(F1, D1, D3) Bound State, Its Scaling Limits and $SL(2, \mathbb{Z})$ Duality

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Abstract

We discuss the properties of the bound state (F1, D1, D3) in IIB supergravity in three different scaling limits and the $SL(2, \mathbb{Z})$ transformation of the resulting theories. In the simple decoupling limit with finite electric and magnetic components of NS $B$ field, the worldvolume theory is the $\mathcal{N}=4$ super Yang-Mills (SYM) and the supergravity dual is still the $AdS_5 \times S^5$. In the large magnetic field limit with finite electric field, the theory is the noncommutative super Yang-Mills (NCSYM), and the supergravity dual is the same as that without the electric background. We show how to take the decoupling limit of the closed string for the critical electric background and finite magnetic field, and that the resulting theory is the noncommutative open string (NCOS) with both space-time and space-space noncommutativities. It is shown that under the $SL(2, \mathbb{Z})$ transformation, the SYM becomes itself with a different coupling constant, the NCSYM is mapped to a NCOS, and the NCOS in general transforms into another NCOS and reduces to a NCSYM in a special case.

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1 Introduction

Over the past few years, one of the important progresses in string and field theories is the observation of Maldacena [1] that open string excitations on D-branes decouple from gravity in the appropriate low energy limit, the so-called decoupling limit and that string/M theories on the anti-de Sitter space (AdS) are dual to a certain large $N$ conformal field theory (CFT) which lives on the boundary of the AdS. One important example of this AdS/CFT correspondence is that IIB string theory on the $AdS_5 \times S^5$ is believed to be dual to the $\mathcal{N}=4$ super Yang-Mills theory (SYM). The geometry $AdS_5 \times S^5$ comes from the decoupling limit of D3-branes (hereafter referred to as the SYM limit). Through this “duality” one can learn something about the large $N$ SYM with strong ’t Hooft coupling from the low energy limit of the superstring/M theory, supergravity. Indeed a lot of knowledge has been acquired from the correspondence, which is consistent with our expectation.

Recently it has been noticed that when a constant NS $B$ field is present, the worldvolume coordinates of D-branes become noncommutative [2, 3, 4]. In an appropriate limit the worldvolume theory of D-branes with nonvanishing spatial components of the NS $B$ field is a noncommutative SYM (NCSYM) [3]-[12] (hereafter called the NCSYM limit) with space-space noncommutativity. More recently when an electric background is introduced, the resulting theory has been found to be a noncommutative open string (NCOS) theory with space-time noncommutativity in an appropriate limit (hereafter NCOS limit) [13, 14]. For related discussions, see also Refs. [15]-[23].

In this paper we consider the above three limits for the bound state (F1, D1, D3) with both electric and magnetic $B$ fields [24, 25, 26]. The SYM limit for this bound state has been discussed in Ref. [26]. For completeness and to compare with the other two cases, we also review the SYM limit in this paper. In the simple $\alpha' \to 0$ limit, the resulting theory is still the $\mathcal{N}=4$ SYM in ordinary spacetime. In the NCSYM limit the theory is the NCSYM with space-space noncommutativity again, namely without space-time noncommutativity. In the NCOS limit the resulting theory is most nontrivial and is the NCOS with not only space-time noncommutativity but also space-space noncommutativity. Another subject we discuss is the $SL(2, \mathbb{Z})$ duality of these solutions. The S-duality has been discussed
for theories with either an electric or magnetic background \[14\], but full understanding of
the behavior of these theories with both types of backgrounds under the general \(SL(2, \mathbb{Z})\)
duality has not been obtained. In the NCOS theory with both space-time and space-space
noncommutativities, there remains in general a nonvanishing axion field and it seems more
appropriate to use the \(SL(2, \mathbb{Z})\) transformation to get clear understanding of the relations
of these theories. We show that these theories are nicely related with each other by the
\(SL(2, \mathbb{Z})\) transformation of IIB supergravity.

As a preparation for our following discussions, in sect. 2 we review the Seiberg-Witten
relation between the open and closed string moduli when both the electric and magnetic
components of the NS \(B\) field are present. In sect. 3 we discuss supergravity duals for
three different limits. In particular we show how we are uniquely lead to the nonvanishing
decoupling limit for space-time and space-space noncommutativities. In sect. 4, we dis-
cuss general \(SL(2, \mathbb{Z})\) transformations of these solutions and show that under the \(SL(2, \mathbb{Z})\)
transformation, the NCSYM is mapped into a NCOS with space-time and space-space noncommutativities, but the NCOS transforms in general into another NCOS with differ-
ent parameters and reduces to a NCSYM in a special case, depending on the asymptotic
value of the axion. Concluding remarks are given in sect. 5.

In the course of our work, two papers \[27, 28\] have appeared in which similar topics
were discussed. The authors of Ref. \[27\] have discussed, in the setting of a vanishing
asymptotic value of the axion, the S-duality of the NCOS theory and NCSYM, from both
points of view of string theory and supergravity. In Ref. \[28\], the authors have discussed
the \(SL(2, \mathbb{Z})\) transformation of NCOS from the open string point of view.\(^1\) Our discussion
is in terms of the dual supergravity description.

\section{Seiberg-Witten relation}

When a constant NS \(B\) field is present, an open string ending on a D-brane has the
following boundary conditions:

\[ g_{ij} \partial_{\sigma} X^j + 2\pi \alpha' B_{ij} \partial_{\tau} X^j = 0, \quad \delta X^a = 0. \]  

\[ (2.1) \]

\(^1\)More recently, related discussion on the S-duality of NCSYM has been given in Refs. \[29, 30\].
The open string moduli appear in the disk correlators on the open string worldsheet boundaries

\[ \langle X^i(\tau)X^j(0) \rangle = -\alpha'G^{ij}\ln(\tau)^2 + i\frac{\Theta^{ij}}{2}\epsilon(\tau). \]  

The open and closed string moduli are connected by the Seiberg-Witten relation \[ G_{ij} = g_{ij} - (2\pi\alpha')^2(B^{-1}g_B)_{ij}, \]
\[ \Theta^{ij} = 2\pi\alpha' \left( \frac{1}{g + 2\pi\alpha'B} \right)^{ij}_A, \]
\[ G^{ij} = \left( \frac{1}{g + 2\pi\alpha'B} \right)^{ij}_S, \]
\[ G_s = g_s \left( \frac{\det G_{ij}}{\det(g_{ij} + 2\pi\alpha'B_{ij})} \right)^{1/2}, \]  

where \((\_)_A\) and \((\_)_S\) denote the antisymmetric and symmetric parts, respectively.

The constant NS B field is equivalent to a gauge field on the worldvolume of the D-brane because only \(F_{ij} = B_{ij} + F_{ij}\) is gauge invariant. Therefore the electric and magnetic components of the B field can always be rotated so that they are parallel to each other. On the other hand, we are mainly concerned with the D3-brane case in this paper, restricting ourselves to the worldvolume of D3-branes. Suppose we have the closed string metric

\[ g_{ij} = g_1(\delta_i^1\delta_j^1 - \delta_i^0\delta_j^0) + g_2(\delta_i^2\delta_j^2 + \delta_i^3\delta_j^3), \]  

and the constant B field has components

\[ B_{ij} = E(-\delta_i^1\delta_j^0 + \delta_i^0\delta_j^1) + B(\delta_i^2\delta_j^3 - \delta_i^3\delta_j^2). \]

Defining

\[ e = \frac{E}{E_{\text{crit}}}, \quad b = \frac{B}{B_0}, \]  

where

\[ E_{\text{crit}} = \frac{g_1}{2\pi\alpha'}; \quad B_0 = \frac{g_2}{2\pi\alpha'}, \]  

and using the Seiberg-Witten relation, we have the open string metric

\[ G^{ij} = \frac{1}{g_1(1 - e^2)}(-\delta_i^0\delta_j^0 + \delta_i^1\delta_j^1) + \frac{1}{g_2(1 + b^2)}(\delta_i^2\delta_j^2 + \delta_i^3\delta_j^3), \]
the noncommutativity matrix
\[ \Theta^{ij} = \frac{2\pi\alpha'}{g_1(1 - e^2)}(\delta_0^i\delta_1^j - \delta_1^i\delta_0^j) + \frac{2\pi\alpha'\beta}{g_2(1 + b^2)}(-\delta_2^i\delta_3^j + \delta_3^i\delta_2^j), \] (2.9)
and the open string coupling constant
\[ G_s = g_s\sqrt{(1 - e^2)(1 + b^2)}. \] (2.10)

Let us now consider three different limits.

(1) SYM limit. From Eqs. (2.8), (2.9) and (2.10), we see that when \( \alpha' \to 0 \) with finite electric \( E \) and magnetic components \( B \), the open and closed string moduli are equal. The noncommutativity matrix \( \Theta^{ij} \) vanishes, so that the entire spacetime becomes an ordinary commutative one. In this case, the oscillation modes of an open string and gravity are decoupled, and the worldvolume theory is the \( \mathcal{N}=4 \) SYM in the low energy limit. Note that the constant \( B \) field is converted into a constant part of the gauge field on the worldvolume of the D3-brane and remains in that limit. If the D3-brane worldvolume is noncompact, the constant part of the gauge field is physically unmeasurable in the flat infinite space \cite{[14]}\). On the other hand, if the worldvolume is compact, the constant part is quantized and the resulting low energy theory is the \( \mathcal{N}=4 \) SYM with both quantized electric and magnetic fluxes.

(2) NCSYM limit. Taking the limit \( \alpha' \to 0 \) with \( g_1 = 1, g_2 = (2\pi\alpha'B)^2 \), and \( g_s = 2\pi\alpha'BG_s \) while keeping \( G_s, E \) and \( B \) finite, we can obtain
\[ G^{ij} = \eta^{ij}, \quad \Theta^{ij} = \frac{1}{B}(-\delta_2^i\delta_3^j + \delta_3^i\delta_2^j). \] (2.11)
In this case, the resulting theory is the noncommutative SYM (because \( \alpha'G^{ij} = 0 \), that is, massive open string modes are decoupled) with space-space noncommutativity \( \Theta^{23} \neq 0 \); the Yang-Mills coupling constant is \( g_{YM}^2 = 2\pi G_s \). The magnetic background gives rise to the space-space noncommutativity. The constant electric component of the NS \( B \) field is converted into a constant electric part of the gauge field, which is physically unmeasurable if the worldvolume is noncompact again. If the worldvolume is compact, the resulting theory is the NCSYM with quantized electric flux.

(3) NCOS limit. In contrast to the magnetic component \( B \), on which there is no restriction, the electric component cannot be beyond its critical value \( E_{\text{crit}} \). When the
electric field approaches its critical value in a certain manner, one may obtain a noncritical NCOS theory, from which the closed string sector is decoupled \cite{13, 14}, in spacetime with the space-time noncommutativity. If a finite magnetic component is also present, the space-space coordinates are also noncommutative. This has been noted in Refs. \cite{19} and \cite{27}. For example, taking the scaling limit

\[ e = 1 - \alpha'^{m}e_{0}/2, \quad (n > 0), \]
\[ g_{1} = \frac{1}{\alpha'_{\text{eff}}e_{0}\alpha'^{m-1}}, \quad g_{2} = \frac{2\pi\alpha' b}{(1 + b^{2})\theta_{1}}, \]
\[ g_{s} = \frac{G_{s}}{\alpha'^{m/2}\sqrt{e_{0}(1 + b^{2})}}, \quad (2.12) \]

with \( e_{0} \) constant, while keeping \( G_{s} \) and \( B \) constant, one may get

\[ \alpha'_{\text{eff}}G^{ij} = - (\delta^{i}_{0}\delta^{j}_{0} - \delta^{i}_{1}\delta^{j}_{1}) + \frac{\theta_{1}}{\theta_{0}b}(\delta^{i}_{2}\delta^{j}_{2} + \delta^{i}_{3}\delta^{j}_{3}), \quad (2.13) \]
\[ \Theta^{ij} = \theta_{0}(\delta^{i}_{0}\delta^{j}_{1} - \delta^{i}_{1}\delta^{j}_{0}) + \theta_{1}(-\delta^{i}_{2}\delta^{j}_{3} + \delta^{i}_{3}\delta^{j}_{2}), \quad (2.14) \]

where \( \alpha'_{\text{eff}} = \theta_{0}/(2\pi) \). In this case, the critical electric field is given by \( E_{\text{crit}} = \frac{1}{e_{0}\theta_{0}\alpha'^{m}} \).

Although the closed string coupling constant is divergent in this limit, the open string metric and coupling constant are well defined.

It is rather nontrivial to derive the dual gravity description of this NCOS theory. One of our purposes in this paper is to elucidate this problem. This is discussed in the next section.

3 Supergravity duals

When both the electric and magnetic components of the NS B field are present on a D3-brane, the D3-brane becomes a (F1, D1, D3) bound state. The supergravity configuration of the bound state has been constructed in Ref. \cite{26}. It can also be constructed as follows \cite{3}. Starting from a D3-brane without an NS B field with worldvolume coordinates \((x_{0}, x_{1}, x_{2} \text{ and } x_{3})\), and making a T-duality along \( x_{3} \), one gets a D2-brane with a

\footnote{Here \( n \) is a positive parameter. It was chosen to be 2 in the S-duality consideration in Ref. \cite{14}, and 1 in Ref. \cite{21}. We will show in the next section that this freedom is allowed in this limit.}
smeared coordinate \( x_3 \). Uplifting the D2-brane yields an M2-brane in the 11-dimensional supergravity. Performing a coordinate rotation with parameter angles \( \varphi \) and \( \theta \)

\[
\begin{align*}
    x_4 &= x'_4 \cos \varphi - (x'_2 \cos \theta - x'_3 \sin \theta) \sin \varphi, \\
    x_2 &= x'_4 \sin \varphi + (x'_2 \cos \theta - x'_3 \sin \theta) \cos \varphi, \\
    x_3 &= x'_2 \sin \theta + x'_3 \cos \theta,
\end{align*}
\]

and then reducing along \( x'_4 \), one obtains a new D2-brane. Applying T-duality along \( x'_3 \), one reaches a (F1, D1, D3) bound state. For the case of black configurations, the procedure is applicable as well.

Using the above approach, we obtain the non-extremal supergravity solution of (F1, D1, D3) bound state

\[
\begin{align*}
    ds^2 &= F^{-1/2}[h'(-fdx_0^2 + dx_1^2) + h(dx_2^2 + dx_3^2)] + F^{1/2}[f^{-1}dr^2 + r^2d\Omega_5^2], \\
    e^{2\phi} &= g_s^2 hh', \quad \chi = -\frac{1}{g_s F} \tan \varphi \sin \theta, \\
    B_{01} &= H^{-1} \coth \alpha \sin \varphi, \quad A_{01} = -H^{-1} \coth \alpha \sin \theta / g_s \cos \varphi, \\
    B_{23} &= \frac{\tan \theta}{G}, \quad A_{23} = \frac{\tan \varphi \cos \theta}{g_s G}, \\
    F_{0123r} &= \frac{\coth \alpha \cos \theta}{g_s \cos \varphi} hh' \partial_r F^{-1},
\end{align*}
\]

where

\[
\begin{align*}
    f &= 1 - \frac{r_0^4}{r^4}, \quad H = 1 + \frac{r_0^4 \sinh^2 \alpha}{r^4}, \\
    h &= F/G, \quad h' = F/H, \\
    F &= 1 + \cos^2 \varphi \frac{r_0^4 \sinh^2 \alpha}{r^4}, \quad G = 1 + \cos^2 \varphi \cos^2 \theta \frac{r_0^4 \sinh^2 \alpha}{r^4}.
\end{align*}
\]

The bound state solution includes several special cases: when \( \varphi = \theta = 0 \), it reduces to the D3-brane solution; when \( \varphi = \pi/2 \) and \( \theta \) is arbitrary, it goes to the F-string solution with two smeared coordinates; when \( \varphi = 0 \) and \( \theta = \pi/2 \), it becomes the D-string solution with two smeared coordinates; when \( \varphi = 0 \) and \( \theta \) is arbitrary, the solution reduces to the

\[3\text{In this section a factor } 2\pi\alpha' \text{ is absorbed into the } B \text{ field.}\]

\[4\text{There is the freedom to shift the asymptotic value of the axion field } \chi_{\infty} \text{ by the } SL(2, \mathbb{R}) \text{ symmetry of the IIB supergravity. We note that this is a classical symmetry of the IIB supergravity.}\]
(D1, D3) bound state; when \( \theta = 0 \) and \( \varphi \) is arbitrary, it becomes the (F1, D3) bound state; and finally when \( \theta = \pi/2 \) and \( \varphi \) is arbitrary, it goes back to the (F1, D1) bound state with two smeared coordinates.

Some thermodynamic quantities, the ADM mass \( M \), the Hawking temperature \( T \), and the entropy \( S \), associated with the solution (3.2) are

\[
M = \frac{5\pi^3 r_0^4 V_3}{16\pi G_{10}} (1 + \frac{4}{5} \sinh^2 \alpha),
\]

\[
T = \frac{1}{\pi r_0 \cosh \alpha},
\]

\[
S = \frac{\pi^3 r_0^5 V_3}{4G_{10}} \cosh \alpha, \tag{3.4}
\]

where \( V_3 \) is the spatial volume of worldvolume of the bound state. An important feature of these thermodynamic quantities is their independence of the parameter angles \( \varphi \) and \( \theta \). This means that the thermodynamics is the same for all special cases discussed above. It also guarantees the thermodynamic equivalence among the three theories coming from different scaling limits which will be discussed shortly.

For the (F1, D1, D3) bound state, the charges of the three kinds of branes are

\[
Q_{D3} = \frac{4\pi^3 V_3 r_0^4 \sinh \alpha \cosh \alpha}{16\pi G_{10}} \cos \varphi \cos \theta,
\]

\[
Q_{D1} = \frac{4\pi^3 V_3 r_0^4 \sinh \alpha \cosh \alpha}{16\pi G_{10}} \cos \varphi \sin \theta,
\]

\[
Q_F = \frac{4\pi^3 V_3 r_0^4 \sinh \alpha \cosh \alpha}{16\pi G_{10}} \sin \varphi. \tag{3.5}
\]

In the extremal limit that \( \alpha \to \infty, r_0 \to 0 \) with finite charges, we have

\[
M_{\text{ext}}^2 = Q_{D3}^2 + Q_{D1}^2 + Q_F^2, \tag{3.6}
\]

which indicates that the bound state is a non-threshold state. The charges satisfy the relations

\[
\frac{Q_{D1}}{Q_{D3}} = \tan \theta, \quad \frac{Q_F}{Q_{D3}} = \frac{\tan \varphi}{\cos \theta}. \tag{3.7}
\]

Furthermore, in this bound state, the number of D3-branes is

\[
N_3 = \frac{R_0^4 \cos \varphi \cos \theta}{4\pi g_s \alpha'}. \tag{3.8}
\]
the number of D-strings is
\[ N_1 = \frac{R_0^4 \cos \varphi \sin \theta}{4 \pi g_s \alpha'^2} \frac{V_2}{(2\pi)^2 \alpha'}, \]  
(3.9)
and the number of F-strings is
\[ N_F = \frac{R_0^4 \sin \varphi}{4 \pi g_s^2 \alpha'^2} \frac{V_2}{(2\pi)^2 \alpha'}, \]  
(3.10)
where \( R_0^4 = r_0^4 \sinh \alpha \cosh \alpha \) and \( V_2 \) is the area of the worldvolume coordinates \( x_2 \) and \( x_3 \).

We are now going to discuss the various decoupling limits for the supergravity duals, keeping the number \( N_3 \) of D3-branes finite.

### 3.1 SYM limit

In this subsection, we first discuss the SYM limit. Taking the usual decoupling limit:
\[ \alpha' \to 0 : \quad r = \alpha'u, \quad r_0 = \alpha'u_0, \]  
(3.11)
and keeping \( \cos \theta \) and \( \cos \varphi \) finite, we have
\[ ds^2 = \alpha' \left[ \frac{u^2}{R^2} \left( \cos^2 \varphi (-\tilde{f} dx_0^2 + dx_1^2) + \cos^2 \theta (dx_2^2 + dx_3^2) \right) + \frac{\tilde{R}^2}{u^2} (\tilde{f}^{-1} du^2 + u^2 d\Omega_5^2) \right], \]  
(3.12)
where \( \tilde{R}^4 = 4\pi g_s N_3 \cos \varphi / \cos \theta \) and \( \tilde{f} = 1 - u_0^4 / u^4 \). The dilaton, axion and \( B \) fields reduce to
\[ e^{2\phi} = g_s^2 \frac{\cos^2 \varphi}{\cos^2 \theta}, \quad \chi = 0, \]
\[ B_{01} = \alpha'^2 \sin \varphi \cos^2 \varphi u^4 / \tilde{R}^4, \quad B_{23} = \alpha'^2 \tan \theta u^4 / (\tilde{R}^4 \cos^2 \theta). \]  
(3.13)
Obviously, rescaling the closed string coupling constant and worldvolume coordinates
\[ g_s = \frac{\cos \theta}{\cos \varphi} \tilde{g}, \quad x_{0,1} = \frac{1}{\cos \varphi} \tilde{x}_{0,1}, \quad x_{2,3} = \cos \theta \tilde{x}_{2,3}, \]  
(3.14)
we can convert the metric (3.12) into a standard form of \( AdS_5 \times S^5 \):
\[ ds^2 = \alpha' \left[ \frac{u^2}{R^2} (-\tilde{f} dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2) + \frac{\tilde{R}^2}{u^2} (\tilde{f}^{-1} du^2 + u^2 d\Omega_5^2) \right]. \]  
(3.15)
The dilaton, axion and $B$ fields become

\[ e^{2\phi} = \tilde{g}^2, \quad \chi = 0, \]
\[ B_{01} = \alpha'^2 \sin \varphi u^4 / \tilde{R}^4, \quad B_{23} = \alpha'^2 \tan \theta u^4 / \tilde{R}^4. \]  

(3.16)

From the above solution, we see that although both the electric and magnetic components are present in the D3-brane bound state, the resulting theory in the SYM limit is still the $\mathcal{N}=4$ SYM with gauge group $U(N_3)$ without noncommutativity, just as was shown in Ref. [26]. This can also be understood from the boundary conditions (2.1) of the open string. In the SYM limit, the mixed boundary conditions reduce to the ordinary Neumann boundary conditions. The constant $B$ field has no effect on the open string ending on the D3-branes in that limit. It is worthwhile to stress here that there is a significant difference between the D3-brane with finite $B$ field and the case without $B$ field depending on whether the D3-brane is compact or not.

The bound state solution (3.2) implies that there is a constant electric component $B_{01} = \sin \varphi$ and a constant magnetic component $B_{23} = \tan \theta$ of the NS $B$ field on the worldvolume of the D3-brane, which gives a constant part of the worldvolume field strength $F_{ij}$. Although the constant part has no effect on the open string ending on the D3-branes in the SYM limit, it remains in that limit. If the D3-brane is compact on a torus, this part should be quantized. We find that the constant part gives $N_1$ units of magnetic flux and $N_F$ units of electric flux, with $N_1$ and $N_F$ being the numbers of D-strings and F-strings in the bound state, respectively. Thus the resulting low energy theory is the $\mathcal{N}=4$ SYM with both quantized electric and magnetic fluxes. On the other hand, if the worldvolume of the D3-brane is not compact, the constant part is physically unmeasurable in the flat infinite space, as mentioned above [14]. The resulting theory is then just the $\mathcal{N} = 4$ SYM as for the case of D3-branes without a $B$ field.

Thus we conclude that in the SYM limit, the worldvolume theory on the (F1, D1, D3) bound state is the $\mathcal{N}=4$ SYM without noncommutativity [26]. We will see in the next section that under the $SL(2, \mathbb{Z})$ transformation the SYM is mapped again into the SYM with a different coupling constant.
3.2 NCSYM limit

Taking the decoupling limit \[\alpha' \rightarrow 0\] :

\[
\begin{align*}
\tan \theta &= \frac{b}{\alpha'}, \quad x_{0,1} = \bar{x}_{0,1}, \quad x_{2,3} = \alpha' \bar{x}_{2,3}, \\
r &= \alpha' u, \quad r_0 = \alpha' u_0, \quad g_s = \alpha' \bar{g}, \quad x_0, 1 = \bar{x}_0, 1, \\
x_2, 3 = \alpha' \bar{x}_2, 3, \\
r &= \alpha' u, \quad r_0 = \alpha' u_0, \quad g_s = \alpha' \bar{g},
\end{align*}
\]

(3.17)

while keeping \(\cos \varphi\) finite, we have

\[
ds^2 = \alpha' \left[ \frac{u^2}{R_y^2} \left[ \cos^2 \varphi (-\bar{f} d\bar{x}_0^2 + d\bar{x}_1^2) + \bar{h}(d\bar{x}_2^2 + d\bar{x}_3^2) \right] + \frac{R_y^2}{u^2} \left[ \bar{f}^{-1} du^2 + u^2 d\Omega_5^2 \right] \right],
\]

(3.18)

where \(R_y^4 = 4\pi \bar{g} \bar{b} N_3 \cos \varphi\), and

\[
\bar{h}^{-1} = 1 + (au)^4, \quad a^4 = \bar{b}^2 / R_y^4.
\]

(3.19)

Also we have

\[
e^{2\phi} = \bar{g}^2 \bar{b}^2 \bar{h} \cos^2 \varphi, \quad \chi = 0,
\]

\[
B_{01} = \alpha'^2 \sin \varphi \cos^2 \varphi u^4 / R_y^4, \quad B_{23} = \frac{\alpha'}{b} \frac{(au)^4}{1 + (au)^4}.
\]

(3.20)

After further rescaling the string coupling constant \(\bar{g}\) and the coordinates \(\bar{x}_{0,1}\) as

\[
g \rightarrow \bar{g} / \cos \varphi, \quad \bar{x}_{0,1} \rightarrow \frac{1}{\cos \varphi} \bar{x}_{0,1},
\]

(3.21)

we reach

\[
ds^2 = \alpha' \left[ \frac{u^2}{R^2} \left[ (-\bar{f} d\bar{x}_0^2 + d\bar{x}_1^2) + \bar{h}(d\bar{x}_2^2 + d\bar{x}_3^2) \right] + \frac{R^2}{u^2} \left[ \bar{f}^{-1} du^2 + u^2 d\Omega_5^2 \right] \right],
\]

(3.22)

and

\[
e^{2\phi} = \bar{g}^2 \bar{b}^2 \bar{h}, \quad \chi = 0,
\]

\[
B_{01} = \alpha'^2 \sin \varphi u^4 / R^4, \quad B_{23} = \frac{\alpha'}{b} \frac{(au)^4}{1 + (au)^4},
\]

(3.23)

where \(a^4 = \bar{b}^2 / R^4\) and \(R^4 = 4\pi \bar{g} \bar{b} N_3\). We note that the geometry (3.22) is completely the same as in the case of black D3-branes with only spatial component of \(B\) field; for the latter see Refs. [3, 4]. It has been claimed that the geometry (3.22) is the gravity dual
configuration of the $\mathcal{N}=4$ NCSYM with gauge group $U(N_3)$ and space-space noncommutativity $[\tilde{x}_2, \tilde{x}_3] = i\tilde{b}$.

In the NCSYM limit, an infinitely large magnetic field gives rise to the noncommutativity of space-space while the electric field is kept finite: $F_{01} = \sin \varphi$. The electric field has no effect on the field theory limit of open string. (It gives rise to a quantized electric flux if the worldvolume is compact.) We do not find well-defined field theory limit with space-time noncommutativity. This is in accordance with the belief that the field theory with space-time noncommutativity may not be unitary [18]. As a result, the low energy field theory of the bound state $(F_1, D_1, D_3)$ in the NCSYM limit is the $\mathcal{N}=4$ NCSYM without space-time noncommutativity, and only the spatial coordinates $(\tilde{x}_{2,3})$ are noncommutative. In addition, we note that the NCSYM limit implies that $\theta \to \pi/2$ and $\varphi$ is arbitrary in this case. Hence the decoupling geometry goes to that of the bound state $(F_1, D_1)$ with two smeared coordinates as $au \gg 1$.

For low energies, $au \ll 1$, the geometry is that of ordinary SYM, $AdS_5 \times S_5$. It significantly deviates from this geometry for high energies, and the deviation appears at the scale of $u \sim 1/a = R/\sqrt{\tilde{b}}$.

### 3.3 NCOS limit

The NCOS limit in the dual supergravity is drastically different from the NCSYM limit, and we study how such a limit can be uniquely determined in our setting.

In order to get the NCOS limit, we should keep $\Theta^{01}$ finite. This means that the electric field $e$ should tend to its critical value and other quantities should scale as given in Eq. (2.12). Note that $\alpha' G^{ij} \neq 0$ in this limit and the oscillating modes of an open string do not decouple. This critical behavior is translated in the supergravity solution (3.2) into $\sin \varphi = 1 - \alpha^m e_0/2$ or

$$\cos \varphi = \left( \frac{\alpha'}{\tilde{b}} \right)^{n/2},$$

with $\theta$ kept finite.

Next suppose the scaling behavior of $r$ is given as

$$r = \alpha^m u.$$ (3.25)
We would like to make all our metric in (3.2) scale as $\alpha'$. First consider the $dr^2$ term. This is transformed into

$$F^{1/2}dr^2 = \left(1 + \frac{R^4}{\alpha'^4m^{-2}u^4}\right)^{1/2} \alpha'^2m du^2,$$

(3.26)

where $R^4 = 4\pi \tilde{g}N_3/(\tilde{b}^{n/2}\cos\theta)$ and $g_s = \tilde{g}\alpha'^{-n/2}$. We thus see that as long as $m \geq \frac{1}{2}$, this scales as

$$\alpha' \left(1 + \frac{R^4}{u^4}\right)^{1/2} du^2, \quad \text{for } m = \frac{1}{2},$$

$$\alpha' \frac{R^2}{u^2} du^2, \quad \text{for } m > \frac{1}{2}.$$  

(3.27)

Thus all values $m \geq \frac{1}{2}$ are allowed at this point.

We next examine the behavior of other components of the metric.

$$H \sim \frac{\tilde{b}^n}{\alpha'^4m^{-2+n}} \frac{R^4}{u^4},$$

$$G = 1 + \frac{\cos^2\theta R^4}{\alpha'^4m^{-2}}.$$  

(3.28)

From Eqs. (3.27) and (3.28), we see that all nontrivial functions of $u$ disappear from the solution for $m > \frac{1}{2}$, and the resulting metric is $AdS_5 \times S_5$ up to a rescaling of the coordinates. This is believed to correspond to the SYM limit and not the limit we are looking for. Thus we are uniquely lead to the special scaling $m = \frac{1}{2}$:

$$r = \sqrt{\alpha' u}.$$  

(3.29)

Note that the parameter $n$ remains arbitrary as long as it is positive.

Having understood how everything scales, we find that the dual gravity solution corresponding to the NCOS is given by

$$ds^2 = \alpha' \tilde{F}^{1/2} \left[\frac{u^4}{R^4}(-\tilde{f} d\tilde{x}_0^2 + d\tilde{x}_1^2) + \frac{1}{G}(d\tilde{x}_2^2 + d\tilde{x}_3^2) + \tilde{f}^{-1} du^2 + u^2 d\Omega_5^2\right],$$

(3.30)

where

$$\tilde{F} = 1 + \frac{R^4}{u^4}, \quad \tilde{G} = 1 + \frac{R^4 \cos^2\theta}{u^4}.$$  

(3.31)

and the coordinates are rescaled as

$$x_{0,1} = \frac{\tilde{b}^{n/2}}{\alpha'^{(n-1)/2}} \tilde{x}_{0,1}, \quad x_{2,3} = \sqrt{\alpha'} \tilde{x}_{2,3}.$$  

(3.32)
The dilaton, axion and $B$ fields are
\[ e^{2\phi} = g^2 \frac{\tilde{F}^2}{\tilde{G}} \frac{u^4}{b^\mu R^4}, \quad \chi = -\frac{\tilde{b}^{\mu/2} \sin \theta}{\tilde{g} \tilde{F}}, \]
\[ B_{01} = \alpha' u^4 / R^4, \quad B_{23} = \alpha' \tan \theta / \tilde{G}. \] (3.33)

We note that the axion field is nonvanishing here, which is quite different from other solutions.

The geometry (3.30) is considered to be the supergravity dual of the NCOS theory with both space-time and space-space noncommutativities. It can be regarded as an extension of Ref. [14], in which the supergravity dual of NCOS has been given with only space-time noncommutativity by applying the S-duality to the supergravity dual of NCSYM. In the NCOS limit, the electric field approaches its critical value while the magnetic field remains finite. The critical electric field leads to the space-time noncommutativity. The magnetic field also gives rise to space-space noncommutativity although it is finite.

Again the geometry is $AdS_5 \times S_5$, that of ordinary SYM for small $u$, which indicates the low energy limit of NCOS is also the ordinary SYM. For large $u$, it deviates from this geometry in two ways, one (due to $\tilde{F}$) involving space-time coordinates, and the other (due to $\tilde{G}$) involving space-space coordinates. These are reflections of the space-time and space-space noncommutativities, and they arise at the scales of $R$ and $R \sqrt{\cos \theta}$, respectively.

4 $SL(2, \mathbb{Z})$ duality

It is well known that IIB superstring has the $SL(2, \mathbb{Z})$ symmetry and its low energy approximation, IIB supergravity, has also this symmetry. It is interesting to consider the relations of the above theories in different decoupling limits under the $SL(2, \mathbb{Z})$ transformation.

As it is already pointed out in Ref. [14], the S-dual of the NCSYM gives a NCOS theory without space-space noncommutativity. This case is simple because the axion field $\chi$ vanishes for the supergravity dual of NCSYM and the S-duality is achieved just by taking a simple inverse of the dilaton field. This relation is also natural because the NCSYM theory is affected only by the magnetic component of the $B$ field, while the
NCOS with $\theta = 0$ has only the electric background. The question we are asking here is what is the general situation. It might appear that this is again simple, but the above consideration suggests that these two theories might not be just the S-duals of each other since the axion field in general does not vanish for the NCOS theory. Indeed we find that the relation is rather nontrivial.

Here we consider the relations among the three theories discussed above by using the more general $SL(2, \mathbb{Z})$ transformation. We show that according to the supergravity description, the NCSYM is always mapped into a NCOS with space-time and space-space noncommutativities in general. The converse is rather nontrivial. Under the $SL(2, \mathbb{Z})$ transformation, we find that the NCOS theory transforms in general into another NCOS, and for a special case this reduces to a NCSYM theory.

Our IIB supergravity has an $SL(2, \mathbb{Z})$ invariance as an effective theory of IIB superstring. The metric in the Einstein frame is inert under this transformation, so that two solutions after a general $SL(2, \mathbb{Z})$ transformation are related as

$$ds^2_E = e^{-\phi/2} ds^2_{st,1} = e^{-\phi'/2} ds^2_{st,2}, \quad (4.1)$$

where the latter two expressions are two string-frame metrics related by $SL(2, \mathbb{Z})$ transformations, and $\phi$ and $\phi'$ are the dilatons for each solution. Using the notation $\tau = \chi + ie^{-\phi}$, they are related by

$$\tau' = \frac{a\tau + b}{c\tau + d}, \quad ad - bc = 1, \quad (4.2)$$

for integers $a, b, c$ and $d$. (The usual S-duality corresponds to the case $\tau' = -1/\tau$.) This gives us

$$e^{-\phi'} = \frac{e^{-\phi}}{|c\tau + d|^2}, \quad (4.3)$$

resulting in

$$ds^2_{st,2} = |c\tau + d| ds^2_{st,1}. \quad (4.4)$$

Let us first discuss the $SL(2, \mathbb{Z})$ transformation of the SYM theory $\{3.12\}$. Since the axion field vanishes and the dilaton is a constant, the supergravity dual $AdS_5 \times S^5$ is still of the form $AdS_5 \times S^5$ after the $SL(2, \mathbb{Z})$ transformation. This implies that the ordinary SYM theory is mapped into itself again, but with a different coupling constant

$$g' = \tilde{g}(d^2 + c^2 \tilde{g}^{-2}). \quad (4.5)$$
Next, for the NCSYM (3.22), the axion is again zero and
\[ \tau = i/\tilde{h}^{1/2}\tilde{g}\tilde{b}. \] (4.6)

The \( SL(2, \mathbb{Z}) \) transformation then leads to
\[
\begin{align*}
\tilde{d}s^2 &= \alpha' \tilde{h}^{-1/2} \left[ \frac{u^2}{R^2} \left( \tilde{f} d\tilde{x}_0^2 + d\tilde{x}_1^2 \right) + \tilde{h}(d\tilde{x}_2^2 + d\tilde{x}_3^2) \right]
+ \frac{R^2}{u^2} \left( f^{-1} du^2 + u^2 d\Omega_5^2 \right), \\
\tilde{h}^{-1} &= d^2 + \frac{c^2}{\tilde{g}^2 \tilde{b}^2} \tilde{h}^{-1}, \quad e^{2\phi'} = \tilde{g}^2 \tilde{b}^2 \tilde{h}^{-2}.
\end{align*}
\] (4.7)

This metric can be cast into the form (3.30) with
\[ \cos^2 \theta' = \frac{c^2}{c^2 + (dbg)^2}, \] (4.8)
up to a coordinate and parameter rescaling. This means that the NCSYM transforms into a NCOS with both space-time and space-space noncommutativities after the transformation. When \( d = 0 \) (the case of S-duality), the space-space noncommutativity then disappears.

Finally let us consider the NCOS theory (3.30) with both space-time and space-space noncommutativities where the axion field does not vanish. In the above two cases, the simple S-duality \( \tau \rightarrow -\frac{1}{\tau} \) was useful to get information on their relations, but here we consider the \( SL(2, \mathbb{Z}) \) transformation with
\[ \tau = -\frac{\tilde{b}^{n/2} \sin \theta}{\tilde{g}\tilde{F}} + i \frac{\tilde{b}^{n/2} \tilde{G}^{1/2} R^2}{\tilde{g}\tilde{F} u^2}. \] (4.9)

The factor in Eq. (4.4) then becomes
\[
|c\tau + d| = \left[ (c\tilde{b}^{n/2} \sin \theta - d\tilde{g}\tilde{F})^2 + c^2 \tilde{b}^{n}\tilde{G} \frac{R^4}{u^4} \right]^{1/2} \frac{1}{\tilde{g}\tilde{F}},
\] (4.10)
which, with the help of Eq. (3.31), is transformed into
\[
\frac{\hat{F}^{1/2}}{F^{1/2}},
\] (4.11)
where
\[ \hat{F} \equiv \left( d - \frac{c\tilde{b}^{n/2}}{\tilde{g}} \sin \theta \right)^2 + \left( d^2 + \frac{c^{2}\tilde{b}^{n}}{\tilde{g}^2} \cos^2 \theta \right) \frac{R^4}{u^4}. \] (4.12)
When Eq. (4.11) is used in Eqs. (4.4) and (3.30), we find that the $SL(2, \mathbb{Z})$-transformed solution is the same as the original one (3.30) with $\tilde{F}$ replaced by $\hat{F}$. The new $\theta'$ is given by

$$
\cos^2 \theta' = \frac{(d \tilde{g} - c \tilde{b}^{n/2} \sin \theta)^2}{d^2 \tilde{g}^2 + c^2 \tilde{b}^n \cos^2 \theta} \cos^2 \theta.
$$

(4.13)

We thus find that a NCOS theory transforms into another NCOS theory with a different noncommutativity parameter under the $SL(2, \mathbb{Z})$ transformation. In general both the NCOSs have space-time and space-space noncommutativities. However, in a special case when the first term in Eq. (4.12) vanishes,

$$
d - c \tilde{b}^{n/2} \sin \theta = 0,
$$

(4.14)

we find that the $SL(2, \mathbb{Z})$-transformed solution reduces to the form (3.22), which describes a NCSYM. This is possible only for the case in which the asymptotic value of the axion $\chi_\infty = -\tilde{b}^{n/2} \sin \theta / \tilde{g}$ is a rational number, in agreement with the conclusion in Ref. [28] derived from the open string point of view. This result is also consistent with that in Ref. [27]. This also includes the case $\theta = 0$ discussed in Ref. [14]. In this case one can also reach the same conclusion by using a simple S-duality.

5 Conclusions and discussions

In this paper, we have considered in the framework of IIB supergravity three different scaling limits for the bound state (F1, D1, D3) and $SL(2, \mathbb{Z})$ transformations of the resulting theories. These three theories are, respectively, (1) the ordinary $\mathcal{N}=4$ SYM with or without $N_F$ units of electric flux and $N_1$ units of magnetic flux depending on whether the worldvolume of the D3-brane is compact or not (here $N_1$ and $N_F$ are, respectively, the numbers of D-strings and F-strings in the bound state); (2) the NCSYM with or without $N_F$ units of electric flux; and (3) the NCOS with both space-time and space-space noncommutativities. Under a general $SL(2, \mathbb{Z})$ transformation, the gravity dual $AdS_5 \times S^5$

\footnote{If we used the classical $SL(2, \mathbb{R})$ symmetry in the original solution (3.2), we could shift the asymptotic value $\chi_\infty$ of the axion. In particular, we could set it to zero. This is the case discussed in Ref. [27]; the NCOS transforms into a NCSYM after the S-duality. It was argued there that the S-dual of NCSYM is not always NCOS, but that limit is different from the well-defined NCOS limit.}

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of the SYM still has the same form up to the rescaling of parameters and coordinates. This implies that the SYM becomes another SYM with different coupling constants after the transformation. The gravity dual of NCSYM takes the form of the NCOS in general with both the space-time and space-space noncommutativities. This means that a NCSYM transforms into a NCOS under the $SL(2,\mathbb{Z})$ transformation. Finally, the gravity dual of NCOS with both space-time and space-space noncommutativities remains in the same form after the $SL(2,\mathbb{Z})$ transformation. Our result implies that in general a NCOS transforms into another NCOS with different noncommutativity parameter. In the special case when the asymptotic value of the axion is a rational number, it is possible to transform a NCOS into a NCSYM, and when the asymptotic value vanishes, one can reach this conclusion by using a simple S-duality alone.

As mentioned before, the thermodynamic quantities (3.4) are independent of the parameter angles $\varphi$ and $\theta$. This implies that the thermodynamics is the same for three different theories, SYM, NCSYM, and NCOS, in this supergravity approximation. Also the thermodynamics remains unchanged under the $SL(2,\mathbb{Z})$ transformation, since the latter does not change the form of Einstein metric.

From the solution (3.2) we see that when $\theta = \pi/2$ and $\varphi$ is arbitrary, the bound state solution goes to that for the (F1, D1) bound state. Taking the critical electric field limit, one has a 2-dimensional NCOS from the bound state (F1, D1) [14, 22]. From our previous discussions on the $SL(2,\mathbb{Z})$ transformation, one can see that in general the 2-dimensional NCOS becomes another 2-dimensional NCOS with different noncommutativity parameters, as in the case of 4 dimensions. When the asymptotic value of the axion is a rational number, it is possible to transform the 2-dimensional NCOS into an ordinary 2-dimensional SYM with quantized electric flux [22]. Of course, when the asymptotic value of the axion vanishes, one can transform the 2-dimensional NCOS into a 2-dimensional SYM only by using S-duality, which is precisely the case in Ref. [22].

An interesting extension of our work is to consider more general solutions like (F1, D1, NS5, D5) bound states and their scaling limits. We suspect that in the bound state, SYM, NCSYM, NCOS and the so-called little string theory are connected with each other through the $SL(2,\mathbb{Z})$ transformation. This conjecture is currently under investigation.
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