Nonlinear dynamics of rotating blades with variable cross-section

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Abstract. In this chapter, nonlinear dynamic behaviors of the variable cross-section rotating blades are studied. The blade is considered to be a rotating cantilever plate model with variable cross-section. Considering the influence of centrifugal force, variable rotating speed and cross-section warp. In the light of the Hamilton's principle, the von Karman deformation theory and the third-order shear deformation theory, we can derive the ordinary differential equation by using Galerkin method from the nonlinear partial differential equations. The multiple scales is applied to get the averaged equations with the 1:3 internal resonance of the rotating blades. Using numerical simulation to study the effects of aerodynamic forces and disturbance amplitude of the rotating blades on nonlinear dynamic behaviors. It shows that the rotating blades performs complex nonlinear dynamic behaviors, such as single periodic, chaotic motions, multiple periodic and quasi-periodic.

Keywords: Rotating blade; Variable cross-section; Variable rotating speed; Aerodynamic force;

1. Introduction

Aero-engine compressor blade is a high-speed rotating mechanism. Affected by the transverse shear and aerodynamic forces, it will produce large amplitude nonlinear vibrations, which will reduce the reliability of the blade and directly affect the safety of the engine and aircraft. Therefore, the study of nonlinear dynamic characteristics of rotating blades has important theoretical and practical value for engineering design.

Many scholars have done research in this field. Because the structure of beam model is relatively simple, many scholars mainly simplified the blade into the beam model at the beginning. Yang and Tsao [1] studied dynamic behaviors of a pre-bending rotating beam with variable rotational speed. On the basis of geometric nonlinear coupling and shear deformation, Machado [2] investigated the thin-walled beams of static stability. Kandil and El-Gohary [3,4] analyzed the influence of time delay on reducing oscillation performance of rotating beams at different speeds by using a proportional-derivative (PD) controller. For purpose of improving the accuracy of the actual rotating blades, many scholars have simplified the rotating blade as the plate model and shell model. Zhang [5] analyzed orbits and chaotic dynamics and the multipulse Shilnikov of the plate. Yao et al. [6,7] investigated the nonlinear dynamics of the rotating pretwisted plate and cylindrical panel. Young et al. [8] established a cantilever plate model to reveal the impact of Coriolis speed on vibrations of rotating blades by using the Lagrange’s method. Zhang et al [9] investigated the influences of damping coefficients, pre-deformation amplitude, rotational speed, thermal gradient and gas pressure on the amplitude-
frequency characteristics of rotating blades of 2:1 internal resonance. However, all the above studies aim at the static or dynamic modeling and analysis of the blade model with uniform section. In fact, the actual blade cross-section is not uniform. Therefore, variable cross-section models have attracted the attention of scholars. Carmakar and Sinha [10] used the methods of finite element to study the vibration characteristics of rotating cantilever plate with varying thickness and variable width. Ma et al. [11] took use of ANSYS to study a rotating variable cross-section twisted blade. Yutaek et al. [12] utilized the methods of Rayleigh-Ritz and Kane to study the vibration characteristics of the rotating blades consist of FGMs.

Through the research of the above scholars, we find that there are few investigations on the nonlinear dynamics of cantilever plates with variable cross-section. In this article, the complex nonlinear dynamics of the cantilever plate with variable cross-section is studied. In terms of von Karman large deformation theory and Hamilton's principle, we can derive the nonlinear partial differential equations, then obtain the nonlinear ordinary differential equations by employing the Galerkin method. The method of multiple scales is yield to get the averaged equations with the 1:3 internal resonance. Numerical simulation are performed to discuss the influence of aerodynamic force and the disturbance amplitude on nonlinear dynamics of the rotating blade.

2. Formulation
The rotating blade is simplify as a variable cross-section cantilever plate fixed on the rigid rotating hub with radius $R_0$, as shown in Figure 1.

Figure 1. Variable section cantilever plate model

The geometric parameters of the rotating blade are given, which the length is $L$, the width on the root and tip ends are $b_r$ and $c$, the thickness is $h$, the pre-setting angle is $\theta_p$, the pre-twist angle at the tip end is $\theta_t$, The pre-twist angle at any point along the spanwise direction of the blade is given by

$$\theta = \theta_p + \beta x,$$

where

$$\beta = \frac{\theta_t - \theta_p}{L}.$$

The width of the blade is derived by

$$b(x) = k_i x + c,$$

where $k_i$ is the coefficient of the width variation.
The rotating speed is assumed to be perturbed by the periodic gas flow, which can be expressed as

\[ \Omega = \Omega_0 + f \cos \Omega t , \tag{4} \]

where \( \Omega_0 \) is the steady rotating speed, \( f \) is the disturbance amplitude and \( \Omega t \) is disturbance frequency of the rotating speed.

In the light of the third-order shear deformation theory, the displacement fields is described as

\[
\begin{align*}
    u(x, y, z, t) &= u_0(x, y, t) + z \phi_x(x, y, t) - \frac{4}{3h^2} z^3 \left( \phi_x + \frac{\partial w_0}{\partial x} \right) - z \beta^2 x w, \tag{5a} \\
v(x, y, z, t) &= v_0(x, y, t) + z \phi_y(x, y, t) - \frac{4}{3h^2} z^3 \left( \phi_y + \frac{\partial w_0}{\partial y} \right), \tag{5b} \\
w(x, y, z, t) &= w_0(x, y, t), \tag{5c}
\end{align*}
\]

where \( u_0 \), \( v_0 \) and \( w_0 \) are the displacement of any point on the mid-surface of the cantilever plate along the \( x \)-axis, \( y \)-axis and \( z \)-axis, respectively, \( \phi_x \) and \( \phi_y \) are the rotation angle about the \( y \)-axis and \( x \)-axis, respectively.

In the light of the von Karman large deformation theory, the strain-displacement relationships can be expressed as

\[
\begin{align*}
\varepsilon_{xx} &= \frac{\partial u}{\partial x} + \frac{1}{2} \left( \frac{\partial w}{\partial x} \right)^2, & \varepsilon_{yy} &= \frac{\partial v}{\partial y} + \frac{1}{2} \left( \frac{\partial w}{\partial y} \right)^2, & \gamma_{xz} &= \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \right), & \varepsilon_{zz} &= \frac{\partial w}{\partial z}, \tag{6}\end{align*}
\]

The constitutive equation of material can be expressed by

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xz} \\
\tau_{yz}
\end{bmatrix}
= \begin{bmatrix}
Q_{11} & Q_{12} & 0 & 0 & 0 \\
Q_{21} & Q_{22} & 0 & 0 & 0 \\
0 & 0 & Q_{44} & 0 & 0 \\
0 & 0 & 0 & Q_{55} & 0
\end{bmatrix}
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xz} \\
\gamma_{yz}
\end{bmatrix},
\tag{7}
\]

where

\[
\begin{align*}
(A_i, B_j, D_{ij}, E_{ij}, F_{ij}, H_{ij}) &= \int_0^h Q_j (1, z, z^2, z^3, z^4, z^5) dz, (i, j = 1,2,6), \\
(A_i, D_{ij}, F_{ij}) &= \int_0^h Q_j (1, z^2, z^4) dz, (i, j = 4,5). \tag{8}
\end{align*}
\]

In this paper, we assume that the aerodynamic force is perpendicular to the blade surface, which can be described as follow

\[ P = q_1 + q_2 \cos \alpha \phi . \tag{9} \]

In the light of the Hamilton principle, the equation of motion of the blade is derived by

\[ \int_0^l (\partial K - \partial U + \partial W) dt = 0 . \tag{10} \]

For the convenience of calculation, we introduce the dimensionless parameters, the dimensionless equation is expressed as

\[
a_{10} \frac{\partial^2 \tilde{u}_0}{\partial x^2} + a_{11} \frac{\partial^2 \tilde{u}_0}{\partial y^2} + a_{12} \frac{\partial^2 \tilde{u}_0}{\partial x \partial y} + a_{13} \frac{\partial \tilde{w}_0}{\partial x} + a_{14} \frac{\partial^2 \tilde{w}_0}{\partial x^2} + a_{15} \frac{\partial \tilde{w}_0}{\partial y} + a_{16} \frac{\partial^2 \tilde{w}_0}{\partial x \partial y} + a_{17} \frac{\partial^3 \tilde{w}_0}{\partial x^3} + a_{18} \frac{\partial^3 \tilde{w}_0}{\partial x^2 \partial y} + a_{19} \frac{\partial^3 \tilde{w}_0}{\partial x \partial y^2} + a_{20} \frac{\partial^3 \tilde{w}_0}{\partial y^3} = 0 . \tag{11}
\]
\[ + a_{18} \frac{\partial^2 \phi_x}{\partial x^2} + a_{19} \frac{\partial^2 \phi_y}{\partial y^2} + a_{30} \frac{\partial \phi_z}{\partial x} \frac{\partial W_0}{\partial y} + a_{21} \frac{\partial W_0}{\partial x} + a_{25} \frac{\partial^2 W_0}{\partial x^2} + a_{23} \frac{\partial W_0}{\partial y} \right) + a_{24} \frac{\partial \phi_z}{\partial y} + a_{25} \frac{\partial W_0}{\partial y} \\
+ a_{26} \frac{\partial^2 \phi_z}{\partial x \partial y} + a_{27} \frac{\partial^2 \phi_x}{\partial y^2} + a_{28} \frac{\partial^2 \phi_y}{\partial x \partial y} + a_{29} \frac{\partial^2 \phi_z}{\partial x \partial y} + a_{30} \frac{\partial^2 \phi_z}{\partial y^2} + a_{31} \frac{\partial \phi_z}{\partial y} + a_{32} \frac{\partial^2 \phi_z}{\partial y^2} \\
+ a_{33} \frac{\partial^2 \phi_z}{\partial x \partial y} + a_{34} \frac{\partial^2 \phi_z}{\partial x \partial y} \left( \frac{\beta}{4} (x) - \frac{y^2}{4} \right) \frac{\partial^2 W_0}{\partial y^2} = a_{35} \phi_x + a_{36} \phi_y + a_{37} \frac{\partial \phi_z}{\partial y} \right), \]

\[ b_{10} \frac{\partial^2 \phi_x}{\partial x^2} + b_{11} \frac{\partial^2 \phi_y}{\partial y^2} + b_{12} \frac{\partial^2 \phi_z}{\partial y^2} + b_{13} \frac{\partial^2 \phi_z}{\partial y^2} + b_{14} \frac{\partial^2 \phi_z}{\partial x^2} + b_{15} \frac{\partial^2 \phi_z}{\partial x^2} + b_{16} \frac{\partial^2 \phi_z}{\partial x^2} + b_{17} \frac{\partial^2 \phi_z}{\partial x^2} \]

\[ + b_{18} \frac{\partial^2 \phi_z}{\partial x \partial y} + b_{19} \frac{\partial^2 \phi_z}{\partial x \partial y} + b_{20} \frac{\partial^2 \phi_z}{\partial x \partial y} + b_{21} \frac{\partial^2 \phi_z}{\partial x \partial y} + b_{22} \frac{\partial^2 \phi_z}{\partial x \partial y} + b_{23} \frac{\partial^2 \phi_z}{\partial x \partial y} + b_{24} \frac{\partial^2 \phi_z}{\partial x \partial y} + b_{25} \frac{\partial^2 \phi_z}{\partial x \partial y} \]

\[ + b_{26} \frac{\partial^2 \phi_z}{\partial x \partial y} + b_{27} \frac{\partial^2 \phi_z}{\partial x \partial y} + b_{28} \frac{\partial^2 \phi_z}{\partial x \partial y} + b_{29} \frac{\partial^2 \phi_z}{\partial x \partial y} + b_{30} \frac{\partial^2 \phi_z}{\partial x \partial y} + b_{31} \frac{\partial^2 \phi_z}{\partial x \partial y} + b_{32} \frac{\partial^2 \phi_z}{\partial x \partial y} \]

\[ + b_{33} \frac{\partial^2 \phi_z}{\partial x \partial y} + b_{34} \frac{\partial^2 \phi_z}{\partial x \partial y} + b_{35} \frac{\partial^2 \phi_z}{\partial x \partial y} \]
By choosing appropriate modal factors, the cross lines in dimensionless space and all the inertial terms of equations \( (11a), (11b), (11d), (11e) \) are removed. Substituting equation (12) into equations (11a), (11b), (11d) and (11e), then \( u_o, v_o, \phi_x, \phi_y \) are represented by \( w_0 \) respectively. Substituting equation (12c) into the third modal function
equation (11c), then, using Galerkin method, the expression of the nonlinear ordinary differential equation can be expressed

\[
\ddot{w}_1 + \left(\omega_1^2 + g_0 f^2 \cos^2 \Omega t + g_1 f \cos \Omega t \right) \ddot{w}_1 + \left(\frac{3}{2} g_0 \sigma_1 + \frac{1}{2} g_0 f^2 \right) x_1 + \frac{3}{8} g_6 x_4 \left( x_1^2 - x_2^2 \right) - \frac{3}{4} g_6 x_1 x_2 x_3 \\
+ \frac{3}{4} g_2 x_1 x_3 \left( x_3^2 + x_4^2 \right) + \frac{9}{8} g_6 x_2 \left( x_1^2 + x_4^2 \right),
\]

\[
\ddot{w}_2 + \left(\omega_2^2 + h_0 f^2 \cos \Omega t + h_1 f \cos \Omega t \right) \ddot{w}_2 + \left(\frac{3}{2} h_0 \sigma_1 + \frac{1}{2} h_0 f^2 \right) x_2 + \frac{3}{8} h_6 x_4 \left( x_1^2 - x_2^2 \right) - \frac{3}{4} h_6 x_1 x_2 x_3 \\
+ \frac{3}{4} h_2 x_1 x_3 \left( x_3^2 + x_4^2 \right) + \frac{9}{8} h_6 x_2 \left( x_1^2 + x_4^2 \right),
\]

where \( q_1 \) and \( q_2 \) are aerodynamic force parameters, \( f \) is the parameter of the rotating disturbance speed.

3. Averaged Equation and Numerical Results

3.1. Averaged Equation.

The first six natural frequencies of the rotating blade are calculated by ANSYS, as shown in Figure 2. It can be observed that when the aspect ratio \( L/b_1 \) is selected as 4.5, the rotating blade presents 1:3 internal resonance between the first two modes.

![Figure 2. Effect of aspect ratio on the first six natural frequencies of the variable section rotating blade.](image)

Perturbation analysis is applied by using the multiple scales method. Considering the 1:3 internal resonance, given the following resonance relationship

\[
\omega_1^2 \approx \frac{\Omega^2}{9} + \varepsilon \sigma_1, \quad \omega_2^2 \approx \Omega^2 + \varepsilon \sigma_2, \quad \omega = \Omega,
\]

The average equations in the Cartesian coordinate are obtained as

\[
\dot{x}_1 = - \frac{\mu_1}{2} x_1 + \left( \frac{3}{2} \sigma_1 + \frac{1}{2} g_0 f^2 \right) x_2 + \frac{3}{8} g_6 x_4 \left( x_1^2 - x_2^2 \right) - \frac{3}{4} g_6 x_1 x_2 x_3 \\
+ \frac{3}{4} g_2 x_1 x_3 \left( x_3^2 + x_4^2 \right) + \frac{9}{8} g_6 x_2 \left( x_1^2 + x_4^2 \right),
\]

\[
\dot{x}_2 = - \frac{\mu_2}{2} x_2 + \left( \frac{3}{2} \sigma_1 + \frac{1}{2} g_0 f^2 \right) x_1 - \frac{3}{8} g_6 x_3 \left( x_1^2 - x_2^2 \right) - \frac{3}{4} g_6 x_1 x_2 x_4
\]
Based on equation (16), the fourth-order Runge-Kutta algorithm is applied to carry out numerical simulations. In this section, we investigate the effects of disturbance amplitude of rotating speed and aerodynamic force on complicated nonlinear dynamic behaviours. Firstly, the effect of the disturbance amplitude on the nonlinear dynamic behaviors of rotating blades is studied by changing the magnitude of the aerodynamic force. The initial condition is $x_0 = [3.1385 \ 0.055 \ 0.35 \ 0.183]$. Other parameters are selected as $\mu_1 = 0.005 \ , \ \mu_2 = 0.005 \ , \ \sigma_1 = -4.275 \ , \ \sigma_2 = -2.012 \ , \ g = 20.01 \ , \ g_6 = 38.9808 \ , \ g_7 = -2 \ , \ g_8 = -2.167 \ , \ h = 0.097 \ , \ h_1 = 0.097 \ , \ h_2 = 0.097 \ , \ h_3 = 14.09 \ , \ h_4 = 1$. The bifurcation diagram can be found that with the increase of the disturbance amplitude from 0 to 2, the first mode of the rotating changes from the periodic-1 motion to the chaotic motion, the second mode exhibits the periodic-1 motion, followed by a periodic-2 motion, and then enters into the chaotic motion.

\[
\begin{align*}
\dot{x}_3 &= \frac{\mu_2}{2} x_3 + \left(\frac{1}{2} \sigma_2 + \frac{3}{8} h_0 f^2\right) x_4 + \frac{1}{8} h_0 x_4 x_3^2 + \frac{1}{8} h_0 x_3 x_4^2 + \frac{3}{8} h_0 x_4 x_3^2 + \frac{1}{4} h_0 x_3 x_4^2 + \frac{1}{2} h_4 f , \quad (16c) \\
\dot{x}_4 &= -\frac{\mu_2}{2} x_4 - \left(\frac{1}{2} \sigma_2 + \frac{3}{8} h_0 f^2\right) x_3 + \frac{1}{8} h_0 x_3 x_4^2 + \frac{1}{8} h_0 x_3 x_4^2 + \frac{3}{8} h_0 x_3 x_4^2 + \frac{1}{2} h_4 q_2. \quad (16d)
\end{align*}
\]

3.2. Numerical Results.

Figure 3 Bifurcation diagrams of the rotating blade for parameter $f$,

(a) Bifurcation diagram of the first mode; (b) Bifurcation diagram of the second mode;
Figure 4 The periodic-1 motion when $f = 0.2$, (a) The waveform on plane $(t, x_1)$; (b) The phase portrait on plane $(x_1, x_2)$; (c) The spectrum diagram; (d) The waveform on plane $(t, x_3)$; (e) The phase portrait on plane $(x_3, x_4)$; (f) The three-dimensional phase portrait in the space $(x_1, x_2, x_3)$.

Figure 5 The chaotic motion when $f = 1.6$, (a) The waveform on plane $(t, x_1)$; (b) The phase portrait on plane $(x_1, x_2)$; (c) The spectrum diagram; (d) The waveform on plane $(t, x_3)$; (e) The phase portrait on plane $(x_3, x_4)$; (f) The three-dimensional phase portrait in the space $(x_1, x_2, x_3)$.

Then, we study the influences of the aerodynamic force on nonlinear dynamics. From the bifurcation diagram we can see that when the aerodynamic force $q_z$ varies within a certain range $(300, 620)$, the system exhibits the chaotic motion at first, and then exhibits the periodic motion, as shown in Figure 6.
Figure 6 Bifurcation diagrams of the variable section rotating blade are obtained for parameter $q_2$, (a) Bifurcation diagram of the first mode; (b) Bifurcation diagram of the second mode;

4. Conclusions
In this article, we mainly analyze nonlinear vibrations of the variable cross-section rotating cantilever plate. Considering the centrifugal force, varying rotating speed, cross-section warping and other factors, we can establish the nonlinear the ordinary differential equations. The average equation of 1:3 internal resonance is derived by multi-scale method. Numerical simulations are applied to discuss complex nonlinear dynamic behaviors by discussing the effect of the aerodynamic force and the disturbance amplitude of rotating speed. The results demonstrate that the rotating blades exhibits complicated dynamics features under the effect of the aerodynamic force and the disturbance amplitude of rotating. The system has the periodic, multi-periodic and chaotic motions.

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