Probabilistic Inventory Model with Expiration Date and All-Units Discount

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Abstract. Inventory is a substantial factor for company to ensure the continuity of its production process. In the inventory management, products usually have expiration date that should be considered in the inventory model, especially in chemical or food industry. Another factor that has an influence to the inventory model is discount factor offered by the supplier. A retailer can accept this offer in order to reduce its total inventory cost. Although buying in a large quantity can decrease purchase cost per unit and set up cost, but it can increase holding cost and expiration cost which all directly has an impact in the total inventory cost. Inventory model with expiration date and all-units discount for probabilistic demand is the focus of this paper. From the mathematical model, an optimal order quantity that minimize the total cost of inventory can be obtained. In building the mathematical model, we assume that lead time is constant and the lead time demand follows Gamma distribution. Numerical examples are given to illustrate our model and algorithm to find the optimal solution. Sensitivity analysis for parameters used in the model is also performed in order to see the impact on the optimal order quantity and total inventory cost.

Keywords: Probabilistic inventory model, expiration date, all-units discount, lead time demand.

1. Introduction
There are a lot of inventory models analyzed and discussed in many books and literatures. One of the simplest model as mentioned in Tersine [1] is the Economic Order Quantity (EOQ) model. This EOQ model has become a basic model for development of other complex inventory models in the last decade. However, among those models, only a few that discuss inventory models that consider expiration date or deteriorated items or perishable items. For a retailer dealing with perishable items such as food or chemicals, expiration date is an important factor to consider in the inventory management. A good quality items will improve the comfort and security when they are consumed. When the expiration date has passed or nearly reached, those items cannot be used anymore. The selling price will gradually decrease when the items are bear to expiration date and even do not have price when the expiration date has passed. Many papers that have developed and discussed perishable inventory models have different aspects in dealing with the problem. In Bukhari [1] a model for production system control for one item has been introduced with uncertain deterioration rate while Ferguson et.al. [2] has extended the EOQ model for perishable items by treating holding cost as a nonlinear function of time. In Muckstadt and Sapra [6] and Zhang and Wang...
[9], a multi item model has been proposed for perishable items with limited storage capacity, while Hariga et.al [3] developed an optimization model to determine product replenishment in the shelf space to maximize the profit. While Kasthuri et.al.[4] developed a multi item model by considering storage capacity and production cost in a fuzzy environment and Zhang [8] proposed a multi-product newsboy system to meet the uncertain demand. In this paper, we propose a probabilistic inventory model by considering expiration date and all-units discount where lead time demand follows Gamma distribution. We choose Gamma distribution since this distribution can be used to approximate other distributions such as exponential and normal by changing values of its parameters.

2. A Probabilistic Inventory Model with Deterioration and All-Units Discount.

The problems often faced by retailers related to inventory are how many items to be ordered and when to order. The first question related to the optimal order quantity while the second related to the “best” time for the retailer to order items to its supplier. In doing so, retailer should also consider lead time and other factors so that problems such as shortages and the potential loss due to shortage can be minimized. To help retailer with such a problem, a mathematical model is needed that can describe the situation and give an optimal ordering quantity for the retailer.

A probabilistic inventory model based on an EOQ model is a choice that can describe the real situation. In daily life, demand fluctuates from time to time, and also lead time. With this situation shortage can occur, especially during lead time. To overcome this problem, we propose a probabilistic inventory problem with fluctuation in lead time demand but the lead time is constant. To make our model more realistic, we also consider discount offered by the supplier and expiration date for the items. The discount offered by the supplier is all-units discount that can trigger the supplier to order items in bulk to benefit from the offer. However, holding cost should become consideration for the retailer when accepting the offer. Another feature we include in the model is expiration date (deterioration) in the sense that the quality of items is decreasing with time.

Some assumptions in developing our model are as follows.

1. Lead time demand follows Gamma distribution.
2. Unlimited storage capacity.
3. Lead time is known and constant.
4. Shortages can only occur during lead time.
5. Unfilled demand during lead time will be filled in the next period.
6. The number of deteriorated items are known in terms of the percentage of good items.
7. All items that about to deteriorate will be solely sold with discount and the retailer order another items with higher price.
8. There is no lead time to order items to replace the deteriorated items.
9. The order quantity is always the same for each procurement.
10. Holding cost is calculated based on the average number of stored items.

Notations used in our model are the following.

\[ D = \text{Average demand in one planning period.} \]
\[ P_i = \text{Purchase price per unit.} \]
\[ Q = \text{Order quantity.} \]
\[ Q_k = \text{The number of deteriorated items.} \]
\[ S = \text{Ordering cost per order.} \]
\[ h = \text{Fraction of holding cost per unit per planning period.} \]
\[ \pi = \text{Shortage cost per shortage.} \]
\[ R = \text{Reorder point.} \]
\[ J = \text{Selling price for deteriorated items.} \]
\[ A = \text{Purchase cost per unit from another supplier.} \]
\[ \theta = \text{Fraction of good items.} \quad (0 < \theta < 1) \]
The basic concept of the inventory model comes from probabilistic EOQ model as explained in [5].

Figure 1. A probabilistic inventory model with deterioration

Figure 1 shows that retailer order a number of $Q$ units each time the number of inventory in the warehouse reach $R$ units, with the number of deteriorated item of $Q_k$ units at $t_i$. At $t_i$ retailer sells all the (almost) deteriorated items with cheaper price and buy the same amount from another supplier at higher price. In developing the model, we assume that one planning period in Figure 1 is one year. These four cost components that affect the total inventory cost as mentioned in [5] still be considered in our model. However, apart from these four cost components there is another cost, that is deterioration cost that also affect the total inventory cost.

1. **Purchase cost**: is the cost to buy items from the supplier. Since in this model there are discount offered by the supplier, then the purchase cost per unit can be defined as follows.

   \[
   P_j = \begin{cases} 
   P_0 & \text{for } U_0 \leq Q < U_1 \\
   P_1 & \text{for } U_1 \leq Q < U_2 \\
   \vdots \\
   P_j & \text{for } U_j \leq Q < U_{j+1}
   \end{cases}
   \]

   where $P_j > P_{j+1}$, $j = 0, 1, 2, 3, \ldots$ for per unit items.

   If the average annual demand is $D$ unit, then then annual purchasing cost is $P_j D$.

2. **Ordering cost**: occur in every item procurement from the retailer to the supplier. If the cost that has to be borne by the retailer for each procurement is $S$, then the ordering cost in a year is $\frac{SD}{Q}$. 

\[
1-\theta = \text{Fraction of deteriorated items.}
\]
\[
f(x) = \text{density function for lead time demand.}
\]
\[
TAC = \text{Total inventory cost.}
\]
\[
U = \text{The minimum number of order quantity allowed by the supplier for different price.}
\]

The basic concept of the inventory model comes from probabilistic EOQ model as explained in [5].
3. Holding cost usually occurs for maintenance, rent or insurance for items. If the amount of holding cost per unit per year can be expressed as a fraction of purchase cost per unit, that is \( P_i h \), then the holding cost in a year can be formulated as \( P_i h \left( \frac{Q}{2} + R - E(X) \right) \), where \( X \) is a random variable for lead time demand.

4. Shortage cost (penalty cost) is cost because retailer has no items when demand arrives. Thus shortage occurs only during lead time, when lead time demand is greater than the stock in the retailer’s warehouse \( (x > R) \). If the shortage cost is \( \pi \) for every shortage, the shortage cost per year is given by

\[
\int_{R}^{\infty} (x-R) f(x) dx.
\]

5. Deterioration cost is the cost occurs when items is near or over their expiration date. In this situation, retailer will sell all deteriorated items with cheaper price at \( i_t \) and will order the same amount to another supplier with higher price. If the selling price at \( i_t \) is \( J \) and purchase price for new items is \( A \), then the deteriorated cost per year is given by

\[
\int_{R}^{\infty} f(x) dx = \frac{P_i h Q}{\pi D}.
\]

Therefore, the total inventory cost in one planning period (one year) from this model is given by

\[
TAC(Q, R) = P_i D + \frac{SD}{Q} + P_i h \left( \frac{Q}{2} + R - E(X) \right) + \frac{\pi D}{Q} \left[ \int_{R}^{\infty} (x-R) f(x) dx \right] + (1-\theta)(P_i - J + A)D.
\]

The next step is finding the minimum total inventory cost based on the sufficient conditions \( \frac{\partial TAC}{\partial Q} = 0 \) and \( \frac{\partial TAC}{\partial R} = 0 \). The condition \( \frac{\partial TAC}{\partial Q} = 0 \) gives

\[
Q = \frac{\sqrt{2D \left( S + \frac{\pi}{D} \int_{R}^{\infty} f(x) dx \right)}}{P_i h}
\]

And from condition \( \frac{\partial TAC}{\partial R} = 0 \), we have

\[
\int_{R}^{\infty} f(x) dx = \frac{P_i h Q}{\pi D}.
\]

We have developed an algorithm to find the optimal order quantity and reorder point based on the model as follows. For each price break offered by the supplier:

1. Calculate \( Q \) using the Wilson formula (EOQ), that is \( Q = \sqrt{\frac{2SD}{P_i h}} \).

2. Substitute the value of \( Q \) from (1) to \( \int_{R}^{\infty} f(x) dx = \frac{P_i h Q}{\pi D} \) to find the value of \( R \).
3. Substitute $R$ to the equation $Q = \left[ \frac{2D}{P_i h} \left( S + \pi \int_{R}^{\infty} (x - R) f(x) dx \right) \right]$ to find the value of $Q$.

4. Substitute $Q$ from (3) to $\int_{R}^{\infty} f(x) dx = \frac{P_i h Q}{\pi D}$ to obtain $R$.

5. Repeat steps (3) and (4) until we get the stable values of $Q$ and $R$.

6. Compare $Q$ with $U$. If $Q$ lies in the interval of $U (U_j \leq Q < U_{j+1})$, then $Q$ is valid.

7. If $Q$ is not valid, then
   (i) For $Q$ that is less than $U_j$, set $Q = U_j$.
   (ii) For $Q$ that is greater than $U_j$, set $Q = U_{j+1}$.

8. Adjust the value of $R$ in $\int_{R}^{\infty} f(x) dx = \frac{P_i h Q}{\pi D}$ for the value of $Q$ obtained in (7).

9. Calculate $Q_k$, where $Q_k = (1 - \theta)Q$

10. Calculate $TAC$ for each valid $Q$ and all possible values of $U$.

11. Choose the optimal order quantity ($Q$) that gives minimum $TAC$.

3. Numerical Examples

Consider a food retailer that require an average 500 units of item in a year with ordering cost of Rp. 150,000.00 per order and fraction of holding cost of 0.2 of the purchase coat. It is also known that 15% of the ordered item will deteriorate near the end of the period. When this happens, the retailer will sell this item for Rp. 9,000.00 per unit and order (the same amount with the deteriorated items) to other supplier at the price of Rp. 12,000.00 per unit. Lead time demand follows Gamma distribution with positive parameters $\alpha$ and $\beta$ and there is a shortage cost of Rp. 5,000.00 for each shortage.

All-units discount offered by the supplier is as follows.

| Units      | Price/Unit |
|------------|------------|
| $\leq 160$ | Rp. 11,500.00 |
| 161 – 180  | Rp. 11,000.00 |
| 181 – 200  | Rp. 10,500.00 |
| $> 200$    | Rp. 10,000.00 |

In this example we will determine the optima order quantity that minimize the total inventory cost for $\alpha = 1$ and $\beta = 1$, $\alpha = 5$ and $\beta = 1$, $\alpha = 10$ and $\beta = 1$, $\alpha = 15$ and $\beta = 1$.

For $\alpha = 1$ and $\beta = 1$, using the above algorithm, we found the optimal order quantity, reorder point, the number of deteriorated item and total inventory cost for different purchase price as summarized in Table 1.
From Table 1, it is optimal to order 275 units, with 42 units deteriorated items, reorder point at 2 units and total inventory cost of Rp. 6.544.453,00. With the same method, we found the optimal values of $R$, $Q$, $Q_k$, and total inventory cost for $\alpha = 5$ and $\beta = 1$, $\alpha = 10$ and $\beta = 1$, $\alpha = 15$ and $\beta = 1$ as depicted in Table 2.

### Table 1. Total inventory cost for different purchase cost for $\alpha = \beta = 1$

| Units  | Price/Unit  | $Q$ (units) | $R$ (units) | $Q_k$ (units) | TAC          |
|--------|-------------|-------------|-------------|---------------|--------------|
| $\leq 160$ | Rp. 11.500,00 | 160         | 2           | 24            | Rp. 7.496.234,00   |
| 161 – 180 | Rp. 11.000,00 | 180         | 2           | 27            | Rp. 7.170.069,00   |
| 181 – 200 | Rp. 10.500,00 | 200         | 2           | 30            | Rp. 6.852.416,00   |
| $> 200$   | Rp. 10.000,00 | 275         | 2           | 42            | Rp. 6.544.453,00   |

### Table 2. $Q$, $R$, $Q_k$, and Total Inventory Cost for Different Purchase Cost

| $\alpha$ | $\beta$ | Units  | Price/Unit | $Q$ (units) | $R$ (units) | $Q_k$ (units) | TAC          |
|---------|---------|--------|------------|-------------|-------------|---------------|--------------|
| 5       | 1       | $\leq 160$ | Rp. 11.500,00 | 160         | 8           | 24            | Rp. 7.503.670,00   |
|         |         | 161 – 180  | Rp. 11.000,00 | 180         | 8           | 27            | Rp. 7.176.915,00   |
|         |         | 181 – 200  | Rp. 10.500,00 | 200         | 8           | 30            | Rp. 6.858.749,00   |
|         |         | $> 200$    | Rp. 10.000,00 | 276         | 7           | 42            | Rp. 6.544.356,00   |
| 10      | 1       | $\leq 160$ | Rp. 11.500,00 | 160         | 14          | 24            | Rp. 7.508.027,00   |
|         |         | 161 – 180  | Rp. 11.000,00 | 180         | 14          | 27            | Rp. 7.180.889,00   |
|         |         | 181 – 200  | Rp. 10.500,00 | 200         | 13          | 30            | Rp. 6.860.296,00   |
|         |         | $> 200$    | Rp. 10.000,00 | 277         | 13          | 42            | Rp. 6.543.874,00   |
| 15      | 1       | $\leq 160$ | Rp. 11.500,00 | 160         | 14          | 24            | Rp. 7.498.102,00   |
|         |         | 161 – 180  | Rp. 11.000,00 | 180         | 19          | 27            | Rp. 7.182.249,00   |
|         |         | 181 – 200  | Rp. 10.500,00 | 200         | 19          | 30            | Rp. 6.863.583,00   |
|         |         | $> 200$    | Rp. 10.000,00 | 277         | 18          | 42            | Rp. 6.544.702,00   |

Illustrations for the above example with different shortage cost of Rp. 6.000 and Rp. 7.000 for $\alpha = 1$ and $\beta = 1$, $\alpha = 5$ and $\beta = 1$, $\alpha = 10$ and $\beta = 1$, $\alpha = 15$ and $\beta = 1$ are provided in Table 3 and Table 4.
### Tabel 3. $Q$, $R$, $Q_s$, and Total Inventory Cost for Different Purchase Cost with Shortage Cost of Rp. 6.000,00

| $\alpha$ | $\beta$ | Units | Price/Unit | $Q$ (units) | $R$ (units) | $Q_s$ (units) | $TAC$          |
|----------|---------|-------|------------|-------------|-------------|---------------|----------------|
| 1        | 1       | $\leq 160$ | Rp. 11.500,00 | 160         | 3           | 24            | Rp. 7.498.534,00 |
|          |         | 161–180 | Rp. 11.000,00 | 180         | 3           | 27            | Rp. 7.172.270,00 |
|          |         | 181–200 | Rp. 10.500,00 | 200         | 2           | 30            | Rp. 6.852.417,00 |
|          |         | $> 200$  | Rp. 10.000,00 | 275         | 2           | 42            | Rp. 6.544.453,00 |
| 5        | 1       | $\leq 160$ | Rp. 11.500,00 | 160         | 8           | 24            | Rp. 7.503.504,00 |
|          |         | 161–180 | Rp. 11.000,00 | 180         | 8           | 27            | Rp. 7.176.773,00 |
|          |         | 181–200 | Rp. 10.500,00 | 200         | 8           | 30            | Rp. 6.858.627,00 |
|          |         | $> 200$  | Rp. 10.000,00 | 276         | 7           | 42            | Rp. 6.544.272,00 |
| 10       | 1       | $\leq 160$ | Rp. 11.500,00 | 160         | 14          | 24            | Rp. 7.507.747,00 |
|          |         | 161–180 | Rp. 11.000,00 | 180         | 14          | 27            | Rp. 7.180.652,00 |
|          |         | 181–200 | Rp. 10.500,00 | 200         | 14          | 30            | Rp. 6.862.193,00 |
|          |         | $> 200$  | Rp. 10.000,00 | 277         | 13          | 42            | Rp. 6.543.732,00 |
| 15       | 1       | $\leq 160$ | Rp. 11.500,00 | 160         | 20          | 24            | Rp. 7.511.538,00 |
|          |         | 161–180 | Rp. 11.000,00 | 180         | 20          | 27            | Rp. 7.184.138,00 |
|          |         | 181–200 | Rp. 10.500,00 | 200         | 20          | 30            | Rp. 6.865.415,00 |
|          |         | $> 200$  | Rp. 10.000,00 | 277         | 18          | 42            | Rp. 6.544.517,00 |

### Tabel 4. $Q$, $R$, $Q_s$, and Total Inventory Cost for Different Purchase Cost with Shortage Cost of Rp. 7.500,00

| $\alpha$ | $\beta$ | Units | Price/Unit | $Q$ (units) | $R$ (units) | $Q_s$ (units) | $TAC$          |
|----------|---------|-------|------------|-------------|-------------|---------------|----------------|
| 1        | 1       | $\leq 160$ | Rp. 11.500,00 | 160         | 3           | 24            | Rp. 7.498.534,00 |
|          |         | 161–180 | Rp. 11.000,00 | 180         | 3           | 27            | Rp. 7.172.269,00 |
|          |         | 181–200 | Rp. 10.500,00 | 200         | 3           | 30            | Rp. 6.854.516,00 |
|          |         | $> 200$  | Rp. 10.000,00 | 275         | 2           | 42            | Rp. 6.544.454,00 |
| 5        | 1       | $\leq 160$ | Rp. 11.500,00 | 160         | 8           | 24            | Rp. 7.503.330,00 |
|          |         | 161–180 | Rp. 11.000,00 | 180         | 8           | 27            | Rp. 7.176.625,00 |
|          |         | 181–200 | Rp. 10.500,00 | 200         | 8           | 30            | Rp. 6.858.499,00 |
|          |         | $> 200$  | Rp. 10.000,00 | 276         | 7           | 42            | Rp. 6.544.183,00 |
| 10       | 1       | $\leq 160$ | Rp. 11.500,00 | 160         | 15          | 24            | Rp. 7.509.756,00 |
|          |         | 161–180 | Rp. 11.000,00 | 180         | 14          | 27            | Rp. 7.180.404,00 |
|          |         | 181–200 | Rp. 10.500,00 | 200         | 14          | 30            | Rp. 6.861.977,00 |
|          |         | $> 200$  | Rp. 10.000,00 | 277         | 13          | 42            | Rp. 6.545.583,00 |
| 15       | 1       | $\leq 160$ | Rp. 11.500,00 | 160         | 20          | 24            | Rp. 7.511.158,00 |
|          |         | 161–180 | Rp. 11.000,00 | 180         | 20          | 27            | Rp. 7.183.815,00 |
|          |         | 181–200 | Rp. 10.500,00 | 200         | 20          | 30            | Rp. 6.865.134,00 |
|          |         | $> 200$  | Rp. 10.000,00 | 277         | 18          | 42            | Rp. 6.548.324,00 |

If we compare Table 1 and Table 2, it can be seen that the optimal order quantity for different values of $\alpha$ and $\beta$ are around 275-277 units, but the reorder points are different. This is due to the different density function of lead time demand for different values of $\alpha$ and $\beta$. 
4. Conclusion
In this paper we have developed a probabilistic inventory model for one item with deterioration and all-units discount. Analysis was conducted by considering gamma distribution for lead time demand. From the sensitivity analysis we found that as the shortage cost increases the total inventory cost also increases. A further research direction is to consider a multi item problem to improve and make the model more realistic.

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