Space-time distributions

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Abstract

The space-time foliation $\Sigma$ compatible with the gravitational field $g$ on a 4-manifold $M$ determines a fibration $\pi$ of $M$, $\pi : M \rightarrow N$ is a surjective submersion over the 1-dimensional leaves space $N$. $M$ is then written as a disjoint union of the leaves of $\Sigma$, which are 3-dimensional spacelike surfaces on $M$.

The decomposition, $TM = \Sigma \oplus T^0M$, also implies that we can define a lift of the curves on $N$ to curves (non-spacelike) on $M$.

The stable causality condition $M$ coincides with $\Sigma$ being a causal space-time distribution, generated by an exact timelike 1-form $\omega^0 = dt$ where $t$ is some real function on $M$. In this case $M$ is written as a disjoint union of a family of spacelike 3-surfaces of constant $t$, which cover $D^+(S)$ of a initial 3-surface $S$ of $M$.

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1 Introduction

**Note:** We will denote by small Latin indices $a, b, \ldots$ numbers ranging from 0 to 3 and the small Greek indices $\alpha, \beta, \ldots$ numbers ranging from 1 to 3.

The space-time structure on a 4-dimensional manifold $M$ is uniquely defined by a triplet $(g^R, g, \Sigma)$, where $g$ and $g^R$ represent a Riemannian, respectively a Lorentzian metric on $M$ and $\Sigma \subset \tau_M$, the space-time distribution on $M$ compatible with the gravitational field $g$, is a 3-dimensional spatial tangent subbundle on $M$ orthocomplement (with respect both $g$ and $g^R$) of the 1-dimensional timelike subbundle $T^0M$ generated by the timelike globally defined vector field $h^0 \in \mathcal{X}(M)$, $g^R(h^0, h^0) = -g(h^0, h^0) = 1$ which indicates the time orientation (locally) at every point of $M$.

If $g^R$ is a Riemannian metric on $M$ (which exists on every paracompact space) and a time-orientation defined by $h^0 \in \mathcal{X}(M)$ then we have a Lorentz metric on $M$ determined by the expression:

$$g(X, Y) = g^R(X, Y) - 2g^R(X, h^0)g^R(Y, h^0) \quad \forall X, Y \in \mathcal{X}(M)$$

Locally, relating to an orthonormal frame $\{e_a\} = \{h^\alpha, h^0\}$ we find:

$$g_{ab} = g^R_{ab} - 2\omega^0_a\omega^0_b,$$

where $\omega^0 = g^R(h^0, \cdot)$ is the unitary timelike 1-form.

The Pfaff equation $\omega^0 = 0$ generates the 3-dimensional spatial spaces $\Sigma_p$ in each fibre $T_pM$ of the tangent bundle $\tau_M$ and these subspaces are orthogonal to the timelike field $h^0$ in $p \in M$.

The tensor

$$h_{ab} = g_{ab} + \omega^0_a\omega^0_b$$

defines a Riemannian metric on the tangent subbundle $\Sigma$.

We have the following decomposition of tangent bundle:

$$TM = \Sigma \oplus T^0M$$

So, given a Riemannian metric $g^R_{ab}$ on $M$, each 3-dimensional distribution $\Sigma$ on $M$ generated by a nonzero 1-form $\sigma$ on $M$ represents a space-time distribution compatible with a gravitational field $g$ defined by the expression (2), where we take $\omega^0 = -\frac{\sigma}{g^R(\sigma, \sigma)}$.

Inverse, given a gravitational field $g$ on $M$ we find a spacetime distribution $\Sigma$ compatible with $g$ and a Riemannian metric $g^R$ which satisfy the expression (3).
Two such different triplets \((g_i, g_i^R, \Sigma_i), \ i = 1, 2\) will describe the same space-time structure on \(M\) if the space-time distributions coincide, \(\Sigma_1 = \Sigma_2\).

The geometric equivalence principle — which states that we may introduce an orthonormal tetrad field \(\{h^\alpha, h^0\}\) at each point of \(M\) — is also sufficient for defining the space-time structure on a 4-manifold \(M\).

The metric \(g_{ab}\) can be transformed to
\[
g_{ab} = \eta_{ab} = \text{diag}(1, 1, 1, -1)
\]
at any point of \(M\).

Also for any point \(p \in M\) and its normal convex neighbourhood \(U_p \subset M, \ U_p = \exp_p W_0, \) where \(W_0 \subset T_p M\) is an open neighbourhood of the origin of \(T_p M\), we can have a causal structure; all events \(q\) in \(U_p\) are divided into three groups according whether the causal relation to \(p\) is timelike, null or spacelike and these relations are independent of the choice of coordinate system. But we don’t know how to split an arbitrary space-time in space and time globally.

An integrable distribution
\[
\Sigma = \bigcup_{p \in M} \Sigma_p
\]
is a spacetime foliation of spatial hypersurfaces \(S\) of \(M\); \(\Sigma_p = T_p S, \ \forall p \in S\).

In this case, at every point \(p \in M\) we can define a local coordinate neighbourhood \(U_i \subset M\) with local adapted coordinates \((x^a)\) so that the spacelike vector fields \(\{\frac{\partial}{\partial x^a}\}\) generate \(\Sigma\) locally.

The Pfaff equation \(x^0 = \text{const.}\) locally defines the spatial hypersurfaces \(S\) of \(M\) (i.e. the maximal integral manifolds or the leaves of \(\Sigma\)).

We have,
\[
U = \{U_i, \ t_i : \tau^{-1}_M(U_i) \longrightarrow U_i \times \mathbb{R}^4\}
\]
a bundle atlas on \(TM\) which is compatible with the subbundle \(\Sigma \subset \tau_M\), so that
\[
U^\Sigma = \{U_i, \ t_i^{-1} : \tau^{-1}_M(U_i) \cap \Sigma \longrightarrow U_i \times \mathbb{R}^3\}
\]
is a bundle atlas on \(\Sigma\), \(t_i(\tau^{-1}_M(U_i) \cap \Sigma) = U_i \times \mathbb{R}^3\).

We can have directly: for any adapted local coordinate system \((U_i, (x^a))\), then \((\dot{x}^a, \ddot{x}^a)\) are local bundle coordinates on \(TM\) and \(\{\frac{\partial}{\partial \dot{x}^a}\}\) are the dual sections. Then, from the local trivialisation of \(\tau_M\)
\[
t_i = (\tau_M|_{\tau^{-1}_M(U_i)}, \dot{x}^a) : \tau^{-1}_M(U_i) \longrightarrow U_i \times \mathbb{R}^4
\]
we obtain a local trivialisation on subbundle \(\Sigma\)
\[
t^\Sigma_i = (\tau_M|_{\tau^{-1}_M(U_i) \cap \Sigma}, \dot{x}^a) : \tau^{-1}_M(U_i) \cap \Sigma \longrightarrow U_i \times \mathbb{R}^3.
\]
2 Family of spatial hypersurfaces of spacetime

A space-time foliation \( \Sigma \) on \( M \) generates a fibration \( \pi \) as follows: we define an equivalence relation on \( M \), \( x, y \in M \), \( x \sim y \) if \( x, y \) are in the same leaf of the foliation.

We denote by \( \mathcal{N} = M_{\sim} \) the space of leaves which has the natural quotient structure of 1-dimensional differentiable manifold, so that \( \pi : M \rightarrow \mathcal{N} \) is a surjective submersion. The leaves of the foliation \( \Sigma \) coincide with the fibres of \( \pi \), \( \pi^{-1}(\nu) = S_{\nu} \) and \( \Sigma = \text{Ker}\pi_{\ast} \). We also may suppose that the fibres of \( \pi \) are simply connected even if it is not a necessary condition.

We can define an vertical lift of the tangent vectors on \( \mathcal{N} \) to tangent vectors on \( M \) which belong to \( \Sigma \).

For \( \bar{X} \in T_{\nu}\mathcal{N} \) its vertical lift is \( X^{\nu} \in T_{p}M \), \( X^{\nu a} = h^{\nu}_{b}(p)X^{b}, \forall X \in T_{p}M \), where \( \pi_{\ast}X = \bar{X} \).

Here

\begin{equation}
(8) \quad h = (h^{\nu}_{a}) : TM \rightarrow TM \\
h^{\nu}_{a} = g^{ac}h_{cb}
\end{equation}

is an endomorphism on \( \tau_{M} \), a projection operator, i.e. \( h^{\nu}_{b}h^{b}_{c} = h^{\nu}_{c} \), which projects any vector \( X \in T_{p}M \) into its part lying in the subspace \( \Sigma_{p} \).

\begin{equation}
(9) \quad X^{a} = h^{a}_{b}X^{b} - h^{0a}_{b}\omega^{0}_{b}X^{b}
\end{equation}

and any \( X \in \mathcal{X}(\mathcal{N}) \) then \( X^{\nu} \in \mathcal{V}(\Sigma) \) defined by \( X^{\nu} = h(X) \) for any \( X \in \mathcal{X}(M) \) so that \( \pi_{\ast}X = \bar{X} \).

Here \( \mathcal{V}(\Sigma) = \{ X \in \mathcal{X}_{\text{loc}}(M) \mid X_{p} \in \Sigma_{p}, \forall p \} \).

The Lorentzian (metric) connection \( \nabla \) on \( M \) induces by restriction a connection \( \nabla^{\Sigma} \) on each leaf of \( \Sigma \),

\begin{equation}
(10) \quad \nabla_{X^{\nu}}Y^{\nu} = \nabla^{\Sigma}_{X^{\nu}}h(Y) = h(\nabla_{X^{\nu}}Y) \in \Sigma.
\end{equation}

We used

\begin{equation}
(11) \quad 0 = (\nabla_{X^{\nu}}h)Y = \nabla_{X^{\nu}}h(Y) - h(\nabla_{X^{\nu}}Y)
\end{equation}

and

\begin{equation}
(12) \quad h_{ab,\nu} = (g_{ab} + \omega^{0}_{a}\omega^{0}_{b})_{\nu} = 0
\end{equation}

It is worth notice that the space-time is a disjoint union of the leaves of the foliation \( \Sigma \), i.e.

\begin{equation}
M = \bigcup_{\nu \in \mathcal{N}} S_{\nu}
\end{equation}
where \( S_\nu = \pi^{-1}(\nu) \) represents a family of spatial hypersurfaces of \( M \).

From decomposition (4) we can define a transport of the those fibres \( S_\nu \) along the curves \( \gamma(\tau) \) on \( N \). That is to say, a curve \( \gamma(\tau) : [0, 1] \rightarrow N \), \( \gamma(0) = \nu_0 \), \( \gamma(1) = \nu_1 \) could be lifted on \( M \) from each point of the fibre \( x_i \in \pi^{-1}(\nu_0) = S \) to a nonspatial future directed curve on \( M \) of end-point \( y_i \in \pi^{-1}(\nu_1) = S_{\nu_1} \).

This way we obtain an 1:1 correspondence between \( S \) and \( S_{\nu_1} \). Then we have a family of maps (homeomorphisms)

\[
\alpha_\gamma : S_{\gamma(0)} \rightarrow S_{\gamma(1)}, \quad \alpha_\gamma : x_i \mapsto y_i
\]

satisfying natural conditions: \( \alpha_{\gamma_1 \gamma_2} = \alpha_{\gamma_1} \alpha_{\gamma_2}, \quad \alpha_{\gamma^{-1}} = (\alpha_\gamma)^{-1}, \quad \alpha_{\gamma} = \text{id for } \gamma = \text{const.}, \quad \alpha_\gamma \) is independent of the choice of the parameter \( \tau \) on \( \gamma \).

On the other hand \( N \) is 1-dimensional (connected) manifold. This implies that \( N \) must be diffeomorphic with an interval \([0, 1]\) or with circle \( S^1 \).

Supposing \( N \) has a fixed orientation given by the unique unitary \( \bar{X} \in \mathcal{X}(N) \), then \( N = \gamma_X([0, 1]) \), where \( \gamma_X : [0, 1] \rightarrow N \) is the integral curve of \( \bar{X} \). We will have the two cases: when \( \gamma_X \) is an injective integral curve, \( N = [0, 1] \) and when \( \gamma_X \) admits cross-points (i.e. \( \gamma_X \) may intersect itself), \( N = S^1 \).

Lifting \( \gamma_X \) from each point \( x_i \in \pi^{-1}(\nu_0) = S \) we obtain a family of timelike future directed curves \( \hat{\gamma}_X : [0, 1] \rightarrow M, \quad \hat{\gamma}_X(\tau) \in T^0_p M, \quad \pi(p) = \nu \) which start on \( S \) and intersect each leaf \( S_\nu \) orthogonally.

The stable causality condition holds on \( M \) means that the space-time foliation \( \Sigma \) is causal, being generated by a globally defined timelike 1-form \( \omega^0 \in \Lambda^1 M \) which is exact, i.e. \( \omega^0 = dt \) where \( t \) is some real function on \( M \).

In this case, the leaves of \( \Sigma \) are spatial hypersurfaces of \( M \) of constant \( t \), and

\[
M = \bigcup_{\nu \in N} S_\nu
\]
is the disjoint union of a family of spatial hypersurfaces \( S_\nu \) of constant \( t \) on \( M \).

If we suppose that the initial surface \( S \) is a partial Cauchy surface of \( M \) we obtain \( M \) as a disjoint union of such homeomorphic partial Cauchy surfaces \( S_\nu \), which in addition satisfy the following conditions: for \( \nu_i < \nu_j \) then \( S_{\nu_i} \) lies to the future of \( S_{\nu_j} \) and they cover \( \mathcal{D}^+(S) \).

\[\text{This condition of stability is equivalent to the existence of a function } f \text{ on } M, \text{ whose } \nabla f \text{ is everywhere timelike.}\]
In the case when $N$ is diffeomorphic with $S^1$, provided that $S$ is a partial Cauchy surface on $M$, a curve $\gamma_{\tilde{X}} : [0, 1] \rightarrow N$, $\gamma_{\tilde{X}}(0) = \gamma_{\tilde{X}}(1) \in S^1$ is lifted from a point $x_i$ of $S$ to a closed timelike curve $\tilde{\gamma}$ on $M$ with $\tilde{\gamma}(0) = x_i = \tilde{\gamma}(1)$.

3 Some discussion

Physically, it seems more convenient to choose the surfaces $S_\nu$ so that they intersect $I^+$. This means that every $S_\nu$ tend asymptotically to null surfaces.

Also by assumption of strong asymptotic predictability (i.e. there exists a partial Cauchy surface $S$ of $M$ such that (a) $I^+$ lies in the boundary of $D^+(S)$ and (b) $J^+(S) \cap \partial J^-(I^+)$ lies in $D^+(S)$ ) for some $\nu$, the surfaces $S_{\nu_i}$ for $\nu_i > \nu$ will intersect the event horizon $\partial J^-(I^+)$. Then a connected component of the nonempty set $S_{\nu_i} \setminus J^-(I^+)$ will represent a black hole on the surface $S_{\nu_i}$.

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