Maxwell field with Torsion

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Abstract

We propose a generalizing gauge-invariant model of propagating torsion which couples to the Maxwell field and to charged particles. As a result we have an Abelian gauge invariant action which leads to a theory with nonzero torsion and which is consistent with available experimental data.

1 Introduction

In gravitation theories the Minimal Coupling Procedure (MCP) can be simply stated as a procedure which, starting from a theory in flat spacetime, substitutes all partial derivatives by covariant derivatives and the flat metric by the Riemannian metric. The impossibility of achieving simultaneously the usual gauge invariance of electromagnetism and MCP of the gauge field to torsion has led many authors to abandon the MCP, keeping usual partial derivatives (as opposed to the covariant ones) in the definition of the electromagnetic field tensor [1]. However, perturbative results from QFT of the photon self-energy suggest corrections to the Maxwell equations coming from the coupling between spinors

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and an external torsion field [2]. Together with the fact that in Riemann-Cartan
spacetime the particle spin also works as a source for torsion [3], one is tempted
to modify the electromagnetic field tensor in a minimal way.

In [4] only the trace part of the torsion tensor minimally couples to the
electromagnetic field tensor. Charge conservation and gauge invariance constrain
the torsion trace to be a gradient of a constant scalar field, thus leading to a
trivial theory with non-propagating torsion. Among the “minimal” non-trivial
modifications, we single out two formalisms. The first one amounts to a modi-
fication of the gauge transformation, keeping the usual MCP [5, 6]:

$$\delta_A \mu = e^\phi \partial_\mu \epsilon,$$

(1)

where $\phi (x)$ is a scalar field whose gradient gives the trace part of the
torsion tensor. This formalism is known as the Hojman-Rosenbaum-Ryan-Shepley
propagating torsion (HRRS model) and has been developed, in particular, for
non-Abelian gauge theories in [7].

The second formalism, which we refer to as Saa’s theory, represents a change
in the MCP procedure by modification of the definition of the invariant volume
form [8, 9, 10]. This formalism, which has been adopted by some authors [11],
does not apply MCP for the electromagnetic field tensor, keeping the usual defi-
nition in terms of the exterior derivative of the Maxwell gauge connection. As a
result, it does not violate usual $U(1)$ gauge symmetry. Later in [12], by requir-
ing that equivalence classes of Lagrangians be related by MCP, it was shown
that the trace of the torsion tensor must be a gradient whose scalar deforms the
invariant volume form, exactly as was proposed before on geometrical grounds.

Both formalisms provide a wave equation for a scalar whose gradient is the
trace part of the torsion tensor and which additionally involves the electromagnetic
field. These models, where the torsion’s dynamics is given by a wave
equation, are called propagating torsion models. In [11] a conformal transforma-
tion is made in order to remove the negative sign of the kinetic term arising from
the torsionic scalar in Saa’s theory. As a result, the invariant volume simply
becomes the square root of the determinant of the conformal metric, and MCP
is performed to the Maxwell field tensor in the sense of HRRS theory. Despite
their elegance, both formalisms are incompatible with experimental results. In
the case of HRRS theory, experimental data from the solar system invalidate
any predicted geodesic deviation of atoms with different electromagnetic con-
tent [13]. Fiziev et al. point out in [14, 15] that Saa’s theory violates basic
experimental gravitational and solar system data.

The absence of any free parameters both in Saa’s theory and in the HRRS
model, together with the experimental data, invalidates both models. In this
work we couple electromagnetism to torsion by a generalizing Abelian gauge
invariance to obtain MCP at the Action level. Here we propose an alternate
model to HRRS’s, where compatibility of a new gauge principle with MCP re-
quires the introduction of a free parameter, which is constrained by the available
experimental data. As a result we have an Abelian gauge invariant Action which
leads to a theory of propagating torsion with nonzero trace part and which is
consistent with available experimental data. Besides, the theory we present also admits a formulation in terms of Semi-Minimal Coupling, i.e., MCP at the level of differential forms. Therefore, our proposal effectively rehabilitates a model of propagating torsion closely related to previous attempts.

The work is organized as follows: in Section 2 we introduce notation and definitions used throughout this work and also describe in detail the compatibility between gauge invariance and MCP. In Section 3 we present a new gauge transformation which provides an invariant Maxwell-like action and we propose a redefinition of the trace part of the torsion tensor in order to achieve MCP at the action level. In this connection we also discuss coupling to scalar fields and the Newtonian limit. In Section 4 we conclude with some final remarks and perspectives. In the Appendix we provide the semi-minimal coupling procedure formalism, and we apply it to obtain sources to the Maxwell equations.

Throughout this work we use units in which \( c = G = \hbar = 1 \) and metric signature \((+,-,-,-)\).

2 Gauge invariance and MCP compatibility

We use the following definitions for torsion and covariant derivative. Let \( \nabla \) be an affine connection compatible with the Lorentzian metric \( g \). The covariant derivative of a vector field \( \upsilon \) along \( \partial_\mu \) in a local chart \( x^\mu \) is

\[
\nabla_\mu \upsilon^\nu = \partial_\mu \upsilon^\nu + \Gamma^\nu_{\mu\sigma} \upsilon^\sigma .
\]

(2)

The tensor components of the torsion of the connection \( \nabla \) can be given in terms of connection coefficients as

\[
T^\mu_{\nu\sigma} = \Gamma^\mu_{\nu\sigma} - \Gamma^\nu_{\mu\sigma} .
\]

The contorsion tensor is defined by

\[
K_{\mu\nu\sigma} = \frac{1}{2} (T_{\mu\nu\sigma} + T_{\sigma\nu\mu} + T_{\sigma\mu\nu}) = -K_{\mu\sigma\nu} \Rightarrow T_{\mu\nu\sigma} = K_{\mu\nu\sigma} .
\]

In terms of the contorsion tensor, one can split the covariant derivative (2) in two parts, one involving the Levi-Civita connection coefficients \( \bar{\Gamma} \) of the Levi-Civita connection \( \bar{\nabla} \) and another involving the contorsion tensor,

\[
\nabla_\mu \upsilon^\nu = \partial_\mu \upsilon^\nu + \bar{\Gamma}^\nu_{\mu\sigma} \upsilon^\sigma + K^\mu_{\nu\sigma} \upsilon^\sigma
\]

\[
= \nabla_\mu \upsilon^\nu + K^\mu_{\nu\sigma} \upsilon^\sigma .
\]

(3)

We note that unlike in Saa’s use of a transposed connection in [9], we adopt the usual definition for the covariant derivative of a scalar density \( S \) of weight \( w \), i.e.,

\[
\nabla_\mu S = \partial_\mu S + w \Gamma^\mu_{\nu\sigma} S = \partial_\mu S + w \bar{\Gamma}^\mu_{\nu\sigma} S,
\]

since the contortion tensor is antisymmetric in the last two indices. Therefore, we do not change the volume element, \( \sqrt{-g} \) being a density of weight \(-1\),

\[
\nabla_\mu \sqrt{|\det g|} = 0 .
\]
Quite generally, the torsion tensor can be divided in three parts,

$$T_{\mu\nu\sigma} = \frac{1}{3} (g_{\mu\sigma} T_{\nu} - g_{\nu\sigma} T_{\mu}) + \frac{1}{6} \varepsilon_{\mu\nu\sigma\rho} S^\rho + Q_{\mu\nu\sigma},$$  \hspace{1cm} (4)$$
such that $S_\mu$ is the axial part (also known as pseudo-trace), $T_\mu$ is the trace part, and $Q_{\mu\nu\sigma}$ is tensor with vanishing trace and pseudo-trace, i.e., \(\varepsilon^{\mu\nu\sigma\kappa} Q_{\nu\sigma\kappa} = 0\). Therefore, the trace part $T_\mu$ is given by

$$T_\mu = T_{\sigma\mu} = K_{\sigma\mu},$$  \hspace{1cm} (5)$$
while the axial part or pseudo-trace is given by

$$S^\mu = \varepsilon^{\mu\nu\sigma\kappa} T_{\nu\sigma\kappa}.$$

Restricting, for now, the analysis to the homogeneous Maxwell equations (without sources), the MCP leads to

$$\int d^4 x f_{\mu\nu} f^{\mu\nu} \to \int d^4 x \sqrt{-g} \tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu},$$
where $f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the usual electromagnetic field tensor and

$$\tilde{F}_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu,$$  \hspace{1cm} (6)$$
is the one obtained via MCP. Using the definition of covariant derivative in (3) and the expression (5) we have

$$\tilde{F}_{\mu\nu} = f_{\mu\nu} - T_{\mu\nu} A_\sigma.$$

The expression for the minimally-coupled field tensor shows explicit dependence on the potential, making MCP incompatible with usual $U(1)$ gauge invariance in the presence of torsion. In addition, in the case where the torsion tensor is completely anti-symmetric, i.e., in the presence of the torsion pseudo-trace alone, the homogeneous Maxwell equation $\nabla_\mu \tilde{F}^{\mu\nu} = 0$ reduces to

$$\nabla_\mu \tilde{F}^{\mu\nu} = \tilde{\nabla}_\mu f^{\mu\nu} - \frac{1}{12} \varepsilon^{\mu\nu\rho\sigma} (\tilde{\nabla}_\mu S_\rho - \tilde{\nabla}_\rho S_\mu) A_\sigma - \frac{g}{36} (g^{\mu\sigma} S^2 - S^\sigma S^\nu) A_\sigma + \frac{1}{6} \varepsilon^{\mu\nu\rho\sigma} f_{\mu\rho} S_\sigma = 0.$$  \hspace{1cm} (7)$$

The non-invariant terms, linear in the gauge potential,

$$- \frac{1}{12} \varepsilon^{\mu\nu\rho\sigma} (\tilde{\nabla}_\mu S_\rho - \tilde{\nabla}_\rho S_\mu) A_\sigma - \frac{g}{36} (g^{\mu\sigma} S^2 - S^\sigma S^\nu) A_\sigma,$$

must vanish independently of $A_\mu$ and of each other. If one contracts them with $A_\nu$, one finds a relation between torsion and the gauge potential which is furthermore not invariant, namely,

$$S^2 A^2 = (S_\mu A^\mu)^2.$$
Therefore, the symmetric matrix $g^{\nu\sigma} S^2 - S^\sigma S^\nu$ must vanish. Since it can be diagonalized to $\text{diag}(0, S^2, S^2, S^2)$, one arrives at the conclusion that $S^2 = 0$. Substituting this condition back into the matrix, one is left with the trivial solution $S_\mu = 0$. As a result, it is not possible to perform MCP and maintain $U(1)$ gauge invariance for non vanishing torsion pseudo-trace.

Even in the case where the pseudo-trace vanishes, from (4) we thus come to the conclusion that is not possible to attain MCP, unless one is willing to modify either the usual $U(1)$ gauge invariance of electromagnetism or the notion of MCP. In effect, in [8], a different notion of MCP was used, where one only demands MCP at the level of differential forms, with the exterior derivative being replaced by a covariant exterior derivative $(d \rightarrow D)$, called “Semi-Minimal Coupling Procedure” (SMCP). While in [5], MCP at the Action level was retained at the cost of modifying the gauge invariance. In the Appendix A we discuss a version of the SMCP applied to our theory.

3 New gauge transformation and minimal coupling

Now consider the (global) gauge transformation with constant parameter $\epsilon$ given by

$$\delta_\epsilon A_\mu = T_\mu \epsilon, \quad (8)$$

where $T_\mu$ is in principle any four-vector. One can easily see that in order that (8) be a global symmetry of the Maxwell action

$$A_0 = \frac{1}{4} \int d^4 x \sqrt{-g} f^{2}, \quad f_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

it is necessary that $T_\mu$ be the gradient of a scalar function, i.e.,

$$T_\mu = \partial_\mu \phi, \quad (9)$$

since $\delta_\epsilon f_{\mu\nu} = (\partial_\mu T_\nu - \partial_\nu T_\mu) \epsilon$. In order to obtain a local transformation, we promote $\epsilon$ to a function $\epsilon(x)$ of the spacetime coordinates. Now the Maxwell action $A_0$ is no longer invariant, and the non invariant term is

$$\delta_\epsilon A_0 = \int d^4 x \sqrt{-g} f^{\mu\nu} \partial_\nu \phi \partial_\mu \epsilon.$$

The appearance of the derivatives $\partial_\mu \epsilon$ can be compensated by a proper modification of the original gauge transformation (8). In what follows we use the well-known Noether technique in constructing supergravity theories [16], in this case weaving together the usual $U(1)$ transformation with the torsion field $T_\mu$. We then compensate non-invariant terms order by order in a new parameter, in order to achieve an invariant Lagrange functional. Now let

$$\delta_\epsilon A_\mu = \frac{1}{\alpha} \partial_\mu \epsilon + \partial_\mu \phi \epsilon, \quad (10)$$
be a local gauge transformation with parameter $\epsilon(x)$, and some coupling constant $\alpha$. To order $\alpha^0$, the action

$$A_1 = A_0 - \alpha \int d^4x \sqrt{-g} f^{\mu\nu} \partial_{\nu} \varphi A_{\mu}$$

is invariant under (10), and the non invariant contribution is

$$\delta \epsilon A_1 = -\alpha \int d^4x \sqrt{-g} (\partial^\nu \varphi \partial_\nu \varphi A_{\mu} - \partial^\nu \varphi A_{\nu} \partial_\mu \varphi) \partial^\mu \epsilon .$$

We compensate this term with the addition of a new term to the action $A_1$, giving the total action

$$A_2 = A_1 + \frac{\alpha^2}{2} \int d^4x \sqrt{-g} (\partial^\nu \varphi \partial_\nu \varphi A_{\mu} - \partial^\nu \varphi A_{\nu} \partial_\mu \varphi) .$$

One can now check that action $A_2$ is invariant under (10) to all orders of $\alpha$, i.e., $\delta \epsilon A_2 = 0$. Action $A_2$ can be conveniently written as

$$A_2 = \frac{1}{4} \int d^4x \sqrt{-g} F^2 , \quad (11)$$

where

$$F_{\mu\nu} = f_{\mu\nu} + \alpha (\partial_\mu \varphi A_{\nu} - \partial_\nu \varphi A_{\mu}) . \quad (12)$$

Invariance of $A_2$ can be easily seen from the identity $\delta \epsilon F_{\mu\nu} = 0$. In addition, the equations of motion that follow from action $A_2$ are

$$\left(\overline{\nabla}_\mu \alpha T_{\mu} \right) F^{\mu\nu} = 0 , \quad (13)$$

where $\overline{\nabla}$ is the Levi-Civita connection (no torsion).

The theory with action (11) is gauge-invariant by the local Abelian transformation (10) and generally covariant. However, at this point it is not clear that the vector field $T_{\mu}$ plays the role of torsion. Below we provide the necessary interpretation.

In order to give the interpretation that the model proposed in this section actually describes a coupling between electromagnetism and torsion, we turn to the definition of the Maxwell field tensor given in terms of MCP (6), i.e.,

$$\tilde{F}_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu .$$

We wish to identify $\tilde{F}_{\mu\nu}$ with $F_{\mu\nu}$ from (12). This can be achieved by rescaling the trace components of the torsion tensor, such that now, instead of (4), one has for the trace part:

$$T_{\mu\nu\sigma} = \alpha (g_{\mu\sigma} T_{\nu} - g_{\nu\sigma} T_{\mu}) . \quad (14)$$

As a result, one has $\nabla_\mu A_\nu - \nabla_\nu A_\mu = F_{\mu\nu}$, with $F_{\mu\nu}$ given by (12). Therefore, by demanding MCP at the level of the Action, one is led to a nontrivial coupling between torsion and electromagnetism, which manifests itself in the very definition of the torsion tensor by means of the introduction of the constant $\alpha$. 

6
3.1 Action functional with matter fields

Given the local transformation (10), let us obtain a unitary realization of $U(1)$ on scalar fields $\phi, \phi' = \gamma \phi, \gamma^\dagger = \gamma^{-1}$, such that the covariant derivative of the scalar field $\phi$ transforms in the same representation,

$$D_\mu \phi \rightarrow \gamma D_\mu \phi.$$ 

We search for a solution in the form

$$\gamma = e^{iqf(\varphi)\epsilon}, \quad D_\mu \phi = (\partial_\mu - ig(q(\varphi) A_\mu) \phi,$$

where $f(\varphi)$ and $g(\varphi)$ are functions of $\varphi$, $T_\mu = \partial_\mu \varphi$, $q$ is the charge and $\epsilon$ is an arbitrary transformation parameter. Thus

$$(D_\mu \phi)' = \partial_\mu \phi' - igg(q(\varphi) A'_\mu \phi'$$

$$= \gamma \partial_\mu \phi + ig\gamma(\epsilon \partial_\mu f + f \partial_\mu \epsilon) \phi - i\gamma g(q(\varphi)(A_\mu + \frac{1}{\alpha} \partial_\mu \epsilon + \partial_\mu \varphi \epsilon) \phi).$$

In order to have $(D_\mu \phi)' = \gamma D_\mu \phi$, $f$ and $g$ must obey

$$f = \frac{1}{\alpha} g, \quad \partial_\mu f = g \partial_\mu \varphi.$$ 

The solutions are either $f = \alpha^{-1} e^{\alpha \varphi}$ and $g = e^{\alpha \varphi}$ or $f = e^{\alpha \varphi}$ and $g = \alpha e^{\alpha \varphi}$. The second pair is favored, since it provides the correct vanishing torsion limit for the charge current density, as we explain below. Therefore, the scalar fields $\phi$ must transform as

$$\phi' = \exp (iqe^{\alpha \varphi} \epsilon) \phi$$

and the covariant derivative is given by

$$D_\mu \phi = (\partial_\mu - iq\alpha e^{\alpha \varphi} A_\mu) \phi.$$ 

An action functional invariant by (10) has the form:

$$S_\phi = \int d^4x \sqrt{-g} D_\mu \phi D^\mu \phi.$$ 

The equations of motion are

$$\frac{1}{\sqrt{-g}} D_\mu (\sqrt{-g} D^\mu \phi) = 0.$$ 

By Noether's theorem one can write the current density

$$\epsilon j_\mu^\phi = \frac{\partial L}{\partial \partial_\mu \phi} \delta_\epsilon \phi + \frac{\partial L}{\partial \partial_\mu \phi} \delta_\epsilon \phi,$$
where $\delta_\epsilon \phi = i q e^{\alpha \varphi} \epsilon \phi$ and $\delta_\epsilon \phi = -i q e^{\alpha \varphi} \epsilon \phi$.

Therefore, the gauge invariant current is

$$j_\mu^\phi = i q \sqrt{-g} e^{\alpha \varphi} \phi D_\mu \phi - i q \sqrt{-g} e^{\alpha \varphi} \phi D^\mu \phi = -i q \sqrt{-g} e^{\alpha \varphi} \phi \phi D_\mu \phi$$

$$= -i \sqrt{-g} q e^{\alpha \varphi} (\phi \partial^\mu \phi - \partial^\mu \phi \phi) - 2 \sqrt{-g} q^2 e^{2 \alpha \varphi} A^\mu \phi \phi.$$ (15)

It is easily seen that

$$(\partial_\mu - \alpha T_\mu) j_\mu^\phi = -i q e^{\alpha \varphi} \partial_\mu \left( \sqrt{-g} \phi \phi D_\mu \phi \right).$$

From the equations of motion for $\phi$ and $\phi$, one can show that $\partial_\mu \left( \sqrt{-g} \phi \phi D_\mu \phi \right) = 0$. Thus, on-shell, one has the generalized continuity equation, see equation (24) in the Appendix.

### 3.2 Physical fields and the action

We also note that the action (11) has symmetry (1), which, with the notation introduced in this section, reads

$$\delta_\epsilon' A_\mu = e^{-\alpha \varphi} \partial_\mu \epsilon, \quad \delta_\epsilon' F_{\mu \nu} = 0.$$ (16)

This symmetry, proposed in [5, 6] is actually the usual $U(1)$-symmetry for the redefined vector potential

$$B_\mu = e^{\alpha \varphi} A_\mu,$$ (16)

since $\delta_\epsilon' B_\mu = \partial_\mu \epsilon$ and $F_{\mu \nu} = e^{-\alpha \varphi} H_{\mu \nu}$, where $H_{\mu \nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$.

For the proposed new gauge symmetry, one can similarly redefine $A_\mu$, so as to obtain a $U(1)$-like symmetry instead of (10). In this case, one has

$$b_\mu = \alpha e^{\alpha \varphi} A_\mu, \quad \delta_\epsilon b_\mu = \partial_\mu \left( e^{\alpha \varphi} \epsilon \right).$$ (17)

It is clear this is not the usual $U(1)$-symmetry, since the torsion scalar $\varphi$ participates in the transformation. Nonetheless, we can also write $h_{\mu \nu} = \alpha e^{\alpha \varphi} F_{\mu \nu}$, where $h_{\mu \nu} = \partial_\mu b_\nu - \partial_\nu b_\mu$ is the usual Maxwell field tensor.

The current (15) has clearer meaning in terms of the $b$-field (17), since then $j_\mu^\phi$ becomes

$$j_\mu^\phi = \sqrt{-g} e^{\alpha \varphi} \left[ -i q \left( \phi \partial^\mu \phi - \partial^\mu \phi \phi \right) - 2 q^2 b^\mu \phi \phi \right].$$

In the above form the geometrical contributions are clearly separated, and one attains the interpretation of a field $\phi$ of charge $q$ coupled to the Maxwell connection $b_\mu$. In the limit of vanishing torsion, $\varphi \to 0$, the current becomes the usual charged Klein-Gordon field minimally coupled to $b_\mu$, which is invariant under the usual $U(1)$ gauge transformation. Unlike in the HRRS model, the current density is affected by torsion even in the absence of electromagnetic fields.

From the above considerations, we take the physical electromagnetic four-vector to be $b_\mu$ and the gauge-invariant physical electromagnetic field strength
tensor to be $h_{\mu\nu}$. This will have important consequences, as we show in the next section on the Newtonian limit.

Apart from a total divergence, the Einstein action is

$$S_g = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left( \hat{R} - 6\alpha^2 T^2 \right),$$

where $\hat{R}$ is the curvature scalar defined in terms of the Levi-Civita connection $\hat{\nabla}$, and we have definition (14) for the trace part of the torsion tensor. Therefore, the total action encompassing gravitation, electromagnetism and matter fields is given by the integral of the Lagrange density

$$\mathcal{L} = \sqrt{-g} \left( \hat{R} - 6\alpha^2 \partial_\mu \phi \partial^\mu \phi - \frac{e^{-2\alpha \phi}}{\alpha^2} h_{\mu\nu}^2 - 4D_\mu \phi D^\mu \phi \right).$$ (18)

We note that the kinetic term for the torsionic scalar field has the right sign, according to [11]. The Euler-Lagrange equations following (18) are

$$\delta b_\mu : (\hat{\nabla}_\mu - \alpha \partial_\mu \phi) \frac{e^{-2\alpha \phi}}{\alpha} h^{\mu\nu} + iq\alpha e^{\alpha \phi} (\hat{\bar{D}}^\nu \phi \hat{D}^\nu \phi) = 0, \quad (19)$$

$$\delta \phi : 3\alpha^2 \Box \phi + \frac{1}{\alpha} \hat{\nabla}_\mu \left( e^{-2\alpha \phi} h_{\mu\nu} b_{\nu} \right) - i\alpha q b^\mu \left( \hat{\bar{D}}^\mu \phi - \hat{D}^\mu \phi \right) = 0, \quad (20)$$

$$\delta \phi : \hat{\nabla}^\mu D_\mu \phi - iq b_\mu D^\mu \phi = 0, \quad (21)$$

$$\delta \phi : \hat{\nabla}^\mu D_\mu \phi - iq b_\mu D^\mu \phi = 0. \quad (22)$$

Using equation (19) in the equation (20) for the torsionic scalar $\phi$ gives

$$3\alpha^2 \Box \phi + \frac{1}{2} \frac{e^{-2\alpha \phi}}{\alpha} h_{\mu\nu} h^{\mu\nu} = 0.$$

We note that taking $\phi = 0$, all torsion-dependent terms in the Lagrangian density (15) vanish and one recovers the usual equations for electromagnetism and matter fields in general relativity in terms of $f_{\mu\nu}$.

For $\alpha = -1$ the above expressions are identical to the ones derived from the HRRS model. Therefore, at first sight, it seems that our theory is equivalent to the HRRS theory by the transformation

$$\alpha \phi \to -\phi, \quad \alpha q \to q.$$

Even the new gauge transformation (10) can be made $\alpha$-independent by means of the rescaling $\frac{\epsilon}{\alpha} \to \epsilon$,

$$\delta \epsilon A_\mu = \partial_\mu \epsilon - \partial_\mu \phi \epsilon.$$

However, we must stress two points: First, we take (10) as the gauge transformation defining measurable quantities $b_\mu$ and $h_{\mu\nu}$. As a result, measurable fields and gauge-invariant fields in our approach differ from those in the HRRS (16). The second point is that making the above rescalings in order to eliminate $\alpha$ is tantamount to setting it to $-1$, which is not acceptable according to available experimental data. Thus, this constant needs to be determined by experiment. We show in the next section that the value $\alpha = -1$ can be excluded and the present theory for $\alpha \neq -1$ is nonequivalent to the HRRS one.
3.3 Newtonian Limit and comparison with experimental results

In this section we quote results from [13] with regard to test body accelerations in the Newtonian limit of the HRRS model. From a merely formal viewpoint, their results can be translated into our model by means of the map \( \varphi \mapsto -\alpha \varphi \) and the definition of the (measurable) field \( b_\mu \) from [17] and its associated strength tensor \( h_{\mu\nu} = \partial_\mu b_\nu - \partial_\nu b_\mu \). By solving the equations of motion for \( \varphi \) in the weak field approximation, as well as considering the background electric and magnetic fields of the sun, one arrives at a relation between \( \varphi \) and the sun’s gravitational potential \( U \). On the other hand, the test body acceleration in a local inertial frame can be computed from the Lagrangian density (18) as

\[
\ddot{X}_\mu = -\frac{2}{\alpha} \frac{E_e - E_m}{m} \partial_\mu \varphi ,
\]

where \( m, E_m \) and \( E_e \) are the mass, the total electric and magnetic energy of the test body. Combining the two results, and using typical values of \( E_e \) for platinum and aluminum atoms, one gets the deviation in the acceleration of platinum (\( \ddot{X}^{\text{Pt}}_i \)) and aluminum (\( \ddot{X}^{\text{Al}}_i \)) atoms:

\[
\ddot{X}^{\text{Pt}}_i - \ddot{X}^{\text{Al}}_i = \alpha^{-4} 2 \times 10^{-7} \partial_i U .
\]

The above deviation vanishes with a precision of 1 part in \( 10^{12} \) of \( \partial_i U \) according to [17]. It is clear that the HRRS model is in disagreement with experimental data, since it corresponds to \( \alpha = -1 \). In the case of our theory, the above experimental result can be used to establish a lower bound for \( \alpha \),

\[
(2 \times 10^{-7}) \alpha^{-4} < 10^{-12} \Rightarrow |\alpha| \gtrsim 20 .
\]

It is important to note that making the rescalings indicated at the end of section (3.2) and proceeding as above to obtain the Newtonian limit, the constant \( \alpha \) disappears. This is a consequence of the choice of the physical fields [17].

4 Final remarks and perspectives

We have proposed a gauge-invariant model of propagating torsion which couples to the Maxwell field and to charged particles. Our model requires the introduction of a constant \( \alpha \) into the definition of the trace part of the torsion tensor, so that minimal coupling can be achieved at the level of the Action. We provide in the Appendix a realization of the equations of motion in terms of the semiminimal coupling (coupling at the level of differential forms), in which case the Maxwell equations are formally identical to the torsionless Maxwell equations by the substitution of the exterior derivative \( d \) by an appropriate map \( D \).

The fact that we do not have MCP when applied to the equations of motion does not represent a setback, since, as was pointed out in [18], MCP can
only be safely applied at the Action level, because of the possible appearance of curvature-dependent terms in the equations of motion, violating the equivalence principle. We can expect that, as in the case of the classical limit of quantum theories, different theories with torsion correspond to the same flat space limit. This is a different viewpoint from the one adopted in [12], where MCP is restricted to map equivalent theories in flat space to equivalent theories in curved space.

Despite the formal identification between the HRRS model and our present proposal, which can be achieved by mapping the torsion scalar $\varphi$ to $-\alpha \varphi$ and the charge $q$ to $\alpha q$, in our construction $\alpha$ naturally appears as coupling constant in the construction of the gauge-invariant Maxwell-like action (11), in terms of the new gauge transformation (10). In this sense, the HRRS model can be seen as a particular case when $\alpha = -1$. However, current experimental tests rule out the value $\alpha = -1$, and set a lower bound on $\alpha$, $|\alpha| > 20$.

Besides, in the case of HRRS theory, one can see that the proposed gauge transformation (1) can be recast in familiar terms by means of a field redefinition, $B_\mu = e^{-\varphi} A_\mu \Rightarrow \delta B_\mu = \partial_\mu \epsilon$; while the gauge transformation proposed here is nontrivial in the sense that the gauge parameter involves the torsion field, $b_\mu = \alpha e^{\alpha \varphi} A_\mu \Rightarrow \delta b_\mu = \partial_\mu (e^{\alpha \varphi} \epsilon)$. We also note that the $\alpha^{-1}$ term in the gauge transformation (10) is not essential, since, as is the case in usual Maxwell theory, one can redefine the gauge field $A_\mu$ such that the gauge transformation becomes $\delta A_\mu = \partial_\mu \epsilon + \alpha \partial_\mu \varphi$. Then, as a result, the Maxwell action will be multiplied by a factor of $\alpha^{-2}$. Perhaps in this sense $\alpha$ might be construed to be some sort of charge carried by torsion, which is related to the interaction between the torsion field and other fields in the model. In fact, by setting $\alpha$ to zero, one negates any and all torsionic effects.

Finally, we wish to note that, albeit the introduction of the constant $\alpha$ in the definition of the gauge transformations might appear as a trivial modification in both HRRS and Saa’s theory, this modification, without the introduction of the new gauge transformation, is not enough to allow both theories to dodge the available experimental restrictions. In addition, in our case the introduction of $\alpha$ is a matter of principle, that led to a nontrivial coupling between torsion and electromagnetism and without which the MCP is ill-defined (see Section 3).

A natural sequel to this work would be to apply the theory here presented to a specific cosmological model, in order to obtain testable physical predictions and hopefully an upper bound for $\alpha$, and also to generalize the gauge principle to non-Abelian gauge groups.

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A Semi-minimal coupling

In the following we consider the general gauge transformation and modified Maxwell field tensor given by (10) and (12), respectively. For arbitrary \( \alpha \), we show that a minimal coupling on the level of differential forms can be achieved which circumvents issues related to restrictions on the connection coefficients and the failure of MCP when applied to the homogeneous equations encountered in [8].

The Maxwell field tensor (12) can be written invariantly in terms of the differential two-form
\[
F = dA - \alpha T \wedge A ,
\]
where \( A = A_{\mu} dx^\mu \) and \( T = T_{\mu} dx^\mu \) are the vector potential and torsion trace one-forms. Consider the map \( D : \Lambda^p \to \Lambda^{p+1} \) defined in the space of \( p \)-forms \( \omega \) such that\(^2\)
\[
D \omega = d \omega - \alpha T \wedge \omega .
\]
One can show that for an arbitrary \( p \)-form \( \omega \)
\[
D^2 \omega = 0
\]
since \( T \) is exact, \( T = d \varphi \). From the nilpotency of the map \( D \) it follows that
\[
DF = 0 ,
\]
which is the analog of the homogeneous Maxwell equations \( df \equiv 0 \), where \( f = dA \). The homogeneous equations in a local coordinate map are
\[
(\partial_\mu - \alpha T_\mu) F_{\nu \sigma} + (\partial_\sigma - \alpha T_\sigma) F_{\mu \nu} + (\partial_\nu - \alpha T_\nu) F_{\sigma \mu} \equiv 0 .
\]
Thus, the homogeneous equations are identically zero, and no restriction on either \( F \) or \( T \) arise.

The analog of the inhomogeneous Maxwell equations without sources, \( \ast d \ast f = 0 \) is
\[
\ast D \ast F = \left( -\tilde{\nabla}_\mu F^{\mu \nu} + \alpha T_\mu F^{\mu \nu} \right) dx_\nu = 0 ,
\]
which coincides with the equations of motion (13).

We have thus shown that the semi-minimal coupling given by the substitution of the de-Rahm exterior product \( d \) by the map \( D \) provides the Maxwell equations coupled to the trace part of the torsion tensor:
\[
f = dA \to F = DA ,
\]
\[
df \equiv 0 \to DF \equiv 0 ,
\]
\[
\ast d \ast f = 0 \to \ast D \ast F = 0 .
\]

\( ^1 \)We note that \( D \) is not a graded derivation, i.e., one does not have \( D (\omega_p \wedge \omega_q) = D\omega_p \wedge \omega_q + (-1)^p \omega_p \wedge D\omega_q \), where \( \omega_p \) is a \( p \)-form and \( \omega_q \) is a \( q \)-form.
B Maxwell Equations with sources

In this section we apply the formalism presented in the previous section in order to calculate conserved currents.

Let us introduce the current density 3-form

\[ j = \frac{1}{3!} j^\mu \varepsilon_{\mu\nu\sigma} dx^\nu \wedge dx^\sigma \wedge dx^\tau, \]

such that the inhomogeneous Maxwell equations become

\[ D \ast F = j. \quad (23) \]

Since \( D^2 = 0 \), one has the condition \( Dj = 0 \) to ensure consistency of the Maxwell equations, which in a local chart reads

\[ (\nabla_\mu - \alpha T_\mu) j^\mu = 0. \quad (24) \]

The interaction term in the action is gauge invariant provided the current satisfies the conservation equation (24):

\[ \delta \int d^4x \sqrt{-g} j^\mu A_\mu = -\frac{1}{\alpha} \int d^4x \sqrt{-g} \left( \nabla_\mu - \alpha T_\mu \right) j^\mu. \]

Now consider the three-form \( \tau_T = \ast (i_T \ast F) \), where \( i_T \) is the interior derivative along the vector field \( T \), which in coordinates is given by

\[ \tau_T = \frac{\alpha}{3!} \left( T_\mu f_{\nu\sigma} + T_\sigma f_{\mu\nu} + T_\nu f_{\sigma\mu} \right) dx^\mu \wedge dx^\nu \wedge dx^\sigma. \]

This three-form is covariantly conserved:

\[ D\tau_T = d\tau_T - \alpha T \wedge \tau_T \equiv 0, \]

which is consistent with \( DF \equiv 0 \). Since \( T \wedge \tau_T \) vanishes, it follows that \( \tau_T \) is a closed form,

\[ d\tau_T = 0. \quad (25) \]

Thus one can construct a conserved quantity, the gauge invariant one-form \( j_T = \ast \tau_T \), which has the local expression

\[ j_T^\mu = -\frac{\alpha}{2} \varepsilon_{\mu\nu\rho\kappa} T^\nu F^{\rho\kappa} = -\frac{\alpha}{2} \varepsilon_{\mu\nu\rho\kappa} T^\nu f^{\rho\kappa}. \]

Following (25), one has \( d \ast j_T = 0 \):

\[ \partial_\mu j_T^\mu \equiv \nabla_\mu j_T^\mu \equiv 0. \]

Thus, if \( \Sigma \) is space-like hypersurface, one has the conserved quantity

\[ Q = \alpha \int_\Sigma d^3x \sqrt{\tilde{g}} T \cdot B, \]

where \( \tilde{g}_{\mu\nu} \) is the induced metric on \( \Sigma \), and \( T \) and \( B \) are torsion and magnetic field vectors. One can show that the total charge \( Q \) is a boundary term and vanishes at infinity in case the fields have vanishing boundary values at infinity, as in the case of propagating torsion theory. From \( j_T \) one can construct a dual torsionic source for electromagnetism, called “electric current” in [5], \( j_E = i_T F \), which in coordinates reads \( j_E^\mu = -\alpha F^{\mu\nu} T_\nu \).
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