QUASI-ISOMETRIC CO-HOPFICITY OF NON-UNIFORM LATTICES IN RANK-ONE SEMI-SIMPLE LIE GROUPS

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Abstract. We prove that if $G$ is a non-uniform lattice in a rank-one semi-simple Lie group $\neq \text{Isom}(\mathbb{H}^2_\mathbb{R})$ then $G$ is quasi-isometrically co-Hopf. This means that every quasi-isometric embedding $G \to G$ is coarsely surjective and thus is a quasi-isometry.

1. Introduction

The notion of co-Hopficity plays an important role in group theory. Recall that a group $G$ is said to be co-Hopf if $G$ is not isomorphic to a proper subgroup of itself, that is, if every injective homomorphism $G \to G$ is surjective. A group $G$ is almost co-Hopf if for every injective homomorphism $\phi : G \to G$ we have $[G : \phi(G)] < \infty$. Clearly, being co-Hopf implies being almost co-Hopf. The converse is not true: for example, for any $n \geq 1$ the free abelian group $\mathbb{Z}^n$ is almost co-Hopf but not co-Hopf.

It is easy to see that any freely decomposable group is not co-Hopf. In particular, a free group of rank at least 2 is not co-Hopf. It is also well-known that finitely generated nilpotent groups are always almost co-Hopf and, under some additional restrictions, also co-Hopf [1]. An important result of Sela [17] states that a torsion-free non-elementary word-hyperbolic group $G$ is co-Hopf if and only if $G$ is freely indecomposable. Partial generalizations of this result are known for certain classes of relatively hyperbolic groups, by the work of Belegradek and Szczepański [2]. Co-Hopficity has also been extensively studied for 3-manifold groups and for Kleinian groups. Delzant and Potyagailo [9] gave a complete characterization of co-Hopfian groups among non-elementary geometrically finite Kleinian groups without 2-torsion.

A counterpart algebraic notion is that of Hopficity. A group $G$ is said to be Hopfian if every surjective endomorphism $G \to G$ is necessarily injective, and hence is an automorphism of $G$. This notion is also extensively studied in geometric group theory. In particular, an important result of Sela [18] shows that every torsion-free word-hyperbolic group is Hopfian. The notion of Hopficity admits a number of interesting “virtual” variations. Thus a group $G$ is called cofinitely Hopfian if every endomorphism of $G$ whose image is of finite index in $G$, is an automorphism of $G$, see, for example [7].

A key general theme in geometric group theory is the study of “large-scale” geometric properties of finitely generated groups. Recall that if $(X,d_X)$ and $(Y,d_Y)$ are metric spaces, a map $f : X \to Y$ is called a coarse embedding if there exist
monotone non-decreasing functions $\alpha, \omega : [0, \infty) \to \mathbb{R}$ such that $\alpha(t) \leq \omega(t)$, that $\lim_{t \to \infty} \alpha(t) = \infty$, and such that for all $x, x' \in X$ we have

\begin{equation}
\alpha(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \omega(d_X(x, x')).
\end{equation}

If $d_X$ is a path metric, then for any coarse embedding $f : X \to Y$ the function $\omega(t)$ can be chosen to be affine, that is, of the form $\omega(t) = at + b$ for some $a, b \geq 0$.

A coarse map $f$ is called a coarse equivalence if $f$ is coarsely surjective, that is, if there is $C \geq 0$ such that for every $y \in Y$ there exists $x \in X$ with $d_Y(y, f(x)) \leq C$. A map $f : X \to Y$ is called a quasi-isometric embedding if $f$ is a coarse embedding and the functions $\alpha(t), \omega(t)$ in (*) can be chosen to be affine, that is, of the form $\alpha(t) = \frac{1}{\lambda} t - \epsilon$, $\omega(t) = \lambda t + \epsilon$ where $\lambda \geq 1$, $\epsilon \geq 0$. Finally, a map $f : X \to Y$ is a quasi-isometry if $f$ is a quasi-isometric embedding and $f$ is coarsely surjective.

The notion of co-Hopficity has the following natural counterpart for metric spaces. We say that a metric space $X$ is quasi-isometrically co-Hopf if every quasi-isometric embedding $X \to X$ is coarsely surjective, that is, if every quasi-isometric embedding $X \to X$ is a quasi-isometry. More generally, a metric space $X$ is called coarsely co-Hopf if every coarse embedding $X \to X$ is coarsely surjective. Clearly, if $X$ is coarsely co-Hopf then $X$ is quasi-isometrically co-Hopf. If $G$ is a finitely generated group with a word metric $d_G$ corresponding to some finite generating set of $G$, then every injective homomorphism $G \to G$ is a coarse embedding. This easily implies that if $(G, d_G)$ is coarsely co-Hopf then the group $G$ is almost co-Hopf.

**Example 1.1.** The real line $\mathbb{R}$ is coarsely co-Hopf (and hence quasi-isometrically co-Hopf). This follows from the fact that any coarse embedding must send the ends of $\mathbb{R}$ to distinct ends. Since $\mathbb{R}$ has two ends, a coarse embedding induces a bijection on the set of ends of $\mathbb{R}$. It is then not hard to see that a coarse embedding from $\mathbb{R}$ to $\mathbb{R}$ must be coarsely surjective. See [3] for the formal definition of ends of a metric space.

**Example 1.2.** The rooted regular binary tree $T_2$ is not quasi-isometrically co-Hopf.

We can identify the set of vertices of $T_2$ with the set of all finite binary sequences. The root of $T_2$ is the empty binary sequence $\epsilon$ and for a finite binary sequence $x$ its left child is the sequence $0x$ and the right child is the sequence $1x$. Consider the map $f : T_2 \to T_2$ which maps $T_2$ isometrically to a copy of itself that "hangs below" the vertex 0. Thus $f(x) = 0x$ for every finite binary sequence $x$. Then $f$ is an isometric embedding but the image $f(T_2)$ is not co-bounded in $T_2$ since it misses the entire infinite branch located below the vertex 1.

**Example 1.3.** Consider the free group $F_2 = F(a, b)$ on two generators. Then $F_2$ is not quasi-isometrically co-Hopf.

The Cayley graph $X$ of $F_2$ is a regular 4-valent tree with every edge of length 1. We may view $X$ in the plane so that every vertex has one edge directed upward, and three downward. Picking a vertex $v_0$ of $X$, denote its left branch by $X_1$ and the remainder of the tree by $X_2$. We have $X_1 \cup X_2 = X$, and $X_1$ is a rooted ternary tree. Define a quasi-isometric embedding $f : X \to X$ by taking $f$ to be a shift on $X_1$ (defined similarly to Example 1.2) and the identity on $X_2$. The map $f$ is not coarsely surjective, but it is a quasi-isometric embedding. Moreover, for any vertices $x, x'$ of $X$ we have $|d(f(x), f(x')) - d(x, x')| \leq 1$.

One can also see that $F_2 = F(a, b)$ is not quasi-isometrically co-Hopf for algebraic reasons. Let $u, v \in F(a, b)$ with $[u, v] \neq 1$. Then there is an injective homomorphism ...
\[ h : F(a, b) \to F(a, b) \] such that \( h(a) = u \) and \( h(b) = v \). This homomorphism \( f \) is always a quasi-isometric embedding of \( F(a, b) \) into itself.

If, in addition, \( u \) and \( v \) are chosen so that \( \langle u, v \rangle \neq F(a, b) \) then \([F(a, b) : h(F(a, b))] = \infty\) and the image \( h(F(a, b)) \) is not co-bounded in \( F(a, b) \).

Thus, the group \( F_2 \) is not almost co-Hopf and not quasi-isometrically co-Hopf.

**Example 1.4.** There do exist finitely generated groups that are algebraically co-Hopf but not quasi-isometrically co-Hopf. The simplest example of this kind is the solvable Baumslag-Solitar group \( B(1, 2) = \langle a, t \mid t^{-1}a^{-1}t = a^2 \rangle \). It is well-known that \( B(1, 2) \) is co-Hopf.

To see that \( B(1, 2) \) is not quasi-isometrically co-Hopf we use the fact that \( B(1, 2) \) admits an isometric properly discontinuous co-compact action on a proper geodesic metric space \( X \) that is “foliated” by copies of the hyperbolic plane \( \mathbb{H}_\mathbb{R}^2 \). We refer the reader to the paper of Farb and Mosher [12] for a detailed description of the space \( X \), and will only briefly recall the properties of \( X \) here.

Topologically, \( X \) is homeomorphic to the product \( \mathbb{R} \times T_3 \) where \( T_3 \) is an infinite 3-regular tree (drawn upwards): there is a natural projection \( p : X \to T_3 \) whose fibers are homeomorphic to \( \mathbb{R} \). The boundary of \( T_3 \) is decomposed into two sets: the “lower boundary” consisting of a single point \( u \) and the “upper boundary” \( \partial_u X \) which is homeomorphic to the Cantor set (and can be identified with the set of dyadic rationals). For any bi-infinite geodesic \( \ell \) in \( T_3 \) from \( u \) to a point of \( \partial_u X \) the full-\( p \)-preimage of \( \ell \) in \( X \) is a copy of the hyperbolic plane \( \mathbb{H}_\mathbb{R}^2 \) (in the upper-half plane model). The \( p \)-preimage of any vertex of \( T_3 \) is a horizontal horocycle in the \( \mathbb{H}_\mathbb{R}^2 \)-“fibers”. Any two \( \mathbb{H}_\mathbb{R}^2 \)-fibers intersect along a complement of a horoball in \( \mathbb{H}_\mathbb{R}^2 \).

Similar to the above example for \( F(a, b) \), we can take a quasi-isometric embedding \( f : T_3 \to T_3 \) whose image misses an infinite subtree in \( T_3 \) and such that \( |d(x, x') - d(f(x), f(x'))| \leq 1 \) for any vertices \( x, x' \) of \( T_3 \). It is not hard to see that this map \( f \) can be extended along the \( p \)-fibers to a map \( \tilde{f} : X \to X \) such that \( \tilde{f} \) is a quasi-isometric embedding but not coarsely surjective. Since \( X \) is quasi-isometric to \( B(1, 2) \), it follows that \( B(1, 2) \) is not quasi-isometrically co-Hopf.

**Example 1.5.** Grigorchuk’s group \( G \) of intermediate growth provides another interesting example of a group that is not quasi-isometrically co-Hopf. This group \( G \) is finitely generated and can be realized as a group of automorphisms of the regular binary rooted tree \( T_2 \). The group \( G \) has a number of unusual algebraic properties: it is an infinite 2-torsion group, it has intermediate growth, it is amenable but not elementary amenable and so on. See Ch. VIII in [15] for detailed background on the Grigorchuk group. It is known that there exists a subgroup \( K \) of index 16 in \( G \) such that \( K \times K \) is isomorphic to a subgroup of index 64 in \( G \). The map \( K \to K \times K, k \mapsto (k, 1) \) is clearly a quasi-isometric embedding which is not coarsely surjective. Since both \( K \) and \( K \times K \) are quasi-isometric to \( G \), it follows that \( G \) is not quasi-isometrically co-Hopf.

For Gromov-hyperbolic groups and spaces quasi-isometric co-Hopficity is closely related to the properties of their hyperbolic boundaries. We say that a compact metric space \( K \) is *topologically co-Hopf* if \( K \) is not homeomorphic to a proper subset of itself. We say that \( K \) is *quasi-symmetrically co-Hopf* if every quasi-symmetric map \( K \to K \) is surjective. Note that for a compact metric space \( K \) being topologically co-Hopf obviously implies being quasi-symmetrically co-Hopf.
Example 1.6. A recent important result of Merenkov [15] shows that the converse implication does not hold. He constructed a round Sierpinski carpet $S$ such that $S$ is quasi-symmetrically co-Hopf. Since $S$ is homeomorphic to the standard “square” Sierpinski carpet, clearly $S$ is not topologically co-Hopf.

It is well-known (see, for example, [3]) that if $X, Y$ are proper Gromov-hyperbolic geodesic metric spaces, then any quasi-isometric embedding $f : X \to Y$ induces a quasi-symmetric topological embedding $\partial f : \partial X \to \partial Y$ between their hyperbolic boundaries. It is then not hard to see that if $G$ is a word-hyperbolic group whose hyperbolic boundary $\partial G$ is quasi-symmetrically co-Hopf (e.g. if it is topologically co-Hopf), then $G$ is quasi-isometrically co-Hopf. This applies, for example, to any word-hyperbolic groups whose boundary $\partial G$ is homeomorphic to an $n$-sphere (with $n \geq 1$), such as fundamental groups of closed Riemannian manifolds with all sectional curvatures $\leq -1$.

The main result of this paper is the following:

**Theorem 1.7.** Let $G$ be a non-uniform lattice in a rank-one semi-simple real Lie group other than $\text{Isom}(\mathbb{H}^2 \mathbb{R})$. Then $G$ is quasi-isometrically co-Hopf.

Thus, for example, if $M$ is a complete finite volume non-compact hyperbolic manifold of dimension $n \geq 3$ then $\pi_1(M)$ is quasi-isometrically co-Hopf. Note that if $G$ is a non-uniform lattice in $\text{Isom}(\mathbb{H}^2 \mathbb{R})$ then the conclusion of Theorem 1.7 does not hold since $G$ is a virtually free group.

If $G$ is a uniform lattice in a rank-one semi-simple real Lie group (including possibly a lattice in $\text{Isom}(\mathbb{H}^2 \mathbb{R})$) then $G$ is Gromov-hyperbolic with the boundary $\partial G$ being homeomorphic to $\mathbb{S}^n$ (for some $n \geq 1$). In this case it is easy to see that $G$ is also quasi-isometrically co-Hopf since every topological embedding from $\mathbb{S}^n$ to itself is necessarily surjective.

**Convention 1.8.** From now on and for the remainder of this paper let $X \neq \mathbb{H}^2 \mathbb{R}$ be a rank-one negatively curved symmetric space with metric $d_X$ (or just $d$ in most cases). Namely, $X$ is isometric to a hyperbolic space $\mathbb{H}^n \mathbb{R}$ (with $n \geq 3$), $\mathbb{H}^n \mathbb{C}$ (with $n \geq 2$), $\mathbb{H}^n \mathbb{O}$ over the reals, complexes, or quaternions, or to the octonionic plane $\mathbb{H}^2 \mathbb{O}$.

If $G$ is as in Theorem 1.7 then $G$ acts properly discontinuously (but with a non-compact quotient) by isometries on such a space $X$ and there exists a $G$-invariant collection $\mathcal{B}$ of disjoint horoballs in $X$ such that $(X \setminus \mathcal{B})/G$ is compact. The “truncated” space $\Omega = X \setminus \mathcal{B}$, endowed with the induced path-metric $d_\Omega$ is quasi-isometric to the group $G$ by the Milnor-Schwartz Lemma. Thus it suffices to prove that $(\Omega, d_\Omega)$ is quasi-isometrically co-Hopf.

Richard Schwartz [16] established quasi-isometric rigidity for non-uniform lattices in rank-one semi-simple Lie groups and we use his proof as a starting point.

First, using coarse cohomological methods (particularly techniques of Kapovich-Kleiner [14]), we prove that spaces homeomorphic to $\mathbb{R}^n$ with “reasonably nice” metrics are coarsely co-Hopf. This result applies to the Euclidean space $\mathbb{R}^n$ itself, to simply connected nilpotent Lie groups, to the rank-one symmetric spaces $X$ mentioned above, as well as to the horospheres in $X$. Let $f : (\Omega, d_\Omega) \to (\Omega, d_\Omega)$ be a quasi-isometric embedding. Schwartz’ work implies that for every peripheral horosphere $\sigma$ in $\Omega$ there exists a unique peripheral horosphere $\sigma'$ of $X$ such that $f(\sigma)$ is contained in a bounded neighborhood of $\sigma'$. Using coarse co-Hopficity of
horospheres, mentioned above, we conclude that \( f \) gives a quasi-isometry (with controlled constants) between \( \sigma \) and \( \sigma' \). Then, following Schwartz, we extend the map \( f \) through each peripheral horosphere to the corresponding peripheral horoball \( B \) in \( X \). We then argue that the extended map \( \hat{f} : X \to X \) is a coarse embedding. Using coarse co-Hopficity of \( X \), it follows that \( \hat{f} \) is coarsely surjective, which implies that the original map \( f : (\Omega, d_\Omega) \to (\Omega, d_\Omega) \) is coarsely surjective as well.

It seems likely that the proof of Theorem 1.7 generalizes to an appropriate subclass of relatively hyperbolic groups. However, a more intriguing question is to understand what happens for higher-rank lattices:

**Problem 1.9.** Let \( G \) be a non-uniform lattice in a semi-simple real Lie group of rank \( \geq 2 \). Is \( G \) quasi-isometrically co-Hopf?

Unlike the groups considered in the present paper, higher-rank lattices are not relatively hyperbolic. Quasi-isometric rigidity for higher-rank lattices is known to hold, by the result of Eskin [11], but the proofs there are quite different from the proof of Schwartz in the rank-one case.

Another natural question is:

**Problem 1.10.** Let \( G \) be as in Theorem 1.7. Is \( G \) coarsely co-Hopf?

Our proof only yields quasi-isometric co-Hopficity, and it is possible that coarse co-Hopficity actually fails in this context.

The result of Merenkov (Example 1.6) produces the first example of a compact metric space \( K \) which is quasi-symmetrically co-Hopf but not topologically co-Hopf. Topologically, \( K \) is homeomorphic to the standard Sierpinski carpet and there exists a word-hyperbolic group (in fact a Kleinian group) with boundary homeomorphic to \( K \). However, the metric structure on the Sierpinski carpet in Merenkov’s example is not “group-like” and is not quasi-symmetric to the visual metric on the boundary of a word-hyperbolic group.

**Problem 1.11.** Does there exist a word-hyperbolic group \( G \) such that \( \partial G \) (with the visual metric) is quasi-symmetrically co-Hopf (and hence \( G \) is quasi-isometrically co-Hopf), but such that \( \partial G \) is not topologically co-Hopf? In particular, do there exist examples of this kind where \( \partial G \) is homeomorphic to the Sierpinski carpet or the Menger curve?

The above question is particularly interesting for the family of hyperbolic buildings \( I_{p,q} \) constructed by Bourdon and Pajot [5, 4]. In their examples \( \partial I_{p,q} \) is homeomorphic to the Menger curve, and it turns out to be possible to precisely compute the conformal dimension of \( \partial I_{p,q} \). Note that, similar to the Sierpinski carpet, the Menger curve is not topologically co-Hopf.

**Problem 1.12.** Are the Burdon-Pajot buildings \( I_{p,q} \) quasi-isometrically co-Hopf? Equivalently, are their boundaries \( \partial I_{p,q} \) quasi-symmetrically co-Hopf?

It is also interesting to investigate quasi-isometric and coarse co-Hopficity for other natural classes of groups and metric spaces. In an ongoing work (in preparation), Jason Behrstock, Alessandro Sisto, and Harold Sultan study quasi-isometric co-Hopficity for mapping class groups and also characterize exactly when this property holds for fundamental groups of 3-manifolds.

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2. Geometric Objects

2.1. Horoballs. Recall that, by Convention 1.8, $X$ is a rank one symmetric space different from $\mathbb{H}^2_\mathbb{R}$. Namely, $X$ is isometric to a hyperbolic space $\mathbb{H}^n_\mathbb{R}$ (with $n \geq 3$), $\mathbb{H}^n_\mathbb{C}$ (with $n \geq 2$), $\mathbb{H}^n_\mathbb{H}$ over the reals, complexes, or quaternions, or to the octonionic plane $\mathbb{H}^2_\mathbb{O}$. We recall some properties of $X$. See [1], Chapter II.10, for details.

Definition 2.1. Let $0 \in X$ be a basepoint and $\gamma$ a geodesic ray starting at 0. The associated function $b : X \to \mathbb{R}$ given by

$$
(2.1) \quad b(x) = \lim_{s \to \infty} d(x, \gamma(s)) - s
$$

is known as a Busemann function on $X$. A horosphere is a level set of a Busemann function. The set $b^{-1}(t_0, \infty) \subset X$ is a horoball. Up to the action of the isometry group on $X$, there is a unique Busemann function, horosphere, and horoball.

A Busemann function $b(x)$ provides a decomposition of $X$ into horospherical coordinates, a generalization of the upper-halfspace model. Namely, let $\sigma = b^{-1}(0)$ and decompose $X = \sigma \times \mathbb{R}^+$ as follows: given $x \in X$, flow along the gradient of $b$ for time $b(x)$ to reach a point $s \in \sigma$, and write $x = (s, e^{b(x)})$. In horospherical coordinates, the $\sigma$-fibers $\{s\} \times \mathbb{R}^+$ are geodesics, the $R^+$-fibers $\sigma \times \{t_0\}$ are horospheres, and the sets $\sigma \times (t_0, \infty)$ are horoballs. Other horoballs appear as closed balls tangent to the boundary $\sigma \times \{0\}$.

If $(M, d)$ is a metric space and $C \geq 0$, a path $\gamma : [a, b] \to M$, parameterized by arc-length, is called a $C$-rough geodesic in $M$, if for any $t_1, t_2 \in [a, b]$ we have

$$
(2.2) \quad |d(\gamma(t_1), \gamma(t_2)) - |t_1 - t_2|| \leq C.
$$

If $Y, Y'$ are metric spaces, a map $f : Y \to Y'$ is coarsely Lipschitz if there exists $C > 0$ such that for any $y_1, y_2 \in Y$ we have $d_{Y'}(f(y_1), f(y_2)) \leq C d_Y(y_1, y_2)$. If $Y$ is a path metric space then it is easy to see that $f : Y \to Y'$ is coarsely Lipschitz if and only if there exist constants $C, C' > 0$ such that for any $y_1, y_2 \in Y$ with $d_Y(y_1, y_2) \leq C$ we have $d_{Y'}(f(y_1), f(y_2)) \leq C'$.

The following two lemmas appear to be well known folklore facts:

Lemma 2.2. There exists $C > 0$ with the following property: Let $\mathcal{B}$ be a horoball in $X$, $x_1 \in X \setminus \mathcal{B}$ and $x_2 \in \mathcal{B}$. Let $b$ be the point in $\mathcal{B}$ closest to $x_1$. Then the piecewise geodesic $[x_1, b] \cup [b, x_2]$ is a $C$-rough geodesic.

Proof. Acting by isometries of $X$, we may assume that $\mathcal{B}$ is a fixed horoball that is tangent to the boundary of $X$ in the horospherical model. We may also assume that $b$ is the top-most point of $\mathcal{B}$, so that $x_1$ lies in the vertical geodesic passing through $b$. See Figure 1.

Consider the “top” of $\mathcal{B}$, i.e. the maximal subset of $\partial \mathcal{B}$ that is a graph in horospherical coordinates. Considering the Riemannian metric on $X$ in horospherical coordinates, one sees that the geodesic $[x_1, x_2]$ must pass through the top of $\mathcal{B}$. Setting $C$ to be the radius of the top of $\mathcal{B}$, centered at $b$, completes the proof. \(\square\)

Lemma 2.3. Let $\mathcal{B}_1, \mathcal{B}_2$ be disjoint horoballs, and $x_1 \in \mathcal{B}_1, x_2 \in \mathcal{B}_2$. Let $[b_1, b_2]$ be the minimal geodesic between $\mathcal{B}_1$ and $\mathcal{B}_2$. Then $[x_1, b_1] \cup [b_1, b_2] \cup [b_2, x_2]$ is a $C$-rough geodesic, for the value of $C$ in Lemma 2.2.

Proof. The proof is analogous to that of Lemma 2.2. We may normalize the horoballs $\mathcal{B}_1, \mathcal{B}_2$ as in Figure 2. The normalization depends only on the distance...
Lemma 2.2. for $X = \mathbb{H}^2_R$.

$\text{Figure 1}$

Lemma 2.3. for $X = \mathbb{H}^2_R$.

$\text{Figure 2}$

$d(B_1, B_2)$. Any geodesic $[x_1, x_2]$ must then pass through compact regions near $b_1$ and $b_2$. Let $C(B_1, B_2)$ be the radius of this region in $B_1$. Fixing $B_1$ and varying $B_2$, set $C = \sup C(B_1, B_2)$. The value $C(B_1, B_2)$ remains bounded if the distance between the horoballs goes to infinity (converging to the constant $C$ in Lemma 2.2). Thus, the infimum is attained and $C < \infty$. This completes the proof.

Lemma 2.4. Let $B_1, B_2$ be disjoint horoballs, $x_1 \in B_1$, $x_2 \in B_2$. Denote the minimal geodesic between $B_1$ and $B_2$ by $[b_1, b_2]$. Then $d(x_1, b_1) \leq d(x_1, x_2)$. Proof. Fix $D > 0$ and allow $B_1, B_2, x_1 \in B_1$, and $x_2 \in B_2$ to vary with the restriction $d(x_1, x_2) = D$. Define a function $f$ on the interval $[0, D]$ by

$$f(t) = \sup \{d(x_1, b_1) : d(B_1, B_2) = t\},$$

where the supremum is over all combinations of the variables with the restriction stated above, and $b_1$ denotes the closest point of $B_1$ to $B_2$. Then $f$ is a decreasing function, since increasing $t$ pushes the horoballs farther apart and forces $x_1$ closer to $x_2$. In particular, $f(D) = 0$ since necessarily $x_1 = b_1$. Conversely, $f(0) = D$, taking $x_2 = b_1 = b_2$. We then have for any choice of disjoint $B_1, B_2$ and $x_1, x_2$ in the corresponding horoballs, that

$$d(x_1, b_1) \leq f(d(B_1, B_2)) \leq d(x_1, x_2) = D,$$

as desired.

2.2. Truncated Spaces.

Definition 2.5. Let $X \neq \mathbb{H}^2_R$ be a negatively curved rank one symmetric space. A truncated space $\Omega$ is the complement in $X$ of a set of disjoint open horoballs. A truncated space is equivariant if there is a (non-uniform) lattice $\Gamma \subset \text{Isom}(X)$ that leaves $\Omega$ invariant, with $\Omega/\Gamma$ compact.

We will consider $\Omega$ with the induced path metric $d_\Omega$ from $X$. Under this metric, curvature remains negative in the interior of $\Omega$. The curvature on the boundary need not be negative. For an extensive treatment of truncated spaces, see [16].
Remark 2.6. Note that truncated spaces are, in general, not uniquely geodesic. Specifically, if $X$ is not a real hyperbolic space, then components of $\partial \Omega$ (which come from horospheres in $X$) are isometrically embedded in $(\Omega, d_\Omega)$ copies of non-uniquely-geodesic Riemannian metrics on certain nilpotent groups. In particular, $(\Omega, d_\Omega)$ is not necessarily a CAT(0)-space.

Remark 2.7. Let $X$ be a negatively curved rank one symmetric space and $\Gamma \subset \text{Isom}(X)$ a non-uniform lattice. Then $X/\Gamma$ is a finite-volume manifold with cusps. In $X$, each cusp corresponds to a $\Gamma$-invariant family of horoballs. Removing the horoballs produces an equivariant truncated space $\Omega$ whose quotient $\Omega/\Gamma$ is the compact core of $X/\Gamma$.

Proposition 2.8. Let $X$ be a negatively curved rank one symmetric space and $\Omega \subset X$ an equivariant truncated space. Then the inclusion $\iota: (\Omega, d_\Omega) \hookrightarrow (X, d_X)$ is a coarse embedding.

Proof. Since $d_\Omega$ and $d_X$ are path metrics with the same line element, we have

$$d_X(x, y) \leq d_\Omega(x, y)$$

To get the lower bound, define an auxiliary function

$$\beta(s) = \max \{d_\Omega(x, y) : x, y \in \Omega \text{ and } d_X(x, y) \leq s\}.$$  

Let $K$ be a compact fundamental region for the action of $\Gamma$ on $\Omega$. Because $\Gamma$ acts on $\Omega$ by isometries with respect to both metrics $d_X$ and $d_\Omega$, we may equivalently define $\beta(s)$ by

$$\beta(s) = \max \{d_\Omega(x, y) : x \in K, y \in \Omega \text{ and } d_X(x, y) \leq s\}.$$  

Because $K$ is compact and the metrics $d_X$, $d_\Omega$ are complete, $\beta(s) \in (0, \infty)$ for $s \in (0, \infty)$. Furthermore, $\beta: [0, \infty] \to [0, \infty]$ is continuous and increasing, with $\beta(0) = 0$. Because horospheres have infinite diameter for both $d_X$ and $d_\Omega$ (they are isometric to appropriate nilpotent Lie groups with left-invariant Riemannian metrics, see [16]), we also have $\beta(\infty) = \infty$.

Let $\beta'$ be an increasing homeomorphism of $[0, \infty]$ with $\beta'(s) \geq \beta(s)$ for all $s$ and consider its inverse $\alpha(t)$. For $x, y \in \Omega$ we then have

$$d_\Omega(x, y) \leq \beta(d_X(x, y)) \leq \beta'(d_X(x, y)),$$

$$\alpha(d_\Omega(x, y)) \leq d_X(x, y).$$

This concludes the proof. □

Remark 2.9. A more precise quantitative version of Proposition 2.8 can be obtained by studying geodesics in $\Omega$, see [10].

2.3. Mappings between truncated spaces. For this section, let $\Omega \subset X$ be a truncated space, with $X \neq \mathbb{H}^2_\mathbb{R}$, and $f: \Omega \to \Omega'$ a $d_\Omega$-quasi-isometric embedding. To ease the exposition, we refer to the target truncated space as $\Omega' \subset X'$.

Lemma 2.10 (Schwartz [15]). There exists $C > 0$ so that for every boundary horosphere $\sigma$ of $\Omega$, there exists a boundary horosphere $\sigma'$ of $\Omega'$ such that $f(\sigma)$ is contained in a $C$-neighborhood of $\sigma'$.

Using nearest-point projection, we may assume $f(\sigma) \subset \sigma'$. 
Definition 2.11. Let $\mathcal{B}, \mathcal{B}'$ be horoballs with boundaries $\sigma, \sigma'$. A point in $\sigma$ corresponds, in horospherical coordinates, to a geodesic ray in $\mathcal{B}$. A map $\sigma \to \sigma'$ then extends to a map $\mathcal{B} \to \mathcal{B}'$ in the obvious fashion.

In view of Lemma 2.10 a $d_{UL}$-quasi-isometric embedding $f : \Omega \to \Omega'$ likewise extends to a map $f : X \to X'$ by filling the map on each boundary horoball.

Lemma 2.12 (Schwartz [16]). A quasi-isometry $f : \sigma \to \sigma'$ induces a quasi-isometry $\mathcal{B} \to \mathcal{B}'$, with uniform control on constants.

Idea of proof. One considers the metric on the horospheres of $\mathcal{B}$ parallel to $\sigma$, or alternately fixes a model horosphere and varies the metric. One then shows that if $f$ is a quasi-isometry with respect to one of the horospheres, it is also a quasi-isometry with respect to the horospheres at other horo-heights. One then decomposes the metric on $\mathcal{B}$ into a sum of the horosphere metric and the standard metric on $\mathbb{R}$, in horospherical coordinates. This replacement is coarsely Lipschitz, so the extended map is also coarsely Lipschitz. Taking the inverse of $f$ completes the proof. □

3. Compactly Supported Cohomology

Definition 3.1. Let $X$ be a simplicial complex and $K_i \subset X$ nested compacts with $\bigcup_i K_i = X$. Compactly supported cohomology $H^*_c(X)$ is defined by

$$H^*_c(X) = \lim\limits_{\to} H^*(X, X \setminus K_i).$$

For a compact space $X$, $H^*_c(X) = H^*(X)$ but the two do not generally agree for unbounded spaces. We have $H^n_c(\mathbb{R}^n) = \mathbb{Z}$ and $H^n_c(\Omega) = 0$ for a non-trivial truncated space $\Omega$. In fact, one has the following lemma.

Lemma 3.2. Let $Z \subset \mathbb{R}^n$ be a closed subset. Then $H^n_c(Z) \neq 0$ if and only if $Z = \mathbb{R}^n$.

Proof. It is well-known that the choice of nested compact sets does not affect $H^n_c(Z)$. Choose the sequence $K_i = \overline{B(0,i)} \cap Z$, the intersection of a closed ball and $Z$. With respect to the subset topology of $Z$, the boundary of $K_i$ is given by $\partial_Z K_i := \partial K_i \cap \partial \overline{B(0,i)}$. We have by excision

$$H^n(Z, K_i) = H^n(K_i, \partial_Z K_i) = \tilde{H}^n(K_i/\partial_Z K_i).$$

Note that $K_i \subset \overline{B(0,i)}$ and $\partial_Z K_i \subset \partial \overline{B(0,i)}$, so $K_i/\partial_Z K_i \subset \overline{B(0,i)}/\partial B(0,i)$. Thus, if $K_i \neq \overline{B(0,i)}$, then $K_i/\partial_Z K_i \subset S^n \setminus \{+\}$. That is, $K_i/\partial_Z K_i$ is a compact set in $\mathbb{R}^n$, and $H^n(K_i/\partial_Z K_i) = 0$. Thus, if $Z = \mathbb{R}^n$, we have $H^n_c(Z) = \mathbb{Z}$. Otherwise, $H^n_c(Z) = 0$. □

Compactly supported cohomology is not invariant under quasi-isometries or uniform embeddings. The remainder of this section is distilled from [14], where compactly supported cohomology is generalized to a theory invariant under uniform embeddings. For our purposes, the basic ideas of this theory, made explicit below, are sufficient.

Definition 3.3. Let $X$ be a simplicial complex with the standard metric assigning each edge length 1. Recall that a chain in $X$ is a formal linear combination of simplices. The support of a chain is the union of the simplices that have non-zero coefficients in the chain. The diameter of a chain is the diameter of its support.
An acyclic metric simplicial complex $X$ is $k$-uniformly acyclic if there exists a function $\alpha$ such that any closed chain with diameter $d$ is the boundary of a $k+1$-chain of diameter at most $\alpha(d)$. If $X$ is $k$-uniformly acyclic for all $k$, we say that it is uniformly acyclic.

Likewise, we say that a metric simplicial complex $X$ is $k$-uniformly contractible if there exists a function $\alpha$ such that every continuous map $S^k \to X$ with image having diameter $d$ extends to a map $B^{k+1} \to X$ with diameter at most $\alpha(d)$. If $X$ is $k$-uniformly contractible for all $k$, we say it is uniformly contractible.

**Remark 3.4.** Rank one symmetric spaces and nilpotent Lie groups (with left-invariant Riemannian metrics) are uniformly contractible and uniformly acyclic.

**Lemma 3.5.** Let $X$, $Y$ be uniformly contractible and geometrically finite metric simplicial complexes and $f : X \to Y$ a uniform embedding. Then there exists an iterated barycentric subdivision of $X$ and $R > 0$ depending only on the uniformity constants of $f, X$, and $Y$ such that $f$ is approximated by a continuous simplicial map with additive error of at most $R$.

**Proof.** We first approximate $f$ by a continuous (but not simplicial) map by working on the skeleta of $X$. Starting with the 0-skeleton, adjust the image of each vertex by distance at most 1 so that the image of each vertex of $X$ is a vertex of $Y$. Next, assuming inductively that $f$ is continuous on each $k$-simplex of $X$, we now extend to the $k+1$ skeleton using the uniform contractibility of $Y$. Since error was bounded on the $k$-simplices, it remains bounded on the $k+1$-skeleton.

Now that $f$ has been approximated by a continuous map, a standard simplicial approximation theorem replaces $f$ by a continuous simplicial map, with bounded error depending only on the geometry of $X$ and $Y$ (see for example the proof of Theorem 2C.1 of [13]).

**Lemma 3.6.** Let $X$ and $Y$ be uniformly acyclic simplicial complexes and $f : X \to Y$ a uniform embedding. Suppose furthermore that $f$ is a continuous simplicial map. Then if $H^n_c(X) \cong H^n_c(fX)$.

**Proof.** We first construct a left inverse $\rho$ of the map $f_* : C_*(X) \to C_*(fX)$ induced by $f$ on the chain complex of $X$, up to a chain homotopy $P$. That is, $P$ will be a map $C_*(X) \to C_{*+1}(X)$ satisfying, for each $c \in C_*(X)$, the homotopy condition

$$\partial P c = c - \rho f_* c - P \partial c$$

and furthermore with diameter of $Pc$ controlled uniformly by the diameter of $c$.

We start with the 0-skeleton. Each vertex $v' \in fX$ is the image of some vertex $v \in X$ (not necessarily unique). Set $\rho(v') = v$, and extend by linearity to $\rho : C_0(fX) \to C_0(X)$. To define $P$, let $v$ be an arbitrary vertex in $X$ and note that $\partial v = 0$. We have to satisfy $\partial P v = v - \rho f_* v$. Since $X$ is acyclic, there exists a 1-chain $P v$ satisfying this condition. Furthermore, note that $\rho f_* v$ is, by construction, a vertex such that $f(\rho f_* v) = f(v)$. Since $f$ is a uniform embedding, $d(\rho f_* v, v)$ is uniformly bounded above. Thus, $P v$ may be chosen using uniform acyclicity so that its diameter is also uniformly bounded above.

Assume next that $\rho$ and $P$ are defined for all $i < k$ with uniform control on diameters. Let $\sigma$ be a $k$-simplex in $X$. Then $\partial \rho f_* \sigma$ is a chain in $X$ whose diameter is bounded independently of $\sigma$. Then, by uniform acyclicity there is a chain $\sigma'$ with
\[ \partial \sigma' = \partial pf_\ast \sigma. \] We define \( \rho(\sigma) = \sigma' \). As before, we need to link \( \sigma' \) back to \( \sigma \). We have
\[ (3.3) \quad \partial(\sigma - \sigma' - \partial P \sigma) = \partial \sigma - \rho f_\ast \partial \sigma - \partial Pf \sigma. \]
By the homotopy condition 3.2 we further have
\[ (3.4) \quad \partial(\sigma - \sigma' - \partial P \sigma) = \partial \sigma - \rho f_\ast \partial \sigma - (\partial \sigma - \rho f_\ast \partial \sigma - \partial Pf \sigma) = 0. \]
Thus, by bounded acyclicity there is a \( k + 1 \) chain \( P \sigma \) such that
\[ (3.5) \quad \partial P \sigma = \sigma - \sigma' - \partial P \sigma, \]
as desired. We extend both \( \rho \) and \( P \) by linearity to all of \( C_k(fX) \) and \( C_k(X) \), respectively.

To conclude the argument, let \( K \) be a compact subcomplex of \( X \) and consider the complex \( X/(X \setminus K) = K/\partial K \). The maps \( P \) and \( \rho \circ f_\ast \) on \( C_\ast(X) \) induce maps on \( C_\ast(K/\partial K) \), and the condition \( \partial Pf \ast + \rho \partial \sigma = c - \rho f_\ast c \) remains true for the induced maps and chains.

Because chain-homotopic maps on \( C_\ast \) induce the same maps on homology, we have, for \( h \in H_\ast(K/\partial K) \), \( h = \rho f_\ast h \). Conversely, \( f_\ast \rho \) is the identity on cell complexes, so still the identity on homology. Thus, \( H_\ast(K/\partial K) \cong H_\ast(fK/\partial fK) \). By duality, \( H^\ast(fK/\partial fK) \cong H^\ast(K/\partial K) \).

Taking \( K_i \) to be an exhaustion of \( X \) by compact subcomplexes and taking a direct limit, we conclude that \( H^\ast_c(X) \cong H^\ast_c(fX) \).

**Corollary 3.7.** Let \( X \) and \( Y \) be uniformly acyclic simplicial complexes and \( f : X \rightarrow Y \) a uniform embedding. There exists an \( R > 0 \) depending only on the uniformity constants of \( f, X, \) and \( Y \) so that \( H^n_c(N_R(fX)) \cong H^n_c(X) \).

**Proof.** Lemma 3.5 approximates \( f \) by a continuous simplicial map, within uniform additive error. Lemma 3.6 shows that the resulting approximation induces an isomorphism on compactly supported cohomology.

**Theorem 3.8** (Coarse co-Hopficity). Let \( (X, d_X) \) be a manifold homeomorphic to \( \mathbb{R}^n \), with \( d_X \) a path metric that is uniformly acyclic and uniformly contractible. For each pair of non-decreasing functions \( \alpha, \omega : [0, \infty) \rightarrow \mathbb{R} \) with \( \alpha(t) < \omega(t) \) and \( \lim_{t \rightarrow \infty} \alpha(t) = \infty \), there exists a \( C' \) such that any \((\alpha, \omega)\)-coarse embedding \( f : X \rightarrow \tilde{X} \) is \( C' \)-coarsely surjective.

**Proof.** By Corollary 3.7, there is a uniform \( R > 0 \) such that \( H^n_c(N_R(fX)) \cong H^n_c(X) \cong \mathbb{Z} \). By Lemma 3.4, \( N_R(fX) = X \). Taking \( C' = C + 2R \) completes the proof.

4. MAIN RESULT

**Theorem 4.1** (Quasi-Isometric co-Hopficity). Let \( \Omega \subset X \) and \( \Omega' \subset X' \) be equivariant truncated spaces and \( f : (\Omega, d_\Omega) \rightarrow (\Omega', d_{\Omega'}) \) a quasi-isometric embedding. Then \( f \) is coarsely surjective with respect to the truncated metric \( d_\Omega \).

**Proof.** By Lemma 2.11, we may assume that \( f \) maps boundary horospheres of \( \Omega \) to boundary horospheres of \( \Omega' \). By Theorem 3.8, \( f \) is a surjection up to a constant independent of the boundary horosphere in question. We then have an extension \( F : X \rightarrow X' \), as in Definition 2.11.

\[ \]
By Lemma 2.12, for each boundary horoball $B$, the restriction $F|_B$ is a quasi-isometry. By assumption, $F|_\Omega$ is a $d_\Omega$-quasi-isometry, so $F|_\Omega$ is a $d$-uniform embedding by Proposition 2.8. Since $X$ is a path metric space, $F$ is then coarsely Lipschitz on all of $X$.

We now show that $F$ is a uniform embedding by establishing a lower bound for distances between image points. Recall that all distances are measured with respect to $d = d_X$ unless another metric is explicitly mentioned.

Let $L \gg 2$ so that $F$ is coarsely $L$-Lipschitz and $F|_B$ is coarsely $L$-co-Lipschitz for every boundary horoball $B$. Let $\alpha, \omega$ be increasing proper functions so that $f$ is an $(\alpha, \omega)$-uniform embedding.

Let $x_1, x_2 \in X$ with $d(x_1, x_2) \gg 0$. We need to provide a lower bound for $d(Fx_1, Fx_2)$ in terms of $d(x_1, x_2)$. Clearly, the lower bound will go to $\infty$ since $F$ is an isometry along vertical geodesics in horoballs. There are four cases to consider; in all cases we can ignore additive noise by working with sufficiently large $d(x_1, x_2)$ and slightly increasing $L$.

1. Let $x_1, x_2 \in B$ for the same horoball $B$. Then $d(f(x_1, x_2)) > d(x_1, x_2)/L$.
2. Let $x_1, x_2 \in \Omega$. This case is controlled by the uniform embeddings $\Omega \hookrightarrow X$ and $\Omega' \hookrightarrow X'$ (Proposition 2.8) and the $d_\Omega$-quasiisometry constants of $f$.
3. Let $x_1 \in \Omega, x_2 \in B$ for a horoball $B$. Let $b \in B$ be the closest point to $x_1$. Then by Lemma 2.2 $[x_1, b] \cup [b, x_2]$ is a $C$-quasi-geodesic for a universal $C$ depending only on $X$ and $X'$ (see also Figure 1). We consider two sub-cases: Suppose that $d(x_1, b) > d(x_1, x_2)/L^3$. Let $b' \in fB$ be the closest point to $f x_1$. Then by definition of $b$, we have

$$d(f^{-1} b', x_1) \geq d(b, x_1) \geq d(x_1, x_2)/L^3.$$ 

Using Lemma 2.4, we conclude

$$d(f(x_1, x_2)) \geq d(b', f x_2) \geq \alpha(d(f^{-1}, x_2)) \geq \alpha(d(x_1, x_2)/L^3).$$

Suppose, instead, that $d(x_1, b) \leq d(x_1, x_2)/L^3$. Then we have the estimate $d(f(x_1, f b)) \leq d(x_1, x_2)/L^2$. We also have $d(x_2, b) \sim d(x_1, x_2)$, so $d(f x_2, f b) \geq d(x_1, x_2)/L$. Consider now $b' \in B$, the closest point to $f x_1$. By Lemma 2.2 $d(f b, f b') \leq d(f b, f x_1)$. Thus,

$$d(f(x_1, x_2)) \geq d(x_1, x_2)/L - d(x_1, x_2)/L^2.$$ 

4. Let $x_1 \in B_1, x_2 \in B_2$ be in disjoint horoballs. This case is identical to the previous one, except one uses Lemma 2.2 rather than 2.2.

We have then provided a lower bound for $d(Fx_1, Fx_2)$ for any pair of points $x_1, x_2 \in X$. Thus, the extended map $F$ is a coarse embedding. By Theorem 3.8 $F$ is then coarsely surjective. Namely, there exists $R > 0$ so that $N_R(F(X)) = X'$ (the neighborhood is taken with respect to $d$).

We now show that the coarse surjectivity of $F$ with respect to $d$ implies the coarse surjectivity of $f$ with respect to $d_\Omega$.

Let $\omega' \in \Omega'$ be an arbitrary point. Since $F$ is coarsely surjective, there exists $x \in X$ so that $d_X(f(x), \omega') \leq R$. If $x \in \Omega$, then we have shown that $\omega' \in N_R(f(\Omega))$. Otherwise, $x$ is contained in a horoball associated with $\Omega$. In appropriate horospherical coordinates, the horoball is given by $S \times (t_0, \infty)$ and $x$ can be written as $(s_1, t_1)$, with $t_1 > t_0$. Likewise, $f(x)$ has coordinates $(s'_1, t'_1)$, with $(t'_1 > t'_0)$. Furthermore, we have $f(s_1, t_0) = (s'_1, t_0)$. Now, $\omega' \in \Omega'$, so it has horospherical coordinates
\[(s'_1, t'_2) \text{ with } t'_2 < t'_0. \text{ It is easy to see that} \]

\[
R \geq d_{X'}(\omega', (s'_1, t'_0)) \geq d_{X'}(\omega', (s'_1, t'_0)) = d_{X'}(\omega', f(s_1, t_0)) \geq d_{X'}(\omega', f(\Omega)).
\]

Thus, for an arbitrary \(\omega' \in \Omega'\) we have \(d_{X'}(\omega', f(\Omega)) \leq R\). Because \(\Omega' \hookrightarrow X'\) is a uniform embedding, this implies that \(f : \Omega \to \Omega'\) is coarsely surjective. \(\square\)

**References**

1. I. Belegradek, *On co-Hopfian nilpotent groups*, Bull. London Math. Soc. **35** (2003), no. 6, 805–811. MR 2000027 (2004i:20060)

2. I. Belegradek and A. Szczepański, *Endomorphisms of relatively hyperbolic groups*, Internat. J. Algebra Comput. **18** (2008), no. 1, 97–110, With an appendix by Oleg V. Belegradek. MR 2394723 (2009a:20069)

3. M. Bonk, and O. Schramm, *Embeddings of Gromov hyperbolic spaces*. Geom. Funct. Anal. **10** (2000), no. 2, 266–306, MR 1771428 (2001g:53077)

4. H. Bourdon, and M. Pajot, *Poincaré inequalities and quasiconformal structure on the boundary of some hyperbolic buildings*. Proc. Amer. Math. Soc. **127** (1999), no. 8, 2315–2324. MR 1610912 (99j:30024)

5. M. Bourdon, *Immeubles hyperboliques, dimension conforme et rigidité de Mostow*, Geom. Funct. Anal. **7** (1997), no. 2, 245–268. MR 1445387 (98c:20056)

6. M. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 319, Springer-Verlag, Berlin, 1999. MR 1744486 (2000k:53038)

7. M. R. Bridson, D. Groves, J. A. Hillman, and G. J. Martin, *Confinitely Hopfian groups, open mappings and knot complements*. Groups Geom. Dyn. **4** (2010), no. 4, 693–707. MR 2727659 (2011j:20103)

8. P. de la Harpe, *Topics in geometric group theory*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 2000. MR 1786869 (2001i:20081)

9. T. Delzant and L. Potyagailo, *Endomorphisms of Kleinian groups*, Geom. Funct. Anal. **13** (2003), no. 2, 396–436. MR 1982149 (2004c:20087)

10. D. Epstein, J. Cannon, D. Holt, S. Levy, M. Paterson, and W. Thurston, *Word processing in groups*, Jones and Bartlett Publishers, Boston, MA, 1992. MR 1161094 (93e:20036)

11. A. Eskin, *Quasi-isometric rigidity of nonuniform lattices in higher rank symmetric spaces*, J. Amer. Math. Soc. **11** (1998), no. 2, 321–361. MR 1475886 (98g:22005)

12. A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002. MR 1867354 (2002k:55001)

13. M. Kapovich and B. Kleiner, *Coarse Alexander duality and duality groups*, J. Differential Geom. **69** (2005), no. 2, 279–352. MR 2168506 (2007c:57033)

14. S. Merenkov, A *Sierpiński carpet with the co-Hopfian property*, Invent. Math. **180** (2010), no. 2, 361–388. MR 2609245 (2011c:30054)

15. R. E. Schwartz, *The quasi-isometry classification of rank one lattices*, Inst. Hautes Études Sci. Publ. Math. (1995), no. 82, 133–168 (1996). MR 1383215 (97c:22014)

16. Z. Sela, *Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups. II*, Geom. Funct. Anal. **7** (1997), no. 3, 561–593. MR 1466338 (98j:20044)

17. Z. Sela, *Endomorphisms of hyperbolic groups. I. The Hopf property*, Topology **38** (1999), no. 2, 301–321. MR 660337 (99m:20081)
