STRUCTURE OF ONE-PHASE FREE BOUNDARIES IN THE PLANE

DAVID JERISON AND NIKOLA KAMBUROV

Abstract. We study classical solutions to the one-phase free boundary problem in which the free boundary consists of smooth curves and the components of the positive phase are simply-connected. We show that if two components of the free boundary are close, then the solution locally resembles an entire solution discovered by Hauswirth, Hélein and Pacard, whose free boundary has the shape of a double hairpin. Our results are analogous to theorems of Colding and Minicozzi characterizing embedded minimal annuli, and a direct connection between our theorems and theirs can be made using a correspondence due to Traizet.

1. Introduction.

The one-phase free boundary problem in a disk $B \subseteq \mathbb{R}^2$,

\begin{align*}
&u \geq 0 \quad \text{in} \quad B \\
&\Delta u = 0 \quad \text{in} \quad B^+(u) := \{ x \in B : u(x) > 0 \} \\
&|\nabla u| = 1 \quad \text{on} \quad F(u) := \partial B^+(u) \cap B
\end{align*}

arises as the Euler-Lagrange equation for the functional

\begin{equation}
I(u, B) = \int_B |\nabla u|^2 + 1_{\{u > 0\}} \, dx \quad u : B \to [0, \infty)
\end{equation}

and appears in a variety of applications (e.g. jet flows in hydrodynamics, see [CS05]). The interior regularity theory of minimizers of the functional $I(u, B)$ with fixed boundary conditions on $\partial B$ is well understood. Alt and Caffarelli [AC81] proved that the free boundary $F(u)$ is locally a graph of a $C^\infty$ function (and hence analytic by [KN77]). Alt and Caffarelli also proved partial regularity of free boundaries in higher dimensions and established a strong analogy between the theory of free boundaries and the theory of minimal surfaces.

In keeping with [AC81] and many subsequent results ([ACF84, Caf87, Caf89, Wei98, CJK04, DSJ11, JS]) one should expect that most theorems about minimal surfaces have counterparts in the theory of free boundaries and vice versa. Here we consider classical solutions to (1) that are higher critical points rather than minimizers of the functional $I(u, B)$ with one additional purely topological assumption, namely that

\begin{equation}
\text{no connected component of } F(u) \text{ is compact in the open disk } B.
\end{equation}

By classical solution we mean one for which $F(u)$ is a finite union of analytic curves. The topological assumption is equivalent to saying that the connected components of the positive phase are simply-connected. It is also equivalent to saying that the analytic curves, although they may become tangent at interior points, end at $\partial B$.

Our work is inspired by the groundbreaking work of Colding and Minicozzi on the structure of limits of sequences of embedded minimal surfaces of fixed genus in a ball in $\mathbb{R}^3$ ([CM04a, CM04b, CM04c, CM04d]). As it turns out, because of recent work of Traizet [Tra14], there is a direct overlap between our a priori estimates and rigidity results for families of solutions to (1) and the description of embedded minimal topological annuli due to Colding and Minicozzi.

Our starting point is the family of simply-connected planar regions $\Omega_a = a\Omega_1$, discovered by Hauswirth, Hélein, and Pacard [HHP11], which solve the free boundary problem (1). They are defined by

\begin{equation}
\Omega_a := \{ (x_1, x_2) \in \mathbb{R}^2 : |x_1/a| < \pi/2 + \cosh(x_2/a) \}, \quad a > 0.
\end{equation}

2010 Mathematics Subject Classification. 35R35, 35N25, 35Q35, 35BXX, 49Q05.

Key words and phrases. one-phase free boundary problem, overdetermined elliptic problem, minimal surfaces, compactness and singular limits.

The first author was supported by NSF grant DMS 1069225 and the Stefan Bergman Trust.
The boundary $\partial \Omega_a$ consists of two curves that we will refer to as hairpins. Hauswirth et al found a positive harmonic function $H_a(x) = aH_1(x/a)$ on $\Omega_a$ that satisfies the free boundary conditions $H_a = 0$ and $|\nabla H_a| = 1$ on $\partial \Omega_a$. Extending $H_a$ to be zero in the complement of $\Omega_a$, we have an entire solution to $F$. (See Section 10 for the explicit formula for $H_a$ using conformal mapping.)

Our first main result characterizes blow-up limits of classical solutions with simply-connected positive phase.

**Theorem 1.1.** Let $u_k$ be a sequence of classical solutions of (1) in the disk $B_{R_k} = B_{R_k}(0)$, with radius $R_k \nearrow \infty$, satisfying $0 \in F(u_k)$ and (3). Then a subsequence converges uniformly on compact subsets of $\mathbb{R}^2$ to some rigid motion of one of the following

(a) $P(x) := x_1^2 + x_2$, a half-plane solution,

(b) $W_b(x) := x_1^2 + (x_2 + b)^2$, for some $b \geq 0$, a two-plane solution, or

(c) $H_n(x)$, for some $n > 0$, a hairpin solution as mentioned above and defined in (27) of Section 10.

Note that unlike property (3), connectivity of the positive phase is not inherited in the limit. For example, blow-up limits of suitable translates and dilates of $H_1$ are two-plane solutions.

Theorem 1.1 is closely related to earlier classifications of entire solutions with simply-connected positive phase due to Khavinson, Lundberg and Teodorescu [KLT13] and Traizet [Tra14]. Traizet showed that classical entire solutions satisfying (3) must be of the form (a), (b), or (c). Khavinson et al showed that the same conclusion is true under a natural, weak regularity assumption on the free boundary known as the Smirnov property. We were not able to use this result to prove our theorem, and this is a central technical difficulty of the paper. Instead, we define another notion of weak solution that we can show is preserved for some rotation $f$, $g$.

Our next result says that near points where the curvature of the free boundary is large, the boundary consists of four curves which are graphs in some common direction with small Lipschitz constants separately on each dyadic annulus, $2^{k-1} \leq |x - z_j| < 2^k$ for $a_j/\epsilon < 2^k < r_0$. What part (c) rules out is the possibility of a spiral. It can be viewed as a quantitative version of the flat $\Rightarrow$ Lipschitz step, in which no information is used about the solution in a neighborhood $|x - z_j| < 50 a_j$. Colding and Minicozzi call the analogous bound in the setting of minimal surfaces an effective removable singularities theorem [CM04, Theorem 0.3]. This crucial estimate plays a large role elsewhere in their work as well.

Our next result says that near points where the curvature of the free boundary is large, the boundary consists of four curves which are graphs in some common direction with small Lipschitz norms. More precisely, there exist $f$, $g : \mathbb{R} \to \mathbb{R}$ such that $f < g$,

$$\|f\|_{L^\infty} + \|g\|_{L^\infty} \leq \delta r, \quad \|f'\|_{L^\infty} + \|g'\|_{L^\infty} \leq \delta,$$

and

$$\{u = 0\} \cap (B_{2r}(z_j))_{B_{a_j/\epsilon}(z_j)} = z_j + \rho_j\{x : f(x_1) \leq x_2 \leq g(x_1)\} \cap \{x : a_j/\epsilon < |x| < 2r\}.$$
Theorem 1.3. There are absolute constants $r_0$, $\kappa_0$, and $N_0$ such that if $u$ is a classical solution to (1) in $B_1$ satisfying (3), then there is $N$, $0 \leq N \leq N_0$ and $N$ saddle points $\{z_j\}_{j=1}^N$ of $u$ with the following properties:

(a) $F(u)$ has curvature at most $\kappa_0$ on $F(u) \cap B_1/2 \setminus \bigcup_{j=1}^N B_{r_0}(z_j)$.

(b) For each $j$, $a_j := u(z_j) \leq r_0/100$, and there is an injective conformal mapping

$$\phi_j : B_{2r_0} \cap \Omega_{a_j} \to \mathbb{R}^2$$

such that $\phi_j(0) = z_j$, and $B_{r_0}(z_j)^+(u) \subset \phi_j(B_{2r_0} \cap \Omega_{a_j}) \subset B_{4r_0}(z_j)^+(u)$.

Moreover, there is $\theta_j \in \mathbb{R}$ such that for all $z \in B_{2r_0} \cap \Omega_{a_j}$,

$$|\phi_j'(z) - e^{i\theta_j}| \leq |z|/(100r_0); \quad |\phi_j''(z)| \leq 1/(100r_0).$$

(c) If $\kappa$ denotes the curvature of $F(u)$ and $\kappa_a$ denotes the curvature of $\partial \Omega_a$, then

$$|\kappa(\phi_j(z)) - \kappa_{a_j}(z)| \leq 1/(100r_0), \quad z \in B_{2r_0} \cap \partial \Omega_{a_j}.$$ 

To interpret part (c) of this theorem, note that

$$\kappa_a(z) \sim a/|z|^2, \quad z \in \partial \Omega_a$$

Hence

$$|z| \leq \sqrt{a r_0} \implies \kappa_a(z) \geq \frac{1}{100r_0}$$

Furthermore, $a$ is comparable to the separation distance between the two hairpins. Thus, for points closer to $z_j$ than the geometric mean of the separation distance between the two hairpins and the distance $r_0$, the bound in part (c) says that the curvature of the approximate hairpins is close to that of the standard model. In particular, the two components of the zero set are convex in this range. At distances significantly larger than this geometric mean, one can no longer guarantee that $\kappa(\phi_j(z))$ is positive, but the bound in part (c) still implies that $\kappa(\phi_j(z)) \leq 1/(50r_0)$. This is a nontrivial bound. At the largest scale, $r_0 < |z| < 2r_0$ it is the same as the standard interior 2nd derivative bounds for flat free boundaries, but at smaller dyadic scales it is a stronger curvature constraint.

In [Tra14], Traizet found a remarkable change of variables that converts the free boundary problem into a problem about minimal surfaces with a plane of symmetry. If $|\nabla u| < 1$, then the minimal surfaces are embedded, and otherwise they are immersed. This means that although neither problem is strictly contained in the other, there is direct overlap between the results of Colding and Minicozzi and the results proved here. The extra hypothesis $|\nabla u| < 1$ removes nearly all the difficulties from the free boundary classification problem we are considering because in that case the zero set of $u$ consists of convex components. Nevertheless, in this simple overlapping case Traizet’s change of variables allows us to make a direct comparison with results of [CM02].

Under Traizet’s correspondence, the standard double hairpin becomes the standard catenoid,

$$\Sigma_\rho = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_2/\rho)^2 + (x_3/\rho)^2 = \cosh^2(x_1/\rho)\}, \quad \rho > 0.$$ 

Denote $B_\rho := \{x \in \mathbb{R}^3 : |x| < \rho\}$.

Corollary 1.4. Let $M \subseteq B_R$ be an embedded minimal surface, homeomorphic to an annulus, with $\partial M \subseteq \partial B_R$. Suppose that $M$ is symmetric with respect to the reflection $x_3 \mapsto -x_3$ and that $M^+ = M \cap \{x_3 > 0\}$ is a simply-connected graph over the $x_1x_2$-plane. Suppose that the shortest closed geodesic of $M$ has length $\epsilon$ and passes through the origin in $B_R$. There are absolute constants $R_0 < \infty$ and $\epsilon_0 > 0$ such that if $R \geq R_0$ and $\epsilon \leq \epsilon_0$, then there exists $\rho > 0$,

$$|2\pi \rho - \epsilon| \leq \epsilon/100$$

and an injective conformal mapping $\phi : \Sigma_\rho \cap B_1 \to M$ that is isometric up to a factor $1 \pm |x|/100$, and the Gauss curvatures $K$ of $M$ and $K_\rho$ of $\Sigma_\rho$ are related by

$$|K(\phi(x)) - K_\rho(x)| \leq \begin{cases} (1/100)(\epsilon/|x|^2), & |x| \leq \sqrt{\epsilon} \\ 1/100, & \sqrt{\epsilon} \leq |x| \leq 1 \end{cases}$$

and an injective conformal mapping $\phi : \Sigma_\rho \cap B_1 \to M$ that is isometric up to a factor $1 \pm |x|/100$, and the Gauss curvatures $K$ of $M$ and $K_\rho$ of $\Sigma_\rho$ are related by

$$|K(\phi(x)) - K_\rho(x)| \leq \begin{cases} (1/100)(\epsilon/|x|^2), & |x| \leq \sqrt{\epsilon} \\ 1/100, & \sqrt{\epsilon} \leq |x| \leq 1 \end{cases}$$

and an injective conformal mapping $\phi : \Sigma_\rho \cap B_1 \to M$ that is isometric up to a factor $1 \pm |x|/100$, and the Gauss curvatures $K$ of $M$ and $K_\rho$ of $\Sigma_\rho$ are related by

$$|K(\phi(x)) - K_\rho(x)| \leq \begin{cases} (1/100)(\epsilon/|x|^2), & |x| \leq \sqrt{\epsilon} \\ 1/100, & \sqrt{\epsilon} \leq |x| \leq 1 \end{cases}$$
Note that because $K_p(x) \sim -\rho^2/|x|^4$ and $\epsilon \approx \rho$, in the range $|x| < \sqrt{\epsilon}$, the curvatures are close. This is the same bound as (but in much less generality than) the sharpest result of Colding and Minicozzi (see [CM02, Remark 3.8]). On the other hand, our corollary gives nontrivial rigidity for both distance distortion and curvature in the range $\sqrt{\epsilon} \ll |x| \ll 1$. This range is not addressed in [CM02], and the present result suggests that there may be interpolating rigidity estimates all the way to unit scale that are valid in the case of general embedded minimal annuli.

1.1. Outline of the paper. The first seven sections of the paper are devoted to the proof of Theorem 1.1. In Section 3 we establish the universal Lipschitz and nondegeneracy bounds enjoyed by the sequence of solutions $u_k$. Section 4 describes the two notions of weak solutions — viscosity and variational — that are preserved under the limit. In Section 5 we recall the Weiss Monotonicity Formula [Wei98] and use it to characterize the blow-up/blow-down limit of a weak nondegenerate solution; there are two possibilities (up to rigid motion): the half-plane solution $P$ (this scenario will ultimately lead to $u$ resembling a hairpin solution); otherwise, the free boundary has bounded curvature at $p$. Sections 6 and 7 are devoted to exploring the first branch of the dichotomy. In Section 8 we start exploring the local structure of a solution $u$, satisfying (3), in the unit disk $B_1$. We delineate a dichotomy — if near a point $p$ of the free boundary there are two connected components of the zero phase close enough to each other at a distance $O(\epsilon)$, then $u$ resembles $|x_2|$ (up to a rigid motion) in a unit-size neighborhood $B_{\epsilon_0}(p)$ (this scenario will ultimately lead to $u$ resembling a hairpin solution); otherwise, the free boundary has bounded curvature at $p$. Sections 9 and 10 are devoted to exploring the first branch of the dichotomy. In Section 9 we show that the free boundary from scale $r_0$ all the way down to scale $O(\epsilon)$ consists of four curves that have bounded turning in the outer scales. In the penultimate Section 10 we finally see the hairpin arising in the inner scale and we systematically treat both scales by constructing an injective holomorphic map (Lemma 10.5) from the positive phase of $u$ in $B_{\epsilon_0}(p)$ to the positive phase of an appropriate hairpin solution $H_\omega$. Obtaining estimates on the second derivative of the map in Lemma 10.6 allows us to relate the curvature of $F(u)$ to the curvature of $F(H_\omega)$ of a model hairpin solution.

In the last Section 11 we exploit the Traizet correspondence to prove Corollary 1.4.

2. Notation.

The disk of radius $r$ centered at $x = (x_1, x_2) \in \mathbb{R}^2$ will be denoted by $B_r(x)$. When the argument is absent, we are referring to the disk centered at the origin, $B_r := B_r(0)$. The unit vectors along $x_1$ and $x_2$ will be denoted by $e_1$ and $e_2$, respectively. The three-dimensional ball of radius $r$, centered at $p \in \mathbb{R}^3$, will be denoted by $B_r(p)$.

If $\Omega$ is an open set of $\mathbb{R}^2$ and $u : \Omega \to \mathbb{R}$ is a non-negative function, define the positive phase of $u$ to be $\Omega^+(u) := \{x \in \Omega : u(x) > 0\}$ and its free boundary $F(u) := \partial \Omega^+(u) \cap \Omega$.

If $\mathfrak{S} \subseteq \mathbb{R}^2$, a $\delta$-neighborhood of $\mathfrak{S}$ will be denoted by $\mathcal{N}_\delta(\mathfrak{S}) := \bigcup_{x \in \mathfrak{S}} B_\delta(x)$.

Denote the distance between two non-empty sets $U, V$ by $d(U, V) = \inf\{|p-q| : p \in U, q \in V\}$, while the Hausdorff distance between two compact subsets $K_1, K_2$ of $\mathbb{R}^2$ will be denoted by $d_H(K_1, K_2) = \inf\{\delta > 0 : K_1 \subseteq \mathcal{N}_\delta(K_2) \text{ and } K_2 \subseteq \mathcal{N}_\delta(K_1)\}$.

By $\mathcal{H}^1$ we shall refer to the one-dimensional Hausdorff measure.
In all that follows, $C, c, c', \tilde{c}, c_0, c_1, c_2$, etc. will denote positive numerical constants. The constants in the $O$-notation, wherever used, are also meant to be numerical.

3. Preliminaries.

Let $u$ be a solution of (1) in a disk $B \subseteq \mathbb{R}^2$ that satisfies (3). In our forthcoming arguments we shall often be working with some connected component $U$ of $[B_r(x)]^\gamma$ such that $B_r(x) \subseteq K \subseteq B$ for some compact set $K$. Claim that $U$ is a piecewise smooth domain; that will provide us with enough regularity to apply the Divergence Theorem in $U$. It suffices to show that only finitely many connected components of $F(u)$ intersect $\partial B_r(x)$ and that each intersects it only a finite number of times. Let $\gamma$ be any connected component of $F(u)$ intersecting $K$. Since for each $p \in F(u)$, $F(u) \cap B_{r(p)}(p)$ is locally the graph of a smooth function when $r(p)$ is small enough, the compact $\gamma \cap K$ has a finite subcover $\{B_{r(p_i)}(p_i), p_i \in \gamma \cap K\}_{i=1}^N$, so that

$$d(\gamma \cap K, (F(u) \setminus \gamma) \cap K) \geq \delta(\gamma) := \frac{1}{2} \min \{\epsilon_{p_i}\}_{i=1}^N.$$  \hfill (4)

But $\{N_{\delta(\gamma)}(\gamma \cap K)\}_{\gamma}$, where $\gamma$ ranges over all connected curves of $F(u)$ intersecting $K$, is a cover of the compact $F(u) \cap K$, so it has a finite subcover $\{N_{\delta(\gamma_j)}(\gamma_j \cap K)\}_{j=1}^M$. Because of (4), each element of the subcover contains only $\gamma_j \cap K$ and nothing else from $F(u) \cap K$, so there are only finitely many curves $\gamma$ intersecting $K$ and thus $B_r(x)$. Each such $\gamma$ intersects $\partial B_r(x)$ only a finite number of times, because by the classical result of [KN77], the free boundary $F(u)$ is real analytic.

We shall now prove two fundamental regularity properties that classical solutions of (1) given (3) satisfy: universal Lipschitz bound and universal non-degeneracy away from the free boundary. To elucidate the latter part of our claim, let us state the relevant definition.

**Definition 3.1.** A non-negative function $u : \Omega \to \mathbb{R}$ is **non-degenerate** if there exists a constant $c > 0$, such that

$$\sup_{B_r(x)} u \geq cr$$

for every $B_r(x) \subseteq \Omega$ centered at a point $x_0 \in F(u)$.

First, let us show that classical solutions enjoy a universal Lipschitz bound.

**Proposition 3.1** (Lipschitz bound). Let $u$ be a classical solution of (1) in $B_R(0)$. If the largest disk in $B_R^+(u)$, centered at $x$, touches $F(u)$, then

$$|\nabla u|(x) \leq C,$$

for some numerical constant $C > 0$. In particular, if $0 \in F(u)$

$$\|\nabla u\|_{L^\infty(B_{R/2})} \leq C. \hfill (5)$$

**Proof.** If $u(x) = m$, then by Harnack’s inequality $c_1m \leq u(y) \leq c_2m$ on $\partial B_{r/2}(x)$. Let $h$ be the harmonic function in the annulus $A_r(x) := B_r(x) \setminus B_{r/2}(x)$, whose boundary values are:

$$h = c_1m \quad \text{on} \quad \partial B_{r/2}(x)$$

$$h = 0 \quad \text{on} \quad \partial B_r(x).$$

By the maximum principle $h \leq u$ in $A_r$ and so by the Hopf lemma,

$$h_{\nu}(p) \leq u_{\nu}(p) = 1,$$

where $\nu$ denotes the inner-normal to $B_R^+$ and $p \in F(u)$ is a point of touching between $F(u)$ and $B_r(x)$. On the other hand, $u_{\nu}(p) \geq c'm/r$, thus

$$m \leq C'r.$$ 

Thus,

$$|\nabla u|(x) \leq \frac{c_0}{r} \int_{\partial B_{r/2}} u \, d\mathcal{H}^1 \leq \frac{c_0c_2m}{r} \leq C,$$

for some numerical constant $C$.

Statement (5) follows once we point out that for $x \in B_{R/2}$ the largest ball contained in $B_R^+(u)$ and centered at $x$, will certainly touch $F(u)$. \hfill $\square$

The universal nondegeneracy property is established through the following proposition.
Proposition 3.2. Let $u$ be a classical solution of \( \square \) in $B_R(0)$, for which \( \square \) is satisfied. Assume further that $0 \in F(u)$. Then
\[
\sup_{B_r(0)} u = \max_{\partial B_r(0)} u \geq \frac{1}{2\pi} r \quad \text{for all} \quad 0 < r < R.
\]

**Proof.** Since $u$ is continuous and subharmonic, the maximum principle implies $\sup_{B_r(0)} u = \max_{\partial B_r(0)} u$. Let $\tilde{u}(x) := r^{-1}u(rx)$ denote the $r$-rescale of $u$. It suffices to show that $\sup_{\partial B_r(0)} \tilde{u} \geq 1/2\pi$.

Let $\phi : [0, 1] \to \mathbb{R}$ be the function
\[
\phi(t) = \begin{cases} 
\frac{1}{2} & \frac{1}{2} < t < 1 \\
1 - t & 0 \leq t \leq \frac{1}{2} 
\end{cases}
\]
and let $\psi(x) = \phi(|x|)$. Let $U$ be the component of $B_{r/2}(\tilde{u}) = r^{-1}B_{r}(u)$ in $B_1$ whose boundary contains the origin. Then if $\tilde{u}_\nu$ denotes the inner normal to $U$,
\[
- \int_U \nabla \psi \cdot \nabla \tilde{u} \, dx = - \int_U \text{div}(\psi \nabla \tilde{u}) \, dx = \int_{\partial U \cap B_1} \psi \tilde{u}_\nu \, d\mathcal{H}^1 = \int_{\partial U \cap B_1} \psi \, d\mathcal{H}^1.
\]
On the other hand, if $\hat{r}$ denotes the unit vector field in the radial direction,
\[
- \int_U \nabla \psi \cdot \nabla \tilde{u} \, dx = \int_{U \setminus B_{1/2}} \text{div}(\tilde{u} \hat{r}) - \text{div}(\tilde{r}) \tilde{u} \, dx = \int_{\partial B_1 \cap U} \tilde{u} \, d\mathcal{H}^1 - \int_{\partial B_{1/2} \cap U} \tilde{u} \, d\mathcal{H}^1 - \int_{U \setminus B_{1/2}} \frac{\tilde{u}}{|x|} \, dx.
\]
Therefore, as $\mathcal{H}^1(\partial U \cap B_1) \geq 2$,
\[
\int_{\partial B_1 \cap U} \tilde{u} \, d\mathcal{H}^1 \geq \int_{F(\tilde{u}) \cap U} \psi \, d\mathcal{H}^1 \geq \frac{1}{2} \mathcal{H}^1(\partial U \cap B_1) \geq 1.
\]
Hence, $\sup_{\partial B_1 \cap U} \tilde{u} \geq 1/2\pi$. \( \square \)

4. Weak solutions.

In this section we define the two notions of weak solutions that will be useful in classifying the limits of sequences of classical solutions. Let
\[
I[u, \Omega] = \int_{\Omega} |\nabla u|^2 + 1_{\{u > 0\}} \, dx \quad \Omega \subseteq \mathbb{R}^2
\]
be the one-phase energy functional whose Euler-Lagrange equation is the free boundary problem \( \square \).

**Definition 4.1.** The function $u \in H^1_{\text{loc}}(\Omega)$ is a variational solution of \( \square \) if $u \in C(\Omega) \cap C^2(\Omega^+(u))$ and
\[
0 = L[u](\phi) := \frac{d}{d\epsilon} \Big|_{\epsilon=0} I[u(x + \epsilon \phi(x))] = \int_{\Omega} (|\nabla u|^2 + 1_{\{u > 0\}}) \text{div} \phi - 2\nabla u D\phi(\nabla u)^T \, dx
\]
for any $\phi \in C_c^\infty(\Omega; \mathbb{R}^2)$.

The next proposition is standard and says that any globally defined limit of uniformly convergent variational solutions that are uniformly Lipschitz continuous and uniformly non-degenerate, inherits the same properties.

**Proposition 4.1.** Let $\{u_k\} \in H^1_{\text{loc}}(B_{R_k})$, $R_k \nearrow \infty$, be a sequence of variational solutions of \( \square \) which satisfies
- (Uniform Lipschitz continuity) There exists a constant $C$, such that $\|\nabla u_k\|_{L^\infty(B_{R_k})} \leq C$;
- (Uniform non-degeneracy) There exists a constant $c$, such that $\sup_{B_r(x)} u_k \geq cr$ for every $B_r(x) \subseteq B_{R_k}$, centered at a free boundary point $x \in F(u_k)$.

Then any limit $u \in H^1_{\text{loc}}(\mathbb{R}^2)$ of a uniformly convergent on compacts subsequence $u_k \to u$ satisfies
- (a) $\{u_k > 0\} \to \{u > 0\}$ and $F(u_k) \to F(u)$ locally in the Hausdorff distance;
- (b) $1_{\{u_k > 0\}} \to 1_{\{u > 0\}}$ in $L^1_{\text{loc}}(\mathbb{R}^2)$;
- (c) $\nabla u_k \to \nabla u$ a.e.

Moreover, $u$ is a Lipschitz continuous, non-degenerate variational solution of \( \square \).
Definition 4.2. A viscosity supersolution (resp. subsolution) of \( u \) in \( \Omega \) is a non-negative continuous function \( w \) in \( \Omega \) such that

\begin{itemize}
  \item \( \Delta w \leq 0 \) (resp. \( \Delta w \geq 0 \)) in \( \Omega^+(w) \);
  \item If \( x_0 \in F(w) \) and there is a disk \( B \subseteq \Omega^+(w) \) (resp. \( B \subseteq \{ w = 0 \} \)) that touches \( F(w) \) at \( x_0 \), then near \( x_0 \) in \( B \) (resp. \( B^c \)), in every non-tangential region,
    \[
    w(x) = \alpha(x - x_0, \nu)^+ + o(|x - x_0|) \quad \text{for some } \alpha \leq 1 \text{ (resp. } \alpha \geq 1),
    \]
    where \( \nu \) denotes the inner (resp. outer) unit normal to \( \partial B \) at \( x_0 \).
\end{itemize}

A function \( w \) is a viscosity solution if \( w \) is both a viscosity super- and subsolution.

The class of viscosity solutions is well-suited for taking uniform limits in compact sets.

Lemma 4.2 (Limit of viscosity solutions). Let \( u_k \in C(\Omega) \) be a sequence of viscosity solutions of \( (1) \) in \( \Omega \) such that \( u_k \to u \) uniformly and \( u \) is Lipschitz continuous. Then \( u \) is also a viscosity supersolution of \( (1) \) in \( \Omega \). If, in addition, \( \Omega^+(u_k) \to \Omega^+(u) \) locally in the Hausdorff distance, then \( u \) is a viscosity subsolution, as well.

Proof. Clearly \( \Delta u = 0 \) in \( \Omega^+(u) \), so we only need to check that the appropriate free boundary conditions are satisfied.

Let us show that \( u \) satisfies the viscosity supersolution condition at the free boundary. Assume there is a disk \( B \) touching \( x_0 \in F(u) \) from the positive phase. Without loss of generality, \( x_0 = 0 \) and the unit normal of \( \partial B \) at \( 0 \) is \( \nu = e_2 \). According to [CS05, Lemma 11.17], \( u \) has the linear behaviour:

\[
u(x) = \alpha x_2 + o(|x|) \quad \text{in non-tangential regions of } B
\]

for some \( 0 < \alpha < \infty \) where \( \nu \) denotes the inner unit normal to \( \partial B \) at \( x_0 \). Claim that \( \alpha \leq 1 \). Fix \( \epsilon > 0 \) small.

If we blow up at 0,

\[
u(x) := \lambda^{-1} u(x_0 + \lambda x) \to \alpha x_2 \quad \text{in } B_1 \cap \{ x_2 > \epsilon \} \quad \text{uniformly as } \lambda \to 0.
\]

Denote \( (u_k)_{\lambda}(x) = \lambda^{-1} u(\lambda x) \) the dilate of \( u_k \) at 0. By the uniform convergence of \( u_k \) to \( u \), for some fixed small enough \( \lambda > 0 \)

\[
|(u_k)_{\lambda}(x) - \alpha x_2| < \alpha \epsilon/2 \quad \text{in } B_1 \cap \{ x_2 > \epsilon \} \quad \text{for all large enough } k.
\]

Consider the perturbation \( D_\epsilon \) of the domain \( B_1 \cap \{ x_2 > \epsilon \} \) defined by

\[
D_\epsilon = \{ x \in B_1 : x_2 > \epsilon - t \eta(x_1) \},
\]

where \( 0 \leq \eta(x_1) \leq 1 \) is a smooth bump function supported in \( |x_1| < 1/2 \) with \( \eta(x_1) = 1 \) for \( |x_1| \leq 1/4 \). We know that \( D_0 \subseteq \Omega^+(u_{k,\lambda}) \) and since \( 0 \in F(u_{\lambda}) \),

\[
F((u_k)_{\lambda}) \cap B_1 \neq \emptyset.
\]

(7)

for all large enough \( k \). Pick a \( k \) such that both (6) and (7) hold. Then for some \( 0 < t_0 < 2 \epsilon \) the domain \( D_{t_0} \subseteq \Omega^+_p((u_k)_{\lambda}) \) will touch \( F((u_k)_{\lambda}) \) at some \( p \in F((u_k)_{\lambda}) \cap \{ x_1 \leq 1/2 \} \). Define a harmonic function \( \nu \) in \( D_{t_0} \) with boundary values

\[
u(x) = \begin{cases}
\alpha x_2 - \alpha \epsilon & \text{on } \partial B_1 \cap \{ x_2 > \epsilon \} \\
0 & \text{on } B_1 \cap \{ x_2 = \epsilon - t_0 \eta(x_1) \}
\end{cases}
\]
Thus, by the maximum principle $v \leq (u_k)_\lambda$ in $D_{t_0}$, so that near $p$ in non-tangential regions of $D_{t_0}$, for some $\hat{\alpha} \leq 1$

$$v(x) \leq (u_k)_\lambda(x) = \hat{\alpha}(x - p, \nu(p)) + o(|x - x_0|),$$

where $\nu(p)$ is the inner normal to $\partial D_{t_0}$ at $p$. On the other hand, a standard perturbation argument gives $v_\nu(p) = \alpha + O(\epsilon)$. Since $\epsilon$ is arbitrary, we conclude $\alpha \leq 1$.

Let us now assume that $\Omega^+(u_k) \to \Omega^+(u)$ in the Hausdorff distance and show that $u$ satisfies the viscosity subsolution condition at the free boundary. Let there be a disk $B$ touching $F(u)$ at $x_0$ from the zero phase. Without loss of generality, $x_0 = 0$ and the unit outer normal at $\partial B$ is $\nu_2$. According to [CS02] Lemma 11.17, for some $0 \leq \beta < \infty$

$$u(x) \leq \beta x_2^+ + o(|x|).$$

Given $\epsilon > 0$ we can dilate $u$ and $u_k$ near 0 sufficiently, so that

$$(u_k)_\lambda(x) \leq u_\lambda(x) + \epsilon/2 \leq \beta x_2^+ + \epsilon \quad \text{in } B_1$$

for some fixed large $\lambda$ and all large enough $k$. Moreover, since $\Omega^+((u_k)_\lambda) \to \Omega^+(u_\lambda)$, we can choose $k$ large enough such that

$$\Omega^+((u_k)_\lambda) \cap B_1 \Subset \{x_2 > -\epsilon/2\} \quad \text{and} \quad F((u_k)_\lambda) \cap B_{\epsilon/2} \neq \emptyset.$$

Let $E_\epsilon$ be the domain

$$E_\epsilon = \{x \in B_1 : x_2 > -\epsilon + t_\nu(x_1)\}$$

and note that for some $0 < t_0 < 2\epsilon$, $E_{t_0} \supset \Omega^+((u_k)_\lambda) \cap B_1$ and $\partial E_{t_0}$ touches $F((u_k)_\lambda) \cap B_1$ at some point $q \in F((u_k)_\lambda) \cap \{|x_1| < 1/2\}$. Define a harmonic function $w$ in $E_{t_0}$ having boundary values:

$$w(x) = \begin{cases} 
\beta x_2^+ + \min\left((2(x_2 + \epsilon)^+), \epsilon\right) & \text{on } \partial B_1 \cap \{x_2 > -\epsilon\} \\
0 & \text{on } B_1 \cap \{x_2 = -\epsilon + t_0\nu(x_1)\}
\end{cases}$$

Thus, the maximum principle implies that near $q$, in non-tangential regions of $\Omega^+((u_k)_\lambda)$,

$$w(x) \geq (u_k)_\lambda(x) = \hat{\beta}(x - q, \nu(q)) + o(|x - x_0|),$$

for some $\hat{\beta} \geq 1$. Hence, $w_\nu(q) \geq \hat{\beta} \geq 1$. On the other hand, a standard perturbation argument gives $w_\nu(q) = \beta + O(\epsilon)$. Since $\epsilon$ is arbitrary, we conclude that $\beta \geq 1$. \hfill $\square$

5. Characterization of blow-downs and blow-ups.

The notion of a variational solution is incredibly useful precisely because it admits the application of the powerful Weiss Monotonicity Formula.

**Lemma 5.1** (Weiss’ Monotonicity Formula, Theorem 3.1 in [Wei98]). Let $u$ be a variational solution of $[1]$ in $\Omega \subseteq \mathbb{R}^n$ and that $B_R(x_0) \subseteq \Omega$. Then

$$\Phi(u, r) := r^{-n} \int_{B_r(x_0)} (|\nabla u|^2 + 1_{\{u > 0\}}) \ dx - r^{-n-1} \int_{\partial B_r(x_0)} u^2 \ d\mathcal{H}^{n-1} \quad (8)$$

satisfies the monotonicity formula

$$\Phi(u, r_2) - \Phi(u, r_1) = \int_{B_{r_2}(x_0) \setminus B_{r_1}(x_0)} 2|x|^{-n-2} (\nabla u \cdot (x - x_0) - u)^2 \ dx \geq 0 \quad (9)$$

for $0 < r_1 < r_2 < R$.

**Lemma 5.2.** Let $u$ be a variational solution of $[1]$ in $\mathbb{R}^n$ which is globally Lipschitz. Assume $0 \in F(u)$ and let $v$ be any limit of a uniformly convergent on compacts subsequence of

$$v_j(x) = R_j^{-1}u(R_jx)$$

as $R_j \to \infty$. Then $v$ is Lipschitz continuous and homogeneous of degree one.
Proof. Denote $v_j(x) = R_j^{-1}u(R_jx)$ and note that $v_j$ are also global variational solutions of \([1]\) and $\Phi(v_j, r) = \Phi(u, r R_j)$. According to Lemma 5.1, the quantity $\Phi(u, R)$ is non-decreasing as $R \to \infty$ and, moreover, it is uniformly bounded since $u$ is Lipschitz continuous. Hence, for any fixed $0 < r_1 < r_2$

$$0 = \lim_{j \to \infty} (\Phi(u, r_2 R_j) - \Phi(u, r_1 R_j)) = \lim_{j \to \infty} (\Phi(v_j, r_2) - \Phi(v_j, r_1))$$

and \([3]\) yields

$$\lim_{j \to \infty} \int_{B_{r_2} \setminus B_{r_1}} 2|x|^{-n-2}(\nabla v_j \cdot x - v_j)^2 \, dx = 0.$$ 

Possibly passing to a subsequence such that $\nabla v_j \rightharpoonup \nabla v$ weakly in $L^2$, the lower semicontinuity of the $L^2$-norm with respect to weak convergence implies

$$\int_{B_{r_2} \setminus B_{r_1}} 2|x|^{-n-2}(\nabla v \cdot x - v)^2 \, dx = 0.$$ 

Thus, $\nabla v \cdot x = v$ a.e. whence it is a standard exercise to conclude that $v$ is homogeneous of degree one.

\[ \Box \]

**Proposition 5.3** (Characterization of blowdowns). Let $u$ be both a viscosity and a variational solution of \([1]\) in $\mathbb{R}^2$, which is Lipschitz-continuous and non-degenerate. Assume $0 \in F(u)$ and let $v$ be any limit of a uniformly convergent on compacts subsequence of

$$v_j(x) = R_j^{-1}u(R_jx)$$

as $R_j \to \infty$. Then $v$ is either $V_1(x) = x_2^+$ or $V_2(x) = s|x_2|$ for some $0 < s \leq 1$ in an appropriately chosen Euclidean coordinate system.

Proof. As a consequence of Proposition 4.1, Lemma 4.2 and Lemma 5.2 applied to the sequence $v_j$ we conclude that $v$ is a Lipschitz continuous, non-degenerate, viscosity and variational solution of \([1]\), which is homogeneous of degree 1. Thus, after possibly rotating the coordinate axes

$$v(x) = c_1 x_2^+ + c_2 x_2^-,$$

where $c_1 \geq c_2 \geq 0$. We have the following two cases.

**Case 1** ($c_2 = 0$). By non-degeneracy we must have $c_1 > 0$ and since every point $x_0 \in F(v) = \{x_2 = 0\}$ has a tangent disk from both the positive and zero set of $v$, then $c_1 = 1$.

**Case 2** ($c_2 > 0$). Every point $x_0 \in F(v) = \{x_2 = 0\}$ has a tangent disk from the positive phase of $v$ only, so from the fact that $v$ is a viscosity solution we can just conclude that $1 \geq c_1 \geq c_2$. On the other hand, $v$ is also a variational solution and an easy computation gives

$$0 = L[v](\phi) = (c_1^2 - c_2^2) \int_{\mathbb{R}} \phi_2(x_1, 0) dx_1$$

for any $\phi = (\phi_1, \phi_2) \in C_c^1(\mathbb{R}^2; \mathbb{R}^2)$. Thus, $c_1 = c_2 = s$. \[ \Box \]

Exactly analogous arguments apply to blow-up limits of Lipschitz continuous weak solutions, so we have the analogous characterization:

**Proposition 5.4** (Characterization of blow-ups). Let $u$ be both a viscosity and a variational solution of \([1]\) in $\Omega \subseteq \mathbb{R}^2$, which is Lipschitz-continuous and non-degenerate. Assume $0 \in F(u)$ and let $v : \mathbb{R}^2 \to \mathbb{R}$ be any limit of a uniformly convergent on compacts subsequence of

$$v_j(x) = \epsilon_j^{-1}u(\epsilon_j x)$$

as $\epsilon_j \to 0$. Then $v$ is either $V_1(x) = x_2^+$ or $V_2(x) = s|x_2|$ for some $0 < s \leq 1$ in an appropriately chosen Euclidean coordinate system.
Lemma 6.1. Let $u$ be a classical solution of (1) in a domain $B_2$, which has Lipschitz norm $L$ and such that

$$|u(x) - s|x_2|| < \epsilon \quad \text{in} \quad B_2$$

(10)

for some $0 < s \leq 1$ and some small $\epsilon > 0$. Then there exists a universal constant $c > 0$ such that

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq 1 + cL\sqrt{\epsilon}.$$  

Proof. Assumption (10) implies $B_2^+(u_R) \subset \{|x_2| > \epsilon/s\}$. Thus, at any $p \in \partial B_1 \cap \{|x_2| > 2M\epsilon/s\}$ for a large $M \leq s/2\epsilon$, we have $B_{M\epsilon/s}(p) \subset B_2^+(u)$ so that $u - s|x_2|$ is harmonic in $B_{M\epsilon/s}(p)$. Hence,

$$|\nabla u(p) - s\nabla|x_2|(p)| \leq \frac{c'}{M\epsilon/s} \int_{\partial B_{M\epsilon/s}} |u - s|x_2|| d\mathcal{H} \leq \frac{c'}{M\epsilon/s},$$

which in turn leads to

$$|\nabla u(p)|^2 \leq \left(s + \frac{c'}{M\epsilon/s}\right)^2 \leq 1 + \frac{3c'}{M\epsilon/s}. \quad (11)$$

Define the function $v : B_2 \to \mathbb{R}$

$$v = \begin{cases} |\nabla u|^2 - 1 & \text{in } B_2^+(u) \\ 0 & \text{otherwise.} \end{cases}$$

Then $v$ is continuous in $B_2$ and since

$$\Delta |\nabla u|^2 = 2|D^2 u|^2 + 2\nabla(\Delta u) \cdot \nabla u = 2|D^2 u|^2 \geq 0 \quad \text{in } B_2^+(u),$$

$v$ is subharmonic in $B_2^+(u)$. Let $v_h : \overline{B_1} \to \mathbb{R}$ be the harmonic function whose boundary values on $\partial B_1$ are given by

$$v_h(x) = \max\{v(x), 3c's/M\} \quad x \in \partial B_1.$$  

By the maximum principle, $v_h > 0$ in $B_1$ and $v_h \geq v$ in $B_1^+(u)$, whence $v \leq v_h$ in $B_1$. By Poisson’s formula, for any $x \in B_{1/2}$

$$v_h(x) \leq c \int_{\partial B_1} v_h \ d\mathcal{H} = c \left( \int_{\partial B_1 \cap \{|x_2| \leq M\epsilon/s\}} v_h \ d\mathcal{H} + \int_{\partial B_1 \cap \{|x_2| > M\epsilon/s\}} v_h \ d\mathcal{H} \right) \leq \frac{cL^2\epsilon M/s + \frac{c'}{M\epsilon/s}}{1}, \quad (12)$$

where the last inequality is a consequence of (11) and the Lipschitz control of $u$. Choosing $M = s/(\sqrt{\epsilon}L)$ yields

$$v \leq v_h \leq 2\epsilon L\sqrt{\epsilon} \quad \text{in } B_{1/2}$$

which is the desired estimate. \hfill \Box

Lemma 6.2. Let $u_k$ be a sequence of classical solutions of (1) in $B_{R_k}$, $R_k \to \infty$ that are uniformly Lipschitz continuous and assume the sequence converges uniformly on compact subsets of $\mathbb{R}^2$ to $u : \mathbb{R}^2 \to \mathbb{R}$ with $0 \in F(u)$. If a blowdown limit of $u$

$$u_{R_j}(x) = R_j^{-1}u(R_jx) \to s|x_2| \quad \text{uniformly on compacts as } R_j \to \infty,$$

for some $0 < s \leq 1$, then

$$|\nabla u| \leq 1 \quad \text{a.e.}$$

Proof. Fix $\epsilon > 0$ and find $j$ large enough so that

$$|u_{R_j} - s|x_2|| < \epsilon/2 \quad \text{in } B_2.$$  

Then for all large enough $k$, such that $|u_{R_j} - (u_k)_{R_j}| < \epsilon/2$ in $B_2$, we have

$$|(u_k)_{R_j} - s|x_2|| < \epsilon,$$

so that Lemma 6.1 yields the estimate

$$\|\nabla u_k\|_{L^\infty(B_{1/2})} = \|\nabla (u_k)_{R_j}\|_{L^\infty(B_{1/2})} \leq 1 + C\sqrt{\epsilon},$$

where $C$ is a universal constant. \hfill \Box
where $C$ is a bound on the Lipschitz norm of $u_k$. At every $x \in \{ u > 0 \}$, for $d = u(x)$, the disk $B_{d/2C}(x) \subseteq \{ u > 0 \}$ as well as $B_{d/2C}(x) \subseteq \{ u_k > 0 \}$ for all $k$ large enough. Since $u_k \to u$ uniformly in $B_{d/2C}(x)$, where the functions are harmonic, we also get $\nabla u_k(x) \to \nabla u(x)$ as $k \to \infty$. Since $\nabla u(x) = 0$ a.e. $x \in \{ u = 0 \}$,

$\|\nabla u\|_{L^\infty(B_{R_j/2})} = \lim_{k \to \infty} \|\nabla u_k\|_{L^\infty(B_{R_j/2})} \leq 1 + C\sqrt{\epsilon}$.

Letting $R_j \to \infty$, followed by $\epsilon \to 0$, yields the result. \hfill \square

**Lemma 6.3.** Let $u$ be a classical solution of \( \square \) in $\Omega \subseteq \mathbb{R}^2$. Then the signed curvature $\kappa$ of $F(u)$ is given by

$$\kappa = -\frac{1}{2} \frac{\partial(|\nabla u|^2)}{\partial \nu},$$

where $\nu$ is the unit normal pointing towards $\Omega^+(u)$.

**Proof.** The curvature of a level set of a function $v$ at a point where $|\nabla v| \neq 0$ is given by

$$\kappa = \text{div} \left( \frac{\nabla v}{|\nabla v|} \right).$$

(Note that $\kappa > 0$ when the curvature vector points in the direction of $-\nabla v$, e.g. the curvature of the 0-level set of $v(x, y) = \log(x^2 + y^2)$ is positive 1). Since in a small enough neighborhood $U$ of each $x \in F(u)$ we can define a harmonic $v$ which agrees with $u$ on $U \cap \Omega^+(u)$, the curvature of $F(u)$ is given by the curvature of the $v = 0$ level set. Using in addition $|\nabla v|(x) = 1$, we compute

$$\kappa = \frac{\Delta v}{|\nabla v|} - \frac{\nabla v \cdot \nabla |\nabla v|}{|\nabla v|^2} = -\frac{(|\nabla v|^2)_{\nu}}{2|\nabla v|^2} = -\frac{(|\nabla u|^2)_{\nu}}{2|\nabla u|^2}. \hfill \square$$

**Remark 6.4.** If $u$ is a classical solution of \( \square \) in $\Omega \subseteq \mathbb{R}^2$ with $|\nabla u| < 1$ in $\Omega^+(u)$, then $F(u)$ has strictly positive curvature.

**Proof.** The result follows immediately from Lemma 6.3 after an application of the Hopf Lemma to $|\nabla u|^2$, which is subharmonic in $\Omega^+(u)$:

$$\Delta |\nabla u|^2 = 2|D^2 u|^2 + 2|\nabla u| \cdot \nabla (\Delta u) = 2|D^2 u|^2 \geq 0 \quad \text{in} \quad \Omega^+(u). \hfill \square$$

**Lemma 6.5.** Let $u$ be a classical solution of \( \square \) in $\Omega \subseteq \mathbb{R}^2$, whose Lipschitz norm is $L < \infty$. If $V \subseteq \Omega^+(u)$, $V \subseteq \Omega$ is a bounded, piecewise $C^1$ domain, then

$$L^{-1}H^1(\partial U \cap F(u)) \leq H^1(\partial U \setminus F(u)).$$

**Proof.** Applying the Divergence Theorem in $U$:

$$0 = -\int_U \Delta u \, dx = \int_{\partial U \cap F(u)} u_{\nu} \, dH^1 + \int_{\partial U \setminus F(u)} u_{\nu} \, dH^1 = H^1(\partial U \cap F(u)) + \int_{\partial U \setminus F(u)} u_{\nu} \, dH^1,$n

where $u_{\nu}$ is the inner unit normal to $\partial U$. The result then follows from $|u_{\nu}| \leq L$ $H^1$-a.e. on $\partial U$. \hfill \square

**Lemma 6.6.** Let $u$ be a classical solution of \( \square \) in $B_3$, which is Lipschitz continuous with norm $L$ and for which assumption \( \mathcal{B} \) is satisfied. There exists $\delta = \delta(L) > 0$ small enough such that if

$$\{ u = 0 \} \subset B_3 \cap \{|x| < \delta\}$$

there are at most two connected components of $B_3^+(u) \cup$ which intersect $B_1$, namely the connected component(s) containing $N = (0, 1)$ and $S = (0, -1)$.

**Proof.** Consider a connected component $U$ of $B_3^+(u)$ that contains neither $N$ nor $S$; then it must be that $U \subseteq B_2 \setminus \{|x| < \delta\}$. Assuming that $U$ intersects $B_1$, by assumption $\mathcal{B}$ we have $H^1(\partial U \cap F(u)) \geq 2$. On the other hand, $U$ is a piecewise $C^1$ domain with $\partial U \setminus F(u) = \partial B_2 \cap \{|x| < \delta\}$, so that $H^1(\partial U \setminus F(u)) \leq c\delta$ for some numerical constant $c > 0$. But then by Lemma 0.3,

$$2/L \leq L^{-1}H^1(\partial U \cap F(u)) \leq H^1(\partial U \setminus F(u)) \leq c\delta,$n

which would be impossible if $\delta < 2/L$. \hfill \square
Lemma 6.7. Let \( u \) be a classical solution of \( \bar{1} \) in \( B_1 \) for which assumption \( 3 \) is satisfied. Assume further that \((0,1)\) and \((0,-1)\) belong to two separate connected components of \( B_2^+ (u) \). Then there exists \( \delta_0 > 0 \) small enough such that if for any \( 0 < \delta \leq \delta_0 \)

\[
\{ u = 0 \} \subseteq \{ |x_2| < \delta \},
\]

the free boundary \( F(u) \) inside \( \{|x_1| < 1/2\} \) consists of two disjoint graphs:

\[
F(u) \cap \{|x_1| < 1/2\} = \{ x_2 = \phi_+(x_1) : |x_1| < 1/2 \} \cup \{ x_2 = \phi_-(x_1) : |x_1| < 1/2 \},
\]

for which \( \phi_+ > \phi_- \) and

\[
\|\phi_+\|_{C^{1,\alpha}(-1/2,1/2)} \leq C\delta
\]

for some numerical positive constants \( C, 0 < \alpha < 1 \).

Proof. By Proposition \( 3.1 \) \( \|\nabla u\|_{L^{\infty}(B_2)} \leq L \) for some numerical constant \( L \), so that by Lemma \( 6.6 \) there exists a small enough \( \delta_0 > 0 \) such that it is precisely the connected components \( U_+ \) and \( U_- \) of \( B_2^+ (u) \), containing \((0,1)\) and \((0,-1)\) respectively, that intersect \( B_1 \). Define the two functions \( u_+ \) and \( u_- \) on \( B_1 \) by \( u_+ = u|_{U_+ \cap B_1} \). Then each \( u_\pm \) is a classical solution of \( \bar{1} \) in \( B_1 \) whose free boundary \( F(u_\pm) \) is contained in a flat strip \( |x_2| < \delta \) with \( u_+ = 0 \) in \( B_1 \cap \{|x_2| < -\delta\} \) and \( u_- = 0 \) in \( B_1 \cap \{|x_2| > \delta\} \). By the classical result of Alt and Caffarelli [ACS1], in \( |x_1| < 1/2 \) the free boundary \( F(u_\pm) \) is the graph of a function \( \phi_\pm : (-1/2,1/2) \to \mathbb{R} \), which satisfies

\[
\|\phi_\pm\|_{C^{1,\alpha}(-1/2,1/2)} \leq C\delta
\]

for some \( \alpha > 0, C > 0 \). Noting that \( F(u) \cap B_1 = F(u_+) \cup F(u_-) \), we are done. \( \Box \)

7. Characterization of the limit.

Recall the setup. We have a sequence \( \{u_k\} \) of classical solutions of \( \bar{1} \) in expanding disks \( B_{R_k}, R_k \nearrow \infty \) with \( 0 \in F(u_k) \). Because of Proposition \( 3.1 \) \( u_k \) are uniformly Lipschitz on compact subsets of \( \mathbb{R}^2 \), so that up to a subsequence, \( u_k \) converges uniformly on compacts to some \( u : \mathbb{R}^2 \to \mathbb{R} \). Moreover, since \( u_k \) are uniformly non-degenerate by Proposition \( 3.2 \) and, trivially also, weak solutions (variational and viscosity), then by Proposition \( 4.1 \) and Lemma \( 4.2 \) \( u \) is a global weak solution, which is Lipschitz continuous and non-degenerate. Thus, by Propositions \( 5.3 \) and \( 5.4 \) \( u \) blows down/blows up at a free boundary point to a half-plane or a wedge solution.

We shall show that, in for appropriately chosen Euclidean coordinates, \( u \) has to be one of the four:

- a half-plane solution \( P(x) = x_2^+ \)
- a two-plane solution \( W_b(x) = x_2^+ + (x_2 - b)^- \), for some \( b < 0 \)
- a wedge solution \( W_0(x) = |x_2| \)
- a hairpin solution \( H_a \), whose \( \{ H_a > 0 \} = \{(x_1, x_2) : |ax_1 - (a + \pi/2)| < \pi/2 + \text{cosh} (ax_2)\} \) for some \( a > 0 \).

The proof of the classification Theorem \( 1.1 \) is realized in a sequence of propositions. Proposition \( 7.1 \) covers the scenario when \( u \) blows down to a half-plane solution, while Proposition \( 7.2 \) covers the case when the blowdown is a wedge solution. In the latter there is a dichotomy, \( u \) can be either a two-plane solution or satisfy \( |\nabla u| < 1 \) globally in its positive phase. The second scenario is the more subtle one and its treatment is carried out in several steps assembled under Proposition \( 7.3 \) in the first we employ a 2-point simultaneous blow-up to show that the free boundary is smooth everywhere but possibly one point; in the second step we prove the free boundary is actually smooth everywhere using the strong geometric constraint of positive free boundary curvature to argue that the zero phase contains a nontrivial vertex with vertex at the exceptional point (cf. Lemmas \( 7.5 \) and \( 7.6 \)); in the final step we establish that the free boundary must consist of exactly two smooth proper arcs, so that \( u \) has to be a hairpin solution, according to [Tra14 Theorem 12].

Let us commence with

Proposition 7.1. Let \( u_k, u \) be as in Theorem \( 1.1 \) and assume \( 0 \in F(u) \). If a blowdown limit of \( u \) at 0 is a half-plane solution:

\[
R_j^{-1} u(R_j x) \to x_2^+ \quad \text{as} \quad R_j \nearrow \infty,
\]

(with coordinates chosen appropriately), then \( u = x_2^+ \) itself.
Proof. Since $u_{R_j} \to x_2^+$ uniformly on compacts, the free boundary $F(u)$ is asymptotically flat, i.e.

$$F(u) \cap B_{R_j} \subseteq \{|x_2| \leq \epsilon_j\}$$

with the aspect ratio $\epsilon_j/R_j \to 0$ as $j \to \infty$. By the classical theorem of Alt and Caffarelli [AC81], $F(u)$ has to be the straight line $\{x_2 = 0\}$ and $u = x_2^+$.

Proposition 7.2. Let $u_k, u$ be as in Theorem 7.1 and assume $0 \in F(u)$ and that a blowdown limit of $u$ at 0 is a wedge solution:

$$R_j^{-1} u(R_j x) \to s|x_2| \quad \text{as} \quad R_j \to \infty,$$

(with coordinates chosen appropriately) for some $0 < s \leq 1$. Then either $u = x_2^+ + (x_2 - b)^-$ for some $b \leq 0$, or

$$\{\nabla u \mid < 1 \quad \text{in} \quad \{u > 0\}.}$$

Proof. From Lemma 6.2 we have the bound $|\nabla u| \leq 1$ a.e. Noting that $|\nabla u|^2$ is a smooth subharmonic function in $\{u > 0\}$:

$$\Delta |\nabla u|^2/2 = 2|D^2 u|^2 \geq 0,$$

the Strong Maximum Principle entails that if $|\nabla u|^2(x_0) = 1$ at some $x_0 \in \{u > 0\}$, then $|\nabla u|^2 \equiv 1$ in the entire connected component $U$ of $x_0$. Equation (13) implies that $|D^2 u|^2 = 0$ in $U$, so that $u$ is an affine function in $U$. Thus $U$ is a half-plane, say $U = \{x_2 < b\}$ for some $b \leq 0$ and $u = (x_2 - b)^-$ in $U$. The latter also implies that $u$ has to blow down precisely to $|x_2|$, i.e. $s = 1$.

We shall now show that $v = u_{123} \mid_{U}$ is a viscosity solution itself. Once this is established, we can apply the previous Proposition 7.1 to $v$ (as $v$ has to blow down to $x_2^+$), so that $v$ will have to be $v = x_2^+$ itself. We will then be able to conclude that $u = v + (x_2 - b)^- = x_2^+ + (x_2 - b)^-$.

Note that $v$ is trivially a viscosity supersolution (as $u$ is), so let us focus on showing that $v$ is also a viscosity subsolution. Let $p \in F(v)$ and let $B \subseteq \{v = 0\}$ be a touching disk to $F(v)$ at $p$ from the zero phase. If there exists a disk $B' \subseteq B$ such that $\partial B \cap \partial B' = p$ and $B' \subseteq \{u = 0\}$, then the subsolution condition for $v$ will be inherited from $u$. Otherwise, every $B' \subseteq B$ with $\partial B \cap \partial B' = p$ will have to intersect the half-plane $U = \{x_2 < b\}$, and thus $B \subseteq U$ and $p \in \partial U$. But then, applying Proposition 5.4, we see that any blowup of $u$ at $p$ will have to equal $|\{x - p\} \cdot e_2|$, where $e_2$ is a unit vector in the direction of $x_2$. Hence, near $p$

$$v(x) = ((x - p) \cdot e_2)^+ + o(|x - p|) \quad \text{in any nontangential region of } B^c,$$

so that the subsolution condition is satisfied again.

Proposition 7.3. Let $u_k, u$ be as in Theorem 7.1 Further assume that $|\nabla u| < 1$ in $\{u > 0\}$. Then $u$ is a hairpin solution.

Proof. We shall prove the proposition in three steps. In the first we show that $F(u)$ is smooth everywhere but possibly one point. In the second step we invoke Lemma 7.5 to establish that $F(u)$ is, in fact, smooth everywhere. In the final step we show that $F(u)$ consists of two disjoint proper arcs, so that by a result of Traizet [Tra14] Theorem 12] $u$ has to be a hairpin solution.

Step 1. In order to establish the claim of the first step above, we have to prove that for no two distinct points $P_1, P_2 \in F(u)$ it can happen that $u$ blows up to wedge solutions at both $P_1$ and $P_2$. From this it follows that at every point of $F(u)$ but possibly one, $u$ has to blow up to the only other alternative – a half-plane solution (according to Proposition 5.4) so that $F(u)$ is smooth there.

Assume the contrary. Denote $u_\epsilon(x) := \epsilon^{-1} u(\epsilon x)$ and $(u_k)_\epsilon(x) := \epsilon^{-1} u_k(\epsilon x)$. If $u$ blows up to wedge solutions at $P_1$ and $P_2$, Proposition 4.1a implies there exist some unit vectors $a_1$ and $a_2$ such that given an arbitrary small $\lambda > 0$, one can find a sequence $\epsilon_j \searrow 0$ small enough such that for any $\epsilon = \epsilon_j$ small enough

$$d_H(\{u_\epsilon = 0\} \cap B_{\epsilon_4}(P_1/\epsilon), \{P_1/\epsilon + t a_i : |t| \leq 4\}) < \lambda/2 \quad i = 1, 2.$$

Further, for that particular fixed $\epsilon$, Proposition 4.1a implies that for all $k$ large enough

$$d_H(\{(u_k)_\epsilon = 0\} \cap B_{\epsilon_4}(P_2/\epsilon), \{u_\epsilon = 0\} \cap B_{\epsilon_4}(P_2/\epsilon)) < \lambda/2 \quad i = 1, 2,$$

As a consequence, for all $k$ large enough

$$d_H(\{(u_k)_\epsilon = 0\} \cap B_{\epsilon_4}(P_2/\epsilon), \{P_2/\epsilon + t a_i : |t| \leq 4\}) < \lambda \quad i = 1, 2. \quad (14)$$
For ease of notation, denote \( v := u_\epsilon \), \( v_k := (u_k) \), and \( Q_i = P_i / \epsilon, i = 1, 2 \); let \( b_i \) be the vector \( a_i \) rotated by \( \pi/2 \). According to Lemma 6.6, there are at most two connected components of \( B_2(Q_i)^+(v_k) \) that intersect \( B_i(Q_i) \) – the one(s) that contain \( N_i = Q_i + b_i \) and \( S_i = Q_i - b_i \). We shall now show that there has to be just one if \( \lambda \) is small enough and \( k \) is large enough.

Assume \( N_1 \) and \( S_1 \) belong to two separate connected components \( U_{+,k} \) and \( U_{-,k} \) of \( B_2(Q_1)^+(v_k) \). Combining this with \([14]\) allows us to invoke Lemma 6.7 and conclude that \( F(v_k) \cap \{(x - Q_1) \cdot a_1 < 1/2\} \) consists of the graphs of some functions \( \phi_{+,k} > \phi_{-,k} \) over the line segment \( I = \{Q_1 + t a_1 : |t| < 1/2\}: \)

\[
F(v_k) \cap \{(x - Q_1) \cdot a_1 < 1/2\} = \Sigma_{+,k} \cup \Sigma_{-,k} \quad \text{where} \quad \Sigma_{\pm,k} = \{y + \phi_{\pm,k}(yb_1 : y \in I)\}.
\]

Moreover, the functions \( \phi_{\pm,k} \) satisfy the uniform bound

\[
\|\phi_{\pm,k}\|_{C^{1,\alpha}(I)} \leq C\lambda.
\]

Hence, there exist \( C^{1,\alpha} \) functions \( \phi_{\pm} : I \to \mathbb{R} \) and a subsequence \( \phi_{\pm,k_i} \) such that

\[
\phi_{\pm,k_i} \to \phi_{\pm} \quad \text{in} \quad C^1(I) \quad \text{as} \quad i \to \infty.
\]

But since \( F(v_k) \to F(v) \) locally in the Hausdorff distance, it must be that \( F(v) \cap \{(x - Q_1) \cdot a_1 < 1/2\} \) consists precisely of the \( C^{1,\alpha} \) graphs

\[
\Sigma_{\pm} = \{y + \phi_{\pm}(yb_1 : y \in I)\}
\]

and

\[
B_2(Q_1)^+(v) \cap \{(x - Q_1) \cdot a_1 < 1/2\} = U_+ \cup U_-,
\]

where \( U_+ = \{y + t b_1 : t > \phi_+(y) : y \in I\} \cup B_2(Q_1) \) and \( U_- = \{y + t b_1 : t < \phi_-(y) : y \in I\} \cup B_2(Q_1). \)

Moreover, since \( v_{k_1}1_{U_{\pm,k}} \) are viscosity solutions in \( B_2(Q_1) \cap \{(x - Q_1) \cdot a_1 < 1/2\} \) and \( v_{k_1}1_{U_{\pm,k}} \to v1_{U_{\pm}} \) uniformly there, Lemma 6.2 implies that \( v1_{U_{\pm}} \) are viscosity solutions there as well. As their free boundary is \( C^{1,\alpha} \), they are in fact classical solutions. But \( |\nabla v| < 1 \) in \( \{v > 0\} \), so that according to Corollary 6.4 both \( \Sigma_+ \) and \( \Sigma_- \) have positive curvature. However, this is impossible, because \( 0 \in \Sigma_+ \cap \Sigma_- \), as \( v \) blows up at \( 0 \) to a wedge solution.

Hence \( N_1 \) and \( S_1 \) belong to the same connected component of \( B_2(Q_1)^+(v_k) \) and similarly \( N_2 \) and \( S_2 \) belong to the same connected component of \( B_2(Q_2)^+(v_k) \) for \( \lambda \) small enough and all \( k \) large enough.

Let

\[
\alpha_L = \partial B_2(Q_1) \cap \{x : (x - Q_1) \cdot b_1 < \lambda, (x - Q_1) \cdot a_1 < 0\}
\]

\[
\alpha_R = \partial B_2(Q_1) \cap \{x : (x - Q_1) \cdot b_1 < \lambda, (x - Q_1) \cdot a_1 > 0\}.
\]

By our topological assumption \( F(v_k) \cap \overline{B_2(Q_1)} \) consists of arcs whose ends lie on \( \alpha_L \cup \alpha_R \). Define \( F_L \) (resp. \( F_R \)) to be the set of points of \( F(v_k) \cap \overline{B_2(Q_1)} \) that lie on arcs whose both ends are on \( \alpha_L \) (resp. \( \alpha_R \)). Then

\[
F(v_k) \cap \overline{B_2(Q_1)} = F_L \cup F_R.
\]

This is so, because the existence of an arc which has one end on \( \alpha_L \) and the other on \( \alpha_R \) would contradict the fact that \( N_1 \) from \( S_1 \) belong to the same connected component of \( B_2(Q_1)^+(v_k) \).

Note that \( F(v_k) \cap \overline{B_2(Q_1)} \) consists of a finite number of connected arcs. This follows from the analyticity of \( F(v_k) \) which implies that only finitely many connected arcs of \( F(v_k) \) can intersect \( \partial B_2(Q_1) \), each intersecting it finitely many times. As a consequence, the sets \( F_L \) and \( F_R \) are closed and being bounded, they are compact. Hence, there exists a point \( p \in F_L \) and a point \( q \in F_R \) such that

\[
|p - q| = \text{dist}(F_L, F_R) < 2\lambda,
\]

where the bound follows from \([14]\). Denote by \( \gamma_L \) the arc of \( F(v_k) \cap \overline{B_2(Q_1)} \) containing \( p \), and by \( \gamma_R \) – the arc containing \( q \).

Claim that the straight line segment \( \tau_i := \{(1 - t)p + t q : 0 < t < 1\} \), connecting \( p \) to \( q \), lies in \( B_2(Q_1)^+(v_k) \). Since \( p \) and \( q \) realize the distance between \( F_L \) and \( F_R \), it must be that \( \tau_i \cap F(v_k) = \emptyset \), so that either \( \tau_i \subseteq B_2(Q_1)^+(v_k) \) or \( \tau_i \subseteq \{v_k = 0\} \cap B_2(Q_1) \). The latter alternative, however, is impossible, since \( \gamma_L \cap \tau_i \cup \gamma_R \) would disconnect \( N_1 \) from \( S_1 \) in \( B_2(Q_1)^+(v_k) \).

Let us look locally at the connected arc(s) of \( F(v_k) \) that contain \( p \) and \( q \). One possibility is that \( p \) and \( q \) belong to the same arc. Let us argue that this is not the case. Denote by \( \gamma \) the arc of \( F(v_k) \) with ends \( p \) and \( q \).
and \( q \). Then \( \gamma \cup \tau \) is a simple closed arc and it encloses a piecewise \( C^1 \) Jordan domain in the positive phase. Applying Lemma 6.5 to it, we see that, as \( H^1(\tau_1) < 2\lambda \),

\[
2\lambda L > H^1(\tau_1)L \geq H^1(\gamma),
\]

where \( L \) denotes the Lipschitz norm of \( v_k \). However, since \( \gamma_L \cup \tau_1 \cup \gamma_R \subseteq \gamma \cup \tau_1 \) connects \( \alpha_L \) to \( \alpha_R \),

\[
H^1(\gamma) + H^1(\tau_1) \geq 2\sqrt{4^2 - \lambda^2},
\]

so that \( H^1(\gamma) > 7 \) for all \( \lambda \) small. Since \( L \) is bounded above by a universal constant, taking \( \lambda \) small enough leads to a contradiction in \([15]\).

Thus, we may assume that \( p \) and \( q \) belong to distinct arcs of \( F(v_k) \). We know that \( v \) blows down to a wedge solution (otherwise if \( |\nabla v(x)| = 1 \) at some \( x \)) and without loss of generality, we may assume the blowdown is exactly \( s|x_2| \). Thus, for any \( \delta > 0 \) there exists \( M = M(\delta) \) large enough such that for all \( k \) large enough

\[
\{v_k = 0\} \cap B_M \subseteq \{|x_2| < \delta M\}.
\]

Note that we may take \( M \) large enough so that both \( Q_1 \) and \( Q_2 \) belong to \( B_{M/3} \) and \( \tau_1 \subseteq B_{M/2}^+(v_k) \). Denote

\[
\alpha_{L,M} = \partial B_M \cap \{x_1 < 0, |x_2| < \delta M\}, \quad \alpha_{R,M} = \partial B_M \cap \{x_1 > 0, |x_2| < \delta M\},
\]

and let \( \gamma_p \) be the arc of \( F(v_k) \cap B_M \) containing \( p \), and \( \gamma_q \) be the arc of \( F(v_k) \cap B_M \) containing \( q \). Let us show that \( \gamma_p \) and \( \gamma_q \) cannot both have an end on \( \alpha_{L,M} \) (and, similarly, they cannot both have an end on \( \alpha_{R,M} \)). Assume they do: let \( \tilde{\gamma}_p \subseteq \gamma_p \) be the subarc connecting \( \alpha_{L,M} \) to \( p \) and \( \tilde{\gamma}_q \subseteq \gamma_q \) be the subarc connecting \( \alpha_{L,M} \) to \( q \) and let \( \tilde{\alpha} \) be (smaller) circular arc on \( \partial B_M \) from the end \( \partial B_M \cap \tilde{\gamma}_p \) to the end \( \partial B_M \cap \tilde{\gamma}_q \). Then \( \tilde{\alpha} \cup \tilde{\gamma}_p \cup \tilde{\gamma}_q \cup \tau_1 \) encloses a Jordan domain \( \tilde{O} \) and let \( O \subseteq \tilde{O}^+(v_k) \) be the connected component of \( \tilde{O}^+(v_k) \) whose boundary contains \( \tau_1 \). Applying Lemma 6.3 to \( O \), we quickly reach a contradiction for small \( \delta \), as

\[
H^1(F(v_k) \cap \partial O) \geq H^1(\tilde{\gamma}_p) + H^1(\tilde{\gamma}_q) \geq M/2 + M/2 = M
\]

while

\[
H^1(\partial O \setminus F(v_k)) \leq H^1(\tau_1) + H^1(\tilde{\alpha}) < 2\lambda + c\delta M < 1 + c\delta M.
\]

Therefore, it must be that \( \gamma_p \) has both its ends on \( \alpha_{L,M} \) while \( \gamma_q \) has both its ends on \( \alpha_{R,M} \). Of course, an analogous scenario holds near \( Q_2 \) as well: we can find a straight line segment \( \tau_2 \subseteq B_M^+(Q_2(v_k)) \subseteq B_M^+(v_k) \) of length \( H^1(\tau_2) < 2\lambda \), one end of which belongs to a free boundary arc with ends on \( \alpha_{L,M} \) and the other contained in a free boundary arc with ends on \( \alpha_{R,M} \). Moreover, \( d(\tau_1, \tau_2) \approx d(Q_1, Q_2) = \epsilon^{-1}d(P_1, P_2) \) can be taken to be of at least unit size

\[
d(\tau_1, \tau_2) \geq 1
\]

if \( \epsilon \) is initially taken small enough. However, we can now appeal to Lemma 7.4 below to rule out the arising picture when \( \lambda \) and \( \delta \) are small enough. This completes the first step of the proof.

**Step 2.** We have just established that \( F(u) \) is smooth everywhere but possibly one point – without loss of generality, this exceptional point sits at the origin. Then each component of \( F(u) \setminus 0 \) is a smooth submanifold of \( \mathbb{R}^2 \), hence diffeomorphic to either the circle \( S^1 \) or the real line \( \mathbb{R} \). Let us establish that the former possibility does not arise. Assume that there is a connected component of \( F(u) \setminus 0 \) that is a smooth, simple closed curve \( \alpha \). Choose \( \delta > 0 \) small enough, such that \( N_{2\delta}(\alpha) \cap (F(u) \setminus \alpha) = \emptyset \) (such a \( \delta \) exists since \( \alpha \) is compact and \( F(u) \setminus \alpha \) is closed). But since \( F(u_k) \to F(u) \) locally in the Hausdorff distance, for \( K := N_{3\delta/2}(\alpha) \),

\[
F(u_k) \cap K \subseteq N_{\delta}(F(u) \cap K) = N_{\delta}(\alpha)
\]

for all \( k \) large enough. However, this is impossible, since by the topological assumption \([3]\), the free boundary of \( u_k \) has to “exit” \( N_{\delta}(\alpha) \):

\[
(F(u_k) \cap K) \setminus N_{\delta}(\alpha) = F(u_k) \cap \left(N_{3\delta/2}(\alpha) \setminus N_{\delta}(\alpha)\right) \neq \emptyset.
\]

This places us in the assumptions of Lemma 7.5 below, through which we establish the smoothness of \( F(u) \) everywhere.
Step 3. Each connected component of $F(u)$ is diffeomorphic to $\mathbb{R}$ and thus, the image of some embedding $\beta : \mathbb{R} \to \mathbb{R}^2$. The embedding has to be proper, i.e. $\lim_{t \to \pm \infty} \beta(t) = \infty$. Otherwise, there exists a sequence, say $t_i \to \infty$, such that $\lim_{i \to \infty} \beta(t_i) = Q \in \mathbb{R}^2$. But $Q \in F(u)$ as $F(u)$ is closed, and since $F(u)$ is smooth at $Q$, for a small enough $r > 0$ $F(u) \cap B_r(Q)$ is a connected arc $\tilde{\beta}$ that contains $Q$ in its interior. But then it has to be that $\tilde{\beta} \cap \beta \neq \emptyset$, so that by connectedness $\tilde{\beta} \subseteq \beta$. The last statement contradicts the finite limit of $\{\gamma(t_i)\}$. Therefore, each connected component of $F(u)$ is a smooth curve, which is the image of a proper embedding of $\mathbb{R}$ into the plane – we shall call such curves “proper arcs”. Furthermore, each proper arc of $F(u)$ has strictly positive curvature.

In this last step of the proof of the Proposition, we shall show that $F(u)$ consists of precisely two proper arcs. Then we can invoke [Tra14, Theorem 12] to conclude that $u$ is the hairpin solution.

As we know, $u$ blows down to a wedge solution $s|x_2|$, so for a sequence of $\delta_j \searrow 0$ we can find a sequence $R_j \nearrow \infty$, so that $F(u) \cap B_{R_j} \subseteq \{|x_2| < \delta_j R_j\}$ and $\{x_2 = 0\} \cap B_{R_j} \subseteq N_{\delta_j R_j}(F(u))$. Define

$$
\alpha_{L,j} = \partial B_{R_j} \cap \{|x_1 < 0, |x_2| < \delta_j R_j\}, \quad \alpha_{R,j} = \partial B_{R_j} \cap \{|x_1 > 0, |x_2| < \delta_j R_j\}.
$$

Claim that each connected arc $\gamma \in F(u)$ that intersects $B_{R_j}$ “enters and exits” $B_{R_j}$ either through $\alpha_{L,j}$ or $\alpha_{R,j}$ if $R_j$ is large enough, i.e.

$$
\partial B_{R_j} \cap \gamma \subseteq \alpha_{L,j} \quad \text{or} \quad \partial B_{R_j} \cap \gamma \subseteq \alpha_{R,j}.
$$

If not, then let $U$ be the connected component of $\{u > 0\}$ such that $\gamma \subseteq \partial U$. Then it’s easy to see that $u 1_U$ is a viscosity solution of (1) whose free boundary is asymptotically flat, and thus $u 1_U$ has to be a half-plane solution, which is impossible since $|\nabla u| < 1$.

As a consequence of the above argument, combined with the fact that $F(u_{R_j}) \to \{x_2 = 0\}$ locally in the Hausdorff distance, it must be that $F(u)$ consists of at least two proper arcs. Assume that $F(u)$ has at least three: $\gamma_1, \gamma_2$ and $\gamma_3$, and let $j_0$ be large enough such that $\partial B_{R_{j_0}} \cap \gamma_i \subseteq \alpha_{L,j_0}$ or $\partial B_{R_{j_0}} \cap \gamma_i \subseteq \alpha_{R,j_0}$, $i = 1, 2, 3$. Note that because $\gamma_i$ has positive curvature, if $\partial B_{R_{j_0}} \cap \gamma_i \subseteq \alpha_{L,j_0}$ then $\partial B_{R_{j_0}} \cap \gamma_i \subseteq \alpha_{L,j}$ for all $j \geq j_0$ (similarly, if $\gamma_i$ “enters and exits from the right” at scale $R_{j_0}$, it will do so at any larger scale). It has to be that at least two of these arcs “enter and exit” from the same side, say $\gamma_1$ and $\gamma_2$ “enter and exit” from the right. Let $0 < M < R_j/3$ be large enough such that $\{x_1 = M\}$ intersects both $\gamma_1$ and $\gamma_2$, and consider any connected component $V$ of

$$
\{u > 0\} \cap \{M < x_1 < M + R_j/3\}.
$$

Applying Lemma 6.5 to the piecewise $C^1$ Jordan domain $V$,

$$\mathcal{H}^1(\partial V \cap F(u)) \leq C \mathcal{H}^1(\partial V \setminus F(u)) \leq 4\delta_j R_j,
$$

while clearly $\mathcal{H}^1(\partial V \cap F(u)) \geq 2R_j/3$. This leads to a contradiction when $j \to \infty$ as $\delta_j \to 0$.

The proof of the proposition is complete modulo the following three technical lemmas.

**Lemma 7.4.** Let $u$ be a classical solution of (1) in $B_{2M}$ for some large $M$, whose Lipschitz norm is $L$. Assume that (3) is satisfied and that

$$
\{u = 0\} \cap \overline{B_M} \subseteq \{|x_2| < \delta M\}
$$

for some $\delta > 0$. Assume further that $F(u) \cap \overline{B_M}$ consists of arcs, each having its two ends either in $\alpha_L$ or in $\alpha_R$, where

$$\alpha_L = \partial B_M \cap \{x_1 < 0, |x_2| < \delta M\}, \quad \alpha_R = \partial B_M \cap \{x_1 > 0, |x_2| < \delta M\}.
$$

Let $F_L$ (resp. $F_R$) denote the set of points of $F(u) \cap \overline{B_M}$ that lie on arcs both whose ends belong to $\alpha_L$ (resp. $\alpha_R$). Then there exist small positive $\delta_0 = \delta_0(L)$ and $\lambda = \lambda(L) < 1$ such that if $0 < \delta < \delta_0$, one cannot find two straight-line open segments $\tau_1$ and $\tau_2$ of length less that $\lambda$ in $B_{M/2}^+(u)$, each having one end in $F_L$ and one in $F_R$, and such that $\text{dist}(\tau_1, \tau_2) \geq 1$.

**Proof.** Assume that for some $\delta, \lambda$ such segments exist; we’ll derive a contradiction by taking $\delta$ and $\lambda$ small enough and universal. Let $\tau_1$ have ends $p_L \in F_L$ and $p_R \in F_R$, and $\tau_2$ connect $q_L \in F_L$ to $q_R \in F_R$. The following three scenarios regarding the relation between these points may hold.
**Scenario 1.** The points \( p_L, p_R, q_L, q_R \) belong to distinct arcs in \( F_L \) and \( F_R \): \( \gamma_{p_L}, \gamma_{p_R}, \gamma_{q_L} \) and \( \gamma_{q_R} \), respectively. Each of these arcs is divided into two subarcs by its respective point — that start on the point and end on \( \partial B_M \); let us choose one of these two and say our choice of a subarc on \( \gamma_{p_L} \) will be denoted by \( \gamma'_{p_L} \). Then note that \( \Gamma_p := \gamma'_{p_L} \cup \tau \cup \gamma'_{p_R} \) and \( \Gamma_q := \gamma'_{q_L} \cup \tau \cup \gamma'_{q_R} \) are disjoint simple curves and \( B_M \setminus (\Gamma_p \cup \Gamma_q) \) consists of three connected components, only one of which is contained in \( \{|x_2| < \delta M\} \cap B_M \); let us call it \( D \). Consider a connected component \( U \) of \( D^+(u) \) that has \( \tau_1 \) as part of its boundary. Obviously, \( U \) is piecewise \( C^1 \) with

\[
\mathcal{H}^1(F(u) \cap \partial U) \geq 2M - \mathcal{H}(\tau_1) \geq 2M - \lambda \quad \text{and} \quad \mathcal{H}^1(\partial U \setminus F(u)) \leq 4\arcsin(\delta)M
\]

Applying Lemma 6.5 to \( U \), we reach the inequality

\[
2M - \lambda \leq 4\arcsin(\delta)M,
\]

which cannot be satisfied if \( \delta \) and \( \lambda \) are small enough.

**Scenario 2.** Two of the points that “connect to one side” belong to the same arc, while their counterparts belong to distinct arcs: say the points \( p_L \) and \( q_L \) belong to the same arc \( \gamma_L \) in \( F_L \), while \( p_R \) and \( q_R \) belong to two distinct arcs \( \gamma_{p_R} \) and \( \gamma_{q_R} \), respectively, in \( F_R \). This time let \( \gamma'_L \) denote the subarc of \( \gamma_L \) whose ends are \( p_L \) and \( q_L \) and let \( \gamma'_{p_R}, \gamma'_{q_R} \) be determined in the same fashion as in Scenario 1 above. Then \( \Gamma := \gamma'_{p_R} \cup \tau \cup \gamma'_L \cup \tau \cup \gamma'_{q_R} \) is a simple curve in \( B_M \) with ends on \( \alpha_R \), so that \( B_M \setminus \Gamma \) consists of two connected components, one of which is contained in \( \{|x_2| < \delta M\} \); let us again call it \( D \). Consider a connected component \( U \) of \( D^+(u) \) that has \( \tau_1 \) as part of its boundary (and thus \( \tau_2 \) and \( \gamma'_L \)). Then \( U \) is piecewise \( C^1 \) with

\[
\mathcal{H}^1(F(u) \cap \partial U) \geq \mathcal{H}^1(\gamma'_L) + \min\{\mathcal{H}^1(\gamma'_{p_R}), \mathcal{H}^1(\gamma'_{q_R})\} + \min\{\mathcal{H}^1(\gamma'_{q_R}), \mathcal{H}^1(\gamma'_{q_R})\}
\]

since it is either \( \gamma'_{p_R} \) or \( \gamma'_{q_R} \) (and similarly \( \gamma'_{q_R} \) or \( \gamma'_{q_R} \)) that belongs to \( \partial U \). But each of these curves intersects both \( \partial B_{M/2} \) and \( \partial B_M \), so that its length is at least \( M/2 \). As \( \mathcal{H}^1(\gamma'_L) \geq \text{dist}(\tau_1, \tau_2) \geq 1 \), we get

\[
\mathcal{H}^1(F(u) \cap \partial U) \geq 1 + M,
\]

while on the other hand

\[
\mathcal{H}^1(\partial U \setminus F(u)) \leq 2\arcsin(\delta)M.
\]

Applying Lemma 6.5 to \( U \) we see that

\[
1 + M \leq 2\arcsin(\delta)M,
\]

which is violated when \( \delta \) is small enough.

**Scenario 3.** In this last scenario, \( p_L \) and \( q_L \) belong to the same arc \( \gamma_L \) of \( F_L \), and \( p_R \) and \( q_R \) belong to the same arc \( \gamma_R \) of \( F_R \). Let \( \gamma'_L \) denote the subarc of \( \gamma_L \) with ends \( p_L, q_L \) and \( \gamma'_R \) denote the subarc of \( \gamma_R \) with ends \( p_R, q_R \). Then \( \Gamma := \tau \cup \gamma'_L \cup \tau \cup \gamma'_R \) is a simple closed curve that encloses a piecewise \( C^1 \) Jordan domain \( U \subseteq B_M^+(u) \) with

\[
\mathcal{H}^1(F(u) \cap \partial U) = \mathcal{H}^1(\gamma'_L) + \mathcal{H}^1(\gamma'_R) \geq 2\text{dist}(\tau_1, \tau_2) \geq 2,
\]

while

\[
\mathcal{H}^1(\partial U \setminus F(u)) = \mathcal{H}^1(\tau_1) + \mathcal{H}^1(\tau_2) \leq 2\lambda.
\]

Lemma 6.5 then yields

\[
2 \leq 2\lambda L,
\]

which is impossible if \( \lambda \) is small enough.

\[\square\]

**Lemma 7.5.** Let \( u : \mathbb{R}^2 \to \mathbb{R} \) be a non-degenerate viscosity and variational solution of \( F \) with \( |\nabla u| < 1 \) in \( \{u > 0\} \). Assume further that \( F(u) \) is smooth everywhere but possibly the origin \( 0 \in F(u) \) and that \( F(u) \setminus 0 \) has no connected component that is a closed curve. Then \( F(u) \) is smooth at the origin as well, and \( u \) is a classical solution of \( F \) globally.

**Proof.** Every connected component \( \gamma \) of \( F(u) \setminus 0 \) is a smooth connected submanifold of \( \mathbb{R}^2 \), and since it is not diffeomorphic to circle \( S^1 \) by hypothesis, it has to be diffeomorphic to the real line \( \mathbb{R} \). Thus, it is the image of an embedding \( \gamma : \mathbb{R} \to \mathbb{R}^2 \) and it must be that for any sequence \( t_n \to \infty \) (or \( t_n \to -\infty \)) \( \lim_{t_n \to \gamma(t_n)} q \) is either \( \infty \) or 0. Otherwise, there would be some sequence \( t_n \to \pm \infty \) such that \( \gamma(t_n) \) converges to some finite limit \( q \in F(u) \), where \( q \neq 0 \), because \( F(u) \) is closed. But \( F(u) \) is smooth at \( q \), so that for a small enough
Let Lemma 7.6.

\( \alpha \) that contains \( q \) in its interior. Since \( \beta \cap \gamma \neq \emptyset \), it follows that \( \beta \subseteq \gamma \) by connectedness, which contradicts the convergence \( \gamma(t_n) \to q \).

As a first step, we claim that \( F(u) \cap \partial B_1 \) consists of finitely many points. Otherwise, there would exist a sequence of points \( p_n \in F(u) \cap \partial B_1 \) that converges to some \( p \in F(u) \cap \partial B_1 \); denote by \( \gamma \) the connected component of \( F(u) \setminus 0 \) containing \( p \). Since \( F(u) \) is smooth at \( p \) it would have to be that for all large enough \( n \), \( p_n \in \gamma \). However, that contradicts the fact that \( \gamma \) is an analytic curve different from the circle \( \partial B_1 \).

Second, assume that \( F(u) \setminus 0 \) has a connected component \( \alpha \), an image of a smooth \( \alpha : \mathbb{R} \to \mathbb{R}^2 \), such that \( \lim_{t \to \pm \infty} \alpha(t) = 0 \). Then \( \alpha \) is a simple closed curve, so that it encloses a bounded connected domain \( U \). Obviously \( U \subseteq \{ u = 0 \} \), as the Strong Maximum Principle prevents \( U \) from containing points \( x \) where \( u(x) > 0 \). Because of Corollary 6.4 \( \alpha \) has positive curvature, so we can apply Lemma 7.6 to conclude that \( \{ u = 0 \} \supseteq U \) contains a non-trivial sector based at \( 0 \). As a result, the blow-up of \( u \) at \( 0 \) cannot be the wedge solution and can only be the one-plane solution. Therefore \( F(u) \) has to be smooth at \( 0 \).

Thus, we may assume that for each connected component \( \gamma \) of \( F(u) \setminus 0 \), \( \gamma(t_n) \to \infty \) for some sequence \( t_n \to \infty \) or \( t_n \to -\infty \).

Next we note that if \( u \) cannot be an isolated point of \( F(u) \), so there exists a sequence of points \( p_n \in F(u) \) such that \( p_n \to 0 \). Since there are only finitely many connected components of \( F(u) \setminus 0 \) intersecting \( B_1 \), there exists a subsequence \( p_{n_k} \) that belongs to a single connected component \( \gamma_1 \), so that \( \lim_{t \to \infty} \gamma_1(t) = 0 \).

Then it must be \( \lim_{t \to -\infty} \gamma_1(t) = \infty \), so the latest “entry time” for \( \gamma_1 \) into \( B_1 \),

\[ T := \sup \{ t : \gamma_1(t) \in (B_1)^c \} \]

satisfies \( |T| < \infty \). Consider the connected component \( V \) of \( \{ u = 0 \} \cap B_1 \) having \( \gamma_1([T, \infty]) \) as part of its boundary. Obviously, \( 0 \in \partial V \) and claim that there exists another curve \( \gamma_2 : (-\infty, 0) \to \mathbb{R}^2 \) such that \( \gamma_2(-\infty, 0) \subset \partial V \cap (F(u) \setminus 0) \) with \( \lim_{t \to -\infty} \gamma_2(t) = 0 \). If not, \( \partial V \cap (F(u) \setminus 0) \) consists of finitely many free boundary arcs with ends \( p_{2i} \) and \( p_{2i+1} \) on the unit circle \( \partial B_1 \), \( k = 0, 1, 2, \ldots, l \). Here we have chosen the enumeration of the points \( \{ p_i \} \) in such a way that the shorter circular arc \( \gamma_1(t) \) intersects \( \gamma_2(t) \), and has \( p_{i+1} \) following \( p_i \) in the direction (clockwise or counterclockwise) set by \( p_1 \) and \( p_2 \). In this way, the circular arcs \( \gamma_1(t) \) \( \partial V \), \( k = 0, 1, \ldots, l - 1 \). Let \( q \in F(u) \cap \partial B_1 \) be the next point after \( p_{2i+1} \) on the unit circle as we traverse it in the same direction. Then it must be that \( \gamma_2(t) \subset \partial V \), and that the connected component of \( F(u) \setminus 0 \) \( \gamma_2(t) \), having \( q \) as one of its ends, is also part of \( \partial V \). Since the other end of that component cannot lie on either \( \partial B_1 \) or 0, it has to be that \( q = p_0 \). This is, however, impossible as \( u \) cannot be zero on both sides of \( \gamma_1 \). So, there exists a free boundary curve \( \gamma_2 \subset \partial V \cap (F(u) \setminus 0) \), disjoint from \( \gamma_1 \), with \( \gamma_2(-\infty) = 0 = \gamma_1(\infty) \). From here it is not hard to see that \( V \) is a Jordan domain. Again, Corollary 6.4 says that \( \gamma_1 \) and \( \gamma_2 \) have positive curvature, and we can invoke Lemma 7.6 to establish that \( V \) contains a non-trivial sector based at 0. As before, the blow-up limit of \( u \) at \( 0 \) has to be the half-plane solution, so that \( F(u) \) is smooth.

**Lemma 7.6.** Let \( U \subseteq \mathbb{R}^2 \) be a Jordan domain with \( 0 \in \partial U \) and let \( \gamma_1 \in C^2([0, \infty), \mathbb{R}^2) \) and \( \gamma_2 \in C^2((-\infty, 0], \mathbb{R}^2) \) be some regular parameterizations (\( \gamma'_i \neq 0 \), \( i = 1, 2 \)) of two simple disjoint subarcs of \( \partial U \), for which \( \lim_{t \to \infty} \gamma_1(t) = \lim_{t \to -\infty} \gamma_2(t) = 0 \), and such that traversing \( \partial U \) in the counterclockwise direction corresponds to \( t \) increasing. Assume further that their curvatures are strictly positive. Then \( B_r \cap U \) contains a non-trivial sector of \( B_r \).

**Proof.** First let us introduce some notation. For a point \( p \) in \( \gamma_i \), \( i = 1, 2 \), let \( L(p) \) be the tangent line to \( \gamma_i \) at \( p \). If \( p = \gamma_i(t_0) \) for some \( t_0 \), let \( \tau(p) = \gamma_i'(t_0)/|\gamma_i'(t_0)| \) be the unit tangent vector to \( p \) in the direction of \( \gamma_i'(t_0) \); let \( \nu(p) \) be the unit normal vector to \( \gamma_i \) at \( p \) that one gets by rotating \( \tau(p) \) by \( \pi/2 \). Denote by \( H^+(p) \) and \( H^-(p) \) the two half-planes:

\[ H^\pm(p) = \{ x \in \mathbb{R}^2 : (x - p) \cdot (\pm \nu(p)) > 0 \} \]

For any two points \( p = \gamma_i(t_1) \) \( q = \gamma_i(t_2) \), \( t_1 < t_2 \), define \( \theta(p, q) \) to be the angle \( \gamma_i'(t) \) sweeps as \( t \) increases from \( t_1 \) to \( t_2 \). Then the fact that \( \gamma_i \) has positive curvature is equivalent to \( \theta(\gamma_i(t), \gamma_i(t + s)) \) being a positive,
strictly increasing function of $s$ for $s > 0$ and any fixed $t$. For any $p \in \gamma_1(t)$ and $\alpha > 0$, denote by $T^\alpha(p)$ the point $q \in \gamma_1$ such that $\theta(p,q) = \alpha$, if it exists. Let $s(p,q)$ be the open segment of $\gamma_1$ with ends $p, q \in \gamma_1$.

Now we proceed with the argument. We shall show that $\theta(\gamma_1(0),\gamma_1(t))$ is bounded from above as a function of $t$. Assume not; then $\tilde{T}^\alpha(p)$ exists for any $p \in \gamma_1$ and any $\alpha > 0$. Claim that there exists a $p_0 \in \gamma_1$ such that $T^2\pi(p_0) \in H^+(p_0)$. If not, then for any $p \in \gamma_1$ and $k \in \mathbb{N}$,

$$T^{(2k+2)\pi}(p) \in H^-(T^{2k\pi}(p)) \subset H^-(p) \quad \text{as well as}$$

$$T^{(2k+3)\pi}(p) \in H^-(T^{(2k+1)\pi}(p)) \subset H^-(T^\pi(p)).$$

However, note that because $\gamma_1$ has positive curvature, we have $s(p,T^\pi(p)) \subset H^+(p)$, as the smallest $\alpha > 0$ for which $s(p,T^\alpha(p))$ can intersect $L(p)$ must be greater than $\pi$. Thus, $H^-(T^\pi(p)) \subset H^+(p)$. But since $\gamma_1(t) \to 0$ as $t \to \infty$ and $\gamma_1' \neq 0$, then both

$$T^{(2k+2)\pi}(p) \to 0 \quad \text{and} \quad T^{(2k+3)\pi}(p) \to 0 \quad \text{as} \quad k \to \infty.$$ 

That contradicts the fact that $T^{(2k+2)\pi}(p) \subset H^-(p)$ whereas $T^{(2k+3)\pi}(p) \subset H^+(p)$.

Thus, for some $p_0$, $T^2\pi(p_0) \subset H^+(p_0)$, so that the whole segment $s(p_0,T^{2\pi}(p_0)) \subset H^+(p_0)$. Denote

$$p_j := T^j\pi(p_0) \quad j \in \mathbb{N}.$$ 

Claim we then have $s(p_{2k-2},p_{2k}) \subset H^+(p_{2k-2})$ for all $k \in \mathbb{N}$. Argue by induction. Note that since $s(p_{2k-1},p_{2k}) \subset H^+(p_{2k-1}) \cap H^+(p_{2k})$, there are exactly two intersection points between $\pi(p_{2k-2},p_{2k})$ and $L(p_{2k})$; namely $p_{2k}$ and a point $q_{2k} \in \pi(p_{2k-2},p_{2k-1})$ (see Figure 1). If it were the case that $p_{2k+2} \subset H^-(p_{2k})$, the segment $s(p_{2k},p_{2k+2})$ would have to leave the convex domain $D_{2k} \subset H^+(p_{2k}) \cap H^+(p_{2k-1})$, enclosed by $s(q_{2k},p_{2k})$ and the straight-line segment $s_{2k} := p_{2k}q_{2k}$. But obviously $s(p_{2k},p_{2k+1}) \subset D_{2k}$, so it would have to be $s(p_{2k+1},p_{2k+2})$ that exits $D_{2k}$. Construct as above the point $q_{k+1} \in s(p_{2k+1},p_{2k})$ being the second intersection point of $L(p_{2k+1})$ with $\pi(p_{2k-1},p_{2k+1})$ and let $s_{2k+1} \subset D_{2k}$ be the straight-line segment $p_{2k+1}q_{2k+1}$. Then the convex domain $D_{2k+1}$, enclosed by $s_{2k+1}$ and $s(q_{2k+1},p_{2k+1})$, is contained within the convex $D_{2k}$, so that $s(p_{2k+1},p_{2k+2})$ which enters $D_{2k-1}$ would have to exit $D_{2k+1}$ before it exits $D_{2k}$. That is, however, impossible as $s(p_{2k+1},p_{2k+2}) \subset D_{2k+1}$. The induction step is complete.

Let now $K \in \mathbb{N}$ be large enough such that for all $k \geq K$, $s(p_{2k-2},p_{2k}) \subset B_0(0)$ where $\delta > 0$ is small enough, such that $\gamma_2(0) \subset B_0(0)^c$ (such a $K$ exists since $\gamma_1(t) \to 0$ as $t \to \infty$). Since $\gamma_2(t) \subset D_{2k+1}$ for all large $|t|$, its last ‘time of exit’ from $D_{2k+1}$

$$T := \sup\{t : \gamma_2(t) \in D_{2k+1}\}$$
exists and must satisfy \( T < 0 \). Obviously, \( \gamma_2(T) \) must belong to \( \sigma_{2k+1} \) and \( \gamma_2 \) must intersect \( \sigma_{2k+1} \) transversally, for otherwise the fact that \( \gamma_2 \) has positive curvature would imply that for some small \( \varepsilon < 0 \) \( \gamma((T-\varepsilon,T+\varepsilon]) \) would lie on one side of \( \sigma_{2k+1} \) which contradicts the definition of \( T \). Let \( \tilde{q}_{2k+1} \) denote the intersection point of \( L(q_{2k+1}) \) and \( s(p_{2k-2},q_{2k-1}) \), and let \( \tilde{\sigma}_{2k+1} \) be the straight-line segment \( q_{2k+1}\tilde{q}_{2k+1} \). Note that since \( \gamma_2 \) intersects \( \sigma_{2k+1} \) transversally, \( \tilde{\sigma}_{2k+1} \subseteq H^- (\gamma_2(T)) \). Also, since \( \gamma_2(0) \notin B_\delta(0) \), \( \gamma_2([T,0]) \) must exit the domain \( \tilde{D}_{2k+1} \subseteq B_\delta(0) \), enclosed by \( s(q_{2k+1},\tilde{q}_{2k+1}) \) and the straightline segment \( q_{2k+1}\tilde{q}_{2k+1} \), having once entered it. Thus \( \gamma_2([T,0]) \) intersects \( \tilde{\sigma}_{2k+1} \subseteq H^- (\gamma_2(T)) \), so that \( \gamma_2((T,0]) \) must cross \( L(\gamma_2(T)) \). Let \( T_1 \) be the first time \( \gamma((T,0]) \) crosses \( L(\gamma_2(T)) \):

\[
T_1 := \inf \{ t > T, \gamma_2(t) \in L(\gamma_2(T)) \}.
\]

Note that \( T_1 > T \) as \( \gamma_2((T,T+\delta)) \subseteq H^+(\gamma_2(T)) \) for all small enough \( \delta > 0 \). Furthermore, it must be that \( \theta(\gamma_2(T),\gamma_3(T_1)) \geq \pi \) but \( \theta(\gamma_2(T),\gamma_2(T_1)) \leq 2\pi \). The former bound is obvious; the latter is true for otherwise \( T^2(\gamma_2(T)) \in H^+(\gamma_2(T)) \), so that by the same argument as before we would have all of \( \gamma_2((T,0]) \subseteq H^+(\gamma_2(T)) \), which would prevent it from crossing \( L(\gamma_2(T)) \). As a result, it must be that \( (\gamma_2(T_1) - \gamma_2(T)) \cdot \gamma_2(T) < 0 \), which in turn implies that \( \gamma_2((T,T_1]) \) must cross the straightline segment \( \gamma_2(T)\tilde{q}_{2k+1} \), which is impossible.

Therefore, \( \theta(\gamma_1(0),\gamma_1(t)) \) is bounded from above, so that \( \tau_1 := \lim_{t \to \infty} \gamma_1(t)/\gamma_1(t) \) exists. Exchanging the roles of \( \gamma_1 \) and \( \gamma_2 \) in the argument above, we can show that \( \tau_2 := \lim_{t \to \infty} \gamma_2(t)/\gamma_2(t) \) exists, as well. As a result, for all small enough \( r > 0 \), \( \gamma_1 \cap B_r \) and \( \gamma_2 \cap B_r \) are flat graphs over the radii along \( \tau_1 \) and \( \tau_2 \), respectively. Let \( A_i = \partial B_r \cap \gamma_i, i = 1,2 \) be the points of intersection of \( \partial B_r \) with \( \gamma_1 \) and \( \gamma_2 \). Because of the positivity of the curvature, the open straight-line segments connecting \( 0 \) to \( A_1 \) and \( 0 \) to \( A_2 \) are contained in \( U \) for \( r \) small enough. Also, since \( \partial B_r \cap \partial U = \{A_1, A_2\} \), the whole open circular arc \( \widehat{A_2A_1} \) (as we trace the circle in the counter-clockwise direction) must be contained in the Jordan domain \( U \). Thus, \( U \) contains the entire open circular sector with vertex \( 0 \) and arc \( \widehat{A_2A_1} \).

\[ \square \]

8. Local structure.

In this section we shall study the shape of the free boundary of solutions of (1), defined in the unit disk, satisfying the topological assumption (3). This will be carried out by examining blow-up limits of sequences of solutions in \( B_1 \), for which exact purpose the classification Theorem (1) was developed. We encounter the following dichotomy: if a component of the zero phase is well separated by the rest of the zero phase, its boundary has bounded curvature (in terms of the separation) – this is the content of Proposition 8.2 below. Once the separation becomes small enough relative a certain universal scale, we shall see the signs of a hairpin-like structure arising – this is described in Propositions 8.3 and 8.4.

Let us make the following definition for ease of reference:

**Definition 8.1.** We shall call the free boundary \( F(u) \) of a solution \( u \) of (1) \( \delta \)-flat in \( B = B_r(p) \) if for some rotation \( \rho \),

\[
|u(p + \rho x) - x_2^2| \leq \delta r \quad \text{for} \quad x \in B_r(0).
\]

**Remark 8.1.** Denote by \( \delta_0 \) the small universal constant, such that if \( 0 < \delta < \delta_0 \) small enough, the Alt-Caffarelli regularity theory [ACS1] states that \( F(u) \cap B_{r/2} \) is a graph in the direction of \( \rho(e_2) \) with Lipschitz norm at most \( C\delta \). For such \( \delta_0 \) we also have the bound

\[
\|\rho \nabla u - e_2\|_{L^{\infty}(B_{r/2}(u))} + \|D^2u\|_{L^{\infty}(B_{r/2}(u))} \leq c\delta.
\]

It implies, in particular, that the curvature of \( F(u) \) in \( B_{r_0/2} \) is \( O(\delta) \).

The next proposition treats the scenario where a point of the free boundary \( F(u) \) is distance at least \( s \) away from all other components of the zero phase; then we expect a curvature bound on \( F(u) \) at the point.

**Proposition 8.2.** Let \( u \) be a classical solution of (1) in \( B_1 \) that satisfies (3) and assume \( 0 \in F(u) \). Denote by \( Z \) the connected component of \( 0 \) in \( \{u = 0\} \). For any \( 0 < s < 1 \) there exists \( \kappa = \kappa(s) < \infty \) such that if

\[
d(0, \{u = 0\} \setminus Z) \geq s
\]

then the curvature of \( F(u) \) at 0 is at most \( \kappa \).
Proof. Assume the proposition is false. Then we have a sequence of counterexamples \( u_l \) for which the curvature \( \kappa_l \) of \( F(u_l) \) at zero is
\[
\kappa_l \geq l^2.
\]
Define the rescales
\[
\tilde{u}_l(x) := lu_l(x/l) \quad \text{for} \quad x \in B_l.
\]
Then the curvature \( \tilde{k}_l \) of \( F(\tilde{u}_l) \) at 0 satisfies
\[
\tilde{k}_l = \kappa_l/l \geq l.
\] (16)
By our classification Theorem 1.1 we see that, up to taking a subsequence, the \( \tilde{u}_l \) converge uniformly on compact subsets to a global solution \( \tilde{u} \) that is either a one-plane, a two-plane, a hairpin or a wedge solution.

Let \( \delta_0 > 0 \) be the small universal constant defined in Remark 8.1. If \( \tilde{u} \) is a one-plane solution, then for all large enough \( l \), in some Euclidean coordinates
\[
|\tilde{u}_l - x_l^+| < \delta_0/2 \quad \text{in} \quad B_1,
\]
hence \( F(\tilde{u}_l \cap B_1) \) is \( \delta_0 \)-flat and \( \tilde{k}_l \leq C\delta_0 \) which contradicts (16). Similarly, if \( \tilde{u} \) is a two-plane solution, for some \( b < 0 \) and all large enough \( l \)
\[
|\tilde{u}_l - (x_l^+ + (x_2 - b)^-) < \min\{\delta_0/2, b/10\} \quad \text{in} \quad B_1.
\]
Thus,
\[
v_l := \tilde{u}_l 1_{B_1 \cap \{x_2 > b/2\}}
\]
is a classical solution of (1) in \( B_1 \), whose free boundary is \( \delta_0 \)-flat in \( B_1 \), so \( \tilde{k}_l \leq C\delta_0 \) — a contradiction.

Analogously, we can rule out \( \tilde{u} \) being a hairpin solution. Assume that it is; then we can find a scale \( s_0 \) such that for every \( p \in F(\tilde{u}) \)
\[
d_H(F(\tilde{u}) \cap B_{s_0}(p), L(x) \cap B_{s_0}(p)) < \delta_0 s_0/2,
\]
where \( L(p) \) denotes the tangent line to \( F(\tilde{u}) \) at \( p \). Now for all large enough \( l \),
\[
d_H(F(\tilde{u}_l) \cap B_{s_0}(0) \cap B_{s_0}(p)) < \delta_0 s_0
\]
so that \( w_l(y) := \tilde{u}_l(s_0 y)/s_0 \) has a \( \delta_0 \)-flat free boundary in \( B_1 \) and the curvature of \( F(w_l) \) at 0 is bounded by \( C\delta_0 \). Thus, the curvature of \( F(\tilde{u}_l) \) at 0
\[
\kappa_l \leq C\delta_0/s_0,
\]
which again contradicts (16).

Finally, assume that \( \tilde{u} = |x_2| \) is the wedge-solution. Then for all \( l \) large enough
\[
d_H(F(\tilde{u}_l) \cap \overline{B_r}, \{x_2 = 0\} \cap \overline{B_r}) \leq \delta_0.
\] (17)
Let \( N = (0, 1) \) and \( S = (0, -1) \). Note that \( N \) and \( S \) cannot belong to two separate components of \( B_1^+ (\tilde{u}_l) \), for according to Lemma 6.7, \( F(\tilde{u}_l) \cap \{|x_1| < 1/2\} \cap \partial B_1 \) consists of two graphs of Lipschitz norm at most \( c\delta_0 \), so that we again get an upper bound for \( \tilde{k}_l \) for all large \( l \). This means that if \( F(\tilde{u}_l) \cap \partial B_3 \) consists of finitely many arcs, each of which “attaches” either to \( \alpha_L \) or \( \alpha_R \), where
\[
\alpha_L = \partial B_3 \cap \{x_1 < 0, |x_2| < \delta_0\} \quad \alpha_R = \partial B_3 \cap \{x_1 > 0, |x_2| < \delta_0\}.
\]
Thus, if \( F_L \) (\( F_R \)) denotes the union of the arcs of \( F(\tilde{u}_l) \cap \partial B_3 \) that attach to \( \alpha_L \) (\( \alpha_R \)), then \( F_L \) and \( F_R \) are disjoint compact sets and so \( d(F_L, F_R) \) is realized for some \( p \in F_L \) and \( q \in F_R \). Moreover, the straight line (open) segment \( \tau \) with ends \( p \) and \( q \) is contained in \( B_3^+ (\tilde{u}_l) \) and because of (17), we have
\[
|p - q| = \mathcal{H}^1(\tau) \leq 6\delta_0.
\]
On the other hand, note that if \( \tilde{Z}_l \) denotes the connected component of 0 in \( \{\tilde{u}_l = 0\} \) in \( B_1 \), we have by assumption
\[
d(0, \{\tilde{u}_l = 0\} \setminus \tilde{Z}_l) \geq l s \gg 1,
\]
hence it must be that both \( p \) and \( q \) belong to the same boundary arc of \( \partial \tilde{Z}_l \) (they cannot belong to different boundary arcs of \( \partial \tilde{Z}_l \), for \( p \) and \( q \) would have to lie on the boundary of two different connected components of \( B_1^+ (\tilde{u}_l) \)). Let \( \beta \subseteq F(\tilde{u}_l) \) denote the arc connecting \( p \) to \( q \). Then \( \beta \cap \tau \) encloses a piecewise-\( C^2 \) Jordan domain \( V \subseteq B_1^+ (\tilde{u}_l) \) and applying Lemma 6.5 to \( V \), we find that
\[
6\delta_0 \geq \mathcal{H}^1(\tau) \geq c\mathcal{H}^1(\beta).
\]
which is impossible for small $\delta_0$ as $H^1(\beta) \geq 2$. This completes the proof. \hfill \Box

**Proposition 8.3.** Let $u$ be a classical solution of \((1)\) in $B_1$ that satisfies \((3)\) and assume $0 \in F(u)$. Denote by $Z$ the connected component of $0$ in $\{u = 0\}$. Then for any $0 < \delta < \delta_0$ small enough there exist $0 < \epsilon_0 \ll 1$ and $r_0 > 0$ such that if for any one $0 < r \leq r_0$

\[ d(0, \{u = 0\} \setminus Z) < \epsilon_0 r \]

then for some rotation $\rho$,

\[ |u(\rho x) - |x_2|| < \delta r \text{ in } B_r. \]

**Proof.** Fix $0 < \delta < \delta_0$. By the scale-invariance of the problem it suffices to show the conclusion of the proposition holds only for $r = r_0$. Assume not; then for any sequences of $\tilde{u}_k \rightarrow 0$, $r_k \rightarrow 0$, there exists a corresponding sequence of counterexamples $u_k$ in $B_1$: namely, if $Z_k$ denotes the component of $0$ in $\{u_k = 0\}$, we have $d(0, \{u_k = 0\} \setminus Z_k) \leq \epsilon_k r_k$, but

\[ \|u_k(\rho x) - |x_2||_{L^\infty(B_{r_k})} > \delta r_k \]

for all rotations $\rho$. Define the rescaled

\[ \tilde{u}_k(x) := u_k(r_k x)/r_k \text{ in } B_1/r_k. \]

According to the Classification Theorem \([1]\) up to taking a subsequence, $\tilde{u}_k$ converge uniformly on compact subsets of $\mathbb{R}^2$ to $\tilde{u}$, being either the half-plane, the wedge, a two-plane or a hairpin solution.

If $\tilde{u} = x_2^+$ in an appropriate coordinate system, then for all large enough $k$

\[ \{\tilde{u}_k = 0\} \cap B_1 \cap \{|x_1| < 1/2\} = \{x \in B_1 : |x_1| < 1/2, x_2 < \phi(x_1)\} \]

for some $C\delta_0$–Lipschitz function $\phi : (-1/2, 1/2) \rightarrow \mathbb{R}$. In particular $\{\tilde{u}_k = 0\} \cap B_{1/2}$ consists of a single component (the one containing $0$). Hence, going back to the original scale,

\[ d(0, \{u_k = 0\} \setminus Z_k) \geq r_k/2 > \epsilon_k r_k \]

for $k$ large enough, which contradicts our assumption. Similarly, we rule out the case when $\tilde{u}$ is the two-plane solution. If $\tilde{u}$ is a hairpin, we can find a scale $s_0$, such that for every $x \in F(\tilde{u})$

\[ d_H(F(\tilde{u}) \cap B_{s_0}(x), L(x) \cap B_{s_0}(x)) < \delta_0 s_0/2, \]

where $L(x)$ denotes the tangent line through $x$ to the hairpin $F(\tilde{u})$. Then, for all large enough $k$,

\[ d_H(F(\tilde{u}_k) \cap B_{s_0}, L(0) \cap B_{s_0}) \leq d_H(F(\tilde{u}) \cap B_{s_0}, L(0) \cap B_{s_0}) + d_H(F(\tilde{u}_k) \cap B_{s_0}, F(\tilde{u}) \cap B_{s_0}) \leq s_0 \delta_0/2 + s_0 \delta_0/2 = s_0 \delta_0, \]

so that in $B_{s_0/2}$, $\{\tilde{u}_k = 0\} \cap B_{s_0/2}$ consists of a single component. Going back to scale $r_k$, we see that

\[ d(0, \{u_k = 0\} \setminus Z_k) \geq s_0 r_k/2 > \epsilon_k r_k \]

which is a contradiction when $k$ is large enough.

Therefore, $\tilde{u}$ must be the wedge solution: $\tilde{u} = |x_2|$ in an appropriately rotated coordinate system. This leads to a contradiction with \((18)\), however, because it implies that for all $k$ large enough, we actually have

\[ \|u_k(x) - |x_2||_{L^\infty(B_{r_k})} \leq \delta r_k. \]

\hfill \Box

**Proposition 8.4.** For any given $0 < \delta < \delta_0$, let $\epsilon_0$, $r_0$ and $u : B_1 \rightarrow \mathbb{R}$ be as in Proposition 8.3. Let $Z$ denote the component of $0$ in $\{u = 0\}$. Then for any $0 < r \leq r_0$ such that

\[ d(0, \{u = 0\} \setminus Z) < \epsilon_0 r, \]

the free boundary $F(u) \cap B_{r/2}$ consists of exactly two arcs $F_L \subseteq Z$ and $F_R \subseteq \{u = 0\} \setminus Z$. Those are contained in $\rho(S_{r/2, \delta r})$ for an appropriate rotation $\rho = \rho_r$, where

\[ S_{r,t} := \{x \in \mathbb{R}^2 : |x_1| \leq r, |x_2| \leq t\}, \]

with the two ends of $F_L$ in $\rho(\alpha_{L,r/2})$ and the two ends of $F_R$ in $\rho(\alpha_{R,r/2})$, where

\[ \alpha_{L,r} = \{x \in \partial B_r : x_1 < 0, |x_2| < \delta r\} \quad \text{and} \quad \alpha_{R,r} = \{x \in \partial B_r : x_1 > 0, |x_2| < \delta r\}. \]

Moreover, the minimum distance between the corresponding two components of $\{u = 0\} \cap B_{r/2}$ is realized for some points $p \in F_L$, $q \in F_R$ with both $p, q \in \rho(S_{r/3, \delta r})$. 

\[ \square \]
Proof. Fix $r$ and choose Euclidean coordinates appropriately so that $\rho_r$ is the identity. Let $\gamma$ be the arc of $F(u) \cap B_r$, containing 0. Claim that the two ends of $\gamma$ both belong to either $\alpha_{L,r}$ or $\alpha_{R,r}$. Assume not. Then the points $N = (0, r/2)$ and $S = (0, r/2)$ belong to two distinct connected components of $B_r^+(u)$, so that according to Lemma 6.6, $F(u) \cap B_r \cap \{ |x_1| < r/4 \}$ consists of two disjoint graphs $\Sigma_{\pm} = \{ x_2 = \phi_{\pm}(x_1) : |x_1| < r/4 \}$ of Lipschitz norm at most $C\delta$ and

$$u(x) = 0 \text{ for } x \in \{ \phi_{-}(x_1) \leq x_2 \leq \phi_{+}(x_1) : |x_1| < r/4 \}$$

But then $d(0, \{ u = 0 \} \setminus Z) \geq r/4 > \epsilon_0 r$, which contradicts our hypothesis. Hence, we may assume that $\gamma$ attaches on $\alpha_{L,r}$.

Look now at the free boundary in $B_{r/3}(P)$, where $P = (0, r/2, 0)$. Since $\gamma \subseteq S_{r,\delta}$ connects $\alpha_L$ to 0, it must be that $\gamma$ disconnects $\tilde{N}_L := P + (0, r/6)$ from $\tilde{S}_L := P + (0, -r/6)$ in $B_{r/2}(P)^+(u)$. Invoking Lemma 6.7 again, we see that $F(u) \cap B_{r/3}(P) \cap \{ |x_1 + r/2| < r/6 \}$ consists of two graphs of Lipschitz norm at most $C\delta$. As a result, the connected component $\tilde{Z}_L$ of 0 in $\{ u = 0 \} \cap B_{r/2}$ is bounded by a single free boundary arc $F_L$ and a circular subarc of $\alpha_{L,r/2}$ that share ends. Another even more significant consequence is that $F(u) \cap B_r$ contains no other arcs besides $\gamma$ that intersect $V_L := B_r \cap \{ |x_1 + r/2| < r/6 \}$. Since $\{ u = 0 \}$ has a component different from $Z$ that is at most $\epsilon_0 r$ away from 0, $F(u) \cap B_r$ contains at least one more arc $\tilde{\gamma} \neq \gamma$.

According to the observation above, $\tilde{\gamma}$ doesn’t cross into the region $V$, so it has to attach on $\alpha_{R,r}$. Consider $F(u) \cap B_{r/3}(Q)$, where $Q = (r/2, 0)$. Since $\tilde{\gamma} \cap B_{r/2} \neq 0$, it must be that $\tilde{\gamma}$ disconnects $\tilde{N}_R = Q + (0, r/6)$ from $\tilde{S}_R = Q + (0, -r/6)$ in $B_{r/3}(Q)^+(u)$. Thus, by Lemma 6.7, $F(u) \cap B_{r/3}(Q) \cap \{ |x_1 - r/2| < r/6 \}$ consists of two graphs of Lipschitz norm at most $C\delta$. Hence, $\{ u = 0 \} \cap B_{r/2}$ has only one other connected component $\tilde{Z}_R \cap \partial Z_R \cap F(u)$ consists of a single free boundary arc $F_R$. As $F_R$ cannot intersect $V_L$ and, similarly, $F_L$ cannot intersect $V_R := B_r \cap \{ |x_1 - r/2| < r/6 \}$, it must be that the minimum distance between $\tilde{Z}_L$ and $\tilde{Z}_R$ is realized for some points $p \in F_L$ and $q \in F_R$ with $|x_1(p)| < r/2 - r/6 = r/3$ and $|x_1(q)| < r/3$. □

Remark 8.5. Assume we are in the situation of Propositions 8.3 and 8.4 above for some fixed small $0 < \delta < \delta_0$. Then $F(u) \cap B_{r_0/2}$ consists of two arcs $F_L$ and $F_R$, and the minimum distance $s = d(F_L, F_R)$ is realized for some points $p \in F_L$, $q \in F_R$ with both $p, q \in \rho_{r_0}(S_{r_0/3, \delta, r_0})$. Now apply again Propositions 8.3 and 8.4 to the translate $\tilde{u}(y) := u(p + y)$ $y \in B_{1/2}$.

Call $\tilde{Z}$ the connected component of $\{ \tilde{u} = 0 \}$ containing 0. We establish that for every $r$ such that $s/\epsilon_0 < r \leq r_0$, there is a rotation $\tilde{\rho} = \tilde{\rho}_r$ such that

$$|\tilde{u}(\tilde{\rho} y) - |y_2| | < \delta r \quad \text{in } B_r$$

and the free boundary in $B_{r/2}$, $F(\tilde{u}) \cap B_{r/2} \subseteq \tilde{\rho}(S_{r/2, \delta, r})$ consists of two arcs $\tilde{F}_L \subseteq \tilde{Z}$ and $\tilde{F}_R \subseteq \{ \tilde{u} = 0 \} \setminus Z$, the minimum distance between which is realized for $0 \in \tilde{F}_L$ and $q - p \in \tilde{F}_R$.

9. Lipschitz Bound of Free Boundary Strands.

In this section we shall further elaborate on the finer-scale structure of the free boundary of a solution that falls under the scenario of Proposition 8.3. More specifically, we shall show that if the separation $s$ between two components of the zero phase becomes small enough, it forces the free boundary outside that scale to be the union of four graphs of small Lipschitz constant over a common line.

Theorem 9.1. For any given small $0 < \delta < \delta_0$, there exist $r_0 > 0$, $\epsilon_0 > 0$ such that if $u$ is a classical solution of 1 in $B_1$, satisfying 3, with $0 \in F(u)$ and

$$\text{dist}(0, \{ u = 0 \} \setminus Z) < \epsilon_0 r_0,$$

then for some $p \in B_{r_0/3}$, $B_{r_0/2}(p) \cap F(u)$ consists of two free boundary arcs $F_L$ and $F_R$, the shortest segment between which is centered at $p$, the separation

$$s := \text{dist}(F_L, F_R) < \epsilon_0 r_0,$$

and for some rotation $\rho$ and functions $f, g : \mathbb{R} \to \mathbb{R}$ with $f < g$

$$\{ u = 0 \} \cap (B_{r_0/2}(p) \setminus B_{4s/\epsilon_0}(p)) = p + \rho \{ 4s/\epsilon_0 < x < r_0/2 : f(x) \leq |x| \leq g(x) \}$$

where $\| f \|_{L^\infty} + \| g \|_{L^\infty} \leq \delta$, $\| f' \|_{L^\infty} + \| g' \|_{L^\infty} \leq \delta$. 


Step 1

In view of Remark 8.5, we may assume that we are dealing with a solution of (1) in \( B_{r_0} \), which satisfies:

- \( F(u) \cap B_{r_0/2} \) consists of two arcs \( F_L \) and \( F_R \); for some rotation \( \rho_{r_0} \) the ends of \( F_L \) belong to \( \rho_{r_0}(\alpha_{L,r_0}/2) \) and the ends of \( F_R \) belong to \( \rho_{r_0}(\alpha_{R,r_0}/2) \), where
  \[ \alpha_{L,r} = \{ x \in \partial B_r : x_1 < 0, |x_2| < \delta r \} \quad \text{and} \quad \alpha_{R,r} = \{ x \in \partial B_r : x_1 > 0, |x_2| < \delta r \}. \]

- The minimum distance \( d(F_L, F_R) = s \) is realized for \( 0 \in F_L \) and some point \( q \in F_R \) with \( 0 < s < \epsilon_0 r_0 \).
- For every \( s/\epsilon_0 < r \leq r_0 \),
  \[ |u(p) - y_2|| < \delta r \quad \text{in} \ B_r \quad \text{for some rotation} \ \rho = \rho_r. \]
- For every \( s/\epsilon_0 < r \leq r_0/2, \) \( F(u) \cap \bar{B}_r \) consists of two arcs \( F_L(r) \) and \( F_R(r) \) that attach on \( \rho_r(\alpha_{L,r_0}) \) and \( \rho_r(\alpha_{R,r_0}). \)

Set

\[ r_k := 2^{-k}s/\epsilon_0 \quad k \in \mathbb{N} \]

and let \( k_0 = \lfloor \log_2(\epsilon_0 r_0/\epsilon_0) \rfloor \), so that \( r_{k_0} \approx r_0/2 \). Define \( F_N \) and \( F_S \) to be the two (closed) subarcs of \( F_L(r_{k_0}) \) that \( 0 \) divides \( F_L(r_{k_0}) \) into: with \( F_N \) being the one such that the end point \( \rho_{r_{k_0}}^{-1}(F_N) \cap \alpha_{L,r_{k_0}} \) has the greater \( x_2 \)-coordinate than the end point \( \rho_{r_{k_0}}^{-1}(F_S) \cap \alpha_{L,r_{k_0}} \). Define \( F_N \) and \( F_S \) to be the two subarcs of \( F_R(r_{k_0}) \) that \( q \) divides \( F_R(r_{k_0}) \) into, analogously. Let \( \tau \) be the straight-line close segment connecting 0 to \( q \), and let \( \beta_N \) and \( \beta_S \) be the two circular arcs of \( \partial B_{r_{k_0}} \cap \{ u > 0 \} \) with \( \beta_N \) containing \( \rho_{r_{k_0}}((0, r_{k_0})) \) and \( \beta_S \) containing \( \rho_{r_{k_0}}((0, -r_{k_0})). \) Then \( \tau \) splits \( B_{r_{k_0}}^+ \) into two simply-connected regions — the “top” \( \Omega_N \), bounded by \( \beta_N, F_N^+, \tau, F_N^- \); and the “bottom” \( \Omega_S \), bounded by \( \beta_S, F_S^+, \tau, F_S^- \).

We may choose the coordinate system so that \( \rho_{r_0} \) is the identity. In the following series of arguments we shall adopt complex notation: denoting the point \( (x_1, x_2) \in \mathbb{R}^2 \) by the complex \( z = x_1 + ix_2 \in \mathbb{C} \).

Let \( z_k \in \mathbb{C} \) be the unique point of intersection between \( \partial B_{r_k} \) and \( F_N^+ \), \( k = 1, 2, \ldots, k_0 \). The region \( \Omega_N \) is simply connected with piece-wise smooth boundary, so we may define the harmonic conjugate \( v : \Omega_N \rightarrow \mathbb{R} \) of \( u \), such that \( v \) is continuous up to the boundary \( \partial \Omega_N \) and has the normalization

\[ v(z_2) = -|z_2|. \]

Now define the holomorphic map \( U : \Omega_N \rightarrow \mathbb{C} \) by

\[ U := iu - v. \]

**Lemma 9.2.** The map \( U \) constructed above is injective on \( \Omega_N \setminus B_{r_2} \) and its image

\[ U(\Omega_N \setminus B_{r_2}) \supseteq \{ \xi \in \mathbb{C} : \operatorname{Im}(\xi) \geq 0, r_2(1 + C\delta) \leq |\xi| \leq r_0(1 - C\delta) \} \]

for some numerical constant \( C \).

**Proof.** First, let us note that for \( k = 2, \ldots, k_0 - 1 \), the free boundary in each dyadic annulus \( F(u) \cap B_{r_{k+1}} \setminus B_{r_k} \) consists of four graphs of Lipschitz norm at most \( c' \delta \) for some numerical constant \( c' > 0 \). This is a direct consequence of Lemma 6.7 applied to \( u \) in \( B_{r_k}(\pm p_k) \), where \( p_k = \rho_{3r_k}((3r_k/2, 0)) \), since the zero phase of \( u \) in \( B_{3r_k} \supseteq B_{r_k}(\pm p_k) \) is contained in a \( (3 \delta r_k) \)-strip that disconnects \( B_{r_k}(\pm p_k)^+ \) into two components. As a result, if we represent the rotation \( \rho_{r_k} \) as a complex phase \( e^{i\theta_k} \), we must have

\[ |e^{i\theta_{k+1}} - e^{i\theta_k}| \leq c\delta, \]

for the Lipschitz graph pieces \( F(u) \cap B_{r_{k+1}} \setminus B_{r_k} \) to be appropriately aligned in successive dyadic annuli.

We shall carry out the proof of the lemma in a couple of steps.

**Step 1.** Define \( A_k := \Omega_N \cap B_{r_{k+1}} \setminus B_{r_k} \). We shall show that

\[ |U(e^{i\theta_k} \zeta) - \zeta| \leq C\delta |\zeta| \quad \text{for} \quad \zeta \in \hat{A}_k := e^{-i\theta_k} A_k \quad k = 2, 3, \ldots, k_0 - 1. \]

Define \( \hat{U}(\zeta) := U(e^{i\theta_k} \zeta) \) and let \( \hat{u} := \operatorname{Im}(\hat{U}) \). First, claim that

\[ |\hat{U}'(\zeta) - 1| \leq c\delta \quad \text{for} \quad \zeta \in \hat{A}_k. \]
The Cauchy-Riemann equations say
\[ \hat{U}'(\zeta) = i\partial_{y_1} \hat{u} + \partial_{y_2} \hat{u}, \quad \zeta = y_1 + iy_2, \]
so it suffices to show that
\[ \nabla_y \hat{u} = e_2 + O(\delta) \quad \text{in} \quad \hat{A}_k, \]
where \( e_2 \) is the unit vector in the direction of \( y_2 \). This is a straightforward corollary of
\[ |\hat{u} - y_2^1| < 3\delta r_k \quad \text{in} \quad e^{-i\theta_k}(\Omega_N \cap (B_{3r_k} \setminus B_{r_k/2})) \supseteq \hat{A}_k, \]
and the fact that \( F(\hat{u}) \cap (B_{3r_k} \setminus B_{r_k/2}) \) consists of two graphs of Lipschitz norm at most \( c'\delta \).

Going back to the complex coordinate \( z = e^{i\theta_k} \zeta \), we see that \( \overline{21} \) becomes
\[ |U'(z) - e^{-i\theta_k}| = |U'(z)e^{i\theta_k} - 1| \leq c\delta \quad \text{for} \quad z \in A_k. \tag{23} \]
Let \( z_k \) be defined as the unique intersection point between \( \partial B_{r_k} \) and \( F^N_k \) for \( k = 1, 2, \ldots, k_0 \), as above. Since there is a piece-wise smooth curve \( \gamma(z_k, z) \subseteq A_k \) of length \( O(r_k) \) connecting \( z_k \) to any other point \( z \in A_k \), integrating \( U'(s) - e^{-i\theta_k} \) along \( \gamma(z_k, z) \), we obtain using \( \overline{23} \)
\[ |U(z) - e^{-i\theta_k}z - (U(z_k) - e^{-i\theta_k}z_k)| \leq C'\delta r_k \quad z \in A_k. \tag{24} \]
In order to establish \( \overline{21} \), it suffices therefore to show that for some large enough numerical constant \( \tilde{c} \)
\[ |U(z_k) - e^{-i\theta_k}z_k| \leq \tilde{c}r_k, \quad k = 2, 3, \ldots, k_0 - 1. \]
We shall use induction. Without of loss of generality, the complex coordinate \( z \) is chosen so that \( \theta_2 = 0 \). Then, since \( z_2 \in \alpha_{R, \delta r_2} \),
\[ |U(z_2) - e^{-i\theta_2}z_2| = |v(z_2) - z_2| = ||z_2| - z_2|| \leq 2\delta r_2. \]
Assume the statement is true for \( k \). Applying \( \overline{24} \) for \( z = z_{k+1} \in \overline{A}_k \)
\[ |U(z_{k+1}) - e^{-i\theta_k}z_{k+1}| \leq C'\delta r_k + |U(z_k) - e^{-i\theta_k}z_k| \leq (C' + \tilde{c})\delta r_k. \]
Taking into account \( \overline{20} \), we see that
\[ |U(z_{k+1}) - e^{-i\theta_k}z_{k+1}| \leq (C' + \tilde{c})\delta r_k + |e^{-i\theta_k} - e^{-i\theta_k}| |z_{k+1}| \leq (C'/2 + \tilde{c}/2 + c)\delta r_{k+1}. \]
and the induction step is complete once we pick \( \tilde{c} = \max\{2, C'/2 + 2c\} \).

**Step 2.** We are now ready to show that \( U \) is injective on \( \overline{\Omega}_N \). Let \( w_1, w_2 \in \overline{\Omega}_N \setminus \partial \Omega \) be such that \( U(w_1) = U(w_2) \); without loss of generality \( \|w_1\| \leq |w_2| \). Because of \( \overline{21} \), we have
\[ |U(w_1)| \geq (1 - C\delta)|w_2|, \quad |U(w_2)| \leq (1 + C\delta)|w_1|. \]
Hence,
\[ 1 \leq \frac{|w_2|}{|w_1|} \leq \frac{|U(w_2)|}{|U(w_1)|} \leq \frac{(1 - C\delta)}{(1 + C\delta)} = \frac{1 + C\delta}{1 - C\delta} < 2, \]
so it has to be the case that both \( w_1, w_2 \) belong to \( A_{k-1} \cup A_k \) for some \( k \). Because of \( \overline{23} \) and \( \overline{20} \), we have
\[ |U'(z) - e^{-i\theta_k}| \leq c'\delta \quad \text{for} \quad z \in A_{k-1} \cup A_k. \]
Let \( \gamma(w_1, w_2) \) be a piece-wise smooth curve in \( D_k := A_{k-1} \cup A_k \) connecting \( w_1 \) to \( w_2 \). It is not hard to see that, because \( \partial D_k \) can be locally represented as a graph of a Lipschitz function with Lipschitz norm bounded by some universal constant \( L \), \( \gamma(w_1, w_2) \) can be taken such that
\[ \mathcal{H}^1(\gamma(w_1, w_2)) \leq \sqrt{1 + L^2}|w_1 - w_2|. \]
Then
\[ 0 = U(w_2) - U(w_1) = \int_{\gamma(w_1, w_2)} U'(z)dz = e^{-i\theta_k}(w_2 - w_1) + \int_{\gamma(w_1, w_2)} (U'(z) - e^{-i\theta_k})dz, \]
so that
\[ |w_1 - w_2| = \left| \int_{\gamma(w_1, w_2)} (U'(z) - e^{-i\theta_k})dz \right| \leq c'\delta \mathcal{H}^1(\gamma(w_1, w_2)) \leq c'\sqrt{1 + L^2}\delta|w_1 - w_2|, \]
which implies that \( w_1 = w_2 \) when \( \delta \) is small enough.

**Step 3.** Finally, we see that \( \overline{19} \) follows from \( \overline{21} \) and the fact that \( \text{Im}(U) = u \geq 0 \) with \( \text{Im}(U)(z) = 0 \) precisely when \( z \in F(u) \cap \overline{\Omega}_N \). \( \square \)
Lemma 9.3. The two curves \( F(u) \cap \Omega_N \setminus B_{r_2} \) are graphs over the line \( \rho_{r_0} \{ y_2 = 0 \} \) with Lipschitz norm at most \( c \delta \) for some numerical constant \( c \).

Proof. From Lemma 9.2 we know that the inverse of \( U \) is well defined on the annulus
\[
A := \{ \xi \in \mathbb{C} : \text{Im}(\xi) \geq 0, r_2(1 + C\delta) \leq |\xi| \leq r_{k_0}(1 - C\delta) \}.
\]
Then \( U^{-1} \circ \exp \) maps the strip
\[
S = \{ z \in \mathbb{C} : 0 \leq \text{Im}(z) \leq \pi, \log (r_2(1 + C\delta)) \leq \text{Re}(z) \leq \log (r_{k_0}(1 - C\delta)) \}
\]
conformally onto its image in \( \Omega_N \setminus \overline{B_{r_2}} \) with \( S \cap \{ \text{Im}(z) = 0 \} \) parameterizing a subarc of the “right” strand \( F_R^N \), and \( S \cap \{ \text{Im}(z) = \pi \} \) parameterizing a subarc of the “left” strand \( F_L^N \), under \( U^{-1} \circ \exp \) (see the discussion at the beginning of the section for definitions). Since \( U' \neq 0 \) on \( \Omega_N \setminus B_{r_2} \) and \( \Omega_N \setminus B_{r_2} \) is simply-connected, one may define a branch of its logarithm \( \log U' \). Finally, define the holomorphic function \( F : S \rightarrow \mathbb{C} \) via:
\[
F := \log U' \circ U^{-1} \circ \exp,
\]
and let
\[
f = \text{Re}(F) \quad \text{and} \quad g = \text{Im}(F).
\]
Since for \( \zeta_1, \zeta_2 \in F(u) \), \( |\text{Im} (\log U' (\zeta_2) - \log U'(\zeta_1))| \) measures the angle of turning of \( \nabla u \) along \( F (u) \) from \( \zeta_1 \) to \( \zeta_2 \), we are going to be interested in estimating the oscillations
\[
\omega_{g,L} := \text{osc} \{ g(z) : z \in \tilde{S} \cap \{ \text{Im}(z) = \pi \} \} \quad \omega_{g,R} := \text{osc} \{ g(z) : z \in \tilde{S} \cap \{ \text{Im}(z) = 0 \} \}
\]
where
\[
\tilde{S} = \{ z \in \mathbb{C} : 0 \leq \text{Im}(z) \leq \pi, \log r_2 + c_0 \leq \text{Re}(z) \leq \log r_{k_0} - c_0 \} \subseteq S \quad c_0 = \log 2.
\]
We would like to show that both \( \omega_{g,L} \) and \( \omega_{g,R} = O(\delta) \), as this would imply that the amount of turning of \( \nabla u \) along \( F_R^N (F_R^N) \), from \( \partial B_{r_2} \) to \( \partial B_{r_{k_0}/2} \), is \( O(\delta) \), which, in turn, would be enough to conclude that \( F_R^N \cap B_{r_{k_0}} \setminus B_{r_2} \) and \( F_R^N \cap B_{r_{k_0}} \setminus B_{r_2} \) are in fact graphs of Lipschitz constant \( O(\delta) \) (as we already know that \( F_R^N \cap B_{r_{k_0}} \setminus B_{r_{k_0}/2} \) and \( F_R^N \cap B_{r_{k_0}} \setminus B_{r_{k_0}/2} \) are graphs of Lipschitz constant \( O(\delta) \)). To that goal we would like to obtain estimates on \( |\nabla g| \) in \( \partial S \), which by the Cauchy–Riemann equations satisfies
\[
|\nabla g| = |\nabla f| \quad \text{in} \ S.
\]
For convenience, define the following coordinates on \( S \)
\[
t = \text{Re}(z) - (\log (r_2(1 + C\delta)) + A) \quad \theta = \text{Im}(z) - \pi/2,
\]
where
\[
2A = \log (r_{k_0}(1 - C\delta)) - \log (r_2(1 + C\delta)) = \log \left( \frac{1 - C\delta}{1 + C\delta} \right) + (k_0 - 2) \log 2 = (k_0 - 2) \log 2 + O(\delta)
\]
by translating the coordinates \( (\text{Re}(z), \text{Im}(z)) \) appropriately, so that \( S \) is parameterized by
\[
S = \{ |t| \leq A, |\theta| \leq \pi/2 \}.
\]
Note that since \( |U'| = |\nabla u| \) on \( F(u) \cap (\Omega_N \setminus B_{r_2}) \), we have
\[
|f(t, \pm \pi/2)| = |\log F| = \log 1 = 0 \quad \text{for} \quad |t| \leq A.
\]
Also, by the estimate [22] of Lemma 9.2 we have
\[
|f(\pm A, \theta)| \leq c\delta \quad \text{for} \quad |\theta| \leq \pi/2.
\]
Applying Schwarz reflection across \( \theta = -\pi/2 \) and \( \theta = \pi/2 \), we can extend \( f \) to a harmonic function on \( \tilde{S} := \{ |t| \leq A, |\theta| \leq 3\pi/2 \} \). By the maximum principle \( |f| \leq \bar{c} \delta \) in \( \tilde{S} \). Denote \( \bar{A} := A - c_0/2 \). Interior estimates for \( f \) then yield
\[
|\nabla f(\pm \bar{A}, \theta)| \leq \bar{c}\| f \|_{L^\infty(\tilde{S})} \leq \bar{C}\delta \quad \text{for} \quad |\theta| \leq \pi/2,
\]
which we can integrate to get
\[
|f(\pm \bar{A}, \theta)| \leq \bar{C}\delta \cos \theta \quad \text{for} \quad |\theta| \leq \pi/2.
\]
Using multiples of \( \cosh t \cos \theta \) as upper and lower barriers for \( f \), we have the bound
\[
-(c\delta / \cosh \bar{A}) \cosh t \cos \theta \leq f \leq (c\delta / \cosh \bar{A}) \cosh t \cos \theta \quad \text{in} \ \tilde{S}.
\]
so that, by the Hopf Lemma, we can conclude
\[
|\nabla f(t, \pm \pi/2)| = |\partial_t f(t, \pm \pi/2)| \leq (c\delta / \cosh \tilde{A}) \cosh t \quad \text{for} \quad |t| \leq \tilde{A}.
\]  
This in turn implies the desired
\[
\omega_{g,L} \leq \int_{-\tilde{A}}^{\tilde{A}} |\nabla g(t, \pi/2)| \, dt \leq 2c\delta, \quad \omega_{g,R} \leq \int_{-\tilde{A}}^{\tilde{A}} |\nabla g(t, -\pi/2)| \, dt \leq 2c\delta.
\]
\[\square\]

10. Curvature bounds.

Let us describe the family of hairpin solutions explicitly. Define
\[
\varphi(\zeta) = i(\zeta + \sinh \zeta).
\]
Then \( \varphi \) maps the strip \( S = \{ |\text{Im}\zeta| < \pi/2 \} \) conformally onto the domain
\[
\Omega_1 := \{ z \in \mathbb{C} : |\text{Re}z| < \pi/2 + \cosh(\text{Im}z) \}
\]
which supports the positive phase of the hairpin solution of \( \Omega \)
\[
H(z) = \begin{cases} 
\text{Re}(V(z)) & \text{when } z \in \Omega_1 \\
0 & \text{otherwise.}
\end{cases}
\]
\[\text{(27)}\]
where \( V(z) := \cosh(\varphi^{-1}(z)) \). The dilates
\[
H_a(z) = aH(z/a)
\]
complete the family of hairpin solution (up to rigid motions). Denote by \( \Omega_a = a\Omega_1 \).

We note a couple of geometric features of these solutions.

- The zero phase \( \{ H_a = 0 \} \) consists of two connected components separated by distance \( s = a(2+\pi) \).
- \( H_a |_{\Omega} \) has a unique critical point (a non-degenerate saddle) and it is situated at the origin. Indeed, to verify this, we have to simply check this is the obviously the case for
\[
H(\varphi(\zeta)) = \text{Re} \cosh(\zeta) = \cosh(y_1) \cos(y_2) \quad \text{when} \quad \zeta = y_1 + iy_2 \in S.
\]
The value of \( H_a \) at the saddle is precisely \( H_a(0) = a \).

- The segments \( \tau_{a,L} := [-s/2, 0] \subseteq \mathbb{C} \) and \( \tau_{a,R} := [0, s/2] \subseteq \mathbb{C} \) are the steepest descent paths from 0 to each of the two components of \( \{ H_a = 0 \} \), respectively. We shall denote \( \tau_a := \tau_{a,L} \cup \tau_{a,R} \).

The following information about the gradient \( \nabla H \) will also be useful.

Lemma 10.1. For some numerical constant \( c_0 > 0 \), the gradient \( \nabla H \) satisfies
\[
|\nabla H(x)| \geq \min(1/2, c_0|x|) \quad \text{when} \quad x \in \Omega_1.
\]

Proof. We have
\[
|\nabla H|(\varphi(\zeta)) = \left| \frac{i \sinh \zeta}{\varphi'(\zeta)} \right| = \sqrt{\frac{\sinh^2 y_1 + \sinh^2 y_2}{(1 + \cosh y_1 \cos y_2)^2 + \sinh^2 y_1 \sin^2 y_2}} = \frac{\sqrt{\sinh^2 y_1 + \sinh^2 y_2}}{\cosh y_1 + \cos y_2} \quad \zeta = y_1 + iy_2 \in S.
\]
Thus,
\[
|\nabla H|(\varphi(\zeta)) \geq \frac{|\sinh y_1|}{1 + \cosh y_1} = |\tanh(y_1/2)| \geq 1/2 \quad \text{when} \quad |y_1| \geq 1.2
\]
We'll be done once we show that
\[
|\nabla H|(\varphi(\zeta)) \geq c_0|\varphi(\zeta)| \quad \text{when} \quad |\text{Re}\zeta| = |y_1| < 1.2.
\]
Since for some numerical constants \( 0 < c_1 < c_2 \)
\[
c_1|\zeta| \leq |\sinh \zeta| \leq c_2|\zeta| \quad \text{when} \quad |\text{Re}\zeta| = |y_1| < 1.2
\]
we have for \( |\text{Re}\zeta| < 1.2 \),
\[
|\nabla H|(\varphi(\zeta)) \geq \frac{|\sinh \zeta|}{\cosh y_1 + \cos y_2} \geq \frac{c_1|\zeta|}{\cosh(1.2) + 1} \geq \tilde{c}_1|\zeta|.
\]
Noting that \( |\varphi(\zeta)| \leq |\zeta| + |\sinh z| \leq (1 + c_2)|\zeta| \) when \( |\text{Re}\zeta| < 1.2 \), we complete the proof of the lemma.  \( \square \)
Remark 10.2. Finally, we would like to make the following remark regarding the mapping properties of $V_a(z) := aV(z/a)$ on $\Omega_a$. Claim that $V_a$ maps both $\Omega_a^+ := \Omega_a \cap \{x_2 > 0\}$ and $\Omega_a^- := \Omega_a \cap \{x_2 < 0\}$ conformally onto 
\[
\mathbb{H}_a := \{\xi : \text{Re}(\xi) > 0\} \setminus (0, a).
\]
Indeed, let $S_\pm = \{\zeta \in \mathbb{C} : \pm \text{Re} \zeta > 0, |\text{Im} \zeta| < \pi/2\}$. Then $(a\varphi)$ is a conformal map from $S_\pm$ onto $\Omega_a^\pm$ and $V_a(a\varphi(\zeta)) = a \cosh(\zeta) = a \cosh y_1 \cos y_2 + ia \sinh y_1 \sin y_2$ when $\zeta = y_1 + iy_2 \in S_\pm$.

Write 
\[
V_a(a\varphi(\zeta)) = r(\zeta)e^{i\theta(\zeta)}
\]
where
\[
r(\zeta)^2/a^2 = |V_a(\varphi(\zeta))|^2 = \sinh^2 y_1 + \cos^2 y_2 \quad \tan \theta(\zeta) = \tanh y_1 \tan y_2.
\]
If $\theta_0$ is any angle in $(-\pi/2, 0) \cap (0, \pi/2)$ and $c := \tan \theta_0$, then $\tan \theta(\zeta) = c$ whenever $y_2 = c \coth y_1$, so that for these values of $\zeta$,
\[
r(\zeta)^2/a^2 = \sinh^2 y_1 + 1/(\tan^2 y_2 + 1) = \sinh^2 y_1 + 1/(1 + c^2 \coth^2(y_1)).
\]
We see that $r(\zeta) \to 0$ as $y_1 \to 0$ and $r(\zeta) \to \infty$ as $y_1 \to \pm \infty$, so $V_a(a\varphi)|_{S_\pm}$ is onto $\mathbb{H}_\infty$. When $\theta_0 = 0$, i.e. $\text{Im}(V_a(a\varphi)) = 0$, so it must be that $y_2 = 0$ and thus, $r(\zeta) = \cosh y_1$ which ranges in $(a, \infty)$. Hence, $V_a(a\varphi)|_{S_\pm}$ is surjective onto $\mathbb{H}_a$. To show that $V_a(a\varphi)|_{S^+}$ is injective, assume that for some $y_1 + iy_2, \tilde{y}_1 + i\tilde{y}_2 \in S^+$ with $y_1 \leq \tilde{y}_1$,
\[
cosh y_1 \cos y_2 = \cosh \tilde{y}_1 \cos \tilde{y}_2 \quad \text{and} \quad \sinh y_1 \sin y_2 = \sinh \tilde{y}_1 \sin \tilde{y}_2.
\]
Divide the second equation by the first to get 
\[
\tanh y_1 \tan y_2 = \tanh \tilde{y}_1 \tan \tilde{y}_2.
\]
We see that if $y_1 = \tilde{y}_1$ we must have $y_2 = \tilde{y}_2$ too. Assume $y_1 < \tilde{y}_1$; then either $y_2 = \tilde{y}_2 = 0$, which then implies $\cosh y_1 = \cosh \tilde{y}_1$ contradicting the assumption $0 < y_1 < \tilde{y}_1$, or $|\tan y_2| > |\tan \tilde{y}_2|$ which implies $\cos y_2 < \cos \tilde{y}_2$. But in the latter case,
\[
cosh y_1 \cos y_2 < \cosh \tilde{y}_1 \cos \tilde{y}_2,
\]
so we get a contradiction again. Similarly, we show the injectivity of $V_a(a\varphi)|_{S^-}$.

Having amassed enough information about the model hairpin solutions let us explore how well they approximate classical solutions of (1) whose zero phase has two connected components that are sufficiently close to each other.

Proposition 10.3. Let $u$ be a classical solution of (1) in $B_1$ that satisfies (3). Assume that $\{u = 0\}$ consists of two connected components $Z_L$ and $Z_R$ and that 0 is the midpoint of a shortest segment between $Z_L$ and $Z_R$. For any given $\delta_1 > 0$ and every $M > 0$, there exists $s_1 > 0$ such that if
\[
s := d(Z_L, Z_R) \leq s_1,
\]
then after a rotation
\[
|u(ax)/a - H(x)| \leq \delta_1 \quad \text{for all} \quad |x| \leq M,
\]
where $a = s/(2 + \pi)$.

Proof. Fix $\delta_1$ and $M$ and assume no such $s_1$ exists that makes the proposition valid. Then for some sequence of $s_k \to 0$, there is a sequence of counterexamples $u_k$ with the separation between the two components of $\{u = 0\}$ being $s_k$. Set $a_k = s_k/(2 + \pi)$. We can then define the rescales
\[
\tilde{u}_k(x) := u(a_kx)/a_k \quad \text{for} \quad x \in B_1/s_k
\]
so that the separation between the two components of the zero phase of $\tilde{u}_k$ is precisely $(2 + \pi)$ and 0 is at the midpoint of a shortest segment connecting them. A subsequence $u_{kj}$ converges uniformly on $\overline{B_M}$ to a global solution $\tilde{u}$ and since
\[
d_H((\overline{B_M})^+(\tilde{u}_{kj}), (\overline{B_M})^+(\tilde{u})) \to 0 \quad \text{as} \quad j \to \infty,
\]
it has to be the case that $\tilde{u}$ is a hairpin solution, with separation between the two components of $\{\tilde{u} = 0\}$ precisely $(2 + \pi)$ and 0 at the midpoint of the shortest segment. Thus, in a rotated coordinate system
\[
\tilde{u} = H,
\]
so we have for all \(j\) large enough
\[
|u_{k_j}(ak_jx)/ak_j - H(x)| \leq \delta_1 \quad \text{in} \quad \overline{B_M}.
\]
This contradicts the assumption that the \(u_{k_j}\) are counterexamples.

The next corollary is a direct consequence of interior estimates applied to the proposition above.

**Corollary 10.4.** Let \(u\) be as in Proposition [10.3] and let \(H\) be the hairpin solution defined above. For every \(M > 0\), any compact domains \(K, K'\) such that \(K \Subset K' \Subset (\overline{B_M})^+(H)\) and any \(\delta_1 > 0\) such that \(N_{2\delta_1}(K') \subseteq (\overline{B_M})^+(H)\), there exists \(s_1 > 0\), such that if the separation between the two components \(Z_L\) and \(Z_R\) of \(\{u = 0\}\) satisfies
\[
s := d(Z_L, Z_R) \leq s_1,
\]
then for \(a = s/(2 + \pi)\), in some rotated coordinate system, the rescale \(u_a := u(ax)/a\) satisfies
\[
\|u_a - H\|_{C^2(K')} \leq C_{K,K'}\delta_1
\]
for some constant \(C_{K,K'}\), dependent on \(K\) and \(K'\). Furthermore, if \(\delta_1 = \delta_1(H)\) is small enough, \(u\) has a unique critical point \(x_0\) in \(B^+_a(M/2)(u)\) which is a non-degenerate saddle point with \(|x_0| = O(\delta_1s)\), and the steepest descent paths \(\beta_L, \beta_R\) for \(u\) from \(x_0\) to \(Z_L\) and \(Z_R\), respectively, are contained in an \(O(\delta_1s)\)-neighborhood of \(\tau_a\) (defined above).

**Proof.** Let \(s_1\) be such that according to (28)
\[
|u_a - H(x)| \leq c\delta_1 \quad \text{for all} \quad |x| \leq M,
\]
where we pick \(c\) such that \(H(x) \geq c d(x, F(H))\) (such a \(c\) exists because of the non-degeneracy of the hairpin solution). Then for all \(x \in K'\)
\[
u_a(x) \geq H(x) - c\delta_1 \geq c d(x, F(H)) - c\delta_1 > 2c\delta_1 - c\delta_1 > 0.
\]
Hence \(v = u_a - H\) is harmonic in \(K'\) and we get (29) by standard interior estimates.

Let us use this to show that \(u\) has a unique critical point in \(B^+_a(M/2)(u)\) if \(\delta_1\) is small enough. Fix \(\delta_0 > 0\) small and find a scale \(r_0 = r_0(\delta_0, H)\) such that for every \(p \in F(H) \cap \overline{B_{M/2}}\)
\[
d_H(F(H) \cap B_{r_0}(p), L(p) \cap B_{r_0}(p)) < \delta_0r_0,
\]
where \(L(p)\) denote the straight line tangent to \(F(H)\) at \(p\). Now, for all small enough \(\delta_1 < \delta_0r_0\)
\[
d_H(F(u_a) \cap B_{r_0}(p), L(p) \cap B_{r_0}(p)) < c'\delta_0r_0.
\]
Hence, \(F(u_a)\) is \(c'\delta_0\)-flat in \(B_{r_0}(p)\) and it must be that for a small enough \(\delta_0\),
\[
|\nabla u_a - \nabla H(p)| \leq C\delta_0 \quad \text{in} \quad B_{r_0/2}(p)^+(u_a) \quad \text{for any} \quad p \in F(H) \cap \overline{B_{M/2}}\]
by the classical Alt-Caffarelli theory [ACS1]. Thus,
\[
|\nabla u_a| \geq 1 - C\delta_0 \quad \text{in} \quad B_{r_0/2}(p)^+(u_a)
\]
and so it suffices to show that \(u_a\) has a unique critical point in
\[
K := \{x \in B^+_a(H) : d(x, \partial \Omega_1) \geq r_0/4\}.
\]
Set \(K' := \{x \in B^+_a(2M/3)(H) : d(x, \partial \Omega_1) \geq r_0/8\}\). We would like to show that
\[
0 = \nabla u_a = \nabla H + \nabla v
\]
has a unique solution \(x_{0,a} \in K\). Since the Jacobian of \(\nabla H = D^2 H\) is invertible at 0, the Inverse Function Theorem implies that for some \(c_1 > 0\), \(\nabla H\) maps \(B_{c_1}\) diffeomorphically onto a neighborhood \(O\) of 0. As \(|\nabla v| \leq C_{K,K'}\delta_1\) in \(K\), we can choose \(\delta_1\) small enough such that \(\nabla v \in O\), whence
\[
\nabla H(x) = -\nabla v(x)
\]
has a unique solution \(x = x_{0,a} \in B_{c_1}\). Applying Lemma [10.1], we obtain
\[
|x_{0,a}| \leq c^{-1}_0|\nabla H(x_{0,a})| = c^{-1}_0|\nabla v(x_{0,a})| \leq c^{-1}_0C_{K,K'}\delta_1.
\]
Furthermore, the equation cannot have another solution in \(K\) if \(\delta_1\) is small enough, because Lemma [10.1] implies that
\[
|\nabla H(x)| \geq \min(1/2, cc_1) > C_{K,K'}\delta_1 \geq |\nabla v(x)| \quad \text{for all} \quad x \in K \setminus B_{c_1}.
\]
Thus, whenever \( \delta_1 \) is small enough, \( u_a \) has a unique critical point \( x_{0,a} \) in \( K \) and since
\[
|D^2u_a(x_{0,a}) - D^2H(0)| \leq |D^2u_a(x_{0,a}) - D^2H(x_{0,a})| + |D^2H(x_{0,a}) - D^2H(0)|
= O(\delta_1) + O(|x_{0,a}|) = O(\delta_1),
\]
x_{0,a} is a non-degenerate saddle point.

The \( O(\delta_1) \) proximity between the steepest descent paths for \( u_a \) and \( H \) from \( x_{0,a} \) and 0, respectively, to their zero sets, follows from the \( O(\delta_1) \) bound for \( \nabla(u_a - H) \) in \( K \) and (30).

Let \( \delta_0 > 0 \) be small constant from Remark 8.1. Let us present the set-up that we shall be working in for the rest of the section. The object of interest is

1. A classical solution \( u \) of (3) such that \( \{u = 0\} \) consists of two connected components \( Z_L \) and \( Z_R \) with 0 being at the midpoint of the shortest segment between the two.
2. The free boundary \( F(u) \) consists of two arcs \( F_L := F(u) \cap \partial Z_L \) and \( F_R = F(u) \cap \partial Z_R \).
3. We assume that \( \delta_1 \leq \delta < \delta_0 \), \( \beta, \rho \) and \( s \) are as in Proposition 10.3 and Corollary 10.4. i.e. the fact that \( d(Z_L,Z_R) = s < s_1 \) implies
\[
|u - H| \leq \delta s \quad \text{in} \quad B_{aM} \quad \text{where} \quad a = (2 + \pi)s
\]

and \( u \) has a unique critical point \( x_0 \) in \( B_{aM/2}^+(u) \). We denote by \( \beta_L \) be the steepest descent path for \( u \) that connects \( x_0 \) to some \( p \in F_L \) and by \( \beta_R \) be the steepest descent path from \( x_0 \) to some \( q \in F_R \). Then \( \beta := \beta_L \cup \beta_R \) is a smooth arc connecting \( F_L \) to \( F_R \) and \( \beta \equiv N_{a\delta_1}(\tau) \). Without loss of generality, we may assume that our coordinate system is chosen in such a way that
\[
\nabla u(p) = e_1.
\]

4. Furthermore, for some rotation \( \rho \)
\[
|u(\rho x) - |x_2|| \leq \delta \quad \text{in all of} \quad B_1.
\]

Let \( \partial B_{1/2} \) intersect \( F_L \) at the two points \( p_N \) and \( p_S \) (subscripts \( N \) and \( S \) are determined by \( x_2(\rho^{-1}(p_N)) > x_2(\rho^{-1}(p_S)) \)) and similarly \( \partial B_{1/2} \) intersects \( F_R \) at the two points \( q_N \) and \( q_S \). Define \( \Omega_N \subseteq B_{1/2}^+(u) \) to be the domain bounded by the subarc of \( F_L \) from \( p_N \) to \( p \), the arc \( \beta \), the subarc of \( F_R \) from \( q \) to \( q_N \) and by the circular arc of \( \partial B_{1/2} \) with ends \( p_N \) and \( q_N \), which contains \( \rho(0,1/2) \). Analogously, define \( \Omega_S \) to be the domain bounded by subarc of \( F_L \) from \( p_S \) to \( p \), the arc \( \beta \), the subarc of \( F_R \) from \( q \) to \( q_S \) and by the circular arc of \( \partial B_{1/2} \) with ends \( p_S \) and \( q_S \), which contains \( \rho(0,-1/2) \). Then
\[
B_{1/2}^+(u) = \Omega_N \cup \beta \cup \Omega_S.
\]

Let \( v \) be the harmonic conjugate of \( u \) in the simply-connected \( B_1^+(u) \) where we choose the normalization
\[
v(x_0) = 0.
\]

Note that this implies \( v = 0 \) on all of \( \beta \), as \( \nabla v \) is a rotation by \( \pi/2 \) of \( \nabla u \) which itself is tangent to \( \beta \). Furthermore \( v \) is increasing (decreasing) at unit speed along \( F_L \cap \Omega_N \) (\( F_L \cap \Omega_S \)) and decreasing (increasing) at unit speed along \( F_R \cap \Omega_N \) (\( F_R \cap \Omega_S \)) as we move towards \( \partial B_{1/2} \).

Define the holomorphic map \( U : B_1^+(u) \to \mathbb{C} \) via
\[
U = u + iv.
\]

The next lemma confirms that the mapping properties of \( U \) are similar to those of \( V_a \) (defined in Remark 10.2), which allows us to construct an injective holomorphic map from \( B_{1/2}^+(u) \) to \( \Omega_{a_0} \) for some \( a_0 > 0 \).

**Lemma 10.5.** Provided \( \delta \) and \( \delta_1 \) small enough, \( U \) is injective on each of \( \Omega_N \) and \( \Omega_S \) and maps each of \( \beta_L \) and \( \beta_R \) injectively onto \( i[0,a_0] \), where \( a_0 := u(x_0) = a(1 + O(\delta_1)) \). Then the map \( \tilde{\psi} : B_{1/2}^+(u) \setminus \beta \to \Omega_{a_0} \) given by
\[
\tilde{\psi}(z) := \begin{cases} (V_{a_0} \mid \Omega_{a_0}^+) \circ U(z) & \text{when} \quad z \in \Omega_N \\ (V_{a_0} \mid \Omega_{a_0}^-) \circ U(z) & \text{when} \quad z \in \Omega_S \end{cases}
\]
is injective and in fact extends continuously to $\beta$. The extension $\psi : B_{1/2}^+(u) \to \Omega_{a_0}$ defines, therefore, an injective holomorphic map whose image contains

$$\psi(B_{1/2}^+(u)) \supseteq \Omega_{a_0} \cap B_{1/4}$$

Proof. Let us first show that $U$ is injective in $\Omega_N$. Since $U$ maps each of $\beta_L$ and $\beta_R$ injectively onto $[0,a_0]$ and since near $x_0$

$$U(z) = a_0 + c(z - x_0)^2 + O(|z - x_0|^3)$$

by the smoothness of $U$, for every small enough $\epsilon > 0$ we can find an arc $\beta_\epsilon \subset \Omega_N \cap \mathcal{N}(\beta)$ connecting $p_\epsilon \in F_L$ to $q_\epsilon \in F_R$, such that $U$ maps it bijectively onto an arc $\gamma_\epsilon \subset \mathbb{H}_a$ with ends $U(p_\epsilon)$ and $U(q_\epsilon)$, where

$$\text{Re}(U(p_\epsilon)) = \text{Re}(U(q_\epsilon)) = 0 \quad \text{and} \quad \text{Im}(U(p_\epsilon)) = v(p_\epsilon) > 0 > v(q_\epsilon) = \text{Im}(U(q_\epsilon))$$

Let $\Omega_{N,\epsilon}$ be the domain bounded by the subarc of $F_L$ with ends $p_N$ and $p_\epsilon$, the subarc of $F_R$ with ends $q_\epsilon$ and $q_N$, and the corresponding circular arc $\overline{p_Nq_N}$ of $\partial B_{1/2}$. Claim that $U$ is injective on the closed Jordan arc $\partial \Omega_{N,\epsilon}$. We can easily see that $U$ maps $(F(u) \cap \partial \Omega_{N,\epsilon}) \cup \beta_\epsilon$ injectively onto

$$\Gamma_\epsilon := \gamma_\epsilon \cup \{y_1 = 0, y_2 \in [-L,R] \} \setminus \{y_1 = 0, y_2 \in (v(q_\epsilon), v(p_\epsilon))\}$$

where

$$L := \mathcal{H}^1(\partial \Omega_N \cap F_L) \geq 2/5 \quad \text{and} \quad R := \mathcal{H}^1(\partial \Omega_N \cap F_R) \geq 2/5.$$ 

It remains to confirm that $U$ is injective on $\overline{p_Nq_N}$ and that $U(\overline{p_Nq_N}) \cap \gamma_\epsilon = U(\overline{p_Nq_N}) \cap \gamma = \emptyset$. Those follow easily from the fact that

$$|U'(z)e^{i\theta} - (-i)| \leq c\delta \quad z \in \overline{p_Nq_N}$$

where $e^{i\theta}$ represents the rotation $\rho$.

Since $U$ is injective on $\partial \Omega_{N,\epsilon}$, $U(\partial \Omega_{N,\epsilon})$ is a closed Jordan arc that divides $\mathbb{C}$ into a bounded domain $D_b$ and an unbounded domain $D_a$. For $\xi \notin U(\partial \Omega_{N,\epsilon})$,

$$Q(\xi_0) := \frac{1}{2\pi i} \int_{\partial \Omega_N} \frac{dz}{U(z) - \xi_0} = \frac{1}{2\pi i} \int_{U(\partial \Omega_{N,\epsilon})} \frac{d\xi}{\xi - \xi_0}$$

equals the winding number of the closed Jordan arc $U(\partial \Omega_{N,\epsilon})$ around $\xi_0$, i.e. $Q(\xi_0) = 1$ when $\xi_0 \in D_b$ and $Q(\xi_0) = 0$ when $\xi \in D_a$. On the other hand, by the Argument Principle, $Q(\xi)$ equals the number of zeros (with multiplicities) of $U(z) = \xi_0$ in $\Omega_{N,\epsilon}$. We can thus conclude that $U$ is injective on $\Omega_{N,\epsilon}$.

Taking a sequence $\epsilon_k \to 0$ we construct a sequence of domains $\Omega_{N,\epsilon_k}$ such that $\Omega_N = \bigcup_k \Omega_{N,\epsilon_k}$ with $U$ injective on each $\Omega_{N,\epsilon_k}$. Therefore, $U$ is injective on all of $\Omega_N$. Analogously, we establish the injectivity of $U$ on $\Omega_S$.

Finally, let’s show that $\psi$ extends continuously to $\beta$. Let $z$ belong to the interior of the arc $\beta_L$, and let $\{z_{N,k}\} \subset \Omega_N$, $\{z_{S,k}\} \subset \Omega_S$ be two sequences such that both

$$z_{N,k} \to z \quad z_{S,k} \to z.$$ 

Denote $\xi_{N,k} := U(z_{N,k})$ and $\xi_{S,k} := U(z_{S,k})$. Then we see that both

$$\xi_{N,k} \to u(z) + i0^+ \quad \text{and} \quad \xi_{S,k} \to u(z) + i0^+$$

with $u(z) \in (0,a_0)$. Then if $\xi_{N,k} = V_{a_0}^{-1}(\xi_{N,k})$ and $\xi_{S,k} = V_{a_0}^{-1}(\xi_{S,k})$, we can easily verify that

$$\xi_{N,k} \to b + i0^+ \quad \text{and} \quad \xi_{N,k} \to b + i0^-$$

with $b \in \tau_{a_0,L}$ being the unique point of $\tau_{a_0,L}$ that $V_{a_0}$ maps to $u(z) \in (0,a_0)$. Hence, $\psi$ can be continuously extended on the interior of $\beta_L$ and similarly, onto the interior of $\beta_R$. Since this extension is bounded in the vicinity of $x_0$, it further extends to a holomorphic function $\psi$ in all of $B_{1/2}^+(u)$ with $\psi(x_0) = 0$. Since $\psi$ maps $\Omega_N$ injectively into $\Omega_{a_0}^+$ and $\Omega_S$ injectively into $\Omega_{a_0}^-$, $\beta_L^0$ injectively into $(\tau_{a_0,L})^0$ and $\beta_R^0$ injectively into $(\tau_{a_0,R})^0$, we conclude that $\psi$ is injective on all of $B_{1/2}^+(u)$.

Lastly, we point out that since $U$ maps $\partial \Omega_N \cap \mathcal{F}(u)$ onto $[L,L,R]$ and maps $\partial \Omega_N \cap \partial B_{1/2}$ into a curve that is $O(\delta_1)$-close to a half-circle of radius $1/2$, according to [31], it has to be that

$$U(\Omega_N) \supseteq \mathbb{H}_{a_0} \cap B_{1/3}.$$ 

Thus for all small $a_0$, $\psi(\Omega_N) = (V_{a_0})^{-1}U'(\Omega_N) \supseteq \Omega_{a_0}^+ \cap B_{1/4}$. After applying the same argument for $\Omega_S$, we establish the full statement $\psi(B_{1/2}^+(u)) \supseteq \Omega_{a_0} \cap B_{1/4}$.
We shall now use the map $\psi$ to obtain curvature bounds of $F(u)$ in $B_{1/4}$. On the road to do so, we will obtain the following crucial estimates on $\psi'$ and $\psi''$.

**Lemma 10.6.** The injective holomorphic map $\psi : B_{1/2}^+(u) \to \Omega_a$ constructed in Lemma 10.5 satisfies:

$$|\psi''(z)| \leq C\delta \quad \text{and} \quad |\psi'(z) - 1| \leq C\delta(|z| + a_0) \quad \text{for} \quad z \in B_{1/4}^+(u).$$

**Proof.** We know that for $z \in \partial B_{1/2} \cap \partial B_{1/2}^+(u)$

$$|\psi'(z)| = \frac{|U'(z)|}{V_{a_0}'(\psi(z))} = 1 + O(\delta)$$

because $|U'(z)| = 1 + O(\delta)$ for $\partial B_{1/2} \cap \partial B_{1/2}^+(u)$ and

$$|V_{a_0}'(\psi(z))| = |V'(\psi(z)/a_0)| = 1 + O(\delta) \quad z \in \partial B_{1/2} \cap \partial B_{1/2}^+(u)$$

for all $a_0 = a(1 + O(\delta_1))$ small enough (depending on $\delta$), because according to Lemma 10.5

$$|\psi(z)| \geq 1/4 \quad \text{when} \quad z \in \partial B_{1/2} \cap \partial B_{1/2}^+(u).$$

Furthermore, $|\psi'| = 1$ on $F(u) \cap B_{1/2}$, so that by the maximum (and minimum) modulus principle,

$$|\psi'| = 1 + O(\delta) \quad \text{in} \quad B_{1/4}^+(u). \quad (32)$$

Since $B_{1/2}^+(u)$ is simply-connected and since $\psi' \neq 0$ as $\psi$ is conformal, we can write

$$\psi' = e^{G}$$

for some holomorphic function $G$ on $B_{1/2}^+(u)$. Then

$$\psi'' = G'\psi'$$

and in view of (32), we shall have $\psi'' = O(\delta)$ in $B_{1/4}^+(u)$ once we establish

$$|G'| \leq c\delta \quad \text{in} \quad B_{1/4}^+(u).$$

Let $g = \text{Re}(G)$; as $|G'| = |\nabla g|$ it suffices to obtain bounds on $|\nabla g|$ and we know that

$$g(z) = \log|\psi'(z)| = \begin{cases} 0 & z \in F(u) \cap B_{1/2} \\ O(\delta) & z \in B_{1/2}^+(u) \end{cases}$$

In particular $g$ vanishes on $F(u) \cap B_{1/2}$ and we can apply the boundary Harnack inequality in the $C\delta$-Lipschitz domains $B_{1/4}(z_\pm)^+(u)$, where $z_\pm := \rho(\pm 1/4, 0)$, in order to establish that

$$|g(z)| \leq c\delta u(z)/u(z_\pm \pm i/8) \leq c'\delta u(z) \quad \text{in} \quad B_{1/8}(z_\pm)^+(u)$$

(since by assumption (4), $u(z_\pm \pm i/8) \approx 1/8$). Because we have

$$u \geq 1/8 - \delta \geq 1/10 \quad \text{on} \quad \partial B_{1/4} \setminus (B_{1/8}(z_+) \cup B_{1/8}(z_-)),$$

we see that $|g| \leq C\delta u$ on $\partial B_{1/4} \cap B_+^+(u)$ and thus by the maximum principle,

$$|g| \leq C\delta u \quad \text{in all of} \quad B_{1/4}^+(u).$$

An application of the Hopf Lemma yields

$$|\nabla g| \leq C\delta|\nabla u| = C\delta \quad \text{on} \quad F(u) \cap B_{1/4}.$$

Finally, we have

$$|\nabla g| = \frac{|\nabla |\psi'|^2|}{2|\psi'|^2} \leq C\delta \quad \text{on} \quad B_{1/4}^+(u) \cap \partial B_{1/4}$$

because of (32) and the fact that on $B_{1/4}^+(u) \cap \partial B_{1/4}$

$$\nabla |\psi'|^2 = 2\text{Re}(\nabla (U'/V_{a_0}' \circ \psi)) = O(|\psi''| + |V_{a_0}' \circ \psi|) = O(\delta).$$

Hence $|\nabla g| \leq C\delta$ in all of $B_{1/4}^+(u)$ as desired.
To get the first derivative bound, we integrate the second derivative bound along a curve $\gamma \subseteq B_{1/4}(u)$ connecting $p \in F_L \cap \beta$ (the “left” end of the steepest path) to $z$:

$$\psi'(z) = \psi'(x_0) + \int_\gamma \psi''(\zeta) \, d\zeta = \psi'(p) + O(\delta H^1(\gamma)).$$

Since by definition $V'(\psi(p)) = U'(p) = 1$ (as $\nabla u(p) = e_1$)

$$\psi'(p) = U'(p)/V'(\psi(p)) = 1$$

As $\gamma$ can be taken to be of length $O(|z| + a_0)$, we obtain the desired bound

$$|\psi'(z)| - 1 \leq C\delta(|z| + a_0) \quad z \in B_{1/4}(u).$$

\[ \square \]

**Theorem 10.7.** Given $\delta > 0$ small enough, there exist $r_0 > 0$, $\epsilon_1 > 0$ such that if $u$ is a classical solution of (1) in $B_1$, $0 \in F(u)$ and

$$\text{dist}(0, \{u = 0\} \setminus Z) < \epsilon_1 r_0$$

then there exists a point $p \in B_{r_0/3}$ such that $B_{r_0/2}(p) \cap F(u)$ consists of two free boundary arcs $F_L$ and $F_R$, the shortest segment between which is centered at $p$, the separation

$$s := \text{dist}(F_L, F_R) < \epsilon_1 r_0.$$  

Furthermore, $u$ has a unique saddle point $x_0$ in $B_{r_0/2}(p)$ and there is an injective holomorphic map

$$\psi : B_{r_0/2}(p)^+(u) \to \Omega_a \quad \text{where} \quad a = u(x_0)$$

that extends continuously to $\partial B_{r_0/2}(p)^+(u)$, mapping $\psi(x_0) = 0$ and $F(u) \cap B_{r_0/2}(p)$ into $\partial \Omega_a$ and satisfying

$$|\psi'| \leq C\delta/r_0 \quad |\psi' - e^{i\theta}| < C\delta(|z| + a)/r_0 \quad \text{in} \quad B_{r_0/2}(p)^+(u)$$

for some $\theta \in \mathbb{R}$. It relates the curvature $\kappa$ of $F(u)$ in $B_{r_0/2}(p)$ to the curvature $\kappa_a$ of $\partial \Omega_{a\epsilon_0}$ via

$$|\kappa(z) - \kappa_a(\psi(z))| \leq C\delta/r_0 \quad z \in F(u) \cap B_{r_0/2}(p)$$

for some numerical constants $C, c > 0$.

**Proof.** For $\delta$ fixed we find $r_0$, $\epsilon_0$ as in Theorem 9.1. Set $\delta_1 = \delta$, $M = 8/\epsilon_0$, apply Proposition 10.3 to find $s_1 = s_1(\delta, M)$ and set $\epsilon_1 = \min(\epsilon_0, s_1)$. Then $\hat{u}(y) := (2/r_0)u(p + ry)/(2)$, defined in $B_1$ for some appropriate rotation $\rho \sim e^{i\theta}$, falls under the set-up (1-4) and we construct the injective holomorphic map $\psi$ as in Lemma 10.5 satisfying the estimates of Lemma 10.6, whose rescaled statement is precisely (33).

Let $\hat{U}$ and $V_{2a/r_0}$ be the holomorphic extensions of $\hat{u} = \text{Re}(\hat{U})$ and $H_{2a/r_0} = \text{Re}(V_{2a/r_0})$ such that

$$\hat{U}(z) = V_{2a/r_0}(\hat{\psi}(z)) \quad \text{in} \quad B_{1/2}.$$

Since the curvature of $F(\hat{u})$ at $z$

$$\kappa(z) = \text{div} \frac{\nabla \hat{u}}{|\nabla \hat{u}|} = -\frac{\nabla \hat{u}}{2|\nabla \hat{u}|^3} \cdot \text{Re}(\hat{U}''(\hat{U})^2) = -\frac{1}{2} \text{Re}(\hat{U}''(\hat{U})^2)$$

we have, in view of $|\hat{\psi}'(z)| = 1 = |V_{2a/r_0}(\hat{\psi}(z))|$

$$\kappa = -\frac{1}{2} \text{Re}\left( (V_{2a/r_0}'' \hat{\psi}^2 + V_{2a/r_0}V_{2a/r_0}'' \hat{\psi}^2) \left(V_{2a/r_0}'' \hat{\psi}^2 + V_{2a/r_0}'' \hat{\psi}^2\right) \right) = -\frac{1}{2} \text{Re}(V_{2a/r_0}''(V_{2a/r_0}^2 + \hat{\psi}''V_{2a/r_0}^2))

= \kappa_{2a/r_0} \circ \hat{\psi} + O(\delta),$$

which is the rescaled version of (34).

We now have all the ingredients for Theorems 1.2 and Theorem 1.3

**Proof of Theorem 1.3.** Fix $\delta > 0$ to be smaller than the flatness constant $\delta_0$ (Remark 8.1). Let $r_0$, $\epsilon_0$ be as in Theorem 9.1. Set $\delta_1 = \delta$, $M = 8\epsilon_0$ and apply Proposition 10.3 to find $s_1 = s_1(\delta, M)$. Finally, set $\epsilon_1 = \min(\epsilon_0, \epsilon_0)$. For any point $q \in F(u)$, let $Z_q$ be the component of the zero phase to which $q$ belongs. Define the set of points

$$C_{\text{prox}} = \{q \in F(u) \cap B_{1/2} : \text{dist}(q, \{u = 0\} \setminus Z_q) < \epsilon_1 r_0\}$$
in whose neighborhood we expect to see a hairpin structure. According to Theorem 9.1, for every $q \in \mathcal{C}_{\text{prox}}$ there exists a $z(q) \in B_{r_0/2}(q)$ such that $F(u) \cap B_{r_0/2}(z)$ is an approximate hairpin centered at $z = z(q)$ in the sense that:

- $F(u) \cap B_{r_0/2}(z)$ consists of two arcs $F_L$ and $F_R$;
- if $s = \text{dist}(F_L, F_R)$, we have for some rotation $\rho$ and functions $f, g : \mathbb{R} \to \mathbb{R}$ with $f < g$,
  \[
  \{u = 0\} \cap \left( B_{r_0/2}(z) \setminus B_{|s|/\epsilon_0}(z) \right) = z + \rho \{4s/\epsilon_0 < |x| < r_0/2 : f(x) \leq |x| \leq g(x)\} \\
  \text{where } \|f\|_{L^\infty} + \|g\|_{L^\infty} \leq \delta r, \quad \|f'\|_{L^\infty} + \|g'\|_{L^\infty} \leq \delta.
  \]

At the same time, Proposition 10.3 gives: a unique saddle point $\{u = 0\} \cap F_{r_0/2}(z)$.

11. THE MINIMAL SURFACE ANALOGUE

In [Tra14] (see Theorems 9 and 10) Traizet discovered a remarkable correspondence between global solutions of (1) with $|\nabla u| < 1$ and complete embedded minimal bigraphs (minimal surfaces symmetric with respect to a plane with the two halves, “above” and “below” the plane, being graphical). The correspondence is expressed via the Weierstrass representation formula for immersed minimal surfaces. Recall, if $X : M \subseteq \mathbb{R}^3$ denotes the minimal immersion, the coordinate $X_3$ is a harmonic function on $M$ and one can locally define a harmonic conjugate $X_3^*$, so that

$$dh = dX_3 + idX_3^*$$

is a well-defined holomorphic differential on $M$ (viewed as a Riemann surface), the so-called height differential. Furthermore, the stereographically projected Gauss map $g : M \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function on $M$. The pair $(g, dh)$ is called the Weierstrass data of the minimal surface and the minimal immersion $X$ is given, up to translation, by

$$X(p) = (X_1(p), X_2(p), X_3(p)) = \text{Re} \int_{p_0}^{p} \left( \frac{1}{2} (g^{-1} - g)dh, -\frac{1}{2} (g^{-1} + g)dh, dh \right)$$

where $p_0$ is a fixed point in $M$. Conversely, if $M$ is a Riemann surface, and $(g, dh)$ is a pair of a meromorphic function and a holomorphic 1-form on $M$, satisfying certain compatibility conditions ([Oss64]), then (35) defines a minimal immersion of $M$ in $\mathbb{R}^3$.

Traizet’s brilliant insight was to define

$$g = 2 \frac{\partial u}{\partial z} \text{ and } dh = 2 \frac{\partial u}{\partial z} dz$$

in terms of a solution $u$ of (1), and show that, under certain conditions, the Weierstrass data $(g, dh)$ give rise to the upper half ($X_3 > 0$) of a minimal bigraph. Conversely, a solution $u$ of (1) can be constructed using the Weierstrass data of a complete embedded minimal bigraph.
We have used Traizet’s correspondence to state Corollary 1.4, the minimal surface version of Theorem 1.3. We can now turn to the proof.

**Proof of Corollary 1.4.** Following the argument of [Tra14, Theorem 10], we shall construct a solution of (1), corresponding to the minimal bigraph \( M \). Let \( \zeta \) be a complex coordinate on \( M \), let \( g \) be the stereographically projected Gauss map and \( dh = (2\partial X_3/\partial \zeta) \, d\zeta \) be the height differential. Note that \( |g| = 1 \) on \( M \cap \{ X_3 = 0 \} \) as the normal points horizontally there and we may assume that the orientation of \( M \) is chosen so that the normal points down in \( M^+ \), i.e. \( |g| < 1 \) in \( M^+ \). Furthermore \( g \) has the same zeros and poles as \( dh \) (with same multiplicities), thus \( g^{-1} dh \) defines a holomorphic non-vanishing one-form on \( M^+ \). Since \( M^+ \) is simply-connected,

\[
\varphi(p) = \int_0^p g^{-1} dh \quad p \in M^+
\]
defines a holomorphic function on \( M^+ \) (recall \( 0 \in M \)). Claim that \( \varphi \) is injective. Define

\[
\Xi := X_1 + iX_2
\]
on \( M^+ \) and let \( \hat{\Omega} = \Xi(M^+) \) be the projection of \( M^+ \) down to the horizontal plane \( \{ X_3 = 0 \} \). Since \( M^+ \) is a graph, \( \Xi \) is a diffeomorphism from \( \overline{M^+} \) to \( \hat{\Omega} \), so \( \varphi \) will be injective if and only if \( \phi := \varphi \circ \Xi^{-1} \) is injective on \( \hat{\Omega} \). Let \( a, b \) be arbitrary points of \( \hat{\Omega} \) and let \( [a, b] \subseteq \mathbb{C} \) denote the straight-line closed segment from \( a \) to \( b \). Then for some \( N \in \mathbb{N} \) we can write

\[
[a, b] = \bigcup_{k=1}^N \{ z_{2k-1}, z_{2k} \} \cup \bigcup_{k=1}^{N-1} \{ z_{2k}, z_{2k+1} \},
\]
where \( z_1 = a, z_{2N} = b \), the interior of \( [z_{2k-1}, z_{2k}] \) belongs to \( \hat{\Omega} \), while \( z_{2k} \) and \( z_{2k+1} \) belong to the same connected component of \( \partial \hat{\Omega} \). Claim that

\[
\langle \phi(z_{2k}) - \phi(z_{2k-1}), \frac{b-a}{|b-a|} \rangle > |z_{2k} - z_{2k-1}|, \tag{36}
\]
where \( \langle w_1, w_2 \rangle := \text{Re}(\overline{w_1}w_2) \) denotes the standard inner product on \( \mathbb{C} \). Let \( \alpha : [0, 1] \to M^+ \) be such that \( \Xi \circ \alpha \) is the constant speed parameterization of \( [z_{2k-1}, z_{2k}] \). For each fixed time \( t \in (0, 1) \), denote

\[
v := \frac{1}{2} g^{-1} dh(\alpha'(t)) \quad w := -\frac{1}{2} g dh(\alpha'(t)),
\]
we have \( |v| > |w| \) because \( |g| < 1 \). Since \( d\varphi(\alpha'(t)) = g^{-1} dh(\alpha'(t)) = 2v \),

\[
z_{2k} - z_{2k-1} = d\Xi(\alpha'(t)) = (dX_1 + i dX_2)(\alpha'(t)) = \frac{1}{2} g^{-1} dh(\alpha'(t)) - \frac{1}{2} g dh(\alpha'(t)) = v + w,
\]
we have

\[
\langle d\tilde{\varphi}(\alpha'(t)), \frac{b-a}{|b-a|} \rangle = |z_{2k} - z_{2k-1}|^{-1} \langle 2v, v + w \rangle > |z_{2k} - z_{2k-1}|^{-1} |v + w|^2 = |z_{2k} - z_{2k-1}|
\]
which leads to (36) once we integrate in \( t \) from 0 to 1. On the other hand,

\[
\langle \phi(z_{2k+1}) - \phi(z_{2k}), \frac{b-a}{|b-a|} \rangle = |z_{2k+1} - z_{2k}|. \tag{37}
\]
This is the case, because on the component \( \beta \) of \( M \cap \{ X_3 = 0 \} \), to which \( \Xi^{-1}(z_{2k+1}) \) and \( \Xi^{-1}(z_{2k}) \) belong, we know \( g^{-1} = \bar{g} \) and \( d\bar{h} = -dh \), so that

\[
d\tilde{\varphi}(\beta') = g^{-1} dh(\beta') = -g dh(\beta') = \frac{1}{2} g^{-1} dh(\beta') - \frac{1}{2} g dh(\beta') = d\Xi(\beta')
\]
and thus, \( \phi(z_{2k+1}) - \phi(z_{2k}) = z_{2k+1} - z_{2k} \). Adding up (36) and (37) from \( k = 1 \) to \( N \), we derive

\[
\langle \phi(b) - \phi(a), (b-a)/|b-a| \rangle > |b-a|
\]
from which the injectivity of \( \phi \) follows.

We can now define the function

\[
u = X_3 \circ \phi^{-1}
\]
on the domain \( \Omega = \varphi(M^+) \), and we can easily verify that \( u \) is a positive, harmonic function in \( \Omega \) that vanishes on \( \partial\Omega \cap B_R \) where, for \( z = \varphi(\zeta) \in F(u) \)

\[
|\nabla u(z)| = \left|2 \frac{\partial X_3}{\partial \zeta} \frac{1}{\varphi'(\zeta)}\right| = |g(\zeta)| = 1.
\]

Furthermore, the metric induced on \( \Omega \) by the conformal immersion \( X \circ \varphi^{-1} \) is given by the standard formula

\[
ds = \frac{1}{2} \left(|g||dh| + |g|^{-1}|dh|\right) = \frac{1}{2} (|g|^2 + 1)|dz| = \lambda(z)|dz|
\]

where \( \frac{1}{2} \leq \lambda(z) \leq 1 \), as \( |g| \leq 1 \) on \( M^+ \). So, if \( \gamma^+ = \gamma \cap M^+ \) denotes the piece of the shortest geodesic lying in \( M^+ \), it is mapped by \( \varphi \) to a curve \( \tilde{\gamma} = \varphi(\gamma^+) \subseteq \Omega \) with Euclidean length \( O(\mathcal{H}^1(\gamma^+)) = O(\epsilon) \) which connects the two pieces of \( \partial\Omega \).

Fix \( \delta < 1/1000 \) a small positive numerical constant and let \( r_0, \epsilon_1 \) be as in Theorem 10.7. Set \( R_0 = 1/r_0 \) and \( \epsilon_0 = \epsilon_1 \). Extend \( u \) by zero in \( B_{R_0} \setminus \Omega \). Then \( u \) is a solution of (1) in \( B_{R_0} \), satisfying (3), so Theorem 10.7 gives us an injective conformal map \( \tilde{\psi} : B^+_1(u) \to \Omega_a \) for some appropriate \( a = O(\epsilon) \), such that

\[
\tilde{\psi}'(z) = 1 + O(\delta(|z| + a)), \quad \tilde{\psi}''(z) = O(\delta) \quad \text{for} \quad z \in B^+_1(u) \quad (38)
\]

and \( U := V_a \circ \tilde{\psi} \) is a holomorphic extension of \( u \) in \( B^+_1(u) \) (recall \( V_a \) is the holomorphic extension of \( H_a \) given in Section 10). It’s easy to see that \( \tilde{\psi} \) gives rise to an injective conformal map from \( M^+ \cap B_2 \) into the standard catenoid \( \Sigma^* := \Sigma_{\rho} \cap \{X_3 > 0\} \) (the counterpart to \( \Omega \) in the Traizet correspondence), which then extends by symmetry to a conformal map \( \psi \) on all of \( M \cap B_2 \). The metric on \( \Omega_a \) induced by its immersion as \( \Sigma_{\rho} \) is

\[
ds_{\text{cat}} = (1 + |V_a'|^2)|dz|
\]

while the metric on \( B^+_1(u) \) is

\[
ds = (1 + |U'|^2)|dz|
\]

and we check that the pull-back metric \( \tilde{\psi}^*(\nds_{\text{cat}}) \) satisfies

\[
\tilde{\psi}^*(\nds_{\text{cat}}) = (1 + |U'|^2/|\tilde{\psi}'|^2)|\tilde{\psi}'|dz = (1 + O(\delta(|z| + a)))ds.
\]

Since \( a \sim \rho \), the induced conformal map \( \psi \) is an isometry up to a factor of \( (1 + O(\delta(|x| + \rho))) \) and

\[
\epsilon = \mathcal{H}^1(\gamma) = (1 + O(\delta))2\pi\rho \quad \implies \quad |\epsilon - 2\pi\rho| = O(\delta^2) < \epsilon/100.
\]

Furthermore, the Gauss curvature of \( M \) is given by the standard formula for the curvature of a conformal metric \( \lambda(z)|dz| = (1 + |U'|^2)|dz|

\[
K = -\frac{\Delta \log \lambda(z)}{\lambda^2(z)} = -\frac{4|U''|^2}{(1 + |U'|^2)^2}
\]

Plugging in \( U(z) = V_a(\tilde{\psi}(z)) \) and applying the estimates (38), we get

\[
K = -\frac{4|V_a''(\tilde{\psi})|^2 + V_a^2\tilde{\psi}''|^2}{(1 + |V_a'|^2)^2|\tilde{\psi}'|^2} = -\frac{4|V_a''|^2}{(1 + |V_a'|^2)^4}(1 + O(\delta(|z| + \rho))) + O\left(\delta^2 \frac{2|V_a''|^2}{(1 + |V_a'|^2)^2} \right) + O(\delta^2) = K_{\rho} + O(\delta(r + \rho)K_{\rho}) + O(\delta \sqrt{|K_{\rho}|}) + O(\delta^2)
\]

Noting that

\[
\sqrt{|K_{\rho}(q)|} = O(\rho/r(q)^2)
\]

and that \( |r(\psi(p)) - r(p)| \sim \rho + \delta r(p) \) we obtain the desired estimate

\[
K(p) = K_{\rho}(\psi(p)) + O\left(\delta \frac{\rho}{r + \rho} \sqrt{|K_{\rho}|}\right) + O\left(\delta \sqrt{|K_{\rho}|}\right) + O(\delta^2) = K_{\rho} + O\left(\delta + \sqrt{|K_{\rho}|}\right) \delta.
\]

\[\square\]
References

[AC81] H. W. Alt and L. A. Caffarelli. Existence and regularity for a minimum problem with free boundary. J. Reine Angew. Math., 325:105–144, 1981.

[ACF84] Hans Wilhelm Alt, Luis A. Caffarelli, and Avner Friedman. Variational problems with two phases and their free boundaries. Trans. Amer. Math. Soc., 282(2):431–461, 1984.

[Caf87] Luis A. Caffarelli. A Harnack inequality approach to the regularity of free boundaries. I. Lipschitz free boundaries are $C^{1, \alpha}$. Rev. Mat. Iberoamericana, 3(2):139–162, 1987.

[Caf89] Luis A. Caffarelli. A Harnack inequality approach to the regularity of free boundaries. II. Flat free boundaries are Lipschitz. Comm. Pure Appl. Math., 42(1):55–78, 1989.

[CKJ04] Luis A. Caffarelli, David Jerison, and Carlos E. Kenig. Global energy minimizers for free boundary problems and full regularity in three dimensions. In Noncompact problems at the intersection of geometry, analysis, and topology, volume 350 of Contemp. Math., pages 83–97. Amer. Math. Soc., Providence, RI, 2004.

[CM02] Tobias H Colding and William P Minicozzi. On the structure of embedded minimal annuli. International Mathematics Research Notices, 2002(29):1539–1552, 2002.

[CM04a] Tobias H. Colding and William P. Minicozzi, II. The space of embedded minimal surfaces of fixed genus in a 3-manifold. I. Estimates off the axis for disks. Ann. of Math. (2), 160(1):27–68, 2004.

[CM04b] Tobias H. Colding and William P. Minicozzi, II. The space of embedded minimal surfaces of fixed genus in a 3-manifold. II. Multi-valued graphs in disks. Ann. of Math. (2), 160(1):69–92, 2004.

[CM04c] Tobias H. Colding and William P. Minicozzi, II. The space of embedded minimal surfaces of fixed genus in a 3-manifold. III. Planar domains. Ann. of Math. (2), 160(2):523–572, 2004.

[CM04d] Tobias H. Colding and William P. Minicozzi, II. The space of embedded minimal surfaces of fixed genus in a 3-manifold. IV. Locally simply connected. Ann. of Math. (2), 160(2):573–615, 2004.

[CS05] Luis Caffarelli and Sandro Salsa. A geometric approach to free boundary problems, volume 68 of Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2005.

[DSJ11] Daniela De Silva and David Jerison. A gradient bound for free boundary graphs. Comm. Pure Appl. Math., 64(4):538–555, 2011.

[HHP11] Laurent Hauswirth, Frédéric Hélein, and Frank Pacard. On an overdetermined elliptic problem. Pacific J. Math., 250(2):319–334, 2011.

[JS] David Jerison and Ovidiu Savin. Some remarks on stability of cones for the one-phase free boundary problem. preprint. arXiv:1410.7463.

[KLT13] Dmitry Khavinson, Erik Lundberg, and Razvan Teodorescu. An overdetermined problem in potential theory. Pacific J. Math., 265(1):85–111, 2013.

[KN77] D. Kinderlehrer and L. Nirenberg. Regularity in free boundary problems. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 4(2):373–391, 1977.

[Oss64] Robert Osserman. Global properties of minimal surfaces in e3 and en. Annals of Mathematics, pages 340–364, 1964.

[Wei98] Georg S. Weiss. Partial regularity for weak solutions of an elliptic free boundary problem. Comm. Partial Differential Equations, 23(3-4):439–455, 1998.

David Jerison, Department of Mathematics, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139, USA
E-mail address: jerison@math.mit.edu

Nikola Kamburov, Department of Mathematics, The University of Arizona, 617 N Santa Rita Ave, P.O. Box 210089, Tucson AZ 85721, USA
E-mail address: kamburov@math.arizona.edu