fano 4-fold with \( \rho_X = 5 \) that has a prime divisor \( D \subset X \) with \( \rho_D = 2 \). As we pointed out in \( \S 2.2 \) one has \( \dim \mathcal{N}_1(D, X) \leq \rho_D \) and being \( \rho_X = 5 \) we deduce that \( \delta_X \geq 3 \). We observe that \( \delta_X = 3 \).

Indeed, if \( \delta_X > 4 \), then [5, Theorem 1.1] gives \( X \cong S_1 \times S_2 \) with \( S_i \) del Pezzo surfaces, such that \( \delta_X = \max\{\rho_{S_1} - 1, \rho_{S_2} - 1\} \) (see [5, Example 3.1]). Without loss of generality we can assume that \( \rho_{S_1} = \delta_X + 1 \). Using that \( X \) is simply connected, one can compute

\[
\rho_X = \rho_{S_1} + \rho_{S_2} = \delta_X + 1 + \rho_{S_2} \geq \delta_X + 2
\]

and being \( \rho_X = 5 \) we get \( \delta_X \leq 3 \), hence a contradiction.

At this point applying Theorem 1.0.1 it follows that \( X \) is toric of combinatorial type \( K \), hence the statement.

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