A study on linear and non-linear parton evolution equations

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In the high energy regime, the proton structure consists of a very large number of particles called partons (quarks and gluons) that interact with each other, according to the theory of strong interactions, the Quantum Chromodynamics (QCD). Through QCD, the number of partons in the proton is described by equations of parton evolution that depend on kinematic variables $x$ and $Q^2$. These equations can be linear, the DGLAP equations, and nonlinear, the GLR-MQ equation. We have studied some analytic solutions of the GLR-MQ equation. In order to generate the preliminary results, we used an ansatz for the solution of the equations of evolution of the gluon distribution, and comparing with results of parametrizations of Parton Distributions Functions (PDFs). In the future, we planned to applied the another method to solve the non-linear equations, using the Laplace transform.

Keywords: Quantum Chromodynamics, Evolution equations, Saturation, Non-linear effects.

I. INTRODUCTION

The Standard Model of the Elementary Particles is the theory that describes the constituents of the matter. In this model, we have that the constituents of all matter are the fermions, which are separated into quarks and leptons, and the bosons are intermediary particles, which perform the interaction between fermions. In this work, we focus in the proton structure, which is formed by quarks and gluons. The last one is the boson that performs the interaction between quarks associated with strong interaction. The theory that studies the strong interaction is the Quantum Chromodynamics (QCD). The particles bearing the color charge are not freely observed in nature, then the quarks and gluons are only observed as bound states, forming particles called hadrons, which are classified according to the number of quarks that constitute them. Therefore, the proton is a hadron that have three quarks in its structure. The experimental study of the structure of the proton is made through the deep inelastic scattering (DIS), which consists of a lepton scattering with a proton, where the lepton emits a photon with virtuality $Q^2$. Other
kinematic variables involved in this process are

\[ x = \frac{Q^2}{Q^2 + W^2 - m_N^2}, \quad y = \frac{W^2 + Q^2 - m_N^2}{(l + P)^2 - m_N^2}, \]

where \( x \) is the Bjorken variable, related to the fraction of momentum of hadron carried by its constituents; \( y \) is the inelasticity, related to energy transferred from the lepton to the final state; \( W^2 \) is the square of center of mass energy of the photon-nucleon system and \( m_N \) is the nucleon mass.

\section{Equations of Evolution of the Parton Densities}

In a low energy regime, the proton has its structure established by three valence quarks. At high energies, the proton structure becomes more complex: due the gluon radiation by the quarks, the gluons are predominant. It is observed that at small \( x \) (large \( W^2 \) with fixed \( Q^2 \)) there is an increase in the density of partons, where sea quarks and gluons appear, with the increase of the gluon density much greater than the quarks one. Using QCD, we can describe this behavior with the linear DGLAP evolution equations (Dokshitzer, V. Gribov, Lipatov, Altarelli e Parisi) (to a complete discussion see [1]), which allow us to determine the evolution of the parton distributions if we do not consider the gluon recombination. This coupled system of integral-differential equations reads

\[
\frac{\partial}{\partial \ln Q^2} \left( \frac{\Sigma(x, Q^2)}{G(x, Q^2)} \right) = \frac{\alpha_s}{2\pi} \int_x^1 dz \left( \frac{P_{qq}(z)}{P_{gq}(z)} \right) \left( \frac{\Sigma(x/z, Q^2)}{G(x/z, Q^2)} \right),
\]

where

\[ \Sigma(x, Q^2) = \sum_{i=1}^{n_f} \left[ x q_i(x, Q^2) + x \bar{q}_i(x, Q^2) \right], \]

\[ G(x, Q^2) = x g(x, Q^2), \]

whith \( \Sigma(x, Q^2) \) is singlet quark distribution, \( G(x, Q^2) \) is the gluon distribution and \( P_{ij}(z) \) are the splitting functions meaning the probability of a parton \( i \) issuing another parton \( j \) with a fraction of momentum \( z \).

\section{Saturation and the GLR-MQ Equation}

The DGLAP equation predicts that the gluon distribution function has a strong growth in the small \( x \) region. This growth and its theoretical description are related to the violation of the so-called Froissard-Martin boundary and the unitarity of the cross section. This is due to the fact that one gluon can emit other gluons. Therefore, another process must be considered in order to reverse the effect described by DGLAP: the recombination of two gluon in one. At a certain energy, the gluons recombine, and this causes the gluon distribution functions to saturate, thus decreasing the density of gluons. The parton evolution equations was modified to include the saturation effect by Gribov, Levin and Ryskin [2] and later by Mueller and Qiu [3]. Thus, this proposal brought a correction in the parton evolution equations, presenting a nonlinear term in the equations, constituting the so-called GLR-MQ equation. In small \( x \), the quark contribution can be neglected, thus the GLR-MQ equation for the gluon distribution function \( G(x, Q^2) = x g(x, Q^2) \) reads

\[
\frac{\partial G(x, Q^2)}{\partial \ln Q^2} = \frac{\partial G_{\text{DGLAP}}(x, Q^2)}{\partial \ln Q^2} - \frac{\gamma}{R^2 Q^2} \int_x^1 \frac{dy}{y} G(y, Q^2),
\]

where the first term comes from the DGLAP equation (also without the quark contribution) and \( R \) is an effective radius of gluon interaction and \( \gamma = 81/16 \).

\section{Solutions for Nonlinear Evolution Equations}

The equations for the gluon distribution \( G(x, Q^2) \) can be analytically solved at small \( x \) assuming an ansatz for functional dependence of \( x \) and \( Q^2 \), based in the Regge theory: \( G(x, Q^2) = x^{-\lambda} H(Q^2) \). After a simple algebra, Eq. \( \text{(6)} \) became \( \text{(6)} \),

\[
\frac{dH(t)}{dt} = \gamma_1(x) \frac{H(t)}{t} - \gamma_2(x) \frac{1}{e^2 t^2} H^2(t),
\]
where \( t = \log(Q^2/\Lambda^2) \) with \( \Lambda^2 \) is the QCD energy scale and

\[
\gamma_1(x) = \frac{12}{\beta_0} \left( \frac{11}{12} - N_f \right) + \ln(1 - x) + \int_x^1 \left( \frac{z^{\lambda+1} - 1}{1 - z} + \left( 1 + \frac{1 - z}{z} \right) z^\lambda dz \right),
\]

and

\[
\gamma_2(x) = \gamma_2 - \frac{16\pi^2}{R^2 \lambda^2} x^{-\lambda} \int_x^1 dz z^{2\lambda - 1},
\]

with \( \beta_0 = 25/3 \).

The above equation can be solved by usual methods and the solutions are

\[
H(t) = \frac{t^{\gamma_1}}{H_0 - \gamma_2 \Gamma(\gamma_1 - 1, t)},
\]

where \( \Gamma(a, z) \) is the incomplete gamma function. Using the properties for gamma function and after algebraic manipulations, we obtain, including the initial condition,

\[
H(t) = \frac{H(t_0) e^{\gamma_2(x)} \Gamma(\gamma_1(x) - 1, t) - \Gamma(\gamma_1(x) - 1, t)}{t_0^{\gamma_2(x)} + \gamma_2(x) H(t_0)[\Gamma(\gamma_1(x) - 1, t) - \Gamma(\gamma_1(x) - 1, t)]}.
\]

Thus, the distribution of gluons will be

\[
G(x, Q^2) = x^{-\lambda_g} H \left( \ln \frac{Q^2}{\Lambda^2} \right).
\]

From the results obtained analytically, we developed a computational code to calculate the distribution of gluons. We choose as initial condition for gluon distribution as the same value of the parton parametrizations found in the LHAPDF interface\(^\text{[5]}\) for the parametrizations (we use CTEQ6, MMHT2014, CT14, HERAPDF15 paramaterizations) and we compared the results obtained by the non-linear evolution equation and the results of the above (linear) parton distributions. We obtained the results for the gluon distribution as a function of \( Q^2 \) presented in the Figs. \(^\text{[2]}\) and \(^\text{[3]}\) for different choices of \( x \) and fixed values of \( \lambda_g = 0.5 \) and \( R = 5.0 \text{ GeV}^{-1} \).

The results show, as expected, that with the decrease of \( x \), the distribution of gluons increases considerably. When \( Q^2 \) increases, the linear result exhibits rapid growth, while the non-linear result has its increase drastically suppressed. Note that the linear result comes from a parameterization that includes the contribution of quarks and, because of our choice of the initial condition, the linear and non-linear results starts from the same point. We obtain the desirable behavior for the distribution of gluons through a simple functional form that factorized the dependence on \( x \) and \( Q^2 \) for the gluon distribution. However, this form may be too simple, since \( \lambda \) may depend on the virtuality of the photon.
In this work, we revised the results to the analytical solution of the GLR-MQ evolution equations in a high energy regime through a simple ansatz for the functional form. The final analytical result is easily implementable and its results show that, in fact, the distributions of gluons have their growth in the region of small $x$ substantially diminished.

As future works, we intend to study the variation of $\lambda_\sigma$ with $Q^2$ and use the Laplace transform method [6, 7] to solve the linear and nonlinear evolution equations, comparing their results.

**V. CONCLUSIONS AND FUTURE WORKS**

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