A Realization of $N = 1 \mathcal{SW}(3/2, 2)$ Algebras

with Wolf Spaces

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Abstract

We find out that some unitary minimal models of the $N = 1 \mathcal{SW}(3/2, 2)$ superconformal algebra can be realized as the level one coset models based on the Wolf spaces $SU(n)/(SU(n - 2) \times SU(2))$. We obtain the expression of the fermionic current with the conformal weight $5/2$ in the algebra. Then, these models are twisted to give the topological conformal field theories.
1 Introduction

Study of representations of super $\mathcal{W}$ algebras is one of fundamental problems in two-dimensional conformal field theories. The super $\mathcal{W}$ algebras would have their applications to provide an exact treatment of the superstring compactifications. Then we need a systematic understanding of the representations. At present such study is quite difficult to overcome, but Gepner and Noyvert [1] studied a construction of the unitary representations of the simplest $N = 1$ super $\mathcal{W}$ algebra, $\mathcal{SW}(3/2, 2)$ [2, 3]. This super $\mathcal{W}$ algebra was expected to play a fundamental role in the string compactifications on $Spin(7)$ holonomy manifolds [4]. Meanwhile, there was much progress on the study of the string compactifications on exceptional holonomy manifolds [5, 6, 7, 8].

It is well known that topological aspects of $N = 2$ superconformal models are often relevant to geometrical information on Calabi-Yau manifolds. Shatashvili and Vafa [4] suggested a construction of the topological twisting of the $c = 12$ $\mathcal{SW}(3/2, 2)$ algebra, and its implication to the conjectured mirror symmetry for $Spin(7)$ holonomy manifolds. Their proposal is quite intriguing by itself because it hints generic structure of the topological counterparts of the $N = 1$ superconformal field theories based on the super $\mathcal{W}$ algebras. However, nobody has succeeded to provide an exact proof on the topological twisting. Motivated by this problem, we arrived at an observation on the $N = 1$ $\mathcal{SW}(3/2, 2)$ algebras with the central charges different from $c = 12$. The result arose from following two facts.

Let us begin with the result in [1] motivated by [9]. The $N = 1$ $\mathcal{SW}(3/2, 2)$ algebra consists of the two commuting Virasoro algebras, and the fermionic spin $3/2, 5/2$ currents. Then, we impose that one of the stress tensors should yield the $N = 0$ minimal model. This tightly restricts possible central charges of the algebra. The allowed values of the central charge are summarized in two discrete series:

$$c_p^{(1)} = 6 - \frac{18}{p + 1}, \quad c_p^{(2)} = 6 + \frac{18}{p},$$

where $p$ is an integer such that $p \geq 3$. I.e., $3/2 \leq c_p^{(1)} \leq 6$ and $6 \leq c_p^{(2)} \leq 12$. At these values, we have the unitary minimal model which contain the representation of the $N = 0$ minimal model of $c = 1 - 6/(p(p + 1))$ as the subrepresentation of that of the $N = 1$ algebra. Then, the unitary model of $c_p^{(1)}$ has the spin $3/2$ supercurrent which is primary with the conformal weight $h_{1,2}$ under the Virasoro current of the $N = 0$ minimal model. The conformal weights of the $N = 0$ minimal model are given by $h_{r,s} = ([rp - s(1 + p)]^2 - 1)/4p(1 + p)$, $(r = 1 \ldots p, \ s = 1 \ldots p - 1)$. Here, the point is that the fermionic spin $5/2$ current of the $N = 1$ algebra is descendant with respect to the spin $3/2$ supercurrent under the stress tensor of the $N = 0$ minimal model. Furthermore, it is possible to reconstruct the whole $N = 1$ $\mathcal{SW}(3/2, 2)$ algebra from these facts in the $N = 0$ minimal model. Closure of the operator product expansions of the $N = 1$ algebra determines the value of the central charge of the $N = 1$ algebra. Subsequently, the unitary representations of the $N = 1$ algebra were determined.

The other ingredient is the result in [10]. We noticed that the central charge $c_p^{(1)}$
coincides with the central charge of the level one coset model based on the Wolf space $G/(H \times SU(2))$ [10]:

\[ c = 6 - \frac{18}{g + 1}, \]

where $g$ is the dual Coxeter number of the Lie group $G$. In fact, we found out that the construction of the two Virasoro currents and the $N = 1$ spin $3/2$ superconformal current of the coset model is the same as that of the $N = 1$ $\mathcal{SW}(3/2, 2)$ algebra in [1]. This observation enabled us to write down the remaining fermionic spin $5/2$ current of the $N = 1$ $\mathcal{SW}(3/2, 2)$ algebra through the coset realization. We will provide the explicit expression in the case with $G = SU(n)$, but it should be straightforward to do with the other choices of $G$. Furthermore, we arrived at topological conformal field theories of the $N = 1$ $\mathcal{SW}(3/2, 2)$ algebra in the same way as [10, 11]. The BRST-exact expression of the twisted stress energy tensor of the topological theories follows from the $c = 0$ coset models of $(G \times G)/G$. The BRST charge is given by the double contour integral of the two spin $3/2$ supercurrents. We expect that the structure of the topological twisting might shed some light on the suggestion in [4].

Along these lines, we found out that the coset realization of the $N = 1$ $\mathcal{SW}(3/2, 2)$ algebras via the Wolf spaces $SU(n)/(SU(n - 2) \times SU(2))$. Then, we were led to the topological conformal field theories. We will state these observations in section 2. We will also include a relation of the $N = 1$ coset models to the $N = 2$ coset models. Section 3 will include conclusions and future problems arising from this note.

2 Construction of the $N = 1$ $\mathcal{SW}(3/2, 2)$ Algebras

We begin with the level one $SU(n)$ WZW theory using the Coulomb gas representation with the $n - 1$ free scalar fields $\phi = (\phi_1, \phi_2, \ldots, \phi_{n-1})$, and choose its Coulomb gas parameters as:

\[ \alpha_+ = \sqrt{\frac{n+1}{n}}, \quad \alpha_- = -\frac{1}{\alpha_+}, \quad \alpha_0 = \alpha_+ + \alpha_- = \frac{1}{n\alpha_+}. \]

We also introduce the highest root and the Weyl vector of $SU(n)$:

\[ \theta = \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1}, \quad \rho_{SU(n)} = \sum_{i=1}^{n-1} \frac{i(n-i)}{2} \alpha_i, \]

where $\alpha_i$ ($i = 1, \ldots, n - 1$) are the simple roots of the $SU(n)$.

Let us define the $SU(2)$ subgroup of $SU(n)$ by $E_\theta, E_{-\theta}$ and $[E_\theta, E_{-\theta}]$ where $E_\theta, E_{-\theta}$ are the lowering, raising operators. This gives the subgroup $SU(n - 2)$ whose root vectors

\[ c_p(2) \]

may be obtained by substituting the level of the current algebra into negative values formally. We will not discuss this case in this note.
span the orthogonal subspace to $\theta$ in the root space of $SU(n)$. Then, we obtain the Wolf space $SU(n)/(SU(n-2) \times SU(2))$ which is a symmetric space with a quaternionic structure [12]. Let us decompose the Weyl vector according to the Wolf space:

$$\rho_{SU(2)} = -\frac{\theta}{2}, \quad \rho_{SU(n)} = \rho_{SU(n-2)} + (1-n)\rho_{SU(2)},$$

which leads to the $N = 1$ superconformal model of $c = 6 - 18/(n+1)$ [10]. Now, we will arrive at the $N = 1 \mathcal{SW}(3/2, 2)$ algebra. Let us recall that the generating currents of the algebra consist of the following four currents [2, 3]: the two bosonic spin 2 stress tensors $T_{N=1}, T_{SU(2)}$, and the fermionic spin 3/2, 5/2 currents $G, U$. The stress tensors are written down as

$$T_{N=1}(z) = -\frac{1}{2} (\partial \phi)^2(z) + i\alpha_0 (\rho_{SU(n-2)} + \rho_{SU(2)}) \partial^2 \phi(z),$$

$$T_{SU(2)}(z) = -\frac{1}{2} \left( \sqrt{2} \rho_{SU(2)} \partial \phi \right)^2(z) + \frac{i}{\sqrt{2}} \alpha_0 \left( \sqrt{2} \rho_{SU(2)} \partial^2 \phi \right)(z).$$

Comparing with the construction in [1], $T_{N=1}$ (resp. $T_{SU(2)}$) is the stress tensor of the whole $N = 1 \mathcal{SW}(3/2, 2)$ algebra (resp. the $N = 0$ minimal model). The fermionic currents $G, U$ turn out to be

$$G(z) = \psi_{SU(2)} \left( \psi_{SU(n-2),1} + \psi_{SU(n-2),2} \right)(z),$$

$$= e^{i\alpha_0 + \alpha_1 \phi(z)} + e^{i\alpha_0 + \alpha_2 \phi(z)},$$

$$U(z) = -\frac{2}{3} h_{1,2} \psi_{SU(2)} \partial(\psi_{SU(n-2),1} + \psi_{SU(n-2),2})(z)$$

$$+ \left( 1 - \frac{2}{3} h_{1,2} \right) \left( \partial \psi_{SU(2)} \right)(\psi_{SU(n-2),1} + \psi_{SU(n-2),2})(z),$$

where $h_{1,2} = (n+3)/4n$ is the conformal weight in the Kac’s table of the $N = 0$ minimal model. Here, we have introduced the following vertex operators

$$\psi_{SU(2)} = e^{-i\alpha_0 \rho_{SU(2)} \phi},$$

$$\psi_{SU(n-2),1} = e^{i\alpha_0 - \alpha_2 \phi(z)} \phi, \quad \psi_{SU(n-2),2} = e^{i\alpha_0 - \alpha_2 \phi(z)} \phi.$$

The complete expression of the currents of the $N = 1 \mathcal{SW}(3/2, 2)$ algebra of $c_{n}^{(1)} = 6 - 18/(n+1)$ through the Coulomb gas representation based on the Wolf space $SU(n)/(SU(n-2) \times SU(2))$ is the main result in this note.

In fact, the coset models have their topological counterparts [10, 11]. The topological stress tensor is defined as

$$T_{c=0}(z) = T_{N=1}(z) - i\alpha_0 n \rho_{SU(2)} \partial^2 \phi(z),$$

$$= -\frac{1}{2} (\partial \phi)^2(z) + i\alpha_0 \rho_{SU(n)} \partial^2 \phi(z).$$
This topological stress tensor can be written in the BRST-exact form

\[ T_{c=0}(z) = \{Q_{\text{BRST}}, e^{-i\alpha_+ \theta \phi(z)} \}. \]

Here, the BRST charge \( Q_{\text{BRST}} \) is defined as the double contour integral of the two spin 3/2 superconformal currents \( G \) using a suitable path of the integration

\[ Q_{\text{BRST}} = \int \int dz\, dw\, G(z)G(w), \]

and satisfies the nilpotency: \( Q_{\text{BRST}}^2 = 0 \). It is possible to prove this nilpotency by checking poles in the integral. The BRST-exactness of the topological stress tensor is also proven by inserting the screening charge \( h_{c=0} \) of the fermionic current \( G \) (resp. \( U \)) under the topological stress tensor \( T_{c=0} \) is an integer, i.e., one (resp. two). In [4], the fermionic spin 5/2 current \( U(z) \) is suggested to be BRST-equivalent to the anti-ghost field \( e^{-i\alpha_+ \theta \phi(z)}: U(z) = e^{-i\alpha_+ \theta \phi(z)} \{Q_{\text{BRST}}, \ast\} \). We do not understand a role of the current \( U(z) \) in the present topological model.

Finally, we discuss that the \( N = 1 \) superconformal coset models have a relation to the standard \( N = 2 \) superconformal coset models [13] (The same argument was given in [14].). Let us explain it with the Wolf space \( SU(4)/(SU(2) \times SU(2)) \). There is the cyclic \( \mathbb{Z}_4 \) symmetry in the root system of the affine Lie algebra \( \hat{su}(4) \): \( (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \rightarrow (\alpha_3, \alpha_0, \alpha_1, \alpha_2) \) (The reader should not confuse the root \( \alpha_0 \) with the Coulomb gas parameter.). Then, the \( N = 1 \) superconformal currents \( T^{N=1}, G^{N=1} \) are mapped into the \( N = 2 \) superconformal currents \( T^{N=2}, G^{\pm, N=2} \) of the coset model of the symmetric space \( SU(4)/(SU(2)^2 \times U(1)) \):

\[
T^{N=1}_{SU(4)/(SU(2) \times SU(2))} = -\frac{1}{2} (\partial \phi)^2 + i\alpha_0 \left( -\frac{\alpha_0 + \alpha_2}{2} \right) \partial^2 \phi,
\]

\[
= -\frac{1}{2} (\partial \phi)^2 + i\alpha_0 \left( \frac{\alpha_1 + \alpha_3}{2} \right) \partial^2 \phi,
\]

\[
= T^{N=2}_{SU(4)/(SU(2)^2 \times U(1))},
\]

\[
G^{N=1}_{SU(4)/(SU(2) \times SU(2))} = e^{i\alpha_+ \alpha_0 \phi} + e^{i\alpha_+ \alpha_2 \phi},
\]

\[
= e^{-i\alpha_+ \theta \phi} + e^{i\alpha_+ \alpha_2 \phi},
\]

\[
= G^{\pm, N=2}_{SU(4)/(SU(2)^2 \times U(1))} + G^{+,- N=2}_{SU(4)/(SU(2)^2 \times U(1))}.
\]

Here, the part of the vertex operators with the imaginary root \( \delta = \sum_{i=0}^{n-1} \alpha_i \) can be ignored in the operator product expansions of these currents due to the orthogonality: \( (\delta, \alpha_i) = 0 \).

Further, it is possible to write down the \( U(1) \) current of the \( N = 2 \) model

\[
J^{N=2} = 2i\alpha_0 \left( \rho_{SU(4)} - \rho_{SU(2)^2} \right) \partial \phi.
\]

In this way, we arrive at the relation of the \( N = 1 \) coset models to the \( N = 2 \) coset models of the \( SU(n) \) group with the \( SU(2) \) factor in the denominator. Here, we do not have a role of the fermionic current \( U \) in the \( N = 2 \) coset models. Alternatively, it is straightforward to obtain another class of the \( N = 2 \) coset models. For example, to obtain the coset model of the symmetric space \( SU(4)/(SU(3) \times U(1)) \), we delete the simple roots \( \alpha_1, \alpha_2 \) of \( SU(4) \) in the construction of [10].
3 Conclusion and Outlook

We pointed out that the \( N = 1 \) superconformal \( SW(3/2, 2) \) algebras with the specific central charge are realized by the Wolf spaces \( SU(n)/(SU(n - 2) \times SU(2)) \). Then, we discussed some consequences of this observation: the existence of the topological conformal field theories and the relation to the \( N = 2 \) superconformal coset models. Through the realization, we confirmed the existence of the topological conformal field theories based on the \( N = 1 \) super-W algebras.

Based on the result in this note, a lot of things should be considered:

(1) It is desirable to analyze in detail the spectrum in the \( N = 1 \) superconformal coset models. We will dwell only on the states in the NS sector of the \( c = 3/2 \) \((p = n = 3)\) model. The primary fields in the NS sector are labelled by the conformal weights \( h, a \) under the stress tensors \( T_{N=1}, T_{SU(2)} \). Then, we arrive at the vertex operators \( 1, e^{i\alpha \alpha_1 \phi}, e^{i\alpha_0 (2\alpha_1 + 2\alpha_2) \phi} \). Here, we have imposed that the conformal weight \( a \) should exist in the Kac’s table of the Ising model of the stress tensor \( T_{SU(2)} \). We write these states as \( |0, 0\rangle, |1/16, 1/16\rangle, |1/2, 0\rangle \) in the way that the state with the conformal weights \( h, a \) is denoted by \( |a, h - a\rangle \) (In fact, these states coincide with those of the minimal model of \( c^{(1)}_{p=3} = 3/2 \) in [1]). Meanwhile, when we move into the topological conformal field theory, these states turn out to be the BRST invariant states in the sense that \( Q_{BRST} |\Psi\rangle = 0 \) where \( |\Psi\rangle \) is the state corresponding to the vertex operator. Subsequently, we should proceed to the highest weight states in the Ramond sector of the \( N = 1 \) coset model. Then, it would be very interesting to consider a geometrical interpretation of the spectrum through the Wolf space. It also might be of use to visit the gauged WZW model on the Wolf space [15].

(2) It would be very interesting to clarify a relation between the present result and the result in [1]. The unitary spectrum in [1] has the continuous parts. The \( N = 1 \) coset model and the \( \mathbb{Z}_2 \) orbifold of the coset model of the \( N = 2 \) super \( W_3 \) algebra [9] should provide the special realization of the continuous spectra. Then, this might suggest a new kind of mirror symmetry because it is likely that the Wolf spaces and the orbifolds of the hermitian symmetric spaces are geometrically different even if the corresponding \( N = 1 \) superconformal field theories are the same.

(3) Extension of the result into the Lie groups other than \( G = SU(n) \) is of interest (see [11]). Along the quaternionic structure of the Wolf spaces, it would be interesting to consider the extension with \( G = Sp(n) \). Then, it is natural to ask how to obtain the coset realization of the \( N = 1 \) super-W algebra through a Hamiltonian reduction along the well-known story in the \( N = 2 \) coset model [16, 3]. In a related direction, the \( N = 1 \) \( SW(3/2, 3/2, 2) \) algebra was studied using the \( (SU(2) \times SU(2))/SU(2) \) coset model motivated by an application to the string compactifications on \( G_2 \) holonomy manifolds [7]. The result might be understood through a Hamiltonian reduction as in [17].
We hope to report further developments on these problems elsewhere.

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