Definition of Mass for Asymptotically AdS space-times for Gravities Coupled to Matter Fields

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We give a general definition of mass for gravities coupled to matter fields. We study the gravity minimally coupled to a scalar field with weakened boundary conditions, with the masses calculated by the Hamiltonian formula and Wald's formula. We show that the masses calculated by these two formulas are equivalent to each other in this case, but are non-integrable. We then discuss the illness of this non-integrable mass and its failure to give interpretation to the entropy of some special solutions. We also show that Wald's formula cannot give the right mass for RN-AdS black holes. To solve these problems we introduce a not conserved scalar charge and develop a new definition for mass based on Wald's formula. The new definition is similar in spirit to both Wald's formula and the Hamiltonian formula, with the difference that we require the variation of the mass to have no contribution from the variation of the matter charges. This new definition is also valid for gravities coupled to matter fields with other charges.

Keywords:

I. INTRODUCTION

Although asymptotically AdS spacetimes have attracted remarkable interest in the last two decades, our understanding of their conserved charges, like energy, is still not clear enough, especially when gravity is coupled to matter fields and the boundary conditions don't preserve the full AdS symmetry. Several methods have been developed to define conserved charges in asymptotically AdS spacetimes: the Hamiltonian definition developed by Henneaux and Teitelboim [1], the AMD's (Ashtekar-Magnon-Das) conformal method [2, 3], the “counterterm subtraction method” [4–13]. There are also other methods like the KBL method [14, 15], the spinor method [15, 16] and the “pseudotensor” method [17], which we will not consider further in this paper. Based on the general “covariant phase space formalism” developed by Wald et. al. [18, 19], Hollands, Ishibashi and Marolf developed another method [20] to calculate the conserved charges in asymptotically AdS-spacetimes, and made a comparison between those methods mentioned above. They showed that, with asymptotically AdS boundary conditions and the matter fields decreasing fast enough when approaching the boundary, those methods are all equivalent to each other, except when the energy calculated by “counterterm subtraction method” has an additional constant term in some cases, which can be interpreted as the Casimir energy of the dual CFT.

These methods all have strict requirements – not only the boundary conditions have to be asymptotically AdS, but also the stress-tensor or matter fields should decrease fast enough when approaching boundary. However, in general these requirements can not be easily satisfied and the boundary conditions are usually weakened. For example, consider gravity minimally coupled to a scalar field \( \phi \) with a scalar mass \( m \) above or saturating the BF bound [21], with the Lagrangian (note that in this paper we use the convention \( 16\pi G = 1 \)) expressed as

\[
\mathcal{L} = \sqrt{-g} \left( R - \frac{1}{2} (\partial \phi)^2 - V(\phi) \right). \tag{1}
\]

In this case the AdS symmetry of the boundary conditions is in general not fully preserved which will be shown below, and the scalar field also decreases slower than the requirements mentioned above. One can find many examples in [22], where the asymptotic behaviour of the metric and scalar field are studied in different conditions. As mentioned in [23], when we use the static spherical metric ansatz,

\[
ds^2 = -h(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_{d-2}^2, \quad \phi = \phi(r), \tag{2}
\]

where \( d \) is the spacetime dimension and \( d\Omega_{d-2}^2 \) is the metric on the unit \((d-2)\)-sphere, the asymptotic scale invariance of the boundary conditions can’t be preserved when \( d \geq 4 \), which breaks the AdS symmetries on the boundary. With their requirements not satisfied, the methods mentioned above begin to give different masses, although they still give finite results in many cases. We can not trust their results when a scalar field is included, for details the reader can consult [22].

Under such weakened boundary conditions, some work [24–26] was done to calculate the conserved charges in Einstein-scalar gravities using the the Hamiltonian formula (or the Henneaux-Teitelboim construction). Soon after, the same results [27, 28] were obtained using Wald’s formula [18, 19]. The only requirement on boundary conditions for the Hamiltonian formula and Wald’s formula is to generate finite conserved charges, so it seems their results can be trusted. The energy has contributions from both the gravity sector and the scalar field.
sector. As the boundary conditions are weakened, both of the two contributions have divergent terms. However, when we add them together, the divergent terms cancel with each other, so a finite result comes out in the end. In the next section we give a brief review on the Hamiltonian formula and Wald’s formula and prove they are equivalent to each other in Einstein-scalar gravity.

However, the masses calculated in these two ways are not integrable, or in other words the variation of the mass can’t be written as a total variation, which makes the energy ill-defined. We will explain why we call it ill-defined in section III. We also discuss another problem of these definitions of mass, which is related to the interpretation of the entropy of some soliton solutions and some exact massless solutions with scalar hair. The validity of Wald’s formula to calculate the conserved energy of scalar hairy solutions was discussed in [27, 28], however we do not agree with their conclusions here. We use the RN-AdS (Reissner-Nordstrom) black holes as an example to prove that Wald’s formula is in fact not used properly for the scalar hairy solutions.

Wald’s formula is a very powerful method to study thermodynamics in gravities, especially when the gravities have coupled matter fields and weakened boundary conditions. Based on Wald’s formula, with a little modification, we give a new definition of mass for these theories including Einstein-scalar gravities, which is different from [27, 28]. In order to make this definition well defined we need to introduce a new scalar charge (section IV) whose existence has already been discussed previously from the black hole thermodynamics point of view [22, 29–31]. This definition gives the same results with the methods mentioned in the first paragraph when their requirements are strictly satisfied, and when the boundary conditions are weakened, the results are different from the definitions given by [24–28] and do not have the problems mentioned in the last paragraph.

II. HAMILTONIAN FORMULA AND WALD’S FORMULA

A. The Hamiltonian formula

The Hamiltonian formula is directly based on the construction of the Hamiltonian [32], which is the usual Hamiltonian supplemented by an addition of a surface integral on the boundary

$$\mathbf{H} = \int_\Sigma \xi^\mu \mathcal{H}_\mu + \text{surface integral terms}$$

(3)

where $\Sigma$ is a spacelike surface and $\mathcal{H}_\mu$ are the constraints. Imposing the constraints

$$\mathcal{H}_\mu = 0,$$

(4)

the energy is just given by the surface integral terms. The surface integral terms are defined on a variational level, and have contributions from both the gravity $\delta Q_G$ and the scalar field $\delta Q_\phi$. If we consider the Lagrangian (1), these two contributions are given by

$$\delta Q_G = \int dS_i G^{ijkl}(\xi^l D_j h_{kl} - \delta h_{kl} D_j \xi^l)$$

$$\delta Q_\phi = -\int dS_i \xi^l \delta \phi D^i \phi$$

(5)

(6)

where $G^{ijkl} = \frac{1}{2} g^{1/2}(g^{ik} g^{jl} + g^{il} g^{jk} - 2 g^{ij} g^{kl})$, $h_{ij} = g_{ij} - \bar{g}_{ij}$ is the deviation from the pure AdS spatial metric and $\xi^l = \xi \cdot n$ with $n$ the unit normal to $\Sigma$.

Both of $\delta Q_G$ and $\delta Q_\phi$ have divergent terms as the scalar field decays too slow when approaching the boundary, however the divergent terms cancel with each other when adding them together [24–26], thus we get a finite variation of energy.

In the next section we give a brief review on the Hamiltonian supplemented by an addition of a surface integral on the horizon

$$\mathbf{H} = \int H(\xi^\mu \mathcal{H}_\mu + \text{surface integral terms})$$

(7)

where $H = \frac{1}{\sqrt{-g}} \int d^d x \sqrt{-g} |\frac{\partial f + f \delta \phi}{r}| r \to \infty$, (7)

which gives the energy of black holes or solitons.

B. Wald’s formula

Here we write the Lagrangian as a $d$-form $\mathbf{L}$, and its variation can always be written in the form

$$\delta \mathbf{L} = \mathbf{E} : \delta \varphi + \delta \Theta,$$

(8)

where $\Theta$ is a $(d-1)$-form, $\varphi$ represents all the metric functions and the scalar field and $\mathbf{E}$ represents the equations of motion. The Noether current

$$\mathbf{J} = \Theta - \xi \cdot \mathbf{L}$$

(9)

can be shown [34] to satisfy

$$d\mathbf{J} = -\mathbf{E} \mathcal{L}_\xi \varphi,$$

(10)

which means that $\mathbf{J}$ is conserved when $\varphi$ satisfies the equations of motion. Hence $\mathbf{J}$ is a closed form and can be written as

$$\mathbf{J} = dQ.$$

(11)

It has been shown in [18] that when $\xi$ is a Killing vector, we have

$$d\delta Q - d(\xi \cdot \Theta) = 0,$$

(12)

which indicates the $d-2$-form $\delta Q - (\xi \cdot \Theta)$ is closed.

Taking $\xi = \partial / \partial t$ and applying this to our Lagrangian, the Stokes’ theorem indicates that the integral of $(\delta Q_\xi - \xi \cdot \Theta)$ over a spherical surface is independent of the radius. The integral on the horizon $S_r$ and on the boundary $S_\infty$ are the same, ie

$$\int_{S_r} (\delta Q_\xi - \xi \cdot \Theta) = \int_{S_\infty} (\delta Q_\xi - \xi \cdot \Theta).$$

(13)
If one is familiar with Wald’s formula, one would know that the equation above usually gives the thermodynamic first law of the solutions. The right hand side of (13) gives \( \delta M \) while the left hand side gives \( T \delta S \), so (13) just gives the standard first law \( T \delta S = \delta M \). The Wald’s formula definition of mass is just

\[
\delta M_W = \int_{S_\infty} (\delta Q - \xi \cdot \Theta). \tag{14}
\]

Here \( M_W \) is defined as the conjugate charge of the Killing vector \( \xi = \partial / \partial t \), its existence require the existence of a \((d - 1)\)-form \( B \) which satisfies

\[
\delta \int_{S_\infty} \xi \cdot B = \int_{S_\infty} \xi \cdot \Theta, \tag{15}
\]

so the variation can be written as a total variation. This makes sure that the mass can be integrated and is independent of the integral path. In [36], this requirement is reinterpreted as, for any variation \( \delta_1 \) and \( \delta_2 \) tangent to the space of solutions, the equation

\[
\int_{S_\infty} \xi \cdot (\delta_1 \Theta(\varphi, \delta_2 \varphi) - \delta_2 \Theta(\varphi, \delta_1 \varphi)) = 0, \tag{16}
\]

should be satisfied.

Let us now consider the Lagrangian (1). The \( d - 2 \) form \( \xi \cdot \Theta \) also has contributions from both the gravity \( \xi \cdot \Theta^G \) and the scalar field \( \xi \cdot \Theta^\phi \), so we have

\[
\xi \cdot \Theta^G_{i_1 \ldots i_{d-2}} = \epsilon_{i_1 \ldots i_{d-2} \mu \nu} (g^{\mu \nu} g^{r \theta} - g^{\mu r} g^{\nu \theta}) D_{\mu} \delta g_{\nu \theta}, \tag{17}
\]

\[
\xi \cdot \Theta^\phi_{i_1 \ldots i_{d-2}} = -\epsilon_{i_1 \ldots i_{d-2} \mu} (D_{\mu} \phi \delta \phi), \tag{18}
\]

\[
Q_{i_1 \ldots i_{d-2}} = \epsilon_{i_1 \ldots i_{d-2} \mu \nu} D_{\mu} \xi_{\nu}, \tag{19}
\]

where \( \epsilon \) is the Levi-Civita tensor. Applying the metric ansatz (2), we have [22, 31]

\[
\delta M_W = -\omega_{d-2} \mu^{d-2} \sqrt{\frac{h}{f}} \left( \frac{d - 2}{r} \delta f + f \delta \phi \right) \bigg|_{r \to \infty}, \tag{20}
\]

which is the same as (7).

The Wald’s formula definition of mass (14) gives the same result as the Hamiltonian formula for Einstein-scalar gravities.

\section*{III. ILL-DEFINED MASS}

\subsection*{A. Problem with a non-integrable mass}

We use the example in [35] to show this problem. Consider a 4-dimensional gravity (1) with a scalar mass \( m^2 = -2 \), which is above the BF bound. The scalar potential \( V(\phi) \) is given by

\[
V(\phi) = -2g^2 (\cosh \phi + 2), \tag{21}
\]

where \( g \) is the inverse of AdS radius \( g = 1/\ell \). By solving the linearised equations of motion of the scalar field, the large radius decay of the scalar field should be

\[
\phi(r) = \frac{\phi_1}{r} + \frac{\phi_2}{r^2} + \cdots. \tag{22}
\]

For soliton solutions, if we impose some metric ansatz like (2), the boundary condition \( \phi_2(\phi_1) \) of the scalar field is totally determined by the equations of motion. Boundary conditions of black hole solutions also depend on the radius of event horizon \( r_0 \), which means \( \phi_2 = \phi_2(\phi_1, r_0) \). On the plane of \((\phi_1, \phi_2)\) there is a region which contains all the solutions of the theory, which we call the space of solutions. Inside this space we can chose boundary conditions \( \phi_2(\phi_1) \) freely. This can be shown by Fig. (1), where the solutions with different \( r_0 \) are given. One thing we should keep in mind is that some points on \((\phi_1, \phi_2)\) plane may represent many different solutions, for example the origin \((0, 0)\) represents all Schwarzschild-AdS solutions with no scalar hair.

The energy \( M_H \) of these soliton solutions is calculated in [35] with the Hamiltonian formula on a variational level. Here we use (7) and get

\[
\delta M_H = 4\pi \left( -2\alpha - \delta \left( \frac{1}{3} \phi_1 \phi_2 + \phi_2 \delta \phi_1 \right) \right), \tag{23}
\]

where \( \alpha \) is defined by

\[
g_{tt} = \frac{r^2}{\ell^2} + 1 + \frac{\alpha}{r} + \cdots. \tag{24}
\]

From (23) we see that \( \delta M_H \) can’t be written as a total variation, so \( M_H \) cannot be expressed as a function of only \( \phi_1 \), \( \phi_2 \) and the metric. For a specific solution with specific metric and scalar field we cannot integrate out
its mass unless we also give a specific boundary condition \( \phi_2(\phi_1) \). A specific solution can be represented as a point on the \((\phi_1, \phi_2)\) plane and the boundary conditions can be represented as lines going through that point. Since there can be an infinite number of different lines passing through a point, talking about the mass of some specific solutions without boundary condition is meaningless if we define mass with (23). The choice of boundary conditions in the space of solutions can be quite arbitrary, so the mass of a specific solution is also arbitrary. This contradicts with our usual understanding of mass.

We can also use Wald’s formula to calculate the mass. Applied to our example, the variation of energy calculated by Wald’s formula (14) agrees with (23). The left hand side of (16) is also calculated in [27, 28] if we assume a boundary condition \( \phi_2(\phi_1) \). Here we just cite the result, which is proportional to

\[
\frac{\partial \phi_2(\phi_1)}{\partial \phi_1}(\delta_1 \phi_2 \phi_1 - \delta_2 \phi_1 \phi_1) = 0. \tag{25}
\]

This means only the variation along \( \phi_1 \) contributes and Eq. (16) holds. However this does not mean the energy is independent of the integral path because when we chose a boundary condition \( \phi_2(\phi_1) \), we have already chosen an integral path in the space of solutions. Hence we do not agree that satisfying Eq. (25) after choosing a boundary condition would make the definition of mass (14) valid.

To explain our arguments, we give a more convincing example, which is calculating the mass of 4-dimensional electrically charged RN-AdS black holes (we set the magnetic charge to be zero for simplicity). These RN black holes have two integration constants, which we can take as the mass \( M \) and the electric charge \( Q \), so the entropy now is a function of these two parameters. The black hole metric is given by

\[
ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\Omega_2^2, \tag{26}
\]

\[
f(r) = \frac{r^2}{\ell^2} + 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, \tag{27}
\]

where \( M \) is acknowledged as the black hole mass and \( Q \) is the electric charge. We can make a gauge choice such that the gauge field \( A \) vanishes on the horizon. Applying Wald’s formula to these charged black holes we have (for details, see [22, 37])

\[
\int_{S_{r_0}} (\delta Q \xi - \xi \cdot \Theta) = T \delta S, \tag{28}
\]

\[
\int_{S_{\infty}} (\delta Q \xi - \xi \cdot \Theta) = \delta M - \Phi(M, Q) \delta Q = \delta M_W. \tag{29}
\]

Wald’s formula gives the standard thermodynamic first law \( T \delta S = \delta M - \Phi \delta Q \). When we fix \( r_0 \), ie \( \delta r_0 = 0 \), only one independent parameter remains. If we take this parameter as the electric charge \( Q \), then \( \Phi \) and \( M \) are both functions of \( Q \). In this situation, the integral function on the left hand side of (16) is

\[
\frac{\partial \Phi(Q)}{\partial Q}(\delta_1 Q \delta_2 Q - \delta_2 Q \delta_1 Q) = 0, \tag{30}
\]

which means Eq.(16) holds. If we follow the logic of [27, 28], then the mass definition (14) is valid here, and surface integration (29) should be the variation of the mass. However, in fact, the integration result is \( \delta M - \Phi(Q) \delta Q \), rather than the variation of the right mass \( \delta M \). This means the mass definition (14) does not work here.

In our opinion, fixing the horizon radius \( r_0 \) equals to picking out an integration path in the space of solutions, which we should not do before we check Eq. (16). If we do not fix \( r_0 \), then the integral function on the left hand side of (16) is

\[
\frac{\partial \Phi(Q, M)}{\partial M} \delta M \delta Q \neq 0, \tag{31}
\]

where we have defined \( \delta_i \) to be a variation along \( M \) and \( \delta_2 \) along \( Q \). We can bow say that the mass definition (14) based on Wald’s formula does not work for these RN black holes, and also does not work for the scalar hairy solutions.

B. The entropy problem of some massless solutions

The Hamiltonian formula and Wald’s formula definitions of mass have another problem, which is its failure to interpret the entropy of some scalar hairy massless solutions. If the mass is the only independent charge of the solutions, then the entropy \( S(M) \) counts the number of microscopic states which have a total mass \( M \). On the other hand \( S \) is proportional to the area \( A \) of the event horizon \( S = \frac{A}{4G} \), which can be determined by the metric. If the definition of mass is wrong, the entropy calculated in these two ways may not be consistent with each other.

We find that there are theories which admit solutions with zero mass calculated with Wald’s formula or the Hamiltonian formula, and zero entropy calculated by \( S = \frac{A}{4G} \), however they also contain an integration constant which is not related to the mass and entropy. Each solution should contain at least one microscopic state, which means that there is an infinite number of microscopic massless states while the entropy is zero. The entropy proportional to the horizon area and the entropy counting the number of microscopic states are now different with each other, which indicates that the definition of mass might not be right.

The first example is the class of 4-dimensional static spherical soliton solutions in [35]. They impose the metric ansatz (2) to the theory (1), fix the radius of event horizon to \( r_0 = 0 \), then numerically solve the equations of motion. Adjusting the value of the scalar field at origin \( \phi(0) \) gives different solitons, which can be represented as a soliton line (the solid line in Fig.(1)) \( \phi_2(\phi_1) \) on the \((\phi_1, \phi_2)\) plane. If we chose this line as the boundary condition, Wald’s formula (13) gives

\[
\delta_s M_W = 4\pi \left(-2\delta_1 \alpha - \delta_2 \left(\frac{1}{3} \phi_1 \phi_2 + \phi_2 \delta_1 \phi_1\right)\right) = 0, \tag{32}
\]

where \( \delta_s \) means variation along the soliton line. Since the soliton line goes through the \((0,0)\) point on \((\phi_1, \phi_2)\)
plane, which is the pure vacuum $AdS_4$ spacetime, we find that according to (32) all the solutions on the soliton line are massless (if we define the $AdS_4$ vacuum as massless).

It is surprising that the scalar hair does not affect the mass of solitons at all. Furthermore, the number of states (soliton solutions) with zero mass is infinite, while the entropy is zero, this contradicts with the statistical understanding of entropy.

Another example is given by 3-dimensional scalar hairy massless black holes (their line elements share the same causal structure with the standard massless 3-d black hole) constructed in [23]. Theories (1) with scalar potentials

\[ V(\phi) = g^2 \left( \cosh^{4\mu} \psi \right) \left( \mu \tanh^2 \psi - 2 \right), \]

\[ \psi = \frac{\phi}{2\sqrt{2\mu}}, \]

admit massless (according to the Hamiltonian formula or Wald’s formula) black hole solutions

\[ \phi(r) = 2\sqrt{2\mu} \frac{\text{Arctanh} \frac{1}{\sqrt{1 + r/q}}}{}, \]

\[ ds^2 = -g^2 r^2 dt^2 + \frac{1}{g^2 r^2 (q/r + 1)^2} dr^2 + r^2 d\theta^2, \]

where $\mu$ is a positive constant which marks different theories and $q$ is the only integration constant which describes the scalar hair. We need to point out that solutions described by (34) only represent a line in the space of solutions. If we only consider the solutions (34), we have already picked out an integration path, so we can integrate (14) and get the mass defined by Wald’s formula $M_W = 0$.

Just like in the previous soliton solutions, there is an integration constant $q$ which is not related to the mass, which means the massless state is degenerate and should have a non-zero entropy, however these black hole solutions all have $r_0 = 0$ so have vanishing entropy $S = 0$.

Hence we think that the mass calculated by (14) or the Hamiltonian formula is not right, as the mass should depend on the scalar hair in such a way that these solutions are not all indistinguishable massless black holes. Besides the mass we need another quantity to determine the macroscopic thermodynamic properties of these solutions, which we call a scalar charge.

IV. INTRODUCING A NEW SCALAR CHARGE

A. The existence of a new charge attached to scalar field

The existence of the scalar charge has already been discussed not only in asymptotically AdS spacetimes with weakened boundary conditions [22, 30, 31] but also in asymptotically flat spacetimes [29]. The authors of these papers assume the existence of such a charge because their definitions of mass are different from the Hamiltonian formula and Wald’s formula.

For example, consider again the 4-dimensional theory (1) with the scalar potential (21) in section III A. If we define the AMD mass $M_A$ as the right mass (as in [22]), then the variation of the non-integrable mass $M_W$ or $M_W$ can be expressed as

\[ \delta M_W = \delta M_A + \frac{4\pi}{3m^2} (2\phi_2 \delta \phi_1 - \phi_1 \delta \phi_2), \]

This looks just like the non-integrable $\delta M_W$ (29) of the RN black holes. It is thus natural to consider the second term of the right hand side of Eq. (35) as a contribution from a scalar charge

\[ \frac{4\pi}{3m^2} (2\phi_2 \delta \phi_1 - \phi_1 \delta \phi_2) = \Phi_S \delta Q_S, \]

where $Q_S$ is defined as the new scalar charge and $\Phi_S$ as the scalar potential. Wald’s formula (13) then gives the familiar first law

\[ T\delta S = \delta M_A + \Phi_S \delta Q_S, \]

which have the same formula as the first law of electrically charged RN black holes.

One interesting fact is that, as mentioned in [22, 30, 38], when the boundary conditions preserve all the AdS symmetries, i.e. $\phi_2 = 0, \phi_1 = 0$ or $\phi_2 = c\phi_1^2$ [25] (where $c$ is an arbitrary constant), the contribution from the scalar charge $\Phi_S \delta Q_S$ in (36) vanishes. We can see this directly from (35) and get $\delta M_W = \delta M_A$. It seems that these special boundary conditions makes $\delta M_W$ integrable and trustful, however in these cases the boundary conditions are also chosen before we calculate the mass, so $M_W$ still depends on the integration path.

The non-integrability of $\delta M_W$ not only happens in such Einstein-scalar gravities and RN-black holes, but also in many other gravities coupled to other matter fields [39–43]. These theories clearly show that when there exist charges which are not the Noether charge of any transformation along Killing vectors, the integral (14) would be non-integrable, which means the Hamiltonian conjugate to time translation does not exist. Compared with these theories, assuming the existence of a scalar charge is reasonable.

Another reason to assume the existence of the scalar charge $Q_S$ is that the full space of solutions has two independent integration constants, so it is hard to believe that energy is the only charge. If the scalar charge $Q_S$ exists, the macroscopic thermodynamic properties of the solutions of gravity should be described by two parameters $M$ and $Q_S$, so the entropy $S(M, Q_S)$ would count the microscopic states which have mass $M$ and scalar charge $Q_S$. We will see this entropy can be consistent with the entropy calculated by $S = \frac{A}{4\pi^2}$.

B. Solutions with a manifest scalar charge

We now show a class of 3-dimensional scalar hairy soliton like solutions (the metric is regular everywhere, while the scalar field has a log divergence at the origin) constructed in [23], which have a manifest scalar charge term
in the thermodynamic first law. The Lagrangian (1) with a scalar potential
\[ V(\phi) = \frac{g^2 \cosh^{\phi} \psi}{64} \times \]
\[ \{ \cosh[6\psi] - 81 \cosh[2\psi] - 6(7 + \cosh[4\psi]) \}, \quad (38a) \]
\[ \psi = \frac{\phi}{2\sqrt{6}}, \quad (38b) \]
admits the solutions
\[ ds^2 = -\frac{g^2 r^2 (q^2 + 4qr + 2r^2)}{2(q + r)^2} dt^2 \]
\[ + \frac{2r^4}{g^2 (q + r)^2 (q^2 + 4qr + 2r^2)} dr^2 + r^2 d\theta^2, \quad (39a) \]
\[ \phi(r) = 2\sqrt{6} \arctanh \frac{1}{\sqrt{1 + r/q}}. \quad (39b) \]
Again, these solutions are not the whole space of solutions, they are in fact on an integration path when we integrate \( \delta M_W \). Applying Wald’s formula we have
\[ \delta M_W = \int_{S_{\infty}} (\delta Q_\xi - \xi \cdot \Theta) = \frac{2\pi}{\ell^2} q \delta q, \quad (40) \]
while the integral on the infinitesimally small ball around the origin is also
\[ \int_{S_{\infty}} (\delta Q_\xi - \xi \cdot \Theta) = \frac{2\pi}{\ell^2} q \delta q. \quad (41) \]
The AMD masses \( M_A \) of these solutions are also finite and equal to the masses calculated by (14), \( M_A = M_W \). It has been mentioned in [23] that for solitons, the non-vanishing of the integral (41) only happens in 3-dimensional spacetime. As the solutions have no event horizon, the integral (41) cannot be interpreted as \( T \delta S \), so we interpret it as a contribution from scalar charge \( \Phi_S \delta Q_S \) instead. This charge term is not an integral on the boundary, so even if we can accept a non-integrable mass, it cannot be absorbed by \( \delta M_W \). Hence this scalar charge term is always manifest in the first law
\[ \delta M_W = \Phi_S \delta Q_S. \quad (42) \]
The existence of these solutions gives further evidence to the existence of scalar charge.

V. A NEW MASS DEFINITION

In section III we discussed the problems of the masses defined by the Hamiltonian formula and Wald’s formula (14). They all define the mass on a variational level, and the results are non-integrable when matter fields are included. Choosing an integration path, \( \delta M_W \) can be integrated and can give a finite mass, however the integrated mass depends on the integration path (or the boundary condition) we choose. The entropy problem of some solutions also indicates the masses defined in these ways are not right. We think Wald’s formula is not used properly by the definition (14). In this section, based on Wald’s formula, we propose a new way to define the mass for theories including matter fields.

We first need to find out why the variation of mass calculated by (14) becomes non-integrable when matter fields are included. Again we use the electrically charged RN-AdS black holes (26) as an example. Unlike the energy \( M \) and angular momentum \( J \), the electric charge \( Q \) is not a Noether charge conjugated to a Killing vector, instead it’s conservation is based on the Bianchi identity. Hence when we calculate \( \delta M_W \), \( Q \) would be mixed in as a pure number, while \( J \) won’t appear. The variation \( \delta \) in (14) is general and \( \delta Q \neq 0 \), so the term proportional to \( \delta Q \) will appear in \( \delta M_W \), which can be clearly seen in (29).

However, if we take \( M_W \) as the right mass, Eq. (29) means that the variation along the mass has a contribution from the variation along the electric charge, which is wrong because we take the mass \( M \) and electric charge \( Q \) as two independent parameters in the space of solutions. We can define a variation \( \tilde{\delta} \) which is only along the \( M \) direction, then we should have
\[ \delta Q = 0. \quad (43) \]
If \( M_W \) is the right mass \( M \), it should satisfy
\[ \delta M_W = \delta M_W, \quad (44) \]
which means that (29) is then modified to
\[ \tilde{\delta} M_W = \delta M - \Phi(M, Q)\delta Q = \tilde{\delta} M. \quad (45) \]
So after we confine the \( \delta \) in (14) to be \( \tilde{\delta} \), Wald’s formula can give the right mass \( M \) for these RN black holes.

We now arrive at our new definition of mass \( M_N \),
\[ \tilde{\delta} M_N = \int_{S_{\infty}} (\delta Q_\xi - \xi \cdot \Theta(\varphi, \tilde{\phi} \varphi)). \quad (46) \]
with the variation \( \tilde{\delta} \) only along \( M_N \) in the space of solutions. It is quit natural to define the mass in this way because it is totally independent from the other charges, so its variation should not have any contribution from the variations of the other charges. This definition obviously gives the right mass for the RN-AdS black holes.

To test this new definition, we can use it to calculate the mass of the scalar hairy black holes. For these black holes we do not know what the real mass is (we do not agree with the mass defined by the Hamiltonian formula and (14)), so we just write the right hand side of (46) as a total variation plus a charge term \( \Phi_S \delta Q_S \) and apply \( \delta Q_S = 0 \) and then we take the remaining total variation as the variation of the real mass.

Based on the analysis in [22], we consider \( d \)-dimensional gravities minimally coupled to a scalar field with a scalar mass \( m^2 \) such that
\[ -\frac{1}{4\ell^2}(d-1)^2 < m^2 < -\frac{1}{4\ell^2}(d-1)^2 + \frac{1}{4\ell^2}, \quad (47) \]
and a general scalar potential \( V(\phi) \) admitting a Taylor expansion of the form
\[ V(\phi) = -\frac{(d-1)(d-2)}{\ell^2} + \frac{1}{2} m^2 \phi^2 + \gamma_3 \phi^3 + \gamma_4 \phi^4 + \cdots. \quad (48) \]
Let us apply (2) as the metric ansatz and define
\[ \sigma = \sqrt{4\ell^2 m^2 + (d - 1)^2}. \] (49)

The asymptotics of the metric functions and scalar field will take the form
\[ \phi = \frac{\phi_1}{r^{(d - 1 - \sigma)/2}} + \frac{\phi_2}{r^{(d - 1 + \sigma)/2}} + \cdots, \] (50a)
\[ h = g^2 r^2 + 1 - \delta_{d,3} + \frac{\alpha}{r^{d - 3}} + \cdots, \] (50b)
\[ f = g^2 r^2 + 1 - \delta_{d,3} + \frac{b}{r^{d - 3 - \sigma}} + \frac{\beta}{r^{d - 3}} + \cdots. \] (50c)

Substituting the expansions (50) into the equations of motion and solving for the first few coefficients, we find that
\[ b = \frac{(d - 1 - \sigma) \phi_1^2}{4(d - 2)\ell^2}, \] (51a)
\[ \beta = \alpha + \frac{(d - 1)^2 - \sigma^2}{2(d - 1)(d - 1 - \sigma)} \phi_1 \phi_2. \] (51b)

Using Eq.(20), the integration \( \int_{S} (\delta Q - \xi \cdot \Theta(\varphi, \delta \varphi)) \) on the boundary gives
\[ \delta M_N = \omega_{d-2} \left[ -(d - 2)\delta \alpha + \frac{\sigma}{2(d - 1)\ell^2} [(d - 1 + \sigma) \phi_2 \delta \phi_1 - (d - 1 - \sigma) \phi_1 \delta \phi_2] \right] \\
= \omega_{d-2} \left[ \delta \left( -(d - 2)\alpha - \frac{\sigma(d - 1 + \sigma)}{2(d - 1)\ell^2} \phi_2 \phi_1 \right) + \frac{\sigma}{\ell^2} \phi_2 \delta \phi_1 \right] \\
= \delta \left[ \omega_{d-2} \left( -(d - 2)\alpha - \frac{\sigma(d - 1 - \sigma)}{2(d - 1)\ell^2} \phi_1 \phi_2 \right) \right]. \] (52)

The new integrable mass \( M_N \) is given by
\[ M_N = \omega_{d-2} \left( -(d - 2)\alpha - \frac{\sigma(d - 1 - \sigma)}{2(d - 1)\ell^2} \phi_1 \phi_2 \right). \] (53)

This new mass \( M_N \) is different from the AMD mass
\[ M_A = -\omega_{d-2}(d - 2)\alpha. \] (54)

The mass \( MC \) calculated by the “counterterm subtraction method” is also different from our new mass. It has a free parameter, which means this method cannot give a unique mass. Under the condition (47) the free parameter can be chosen to give a mass equal to the AMD mass \( M_A \) or our new mass \( M_N \). For other \( m^2 \), the solutions may contain logarithmic functions, but under these conditions, the free parameter can be fixed by the cancellation of the logarithmic divergent terms in the action, however the mass \( MC \) is then different from both \( M_A \) and \( M_N \). The details for the calculation of \( MC \) can be found in [22].

The thermodynamic first law (14) is now given by
\[ T \delta S = \delta M_N + \frac{\omega_{d-2}\sigma}{\ell^2} \phi_2 \delta \phi_1 \]
\[ = \delta M_N - \Phi_S \delta Q_S \] (55)
where we define
\[ Q_S = \phi_1 \quad \Phi_S = -\frac{\omega_{d-2}}{\ell^2} \sigma \phi_2. \] (56)

The 3-dimensional massless black holes described by (33) and (34) have a scalar mass \( m^2 = -\frac{1}{\ell^2} (d - 1)^2 + \frac{1}{\ell^2} = -\frac{3}{\ell^2} \) which is outside (47), but the solutions (34) can still be described by (50) and do not contain any logarithmic functions, so the general analysis above is still valid here. The solutions (34) correspond to
\[ \phi_1 = 2\sqrt{2\mu q} \quad \phi_2 = -\frac{\sqrt{2\mu q^3}}{3} \quad \alpha = 0, \] (57)
so according to (53), the masses we defined are
\[ M_N = -\frac{\pi}{2\ell^2} \phi_1 \phi_2 = \frac{2\pi \mu q^2}{3\ell^2}. \] (58)

Note that \( \mu \) is a parameter in the Lagrangian, so is not an integration constant. Using our definition of mass (46) we see the solutions (34) are actually massive black holes, and their mass is related to the scalar hair. The solution with
\[ M_N = \frac{2\pi \mu q^2}{3\ell^2} \quad Q_S = 2\sqrt{2\mu q}, \] (59)
is now unique, and could correspond to a quantum state, thus give \( S = 0 \). This solves the entropy problem we presented in Section III.

Solutions described by (38) and (39) also have masses \( M_N \) different from \( M_W \) and \( M_A \).

The newly defined mass \( M_N \) is integrable and does not have the entropy problem mentioned above. We just require the variation \( \delta \) in (14) to be \( \delta \), thus weaken the requirements (15) and (16) for the existence of a well defined energy. Our definition (46) is consistent with the spirit of Wald’s construction. The Hamiltonian formula can also be modified in this way as it is also defined on a variational level.

For pure gravities, our definition will reduce to be the same as the other definitions. On the other hand, keeping the variation \( \delta \) general in (13) still gives the thermodynamic first law.
VI. DISCUSSION

When gravity is coupled to more than one matter field $A_i$, Wald’s formula will inevitably have terms proportional to the variation of the matter charges $\delta Q_i$, which make the variation $\delta M_W$ non-integrable (the same will happen to the variation of mass $\delta M_H$ defined by the Hamiltonian formula), because the matter charges are not a Noether charge generated by Killing vectors and the variation $\delta$ is general. So for these gravities Wald’s (20) formula will give

$$T\delta S = \delta M(\phi) - \sum_i \Phi_i \delta Q_i$$  \hspace{1cm} (60)$$

where $M$ is a function of the metric and matter fields, and $\delta M_W$ becomes non-integrable

$$\delta M_W = \delta M(\phi) - \sum_i \Phi_i \delta Q_i.$$  \hspace{1cm} (61)

Hence we need much more restrictive boundary conditions to make $\delta M_W$ integrable and get a much more complex mass. If we do not give any boundary condition, then the non-integrability of $\delta M_W$ indicates that a well defined energy does not exist. However, as $\partial / \partial t$ is still a Killing vector and every term in the thermodynamic first law (Wald’s formula) (60) is finite, we believe the well-defined mass should exist. The real mass should be a charge totally independent from the other charges so we need to replace $\delta$ with $\tilde{\delta}$, giving

$$\tilde{\delta} M_N = \tilde{\delta} M(\phi)$$  \hspace{1cm} (62)

which means $M(\phi)$ is the right mass.

Although we use a spherically symmetrical metric ansatz (2) through out this paper, our definition should also work for other geometries. These arguments can be checked by looking at the results of some recent papers [39–43], which study the thermodynamics of some multi-charged gravities with different geometries. Our definition is also consistent with the fact that Wald’s formula (20) gives the thermodynamic first law (60) of the black holes.

For Einstein-scalar gravities with weakened boundary conditions, we need to introduce a scalar charge to calculate the mass $M_N$. However, as mentioned in [29], this scalar charge is not conserved, as we cannot construct a conserved current from a scalar field the way we construct the Bianchi identity from the gauge fields.

In [44–46] the free energy of some exact scalar hairy black-holes has been calculated and compared with the free energy of the corresponding Schwarzschild AdS black holes at the same temperature. The results show that the Schwarzschild AdS black hole solutions are always thermodynamically preferred, which means the scalar hairy black holes will always decay into Schwarzschild AdS black holes at last. This is not surprising as we have mentioned that the scalar charge is not conserved.

The calculation would be straightforward if all the matter charges were conserved (for example the RN black holes) and could be calculated precisely from the solution of the metric and matter fields. We can always write Wald’s formula in the form (60) with some Legendre-like transformations, then read the mass $M_N$ and charge potentials $\Phi_i$ from Wald’s formula. However when there are non-conserved charges (for example the scalar hair), we can chose the parameter which describes the hair as the charge, just as we did in the previous section for the scalar hairy black holes.

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