RIESZ TRANSFORM, GAUSSIAN BOUNDS AND THE METHOD OF WAVE EQUATION

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Abstract. For an abstract self-adjoint operator \( L \) and a local operator \( A \) we study the boundedness of the Riesz transform \( AL^{-\alpha} \) on \( L^p \) for some \( \alpha > 0 \). A very simple proof of the obtained result is based on the finite speed propagation property for the solution of the corresponding wave equation. We also discuss the relation between the Gaussian bounds and the finite speed propagation property. Using the wave equation methods we obtain a new natural form of the Gaussian bounds for the heat kernels for a large class of the generating operators. We describe a surprisingly elementary proof of the finite speed propagation property in a more general setting than it is usually considered in the literature.

As an application of the obtained results we prove boundedness of the Riesz transform on \( L^p \) for all \( p \in (1, 2] \) for Schrödinger operators with positive potentials and electromagnetic fields. In another application we discuss the Gaussian bounds for the Hodge Laplacian and boundedness of the Riesz transform on \( L^p \) of the Laplace-Beltrami operator on Riemannian manifolds for \( p > 2 \).

1. Introduction

Let \( \Delta = -\sum_{i=1}^n \partial_i^2 \) be the standard Laplace operator acting on \( \mathbb{R}^n \). Then the corresponding Riesz transform is defined by the formula \( \partial_j \Delta^{-1/2} \). The \( L^p \) continuity of the Riesz transform for all \( p \in (1, \infty) \) is one of the most important and celebrated results in analysis. Papers devoted to the study of the Riesz transform and its generalizations are too numerous to list here. Hence we would like to mention only a few most relevant works [1, 3, 9, 10, 17, 18, 21, 26, 27, 33, 34, 35, 38, 39, 42].

The operator \( \nabla L^{-1/2} \), where \( \nabla \) is the gradient and \( L \) is the Laplace-Beltrami operator on a Riemannian manifold \( M \), is a natural generalization of the classical Riesz transform. \( L^2 \) boundedness of the Riesz transform \( \nabla L^{-1/2} \) is a consequence of the equality \( \| \nabla f \|_{L^2} = \| L^{1/2} f \|_{L^2} \), which is actually the definition of the operator \( L \). In [42] Strichartz asked whether one could extend \( L^p \) continuity of the classical Riesz transform to the setting of Laplace-Beltrami operators described above. An answer to this question was given in [9] for \( 1 \leq p \leq 2 \). In [9] Coulhon and Duong proved that if \( M \) is a complete Riemannian manifold which satisfies the doubling volume property (see Assumption [1]), \( L \) is the Laplace-Beltrami operator on \( M \) and the heat kernel corresponding to \( L \) satisfies the Gaussian bounds then the Riesz transform \( \nabla L^{-1/2} \) is of weak type \( (1, 1) \) and so bounded on \( L^p \) for all \( p \in (1, 2] \). Note that \( (\partial_j \Delta^{-1/2})^* = -\partial_j \Delta^{-1/2} \) so the boundedness of the standard Riesz transform \( \partial_j \Delta^{-1/2} \) for \( p \in (1, 2] \) implies continuity of the standard Riesz transform for all \( p \in (1, \infty) \).

Surprisingly in general the Riesz transform corresponding to the Laplace-Beltrami operator

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\( \nabla L^{-1/2} \) is no longer necessarily continuous for \( p > 2 \) even under the above assumptions (see [9] for a counterexample).

One of the main aims of this paper is to remove any assumptions about the nature of the operator \( L \) from the result obtained in [9]. In Theorem 5 below we consider an abstract self-adjoint positive definite operator. We show that if an operator \( L \) satisfies finite speed propagation property for the solutions of the corresponding wave equation, \( A \) is a local operator and \( AL^{-\alpha} \) is bounded on \( L^2 \) for some \( \alpha > 0 \) then the operator \( AL^{-\alpha} \) is automatically bounded on all \( L^p \) for \( p \in (1, 2] \) and of weak type \((1, 1)\). Thus, it turns out that one does not have to assume that \( L \) is the Laplace-Beltrami operator in [9] and that the finite speed propagation property is the only essential assumption in [9]. Removing assumptions about the nature of the operator \( L \) allows us to study the Riesz transform for Schrödinger operators with positive potentials and electromagnetic fields. Such Riesz transforms were investigated in [26, 31, 34, 35]. Results which we describe here generalize and strengthen a part of the results described in [26, 31, 34, 35].

We start our discussion with a description of the equivalence of the finite speed propagation property and the \( L^2 \) version of the Gaussian bounds (see condition (2.21)). This allows us to obtain a very elegant and straightforward proof of the finite speed propagation property (see Theorem 6) in a more general setting than usually seen in the literature (see for example [20, 43]). Theorem 6 describes a large and natural class of examples of possible applications of our main results.

Another goal of this paper is to prove that on-diagonal bounds for the heat kernel and finite speed propagation property imply off-diagonal Gaussian bounds (see Theorem 4). It is well known that for the Laplace-Beltrami operator, for the sublaplacians acting on Lie groups, and more generally for diffusion semigroups on-diagonal bounds imply Gaussian off-diagonal bounds (see for example [8, 14, 22]). The advantage of our approach is that Theorem 4 again holds without any assumptions about the nature of the semigroup generator. Our only assumption is the finite propagation speed. And so for example Theorem 4 together with Theorem 6 show that on-diagonal estimates imply sharp Gaussian off-diagonal estimates for the heat kernels generated by the Hodge Laplacian acting on \( p \)-forms (see Corollary 5). The proofs from [8, 14, 22] do not easily generalize to the De Rham-Hodge Laplacian setting. It turns out that the understanding of the behavior of the heat kernel generated by the Hodge Laplacian is a useful tool in the study of the \( L^p \) boundedness of the Riesz transform for \( p > 2 \) (see [10] and [11]). In Theorem 10 we describe a natural generalization of the main result from [10] and [11, Theorem 5.5]. We obtain Theorem 10 as a straightforward consequence of Theorem 5 and Theorem 6. The main idea of the proof of Theorem 4 comes from [36]. However, here we significantly simplify the proof. We also state the result in a substantially more general and natural setting.

## 2. Assumptions and notation

Before we state our main results we have to introduce some notation and describe our set of assumptions.

**Assumption 1.** Let \( X \) be a metric measurable space equipped with a Borel measure \( \mu \) and metric \( \rho \). Next let \( B(x, r) = \{ y \in X, \rho(x, y) \leq r \} \) be the open ball with center at \( x \) and radius \( r \). We suppose throughout that \( X \) satisfies the doubling property, i.e. there exists a
constant $C$ such that
\begin{equation}
\mu(B(x, 2r)) \leq C \mu(B(x, r))
\end{equation}
uniformly for all $x \in X$ and for all $r > 0$.

Note also that (2.1) implies that there exist positive constants $C$ and $D$ such that
\begin{equation}
\mu(B(x, \gamma r)) \leq C(1 + \gamma)^D \mu(B(x, r)) \quad \forall \gamma > 0, x \in X, r > 0.
\end{equation}

In the sequel the value $D$ always refers to the constant in (2.2).

Next suppose that $TX$ is a continuous vector bundle with the base $X$, fibers $C^l$ and with measurable (with respect to $x$) scalar product $( \cdot, \cdot)_x$. For $f(x) \in T_x X$ we put $|f(x)|_x^2 = (f(x), f(x))_x$. To simplify the notation we will write $(\cdot, \cdot)$ and $|\cdot|$ instead of $(\cdot, \cdot)_x$ and $|\cdot|_x$. Now for sections $f$ and $g$ of $TX$ we put
\[
\|f\|_{L^p(TX)} = \int_X |f(x)|^p \, d\mu(x) \quad \text{and} \quad \langle f, g \rangle = \int_X (f(x), g(x)) \, d\mu(x).
\]

By $L^p(TX, \mu)$ we denote the Banach spaces corresponding to this norms. $L^2(TX, \mu)$ is a Hilbert space with the scalar product $(\cdot, \cdot)$.

Now suppose that $L$ is a self-adjoint positive definite operator acting on $L^2(TX, \mu)$. Such an operator admits a spectral decomposition $E_L(\lambda)$ and for any bounded Borel function $F : [0, \infty) \to \mathbb{C}$, we define the operator $F(L) : L^2(TX) \to L^2(TX)$ by the formula
\begin{equation}
F(L) = \int_0^\infty F(\lambda) \, dE_L(\lambda).
\end{equation}

Suppose that $S$ is a bounded operator from $L^p(TX)$ to $L^q(TX)$. We write $\|S\|_{L^p(TX) \to L^q(TX)}$ for the usual operator norm of $S$. If $S$ is of weak type $(1, 1)$, i.e., if
\[
\mu(\{x \in X : |Sf(x)| > \lambda\}) \leq C \frac{\|f\|_{L^1(TX)}}{\lambda} \quad \forall \lambda \in \mathbb{R}^+, \forall f \in L^1(TX),
\]
then we write $\|S\|_{L^1 \to L^{1, \infty}}$ for the least possible value of $C$ in the above inequality; this is often called the “operator norm”, though in fact it is not a norm.

Now let us describe the notion of integral operators. For any point $(x, y) \in X^2$ we consider the space $T_y^* \otimes T_x$. The space $T_y^* \otimes T_x$ is canonically isomorphic to $\text{Hom}(T_y, T_x)$, the space of all linear homeomorphisms from $T_y$ to $T_x$. $T_x$ and $T_y$ are equipped with scalar product and one can consider two natural norms on $T_y^* \otimes T_x$. These norms are: the Hilbert-Schmidt norm $|\cdot|_{HS}$ and the operator norm $|\cdot|$. Let us recall that if $K(x, y) : T_y \to T_x$ is a bounded operator from $T_y$ to $T_x$, then $|K(x, y)|_{HS}^2 = \operatorname{Tr} K(x, y) K(x, y)^* = \sum_{i=1}^l \sum_{j=1}^l |(K(x, y) e_i^y, e_j^x)|^2$, where $K(x, y)^*$ is the adjoint of $K(x, y)$ and $e_1^y, \ldots, e_l^y$ are arbitrary orthonormal bases for $T_y$, $z = x$ or $z = y$. Note that
\begin{equation}
|K(x, y)| \leq |K(x, y)|_{HS} \leq l^{1/2} |K(x, y)|
\end{equation}
for all $K(x, y) \in T_y^* \otimes T_x$. By $(T^* \otimes T)X^2$ we denote the continuous bundle with the base space equal to $X^2$ and with the fiber over the point $(x, y)$ equal to $T_y^* \otimes T_x$. If there is a section $K_S$ of $(T^* \otimes T)X^2$ such that $|K_S|$ is a locally integrable function on $(X^2, \mu \times \mu)$ and
\[
\langle Sf_1, f_2 \rangle = \int_X (Sf_1, f_2) \, d\mu = \int_X (K_S(x, y) f_1(y), f_2(x)) \, d\mu(y) \, d\mu(x)
\]
for all sections $f_1$ and $f_2$ in $C_c(TX)$, then we say that $S$ is an integral operator with kernel $K_S$. Note that if for some $q \in [1, \infty)$ and for every $x \in X$ there exists a constant $C_x$ such that 

$$
(2.5) \quad \sup_{f \neq 0} \frac{|Sf(x)|}{\|f\|_{L^q(TX)}} = C_x < \infty,
$$

then by Riesz representation theorem (see [15, Theorem 1, p. 286]) $S$ is an integral operator and

$$
(2.6) \quad l^{-1} \|K_S(x, \cdot)\|_{L^q'(X)} \leq l^{-1} \|K_S(x, \cdot)\|_{HS}_{L^q'(X)} \leq C_x \leq \|K_S(x, \cdot)\|_{L^q'(X)} \leq \|K_S(x, \cdot)\|_{HS}_{L^q'(X)},
$$

where $1/q + 1/q' = 1$ and $1 \leq q < \infty$. Thus if $S_1$ satisfies (2.5) for $q = 2$ and $S_2$ is a bounded operator on $L^2(TX, \mu)$ then $S_1S_2$ is an integral operator. Moreover,

$$
(2.7) \quad \|K_{S_1S_2}(x, \cdot)\|_{HS}_{L^2(X)} \leq \|S_2\|_{L^2(TX) \to L^2(TX)} \|K_{S_1}(x, \cdot)\|_{HS}_{L^2(X)}.
$$

In particular if $F_1(L)$ and $F_2(L)$ are the operators defined by (2.3), then

$$
(2.8) \quad \|K_{F_1F_2(L)}(x, \cdot)\|_{HS}_{L^2(X)} \leq \|F_1\|_{L^\infty} \|K_{F_2(L)}(x, \cdot)\|_{HS}_{L^2(X)}.
$$

Note also that

$$
(2.9) \quad \|K_{F_1(L)}(x, \cdot)\|_{L^2(X)} = \|K_{F_1(L)}(\cdot, x)\|_{L^2(X)},
$$

$$
(2.10) \quad \|K_{F_1(L)}(x, \cdot)\|_{HS}_{L^2(X)} = \|K_{F_1(L)}(\cdot, x)\|_{HS}_{L^2(X)}
$$

and

$$
(2.11) \quad |K_{F_1F_2(L)}(x, y)| \leq \|K_{F_1(L)}(x, \cdot)\|_{L^2(X)} \|K_{F_2(L)}(\cdot, y)\|_{L^2(X)}.
$$

Next

$$
(2.12) \quad \text{Tr} K_{F_1^2(L)}(x, x) = \|K_{F_1(L)}(x, \cdot)\|_{HS}_{L^2(X)}^2 = \|K_{F_1(L)}(\cdot, x)\|_{HS}_{L^2(X)}^2
$$

and so

$$
(2.13) \quad \text{Tr} K_{\exp(-2tL)}(x, x) = \|K_{\exp(-tL)}(x, \cdot)\|_{HS}_{L^2(X)}^2 = \|K_{\exp(-tL)}(\cdot, x)\|_{HS}_{L^2(X)}^2.
$$

Finally note that by (2.6)

$$
(2.14) \quad l^{-1} \sup_{x \in X} \|K_S(x, \cdot)\|_{L^q'(X)} \leq \|S\|_{L^q(X) \to L^\infty(X)} \leq \sup_{x \in X} \|K_S(x, \cdot)\|_{L^q'(X)}.
$$

Hence (see [15, Theorem 6, p. 503]) if $S$ is bounded from $L^1(TX)$ to $L^q(TX)$, where $q > 1$, then $S$ is an integral operator, and

$$
(2.15) \quad l^{-1} \sup_{y \in X} \|K_S(\cdot, y)\|_{L^q(X)} \leq \|S\|_{L^1(TX) \to L^q(TX)} \leq \sup_{y \in X} \|K_S(\cdot, y)\|_{L^q(X)};
$$

vice versa, if $S$ is an integral operator and the right hand side of the above inequality is finite, then $S$ is bounded from $L^1(TX)$ to $L^q(TX)$, even if $q = 1$.

**Theorem 1.** Let $X$ be a measurable metric space with the doubling condition and let $L$ be a self-adjoint positive definite operator. The following conditions are equivalent:

$$
(2.16) \quad \|K_{\exp(-tL)}(x, \cdot)\|_{L^2(X)} \leq C \mu(B(x, t^{1/2}))^{-1} \quad \forall t > 0, x \in X;
$$

$$
(2.17) \quad \|K_{(I+tL)^{-m/4}}(x, \cdot)\|_{L^2(X)}^2 \leq C_m \mu(B(x, t^{1/2}))^{-1} \quad \forall t > 0, x \in X
$$

$^1$We assume that $C_x$ is locally integrable as a function of $x$. 
for any \( m > D \), where \( D \) is the constant from condition (2.2).

**Proof.** Note that

\[
(I + (tL))^{-m/4} = \frac{1}{\Gamma(m/4)} \int_0^\infty e^{-s} s^{m/4-1} \exp(-s(tL)) \, ds.
\]

Hence by (2.2)

\[
\|K_{(I+tL)^{-m/4}}(x, \cdot)\|_{L^2(X)} \leq \frac{1}{\Gamma(m/4)} \int_0^\infty e^{-s} s^{m/4-1} \|K_{\exp(-tsL)}(x, \cdot)\|_{L^2(X)} \, ds
\]

\[
\leq \frac{1}{\Gamma(m/4)} \int_0^\infty e^{-s} s^{m/4-1} \mu(B(x, (st)^{1/2}))^{-1/2} \, ds
\]

\[
\leq \frac{1}{\Gamma(m/4)} \mu(B(x, t^{1/2}))^{-1/2} \int_0^\infty e^{-s} s^{m/4-1}(1 + 1/s)^{D/4} \, ds
\]

\[
= C\mu(B(x, t^{1/2}))^{-1/2}.
\]

To prove that (2.17) implies (2.16) we note that by (2.8) and (2.4)

\[
\|K_{\exp(-tL)}(x, \cdot)\|_{L^2(X)} \leq \|\exp(-tL)(1 + tL)^m\|_{L^2 \to L^2} \|K_{(I+tL)^{-m}}(x, \cdot)\|_{L^2(X)}
\]

(2.18) \leq \frac{1}{\lambda} \sup_{\lambda \in \mathbb{R}^+} e^{-t\lambda}(1 + t\lambda)^m \|K_{(I+tL)^{-m}}(x, \cdot)\|_{L^2(X)} \leq C\|K_{(I+tL)^{-m}}(x, \cdot)\|_{L^2(X)}.
\]

**Remarks.** 1. Note that Theorem 1 remains valid if we replace \( \mu(B(x, t^{1/2}))^{-1/2} \) by \( v_x(t) \)

for any decreasing function \( v_x \). (2.17) implies (2.16) without any assumptions on \( v_x \) or \( \mu \).

To show the inverse implication one has to assume that \( v_x(t) \leq C v_x(t)(1 + 1/s)^{D/4} \). For

\( v_x(t) = \mu(B(x, t^{1/2}))^{-1/2} \) this means that \( \mu \) satisfies condition (2.2).

2. Note that in virtue of (2.13) and (2.21) it is enough to know the value of \( \text{Tr } K_{\exp(-2tL)}(x, x) \)

to verify condition (2.16). Therefore condition (2.16) is often called on-diagonal bounds of a heat kernel. Condition (2.16) is well understood. On-diagonal bounds are very often used as a basic assumption in theorems concerning the heat kernels and boundedness of the Riesz transforms (see for example [9, 11, 22]). There are many examples of operators satisfying condition (2.16) and there are efficient techniques to obtain condition (2.16) for some particular class of operators in the scalar case. For example it is known that for

the Laplace-Beltrami operators condition (2.16) is equivalent to a relative Faber-Krahn inequality (see [23]).

The related literature is too large to be listed here, so we refer reader to [7, 12, 23, 32, 44] for the related theory, examples of operators satisfying (2.16) and for further references.

Now we set

\[
\mathcal{D}_r = \{(x, y) \in X \times X : \rho(x, y) \leq r\}.
\]

Given an operator \( S \) from \( L^p(TX) \) to \( L^q(TX) \), we write

(2.19) \[ \text{supp } K_S \subseteq \mathcal{D}_r \]

if \( \langle Sf_1, f_2 \rangle = 0 \) whenever \( f_n \) is in \( C(TX) \) and \( \text{supp } f_n \subseteq B(x_n, r_n) \) when \( n = 1, 2, \) and \( r_1 + r_2 + r < \rho(x_1, x_2) \). This definition makes sense even if \( S \) is not an integral operator, in the sense of the previous definition. If \( S \) is an integral operator with the kernel \( K_S \), then
is equivalent to the standard meaning of supp $K_s \subseteq D_r$, that is $K_s(x, y) = 0$ for all $(x, y) \notin D_r$.

**Theorem 2.** Let $L$ be a self-adjoint positive definite operator acting on $L^2(X)$. The following conditions are equivalent:

\[(2.20) \quad \text{supp } K_{\cos(t\sqrt{L})} \subseteq D_t \quad \forall t \geq 0;\]

\[(2.21) \quad |\langle \exp(-tL)f_1, f_2 \rangle| \leq Ce^{-\frac{t^2}{4\pi}}\|f_1\|_{L^2(TX)}\|f_2\|_{L^2(TX)} \quad \forall t > 0,\]

whenever $f_n$ is in $C(TX)$ and supp $f_n \subseteq B(x_n, r_n)$ when $n = 1, 2$, and $0 \leq r < \rho(x_1, x_2) - (r_1 + r_2)$.

**Remark.** The connection of the heat and the wave equation has a long history (see [4, 29] see also [36] and the third proof of [23 Theorem 3.2, p. 157]). For the origin of the $L^2$ Gaussian estimates (2.21) so-called the Davies or the Davies-Gaffney estimates see [13].

**Proof.** Suppose that supp $f_n \subseteq B(x_n, r_n)$ for $n = 1, 2$, and that $0 \leq r < \rho(x_1, x_2) - (r_1 + r_2)$. Put

$$u(z) = \langle \exp(-L/4z) \rangle f_1, f_2 >.$$  

$L$ is a self-adjoint positive definite operator so $u$ is an analytic function on the complex half-plane Re $z > 0$, continuous and bounded on the set $\{z \in \mathbb{C}: \text{Re } z \geq 0, z \neq 0\}$, and

$$\sup_{\text{Re } z = 0} |e^{r^2z}u(z)| \leq \|f_1\|_{L^2(TX)}\|f_2\|_{L^2(TX)}.\]

By (2.21)

$$\sup_{z \in \mathbb{R}_+} |e^{r^2z}u(z)| \leq C\|f_1\|_{L^2(TX)}\|f_2\|_{L^2(TX)}.\]

Hence, by Phragmén-Lindelöf theorem for an angle (see [28 Theorem 7.5, p.214, vol. II] or [41 Lemma 4.2, p.107])

$$|e^{r^2z}u(z)| \leq \|f_1\|_{L^2(TX)}\|f_2\|_{L^2(TX)}$$

and

\[(2.22) \quad |u(z)| \leq \exp(-r^2\text{Re } z)\|f_1\|_{L^2(TX)}\|f_2\|_{L^2(TX)}\]

for all $z$ such that $\text{Re } z > 0$. Next we note that

\[(2.23) \quad \langle \exp(-sL)f_1, f_2 \rangle = \int_0^\infty \langle \cos(t\sqrt{L})f_1, f_2 \rangle \frac{2}{\pi s} e^{-\frac{t^2}{4}} dt.\]

By change of variable $t := \sqrt{t}$ in integral (2.23) and putting $s := 1/(4s)$ we get

\[(2.24) \quad s^{-1/2} \langle -\frac{L}{4s} f_1, f_2 \rangle = 2 \int_0^\infty (\pi t)^{-1/2} < \cos(\sqrt{t\sqrt{L}})f_1, f_2 > e^{-st} dt,\]

so the function $v(z) = z^{-1/2}u(z)$ is a Fourier-Laplace transform of the function $w(t) = (\sqrt{\pi t})^{-1} < \cos(\sqrt{t\sqrt{L}})f_1, f_2 >$. Now by (2.22) and the Paley-Wiener Theorem (Theorem 7.4.3 [25])

\[(2.25) \quad \text{supp } w \subseteq [r^2, \infty).\]
Then, there exists a constant $C$ such that
\begin{align}
| \langle \exp(-sL)f_1, f_2 \rangle | & \leq \int_0^\infty | \langle \cos(t\sqrt{L})f_1, f_2 \rangle | \frac{2}{\sqrt{\pi s}} e^{-\frac{t^2}{2s}} \, dt \\
& = \int_r^\infty | \langle \cos(t\sqrt{L})f_1, f_2 \rangle | \frac{2}{\sqrt{\pi s}} e^{-\frac{t^2}{2s}} \, dt \\
& \leq \|f_1\|_{L^2(TX)} \|f_2\|_{L^2(TX)} \int_r^\infty \frac{2}{\sqrt{\pi s}} e^{-\frac{t^2}{2s}} \, dt \\
& \leq e^{-\frac{r^2}{8}} \|f_1\|_{L^2(TX)} \|f_2\|_{L^2(TX)}.
\end{align}

The following lemma is a very simple but useful consequence of (2.20).

**Lemma 3.** Assume that $L$ satisfies (2.20) and that $\hat{F}$ is the Fourier transform of an even bounded Borel function $F$ with $\text{supp} \hat{F} \subseteq [-r, r]$. Then $\text{supp} K_{F(\sqrt{L})} \subseteq D_r$.

**Proof.** Since $F$ is even, by the Fourier inversion formula,
\[
F(\sqrt{L}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{F}(t) \cos(t\sqrt{L}) \, dt.
\]
But $\text{supp} \hat{F} \subseteq [-r, r]$ and Lemma 3 follows from (2.20). \hfill \square

3. **Main results**

We are now in a position to state our two main results.

**Theorem 4.** Suppose that for some number $N \in \mathbb{N}$ and points $x, y \in X$ there exist functions $V_x, V_y : \mathbb{R}^+ \to \mathbb{R}$ such that
\[
\|K_{(1+\varepsilon L)^{-N/4}(z, \cdot)}\|_{L^2(X)} \leq V_z(t) \quad \forall t > 0, \ z = x, y.
\]
Then, there exists a constant $C_N$ such that for all $t < \rho(x, y)^2$
\begin{align}
|K_{\exp(-tL)}(x, y)| & \leq C_N V_x \left( \frac{t}{\rho(x, y)} \right) V_y \left( \frac{t}{\rho(x, y)} \right) \exp \left( \frac{-\rho(x, y)^2}{4t} \right) .
\end{align}

Thus if $L$ satisfies (2.16) or (2.17), then
\begin{align}
|K_{\exp(-tL)}(x, y)| & \leq C_\mu \left( B \left( x, \frac{t}{\rho(x, y)} \right) \right)^{-\frac{1}{2}} \mu \left( B \left( y, \frac{t}{\rho(x, y)} \right) \right)^{-\frac{1}{2}} \exp \left( \frac{-\rho(x, y)^2}{4t} \right) \\
& \quad \quad \text{for all } t < \rho(x, y)^2.
\end{align}

**Theorem 5.** Suppose that $X$ is a measurable metric space satisfying Assumption 7 and that $TX$ and $T'X$ are vector bundles with measurable scalar products. Next assume that an operator $L$ acting on $L^2(TX)$ satisfies (2.17) and (2.20). Assume also that $A : D(A) \to L^2(T'X)$ is a local operator, which means that for any $f \in D(A) \subset L^2(TX)$
\[
\text{supp } Af \subseteq \text{supp } f.
\]
Finally assume that $D(L^\alpha) \subset D(A)$ and $AL^{-\alpha} : L^2(TX) \to L^2(T'X)$ is bounded for some $\alpha > 0$. Then the operator $AL^{-\alpha}$ is of weak type $(1, 1)$ and bounded as an operator from $L^p(TX, \mu)$ to $L^p(T'X, \mu)$ for all $p \in (1, 2]$. 

Remarks. 1. Note that by (2.11) and (2.18) \( |K_{\exp(-2tL)}(x,y)| \leq V_y(\sqrt{t})V_y(\sqrt{t}) \). For \( t \geq \rho(x,y)^2 \) this obvious estimate is sharp even for the standard Laplace operator. Therefore the discussion of Gaussian bounds for \( t \geq \rho(x,y)^2 \) is straightforward so we do not include description of the heat kernel bounds for this case in the statement of Theorem 4.

2. Theorem 4 holds without the doubling volume property. One needs Assumption 1 only to prove that (2.16) and (2.17) are equivalent. In our proof that (3.1) implies (3.2) and that finite speed propagation property for a large class of operators.

3. Note that if Assumption 1 holds then \( \mu(B(x,t/\rho(x,y)))^{-\frac{1}{2}} \leq \mu(B(x,\sqrt{t}))^{-\frac{1}{2}}(\rho(x,y)/\sqrt{t})^{D/2} \) and by (3.3)

\[
|K_{\exp(-tL)}(x,y)| \leq C \mu(B(x,\sqrt{t}^{-\frac{1}{2}}\mu(B(y,\sqrt{t}))^{-\frac{1}{2}}(\rho(x,y)/\sqrt{t})^{(D-1)/2}\exp\left(-\frac{\rho(x,y)^2}{4t}\right).
\]

In [30] Molchanov proved that if \( N \) is the north pole and \( S \) is the south pole of the \( D \)-dimensional unit sphere and \( L \) is the Laplace-Beltrami operator on the sphere, then

\[
K_{\exp(-tL)}(N,S) \sim t^{-D/2}(1 + \rho(S,N)/\sqrt{t})^{D-1}\exp\left(-\frac{\rho(N,S)^2}{4t}\right) \quad \text{as } t \downarrow 0.
\]

This shows that estimates (3.2) and (3.3) are sharp (see also [23, Theorem 5.9] and [36]).

4. Finite speed propagation

Finite speed propagation property for the solution of the wave equation is one of our main assumptions. Hence for the sake of completeness we describe the proof of finite speed propagation property for a large class of operators.

Finite speed propagation property for the wave propagator is well known (see for example [20, Theorem (5.3)], [13, Theorem 6.1]). However, the statement of Theorems 4.1 below is more general than what is usually found in the literature. Moreover, the proof given here is simpler than other proofs known to the author.

Suppose that \( M \) is a complete Riemannian manifold and \( \mu \) is an absolutely continuous measure with a smooth density not equal to zero at any point of \( M \). By \( \Lambda^kT^*M \) we denote the bundle of \( k \)-forms on \( M \). For fixed \( \beta, \beta_* \in L^2(\Lambda^1T^*M) \) and \( \gamma \in L^2(\Lambda^kT^*M) \) we define the operator \( L \) (\( L = L_{\beta, \beta_*, \gamma} \)) acting on \( L^2(\Lambda^kT^*M) \) by the formula

\[
\langle L\omega, \omega \rangle = \int_M |d_k\omega + \omega \wedge \beta|^2 + |d_{n-k} \ast\omega + \ast\omega \wedge \beta_*|^2 + |\ast\omega \wedge \gamma|^2 \, d\mu(x),
\]

where \( \omega \) is a smooth compactly supported \( k \)-form and \( \ast \) is the Hodge star operator. With some abuse of notation we also denote by \( L \) its Friedrichs extension. Note that for example the Hodge Laplacian (see § 7) and Schrödinger operators with electromagnetic fields (see § 8.1) can be defined by (4.1).

**Theorem 6.** The operator \( L \) defined by (4.1) satisfies (2.20) and (2.21).
Proof. We put \( \omega_t(x) = \omega(t, x) = \exp(-tL)\omega \). Then we fix some function \( \xi \in C^\infty(M) \) such that \( |d\xi| \leq \kappa \) and we consider the integral

\[
E(t) = \int_M (\omega(t, x), \omega(t, x)) e^{\xi(x)} \, d\mu(x).
\]

Next we note that for every \( k \)-form \( \eta \) and 1-form \( \zeta \) we have \( |\zeta \wedge \eta|^2 + |\zeta \wedge *\eta|^2 = |\eta|^2|\zeta|^2 \) and

\[
\frac{E'(t)}{2} = \text{Re} \int_M (\partial_t \omega(t, x), \omega(t, x)) e^{\xi} \, d\mu(x) = -\text{Re} \int_M (L\omega_t, \omega_t e^{\xi}) \, d\mu
\]

Next assume that \( 0 \leq \rho(x_1, x_2) < \rho \). Then

\[
\int_M [(d_0 \omega_t + \omega_t \wedge \beta) + (\omega_t \wedge \gamma) + (*\omega_t \wedge \gamma) e^\xi] \, d\mu
\]

Hence \( E(t) \leq \exp(\kappa^2 t/2) E(0) \). Now we say that \( \xi \in \Theta_\kappa \subseteq C^\infty(M) \) if \( \xi(x) = 0 \) for \( x \in B(x_1, r_1) \) and \( |d\xi| \leq \kappa \). Next assume that \( 0 \leq r < \rho(x_1, x_2) - (r_1 + r_2) \). Then

\[
\sup_{\xi \in \Theta_\kappa} \int_{B(x_2, r_2)} |\xi|^2 e^{\xi} \, d\mu \geq e^{r\kappa} \int_{B(x_2, r_2)} |\xi|^2 \, d\mu.
\]

Hence if \( \text{supp} \, \omega_0 \subseteq B(x_1, r_1) \) then

\[
\int_{B(x_2, r_2)} |\omega|^2 \, d\mu \leq \exp \left( \frac{\kappa^2 t}{2} - r\kappa \right) \int_{B(x_2, r_2)} |\omega_0|^2 \, d\mu
\]

Putting \( \kappa = r/t \) we get

\[
\int_{B(x_2, r_2)} |\omega|^2 \, d\mu \leq \exp \left( \frac{-r^2}{2t} \right) \int_M |\omega_0|^2 \, d\mu = \exp \left( \frac{-r^2}{2t} \right) \int_{B(x_2, r_2)} |\omega_0|^2 \, d\mu
\]

Now \( (2.21) \) is a straightforward consequence of \( (4.2) \).

5. Off-diagonal Gaussian bounds, proof of Theorem \( 11 \)

Proof. For \( s > 1 \), we define the family of functions \( \phi_s \) by the formula

\[
\phi_s(x) = \psi(s(|x| - s)),
\]
where \( \psi \in C^\infty(\mathbb{R}) \) and
\[
\psi(x) = \begin{cases} 
0 & \text{if } x \leq -1 \\
1 & \text{if } x \geq -1/2.
\end{cases}
\]

Finally we define functions \( F_s \) and \( R_s \) by the following formula
\[
F_s(x) = \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{x^2}{4}\right) - R_s(x) = \phi_s(x) \frac{1}{\sqrt{4\pi}} \exp\left(-\frac{x^2}{4}\right)
\]
so that \( \hat{F}_s(\lambda) + \hat{R}_s(\lambda) = \exp(-\lambda^2) \) and
\begin{equation}
(5.1) \quad \hat{F}_s(\sqrt{tL}) + \hat{R}_s(\sqrt{tL}) = \exp(-tL).
\end{equation}

Integration by parts \( N \) times yields
\[
\int \phi_s(x) e^{-\frac{x^2}{4}} e^{-ix\lambda} = \int \left( \frac{1}{x/2 + i\lambda} \cdots \frac{1}{x/2 + i\lambda} \phi_s(x) \right) e^{-\frac{x^2}{4}} e^{-ix\lambda} dx.
\]

Hence for any natural number \( N \) and \( s > 1 \)
\begin{equation}
(5.2) \quad |\hat{F}_s(\lambda)| \leq C'_N \frac{1}{s(1 + \lambda^2/s^2)^{N/2}} e^{-\frac{s^2}{4}},
\end{equation}
where \( C'_N \) is a constant depending only on \( N \). Next we note that \( \text{supp } R_s \subseteq [-s + \frac{1}{2\pi}, s - \frac{1}{2\pi}] \), so if we put \( s_{xy} = \rho(x, y)t^{-1/2} \), then Lemma \( 3 \) \( K_{\hat{F}_{s_{xy}}(\sqrt{tL})}(x, y) = 0 \). Hence by \( 5.1 \)
\begin{equation}
(5.3) \quad K_{\exp(-tL)}(x, y) = K_{\hat{F}_{s_{xy}}(\sqrt{tL})}(x, y).
\end{equation}

Now let \( J_{s_{xy}} \) be a function such that \( J_{s_{xy}}(\lambda) = \hat{F}_{s_{xy}}(t^{1/2}\lambda) \). By \( 5.2 \)
\[
\sup_{\lambda \geq 0} \left| J_{s_{xy}}(\lambda) \left( 1 + \frac{\lambda^2 t^2}{\rho(x, y)^2} \right)^{N/4} \right| \leq C \frac{\exp\left(-\frac{\rho(x, y)^2}{st}\right)}{\sqrt{\rho(x, y)^2 t^{-1/2}}}.
\]

Hence by \( 2.8 \) and \( 2.4 \)
\begin{equation}
(5.4) \quad \|K_{J_{s_{xy}}(\sqrt{tL})}(x, \cdot)\|_{L^2(X)} \leq C \frac{\exp\left(-\frac{\rho(x, y)^2}{st}\right)}{\sqrt{\rho(x, y)^2 t^{-1/2}}} \left\| K_{(t^{1/2}L)_{\rho(x, y)^2}^{-1}}(x, \cdot) \right\|_{L^2(X)}.
\end{equation}

Finally by \( 2.11 \)
\begin{equation}
(5.5) \quad |K_{\exp(-tL)}(x, y)| = |K_{\hat{F}_{s_{xy}}(\sqrt{tL})}(x, y)| \leq \|K_{J_{s_{xy}}(\sqrt{tL})}(x, \cdot)\|_{L^2(X)} \left\| K_{J_{s_{xy}}(\sqrt{tL})}(y, \cdot) \right\|_{L^2(X)}
\end{equation}
and \( 3.2 \) follows from \( 5.4 \) and \( 5.5 \). 

\( 5.3 \) shows that the remainder \( \hat{R}_{s_{xy}}(\sqrt{tL}) \) does not contribute to the value of the heat kernel \( K_{\exp(-tL)}(x, y) \). Subtracting the remainder from the heat propagator is the main idea of the proof.
6. Riesz Transform, proof of Theorem 5

We fix an even function \( \Phi \) in the Schwartz space \( \mathcal{S}(\mathbb{R}) \) such that \( \Phi(0) = 1 \), whose Fourier transform \( \hat{\Phi} \) is supported in \([-1, 1]\); we let \( \Phi_r \) denote the dilated function \( \Phi(r \cdot) \) and \( \Phi^{(l)} \) denote the \( l \)th derivative of \( \Phi \). For later purposes, note that for any fixed positive integer \( K \), one may assume that \( \Phi^{(l)}(0) = 0 \) when \( 1 \leq l \leq K \).

Lemma 7. Let \( \Phi \) in \( \mathcal{S}(\mathbb{R}) \) be chosen as above. If (2.17) and (2.20) hold, then the kernel \( K_{\Phi_r(\sqrt{L})} \) of the self-adjoint operator \( \Phi_r(\sqrt{L}) \) satisfies

\[
\text{supp } K_{\Phi_r(\sqrt{L})} \subseteq D_r
\]

and

\[
\int |K_{\Phi_r(\sqrt{L})}(x,y)|^2 \, d\mu(x) = \int |K_{\Phi_r(\sqrt{L})}(y,x)|^2 \, d\mu(x) \leq C \mu(B(y,r))^{-1}
\]

for all \( r > 0 \) and \( y \in X \).

Proof. (6.1) follows from Lemma 3. Next by (2.7)

\[
\|K_{\Phi_r(\sqrt{L})}(\cdot,y)\|_{L^2(X)} \leq l^{1/2} \|(I + r^2 L)^m \Phi_r(\sqrt{L})\|_{L^2(TX) \to L^2(TX)} \|K((I+r^2L)^{-m}(\cdot,y))\|_{L^2(X)}.
\]

The \( L^2 \) operator norm of the first term is equal to the \( L^\infty \) norm of the function \((1 + r^2 \lambda)^{2m} \Phi(r \sqrt{\lambda})\) which is uniformly bounded in \( r > 0 \) for any fixed \( m \in \mathbb{N} \) and so (6.2) follows from (2.17). □

We now recall the Calderón-Zygmund decomposition. (see e.g. [5, 6, 40]).

Theorem 8. There exists \( C \) such that, given \( f \in L^1(TX, \mu) \) and \( \lambda > 0 \), one can decompose \( f \) as

\[
f = g + b = g + \sum b_i
\]

so that

(a) \(|g(x)| \leq C\lambda, \text{ a.e. } x \) and \( \|g\|_{L^1(TX)} \leq C \|f\|_{L^1(TX)} \).

(b) There exists a sequence of balls \( B_i = B(x_i, r_i) \) such that the support of each \( b_i \) is contained in \( B_i \) and

\[
\int |b_i(x)| \, d\mu(x) \leq C\lambda \mu(B_i).
\]

(c) \( \sum \mu(B_i) \leq C \frac{1}{\lambda} \int |f(x)| \, d\mu(x) \).

(d) There exists \( \sigma \in \mathbb{N} \) such that each point of \( X \) is contained in at most \( \sigma \) of the balls \( B(x_i, 2r_i) \).

The proof of Theorem 8 is a variant of standard arguments, for which see, e.g. [5, p. 66], [6] or [40, p. 8].

We are now in a position to prove Theorem 5.

\[\text{Note that we do not have to assume that } \int b_i = 0 \text{ which could be difficult to achieve for the vector bundle version which we consider here (see [11]).}\]
Proof of Theorem We start by decomposing \( f \) into \( g + \sum b_i \) at level \( \lambda \) according to Theorem. We will follow the idea of using more information of the \( L^2 \) operator norm (in our case, \( \| AL^{-\alpha} \|_{L^2(T^*X) \to L^2(T^*X)} < \infty \)) by smoothing out the bad functions \( b_i \) at a scale of the size of their support and considering this part of the good function where \( L^2 \) estimates can be used (see [2,16,19,24] for other variants of this).

In our case let \( G = g + \sum \Phi_{r_i}(\sqrt{L})b_i \) be the modified good function.\(^4\) Hence \( f = G + B \), where \( B = \sum (I - \Phi_{r_i}(\sqrt{L}))b_i \) and we write

\[
\lambda \mu(\{|AL^{-\alpha}f(x)| \geq \lambda \}) \leq \lambda \mu(\{|AL^{-\alpha}G(x)| \geq \lambda/2 \}) + \lambda \mu(\{|AL^{-\alpha}B(x)| \geq \lambda/2 \}).
\]

The first term is less than \( 4/\lambda \| AL^{-\alpha}G \|^2_{L^2(T^*X)} \leq C/\lambda \| G \|^2_{L^2(T^*X)} \). However, according to Lemma \( \| \Phi_{r_i}(\sqrt{L})b_i \|_{L^2(T^*X)} \leq C \sum_{y \in B(x_i, 2r_i)} \int |K_{\Phi_{r_i}(\sqrt{L})}(x, y)|^2 \mu(y) |b_i(y)|^2_{L^1(T^X)} \leq C \mu(B(x_i, r_i))^{-1} \| b_i \|^2_{L^1(T^X)} \leq C \lambda \| b_i \|_{L^1(T^X)} \).

Hence by Theorem again

\[
\| G \|^2_{L^2(T^X)} \leq C \left( \| g \|^2_{L^2(T^X)} + \sigma \lambda \sum_i \| b_i \|^2_{L^1(T^X)} \right) \leq C \lambda \| f \|_{L^1(T^X)}.
\]

and so the first term in (6.3) is bounded by \( C \| f \|_{L^1(T^X)} \).

Since \( \mu(\cup B(x_i, 2r_i)) \leq C \sum \mu(B_i) \leq C \| f \|_{L^1(T^X)} / \lambda \), then to bound the second term in (6.3), it suffices to show

\[
\int_{x \notin \cup B_i} |AL^{-\alpha}B(x)| \mu(x) \leq C \| f \|_{L^1(T^X)},
\]

where \( B_i = B(x_i, 2r_i) \). The left side of (6.4) is less than

\[
\sum_i \int_{x \notin \cup B_i} \left| \int K_{AL^{-\alpha}(1-\Phi_{r_i})(\sqrt{L})}(x, y)b_i(y) \mu(y) \right| \mu(x) \leq \sum_i \int |b_i(y)| \int_{x \notin B_i} |K_{AL^{-\alpha}(1-\Phi_{r_i})(\sqrt{L})}(x, y)| \mu(y) \mu(x).
\]

By (2.15) (6.4) follows from Theorem once we establish

\[
\sup_{y, r \geq r} \int_{\rho(x, y) \geq r} |K_{AL^{-\alpha}(1-\Phi_{r})(\sqrt{L})}(x, y)| \mu(x) \leq C.
\]

By the Cauchy-Schwartz inequality,

\[
\int_{\rho(x, y) \geq r} |K_{AL^{-\alpha}(1-\Phi_{r})(\sqrt{L})}(x, y)| \mu(x) \leq \sum_{j \geq 1} \int_{2^j r \geq \rho(x, y) \geq 2^{j-1} r} |K_{AL^{-\alpha}(1-\Phi_{r})(\sqrt{L})}(x, y)| \mu(x)
\]

\[
\leq \sum_{j \geq 1} \mu(B(y, 2^j r))^{1/2} \left( \int_{\rho(x, y) \geq 2^{j-1} r} |K_{AL^{-\alpha}(1-\Phi_{r})(\sqrt{L})}(x, y)|^2 \mu(x) \right)^{1/2}.
\]

\(^4\) The \( \Phi_{r_i} \) is the function from Lemma.
Fix a nonnegative even $\varphi \in C^\infty_c(\mathbb{R})$ such that $\varphi = 1$ on $[-1/4, 1/4]$ and $\varphi = 0$ on $\mathbb{R}\setminus [-1/2, 1/2]$. Set $\varphi_s(\lambda) = \varphi(s\lambda)$ and let denote by $\hat{\varphi}_s$ the inverse Fourier transform of $\varphi_s$. We put $H^\alpha(\lambda) = \lambda^{-2\alpha}$. Note that $H^\alpha(1 - \Phi_r)(\lambda) = r^{2\alpha} H^\alpha(1 - \Phi_1)(r\lambda)$. We define functions $F_j^\alpha$ and $R_j^\alpha$ by the formula

$$r^{2\alpha} R_j^\alpha(r\lambda) = [H^\alpha(1 - \Phi_r)] * \varphi_{2^{-j}/r}(\lambda) = [H^\alpha(1 - \Phi_r)] - r^{2\alpha} F_j^\alpha(r\lambda).$$

Then

$$\text{supp } \hat{R}_j^\alpha \subset [-2^{j-1}, 2^{j-1}].$$

Hence by (3.4) and Lemma 3 the kernels of $A H^\alpha(1 - \Phi_r)(\sqrt{L})$ and $r^{2\alpha} A F_j^\alpha(r\sqrt{L})$ coincide on the set $X^2 \setminus D_{r,2^j-1}$ and

$$\int_{r^{2j-1} < \rho(x,y)} |K_{A H^\alpha(1 - \Phi_r)(\sqrt{L})}(x,y)|^2 \, d\mu(x) = \int_{r^{2j-1} < \rho(x,y)} |K_{A r^{2\alpha} F_j^\alpha(r\sqrt{L})}(x,y)|^2 \, d\mu(x) \leq \int_X |K_{A r^{2\alpha} F_j^\alpha(r\sqrt{L})}(x,y)|^2 \, d\mu(x) = \int_X |K_{A l^{-\alpha}(r^2 L)^\alpha F_j^\alpha(r\sqrt{L})}(x,y)|^2 \, d\mu(x).$$

However, we assume that $L$ satisfies (2.17) and by (2.7) and (2.8)

$$\int_X |K_{A l^{-\alpha}(r^2 L)^\alpha F_j^\alpha(r\sqrt{L})}(x,y)|^2 \, d\mu(x) \leq \|AL^{-\alpha}\|_{L^2(X) \rightarrow L^2(X)}^2 \|K_{(r^2 L)^\alpha F_j^\alpha(r\sqrt{L})}(\cdot, y)\|_{L^2(X)}^2 \leq C\||J_j^\alpha(r\sqrt{L})|^2\|_{L^2(X) \rightarrow L^2(X)} \|K_{(1+2r^2 L)^{-m}}(\cdot, y)\|_{L^2(X)}^2 \leq C\|J_j^\alpha\|_{L^\infty}^2 \mu(B(y, 2^j r))^{-1},$$

where $J_j^\alpha(r\lambda) = (1 + 2^{2j} r^2 \lambda^2)^m (r\lambda)^{-2\alpha} F_j^\alpha(r\lambda)$. Now to prove that the sum in (6.6) is bounded it is enough to show the following elementary estimate

$$(6.7) \quad \|J_j^\alpha(\lambda)\|_{L^\infty} = \sup_{\lambda \in \mathbb{R}} \|1 + 2^{2j} \lambda^2 \|^m (\lambda)^{-2\alpha} F_j^\alpha(\lambda) \| \leq 2^{-j}.$$ 

As we noted before for any fixed natural number $K \in \mathbb{N}$ we may assume that $\Phi^{(l)}(0) = 0$ when $1 \geq l \geq K$. Now for any natural number $N$ we choose $K$ large enough so that the Fourier transform of $H^\alpha(1 - \Phi_1)$ is in $C^\infty(\mathbb{R} - \{0\})$ and it has a polynomial decay of order $N$. Next $\hat{F}_j^\alpha(\lambda) = [H^\alpha(1 - \Phi_1)](\lambda)(1 - \varphi(2^{-j}\lambda))$ and it is not difficult to note that for any nonnegative integers $m_1$ and $m_2$ we can choose $K$ large enough so that $\sup_{\lambda \in \mathbb{R}} |\lambda^{2m_1} F_j^\alpha(\lambda)| \leq 2^{-m_2j}$ and (6.7) follows.

\[\square\]

7. Hodge Laplacian and Riesz transform for $p > 2$

Let $L_k$ be the Hodge-Laplace operator acting on $L^2(\Lambda^k T^* M)$, where $M$ is $n$-dimension complete Riemannian manifolds. That is

$$\langle L_k \omega, \omega \rangle = \int_M \left[ |d_k \omega|^2 + |d_{k-1} \omega|^2 \right] d\mu(x),$$

where $d_k^*$ is the adjoint operator of $d_k$ (when $k = -1$ or $k = n$ one should interpret $d_k \omega$ and $d^*_k \omega$ as $0$). Note that if $d\mu = \nu(x) \, dx$, where $dx$ is the Riemannian measure then

$$(7.1) \quad d^* \omega \, d\mu = (-1)^{n(k+1)+1} * d * (\nu \omega) \, dx = (-1)^{n(k+1)+1} * (d * \omega + \frac{d\nu}{\nu} \wedge \omega) \, d\mu$$

$^5$We estimate the last integral in (6.6) in a similar way to the proof of theorem 4.
Proof. First we note that transformed $H$ hence ($\equiv$ for some $2 \leq \lambda \leq L$ if satisfies the expected on-diagonal bounds then the Riesz transform $L$ volume property i.e. Assumption 1 and that

8.1. Schrödinger operators. Let $L_k$ be the Hodge-Laplace operator acting on $L^2(\Lambda^k T^* M)$, where $M$ is $n$-dimension complete Riemannian manifolds. Suppose that for some number $N \in \mathbb{N}$ and points $x, y \in X$ there exist functions $V_x, V_y : \mathbb{R}^+ \mapsto \mathbb{R}$ such that

$$\|K_{(I+\epsilon^2 L_k)}(z, \cdot)\|_{L_2(M)} \leq V_z(t) \quad \forall t > 0, z = x, y.$$  

Then, there exists a constant $C$ such that for all $t < \rho(x, y)^2$

$$|K_{\exp(-t L_k)}(x, y)| \leq CV_x \left( \frac{t}{\rho(x, y)} \right) V_y \left( \frac{t}{\rho(x, y)} \right) \exp \left( \frac{-\rho(x, y)^2}{4t} \right) \left( \frac{\rho(x, y)}{t} \right)^{n/2}. $$

Remark. Note that our proof of Theorem 5, contrary to some other available arguments (see [8, 13, 22]), does not use the positivity of the heat kernel. Hence it works for operator acting on fiber bundles and for the Hodge Laplacian in particular.

As we mentioned in introduction one cannot expect in general the Riesz transform $dL_0^{-1/2}$ to be bounded on $L^p$ for $p > 2$. However, the following theorem shows that if $K_{\exp(-t L_1)}(x, y)$ satisfies the expected on-diagonal bounds then the Riesz transform $dL_0^{-1/2}$ is bounded for all $p \in [2, \infty)$.

Theorem 10. Suppose that $M$ is a complete Riemannian manifold satisfying the doubling volume property i.e. Assumption 1 and that $L_1$ satisfies condition (2.10). Then the Riesz transform $d_0 L_0^{-1/2}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for all $2 \leq p < \infty$.

Proof. First we note that $d_0 L_0 = d_0 d_0^* d_0 = d_1^* d_1 d_0 + d_0 d_0^* d_0 = L_1 d_0$ so $d_0^* L_1 = L_0 d_0^*$ and

$$d_0^* L_1^{-1/2} = L_0^{-1/2} d_0^*.$$  

Hence $(d_0 L_0^{-1/2})^* = L_0^{-1/2} d_0^* = d_0^* L_1^{-1/2}$. Now $d_0 L_0^{-1/2}$ is bounded from $L^p(M)$ to $L^p(\Lambda^1 T^* M)$ for some $2 \leq p < \infty$ if and only if $(d_0 L_1^{-1/2})^* = d_0^* L_1^{-1/2}$ is bounded from $L^p(\Lambda^1 T^* M)$ to $L^{p'}(M)$ for $1/p + 1/p' = 1$. However, by (2.11) the operator $d_0^*$ is local and we assume that $L_1$ satisfies (2.11) so continuity of $d_0^* L_1^{-1/2}$ on $L^{p'}$ for $1 < p' \leq 2$ follows from Theorems 5 and 6.

Theorem 10 generalizes results described in [10, 11] to a very natural and somehow optimal setting.

8. Other applications

8.1. Schrödinger operators. Let $M$ be a connected and complete Riemannian manifold. The Riemannian metric give us canonical isomorphisms $\Lambda^1 T^*_x M \cong \Lambda^1 T_x M$. We denote this isomorphism by $\tilde{\omega}$, so if $\omega \in \Lambda^1 T^*_x M$ then $\tilde{\omega}$ is the corresponding dual element in $\Lambda^1 T_x M$ and if $Y \in \Lambda^1 T_x M$ then $\tilde{Y}$ is the corresponding dual element in $\Lambda^1 T^*_x M$. Then if $f$ is a function on $M$, its gradient is the vector $\nabla f = df$. We consider the operator $L_{Y,V}$ given by
the formula
\[
\langle L_{Y,V} f, f \rangle = \int_M \left( |\nabla f(x) + if(x)Y|^2 + V^2(x)|f(x)|^2 \right) d\mu(x)
\]
\[
= \int_M \left( |df + if\tilde{Y}|^2 + |f(x) \wedge V|^2 \right) d\mu(x),
\]
where \( f \in C^\infty_c(M) \), \( Y \) is a real vector field such that \( |Y|^2 \in L^1_{\text{loc}}(M) \), \( V \in L^2_{\text{loc}}(M) \).

**Theorem 11.** Suppose that the manifold \( M \) satisfies Assumption 1 and that the operator \( L_{Y,V} \) satisfies (2.17). Then the operators \( V L_{Y,V}^{-1/2} \) and \( (\nabla - iY)L_{Y,V}^{-1/2} \) are bounded on \( L^p \) for all \( p \in (1,2] \) and of weak type \( (1,1) \).

**Proof.** Theorem 11 is a straightforward consequence of Theorems 5 and 6. \( \Box \)

**Remarks.**
1. In the case \( M = \mathbb{R}^n \) and \( L_{0,0} = \Delta \) Theorem 11 was obtained independently in [31]. In the same setting, i.e \( M = \mathbb{R}^n \) and \( L_{0,0} = \Delta \) the Riesz transform was studied in [34, 35]. Theorem 11 applied to this setting yields an interesting variation of Shen’s result [34, Theorem 0.5] without any assumption concerning regularity of the potential \( V \). The counterexample investigated in [34], \( (V(x) = |x|^{2-\varepsilon}) \) shows that the operator \( \nabla(\Delta + V)^{-1/2} \) is not necessarily bounded for \( p > 2 \). However, boundedness of the Riesz transforms for a larger range of \( p \) can be obtained if one imposes an additional regularity conditions for the potential \( V \) (see again [34]).

2. It follows from [37, Theorem 2.3] and [2, Theorem 4.2, p. 470] that
\[
|K_{\exp - tL_{Y,V}}(x,y)| \leq K_{\exp - tL_{0,0}}(x,y).
\]
Hence if \( L_{0,0} \) satisfies (2.17) (or (2.10)) then \( L_{Y,V} \) also satisfies this assumption.

### 8.2. Sub-elliptic operators acting on Lie groups

Now let me describe another application of Theorem 5. Let \( G \) be a Lie group with polynomial growth. For a system of left-invariant vector fields \( X_1, \ldots, X_k \) satisfying Hörmander condition, a function \( V \in L^2_{\text{loc}}(G) \) and a family of functions \( Y_1, \ldots, Y_k \in L^2(G) \) we define the operator \( L_{V,Y} \) by the formula
\[
L_{V,Y} = - \sum_{j=1}^k (X_j - iY_j)^2 + |V|^2.
\]

One can easily notice that the proof of Theorem 5 works for the operator \( L_{V,Y} \). Hence \( L_{V,Y} \) satisfies the finite speed propagation theorem with respect to the optimal control metrics corresponding to the system \( X_1, \ldots, X_k \) (see e.g. [44, §III.4, p. 39] for the definition). And so the following theorem is again a straightforward consequence of Theorem 5.

**Theorem 12.** The Riesz transforms
\[
(8.4) \quad VL_{V,Y}^{-1/2} \quad \text{and} \quad (X_j - iY_j)L_{V,Y}^{-1/2} \quad \text{for} \quad j = 1, \ldots, k
\]
are bounded on \( L^p \) for all \( p \in (1,2] \) and of weak type \((1,1)\).
Theorem 12 is related to results described in [26]. In [26, Theorem C] Li proved the continuity of the Riesz transforms (8.4) with additional assumptions about the regularity of the potential $V$ in the case $Y_j = 0$ for all $j$.

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