THE Z-MEASURES ON PARTITIONS, PFaffian Point Processes, AND THE MAtrix HYPERGEOMETRIC KERNEL

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Abstract. We consider a point process on one-dimensional lattice originated from the harmonic analysis on the infinite symmetric group, and defined by the z-measures with the deformation (Jack) parameter 2. We derive an exact Pfaffian formula for the correlation function of this process. Namely, we prove that the correlation function is given as a Pfaffian with a 2 × 2 matrix kernel. The kernel is given in terms of the Gauss hypergeometric functions, and can be considered as a matrix analogue of the Hypergeometric kernel introduced by A. Borodin and G. Olshanski [5]. Our result holds for all values of admissible complex parameters.

Keywords. Random partitions, Young diagrams, correlation functions, Pfaffian point processes, the Meixner orthogonal polynomials

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1. Introduction

It is well-known that determinantal point processes appear in different areas of mathematical physics, probability theory and statistical mechanics. The theory of random Hermitian matrices (see, for example, Deift [12]), random growth models (Johansson [18, 19]), the theory of random power series (Peres and Virág [32]) are among numerous topics of current research where the main problems are reduced to investigation of determinantal point processes. We refer the reader to surveys by Soshnikov [33], and by Hough, Krishnapur, Peres, and Virág [15] for definitions and for different properties of determinantal point processes.

Representation theory and the harmonic analysis on the infinite symmetric and the infinite-dimensional unitary groups is yet another area of mathematics where the determinantal processes play a crucial role. The relation between determinantal processes and representation theory of such groups was discovered by Borodin and Olshanski in the series of papers, see Refs. [4, 2, 5, 30]. Let us briefly describe this relation.

Let $S(\infty)$ denote the group whose elements are finite permutations of $\{1, 2, 3, \ldots\}$. The group $S(\infty)$ is called the infinite symmetric group, and it is a model example of a "big" group. Set

$$G = S(\infty) \times S(\infty),$$

$$K = \text{diag} S(\infty) = \{(g, g) \in G \mid g \in S(\infty)\} \subset G.$$ 

Then $(G, K)$ is an infinite dimensional Gelfand pair in the sense of Olshanski [29]. It can be shown that the biregular spherical representation of $(G, K)$ in the space $\ell^2(S(\infty))$ is irreducible. Thus the conventional scheme of noncommutative harmonic analysis is not applicable to the case of the infinite symmetric group.

In 1993, Kerov, Olshanski and Vershik [21] (Kerov, Olshanski and Vershik [22] contains the details) constructed a family $\{T_z : z \in \mathbb{C}\}$ of unitary representations of the bisymmetric infinite group $G = S(\infty) \times S(\infty)$. Each representation $T_z$ acts in the Hilbert space $L^2(\mathcal{G}, \mu_t)$, where $\mathcal{G}$ is a certain compact space called the space of virtual permutations, and $\mu_t$ is a distinguished $G$-invariant probability measure on $\mathcal{G}$ (here $t = |z|^2$). The representations $T_z$ (called the generalized regular representations) are reducible. Moreover, it is possible to extend the definition of $T_z$ to the limit values $z = 0$ and $z = \infty$, and it turns out that $T_\infty$ is equivalent to the biregular representation of $S(\infty) \times S(\infty)$. Thus, the family $\{T_z\}$ can be viewed as a deformation of the biregular representation. Once the representations $T_z$ are constructed, the main problem of the harmonic analysis on the infinite symmetric group is in decomposition of the generalized regular representations $T_z$ into irreducible ones.
One of the initial steps in this direction can be described as follows. Let \(1\) denote the function on \(S\) identically equal to 1. Consider this function as a vector of \(L^2(\mathfrak{S}, \mu_t)\). Then \(1\) is a spherical vector, and the pair \((T, 1)\) is a spherical representation of the pair \((G, K)\), see, for example, Olshanski \([30]\), Section 2. The spherical function of \((T, 1)\) is the matrix coefficient \((T(g_1, g_2) 1, 1)\), where \((g_1, g_2) \in S(\infty) \times S(\infty)\).

Set \(\chi_z(g) = (T_z(g, e) 1, 1)\), \(g \in S(\infty)\).

The function \(\chi_z\) can be understood as a character of the group \(S(\infty)\) corresponding to \(T_z\). Kerov, Olshanski and Vershik \([21, 22]\) found the restriction of \(\chi_z\) to \(S(n)\) in terms of irreducible characters of \(S(n)\). Namely, let \(Y_n\) be the set of Young diagrams with \(n\) boxes. For \(\lambda \in Y_n\) denote by \(\chi^\lambda\) the corresponding irreducible character of the symmetric group \(S(n)\) of degree \(n\). Then for any \(n = 1, 2, \ldots\) the following formula holds true

\[
\chi_z \bigg|_{S(n)} = \sum_{\lambda \in Y_n} M^{(n)}_{z, \bar{z}}(\lambda) \frac{\chi^\lambda}{\chi^\lambda(e)}. \tag{1.1}
\]

In this formula \(M^{(n)}_{z, \bar{z}}\) is a probability measure (called the \(z\)-measure) on the set of Young diagrams with \(n\) boxes, or on the set of integer partitions of \(n\). Formula (1.1) defines the \(z\)-measure \(M^{(n)}_{z, \bar{z}}\) as a weight attached to the corresponding Young diagram in the decomposition of the restriction of \(\chi_z\) to \(S(n)\) in irreducible characters of \(S(n)\). Expression (1.1) enables to reduce the problem of decomposition of \(T_z\) into irreducible components to the problem on the computation of spectral counterparts of \(M^{(n)}_{z, \bar{z}}\).

Using a distribution on \(\{0, 1, 2, \ldots\}\) defined by

\[\text{Prob}\{n\} = (1 - \xi)^\frac{z \bar{z}}{n!} \xi^n, \quad \xi > 0\]

(where \((a)_n\) stands for \(a(a + 1) \cdots (a + n - 1)\)) it is possible to mix distributions \(M^{(n)}_{z, \bar{z}}\), and to obtain a distribution \(M_{z, \bar{z}, \xi}\) on the set of all Young diagrams.

It was shown by Borodin and Olshanski in Ref. \([6]\), that \(M_{z, \bar{z}, \xi}\) defines a determinantal point process on one-dimensional lattice. The kernel of this process has the integrable form in the sense of Its, Izergin, Korepin, and Slavnov \([16]\), and can be written in terms of the Gauss hypergeometric functions. This fact was proved in many ways in a variety of papers (see, for example, Okounkov \([28]\), Borodin, Olshanski, and Strahov \([10]\), Borodin and Olshanski \([7]\), and references therein). The relation between representation theory of big groups and determinantal point processes gave rise to numerous applications from enumerative combinatorics and random growth models to the theory of Painlevé equations, see Borodin and Deift \([3]\).

It is known that if \(z, z' \to \infty\) and \(\xi = \frac{\eta z}{z'} \to 0\), where \(\eta > 0\) is fixed, then \(M_{z, z', \xi}\) tends to Poissonized Plancherel distribution studied in many papers (see, for example, Baik, Deift and Johansson \([1]\)). In particular, it was demonstrated that the Poissonized Plancherel distribution is similar to the Gaussian unitary ensemble (GUE) of random matrix theory, which is an example of an ensemble from the \(\beta = 2\) symmetry class. On
the other hand, in addition to ensembles of $\beta = 2$ symmetry class, random matrix theory deals with ensembles of $\beta = 1$ and $\beta = 4$ symmetry classes. Note that ensembles from both $\beta = 1$ and $\beta = 4$ symmetry classes (in contrast to those from $\beta = 2$ symmetry class) lead to Pfaffian point processes, and analogy between random partitions and random matrices naturally motivates a search for Pfaffian point processes originated from the representation theory of the infinite symmetric group.

It is the purpose of the present paper to construct and to investigate Pfaffian point processes relevant for the representation theory and for the harmonic analysis on the infinite symmetric group. It turns out that such processes are determined by $z$-measures with the Jack parameters $\theta = 2$ and $\theta = 1/2$. The fact that these measures play a role in the harmonic analysis was established by Olshanski [31], and the detailed explanation of this representation-theoretic aspect can be found in Strahov [35]. Due to the fact that $z$-measures with the Jack parameters $\theta = 2$ and $\theta = 1/2$ are related to each other in a very simple way (see Proposition 2.2), it is enough to consider a point process defined by the $z$-measure with the Jack parameter $\theta = 2$. The main new result of the present paper is in explicit computation of the correlation functions for this measure. We prove that the correlation functions of the processes are given by Pfaffian formulas with $2 \times 2$ matrix valued kernel. The kernel is constructed in terms of the Gauss hypergeometric functions. Our result holds for all values of admissible complex parameters $z, z'$.

Once the relation with the harmonic analysis on the infinite symmetric group is the main motivation behind this work, we expect different applications of our results in enumerative combinatorics and statistical physics similar to the case of the $z$-measures with the Jack parameter $\theta = 1$ studied by Borodin and Olshanski.

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2. Definitions and the main result

2.1. The $z$-measures on partitions with the general parameter $\theta > 0$. We use Macdonald [24] as a basic reference for the notations related to integer partitions and to symmetric functions. In particular, every decomposition

$$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) : n = \lambda_1 + \lambda_2 + \ldots + \lambda_l,$$

where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_l$ are positive integers, is called an integer partition. We identify integer partitions with the corresponding Young diagrams. The set of Young diagrams with $n$ boxes is denoted by $\mathbb{Y}_n$.

Following Borodin and Olshanski [8], Section 1, and Kerov [20] let $M_{z,z',\theta}^{(n)}$ be a complex measure on $\mathbb{Y}_n$ defined by

$$M_{z,z',\theta}^{(n)}(\lambda) = \frac{n!(z)_{\lambda,\theta}(z')_{\lambda,\theta}}{(t)_n H(\lambda, \theta) H'(\lambda, \theta)},$$

where $n = 1, 2, \ldots$, and where we use the following notation
THE z-MEASURES ON PARTITIONS AND PFAFFIAN PROCESSES

• $z, z' \in \mathbb{C}$ and $\theta > 0$ are parameters, the parameter $t$ is defined by
  $t = \frac{zz'}{\theta}$.

• $(t)_n$ stands for the Pochhammer symbol,
  $$(t)_n = t(t + 1) \ldots (t + n - 1) = \frac{\Gamma(t + n)}{\Gamma(t)}.$$  

• $(z)_{\lambda, \theta}$ is a multidimensional analogue of the Pochhammer symbol defined by
  $$(z)_{\lambda, \theta} = \prod_{(i,j) \in \lambda} (z + (j - 1) - (i - 1)\theta) = \prod_{i=1}^{l(\lambda)} (z - (i - 1)\theta)_{\lambda_i}.$$  

Here $(i, j) \in \lambda$ stands for the box in the $i$th row and the $j$th column of the Young diagram $\lambda$, and we denote by $l(\lambda)$ the number of nonempty rows in the Young diagram $\lambda$.

• $H(\lambda, \theta) = \prod_{(i,j) \in \lambda} ((\lambda_i - j) + (\lambda'_j - i)\theta + 1)$,
  $H'(\lambda, \theta) = \prod_{(i,j) \in \lambda} ((\lambda_i - j) + (\lambda'_j - i)\theta + \theta)$,

where $\lambda'$ denotes the transposed diagram.

**Proposition 2.1.** The following symmetry relations hold true

$$H(\lambda, \theta) = \theta|\lambda| H'(\lambda', \frac{1}{\theta}), \quad (z)_{\lambda, \theta} = (-\theta)|\lambda| \left(-\frac{z}{\theta}\right)_{\lambda'}.$$  

Here $|\lambda|$ stands for the number of boxes in the diagram $\lambda$.

**Proof.** These relations follow immediately from definitions of $H(\lambda, \theta)$ and $(z)_{\lambda, \theta}$. $\square$

**Proposition 2.2.** We have

$$M_{z, z', \theta}(\lambda) = M_{-z/\theta, -z'/\theta, 1/\theta}(\lambda).$$

**Proof.** Use definition of $M_{z, z', \theta}(\lambda)$, equation (2.1), and apply Proposition 2.1 $\square$

**Proposition 2.3.** We have

$$\sum_{\lambda \in \mathbb{Y}_n} M_{z, z', \theta}(\lambda) = 1.$$  

**Proof.** See Kerov [20], Borodin and Olshanski [8, 9]. $\square$

**Proposition 2.4.** If parameters $z, z'$ satisfy one of the three conditions listed below, then the measure $M_{z, z', \theta}$ defined by expression (2.1) is a probability measure on $\mathbb{Y}_n$. The conditions are as follows.
• Principal series: either $z \in \mathbb{C} \setminus (\mathbb{Z}_{<0} + \mathbb{Z}_{>0} \theta)$ and $z' = \bar{z}$.
• The complementary series: the parameter $\theta$ is a rational number, and both $z, z'$ are real numbers lying in one of the intervals between two consecutive numbers from the lattice $\mathbb{Z} + \mathbb{Z} \theta$.
• The degenerate series: $z, z'$ satisfy one of the following conditions
  1. $(z = m \theta, z' > (m - 1) \theta)$ or $(z' = m \theta, z > (m - 1) \theta)$;
  2. $(z = -m, z' < -m + 1)$ or $(z' = -m, z < m - 1)$.

Proof. See Propositions 1.2, 1.3 in Borodin and Olshanski [8]. □

Thus, if the conditions in the Proposition above are satisfied, then $M_{z,z',\theta}^{(n)}$ is a probability measure defined on $\mathbb{Y}_n$, as follows from Proposition 2.3.

Remark 2.5. When both $z, z'$ go to infinity, expression (2.1) has a limit

\[
M_{\infty,\infty,\theta}^{(n)}(\lambda) = \frac{n! \theta^n}{H(\lambda, \theta) H'(\lambda, \theta)} \tag{2.2}
\]

called the Plancherel measure on $\mathbb{Y}_n$ with general $\theta > 0$. Statistics of the Plancherel measure with the general Jack parameter $\theta > 0$ is discussed in many papers, see, for example, a very recent paper by Matsumoto [25], and references therein. Matsumoto [25] compares limiting distributions of rows of random partitions with distributions of certain random variables from a traceless Gaussian $\beta$-ensemble.

It is convenient to mix all measures $M_{z,z',\theta}^{(n)}$, and to define a new measure $M_{z,z',\xi,\theta}$ on $\mathbb{Y} = \mathbb{Y}_0 \cup \mathbb{Y}_1 \cup \ldots$. Namely, let $\xi \in (0, 1)$ be an additional parameter, and set

\[
M_{z,z',\xi,\theta}(\lambda) = (1 - \xi)^t |\lambda| \frac{(z)_{\lambda, \theta}(z')_{\lambda, \theta}}{H(\lambda, \theta) H'(\lambda, \theta)} \tag{2.3}
\]

Proposition 2.6. We have

\[
\sum_{\lambda \in \mathbb{Y}} M_{z,z',\xi,\theta}(\lambda) = 1.
\]

Proof. Follows immediately from Proposition 2.3. □

If conditions on $z, z'$ formulated in Propositions 1.2, 1.3 in Borodin and Olshanski [7] are satisfied, then $M_{z,z',\xi,\theta}(\lambda)$ is a probability measure on $\mathbb{Y}$. We will refer to $M_{z,z',\xi,\theta}(\lambda)$ as to the $z$-measure with the deformation (Jack) parameter $\theta$.

2.2. A basis in the $l^2$ space on the lattice $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$. In this Section we describe a basis in the $l^2$ space on the 1-dimensional lattice $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$ introduced in Borodin and Olshanski [7], and define certain operators acting in the space $l^2$. Elements of $\mathbb{Z}' = \mathbb{Z} + \frac{1}{2}$ will be denoted by $x, y$. Introduce the principal series, the complementary series, and the degenerate series as in Proposition 2.3 with $\theta = 1$. Assume that parameters $z, z'$ are in the principal series or in the complementary series, but not in the degenerate series. Therefore, the conditions on $z, z'$ are as follows.
The numbers \( z, z' \) are not real and are conjugate to each other (principal series).

Both \( z, z' \) are real and are contained in the same open interval of the form \((m, m + 1)\), where \( m \in \mathbb{Z} \).

In this case we say that parameters \( z, z' \) are admissible. In particular, \( z, z' \) are not integers. Introduce a family of functions on \( \mathbb{Z}' = \mathbb{Z} + \frac{1}{2} \) depending on a parameter \( a \in \mathbb{Z}' \), and also on the parameters \( z, z', \xi \):

\[
\psi_a(x; z, z', \xi) = \left( \frac{\Gamma(x + z + \frac{1}{2})}{\Gamma(x - a + \frac{1}{2})} \right)^{1/2} \xi^{\frac{x+a}{2}} (1 - \xi)^{\frac{x+a}{2} - a} \\
\times \frac{F\left(-z + a + \frac{1}{2}, -z' + a + \frac{1}{2}; x + a + 1; \frac{\xi}{\xi-1}\right)}{\Gamma(x + a + 1)}, \quad x \in \mathbb{Z}',
\]

where \( F(A, B; C; w) \) is the Gauss hypergeometric function. As it is explained in Borodin and Olshanski \([7]\), Section 2, the above expression makes sense, and the functions \( \psi_a(x; z, z', \xi) \) are real-valued. In particular, the assumptions on \((z, z')\) imply that \( \Gamma(x + z + \frac{1}{2}) \) and \( \Gamma(x + z' + \frac{1}{2}) \) have no singularities for \( x \in \mathbb{Z}' \), and that

\[
\Gamma(x + z + \frac{1}{2})\Gamma(x + z' + \frac{1}{2}) > 0, \quad \Gamma(z - a + \frac{1}{2})\Gamma(z' - a + \frac{1}{2}) > 0.
\]

so we can take the positive values of the square roots in equation (2.4).

**Proposition 2.7.**

a) Introduce a second order difference operator \( D(z, z', \xi) \) on the lattice \( \mathbb{Z}' \), depending on parameters \( z, z', \xi \) and acting on functions \( f(x) \) (where \( x \) ranges over \( \mathbb{Z}' \)) as follows

\[
D(z, z', \xi)f(x) = \sqrt{\xi(x + z + \frac{1}{2})(z' + x + \frac{1}{2})} f(x + 1) \\
+ \sqrt{\xi(z + x + \frac{1}{2})(z' + x + \frac{1}{2})} f(x - 1) - (x + \xi(z + z' + x)) f(x).
\]

Then the functions \( \psi_a(x; z, z', \xi), \) where \( a \) ranges over \( \mathbb{Z}' \), are eigenvalues of the operator \( D(z, z', \xi) \),

\[
D(z, z', \xi)\psi_a(x; z, z', \xi) = a(1 - \xi)\psi_a(x; z, z', \xi).
\]

b) The functions \( \psi_a(x; z, z', \xi), \) where \( a \) ranges over \( \mathbb{Z}' \), form an orthonormal basis in the Hilbert space \( l^2(\mathbb{Z}') \).

**Proof.** See Borodin and Olshanski \([7]\), Section 2.

**Proposition 2.8.** For any \( A, B \in \mathbb{C}, M \in \mathbb{Z}, \) and \( \xi \in (0, 1) \) we have

\[
\frac{1}{2\pi i} \oint_{\{w\}} (1 - \sqrt{\xi}w)^{A-1}(1 - \frac{\sqrt{\xi}}{w})^{-B} \frac{dw}{w^{M+1}} = \xi^{\frac{A}{2}}(1 - \xi)^{-B} \frac{\Gamma(-A + M + 1)}{\Gamma(-A + 1)\Gamma(M + 1)} F(A, B; M + 1; \frac{\xi}{\xi-1}).
\]
Here $\xi \in (0, 1)$ and $\{w\}$ is an arbitrary simple contour which goes around the points $0$ and $\xi$ in the positive direction leaving $1/\sqrt{\xi}$ outside.

**Proof.** See Borodin and Olshanski [7], Lemma 2.2. \hfill \square

**Proposition 2.9.** We have the following integral representations

$$
\psi_a(x; z, z', \xi) = \left( \frac{\Gamma(x + z + \frac{1}{2})\Gamma(x + z' + \frac{1}{2})}{\Gamma(z - a + \frac{1}{2})\Gamma(z' - a + \frac{1}{2})} \right)^{1/2} \frac{\Gamma(z' - a + \frac{1}{2})}{\Gamma(x + z' + \frac{1}{2})} (1 - \xi)^{z' - a + 1} \times \frac{1}{2\pi i} \oint_{\{w\}} (1 - \sqrt{\xi}w)^{-z' + a - \frac{1}{2}} (1 - \frac{\sqrt{\xi}}{w})^{z - a - \frac{1}{2}} \frac{dw}{w},
$$

where $\{w\}$ is an arbitrary simple loop, oriented in positive direction, surrounding the points $0$ and $\sqrt{\xi}$, and leaving $1/\sqrt{\xi}$ outside.

**Proof.** Follows immediately from equation (2.4), and from Proposition 2.8. \hfill \square

Let $K_{z, z', \xi}$ be the orthogonal projection operator in $l^2(\mathbb{Z}')$ whose range is the subspace spanned by the basis vectors $\psi_a$ with indexes $a \in \mathbb{Z}_+ \subset \mathbb{Z}'$. If $K_{z, z', \xi}(x, y)$ is the matrix of $K_{z, z', \xi}$, then

$$
K_{z, z', \xi}(x, y) = \sum_{a \in \mathbb{Z}_+} \psi_a(x; z, z', \xi) \psi_a(y; z, z', \xi).
$$

**Proposition 2.10.** The function $K_{z, z', \xi}(x, y)$ can be written in the form

$$
K_{z, z', \xi}(x, y) = \frac{1}{(2\pi i)^2} \sqrt{\frac{\Gamma(x + z + \frac{1}{2})\Gamma(x + z' + \frac{1}{2})}{\Gamma(y + z' + \frac{1}{2})}} \times \oint_{\{w_1\}} \oint_{\{w_2\}} \left(1 - \sqrt{\xi}w_1\right)^{-z' - \frac{1}{2}} \left(1 - \frac{\sqrt{\xi}}{w_1}\right)^{z - \frac{1}{2}} \frac{dw_1}{w_1} \frac{dw_2}{w_2} \frac{1}{w_1^{x + \frac{1}{2}} w_2^{y + \frac{1}{2}}},
$$

where $\{w_1\}$ and $\{w_2\}$ are arbitrary simple contours satisfying the following conditions

- both contours go around $0$ in positive direction;
- the point $\xi^{1/2}$ is in the interior of each of the contours while the point $\xi^{-1/2}$ lies outside the contours;
- the contour $\{w_1^{-1}\}$ is contained in the interior of the contour $\{w_2\}$ (equivalently, $\{w_2^{-1}\}$ is contained in the interior of $\{w_1\}$).

**Proof.** See Borodin and Olshanski [7], Theorem 3.3. \hfill \square
2.3. The main result: the formula for the correlation function of the z-measure with the Jack parameter $\theta = 2$. The matrix hypergeometric kernel. By a point configuration in $Z'$ we mean any subset of $Z'$. Let $\text{Conf}(Z')$ be the set of all point configurations, and assume that we are given a probability measure on $\text{Conf}(Z')$. Then we can speak about random point configurations in $\text{Conf}(Z')$. The $n$th point correlation function of the given probability measure is defined by

$$\varrho_n(x_1, \ldots, x_n) = \text{Prob}\{\text{the random configuration contains } x_1, \ldots, x_n\}.$$ 

Here $n = 1, 2, \ldots$, and $x_1, \ldots, x_n$ are pairwise distinct points in $Z'$.

We say that a given probability measure defines a Pfaffian point process on $\text{Conf}(Z')$ if there exists a $2 \times 2$ matrix valued kernel $K(x, y)$ on $Z' \times Z'$ such that

$$\varrho_n(x_1, \ldots, x_n) = \text{Pf}\{K(x_i, x_j)\}_{i,j=1}^n, \quad n = 1, 2, \ldots.$$ 

The kernel $K(x, y)$ is referred to as the correlation kernel of the Pfaffian point process under considerations.

Set $D_2(\lambda) = \{\lambda_i - 2i + \frac{1}{2}\}$. Thus $D_2(\lambda)$ is an infinite subset of $Z'$ corresponding to the Young diagram $\lambda$. Let $X = (x_1, \ldots, x_n)$ be a subset of $Z'$ consisting of $n$ pairwise distinct points, and define

$$\varrho_n^{(z', \xi, \theta = 2)}(x_1, \ldots, x_n) = M_{z', \xi, \theta = 2}(\{\lambda X \subset D_2(\lambda)\}).$$

If $M_{z', \xi, \theta = 2}$ is positive, then it is a probability measure defined on $\mathcal{Y}$, and $\varrho^{(z', \xi, \theta)}(x_1, \ldots, x_n)$ is the probability that the random point configuration $D_2(\lambda)$ contains the fixed $n$-point configuration $X = (x_1, \ldots, x_n)$. The function $\varrho_n^{(z', \xi, \theta = 2)}(x_1, \ldots, x_n)$ can be understood as the correlation function of the point process defined by the measure $M_{z', \xi, \theta = 2}$.

The main result of the present paper is in explicit computation of $\varrho^{(z', \xi, \theta)}(x_1, \ldots, x_n)$ for admissible parameters $z$ and $z'$, for which both the measure $M_{z', \xi, \theta = 2}$ is positive, and the functions $\psi_a(x; z, z', \xi)$, $K_{z, z', \xi}(x; y)$ are well defined by equations (2.4), (2.5) correspondingly. In order to present our result, let us introduce the functions $(E(z, z')K_{z, z', \xi})(x; y)$, $(E(z, z')\psi_{\frac{1}{2}})(x; z, z', \xi)$, and $(E(z, z')\psi_{\frac{3}{2}})(x; z, z', \xi)$. We first define these functions in terms of infinite series. Namely, for any admissible $z, z'$ (i.e. for $z, z'$ from the principal or complementary series defined as in Proposition [2.3] with $\theta = 1$) these functions are given by the following formulae. If $x - \frac{3}{2}$ is an even integer, then

$$(E(z, z')K_{z, z', \xi})(x; y) = - \sum_{l=0}^{\infty} \sqrt{\frac{(x + z + \frac{3}{2})_{l,2}}{(x + z + \frac{5}{2})_{l,2}} \frac{(x + z' + \frac{3}{2})_{l,2}}{(x + z' + \frac{5}{2})_{l,2}}} K_{z, z', \xi}(x + 2l + 1; y),$$

and

$$(E(z, z')\psi_{\frac{3}{2}})(x; z, z', \xi) = - \sum_{l=0}^{\infty} \sqrt{\frac{(x + z + \frac{3}{2})_{l,2}}{(x + z + \frac{5}{2})_{l,2}} \frac{(x + z' + \frac{3}{2})_{l,2}}{(x + z' + \frac{5}{2})_{l,2}}} \psi_{\frac{3}{2}}(x + 2l + 1; z, z', \xi).$$
Otherwise, if \( x - \frac{1}{2} \) is an odd integer, then \((E(z, z')K_{x,z',\xi}) (x; y), (E(z, z')\psi_{\pm \frac{1}{2}}) (x; z, z', \xi)\), and \((E(z, z')\psi_{\pm \frac{1}{2}}) (x; z, z', \xi)\) are defined by

\[
(E(z, z')K_{x,z',\xi}) (x, y) = \sum_{l=1}^{\infty} \sqrt{\frac{(-x - z - \frac{1}{2})_{l,2}(-x - z' - \frac{1}{2})_{l,2}}{(-x - z + \frac{1}{2})_{l,2}(-x - z' + \frac{1}{2})_{l,2}}} K_{x,z',\xi}(x - 2l + 1; y),
\]

and

\[
(E(z, z')\psi_{\pm \frac{1}{2}}) (x; z, z', \xi) = \sum_{l=1}^{\infty} \sqrt{\frac{(-x - z - \frac{1}{2})_{l,2}(-x - z' - \frac{1}{2})_{l,2}}{(-x - z + \frac{1}{2})_{l,2}(-x - z' + \frac{1}{2})_{l,2}}} \psi_{\pm \frac{1}{2}}(x - 2l + 1; z, z', \xi).
\]

Here \((x)_{n,k}\) denotes the Pochhammer \( k \)-symbol,

\[
(x)_{n,k} = x(x + k)(x + 2k) \ldots (x + (n - 1)k).
\]

Let us explain why the formulae above make sense. Once the parameters \( z, z' \) are admissible, all the expressions inside square roots are strictly positive, so we can take the positive values of the square roots in equations for \((E(z, z')K_{x,z',\xi}) (x, y), (E(z, z')\psi_{\pm \frac{1}{2}}) (x; z, z', \xi)\), and \((E(z, z')\psi_{\pm \frac{1}{2}}) (x; z, z', \xi)\) just written above. Using Propositions [2.9] and [2.10] we can represent these functions as well-defined contour integrals, finite for all \( x, y \in \mathbb{Z}' \). For example, if \( x - \frac{1}{2} \) is an even integer, then

\[
(E(z, z')K_{x,z',\xi}) (x, y) = -\frac{1}{(2\pi i)^{2}} \frac{\Gamma(x + z + \frac{3}{2})\Gamma(y + z' + \frac{1}{2})}{\Gamma(x + z + \frac{3}{2})\Gamma(x + z' + \frac{3}{2})\Gamma(y + z + \frac{1}{2})\Gamma(y + z' + \frac{1}{2})} \times \int_{w_{1}} \int_{w_{2}} \frac{(1 - \sqrt{w_{1}})^{z'}(1 - \sqrt{w_{2}})^{z'}(1 - \sqrt{w_{1}})^{2}(1 - \sqrt{w_{2}})^{2}}{w_{1}w_{2} - 1} \times F \left( \frac{x + z + \frac{3}{2}}{2}, 1; \frac{x + z' + \frac{5}{2}}{2}, \frac{1}{w_{1}} \right) \frac{dw_{1}}{w_{1}^{2 + \frac{1}{2}}} \frac{dw_{2}}{w_{2}^{y + \frac{1}{2}}},
\]

where the contours \( \{w_{1}\}, \{w_{2}\} \) are chosen as in Proposition [2.10] and are contained in the domain \(|w| > 1\). In the domain \(|w_{1}| > 1\) the Gauss hypergeometric function inside the integral is an analytic function of \( w_{1} \), and can be represented by the uniformly convergent series,

\[
F \left( \frac{x + z + \frac{3}{2}}{2}, 1; \frac{x + z' + \frac{5}{2}}{2}, \frac{1}{w_{1}} \right) = \sum_{l=0}^{\infty} \frac{(x + z + \frac{4}{2})_{l}}{(x + z' + \frac{2}{2})_{l}} \frac{1}{w_{1}^{2l}}.
\]

Once the contour \( \{w_{1}\} \) is chosen in the domain where the series above is uniformly convergent, we can interchange summation and integration, and, expressing each integral
in the sum in terms of function $K_{z,z',\xi}(x,y)$, we arrive to the series in the definition of $(E(z,z')K_{z,z',\xi})(x,y)$. Since the righthand side in equation (2.6) is finite for all $x, y \in \mathbb{Z}'$, we conclude that the series in the definition of $(E(z,z')K_{z,z',\xi})(x,y)$ is convergent for all $x, y \in \mathbb{Z}'$.

Now we are in position to formulate the main result of this work.

**Theorem 2.11.** For any admissible $z, z'$ the $z$-measure with the Jack parameter $\theta = 2$ defines a Pfaffian point process. Namely, for any admissible $z, z'$ the $n$-point correlation function of $M_{z,z',\xi,\theta=2}(\lambda)$ is given by

\begin{equation}
E(z,z')K_{z,z',\xi,\theta=2}(x,
\end{equation}

where the correlation kernel $K_{z,z',\xi,\theta=2}(x,y)$ has the following form

\[
\begin{bmatrix}
S_{z,z',\xi,\theta=2}(x,y) & -SD_{z,z',\xi,\theta=2}(x,y) \\
-D_{+}S_{z,z',\xi,\theta=2}(x,y) & D_{+}SD_{z,z',\xi,\theta=2}(x,y)
\end{bmatrix}.
\]

In the formula above

\[
S_{z,z',\xi,\theta=2}(x,y) = \sqrt{(z+y+\frac{1}{2})(z'+y+\frac{1}{2})(E(z,z')K_{z,z',\xi})(x,y) + \sqrt{zz'}(E(z,z')\psi_{\frac{1}{2}})(x,z,z',\xi)\left(E(z,z')\psi_{\frac{1}{2}}\right)(y,z,z',\xi),
\]

\[
D_{+}S_{z,z',\xi,\theta=2}(x,y) = \frac{1}{\sqrt{(z+x+\frac{3}{2})(z'+x+\frac{3}{2})}}S_{z,z',\xi,\theta=2}(x+1,y),
\]

\[
SD_{-z,z',\xi,\theta=2}(x,y) = \frac{1}{\sqrt{(z+y+\frac{3}{2})(z'+y+\frac{3}{2})}}S_{z,z',\xi,\theta=2}(x,y+1),
\]

and

\[
D_{+}SD_{-z,z',\xi,\theta=2}(x,y) = \frac{1}{\sqrt{(z+x+\frac{3}{2})(z'+x+\frac{3}{2})(z+y+\frac{3}{2})(z'+y+\frac{3}{2})}}S_{z,z',\xi,\theta=2}(x+1,y+1).
\]

2.4. Remarks on Theorem 2.11

2.4.1. All matrix elements of $K_{z,z',\xi,\theta=2}(x,y)$ are constructed in terms of the Gauss hypergeometric functions, so it is natural to refer to $K_{z,z',\xi,\theta=2}(x,y)$ as to the matrix hypergeometric kernel.

2.4.2. All matrix elements of $K_{z,z',\xi,\theta=2}(x,y)$ are symmetric with respect to $z \longleftrightarrow z'$. This implies that the correlation function is symmetric with respect to $z \longleftrightarrow z'$ as well. The fact that this symmetry relation must be satisfied is evident from the symmetry of the $z$-measure under considerations under $z \longleftrightarrow z'$. 
2.4.3. It is possible to present the function $S_{z,z',\xi,\theta=\pm}(x,y)$ (which defines the matrix kernel $\mathbb{K}_{z,z',\xi,\theta=\pm}(x,y)$) in a form which is manifestly antisymmetric with respect to $x \leftrightarrow y$. For this purpose let us introduce the functions $\mathcal{P}(x,w,z,z')$ and $\mathcal{Q}(x,w,z,z')$. If $x - \frac{1}{2}$ is even, these functions are defined by

$$\mathcal{P}(x,w,z,z') = \frac{\Gamma(x+z+\frac{1}{2})}{\sqrt{\Gamma(x+z+\frac{3}{2})\Gamma(x+z'+\frac{3}{2})}} \times F\left(\frac{x+z+\frac{3}{2}}{2}, 1; \frac{x+z'+\frac{5}{2}}{2}; \frac{1}{w^2}\right) w^{-x-\frac{1}{2}},$$

(2.8)

and by

$$\mathcal{Q}(x,w,z,z') = \frac{\Gamma(x+z'+\frac{3}{2})}{\sqrt{\Gamma(x+z+\frac{3}{2})\Gamma(x+z'+\frac{3}{2})}} \times \left[F\left(-\frac{x+z'+\frac{3}{2}}{2}, 1; -\frac{x+z'-\frac{1}{2}}{2}; \frac{1}{w^2}\right) - 1\right] w^{-x+\frac{1}{2}}.$$

(2.9)

If $x - \frac{1}{2}$ is odd, these functions are defined by

$$\mathcal{P}(x,w,z,z') = -\frac{\Gamma(x+z+\frac{1}{2})}{\sqrt{\Gamma(x+z+\frac{3}{2})\Gamma(x+z'+\frac{3}{2})}} \times \left[F\left(-\frac{x+z+\frac{3}{2}}{2}, 1; -\frac{x+z'-\frac{1}{2}}{2}; \frac{1}{w^2}\right) - 1\right] w^{-x+\frac{1}{2}},$$

(2.10)

and by

$$\mathcal{Q}(x,w,z,z') = -\frac{\Gamma(x+z'+\frac{3}{2})}{\sqrt{\Gamma(x+z+\frac{3}{2})\Gamma(x+z'+\frac{3}{2})}} \times F\left(-\frac{x+z-\frac{3}{2}}{2}, 1; -\frac{x+z'-\frac{1}{2}}{2}; \frac{1}{w^2}\right) w^{-x-\frac{1}{2}}.$$

(2.11)

Set

$$\mathcal{S}_{z,z',\xi}(x,y) = \frac{1}{(2\pi i)^2} \oint_{\{w_1\}} \oint_{\{w_2\}} \frac{(1 - \sqrt{\xi} w_1)^{z'}(1 - \sqrt{\xi} w_2)^{-z}(1 - \sqrt{\xi} w_2)^{-z}(1 - \sqrt{\xi} w_2)^{z'}}{w_1 w_2 - 1} dw_1 dw_2 \times \mathcal{P}(x,w_1,z,z') \mathcal{Q}(y,w_2,z,z'),$$

(2.12)

where the contours $\{w_1\}, \{w_2\}$ are chosen as in the statement of Proposition 2.10 with the following additional conditions. If $x - \frac{1}{2}$ is an even integer, then $\{w_1\}$ lies in the domain $|w| > 1$. If $x - \frac{1}{2}$ is an odd integer, then $\{w_1\}$ lies in the domain $|w| < 1$. The same
condition is imposed on \( \{ w_{2} \} \): if \( y - \frac{1}{2} \) is an even integer, then \( \{ w_{2} \} \) lies in the domain \( |w| > 1 \), and if \( x - \frac{1}{2} \) is an odd integer, then \( \{ w_{1} \} \) lies in the domain \( |w| < 1 \). Then the following Proposition holds true

**Proposition 2.12.** The function \( S_{x, z', \xi, \theta = 2}(x, y) \) in the definition of the matrix kernel \( K_{x, z', \xi, \theta = 2}(x, y) \) is antisymmetric with respect to \( x \leftrightarrow y \), and it can be written as

\[
S_{x, z', \xi, \theta = 2}(x, y) = \tilde{S}_{x, z', \xi, \theta = 2}(x, y) - \tilde{S}_{x, z', \xi, \theta = 2}(y, x),
\]

where the function \( \tilde{S}_{x, z', \xi, \theta = 2}(x, y) \) is defined by equations (2.8)-(2.12).

2.4.4. Let us drop assumption that the parameters \( z, z' \) are admissible. Denote by \( \rho_{n}(z, z', \xi, \theta = 2)(x_{1}, \ldots, x_{n}) \) the function which is obtained from \( \rho_{n}(z, z', \xi, \theta = 2)(x_{1}, \ldots, x_{n}) \) by setting \( z' = z - 1 \) in the formulae of Theorem 2.11.

**Proposition 2.13.** The function \( \rho_{n}(z, z', \xi, \theta = 2)(x_{1}, \ldots, x_{n}) \) takes the form

\[
\rho_{n}(z, z', \xi, \theta = 2)(x_{1}, \ldots, x_{n}) = \text{Pf} \left[ \frac{1}{(2\pi i)^{2}} \oint_{w_{1}} \oint_{w_{2}} \left( 1 - \sqrt{\xi} w_{1} \right)^{-z'} (1 - \sqrt{\xi} w_{2})^{z} \frac{1 - \sqrt{\xi} y}{w_{1} w_{2} - 1} \right]_{i,j=1}^{n},
\]

where

\[
\left( \frac{1 - \sqrt{\xi} y}{w_{1} w_{2} - 1} \right)^{-z'} \times \frac{(w_{2} - w_{1})}{(w_{1}^{2} - 1)} \frac{dw_{1}}{w_{1}^{x-\frac{1}{2}}} \frac{dw_{2}}{w_{2}^{y-\frac{1}{2}}}
\]

where \( \{ w_{1} \} \) and \( \{ w_{2} \} \) are arbitrary simple contours satisfying the following conditions

- both contours go around 0 in positive direction;
- the point \( \xi^{1/2} \) is in the interior of each of the contours while the point \( \xi^{-1/2} \) lies outside the contours;
- Both contours \( \{ w_{1} \} \) and \( \{ w_{2} \} \) lie in the domain \( |w| > 1 \).

Formula (2.15) is equivalent to the result of Theorem 3.1 a) in Strahov [34]. Theorem 3.1 a) in Strahov [34] was obtained by a completely different method, and this comparison provides a check of validity for Theorem 2.11.
2.5. The method: analytic continuation of the Meixner symplectic ensemble. It was shown in Borodin and Strahov [11] that the z-measures with parameters $z = 2N, z' = 2N + \beta - 2$, and $\theta = 2$ turns into an ensemble of $N$ particles on $\mathbb{Z}_{\geq 0}$ called in Borodin and Strahov [11] the Meixner symplectic ensemble. It was shown in [11] that this discrete ensemble is integrable in the sense that the correlation function can be expressed explicitly in terms of known functions. Namely, a discrete version of the method developed by Tracy and Widom [36], Widom [37] works for the Meixner symplectic ensemble, and correlation functions are expressible in terms of Pfaffians of $2 \times 2$ matrix kernels. The matrix elements of these kernels can be written in terms of the classical Meixner orthogonal polynomials. In the present paper we provide contour integral representations for the elements of the correlation kernel (see Theorem 4.1), which is the result of an independent interest.

We regard the z-measures with the Jack parameter $\theta = 2$ as the result of analytic continuation of the Mexiner symplectic ensemble in parameter $N$ (number of particles). The procedure of the analytic continuation is a natural extension of the approach developed in Borodin and Olshanski [7] to much more complicated situation of the matrix correlation kernels.

3. The relation between the z-measure with the parameter $\theta = 2$ and the Meixner symplectic ensemble. The correlation function for the Meixner symplectic ensemble.

We define the Meixner symplectic ensemble in the same way as in Borodin and Strahov [11], Section 2. Elements of $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \ldots\}$ will be denoted by letters $\tilde{x}, \tilde{y}$. (Recall that the elements of $\mathbb{Z}'$ were denoted by letters $x, y$.)

Let $w(\tilde{x})$ be a strictly positive real valued function defined on $\mathbb{Z}_{\geq 0}$ with finite moments, i.e. the series $\sum_{\tilde{x} \in \mathbb{Z}_{\geq 0}} w(\tilde{x}) \tilde{x}^j$ converges for all $j = 0, 1, \ldots$.

**Definition 3.1.** The $N$-point discrete symplectic ensemble with the weight function $w$ and the phase space $\mathbb{Z}_{\geq 0}$ is the random $N$-point configuration in $\mathbb{Z}_{\geq 0}$ such that the probability of a particular configuration $\tilde{x}_1 < \ldots < \tilde{x}_N$ is given by

$$\text{Prob}\{\tilde{x}_1, \ldots, \tilde{x}_N\} = Z_{N4}^{-1} \prod_{i=1}^{N} w(\tilde{x}_i) \prod_{1 \leq i < j \leq N} (\tilde{x}_i - \tilde{x}_j)^2(\tilde{x}_i - \tilde{x}_j - 1)(\tilde{x}_i - \tilde{x}_j + 1).$$

Here $Z_{N4}$ is a normalization constant which is assumed to be finite.

In what follows $Z_{N4}$ is referred to as the partition function of the discrete symplectic ensemble under considerations.

We consider the particular case when $w(\tilde{x})$ is the Meixner weight given by the formula

$$W_{\beta, \xi}^{\text{Meixner}}(\tilde{x}) = \frac{(\beta)_{\tilde{x}}}{\tilde{x}!} \xi^{\tilde{x}}, \quad \tilde{x} \in \mathbb{Z}_{\geq 0},$$

where $\beta$ is a strictly positive real parameter, and $0 < \xi < 1$. In this situation we say that we are dealing with the Meixner symplectic ensemble.
Proposition 3.2. For $N = 1, 2, \ldots$ let $\mathcal{Y}(N) \subset \mathcal{Y}$ denote the set of diagrams $\lambda$ with $l(\lambda) \leq N$ (where $l(\lambda)$ is the number of rows in $\lambda$). Under the bijection between diagrams $\lambda \in \mathcal{Y}(N)$ and $N$-point configurations on $\mathbb{Z}_{\geq 0}$ defined by

$$
\lambda \longleftrightarrow \tilde{x}_{N-i+1} = \lambda_i - 2i + 2N \quad (i = 1, \ldots, N)
$$

the $z$-measure with parameters $z = 2N, \theta = 2, z' = 2N + \beta - 2$ turns into

$$
\text{Prob}^{\text{Meixner}} \{\tilde{x}_1, \ldots, \tilde{x}_N\} = \text{const} \cdot \prod_{i=1}^{N} \frac{(\beta)_{\tilde{x}_i}}{\tilde{x}_i!} \prod_{1 \leq i < j \leq N} (\tilde{x}_i - \tilde{x}_j)^2(\tilde{x}_i - \tilde{x}_j - 1)(\tilde{x}_i - \tilde{x}_j + 1),
$$

which is precisely the discrete symplectic ensemble with the Meixner weight in the sense of Definition 3.1.

Proof. The proof is a straightforward computation based on the application of the explicit formulae for $H(\lambda; 2)H'(\lambda; 2)$, see the proof of Lemma 3.5 in [7], and $(z)_{\lambda, \theta}$, see Section 1 in [7].

We employ the same notation for the Meixner polynomials as in Borodin and Olshanski [7]. Thus the Meixner polynomials are denoted by $\widetilde{M}_n(\tilde{x}; \beta, \xi)$. We use the same normalization for these polynomials as in Koekoek and Swarttouw [23]). Note that in Koekoek and Swarttouw [23]) the parameter $\xi$ in the definition of the Meixner weight is denoted as $c$. For basic properties of the classical discrete orthogonal polynomials, and, in particular, the Meixner polynomials, see Ismail [17].

As in Borodin and Olshanski [7], we set

$$
\widetilde{M}_n(\tilde{x}; \beta, \xi) = (-1)^n \frac{\widetilde{M}_n(\tilde{x}; \beta, \xi)}{|\widetilde{M}_n(\cdot; \beta, \xi)|} \sqrt{W_{\beta, \xi}^{\text{Meixner}}(\tilde{x})}, \quad \tilde{x} \in \mathbb{Z}_{\geq 0},
$$

where

$$
|\widetilde{M}_n(\cdot; \beta, \xi)|^2 = \sum_{\tilde{x}=0}^{\infty} \frac{\widetilde{M}_n^2(\tilde{x}; \beta, \xi)W_{\beta, \xi}^{\text{Meixner}}(\tilde{x})}{||\widetilde{M}_n(\cdot; \beta, \xi)||^2}.
$$

Let $\mathcal{H}^{\text{Meixner}}$ be the space spanned by functions $\widetilde{M}_0, \widetilde{M}_1, \widetilde{M}_2, \ldots$, that is, each element of $\mathcal{H}^{\text{Meixner}}$ is a linear combination of $\widetilde{M}_0, \widetilde{M}_1, \widetilde{M}_2, \ldots$. We introduce the operators $D_+^{\text{Meixner}}, D_-^{\text{Meixner}}$, and $E^{\text{Meixner}}$ which act on the elements of the space $\mathcal{H}^{\text{Meixner}}$. The first and the second operators, $D_+^{\text{Meixner}}$ and $D_-^{\text{Meixner}}$, are defined by the expression:

$$
(D_+^{\text{Meixner}} f)(\tilde{x}) = \sum_{\tilde{y}=0}^{\infty} D_+^{\text{Meixner}}(\tilde{x}, \tilde{y}) f(\tilde{y}),
$$

where the kernels $D_+^{\text{Meixner}}(\tilde{x}, \tilde{y})$ are given explicitly by

$$
D_+^{\text{Meixner}}(\tilde{x}, \tilde{y}) = \frac{1}{\sqrt{\xi}} \sqrt{\frac{1 + \tilde{x}}{\beta + \tilde{x}}} \delta_{\tilde{x}+1, \tilde{y}}, \quad \tilde{x}, \tilde{y} \in \mathbb{Z}_{\geq 0},
$$

(3.2)
Proposition 3.3. The correlation function of the $N$-point Meixner symplectic ensemble can be written as a Pfaffian of $2 \times 2$ matrix valued kernel,

$$\varrho_{n,\text{Meixner}}^{(N,\beta,\xi)}(\vec{x},\ldots,\vec{x}_n) = \text{Pf} \left[ K_{2N}^{\text{Meixner}}(\vec{x}_i,\vec{x}_j) \right]_{i,j=1}^{n},$$

where

$$K_{2N}^{\text{Meixner}}(\vec{x},\vec{y}) = \sum_{k=0}^{2N-1} \tilde{\mathcal{M}}_k(\vec{x}) \tilde{\mathcal{M}}_k(\vec{y}).$$

The third operator, $E_{\text{Meixner}}$, is defined by the formula

$$E_{\text{Meixner}} f(x) = \begin{cases} -\sqrt{\xi} \sum_{y=0}^{\infty} \sqrt{\frac{(\beta+\bar{x})_{y+1,2}(2+\bar{x})_{y,2}}{(1+\bar{x})_{y+1,2}(1+\beta+\bar{x})_{y,2}}} f(\bar{x} + 2\bar{y} + 1), & \bar{x} \text{ is even;} \\ \sqrt{\xi} \sum_{y=0}^{\infty} \sqrt{\frac{(1-\beta-\bar{x})_{y+1,2}(1-\bar{x})_{y,2}}{(-\bar{x})_{y+1,2}(2-\beta-\bar{x})_{y,2}}} f(\bar{x} - 2\bar{y} - 1), & \bar{x} \text{ is odd.} \end{cases}$$

Note that the sum in the case of an odd $\bar{x}$ actually runs from 0 to $\frac{\bar{x}-1}{2}$, so $\bar{x} - 2\bar{y} - 1 \in \mathbb{Z}_{\geq 0}$ in the argument of the function $f$. It is explained in Borodin and Strahov [11] that the series defining $(E_{\text{Meixner}} f)(x)$ converges for any $f$ from $\mathcal{H}_{\text{Meixner}}$, i.e. $E_{\text{Meixner}} f$ is well defined, see the discussion after equation (2.3) in Borodin and Strahov [11], Section 2.

Let $\mathcal{H}_{2N}^{\text{Meixner}}$ be the subspace of $\mathcal{H}_{\text{Meixner}}$ spanned by the functions $\tilde{\mathcal{M}}_0, \tilde{\mathcal{M}}_1, \ldots, \tilde{\mathcal{M}}_{2N-1}$. We denote by $K_{2N}^{\text{Meixner}}$ the projection operator onto $\mathcal{H}_{2N}^{\text{Meixner}}$. Its kernel is

$$K_{2N}^{\text{Meixner}}(\vec{x},\vec{y}) = \sum_{k=0}^{2N-1} \tilde{\mathcal{M}}_k(\vec{x}) \tilde{\mathcal{M}}_k(\vec{y}).$$

In addition, we introduce the operator $S_{2N}^{\text{Meixner}}$ by the formula

$$S_{2N}^{\text{Meixner}} = E_{\text{Meixner}} K_{2N}^{\text{Meixner}} + K_{2N}^{\text{Meixner}} E_{\text{Meixner}} - E_{\text{Meixner}} K_{2N}^{\text{Meixner}} D_{\text{Meixner}} K_{2N}^{\text{Meixner}} E_{\text{Meixner}},$$

where

$$D_{\text{Meixner}} = D_+^{\text{Meixner}} - D_-^{\text{Meixner}}.$$
This kernel, \( K_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) \), has the following representation
\[
K_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) = \begin{bmatrix}
S_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) & -S_{2N}^{\text{Meixner}}D_{-}^{\text{Meixner}}(\tilde{x}, \tilde{y}) \\
-D_{+}^{\text{Meixner}}S_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) & D_{+}^{\text{Meixner}}S_{2N}^{\text{Meixner}}D_{-}^{\text{Meixner}}(\tilde{x}, \tilde{y})
\end{bmatrix},
\]
where the matrix entries are the kernels of the operators \( S_{2N}^{\text{Meixner}}, -S_{2N}^{\text{Meixner}}D_{-}^{\text{Meixner}}, -D_{+}^{\text{Meixner}}S_{2N}^{\text{Meixner}} \), and \( D_{+}^{\text{Meixner}}S_{2N}^{\text{Meixner}}D_{-}^{\text{Meixner}} \).

Proof. The representation for \( K_{2N}^{\text{Meixner}} \) follows from Theorem 2.4, Theorem 2.9 and Proposition 14.3 in Borodin and Strahov [11]. □

Lemma 3.4. For \( N = 1, 2, \ldots \) let \( z = 2N \) and \( z' = 2N + \beta - 2 \) with \( \beta > 0 \). Assume that \( x_1, \ldots, x_n \) lie in the subset \( \mathbb{Z}_{2N} = \{ x \} \in \mathbb{Z} \) such that the points \( \tilde{x}_i = x_i + 2N - 1/2 \) are in \( \mathbb{Z}_{2N} \). Then
\[
\theta_n^{(z, z', \xi, \theta=2)}(x_1, \ldots, x_n) = \text{Pf} \left[ K_{2N}^{\text{Meixner}} \left( x_i + 2N - \frac{1}{2}, x_j + 2N - \frac{1}{2} \right) \right]_{i,j=1}^n.
\]

Proof. If \( z = 2N \) and \( z' = 2N + \beta - 2 \), then Proposition 3.2 implies that \( M_{z, z', \xi, \theta=2} \) defines the Meixner symplectic ensemble on the point configurations \( \tilde{X}(\lambda) \) defined by
\[
\tilde{x}_{N-i+1} = \lambda_i - 2i + 2N, \quad i = 1, \ldots, N.
\]
To obtain \( \theta_n^{(z, z', \xi, \theta=2)}(x_1, \ldots, x_n) \) we need to consider random configurations \( D_2(\lambda) \) defined by \( x_i = \lambda_i - 2i + 1/2 \), where \( i = 1, \ldots, N \). On the other hand, there is a bijective correspondence between the set of all configurations \( \tilde{X}(\lambda) \), and the set of all configurations \( D_2(\lambda) \) defined by
\[
(3.7) \quad \tilde{x}_{N-i+1} = x_i - \frac{1}{2} + 2N, \quad i = 1, \ldots, N.
\]
Note that two configurations \( \tilde{X}(\lambda) \) and \( D_2(\lambda) \) related by (3.7) have the same probability. The statement of the Lemma immediately follows from this observation, and from Proposition 3.3. □

4. The contour integral representation for \( S_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) \)

The aim of this Section is to obtain an explicit formula for the function \( S_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) \) which completely determines the correlation function for the Meixner symplectic ensemble via Proposition 3.3. Namely, we provide a contour integral representation for \( S_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) \).

Theorem 4.1. The function \( S_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) \) (which is the kernel of the operator \( S_{2N}^{\text{Meixner}} \)) admits the following contour integral representation
a) If both $\tilde{x}$ and $\tilde{y}$ are even, then

$$S_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) = -\frac{\sqrt{\xi}}{(2\pi i)^2} \sqrt{\frac{\Gamma(\tilde{x} + 1)\Gamma(\tilde{y} + \beta)}{\Gamma(\tilde{x} + \beta)\Gamma(\tilde{y} + 1)}} \left[ \int_{\{w_1\}} \int_{\{w_2\}} \frac{(1 - \sqrt{\xi}w_1)^{-2N-\beta+1}(1 - \frac{\sqrt{\xi}}{w_1})^{2N}(1 - \sqrt{\xi}w_2)^{-2N}(1 - \frac{\sqrt{\xi}}{w_2})^{2N+\beta-1}}{w_1 w_2 - 1} dw_1 dw_2 \right]$$

$$F\left(\frac{\tilde{x} + 2}{2}, 1; \frac{\tilde{x} + \beta + 1}{2}; \frac{1}{w_1^2}\right) \frac{dw_1}{w_1^{\tilde{x}-2N+2}} \frac{dw_2}{w_2^{\tilde{y}-2N+1}} - \frac{\sqrt{\xi}}{w_1^{\tilde{x}-2N+2}} \frac{\sqrt{\xi}}{w_2^{\tilde{y}-2N+1}} \left[ F\left(\frac{\tilde{y} + \beta}{2}, 1; \frac{\tilde{y} + 1}{2}; \frac{1}{w_2^2}\right) - 1 \right] \frac{dw_1}{w_1^{\tilde{x}-2N+2}} \frac{dw_2}{w_2^{\tilde{y}-2N+1}} \right].$$

In the formula just written $\{w_1\}$ and $\{w_2\}$ are arbitrary simple contours satisfying the following conditions

- both contours go around 0 in positive direction;
- the point $\xi^{1/2}$ is in the interior of each of the contours while the point $\xi^{-1/2}$ lies outside the contours;
- the contour $\{w_1^{-1}\}$ is contained in the interior of the contour $\{w_2\}$ (equivalently, $\{w_2^{-1}\}$ is contained in the interior of $\{w_1\}$);
- both contours $\{w_1\}, \{w_2\}$ lie in the domain $|w| > 1$.

b) In the case when both $\tilde{x}$ and $\tilde{y}$ are odd positive integers we have

$$S_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) = \frac{\sqrt{\xi}}{(2\pi i)^2} \sqrt{\frac{\Gamma(\tilde{x} + 1)\Gamma(\tilde{y} + \beta)}{\Gamma(\tilde{x} + \beta)\Gamma(\tilde{y} + 1)}} \left[ \int_{\{w_1\}} \int_{\{w_2\}} \frac{(1 - \sqrt{\xi}w_1)^{-2N-\beta+1}(1 - \frac{\sqrt{\xi}}{w_1})^{2N}(1 - \sqrt{\xi}w_2)^{-2N}(1 - \frac{\sqrt{\xi}}{w_2})^{2N+\beta-1}}{w_1 w_2 - 1} dw_1 dw_2 \right]$$

$$F\left(\frac{\tilde{x} + 2}{2}, 1; \frac{\tilde{x} + \beta + 1}{2}; \frac{1}{w_1^2}\right) \frac{dw_1}{w_1^{\tilde{x}-2N+2}} \frac{dw_2}{w_2^{\tilde{y}-2N+1}} \left[ F\left(\frac{\tilde{y} + \beta}{2}, 1; \frac{\tilde{y} + 1}{2}; \frac{1}{w_2^2}\right) - 1 \right] \frac{dw_1}{w_1^{\tilde{x}-2N+2}} \frac{dw_2}{w_2^{\tilde{y}-2N+1}} \right].$$
where the contours $\{w_1\}, \{w_2\}$ are arbitrary simple contours satisfying the first three conditions of a) that lie in the domain $|w| < 1$.

c) If $\tilde{x}$ is even positive integer, and $\tilde{y}$ is an odd positive integer, then

$$S_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) = -\frac{\sqrt{\xi}}{(2\pi i)^2} \sqrt{\frac{\Gamma(\tilde{x}+1)\Gamma(\tilde{y}+\beta)}{\Gamma(\tilde{x}+\beta)\Gamma(\tilde{y}+1)}}$$

$$\left[ \oint_{w_1} \oint_{w_2} (1 - \sqrt{\xi} w_1)^{-2N-\beta+1} (1 - \frac{\xi}{w_1})^{2N} (1 - \sqrt{\xi} w_2)^{-2N} (1 - \frac{\xi}{w_2})^{2N+\beta-1} \right] \frac{dw_1}{w_{1}^{2N+2}} \frac{dw_2}{w_{2}^{2N+1}}$$

$$+ \left[ \oint_{w_1} \oint_{w_2} (1 - \sqrt{\xi} w_1)^{-2N-\beta+1} (1 - \frac{\xi}{w_1})^{2N} (1 - \sqrt{\xi} w_2)^{-2N} (1 - \frac{\xi}{w_2})^{2N+\beta-1} \right] \frac{dw_1}{w_{1}^{2N+2}} \frac{dw_2}{w_{2}^{2N+1}}$$

where the contours $\{w_1\}, \{w_2\}$ are arbitrary simple contours satisfying the first three conditions of a). Moreover, the first contour, $\{w_1\}$, lies in the domain $|w| > 1$, and the second contour, $\{w_2\}$, lies in the domain $|w| < 1$.

d) Finally, if $\tilde{x}$ is an odd integer, and $\tilde{y}$ is even integer, then

$$S_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) = \frac{\sqrt{\xi}}{(2\pi i)^2} \sqrt{\frac{\Gamma(\tilde{x}+1)\Gamma(\tilde{y}+\beta)}{\Gamma(\tilde{x}+\beta)\Gamma(\tilde{y}+1)}}$$

$$\left[ \oint_{w_1} \oint_{w_2} (1 - \sqrt{\xi} w_1)^{-2N-\beta+1} (1 - \frac{\xi}{w_1})^{2N} (1 - \sqrt{\xi} w_2)^{-2N} (1 - \frac{\xi}{w_2})^{2N+\beta-1} \right] \frac{dw_1}{w_{1}^{2N+2}} \frac{dw_2}{w_{2}^{2N+1}}$$

$$- \left[ \oint_{w_1} \oint_{w_2} (1 - \sqrt{\xi} w_1)^{-2N-\beta+1} (1 - \frac{\xi}{w_1})^{2N} (1 - \sqrt{\xi} w_2)^{-2N} (1 - \frac{\xi}{w_2})^{2N+\beta-1} \right] \frac{dw_1}{w_{1}^{2N+2}} \frac{dw_2}{w_{2}^{2N+1}}$$

where the contours $\{w_1\}, \{w_2\}$ are arbitrary simple contours satisfying the first three conditions of a), which both lie in the domain $|w| < 1$.

**Proof.** We start from equation (1). The operators $E^{\text{Meixner}}, D^{\text{Meixner}}$ are defined explicitly by equations (3.2), (3.3), (3.4), and (3.6). The contour integral representation for the
kernel $K_{2N}^{\text{Meixner}}$ can be obtained immediately from Proposition 2.10. Indeed, Theorem 3.2 and Proposition 2.8 in Borodin and Olshanski [7] imply the relation

$$K_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) = \frac{1}{\sqrt{\xi}} \sum_{m=\infty}^{+\infty} K_{2N}^{\text{Meixner}}(\tilde{x}, m) \sqrt{\frac{m + 1}{m + \beta}} K_{2N}^{\text{Meixner}}(m + 1, \tilde{y}).$$

Once the contour integral representation of $K_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y})$ is given it is straightforward to derive the contour integral representations for the kernels of the operators $E_{2N}^{\text{Meixner}}K_{2N}^{\text{Meixner}}$ and $K_{2N}^{\text{Meixner}}E_{2N}^{\text{Meixner}}$. Next we need to derive the contour integral representation for the kernel of the operator $D_{2N}^{\text{Meixner}}K_{2N}^{\text{Meixner}}$ (this is the most nontrivial part of these calculations). We have

$$K_{2N}^{\text{Meixner}}D_{2N}^{\text{Meixner}}K_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) = K_{2N}^{\text{Meixner}}D_{+}^{\text{Meixner}}K_{2N}^{\text{Meixner}} - K_{2N}^{\text{Meixner}}D_{-}^{\text{Meixner}}K_{2N}^{\text{Meixner}}.$$

Let us first derive the contour integral representation for $K_{2N}^{\text{Meixner}}D_{+}^{\text{Meixner}}K_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y})$. Taking into account the definition of $D_{+}^{\text{Meixner}}$ we can write

$$K_{2N}^{\text{Meixner}}D_{+}^{\text{Meixner}}K_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) = \frac{1}{(2\pi i)^4 \sqrt{\xi}} \frac{\Gamma(\tilde{x} + 1)\Gamma(\tilde{y} + 1)}{\Gamma(\tilde{x} + \beta)\Gamma(\tilde{y} + \beta)} \times \oint \{w_1\} \oint \{w_2\} \oint \{w_3\} \oint \{w_4\} \Phi(w_1, w_2)\Phi(w_3, w_4) \left( \sum_{m=0}^{+\infty} \frac{1}{(w_2w_3)^m} \right) \frac{dw_1dw_2dw_3dw_4}{w_1^{2-2N+1}w_2^{2-2N+1}w_3^{2-2N+2}w_4^{2-2N+1}}.$$

where

$$\Phi(w_1, w_2) = \frac{(1 - \sqrt{\xi}w_1)^{-2N}(1 - \sqrt{\xi}w_2)^{-2N+\beta-1}(1 - \sqrt{\xi}w_2)^{-2N-\beta+1}(1 - \sqrt{\xi}w_2)^{2N}}{w_1w_2 - 1}.$$

We observe that the integral above remains unchanged if we replace the sum in the integrand by

$$\sum_{m=\infty}^{+\infty} \frac{1}{(w_2w_3)^m}.$$

We split this sum into two parts,

$$\sum_{m=\infty}^{+\infty} \frac{1}{(w_2w_3)^m} = \sum_{m=0}^{+\infty} \frac{1}{(w_2w_3)^m} + \sum_{m=\infty}^{-1} \frac{1}{(w_2w_3)^m}.$$
If $|w_2w_3| > 1$, then the first sum in the righthand side of the equation converges, and it equals

$$
\sum_{m=0}^{+\infty} \frac{1}{(w_2w_3)^m} = \frac{w_2}{w_2 - w_3}.
$$

If $|w_2w_3| < 1$, then the second sum can be written as

$$
\sum_{m=-\infty}^{-1} \frac{1}{(w_2w_3)^m} = \frac{w_2}{w_3 - w_2}.
$$

This gives us

$$
K_{2N}^{\text{Meixner}} D_+^{\text{Meixner}} K_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) = \frac{1}{(2\pi i)^4 \sqrt{\xi}} \frac{\Gamma(\tilde{x} + \beta)\Gamma(\tilde{y} + 1)}{\Gamma(\tilde{x} + 1)\Gamma(\tilde{y} + \beta)} \times \left[ \oint \oint \oint \oint \Phi(w_1, w_2)\Phi(w_3, w_4) \frac{w_2}{w_2 - w_3 - 1} \frac{dw_1dw_2dw_3dw_4}{w_1^{2-2N+1}w_2^{2N+1}w_3^{2N+2}w_4^{y-2N+1}} + \oint \oint \oint \oint \Phi(w_1, w_2)\Phi(w_3, w_4) \frac{w_2}{w_3 - 1} \frac{dw_1dw_2dw_3dw_4}{w_1^{2-2N+1}w_2^{2N+1}w_3^{2N+2}w_4^{y-2N+1}} \right].
$$

(4.3)

Here we take as the contours concentric circles $\{w_1\}, \{w_2\}, \{w_3\}, \{w_4\}$. The contours $\{w_1\}$ and $\{w_2\}$ satisfy the same conditions as in the statement of Proposition 2.10 in particular we can agree that $\frac{1}{|w_2|} < |w_1|$. We also agree that the contours $\{w_3\}$ and $\{w_4\}$ satisfy the conditions of Proposition 2.10 and that $\frac{1}{|w_3|} < |w_4|$. In addition, we require that $|w_3| > \frac{1}{|w_2|}$ in first integral which corresponds to the sum over $Z_{>0}$. In the second integral (corresponding to the sum over $Z_{<0}$) we chose contours in such a way that $|w_3| < \frac{1}{|w_2|}$.

We transform the first integral: keeping the contours $\{w_1\}, \{w_3\}, \{w_4\}$ unchanged we move $\{w_2\}$ inside the circle of the radius $\frac{1}{|w_3|}$. Then we obtain an integral which cancels the second integral in equation (4.3), plus an integral arising from the residue of the function $w_2 \rightarrow (w_2 - w_3^{-1})^{-1}$. This gives us

$$
K_{2N}^{\text{Meixner}} D_+^{\text{Meixner}} K_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) = \frac{1}{(2\pi i)^3 \sqrt{\xi}} \frac{\Gamma(\tilde{x} + \beta)\Gamma(\tilde{y} + 1)}{\Gamma(\tilde{x} + 1)\Gamma(\tilde{y} + \beta)} \oint \oint \oint \Phi(w_1, w_3^{-1})\Phi(w_3, w_4) \frac{dw_1dw_3dw_4}{w_1^{2-2N+1}w_3^{2N+2}w_4^{y-2N+1}}.
$$
We find
\[ \Phi(w_1, w_3^{-1})\Phi(w_3, w_4) = \frac{(1 - \sqrt{\xi} w_1)^{-2N}(1 - \frac{\sqrt{\xi}}{w_1})^{2N+\beta-1}(1 - \sqrt{\xi} w_4)^{-2N-\beta+1}(1 - \frac{\sqrt{\xi}}{w_4})^{2N}}{(w_1 w_3^{-1}) (w_3 w_4 - 1)}. \]

Now we integrate over \{w_3\}. Note that the contour \{w_3\} can always be chosen inside the circle \{w_1\}. Therefore the integration over \{w_3\} reduces to the computation of the residue of the function \(w_3 \to (w_1 - w_3)^{-1}\) in the situation when \{w_3\} lies inside \{w_1\}. The result is
\[
K_{2N}^{\text{Meixner}} D_+^{\text{Meixner}} K_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) = \frac{1}{(2\pi i)^2} \frac{\sqrt{\Gamma(\tilde{x} + \beta)\Gamma(\tilde{y} + 1)}}{\Gamma(\tilde{x} + 1)\Gamma(\tilde{y} + \beta)} \times \int_{\{w_1\}} \int_{\{w_2\}} \frac{(1 - \sqrt{\xi} w_1)^{-2N}(1 - \frac{\sqrt{\xi}}{w_1})^{2N+\beta-1}(1 - \sqrt{\xi} w_2)^{-2N-\beta+1}(1 - \frac{\sqrt{\xi}}{w_2})^{2N}}{w_1 w_4 - 1} \frac{dw_1}{w_1^{2N+2}} \frac{dw_2}{w_2^{2N+2}}.
\]

Note that as soon as \{w_3\} is chosen to be inside \{w_1\}, and \(|w_3| > |w_4|^{-1}\), we have \(|w_3| > |w_4|^{-1}\) in the integral above. Thus \(w_4^{-1}\) is contained in the interior of \{w_1\}. To obtain formula for \(K_{2N}^{\text{Meixner}} D_+^{\text{Meixner}} K_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y})\) we can use the relation
\[
K_{2N}^{\text{Meixner}} D_+^{\text{Meixner}} K_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) = K_{2N}^{\text{Meixner}} D_-^{\text{Meixner}} K_{2N}^{\text{Meixner}}(\tilde{y}, \tilde{x})
\]
which follows immediately from the definitions of the involved operators. Thus we arrive to the formula
\[(4.4)\]
\[
K_{2N}^{\text{Meixner}} D_+^{\text{Meixner}} K_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) = \frac{1}{(2\pi i)^2} \frac{\sqrt{\Gamma(\tilde{x} + \beta)\Gamma(\tilde{y} + 1)}}{\Gamma(\tilde{x} + 1)\Gamma(\tilde{y} + \beta)} \times \int_{\{w_1\}} \int_{\{w_2\}} \frac{(1 - \sqrt{\xi} w_1)^{-2N}(1 - \frac{\sqrt{\xi}}{w_1})^{2N+\beta-1}(1 - \sqrt{\xi} w_2)^{-2N-\beta+1}(1 - \frac{\sqrt{\xi}}{w_2})^{2N}}{w_1 w_2 - 1} \frac{dw_1}{w_1^{2N+2}} \frac{dw_2}{w_2^{2N+2}}
\]
\[- \frac{1}{(2\pi i)^2} \frac{\sqrt{\Gamma(\tilde{x} + 1)\Gamma(\tilde{y} + \beta)}}{\Gamma(\tilde{x} + \beta)\Gamma(\tilde{y} + 1)} \times \int_{\{w_1\}} \int_{\{w_2\}} \frac{(1 - \sqrt{\xi} w_1)^{-2N}(1 - \frac{\sqrt{\xi}}{w_1})^{2N+\beta-1}(1 - \sqrt{\xi} w_2)^{-2N-\beta+1}(1 - \frac{\sqrt{\xi}}{w_2})^{2N}}{w_1 w_2 - 1} \frac{dw_1}{w_1^{2N+2}} \frac{dw_2}{w_2^{2N+2}},
\]
where the contours \{w_1\}, \{w_2\} are chosen in the same way as in the statement of Proposition 2.11. Exploiting the fact that \(K_{2N}^{\text{Meixner}}(\tilde{x}, \tilde{y}) = K_{2N}^{\text{Meixner}}(\tilde{y}, \tilde{x})\) we can rewrite this as a double integral. Applying the operators \(E^{\text{Meixner}}\) we obtain the contour integral representation for the kernel of the operator \(E^{\text{Meixner}} K_{2N}^{\text{Meixner}} D_+^{\text{Meixner}} K_{2N}^{\text{Meixner}} E^{\text{Meixner}}\). Adding
expressions for the kernels of $E^{\text{Meixner}}_2 K^{\text{Meixner}}_N$ and $K^{\text{Meixner}}_N E^{\text{Meixner}}_2$ we arrive to formulae in the statement of the Theorem.

It is possible to represent the function $S^{\text{Meixner}}_{2N}(\tilde{x}, \tilde{y})$ in the form which is manifestly antisymmetric with respect to $\tilde{x} \leftrightarrow \tilde{y}$. With this purpose in mind we introduce functions $P^{\text{Meixner}}(\tilde{x}, w, \beta)$ and $Q^{\text{Meixner}}(\tilde{x}, w, \beta)$ as follows. If $\tilde{x}$ is an even integer, then the functions $P^{\text{Meixner}}(\tilde{x}, w, \beta)$ and $Q^{\text{Meixner}}(\tilde{x}, w, \beta)$ are defined by the formulae

$$
P^{\text{Meixner}}(\tilde{x}, w, \beta) = \sqrt{\frac{\Gamma(\tilde{x}+1)}{\Gamma(\tilde{x}+\beta)}} F\left(\frac{\tilde{x}+2}{2}, 1; \frac{\tilde{x}+\beta+1}{2}; \frac{1}{w^2}\right) \frac{1}{w^{\tilde{x}+1}},
$$
and

$$
Q^{\text{Meixner}}(\tilde{x}, w, \beta) = \sqrt{\frac{\Gamma(\tilde{x}+\beta)}{\Gamma(\tilde{x}+1)}} F\left(\frac{\tilde{x}+\beta+1}{2}; 1; \frac{1}{w^2}\right) - 1 \frac{1}{w^{\tilde{x}}},
$$

If $\tilde{x}$ is an odd integer, then $P^{\text{Meixner}}(\tilde{x}, w, \beta)$ and $Q^{\text{Meixner}}(\tilde{x}, w, \beta)$ are defined by

$$
P^{\text{Meixner}}(\tilde{x}, w, \beta) = -\sqrt{\frac{\Gamma(\tilde{x}+1)}{\Gamma(\tilde{x}+\beta)}} F\left(-\frac{\beta+\tilde{x}-1}{2}; 1; -\frac{\tilde{x}}{2}; w^2\right) - 1 \frac{1}{w^{\tilde{x}+1}},
$$
and

$$
Q^{\text{Meixner}}(\tilde{x}, w, \beta) = -\sqrt{\frac{\Gamma(\tilde{x}+\beta)}{\Gamma(\tilde{x}+1)}} F\left(-\frac{\beta+\tilde{x}-2}{2}; 1; -\frac{\tilde{x}}{2}; w^2\right) \frac{1}{w^{\tilde{x}}}.
$$

In addition, set

$$
\tilde{S}^{\text{Meixner}}_{2N}(\tilde{x}, \tilde{y}) = \frac{\sqrt{\pi}}{(2\pi i)^2} \oint \oint (1 - \sqrt{\xi} w_1)^{-2N-\beta+1} (1 - \sqrt{\xi} w_2)^{-2N} \frac{(1 - \sqrt{\xi} w_1)^{-2N} (1 - \sqrt{\xi} w_2)^{-2N}}{w_1 w_2 - 1} \frac{dw_1}{w_1^{2N+1}} \frac{dw_2}{w_2^{2N+1}} \times P^{\text{Meixner}}(\tilde{x}, w, \beta) Q^{\text{Meixner}}(\tilde{x}, w, \beta) \frac{dw_1}{w_1^{2N+1}} \frac{dw_2}{w_2^{2N+1}}.
$$

where the contours $\{w_1\}, \{w_2\}$ are chosen as in the statement of Proposition 2.10 with the following additional conditions. If $\tilde{x}$ is an even integer, then $\{w_1\}$ lies in the domain $|w| > 1$. If $\tilde{x}$ is an odd integer, then $\{w_1\}$ lies in the domain $|w| < 1$. The same condition is imposed on $\{w_2\}$: if $\tilde{x}$ is an even integer, then $\{w_2\}$ lies in the domain $|w| > 1$, and if $\tilde{x}$ is an odd integer, then $\{w_1\}$ lies in the domain $|w| < 1$.

**Proposition 4.2.** The function $S^{\text{Meixner}}_{2N}(\tilde{x}, \tilde{y})$ can be written as

$$
S^{\text{Meixner}}_{2N}(\tilde{x}, \tilde{y}) = S^{\text{Meixner}}_{2N}(\tilde{x}, \tilde{y}) - S^{\text{Meixner}}_{2N}(\tilde{y}, \tilde{x}),
$$
where the function $S^{\text{Meixner}}_{2N}(\tilde{x}, \tilde{y})$ is defined by equation (4.9).
Remark 4.3. For our purposes, in particular, for the analytic continuation of the Meixner symplectic ensemble the expression for $S_{2N}^{Meixner}$ is convenient.

Proof. Using the fact that the operators $D^{Meixner}$ and $E^{Meixner}$ are mutually inverse we obtain from equation (4.1) the relation

$$D^{Meixner} S_{2N}^{Meixner} D^{Meixner} = D^{Meixner} K_{2N}^{Meixner} + K_{2N}^{Meixner} D^{Meixner} - K_{2N}^{Meixner} D^{Meixner} K_{2N}^{Meixner}. $$

This relation (together with formulae (4.1), (4.4), and Proposition 2.10) enables us to find an explicit formula for the kernel of the operator $D^{Meixner} S_{2N}^{Meixner} D^{Meixner}$. Namely,

$$D^{Meixner} S_{2N}^{Meixner} D^{Meixner} (x, y) = \frac{1}{(2\pi i)^2 \sqrt{\xi}} \left[ \frac{\Gamma(\tilde{x} + \beta) \Gamma(\tilde{y} + 1)}{\Gamma(\tilde{x} + 1) \Gamma(\tilde{y} + \beta)} \right] \times \oint_{\{w_1\}} \oint_{\{w_2\}} \frac{(1 - \sqrt{\xi} w_1)^{-2N - \beta + 1} (1 - \sqrt{\xi} w_2)^{-2N}}{w_1 w_2 - 1} \frac{dw_1}{w_1^{2N+1}} \frac{dw_2}{w_2^{2N+1}}$$

- $(\tilde{x} \longleftrightarrow \tilde{y})$.

Applying $E^{Meixner}$ to the both sides of the formula above we obtain the representation for $S_{2N}^{Meixner}(x, y)$ in the manifestly antisymmetric form. \hfill \Box

Remark 4.3. For our purposes, in particular, for the analytic continuation of the Meixner symplectic ensemble the expression for $S_{2N}^{Meixner}(x, y)$ given in Theorem 4.1 is more convenient.

5. Proof of Theorem 2.11 for special values of parameters $z$ and $z'$

The aim of the this Section is to show that for $N = 1, 2, \ldots, z = 2N$, and $z' = 2N + \beta - 2$ with $\beta > 0$ the formula for the correlation function $\varrho_{n}^{(z,z',\xi,\theta=2)}(x_1, \ldots, x_n)$ obtained in Lemma 3.4 is equivalent to the formula for the correlation function $\varrho_{n}^{(z,z',\xi,\theta=2)}(x_1, \ldots, x_n)$ stated in Theorem 2.11. Once we show this equivalence, we prove Theorem 2.11 for special values of $z$ and $z'$. The transformation from the formula in Lemma 3.4 (where the kernel $K_{2N}^{Meixner}$ is given by Proposition 3.3 together with formula (1)) to the formula in Theorem 2.11 is achieved by a set of nontrivial and rather technically complicated algebraic manipulations. To motivate these manipulations recall that the $z$-measure $M_{z,z',\xi,\theta=2}$ is manifestly symmetric with respect to $z \longleftrightarrow z'$. Therefore, the final formula for the correlation function $\varrho_{n}^{(z,z',\xi,\theta=2)}(x_1, \ldots, x_n)$ must be manifestly symmetric with respect to $z \longleftrightarrow z'$ as well.

It is convenient to introduce three functions on $Z \times Z$, namely $I_{z,z',\xi,\theta=2}(x,y)$, $A_{z,z',\xi,\theta=2}(x,y)$, and $B_{z,z',\xi,\theta=2}(x,y)$. We will define these functions in terms of contour integrals. Let $\{w_1\}$ and $\{w_2\}$ be arbitrary simple contours satisfying the conditions

- both contours go around 0 in positive direction;
- the point $\xi^{1/2}$ is in the interior of each of the contours while the point $\xi^{-1/2}$ lies outside the contours;
- the contour $\{w_1^{-1}\}$ is contained in the interior of the contour $\{w_2\}$ (equivalently, $\{w_2^{-1}\}$ is contained in the interior of $\{w_1\}$).
If $x - \frac{1}{2}$ is an even integer, and $y - \frac{1}{2}$ is an arbitrary integer, then the first function, $I_{z, z', \xi, \theta=2}(x, y)$, is defined by

$$I_{z, z', \xi, \theta=2}(x, y) = -\frac{1}{(2\pi i)^2} \oint_{\{w_1\}} \oint_{\{w_2\}} \frac{(1 - \sqrt{\xi} w_1)^{-z'} (1 - \sqrt{\xi} w_2)^z}{w_1 w_2 - 1} \times F\left(\frac{x + z - \frac{1}{2}}{2}, 1; \frac{x + z' + \frac{1}{2}}{2}; w_1^2\right) \frac{dw_1}{w_1^{-\frac{1}{2}}} \frac{dw_2}{w_2^{-\frac{1}{2}}},$$

(5.1)

where the contours are chosen as described above with an additional condition that both contours lie in the domain $|w| > 1$.

If $x - \frac{1}{2}$ is an odd integer, and $y - \frac{1}{2}$ is an arbitrary integer, then $I_{z, z', \xi, \theta=2}(x, y)$ is defined by

$$I_{z, z', \xi, \theta=2}(x, y) = \frac{1}{(2\pi i)^2} \oint_{\{w_1\}} \oint_{\{w_2\}} \frac{(1 - \sqrt{\xi} w_1)^{-z'} (1 - \sqrt{\xi} w_2)^z}{w_1 w_2 - 1} \times \left[F\left(-\frac{x + z - \frac{3}{2}}{2}, 1; -\frac{x + z - \frac{5}{2}}{2}; w_1^2\right) - 1\right] \frac{dw_1}{w_1^{-\frac{1}{2}}} \frac{dw_2}{w_2^{-\frac{1}{2}}},$$

(5.2)

where the contours are chosen as described above with an additional condition that both contours lie in the domain $|w| < 1$.

Next let us define the second function, namely $A_{z, z', \xi, \theta=2}(x)$. If $x - \frac{1}{2}$ is an even integer, then $A_{z, z', \xi, \theta=2}(x)$ is defined by the contour integral

$$A_{z, z', \xi, \theta=2}(x) = -\frac{1}{2\pi i} \oint_{\{w\}} (1 - \sqrt{\xi} w)^{-z'-1} (1 - \sqrt{\xi} w)^z F\left(-\frac{x + z - \frac{3}{2}}{2}, 1; \frac{x + z' + \frac{1}{2}}{2}; w^2\right) \frac{dw}{w^{x-\frac{1}{2}}},$$

(5.3)

where $\{w\}$ is an arbitrary simple contour going around 0 in the positive direction, and such that it lies in the domain $|w| > 1$.

If $x - \frac{1}{2}$ is an odd integer, then we set

$$A_{z, z', \xi, \theta=2}(x) = \frac{1}{2\pi i} \oint_{\{w\}} (1 - \sqrt{\xi} w)^{-z'-1} (1 - \sqrt{\xi} w)^z \times \left[F\left(-\frac{x + z - \frac{3}{2}}{2}, 1; -\frac{x + z - \frac{5}{2}}{2}; w^2\right) - 1\right] \frac{dw}{w^{x-\frac{1}{2}}},$$

(5.4)

where $\{w\}$ is an arbitrary simple contour going around 0 in the positive direction, and such that it lies in the domain $|w| < 1$. 


Finally, we define $B_{z,z',\xi,\theta=2}(y)$. This function has the following contour integral representation. If $y - \frac{1}{2}$ is an even integer, then

$$
B_{z,z',\xi,\theta=2}(y) = \frac{1}{2\pi i} \oint_{\{w\}} (1 - \sqrt{\xi w})^{-\frac{y}{2}}(1 - \frac{\sqrt{\xi}}{w})^{\frac{y}{2}} \left(1 - (1 - \frac{\sqrt{\xi}}{w})F\left(\frac{y + z'+ \frac{1}{2}}{2}, 1; \frac{y + z - \frac{1}{2}}{2}; \frac{1}{w^2}\right)\right) \frac{dw}{w^{y - \frac{1}{2}}}. 
$$

(5.5)

Here $\{w\}$ is an arbitrary simple contour going around 0 in the positive direction, and such that it lies in the domain $|w| > 1$.

If $y - \frac{1}{2}$ is an odd integer, then

$$
B_{z,z',\xi,\theta=2}(y) = \frac{1}{2\pi i} \oint_{\{w\}} (1 - \sqrt{\xi w})^{-\frac{y}{2}}(1 - \frac{\sqrt{\xi}}{w})^{\frac{y}{2}} \left(1 - (1 - \frac{\sqrt{\xi}}{w})F\left(\frac{y + z'+ \frac{1}{2}}{2}, 1; \frac{y + z - \frac{1}{2}}{2}; \frac{1}{w^2}\right)\right) \frac{dw}{w^{y - \frac{1}{2}}}. 
$$

(5.6)

Here $\{w\}$ is an arbitrary simple contour going around 0 in the positive direction, and such that it lies in the domain $|w| > 1$.

**Proposition 5.1.** 1) For $N = 1, 2, \ldots$ let $z = 2N$ and $z' = 2N + \beta - 2$ with $\beta > 0$. Assume that $x_1, \ldots, x_n$ lie in the subset $\mathbb{Z}_{\geq 0} - 2N + \frac{1}{2} \subset \mathbb{Z}'$, so that the points $\bar{x}_i = x_i + 2N - 1/2$ are in $\mathbb{Z}_{\geq 0}$. Then

$$
\varrho_n^{(z,z',\xi,\theta=2)}(x_1, \ldots, x_n) = \text{Pf} [\mathbb{K}_{z,z',\xi,\theta=2}(x_i, x_j)]_{i,j=1}^n,
$$

where the correlation kernel $\mathbb{K}_{z,z',\xi,\theta=2}(x,y)$ has the following form

$$
\mathbb{K}_{z,z',\xi,\theta=2}(x,y) = \begin{bmatrix}
S_{z,z',\xi,\theta=2}(x,y) & -SD_{z,z',\xi,\theta=2}(x,y) \\
-D+S_{z,z',\xi,\theta=2}(x,y) & D_+SD_{z,z',\xi,\theta=2}(x,y)
\end{bmatrix}.
$$

2) The functions $SD_{z,z',\xi,\theta=2}(x,y)$, $D_+S_{z,z',\xi,\theta=2}(x,y)$, and $D_+SD_{z,z',\xi,\theta=2}(x,y)$ are expressible in terms of the function $S_{z,z',\xi,\theta=2}(x,y)$ as follows

$$
SD_{z,z',\xi,\theta=2}(x,y) = S_{z,z',\xi,\theta=2}(x,y) \sqrt{\frac{y + z + \frac{1}{2}}{y + z' + \frac{3}{2}}},
$$

$$
D_+S_{z,z',\xi,\theta=2}(x,y) = \sqrt{\frac{x + z + \frac{1}{2}}{x + z' + \frac{3}{2}}} S_{z,z',\xi,\theta=2}(x + 1, y),
$$

and

$$
D_+SD_{z,z',\xi,\theta=2}(x,y) = \sqrt{\frac{x + z + \frac{1}{2}}{x + z' + \frac{3}{2}}} S_{z,z',\xi,\theta=2}(x + 1, y + 1) \sqrt{\frac{y + z + \frac{1}{2}}{y + z' + \frac{3}{2}}}.
$$
3) The function $S_{z,z',\xi,\theta=2}(x,y)$ can be written as
\[
S_{z,z',\xi,\theta=2}(x,y) = \sqrt{\frac{\Gamma(x + z + \frac{1}{2})\Gamma(y + z' + \frac{1}{2})}{\Gamma(x - \frac{1}{2})\Gamma(y + z' + \frac{1}{2})}} 
\times [I_{z,z',\xi,\theta=2}(x + 2, y + 1) + A_{z,z',\xi,\theta=2}(x + 2)B_{z,z',\xi,\theta=2}(y + 1)],
\]
and it is related with the kernel $S_{2N}^{Meixner}(\tilde{x}, \tilde{y})$ as
\[
S_{2N}^{Meixner}(x + 2N - \frac{1}{2}, y + 2N - \frac{1}{2}) = \sqrt{\xi} S_{z,z',\xi,\theta=2}(x,y).
\]

**Proof.** Follows immediately from Theorem 4.1, Lemma 3.4, and Proposition 3.3. \qed

**Proposition 5.2.** If $x - \frac{1}{2}$ is an even integer, then the function $I_{z,z',\xi,\theta=2}(x,y)$ (defined by equation (5.7)) can be written as
\[
I_{z,z',\xi,\theta=2}(x,y) = -\sqrt{\frac{\Gamma(x + z' - \frac{1}{2})\Gamma(y + z - \frac{1}{2})}{\Gamma(x + z - \frac{1}{2})\Gamma(y + z' - \frac{1}{2})}} 
\times \sum_{l=0}^{\infty} \sqrt{\frac{(x + z - \frac{1}{2})_{l,2}(x + z' - \frac{1}{2})_{l,2}}{(x + z + \frac{1}{2})_{l,2}(x + z' + \frac{1}{2})_{l,2}}} K_{z,z'}(x + 2l - 1, y - 1),
\]
where the function $K_{z,z'}(x,y)$ is defined by equation (2.5). If $x - \frac{1}{2}$ is an odd integer, then the function $I_{z,z',\xi,\theta=2}(x,y)$ (defined by equation (5.7)) can be written as
\[
I_{z,z',\xi,\theta=2}(x,y) = \sqrt{\frac{\Gamma(x + z' - \frac{1}{2})\Gamma(y + z - \frac{1}{2})}{\Gamma(x + z - \frac{1}{2})\Gamma(y + z' - \frac{1}{2})}} 
\times \sum_{l=1}^{\infty} \sqrt{\frac{(-x - z + \frac{3}{2})_{l,2}(-x + z' - \frac{3}{2})_{l,2}}{(-x + z + \frac{3}{2})_{l,2}(-x + z' + \frac{3}{2})_{l,2}}} K_{z,z'}(x + 2l - 1, y - 1).
\]

**Proof.** Rewrite the Gauss hypergeometric functions inside the integrals in equations (5.1), (5.2) as infinite sums. These sums are uniformly convergent in the domains where the contours of integration are chosen. Therefore we can interchange summation and integration. The integrals inside the sums can be expressed in terms of the function $K_{z,z'}(x,y)$ as it follows from Proposition 2.10. \qed

**Proposition 5.3.** If $x - \frac{1}{2}$ is an even integer, then
\[
A_{z,z',\xi,\theta=2}(x) = -(1 - \xi)^{\frac{x - z' - 1}{2}} \sqrt{\frac{\Gamma(z + 1)\Gamma(x + z' - \frac{1}{2})}{\Gamma(z' + 1)\Gamma(x + z - \frac{1}{2})}} 
\times \sum_{l=0}^{\infty} \sqrt{\frac{(z + x - \frac{1}{2})_{l,2}(z' + x - \frac{1}{2})_{l,2}}{(z + x + \frac{1}{2})_{l,2}(z' + x + \frac{1}{2})_{l,2}}} \psi_{\frac{1}{2}}(x + 2l - 1; z, z', \xi),
\]
where the function $\psi_{-\frac{1}{2}}(x; z, z', \xi)$ is defined by equation (2.4). If $x - \frac{1}{2}$ is an odd integer, then

$$A_{z,z',\xi,\theta=2}(x) = (1 - \xi)^{\frac{x'-\frac{1}{2}}{2}} \sqrt{\frac{\Gamma(z+1)\Gamma(x + z' - \frac{1}{2})}{\Gamma(z'+1)\Gamma(x + z - \frac{1}{2})}} \times \sum_{l=1}^{\infty} \sqrt{\frac{(-x - z + \frac{3}{2})_{l,2} (-x - z' + \frac{3}{2})_{l,2}}{(-x - z + \frac{3}{2})_{l,2} (-x - z' + \frac{3}{2})_{l,2}}} \psi_{-\frac{1}{2}}(x - 2l - 1; z, z', \xi).$$

**Proof.** The function $A_{z,z',\xi,\theta=2}(x)$ is defined by equations (5.3), (5.5). As in the proof of the previous Proposition represent the Gauss hypergeometric functions inside the integrals as infinite sums, and interchange summation and integration. Then use Proposition 2.4 to rewrite the integrals inside the sums in terms of $\psi_{-\frac{1}{2}}(x; z, z', \xi)$. □

**Proposition 5.4.** If $y - \frac{1}{2}$ is an even integer, then the function $B_{z,z',\xi,\theta=2}^1(y)$ (defined by equation (5.5)) can be represented as

$$B_{z,z',\xi,\theta=2}^1(y) = -(1 - \xi)^{\frac{y'-\frac{1}{2}}{2}} \sqrt{\frac{\Gamma(z+1)\Gamma(y + z' - \frac{1}{2})}{\Gamma(z'+1)\Gamma(y + z - \frac{1}{2})}} \times \sum_{l=0}^{\infty} \sqrt{\frac{(y + z + \frac{1}{2})_{l,2} (y + z' + \frac{1}{2})_{l,2}}{(y + z + \frac{5}{2})_{l,2} (y + z' + \frac{5}{2})_{l,2}}} \psi_{1/2}(y + 2l; z, z', \xi).$$

If $y - \frac{1}{2}$ is an odd integer, then the function $B_{z,z',\xi,\theta=2}^1(y)$ (defined by equation (5.6)) can be represented as

$$B_{z,z',\xi,\theta=2}^1(y) = (1 - \xi)^{\frac{y'-\frac{1}{2}}{2}} \sqrt{\frac{\Gamma(z+1)\Gamma(y + z' - \frac{1}{2})}{\Gamma(z'+1)\Gamma(y + z - \frac{1}{2})}} \times \sum_{l=1}^{\infty} \sqrt{\frac{(-y - z + \frac{1}{2})_{l,2} (-y - z' + \frac{1}{2})_{l,2}}{(-y - z + \frac{3}{2})_{l,2} (-y - z' + \frac{3}{2})_{l,2}}} \psi_{1/2}(y - 2l; z, z', \xi).$$

**Proof.** Consider first the case when $y - \frac{1}{2}$ is an even integer. In this case the function $B_{z,z',\xi,\theta=2}^1(y)$ is defined by equation (5.5). We use the identity

$$(1 - \sqrt{\xi} w)[1 - (1 - \sqrt{\xi} w)F] = 1 - (1 + \xi)F + \sqrt{\xi}[\frac{F}{w} + w(F - 1)]$$

to rewrite $B_{z,z',\xi,\theta=2}^1(y)$ as a sum of three terms each of which is defined by contour integrals. Namely, we have

$$\hat{B}_{z,z',\xi,\theta=2}^1(y + \frac{1}{2}) = T_1(y) - (1 + \xi)T_2(y) + \sqrt{\xi}T_3(y).$$
where

\[
T_1(y) = \frac{1}{2\pi i} \oint_{\{w\}} (1 - \sqrt{\xi w})^{-z-1}(1 - \sqrt{\xi w})^{-z} dw \frac{df}{w^y},
\]

\[
T_2(y) = \frac{1}{2\pi i} \oint_{\{w\}} (1 - \sqrt{\xi w})^{-z-1}(1 - \sqrt{\xi w})^{-z} F \left( \frac{y + z' + 1}{2}, 1; \frac{y + z}{2}; \frac{1}{w^2} \right) dw \frac{1}{w^{y+1}},
\]

and

\[
T_3(y) = \frac{1}{2\pi i} \oint_{\{w\}} (1 - \sqrt{\xi w})^{-z-1}(1 - \sqrt{\xi w})^{-z} F \left( \frac{y + z' + 1}{2}, 1; \frac{y + z}{2}; \frac{1}{w^2} \right) dw \frac{1}{w^{y+1}}.
\]

In the formulae for \(T_1(y), T_2(y),\) and \(T_3(y)\) written above the contour \(\{w\}\) lies in the domain \(|w| > 1\). We use Proposition 2.8 to represent \(T_1(y)\) in terms of the Gauss hypergeometric function, namely we obtain

\[
T_1(y) = (1 - \xi)^{z'} \xi^{\frac{z+1}{2}} \Gamma(z+y) \frac{\Gamma(\frac{z+y}{2})}{\Gamma(y)}.
\]

Consider the expression for the function \(T_2(y)\). We represent the hypergeometric function inside the integral as the infinite series,

\[
F \left( \frac{y + z' + 1}{2}, 1; \frac{y + z}{2}; \frac{1}{w^2} \right) = \sum_{l=0}^{\infty} \left( \frac{y+z+1}{2} \right)_l \frac{1}{w^{2l}}.
\]

Again, this series is uniformly convergent on the integration contour. Therefore, we can interchange the summation and integration, and compute the integrals in terms of the Gauss hypergeometric function using Proposition 2.8. The result is

\[
T_2(y) = (1 - \xi)^{z'} \frac{\Gamma(z+y)}{\Gamma(z+1)} \times \sum_{l=0}^{\infty} \left( \frac{y+z+1}{2} \right)_l \left( \frac{y+z}{2} \right)_l \xi^{\frac{2l+y-1}{2}} \frac{F \left( -z, -z'; 2l + y; \frac{\xi}{\xi-1} \right)}{\Gamma(2l+y)}.
\]
In a similar way we find after some calculations that

\[ T_3(y) = (1 - \xi)^{z'} \frac{\Gamma(z + y)}{\Gamma(z + 1)} \sum_{l=0}^{\infty} \left( \frac{z' + y + 1}{2} \right)_l \left( \frac{z + y + 1}{2} \right)_l \xi^{2l+y} 2^{2l} \]

\[ \times (2y + z + z' + 4l + 1) \frac{F\left(-z, -z'; 2l + y + 1; \frac{\xi}{\xi-1}\right)}{\Gamma(2l + y + 1)} \].

This gives the following expression for \( B_{z,z',\xi,\theta=2}(y + \frac{1}{2}) \)

\[ B_{z,z',\xi,\theta=2}(y + \frac{1}{2}) = -(1 - \xi)^{z'} \frac{\Gamma(z + y)}{\Gamma(z + 1)} \sum_{l=0}^{\infty} \left( \frac{z + y + 1}{2} \right)_l \left( \frac{z' + y + 1}{2} \right)_l \xi^{2l+y+1} 2^{2l} \]

\[ \times \left[ (z + y + 1 + 2l)(z' + y + 1 + 2l) \frac{F\left(-z, -z'; 2l + y + 2; \frac{\xi}{\xi-1}\right)}{\Gamma(2l + y + 2)} + \frac{F\left(-z, -z'; 2l + y; \frac{\xi}{\xi-1}\right)}{\Gamma(2l + y)} \right] \]

\[ - (2y + z + z' + 4l + 1) \frac{F\left(-z, -z'; 2l + y + 1; \frac{\xi}{\xi-1}\right)}{\Gamma(2l + y + 1)} \].

Set

\[ A = -z, \ B = -z', \ C = 2l + y + 1. \]

Then

\[ 2y + z + z' + 4l + 1 = 2C - A - B - 1, \ z + y + 1 + 2l = C - A, \ z' + y + 1 + 2l = C - B, \]

and the sum of three terms in the brackets in the expression for \( B_{z,z',\xi,\theta=2}(y + \frac{1}{2}) \) can be rewritten as

\[ (C - A)(C - B) \frac{F(A, B; C + 1; w)}{\Gamma(C + 1)} + \frac{F(A, B; C - 1; w)}{\Gamma(C - 1)} - (2C - A - B - 1) \frac{F(A, B; C; w)}{\Gamma(C)}. \]

Here \( w = \frac{\xi}{\xi-1} \). The following relation for the Gauss hypergeometric function holds true

\[ (C - A)(C - B)F(A, B; C + 1; w) + C(C - 1)F(A, B; C - 1; w) \]

\[ - C(2C - A - B - 1)F(A, B; C; w) = ABF(A + 1, B + 1; C + 1; w). \]

Using this relation, we rewrite \( B_{z,z',\xi,\theta=2}(y + \frac{1}{2}) \) as

\[ B_{z,z',\xi,\theta=2}(y + 1/2) = -(1 - \xi)^{z'} \frac{\Gamma(z + y)}{\Gamma(z + 1)} \]

\[ \times \sum_{l=0}^{\infty} \left( \frac{z + y + 1}{2} \right)_l \left( \frac{z' + y + 1}{2} \right)_l \xi^{2l+y+1} 2^{2l} \frac{F\left(-z, -z'; 2l + y; \frac{\xi}{\xi-1}\right)}{\Gamma(2l + y)} \].
Finally, Proposition 2.4 enables us to represent $B_{z,z',\xi,\theta=2}(y)$ as in the statement of Proposition 5.4. The formula for $B_{z,z',\xi,\theta=2}(y)$ in the case of an odd $y - \frac{1}{2}$ is obtained by a similar calculation.

**Proposition 5.5.** For $N = 1, 2, \ldots$ let $z = 2N$ and $z' = 2N + \beta - 2$ with $\beta > 0$. Assume that $x_1, \ldots, x_n$ lie in the subset $\mathbb{Z}_{\geq 0} - 2N + \frac{1}{2} \subset \mathbb{Z}'$, so that the points $\bar{x}_i = x_i + 2N - 1/2$ are in $\mathbb{Z}_{\geq 0}$. Then the correlation function $\varrho_n^{(z,z',\xi,\theta=2)}(x_1, \ldots, x_n)$ is determined by the formulae of Theorem 2.11.

**Proof.** Let the parameters $z, z'$, and the points $x_1, \ldots, x_n$ be chosen as in the statement of the Proposition. Then a straightforward application of Propositions 5.1-5.4 gives the correlation function $\varrho_n^{(z,z',\xi,\theta=2)}(x_1, \ldots, x_n)$ as a Pfaffian of a $2 \times 2$ block matrix defined by the kernel $K_{z,z',\xi,\theta=2}(x,y)$, and the explicit formulae for the matrix entries of $K_{z,z',\xi,\theta=2}(x,y)$. It can be checked that the kernel $K_{z,z',\xi,\theta=2}(x,y)$ is equivalent to the kernel $K_{z,z',\xi,\theta=2}(x,y)$ of Theorem 2.11, i.e.

$$\text{Pf} \left[ K_{z,z',\xi,\theta=2}(x_i, x_j) \right]_{i,j=1}^n = \text{Pf} \left[ K_{z,z',\xi,\theta=2}(x_i, x_j) \right]_{i,j=1}^n,$$

and that the functions $S_{z,z',\xi,\theta=2}$ and $S_{z,z',\xi,\theta=2}$ that define the matrix kernels $K_{z,z',\xi,\theta=2}(x,y)$ and $K_{z,z',\xi,\theta=2}(x,y)$ are related as

$$S_{z,z',\xi,\theta=2}(x,y) = \sqrt{(x + \frac{1}{2})(y + \frac{1}{2})} S_{z,z',\xi,\theta=2}(x,y).$$

□

6. Analytic continuation

6.1. Analytic properties of correlation functions.

**Proposition 6.1.** Fix an arbitrary set of Young diagrams $\mathcal{D} \subset \mathcal{Y}$. For any fixed admissible pair of parameters $(z, z')$, and for $\theta > 0$, the function

$$\xi \rightarrow \sum_{\lambda \in \mathcal{D}} M_{z,z',\xi,\theta}(\lambda)$$

which is initially defined on the interval $(0, 1)$ can be extended to a holomorphic function in the unit disk $|\xi| < 1$.

**Proof.** Comparing equations (2.3) and (2.5) we obtain

$$M_{z,z',\xi,\theta}(\lambda) = (1 - \xi)^t \frac{(t)_n}{n!} \xi^n M_{z,z',\theta}(\lambda), \ n = |\lambda|.$$

Set $\mathcal{D}_n = \mathcal{D} \cap \mathcal{Y}_n$. Using equation (6.1) we can write

$$\sum_{\lambda \in \mathcal{D}} M_{z,z',\xi,\theta}(\lambda) = (1 - \xi)^t \sum_{n=0}^\infty \left( \sum_{\lambda \in \mathcal{D}_n} M_{z,z',\theta}^{(n)}(\lambda) \right) \frac{(t)_n \xi^n}{n!}. $$
The interior sum is nonnegative, and does not exceed 1. On the other hand
\[ \sum_{n=0}^{\infty} \frac{(t)_n \xi^n}{n!} = \sum_{n=0}^{\infty} \frac{(t)_n |\xi|^n}{n!} < \infty, \quad \xi \in \mathbb{C}, \quad |\xi| < 1. \]

This shows that the function \( \xi \to \sum_{\lambda \in \mathcal{D}} M_{z,z',\xi,\theta}(\lambda) \) can be represented as a power series in \( \xi \), which is convergent in the unit disk \(|\xi| < 1\). Therefore, the function \( \xi \to \sum_{\lambda \in \mathcal{D}} M_{z,z',\xi,\theta}(\lambda) \) is holomorphic in \(|\xi| < 1\). \( \square \)

**Proposition 6.2.** Fix an arbitrary set of Young diagrams \( \mathcal{D} \subset \mathcal{Y} \). Consider the Taylor expansion of the function
\[ \xi \to \sum_{\lambda \in \mathcal{D}} M_{z,z',\xi,\theta}(\lambda) \]
at \( \xi = 0 \),
\[ \sum_{\lambda \in \mathcal{D}} M_{z,z',\xi,\theta}(\lambda) = \sum_{k=0}^{\infty} G_{k,\mathcal{D}}^{(\theta)}(z, z') \xi^k. \]

Then the coefficients \( G_{k,\mathcal{D}}^{(\theta)}(z, z') \) are polynomial functions in \( z, z' \).

**Proof.** By (2,3)
\[ \sum_{\lambda \in \mathcal{D}} M_{z,z',\xi,\theta}(\lambda) = (1 - \xi)^{zz'} \sum_{n=0}^{\infty} \sum_{\lambda \in \mathcal{D}_n} (z)_{\lambda,\theta}(z')_{\lambda,\theta} \xi^n \frac{1}{H(\lambda, \theta) H'(\lambda, \theta)}. \]
Recall that \((z)_{\lambda,\theta}\) and \((z')_{\lambda,\theta}\) are polynomials in variables \( z \) and \( z' \) correspondingly. We have
\[ (1 - \xi)^{zz'} = \sum_{m=0}^{\infty} \frac{(-\frac{zz'}{\theta})^m \xi^m}{m!}. \]
Inserting this expansion into the righthand side of the formula for \( \sum_{\lambda \in \mathcal{D}} M_{z,z',\xi,\theta}(\lambda) \) written above we find
\[ G_{k,\mathcal{D}}^{(\theta)}(z, z') = \sum_{k=0}^{\infty} \sum_{\lambda \in \mathcal{D}_n} \frac{(-\frac{zz'}{\theta})^k (-\frac{zz'}{\theta})^n (z)_{\lambda,\theta}(z')_{\lambda,\theta} \xi^n}{(k - n)!} \frac{1}{H(\lambda, \theta) H'(\lambda, \theta)}. \]
Since each \( \mathcal{D}_n \) is a finite set, this expression is a polynomial in variables \( z, z' \). \( \square \)

Set
\[ K_{z,z',\xi}(x, y) = \varphi_{z,z'}(x, y) \overline{K}_{z,z',\xi}(x, y), \]
where
\[ \varphi_{z,z'}(x, y) = \frac{\sqrt{\Gamma(x+z+\frac{1}{2})\Gamma(x+z'+\frac{1}{2})\Gamma(y+z+\frac{1}{2})\Gamma(y+z+\frac{1}{2})}}{\Gamma(x+z+\frac{1}{2})\Gamma(y+z+\frac{1}{2})}. \]
Then Proposition 2.10 implies that $\hat{K}_{x,z',\xi}(x,y)$ is representable as a double contour integral involving elementary functions only. Namely, we have

\begin{equation}
\hat{K}_{x,z',\xi}(x,y) = \frac{1}{(2\pi i)^2} \int_{\{w_1\}} \int_{\{w_2\}} \frac{(1 - \sqrt{\xi}w_1)^{-z'}(1 - \frac{\xi}{w_1})^{-z}(1 - \sqrt{\xi}w_2)^{-z'}(1 - \frac{\xi}{w_2})^{-z}}{w_1w_2 - 1}
\times w_1^{-y - \frac{1}{2}} w_2^{-y + \frac{1}{2}} dw_1 dw_2,
\end{equation}

where the contours are chosen in the same way as in the statement of Proposition 2.10.

We also introduce the functions $\hat{\psi}_{\frac{1}{2}}(x; z, z', \xi)$ and $\hat{\psi}_{-\frac{1}{2}}(x; z, z', \xi)$ that are closely related to the functions $\psi_{\frac{1}{2}}(x; z, z', \xi)$ and $\psi_{-\frac{1}{2}}(x; z, z', \xi)$. The functions $\hat{\psi}_{\frac{1}{2}}(x; z, z', \xi)$ and $\hat{\psi}_{-\frac{1}{2}}(x; z, z', \xi)$ are defined by equations

\begin{equation}
\hat{\psi}_{\frac{1}{2}}(x; z, z', \xi) = f_{x,z',\frac{1}{2}}(x)(1 - \xi)^{-\frac{z'}{2} + \frac{1}{2}} \hat{\psi}_{\frac{1}{2}}(x; z, z', \xi),
\end{equation}

and

\begin{equation}
\hat{\psi}_{-\frac{1}{2}}(x; z, z', \xi) = f_{x,z',-\frac{1}{2}}(x)(1 - \xi)^{-\frac{z'}{2} + \frac{1}{2}} \hat{\psi}_{-\frac{1}{2}}(x; z, z', \xi).
\end{equation}

Here $f_{x,z',\frac{1}{2}}$ and $f_{x,z',-\frac{1}{2}}$ are gamma prefactors given by

\begin{equation}
f_{x,z',\frac{1}{2}}(x) = \frac{\Gamma(z)}{\sqrt{\Gamma(z)\Gamma(z')}} \frac{\sqrt{\Gamma(x + z + \frac{1}{2})\Gamma(x + z' + \frac{1}{2})}}{\Gamma(x + z + \frac{1}{2})},
\end{equation}

and

\begin{equation}
f_{x,z',-\frac{1}{2}}(x) = \frac{\Gamma(z' + 1)}{\sqrt{\Gamma(z + 1)\Gamma(z' + 1)}} \frac{\sqrt{\Gamma(x + z + \frac{1}{2})\Gamma(x + z' + \frac{1}{2})}}{\Gamma(x + z' + \frac{1}{2})}.
\end{equation}

Proposition 2.9 implies that the functions $\hat{\psi}_{\frac{1}{2}}(x; z, z', \xi)$ and $\hat{\psi}_{-\frac{1}{2}}(x; z, z', \xi)$ have the contour integral representations

\begin{equation}
\hat{\psi}_{\frac{1}{2}}(x; z, z', \xi) = \frac{1}{2\pi i} \oint_{\{w\}} (1 - \sqrt{\xi}w)^{-z}(1 - \frac{\xi}{w})^{-1} w^{-\frac{1}{2}} dw w^{-\frac{1}{2}},
\end{equation}

and

\begin{equation}
\hat{\psi}_{-\frac{1}{2}}(x; z, z', \xi) = \frac{1}{2\pi i} \oint_{\{w\}} (1 - \sqrt{\xi}w)^{-z'}(1 - \frac{\xi}{w})^{-1} w^{-\frac{1}{2}} dw w^{-\frac{1}{2}}.
\end{equation}

Here the contour $\{w\}$ is chosen as in Proposition 2.9. Note that the functions $\hat{\psi}_{\frac{1}{2}}(x; z, z', \xi)$ and $\hat{\psi}_{-\frac{1}{2}}(x; z, z', \xi)$ are defined by contour integrals involving elementary functions only.
It is convenient to introduce the functions $\left(\mathcal{E}\widehat{K}_{z,z',\xi}\right)(x, y)$, $\left(\mathcal{E}\widehat{\psi}_1\right)(x; z, z', \xi)$, and $\left(\mathcal{E}\widehat{\psi}_{-1}\right)(x; z, z', \xi)$. If $x - \frac{1}{2}$ is an even integer, then we set

\begin{equation}
\left(\mathcal{E}\widehat{K}_{z,z',\xi}\right)(x, y) = -\sum_{l=0}^{\infty} \frac{(z + x + \frac{3}{2})l, 2}{(z' + x + \frac{5}{2})l, 2} \widehat{K}_{z,z',\xi}(x + 2l + 1, y),
\end{equation}

(6.5)

\begin{equation}
\left(\mathcal{E}\widehat{\psi}_{\pm\frac{1}{2}}\right)(x; z, z', \xi) = -\sum_{l=0}^{\infty} \frac{(z + x + \frac{3}{2})l, 2}{(z' + x + \frac{5}{2})l, 2} \widehat{\psi}_{\pm\frac{1}{2}}(x + 2l + 1; z, z', \xi).
\end{equation}

(6.6)

If $x - \frac{1}{2}$ is an odd integer, then we set

\begin{equation}
\left(\mathcal{E}\widehat{K}_{z,z',\xi}\right)(x, y) = \sum_{l=1}^{\infty} \frac{(-z' - x - \frac{1}{2})l, 2}{(-z - x + \frac{1}{2})l, 2} \widehat{K}_{z,z',\xi}(x - 2l + 1, y),
\end{equation}

(6.7)

\begin{equation}
\left(\mathcal{E}\widehat{\psi}_{\pm\frac{1}{2}}\right)(x; z, z', \xi) = \sum_{l=1}^{\infty} \frac{(-z' - x - \frac{1}{2})l, 2}{(-z - x + \frac{1}{2})l, 2} \widehat{\psi}_{\pm\frac{1}{2}}(x - 2l + 1; z, z', \xi).
\end{equation}

(6.8)

Finally, let us introduce the function $\widehat{S}_{z,z',\xi,\theta=2}(x, y)$ by the formula

\begin{equation}
\widehat{S}_{z,z',\xi,\theta=2}(x, y) = \left(z + y + \frac{1}{2}\right) \left(\mathcal{E}\widehat{K}_{z,z',\xi}\right)(x, y) + (1-\xi)z' \left(\mathcal{E}\widehat{\psi}_{-\frac{1}{2}}\right)(x; z, z', \xi) \left(\mathcal{E}\widehat{\psi}_{\frac{1}{2}}\right)(y; z, z', \xi).
\end{equation}

(6.9)

**Proposition 6.3.** The functions $\left(\mathcal{E}\widehat{K}_{z,z',\xi}\right)(x, y)$, $\left(\mathcal{E}\widehat{\psi}_{\pm\frac{1}{2}}\right)(x; z, z', \xi)$ and $\widehat{S}_{z,z',\xi,\theta=2}(x, y)$ defined by equations (6.5)-(6.9) are real analytic functions of $\sqrt{\xi}$, $0 < \sqrt{\xi} < 1$, that admit holomorphic extension to the open unit disk. The Taylor coefficients of these functions are rational in variables $z$ and $z'$.

**Proof.** Formulae (6.2), (6.3), (6.4), and (6.5)-(6.8) enable us to obtain contour integral representations for the functions $\left(\mathcal{E}\widehat{K}_{z,z',\xi}\right)(x, y)$ and $\left(\mathcal{E}\widehat{\psi}_{\pm\frac{1}{2}}\right)(x; z, z', \xi)$. From these representations, and from equation (6.9) the statement of the Proposition follows immediately.

Now we are in position to complete the proof of Theorem 2.11.

6.2. **Proof of Theorem 2.11** It was shown (see Section 5) that the formula for the correlation function $\varrho_n(z, z', \xi, \theta=2)(x_1, \ldots, x_n)$ holds true for $z = 2N$, $z' = 2N + \beta - 2$, where $N = 1, 2, \ldots$, and $\beta > 0$. We want to extend this formula for all admissible values of parameters $(z, z')$. Assume that $z = 2N$ and $z' = 2N + \beta - 2$. Then a straightforward
algebra (and the fact that $S_{z,z',\xi}(x, y)$ is antisymmetric for $z = 2N$ and $z' = 2N + \beta - 2$) gives the following expressions for the matrix elements of $K_{z,z',\xi}(x, y)$:

$$S_{z,z',\xi,\theta=2}(x, y) = \varphi_{z,z'}(x + 1, y + 1) \widehat{S}_{z,z',\xi,\theta=2}(x, y),$$

$$D + S_{z,z',\xi,\theta=2}(x, y) = \frac{\varphi_{z,z'}(x + 1, y + 1)}{z' + x + \frac{3}{2}} \widehat{S}_{z,z',\xi,\theta=2}(x + 1, y),$$

$$SD_{z,z',\xi,\theta=2}(x, y) = -\frac{1}{(z' + y + \frac{3}{2})\varphi_{z,z'}(x + 1, y + 1)} \widehat{S}_{z,z',\xi,\theta=2}(y + 1, x),$$

and

$$D + SD_{z,z',\xi,\theta=2}(x, y) = -\frac{1}{(z' + x + \frac{3}{2})(z' + y + \frac{3}{2})\varphi_{z,z'}(x + 1, y + 1)} \widehat{S}_{z,z',\xi,\theta=2}(y + 1, x + 1).$$

Computing the Pfaffian in the righthand side of equation (2.7) we see that the function $\varphi_{z,z'}(x, y)$ (which is the gamma prefactor) is completely cancelled out. Therefore, the righthand side of equation (2.7) has the same property as the function $\widehat{S}_{z,z',\xi,\theta=2}(x, y)$: it is a real analytic function in $\sqrt{\xi}$, $0 < \sqrt{\xi} < 1$, that admits a holomorphic extension to the open unit disk. Moreover, the Taylor coefficients of this function are rational in $z, z'$. On the other hand, Propositions 6.1, 6.2 imply that the left-hand side of equation (2.7) has the same property, with $\sqrt{\xi}$ replaced by $\xi$. Thus, both sides of equation (2.7) can be viewed as holomorphic functions with Taylor coefficients rational in $z$ and $z'$. Since the set

$$\{(z, z') : z \text{ is a large natural number } 2N \text{ and } z' > 2N - 2\}$$

is a set of uniqueness of rational functions in two variables $z, z'$, we conclude that equation (2.7) holds true for all admissible $z, z'$. □

7. Proof of Propositions 2.12 and 2.13

7.1. Proof of Proposition 2.12. For $N = 1, 2, \ldots$, let $z = 2N$ and $z' = 2N + \beta - 2$ with $\beta > 0$. Assume that $x, y$ lie in the subset $\mathbb{Z}_{\geq 0} - 2N + \frac{1}{2} \in \mathbb{Z}'$. Then the formula for $S_{z,z',\xi,\theta=2}(x, y)$ is obtained from Proposition 6.2 and equations (5.7), (5.8). Thus the Proposition is proved for these specific values of the parameters $z$ and $z'$. Now we claim

that the expression in the righthand side of the formula for $S_{z,z',\xi,\theta=2}(x, y)$ in the statement of the Proposition is identically equal to the expression in the righthand side of formula
for $S_{z,z',\xi,\theta=2}(x,y)$ in the statement of Theorem 2.11. Indeed, using the identity

$$\frac{1}{(2\pi i)^2} \sqrt{\frac{\Gamma(x+z+\frac{1}{2})\Gamma(y+z'+\frac{1}{2})}{\Gamma(x+z'+\frac{1}{2})\Gamma(y+z+\frac{1}{2})}} \times \int \int_{\{w_1\} \backslash \{w_2\}} \frac{(1-\sqrt{w_1})^z(1-\sqrt{w_1})^{z'}(1-\sqrt{w_2})^{-z}(1-\sqrt{w_2})^{-z'}}{w_1w_2-1} dw_1 dw_2$$

$$= \frac{1}{(2\pi i)^2} \sqrt{\frac{\Gamma(y+z+\frac{1}{2})\Gamma(x+z'+\frac{1}{2})}{\Gamma(y+z'+\frac{1}{2})\Gamma(x+z+\frac{1}{2})}} \times \int \int_{\{w_1\} \backslash \{w_2\}} \frac{(1-\sqrt{w_1})^z(1-\sqrt{w_1})^{z'}(1-\sqrt{w_2})^{-z}(1-\sqrt{w_2})^{-z'}}{w_1w_2-1} dw_1 dw_2$$

(which follows from Proposition 2.10 and from the fact that $K_{z,z',\xi}(x,y) = K_{z,z',\xi}(y,x)$) we can rewrite the righthand side of equation (2.13) as a double contour integral. In this way we arrive to formula (5.8) for $S_{z,z',\xi,\theta=2}(x,y)$, where the function $S_{z,z',\xi,\theta=2}(x,y)$ is given in Proposition 5.1.3. The expressions for $S_{z,z',\xi,\theta=2}(x,y)$ and $S_{z,z',\xi,\theta=2}(x,y)$ hold now for all admissible $z$ and $z'$. Repeating the algebraic calculations as in Propositions 5.2 and 5.4 we obtain the formula for $S_{z,z',\xi,\theta=2}(x,y)$ stated in Theorem 2.11.

7.2. Proof of Proposition 2.13. Using formulae of Theorem 2.11 we check by direct calculations that $g_n^{(z,z'-1,\xi,\theta=2)}(x_1, \ldots, x_n)$ can be written as in the statement of Proposition 2.13. In particular, $g_n^{(z,z'-1,\xi,\theta=2)}(x_1, \ldots, x_n)$ is determined by the kernel $S_{z,z',\xi,\theta=2}(x,y)$ defined by equation (2.15). However, the contours $\{w_1\}$ and $\{w_2\}$ in equation (2.15) still must be chosen according to the parities of $x - \frac{1}{2}$ and $y - \frac{1}{2}$. In particular, we can choose $\{w_1\}$, $\{w_2\}$ to be circular contours such that $|w_1| > 1$ in the case when $x - \frac{1}{2}$ is even, $|w_2| > 1$ in the case when $x - \frac{1}{2}$ is odd, $|w_2| > 1$ in the case of an even $y - \frac{1}{2}$, and $|w_2| < 1$ in the case of an odd $y - \frac{1}{2}$. In addition, we require that all three conditions in Proposition 2.10 on the contours $\{w_1\}$, $\{w_2\}$ are satisfied. In particular, in the case when both $x - \frac{1}{2}$, $y - \frac{1}{2}$ are even, we choose $|w_1| > 1$ and $|w_2| > 1$, so the statement of the Proposition holds true for even $x - \frac{1}{2}$ and $y - \frac{1}{2}$.

Now we are going to show that equation (2.15) with $|w_1| > 1$ and $|w_2| > 1$ holds true no matter what parities of $x - \frac{1}{2}$ and $y - \frac{1}{2}$ are. Assume, for example, that $x - \frac{1}{2}$ is even, and $y - \frac{1}{2}$ is odd. Then the contours $\{w_1\}$ and $\{w_2\}$ must be chosen in formula (2.15) such that $|w_1| > 1$, and $|w_2| < 1$. Let us transform the integral in the righthand side of equation (2.15): keeping the contour $\{w_1\}$ unchanged we move $\{w_2\}$ outside the circle of the radius 1. As a result contributions from the residues of the function $w_2 \to \frac{1}{w_2-1}$ will
arise. These contributions are

\[
\frac{1}{2\pi i} \oint_{\{w_1\}} \frac{(1 - \sqrt{\xi w_1})^{-z}(1 - \frac{\sqrt{\xi}}{w_1})^z (1 - w_1)}{w_1 - 1} \frac{dw_1}{2(w_1^2 - 1) w_1^{x-1/2}}
\]

\[
+ \frac{(-1)^{y-1/2}}{2\pi i} \oint_{\{w_1\}} \frac{(1 - \sqrt{\xi w_1})^{-z}(1 - \frac{\sqrt{\xi}}{w_1})^z (-1 - w_1)}{-w_1 - 1} \frac{dw_1}{-2(w_1^2 - 1) w_1^{x-1/2}} = 0,
\]

since \( y - 1/2 \) is odd. Other cases can be considered in the same way. \( \square \)

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