Preference elicitation and robust optimization with multi-attribute quasi-concave choice functions

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Abstract

Decision maker’s preferences are often captured by some choice functions which are used to rank prospects. In this paper, we consider ambiguity in choice functions over a multi-attribute prospect space. Our main result is a robust preference model where the optimal decision is based on the worst-case choice function from an ambiguity set constructed through preference elicitation with pairwise comparisons of prospects. Differing from existing works in the area, our focus is on quasi-concave choice functions rather than concave functions and this enables us to cover a wide range of utility/risk preference problems including multi-attribute expected utility and S-shaped aspirational risk preferences. The robust choice function is increasing and quasi-concave but not necessarily translation invariant, a key property of monetary risk measures. We propose two approaches based respectively on the support functions and level functions of quasi-concave functions to develop tractable formulations of the maximin preference robust optimization model. The former gives rise to a mixed integer linear programming problem whereas the latter is equivalent to solving a sequence of convex risk minimization problems. To assess the effectiveness of the proposed robust preference optimization model and numerical schemes, we apply them to a security budget allocation problem and report some preliminary results from experiments.

Keywords: preference elicitation; quasi-concave choice functions; multi-attribute decision-making; preference robust optimization

1 Introduction

Decision making under uncertainty is a universal theme in the stochastic and robust optimization communities. Much of the focus has been on exogenous uncertainties which go beyond the control of the decision maker such as market demand and climate change. In practice, however, there is often significant endogenous uncertainty which arises from ambiguity about the decision maker’s preferences; either because the decision making involves several stakeholders who are unable to reach a consensus, or there is inadequate information for the decision maker to identify a unique utility/risk function which precisely characterizes his preferences. Preference robust optimization (PRO) models are subsequently proposed where the optimal decision is based on the worst preference from an ambiguity set of utility/risk preference functions constructed through available partial information. The robustness is aimed to mitigate the risk arising from ambiguity about the decision maker’s preferences.

Armbruster and Delage [5] study a PRO model for utility maximization problems. Specifically, they model decision maker’s ambiguity about utility preferences by incorporating various possible properties of the utility function such as monotonicity, concavity, and S-shapedness, along with preference elicitation information from pairwise comparisons. An important component their research is to derive tractable formulations of the resulting maximin problem and they manage to do so by exploiting linear envelopes of convex/concave

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functions. Delage and Li [19] extend this research to risk management problem in finance where the investor’s choice of a risk measure is ambiguous. As in [5], they consider the ambiguity set of risk measures primarily via pairwise elicitation but also featured with important properties such as convexity, coherence, and law invariance, and they develop tractable formulations accordingly.

Hu and Mehrotra [35] approach the PRO model in a different manner. First, they propose a moment-type approach which allows one to define the ambiguity set for a decision maker’s utility preferences via the certainty equivalent method, pairwise comparisons, upper and lower bounds of the trajectories of the utility functions, and bounds on their derivatives at specified grid points; second, they consider a probabilistic representation of the class of increasing convex utility functions by confining them to a compact interval and scaling them to being bounded by one; third, by constructing a piecewise linear approximation of the trajectories of the utility bounds, they derive a tractable reformulation of the resulting PRO as a linear programming problem. Qualitative convergence analysis is presented to justify the piecewise linear approximation.

Hu and Mehrotra’s approach is closely related to stochastic dominance, a subject which has been intensely studied over the past few decades, see the monographs [46, 55] for a comprehensive treatment of the topic and [20, 21] for the optimization models with stochastic dominance constraints. Indeed, when the preference of a decision maker satisfies certain axioms including completeness, transitivity, continuity and independence, Von Neumann and Morgenstern’s expected utility theory ([63]) guarantees that any set of preferences that the decision maker may have among uncertain/risky prospects can be characterized by an expected utility measure. The issue that we are looking at here is that the decision maker does not necessarily have complete information about his preferences, and prospects associated with decisions do not necessarily have definitive stochastic dominance relationships - which means that the existing models based on stochastic dominance are not applicable.

In a more recent development, Haskell et al. [32] consider a stochastic optimization problem where the decision maker faces both exogenous uncertainty and endogenous uncertainty associated with decision maker’s risk attitude. They propose a PRO model where the ambiguity is constructed in the product space of exogenous uncertainty and utility functions. By using Lagrangian duality, they derive an exact tractable reformulation for some special cases and a tractable relaxation in the general setting. Delage et al. [18] propose a robust shortfall risk measure model to tackle the case where investors are ambiguous about their utility loss functions. They study viable ways to identify the ambiguity set of loss functions via pairwise comparison for utility risk measures with respective features such as coherence, convexity, and boundedness, and they derive tractable linear program reformulation for the resulting optimization problems with robust shortfall risk constraints.

In this paper, we take on this stream of research but with a different focus in terms of both modeling and tractable reformulations. Specifically, we consider a class of so-called choice functions applicable to multi-attribute decision making which are monotonically increasing along some specified direction, quasi-concave, but not necessarily convex/concave or translation invariant. We tackle the subsequent PRO model via “hockey-stick” type support functions and level functions. Monotonicity ensures that the decision maker universally prefers more to less. Quasi-concavity is a general form of risk aversion, the axiom of quasi-concavity is further supported by the study in [12] on aspirational preferences. Moreover, by dropping translation invariance) which is an key axiomatic property of convex risk measures), we allow our PRO model to cover a wider range of problems where translation invariance may not hold [12].

Another important departure from existing research is that our PRO model is applicable to multi-attribute decision making problems. These problems are ubiquitous in practical applications but the existing PRO models mainly emphasize the single attribute setting. For instance, in management research of healthcare, it is typical to use several metrics rather than just one to measure the quality of life [58, 24, 56]. Similar problems can be found in network management problems [15], scheduling, [69] design [59, 22] and portfolio optimization [27]. Indeed, over the past few decade, there have been significant research on multi-attribute expected utility [64, 25, 65, 60, 61, 62] and multi-attribute risk management [66, 11, 31, 29]. In a more recent development, research on robust multi-attribute choice models has also emerged, see [69, 23, 49] for instance. In particular, [19] considers optimization with a general class of scalarization functions (in particular, the class of min-biaffine functions), where the weights of the scalarization lie in a convex ambiguity set.

We summarize the main contributions of our present paper as follows.
• **Multi-attribute quasi-concave PRO model.** We propose a robust choice model for preference ambiguity where the underlying choice function is monotonic and quasi-concave. By replacing concavity with quasi-concavity and dropping translation invariance, we extend the existing PRO model so that it is easier to incorporate preference elicitation information. Moreover, the new model framework covers a number of well-known preference models (such as expected utility and aspirational preferences) and can be applied to a wider range of PRO problems where the decision maker’s choice function is merely increasing and quasi-concave. Of course, it also poses new challenges to tractable reformulation which so far have depended on convexity [5, 33, 19]. Our model’s support for multi-objective stochastic decision making problems makes the model applicable to an even broader class of problems.

• **New forms of tractable formulations.** We propose two schemes for tractable reformulation of the proposed new PRO model. One is based on the support functions of quasi-concave functions and the other exploits approximation of quasi-concave function by a sequence of convex level functions. While both approaches are well known in the literature of generalized convex optimization, they are applied to PRO here for the first time. The support function approach takes the PRO model to a mixed-integer linear program as opposed to a linear program as in earlier work. The level set representation is a generalization of the representation results in [13, 12] to the case of preference ambiguity. We are able to explicitly derive the connection between these two approaches by using the special form of piecewise linear support functions for quasi-concave functions. This framework, based on representing any monotonic diversification favoring choice function in terms of a family of risk functions, leads to a unifying framework for representing multi-attribute choice functions. Moreover, this framework naturally converts a decision-making problem with a quasi-concave choice function into a sequence of convex optimization problems, yielding a viable computational recipe. This development is related to the methods in [13, 12] and extends these methods to the multi-attribute setting. The level set representation further builds on the managerial insights from [13, 12]. It reveals that multi-attribute choice functions can in general be understood in terms of a family of multi-attribute risk functions and the decision maker’s desired satiation levels.

• **Level function method.** In the case where the attributes depend nonlinearly on the decision variables, we propose an algorithm for solving the PRO by using the level function method from [68]. The algorithm is an iterative regime where at each iterate we solve a mixed-integer linear program based on the support function representation and identification of a level function, which is closely related to the level set representation. To examine the performance of the model and numerical schemes, we apply them to a homeland security problem considered by Hu and Mehrotra [34]. Our numerical experiments for this problem illustrate how our PRO model captures diversification favoring behavior, and also how it depends on the elicited comparison data set.

The rest of the paper is organized as follows. In Section 2, we review some preliminary materials related to choice functions. In Section 3, we formally describe the robust choice function model including the definition of the class of choice functions to be considered, specification of the ambiguity set, and characterization of the robust choice function and the corresponding maximin optimization problem. Section 4 details the support function approach for tractable reformulation of the robust choice model and its underlying theory. Next in Section 5, we discuss an alternative level set representation for quasi-concave choice functions. This development leads to a general representation formula for quasi-concave choice functions. Here, we also discuss the role of “targets” which feature prominently in the decision analysis literature. Section 6 builds on our PRO model and explains how to solve optimization problems in the presence of preference ambiguity, and Section 7 applies this methodology to a budget allocation problem for homeland security. The paper concludes in Section 8 with a discussion of potential impacts and future research directions.

2 Preliminaries

This section presents some preliminary materials on choice functions. We begin with a set of states of nature Ω endowed with a σ-algebra $\mathcal{B}$. Let $\mathcal{L}$ be an admissible space of measurable mappings $X : \Omega \to \mathbb{R}^n$, equipped with the supremum norm topology. We generally treat $\mathcal{L}$ as a space of multi-attribute prospects with $n \geq 2$ characteristics, although the case $n = 1$ is also covered. The inequality $X \leq Y$ for $X, Y \in \mathcal{L}$ is
understood to mean $X(\omega) \leq Y(\omega)$ component-wise for all $\omega \in \Omega$. We adopt the convention that prospects in $\mathcal{L}$ represent rewards/gains so that larger values of all attributes are preferred to smaller values.

Let $\mathbb{R} \doteq \mathbb{R} \cup \{-\infty, \infty\}$ be the extended-valued real line. A choice function is a mapping $\rho : \mathcal{L} \to \mathbb{R}$ that gives numerical values to prospects in $\mathcal{L}$ to evaluate their fitness (e.g., see [12]). When $\rho(X) \geq \rho(Y)$, $X$ is said to be weakly preferred to $Y$ and when $\rho(X) > \rho(Y)$, $X$ is said to be strongly preferred to $Y$. The following definition specifies the key properties of choice functions that appear frequently in the literature (see [24, 13] for example).

**Definition 2.1** (Properties of the choice function). (i) (Upper semi-continuity) For all $X \in \mathcal{L}$, $\lim \sup_{Y \to X} \rho(Y) = \rho(X)$.

(ii) (Monotonicity) For all $X, Y \in \mathcal{L}$, $X \leq Y$ implies $\rho(X) \leq \rho(Y)$.

(iii) (Quasi-concavity) For all $X, Y \in \mathcal{L}$, $\rho(\lambda X + (1 - \lambda) Y) \geq \min \{\rho(X), \rho(Y)\}$ for all $\lambda \in [0, 1]$.

(iv) (Completeness) For all $X, Y \in \mathcal{L}$, either $\rho(X) \geq \rho(Y)$ or $\rho(X) \leq \rho(Y)$.

Upper semi-continuity is a common technical condition (see [8, 12, 65]). Monotonicity means that the decision maker always prefers more reward to less - this property is universally accepted. Quasi-concavity means that diversification does not decrease reward, where the convex combination $\lambda X + (1 - \lambda) Y$ is understood as a mixture of the prospects $X$ and $Y$, see [12, 13]. Properties (i) - (iii) appear in much of the decision theory literature (e.g., [12]). Since the choice function is real-valued, it is automatically transitive: for all $X, Y, Z \in \mathcal{L}$, if $\rho(X) \geq \rho(Y)$ and $\rho(Y) \geq \rho(Z)$, then $\rho(X) \geq \rho(Z)$. Property (iv) and transitivity ensure that the choice functions we consider are “rational”.

We now give some examples of multi-attribute choice functions to motivate our discussion and to point out related work on multi-attribute prospects. Here and throughout, the Euclidean inner product is denoted by $\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$.

**Example 2.2.** Let $u_i$ for $i = 1, \ldots, n$ be univariate utility functions.

(i) The assumption of mutual utility independence (see [3] for example) gives rise to additive utility functions $u(x_1, \ldots, x_n) = \sum_{i=1}^{n} \kappa_i u_i(x_i)$ for $\kappa_1, \ldots, \kappa_n \geq 0$. The expected utility of a random vector $X = (X_1, \ldots, X_n)$ is then $\rho(X) = \sum_{i=1}^{n} \kappa_i \mathbb{E}[u_i(X_i)]$.

(ii) In [37], an alternative independent utility aggregation model is proposed with

$$u(x_1, \ldots, x_n) = \frac{1}{K'} \left\{ \left[ \prod_{i=1}^{n} (K' \kappa_i u_i(x_i) + 1) \right] - 1 \right\},$$

where $\sum_{i=1}^{n} \kappa_i \neq 1$, $K' > -1$ and $K' + 1 = \prod_{i=1}^{n} (\kappa_i K' + 1)$. The expected utility is then

$$\rho(X) = \frac{1}{K'} \left\{ \left[ \prod_{i=1}^{n} (K' \kappa_i \mathbb{E}[u_i(X_i)] + 1) \right] - 1 \right\}.$$

(iii) For a general utility function $u : \mathbb{R}^n \to \mathbb{R}$, the expected utility $\rho(X) = \mathbb{E}[u(X)]$ is a choice function.

(iv) Let $C : [0, 1]^n \to [0, 1]$ be a multivariate copula (i.e., $C$ is the joint cumulative distribution function of an $n$-dimensional random vector on the unit cube $[0, 1]^n$ with uniformly distributed marginals), then $\rho(X) = C(u_1(X_1), \ldots, u_n(X_n))$ is a choice function (see [2]).

(v) The conditional value-at-risk (CVaR) of a univariate random variable $X$ at level $\alpha \in (0, 1)$ is

$$\text{CVaR}_\alpha(X) \doteq \inf_{\eta \in \mathbb{R}} \left\{ \eta + (1 - \alpha)^{-1} \mathbb{E} \left[ (X - \eta)_+ \right] \right\}.$$

A multivariate version of the conditional value-at-risk (CVaR) is developed in [48] based on linear scalarization. Given a vector of weights $w \in \mathbb{R}^n_+$, we may consider the choice function $\rho(X) = \text{CVaR}_\alpha\left((w, X)\right)$ on $\mathcal{L}$.

(vi) More generally, as in [49], we may take any univariate risk measure $\vartheta$ (such as a mean-deviation risk measure, see [54]) and then consider the choice function $\rho(X) = \vartheta(\varphi(w, X))$ where $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a scalarization function. In [49], the authors focus on the computationally tractable class of “min-biaffine” scalarization functions $\varphi$.

4
3 Robust preferences model

Now we come to the main problem under consideration in this paper. To begin, we introduce the set of all upper semi-continuous, monotonic, and quasi-concave choice functions

\[ \mathcal{R}_{iqv} \triangleq \{ \text{upper semi-continuous, increasing, and quasi-concave } \rho : \mathcal{L} \to \mathbb{R} \} . \]

This set characterizes our decision makers of interest. Since concave functions are quasi-concave, this class of functions consists of all continuous increasing concave utility functions in the literature.

In practice it is difficult to elicit a precise functional form for \( \rho \). This difficulty is exacerbated in the multi-attribute setting. First, when multi-attribute prospects are in play, it is not obvious how to characterize the marginal dependencies of the variety of attributes, i.e., it is not always clear how much an increase in the value of one asset should depend on the levels of the other assets. Second, it is hard to specify a choice function in group decision making where the group must come to a consensus. Third, we may have only a few observations of the decision maker’s behavior which makes it impossible to precisely specify preferences.

To circumvent these difficulties, we design a robust choice function. To this end, we will first need to construct a preference ambiguity set \( \mathcal{R} \subset \mathcal{R}_{iqv} \) which contains a range of possible choice functions. Then, given this preference ambiguity set \( \mathcal{R} \), we will set up a framework which chooses a “robust choice function” as:

\[ \psi (X; \mathcal{R}) \triangleq \inf_{\rho \in \mathcal{R}} \rho (X) , \forall X \in \mathcal{L} . \] (1)

In the upcoming definition, we generalize formulation (1) by taking a “benchmark” prospect \( Y \in \mathcal{L} \). Benchmark prospects have a long history in the field of optimization with stochastic constraints, see for instance [21, 48, 5, 49].

**Definition 3.1** (Robust choice function). Let \( \mathcal{R} \subset \mathcal{R}_{iqv} \) and \( Y \in \mathcal{L} \) be given, then \( \psi (\cdot ; \mathcal{R}, Y) : \mathcal{L} \to \mathbb{R} \) defined via

\[ \psi (X; \mathcal{R}, Y) \triangleq \inf_{\rho \in \mathcal{R}} \{ \rho (X) - \rho (Y) \} , \forall X \in \mathcal{L} , \] (2)

is the robust choice function corresponding to \( \mathcal{R} \) and \( Y \).

When the benchmark \( Y \) is a constant act and all \( \rho \in \mathcal{R} \) are normalized to have the same value at \( Y \), we recover formulation (1) from (2). The robust formulation is based on the minimal excess value of \( X \) over the value of the benchmark prospect \( Y \) for the whole set of choice functions specified in \( \mathcal{R} \). This kind of conservatism is to be used in decision making for countering risks arising from ambiguity about the true preferences. It is also very much in line with the philosophy of robust optimization. In both [5] and [19], the “worst-case utility function” and “worst-case risk measure” are considered in the same manner. This kind of framework is particularly relevant in the context of group decision making whereby the least favorable utility function from a member of the group is to be used for the holistic decision making process.

**Remark 3.2.** In our upcoming development, one may omit the benchmark \( Y \) and just consider the function \( \inf_{\rho \in \mathcal{R}} \rho (X) \) with only minor modification. We include the benchmark to stay consistent with [5] and the wider literature on stochastic dominance constrained optimization. The presence or absence of the benchmark does not materially affect our main development.

The following proposition shows that the robust choice function \( \psi (\cdot ; \mathcal{R}, Y) \) itself belongs to \( \mathcal{R}_{iqv} \) whenever \( \mathcal{R} \subset \mathcal{R}_{iqv} \).

**Proposition 3.3.** For any \( \mathcal{R} \subset \mathcal{R}_{iqv} \) and \( Y \in \mathcal{L} \), \( \psi (\cdot ; \mathcal{R}, Y) \) is upper semi-continuous, increasing, and quasi-concave.

**Proof.** Upper semi-continuity: Upper semi-continuity of a set of functions is preserved by taking the pointwise infimum of the collection.

Monotonicity: Monotonicity follows by

\[ \psi (X; \mathcal{R}, Y) = \inf_{\rho \in \mathcal{R}} \{ \rho (X) - \rho (Y) \} \leq \inf_{\rho \in \mathcal{R}} \{ \rho (Z) - \rho (Y) \} = \psi (Z; \mathcal{R}, Y) , \]

since \( \rho (X) \leq \rho (Z) \) for all \( \rho \in \mathcal{R} \) whenever \( X \leq Z \).
Quasi-concavity: Quasi-concavity follows since
\[
\psi(\lambda X + (1 - \lambda) Z; \mathcal{R}, Y) = \inf_{\rho \in \mathcal{R}} \{ \rho(\lambda X + (1 - \lambda) Z) - \rho(Y) \} \\
\geq \inf_{\rho \in \mathcal{R}} \{ \min \{ \rho(X), \rho(Z) \} - \rho(Y) \} \\
= \inf_{\rho \in \mathcal{R}} \min \{ \rho(X) - \rho(Y), \rho(Z) - \rho(Y) \} \\
= \min \left\{ \inf_{\rho \in \mathcal{R}} \{ \rho(X) - \rho(Y) \}, \inf_{\rho \in \mathcal{R}} \{ \rho(Z) - \rho(Y) \} \right\} \\
= \min \left\{ \psi(X; \mathcal{R}, Y), \psi(Z; \mathcal{R}, Y) \right\},
\]
for any \( X, Z \in \mathcal{L} \) with \( \lambda \in [0, 1] \), where the inequality follows by quasi-concavity of all \( \rho \in \mathcal{R} \), and the third equality follows by interchanging the order of minimization.

We now turn to discuss specification of the ambiguity set \( \mathcal{R} \). This set will have the following characteristics:

- **Preference elicitation**: For a sequence of pairs of prospects \( \{(W_i, Y_i)\}_{i \in \mathcal{I}} \), where \( \mathcal{I} \) is a finite index set, the decision maker prefers \( W_i \) to \( Y_i \) for all \( i \in \mathcal{I} \). In this case, all admissible choice functions in \( \mathcal{R}_{iqv} \) that are consistent with the decision maker’s observed behaviors must satisfy \( \rho(W_i) \geq \rho(Y_i) \) for all \( i \in \mathcal{I} \). This form of preference elicitation also appears in \([5, 19]\).

- **Normalization**: The decision maker’s choice function satisfies \( \rho(0) = 0 \).

- **Lipschitz continuity**: the decision maker’s choice function \( \rho \) is Lipschitz continuous. Lipschitz continuity ensures that the choice function does not vary too rapidly. Additionally, this technical condition is necessary to apply a key representation result for quasi-concave functions that appears in the next section. Since \( \psi(X; \alpha \mathcal{R}, Y) = \alpha \psi(X; \mathcal{R}, Y) \) for all \( \alpha \geq 0 \) and any \( \mathcal{R} \subset \mathcal{R}_{iqv} \), we may specify the Lipschitz constant \( L \) of \( \rho \) arbitrarily.

Our resulting specific ambiguity set is then
\[
\mathcal{S} \triangleq \{ \rho \in \mathcal{R}_{iqv} : \rho(W_i) \geq \rho(Y_i), \forall i \in \mathcal{I}; \rho(0) = 0; \rho \text{ is } L-\text{Lipschitz continuous} \}.
\]
We impose the \( L-\) Lipschitz condition since otherwise the set \( \mathcal{S} \) is a cone, in which case \( \psi(X; \mathcal{S}, Y) \) may not be finite-valued. In the next section we will develop a computational recipe for evaluating \( \psi(X; \mathcal{S}, Y) \).

We conclude this section by introducing our robust choice model. Let \( \mathcal{Z} \subset \mathbb{R}^m \) be a set of available decisions and let \( G : \mathcal{Z} \to \mathcal{L} \) be a random-variable-valued mapping with realizations denoted \([G(z)](\omega)\) for all \( \omega \in \Omega \). The mapping \( G \) captures the randomness inherent in the underlying decision-making problem. In general, we are interested in solving
\[
\max_{z \in \mathcal{Z}} \psi(G(z); \mathcal{S}, Y) \triangleq \max_{z \in \mathcal{Z}} \inf_{\rho \in \mathcal{R}} \{ \rho(g(Z)) - \rho(Y) \}. \tag{3}
\]
We make the following two key convexity assumptions on the problem data \( \mathcal{Z} \) and \( G \).

**Assumption 3.4.** (i) \( \mathcal{Z} \) is closed and convex.

(ii) \( G : \mathcal{Z} \to \mathcal{L} \) is concave in the sense that \([G(z)](\omega) : \mathcal{Z} \to \mathbb{R}^n \) is concave in \( z \in \mathcal{Z} \) for \( P \)-almost all \( \omega \in \Omega \).

Assumption 3.4 ensures that our upcoming optimization problems are convex, and the next proposition reveals the subsequent key structural results of Problem (3).

**Proposition 3.5.** Problem (3) is a quasi-concave maximization problem.

**Proof.** Let \( z_1, z_2 \in \mathcal{Z} \) and \( \lambda \in [0, 1] \). For any \( \rho \in \mathcal{R}_{iqv} \), we have
\[
\rho(\lambda G(z_1) + (1 - \lambda) G(z_2)) \geq \rho(\lambda G(z_1) + (1 - \lambda) G(z_2)) \\
\geq \min \{ \rho(G(z_1)), \rho(G(z_2)) \},
\]
where the first inequality follows by monotonicity of \( \rho \) and concavity of \( G \), and the second inequality follows by quasi-concavity of \( \rho \). The conclusion then follows by the previous part, the fact that the infimum of quasi-concave functions is quasi-concave, and the fact that the feasible region of Problem (3) is convex. \( \square \)
4 Support function representation

In this section we turn to the primary issue of numerical evaluation of the robust choice function \( \psi (X; S, Y) \). At first glance, it is evident that \( \psi (X; S, Y) \) cannot be evaluated with convex optimization because quasi-concavity is not preserved under convex combination (and \( \psi (X; S, Y) \) calls a minimization problem over a subset of quasi-concave functions). In contrast, the robust choice functions in [5] and [19] are amenable to convex optimization because concave utility functions and convex risk measures are preserved under convex combination. Despite this new difficulty, we can build on the support function technique used in [5, 19] and augment it for our present setting in \( R_{iqv} \).

We begin with some preliminary definitions and facts associated with support functions. For emphasis, in the following definition and throughout we consider upper support functions which dominate a target function from above rather than below (as in the convex and quasi-convex cases). We remind the reader that \( \langle a, b \rangle \) denotes the Euclidean inner product.

**Definition 4.1.** Let \( f : \mathbb{R}^d \to \mathbb{R} \). Recall that \( f \) is said to be **majorized** by a function \( g \) if

\[
f (x) \leq g (x), \forall x \in \text{dom } f,
\]

and \( g \) is an upper **support function** of \( f \) at \( x \in \mathbb{R}^d \) if \( f (x) = g (x) \). Here and later on, \( \text{dom } f \) denotes the domain of \( f \). A vector \( s \in \mathbb{R}^d \) is called a **subgradient** of \( f \) at \( x \in \mathbb{R}^d \) if

\[
f (y) \leq f (x) + \langle s, y - x \rangle, \forall y \in \text{dom } f.
\]

We denote the set of subgradients of \( f \) at \( x \) by \( \partial f (x) \) and call the latter **subdifferential**. A vector \( s \in \mathbb{R}^d \) is an **upper subgradient** of \( f \) at \( x \in \mathbb{R}^d \) if

\[
f (y) \leq f (x) + \langle s, y - x \rangle, \forall y \in \{ y \in \text{dom } f : f (y) \geq f (x) \}.
\]

We denote the subdifferential of \( f \) at \( x \) by \( \partial^+ f (x) \).

When \( f \) is concave and subdifferentiable at \( x \), i.e., \( \partial f (x) \neq \emptyset \), the linear function

\[
l (y) = f (x) + \langle a, y - x \rangle
\]

is a support function of \( f \) at \( x \) for any \( a \in \partial f (x) \). The following theorem gives rise to a characterization of concave functions by their support functions.

**Theorem 4.2.** Let \( f : \mathbb{R}^d \to \mathbb{R} \). The following assertions hold.

(i) \( f \) is a concave function if and only if there exists an index set \( J \) such that

\[
f (x) = \inf_{j \in J} l_j (x), \forall x \in \text{dom } f,
\]

where \( J \) is possibly infinite and \( l_j (x) = \langle a_j, x \rangle + b_j \) for all \( j \in J \).

(ii) For any finite set \( \Theta \subset \mathbb{R}^d \) and values \( \{ v_\theta \}_{\theta \in \Theta} \subset \mathbb{R} \), \( \hat{f} : \mathbb{R}^d \to \mathbb{R} \) defined by

\[
\hat{f} (x) = \min_{a, b} \{ \langle a, x \rangle + b : \langle a, \theta \rangle + b \geq v_\theta, \forall \theta \in \Theta \}
\]

is concave. Moreover, for all concave functions \( \hat{f} \) with \( \hat{f} (\theta) \geq v_\theta, \hat{f} \leq \hat{f} \).

**Proof.** Part (i). These results are well known, see for instance [11] Section 3.2 and [53].

Part (ii). It is immediate that \( \hat{f} \) as defined is concave, as it is of the form \( \hat{f} \). Note that the hypograph of \( \hat{f} \) is the convex hull of \( \{ \theta, v_\theta \}_{\theta \in \Theta} \). The convex hull of a set of points is by definition the smallest convex set containing these points. See also [11] Section 6.5.5.
Theorem 4.2 says that one can recover concave functions by taking the infimum of their support functions. Moreover, it shows that support functions can be used for the construction of the “lowest” concave function that dominates a fixed set of values. The results provide the basis for tractable formulations of PRO models in [5, 35, 10]. For further details on other applications of this result, we refer the reader to [11, Section 6.5.5] for a discussion of interpolation with convex functions.

When \( f \) is quasi-concave and upper subdifferentiable at \( x \), the piecewise linear function

\[
    h(x) = \max \{ f(x) + \langle a, y - x \rangle, f(x) \}
\]

is a support function of \( f \) at \( x \) for any \( a \in \partial^+ f(x) \). Note that these functions \( h \) are the maximum of two linear functions and so open upwards. We informally refer to such \( h \) as “hockey stick” functions in recognition of this shape. Note that these functions are quasi-concave themselves.

For any \( a \in \mathbb{R}^d \) and \( b, c \in \mathbb{R} \), the function \( h : \mathbb{R}^d \to \mathbb{R} \) defined by

\[
    h(x) = \max \{ \langle a, x \rangle + b, c \}
\]

is quasi-concave since its upper level sets are convex. In particular, it is easy to verify that for \( t \leq c \),

\[
    \{ x \in \mathbb{R}^d : h(x) \geq t \} = \mathbb{R}^d
\]

and for \( t > c \),

\[
    \{ x \in \mathbb{R}^d : h(x) \geq t \} = \{ x \in \mathbb{R}^d : \langle a, x \rangle + b \geq t \}
\]

which is convex (it is a half-space) since it is the upper level set of an affine function.

The following result is the analog of Theorem 4.2 for quasi-concave functions, it characterizes quasi-concave functions via this class of “hockey stick” support functions. The result will provide a theoretical foundation for tractable reformulation of our quasi-concave robust choice function. We note that the first part of the following theorem requires the stronger assumption that our function of interest \( f \) is \( L \)-Lipschitz continuous.

**Theorem 4.3.** Let \( f : \mathbb{R}^d \to \mathbb{R} \). The following assertions hold.

(i) Suppose that \( f \) is quasi-concave and \( L \)-Lipschitz continuous. Then

\[
    f(x) = \inf_{j \in J} h_j(x), \forall x \in \text{dom} f,
\]

where \( J \) is possibly infinite and \( h_j(x) = \max \{ \langle a_j, x \rangle + b_j, c_j \} \) for \( \|a_j\|_2 \leq L, j \in J \).

(ii) If \( f \) has a representation (5), then it is quasi-concave.

(iii) For any finite set \( \Theta \subset \mathbb{R}^d \) and values \( \{v_\theta\}_{\theta \in \Theta} \subset \mathbb{R} \), \( \hat{f} : \mathbb{R}^d \to \mathbb{R} \) defined by

\[
    \hat{f}(x) \triangleq \inf_{a,b,c} \max \{ \langle a, x \rangle + b, c \}
\]

\[
    \text{s.t. } \max \{ \langle a, \theta \rangle + b, c \} \geq v_\theta, \forall \theta \in \Theta,
\]

is quasi-concave. Furthermore, the graph of \( \hat{f} \) is the quasi-concave envelope for the set of points \( \{ (\theta, v_\theta) : \theta \in \Theta \} \).

**Proof.** Part (i). By [11] Theorem 2.3], any Lipschitz continuous quasi-concave function is upper subdifferentiable on its domain. Thus \( \partial^+ f(x) \neq \emptyset \) for any \( x \in \text{dom} f \) under the assumption that \( f \) is \( L \)-Lipschitz, and for any \( s_x \in \partial^+ f(x) \), \( \| s_x \|_2 \leq L \). Moreover, for any \( x \in \text{dom} f \) and \( s_x \in \partial^+ f(x) \), \( f \) is supported by \( h_x(y) = \max \{ f(x) + \langle s_x, y - x \rangle, f(x) \} \) at \( x \). By taking the infimum of all support functions defined as such, we have

\[
    f(y) \leq \inf_{x \in \text{dom} f, s_x \in \partial f(x)} \max \{ f(x) + \langle s_x, y - x \rangle, f(x) \}, \forall y \in \text{dom} f.
\]

Furthermore, for any \( y \in \text{dom} f \),

\[
    f(y) = h_y(y) \geq \inf_{x \in \text{dom} f} h_x(y)
\]
since \( f(y) = \max \{ f(y) + \langle s_y, y - y \rangle, f(y) \} = h_y(y) \).

Part (ii). For any \( t \in \mathbb{R} \) we have

\[
\{ x \in \mathbb{R}^d : f(x) \geq t \} = \left\{ x \in \mathbb{R}^d : \inf_{j \in J} h_j(x) \geq t \right\} = \bigcap_{j \in J} \{ x \in \mathbb{R}^d : h_j(x) \geq t \},
\]

each \( \{ x \in \mathbb{R}^d : h_j(x) \geq t \} \) is convex by quasi-concavity of \( h \) for all \( j \in J \), and the intersection of convex sets is convex.

Part (iii). The quasi-concavity of \( \hat{f} \) follows from the fact that each function \( \max \{ \langle a, x \rangle + b, c \} \) is quasi-concave and the infimum of such functions is also quasi-concave. In what follows, we prove the second part of the statement. For any \( t \in \mathbb{R} \), each \( \theta \in \Theta \) with \( v_\theta \geq t \) belongs to \( \{ x \in \mathbb{R}^d : \hat{f}(x) \geq t \} \) because by definition \( \hat{f}(\theta) \geq v_\theta \geq t \). Since \( \hat{f} \) is quasi-concave and has convex upper level sets, we must then have

\[
\text{conv} \{ \theta \in \Theta : v_\theta \geq t \} \subseteq \{ x \in \mathbb{R}^d : \hat{f}(x) \geq t \}, \tag{6}
\]

where “conv” denotes the convex hull of a set.

For \( x \notin \text{conv} \{ \theta \in \Theta : v_\theta \geq t \} \), we may choose appropriate parameters \( a \in \mathbb{R}^d \), \( b \in \mathbb{R} \) and \( c \in \mathbb{R} \) such that \( \max \{ \langle a, \theta \rangle + b, c \} \geq v_\theta \) for all \( \theta \in \Theta \) and \( \max \{ \langle a, x \rangle + b, c \} < t \). To construct such a hockey-stick function, first let \( y \in \mathbb{R}^d \) be the projection of \( x \) onto the convex set \( \text{conv} \{ \theta \in \Theta : v_\theta \geq t \} \). Next, by virtue of the separation theorem in convex analysis, there exists \( \hat{a} \in \mathbb{R}^d \) such that \( \langle \hat{a}, y - x \rangle \geq \max_{\theta \in \Theta} v_\theta \) and \( \langle \hat{a}, \theta - y \rangle \geq 0 \) for all \( \theta \in \{ \theta \in \Theta : v_\theta \geq t \} \). Let \( \hat{c} = \max_{\theta \in \Theta} \{ v_\theta : v_\theta < t \} < t \) and \( \hat{b} = -\langle \hat{a}, x \rangle \). Then

\[
\max \{ \langle \hat{a}, \theta \rangle + \hat{b}, \hat{c} \} = \max \{ \langle \hat{a}, \theta - x \rangle, \hat{c} \} = \max \{ \langle \hat{a}, \theta - y \rangle + \langle \hat{a}, y - x \rangle, \hat{c} \} \geq \max \{ \langle \hat{a}, \theta - y \rangle, \hat{c} \}, \quad \text{for} \ \theta \in \{ \theta \in \Theta : v_\theta \geq t \} \text{ in any case}
\]

By the definition of \( \hat{f}(x) \), the inequality above implies that \( \hat{f}(x) \leq \max \{ \langle \hat{a}, \theta \rangle + \hat{b}, \hat{c} \} \). On the other hand, it is easy to verify that \( \max \{ \langle \hat{a}, x \rangle + \hat{b}, \hat{c} \} = \hat{c} \). Thus, we arrive at \( \hat{f} < t \), which enables us to deduce that \( x \notin \{ x \in \mathbb{R}^d : \hat{f}(x) \geq t \} \) and subsequently

\[
\{ x \in \mathbb{R}^d : \hat{f}(x) \geq t \} \subseteq \text{conv} \{ \theta \in \Theta : v_\theta \geq t \}. \tag{7}
\]

A combination of (6) and (7) yields

\[
\text{conv} \{ \theta \in \Theta : v_\theta \geq t \} = \{ x \in \mathbb{R}^d : \hat{f}(x) \geq t \}.
\]

Since the convex hull is the smallest convex set containing a set of points, it must be that

\[
\{ x \in \mathbb{R}^d : \hat{f}(x) \geq t \} = \text{conv} \{ \theta \in \Theta : v_\theta \geq t \} \subseteq \{ x \in \mathbb{R}^d : \hat{f}(x) \geq t \}
\]

for any other quasi-concave \( \hat{f} \) with \( \hat{f}(\theta) \geq v_\theta \) for all \( \theta \in \Theta \). This same reasoning holds for all \( t \in \mathbb{R} \), so

\[
\{ x \in \mathbb{R}^d : \hat{f}(x) \geq t \} \subseteq \{ x \in \mathbb{R}^d : \hat{f}(x) \geq t \} \text{ for all } t \in \mathbb{R} \text{ and } \hat{f} \leq \hat{f}.
\]

We note that we may use any norm (not necessarily the Euclidean norm) to enforce Lipschitz continuity in part (i) of Theorem 4.3, since all norms on \( \mathbb{R}^d \) are equivalent. In parallel to Theorem 4.2, Theorem 4.3 gives conditions where a quasi-concave function can be recovered by taking the infimum of its support functions (which are hockey stick functions in this case). Moreover, Theorems 4.2 and 4.3 give conditions for constructing the “lowest” quasi-concave function that contains a fixed set of values.
4.1 Reformulation as a mixed-integer linear program

Theorem 4.3 gives an explicit form for the “worst-case” quasi-concave function that dominates a set of values \( \{v_\theta\}_{\theta \in \Theta} \) over a finite set \( \Theta \). In fact, this is exactly what we need to derive a tractable reformulation of \( \psi(X; S, \tilde{Y}) \). The remaining challenge is to put the correct conditions on the values \( \{v_\theta\}_{\theta \in \Theta} \), which will take the form of an optimization problem. To this effect, for the remainder of this section we introduce the major assumption that the underlying sample space is finite.

**Assumption 4.4.** The sample space \( \Omega \) is finite.

Assumption 4.4 also appears in [5, 33, 19] where it is used for obtaining tractable optimization formulations. However, in the case when \( \Omega \) is continuous, it is possible to develop a discrete approximation (see [18, Section 5] and [33]).

Under Assumption 4.4 we adopt the convention that a prospect \( X \in \mathcal{L} \) may be identified with the vector of its realizations

\[
X = (X(\omega))_{\omega \in \Omega} \in \mathbb{R}^{|\Omega|}.
\]

This convention first appeared in [19] and depends on a finite sample space \( \Omega \). In this way, there is a one-to-one correspondence between elements of \( \mathcal{L} \) and \( \mathbb{R}^{|\Omega|} \). We now define

\[
\Theta \triangleq \{\emptyset\} \cup \{\tilde{W}_i\}_{i \in \mathcal{I}} \cup \{Y_i\}_{i \in \mathcal{I}} \cup \{\tilde{Y}\}
\]

to be the union of all the prospects used in the definition of \( \mathcal{S} \) (including the constant prospect \( \emptyset \) which is used in the normalization condition) along with the benchmark \( Y \).

To proceed on, we let

\[
S(v) \triangleq \{\rho : \mathcal{L} \to \mathbb{R} \text{ s.t. } \rho(\theta) = v_\theta, \forall \theta \in \Theta, \rho \text{ is law invariant}\}
\]

denote the set of choice functions that take the values \( \{v_\theta\}_{\theta \in \Theta} \) on the set \( \Theta \). We now evaluate the worst-case choice function \( \psi(X; S, Y) \) with the following procedure:

1. Set the values \( \{v_\theta\}_{\theta \in \Theta} \) of \( \rho \) on \( \Theta \). We only consider these values at first because they give sufficient information to construct \( \rho \) on the rest of \( \mathcal{L} \), as we will show.

   a. The values \( \{v_\theta\}_{\theta \in \Theta} \) must satisfy the majorization characterization for quasi-concave functions (a quasi-concave function is majorized by its hockey stick support function at every point on its graph). This is equivalent to determining if \( S(v) \cap \mathcal{R}_{\rho \theta} \neq \emptyset \).

   b. The values \( \{v_\theta\}_{\theta \in \Theta} \) must satisfy the preference elicitation condition given in the definition of \( \mathcal{S} \).

   c. The values \( \{v_\theta\}_{\theta \in \Theta} \) must satisfy the \( L \)-Lipschitz continuity condition given in the definition of \( \mathcal{S} \).

2. Once the values of \( \rho \) are fixed on \( \Theta \) satisfying the above conditions, interpolate using hockey stick support functions to determine the value at any \( X \in \mathcal{L} \).

This procedure results in the following optimization problem:

\[
\min_{a, s, b, c, v} \max \left\{ (a, X') + b, c \right\} - v_Y \quad \text{s.t.} \quad \max \{s_\theta + (s_\theta, \theta' - \theta), v_\theta\} \geq v_{\theta'}, \quad \forall \theta \neq \theta'; \theta, \theta' \in \Theta, \quad v_\emptyset = 0, \quad v_{\tilde{W}_i} \geq v_Y, \quad \forall i \in \mathcal{I}, \quad s_\theta \geq 0, \quad \|s_\theta\|_{\infty} \leq L, \quad \forall \theta \in \Theta, \quad \max \{(a, \theta) + b, c\} \geq v_\theta, \quad \forall \theta \in \Theta, \quad a \geq 0, \quad \|a\|_{\infty} \leq L. \tag{10}
\]

Intuitively, constraint (9) is the majorization characterization for the values \( v_\theta = \rho(\theta) \) for all \( \theta \in \Theta \); constraint (10) is the normalization constraint; constraint (11) corresponds to the preference elicitation requirement.
in the definition of $\mathcal{S}$; constraint $[12]$ requires the support functions used to characterize $\{v_{\theta}\}_{\theta \in \Theta}$ to be increasing and Lipschitz continuous; constraint $[13]$ requires the support function used to determine the value of $\rho$ at $X$ to majorize $\rho$; and finally constraint $[14]$ requires the support function used to determine the value of $\rho$ at $X$ to be increasing and Lipschitz continuous.

Problem $[8]$ - $[14]$ has several features in common with $[5, 19]$. In particular, all of these formulations have a support function characterization that ensures convexity/concavity/quasi-concavity, and all of these formulations have constraints corresponding to preference elicitation. The main difference is that the formulations in $[5, 19]$ are based on linear functions while our formulation is built on hockey stick functions.

Remark 4.5. The convention $\bar{X} = (X(\omega))_{\omega \in \Omega}$ avoids the requirement that the convex hull of $\Theta$ contain the support of $X$ in Problem $[8]$ - $[14]$ as in $[5] [18]$.

The next theorem formally verifies the correctness of this formulation.

Theorem 4.6. Suppose Assumption $[4, 4]$ holds. Given $X \in \mathcal{L}$, the optimal value of Problem $[8]$ - $[14]$ is equal to $\psi (X; S, Y)$.

Proof. To begin, we may partition the set of choice functions by their values on the finite set $\Theta$. We have the equivalence

$$\psi (X; S, Y) = \inf_{\rho \in \mathcal{S}} \{\rho (X) - \rho (Y)\}$$

$$= \min_{v \in [0, 1]} \inf_{\rho \in \mathcal{S}(v) \cap \mathcal{S}} \{\rho (X) - \rho (Y)\}$$

$$= \min_{v \in [0, 1]} \{\psi (X; S(v) \cap S, Y) : S(v) \cap S \neq \emptyset\},$$

where we use $S = \bigcup_{v \in [0, 1]} \{S(v) \cap S\}$, and $|\Theta|$ denotes the cardinality of $\Theta$. This condition is simply saying that $\psi (X; S, Y)$ can be understood as either minimizing over $\mathcal{S}$ directly, or first fixing the relevant values $\{v_{\theta}\}_{\theta \in \Theta}$ on $\Theta$ and then minimizing over functions in $\mathcal{S}$ that coincide with those values.

Next define the sets

$$S_N \triangleq \{\rho : \mathcal{L} \to \mathbb{R} : \rho (0) = 0\},$$

$$S_E \triangleq \{\rho : \mathcal{L} \to \mathbb{R} \text{ s.t. } \rho (W_i) \geq \rho (Y_i), \forall i \in I\},$$

and

$$S_L \triangleq \{\rho : \mathcal{L} \to \mathbb{R} \text{ s.t. } \rho \text{ is } L\text{-Lipschitz continuous}\}. $$

Then, using the fact that $S = S_N \cap S_E \cap S_L \cap \mathcal{R}_{\mathrm{iqv}}$, we have

$$\psi (X; S; Y) = \min_{v \in [0, 1]} \psi (X; S(v) \cap S, Y)$$

s.t. $S(v) \subseteq S_N$, $S(v) \subseteq S_E$, $S(v) \subseteq S_L$, $S(v) \cap \mathcal{R}_{\mathrm{iqv}} \neq \emptyset$.

Constraint $S(v) \subseteq S_N$ is just the preference elicitation condition $[11]$ and constraint $S(v) \subseteq S_L$ is just the Lipschitz continuity requirement $[12]$. Both of these conditions only constrain the values of $\rho$ on $\Theta$ via $v$. Constraint $S(v) \cap \mathcal{R}_{\mathrm{iqv}} \neq \emptyset$ states that there must exist a function in $\mathcal{R}_{\mathrm{iqv}}$ that takes the values $\{v_{\theta}\}_{\theta \in \Theta}$ on $\Theta$. This requirement is enforced by: (i) constraint $[9]$, the majorization characterization of the quasi-concavity of $\{v_{\theta}\}_{\theta \in \Theta}$; and (ii) constraint $[12]$, the requirement that the support functions that majorize $\rho$ on $\Theta$ be increasing and Lipschitz continuous.

So, it remains to evaluate $\psi (X; S(v) \cap S, Y)$ for fixed $v \in [0, 1]$, which is explicitly

$$\inf_{\rho \in S(v) \cap S} \{\rho (X) - \rho (Y)\} = \inf_{\rho \in S(v) \cap S} \rho (X) - \rho (Y)$$

since the value $\rho (Y)$ is fixed for $\rho \in S(v) \cap S$ by construction of $S(v)$. Over all increasing quasi-concave functions in $S(v)$, the one minimizing $\inf_{\rho \in S(v) \cap S} \rho (X)$ attains the value

$$\min \left\{ \max \left\{ \langle a, \bar{X} \rangle + b, c \right\} : \max \left\{ \langle a, \theta \rangle + b, c \right\} \geq v_{\theta}, \forall \theta \in \Theta \right\},$$

which is captured by constraint $[13]$ that requires the support function $\max \left\{ \langle a, \bar{X} \rangle + b, c \right\}$ to majorize $\{v_{\theta}\}_{\theta \in \Theta}$. \qed
Problem (8) - (14) is finite-dimensional, but it is not a linear programming problem (or even a convex optimization problem) due to constraints (9) and (13) which require a convex function to be greater than a linear term. However, it is possible to transform Problem (8) - (14) into a mixed-integer linear programming optimization problem due to constraints (9) and (13) which require a convex function to be greater than a linear term. We obtain the following MILP where we use a new constant linear term. However, it is possible to transform Problem (8) - (14) into a mixed-integer linear programming problem (MILP) using standard techniques. We obtain the following MILP where we use a new constant $M \gg 0$:

$$\begin{align*}
\text{min} & \quad t - v_\varphi \\
\text{s.t.} & \quad t \geq \langle a, \bar{X} \rangle + b, \\
& \quad t \geq c, \\
& \quad \langle s_\theta, \theta' - \theta \rangle + v_\theta + M x_\theta, \theta' \geq v_{\theta'}, \quad \forall \theta \neq \theta'; \theta, \theta' \in \Theta, \\
& \quad v_\theta + M (1 - x_\theta, \theta') \geq v_{\theta'}, \quad \forall \theta \neq \theta'; \theta, \theta' \in \Theta, \\
& \quad v_\bar{\varphi} = 0, \\
& \quad v_\bar{W}_i \geq v_{\bar{Y}_i}, \quad \forall i \in \mathcal{I}, \\
& \quad s_\theta \geq 0, \quad \|s_\theta\|_\infty \leq L, \quad \forall \theta \in \Theta, \\
& \quad \langle a, \theta \rangle + b + M y_\theta \geq v_\theta, \quad \forall \theta \in \Theta, \\
& \quad c + M (1 - y_\theta) \geq v_\theta, \quad \forall \theta \in \Theta, \\
& \quad a \geq 0, \quad \|a\|_\infty \leq L, \\
& \quad x_\theta, \theta' \in \{0, 1\}, \quad \forall \theta \neq \theta'; \theta, \theta' \in \Theta, \\
& \quad y_\theta \in \{0, 1\}, \quad \forall \theta \in \Theta.
\end{align*}$$

In explanation, constraints (16) and (17) replace the term $\max\left(\langle a, \bar{X} \rangle + b, c \right)$ that appears in the objective of Problem (8) - (14) with linear terms via the epigraphical transformation. Constraints (18) and (19) along with the binary constraints on $x$ replace constraints (9) with disjunctive constraints; likewise, constraints (23) and (24) along with the binary constraints on $y$ replace constraints (13) with disjunctive constraints.

**Remark 4.7.** In terms of computation, Problem (15) - (25) can be effectively solved by Bender’s decomposition as proposed in [16, 52, 57]. In [17], the authors explain that the linear programming relaxation of the MILP is typically a poor approximation due to the big-$M$ coefficients. In fact, the binary solutions of LP relaxations are only marginally affected by the addition of continuous variables and the associated constraints. Different choices of $M$ will affect the branching process in the algorithm, as well as the constraints in the LP relaxation. In recognition of this difficulty, these authors introduce “Combinatorial Bender’s cuts” that can generate more effective cuts for the master problem and that also avoid the difficulty of choosing the constant $M$. Alternatively, based on [33], we could use a convex hull reformulation for each disjunction in Problem (15) - (25). This reformulation will always give a tighter bound than the big-$M$ formulation, at the expense of a large number of variables and constraints. Both of these approaches converge to the global optimum after finitely many iterations, and can improve upon basic methods for solving Problem (15) - (25).

### 4.2 The worst-case choice function

We conclude this section by giving further details on the explicit form of the worst-case choice function. In [5], the worst-case utility function is shown to be piecewise linear concave by using a support function argument. This is possible because the worst-case utility function in question is a mapping from $\mathbb{R}$ to $\mathbb{R}$. Our present setting is more complicated because: (i) we deal with multi-attribute prospects and (ii) we are concerned with quasi-concave functions. Yet, as we will see shortly, our worst-case choice function can also be constructed explicitly. The Delaunay triangulation, defined next, is the key to this approach.

**Definition 4.8.** (i) A simplex $\sigma$ is a polytope in $\mathbb{R}^d$ such that $\sigma$ is the convex hull of $d+1$ affinely independent points.

(ii) A simplicial complex $\mathcal{C}$ is a finite collection of simplices such that: $\forall \sigma \in \mathcal{C}$, $\sigma$ is a simplex; $\sigma_1 \in \mathcal{C}$ and $\sigma_1 \subset \sigma_2$ imply $\sigma_2 \in \mathcal{C}$; and for any $\sigma_1, \sigma_2 \in \mathcal{C}$, either $\sigma_1 \cap \sigma_2 = \emptyset$ or $\sigma_1 \cap \sigma_2 \in \mathcal{C}$. 

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Theorem 4.10. Choose theorem 4.3(iii), only now the worst-case choice function is piecewise linear.

Definition 5.2. A function \( \mu : \mathcal{L} \to \mathbb{R} \) is a convex risk measure if it satisfies

(i) Monotonicity: If \( X \leq Y \) then \( \mu (X) \geq \mu (Y) \).
(ii) Normalization: \( \mu (0) = 0 \).
(iii) Convexity: for any \( X, Y \in \mathcal{L} \) and \( \lambda \in [0, 1] \), \( \mu (\lambda X + (1 - \lambda) Y) \leq \lambda \mu (X) + (1 - \lambda) \mu (Y) \).

Since we treat prospects in \( \mathcal{L} \) as gains/rewards, our definition of monotonicity above is the opposite of the typical definition of monotonicity for losses. Convexity of risk measures is extremely important for our considerations because of the prominent role of convexity in optimization. Note that we do not yet stipulate a property of translation invariance for the multi-attribute setting, we give further commentary on this issue later.

Any risk measure \( \mu : \mathcal{L} \to \mathbb{R} \) induces a set of “acceptable” prospects in the sense that a prospect \( X \) is acceptable if \( \mu (X) \leq 0 \), i.e. it has nonpositive risk according to \( \mu \). We formalize this idea in the following definition.

Definition 5.2. Let \( \mu : \mathcal{L} \to \mathbb{R} \) be a risk measure, the set \( \mathcal{A}_\mu \triangleq \{ X \in \mathcal{L} : \mu (X) \leq 0 \} \) is the acceptance set associated with \( \mu \).
Note that $\mathcal{A}_k$ is a convex set in $\mathcal{L}$ whenever $\mu$ is a convex risk measure. Conversely, given an acceptance set $\mathcal{A} \subset \mathcal{L}$ we can specify a risk measure $\mu_{\mathcal{A},d}(X) \triangleq \inf_{\alpha \in \mathbb{R}} \{X + \alpha \cdot d \in \mathcal{A}\}$ for some $d \in \mathbb{R}^n$ with $d > 0$. We interpret $\mu_{\mathcal{A},d}(X) \cdot d$ as the vector-valued amount that must be added to $X$ to make $X$ acceptable to the decision maker.

For a choice function $\rho \in \mathcal{R}_{iqv}$ which evaluates the fitness of prospects in $\mathcal{L}$, we seek a family of convex risk functions $\{\mu_k\}_{k \in \mathbb{R}}$ such that:

$$\{X \in \mathcal{L} : \rho(X) \geq k\} = \{X \in \mathcal{L} : \mu_k(X) \leq 0\}, \forall k \in \mathbb{R}. \quad (26)$$

Relationship (26) means that the upper level sets of the choice function $\rho$ can be characterized by the acceptance sets of a family of convex risk functions. The acceptance sets are closed and convex since $\rho$ is assumed to be upper-semicontinuous and quasi-concave. From a practical point of view, if a decision maker with choice function $\rho$ selects satisfaction level $k$, then the prospect of exceeding $k$ is equivalent to the prospect of the risk being less than or equal to zero under measure $\mu_k$. This perspective is related to the notion of satisficing measures developed in [13]. We interpret (26) to mean that the set of satisfiable prospects can be characterized by the acceptance sets of a sequence of risk measures.

In the case of relationship (26), we have the representation

$$\rho(X) = \sup \{k \in \mathbb{R} : \mu_k(X) \leq 0\}, \forall X \in \mathcal{L}. \quad (27)$$

This equivalence is established in Proposition A.3 in the Appendix. Representation (27) is valuable both from a theoretical perspective and a computational one. Theoretically, it reveals the connection between quasi-concave choice functions and convex risk measures. Computationally, it allows us to evaluate quasi-concave choice functions with a sequence of convex feasibility problems and a bisection algorithm.

**Remark 5.3.** A related form of representation (27) is considered for univariate prospects in [12]. As an illustrative example, [12, Example 5] considers the case where the $\{\mu_k\}_{k \in \mathbb{R}}$ are given by CVaR. Furthermore, the univariate form of Representation (27) can be viewed as a generalization of the shortfall risk measure from [23, Section 4.3], [41, 64 Section 3], and [30]. In this case, for a convex loss function $l : \mathbb{R} \to \mathbb{R}$ we may take $\mu_k(X) = \mathbb{E}[l(X - k)]$ for all $k \in \mathbb{R}$.

In representation (27), we may also take $\mu_k$ to be any support function of $\rho$ at $\rho(X) = k$. We now consider the family $\{\mu_k\}_{k \in \mathbb{R}}$ more carefully, and we require it to satisfy the following assumptions.

**Assumption 5.4.** $\{\mu_k\}_{k \in \mathbb{R}}$ is a family of risk measures which satisfies the following:

(i) for each fixed $k \in \mathbb{R}$, $\mu_k(\cdot)$ is monotonically decreasing (non-increasing) on $\mathcal{L}$;

(ii) for each fixed $k \in \mathbb{R}$, $\mu_k(\cdot)$ is convex on $\mathcal{L}$;

(iii) for each fixed $k \in \mathbb{R}$, $\mu_k(\cdot)$ has closed acceptance sets;

(iv) for each fixed $X \in \mathcal{L}$, $\mu_k(X)$ is monotonically increasing (non-decreasing) in $k$ over $\mathbb{R}$.

Property (iv) means that for fixed $X \in \mathcal{L}$, $\mu_{k_1}(X) \leq \mu_{k_2}(X)$, that is, $\mu_{k_2}(\cdot)$ assigns a higher risk value $\mu_{k_1}(\cdot)$ for $X$.

Our next technical result shows that we can use the set of risk measures $\{\mu_k\}_{k \in \mathbb{R}}$ satisfying Assumption 5.4 to construct a choice function $\rho \in \mathcal{R}_{iqv}$. Specifically, we may define $\rho(X)$ to be the highest index level $k$ such that the corresponding risk of $X$ is acceptable, i.e. $\mu_k(X) \leq 0$. The conclusion follows directly from Proposition A.2 in the Appendix.

**Theorem 5.5.** Suppose Assumption 5.4 holds for $\{\mu_k\}_{k \in \mathbb{R}}$ and let

$$\vartheta(X) \triangleq \sup \{k \in \mathbb{R} : \mu_k(X) \leq 0\}, \forall X \in \mathcal{L}. \quad (28)$$

Then $\vartheta$ is upper semi-continuous, increasing, and quasi-concave.

In the case when $\mu_k(X) \leq 0$ for all $k \in \mathbb{R}$, we have $\rho(X) = +\infty$. In order for $\rho(X)$ to be finite valued, $\mu_k(X)$ must increase substantially w.r.t. increase of $k$. This situation is similar to the case of utility shortfall risk measures where the underlying loss function must be strictly increasing from some point, see [23].

We now consider the reverse implication of the previous theorem, which shows that there is such a representation (27) for any choice function $\rho \in \mathcal{R}_{iqv}$. As we will see in the following proof, the choice of this representation is not unique.
Theorem 5.6. For any $\rho \in \mathcal{R}_{sqv}$, there exists a family $\{\mu_k\}_{k \in \mathbb{R}}$ satisfying Assumption 5.4 such that 

$$\rho(X) = \sup \{ k \in \mathbb{R} : \mu_k(X) \leq 0 \}, \forall X \in \mathcal{L}.$$ 

Proof. We provide a constructive proof for the claim and demonstrate that $\mu_k$ can be constructed in three different ways.

(i) Let

$$\mu_k(X) = \begin{cases} 
0 & \rho(X) \geq k, \\
\infty & \text{otherwise.}
\end{cases}$$

Then

$$\{X \in \mathcal{L} : \rho(X) \geq k\} = \{X \in \mathcal{L} : \mu_k(X) \leq 0\},$$

for all $k \geq 0$, and $\rho(X) = \sup \{ k \in \mathbb{R} : \mu_k(X) \leq 0 \}$. It is immediate that the proposed $\{\mu_k\}_{k \in \mathbb{R}}$ are increasing in $k$, decreasing by monotonicity of $\rho \in \mathcal{R}_{sqv}$, are convex, and have closed acceptance sets.

(ii) Let

$$\mu_k(X) = \text{dist}(X, \{X \in \mathcal{L} : \rho(X) \geq k\}).$$

Then $\{X \in \mathcal{L} : \rho(X) \geq k\} = \{X \in \mathcal{L} : \mu_k(X) \leq 0\}$ and $\rho(X) = \sup \{ k \in \mathbb{R} : \mu_k(X) \leq 0 \}$. Again, it is immediate that the proposed $\{\mu_k\}_{k \in \mathbb{R}}$ satisfy the criteria of Assumption 5.4 by choice of $\rho \in \mathcal{R}_{sqv}$.

(iii) Let $d \in \mathbb{R}^n$ with $d > 0$ (component-wise strict inequality), and let

$$\mu_k(X) = \inf \{ a \in \mathbb{R} : \rho(X + a) \geq k \}.$$ 

Then $\{X \in \mathcal{L} : \rho(X) \geq k\} = \{X \in \mathcal{L} : \mu_k(X) \leq 0\}$ and $\rho(X) = \sup \{ k \in \mathbb{R} : \mu_k(X) \leq 0 \}$. This choice of $d$ has an important interpretation in multi-attribute decision-making, specifically, this $d$ reveals the decision maker’s endogenous weights among the multiple attributes. The desired properties of the proposed $\{\mu_k\}_{k \in \mathbb{R}}$ then follow from Proposition A.1.

It follows from the previous two theorems that there is an equivalence between $\rho \in \mathcal{R}_{sqv}$ and families $\{\mu_k\}_{k \in \mathbb{R}}$ satisfying Assumption 5.4. Theorems 5.5 and 5.6 together complete what we call the “level set representation” for a choice function $\rho \in \mathcal{R}_{sqv}$. These theorems are closely related to the results in [13, 12] for univariate choice functions, we discuss this relationship further later in the paper.

5.2 Connection with support functions

In this subsection we briefly comment on the connection between our earlier support function representation and our level set representation.

Theorem 5.7. Let $\rho(X) = \inf_{j \in \mathcal{J}} h_j(X)$ where $h_j(X) = h_j\left(\vec{X}\right) = \max \left\{ (a_j, \vec{X}) + b_j, c_j \right\}$ for $a_j \in \mathbb{R}^{n \mid \Omega}$ and $a_j \geq 0$ for all $j \in \mathcal{J}$. Assume that there exists $d \in \mathbb{R}^n$ such that $\langle a_j, d \rangle > 0$ for all $j \in \mathcal{J}$ and let

$$\mu_k(X) \triangleq \sup_{j \in \mathcal{J}} \inf \{ t \in \mathbb{R} : h_j(X + t d) \geq k \}, \forall k \in \mathbb{R}. \quad (29)$$

The following assertions hold.

(i) $\rho(X) = \sup \{ k \in \mathbb{R} : \mu_k(X) \leq 0 \}$ for all $X \in \mathcal{L}$;

(ii) $\mu_k$ is convex on $\mathcal{L}$ for all $k \in \mathbb{R}$.

Proof. Observe first that $h_j(X + t d)$ is strictly increasing in $t$ for each $j \in \mathcal{J}$ under the specified choice of $d$.

Part (i). By the proof of Theorem 5.6, $\rho(X) = \sup \{ k \in \mathbb{R} : \mu_k(X) \leq 0 \}$, where

$$\mu_k(X) \triangleq \inf \{ t \in \mathbb{R} : \rho(X + t d) \geq k \}.$$ 

It suffices to show that $\mu_k(X) = \mu_k(X)$ where the latter is defined by (29). By definition

$$\inf \{ t \in \mathbb{R} : \rho(X + t d) \geq k \} = \inf \left\{ t \in \mathbb{R} : \inf_{j \in \mathcal{J}} h_j(X + t d) \geq k \right\}.$$
Thus, we are left to show
\[
\inf \left\{ t \in \mathbb{R} : \inf_{j \in J} h_j(X + td) \geq k \right\} = \sup \inf \left\{ t \in \mathbb{R} : h_j(X + td) \geq k \right\}.
\]

Let \( t^\ast \) and \( \hat{t} \) denote respectively the optimal value on the left hand side and right hand side of equation (30), let \( t_j = \inf \{ t \in \mathbb{R} : h_j(X + td) \geq k \} \). If \( t^\ast = +\infty \), then \( \hat{t} = +\infty \). To see this, assume for the sake of a contradiction that \( \hat{t} < +\infty \). Then \( t_j < +\infty \) for every \( j \in J \). Since \( h_j(X + td) \) is strictly increasing in \( t \), then for any positive number \( \epsilon \),
\[
h_j \left( X + (\hat{t} + \epsilon) d \right) \geq h_j \left( X + (t_j + \epsilon) d \right) > h_j(X + t_j d) \geq k, \forall j \in J,
\]
which implies \( \hat{t} + \epsilon \) is a feasible solution of the minimization problem at the left hand side of (30) and hence \( t^\ast < +\infty \), which gives the desired contradiction.

We now consider the case when \( t^\ast < +\infty \). For any \( \epsilon > 0 \),
\[
\inf_{j \in J} h_j \left( X + (t^\ast + \epsilon) d \right) \geq k.
\]
Driving \( \epsilon \) to 0, we have that \( \inf_{j \in J} h_j \left( X + t^\ast d \right) \geq k \) and hence
\[
h_j \left( X + t^\ast d \right) \geq k, \forall j \in J.
\]
This enables us to deduce that \( t_j \leq t^\ast \) for each \( j \in J \) and subsequently \( \hat{t} \leq t^\ast \). To show the reverse inequality, we note that for any \( \epsilon > 0 \),
\[
h_j \left( X + (\hat{t} + \epsilon) d \right) \geq k, \forall j \in J,
\]
and hence \( \inf_{j \in J} h_j \left( X + \hat{t} d \right) \geq k \). The latter implies \( \hat{t} \) is a feasible solution of the minimization problem on the left hand side of equation (30) and hence \( t^\ast \leq \hat{t} \). The proof is complete.

Part (ii). We first note that \( h_j \) is quasi-concave as the maximum of a linear function and a constant function (\( h_j \) is automatically convex, but it is also quasi-concave in this special case). For each \( j \in J \) and \( k \in \mathbb{R} \) we define
\[
\hat{\mu}_{j,k}(X) \triangleq \inf \{ t \in \mathbb{R} : h_j(X + td) \geq k \}.
\]
Convexity of \( \hat{\mu}_{j,k} \) follows by Proposition A.1. The desired conclusion follows by noting that \( \mu_k \) itself is the supremum of the convex functions \( \hat{\mu}_{j,k} \), which preserves convexity.

Theorem 5.7 shows that we may construct the level set representation \([27]\) for \( \rho \in \mathcal{R}_{ivq} \) by using the support functions of \( \rho \). However, from computational point view, it might be difficult to identify the optimal (lowest) support function which majorizes \( \rho(X) \) when \( \rho \) has a general structure. This motivates us to consider a lower approximation of \( \mu_k(X) \) constructed by using a single piece of a support function, which is not necessarily optimal, but that can be obtained by calculating an upper sub-gradient of \( \rho \). Let us denote this function by \( h_j(X) = \max \left( (a_j, \bar{X}) + b_j, c_j \right) \) and define
\[
\nu_k(X) \triangleq \inf \{ t \in \mathbb{R} : h_j(X + td) \geq k \}.
\]
Obviously \( \nu_k(X) \leq \mu_k(X) \) and through \([26]\), we have
\[
\{ X \in \mathcal{L} : \rho(X) \geq k \} = \{ X \in \mathcal{L} : \mu_k(X) \leq 0 \} \subset \{ X \in \mathcal{L} : \nu_k(X) \leq 0 \}, \forall k \in \mathbb{R},
\]
which implies that the upper level set of \( \rho \) at level \( k \) is contained by the lower level set of \( \nu_k \) at level \( 0 \). By developing a sequence of functions \( \{\nu_k\}_{k \in \mathbb{R}} \) appropriately, we will be able to use the intersection of the lower level sets of \( \nu_k, k = 1, \cdots, \) to approximate the maximizer of \( \rho \) (the upper level set of \( \rho \) at its maximum). This procedure is closely related to the idea of the so-called level function method in [68] which we will discuss in detail in Section 6 for solving Problem \([2]\). Note also that in the case when \( k = c \), we can obtain an explicit form of \( \nu_k(X) \) with \( \nu_c(X) = (a_j, X) + b_j - c_j \) for each \( a_j, b_j, c_j \). So, if we set \( a_j, d_j = 1 \), then we obtain \( \nu_c(X) = (a_j, X) + b_j - c_j \). Moreover, by setting \( b_j = c_j - \langle a_j, X_c \rangle \), we have \( \nu_c(X) = \langle a_j, X - X_c \rangle \) which is a level function (see Definition 6.1).
5.3 The role of targets

In this section we explore the role of targets in expressing decision maker preferences. This issue is originally investigated in [12], and here we extend it to the multi-attribute setting. A target is a benchmark (typically a monetary amount or level of reward) that the decision maker wishes to achieve. We can assess a prospect $X$ in terms of its ability to meet a target. Target-based decision making is further elaborated upon in [40, 44, 25] and forms the basis for the satisficing and aspirational preferences in [13, 12]. We remark that this notion of target is different from the benchmarks that appear in the literature on stochastic dominance constraints. Here, we have a continuum of scalar targets for all satiation levels, while in the case of stochastic dominance constraints the benchmark is a single fixed prospect.

Next we formally define a target and related concepts for prospects $X \in \mathcal{L}$.

**Definition 5.8.** (i) A target $\tau \in \mathbb{R}^n$ is a desired level of gain/reward.

(ii) For a prospect $X$, $X - \tau$ is the target premium.

(iii) For a risk measure $\mu : \mathcal{L} \to \mathbb{R}$, $\mu(X - \tau)$ is the risk associated with the target premium $X - \tau$.

The next two theorems parallel Theorems 5.5 and 5.6, except that now targets appear explicitly. The following result shows that given risk measures and targets, we may construct a choice function in $\mathcal{R}_{iqv}$. The proof of the following theorem is based on Proposition A.2 in the Appendix. In the following theorem and throughout this section, we let $\{\tau(k)\}_{k \in \mathbb{R}} \subset \mathbb{R}^n$ denote a family of targets indexed by $k \in \mathbb{R}$. In this sense we can view the target mapping $\tau(k)$ as a function of $k$, i.e. $\tau : \mathbb{R} \to \mathbb{R}^n$.

**Theorem 5.9.** Suppose we are given a collection $\{\mu_k\}_{k \in \mathbb{R}}$ of convex risk measures with closed acceptance sets and targets $\{\tau(k)\}_{k \in \mathbb{R}} \subset \mathbb{R}^n$. We define $\tilde{\mu}_k(X) \triangleq \mu_k(X - \tau(k))$ and suppose:

(i) all $\{\tilde{\mu}_k\}_{k \in \mathbb{R}}$ have closed acceptance sets;

(ii) for any $X \in \mathcal{L}$, $\tilde{\mu}_k(X)$ as a function of $k$ is non-decreasing and left continuous;

(iii) for any $X \in \mathcal{L}$, there is $k \in \mathbb{R}$ such that $\tilde{\mu}_k(X) \leq 0$.

Then

$$\psi(X) \triangleq \sup \{k \in \mathbb{R} : \mu_k(X - \tau(k)) \leq 0\}, \forall X \in \mathcal{L},$$

(32)

is upper semi-continuous, monotonic, and quasi-concave.

The next result is the reverse implication of Theorem 5.9, it shows how to construct risk measures and targets so that we may recover any choice function $\rho \in \mathcal{R}_{iqv}$ in the form of representation (32). Now, we allow the targets to be any measurable selection from the level sets of $\rho$. In the following, let $bd(\cdot)$ denote the boundary of a set in $\mathcal{L}$ with respect to the supremum norm topology.

**Theorem 5.10.** Let $\rho \in \mathcal{R}_{iqv}$ and choose

$$\tau(k) \in bd \{X \in \mathcal{L} : \rho(X) \geq k\}, \forall k \in \mathbb{R}.$$

Let

$$\mu_k(X) \triangleq \inf \{\alpha \in \mathbb{R} : \rho(X + \alpha \tau(k)) \geq k\} - 1, \forall X \in \mathcal{L}.$$

Then the following assertions hold.

(i) $\{\mu_k\}_{k \in \mathbb{R}}$ are monotonic, translation invariant (along $\tau(k)$), normalized, convex, and have closed acceptance sets.

(ii) $\rho(X) = \sup \{k \in \mathbb{R} : \mu_k(X - \tau(k)) \leq 0\}$ for all $X \in \mathcal{L}$.

**Proof.** (i) The desired properties again follow from Proposition A.1 with some minor variations.

Translation invariance: For any $c \in \mathbb{R}$, we have

$$\mu_k(X + c \tau(k)) = \inf \{\alpha \in \mathbb{R} : \rho(X + c \tau(k) + \alpha \tau(k)) \geq k\} - 1 = \inf \{\alpha - c : \rho(X + \alpha \tau(k)) \geq k\} - 1 = \inf \{\alpha \in \mathbb{R} : \rho(X + \alpha \tau(k)) \geq k\} - 1 - c = \mu_k(X) - c.$$
Normalization: Normalization follows immediately by definition since:

\[
\mu_k(0) = \inf \{ \alpha \in \mathbb{R} : \rho(\alpha \tau(k)) \geq k \} - 1 = 0,
\]

since \(\inf \{ \alpha \in \mathbb{R} : \rho(\alpha \tau(k)) \geq k \} = 1\) by choice of \(\tau(k)\) in \(\{ d \in \mathbb{R}^n : \rho(d) = k \}\).

(ii) We first verify that

\[
\{ X \in \mathcal{L} : \rho(X) \geq k \} = \{ X \in \mathcal{L} : \mu_k(X - \tau(k)) \leq 0 \}, \quad \forall k \in \mathbb{R}. \tag{33}
\]

If \(\rho(X) \geq k\) then

\[
\mu_k(X - \tau(k)) = \inf \{ \alpha \in \mathbb{R} : \rho(X + \alpha \tau(k)) \geq k \} + 1 - 1 \leq 0,
\]

where the equality follows by translation invariance of \(\mu_k\) along \(\tau(k)\), and the inequality follows since \(\inf \{ \alpha \in \mathbb{R} : \rho(X + \alpha \tau(k)) \geq k \} \leq 0\) when \(\rho(X) \geq k\). Conversely, suppose \(\mu_k(X - \tau(k)) \leq 0\), then we must have \(\inf \{ \alpha \in \mathbb{R} : \rho(X + \alpha \tau(k)) \geq k \} \leq 0\) which implies \(\rho(X) \geq k\). Now we show that the desired conclusion follows from equivalence [33] by Proposition [A.3] in the Appendix.

We now develop some concrete examples of our main development in this section. As our first example, we extend some univariate expected utility choice functions to the multi-attribute setting.

Example 5.11. We first take utility functions \(u_i : \mathbb{R} \to \mathbb{R}\) which are continuous, monotonically nondecreasing, concave, and satisfy \(u_i(0) = 0\) for \(i = 1, \ldots, n\). For weights \(\kappa_1, \ldots, \kappa_n \geq 0\), we may then consider the (additive expected utility) choice function \(\rho(X) = \sum_{i=1}^{n} \kappa_i \mathbb{E}[u_i(X_i)]\) on \(\mathcal{L}\) where the expectation \(\mathbb{E}[\cdot]\) is taken with respect to some fixed distribution on \(\Omega\). We may then leverage [12, Example 1] for each component \(\mathbb{E}[u_i(X_i)]\) for \(i = 1, \ldots, n\).

As a generalization, we now take a multi-attribute utility function \(u : \mathbb{R}^n \to \mathbb{R}\) that is continuous, monotonically nondecreasing, concave, and satisfies \(u(0) = 0\). The resulting choice function is the subjective expected utility \(\rho(X) = \mathbb{E}[u(X)]\), which gives either way of constructing corresponding risk functions. We have

\[
\tau(k) = \inf \{ a \in \mathbb{R} : u(ad) \geq k \},
\]

and

\[
\mu_k(X) = \inf \{ a \in \mathbb{R} : \mathbb{E}[u(X + ad)] \geq k \} - \tau(k) = - (\sup \{ a \in \mathbb{R} : \mathbb{E}[u(X - ad)] \geq k \} + \tau(k)).
\]

We may apply this same reasoning to some classical risk measures as follows.

Example 5.12. The Optimized Certainty Equivalent (OCE) (which includes conditional value-at-risk as a special case) is defined in [9] as follows

\[
S_u(X) \triangleq \sup_{\eta \in \mathbb{R}} \{ \eta + \mathbb{E}[u(X - \eta)] \},
\]

where \(u\) is any continuous, monotonically nondecreasing, and concave utility function that satisfies \(u(0) = 0\). OCE has the following interpretation: suppose a decision maker faces a future uncertain income of \(X\) dollars, and can consume part of \(X\) at present. If he chooses to consume \(\eta\) dollars, then the resulting present value of \(X\) is \(\eta + \mathbb{E}[u(X - \eta)]\). Thus, the sure (present) value of \(X\) is \(S_u(X)\).

To extend OCE to multi-attribute case, we now take a multi-attribute utility function \(u : \mathbb{R}^n \to \mathbb{R}\) satisfying the same conditions as for the univariate case. We may then define the targets

\[
\tau(k) = \inf \left\{ \alpha \in \mathbb{R} : \sup_{\eta \in \mathbb{R}} \{ \eta + u(\alpha d - \eta) \} \geq k \right\},
\]

and risk measures

\[
\mu_k(X) = \inf \left\{ \alpha \in \mathbb{R} : \sup_{\eta \in \mathbb{R}} \{ \eta + \mathbb{E}[u(X + \alpha d - \eta)] \} \geq k \right\} - \tau(k)
\]

to use in our representation.
Some risk measures, such as ratio type risk measures, naturally lead to a representation of the form (27).

**Example 5.13.** A robust variant of reward-risk ratio measures is developed in [42]. The classical reward-risk ratio measure (on $\mathcal{L}$ for $n = 1$) which penalizes downward variations is

$$\rho(X) = \frac{\mathbb{E}|X - Y|}{\mathbb{E}[|Y - X|_+]}$$

where $Y$ is a benchmark return. The epigraphical formulation of this risk measure is

$$\rho(X) = \sup_{\tau \in \mathbb{R}} \{\tau : \mathbb{E}[X - Y - \tau (Y - X)_+] \geq 0\},$$

which is quite similar to our representation (27) with $\mu_\tau(X) = \mathbb{E}[X - Y - \tau (Y - X)_+]$. In [12], the authors focus on distributional ambiguity, while we can adapt their technique to deal with multi-attribute prospects on $\mathcal{L}$ for $n \geq 2$. In particular, for any scalarization function $\varphi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ (such as $\varphi(w, X) = \sum_{i=1}^n w_i X_i$) we may consider the choice function

$$\rho(X) = \inf_{w \in \mathcal{W}} \mathbb{E}[\varphi(w, X) - \varphi(w, Y)]$$

where $\mathcal{W} \subset \mathbb{R}^n$ is a set of weights. Then, using a similar analysis as in [12] Proposition 2.1, we can obtain

$$\rho(X) = \sup_{\tau \in \mathbb{R}} \tau$$

s.t. $\inf_{w \in \mathcal{W}} \mathbb{E}[\varphi(w, X) - \varphi(w, Y) - \tau (\varphi(w, Y) - \varphi(w, X))_+] \geq 0$.

In this case, the family of risk measures in representation (27) is

$$\mu_\tau(X) = \inf_{w \in \mathcal{W}} \mathbb{E}[\varphi(w, X) - \varphi(w, Y) - \tau (\varphi(w, Y) - \varphi(w, X))_+]$$

which includes the inner minimization over weights.

In [13,12], the authors use a specific construction of the targets. In our case, we require a “distinguished” strictly positive (component-wise) direction $d \in \mathbb{R}^n$ along which we may increase or decrease prospects in $\mathcal{L}$. No such direction was needed because this work lies in the univariate setting. This distinguished direction leads to a notion of translation invariance for multivariate risk measures. We use the notation $\{\nu(k)\}_{k \in \mathbb{R}} \subset \mathbb{R}$ to emphasize that we are only referring to aspiration levels along this distinguished direction $d$.

**Theorem 5.14.** Let $d \in \mathbb{R}^n$ with $d > 0$ (component-wise). Given $\rho \in \mathcal{R}_{iqv}$, define

$$\nu(k) \triangleq \inf \{\alpha \in \mathbb{R} : \rho(\alpha d) \geq k\},$$

and $\mu_k : \mathcal{L} \to \mathbb{R}$ via

$$\mu_k(X) \triangleq \inf \{\alpha \in \mathbb{R} : \rho(X + \alpha d) \geq k\} - \nu(k), \forall X \in \mathcal{L}.$$

Then we have:

(i) $\{\nu(k)\}_{k \in \mathbb{R}}$ are monotonic, translation invariant (along $d$), normalized, convex, and have closed acceptance sets.

(ii) $\rho(X) = \sup \{k \in \mathbb{R} : \mu_k(X - \nu(k) d) \leq 0\}$ for all $X \in \mathcal{L}$.

**Proof.** Part (i). All of the desired properties of $\{\mu_k\}_{k \in \mathbb{R}}$ follow from Proposition A.1 except for translation invariance, which is new. For any $c \in \mathbb{R}$, we have

$$\mu_k(X + cd) = \inf \{\alpha \in \mathbb{R} : \rho(X + cd + \alpha d) \leq k\} - \nu(k) = \inf \{\alpha - c : \rho(X + \alpha d) \leq k\} - \nu(k) = \inf \{\alpha \in \mathbb{R} : \rho(X + \alpha d) \leq k\} - c - \nu(k) = \mu_k(X) - c.$$
Part (ii). Now we verify that \( \{ X \in \mathcal{L} : \rho(X) \geq k \} \) \( \leftrightarrow \) \( \{ X \in \mathcal{L} : \mu_k(X - v(k)d) \leq 0 \} \). If \( \rho(X) \geq k \) then
\[
\mu_k(X - v(k)d) = \mu_k(X) + v(k)
= \inf \{ \alpha \in \mathbb{R} : \rho(X + \alpha d) \geq k \} - v(k) + v(k)
\leq 0,
\]
since \( \inf \{ \alpha \in \mathbb{R} : \rho(X + \alpha d) \geq k \} \leq 0 \) when \( \rho(X) \geq k \). Conversely, suppose \( \mu_k(X - v(k)d) \leq 0 \), then we must have \( \inf \{ \alpha \in \mathbb{R} : \rho(X + \alpha d) \geq k \} \leq 0 \) which implies \( \rho(X) \geq k \). \( \square \)

### 6 Optimization of quasi-concave choice functions

With the preceding discussions on the support function representation of the robust choice function in Section 4 and the level set representation in Section 5, we are now ready to develop a numerical procedure for the maximin robust preference optimization Problem (3).

\[
\max_{z \in \mathcal{Z}} \psi(G(z); \mathcal{S}, Y) = \max \inf_{z \in \mathcal{Z}, \rho \in \mathcal{S}} \{ \rho(G(z)) - \rho(Y) \}. \tag{34}
\]

Observe that if \( G \) is a linear function of \( z \) in every scenario, then we will be able to solve (3) by solving a mixed-integer linear programming problem as we outlined in Section 4. Here we concentrate on the case where \( G \) is convex in \( z \) component-wise almost surely. It is easy to verify that \( \psi(G(z); \mathcal{S}, Y) \) is then quasi-concave in \( z \). We will combine the support function approach and level set representation approach and put them in the framework of the so-called level function method \([68]\). We start with a formal definition of level functions of a quasi-concave function.

**Definition 6.1 (Level function).** Let \( f : \mathbb{R}^n \rightarrow \mathbb{R} \) be a continuous real-valued quasi-concave function. A function \( \sigma : \mathcal{L} \rightarrow \mathbb{R} \) is called a level function of \( \rho \) at \( x \) if it satisfies:

(i) \( \sigma(x) = 0 \);
(ii) \( \sigma \) is a continuous concave function;
(iii) \( T_f(f(x)) \subset T_x(0) \), where \( T_f(\alpha) = \{ x \in \mathbb{R}^d : f(x) > \alpha \} \) denotes the strict upper level set of \( f \) at level \( f(x) \).

The basic idea of the level function method can be described as follows. At starting point \( x_0 \), we calculate a level function \( \sigma_0 \) at the point. This can be achieved by calculating an upper sub-gradient \( g_0 \in \partial^+ f(x_0) \) and define \( \sigma_0(x) \triangleq \langle g_0, x - x_0 \rangle \). We can then minimize \( \sigma_0(x) \) over the feasible set of the associated maximization problem (which of course must be compact and convex) and denote the minimizer as the next iterate \( x_1 \). Next, we calculate a level function of \( f \) at \( x_1 \), minimize \( \max \{ \sigma_0(x), \sigma_1(x) \} \) and use the minimizer as the next iterate. In this process, we will have to evaluate \( f \) at each iterate in order to calculate a sub-gradient of \( f \) and construct a level function there.

To see how this process can be applied to solve (34), we need to discuss how to construct a level function at each iterate. Let \( (a_i, b_i, c_i) \) denote an optimal solution of Problem (8) - (14) at \( z_i \). Then the function \( \max \{ \langle a_i, G(z) \rangle + b_i, c_i \} \) majorizes \( \psi(G(z); \mathcal{S}, Y) \) and they coincide at \( z = z_i \), that is, \( c_i = \psi(G(z_i); \mathcal{S}, Y) \). Let \( \sigma_{z_i}(z) \triangleq \langle a_i, G(z) - G(z_i) \rangle \), then \( \sigma_{z_i}(z_i) = 0 \) and it is a convex function. Moreover, since
\[
\max \{ \langle a_i, G(z) \rangle + b_i, c_i \} \geq \max \{ \langle a_i, G(z_i) \rangle + b_i, c_i \} = \psi(G(z_i); \mathcal{S}, Y)
\]
for all \( z \in \mathcal{Z} \), we have that
\[
\sigma_{z_i}(z) \triangleq \langle a_i, G(z) - G(z_i) \rangle \geq 0, \forall z \in T_{\psi(G(\cdot); \mathcal{S}, Y)}(c_i).
\]

This computation shows that \( \sigma_{z_i} \) is a level function of \( \psi(G(\cdot); \mathcal{S}, Y) \) at \( z_i \). Note that the way level functions are used here is closely linked to the level set representation that we discuss in Section 5.

We are now ready to present an algorithm for solving (34) based on the projected level function method \([68]\) Algorithm 3.3.

In Algorithm 1, when \( \lambda = 1 \), \( Q_i \) is the set of minimizers of \( \sigma_i \) over \( \mathcal{Z} \). Generally, we assume that the constant \( \lambda \) belongs to \( (0, 1) \). The worst-case choice function at point \( z_i \in \mathcal{Z} \) i.e. the minimizer in \( \psi(G(z_i); \mathcal{S}, Y) \), is modeled given by the solution of Problem (8) - (14) at \( z_i \).
These four attributes also appear in the case study in [34] and give a comprehensive measure of the health function $B$ and attributes for measuring the effect of a terrorist attack:

We remark on the probability distribution over these scenarios later in this section. There are $n$ terrorist attacks:

We begin by introducing the problem setup and notations. We have $m = 10$ cities among which we need to allocate the budget. There are three possible underlying loss scenarios corresponding to different levels of terrorist attacks:

$\{\text{reduced loss, standard loss, increased loss}\}$.

We remark on the probability distribution over these scenarios later in this section. There are $n = 4$ attributes for measuring the effect of a terrorist attack:

- **property loss** measures the impact on economic structures and individual property;
- **fatalities** measures the loss of human lives;
- **air departures** measures the number of outgoing flights from the city’s major airport;
- **bridge traffic** measures vehicle movement on the city’s major bridges.

These four attributes also appear in the case study in [34] and give a comprehensive measure of the health of a city.

We denote the budget allocated to city $j$ corresponding to attribute $i$ by $z_{ij}$, and allocations $z = (z_{ij})$ must satisfy $z \in Z \subseteq \mathbb{R}^{(n \times m)}$, where $Z$ represents the set of feasible resource allocations:

$$Z \triangleq \left\{ z \in \mathbb{R}^{(n \times m)} : \sum_{i=1}^{n} \sum_{j=1}^{m} z_{ij} \leq B \right\},$$

and $B$ is the overall budget. Based on the budget allocation models in [47, 43], we may use an exponential function $g : \mathbb{R} \rightarrow \mathbb{R}$ to measure the effectiveness of investment. Let $v_{ij}$ denote the target for city $j$ and

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**Algorithm 1** Projected Level Function Method

**Step 1:** Select a starting point $z_0 \in Z$, set $i = 0$; specify tolerance level $\epsilon$;

**Step 2:** Calculate a level function $\sigma_i(z)$ of $\psi(G(\cdot); S, Y)$, and set $\sigma_i(z) := \inf \{ \sigma_{i-1}(z), \sigma_{z}(z) \}$, where $\sigma_{-1}(z) := \infty$. Let

$$z_i = \arg \max_{z \in Z} \{ \psi(G(z_j); S, Y) : j = 1, ..., i \},$$

and $z_{i+1} \in \Pi_{Qi}[z_i]$, where $Q_i := \{ z \in Z : \sigma_i(z) \geq \lambda \Delta(i) \}$, $\Delta(i) = \max_{z \in Z} \sigma_i(z)$, and $\Pi_{Qi}[\cdot]$ denotes the projection operator onto a set $Q$.

**Step 3:** If $\Delta(i) \leq \epsilon$, stop; otherwise, set $i := i + 1$, go to Step 2.

Based on [68, Proposition 3.1], if Algorithm 1 terminates at iteration $i$, then the global maximum of $\psi((G(\cdot); S, Y)$ has been obtained. Let $\{z_i\}_{i \geq 0}$ be the sequence generated by Algorithm 1. By [68, Proposition 3.4], if $\lim_{i \rightarrow \infty} \Delta(i) = 0$, then there exists a subsequence of $\{z_i\}_{i \geq 0}$ converging to a global maximizer of $\psi((G(\cdot); S, Y)$ over $Z$. Further, we have the convergence rate of Algorithm 1 presented next.

**Theorem 6.2.** [68, Theorem 3.3] Let $\{z_i\}_{i \geq 0}$ be a sequence generated by Algorithm 1. Assume that the sequence of level functions $\{\sigma_i(z)\}_{i \geq 0}$ is uniformly Lipschitz on $Z$ with constant $K$. Then for any $\epsilon > 0$, $\Delta(i) \leq \epsilon$ for $i > K^2D^2\epsilon^{-2}\lambda^{-2}(1 - \lambda^2)^{-1}$ where $D$ is the diameter of convex set $Z$.

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7 Application to homeland security

In [50], the authors highlight the need for multiple criteria when investing limited resources. In this section, we apply our methods to a real world multi-attribute budget allocation problem in homeland security where we are investing resources to protect against terrorist threats. This application is based on the case study originally investigated in [34]. Specifically, we want to allocate resources among ten major cities in the United States under the Urban Areas Security Initiative (UASI) of the Department of Homeland Security (DHS). DHS is the principle decision maker in this application.

7.1 Problem description

We begin by introducing the problem setup and notations. We have $m = 10$ cities among which we need to allocate the budget. There are three possible underlying loss scenarios corresponding to different levels of terrorist attacks:

$\{\text{reduced loss, standard loss, increased loss}\}$.

We remark on the probability distribution over these scenarios later in this section. There are $n = 4$ attributes for measuring the effect of a terrorist attack:

- **property loss** measures the impact on economic structures and individual property;
- **fatalities** measures the loss of human lives;
- **air departures** measures the number of outgoing flights from the city’s major airport;
- **bridge traffic** measures vehicle movement on the city’s major bridges.

These four attributes also appear in the case study in [34] and give a comprehensive measure of the health of a city.

We denote the budget allocated to city $j$ corresponding to attribute $i$ by $z_{ij}$, and allocations $z = (z_{ij})$ must satisfy $z \in Z \subseteq \mathbb{R}^{(n \times m)}$, where $Z$ represents the set of feasible resource allocations:

$$Z \triangleq \left\{ z \in \mathbb{R}^{(n \times m)} : \sum_{i=1}^{n} \sum_{j=1}^{m} z_{ij} \leq B \right\},$$

and $B$ is the overall budget. Based on the budget allocation models in [47, 43], we may use an exponential function $g : \mathbb{R} \rightarrow \mathbb{R}$ to measure the effectiveness of investment. Let $v_{ij}$ denote the target for city $j$ and
attribute \(i\) (for instance, this target can be chosen to be the possible maximum value of loss for this attribute - indeed, we take the target to be this maximum value in our upcoming experiments). Let \(\delta \in (0, 1]\) denote the effectiveness ratio of investment to obtain the function \(g_{ij}\) for each \(z_{ij}\) defined as

\[
g_{ij}(z_{ij}) \triangleq \nu_{ij}(1 - \exp(-\delta z_{ij})) ,
\]

(36)

for all attributes \(i\) and cities \(j\).

The random loss at city \(j\) with respect to attribute \(i\) is denoted by \(C_{ij}\). We have an \((n \times m)\) random loss matrix \(C\) that captures the loss for all attributes in all cities. Given a particular budget allocation \(z \in \mathbb{Z}\), the shortfall with respect to attribute \(i\) is measured by the function

\[
C_i(z) \triangleq \sum_{j=1}^{m} (C_{ij} - g(z_{ij}))_+ ,
\]

where \((\cdot)_+ \triangleq \max\{\cdot, 0\}\). The quantity \(C_i(z)\) can be viewed as the sum of the shortfall in attribute \(i\) over all cities. If \(g(z_{ij}) \geq C_{ij}\), then the allocated resources exceed each cities’ need and there is no shortfall. We combine the shortfall for each attribute into the vector-valued mapping

\[
C(z) = (C_1(z), \ldots, C_n(z)) .
\]

The negative value of the shortfall, i.e. \(-C(z)\) can be viewed as the “reward”. It is clear that \(-C(z)\) is component-wise concave in \(z\) (since each component \(-C_i(z)\) is a concave function of \(z\)).

We consider two models in this case study. The first is risk-neutral and minimizes expected shortfall:

\[
\min_{z \in \mathbb{Z}} E \left[ \sum_{i=1}^{n} \sum_{j=1}^{m} (C_{ij} - g(z_{ij}))_+ \right] .
\]

(37)

The second is our robust preference model with the special ambiguity set \(\mathcal{S}\):

\[
\max \inf_{z \in \mathbb{Z}} \rho(-C(z)) .
\]

(38)

Problem (38) is especially relevant at the governmental level where the preferences of many different stakeholders must be accommodated. In [1], the stakeholders are noted to include: federal agencies; department and component officials; state, local, and tribal governments; the private sector; academics; and policy experts.

7.2 Data

We now describe the data set used in our experiments. First we describe the data for our four attributes: (i) property loss; (ii) fatalities; (iii) air departures; and (iv) bridge traffic. Table 1 represents the data obtained from [67] and [10] (these data were previously applied in [34]). The ten cities in Table 1 received 40% of the total UASI budget in 2004 and 60% of the total UASI budget in 2009. The number of air departures in these ten urban areas accounts for roughly one third of the total air departures in the United States (see [10]). In Table 1, the data on property loss and fatalities are recorded as random losses in millions of dollars. The data on air departures and bridge traffic are recorded as daily averages.

We follow [34] and assume that both daily air departures and daily bridge traffic have a log-uniform distribution (the log-uniform distribution is also used in [67]). We let the random variable \(U\) have distribution:

\[
P(U = -1) = P(U = 0) = P(U = 1) = 1/3 .
\]

For attributes \(i = 3, 4\) (air departures and bridge traffic), we set \(T_{ij}\) to be the averages from columns 3 and 4 of Table 1. For a constant \(\gamma > 1\) (which controls volatility), we set

\[
T_{ij} = \left( \frac{2\gamma T_{ij} \ln \gamma}{(\gamma^2 - 1)} \right)^\gamma U
\]
| Urban area   | Property losses ($ million) | Fatalities ($ million) | Average daily air departures | Average daily bridge traffic |
|-------------|-----------------------------|------------------------|-----------------------------|----------------------------|
|             | Standard | Reduced | Increased | Standard | Reduced | Increased | Standard | Reduced | Increased | Standard | Reduced | Increased |
| New York    | 413      | 265     | 550       | 304      | 221     | 401       | 23599    | 596400  |
| Chicago     | 115      | 77      | 150       | 54       | 38      | 73        | 39949    | 318800  |
| San Francisco | 57     | 38      | 81        | 24       | 16      | 36        | 19142    | 277700  |
| Washington  | 36       | 21      | 59        | 29       | 16      | 48        | 17253    | 254975  |
| Los Angeles | 34       | 16      | 58        | 17       | 7       | 31        | 28816    | 336000  |
| Philadelphia| 21       | 8       | 28        | 9        | 5       | 13        | 13640    | 192204  |
| Boston      | 18       | 8.3     | 26        | 12       | 8       | 17        | 11625    | 669000  |
| Houston     | 11       | 6.7     | 15        | 9        | 6       | 12        | 20979    | 308600  |
| Newark      | 7.3      | 0.8     | 12        | 4        | 0.1     | 9         | 12827    | 518100  |
| Seattle     | 6.7      | 4       | 10        | 4        | 3       | 6         | 13578    | 212000  |

Table 1: Terrorism losses, average daily air departures, and average daily bridge traffic

| Urban area   | DAD ($ million) | DBT ($ million) |
|-------------|-----------------|-----------------|
|             | Standard | Reduced | Increased | Standard | Reduced | Increased | Standard | Reduced | Increased |
| New York    | 10.71    | 11.78   | 12.96     | 162.41   | 178.65  | 196.51    |
| Chicago     | 18.13    | 19.94   | 21.94     | 86.81    | 95.50   | 105.04    |
| San Francisco | 8.69  | 9.56    | 10.51     | 75.62    | 83.18   | 91.50     |
| Washington  | 7.83     | 8.61    | 9.47      | 69.43    | 76.38   | 84.01     |
| Los Angeles | 13.08    | 14.39   | 15.82     | 91.50    | 100.65  | 110.71    |
| Philadelphia| 6.19     | 6.81    | 7.49      | 52.34    | 57.57   | 63.3314   |
| Boston      | 5.28     | 5.80    | 6.38      | 182.18   | 200.40  | 220.44    |
| Houston     | 9.52     | 10.47   | 11.52     | 84.04    | 92.44   | 101.68    |
| Newark      | 5.82     | 6.40    | 7.04      | 141.09   | 155.19  | 170.71    |
| Seattle     | 6.16     | 6.78    | 7.46      | 57.37    | 63.50   | 69.85     |

Table 2: Air departures and bridge traffic
Table 3: Targets

| Urban area     | Targets ($ million) |
|----------------|---------------------|
| New York       | 500                 |
| Chicago        | 450                 |
| San Francisco  | 400                 |
| Washington     | 350                 |
| Los Angeles    | 300                 |
| Philadelphia   | 250                 |
| Boston         | 200                 |
| Houston        | 150                 |
| Newark         | 100                 |
| Seattle        | 50                  |

to be the random number of daily incidents in each city \( j \) for \( i = 3, 4 \). Finally, we let \( C_{ij} = c_i T_{ij} \) denote the random costs corresponding to attributes \( i = 3, 4 \), where \( c_i \) is the economic loss per incident of attribute \( i \).

In our experiments, we set \( \gamma = 1.1 \), \( c_3 = $500 \), and \( c_4 = $300 \). Table 2 shows the random losses for daily air departures (DAD) and daily bridge traffic (DBT) under each scenario using this construction.

The data in Tables 1 and 2 are used to form the random loss matrix \( C \).

7.3 The Experiments

In this section, we describe the details of our experiments and present the results. We start by describing the preference elicitation procedure, where we simulate the preferences of DHS by using the multivariate certainty equivalent. First, we take a continuous, monotonically increasing, and concave univariate utility function \( u : \mathbb{R} \to \mathbb{R} \) with \( u(0) = 0 \). Then, we choose weights \( w \in \mathbb{R}^n \) with \( w \geq 0 \) and \( \|w\|_2 = 1 \) to define the multivariate certainty equivalent

\[
\rho(X) = u^{-1}\left\{E\left[u\left(\langle w, -X \rangle\right)\right]\right\}, \quad X \in \mathcal{L},
\]

where \( u^{-1} \) is the inverse of \( u \). Here, we use the exponential utility function \( u(y) = -\exp(-\kappa y) \) with \( \kappa = 0.05 \).

The data set used to elicit preferences consists of many pairs of prospects. DHS is asked to choose the preferred prospect from each pair of prospects (the preferred prospect is denoted \( W_i \) and the other is denoted \( Y_i \)), and the choice is made using the choice function (39). We refer to this data set as the “elicited comparison data set”.

Remark 7.1. We emphasize that we select the choice function (39) and the value of \( \kappa \) artificially for our experiments. It is used to elicit consistent preferences on the elicited comparison data set. Our robust preference model (38) and our algorithm do not have knowledge of this choice function.

In the experiments, we set the overall budget to be \( B = $400 \) million and the effectiveness ratio of investment to be \( \delta = 0.05 \) (these are the same settings as in [47]). For computation tractability, we use a piecewise linear function to approximate the original effectiveness function (36). This piecewise linear function is constructed from a set of segments with ten breakpoints \( \{5, 10, 20, 30, 40, 50, 75, 100, 150, 200\} \).

We assume that the budget allocated to each city \( j \) corresponding to attribute \( i \) cannot be less than $1 million, i.e. \( z_{ij} \geq 1 \) for all \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \). In Problem (15) - (25), we take the modulus of Lipschitz continuity to be \( L = 1/12 \) (this choice is without loss of generality since the robust choice function \( \psi(\cdot; \mathcal{S}, Y) \) will preserve the same preferences even if we change the modulus of continuity of the choice functions in \( \mathcal{S} \)). For implementation of Algorithm 1, we set \( \lambda = 0.8 \) and \( \epsilon = 0.0001 \). It should be noted that, when solving Problem (38), the value of the function \( \psi(-C(\cdot); \mathcal{S}, Y) \) is obtained by solving the MILP (15) - (25). The optimal solution from Problem (35) in Algorithm 1 is obtained by solving a sequence of MILPs. Finally, we set the DHS targets \( v_{ij} \) for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \) as shown in Table 3. These targets vary from city to city since they are related to the economic condition of the corresponding city.
| Urban area | Property losses ($ million) | Fatalities ($ million) | Air departures ($ million) | Average daily bridge traffic ($ million) |
|------------|-----------------------------|------------------------|-----------------------------|-----------------------------------------|
| New York   | 62.472                      | 32.253                 | 1.000                       | 9.985                                   |
| Chicago    | 8.017                       | 3.485                  | 1.000                       | 5.314                                   |
| San Francisco | 4.520                       | 1.631                  | 1.000                       | 5.194                                   |
| Washington | 3.648                       | 2.841                  | 1.000                       | 5.484                                   |
| Los Angeles | 4.284                       | 1.973                  | 1.000                       | 9.194                                   |
| Philadelphia | 2.196                       | 1.000                  | 1.000                       | 5.825                                   |
| Boston     | 2.658                       | 1.502                  | 1.000                       | 59.961                                  |
| Houston    | 1.888                       | 1.374                  | 1.000                       | 22.488                                  |
| Newark     | 2.401                       | 1.631                  | 1.000                       | 59.961                                  |
| Seattle    | 4.456                       | 2.401                  | 1.000                       | 59.961                                  |

Table 4: Optimal UASI budget allocation: risk-neutral

| Urban area | Property losses ($ million) | Fatalities ($ million) | Air departures ($ million) | Average daily bridge traffic ($ million) |
|------------|-----------------------------|------------------------|-----------------------------|-----------------------------------------|
| New York   | 78.829                      | 32.253                 | 1.000                       | 9.985                                   |
| Chicago    | 8.017                       | 3.485                  | 1.000                       | 5.314                                   |
| San Francisco | 4.520                       | 1.631                  | 1.000                       | 5.194                                   |
| Washington | 3.648                       | 2.841                  | 1.000                       | 5.484                                   |
| Los Angeles | 4.284                       | 1.973                  | 1.000                       | 9.194                                   |
| Philadelphia | 2.196                       | 1.000                  | 1.000                       | 5.825                                   |
| Boston     | 2.658                       | 1.502                  | 1.000                       | 59.961                                  |
| Houston    | 1.888                       | 1.374                  | 1.000                       | 22.488                                  |
| Newark     | 2.401                       | 1.631                  | 1.000                       | 59.961                                  |
| Seattle    | 4.456                       | 2.401                  | 1.000                       | 59.961                                  |

Table 5: Optimal UASI budget allocation: robust choice

7.3.1 Experiment I: Risk-neutral vs. Robust preference

In our first experiment, we compare the solutions of Problems (37) and (38). The elicited comparison data set used for this experiment is presented in Appendix B, it contains five pairs of prospects in total. We define these prospects over three scenarios where each scenario has an equal one-third probability of being realized. For each prospect, we generate the random loss level from the uniform distribution on $[-1000, 0]$. In this experiment, Algorithm 1 terminates in iteration $i = 12$ when it finds the optimal solution of Problem (38). When solving Problems (37) and (38), we obtain the optimal UASI budget allocations as shown in following tables. Table 4 gives the optimal risk-neutral UASI budget allocation and Table 5 gives the optimal UASI budget allocation for our robust choice model.

We test the performance of these two budget allocation plans in simulation with 1000 i.i.d samples. Figure 1 shows the histogram of costs after 1000 simulations for both the risk-neutral and robust choice budgets. From Figure 1, we observe that the loss distribution of the optimal budget allocation from Problem (38) generally has a slightly higher mean but lower variance compared to the optimal budget allocation from Problem (37). To make the comparison more apparent, we fit a normal distribution to the simulation output. Figure 2 shows that the random loss distribution of the optimal budget allocation from our robust preference model second-order stochastically dominates the random loss distribution of the optimal budget allocation of the risk-neutral model. Let $F_R$ and $F_N$ denote the cumulative distribution function of random loss distribution of the optimal budges from the robust preference and risk-neutral models, respectively. We can explicitly verify that second-order stochastic dominance is satisfied by verifying that \[
\int_{-\infty}^{x} |F_R(t) - F_N(t)| dt \geq 0.0288
\]
for all real numbers $x$.

We also estimate the mean and variance of the loss distribution, as well as compute its 5% and 10%-level CVaR, under these two optimal budget allocation plans. The results in Table 6 show explicitly that the optimal budget allocation from Problem (38) has a slightly higher mean but that it also has a lower
| Budget allocation   | Mean ($ million) | Variance | 10%-CVaR ($ million) | 5%-CVaR ($ million) |
|--------------------|------------------|----------|----------------------|---------------------|
| Risk-neutral       | 112.088          | 1404.362 | 177.273              | 183.806             |
| Robust preference  | 116.213          | 1083.492 | 174.264              | 180.884             |

Table 6: Optimal UASI budget allocation: robust choice

| Effectiveness ratio $\delta$ | 0.01 | 0.02 | 0.04 | 0.05 |
|------------------------------|------|------|------|------|
| Expected loss ($\$ million$) | 1034.162 | 483.320 | 145.784 | 111.525 |

Table 7: Sensitivity analysis $\delta$

variance, 5%-CVaR, and 10%-CVaR compared to the optimal budget allocation from Problem (37). This table also reveals that the budget allocation from our robust choice model induces a lower probability of suffering extremely high losses. Thus, we claim that the optimal budget allocation from Problem (38) is more resilient against loss from terror attacks. These experimental results are in line with what we expect from the diversification favoring behavior of quasi-concave choice functions.

7.3.2 Experiment II: Sensitivity analysis

In our second experiment, we conduct sensitivity analysis on the effectiveness ratio $\delta$ in (36) in our robust choice model. Also, we study how sensitive the optimal budget allocation is to different elicited comparison data sets. In this experiment, we fix $\kappa = 0.05$.

Table 7 shows that as the effectiveness ratio $\delta$ increases, the optimal budget allocation becomes less sensitive, and the expected loss decreases. One possible explanation is that if $\delta$ is extremely low, putting all of the budget in a single city and on a single attribute cannot achieve an investment close to the corresponding economic target. However, when $\delta$ is sufficiently large, the investment triggered by the budget allocated in each city on each attribute is able to achieve its economic target. In this case, the optimal budget allocation will be insensitive to changes in $\delta$ and the expected loss becomes stable.

Table 8 shows how sensitive the optimal budget allocation is to different elicited comparison data sets in our robust choice model. In this experiment, we fix $\kappa = \delta = 0.05$. We run the experiments for twenty different elicited comparison data sets with the same number of pairs. The “variance of the optimal budget allocation” is computed by summing the variances of the optimal budget allocations in each city on each attribute among twenty groups. The “variance of expected loss” is the variance of the expected loss from the optimal budget allocation among twenty groups. As the size of elicited comparison data set increases, these two variance measures will increase first and then decrease. We hypothesize that when the number of pairs is small, the preference estimation error is high. When the size of the elicited comparison data set is large, the preference estimation error is reduced and differences in elicited comparison data sets will not affect the optimal budget allocation.

7.3.3 Experiment III: Out-of-sample performance

In our third experiment, we conduct out-of-sample tests on the effectiveness of our robust choice model (15)-(25), in terms of eliciting the true preferences of the decision maker, when the true choice function is (39). Two data sets are constructed for this experiment: one is the elicited comparison data set with 100 pairs of prospects, the other is a test data set with twenty pairs of prospects. Both data sets are generated following the same procedures as for Experiment I. In each run, we select the first $I$ pairs of prospects from

| Number of pairs | 1     | 3     | 5     | 10    | 25    | 50    |
|-----------------|-------|-------|-------|-------|-------|-------|
| Variance of optimal budget allocation | 2.45E-29 | 2.45E-29 | 3.50E-07 | 3.664429998 | 6.38E-06 | 4.87E-08 |
| Variance of expected loss | 8.50E-28 | 8.50E-28 | 3.11E-10 | 0.139151122 | 5.80E-09 | 2.23E-11 |

Table 8: Sensitivity analysis: elicited comparison data set
Figure 1: Simulation results
to form the elicited comparison data set, and then we solve Problem (15)-(25) to compare the robust preference of each pair of prospects in the test data set. We compare the robust preference with the true preference derived by the true choice function (39), and record the number of violated orders in the test data set. Table 9 shows that as $|I|$ increases, the number of violated orders in the test data set diminishes. This phenomenon confirms that, as $|I|$ increases, the model (15)-(25) with extra preference information provides a better estimate of the preferences induced by the true choice function (39).

### 7.3.4 Experiment IV: Runtime

In our fourth experiment, we study how the solution time of Problem (38) depends on the size of the elicited comparison data set. We consider preference elicitation with $I \in \{1, 2, 4, 6, 8, 10\}$ pairs of prospects. For each $I$, we generate the appropriate elicited comparison data set following the same procedure as in Experiment I, and then we solve the corresponding instance of Problem (38). The experiments were performed on a generic laptop with Intel Core i7 processor, 8GM RAM, on a 64-bit Windows 8 operating system via Matlab R2015a and CPLEX Studio 12.5. Table 10 and Figure 2 show that the runtime of Problem (38) grows gracefully as a function of the size of the elicited comparison data set.

### 8 Conclusion

This paper attempts to take on the simultaneous challenges of multi-attribute prospects and non-convexity in robust utility/risk choice models. It advances the existing research of preference robust optimization with
a more general mathematical model which complements the existing convex optimization models, tractable computational schemes, and structural insights for modern decision-making problems.

As our first main result, we show how to optimize our robust choice model using support functions. We give a tractable procedure based on solving a sequence of MILPs. As our second main result, we give a common representation for multi-attribute quasi-concave choice functions. This representation result shows that all multi-attribute quasi-concave choice functions can be expressed in terms of a family of convex risk functions and satiation levels (a.k.a. targets). This result is analogous to the main result in [12]. We develop a case study in homeland security which shows that our model is both: (i) highly expressive and can incorporate decision maker preference information; and (ii) provides solutions with more favorable risk properties compared to other models.

In future research, we will consider further computational issues of our model, in particular application to general probability spaces (where the sample space is not necessarily finite). In parallel, we will explore the connection between our present work and multi-attribute stochastic dominance models.

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A Basic results

The following proposition is used to show that several of the risk measures constructed in this paper are convex.

Proposition A.1. Let $\rho \in \mathcal{R}_{\text{inv}}$, and for $d \geq 0$, define $\mu_k : \mathcal{L} \to \mathbb{R}$ for all $k \in \mathbb{R}$ by

$$\mu_k(X) \triangleq \inf \{\alpha \in \mathbb{R} : \rho(X + \alpha d) \geq k\}, \forall X \in \mathcal{L}.$$ 

If $\rho(X + \alpha d)$ is increasing w.r.t. $\alpha$, then the following hold.

(i) For each $X \in \mathcal{L}$,

$$\mu_k(X) \leq 0 \iff \rho(X) \leq 0;$$

(ii) $\mu_k(\cdot)$ is monotone and convex over $\mathcal{L}$ and has closed acceptance sets;

(iii) $\mu_k(\cdot)$ is monotonically increasing in $k$.

Proof. Part (i). Let

$$\mathcal{F}(X, k) \triangleq \{\alpha \in \mathbb{R} : \rho(X + \alpha d) \geq k\}.$$ 

If $\rho(X) \leq k$, then $0 \in \mathcal{F}(X, k)$ and consequently $\mu_k(X) \leq 0$. Conversely, if $\mu_k(X) \leq 0$, then for any positive number $\epsilon$, the increasing property of $\rho(X + \alpha d)$ in $\alpha$ implies $0 + \epsilon \in \mathcal{F}(X, k)$, that is,

$$\rho(X + \epsilon d) \geq k.$$ 

Driving $\epsilon$ to 0, we deduce $\rho(X) \geq k$.

Part (ii). The closedness of the acceptance sets follows from Part (i) and closedness of the upper level set of $\rho(\cdot)$. So we are left to prove monotonicity and convexity.

For any $X, Y \in \mathcal{L}$ with $X \leq Y$, since $\rho(\cdot)$ is non-decreasing,

$$\rho(X + \alpha d) \leq \rho(Y + \alpha d), \forall \alpha \in \mathbb{R},$$

which implies $\mathcal{F}(X, k) \subset \mathcal{F}(Y, k)$ and subsequently $\mu_k(X) \geq \mu_k(Y)$. Moreover, for $\lambda \in [0, 1]$, and arbitrarily small positive number $\epsilon > 0$, it follows by the definition of $\mu_k$

$$\rho(X + (\mu_k(X) + \epsilon) d) \geq k \text{ and } \rho(Y + (\mu_k(Y) + \epsilon) d) \geq k.$$ 

Let $a_\lambda = \lambda \mu_k(X) + (1 - \lambda) \mu_k(Y)$. Then

$$\rho(\lambda X + (1 - \lambda) Y + (a_\lambda + \epsilon) d) = \rho(\lambda X + (\mu_k(X) + \epsilon) d) + (1 - \lambda) (Y + (\mu_k(Y) + \epsilon) d) \geq \min \{\rho(X + (\mu_k(X) + \epsilon) d), \rho(Y + (\mu_k(Y) + \epsilon) d)\},$$

$$\geq k,$$
where the first inequality follows from quasi-concavity of \( \rho \in \mathcal{R}_{eqv} \), and the second inequality follows from the definition of \( \mu_k (X) \) and \( \mu_k (Y) \). The above inequality then gives rise to

\[
\mu_k (\lambda X + (1-\lambda) Y) = \inf \{ a \in \mathbb{R} : \rho (\lambda X + (1-\lambda) Y + a d) \geq k \} \\
\leq a_\lambda \\
= \lambda \mu_k (X) + (1-\lambda) \mu_k (Y),
\]

and the desired conclusion follows.

Part (iii). For any \( X \in \mathcal{L} \), we must have \( \mu_k (X) \leq \mu_k^* (X) \) for \( k \leq k' \) since \( \rho (X + (\mu_k^* (X) + \epsilon) d) \geq k' \geq k \) for all \( \epsilon > 0 \).

The next proposition is used to show that several of the choice functions constructed from risk measures in this paper belong to \( \mathcal{R}_{eqv} \).

**Proposition A.2.** Let \( \{ \mu_k \}_{k \in \mathbb{R}} \) be a class of convex risk measures. Assume that:

(i) all \( \{ \mu_k \}_{k \in \mathbb{R}} \) have closed acceptance sets;

(ii) for any \( X \in \mathcal{L} \), \( \mu_k (X) \) as a function of \( k \) is non-decreasing and left continuous;

(iii) for any \( X \in \mathcal{L} \), there is \( k \in \mathbb{R} \) such that \( \mu_k (X) \leq 0 \).

Define \( \vartheta : \mathcal{L} \rightarrow \mathbb{R} \) by

\[
\vartheta (X) \triangleq \sup \{ k \in \mathbb{R} : \mu_k (X) \leq 0 \}, \forall X \in \mathcal{L}. \tag{40}
\]

Then \( \vartheta \) is upper semi-continuous, monotonic, and quasi-concave.

**Proof.** Upper semi-continuity: We first show that \( \vartheta \) is upper semicontinuous, or equivalently that \( \{ X \in \mathcal{L} : \vartheta (X) \geq k \} \) is closed for all \( k \in \mathbb{R} \). Let \( \{ X_i \}_{i \geq 0} \subset \{ X \in \mathcal{L} : \vartheta (X) \geq k \} \) and suppose \( \lim_{i \to \infty} X_i \to X \). Since \( \vartheta (X_i) \geq k \), it follows that \( \mu_k (X_i) \leq 0 \) for all \( i \geq 0 \) and thus the sequence \( \{ X_i \}_{i \geq 0} \) belongs to the acceptance set \( \mathcal{A}_{\mu_k} \triangleq \{ X \in \mathcal{L} : \mu_k (X) \leq 0 \} \). Since \( \mathcal{A}_{\mu_k} \) is closed we have \( X \in \mathcal{A}_{\mu_k} \), and so \( \vartheta (X) \geq k \) and thus \( X \in \{ X \in \mathcal{L} : \vartheta (X) \geq k \} \).

Monotonicity: Choose \( X, Y \in \mathcal{L} \) with \( X \leq Y \). Each \( \mu_k \) is monotonic, so if \( \mu_k (X) \leq 0 \) then \( \mu_k (Y) \leq 0 \) also holds, and thus \( \vartheta (X) \leq \vartheta (Y) \).

Quasi-concavity: Choose \( X, Y \in \mathcal{L} \) and \( \lambda \in [0, 1] \). Then we have

\[
\vartheta (\lambda X + (1-\lambda) Y) = \sup \{ k \in \mathbb{R} : \mu_k (\lambda X + (1-\lambda) Y) \leq 0 \} \\
\geq \sup \{ k \in \mathbb{R} : \lambda \mu_k (X) + (1-\lambda) \mu_k (Y) \leq 0 \} \\
\geq \min \{ \vartheta (X), \vartheta (Y) \},
\]

where the first inequality uses convexity of \( \{ \mu_k \}_{k \in \mathbb{R}} \), and the second inequality uses the fact that \( \lambda \mu_k (X) + (1-\lambda) \mu_k (Y) \geq 0 \) for \( k^* = \min \{ \vartheta (X), \vartheta (Y) \} \) since both \( \mu_k (X) \geq 0 \) and \( \mu_k (Y) \geq 0 \) hold for \( k^* \) by left continuity of \( \mu_k (X) \) in \( k \) for all \( X \in \mathcal{L} \). \( \square \)

The next proposition is used to establish our level set representation.

**Proposition A.3.** Suppose

\[
\{ X \in \mathcal{L} : \rho (X) \geq k \} = \{ X \in \mathcal{L} : \mu_k (X) \leq 0 \}, \forall k \in \mathbb{R},
\]

then

\[
\rho (X) = \sup \{ k \in \mathbb{R} : \mu_k (X) \leq 0 \}, \forall X \in \mathcal{L}.
\]

**Proof.** By construction of \( \{ \mu_k \}_{k \in \mathbb{R}} \), we have: (i) if \( \rho (X) \geq k \) then \( \mu_k (X) \leq 0 \); (ii) if \( \mu_k (X) > 0 \) then \( \rho (X) < k \). We have \( \mu_k (X) \leq 0 \) implies \( \rho (X) \geq k \) for all \( k \), so sup \( \{ k \in \mathbb{R} : \mu_k (X) \leq 0 \} \leq \rho (X) \). Conversely, we have \( \mu_{\rho (X)} (X) \leq 0 \) and so sup \( \{ k \in \mathbb{R} : \mu_k (X) \leq 0 \} \geq \rho (X) \). \( \square \)
## B Elicited comparison data set

We show the elicited comparison data set mentioned in Section 7.2 in the following tables. Losses are recorded here as negative values.

| Attribute                  | Loss ($ million) |
|----------------------------|------------------|
|                            | Scenario I | Scenario II | Scenario III |
| **Property losses**        | -185       | -368        | -42          |
| **Fatalities**             | -94        | -903        | -35          |
| **Air Departures**         | -873       | -722        | -843         |
| **Average daily bridge traffic** | -86        | -453        | -29          |

| Attribute                  | Loss ($ million) |
|----------------------------|------------------|
|                            | Scenario I | Scenario II | Scenario III |
| **Property losses**        | -917       | -529        | -331         |
| **Fatalities**             | -25        | -439        | -794         |
| **Air Departures**         | -348       | -731        | -346         |
| **Average daily bridge traffic** | -769       | -251        | -928         |

| Attribute                  | Loss ($ million) |
|----------------------------|------------------|
|                            | Scenario I | Scenario II | Scenario III |
| **Property losses**        | -597       | -496        | -593         |
| **Fatalities**             | -878       | -353        | -333         |
| **Air Departures**         | -732       | -692        | -66          |
| **Average daily bridge traffic** | -742       | -862        | -189         |

| Attribute                  | Loss ($ million) |
|----------------------------|------------------|
|                            | Scenario I | Scenario II | Scenario III |
| **Property losses**        | -699       | -524        | -515         |
| **Fatalities**             | -848       | -638        | -243         |
| **Air Departures**         | -652       | -212        | -583         |
| **Average daily bridge traffic** | -879       | -219        | -28          |

| Attribute                  | Loss ($ million) |
|----------------------------|------------------|
|                            | Scenario I | Scenario II | Scenario III |
| **Property losses**        | -115       | -331        | -12          |
| **Fatalities**             | -906       | -867        | -135         |
| **Air Departures**         | -70        | -979        | -611         |
| **Average daily bridge traffic** | -601       | -440        | -545         |

| Attribute                  | Loss ($ million) |
|----------------------------|------------------|
|                            | Scenario I | Scenario II | Scenario III |
| **Property losses**        | -953       | -699        | -754         |
| **Fatalities**             | -658       | -60         | -215         |
| **Air Departures**         | -264       | -19         | -117         |
| **Average daily bridge traffic** | -205       | -714        | -86          |

| Attribute                  | Loss ($ million) |
|----------------------------|------------------|
|                            | Scenario I | Scenario II | Scenario III |
| **Property losses**        | -455       | -199        | -442         |
| **Fatalities**             | -314       | -103        | -401         |
| **Air Departures**         | -106       | -402        | -851         |
| **Average daily bridge traffic** | -946       | -116        | -100         |
| Attribute                          | Scenario I | Scenario II | Scenario III |
|-----------------------------------|------------|-------------|--------------|
| Property losses                   | -697       | -56         | -550         |
| Fatalities                        | -954       | -451        | -795         |
| Air Departures                    | -805       | -271        | -100         |
| Average daily bridge traffic      | -280       | -423        | -373         |

| Attribute                          | Scenario I | Scenario II | Scenario III |
|-----------------------------------|------------|-------------|--------------|
| Property losses                   | 0          | -334        | -100         |
| Fatalities                        | -356       | -352        | -133         |
| Air Departures                    | -293       | -469        | -475         |
| Average daily bridge traffic      | -268       | -351        | -464         |

| Attribute                          | Scenario I | Scenario II | Scenario III |
|-----------------------------------|------------|-------------|--------------|
| Property losses                   | -244       | -451        | -190         |
| Fatalities                        | -459       | -478        | -320         |
| Air Departures                    | -140       | -221        | -121         |
| Average daily bridge traffic      | -1         | -113        | -293         |