Global Hilbert Expansion for the Vlasov-Poisson-Boltzmann System

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**Abstract**

We study the Hilbert expansion for small Knudsen number $\epsilon$

\[
F^\epsilon = \sum_{n=0}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^\epsilon; \quad \nabla \phi^\epsilon = \sum_{n=0}^{2k-1} \varepsilon^n \nabla \phi_n + \varepsilon^k \nabla \phi_R^\epsilon \quad (k \geq 6)
\]

for the Vlasov-Boltzmann-Poisson system for an electron gas:

\[
\partial_t F^\epsilon + v \cdot \nabla_x F^\epsilon + \nabla_x \phi^\epsilon \cdot \nabla_v F^\epsilon = \frac{1}{\varepsilon} Q(F^\epsilon, F^\epsilon), \quad \Delta \phi^\epsilon = \int_{\mathbb{R}^3} F^\epsilon dv - \bar{\rho}, \quad |\phi^\epsilon| \to 0 \text{ as } |x| \to \infty.
\]

The zeroth order term local Maxwellian takes the form:

\[
F_0(t, x, v) = \frac{\rho_0(t, x)}{(2\pi \theta_0(t, x))^{3/2}} e^{-\frac{|v-u_0(t, x)|^2}{2\theta_0(t, x)}}, \quad \theta_0(t, x) = K \rho_0^{5/3}(t, x).
\]

Our main result states that if (1) is valid at $t = 0$, with smooth irrotational velocity $\nabla \times u_0(0, x) = 0$, $\int \rho_0(0, x) - \bar{\rho} dx = 0$, and $u_0(0, x)$ and $\rho_0(0, x) - \bar{\rho}$ sufficiently small, then (1) is valid for $0 \leq t \leq \varepsilon^{-\frac{3}{2k-2}}$, where $\rho_0(t, x)$ and $u_0(t, x)$ satisfy the Euler-Poisson system

\[
\partial_t \rho_0 + (u_0 \cdot \nabla) \rho_0 + \rho_0 \nabla \cdot u_0 = 0, \quad \rho_0 \partial_t u_0 + \rho_0 (u_0 \cdot \nabla) u_0 + \nabla K \rho_0^{5/3} - \rho_0 \nabla \phi_0 = 0, \quad \Delta \phi_0 = \rho_0 - \bar{\rho}.
\]

1 Introduction and Formulation

The dynamics of electrons in the absence of a magnetic field can be described by the Vlasov-Poisson-Boltzmann system (2) where $F(t, x, v) \geq 0$ is the number density function for the electron at time $t \geq 0$, position $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and velocity $v = (v_1, v_2, v_3) \in \mathbb{R}^3$. The self-consistent electric potential $\phi(t, x)$ is coupled with the distribution function $F$ through the Poisson equation. The constant ion background charge is denoted by $\bar{\rho} > 0$. The collision between particles is given by the standard Boltzmann collision operator $Q(G_1, G_2)$ with hard-sphere interaction:

\[
Q(G_1, G_2) = \int_{\mathbb{R}^3 \times S^2} |(u - v) \cdot \omega| \{G_1(v') G_2(u') - G_1(v) G_2(u)\} dud\omega,
\]
where \( v' = v - [(v-u) \cdot \omega] \omega \) and \( u' = u + [(v-u) \cdot \omega] \omega \). On the other hand, at the hydrodynamic level, the electron gas obeys the Euler-Poisson system (1), which is an important ‘two-fluid’ model for a plasma.

The purpose of this article is to derive the Euler-Poisson system (1) from the Vlasov-Poisson-Boltzmann system (2) as the Knudsen number (the mean free path) \( \varepsilon \) tends to zero. We consider the truncated Hilbert expansion (1). To determine the coefficients \( F_0(t, x, v), ..., F_{2k-1}(t, x, v) \); \( \phi_0(t, x, v), ..., \phi_{2k-1}(t, x, v) \), we plug the formal expansion (1) into the rescaled equations (2):

\[
\begin{align*}
\partial_t \big( \sum_{n=0}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^\varepsilon \big) + v \cdot \nabla_x \big( \sum_{n=0}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^\varepsilon \big) + \nabla_x \big( \sum_{n=0}^{2k-1} \varepsilon^n \phi_n + \varepsilon^k \phi_R^\varepsilon \big) \cdot \nabla_x \big( \sum_{n=0}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^\varepsilon \big) &= \frac{1}{\varepsilon} Q \big( \sum_{n=0}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^\varepsilon, \sum_{n=0}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^\varepsilon \big), \\
\Delta \big( \sum_{n=0}^{2k-1} \varepsilon^n \phi_n + \varepsilon^k \phi_R^\varepsilon \big) &= \int_{\mathbb{R}^3} \big( \sum_{n=0}^{2k-1} \varepsilon^n F_n + \varepsilon^k F_R^\varepsilon \big) dv - \bar{\rho}.
\end{align*}
\]

(1.1)

Now we equate the coefficients on both sides of the equation (1.1) in front of different powers of the parameter \( \varepsilon \) to obtain:

\[
\begin{align*}
\frac{1}{\varepsilon} : Q(F_0,F_0) &= 0, \\
\varepsilon^0 : \partial_t F_0 + v \cdot \nabla_x F_0 + \nabla_x \phi_0 \cdot \nabla_v F_0 &= Q(F_1,F_0) + Q(F_0,F_1), \\
\Delta \phi_0 &= \int_{\mathbb{R}^3} F_0 dv - \bar{\rho}, \\
&... \\
\varepsilon^n : \partial_t F_n + v \cdot \nabla_x F_n + \nabla_x \phi_0 \cdot \nabla_v F_n + \nabla_x \phi_n \cdot \nabla_v F_n &= \sum_{i+j=n+1}^{i+j=0} Q(F_i,F_j) - \sum_{i+j=n}^{i+j=1} \nabla_x \phi_i \cdot \nabla_v F_j, \\
\Delta \phi_n &= \int_{\mathbb{R}^3} F_n dv.
\end{align*}
\]

The remainder equations for \( F_R^\varepsilon \) and \( \phi_R^\varepsilon \) are given as follows:

\[
\begin{align*}
\partial_t F_R^\varepsilon + v \cdot \nabla_x F_R^\varepsilon + \nabla_x \phi_0 \cdot \nabla_v F_R^\varepsilon + \nabla_x \phi_R^\varepsilon \cdot \nabla_v F_R^\varepsilon - \frac{1}{\varepsilon} \{ Q(F_0,F_R^\varepsilon) + Q(F_R^\varepsilon,F_0) \} &= \varepsilon^{k-1} Q(F_R^\varepsilon,F_R^\varepsilon) + \sum_{i=1}^{2k-1} \varepsilon^{i-1} \{ Q(F_i,F_R^\varepsilon) + Q(F_R^\varepsilon,F_i) \} - \varepsilon^k \nabla_x \phi_R^\varepsilon \cdot \nabla_v F_R^\varepsilon \\
&- \sum_{i=1}^{2k-1} \varepsilon^i \{ \nabla_x \phi_i \cdot \nabla_v F_R^\varepsilon + \nabla_x \phi_R^\varepsilon \cdot \nabla_v F_i \} + \varepsilon^{k-1} A, \\
\Delta \phi_R^\varepsilon &= \int_{\mathbb{R}^3} F_R^\varepsilon dv,
\end{align*}
\]

where

\[
A = \sum_{i+j=2k, 1 \leq i,j \leq 2k-1} \varepsilon^{i+j-2k} Q(F_i,F_j) - \sum_{i+j=2k+1, 0 \leq i,j \leq 2k-1} \varepsilon^{i+j-2k+1} \nabla_x \phi_i \cdot \nabla_v F_j - \{ \partial_t F_{2k-1} + v \cdot \nabla_x F_{2k-1} \}.
\]

(1.4)

From the first condition, the \( \frac{1}{\varepsilon} \) step, in (1.2), we deduce that the first coefficient \( F_0 \) should be a local Maxwellian \( \omega = F_0 \) as given in (3), where \( \rho_0(t,x), u_0(t,x) \) and \( \theta_0(t,x) \) represent the macroscopic
density, velocity, and temperature fields respectively. Note that

\[ \int_{\mathbb{R}^3} F_0 dv = \rho_0, \quad \int_{\mathbb{R}^3} v F_0 dv = \rho_0 u_0, \quad \int_{\mathbb{R}^3} |v|^2 F_0 dv = \rho_0 |u_0|^2 + 3 \rho_0 \theta_0. \]

Projecting the equation of \( F_0 \) from the \( \varepsilon^0 \) step in (1.2) onto \( 1, v, \frac{|v|^2}{2} \), which are five collision invariants for the Boltzmann collision operator \( Q \), we obtain the equations for \( \rho_0, u_0, \theta_0 \):

\[
\begin{align*}
\partial_t \rho_0 + \nabla \cdot (\rho_0 u_0) &= 0, \\
\partial_t (\rho_0 u_0) + \nabla (\rho_0 u_0 \otimes u_0) + \nabla (\rho_0 \theta_0) - \rho_0 \nabla \phi_0 &= 0, \\
\partial_t (\rho_0 |u_0|^2 \gamma_2 + 3 \rho_0 \theta_0) + \nabla \cdot (\frac{5 \rho_0 \theta_0 u_0}{2} + \rho_0 |u_0|^2 u_0) - \rho_0 \nabla \phi_0 \cdot u_0 &= 0, \\
\Delta \phi_0 &= \rho_0 - \theta.
\end{align*}
\]

Setting \( p_0 = \rho_0 \theta_0 \): the equation of state, these equations form the repulsive Euler-Poisson system for an ideal perfect electron gas. This ideal gas law and an internal energy of \( \varepsilon \) for an ideal perfect gas. In order to see that, we first note that for smooth solutions, the system (1.5) can be written as follows:

\[
\begin{align*}
\partial_t \rho_0 + (u_0 \cdot \nabla) \rho_0 + \rho_0 \nabla \cdot u_0 &= 0, \\
\rho_0 \partial_t u_0 + \rho_0 (u_0 \cdot \nabla) u_0 + \nabla (\rho_0 \theta_0) - \rho_0 \nabla \phi_0 &= 0, \\
\partial_t \theta_0 + (u_0 \cdot \nabla) \theta_0 + \frac{2}{3} \theta_0 \nabla \cdot u_0 &= 0, \\
\Delta \phi_0 &= \rho_0 - \theta.
\end{align*}
\]

The continuity equation \( (\rho_0) \) and the temperature equation \( (\theta_0) \) are equivalent when \( \theta_0 \sim \frac{2}{3} \rho_0 \). Therefore, letting \( \theta_0 \equiv K \rho_0^\frac{2}{3} \), we can recover the isentropic Euler-Poisson flow for monatomic gas where the adiabatic exponent \( \gamma = \frac{5}{3} \) from (1.5): where \( \rho_0 = K \rho_0^\frac{2}{3} \).

We define the linearized Boltzmann operator at \( \omega \) as

\[
Lg = -\frac{1}{\sqrt{\omega}} (Q(\sqrt{\omega} g, \omega) + Q(\omega, \sqrt{\omega} g)) = \nu(\omega) g - \nu g, \\
\Gamma(g_1, g_2) = \frac{1}{\sqrt{\omega}} Q(\sqrt{\omega} g_1, \sqrt{\omega} g_2).
\]

We recall \( L \geq 0 \) and the null space of \( L \) is generated by

\[
\chi_0(v) = \frac{1}{\sqrt{\rho_0}} \sqrt{\omega}, \\
\chi_i(v) = \frac{v^i - u_0^i}{\sqrt{\rho_0 \theta_0}} \sqrt{\omega}, \text{ for } i = 1, 2, 3, \\
\chi_4(v) = \frac{1}{\sqrt{6 \rho_0 \theta_0}} \left\{ \frac{|v - u_0|^2}{\theta_0} - 3 \right\} \sqrt{\omega}.
\]

One can easily check \( \langle \chi_i, \chi_j \rangle = \delta_{ij} \) for \( 0 \leq i, j \leq 4 \). We also define the collision frequency \( \nu \):

\[
\nu(t, x, v) = \int_{\mathbb{R}^3} |v - v^*| \omega(v^*) dv^*.
\]

We shall use \( \| \cdot \| \) and \( \| \cdot \|_\nu \) to denote \( L^2 \) norms corresponding to \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle_\nu \). We define \( P \) as the \( L^2(\mathbb{R}^3) \) orthogonal projection on this null space. Then we have

\[
\langle Lg, g \rangle \geq \delta_0 \| (I - P) g \|_\nu^2.
\]
for some positive constant $\delta_0 > 0$. Write
\[ F_R^\varepsilon = \sqrt{\omega} f^\varepsilon. \] (1.6)

By introducing a global Maxwellian
\[ \omega_M = \frac{1}{(2\pi \theta_M)^{3/2}} \exp \left\{ -\frac{|v|^2}{2\theta_M} \right\}, \]
where $\theta_M = K \bar{\rho}^{3/2}$, we further define
\[ F_R^\varepsilon = \{1 + |v|^2\}^{-\beta} \sqrt{\omega_M} h^\varepsilon \equiv \frac{\sqrt{\omega_M}}{w(v)} h^\varepsilon \] (1.7)
where $w(v) \equiv \{1 + |v|^2\}^\beta$ for some $\beta \geq 7/2$.

It is standard to construct the coefficients $F_i$ ($1 \leq i \leq 2k - 1$), and the key is to control the remainder $F_R^\varepsilon$ in the nonlinear dynamics. We now state the main result of this article.

**Theorem 1.1.** Let $F_0 = \omega$ as in [3] and let $u_0(0, x)$ and $\rho_0(0, x)$ satisfy
\[ \nabla \times u_0(0, x) = 0 \ (\text{irrotationality}), \quad \int_{\mathbb{R}^3} \{\rho_0(0, x) - \bar{\rho}\} dx = 0 \ (\text{neutrality}), \]
be smooth, and $u_0(0, x)$ and $\rho_0(0, x) - \bar{\rho}$ be sufficiently small such that global solution $[u_0(t, x), \rho_0(t, x)]$ to the Euler-Poisson equations [1] can be constructed as in [4]. Then for the remainder $F_R^\varepsilon$ in (1) there exist an $\varepsilon_0 > 0$ and a constant $C > 0$ independent of $\varepsilon$ such that for $0 \leq \varepsilon \leq \varepsilon_0$,
\[
\sup_{0 \leq t \leq \varepsilon^{-m}} \left\{ \varepsilon^2 \|\frac{1 + |v|^2\beta + 1}{\sqrt{\omega_M}} F_R^\varepsilon(t)\|_\infty + \varepsilon^2 \|\nabla \phi_R\|_\infty \right\} + \sup_{0 \leq t \leq \varepsilon^{-m}} \left\{ \varepsilon^5 \|\nabla_x \{1 + |v|^2\beta F_R^\varepsilon(t)\}\|_\infty \right\} + \sup_{0 \leq t \leq \varepsilon^{-m}} \left\{ \varepsilon^5 \|\nabla_x \phi_R\|_\infty \right\} \leq C \left( \varepsilon^4 \|\frac{1 + |v|^2\beta + 1}{\sqrt{\omega_M}} F_R^\varepsilon(0)\|_\infty + \varepsilon^5 \|\nabla_x (1 + |v|^2\beta F_R^\varepsilon(0))\|_\infty \right) + \|\frac{F_R^\varepsilon(0)}{\sqrt{\omega(0)}}\| + \|\nabla \phi_R(0)\| + 1 \right\}
\]
for all $0 < m \leq \frac{12k-3}{2k-2}$.

**Remark 1.2.** While we get a uniform in $\varepsilon$ estimate for the $L^2$ norm of the remainders, for the weighted $W^{1,\infty}$ norms, we only obtain a uniform estimate of $\varepsilon F_R^\varepsilon$ and $\varepsilon^5 \nabla_x \phi_R$, which is why we need higher order expansion $k \geq 6$ in [1]. With these higher order Hilbert expansions, our uniform estimates lead to the Euler-Poisson limit $\sup_{0 \leq t \leq \varepsilon^{-m}} \|F^\varepsilon - \omega\| = O(\varepsilon)$ for all $0 < m \leq \frac{12k-3}{2k-2}$.

Our result provides a rare example such that the Hilbert expansion is valid for all time. In the absence of the electrostatic interaction, it is well-known [2] that similar Hilbert expansion is only valid local in time, before shock formations in the pure compressible Euler flow, for example, see [9]. By a classical result [10], it is well-known that even for arbitrary small perturbations of a motionless steady state, singularity does form in finite time for the Euler system for a compressible fluid. In contrast, the validity time in the Euler-Poisson limit is $\varepsilon^{-\frac{2k-3}{2k-2}}$ for irrotational flow, which implies global in-time convergence from the Vlasov-Poisson-Boltzmann to the Euler-Poisson system [11]. The key difference, in the presence of electrostatic interaction, is that small irrotational flows exist forever without any shock formation for the Euler-Poisson [11], see [4]. Such a surprising result is due to extra dispersive effect in the presence of a self-consistent electric field, which is characterized by so-called ‘plasma frequency’ in the physics literature. This leads to a ‘Klein-Gordon effect’ which enhances the linear decay rate and destroys the possible shock formation.

Our method of proof relies on a recent $L^2-L^\infty$ approach to study the Euler limit of the Boltzmann equation [12] [8]. The improvement over the classical Caflisch’s paper is that now the positivity of the
initial datum can be guaranteed. The main idea of our approach is to use the natural $L^2$ energy estimate as the first step. The most difficult term in the energy estimate is

\[
\frac{1}{2} (\partial_t + v \cdot \nabla_x) \theta_0 f^\epsilon - \theta_0 \left\{ \frac{\partial_t + v \cdot \nabla_x + \nabla_x \phi_0 \cdot \nabla_v}{\sqrt{\omega}} \right\} \sqrt{\omega} f^\epsilon,
\]

which involves cubic power of $|v|$, and it is hard to control by an $L^2$ type of norm only. We introduce a new weighted $L^\infty$ space to control such a term. The second step is to estimate such a weighted $L^\infty$ norm along the trajectory, based on the $L^2$ estimate in the first step, but with a singular negative power of $\varepsilon$. Such a simple interplay between $L^2$ and $L^\infty$ norms fails to yield a closed estimate in our study, unlike compressible Euler limit [5]. The new analytical difficulty to overcome in the present work is a delicate point-wise estimate of the distribution function in the presence of a curved trajectory caused by the self-consistent electric field. It turns out that, due to the Poisson coupling, we need to further estimate the $W^{1,\infty}$ norm along the trajectory to close our estimate. This requires higher expansion (1) to compensate more singular power of $\varepsilon$ for the $W^{1,\infty}$ estimate. In order to obtain the uniform estimate over the time scale of $\varepsilon^{-\frac{1}{2} \frac{3+\epsilon}{3+2\epsilon}}$, we must carefully analyze the decay and growth of the coefficients $F_j$.

Recently, there has been quite some mathematical study of the Vlasov-Poisson-Boltzmann system: [2] for $\varepsilon = 1$. Among others, global solutions near a Maxwellian were constructed [6] in a periodic box. In [12] [13], global solutions near a Maxwellian were constructed for the whole space. In [11], a self-consistent magnetic effect were also included.

Our paper is organized as follows. In section 2, we construct the coefficients $F_i$ for the Hilbert expansion (1), starting with the global smooth irrotational solution to the Euler-Poisson system (4) constructed in [4]. In particular, we study carefully the growth in time $t$ for $F_i$. In section 3, we use the $L^2$ energy estimate for the remainder $F^c_R$ around the local Maxwellian $F_0$ (3), in terms of the weighted $L^\infty$ norm of $h^\epsilon$. Section 4 is a study of the curved trajectory. Section 5 is the main technical part of the paper, in which $L^\infty$ and $W^{1,\infty}$ norms of $h^\epsilon$ are estimated along the curved trajectories in terms of the $L^2$ energy to close the whole argument. Section 6 is a direct proof of our main theorem based on the $L^2 - L^\infty$ estimates. Throughout this paper, we use $C$ to denote possibly different constants but independent of $t$ and $\varepsilon$.

## 2 Coefficients of the Hilbert Expansion

In this section, we discuss the existence and regularity of $F_i$, inherited from $F_0 = \omega$ as defined in (3). Write $\frac{F_i}{\sqrt{\omega}}$ as the sum of macroscopic and microscopic parts as follows: for each $i \geq 1$,

\[
\frac{F_i}{\sqrt{\omega}} = P \left( \frac{F_i}{\sqrt{\omega}} \right) + \{ I - P \} \left( \frac{F_i}{\sqrt{\omega}} \right)
\]

\[
\equiv \left\{ \frac{\rho_i}{\sqrt{\rho_0}} \sqrt{\theta_0} \chi_0 + \sum_{j=1}^{3} \sqrt{\frac{\rho_0}{\theta_0}} u_i^j \cdot \chi_j + \sqrt{\frac{\rho_0}{\theta_0}} \frac{\theta_i}{6} \chi_4 \right\} + \{ I - P \} \left( \frac{F_i}{\sqrt{\omega}} \right) \quad \text{(2.1)}
\]

$F_i$’s will be constructed inductively as follows:

**Lemma 2.1.** For each given nonnegative integer $k$, assume $F_k$’s are found. Then the microscopic part of $\frac{F_{k+1}}{\sqrt{\omega}}$ is determined through the equation for $F_k$ in (1.2):

\[
\{ I - P \} \left( \frac{F_{k+1}}{\sqrt{\omega}} \right) = L^{-1} \left( - \frac{\left\{ \partial_t + v \cdot \nabla_x \right\} F_k + \sum_{i,j \geq 0} \frac{i+j=k}{\sqrt{\omega}} \nabla_x \phi_i \cdot \nabla_v F_j - \sum_{i,j \geq 1} \frac{i+j=k+1}{\sqrt{\omega}} Q(F_i, F_j) \right).
\]
For the macroscopic part, $\rho_{k+1}, u_{k+1}, \theta_{k+1}$ satisfy the following:

$$\partial_t \rho_{k+1} + \nabla \cdot (\rho_{k+1} u_{k+1} + \rho_{k+1} u_0) = 0,$$

$$\rho_0 \{ \partial_t u_{k+1} + (u_{k+1} \cdot \nabla) u_0 + (u_0 \cdot \nabla) u_{k+1} - \nabla \phi_{k+1} \} - \frac{\rho_{k+1}}{\rho_0} \nabla (\rho_0 \theta_0) + \nabla \left( \frac{\rho_0 \theta_{k+1} + 3 \theta_0 \rho_{k+1}}{3} \right) = f_k,$$

$$\rho_0 \{ \partial_t \theta_{k+1} + \frac{2}{3} (\theta_{k+1} \nabla \cdot u_0 + 3 \theta_0 \nabla \cdot u_{k+1}) + u_0 \cdot \nabla \theta_{k+1} + 3 u_{k+1} \cdot \nabla \theta_0 \} = g_k,$$

$$\Delta \phi_{k+1} = \rho_{k+1}, \quad (2.2)$$

where

$$f_k = -\partial_j \int \{(v^i - u_0^i)(v^j - u_0^j) - \delta_{ij} \frac{|v - u_0|^2}{3}\} F_{k+1} dv + \sum_{i+j=k+1} \rho_j \nabla \phi_i$$

$$g_k = -\partial_i \{ \int (v^i - u_0^i)(|v - u_0|^2 - 5 \theta_0) F_{k+1} dv + 2 u_0^i \int \{(v^i - u_0^i)(v^j - u_0^j) - \delta_{ij} \frac{|v - u_0|^2}{3}\} F_{k+1} dv \}

- 2 u_0 \cdot f_k + \sum_{i+j=k+1} (\rho_0 u_j + \rho_j u_0) \nabla \phi_i$$

Here we use the subscript $k$ for forcing terms $f$ and $g$ in order to emphasize that the right hand sides depend only on $F_1$’s and $\nabla \phi_i$’s for $0 \leq i \leq k$.

**Proof. of Lemma 2.1:** We shall only derive the equations for $F_1$. From the coefficient of $\epsilon^0$ in (1.2), the microscopic part of $F_1$ should be

$$\{\mathbf{I} - \mathbf{P}\} \left( \frac{F_1}{\sqrt{\omega}} \right) = L^{-1} \left(- \frac{\partial_i + v \cdot \nabla_x + \nabla_x \theta \cdot \nabla_x \omega}{\sqrt{\omega}} \right). \quad (2.3)$$

Since $L^{-1}$ preserves decay in $v$,

$$||\{\mathbf{I} - \mathbf{P}\} \left( \frac{F_1}{\sqrt{\omega}} \right)|| \leq (||\partial \rho_0||_\infty + ||\partial u_0||_\infty + ||\partial \theta_0||_\infty + ||\nabla \phi_0||_\infty)(1 + |v|^3) \sqrt{\omega}, \quad (2.4)$$

where $\partial$ is either $\partial_t$ or $\nabla_x$. For macroscopic variables $\rho_1, u_1, \theta_1$ of $F_1$ in (2.1), note that

$$\int F_1 dv = \rho_1,$n \int (v - u_0) F_1 dv = \rho_0 u_1, \quad \int v F_1 dv = \rho_0 u_1 + \rho_1 u_0,$$n

$$\int |v - u_0|^2 F_1 dv = \int (|v - u_0|^2 - 3 \theta_0) F_1 dv + 3 \theta_0 \rho_1 = \rho_0 \theta_1 + 3 \theta_0 \rho_1,$$n

$$\int |v|^2 F_1 dv = \int |v - u_0|^2 F_1 dv = \rho_0 \theta_1 + 3 \theta_0 \rho_1 + \rho_1 |u_0|^2 + 2 \rho_0 u_0 u_1,$$n

$$\int v^i v^j F_1 dv = \int \{(v^i - u_0^i)(v^j - u_0^j) - \delta_{ij} \frac{|v - u_0|^2}{3}\} F_1 dv + \rho_0 u_0^i u_0^j + \rho_0 u_0^i u_0^j + u_0^i u_0^j \rho_1 + \delta_{ij} \frac{\rho_0 \theta_1 + 3 \theta_0 \rho_1}{3},$$n

$$\int v^i |v|^2 F_1 dv = \int (v^i - u_0^i)|v|^2 F_1 dv + (\rho_0 \theta_1 + 3 \theta_0 \rho_1 + \rho_1 |u_0|^2 + 2 \rho_0 u_0 u_1) u_0$$n

$$= \int (v^i - u_0^i)(|v - u_0|^2 - 5 \theta_0) F_1 dv + 2 u_0^i \int \{(v^i - u_0^i)(v^j - u_0^j) - \delta_{ij} \frac{|v - u_0|^2}{3}\} F_1 dv + (5 \theta_0 + |u_0|^2) \rho_0 u_1 + \frac{2}{3} (\rho_0 \theta_1 + 3 \theta_0 \rho_1) u_0 + (\rho_0 \theta_1 + 3 \theta_0 \rho_1 + \rho_1 |u_0|^2 + 2 \rho_0 u_0 u_1) u_0.$$n

Project the equation for $F_1$ in (1.2) onto $1, v, |v|^2$ to get equations of $\rho_1, u_1, \theta_1$ with forcing terms as
Letting and define we obtain

Similarly, the equation for follows:

By using the equations for \( \rho_0, u_0 \) and \( \rho_1 \), the equation for \( u_1 \) can be reduced to

where

Here

and for the last term we have used the coefficient of \( \varepsilon^0 \) in (1.2):

Letting

we obtain

and define

Similarly, the equation for \( \theta_1 \) can be reduced to

We rewrite the fluid equations for the first order coefficients \( \rho_1, u_1, \theta_1, \phi_1 \):

\[
\rho_0 \{ \partial_t u_1 + (u_1 \cdot \nabla) u_0 + (u_0 \cdot \nabla) u_1 - \nabla \phi_1 \} - \frac{\rho_1}{\rho_0} \nabla (\rho_0 \theta_1) + \nabla (\rho_0 \theta_1 + \frac{\rho_0 \theta_1}{\rho_0} + \frac{\rho_0 \theta_1}{\rho_0} \rho_1) = \partial_j (\mu(\theta_0) \partial_j u_1^1) \]

(2.5)

\[ \Delta \phi_1 = \rho_1. \]
Here, $\mu(\theta_0)$ an $\kappa(\theta_0)$ represent the viscosity and heat conductivity coefficients respectively. This is reminiscent of the derivation of compressible Navier-Stokes equations from the Boltzmann equation. We refer to [1] for more details. This completes the proof for $F_1$ and higher expansion coefficients $F_k$’s can be found in the same way.

Since $p_1, u_1, \theta_1, \phi_1$ solve linear equations with coefficients and forcing terms coming from the smooth functions $\rho_0, u_0, \theta_0$, the initial value problem for (2.5) is well-posed in the Sobolev spaces, and moreover, we will show in Lemma 2.2 that

$$|F_1(t, x, v)| \leq C(1 + |v|^3)\omega,$$

where $C$ only depends on the regularity of $\rho_0, u_0, \theta_0, \phi_0$ and the given initial data $p_1(0), u_1(0), \theta_1(0)$. From (2.1) and (2.3), we also deduce that

$$|\nabla_{x} F_1(t, x, v)| \leq C(1 + |v|^4)\omega \quad \text{and} \quad |\nabla_{v} F_1(t, x, v)| \leq C(1 + |v|^5)\omega.$$

Recall [1] that the Euler-Poisson system (1) for a perfect gas admits smooth small global solutions $\rho_0, u_0, \nabla \phi_0$ with the following point-wise uniform-in-time decay:

$$\|\rho_0 - 7\|_{W^{s, \infty}} + \|u_0\|_{W^{s, \infty}} + \|\nabla \phi_0\|_{W^{s, \infty}} \leq \frac{C}{(1 + t)^p} \quad (2.6)$$

for any $1 < p < \frac{3}{2}$ and for each $s \geq 0$. In the next lemma, we show that the corresponding Hilbert expansion coefficients $F_i$ cannot grow arbitrarily in time.

**Lemma 2.2.** Let smooth global solutions $\rho_0, u_0, \nabla \phi_0$ to the Euler-Poisson system (1) be given and let $\theta_0 = K\rho_0^3$. For each $k \geq 0$, let $\rho_{k+1}(0, x), u_{k+1}(0, x), \theta_{k+1}(0, x) \in H^s, s \geq 0$ be given initial data to (2.2). Then the linear system (2.2) is well-posed in $H^s$, and furthermore, there exists a constant $C > 0$ depending only on the initial data (independent of $t$) such that for each $t$,

$$|F_i| \leq C(1 + t)^{-i-1}(1 + |v|^{3i})\omega, \quad |\nabla_{x} F_i| \leq C(1 + t)^{-i+1}, \quad |\nabla_{v} F_i| \leq C(1 + t)^{-i+1}(1 + |v|^{3i+2})\omega, \quad \text{and} \quad |\nabla_{v} \nabla_{x} F_i| \leq C(1 + t)^{-i+1}(1 + |v|^{3i+3})\omega. \quad (2.7)$$

**Proof.** The well-posedness easily follows from the linear theory, for instance see [3]. Here we provide the a priori estimates for (2.7). The proof relies on the induction on $i$. We first prove for $F_1$. Write the linear system (2.5) as a symmetric hyperbolic system with the following symmetrizer $A_0$:

$$A_0 \{\partial_t U - V\} + \sum_{i=1}^3 A_i \partial_i U + BU = F \quad (2.8)$$

where $U$, $V$, $A_0$, and $A_i$’s are given as follows:

$$U = \begin{pmatrix} \frac{p_1}{(u_1)^t} \\ \frac{\theta_1}{(u_1)^t} \end{pmatrix}, \quad V = \begin{pmatrix} 0 \\ (\nabla \phi_1)^t \end{pmatrix}, \quad A_0 = \begin{pmatrix} \frac{(\rho_0)^2}{\rho_0} & 0 & 0 \\ 0 & \frac{(\rho_0)^2}{\rho_0} & 0 \\ 0 & 0 & \frac{(\rho_0)^2}{6} \end{pmatrix},$$

$$A_i = \begin{pmatrix} \frac{(\rho_0)^2 u_i^0}{\rho_0} & \frac{(\rho_0)(\rho_0)^2 N}{(\rho_0)^2 N} e_i & 0 \\ \frac{(\rho_0)(\rho_0)^2 N}{(\rho_0)^2 N} e_i & \frac{(\rho_0)(\rho_0)^2 N}{(\rho_0)^2 N} e_i & 0 \\ 0 & 0 & \frac{(\rho_0)^2}{3} \end{pmatrix}.$$

$(\cdot)^t$ denotes the transpose of row vectors, $e_i$’s for $i = 1, 2, 3$ are the standard unit (row) base vectors in $\mathbb{R}^3$, and $I$ is the $3 \times 3$ identity matrix. $B$ and $F$, which consist of $\rho_0, u_0, \theta_0$, and first derivatives of $\rho_0, u_0, \theta_0$, can be easily written down. In particular, we have $\|B\|_{W^{s, \infty}} + \|F\|_{W^{s, \infty}} \leq \frac{C}{(1 + t)^p}$ for any $1 < p < \frac{3}{2}$ and any $s \geq 0$. Note that (2.8) together with $\Delta \phi_1 = \rho_1$ is strictly hyperbolic and thus we
can apply the standard energy method of the linear symmetric hyperbolic system to (2.8) to obtain the following energy inequality: for each $s \geq 0$,

$$
\frac{d}{dt} \{ \|U\|_{H^s}^2 + \|V\|_{H^s}^2 \} \leq \frac{C}{(1+t)^p} \{ \|U\|_{H^s}^2 + \|V\|_{H^s}^2 \} + \frac{C}{(1+t)^p} \{ \|U\|_{H^s} + \|V\|_{H^s} \}.
$$

(2.9)

Hence, we obtain $\|U\|_{H^s} + \|V\|_{H^s} \leq C$ and therefore, from (2.7) and (2.8), the inequality (2.7) for $i = 1$ follows. Now suppose (2.7) holds for $1 \leq i \leq n$. For $i = n + 1$, we first note that from the coefficient of $\varepsilon^n$ in (1.2), the microscopic part of $\frac{F_{n+1}}{\sqrt{\omega}}$ is bounded by

$$
\| (I - P) \{ \frac{F_{n+1}}{\sqrt{\omega}} \} \| \leq C(1 + t)^n(1 + |v|^{3(n+1)}) \sqrt{\omega}
$$

by the induction hypothesis. For the macroscopic part, we project the equation for $F_{n+1}$ in (1.2) onto $1, v, |v|^2$ as in $F_1$ to obtain fluid equations for $\rho_{n+1}, u_{n+1}, \theta_{n+1}, \nabla \phi_{n+1}$. See (2.2). Since the structure of the left hand side of (2.2) is the same as in (1.2), one can write the equations for $\rho_{n+1}, u_{n+1}, \theta_{n+1}, \nabla \phi_{n+1}$ as the linear symmetric hyperbolic system. The difference is that there are extra terms coming from $\sum_{i,j=1}^{n+1} \nabla x \phi_i \cdot \nabla v F_j$. From the induction hypothesis, one can get the following corresponding inequality for $U, V$ as in (2.9)

$$
\frac{d}{dt} \{ \|U\|_{H^s}^2 + \|V\|_{H^s}^2 \} \leq \frac{C}{(1+t)^p} \{ \|U\|_{H^s}^2 + \|V\|_{H^s}^2 \} + C(1 + t)^n \{ \|U\|_{H^s} + \|V\|_{H^s} \}.
$$

By Gronwall inequality, we obtain

$$
\|U\|_{H^s} + \|V\|_{H^s} \leq C(1 + t)^{n+1},
$$

and this verifies (2.7) for $i = n + 1$.

\section{L2 Estimates for Remainder $F^\varepsilon_R$}

In this section, we perform the $L^2$ energy estimates of remainders $f^\varepsilon = \frac{F^\varepsilon}{\sqrt{\omega}}$ and $\nabla x \phi^\varepsilon_R$. Here is the main result of this section.

\begin{proposition}
There exists a constant $C$ independent of $t, \varepsilon$ such that for each $t$ and $\varepsilon$,

$$
\frac{d}{dt} \{ \| \sqrt{\theta_0} f^\varepsilon \|^2 + \| \nabla \phi^\varepsilon_R \|^2 \} + \frac{\delta_0}{2\varepsilon} \theta_M \| (I - P) f^\varepsilon \|^2 \\
\leq C \{ \varepsilon^2 \| h^\varepsilon \|_\infty \| f^\varepsilon \|^2 + \varepsilon^{k-1} \| h^\varepsilon \|_\infty \| f^\varepsilon \|^2 + \varepsilon^k \| h^\varepsilon \|_\infty \| f^\varepsilon \| \| \nabla \phi^\varepsilon_R \| \} \\
+ \frac{C}{(1+t)^p} \{ \| f^\varepsilon \|^2 + \| \nabla \phi^\varepsilon_R \|^2 \} + C \mathcal{I}_1 \{ \varepsilon \| f^\varepsilon \|^2 + \varepsilon \| \nabla \phi^\varepsilon_R \|^2 \} + C \mathcal{I}_2 \varepsilon^{k-1} \| f^\varepsilon \|,
$$

where $\mathcal{I}_1$ and $\mathcal{I}_2$ are given as follows:

$$
\mathcal{I}_1 = \sum_{i=1}^{2k-1} \varepsilon(1+t)^{i-1} + \left( \sum_{i=1}^{2k-1} \varepsilon(1+t)^{i-1} \right)^2; \quad \mathcal{I}_2 = \sum_{2k \leq i+j} \varepsilon^{i+j-2k}(1+t)^{i+j-2}.
$$

(3.1)
\end{proposition}
Proof. First we write the equation for $f^\varepsilon$ from (1.3):

$$
\partial_t f^\varepsilon + v \cdot \nabla x f^\varepsilon + \nabla x \phi_0 \cdot \nabla v f^\varepsilon - \frac{v - u_0}{\theta_0} \sqrt{\omega} \cdot \nabla x \phi_R^\varepsilon + \frac{1}{\varepsilon} L f^\varepsilon \\
= - \left\{ \partial_t + v \cdot \nabla x + \nabla x \phi_0 \cdot \nabla v \right\} \sqrt{\omega} f^\varepsilon + \varepsilon^{k-1} \Gamma(f^\varepsilon, f^\varepsilon) + \sum_{i=1}^{2k-1} \varepsilon^i \left\{ \Gamma(\frac{F_i}{\sqrt{\omega}}, f^\varepsilon) + \Gamma(f^\varepsilon, \frac{F_i}{\sqrt{\omega}}) \right\} \\
- \varepsilon^k \nabla x \phi_R^\varepsilon \cdot \nabla v f^\varepsilon + \varepsilon^k \nabla x \phi_R^\varepsilon \cdot \frac{v - u_0}{2\theta_0} f^\varepsilon - \sum_{i=1}^{2k-1} \varepsilon^i \left\{ \nabla x \phi_i \cdot \nabla v f^\varepsilon + \nabla x \phi_R^\varepsilon \cdot \nabla v F_i \right\} \\
+ \sum_{i=1}^{2k-1} \varepsilon^i \nabla x \phi_i \cdot \frac{v - u_0}{2\theta_0} f^\varepsilon + \varepsilon^{k-1} \mathbf{A},
$$

where $\mathbf{A} = \frac{A}{\sqrt{\omega}}$. Note that $\nabla_v \omega = -\frac{v - u_0}{\theta_0} \omega$. Take $L^2$ inner product with $\theta_0 f^\varepsilon$ on both sides to get

$$
\frac{1}{2} \frac{d}{dt} \| \theta_0 f^\varepsilon \|^2 \leq - \int \left( (v - u_0) \sqrt{\omega} f^\varepsilon dv \cdot \nabla x \phi_R^\varepsilon dx + \frac{\delta_0}{\varepsilon} \inf \theta_0 \| \{ I - P \} f^\varepsilon \|^2 \right) \\
\leq \frac{1}{2} \left\langle (\partial_t + v \cdot \nabla x) \theta_0 f^\varepsilon, f^\varepsilon \right\rangle - \left\langle \theta_0 \left\{ \partial_t + v \cdot \nabla x + \nabla x \phi_0 \cdot \nabla v \right\} \sqrt{\omega} f^\varepsilon, f^\varepsilon \right\rangle \\
+ \varepsilon^{k-1} \left\langle \theta_0 \Gamma(f^\varepsilon, f^\varepsilon), f^\varepsilon \right\rangle + \left\langle \theta_0 \sum_{i=1}^{2k-1} \varepsilon^i \left\{ \Gamma\left(\frac{F_i}{\sqrt{\omega}}, f^\varepsilon\right) + \Gamma(f^\varepsilon, \frac{F_i}{\sqrt{\omega}})\right\}, f^\varepsilon \right\rangle \\
+ \varepsilon^k \left\langle \nabla x \phi_R^\varepsilon \cdot \frac{v - u_0}{2} f^\varepsilon, f^\varepsilon \right\rangle - \left\langle \theta_0 \sum_{i=1}^{2k-1} \varepsilon^i \nabla x \phi_i \cdot \nabla v F_i \right\rangle \\
+ \left\langle \sum_{i=1}^{2k-1} \varepsilon^i \nabla x \phi_i \cdot \frac{v - u_0}{2} f^\varepsilon, f^\varepsilon \right\rangle + \varepsilon^{k-1} \left\langle \theta_0 \mathbf{A}, f^\varepsilon \right\rangle. (3.2)
$$

From $\Delta \phi_R^\varepsilon = \int_{\mathbb{R}^3} f^\varepsilon \sqrt{\omega} dv$ and (1.3), we obtain

$$
-\Delta \partial_t \phi_R^\varepsilon = - \int_{\mathbb{R}^3} \partial_t F_R^\varepsilon dv = \int v \cdot \nabla x (\sqrt{\omega} f^\varepsilon) dv.
$$

Take $L^2$ inner product with $\phi_R^\varepsilon$ on both sides to get

$$
\frac{1}{2} \frac{d}{dt} \| \phi_R^\varepsilon \|^2 = \int -\Delta \partial_t \phi_R^\varepsilon \cdot \phi_R^\varepsilon \ dx = \int \int v \cdot \nabla x (\sqrt{\omega} f^\varepsilon) \phi_R^\varepsilon vdx \ dx = - \int \int v \sqrt{\omega} f^\varepsilon dv \cdot \nabla x \phi_R^\varepsilon dx. (3.3)
$$

Combining (3.2) and (3.3), we obtain

$$
\frac{1}{2} \frac{d}{dt} \left( \| \theta_0 f^\varepsilon \|^2 + \| \nabla \phi_R^\varepsilon \|^2 \right) + \frac{\delta_0}{\varepsilon} \theta_M \| \{ I - P \} f^\varepsilon \|^2 \leq - \int u_0 \int \sqrt{\omega} f^\varepsilon dv \cdot \nabla x \phi_R^\varepsilon dx \\
+ \frac{1}{2} \left\langle (\partial_t + v \cdot \nabla x) \theta_0 f^\varepsilon, f^\varepsilon \right\rangle - \left\langle \theta_0 \left\{ \partial_t + v \cdot \nabla x + \nabla x \phi_0 \cdot \nabla v \right\} \sqrt{\omega} f^\varepsilon, f^\varepsilon \right\rangle \\
+ \varepsilon^{k-1} \left\langle \theta_0 \Gamma(f^\varepsilon, f^\varepsilon), f^\varepsilon \right\rangle + \left\langle \theta_0 \sum_{i=1}^{2k-1} \varepsilon^i \left\{ \Gamma\left(\frac{F_i}{\sqrt{\omega}}, f^\varepsilon\right) + \Gamma(f^\varepsilon, \frac{F_i}{\sqrt{\omega}})\right\}, f^\varepsilon \right\rangle \\
+ \varepsilon^k \left\langle \nabla x \phi_R^\varepsilon \cdot \frac{v - u_0}{2} f^\varepsilon, f^\varepsilon \right\rangle - \left\langle \theta_0 \sum_{i=1}^{2k-1} \varepsilon^i \nabla x \phi_i \cdot \nabla v F_i \right\rangle \\
+ \left\langle \sum_{i=1}^{2k-1} \varepsilon^i \nabla x \phi_i \cdot \frac{v - u_0}{2} f^\varepsilon, f^\varepsilon \right\rangle + \varepsilon^{k-1} \left\langle \theta_0 \mathbf{A}, f^\varepsilon \right\rangle. (3.4)
$$
The first term in the right hand side is controlled by

$$-\int u_0 \int \sqrt{\omega} f^e dv \cdot \nabla_x \phi_R^e dx = -\int u_0 \Delta_x \phi_R^e \cdot \nabla_x \phi_R^e dx = -\int u_0 \partial_j \partial_j \phi_R^e \partial_i \phi_R^e dx = \int \partial_j u_0^i \partial_j \phi_R^e \partial_i \phi_R^e dx - \frac{1}{2} \int \partial_i u_0^i |\partial_j \phi_R^e|^2 dx \leq \frac{C}{(1+t)^p} \|\nabla \phi_R^e\|^2 \text{ from (2.6).}$$

The key difficult term $\frac{1}{2}(\partial_t + v \cdot \nabla_x)\theta_0 - \theta_0 \frac{1}{\sqrt{\omega}} (1 + |v|^2)^{3/2} f^e$ is a cubic polynomial in $v$, and since $\{1 + |v|^2\}^{3/2} f^e \leq (1 + |v|^2)^{-1} h^e$, for $\beta \geq 7/2$ in (1.7), the second line in (3.3) can be estimated as follows: from the uniform bounds for $\rho_0, u_0, \theta_0, \nabla \phi_0$ again in (2.6),

$$\frac{1}{2}\{(\partial_t + v \cdot \nabla_x)\theta_0 f^e, f^e\} - \langle \theta_0 \frac{1}{\sqrt{\omega}} (1 + |v|^2)^{3/4} P f^e, f^e \rangle$$

$$= \int_{|v| \geq \frac{\delta_t}{\delta \gamma}} + \int_{|v| \leq \frac{\delta_t}{\delta \gamma}}$$

$$\leq C\{(\partial \rho_0) + |\partial u_0| + |\partial \theta_0| + |\nabla \phi_0|\} \times \{(1 + |v|^2)^{3/4} f^e 1_{|v| \geq \frac{\delta_t}{\delta \gamma}} \|f^e\| \}$$

$$+ C\{(\partial \rho_0)_{\infty} + |\partial u_0|_{\infty} + |\partial \theta_0|_{\infty} + |\nabla \phi_0|_{\infty}\} \times \{(1 + |v|^2)^{3/4} f^e 1_{|v| \leq \frac{\delta_t}{\delta \gamma}} \|f^e\| \}$$

$$\leq C \varepsilon^2 \|h^e\|_{\infty} \|f^e\|^2 + \frac{C}{(1+t)^p} \{(1 + |v|^2)^{3/4} \|f^e\|_{|v| \leq \frac{\delta_t}{\delta \gamma}} \|^2 + \|(1 + |v|^2)^{3/4} (I - P) f^e 1_{|v| \leq \frac{\delta_t}{\delta \gamma}} \|^2 \}$$

By applying Lemma 2.3 in [5] and (1.7), the third line in (3.3) can be estimated as follows:

$$\varepsilon^{k-1} \langle \theta_0 \Gamma (f^e, f^e), f^e \rangle \leq C \varepsilon^{k-1} \|\nu f^e\|_{\infty} \|f^e\|^2 \leq C \varepsilon^{k-1} \|h^e\|_{\infty} \|f^e\|^2.$$

From collision symmetry, we get

$$\langle \theta_0 \varepsilon^{i-1} \Gamma \left( \frac{F_i}{\sqrt{\omega}}, f^e \right) + \Gamma \left( f^e, \frac{F_i}{\sqrt{\omega}} \right), f^e \rangle$$

$$= \langle \theta_0 \varepsilon^{i-1} \Gamma \left( \frac{F_i}{\sqrt{\omega}}, f^e \right) + \Gamma \left( f^e, \frac{F_i}{\sqrt{\omega}} \right), (I - P) f^e \rangle$$

$$\leq \sum_{i=1}^{2k-1} \varepsilon^{i-1} \langle \theta_0 \int (1 + |v|) \frac{F_i}{\sqrt{\omega}} dv \|f^e\|_{\infty} \|f^e\|_{\nu} \rangle$$

$$\leq \left( \sum_{i=1}^{2k-1} \varepsilon^{i-1} \langle \theta_0 \int v \frac{F_i}{\sqrt{\omega}} dv \|f^e\|_{\infty} \rangle \right)^2 \frac{\varepsilon}{\kappa^2} \|f^e\|^2 + \frac{\kappa^2}{\varepsilon} \|\nu \nu \nu \| (I - P) f^e \|^2$$

$$\leq C \varepsilon \sum_{i=1}^{2k-1} \varepsilon^{(1+t)^{i-1}} \varepsilon \|f^e\|^2 + \frac{\kappa^2}{\varepsilon} \|\nu \nu \nu \| (I - P) f^e \|^2 \text{ from (2.7).}$$
Next we estimate $\varepsilon^k \langle \nabla_x \phi_R^\varepsilon \cdot \frac{v-u_0}{2} f^\varepsilon, f^\varepsilon \rangle$:

$$
\varepsilon^k \langle \nabla_x \phi_R^\varepsilon \cdot \frac{v-u_0}{2} f^\varepsilon, f^\varepsilon \rangle \leq \varepsilon^k \| \nabla \phi_R^\varepsilon \| \left( \int \int |v-u_0| |f^\varepsilon|^2 dv \right)^{\frac{1}{2}} \\
= \varepsilon^k \| \nabla \phi_R^\varepsilon \| : \left( \int \int |v-u_0| f^\varepsilon \frac{\sqrt{\omega M}}{w(v) \sqrt{\omega}} h^\varepsilon dv \right)^{\frac{1}{2}} \\
\leq \varepsilon^k \left( \int \int |v-u_0|^2 \omega M \frac{1}{w(v) \omega} dv \right)^{\frac{1}{2}} \| h^\varepsilon \|_\infty \| \nabla \phi_R^\varepsilon \| \cdot \| f^\varepsilon \| \\
\leq C \varepsilon^k \| h^\varepsilon \|_\infty \| \nabla \phi_R^\varepsilon \| \cdot \| f^\varepsilon \|
$$

From (2.7),

$$
-\langle \theta_0 \sum_{i=1}^{2k-1} \varepsilon^i \nabla_x \phi_i^\varepsilon \cdot \frac{\nabla_v F_i}{\sqrt{\omega}}, f^\varepsilon \rangle \leq \sum_{i=1}^{2k-1} \varepsilon^i \| \theta_0 \left( \frac{\nabla_v F_i}{\omega} \right)^{\frac{1}{2}} \|_{\infty} \| \nabla \phi_i^\varepsilon \| \cdot \| f^\varepsilon \| \\
\leq C \left( \sum_{i=1}^{2k-1} \varepsilon(1+t)^{i-1} \right) \{ \varepsilon \| \nabla \phi_i^\varepsilon \|^2 + \varepsilon \| f^\varepsilon \|^2 \}.
$$

Next,

$$
\langle \sum_{i=1}^{2k-1} \varepsilon^i \nabla_x \phi_i \cdot \frac{v-u_0}{2} f^\varepsilon, f^\varepsilon \rangle \leq \left( \sum_{i=1}^{2k-1} \varepsilon^i \| \nabla_x \phi_i \|^2 \right)^{\frac{1}{2}} \varepsilon \| f^\varepsilon \|^2 + \frac{\kappa^2}{\varepsilon} \{ I - P \} f^\varepsilon \right|^2 \\
\leq C \left( \sum_{i=1}^{2k-1} \varepsilon^i (1+t)^{i-1} \right)^2 \varepsilon \| f^\varepsilon \|^2 + \frac{\kappa^2}{\varepsilon} \| \{ I - P \} f^\varepsilon \|^2
$$

Lastly, by recalling (1.34) and from (2.7),

$$
\varepsilon^{k-1} \langle \theta_0, f^\varepsilon \rangle \leq C \left( \sum_{2k \leq i+j \leq 4k-2} \varepsilon^{i+j-2k} (1+t)^{i+j-2} \right) \varepsilon^{k-1} \| f^\varepsilon \|.
$$

We choose $\kappa$ sufficiently small to absorb $\frac{\varepsilon^2}{\varepsilon} \| \{ I - P \} f^\varepsilon \|^2$ terms into the dissipation in the left hand side of (3.2). Together with Lemma 2.2, we complete the proof of our proposition.

## 4 Characteristics

In this section, we study the curved trajectory for the Vlasov-Poisson-Boltzmann system (2). For any function $\phi \in L^\infty([0, T]; C^{2,\alpha})$, we define the characteristics $[X(\tau; t, x, v), V(\tau; t, x, v)]$ passing through $(t, x, v)$ such that

$$
\frac{dX(\tau; t, x, v)}{d\tau} = V(\tau; t, x, v), \quad X(t; t, x, v) = x \\
\frac{dV(\tau; t, x, v)}{d\tau} = \nabla_x \phi^\varepsilon (\tau, X(\tau; t, x, v)), \quad V(t; t, x, v) = v.
$$

**Lemma 4.1.** Assume $0 \leq T \leq \frac{1}{\varepsilon}$ and

$$
\sup_{0 \leq \tau \leq T} \varepsilon^k \| h^\varepsilon(\tau) \|_{W^{1,\infty}} \leq \sqrt{\varepsilon}.
$$

Then we have

$$
\sup_{0 \leq t \leq T} \{ \| \partial_x X(t) \|_\infty + \| \partial_t V(t) \|_\infty \} \leq C,
$$

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where $C$ is independent of $t$, $\varepsilon$. Moreover, there exists $0 < T_0 \leq T$ such that for $0 \leq \tau \leq t \leq T_0$

\[
\frac{1}{2}|t - \tau|^3 \leq \left| \det \left( \frac{\partial X(\tau)}{\partial v} \right) \right| \leq 2|t - \tau|^3, \quad (4.4)
\]

\[
\frac{1}{2} \leq \left| \det \left( \frac{\partial V(\tau)}{\partial v} \right) \right| \leq 2, \quad \frac{1}{2} \leq \left| \det \left( \frac{\partial X(\tau)}{\partial x} \right) \right| \leq 2, \quad (4.5)
\]

\[
\sup_{0 \leq \tau \leq T_0, x_0 \in \mathbb{R}^3, |v| \leq N} \left\{ \int_{|x-x_0| \leq C_N} \left( |\partial_{xx}X(\tau; t, x, v)|^2 + |\partial_{vc}X(\tau; t, x, v)|^2 \right) dx \right\}^{1/2} \leq C_N \text{ for } N \geq 1. \quad (4.6)
\]

\[
\begin{equation}
\Delta \phi^\varepsilon = \int (\omega + \sum_{i=1}^{2^{k-1}} \varepsilon^i F_i) dv + \varepsilon^k \int \frac{\chi_{\varepsilon^k}}{\varepsilon^k} h^\varepsilon dv - \bar{p} \quad \text{and by assumption (4.2), we obtain that for any } 0 < \alpha < 1, \text{ for } 0 \leq T \leq \frac{1}{\varepsilon},
\end{equation}
\]

\[
\| \nabla_x \phi^\varepsilon \|_{C^{1,\alpha}} \leq C\|h^\varepsilon\|_{W^{1,\infty}} \quad \text{and} \quad \| \nabla_x \phi^\varepsilon \|_{C^{1,\alpha}} \leq C + C\varepsilon I_1 + C\varepsilon^k\|h^\varepsilon\|_{W^{1,\infty}} \leq C. \quad (4.8)
\]

Noting that these characteristics are uniquely determined under the Lipschitz continuity condition of $\nabla_x \phi^\varepsilon$, we have for $\partial = \partial_x$ or $\partial_v$

\[
\frac{d^2 \partial X(\tau; t, x, v)}{dt^2} = \nabla_{xx} \phi^\varepsilon (\tau, X(\tau; t, x, v)) \partial X. \quad (4.9)
\]

By integrating in time, from the H"older continuity of $\nabla_x \phi^\varepsilon$ as in (4.8), one can deduce that for each $1 \leq i, j \leq 3$,

\[
\left\| \frac{\partial X^i}{\partial v} \right\|_{\infty} + \left\| \frac{\partial X^j}{\partial x_i} \right\|_{\infty} + \left\| \frac{\partial V^j}{\partial x_i} \right\|_{\infty} + \left\| \frac{\partial V^i}{\partial x_i} \right\|_{\infty} \leq C(1 + \varepsilon I_1) + C\varepsilon^k\|h^\varepsilon\|_{W^{1,\infty}} \quad (4.10)
\]

so that (4.3) follows.

In order to see the Jacobian of the change of variables: $v \rightarrow X(\tau)$, consider Taylor expansion of $\frac{\partial X(\tau)}{\partial v}$ in $\tau$ around $t$:

\[
\frac{\partial X(\tau)}{\partial v} = \frac{\partial X(t)}{\partial v} + (\tau - t) \frac{d}{d\tau} \left( \frac{\partial X(\tau)}{\partial v} \right)_{\tau=t} + \frac{(\tau - t)^2}{2} \frac{d^2}{d\tau^2} \left( \frac{\partial X(\tau)}{\partial v} \right)_{\tau=t} = (\tau - t)I + \frac{(\tau - t)^2}{2} \frac{d^2}{d\tau^2} \left( \frac{\partial X(\tau)}{\partial v} \right)_{\tau=t} \quad (4.11)
\]

for $\tau \leq \tilde{\tau} \leq t$. The Jacobian matrix $\left( \frac{\partial X(\tau)}{\partial v} \right)$ is given by

\[
\left( \frac{\partial X(\tau)}{\partial v} \right) = (\tau - t) \left\{ I + \frac{t-t}{2} \left( \frac{d^2}{d\tau^2} \left( \frac{\partial X(\tau)}{\partial v} \right) \right) \right\}
\]

or

\[
\left( \frac{\partial X(\tau)}{\partial v} \right) = \begin{pmatrix}
\partial_{v_1} X^1(\tau) & \partial_{v_2} X^1(\tau) & \partial_{v_3} X^1(\tau) \\
\partial_{v_1} X^2(\tau) & \partial_{v_2} X^2(\tau) & \partial_{v_3} X^2(\tau) \\
\partial_{v_1} X^3(\tau) & \partial_{v_2} X^3(\tau) & \partial_{v_3} X^3(\tau)
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\tau - t + \frac{(\tau - t)^2}{2} \frac{d^2}{d\tau^2} \partial_{v_1} X^1(\tau_1) & \frac{(\tau - t)^2}{2} \frac{d^2}{d\tau^2} \partial_{v_2} X^1(\tau_2) & \frac{(\tau - t)^2}{2} \frac{d^2}{d\tau^2} \partial_{v_3} X^1(\tau_3) \\
\frac{(\tau - t)^2}{2} \frac{d^2}{d\tau^2} \partial_{v_1} X^2(\tau_4) & \tau - t + \frac{(\tau - t)^2}{2} \frac{d^2}{d\tau^2} \partial_{v_2} X^2(\tau_5) & \frac{(\tau - t)^2}{2} \frac{d^2}{d\tau^2} \partial_{v_3} X^2(\tau_6) \\
\frac{(\tau - t)^2}{2} \frac{d^2}{d\tau^2} \partial_{v_1} X^3(\tau_7) & \frac{(\tau - t)^2}{2} \frac{d^2}{d\tau^2} \partial_{v_2} X^3(\tau_8) & \tau - t + \frac{(\tau - t)^2}{2} \frac{d^2}{d\tau^2} \partial_{v_3} X^3(\tau_9)
\end{pmatrix}
\]

We claim that if $T_0$ is sufficiently small, the determinant of the Jacobian is bounded from below and
above by $|t - \tau|^3$. Note that from (4.9)

\[
\frac{d^2}{d\tau^2} \frac{\partial X(\tau)}{\partial \tau} = \left| \frac{\partial}{\partial \tau} \nabla_x \phi^\tau(\tau, X(\tau, t, x, v)) \right| \leq |\nabla_x \nabla_x \phi^\tau||\frac{\partial X(\tau)}{\partial \tau}|. \tag{4.12}
\]

Now $\|\nabla_x \phi^\tau\|_{C^{1,0}} \leq C(1 + \varepsilon \|h^\tau\|_{W^{1,\infty}}) + C\varepsilon T_1 \leq C$ for $t \leq \frac{1}{\varepsilon}$, and thus by (4.10), we can choose $T_0$ sufficiently small so that

\[
\left| \frac{(\tau - t)}{2} \frac{d^2}{d\tau^2} \frac{\partial X(\tau)}{\partial \tau} \right| \leq \frac{CT_0}{2} \leq \frac{1}{8},
\]

and in turn

\[
\left| \frac{|t - \tau|^3}{2} \right| \leq \left| \frac{\partial X(\tau)}{\partial \tau} \right| \leq \frac{3|t - \tau|^3}{2}.
\]

We then deduce both (4.4) and (4.5).

On the other hand, consider Taylor expansion of $\frac{\partial X(\tau)}{\partial \tau}$ in $\tau$ around $t$:

\[
\frac{\partial X(\tau)}{\partial \tau} = \frac{\partial X(t)}{\partial \tau} + (\tau - t) \frac{d}{d\tau} \frac{\partial X(\tau)}{\partial \tau} \bigg|_{\tau = t} + \frac{(\tau - t)^2}{2} \frac{d^2}{d\tau^2} \frac{\partial X(\tau)}{\partial \tau}
\]

\[
= I + \frac{(\tau - t)^2}{2} \frac{d^2}{d\tau^2} \frac{\partial X(\tau)}{\partial \tau}, \tag{4.13}
\]

for $\tau \leq \bar{\tau} \leq t$. Note that from (4.9), we have

\[
\left| \frac{d^2}{d\tau^2} \frac{\partial X(\tau)}{\partial \tau} \right| = \left| \frac{\partial}{\partial \tau} \nabla_x \phi^\tau(\tau, X(\tau, t, x, v)) \right| \leq |\nabla_x \nabla_x \phi^\tau||\frac{\partial X(\tau)}{\partial \tau}|
\]

and thus (4.6) is valid for $\frac{\partial X(\tau)}{\partial \tau}$ and for $T_0$ small. We also have for $\tau \leq \bar{\tau} \leq t$

\[
\frac{\partial V(\tau)}{\partial \tau} = \frac{\partial V(t)}{\partial \tau} + \frac{d}{d\tau} \frac{\partial V(\bar{\tau})}{\partial \tau} (\tau - t)
\]

\[
= I + \left| \frac{\partial}{\partial \tau} \nabla_x \phi^\tau(\bar{\tau}) \frac{\partial X(\bar{\tau})}{\partial \tau} \right| (\tau - t).
\]

By (4.5), (4.6) is true for $\frac{\partial X(\tau)}{\partial \tau}$ for $T_0$ sufficiently small.

To show (4.7), we take one more derivative $\partial = \partial_x$ or $\partial_v$ of (4.11) to get

\[
\frac{d}{d\tau} \frac{\partial}{\partial \tau} \frac{\partial X(\tau)}{\partial \tau} = \frac{\partial}{\partial \tau} \left\{ \nabla_x \phi^\tau(\tau, X(\tau, t, x, v)) \partial_v X \right\}
\]

\[
= \nabla_x^3 \phi^\tau(\tau, X(\tau, t, x, v)) \{ \partial_v X \} \partial X
\]

\[
+ \nabla_x^2 \phi^\tau(\tau, X(\tau, t, x, v)) \{ \partial_v \partial_v X \} \partial X.
\]

We thus conclude that by integrating twice in time:

\[
\| \partial_v X(\tau) \|_{L^2( |x - x_0| \leq N)} \leq \frac{T^2_0 \| \nabla_x \phi^\tau \|_{C^{1,0}}^2}{2} \sup_{0 \leq \tau \leq T_0} \| \nabla_x^3 \phi^\tau(\tau, X(\tau, t, x, v)) \|_{L^2( |x - x_0| \leq N)}
\]

\[
+ \frac{T^2_0 \| \nabla_x^2 \phi^\tau \|_{C^{1,0}}^2}{2} \sup_{0 \leq \tau \leq T_0} \| \partial_v \partial_v X(\tau) \|_{L^2( |x - x_0| \leq N)}.
\]
We note that for $|v| \leq N$, from the characteristic equation,

\[
|X(\tau; t, x, v) - x_0| \leq |X(\tau; t, x, v) - x| + |x - x_0| \leq |v|(t - \tau) + \int_\tau^t \int_\tau^{s_1} |\nabla_x \phi^\varepsilon(s)| ds ds_1 + CN
\]

\[
\leq T_0 N + CT_0^2 + N \leq CN,
\]

for $T_0 \leq 1$ sufficiently small and $N \geq 1$. From boundedness of $||\nabla^2_x \phi^\varepsilon||_\infty$ and (4.6), we make a change of variables $x \to X(\tau; t, x, v)$ in $\nabla^3_x \phi^\varepsilon$ to get

\[
\sup_{0 \leq \tau \leq T_0, x_0 \in \mathbb{R}^3, |v| \leq N} ||\partial_{\tau} X(\tau)||_{L^2(|x - x_0| \leq N)} \leq C \sup_{0 \leq \tau \leq T_0, x_0 \in \mathbb{R}^3, |v| \leq N} ||\nabla^3_x \phi^\varepsilon(\tau, X(\tau; t, x, v))||_{L^2(|X(\tau; t, x, v) - x_0| \leq CN)} \leq C \sup_{0 \leq \tau \leq T_0, x_0 \in \mathbb{R}^3, |v| \leq N} ||\nabla^3_x \phi^\varepsilon(\tau)||_{L^2(|X(\tau) - x_0| \leq CN)},
\]

for $T_0$ small. To control $||\nabla^3_x \phi^\varepsilon(\tau)||_{L^2(|X(\tau) - x_0| \leq CN)}$, we make use of the Poisson equation $\Delta \phi^\varepsilon = \int (\omega + \sum_{i=1}^{2k-1} \varepsilon^i F_i) dv + \varepsilon^k \int \frac{\omega_M}{w} h^\varepsilon dv - \bar{\rho}$. Note

\[
\Delta \partial_x \phi^\varepsilon = \int (\partial_x \omega + \sum_{i=1}^{2k-1} \varepsilon^i \partial_x F_i) dv + \varepsilon^k \int \frac{\omega_M}{w} \partial_x h^\varepsilon dv.
\]

Letting $\chi$ be a smooth cutoff function of $|x - x_0| \leq CN + 1$, we have

\[
\Delta \partial_x \{\chi \phi^\varepsilon\} = \chi \int (\partial_x \omega + \sum_{i=1}^{2k-1} \varepsilon^i \partial_x F_i) dv + \varepsilon^k \int \frac{\omega_M}{w} \partial_x h^\varepsilon dv + \sum_{|\alpha + \beta| = 3, |\beta| \leq 2} \partial^\alpha \chi \partial^\beta \phi^\varepsilon.
\]

It thus follows that, from the assumption (4.2), and the fact $\partial_x \omega, \partial_x F_i \in L^2$, we conclude

\[
||\nabla^3_x \phi^\varepsilon(\tau, x)||_{L^2(|x - x_0| \leq CN)} \leq C + CN^{3/2} \varepsilon^k ||h^\varepsilon||_{W^{1, \infty}} + C ||\phi^\varepsilon||_C N^{3/2} \leq CN^{3/2}.
\]

We then complete the proof of (4.7).

\[\square\]

5 $W^{1, \infty}$ Estimates for Remainder $F^\varepsilon_R$

In this section we establish $W^{1, \infty}$ estimate for $h^\varepsilon$ with suitable factors of $\varepsilon$. To be more precise, we will show that for sufficiently small $\varepsilon$, $||\varepsilon^3/2 h^\varepsilon||_\infty$ and $||\varepsilon^{9/2} \nabla_{x,v} h^\varepsilon||_\infty$ are bounded by $||f^\varepsilon||$ and initial data. Recall $I_1$ and $I_2$ in (3.1).

We now turn to the main estimates of $h^\varepsilon$. As the first preparation, we define

\[
L_M g = -\frac{1}{\sqrt{\omega_M}} \{Q(\omega, \sqrt{\omega_M} g) + Q(\sqrt{\omega_M} g, \omega)\} = \{\nu(\omega) + K_M\} g
\]
as in [2]. Letting $K_{M,w} g \equiv wK_M(\frac{h}{w})$, from [3] and [4.17], we obtain

$$\partial_t h^\varepsilon + v \cdot \nabla_x h^\varepsilon + \nabla_x \phi^\varepsilon \cdot \nabla_x h^\varepsilon + \frac{\nu(\omega)}{\varepsilon} h^\varepsilon + \frac{1}{\varepsilon} K_{M,w} h^\varepsilon$$

$$= \frac{\varepsilon^{k-1}}{\sqrt{\omega_M}} Q(h^\varepsilon \sqrt{\omega_M} w, h^\varepsilon \sqrt{\omega_M} w)^2 + \sum_{i=1}^{2k-1} \varepsilon^{i-1} \frac{w}{\sqrt{\omega_M}} \{Q(F_i, - \frac{h^\varepsilon \sqrt{\omega_M} w}{w}) + Q(h^\varepsilon \sqrt{\omega_M} w, F_i)\}$$

$$- \nabla_x \phi^\varepsilon \cdot \frac{w}{\sqrt{\omega_M}} \nabla_v (\sqrt{\omega_M} w) h^\varepsilon - \nabla_x \phi^\varepsilon \cdot \frac{w}{\sqrt{\omega_M}} \nabla_v (\omega + \sum_{i=1}^{2k-1} \varepsilon^i F_i) + \varepsilon^{k-1} \frac{w}{\sqrt{\omega_M}} A$$

(5.2)

where $\nabla \phi^\varepsilon = \sum_{n=0}^{2k-1} \varepsilon^{n} \nabla \phi_n + \varepsilon^{k} \nabla \phi^\varepsilon_R$. Our main task is to derive $W^{1,\infty}$ estimates of $h^\varepsilon$

**Proposition 5.1.** Let $0 < T \leq \frac{1}{2}$ be given and the electric fields $\nabla \phi^\varepsilon_R$ and $\nabla \phi^\varepsilon$ satisfy the estimates (4.8). For all $\varepsilon$ sufficiently small, there exists a constant $C > 0$ independent of $T$ and $\varepsilon$ such that

$$\sup_{0 \leq s \leq T} \{\varepsilon^{3/2} \|h^\varepsilon(s)\|_\infty \leq C \{\|e^{3/2} h_0\|_\infty + \sup_{0 \leq s \leq T} \|f^\varepsilon(s)\| + \varepsilon^{(2k+1)/2}\},$$

(5.3)

as well as

$$\sup_{0 \leq s \leq T} \{\varepsilon^{3/2} \|h^\varepsilon(s)\|_\infty \leq \varepsilon^n \nabla \phi_n + \varepsilon^k \nabla \phi^\varepsilon_R\}$$

$$\leq C \{\|e^{3/2} (1 + |v|) h_0\|_\infty + \varepsilon^n \nabla \phi_n + \varepsilon^k \nabla \phi^\varepsilon_R \}$$

(5.4)

The proof of the proposition relies on the following two lemmas:

**Lemma 5.2.** Assume (4.14). There exists a $T_0 > 0$ such that $0 \leq T_0 \leq T \leq \frac{1}{2}$ for all $\varepsilon$ sufficiently small,

$$\sup_{0 \leq s \leq T_0} \{\varepsilon^{3/2} \|h^\varepsilon(s)\|_\infty \leq C \{\|e^{3/2} h_0\|_\infty + \sup_{0 \leq s \leq T} \|f^\varepsilon(s)\| + \varepsilon^{(2k+1)/2}\}$$

(5.5)

and moreover,

$$\varepsilon^{3/2} \|h^\varepsilon(T_0)\|_\infty \leq \frac{1}{2} \|e^{3/2} h_0\|_\infty + C \{\sup_{0 \leq s \leq T} \|f^\varepsilon(s)\| + \varepsilon^{(2k+1)/2}\}$$

(5.6)

**Lemma 5.3.** For $T_0 > 0$ obtained in Lemma 5.2, there exists a sufficiently small $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$,

$$\sup_{0 \leq s \leq T_0} \{\varepsilon^n \|Dh^\varepsilon(s)\|_\infty + \varepsilon^n \|D_v h^\varepsilon(s)\|_\infty \leq C \{\varepsilon^n \|(1 + |v|) h_0\|_\infty + \varepsilon^5 \|D h_0\|_\infty + \varepsilon^5 \|D_v h_0\|_\infty$$

$$+ \varepsilon^{3/2} \|h_0\|_\infty + \varepsilon^{1/2} \sup_{0 \leq s \leq T} \|f^\varepsilon(s)\| + \varepsilon^{k+1}\}$$

(5.7)

and moreover,

$$\varepsilon^n \|Dh^\varepsilon(T_0)\|_\infty + \varepsilon^n \|D_v h^\varepsilon(T_0)\|_\infty \leq \frac{1}{2} (\varepsilon^n \|(1 + |v|) h_0\|_\infty + \varepsilon^5 \|D h_0\|_\infty + \varepsilon^5 \|D_v h_0\|_\infty + \varepsilon^{3/2} \|h_0\|_\infty$$

$$+ C \{\varepsilon^{1/2} \sup_{0 \leq s \leq T} \|f^\varepsilon(s)\| + \varepsilon^{k+1}\}.$$

(5.8)

Once we establish Lemma 5.2 and 5.3 by bootstrapping the time interval into the given time $T$, we can readily conclude Proposition 5.1. We also remark that in light of the estimate in Proposition 5.1, the assumption (4.12) will be automatically satisfied by a continuity argument.

**Proof of Proposition 5.1.** If $t \leq T_0$, the conclusion directly follows from Lemma 5.2 and Lemma 5.3. Assume that $T_0 \leq t \leq T$. Then there exists a positive integer $n$ so that $t = nT_0 + \tau$ where $0 \leq \tau \leq T_0$. 

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Apply \((5.6)\) in Lemma \(5.2\) repeatedly to get for each \(n\),
\[
\varepsilon^{3/2} \| h(t) \|_\infty \leq \frac{1}{2} \| \varepsilon^{3/2} h\(\{n-1\}T_0 + \tau\) \|_\infty + C\{ \sup_{0 \leq s \leq T} \| f^\varepsilon(s) \| + \varepsilon^{(2k+1)/2}\}
\]
\[
\leq \frac{1}{4} \| \varepsilon^{3/2} h\(\{n-2\}T_0 + \tau\) \|_\infty + \left\{ \frac{C}{2} + C\right\} \{ \sup_{0 \leq s \leq T} \| f^\varepsilon(s) \| + \varepsilon^{(2k+1)/2}\}
\]
\[
\leq \ldots
\]
\[
\leq \frac{1}{2^n} \| \varepsilon^{3/2} h(\tau) \|_\infty + 2C\{ \sup_{0 \leq s \leq T} \| f^\varepsilon(s) \| + \varepsilon^{(2k+1)/2}\}
\]
since \(1 + \frac{1}{2} + \frac{1}{4} + \ldots + \frac{1}{2^n} \leq 2\) for each \(n\). From \((5.3)\), the estimate \((5.3)\) follows. Similarly, one can deduce \((5.4)\) from the above two lemmas.

In the following two subsections, we prove the above two lemmas.

5.1 \(L^\infty\) bound : Proof of Lemma \(5.2\)

Since \(\frac{d}{ds} h(s, X(s; t, x, v), V(s; t, x, v)) = \partial_t h + \nabla_x h \cdot \frac{dX}{ds} + \nabla_v h \cdot \frac{dV}{ds}\), the solution to the following transport equation
\[
\partial_t h^\varepsilon + v \cdot \nabla_x h^\varepsilon + \nabla_x \phi^\varepsilon \cdot \nabla_x h^\varepsilon + \frac{\nu(\omega)}{\varepsilon} h^\varepsilon = 0
\]
can be written as \(h^\varepsilon(t, v, x) = \exp\{-\frac{1}{\varepsilon} \int_0^t \nu(\tau) d\tau\} h^\varepsilon(0, X(0; t, x, v), V(0; t, x, v))\). Thus for any \((t, x, v)\), integrating along the backward trajectory \([4.1]\), by the Duhamel’s principle, the solution \(h^\varepsilon(t, x, v)\) of the original nonlinear equation \((5.2)\) can be written as follows:

\[
\begin{align*}
&h^\varepsilon(t, x, v) = \exp\{-\frac{1}{\varepsilon} \int_0^t \nu(\tau) d\tau\} h^\varepsilon(0, X(0; t, x, v), V(0; t, x, v)) \\
&- \int_0^t \exp\{-\frac{1}{\varepsilon} \int_s^t \nu(\tau) d\tau\} \left( \frac{1}{\varepsilon} K_{M, w} h^\varepsilon \right) (s, X(s), V(s)) ds \\
&+ \int_0^t \exp\{-\frac{1}{\varepsilon} \int_s^t \nu(\tau) d\tau\} \left( \frac{\varepsilon^{k-1} w}{\sqrt{\omega_M}} Q\left(h^\varepsilon \sqrt{\omega_M}, h^\varepsilon \sqrt{\omega_M} \right) \right) (s, X(s), V(s)) ds \\
&+ \int_0^t \exp\{-\frac{1}{\varepsilon} \int_s^t \nu(\tau) d\tau\} \left( \sum_{i=1}^{2k-1} \varepsilon^{i-1} w \sqrt{\omega_M} Q(F_i, h^\varepsilon \sqrt{\omega_M}, \sqrt{\omega_M} w) \right) (s, X(s), V(s)) ds \\
&- \int_0^t \exp\{-\frac{1}{\varepsilon} \int_s^t \nu(\tau) d\tau\} \left( \nabla_x \phi^\varepsilon \cdot \frac{w}{\sqrt{\omega_M}} \nabla_v \left( \sqrt{\omega_M} w \right) h^\varepsilon \right) (s, X(s), V(s)) ds \\
&- \int_0^t \exp\{-\frac{1}{\varepsilon} \int_s^t \nu(\tau) d\tau\} \left( \nabla_x \phi^\varepsilon \cdot \frac{w}{\sqrt{\omega_M}} \nabla_v \left( \sqrt{\omega_M} w \right) \right) (s, X(s), V(s)) ds \\
&+ \int_0^t \exp\{-\frac{1}{\varepsilon} \int_s^t \nu(\tau) d\tau\} \left( \varepsilon^{k-1} w \sqrt{\omega_M} A \right) (s, X(s), V(s)) ds.
\end{align*}
\]

We will prove only \((5.6)\). The estimate of \((5.5)\) can be obtained in the same way by directly estimating \(\| h^\varepsilon \|_\infty\) rather than \(\frac{d}{ds} \| h^\varepsilon \|_\infty\) in \((5.22)\) as done in \([7, 8]\).

Since \(\| \frac{d}{ds} Q\left(h^\varepsilon \sqrt{\omega_M}, h^\varepsilon \sqrt{\omega_M} \right) \| \leq C \nu(\omega) \| h^\varepsilon \|_2^2 \leq C \varepsilon \) from Lemma 10 in \([6]\), and since
\[
\nu(\omega) \sim \int |v - u| \nu dv \sim (1 + |v|) \rho_0(t, x) \sim \nu_M(v), \quad \nu(\omega) \geq 2\nu_0 > 0
\]
Since \( \nu \sim \nu_M \), we bound the second term by
\[
\frac{1}{\varepsilon} \int_0^t \exp\left\{-\frac{1}{\varepsilon} \int_s^t \nu(\tau)d\tau\right\} \nu(\omega)e^{-\frac{\nu_M(t-s)}{\varepsilon}} ds \leq C e^{-\frac{\nu_M t}{\varepsilon}} \int_0^t \exp\left\{-\frac{\nu_M(t-s)}{\varepsilon}\right\} \nu_M ds \leq C \varepsilon e^{-\frac{\nu_M t}{\varepsilon}},
\]
the third line in (5.9) is bounded by
\[
C \varepsilon^{k-1} \int_0^t \exp\left\{-\frac{1}{\varepsilon} \int_s^t \nu(\tau)d\tau\right\} \nu(\omega)\|h^\varepsilon(s)\|_\infty^2 ds \leq C \varepsilon^k e^{-\frac{\nu_M t}{\varepsilon}} \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_M}{\varepsilon}} \|h^\varepsilon(s)\|_\infty \right\}^2.
\]
From Lemma 10 in [6] again,
\[
\sum_{i=1}^{2k-1} e^{i-1} \frac{\omega}{\sqrt{\omega_M}} \{Q(F_i, h^\varepsilon \sqrt{\omega_M} \omega) + Q(h^\varepsilon \sqrt{\omega_M} \omega, F_i)\} \leq \nu_M \|h^\varepsilon\|_\infty \frac{\omega}{\sqrt{\omega_M}} \sum_{i=1}^{2k-1} e^{i-1} F_i \|_\infty,
\]
so that the fourth and fifth lines in (5.9) are bounded by
\[
\int_0^t \exp\left\{-\frac{\nu_M(t-s)}{\varepsilon}\right\} \nu_M \|h^\varepsilon(s)\|_\infty ds \leq C \varepsilon \mathcal{I}_1 e^{-\frac{\nu_M t}{\varepsilon}} \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_M}{\varepsilon}} \|h^\varepsilon(s)\|_\infty \right\}.
\]
Since \( \left| \frac{\omega}{\sqrt{\omega_M}} \nabla_v (\sqrt{\omega_M} \omega) \right| \leq C(1 + |v|) \) and \( \left| \frac{\omega}{\sqrt{\omega_M}} \nabla_v (\omega + \sum_{i=1}^{2k-1} e^{i} F_i) \right| \leq C + \varepsilon \mathcal{I}_1 \), the sixth and seventh lines in (5.9), from (1.3), are bounded by \( (C + \varepsilon \mathcal{I}_1) \varepsilon e^{-\frac{\nu_M t}{\varepsilon}} \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_M}{\varepsilon}} \|h^\varepsilon(s)\|_\infty \right\} \). The last line in (5.9) is clearly bounded by \( C \mathcal{I}_2 \varepsilon^k \).

We shall mainly concentrate on the second term on the right hand side of (5.9). Let \( l_M (v, v') \) be the corresponding kernel associated with \( K_M \) in [2], we have
\[
|l_M (v, v')| \leq C \{ |v - v'| + \frac{1}{|v - v'|} \} \exp\left\{-c|v - v'|^2 - c \frac{|v|^2 - |v'|^2}{|v - v'|^2} \right\} = (5.10)
\]
Since \( \nu(\omega) \sim \nu_M \), we bound the second term by
\[
\frac{1}{\varepsilon} \int_0^t \exp\left\{-\frac{1}{\varepsilon} \int_s^t \nu(\tau)d\tau\right\} \int_{\mathbb{R}^3} |l_M (v, v')| h^\varepsilon (s, X(s), v') dv' ds, \tag{5.11}
\]
where \( l_M (v, v') = \frac{w(\tilde{v}, v')}{w(\tilde{v}, v')} l_M (\tilde{v}, v') \). We now use (5.9) again to evaluate \( K_M u h^\varepsilon \) in (5.11). Denoting
\[
[X(s_1), V(s_1)] = [X(s_1; s, X(s; t, x, v), v'), V(s_1; s, X(s; t, x, v), v')],
\]
we can further bound (5.11) by
\[
\frac{1}{\varepsilon} \int_{0}^{t} \exp \left\{ -\frac{1}{\varepsilon} \int_{s}^{t} \nu(\tau) d\tau \right\} \int_{\mathbb{R}^3} \left| l_{M,w}(V(s), v') h^\varepsilon(0, X(0), V(0)) \right| dv' ds \\
+ \frac{1}{\varepsilon^2} \int_{0}^{t} \int_{0}^{s} \exp \left\{ -\frac{1}{\varepsilon} \int_{s}^{t} \nu(\tau) d\tau \right\} \int_{\mathbb{R}^3} \left| l_{M,w}(V(s), v') l_{M,w}(V(s_1), v'') \right| h^\varepsilon(s_1, X(s_1), v'') dv'' ds_1 ds \\
+ \frac{1}{\varepsilon} \int_{0}^{t} \int_{0}^{s} \exp \left\{ -\frac{1}{\varepsilon} \int_{s}^{t} \nu(\tau) d\tau \right\} \int_{\mathbb{R}^3} \left| l_{M,w}(V(s), v') \right| \left( \frac{\varepsilon^{-1} w}{\sqrt{\omega_M}} Q \left( \frac{h^\varepsilon \sqrt{\omega_M}}{w}, \frac{h^\varepsilon \sqrt{\omega_M}}{w} \right) \right) (s_1, X(s_1), V(s_1)) dv'' ds_1 ds \\
+ \frac{1}{\varepsilon} \int_{0}^{t} \int_{0}^{s} \exp \left\{ -\frac{1}{\varepsilon} \int_{s}^{t} \nu(\tau) d\tau \right\} \int_{\mathbb{R}^3} \left| l_{M,w}(V(s), v') \right| \left( \sum_{i=1}^{2k-1} \frac{w}{\sqrt{\omega_M}} Q(F_i, h^\varepsilon \sqrt{\omega_M} \frac{w}{w}) + Q \left( h^\varepsilon \sqrt{\omega_M} \frac{w}{w}, F_i \right) \right) (s_1, X(s_1), V(s_1)) dv'' ds_1 ds \\
+ \frac{1}{\varepsilon} \int_{0}^{t} \int_{0}^{s} \exp \left\{ -\frac{1}{\varepsilon} \int_{s}^{t} \nu(\tau) d\tau \right\} \int_{\mathbb{R}^3} \left| l_{M,w}(V(s), v') \right| \left( \nabla_x \phi^\varepsilon \cdot \frac{w}{\sqrt{\omega_M}} \nabla_v (\sqrt{\omega_M} h^\varepsilon) + \nabla_x \phi_R^\varepsilon \cdot \frac{w}{\sqrt{\omega_M}} \nabla_v (\omega + \sum_{i=1}^{2k-1} \varepsilon^i F_i) \right) (s_1, X(s_1), V(s_1)) dv'' ds_1 ds \\
+ \frac{1}{\varepsilon} \int_{0}^{t} \int_{0}^{s} \exp \left\{ -\frac{1}{\varepsilon} \int_{s}^{t} \nu(\tau) d\tau \right\} \int_{\mathbb{R}^3} \left| l_{M,w}(V(s), v') \right| \left( \frac{\varepsilon^{-1} w}{\sqrt{\omega_M}} A \right) (s_1, X(s_1), V(s_1)) dv'' ds_1 ds.
\] (5.12)

Since \( \sup_{\mathbb{R}^3} \left| l_{M,w}(\tilde{v}, v') \right| dv' < +\infty \) from Lemma 7 in [6], and by the previous estimates, there is an upper bound except for the second term as
\[
\frac{t}{\varepsilon} e^{-\frac{\eta t}{\varepsilon}} \left| h^\varepsilon(0) \right|_\infty + \varepsilon^k e^{-\frac{\eta t}{\varepsilon}} \sup_{0 \leq s \leq t} \left\{ e^{\frac{\eta t}{\varepsilon}} \| h^\varepsilon(s) \|_\infty \right\}^2 + (1 + T_1) e^{-\frac{\eta t}{\varepsilon}} \sup_{0 \leq s \leq t} \left\{ e^{\frac{\eta t}{\varepsilon}} \| h^\varepsilon(s) \|_\infty \right\} + T_2 \varepsilon^k
\]
up to a constant. We now concentrate on the second term in (5.12) and we follow the same spirit of the proof of Theorem 20 in [6]. From (4.18) and (4.11), fix \( N > 0 \) large enough so that
\[
\frac{N}{2} \geq \sup_{0 \leq t \leq T, 0 \leq s \leq T} |V(s) - v|.
\]

Note that by Lemma 7 in [6] (Grad estimate),
\[
\int \int \left| l_{M,w}(V(s), v') l_{M,w}(V(s_1), v'') \right| dv' dv'' \leq \frac{C}{1 + |V(s)|}.
\] (5.13)

We divide into four cases according to the size of \( v, v', v'' \) and for each case, an upper bound of the second term in (5.12) will be obtained.

**CASE 1:** \( |v| \geq N \). In this case, since \( |V(s)| \geq \frac{N}{2} \), (5.13) implies that
\[
\int \int \left| l_{M,w}(V(s), v') l_{M,w}(V(s_1), v'') \right| dv' dv'' \leq \frac{C}{N},
\]
and thus we have the following bound
\[
\begin{align*}
\frac{C}{\varepsilon^2N} \int_0^t \int_0^s \exp\left(-\frac{\nu_M(t-s)}{\varepsilon}\right) \exp\left(-\frac{\nu_M(s-s_1)}{\varepsilon}\right) \|h^\varepsilon(s_1)\|_\infty ds_1 ds 
\leq \frac{C}{N} e^{-\frac{\nu_M}{\varepsilon}} \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_M}{\varepsilon}} \|h^\varepsilon(s)\|_\infty \right\}.
\end{align*}
\]

**CASE 2:** $|v| \leq N$, $|v'| \geq 2N$, or $|v'| \leq 2N$, $|v''| \geq 3N$. Observe that
\[
\begin{align*}
|V(s) - v'| &\geq |v' - v| - |V(s) - v| \geq |v'| - |v| - |V(s) - v| \\
|V(s_1) - v'| &\geq |v'' - v'| - |V(s_1) - v'| \geq |v''| - |v'| - |V(s_1) - v'|
\end{align*}
\]
thus we have either $|V(s) - v'| \geq \frac{N}{2}$ or $|V(s_1) - v'| \geq \frac{N}{2}$, and either one of the following are valid correspondingly for $\eta > 0$:
\[
\begin{align*}
|l_{M,w}(V(s), v')| &\leq e^{-\frac{\nu_M}{\varepsilon} N^2} |l_{M,w}(V(s), v') e^{\frac{\nu_M}{\varepsilon} |V(s) - v'|^2}|, \\
|l_{M,w}(V(s_1), v'')| &\leq e^{-\frac{\nu_M}{\varepsilon} N^2} |l_{M,w}(V(s_1), v'') e^{\frac{\nu_M}{\varepsilon} |V(s_1) - v''|^2}|.
\end{align*}
\] (5.14)

From Lemma 7 in [6], both $\int l_{M,w}(V(s), v') e^{\frac{\nu_M}{\varepsilon} |V(s) - v'|^2} dv'$ and $\int l_{M,w}(V(s_1), v'') e^{\frac{\nu_M}{\varepsilon} |V(s_1) - v''|^2} dv''$ are still finite for sufficiently small $\eta > 0$. We use (5.14) to combine the cases of $|V(s) - v'| \geq \frac{\eta}{2}$ or $|V(s_1) - v''| \geq \frac{\eta}{2}$ to get the following bound
\[
\begin{align*}
\int_0^t \int_0^s \left\{ \int_{|v'| \leq 2N, |v''| \geq \frac{\eta}{2}} \right\} + \int_{|v'| \leq 2N, |v''| \geq \frac{\eta}{2}} \left\{ \int_0^s \exp\left(-\frac{\nu_M(t-s)}{\varepsilon}\right) \exp\left(-\frac{\nu_M(s-s_1)}{\varepsilon}\right) \|h^\varepsilon(s_1)\|_\infty ds_1 ds 
\leq C_\eta e^{-\frac{\nu_M}{\varepsilon} N^2} \int_0^s \int_0^s \exp\left(-\frac{\nu_M(t-s)}{\varepsilon}\right) \exp\left(-\frac{\nu_M(s-s_1)}{\varepsilon}\right) \|h^\varepsilon(s_1)\|_\infty ds_1 ds 
\leq C_\eta e^{-\frac{\nu_M}{\varepsilon} N^2} \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_M}{\varepsilon}} \|h^\varepsilon(s)\|_\infty \right\}.
\end{align*}
\] (5.15)

**CASE 3a:** $|v| \leq N$, $|v'| \leq 2N$, $|v''| \leq 3N$. This is the last remaining case because if $|v'| > 2N$, it is included in Case 2; while if $|v''| > 3N$, either $|v'| \leq 2N$ or $|v'| > 2N$ are also included in Case 2. We further assume that $s - s_1 \leq \varepsilon \kappa$, for $\kappa > 0$ small. We bound the second term in (5.12) by
\[
\begin{align*}
\frac{C_N}{\varepsilon^2} \int_0^t \int_{s-\varepsilon \kappa}^s \exp\left(-\frac{\nu_M(t-s)}{\varepsilon}\right) \exp\left(-\frac{\nu_M(s-s_1)}{\varepsilon}\right) \|h^\varepsilon(s_1)\|_\infty ds_1 ds 
\leq C_N e^{-\frac{\nu_M}{\varepsilon} \kappa} \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_M}{\varepsilon} \kappa} \|h^\varepsilon(s)\|_\infty \right\} \left( \frac{1}{\varepsilon} \int_0^t \exp\left(-\frac{\nu_M(t-s)}{2\varepsilon}\right) ds \right) \left( \int_{s-\varepsilon \kappa}^s \frac{1}{\varepsilon} ds \right) 
\leq \kappa C_N e^{-\frac{\nu_M}{\varepsilon} \kappa} \sup_{0 \leq s \leq t} \left\{ e^{\frac{\nu_M}{\varepsilon} \kappa} \|h^\varepsilon(s)\|_\infty \right\}.
\end{align*}
\] (5.16)

**CASE 3b:** $|v| \leq N$, $|v'| \leq 2N$, $|v''| \leq 3N$, and $s - s_1 \geq \varepsilon \kappa$. We now can bound the second term in (5.12) by
\[
\begin{align*}
\frac{C}{\varepsilon^2} \int_0^t \int_B \int_0^s e^{-\frac{\nu_M(t-s)}{\varepsilon}} e^{-\frac{\nu_M(s-s_1)}{\varepsilon}} |l_{M,w}(V(s), v') l_{M,w}(V(s_1), v'') h^\varepsilon(s_1, X(s_1), v'')| 
\end{align*}
\]
where $B = \{|v'| \leq 2N$, $|v''| \leq 3N\}$. By (5.10), $l_{M,w}(v, v')$ has possible integrable singularity of $\frac{1}{|v-v'|}$, we can choose $l_N(v, v')$ smooth with compact support such that
\[
\sup_{|p| \leq 3N, |v'| \leq 3N} \int_{|v'| \leq 3N} \left| l_N(p, v') - l_{M,w}(p, v') \right| dv' \leq \frac{1}{N}.
\] (5.17)
we can use such an approximation (5.17) to bound the above \( s_1, s \) integration by

\[
\frac{C}{N} \sup_{0 \leq s \leq \tau} \| h^\varepsilon (s) \|_\infty \cdot \left\{ \sup_{|v'| \leq 2N} \int |l_{M,w}(V(s), v')| dv' + \sup_{|v'| \leq N} \int |l_N(V(s), v')| dv' \right\} \\
+ \frac{C}{\varepsilon^2} \int_0^t \int_0^{s-\kappa\varepsilon} e^{-\nu(v') \epsilon} e^{-\nu(v'' \epsilon)} |l_N(V(s), v') l_N(V(s), v'') h^\varepsilon (s_1, X(s_1), v'')| dv' dv'' ds_1 ds.
\]

Introduce a new variable

\[
y = X(s_1) = X(s_1; s, X(s; t, x, v), v')
\]

such that

\[
|y - X(s)| = |X(s_1) - X(s)| \leq C(s - s_1).
\]

We now apply Lemma [4.1] to \( X(s_1; s, X(s; t, x, v), v') \) with \( x = X(s; t, x, v), \tau = s_1, t = s \). By (4.4), we can choose small but fixed \( T_0 \) > 0 such that for \( s - s_1 \geq \kappa\varepsilon \),

\[
\frac{dy}{dv'} \geq \frac{\kappa^3 \varepsilon^3}{2}.
\]

Since \( l_N(V(s), v') l_N(V(s_1), v'') \) is bounded, we first integrate over \( v' \) to get

\[
C_N \int_{|v'| \leq 2N} |h^\varepsilon (s_1, X(s_1), v'')| dv'
\]

\[
\leq C_N \left\{ \int_{|v'| \leq 2N} 1\Omega(X(s_1)) |h^\varepsilon (s_1, X(s_1), v'')|^2 dv' \right\}^{1/2}
\]

\[
\leq \frac{C_N}{\kappa^{3/2} \varepsilon^{3/2}} \left\{ \int_{|y - X(s)| \leq C(s - s_1)N} |h^\varepsilon (s_1, y, v'')|^2 dy \right\}^{1/2}
\]

\[
\leq \frac{C_N}{\kappa^{3/2} \varepsilon^{3/2}} \left\{ \int_{\mathbb{R}^3} |h^\varepsilon (s_1, y, v'')|^2 dy \right\}^{1/2}.
\]

By (1.7) and (1.0), we then further control the last term in (5.18) by:

\[
\frac{C_N \kappa}{\varepsilon^{7/2} T_0^{3/2}} \int_0^{s-\kappa\varepsilon} e^{-\nu(v') \epsilon} e^{-\nu(v'' \epsilon)} \int_{|v'| \leq 3N} \left\{ \int_{\mathbb{R}^3} |h^\varepsilon (s, y, v'')|^2 dy \right\}^{1/2} dv'' ds_1 ds
\]

\[
\leq \frac{C_N \kappa}{\varepsilon^{7/2} T_0^{3/2}} \int_0^{s-\kappa\varepsilon} e^{-\nu(v') \epsilon} e^{-\nu(v'' \epsilon)} \left\{ \int_{|v'| \leq 3N} \int_{\mathbb{R}^3} |f^\varepsilon (s, y, v'')|^2 dy dv'' \right\}^{1/2} ds_1 ds
\]

\[
\leq \frac{C_N \kappa}{\varepsilon^{9/2}} \sup_{0 \leq s \leq T} \| f^\varepsilon (s) \|.
\]
In summary, we have established, for any \( \kappa > 0 \) and large enough \( N > 0 \),

\[
\sup_{0 \leq s \leq T_0} \{ e^{\frac{\nu s}{\varepsilon}} \| h^\varepsilon(s) \|_\infty \} 
\leq C \sup_{0 \leq s \leq T_0} \{ (1 + \frac{s}{\varepsilon}) e^{-\frac{\nu s}{2\varepsilon}} \} \| h_0 \|_\infty + \{ C(1 + I_1(T_0)) \varepsilon + C\kappa + \frac{C\nu}{N} \} \sup_{0 \leq s \leq T_0} \{ e^{\frac{\nu s}{2\varepsilon}} \| h^\varepsilon(s) \|_\infty \} \tag{5.22}
\]

\[+ C\varepsilon^k \sup_{0 \leq s \leq T_0} \{ e^{\frac{\nu s}{3\varepsilon}} \| h^\varepsilon(s) \|_\infty \}^2 + \frac{C_{N,\kappa}}{\varepsilon^{3/2}} \sup_{0 \leq s \leq T} \| f_\varepsilon(s) \| + C I_2(T_0) \frac{\nu T_0}{\varepsilon}\varepsilon^k \].

Note that \((1 + \frac{s}{\varepsilon}) e^{-\frac{\nu s}{2\varepsilon}}\) is uniformly bounded in \( s \) and \( \varepsilon \) and \( I_1(T_0) \) and \( I_2(T_0) \) are uniformly bounded in \( \varepsilon \). For sufficiently small \( \varepsilon > 0 \), first choosing \( \kappa \) small, then \( N \) sufficiently large so that \( \{ C(1 + I_1(T_0)) \varepsilon + C\kappa + \frac{C\nu}{N} \} < \frac{1}{2} \), we obtain, in light of assumption \([12]\),

\[
\sup_{0 \leq s \leq T_0} \{ e^{\frac{\nu s}{\varepsilon}} \| h^\varepsilon(s) \|_\infty \} \leq C \| h_0 \|_\infty + \frac{C_{N,\kappa}}{\varepsilon^{3/2}} \frac{m_0}{\varepsilon^T} \sup_{0 \leq s \leq T} \| f_\varepsilon(s) \| + C I_2(T_0) \frac{\nu T_0}{\varepsilon}\varepsilon^k \].
\]

Letting \( s = T_0 \) in the above and multiplying by \( \varepsilon^{3/2} e^{-\frac{\nu T_0}{2\varepsilon}} \), we obtain for sufficiently small \( \varepsilon \),

\[\varepsilon^{3/2} \| h^\varepsilon(T_0) \|_\infty \leq \frac{1}{2} \varepsilon^{3/2} \| h_0 \|_\infty + C \sup_{0 \leq s \leq T} \| f_\varepsilon(s) \| + C \varepsilon^{(2k+3)/2}.
\]

### 5.2 \( W^{1,\infty} \) bound: Proof of Lemma 5.3

We will prove only \([5.3]\). The estimate \([5.7]\) can be done in the same way. Let \( D_x \) be any \( x \) derivative. We now take \( D_x \) of the equation \([5.22]\) to get

\[
\partial_t (D_x h^\varepsilon) + v \cdot \nabla_x (D_x h^\varepsilon) + \nabla_x \phi^\varepsilon \cdot \nabla_v (D_x h^\varepsilon) + \frac{\nu(\omega)}{\varepsilon} D_x h^\varepsilon
\]

\[= -\nabla_x (D_x \phi^\varepsilon) \cdot \nabla_v h^\varepsilon - \frac{D_x \nu(\omega)}{\varepsilon} h^\varepsilon - \frac{1}{\varepsilon} D_x (K_{M,w} h^\varepsilon)
\]

\[+ \frac{\varepsilon^{k-1}}{\sqrt{\omega M}} D_x (\bar{Q}(\frac{h^\varepsilon}{\sqrt{\omega M}}, \frac{h^\varepsilon}{\sqrt{\omega M}})) + \sum_{i=1}^{2k-1} \varepsilon^{i-1} \frac{w}{\sqrt{\omega M}} \{ D_x (Q(F_i, \frac{h^\varepsilon}{\sqrt{\omega M}})) + Q(\frac{h^\varepsilon}{\sqrt{\omega M}}, F_i) \}
\]

\[- D_x (\nabla_x \phi^\varepsilon \cdot \frac{w}{\sqrt{\omega M}} \nabla_v (\frac{\omega_M}{w}) h^\varepsilon) - D_x (\nabla_x \phi_R^\varepsilon \cdot \frac{w}{\sqrt{\omega M}} \nabla_v (\omega + \sum_{i=1}^{2k-1} \varepsilon^i F_i)) + \varepsilon^{k-1} \frac{w}{\sqrt{\omega M}} (D_x A).
\tag{5.23}
\]
Thus the solution $D_x h^\varepsilon$ of the equation (5.23) can be expressed as follows:

$$
D_x h^\varepsilon(t, x, v) = \exp\left\{-\frac{1}{\varepsilon} \int_0^t \nu(\tau) d\tau\right\} D_x h^\varepsilon(0, X(0; t, x, v), V(0; t, x, v))
- \int_0^t \exp\left\{-\frac{1}{\varepsilon} \int_0^t \nu(\tau) d\tau\right\} \left(\nabla_x (D_x \phi^\varepsilon) \cdot \nabla_v h^\varepsilon\right)(s, X(s), V(s)) ds
- \int_0^t \exp\left\{-\frac{1}{\varepsilon} \int_0^t \nu(\tau) d\tau\right\} \left(\frac{D_x \nu(\omega)}{\varepsilon} h^\varepsilon\right)(s, X(s), V(s)) ds
- \int_0^t \exp\left\{-\frac{1}{\varepsilon} \int_0^t \nu(\tau) d\tau\right\} \left(\frac{1}{\varepsilon} D_x (K_M, w) h^\varepsilon\right)(s, X(s), V(s)) ds
+ \int_0^t \exp\left\{-\frac{1}{\varepsilon} \int_0^t \nu(\tau) d\tau\right\} \left(\varepsilon^{k-1} w \frac{\varepsilon}{\sqrt{\omega M}} D_x \left(Q\left(\frac{h^\varepsilon \sqrt{\omega M}}{w}, h^\varepsilon \sqrt{\omega M}\right)\right)\right)(s, X(s), V(s)) ds
+ \int_0^t \exp\left\{-\frac{1}{\varepsilon} \int_0^t \nu(\tau) d\tau\right\} \left(\sum_{i=1}^{2k-1} \frac{w}{\varepsilon} D_x \left(1 + \sum_{i=1}^{2k-1} \varepsilon^i F_i\right)\right)(s, X(s), V(s)) ds
+ \int_0^t \exp\left\{-\frac{1}{\varepsilon} \int_0^t \nu(\tau) d\tau\right\} \left(\varepsilon^{k-1} w \frac{\varepsilon}{\sqrt{\omega M}} (D_x A)\right)(s, X(s), V(s)) ds.
$$

(5.24)

Note that since $\omega$ is a local Maxwellian depending on $t$, $x$, and $v$, the right hand side contains not only $Dh^\varepsilon$ terms but also $h^\varepsilon$ terms coming from commutators. In addition, there is a $\nabla_v h^\varepsilon$ term coming from forcing, which we will estimate afterwards. The terms involving $D_x h^\varepsilon$ can be estimated similarly as done in $\|h^\varepsilon\|_\infty$ estimate. The terms from commutators are lower order, but they carry extra weight $1 + |v|^2$; they will be either controlled by $L^\infty$ norm of $(1 + |v|)h^\varepsilon$ or absorbed by the stronger exponential decay factor $\omega$. We will estimate line by line as in the previous section.

It is easy to see that the second line in (5.24) is bounded by

$$
\varepsilon e^{-\frac{1}{2\varepsilon}} \left\{CT\varepsilon + C(1 + \varepsilon^h)\|h^\varepsilon\|_{W^{1,\infty}}\right\} \sup_{0 \leq s \leq t} \left\{e^{-\frac{1}{2\varepsilon}} \|\nabla_v h^\varepsilon\|_\infty\right\}
$$

where we have used the elliptic regularity [4,8]. Since $|D_x \nu(\omega)| \leq C\nu(\omega)$, the third line is bounded by

$$
Ce^{-\frac{1}{2\varepsilon}} \sup_{0 \leq s \leq t} \left\{e^{-\frac{1}{2\varepsilon}} \|h^\varepsilon\|_\infty\right\}
$$

In order to estimate the fifth line, first write the term $D_x Q\left(\frac{h^\varepsilon \sqrt{\omega M}}{w}, \frac{h^\varepsilon \sqrt{\omega M}}{w}\right)$ as

$$
(D_x Q)\left(\frac{h^\varepsilon \sqrt{\omega M}}{w}, \frac{h^\varepsilon \sqrt{\omega M}}{w}\right) + Q\left(\frac{D_x h^\varepsilon \sqrt{\omega M}}{w}, \frac{h^\varepsilon \sqrt{\omega M}}{w}\right) + Q\left(\frac{h^\varepsilon \sqrt{\omega M}}{w}, \frac{D_x h^\varepsilon \sqrt{\omega M}}{w}\right)
$$

where $D_x Q$ is a commutator which consists of the terms that are given rise to when the derivative hits other than $\frac{h^\varepsilon \sqrt{\omega M}}{w}$. Note that $|(D_x Q)\left(\frac{h^\varepsilon \sqrt{\omega M}}{w}, \frac{h^\varepsilon \sqrt{\omega M}}{w}\right)(v)| \leq C(1 + |v|^2)|Q\left(\frac{h^\varepsilon \sqrt{\omega M}}{w}, \frac{h^\varepsilon \sqrt{\omega M}}{w}\right)(v)|$.

By Lemma 10 in [6],

$$
|\frac{w}{\sqrt{\omega M}} D_x Q\left(\frac{h^\varepsilon \sqrt{\omega M}}{w}, \frac{h^\varepsilon \sqrt{\omega M}}{w}\right)| \leq C\nu(\omega)\{1(1 + |v|)h^\varepsilon\|_\infty^2 + \|h^\varepsilon\|_\infty\|D_x h^\varepsilon\|_\infty\}
$$
and hence the fifth line in (5.24) is bounded by
\[
C\varepsilon^k e^{-\frac{\varepsilon}{\nu_0} t} \sup_{0 \leq s \leq t} \{ (e^{\frac{\varepsilon}{\nu_0} s} \| (1 + |v|)\tilde{h}^\varepsilon \|_\infty)^2 + (e^{\frac{\varepsilon}{\nu_0} s} \| \tilde{h}^\varepsilon \|_\infty)(e^{\frac{\varepsilon}{\nu_0} s} \| D_x \tilde{h}^\varepsilon \|_\infty) \}
\]
Commutators in the sixth and seventh lines also have the extra weight \((1 + |v|^2)\), but this weight can be absorbed into the exponential decay of \(F_i\)'s. Thus the sixth and seventh lines in (5.24) are bounded by
\[
CT_1 \varepsilon e^{-\frac{\varepsilon}{\nu_0} t} \sup_{0 \leq s \leq t} \{ e^{\frac{\varepsilon}{\nu_0} s} \| \tilde{h}^\varepsilon \|_{W^{1,\infty}} \}
\]
Similarly, one can deduce that the eighth through tenth lines are bounded by
\[
\varepsilon(C + CT_1 \varepsilon) e^{-\frac{\varepsilon}{\nu_0} t} \sup_{0 \leq s \leq t} \{ e^{\frac{\varepsilon}{\nu_0} s} \| \tilde{h}^\varepsilon \|_{W^{1,\infty}} \} + C\varepsilon^k
\]
We shall concentrate on the fourth line in (5.24). Write \(D_x(K_{M,w} h^\varepsilon)\) as
\[
D_x(K_{M,w} h^\varepsilon)(v) = \int (D_x l_{M,w})(v, v') h^\varepsilon(v') dv' + \int l_{M,w}(v, v')(D_x h^\varepsilon)(v') dv'
\]
where \(l_{M,w}\) is the corresponding kernel associated with \(K_{M,w}\). Note that
\[
\| (D_x l_{M,w})(v, v') \| \leq C(1 + |v|)(1 + |v - v'|) l_{M,w}(v, v')(1 + |v'|) \leq C_k(1 + |v - v'|) l_{M,w}(v, v')(1 + |v'|)
\]
due to the dependence of \(l_{M,w}\) on the local Maxwellian \(\omega\). Thus we can bound the fourth line in (5.24) by
\[
\frac{1}{\varepsilon} \int_0^t \exp\{-\frac{1}{\varepsilon} \int_s^t \nu(\tau) d\tau\} \nu_M \int_{\mathbb{R}^3} |(1 + |V(s) - v'|) l_{M,w}(V(s), v')(1 + |v'|) h^\varepsilon(s, X(s), v')| dv' ds
\]
\[
+ \frac{1}{\varepsilon} \int_0^t \exp\{-\frac{1}{\varepsilon} \int_s^t \nu(\tau) d\tau\} \int_{\mathbb{R}^3} l_{M,w}(V(s), v')(D_x h^\varepsilon)(s, X(s), v') dv' ds \equiv (I) + (II)
\]
Letting \(\tilde{h}^\varepsilon \equiv (1 + |v|)h^\varepsilon\), \(\tilde{h}^\varepsilon\) satisfies the equation (5.2) with a different weight \(w_1 \equiv (1 + |v|)w(v)\). We now use the Duhamel equation (5.3) for \(\tilde{h}^\varepsilon\) with the weight \(w_1\) to evaluate (I). Recall (5.12). Note that the \(L^\infty\) estimates of \(\tilde{h}^\varepsilon\) do not depend on the strength of the weight, and also both \(l_{M,w}\) and \(l_{M,w_1}\) inherit Grad estimates (5.11). Thus we can follow the previous estimates to obtain the bound for (I)
\[
(I) \leq C \varepsilon^k e^{-\frac{\varepsilon}{\nu_0} t} \| \tilde{h}^\varepsilon(0) \|_\infty + C \varepsilon^k \sum_{0 \leq s \leq t} \{ e^{\frac{\varepsilon}{\nu_0} s} \| \tilde{h}^\varepsilon(s) \|_\infty \}^2
\]
\[
+ (C(1 + T_1)\varepsilon + C_k + \frac{C_N}{N}) e^{-\frac{\varepsilon}{\nu_0} t} \sum_{0 \leq s \leq t} \{ e^{\frac{\varepsilon}{\nu_0} s} \| \tilde{h}^\varepsilon(s) \|_\infty \} + \frac{C_{N,N_k}}{\varepsilon^{3/2}} \sum_{0 \leq s \leq T_0} \| f^\varepsilon(s) \| + CT_2 \varepsilon^k
\]
And we use the equation (5.24) to evaluate (II). The estimation is again exactly same as \(L^\infty\) bound except the very last part where \(\| f^\varepsilon \|\) comes up. Here we will only present this last case: recall the CASE 3b. in the previous section and see (5.15)
\[
\frac{C}{\varepsilon^2} \int_0^t \int_{\mathbb{R}^3} \int_B \exp\{-\frac{1}{\varepsilon} \int_s^t \nu(\tau) d\tau - \frac{1}{\varepsilon} \int_{s_1}^s \nu(\tau) d\tau\} l_N(V(s), v') l_N(V(s_1), v'')
\]
\[
D_x h^\varepsilon(s_1, X(s_1), v''') dv''dv''' ds_1 ds.
\]
As before in (5.19), we introduce a new variable \(y = X(s_1) = X(s_1; s, X(s; t, x, v), v')\) such that \(|y - X(s)| = |X(s_1) - X(s)| \leq C(s - s_1)\). We now apply Lemma 4.1 to \(X(s_1; s, X(s; t, x, v), v')\) with
\[ x = X(s; t, x, v), \tau = s_1, t = s. \] Make a change of variables from \( v' \) to \( y \) and integrate by parts:

\[
\frac{1}{\epsilon^2} \int_0^t \int_0^{s-\kappa \epsilon} \int_B \exp\left\{-\frac{1}{\epsilon} \int_s^t \nu(\tau)d\tau - \frac{1}{\epsilon} \int_{s_1}^s \nu(\tau)d\tau \right\} l_N(V(s), v') l_N(V(s_1), v'') \]

\[
D_x h^\epsilon(s_1, y, v'') \frac{dv'}{dy} dy dv'' ds_1 ds \]

\[
\leq -\frac{1}{\epsilon^2} \int_0^t \int_0^{s-\kappa \epsilon} \int_B \exp\left\{-\frac{1}{\epsilon} \int_s^t \nu(\tau)d\tau - \frac{1}{\epsilon} \int_{s_1}^s \nu(\tau)d\tau \right\} D_x (l_N(V(s), v') l_N(V(s_1), v'')) \]

\[
h^\epsilon(s_1, y, v'') \frac{dv'}{dy} dy dv'' ds_1 ds \quad \text{(5.25)}
\]

\[
- \frac{1}{\epsilon^2} \int_0^t \int_0^{s-\kappa \epsilon} \int_B \exp\left\{-\frac{1}{\epsilon} \int_s^t \nu(\tau)d\tau - \frac{1}{\epsilon} \int_{s_1}^s \nu(\tau)d\tau \right\} l_N(V(s), v') l_N(V(s_1), v'') \]

\[
h^\epsilon(s_1, y, v'') D_x \left(\frac{dv'}{dy}\right) dy dv'' ds_1 ds
\]

\[
+ \frac{C_{N, \kappa}}{\epsilon^3} \| h^\epsilon \|_{\infty} \text{ (boundary contribution)},
\]

where \( \tilde{B} = \{ |y - X(s)| \leq C(s - s_1)N, |v'| \leq 3N \} \). For the first term in the right hand side, since \( D_x (l_N(V(s), v') l_N(V(s_1), v'')) \) is bounded, and by \( \left[ 21 \right] \), following the same argument in \( L^\infty \) bound, one can deduce that it is bounded by

\[
\frac{C_{N, \kappa}}{\epsilon^3} \sup_{0 \leq s \leq T} \| f^\epsilon(s) \|.
\]

For the second term, we need to estimate \( D_x \left(\frac{dv'}{dy}\right) \). First note that

\[
D_x (\det \left(\frac{dv'}{dy}\right)) = D_x (\frac{1}{\det \left(\frac{dv'}{dy}\right)^2}) = -\frac{1}{\det \left(\frac{dv'}{dy}\right)^2} D_x (\det \left(\frac{dy}{dv'}\right)),
\]

where

\[
\frac{1}{4(s_1 - s)^6} \leq \frac{1}{\det \left(\frac{dv'}{dy}\right)^2} \leq \frac{4}{(s_1 - s)^6} \text{ by \left[ 4.4 \right] in Lemma 4.1}
\]

Since

\[
\left(\frac{dy}{dv'}\right) = \begin{pmatrix}
\partial_{v'_1} X^1(s_1) & \partial_{v'_2} X^1(s_1) & \partial_{v'_3} X^1(s_1) \\
\partial_{v'_1} X^2(s_1) & \partial_{v'_2} X^2(s_1) & \partial_{v'_3} X^2(s_1) \\
\partial_{v'_1} X^3(s_1) & \partial_{v'_2} X^3(s_1) & \partial_{v'_3} X^3(s_1)
\end{pmatrix},
\]

by the product rule and \( \left[ 4.5 \right] \) in Lemma 4.1 we get

\[
|D_x (\det \left(\frac{dy}{dv'}\right))| \leq C|\partial_{v'} X(s_1)|^2 |D_x \partial_{v'} X(s_1)| \leq C(s_1 - s)^2 |\partial_x \partial_{v'} X(s_1)|.
\]

Thus,

\[
|D_x (\det \left(\frac{dv'}{dy}\right))| = \frac{1}{\det \left(\frac{dv'}{dy}\right)^2} |D_x (\det \left(\frac{dy}{dv'}\right))| \leq \frac{C|\partial_x \partial_{v'} X(s_1)|}{(s_1 - s)^4} \leq \frac{C|\partial_x \partial_{v'} X(s_1)|}{\kappa^4 \epsilon^4}.
\]
Therefore, we obtain

\[
\|D_x(\frac{dy}{dv'}())\|_{L^2(B)}^2 \leq \frac{C_\kappa}{\varepsilon^4} \|\partial_x \partial_{v'} X(s, 1)\|_{L^2(B)}^2 \\
= \frac{C_\kappa}{\varepsilon^4} \int_B \left| \partial_x \partial_{v'} X(s, X(s, x, v), v') \right|^2 dx \left\{ \frac{\partial X(s, X(s, x, v), v')}{\partial X(s, x, v)} \right\} dxdv'ds \\
\leq \frac{C_{\kappa, T_0, N}}{\varepsilon^4} \int_{|x-x'| \leq T_0} \left| \partial_x \partial_{v'} X(s, z, v') \right|^2 dz \\
\leq \frac{C_{\kappa, T_0, N}}{\varepsilon^4} \int_{|x-x'| \leq T_0} \left| \partial_x \partial_{v'} X(s, z, v') \right|^2 dz 
\]

where we have used (5.20) and (5.24). Hence, by Cauchy-Schwarz's inequality, the second integral in (5.25) is bounded by

\[
\frac{C_{N, \kappa, T_0}}{\varepsilon} \sup_{0 \leq s \leq T} \|f^\varepsilon(s)\|.
\]

In summary, for any \( x \) derivative \( D_x \), we have shown that for any \( \kappa > 0 \) and large enough \( N > 0 \),

\[
\sup_{0 \leq s \leq T} \left\{ e^{\frac{\nu s}{\varepsilon}} \|D_x h^\varepsilon(s)\|_{\infty} \right\} \leq C \left\{ \|1 + |v|\|h^\varepsilon(0)\|_{\infty} + \|D_x h(0)\|_{\infty} \right\} \\
+ C \varepsilon \sup_{0 \leq s \leq T_0} \left\{ e^{\frac{\nu s}{\varepsilon}} \|\nabla_v h^\varepsilon(s)\|_{\infty} \right\} + C \varepsilon \sup_{0 \leq s \leq T_0} \left\{ e^{\frac{\nu s}{\varepsilon}} \|h^\varepsilon(s)\|_{\infty} \right\} \\
+ \left\{ C \varepsilon + C \kappa + \frac{C_\kappa}{N} \right\} \sup_{0 \leq s \leq T_0} \left\{ e^{\frac{\nu s}{\varepsilon}} \|D_x h^\varepsilon(s)\|_{\infty} + e^{\frac{\nu s}{\varepsilon}} \|(1 + |v|)h^\varepsilon(s)\|_{\infty} \right\} \\
+ C \varepsilon^k \sup_{0 \leq s \leq T_0} \left\{ e^{\frac{\nu s}{\varepsilon}} \|D_x h^\varepsilon(s)\|_{\infty} \right\} + (\frac{C}{N} \sup_{0 \leq s \leq T} \|f^\varepsilon(s)\| + C e^{\frac{\nu s}{\varepsilon}} \varepsilon^{k-1} \right\}.
\]

As a final step, it now remains to estimate \( \|\nabla_v h^\varepsilon\|_{\infty} \) that appears in (5.26) – the estimate of \( \|D_x h^\varepsilon\|_{\infty} \). Let \( D_v \) be any \( v \) derivative. Take \( D_v \) of the equation (5.22) to get

\[
\partial_t(D_v h^\varepsilon) + v \cdot \nabla_x (D_v h^\varepsilon) + \nabla_x \phi^\varepsilon \cdot \nabla_v (D_v h^\varepsilon) + \frac{\nu(\omega)}{\varepsilon} D_v h^\varepsilon \\
= -D_x h^\varepsilon - \frac{D_v \nu(\omega)}{\varepsilon} h^\varepsilon - \frac{1}{\varepsilon} D_v(K_M, w h^\varepsilon) \\
+ D_v \left( \frac{\varepsilon^{-1} w}{\sqrt{\omega M}} Q \left( \frac{h^\varepsilon \sqrt{\omega M}}{w}, \frac{\omega M}{w} \right) \right) + \sum_{i=1}^{2k-1} \varepsilon^{i-1} D_v \left( \frac{w}{\sqrt{\omega M}} \{Q(F_i, \frac{h^\varepsilon \sqrt{\omega M}}{w}) + Q \left( \frac{h^\varepsilon \sqrt{\omega M}}{w}, F_i \right) \} \right) \\
- \nabla_x \phi^\varepsilon \cdot D_v \left( \frac{w}{\sqrt{\omega M}} \nabla_v \left( \frac{\sqrt{\omega M}}{w} \right) \right) - \nabla_x \phi_R^\varepsilon \cdot D_v \left( \frac{w}{\sqrt{\omega M}} \nabla_v (\omega + \sum_{i=1}^{2k-1} \varepsilon^{i} F_i) \right) + \varepsilon^k D_v \left( \frac{w}{\sqrt{\omega M}} A \right)
\]

where \( D_x \) is a spatial derivative obtained from \( D_v(v) \cdot \nabla_x \). By Duhamel principle, the solution \( D_v h^\varepsilon \)
of the equation (5.27) can be expressed as follows:

$$D_{v}h^{\varepsilon}(t, x, v) = \exp\left\{-\frac{1}{\varepsilon} \int_{0}^{t} \nu(\tau) d\tau\right\} D_{v}h^{\varepsilon}(0, X(0; t, x, v), V(0; t, x, v))$$

$$- \int_{0}^{t} \exp\left\{-\frac{1}{\varepsilon} \int_{s}^{t} \nu(\tau) d\tau\right\} (D_{v}h^{\varepsilon})(s, X(s), V(s)) ds$$

$$- \int_{0}^{t} \exp\left\{-\frac{1}{\varepsilon} \int_{s}^{t} \nu(\tau) d\tau\right\} \left(\frac{D_{v}v(\omega)}{\varepsilon} h^{\varepsilon}\right)(s, X(s), V(s)) ds$$

$$- \int_{0}^{t} \exp\left\{-\frac{1}{\varepsilon} \int_{s}^{t} \nu(\tau) d\tau\right\} \left(\frac{1}{\varepsilon} D_{v}(K_{M,w}h^{\varepsilon})\right)(s, X(s), V(s)) ds$$

$$+ \int_{0}^{t} \exp\left\{-\frac{1}{\varepsilon} \int_{s}^{t} \nu(\tau) d\tau\right\} D_{v}\left(\int_{1}^{\varepsilon} w_{s} Q(h^{\varepsilon}/\sqrt{\omega_{M}}, h^{\varepsilon}/\sqrt{\omega_{M}})\right)(s, X(s), V(s)) ds$$

$$+ \int_{0}^{t} \exp\left\{-\frac{1}{\varepsilon} \int_{s}^{t} \nu(\tau) d\tau\right\} D_{v}\left(\sum_{i=1}^{2k-1} w_{s} Q(h^{\varepsilon}/\sqrt{\omega_{M}}, h^{\varepsilon}/\sqrt{\omega_{M}}, F_{i})\right)(s, X(s), V(s)) ds$$

$$+ \int_{0}^{t} \exp\left\{-\frac{1}{\varepsilon} \int_{s}^{t} \nu(\tau) d\tau\right\} \left(\nabla x \phi^{\varepsilon} \cdot D_{v}\left(\frac{w}{\sqrt{\omega_{M}}} \nabla_{v}(\sqrt{\omega_{M}})h^{\varepsilon}\right)\right)(s, X(s), V(s)) ds$$

$$- \int_{0}^{t} \exp\left\{-\frac{1}{\varepsilon} \int_{s}^{t} \nu(\tau) d\tau\right\} \left(\nabla x \phi^{R} \cdot D_{v}\left(\frac{w}{\sqrt{\omega_{M}}} \nabla_{v}(\omega + \sum_{i=1}^{2k-1} \varepsilon^{i} F_{i})\right)\right)(s, X(s), V(s)) ds$$

$$+ \int_{0}^{t} \exp\left\{-\frac{1}{\varepsilon} \int_{s}^{t} \nu(\tau) d\tau\right\} \left(\varepsilon^{k-1} D_{v}\left(\frac{w}{\sqrt{\omega_{M}}} A\right)\right)(s, X(s), V(s)) ds.$$

As in the spatial derivative ($D_{x}h^{\varepsilon}$) case, the right hand side contains not only $D_{v}h^{\varepsilon}$ terms but also $h^{\varepsilon}$ terms coming from commutators. But this time the terms from commutators carry the weight $1 + |v|$ at most since they are from $v$ derivatives. The estimates will be almost same as in the spatial derivative case, so we won’t present every detail. We rather give some brief explanations.

For instance, since $|D_{v}v(\omega)| \leq C$, the third term in the right hand side of (5.28) is bounded by $Ce^{-\frac{\omega}{2\varepsilon}} \sup_{0 \leq t \leq s} \{e^{\frac{\omega t}{2\varepsilon}} ||h^{\varepsilon}||_{\infty}\}$, and since $|D_{v}\left(\frac{w}{\sqrt{\omega_{M}}} Q(h^{\varepsilon}/\sqrt{\omega_{M}}, h^{\varepsilon}/\sqrt{\omega_{M}})\right)| \leq C\nu(\omega)||h^{\varepsilon}||_{\infty} \{||1+|v||h^{\varepsilon}||_{\infty} + ||D_{v}h^{\varepsilon}||_{\infty}\}$, the fifth line is bounded by

$$Ce^{\varepsilon}e^{-\frac{\omega t}{2\varepsilon}} \sup_{0 \leq t \leq s} \{e^{\frac{\omega t}{2\varepsilon}} ||h^{\varepsilon}||_{\infty}\}(e^{\frac{\omega t}{2\varepsilon}} ||(1+|v|)h^{\varepsilon}||_{\infty} + e^{\frac{\omega t}{2\varepsilon}} ||D_{v}h^{\varepsilon}||_{\infty})$$

Other terms except the fourth line can be estimated in the same way as before.

For the intriguing term in the fourth line, we need to control $(\frac{1}{\varepsilon} D_{v}(K_{M,w}h^{\varepsilon}))(s, X(s), V(s))$. First note for $T_{0}$ sufficiently small, by (140), $\frac{\partial V(s)}{\partial v}$ is non-singular. We therefore can write

$$(D_{v}(K_{M,w}h^{\varepsilon}))(s, X(s), V(s)) = \left[\frac{\partial V(s)}{\partial v}\right]^{-1} D_{v}(s)(K_{M,w}h^{\varepsilon})(s, X(s), V(s)).$$

But for $D_{v}(s)(K_{M,w}h^{\varepsilon})(s, X(s), V(s))$, we can employ Lemma 2.2 in [3] so that

$$D_{v}(s)(K_{M,w}h^{\varepsilon})(s, X(s), V(s)) = (K_{M,w}h^{\varepsilon})(s, X(s), V(s)) + (K_{M,w}^{2} \partial_{v}h^{\varepsilon})(s, X(s), V(s))$$

in which the kernels in both $K_{M,w}^{1}$ and $K_{M,w}^{2}$ satisfy the Grad estimate (5.10). We then can repeat the same procedure to $K_{M,w}^{1}$ and $K_{M,w}^{2}$. We use integration by parts in $v''$ so that we do not need to take
derivatives for the determinant of \( \frac{d\mu}{d\nu} \) which is independent of \( v'' \) for \( (K_{M,w}^2 \partial_{w} h)(s, X(s), V(s)) \). Therefore, we have established the following \( W^{1,\infty} \) estimates:

\[
\sup_{0 \leq s \leq T_0} \{ e^{\frac{\nu}{\epsilon}} \| \nabla_x v h^\epsilon(s) \|_\infty \} \leq C\{ \| (1 + |v|) h^\epsilon(0) \|_\infty + \| \nabla_x v h^\epsilon(0) \|_\infty \}
\]

\[
+ \{ C \epsilon + C \kappa + \frac{C_N}{\epsilon} \} \sup_{0 \leq s \leq T_0} \{ e^{\frac{\nu}{\epsilon}} \| \nabla_x v h^\epsilon(s) \|_\infty + e^{\frac{\nu}{\epsilon}} \| (1 + |v|) h^\epsilon(s) \|_\infty \}
\]

\[
+ C \epsilon^k \sup_{0 \leq s \leq T_0} \{ \{ e^{\frac{\nu}{\epsilon}} \| \nabla_x v h^\epsilon(s) \|_\infty \}^2 + \{ e^{\frac{\nu}{\epsilon}} \| (1 + |v|) h^\epsilon(s) \|_\infty \}^2 \}
\]

\[
+ C \epsilon^\nu \sup_{0 \leq s \leq T_0} \{ e^{\frac{\nu}{\epsilon}} \| h^\epsilon(s) \|_\infty \} + \frac{C \epsilon^\nu k \epsilon}{\nu} \sup_{0 \leq s \leq T} \| f^\epsilon(s) \| + C e^{\frac{\nu}{\epsilon}} \epsilon^{k-1}
\]

From Lemma 5.2, we have the estimates of \( \sup_{0 \leq s \leq T_0} \epsilon^{3/2} \| h^\epsilon(s) \|_\infty \). Due to the singular term, the first term in the fourth line in (5.29), we first multiply both sides by \( \epsilon^5 \) and combine this estimate with \( L^\infty \) bound of \( h^\epsilon \equiv (1 + |v|) h^\epsilon \) in Lemma 5.2 and choose \( \kappa \) small, \( N \) large to deduce that for sufficiently small \( \epsilon \),

\[
\epsilon^5 \| \nabla_x v h^\epsilon(T_0) \|_\infty \leq \frac{1}{2} \{ \epsilon^5 \| (1 + |v|) h^\epsilon(0) \|_\infty + \epsilon^5 \| \nabla_x v h(0) \|_\infty + \epsilon^{3/2} \| h(0) \|_\infty \}
\]

\[
+ C \{ \epsilon^{1/2} \sup_{0 \leq s \leq T} \| f^\epsilon(s) \| + \epsilon^{k+1} \}.
\]

6 Proof of Theorem 1.1

**Proof of Theorem 1.1** Combining Proposition 3.1 and Proposition 5.1 we deduce

\[
\frac{d}{dt} (\| f^\epsilon \|^2 + \| \nabla \phi_R^\epsilon \|^2) + \frac{C_0}{\epsilon} \| (I - P) f^\epsilon \|^2
\]

\[
\leq C \epsilon \left[ \epsilon \| \epsilon^2 h_0 \|_\infty + \sup_{0 \leq s \leq T} \| f^\epsilon \| + \frac{\epsilon^{2k+1}}{2} \right] \left[ \| f^\epsilon \| + \epsilon^{k-3} \| f^\epsilon \|^2 + \epsilon^{k-2} \| f^\epsilon \| \| \nabla \phi_R^\epsilon \| \right]
\]

\[
+ C \left( \frac{1}{(1 + t)^p} + I_1 \epsilon \right) \{ \| f^\epsilon \|^2 + \| \nabla \phi_R^\epsilon \|^2 \} + C I_2 \epsilon^{k-1} \| f^\epsilon \|.
\]

Gronwall inequality yields

\[
\| f^\epsilon(t) \| + \| \nabla \phi_R^\epsilon(t) \| + 1 \leq (\| f^\epsilon(0) \| + \| \nabla \phi_R^\epsilon(0) \| + 1)
\]

\[
\exp \left\{ \int_0^t C \{ \sqrt{\epsilon} \| \epsilon^2 h_0 \|_\infty + \sup_{0 \leq s \leq T} \| f^\epsilon \| + (1 + s)^{-p} + I_1 \epsilon + I_2 \epsilon^{k-1} \} ds \right\}
\]

\[
\leq (\| f^\epsilon(0) \| + \| \nabla \phi_R^\epsilon(0) \| + 1)
\]

\[
\exp \left\{ C + \epsilon C \sqrt{\epsilon} \| \epsilon^2 h_0 \|_\infty + \sup_{0 \leq s \leq T} \| f^\epsilon \| + C I_1 t \epsilon + C I_2 t \epsilon^{k-1} \right\},
\]

where we have used \( \int_0^t \frac{1}{(1 + s)^p} ds < +\infty \). Now for \( 0 \leq t \leq \epsilon^{-m} \), where \( 0 < m \leq \frac{1}{2} \) and \( \frac{2k-3}{2k-2} \) \( (\frac{1}{2} \)

\[
I_1 \leq 2 \sum_{i=1}^{2k-1} \epsilon^{i-1} (1 + t)^{i-1} \leq C \sum_{i=1}^{2k-1} (\epsilon + \epsilon^{1-m})^{i-1} \leq C,
\]

\[
I_2 = \sum_{2k \leq i+j \leq 4k-2} \epsilon^{i+j-2k} (1 + t)^{i+j-2} \leq C (1 + \epsilon^{-m})^{2k-2} \leq C \epsilon^{-m(2k-2)}.
\]

Thus we obtain

\[
I_1 t \epsilon + I_2 t \epsilon^{k-1} \leq C (\epsilon^{1-m} + \epsilon^{k-1-m(2k-1)}) \leq C \epsilon^{3-m}
\]
and hence, for sufficiently small $\varepsilon$,

$$\|f^\varepsilon(t)\| + \|\nabla \phi^\varepsilon_R(t)\| \leq C(\|f^\varepsilon(0)\| + \|\nabla \phi^\varepsilon_R(0)\| + 1) \left\{ 1 + \varepsilon^{\frac{1}{2} - m} \|\varepsilon^\frac{1}{2} h_0\|_{\infty} + \varepsilon^{\frac{1}{2} - m} \sup_{0 \leq \varepsilon \leq T} \|f^\varepsilon\| \right\}$$

For $t \leq T (= \varepsilon^{-m})$, since $m < 1/2$, letting $\varepsilon$ small, we conclude the proof of our theorem as

$$\sup_{0 \leq \varepsilon \leq \varepsilon^{-m}} \{ \|f^\varepsilon\| + \|\nabla \phi^\varepsilon_R(t)\| \} \leq C\{ 1 + \|f^\varepsilon(0)\| + \|\nabla \phi^\varepsilon_R(0)\| + \|\varepsilon^\frac{1}{2} h_0\|_{\infty} \}.$$  

Note that $C$ is independent of $\varepsilon$. \hfill \Box

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