Determination of the Parameters of the Earth’s Gravitational Field from Gradiometric Measurements

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Abstract—The paper discusses the goals and objectives of the GOCE project, the measurement information and the data processing strategy used in determining the parameters of the Earth’s gravitational field, as well as the products obtained as a result of mathematical processing of data at various levels. In addition, the time and coordinate systems used in the processing of measurement information were considered. Further, the analysis of the correction equation for gradiometric measurements was performed, and an algorithm for calculating the coefficients and free terms of the correction equations was presented for the case of using the direct approach in the mathematical processing of measurement data in order to determine the parameters of the Earth’s gravitational field.

Keywords: Earth’s gravitational field, gravitational potential tensor, coordinate system, gradiometer, mathematical processing of measurements, least squares method

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1. INTRODUCTION

A deep understanding of the processes taking place in the “Earth” system cannot be achieved without a sufficiently large number of observations of the parameters that characterize these processes. Recently, there has been significant progress in these studies due to a comprehensive approach to the study of processes in the “Earth” system using various data, including the Earth’s gravitational field data obtained by satellite onboard sensor systems.

With the beginning of the space age, the determination of the parameters of the Earth’s global gravitational field model has become one of the main tasks of fundamental science. The first period of determining the parameters of the Earth’s global gravitational field spans about 40 years and is characterized by a combination of satellite observations (optical, laser, Doppler) and ground-based gravimetric observations. Since 2000, a new era has begun in the field of studies of the Earth’s gravitational field. The practical implementation of the CHAMP, GRACE, and GOCE projects can be considered the beginning of this era. As a result of the practical implementation of these projects, consumers have received global high-precision models of the Earth’s gravitational field with high spatial resolution.

Distinctive features of this period of studies of the Earth’s gravitational field are as follows:

(1) the use of specialized spaceborne systems to determine the parameters of the Earth’s gravitational field;
(2) the determination of the parameters of the Earth’s gravitational field using only the data from onboard sensor systems;
(3) the use of new observation concepts (gradiometer, intersatellite tracking in high–low and low–low options);
(4) the determination of high-precision orbits of low-orbiting spacecraft from the observations of GNSS navigation satellites;
(5) the use of new methods and approaches for measurement processing to determine the parameters of the Earth’s gravitational field (energy balance method, semi-analytical method, time-wise and space-wise methods).

2. OBJECTIVES OF THE GOCE PROJECT

The GOCE project is the first satellite project aimed at studying the geodynamics of the Earth. The scientific objectives of the project in the geodetic part are as follows:

(1) creation of a high-precision model of the Earth’s gravitational field (1 cm in geoid heights and 1 mGal in gravity anomalies) with high spatial resolution (100 km);
(2) establishment of a global high-precision system of altitudes to be used as a basis to study the evolution of the polar ice caps and the sea topographic surface.

Achieving these unique requirements will facilitate solving the following interdisciplinary problems:

(1) a new understanding of the physics of external processes associated with the geodynamics of the lithosphere, mantle, rheology, and subduction processes;

(2) accurate assessment of the marine geoid required for quantification of absolute ocean circulation and ocean mass movement in combination with satellite altimetry;

(3) estimation of polar ice thickness by combining data on the thickness of the polar ice layer determined with the aid of space gravimetry methods and data obtained from mathematical processing of satellite altimetry measurements.

The target tasks of the GOCE project are solved on the basis of mathematical processing of the measurement data of the following sensor systems installed on board the spacecraft:

- **gradiometer**, which measures acceleration differences in three spatial directions between the test masses of the accelerometer ensemble;

- **satellite navigation equipment**, from the measurements of which the high-precision orbit of the GOCE satellite is determined;

- **stellar camera**, from the measurements of which the orientation of the GRF relative to the inertial reference frame is determined.

The non-gravitational acceleration of the spacecraft caused by atmospheric drag and light pressure affects all the accelerometers in the same way and becomes negligible when acceleration differences are formed.

The parameters of the Earth’s gravitational field model in the GOCE project are determined from the measurements of the acceleration differences in three spatial directions between the test masses of the accelerometer ensemble.

### 2.1. GOCE Project Data

The information basis for solving the target tasks of the GOCE project is three main levels of data [1]: level 0 data; level 1b data; and level 2 data.

- **Level 0 data** are the “raw” data received by the sensor systems.
- **Level 1b data** are acquired from level 0 data by converting the time series of level 0 data into engineering units of level 1b data. These include:
  
  - (1) gravitational gradients in the GRF;
  
  - (2) reference frame transformation matrices;

  - (3) linear accelerations, angular velocities, and accelerations;

  - (4) SST measurements (including files in RINEX format);

  - (5) orbital data (positions, velocities at time points).

- **Level 2 data** are generated using level 1b data; they include:
  
  - (1) preprocessed, externally calibrated and corrected gravity gradients in the GRF and terrestrial reference frame (TRF);

  - (2) “rapid” and precise (PSO) orbits;

  - (3) spherical harmonic coefficients, covariance matrix and determined quantities (geoid heights, gravity anomalies, and plumb-line deviations) obtained from the solution.

Level 1b data and level 2 data are available to users.

The determination of the parameters of the Earth’s gravitational field from gradiometric measurements involved three approaches: direct, time-wise, and space-wise [1].

The **direct approach** for determining the parameters of the Earth’s gravitational field is based on the use of the orbital dynamic method of space geodesy. Mathematical processing of measurements along the satellite-to-satellite line and gradiometric measurements is performed separately; matrices of normal equations compiled on their basis are combined. The combined system of normal equations is solved by the least squares method with respect to the specified coefficients of spherical harmonics.

The **time-wise approach** is based on a semi-analytical method for the analysis of the Earth’s gravitational field developed by N. Sneeuw. In this method, the gradients and satellite-to-satellite observations in the high–low variant are represented as Fourier time series along the orbital arc. The parameters of the Earth’s gravitational field are determined using the two-dimensional Fourier transform and the least squares method.

In the **space-wise approach**, the coefficients of spherical harmonics are determined from the measurements, which are transformed into a regular grid on the reference surface or into a spatial grid. In this case, the coefficients of spherical harmonics are determined with the use of the fast collocation method.

### 3. REFERENCE FRAMES USED IN THE GOCE PROJECT

The mathematical processing of gradiometric measurements involves the following main reference frames:

- inertial geocentric reference frame, in which the satellite orbit is determined;

- terrestrial geocentric reference frame, in which the determined parameters of the gravitational field model are associated;

- local north-oriented reference frame, in which the components of the geopotential tensor are calculated;
DETERMINATION OF THE PARAMETERS

The parameters of the Earth’s gravitational field are determined using level 2 data. In this case, the components of the gravitational potential gradient tensor refer to the gradiometer reference frame (GRF), and the sought-for parameters refer to the Earth-fixed reference frame (EFRF). Therefore, to solve the problem of determining the sought-for parameters, it is necessary to transform the gravitational potential gradient tensor from one reference frame to another.

4. ALGORITHM FOR DETERMINING THE PARAMETERS OF THE EARTH’S GRAVITATIONAL FIELD USING THE DIRECT APPROACH

When the first option is implemented, it will be necessary to transform the geopotential gradient tensor from GRF to EFRF [3]. However, in this case, there is a problem that accelerometers have two super-sensitive and one less sensitive axes, so four components of the gravitational potential tensor \( V_{XX}, V_{YY}, V_{ZZ}, \) and \( V_{XY} \) have high accuracy, and two components \( V_{YX} \) and \( V_{YZ} \) are less accurate. In this case, direct rotation of the gravitational gradient tensor will cause the projection of the least accurate components of the geopotential gradient tensor (\( V_{YX} \) and \( V_{YZ} \)) to result in errors in the geopotential gradient tensor in the EFRF. Therefore, it is more reasonable to choose the GRF as the working reference frame.

The correction equation for gradiometric measurements for determining the parameters of the Earth’s gravitational field with the direct approach can then be represented as follows:

\[
\mathbf{v} = \frac{\partial \mathbf{V}_{\text{GRF}}}{\partial (\mathbf{C}_{nm}, \mathbf{S}_{nm})} \left( \Delta \mathbf{C}_{nm} \right) - (\mathbf{V}_{\text{GRF}} - \mathbf{V}_{\text{GRF}}^0),
\]

where \( \mathbf{v} \) is the vector of corrections to the measured values; \( \Delta \mathbf{C}_{nm} \) are the coefficients of spherical harmonics; \( \Delta (\mathbf{C}S)_{nm} \) are the corrections to the coefficients of spherical harmonics; \( \mathbf{V} = (\mathbf{V}_{\text{GRF}} - \mathbf{V}_{\text{GRF}}^0) \) is the free term of the correction equations.

Now it is necessary to convert the coefficients of the system of equations (1) and the gravitational potential tensor calculated from the a priori model of the Earth’s gravitational field \( \mathbf{V}_{\text{GRF}}^0 \) into the GRF. This transformation is carried out using the relations

\[
\mathbf{V}_{\text{GRF}} = (\mathbf{C}_{\text{GRF}}^T \cdot \mathbf{C}_{\text{GRF}}^e) \cdot \mathbf{V}_{\text{EFRF}},
\]

\[
\frac{\partial \mathbf{V}_{\text{GRF}}^0}{\partial (\mathbf{C}_{nm}, \mathbf{S}_{nm})} = (\mathbf{C}_{\text{GRF}}^T \cdot \mathbf{C}_{\text{GRF}}^e) \cdot \frac{\partial \mathbf{V}_{\text{EFRF}}^0}{\partial (\mathbf{C}_{nm}, \mathbf{S}_{nm})} \cdot (\mathbf{C}_{\text{GRF}}^T \cdot \mathbf{C}_{\text{GRF}}^e)^T.
\]

The gravitational potential tensor \( \mathbf{V}_{\text{EFRF}}^0 \) is calculated by double differentiation in rectangular spatial coordinates of the Earth’s gravitational potential formula

\[
V(r, \phi, \lambda) = \frac{GM_e}{r} + \frac{GM_e}{r} \sum_{n=2}^{L_{\text{max}}} \left( \frac{a_n}{n^2} \right)^2 \times \sum_{m=0}^{n} (\mathbf{C}_{nm} \cos m\lambda + \mathbf{S}_{nm} \sin m\lambda) P_n^m(\sin \phi),
\]

where \( GM_e \) is the geocentric gravitational constant; \( a_e \) is the semimajor axis of the common Earth ellipsoid; \( r, \phi, \lambda \) are the geocentric coordinates of the satellite; \( L_{\text{max}} \) is the maximum order of expansion of the geopotential into a series of spherical functions; \( \mathbf{C}_{nm}, \mathbf{S}_{nm} \) are the normalized coefficients of spherical harmonics; \( P_n^m \) are the normalized associated Legendre functions of degree \( n \) and order \( m \).

In formulas (2) and (3), \( \mathbf{C}_{nm}^e \) is the transformation matrix from the terrestrial reference frame to the inertial reference frame; it is expressed in terms of quaternions and has the following form:

\[
\mathbf{C}_{nm}^e = \begin{pmatrix}
q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2(q_1q_3 + q_2q_4) & 2(q_1q_3 - q_2q_4) \\
2(q_1q_3 - q_2q_4) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2(q_2q_3 + q_4q_4) \\
2(q_1q_3 + q_2q_4) & 2(q_2q_3 - q_4q_4) & -q_2^2 + q_3^2 + q_4^2
\end{pmatrix}_{\text{int}},
\]

and \( \mathbf{C}_{\text{GRF}}^e \) is the transformation matrix from the inertial reference frame to the gradiometric reference frame; it is expressed in terms of quaternions and is calculated in accordance with formula (6) [4]:

\[
\mathbf{C}_{\text{GRF}}^e = \begin{pmatrix}
q_1^2 + q_2^2 - q_3^2 - q_4^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_2 + q_0q_3) \\
2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_4) \\
2(q_1q_2 - q_0q_3) & 2(q_2q_3 + q_0q_4) & q_0^2 - q_1^2 - q_2^2 + q_3^2
\end{pmatrix}.
\]
In formula (6), \( q_0 = q_4 \).

In addition, the quaternions in formula (4) must be interpolated to the time points of gradiometric measurements in accordance with the procedure given in [5].

The quaternion components of matrix (5) are contained in the products SST_PSO_2 and SST_PRM_2, and the quaternion components of matrix (6) are contained in the product EGG_NOM_2.

Further, we transform the system of correction equations (1) into a system of normal equations; solving it with the use of the least squares method, we determine the corrections to the Earth gravitational field’s coefficients of spherical harmonics of the a priori model

\[
\begin{bmatrix}
\Delta C_{lm} \\
\Delta S_{lm}
\end{bmatrix} = (A^T PA + \alpha K)^{-1}(A^T P T)
\]

(7)

where \( (A^T PA) \) is the coefficient matrix of the system of normal equations; \( \alpha \) is the regularization parameter; \( K \) is the regularization matrix; \( (A^T P T) \) is the vector of free members of the normal system; and \( P \) is the weight matrix of measurements.

The accuracy of the obtained solution is estimated by the formula

\[
Q = \mu^2 (A^T PA + \alpha K)^{-1},
\]

(8)

\[
\mu^2 = \frac{v^T P v}{n - u},
\]

(9)

where \( Q \) is the covariance matrix of the solution; \( \mu \) is the root mean square error of the weight unit; \( v \) is the vector of corrections to the measured values, which is calculated in accordance with formula (1); \( P \) is the weight matrix of measurements; \( n \) is the number of measurements; and \( u \) is the number of determined parameters.

However, it should be noted that the practical implementation of determining the parameters of the gravitational field model using the direct approach is associated with great computational difficulties due to the large dimension of the vector of determined parameters (for example, when considering the expansion of the geopotential into a series of spherical functions up to 300th order, the number of refined parameters can exceed 90000). In this regard, the determination of the parameters of the Earth’s gravitational field model requires using regularization methods and a parallel computing strategy.

**CONFLICT OF INTEREST**

The author declares that he has no conflicts of interest.

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