ASSOCIATIVITY AND OPERATOR HAMILTONIAN QUANTIZATION OF GAUGE THEORIES

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ABSTRACT. We show that the associative algebra structure can be incorporated in the BRST quantization formalism for gauge theories such that extension from the corresponding Lie algebra to the associative algebra is achieved using operator quantization of reducible gauge theories. The BRST differential that encodes the associativity of the algebra multiplication is constructed as a second-order quadratic differential operator on the bar resolution.

1. INTRODUCTION

1.1. BRST quantization methods for constrained systems (synonymously, gauge theories) [1,2,3] are recognized as a powerful approach reaching beyond the contexts in which it was originally created. In the Hamiltonian formalism, the BRST quantization of a first-class constrained system amounts to constructing an odd differential (the “BRST” operator) in the constrained system extended by ghosts and the conjugate momenta (which are recognized as Koszul–Tate variables at the classical level [4]). But the formalism has been limited to algebraic structures built on
(graded) antisymmetric operations (generated by commutators or Poisson brackets).

In this paper, we show that the Hamiltonian BRST formalism is also applicable to graded associative algebras and is therefore not limited to Lie-like structures. For a given associative algebra $A$, we construct a differential $\Omega$ such that the relation $\Omega^2 = 0$ is a “BRST encoding” of the associativity of multiplication in $A$.

The construction has a “trivial” part, the associated (graded) Lie algebra $(A, [~,~])$, which is treated by the standard BRST methods, and a “difficult” part, the extension of the BRST scheme to the associative algebra. This extension is constructed using the machinery of reducible gauge theories [5], with the bar resolution of $A$ viewed as the data defining a reducible gauge theory. In accordance with the BRST ideology, we introduce ghosts for each term in the bar resolution, quantize them, and seek the BRST differential $\Omega$ of ghost number 1; we also require $\Omega$ to be at most quadratic in the ghosts and at most bilinear in the momenta, with no individual momentum being squared in $\Omega$. The BRST differential is therefore a quadratic second-order differential operator on the bar resolution; moreover, its purely quadratic part is an operator with scalar coefficients. The exact statement is formulated in 3.4 below.

1.2. Although our construction of the differential $\Omega$ is motivated by BRST methods, similarities with the known BRST formalism for reducible gauge theories are somewhat limited because the bar resolution is infinite, and the corresponding reducible gauge theory is therefore of an infinite reducibility degree. Despite some effort [6, 5, 7], additionally motivated by possible string theory applications [8], infinitely reducible gauge theories have not been given a complete formulation that would allow proving the existence of the BRST differential (or a solution to the master equation in the Lagrangian formulation). The standard inductive argument applicable to finitely reducible gauge theories fails in the infinitely reducible case, and we must therefore independently prove the existence in the associative algebra setting; for this, we construct a recursive procedure that yields a solution of the equation $\Omega^2 = 0$. This also bypasses another complication: in contrast to “genuine” gauge theories, where first-class constraints satisfy the so-called involution relations (which become Poisson-bracket relations as $\hbar \to 0$)

\[ [T_\alpha, T_\beta] = i\hbar \sum_\gamma U^\gamma_{\alpha\beta} T_\gamma \]

with some operators $U^\gamma_{\alpha\beta}$, the commutators in $A$ do not involve a Planck constant $\hbar$. In other words, the Planck constant is equal to 1 (in fact, to $-i$) in the formalism
proposed here. Hence, there is no classical limit of the corresponding gauge theory, and the BRST formalism must be applied directly at the operator level. Another consequence of ℏ being equal to 1 is a problem (which we entirely ignore) in interpreting infinite series that are no longer formal series in ℏ.

1.3. The paper is organized as follows. For completeness, we recall basic facts about reducible gauge theories in Sec. 2. In 3.1 we explain the relation between Sec. 2 and the main part of Sec. 3. We introduce ghosts and construct a differential in 3.3 as the BRST operator in a specific (infinitely) reducible gauge theory. The main result that Ω² = 0 is formulated in 3.4 and is proved in 3.5–3.7. Several additional remarks are collected in Sec. 4.

2. REDUCIBLE CONSTRAINED SYSTEMS

In this section, we summarize the main points of the Hamiltonian BRST quantization of constrained systems. We use it as a motivation for what follows, even though direct application of the basic BRST formalism to the associative algebra case is hindered by the infinite reducibility and the absence of the classical limit. The actual link between the BRST formalism and our main construction is explained in 3.1.

2.1. Classical reducible constrained systems. A classical reducible constrained system [5] consists of

– a symplectic manifold X (in fact, a symplectic vector space) and (sufficiently smooth) functions T_α₀, α₀ ∈ I₀, on X, called constraints, whose zero locus is called the constraint “surface” S ⊂ X and is viewed by physicists as something very close to a smooth manifold,
– the functions Z^α_α−1, α_i ∈ I_i, on X satisfying the rank assumption and “zero-mode” equations (2.1) and (2.2) given below.

We now consider these ingredients in more detail. If the given constraints are linearly independent over C∞(X), the theory is said to be irreducible; if they are not, there exist functions Z^α_α, α₁ ∈ I₁, on X such that

\[ \sum_{α₀ ∈ I₀} Z^α_{α₀} T_{α₀} = 0. \]

(2.1)

The functions Z^α_{α₀} can in turn be linearly dependent over C∞(X), which gives rise to Z^α_{α₁}, and so on. More precisely, each subsequent linear dependence is only
required in a “weak” form, i.e., modulo the ideal \( \mathcal{J} \) generated by \( \{ T_{\alpha_0} \} \),

(2.2) \[ \sum_{\alpha_n}^{n} Z_{\alpha_n}^{\alpha_{n-1}} Z_{\alpha_n}^{\alpha_{n-2}} \in \mathcal{J}, \quad \alpha_i \in \mathcal{I}_i, \quad n \geq 2 \]

(we often omit top labels \( n \) from the notation in what follows). Upon restriction to the constraint surface, this gives a complex, which is exact by definition.\(^1\) With each \( Z_{\alpha_n}^{\alpha_{n-1}} \) viewed as a rectangular matrix, the ranks of their restrictions to \( S \) must therefore satisfy

\[
\text{rank}( Z_{\alpha_n}^{\alpha_{n+1}} ) \bigg|_S + \text{rank}( Z_{\alpha_n}^{\alpha_{n-1}} ) \bigg|_S = \text{card} \mathcal{I}_n, \quad n \geq 1,
\]

and in addition,

\[
\text{rank}( Z_{\alpha_0}^{\alpha_1} ) \bigg|_S + \text{rank}( \frac{\partial T_{\alpha_0}}{\partial x^i} ) \bigg|_S = \text{card} \mathcal{I}_0,
\]

where \( \{ x^i \} \) is any local coordinate system in a neighborhood of \( S \) in \( X \). These ranks are assumed to be constant in some neighborhood of \( S \) in \( X \) \([5, 3]\). We generally refer to \( Z \) as “zero modes.”

2.1.1. Definition. A constrained system is said to be \( \ell \)-reducible if \( \ell + 1 \) \( Z = 0 \), but \( \ell \neq 0 \). In particular, a \( 0 \)-reducible theory is irreducible (relations (2.1) are already absent).

2.1.2. A constrained system can be extended by auxiliary variables, called ghosts and the conjugate momenta, such that there exists an odd Hamiltonian vector field \( \{ \Omega, - \} \) (the BRST operator) whose square is zero and the lowest terms in the expansion of \( \Omega \) in the ghosts involve the constraints and the zero modes \( Z \). To avoid repetition, we consider the details in the quantum setting.

2.2. Quantum reducible constrained systems. The mathematically rigorous existence of the quantum theory is a subtle issue, and the following statements may be sensitive to the chosen quantization. Deformation quantization \([9, 10, 11]\) alone does not automatically allow speaking of operator relations, but we proceed in terms of these to recapitulate the basic gauge-theory folklore.

\(^1\)The entire complex, including a specific choice of \( Z \), makes part of the definition of a reducible constrained system (reducible gauge theory): although the exact sequence furnished by \( \check{Z} \) splits in physical applications, a given splitting is not canonical and in realistic theories, moreover, typically violates some important symmetries (e.g., the Lorentz covariance) or locality.
2.2.1. 
Quantum-mechanically, $T_{\alpha_0}$ and $Z_{\alpha_{n-1}}$ become operators, i.e., elements of an algebra $\mathbb{A}^\hbar$. Equation (2.1) then retains its form in terms of elements of $\mathbb{A}^\hbar$, and Eqs. (2.2) become

\[
\sum_{\alpha_{n-1}} Z_{\alpha_{n-1}}^{\alpha_{n-2}} Z_{\alpha_{n-1}}^{\alpha_0} = \sum_{\alpha_0} \Pi_{\alpha_{n-20}} T_{\alpha_0} + \hbar A^\hbar, \quad \alpha_i \in I_i,
\]

with some $\Pi_{\alpha_{n-20}} \in \mathbb{A}^\hbar$ (the relations $Z \cdot Z \equiv 0 \mod I$ are therefore reproduced only as $\hbar \to 0$).

2.2.2. Ghost content. 
The algebra $\mathbb{A}^\hbar$ is extended to $\mathbb{A}_{\text{gh}}^\hbar$ by a set of operators $\{C_A\}$ (ghosts) and $\{\bar{P}_A\}$ (conjugate momenta) satisfying the canonical graded commutation relations

\[
[\bar{P}_B, C^A] \equiv \bar{P}_B C^A - (-1)^{\varepsilon(\bar{P}_B)\varepsilon(C^A)} C^A \bar{P}_B = i\hbar \delta^A_B.
\]

Here,

- $A$ is a collection of (multi)indices, $A = \{\alpha_0, \alpha_1, \ldots, \alpha_\ell\}$, $\alpha_i \in I_i$, and the ghosts are therefore a collection $\{C^A\} = \{C^{\alpha_0}, \ldots, C^{\alpha_\ell}\}$, $\alpha_i \in I_i$, and similarly for the momenta, $\{\bar{P}_B\} = \{\bar{P}_{\beta_0}, \ldots, \bar{P}_{\beta_\ell}\}$, $\beta_i \in I_i$.
- $\mathbb{Z}_2$-gradings of the ghosts and the momenta are

\[
\varepsilon(C^{\alpha_0}) = \varepsilon(\bar{P}_{\alpha_0}) = \varepsilon_{\alpha_0} + n + 1;
\]

- ghost-number assignments for the ghosts and the momenta are

\[
\text{gh } C^{\alpha_0} = n + 1, \quad \text{gh } \bar{P}_{\alpha_0} = -n - 1.
\]

The following statement gives an operational tool for the Hamiltonian quantization of constrained systems (see comments in 2.2.4 below).

2.2.3. Theorem (Batalin–Fradkin). 
In a quantum $\ell$-reducible constrained system, there exists a 2-nilpotent odd operator $\Omega$ with ghost number 1 (the BRST differential),

\[
\varepsilon(\Omega) = 1, \quad \text{gh } \Omega = 1, \quad \Omega^2 = 0,
\]

of the form

\[
\Omega = \sum_{n \geq 1} \sum_{m \geq 0} \sum_{A_1, \ldots, A_m} C^{A_m} \ldots C^{A_1} V_{A_1 \ldots A_m}^{B_1 \ldots B_n} \bar{P}_{B_1} \ldots \bar{P}_{B_n} \in \mathbb{A}_{\text{gh}}^\hbar,
\]

where $V_{A_1 \ldots A_m}^{B_n \ldots B_1}$ are operators in $\mathbb{A}^\hbar$ such that

\[
V_{\alpha_0} = T_{\alpha_0}, \quad V_{\alpha_{n-1}}^{\alpha_{n-1}} = Z_{\alpha_{n-1}}^{\alpha_{n-1}},
\]
and as $\hbar \to 0$, $T^\alpha_0$ and $\bar{Z}^\alpha_{n-1}$ become the respective data specifying a classical $\ell$-reducible gauge theory and $\Omega$ becomes an odd function satisfying the Poisson-bracket relation $\{\Omega, \Omega\} = 0$ (with the Poisson bracket between the ghosts and the momenta determined by the above commutator).

**Remarks.**

2.2.4. The theorem claims the existence of the operators $V^B_{A_1...A_m}$ in addition to those fixed by “boundary conditions” (2.5). In other words, given the constraints and the $\bar{Z}$ operators, a nilpotent $\Omega$ with the ghost number 1 can be constructed as a formal series in the ghosts and momenta starting with the lower-order terms

$$\Omega = \sum_{\alpha_0} C^\alpha_{\alpha_0} T^\alpha_{\alpha_0} + \ell \sum_{n=1} \sum_{\alpha_n, \alpha_{n-1}} C^{\alpha_n} \bar{Z}^\alpha_{n-1} \bar{P}^\alpha_{n-1} + \ldots.$$  

(2.6)

The existence of a quantum BRST operator given by a series in the ghosts has never been proved explicitly, however. Its classical counterpart is known to exist if $\mathcal{X}$ is a symplectic linear space and the homology of the Koszul–Tate operator is concentrated in ghost number zero (this is the actual significance of the rank assumptions). The classical BRST operator can then be subjected to deformation quantization, and its quantization can be shown to exist in terms of symbols (formal series in $\hbar$) if the classical operator $\{\Omega, -\}$ has trivial homology in ghost number 1.

2.2.5. The coefficients in (2.4) are usually redefined as

$$V^B_{A_1...A_m} = (-1)^{E^B_{A_1...A_m}} \frac{1}{m!n!} U^B_{A_1...A_m},$$

where the sign factors defined by

$$E^B_{A_1...A_m} = \left[ \frac{\left[ \begin{array}{c} m \\ 2 \end{array} \right]}{2} \right] \varepsilon(C^{A_{2k}}) + \left[ \frac{\left[ \begin{array}{c} n \\ 2 \end{array} \right]}{2} \right] \varepsilon(\bar{P}^{B_{2k}})$$

$$+ \left[ \frac{\left[ \begin{array}{c} m \\ 2 \end{array} \right]}{2} \right] \max gh(C^{A_1}, \ldots, C^{A_m}) + \left[ \frac{\left[ \begin{array}{c} n \\ 2 \end{array} \right]}{2} \right] \max gh(\bar{P}^{B_1}, \ldots, \bar{P}^{B_n})$$

(with $\left[ \right]$ denoting the integer part) are chosen such that

$$U^B_{A_1...A_k, A_{k+1}...A_m} = -(-1)^{(\varepsilon_{A_k}+1)(\varepsilon_{A_{k+1}}+1)} U^B_{A_1...A_k, A_{k+1}A_{k+1}A_{k+1}...A_m},$$

$$U^B_{A_1...A_m} = -(-1)^{(\varepsilon_{B_k}+1)(\varepsilon_{B_{k+1}}+1)} U^B_{A_1...A_m},$$

where $\varepsilon_{A_k} = \varepsilon(C^{A_k})$. This minimizes the number of explicit sign factors when the relations generated by Eqs. (1.1) and the reducibility relations are written in terms of $U^B_{A_1...A_k, A_{k+1}...A_m}$; (inevitable) sign factors then occur only in cyclic-permutation sums.
2.2.6. In what follows, we specialize to the case where \( \varepsilon_{\alpha_i} = 0\), and therefore the \( \mathbb{Z}_2\)-gradings of the ghosts are
\[
\varepsilon(C^{\alpha_i}) = i + 1, \quad i = 0, \ldots, \ell,
\]
and all \( U^{B_1 \ldots B_l}_{A_1 \ldots A_m} \) are even, \( \varepsilon(U^{B_1 \ldots B_l}_{A_1 \ldots A_m}) = 0 \). The lower equations following from \( \Omega \Omega = 0 \) are given by involution relations (1.1), “zero-mode” conditions (2.1), and (with summation over repeated indices understood)
\[
\begin{align*}
(i\hbar)^{-1} & \left[ T_{\alpha_0}, Z_{\alpha_1}^{\alpha_0} \right] - Z_{\alpha_1}^{\gamma_0} U_{\gamma_0 \alpha_0}^{\beta_0} = -U_{\alpha_0 \alpha_1}^{\gamma_0 \beta_0} \Pi_{\gamma_0 \delta_0}^{\beta_0 \delta_0} - U_{\alpha_0 \alpha_1}^{\beta_0} Z_{\delta_0}^{\beta_0}, \\
2\Pi_{\alpha_0 \beta_0}^{\gamma_0} & = T_{\alpha_0 \delta_0}^{\gamma_0} - i\hbar U_{\alpha_0 \beta_0}^{\gamma_0},
\end{align*}
\]
(2.7)
\[
(i\hbar)^{-1} \left[ Z_{\alpha_0}^{\alpha_0}, Z_{\alpha_1}^{\beta_0} \right] - Z_{\alpha_1}^{\gamma_0} U_{\gamma_0 \alpha_0}^{\alpha_0 \beta_0} = -U_{\alpha_0 \alpha_1}^{\gamma_0 \beta_0} \Pi_{\gamma_0 \delta_0}^{\beta_0 \delta_0} - U_{\alpha_0 \alpha_1}^{\alpha_0 \beta_0} Z_{\delta_0}^{\beta_0} \\
& \quad + U_{\alpha_0 \alpha_1}^{\gamma_0 \beta_0} U_{\alpha_2}^{\alpha_2 \delta_0} + \frac{i\hbar}{2} U_{\alpha_0 \alpha_1}^{\gamma_0 \beta_0} U_{\alpha_0 \delta_0}^{\alpha_0 \beta_0},
\]
(2.8)
\[
(i\hbar)^{-1} \left[ Z_{\alpha_1}^{\alpha_0}, Z_{\beta_1}^{\beta_0} \right] - Z_{\alpha_1}^{\gamma_0} U_{\gamma_0 \alpha_0}^{\alpha_0 \beta_0} = -U_{\alpha_1 \beta_0}^{\gamma_0 \beta_0} \Pi_{\gamma_0 \delta_0}^{\beta_0 \delta_0} - U_{\alpha_1 \beta_0}^{\beta_0 \gamma_0} Z_{\delta_0}^{\beta_0},
\]
(2.9)
\[
\begin{align*}
\sum_{A} C^{A} & \tau_{A} + \frac{1}{2} \sum_{A, B} (-1)^{\varepsilon(C^{B})+1} C^{B} C^{A} U_{AB},
\end{align*}
\]
(2.11)

2.3. Reducible closed-algebra theories.

2.3.1. Definition. First-class constraints \( T_{\alpha_0}, \alpha_0 \in I_0 \), are said to generate a closed gauge algebra if \( \Omega \) is at most quadratic in the ghosts.

The issue of a “closed algebra” is more subtle in the Hamiltonian than in Lagrangian BRST formalism. In what follows, we consider closed-algebra theories satisfying an additional assumption that the BRST differential \( \Omega \) is at most bilinear in the momenta, with none individual momentum \( \bar{P}_{\alpha} \), entering squared (in reducible theories, this is not a consequence of the closed-algebra condition, but in all known examples where both conditions are satisfied, the corresponding gauge generators in the Lagrangian formulation generate a closed gauge algebra).

2.3.2. The “gauge theory” considered in Sec. 3 has a closed gauge algebra but is infinitely reducible, and we therefore allow sums over infinitely many ghosts. We use indices \( A, B, \ldots \) to label all ghosts (i.e., \( C^{\alpha_0}, C^{\alpha_1}, \ldots \)) in terms of 2.2.2. In a closed-algebra theory, the BRST differential can be written as
\[
\begin{align*}
\Omega & = \sum_{A} C^{A} \tau_{A} + \frac{1}{2} \sum_{A, B} (-1)^{\varepsilon(C^{B})+1} C^{B} C^{A} U_{AB},
\end{align*}
\]
(2.11)
where \( \tau_{A} \) and \( U_{AB} \) are functions of the ghost momenta \( \bar{P}_{C} \). In the notation of 2.2.2, \( \tau_{\alpha_0} = T_{\alpha_0} \) are obviously the original constraints; all \( \tau_{\alpha_n} \) with \( n \geq 1 \) are \( \bar{P}_{C} \)-
dependent. The equation $\Omega^2 = 0$ then amounts to independent vanishing conditions for the terms of the first, second, and third degree in $C^A$. Anticipating the structure of the BRST differential in Sec. 3 (where $U_{AB}$ are scalar-valued), we write these equations in the case where

\begin{align}
&\text{(2.12)}
[U_{AB}, U_{CD}] = 0 \quad \text{and} \quad [\tau_A, U_{BC}] = 0.
\end{align}

In the linear order in $C^A$, the equations are given by (with summations over repeated indices understood)

\begin{align}
&\text{(2.13)}
[\tau_A, C^B] \tau_B + \frac{1}{2}[[\tau_A, C^C], C^B] U_{BC} (-1)^{\varepsilon(C^A)+1} = 0.
\end{align}

In the quadratic and cubic orders, the respective equations are

\begin{align}
&\text{(2.14)}
[\tau_A, \tau_B] = [U_{AB}, C^C] \tau_C - [\tau_A, C^C] U_{CB} (-1)^{\varepsilon(C^B)+1}
+ [\tau_B, C^C] U_{CA} (-1)^{(\varepsilon(C^A)+1)\varepsilon(C^B)} + \frac{1}{2}[[U_{AB}, C^D], C^C] U_{CD} (-1)^{\varepsilon(C^D)+1}
\end{align}

and

\begin{align}
&\text{(2.15)}
[U_{AB}, C^D] U_{DC} (-1)^{(\varepsilon(C^A)+1)\varepsilon(C^C)+1} + \text{cycle}(A, B, C) = 0.
\end{align}

3. The BRST Differential

In this section, we construct the BRST differential $\Omega$ as a differential operator on the bar resolution of a given associative algebra $\mathcal{A}$. In Sec. 3.3, we introduce the necessary structures in the physical context, in the guise of ghosts in a reducible gauge theory. The main result that $\Omega$ is a differential is formulated in Theorem 3.4, a recursive solution to $\Omega \Omega = 0$ is constructed in Sec. 3.5, and the proof that $\Omega \Omega = 0$ extends to Sec. 3.7.

3.0. Tensor algebra preliminaries. Let $\mathcal{A}$ be an associative graded algebra with a unit. We write

\begin{align}
&\text{m}_{1,i} : \mathcal{A}^\otimes n \to \mathcal{A}^\otimes (n-1)
\quad a_1 \otimes \ldots \otimes a_n \mapsto (-1)^{\varepsilon(a_i)(\varepsilon(a_2)+\ldots+\varepsilon(a_{i-1}))} a_1 a_i \otimes a_2 \otimes \ldots \otimes a_n
\end{align}

for $2 \leq i \leq n$ and use the somewhat redundant notation $m^*_{i,1}$ (with both the transposed subscripts and the asterisk) for the reversed-order multiplication (as before, placed in the first tensor factor)

\begin{align}
&\text{m}^*_{i,1} : \mathcal{A}^\otimes n \to \mathcal{A}^\otimes (n-1)
\quad a_1 \otimes \ldots \otimes a_n \mapsto (-1)^{\varepsilon(a_i)(\varepsilon(a_1)+\ldots+\varepsilon(a_{i-1}))} a_i a_1 \otimes a_2 \otimes \ldots \otimes a_n.
\end{align}

We also use the notation

\begin{align}
&\text{ad}_{1, a'} \cdot a_0 \otimes \ldots \otimes a_n = [a', a_0] \otimes a_1 \otimes \ldots \otimes a_n,
\end{align}

\begin{align}
&\text{ad}_{a'} \cdot a_0 \otimes \ldots \otimes a_n = a' a_0 \otimes a_1 \otimes \ldots \otimes a_n + \ldots + a_0 a' \otimes a_1 \otimes \ldots \otimes a_n.
\end{align}
and let $s$ be the “right shift” mapping

$$s : \mathcal{A}^\otimes n \rightarrow \mathcal{A}^\otimes (n+1)$$

$$a_1 \otimes \ldots \otimes a_n \mapsto 1 \otimes a_1 \otimes \ldots \otimes a_n.$$  

Finally, we need the operation of moving the $i$th tensor factor to the first position,

$$P_{1,i} : a_1 \otimes \ldots \otimes a_n \mapsto (-1)^{\varepsilon(a_i)(\varepsilon(a_1)+\ldots+\varepsilon(a_{i-1}))} a_i \otimes a_1 \otimes a_2 \otimes \ldots \otimes a_{i-1} \otimes a_{i+1} \otimes \ldots \otimes a_n,$$

and the inverse operation

$$P_{i,1} : a_1 \otimes \ldots \otimes a_n \mapsto (-1)^{\varepsilon(a_1)(\varepsilon(a_2)+\ldots+\varepsilon(a_i))} a_2 \otimes \ldots \otimes a_i \otimes a_1 \otimes a_{i+1} \otimes \ldots \otimes a_n.$$  

### 3.1. The bar resolution and the “zero modes”.

We consider the bar resolution of the algebra $\mathcal{A}$ by free $\mathcal{A}$-bimodules

$$\ldots \to \mathcal{A} \otimes \bar{\mathcal{A}} \otimes \mathcal{A} \to \mathcal{A} \otimes \bar{\mathcal{A}} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \to 0,$$

where $\bar{\mathcal{A}} = \mathcal{A}/C$ and the differential is given by

$$b' a_0 \otimes a_1 \otimes \ldots \otimes a_n = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \ldots \otimes a_n$$

(in particular, $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is simply the multiplication $\mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$). This complex is contractible with the contracting homotopy given by

$$s b' + b' s = \text{id}.$$  

To establish connection with Sec. 2, we define linear mappings $Z^n : \mathcal{A}^\otimes n \rightarrow \mathcal{A}^\otimes n$ as

$$Z^n = b' s.$$  

They are to be viewed as the (quantum) $\hat{Z}$ mappings in Sec. 2. Explicitly,

$$Z^2 a \otimes b = a \otimes b - 1 \otimes ab,$$

$$Z^3 a \otimes b \otimes c = a \otimes b \otimes c - 1 \otimes ab \otimes c + 1 \otimes a \otimes bc,$$

and so on. For future use, we also introduce the notation

$$Z^j_{1,i} = P_{1,i} \circ (\text{id}^\otimes (i-1) \otimes Z^j) : \mathcal{A}^\otimes (i+j-1) \rightarrow \mathcal{A}^\otimes (i+j-1).$$

Obviously, $m \circ Z^2 = 0$; in fact, there is the exact sequence

$$0 \to \mathcal{A} \otimes \bar{\mathcal{A}} \to \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \to 0.$$  

To obtain a similar vanishing statement for higher $Z^n$, we compose $Z^n$ and $Z^{n-1}$ by “padding” the lower one from the left with the identity mapping.
3.1.1. Lemma. For $n \geq 2$,

$$Z^n \circ (\text{id} \otimes Z^{n-1}) = (\text{id} \otimes Z^{n-1}) \circ Z^n = (Z^2 \otimes \text{id}^{\otimes (n-2)}) \circ (\text{id} \otimes Z^{n-1}) .$$

Hence, the through mappings

$$\mathcal{A} \otimes_n \xrightarrow{\text{id} \otimes Z^{n-1}} \mathcal{A} \otimes_n \xrightarrow{Z^n} \mathcal{A} \otimes_m \xrightarrow{\text{id} \otimes Z^{n-1}} \mathcal{A} \otimes (n-1)$$

and

$$\mathcal{A} \otimes_n \xrightarrow{Z^n} \mathcal{A} \otimes_n \xrightarrow{\text{id} \otimes Z^{n-1}} \mathcal{A} \otimes_m \xrightarrow{\text{id} \otimes Z^{n-1}} \mathcal{A} \otimes (n-1)$$

are identically zero.

This lemma explains the gauge-theory interpretation of $Z^n$ as the “zero modes” $\tilde{Z}$ in Sec. 2. Let $t_a$ be a basis in $\mathcal{A}$ and

$$(3.4) \quad t_a t_b = \sum_c f_{ab}^c t_c$$

be the algebra multiplication table. Then the analogue of Eqs. (1.1) is obviously given by

$$(3.5) \quad [t_a, t_b] = \sum_c f_{[ab]}^c t_c .$$

Next, for elements $a = \sum_c t_c a^c$, we write

$$Z^n(a_1 \otimes \ldots \otimes a_n) = Z_{b_1 \ldots b_n}^{c_1 \ldots c_{n-1}} \otimes t_{c_1} \otimes \ldots \otimes t_{c_{n-1}} a_1^{b_1} \ldots a_n^{b_n} ,$$

where summation over repeated indices is understood. The first tensor factor in the image of $Z^n$ is therefore separated from the others, with each $Z_{b_1 \ldots b_n}^{c_1 \ldots c_{n-1}}$ being an element in this copy of $\mathcal{A}$ (consistently with the language of noncommutative differential forms used in what follows).

In this component form, $Z^2$ is represented by $Z_{bc}^a = t_b \delta_c^a - f_{bc}^a$ and the above relation $m \circ Z^2 = 0$ becomes

$$(3.6) \quad \sum_c Z_{ab}^c t_c = 0 ,$$

which is to be viewed as the “reducibility” equation (2.1). Further, the “reducibility relations” for the higher $Z^n$ mappings — analogues of (2.2) — are the vanishing conditions in 3.1.1 which in the component notation become

$$(3.7) \quad \sum_{a_1, \ldots, a_n} Z_{a_1 \ldots a_{n+1}}^{a_1 \ldots a_n} Z_{a_1 \ldots a_{n-1}}^{a_1 \ldots a_n} = 0$$
(this involves multiplication in \( \mathcal{A} \)). More precisely, Eqs. (3.7) are interpreted as "quantum" equations (2.3), in the particular case where \( \Pi = 0 \) and no \( \hbar \) terms arise in the right-hand side (consistently with the fact that \( i\hbar \) is "genuinely" equal to 1). The component representation of the \( Z^n \) mappings is easily obtained from

\[
Z_{b_1 \ldots b_{n+1}}^{a_1 \ldots a_n} = Z_{b_1 \ldots b_n}^{a_1 \ldots a_{n-1}} \delta_{a_n}^{b_{n+1}} + (-1)^n \delta_{b_1}^{a_1} \ldots \delta_{b_{n-1}}^{a_{n-1}} f_{a_n b_{n+1}}.
\]

### 3.2. Noncommutative differential forms.

We recall the interpretation of \( \Omega^n = \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes n} \) as noncommutative differential forms \([12, 13, 14]\). The algebra of noncommutative differential forms \( \Omega^* = \Omega^* \mathcal{A} \) over \( \mathcal{A} \) is the universal differential graded algebra generated by \( \mathcal{A} \) and the symbols \( da, a \in \mathcal{A} \), such that \( da \) is linear in \( a \), the Leibnitz rule \( d(ab) = d(a)b + adb \) is satisfied, and \( d1 = 0 \). The isomorphism \( \mathcal{A} \otimes \bar{\mathcal{A}}^{\otimes n} \to \Omega^n \) is given by

\[
a_0 \otimes a_1 \otimes \ldots \otimes a_n \mapsto a_0 da_1 \ldots da_n.
\]

Under this isomorphism, the action of \( d \) becomes \( d(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = 1 \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_n \). Noncommutative differential forms are a bimodule over \( \mathcal{A} \). The left action is obvious, and the rule to define the right action is as follows: on 1-forms, an element \( c \in \mathcal{A} \) acts as \( a_0 da_1 \cdot c = a_0 d(a_1 c) - a_0 a_1 dc \), and similarly for higher-degree forms, starting from the right end and "propagating" to the left using the Leibnitz rule until all terms take the form \( b_0 db_1 \ldots db_n \), for example

\[
a_0 da_1 da_2 \cdot c = a_0 da_1 d(a_2 c) - a_0 d(a_1 a_2) dc + a_0 a_1 da_2 dc.
\]

For \( D \in \text{Hom}(\mathcal{A}, \mathcal{A}) \), we let \( L_D \) denote the Lie derivative acting on noncommutative differential forms as \([14]\)

\[
L_D(a_0 da_1 \ldots da_n) = D(a_0) da_1 \ldots da_n
+ a_0 dD(a_1) da_2 \ldots da_n + \cdots + a_0 da_1 \ldots da_{n-1} dD(a_n)
\]

and also let \( I_D \) be the contraction

\[
I_D(a_0 da_1 \ldots da_n) = a_0 D(a_1) da_2 \ldots da_n
- a_0 da_1 \cdot D(a_2) da_3 \ldots da_n + \cdots + (-1)^{n-1} a_0 da_1 \ldots da_{n-1} \cdot D(a_n).
\]

The Cartan homotopy formula then holds,

\[
L_D = I_D d + d I_D.
\]
For $a \in \mathcal{A}$ and the inner derivation $[a, -]$ of $\mathcal{A}$, we abuse the notation by writing $L_a = L_{[a, -]}$ and $l_a = l_{[a, -]}$.

### 3.2.1. Lemma.

If $D \in \text{Hom}(\mathcal{A}, \mathcal{A})$ is a derivation, then

$$I_D b' + b' I_D = 0 \quad \text{and} \quad b' L_D - L_D b' = 0.$$  

The first relation is verified directly. Combined with the above homotopy formula, it then immediately implies the second relation. In what follows, this lemma is used for inner derivations $D = [a, -]$; in particular, we have $b' L_a = L_a b'$ for $a \in \mathcal{A}$.

We extend the above relations to the bar resolution $\Omega^* \otimes \mathcal{A}$.

### 3.3. Ghosts and the BRST differential $\Omega$.

We now view the bar resolution and the mappings $Z^n$ as a reducible gauge theory with $t_a$ (a chosen basis in $\mathcal{A}$) playing the role of constraints and with the reducibility relations given by (3.6) and (3.7).

#### 3.3.1. The ghost content.

We introduce ghosts for each term in the bar resolution,

$$C_n \in \Omega^{n-2} \otimes \mathcal{A}, \quad n \geq 2,$$

$$C_1 \in \mathcal{A}$$

and also introduce the conjugate momenta

$$P_n \in \text{Hom}(\bar{\mathcal{A}} \otimes (n-1) \otimes \mathcal{A}, \mathbb{C}), \quad n \geq 2,$$

$$P_1 \in \text{Hom}(\mathcal{A}, \mathbb{C})$$

with the $\mathbb{Z}_2$ gradings $\varepsilon(C_n) = \varepsilon(P_n) \equiv n \mod 2$. The ghost number assignments are as follows:

$$\text{gh} C_n = n, \quad \text{gh} P_n = -n.$$  

The $\mathbb{Z}_2$ grading therefore coincides with the ghost-number grading considered modulo 2; we nevertheless explicitly specify the $\mathbb{Z}_2$-grading along with the ghost-number grading.

With the elements $a_i \in \mathcal{A}$ assigned $\mathbb{Z}_2$-gradings $\varepsilon(a_i)$, we set $\varepsilon(a_1 \otimes \ldots \otimes a_n) = \varepsilon(a_1) + \cdots + \varepsilon(a_n)$ and often write $\varepsilon(a)$ for $\varepsilon(a_1 \otimes \ldots \otimes a_n)$.

We write the canonical coupling as

$$\langle \bar{\mathcal{A}} \otimes \mathcal{A} \rangle \otimes \text{Hom}(\bar{\mathcal{A}} \otimes \mathcal{A}, \mathbb{C}) \rightarrow \mathbb{C},$$

$$(a_1 \otimes \ldots \otimes a_n \otimes a') \otimes D \mapsto \langle a_1 \otimes \ldots \otimes a_n \otimes a', D \rangle.$$  

$^3$ Relation with the notation for the conjugate momenta used in Sec.2 and in the literature on constrained Hamiltonian systems in general is $(P_n)_{a_1, \ldots, a_n} = (-1)^{n-1} \tilde{p}_{a_{n-1}}$, see footnote2.
This also induces the contraction
\[ (A \otimes \overline{A}^\otimes n \otimes A) \otimes \operatorname{Hom}(\overline{A}^\otimes n \otimes A, C) \to A \]
(3.9)
\[ (a_0 \otimes \ldots \otimes a_n \otimes a', D) \mapsto \iota(a_0 \otimes \ldots \otimes a_n \otimes a', D), \]
where
\[ \iota(a_0 \otimes \ldots \otimes a_n \otimes a', D) = a_0 \langle a_1 \otimes \ldots \otimes a_n, D \rangle. \]

### 3.3.2. Differential operators on the bar resolution and Ω
The Hamiltonian quantization prescription involves canonical quantization of the ghosts \[2\]; in our case, this amounts to considering differential operators on the bar resolution. With each pair \(C^n, P_n\) subject to the canonical commutation relations (mnemonically, \([C^n, P_m] = \delta^n_m\), and hence, \([P_n, C^m] = (-1)^{n+1}\delta^n_m\)), commuting \(C^n\) through \(P_n\) evaluated on \(a_1 \otimes \ldots \otimes a_n\) in accordance with (3.8) gives
\[ C^n \langle a_1 \otimes \ldots \otimes a_n, P_n \rangle = (-1)^{n \varepsilon(a)} a_1 \otimes \ldots \otimes a_n \]
\[ + (-1)^{(n(\varepsilon(a)+1))} \langle a_1 \otimes \ldots \otimes a_n, P_n \rangle C^n, \]
(3.10)
or in other words,
\[ [C^n, \langle a_1 \otimes \ldots \otimes a_n, P_n \rangle] = (-1)^{n \varepsilon(a)} a_1 \otimes \ldots \otimes a_n. \]

We similarly evaluate commutators involving the contraction in (3.9), e.g.,
\[ [C^1, \iota(a_0 \otimes a_1) P_1] = [C^1, a_0] \langle a_1, P_1 \rangle + (-1)^{\varepsilon(a_0)+\varepsilon(a_1)} a_0 a_1, \]
(3.11)
where the first term in the right-hand side involves a commutator in \(A\) and the second term involves multiplication in \(A\).

We use the experience with the BRST formalism to seek the differential \(\Omega\) in the form
\[ \Omega = \Omega_0 + \Omega_A, \quad \gh \Omega = 1, \]
(3.12)
where
\[ \Omega_A = C^1 + \sum_{n \geq 1} \iota(Z^{n+1} C^{n+1}) P_n \]
(3.13)
includes the “boundary” terms explicitly written in (2.6) (although it pertains to the infinitely reducible case \(\ell = \infty\)) and where
\[ \Omega_0 = \sum_{j \geq i \geq 1} \langle U_{i,j}^{i+j-1}(C^i \otimes C^j), P_{i+j-1} \rangle \]
\[ + \sum_{j \geq i \geq 1} \sum_{m=1}^{[i+j-2]} \langle U_{i,j}^{m,i+j-m-1}(C^i \otimes C^j), P_m \otimes P_{i+j-m-1} \rangle \]
(3.14)
with the “coefficients” $U$ to be determined. This is a BRST differential for a closed-algebra (reducible) gauge theory (see the remarks below for its special properties).

We also write $\Omega_A$ in the component notation (see (3.4)–(3.6)),

$$\Omega_A = t_a(C^1)^a + Z_{ab}^{2ab}C^{2ab}P_{1c} + Z_{abc}^{3abc}C^{3abc}P_{2de} + \ldots.$$  

3.4. **Theorem.** There exist mappings

$$U_{m,m'}^{n,n'} \in \text{Hom}(A^m \otimes A^{n'}, A^{n} \otimes A^{n'}), \quad m + m' = n + n' + 1, \quad 1 \leq n < n', \quad 1 \leq m \leq m',$$

$$U_{m,m'}^{m+m'-1} \in \text{Hom}(A^m \otimes A^{n'}), \quad 1 \leq m \leq m',$$

such that the operator $\Omega$ in (3.12)–(3.14) satisfies

(3.15) $$\Omega \Omega = 0.$$  

**Remarks.**

3.4.1. The mappings $U_{m,m'}^{n,n'}$ whose labels do not satisfy the restrictions in the Theorem can be considered vanishing. It is a matter of convention that $U_{m,m'}^{n,n'} = 0$ for $n > n'$ (an alternative would be to impose graded symmetry with respect to the transposition of indices), but the condition $U_{m,m'}^{n,n} = 0$ is essential: it implies that no ghost momentum is squared in the BRST differential. Another conventional condition is that $U_{m,m'}^{n,n} = 0$ for $m > m'$. Because we only have $U_{m,m'}^{m,m}$ with $m + n = m' + n' + 1$, one of the four labels is redundant, but the notation is more transparent when all labels are kept.

3.4.2. We note that the operator $\Omega_A$ in (3.13) is linear in both ghosts and momenta, with the coefficients given by $Z^n$ mappings (3.2); it is therefore a differential operator with coefficients in $A$, cf. the text after Lemma 3.1.1 (and is nothing but the “boundary terms,” cf. (2.6)). On the other hand, $\Omega_0$ is bilinear in the ghosts and (separately) in the momenta but is an operator with scalar coefficients because of the $\langle - , P \rangle$ and $\langle - , P \otimes P \rangle$ contractions in it. The general structure of $\Omega$ is the same as described in (2.11) with additional properties that the $\tau_A$ are linear and the $U_{AB}$ are at most bilinear in the ghost momenta. Conditions (2.12) are satisfied because $\tau_A$ and $U_{AB}$ are functions of only the momenta and $U_{AB}$ are scalar as noted above.

3.4.3. For the differential $\Omega$ in Theorem 3.3, $\Omega_0$ is not a differential.

3.5. **Solution for $U_{m,m'}^{n,n'}$ and $U_{m,m'}^{m+m'-1}$ in Theorem 3.4.** We determine the $U$-mappings by solving a part of the equations following from $\Omega \Omega = 0$ and then show that the remaining equations are also satisfied. We begin with evaluating $\Omega \Omega$ for the operator in (3.12)–(3.14).
3.5.1. Calculating $\Omega^2$. A simple calculation using (3.10)–(3.11) shows that

$$\Omega, \Omega = C^1 C^1 + \sum_{m \geq 1} \ell(Z_m)_{p_m} + \sum_{n \geq m \geq 1} \ell(Z_{mn})_{p_m \otimes p_n},$$

where the first term in the right-hand side involves the algebra multiplication and the terms in the series are given by $\ell(\cdot)_{p}$-contractions of

$$Z_m = \text{ad}_{C^1} \cdot Z^{m+1}(C^{m+1}),$$

$$Z_{nn} = \zeta_n(C^{n+1} \otimes C^{n+1}),$$

$$Z_{mn} = (-1)^{m+m}(m_{1,m+2} - m^*_{m+2,1})(Z^{m+1}C^{m+1} \otimes Z^{n+1}C^{n+1}), \quad m < n,$$

with $\zeta_n$ given by

$$\zeta_n(a_1 \otimes \ldots \otimes a_{n+1} \otimes b_1 \otimes \ldots \otimes b_{n+1}) =$$

$$= \frac{1}{n!}([a_1, b_1] \otimes a_2 \otimes \ldots \otimes a_{n+1} \otimes b_2 \otimes \ldots \otimes b_{n+1}$$

$$- (-1)^n [a_1, b_1] \otimes b_2 \otimes \ldots \otimes b_{n+1} \otimes a_2 \otimes \ldots \otimes a_{n+1}).$$

Further using (3.10)–(3.11), we next obtain

$$[\Omega_0, \Omega, \Omega] = \mathcal{U}^1_{1,1} (C^1 \otimes C^1) + \sum_{m \geq 1} \ell(\mathcal{Y}_m)_{p_m} + \sum_{n \geq m \geq 1} \ell(\mathcal{Y}_{mn})_{p_m \otimes p_n},$$

where the terms in the series are $\ell(\cdot)_{p}$-contractions of

$$\mathcal{Y}_{mn} = (-1)^n Z^{n+1}_{1,m+1} \circ \sum_{j \geq i \geq 0} U^m_{i+1,j+1} (C^{i+1} \otimes C^{j+1})$$

$$+ \sum_{j > i \geq 0} (-1)^{m+n} (\text{id} \otimes U^m_{i+1,j}) \circ Z_{1,i+2}^{j+1} (C^{i+1} \otimes C^{j+1}) +$$

$$+ \sum_{j+1 > i \geq 0} (-1)^{i+1} (\text{id} \otimes U^m_{i,j+1}) (Z_{i+1}^{i+1} C^{i+1} \otimes C^{j+1})$$

$$+ (-1)^{m+n} (Z^{m+1} \otimes \text{id}^n) \circ \sum_{j \geq i \geq 0} U^{m+1,n}_{i+1,j+1} (C^{i+1} \otimes C^{j+1}), \quad m < n,$$

and

$$\mathcal{Y}_{nn} = (-1)^n S^{(n)}_{2,n+1} \circ \sum_{i=0}^n U^{n,n+1}_{i+1,2n-i+1} (C^{i+1} \otimes C^{2n-i+1}).$$

We here use the notation (3.3), and the operator $S^{(n)}_{p,q,r,s}$, with $n$ considered modulo 2 and $p \leq q < r \leq s$ such that $s-r = q-p$, performs graded symmetrization (for n
even) or antisymmetrization (for $n$ odd) of tensor factors in the positions $[p, \ldots, q]$ and $[r, \ldots, s]$, for example,

$$
S_{2,3;4,5}^{(n)}(a \otimes b \otimes c \otimes d \otimes e) = \frac{1}{2} a \otimes b \otimes c \otimes d \otimes e
+ \left((-1)^n \cdot (-1)^{(\varepsilon(b) + \varepsilon(c))(\varepsilon(d) + \varepsilon(e))} \right) \frac{1}{2} a \otimes d \otimes e \otimes b \otimes c.
$$

We also find that $\mathcal{V}_m = \mathcal{V}_{0,m}$ if we set

$$
(3.16) \quad U^{0,\ell}_{i,j} = U^\ell_{i,j}.
$$

Finally, calculating $\Omega_0\Omega_0$ gives

$$
\Omega_0\Omega_0 = \sum_{m \geq 2} \langle \mathcal{X}_m, P_m \rangle + \sum_{m \geq 1 \ n \geq 1} \langle \mathcal{X}_{mn}, P_m \otimes P_n \rangle + \mathcal{X}^{(3)},
$$

where

$$
\mathcal{X}_{mn} = \sum_{i \geq 2 \ j \geq 1} (-1)^{(i+1)(m+n)+1} U_{i,m+n+1-i}^{m,n} \otimes U_{j,m+n+2-j}^{i,m+n+1-i} (C^j \otimes C^{m+n+2-j}),
$$

$\mathcal{X}_m = \mathcal{X}_{0,m}$, and $\mathcal{X}^{(3)}$ denotes third-order terms in the $C^n$ ghosts. The square of $\Omega$ is therefore given by

$$
(3.17) \quad \Omega \Omega = Z_0 + \sum_{m \geq 1} \tau(\mathcal{Z}_m + \mathcal{V}_m + 1 \otimes \mathcal{X}_m) P_m
+ \sum_{n \geq m \geq 1} \tau(\mathcal{Z}_{mn} + \mathcal{V}_{mn} + 1 \otimes \mathcal{X}_{mn}) P_m \otimes P_n + \mathcal{X}^{(3)},
$$

where $Z_0 = \mathcal{C}^1 \mathcal{C}^1 + \bigcup_{i=1} U^{1,1}_{i,1} (C^1 \otimes C^1)$ and $\mathcal{Z}_m$, $\mathcal{V}_m$, $\mathcal{X}_m$, $\mathcal{Z}_{mn}$, $\mathcal{V}_{mn}$, $\mathcal{X}_{mn}$ with $n \geq m \geq 1$ are given above (as we have seen, $\mathcal{X}_1 = 0$ and also $\mathcal{X}_{mn} = 0$). Expanding (3.17) in the ghosts and momenta and equating each power to zero, we obtain a specific form of Eqs. (2.13–2.15). In particular, the equations that are cubic in $C^n$, $\mathcal{X}^{(3)} = 0$, are nothing but a rewriting of (2.15). The other terms in (3.17) are at most quadratic in the $C^n$ ghosts. Their dependence on the ghost momenta is shown in (3.17) explicitly, and we therefore first expand in the momenta. The equation $\Omega \Omega = 0$ is then equivalent to the set of equations $\mathcal{X}^{(3)} = 0$ and

$$
(3.18) \quad S_{2,n+1;2,2n+1}^{(n)}(\mathcal{Z}_{mn} + \mathcal{V}_{mn} + 1 \otimes \mathcal{X}_{mn}) = 0, \quad 1 \leq m < n,
$$

Further expanding each of these in the $C^n$ ghosts gives an infinite list of equations starting with (2.7–2.10). Speaking about the equations $\mathcal{Z}_{mn} + \mathcal{V}_{mn} + 1 \otimes \mathcal{X}_{mn} = 0$ in general, we often mean the $S_{2,n+1;2,2n+1}^{(n)}$ symmetrization of the $(m = n)$ equations.
without explicitly specifying it in the notation. We now solve Eqs. (3.18) by taking a certain projection of these equations that yields a set of recursive relations for the sought mappings $U_{**}$.

3.5.2. Finding the lowest mappings. The lowest-order terms $Z_0$ in (3.17) vanish if $U_{1,1}^1 = -m$, minus the multiplication in $A$. Because $C^1$ is $\mathbb{Z}_2$-odd, only the antisymmetric part of the multiplication actually contributes, and we have

$$U_{1,1}^1 = -\frac{1}{2}[-,-]$$

(which is a totally standard BRST fact that the lowest coefficient in the BRST operator involves Lie algebra structure constants). In the order $P_1$, we readily find that

$$Z_1 + \mathcal{Y}_1 = L_{C^1} \cdot Z^2 \circ C^2 - Z^2 \circ U_{1,2} (C^1 \otimes C^2)$$

(the term $1 \otimes X'_{1}$ vanishes). But

$$L_a \circ Z^2 = L_a b' \cdot d = b'(d(l_a)) d = b'(d(l_a) + d l_a) = Z^2 \circ L_a,$$

which shows that $Z_1 + \mathcal{Y}_1 = 0$ if

$$U_{1,2} (C^1 \otimes C^2) = L_{C^1} C^2.$$ We note that this solves Eq. (2.7) in the list of equations in 2.2.6.

3.5.3. Establishing a recursion. With the lowest two mappings $U_{1,1}^1$ and $U_{1,2}$ thus determined, we write the $Z^n$ mappings as $Z = \text{id} - 1 \otimes b'$ and consider the terms that do not have the form $1 \otimes \ldots$ in each equation in (3.18). The operation $P_f$ of projecting onto such terms amounts to dropping all explicit occurrences of $1 \otimes \ldots$ (in particular, of $1 \otimes X_{mn}$) and replacing $Z^n$ by $\text{id}^{\otimes n}$ and $Z^n_{1,i} \rightarrow P_{1,i}$. Equations (3.18) thus imply the equations

$$\sum_{i=0}^{m+n} U_{i+1,m+n-i+1}^{m,n+1} (C^{i+1} \otimes C^{m+n-i+1}) =$$

$$-(-1)^{m+n} P_{m+1,1} \circ (m_{1,m+2} - m_{m+2,1}^*) (C^{m+1} \otimes C^{n+1})$$

$$- (-1)^m \sum_{i=0}^{m+n-1} P_{m+1,i} \circ (\text{id} \otimes U_{i+1,m+n-i}^m) \circ P_{1,i+2} (C^{i+1} \otimes C^{m+n-i+1})$$

$$+ \sum_{i=1}^{m+n} (-1)^{i+n} P_{m+1,1} \circ (\text{id} \otimes U_{i,m+n-i+1}^m) (C^{i+1} \otimes C^{m+n-i+1})$$

$$- (-1)^m \sum_{i=0}^{m+n} P_{m+1,1} \circ U_{i+1,m+n-i+1}^{m+1,n} (C^{i+1} \otimes C^{m+n-i+1}),$$
where \(0 \leq m \leq n\) (and we recall Eq. (3.16) for \(m = 0\)). Each of these relations amounts to \(\left\lfloor \frac{m+n}{2} \right\rfloor + 1 - m\) independent equations obtained by extracting each ghost pair \((C^a, C^b)\), but to save space, we keep them in the above form of “generating relations”. We also remark that in applying mappings to \(C^c \otimes C^d\), graded symmetrization with respect to the two “halves” of the tensor argument must also be performed in accordance with \(\varepsilon(C^c) = \ell\).

To see that Eqs. \((m,n)\) are in fact a set of recursive relations, we fix \(m + n = N\) with a positive integer \(N\) and arrange the \(\frac{1}{2} \left( \left\lfloor \frac{N}{2} \right\rfloor + 1 \right) \left( \left\lfloor \frac{N}{2} \right\rfloor + 2 \right)\) equations following from \((m,n)\) in the order specified by
\[
\begin{align*}
(m, n) &= \left( \left\lfloor \frac{N}{2} \right\rfloor, \left\lfloor \frac{N}{2} \right\rfloor + 1 \right) \quad (1 \text{ equation}),
(m, n) &= \left( \left\lfloor \frac{N}{2} \right\rfloor - 1, \left\lfloor \frac{N}{2} \right\rfloor + 1 \right) \quad (2 \text{ equations}), \\
\ldots
(m, n) &= (1, N - 1) \quad \left( \left\lfloor \frac{N}{2} \right\rfloor \right) \text{ equations),}
(m, n) &= (0, N) \quad \left( \left\lfloor \frac{N}{2} \right\rfloor + 1 \right) \text{ equations).}
\end{align*}
\]

The underlying partial ordering is therefore given by
\[
U_{m, n}^{m', n'} < U_{m_1, n_1}^{m', n'} \quad \text{if} \quad \text{either } m' + n' < m'_1 + n'_1 \quad \text{or} \quad m' + n' = m'_1 + n'_1 \quad \text{and} \quad n' - m' < n'_1 - m'_1.
\]

It follows that for each integer \(N \geq 4\), the vanishing \(U_{m, n}^{m', n'}\) mappings are given by
\[
U_{\left\lfloor \frac{N}{2} \right\rfloor - j, \left\lfloor \frac{N}{2} \right\rfloor + j - 1}^{\left\lfloor \frac{N}{2} \right\rfloor - i, \left\lfloor \frac{N}{2} \right\rfloor + i} = 0, \quad 1 \leq j \leq i \leq \left\lfloor \frac{N}{2} \right\rfloor - 1.
\]

In other words,
\[
U_{m, n}^{m', m + n - m' - 1} = 0 \quad \text{if} \quad m' \geq m \quad \text{and} \quad n - m \geq 2.
\]

The recursive relations are now written most conveniently for odd and even \(m + n\) separately.

### 3.5.4. Odd \(m + n\).

For \(m + n = 2k - 1\), Eqs. \((m,n)\), which can then be labeled as \((2k - 1; m)\) with \(0 \leq m \leq k - 1\), become
\[
(2k - 1; m) \sum_{i=m}^{k-1} U_{i+1, 2k-i}^{m, 2k-m} (C^i+1 \otimes C^{2k-i}) = \]

In more detail, we first choose the top value \( m = k - 1 \) in accordance with the ordering. The last sum in the right-hand side is then absent, and \( \sum_{i=0}^{k+1} \) is therefore expressed through \( \sum_{i=0}^{m', n'} \) with \( m' + n' = 2k - 1 < 2k \), namely,

\[
(\text{3.25}) \quad U_{k,k+1}^{k-1,k+1}(C^k \otimes C^{k+1}) = P_{k,1} \circ \left( (m_1, k+1 - m_{k+1,1}) (C^k \otimes C^{k+1}) + (-1)^k (\id \otimes U_{k,k}^{k-1,k}) \circ P_{1,k}(C^k \otimes C^{k+1}) + (\id \otimes U_{k,k}^{k-1,k})(C^{k+1} \otimes C^k) \right),
\]

Next, setting \( m = k - 2 \) gives the two equations

\[
(\text{3.26}) \quad U_{k-1,k+2}^{k-2,k+2} = P_{k-1,1} \circ \left( (m_{2,1} - m_{1,2} - (-1)^k (\id \otimes U_{k-1,k+1}^{k-2,k+1})) \circ P_{1,k} \right)
\]

and

\[
(\text{3.27}) \quad U_{k,k+1}^{k-2,k+2}(C^k \otimes C^{k+1}) = P_{k-1,1} \circ \left( (-1)^k U_{k,k+1}^{k-1,k+1}(C^k \otimes C^{k+1}) + (\id \otimes U_{k,k+1}^{k-2,k+1})(C^k \otimes C^{k+1}) + (\id \otimes U_{k,k+1}^{k-2,k+1})(C^{k+1} \otimes C^k) \right),
\]

and so on: for each \( m, 0 \leq m \leq k - 2 \), the equations for “generic” values of \( i \), i.e., \( m + 1 \leq i \leq k - 1 \), are given by

\[
(\text{3.28}) \quad U_{i+1,2k-i}^{m,2m-m}(C^k \otimes C^{k+1}) = P_{m+1,1} \circ \left( (-1)^m (\id \otimes U_{i+1,2k-i}^{m,2m-m-1}) \circ P_{1,i+2} \right)
\]

The “boundary” equations (those with \( i = m \) and \( i = k \)) are also easily extracted from \( (2k - 1; m) \) (the \( i = m \))-equation involves the “inhomogeneous” contribution \( P_{m+1,1} \circ (m_{1,m+2} - m_{m+2,1}^*) \) in the right-hand side).
3.5.5. Even $m + n$. For $m + n = 2k$, Eqs. $(m,n)$, now labeled as $(2k;m)$ with $0 \leq m \leq k$, become

$$(2k;m) \sum_{i=m}^{k} U_{i+1,2k-i+1}^{m,2k-m-1}(C^{i+1} \otimes C^{2k-i+1}) =$$

$$= -(-1)^m P_{m+1,1} \circ \left( (m_1,m+2 - m_2)^{(m+1) \otimes C^{2k-m+1}} \right)$$

$$- (-1)^m \sum_{i=m}^{k-1} (1 \otimes U_{i+1,2k-i}^{m,2k-m}) \circ P_{1,i+2}(C^{i+1} \otimes C^{2k-i+1})$$

$$+ \sum_{i=m+1}^{k} (-1)^{i+m} (1 \otimes U_{i,2k-i+1}^{m,2k-m})(C^{i+1} \otimes C^{2k-i+1})$$

$$- (-1)^m \sum_{i=m+1}^{k} U_{i+1,2k-i+1}^{m+1,2k-m}(C^{i+1} \otimes C^{2k-i+1}) \right).$$

These equations must also be considered in the order specified by consecutively taking $m = k$, $m = k-1$, $\ldots$, $m = 1$. The top value $m = k$ is somewhat special here: on one hand, all the $U_{m,n}^{m',n'}$ mappings drop from the right-hand side and only the first term survives, thereby giving an explicit expression for $U_{k+1,k+1}^{k,k+1}$; on the other hand, the equation must only be satisfied after the symmetrization (see the remark at the end of 3.5.1 and with the proper symmetrization of the ghost argument $C^{k+1} \otimes C^{k+1}$ (see the remark after equation $(m,n)$)). This leaves some freedom in determining $U_{k+1,k+1}^{k,k+1}$, which can be fixed using the full equation $S_{2k+1;2k+2}^{(k)} (Z_{kk} + Y_{kk} + 1 \otimes X_{kk}) = 0$ for $U_{k,k}^{k-1,k}$. It follows that

$$(3.29) \quad U_{k,k}^{k-1,k} = (-1)^{k+1} b' \otimes 1^{\otimes k} + (-1)^k (1^{\otimes (k-1)} \otimes b') \circ P_{1,k},$$

where $P_{1,k}$ denotes the operator inverse to $P_{1,k}$ and graded symmetrization with respect to the two “halves” of the tensor argument must be performed in evaluating $U_{k,k}^{k-1,k}$ on $C^k \otimes C^k$, for example (omitting sign factors due to the $\mathbb{Z}_2$-grading),

$$(3.30) \quad U_{2,2}^{1,2} a \otimes b \otimes a' \otimes b' = \frac{1}{2} (-ab \otimes a' \otimes b' + b \otimes a a' \otimes b' - b \otimes a \otimes a' b'$$

$$- a b' \otimes a \otimes b + b' \otimes a' a \otimes b - b' \otimes a' \otimes ab)$$

(which actually solves Eq. (2.9)).

The next (two) equations following from $(2k;m)$, which correspond to $m = k - 1$, allow expressing $U_{k,k+1}^{k-1,k+2}$ through $U_{k,k+1}^{k-1,k+1}$ and $U_{k,k+1}^{k-1,k+2}$ through $U_{k,k+1}^{k-1,k+1}$ and $U_{k+1,k+1}^{k,k+1}$; all these equations are easily written out similarly to (3.28). In applying the recursion further, we must only take into account that for $0 \leq m \leq k - 1$, the equations
relations above. For formulas for several lowest mappings that can easily be derived from the recursive relations implied by \((3.5.7)\). Examples.

\[ \begin{align*}
(3.32) & \quad U_{m,2k-m+1}^{m,2k-m+1}(C^{k+1} \otimes C^{k+1}) = \\
& \quad = (-1)^{k+m} p_{m+1,1} \circ (\text{id} \otimes U_{k,k+1}^{m,2k-m})(C^{k+1} \otimes C^{k+1}) \\
& \quad - (-1)^m p_{m+1,1} \circ U_{k+1,k+1}^{m+1,2k-m}(C^{k+1} \otimes C^{k+1}),
\end{align*} \]

implied by \((2k; m)\) are evaluated on \(C^{k+1} \otimes C^{k+1}\) and the mappings in the right-hand side must therefore be symmetrized appropriately with respect to the two “halves” of the tensor argument (i.e., antisymmetrized for \(k + 1\) odd and symmetrized for \(k + 1\) even).

Equations \((3.19), (3.20), (3.24), (3.32)\), \((2k-1; m)\), and \((2k; m)\) determine all the mappings involved in the differential \(\Omega\).

3.5.6. Remark. It follows that the structure in Eq. \((3.20)\) propagates through the recursive relations to

\[ \begin{align*}
(3.32) & \quad U_{1,n}^n a_0 \otimes a_1 \otimes \ldots \otimes a_n = (-1)^n L_{a_0} a_1 \otimes \ldots \otimes a_n.
\end{align*} \]

3.5.7. Examples. In addition to \((3.19), (3.20), (3.30)\) (and \((3.32)\)), we give explicit formulas for several lowest mappings that can easily be derived from the recursive relations above. For \(U_{m_1,n_1}^{m',n'}\) with \(m' + n' = 3\), we have, along with \((3.30)\),

\[ \begin{align*}
2 U_{2,2}^2 a \otimes b \otimes a' \otimes b' = & \quad a \otimes a' \otimes b b' - a \otimes a' \otimes b b' + a \otimes a' \otimes b' - a \otimes b a' \otimes b' \\
& + a' \otimes a \otimes b b' - a' \otimes a \otimes b b' + a' \otimes a \otimes b b' - a' \otimes b a' \otimes b' \\
& - b \otimes a a' \otimes b' + b' \otimes a' \otimes a b - b' \otimes a' \otimes a b + ab \otimes a' \otimes b' + a b' \otimes a \otimes b.
\end{align*} \]

One of the \(U_{m_1,n_1}^{m',n'}\) mappings with \(m' + n' = 4\) is given by

\[ \begin{align*}
U_{2,3}^{1,3} a \otimes b \otimes a' \otimes b' \otimes c' = & \quad c' \otimes a' \otimes b' a \otimes b - c' \otimes a' \otimes b' \otimes a b \\
& + b \otimes a a' \otimes b' \otimes c' + b \otimes a' \otimes a b' \otimes c' - b \otimes a' \otimes a b' c' \\
& + b \otimes a a' \otimes b' \otimes c' - ab \otimes a' \otimes b' \otimes c' - b' c' \otimes a' \otimes a \otimes b,
\end{align*} \]

or in the component notation (see \((3.4)--(3.6)\)), by the tensor

\[ \begin{align*}
U_{abcde}^{g,hij} = & \quad \delta_e^g \delta_c^h f^{e_i} \delta_b^j - \delta_e^g \delta_c^h \delta_b^i \delta_d^j f^{e} - \delta_b^g \delta_h^f \delta_a^i \delta_d^j \\
& + \delta_b^g \delta_h^f \delta_a^i \delta_d^j - \delta_b^g \delta_h^f \delta_a^i \delta_d^j + \delta_b^g \delta_h^f \delta_a^i \delta_d^j - \delta_b^g \delta_h^f \delta_a^i \delta_d^j.
\end{align*} \]

3.6. The end of the proof of Eqs. \((3.18)\). With the \(U_{m,n}^{m,n-\ell-1}\) mappings found recursively from the \(P_f\)-projections of Eqs. \((3.18)\), we must next show that the remaining parts of the equations

\[ \begin{align*}
Z_{mn} + Y_{mn} + 1 \otimes X_{mn} = 0, \quad 0 \leq m < n
\end{align*} \]
are satisfied. Explicitly, the equations to be verified are obtained by replacing \( Z \to -1 \otimes b' \); all tensor terms then acquire the form \( 1 \otimes X \), and the resulting equations \( X = 0 \) (as previously, written as “generating equations”) are given by

\[
(3.33) \quad (-1)^m (\text{id}^\otimes m \otimes b') \circ \sum_{i=m}^{\frac{m+n}{2}} U_{i+1,m+n-i+1}^{m,n+1} (C^{i+1} \otimes C^{m+n-i+1})
\]

\[
+ (b' \otimes \text{id}^\otimes n) \circ \sum_{i=m+1}^{\frac{m+n}{2} - 1} U_{i+1,m+n-i+1}^{m,n} (C^{i+1} \otimes C^{m+n-i+1})
\]

\[
+ \sum_{i=m}^{\frac{m+n}{2} - 1} U_{i+1,m+n-i+1}^{m,n} (C^{i+1} \otimes b'C^{m+n-i+1})
\]

\[
+ \sum_{j=m+1}^{\frac{m+n}{2}} \sum_{i=j}^{j(m+n)} U_{j,m+n-j}^{m,n} \circ U_{i+1,m+n-j-1}^{i,m+n-j} (C^{i+1} \otimes C^{m+n-j-1}) = 0
\]

for \( 0 \leq m < n \). We must show that they are satisfied with the mappings \( U_{m,n}^{\ell,m+n-\ell-1} \) determined by the above recursion.

3.6.0. The strategy. Equations \((3.33)\) are proved by (somewhat tedious) induction on the order of the \( U \) mappings introduced in \((3.22)\). We consider Eqs. \((3.33)\) with \( m + n = 2k - 1 \) and assume that all equations with \( m + n \leq 2k - 2 \) are satisfied (Eqs. \((3.33)\) with \( m + n = 2k \) are proved similarly). The equations that we must verify for \( m + n = 2k - 1 \) are as follows. Setting \( m = k - 1 - \ell \) and \( n = k + \ell \), we have \( \ell + 1 \) equations for each \( \ell \in \{0, \ldots, k-1\} \); for \( \ell \geq 1 \), these are

\[
(3.34) \quad (-1)^{k+\ell} (\text{id}^\otimes (k-\ell-1) \otimes b') \circ U_{k,k+1}^{\ell-1,k+\ell+1} (C^k \otimes C^{k+1})
\]

\[
- (b' \otimes \text{id}^\otimes (k+\ell)) \circ U_{k,k+1}^{\ell-1,k+\ell} (C^k \otimes C^{k+1}) - U_{k,k}^{\ell-1,k+\ell} (C^k \otimes b'C^{k+1})
\]

\[
+ (-1)^k U_{k+1,k+1}^{\ell-1,k+\ell} (b'C^k \otimes C^{k+1}) - (-1)^k U_{k,k}^{\ell-1,k+\ell} (b'C^{k+1} \otimes C^k)
\]

\[
- \sum_{j=0}^{\ell-1} (-1)^{j+1} U_{k-k-j+1,k+1}^{\ell-1,k+\ell} \circ U_{k,k+1}^{\ell-1,k+\ell+1} (C^k \otimes C^{k+1}) = 0,
\]

\[
(3.35) \quad (-1)^{k+\ell} (\text{id}^\otimes (k-\ell-1) \otimes b') \circ U_{k,k+1}^{\ell-1,k+\ell+1} - U_{k,k+1}^{\ell-1,k+\ell+1} \circ (\text{id}^\otimes (k-\ell) \otimes b') = 0,
\]
and

\[(3.36) \ (-1)^{k+\ell} (id^\otimes (k-\ell-1) \otimes b') \circ U_{k-1,k+1}^{k-1,k+1} - (id \otimes id^\otimes (k+\ell)) \circ U_{k-1,k+i+1}^{k-1,k+i+1} - U_{k-1,k+i+1}^{k-1,k+i+1} \circ (id^\otimes (k+1+i)) \]

\[- \sum_{j=i}^{\ell-1} (-1)^{k+j+1} U_{k-j-1,k+j+1}^{k-1,k+i+1} \circ U_{k-1,k+i+1}^{k-1,k+i+1} = 0, \quad 1 \leq i \leq \ell - 1.\]

For \(\ell = 0\), the only equation is

\[(3.37) \ (id^\otimes (k-1) \otimes b') \circ U_{k,k+1}^{k-1,k+1} (C^k \otimes C^{k+1}) =
\[- (id^\otimes (k-1) \otimes b') \circ P_{k,1} (C^k \otimes b'C^{k+1} + (-1)^k b'C^{k+1} \otimes C^k).\]

For a fixed \(k\), we proceed by induction on \(\ell\), which is actually part of the induction on the order of the \(U\) mappings in (3.22) — \(\ell\) labels rows in (3.21). At each step, we use the defining recursive relations and the lower (previously proved) equations (3.34)–(3.37).

3.6.1. We begin with the \(\ell\)-induction base \(\ell = 0\) and show that (3.37) is satisfied. With (3.29), Eq. (3.37) becomes

\[(3.38) \ (id^\otimes (k-1) \otimes b') \circ U_{k,k+1}^{k-1,k+1} (C^k \otimes C^{k+1}) =
\[- b'C^k \otimes b'C^{k+1} + (id^\otimes (k-1) \otimes b') \circ P_{k,1} (C^k \otimes b'C^{k+1} + (-1)^k b'C^{k+1} \otimes C^k).\]

Using the obvious identity

\[(3.39) \ (id^\otimes (k-1) \otimes b') \circ P_{k,1} = P_{k,1} \circ (m_{1,k+1} - (id^\otimes k \otimes b')) ,\]

we further rewrite (3.38) as

\[(3.40) \ (id^\otimes (k-1) \otimes b') \circ U_{k,k+1}^{k-1,k+1} (C^k \otimes C^{k+1}) =
\[- b'C^k \otimes b'C^{k+1} + (-1)^k P_{k,1} (b'C^{k+1} \otimes b'C^k)
\quad + P_{k,1} \circ m_{1,k+1} (C^k \otimes b'C^{k+1}) + (-1)^k P_{k,1} \circ m_{1,k+1} (b'C^{k+1} \otimes C^k).\]

On the other hand, the mapping \(U_{k,k+1}^{k-1,k+1}\) is expressed via recursive relation (3.25), and therefore (applying (3.39) again), we can rewrite the left-hand side of (3.40) as

\[(id^\otimes (k-1) \otimes b') \circ U_{k,k+1}^{k-1,k+1} (C^k \otimes C^{k+1}) =\]
where each of the three groups of terms (labeled with 1, 2, and 3) vanish separately. Equation (3.40) is thus proved.

### 3.6.2

The subsequent calculations are straightforward but quite tiresome. To keep the presentation reasonably compact, we give the details only for $\ell = 1$. This representatively illustrates the general case because the quadratic term $U \circ U$ is already present in the corresponding equation (3.34) (compared with (3.36), Eqs. (3.34) involve an additional complication due to the graded symmetrization $F(C^k \otimes b'C^{k+1}) + (-1)^k F(b'C^{k+1} \otimes C^k)$).

For $\ell = 1$, the two equations that we must prove are given by

\begin{equation}
(3.41) \quad \text{id}^{\otimes (k-2)} \otimes b' \circ U_{k,k+1}^{k-2,k+2} (C^k \otimes C^{k+1}) \\
+ (-1)^k (b' \otimes \text{id}^{\otimes (k+1)}) \circ U_{k,k+1}^{k-1,k+1} (C^k \otimes C^{k+1}) - U_{k-1,k+1}^{k-2,k+1} (b'C^k \otimes C^{k+1}) \\
+ (-1)^k U_{k,k}^{k-2,k+1} (C^k \otimes b'C^{k+1}) + U_{k,k}^{k-2,k+1} (b'C^{k+1} \otimes C^k) = \\
= U_{k-1,k+1}^{k-2,k+1} \circ U_{k,k+1}^{k-1,k+1} (C^k \otimes C^{k+1})
\end{equation}
and

\[(3.42) \quad (-1)^k (\text{id} \otimes (k-2) \otimes b') \circ U_{k-1,k+1}^{k-2,k+2} (C^{k-1} \otimes C^{k+2}) + U_{k-1,k+1}^{k-2,k+1} (C^{k-1} \otimes C^{k+2}) = 0.\]

To begin with (3.42), we recall recursive relation (3.26). We also use the recursive relation

\[(3.43) \quad U_{k-1,k+1}^{k-2,k+1} = (-1)^k P_{k-1,1} \circ \left( m_{2,1} - m_{1,2} + (\text{id} \otimes U_{k-1,k}^{k-2,k}) \right) \circ P_{1,k} \]

that follows from (2k-2; k-2) (i.e., from Eq. (2k; m) where we replace \(k \to k-1\) and set \(m = k-2\)). We insert Eqs. (3.26) and (3.43) in (3.42) but keep \(U_{k-1,k+1}^{k-2,k+1}\) “unevaluated” in the combination \(\text{id} \otimes U_{k-1,k+1}^{k-2,k+1}\) that arises from (3.26). As before, we use identity (3.39) (with \(k \to k-1\)). Equation (3.42) that we must verify then becomes

\[(3.44) \quad (-1)^k m_{1,k} \circ (m_{2,1} - m_{1,2}) \circ P_{1,k} (C^{k-1} \otimes C^{k+2}) + (-1)^{k+1} (\text{id} \otimes (k-1) \otimes b') \circ (m_{2,1} - m_{1,2}) \circ P_{1,k} (C^{k-1} \otimes C^{k+2}) + (\text{id} \otimes U_{k-1,k+1}^{k-2,k+1}) \circ P_{1,k} (C^{k-1} \otimes C^{k+2}) + (-1)^k m_{1,k} \circ (m_{2,1} - m_{1,2}) \circ P_{1,k} (C^{k-1} \otimes b'C^{k+2}) + (-1)^{k+1} (\text{id} \otimes U_{k-1,k+1}^{k-2,k+1}) \circ P_{1,k} (C^{k-1} \otimes b'C^{k+2}) = 0.\]

We now apply the induction hypothesis: for the preceding mapping \(U_{k-1,k+1}^{k-2,k+1}\), Eqs. (3.33) imply that

\[(3.45) \quad (\text{id} \otimes (k-2) \otimes b') \circ U_{k-1,k+1}^{k-2,k+1} = -(-1)^k U_{k-1,k}^{k-2,k} \circ (\text{id} \otimes (k-1) \otimes b').\]

Using this together with a simple identity

\[(\text{id} \otimes (k-1) \otimes b') \circ (m_{2,1} - m_{1,2}) + (m_{1,k} - m_{k,1}) \circ (\text{id} \otimes (k-1) \otimes b') \circ P_{1,k} - m_{1,k} \circ (m_{2,1} - m_{1,2}) = m_{1,k} \circ m_{1,2} - m_{1,2} \circ m_{1,k+1},\]

we reduce Eq. (3.44) to

\[(3.46) \quad (-1)^{k+1} (m_{1,k} \circ m_{1,2} - m_{1,2} \circ m_{1,k+1}) - m_{1,k} \circ (\text{id} \otimes U_{k-1,k+1}^{k-2,k+1}) - (-1)^k (\text{id} \otimes U_{k-1,k}^{k-2,k+1}) \circ m_{1,k+1} = 0.\]

Writing \(U_{k-1,k+1}^{k-2,k+1} = P_{k-1,1} \circ \tilde{U}\), we have \(m_{1,k} \circ (\text{id} \otimes U_{k-1,k+1}^{k-2,k+1}) = m_{1,2} \circ (\text{id} \otimes \tilde{U})\). Recalling the actual form of \(\tilde{U}\) from (3.43), we then see that Eq. (3.46) is identically satisfied (independently of any properties of \(U_{k-1,k+1}^{k-2,k+1}\)). Hence, Eq. (3.42) is proved.
We next prove Eq. (3.41). In its left-hand side, $U^{k-2,k+2}_{k,k+1}$ is expressed from recursive relation (3.27). We also express $U^{k-1,k+1}_{k,k+1}$ using Eq. (3.25), except for the occurrences of $U^{k-1,k+1}_{k,k+1}$ originating from (3.27); instead, we use Eq. (3.37) to rewrite the thus arising combination \((id \otimes (k-1) \otimes b') \circ U^{k-1,k+1}_{k,k+1}\) as

\[
(id \otimes (k-1) \otimes b') \circ U^{k-1,k+1}_{k,k+1}(C^k \otimes C^{k+1}) = \]

\[
= (-1)^k U^{k-1,k}_{k,k} C^k \otimes C^{k+1} + U^{k-1,k}_{k,k} (b' C^{k+1}) + U^{k-1,k}_{k,k} (b' C^{k+1} \otimes C^k).
\]

In the right-hand side of (3.41), we use recursive relations (3.25) and (3.43) and then apply the induction hypothesis, which implies the equation

\[
(3.47) \quad (-1)^k U^{k-2,k}_{k-1,k} (b' C^k \otimes C^k) + U^{k-2,k}_{k-1,k} \circ U^{k-1,k}_{k,k} (C^k \otimes C^k)
\]

\[
+ (-1)^k (id \otimes (k-2) \otimes b'_{(k+1)}) \circ U^{k-2,k+1}_{k,k} (C^k \otimes C^k)
\]

\[
+ (b'_{(k-1)} \otimes id \otimes (k-2)) \circ U^{k-1,k}_{k,k} (C^k \otimes C^k) = 0,
\]

which we now use to eliminate the term bilinear in \(U\) (as noted above, equations evaluated on \(C^k \otimes C^k\) require graded symmetrization when the argument is stripped off). Straightforward manipulations involving (3.39) and another obvious identity,

\[
(b'_{(k-1)} \otimes id \otimes (k-2)) \circ P_{k,1} = P_{k-1,1} \circ (id \otimes b'_{(k-1)} \otimes id \otimes (k-2))
\]

then show that the terms arising from the second and third lines in (3.47) cancel against similar terms in the left-hand side of (3.41), and Eq. (3.41) therefore becomes (where in the hope to keep the derivation traceable, we have not yet used any relations for $U^{k-2,k+1}_{k-1,k+1}$ and $U^{k-2,k+1}_{k,k}$)

\[
(3.48) \quad U_{k,k}^{k-1,k}(C^k \otimes b' C^{k+1} + (-1)^k b' C^{k+1} \otimes C^k)
\]

\[
+ m_{1,k} \circ \left( (id \otimes U_{k,k}^{k-2,k+1})(C^k \otimes C^{k+1}) - (id \otimes U_{k,k}^{k-2,k+1})(C^{k+1} \otimes C^k) - (-1)^k (id \otimes U_{k,k}^{k-2,k+1}) \circ P_{1,k+1}(C^k \otimes C^{k+1}) \right)
\]

\[
- (id \otimes (k-1) \otimes b') \circ (id \otimes U_{k,k}^{k-2,k+1})(C^k \otimes C^{k+1}) - P_{1,k-1} \circ U_{k,k}^{k-2,k+1} (b' C^k \otimes C^{k+1})
\]

\[
+ (-1)^k (id \otimes b'_{(k-1)} \otimes id \otimes (k-2)) \circ (m_{1,k+1} - m_{k+1,1})(C^k \otimes C^{k+1})
\]

\[
+ (-1)^k P_{1,k-1} \circ U_{k,k}^{k-2,k+1}(C^k \otimes b' C^{k+1}) + P_{1,k-1} \circ U_{k,k}^{k-2,k+1}(b' C^{k+1} \otimes C^k) =
\]

\[
= (-1)^k m_{1,2} \circ \left( (m_{1,k+1} - m_{k+1,1})(C^k \otimes C^{k+1}) + (id \otimes U_{k,k}^{k-1,k})(C^{k+1} \otimes C^k)
\]

\[
+ (-1)^k (id \otimes U_{k,k}^{k-1,k}) \circ P_{1,k+1}(C^k \otimes C^{k+1}) \right)
\]

\[
+ (-1)^k (id \otimes U_{k,k}^{k-2,k})(m_{1,k+1} - m_{k+1,1})(C^k \otimes C^{k+1})
\]

\[
+ 2(id \otimes (U_{k,k}^{k-2,k} \circ (b'_{(k)} \otimes id \otimes (k-2)))) \circ \mathcal{S}^{(k)}_{2,k+1,2,2k+1}(C^{k+1} \otimes C^k).
\]
The $S^{(k)}_{2, k+1; k+2, k+1}$ symmetrization arises here because of the symmetry of the argument in (3.47).

Next, using the recursive relations for $U_{k-1, k+1}^{k-2, k+1}$ and $U_{k, k}^{k-2, k+1}$ in the left-hand side of (3.48), we obtain in accordance with (3.43) that

$$
(3.49) \quad -P_{1,k-1} \circ U_{k-1,k+1}^{k-2,k+1}(b'C^k \otimes C^{k+1}) = \\
= (-1)^k (m_{1,k} - m_{1,1}^*) (b'C^k \otimes C^{k+1}) + (-1)^k (id \otimes U_{k-1,k}^{k-2,k}) \circ P_{1,k} (b'C^k \otimes C^{k+1}),
$$

where the last term is readily seen to cancel one of the two terms produced by the symmetrization in the right-hand side of (3.48). Next, recalling the recursive relation

$$
(3.50) \quad U_{k,k}^{k-2,k+1}(C^k \otimes C^k) = \\
= -P_{1,1} \circ (id \otimes U_{k-1,k}^{k-2,k})(C^k \otimes C^k) - (-1)^k P_{k-1,1} \circ U_{k,k}^{k-1,k}(C^k \otimes C^k)
$$

(which is Eq. (3.31) for $k \rightarrow k-1$ and $m = k-2$), we see that the terms in the right-hand side cancel other occurrences of $U_{k-1,k+1}^{k-2,k+1}$ and $U_{k,k}^{k-1,k+1}$ in (3.48). In the left-hand side of (3.48), we also evaluate

$$
(3.51) \quad (-1)^k (id \otimes b'_{(k-1)} \otimes id^\otimes k) \circ (m_{1,k+1} - m_{1,1}^*) (C^k \otimes C^{k+1}) \\
= (-1)^k (m_{1,k} - m_{1,1}^*) ((id \otimes b') C^k \otimes C^{k+1}),
$$

which cancels most of the first term in the right-hand side of (3.49). For the combination $(id^\otimes (k-1) \otimes b') \circ (id \otimes U_{k-1,k+1}^{k-2,k+1})$ in (3.48), we again use Eq. (3.45) implied by the induction hypothesis.

Equation (3.48) thus becomes

$$
(3.52) \quad m_{1,k} \circ \left( (id \otimes U_{k-1,k+1}^{k-2,k+1})(C^k \otimes C^{k+1}) - (id \otimes U_{k,k}^{k-2,k+1})(C^{k+1} \otimes C^k) \\
- (-1)^k (id \otimes U_{k,k}^{k-2,k+1}) \circ P_{1,k+1}(C^k \otimes C^{k+1}) \\
+ (-1)^k (m_{1,k} - m_{1,1}^*) (m_{1,2} C^k \otimes C^{k+1}) \right) = \\
= (-1)^k m_{1,2} \circ \left( (m_{1,k+1} - m_{1,1}^*)(C^k \otimes C^{k+1}) + (id \otimes U_{k,k}^{k-1,k})(C^{k+1} \otimes C^k) \\
+ (-1)^k (id \otimes U_{k,k}^{k-1,k}) \circ P_{1,k+1}(C^k \otimes C^{k+1}) \\
+ (-1)^k (id \otimes U_{k,k}^{k-2,k})(m_{1,k+1} - m_{1,1}^*)(C^k \otimes C^{k+1}) \\
+ (id \otimes U_{k-1,k}^{k-2,k}) \circ m_{1,2} (C^{k+1} \otimes C^k) \right)
$$

It can be immediately seen that the underlined terms can be replaced with

$$
(3.53) \quad (-1)^k (m_{1,k} \circ m_{1,2} - m_{1,2} \circ m_{1,k+1})
$$
in the left-hand side. Next, recursive relation \( (3.43) \) allows us to calculate

\[
(3.54) \quad m_{1,k} \circ (\text{id} \otimes U_{k-1,k+1}^{k-2,k+1}) = \\
= (-1)^{k+1} m_{1,k} \circ (\text{id} \otimes (P_{k-1,1} \circ (m_{2,1}^* - m_{1,2} + (\text{id} \otimes U_{k-1,k}^{k-2,k}) \circ P_{1,k}))) \\
= (-1)^{k+1} m_{1,k} \circ P_{k,2} \circ (m_{3,2}^* - m_{2,3} + (\text{id} \otimes U_{k-1,k}^{k-2,k}) \circ P_{2,k+1}) \\
= (-1)^{k+1} (m_{1,k} \circ m_{1,2} - m_{1,2} \circ m_{1,k+1}) + (-1)^{k+1} (\text{id} \otimes U_{k-1,k}^{k-2,k}) \circ m_{1,k+1}.
\]

Again, these manipulations do not rely on any properties of \( U_{k-1,k+1}^{k-2,k} \), see Fig. 1 for the corresponding rearrangement of the term involving \( U_{k-1,k}^{k-2,k} \); the terms involving \( m_{3,2}^* \) and \( m_{2,3} \) are transformed similarly. The last term cancels a similar term in the right-hand side of \( (3.52) \), the \( m \circ m \) terms cancel those in \( (3.53) \), and Eq. \( (3.52) \) therefore becomes

\[
-m_{1,k} \circ \left( (\text{id} \otimes U_{k-1,k}^{k-2,k+1})(C^{k+1} \otimes C^k) + (-1)^{k} (\text{id} \otimes U_{k,k}^{k-2,k+1}) \circ P_{1,k+1}(C^k \otimes C^{k+1}) \right) \\
= m_{1,2} \circ \left( (-1)^{k} (\text{id} \otimes U_{k,k}^{k-1,k})(C^{k+1} \otimes C^k) + (\text{id} \otimes U_{k,k}^{k-1,k}) \circ P_{1,k+1}(C^k \otimes C^{k+1}) \right) \\
+ (-1)^{k} (\text{id} \otimes U_{k-1,k}^{k-2,k}) \circ m_{3,1}^* (C^k \otimes C^{k+1}) + (\text{id} \otimes U_{k-1,k}^{k-2,k}) \circ m_{1,2} (C^{k+1} \otimes C^k).
\]

This is readily seen to be an identity in view of recursive relation \( (3.50) \), via an argument entirely similar to the one applied to \( (3.54) \) above. Equation \( (3.41) \) is proved.

3.7. The end of the proof of Theorem 3.4 It remains to prove Eqs. \( (2.15) \), or \( \chi^{(3)} = 0 \), for the BRST differential \( \Omega \). The proof involves algebraic consequences
of Eqs. (2.14), which are already established. Some details are as follows. We write \( \Omega \) as in (2.11) in the case where \( \tau_A \) and \( U_{AB} \) have the form (with summations over repeated indices)

\[
\tau_A = t_A + Z^B_A \bar{P}_B, \\
U_{AB} = U^C_{AB} \bar{P}_C + \frac{1}{2}(-1)^{\varepsilon_C} U^{CD}_{AB} \bar{P}_D \bar{P}_C
\]

with \( \bar{P} \)- and \( C \)-independent \( t_A, Z^B_A, U^C_{AB}, \) and \( U^{CD}_{AB} \). Here, \( \varepsilon(C^A) = \varepsilon(\bar{P}_A) = \varepsilon_A + 1 \). In general, the \( \mathbb{Z}_2 \)-gradings are given by \( \varepsilon(t_A) = \varepsilon_A, \varepsilon(Z^B_A) = \varepsilon_A + \varepsilon_B + 1, \varepsilon(U^C_{AB}) = \varepsilon_A + \varepsilon_B + \varepsilon_C + \varepsilon_D + 1 \), but in our specific case, these are all even. Expanding Eqs. (2.13)–(2.15) in the ghost momenta, we then, in addition to the basic relations \( Z^B_A t_B = 0, Z^B_A Z^C_A = 0, \) and \( [t_A, t_B] = U^C_{AB} t_C \), obtain the commutators involving \( Z^B_A \),

\[
[t_A, Z^B_B] - (-1)^{\varepsilon_A \varepsilon_B} [t_B, Z^B_A] = U^C_{AB} Z^D_C - U^{DC}_{AB} t_C - (-1)^{\varepsilon_B} Z^C_A U^D_{CB} + (-1)^{\varepsilon_A(1+\varepsilon_B)} Z^C_B U^D_{CA} + \frac{1}{2} U^E_{AB} U^D_{CE},
\]

\[
(-1)^{(\varepsilon_B+1)(\varepsilon_D+1)} [Z^D_A, Z^B_B] - (-1)^{(\varepsilon_B+1)(\varepsilon_D+1) + \varepsilon_D} [Z^E_A, Z^B_B] = (-1)^{\varepsilon_D} U^{DC}_{AB} Z^E_C - (-1)^{\varepsilon_D(1+\varepsilon_D)} U^{EC}_{AB} Z^D_C + (-1)^{\varepsilon_B} Z^C_B U^D_{CB} - (-1)^{\varepsilon_A(1+\varepsilon_B)} Z^C_B U^D_{CA} - \frac{1}{2} U^F_{AB} U^D_{CF},
\]

and (from (2.15)) the Jacobi identity

\[
(-1)^{\varepsilon_A \varepsilon_C} U^{D}_{AB} U^{E}_{DC} + \text{cycle}(A, B, C) = 0
\]

and its generalizations involving the four-indexed \( U^{CD}_{AB} \),

\[
(-1)^{\varepsilon_A \varepsilon_C} \left( U^{D}_{AB} U^{E}_{DC} + (-1)^{(\varepsilon_E+1)\varepsilon_C} U^{ED}_{AB} U^{F}_{DC} \right) + \text{cycle}(A, B, C) = 0
\]

and

\[
(-1)^{(\varepsilon_E+1)(\varepsilon_C+1) + \varepsilon_A \varepsilon_C + \varepsilon_E \varepsilon_G} U^{ED}_{AB} U^{FG}_{DC} + \text{cycle}(A, B, C) + \text{cycle}(E, F, G) = 0.
\]

The tedious proof in (3.5)–(3.6) is nothing but the demonstration of Eqs. (3.56) and (3.57). The remaining equations (3.58)–(3.60) can be shown by studying the algebraic consistency conditions for (3.56) and (3.57) and using specific properties of \( Z^B_A \) and “selection rules” for \( U^C_{AB} \) and \( U^{CD}_{AB} \). For this, we evaluate the left-hand side of the Jacobi identity \( (-1)^{\varepsilon_A \varepsilon_C} [[\tau_A, \tau_B], \tau_C] + \text{cycle}(A, B, C) = 0 \). The double commutator involves terms of zeroth, first, and second orders in the ghost momenta,

\[
[[\tau_A, \tau_B], \tau_C] = J_{ABC} + J^H_{ABC} \bar{P}_H + J^{HG}_{ABC} \bar{P}_G \bar{P}_H,
\]
where

\[ J_{ABC} = U_{AB}^D U_{DC}^E t_E \]

and

\[
J_{ABC}^H = -(-1)^{\varepsilon_C} U_{AB}^D U_{DC}^E Z_{ED}^H + (-1)^{\varepsilon_D(\varepsilon_C+1)} U_{AB}^D Z_{CD}^E U_{ED}^H + U_{AB}^D U_{DC}^E Z_{ED}^H \\
+ \frac{1}{2} U_{AB}^D U_{DC}^E U_{KE}^H + (-1)^{\varepsilon_E \varepsilon_H} U_{AB}^D U_{DC}^E U_{KE}^H t_E + (-1)^{\varepsilon_D(\varepsilon_H+1)(\varepsilon_D+\varepsilon_C)} U_{AB}^D U_{DC}^E U_{KE}^H t_E \\
+ (-1)^{\varepsilon_A \varepsilon_C + \varepsilon_B \varepsilon_C + \varepsilon_B [t_C, Z_{DB}^E]} U_{DB}^H - (-1)^{\varepsilon_A \varepsilon_B + \varepsilon_B \varepsilon_C + \varepsilon_A \varepsilon_C + \varepsilon_A [t_C, Z_{DB}^E]} U_{DA}^H
\]

(the expression for \( J_{ABC}^H \) is totally straightforward but rather unwieldy). The vanishing of the cyclic sum of \( (-1)^{\varepsilon_A \varepsilon_C} J_{ABC} \) does not allow us to conclude immediately that the cyclic sum of \( (-1)^{\varepsilon_A \varepsilon_C} U_{AB}^D U_{DC}^E \) vanishes, because as follows from comparing (3.13) and the first equation in (3.55), the \( t_A \) term is present in \( \tau_A \) only in the “irreducible sector,” where \( A = a \), in which case \( t_a \) are elements of a basis in \( A \). We must therefore distinguish between the cases where the indices \( A, B, C \) take the “lower” and “higher” values, i.e., label elements of a basis in \( A \) and elements of a basis in \( A \oplus n \) with \( n \geq 2 \), respectively. But when all three indices take “higher” values, the commutator terms in the formula for \( J_{ABC}^H \) vanish. Moreover, selection rules for the \( U \) mappings (that \( U_{m,n'}^m \) can be nonzero only for \( m + n = m' + n' + 1 \)) show that the terms involving \( t_E \) can also be discarded for generic “higher” values of \( A, B, C \). Among the remaining four terms in \( J_{ABC}^H \), one is proportional to the unit element of the algebra, while the other three are not (when evaluated on generic elements in the corresponding tensor powers of \( A \)). To these three elements (in the first line in the last formula), we apply the same trick of taking the quotient over \( C \cdot 1 \) as in 3.5.3. In the current normalization, this amounts to replacing \( Z_{AB}^E \) with \( (-1)^{\varepsilon_A \delta_B} \). Two of the three terms then cancel each other, while the cyclic sum of the remaining one gives the desired relation (3.58); the remaining equations are shown similarly.

4. ADDITIONAL REMARKS

4.1. “Hamiltonians,” cohomology, and observables. The cohomology problem for the differential \( \Omega \) can in principle be considered on different spaces. As with the general form of \( \Omega \), we borrow one such setting from quantization of gauge theories. In genuine gauge theories, the (quantum) Hamiltonian \( H \in A_{gh}^H \) commutes with \( \Omega \) (and is therefore naturally considered modulo \( \Omega \)-exact terms). A physical requirement is \( gh H = 0 \). In closed-algebra gauge theories, Hamiltonians are typically at most linear in the \( C \) ghosts but can depend on the ghost momenta. We restrict ourself to the at-most-bilinear dependence on the ghost momenta and moreover choose the part depending on the ghost momenta to be “scalar” in the sense
that it commutes with elements of $\mathcal{A}^h$ (the “original” operators of the theory, not involving ghosts). In our setting, such Hamiltonians are therefore given by

$$\mathcal{H} = H + \sum_{r \geq 1} \langle \mathcal{V}_r(C^r), P_r \rangle + \sum_{s > r \geq 1} \langle \mathcal{V}_{r,s}^P (C^{r+s}), P_r \otimes P_s \rangle,$$

where $H \in \mathcal{A}$ and

$$\mathcal{V}_r \in \text{Hom}(\mathcal{A}^\otimes r, \mathcal{A}^\otimes r),$$

$$\mathcal{V}_{r,s}^P \in \text{Hom}(\mathcal{A}^\otimes m, \mathcal{A}^\otimes r \otimes \mathcal{A}^\otimes s).$$

The terms that are linear in the $C$ ghosts in the commutator are given by

$$[\mathcal{H}, \Omega]_{\mathcal{C}^\otimes 1} = [H, C^1] + \mathcal{V}_1(C^1)$$

$$+ \sum_{s \geq 1} t(\text{ad}_H \cdot Z^{s+1} (C^{s+1})) _{P_s} + \sum_{s \geq 2} (-1)^s t(\mathcal{V}^1_{s+1}(C^{s+1})) _{P_s}$$

$$+ \sum_{s \geq 1} (-1)^s t(Z^{s+1} \circ \mathcal{V}_{s+1}(C^{s+1})) _{P_s} + \sum_{s \geq 1} (-1)^s t((\text{id} \otimes \mathcal{V}_s) \circ Z^{s+1}(C^{s+1})) _{P_s}$$

$$+ \sum_{s > r \geq 1} (-1)^{r+s} t(\mathcal{V}^r_{r,s} \circ \mathcal{V}^s_{r,s}(C^{r+s}), P_{r+s-1})$$

$$+ \sum_{r \geq 1} \sum_{s \geq 1} (-1)^{r(s+1)} t((Z^{s+1} \otimes \text{id}^\otimes s) \circ \mathcal{V}^{r+1}_{r+1,s}(C^{r+s+1})) _{P_r \otimes P_s}$$

$$+ \sum_{r \geq 1} \sum_{s \geq 1} (-1)^s t(P_{r+1} (\text{id}^\otimes r \otimes Z^{s+1}) \circ \mathcal{V}^{r+1}_{r+1,s}(C^{r+s+1})) _{P_r \otimes P_s}$$

$$+ \sum_{s > r \geq 1} \sum_{m=1}^{\frac{r+s-2}{2}} (-1)^{r+s} \langle U_{r,s}^m, \mathcal{V}^r_{r,s} (C^{r+s}), P_m \otimes P_{r+s-m-1} \rangle.$$

For any $H \in \mathcal{A}$, there exist mappings $\mathcal{V}_r$ such that $[\mathcal{H}, \Omega] = 0$.$^4$ Indeed, the vanishing of $P_n$-independent terms in the commutator is expressed as

$$[H, C^1] + \mathcal{V}_1(C^1) = 0,$$

and the vanishing of the terms linear in $P_n$ as

$$\text{ad}_H \cdot Z^{n+1} + (-1)^n Z^{n+1} \circ \mathcal{V}_{n+1} + (-1)^n (\text{id} \otimes \mathcal{V}_n) \circ Z^{n+1} = 0,$$

which immediately yields a solution

$$\mathcal{V}_n = (-1)^n L_H.$$

$^4$We note that from the standpoint of constrained systems, a “Hamiltonian” $H \in \mathcal{A}$ is a linear combination of constraints (elements of any chosen basis in $\mathcal{A}$). The associative-algebra counterparts of more general Hamiltonians can be realized by taking $H$ to be an arbitrary endomorphism of a representation space of $\mathcal{A}$.
and it remains to note that $L_H \circ U^{\ell,m+n-1-\ell}_{m,n} - U^{\ell,m+n-1-\ell}_{m,n} \circ L_H = 0$ for any $H \in \mathcal{A}$. Moreover, $H$ of the above form is not $\Omega$-exact, and the cohomology of $\Omega$ on the space of such operators therefore contains the algebra $\mathcal{A}$. The cohomology of $\Omega$ on objects with other, nonzero ghost numbers determines what is called observables in the standard quantum setting.

4.2. Automorphisms. Obviously, the relation $\Omega \Omega = 0$ is preserved by similarity transformations $\Omega \mapsto U \Omega U^{-1}$, where $U$ is an arbitrary invertible operator on the bar resolution; a natural subclass of such operators is given by “inner” automorphisms with $U = e^G$ (assuming that the exponential mapping exists), where $G$, with $[G] = 0$, is an arbitrary operator on the bar resolution. Following the pattern set above, we can further restrict the class of transformations by requiring $G$ to have scalar coefficients except in the “constant” term (even with these operators, the transformed BRST differential would no longer manifestly look like the one corresponding to a closed algebra). In particular, this class contains transformations described in [15] that act “covariantly” on the multiplication $m \in \text{Hom}(\mathcal{A} \otimes \mathcal{A}, \mathcal{A})$, thereby “preserving associativity.” Restricting to transformations that preserve the maximum powers in the ghost and ghost momenta expansion, it may be possible to obtain solutions for the $U$ mappings in which the mappings listed in (3.24) do not necessarily vanish.

4.3. Weyl ordering. The $CP$ ordering of the ghost operators (ghosts to the left and momenta to the right) was chosen in constructing the BRST differential, but other orderings are often preferred in “genuine” quantum systems; most popular are the Weyl (totally symmetric) and Wick orderings. In application to associative algebras, the $CP$ ordering is “nonminimal” in the following sense. As noted above, a totally straightforward part of the construction of $\Omega$ is related to the (graded) antisymmetrized part of the multiplication in $\mathcal{A}$, i.e., to the associated Lie algebra (3.5). The entire reducible gauge theory formalism then serves to incorporate the (graded) symmetrized part of the multiplication, $\{t_a, t_b\} = \sum_c f_{\{ab\}}^c t_c$. But the antisymmetric sector is actually “admixed” to the relations in the “reducible” part of the formalism; for example, the tensor $Z_{ab}$ (see (3.6)) involves an antisymmetric part with respect to its lower indices, and the corresponding projection of the equation for $Z^2$ is satisfied as a consequence of (3.5).

In the Weyl ordering, this redundancy is eliminated, but most of the relations following from $\Omega \Omega = 0$ become somewhat less transparent. Using $Z_{ab}^c$, etc. to denote the coefficients in the Weyl-ordered BRST operator, we then have

$$\sum_c (Z_{ab}^c t_c + t_c Z_{ab}^c) = 0$$
instead of Eq. (3.6), and hence,

\[ Z^c_{ab} = t_a \delta^c_b - \frac{1}{2} f^c_{\{ab\}}. \]

Next, instead of (3.7) with \( n = 2 \), we have an equation with a nonzero right-hand side,

\[ Z_{de}^{ab} Z^{c}{}_{ab} + Z^{ab} Z_{de}^{ab} = -\frac{1}{2} f^b_{\{de\}} f^c_{\{eb\}} \]

with

\[ Z_{de}^{ab} = t_d \delta^a_e \delta^b_f - \frac{1}{2} f^f_{\{de\}} \delta^b_f + \frac{1}{2} \delta^a_d f^b_{\{ef\}}, \]

and so on.

We also note that in the notation similar to that in 2.3, with \( \tau_A \) and \( U_{AB} \) now used to denote the respective coefficients in the Weyl form of the BRST operator (2.11) for a closed-algebra theory, the equations analogous to (2.13)–(2.15) (derived under the same conditions as in (2.12)) begin with zero-order ones in the expansion in the ghosts (with summations over repeated indices understood),

\[ (-1)^{\epsilon(C^A)+1} [[\tau_A, C^B], [\tau_B, C^A]] + (-1)^{\epsilon(C^B)+\epsilon(C^D)} [[\tau_D, C^B], C^A] [U_{AB}, C^D] = 0. \]

Equations corresponding to the first and second order in the ghosts are given by

\[ [\tau_D, C^B] \tau_B + (-1)^{(\epsilon(C^B)+1)(\epsilon(C^D)+1)} \tau_B [\tau_D, C^B] \]

\[ - (-1)^{\epsilon(C^B)} \frac{1}{4} [[U_{DE}, C^B], C^A][U_{AB}, C^E] = 0 \]

and

\[ [\tau_A, \tau_B] = [U_{AB}, C^D] \tau_D + (-1)^{\epsilon(C^B)} [\tau_A, C^D] U_{DB} \]

\[ + (-1)^{(\epsilon(C^A)+1)\epsilon(C^B)} [\tau_B, C^D] U_{DA}. \]

In a theory with a closed algebra, the \( C^3 \)-equations are not modified compared with (2.15).

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