ON CURVATURE PINCHING OF CONIC 2-SPPHERES

HAO FANG AND MIJIA LAI

ABSTRACT. We study metrics on conic 2-spheres when no Einstein metrics exist. In particular, when the curvature of a conic metric is positive, we obtain the best curvature pinching constant. We also show that when this best pinching constant is approached, the conic 2-sphere has an explicit Gromov-Hausdorff limit. This is a generalization of the previous results of Chen-Lin and Bartolucci for 2-spheres with one or two conic points.

1. INTRODUCTION

Due to the recent development in Kähler geometry, there is a lot of interest in the study of metrics with conic singularities along a divisor [D JMR]. The deep connection between algebraic stability and existence of Kähler-Einstein metrics on Fano manifolds is highlighted in the recent solution of the Yau-Tian-Donaldson conjecture [CDS1, CDS2, CDS3, Ti]. Metrics with conic singularities play an essential role in this direction, namely it is more natural to consider Kähler-Einstein metrics with possible conic singularities on Fano manifolds. It is thus interesting to explore further geometric properties of conic manifolds. On the other hand, manifolds satisfying certain unstability conditions do not admit Kähler-Einstein metrics or constant scalar curvature metrics. Therefore, an interesting problem is to search for other “canonical” Kähler metrics (possibly with conical singularities) on such manifolds. In this article, we search for “least-pinched” metrics on conic 2-spheres when no Einstein metrics exist.

For surfaces with conic metric singularities, or conic surfaces, the relation between stability and existence of Einstein metrics can be expressed in explicit forms. In order to study the prescribing curvature problem for conic surfaces, Troyanov [T] has classified conic surfaces into sub-critical, critical and super-critical categories. On the other hand, the logarithmical K-stability [RT] is linked to the coercivity of
twisted Mabuchi $K$-energy functional, which means any conic surface is either logarithmically $K$-stable, semi-stable or unstable. It is shown that these two classifications coincide [RT].

Let us start with some definitions before further elaboration. For a closed Riemann surface $S$, a metric $g$ is said to have a conic singularity of order $\beta \in (-1, \infty)$ at a point $p \in S$ if under a local holomorphic coordinate $\{z\}$ centered at $p$,

$$g = e^{f(z)}|z|^{2\beta}|dz|^2,$$

where $f(z)$ is locally bounded and $C^2$ away from $p$. The conic singularity is modeled on the Euclidean cone: $\mathbb{C}$ with a metric $|z|^{2\beta}|dz|^2$ is isometric to a Euclidean cone of angle $2\pi(\beta + 1)$ with the cone tip at the origin.

In general, we shall use the triple $(S, D, g)$ to denote a closed orientable Riemannian surface $S$ with a conic metric $g$, for which the information of its singularities is encoded in the divisor $D = \sum_{i=1}^{n} \beta_i p_i$ in an obvious manner, i.e., $g$ has conic singularities at $p_i$ of order $\beta_i$. We sometimes say that the conic metric $g$ represents $D$.

For such a triple $(S, D, g)$, let $K = K(g)$ be the Gaussian curvature of $g$ defined on $S \setminus D$. Throughout the paper, we assume $K$ can be extended to a Hölder continuous function on $S$. The collection of all such conic metrics representing $D$ is denoted by $\mathcal{C}(S, D)$. We shall use the pair $(S, D)$ to denote a conic surface when the metric is not specified.

The Gauss-Bonnet formula for the conic surface $(S, D)$ becomes [Tr]

$$\int_{S} K(g) dA(g) = 2\pi \chi(S, D) := 2\pi(\chi + |D|),$$

where $|D| = \sum_{i=1}^{n} \beta_i$ is the degree of the divisor.

In his seminal paper [Tr], Troyanov systematically studies the prescribing curvature problem on the conic surface $(S, D)$. He divided the problem into three cases according to the sign of the Euler characteristic number $\chi(S, D)$. For $\chi(S, D) > 0$, he further classified the problem into the following three cases:

1. subcritical case: $\chi(S, D) < \min\{2, 2 + 2 \min_i \beta_i\}$;
2. critical case: $\chi(S, D) = \min\{2, 2 + 2 \min_i \beta_i\}$;
3. supercritical case: $\chi(S, D) > \min\{2, 2 + 2 \min_i \beta_i\}$.

The constant $\min\{2, 2 + 2 \min_i \beta_i\}$ turns out to be the Trudinger constant [Tr] in the corresponding Moser-Trudinger inequality for conic surfaces, which plays an important role in the prescribing curvature problem. According to Ross-Thomas [RT], a conic surface $(S, D)$ being subcritical, critical or supercritical, respectively can be reinterpreted as
it being logarithmically K-stable, semi-stable or unstable, respectively. For further developments of the prescribing curvature problem, we refer readers to [BDM, BaMa, E] and the references therein.

Now let us examine the Yamabe problem on conic surfaces, namely the existence of constant curvature metrics on $\mathcal{(S,D)}$. If $\chi(S,D) \leq 0$, there always exists a conic metric representing $D$ with constant curvature [Tr]. When $\chi(S,D) > 0$ and all $\beta_i \in (-1,0)$ (which we assume from now on), $S$ is necessarily a 2-sphere. In this situation, $S$ admits a conic metric of positive constant curvature if and only if:

1. either $n = 2$ and $D = \beta_1 p + \beta_2 q$ with $\beta_1 = \beta_2$;
2. or $n \geq 3$ and $D$ is subcritical i.e., $\chi(S,\beta) < \min\{2, 2 + \min_i \beta_i\}$.

Note that surfaces satisfying (1) are critical and are often called (American) footballs, see [CL2] for some classification results. For surfaces satisfying (2), the sufficiency is proved by Troyanov [T], the necessity and uniqueness argument is due to Luo-Tian [LT].

In view of above results, there are two cases of conic 2-spheres which do not carry metrics of constant (positive) curvature:

1. $D$ is supercritical;
2. $D$ is critical and $n \geq 3$.

It is then natural to seek for other “canonical” metrics as substitutes for constant curvature metrics. Since in two-dimension the curvature is a scalar function, we can consider from the viewpoint of “least-pinched” metrics. More precisely, we ask the following question:

**Problem 1.1.** For a conic 2-sphere $(S, D, g)$ with positive Gaussian curvature $K(g)$ (assumed to be extended to a continuous function on $S$), let $K_{\text{max}}$ and $K_{\text{min}}$ denote the maximum and the minimum of $K(g)$, respectively. Define the curvature pinching constant of $g$ as

$$\rho(g) = \frac{K_{\text{min}}}{K_{\text{max}}}.$$  

What is $\sup_{g \in C(S,D)} \{\rho(g)\}$?

If $(S, D)$ admits a constant curvature metric $g$, then clearly $\sup\{\rho(g)\} = 1$, which is attained by constant curvature metrics. So nontrivial cases for this problem are

1. $D$ is supercritical;
2. $D$ is critical and $n \geq 3$.

Problem [11] was first asked by Thurston [Th] in 1978 for 2-spheres with one or two conic points (both are supercritical). Bartolucci [B] answers Thurston’s question based on the analysis of Chen-Lin [ChLi],
who have treated the one conic point case. More precisely, Bartolucci has proved the following

**Theorem 1.2** (Bartolucci). Let \((S^2, D)\) be a conic 2-sphere, with \(D = \alpha p + \beta q\). Suppose \(-1 < \beta < \alpha \leq 0\) (\(\alpha = 0\) corresponds to the case of one conic point), then for all piecewise smooth and \(C^{1,1}\) conic metrics \(g\) on the \(S^2\) representing \(D\),

\[
\rho(g) \leq \frac{(\beta + 1)^2}{(\alpha + 1)^2},
\]

where the equality holds if and only if \((S^2, D, g)\) (up to a Möbius transformation) is the “glued football”.

For \(-1 < \beta \leq \alpha \leq 0\), a glued football \(S^2_{\alpha,\beta}\) is given by the following conformal metric \(g = e^{2u} g_0\) on the 2-sphere, where \(g_0\) is the standard Euclidean metric on \(\mathbb{C}\) and the conformal factor \(u\) is defined as

\[
u = u_{\alpha,\beta} = \begin{cases} 
\ln\left(\frac{2(\alpha+1)^{\alpha}}{1+r^2+2r}\right), & r < 1; \\
\ln\left(\frac{2(\alpha+1)^{\beta}}{1+r^2+2r}\right) & r \geq 1,
\end{cases} \tag{1.3}
\]

with \(r = |z|\).

The glued football \(S^2_{\alpha,\beta}\) has a conic singularity along \(\beta\infty\) when \(\alpha = 0\), or \(\alpha 0 + \beta\infty\), when \(\alpha < 0\).

We note that if \(\alpha = \beta\) this is a smooth conic metric away from \(z = 0, \infty\); it has constant curvature 1. This is the so-called (American) football.

When \(\alpha \neq \beta\), \(S^2_{\alpha,\beta}\) is glued by two footballs of different angles along their equator, and thus has piecewise constant curvature:

\[
K(g) = \begin{cases} 
1, & r < 1; \\
\frac{(\beta+1)^2}{(\alpha+1)^2}, & r > 1.
\end{cases} \tag{1.4}
\]

It follows \(\rho(g) = \frac{(\beta+1)^2}{(\alpha+1)^2}\), which realizes the equality in (1.2).

In this paper, we answer Problem 1.1 in its full generality. Our main result is the following

**Theorem 1.3.** Let \((S, D)\) be a conic 2-sphere where \(D = \sum_{i=1}^n \beta_i p_i\) is supercritical, assume \(\beta_1 = \min_i \beta_i\) and let \(\alpha = |D| - \beta_1\). Then for all \(g \in \mathcal{C}(S, D)\),

\[
\rho(g) < \frac{(1 + \beta_1)^2}{(1 + \alpha)^2} := \rho_0(S, D), \tag{1.5}
\]

moreover \(\rho_0\) is optimal.
Note if $D$ is supercritical, then $\beta_1 < \alpha$, consequently $\rho_0(S, D) < 1$. Hence this gives a qualitative evidence for the non-existence of constant positive curvature on supercritical conic 2-spheres, or it can be viewed as a necessary condition for the Nirenberg problem in the supercritical conic 2-sphere setting. The theorem also recovers Bartolucci’s result if $D$ has one or two conic points.

We find that glued footballs also serve as the extremal geometrically, when $n \geq 3$.

**Theorem 1.4.** Let $(S, D)$ be a conic 2-sphere where $D = \sum_{i=1}^{n} \beta_i p_i$ is supercritical, assume $\beta_1 = \min_i \beta_i$ and let $\alpha = |D| - \beta_1$. Then for any sequence of smooth conic metrics $\{g_i\}_{i=1}^{\infty}$ with $\lim_{i \to \infty} \rho(g_i) = \rho_0$ and $\max K(g_i) = 1$, $\{(S, D, g_i)\}$ converges to the glued football $(S_{\alpha, \beta_1}^2, \tilde{D} = \beta_1 p + \alpha q)$ in the Gromov-Hausdorff sense, moreover $p_1 \to p$ and $p_2, \ldots, p_n \to q$ along the convergence.

For the critical case, following identical arguments, we have

**Theorem 1.5.** Let $(S, D)$ be a conic 2-sphere where $D = \sum_{i=1}^{n} \beta_i p_i$ is critical and $n \geq 3$, assume $\beta_1 = \min \beta_i$ and let $\alpha = |D| - \beta_1$. Then $\forall g \in C(S, D)$,

$$\rho(g) < \frac{(1 + \beta_1)^2}{(1 + \alpha)^2} := \rho_0(S, D) = 1,$$

and the constant $\rho_0(S, D)$ is optimal; moreover, for any sequence of smooth conic metrics $g_i$ with $\lim_{i \to \infty} \rho(g_i) = 1$ and $\max K_i = 1$, $\{(S, D, g_i)\}$ converges to the football $(S_{\alpha, \alpha}^2, \tilde{D} = \alpha p + \alpha q)$ in the Gromov-Hausdorff sense, and $p_1 \to p$ and $p_2, \ldots, p_n \to q$ along the convergence.

Our main tool of the proof is to apply the isoperimetric inequality to obtain sharp differential inequalities. This is inspired by methods first used by Chen-Lin [ChLi] that are later extended by Bartolucci [B]. However, since there can be no simple symmetric rearrangement procedure for the multi-singularity case, their method can not be applied directly. Instead, we apply a similar argument used in our earlier paper [FL] to simplify their proof. In particular, we are able to derive a differential inequality involving only the level sets of the conformal factor without the symmetric rearrangement argument. On the other hand, the characterization of the equality case is obtained in [ChLi, B] by the more involved rearrangement technique. We need to do a finer analysis of the defect of isoperimetric inequalities as in [FL] to prove the convergence result in Theorem 1.4, which indicates all but one conic points merge in the limit procedure.

While previous works of [ChLi, B] focus on $C^{1,1}$ and piecewise smooth metrics, for which glued footballs serve as the unique geometric sharp
examples up to Möbius transformations, it is clear that (1.5) is a strict inequality for smooth conic metrics. Using our previous results in [FL] and some careful analysis, we are able to construct examples in general cases that (1.5) is indeed sharp.

In recent preprints [PSSW1, PSSW2], the Ricci flow on conic 2-spheres is shown to converge in stable, semi-stable and unstable cases. It is shown in [PSSW1] that conic metrics on a sphere with \( n \geq 3 \) conic points in the semi-stable case converge in Gromov-Hausdorff topology to a football along the Ricci flow. Also in [PSSW2] it is proved that the Ricci flow of an unstable conic metric converges geometrically to a Ricci soliton with two conic points. Thus, in the sense of the Ricci flow, Ricci solitons are considered to be canonical metrics for conic 2-spheres.

Comparing the convergence results of Theorem 1.4 and Theorem 1.5 to convergence results of the Ricci flow to Ricci solitons, we observe an interesting common feature: the conic point with the smallest cone angle remains in the limit process, while all other conic points merge to form a new conic point. Their difference is however more obvious: while Ricci solitons are smooth with varied curvature, the glued footballs have piece-wise constant curvature. This is somewhat expected. As the problem of searching for metrics realizing \( \sup_{g \in C(S,D)} \{ \rho(g) \} \) is not variational, and thus the extremal usually loses smoothness.

In general, if a Kähler manifold does not carry metrics of constant scalar curvature in a fixed Kähler class, it would be very interesting to see if the best scalar curvature pinching can be computed. Hopefully we can address this problem in future works.

From a more analytic point of view, as mentioned earlier, Theorem 1.3 gives a necessary condition for the Nirenberg problem, which would be an interesting topic for further discussion. It also indicates a sharp Moser-Trudinger inequality for supercritical conic 2-spheres.

This paper is organized as follows: in Section 2, we present our key estimate for proving Theorem 1.3; in Section 3 and Section 4, we prove Theorems 1.3 and 1.4 respectively.

Acknowledgements: Both authors would like to thank Bartolucci for bringing up his work [B] to their attention. They also thank the referee for pointing out some technical inaccuracy in a previous draft.

2. Key Estimate

In this section we give a proper set-up for Theorem 1.3 and present a key estimate for its proof.

We start with some definitions.
Let $g_0$ be the standard Euclidean metric on $\mathbb{C}$. We identify any point $p \in S^2$ with $z \in \mathbb{C}^*$ via stereographic projection. For any given natural number $n$, consider a divisor $D = \sum_{i=1}^{n} \beta_ip_i$ where $p_i \in S^2$ and $-1 < \beta_i < 0$, $i = 1, \ldots, n$. For simplicity, we assume

$$\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n. \quad (2.1)$$

Remember that

$$|D| = \sum_{i=1}^{n} \beta_i. \quad (2.2)$$

We let

$$\alpha := |D| - \beta_1 = \sum_{i=2}^{n} \beta_i. \quad (2.3)$$

The supercritical condition implies that $\alpha > \beta_1$. Also, without loss of generality, we may assume $z_1 = \infty$.

Thus, a conformal metric $g = e^{2u}g_0$ represents $(S^2, D)$ if the asymptotic behavior of $u$ near $z_i$ is:

- $u \sim \beta_i \ln |z - z_i|$ as $z \to z_i$, $i > 1$;
- $u \sim -(\beta_1 + 2) \ln |z|$ as $|z| \to z_1 = \infty$.

The Gaussian curvature of $g$ is computed as

$$K(g) = -e^{-2u} \Delta u, \quad (2.4)$$

when $z \neq z_i$. Here the Laplacian is with respect to $g_0$.

The main result of this section is the following estimate:

**Proposition 2.1.** Notations as above. Let $K$ be a positive continuous function on $\mathbb{C}$ satisfying (2.4) such that

$$0 < a \leq K \leq b, \quad (2.5)$$

then

$$\frac{a}{b} \leq \rho_0(S, D) = \frac{(1 + \beta_1)^2}{(1 + \alpha)^2}. \quad (2.6)$$

**Proof.** We consider the level sets of $u$. Using notations of our earlier works [FL], we define,

$$\Omega(t) := \{ u \geq t \} \subset \mathbb{C}, \quad A(t) := \int_{\Omega_t} Ke^{2u}, \quad B(t) := |\Omega_t|,$$

where integrals are with respect to the Euclidean metric $g_0$ and $|\cdot|$ stands for the Lebesgue measure. In view of the asymptotic behavior
of $u$ near $\infty$, $B(t)$ is finite for any $t \in \mathbb{R}$. The Gauss-Bonnet formula now reads as

$$\int_{\mathbb{R}^2} Ke^{2u} = 2\pi(2 + |D|) = \lim_{t \to -\infty} A(t). \quad (2.7)$$

Moreover, the asymptotic behavior of $u$ at singularities $z_2, \cdots z_n$ implies that $z_i \in \Omega(t)$, for $2 \leq i \leq n$ and for all $t \in \mathbb{R}$. Thus using equation (2.4), we have

$$A(t) = \int_{\Omega(t)} Ke^{2u} = \int_{\Omega(t)} -\Delta u = \int_{\partial\Omega(t)} |\nabla u| + 2\pi \alpha.$$

It is easy to from definition that both $A(t)$ and $B(t)$ are non-increasing. It follows that $A'(t)$ and $B'(t)$ exist for almost everywhere $t$.

For such a $t$,

$$A'(t) = \lim_{s \to t^-} \frac{\int_{\Omega(s) \setminus \Omega(t)} Ke^{2u}}{s - t} \geq \lim_{s \to t^-} \frac{be^{2u(z^*)}|\Omega(s) \setminus \Omega(t)|}{s - t} = be^{2t}B'(t),$$

where we have applied mean value theorem with some $z^* \in \Omega(s) \setminus \Omega(t)$ and (2.5). Similarly, $A'(t) \leq ae^{2t}B'(t)$ for almost everywhere $t$, so

$$\frac{1}{b} \leq \frac{e^{2t}B'(t)}{A'(t)} \leq \frac{1}{a}. \quad (2.8)$$

It follows from the co-area formula (see Lemma 2.3 in [BZ]) that for given $t_1 < t_2$,

$$B(t_1) - B(t_2) = |C \cap u^{-1}((t_1, t_2))] + \int_{t_1}^{t_2} \int_{u=\tau} dH^1 d\tau, \quad (2.9)$$

where $C$ denotes the set of critical points of $u$, i.e., $C = \{z|\nabla u(z) = 0\}$.

We claim $|C \cap u^{-1}((t_1, t_2))] = 0$, for any $t_1 < t_2$, which indicates that $B$ is absolutely continuous. By (2.4) and (2.5), then for any $z_0 = (x_0, y_0) \in \{C \cap u^{-1}((t_1, t_2))]\}$, we may assume, without loss of generality, that $u_{xx}(z_0) \neq 0$. By virtue of the implicit function theorem, there exists some $\rho > 0$ and $g : (y_0 - \rho, y_0 + \rho) \to \mathbb{R}$ such that $\{u_x(x, y) = 0 \cap B_\rho(z_0)\}$ is the graph of the function $x = g(y)$. Clearly,

$$C \cap B_\rho(z_0) \subset \{u_x(x, y) = 0\} \cap B_\rho(z_0),$$

which has 0 measure. We have thus established the claim. $B(t)$ is absolutely continuous on any finite interval $[t_1, t_2]$.

From now on we may assume that computation is done for a generic $t$, i.e., for which $A'(t)$ and $B'(t)$ exist and (2.8) and (2.9) hold.
By the isoperimetric inequality and the Hölder's inequality, we have

\[ 4\pi B(t) \leq \left( \int_{\partial \Omega(t)} 1 \right)^2 \leq \int_{\partial \Omega} |\nabla u| \int_{\partial \Omega(t)} \frac{1}{|\nabla u|} \leq -B'(t)(A(t) - 2\pi \alpha), \]

which leads to

\[ \frac{B(t)}{B'(t)} \geq \frac{-A(t) - 2\pi \alpha}{4\pi}. \]  

(2.10)

Combining (2.8) and (2.11), we get

\[ \frac{d}{dt}[e^{2t}B(t)] = e^{2t}B'(t) + 2e^{2t}B(t) = e^{2t}B'(t)(1 + \frac{2B(t)}{B'(t)}) \]

\[ \leq \frac{e^{2t}B'(t)}{A'(t)}A'(t)(1 - \frac{A - 2\pi \alpha}{2\pi}) \]

\[ \leq \begin{cases} 
\frac{A'_{\alpha}}{b}(1 + \alpha - \frac{A}{2\pi}), & A \leq 2\pi(1 + \alpha); \\
\frac{A'_{\alpha}}{a}(1 + \alpha - \frac{A}{2\pi}), & A > 2\pi(1 + \alpha). 
\end{cases} \]  

(2.12)

Since \( A(-\infty) = 2\pi(2 + |D|) > 2\pi(1 + \alpha) \), there exists \( t_0 \in \mathbb{R} \) such that \( A(t_0) = 2\pi(1 + \alpha) \). For \( t < t_0 < T \), integrating (2.12) from \( t \) to \( T \), and noticing \( e^{2t}B(t) \) is absolutely continuous as well, we get

\[ e^{2T}B(T) - e^{2t}B(t) \leq \frac{A^2(t)}{4a\pi} \frac{1 + \alpha}{a}A(t) + (1 + \alpha)^2(\frac{\pi}{a} - \frac{\pi}{b}). \]  

(2.13)

Since

\[ \int_{\mathbb{R}^2} e^{2u} = 2 \int_{-\infty}^{\infty} B(t)e^{2t}dt < \infty, \]

there exist sequences \( t_n \to -\infty \) and \( T_n \to \infty \), such that \( e^{2t_n}B(t_n) \to 0 \) and \( e^{2T_n}B(T_n) \to 0 \), respectively.

Plugging such two sequences into (2.13), we infer

\[ \frac{2 + |D|}{1 + \alpha} \geq 1 + \sqrt{\frac{a}{b}}, \]

which implies (2.6). We have thus finished the proof. \( \square \)

3. Proof of main result

In this section, we prove that the constant \( \rho_0(S, D) \) obtained in Proposition 2.1 is optimal.

The main result of this section is the following:
Theorem 3.1. For any fixed supercritical conic 2-sphere \((S, D)\) and any \(\epsilon > 0\), there exists a smooth conic metric \(g \in C(S, D)\) such that

\[\rho(g) \geq \rho_0(S, D) - \epsilon.\]

We shall construct a conformal factor \(u_2\) whose pinching constant is close to \(\rho_0(S, D)\). The construction is divided into three steps. First, we construct an approximate conformal factor \(u_0\) based on our earlier work \([FL]\), which has desired singularities along \(D\), but it has discontinuity along \(\{|z| = 1\}\). Second, we run a mollification argument to get a conformal factor \(u_1\), which smooths out the discontinuity of \(u_0\). Finally, we combine \(u_0\) and \(u_1\) using a standard cutoff function to get the desired function \(u_2\).

Proof. We now describe our approximate conformal factor \(u_0\). For \(n = 1, 2\), \(u_0\) can be chosen as the conformal factor of the glued football (up to a constant) with the corresponding cone angles. Namely, by (1.3), we define

\[u_0(z) = \begin{cases} \ln(\frac{2|z|^\alpha}{1+|z|^{2+2\alpha}}), & |z| < 1; \\ \ln(\frac{2|z|^{\beta_1}}{1+|z|^{2+2\beta_1}}), & |z| \geq 1. \end{cases}\] (3.1)

Thus, we have that \(u_0 \in C^1(C)\) for \(n = 1\) and \(u_0 \in C^1(C\setminus\{0\})\) for \(n = 2\).

For the case \(n \geq 3\), to construct \(u_0\), we need to apply the main result in \([FL]\). Given a supercritical divisor \(D\) on \(S = \mathbb{S}^2\), where \(D = \sum_{i=1}^n \beta_i p_i\), assume \(\beta_1 = \min \beta_i\) and let \(\alpha = |D| - \beta_1\). We then have \(\beta_1 < \alpha < 0\). Define a monotone decreasing sequence \(\{\alpha_j\}_{j=1}^\infty\), such that \(\alpha_j < \beta_2\), and

\[\lim_{j \to \infty} \alpha_j = \alpha.\]

We fix \(p_0 = \infty\) and consider divisors

\[(S, D_j)\] is subcritical for each \(j\). Therefore, due to \([11]\), there exists a conic metric \(g_j = e^{2v_j}g_0\) for the pair \((S, D_j)\) such that \(K(g_j) = (\alpha_j + 1)^2\). By the main theorem of \([FL]\), we know that after a suitable normalization and passing to a subsequence if necessary,

\[v_j(z) \to v_\infty(z) = \ln(\frac{2|z|^\alpha}{1+|z|^{2+2\alpha}}), \quad j \to \infty, \quad (3.3)\]

where the convergence is \(C^\infty\) on any compact \(K \subset \mathbb{C}\setminus O\). Note that the convergence \((3.3)\) implies that the singular points of \(v_j\) converge to
the origin $O$ when $j \to \infty$. Without loss of generality, we may assume that $v_j$ is smooth for $1/4 < |z| < \infty$. Define

$$w_j(z) = \begin{cases} v_j(z), & |z| < 1; \\ \ln\left(\frac{2|z|^\beta_1}{1+|z|^2+|z|^4}\right) & |z| \geq 1. \end{cases}$$

(3.4)

Consider a general piecewise smooth function $w$ defined in the region $\{|z| \geq 1/2\}$ with the discontinuity at $\{|z| = 1\}$. Define the following

$$D^0_z(w) = \limsup_{z_1, z_2 \to z} |w(z_1) - w(z_2)|,$$

$$D^1_i(z) = \limsup_{z_1, z_2 \to z} |\nabla w(z_1) - \nabla w(z_2)|.$$

Following (3.3), we get:

**Proposition 3.2.** For any $\epsilon' > 0$, there exists a $J \in \mathbb{N}$ such that for $j \geq J$, $z \in \{2 > |z| > 1/2\}$, $i = 0, 1$, we have

1. $|\nabla^p(v_j(z) - v_\infty(z))| \leq \epsilon' \quad p = 0, 1, 2, 3$;
2. $D^i_j(w_j) \leq \epsilon'$.

We will pick our approximate conformal factor $u_0$ as one of the $w_j$’s, with the choice of $j$ given later.

To summarize properties of $u_0$, we have the following

**Proposition 3.3.** For any $\epsilon' > 0$, there exists a function $u_0 : \mathbb{C} \to \mathbb{R}$, such that

1. $u_0$ is smooth away from the curve $\{|z| = 1\}$, and $z_2, \ldots, z_n \in \{|z| < 1/2\}$;
2. when $u_0$ is smooth, we have
   $$(\beta + 1)^2 \leq -e^{-2u_0} \Delta u_0 \leq (\alpha + 1)^2;$$
3. $u \sim \beta_i \ln |z - z_i|$ as $z \to z_i$, $i > 1$;
4. $u \sim -(\beta_1 + 2) \ln |z|$ as $|z| \to z_1 = \infty$;
5. $D^i(u_0) \leq \epsilon'$, for $|z| > \frac{1}{2}$;
6. There are constants $m_0, M_0$ and $M_1$, depending only on $\alpha$ and $\beta_1$ such that
   $$M_0 \geq \sup_{|z| > 1/2} u_0(z),$$
   $$M_1 \geq \sup_{1/2 \leq |z|, |z| \neq 1} |\nabla u_0(z)|,$$
   $$m \leq \inf_{1/2 \leq |z| \leq 2} u_0(z).$$
Next, we describe our mollification procedure. Define, for any \( z \in \mathbb{C} \) and \( \delta > 0 \),
\[
\varphi(z) = \varphi_\delta(z) := \begin{cases} \frac{c}{\delta} \exp(-\frac{\delta^2}{4|z|^2}), & 0 \leq |z| < \delta, \\ 0, & |z| \geq \delta, \end{cases}
\]
where \( c \in \mathbb{R} \) is chosen such that \( \int_\mathbb{C} \varphi dv = 1 \). There exists a constant \( C > 10 \), such that
\[
|\varphi| \leq \frac{C}{\delta^2}, \quad |\nabla \varphi| \leq \frac{C'}{\delta^3}.
\]
(3.5)
Define, for \( |z| \geq 1/2 \),
\[
u_1(z) = \int_{w \in \mathbb{C}} \varphi(z - w)u_0(w) dv_w.
\]
(3.6)
We prove the following

**Proposition 3.4.** For any \( \epsilon > 0 \), and there exists \( \epsilon' > 0 \) and \( 1/8 > \delta > 0 \), such that if \( u_0 \) is a function satisfying Proposition 3.3, then the function \( u_1 \) defined in (3.6) satisfies the following

1. \( u_1 \) is smooth in the region \( \{z : |z| \geq 5/8\} \);
2. In the region \( \{1/4 \leq ||z| - 1| \leq 3/8\} \),
\[ ||u_1 - u_0||_{C^2} \leq \epsilon; \]
3. For \( 3/4 \leq |z| \leq 5/4 \),
\[ (\beta + 1)^2 - \epsilon \leq -e^{-2\nu_1} \Delta u_1 \leq (\alpha + 1)^2 + \epsilon. \]

*Proof.* The proof of parts (1) and (2) of Proposition 3.4 is standard. We just need to prove part (2).

Notice that
\[
\nabla_z \varphi(z - w) = -\nabla_w \varphi(z - w),
\]
we have the following inequalities for \( |z| \geq 5/8 \),
\[
|u_1(z) - u_0(z)| = |\int \varphi(z - w)(u_0(z) - u_0(w)) dv_w| \leq M_1 \delta, \quad (3.7)
\]
\[
|\Delta u_1(z) - \int \varphi(z - w) \Delta u_0(w) dv_w| \leq \int_{\{|w| = 1, |w - z| \leq \delta\}} |\nabla \varphi(z - w)| D^0_w(u_0) + \varphi D^1_w(u_0) \leq 2\pi \delta (C' \delta^3 + C' \delta^2).
\]
(3.8)
Now we can choose \( \epsilon' = \delta^3 \) in (3.5) to get
\[
|\Delta u_1(z) - \int \varphi(z - w) \Delta u_0(w) dv_w| \leq 4\pi C \delta. \quad (3.9)
\]
Notice that if $|z - w| \leq \delta < 1/8$, $|z| \geq 3/4$, we have $|w| \geq 5/8$, and 
\[ |u_0(z) - u_1(w)| \leq |u_0(w) - u_1(w)| + |u_0(w) - u_0(z)| \leq 2M_1\delta. \]
For $3/4 \leq |z| \leq 5/4$, $|z| \neq 1$,
\[-e^{-2u_1(z)}\Delta u_1(z) \leq e^{-2u_1(z)}[\int \varphi(z - w)(\alpha + 1)^2 e^{2u_0(w)} + 4\pi C\delta] \]
\[ \leq 4\pi C e^{-2m\delta} + (\alpha + 1)^2 e^{4M_0\delta}. \quad (3.10) \]
Similarly, we can prove that
\[-e^{-2u_1(z)}\Delta u_1(z) \geq (\beta_1 + 1)^2 e^{-4M_0\delta} - 4\pi C e^{-2m\delta}. \]
Thus for any given $\epsilon > 0$, we can choose a proper $\delta << 1$ to get (3) for $3/4 \leq |z| \leq 5/4$, $|z| \neq 1$. Now that $u_1$ is smooth for $3/4 \leq |z| \leq 5/4$, we can thus extend this estimate to get (3).

Finally, we describe our smooth conic metric whose curvature pinching is arbitrarily close to $\rho_0$. Define a cut-off function $\chi \in C^\infty(\mathbb{C})$ such that
\[
\chi(z) = \begin{cases} 
1, & z \in R_1 = \{3/4 \leq |z| \leq 5/4\}; \\
0, & z \in R_2 = \{|z| < 5/8, \text{ or } |z| > 11/8\}; 
\end{cases} 
\]
\[ 0 \leq \chi(z) \leq 1, |\nabla \chi| < 16, |\Delta \chi| \leq 256, |z| \in \mathbb{C}\setminus (R_1 \cup R_2). \quad (3.11) \]
We define the following function
\[ u_2(z) = \chi u_1 + (1 - \chi)u_0. \quad (3.12) \]
We prove the following

**Proposition 3.5.** Function $u_2$ satisfies the following:
\begin{enumerate}
\item $u_2$ is smooth away from $z_2, \ldots, z_n \in \{|z| < 1/4\}$;
\item $u_2 \sim \beta_i \ln |z - z_i|$ as $z \to z_i$, for $i = 2, \ldots, n$;
\item $u_2 \sim -(\beta_1 + 2) \ln |z|$ as $|z| \to z_1 = \infty$;
\item In the region where $u_2$ is smooth, there exists a constant $C'' = C''(\alpha, \beta_1) > 0$ such that
\[ (\beta_1 + 1)^2 - C'' \epsilon \leq -e^{-2u_2}\Delta u_2 \leq (\alpha + 1)^2 + C'' \epsilon; \quad (3.13) \]
\end{enumerate}

**Proof.** Since $u_2(z) = u_0(z)$ for $z \in R_2$, and $u_2 = u_1$ for $z \in R_1$, parts (1), (2) and (3) are obvious. Apply (3) of Proposition 3.4, we can prove (4) for $z \in R_1 \cup R_2$. 


Thus, to prove Proposition 3.5, we just need to verify part (4) for $z \in R = \mathbb{C}\setminus(R_1 \cup R_2) = \{5/8 \leq |z| \leq 3/4, \text{or } 5/4 \leq |z| \leq 11/8\}$. Notice that $u_2 - u_0 = \chi(u_1 - u_0)$, thus by (2) and (3.11), we have a finite constant $C' > 0$, for any $z \in \mathbb{R}$ such that $|z| \neq 1$,

$$|\Delta(u_2 - u_0)| = |\chi \Delta(u_1 - u_0) + 2\nabla \chi \nabla(u_1 - u_0) + (\Delta \chi)(u_1 - u_0)| \leq C' \epsilon,$$

which leads to

$$|e^{-2u_2} \Delta u_2 - e^{-2u_0} \Delta u_0| \leq e^{-2u_2} |\Delta(u_2 - u_0)| + |(e^{-2u_0} - e^{-2u_2}) \Delta u_0| \leq e^{-2m+2\epsilon} C' \epsilon + |e^{-2u_0} \Delta u_0(1 - e^{-2(u_2-u_0)})| \leq C'' \epsilon.$$

The last inequality follows from the fact that $-e^{-2u_0} \Delta u_0(z) = (\beta_1 + 1)^2$ or $(\alpha + 1)^2$ for $z \in R$. Thus we have proved part (4) in $R$.

Combine Propositions 3.3, 3.4 and 3.5, we have effectively proved Theorem 3.1. □

Theorem 1.3 thus easily follows Proposition 2.1 and Theorem 3.1. □

4. Convergence

In this section, we prove Theorem 1.4. Tracing the proof of Proposition 2.1, it is easy to check that if the best constant $\rho_0$ is achieved by a conformal factor $u$, level sets of $u$ are concentric round circles; thus, $u$ has to be radially symmetric. A bit further computation shows that $u$ has to be the conformal factor of a glued football, which is $C^{1,1}$ and piece-wise smooth away from one or two singular points. This fact was pointed out by Chen-Lin [ChLi] and Bartolucci [B] for the single and double singular points cases respectively using the symmetric rearrangement argument. In multiple conic points cases, equality case of Proposition 2.1 cannot be expected. We will do a finer analysis on the isoperimetric defect to show that all but one of singular points merge to one conic point when the best pinching constant $\rho_0$ is approximated.

In [FL], we have described exactly this kind of merging behavior for conic 2-spheres with constant curvature metrics. We thus follow the arguments given in [FL], pointing out only the necessary modification for the supercritical case.

First, we prove a technical lemma.

**Lemma 4.1.** Let $u \in C^\infty(\Omega)$ be a solution of the Dirichlet problem in a bounded region $\Omega \subset \mathbb{C}$,

$$\begin{cases}
\Delta u = -Ke^{2u}, & \text{in } \Omega; \\
u = s, & \text{on } \partial \Omega,
\end{cases}$$
where $K$ is a positive continuous function with $0 < a \leq K \leq b$. Let $\Omega_t := \{u > t\} \subset \Omega$, $A(t) = \int_{\Omega_t} Ke^{2u}$, $B(t) = |\Omega_t|$, and $H = \max_{z \in \Omega} u(z)$, then

$$A(t) \geq \frac{4a\pi}{b}(1 - e^{t-H}),$$

moreover for $A(t) \geq \frac{2a\pi}{b}$, we have, for any $t \geq s$,

$$B(t) \geq \frac{4a\pi}{b^2}(e^{-t-H} - e^{-2H}).$$

Proof. We are in a similar but simpler set up as that of Proposition 2.1, as no singularity appears here. In particular, similar to (2.8), we have

$$be^{2t}B' \leq A' \leq ae^{2t}B'.$$  \hspace{1cm} (4.1)

Thus we follow the proof of Lemma 3.4 of [FL] to get

$$(A^2)' = 2(\int_{\partial\Omega(t)} |\nabla u|)(-e^{2t}\int_{\partial\Omega(t)} K|\nabla u|) \leq -2e^{2t}a|\partial\Omega(t)|^2 \leq -8a\pi e^{2t}B.$$  \hspace{1cm} (4.2)

We integrate (4.2) from $t$ to $H$ to get

$$A^2(t) \geq 8a\pi \int_t^H e^{2\mu}B(\mu)d\mu.$$  \hspace{1cm} (4.3)

On the other hand, integrating (4.1), we get

$$-A(t) \geq -\int_t^H 2be^{2\mu}B(\mu)d\mu - be^{2t}B(t).$$  \hspace{1cm} (4.4)

Thus we combine (4.3) and (4.4) to get

$$\frac{b}{4a\pi}A^2 \geq A - be^{2t}B$$

$$\geq \frac{b}{4a\pi}AA' + A,$$  \hspace{1cm} (4.5)

which leads to

$$A(t) \geq \frac{4a\pi}{b}(1 - e^{t-H}).$$

When $A \geq \frac{2a\pi}{b}$, by (4.5),

$$B(t) \geq \frac{1}{b}e^{-2t}A(1 - \frac{b}{4a\pi}A) \geq \frac{4a\pi}{b^2}[e^{-t-H} - e^{-2H}].$$

$\square$
Remark 4.2. As in [FL], Lemma 4.1 is used to show the uniform upper bound for conformal factors in consideration. A more general form of such estimates has been obtained by Brezis-Merle [BM].

We can now start the proof of Theorem 1.4.

Proof. We write
\[ g_i = e^{2u_i}g_0, \]
which has conic singularity along the divisor $D$. We will normalize $u_i$ later. Following notations of our earlier works [FL], we define
\[ \Omega_i(t) := \{ u_i > t \} \subset \mathbb{C}, \quad A_i(t) := \int_{\Omega_i(t)} Ke^{2u_i}, \quad B_i(t) := |\Omega_i(t)|, \]
where all integrals are with respect to the Euclidean metric $g_0$. Note that, under our set-up, the Gauss-Bonnet formula can be written as
\[ \int_{\mathbb{R}^2} K(g_i) e^{2u} = 2\pi(2 + |D|) = \lim_{t \to -\infty} A_i(t); \quad (4.7) \]
while we also have that
\[ A_i(t) = \int_{\Omega_i(t)} Ke^{2u_i} = \int_{\partial \Omega_i(t)} |\nabla u_i| + 2\pi \alpha, \]
for any $t > -\infty$.

According to the proof in Proposition 2.1, we have
\[ B_i'(t) = -\int_{\partial \Omega_i(t)} \frac{1}{|\nabla u_i|}, \quad A'(t) = -e^{2t} \int_{\partial \Omega_i(t)} \frac{K(g_i)}{|\nabla u_i|}, \quad (4.8) \]
for $t$ almost everywhere.

Assuming that
\[ a_i = \min K(g_i) \leq K(g_i) \leq b_i = \max K(g_i). \quad (4.9) \]
By adding a proper constant to each $u_i$, we may assume $b_i = 1$, which leads to
\[ a_i \to \rho_0 = \frac{(1 + \beta_1)^2}{(1 + \alpha)^2}, \quad (4.10) \]
as $i \to \infty$. Thus, combining (4.8) and (4.9), we get
\[ 1 \leq \frac{e^{2t} B_i'(t)}{A_i'(t)} \leq \frac{(1 + \alpha)^2}{(1 + \beta_1)^2}. \quad (4.11) \]
Also by the isoperimetric inequality and the Hölder’s inequality, we have
\[ 4\pi B_i(t) \leq (\int_{\partial \Omega_i(t)} 1)^2 \leq (\int_{\partial \Omega_i(t)} |\nabla u_i|) \int_{\partial \Omega_i(t)} \frac{1}{|\nabla u_i|} = -B_i'(t)(A_i(t) - 2\pi \alpha). \quad (4.12) \]
Similar to the discussion in [FL], due to the non compact conformal transformation group, there are two families of normalization that we can apply to functions \( \{u_i\} \) without changing the geometric setting. Namely,

- scaling: \( u^{\lambda,0}(z) := u(\lambda z) + \ln \lambda \);
- translation: \( u^{0,k}(z) := u(z - k) \).

We choose the normalization so that for a generic \( t_0 \in \mathbb{R} \),

\[
A_i(\ln(\alpha + 1)) = 2\pi(\alpha + 1),
\]

the centroid of \( \Omega_i(t_0) \) is at 0.

Define

\[
D_i(t) = (\int_{\partial\Omega_i(t)} 1)^2 - 4\pi B_i(t)
\]

as the isoperimetric defect for the region \( \Omega_i(t) \). We have the following improvement of (2.11):

\[
4\pi B_i(t) + D_i(t) \leq -B_i'(t)(A_i(t) - 2\pi\alpha),
\]

which means

\[
- \frac{B_i(t)}{B_i'(t)} \leq \frac{A_i(t) - 2\pi\alpha}{4\pi} + \frac{D_i(t)}{4\pi B_i'(t)}.
\]

Thus, similar to (2.12)

\[
\frac{d}{dt}[e^{2t}B_i(t)] = e^{2t}B_i'(t) + 2e^{2t}B_i(t) = \frac{e^{2t}B_i'(t)}{A_i'(t)}A_i'(t)(1 + \frac{2B_i(t)}{B_i'(t)}) \\
\leq \frac{e^{2t}B_i'(t)}{A_i'(t)}A_i'(t)(1 - \frac{A_i - 2\pi\alpha}{2\pi}) - \frac{e^{2t}D_i(t)}{2\pi} \\
\leq \begin{cases} 
\frac{A_i}{a_i}(1 + \alpha - \frac{A_i}{2\pi}) - \frac{e^{2t}D_i(t)}{2\pi}, & A_i \leq 2\pi(1 + \alpha) \\
\frac{A_i}{a_i}(1 + \alpha - \frac{A_i}{2\pi}) - \frac{e^{2t}D_i(t)}{2\pi}, & A_i > 2\pi(1 + \alpha).
\end{cases}
\]

Integrating (4.15) from some \( t \leq \ln(\alpha + 1) \) to \( \infty \), we get

\[
\frac{1}{2\pi} \int_{t}^{\infty} e^{2s}D_i(s)ds - e^{2t}B(t) \leq \frac{1}{4a_i\pi}A^2(t) - \frac{1 + \alpha}{a_i}A(t) + \pi(1+\alpha)^2(\frac{1}{a_i} - 1).
\]

Taking a sequence \( t_n \to -\infty \) with \( e^{2t_n}B(t_n) \to 0 \), and noticing (4.7) and (4.10), we then get

\[
\int_{-\infty}^{\infty} e^{2t}D_i(t)dt \to 0, \quad \text{as } i \to \infty.
\]

Thus, away from a set \( S \) of measure 0, we have

\[
D_i(t) \to 0, \quad i \to \infty.
\]
In particular, we pick our \( t_0 \notin S \). A similar argument as given in Lemma 3.5 of \([FL]\) indicates that
\[
A_i(t) \to A(t), \quad B_i(t) \to B(t),
\]
and all inequalities in the discussion above will be equalities when passing to the limit. Therefore, by (4.14) and (4.15), \( A(t) \) and \( B(t) \) satisfy the following
\[
4\pi B(t) = -B'(t)(A(t) - 2\pi \alpha), \quad (4.16)
\]
\[
e^{2t} B(t) = \begin{cases} 
-\frac{A^2(t)}{4\rho_0^2} + \frac{1+\alpha}{\rho_0} A(t), & t \geq \ln(\alpha + 1); \\
-\frac{1}{4\rho_0^2} A^2(t) + \frac{1+\alpha}{\rho_0} A(t) - \pi(1+\alpha)^2 \left( \frac{1}{\rho_0} - 1 \right), & t < \ln(\alpha + 1),
\end{cases} \quad (4.17)
\]
\[
K(g_u) = \begin{cases} 
1, & t \geq \ln(\alpha + 1); \\
\rho_0, & t < \ln(\alpha + 1).
\end{cases} \quad (4.18)
\]
Combining these with proper initial conditions, we can compute \( A(t) \) and \( B(t) \) precisely. Readers are referred to \([FL]\) for explicit formulae. It is straightforward to see that they are given by the corresponding data of the glued football.

We now follow \([FL]\) to prove the Gromov-Hausdorff convergence. Let \( M_i(t) \) be the connected component of \( \Omega_i(t) \) with the largest area. Since \( D_i(t) \to 0 \), we apply Benneson’s inequality to get
\[
|M_i(t)| \to B(t).
\]
Due to the normalization, the centroid of \( \Omega_i(t_0) \) is 0, we conclude that \( M_i(t_0) \) converge to a round disc in Gromov-Hausdorff sense. Indeed, following Lemma 3.9 of \([FL]\), \( M_i(t) \) converges in Hausdorff distance to a disk \( D(t) \) for almost every \( t \). Let \( p_0 \) be the limit of center of \( D(t) \) as \( t \to \infty \). Passing to a subsequence if necessary, let \( p_1, \ldots, p_{n-1} \) be the possible limit points of \( n-1 \) conic points. Consider any compact set \( K \subset \mathbb{C} \setminus \{p_0, p_1, \ldots, p_{n-1}\} \). Following exactly the argument given in \([FL]\), with Lemma 4.1, we can show that there exist constants \( N \in \mathbb{N} \) and a uniform constant \( C_K \in \mathbb{R} \), such that \( \|u_i\|_{C^0(K)} \leq C_K \) for \( i \geq N \). Thus, by a standard bootstrap argument we have, up to a subsequence,
\[
u_i \longrightarrow u,
\]
in \( C^\infty(K) \) topology. It follows that the limit \( u \) must be radially symmetric, and its associated \( A(t) \) and \( B(t) \) are given by (4.16). We also have that \( p_0 = O \). A straightforward computation shows that \( u = u_{\alpha,\beta_1} \), the conformal factor of the glued football \( S^2_{\alpha,\beta_1} \).
It is now straightforward to see that $p_i = O$ for $i = 1, \cdots, n-1$, which means $n - 1$ of the conic points collapse into one.

The proof of Theorem 1.5 follows exactly the same line of those of Theorems 1.3 and 1.4 thus we omit it here.

References

[B] D. Bartolucci, *On the best pinching constant of conformal metrics on $S^2$ with one and two conical singularities*. J. Geom. Anal. 23 (2013), no. 2, 855-877.

[BDM] D. Bartolucci, F. De Marchis and A. Malchiodi, *Supercritical conformal metrics on surfaces with conical singularities*, Int. Math. Res. Not. IMRN 2011, no. 24, 5625-5643.

[BaMa] D. Bartolucci and F. De Marchis, *On the Ambjorn-Olesen electroweak condensates*, Jour. Math. Phys. 53 (2012), no. 7, 073704, 15 pp.

[BM] H. Brezis and F. Merle *Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions*, Comm. Partial Differential Equations 16 (1991), no. 8-9, 1223-1253.

[BZ] J. Brothers and W. Ziemer, *Minimal rearrangements of Sobolev functions*, J. Reine Angew. Math., 384 (1988), 153-179.

[CDS1] X. Chen, S. Donaldson and S. Song, *Kähler-Einstein metrics on Fano manifolds I: Approximation of metrics with cone singularities*, J. Amer. Math. Soc. 28 (2015), no. 1, 183-197.

[CDS2] X. Chen, S. Donaldson and S. Song, *Kähler-Einstein metrics on Fano manifolds. II: Limits with cone angle less than $2\pi$*, J. Amer. Math. Soc. 28 (2015), no. 1, 199-234.

[CDS3] *Kähler-Einstein metrics on Fano manifolds. III: Limits as cone angle approaches $2\pi$ and completion of the main proof*, J. Amer. Math. Soc. 28 (2015), no. 1, 235-278.

[ChLi] C.C. Chen and C.S. Lin, *A sharp sup+inf inequality for a nonlinear elliptic equation in $R^2$*. Commun. Anal. Geom. 6(1), (1998) 1-19.

[CL1] W. Chen and C. Li, *Prescribing Gaussian curvatures on surfaces with conical singularities*, J. Geom. Anal. 1 (1991), no. 4, 359-372.

[CL2] W. Chen and C. Li, *What kinds of singular surfaces can admit constant curvature?*, Duke Math. J., 78(1995) no.2, 437-451.

[D] S. Donaldson, *Kähler metrics with cone singularities along a divisor*, Essays in mathematics and its applications, 49-79.

[E] A. Eremenko, *Metrics of positive curvature with conical singularities on the sphere*, Proc. Amer. Math. Soc. 132 (2004), no. 11, 3349-3355.

[FL] H. Fang and M. Lai, *On convergence to a football*, 2015, [arXiv:1501.06881](http://arxiv.org/abs/1501.06881), to appear in Math. Ann.

[JMR] T. Jeffres, R. Mazzeo and Y. Rubinstein, *Kähler-Einstein metrics with edge singularities with an appendix by C. Li and Y. Rubinstein*, [arXiv:1105.5216v3](http://arxiv.org/abs/1105.5216v3).

[LT] F. Luo and G. Tian, *Liouville equation and spherical convex polytopes*, Proc. Amer. Math. Soc. 116(1992), no.4, 1119-1129.

[PSSW1] D.H. Phong, J. Song, J. Sturm and X. Wang, *Ricci flow on $S^2$ with marked points*, [arXiv:1407.1118](http://arxiv.org/abs/1407.1118).
[PSSW2] D.H. Phong, J. Song, J. Sturm and X. Wang, Convergence of the conical Ricci flow on $S^2$ to a soliton, arXiv:1503.04488.

[RT] J. Ross and R. Thomas, Weighted projective embeddings, stability of orbifolds and constant scalar curvature Kähler metrics, J. Diff. Geom. 88 (2011) No. 1, 109-159.

[Th] W. Thurston, The Geometry and Topology of Three-Manifolds, Princeton University Press, Princeton (1978). Chap. 13.

[Ti] G. Tian, K-stability and Kähler-Einstein metrics, Comm. Pure Appl. Math. (2015), no.7, 1085-1156.

[Tr] M. Troyanov, Prescribing curvature on compact surfaces with conical singularities, Trans. Amer. Math. Soc., 324(1991), no.2, 793-821.

14 MacLean Hall, University of Iowa, Iowa City, IA, 52242
E-mail address: hao-fang@uiowa.edu

800 Dongchuan RD, Shanghai Jiao Tong University, Shanghai, China, 200240
E-mail address: laimijia@sjtu.edu.cn