Regularized finite difference methods for the logarithmic Klein-Gordon equation

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Abstract

We propose and analyze two regularized finite difference methods for the logarithmic Klein-Gordon equation (LogKGE). Due to the blowup phenomena caused by the logarithmic nonlinearity of the LogKGE, it is difficult to construct numerical schemes and establish their error bounds. In order to avoid singularity, we present a regularized logarithmic Klein-Gordon equation (RLogKGE) with a small regularized parameter $0 < \varepsilon \ll 1$. Besides, two finite difference methods are adopted to solve the regularized logarithmic Klein-Gordon equation (RLogKGE) and rigorous error bounds are estimated in terms of the mesh size $h$, time step $\tau$, and the small regularized parameter $\varepsilon$. Finally, numerical experiments are carried out to verify our error estimates of the two numerical methods and the convergence results from the LogKGE to the RLogKGE with the linear convergence order $O(\varepsilon)$.

Keywords: logarithmic Klein-Gordon equation; regularized logarithmic Klein-Gordon equation; finite difference method; error estimate; convergence order

1. Introduction

The logarithmic Klein-Gordon equation (LogKGE) known as the relativistic version of the logarithmic Schrödinger equation [1] has been introduced in the quantum field theory by Rosen [2] and has the form

\[
\begin{cases}
    u_{tt}(\mathbf{x}, t) - \Delta u(\mathbf{x}, t) + u(\mathbf{x}, t) + \lambda u(\mathbf{x}, t) \ln(\|u(\mathbf{x}, t)\|^2) = 0, & \mathbf{x} \in \mathbb{R}^d, t > 0, \\
    u(\mathbf{x}, 0) = \phi(\mathbf{x}), & \partial_t u(\mathbf{x}, 0) = \gamma(\mathbf{x}), & \mathbf{x} \in \mathbb{R}^d,
\end{cases}
\]

where $\mathbf{x} = (x_1, \ldots, x_d)^T \in \mathbb{R}^d, (d = 1, 2, 3)$ is the spatial coordinate, $t$ is time, $u := u(\mathbf{x}, t)$ is a real-valued scalar field, the parameter $\lambda$ measures the force of the nonlinear interaction. This kind of nonlinearity frequently appears in inflation cosmology and supersymmetric field theories [2, 3, 4]. The LogKGE (1) has been used to describe the spinless particle [5] in optics, nuclear physics and geophysics [6, 7, 8, 9]. Assume that $u(\cdot, t) \in H^1(\mathbb{R}^d)$ and $\partial_t u(\cdot, t) \in L^2(\mathbb{R}^d)$, the LogKGE (1) admits the energy conservation law [10, 11], which is defined as:

\[
E(t) = \int_{\Omega} \left[ (u_t(\mathbf{x}, t))^2 + (\nabla u(\mathbf{x}, t))^2 + (1 - \lambda)u^2(\mathbf{x}, t) + \lambda u^2(\mathbf{x}, t) \ln(\|u(\mathbf{x}, t)\|^2) \right] \, d\mathbf{x} \equiv E(0).
\]

The Klein-Gordon equation with logarithmic potentials posses some special analytical solutions in quantum mechanics [12, 13, 14], and the existence of classical solutions and weak solutions have been investigated in [1, 15]. In the paper [13], the author studies the solutions named Gaussons which represent...
Remark 1.1. The Cauchy problem of the LogKGE (NKGE) and the oscillatory NKGE, various analysis and numerical results have been represented in literature. Along the mathematical front, the derivation, Cauchy problem, well-posedness and dynamical properties have been proposed in [18, 19, 20, 21, 22] and the references therein. Along the numerical aspects, a surge of efficient and accurate numerical methods have been proposed and analyzed for the nonlinear Klein-Gordon equation (NKGE) and the oscillatory NKGE in the literature. For example, the standard finite difference time domain (FDTD) methods including energy conservative/semi-implicit/explicit finite difference time domain methods [23, 24, 25, 26, 27], multiscale time integrator Fourier pseudospectral (MWI-FP) method [28], finite element method [29], exponential wave integrator Fourier pseudospectral (EWI-FP) method [23, 30], asymptotic preserving (AP) [31] method, etc. For numerical comparisons of different numerical methods of the NKGE and the oscillatory NKGE, we refer to [23, 32, 33, 34]. However, due to the singularity of the logarithmic nonlinearity at the origin, these methods cannot be applied to the LogKGE (1) equation directly.

In order to avoid blowup of the LogKGE (1), i.e., \( \log |u| \to -\infty, |u| \to 0 \), we consider a regularized logarithmic Klein-Gordon equation (RLogKGE) with a small regularized parameter \( \varepsilon \leq 1 \),

\[
\begin{aligned}
&\left\{ u_{\varepsilon}^t(x, t) - \Delta u_{\varepsilon}(x, t) + u_{\varepsilon}(x, t) + \lambda u_{\varepsilon}(x, t) \ln \left( \varepsilon^2 + (u_{\varepsilon}(x, t))^2 \right) = 0, \quad x \in \mathbb{R}^d, t > 0, \\
&u_{\varepsilon}(x, 0) = \phi(x), \quad \partial_t u_{\varepsilon}(x, 0) = \gamma(x), \quad x \in \mathbb{R}^d.
\end{aligned}
\]  

(3)

The above RLogKGE (3) is time symmetric or time reversible, i.e., they are invariant if interchanging \( n + 1 \leftrightarrow n - 1 \) and \( \tau \leftrightarrow -\tau \).

Remark 1.1. The Cauchy problem of the LogKGE (1) and the RLogKGE (3), the convergence estimate between the regularized model (3) and the LogKGE (1) will be represented in another paper.

Theorem 1.1. Assume \( u_{\varepsilon}^t(\cdot, t) \in H^1(\mathbb{R}^d) \) and \( \partial_t u_{\varepsilon}^t(\cdot, t) \in L^2(\mathbb{R}^d) \), the RLogKGE (3) conserves energy conservation law, which is defined as:

\[
E_{\varepsilon}(t) = \int_\Omega \left[ (u_{\varepsilon}^t(x, t))^2 + (\nabla u_{\varepsilon}(x, t))^2 + (u_{\varepsilon}(x, t))^2 + \lambda F_{\varepsilon}((u_{\varepsilon}(x, t))^2) \right] dx \equiv E_{\varepsilon}(0),
\]  

(4)

where \( F_{\varepsilon}(\rho) = \int_0^\rho \ln(\varepsilon^2 + s) ds = \rho \ln(\varepsilon^2 + \rho) + \varepsilon^2 \ln(1 + \frac{\rho}{\varepsilon^2}) - \rho, \quad \rho = (u_{\varepsilon}(x, t))^2 \).

Proof.

\[
\frac{d}{dt} E_{\varepsilon}(t) = 2 \int_\Omega \left[ u_{\varepsilon} \cdot u_{\varepsilon}^t + \nabla u_{\varepsilon} \cdot \nabla u_{\varepsilon}^t + u_{\varepsilon}^2 + \lambda F_{\varepsilon}'((u_{\varepsilon}(x, t))^2) \cdot u_{\varepsilon} \cdot u_{\varepsilon}^t \right] (x, t) dx
\]

\[
= 2 \int_\Omega \left[ u_{\varepsilon} \left( u_{\varepsilon}^t - \Delta u_{\varepsilon} + u_{\varepsilon}^2 + \lambda u_{\varepsilon} \ln \left( \varepsilon^2 + (u_{\varepsilon}(x, t))^2 \right) \right) \right] (x, t) dx = 0.
\]  

(5)

This ends the proof.

The main purpose of this work is to analyze two FDTD schemes for the RLogKGE (3) and study the efficiency, accuracy between the LogKGE (1) and the RLogKGE (3) as well as their numerical simulations. The rest of this paper is organized as follows. In Section 2, a semi-implicit and an explicit FDTD schemes are proposed for the RLogKGE (3). Besides, we analyze the stability and solvability of the two schemes. The details of error analysis are established in Section 3. Section 4 is devoted to verifying our error estimates using the numerical experiments. At last, some concluding remarks are drawn in Section 5. Throughout this paper, we denote \( p \lesssim q \) to represent that there exists a generic constant \( C \) which is independent of \( \tau, h, \varepsilon \), such that \( |p| \leq Cq \).

2. FDTD methods and their stability

In this section, we construct two FDTD schemes to approximate the RLogKGE (3) and study their stability, solvability and analyze their error estimates. For simplicity of notations, we set \( \lambda = 1 \) and only
make analysis and construct numerical schemes in one dimensional space \((d = 1)\) for the RLogKGE (3). When \(d = 1\), we truncate the RLogKGE (3) with periodic boundary conditions

\[
\begin{align*}
  u_t(x, t) - \Delta u^r(x, t) + u^c(x, t) + u^e(x, t) \ln \left( \varepsilon^2 + (u^e(x, t))^2 \right) &= 0, \quad x \in \Omega = (a, b), \quad t > 0, \\
  u^c(x, 0) &= \phi(x), \quad \partial_x u^c(x, 0) = \gamma(x), \quad x \in \Omega = [a, b].
\end{align*}
\]  

(6)

2.1. FDTD methods

Choose time step \(\tau := \Delta t\) and time steps \(t_n := n \tau, n = 0, 1, 2, \ldots\); let the mesh size \(h := \frac{b-a}{N}\) with \(N\) being a positive integer and denote the grid points as \(x_j := a + jh, j = 0, 1, \ldots, N\). Define the index sets as:

\[
T_N = \{j | j = 0, 1, 2, \ldots, N - 1\}, \quad T_N^0 = \{j | j = 0, 1, 2, \ldots, N\}.
\]  

(7)

Assume \(u^{ε,n}_j, u^n_j\) are the approximations of the exact solution \(u^ε(x, t_n)\) and \(u(x_j, t_n), j \in T_N^0\) and \(n \geq 0\). Define \(u^{ε,n} = (u^{ε,n}_0, u^{ε,n}_1, \ldots, u^{ε,n}_N)^T, u^n = (u^n_0, u^n_1, \ldots, u^n_N)^T \in \mathbb{C}^{N+1}\) as the numerical solutions vector at time \(t = t_n\). The followings are the finite difference operators:

\[
\begin{align*}
  \delta^+_t u^n_j &= \frac{u^{n+1}_j - u^n_j}{\tau}, & \delta^-_t u^n_j &= \frac{u^n_j - u^{n-1}_j}{\tau}, & \delta^2_t u^n_j &= \frac{u^{n+1}_j - 2u^n_j + u^{n-1}_j}{\tau^2}, \\
  \delta^+_x u^n_j &= \frac{u^{n+1}_j - u^n_j}{h}, & \delta^-_x u^n_j &= \frac{u^n_j - u^{n-1}_j}{h}, & \delta^2_x u^n_j &= \frac{u^{n+1}_j - 2u^n_j + u^{n-1}_j}{h^2}.
\end{align*}
\]

We denote a space of grid functions

\[
X_N = \{u | u = (u_0, u_1, u_2, \ldots, u_N)^T, u_0 = u_N, u_{-1} = u_{N-1} \} \subseteq \mathbb{C}^{N+1}.
\]  

(8)

We define the standard discrete \(l^2\), semi-\(H^1\) and \(l^\infty\) norms and inner product over \(X_N\) as follows

\[
\|u\|_{l^2}^2 = (u, u) = h \sum_{j=0}^{N-1} |u_j|^2, \quad \|\delta^+_x u\|_{l^2}^2 = h \sum_{j=0}^{N-1} |\delta^+_x u_j|^2, \quad \|u\|_{l^\infty} = \sup_{0 \leq j \leq N-1} |u_j|, \quad (u, v) = h \sum_{j=0}^{N-1} u_j v_j,
\]  

(9)

where \(u, v \in X_N\), and \((\delta^2_xu, v) = -(\delta^+_x u, \delta^-_x v)\) \(= (u, \delta^2_x v)\). In the following, we introduce two frequently used FDTD methods for the RLogKGE (3):

1. **Semi-implicit finite difference (SIFD) scheme**

\[
\delta^2_t u^{ε,n}_j - \frac{1}{2} \delta_x^2 (u^{ε,n}_j + u^{ε,n-1}_j) + \frac{1}{2} (u^{ε,n+1}_j + u^{ε,n-1}_j) + u^{ε,n}_j f_\varepsilon ((u^{ε,n}_j)^2) = 0, \quad n \geq 1;
\]  

(10)

2. **Explicit finite difference (EFD) scheme**

\[
\delta^2_t u^{ε,n}_j - \delta_x^2 u^{ε,n}_j + u^{ε,n}_j + u^{ε,n}_j f_\varepsilon ((u^{ε,n}_j)^2) = 0, \quad n \geq 1;
\]  

(11)

where, \(f_\varepsilon(\rho) = \ln(\varepsilon^2 + \rho).\) The initial and boundary conditions are discretized as

\[
u^{ε,n+1}_0 = u^{ε,n+1}_N, u^{ε,n+1}_{N-1} = u^{ε,n+1}_0, \quad n \geq 0, \quad u^{ε,0}_j = \phi(x_j), \quad j \in T_N^0.
\]  

(12)

Using the Taylor expansion we can get the first step solution \(u^{ε,1}_j\),

\[
u^{ε,1}_j = \phi(x_j) + \tau \gamma(x_j) + \frac{\tau^2}{2} \left[ \delta_x^2 \phi(x_j) - \phi(x_j) - \phi(x_j) \ln(\varepsilon^2 + (\phi(x_j))^2) \right], \quad j \in T_N^0.
\]  

(13)

It is easy to prove that the above FDTD schemes are all time symmetric or time reversible.
2.2. Stability analysis

Let $0 < T < T_{\text{max}}$ with $T_{\text{max}}$ being the maximum existence time. Define

$$\sigma_{\text{max}} := \max\{|\ln(\varepsilon^2)|, |\ln(\varepsilon^2 + \|u^{\varepsilon,n}\|_{2,\infty}^2)|\}, \quad 0 \leq n \leq \frac{T}{\tau} - 1.$$  \hspace{1cm} (14)

According to the von Neumann linear stability analysis, we can get the following stability results for the FDTD schemes.

**Theorem 2.1.** For the above FDTD schemes applied to the RLogKGE (3) up to $t = T$, we have:

(i) When $-1 \leq \sigma_{\text{max}} \leq 1$, the SIFD scheme (10) is unconditionally stable; and when $\sigma_{\text{max}} > 1$, it is conditionally stable under the stability condition

$$\tau \leq \frac{2}{\sqrt{\sigma_{\text{max}} - 1}}.$$  \hspace{1cm} (15)

(ii) The EFD scheme (11) is conditionally stable under the stability condition

$$\tau \leq \frac{2h}{\sqrt{(\sigma_{\text{max}} + 1)h^2 + 4}}.$$  \hspace{1cm} (16)

**Proof.** Substituting $u_{j}^{\varepsilon,n-1} = \sum_{l} \hat{U}_{l} e^{2ij\pi/N}$, $u_{j}^{\varepsilon,n} = \sum_{l} \xi_{l} \hat{U}_{l} e^{2ij\pi/N}$, $u_{j}^{\varepsilon,n+1} = \sum_{l} \xi_{l}^2 \hat{U}_{l} e^{2ij\pi/N}$, into (10)-(11), where $\xi_{l}$ is the amplification factor of the $l$th mode in phase space, we can get the characteristic equation with the following structure

$$\xi_{l}^2 - 2\theta_{l}\xi_{l} + 1 = 0, \quad l = -\frac{N}{2}, \ldots, \frac{N}{2} - 1.$$  \hspace{1cm} (17)

where $\theta_{l}$ is invariant with different methods. By the above equation, we get $\xi_{l} = \theta_{l} \pm \sqrt{\theta_{l}^2 - 1}$. The stability of numerical schemes amounts to

$$|\xi_{l}| \leq 1 \iff |\theta_{l}| \leq 1, \quad l = -\frac{N}{2}, \ldots, \frac{N}{2} - 1.$$  \hspace{1cm} (18)

Denote $s_{l} = \frac{2}{h} \sin \left(\frac{\pi l}{N}\right), \quad l = -\frac{N}{2}, \ldots, \frac{N}{2} - 1$, we have

$$0 \leq s_{l}^2 \leq \frac{4}{h^2}.$$  \hspace{1cm} (19)

Firstly, we prove linear stability. We assume $f_{\varepsilon}((u^{\varepsilon})^2) = \alpha$, and $\alpha$ is a constant satisfying $\alpha > -1$, then (10) and (11) are linear.

(i) For the SIFD scheme (10), we have

$$\theta_{l} = \frac{2 - \alpha \tau^2}{2 + \tau^2 (s_{l}^2 + 1)}, \quad l = -\frac{N}{2}, \ldots, \frac{N}{2} - 1.$$  \hspace{1cm} (20)

When $-1 \leq \alpha \leq 1$, it implies that $|\theta_{l}| \leq 1$ and the SIFD scheme (10) is unconditional stable. On the other hand, when $1 < \alpha$, we have

$$2 - \alpha \tau^2 \geq -2 - \tau^2,$$  \hspace{1cm} (21)

it implies that, when $\tau \leq \frac{2}{\sqrt{\alpha - 1}}$, the SIFD scheme (10) is stable.

And when SIFD scheme is nonlinear, with the same method we can get stability condition is

$$\tau \leq \frac{2}{\sqrt{\sigma_{\text{max}} - 1}}.$$  \hspace{1cm} (22)
(ii) For the EFD scheme (11), we have
\[
\theta_l = \frac{2 - \tau^2(1 + \alpha + s_l^2)}{2}, \quad l = -\frac{N}{2}, \ldots, \frac{N}{2} - 1.
\] (24)
By (20), we get
\[
\tau^2(\alpha + 1 + s_l^2) \leq \tau^2(\alpha + 1 + \frac{4}{h^2}) \leq 4, \Rightarrow |\theta_l| < 1.
\] (25)
It implies that, when \(\tau \leq \frac{2h}{\sqrt{(\alpha+1)h^2+4}}\), the EFD scheme (11) is stable. Besides, when the EFD scheme (11) is nonlinear, the stability condition is
\[
\tau \leq \frac{2h}{\sqrt{\sigma_{\text{max}} + 1)h^2 + 4}}
\] (26)
\[\square\]
Remark 2.1. \(\text{Since the scheme SIFD (10) is linear, and the coefficient matrix is strictly diagonal, it is easy to conclude that the SIFD (10) is solvable. In addition, (11) is explicit, so it is evident that there exists a unique solution.}\]

3. Error estimates

3.1. Main results

Motivated by the analytical results in [27, 23, 35, 36, 37], we will establish the error estimates of the FDTD schemes. Assume that the solution \(u^\varepsilon\) is smooth enough over \(\Omega_T : \Omega \times [0, T]\), i.e.
\[
(A) \quad u^\varepsilon \in C\left([0, T]; H^5(\Omega)\right) \cap C^2\left([0, T]; H^4(\Omega)\right) \cap C^4\left([0, T]; H^2(\Omega)\right),
\] (27)
and there exist \(\varepsilon_0 > 0\) and \(C_0 > 0\) independent of \(\varepsilon\) such that
\[
\|u^\varepsilon\|_{L^\infty(0,T;H^5(\Omega))} + \|\partial_t^2 u^\varepsilon\|_{L^\infty(0,T;H^4(\Omega))} + \|\partial_t^4 u^\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} \leq C_0,
\] (28)
is valid uniformly in \(0 \leq \varepsilon \leq \varepsilon_0\).
Denote \(\Lambda = \|u^\varepsilon(x,t)\|_{L^\infty(\Omega_T)}\) and the grid ‘error’ function \(e^\varepsilon,n \in X_N\) \((n \geq 0)\) as
\[
e_j^\varepsilon,n = u^\varepsilon(x_j,t_n) - u_j^\varepsilon,n, \quad j \in \mathcal{T}_N^n, \quad n = 0, 1, 2, \ldots,
\] (29)
where \(u^\varepsilon\) and \(u_j^\varepsilon,n\) are the exact solution and numerical approximation of (6) respectively.

Theorem 3.1. \(\text{Assume } \tau \leq \frac{1}{2}\min\{1, h\} \text{ and under the assumption (A), there exist } h_0 > 0, \tau_0 > 0 \text{ sufficiently small and independent of } \varepsilon, \text{ for any } 0 < \varepsilon \ll 1, \text{ when } 0 < h \leq h_0 \text{ and } 0 < \tau \leq \tau_0 \text{ and under the stability condition (15), the SIFD (10) with (12) and (13) satisfies the following error estimates}\)
\[
\|\delta_x e_j^\varepsilon,n\|_{L^2} + \|e_j^\varepsilon,n\|_{L^2} \lesssim \varepsilon^\alpha (\ln(\varepsilon^2))^2 (\tau^2 + h^2), \quad \|u^\varepsilon,n\|_{L^\infty} \leq \Lambda + 1.
\] (30)

Theorem 3.2. \(\text{Assume } \tau \leq \frac{1}{4}\min\{1, h\} \text{ and under the assumption (A), there exist } h_0 > 0, \tau_0 > 0 \text{ sufficiently small and independent of } \varepsilon, \text{ for any } 0 < \varepsilon \ll 1, \text{ when } 0 < h \leq h_0 \text{ and } 0 < \tau \leq \tau_0 \text{ and under the stability condition (16), the EFD (11) with (12) and (13) satisfies the error estimates}\)
\[
\|\delta_x e_j^\varepsilon,n\|_{L^2} + \|e_j^\varepsilon,n\|_{L^2} \lesssim \varepsilon^\alpha (\ln(\varepsilon^2))^2 (\tau^2 + h^2), \quad \|u^\varepsilon,n\|_{L^\infty} \leq \Lambda + 1.
\] (31)

Remark 3.1. \(\text{[23, 27] Extending to 2 and 3 dimensions, the above Theorems are still valid under the conditions } 0 < h \lesssim \sqrt{C_d(h)}, \quad 0 < \tau \lesssim \sqrt{C_d(h)}. \text{ Besides, the inverse inequality becomes}\)
\[
\|u^\varepsilon,n\|_{L^\infty} \lesssim \frac{1}{C_d(h)} \left(\|\delta_x u^\varepsilon,n\|_{L^2} + \|u^\varepsilon,n\|_{L^2}\right),
\] (32)
where \(C_d(h) = 1/|\ln h|\) when \(d = 2\) and when \(d = 3, C_d(h) = h^{1/2}.\)
3.2. Proof of Theorem 3.1 for the SIFD

Define the local truncation error for the SIFD (10) as

\[
\xi_j^0 := \delta_1^+ u^\varepsilon(x_j, 0) - \gamma(x_j) - \frac{\tau}{2} \left[ \delta_2^+ \phi(x_j) - \phi(x_j) - \phi(x_j) \ln(\varepsilon^2 + (\phi(x_j))^2) \right],
\]

\[
\xi_j^{\varepsilon,n} := \delta_1^2 u^\varepsilon(x_j, t_n) - \frac{1}{2} \delta_2^2 (u^\varepsilon(x_j, t_{n+1}) + u^\varepsilon(x_j, t_{n-1})) + \frac{1}{2} (u^\varepsilon(x_j, t_{n+1}) + u^\varepsilon(x_j, t_{n-1}))
\]

\[
+ u^\varepsilon(x_j, t_n) f_x \left((u^\varepsilon(x_j, t_n))^2\right), \quad j \in T_N, \quad 1 \leq n \leq \frac{T}{\tau} - 1,
\]

then we have the following bounds for the local truncation error.

**Lemma 3.1.** Under the assumption (A), we have

\[
\|\xi_j^0\|_{H^1} \lesssim h^2 + \tau^2, \quad 1 \leq n \leq \frac{T}{\tau} - 1.
\]

\[
\|\xi_j^{\varepsilon,n}\|_{L^2} \lesssim h^2 + \tau^2, \quad 1 \leq n \leq \frac{T}{\tau} - 1.
\]

\[
\|\delta_2^+ \xi_j^{\varepsilon,n}\|_{L^2} \lesssim h^2 + \tau^2, \quad 1 \leq n \leq \frac{T}{\tau} - 1.
\]

**Proof.** By (13) and the Taylor expansion, it leads to

\[
|\xi_j^0| \leq \frac{\tau^2}{6} \|\partial_t^3 u\|_{L^\infty(0,T;L^\infty(\Omega))} + \frac{\tau h}{6} \|\partial_2^3 \phi\|_{L^\infty(\Omega)} \lesssim h^2 + \tau^2.
\]

Similarly, we have

\[
|\delta_2^+ \xi_j^{\varepsilon,n}| \leq \frac{\tau^2}{6} \|\partial_t u\|_{L^\infty(0,T;H^1(\Omega))} + \frac{\tau h}{6} \|\partial_2^3 \phi\|_{L^\infty(\Omega)} \lesssim h^2 + \tau^2, \quad j \in T_N.
\]

Therefore

\[
\|\xi_j^{\varepsilon,n}\|_{H^1} \lesssim h^2 + \tau^2.
\]

Noting that

\[
\xi_j^{\varepsilon,n} := \delta_1^2 u^\varepsilon(x_j, t_n) - \frac{1}{2} \delta_2^2 (u^\varepsilon(x_j, t_{n+1}) + u^\varepsilon(x_j, t_{n-1})) + \frac{1}{2} (u^\varepsilon(x_j, t_{n+1}) + u^\varepsilon(x_j, t_{n-1}))
\]

\[
+ u^\varepsilon(x_j, t_n) f_x \left((u^\varepsilon(x_j, t_n))^2\right)
\]

\[
- \partial_t u^\varepsilon(x_j, t_n) - \partial_x u^\varepsilon(x_j, t_n) + u^\varepsilon(x_j, t_n) + u^\varepsilon(x_j, t_n) f_x \left((u^\varepsilon(x_j, t_n))^2\right)
\]

\[
= \left[ \delta_1^2 u^\varepsilon(x_j, t_n) - \partial_t u^\varepsilon(x_j, t_n) \right] - \left[ \frac{1}{2} \delta_2^2 (u^\varepsilon(x_j, t_{n+1}) + u^\varepsilon(x_j, t_{n-1})) - \partial_2 u^\varepsilon(x_j, t_n) \right]
\]

\[
+ \frac{1}{2} (u^\varepsilon(x_j, t_{n+1}) + u^\varepsilon(x_j, t_{n-1})) - u^\varepsilon(x_j, t_n).
\]

Taking the Taylor expansion, we obtain

\[
\xi_j^{\varepsilon,n} = \frac{\tau^2}{12} \alpha_j^{\varepsilon,n} + \frac{\tau^2}{2} \beta_j^{\varepsilon,n} + \frac{h^2}{12} \eta_j^{\varepsilon,n} + \frac{\tau^2}{2} \phi_j^{\varepsilon,n},
\]

where

\[
\alpha_j^{\varepsilon,n} = \int_{-1}^{1} (1 - |s|)^3 \partial_4^3 u^\varepsilon(x_j + t_n + s\tau) ds,
\]

\[
\beta_j^{\varepsilon,n} = \int_{-1}^{1} (1 - |s|) \partial_2^3 u^\varepsilon_x(x_j + t_n + s\tau) ds,
\]

\[
\eta_j^{\varepsilon,n} = \int_{-1}^{1} (1 - |s|)^3 \partial_2^3 u^\varepsilon(x_j + s\tau, t_{n+1}) + \partial_2^3 u^\varepsilon(x_j + s\tau, t_{n-1}) ds,
\]

\[
\phi_j^{\varepsilon,n} = \int_{-1}^{1} (1 - |s|) \partial_2^3 u^\varepsilon(x_j + s\tau) ds.
\]
Applying Cauchy-Schwarz inequality, we obtain

\[
\|\alpha^{\varepsilon,n}\|^2 = \frac{2}{7} \left[ \int_{-1}^{1} (1 - |s|)^6 ds \sum_{j=1}^{N-1} \int_{-1}^{1} |\partial^4 u^\varepsilon(x_j, t_n + s\tau)|^2 ds \right]
\]

which implies that when \( h \leq 1 \),

\[
\|\alpha^{\varepsilon,n}\|^2 \leq \|\partial^4 u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}.
\]

Similarly, we can get

\[
\|\beta^{\varepsilon,n}\|^2 \leq \|\partial^2 u^\varepsilon\|_{L^\infty(0,T;H^3(\Omega))},
\]

\[
\|\phi^{\varepsilon,n}\|^2 \leq \|\partial^4 u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}.
\]

On the other hand, it can be estimated that

\[
\|u^{\varepsilon,n}\|^2 \leq \frac{h}{7} \sum_{j=1}^{N-1} \int_{-1}^{1} (|\partial^4 u^\varepsilon(x_j, t_n + s\tau)|)^2 ds
\]

which yields \( \|u^{\varepsilon,n}\|^2 \leq 2 \|u^\varepsilon\|_{L^\infty(0,T;H^4(\Omega))} \). Therefore, according to the assumption (A), we get

\[
\|\xi^{\varepsilon,n}\|^2 \leq \frac{\tau^2}{12} \|\partial^4 u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} + \tau^2 \|\partial^2 u^\varepsilon\|_{L^\infty(0,T;H^3(\Omega))} + \frac{h^2}{6} \|u^\varepsilon\|_{L^\infty(0,T;H^4(\Omega))}
\]

\[
\leq \tau^2 + h^2.
\]

Using the same approach, we can get

\[
\|\delta^+_j \xi^{\varepsilon,n}\|^2 \leq \frac{\tau^2}{12} \|\partial^4 u^\varepsilon\|_{L^\infty(0,T;H^2(\Omega))} + \tau^2 \|\partial^2 u^\varepsilon\|_{L^\infty(0,T;H^4(\Omega))} + \frac{h^2}{6} \|u^\varepsilon\|_{L^\infty(0,T;H^5(\Omega))}
\]

\[
\leq \tau^2 + h^2.
\]

This ends the proof. \( \square \)

Subtracting (10) from (33), the error \( e^{\varepsilon,n}_j \) satisfies

\[
\delta^+_j e^{\varepsilon,n}_j - \frac{1}{2} \delta^2 u_{ij} e^{\varepsilon,n+1}_j + \frac{1}{2} \delta^2 u_{ij} e^{\varepsilon,n-1}_j = \xi^{\varepsilon,n}_j - s_j^{\varepsilon,n},
\]

\[
e^{\varepsilon,n}_0 = 0, \quad e^{\varepsilon,n}_{-1} = 0, \quad n = 0, 1, \ldots
\]

\[
e^{\varepsilon,n}_j = \tau \xi^{\varepsilon,n}_j, \quad j \in \mathcal{T}_N,
\]
where
\[
\zeta_j^{ε,m} = u^ε(x_j,t_m)f_ε(\{(u^ε(x_j,t_n))^2\}) - u_j^{ε,n}f_ε(\{(u_j^{ε,n})^2\}).
\] (53)

We define the “energy” for the error vector \(e^{ε,n}\) \((n = 0,1\ldots)\) as
\[
E^n_e := \|δ_x^+ e^{ε,n}\|_2^2 + \frac{1}{2} (\|δ_x^+ e^{ε,n+1}\|_2^2 + \|δ_x^+ e^{ε,n}\|_2^2) + \frac{1}{2} (\|e^{ε,n+1}\|_2^2 + \|e^{ε,n}\|_2^2).
\] (54)

Besides, we can get that
\[
E^n_e := \|\xi^0\|_2^2 + \frac{1}{2} \|\delta_x^+ \xi^0\|_2^2 + \frac{1}{2} \|\xi^0\|_2^2 \lesssim (τ^2 + h^2)^2.
\] (55)

**Proof. (Proof of Theorem 3.1)** When \(k = 1\), under the assumption (A), by Lemma 3.1 we can conclude the errors of the first step discretization (13)
\[
e^{ε,0} = 0, \quad \|e_j^{ε,1}\|_{H^1} \lesssim τ^2 + h^2,
\] (56)
for sufficiently small \(0 < τ < τ_1\) and \(0 < h < h_1\). So it is true for \(k = 0,1\).

Assume (52) is valid for \(k \leq n \leq \frac{T}{h} - 1\). Next, we need to verify (52) is true for \(k = n + 1\). Denote
\[
\zeta_j^{ε,m} = u^ε(x_j,t_m)f_ε(\{(u^ε(x_j,t_m))^2\}) - u_j^{ε,m}f_ε(\{(u_j^{ε,m})^2\}).
\] (57)

When \(|u_j^{ε,m}| \leq |u^ε(x_j,t_m)|\), we get
\[
|\zeta_j^{ε,m}| = |u_j^{ε,m} \ln (\varepsilon^2 + (u_j^{ε,m})^2) - u_j^{ε,m} \ln (\varepsilon^2 + (u^ε(x_j,t_m))^2)\]
\[
\quad + u_j^{ε,m} \ln (\varepsilon^2 + (u^ε(x_j,t_m))^2) - u^ε(x_j,t_m) \ln (\varepsilon^2 + (u^ε(x_j,t_m))^2)|
\]
\[
= e_j^{ε,m} \ln (\varepsilon^2 + (u^ε(x_j,t_m))^2) + u_j^{ε,m} \ln \left(1 + \frac{(u_j^{ε,m})^2 - (u^ε(x_j,t_m))^2}{\varepsilon^2 + (u^ε(x_j,t_m))^2}\right)
\]
\[
\leq |e_j^{ε,m}| \ln \left(\max\{\left|\frac{1}{\varepsilon^2}\right|, |\ln(1 + \varepsilon^2)|\}\right) + 2|e_j^{ε,m}|
\]
\[
= |e_j^{ε,m}| \left(\max\{\left|\frac{1}{\varepsilon^2}\right|, |\ln(1 + \varepsilon^2)|\}\right) + 2|e_j^{ε,m}|.
\] (58)

In addition, when \(|u_j^{ε,m}| \geq |u^ε(x_j,t_m)|\), we obtain
\[
|\zeta_j^{ε,m}| = |u^ε(x_j,t_m) \ln (\varepsilon^2 + (u^ε(x_j,t_m))^2) - u^ε(x_j,t_m) \ln (\varepsilon^2 + (u_j^{ε,m})^2)\]
\[
\quad + u^ε(x_j,t_m) \ln (\varepsilon^2 + (u_j^{ε,m})^2) - u_j^{ε,m} \ln (\varepsilon^2 + (u_j^{ε,m})^2)|
\]
\[
= e_j^{ε,m} \ln (\varepsilon^2 + (u_j^{ε,m})^2) + u^ε(x_j,t_m) \ln \left(\frac{\varepsilon^2 + (u^ε(x_j,t_m))^2}{\varepsilon^2 + (u_j^{ε,m})^2}\right)
\]
\[
\leq |e_j^{ε,m}| \ln \left(\max\{\left|\frac{1}{\varepsilon^2}\right|, |\ln(1 + \varepsilon^2)|\}\right) + 2|e_j^{ε,m}|
\]
\[
= |e_j^{ε,m}| \left(\max\{\left|\frac{1}{\varepsilon^2}\right|, |\ln(1 + \varepsilon^2)|\}\right) + 2|e_j^{ε,m}|.
\] (59)

where we use the assumption \(\|u^{ε,m}\|_{∞} \leq 1 + \Lambda\) above for \(m \leq n\). Since \(ε\) is sufficiently small, we have
\[
\|\zeta^{ε,m}\|_2^2 \lesssim \|e^{ε,m}\|_2^2 (\ln ε^2)^2.
\] (60)
Multiplying both sides of (52a) by $h(e^{\varepsilon,m+1} - e^{\varepsilon,m-1})$, then summing up for $j \in T_N$. And by Young’s inequality, Lemma 3.1, and (60) we can obtain

$$E^m_e - E^{m-1}_e = h \sum_{j=0}^{N-1} (\xi_j^{\varepsilon,m} - \xi_j^{\varepsilon,m})(e_j^{\varepsilon,m+1} - e_j^{\varepsilon,m-1})$$

\[\leq h \sum_{j=0}^{N-1} (|\xi_j^{\varepsilon,m}| + |\xi_j^{\varepsilon,m}|)|e_j^{\varepsilon,m+1} - e_j^{\varepsilon,m-1}| \leq \tau \left( (|\varepsilon^{\varepsilon,m}|^2 + |\xi_j^{\varepsilon,m}|^2 + |\delta_t^+e^{\varepsilon,m}|^2 + |\delta_t^-e^{\varepsilon,m-1}|^2) \right) \leq \tau (\tau^2 + h^2)^2 + (\ln \varepsilon)^2 \|e^{\varepsilon,m}\|^2_{L^2} + E^m_e + E^{m-1}_e, \quad 1 \leq m \leq \frac{T}{\tau} - 1. \]

Therefore, there exists a constant $\tau_2 > 0$ sufficiently small and independent of $\varepsilon$ and $h$, such that when $0 < \tau < \tau_2$, we get

$$E^m_e - E^{m-1}_e \leq \tau (\tau^2 + h^2)^2 + (\ln \varepsilon)^2 \|e^{\varepsilon,m}\|^2_{L^2} + E^m_e + E^{m-1}_e, \quad 1 \leq m \leq \frac{T}{\tau} - 1. \quad (62)$$

Summing above the inequalities up to $n$, and noticing (55), the following holds

$$E^n_e \leq (\tau^2 + h^2)^2 + \tau (\ln \varepsilon)^2 \sum_{m=0}^{n-1} E^m_e, \quad 1 \leq n \leq \frac{T}{\tau} - 1. \quad (63)$$

By applying the discrete Gronwall’s inequality [38], we have

$$E^n_e \leq e^{T(\ln \varepsilon)^2} (\tau^2 + h^2)^2, \quad 1 \leq n \leq \frac{T}{\tau} - 1. \quad (64)$$

Recalling $\|\delta_t^+e^{\varepsilon,n+1}\|^2_{L^2} + \|e^{\varepsilon,n+1}\|^2_{L^2} \leq 2E^n_e$ when $0 < \varepsilon \ll 1$, we can get the error estimate

$$\|\delta_t^+e^{\varepsilon,n+1}\|^2_{L^2} + \|e^{\varepsilon,n+1}\|^2_{L^2} \leq e^{T(\ln \varepsilon)^2} (\tau^2 + h^2)^2. \quad (65)$$

By Sobolev inequality, we obtain

$$\|e^{\varepsilon,n}\|^2_{L^\infty} \leq \|\delta_t^+e^{\varepsilon,n}\|^2_{L^2} + \|e^{\varepsilon,n}\|^2_{L^2} \leq e^{T(\ln \varepsilon)^2} (\tau^2 + h^2)^2. \quad (66)$$

Therefore, there exist $\tau_3 > 0, h_2 > 0$ sufficiently small. When $0 < h < h_2, 0 < \tau < \tau_3$, applying the triangle inequality, it implies that

$$\|u^{\varepsilon,n}\|_{L^\infty} \leq \|u^{\varepsilon,1}\|_{L^\infty(\Omega)} + \|e^{\varepsilon,n}\|_{L^\infty} \leq \Lambda + 1. \quad (67)$$

We complete the proof by choosing $h_0 = \min\{h_1, h_2\}$, $\tau_0 = \min\{\tau_1, \tau_2, \tau_3\}$. \hfill \square

3.3. The proof of Theorem for EFD

Define the local truncation error for the EFD (11) as

$$\xi_j^0 := \delta_t^+u^{\varepsilon}(x_j, 0) - \gamma(x_j) - \frac{\tau}{2} \left[ \delta_t^2 \phi(x_j) - \phi(x_j) - \phi(x_j) \ln(\varepsilon^2 + (\phi(x_j))^2) \right], \quad j \in T_N,$n

$$\xi_j^{\varepsilon,m} := \delta_t^2u^{\varepsilon}(x_j, t_n) - \delta_t^2u^{\varepsilon}(x_j, t_n) + u^{\varepsilon}(x_j, t_n) + u^{\varepsilon}(x_j, t_n) f_{\varepsilon}((u^{\varepsilon}(x_j, t_n))^2), \quad 1 \leq n \leq \frac{T}{\tau} - 1, \quad (68)$$

then we have the following bounds for the local truncation error.

Lemma 3.2. Under the assumption (A), we have

$$\|\xi^{\varepsilon,0}\|_{H^1} \leq h^2 + \tau^2, \quad (69)$$

$$\|\xi^{\varepsilon,n}\|_{L^2} \leq h^2 + \tau^2, \quad (70)$$

$$\|\delta_t^\pm \xi^{\varepsilon,n}\|_{L^2} \leq h^2 + \tau^2, \quad 1 \leq n \leq \frac{T}{\tau} - 1. \quad (71)$$
Proof. According to the Lemma 3.1, we have
\[ \|\xi^{\varepsilon,0}\|_{H^1} \lesssim h^2 + \tau^2. \] (72)

Noting that
\[ \xi^{\varepsilon,n}_j := \delta_t^2 u^\varepsilon(x_j, t_n) - \delta_t^2 u^\varepsilon(x_j, t_n) + u^\varepsilon(x_j, t_n) + u^\varepsilon(x_j, t_n) f^\varepsilon \left((u^\varepsilon(x_j, t_n))^2\right) - \partial_t u^\varepsilon(x_j, t_n) - \partial_{xx} u^\varepsilon(x_j, t_n) + u^\varepsilon(x_j, t_n) f^\varepsilon \left((u^\varepsilon(x_j, t_n))^2\right) \]
\[ = \left[ \delta_t^2 u^\varepsilon(x_j, t_n) - \partial_t u^\varepsilon(x_j, t_n) \right] - \left[ \delta_t^2 u^\varepsilon(x_j, t_n) - \partial_{xx} u^\varepsilon(x_j, t_n) \right]. \] (73)

Taking the Taylor expansion, we obtain
\[ \xi^{\varepsilon,n}_j = \frac{\tau^2}{12} \alpha^{\varepsilon,n}_j + \frac{h^2}{6} \beta^{\varepsilon,n}_j, \] (74)
where
\[ \alpha^{\varepsilon,n}_j = \int_{-1}^{1} (1 - |s|)^3 \partial_t^4 u^\varepsilon(x_j + sh, t_n) ds, \quad \beta^{\varepsilon,n}_j = \int_{-1}^{1} (1 - |s|)^3 \partial_t^4 u^\varepsilon(x_j + sh, t_n) ds. \] (75)

By Cauchy-Schwarz inequality, we obtain
\[
\|\alpha^{\varepsilon,n}\|_2^2 = h \sum_{j=1}^{N-1} |\alpha^{\varepsilon,n}_j|^2 \leq h \int_{-1}^{1} (1 - |s|)^6 ds \sum_{j=1}^{N-1} \int_{-1}^{1} |\partial_t^4 u^\varepsilon(x_j + sh, t_n + s\tau)|^2 ds
\]
\[ = \frac{2}{7} \left[ \int_{-1}^{1} \|\partial_t^4 u^\varepsilon(\cdot, t_n + s\tau)\|_{L^2(\Omega)}^2 ds - \int_{-1}^{1} \sum_{j=0}^{N-1} \int_{x_j}^{x_{j+1}} \int_{x_j}^{\omega} \partial_x |\partial_t^4 u^\varepsilon(\cdot, t_n + s\tau)|^2 d\tau d\omega ds \right] \]
\[ \leq \frac{2}{7} \int_{-1}^{1} \left[ \|\partial_t^4 u^\varepsilon(\cdot, t_n + s\tau)\|_{L^2(\Omega)}^2 + 2h \|\partial_t^4 \partial_{x} u^\varepsilon(\cdot, t_n + s\tau)\|_{L^2(\Omega)} \|\partial_t^4 u^\varepsilon(\cdot, t_n + s\tau)\|_{L^2(\Omega)} \right] ds
\]
\[ \leq \max_{0 \leq t \leq T} \left( \|\partial_t^4 u^\varepsilon\|_{L^2(\Omega)} + h \|\partial_t^4 \partial_{x} u^\varepsilon\|_{L^2(\Omega)} \right)^2, \]
which implies that when \( h \leq 1, \)
\[ \|\alpha^{\varepsilon,n}\|_2 \leq \|\partial_t^4 u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}. \] (77)

On the other hand, it can be estimated that
\[ \|\beta^{\varepsilon,n}\|_2^2 \leq h \int_{-1}^{1} (1 - |s|)^6 ds \sum_{j=1}^{N-1} \int_{-1}^{1} |\partial_t^4 u^\varepsilon(x_j + sh, t_n)|^2 ds
\]
\[ \leq \frac{2h}{7} \sum_{j=1}^{N-1} \int_{-1}^{1} \left|\partial_t^4 u^\varepsilon(x_j + sh, t_n)\right|^2 ds \]
\[ \leq \frac{4}{7} \left( \|\partial_t^4 u^\varepsilon(\cdot, t_n)\|_{L^2(\Omega)}^2 \right)
\]
\[ \leq \|u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}^2, \]
which yields that \( \|\beta^{\varepsilon,n}\|_2 \leq \|u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}. \) Therefore, according to assumption (A), we get
\[ \|\xi^{\varepsilon,n}\|_2 \leq \frac{\tau^2}{12} \|\partial_t^4 u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} + \frac{h^2}{6} \|u^\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}
\]
\[ \lesssim \tau^2 + h^2. \] (79)
With the same method, we have
\[
\|\delta^+_T \xi^{e,n}\|_2 \leq \frac{\tau^2}{12} \|\delta^+_T u^e\|_{L^\infty(0,T;H^2(\Omega))} + \frac{h^2}{6} \|u^e\|_{L^\infty(0,T;H^4(\Omega))}
\]
(80)

This completes the proof.

Subtracting (11) from (68), the error \(e_j^{e,n}\) satisfies
\[
\begin{align*}
\delta^+_T e_j^{e,n} - \delta^+_2 e_j^{e,n} + e_j^{e,n} &= \xi_j^{e,n} - \zeta_j^{e,n}, \\
e_j^{e,n} &= e_N^{e,n}, e_{-1}^{e,n} = e_{N-1}^{e,n}, n = 0, 1, \ldots \\
e_j^{e,0} &= 0, e_j^{e,1} = \tau \xi_j^{0}, j \in T_N,
\end{align*}
\]
(81a-81c)

where
\[
\zeta_j^{e,n} = u^e(x_j, t_n) f_{\xi} ((u^e(x_j, t_n))^2) - u_j^{e,n} f_{\xi} ((u_j^{e,n})^2).
\]
(82)

We define the “energy” for the error vector \(e_j^{e,n}\) (\(n = 0, 1, \ldots\)) as
\[
E_e^n := (1 - \frac{\tau^2}{2} - \frac{\tau^2}{h^2}) \|\delta^+_T e_j^{e,n}\|_2^2 + \frac{1}{2} \|e_j^{e,n+1}\|_2^2 + \frac{1}{2} \sum_{j=0}^{N-1} \left[ (e_j^{e,n+1} - e_j^{e,n})^2 + (e_j^{e,n} - e_j^{e,n+1})^2 \right].
\]
(83)

Besides, we can get that
\[
E_e^0 := (1 - \frac{\tau^2}{2} - \frac{\tau^2}{h^2}) \|\delta^+_T e_j^{e,0}\|_2^2 + (\frac{1}{2} + \frac{1}{h^2}) \|e_j^{e,1}\|_2^2 = \|\xi_j^{0}\|_2^2 \lesssim (\tau^2 + h^2)^2.
\]
(84)

\textbf{Proof. (Proof of Theorem 3.2)} When \(m = 1\), under assumption (A), by Lemma 3.1 we can conclude the first step errors of the discretization (13)
\[
e_j^{e,0} = 0, \|e_j^{e,1}\|_{H^1} \lesssim \tau^2 + h^2,
\]
(85)

for sufficiently small \(0 < \tau < \tau_1\) and \(0 < h < h_1\). So it is true for \(m = 0, 1\). Assume (52) is valid for \(m \leq n \leq \frac{\tau}{\tau} - 1\). Next, we need to verify (52) is true for \(m = n + 1\). Denote
\[
\zeta_j^{e,m} = u^e(x_j, t_m) f_{\xi} ((u^e(x_j, t_m))^2) - u_j^{e,m} f_{\xi} ((u_j^{e,m})^2).
\]
(86)

With the same method in Theorem 3.1, we have
\[
\|\zeta^{e,m}\|_2^2 \lesssim \|e^{e,m}\|_2^2 (\ln \varepsilon)^2.
\]
(87)

Besides, under the assumption \(\tau \leq \frac{1}{2} \min\{1, h\}\), we have \(1 - \frac{\tau^2}{2} - \frac{\tau^2}{h^2} \geq \frac{1}{2} > 0\). By
\[
\|\delta_T^m e^{m+1}\|_2^2 = \frac{1}{h} \sum_{j=0}^{N-1} (e_{j+1}^{m+1} - e_j^{m} - \tau \delta_T^1 e_j^{m})^2 \leq \frac{2}{h} \sum_{j=0}^{N-1} (e_{j+1}^{m+1} - e_j^{m})^2 + \frac{2 \tau^2}{h^2} \|\delta_T^1 e_j^{m}\|_2^2,
\]
(88)

we have
\[
E_e^m \geq \frac{1}{4} \|\delta_T^m e^{m+1}\|_2^2 + \frac{1}{2} (\|e^m\|_2^2 + \|e^{m+1}\|_2^2), \quad 1 \leq m \leq n - 1.
\]
(89)

Similar to the proof of Theorem 3.1, there exists a \(\tau_2 > 0\) sufficiently small, when \(0 < \tau \leq \tau_2\), we get
\[
E_e^m \leq e^{T(\ln \varepsilon)^2} (\tau^2 + h^2)^2, \quad 1 \leq m \leq n - 1.
\]
(90)
Therefore, we can get the \((n+1)th\) error estimate
\[
\|\delta x e^{\epsilon,n+1}\|_2^2 + \|e^{\epsilon,n+1}\|_2^2 \leq e^{T(\ln(\delta))}(\tau^2 + h^2)^2.
\] (91)

By Sobolev inequality, we obtain
\[
\|e^{\epsilon,n}\|_l^2 \leq \|\delta x e^{\epsilon,n}\|_2^2 + \|e^{\epsilon,n}\|_2^2 \leq e^{T(\ln(\delta))}(\tau^2 + h^2)^2.
\] (92)

Applying the triangle inequality, it implies that
\[
\|u^{\epsilon,n}\|_{l_\infty} \leq \|u^\epsilon(\cdot, t_n)\|_{L_\infty(\Omega)} + \|e^{\epsilon,n}\|_{l_\infty} \leq \Lambda + 1.
\] (93)

This ends the proof by choosing \(h_0 = h_1, \tau_0 = \min\{\tau_1, \tau_2\}\). □

4. Numerical results

In this section, we represent some numerical experiments of the EFD (11) scheme to quantify the error bounds. Since the results of the SIFD (10) are similar to the EFD (11), we omit the details here for brevity. Here we take \(d = 1, \lambda = 1\) and we define the error functions as:
\[
\hat{e}^\epsilon(t_n) := u(\cdot, t_n) - u^\epsilon(\cdot, t_n), \quad e^\epsilon(t_n) := u^\epsilon(\cdot, t_n) - u^{\epsilon,n}, \quad \hat{e}^\epsilon(t_n) := u(\cdot, t_n) - u^{\epsilon,n}.
\] (94)

Besides, we denote the error functions:
\[
e^\epsilon_\infty(t_n) := \|u^\epsilon(\cdot, t_n) - u^{\epsilon,n}\|_{l_\infty}, \quad e^\epsilon_2(t_n) := \|u^\epsilon(\cdot, t_n) - u^{\epsilon,n}\|_{l_2},
\] (95)
\[
e^\epsilon_{H^1}(t_n) := \sqrt{(e^\epsilon_2(t_n))^2 + \|\delta x (u^\epsilon(\cdot, t_n) - u^{\epsilon,n})\|_2^2}.
\] (96)

Here \(u, u^\epsilon\) are the exact solutions of the LogKGE (1) and the RLogKGE (3), \(u^n, u^{\epsilon,n}\) are the numerical solutions of the LogKGE (1) and the RLogKGE (3).

Example 1. The initial datum is taken as \(\phi(x) = e^{-\frac{(x-k)^2}{2\sigma^2}}, \gamma(x) = \frac{ck}{\epsilon^2 - k^2}e^{-\frac{(kx-ct)^2}{2\epsilon^2 - k^2}}\), and the Gaussian solitary wave solution is
\[
u(x, t) = e^{-\frac{(kx-ct)^2}{2\epsilon^2 - k^2}},
\] (97)
where \(\epsilon = 2, k = 1\). The RLogKGE (3) is simulated on the domain \(\Omega = [-16, 16]\). The ‘exact’ solution \(u^\epsilon\) is obtained numerically by the EFD (11) scheme with \(\epsilon = 10^{-7}\).

Example 2. We take the initial value as \(\phi(x) = \frac{2}{e^{x^2} + \epsilon^2}, \gamma(x) = 0\). The computation domain is chosen as \(\Omega = [-16, 16]\) with periodic boundary conditions. Since the analytical solution is not available in this example. The ‘exact’ solution \(u^\epsilon\) is obtained by the EFD (11) with a small mesh size \(h = 2^{-10}\), and time step \(\tau = 0.01 \times 2^{-9}\). In addition, the ‘exact’ solution \(u\) is approximated by \(u^\epsilon\) with \(\epsilon = 10^{-7}\).

4.1. Convergence of the regularized model

Here we test the order of accuracy of the regularized model, that is the convergence rate between the solutions of the RLogKGE (3) and the LogKGE (1). Figure 1 represents \(\|\hat{e}^\epsilon\|_{l_2}, \|e^\epsilon\|_{l_\infty}, \|\hat{e}^\epsilon\|_{H^1}\) with the scheme EFD (11) for Example 1 and Example 2. The errors are displayed at \(T = 0.5\).

From Figure 1, we can observe that the solutions of the RLogKGE (3) are linearly convergent to the LogKGE (1) with regard to \(\epsilon\), and the convergence rate is \(O(\epsilon)\) in the \(l^2\)-norm, \(l^\infty\)-norm, \(H^1\)-norm.
4.2. Convergence of FDTD to the RLogKGE

Then we check the convergence patterns of the finite difference scheme: EFD (11) to the RLogKGE (3) for various mesh size $h$, time step $\tau$ under any fixed parameter $0 < \varepsilon \ll 1$ for Example 1 and Example 2.

Firstly, we perform test on the temporal errors with the EFD (11) in the $l^2$-norm, $l^\infty$-norm, $H^1$-norm at $T = 1$, depicted in Figure 3. Due to the stability condition of the EFD (11), we set $0 < \tau < \min\{\frac{1}{k}, \frac{1}{h}\}$, varying the mesh size and time step simultaneously as $\tau_j = 0.01 \times 2^{-j}$, $h_j = 2^{-j}$ for $j = 1, \ldots, 7$.

Secondly, for spatial accuracy of the EFD (11) at $T = 1$, we set time step $\tau = \tau_e = 0.01 \times 2^{-9}$, such that the errors from the time discretization are ignored and solve the RLogKGE (3) with the FDTD schemes versus mesh size $h$. The results are displayed in Figure 5. Figure 5 depict $\|e^\varepsilon\|_2, \|e^\varepsilon\|_\infty, \|e^\varepsilon\|_{H^1}$ with different $h$ of the scheme EFD (11) for Example 1 and Example 2.

From Figure 5, we can make the observations: the scheme EFD (11) are uniformly second order accurate for the RLogKGE (3) for any $0 < \varepsilon \ll 1$ which demonstrate the Theorem 3.2.
Figure 3: The temporal errors $e^\epsilon(1)$ in three different norms for Example 2.

Figure 4: The spatial errors $e^\epsilon(1)$ in three different norms for Example 1.

Figure 5: The spatial errors $e^\epsilon(1)$ in three different norms for Example 2.


4.3. Convergence of FDTD to the LogKGE

We check the convergence rates of the finite difference schemes: SIFD (10) and EFD (11) to the LogKGE (1) for Example 1. Tables 1 and 2 display \(l^2\)-norm, \(l^\infty\)-norm, \(H^1\)-norm of \(\tilde{e}(1)\), respectively, for various mesh size \(h\), time step \(\tau\) and parameter \(\varepsilon\).

4.4. The evolution of the solution

Figure 6 represents the numerical solutions of the EFD (11) at three different time \(T = 1, 5, 9\) for Example 2. We take the step size as \(\tau = 0.01 \times 2^{-7}\), and the mesh size as \(h = 2^{-7}\) at the large domain \([-16, 16]\). From Figure 6, we can see that the numerical solutions of those two schemes are very close different \(\varepsilon\) at fixed times. Besides the number of wave crests increase over time. We can conclude the two discritization schemes are stable under the stability conditions.

![Figure 6: The numerical solution \(u^\varepsilon\) in three different time for the scheme EFD (11).](image)

5. Conclusions

To avoid the singularity of the LogKGE (1) at the origin, we proposed the RLogKGE (3) with a small regularized parameter \(0 < \varepsilon \ll 1\). Two finite difference methods: SIFD, EFD were proposed and analyzed theoretically for the RLogKGE, which showed that the orders of accuracy are all second in both space and time. Besides, The numerical results demonstrated that the solutions of the RLogKGE (3) are linearly convergent to the LogKGE (1) at \(O(\varepsilon)\). In addition, the error bounds of FDTD methods to the LogKGE (1) were numerically investigated and depended on \(\tau, h, \varepsilon\).

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Table 1: The convergence of the SIFD (10) scheme to the LogKGE (1) with different $\tau, h, \varepsilon$

| $\|\hat{e}^\varepsilon(1)\|_{l^\infty}$ | $h = 0.1$ | $h/2$ | $h/2^2$ | $h/2^3$ | $h/2^4$ | $h/2^5$ |
|---|---|---|---|---|---|---|
| $\varepsilon = 10^{-3}$ | $4.03E-03$ | $1.03E-03$ | $7.34E-04$ | $7.66E-04$ | $7.74E-04$ | $7.76E-04$ |
| rate | -- | 1.97 | 0.48 | -0.06 | -0.02 | 0.00 |
| $\varepsilon/4$ | $4.03E-03$ | $1.03E-03$ | $2.59E-04$ | $2.24E-04$ | $2.28E-04$ | $2.29E-04$ |
| rate | -- | 1.97 | 1.99 | 0.21 | -0.03 | -0.01 |
| $\varepsilon/4^2$ | $4.03E-03$ | $1.03E-03$ | $2.59E-04$ | $6.52E-05$ | $6.70E-05$ | $6.75E-05$ |
| rate | -- | 1.97 | 1.99 | 1.99 | -0.04 | -0.01 |
| $\varepsilon/4^3$ | $4.03E-03$ | $1.03E-03$ | $2.59E-04$ | $6.50E-05$ | $1.96E-05$ | $1.98E-05$ |
| rate | -- | 1.97 | 1.99 | 1.99 | 1.73 | -0.01 |

| $\|\hat{e}^\varepsilon(1)\|_{l^2}$ | $h = 0.1$ | $h/2$ | $h/2^2$ | $h/2^3$ | $h/2^4$ | $h/2^5$ |
|---|---|---|---|---|---|---|
| $\varepsilon = 10^{-3}$ | $7.72E-03$ | $2.23E-03$ | $1.30E-03$ | $1.24E-03$ | $1.25E-03$ | $1.25E-03$ |
| rate | -- | 1.80 | 0.78 | 0.06 | -0.01 | 0.00 |
| $\varepsilon/4$ | $7.73E-03$ | $1.99E-03$ | $5.93E-04$ | $3.69E-04$ | $3.56E-04$ | $3.56E-04$ |
| rate | -- | 1.96 | 1.75 | 0.68 | 0.05 | 0.00 |
| $\varepsilon/4^2$ | $7.74E-03$ | $1.98E-03$ | $5.07E-04$ | $1.59E-04$ | $1.05E-04$ | $1.02E-04$ |
| rate | -- | 1.97 | 1.97 | 1.68 | 0.59 | 0.05 |
| $\varepsilon/4^3$ | $7.74E-03$ | $1.98E-03$ | $5.02E-04$ | $1.29E-04$ | $4.25E-05$ | $3.01E-05$ |
| rate | -- | 1.96 | 1.98 | 1.96 | 1.60 | 0.50 |

| $\|\hat{e}^\varepsilon(1)\|_H^2$ | $h = 0.1$ | $h/2$ | $h/2^2$ | $h/2^3$ | $h/2^4$ | $h/2^5$ |
|---|---|---|---|---|---|---|
| $\varepsilon = 10^{-3}$ | $1.08E-02$ | $3.07E-03$ | $1.68E-03$ | $1.59E-03$ | $1.59E-03$ | $1.59E-03$ |
| rate | -- | 1.81 | 0.87 | 0.09 | 0.00 | 0.00 |
| $\varepsilon/4$ | $1.08E-02$ | $2.79E-03$ | $8.26E-04$ | $4.91E-04$ | $4.65E-04$ | $4.64E-04$ |
| rate | -- | 1.95 | 1.76 | 0.75 | 0.08 | 0.00 |
| $\varepsilon/4^2$ | $1.08E-02$ | $2.77E-03$ | $7.10E-04$ | $2.21E-04$ | $1.42E-04$ | $1.36E-04$ |
| rate | -- | 1.97 | 1.96 | 1.68 | 0.64 | 0.07 |
| $\varepsilon/4^3$ | $1.08E-02$ | $2.76E-03$ | $6.99E-04$ | $1.80E-04$ | $5.92E-05$ | $4.10E-05$ |
| rate | -- | 1.97 | 1.98 | 1.96 | 1.60 | 0.53 |
| $\varepsilon/4^4$ | $1.08E-02$ | $2.76E-03$ | $6.98E-04$ | $1.76E-04$ | $4.54E-05$ | $1.60E-05$ |
| rate | -- | 1.97 | 1.99 | 1.99 | 1.95 | 1.51 |
Table 2: The convergence of the EFD (11) scheme to the LogKGE (1) with different $\tau, h, \varepsilon$

| $\|\tilde{e}^\varepsilon(1)\|_{l^\infty}$ | $h = 0.1$ | $h/2$ | $h/2^2$ | $h/2^3$ | $h/2^4$ | $h/2^5$ | $\tau = 0.1$ | $\tau/2$ | $\tau/2^2$ | $\tau/2^3$ | $\tau/2^4$ | $\tau/2^5$ |
|--------------------------------------|-----------|--------|--------|--------|--------|--------|-----------|--------|--------|--------|--------|--------|
| $\varepsilon = 10^{-3}$             | $1.63E-03$| $6.76E-04$| $7.43E-04$| $7.68E-04$| $7.75E-04$| $7.76E-03$| $1.70E-03$| $4.31E-04$| $2.14E-04$| $2.25E-04$| $2.28E-04$| $2.29E-04$|
| $\varepsilon/4$                     | $-1.27$   | $-0.14$| $-0.05$| $-0.01$| $0.00$ | $-1.98$   | $1.01$  | $-0.08$| $-0.02$| $0.00$ |
| $\varepsilon/4^2$                   | $1.71E-03$| $4.37E-04$| $1.10E-04$| $6.58E-05$| $6.72E-05$| $6.76E-05$| $1.97E-05$| $1.99E-05$| $0.48$  | $-0.01$|
| $\varepsilon/4^3$                   | $1.71E-03$| $4.37E-03$| $1.10E-04$| $2.76E-04$| $1.97E-05$| $1.99E-05$| $2.00$   | $2.00$  | $0.25$ |
| $\varepsilon/4^4$                   | $1.71E-03$| $4.37E-04$| $1.10E-04$| $2.76E-05$| $6.90E-06$| $5.82E-06$| $2.00$   | $2.00$  | $0.25$ |

| $\|\tilde{e}^\varepsilon(1)\|_{l^2}$ | $h = 0.1$ | $h/2$ | $h/2^2$ | $h/2^3$ | $h/2^4$ | $h/2^5$ | $\tau = 0.1$ | $\tau/2$ | $\tau/2^2$ | $\tau/2^3$ | $\tau/2^4$ | $\tau/2^5$ |
|--------------------------------------|-----------|--------|--------|--------|--------|--------|-----------|--------|--------|--------|--------|--------|
| $\varepsilon = 10^{-3}$             | $3.73E-03$| $1.43E-03$| $1.23E-03$| $1.24E-03$| $1.25E-03$| $1.25E-03$| $3.71E-03$| $9.42E-03$| $3.72E-03$| $9.43E-03$| $2.38E-04$| $1.70E-03$|
| $\varepsilon/4$                     | $-1.39$   | $0.21$ | $-0.01$| $-0.01$| $0.00$ | $1.98$   | $1.99$  | $2.00$  | $2.00$  | $0.25$ |
| $\varepsilon/4^2$                   | $3.72E-03$| $9.43E-04$| $2.52E-04$| $1.15E-04$| $1.02E-04$| $1.02E-04$| $3.72E-03$| $9.42E-04$| $2.38E-04$| $6.53E-05$| $3.23E-05$| $2.93E-05$|
| $\varepsilon/4^3$                   | $3.72E-03$| $9.42E-04$| $2.38E-04$| $6.53E-05$| $3.23E-05$| $2.93E-05$| $3.72E-03$| $9.42E-04$| $2.38E-04$| $6.53E-05$| $3.23E-05$| $2.93E-05$|
| $\varepsilon/4^4$                   | $3.72E-03$| $9.42E-04$| $2.37E-04$| $5.98E-05$| $1.69E-05$| $9.10E-06$| $1.99$   | $1.99$  | $2.00$  | $2.00$  | $0.25$ |

| $\|\tilde{e}^\varepsilon(1)\|_{H^1}$ | $h = 0.1$ | $h/2$ | $h/2^2$ | $h/2^3$ | $h/2^4$ | $h/2^5$ | $\tau = 0.1$ | $\tau/2$ | $\tau/2^2$ | $\tau/2^3$ | $\tau/2^4$ | $\tau/2^5$ |
|--------------------------------------|-----------|--------|--------|--------|--------|--------|-----------|--------|--------|--------|--------|--------|
| $\varepsilon = 10^{-3}$             | $5.05E-03$| $1.92E-03$| $1.59E-03$| $1.59E-03$| $1.59E-03$| $1.59E-03$| $5.05E-03$| $1.31E-03$| $5.11E-04$| $4.68E-04$| $4.64E-04$| $4.64E-04$|
| $\varepsilon/4$                     | $-1.39$   | $0.27$ | $0.01$ | $0.00$ | $0.00$ | $1.98$   | $1.88$  | $1.12$  | $0.19$  | $0.01$ |
| $\varepsilon/4^2$                   | $4.92E-03$| $1.25E-03$| $3.39E-04$| $1.56E-04$| $1.37E-04$| $1.35E-04$| $4.92E-03$| $1.24E-03$| $3.14E-04$| $8.76E-05$| $4.12E-05$| $3.98E-05$|
| $\varepsilon/4^3$                   | $4.92E-03$| $1.24E-03$| $3.11E-04$| $7.87E-05$| $2.27E-05$| $1.25E-05$| $4.92E-03$| $1.24E-03$| $3.11E-04$| $7.87E-05$| $2.27E-05$| $1.25E-05$|
| $\varepsilon/4^4$                   | $4.92E-03$| $1.24E-03$| $3.11E-04$| $7.87E-05$| $1.96E-05$| $5.92E-06$| $4.92E-03$| $1.24E-03$| $3.11E-04$| $7.87E-05$| $1.96E-05$| $5.92E-06$|
References

[1] K. Bartkowski, P. Górka, One-dimensional Klein–Gordon equation with logarithmic nonlinearities, Journal of Physics A: Mathematical and Theoretical 41 (35) (2008) 355201.

[2] G. Rosen, Dilatation covariance and exact solutions in local relativistic field theories, Physical Review 183 (5) (1969) 1186.

[3] K. Enqvist, J. McDonald, Q-balls and baryogenesis in the MSSM, Physics Letters B 425 (3-4) (1998) 309–321.

[4] A. Linde, Strings, textures, inflation and spectrum bending, Physics Letters B 284 (3-4) (1992) 215–222.

[5] J. J. Sakurai, Advanced quantum mechanics, Pearson Education India, 1967.

[6] H. Buljan, A. Šiber, M. Soljačić, T. Schwartz, M. Segev, D. Christodoulides, Incoherent white light solitons in logarithmically saturable noninstantaneous nonlinear media, Physical Review E 68 (3) (2003) 036607.

[7] S. De Martino, M. Falanga, C. Godano, G. Lauro, Logarithmic Schrödinger-like equation as a model for magma transport, EPL (Europhysics Letters) 63 (3) (2003) 472.

[8] E. F. Hefter, Application of the nonlinear Schrödinger equation with a logarithmic inhomogeneous term to nuclear physics, Physical Review A 32 (2) (1985) 1201.

[9] W. Królikowski, D. Edmundson, O. Bang, Unified model for partially coherent solitons in logarithmically nonlinear media, Physical Review E 61 (3) (2000) 3122.

[10] N. Masmoudi, K. Nakanishi, From nonlinear Klein-Gordon equation to a system of coupled nonlinear Schrödinger equations, Mathematische Annalen 324 (2) (2002) 359–389.

[11] S. Machihara, The nonrelativistic limit of the nonlinear Klein-Gordon equation, Funkcialaj Ekvacioj Serio Internacia 44 (2) (2001) 243–252.

[12] E. M. Maslov, Pulsons, bubbles, and the corresponding nonlinear wave equations in n+ 1 dimensions, Physics Letters A 151 (1-2) (1990) 47–51.

[13] I. Bialynicki-Birula, J. Mycielski, Gaussons: solitons of the logarithmic Schrödinger equation, Physica Scripta 20 (3-4) (1979) 539.

[14] V. A. Koutvitsky, E. M. Maslov, Instability of coherent states of a real scalar field, Journal of mathematical physics 47 (2) (2006) 022302.

[15] P. Gorka, Logarithmic Klein-Gordon equation, Acta Physica Polonica B 40 (2009) 59–66.

[16] A. M. Wazwaz, Gaussian solitary wave solutions for nonlinear evolution equations with logarithmic nonlinearities, Nonlinear Dynamics 83 (1-2) (2016) 591–596.

[17] V. Makhankov, I. Bogolubsky, G. Kummer, A. Shvachka, Interaction of relativistic gaussons, Physica Scripta 23 (5A) (1981) 767.

[18] D. Bainov, E. Minchev, Nonexistence of global solutions of the initial-boundary value problem for the nonlinear Klein–Gordon equation, Journal of Mathematical Physics 36 (2) (1995) 756–762.

[19] P. Brenner, W. von Wahl, Global classical solutions of nonlinear wave equations, Mathematische Zeitschrift 176 (1) (1981) 87–121.
[20] S. Ibrahim, M. Majdoub, N. Masmoudi, Global solutions for a semilinear, two-dimensional Klein-Gordon equation with exponential-type nonlinearity, Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences 59 (11) (2006) 1639–1658.

[21] R. Kosecki, The unit condition and global existence for a class of nonlinear Klein-Gordon equations, Journal of Differential Equations 100 (2) (1992) 257–268.

[22] J. C. Simon, E. Taflin, The Cauchy problem for non-linear Klein-Gordon equations, Communications in Mathematical Physics 152 (3) (1993) 433–478.

[23] W. Bao, X. Dong, Analysis and comparison of numerical methods for the Klein–Gordon equation in the nonrelativistic limit regime, Numerische Mathematik 120 (2) (2012) 189–229.

[24] Q. Chang, G. Wang, B. Guo, Conservative scheme for a model of nonlinear dispersive waves and its solitary waves induced by boundary motion, Journal of Computational Physics 93 (2) (1991) 360–375.

[25] D. Duncan, Sympletic finite difference approximations of the nonlinear Klein–Gordon equation, SIAM Journal on Numerical Analysis 34 (5) (1997) 1742–1760.

[26] L. Zhang, Convergence of a conservative difference scheme for a class of Klein–Gordon–Schrödinger equations in one space dimension, Applied Mathematics and Computation 163 (1) (2005) 343–355.

[27] W. Bao, Y. Feng, W. Yi, Long time error analysis of finite difference time domain methods for the nonlinear Klein-Gordon equation with weak nonlinearity, arXiv preprint arXiv:1903.01133.

[28] W. Bao, Y. Cai, X. Zhao, A uniformly accurate multiscale time integrator pseudospectral method for the Klein–Gordon equation in the nonrelativistic limit regime, SIAM Journal on Numerical Analysis 52 (5) (2014) 2488–2511.

[29] W. Cao, B. Guo, Fourier collocation method for solving nonlinear Klein-Gordon equation, Journal of Computational Physics 108 (2) (1993) 296–305.

[30] W. Bao, X. Dong, X. Zhao, An exponential wave integrator sine pseudospectral method for the Klein–Gordon–Zakharov system, SIAM Journal on Scientific Computing 35 (6) (2013) A2903–A2927.

[31] E. Faou, K. Schratz, Asymptotic preserving schemes for the Klein–Gordon equation in the non-relativistic limit regime, Numerische Mathematik 126 (3) (2014) 441–469.

[32] S. Jiménez, L. Vázquez, Analysis of four numerical schemes for a nonlinear Klein-Gordon equation, Applied Mathematics and Computation 35 (1) (1990) 61–94.

[33] P. Pascual, S. Jiménez, L. Vázquez, Numerical simulations of a nonlinear Klein-Gordon model. applications, in: Third Granada Lectures in Computational Physics, Springer, 1995, pp. 211–270.

[34] W. Bao, X. Zhao, Comparison of numerical methods for the nonlinear Klein-Gordon equation in the nonrelativistic limit regime, Journal of Computational Physics 398 (2019) 108886.

[35] W. Bao, R. Carles, C. Su, Q. Tang, Error estimates of a regularized finite difference method for the logarithmic Schrödinger equation, SIAM Journal on Numerical Analysis 57 (2) (2019) 657–680.

[36] W. Bao, C. Su, Uniform error bounds of a finite difference method for the Klein-Gordon-Zakharov system in the subsonic limit regime, Mathematics of Computation 87 (313) (2018) 2133–2158.

[37] W. Bao, C. Su, Uniform error estimates of a finite difference method for the Klein-Gordon-Schrödinger system in the nonrelativistic and massless limit regimes, Kinetic Related Models 11 (4) (2018) 1037–1062.

[38] J. M. Holte, Discrete Gronwall lemma and applications, in: MAA-NCS meeting at the University of North Dakota, Vol. 24, 2009, pp. 1–7.