Eigenvector and eigenvalue problem for 3D bosonic model.

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Abstract

In this paper we reformulate free field theory models defined on the rectangular $D+1$ dimensional lattices as $D+1$ evolution models. This evolution is in part a simple linear evolution on free ("creation" and "annihilation") operators. Formal eigenvectors of this linear evolution can be directly constructed, and them play the role of the "physical" creation and annihilation operators. These operators being completed by a "physical" vacuum vector give the spectrum of the evolution operator, as well as the trace of the evolution operator give a correct expression for the partition function. As an example, Bazhanov – Baxter’s free bosonic model is considered.
1 Introduction

In this paper we investigate the simplest case of integrable statistical mechanics and field theory models: the free models.

To fix some definitions and terminology, recall in few words, for example, scholar scalar free field theory in four dimensions. The Hamiltonian

$$H_0(\pi, \phi) = \frac{1}{2} \int d\vec{x} \left( \pi(\vec{x})^2 + (\partial^2 \phi(\vec{x}))^2 + m^2 \phi(x)^2 \right),$$

(1.1)
gives the evolution in physical time. In the quantum field theory a matrix element of the evolution operator can be presented in the form of the path integral of a Gaussian exponent. Hamiltonian (1.1) can be diagonalized by the Fourier transformation

$$\phi(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int \left( a^+(\vec{k}) + a(\vec{k}) \right) \exp(ik\vec{x}) \frac{dk}{\sqrt{2\omega}},$$

(1.2)

and

$$\pi(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int \left( a^+(\vec{k}) - a(\vec{k}) \right) \exp(ik\vec{x}) i\sqrt{\frac{\omega}{2}} dk,$$

(1.3)

with the dispersion relation

$$\omega^2 = \vec{k}^2 + m^2.$$  

(1.4)

The Hamiltonian becomes

$$H_0 = \int \omega \ a^+(\vec{k}) \ a(\vec{k}) \ dk, \quad [a(\vec{k}), a^+(\vec{k}')] = \delta(\vec{k} - \vec{k}').$$

(1.5)

The $n$–particle eigenvectors of the Hamiltonian and so of the evolution operator are the Fock states

$$\prod_{j=1}^n a^+(\vec{k}_j) |0> \ , \quad H_0 |0> = 0.$$ 

(1.6)

General free models on regular lattices have the same feature as this scalar model: all them can be diagonalized by proper Fourier transformations [1]. The differences between the scalar model and a general free theory on the rectangular lattice are:

- on the lattice the main object is not a hamiltonian, but an one-step evolution operator in the discrete time.

- dispersion relations arising in the lattice models are little bit more complicated then (1.4), and they give nontrivial partition functions of the lattice models endowed by a phase transition phenomena.

In the well known realm of the two dimensional models one could mention the free fermionic model (F.F.M.) as such case. The partition function for F.F.M. can be represented in a form of a path – type integral over the Grassmanian variables. The partition function is a determinant of large enough matrix which can be block – diagonalized with a help of the Fourier transformation, in this way the well known expression for the partition function for finite lattice can be obtained.

An advantage of a $1 + 1$ evolution formulation for the free bosonic and fermionic models is that one may avoid the path – type integration, considering the evolution of the creation and annihilation operators.

It is very useful also to deal not with the usual transfer matrices as the evolution operators, but with the transfer matrices appearing in the diagonal sections of $2D$ lattice. Such matrices $U$ do not form a commutative family, but they commute with the set of usual transfer matrices $T$, this is provided by the Yang – Baxter equation.

$^1$The same is valid for the $3D$ F.F.M. [1].
2 Formulation

Let $\mathcal{A}$ be an algebra with a left $\mathcal{A}$-module $F$. Let $\mathcal{A}$ as the linear space has a basis $e_\alpha$. Let $\mathcal{G}(\mathcal{A})$ be a group of linear automorphisms of $\mathcal{A}$: for any $G \in \mathcal{G}(\mathcal{A})$

$$e'_\alpha = G \circ e_\alpha = \sum_\beta g_{\alpha,\beta} e_\beta,$$

(2.1)

where $G$ is properly defined on $F$ so as

$$G \circ e \overset{\text{def}}{=} G \cdot e \cdot G^{-1}.$$

(2.2)
Call these $G$ the Gaussian operators on $\mathcal{A}$. To exhibit the definition of $G$ via the matrix $g_{\alpha,\beta}$ we will use the universal notation $G = G(g)$. Also we will omit the indices $\alpha$ denoting the basis of $\mathcal{A}$ via $\vec{e}$ so that

$$G(g) \circ \vec{e} = g \cdot \vec{e}. \quad (2.3)$$

Consider now the tensor product of $\Delta$ copies of $F$:

$$F^{\otimes \Delta} = F \otimes F \otimes \cdots \otimes F \quad \Delta \text{ times} \quad (2.4)$$

This linear space is the left module of the direct sum of $\Delta$ copies of $\mathcal{A}$. Let

$$\vec{e}_k = 1 \otimes 1 \otimes \cdots \otimes \vec{e} \otimes 1 \otimes \cdots \downarrow \quad k-\text{th place} \quad (2.5)$$

The Gaussian operators from $G(\sum \mathcal{A}_k)$,

$$G(g) \circ \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vdots \\ \vec{e}_\Delta \end{pmatrix} = \begin{pmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,\Delta} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,\Delta} \\ \vdots & \vdots & \ddots & \vdots \\ g_{\Delta,1} & g_{\Delta,2} & \cdots & g_{\Delta,\Delta} \end{pmatrix} \cdot \begin{pmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vdots \\ \vec{e}_\Delta \end{pmatrix}, \quad (2.6)$$

we suppose to be properly defined on $F^{\otimes \Delta}$.

Note that the matrices $g$ represent the antihomomorphism of the group of the Gaussian operators $G(g)$:

$$G(g) \cdot G(g') = G(g' \cdot g). \quad (2.7)$$

Suppose for some reason we consider the Gaussian operators of special type: operators which act nontrivially only on some part of $F^{\otimes \Delta}$. Namely, let

$$h = \{k_1, k_2, \ldots, k_M\}, \quad (2.8)$$

then denote

$$G_h \equiv G_{k_1, k_2, \ldots, k_M} \quad (2.9)$$

an operator such that

$$G_h \circ \vec{e}_j = \vec{e}_j, \quad j \not\in h, \quad (2.10)$$

while for $\vec{e}_k$ with $k \in h$

$$G_h \circ \begin{pmatrix} \vec{e}_{k_1} \\ \vdots \\ \vec{e}_{k_M} \end{pmatrix} = \begin{pmatrix} g_{k_1,k_1} & \cdots & g_{k_1,k_M} \\ \vdots & \ddots & \vdots \\ g_{k_M,k_1} & \cdots & g_{k_M,k_M} \end{pmatrix} \cdot \begin{pmatrix} \vec{e}_{k_1} \\ \vdots \\ \vec{e}_{k_M} \end{pmatrix}, \quad (2.11)$$

Such operators appear, for example, when one considers the $D$-simplex equation:

$$G_h \cdot G_{h'} \cdot \ldots \cdot G_{h(D)}^{(D)} = G_{h(D)}^{(D)} \cdot \ldots \cdot G_{h'}^{(D)} \cdot G_h, \quad (2.12)$$

where each $h^{(m)}$ consists on $D$ indices, the whole number of the indices in $\sum_{h^{(m)}} D(D + 1)$ is $\frac{D(D + 1)}{2}$, and denoting

$$h = \{k_1, \ldots, k_D\} \Leftrightarrow k_n = (h)_n, \quad (2.13)$$
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then

\[ (h^{(m)})_n = (h^{(n)})_{m+1}. \]  

(2.14)

Note, \( h = h^{(0)}. \)

All operators \( G_h \) are defined in \( \text{End} \ F^\otimes \Delta, \) \( (2.12) \) is the usual \( D \)-simplex equation for the operators acting in the tensor product. But \( (2.14) \) implies also the corresponding equation for \( g_h: \)

\[ g_{h_1}^{(D)} \cdot \cdots \cdot g_{h_l}^{(D)} = g_h \cdot g_{h_1}^{(D)} \cdot \cdots \cdot g_{h_l}^{(D)}, \]  

(2.15)

and the main difference between \( (2.12) \) and \( (2.14) \) is that \( g_h \)-s act on the direct sum of linear spaces:

\[ g_h \in \text{End} \sum_k A_k. \]  

Such kind of \( D \)-simplex-type equations was investigated in Refs. \[4\]. \( \{ \vec{e}_k \} \) is the formal basis, and \( G(g_h) \)-s are defined on the vectors

\[ \omega = \sum_{k=1}^\Delta \vec{x}_k^T \cdot \vec{e}_k. \]  

(2.16)

Thus there defined the co-action of \( G(g) \) on the amplitudes \( \vec{x}_k: \)

\[ \vec{x}_k \mapsto \vec{x}'_k = (g^T \cdot \vec{x})_k \]  

(2.17)

due to

\[ G(g) \circ \omega = \sum_k (g^T \cdot \vec{x}_k^T) \cdot \vec{e}_k. \]  

(2.18)

These simple facts allows one to formulate and solve eigenvalue and eigenvector problem of a special type. This will be done in the next section.

3 \ 2 + 1 evolution

In this section we’ll formulate some 2 + 1 evolution problem. The formulation is rather dimension independent, as well as the solution. We choose the dimension 2 + 1 because this is not so convenient as 1 + 1 evolution problem and is not so cumbersome as an arbitrary \( D + 1 \) case. Also the examples we’ll give belong to the 2 + 1 case.

Suppose we are intending to investigate a model (statistical mechanics or field theory) defined on the cubic lattice: the cites of the lattice are

\[ \vec{n} = n_x \vec{e}_x + n_y \vec{e}_y + n_z \vec{e}_z, \]  

(3.1)

where \( \vec{e}_x, \vec{e}_y, \vec{e}_z \) are orthogonal normal basis in \( E_3, n_x, n_y, n_z \in \mathbb{Z}_N, N \) is the size of the lattice, the periodical boundary conditions assumed. Instead of dealing with the usual layer – to – layer transfer matrices, i. e. section of the lattice between any \( n_z = k \) and \( n_z = k + 1, \) consider the section of the 3D lattice between \( n_x + n_y + n_z = m \) and \( n_z + n_y + n_z = m + 1. \) Call this section \( n_x + n_y + n_z = m \) as \( Z_m \) (do not mix this with the ring of residues) and define the evolution as the transition \( \ldots Z_m \mapsto Z_{m+1} \ldots. \) Such evolution problem was suggested in Refs. \[4\].

Consider now the section, e. g. \( Z_0, \) in details. The two - dimensional oriented cagome – type lattice \( Z_0 \) is shown in Fig. 2. \( Z_0 \) consists of the oriented triples of the vertices,

\[ Z_0 = \{ \text{Triples} [ V_{\alpha,\beta}, V_{\alpha,\gamma_k}, V_{\beta,\gamma_k} ] \}, \]  

(3.2)

where the numbers are chosen so as \( i + j + k = 0, \) and \( V_{\alpha,\beta} \) stands for a vertex at the intersection of the lines \( \alpha \) and \( \beta \) etc.
Fix some notations for the sake of simplicity and shortness. First, introduce the 3D translations as the re-enumeration operators:

\[
\begin{align*}
\tau_z \circ [V_{\alpha_i,\beta_j}, V_{\alpha_i,\gamma_k}, V_{\beta_j,\gamma_k}] &= [V_{\alpha_{i+1},b_j}, V_{\alpha_{i+1},\gamma_k}, V_{\beta_j,\gamma_k}], \\
\tau_y \circ [V_{\alpha_i,\beta_j}, V_{\alpha_i,\gamma_k}, V_{\beta_j,\gamma_k}] &= [V_{\alpha_i,b_{j+1}}, V_{\alpha_i,\gamma_k}, V_{\beta_{j+1},\gamma_k}], \\
\tau_x \circ [V_{\alpha_i,\beta_j}, V_{\alpha_i,\gamma_k}, V_{\beta_j,\gamma_k}] &= [V_{\alpha_i,b_j}, V_{\alpha_i,\gamma_{k+1}}, V_{\beta_j,\gamma_{k+1}}].
\end{align*}
\]

(3.3)

Pure 2D translations on the kagome lattice are

\[
\begin{align*}
p_\alpha &= \tau_y \tau_x^{-1}, & p_\beta &= \tau_z \tau_x^{-1}, & p_\gamma &= \tau_z \tau_y^{-1},
\end{align*}
\]

(3.4)

so as

\[
p_\beta = p_\alpha \cdot p_\gamma.
\]

(3.5)

Thus the position of any triple \(i,j,k: i+j+k = 0\) can be noted by

\[
p = p_\alpha^{-k} \cdot p_\gamma^k
\]

(3.6)
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because of

\[ p \circ [ V_{\alpha_0, \beta_0}, V_{\alpha_0, \gamma_0}, V_{\beta_0, \gamma_0} ] = [ V_{\alpha_1, \beta_1}, V_{\alpha_1, \gamma_1}, V_{\beta_1, \gamma_1} ] . \] (3.7)

Note, by this notation we chose \( \alpha_0, \beta_0, \gamma_0 \) to be special point in the frame of reference.

Let the sizes of the lattice be \( N \) in each direction:

\[ p_N^\alpha = p_N^\beta = p_N^\gamma = 1 , \] (3.8)

so that the total number of the vertices is \( \Delta = 3 N^2 \).

Assign now to each vertex \( V \) the algebra \( A_V \) with the basis \( \vec{e}_V \) as well as its module \( F_V \). Then for a triple \( [ V_{\alpha_i, \beta_j}, V_{\alpha_i, \gamma_k}, V_{\beta_j, \gamma_k} ] \) introduce a (Gaussian) operator \( R \):

\[ R : F_{V_{\alpha, \beta}} \otimes F_{V_{\alpha, \gamma}} \otimes F_{V_{\beta, \gamma}} \mapsto F_{V_{\alpha, \beta}} \otimes F_{V_{\alpha, \gamma}} \otimes F_{V_{\beta, \gamma}} , \] (3.9)

where it is supposed that in the right hand side the orientation of the vertices \( [ V_{\alpha_i, \beta_j}, V_{\alpha_i, \gamma_k}, V_{\beta_j, \gamma_k} ] \) is changed as it is shown in Fig. 3.

To distinguish the configurations call the left hand side one as L.H.S. triple and the right hand side – as R.H.S. triple. Consider now the evolution of \( Z_0 \) generated by applying \( R \)-s to all L.H.S. triples of \( Z_0 \). As the result we obtain \( Z_1 \) formed by other L.H.S. triples:

\[ Z_1 = \{ \text{L.H.S. Triples } [ V_{\alpha_i, \beta_j}, V_{\alpha_i, \gamma_k}, V_{\beta_j, \gamma_k} : i + j + k = 1 ] \} . \] (3.10)

Applying \( R \)-s to new L.H.S. triples repeatedly, one obtains

\[ Z_0 \rightarrow Z_1 \rightarrow Z_2 \rightarrow Z_3 \rightarrow ... \] (3.11)

where

\[ Z_m = \{ \text{L.H.S. Triples } [ V_{\alpha_i, \beta_j}, V_{\alpha_i, \gamma_k}, V_{\beta_j, \gamma_k} : i + j + k = m ] \} . \] (3.12)

For \( U_m : Z_m \mapsto Z_{m+1} \) one may write the following expression:

\[ U_m = \prod_{i+j+k=m} R_{V_{\alpha_i, \beta_j}, V_{\alpha_i, \gamma_k}, V_{\beta_j, \gamma_k}} . \] (3.13)

---

Fig. 3. The action of \( R \).
Introduce shorter notations for $\vec{e}_V$ and $F_V$ assigned to the vertices $V_{\alpha,\beta}$, $V_{\alpha,\gamma}$ and $V_{\beta,\gamma}$. Namely, mark everything for $V_{\alpha,\beta}$ by $x$, for $V_{\alpha,\gamma}$ by $y$, and $V_{\beta,\gamma}$ by $z$, and the position of a triple $[V_{\alpha,\beta}, V_{\alpha,\gamma}, V_{\beta,\gamma}]$ by its $p$ as in Eq. (3.6):

\[
V_{\alpha,\beta} = p^j_\alpha \cdot p^i_\beta \circ V_{\alpha,\beta_0} = V_x(p^j_\alpha p^i_\beta) \leftrightarrow \vec{e}_x(p^j_\alpha p^i_\beta), \ F_x(p^j_\alpha p^i_\beta), \ (3.14)
\]

\[
V_{\alpha,\gamma} = p^{-k}_\alpha \cdot p^j_\gamma \circ V_{\alpha,\gamma_0} = V_y(p^{-k}_\alpha p^j_\gamma) \leftrightarrow \vec{e}_y(p^{-k}_\alpha p^j_\gamma), \ F_y(p^{-k}_\alpha p^j_\gamma), \ (3.15)
\]

\[
V_{\beta,\gamma} = p^{-k}_\beta \cdot p^{-j}_\gamma \circ V_{\beta,\gamma_0} = V_z(p^{-k}_\beta p^{-j}_\gamma) \leftrightarrow \vec{e}_z(p^{-k}_\beta p^{-j}_\gamma), \ F_z(p^{-k}_\beta p^{-j}_\gamma), \ (3.16)
\]

With these notations the action of $R$, Eq. (3.9), looks like

\[
R : F_x(p) \otimes F_y(p) \otimes F_z(p) \mapsto F_x(\tau_x p) \otimes F_y(\tau_y p) \otimes F_z(\tau_z p). \tag{3.17}
\]

This situation is shown in Fig. 4, analogous to Fig. 3.

Let now all operators $R$ coincide:

\[
R_{\alpha,\beta,\gamma} = R(\vec{e}_x(p), \vec{e}_y(p'), \vec{e}_z(p'')) \tag{3.18}
\]

with the same operator function $R$. Then all $U_m$ differ only by some re-enumeration of the vertices. Choose the re-enumeration operator $\tau = \tau_x$:

\[
\tau \circ V_x(p) = V_x(p),
\tau \circ V_y(p) = V_y(p^{-1}_\alpha),
\tau \circ V_z(p) = V_z(p^{-1}_\beta). \tag{3.19}
\]

Then, obviously,

\[
U_m = \tau^m \circ U_0 = \tau^m \cdot U_0 \cdot \tau^{-m}. \tag{3.20}
\]
Consider now the co-action of $U$ and $H$. Hence

$$U^{(m)} = U_{m-1} \cdot U_{m-2} \cdots U_1 \cdot U_0 = \tau^m \cdot (\tau^{-1} \cdot U_0)^m.$$  \hfill (3.21)

Thus the universal one step evolution operator

$$U = \tau^{-1} \cdot U_0 : \mathcal{Z}_0 \mapsto \mathcal{Z}_0$$  \hfill (3.22)

arises.

Now we are going to diagonalize $U$.

Let further $R$-s (and so $U$) are indeed Gaussian operators. Investigate the action of $R$ and $U$ on the linear space $\sum A_k$ of

$$\bar{e}_x(p), \bar{e}_y(p), \bar{e}_z(p).$$  \hfill (3.23)

Let the nontrivial part of

$$r \in \mathcal{G}(A_x \oplus A_y \oplus A_z),$$  \hfill (3.24)

$R = R(r)$, is given by $3 \times 3$ block - matrix

$$R(r) \circ \begin{pmatrix} \bar{e}_x \\ \bar{e}_y \\ \bar{e}_z \end{pmatrix} = \begin{pmatrix} r_{x,x} & r_{x,y} & r_{x,z} \\ r_{y,x} & r_{y,y} & r_{y,z} \\ r_{z,x} & r_{z,y} & r_{z,z} \end{pmatrix} \cdot \begin{pmatrix} \bar{e}_x \\ \bar{e}_y \\ \bar{e}_z \end{pmatrix}.$$  \hfill (3.25)

The space $\sum p A_x(p) \oplus A_y(p) \oplus A_z(p)$ consists of the vectors

$$\omega = \sum_p \bar{x}_p \cdot \bar{e}_x(p) + \bar{y}_p \cdot \bar{e}_y(p) + \bar{z}_p \cdot \bar{e}_z(p),$$  \hfill (3.26)

where the amplitudes $\bar{x}_p, \bar{y}_p, \bar{z}_p$ may be chosen so that

$$U \circ \omega = \lambda \omega.$$  \hfill (3.27)

Obviously,

$$p_\alpha \cdot U = U \cdot p_\alpha, \quad p_\beta \cdot U = U \cdot p_\beta.$$  \hfill (3.28)

Hence

$$p_\alpha \circ \omega = \sum_p \bar{x}_p \cdot \bar{e}_x(p_\alpha p) + \cdots$$

$$= \sum_p \bar{x}_{p\alpha^{-1}p} \cdot \bar{e}_x(p) + \cdots = k_\alpha \omega,$$  \hfill (3.29)

and

$$p_\beta \circ \omega = k_\beta \omega.$$  \hfill (3.30)

$k_\alpha^N = k_\beta^N = 1$. Thus

$$[\bar{x}, \bar{y}, \bar{z}]_{p\alpha^{-1}p} = k_\alpha [\bar{x}, \bar{y}, \bar{z}]_p, \quad [\bar{x}, \bar{y}, \bar{z}]_{p\beta^{-1}p} = k_\beta [\bar{x}, \bar{y}, \bar{z}]_p.$$  \hfill (3.31)

Consider now the co-action of $U$ on the amplitudes (see Fig. 5): with the eigenvalue $\lambda$

$$\bar{x}_p \lambda = \bar{x}_p \cdot r_{x,x} + \bar{y}_p \cdot r_{y,x} + \bar{z}_p \cdot r_{z,x},$$

$$\bar{y}_p \lambda = \bar{x}_{p\alpha^{-1}p} \cdot r_{x,y} + \bar{y}_{p\alpha^{-1}p} \cdot r_{y,y} + \bar{z}_{p\alpha^{-1}p} \cdot r_{z,y},$$

$$\bar{z}_p \lambda = \bar{x}_{p\beta^{-1}p} \cdot r_{x,z} + \bar{y}_{p\beta^{-1}p} \cdot r_{y,z} + \bar{z}_{p\beta^{-1}p} \cdot r_{z,z}.$$  \hfill (3.32)
Take into account the translation symmetry, then the finite eigenvector and eigenvalue problem arises:

\[
\begin{pmatrix}
    r_{x,x} - \lambda, & r_{x,y}, & r_{x,z} \\
    r_{y,x}, & r_{y,y} - \lambda k^{-1}_a, & r_{y,z} \\
    r_{z,x}, & r_{z,y}, & r_{z,z} - \lambda k^{-1}_\beta
\end{pmatrix}^T \cdot \begin{pmatrix}
    \frac{\vec{x}_p}{\xi} \\
    \frac{\vec{y}_p}{\xi} \\
    \frac{\vec{z}_p}{\xi}
\end{pmatrix} = 0.
\]

(3.33)

Solving this problem, one obtains the eigenvalues of \(U\) as the roots of the characteristic polynomial (the dispersion relation)

\[
\chi(\lambda, \lambda k^{-1}_a, \lambda k^{-1}_\beta | r) = 0 \iff \lambda_i = \lambda_i(k_a, k_\beta), \; i = 1, ..., \text{dim}\mathcal{A},
\]

(3.34)

where

\[
\chi(\lambda_1, \lambda_2, \lambda_3 | r) = \det \left( \begin{pmatrix}
    \lambda_1 & 0 & 0 \\
    0 & \lambda_2 & 0 \\
    0 & 0 & \lambda_3
\end{pmatrix} - r \right).
\]

(3.35)

Let further

\[
|\Omega\rangle \in F^{\oplus \Delta}
\]

(3.36)

be the vacuum vector for \(U\), i.e.

\[
U \cdot |\Omega\rangle = |\Omega\rangle
\]

(3.37)

and \(|\Omega\rangle\) is the cyclic vector in \(F^{\oplus \Delta}\). Suppose \(\vec{e}\) are free elements of \(\mathcal{A}\), then the set of eigenvectors of \(U\) is the following set of the Fock – type vectors

\[
|n_1, ..., n_{\Delta}\rangle = \omega^{n_1}_{\lambda_1} \cdots \omega^{n_{\Delta}}_{\lambda_{\Delta}} \cdot |\Omega\rangle
\]

(3.38)

with the eigenvalues

\[
U \cdot |n_1, ..., n_{\Delta}\rangle = \lambda^{n_1}_{\lambda_1} \cdots \lambda^{n_{\Delta}}_{\lambda_{\Delta}} |n_1, ..., n_{\Delta}\rangle.
\]

(3.39)

Then formally

\[
\text{Trace } U^M = \prod_{\lambda} \frac{1}{1 - \lambda^M},
\]

(3.40)
so for the partition function per cite \( z_0(r) \) the bulk free energy \( k_0(r) = \log z_0(r) \) in the thermodynamic limit is given by

\[
k_0(r) = - \lim_{N,M \to \infty} \frac{1}{N^2 M} \sum_{k_a, k_b = 0}^{N-1} \log \prod_j (1 - \lambda_j^M(k_a, k_b))
\]

\[
= - \frac{1}{(2\pi)^3} \int \int \int_0^{2\pi} d\phi_1 \, d\phi_2 \, d\phi_3 \, \log \chi(e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3} | r) \quad (3.41)
\]

In general vectors (3.38) are not all independent, in this case one should take into account the difference between “the number of the degrees of freedom” and the dimension of “phase space” multiplying (3.41) by an appropriate constant \( k_0 \mapsto \eta k_0 \). More precisely, for a given representation of \( \vec{e}_k \) on \( F \) one should extract from the set of the eigenvectors \( \omega_\lambda = \omega_i(k_\alpha, k_\beta) \) the operators, cyclic with respect to \( |\Omega> \), and sum the power series of their eigenvalues.

Thus for a Gaussian local evolution operators one can easily construct the eigenvectors as well as the free energy for the thermodynamic limit. Note that the local integrability conditions (D – simplex equations) we have not used at all, because the diagonalization of our \( U \) and the existence of a set of commutative operators are not linked at all. Nevertheless, the cases of the integrable models are more interesting and we’ll deal with the evolution operators \( U \) constructed with the help of \( R \)-s solving the Tetrahedron equation.

### 4 Functional \( R \) – operator

The interesting example of the Gaussian operators connected with integrable models can be obtained from the functional \( R \) – operators solving the functional tetrahedron equation (F.T.E.) \( \mathfrak{F} \).

Let a functional operator \( R \):

\[
R_{1,2,3} \circ [x_1, x_2, x_3] = [x'_1, x'_2, x'_3] \quad (4.1)
\]

where

\[
x'_1 = r_1(x_1, x_2, x_3) \quad x'_2 = r_2(x_1, x_2, x_3) \quad x'_3 = r_3(x_1, x_2, x_3) \quad (4.2)
\]

solves F.T.E. Examples of such \( R \)-s one can find in Ref. \( \mathfrak{F} \). Mention now a very useful aspect of the functional maps. Namely, let \( x_k \in S_k \). Then actually \( R \) can be upgraded to \( \widehat{R} \) giving an isomorphism on a bundle of \( S_1 \otimes S_2 \otimes S_3 \). Partially, on the tangent bundle,

\[
\widehat{R} \circ [x_1, x_2, x_3 | dx_1, dx_2, dx_3] = [x'_1, x'_2, x'_3 | dx'_1, dx'_2, dx'_3] \quad (4.3)
\]

where

\[
dx'_i = \sum \frac{\partial x'_i}{\partial x_k} dx_k \quad (4.4)
\]

so as we can introduce the Gaussian operator for the differentials on \( S \):

\[
r = \widehat{R} \cdot R^{-1} \quad (4.5)
\]

where, using the symbol \( a_i \) instead of \( dx_i \),

\[
r \circ a_i = a'_i = \sum r_{i,k}(\vec{x}) a_k \quad r_{i,k}(\vec{x}) = R^{-1} \circ \frac{\partial x'_i}{\partial x_k} \quad (4.6)
\]
Thus for each functional operator $R : S^{\otimes 3} \rightarrow S^{\otimes 3}$ there appears the gaussian action $G(r) : \sum dS \rightarrow \sum dS$ for a still commutative algebra $dS$. $dS$ can be raised to nontrivial $dS \oplus d^* S$,

$$r \circ a_i^* = \sum a_k^* (r^{-1})_{k,i},$$

with the conserving Weyl algebra $[a_i, a_k^*] = \delta_{i,k}$.

Because of $R$ solves F.T.E., $G(r)$ solves a local T.E. in the direct sum of $dS \oplus dS \oplus \ldots$. Thus the previous considerations give the solution of the eigenvalue and eigenfunction problem for the $2 + 1$ evolution governed by the properly quantized $R$.

Among other models one could mention several free bosonic or fermionic models, namely Bazhanov - Stroganov's free fermionic model and Bazhanov - Baxter's free bosonic model.

### 4.1 Bazhanov-Baxter’s free bosonic model

Recall in few words the form of $R$ – operator connected with the complex of the descendants of the Zamolodchikov – Bazhanov – Baxter model [7, 3, 8, 9, 10, 11]. Let

$$u \cdot v = q v \cdot u, \quad w = -q^{-1/2} u \cdot v.$$  

Then for the quantum dilogarithm function [9, 10]

$$\psi(x) = \prod_{n=0}^{\infty} (1 - q^{1+2n} x)$$

over $u, v, w$ there exists the Pentagon equation

$$\psi(v) \cdot \psi(u) = \psi(u) \cdot \psi(w) \cdot \psi(v),$$

and as the sequence of it the Tetrahedron equation for

$$R_{x,y,z} = \psi(w_y^{-1} w_z) \psi(w_x^{-1} u_z) \Pi_{x,y,z} \psi(w_x u_z^{-1})^{-1} \psi(w_y w_z^{-1})^{-1},$$

with

$$\Pi_{x,y,z} \circ \{v_x, v_y, v_z\} = \{v_x v_y v_z, v_x^{-1} v_y\},$$

$$\Pi_{x,y,z} \circ \{u_x, u_y, u_z\} = \{u_x w_y w_z^{-1}, v_x u_z, v_x u_y\},$$

$$\Pi_{x,y,z} \circ \{w_x, w_y, w_z\} = \{w_x w_y w_z^{-1}, w_z, w_y\}.$$  

Profs. Bazhanov and Baxter in 1992 established the correspondence between the pair $u, v$ and the Gaussian exponents of the canonical Weyl pair $a, a^*$. First, show as the Gaussian action like (4.3)

$$\Phi(a) = (-i) \omega a^2 / 2 \rightarrow \Phi(a) = \exp(-i \omega a^2),$$

$$\Delta(v)^N = 1 - v^N \rightarrow \Delta(v) = 1 - v.$$  

With this correspondence all the key relations in the Zamolodchikov – Bazhanov – Baxter model conserve, $W$ – type model is well defined and its partition function is proportional to the partition function of usual finite – states ZBB model.
arises in the language of the quantum dilogarithms. First, easily,
\[
\psi(u) \circ v = v \cdot (1 - q^{1/2}u)^{-1} \\
\psi(v) \circ u = u \cdot (1 - q^{-1/2}v)
\]  
(4.14)
In the limit when \(q^{1/2} \to 1\) \(\psi(u)\) and \(\psi(v)\) become functional operators \(\psi_f(u)\) and \(\psi_f(v)\) and
\[
\hat{\psi}(u)_f = L_u \cdot \psi(u)_f, \quad \hat{\psi}(v)_f = L_v \cdot \psi(v)_f, 
\]  
(4.15)
where for \(a^+ = \frac{du}{u}\) and \(a = \frac{dv}{v}\),
\[
a \cdot a^+ = a^+ \cdot a + \lambda. 
\]  
(4.16)

'+' is not h.c., \(q = \exp(\lambda)\), the Gaussian operators
\[
G : \left( \begin{array}{c} a \\ a^+ \end{array} \right) \mapsto \left( \begin{array}{c} a' \\ a'^+ \end{array} \right) = \left( \begin{array}{cc} \alpha & \beta \\ \gamma & \delta \end{array} \right) \cdot \left( \begin{array}{c} a \\ a^+ \end{array} \right), 
\]  
(4.17)
where from the automorphism condition \([a', a^+] = \lambda\) it follows that \(\alpha \delta - \beta \gamma = 1\), arise. Eqs. (4.15)
with
\[
L_v = \exp(\frac{p^2}{2 \lambda} a^2) \quad \text{and} \quad L_u = \exp(\frac{q^2}{2 \lambda} a^+^2) 
\]  
(4.18)
give
\[
\exp(\frac{p}{2 \lambda} a^2) \circ \left( \begin{array}{c} a \\ a^+ \end{array} \right) = \left( \begin{array}{cc} 1 & 0 \\ \frac{p}{\lambda} & 1 \end{array} \right) \cdot \left( \begin{array}{c} a \\ a^+ \end{array} \right), \quad \exp(\frac{q}{2 \lambda} a^+^2) \circ \left( \begin{array}{c} a \\ a^+ \end{array} \right) = \left( \begin{array}{cc} 1 & -q \\ 0 & 1 \end{array} \right) \cdot \left( \begin{array}{c} a \\ a^+ \end{array} \right). 
\]  
(4.19)
These operators should be completed by the diagonal one,
\[
: \exp(\frac{r}{\lambda} a^+ \cdot a) : \circ \left( \begin{array}{c} a \\ a^+ \end{array} \right) = \left( \begin{array}{cc} (1+r)^{-1} & 0 \\ 0 & (1+r) \end{array} \right) \cdot \left( \begin{array}{c} a \\ a^+ \end{array} \right),
\]  
(4.20)
where the sign :: of the normal ordering is used: formally
\[
: \exp(\frac{r}{\lambda} a^+ \cdot a) : = \sum_{n=0}^{\infty} \frac{(r/\lambda)^n}{n!} a^n \cdot a^n. 
\]  
(4.21)
These three operators give the Gaussian triangular expansion of \(CP(2)\). Mention one more operator,
\[
\exp(\frac{r}{2 \lambda}(a + a^+)^2) \circ \left( \begin{array}{c} a \\ a^+ \end{array} \right) = \left( \begin{array}{cc} 1 - r & -r \\ r & 1 + r \end{array} \right) \cdot \left( \begin{array}{c} a \\ a^+ \end{array} \right). 
\]  
(4.22)

The exponential forms (these Gaussian exponents define our terminology) are rather formal, because their explicit forms depend actually on a representation, and the following integral form is more useful sometimes then the series decomposition:
\[
\exp(\frac{p}{2 \lambda} a^2) = \int d\zeta \exp(\frac{\lambda p}{2} \zeta^2 + ip \zeta a)
\]  
(4.23)
equal.
There are two remarkable relations for these Gaussian operators:
first, Kashaev’s relation \[12\]:
\[
\exp(\frac{x}{2} a^2) \cdot \exp(\frac{y}{2} a^+^2) \cdot \exp(\frac{z}{2} a^2) = \exp(\frac{x'}{2} a^+^2) \cdot \exp(\frac{y'}{2} a^2) \cdot \exp(\frac{z'}{2} a^+^2), 
\]  
(4.24)
where the electric network transformation arises
\[ x' = \frac{xy}{x + z - \lambda^2 xyz}, \]
\[ y' = \frac{x + z - \lambda^2 xyz}{x + z - \lambda^2 xyz}, \]
\[ z' = \frac{yz}{x + z - \lambda^2 xyz}. \] (4.26)

Second, Bazhanov’s “restricted star – triangle” for the bosonic 3D model [3]:
\[ \exp\left(\frac{p}{2\lambda}a^2\right) \cdot \exp\left(\frac{q}{2\lambda}a^+a^2\right) = \exp\left(\frac{q'}{2\lambda}(a^+)^2\right) \cdot \exp\left(\frac{r'}{2\lambda}(a^+ + a)^2\right) \cdot \exp\left(\frac{p'}{2\lambda}a^2\right), \] (4.27)

where
\[ q' = q \frac{1-p}{1-pq}, \]
\[ p' = p \frac{1-q}{1-pq}, \]
\[ r' = pq. \] (4.28)

This is the usual pentagon relation connected with eq. (4.10). Although we can’t extract \( \exp a^2 \) directly from the quantum dilogarithm, the identification of the spectral parameters of \( \exp \)-s with the functional transformation generated by the quantum dilogarithm \( \psi(v) \) when \( q^{1/2} \to 1 \) and \( v \mapsto v \exp(a) \), following from eq. (4.15), gives the correspondence
\[ \psi(v) \to \left[ \text{Func. Part} \, \psi_f(u) \right] \cdot \exp\left(-\frac{1}{2\lambda} \frac{v}{1-v} a^2\right), \] (4.29)

Give just an example a representation of eq. (4.27). Let \( \lambda \) be pure imaginary so that the pair \( a \) and \( a^+ \) is the pair coordinate – momentum. Take the representation when
\[ a \cdot |x> = (\lambda x - \frac{\partial}{\partial x}) |x>, \quad a^+ \cdot |x> = \frac{\partial}{\partial x} |x>, \]
\[ <x| \cdot a = <x| \frac{\partial}{\partial x}, \quad <x| \cdot a^+ = <x| (\lambda x - \frac{\partial}{\partial x}), \] (4.30)

so that
\[ <y|x> = \exp\left(\frac{\lambda}{2} x^2\right) \delta(x-y), \] (4.31)

and
\[ 1 = \int |x> \exp\left(-\frac{\lambda}{2} x^2\right) dx <x|. \] (4.32)

Then
\[ <x|\exp\left(\frac{p}{2\lambda}a^2\right)|y> = \exp\left(-\frac{\lambda}{2p} (x-y)^2 + \frac{\lambda}{2} y^2\right), \]
\[ <x|\exp\left(\frac{q}{2\lambda}a^+a^2\right)|y> = \exp\left(-\frac{\lambda}{2q} (x-y)^2 + \frac{\lambda}{2} x^2\right), \]
\[ <x|\exp\left(\frac{r}{2\lambda}(a^+ + a)^2\right)|y> = \exp\left(\frac{\lambda}{2} (1+r) y^2\right) \delta(x-y). \] (4.33)

In this representation there appears exactly Bazhanov – Baxter’s bosonic restricted star – triangle relation.
Eigenvector and eigenvalue problem ...

Return now to \( R \)– matrix (4.11). We’ll identify
\[
v \to v \exp(a - a^+), \quad u \to u \exp(a^+), \quad w \to w \exp(a).
\]
(4.34)
We’ll deal with the endomorphisms of the direct sum of several pairs of
\[
\tilde{a}_k = \left( \begin{array}{c} a_k \\ a_k^+ \end{array} \right),
\]
(4.35)
The equivalence between \( \psi \) and \( \exp a^2 \) is given by (4.29).

As it was described in the previous sections, for the \( R \)– matrix being treated as the Gaussian operator \( R = G(r) \), we are interested in the matrix \( r \) and the evolution governed by this \( r \). Thus for \( \Pi_{x,y,z} \)
\[
\pi_{x,x} = \pi_{y,z} = \pi_{z,y} = 1, \quad \pi_{y,y} = \pi_{z,z} = 0,
\]
(4.36)
and
\[
\pi_{x,y} = -\pi_{x,z} = \left( \begin{array}{cc} 1 & 0 \\ 1 & 0 \end{array} \right), \quad \pi_{y,x} = -\pi_{z,x} = \left( \begin{array}{cc} 0 & 0 \\ -1 & 1 \end{array} \right).
\]
(4.37)

Restore all other exp-s:
\[
R_{x,y,z} = \exp\left( \frac{p_1}{2\lambda}(a_y - a_z)^2 \right) \cdot \exp\left( \frac{p_2}{2\lambda}(a_x - a_z^+)^2 \right),
\]
\[
\pi_{x,y,z} \cdot \exp\left( -\frac{p_3}{2\lambda}(a_x - a_z^+)^2 \right) \cdot \exp\left(-\frac{p_4}{2\lambda}(a_y - a_z)^2 \right),
\]
(4.38)
where from eq. (4.11) it follows that
\[
p_1(1 - p_2)(1 - p_4) = p_4(1 - p_3)(1 - p_1).
\]
(4.39)
Two matrix functions arise:
\[
\exp\left( \frac{p}{2\lambda}(a_x - a_z^+)^2 \right) \to \sigma(p)
\]
(4.40)
and
\[
\exp\left( \frac{p}{2\lambda}(a_y - a_z)^2 \right) \to \sigma'(p).
\]
(4.41)
Then
\[
\sigma(p) = \left( \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ p & 1 & 0 & 0 & 0 & -p \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ p & 0 & 0 & 0 & 1 & -p \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right),
\]
(4.42)
and
\[
\sigma'(p) = \left( \begin{array}{cccccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & p & 1 & -p & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -p & 0 & p & 1 \end{array} \right),
\]
(4.43)
and
\[
r = \sigma'(-p_4) \cdot \sigma(-p_3) \cdot \pi_{x,y,z} \cdot \sigma(p_2) \cdot \sigma'(p_1).
\]
(4.44)
Thus the eigenvector problem for the bosonic model reduces to the six-dimensional eigenvector problem. Let amplitudes for eq. (4.33) be

\[ x^T_p = (x_p, x^*_p) \quad \text{etc.} \quad (4.45) \]

(do not mix \( p \) as a point on the lattice with \( p \) as a parameter). Then the amplitudes of the eigenvectors are given by eq. (3.33), where the final expression for \( r \) (4.44) is

\[
\begin{pmatrix}
1 - P_1 & 0 & Q_2 & 0 & -Q_2 & P_1 \\
0 & 1 - Q_1 & P_3 & Q_1 & -P_3 & 0 \\
P_1 & 0 & 1 - Q_2 & 0 & Q_2 & -P_1 \\
-Q_3 & P_2 & 0 & 1 - P_2 & 0 & Q_3 \\
0 & -Q_1 & P_3 & Q_1 & 1 - P_3 & 0 \\
Q_3 & -P_2 & 0 & P_2 & 0 & 1 - Q_3
\end{pmatrix},
\]

\quad (4.46)

\[ P_1 = p_2, \quad P_2 = 1 - p_3 p_4, \quad P_3 = 1 - p_3(1 - p_1), \]
\[ Q_1 = p_3, \quad Q_2 = 1 - p_1 p_2, \quad Q_3 = 1 - p_2(1 - p_4). \quad (4.47) \]

The characteristic polynom (3.35) for \( r \)-matrix (4.43) does not factorize except some special limits, and the direct calculations for the integral (3.41) give the expression

\[ z_0(p_1, p_2, p_3, p_4) = B(p_1) B(p_2) B(p_3) B(p_4), \quad (4.48) \]

where

\[ B(p) = \exp \frac{2i}{\pi} \varepsilon(p) [\text{Li}(p) - \text{Li}(1/2)], \quad (4.49) \]

and Roger’s dilogarithm

\[ \text{Li}(p) = -\frac{1}{2} \int_0^p \left( \frac{\log x}{1 - x} + \frac{\log 1 - x}{x} \right) \, dx. \quad (4.51) \]

These calculations are cumbersome, but they can be simplified significantly in the limit when \( p_1 = p_4 = 0 \). In this case the characteristic polynom factorizes and there appeared the integral definition of \( B \):

\[
B(p) = \exp - \frac{1}{(2\pi)^3} \int \int \int_0^{2\pi} d\phi_1 d\phi_2 d\phi_3 \cdot 
\log(1 - p(e^{i\phi_1} + e^{i(\phi_2 + \phi_3)}) - (1 - p)(e^{i\phi_2} + e^{i(\phi_1 + \phi_3)}) + e^{i(\phi_1 + \phi_2 + \phi_3))}
\]

\quad (4.52)

This integral can be taken easily and gives (4.49). Match the expression for \( z_0 \) with Baxter’s results for Zamolodchikov model \( z_B \) and Bazhanov’s result for the bosonic model \( z_B \):\[ z_0 = z_2^{-4} = z_B^{-2}. \quad (4.53) \]

The origin of the difference between \( z_0 \) and Bazhanov’s \( z_B \) is that up to the proper representation our system of the eigenvectors is overdefined twice (see the remark at the end of Section 3).

Few words concerning the eigenvectors of \( U \). First, the characteristic polynom (3.35) for \( r \) (4.46) has the following property:

\[ \chi(x, y, z | r) = x^2 y^2 z^2 \chi(x^{-1}, y^{-1}, z^{-1} | r) \].
\[ \chi(x, y, z | r) \]

Thus for the isotropic eigenvectors \( x = y = z \) six roots of \( \chi = 0 \) divide into two subsets:

\[ \{\lambda_1, \lambda_2, \lambda_3\} \quad \text{and} \quad \{\lambda_4 = \lambda_1^{-1}, \lambda_5 = \lambda_2^{-1}, \lambda_6 = \lambda_3^{-1}\}, \quad (4.55) \]
shuch that $|\lambda_{1,2,3}| < 1$ (note, $|\lambda| = 1$ only if all $p$-s are real). This property remains for arbitrary quasimomenta $k_\alpha, k_\beta$:

$$|\lambda_{1,2,3}(k_\alpha, k_\beta)| < 1 \quad \text{and} \quad \lambda_{4,5,6}(k_\alpha, k_\beta) = \lambda_{1,2,3}(k_\alpha^{-1}, k_\beta^{-1})^{-1}.$$ (4.56)

In order to obtain the convergency of Trace $U^M$, suppose the eigenvectors for the linear evolution operator $\omega_{1,2,3}(k_\alpha, k_\beta)$ to be the creation operators. so the vacuum vector is to be defined

$$\omega_{4,5,6}(k_\alpha, k_\beta) |\Omega> = 0.$$ (4.57)

Due to Eq. (4.51), the structure of the bulk free energy formula (3.41) for $k_0$ does not change, but the extra multiplier $\frac{1}{2}$ appears: $k = \frac{1}{2} k_0$. Recall, in the case then $p_1, p_2, p_3, p_4$ are real, all $\lambda$-s belong to the unit circle, hence when one changes the signs of Im $p$, the structure of the physical vacuum changes, so $B(p)$ depends on the sign of Im $p$ so drastically.

Note, the derivation of the partition function via the three dimensional integral was suggested originally for the free bosonic model by V. V. Bazhanov \[1\].

5 Summary

In this paper we have formulated the evolution problem for $D + 1$ dimensional free models. The evolution $U$ – type operators act linearly on the algebra of the creation and annihilation operators, and so due to the locality of $U$ the eigenvector problem can be solved directly up to a finite eigenvector problem. Appeared physical creation and annihilation operators depending on $D$ momenta diagonalize the evolution operator. In the case of integrable models, when there exists a set of $T$ – type transfer matrices commuting with $U$, the eigenvectors diagonalize these $T$ as well. The existence of the set $T$ for the given $U$ is provided by the $D + 1$ simplex equation \[3\].

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\[3\]Note that one can construct a $T$ – type commutative set without $D + 1$ simplex equation for $D > 1$. This fact was mentioned by Bazhanov and Stroganov in Ref. \[1\].
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