A Generalised Harbourne-Hirschowitz Conjecture

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Abstract

We give a generalised Harbourne-Hirschowitz conjecture which suggests a test for determining when a linear system on a generic rational surface separates $k$-clusters. In particular when it is base point free or very ample.

1 Introduction

Recall the general context in which the Harbourne-Hirschowitz conjecture is formulated. This is best presented as a conjecture for generic rational surfaces which we define as follows. Fix a field $k$ and for an integer $r \geq 0$, let $k_r$ be the residual field at the generic point of $(\mathbb{P}_k^2)^r$. We then have the blowing-up $\pi_r : \mathbb{P}_r \rightarrow \mathbb{P}_{k_r}$ in the tautological sequence of points $x_1, \ldots, x_r$ in the projective plane $\mathbb{P}_{k_r}$. One says that $\mathbb{P}_r$ is the generic rational surface of rank $r$, or the blowing-up of $\mathbb{P}_k^2$ in $r$ generic points.

As usual, a curve $E$ on $\mathbb{P}_r$ with $E \cong \mathbb{P}^1$ and $E^2 = -1$ is called an exceptional curve on $\mathbb{P}_r$. Since $\mathbb{P}_r$ is rational, for any divisor class $H$ on $\mathbb{P}_r$, either $h^0(\mathbb{P}_r, O(H)) = 0$ or $h^2(\mathbb{P}_r, O(H)) = 0$. One says that an effective divisor class $H$ is non-special if $h^1(\mathbb{P}_r, O(H)) = 0$.

In slightly different forms, [13] Harbourne and [9] Hirschowitz conjectured that (see [1], [4], [5], [15] for work on the conjecture)

**Conjecture (H-H) (Harbourne-Hirschowitz)** The effective divisors $H$ on $\mathbb{P}_r$ that are non-special are those that satisfy $H.E \geq -1$ for all exceptional curves $E$ on $\mathbb{P}_r$.

This can also be stated in the following form: if $H$ is an effective, special divisor on $\mathbb{P}_r$, then $H = 2E + H'$ where $H'.E \leq 0$. In particular $H$ is non-reduced.

**Remark.** More recently, it has been pointed out by Ciliberto and Miranda that this conjecture is in fact equivalent to an older conjecture of Segre which says that a special effective divisor is non-reduced.

Our purpose here is to propose a conjecture in the same vein for separation properties such as base point freeness and very ampleness. In [8], when studying certain classes of very ample and base point free divisors, the authors tried unsuccessfully to propose a suitable conjecture. In [12], [13] and [7], the respective authors showed that in the case $r \leq 9$, base point freeness, very ampleness and more general separation properties were characterised by intersection numbers on a small class of curves, but no general conjecture was forthcoming.

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In accordance with known results for \( r \leq 9 \) the following conjecture says that if a divisor class \( H \) with sufficiently many sections does not separate \( k \)-clusters then there is an integral (even smooth and irreducible) curve \( E \) such that \( H \) does not separate some \( k \)-cluster in \( E \). Furthermore, evidence would suggest that this failure must be detectable numerically. Before formulating the conjecture we need the following

**Definition. 1.1** A \( k \)-cluster \( Z \) on \( X_r \) is any finite closed subscheme of \( X \) of length (i.e. degree) \( k \). We say that an effective divisor class \( H \) on \( X \) separates \( k \)-clusters if \( h^0(\mathcal{I}_Z(H)) = 0 \).

**Conjecture 1.** Let \( k \geq 0 \) be an integer. If \( H \) is an effective divisor class on \( X_r \) such that \( \chi(X_r, \mathcal{O}(H)) \geq 3k \), then \( H \) separates \( k \)-clusters if the following (necessary) conditions are satisfied: for all integral curves \( E \) of genus \( a \), if \( 0 \leq a \leq k \) then \( H.E \geq 2a - 1 + k \) and if \( k < a \leq \frac{4k}{3} \) then \( H.E \geq a + 2k - 1 \).

A weakness in this formulation of the conjecture is that it would appear to be necessary to test on all integral classes of low genus. In fact for \( r \geq 3 \) one can use the Weyl group action on \( X_r \) to obtain a standard representative of classes \( H \) that satisfy \( H.E \geq 0 \) for all exceptional classes \( E \). One can then show that the conjecture is equivalent to testing on a finite set of isolated, integral, standard classes of low genus.

In \S 2, we give the structure theorem for effective divisors on \( X_r \) as implied by H-H. In \S 3, we give the motivation and assumptions underlying the conjecture. In \S 4, we review the Weyl group action and the notion of a standard class. While the notion is not new, I have not seen mention in the literature of the characterisation of these classes as minimising for the intersection product in all semi-standard orbits (prop. 4.12). This observation allows the reformulation of the conjecture given in \S 4.3.

In the remainder of the introduction we make some preliminary remarks.

### 1.1 Preliminary remarks

1. If \( k = 0 \), conjecture 1 is just the Harbourne-Hirshowitz conjecture for the natural cohomologie of \( H \). For \( k > 0 \), it corresponds to the surjectivity of the canonical map

   \[
   H^0(X_r, \mathcal{O}(H)) \longrightarrow H^0(X_r, \mathcal{O}_Z(H))
   \]

   for all \( k \)-clusters, so that it is equivalent to \( |H| \) being base point free (resp. very ample) when \( k = 1 \) (resp. \( k = 2 \)). Note that in the latter cases \((k = 1, 2)\) we only require that \( H.E \leq 2a - 1 + k \) for \( 0 \leq a \leq k \).

2. The conjecture is true for \( r \leq 9 \). This is best seen using the equivalence of conjectures 1 and 2. When \( k = 0, 1, 2 \) this was proven in [12] Harbourne and, for \( k \geq 3, r \leq 8 \) in [4] Di Rocco, but any reader will see for themselves that her proof using the general vanishing theorem [S] goes over to \( r = 9 \).
3. The condition $\chi(X_r, \mathcal{O}(H))$ is the natural condition for a general $H$ to separate $k$-clusters in the following sense. Consider the canonical diagramme

$$
Z \hookrightarrow \text{Hilb}^k(X_r/\Lambda) \times_{\Lambda} X_r \xrightarrow{p} X_r
$$

where $Z$ is the tautological subscheme. We then have the canonical vector bundle map

$$
H^0(X_r, \mathcal{O}(H)) \otimes \mathcal{O}_{\text{Hilb}^k(X_r)} \rightarrow q_* \mathcal{O}_Z(p^*H)
$$

giving the map (1) at each point of the Hilbert scheme. The left hand side has rank $n = h^0(X_r, \mathcal{O}(H))$ and the right hand side has rank $k$, so that the degeneracy locus is empty or has codimension $\leq n - k + 1$. If the map is to be nowhere degenerate in general (i.e. without further numerical conditions being applied) then we need $n - k + 1 > \dim \text{Hilb}^k(X_r) = 2k$, giving the condition $h^0(X_r, \mathcal{O}(H)) \geq 3k$. In view of the Harbourne-Hirschowitz conjecture, we should have $h^1(X_r, \mathcal{O}(H)) = 0$ and since $h^2(X_r, \mathcal{O}(H)) = 0$ for an effective divisor on the rational surface $X_r$, this should be the same thing as $\chi(X_r, \mathcal{O}(H)) \geq 3k$.

2 About isolated curves

2.1 Harbourne-Hirschowitz and the structure of effective classes

**Definition. 2.1** An effective divisor $E$ on $X_r$ satisfying $E^2 = a - 1 = E.K; a \geq 0$; will be called an isolated curve of genus $a$. *Note that exceptional curves are just isolated curves of genus zero.*

If one supposes that H-H holds, one gets a precise description of the effective classes. In particular, one gets that for $a \geq 2$, an isolated curve of genus $a$ is reduced and irreducible, and isolated in its linear system.

**Proposition. 2.2** (Bossini) If H-H holds, then for any effective divisor class $H$ on $X_r$, the generic curve $C$ in $|H|$ can be decomposed as an orthogonal sum $A + n_1F_1 + \cdots + n_rF_r$, where each $F_i$ is an exceptional curve, $n_i > 0$ and $A$ is effective and satisfies $p(A) \geq 0$, $A^2 \geq p(C) - 1$ and $A^2 \geq 0$ if $p(A) = 0$; $p$ being the arithmetic genus. If $A^2 > 0$ then $A$ is reduced and irreducible. If $A^2 = 0$ then we have one of the following two cases

1. $A^2 = 0, A.K = -2b (b > 0)$ in which case $A = bE$, where $E$ is a pencil class (definition 4.1)
2. $A^2 = 0 = A.K$ in which case $A = mE$ ($m > 0$) where $E$ is a smooth irreducible elliptic classe which, with respect to a suitable exceptional configuration (see §4), can be written $E \equiv 3E_0 - E_1 - \cdots - E_9$. 

**Proof.** One can write \( C = A + \sum_{i=1}^{s} n_i E_i \), where the \( E_i \) are the reduced, irreducible components of \( C \) which are rational of self-intersection \(< 0 \). Now, it is known that the only integral rational curves on \( X_r \) having self-intersection \(< 0 \) are the exceptional curves. In fact if \( W \) is such a curve with \( W^2 \leq 2 \), then \( W(-K) < 0 \) and, specialising to the situation where all points are generic on a smooth plane cubic, \( W \) must specialise to an effective divisor having a smooth elliptic component, but this is impossible for \( W \) rational. We conclude that each \( E_i \) is exceptional.

Since each \( E_i \) is isolated we have \( E_i.E_j = 0 \) otherwise \( \dim |E_i + E_j| > 0 \) and the generic curve in \( |E_i + E_j| \) does not decompose. As well \( A \) is non-special by H-H, because it is positive on all exceptional curves. The isolation of \( \sum_{i=1}^{s} n_i E_i \) in \( |H| \) gives \( h^0(X_r, \mathcal{O}(A)) = h^0(X_r, \mathcal{O}(C)) = h^0(X_r, \mathcal{O}(A + E_i)) \) which shows that \( h^0(E_i, \mathcal{O}_{E_i}(A + E_i)) = 0 \) and finally that \( A.E_i \leq 0 \), but \( A.E_i \geq 0 \) in any case.

Now if \( A = A_1 + A_2 \) where \( A_2 \) is a reduced and irreducible curve and \( A_1 \) is an effective divisor, then \( A_2 \), not being rational of negative self-intersection, is non-special by H-H. This implies that \( A_2^2 \geq p(A_2) - 1 \geq 0 \) if \( p(A_2) > 0 \) and \( A_2^2 \geq 0 \) if \( p(A_2) = 0 \), where \( p \) denotes the arithmetic genus. If \( A_2^2 = 0 \) then \( A_2 \) is a pencil class or an isolated integral elliptic class.

Since this is true for all integral components \( A_2 \) of \( A \), we conclude that \( A^2 \geq 0 \) and even \( A^2 \geq p(A) - 1 \) (using the standard formulas for \( p \)). As such \( A^2 \geq 0 \) and \( A^2 > 0 \) unless \( A \) is a pencil class or an isolated elliptic class.

Now using the fact that \( A_1 \) and \( A_2 \) are non-special (H-H), and the surjectivity of the map \( |A_1| \times |A_2| \rightarrow |A_1 + A_2| \), we find by H-H

\[
\dim |A_1| + \dim |A_2| \geq \dim |A_1 + A_2| \\
= \chi(X_r, \mathcal{O}(A_1 + A_2)) - 1 \\
= \chi(X_r, \mathcal{O}(A_1)) - 1 + \chi(X_r, \mathcal{O}(A_2)) - 1 + A_1A_2 \\
= \dim |A_1| + \dim |A_2| + A_1A_2
\]

so that \( A_1A_2 = 0 \).

By the algebraic index theorem, since we know that \( A_i^2 \geq 0 \), \( i = 1, 2 \), we must have \( A_1^2 = A_2^2 = A_1A_2 = 0 \) and \( C = bA_2 \) (by the Cauchy-Schwartz inequality after writing \( A_1 \) and \( A_2 \) in an exceptional configuration) with \( A_2 \) a pencil class, or \( A_2 \) an isolated integral elliptic class, hence equivalent to \( 3E_0 - E_1 - \cdots - E_9 \) by lemma 4.9. \( \square \)

It is conjectured that all isolated curves \( E \) of genus \( a \geq 2 \) are in fact smooth as well (see remark 4.9).

### 2.2 The ample classes

**Proposition.** 2.3 If H-H holds then a divisor class \( H \) on \( X_r \) is ample if and only if \( H^2 > 0 \) and \( H.E > 0 \) for all exceptional curves \( E \) on \( X_r \). If \( H \) is standard and \( H^2 > 0 \), and \( H.E \leq 0 \) for some reduced and irreducible curve \( E \) on \( X_r \), then \( E \) is exceptional and \( H.E = 0 \).

**Proof.** If \( H \) is standard then \( h^2(X_r, \mathcal{O}(H)) = 0 \) and the condition \( H^2 > 0 \) shows that \( aH \) is effective for \( a \gg 0 \). We can thus suppose \( H \) effective, hence reduced and irreducible by
If $E$ is a reduced and irreducible curve then $H.E \geq 0$, but if $H.E = 0$ then $E^2 < 0$ by the algebraic index theorem and by (2.2) again, this implies that $E$ is exceptional. □

In this direction one might consult [16].

3 The conditions of the conjecture

3.1 The conditions $H.E \geq 2a - 1 + k$, $a \leq k$ and $H.E \geq a - 1 + 2k$, $k < a \leq \frac{4}{3}k$ of the conjecture

Throughout this section let $H$ be an effective divisor class on $X_r$ ($r \geq 3$) such that $\chi(\mathcal{O}(H)) \geq 3k$ and $H.F \geq k - 1$ for all exceptional curves $F$ sur $X_r$.

It has already been said that the underlying assumption in conjecture 1. is that if $|H|$ does not separate $k$-clusters then this because it does not separate some $k$-cluster in an integral (even smooth) curve on $X_r$. It is also supposed that if the map

$$(2) \quad \rho : H^0(X_r, \mathcal{O}(H)) \longrightarrow H^0(X_r, \mathcal{O}_E(H))$$

has rank $\geq 2k$ then $|H|$ separates $k$-clusters in $E$. We will see that if H-H holds and $\mathcal{O}_E(H)$ is special, or if $H - E$ is effective and special, then (2) always has rank $\geq 2k$. In the remaining cases (2) is surjective and there are two possibilities. Firstly, if $k > a$ then it is well known that $\mathcal{O}_E(H)$ separates $k$-clusters if and only if $H.E \geq 2a - 1 + 2k$, but in this case $h^0(\mathcal{O}_E(H))$ can be less than $2k$. Secondly, if $k \leq a$, a necessary condition for $\mathcal{O}_E(H)$ to separate $k$-clusters is that $H.E \geq a - 2 + 2k$, and this is sufficient if $\mathcal{O}_E(H)$ is a general invertible sheaf of degree $H.E$. We are thus supposing in the conjecture that $\mathcal{O}_E(H)$ is general in this sense.

This said we must prove the

**Proposition. 3.1** Suppose that H-H holds. Then the image of (2) has dimension $< 2k$ only if $\mathcal{O}_E(H)$ is non-special. When $\mathcal{O}_E(H)$ is non-special, $a > k$ and $H.E \leq a - 2 + 2k$ one has $4k \geq 3a$.

**Proof.** Firstly, if H-H holds, $E$ is non-special.

There are two more or less obvious cases where the image of (2) has dimension $\geq 2k$. These are when $H - E$ is not effective (obvious) and when $H - E$ is effective and special. In fact, in the second case, if H-H holds, then by (2.2) we can write $H - E = A + \sum_i n_iF_i$ as an orthogonal sum of an effective standard divisor $A$ and multiples $n_i > 0$ of exceptional curves $F_i$ with at least one of the $n_i \geq 2$. Since $k - 1 \leq H.F_i = E.F_i - n_i$ one finds $E.F_i \geq n_i + k - 1$
and

\[ h^0(X, \mathcal{O}(H)) - h^0(X, \mathcal{O}(H - E)) = \overline{\chi}(H) - \overline{\chi}(A) = \overline{\chi}(A) + \overline{\chi}(E) + \sum_i n_i \chi_i + A.E - \overline{\chi}(A) = -\sum_i n_i(n_i - 1)/2 + \sum_i n_i E_i + A.E \geq \sum_i n_i(n_i - 1)/2 + k(\sum_i n_i) + A.E \geq 2k \]

as required.

Moreover, since \( H.F \geq 0 \) for all exceptional curves \( F \) on \( X \), \( H \) is non-special. Also if \( \mathcal{O}_E(H) \) is special, \( h^2(\mathcal{O}_E(H - E)) > 0 \), implying that \( H - E \) is not effective, so that the image of \((4)\) is at least \( 2k \) dimensional in this case as well.

We can thus suppose henceforth that \( H - E \) and \( \mathcal{O}_E(H) \) are effective and non-special and we will show that if \( a > k \) and \( H.E \leq a - 2 + 2k \) one has \( 4k \geq 3a \).

**Claim.** In this case, \( H - 3E \) is not effective, \( \mathcal{O}_E(H - E) \) is non-special, and \( \mathcal{O}_E(H - 2E) \) is special.

This gives the desired inequality as follows using Riemann-Roch and Clifford’s theorem

\[
\frac{1}{2}(E.(E - 2H)) + 1 \geq h^0(\mathcal{O}_E(H - 2E)) \geq h^0(\mathcal{O}(H - 2E)) \geq h^0(\mathcal{O}(H)) - h^0(\mathcal{O}_E(H)) - h^0(\mathcal{O}_E(H - E)) = h^0(\mathcal{O}(H)) - 2E.H + E^2 + 2(a - 1)
\]

so that

\[ 5E.H + 2 \geq 2h^0(\mathcal{O}(H)) + 4E^2 + 4(a - 1) \]

and using \( E^2 \geq a - 1 \) \((4.7)\) and \( a - 2 + 2k \geq H.E \), we get \( 4k \geq 3a \).

**Proof of the claim.** Since \((H - 3E).E = H.E - 3E^2 \leq a - 2 + 2k - 3(a - 1) = 2(k - a) + 1 < 0 \) and \( E^2 \geq 0 \) it follows that \( H - 3E \) is not effective. If \( \mathcal{O}_E(H - E) \) is special, then \( h^2(\mathcal{O}(H - 2E)) = h^1(\mathcal{O}_E(H - E)) \neq 0 \), so that \( H - 2E \) is not effective and by Clifford’s theorem

\[ k+1 \leq h^0(\mathcal{O}(H - E)) \leq h^0(\mathcal{O}_E(H - E)) \leq \frac{1}{2}(E.H - E^2) + 1 \leq \frac{1}{2}(a - 2 + 2k - a + 1) + 1 = k + \frac{1}{2} \]

From this contradiction we conclude that \( \mathcal{O}_E(H - E) \) is non-special. Finally,

\[
\overline{\chi}(\mathcal{O}(H - 3E)) = \overline{\chi}(\mathcal{O}(H)) - 3E.(H - 3E) = \overline{\chi}(\mathcal{O}(H)) - 3E.(H - E) + 6E^2 - 3E.H \geq 3k + 6(a - 1) - 3(a - 2 + 2k) = 3(a - k) \geq 0
\]

so that \( h^2(\mathcal{O}(H - 3E)) > 0 \) and \( \mathcal{O}_E(H - 2E) \) is special. \( \square \)
4 Translation into standard divisors

As we will see below, standard divisors are essentially those that test positive on all exceptional curves and include all integral classes that are not exceptional. The more general semi-standard classes include all effective classes on an $X_r$. What we want here is threefold: (1) an intrinsic characterisation of the (semi)-standard classes, (2) a standard way of writing such a class with respect to a suitable exceptional configuration, (3) the unicity of this expression (but not of the exceptional configuration). I have found no reference in the literature to parts (1) and (3), but (2) has been used extensively since Noether. We thus begin by defining the classic $E$-(semi)-standard classes for a fixed exceptional configuration $E$ (definition 4.2), and then show that this is equivalent to an intrinsic definition (proposition 4.7). Finally in proposition 4.10, we deal with the unicity via a minimising characterisation of the exceptional configurations that give $E$-(semi)-standard representations of (semi)-standard classes. This gives corollary 4.12 which is essential for the reformulation of the conjecture.

Before giving the definitions let us recall [10] that the Weyl groupe $W_r$ of the surface $X_r$ (see [13] and [8]) can be viewed as a groupe of $k$-automorphismes of $X_r$. There is an induced action of $W_r$ on Pic($X_r$) which stabilises the intersection form and fixes the canonical class. The latter action is faithful so that $W_r$ is identified with a subgroup of the orthogonal group $O_r$ of (Pic($X_r$), $\langle , \rangle$) where $\langle , \rangle$ is the intersection form.

This induced action on Pic($X_r$) is simply transitive on the exceptional configurations which are by definition the orthogonal sequences of effective, irreducible classes $E=(E_0,E_1,\ldots,E_r)$ satisfying

$$E_0^2=1 \quad , \quad E_0.K_r=-3 \quad , \quad E_j^2=-1 = E.K_r \quad , \quad E_i.E_j = 0 \quad , \quad 0 \leq i < j \leq r$$

In particular $E_i \simeq \mathbb{P}^1$ for $i=0,1,\ldots,r$. For any fixed exceptional configuration $E$ as above, the group $W_r$ is generated by the orthogonal reflexions $\sigma_i = \text{id} + \langle r_i, \rangle r_i$ where

$$r_0 = E_0 - E_1 - E_2 - E_3, \quad r_i = E_i - E_{i+1}, \quad i = 1,\ldots,r-1$$

Central to the proof of these results is Noether’s inequality (see [3] 5), which says that if $r \geq 3$, then a sequence of non-negative integers $d, m_1 \geq m_2 \geq \cdots \geq m_r$ satisfying

$$d^2 - m_1^2 - \cdots - m_r^2 = -1 = -3d + m_1 + \cdots + m_r$$

also satisfies $d < m_1 + m_2 + m_3$. This inequality can also be used to show that for $r \geq 3$ any irreducible rational class $E$ (i.e. $E$ is irreducible, effective and $E^2 = -2 = E.K_r \geq -1$ lies in the $W_r$ orbit of one and one only of the following classes

(3) $E_r, E_0 - E_1, E_0, 2E_0, dE_0 - (d-1)E_1 - E_2, dE_0 - (d-1)E_1, \quad d \geq 2$

Note that these are uniquely determined by $E^2$ except for the pair $2E_0$ and $3E_0 - 2E_1 - E_2$. 

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Definition. 4.1 An irreducible rational classe $E$ on $X_r$ with $E^2 = 0$ (resp. $E^2 = 1$, resp. $E^2 = 2$) will be called a pencil class (resp. line class, resp. quadratic class). Note that this is $W_r$-equivalent to $E_0 - E_1$ (resp. $E_0$, resp. $2E_0 - E_1 - E_2$).

It is also true that $W_r$ acts transitively on the orthogonal sequences of exceptional curves of length $\neq r - 1$. When the length is $r - 1$ there are two orbits which, for a fixed exceptional configuration $E = (E_0, E_1, \ldots, E_r)$, have representatives of the form $(E_2, \ldots, E_r)$ et $(E_0 - E_1 - E_2, E_3, \ldots, E_r)$. These correspond respectively to sequences of length $r - 1$ that can be extended and those that cannot. If we forget the ordering, then the non-extendable orthogonal sequences of exceptional curves correspond to quadratic classes $E$ which induce birational morphisms of $X_r$ to a smooth quadrique $Q$ in $\mathbb{P}^3$, contracting the exceptional curves in such a non-extendable sequence.

We define standard classes as follows

Definition. 4.2 A divisor class $H$ on $X_r$ is said to be $E$-standard for an exceptional configuration $E = (E_0, E_1, \ldots, E_r)$ if

$$H \equiv dE_0 - m_1E_1 - \cdots - m_rE_r$$

where $d \geq m_1 \geq \cdots \geq m_r \geq 0$, $d \geq m_1 + m_2$ and $d \geq m_1 + m_2 + m_3$.

We say that $H$ is $E$-semi-standard if $d \geq 0$, $d \geq m_1 + m_2 + m_3$ et $d \geq m_1 \geq m_2 \geq \cdots \geq m_r$ (but some $m_i$ may be $< 0$).

We will say that a divisor class $H$ on $X_r$ is standard (resp. semi-standard) if it is $E$-standard (resp. $E$-semi-standard) for some exceptional configuration $E$ on $X_r$.

Example 4.3 The class $5E_0 - 2E_1 - 2E_2 - 2E_3$ is not $E$-standard, but is standard as one can see by making the standard quadratic transformation by the reflection with root $r_0$.

4.1 generating classes

It is obvious and was pointed out by Harbourne that the $E$-standard classes are precisely those that can be expressed as non-negative sums

$$H \equiv aE_0 + b(E_0 - E_2) + b(2E_0 - E_1 - E_2) + \sum_{i=3}^{r} \alpha_i C_i$$

where $a, b, c, \alpha_i \geq 0$ and $C_i = -K_r + E_{i+1} + \cdots + E_r$ ($i \geq 3$).

Definition. 4.4 We say that the classes $E_0$, $E_0 - E_1$, $2E_0 - E_1 - E_2$, $C_i$ ($i \geq 3$) are the (standard) generating classes of the exceptional configuration $E$. The first three generating classes are called the rational generating classes while the $C_i$, which have arithmetic genus one, are called the elliptic generating classes.
Remark. 4.5 As we saw above the rational generating classes of self intersection 0,1,2 each form a single orbit under the Weyl group action. The same is true for the elliptic generating classes for each self intersection number $\leq 6$ since $W_r$ fixes $K_r$ and acts transitively on the orthogonal sequences of exceptional curves of length $\neq r - 1$.

Lemma. 4.6 (Bossini) If $C$ is an $E$-standard class satisfying $C^2 = 0 = C.K_r$, then $C \equiv \alpha_9 C_9$.

Proof. The class $C$ is effective and it is clear that $C.E > 0$ on all rational generating classes and $C.C_i \geq 0$ since $C.C_r = C - K_r = 0$. As well $C.C_i > 0$ for $i \leq 8$ since any effective divisor that is a proper component of a curve in $|C_i|$ is rational. As such, $C \equiv \sum_{i \geq 9} \alpha_i C_i$. If $C \neq \alpha_9 C_9$ then $C^2 < 0$.

4.2 semi-standard classes

An $E$-semi-standard class $H$ has one of two forms reflecting the two orbits of orthogonal sequences of exceptional curves of length $r - 1$. In fact, either $H = (dE_0 - m_1E_1 - \cdots - m_sE_s) + n_{s+1}E_{s+1} + \cdots + n_rE_r$, where the first part is standard and $n_i = -m_i > 0$, or $m_3 < 0$, $m_1, m_2 \geq 0$, $d < m_1 + m_2$ and

$$H \equiv (2d - m_1 - m_2)E_0 - (d - m_2)E_1 - (d - m_1)E_2 + (m_1 + m_2 - d)(E_0 - E_1 - E_2) + n_3E_3 + \cdots + n_rE_r$$

where $n_i = -m_i > 0$. As such any $E$-semi-standard class, can be written in the form $A + \sum n_iF_i$, where $A$ is $E$-standard, $n_i > 0$ and the $F_i$ are a family of orthogonal exceptional curves.

Proposition. 4.7 For $r \geq 2$, a divisor class $H$ on $X_r$ is standard (resp. semi-standard) if and only if $H.E \geq 0$ for all exceptional curves $E$ on $X_r$ (resp. $H.E \geq 0$ for all line and pencil classes $E$ on $X_r$).

A classe $H$ is semi-standard if and only if it is an orthogonal sum of the form $H = A + \sum n_iF_i$, where $A$ is standard, $n_i > 0$, and each $F_i$ is an exceptional curve.

Proof. Firstly, if $H$ is non-negative on all exceptional classes, then choosing any line class $E_0$, we obtain an exceptional configuration $E$, with $H = dE_0 - m_1E_1 - \cdots - m_rE_r$, $m_i \geq 0$ and $(d - m_1 - m_2) = H.(E_0 - E_1 - E_2) \geq 0$, so that $H.E_0 = d \geq 0$ on all line classes $E_0$.

Suppose then that $H$ is non-negative on all line classes. In this case we can choose a line class $E_0$ such that $H.E_0 = d$ is the minimum for all line classes on $X_r$ and by suitably ordering the exceptional curves contracted by $E_0$ we obtain an exceptional configuration $E = (E_0, E_1, \ldots, E_r)$ with

$$H = dE_0 - m_1E_1 - \cdots - m_rE_r$$

$m_1 \geq \cdots \geq m_r$ and, if $r \geq 3$, $2d - m_1 - m_2 - m_3 = H.(2E_0 - E_1 - E_2 - E_3) \geq H.E_0 = d$ by the minimality of $H.E_0$ over all line classes.
When $H$ is positive on all exceptional curves we have $m_r \geq 0$, $d \geq m_1 + m_2 \geq m_1$ and when $H$ is positive on all pencil classes we have $d \geq m_1$.

Conversely, if $H = dE_0 - m_1E_1 - \cdots - m_rE_r$ is $E$-standard, we can write

$$H \equiv aE_0 + b(E_0 - E_2) + b(2E_0 - E_1 - E_2) + \sum_{i=3}^{r} \alpha_i C_i$$

as in (4) and it suffices to note that each of the generating classes is positive on all exceptional curves.

If $H$ is $E$-semi-standard, then as we saw above, $H$ is an orthogonal sum $H \equiv A + \sum_i n_i F_i$ where $A$ is $E$-standard, $n_i > 0$ and the $F_i$ are exceptional. Conversely every such class is non-negative on all line and pencil classes. □

**Corollary. 4.8** On $X_r$ every effective class is semi-standard and every integral class that is not exceptional is standard.

**Remark. 4.9** We have defined isolated curves of genus $a \geq 1$ to be reduced and irreducible curves $E$ with $E^2 = a - 1 = E.K_r$. The preceding proposition says that if $H$ holds then for $a \geq 2$, we could define these to be standard classes with $E^2 = a - 1 = E.K_r$.

The isolated curves of genus $a \leq 4$ were completely classified in [3] and [14]. The only standard classe with $E^2 = 0 = E.K_r$ is $C_9$, while the only standard classes with $E^2 = 1 = E.K_r$ are

$$G_1 = 4E_0 - 2E_1 - E_2 - \cdots - E_{12} \quad (5)$$
$$G_2 = 6E_0 - 2E_1 - \cdots - 2E_8 - E_9 - E_{10} - E_{11}$$
$$G_3 = 9E_0 - 3E_1 - \cdots - 3E_8 - 2E_9 - 2E_{10}$$

and it has been proven that for $a = 2, 3, 4$ all standard classes with $E^2 = a - 1 = E.K_r$ contain a unique curve and it is smooth and irreducible.

**Proposition. 4.10** Let $H \equiv dE_0 - m_1E_1 - \cdots - m_rE_r$ be an $E$-standard class, then

1. for $i \geq 1$, $m_i + \cdots + m_r$ is the minimum value of $H.(G_i + \cdots + G_r)$ for all extendable orthogonal sequences of exceptional curves $G_i, \ldots, G_r$. If $i \geq 3$ this is the minimum value on all sequences of exceptional curves, orthogonal or not.

2. $d - m_1 - m_2 + m_3 + \cdots + m_r$ is the minimum value of $H.(G_2 + \cdots + G_r)$ on all unextendable orthogonal sequences of exceptional curves $G_2, \ldots, G_r$.

In particular, if $H$ is a standard class which can be written in the $E$-standard form $dE_0 - m_1E_1 - \cdots - m_rE_r$, then the sequence $d, m_1, \ldots, m_r$ is unique, independent of the exceptional configuration $E$ in which $H$ is $E$-standard.
Proof. As we saw in (4), we can write $H$ uniquely in the form

$$H \equiv aE_0 + b(E_0 - E_2) + c(2E_0 - E_1 - E_2) + \sum_{i=3}^{r} \alpha_i C_i$$

For $i \geq 3$ we have $m_i = \alpha_i + \cdots + \alpha_r$. If $E$ is an exceptional curve

$$H.E \geq \alpha_i E.C_i + \cdots + \alpha_r E.C_r$$

and $E.C_i = 1 + E(E_{i+1} + \cdots + E_r)$, so that if $H.E < m_i$ then $E = E_j$ for some $j > i$. This shows that for all sequences $G_i, \ldots, G_r$ ($3 \leq i \leq r$) of exceptional curves, $H.(G_i + \cdots + G_r) \geq H.(E_i + \cdots + E_r)$.

If $E$ is an exceptional curve and $m_3 < H.E < m_2 = m_3 + c$ then $c > 0$, $E.(2E_0 - E_1 - E_2) = 0$ et $E \neq E_i$ for $i = 3, \ldots, r$. As such $E = E_0 - E_1 - E_2$. It follows that for extendable (resp. unextendable) sequences $G_2, \ldots, G_r$ of orthogonal exceptional curves, the minimum value of $H.(G_2 + \cdots + G_r)$ is attained on the sequence $G_i = E_i; i = 3, \ldots, r$.

If $E$ is an exceptional curve and $m_2 < H.E < m_1 = m_2 + b$, then $b > 0$, $E \neq E_j; j = 2, \ldots, r$; et $(E_0 - E_1)E = 0$. In this case $E = E_0 - E_1 - E_j$ for some $j > 1$ and $E, E_2, \ldots, E_r$ is not an orthogonal sequence. It follows that for orthogonal sequences of length $r$, $G_1, \ldots, G_r$ of exceptional curves, the minimum value of $H.(G_1 + \cdots + G_r)$ is attained on the sequence $(E_1, \ldots, E_r)$. □

Corollary. 4.11 The decomposition of a semi-standard class $D$ as an orthogonal sum $A + \sum_i n_i F_i$, where $A$ is standard, the $F_i$ are exceptional and $n_i > 0$, is unique.

Proof. If $B + \sum_i m_i G_i$ is another such decomposition, then, $B$ being standard, $D.F_i < 0$ implies that $F_i, G_j < 0$ for some $j$. Since the $F_i$ and $G_i$ are smooth and irreducible classes, we can suppose that $F_i = G_i$, then $n_i = D.F_i = m_i$. □

Proposition. 4.12 Fix an exceptional configuration $E = (E_0, E_1, \ldots, E_r)$ on $X_r$. Let $H \equiv dE_0 - m_1 E_1 - \cdots - m_r E_r$ be an $E$-standard divisor and let $D$ be a semi-standard divisor on $X_r$. Then for $\sigma \in W_r$, the minimum value of $H.\sigma(D)$ occurs exactly when $\sigma(D)$ is $E$-semi-standard. Otherwise said, if $F = (F_0, F_1, \ldots, F_r)$ is an exceptional configuration for which $D = d'F_0 - m'_1 F_1 - \cdots - m'_r F_r$ is $F$-semi-standard and $D' = d'E_0 - m'_1 E_1 - \cdots - m'_r E_r$ then

$$H.D \geq H.D'$$

Proof. Write $H$ as

$$H \equiv aE_0 + b(E_0 - E_2) + c(2E_0 - E_1 - E_2) + \sum_{i=3}^{r} \alpha_i C_i$$

and $D$ as an orthogonal sum $D = A + \sum_i n_i G_j$ ($i \geq 2$) where $A$ is $F$-standard, $n_i > 0$ and $G_i, \ldots, G_r$ is an orthogonal sequence of exceptional curves with either $F_j = E_j$ for $j = i, \ldots, r$, or $i = 2$, $G_2 = F_0 - F_1 - F_2$, $G_j = F_j$ for $j = 3, \ldots, r$. 11
By (4.10), the proposition holds when \( A = 0 \), so it suffices to prove the proposition for \( D \) standard and even for \( D \) one of the \( F \)-standard generating classes. By (4.10), the proposition holds for \( D = -K_r - E_i - \cdots - E_r \) (\( i \geq 3 \)) and by the symmetry of the proposition it suffices to suppose that \( H \) (resp. \( D \)) is one of the rational generating classes.

The system \(|E_0|\) (resp. \(|2E_0 - E_1 - E_2|\)) is positive on all effective classes other than exceptional curves, so we need only show that \((2E_0 - E_1 - E_2).F_0 \geq 2 \) and \((2E_0 - E_1 - E_2)(2F_0 - F_1 - F_2) \geq 2 \). This follows from the fact that the only effective divisors with \((2E_0 - E_1 - E_2).G \leq 1\) move in a linear system of projective dimension at most one. \( \square \)

When \( r \leq 1 \), there is only one exceptional configuration so that \( E \)-standard and standard are equivalent.

**Remark. 4.13** It follows from proposition 4.10 that if \( r \geq 3 \) then \( H.E_r \) is the minimum value of \( H.E \) on all exceptional curves \( E \). It then follows from proposition 4.12 and the list of effective \( E \)-standard rational classes (3), that \( H.E_r \) is the minimum on all effective rational classes.

### 4.3 reformulation of the conjecture

**Definition. 4.14** For an exceptional configuration \( E = (E_0, E_1, \ldots, E_r) \) on \( X_r \) we extend the definition of \( E \)-standard class to include, for any \( \delta \geq 0 \), those classes \( E \) on \( X_{r+\delta} \) which for the canonical extension \( E' = (E_0, E_1, \ldots, E_r, E_{r+1}, \ldots, E_{r+\delta}) \) of the exceptional configuration to \( X_{r+\delta} \) is \( E' \) standard on \( X_{r+\delta} \). An \( E \)-standard isolated curve of genus \( a \) will then be an integral \( E \)-standard class (for this extended definition) \( E \) such that \( E^2 = a - 1 = E.K_{r+\delta} \).

**Remark. 4.15** Note that if \( H \) is a semi-standard or standard class on \( X_r \) then so is its pull back (also denoted by \( H \)) to \( X_{r+\delta} \).

We can now reformulate conjecture 1. in the following form

**Conjecture 2.** Let \( k > 0 \) be an integer and let \( E = (E_0, E_1, \ldots, E_r) \) be an exceptional configuration on \( X_r \); \( r \geq 3 \); and let \( H \equiv dE_0 - m_1E_1 - \cdots - m_rE_r \) be an \( E \)-standard class satisfying \( \chi(X_r, \mathcal{O}(H)) \geq 3k \) and \( m_r \geq k - 1 \), then \( H \) is non-special and separates \( k \)-clusters if the following necessary conditions are satisfied: \( H.E \geq 2a - 1 + k \) for all \( E \)-standard isolated curves of genus \( a \) (\( 1 \leq a \leq k \)) and \( H.E \geq a - 1 + 2k \) for all \( E \)-standard isolated curves of genus \( a \) (\( \frac{4}{3}k \geq a \geq k \)).

**Proposition. 4.16** Conjecture 2 is equivalent to conjecture 1.

**Proof.** We have already noted in 4.13 that for \( r \geq 3 \), \( H.E_r \) is the minimum value of \( H.E \) for all effective rational classes on \( X_r \). It therefore suffices to show that if the stated condition holds on all \( E \)-standard isolated curves of genus \( a \), \( 1 \leq a \leq k \) and \( \frac{4}{3}k \geq a \geq k \), then it holds for all integral curves of such genus.

An integral curve \( F \) on \( X_r \) of genus \( a \geq 1 \) is standard by 4.8 and \( H.F \) is minimum when \( F \) is \( E \)-semi-standard. If \( F \) is \( E \)-standard, but not isolated, then letting
$j = \min \{ i | 1 \leq i \leq r \text{ and } F.E_i = 0 \}$, the class $F' = F - E_i - \cdots - E_{i+s}$ is an $E$-standard isolated curve of genus $a > 0$ for some $s \geq 0$ and satisfies $F'.H \leq F.H$. Hence $H.F$ takes its minimum value on the $E$-standard isolated curves of genus $a$.

Remark. 4.17 1. To see that one does have to test in the range $\frac{4}{3}k \geq a > k$ when $k \geq 3$, consider $H = 13E_0 - 9E_1 - 2E_2 - \cdots - 2E_{18}$ which is constructed from the isolated hyperelliptic curve of genus 4, $E = 6E_0 - 4E_1 - E_2 - \cdots - E_{18}$. We have $\chi(\mathcal{O}(H)) = 3.3$ and $H.E = 8 < 4 - 1 = 2.3$.

2. The following example shows that it is necessary to consider isolated curves which may not lie on $X_r$. It suffices to consider $kC_8$ on $X_8$. For any $k \geq 6$, $\chi(kC_8) \geq 3k$, but $(kC_8).C_8 = k < k + 1$. However with the definitions introduced in (4.14) this can be detected on the isolated curve $C_9$ on $X_9$.

5 Further motivation

Here we show that the general adjunction theorems imply the conjecture under much heavier restrictions on $H$.

Recall the

Proposition. 5.1 (see [4], Theorem 2.1) Let $H$ be a divisor class on a smooth surface $S$ such that $H - K$ is nef and big and $(H - K)^2 \geq 4k + 1$. Then $H$ separates $k$-clusters unless there exists an effective divisor $D$ on $S$ of arithmetic genus $p$ such that $H - K - 2D$ is effective,

(6) \hspace{1cm} H.D \leq 2p - 2 + k \quad \text{and} \quad 2p - 2 + D^2 < H.D < 2k + D.K

Now suppose that $H.H$ holds and let $H$ be a divisor class on $X_r$ such that $\chi(X_r, \mathcal{O}(H)) \geq 3k$, $H.F \geq k - 1$ for all exceptional curves and $H.F \geq k + 1$ for all integral elliptic curves $F$. Suppose further that $H - K_r$ is nef and big and that $(H - K_r)^2 \geq 4k + 1$. Let $D$ be an effective divisor on $X_r$ satisfying (6) so that $D = E + \sum_{i=1}^s n_i F_i$ is an orthogonal sum of a standard class $E$ and multiples $n_i > 0$ of exceptional curves $F_i$. By (6), we find

$$H.E + \sum_{i=1}^s n_i H.F_i \leq 2(p(E) - n_i(n_i + 1)/2 - 1) - 2 + k$$

so that $H.E \leq 2p(E) - 2$ and $E$ is neither elliptic nor rational, hence is integral of genus $a \geq 2$. As well, using the effectiveness of $H - K - 2D$, we get $(H - K - 2E - 2(\sum_{i=1}^s n_i F_i)).E \geq 0$ so that $(H - K).E \geq 2E^2$. Now by the last part of (6) $2k > (H - K).E + (H - K).E \geq 2E^2$, so that $k \geq a$ in accordance with the conjecture.
6 The not very ample standard classes

In this section we look at the $E$-standard classes $H$ on $X_r$ with $m_r \geq 1$, $\chi(X_r, \mathcal{O}(H)) \geq 6$ and $H.E < 2a - 1 + k$ for some isolated standard class $E$ of genus $a = 1$ or 2 for $k = 1, 2$; i.e. the standard classes which have sufficient sections and test positive on all exceptional curves, but are not base point free or not very ample because this is not the case for their restriction to an isolated curve of genus one or two. We will use the list (3) of isolated, genus two curves $G_i$.

Let $H \equiv dE_0 - m_1 E_1 - \cdots - m_r E_r$ be an $E$-standard class with $m_r > 0$ and $\chi(X_r, \mathcal{O}(H)) \geq 6$.

The only isolated $E$-standard curve of genus one is $C_9 = 3E_0 - E_1 - \cdots - E_9$. To determine the required classes we can suppose that $r = 9$. One easily sees that the standard classes with $H.C_9 \leq 1$ and $\chi(X_r, (H)) \geq 3$ is $H \equiv mC_9 + E_9$ which has $\chi = m$, and that those with $\chi(X_r, \mathcal{O}(H)) \geq 6$ et $H.C_9 \leq 2$, are

$$H \equiv mC_9 + E_9; \quad H \equiv mC_9 + 2E_9; \quad H \equiv mC_9 + E_8 + E_9$$

which have $\chi, m + 1, 2m$ and $2m + 1$ respectively.

Let $E$ be an $E$-standard isolated curves of genus 2 (3) such that $H.E \leq 4$. If $H.E \leq 3$ then $H - 4E$ is effective and negative on $E$ which is an ample divisor! We can thus suppose $H.E = 4$. In this case $H - 2E$ is effective.

If $H - 3E$ is effective, then $H = 3E + A$ with $1 = E.A > A^2$. As such, either $H = 3E + F$ where $F$ is exceptional and $E.F = 1$, or $H - 3E$ is standard and isolated. In the; latter case, either $H = 4E$ or $H = 3E + A$ where $A$ is an isolated elliptic curve with $E.A = 1$ (eg. $E = 6E_0 - 2E_1 - \cdots - 2E_8 - E_9 - E_10 - E_11, A = C_9$).

If $H - 3E$ is not effective, then $H = 2E + A$ where $E.A = 2$. There are two cases. In the first case, $A$ can be decomposed as a sum $A = A_1 + A_2$ of reduced and irreducible divisors $A_i$ with $1 = E.A_i > A_i^2 \geq -1$ so that the $A_i$ are exceptional or isolated and elliptic (eg. $E = 6E_0 - 2E_1 - \cdots - 2E_8 - E_9 - E_{10} - E_{11}$ and

$$A_1 = C_9, A_2 = E_{10} \quad \text{or} \quad A_1 = E_9, A_2 = E_{10}$$

In the second case $H = 2E + A$ where $A$ is reduced and irreducible, and $4 = (E.A)^2 > A^2 \geq -1$, in which case, $A$ is an exceptional curve or an isolated curve of genus $a = 1, 2$ or 3. One has examples with $(E = G_1, A = C_9), (E = G_2, A = G_3)$ and the pair $E = G_2$

$$A = 12E_0 - 4E_1 - \cdots - 4E_8 - 2E_9 - 2E_{10} - 2E_{11}$$

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