On the parameters of Lewis metric
for the Lewis class

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Abstract

The physical and geometrical meaning of the four parameters of the Lewis metric for the Lewis class are investigated. Matching this spacetime to a completely anisotropic, rigidly rotating, fluid cylinder, we obtain from the junction conditions that the four parameters are related to the vorticity of the source. Furthermore it is shown that one of the parameters must vanish if one wishes to reduce the Lewis class to a locally static spacetime. Using the Cartan scalars it is shown that the Lewis class does not include globally Minkowski as special class, and that it is not locally equivalent to the Levi-Civita metric. Also it is shown that, in contrast with the Weyl class, the parameter responsible for the vorticity appears explicitly in the expression for the Cartan scalars. Finally, to enhance our understanding of the Lewis class, we analyse the van Stockum metric.

1 Introduction

In a recent paper [1], we have discussed about the physical meaning of the four parameters of the Lewis metric for the Weyl class. In this work we endeavour to extend such discussion to the Lewis class. Such an effort will be justified, in part, by the deep differences, exhibited below, between the two classes.

Matching the Lewis class to a cylindrical fluid, a relationship linking the vorticity of the source with three of the real constants entering into
the definition of the four complex parameters of the Lewis class is found. Furthermore it will be shown that the vanishing of the vorticity implies the vanishing of the parameter responsible for the non-staticity of the metric.

From the study of the Cartan scalars it will be shown that, in contrast with the Weyl class, the Lewis class is locally distinguishable from the Levi-Civita metric. Also it will be shown the Lewis class does not include locally flat spacetime as special class. Therefore topological strings (in locally flat spacetime) cannot be associated with this metric, hindering thereby the topological interpretation of some of the parameters, in contrast with the Weyl class [1].

The three classes of van Stockum metric [2, 3] provide an excellent example for our discussion. We shall propose a lower limit for the linear mass density, representing the frontier between the Lewis class and the Weyl class for the van Stockum metric.

2 Spacetime

The spacetime is divided into two regions: the interior, with $0 \leq r \leq R$, to a cylindrical $\Sigma$ surface of radius $R$ centered along $z$; and the exterior, with $R \leq r < \infty$. Both regions are described by the general cylindrically symmetric stationary metric

$$ds^2 = -f dt^2 + 2k dt d\varphi + e^\mu \left( dr^2 + dz^2 \right) + ld \varphi^2,$$  \hspace{1cm} (2.1)
where \( f, k, \mu \) and \( l \) are functions only of \( r \), and the ranges of the coordinates \( t, z \) and \( \varphi \) are

\[
-\infty < t < \infty, \quad -\infty < z < \infty, \quad 0 \leq \varphi \leq 2\pi,
\]

(2.2)

with the hypersurfaces \( \varphi = 0 \) and \( \varphi = 2\pi \) being identified. The coordinates are numbered

\[
x^0 = t, \quad x^1 = r, \quad x^2 = z, \quad x^3 = \varphi.
\]

(2.3)

Einstein’s field equations

\[
R_{\mu \nu} = \kappa \left( T_{\mu \nu} - \frac{1}{2} g_{\mu \nu} T \right)
\]

(2.4)

will be imposed to the metric (2.1). The components of the Ricci tensor \( R_{\mu \nu} \) for (2.1) are given in the reference [1]. The exterior spacetime is constituted of vacuum, hence Einstein’s equations (2.4) reduce to

\[
R_{\mu \nu} = 0.
\]

(2.5)

The general solution of (2.5) for (2.1) is the stationary Lewis metric [1], which can be written as [2]

\[
f = a r^{-n+1} - \frac{c^2}{n^2 a} r^{n+1},
\]

(2.6)
\[ k = -Af, \quad (2.7) \]
\[ l = \frac{r^2}{f} - A^2f, \quad (2.8) \]
\[ e^\mu = r^{\frac{1}{2}}(n^2-1), \quad (2.9) \]

with
\[ A = \frac{cr^{n+1}}{naf} + b. \quad (2.10) \]

The constants \( n, a, b \) and \( c \) can be either real or complex, the corresponding solutions belong to the Weyl class or Lewis class, respectively. The Weyl class was studied in [1]. Here we restrict our study to the Lewis class. In this case, these constants are given by

\[ n = im, \quad (2.11) \]
\[ c = \frac{m}{2} \left( a_1^2 + b_1^2 \right), \quad (2.12) \]
\[ a = \frac{1}{2} \left( a_1^2 - b_1^2 \right) + ia_1b_1, \quad (2.13) \]
\[ b = \frac{a_1a_2 + b_1b_2}{a_1^2 + b_1^2} + \frac{i}{a_1^2 + b_1^2}, \quad (2.14) \]

where \( m, a_1, b_1, a_2 \) and \( b_2 \) are real constants and satisfy

\[ a_1b_2 - a_2b_1 = 1. \quad (2.15) \]

The equations \((2.11)-(2.15)\) reveal us that if it is known the value of the parameters \( n \) and \( a \), or \( n \) and \( b \), we can obtain the parameter \( c \). However knowing \( n \) and \( c \) we cannot obtain \( a \) and \( b \). The metric coeffcients \((2.6)-(2.9)\)
with (2.11)–(2.14) become [4]

\[
f = r \left( a_1^2 - b_1^2 \right) \cos (m \ln r) + 2ra_1b_1 \sin (m \ln r), \quad (2.16)
\]

\[
k = -r (a_1a_2 - b_1b_2) \cos (m \ln r) - r (a_1b_2 + a_2b_1) \sin (m \ln r), \quad (2.17)
\]

\[
l = -r (a_2^2 - b_2^2) \cos (m \ln r) - 2ra_2b_2 \sin (m \ln r), \quad (2.18)
\]

\[
e^\mu = r^{-\frac{1}{2}}(m^2+1). \quad (2.19)
\]

In fact, this metric is a subclass of the Kasner type metrics, as pointed out in [6].

### 3 Vorticity

Using the transformation

\[
d\varphi = d\bar{\varphi} + \omega dt, \quad \text{where} \quad \omega = -\frac{k}{l}, \quad (3.1)
\]

the metric (2.1) can be diagonalized. In order to have an integral coordinate transformation \(\omega\) must be constant, therefore from equations (2.17)–(2.18), \(m = 0\). This implies, from (2.11)–(2.12), that \(n = 0\) and \(c = 0\). Thus the line element becomes

\[
ds^2 = \frac{r}{a_2^2 - b_2^2} dt^2 + r^{-\frac{1}{2}} \left( dr^2 + dz^2 \right) - r \left( a_2^2 - b_2^2 \right) d\bar{\varphi}^2. \quad (3.2)
\]
This is a particular case of the static Levi-Civita metric with the energy density per unit length $\sigma$, given by (5.26), equals $\frac{1}{4}$. Nevertheless the transformation (3.1) is not global, since the new coordinate $\bar{\phi}$ ranges from $-\infty$ to $\infty$ instead of ranging from 0 to $2\pi$ [1, 7].

Considering the interior spacetime of the cylinder, $0 \leq r \leq R$, filled with anisotropic fluid, then we can integrate one of the Einstein’s equations (2.4) and obtain

$$\xi r = f k' - k f',$$

(3.3)

where $\xi$ is a constant. $\xi$ measures the vorticity of the source, since a straightforward calculation [1] shows that the magnitude of the vorticity tensor is $\xi/(2fe^2)$. Considering $f$ and $k$ given by (2.16) and (2.17), we have

$$\xi = -m \left( a_1^2 + b_1^2 \right).$$

(3.4)

So,

$$c = -\frac{\xi}{2}.$$  

(3.5)

Hence in order to have the vorticity equals to zero, i.e. $\xi = 0$, we need $m = 0$ since $a_1^2 + b_1^2 \neq 0$.

Observe the difference, at this point, between the Weyl class [1] and the Lewis class. In the latter the vanishing of the vorticity yields a locally Levi-Civita spacetime, whereas in the former the vanishing of vorticity does not, necessarily, implies that the metric can be reduced to a globally or locally
4 The Cartan scalars

It is known [8] that the so-called 14 algebraic invariants (and even all the polynomial invariants of any order) are not sufficient for locally characterizing a spacetime, in the sense that two metrics may have the same set of invariants and be not equivalent. As an example, all these invariants vanish for both Minkowski and plane-wave [4, 8] spacetimes and they are not the same. A complete local characterization of spacetimes may be done by the Cartan scalars. Briefly, the Cartan scalars are the components of the Riemann tensor and its covariant derivatives (up to possibly the $10^{th}$ order) calculated in a constant frame. For a review, see [1] and references therein. In practice, the Cartan scalars are calculated using the spinorial formalism. For the purpose here, the relevant quantities are the Weyl spinor $\Psi^A$, and its first covariant symmetrized derivative $\nabla \Psi_{AB'}$, which represent the Weyl tensor and its covariant derivative. It should be stressed that, although the Cartan scalars provide a local characterization of the spacetime, global properties such as topological defects do not probably appear in them.

In a previous paper [1] the Cartan scalars for the Weyl class of Lewis metric are given. The Lewis class metric may be obtained from the Weyl class metric by considering the constants $a$, $b$, $c$ and $n$ as complex, subjected to side relations (2.11)–(2.15) to assure that the metric components remain real.
Therefore, the Cartan scalars for the Lewis class can be obtained from those of the Weyl class by a proper redefinition of the constants. As in the Weyl class, only the constant \( n \) appears in the Cartan scalars. Nevertheless, here, \( n \) must be substituted by its complex value \((2.11) i m\). The nonvanishing Cartan scalars are:

\[
\begin{align*}
\Psi_2 & = \frac{1}{8}(m^2 + 1)r^{\frac{3}{2}}(m^2 - 3), \\
\Psi_0 = \Psi_4 & = -i m \Psi_2, \\
\nabla \Psi_{01'} = \nabla \Psi_{50'} & = -\frac{\sqrt{2}}{16} i m (m^4 - 1)r^{\frac{3}{2}}(m^2 - 3), \\
\nabla \Psi_{10'} = \nabla \Psi_{41'} & = \frac{\sqrt{2}}{8} i m (m^2 + 1)r^{\frac{3}{2}}(m^2 - 3), \\
\nabla \Psi_{21'} = \nabla \Psi_{30'} & = \frac{\sqrt{2}}{32} (m^2 - 3)(m^2 + 1)r^{\frac{3}{2}}(m^2 - 3).
\end{align*}
\]

This provides an invariant criterion distinguishing the Lewis and the Weyl classes, since for the Lewis class, for instance, \( \Psi_0 = -i m \Psi_2 \) while, for the Weyl class [1], \( \Psi_0 = -n \Psi_2 \), where \( m \) and \( n \) are arbitrary real constants.

Contrary to the Weyl class [1], the Cartan scalars for the Lewis class are distinguishable from those of the Levi Civita metric, except for \( m = 0 \) (cf. section 3). Furthermore there is no value of \( m \) for which the Cartan scalars are all zero, implying at once that the Lewis class does not include Minkowski as special class.

On the other hand, the van Stockum exterior solution [2, 3] (case I), which is a particular case of the Lewis metric, contains the globally Minkowski spacetime as special case.\footnote{Although in the form presented in this paper, it is not obvious that this special case}
be a particular case of the Weyl class. Since the case I of the van Stockum exterior solution cannot be reduced to the globally static Levi-Civita metric (neither case II and III), it is clear that it is a particular case of the Weyl class with \( b \neq 0 \) and \( c \neq 0 \), since for \( b = 0 \) and \( c = 0 \) the Weyl class can be globally reduced to the static Levi-Civita metric. In the next section we shall make a more detailed analysis of the van Stockum solution, which will help us to improve our comprehension on the parameters of the Lewis metric.

5 The van Stockum’s metric classification

In 1937 van Stockum [2] solved the problem of a rigidly rotating infinite cylinder filled with dust and matched it to the vacuum Lewis solution. The solution depends on the parameter \( wR \), related to the mass per unit length of the dust cylinder (note that in [2, 3], the letter \( a \) is used instead of \( w \)), and is given by, for \( wR < \frac{1}{2} \) (case I),

\[
\begin{align*}
  f &= -r \left[ 2 \beta \cosh (2N \ln r) + \frac{\alpha^2 + \beta^2}{\alpha} \sinh (2N \ln r) \right], \\
  k &= -r \left[ \cosh (2N \ln r) + \frac{\beta}{\alpha} \sinh (2N \ln r) \right], \\
  l &= \frac{r}{\alpha} \sinh (2N \ln r), \\
  e^\mu &= \lambda \left( \frac{r}{R} \right)^{(2N^2 - \frac{1}{2})},
\end{align*}
\]

occurs, it is quite simple to obtain this limit in the original form [3].
and for $wR > \frac{1}{2}$ (case III),

\[ f = r \left[ 2\beta \sin (2N \ln r) + \frac{\alpha^2 - \beta^2}{\alpha} \cos (2N \ln r) \right], \quad (5.5) \]

\[ k = r \left[ \sin (2N \ln r) - \frac{\beta}{\alpha} \cos (2N \ln r) \right], \quad (5.6) \]

\[ l = \frac{r}{\alpha} \cos (2N \ln r), \quad (5.7) \]

\[ e^\mu = \lambda \left( \frac{r}{R} \right)^{-2N^2 + \frac{1}{2}}, \quad (5.8) \]

The constants $\alpha$, $\beta$, $N$ and $\lambda$ are given by, for case I,

\[ N = \frac{1}{2} \sqrt{1 - 4w^2R^2}, \quad (5.9) \]

\[ \alpha = \frac{\sqrt{1 - 4w^2R^2}}{2w^3R^4}, \quad (5.10) \]

\[ \beta = -\frac{1 - 2w^2R^2}{2w^3R^4}, \quad (5.11) \]

\[ \lambda = e^{-w^2R^2}, \quad (5.12) \]

and for case III,

\[ N = \frac{1}{2} \sqrt{4w^2R^2 - 1}, \quad (5.13) \]

\[ \alpha = \frac{\sqrt{4w^2R^2 - 1}}{2w^3R^4}, \quad (5.14) \]

\[ \beta = \frac{2w^2R^2 - 1}{2w^3R^4}, \quad (5.15) \]

\[ \lambda = e^{-w^2R^2}. \quad (5.16) \]
Case II, i.e., \( wR = \frac{1}{2} \) is defined by van Stockum \[2\] by a limiting process of case I. We add that it is also a limit of case III. Nevertheless, it should be stressed that the direct substitution of \( wR = \frac{1}{2} \) in cases I or III does not give the proper result. The van Stockum solution I belongs to the Weyl class, where the real parameters \( n, a, b \) and \( c \) assume the following values,

\[
\begin{align*}
n &= \sqrt{1 - 4w^2R^2}, \\
a &= \frac{(\alpha - \beta)^2}{2\alpha}, \\
b &= \pm\frac{1}{\alpha - \beta}, \\
c &= \frac{(\alpha^2 - \beta^2)}{\alpha}N,
\end{align*}
\]

while the van Stockum solution III belongs to the Lewis class, where the real parameters \( m, a_1, b_1, a_2 \) and \( b_2 \) assume the following values,

\[
\begin{align*}
m &= \sqrt{4w^2R^2 - 1}, \\
b_2 &= 0, \\
a_2 &= -\frac{1}{b_1}, \\
a_1 &= \frac{\beta}{b_1}, \\
b_1^2 &= -\alpha.
\end{align*}
\]

In order to understand better the Lewis class metric let us consider shortly the Weyl class metric. In \[3\] it is shown that the Newtonian mass per unit
length is given by

\[ \sigma = \frac{1}{4} (1 - n), \quad (5.26) \]

with \( n \) being a real constant. So, from (5.17) and (5.26), we have

\[ \sigma = \frac{1}{4} \left[ 1 - \sqrt{1 - 4w^2R^2} \right] \quad (5.27) \]

for case I. For \( w^2R^2 \ll 1 \), this expression reduces to

\[ \sigma = \frac{1}{2} w^2R^2, \quad (5.28) \]

This is the same value obtained by Bonnor [3] in this approximation. Using this result, he establishes a lower limit for the linear mass density in case III \( (wR > \frac{1}{2}) \), obtaining \( \frac{1}{8} \). We believe that a better lower limit would be given directly by (5.27), which is \( \sigma = \frac{1}{4} \). Then \( \sigma = \frac{1}{4} \) represents the frontier between the Weyl class metric and the Lewis class metric at least for the particular case of van Stockum.

Returning to the Lewis class metric, it is important to note that the Cartan scalars do not admit Minkowski spacetime. This is in accordance with the existence of a lower limit for \( \sigma \) in the van Stockum solution III, since with this lower limit the source cannot be made vacuum and therefore the exterior solution cannot be Minkowski.
The Cartan scalars impose a superior limit to the parameter $m$, given by

$$m \leq \sqrt{3}, \quad (5.29)$$

since for $m$ larger than this value, the singularity is at $r = \infty$, not in $r = 0$. When we substitute this value in (5.21), considering the equality, we have that $wR = 1$, which agrees with Bonnor’s result [3].

6 Conclusion

It is known that the Lewis metric comprises two different families called Weyl class and Lewis class. The first one occurs when we consider that all four parameters appearing in the metric are real. On the other hand, in the second one, some of the parameters may be complex. On figure 1 we present a diagram showing some of the subclasses of the Lewis metric, according to the value of the parameters.

In a previous paper [1] we obtained physical interpretations for the four real parameters $n, a, b$ and $c$ which characterize the Lewis metric for the Weyl class. The parameters $b$ and $c$ are related to the non-staticity of the spacetime, since when we take $b = 0$ and $c = 0$ the Weyl class reduces to the static Levi-Civita metric. While the parameter $n$ is associated to the Newtonian mass per unit length of an uniform line mass $\sigma$ when it produces low densities. So if we assume $n = 1$, which means $\sigma = 0$, the Weyl class metric becomes a static locally flat spacetime or a stationary locally flat
spacetime if we take \( b = 0 \) and \( c = 0 \) or \( b \neq 0 \) and \( c \neq 0 \), respectively. The parameter \( c \) measures the vorticity of the source of the Weyl class metric when it is matched to a general stationary completely anisotropic fluid. Finally, the parameter \( a \) was interpreted as the constant arbitrary potential that exists in the corresponding Newtonian solution and a linear energy density along a string in the locally flat static limit of the Weyl class. When \( a = 1 \) the locally flat static spacetime reduces to the globally Minkowski spacetime.

In this paper we give more attention to the Lewis class metric. We show that, as in the Weyl class, the parameter \( c \) is also proportional to the vorticity of the source of the Lewis class metric if we match this spacetime to the same general stationary completely anisotropic fluid. Moreover for the Lewis class we find that the vorticity is also directly associated with the parameter \( n \) and indirectly related with the parameters \( a \) and \( b \) and, in fact, the vorticity vanishes if \( m = 0 \). We verify that the van Stockum solution for vacuum is a particular case of the Lewis metric. Indeed, the van Stockum case I is a subclass of the Weyl class metric while the van Stockum case III is a subclass of the Lewis class metric. The case II of the van Stockum solution can be obtained from the case I or from the case III by a limiting process when \( n \to 0 \). As can be seen in the Cartan scalars, the Lewis class can never be reduced to the locally flat spacetime.
Figure 1: Lewis metrics and their subclasses. Note that flat means locally flat. * means that equations (5.17)–(5.20) should be satisfied. ** means that equations (5.21)–(5.25) should be satisfied.

7 Acknowledgment

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A Appendix

We present below the components of the Riemann tensor for the Lewis class, in the same Lorentz frame used in [1]

\begin{align}
R_{0101} = -R_{2323} &= \frac{1}{4}(m^2 + 1)r^{\frac{3}{2}}(m^2-3) \\
R_{1313} = -R_{0202} &= \frac{1}{8}(im + 1)(m^2 + 1)r^{\frac{3}{2}}(m^2-3) \\
R_{0303} = -R_{1212} &= \frac{1}{8}(im - 1)(m^2 + 1)r^{\frac{3}{2}}(m^2-3)
\end{align}

(A.1)

Note that some of these components are complex. This present no problem since the frame used becomes complex for the Lewis class. For completeness, we list the nonvanishing algebraic invariants, which are of course real:

\begin{align}
R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} &= -\frac{1}{4}(m^2 - 3)(m^2 + 1)^2 r^{(m^2-3)} \\
R_{\alpha\beta\gamma\delta}R^{\gamma\delta\mu\nu}R_{\mu\nu}^{\alpha\beta} &= \frac{3}{16}(m^2 + 1)^4 r^{\frac{3}{2}}(m^2-3)
\end{align}

(A.2)

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