Asymptotic solutions of scalar integro-differential equations with partial derivatives and with fast oscillating coefficients

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Abstract. In the paper, ideas of the Lomov regularization method are generalized to the Cauchy problem for a singularly perturbed partial integro-differential equation in the case when the integral term contains a rapidly varying kernel. Regularization of the problem is carried out, the normal and unique solvability of general iterative problems is proved.

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1. Introduction

In the paper, we consider the Cauchy problem for the integro-differential equation with partial derivatives:

\[ L_\varepsilon y(x,t,\varepsilon) \equiv \varepsilon \frac{\partial y}{\partial x} = a(x)y + \int_{x_0}^{x} K(x,t,s) y(s,t,\varepsilon) ds + h(x,t) + \varepsilon g(x) \cos \beta(x)\varepsilon y, \quad y(x_0, t, \varepsilon) = y^0(t) \quad (x, t) \in [x_0, X] \times [0, T], \]

where \( \beta'(x) > 0, g(x), a(x) \) is a scalar functions, \( y^0(t) \) constant, \( \varepsilon > 0 \) is a small parameter. The problem of constructing a regularized asymptotic solution \[1\] of the problem \((1)\) is posed. Earlier, in \[2\], \[3\], \[4\], \[5\], \[6\], \[7\], systems for ordinary integro-differential equations were mainly considered. In this paper we consider a partial integro-differential equations. Construction of asymptotic solutions for singularly perturbed integro-differential equations with partial derivatives in the case when integral operators change rapidly was first investigated in the works \[8\], \[9\], \[10\]. Construction of asymptotical solutions for

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ordinary integro-differential equations with fast oscillating coefficients from the position of the regularization method are considered in [11].

Denote by \( \lambda_1(x) = -a(x) \), \( \beta'(x) \) is a frequency of fast oscillating cosine. In the following, functions \( \lambda_2(x) = -i \beta'(x) \), \( \lambda_3(x) = +i \beta'(x) \) will be called the spectrum of a fast oscillating coefficient.

We assume that the conditions are fulfilled:

(i) \( K(x, t, s) \in C^\infty \{ x_0 < x < s < X, 0 < t < T \}, h(x, t) \in C^\infty ([x_0, X] \times [0, T]), a(x), g(x), \beta(x) \in C^\infty [x_0, X] \),

(ii) \( \lambda_1(x) \neq \lambda_j(x), \quad j = 2, 3, \quad \lambda_i(x) \neq 0, \quad (\forall x \in [x_0, X]), \quad i = 1, 2, 3; \)

(iii) \( \text{Re} \lambda_1(x) \leq 0, \quad (\forall x \in [x_0, X]); \)

(iv) for \( \forall x \in [x_0, X] \) and \( n_2 \neq n_3 \) inequalities

\[
n_2 \lambda_2(x) + n_3 \lambda_3(x) \neq \lambda_1(x), \quad \lambda_1(x) + n_2 \lambda_2(x) + n_3 \lambda_3(x) \neq \lambda_1(x), \quad (\forall x \in [x_0, X])
\]

for all multi-indices \( n = (n_2, n_3) \) with \( |n| \equiv n_2 + n_3 \geq 1 \) (\( n_2 \) and \( n_3 \) are non-negative integers) are holds.

We will develop an algorithm for constructing a regularized [1] asymptotic solution of problem (1).

2. Regularization of the problem

Denote by \( \sigma_j = \sigma_j(\epsilon) \) independent of magnitude \( \sigma_1 = e^{-\frac{\epsilon}{2} \beta(t_0)}, \sigma_2 = e^{+\frac{\epsilon}{2} \beta(t_0)} \), and rewrite system (1) as

\[
\varepsilon \frac{\partial y}{\partial x} = a(x)y + \varepsilon g(x) + \frac{\partial^2 y}{\partial x^2} \left( -\frac{1}{2} \int_0^t \beta'(\theta) d\theta \frac{\sigma_1}{\sigma_2} + \frac{1}{2} \frac{\partial}{\partial x} \beta'(\theta) d\theta \frac{\sigma_1}{\sigma_2} \right) y + \int_{x_0}^{x} K(x, t, s)y(s, t, \epsilon) ds + h(x, t), \quad y(x_0, t, \epsilon) = y^0.
\]

Introduce the regularized variables:

\[
\tau_j = \frac{1}{\varepsilon} \int_{x_0}^{x} \lambda_j(\theta) d\theta \equiv \frac{\psi_j(x)}{\varepsilon}, \quad j = 1, 3
\]

and instead of problem (2), consider the problem

\[
\varepsilon \frac{\partial \tilde{y}}{\partial x} + \sum_{j=1}^{3} \lambda_j(x) \frac{\partial \tilde{y}}{\partial \tau_j} - a(x)\tilde{y} - \int_{x_0}^{x} K(x, t, s)\tilde{y}(s, t, \frac{\psi(s)}{\varepsilon}, \epsilon) ds - \varepsilon g(x) (e^{\epsilon_2 \sigma_1} + e^{\epsilon_3 \sigma_2})\tilde{y} = h(x, t), \quad \tilde{y}(x_0, t, 0, \epsilon) = y^0,
\]

for the function \( \tilde{y} = \tilde{y}(x, t, \tau, \epsilon) \) where is indicated: \( \psi = (\psi_1, \psi_2, \psi_3) \). It is clear that if \( \tilde{y} = \tilde{y}(x, t, \tau, \epsilon) \) is a solution of the problem (3), then the function is \( \tilde{y} = \tilde{y}(x, t, \frac{\psi(x)}{\varepsilon}, \epsilon) \) an
exact solution to problem (2), therefore, problem (3) is extended with respect to problem (2). However, it cannot be considered fully regularized, since it does not regularize the integral
\[ J\tilde{y} = \int_{x_0}^{x} K(x, t, s)\tilde{y}(s, \psi(s, \varepsilon), \varepsilon)ds. \]

**Definition.** A class \( M_\varepsilon \) is said to be asymptotically invariant (with \( \varepsilon \to +0 \)) with respect to an operator \( P_0 \) if the following conditions are fulfilled:
1) \( M_\varepsilon \subset D(P_0) \) for each fixed \( \varepsilon > 0 \);
2) the image \( P_0\mu(x, t, \varepsilon) \) of any element \( \mu(x, t, \varepsilon) \in M_\varepsilon \) decomposes in a power series
\[ P_0\mu(x, t, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \mu_n(x, t, \varepsilon)(\varepsilon \to +0, \ \mu_n(x, t, \varepsilon) \in M_\varepsilon, \ n = 0, 1, \ldots), \]
convergent asymptotically for \( \varepsilon \to +0 \) (uniformly with \( \varepsilon \in [t_0, T] \)).

From this definition it can be seen that the class \( M_\varepsilon \) depends on the space \( U \), in which the operator \( P_0 \) is defined. In our case \( P_0 = J \). For the space \( U \) we take the space of vector functions \( y(x, t, \tau) \), represented by sums
\[ y(x, t, \tau, \sigma) = \sum_{i=1}^{3} y_i(x, t, \tau) e^{s\tau_i} + \sum_{2 \leq |m| \leq N_\psi} y^m(x, t, \sigma) e^{(m, \tau)}, \]
\[ y_0(x, t, \sigma) + \sum_{1 \leq |m| \leq N_\psi} y^{e_1+m}(x, t, \sigma) e^{(e_1+m, \tau)}, \]
\[ y^{m}(x, t, \sigma), \ y^{e_1+m}(x, t, \sigma) \in C^\infty([x_0, X] \times [0, T]), \]
\[ 1 \leq |m| \equiv m_2 + m_3 \leq N_\psi, \ i = 0, 3, \ m = (0, m_2, m_3). \]
where is denoted: \( (m, \lambda(x)) = m_2 \lambda_2(x) + m_3 \lambda_3(x), \ (e_1 + m, \lambda(x)) = \lambda_1(x) + m_2 \lambda_3(x) + m_3 \lambda_3(x) \); an asterisk * above the sum sign indicates that the summation for \( |m| \geq 1 \) it occurs only over multi-indices \( m = (0, m_2, m_3) \) with \( m_2 \neq m_3, e_1 = (1, 0, 0), \sigma = (\sigma_1, \sigma_2) \).

Note that here the degree \( N_\psi \) of the polynomial \( y(x, t, \tau) \), relative to the exponentials \( e^{s\tau_i} \) depends on the element \( y \). In addition, the elements of space \( U \) depend on bounded in \( \varepsilon > 0 \) terms of constants \( \sigma_1 = \sigma_1(\varepsilon) \) and \( \sigma_2 = \sigma_2(\varepsilon) \) and which do not affect the development of the algorithm described below, therefore, in the record of element (4) of this space \( U \), we omit the dependence on \( \sigma = (\sigma_1, \sigma_2) \) for brevity. We show that the class \( M_\varepsilon = U|_{\tau=\psi(t)/\varepsilon} \) is asymptotically invariant with respect to the operator \( J \).

The image of the integral operator \( J \) on an arbitrary element \( y(x, t, \tau) \), of the space \( U \) has the form
\[ Jy(x, t, \tau) = \int_{x_0}^{x} K(x, t, s)y_0(s, t)ds + \sum_{i=1}^{3} \int_{x_0}^{x} K(x, t, s)y_i(s, t)e^{s\tau_i} \int_{x_0}^{s} \lambda_1(\theta)d\theta ds + \int_{x_0}^{x} K(x, t, s)y^{m}(s, t)e^{s\tau} \int_{x_0}^{s} (m, \lambda(\theta))d\theta ds + \]
\[ + \sum_{2 \leq |m| \leq N_\psi} \int_{x_0}^{x} K(x, t, s)y^{e_1+m}(s, t)e^{s\tau} \int_{x_0}^{s} (e_1+m, \lambda(\theta))d\theta ds. \]
\[ + \sum_{1 \leq |m| \leq N} \int_{x_0}^{x} K(x, t, s) y^{e_1 + m}(s, t)e^{\frac{1}{2} \int_{x_0}^{s} (e_1 + m, \lambda(\theta))d\theta} ds. \]

Apply the operation of integration by parts to the first term.

\[
J_i(x, t, \varepsilon) = \int_{x_0}^{x} K(x, t, s)y_i(s, t) e^{\frac{1}{2} \int_{x_0}^{s} \lambda_i(\theta)d\theta} ds = \varepsilon \int_{x_0}^{x} \frac{K(x, t, s)y_i(s, t)}{\lambda_i(s)}ds e^{\frac{1}{2} \int_{x_0}^{s} \lambda_i(\theta)d\theta} ds =
\]

\[
= \varepsilon \left[ \frac{K(x, t, s)y_i(s, t)}{\lambda_i(s)} e^{\frac{1}{2} \int_{x_0}^{s} \lambda_i(\theta)d\theta} - \int_{x_0}^{x} \left( \frac{\partial}{\partial s} \frac{K(x, t, s)y_i(s, t)}{\lambda_i(s)} \right) e^{\frac{1}{2} \int_{x_0}^{s} \lambda_i(\theta)d\theta} ds \right] =
\]

\[
= \varepsilon \left[ \frac{K(x, t, x)y_i(x, t)}{\lambda_i(x)} e^{\frac{1}{2} \int_{x_0}^{x} \lambda_i(\theta)d\theta} - \frac{K(x, t, x_0)y_i(x_0, t)}{\lambda_i(x_0)} \right] - \varepsilon \int_{x_0}^{x} \left( \frac{\partial}{\partial s} \frac{K(x, t, s)y_i(s, t)}{\lambda_i(s)} \right) e^{\frac{1}{2} \int_{x_0}^{s} \lambda_i(\theta)d\theta} ds .
\]

Continuing this process, we obtain the series

\[
J_i(x, t, \varepsilon) = \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} \left[ (I_i^\nu (K(x, t, s)y_i(s, t)))_{s=x} e^{\frac{1}{2} \int_{x_0}^{s} \lambda_i(\theta)d\theta} - (I_i^{\nu+1} (K(x, t, s)y_i(s, t)))_{s=x_0} \right],
\]

where \( I_i^0 = \frac{1}{\lambda_i(x)} \), \( I_i^\nu = \frac{1}{\lambda_i(x)} I_i^{\nu-1} \) (\( \nu \geq 1, i = 1, 3 \)).

Applying the integration operation in parts to integrals

\[
J_m(x, t, \varepsilon) = \int_{x_0}^{x} K(x, t, s)y^m(s, t)e^{\frac{1}{2} \int_{x_0}^{s} (m, \lambda(\theta))d\theta} ds,
\]

\[
J_{e_1 + m}(x, t, \varepsilon) = \int_{x_0}^{x} K(x, t, s)y^{e_1 + m}(s, t)e^{\frac{1}{2} \int_{x_0}^{s} (e_1 + m, \lambda(\theta))d\theta} ds,
\]

we note that for all multi-indices \( m = (0, m_2, m_3), m_2 \neq m_3 \), inequalities

\[
(m, \lambda(x)) \equiv m_2 \lambda_2(x) + m_3 \lambda_3(x) \neq 0 \quad \forall x \in [x_0, X], \quad m_2 + m_3 \geq 2
\]

are satisfied. In addition, for the same multi-indices we have

\[
(e_1 + m, \lambda(x)) \neq 0 \quad \forall x \in [x_0, X], \quad m_2 \neq m_3, |m| = m_2 + m_3 \geq 1.
\]
Indeed, if $(\epsilon_1 + m, \lambda(x)) = 0$ for some $x \in [x_0, X]$ and $m_2 \neq m_3$, $m_2 + m_3 \geq 1$, then $m_2 \lambda_2(x) + m_3 \lambda_3(x) = -\lambda_1(x)$, $m_2 + m_3 \geq 1$, which contradicts condition (iv). Therefore, integration by parts in integrals $J_m(t, \varepsilon), J_{\epsilon_1 + m}(t, \varepsilon)$ is possible. Performing it, we will have:

$$J_m(x, t, \varepsilon) = \int_{t_0}^{x} K(x, t, s)y^m(s, t)e^{\frac{1}{\varepsilon} \int_{t_0}^{s} (m, \lambda(\theta))d\theta} ds = \varepsilon \int_{x_0}^{x} \frac{K(x, t, x_0)y^m(s, t)}{(m, \lambda(s))} e^{\frac{1}{\varepsilon} \int_{t_0}^{s} (m, \lambda(\theta))d\theta} ds =$$

$$= \varepsilon \left[ \frac{K(x, t, x)y^m(x, t)}{(m, \lambda(x))} e^{\frac{1}{\varepsilon} \int_{x_0}^{x} (m, \lambda(\theta))d\theta} - \frac{K(x, t, x_0)y^m(x_0, t)}{(m, \lambda(x_0))} \right] -$$

$$-\varepsilon \int_{x_0}^{x} \left( \frac{\partial K(x, t, s)y^m(s, t)}{(m, \lambda(s))} \right) e^{\frac{1}{\varepsilon} \int_{t_0}^{s} (m, \lambda(\theta))d\theta} ds =$$

$$= \sum_{\nu = 0}^{\infty} (-1)^{\nu} e^{\nu+1} \left[ \left( I_m^\nu (K(x, t, s)y^m(s, t)) \right)_{s = t} e^{\frac{1}{\varepsilon} \int_{t_0}^{s} (m, \lambda(\theta))d\theta} - \left( I_m^\nu (K(x, t, s)y^m(s, t)) \right)_{s = t_0} \right],$$

where $I_m^0 = \frac{1}{(m, \lambda(s))} \cdot I_m^\nu = \frac{1}{(m, \lambda(s))} \frac{\partial}{\partial s} I_m^{\nu-1} (\nu \geq 1, |m| \geq 2)$,

$$J_{\epsilon_1 + m}(x, t, \varepsilon) = \int_{x_0}^{x} K(x, t, s)y^{\epsilon_1 + m}(s, t)e^{\frac{1}{\varepsilon} \int_{t_0}^{s} (\epsilon_1 + m, \lambda(\theta))d\theta} ds =$$

$$= \varepsilon \int_{x_0}^{x} K(x, t, s)y^{\epsilon_1 + m}(s, t)e^{\frac{1}{\varepsilon} \int_{t_0}^{s} (\epsilon_1 + m, \lambda(\theta))d\theta} ds -$$

$$\int_{x_0}^{x} \left( \frac{\partial K(x, t, s)y^{\epsilon_1 + m}(s, t)}{(\epsilon_1 + m, \lambda(s))} \right) e^{\frac{1}{\varepsilon} \int_{t_0}^{s} (\epsilon_1 + m, \lambda(\theta))d\theta} ds =$$

$$= \sum_{\nu = 0}^{\infty} (-1)^{\nu} e^{\nu+1} \left[ \left( I_{\epsilon_1 + m}^\nu (K(x, t, s)y^{\epsilon_1 + m}(s, t)) \right)_{s = t} e^{\frac{1}{\varepsilon} \int_{t_0}^{s} (\epsilon_1 + m, \lambda(\theta))d\theta} - \left( I_{\epsilon_1 + m}^\nu (K(x, t, s)y^{\epsilon_1 + m}(s, t)) \right)_{s = t_0} \right],$$

where $I_{\epsilon_1 + m}^0 = \frac{1}{(\epsilon_1 + m, \lambda(s))} \cdot I_{\epsilon_1 + m}^\nu = \frac{1}{(\epsilon_1 + m, \lambda(s))} \frac{\partial}{\partial s} I_{\epsilon_1 + m}^{\nu-1} (\nu \geq 1, |m| \geq 1)$.
Therefore, the image of the operator $J$ on the element (5) of the space $U$ is represented as a series

$$Jy(x,t,\tau) = \int_{x_0}^{x} K(x,t,s)y_0(s,t)ds +$$

$$+ \sum_{i=1}^{3} \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} \left[ \left( I_{i}^{\nu} (K(x,t,s)y_i(s,t)) \right)_{s=t} e^{\frac{1}{2} \int_{s_0}^{s} \lambda(\theta)d\theta} - \right.$$  

$$\left. - \left( I_{m}^{\nu} (K(x,t,s)y_m(s,t)) \right)_{s=t_0} \right] +$$

$$+ \sum_{2 \leq |m| \leq N_{Y}} \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} \left[ \left( I_{m}^{\nu} (K(x,t,s)y^m(s,t)) \right)_{s=t} e^{\frac{1}{2} \int_{s_0}^{s} \lambda(\theta)d\theta} - \right.$$  

$$\left. - \left( I_{m}^{\nu} (K(x,t,s)y^m(s,t)) \right)_{s=t_0} \right] +$$

$$+ \sum_{1 \leq |m| \leq N_{Y}} \sum_{\nu=0}^{\infty} (-1)^\nu \varepsilon^{\nu+1} \left[ \left( I_{e_1+m}^{\nu} (K(x,t,s)y^{e_1+m}(s,t)) \right)_{s=t} \right.$$  

$$\left. \times e^{\frac{1}{2} \int_{s_0}^{s} (e_1+m,\lambda(\theta))d\theta} - \left( I_{e_1+m}^{\nu} (K(x,t,s)y^{e_1+m}(s,t)) \right)_{s=t_0} \right].$$

It is easy to show (see, for example, [12], pp. 291-294) that this series converges asymptotically for $\varepsilon \rightarrow +0$ (uniformly in $x,t \in [x_0,X] \times [0,T]$). This means that the class $M_\varepsilon$ is asymptotically invariant (for $\varepsilon \rightarrow +0$) with respect to the operator $J$.

We introduce operators $R_{\nu}$: $U \rightarrow U$, acting on each element $y(x,t,\tau) \in U$ of the form (5) according to the law:

$$R_{0}y(x,t,\tau) = \int_{x_0}^{x} K(x,t,s)y_0(s,t)ds,$$  \hspace{1cm} (\text{60})

$$R_{1}y(x,t,\tau) = \sum_{i=1}^{3} \left[ \left( I_{i}^{0} (K(x,t,s)y_i(s,t)) \right)_{s=x} e^{\tau_i} - \left( I_{i}^{0} (K(x,t,s)y_i(s,t)) \right)_{s=x_0} \right] +$$

$$+ \sum_{1 \leq |m| \leq N_{Y}} \left[ \left( I_{m}^{0} (K(x,t,s)y^m(s,t)) \right)_{s=x} e^{(m,\tau)} - \left( I_{m}^{0} (K(x,t,s)y^m(s,t)) \right)_{s=x_0} \right] +$$

$$+ \sum_{1 \leq |m| \leq N_{Y}} \left[ \left( I_{e_1+m}^{0} (K(x,t,s)y^{e_1+m}(s,t)) \right)_{s=x} e^{(e_1+m,\tau)} - \right.$$  

$$\left. - \left( I_{e_1+m}^{0} (K(x,t,s)y^{e_1+m}(s,t)) \right)_{s=x_0} \right],$$  \hspace{1cm} (\text{61})
Now let \( \tilde{y}(x, t, \tau, \epsilon) \) be an arbitrary continuous function on \((x, t, \tau, \epsilon) \in [x_0, X] \times [0, T] \times \{ \tau : \Re \tau_j, j = \overline{1, 3} \}, \) with asymptotic expansion

\[
\tilde{y}(x, t, \tau, \epsilon) = \sum_{k=0}^{\infty} \epsilon^k y_k(x, t, \tau), \quad y_k(x, t, \tau) \in U,
\]

converging as \( \epsilon \to +0 \) (uniformly in \((x, t, \tau) \in [x_0, X] \times [0, T] \times \{ \tau : \Re \tau_j, j = \overline{1, 3} \})\). Then the image \( J \tilde{y}(x, t, \tau, \epsilon) \) of this function is decomposed into an asymptotic series

\[
J \tilde{y}(x, t, \tau, \epsilon) = \sum_{k=0}^{\infty} \epsilon^k Jy_k(x, t, \tau) = \sum_{r=0}^{\infty} \epsilon^r \sum_{s=0}^{r} R_{r-s} y_s(x, t, \tau)|_{\tau=\psi(t)/\epsilon}.
\]

This equality is the basis for introducing an extension of an operator \( J \) on series of the form (7):

\[
\tilde{J} \tilde{y} = \tilde{J} \left( \sum_{k=0}^{\infty} \epsilon^k y_k(x, t, \tau) \right) = \sum_{r=0}^{\infty} \epsilon^r \left( \sum_{k=0}^{r} R_{r-k} y_k(x, t, \tau) \right).
\]

Although the operator \( \tilde{J} \) is formally defined, its utility is obvious, since in practice it is usual to construct the \( N \)-th approximation of the asymptotic solution of the problem (2), in which impose only \( N \)-th partial sums of the series (7), which have not a formal, but a true meaning. Now you can write a problem that is completely regularized with respect to the original problem (2):

\[
L_\epsilon \tilde{y}(x, t, \tau, \epsilon) = \epsilon \frac{\partial^2 \tilde{y}}{\partial \tau^2} + 3 \sum_{j=1}^{3} \lambda_j(x) \frac{\partial y}{\partial \tau_j} - a(x) \tilde{y} - \tilde{J} \tilde{y} - \epsilon \frac{g(x)}{2}(e^{\tau_1} \sigma_1 + e^{\tau_2} \sigma_2) \tilde{y} = h(x, t), \quad \tilde{y}(x_0, 0, \epsilon) = y_0, \quad ((x, t) \in [x_0, X] \times [0, T]). \tag{8}
\]

3. Solvability of iterative problems

Substituting the series (7) into (8) and equating the coefficients of the same powers of \( \epsilon \), we obtain the following iterative problems:

\[
L_{y_0} = \sum_{j=1}^{3} \lambda_j(x) \frac{\partial y_0}{\partial \tau_j} - a(x) y_0 - R_0 y_0 = h(x, t), \quad y_0(x_0, 0, \epsilon) = y_0; \tag{9_0}
\]

\[
L y_1 = - \frac{\partial y_0}{\partial x} + \frac{g(x)}{2} (e^{\tau_1} \sigma_1 + e^{\tau_2} \sigma_2) y_0 + R_1 y_0, \quad y_1(x_0, 0, \epsilon) = 0; \tag{9_1}
\]

\[
L y_2 = - \frac{\partial y_1}{\partial x} + \frac{g(x)}{2} (e^{\tau_1} \sigma_1 + e^{\tau_2} \sigma_2) y_1 + R_1 y_1 + R_2 y_0, \quad y_2(x_0, 0, \epsilon) = 0; \tag{9_2}
\]

\[
L y_k = - \frac{\partial y_{k-1}}{\partial x} + \frac{g(x)}{2} (e^{\tau_1} \sigma_1 + e^{\tau_2} \sigma_2) y_{k-1} + R_k y_0 + R_1 y_{k-1}, \quad y_k(x_0, 0, \epsilon) = 0, \quad k \geq 1. \tag{9_k}
\]
Each iterative problem \((9_k)\) has the form
\[
Ly \equiv \sum_{j=1}^{3} \lambda_j(x) \frac{\partial y}{\partial \tau_j} - a(x)y - R_0 y = H(x, t, \tau), \quad y(x_0, t, 0) = y_*,
\]
where \(H(x, t, \tau) \in U\), is the known vector function of space \(U\), \(y_*\) is the known constant vector of the complex space \(C\), and the operator \(R_0\) has the form (see (6_0))
\[
R_0 y(x, t, \tau) \equiv R_0 \left[ y_0(x, t) + \sum_{i=1}^{3} y_i(x, t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_y}^* y^m(x, t) e^{(m, \tau)} + \right.
\]
\[
+ \left. \sum_{1 \leq |m| \leq N_y}^* y^{e_1+m}(x, t) e^{(e_1+m, \tau)} \right] \Delta \int_{x_0}^{x} K(x, t, s) y_0(s, t) ds.
\]
We introduce scalar (for each \(x \in [x_0, X]\)) product in space \(U\):
\[
< u, w > \equiv < u_0(x, t) + \sum_{i=1}^{3} u_i(x, t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_y}^* u^m(x, t) e^{(m, \tau)} + \]
\[
+ \sum_{1 \leq |m| \leq N_y}^* u^{e_1+m}(x, t) e^{(e_1+m, \tau)}, w_0(x, t) + \sum_{i=1}^{3} w_i(x, t) e^{\tau_i} + \]
\[
+ \sum_{2 \leq |m| \leq N_w}^* w^m(x, t) e^{(m, \tau)} + \sum_{1 \leq |m| \leq N_w}^* w^{e_1+m}(x, t) e^{(e_1+m, \tau)} > \Delta
\]
\[
\Delta \equiv (u_0(x, t), w_0(x, t)) + \sum_{i=1}^{3} (u_i(x, t), w_i(x, t)) + \sum_{2 \leq |m| \leq \min(N_y, N_w)}^* (u^m(x, t), w^m(x, t)) + 
\]
\[
+ \sum_{1 \leq |m| \leq \min(N_y, N_w)}^* (u^{e_1+m}(x, t), w^{e_1+m}(x, t)),
\]
where we denote by \((*, *)\) the usual scalar product in the complex space \(C\). Let us prove the following statement.

**Theorem 1.** Let conditions (i)-(ii), (iv) be fulfilled and the right-hand side \(H(x, t, \tau)\) of system (10) belongs to the space \(U\). Then the system (10) is solvable in \(U\), if and only if
\[
H_1(x, t, \tau) \equiv 0, \quad \forall x \in [x_0, X].
\]

**Proof.** We will determine the solution of system (10) as an element (5) of the space \(U\):
\[
y(x, t, \tau) = y_0(x, t) + \sum_{i=1}^{3} y_i(x, t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_y}^* y^m(x, t) e^{(m, \tau)} + 
\]
\[
+ \sum_{1 \leq |m| \leq N_y}^* y^{e_1+m}(x, t) e^{(e_1+m, \tau)}.
\]
Due to the smoothness of the kernel $-a^{-1}(x)K(x, t, s)$ and heterogeneity $-a^{-1}(x)H_0(x, t)$, this Volterra integral equation has a unique solution $z_0(x, t) \in C^\infty ([x_0, X] \times [0, T])$. The equations (13$_2$) and (13$_3$) also have unique solutions

\[ z_i(x, t) = [\lambda_i(x) - a(x)]^{-1} H_i(x, t) \in C^\infty ([x_0, X] \times [0, T]), \quad i = 2, 3. \]
Equation (13) are solvable in space $C^\infty ([x_0, X] \times [0, T])$ if and only if there are identities

$$H_1(x, t) \equiv 0 \quad \forall x \in [x_0, X],$$

It is not difficult to see that these identities coincide with identities (11).

Further, since $(m, T(x)) \equiv m_2 \lambda_2(x) + m_3 \lambda_3(x) \neq \lambda_1(x), \ |m| = m_2 + m_3 \geq 2$ (see condition (iv)) the absence of resonance), the equation system (13) has a unique solution

$$z^m(x, t) = [(m, T(x)) - a(x)]^{-1} H^m(x, t), \ 2 \leq |m| \leq N_y \in C^\infty ([x_0, X] \times [0, T]).$$

We now consider equation (14). Let $(m^1, T(x)) = \lambda_1(x), \ |m^1| \geq 2$. Then

$$\lambda_1(x) + m_2 \lambda_2(x) + m_3 \lambda_3(x) = \lambda_1(x) \Leftrightarrow$$

$$\Leftrightarrow m_2 \lambda_2(x) + m_3 \lambda_3(x) = 0 \Leftrightarrow m_2 \neq m_3, \ m_2 + m_3 \geq 1,$$

which cannot be (see definition of class $U$). Unique solution of equation (18) for $|m^1| \geq 2$ in the class $C^\infty ([x_0, X] \times [0, T])$:

$$z^{m^1}(x, t) = [(m^1, T(x)) - a(x)]^{-1} H^{m^1}(x, t), \ 2 \leq |m^1| \leq N_y.$$

Thus, condition (11) is necessary and sufficient for the solvability of equation (10) in the space $U$. The theorem is proved.

**Remark.** If identity (11) holds, then under conditions (i)-(ii) and (iv), equation (10) has the following solution in the space $U$:

$$y(x, t, \tau) = y_0(x, t) + \alpha_1(x, t)e^{\tau_1} + \sum_{i=2}^{3} \left[ \lambda_i(x) - a(x) \right]^{-1} H_i(x, t)e^{\tau_i} +$$

$$+ \sum_{2 \leq |m| \leq N_y} [(m, T(x)) - a(x)]^{-1} H^m(x, t)e^{(m, \tau)} +$$

$$+ \sum_{1 \leq |m| \leq N_y} [(e_1 + m, T(x)) - a(x)]^{-1} H^{e_1 + m}(x, t)e^{(e_1 + m, \tau)},$$

where $\alpha_1(x, t) \in C^\infty ([x_0, X] \times [0, T])$ are arbitrary function, $y_0(x, t)$ is the solution of an integral equation (13), $m \equiv (0, m_2, m_3), \ m_2 \neq m_3, \ |m| = m_2 + m_3 \geq 1.$

4. The unique solvability of the general iterative problem in the space $U$. Residual term theorem

Let us proceed to the description of the conditions for the unique solvability of equation (10) in space $U$. Along with problem (10), we consider the equation

$$L y(x, t, \tau) = -\frac{\partial y}{\partial x} + \frac{g(x)}{2} (e^{2\sigma_1} + e^{3\sigma_2}) y + Q(x, t, \tau),$$

(15)
where \( y = y(x, t, \tau) \) is the solution (14) of the equation (10), \( Q(x, t, \tau) \in U \) is the well-known function of the space \( U \). The right part of this equation:

\[
G(x, t, \tau) \equiv -\frac{\partial y}{\partial x} + \frac{g(x)}{2} (e^{\tau_2 \sigma_1} + e^{\tau_3 \sigma_2}) y + Q(x, t, \tau) =
\]

\[
= -\frac{\partial}{\partial x} \left[ y_0(x, t) + \sum_{i=1}^{3} y_i(x, t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_y}^{*} y^m(x, t) e^{(m, \tau)} + \right]
\]

\[
+ \sum_{1 \leq |m| \leq N_y}^{*} y^{e_1+m}(x, t) e^{(e_1+m, \tau)} \right] + +
\]

\[
+ \frac{g(x)}{2} (e^{\tau_2 \sigma_1} + e^{\tau_3 \sigma_2}) \left[ y_0(x, t) + \sum_{i=1}^{3} y_i(x, t) e^{\tau_i} + \sum_{2 \leq |m| \leq N_y}^{*} y^m(x, t) e^{(m, \tau)} + \right]
\]

\[
+ \sum_{1 \leq |m| \leq N_y}^{*} y^{e_1+m}(x, t) e^{(e_1+m, \tau)} \right] + Q(x, t, \tau),
\]

may not belong to space \( U \), if \( y = y(x, t, \tau) \in U \). Indeed, taking into account the form (14) of the function \( y = y(x, t, \tau) \in U \), we will have

\[
Z(x, t, \tau) \equiv G(x, t, \tau) + \frac{\partial y}{\partial x} - \frac{g(x)}{2} (e^{\tau_2 \sigma_1} + e^{\tau_3 \sigma_2}) \left[ y_0(x, t) + \sum_{i=1}^{3} y_i(x, t) e^{\tau_i} + \right]
\]

\[
+ \sum_{2 \leq |m| \leq N_y}^{*} y^m(x, t) e^{(m, \tau)} + \sum_{1 \leq |m| \leq N_y}^{*} z^{e_1+m}(x, t) e^{(e_1+m, \tau)} \right] =
\]

\[
= \frac{g(x)}{2} y_0(x, t) (e^{\tau_2 \sigma_1} + e^{\tau_3 \sigma_2}) + \sum_{i=2}^{3} \frac{g(x)}{2} y_i(x, t) \left( e^{\tau_i+\tau_2 \sigma_1} + e^{\tau_i+\tau_3 \sigma_2} \right) +
\]

\[
+ \frac{g(x)}{2} y_1(x, t) \left( e^{\tau_1+\tau_2 \sigma_1} + e^{\tau_1+\tau_3 \sigma_2} \right) + \frac{g(x)}{2} (e^{\tau_2 \sigma_1} + e^{\tau_3 \sigma_2}) \left[ \sum_{2 \leq |m| \leq N_y}^{*} y^m(x, t) e^{(m, \tau)} + \right]
\]

\[
+ \sum_{1 \leq |m| \leq N_y}^{*} z^{e_1+m}(x, t) e^{(e_1+m, \tau)} \right] + Q(x, t, \tau).
\]

Here are terms with exponents

\[
e^{\tau_3 + \tau_2} = e^{(m, \tau)} |_{m=(0,1,1)},
\]

\[
e^{\tau_2 + (m, \tau)} \ (\text{if} \ m_2 + 1 = m_3), \ e^{\tau_3 + (m, \tau)} \ (\text{if} \ m_3 + 1 = m_2), \quad (*)
\]
do not belong to space $U$, since in multi-index $m = (0, m_2, m_3)$ of the space $U$ must be $m_2 \neq m_3, m_2 + m_3 \geq 1$. Then, according to the well-known theory (see [1], p. 234), we embed these terms in the space $U$ according to the following rule (see (\ref{eq:*})):

$$
\overline{e^{\tau_2 + (e_1 + m, \tau)}} (\text{if } m_2 + 1 = m_3) m_3 + 1 = m_2,
$$

In $Z(x, t, \tau)$ need of embedding only the terms

$$
M(x, t, \tau) = \frac{g(x)}{2} y_1(x, t) (e^{\tau_1 + \tau_2} \sigma_1 + e^{\tau_1 + \tau_3} \sigma_2) + \frac{g(x)}{2} y_1(x, t) (e^{\tau_1 + \tau_2} \sigma_1 + e^{\tau_1 + \tau_3} \sigma_2),
$$

$$
S(x, t, \tau) = \frac{g(x)}{2} (e^{\tau_1} \sigma_1 + e^{\tau_3} \sigma_2) \left[ \sum_{2 \leq |m| \leq N_p} y^m(x, t) e^{(m, \tau)} + \sum_{1 \leq |m| \leq N_p} y^{e_1 + m}(x, t) e^{(e_1 + m, \tau)} \right].
$$

We describe this embedding in more detail, taking into account formulas (\ref{eq:*}):

$$
M(x, t, \tau) = \frac{g(x)}{2} y_1(x, t) (e^{\tau_1 + \tau_2} \sigma_1 + e^{\tau_1 + \tau_3} \sigma_2) + \frac{g(x)}{2} y_1(x, t) (e^{\tau_1 + \tau_2} \sigma_1 + e^{\tau_1 + \tau_3} \sigma_2) =
$$

$$
= \frac{g(x)}{2} \left[ y_1(x, t) e^{\tau_1 + \tau_2} \sigma_1 + y_1(x, t) e^{\tau_1 + \tau_3} \sigma_2 + y_2(x, t) e^{2 \tau_2} \sigma_1 + y_2(x, t) e^{2 \tau_3} \sigma_2 +
\right.
$$

$$
\left. + y_3(x, t) e^{2 \tau_3} \sigma_1 + y_3(x, t) e^{2 \tau_3} \sigma_2 \right] \Rightarrow
$$

$$
\overline{M}(x, t, \tau) = \frac{g(x)}{2} \left[ y_1(x, t) e^{\tau_1 + \tau_2} \sigma_1 + y_1(x, t) e^{\tau_1 + \tau_3} \sigma_2 + y_2(x, t) e^{2 \tau_2} \sigma_1 +
\right.
$$

$$
\left. + y_2(x, t) e^{2 \tau_2} \sigma_1 + y_3(x, t) e^{2 \tau_3} \sigma_1 + y_3(x, t) e^{2 \tau_3} \sigma_2 \right],
$$

(note that in $\overline{M}(x, t, \tau)$ there are no members containing $e^{\tau_1}$, measurement exponents $|m| = 1$):

$$
S(x, t, \tau) = \frac{g(x)}{2} (e^{\tau_2} \sigma_1 + e^{\tau_3} \sigma_2) \left[ \sum_{2 \leq |m| \leq N_p} y^m(x, t) e^{(m, \tau)} + \sum_{1 \leq |m| \leq N_p} y^{e_1 + m}(x, t) e^{(e_1 + m, \tau)} \right] =
$$

$$
= \frac{g(x)}{2} \left[ \sum_{2 \leq |m| \leq N_p} y^m(x, t) \left( e^{\tau_2 + (m, \tau)} \sigma_1 + e^{\tau_3 + (m, \tau)} \sigma_2 \right) +
\right.
$$

$$
\left. + \sum_{1 \leq |m| \leq N_p} y^{e_1 + m}(x, t) \left( e^{(e_1 + m, \tau) + \tau_2} \sigma_1 + e^{(e_1 + m, \tau) + \tau_3} \sigma_2 \right) \right] \Rightarrow
$$
\[ \Rightarrow \hat{S}(x,t,\tau) = \frac{g(x)}{2} \left[ \sum_{2 \leq |m| \leq N_y, \atop m_2 + 1 = m_3} y^m(x,t)\sigma_1 + \sum_{2 \leq |m| \leq N_y, \atop m_3 + 1 = m_2} y^m(x,t)\sigma_2 + \sum_{2 \leq |m| \leq N_y, \atop m_2 + 1 \neq m_3, m_3 + 1 \neq m_2} y^m(x,t)e^{(m,\tau)} + \right] \]

\[ + \left[ \sum_{1 \leq |m| \leq N_g, \atop m_2 + 1 = m_3} y^{e_1+m}(x,t)\sigma_1 + \sum_{1 \leq |m| \leq N_g, \atop m_3 + 1 = m_2} y^{e_1+m}(x,t)\sigma_2 \right] e^{\tau_1} + \]

\[ + \left[ \sum_{1 \leq |m| \leq N_g, \atop m_2 + 1 \neq m_3, m_3 + 1 \neq m_2} y^{e_1+m}(x,t)e^{(e_1+m,\tau)} \right] \]

After embedding, the right-hand side of system (15) will look like

\[ \hat{G}(x,t,\tau) = -\frac{\partial}{\partial x} \left[ y_0(x,t) + \sum_{i=1}^{3} y_i(x,t)e^{\tau_i} + \sum_{2 \leq |m| \leq N_y} y^m(x,t)e^{(m,\tau)} \right] - \]

\[ -\frac{\partial}{\partial x} \left[ \sum_{1 \leq |m| \leq N_g} y^{e_1+m}(x,t)e^{(e_1+m,\tau)} \right] + \hat{M}(x,t,\tau) + \hat{S}(x,t,\tau) + Q(x,t,\tau), \]

moreover, in \( \hat{S}(x,t,\tau) \) the coefficients at \( e^{\tau_i} \) do not depend on \( z_1(x,t) \). As indicated in [1], the embedding \( G(x,t,\tau) \rightarrow \hat{G}(x,t,\tau) \) will not affect the accuracy of the construction of asymptotic solutions of problem (2), since \( G(x,t,\tau) \rightarrow \hat{G}(x,t,\tau) \).

**Theorem 2.** Let conditions (i)-(ii), (iv) be fulfilled and the right-hand side \( H(x,t,\tau) \in U \) of equation (10) satisfy condition (11). Then problem (10) under additional conditions

\[ \hat{G}(x,t,\tau) \equiv 0 \forall t \in [x_0,X], \]

(16)

where \( Q(x,t,\tau) \) is the known vector function of space \( U \), is uniquely solvable in \( U \).

**Proof.** Since the right-hand side of equation (10) satisfies condition (11), this equation has a solution in space \( U \) in the form (14), where \( \alpha_1(x,t) \in C^\infty([x_0,X] \times [0,T]) \) are arbitrary function so far. Submit (14) to the initial condition \( y(x_0,t,0) = y^* \). We get \( \alpha_1(x_0,t) = y_* \), where denoted

\[ y_* = y^* + a^{-1}(x_0)H_0(x_0,t) - \sum_{i=2}^{3} [\lambda_i(x_0) - a(x_0)]^{-1} H_i(x_0,t) - \]
\[- \sum_{2 \leq |m| \leq N}^* \left( (m, \lambda(x_0)) - a(x_0) \right)^{-1} H^m(x_0, t) - \]
\[- \sum_{1 \leq |m^k| \leq N}^* \left( (m^k, \lambda(x_0)) - a(x_0) \right)^{-1} H^{m^k}(x_0, t).\]

where do we find the values $\alpha_1(x_0, t) = y_*$. Then condition (16) takes the form

\[
- \frac{\partial}{\partial x} \alpha_1(x, t) e^{\tau_1} +
\]
\[
+ \left[ \sum_{1 \leq |m| \leq N} \frac{y^{e_1+m}(x, t) \sigma_1}{m_2 + 1 = m_3} + \sum_{1 \leq |m| \leq N} \frac{y^{e_1+m}(x, t) \sigma_2}{m_3 + 1 = m_2} \right] e^{\tau_1} +
\]
\[+ Q_1(x, t) e^{\tau_1} \equiv 0 \quad \forall(x, t) \in [x_0, X] \times [0, T]. \]

We obtain linear ordinary differential equations with respect to the function $\alpha_1(x, t)$, involved in the solution (14) of equation (10). Attaching to them the initial conditions $\alpha_1(t_0) = y_*$ computed earlier, we find uniquely the function $\alpha_1(x_0, t) = y_*$ and, therefore, we construct solution (14) in the space in a unique way. The theorem 2 is proved.

Applying Theorems 1 and 2 to iterative problems (9\(_k\)) (in this case, the right-hand sides $H^{(k)}(x, t, \tau)$ of these problems are embedded in the space $U$, i.e. $H^{(k)}(x, t, \tau)$ we replace with $\hat{H}^{(k)}(x, t, \tau) \in U), we find uniquely their solutions in space $U$ and construct series (7). Just as in [1], we prove the following statement.

**Theorem 3.** Suppose that conditions (i)-(ii), (iv) are satisfied for problem (2). Then, when $\varepsilon \in (0, \varepsilon_0]$ ($\varepsilon_0 > 0$ is sufficiently small), problem (2) has a unique solution $y(x, t, \varepsilon) \in C^1([x_0, X] \times [0, T]),$ in this case, the estimate

\[
||y(x, t, \varepsilon) - y_{\varepsilon N}(x, t)||_{C[x_0, X] \times [0, T]} \leq c_N \varepsilon^{N+1},
\]

holds true, where $z_{\varepsilon N}(x, t)$ is the restriction (for $\tau = \frac{\psi(t)}{\varepsilon}$) of the $N$-partial sum of series (7) (with coefficients $y_k(x, t, \tau) \in U$, satisfying the iteration problems (9\(_k\)), and the constant $c_N > 0$ does not depend on $\varepsilon \in (0, \varepsilon_0]$.

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