EMBEDDINGS OF 4–MANIFOLDS IN $\mathbb{C}P^3$

ABHIJEET GHANWAT AND DISHANT M. PANCHOLI

ABSTRACT. In this article we show that every closed orientable smooth 4–manifold admits a smooth embedding in the complex projective 3–space.

1. INTRODUCTION

A basic question in the field of geometric topology which concerns embeddings of manifolds, can be stated as follows: Given a pair of manifolds $M$ and $N$, how many smooth embeddings of $M$ exist in $N$?

Detailed investigations in this regard have lead to the discovery of interesting invariants of manifolds. One of the earliest seminal results in this context is due to H. Whitney who showed that every closed manifold of dimension $n$ admits an embedding in $\mathbb{R}^{2n}$. Subsequently, this result has been extensively generalized. Most notably, M. Hirsch showed \cite{13} that every closed orientable odd–dimensional manifold $M^{2n-1}$ admits a smooth embedding in $\mathbb{R}^{4n-3}$. This result together with those by Wall and Rokhlin implies that every closed 3–manifold admits an embedding in $\mathbb{R}^5$. In general, an even–dimensional manifold $M^{2n}$ does not smoothly embed in $\mathbb{R}^{4n-1}$. However, for 4–manifolds it was shown by M. Hirsch \cite{14} and C. T. C. Wall\footnote{M. Hirsch has mentioned in \cite{14} that C. T. C. Wall had independently proved this result.} that every orientable PL 4–manifold admits a PL embedding in $\mathbb{R}^7$.

It is usually possible to construct an invariant of a manifold $M$ using its embeddings in a manifold $N$, provided that (1) the topology of $N$ is relatively simple and (2) the co-dimension of the embedding of $M$ in $N$ is small. The importance of these two conditions is evident even from the examples of embeddings of surfaces. We recall that there exists an embedding of a closed smooth surface $\Sigma$ in $\mathbb{R}^3$ if and only if $\Sigma$ is orientable. This clearly shows that the orientability of a smooth closed surface can be captured by its embeddability in Euclidean 3–space. Further, the embeddability of every closed surface in $\mathbb{R}^4$ demonstrates the importance of lower co-dimension of embeddings, while the fact that $\mathbb{R}P^3 \# \mathbb{R}P^3$ admits an embedding of every closed surface shows the need for a relatively simple topology for the target space.

It was shown by S. Cappell and J. Shaneson \cite{6} that a smooth 4–manifold admits a smooth embedding in $\mathbb{R}^6$ if and only if it admits a spin structure. We know that a closed orientable 4–manifold is spin if and only if the second Stiefel-Whitney class $w_2(M)$ is zero. In particular, this implies that $\mathbb{C}P^2$ does not smoothly embed in $\mathbb{R}^6$. In this article, we investigate whether there exist topologically simple closed 6–dimensional manifolds which admit embeddings of all smooth 4–manifolds.

Two important classes of closed orientable smooth 4–manifolds are symplectic 4–manifolds and smooth algebraic surfaces. Their embeddings in various complex projective spaces have been extensively examined (see, for instance \cite{2}, \cite{8}, and \cite{9}), and the question of their embeddability in $\mathbb{C}P^3$ is very important. Furthermore, the topology of $\mathbb{C}P^3$ is very simple and $\mathbb{C}P^2$ naturally embeds in $\mathbb{C}P^3$. We therefore investigate embeddings of 4–manifolds in $\mathbb{C}P^3$ and establish the following:

**Theorem 1.1.** Every closed orientable smooth 4–manifold admits a smooth embedding in $\mathbb{C}P^3$.

The central idea for the proof of Theorem 1.1 is drawn from a well–known fact that given a projective embedding of a smooth algebraic surface, the standard Lefschetz pencil of the complex projective space generically induces a Lefschetz pencil structure on the surface. It was established by I. Baykur and S. Osamu \cite{5} that every smooth 4–manifold admits a simplified broken Lefschetz fibration (SBLF), which can
be regarded as a natural generalization of the Lefschetz pencil for an arbitrary smooth 4–manifold. This decomposition allows us to express any smooth 4–manifold as a singular fiber bundle over \( \mathbb{C}P^1 \) with a finite number of Lefschetz singularities and a unique fold singularity. The advantage of this decomposition is that we can associate with any smooth 4–manifold certain data which comprise two constituents. These are: (1) an element of the mapping class group of a closed orientable surface of genus \( g \) expressed as a product of Dehn twists, corresponding to Lefschetz singularities, and (2) a round handle attachment corresponding to the fold singularity.

Given a closed orientable smooth 4–manifold \( M \), we first consider the manifold \( M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \) together with any given SBLF. Then, we produce an embedding \( f \) of \( M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \) in \( \mathbb{C}P^2 \times \mathbb{C}P^1 \) such that the trivial product fibration \( \pi_2 : \mathbb{C}P^2 \times \mathbb{C}P^1 \rightarrow \mathbb{C}P^1 \) of \( \mathbb{C}P^2 \times \mathbb{C}P^1 \) induces the given SBLF.

The three important steps for constructing the embedding \( f \) are the following: In the first step, using an appropriate generalization of techniques from [18], and a specific local embedding model for a given Lefschetz singularity, we provide an embedding of genus \( g + 1 \) Lefschetz sub–fibration over a disk \( D^2 \) in \( \mathbb{C}P^2 \times D^2 \), which is associated with the given SBLF. This embedding is such that the trivial product fibration \( \pi_2 : \mathbb{C}P^2 \times D^2 \rightarrow D^2 \) induces the given Lefschetz fibration. This is the most important step in the proof, and is detailed in Section 4. In fact, in Section 4 we show how to embed any Lefschetz fibration over a disk or \( \mathbb{C}P^1 \) in a trivial fibration over \( \mathbb{C}P^1 \) with fiber \( \mathbb{C}P^2 \).

Next, we use a local embedding model for fold singularities to produce an embedding of a sub-manifold \( (\tilde{M}, \partial \tilde{M}) \subset M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \) (having two disjoint boundary components) in \( \mathbb{C}P^2 \times I \times S^1 \). This embedding is constructed such that it agrees with the embedding in the first step near one of the boundary components of \( \tilde{M} \), and is a trivial fibration \( \Sigma_g \times S^1 \) near the other boundary component of \( \tilde{M} \). Here, \( \Sigma_g \) denotes a surface of genus \( g \). This provides us with a fiber preserving embedding of \( M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \) \( \Sigma_g \times D^2 \) in \( \mathbb{C}P^2 \times D^2 \). Finally, we extend the embedding of \( M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \) \( \Sigma_g \times D^2 \) in \( \mathbb{C}P^2 \times D^2 \) using an embedding of \( \Sigma_g \times D^2 \) in \( \mathbb{C}P^2 \times D^2 \) to obtain the embedding \( f : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \rightarrow \mathbb{C}P^2 \times \mathbb{C}P^1 \). These two steps are discussed in Section 5. The general result that we manage to prove in Section 5 is Theorem 5.4.

In order to embed the given \( M \) in \( \mathbb{C}P^3 \), we first construct a specific SBLF \( M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \). We then embed \( M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \subset \mathbb{C}P^2 \times \mathbb{C}P^1 \) using this specific SBLF. Next, we note that the blowup of \( \mathbb{C}P^3 \) along \( \mathbb{C}P^1 \) is diffeomorphic to \( \mathbb{C}P^2 \times \mathbb{C}P^1 \). This means that there exists an embedding of \( M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \) in the blowup of \( \mathbb{C}P^3 \) along \( \mathbb{C}P^1 \). By slightly augmenting the argument for producing the embedding of \( M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \) in the blowup of \( \mathbb{C}P^3 \) along \( \mathbb{C}P^1 \), we notice that the embedding of \( M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \) can be produced such that when we blowdown the blowup of \( \mathbb{C}P^3 \), we produce a \( \mathbb{C}P^4 \) that has \( M \) as its embedded sub-manifold. The construction of the specific SBLF, blowup and blowdown procedure, and the proof of Theorem 5.4 are discussed in the final section.

The mathematical preliminaries to carry out these steps are given in Sections 2 and 3. In particular, we discuss relevant aspects of broken Lefschetz fibrations in Section 2 and of mapping class groups in Section 3.

Finally, a few remarks on conventions used in this article. By a manifold we mean a compact orientable manifold with or without boundary. We denote manifolds by capital letters \( M, N \), etc. When we need to emphasize that we are working with a manifold with boundary, we use the notation \( (M, \partial M) \) consisting of the pair \( M \) and the boundary \( \partial M \) of \( M \). The notation \( \Sigma_g \) is used for denoting an orientable surface of genus \( g \).

1.1. Acknowledgment.

We are thankful to S. Lakshmibala for critical comments and suggestions regarding the presentation. Dishant M. Pancholi is thankful to the Simons’ Foundation and ICTP, Trieste, Italy, for award of the Simons Associateship, which allowed him to travel to ICTP, Trieste, Italy, where a part of the work related to this article was carried out.

2. Review of Broken Lefschetz fibrations

Broken Lefschetz fibrations (BLF) were introduced by D. Auroux, S. K. Donaldson, and L. Katzarkov in [1]. These are generalized Lefschetz fibrations. I. Baykur [9] established that every smooth orientable
4–manifold admits a broken Lefschetz fibration. The purpose of this section is to review few definitions and result related to BLF. We refer \[3\] and \[5\] for a detailed discussion on BLF. First we recall the definitions of Lefschetz singularity.

**Definition 2.1** (Lefschetz singularity). Let \(M\) be an orientable 4–manifold and \(\Sigma\) be a surface. Let \(f: M \to \Sigma\) be a smooth map. A point \(x \in M\) is said to have a Lefschetz singularity at \(x\) provided there is an orientation preserving parametrization \(\phi: U \to \mathbb{C}^2\) and orientation preserving parametrization \(\psi: V \subset \Sigma \to \mathbb{C}\) such that the following properties are satisfied:

1. \(x \in U\) and \(\phi(x) = (0, 0) \in \mathbb{C}^2\).
2. \(f(x) \in V\) and \(\psi(f(x)) = 0 \in \mathbb{C}\).
3. For the map \(g: \mathbb{C}^2 \to \mathbb{C}\) given by \(g(z_1, z_2) = z_1 \cdot z_2\), the following diagram commutes:

\[
\begin{align*}
U & \xrightarrow{\phi} \mathbb{C}^2 \\
\downarrow f & \quad \downarrow g \\
V & \xrightarrow{\psi} \mathbb{C}.
\end{align*}
\]

**Remark 2.2.**

(a) Observe that both \(M\) as well as \(\Sigma\) can have non-empty boundary, however, it follows from Definition 2.1 that the critical point \(c\) belongs to the interior \(M\) of \(M\), and \(c \in \Sigma\).

(b) In case, we do not put any condition regarding preservation of the orientation by the parametrization around \(x\) or \(f(x)\) in Definition 2.1 above, the singularity is termed as achiral Lefschetz singularity.

(c) Let \(f: M \to Z\) be a map with an isolated Lefschetz singularity at \(c \in M\). It is well known that the fiber over \(f(c)\) is obtained by pinching a simple closed curve \(\gamma\) on any nearby smooth fiber \(\Sigma_g\) to a point. The curve \(\gamma\) is known as vanishing cycle.

(d) If we take a small closed disk \(D\) around \(f(c)\) not containing any other critical value, then the \(f^{-1}(\partial D)\) is a mapping torus over the smooth fiber \(\Sigma_g\) with monodromy a positive Dehn twist along the vanishing cycle \(\gamma\). In case, of a Achiral Lefschetz singularity the monodromy could be a positive or a negative Dehn twist along \(\gamma\).

Next, we recall the notion of 1–fold singularity.

**Definition 2.3** (1–fold Singularity). Let \(M\) be an orientable 4–manifold and \(\Sigma\) be a surface. Let \(f: M \to \Sigma\) be a smooth surjective map. A point \(x \in M\) is said to have a 1–fold singularity at \(x\) provided there is an orientation preserving parametrization \(\phi: U \to \mathbb{R}^4\) and orientation preserving parametrization \(\psi: V \subset \Sigma \to \mathbb{R}^2\) such that the following properties are satisfied:

1. \(x \in U\) and \(\phi(x) = (0, 0) \in \mathbb{R}^4\).
2. \(f(x) \in V\) and \(\psi(f(x)) = 0 \in \mathbb{R}^2\).
3. For the map \(h: \mathbb{R}^4 \to \mathbb{R}^2\) given by \(h(t, x_1, x_2, x_3) = (t, -x_1^2 + x_2^2 + x_3^2)\), the following diagram commutes:

\[
\begin{align*}
U & \xrightarrow{\phi} \mathbb{R}^4 \\
\downarrow f & \quad \downarrow h \\
V & \xrightarrow{\psi} \mathbb{R}^2.
\end{align*}
\]

**Remark 2.4.**

(a) Notice that a map \(f: M \to \Sigma\) has a 1–fold singularity at \(x\), then \(x \in \bar{M}\) and \(f(x) \in \bar{\Sigma}\).

(b) When the map \(h\) in the definition of 1-fold singularity is allowed to have the local model:

\[(t, x_1, x_2, x_3) \to (t, \pm x_1^2 \pm x_2^2 \pm x_3^2),\]

then the singularity is termed as a fold singularity. In this article, we will only need the local model around 1–fold singularity.
(c) A local singularity model for a smooth function of the form:
\[(t, x_1, x_2, x_3) \mapsto (t, x_1^3 + tx_1 \pm x_2^2 \pm x_3^3),\]

is known as a *cusp* singularity.

We are now in a position to recall the notion of a broken Lefschetz fibration (BLF).

**Definition 2.5 (Broken Lefschetz fibration).** Let \( M \) a smooth oriented 4–manifold. By a *broken Lefschetz fibration* of \( M \) we mean a smooth map \( f : M \to \mathbb{C}P^1 \) such that \( f \) has only 1-fold or Lefschetz singularity.

**Remark 2.6.**
(a) Given a BLF \( f : M \to \mathbb{C}P^1 \) the inverse image \( f^{-1}(y) \) for any regular value \( y \) is called a fiber of BLF.
(b) The image set of a 1–fold singularity on \( \Sigma \) is generically an immersed circle in \( \Sigma \).

A BLF without 1-fold singularity is called Lefschetz fibration. These singularities are extremely useful in algebraic and symplectic geometry. See, for example, [9]. Let us now formally define a Lefschetz fibration.

**Definition 2.7 (Lefschetz fibration).** Let \( M \) be a smooth 4–manifold. A smooth map \( f : M \to \Sigma \), where \( \Sigma \) is a surface, having its singular points modeled only on Lefschetz singularities is called a *Lefschetz fibration* associated to \( M \).

**Remark 2.8.**
(a) Unlike a fiber bundle or Lefschetz fibration the fibers of a BLF are not diffeomorphic. In fact, the 1-fold singularity in the definition of BLF corresponds to a round 1–handle attachment. See, for example, [10]. Hence, if BLF has points having fold singularity, then the genus of fibers change as we cross the image of an immersed circle coming from a 1–fold singularity.
(b) The fibers of BLF need not be connected. However, it can be shown that every 4–manifold admits a BLF with connected fibers having genus at least 2. This follows, for example, from [3].

Observe that a BLF provides us a decomposition of a smooth manifold into simple pieces. A more simplified form of this decomposition of smooth 4–manifold is what we will need for this article. This simplification was introduced by I. Baykur and S. Osamu in [5]. This decomposition is known as simplified broken Lefschetz fibration. Let us recall the definition of this:

**Definition 2.9 (Simplified broken Lefschetz fibration (SBLF)).** Let \( f : M \to \mathbb{C}P^1 \) be a BLF. We say that this BLF is *simplified broken Lefschetz fibration* (SBLF) provided the function \( f \) satisfies the following additional properties:
(1) The set \( Z_f \) of all \( x \in M \) admitting a 1-fold singularity model is connected.
(2) All fibers are connected.
(3) The map \( f \) is injective when restricted to \( Z_f \) as well as when restricted to the set \( C_f \) of Lefschetz singular points.

**Remark 2.10.**
(a) A SBLF having no fold singularity will be called *simplified Lefschetz fibration* or SLF in short.
(b) Observe that SBLF implies that there exists a disk \( D \) contained in \( \mathbb{C}P^1 \) such that every \( y \in D \) is a regular value and the genus of the fiber over \( y \) is minimum among all fibers of SBLF. We call this fiber *lower genus fiber*.
(c) Topologically the unique 1–fold singularity of SBLF corresponds to adding 1–handle to a circle worth of lower genus fibers over \( \partial D \) – which corresponds to addition of a round 1–handle to \( f^{-1}(D) \) – such that a generic fiber of SBLF over \( \mathbb{C}P^1 \setminus D \) has genus one more than the fibers over \( D \).

In [5], it was shown that every orientable smooth 4–manifold admits a SBLF. To state their theorem precisely, let us recall the definition of an *indefinite fibration*. 
**Definition 2.11 (Indefinite fibration).** Let $M$ be a compact manifold. A smooth map $f$ from $M$ to a surface $\Sigma$ is said to be an *indefinite* fibration provided it has only fold or cusp singularities.

**Theorem 2.12 (I. Baykur, S. Osamu [5]).** Let $(M, \partial M)$ be an orientable smooth 4–manifold. If $g : (M, \partial M) \rightarrow (\mathbb{D}^2, \partial \mathbb{D}^2)$ is fibration, which is a fibration when restricted to a collar neighborhood of $\partial M$ and send $\partial M$ to $\partial \mathbb{D}$, then there is a smooth map $f : (M, \partial M) \rightarrow \mathbb{D}^2$ homotopic to $g$ which satisfies the following:

1. $f$ defines SBLF on $(M, \partial M)$.
2. Lower genus fibers of $f$ have genus bigger than 1.
3. $f = g$ in a neighborhood of $\partial M$.

We would like to point out that Theorem 2.12 is not stated in the precise form that is presented above. I. Baykur and S. Osamu proved a similar result regarding smooth maps between 4–manifolds and arbitrary surfaces. Theorem 2.12 is a relatively straightforward consequence of their main result [5].

### 3. Surfaces and the mapping class groups

In this section, we review all results related to mapping class groups of orientable genus $g$ surfaces that is required for this article. Good references for the topics discussed in this section are [4], and [16]. Let us begin by recalling the definition of the mapping class group.

**Definition 3.1 (Mapping class group).** Let $\Sigma$ be an orientable surface. By the *mapping class group* of $\Sigma$ we mean the group of orientation preserving self diffeomorphisms of $\Sigma$ up to isotopy. In case, $\Sigma$ has a non-empty boundary then diffeomorphisms are always assumed to be the identity in a collar neighborhood of the boundary.

We denote the mapping class group of a surface $\Sigma$ by $\mathcal{MCG}(\Sigma)$ or sometimes by $(\mathcal{MCG}(\Sigma, \partial \Sigma))$, when $\partial \Sigma$ is non-empty. Next, let us discuss the notion of Dehn twist along a simple closed curve embedded in a surface $\Sigma$. We refer [4] for a more detailed discussion on Dehn twist diffeomorphisms.

![Figure 1](image.png)

**Figure 1.** The figure is a pictorial description of the Dehn twist diffeomorphism $\tau_\beta$ restricted to the neighborhood $A(\beta)$. Picture on the right depicts the image of the arc $c$ under the diffeomorphism $\tau_\beta$.

**Definition 3.2 (Dehn twist).** Let $\Sigma$ be an orientable surface. Let $\beta$ be a simple closed curve embedded in the interior of $\Sigma$. By a *Dehn twist* along $\beta$ we mean a diffeomorphism which is identity outside a small annulus neighborhood $A(\beta)$ of $\beta$ in $\Sigma$ and $\tau_\beta$ on $A(\beta)$ as depicted in Figure 1.

M. Dehn [7] and W. Lickorish [16] independently established that the mapping class group of an orientable genus $g$ surface $\Sigma_g$ is generated by Dehn twists along simple closed curves embedded in $\Sigma_g$. He further strengthened this result in [17] to show that the mapping class group of a closed orientable surface $\Sigma_g$ is generated by Dehn twists along the curves $a_i$’s , $b_j$’s and $c_k$’s as depicted in Figure 2. Following [18], we will call these curves as *Lickorish generators*.

We end this section with a proposition which is a consequence of Lemma-3 established in [16]. In order to state this proposition we need a few terminologies from [16].
Figure 2. In this figure the Dehn twist along curves $a_1$'s, $b_j$'s and $c_k$'s depicts the standard generators for the mapping class group of an orientable genus $g$ surface.

Figure 3. The figure shows surface of genus $g$ embedded in $\mathbb{R}^3$ as a boundary of a genus $g$ handlebody considered as a unit ball with $g$ 1–handles attached to it.

Let us regard a surface $\Sigma_g$ of genus $g$ as the boundary of a standard handle-body $H_g$ obtained by adding $g$ 1–handles to the unit 3–ball in $\mathbb{R}^3$ as depicted in Figure 3.

Consider a typical handle $H_k$, as shown in Figure 3. Following [16], we say that a simple closed curve $p$ does not meet the handle $H_k$ provided it does not intersect the curve $a_k$ depicted Figure 3.

**Proposition 3.3** (Lickorish: Lemma-3 [16]). Let $p$ be any simple closed curve on $\Sigma_g$. Then there exists a diffeomorphism $\phi: \Sigma_g \to \Sigma_g$ such that $\phi(p)$ is one of the following:

1. $\phi(p)$ is a Lickorish generator.
2. $\phi(p)$ does not meet any handle of $\Sigma_g$.

### 4. LEFSCHETZ FIBRATION EMBEDDING

Recall that if SBLF $(M, \pi)$ does not have fold singularities, then the SBLF is known as simplified Lefschetz fibration (SLF). In this section, we show that there exists Lefschetz fibration embedding of any SLF into $(\mathbb{C}P^2 \times \mathbb{C}P^1, \pi_2)$, which is fiber preserving in the sense defined in Definition 4.4. This result, i.e., Theorem 4.5, can be regarded as the first step towards establishing Theorem 1.1.

#### 4.1. Flexible embedding in standard position.

We begin by reviewing a few notions from [15] and [18]. First notion that we recall is the definition of flexible embedding.

**Definition 4.1** (Flexible embedding). Let $M$ be an orientable closed smooth manifold. A smooth embedding $\phi: \Sigma_g \hookrightarrow M$ of a closed orientable surface $\Sigma_g$ is said to be flexible provided, for every $f \in \text{MCG}(\Sigma_g)$ there exists a diffeomorphism $\psi$ of $M$ isotopic to the identity which maps $\Sigma_g$ to itself and satisfies $\phi^{-1} \circ \psi \circ \phi = f$.

Next, we state a lemma regarding a flexible embedding of any surface of genus $g$ in $\mathbb{C}P^2$. In order to state this lemma, we need to introduce the following:

**Definition 4.2** (Standard position). An embedding $\phi: \Sigma_g \hookrightarrow \mathbb{C}P^2$ is said to be in a standard position provided the following properties are satisfied:
Every simple closed curve $\gamma$ on $\phi(\Sigma)$ is a boundary of a $2$–disk $B^2$ intersecting $\phi(\Sigma_g)$ only in $\gamma$. There exists a tubular neighborhood $N(\mathbb{D})$ of the disk $\mathbb{D}$ having the boundary $\gamma$ such that $N(\mathbb{D})$ is the image of a coordinate chart $\phi_\gamma : \mathbb{C}^2 \to N(\mathbb{D})$ satisfying the following:

$$\phi_\gamma^{-1}(\phi(\Sigma_g) \cap N(\mathbb{D})) = g^{-1}(1),$$

where $g : \mathbb{C}^2 \to \mathbb{C}$ is the polynomial map $g(z_1, z_2) = z_1 z_2$.

Lemma 4.3. There exists an embedding $\phi$ of any orientable surface $\Sigma_g$ of genus $g$ which satisfies the following:

1. The embedding is flexible.
2. The embedding is in a standard position.

Before, we establish this lemma, we would like to point out that the flexible embedding of $\Sigma_g$ in $\mathbb{C}P^2$ was first provided by S. Hirose and A. Yasuhara in [15]. Our main observation is that we can achieve the additional property of the embedding being in a standard position, provided we use Proposition 3.3 established by Lickorish in [16] in conjunction with the techniques from [15].

Proof of Lemma 4.3. We want to construct an embedding of $\Sigma_g$ which is both flexible and in a standard position. To begin with, we regard $\mathbb{C}P^2$ as a handle-body with the $0$–handle $H_0$ corresponding to $B^4(0, 2)$ – the $4$–ball of radius $2$ in $\mathbb{C}^2$ with its center at the origin – to which a $2$–handle $H_2$ is attached along an unknot with framing $+1$. Finally a $4$–handle $H_4$ is attached to the $4$–manifold, which is the union of the $0$–handle $B^4(0, 2)$ and the $2$–handle $H_2$. See, for example [12] page–126, for this particular handle decomposition of $\mathbb{C}P^2$. We will also regard $S^3 \times [1, 2]$ as the collar $B^4(0, 2) \setminus B^4(0, 1)$ contained in $\mathbb{C}P^2$.

Next, embed a genus $g$ surface $\Sigma_g$ in $S^3 \times \{\frac{3}{2}\} \subset S^3 \times [1, 2] \subset \mathbb{C}P^2$ as the boundary of standard genus $g$ handle body $H_g$ as depicted in Figure 3. From the embedded surface of genus $g$, we remove an open disc $\mathbb{D}$ and attach a full twisted band along $\partial \mathbb{D}$ to obtain a surface $\Sigma_g$ with two boundary components as shown in Figure 4. Let us denote this embedding – after smoothing the corners – by $\phi$. For a pictorial description of the embedding $\phi$ we refer the reader to Figure 4. We claim that the embedding $\phi : \Sigma_g \to \mathbb{C}P^2$ is both flexible and in standard position. Let us now establish this claim.

The claim that the embedding is flexible is already established in [13] Theorem: 3.1. Let us briefly review the argument. First of all, notice that every Lickorish generator $\gamma$ of $\Sigma_g$ embedded in $\mathbb{C}P^2$ via $\phi$ has – up to an isotopy – a Hopf annulus neighborhood which is contained in $S^3 \times \{\frac{3}{2}\} \subset \mathbb{C}P^2$. Next, recall that the mapping class group is generated by a product of Dehn twists along Lickorish generators, and in $S^3$ there exists a diffeomorphism isotopic to the identity which induces a Dehn twist on a given Hopf annulus fixing its boundary point wise. In the proof of [13] Theorem: 3.1 it is shown that this implies that there exists a diffeomorphism of $\mathbb{C}P^2$ isotopic to the identity which induces a Dehn twist along a Lickorish generator of $\phi(\Sigma_g)$. The claim now follows by successive application of isotopies of $\mathbb{C}P^2$ inducing a Dehn twists on Lickorish generators. See also [13] for the necessary details.

Let us now show that the embedding is in a standard position. First of all notice that, by very construction, any simple closed curve on $\phi(\Sigma_g)$ can be isotoped on the surface $\phi(\Sigma_g)$ so that it is contained in $\phi(\Sigma) \cap S^3 \times \{\frac{3}{2}\}$. We claim that any Lickorish generator of $\phi(\Sigma_g)$ as well as any curve which does not meet handles of $\phi(\Sigma_g)$ satisfy both the properties necessary for an embedding to be in a standard position. This is because:

1. All curves mentioned in the claim are unknots in $S^3 \times \{\frac{3}{2}\}$ hence they bound a disk in $S^3 \times [1, \frac{3}{2}]$, that meets $\phi(\Sigma)$ only in the given curve.
2. Any curve $\gamma$ mentioned in the claim admits a neighborhood $N(\mathcal{C})$ in $\phi(\Sigma_g)$ which is a Hopf band in $S^3 \times \{\frac{3}{2}\}$.\(^2\)

\(^2\) Recall that, we say that a simple closed curve $p$ does not meet the handle $H_k$ provided it does not intersect the curve $a_k$ depicted Figure 3.
Let us first provide a proof of the theorem, when $\Sigma = \emptyset$ is not meeting any handle, satisfies both the properties necessary for a surface to be in the standard position.

Definition 4.4: Let $(M, \pi : M \to \Sigma)$ be a Lefschetz fibration. An embedding $f : M \to \mathbb{CP}^2 \times \mathbb{CP}^1$ of a manifold $M$ into a manifold $\mathbb{CP}^2 \times \mathbb{CP}^1$ is said to be a **Lefschetz fibration embedding** provided $\pi_2 \circ f = i \circ \pi$, where $i$ is an inclusion of $\mathbb{CP}^2$ in $\mathbb{CP}^1$ when $\partial M \neq \emptyset$, otherwise it is the identity.

Figure 4. The figure depicts the embedding of the surface $\Sigma_g$ which is flexible as well as in the standard position. Figure depicts the collar $S^3 \times [1, 2] \subset \mathbb{CP}^2$ with dashed lines representing $S^3$ at levels 1, 2 and $\frac{3}{2}$. Notice that the surface is embedded disjoint from the zero section which in the figure consist of the red dashed cylinder $I \times [1, 2]$ with two caps. The disk $D_1$ - depicted in the figure in green color - contained in $S^3 \times \{1\}$ and the core of 1–handle attached along $U \times \{2\}$.

It follows from both the properties listed above that any curve $C$, which is either a Lickorish generator or is not meeting any handle, satisfies both the properties necessary for a surface to be in the standard position.

Now, according to Proposition 4.3, given any curve $C$, there exists a diffeomorphism of $\phi(\Sigma_g)$ which send $C$ to a curve which is either a Lickorish generator or it does not meet any handle. Since the embedding $\phi$ of $\Sigma_g$ is flexible in $\mathbb{CP}^2$, the claim that the embedding is also in a standard position follows.

4.2. The existence of Lefschetz fibration embedding.

We are now in a position to state and prove our main result regarding **Lefschetz fibration embeddings**.

We denote the map $\mathbb{CP}^2 \times \mathbb{CP}^1$ to $\mathbb{CP}^1$ corresponding to the projection on the second factor by $\pi_2$.

**Definition 4.4** (Lefschetz fibration embedding). Let $(M, \pi : M \to \Sigma)$ be a Lefschetz fibration, where $\Sigma$ is 2–disk or $\mathbb{CP}^1$. An embedding $f : M \to \mathbb{CP}^2 \times \mathbb{CP}^1$ of a manifold $M$ into a manifold $\mathbb{CP}^2 \times \mathbb{CP}^1$ is said to be a **Lefschetz fibration embedding** provided $\pi_2 \circ f = i \circ \pi$, where $i$ is an inclusion of $\mathbb{CP}^2$ in $\mathbb{CP}^1$ when $\partial M \neq \emptyset$, otherwise it is the identity.

**Theorem 4.5.** Let $M$ be an orientable smooth 4–manifold. If $\pi : M \to \Sigma$, where $\Sigma = \mathbb{CP}^1$ or a 2–disk $\mathbb{CP}^2$, is a simplified Lefschetz fibration (SLF) of $M$ having genus of generic fiber bigger than 1, then there exists a Lefschetz fibration embedding of $(M, \pi)$ in $(\mathbb{CP}^2 \times \mathbb{CP}^1, \pi_2)$.

**Proof.** Let us first provide a proof of the theorem, when $\Sigma = \mathbb{CP}^1$. In this case, $M$ is a closed orientable manifold admitting a SLF $\pi : M \to \mathbb{CP}^1$.

Let $c_1, c_2, \ldots, c_k$ be $k$ critical points of the Lefschetz fibration $(M, \pi)$. Since the Lefschetz fibration is simple, $\pi(c_1) = p_1, \pi(c_2) = p_2, \ldots$, and $\pi(c_k) = p_k$ are distinct points on $\mathbb{CP}^1$. Also, recall that that the genus $g$ of the generic fiber is bigger than or equal to 2. Let $\gamma_i$ be the vanishing cycle corresponding to the critical point $c_i$ on a generic fiber $\Sigma_g$ of the SLF.

Let $U_i$ be the open ball in $M$ around $c_i$ such that on $U_i$ we have co-ordinates $(z_1, z_2)$ such that $\pi$ in this co-ordinates is given by $(z_1, z_2) \to z_1, z_2$. Let $\overline{D}_i = \pi(U_i) \subset \mathbb{CP}^1$. Let $D_i$ be an open disk containing $p_i$ with $\overline{D}_i \subset \overline{D}_i$. 
Embeddings of 4–Manifolds in \( \mathbb{C}P^3 \)

The figure depicts part of a Lefschetz fibration \((M, \pi)\) over a disk embedded as Lefschetz fibration in the (Lefschetz) fibration \( \mathbb{C}P^2 \times D^2 \rightarrow D^2 \). The embedding is such that the generic fiber of \((M, \pi)\) is a flexible embedding in the standard position in \( \mathbb{C}P^2 \setminus \mathbb{C}P^1 \). The curves on the surface depicts the vanishing cycles \( \gamma_i \)'s.

First of all consider an embedding \( \phi \) of the fiber \( \Sigma_g \) in \( \mathbb{C}P^2 \) which is flexible and in a standard position. Recall that the existence of such an embedding is the content of Lemma 4.3.

Using the flexibility of the embedding \( \phi \), we first produce an embedding \( \hat{f} \) of \( M \sqcup \cup_{i=1}^k \pi^{-1}(D_i) \) in the manifold \( \mathbb{C}P^2 \times (\mathbb{C}P^1 \setminus \cup_{i=1}^k D_i) \) such that the following diagram commutes:

\[
\begin{array}{ccc}
M \setminus \cup_{i=1}^k \pi^{-1}(D_i) & \xrightarrow{\hat{f}} & \mathbb{C}P^2 \times (\mathbb{C}P^1 \setminus \cup_{i=1}^k D_i) \\
\downarrow \pi & & \downarrow \pi_2 \\
\mathbb{C}P^1 \setminus \cup_{i=1}^k D_i & \xrightarrow{Id} & \mathbb{C}P^1 \setminus \cup_{i=1}^k D_i.
\end{array}
\]

In order to do this, we observe that the embedding of \( \Sigma_g \) in \( \mathbb{C}P^2 \) is flexible. This implies given an element \( \psi \in \text{MCG}(\Sigma_g) \) there exists an embedding \( \psi \) of the mapping torus, \( \mathcal{M}T(\Sigma_g, \psi) \), in the trivial fiber bundle \( \pi_2 : \mathbb{C}P^2 \times S^1 \rightarrow S^1 \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{M}T(\Sigma_g, \psi) & \xrightarrow{\psi} & \mathbb{C}P^2 \times S^1 \\
\downarrow \pi & & \downarrow \pi \\
S^1 & \xrightarrow{Id} & S^1.
\end{array}
\]

Next, considering \( \partial D_i \subset \mathbb{C}P^1 = S^1 \) then it follows from the existence of an embedding \( \phi \) satisfying the diagram (2) that there is an embedding of the mapping torus \( \mathcal{M}T(\Sigma_g, \tau_{\gamma_i}) \) in \( \mathbb{C}P^2 \times \partial D_i \), where \( \tau_{\gamma_i} \) denotes the Dehn twist along the curve \( \gamma_i \). Now take arcs connecting a point on \( \partial D_i \) to a fixed regular point \( p \) for the map \( \pi \) in \( \mathbb{C}P^1 \) as depicted in Figure 5.

Since the Lefschetz fibration \((M, \pi)\) restricted to a regular neighborhood \( \mathcal{N} \) of \( D_i \)'s together with arcs connecting them satisfies the following:

(1) \( \pi^{-1}(\partial D_i) \) is the mapping torus \( \mathcal{M}T(\Sigma_g, \tau_{\gamma_i}) \),
(2) $M$ restricted to $\partial N$ is the manifold $\Sigma_g \times S^1$ as $\prod_{i=1}^k \tau_{\gamma_i} = Id$ in $\text{MCG}(\Sigma_g)$,

(3) the complement of $N$ is a disk in $\mathbb{C}P^1$,

(4) and the genus $g \geq 2$,

we get the required embedding $\hat{f}$ such that the diagram (1) commutes.

Our next step is to show how to extend this embedding to produce a Lefschetz fibration embedding of $f$ of $M$ in $\mathbb{C}P^2 \times \mathbb{C}P^1$. For this the property that the embedding $\phi$ of $\Sigma_g$ is also in the standard position is required.

Since the embedding $\phi$ is in a standard position – by the definition of an embedding in a standard position given in [4,2] – there exists an embedding of $\phi_{\gamma_i} : \mathbb{C}^2 \hookrightarrow \mathbb{C}P^2$ which satisfies the second property listed in Definition [4,2].

Next, for each critical point $c_i$, we claim that, we have following commutative diagram:

\[
\begin{align*}
U_i & \subset M \quad \phi_i \to \mathbb{C}^2 \quad i \to \mathbb{C}^2 \times \mathbb{C} \quad f_{c_i} \to \mathbb{C}P^2 \times \mathbb{C}P^1 \\
\pi & \downarrow \quad \phi \downarrow \quad g \downarrow \quad \pi_2 \downarrow \\
\hat{D}_i & \quad \hat{D}_i
\end{align*}
\]

Where,

(1) $\phi_i : U_i \subset M \to \mathbb{C}^2$ and $\phi : \hat{D}_i \subset \mathbb{C}P^1 \to \mathbb{C}$ are orientation preserving parametrizations around critical point $c_i$ of $\pi$ and $\pi(c_i)$ respectively such that left square commutes in the diagram above,

(2) $i : \mathbb{C}^2 \to \mathbb{C}^2 \times \mathbb{C}$ and $g : \mathbb{C}^2 \to \mathbb{C}$ are defined as $i(z_1, z_2) = (z_1, z_2, 0)$ and $g(z_1, z_2) = z_1z_2$,

(3) $f_{c_i} : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}P^2 \times \mathbb{C}P^1$ and $P : \mathbb{C}^2 \times \mathbb{C} \to \mathbb{C}$ are defined as $f_{c_i}(z_1, z_2, z_3) = (\phi_{\gamma_i}(z_1, z_2), \phi^{-1}(z_1z_2 + z_3))$ and $P(z_1, z_2, z_3) = z_1z_2 + z_3$.

The commutativity of the middle square is follows directly from definitions of maps $g, i$ and $P$. Also the commutativity of the last square is clear by the definition of the map $f_{c_i}$. Next, we observe that the commutative diagram allows us to extend the embedding $\hat{f}$ to the embedding $\hat{f}_{c_i}$ of $M \setminus \bigcup_{i=1}^k \pi^{-1}(D_i) \cup U_i$. This is possible because $\hat{f}$ and $f_{c_i} \circ i \circ \phi_i$ agree on the overlapping region of the domain. Hence, $\hat{f}$ and $f_{c_i} \circ i \circ \phi_i$ together defines a map $\hat{f}_{c_i}$.

Let us now notice that this allows us to extend the embedding $\hat{f}_{c_i}$ to an embedding $\hat{f}_{c_j}$ of $W_{c_j} = M \setminus \left(\bigcup_{i=1}^{i-1} \pi^{-1}(D_i) \bigcup_{l=i+1}^k \pi^{-1}(D_l)\right)$ in $\mathbb{C}P^2 \times \mathbb{C}P^1$ such that the following diagram commutes:

\[
\begin{align*}
W_{c_i} & \quad \hat{f}_{c_i} \to \hat{f}_{c_j}(W_{c_j}) \subset \mathbb{C}P^2 \times \mathbb{C}P^1 \\
\pi & \quad \pi_2 \downarrow \\
\pi(W_{c_i}) & \subset \mathbb{C}P^1 \quad \text{Id} \to \pi_2(\hat{f}_{c_j}) = \pi(W_{c_j}).
\end{align*}
\]

Observe that by construction the embeddings $\hat{f}_{c_i}$ and $\hat{f}_{c_j}$ agree on on $W_{c_i} \cap W_{c_j}$. Since $M = \bigcup_{i=1}^k W_{c_i}$, we get an embedding $f$ of $M$ with required properties. This completes our argument in case when $\Sigma = \mathbb{C}P^1$.

The case, when $\Sigma = \mathbb{D}^2$ the argument is essentially same. The only difference is that the product $\prod_{i=1}^k \tau_{\gamma_i}$ need not be the identity. However, notice that since $\Sigma = \mathbb{D}^2$ the same argument produces an embedding such that the monodromy along $\partial \mathbb{D}^2$ is precisely $\prod_{i=1}^k \tau_{\gamma_i}$.
5. Embedding of orientable 4-manifolds in $\mathbb{CP}^2 \times \mathbb{CP}^1$ via SBLF

The purpose of this section is to establish Theorem 1.1. As mentioned earlier, we will use the SBLF decomposition of a closed orientable smooth 4-manifold for the same. We first need the following:

**Definition 5.1** (1-fold simple singular fibration). Let $(M, \partial M)$ be an orientable smooth 4-manifold with boundary and let $f : M \to [-1, 1] \times S^1$ be a smooth surjective map which satisfies the following:

1. There exists a unique embedded circle $Z_f$ in $M$ of 1-fold singularity for $f$ such that $f(Z_f)$ is an embedded circle in $[-1, 1] \times S^1$ which is ambiently isotopic to the circle $\{0\} \times S^1$.
2. Every $x \in M \setminus Z_f$ is a regular value for the map $f$.

Then, we say that $f : M \to [-1, 1] \times S^1$ is a 1-fold simple singular fibration.

**Remark 5.2.**

(a) Since $f : M \to [-1, 1] \times S^1$ has a unique embedded singular locus $Z_f$ which projects to a circle $C$ isotopic to $\{0\} \times S^1$ the inverse image of any regular value is a closed surface $\Sigma$ whose genus is either $k$ or $k + 1$ for some $k \in \mathbb{N} \cup \{0\}$. We call a fiber with genus $k$ as a lower genus fiber.

(b) Observe that as we cross the $f(Z_f)$ a round 1-handle is added to a manifold $N$ diffeomorphic to $\Sigma \times A$ where $A$ is some annulus.

(c) We will always use the convention that fiber over $\{-1\} \times S^1$ have lower genus.

**Lemma 5.3.** Let $(M, \partial M)$ be an orientable smooth 4-manifold with boundary and $f : M \to [-1, 1] \times S^1$ be a simple singular fibration. Then there exists an embedding $\psi : M \to \mathbb{CP}^2 \times [-1, 1] \times S^1$ such that following diagram commutes:

$$
\begin{array}{ccc}
M & \xrightarrow{\psi} & \mathbb{CP}^2 \times [-1, 1] \times S^1 \\
\downarrow f & & \downarrow \pi_2 \\
[-1, 1] \times S^1 & \xrightarrow{Id} & [-1, 1] \times S^1.
\end{array}
$$

Moreover, an embedding $\psi$ can be chosen such that restriction of $\psi$ to a regular fiber of $f$ gives an embedding which is in the standard position.

**Proof.** Let $(x, y, z, \theta)$ be co-ordinates on a tubular neighborhood $B^3 \times S^1$ of the singular locus $Z_f$ of $f$ such that the map $f$ sends $(x, y, z, \theta)$ to $(-x^2 + y^2 + z^2, \theta)$. As before, let $B^4(0, 1) \subset B^4(0, 2)$ be the unique 0-handle in the standard handle decomposition of $\mathbb{CP}^2$ which consist of the unique 0-handle $B(0, 2)$ on which a 2-handle is added along an unknot in $\partial B(0, 2)$ with framing +1 and a unique 4-handle attached to the boundary of $B(0, 2)$ union the 2-handle. Recall that this is the handle decomposition of $\mathbb{CP}^2$ we work with to establish Lemma 4.3.

Let us embed $B^3 \times S^1$ in $B^4(0, 1) \times [-1, 1] \times S^1$. The embedding $\hat{\psi}_1 : B^3 \times S^1 \to B^4(0, 1) \times [-1, 1] \times S^1$ is defined as $\hat{\psi}_1(x, y, z, \theta) = (x, y, z, 0, -x^2 + y^2 + z^2, \theta)$. We can see $\hat{\psi}_1$ is defined such that following diagram commutes:

$$
\begin{array}{ccc}
B^3 \times S^1 \subset M & \xrightarrow{\hat{\psi}_1} & B^4(0, 1) \times [-1, 1] \times S^1 \subset \mathbb{CP}^2 \times [-1, 1] \times S^1 \\
\downarrow f & & \downarrow \pi_2 \\
[-1, 1] \times S^1 & \xrightarrow{Id} & [-1, 1] \times S^1.
\end{array}
$$

Next, consider $\mathbb{CP}^2 \setminus B^4(0, 1)$. Let $g$ be the lower genus fiber for the fibration associated to $M$. Embed the surface $\Sigma_g$ of genus $g$ with two boundary components in $\mathbb{CP}^2$ such that the embedded surface satisfies the following:

1. The embedding is both flexible and in the standard position.
2. $\Sigma_g \cap (\partial \mathbb{CP}^2 \setminus B^4(0, 1)) = \{(x, y, z, 0) \in \partial B^4(0, 1)| -x^2 + y^2 + z^2 = 0\}$. 


Remark 5.5

Let $M$ implies that we have an embedding of that embedding of each piece can be arranged such that in the overlapping region they agree. This clearly

\[\begin{align*}
\text{Theorem 5.4.} & \quad \text{Let } M \text{ be an orientable closed smooth 4–manifold. Then there exists an embedding } \psi : M \to CP^2 \times CP^1. \\
\text{Proof.} & \quad \text{Let } M \text{ be a closed oriented 4–manifold. By Theorem 2.12 there exists a smooth map } f : M \to CP^1 \text{ which defines SBLF on } M \text{ and lower genus fiber } \Sigma_g \text{ of } f \text{ have genus bigger than 1. Therefore, We have a decomposition of } M, M = X_1 \cup X_2 \cup \Sigma_g \times D_2 \text{ satisfying the following:} \\
& \quad \text{(1) } X_1 = f^{-1}(D_1) \text{ with where } D_1 \text{ is a disc in } CP^1 \text{ such that in the interior of } D_1 \text{ contains all Lefschetz critical values of } f. \\
& \quad \text{(2) } f \text{ restricted to } X_2 \text{ is 1-fold singular fibration.} \\
& \quad \text{(3) } \Sigma_g \times D_2 = f^{-1}(D_2), \text{ where } D_2 \text{ is a disc in } CP^1 \text{ containing no critical points of } f \text{ with } \{-1\} \times S^1 = \partial D_2. \\
\text{It follows from Theorem 4.5 and Lemma 5.5 that each piece of } M \text{ embeds in } CP^2 \times CP^1. \text{ Also, it is clear that embedding of each piece can be arranged such that in the overlapping region they agree. This clearly implies that we have an embedding of } M \text{ in } CP^2 \times CP^1 \text{ as claimed.} \square
\end{align*}\]

Remark 5.5.

(a) The embedding $\psi : M \to CP^2 \times CP^1$ produced in Theorem 5.4, satisfies $\psi \circ \pi_2 = f$, where $f : M \to CP^1$ is SBLF associated to $M$ and $\pi_2 : CP^2 \times CP^1 \to CP^1$ is projection onto second factor of $CP^2 \times CP^1$. In this case, the embedding $\psi$ is termed as SBLF embedding.

Figure 6. Simple Lefschetz fibration embedding

Since the surface $\Sigma_g$ is flexible in $CP^2 \setminus B^4(0, 1)$ there exist an embedding of any fiber bundle $V^4$ over $[-1, 1] \times S^1$ with fiber $\Sigma_g$ in $CP^2 \times [-1, 1] \times S^1$. This is because, as before, such fiber bundles are determined by an element of the mapping class group of $\Sigma_g$ and $\Sigma_g$ being flexible implies that this element of the mapping class group is conjugated by a diffeomorphism of $CP^2 \setminus B^4(0, 1)$ which is isotopic to the identity.

Observe that this produces an embedding of $M \setminus B^3 \times S^1$ in $CP^2 \setminus B^3(0, 1) \times [-1, 1] \times S^1$. Call this embedding $\widehat{F}$.

Now observe that because of the property (2) of embedding of the surface $\Sigma_g$ listed above there exists and embedding $\psi$ of $M$ in $CP^2 \times [-1, 1] \times S^1$ which satisfies the following:

1. $\psi$ agrees with $\widehat{F}$ restricted to the complement of a collar neighborhood of $\partial B^3 \times S^1$, where we regard $\partial B^3 \times S^1$ as the boundary component of $M \setminus B^3 \times S^1$ arising when we remove $B^3 \times S^1$ from the interior of $M$.
2. $\psi$ agrees with $\widehat{\psi}$ when restricted to the neighborhood $B^3 \times S^1$ of $Z_f$.

Clearly, $\psi$ is the required embedding establishing the claim. \hfill \square
6. Embeddings in \( \mathbb{C}P^3 \)

Let us now establish Theorem 1.1. As mentioned in the introduction, the first step of the proof involves construction of a specific SBLF on \( M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \). We then use this SBLF to produce an embedding of \( M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \) in \( \mathbb{C}P^2 \times \mathbb{C}P^1 \) using Theorem 5.4. Finally, we observe that \( \mathbb{C}P^2 \times \mathbb{C}P^1 \) is diffeomorphic to the blowup \( B_{\mathbb{C}P^1}(\mathbb{C}P^3) \) of \( \mathbb{C}P^3 \) along \( \mathbb{C}P^1 \) hence there is an embedding of \( M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \) in the blowup of \( \mathbb{C}P^3 \) along \( \mathbb{C}P^1 \). Furthermore, we show that this embedding can be constructed such that when we blowdown \( B_{\mathbb{C}P^1}(\mathbb{C}P^3) \), we get an embedding of \( M \) in \( \mathbb{C}P^3 \).

Let us begin the section by reviewing notions related to blowup and blowdown.

6.1. Lefschetz pencil and blowup and blowdown of 4–manifolds.

**Definition 6.1** (Generalized Lefschetz pencil). Let \( M \) be a smooth 4–manifold. A generalized Lefschetz pencil associated to \( M \) is a map \( \pi : M \setminus B \to \mathbb{C}P^1 \) such that the following properties are satisfied:

1. \( B \) is finite.
2. \( \pi : M \setminus B \to \mathbb{C}P^1 \) is a Lefschetz fibration.
3. For every point \( b \in B \) there are parametrizations – not necessarily preserving orientations – \( \phi : U \subset M \to \mathbb{C}^2 \) and \( \psi : V \subset \mathbb{C}P^1 \to \mathbb{C} \) that satisfies the following:
   a. \( b \in U \) and \( \phi(b) = 0 \in \mathbb{C}^2 \)
   b. \( f(b) \in V \) and \( \psi(f(b)) = 0 \in \mathbb{C} \)
   c. For the map \( g : \mathbb{C}^2 \setminus \{0\} \to \mathbb{C} \setminus \{0\} \) given by \( g(z_1, z_2) = \frac{\overline{z}_2}{z_1} \), the following diagram commutes:

\[
\begin{array}{ccc}
U \setminus \{b\} & \xrightarrow{\phi} & \mathbb{C}^2 \setminus \{0\} \\
\downarrow{\pi} & & \downarrow{g} \\
V \setminus \{f(b)\} & \xrightarrow{\psi} & \mathbb{C} \setminus \{0\}
\end{array}
\]

In this case, we call \( B \) as a base locus of a generalized Lefschetz pencil associated to \( M \).

**Remark 6.2.**

(a) We would like to emphasis that the notion of generalized Lefschetz pencil defined above is weaker than the notion of Lefschetz pencil well known in the literature. Generally one demands that the parametrizations \( \phi : U \subset M \to \mathbb{C}^2 \) and \( \psi : V \subset \mathbb{C}P^1 \to \mathbb{C} \) are orientation preserving in Definition 6.1.

(b) If the fibration \( \pi : M \setminus B \to \mathbb{C}P^1 \) is simplified Lefschetz fibration, the pencil is termed as generalized simplified Lefschetz pencil.

(c) In case, we allow the fibration \( \pi : M \setminus B \to \mathbb{C}P^1 \) to be achiral Lefschetz fibration, the pencil is termed as generalized achiral Lefschetz pencil.

(d) It is clear that we can define generalized simplified achiral Lefschetz pencil and generalized simplified broken achiral Lefschetz pencil in a similar way (see [5]).

(e) If the fibration \( \pi : M \setminus B \to \mathbb{C}P^1 \) is simplified broken Lefschetz fibration and the parametrizations \( \phi : U \subset M \to \mathbb{C}^2 \) and \( \psi : V \subset \mathbb{C}P^1 \to \mathbb{C} \) are orientation preserving, the pencil is termed as simplified broken Lefschetz pencil (SBLP).

**Definition 6.3** (Generalized blowup and blowdown). Let \( M \) and \( \widehat{M} \) be two oriented smooth 4–manifolds. Then we say that \( \widehat{M} \) is a blowup of \( M \) at point \( p \in M \) provided:

1. There is an embedded \( \mathbb{C}P^1 \) in \( \widehat{M} \) such that the self intersection of \( \mathbb{C}P^1 \) is \( \pm 1 \).
2. There is a orientation preserving smooth map \( f : \widehat{M} \to M \) such that \( f(\mathbb{C}P^1) = \{p\} \) and \( f : \widehat{M} \setminus \mathbb{C}P^1 \to M \setminus \{p\} \) is a diffeomorphism.

The manifold \( M \) is termed as a blowdown of \( \widehat{M} \) along \( \mathbb{C}P^1 \). We call the \( \mathbb{C}P^1 \) embedded in \( \widehat{M} \) as exceptional divisor.

**Remark 6.4.**
(a) In the operation of standard blowdown of \(\widehat{M}\) along \(CP^1\), the self intersection \(CP^1\cdot CP^1 = -1\) is necessary.

(b) Topologically \(\widehat{M}\) can be thought as a connected sum \(M\#CP^2\) and \(M\) can be thought as removing a tubular neighborhood \(N(CP^1)\) of \(CP^1\) from \(\widehat{M}\) and gluing a closed 4-ball \(B^4\).

(c) It is well known that the Lefschetz pencil \(\pi : M \setminus B \to CP^1\) extends to a Lefschetz fibration on the blowup \(\widehat{M} = M\#CP^2\).

(d) Given a \(M\#CP^2\) and the \(CP^1 \subset M\#CP^2\) corresponding to the standard \(CP^1 \subset CP^2\) we can obtain \(M\) from \(M\#CP^2\) by removing \(CP^1\) and replacing it by a 4-ball to obtain \(M\). By slight abuse of notation, we will also term this as a blowdown of \(M\#CP^2\) along \(CP^1\).

(e) It is easy to see that given a simplified achiral Lefschetz pencil \(\pi : M \setminus B \to CP^1\) with base locus \(B\), blowups of the pencil along the base locus produces a simplified achiral Lefschetz fibration of \(M\#CP^2\). Same statement holds for a simplified achiral broken Lefschetz pencil.

6.2. Blowup and blowdown of \(CP^3\) along \(CP^1\).

Consider \(CP^3\) and a standard \(CP^1\) embedded in it. Fix a local trivialization \(D^2 \times C^2\) of the normal bundle 
\(N(CP^1)\) of \(CP^1\) in \(CP^3\).

Now consider \(D^2 \times C^2 \times CP^1\) and a subset \(V\) of \(D^2 \times C^2 \times CP^1\) given by,

\[V = \{(w, z_1, z_2, l)| \|z_1\|^2 + \|z_2\|^2 \leq 1 \text{ and } (z_1, z_2) \in l\}\]

where a point \(l\) in \(CP^1\) is identified with the complex linear subspace corresponding to that point.

Now, observe that the complement of \(D^2 \times \{(0, 0)\} \times CP^1\) in \(V\) can be identified with the complement of \(D^2 \times \{(0, 0)\}\) in \(D^2 \times C^2\). By the (topological) blowup of \(CP^3\) along \(CP^1\) we mean the operation of removing \(D^2 \times \{(0, 0)\}\) and replacing it with the interior of \(V\) for a finite collection \(V_s\) of trivializations of the bundle 
\(N(CP^1)\). \n
Remark 6.5.

(a) It is easy to see that the manifold \(B_{CP^1}(CP^3)\) is diffeomorphic to \(CP^2 \times CP^1\). Hence, from now on we will regard \(CP^2 \times CP^1\) as a blowup of \(CP^3\) along \(CP^1\).

(b) Exceptional divisor of \(B_{CP^1}(CP^3)\) is the union of \(D^2 \times \{(0, 0)\} \times CP^1\) over a finite collection \(V_s\) of trivializations of the bundle 
\(N(CP^1)\). It can be shown that the exceptional divisors is diffeomorphic to \(CP^1 \times CP^1\).

(c) The above notion of blowup is a particular case of blowup of a manifold along a submanifold. We refer [11, p. 196,602] for a detailed discussion on blowups.

By a blowdown of \(B_{CP^1}(CP^3) = CP^2 \times CP^1\) we will mean the process of removing the interior of \(V\) and replacing it by \(N(CP^1)\) to obtain \(CP^3\) from \(B_{CP^1}(CP^3)\). Since \(CP^2 \times CP^1\) is diffeomorphic to \(B_{CP^1}(CP^3)\) the following process will also be termed as (topological) blowdown:

Consider \(CP^1 \times CP^1\) embedded in \(CP^2 \times CP^1\) via the standard embedding of \(CP^1\) in \(CP^2\). Observe that a tubular neighborhood \(N(CP^1 \times CP^1)\) in \(CP^2 \times CP^1\) is diffeomorphic to \(V\) by a diffeomorphism that sends \(CP^1 \times CP^1\) contained in \(N(CP^1 \times CP^1)\) to the exceptional divisor \(CP^1 \times CP^1\) contained in \(V\). Hence we can remove a tubular neighborhood of \(N(CP^1 \times CP^1)\) form \(CP^2 \times CP^1\) and replace it by \(N(CP^1)\) to get \(CP^3\) from \(CP^2 \times CP^1\).

We say that \(CP^3\) is obtained from \(CP^2 \times CP^1\) by blowing down along a \(CP^1 \times CP^1\) provided we perform a blowdown of \(CP^2 \times CP^1\) as described above to obtain \(CP^3\).

We end this subsection with the following:

Lemma 6.6. Let \(M\#CP^2\#CP^2\) be a smooth manifold. Let \(\pi : M\#CP^2\#CP^2 \to CP^1\) be a SBLF which satisfies that the fibration agrees with the standard fibration in a tubular neighborhood of exceptional divisors. If there exists a SBLF embedding of \(M\#CP^2\#CP^2\) in \(B_{CP^1}(CP^3)\) such that each fiber of SBLF intersects the standard \(CP^1\) of the fiber \(CP^2\) in two fixed points, then there exist an embedding of \(M\) in \(CP^3\) such that the standard generalized pencil of \(CP^3\) induces the SBLF of \(M\#CP^2\#CP^2\).
Proof. Let $P^+$ and $P^−$ be two standard embeddings of $\mathbb{C}P^1$'s in manifold $\hat{M} = M \# \mathbb{C}P^2 \# \mathbb{C}P^2$ corresponding to $\mathbb{C}P^2$ and $\overline{\mathbb{C}P^2}$ respectively. Consider $\hat{M}$ as a blowup of $M$ done at two distinct points $p_1$ and $p_2$. This implies that $P^+$ and $P^−$ can be regarded as exceptional divisors of $\hat{M}$. Let $\psi : M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \to B_{\mathbb{C}P^1}(\mathbb{C}P^3)$ be a SBLF embedding satisfying the hypothesis of the lemma. Therefore, $\psi(P^+)$ and $\psi(P^−)$ are two distinct exceptional divisors of $B_{\mathbb{C}P^1}(\mathbb{C}P^3)$ corresponding to two distinct points on $\mathbb{C}P^3$ along which we have blown up $\mathbb{C}P^3$. Thus the blowdown of $B_{\mathbb{C}P^1}(\mathbb{C}P^3)$ along the exceptional divisor $\mathbb{C}P^1 \times \mathbb{C}P^1$ induces blowdown of $\hat{M}$ along $P^+$ and $P^−$. This clearly produces desired embedding of $M$ into $\mathbb{C}P^3$. \hfill $\square$

6.3. Proof of Theorem 1.1

Proof. Let $M$ be the given closed orientable 4-manifold. Consider the manifold $\hat{M} = M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ thought as a blowup of $M$ done at two distinct points $p_1$ and $p_2$. It is clear that $\hat{M}$ admits a pair of $\mathbb{C}P^1$'s – say $P^+$ and $P^−$ such that $P^+ \cap P^+ = 1$ while $P^− \cap P^− = −1$.

It is easy to see that there exists a surface $\Sigma$ of genus $g$ embedded in $\hat{M}$ which satisfies the following:

1. The genus $g$ of $\Sigma$ is bigger than 1
2. $\Sigma \cap P^\pm = \pm 1$
3. $\Sigma$ is a connected surface.
4. $\Sigma \cap \Sigma = 0$

Consider a smooth function $f : \hat{M} \to \mathbb{C}P^1$ which satisfies the following:

1. $f$ near tubular neighborhoods of $P^\pm$ is the standard bundle projections associated to the tubular neighborhoods.
2. There exists a regular value $p$ of $f$ in $\mathbb{C}P^1$ such that $f^{-1}\{p\} = \Sigma$

Since $p$ is a regular value for $f$ there exist an open disk neighborhood $U$ of $p$ in $\mathbb{C}P^1$ such that every point $q \in U$ is a regular value. Remove the neighborhood from $\mathbb{C}P^1$ to produce a map $\widetilde{g} : \hat{M} \to (\mathbb{D}^2, \partial \mathbb{D}^2)$, where $\hat{M} = M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \setminus f^{-1}(U)$, which is fibration sending the boundary of $\hat{M}$ to the boundary of $\mathbb{D}^2$.

Next, apply the relative version of Thom transversality result to convert the map $\widetilde{g}$ into an indefinite fibration $\tilde{h}$ which agrees with $\widetilde{g}$ near the boundary $\partial \hat{M}$.

Now observe that Theorem 2.12 implies that there exists a SBLF $\pi_{\hat{M}} : \hat{M} \to (\mathbb{D}, \partial \mathbb{D})$, where $\pi_{\hat{M}}$ agrees with $\tilde{h}$ near the boundary. Extend this SBLF to a SBLF $\pi$ on $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ such that the following properties are satisfied:

1. $\pi$ agrees with the standard fibration near $P^\pm$.
2. $\pi$ restricted to $f^{-1}(U) = f$.

Now, by Theorem 5.4 there exist SBLF embedding of $\hat{M}$ in $\mathbb{C}P^2 \times \mathbb{C}P^1$. Since $\mathbb{C}P^2 \times \mathbb{C}P^1$ is diffeomorphic to $B_{\mathbb{C}P^1}(\mathbb{C}P^3)$, we get an embedding of $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ in $B_{\mathbb{C}P^1}(\mathbb{C}P^3)$.

In addition, notice that we can ensure that each fiber of the SBLF associated to $M \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ intersects the standard $\mathbb{C}P^1$ of a fiber $\mathbb{C}P^2$ of the trivial fibration $B_{\mathbb{C}P^1}(\mathbb{C}P^3) \to \mathbb{C}P^1$ in a pair of canceling points.

Finally, blowdown $B_{\mathbb{C}P^1}(\mathbb{C}P^3)$ along its exceptional divisor. Observe that Lemma 6.6 implies that blowdown produces an embedding of $M$ in $\mathbb{C}P^3$ such that the standard Lefschetz pencil of $\mathbb{C}P^3$ induces a SBLP on $M$. \hfill $\square$

References

1. D. Auroux, S. K. Donaldson, L. Katzarkov. Singular Lefschetz pencils. Geom. Topol. Vol. 9 (2005) 1043–1114.
2. W. Barth, W. Hulek, C. Peters, and A. van de Van. Compact complex surfaces. Springer-Verlag, second edition, (2004).
3. I. Baykur. Existence of broken Lefschetz fibrations. Int. Math. Res. Not. IMRN 2008, Art. ID rnn 101, 15 pp.
4. F. Benson and D. Margalit. A primer on mapping class groups, Princeton Mathematical series, Vol. 49, Princeton University Press, (2012).
5. I. Baykur and S. Osamu, Simplified broken Lefschetz fibrations and trisections of 4-manifolds. Proc. Natl. Acad. Sci. USA 115 Vol. 43 (2018) 1089410900.
6. S. Cappell and J. Shaneson. Embedding and immersion of four-dimensional manifolds in $\mathbb{R}^6$. Geometric Top. Proc. of the 1977 Georgia Topology Conf. Academic Press, New York (1979), 301–305.
7. M. Dehn. *Die Gruppe der Abbildungsklassen*. Acta Math., 69(1), 135–206, 1938.
8. S. K. Donaldson. *Symplectic sub-manifolds and almost complex geometry*. Journal of Differential Geometry, Vol. 44 (4), (1996), 666–705.
9. S. K. Donaldson. *Lefschetz pencils on symplectic manifolds*. Journal of Differential geometry, Vol. 55 (1999), 205–236.
10. D. Gay and R. Kirby. *Constructing Lefschetz type fibration on 4–manifolds*, Geometry and Topology, Vol. 11(4), (2005), 2075–2115.
11. P. Griffiths and J. Harris. *Principles of algebraic geometry*. Reprint of the 1978 original, Wiley Classics Library, John Wiley and Sons, Inc., New York, 1994.
12. R. Gompf and A. I. Stipsicz. 4–manifolds and Kirby calculus. Graduate Studies in Mathematics, Vol. 20, American Mathematical Society, Providence, RI, (1999).
13. M. W. Hirsch. *On imbedding differentiable manifolds in euclidean space*. Ann. of Math. Vol. 73(2), (1961), 566–571.
14. M. W. Hirsch. *On embedding 4–manifolds in R7*. Proc. Cambridge Philos. Soc., Vol. 61, (1965), 657–658.
15. S. Hirose and A. Yasuhara. *Surfaces in 4–manifolds and their mapping class groups*. Topology, Vol. 47(1) (2008), 41–50.
16. W. B. R. Lickorish. *A representation of orientable combinatorial 3–manifolds*. Ann. of Math., Vol. 76(2), (1962), 531540.
17. W. B. R. Lickorish. *A finite set of generators for the homeotopy group of a 2–manifold*. Proc. Cambridge Philos. Soc., Vol.60, (1964), 769–778.
18. D. M. Pancholi, S. Pandit and K. Saha. *Embeddings of 3–manifolds via open books*, [arXiv:1806.09784v2 [math.GT]].

Chennai Mathematical Institute, H1, SIPCOT IT Park, Kelambakkam (Chennai), India.
E–mail address: abhiyect@cmi.ac.in

The Institute of Mathematical Sciences, No. 8, CIT Campus, Taramani, Chennai 600113, India.
E–mail address: dishant@imsc.res.in
$\partial D^4 \times [1, 2]$