VANISHING RESULTS IN CHOW GROUPS FOR THE MODIFIED DIAGONAL CYCLES

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Abstract. We prove a sufficient condition for the vanishing of the modified diagonal cycle in the Chow group (with \( \mathbb{Q} \)-coefficients) of the triple product of a curve over \( \mathbb{C} \). We exhibit infinitely many non-hyperelliptic curves, including the Fricke–Macbeath curve, the Bring curve, and two 1-dimensional families parameterized by certain Hurwitz spaces, for which our condition is satisfied.

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1. Introduction

1.1. The modified diagonal cycle in the Chow group. For a smooth projective variety \( X \) over a field \( F \), let \( \text{Ch}^i(X) \) be the Chow group of \( X \) (the group of codimension \( i \) cycles with \( \mathbb{Q} \)-coefficients, modulo rational equivalence). The diagonal cycle on \( X \times X \) plays an important role in many aspects. For the multi-folded product \( X^n = X \times \cdots \times X \) (\( n \)-times), the diagonal cycle \( \Delta \) (sometimes called the small diagonal when \( n \geq 3 \)) has also been studied by many authors. Of particular interest is the modified diagonal cycle in the case \( n = 3 \) and \( \dim X = 1 \), also called the Gross–Schoen cycle \([10, 32]\). This will be the main focus of this paper, and we refer to \([2, 17, 29]\) for some other classes of \( X \) and more general \( n \). For example, by \([17, \text{Proposition 2.4}]\), the vanishing of the modified diagonal cycle in the case \( n = 3 \) implies the vanishing for \( n \geq 3 \).

Throughout this paper, a “curve” always means a “smooth projective curve” over a field \( F \). Let \( X \) be a curve over \( F \). For any \( e \in \text{Ch}^1(X) \) with \( \deg e = 1 \), there is the modified diagonal cycle \([\Delta_e] \in \text{Ch}^2(X^3)\) defined in \([10]\). If \( e = [p] \) the class of an \( F \)-point \( p \), we set

\[
\begin{align*}
\Delta_{12} &= \{(x, x, p) : x \in X\}, \\
\Delta_{23} &= \{(p, x, x) : x \in X\}, \\
\Delta_{31} &= \{(x, p, x) : x \in X\}, \\
\Delta_1 &= \{(x, p, p) : x \in X\}, \\
\Delta_2 &= \{(p, x, p) : x \in X\}, \\
\Delta_3 &= \{(p, p, x) : x \in X\}.
\end{align*}
\]

Then \([\Delta_e]\) is the class in \( \text{Ch}^2(X^3) \) of the algebraic cycle

\begin{equation}
\Delta - \Delta_{12} - \Delta_{23} - \Delta_{31} + \Delta_1 + \Delta_2 + \Delta_3.
\end{equation}

In general, we can define \([\Delta_e] \) using (2.5) (a formula similar to (1.1) for \([\Delta_e] \) is given in \([32, (1.1)]\)).

The modified diagonal \([\Delta_e] \) has trivial cohomology class, by Proposition 2.3.1. When \( X \) is a hyperelliptic curve, Gross and Schoen showed that \([\Delta_e] \) vanishes in \( \text{Ch}^2(X^3) \) if \( e = [p] \) for any Weierstrass point \( p \). Below,
we always assume that a curve has genus at least 2. By Proposition 2.3.2, the class \([\Delta_c]\) does not vanish unless \(e\) is equal to
\[
(1.2) \quad \xi = \xi_X := \frac{1}{\deg c_1(X)} c_1(X) \in \text{Ch}^1(X).
\]

The modified diagonal cycle is closely related to the Ceresa cycle \([6]\), i.e., \([X] - (-1)^* [X] \in \text{Ch}^{g-1}(J_X)\), where \(J_X\) is the Jacobian variety of \(X\) and we embed \(X\) into \(J_X\) by \(x \mapsto x - \xi\). The vanishing of \([\Delta_\xi] \in \text{Ch}^2(X^3)\) is equivalent to the vanishing of the Ceresa cycle in \(\text{Ch}^{g-1}(J_X)\) (see \([32, \text{Theorem 1.5.5}]\)). Ceresa \([6]\) proved that the class \([X] - (-1)^* [X] \in \text{Ch}^{g-1}(J_X)\), hence \([\Delta_\xi] \in \text{Ch}^2(X^3)\), does not vanish for a general curve \(X\) over \(\mathbb{C}\). We recall another related “non-vanishing result”: by the “Northcott property” of S. Zhang \([32, \text{Theorem 1.3.5}]\), it is more often for the class \([\Delta_\xi] \in \text{Ch}^2(X^3)\) to be non-vanishing in a proper (non-isotrivial) family of curves defined over a number field.

However, it was not known whether there exists any non-hyperelliptic curve \(X\) such that \([\Delta_\xi] \in \text{Ch}^2(X^3)\) vanishes. The analogous question with the \(\ell\)-adic intermediate Jacobian, the complex intermediate Jacobian, and the Griffiths group (Chow group modulo algebraic equivalence) respectively in place of the Chow group, has been recently answered affirmatively by Bisogno–Li–Litt–Srinivasan \([4]\), Lilienfeldt \([13]\), and Beauville–Schoen \([1]\) respectively, using different curves. In this paper we will provide many examples, including two 1-dimensional families, of non-hyperelliptic \(X\) over \(\mathbb{C}\). In particular, by S. Zhang’s theorem of successive minima (see \([31, \S 2]\)), this positivity in turn implies the Bogomolov conjecture for \(X\).

1.2. A vanishing theorem. Our result relies on the following sufficient condition for the vanishing of \([\Delta_\xi]\).

From now on we work with the ground field \(F = \mathbb{C}\), except in \(\S 1.6\). Let \(H^1(X)\) denote the first Betti cohomology \(H^1(X, \mathbb{C})\). Let \(G\) be a finite group acting on \(X\) by automorphisms. Then there is the induced action of \(G\) on \(H^1(X)\). On \(H^1(X)^{\oplus g}\), there is the induced diagonal \(G\)-action. We have the following theorem, proved in 2.4.

**Theorem 1.2.1.** If \((H^1(X)^{\oplus 3})^G = 0\), then the class \([\Delta_\xi]\) vanishes in \(\text{Ch}^2(X^3)\).

Some remarks are in order. A necessary condition for \((H^1(X)^{\oplus 3})^G = 0\) is \(H^1(X)^G = 0\). Hence the quotient curve \(X/G\) has genus zero.

A special case of Theorem 1.2.1 is when \(X\) is hyperelliptic and \(G = \mathbb{Z}/2\mathbb{Z}\) is generated by an involution that gives rise to a degree two map \(X \rightarrow \mathbb{P}^1\). Then \(H^1(X)^G = 0\) and hence \(G\) acts on \(H^1(X)\) by the unique nontrivial character. It follows that \((H^1(X)^{\oplus 3})^G = 0\). This gives a new proof of the aforementioned theorem of Gross and Schoen.

If \(X\) is defined over a number field, Theorem 1.2.1 is also predicted by the conjectural injectivity of the Abel–Jacobi map (due to Beilinson and Bloch).

To the best of our knowledge, the only other known result (besides the trivial ones for varieties such as \(\mathbb{P}^2\)) on the vanishing (in Chow group) of the modified diagonal cycle on the triple product of a variety is due to Beauville and Voisin \([2]\), who showed that when \(X\) is a K3 surface, \(e = \deg c_2(X) c_2(X) \in \text{Ch}^2(X)\) (note that \(\deg c_2(X) = 24\)) is the class of an \(F\)-point, and the class of \([\Delta_c]\) similarly defined by \((1.1)\) vanishes in \(\text{Ch}^4(X^3)\).
1.3. Hurwitz curves. A curve over $\mathbb{C}$ of genus $g$ is called a Hurwitz curve if its automorphism group achieves the maximal possible order $84(g - 1)$. There is a unique (up to isomorphism) Hurwitz curve $X$ of genus 7 over $\mathbb{C}$, known as the Fricke–Macbeath curve [14]. Its automorphism group is isomorphic to $\text{PGL}_2(\mathbb{F}_5)$. Shimura [24] identified it as a Shimura curve.

By a theorem of Bisogno, Li, Litt, Srinivasan [4] (see also Gross [11]), we have $(H^1(X)^{\otimes 3})^{\text{Aut}(X)} = 0$ for the Fricke–Macbeath curve $X$. We thus obtain

**Theorem 1.3.1.** Let $X$ be the Fricke–Macbeath curve. Then the class $[\Delta_X]$ vanishes in $\text{Ch}^2(X^3)$.

We will also provide an independent proof of $(H^1(X)^{\otimes 3})^{\text{Aut}(X)} = 0$ for the Fricke–Macbeath curve $X$ in §3.3. This will also help us find another example of similar sort, the Bring curve in §4.1.2, which has the automorphism group $\text{PGL}_2(\mathbb{F}_5)$, the largest possible automorphism group for genus 4 curves.

We make the following conjecture.

**Conjecture 1.3.2.** The cycle $[\Delta_X]$ vanishes in $\text{Ch}^2(X^3)$ for only finitely many Hurwitz curves $X$.

Note that Hurwitz curves are never hyperelliptic, cf. Lemma 3.1.2. The authors would not be surprised if the Fricke–Macbeath curve turns out to be the only Hurwitz curve with vanishing $[\Delta_X]$ in the Chow group.

As an evidence, we prove that, among those Hurwitz curves with $G = \text{Aut}(X) \simeq \text{PSL}_2(\mathbb{F}_5)$, the Fricke–Macbeath curve is the only one satisfying the condition $(H^1(X)^{\otimes 3})^G = 0$, see Theorem 3.4.1.

1.4. More examples. In addition to the Fricke–Macbeath curve, we have also found other examples satisfying $(H^1(X)^{\otimes 3})^G = 0$ for every genus $g \in \{3, 4, 5\}$. We have also found an 1-dimensional family in genus 4 and 5 respectively. See §4. We use the data obtained in [16].

Based on our computation, we would like to make the following two conjectures. Let $\mathcal{M}_g$ be the moduli stack of genus $g$ curves over $\mathbb{C}$.

**Conjecture 1.4.1.** (a) For $g > 2$, there exists a genus $g$ non-hyperelliptic curve $X$ with $[\Delta_X] = 0$ in $\text{Ch}^2(X^3)$.

(b) For any integer $d$, there exist an integer $g$ and a connected smooth family of genus $g$ curves $f : X \to B$ such that under the map $B \to \mathcal{M}_g$ induced by $f$, the image of $B$ has dimension at least $d$, and such that, for every $b \in B(\mathbb{C})$, the fiber $X_b$ of $f$ over $b$ is non-hyperelliptic and has $[\Delta_X] = 0$ in $\text{Ch}^2(X^3_b)$.

(c) The image of $B$ in $\mathcal{M}_g$ in part (b) has dimension at most $2g - 1$ (the dimension of the hyperelliptic loci in $\mathcal{M}_g$).

**Conjecture 1.4.2.** (a) There exist non-hyperelliptic curves $X$ of arbitrary large genera with

$$\left(H^1(X)^{\otimes 3}\right)^{\text{Aut}(X)} = 0.$$

(b) For any integer $d$, there exist an integer $g$ and a connected smooth family of genus $g$ curves $f : X \to B$ such that under the map $B \to \mathcal{M}_g$ induced by $f$, the image of $B$ has dimension at least $d$, and such that, for every $b \in B(\mathbb{C})$, the fiber $X_b$ of $f$ over $b$ is non-hyperelliptic and has $(H^1(X_b)^{\otimes 3})^{\text{Aut}(X_b)} = 0$.

For the first open case in Conjecture 1.4.1 (a), we have not found any example of genus $g = 6$ curve with vanishing $[\Delta_X]$. In fact, we expect that all non-hyperelliptic genus 6 curves $X$ have $(H^1(X)^{\otimes 3})^{\text{Aut}(X)} \neq 0$ (which also explains the formulation of Conjecture 1.4.2 (a)). This is indeed the case for non-hyperelliptic genus 6 curves in [16]. And those $X$ not in [16] by definition have “small” automorphism groups so that $(H^1(X)^{\otimes 3})^{\text{Aut}(X)}$ are less likely to be 0.

As suggested by part (a)’s of these two conjectures, $(H^1(X)^{\otimes 3})^{\text{Aut}(X)} = 0$ should not be a necessary condition for the vanishing of $[\Delta_X]$. This can already be seen in genus 3, for example, from the genus 3 quotient of the Fricke–Macbeath curve (Corollary 3.3.2 and Lemma 3.3.3). As another example, if we assume a conjecture of Beilinson and Bloch relating Chow groups and $L$-functions, then the genus 3 curve used by Beauville and Schoen [1] should have vanishing $[\Delta_X]$ in the Chow group, even though it does not satisfy $(H^1(X)^{\otimes 3})^{\text{Aut}(X)} = 0$. In our next paper [22], we will provide more examples using Shimura curves.
1.5. Chow–K"unneth modified diagonal cycle and Faber–Pandharipande cycle. Our proof of Theorem 1.2.1 on the vanishing of the Gross–Schoen modified diagonal cycle is based on the study of the Chow–K"unneth modified diagonal cycle on $X^3$ (2.6). It coincides with $[\Delta_e]$ when $e$ is the class of an $F$-point, and in general differs from $[\Delta_e]$ by pullbacks of the following Faber–Pandharipande cycle (see (2.7)).

Let $\delta : X \to X^2$ be the diagonal embedding. Following a construction of Faber and Pandharipande, define a zero-cycle (see for example [9])

$$z_e := e \times e - \delta_* e \in \text{Ch}^2(X^2).$$

For $e = \xi$, the class $z_\xi$ is known to vanish if $X$ is hyperelliptic. Faber and Pandharipande proved $z_\xi = 0$ if the genus of $X$ is 3; Green and Griffiths [9] proved that $z_\xi \neq 0$ if $X$ is a general curve of genus $g \geq 4$.

When $e = \xi$, the vanishing of either modified diagonal cycle implies the vanishing of the other modified diagonal cycle as well as $z_\xi$, by Proposition 2.3.2. We prove the vanishing of the Chow–K"unneth modified diagonal cycle assuming $(H^1(X)^{\otimes 3})^G = 0$ (a special case of Theorem 2.4.1). Then we have Theorem 1.2.1 and the following theorem, proved in §2.4.

**Theorem 1.5.1.** If $(H^1(X)^{\otimes 3})^G = 0$, then the class $z_\xi$ vanishes in $\text{Ch}^2(X^2)$.

1.6. Finite fields. Finally in this introduction, we let $F$ be a finite field. By a theorem of Soulé [25, Theorem 3], on a product of curves over $F$, Chow 1-cycles with $\mathbb{Q}_F$-coefficients coincide (via the cycle class map) with Tate 1-cycles with $\mathbb{Q}_F$-coefficients. Here, $\ell$ is different from the characteristic of $F$. In particular, since modified diagonal cycles are cohomologically trivial, they vanish in the Chow groups. In the proof of Soulé’s theorem, the Frobenius map plays a similar role to the automorphism group in our result.

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2. Proof of the vanishing theorem

In this section we prove Theorem 1.2.1. The key observation is that one can explicitly describe self-correspondences on a curve annihilating the cohomology of a particular degree.

2.1. Divisorial correspondences. Let $X$ be a curve over $\mathbb{C}$. Consider the cycle class map,

$$\text{cl} : \text{Ch}^1(X^2) \to H^2(X^2) \simeq \bigoplus_{i=0}^{2} \text{End}(H^i(X)).$$

Here the isomorphism is defined by K"unneth decomposition and Poincaré duality.

Fix $e \in \text{Ch}^1(X)$ of degree 1.

**Definition 2.1.1.** Let $\text{DC}(X^2)$ be the subspace of $\text{Ch}^1(X^2)$ of $z$’s such that $z_*e \in \mathbb{Q}e$ and $z^*e \in \mathbb{Q}e$. Let $\text{DC}^1(X^2)$ be the subspace of $\text{Ch}^1(X^2)$ of $z$’s such that $z_*e = 0$ and $z^*e = 0$.

We call cycles in $\text{DC}^1(X^2)$ divisorial correspondences with respect to $e$. For notational convenience, we also define $\text{DC}^0(X^2) = \mathbb{Q}(e \times [X])$, $\text{DC}^2(X^2) = \mathbb{Q}([X] \times e)$.

For $z \in \text{DC}(X^2)$, we assign

$$(z^*e \times [X], z - z^*e \times [X] - [X] \times z_*e, [X] \times z_*e)$$
to get the following direct sum decomposition (the sum is clearly direct):\[(2.1)\]
\[\text{DC}(X^2) = \text{DC}^0(X^2) \oplus \text{DC}^1(X^2) \oplus \text{DC}^2(X^2).\]

The following lemma should be well-known. For the sake of completeness we include a proof.

**Lemma 2.1.2.** The restriction $\text{cl}|_{\text{DC}(X^2)}$ is injective.

**Proof.** Note that the image of $\text{cl}|_{\text{DC}^i(X^2)}$ is in $\text{End}(H^i(X))$ for $0 \leq i \leq 2$. Therefore it suffices to show the injectivity of $\text{cl}|_{\text{DC}^i(X^2)}$ for every $i \in \{0, 1, 2\}$. The maps $\text{cl}|_{\text{DC}^0(X^2)}, \text{cl}|_{\text{DC}^0(X^2)}$ are obviously injective. Note that there is a natural isomorphism (for a proof, see [23, Theorem 3.9]) \[\text{DC}^1(X^2) \cong \text{End}^0(J_X),\]
where $J_X$ is the Jacobian variety of $X$, and $\text{End}^0(J_X)$ denotes the $\mathbb{Q}$-endomorphism ring of the abelian variety $J_X$. Since the natural map $\text{End}^0(J_X) \to \text{End}(H^1(X))$ is injective, the map $\text{cl}|_{\text{DC}^1(X^2)}$ is injective and the proof is complete. \[\square\]

Let $\delta : X \to X^2$ be the diagonal embedding.

**Remark 2.1.3.** We may view $\text{Ch}^1(X^2)$ as the ring of self-correspondence on the curve $X$. Then $\text{DC}(X^2)$ is a subring and the identity is the diagonal cycle $[\delta_* X]$. We have a decomposition (as $\mathbb{Q}$-vector spaces) \[\text{Ch}^1(X^2) = \text{DC}(X^2) \oplus (\pi_1^* \text{Ch}^1(X)_0 \oplus \pi_2^* \text{Ch}^1(X)_0),\]
where $\pi_1, \pi_2 : X^2 \to X$ are the two projection maps, and $\pi_1^* \text{Ch}^1(X)_0 \oplus \pi_2^* \text{Ch}^1(X)_0$ is an ideal of the ring $\text{Ch}^1(X^2)$.

For a fixed $e \in \text{Ch}_0(X)$, let $\delta_e$ be the projection of $[\delta_* X]$ to $\text{DC}^1(X^2)$ via (2.1), i.e.,
\[(2.2)\]
\[\delta_e := [\delta_* X] - (e \times [X] + [X] \times e) \in \text{DC}^1(X^2).\]
It is straightforward to check the following lemma.

**Lemma 2.1.4.** The left/right composition by $\delta$ on $\text{DC}(X^2)$ is the projection to $\text{DC}^1(X^2)$ via (2.1).

### 2.2. Product of curves.

Let $X^n$ be the $n$-th fold self-product of $X$. We identify $X^n \times X^n = X^{2n}$, sending $((x_1, \cdots, x_n), (y_1, \cdots, y_n)) \in X^n \times X^n$ to $(x_1, \cdots, x_n, y_1, \cdots, y_n) \in X^{2n}$. Let $\pi_{i,j} : X^{2n} \to X \times X$ be the projection map to the $(i, j)$-th factor. We have a natural map \[\bigotimes_{i=1}^n \text{Ch}^1(X^2) \longrightarrow \text{Ch}^n(X^n \times X^n)\]
which sends a tensor $z_1 \otimes \cdots \otimes z_n$ to the intersection product of divisors $\pi_1, n+1(z_1), \cdots, \pi_{2n}(z_n)$ on $X^{2n}$.

Restricting this map to its subspace and applying the cycle class maps, we have a commutative diagram:
\[(2.3)\]
\[\bigotimes_{i=1}^n \text{DC}^1(X^2) \longrightarrow \text{Ch}^n(X^n \times X^n) \]
\[\bigotimes_{i=1}^n \text{End}(H^1(X)) \longrightarrow \text{End}(H^n(X^n)) \]
where the bottom map is induced by the Künneth decomposition. By Lemma 2.1.2, the left map is injective. By the obvious injectivity of the bottom map, the top map of (2.3) is also injective. We will therefore view an element in $\bigotimes_{i=1}^n \text{DC}^1(X^2)$ as an element in $\text{Ch}^n(X^n \times X^n)$. By the Künneth decomposition, we have the following proposition.

**Proposition 2.2.1.** If $m \neq n$, $\bigotimes_{i=1}^n \text{DC}^1(X^2)$ annihilates $H^m(X^n)$, by either pushforward or pullback.
For a class $z \in DC(X^2)$, we denote
\begin{equation}
(2.4) \quad z^{\otimes n} := \bigotimes_{i=1}^{n} z \in \bigotimes_{i=1}^{n} DC(X^2).
\end{equation}

We will use the Chow–Künneth projector $\delta_e^{\otimes n} \in \bigotimes_{i=1}^{n} DC^1(X^2)$ for $\delta_e$ defined by (2.2). We remind the reader that this is not the projector used by Gross and Schoen [10, Section 2] (in the case that $e$ is the class of an $F$-point), as we now discuss.

2.3. Modified diagonals. We will be mainly interested in the case $n = 3$. The Gross–Schoen modified diagonal cycle with respect to $e$ is
\begin{equation}
(2.5) \quad [\Delta_e] = ([\delta_e X] - [X] \times e)^{\otimes 3} [\Delta] \in \text{Ch}^2(X^3).
\end{equation}

(It is straightforward to check that $[\Delta_e]$ coincides with the one in [32, (1.1)] generalizing (1.1).) Similar to Proposition 2.2.1, we have the following proposition.

**Proposition 2.3.1** ([10, Proposition 3.1]). The cohomology class of $[\Delta_e]$ is 0.

Define the Chow–Künneth modified diagonal cycle to be
\begin{equation}
(2.6) \quad (\delta_e^{\otimes 3})_* [\Delta] \in \text{Ch}^2(X^3),
\end{equation}
whose cohomology class is 0 by Proposition 2.2.1. Let $\pi_{ij} : X^3 \to X^2$, $i < j \in \{1, 2, 3\}$, be the projection to the product of $i$-th and $j$-th $X$. It is straightforward to check
\begin{equation}
(2.7) \quad (\delta_e^{\otimes 3})_* [\Delta] - [\Delta_e] = \sum_{i<j} \pi_{ij}^* z_e.
\end{equation}

Here $z_e$ is the Faber–Pandharipande cycle (1.3). Two immediate consequences of (2.7) are as follows. First, since $z_e$ has cohomology class 0, Proposition 2.2.1 implies Proposition 2.3.1. Second, if $e$ is the class of an $F$-point, then $z_e = 0$ and thus $(\delta_e^{\otimes 3})_* [\Delta] = [\Delta_e]$. In general, we have the following proposition.

**Proposition 2.3.2.** Let the genus $g$ of $X$ be at least 2.

(1) If $[\Delta_e] = 0$, then $e$ is equal to $\xi$ defined by (1.2) and $z_\xi = 0$.

(2) If $(\delta_e^{\otimes 3})_* [\Delta] = 0$, then $z_\xi = 0$.

(3) $(\delta_e^{\otimes 3})_* [\Delta] = 0$ if and only if $[\Delta_e] = 0$.

**Proof.** (1) Consider the intersection product $[\Delta_e] \cdot ([X] \times [\delta_e X])$. By the adjunction formula, we have $[\delta_e X] \cdot [\delta_e X] = \delta_c(X)$. In particular,
\begin{equation}
[\Delta] \cdot ([X] \times [\delta_e X]) = ([\delta_e X] \times [X]) \cdot ([X] \times [\delta_e X]) = \Delta_e c_1(X).
\end{equation}

Here for $\Delta_e c_1(X)$, we abuse notation and understand $\Delta$ as the diagonal embedding of $X$ in $X^3$. Then a direct computation shows that
\begin{equation}
(2.8) \quad [\Delta_e] \cdot ([X] \times [\delta_e X]) = \Delta_e c_1(X) - e \times \delta_e c_1(X) - 2\Delta_e e + 2e \times \delta_e e.
\end{equation}

Its projection to the first $X$ is $\text{deg} c_1(X) (\xi - e)$. Since $\text{deg} c_1(X) = 2 - 2g \neq 0$, $[\Delta_e] = 0$ implies $e = \xi$. Let $e = \xi$. Then the projection of (2.8) to the product $X^2$ of the first two $X$’s is $2g z_\xi$. So $[\Delta_e] = 0$ implies $z_\xi = 0$.

(2) By (2.7) and (2.8), it is straightforward to show that the projection of $(\delta_e^{\otimes 3})_* [\Delta] \cdot ([X] \times [\delta_e X])$ is
\begin{equation}
(2g + 2) z_\xi. \quad \text{So} \quad (\delta_e^{\otimes 3})_* [\Delta] = 0 \text{ implies } z_\xi = 0.
\end{equation}

(3) This now follows from (2.7), the assertions in (1) and (2). \qed
Proposition 2.3.3. Let \( \pi : X \to Y \) be a non-constant morphism of curves of genus at least 2. If \( [\Delta_{\xi X}] = 0 \in \text{Ch}^2(X^3) \), then we have \( \pi_*(\xi_X) = \xi_Y \in \text{Ch}^1(Y) \) and \( [\Delta_{\xi Y}] = 0 \in \text{Ch}^2(Y^3) \).

**Proof.** We note that \( (\pi^3)_*[\Delta_{\xi X}] = \deg(\pi)[\Delta_{\pi_*\xi_X}] \in \text{Ch}^2(Y^3) \). Under the assumption \( [\Delta_{\xi X}] = 0 \), we have \( [\Delta_{\pi_*\xi_X}] = 0 \). This in turn implies that \( \pi_*(\xi_X) = \xi_Y \) by Proposition 2.3.2 (1) and \( [\Delta_{\xi Y}] = 0 \).

**Remark 2.3.4.** It follows that a necessary condition for \( [\Delta_{\xi X}] = 0 \in \text{Ch}^2(X^3) \) is that \( \pi_*(\xi_X) = \xi_Y \in \text{Ch}^1(Y) \) holds for every non-constant morphism \( \pi : X \to Y \) with \( g(Y) \geq 2 \). We are not aware of a more direct characterization of curves \( X \) with the latter property.

2.4. **Proof of the vanishing theorem.** Now let \( G \) be a finite group acting on \( X \) by automorphisms. Let \( \xi \in \text{Ch}^1(X) \) be a \( G \)-invariant divisor class of degree 1. In fact, such a \( \xi \) clearly exists by averaging any degree 1 divisor class, and is in fact unique if \( H^1(X)^G = 0 \). We will simply take \( \xi = \xi_X \) defined by (1.2) when the genus of \( X \) is not 1. For \( g \in G \), let \( \Gamma_g \) be the graph of the automorphism of \( X \) given by \( g \). The \( G \)-invariance of \( \xi \) implies that \( [\Gamma_g] \in \text{DC}(X^2) \), and satisfies

\[
(2.9) \quad [\Gamma_g] \circ \delta_\xi = \delta_\xi \circ [\Gamma_g]
\]

for the correspondence compositions. In particular, \( (\delta_\xi^{\otimes 3})_* \text{Ch}^*(X^3) \) is stable under the \( G^3 \)-action.

**Theorem 2.4.1.** If \( (H^1(X)^{\otimes 3})^G = 0 \) under the diagonal \( G \)-action, then there is no nonzero diagonal-\( G \)-invariant element in \( (\delta_\xi^{\otimes 3})_* \text{Ch}^*(X^3) \), or equivalently, \( \delta_\xi^{\otimes 3} \) annihilates the diagonal-\( G \)-invariant elements in \( \text{Ch}^*(X^3) \).

**Proof.** Consider the following element, by averaging the diagonal \( G \)-action:

\[
z := \frac{1}{|G|} \sum_{g \in G} (\Gamma_g \circ \delta_\xi)^{\otimes 3} \in \text{DC}(X^2)^{\otimes 3}.
\]

By Lemma 2.1.4, we have \( z \in \text{DC}^1(X^2)^{\otimes 3} \). By the assumption \( (H^1(X)^{\otimes 3})^G = 0 \), \( z \) acts on \( H^1(X)^{\otimes 3} \) by the zero map. Therefore the cycle class of \( z \) vanishes under the left map of (2.3). It follows from the injectivity of this map that \( z = 0 \in \text{DC}^1(X^2)^{\otimes 3} \).

Since the diagonal-\( G \)-invariant elements in \( (\delta_\xi^{\otimes 3})_* \text{Ch}^*(X^3) \) are exactly \( z_* \text{Ch}^*(X^3) \), the first assertion follows. The equivalence of the two assertions follows from (2.9) and the proof is complete.

**Remark 2.4.2.** The proof of Theorem 2.4.1 can be easily generalized to show that there is a decomposition of representations of \( G^3 \)

\[
\text{Ch}^*(X^3)_\mathbb{C} = \bigoplus_\Pi \text{Ch}^*(X^3)_\mathbb{C}[\Pi]
\]

where \( \Pi \) runs over all irreducible \( \mathbb{C} \)-representations of \( G^3 \) that appear in \( (H^1(X)^{\otimes 3})^\mathbb{C} \), and \( \text{Ch}^*(X^3)_\mathbb{C}[\Pi] \) is the \( \Pi \)-isotypic subspace of \( \text{Ch}^*(X^3)_\mathbb{C} \). This can then be applied to study the subspace of \( \text{Ch}^*(X^3) \) generated by \( [\Delta_\xi] \) under the \( G^3 \)-action (namely, the twisted diagonal cycles). Moreover, it is also easy to generalize the above discussion to an arbitrary \( n \), instead of \( n = 3 \).

Now we can prove Theorem 1.2.1 and Theorem 1.5.1. Clearly the class \( [\Delta] \in \text{Ch}^2(X^3) \) is diagonal-\( G \)-invariant. Assume \( (H^1(X)^{\otimes 3})^G = 0 \). Then by Theorem 2.4.1, \( (\delta_\xi^{\otimes 3})_* [\Delta] = 0 \). Hence by Proposition 2.3.2 (2), we obtain \( \zeta_\xi = 0 \) and Theorem 1.5.1 is proved. By Proposition 2.3.2 (3), we obtain \( [\Delta_\xi] = 0 \) and Theorem 1.2.1 is proved.
3. Hurwitz curves

3.1. Hurwitz representation. We first recall the Hurwitz space [16, §2, §3]. Fix a finite group $G$ and an ordered tuple $\mathcal{C} = (C_1, \ldots, C_r)$ of non-trivial conjugacy classes $C_i$ of $G$. A $G$-curve is a pair $(X, \iota)$ where $X$ is a curve and $\iota : G \to \text{Aut}(X)$ is an injective group homomorphism. We will suppress $\iota$ and simply say that $X$ is a $G$-curve. Let $Y = X/G$ (a smooth curve) and let $\pi : X \to Y = X/G$ be the covering map. We say that the ramification type of the $G$-curve $X$ is $\mathcal{C} = (C_1, \ldots, C_r)$ if the ramified points on $Y$ can be labeled as $z_1, \ldots, z_r$ such that $C_i$ is the conjugacy class in $G$ of the distinguished inertia group generator over $z_i$, i.e., the generator that acts on the tangent space at $z_i$ by $\exp(2\pi i/e_i)$ where $e_i$ is the ramification index, for $1 \leq i \leq r$. Let $G_i, i = 1, \ldots, r$, be the inertia subgroup of ramified points of the covering map $\pi : X \to Y$, well-defined up to conjugacy so that $e_i = |G_i|$. The signature of the $G$-curve $X$ is by definition the tuple $(e_1, \ldots, e_r)$ which we arrange to be in increasing order.

Since our purpose is to find $X$ with $H^1(X)^G = 0$, from now on we will assume that $H^1(X)^G = 0$, or equivalently the genus of $Y = X/G$ is

\begin{equation}
\label{eq:gY}
g(Y) = 0.
\end{equation}

Assume that $g \geq 2$. Let $\mathcal{H}(g, G, \mathcal{C})$ be the groupoid of $G$-curves $X$ of genus $g$ and ramification type $\mathcal{C}$ satisfying \eqref{eq:gY}, up to the obvious notion of isomorphism. Then $\mathcal{H}(g, G, \mathcal{C})$ has a structure of smooth Deligne–Mumford stack (over $\mathbb{C}$) equipped with a universal family of $G$-curves. We denote this stack by the same notation and will call it the Hurwitz stack (for the data $(g, G, \mathcal{C})$ equipped with a universal family of $G$-curves). We will suppress $\mathcal{C}$ and call it the Hurwitz stack $(g, G, \mathcal{C})$.

Fix a finite group $G$ and let $\mathcal{C}$ be the covering map. We say $\mathcal{C}$ is the Hurwitz stack equipped with a finite morphism $\mathcal{H}(g, G, \mathcal{C}) \to \mathcal{M}_g$ and whenever it is non-empty, every irreducible component has dimension $r - 3$.

For any $G$-curve $X$, the induced representation of $G$ on $H^1(X)$ is determined by the ramification type $\mathcal{C}$ via the following formula in the Grothendieck group of $G$-representations over $\mathbb{C}$ (note that $g(Y) = 0$ by our assumption \eqref{eq:gY})

\begin{equation}
\label{eq:Roch}
H^1(X) = -2(\text{Ind}_{(1)}^G(1)) + \sum_{i=1}^{r} (\text{Ind}_{(1)}^{G_i}(1) - \text{Ind}_{G_i}(1)),
\end{equation}

where every $(1)$ denotes the trivial character of the corresponding group. Note that the right hand side depends only on the ramification type $\mathcal{C} = (C_1, \ldots, C_r)$.

The formula is implicitly in [15, §3] and can be proved using Lefschetz fixed point formula. In particular, evaluating the trace of \eqref{eq:Roch} at the identity of $G$ gives the Riemann–Hurwitz formula

\begin{equation}
\label{eq:RHF}
\frac{(2g - 2)}{|G|} = -2 + \sum_{i=1}^{r} \left(1 - \frac{1}{e_i}\right).
\end{equation}

Note that the equation is equivalent to the assumption $g(Y) = 0$ \eqref{eq:gY}.

We recall the following result on the automorphism group of a hyperelliptic curve, pointed to us by B. Totaro.

\begin{lemma}
Let $X$ be a hyperelliptic curve of genus $\geq 2$. Then $\text{Aut}(X)$ is a central extension of a finite subgroup of $\text{PGL}_2(\mathbb{C})$ by $\mathbb{Z}/2\mathbb{Z}$ (generated by the unique hyperelliptic involution). Moreover, the order of $\text{Aut}(X)$ is at most $\max\{8(g + 1), 120\}$.
\end{lemma}

\begin{proof}
The first part is well-known due to the uniqueness of hyperelliptic involution $\sigma$ and the fact that the quotient $\text{Aut}(X)/\langle \sigma \rangle$ acts faithfully on the quotient $X/\langle \sigma \rangle \simeq \mathbb{P}^1$. A complete list of $\text{Aut}(X)$ is obtained in
[5, Theorem 2.1], from which we deduce that the either \(|\text{Aut}(X)| \leq 8(g + 1)|\) (case (3.b) of Table 2 in loc. cit.) or \(|\text{Aut}(X)| \geq 120\).

\[\square\]

**Corollary 3.1.2.** Hurwitz curves are never hyperelliptic.

**Proof.** For a Hurwitz curve \(X\), its genus \(g \geq 3\) and the order of \(\text{Aut}(X)\) is \(84(g - 1)\). Therefore we have \(|\text{Aut}(X)| \geq \max\{8(g + 1), 120\}\). By Lemma 3.1.1 the proof is complete.

\[\square\]

In §4, we will use Lemma 3.1.1 repeatedly when we need to show that certain curves are non-hyperelliptic.

### 3.2. Trilinear forms for PGL\(_2(\mathbb{F}_q)\).

Let \(G = \text{PGL}_2(\mathbb{F}_q)\), \(U\) the subgroup of unipotent upper triangular matrices, \(S\) the subgroup of diagonal matrices, \(T\) a non-split maximal torus, and \(B = US = SU\). We consider the generic (i.e., the restriction to \(U\) contains one hence every nontrivial character of \(U\)) irreducible representations of \(G\). Let \(S'\) (resp. \(T'\)) be the normalizer of \(S\) (resp. \(T\)) in \(G\). Let \(\pi^{S'}\) (resp. \(\pi^{T'}\)) be the unique representation of \(S'\) (resp. \(T'\)) such that the restriction \(\pi|_{S} - \pi^{S'}|_{S}\) (resp. \(\pi|_{T} + \pi^{T'}|_{T}\), note the sign change) is the regular representation of \(S\) (resp. \(T\)). More explicitly, there are three cases:

(i) if \(\pi\) is a principal series \(\text{Ind}(\eta) := \text{Ind}^{G}_{H}(\eta)\) for a character \(\eta\) of \(S\) (\(\eta \neq \eta^{-1}\)), then \(\pi^{S'} = \text{Ind}^{S'}_{H}(\eta)\) and \(\pi^{T'} = 0\).

(ii) if \(\pi = \text{St} \otimes \eta \circ \det\) is a twist of the Steinberg representation \(\text{St}\) (\(\eta\) is a quadratic character of \(\mathbb{F}_q^\times\)), then \(\pi^{S'} = \eta \circ \det|_{S'}\) and \(\pi^{T'} = \eta \circ \det|_{T'}\).

(iii) if \(\pi\) is a cuspidal representation \(\pi_\chi\) attached to a character \(\chi\) of \(T\) (\(\chi \neq \chi^{-1}\)), then \(\pi^{S'} = 0\) and \(\pi^{T'} = \text{Ind}^{T'}_{F}(\chi)\).

We have an analog over finite field of Prasad’s theorem on trilinear forms over local fields. A special case has appeared in [20, Example 2.5]. For \(H \in \{G, S', T'\}\), we define

\[
m_H(\pi_1, \pi_2, \pi_3) = \dim(\pi_1^H \otimes \pi_2^H \otimes \pi_3^H)^H,
\]

where \(\pi_i^G\) is understood as \(\pi_i\).

**Theorem 3.2.1.** Let \(\pi_1, \pi_2, \pi_3\) be irreducible generic representations of \(\text{PGL}_2(\mathbb{F}_q)\). Then

\[
\sum_{H \in \{G, S', T'\}} \text{sgn}(H)m_H(\pi_1, \pi_2, \pi_3) = 1,
\]

where \(\text{sgn}(G) = \text{sgn}(T') = 1\) and \(\text{sgn}(S') = -1\).

**Proof.** Let \(f\) be a conjugate invariant function on \(G = \text{PGL}_2(\mathbb{F}_q)\). Then we have the following integration formula by summing over all conjugacy classes

\[
\frac{1}{|G|} \sum_{g \in G} f(g) = \frac{1}{|G|} f(1) + \frac{1}{|U|} f(u) + \frac{1}{2|S|} \sum_{s \in S, s \neq 1} f(s) + \frac{1}{2|T|} \sum_{t \in T, t \neq 1} f(t).
\]

We will apply the formula to the product \(f = \chi_{\pi_1} \chi_{\pi_2} \chi_{\pi_3}\) of characters. We use the character table of \(G\), where \(u\) denotes a fixed non-identity element in \(U\):

| conjugacy classes | 1       | \(u\) | \(s \neq 1\) | \(t \neq 1\) |
|-------------------|---------|-------|-------------|-------------|
| principal series \(\text{Ind}(\eta)\) | \(q + 1\) | 1     | \(\eta(s) + \eta^{-1}(s)\) | 0           |
| Steinberg twist \(\text{St} \otimes \eta\) | \(q\)    | 0     | \(\eta \circ \det(s)\) | \(-\eta \circ \det(t)\) |
| cuspidal \(\pi_\chi\) | \(q - 1\) | \(-1\) | 0           | \(-\chi(t) - \chi^{-1}(t)\) |
We claim
\[
\sum_{H \in \{G, S', T\}} \text{sgn}(H) m_H(\pi_1, \pi_2, \pi_3) =
\frac{1}{|G|} \chi_{\pi_1} \chi_{\pi_2} \chi_{\pi_3}(1) + \frac{1}{|U|} \chi_{\pi_1} \chi_{\pi_2} \chi_{\pi_3}(u) - \frac{1}{2|S|} \chi_{\pi_1^{T'}} \chi_{\pi_2^{T'}} \chi_{\pi_3^{T'}}(1) + \frac{1}{2|T|} \chi_{\pi_1^{T'}} \chi_{\pi_2^{T'}} \chi_{\pi_3^{T'}}(1).
\]

In fact, if at least one of the three representations, say \( \pi_i \), is not a twist of Steinberg, then by (i) and (iii), we have (note that \( S \) is a normal subgroup of \( S' \))
\[
\chi_{\pi_i}^{S'}(s) = \begin{cases} 
\chi_{\pi_i}(s), & \text{if } s \in S \setminus \{1\}, \\
0, & \text{if } s \in S' \text{ is not (conjugate to any element) in } S.
\end{cases}
\]

Then we see that
\[
m_{S'}(\pi_1, \pi_2, \pi_3) = \frac{1}{2|S|} \sum_{s \in S} \chi_{\pi_1^{S'}} \chi_{\pi_2^{S'}} \chi_{\pi_3^{S'}}(s)
= \frac{1}{2|S|} \sum_{s \in S \setminus \{1\}} \chi_{\pi_1} \chi_{\pi_2} \chi_{\pi_3}(s).
\]

(For the first equality, the contribution from \( s \in S' \setminus S \) vanishes by the character formula (3.7).) There is a similar identity for \( \chi_{\pi_i^{T'}} \) and the claim follows from (3.5) (for \( f = \chi_{\pi_1} \chi_{\pi_2} \chi_{\pi_3} \)).

If all of the \( \pi_i \)'s are twists of Steinberg, say \( \pi_i = \text{St} \otimes \eta_i \circ \det \), we set \( \eta = \prod_{i=1}^3 \eta_i \). By (ii) and the fact that \( \det|_S \) is surjective onto \( \mathbb{F}_q^* \), we obtain
\[
\frac{1}{2|S|} \sum_{s \in S \setminus S} \chi_{\pi_1^{S'}} \chi_{\pi_2^{S'}} \chi_{\pi_3^{S'}}(s) = \frac{1}{2|S|} \sum_{s \in S} \eta \circ \det(s)
= \begin{cases} 
0, & \text{if } \eta \text{ is nontrivial}, \\
1/2, & \text{if } \eta \text{ is trivial}.
\end{cases}
\]

We have a similar identity for \( T \) replacing \( S \) and hence we obtain
\[
\frac{1}{2|S|} \sum_{s \in S \setminus S} \chi_{\pi_1^{S'}} \chi_{\pi_2^{S'}} \chi_{\pi_3^{S'}}(s) = \frac{1}{2|T|} \sum_{t \in T \setminus T} \chi_{\pi_1^{T'}} \chi_{\pi_2^{T'}} \chi_{\pi_3^{T'}}(t) \in \{0, 1/2\}.
\]

Then the same argument shows the claimed equality (3.6).

Next, to apply (3.6), we note that (e.g., from the character table)
\[
\begin{align*}
\chi_{\pi}(1) &= \chi_{\pi}(u) + q, \\
\chi_{\pi^{S'}}(1) &= \chi_{\pi}(u) + 1, \\
\chi_{\pi^{T'}}(1) &= -\chi_{\pi}(u) + 1.
\end{align*}
\]

Denote \( x_i = \chi_{\pi_i}(u) \). It suffices to show
\[
\frac{(q+x_1)(q+x_2)(q+x_3)}{(q-1)q(q+1)} + \frac{x_1x_2x_3}{q} - \frac{(1+x_1)(1+x_2)(1+x_3)}{2(q-1)} + \frac{(1-x_1)(1-x_2)(1-x_3)}{2(q+1)} = 1.
\]

It is straightforward to check that the above identity holds (in fact it holds as a rational function in the four variables \( q, x_1, x_2, x_3 \)). The proof is complete.

\[\square\]

**Remark 3.2.2.** If any of \( \pi_i \) is cuspidal, then \( m_{S'}(\pi_1, \pi_2, \pi_3) = 0 \) and we obtain
\[
\dim (\pi_1 \otimes \pi_2 \otimes \pi_3)^G + \dim (\pi_1^{T'} \otimes \pi_2^{T'} \otimes \pi_3^{T'})^{T'} = 1.
\]
If any of \( \pi_i \) is a principal series, then \( m_T(\pi_1, \pi_2, \pi_3) = 0 \) and we obtain
\[
\dim (\pi_1 \otimes \pi_2 \otimes \pi_3)^G - \dim (\pi_1^{S'} \otimes \pi_2^{S'} \otimes \pi_3^{S'})^S' = 1.
\]
If all of \( \pi_i \) are twists of Steinberg representation, then by (ii) it is easy to see that \( m_S(\pi_1, \pi_2, \pi_3) = m_T(\pi_1, \pi_2, \pi_3) \in \{0, 1\} \) and we obtain
\[
\dim (\pi_1 \otimes \pi_2 \otimes \pi_3)^G = 1.
\]
These equalities allow us to compute \( \dim (\pi_1 \otimes \pi_2 \otimes \pi_3)^G \) easily in all cases.

**Corollary 3.2.3.** Let \( \pi \) be a generic irreducible representation of \( G = \text{PGL}_2(\mathbb{F}_q) \). Then \( \dim (\pi \otimes \pi \otimes \pi)^G = 1 \) unless \( \pi \) is a cuspidal representation attached to a nontrivial cubic character of \( T \), in which case \( \dim (\pi \otimes \pi \otimes \pi)^G = 0 \).

This result will be applied twice later: the Fricke–Macbeath curve in §3.3 and the Bring curve in §4.1.2.

**Remark 3.2.4.** We indicate the relation to Prasad’s trilinear form for representations of \( \text{PGL}_2 \) over local fields. Let \( F \) be a non-archimedean local field with ring of integers \( \mathcal{O}_F \) and residue field \( k = \mathbb{F}_q \). For irreducible cuspidal representations \( \pi_i, 1 \leq i \leq 3 \) of \( \text{PGL}_2(k) \), by inflation we obtain representations \( \pi_i^0 \) of \( \text{PGL}_2(\mathcal{O}_F) \). Then the compact induction \( \Pi_i = c\text{Ind}_{\text{PGL}_2(\mathcal{O}_F)}^G \pi_i \) is a supercuspidal representation of \( \text{PGL}_2(F) \). Then using the facts after [19, Prop. 6.7] one can show
\[
\dim \text{Hom}_{\text{PGL}_2(F)}(\Pi_1 \otimes \Pi_2 \otimes \Pi_3, \mathbb{C}) = \dim \text{Hom}_{\text{PGL}_2(k)}(\pi_1 \otimes \pi_2 \otimes \pi_3, \mathbb{C}).
\]
Let \( D \) be the division quaternion algebra over \( F \). Then we have \( D^x/F^x \mathcal{O}_D^x \simeq T' \) and we may view \( \pi_i^{T'} \) as a representation of \( PD^x = D^x/F^x \), which is exactly the Jacquet–Langlands correspondence \( \Pi_i' \) of \( \Pi_i \). Then our result translates to Prasad’s theorem [19] that
\[
\dim \text{Hom}_{\text{PGL}_2(F)}(\Pi_1 \otimes \Pi_2 \otimes \Pi_3', \mathbb{C}) + \dim \text{Hom}_{D^x}(\Pi_1' \otimes \Pi_2' \otimes \Pi_3', \mathbb{C}) = 1.
\]

### 3.3. Fricke–Macbeath curve and its genus 3 quotient

**Theorem 3.3.1** (Bisogno–Li–Litt–Srinivasan [4], Gross [11]). There are no nonzero diagonal-\( \text{PGL}_2(\mathbb{F}_8) \)-invariant elements in \( H^1(X)^{\otimes 3} \).

Let \( \pi_{X_3} \) denote the cuspidal representation of \( \text{PGL}_2(\mathbb{F}_8) \) attached to the unique order-3 character \( \chi_3 \) (up to taking its inverse) of the anisotropic torus \( T \simeq \mathbb{F}_8^3/\mathbb{F}_8^3 \simeq \mathbb{Z}/9\mathbb{Z} \). We sketch a “pure thought” proof that
\[
H^1(X) = 2\pi_{X_3},
\]
which implies the desired assertion \( (H^1(X)^{\otimes 3})^G = 0 \) by Corollary 3.2.3.

For \( i = 2, 3, 7 \), let \( H_i \) be a cyclic subgroup of order \( i \), which is unique up to conjugacy. We may assume that \( H_2 \) is generated by any nontrivial element of order 2 in the unipotent radical \( U \simeq \mathbb{F}_8^3 \) of the Borel subgroup, \( H_3 \) is the 3-torsion subgroup of \( T \), and \( H_7 \) is \( S \). Then, by (3.3) and Frobenius reciprocity, for any non-trivial irreducible representation \( \pi \), its multiplicity in \( H^1(X) \) is equal to
\[
m(\pi) = \dim \pi - \dim \pi^{H_2} - \dim \pi^{H_3} - \dim \pi^{H_7}.
\]
There are four 7-dimensional cuspidal representations called \( \pi_{\chi} = \pi_{\chi^{-1}} \), attached to \( \chi : T \to \mathbb{C}^* \) such that \( \chi \neq \chi^{-1} \). Recall (see for example [12, §7]) that the restriction of \( \pi_\chi \) to the torus \( S \) is the regular representation, the restriction to \( T \) is the regular representation minus \( \chi, \chi^{-1} \), and the restriction to the unipotent radical \( U \) is the regular representation minus the trivial representation. It follows that all four cuspidal representations the property: \( \dim \pi^{H_2} = 1, \dim \pi^{H_3} = 4 - 1 = 3, \dim \pi^{H_7} = 3 - 2 \) (resp. \( 3 - 0 \)) if \( \chi|_{H_3} = 1 \) (resp. \( \chi|_{H_3} \neq 1 \), namely \( \chi \) is of order 3 (resp. of order 9). Hence
m(\pi) = 7 - 3 - 1 - 1 = 2 \text{ (resp. } 7 - 3 - 3 - 1 = 0 \text{ ) if } \chi \text{ is of order 3 (resp. of order 9). By dimension reason, we must have } H^1(X) = 2\pi_\chi.

Now by Theorem 2.4.1, we have \([\Delta_\chi] = 0\). Let \(Y\) be the quotient curve of the Fricke–Macbeath curve \(X\) under an involution (all order 2 elements are conjugate).

**Corollary 3.3.2.** The curve \(Y\) is a genus 3 non-hyperelliptic curve such that \(\pi_\ast(\xi_X) = \xi_Y \in \text{Ch}^1(Y)\) and \([\Delta_\xi] = 0 \text{ in } \text{Ch}^2(Y)\).

**Proof.** By [27, §4], \(Y\) is non-hyperelliptic. The rest follows from Proposition 2.3.3. \(\square\)

In particular, \(X\) is non-hyperelliptic, which can also be checked directly since we know \(\text{Aut}(X)\).

**Lemma 3.3.3.** We have \((H^1(Y) \otimes 3)_{\text{Aut}(Y)} \neq 0\).

**Proof.** By a direct computation using [16, §6, Table 1, Table 2], we find that the only genus 3 non-hyperelliptic curve \(C\) with \((H^1(C) \otimes 3)_{\text{Aut}(C)} = 0\) has plane model \(p^4 = x^3 - 1\) (see 4.1.1). By computing Dixmier–Ohno invariants [8, 18], we found that \(Y\) is not isomorphic to \(C\) over \(\mathbb{C}\). The lemma follows. \(\square\)

### 3.4. Hurwitz curves with automorphism groups \(G = \text{PSL}_2(\mathbb{F}_q)\)

We consider all Hurwitz curves with automorphism group \(G = \text{PSL}_2(\mathbb{F}_q)\) with \(q = p^m\) for a rational prime \(p\). By a theorem of Macbeath, the necessary and sufficient condition for \(\text{PSL}_2(\mathbb{F}_q)\) to arise as the automorphism group of a Hurwitz curve is that \(q = 7\), or \(q = p\) with \(p \equiv \pm 1 \mod 7\), or \(q = p^3\) with \(p \equiv \pm 2\), or \(\pm 3 \mod 7\) [7, §3].

**Theorem 3.4.1.** The Fricke–Macbeath curve is the unique Hurwitz curve \(X\) with \(\text{Aut}(X) \simeq \text{PSL}_2(\mathbb{F}_q)\) that has the property \((H^1(X) \otimes 3)_{\text{Aut}(X)} = 0\).

First of all, it is easy to check the assertion case-by-case when \(q < 43\). In general, we note that the group \(\text{PSL}_2(\mathbb{F}_q)\) has the property that there is at most one conjugacy class of cyclic subgroups of a given order. Note that a Hurwitz curve \(X\) must be a \(G = \text{Aut}(X)\)-covering of \(\mathbb{P}^1 \simeq X/G\) of signature \((2, 3, 7)\) by the Riemann–Hurwitz formula (3.4). We let \(G_i\) be a cyclic subgroup of \(G\) of order \(e_i\) (unique to \(i\) to \(\text{conj} u g\) i\(y\)y) for \(e_1 = 2, e_2 = 3, e_3 = 7\). We let Hur denote the representation defined by (3.3). It may or may not arise from the Hurwitz representation for some \(G\)-curve \(X\) (we have not imposed the condition (3.2)). Every Hurwitz representation arising from a Hurwitz curve with automorphism group \(G = \text{PSL}_2(\mathbb{F}_q)\) must be of such form. We prove the following assertion which implies Theorem 3.4.1.

**Proposition 3.4.2.** When \(q \geq 43\) is odd, the representation \(\text{Hur} \circ G = \text{PSL}_2(\mathbb{F}_q)\) has the property \((\text{Hur} \otimes 3)^G \neq 0\).

**Proof.** We list the following properties of the simple group \(\text{PSL}_2(\mathbb{F}_q)\) [15, §4]. (See [12, §2] for more details when \(q = p > 3\) is an odd prime.)

- The order of the group is \(|\text{PSL}_2(\mathbb{F}_q)| = \frac{1}{2}q(q + 1)(q - 1)|.
- For every divisor \(d > 2\) of \((q - 1)/2\) (resp. \((q + 1)/2\)), there are \(\phi(d)/2\) conjugacy classes of order \(d\) in \(G\) with size \(2|G|/(q - 1)\) (resp. \(2|G|/(q + 1)\)).
- When \(d = 2\), there is a unique conjugacy class of order \(d\) with size \(|G|/(q - 1)\) (resp. \(|G|/(q + 1)\)) if \(2|(q - 1)/2\) (resp. \(2|(q + 1)/2\)).
- There are two unipotent conjugacy classes of order \(p\) with size \(|G|/q|.
- Each element is conjugate to one in either the split torus \(S\), the anisotropic torus \(T\), or a cyclic subgroup \(\simeq \mathbb{F}_p\) of the unipotent \(U\).

Let \(g \in \text{PSL}_2(\mathbb{F}_q)\) be an element of order \(d \neq 1\). Then by (3.3) (also computed explicitly in [15, §4]) we have

\[\chi(g) := \text{Tr}(g|\text{Hur}) = 2 - \begin{cases} 
(q - 1) \sum_{d|e_i} \frac{1}{e_i}, & \text{if } d|2^{e_i - 1} \\
(q + 1) \sum_{d|e_i} \frac{1}{e_i}, & \text{if } d|2^{e_i + 1} \bigg) - q(1 - p^{-1}) \sum_{e_i=p} 1, & \text{if } d = p.
\]
It follows that $\chi(g) = 2$ unless $g = 1$ or $g$ is conjugate to an element in $G_i$. We have $\chi(1) = 2 + \frac{|G|}{q-1} > 0$.

The number of the conjugacy class of order $i = 2, 3, 7$ is $1, 1, 3$ respectively and the size is at most $\frac{|G|}{q-1}$, and the value of $\chi$ at such a conjugacy class is bounded below by $-q$. By omitting the other conjugacy classes, we obtain

\[
\sum_{g \in G} \chi(g)^3 > \left(\frac{|G|}{42}\right)^3 - 2\frac{|G|}{q-1} q^3(1+1+3) > \frac{(q-1)^3q^3(1+1)}{84^3} - \frac{(q-1)q(q+1)}{q-1} q^3(1+1+3) > \frac{q^4(q+1)}{84^3}((q-1)^3(q+1) - 5 \cdot 2^3 \cdot 42^3).
\]

If $q \geq 43$ we have $(q-1)^3(q+1) - 5 \cdot 2^3 \cdot 42^3 \geq 42^3 \cdot (44 - 5 \cdot 2^3) > 0$. Therefore $\sum_{g \in G} \chi(g)^3 > 0$ and the space of $G$-invariants $(\text{Hur} \otimes G)^G$ must be non-zero.

\[\square\]

4. More examples

The purpose of this section is to show that our vanishing Theorem 2.4.1 is applicable to a rather large number of examples in genus $3 \leq g \leq 5$ listed in [16], by exhibiting non-hyperelliptic curves there with the property $(H^1(X) \otimes G)^G = 0$. We give “pure thought” proofs of this equality whenever possible (e.g., Bring’s curve in §4.1.2 and the 1-dimensional family in §4.2.1). The proofs for the remaining cases use character tables together with straightforward, albeit tedious, computation, which we feel are more suitable to appear in a separate document [21]. Using Lemma 3.1.1 and Klein’s classification of finite subgroups of $\text{PGL}_2(\mathbb{C})$, it is easy to see that none of the curves below is hyperelliptic.

4.1. 0-dimensional examples. The following examples arise from zero-dimensional Hurwitz stacks $\mathcal{H}(g, G, \mathcal{C})$. (In fact, by [16, §6, 7.1], every example below has $G$ as the full automorphism group.)

4.1.1. $g = 3$: $y^4 = x^3 - 1$. By [16, Table 2], the curve $X$ with plane model $y^4 = x^3 - 1$ has $g = 3$ and is a $G$-curve of signature $(2, 3, 12)$, where $G$ is the non-split central extension by $\mathbb{Z}/4\mathbb{Z}$ of $A_4$, which has Group ID $(48, 33)$ in the Small Groups Library. In fact, this is only genus 3 non-hyperelliptic curve $X$ with $(H^1(X) \otimes G)^{\text{Aut}(X)} = 0$, as [16, §6] exhausts all curves of genus 3 (while for higher genera, [16, §6] only includes curves with large automorphism groups).

4.1.2. $g = 4$: Bring’s curve. Among all genus 4 curves, the Bring curve $X$ is the unique one with the largest possible automorphism group:

\[\text{Aut}(X) = G := S_5 \simeq \text{PGL}_2(\mathbb{F}_5),\]

which has Group ID $(120, 34)$ from the Small Groups Library. It is a $G$-curve of signature $(2, 4, 5)$. It admits an explicit model

\[\left\{ \sum_{i=1}^5 X_i = \sum_{i=1}^5 X_i^2 = \sum_{i=1}^5 X_i^3 = 0 \right\} \subset \mathbb{P}^4\]

on which $S_5$ acts by permuting the coordinates $X_i$. By the same argument after Theorem 3.3.1, we can show

\[H^1(X) = 2\pi_{\chi_3}\]

where $\pi_{\chi_3}$ is the cuspidal representation of $\text{PGL}_2(\mathbb{F}_5)$ attached to the unique order-3 character (up to taking its inverse) of the anisotropic torus $T$. By Corollary 3.2.3, we have $(H^1(X) \otimes G)^G = 0$.

4.1.3. $g = 4$: $y^4 = (x^3 - 1)^2(x^3 + 1)$. By [16, Table 4], there is a unique $G$-curve $X$ of $g = 4$ with $G$ the wreath product $S_3 \wr \mathbb{Z}/2\mathbb{Z}$, which has Group ID $(72, 40)$ from the Small Groups Library. It has signature $(2, 4, 6)$. It is cyclic trigonal and its equation is found in [26].
4.1.4. $g = 5$: Wiman’s curve. By [16, Table 4], there is a unique $G$-curve $X$ of $g = 5$ with $G = (\mathbb{Z}/2\mathbb{Z})^4 \rtimes D_5$, which has Group ID (160, 234) from the Small Groups Library. It has signature $(2, 4, 5)$. It was first studied by Wiman as we learnt from [26]. Moreover, its equation can also be found in [26].

4.1.5. $g = 5$: another Wiman’s curve. By [16, Table 4], there is a unique $G$-curve $X$ of $g = 5$ with $G = GL_2(\mathbb{Z}/4\mathbb{Z})$, which has Group ID (96, 195) from the Small Groups Library. It has signature $(2, 4, 6)$. It was also studied by Wiman and its equation can also be found in [26].

4.2. 1-dimensional families. The following examples arise from 1-dimensional Hurwitz stacks $\mathcal{H}(g, G, \mathcal{C})$. (In fact, from [16, 7.1], it is not hard to deduce that the generic fiber of each example has $G$ as the full automorphism group.) Their equations could be computed as [26] (though not done here or in [26]). In particular, it is not hard to show that they are in fact defined over number fields.

4.2.1. $g = 4$. By [16, Table 4], there is a unique 1-dimensional family of $G$-curves of $g = 4$ and signature $(2, 2, 2, 3)$ where $G = S_3 \times S_3$, which has Group ID (36, 10) from the Small Groups Library. (The existence can be shown by the theory of Hurwitz stack recalled in §3.1; see the next paragraph.)

This is the universal family over the Hurwitz stack $\mathcal{H}(4, G, \mathcal{C})$ for the choice of $\mathcal{C}$ given below. We denote $H = S_3$ and let $H_i$, $i = 2, 3$, be a subgroup of $H$ of order $i$, which is unique up to conjugacy. The four cyclic subgroups of $G = H \times H$ appearing in the Hurwitz representation (3.3) are (up to conjugacy)

$$H_2 \times \{1\}, \{1\} \times H_2, \Delta(H_2), \Delta(H_3).$$

It is easy to see that one can arrange a choice of the generator of a suitable conjugacy of each of the four subgroups such that their product is $1$ and they generate the group $G$. Choosing $\mathcal{C}$ accordingly and then by (3.2), the Hurwitz stack $\mathcal{H}(4, G, \mathcal{C})$ is non-empty and has dimension $r - 3 = 4 - 3 = 1$.

Now we compute the induced representations appearing in (3.3), using $\text{Hom}_G(\text{Ind}_H^G(1), \pi) \simeq \pi^H$ (Frobenius reciprocity) and noting that the regular representation of $H$ decomposes:

$$\text{Ind}_{1}^{S_3}(1) = (1) + \text{sgn} + 2\rho,$$

where $\text{sgn}$ denotes the sign character and $\rho$ is the unique irreducible representation of dimension 2. We enumerate the results:

$$\text{Ind}_{H_2 \times \{1\}}^{H \times H}(1) = \text{Ind}_{H_2}(1) \boxtimes \text{Ind}_{\{1\}}^{H}(1) = ((1) + \rho) \boxtimes ((1) + \text{sgn} + 2\rho)$$

$$\text{Ind}_{\{1\} \times H_2}^{H \times H}(1) = \text{Ind}_{\{1\}}^{H}(1) \boxtimes \text{Ind}_{H_2}(1) = ((1) + \text{sgn} + 2\rho) \boxtimes ((1) + \rho)$$

$$\text{Ind}_{\Delta(H_2)}^{H \times H}(1) = (1) + \text{sgn} \boxplus \text{sgn} + \text{sgn} \boxplus \rho + \rho \boxplus \text{sgn} + (1) \boxplus \rho + \rho \boxplus (1) + 2\rho \boxplus \rho$$

$$\text{Ind}_{\Delta(H_3)}^{H \times H}(1) = (1) + (1) \boxplus \text{sgn} + \text{sgn} \boxplus (1) + \text{sgn} \boxplus \text{sgn} + 2\rho \boxplus \rho.$$

It follows from (3.3) that for any curve $X$ in this family,

$$H^1(X) = 2(\text{sgn} \boxplus \rho + \rho \boxplus \text{sgn}).$$

By the lemma below we have $(H^1(X)^{\otimes 3})^G = 0$.

**Lemma 4.2.1.** We have $((\text{sgn} \boxplus \rho + \rho \boxplus \text{sgn})^{\otimes 3})^{H \times H} = 0$.

**Proof.** By looking at $H_2 \times \{1\}$-action and by $(\text{sgn}^{\otimes 3})^{H_2} = 0$, we have

$$(\text{sgn} \boxplus \rho)^{\otimes 3}H \times H \simeq (\text{sgn}^{\otimes 3})H \boxtimes (\rho^{\otimes 3})H = 0.$$ 

By symmetry we have $((\rho \boxplus \text{sgn})^{\otimes 3})^{H \times H} = 0$. Next, since

$$(\text{sgn} \boxplus \rho)^{\otimes 2} \otimes (\rho \boxplus \text{sgn}) = \rho \boxtimes (\rho^{\otimes 2} \boxplus \text{sgn})$$

and $\rho^{H_3} = 0$, by looking at $H_3 \times \{1\}$-action, we conclude that the left hand side has no $H \times H$-invariant. By symmetry we conclude that $(\text{sgn} \boxplus \rho) \otimes (\rho \boxplus \text{sgn})^{\otimes 2}$ has no $H \times H$-invariant. We have thus shown $((\text{sgn} \boxplus \rho + \rho \boxplus \text{sgn})^{\otimes 3})^{H \times H} = 0$. \qed
4.2.2. \( g = 5 \). By [16, Table 4], there is a unique 1-dimensional family of \( G \)-curves of \( g = 5 \) where \( G = (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z} \), which has Group ID (32, 27) from the Small Groups Library. It has signature (2, 2, 2, 4).

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