On Haar Expansion of Riemann–Liouville process in a critical case *

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Abstract
We show that Haar-based series representation of the critical Riemann–Liouville process $R^{\alpha}$ with $\alpha = 3/2$ is rearrangement non-optimal in the sense of convergence rate in $C[0,1]$.

Key words: Approximation of operators and processes, Riemann-Liouville process, series representation, Haar base.

Introduction and main result

The aim of this note is to solve a problem stated by A. Ayache and W. Linde in their recent work [1]. In their article, the quality of several common series representations of fractional processes is considered. In particular, Ayache and Linde examine the representations of Riemann–Liouville processes $R^{\alpha}$, $\alpha > 1/2$, based on Haar and trigonometric systems. They were able to show that Haar-based representation of $R^{\alpha}$ is optimal w.r.t. the uniform norm when $1/2 < \alpha < 3/2$ while for $\alpha > 3/2$ it is not optimal. Their approach however does not yield an answer in the delicate critical case $\alpha = 3/2$. We will show that Haar-based representation is not optimal for $R^{3/2}$ either.

It is an immense pleasure for the author to stress that the main tool used in this note is due to V.N. Sudakov whose anniversary we celebrate in this volume.

The reader can find recent results on series representations of fractional processes and fields in [2], [3], [4], [5], [6], [9], [10].

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Recall that Riemann-Liouville process $R^\alpha$ is defined by white noise representation

$$R^\alpha(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1}dW(u), \quad 0 \leq t \leq 1.$$ 

This process (RL for short) is known to have continuous sample paths whenever $\alpha > 1/2$. It belongs to the family of so called fractional processes along with more widely known fractional Brownian motion $W^H$ (whenever $\alpha \in (1/2, 3/2)$ and $H = \alpha - 1/2$) and periodic stationary Weil process $I^\alpha$. All these processes differ by very smooth terms and therefore their approximation properties we discuss here are essentially the same, see [1] for details. We also refer to [8] for further properties and stable extensions of RL-process.

The representation of RL-process with $\alpha > 1/2$ generated by Haar base writes as follows,

$$R^\alpha(t) = \xi_{-1} t^\alpha \Gamma(\alpha + 1) + \sum_{j=0}^{\infty} \sum_{k=0}^{2^j-1} \xi_{j,k} (R_\alpha h_{j,k})(t)$$ \tag{1}

where $R_\alpha$ is the classical RL integration operator,

$$R_\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-u)^{\alpha-1}h(u)du,$$

$\{\xi_{-1}, (\xi_{j,k})\}$ is a family of i.i.d. standard normal random variables and $h_{j,k}$’s are the Haar functions

$$h_{j,k}(t) = 2^{j/2} \{1_{[\frac{2k+1}{2^{j+1}}, \frac{2k+2}{2^{j+1}})}(t) - 1_{[\frac{2k+1}{2^{j+1}}, \frac{2k+2}{2^{j+1}})}(t)\}.$$ 

The series converges almost surely uniformly on $t \in [0, 1]$, i.e. it converges in the sense of the uniform norm $\| \cdot \|_\infty$.

Recall that one can evaluate the integrated Haar functions by the formula

$$(R_\alpha h_{j,k})(t) = \frac{2^{j/2}}{\Gamma(\alpha + 1)} \left\{ \left( t - \frac{2k+2}{2^{j+1}} \right)^\alpha - 2 \left( t - \frac{2k+1}{2^{j+1}} \right)^\alpha + \left( t - \frac{2k}{2^{j+1}} \right)^\alpha \right\},$$ \tag{2}

Now we recall the necessary standard notation as well as the notions related to finite rank approximation of Gaussian random functions.

Throughout the article $f_n \succeq g_n$ means $\liminf_n f_n/g_n > 0$ while $f_n \asymp g_n$ means that both $f_n \succeq g_n$ and $g_n \succeq f_n$ hold. We write $\#(B)$ for the number of points in a set $B$, while $|T|$ denotes the length of an interval $T$. Finally, $c$ denotes unspecified positive and finite constants which can be different in each occurrence.
Let $X$ be a centered Gaussian random element of a normed space $(\mathcal{X}, \| \cdot \|)$. The $\ell$-numbers (stochastic approximation numbers) of $X$ are defined as

$$
\ell_n(X) := \inf_{(\xi_i, (\varphi_i))} \left\{ \mathbb{E} \left\| X - \sum_{i=1}^{n-1} \xi_i \varphi_i \right\| \right\} .
$$

Here the infimum is taken over all families of random variables $(\xi_i)$ and all finite deterministic subsets $(\varphi_i) \subset \mathcal{X}$. A series representation

$$
X = \sum_{i=1}^{\infty} \xi_i \varphi_i
$$

is called optimal if

$$
\mathbb{E} \left\| \sum_{i=n}^{\infty} \xi_i \varphi_i \right\| \asymp \ell_n(X) \quad \text{as } n \to \infty.
$$

It is called rearrangement non-optimal, if it can not be rendered optimal by any permutation of $(\varphi_i)$. In particular, for non-optimal representation the approximation error $\mathbb{E} \| \sum_{i=n}^{\infty} \xi_i \varphi_i \|$ tends to zero slower than optimal rate $\ell_n(X)$.

In the particular case of RL-process and uniform norm $\| \cdot \|_\infty$, the optimal approximation rate is well known, see [6]. Namely, for any $\alpha > 1/2$ it is true that

$$
\ell_n(R^\alpha) \asymp n^{-1(\alpha-1/2)} \sqrt{\ln n}, \quad \text{as } n \to \infty.
$$

The optimal rate can be attained, for example, by using modified Daubechies wavelet base, see [10]. The remaining question is to understand which representations provide this rate and which ones perform more poorly.

For the critical case, $\alpha = 3/2$, we have

$$
\ell_n(R^{3/2}) \asymp n^{-1} \sqrt{\ln n}, \quad \text{as } n \to \infty. \quad (3)
$$

Our main result shows that Haar-based representation is rearrangement non-optimal.

**Theorem 1** Let $\{\xi_i \varphi_i\}$ be any rearrangement of the Haar-based series representation [1]. Then

$$
\mathbb{E} \left\| \sum_{i=n}^{\infty} \xi_i \varphi_i \right\|_\infty \asymp n^{-1} \ln n, \quad \text{as } n \to \infty.
$$

We observe a gap of order $\sqrt{\ln n}$ w.r.t. the optimal rate (3).
Proofs

From now on, we consider only the critical case \( \alpha = 3/2 \).

Introduce the function \( H(t) = (t-2)^{3/2} - 2(t-1)^{3/2} + t^{3/2} \) which is the unscaled version of integrated Haar functions (2). Namely, we have

\[
(R_{3/2}h_{j,k})(t) = \frac{H \left( \frac{2^{j+1} (t - \frac{2k}{2^{j+1}})}{2^{3/2+j} \Gamma(5/2)} \right)}{2^{3/2+j} \Gamma(5/2)}. 
\] (4)

Notice that for \( t \geq 2 \)

\[
H(t) = \frac{3}{2} \int_{t-1}^{t} (x^{1/2} - (x - 1)^{1/2}) \, dx \sim \frac{3}{4} \, t^{-1/2}, \quad \text{as} \quad t \to \infty. \] (5)

For any level number \( j \) and a set of positive integers \( K \subset \{0, \ldots, 2^j - 1\} \) we consider a Gaussian process \( \{X_K(t), \, t \in [0, 2^{j+1}]\} \) defined by

\[
X_K(t) = \sum_{k \in K} \xi_k H(t - 2k) = \sum_{k \in K, k \leq t/2} \xi_k H(t - 2k),
\]

where \( \{\xi_k\} \) is a family of i.i.d. standard normal random variables. Clearly, \( X_K \) is the unscaled version of the \( K \)-part of level \( j \) in Haar-based representation (1), since by (4)

\[
\sum_{k \in K} \xi_{j,k} (R_{3/2}h_{j,k})(t) = \frac{X_K(2^{j+1}t)}{2^{3/2+j} \Gamma(5/2)}.
\] (6)

in distribution.

**Proposition 2** There exists a numerical constant \( c \) such that

\[
\mathbb{E} \sup_{t \in [0, 2^{j+1}]} |X_K(t)| \geq c \, j,
\]

for all \( j > 0 \) and all \( K \subset \{0, \ldots, 2^j - 1\} \) such that \( \#(K) \geq 2^{j-1} \).

**Corollary 3** By scaling (6) it follows that

\[
\mathbb{E} \sup_{t \in [0, 1]} \left| \sum_{k \in K} \xi_{j,k}(R_{3/2}h_{j,k})(t) \right| \geq c \, j \, 2^{-j},
\] (7)

where \( h_{j,k} \) are Haar functions.
Proof of Proposition 2. First of all, let us evaluate the natural distance associated to the process $X_K$. Let $0 \leq s < t \leq 2^{j+1}$. We have

$$X_K(t) = \sum_{k \in K: k \leq t/2} \xi_k H(t - 2k)$$

$$= \sum_{k \in K: k \leq s/2} \xi_k H(t - 2k) + \sum_{k \in K: s/2 < k \leq t/2} \xi_k H(t - 2k)$$

$$:= Y(s, t) + Z(s, t).$$

Notice that the variable $X_K(s)$ belongs to $\text{span}(\xi_k, k \in K, k \leq s/2)$ and the same is true for $Y(s, t)$, while $Z(s, t)$ is orthogonal to that span. By using (5), it follows that for $s < t$ we have

$$\mathbb{E}(X_K(t) - X_K(s))^2 \geq \mathbb{E} Z(s, t)^2 = \sum_{k \in K: s/2 < k \leq t/2} H(t - 2k)^2$$

$$\geq c \sum_{k \in K: s/2 < k \leq t/2 - 1} (t - 2k)^{-1}.$$

Suppose now that we found in $[0, 2^{j+1}]$ an ordered subsequence $t_1, \ldots, t_m$, $m \geq 2^{j/2-2}$ such that for any $2 \leq i \leq m$ it is true that

$$\sum_{k \in K: t_{i-1}/2 < k \leq t_i/2 - 1} (t_i - 2k)^{-1} \geq c j. \quad (8)$$

Then, by Sudakov lower bound (see e.g. [7], Section 14)

$$\mathbb{E} \sup_{t \in [0, 2^{j+1}]} |X_K(t)| \geq \mathbb{E} \sup_{1 \leq i \leq m} X_K(t_i)$$

$$\geq c (\ln m)^{1/2} \cdot \inf_{i \neq j} \left[ \mathbb{E}(X(t_i) - X(t_j))^2 \right]^{1/2}$$

$$\geq c j^{1/2} \cdot c j^{1/2} = c j.$$

It remains to find a sequence $(t_i)$ satisfying assumption (8). Would $K$ coincide with the maximal set, $K = \{0, \ldots, 2^j - 1\}$, we could just take $t_i = i2^{j/2}, i = 0, \ldots, 2^{j/2} - 1$. However, the situation is more delicate for general case. We will use the following elementary fact worth to be stated separately.

**Lemma 4** Let the points $s_1 < s_2 < \cdots < s_{2q}$ belong to an interval $T$ in $\mathbb{R}$. Then

$$\max_{1 < r \leq 2q} \frac{1}{s_r - s_j} \sum_{i=1}^{r-1} \frac{1}{s_r - s_i} \geq c \frac{q}{|T|} \ln q.$$
Proof of Lemma 4. Take an integer \( j \) such that \( 1 \leq j \leq q \). We have

\[
\sum_{r=q+1}^{2q} (s_r - s_{r-j}) = \sum_{r=q+1}^{2q} \sum_{l=r-j}^{r-1} (s_{l+1} - s_l) \\
\leq \sum_{l=q+1-j}^{2q-1} j(s_{l+1} - s_l) \leq j|T|.
\]

By convexity of the function \( x \to \frac{1}{x} \) we have

\[
\frac{1}{q} \sum_{r=q+1}^{2q} \frac{1}{s_r - s_{r-j}} \geq \left( \frac{1}{q} \sum_{r=q+1}^{2q} (s_r - s_{r-j}) \right)^{-1} \geq \left( \frac{|T|}{q} \right)^{-1}.
\]

By summing up over \( j \) we get

\[
\frac{1}{q} \sum_{r=q+1}^{2q} \sum_{j=1}^{q} \frac{1}{s_r - s_{r-j}} = \sum_{j=1}^{q} \frac{1}{q} \sum_{r=q+1}^{2q} \frac{1}{s_r - s_{r-j}} \\
\geq \sum_{j=1}^{q} \left( \frac{|T|}{q} \right)^{-1} = \frac{q}{|T|} \sum_{j=1}^{q} j^{-1} \geq cq |T| \ln q.
\]

Clearly,

\[
\max_{1 \leq r \leq 2q} \frac{1}{s_r - s_i} \geq \max_{q+1 \leq r \leq 2q} \sum_{i=r-q}^{r-1} \frac{1}{s_r - s_i} = \max_{q+1 \leq r \leq 2q} \sum_{j=1}^{q} \frac{1}{s_r - s_{r-j}} \\
\geq \frac{1}{q} \sum_{r=q+1}^{2q} \sum_{j=1}^{q} \frac{1}{s_r - s_{r-j}} \geq cq |T| \ln q. \quad \square
\]

We continue the proof of Proposition 2. For the sake of expression simplicity, assume that \( j \) is an even number, thus \( 2^{j/2} \) is an integer. We split the interval \([0, 2^j - 1]\) in \( 2^{j/2} \) blocks \( B_t = [t2^{j/2}, (t + 1)2^{j/2}) \).

Let

\[
I = \left\{ t : \#(K \cap B_t) \geq \frac{1}{4} 2^{j/2} \right\}
\]
and $m = \#(I)$. We show that $m$ is large enough, since

$$2^{j-1} \leq \#(K) = \sum_{i=0}^{2^{j/2}-1} \#(K \cap B_i)$$

$$= \sum_{i \in I} \#(K \cap B_i) + \sum_{i \not\in I} \#(K \cap B_i)$$

$$\leq m 2^{j/2} + (2^{j/2} - m) \frac{1}{4} 2^{j/2}$$

yields

$$\frac{1}{2} \leq \frac{m}{2^{j/2}} + \frac{1}{4} - \frac{m}{2^{j/2}},$$

thus

$$m \geq \frac{1}{3} 2^{j/2} > 2^{j/2-2}.\] For each block $B_i, i \in I$, we apply Lemma [4] with $T = B_i$, $|T| = 2^{j/2}$ and \{s_1, \ldots, s_{2q}\} \subset K \cap B_i and choosing $q$ as large as possible,

$$2q \geq \#(K \cap B_i) - 1 \geq \frac{1}{4} 2^{j/2} - 1.$$ By Lemma [4] we can find an integer point $v_i \in \{s_1, \ldots, s_{2q}\} \subset K \cap B_i such that

$$\sum_{s \in K \cap B_i, s < v_i} \frac{1}{v_i - s} \geq \frac{cq}{T} \ln q \geq c j.$$ By renumbering the points \{2v_i, i \in I\} we get a large collection of $m \geq 2^{j/2-2}$ points $t_i$ with the property

$$\sum_{k \in K: t_i - 2k < t_i/2 - 1} (t_i - 2k)^{-1} = \frac{1}{2} \sum_{k \in K: t_i - 2k < t_i/2 - 1} (t_i/2 - k)^{-1}$$

$$\geq \frac{1}{2} \sum_{k \in K: t_i/2 - k \leq t_i/2} (t_i/2 - k)^{-1} - 1$$

$$\geq \frac{1}{2} \sum_{k \in K \cap B_i, k < v_i} (v_i - k)^{-1} - 1 \geq c j,$$

where the index $i$ is such that $t_i/2 = v_i$. We also used here the fact that all $v_i$'s and all $k$'s are integers, thus we don't lose more than one while dropping one $k$. 7
from the sum. Therefore, the collection \(\{t_i, i \leq m\}\) satisfies (8) and Proposition 2 is proved. □

**Proof of Theorem 1.** Actually our theorem follows from Proposition 2 and Lemma 2.3 in [1] immediately. However, we recall the proof for reader’s convenience.

Let \(\{\xi_i \varphi_i\}\) be any rearrangement of the Haar-based series representation (1). Take an integer \(j \geq 1\), let \(n = 2^{j-1}\) and let \(K \subset [0, \ldots, 2^j - 1]\) be the set of all \(k\) such that \(R_{\alpha}h_{j,k}\) belongs to the set \(\{\varphi_i, i \geq n\}\). Clearly, \(#(K) \geq 2^{j-1}\), and we can write

\[
\sum_{i \geq n} \xi_i \varphi_i = \sum_{k \in K} \xi_{j,k} (R_{\alpha}h_{j,k}) + Y,
\]

where \(Y\) and the sum over \(K\) are independent. By standard arguments based on Anderson inequality for Gaussian processes (see [7]), we obtain

\[
\mathbb{E} \left\| \sum_{i \geq n} \xi_i \varphi_i \right\|_\infty \geq \mathbb{E} \left\| \sum_{k \in K} \xi_{j,k} (R_{\alpha}h_{j,k}) \right\|_\infty.
\]

Now (7) yields

\[
\mathbb{E} \left\| \sum_{i \geq n} \xi_i \varphi_i \right\|_\infty \geq c j 2^{-j} \sim n^{-1} \ln n,
\]

and we are done. □

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