Renormalization group defects for boundary flows

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Abstract

Recently, Gaiotto (2012 arXiv:1201.0767) considered conformal defects which produce an expansion of infrared local fields in terms of the ultraviolet (UV) ones for a given renormalization group flow. In this paper, we propose that for a boundary RG flow in two dimensions there exist boundary condition changing fields (RG defect fields) linking the UV and the infrared (IR) conformal boundary conditions which carry similar information on the expansion of boundary fields at the fixed points. We propose an expression for a pairing between IR and UV operators in terms of a four-point function with two insertions of the RG defect fields. For the boundary flows in minimal models triggered by \( \psi_1 \) perturbation, we make an explicit proposal for the RG defect fields. We check our conjecture by a number of calculations done for the example of \((p, 2) \rightarrow (p - 1, 1) \oplus (p + 1, 1)\) flows.

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1. Introduction

In perturbative renormalization, the operators of perturbed theory are expressed in terms of the operators in the unperturbed theory with counter terms provided. At the level of the deformed OPE algebra, such expressions are free from infrared (IR) divergences and are essentially perturbative under a very broad set of assumptions [2]. The non-perturbative data needed to calculate correlation functions can be put into vacuum expectation values (see e.g. [3, 2, 4]). One can imagine that such expressions hold all the way into the new IR fixed point and that one can express the operators \( \phi^\text{IR}_i \) in the IR fixed point in terms of linear combinations of operators in the ultraviolet (UV) fixed point:

\[
\phi^\text{IR}_i = \sum_j b_{ij} \phi^\text{UV}_j.
\]  

When the two fixed points are near in the parameter space, one can calculate the expansion coefficients \( b_{ij} \) perturbatively. This is the case for renormalization group (RG) flows between neighbouring minimal models \( M_n \) and \( M_{n-1} \) first considered in [5].
The idea to associate a conformal defect (or domain wall) with an RG flow was first put forward in [11] (see also section 5 of [12]) in the context of two-dimensional (2D) quantum field theories. We refer the reader to [6–10] for a definition and discussion of general conformal defects. Recently, a more concrete proposal relating defects to (1.1) was made in [1]. It was proposed that given a flow between two 2D conformal field theories (CFTs) there exists a conformal defect between the fixed point CFTs which allows one to calculate the coefficients $b_{ij}$. The prescription of [1] is as follows. Assume that the fields $\phi^{UV}$ are canonically normalized. Let $|\text{RG}\rangle$ be the conformal boundary state in the tensor product $\text{CFT}^{UV} \otimes \text{CFT}^{IR}$ that represents the RG defect via the folding trick (see e.g. [7]); then

$$b_{ij} = \langle \bar{\phi}_{j}^{IR} \otimes \phi_{i}^{UV} | \text{RG} \rangle,$$

(1.2)

where $\bar{\phi}_{j}^{IR}$ are the fields reflected by the folding (this essentially exchanges the holomorphic and antiholomorphic components). Alternatively, representing the RG defect by an operator $\hat{\text{RG}} : \mathcal{H}^{UV} \rightarrow \mathcal{H}^{IR}$, we have

$$b_{ij} = \langle \phi_{j}^{IR} | \hat{\text{RG}} | \phi_{i}^{UV} \rangle.$$

(1.3)

The last formula corresponds to putting the RG defect on the unit circle and computing a correlator with $\phi_{i}^{UV}$ inserted at the origin and $\phi_{j}^{IR}$ at infinity.

In [1], an explicit algebraic construction was put forward for the RG defect corresponding to the flows between two neighbouring minimal models. It was shown that the leading order mixing coefficients calculated in [5] are reproduced precisely by the proposed RG defect via (1.2).

The idea that such RG defects exist in general seems very attractive. The complicated data of the mapping of fields (1.1) done by the RG flow can be encoded algebraically in the defect boundary field which can be accessed using the techniques of conformal field theory. If one understands better the general properties of such RG defects, then this could lead to selection rules for possible RG flows between known 2D CFTs.

Heuristically, the existence of such RG defects can be argued for using the following construction. Consider a perturbed CFT. Put the perturbed theory on a plane with a non-trivial metric profile in the radial direction: $d\lambda^2 = g^2(r) (dr)^2 + r^2 (d\theta)^2$. Let the scale function $g(r)$ interpolate between a scale $\Lambda^{UV}$ near the origin and the scale $\Lambda^{IR}$ at infinity. An operator $\phi_{i}^{IR}$ defined at scale $\Lambda^{IR}$ can be transported to an operator at scale $\Lambda^{UV}$ near the origin. One can then imagine a limiting process for which the region in which the scale function $g(r)$ changes shrinks to a small neighbourhood of $r = 1$ and the constant scales $\Lambda^{UV}$ and $\Lambda^{IR}$ are sent to 0 and $\infty$, respectively. We obtain a domain wall between the UV and IR fixed points.

Another possible general construction of the RG defect proceeds by perturbing the UV theory on a half-plane and letting it flow with the RG [11]. As discussed in [1], renormalizing such perturbations may require switching on new fields localized on the edge of the half-plane the interpretation of which is unclear.

Both of these heuristic constructions may be extended to the case of the boundary flows in which the bulk CFT is kept fixed and the flow interpolates between two conformal boundary conditions in this bulk theory. Naturally, such defects would be point-like and thus must be represented by some boundary condition changing fields: $\psi_{i}^{UV,IR}$ (with the conjugate counterpart $\bar{\psi}_{i}^{IR,UV}$). By analogy with the bulk picture of figure 1, we may expect the RG pairing for boundary fields to be given by a four-point function

$$\langle \psi_{j}^{IR}(\infty) \psi_{i}^{IR,UV}(-\chi) \psi_{i}^{UV}(0) \psi_{i}^{UV,IR}(\chi) \rangle,$$

(1.4)

see figure 2.
Assuming that the fields $\psi_{\text{UV}}$ and $\psi_{\text{IR}}$ are quasi-primary of dimension $\Delta$ and the fields $\phi_{\text{IR}}$ and $\phi_{\text{UV}}$ are quasi-primaries of dimensions $\Delta_{\text{IR}}$ and $\Delta_{\text{UV}}$, respectively, one has

$$\langle \psi_{\text{IR}}(\infty) \phi_{\text{IR}}(x) \phi_{\text{UV}}(0) \psi_{\text{UV}}(1) \rangle = \left[ x^{\Delta_{\text{IR}} - \Delta_{\text{UV}} - 2\Delta} \right] \left[ \psi_{\text{UV}}(\infty) \phi_{\text{IR}}(x) \phi_{\text{UV}}(0) \right].$$

(1.5)

Stripping off the prefactor, we propose an analogue of the RG pairing (1.2) and (1.3) for the boundary flows to be given by

$$\langle \psi_{\text{IR}}, \psi_{\text{UV}} \rangle_{\text{RG}} \equiv \left[ \psi_{\text{UV}}(\infty) \phi_{\text{IR}}(x) \phi_{\text{UV}}(0) \right].$$

(1.6)

so that for the canonically normalized basis $\psi_{\text{IR}}$

$$\phi_{\text{IR}} = \sum_{j} b_{j} \phi_{\text{IR}}^{j}, \quad b_{j} = \langle \psi_{\text{IR}}, \psi_{\text{IR}}^{j} \rangle_{\text{RG}}.$$

(1.7)

In the rest of the paper, we focus on flows between conformal boundary conditions in minimal A-type unitary models which are triggered by the boundary field $\psi_{(1,3)}$. It was shown in [15] that if one starts with a single Cardy boundary state with labels $(a_{1}, a_{2})$ and switches on $\psi_{(1,3)}$, the end point of the flow is the following superposition of Cardy boundary conditions:

$$\min(a_{1}, a_{2}) \rightarrow \bigoplus_{i=1}^{\min(a_{1}, a_{2})} (a_{1} + a_{2} + 1 - 2i, 1).$$

(1.8)

Since the IR boundary condition has $N = \min(a_{1}, a_{2})$ components, the RG defect field $\psi_{\text{IR}, \text{UV}}$ breaks up into $N$ components as well. We propose that up to normalization the corresponding components are given by the fields $\psi_{(a_{1}, a_{2})}^{(i, 1, a_{1}, a_{2})}$ with $i = a_{1} + a_{2} + 1 - 2l, 1 \leq l \leq N$. Here and elsewhere in the paper, we put the boundary condition labels as superscripts in square brackets.

The rest of the paper is organized as follows. After some preliminaries in section 2 we give a detailed formula for the RG pairing in section 3. In section 4, we analyse the flow $(2, 2) \to (3, 1) \oplus (1, 1)$ for which we calculate the leading order expansions (1.7) for fields of dimensions near 0 and near 1, that is, for weights $h = O(1/m)$ and $h = 1 + O(1/m)$, respectively. The result is shown to match the expansions found by other methods in [13]. In section 5, we calculate the expansions of fields of dimension near 0 for more general flows:
In section 6, we check the first subleading corrections for the expansions found in section 5 against conformal perturbation theory. We conclude with some brief comments in section 7. The appendices contain the expressions for OPE coefficients and a derivation of asymptotic expansions for conformal blocks.

2. Some preliminaries

The unitary Virasoro minimal models $M_m$ have central charges

$$c_m = 1 - \frac{6}{m(m+1)}, \quad (2.1)$$

where $m \geq 3$ is an integer. We assume the diagonal modular invariant. The conformal weights of primaries are given by the values

$$h_{(r,s)} = \frac{(m+1)r - ms)^2 - 1}{4m(m+1)}, \quad (2.2)$$

where $(r, s)$ belong to the Kac table: $(r, s) \in \mathcal{K} = \{(r', s') : 1 \leq r' \leq m - 1, 1 \leq s' \leq m\}$ defined modulo the symmetry $(r, s) \rightarrow (m-r, m+1-s)$. For large values of $m$, we have

$$h_{(r,s)} = \frac{(r-s)^2}{4} + \frac{r^2 - s^2}{4m} + O(m^{-3}). \quad (2.3)$$

The fusion rules for chiral fields $\phi_{(r,s)}$ are

$$\phi_{(r,s)} \times \phi_{(r',s')} = \sum_{r'', s''} \mathcal{N}_{r,s,r',s'}^{(r'',s'')} \phi_{(r'',s'')}, \quad (2.4)$$

$$\mathcal{N}_{r,s,r',s'}^{(r'',s'')} = \mathcal{N}_{r,s,r',s'}^{(r'',s'')} \mathcal{N}_{r,s,r',s'}^{(r'',s'')} (m+1), \quad (2.5)$$

$$\mathcal{N}_{a,b}^{(r,s)} (m) \begin{cases} 1, & |a-b| + 1 \leq c \leq \min(a+b-1, 2m-a-b-1), a+b+c \text{ odd} \\ 0, & \text{otherwise}. \end{cases} \quad (2.6)$$

For large values of $m$, we obtain simpler $SU(2)$ fusion rules

$$\lim_{m \rightarrow \infty} \mathcal{N}_{a,b}^{(r,s)} (m) \begin{cases} 1, & |a-b| + 1 \leq c \leq a+b-1, a+b+c \text{ odd} \\ 0, & \text{otherwise}. \end{cases} \quad (2.7)$$

Irreducible conformal boundary conditions in $M_m$ are labelled by a pair from the Kac table: $(a_1, a_2)$. The spectrum of boundary fields $\psi_{[(a_1,a_2),(b_1,b_2)]}$ which join two such boundary conditions is determined from the following decomposition of the state space:

$$\mathcal{H}_{(a_1,a_2),(b_1,b_2)} = \bigoplus_{(c_1,c_2) \in \mathcal{K}} \mathcal{N}_{(a_1,a_2),(b_1,b_2)}^{(c_1,c_2)} \mathcal{H}_{(c_1,c_2)}. \quad (2.8)$$

The OPE of fields $\psi_{[(a_1,a_2),(b_1,b_2)]}$ has the following form:

$$\psi_{i}^{[ab]}(x) \psi_{j}^{[bc]}(y) \sim \sum_{l} C_{ij}^{[ab]} \psi_{l}^{[ac]}(y) (x-y)^{h_b-h_a-l}, \quad x > y. \quad (2.9)$$

Here, for brevity each index stands for a pair from the Kac table, e.g. $l = (l_1, l_2) \in \mathcal{K}$.

The OPE coefficients for (unnormalized) boundary fields can be expressed in terms of the fusion matrices $[14]$

$$C_{ij}^{[abc]} = F_{kl} \left[ \begin{array}{cc} a & c \\ l & j \end{array} \right]. \quad (2.10)$$

These OPE coefficients satisfy the identities

$$\tilde{C}_{ij}^{[abc]} = C_{ji}^{[cab]}, \quad \tilde{C}_{ij}^{[abc]} \tilde{C}_{kkl}^{[def]} = C_{ijk}^{[def]} C_{iil}^{[def]} \quad (2.11)$$
Normalizing the fields $\psi_{i}^{[ab]}$ so that

$$\psi_{i}^{[ab]}(x)\psi_{i}^{[ab]}(y) = \frac{1}{(x-y)^{2\xi_{i}}} I_{aa} + \cdots,$$

we obtain the normalized OPE coefficients

$$C_{ij}^{(aa)k} = F_{ik\alpha} \left[ \begin{array}{ccc} a & a & a \\ \alpha & \alpha & \alpha \end{array} \right] F_{ij\beta} \left[ \begin{array}{ccc} \beta & \beta & \beta \\ k & k & k \end{array} \right]^{1/2}.$$

(2.12)

More details on the OPE coefficients are given in appendix A.

### 3. RG pairing

In this section, we bring the general prescription (1.6), (1.7) to a more concrete form. We have in mind applications to Virasoro minimal models, but most of the formulae below can be easily generalized to include other theories.

To apply (1.6) and (1.7), we normalize the UV fields $\psi_{i}^{\text{UV}}$ so that

$$C_{ii}^{\text{UV UV UV}} = 1, \quad \langle \psi_{i}^{\text{UV}}(x) \psi_{i}^{\text{UV}}(y) \rangle = \frac{g_{\text{UV}}}{(x-y)^{2\xi_{i}}},$$

(3.1)

where $g_{\text{UV}} = \langle 1_{\text{UV}} \rangle$ is the $g$-factor (ground state degeneracy, [16]) of the UV boundary condition. (Here for simplicity we assume that the UV boundary condition is irreducible.) Let us further consider the expansions

$$\psi_{i}^{\text{IR, UV}}(x) = \sum_{a} \xi_{a} \hat{\psi}_{a}(x), \quad \psi_{i}^{\text{UV, IR}}(x) = \sum_{a} \xi_{a} \hat{\psi}_{a}^{\dagger}(x),$$

(3.2)

where the index $a$ labels the irreducible components of the IR boundary condition and the fields

$$\hat{\psi}_{a} = \hat{\psi}_{a}^{[a,\text{UV}]}, \quad \hat{\psi}_{a}^{\dagger} = \hat{\psi}_{a}^{[a,\text{UV}]}$$

are primaries in the corresponding boundary condition changing sectors which are normalized so that

$$\langle \hat{\psi}_{a}(x) \hat{\psi}_{a}^{\dagger}(y) \rangle = \langle \hat{\psi}_{a}^{\dagger}(x) \hat{\psi}_{a}(y) \rangle = \frac{1}{(x-y)^{2\xi_{a}}}.$$

(3.3)

Using these definitions, we can write the decomposition into conformal blocks for the RG-pairing (1.6)

$$\langle \psi_{j}^{[a,b]}(\infty), \psi_{i}^{\text{UV}} \rangle_{\text{RG}} = \langle \hat{\psi}_{a}^{\dagger}(\infty) \psi_{j}^{[a,b]}(1) \hat{\psi}_{b} \left( \frac{1}{2} \right) \psi_{i}^{\text{UV}}(0) \rangle$$

$$= \xi_{a} \xi_{b} \sum_{p} C_{[^{ab}]_{\text{UV}}^{\text{IR}}}^{[^{ab}]_{\text{UV}}^{\text{IR}}} \langle \psi_{i}^{\text{UV}}(0) \rangle_{[a,b]_{\text{UV}}^{p}} \left( \frac{1}{2} \right),$$

(3.4)

where the indices $\hat{a}$ and $\hat{b}$ label the Virasoro representations corresponding to the fields $\hat{\psi}_{a}$.

For the flows

$$(a_{1}, a_{2}) \rightarrow \bigoplus_{i=1}^{N} (a_{1} + a_{2} + 1 - 2i, 1), \quad N = \min(a_{1}, a_{2}),$$

(3.5)

we label the components by the index $a \in \{ a_{1} + a_{2} + 1 - 2i | i = 1, 2, \ldots, N \}$. We propose that

$$\hat{\psi}_{a} = \psi_{[a_{1}, a_{2}]}^{(a_{1}, a_{2})},$$

(3.6)
with the normalization (3.3). Thus, we have $\hat{a} = \hat{b} = (a_2, a_2)$ in (3.4) which can now be written as

$$\left\{ y_j^{[(a_1, b_1), (a_2, b_2)]}, y_j^{[\text{UV}]} \right\}_{\text{RG}} = \xi_{i\hat{b}} \sum_p C_{j^p}^{[(a_1, b_1), (a_2, b_2)]} (a_2, a_2) C_{j^p}^{[(b_1, a_2), (a_1, a_2)]} (a_2, a_2) F^p_{(a_1)j^p; (b_1)i} \left( \frac{1}{2} \right),$$

(3.7)

where $i = (i_1, i_2)$, $j = (j_1, j_2)$, $p = (p_1, p_2)$.

The pairing (3.7) is now expressed in terms of the OPE coefficients which can be calculated using fusion matrices (2.10) (see appendix A), the minimal model conformal blocks $F^p_{ij;kl}$, and the expansion coefficients $\xi_i$. We will see in the forthcoming sections how one can fix the coefficients $\xi_i$ for particular examples of these flows.

Our proposal (3.6) is essentially a guess. We expect the RG defect to be close to the identity operator so its dimension should go to zero as $m \to \infty$. Equation (3.6) is the simplest possibility. In addition, we offer the following argument that further limits the choices. It is claimed in [8] that given a particular boundary flow in minimal models one can apply to it the bulk topological defect corresponding to the $(r, 1)$ representation and obtain another boundary flow. It seems reasonable to us that the statement of [8] should also extend to the boundary defect operators. Namely, if one knows the RG boundary defect field corresponding to the original flow, then the defect field for the image flow can be obtained via the action of the same bulk topological defect. Consider then the boundary flow triggered by $\psi_1^{[3, 1]}: (1, a_2) \to (a_2, 1)$. In this case, there is only one boundary condition changing primary field between the UV and IR fixed points: $\psi_1^{[(a_2, 1), (a_2, a_2)]}$. Thus, the RG boundary defect field must be built using this field and possibly its descendants. Since the fusion with a topological defect $(a_1, 1)$ cannot change the representation content, we obtain that the RG boundary defect field for a flow from the $(a_1, a_2)$ boundary condition must be built upon the Virasoro representation $(a_2, a_2)$. There is only one such primary field in each boundary sector—the one given by formula (3.6). The simplest possibility is then that the defect fields are given by the appropriately normalized primary components.

4. The flow $(2, 2) \to (3, 1) \oplus (1, 1)$

In this section, we focus on the most simple example of boundary flows considered in [15]—the flow from the $(2, 2)$ boundary condition into the superposition of $(3, 1)$ and $(1, 1)$ boundary conditions. We will investigate in detail the mapping of fields of dimensions near 0 and near 1. For $m = \infty$, a mapping of these fields was worked out in [13]. We will reproduce their answers using our RG pairing (3.4). We will also obtain a prediction for the finite values of $m$.

Let us now list the fields involved and fix the rest of normalizations. For the UV boundary condition, the complete list of primaries is

$$I_{22} \equiv \psi_1^{[(2, 2), (2, 2)]}, \quad \phi \equiv \psi_1^{[(2, 2), (3, 2)]}, \quad \psi = \psi_1^{[(2, 2), (2, 2)]}, \quad \bar{\psi} = \psi_1^{[(2, 2), (2, 2)]}$$

(4.1)

where we use essentially the same notations as in [13]. These fields are normalized as in (3.1). Together with the primaries $\psi$ and $\bar{\psi}$, there is also a descendant $\partial \phi$ which has a dimension near 1. As explained in [13] to account for an apparent jump in the number of null vectors in the $m \to \infty$ limit, one introduces a rescaled field

$$d_1(x) = -\frac{m}{2} \partial \phi(x).$$

(4.2)

Although the state $L_{-1}|(3, 3)\rangle$ becomes null in the $m \to \infty$ limit, the rescaled field $d_1$ retains a finite norm throughout and does not decouple.
In the IR, we have boundary fields
\[ I_{11} \equiv \psi^{[(1,1)(1,1)]}_{(1,1)}, \quad I_{31} \equiv \phi^{[(1,3)(1,1)]}_{(1,1)}, \quad \phi_{31} \equiv \psi^{[(3,1)(3,1)]}_{(3,1)}, \quad \phi_{51} \equiv \psi^{[(3,1)(3,1)]}_{(3,1)}, \]
(4.3)
\[ \hat{\phi}_{31} \equiv \psi^{[(1,1)(3,1)]}_{(3,1)}, \quad \hat{\phi}_{51} \equiv \psi^{[(3,1)(1,1)]}_{(3,1)} \]
(4.4)
The fields \( I_{11}, I_{31}, \phi_{31}, \) and \( \phi_{51} \) are normalized similarly to (3.1), while the fields \( \hat{\phi}_{31} \) and \( \hat{\phi}_{51} \) are normalized so that
\[ \hat{\phi}_{31}(x) \hat{\phi}_{31}^+(y) \sim \frac{1}{(x-y)^{2\alpha_1}} I_{11} + \cdots. \]
(4.5)
We have the following fields of the types \( \psi_{\text{UV,IR}} \) and \( \psi_{\text{UV,IR}}^{\text{IR}} \):
\[ \hat{\psi}_1 \equiv \psi^{[(1,1)(2,2)]}_{(2,2)}, \quad \hat{\psi}_3 \equiv \psi^{[(3,1)(2,2)]}_{(2,2)}, \quad \hat{\psi}_{42} \equiv \psi^{[(3,1)(2,2)]}_{(4,2)} \]
(4.6)
\[ \hat{\psi}_1 \equiv \psi^{[(2,2)(1,1)]}_{(2,2)}, \quad \hat{\psi}_3 \equiv \psi^{[(2,2)(3,1)]}_{(2,2)}, \quad \hat{\psi}_{42} \equiv \psi^{[(2,2)(3,1)]}_{(4,2)}. \]
(4.7)
These fields are normalized as in (3.3). The fields \( \hat{\psi}_1 \) and \( \hat{\psi}_3 \) are the components of the RG defect field (3.2), while the field \( \hat{\psi}_{42} \) arises as an intermediate channel in the conformal block decomposition (3.4).

Since we are dealing with a perturbative RG flow, we expect that only operators of nearby scaling dimensions get mixed. Thus, we can analyse groups of operators with close conformal weights. We will only look at two groups: dimension near 0 and dimension near 1 operators.

Starting with the operators of dimension near 0, we note that the following RG pairings do not require knowledge of any nontrivial conformal blocks and can be expressed as
\[ (I_{11}, I_{22})_{\text{RG}} = (\xi_1(m))^2, \quad (I_{11}, \phi)_{\text{RG}} = (\xi_1(m))^2 \alpha_1, \]
\[ (I_{31}, I_{22})_{\text{RG}} = (\xi_3(m))^2, \quad (I_{31}, \phi)_{\text{RG}} = (\xi_3(m))^2 \alpha_3, \]
(4.8)
where
\[ \alpha_1 = C^{22,33}, \quad \alpha_3 = C^{31,22,22}. \]
(4.9)
In these expressions, we dropped the brackets around the pairs of numbers labelling Virasoro representations as well as some commas. Thus 31 stands for the (3, 1) representation. To avoid clutter, we will use this shorthand notation below in the OPE coefficients and conformal blocks whenever each number in a pair is a digit.

The above expressions hold for a finite \( m \). Here, \( \xi_1(m) \) and \( \xi_3(m) \) are the coefficients in expansion (3.2) which depend on \( m \). To find the exact expressions for these coefficients, we note that formulae (1.7) and (4.8) imply
\[ I_{11} = (\xi_1(m))^2 (I_{22} + \alpha_1 \phi), \quad I_{31} = (\xi_3(m))^2 (I_{22} + \alpha_3 \phi). \]
(4.10)
The OPE algebra for \( I_{11}, I_{31} \) is that of the projector operators:
\[ I_{11} \cdot I_{11} = I_{11}, \quad I_{31} \cdot I_{31} = I_{31}, \quad I_{31} \cdot I_{11} = 0. \]
(4.11)
On the other hand, the relevant part of the deformed OPE algebra for the fields \( I_{22}, \phi \) has the form
\[ \phi(x) \phi(0) \sim \frac{1}{x^{2\Delta_\phi}} D_{\phi,\phi}^\phi(\lambda) I_{22} + \frac{1}{x^{2\Delta_\phi}} D_{\phi,\phi}^\phi(\lambda) \phi(0) + \cdots, \]
(4.12)
where \( \lambda \) is the coupling constant in front of the \( \int \psi(x) \) dx perturbation, \( \Delta_\phi(\lambda) \) is the deformed scaling dimension of \( \phi \) and \( D_{\phi,\phi}^\phi(\lambda) \), \( D_{\phi,\phi}^\phi(\lambda) \) are the deformed OPE coefficients. Let \( \lambda^* \) be
the value of the coupling at the IR fixed point. One can check perturbatively that \( \Delta_\phi (\lambda^*) = 0 \).

Denoting \( D_{\phi,\phi}^1 = D_{\phi,\phi}^1 (\lambda^*) \), \( D_{\phi,\phi}^2 = D_{\phi,\phi}^2 (\lambda^*) \) we obtain from (4.8), (4.11) and (4.12)

\[
D_{\phi,\phi}^1 = \frac{1}{\alpha_1 \alpha_3}, \quad D_{\phi,\phi}^2 = - \left( \frac{1}{\alpha_1} + \frac{1}{\alpha_3} \right),
\]

\[
(\xi_1 (m))^2 = \frac{1}{1 - \frac{2}{\alpha_1}}, \quad (\xi_3 (m))^2 = \frac{1}{1 - \frac{2}{\alpha_1}}.
\]  

The exact expressions for the coefficients \( \alpha_1 \) and \( \alpha_3 \) in terms of Euler’s Gamma functions are given in (A.13) and (A.11), respectively. The expression for their ratio is particularly simple

\[
\frac{\alpha_3}{\alpha_1} = - \frac{\sin \left( \frac{\pi}{m} \right)}{\sin \left( \frac{3\pi}{m} \right)}.
\]  

We note that this coincides up to the sign with the ratio of boundary entropies of the two IR components

\[
\frac{\alpha_3}{\alpha_1} = \frac{g_{11}}{g_{31}},
\]

\[
g_{11} = \left( \frac{8}{m(m + 1)} \right)^{1/4} \left( \sin \left( \frac{\pi}{m} \right) \sin \left( \frac{\pi}{m + 1} \right) \right)^{1/2},
\]

\[
g_{31} = \left( \frac{8}{m(m + 1)} \right)^{1/4} \left( \sin \left( \frac{3\pi}{m} \right) \sin \left( \frac{\pi}{m + 1} \right) \right)^{1/2}.
\]

Thus, the coefficients \( \xi_1 \) and \( \xi_3 \) can be expressed in terms of the boundary entropies as

\[
(\xi_1 (m))^2 = \frac{g_{11}}{g_{11} + g_{31}}, \quad (\xi_3 (m))^2 = \frac{g_{31}}{g_{11} + g_{31}}.
\]

We will show in section 5 that similar expressions hold for more general flows into two IR components. It is tempting to conjecture that similar expressions will hold for all flows of the type considered in [15]. We postpone other checks of this hypothesis to future work.

Asymptotically, one has

\[
\alpha_1 = \sqrt{3} - \left( \frac{2\pi^2}{\sqrt{3}} \right) \frac{1}{m^2} + \left( \frac{2\pi^2}{\sqrt{3}} \right) \frac{1}{m^3} + \mathcal{O}(m^{-4}),
\]

\[
\alpha_3 = - \sqrt{3} - \left( \frac{2\sqrt{3}\pi^2}{9} \right) \frac{1}{m^2} - \left( \frac{2\sqrt{3}\pi^2}{9} \right) \frac{1}{m^3} + \mathcal{O}(m^{-4}).
\]

Substituting these expressions into (4.14) and (4.10), we obtain at \( m = \infty \)

\[
I_{11} = \frac{1}{4} (I_{22} + \sqrt{3} \phi), \quad I_{31} = \frac{3}{4} \left( I_{22} - \frac{1}{\sqrt{3}} \phi \right), \quad m = \infty
\]

which matches with formula (3.31) in [13].

We next calculate the RG pairings that involve the dimension near 1 fields: \( \varphi_{31}, \tilde{\varphi}_{31} \) and \( \tilde{\varphi}_{31} \). These pairings involve contributions from conformal blocks and we will only work out the answer at \( m = \infty \).

We have the following expressions for the RG pairings:

\[
\langle \varphi_{31}, \psi \rangle_{\text{RG}} = (\xi_3)^2 C_{31,22}^{31,12,12} C_{22,13}^{31,22,22} C_{22,31,22}^{22,22,12} \left( \frac{1}{2} \right),
\]

\[1\] The leading order perturbative calculation of the shift in anomalous dimension is subtle because one has to use a correction coming from a four-point function. The details will be published elsewhere [17].
As shown in appendix B.1, the leading contributions from the conformal blocks are
\[ F_{22,31;22,31}^{22,31;22,31} \sim -\frac{m}{2} \xi \phi, \]
\[ F_{22,31;22,31}^{22,31;22,31} \sim -\frac{1}{2} m^2, \]
\[ F_{22,31;22,31}^{22,31;22,31} \sim -\frac{1}{2} m^2, \]
\[ F_{22,31;22,31}^{22,31;22,31} \sim -1. \]

The leading order asymptotics for the OPE coefficients are given in formulae (4.8)–(A.17). Using those formulae along with (4.30) and (4.31), we obtain the leading order expansions
\[ \phi_3 = \sqrt{\frac{2}{3}} (\psi - \psi), \]
\[ \phi_{31} = \frac{s}{4} [\psi - \psi - \sqrt{2} \phi], \]
\[ \phi_{31}^+ = \frac{s}{4} [\psi + \psi + \sqrt{2} \phi], \]
where \( s = -\text{sign}(\xi \phi) \). These expansions match with formula (3.34) from [13]. More precisely, the expansions given in (3.34) of [13] are fixed up to two unknown constants denoted by the authors as \( \lambda_2 \) and \( \lambda_3 \). These constants must satisfy the relation \( \lambda_2 \lambda_3 = \frac{1}{16} \) derived from the OPE algebra. Our result (4.32), (4.33) corresponds to the values \( \lambda_2 = \lambda_3 = -\frac{s}{2} \).
5. Flows from \((p, 2)\) boundary conditions

In this section, we will consider the RG flows

\[
(p, 2) \rightarrow (p - 1, 1) \oplus (p + 1, 1)
\]

with \(p > 2\). We will focus on the dimension near zero sector. The results obtained will be further used in the analysis of \(1/m\) corrections. The normalized UV fields of dimension near zero are \(1_{p,2}\) and \(\phi = \psi_{(p,2)}^{(p_1,p_2)}\). In the IR we have \(1_{p-1,1}, 1_{p+1,1}\). The normalized RG defect fields are

\[
\hat{\psi}_{p-1} = \psi_{(2,2)}^{(p-1,1)(p,2)}, \quad \hat{\psi}_{p+1} = \psi_{(2,2)}^{(p+1,1)(p,2)}.
\]

Similarly to (4.8), we now have

\[
\begin{align*}
(1_{p-1,1}, 1_{p,2}) &\rightarrow (\xi_{p-1}(m))^2, \quad (1_{p-1,1}, \phi) \rightarrow (\xi_{p-1}(m))^2\alpha_{p-1}, \\
(1_{p+1,1}, 1_{p,2}) &\rightarrow (\xi_{p+1}(m))^2, \quad (1_{p+1,1}, \phi) \rightarrow (\xi_{p+1}(m))^2\alpha_{p+1},
\end{align*}
\]

where

\[
\alpha_{p-1} = C_{22,33}^{[(p-1,1)(p,2)]22}, \quad \alpha_{p+1} = C_{22,33}^{[(p+1,1)(p,2)]22}.
\]

The asymptotic expansions for these OPE coefficients are given in (A.21) and (A.22).

Following the same steps as in the previous section, we obtain

\[
D_{\phi,\phi} = -\frac{1}{\alpha_{p-1}\alpha_{p+1}}, \quad D_{\phi,\phi} = -\left(\frac{1}{\alpha_{p-1}} + \frac{1}{\alpha_{p+1}}\right),
\]

\[
(\xi_{p-1}(m))^2 = \frac{1}{1 - \frac{g_{p-1}}{\alpha_{p+1}}}, \quad (\xi_{p+1}(m))^2 = \frac{1}{1 - \frac{g_{p+1}}{\alpha_{p-1}}}.
\]

Using (A.25) and (A.26), we have

\[
(\xi_{p-1}(m))^2 = \frac{g_{p-1}}{g_{p-1} + g_{p+1}}, \quad (\xi_{p+1}(m))^2 = \frac{g_{p+1}}{g_{p-1} + g_{p+1}}.
\]

where \(g_{p+1}\) are the boundary entropies of the IR components. We have the following asymptotics:

\[
\alpha_{p-1} = \left(\frac{p+1}{p-1}\right) \left(1 - \frac{\pi^2 p}{3m^2} + O(m^{-3})\right), \quad \alpha_{p+1} = -\sqrt{\frac{p-1}{p+1}} \left(1 + \frac{p\pi^2}{3m^2} + O(m^{-3})\right).
\]

Hence, at \(m = \infty\) we obtain the following expansions:

\[
1_{p-1,1} = \frac{p-1}{2p} \left[1_{p,2} + \sqrt{\frac{p+1}{p-1}}\right], \quad 1_{p+1,1} = \frac{p+1}{2p} \left[1_{p,2} - \sqrt{\frac{p-1}{p+1}}\right]
\]

that matches with formulae (A.27) and (A.28) of [13].

6. Leading \(1/m\) corrections

We are interested in checking some of the \(1/m\) subleading terms against the RG calculations. We do this for equations (5.4). Recall that the coefficients \(D_{\phi,\phi}^1(\lambda)\) and \(D_{\phi,\phi}^\phi(\lambda)\) are defined to be the OPE coefficients in the theory deformed by the perturbation \(\lambda \int dx \psi(x).\) Renormalization subtracts the logarithmic divergences arising from short distances in the \(m \rightarrow \infty\) limit. For finite \(m\), they manifest themselves as poles in anomalous dimensions, or, equivalently, terms divergent in \(m\). The one-loop beta function is

\[
\beta(\lambda) = (1 - h_{13})\lambda + D\lambda^2, \quad D = C_{13,13}^{(2,p)(2,p)}(2,p)|_{13}.
\]
The fixed point is at
\[
\lambda^* = \frac{h_{13} - 1}{D} = \frac{8}{\sqrt{6m}} + O\left(\frac{1}{m^3}\right). \tag{6.2}
\]
The terms in the beta function from two loops and higher result in subleading \(1/m\) corrections so that the leading asymptotics of \(\lambda^*\) is fixed by (6.2).

Equations (5.4) predict the following asymptotics for the OPE coefficients at the IR fixed point:
\[
D^{1}_{\phi,\phi} = D^{1}_{\phi,\phi}(\lambda^*) = 1 + O\left(\frac{1}{m^3}\right),
\]
\[
D^{\phi}_{\phi,\phi} = D^{\phi}_{\phi,\phi}(\lambda^*) = \frac{2}{\sqrt{p^2 - 1}} - \frac{2\pi^2 p^2}{3\sqrt{p^2 - 1} m^2} + O\left(\frac{1}{m^7}\right). \tag{6.3}
\]
The perturbation theory expansions have the form
\[
D^{1}_{\phi,\phi}(\lambda) = 1 + D^{(1)}_{\phi,\phi} + D^{(2)}_{\phi,\phi} \lambda^2 + \cdots,
\]
\[
D^{\phi}_{\phi,\phi}(\lambda) = D^{(p,2)}_{\phi,\phi} + D^{(1)}_{\phi,\phi} \lambda + D^{(2)}_{\phi,\phi} \lambda^2 + \cdots. \tag{6.4}
\]
From (6.2), we see that terms of the orders \(m^{-1}\) and \(m^{-2}\) in expansions (6.3) can come only from the terms written out in (6.4). Comparing (A.29) with (6.3) and (6.4), we obtain the following predictions for perturbative corrections:
\[
D^{(1)}_{\phi,\phi}(\lambda^*) + D^{(2)}_{\phi,\phi}(\lambda^*)^2 = O\left(\frac{1}{m^7}\right), \tag{6.5}
\]
\[
D^{(1)}_{\phi,\phi}(\lambda^*) + D^{(2)}_{\phi,\phi}(\lambda^*)^2 = O\left(\frac{1}{m^7}\right). \tag{6.6}
\]
In the rest of this section we show that these identities hold. To begin with, it is easy to argue that
\[
D^{(2)}_{\phi,\phi} = O\left(\frac{1}{m}\right), \quad D^{(2)}_{\phi,\phi} = O\left(\frac{1}{m}\right). \tag{6.7}
\]
Such second-order corrections come from the integrals
\[
\int \int \text{d}x_1 \text{d}x_2 \langle \phi \psi \phi \psi \rangle \int \int \text{d}x_1 \text{d}x_2 \langle \phi \psi \phi \psi \rangle. \tag{6.8}
\]
On the \((p, 2)\) boundary conditions, the OPE of \(\psi\) and \(\phi\) contains only \(\psi\) and the corresponding OPE coefficient \(A.30\) goes as \(m^{-1}\). Thus, the correlation functions at hand contain a factor of \(m^{-1}\) as well. Short distance divergences are subtracted by renormalization and hence any possible \(m \rightarrow \infty\) divergences are subtracted as well. We conclude that (6.7) holds.

Thus, we need to show
\[
D^{(1)}_{\phi,\phi} = O\left(\frac{1}{m^2}\right), \quad D^{(1)}_{\phi,\phi} = O\left(\frac{1}{m^2}\right). \tag{6.9}
\]
We follow the method of calculating perturbative corrections to OPE coefficients presented in [2]. It is based on the action principle according to which
\[
\frac{\partial}{\partial \lambda} (\Phi_1 (t_1) \ldots \Phi_n (t_n))_{\lambda} = \int \text{d}x \langle \psi(x) \Phi_1 (t_1) \ldots \Phi_n (t_n) \rangle, \tag{6.10}
\]
for any correlator of renormalized operators \(\Phi_i\) in the deformed theory.

Consider the deformed correlator \(\langle \phi(x) \phi(0) \rangle_x\), where \(x > 0\) and \(\phi\) stands for a renormalized operator. The operator product expansion has the form (4.12). Using this OPE
inside the two-point function at hand, taking a derivative with respect to $\lambda$ and setting $\lambda = 0$ afterwards, we obtain
\[
\int dt \langle \phi(x)\phi(0)\psi(t) \rangle = D^{(1)}_{\phi,\phi} \frac{\lambda}{x^{2h_1}} - 2\partial \Delta_{\phi}(0) \frac{\ln x}{x^{2h_1}} + \cdots,
\]
where the ellipsis stands for terms less singular in the $x \to 0$ limit. Taking the integral of the three-point function, we obtain
\[
\int dt \langle \phi(x)\phi(0)\psi(t) \rangle = C_{13,33}^{(p,2)(p,2)(p,2)33} \frac{\sqrt{\pi} \Gamma \left(h_{13} - \frac{1}{2}\right) \Gamma^2 \left(h_{13} - \frac{1}{2}\right)}{\Gamma^{\frac{3}{2}} \Gamma \left(1 - h_{13}\right)}.
\]
(6.12)
We have the following 1/m expansion:
\[
\frac{1}{x^{2h_1}} \frac{\sqrt{\pi} \Gamma \left(h_{13} - \frac{1}{2}\right) \Gamma^2 \left(h_{13} - \frac{1}{2}\right)}{\Gamma^{\frac{3}{2}} \Gamma \left(1 - h_{13}\right)} = 2m + 4 \ln(x) + O \left(\frac{1}{m}\right).
\]
(6.13)
Renormalization amounts to subtracting the linear divergence in $m$. Using (A.30), we obtain
\[
D^{(1)}_{\phi,\phi} = O \left(\frac{1}{m^2}\right), \quad \partial \Delta_{\phi}(0) = -2C_{13,33}^{(p,2)(p,2)(p,2)33} + O \left(\frac{1}{m^2}\right).
\]
(6.14)
Note that we are expanding in 1/m the OPE coefficient $C_{13,33}^{(p,2)(p,2)(p,2)33}$ only after we have subtracted the linear term in (6.13).

The same (leading order) result can be obtained by introducing a short distance cutoff into the three-point function, taking the $m = \infty$ limit for the integrand first, taking the integral and then subtracting the logarithmic divergences. The short distance cutoff approach is in general computationally simpler.

To analyse the $D^{(1)}_{\phi,\phi}$ correction, we start with the three-point function of deformed theory
\[
\langle \phi(x')\phi(x)\phi(0) \rangle_{\lambda}, \quad x > 0, \quad x' > 0.
\]
Taking $x \to 0$ we can use the deformed OPE (4.12). Differentiating with respect to $\lambda$ and setting $\lambda = 0$, we obtain
\[
\int dt \langle \phi(x')\phi(x)\phi(0)\psi(t) \rangle = D^{(1)}_{\phi,\phi} \frac{\lambda}{x^{2h_1}} + \frac{C_{13,33}^{(p,2)(p,2)(p,2)33}}{x^{h_{13}}} \int dt \langle \phi(x')\phi(0)\psi(t) \rangle
\]
\[
- \ln x \partial \Delta_{\phi}(0) \frac{C_{33,33}^{(p,2)(p,2)(p,2)33}}{x^{h_{13}}(x')^{2h_1}} + \cdots,
\]
(6.15)
where the ellipsis stands for terms less singular in $x$.

Denote
\[
G(x) = \langle \phi(\infty)\phi(1)\phi(x)\psi(0) \rangle, \quad \eta = \frac{(x' - t)x}{(x' - x)t}.
\]
(6.16)
Using global conformal transformations, we obtain
\[
\langle \phi(x')\phi(x)\phi(0)\psi(t) \rangle = \begin{cases} f_1(x, x', t)G \left(\frac{1}{1 - \eta}\right), & t < 0 \text{ or } t > x' \\ f_2(x, x', t)G \left(\frac{1 - \eta}{\eta}\right), & 0 < t < x \\ f_3(x, x', t)G(\eta), & x < t < x', \end{cases}
\]
(6.17)
\footnote{Using finite $m$ as regularization we obtain power divergences in $m$ as $m \to \infty$. Using a short distance cutoff $\epsilon$, we obtain power divergences in $\epsilon$. Keeping $\epsilon$ finite and taking $m \to \infty$ produces logarithmic divergences. Subtracting such logarithms is equivalent to subtracting power divergences in $m$ up to finite terms of order $1/m$.}
where
\[ f_1(x, x', t) = (x' - t)^{h_{33}-h_{11}} (x' - x)^{h_{11}-h_{33}} (x - t)^{-h_{33}-h_{11}} (x')^{-2h_{33}}, \]
\[ f_2(x, x', t) = (x' - t)^{h_{33}-h_{11}} (x')^{h_{11}-h_{33}} (x - t)^{-h_{33}-h_{11}} x^{-2h_{33}}, \]
\[ f_3(x, x', t) = (t - x)^{h_{33}-h_{11}} (h_{33}-h_{11}) (x')^{-2h_{33}}. \]

The four-point function \( G(x) \) decomposes into conformal blocks as
\[ G(x) = X_{13}^3 + X_{21}^3 + X_{33}^3(x) + X_{33}^3 + X_{33}^3 + X_{33}^3 + X_{33}^3(x), \]
where (see formulas \( \text{(A.29)} \)–\( \text{(A.32)} \))
\[ X_{33} = C_{33}^{(p,2),(p,2),(p,2)} C_{33}^{(p,2),(p,2),(p,2)} = \frac{8}{\sqrt{6(p^2 - 1)m}} + O \left( \frac{1}{m^2} \right), \]
\[ X_{31} = C_{33}^{(p,2),(p,2),(p,2)} C_{33}^{(p,2),(p,2),(p,2)} = \frac{4}{\sqrt{6(p^2 - 1)m}} + O \left( \frac{1}{m^2} \right). \]

It is shown in appendix B.2 that the leading asymptotics for the conformal blocks are
\[ F_{33}^{33,33,33,13}(x) = \frac{1}{x} - \frac{1}{2} + O \left( \frac{1}{m} \right), \quad F_{31}^{33,33,33,13}(x) = 1 + O \left( \frac{1}{m} \right). \]

Using \( \text{(6.22)} \) and \( \text{(6.23)} \), we obtain the leading asymptotics
\[ G(x) = \left( \frac{1}{x} \right) \frac{8}{\sqrt{6(p^2 - 1)m}} + O \left( \frac{1}{m^2} \right). \]

From this and \( \text{(6.17)} \), we obtain
\[ \langle \phi(x') \phi(x) \phi(0) \psi(t) \rangle = \begin{cases} X_{33} \left( \frac{x}{x - t} \right) + O \left( \frac{1}{m^2} \right), & t < 0 \ or \ t > x' \\ X_{33} \left( \frac{x}{x - t} \right) + O \left( \frac{1}{m^2} \right), & 0 < t < x \\ X_{33} \left( \frac{x'}{x' - t} \left( t - x' \right) \right) + O \left( \frac{1}{m^2} \right), & x < t < x'. \end{cases} \]

We now proceed by using the short distance cutoff version of renormalization as that is more computationally concise. Substituting \( \text{(6.25)} \) into the integral on the left-hand side of \( \text{(6.15)} \) regulated by a short distance cutoff, we obtain
\[ \int dt \langle \phi(x') \phi(x) \phi(0) \psi(t) \rangle \]
\[ = X_{33} \left[ -6 \ln \epsilon + 2 \ln(x' + \epsilon) + 2 \ln(x - \epsilon) + 2 \ln(x' - x - \epsilon) \right] + O \left( \frac{1}{m^2} \right). \]

Renormalization amounts to (minimally) subtracting the \( \ln \epsilon \) divergences. We thus obtain
\[ \int dt \langle \phi(x') \phi(x) \phi(0) \psi(t) \rangle = 2X_{33} \left[ \ln(x') + \ln(x) + \ln(x' - x) \right] + O \left( \frac{1}{m^2} \right). \]

We also have
\[ \int dt \langle \phi(x') \phi(x) \phi(0) \psi(t) \rangle \]
\[ = C_{33}^{(p,2),(p,2),(p,2)} \left[ 2 \ln(x' - \epsilon) + 2 \ln(x' + \epsilon) - 4 \ln \epsilon \right] + O \left( \frac{1}{m^2} \right). \]
Renormalizing and using (6.14), we obtain
\[
\frac{D_{\phi, \phi}^{(1)}}{x^{b_{33}}(x')^{2b_{33}}} + C^{[(p,2)(p,2)]}_{33,33,33} \int \frac{dx}{x} \Delta_{\phi}(0) \frac{2^{(p,2)(p,2)33}}{x^{b_{33}}(x')^{2b_{33}}} = D_{\phi, \phi}^{(1)} + 4X_{33} \ln(x') + 2X_3 \ln(x) + O\left(\frac{1}{m^2}\right).
\]  
(6.29)

Matching this with the leading asymptotics of (6.27) in the \(x \to 0\) limit, we finally obtain the desired result
\[
D_{\phi, \phi}^{(1)} = O\left(\frac{1}{m^2}\right).
\]  
(6.30)

7. Conclusion

Here, we summarize the main results of the paper and spell out some open questions. Mimicking Gaiotto’s construction [1] we have proposed a general formula for RG pairings for boundary flows (1.6). It is proposed that all information about the mapping of fields is encoded in a special local boundary field—boundary RG defect. For boundary flows in minimal models triggered by the field \(\hat{\psi}_{13}\), we propose a candidate for such a field (3.2), (3.6) which we fix up to relative normalization coefficients \(\xi_a\). For flows from \((p,2)\) boundary conditions, we fix these coefficients up to signs (see (5.6)). Formula (5.6) is suggestive of a general relation in which the squares of coefficients \(\xi_a\) are given by the ratio of the \(g\)-factor of the \(a\)th IR component to the total IR \(g\)-factor. This conjecture needs further checks which we hope to perform in the future. For flows from \((2,2)\) boundary condition, we found using our RG pairing the mapping of fields of dimensions near 0 and 1. The results at the leading order coincide with those obtained in [13] by other methods. Our prescription gives the expansion to all orders in \(1/m\). The terms subleading in \(1/m\) should capture the RG corrections to OPE coefficients.

In section 6, we checked some particular coefficients in the expansions of dimension near 0 fields against conformal perturbation calculations at the first subleading order. We found a precise match. It would be desired to perform more checks of this kind. For that one would need to develop some systematics for the \(1/m\) expansion of the relevant conformal blocks. It is conceivable that there are corrections to our RG defect fields \(\hat{\psi}_a\) proportional to fields of higher dimension. In view of our remarks in the end of section 3, we believe that such fields can only be descendants of \(\hat{\psi}_a\). Such corrections must be suppressed by powers of \(1/m\), but may enter into the game at higher orders.

In this paper, we limited ourselves to boundary flows triggered by the \(\psi_{13}\) field which start from a single Cardy boundary condition. It would be interesting to find candidates for boundary RG defects for other known boundary flows. Work on this and other related questions is currently underway.

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Appendix A. Fusion matrices and structure constants

It was explained in [14] that that for the A-series Virasoro minimal models the boundary structure constants in a certain normalization are given by the fusion matrices

\[ \tilde{C}_{ij}^{abc} = F_{bl} \begin{bmatrix} a \\ i \\ f \end{bmatrix}. \] (A.1)

In this paper, we work with fields normalized as in (3.1) and (3.3). The corresponding OPE coefficients differ from (A.1) by normalization factors which are also expressible via the fusion matrices as in formula (2.12).

The fusion matrices can be calculated recursively as explained in [14] and are expressed via Euler’s Gamma functions. We are indebted to C Schmidt-Colinet for the use of his computer code implementing the recursive procedure of [14]. Below, we list the asymptotic values and expansions of various OPE coefficients used to calculate the RG pairings.

A.1. Flows from (2, 2) boundary conditions

We record the following normalization coefficients with their leading asymptotics in the \( m \to \infty \) limit:

\[ d_{13} = \left( F_{(22)(11)} \begin{bmatrix} (22) \\ (13) \end{bmatrix} \right)^{-1/2} \sim \sqrt{\frac{3}{2m}}, \] (A.2)

\[ d_{31} = \left( F_{(22)(11)} \begin{bmatrix} (22) \\ (31) \end{bmatrix} \right)^{-1/2} \sim \sqrt{\frac{3}{2m}}, \] (A.3)

\[ d_{33} = \left( F_{(22)(11)} \begin{bmatrix} (22) \\ (33) \end{bmatrix} \right)^{-1/2} \]

\[ = \left( -\frac{1}{4} m(2m+1) \Gamma \left( \frac{2}{m+1} \right) \Gamma^2 \left( 1 + \frac{1}{m} \right) \Gamma \left( 1 - \frac{1}{m} \right) \sin \left( \frac{2\pi m}{m+1} \right) \right)^{-1/2} \]

\[ = \sqrt{3} - \left( \frac{2\pi^2}{\sqrt{3}} \right) \frac{1}{m^2} + \left( \frac{2\pi^2}{\sqrt{3}} \right) \frac{1}{m^3} + \mathcal{O}(m^{-4}), \] (A.4)

\[ \tilde{d}_{31} = \left( F_{(31)(11)} \begin{bmatrix} (31) \\ (31) \end{bmatrix} \right)^{-1/2} \sim 2, \] (A.5)

\[ \mu = \left( F_{(31)(11)} \begin{bmatrix} (22) \\ (42) \end{bmatrix} \right)^{1/2} \cdot \left( F_{(31)(11)} \begin{bmatrix} (22) \\ (22) \end{bmatrix} \right)^{-1/2} \sim \frac{\sqrt{3}}{2m}, \] (A.6)

\[ v = \left( F_{(11)(11)} \begin{bmatrix} (22) \\ (22) \end{bmatrix} \right)^{1/2} \cdot \left( F_{(31)(11)} \begin{bmatrix} (22) \\ (22) \end{bmatrix} \right)^{-1/2} \sim \frac{1}{\sqrt{2m}}. \] (A.7)

We next list the normalized OPE coefficients involving the UV fields and the \( \hat{\psi}_a \) fields

\[ C_{22,13}^{[31,22,22]} = d_{13} F_{(22)(22)} \begin{bmatrix} (31) \\ (22) \end{bmatrix} \sim \sqrt{\frac{3}{2}} \cdot \frac{1}{m}, \] (A.8)

\[ C_{22,31}^{[31,22,22]} = d_{31} F_{(22)(22)} \begin{bmatrix} (31) \\ (22) \end{bmatrix} \sim \frac{1}{m\sqrt{6}}. \] (A.9)
where the expansion is up to the terms of order $d$

We have the following normalization factor for the

\[ C^{[31,22,22]}_{22,31} = d_{31} \mu F_{(22)(42)} \begin{bmatrix} (31) \\ (22) \\ (31) \end{bmatrix} \sim -\frac{2\sqrt{2}}{3}. \]  

(A.10)

\[ C^{[31,22,22]}_{22,33} = d_{33} F_{(22)(33)} \begin{bmatrix} (31) \\ (22) \\ (33) \end{bmatrix} = -d_{33} \left( \frac{\sin(\frac{\pi}{2})}{\sin(\frac{2\pi}{3})} \right) \sim -\frac{1}{\sqrt{3}} - \left( \frac{2\sqrt{3}\pi^2}{9} \right) \frac{1}{m^2} - \left( \frac{2\sqrt{3}\pi^2}{9} \right) \frac{1}{m^3} + O(m^{-4}). \]  

(A.11)

\[ C^{[31,22,22]}_{22,33} = d_{33} \mu F_{(22)(42)} \begin{bmatrix} (31) \\ (22) \\ (33) \end{bmatrix} \sim \frac{4}{3m}. \]  

(A.12)

The vertices involving the IR fields and the RG defect fields are as follows:

\[ C^{[31,31,22]}_{31,22} = d_{31} F_{(31)(22)} \begin{bmatrix} (31) \\ (22) \\ (31) \end{bmatrix} \sim 1/m. \]  

(A.14)

\[ C^{[31,42,22]}_{31,42} = d_{31} \mu^{-1} F_{(31)(42)} \begin{bmatrix} (31) \\ (22) \\ (42) \end{bmatrix} \sim -\frac{1}{\sqrt{3}}. \]  

(A.15)

\[ C^{[11,22,22]}_{31,22} = v, \quad C^{[11,42,22]}_{31,42} = v \mu^{-1}. \]  

(A.16)

\[ C^{[31,11,22]}_{31,22} = v^{-1} F_{(11)(22)} \begin{bmatrix} (31) \\ (22) \\ (22) \end{bmatrix} \sim \frac{1}{\sqrt{2}m}. \]  

(A.17)

**A.2. Flows from $p_2$ boundary conditions**

We have the following normalization factor for the $\psi_{(3,3)}^{(p,2)}(p,2)$ field:

\[ d_{33}(p) = F_{(p,2)(11)} \begin{bmatrix} (33) \\ (p,2) \\ (p,2) \end{bmatrix}^{-1/2} \sim \frac{p + 1}{p - 1} \left( 1 - \frac{1}{3m^2} \right) [-6\gamma + \pi^2p - 6(p - 2)\psi(p - 2) + (p - 2)\psi'(p - 2)](A.18) \]

where the expansion is up to the terms of order $m^{-3}$. For the $\psi_{(1,3)}^{(p,2)}(p,2)$ and $\psi_{(3,1)}^{(p,2)}(p,2)$ fields, the similar factors are

\[ d_{13}(p) = F_{(p,2)(11)} \begin{bmatrix} (13) \\ (p,2) \\ (p,2) \end{bmatrix}^{-1/2} \sim \frac{p - 2}{\sqrt{6}}, \quad \text{for } p > 2, \]  

(A.19)

\[ d_{31}(p) = F_{(p,2)(11)} \begin{bmatrix} (31) \\ (p,2) \\ (p,2) \end{bmatrix}^{-1/2} \sim (p - 2) \sqrt{\frac{p + 1}{2(p - 1)}}, \quad \text{for } p > 2. \]  

(A.20)
We record the following expansions for normalized OPE coefficients:
\[
C_{22,33}^{(p-1),(p,2)}(p,2)\] = \(d_{33}(p)F_{(p,2)}(22)\left[\frac{(22)}{(p-1,1)}\frac{(33)}{(p,2)}\right]
\]
\[
= \frac{p+1}{p-1} \left(1 - \frac{\pi^2 p}{3m^2} + O(m^{-3})\right)
\] (A.21)

\[
C_{22,33}^{(p+1),(p,2)}(p,2)\] = \(d_{33}(p)F_{(p,2)}(22)\left[\frac{(22)}{(p+1,1)}\frac{(33)}{(p,2)}\right]
\]
\[
= -\frac{p-1}{p+1} \left(1 + \frac{p\pi^2}{3m^2} + O(m^{-3})\right).
\] (A.22)

The following expressions are exact:
\[
F_{(p,2)}(22)\left[\frac{(22)}{(p-1,1)}\frac{(33)}{(p,2)}\right] = \frac{\Gamma \left(\frac{1}{m+1}\right)}{\Gamma \left(-\frac{1}{m+1}\right)} \frac{\Gamma \left(\frac{2m+1-p-\gamma}{m+1}\right)}{\Gamma \left(-\frac{1}{m+1}\right)} \frac{\Gamma \left(\frac{p+1}{m+1}\right)}{\Gamma \left(\frac{p-1}{m+1}\right)} \frac{\Gamma \left(\frac{m+1}{m}\right)}{\Gamma \left(\frac{m+1-p}{m+1}\right)}
\] (A.23)

\[
F_{(p,2)}(22)\left[\frac{(22)}{(p+1,1)}\frac{(33)}{(p,2)}\right] = -m \frac{\Gamma \left(\frac{2m+1-p-\gamma}{m+1}\right)}{\Gamma \left(-\frac{1}{m+1}\right)} \frac{\Gamma \left(\frac{p+1}{m+1}\right)}{\Gamma \left(-\frac{1}{m+1}\right)} \frac{\Gamma \left(\frac{m+1}{m}\right)}{\Gamma \left(\frac{m+1-p}{m+1}\right)} \frac{\Gamma \left(\frac{1}{m}\right)}{\Gamma \left(\frac{1}{m}\right)}
\] (A.24)

Using these expressions, we find
\[
C_{22,33}^{((p-1),(p,2))22} = \frac{1}{8(m(m+1))^{1/4}} \sin \left(\frac{\pi(p+1)}{m}\right) \sin \left(\frac{\pi(p-1)}{m}\right)
\] (A.25)

where \(g_{p\pm 1}\) are the \(g\)-factors of the Cardy states \((p \pm 1, 1)\):
\[
g_{p\pm 1} = \left(\frac{8}{m(m+1)}\right)^{1/4} \frac{\sin \left(\frac{\pi(p+1)}{m}\right)}{\sin \left(\frac{\pi}{m}\right)} \sin \left(\frac{\pi(p-1)}{m}\right).
\] (A.26)

We further record the asymptotics for the following fusion matrices and OPE coefficients:
\[
C_{13,13}^{((p,2),(p,2))13} = d_{13}F_{(p,2)}(13)\left[\frac{(13)}{(p,2)}\frac{(13)}{(p,2)}\right] \sim -\frac{4}{\sqrt{6}} \quad \text{for any } p \geq 2,
\] (A.27)

\[
F_{(p,2)}(33)\left[\frac{(33)}{(p,2)}\frac{(33)}{(p,2)}\right] = \frac{2}{p+1} - \frac{2}{3(p+1)m^2} \left[\pi^2 p(p-1) + 6(p+\psi(p-2))
\]
\[
+ (p-2)\psi'(p-2))\right] + O\left(\frac{1}{m^3}\right), \quad p > 2.
\] (A.28)

\[
C_{33,33}^{((p,2),(p,2))33} = d_{33}(p)F_{(p,2)}(33)\left[\frac{(33)}{(p,2)}\frac{(33)}{(p,2)}\right]
\]
\[
= \frac{2}{\sqrt{p^2-1}} - \frac{2\pi^2 p^2}{3\sqrt{p^2-1}m^2} + O\left(\frac{1}{m^3}\right), \quad p \geq 2,
\] (A.29)

\[
C_{33,13}^{((p,2),(p,2))33} = d_{13}(p)F_{(p,2)}(33)\left[\frac{(33)}{(p,2)}\frac{(13)}{(p,2)}\right] = \frac{4}{\sqrt{6m}} + O\left(\frac{1}{m^2}\right) \quad p \geq 2.
\] (A.30)
Appendix B. Conformal blocks

In this appendix, \( \phi_i, i = (i_1, i_2) \in K \) denote the Virasoro algebra chiral fields corresponding to the irreducible representations \( \mathcal{H}_p \). Such a representation is obtained from the Virasoro Verma module built on descendants \( L_{-n_1}L_{-n_2} \ldots L_{-n_m}h_i \) by taking the quotient with respect to singular vectors and their descendants. We have the usual \( L_0 \) grading: \( \mathcal{H}_p = \oplus_{m=0}^{\infty} \mathcal{H}^m_p \) so that \( L_0 \) restricted to \( \mathcal{H}^m_p \) equals \( h_p + n \).

A conformal block is formally defined by means of the expansion

\[
\mathcal{F}^p_{ij;KL}(\eta) = \sum_{K,K'} \eta^{h_i-h_j-h_h-h_{K}} \langle \phi_i | \phi_j(1) | \phi_p, K \rangle Q_{K,K'}^{-1}(p) \langle \phi_p, K' | \phi(1) \rangle, \tag{B.1}
\]

where the indices \( K \) and \( K' \) label the elements of a basis \( | \phi_p, K \rangle \) in \( \mathcal{H}^m_p \) with \( |K| = |K'| = n \). We can assume that the vectors \( | \phi_p, K \rangle \) are linear combinations of vectors of the form \( L_{-n_1}L_{-n_2} \ldots L_{-n_m}h_p \) with \( |K| = n_1 + \cdots + n_m \). The matrix \( Q_{K,K'}^{-1}(p) \) is the inverse matrix to

\[
Q_{K,K}(p) = \langle \phi_p, K | \phi_p, K' \rangle, \tag{B.2}
\]

The matrix elements are defined as

\[
\langle \phi_i | \phi_j(\eta) \rangle = \mathcal{N}_{ij}^{KL} \eta^{h_i-h_j}, \tag{B.3}
\]

and the operators \( L_0 \) act as

\[
[L_0, \phi_1(\eta)] = \mathcal{L}_0^h \phi_1(\eta), \tag{B.4}
\]

where

\[
\mathcal{L}_n^h = \eta^{n+1} \partial_\eta + (n + 1) h_i \eta^n. \tag{B.5}
\]

We will use the following general formulas:

\[
\langle \phi_1 | \phi_j(1) L_{-k_1} \ldots L_{-k_l} | \phi_p \rangle = \prod_{i=1}^{n} \left( h_p - h_i + k_i \right), \tag{B.6}
\]

\[
\langle \phi_p | L_{k_1} \ldots L_{k_l} \phi_1(1) \rangle = \prod_{i=1}^{n} \left( h_p - h_i + k_i \right). \tag{B.7}
\]

The minimal model conformal blocks satisfy the following transformation rules:

\[
\mathcal{F}^p_{ij;KL} \left( 1 - \eta \right) = \sum_q F_{pq} \left[ j \atop i \right] K \mathcal{F}^q_{il; jk}(\eta), \tag{B.8}
\]

\[
\mathcal{F}^p_{ij;KL} \left( 1 \atop \eta \right) = \eta^{k_i+h_i-h_j-h_h} \sum_q B_{pq}^{kl} \left[ j \atop i \right] K \mathcal{F}^q_{ik;jl}(\eta), \tag{B.9}
\]

with

\[
B_{pq}^{kl} \left[ j \atop i \right] K = F_{pq} \left[ j \atop i \right] K e^{\pm i \pi (h_i+h_j-h_h-h_h-h_h)}. \tag{B.10}
\]
B.1. Conformal blocks with (2, 2) fields

Consider a conformal block \( F_{22,31,22,13}^{22}(\eta) \). As \( m \to \infty \) the weight \( h_{22} \) goes to zero and the intermediate channel \( \phi_{22} \) develops a zero norm vector \( L_{-1}|\phi_{22}\rangle \). Thus, the leading asymptotics should come from this singular vector and its descendants. We find from (B.6) and (B.7)

\[
\langle \phi_{22}|\phi_{31}(1)\rangle L_{-k_1, \ldots, L_{-k_i}}|\phi_{22}\rangle = \prod_{i=1}^{n} \left( \sum_{k_i} k_i \right) + O \left( \frac{1}{m} \right),
\]

\[
\langle \phi_{22}|L_{k_1}, \ldots, L_{k_i}\phi_{22}(1)|\phi_{13}\rangle = \prod_{i=1}^{n} \left( -1 + \sum_{k_i} k_i \right) + O \left( \frac{1}{m} \right). \tag{B.11}
\]

Since any descendant of \( L_{-1}|\phi_{22}\rangle \) is a linear combination of vectors \( L_{-k_1, \ldots, L_{-k_i}}|\phi_{22}\rangle \) we see from (B.11) that the leading contribution comes from \( L_{-1}|\phi_{22}\rangle \) itself. Thus,

\[
F_{22,31,22,13}^{22}(\eta) = -\frac{h_{31}h_{13}}{2h_{22}} + O(m) = -\frac{2}{3}m^2 + O(m). \tag{B.12}
\]

Analogously, we obtain

\[
F_{22,31,22,31}^{22}(\eta) = -\frac{(h_{31})^2}{2h_{22}} + O(m) = -\frac{2}{3}m^2 + O(m). \tag{B.13}
\]

We next take up the \( F_{22,31,22,31}^{42}(\eta) \) conformal block. We have

\[
\langle \phi_{42}|L_{k_1}, \ldots, L_{k_i}\phi_{22}(1)|\phi_{31}\rangle = \langle h_{42} - h_{31} + k_1h_{22} \rangle \prod_{i=2}^{n} \left( h_{42} - h_{31} + k_ih_{22} + \sum_{j<i} k_i \right) + O \left( \frac{1}{m} \right),
\]

and thus

\[
F_{22,31,22,31}^{42}(\eta) = 1 + O \left( \frac{1}{m} \right). \tag{B.14}
\]

The leading order contribution to \( F_{22,31,22,33}^{22}(\eta) \) comes from the asymptotic singular vector \( L_{-1}|\phi_{22}\rangle \) and its descendants. Let \( L_K = L_{k_1}, \ldots, L_{k_i} \) with \( k_i \geq 1 \). We have

\[
\langle \phi_{22}|L_1L_K\phi_{22}(\eta)L_{-1}|\phi_{33}\rangle = h_{33}(1 - h_{33} + 2h_{22})L_K^{h_{22}}\eta^{-h_{33}} + \langle \phi_{22}|L_1\phi_{22}(\eta)[L_K, L_{-1}]|\phi_{33}\rangle, \tag{B.16}
\]

where

\[
L_K^{h_{22}} = L_{k_1}^{h_{22}} \cdots L_{k_i}^{h_{22}}. \tag{B.17}
\]

Since

\[
[L_K, L_{-1}] = \sum_{|K|=|K|-1} a_K L_K, \tag{B.18}
\]

we have

\[
\langle \phi_{22}|L_1\phi_{22}(\eta)[L_K, L_{-1}]|\phi_{33}\rangle \sim h_{33}L_1\eta^{-h_{33}} = h_{33}(-h_{33} + 2h_{22})\eta^{1-h_{33}}. \tag{B.19}
\]

Since \( h_{22} \sim m^{-2} \) and \( h_{33} \sim m^{-2} \), we conclude that

\[
\langle \phi_{22}|L_1L_K\phi_{22}(\eta)L_{-1}|\phi_{33}\rangle = O \left( \frac{1}{m^2} \right). \tag{B.20}
\]
when $|\mathcal{K}| = k_1 + \cdots + k_n > 0$. Thus, the leading contribution comes from $L_{-1}|\phi_{22}\rangle$ and can be readily evaluated:

$$\tilde{F}_{22,31,22,33}^{22} (\eta) = \frac{h_{31}(2h_{22} h_{33} + h_{33}(1 - h_{33}))}{2h_{22}} \eta^{-h_{33}} + O \left( \frac{1}{m^2} \right) = \frac{4}{3} + O \left( \frac{1}{m} \right).$$ (B.21)

Similarly, we have

$$\langle \phi_{42}|L_{\mathcal{K}}\phi_{22}(\eta)L_{-1}|\phi_{33}\rangle = (h_{22} + h_{33} - h_{42}) h_{h_{22} - h_{33} - h_{33}} + \langle \phi_{42}|\phi_{22}(\eta)|L_{\mathcal{K}},L_{-1}|\phi_{33}\rangle$$

and thus

$$\langle \phi_{42}|L_{\mathcal{K}}\phi_{22}(\eta)L_{-1}|\phi_{33}\rangle = O \left( \frac{1}{m} \right)$$ (B.23)

if $|\mathcal{K}| > 0$. This implies

$$\tilde{F}_{22,31,22,33}^{42} = -1 + O \left( \frac{1}{m} \right).$$ (B.24)

### B.2. Conformal blocks for $1/m$ corrections

In this appendix, we derive the leading asymptotics of $\mathcal{F}_{33,33,33,13}^{33}(\eta)$ and $\mathcal{F}_{33,33,33,13}^{31}(\eta)$. We will use a method different from the method of section B.1. Note that at $m = \infty$ the conformal dimensions of $h_{33}, h_{13}$ and $h_{31}$ become integers so that the leading asymptotics of the conformal blocks at hand should be given by rational functions. The behaviour of these rational functions at $\eta = 0, 1, \infty$ can be obtained using (B.8)–(B.10). We record the following leading asymptotics of the relevant fusion and braiding matrices:

$$F_{(33)(33)}^{(33)(33)} \sim \frac{1}{2}, \quad F_{(33)(31)}^{(33)(33)} \sim \frac{1}{3},$$ (B.25)

$$F_{(33)(35)}^{(33)(33)} \sim \frac{5}{12}, \quad F_{(31)(33)}^{(33)(33)} \sim 1,$$ (B.26)

$$F_{(31)(31)}^{(33)(33)} \sim \frac{1}{3}, \quad F_{(31)(35)}^{(33)(33)} \sim \frac{5}{6},$$ (B.27)

$$B_{(33)(33)}^{(+)} \sim \frac{1}{2}, \quad B_{(33)(31)}^{(+)} \sim \frac{1}{3},$$ (B.28)

$$B_{(33)(35)}^{(+)} \sim \frac{5}{12}, \quad B_{(31)(33)}^{(+)} \sim 1,$$ (B.29)

$$B_{(31)(31)}^{(+)} \sim \frac{1}{3}, \quad B_{(31)(35)}^{(+)} \sim \frac{5}{6}.$$ (B.30)

We have

$$\mathcal{F}_{33,33,33,13}^{33}(\eta) = \eta^{-h_{13}} + \frac{2h_{33} - h_{13}}{2} \eta^{1-h_{13}} + \ldots \sim \frac{1}{\eta} - \frac{1}{2}, \quad \eta \sim 0.$$ (B.31)

Using (B.8)–(B.10), we also obtain

$$\mathcal{F}_{33,33,33,13}^{33}(\eta) = \frac{5}{12} \mathcal{F}_{33,13,33,33}^{35}(1 - \eta) + \frac{1}{3} \mathcal{F}_{33,13,33,33}^{31}(1 - \eta) + \frac{1}{2} \mathcal{F}_{33,13,33,33}^{33}(1 - \eta) + O \left( \frac{1}{m} \right).$$ (B.32)
\[ \mathcal{F}_{33,33,33,13}(\eta) = \eta^{-h_{11} - h_{33}} \left[ \frac{5}{12} \mathcal{F}_{33,33,33,13}^{35} \left( \frac{1}{\eta} \right) + \frac{1}{3} \mathcal{F}_{33,33,33,13}^{31} \left( \frac{1}{\eta} \right) \right. \\
\left. - \frac{1}{2} \mathcal{F}_{33,33,33,13}^{33} \left( \frac{1}{\eta} \right) \right] + \mathcal{O} \left( \frac{1}{m} \right). \] (B.33)

Noting that for \( \eta \sim 1 \) we have an expansion
\[ \mathcal{F}_{33,13,33,33}(1-\eta) = (1-\eta)^{-h_{33}} + \frac{h_{13}}{2} (1-\eta)^{1-h_{33}} + \mathcal{O}((1-\eta)^{2-h_{33}}), \] (B.34)
we find from the above
\[ \mathcal{F}_{33,33,33,13}(\eta) \sim \frac{1}{2} + (1-\eta) + \cdots, \quad \eta \to 1, \]
\[ \mathcal{F}_{33,33,33,13}(\eta) \sim -\eta + \frac{1}{2} + \frac{1}{\eta} + \cdots, \quad \eta \to \infty \] (B.35)
up to terms suppressed by \( 1/m \). It follows from (B.31) and (B.35) that
\[ \mathcal{F}_{33,33,33,13}(\eta) = \frac{1}{\eta} - \frac{1}{2} + \mathcal{O} \left( \frac{1}{m} \right). \] (B.36)

We further find
\[ \mathcal{F}_{33,33,33,13}^{31}(\eta) = \mathcal{F}_{33,33,33,13}^{33}(1-\eta) + \frac{1}{3} \mathcal{F}_{33,13,33,33}^{31}(1-\eta) \]
\[ - \frac{5}{6} \mathcal{F}_{33,13,33,33}^{35}(1-\eta) + \mathcal{O} \left( \frac{1}{m} \right), \] (B.37)
\[ \mathcal{F}_{33,33,33,13}^{33}(\eta) = \eta^{-h_{11} - h_{33}} \left[ \mathcal{F}_{33,33,33,13}^{33} \left( \frac{1}{\eta} \right) - \frac{1}{3} \mathcal{F}_{33,33,33,13}^{31} \left( \frac{1}{\eta} \right) \right. \\
\left. + \frac{5}{6} \mathcal{F}_{33,13,33,33}^{35} \left( \frac{1}{\eta} \right) \right] + \mathcal{O} \left( \frac{1}{m} \right). \] (B.38)
From this, we find that \( \mathcal{F}_{33,33,33,13}^{33}(\eta) \sim 1 \) through the first two orders near \( \eta = 0, 1, \infty \) and thus
\[ \mathcal{F}_{33,33,33,13}^{31}(\eta) = 1 + \mathcal{O} \left( \frac{1}{m} \right). \] (B.39)

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