ON CERTAIN INTEGRALS INVOLVING
THE DIRICHLET DIVISOR PROBLEM

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ABSTRACT. We prove that

\[ \int_1^X \Delta(x) \Delta_3(x) \, dx \ll X^{13/9} \log^{10/3} X, \quad \int_1^X \Delta(x) \Delta_4(x) \, dx \ll \varepsilon X^{25/16 + \varepsilon}, \]

where \( \Delta_k(x) \) is the error term in the asymptotic formula for the summatory function of \( d_k(n) \), generated by \( \zeta^k(s) \) \( (\Delta_2(x) \equiv \Delta(x)) \). These bounds are sharper than the ones which follow by the Cauchy-Schwarz inequality and mean square results for \( \Delta_k(x) \). We also obtain the analogues of the above bounds when \( \Delta(x) \) is replaced by \( E(x) \), the error term in the mean square formula for \( |\zeta(\frac{1}{2} + it)| \).

1. Introduction and statement of results

Let \( d_k(n) \) denote the number of ways \( n \) can be written as a product of \( k \) factors, where \( k \in \mathbb{N} \) is given. Thus \( d_k(n) \) is generated by \( \zeta^k(s) \) \( (\Re s > 1) \), where \( \zeta(s) \) denotes the familiar zeta-function of Riemann. It is seen that \( d_1(n) \equiv 1 \) and \( d_2(n) \equiv d(n) \) is the number of divisors of \( n \). The error term in the asymptotic formula for sums of \( d_k(n) \) is commonly denoted by \( \Delta_k(x) \), namely

\[ \Delta_k(x) = \sum_{n \leq x} d_k(n) - xP_{k-1}(\log x), \]

where \( P_{k-1}(t) \) is a suitable polynomial of degree \( k-1 \) in \( t \), whose coefficients depend on \( k \). Usually one writes \( \Delta_2(x) \equiv \Delta(x) \), and the estimation of this function is called the Dirichlet divisor problem, while the estimation of \( \Delta_k(x) \) for \( k \geq 3 \) is known as the general (or generalized) divisor problem, or the Piltz divisor problem. A comprehensive account on \( \Delta_k(x) \) is to be found in Chapter 12 of E.C. Titchmarsh.
The coefficients of $P_{k-1}(t)$ may be evaluated by using the formula
\[ P_{k-1}(\log x) = \text{Res}_{s=1} x^{s-1} \zeta^k(s) s^{-1}, \]
and the fact that
\[ \zeta(s) = \frac{1}{s-1} + \gamma + \sum_{k=1}^{\infty} \gamma_k (s-1)^k \]
in a neighborhood of $s = 1$, where $\gamma = -\Gamma'(1) = 0.577215\ldots$ is Euler’s constant. In this way one finds that
\[ P_1(t) = t + (2\gamma - 1), \]
\[ P_2(t) = \frac{1}{2} t^2 + (3\gamma - 1) t + (3\gamma^2 - 3\gamma + 3\gamma_1 + 1), \]
and so on.

Following standard notation one defines $\alpha_k$ and $\beta_k$ as the infima of numbers $a_k$ and $b_k$, respectively, for which
\[ (1.2) \quad \Delta_k(x) \ll x^{a_k}, \quad \int_{1}^{x} \Delta_k^2(y) \, dy \ll x^{1+2b_k}. \]
It is obvious that $\beta_k \leq \alpha_k$ ($\forall k$). It is known that $\beta_k \geq (k-1)/(2k)$, and it is conjectured that
\[ (1.3) \quad \alpha_k = \beta_k = \frac{k-1}{2k} \quad (k > 1). \]
This is not yet proved for any $\alpha_k$, and for $\beta_k$ (1.3) is known to hold only for $k = 2, 3, 4$. Currently the best known upper bounds for $\alpha_2, \alpha_3$ and $\alpha_4$ are
\[ \alpha_2 \leq \frac{517}{1648} = 0.31371\ldots, \quad \alpha_3 \leq \frac{43}{96} = 0.447916\ldots, \quad \alpha_4 \leq \frac{1}{2}. \]
The bound for $\alpha_2$ is a recent result of J. Bourgain and N. Watt [BW], the bound for $\alpha_3$ was proved in 1981 by G. Kolesnik [Kol], and the bound for $\alpha_4$ follows easily by the Perron inversion formula.

It is not only that $\beta_2 = 1/2, \beta_3 = 1/3$ is known, but actually one has the asymptotic formulas
\[ (1.4) \quad \int_{1}^{X} \Delta^2(x) \, dx = A_2 X^{3/2} + O(X \log^3 X \log \log X) \left( A_2 = (6\pi^2)^{-1} \sum_{n=1}^{\infty} d^2(n)n^{-3/2} \right), \]
and

$$\int_1^X \Delta_3^2(x) \, dx = A_3 X^{5/3} + O_\varepsilon (X^{14/9 + \varepsilon}) \quad (A_3 = (10\pi^2)^{-1} \sum_{n=1}^{\infty} d_3^2(n)n^{-4/3}).$$

The asymptotic formula (1.4) is due to Y.-K. Lau and K.-M. Tsang [LT], while (1.5) was proved in 1956 by K.-C. Tong [Ton]. On the other hand, \(\Delta_k(x)\) takes large positive and small negative values. Indeed, Theorem 2 of A. Ivić [Iv2] states that, for \(k \geq 2\) fixed, there exist constants \(B, C > 0\) such that for \(T \geq T_0\) the interval \([T, T + CT^{(k-1)/k}]\) contains two points \(t_1, t_2\) for which

$$\Delta_k(t_1) > B t_1^{(k-1)/(2k)}, \quad \Delta_k(t_2) < -B t_2^{(k-1)/(2k)}.$$

Therefore it seems reasonable to expect a lot of cancellation in the integral of \(\Delta_k(x)\Delta_\ell(x)\) \((k \neq \ell; k, \ell > 1)\). In other words, the integral of this function should be of a smaller order of magnitude than what one expects is the order of the integral of \(|\Delta_k(x)||\Delta_\ell(x)|\). We are able to establish two results in this direction when \(k = 2\). We shall prove

**THEOREM 1.** We have

$$\int_1^X \Delta(x)\Delta_3(x) \, dx \ll X^{13/9}(\log X)^{10/3}.$$

**THEOREM 2.** We have

$$\int_1^X \Delta(x)\Delta_4(x) \, dx \ll_\varepsilon X^{25/16 + \varepsilon}.$$

Here and later \(\varepsilon\) denotes arbitrarily small positive constants, not necessarily the same ones at each occurrence. The symbol \(a \ll_\varepsilon b\) means that the implied \(\ll\)-constant depends only on \(\varepsilon\). To assess the strength of the bounds in (1.6) and (1.7) note that, using (1.4), (1.5) and the Cauchy-Schwarz inequality for integrals, it follows that

$$\left| \int_1^X \Delta(x)\Delta_3(x) \, dx \right| \leq \left( \int_1^X \Delta^2(x) \, dx \int_1^X \Delta_3^2(x) \, dx \right)^{1/2} \ll X^{19/12}.$$

Similarly, from (1.4) and \(\beta_4 = 3/8\) we have

$$\int_1^X \Delta(x)\Delta_4(x) \, dx \ll_\varepsilon X^{13/8 + \varepsilon}.$$
Since
\[ 13/9 = 1.4 < 19/12 = 1.583, \quad 25/16 = 1.5625 < 13/8 = 1.625, \]
we show that there is a substantial cancellation in the integrals in (1.6) and (1.7). However, the true order of these integrals remains elusive and represents a difficult problem. We note that the estimation of the integral of \( \Delta(x) \Delta_\ell(x) \) when \( \ell \geq 5 \) seems difficult, and that \( \beta_\ell = (\ell - 1)/(2\ell) \) is not known yet for any \( \ell \geq 5 \); for \( \ell = 5 \) the sharpest bound is \( \beta_5 \leq 9/20 \), due to W.-P. Zhang [Zh], while it is conjectured that \( \beta_5 = 2/5 \).

We note that in [IvZh] it was proved that
\[
\int_0^T \Delta(t)|\zeta(\frac{1}{2} + it)|^2 \, dt \ll T(\log T)^4,
\]
but we remarked that obtaining an asymptotic formula for the integral in (1.8) seems difficult.

The method of proof of Theorem 1 is of a fairly general nature, and works if \( \Delta(x) \) is replaced by a number-theoretic error term which, broadly speaking, has a structure similar to \( \Delta(x) \). Perhaps the most interesting case is when \( \Delta(x) \) is replaced by
\[
E(x) := \int_0^x |\zeta(\frac{1}{2} + it)|^2 \, dt - x\left(\log\left(\frac{x}{2\pi}\right) + 2\gamma - 1\right),
\]
the error term in the mean square formula for \( |\zeta(\frac{1}{2} + it)| \). This is a fundamental function in the theory of \( \zeta(s) \), and the reader is referred to Chapter 15 of [Iv1] and Chapter 3 of [Iv3] for an account of \( E(x) \). We shall prove

**THEOREM 3.** We have
\[
\int_1^X E(x) \Delta_3(x) \, dx \ll X^{3/2} \log^{3/2} X, \quad \int_1^X E(x) \Delta_4(x) \, dx \ll \varepsilon X^{25/16 + \varepsilon}.
\]

There are also several other ways to generalize Theorem 1 and Theorem 2, namely by replacing \( \Delta(x) \) with another suitable number-theoretic error term. These possibilities were already analyzed in detail in [Iv6], where the functions \( P(x) \) and \( A(x) \) were mentioned. One has
\[
P(x) := \sum_{n \leq x} r(n) - \pi x, \quad r(n) = \sum_{n = a^2 + b^2} 1,
\]
and
\[
A(x) := \sum_{n \leq x} a(n),
\]
where \(a(n)\) is the \(n\)-th Fourier coefficient of \(\varphi(z)\), and \(\varphi(z)\) is a holomorphic cusp form of weight \(\kappa\) with respect to the full modular group \(SL(2, \mathbb{Z})\). However, correlating these functions with \(\Delta_k(x)\) seems a little artificial, while the integrals of \(\Delta(x)\Delta_k(x)\) seemed much more natural to investigate.

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2. **The necessary lemmas**

In this section we formulate the lemmas needed for the proof of our results. The first lemma gives a good expression for \(\Delta(x)\) and a mean square estimate involving the “tails” of this expression. It is the lack of such a result for \(\Delta_k(x)\) when \(k \geq 3\) that thwarts the efforts to obtain satisfactory bounds for the integral of \(\Delta_k(x)\Delta_\ell(x)\) when \(3 \leq k < \ell\).

**Lemma 1.** We have, for \(1 \ll N \ll x\),

\[
\Delta(x) = \frac{1}{\pi \sqrt{2}} x^{\frac{3}{4}} \sum_{n \leq N} d(n) n^{-\frac{7}{4}} \cos(4\pi \sqrt{nx} - \frac{1}{4}\pi) + R(x),
\]

where

\[
R(x) = R(x, N) = O_\varepsilon (x^{\frac{3}{4} + \varepsilon} N^{-\frac{1}{2}}).
\]

If \(1 \ll N \ll X \log^{-4} X\), then

\[
\int_X^{2X} R^2(x, N) \, dx \ll X^{3/2} N^{-1/2} \log^3 X.
\]

**Proof.** The formulas (2.1)–(2.2) are the classical truncated Voronoi formula for \(\Delta(x)\). For this see e.g., Chapter 3 of A. Ivić [Iv1] for a proof. To obtain (2.3), note that by a result of T. Meurman [Meu] one has, for \(Q \gg x \gg 1\),

\[
\Delta(x) = \frac{1}{\pi \sqrt{2}} x^{\frac{3}{4}} \sum_{n \leq Q} d(n) n^{-\frac{7}{4}} \cos(4\pi \sqrt{nx} - \frac{1}{4}\pi) + F(x),
\]

where \(F(x) \ll x^{-1/4} \text{ if } ||x|| \gg x^{5/2} Q^{-1/2} \), and we always have \(F(x) \ll_\varepsilon x^\varepsilon\). Here, as usual, \(||x||\) is the distance of the real number \(x\) to the nearest integer.

In (2.4) we take \(Q = X^7, X \leq x \leq 2X\). Then, for \(1 \ll N \ll X\), \(R(x)\) in (2.1) can be written as

\[
R(x) = \frac{1}{\pi \sqrt{2}} x^{\frac{3}{4}} \sum_{N < n \leq Q} d(n) n^{-\frac{7}{4}} \cos(4\pi \sqrt{nx} - \frac{1}{4}\pi) + F(x),
\]
where in our case (2.4) shows that $F(x) \ll x^\varepsilon$ always, and $F(x) \ll x^{-1/4}$ if $||x|| \gg X^{-1}$. Then the integral in (2.3) is

$$X^{1/2} \int_X^{2X} \left| \sum_{N<n \leq Q} d(n)n^{-3/4}e^{4\pi i \sqrt{nx}} \right|^2 \, dx + \int_X^{2X} F^2(x) \, dx.$$  

We have

$$\int_X^{2X} F^2(x) \, dx = \int_{X, ||x|| \gg X^{-1}}^{2X} F^2(x) \, dx + \int_{X, ||x|| \ll X^{-1}}^{2X} F^2(x) \, dx \ll \int_X^{2X} x^{-1/2} \, dx + X^{2\varepsilon} \ll X^{1/2}.$$  

Now squaring out and integrating it follows that

$$\int_X^{2X} \left| \sum_{N<n \leq Q} d(n)n^{-3/4}e^{4\pi i \sqrt{nx}} \right|^2 \, dx \ll X \sum_{n>N} d^2(n)n^{-3/2} + \sum_{N<m \neq n \leq Q} \frac{d(m)d(n)\sqrt{X}}{(mn)^{3/4}|\sqrt{m} - \sqrt{n}|}.$$  

Note that $\sum_{n \leq y} d^2(n) \ll y \log^3 y$ for $y \geq 3$. Thus partial summation gives

$$X \sum_{n>N} d^2(n)n^{-3/2} \ll XN^{-1/2} \log^3 X.$$  

The second sum above does not exceed $S_1 + S_2$, where

$$S_j = \sum_{m} \frac{d(m)d(n)\sqrt{X}}{(mn)^{3/4}|\sqrt{m} - \sqrt{n}|} \quad (j = 1, 2),$$  

and (SC(A) means: summation conditions for $A$)

$$\text{SC} (S_1) : N < m \neq n \leq Q, |\sqrt{m} - \sqrt{n}| \leq (mn)^{1/4}/10,$$

$$\text{SC} (S_2) : N < m \neq n \leq Q, |\sqrt{m} - \sqrt{n}| \geq (mn)^{1/4}/10.$$  

The sum $S_2$ is bounded by

$$\sqrt{X} \sum_{m,n \leq Q} \frac{d(m)d(n)}{mn} \ll \sqrt{X} \log^4 X.$$
by using the well-known estimate \( \sum_{n \leq y} d(n) \ll y \log y \). In \( S_1 \) we have \( m \asymp n \). By \( d(m)d(n) \leq \frac{1}{2}(d^2(m) + d^2(n)) \) we have

\[
S_2 \ll X^{1/2} \sum_{N < m \leq Q} \frac{d^2(m)}{m} \sum_{N < n \leq Q, n \neq m} \frac{1}{|n - m|} \ll X^{1/2} \log X \sum_{N < n \leq Q} \frac{d^2(m)}{m} \ll X^{1/2} \log^5 X.
\]

Collecting the above estimates we get the bound in (2.3), since

\[
X \log^5 X \ll X^{3/2} N^{-1/2} \log^3 X \quad (1 \ll N \ll X \log^{-4} X).
\]

**Lemma 2.** Let \( 1/6 \leq \sigma_0 < 1/2 \) be a constant. Then, for \( k = 3 \) and \( k = 4 \), we have

\[
(2.5) \quad \Delta_k(x) = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} \frac{\zeta^k(s)}{s} x^s \, ds + O_{\varepsilon}(X^{1+\varepsilon}T^{-1}) + O(x^{\sigma_0}T^{k(1/2-\sigma_0)-1}),
\]

where \( X \leq x \leq 2X, 1 \ll T \ll X \).

**Proof.** From the Perron inversion formula (see e.g., (A.10) of [Iv1]) we have, for \( X \leq x \leq 2X, 1 \ll T \ll X \) and sufficiently small \( \varepsilon > 0 \),

\[
\Delta_k(x) = \frac{1}{2\pi i} \int_{1-\varepsilon-iT}^{1-\varepsilon+iT} \frac{\zeta^k(s)}{s} x^s \, ds + O_{\varepsilon}(X^{1+\varepsilon}T^{-1}).
\]

By Cauchy’s theorem we replace the segment of integration \([1 - \varepsilon - iT, 1 - \varepsilon + iT]\) by the segment \([\sigma_0 - iT, \sigma_0 + iT]\). In this process we are making an error which is \( \ll J_1(T) + J_2(T) \), where

\[
J_1(T) := \int_{\sigma_0}^{1/2} \frac{x^\sigma}{T} |\zeta(\sigma + iT)|^k \, d\sigma, \quad J_2(T) := \int_{1/2}^{1} \frac{x^\sigma}{T} |\zeta(\sigma + iT)|^k \, d\sigma.
\]

Then

\[
(2.6) \quad \int_{T}^{2T} J_1(t) \, dt = \int_{\sigma_0}^{1/2} x^\sigma \left( \int_{T}^{2T} \frac{|\zeta(\sigma + it)|^k}{t} \, dt \right) \, d\sigma
\]

\[
\ll \int_{\sigma_0}^{1/2} x^\sigma T^{k(1/2-\sigma)} \left( \int_{T}^{2T} \frac{|\zeta(1-\sigma + it)|^k}{t} \, dt \right) \, d\sigma
\]

\[
\ll \int_{\sigma_0}^{1/2} x^\sigma T^{k(1/2-\sigma)} \, d\sigma \ll X^{1/2} + X^{\sigma_0} T^{k(1/2-\sigma_0)}.
\]
Here we used the functional equation
\begin{equation}
(2.7) \quad \zeta(s) = \chi(s)\zeta(1-s), \quad |\chi(s)| = \left| \frac{\Gamma(\frac{1}{2}(1-s))}{\Gamma(\frac{1}{2}s)} \right| \approx |t|^{1/2-\sigma},
\end{equation}
and the bound (follows from Lemma 5)
\begin{equation}
(2.8) \quad \int_T^{2T} |\zeta(\alpha + it)|^k \, dt \ll T \quad (1/2 < \alpha \leq 1; \ k = 3, 4).
\end{equation}
Note that (2.8) is not known to hold in the whole range $1/2 < \alpha \leq 1$ when $\alpha$ is a constant, and $k > 4$ is an integer, which is one of the reasons why obtaining the analogue of Theorem 1 and 2 for the integral of $\Delta(x)\Delta_k(x)$ when $k > 4$ is difficult.

Similarly to (2.6) we obtain
\begin{align*}
\int_T^{2T} J_2(t) \, dt &\ll \int_{1/2}^{1} x^\sigma \left( \int_T^{2T} \frac{|\zeta(\sigma + it)|^k}{t} \, dt \right) \, d\sigma \\
&\ll \int_{1/2}^{1} x^\sigma \, d\sigma \ll X.
\end{align*}
It follows that
\begin{equation}
\int_T^{2T} (J_1(t) + J_2(t)) \, dt \ll X + X^{\sigma_0} T^{k(1/2-\sigma_0)}.
\end{equation}
This means that there exists $T_0 \in [T, 2T]$ such that
\begin{equation*}
J_1(T_0) + J_2(T_0) \ll X/T_0 + X^{\sigma_0} T_0^{k(1/2-\sigma_0)-1}.
\end{equation*}
If instead of the initial $T$ we take this $T_0$ and call it again $T$, we obtain (2.5).

**Lemma 3.** Let $f(x), \varphi(x)$ be real-valued functions on $[a, b]$ which satisfy
\begin{enumerate}
\item $f^{(4)}(x)$ and $\varphi^{(2)}(x)$ are continuous;
\item there exist numbers $H, U, A, 0 < H, A < U, 0 < b - a \leq U$ such that
\begin{align*}
A^{-1} &\ll f^{(2)}(x) \ll A^{-1}, f^{(3)}(x) \ll A^{-1}U^{-1}, f^{(4)}(x) \ll A^{-1}U^{-2}, \\
\varphi(x) &\ll H, \varphi^{(1)}(x) \ll HU^{-1}, \varphi^{(2)}(x) \ll HU^{-2};
\end{align*}
\item for some $c, a \leq c \leq b, f'(c) = 0$. Then
\begin{equation}
(2.9) \quad \int_a^b \varphi(x) \exp(2\pi if(x)) \, dx = \frac{1 + i}{\sqrt{2}} \cdot \frac{\varphi(c) \exp(2\pi ic)}{\sqrt{f''(c)}} + O(HAU^{-1})
\end{equation}
\begin{align*}
&+ O\left( H \min(|f'(a)|^{-1}, \sqrt{A}) \right) + O\left( H \min(|f'(b)|^{-1}, \sqrt{A}) \right).
\end{align*}
\end{enumerate}
This is a version of the classical result on exponential integrals with a “saddle” point $c$ (see e.g., [Iv1] and [Tit]). The particular version embodied in (2.9) is Lemma 2 on p. 71 of the monograph of A.A. Karatsuba and S.M. Voronin [KV]. The proof actually shows that there is no main term in (2.9) if $c \notin (a, b)$. If $c = a$ or $c = b$, then the respective main term is to be halved.

Lemma 4. For $\frac{1}{2} \leq \sigma < 1$ fixed, $1 \ll x, y \ll t^k$, $s = \sigma + it$, $xy = \left(\frac{s}{2\pi}\right)^k$, $t \geq t_0$ and $k \geq 1$ a fixed integer, we have

\begin{equation}
\zeta^k(s) = \sum_{m=1}^{\infty} \rho(m/x)d_k(m)m^{-s} + \chi^k(s) \sum_{m=1}^{\infty} \rho(m/y)d_k(m)m^{s-1} + O(t^{k(1-\sigma)/3-\frac{1}{2}} + O(t^{k(1/2-\sigma)-2y^\sigma \log^{k-1} t}).
\end{equation}

Here $\chi(s)$ is given by (2.7), and $\rho(u) (\geq 0)$ is a smooth function such that $\rho(0) = 1, \rho(u) = 0$ for $u \geq 2$.

This is Theorem 4.2 of A. Ivić [Iv3]. The explicit construction of $\rho(u)$ is given by Lemma 4.3 therein. The point of smoothing is to have much better error terms than those which can be at present obtained without it; see e.g., Theorem 4.3 of [Iv1].

Lemma 5. For fixed $\sigma$ such that $\frac{1}{2} < \sigma \leq 1$, we have

\begin{equation}
\int_1^T |\zeta(\sigma + it)|^4 \, dt = \frac{\zeta^2(2\sigma)}{\zeta(4\sigma)} T + O(T^{2-2\sigma} \log^3 T).
\end{equation}

This is Theorem 2 of A. Ivić [Iv4].

3. Proof of Theorem 1

We may consider, both in the proof of Theorem 1 and Theorem 2, that the integration is over $[X, 2X]$. Namely if we obtain the bounds in (1.6) and (1.7) for such an integral, then replacing $X$ by $X2^{-j}$, summing over $j$ and adding all the results we easily obtain the assertions of (1.6) and (1.7), respectively.

We begin by noting that the contribution of $R(x)$ in (2.1) to

\begin{equation}
I(X) := \int_X^{2X} \Delta(x) \Delta_3(x) \, dx
\end{equation}

is, by the Cauchy-Schwarz inequality, (1.5) and (2.3),

\begin{equation}
\ll \left( \int_X^{2X} R^2(x) \, dx \int_X^{2X} \Delta_3^2(x) \, dx \right)^{1/2} \ll X^{19/12} N^{-1/4} \log^{3/2} X,
\end{equation}
if \( 1 \ll N \ll X \log^{-4} X \). By (2.1) and (2.2) there remains a multiple of

\[
\int_{X}^{2X} x^{\frac{3}{4}} \sum_{n \leq N} d(n)n^{-\frac{3}{4}} \cos(4\pi \sqrt{nx} - \frac{1}{4} \pi) \Delta_3(x) \, dx.
\]

For \( \Delta_3(x) \) we use (2.5) of Lemma 2 with \( k = 3 \). The contribution of the error terms to (3.3) will be

\[
\ll \varepsilon X^{1/4}(X^{1+\varepsilon}T^{-1} + X^{\sigma_0}T^{1/2-3\sigma_0}) \int_{X}^{2X} \left| \sum_{n \leq N} d(n)n^{-3/4}e^{4\pi i \sqrt{nx}} \right| \, dx.
\]

Henceforth we assume that

\[
T = X, \quad \sigma_0 = \frac{1}{4},
\]

so that \( X^{1+\varepsilon}T^{-1} + X^{\sigma_0}T^{1/2-3\sigma_0} \ll \varepsilon X^{\varepsilon} \). By the method of proof of Lemma 1 it is seen that the sum over \( n \) in (3.3) is \( \ll 1 \) in mean square, if \( N \ll X \). This means that the contribution of the error terms to (3.3) is

\[
\ll \varepsilon X^{5/4+\varepsilon}
\]

if (3.4) holds. Since \( z + \bar{z} = 2\Re z \), we are left with

\[
\sum_{n \leq N} d(n)n^{-3/4} \int_{-X}^{X} \zeta^3(\frac{1}{4} + it) \int_{X}^{2X} x^{1/2+it} \cos(4\pi \sqrt{nx} - \frac{1}{4} \pi) \, dx \, dt
\]

\[
= 2 \sum_{n \leq N} d(n)n^{-3/4} \Re \left\{ \int_{0}^{X} \zeta^3(\frac{1}{4} + it) \int_{X}^{2X} x^{1/2+it} \cos(4\pi \sqrt{nx} - \frac{1}{4} \pi) \, dx \, dt \right\}.
\]

It transpires that integration over \( x \) leads to exponential integrals of the form

\[
\int_{X}^{2X} x^{1/2}e^{iF(x)} \, dx,
\]

where henceforth \( t > 0 \) and

\[
F(x) = F(x; t, n) := 4\pi \sqrt{nx} \pm t \log x.
\]

We first consider the case of the plus sign. In this case the derivative

\[
F'(x) = 2\pi \sqrt{n/x} + t/x
\]
is positive. Thus we can apply the first derivative test (Lemma 2.1 of [Iv1]) to the integral in (3.7). The contribution of (3.7) for \( t < \sqrt{nX} \) is \( \ll Xn^{-1/2} \), and its contribution to (3.6) is

\[
\ll X \sum_{n \leq N} d(n)n^{-5/4} \int_{0}^{\sqrt{nX}} t^{3/4} \left| \frac{\zeta'(3/4 + it)}{1 + it} \right|^3 \, dt
\]

\[
\ll X \sum_{n \leq N} d(n)n^{-5/4}(nX)^{3/8} \ll X^{11/8}N^{1/8} \log X,
\]

where we used (2.7) and (2.8). Similarly, for \( t > \sqrt{nX} \), one may use again the first derivative test to obtain a contribution which is

\[
\ll X^{3/2} \sum_{n \leq N} d(n)n^{-3/4} \int_{\sqrt{nX}}^{X} t^{3/4 - 1} \left| \frac{\zeta'(3/4 + it)}{t} \right|^3 \, dt
\]

\[
\ll X^{3/2} \sum_{n \leq N} d(n)n^{-3/4}(nX)^{-1/8} \ll X^{11/8}N^{1/8} \log X.
\]

From now on we consider in detail the case of the minus sign. We have

\[
F(x) = 4\pi \sqrt{nx} - t \log x,
\]

\[
F'(x) = 2\pi n^{1/2}x^{-1/2} - tx^{-1},
\]

\[
F''(x) = tx^{-2} - \pi n^{1/2}x^{-3/2}.
\]

For \( t < c_2\sqrt{nX} \) (\( c_j \) denotes positive constants) we have \( |F'(x)| \gg n^{1/2}X^{-1/2} \), and as in the previous case the contribution will be

(3.8) \[
\ll X^{11/8}N^{1/8} \log X.
\]

The bound in (3.8) will also hold for the contribution of \( t > c_1\sqrt{nX} \), when \( |F'(x)| \gg t/X \). Actually we can take \( c_2 = 2\pi(1 - \varepsilon), c_1 = 2\sqrt{2\pi}(1 + \varepsilon) \).

The integral in (3.7) may have a saddle point (a solution of \( F''(x) = 0 \)) if

\[
2\pi n^{1/2}x_0^{-1/2} - t/x_0 = 0,
\]

hence when

(3.9) \[
x_0 = \frac{t^2}{4\pi^2 n}.
\]

It is readily checked that \( X \leq x_0 \leq 2X \) for \( 2\pi \sqrt{nX} \leq t \leq 2\pi \sqrt{2nX} \). The integral in (3.7) is evaluated by Lemma 4, where one can take

\[
a = X, b = 2X, U = X, \varphi(x) = x^{1/2}, H = X^{1/2}, f(x) = \frac{F(x)}{2\pi}, A = X^{3/2}n^{-1/2},
\]
and the conditions on \( f^{(3)}(x) \) and \( f^{(4)}(x) \) hold. The total contribution of the error term \( O(HAU^{-1}) \) is

\[
\ll X \sum_{n \leq N} d(n)n^{-5/4} \int_{c_{1} \sqrt{n} X}^{c_{2} \sqrt{n} X} t^{-1/4} |\zeta(t + it)|^{3} \, dt
\]

\[
\ll X \sum_{n \leq N} d(n)n^{-5/4} (nX)^{3/8} \ll X^{11/8} N^{1/8} \log X,
\]

which is similar to (3.8).

We consider now the contribution of the error terms

\[
O \left\{ X^{1/2} \min \left( \frac{X^{3/4}}{n^{1/4}}, \left| \frac{t}{X} - \frac{2\pi \sqrt{n}}{\sqrt{X}} \right|^{-1} \right) \right\} +
\]

\[
O \left\{ X^{1/2} \min \left( \frac{X^{3/4}}{n^{1/4}}, \left| \frac{t}{2X} - \frac{2\pi \sqrt{n}}{\sqrt{2X}} \right|^{-1} \right) \right\}.
\]

They are treated analogously, so only the second one will be considered in detail. Let

\[
\mathcal{I} := [c_{1} \sqrt{n} X, c_{2} \sqrt{n} X]
\]

be the remaining interval in the \( t \)-integral. Let further

\[
\mathcal{I}_{0} := \left\{ (t \in \mathcal{I}) \wedge \left( \frac{X^{3/4}}{n^{1/4}} \leq \left| \frac{t}{2X} - \frac{2\pi \sqrt{n}}{\sqrt{2X}} \right|^{-1} \right) \right\},
\]

and for \( j = 1, 2, \ldots \)

\[
\mathcal{I}_{j} := \left\{ (t \in \mathcal{I}) \wedge \left( \frac{2^{j-1}n^{1/4}}{X^{3/4}} \leq \left| \frac{t}{2X} - \frac{2\pi \sqrt{n}}{\sqrt{2X}} \right| \leq \frac{2^{j}n^{1/4}}{X^{3/4}} \right) \right\},
\]

so that \( j \ll \log X \). It is easy to see \( \mathcal{I} = \cup_{j \geq 0} \mathcal{I}_{j} \) and

\[
|\mathcal{I}_{0}| \leq (nX)^{1/4}, \quad |\mathcal{I}_{j}| \leq 2^{j+1}(nX)^{1/4} \quad (j \geq 1),
\]

where \( |\mathcal{A}| \) denotes the cardinality of the set \( \mathcal{A} \).

The contribution of \( \mathcal{I}_{0} \) to the integral over \( t \) in (3.6) is

\[
\ll \sum_{n \leq N} d(n)n^{-3/4} \int_{\mathcal{I}_{0}} \frac{|\zeta(\frac{3}{4} + it)|^{3}}{t} \cdot \frac{X^{5/4}}{n^{1/4}} \, dt
\]

\[
\ll X^{5/4} \sum_{n \leq N} d(n)n^{-1} \int_{\mathcal{I}_{0}} t^{-1/4} |\zeta(\frac{3}{4} + it)|^{3} \, dt
\]

\[
\ll X^{5/4} \sum_{n \leq N} d(n)n^{-1}(nX)^{-1/8} \int_{\mathcal{I}_{0}} |\zeta(\frac{3}{4} + it)|^{3} \, dt.
\]
Now we invoke Lemma 5 (with $\sigma = 3/4$) and use Hölder’s inequality for integrals to obtain
\[
\int_{I_0} |\zeta(\frac{3}{4} + it)|^3 \, dt \leq \left( \int_{I_0} |\zeta(\frac{3}{4} + it)|^4 \, dt \right)^{3/4} |I_0|^{1/4}
\ll (nX)^{1/4} + (nX)^{1/4} \log X \right)^{3/4} (nX)^{1/16} \ll (nX)^{1/4} (\log X)^{9/4}.
\]
Thus we finally see that the contribution is
\[
\ll X^{11/8} \sum_{n \leq N} d(n)n^{-7/8} (\log X)^{9/4} \ll X^{11/8} N^{1/8} (\log X)^{13/4}.
\]
In a similar vein it is shown that the above bound also holds for the contribution of $\cup_{j \geq 1} I_j$, only with an additional log-factor since $j \ll \log X$.

We turn now to the contribution of the saddle points. By Lemma 3 the main contribution is a multiple of
\[
x_0^{1/2} |F''(x_0)|^{-1/2} e^{iF(x_0)}.
\]
We have, since $x_0 = t^2/(4\pi^2 n)$,
\[
F''(x_0) = -\pi n^{1/2} t^{-3} \cdot 8\pi^3 n^{3/2} + t^{-3} \cdot 16\pi^4 n^2 = 8\pi^4 t^{-3} n^2.
\]
Thus
\[
|F''(x_0)|^{-1/2} = (8\pi^4)^{-1/2} t^{3/2} n^{-1} \asymp X^{3/4} n^{-1/4},
\]
since $t \asymp \sqrt{nX}$ ($a \asymp b$ means that $a \ll b \ll a$). We also have
\[
F(x_0) = 4\pi \sqrt{\frac{nt^2}{4\pi^2 n}} - t \log \left( \frac{t^2}{4\pi^2 n} \right) = 2t - 2t \log t + t \log(4\pi^2 n).
\]
With this in mind, we are left with a multiple of
\[
\sum_{n \leq N} d(n)n^{-3/4} \int_{2\pi \sqrt{nX}}^{2\pi \sqrt{nX}} x_0^{1/2} \frac{e^{iF(x_0)}}{|F''(x_0)|^{1/4} + it} \, dt,
\]
since $x_0 \in (X, 2X)$ exactly for $2\pi \sqrt{nX} < t < 2\pi \sqrt{2nX}$.

At this point we use the approximate functional equation for $\zeta^3(s)$, writing first $\zeta(s) = \chi(s)\zeta(1-s)$ with $s = 1/4 + it$, noting that $\chi(s)\chi(1-s) \equiv 1$ and that in Lemma 4 one has $t = 3s > 0$. Thus by (2.10) with $k = 3$ we obtain
\[
\zeta^3(s) = \chi^3(s)\zeta^3(1-s) = \chi^3(s)\zeta^3(1-\bar{s})
\]
\[
= \chi^3(s) \sum_{m=1}^{\infty} \rho(m/x)d_3(m)m^{s-1} + \sum_{m=1}^{\infty} \rho(m/y)d_3(m)m^{-s} + O(1) + O(t^{-2}y^{3/4} \log^2 X),
\]
where \( \rho(u) (\geq 0) \) is a smooth function such that \( \rho(0) = 1, \rho(u) = 0 \) for \( u \geq 2 \), and

\[
(3.13) \quad xy = \left( \frac{t}{2\pi} \right)^3 \quad (x \gg 1, y \gg 1).
\]

We choose \( x = y = (t/(2\pi))^{3/2} \) in (3.13). Then \( t^{-2}y^{3/4}\log^2 X \ll 1 \), so that the contribution of the error terms in (3.12) to (3.11) will be

\[
\ll X^{1/2} \sum_{n \leq N} d(n)n^{-3/4}X^{3/4}n^{-1/4} \int_{2\pi \sqrt{nX}}^{2\pi \sqrt{2nX}} \frac{dt}{t} \ll X^{5/4}\log^2 X.
\]

There remains

\[
(3.14) \quad \sum_{n \leq N} d(n)n^{-3/4}(I' + I''),
\]

say, where we put \( s = 1/4 + it \)

\[
I' := \int_{2\pi \sqrt{nX}}^{2\pi \sqrt{2nX}} x_0^{1/2} e^{iF(x_0)} |F''(x_0)|^{-1/2} \chi^3(s)s^{-1} \sum_{m \leq 2x} \rho(m/x)d_3(m)m^{s-1} \, dt,
\]

\[
I'' := \int_{2\pi \sqrt{nX}}^{2\pi \sqrt{2nX}} x_0^{1/2} e^{iF(x_0)} |F''(x_0)|^{-1/2} s^{-1} \sum_{m \leq 2x} \rho(m/x)d_3(m)m^{-s} \, dt.
\]

Since, by (2.7), \( \chi(s) \) is essentially a quotient of two gamma-factors, it admits by Stirling’s formula a full asymptotic expansion in term of negative powers of \( t \). Thus the main contribution coming from \( \chi^3(s) \) with \( s = 1/4 + it \) will be (see e.g., (1.9) of [Iv3])

\[
\left( \frac{2\pi}{t} \right)^{3(-1/4+it)} e^{3i(t+\pi/4)}.
\]

Consequently the dominating terms in \( I' \) and \( I'' \) will be a multiple of (we set \( M = \max(2\pi \sqrt{nX}, 2\pi m^{2/3})2^{-2/3} \) for shortness)

\[
(3.15) \quad \sum_{m \leq 2(nX)^{3/4}} d_3(m)m^{-3/4} \int_{M}^{2\pi \sqrt{2nX}} x_0^{1/2} \left( \frac{t}{2\pi} \right)^{3/4} \rho\left( \frac{m}{x} \right) \frac{e^{iH_1(t)}}{s\sqrt{|F''(x_0)|}} \, dt
\]

and

\[
(3.16) \quad \sum_{m \leq 2(nX)^{3/4}} d_3(m)m^{-1/4} \int_{M}^{2\pi \sqrt{2nX}} x_0^{1/2} \rho\left( \frac{m}{x} \right) \frac{e^{iH_2(t)}}{s\sqrt{|F''(x_0)|}} \, dt,
\]
respectively, where

\[ H_1(t) := 3t \log \left( \frac{2\pi e}{t} \right) + t \log m + F(x_0), \]

\[ H_2(t) := -t \log m + F(x_0). \]

We see that (recall (3.10))

\[ H'_1(t) = -5 \log t - 3 + 3 \log(2\pi e) + \log(4\pi^2 mn), \]

\[ H'_2(t) = -2 \log t + \log \left( \frac{4\pi^2 n}{m} \right), \]

so that, in our range of \( m, n \) and \( t \),

\[ H'_1(t) \gg \log t, \quad H'_2(t) \gg \log t. \]

Therefore we can estimate the integrals in (3.15) and (3.16) by the first derivative test to obtain

\[
I' \ll \sum_{m \leq 2(nX)^{3/4}} d_3(m)m^{-3/4}X^{1/2}X^{3/4}n^{-1/4}(nX)^{-1/8}(\log X)^{-1}
\ll (nX)^{3/16}X^{9/8}n^{-3/8} \log X = X^{21/16}n^{-3/16} \log X,
\]

and

\[
I'' \ll \sum_{m \leq 2(nX)^{3/4}} d_3(m)m^{-1/4}X^{1/2}X^{3/4}n^{-1/4}(nX)^{-1/2}(\log X)^{-1}
\ll (nX)^{9/16}X^{3/4}n^{-3/4} \log X = X^{21/16}n^{-3/16} \log X.
\]

This gives for the expression in (3.14)

\[
\sum_{n \leq N} d(n)n^{-3/4}(I' + I'') \ll X^{21/16}N^{1/16} \log^2 X.
\]

Collecting all the estimates we have

\[
I(X) \ll X^{19/12}N^{-1/4} \log^{3/2} X + X^{11/8}N^{1/8} \log^{17/4} X + X^{21/16}N^{1/16} \log^2 X.
\]

So with the choice

\[ N = X^{5/9}(\log X)^{-22/3} \]

it follows that

\[ I(X) \ll X^{13/9}(\log X)^{10/3}, \]

which finishes the proof of Theorem 1.
4. Proof of Theorem 2

The proof of Theorem 2 is on the same lines as the proof of Theorem 1, so we shall indicate only the salient points in the proof. This seemed preferable than considering the general integral of $\Delta(x)\Delta_k(x)$, and then distinguishing between the cases $k = 3$ and $k = 4$. Whenever possible we shall retain the same notation as in the proof of Theorem 1.

We recall first that W.-P. Zhang [Zh] proved that $\beta_4 = 3/8$, in other words that

$$\int_{X}^{2X} \Delta_4^2(x) \, dx \ll \varepsilon X^{7/4+\varepsilon}. \tag{4.1}$$

An asymptotic formula for the integral in (4.1), analogous to (1.4) and (1.5), is not known to hold yet. From (1.4), (4.1) and the Cauchy-Schwarz inequality we have

$$I(X) := \int_{X}^{2X} \Delta(x)\Delta_4(x) \, dx \ll \varepsilon X^{13/8+\varepsilon}, \tag{4.2}$$

and similarly

$$\int_{X}^{2X} \Delta_4(x)R(x) \, dx \ll \varepsilon X^{13/8+\varepsilon}N^{-1/4}, \tag{4.3}$$

where $R(x)$ is as in (2.1)–(2.2). By using (2.1) it is seen that there remains a multiple of

$$\int_{X}^{2X} x^{1/4} \sum_{n \leq N} d(n)n^{-3/4} \cos(4\pi \sqrt{nx} - \pi/4) \, dx. \tag{4.4}$$

For $\Delta_4(x)$ we use Lemma 2 with $k = 4, \sigma_0 = 1/4, T = X$. The error terms in (2.5) are $\ll \varepsilon X^{1/4} \log^4 X$, and their contribution to (4.4) is

$$\ll X^{3/2} \log^5 X. \tag{4.5}$$

Thus we are left with

$$\sum_{n \leq N} d(n)n^{-3/4} \int_{-X}^{X} \frac{\zeta^4(\frac{1}{4} + it)}{\frac{1}{4} + it} \int_{X}^{2X} x^{1/2+it} \cos(4\pi \sqrt{nx} - \pi/4) \, dx \, dt. \tag{4.6}$$

We are led again to exponential integrals of the form

$$\int_{X}^{2X} x^{1/2}e^{iF(x)} \, dx, \quad F(x) = 4\pi \sqrt{nx} \pm t \log x.$$
As in the case of Theorem 1, one should take care only of the case of the minus sign in $F(x)$. Namely, in case of the plus sign we use the first derivative test and see that the total contribution is

\[(4.7) \quad \ll X^{3/2}N^{1/4} \log X,\]

and the relevant contribution of the $t$-integral is for $t \approx \sqrt{nX}$, as in Theorem 1. The saddle point is again

\[x_0 = \frac{t^2}{4\pi^2n}.\]

Lemma 3 is used again. The contribution of the error term $O(HA/U)$ is

\[\ll X \sum_{n \leq N} d(n)n^{-5/4} \int_{c_1}^{c_2\sqrt{nX}} |\zeta(\frac{3}{4} + it)|^4 dt \ll X^{3/2}N^{1/4} \log X,\]

like in (4.7). For the other two error terms in (2.9) we again define the sets $I_j (j \geq 0)$ as in the proof of Theorem 1. The contribution of $I_0$ is

\[\ll \sum_{n \leq N} d(n)n^{-3/4} \int_{I_0} |\zeta(\frac{3}{4} + it)|^4 X^{1/2}X^{3/4}n^{-1/4} dt \ll X^{5/4} \sum_{n \leq N} d(n)n^{-1}(|I_0| + (nX)^{1/4} \log^3 X) \ll X^{3/2}N^{1/4} \log^4 X.\]

This is actually simpler than the analogous portion of the proof in Theorem 1, as there is no need for Hölder’s inequality for integrals, and Lemma 5 can be used directly. The same bound is found to hold for $\sum_{j \geq 1} I_j$, with an additional log-factor.

The main contribution is a multiple of

\[(4.8) \quad \sum_{n \leq N} d(n)n^{-3/4} \int_{2\pi\sqrt{nX}}^{2\pi\sqrt{2nX}} x_0^{1/2}e^{iF(x_0)}|F''(x_0)|^{-1/2} \cdot \zeta^4(\frac{1}{4} + it) \frac{1}{\zeta^4(\frac{1}{4} + it)} dt.\]

With $k = 4, s = \frac{1}{4} + it$ Lemma 4 gives

\[
\zeta^4(s) = \chi^4(s)\zeta^4(1 - s) = \chi^4(s) \sum_{m=1}^{\infty} \rho(m/x)d_4(m)m^{s-1} + \sum_{m=1}^{\infty} \rho(m/y)d_4(m)m^{-s} + O(t^{1/3}) + O(t^{-2}y^{3/4} \log^3 X).
\]
Here $xy = (t/(2\pi))^4$, and we choose $x = y = (t/(2\pi))^2$. The error terms are 
\[ \ll t^{1/3} + t^{-1/2} \log^3 X \ll t^{1/3}, \]
and the contribution to (4.8) is 
\[ \ll X^{1/2} \sum_{n \leq N} d(n)n^{-3/4}X^{3/4}n^{-1/4}(nX)^{1/6} \ll X^{17/12}N^{1/6} \log X. \]

There remains the contribution of 
\[ \sum_{n \leq N} d(n)n^{-3/4}(I' + I''), \]
where $(s = \frac{1}{4} + it)$
\[ I' := \int_{2\pi \sqrt{2nX}}^{2\pi \sqrt{nX}} x_0^{1/2} \frac{e^{iF(x_0)}}{\sqrt{|F''(x_0)|}} \frac{X^4(s)}{s} \sum_{m \leq 2x} \rho(m/x)d_4(m)m^{s-1} \, dt \]
\[ I'' := \int_{2\pi \sqrt{2nX}}^{2\pi \sqrt{nX}} x_0^{1/2} \frac{e^{iF(x_0)}}{s \sqrt{|F''(x_0)|}} \sum_{m \leq 2x} \rho(m/x)d_4(m)m^{-s} \, dt. \]

Estimating, similarly as in the proof of Theorem 1, both $I'$ and $I''$ by the first derivative test we obtain 
\[ I' \ll X^{1/2}X^{3/4}n^{-1/4}(nX)^{1/4} \log^2 X \ll X^{3/2} \log^2 X, \]
\[ I'' \ll X^{1/2}X^{3/4}n^{-1/4}(nX)^{-1/2}(nX)^{3/4} \log^2 X \ll X^{3/2} \log^2 X. \]

This gives 
\[ \sum_{n \leq N} d(n)n^{-3/4}(I' + I'') \ll \sum_{n \leq N} d(n)n^{-3/4}X^{3/2} \log^2 X \ll X^{3/2}N^{1/4} \log^3 X. \]

Collecting all the estimates we infer that 
\[ I(X) \ll \varepsilon X^{13/8 + \varepsilon}N^{-1/4} + X^{3/2}N^{1/4} \log^3 X + X^{3/2 + \varepsilon} + X^{17/12}N^{1/6} \log X. \]

We have $X^{3/2}N^{1/4} = X^{13/8}N^{-1/4}$ for $N = X^{1/4}$, which finally gives 
\[ I(X) \ll \varepsilon X^{25/16 + \varepsilon}, \]
as asserted in (1.7) of Theorem 2.
5. Proof of Theorem 3

Note that, by (1.1), one has
\[
\Delta(x) = \sum_{n \leq x} d(n) - x (\log x + 2\gamma - 1),
\]
so that a comparison with (1.9) shows that the expression for \(E(x)\) has a factor of \(2\pi\) in the logarithm, which (5.1) does not. It was observed first by M. Jutila [Jut], that the analogy between \(\Delta(x)\) and \(E(x)\) becomes more exact if, instead of \(\Delta(x)\), one introduces the function
\[
\Delta^*(x) := -\Delta(x) + 2\Delta(2x) - \frac{1}{2}\Delta(4x) = \frac{1}{2} \sum_{n \leq 4x} (-1)^n d(n) - x (\log x + 2\gamma - 1),
\]
Namely, then the function
\[
E^*(t) := E(t) - 2\pi \Delta^*\left(\frac{t}{2\pi}\right)
\]
is smaller in mean square than either \(\Delta^*(x)\) or \(E(x)\). Namely Jutila [Jut] proved that
\[
\int_T^{T+H} (E^*(t))^2 \, dt \ll \varepsilon HT^{1/3} \log^3 T + T^{1+\varepsilon} \quad (2 \leq H \leq T),
\]
while \(E(x)\) satisfies an asymptotic formula analogous to (1.4). The first author [Iv5] obtained the asymptotic formula
\[
\int_0^T (E^*(t))^2 \, dt = T^{4/3} P_3(\log T) + O_\varepsilon(T^{7/6+\varepsilon}).
\]
A comparison with (1.4) shows that in (5.4) the main term has the exponent smaller by 1/6.

Note that the analogue of Lemma 1 will hold with \(\Delta^*(x)\) in place of \(\Delta(x)\) (see e.g., (15.68) of [Iv1] for the analogue of (2.1)). Further, on using (5.4) we have
\[
\int_1^X E^*(x)\Delta_3(x) \, dx \ll \left\{ \int_1^X (E^*(x))^2 \, dx \int_1^X \Delta_3^2(x) \, dx \right\}^{1/2} \ll X^{3/2} \log^{3/2} X.
\]
Here we have \(13/9 < 3/2\), where \(13/9\) was the exponent in Theorem 1. Similarly it is found that
\[
\int_1^X E^*(x)\Delta_4(x) \, dx \ll \varepsilon X^{37/24+\varepsilon},
\]
but here $37/24 < 25/16$, where the latter is the exponent in our Theorem 2.

Therefore, by (5.4), it remains to check that the integrals

$$
\int_{X}^{2X} \Delta^* \left( \frac{x}{2\pi} \right) \Delta_3(x) \, dx, \quad \int_{X}^{2X} \Delta^* \left( \frac{x}{2\pi} \right) \Delta_4(x) \, dx
$$

satisfy the same bounds as the integrals in Theorem 1 and Theorem 2, respectively. This is analogous as the previous reasoning, with the only real difference that now the saddle point $x_0$ (we retain the same notation) will be different. Indeed, instead of (3.6) we shall have

$$
\sum_{n \leq N} (-1)^n d(n) n^{-3/4} \Re \left\{ \int_{X}^{2X} \zeta^3 \left( \frac{1}{4} + it \right) \frac{x}{2\pi} \right\}
$$

The factor $(-1)^n$ in the sum over $n$ is unimportant. Then we obtain, instead of $F(x)$, the function

$$
F^*(x) := \sqrt{8\pi nx} - t \log x;
$$

$$(F^*(x))' = \sqrt{\frac{2\pi n}{x} - \frac{t}{x}},$$

$$(F^*(x))'' = \frac{t}{x^2} - \sqrt{\frac{\pi n}{2x^3}}.$$

The new saddle point is

$$
x_0 = \frac{t^2}{2\pi n}.
$$

Then

$$
F^*(x_0) = 2t - 2t \log t + t \log(2\pi n).
$$

Instead of the functions $H_1, H_2$, we shall have now the functions $H_1^*, H_2^*$ satisfying

$$
H_1^*(t) = 3t \log \left( \frac{2\pi e}{t} \right) + t \log m + F^*(x_0),$$

$$(H_1^*(t))' = -5 \log t + \log(2\pi)^5 mn,$$

$$
H_2^*(t) = -t \log m + F^*(x_0),$$

$$(H_2^*(t))' = \log \frac{\pi n}{m} - 2 \log t.$$

In the relevant range for $m, n, t$ we then obviously have

$$(H_1^*(t))' \gg \log t, \quad (H_2^*(t))' \gg \log t,$$

similarly as we had for $H_1'(t), H_2'(t)$. Thus the proof goes through as before and finally one ends up with the bounds (1.10), as asserted.
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