ADAPTIVE PROJECTIVE SYNCHRONIZATION OF MEMRISTIVE NEURAL NETWORKS WITH TIME-VARYING DELAYS AND STOCHASTIC PERTURBATION

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(Communicated by Gianmario Tessitore)

Abstract. This paper is concerned with the projective synchronization issue for memristive neural networks with time-varying delays and stochastic perturbations. Based on LaSalle-type invariance principle of stochastic functional-differential equations, by applying Lyapunov functional approach, several sufficient conditions are developed to achieve the projective synchronization between the master-slave systems with time-varying delays under stochastic perturbation and adaptive controller. A numerical example and its simulation is given to show the effectiveness of the theoretical results in this paper.

1. Introduction. For 150 years, the known fundamental elements of an electrical circuit were the capacitor, resistor and inductor, which discovered in 1745, 1827 and 1831, respectively. However, at the beginning of 1970s, when L.Chua, an engineer of the University of California at Berkeley, attempting for the six known different mathematical relationships that connecting pairs of the four fundamental circuit variables, current $i$, voltage $v$, charge $q$ and flux $\phi$, observed that five of these relationships were well-known, two of them are given by the definition of electric current and Faraday’s law, which is

$$i = \frac{dq}{dt} \quad \text{and} \quad v = \frac{d\phi}{dt}.$$  

The other three are given by the basic equations which axiomatically defined the three classical circuit elements, which is

$$R = \frac{dv}{di}, \quad L = \frac{d\phi}{di} \quad \text{and} \quad C = \frac{dq}{dv}.$$  

2010 Mathematics Subject Classification. 34A36, 34K27, 34K50, 34D06, 93C23.
Key words and phrases. Memristor, neural network, time-varying delays, synchronization, stochastic perturbations.
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where $R$, $L$ and $C$ represent the resistance, inductance, and capacitance respectively. On the other hand, Chua realized that one of these relationships were undefined, that is the flux $\phi$ and the charge $q$. Based on this analysis, a seminal paper appeared on 1 May 2008 [12]. Subsequently, a team from HP company announced the fabrication of a nanometer-sizes solid-state two-terminal device called memristor (a crisis for memory resistor) [3]. From then on, worldwide interest has been generated on this passive electronic because of its potential applications in various fields, from new high speed low-power processors [14], filters [4] to new biological models for associative memory [10], learning models of simple organisms [9], to which memristor has been introduced as new technology process nodes.

The physical model of the memristor is shown in Fig. 1, a thin undoped titanium dioxide ($TiO_2$) layer and a thin oxygen-poor titanium dioxide ($TiO_{2-x}$) layer are sandwiched between two platinum electrodes. Pure ($TiO_2$) is of high resistivity while the oxygen vacancies render the ($TiO_{2-x}$) material conductive. When current flows in one direction through the device, the boundary between the two materials moves, causing an increase in the percentage of the conducting ($TiO_{2-x}$) layer. As a result, the $Ti$ resistance of the device decreases. When current flows in the opposite direction, the amount of insulating ($TiO_2$) increases and its resistance increases. When the current is stopped, the oxygen vacancies stop moving and the device retains its last resistance value. In this case the boundary between the two layers remains frozen. In other words, the memristor “remembers” how much current has passed through it. One immediate application offers an enabling low cost technology for non-volatile memories where future computers would turn on instantly without the usual “booting time”.

**Figure 1.** Schematic of HP memristor, $w$ is the time-dependent thickness of the ($TiO_{2-x}$) layer, while $D$ is the thicknesses of the film.

As we know, the recurrent neural networks are very important nonlinear circuit networks due to its wide applications in optimization problems, systems control, data compression, image processing, pattern recognition and associative memory [15], [6], [16], [20], [22], [27], [25], [24], [11], [28]. In consideration of many practical applications of the recurrent neural networks, an interesting issue is to investigate the memristor-based recurrent neural networks, which is an ideal model to mimic the functionalities of the human brain, and provide an in-depth understanding of key design implications of memristive networks.

Since the concept of drive-response synchronization for coupled chaotic systems was proposed in [8], much attention have been focused on control and chaos synchronization for its potential applications, such as secure communication, biological systems, information science, etc. [13], [2]. Therefore, the investigation of synchronization of delayed neural networks are of practical importance. However, a real
system is usually affected by external perturbations which are of great uncertainty and may be treated as random. As pointed by Haykin, in real nervous systems, synaptic transmission is a noisy process brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes. Therefore, the effect of noise should be taken into account in researching the synchronization of chaos systems.

Inspired by the above discussion, in this paper, we treat of the projective synchronization issue of chaotic memristive networks. Based on LaSalle-type invariance principle, an appropriate adaptive controller is proposed for the memristive networks with stochastic perturbation. The obtained results show that the projective synchronization between the master-slave systems could be almost surely achieved even if the networks subjected to stochastic perturbation. It should be noted that in [19]-[17], the following assumption:

\[
\text{co}\{A, \bar{A}\} f(x(t)) - \text{co}\{A, \bar{A}\} f(y(t)) \subseteq \text{co}\{A, \bar{A}\} (f(x(t)) - f(y(t)))
\]

was used in the proof of the main results. It can be easily checked that this assumption holds only when \(f_j(x_j(t))\) and \(f_j(y_j(t))\) have different signs, or \(f_j(x_j(t)) = 0\) or \(f_j(y_j(t)) = 0\); The results in [19]-[17] are independent of the neuron state switching jumps \(T_i, i = 1, 2, \cdots, n\). From the view of mathematics, the results obtained in these works are meaningless to the theory research and application.

The main contribution of this paper lies: 1) We treat projective synchronization, in this case, many kinds of synchronization can be regard as its special cases; 2) A novel adaptive controller is designed to achieve projective goal; 3) The typically assumption in [19]-[17] is abounded and all the results obtained in this paper are related to the switching jumps, which show that our results are more consistent with the actual fact.

Now we are in a position to give a short overview of the paper. In Section 2, the mathematics models of memristor is described, and some preliminaries are introduced. The adaptive function projective synchronization is discussed in Section 3. In Section 4, a numerical example is presented to demonstrate the validity of the proposed results. Section 5 is the conclusions of this paper.

2. Preliminaries.

2.1. Notation. \(\mathbb{R}\) denotes the set of real numbers, \(\mathbb{R}^n\) denotes the n-dimensional Euclidean space. \([\cdot, \cdot]\) represents the interval. \(\text{co}(Q)\) denotes the closure of the convex hull of \(Q\). \(I\) denotes the identity matrix with compatible dimension. A vector or matrix \(A \geq 0\) means that all entries of \(A\) are greater than or equal to zero, \(A > 0\) can be defined similarly. For vectors or matrices \(A\) and \(B\), \(A \geq B\) (or \(A > B\)) means that \(A - B \geq 0\) (or \(A - B > 0\)). Denote \(\lambda_{\max}(P)(\lambda_{\min}(P))\) as the maximum (minimum) eigenvalue of the positive definite matrix \(P\). Given the vectors \(x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n\), \(\|x\|\) denotes the Euclidean vector norm, i.e., \(\|x\| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}\).

2.2. Model description. Memristor-based recurrent neural network can be implemented by very large-scale integration (VLSI) circuits as shown in Fig.2. To better understand of the memristor-based recurrent neural networks, we take the \(i\)th subsystem as the unit of analysis in order to simplify illustration. From Fig.2, the KCL equation of the \(i\)th subsystem can be written as:
Figure 2. Circuit of memristor-based recurrent network.

\[
\dot{x}_i(t) = -\frac{1}{C_i} \left[ \sum_{j=1}^{n} \left( \frac{1}{R_{ij}} + \frac{1}{F_{ij}} \right) \times \text{sign}_{ij} + \frac{1}{R_i} \right] x_i(t) + \frac{1}{C_i} \sum_{j=1}^{n} \frac{f_j(x_j(t))}{F_{ij}} \times \text{sign}_{ij}, \quad t \geq 0, \quad i = 1, 2, \ldots, n, \tag{1}
\]

where \(x_i(t)\) is the voltage of the capacitor \(C_i\), \(R_{ij}\) denotes the resistor between the feedback function \(f_i(x_i(t))\) and \(x_i(t)\), \(F_{ij}\) denotes the resistor between the feedback function \(f_i(x_i(t-\tau(t)))\) and \(x_i(t)\), \(\tau(t)\) is the time-varying delay with \(0 < \tau(t) \leq \tau\), \(\hat{\tau}(t) \leq h < 1\); \(R_i\) represents the parallel-resistor corresponding the capacitor \(C_i\), and

\[
\text{sign}_{ij} = \begin{cases} 1, & i \neq j, \\ -1, & i = j. \end{cases}
\]

By replacing the resistors \(R_{ij}\), \(F_{ij}\) and \(R_i\) in the primitive recurrent neural networks (1) with memristors, whose memductances are \(W_{ij}\), \(M_{ij}\) and \(P_i\), respectively,
then we can construct the memristor-based recurrent neural network with time-varying delays as follows:

\[
\dot{x}_i(t) = -c_i(x_i(t))x_i(t) + \sum_{j=1}^{n} a_{ij}(x_i(t))f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(x_i(t))f_j(x_j(t-\tau(t))),
\]

\[t \geq 0, \quad i = 1, 2, \ldots, n,\]

(2)

where

\[
c_i(x_i(t)) = \frac{1}{C_i} \left[ \sum_{j=1}^{n} (W_{ij} + M_{ij}) \times \text{sign}_{ij} + F_i \right], \quad a_{ij}(x_i(t)) = \frac{W_{ij}}{C_i} \times \text{sign}_{ij},
\]

\[
b_{ij}(x_i(t)) = \frac{M_{ij}}{C_i} \times \text{sign}_{ij}.
\]

According to the feature of memristor, the coefficients in system (2) can be written as

\[
c_i(x_i(t)) = \begin{cases} \hat{c}_i, & |x_i(t)| \leq T_i, \\ \tilde{c}_i, & |x_i(t)| > T_i, \end{cases}
\]

\[
a_{ij}(x_i(t)) = \begin{cases} \hat{a}_{ij}, & |x_i(t)| \leq T_i, \\ \tilde{a}_{ij}, & |x_i(t)| > T_i, \end{cases}
\]

\[
b_{ij}(x_i(t)) = \begin{cases} \hat{b}_{ij}, & |x_i(t)| \leq T_i, \\ \tilde{b}_{ij}, & |x_i(t)| > T_i, \end{cases}
\]

where \(T_i > 0\) is the neuron state switching jump, \(\hat{c}_i > 0, \hat{c}_i > 0, \hat{a}_{ij}, \tilde{a}_{ij}, \hat{b}_{ij}, \tilde{b}_{ij}\), \(i, j = 1, 2, \ldots, n\), are all constants.

2.3. Preliminary.

Definition 2.1. ([1]) Suppose \(E \subseteq \mathbb{R}^n\), then \(x \to F(x)\) is called a set-valued map from \(E \to \mathbb{R}^n\), if for each point \(x \in E\), there exists a nonempty set \(F(x) \subseteq \mathbb{R}^n\). A set-valued map \(F\) with nonempty values is said to be upper semicontinuous (USC) at \(x_0 \in E\), if for any open set \(N\) containing \(F(x_0)\), there exists a neighborhood \(M\) of \(x_0\) such that \(F(M) \subseteq N\). The map \(F(x)\) is said to have a closed (convex, compact) image if for each \(x \in E, F(x)\), is closed (convex, compact).

The initial value associated with system (2) is \(x_i(t) = \phi_i(t) \in C([-\tau, 0]; \mathbb{R})\), \(i = 1, 2, \ldots, n\). Let \(c_i = \min\{\hat{c}_i, \tilde{c}_i\}, \tilde{c}_i = \max\{\hat{c}_i, \tilde{c}_i\}, a_{ij} = \min\{\hat{a}_{ij}, \tilde{a}_{ij}\}, \hat{a}_{ij} = \max\{\hat{a}_{ij}, \tilde{a}_{ij}\}, \hat{b}_{ij} = \min\{\hat{b}_{ij}, \tilde{b}_{ij}\}, \tilde{b}_{ij} = \max\{\hat{b}_{ij}, \tilde{b}_{ij}\}, \hat{a}_{ij} = \max\{|a_{ij}|, |\tilde{a}_{ij}|\}, \tilde{a}_{ij} = \max\{|\hat{a}_{ij}|, |\tilde{a}_{ij}|\}\). Define

\[
\text{co}(c_i(x_i(t))) = \begin{cases} \hat{c}_i, & |x_i(t)| < T_i, \\ \hat{c}_i, & |x_i(t)| = T_i, \\ \tilde{c}_i, & |x_i(t)| > T_i, \end{cases}
\]

\[
\text{co}(a_{ij}(x_i(t))) = \begin{cases} \hat{a}_{ij}, & |x_i(t)| < T_i, \\ \hat{a}_{ij}, & |x_i(t)| = T_i, \\ \tilde{a}_{ij}, & |x_i(t)| > T_i, \end{cases}
\]

\[
\text{co}(b_{ij}(x_i(t))) = \begin{cases} \hat{b}_{ij}, & |x_i(t)| < T_i, \\ \hat{b}_{ij}, & |x_i(t)| = T_i, \\ \tilde{b}_{ij}, & |x_i(t)| > T_i, \end{cases}
\]
Clearly, system (2) is a differential equation with discontinuous right-hand side, its solution in the conventional sense does not exist. Inspired by [23]-[18], we adopt the following definition of the solution in the sense of Filippov for system (2).

**Definition 2.2.** Suppose that \( \phi(s) = (\phi_1(s), \phi_2(s), \cdots, \phi_n(s))^T \in C([s, 0]; \mathbb{R}^n) \) is a continuous function. An absolutely continuous function \( x(t) \) is said to be a solution with initial data \( \phi(s) \) of system (2), if \( x(t) \) satisfies the differential inclusion

\[
\begin{align*}
\dot{x}_i(t) & \in -c_i(x_i(t))x_i(t) + \sum_{j=1}^{n} a_{ij}(x_i(t))f_j(x_j(t)) \\
& \quad + \sum_{j=1}^{n} b_{ij}(x_i(t))f_j(x_j(t - \tau(t))),
\end{align*}
\]

for all \( i = 1, 2, \cdots, n \), or equivalently, there exist \( c_i(t) \in \text{co}(c_i(x_i(t))) \), \( a_{ij}(t) \in \text{co}(a_{ij}(x_i(t))) \) and \( b_{ij}(t) \in \text{co}(b_{ij}(x_i(t))) \), such that

\[
\dot{x}_i(t) = -c_i(t)x_i(t) + \sum_{j=1}^{n} a_{ij}(t)f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}(t)f_j(x_j(t - \tau(t))).
\]

**Remark 1.** From the theoretical point of view, the above parameters \( c_i(t), a_{ij}(t) \) and \( b_{ij}(t) \) in system (2) are measurable functions and depend on the state \( x_i(t) \) and time \( t \).

To investigate synchronization problem, we define the error symbol as

\[
e_i(t) = y_i(t) - \alpha_i(t)x_i(t), \quad i = 1, 2, \cdots, n,
\]

where \( \alpha_i(t) \) is differentiable and bounded, i.e., there exist scalars \( \xi_i > 0 \), such that \( |\alpha_i(t)| \leq \xi_i, \quad i = 1, 2, \cdots, n \).

The corresponding response system of (2) is given by

\[
dy_i(t) = \left[ -c_i(y_i(t))y_i(t) + \sum_{j=1}^{n} a_{ij}(y_i(t))f_j(y_j(t)) + \sum_{j=1}^{n} b_{ij}(y_i(t))f_j(y_j(t - \tau(t))) \\
+ u_i(t) \right] dt + h_i(t, e_i(t), e_i(t - \tau(t)))d\omega_i(t), \quad t \geq 0, \quad i = 1, 2, \cdots, n,
\]

or, in a vector form

\[
dy(t) = \left[ -C(y(t))y(t) + A(y(t))f(y(t)) + B(y(t))f(y(t - \tau(t))) + u(t) \right] dt \\
+ h(t, e(t), e(t - \tau(t)))d\omega(t),
\]

where \( u(t) \) is the controller to be designed, \( \omega(t) \) is a \( n \)-dimensional Brownian motion defined on a complete probability space \( (\Omega, \mathcal{F}, \mathcal{P}) \) with a natural filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) generated by \( \{\omega(s) : 0 \leq s \leq t\} \), where we associate \( \Omega \) with the canonical space generated by \( \omega(t) \), and denote \( \mathcal{F} \) the associated \( \sigma \)-algebra generated by \( \{\omega(t)\} \) with the probability measure \( \mathcal{P} \). Here, the white noise \( d\omega_i(t) \) is independent of \( d\omega_j(t) \), for \( i \neq j \), and \( h : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n} \) is called the noise intensity function.
Lemma 2.3. (8) could be stated as follows.

where

This type of stochastic perturbation can be regarded as a result from the occurrence of external random fluctuation and other probabilistic causes.

In the following discussions, it is assumed that

(H1): There exist unknown positive constants \( l_i, F_i, i = 1, 2, \ldots, n \), such that

\[
0 \leq \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \leq l_i, \quad |f_i(\cdot)| \leq F_i, \quad \forall s_1, s_2 \in \mathbb{R}.
\]

(H2): There exist two real matrices \( G_1 \geq 0, G_2 \geq 0 \), such that

\[
\text{trace}[H^T(t,x,y)H(t,x,y)] \leq x^T(t)G_1x(t) + y^T(t)G_2y(t).
\]

Now, we are in a position to introduce the LaSalle-type invariance principle for stochastic differential delay equations, which plays an key role in the proof of our Theorem.

Consider an \( n \)-dimensional stochastic differential delay equation

\[
dx(t) = f(x(t), x(t - \tau), t)dt + h(x(t), x(t - \tau), t)d\omega(t). \tag{8}
\]

Let \( C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+) \) denote the family of all nonnegative functions \( V(x,t) \) on \( \mathbb{R}^n \times \mathbb{R}_+ \), which are twice continuously differentiable in \( x \) and once in \( t \). For each \( V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+) \), define an operator \( \mathcal{L}V \) from \( \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+ \) by

\[
\mathcal{L}V(x,y,t) = V(x,t) + V_z(x,t)f(x,y,t) + \frac{1}{2}\text{trace}[h^T(t,x,y)V_{xx}h(x,y,t)], \tag{9}
\]

where

\[
\begin{align*}
V_i(t,x) &= \frac{\partial V(t,x)}{\partial x_i}, & V_z(x,t) &= \left( \frac{\partial V(t,x)}{\partial x_1}, \ldots, \frac{\partial V(t,x)}{\partial x_n} \right), \\
V_{xx}(t,x) &= \left( \frac{\partial^2 V(t,x)}{\partial x_i \partial x_j} \right)_{n \times n},
\end{align*}
\]

so, the LaSalle-type invariance principle for stochastic differential delay equations (8) could be stated as follows.

**Lemma 2.3.** Assume that system (8) exists a unique solution \( x(t, \gamma) \) on \( t > 0 \) for any given initial data \( \{x(\theta) : -\tau \leq \theta \leq 0\} = \gamma \in C_{\mathcal{F}_0}^2([-\tau,0];\mathbb{R}^n) \), moreover, both \( f(x,y,t) \) and \( h(x,y,t) \) are locally bounded in \( (x, y) \) and uniformly bounded in \( t \). If there are a function \( V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+) \), \( \beta \in L^1(\mathbb{R}_+, \mathbb{R}_+) \) and \( \omega_1, \omega_2 \in C(\mathbb{R}^n; \mathbb{R}_+) \) such that

\[
\mathcal{L}V(x,y,t) \leq \beta(t) - \omega_1(x) + \omega_2(y), \quad (x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+,
\]

\[
\omega_1(x) > \omega_2(x), \quad \forall x \neq 0,
\]

\[
\lim_{\|x\| \to \infty} \inf_{0 \leq t \leq \infty} V(x,t) = \infty. \tag{10}
\]
Then
\[ \lim_{t \to \infty} x(t, \gamma) = 0 \quad \text{a.s.,} \] (11)
for every \( \gamma \in C^b_{\text{loc}}([-\tau, 0]; \mathbb{R}^n) \).

**Lemma 2.4.** Let scalar \( s > 0 \), \( x, y \in \mathbb{R}^n \), and \( Q \in \mathbb{R}^{n \times n} \), then
\[ 2x^TQy \leq sx^TQ^Tx + s^{-1}y^Ty. \]

**Definition 2.5.** The noise-perturbed response system (6) and the drive system (2) can be function projective synchronized, if there exists a bounded and differentiable scalar function \( \alpha_i(t) \), such that the trivial solution of the error system is asymptotically in mean square, i.e.,
\[ \lim_{t \to +\infty} E\|e(t)\|^2 = \lim_{t \to +\infty} E\|y(t) - \alpha(t)x(t)\|^2 = 0 \] (12)

3. Main results. In this section, based on the adaptive control techniques, some theoretical results are developed to realize projective synchronization between systems (2) and (6).

**Theorem 3.1.** Suppose that the assumptions (H1) and (H2) hold, the two delayed neural networks (2) and (6) can achieve function projective synchronization under the following adaptive controller
\[ u_i(t) = -\sum_{j=1}^{n} \dot{\alpha}_{ij}f_j(\alpha_j(t)x_j(t)) + \alpha_i(t) \sum_{j=1}^{n} \dot{\alpha}_{ij}f_j(x_j(t)) + \alpha_i(t) \sum_{j=1}^{n} \dot{\beta}_{ij}f_j(x_j(t - \tau(t))) - \sum_{j=1}^{n} \dot{\beta}_{ij}f_j(\alpha_j(t - \tau(t))x_j(t - \tau(t))) + \dot{\alpha}_i(t)x_i(t) - \dot{\beta}_i(t)e_i(t) - \lambda_i \text{sign}(e_i(t)), \] (13)
with the adaptive updated law
\[ \dot{\lambda}_i(t) = \varepsilon_i e_i^2(t), \quad i = 1, 2, \ldots, n, \] (14)
where \( \varepsilon_i > 0 \), \( \lambda_i \) are constants, and \( \lambda_i \) satisfies
\[ \lambda_i > \max_i \left\{ \xi_i \sum_{j=1}^{n} l_j |\dot{a}_{ij} - \dot{a}_{ij}|T_j + \sum_{j=1}^{n} |\dot{b}_{ij} - \dot{b}_{ij}|F_j + \xi_i \sum_{j=1}^{n} (l_j |\dot{a}_{ij} - \dot{a}_{ij}|T_j + |\dot{a}_{ij} - \dot{a}_{ij}|T_j + \sum_{j=1}^{n} |\dot{b}_{ij} - \dot{b}_{ij}|F_j, \right. \]
\[ \left. \xi_i |\dot{a}_i - \dot{a}_i|T_i + \sum_{j=1}^{n} l_j |\dot{a}_{ij} - \dot{a}_{ij}|T_j + \sum_{j=1}^{n} |\dot{b}_{ij} - \dot{b}_{ij}|F_j, \right. \]
\[ \xi_i |\dot{a}_i - \dot{a}_i|T_i + \xi_i \sum_{j=1}^{n} l_j |\dot{a}_{ij} - \dot{a}_{ij}|T_j + \xi_i \sum_{j=1}^{n} |\dot{b}_{ij} - \dot{b}_{ij}|F_j \right\}. \]

**Proof.** We use the following Lyapunov functional to derive function projective synchronization criterion:
\[ V(t) = \frac{1}{2} \sum_{i=1}^{n} e_i^2(t) + \frac{1}{2(1-\tau)} \int_{t-\tau(t)}^{t} e^T(s)P e(s)ds + \sum_{i=1}^{n} \frac{1}{2\varepsilon_i} (\dot{\alpha}_i(t) - \dot{\alpha}_i)^2, \] (15)
where $P$ is a positive definite matrix, $\delta_i$ is a constant to be determined. According to the feature of memristive networks, the error system may appear the following four different cases at time $t$.

**Case 1.** If $|x_i(t)| \leq T_i$, $|y_i(t)| \leq T_i$ at time $t$, then the master system (2) and the slave system (6) reduce to the following systems, respectively

\[
dx_i(t) = \left[ -\dot{c}_i x_i(t) + \sum_{j=1}^{n} \hat{a}_{ij} f_j(x_j(t)) + \sum_{i=j}^{n} \hat{b}_{ij} f_j(x_j(t - \tau(t))) \right] dt,
\]

(16)

and

\[
dy_i(t) = \left[ -\dot{c}_i y_i(t) + \sum_{j=1}^{n} \hat{a}_{ij} f_j(y_j(t)) + \sum_{j=1}^{n} \hat{b}_{ij} f_j(y_j(t - \tau(t))) + u_i(t) \right] dt + h_i(t, e_i(t), e_i(t - \tau(t))) d\omega_i(t).
\]

(17)

Correspondingly, the error system can be written as

\[
d\varepsilon_i(t) = \left[ -\dot{c}_i \varepsilon_i(t) + \sum_{j=1}^{n} \hat{a}_{ij} g_j(\varepsilon_j(t)) + \sum_{j=1}^{n} \hat{b}_{ij} g_j(\varepsilon_j(t - \tau(t))) \\
+ \sum_{j=1}^{n} (\hat{a}_{ij} - \hat{a}_{ij}) f_j(\alpha_j(t)x_j(t)) + \alpha_i(t) \sum_{j=1}^{n} (\hat{a}_{ij} - \hat{a}_{ij}) f_j(x_j(t)) \\
+ \alpha_i(t) \sum_{j=1}^{n} (\hat{b}_{ij} - \hat{b}_{ij}) f_j(x_j(t - \tau(t))) - \vartheta_i(t) e_i(t) \\
+ \sum_{j=1}^{n} (\hat{b}_{ij} - \hat{b}_{ij}) f_j(\alpha_j(t - \tau(t))x_j(t - \tau(t))) - \lambda_i \text{sign}(e_i(t)) \right] dt \\
+ h_i(t, e_i(t), e_i(t - \tau(t))) d\omega_i(t),
\]

(18)

where $g_j(\varepsilon_j(t)) = f_j(y_j(t)) - f_j(\alpha_j(t)x_j(t))$, $g_j(\varepsilon_j(t - \tau(t))) = g_j(y_j(t - \tau(t)) - g_j(\alpha_j(t - \tau(t))x_j(t - \tau(t)))$.

According to Itô’s differential formula [5], the stochastic derivative of $V(t)$ can be written as

\[
dV(t) = \mathcal{L}V(t)dt + V_e(t)H(t, e(t), e(t - \tau(t))) d\omega(t).
\]

(19)
Operating the weak infinitesimal $\mathcal{L}V$ along the trajectory of (18) gives

$$\mathcal{L}V(t) = \sum_{i=1}^{n} e_i(t) \left( -\dot{c}_i e_i(t) + \sum_{j=1}^{n} \dot{a}_{ij} g_j(e_j(t)) + \sum_{j=1}^{n} \dot{b}_{ij} g_j(e_j(t)) \right) + \sum_{j=1}^{n} (\dot{a}_{ij} - \dot{a}_{ij}) f_j(x_j(t)) + \alpha_i(t) \sum_{j=1}^{n} (\dot{a}_{ij} - \dot{a}_{ij}) f_j(x_j(t)) + \alpha_i(t) \sum_{j=1}^{n} (\dot{b}_{ij} - \dot{b}_{ij}) f_j(x_j(t) - \tau(t)) + \frac{1}{2(1-\tau)} e^T(t) Pe(t)$$

$$- \frac{1}{2} e^T(t - \tau(t)) Pe(t - \tau(t)) + \sum_{i=1}^{n} (\dot{a}_i(t) - \delta_i) e_i(t)$$

$$+ \frac{1}{2} \text{trace}\left[H^T(t, e(t), e(t - \tau(t))) H(t, e(t), e(t - \tau(t)))\right]$$

$$\leq \sum_{i=1}^{n} \left( -\dot{c}_i e_i^2(t) + \sum_{j=1}^{n} e_i(t) \dot{a}_{ij} g_j(e_j(t)) + \sum_{j=1}^{n} e_i(t) \dot{b}_{ij} g_j(e_j(t)) \right)$$

$$- \delta_i e_i^2(t) \right) + \sum_{i=1}^{n} |e_i(t)| \left( \xi_t \sum_{j=1}^{n} \dot{l}_j |\dot{a}_{ij} - \dot{a}_{ij}| T_j + \sum_{j=1}^{n} |\dot{b}_{ij} - \dot{b}_{ij}| F_j - \lambda_i \right)$$

$$+ \xi_i \sum_{j=1}^{n} \dot{l}_j |\dot{a}_{ij} - \dot{a}_{ij}| T_j + \xi_i \sum_{j=1}^{n} |\dot{b}_{ij} - \dot{b}_{ij}| F_j) + \frac{1}{2(1-\tau)} e^T(t) Pe(t)$$

$$- \frac{1}{2} e^T(t - \tau(t)) Pe(t - \tau(t))$$

$$+ \frac{1}{2} \text{trace}\left[H^T(t, e(t), e(t - \tau(t))) H(t, e(t), e(t - \tau(t)))\right].$$

According to the assumption $(H_1)$ we have the following inequality

$$\sum_{i=1}^{n} \sum_{j=1}^{n} e_i(t) \dot{a}_{ij} g_j(e_j(t)) = e^T(t) \dot{A} g(e(t))$$

$$\leq \frac{1}{2} e^T(t) \dot{A}^T \dot{A} e(t) + \frac{1}{2} g^T(t) g(e(t))$$

$$\leq \frac{1}{2} e^T(t) \dot{A}^T \dot{A} e(t) + \frac{1}{2} e^T(t) \Lambda^T \Lambda e(t),$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} e_i(t) \dot{b}_{ij} g_j(e_j(t)) = e^T(t) \dot{B} g(e(t - \tau(t)))$$

$$\leq \frac{1}{2} e^T(t) \dot{B}^T \dot{B} e(t) + \frac{1}{2} e^T(t - \tau(t)) \Lambda^T \Lambda e(t - \tau(t)),$$

where $\Lambda = \text{diag}(l_1, l_2, \cdots, l_n)$. From $(H_2)$ it follows that

$$\frac{1}{2} \text{trace}\left[H^T(t, e(t), e(t - \tau(t))) H(t, e(t), e(t - \tau(t)))\right]$$

$$\leq \frac{1}{2} e^T(t) G_1 e(t) + \frac{1}{2} e^T(t - \tau(t)) G_2 e(t - \tau(t)).$$
Together with (20)-(22), we can conclude that
\[
\mathcal{L}V(t) \leq -e^T(t) \left( \dot{C} + \nabla - \frac{1}{2} \dot{A}^T \dot{A} - \frac{1}{2} \Lambda^T \Lambda - \frac{1}{2} \dot{B}^T \dot{B} - \frac{1}{2(1-\tau)} P - \frac{1}{2} G_1 \right) e(t)
\]
\[
+ \frac{1}{2} e^T(t - \tau(t)) \left( \Lambda^T \Lambda + G_2 - P \right) e(t - \tau(t)) + \sum_{i=1}^{n} \left( \xi_i \sum_{j=1}^{n} l_j |\dot{a}_{ij} - \dot{a}_{ij}|T_j \right)
\]
\[
+ \sum_{j=1}^{n} |\dot{b}_{ij} - \dot{b}_{ij}|F_j + \xi_i \sum_{j=1}^{n} l_j |\dot{a}_{ij} - \dot{a}_{ij}|T_j + \xi_i \sum_{j=1}^{n} |\dot{b}_{ij} - \dot{b}_{ij}|F_j - \lambda_i |e_i(t)|.
\]
(23)

Letting
\[
P = \Lambda^T \Lambda + G_2,
\]
\[
\nabla = \max \left\{ \frac{1}{2} \lambda_{\text{max}}(\dot{A}^T \dot{A}) + \frac{1}{2} \lambda_{\text{max}}(\Lambda^T \Lambda) + \frac{1}{2} \lambda_{\text{max}}(\dot{B}^T \dot{B}) + \frac{1}{2(1-\tau)} \lambda_{\text{max}}(P)
\]
\[
+ \frac{1}{2} \lambda_{\text{max}}(G_1) + \lambda_{\text{max}}(-\dot{C}) \right\} I + I,
\]
\[
\left\{ \frac{1}{2} \lambda_{\text{max}}(\dot{A}^T \dot{A}) + \frac{1}{2} \lambda_{\text{max}}(\Lambda^T \Lambda) + \frac{1}{2} \lambda_{\text{max}}(\dot{B}^T \dot{B}) + \frac{1}{2(1-\tau)} \lambda_{\text{max}}(P)
\]
\[
+ \frac{1}{2} \lambda_{\text{max}}(G_1) + \lambda_{\text{max}}(-\dot{C}) \right\} I + I \}
\]
\[
\nabla = \text{diag}(\delta_1, \delta_2, \ldots, \delta_n).
\]

Then, it follows from (23) that
\[
\mathcal{L}V(t) \leq -e^T(t) e(t).
\]

**Case 2.** If \(|x_i(t)| > T_i, |y_i(t)| > T_i\) at time \(t\), then the master system (2) and the slave system (6) reduce to the following systems, respectively
\[
dx_i(t) = \left[ -\dot{c}_i x_i(t) + \sum_{j=1}^{n} \dot{a}_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} \dot{b}_{ij} f_j(x_j(t - \tau(t))) \right] dt,
\]
(24)
and
\[
dy_i(t) = \left[ -\dot{c}_i y_i(t) + \sum_{j=1}^{n} \dot{a}_{ij} g_j(y_j(t)) + \sum_{j=1}^{n} \dot{b}_{ij} g_j(y_j(t - \tau(t))) + u_i(t) \right] dt
\]
\[
+ h_i(t, e_i(t), e_i(t - \tau(t))) d\omega_i(t).
\]
(25)

Correspondingly, the error system can be written as
\[
d\varepsilon_i(t) = \left[ -\dot{c}_i e_i(t) + \sum_{j=1}^{n} \dot{a}_{ij} g_j(e_j(t)) + \sum_{j=1}^{n} \dot{b}_{ij} g_j(e_j(t - \tau(t))) - \vartheta_i(t) e_i(t)
\]
\[
- \lambda_i \text{sign}(e_i(t)) \right] dt + h_i(t, e_i(t), e_i(t - \tau(t))) d\omega_i(t).
\]
(26)

Arguing as Case 1, we can obtain
\[
\mathcal{L}V(t) \leq -e^T(t) \left( \dot{C} + \nabla - \frac{1}{2} \dot{A}^T \dot{A} - \frac{1}{2} \Lambda^T \Lambda - \frac{1}{2} \dot{B}^T \dot{B} - \frac{1}{2(1-\tau)} P - \frac{1}{2} G_1 \right) e(t)
\]
\[
+ e^T(t - \tau(t)) \left( \frac{1}{2} \Lambda^T \Lambda + \frac{1}{2} G_2 - \frac{1}{2} P \right) e(t - \tau(t)) - \lambda_i \text{sign}(e_i(t)).
\]
(27)
In consideration of the definition $\nabla$, $P$ and $\lambda_i$, we have the following estimation

$$\mathcal{L}V(t) \leq - e^T(t)e(t).$$

**Case 3.** If $|x_i(t)| > T_i$, $|y_i(t)| \leq T_i$ at time $t$, then the master system (2) and the slave system (6) reduce to (24) and (17). Correspondingly, the error system can be written as

$$de_i(t) = \left[ - \dot{c}_i e_i(t) + \sum_{j=1}^n (\dot{a}_{ij} g_j(e_j(t)) + \sum_{j=1}^n (\dot{b}_{ij} g_j(e_j(t - \tau(t))) + (\dot{d}_i - \dot{d}_i) y_i(t)

+ \sum_{j=1}^n (\dot{a}_{ij} - \dot{a}_{ij}) f_j(y_j(t)) + \sum_{j=1}^n (\dot{b}_{ij} - \dot{b}_{ij}) f_j(y_j(t - \tau(t))) - \vartheta_i(t) e_i(t) \right] dt + h_i(t, e_i(t), e_i(t - \tau(t))) dw_i(t).$$

Similarly, operation $\mathcal{L}V$ along the trajectory of (28), we have

$$\mathcal{L}V(t) \leq \sum_{i=1}^n e_i(t) \left\{ - \dot{c}_i(t) e_i(t) + \sum_{j=1}^n (\dot{a}_{ij} g_j(e_j(t)) + \sum_{j=1}^n (\dot{b}_{ij} g_j(e_j(t - \tau(t)))

+ |\dot{d}_i - \dot{d}_i||y_i(t)| + \sum_{j=1}^n |\dot{a}_{ij} - \dot{a}_{ij}||f_j(y_j(t))| - \lambda_i \text{sign}(e_i(t))

+ \sum_{j=1}^n (\dot{b}_{ij} - \dot{b}_{ij}) |f_j(y_j(t - \tau(t)))| - \vartheta_i(t) e_i(t) \right\} + \frac{1}{2(1 - \tau)} e^T(t)e(t)

- \frac{1}{2} e^T(t - \tau(t))Pe(t - \tau(t)) + \sum_{i=1}^n (\vartheta_i(t) - \delta_i) e_i^2(t)

+ \text{trace} \left[ H^T(t, e(t), e(t - \tau(t)))H(t, e(t), e(t - \tau(t))) \right].$$

Noting that $|y_i(t)| \leq T_i$, one has

$$\mathcal{L}V(t) \leq \sum_{i=1}^n e_i(t) \left\{ - \dot{c}_i(t) e_i(t) + \sum_{j=1}^n (\dot{a}_{ij} g_j(e_j(t)) + \sum_{j=1}^n (\dot{b}_{ij} g_j(e_j(t - \tau(t)))

+ |\dot{d}_i - \dot{d}_i||T_i + \sum_{j=1}^n l_j|\dot{a}_{ij} - \dot{a}_{ij}|T_j + \sum_{j=1}^n (\dot{b}_{ij} - \dot{b}_{ij}) |F_j - \vartheta_i(t)e_i(t)

- \lambda_i \text{sign}(e_i(t)) \right\} + \frac{1}{2} e^T(t)G_1 e(t) + \frac{1}{2} e^T(t - \tau(t))G_2 e(t - \tau(t))

+ \frac{1}{2(1 - \tau)} e^T(t)e(t) - \frac{1}{2} e^T(t - \tau(t))Pe(t - \tau(t)) + \sum_{i=1}^n (\vartheta_i(t)

- \delta_i) e_i^2(t)

\leq - e^T(t) \left( \dot{C} + \nabla - \frac{1}{2} \dot{A}^T \dot{A} - \frac{1}{2} \dot{A}^T A - \frac{1}{2} \dot{B}^T \dot{B} - \frac{1}{2(1 - \tau)} P - \frac{1}{2} G_1 \right) e(t)

+ e^T(t - \tau(t)) \left( \frac{1}{2} \dot{A}^T A + \frac{1}{2} \ddot{G} \right) e(t - \tau(t)) + \sum_{i=1}^n |e_i(t)| \left[ |\dot{d}_i - \dot{d}_i||T_i

+ \sum_{j=1}^n l_j|\dot{a}_{ij} - \dot{a}_{ij}|T_j + \sum_{j=1}^n (\dot{b}_{ij} - \dot{b}_{ij}) |F_j - \lambda_i \right].$$
Corollary 1. For the given scalars $\alpha_i, i = 1, 2, \cdots, n$, the memristive systems (2) and (6) can achieve modified projective synchronization under the following adaptive controller

$$u_i(t) = -\sum_{j=1}^{n} a_{ij}f_j(\alpha_jx_j(t)) + \alpha_i \sum_{j=1}^{n} a_{ij}f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij}f_j(\alpha_jx_j(t - \tau(t))) + \alpha_i \sum_{j=1}^{n} b_{ij}g_j(x_j(t - \tau(t)))$$

$$+ \alpha_i \sum_{j=1}^{n} b_{ij}g_j(x_j(t - \tau(t))) - v_i(t)e_i(t) - \lambda_i \text{sign}(e_i(t)),$$

with adaptive updated law

$$\dot{\dot{e}}_i(t) = \varepsilon_i e_i^2(t), \ i = 1, 2, \cdots, n,$$

Case 4. If $|x_i(t)| \leq T_i, |y_i(t)| > T_i$ at time $t$, then the master system (2) and the slave system (6) reduce to (16) and (25). Correspondingly, the error system can be written as

$$de_i(t) = \left[ -\dot{\dot{e}}_i(t) + \sum_{j=1}^{n} \dot{a}_{ij}g_j(e_j(t)) + \sum_{j=1}^{n} \dot{b}_{ij}g_j(e_j(t - \tau(t))) + \alpha_i(t)(\dot{d}_i - \dot{d}_i)x_i(t) ight. \\
+ \alpha_i(t)\sum_{j=1}^{n} (\dot{a}_{ij} - \dot{a}_{ij})f_j(x_j(t)) + \alpha_i(t)\sum_{j=1}^{n} (\dot{b}_{ij} - \dot{b}_{ij})f_j(x_j(t - \tau(t))) \\
- \dot{\vartheta}_i(t)e_i(t) - \lambda_i \text{sign}(e_i(t)) \right] dt + h_i(t, e_i(t), e_i(t - \tau(t)))d\omega_i(t).

(29)

By using $|x_i(t)| \leq T_i$, we can also have

$$\mathcal{L}V(t) \leq -e^T(t) \left( \dot{C} + \nabla - \frac{1}{2} \dot{A}^T \tilde{A} - \frac{1}{2} \dot{A}^T A - \frac{1}{2} \dot{B}^T \tilde{B} - \frac{1}{2}(1 - \tau) P - \frac{1}{2} G_1 \right) e(t)$$

$$+ e^T(t - \tau(t)) \left( \frac{1}{2} \dot{A}^T A + \frac{1}{2} G_2 - \frac{1}{2} P \right) e(t - \tau(t)) + \sum_{i=1}^{n} \left[ \xi_i \dot{d}_i - \dot{d}_i |T_i \\
+ \sum_{j=1}^{n} l_j \xi_i \dot{a}_{ij} - \dot{a}_{ij} |T_j + \sum_{j=1}^{n} \xi_j \dot{b}_{ij} - \dot{b}_{ij} |T_j - \lambda_i \right] |e_i(t)| \\
\leq -e^T(t) e(t).$$

The LaSalle invariance principle of stochastic differential equation proposed in [28] yields $e(t) \rightarrow 0$, then we can derive that

$$\lim_{t \to +\infty} \mathbb{E}[\|e(t)\|^2] = \lim_{t \to +\infty} \mathbb{E}[\|y(t) - \alpha(t)x(t)\|^2] = 0.$$

It follows from Definition 2.5 that the noise-perturbed response system (6) is projective synchronized with the drive system (2) under the designed controller (13). This completes the proof. \qed

When $\alpha_i(t)$ is a constant, We have the following corollary.

Corollary 1. For the given scalars $\alpha_i, i = 1, 2, \cdots, n$, the memristive systems (2) and (6) can achieve modified projective synchronization under the following adaptive controller

$$u_i(t) = -\sum_{j=1}^{n} a_{ij}f_j(\alpha_jx_j(t)) + \alpha_i \sum_{j=1}^{n} a_{ij}f_j(x_j(t)) - \sum_{j=1}^{n} b_{ij}f_j(\alpha_jx_j(t - \tau(t)))$$

$$+ \alpha_i \sum_{j=1}^{n} b_{ij}g_j(x_j(t - \tau(t))) - v_i(t)e_i(t) - \lambda_i \text{sign}(e_i(t)),$$

with adaptive updated law

$$\dot{\dot{e}}_i(t) = \varepsilon_i e_i^2(t), \ i = 1, 2, \cdots, n,$$

$$\dot{\vartheta}_i(t) = \varepsilon_i \vartheta_i^2(t), \ i = 1, 2, \cdots, n,$$

$$\dot{\lambda}_i = \varepsilon_i \lambda_i^2(t), \ i = 1, 2, \cdots, n.$$
where \( \varepsilon_i > 0 \), \( \hat{\lambda}_i \) are constants, and \( \hat{\lambda}_i \) satisfies

\[
\hat{\lambda}_i > \max_i \left\{ |\alpha_j| \sum_{j=1}^{n} l_j |\dot{a}_{ij} - \dot{\hat{a}}_{ij}|T_j + \sum_{j=1}^{n} |b_{ij} - \dot{\hat{b}}_{ij}|F_j + |\alpha_i| \sum_{j=1}^{n} (l_j |\dot{a}_{ij} - \dot{\hat{a}}_{ij}|T_j + |\alpha_i| \sum_{j=1}^{n} |\dot{b}_{ij} - \dot{\hat{b}}_{ij}|F_j),
\]

\[
|\dot{a}_i - \dot{\hat{a}}_i|T_i + \sum_{j=1}^{n} l_j |\dot{a}_{ij} - \dot{\hat{a}}_{ij}|T_j + \sum_{j=1}^{n} |\dot{b}_{ij} - \dot{\hat{b}}_{ij}|F_j,
\]

\[
|\alpha_i| |\dot{a}_i - \dot{\hat{a}}_i|T_i + |\alpha_i| \sum_{j=1}^{n} l_j |\dot{a}_{ij} - \dot{\hat{a}}_{ij}|T_j + |\alpha_i| \sum_{j=1}^{n} |\dot{b}_{ij} - \dot{\hat{b}}_{ij}|F_j \}\right).
\]

Proof. This proof can be derived directly by taking \( \alpha_i(t) = \alpha_i \) in Theorem 3.1. Thus omitted here. \( \square \)

Remark 2. If the constant number \( \alpha_i \) is chosen as -1 or 1, then anti-synchronization and synchronization would appear, respectively. In short, projective synchronization is a more general form that includes many kinds of synchronization as its special cases.

Remark 3. In this paper, augmented Lyapunov functional is used to analyze the projective synchronization of the memristor-based neural networks. In the Lyapunov functional, both state and activation function are considered, which make sure that the conditions given in this paper are less conservative than the existing results.

4. A numerical example. In this section, one example is offered to illustrate the effectiveness of the results obtained in this paper.

Consider the following two-dimensional memristor-based recurrent neural networks

\[
\begin{aligned}
\dot{x}_1(t) &= -c_1(x_1(t))x_1(t) + a_{11}(x_1(t))f(x_1(t)) + a_{12}(x_1(t))f(x_2(t)) + b_{11}(x_1(t))g(x_1(t-1)) + b_{12}(x_1(t))g(x_2(t-1)) + 1, \\
\dot{x}_2(t) &= -c_2(x_2(t))x_2(t) + a_{21}(x_2(t))f(x_1(t)) + a_{22}(x_2(t))f(x_2(t)) + b_{21}(x_2(t))g(x_1(t-1)) + b_{22}(x_2(t))g(x_2(t-1)) + 1,
\end{aligned}
\]

where

\[
\begin{align*}
c_1(x_1(t)) &= \begin{cases} 1, & |x_1(t)| \leq 0.5, \\ 1.1, & |x_1(t)| > 0.5, \end{cases} & c_2(x_2(t)) &= \begin{cases} 1.1, & |x_2(t)| \leq 0.5, \\ 1, & |x_2(t)| > 0.5, \end{cases} \\
a_{11}(x_1(t)) &= \begin{cases} -2, & |x_1(t)| \leq 0.5, \\ -1.7, & |x_1(t)| > 0.5, \end{cases} & a_{12}(x_1(t)) &= \begin{cases} 1, & |x_1(t)| \leq 0.5, \\ 1.5, & |x_1(t)| > 0.5, \end{cases} \\
a_{21}(x_2(t)) &= \begin{cases} -4.7, & |x_2(t)| \leq 0.5, \\ -5, & |x_2(t)| > 0.5, \end{cases} & a_{22}(x_2(t)) &= \begin{cases} -2.4, & |x_2(t)| \leq 0.5, \\ 2, & |x_2(t)| > 0.5 \end{cases} \\
b_{11}(x_1(t)) &= \begin{cases} -5, & |x_1(t)| \leq 0.5, \\ -4, & |x_1(t)| > 0.5, \end{cases} & b_{12}(x_1(t)) &= \begin{cases} 0.1, & |x_1(t)| \leq 0.5, \\ 0.05, & |x_1(t)| > 0.5, \end{cases} \\
b_{21}(x_2(t)) &= \begin{cases} -4.3, & |x_2(t)| \leq 0.5, \\ -4, & |x_2(t)| > 0.5, \end{cases} & b_{22}(x_2(t)) &= \begin{cases} -3, & |x_2(t)| \leq 0.5, \\ -2.6, & |x_2(t)| > 0.5. \end{cases}
\end{align*}
\]
For the drive system (32), constructing the following response system:

\[
\begin{cases}
  dy_1(t) = \left[ -c_1(y_1(t))y_1(t) + a_{11}(y_1(t))f(y_1(t)) + a_{12}(y_1(t))f(y_2(t)) \\
  \quad + b_{11}(y_1(t))g(y_1(t - 1)) + b_{12}(y_1(t))g(y_2(t - 1)) + 1 \\
  \quad + u_1(t) \right] dt + h_1(t, e_1(t), e_1(t - 1))d\omega_1(t),
  \end{cases}
\]

\[
\begin{cases}
  dy_2(t) = \left[ -c_2(y_2(t))y_2(t) + a_{21}(y_2(t))f(y_1(t)) + a_{22}(y_2(t))f(y_2(t)) \\
  \quad + b_{21}(y_2(t))g(y_1(t - 1)) + b_{22}(y_2(t))g(y_2(t - 1)) + 1 \\
  \quad + u_2(t) \right] dt + h_2(t, e_2(t), e_2(t - 1))d\omega_2(t).
  \end{cases}
\]

(33)

The activation functions are taken as \( f(s) = \frac{1}{2} \tanh s \), it can be verified that \( l_1 = l_2 = 0.5 \). Set

\[
G_1 = \begin{pmatrix} 0.01 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.1 \end{pmatrix}.
\]

Now, we study the projective synchronization of memristive neural networks (32).

In order to demonstrate the adaptive controller (13) can realize complete periodic synchronization of memristor-based neural networks, the scaling matrix is chosen as \( \alpha(t) = 0.85 \), the initial states are chosen as \( x(t) = (\sin(6t), -0.6\cos(5t))^T, y(t) = (0.3\tanh(3t), 0.7\tanh(3t))^T \), \( \vartheta_1(t) = \vartheta_2(t) = 0.2 \) for \( t \in [-1, 0] \). According to Theorem 3.1, it can be calculate that \( \lambda_1 \geq 1.45, \lambda_2 \geq 1.05 \), so we choose \( \lambda_1 = 1.5, \lambda_2 = 1.1 \). Then, the simulation results are given in Fig.3-Fig.8. Among them, Fig.3-4 show that the evolution of variables \( x_1(t), y_1(t), x_2(t), y_2(t) \), Fig.5-6 describes the dynamic behavior of drive system and response system, respectively. The time responses of projective synchronization errors \( e_i(t) = y_i(t) - \alpha_i(t)x_i(t), i = 1, 2 \) are given in Fig.7, which turn to zero quickly as time goes. From Fig.8, one can see that the control parameters \( \vartheta_i(t), i = 1, 2 \), turn out to be constants eventually.

![Figure 3](image.png)

**Figure 3.** The evolution of variables \( x_1(t), y_1(t) \) of coupled neural networks (32) and (33).

5. **Conclusion.** This paper proposed an appropriate adaptive controller to achieve projective synchronization of memristive neural networks subject to stochastic perturbation. By using LaSalle-type invariance principle for stochastic differential delay equations, some new sufficient conditions have been presented, which are dependent
Figure 4. The evolution of variables $x_2(t), y_2(t)$ of coupled neural networks (32) and (33).

Figure 5. Chaotic trajectory of model (32).

Figure 6. The dynamic behavior of response system (33).
on the switching jumps. An example and numerical simulations are given to demonstrate the effectiveness of the theory results, the analysis method in this paper can be possibly used to treat other synchronization issues of the memristive networks in the future.

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