Spin orbit-induced anisotropic conductivity of a disordered 2DEG.

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We present a semi-automated computer-assisted method to generate and calculate diagrams in the disorder averaging approach to disordered 2D conductors with intrinsic spin-orbit interaction (SOI). As an application, we calculate the effect of the SOI on the charge conductivity for disordered 2D systems and rings in the presence of Rashba and Dresselhaus SOI. In an infinite-size 2D system, anisotropic corrections to the conductivity tensor arise due to phase-coherence and the interplay of Rashba and Dresselhaus SOI. The effect is more pronounced in the quasi-one-dimensional case, where the conductivity becomes anisotropic already in the presence of only one type of SOI. The anisotropy further increases if the time-reversal symmetry of the Hamiltonian is broken.

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I. INTRODUCTION

Spin-orbit interaction (SOI) is a promising tool for manipulating spin degrees of freedom via electric field; because of that, it plays an important role in various novel microelectronic devices. In 2D semiconductor microstructures, Rashba$^2$ and Dresselhaus$^3$ types of SOI are the most important ones. In phase-coherent diffusive systems, their dominant effect on the transport properties is the so-called anti-localization$^4,5$—isotropic SOI-induced correction to the conductivity tensor, leading to the sign change of the phase-coherent correction to the conductivity. Rashba and Dresselhaus SOI terms equally and independently contribute to the weak anti-localization correction. The anisotropic contribution to the conductivity tensor is a more subtle effect (arising in the next order in the weak-disorder expansion). In an infinite-size 2D conductor it comes from the interference between Rashba and Dresselhaus SOI$^6$.

A pure Dresselhaus SOI affects 2D electron systems in the same way as pure Rashba SOI with the same amplitude. However, it is known that in a system with mixed (Rashba-Dresselhaus) type of SOI their action is not independent. This becomes most pronounced in the special case when the amplitude of Rashba SOI is equal to the Dresselhaus one$^6,7$. In order to highlight the interference between the Rashba and Dresselhaus SOI, it is convenient to consider effects, which arise entirely due to this interference. An example is the anisotropic contribution to the conductivity of a diffusive (unconfined) 2D electron gas, which is zero in case when only one type of SOI is present in the system$^8$.

Both the (isotropic) antilocalization correction and the SOI-induced anisotropic correction are phase coherent effects; hence it is not surprising that in a fully phase coherent system both of them depend singularly on SOI amplitudes. In the limit of small SOI the weak localization correction diverges both in 2D and quasi-1D cases, while the behavior of the anisotropic correction depends on system’s geometry: in an infinite 2D-system with time-reversal symmetry it remains finite also when SOI is infinitesimal, and in a quasi-1D case it diverges with the vanishing SOI. Moreover, while in an infinite 2D disordered slab, two different types of SOI (Rashba and Dresselhaus) are required in order to make the anisotropic component of the conductivity tensor non-zero, in a quasi-1D geometry (see Sec. X) the SOI-induced anisotropy of the conductivity tensor arises also for the case when the energy spectrum is isotropic (i.e., when only one type of SOI is present), despite the fact that all dimensions of a quasi-1D sample are much larger than the mean-free path $l$ of an electron. Thus, in the phase-coherent regime the macroscopic shape anisotropy of the sample results in the (microscopic) anisotropy of the conductivity tensor. The SOI is still required, but the energy spectrum does not need to be anisotropic.

According to the theorem by Vollhardt and Wölfle (proved in for the spinless case), diffuson propagator poles do not contribute to the conductivity if the system is invariant under time reversal. We extend the validity of this theorem to the spinful case in Appendix D. From this theorem one may expect the appearance of uncompensated diffusion divergences in systems with broken time reversal symmetry, which would then result in enhancement of the SOI-correction to the conductivity tensor. We indeed observe such an enhancement in the example of a ring pierced by a magnetic flux, where the time-reversal symmetry is broken due to presence of the vector potential (see Sec. XII).

We perform our calculations using the disorder averaging diagrammatic techniques$^{10,11,12,13}$. In problems with spin, a summation over spin indices produces huge expressions which cannot be handled manually anymore in a reasonable time. We have overcome this problem by developing a symbolic-calculation program$^{14}$ that (i) generates diagrams having the requested number of loops, (ii) calculates the Hikami-boxes, and finally (iii) performs the integration over the cooperon and diffusion momenta. The first two stages of the program are universal, i.e., can be readily used for other calculations in a diagrammatic approach. The program significantly
facilitates the usage of the diagrammatics, especially in the spinful case.

The first part of the paper is not specific to the problem of anisotropic conductivity. In Sec. II we define the model which we use in our calculation, then we introduce disorder-averaged Green functions (see Sec. III) and derive the Kubo-Greenwood formula in the Keldysh technique (see Sec. IV). Then we derive the loop expansion for the diagrammatic technique in Sec. V and derive expressions for diffusons (cooperons) in Sec. VI. The second part of the paper starts with Sec. VII where we derive the incoherent SOI-correctation to the conductivity tensor. We then proceed with the contribution from the weak-localization diagrams (which remains isotropic at zero frequency) in Sec. VIII. The results for the anisotropic transport in 2D and quasi-1D geometries in the presence of the time-reversal invariance are described in Sec. IX for the case of zero frequency \( \omega = 0 \) and in Sec. X for \( \omega \neq 0 \). Finally, in Sec. XI we give an example of how the effect of SOI-induced anisotropy could be enhanced by the time-reversal symmetry breaking terms.

For convenience we summarize some often used notations in Table I at the end of the paper.

II. THE HAMILTONIAN

Rashba and Dresselhaus spin-orbit interaction terms modify the Hamiltonian as follows:

\[
\hat{H}' = \frac{\hat{p}^2}{2m} + \hat{V}' + U'(\mathbf{r}), \quad V'_s = aV'_k + bV'_D, \tag{1}
\]

where \( \hat{p} \) denotes momentum operator, \( a \) and \( b \) are Rashba and Dresselhaus amplitudes, and \( U'(\mathbf{r}) \) is the disorder potential created by impurities or defects randomly placed in the sample. We assume that \( U'(\mathbf{r}) \) is uncorrelated:

\[
\langle U'(\mathbf{r})U'(\mathbf{r}') \rangle = \frac{\hbar^2}{2\pi\nu\tau} \delta(\mathbf{r} - \mathbf{r}'), \tag{2}
\]

where \( \tau \) is the mean time between collisions of an electron off impurities, \( \nu \) is the density of states (DoS) at the Fermi level, and the over-bar indicates average over the different disorder configurations. The Dresselhaus SOI is invariant under arbitrary rotation in the \((x, y)\)-plane:

\[
V'_k = \hat{z} \cdot [\sigma \times \mathbf{p}] \equiv (\hat{z}, \sigma, \mathbf{p}) = \sigma_1 \hat{p}_y - \sigma_2 \hat{p}_x
\]

\[
= (\hat{z}, R^z_0 \sigma, R^z_0 \mathbf{p}) = \begin{pmatrix} 0 & p_y + ip_x & 0 \\ p_y - ip_x & 0 & 0 \end{pmatrix}, \quad \forall \phi, \tag{3}
\]

where \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) is composed of Pauli matrices, and \( R^z_\phi \) denotes a \( 3 \times 3 \) matrix describing rotation by an angle \( \phi \) around the \( z \)-axis. The Dresselhaus SOI term can be written as

\[
V'_D = (\hat{z}, C_{\sigma, \mathbf{p}}) = \sigma_1 \hat{p}_y - \sigma_2 \hat{p}_x, \tag{4}
\]

\[
C = R^-_{\pi/2} R^z_\phi \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

In the coordinate system (CS), rotated by an arbitrary angle \( \phi \) around the \( z \)-axis with respect to the initial CS, the SOI part of the Hamiltonian is transformed into

\[
V_s(\mathbf{p}, \sigma) = V'_D(R^z_\phi \mathbf{p}, R^z_\phi \sigma) = (\hat{z}, a\sigma + bR^z_\phi C \sigma, \mathbf{p}). \tag{5}
\]

In case when \( \phi = -\pi/4 \), \( V_D \) becomes similar to Rashba-SOI, so that the SOI term can be written as

\[
R^-_{\pi/4} CR^z_{\phi} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{6}
\]

\[
V_s \equiv V_s(\mathbf{p}, \sigma) = (a - b)\sigma_1 \hat{p}_y - (a + b)\sigma_2 \hat{p}_x = \hat{s}\Delta/2, \tag{7}
\]

where the SOI-induced spectrum-splitting \( \hat{\Delta} \) and the helicity (spiralality) operators are defined as

\[
\hat{\Delta} = 2\sqrt{\hat{s}^2(a + b)^2 + \hat{p}_z^2(a - b)^2}, \quad \hat{s} = 2\left[(a - b)\hat{p}_y\sigma_1 - (a + b)\sigma_2\hat{p}_x\right] / \Delta, \quad \hat{s}_z = \mathbb{1}. \tag{8}
\]

The original disorder-free Hamiltonian in the rotated CS may be written in the form \( \hat{p}^2/(2m) + \hat{s}\hat{\Delta}/2 \); it possesses the following eigensystem:

\[
|\mathbf{p}, s\rangle = \left( \frac{\sin\Delta s/2}{(a + b)p_x + (a - b)p_y} \right) |\mathbf{p}\rangle, \quad s = \pm 1, \tag{9}
\]

\[
E_{\mathbf{p}, s} = \langle \mathbf{p}, s | \hat{H}_0 | \mathbf{p}, s \rangle = \frac{\hat{p}^2}{2m} + \frac{s\hat{\Delta}_p}{2}, \tag{10}
\]

\[
\Delta_p = \langle \mathbf{p}, s | \hat{\Delta} | \mathbf{p}, s \rangle = 2\sqrt{\hat{s}^2(a + b)^2 + \hat{p}_z^2(a - b)^2}. \tag{11}
\]

We see that the simultaneous presence of Rashba and Dresselhaus SOI leads to an anisotropy of the energy spectrum. On the other hand, \( \hat{\Delta} \) is symmetric with respect to the exchange \( a \leftrightarrow b \); conversely, the same is true for SOI-induced corrections to the conductivity tensor. We note that helicity is invariant with respect to the time reversal:

\[
\sigma_2 \hat{s}^T_{-\mathbf{p}} = \hat{s}_p \equiv \langle \mathbf{p} | \hat{s} | \mathbf{p} \rangle. \tag{12}
\]

In the rest of the paper we perform calculations in the coordinate system, rotated by \( \pi/4 \) in the \( xy \) plane, where the (unperturbed) Hamiltonian is given by

\[
\hat{H} = \frac{\hat{p}^2}{2m} + \frac{\hat{s}\hat{\Delta}}{2} + U(\mathbf{r}), \tag{13}
\]

where \( U(\mathbf{r}) = U'(R^z_\phi \mathbf{r}) \) is the disorder potential in the rotated coordinate system. Our assumption that \( U'(\mathbf{r}) \)
a δ-correlated random disorder potential is inherited by the disorder potential in the rotated coordinate system:

\[ \frac{U(r)U(r')} = \frac{\hbar^2}{2\pi \nu T} \delta(r - r'). \] (13)

The Hamiltonian (12) defines the velocity operator \( \hat{v} \) together with the “fictitious” vector potential \( \hat{A} \):

\[
\hat{v} = \frac{i}{\hbar} [H, r] = \frac{\hat{p}}{m} - \frac{e}{mc} \hat{A},
\]

\[
\hat{A} = \frac{mc}{e} [(a + b)\sigma_2, (b - a)\sigma_1, 0],
\] (14)

so that

\[
\frac{\hat{p}^2}{2m} + \frac{s\hat{A}}{2} = \frac{m\delta^2}{2} - m(a^2 + b^2).
\] (15)

The strength of the SOI can be characterized by dimensionless Rashba and Dresselhaus amplitudes introduced as follows:

\[ x_a = 2p_F a \tau / \hbar, \quad x_b = 2p_F b \tau / \hbar, \quad p_F = \sqrt{2m\mu}, \] (16)

where \( \mu \) is the (temperature-dependent) chemical potential. Alternatively, the SOI can be characterized by another set of two dimensionless parameters:

\[ x = \sqrt{x_a^2 + x_b^2}, \quad \delta = \frac{2ab}{a^2 + b^2}, \quad -1 \leq \delta \leq 1, \] (17)

which characterizes the “total” SOI amplitude and the anisotropy of the energy spectrum (10) correspondingly. The choice between two parameter sets (16) and (17) becomes important when we have to expand expressions in Taylor series. On the example of the spectrum splitting \( \Delta_p \) defined in (10) we see the advantage of the choice (17): while the Taylor expansion of \( \Delta_p \) in powers of \( (x, \delta) \) is uniform, its expansion in the parameters (16) is non-uniform: the expansion depends on the fact if one expands subsequently in \( x_a, x_b \) or in \( x_b, x_a \).

### III. AVERAGED GREEN FUNCTION IN THE SELF-CONSISTENT BORN APPROXIMATION

In our calculations, we use the electron-gas model of the Fermi liquid\(^{15}\). In the absence of SOI and applied electric field, disorder-averaged Green functions are obtained from the self-consistent Born approximation\(^ {11}\):

\[ g_{rs}^{E\sigma}(p) = \left[ E - \xi_p \pm \frac{i}{2\tau} \right]^{-1}, \quad \hbar \xi_p = \frac{p^2}{2m} - \mu, \] (18)

where \( \tau \) has small \( (\sim E_h/\mu) \) dependence on frequency \( E \). The presence of the SOI changes the expression for Green function (GF) from (18) into

\[ G_{rs}^{E}(p) = g_{rs}^{E\sigma}(p) \sum_{s=0} g_{ss}^{E\sigma}(p)/\hbar \]

\[ = \left[ \sigma_0 \left[ g_{rs}^{E\sigma}(p) \right]^{-1} - s_p^2 \Delta_p / \hbar \right]^{-1} \] (19)

where \( \sigma_0 \) is the 2x2 unity matrix, and

\[ g_{rs}^{E\sigma}(p) = \left( \left[ g_{s\sigma}^{E\sigma}(p) \right]^{-1} \pm \frac{\Delta_p}{\hbar} \right) \] (20)

From (19) we see that averaged GF for the Hamiltonian (12) can be obtained from the GF of the disorder-free Hamiltonian by substituting infinitesimal “epsilon” in the denominator with \((2\tau)^{-1}\). Thus for energies close to \( \mu \), the averaged GFs are very different from GFs of the disorder-free system; the latter are strongly modified due to the disorder. In fact, Eq. (18) is the result of the summation of an infinite perturbation series.

We calculate the universal contribution (37) to the conductivity tensor (see Sec. [V] below): the corresponding momentum integrals converge in the vicinity of the Fermi level, so that the momentum arguments of all Green functions are close to \( p_F \). The assumption \( p \approx p_F \) simplifies the SOI-term \( V_s \) in the GF-expression (19). In the zeroth order (in powers of \( \hbar \xi_p / \mu \ll 1 \))

\[ V_s(p) \approx p_F \left[ (a + b)\sigma_1 \sin \phi - (a - b)\sigma_2 \cos \phi \right], \] (21)

where \( \phi \) is the angular coordinate of \( p \). This approximation is sufficient for the calculation of the weak localization and two-loop correction in Sec. [IX] and [X]. However, in the calculation of the zero-loop contribution (see Sec. [VIII]), higher accuracy is required:

\[ V_s(p) \approx p_F \left[ 1 + \frac{\hbar \xi_p}{2\mu} \right] \times \]

\[ \times \left[ (a + b)\sigma_1 \sin \phi - (a - b)\sigma_2 \cos \phi \right]. \] (22)

### IV. (NON)UNIVERSAL CONTRIBUTIONS TO THE CONDUCTIVITY

We calculate the universal (i.e., independent of the details of the energy spectrum far from the Fermi level) corrections to the conductivity tensor. The latter quantity is derived in linear response in the applied electric field (see Sec. [V]). In the diagrammatic approach, the SOI-induced correction to the conductivity can be graphically represented as a sum of diagrams. A contribution of an individual diagram is initially expressed as an integral in both frequency and momentum over the combination of Green functions and the distribution function. We call such an integral universal if its leading contribution comes from the part of the integration space, where all momentum and frequency arguments of GFs in the integrand are close to the Fermi level. In the momentum space this means \( |p - p_F| \lesssim \hbar |l| \); in the frequency space \( \hbar |E| \lesssim T \). (\( p_F \) is the Fermi momentum, and \( T \) is the temperature in equilibrium or effective temperature in a non-equilibrium case.) Then the integration in momentum space can be performed assuming constant averaged DoS \( \nu \) and approximating \( \int d^2p/(2\pi \hbar)^2 \approx \nu \int_{-\infty}^{\infty} d\xi \).
where in 2D $v = m/(2\pi\hbar)$. According to the Fermi liquid theory, only electrons with energies near the Fermi level behave like free electron gas so that the effect of the interaction between electrons can be disregarded. Thus only the universal corrections are expected to give reasonable physical results. The non-universal contributions (i) cannot be reliably calculated and (ii) cannot cancel universal contributions. In the diagrammatics, arbitrary universal contributions to Hikami boxes can be calculated. Unfortunately, this is not true for non-universal corrections: some of them can be considered within the diagrammatics\(^\text{22}\), others are too complicated to be calculated. The impossibility to take into account all non-universal contributions may leave an impression of imperfection of the diagrammatic technique. However, one should note that the situation is not better in the non-linear $\sigma$-model\(^\text{23}\), where all approximations we use in the diagrammatics are required as well.

V. NON-EQUILIBRIUM KUBO FORMULA IN KELDYSH TECHNIQUE

The (mean) current density in a system, characterized by a one-particle density matrix (DM) $\hat{\rho}$ is given by

$$j(t) = \text{Tr} \left[ \hat{\rho}(t) \hat{j} \right], \quad \hat{j} = e \hat{\Psi},$$

(23)

where $\hat{j}$ and $\hat{\Psi}$ denote current and velocity operators. We proceed with calculations in momentum representation:

$$j(t) = \int \frac{d^2p}{(2\pi\hbar)^2} \int \frac{d^2p'}{(2\pi\hbar)^2} \text{Tr}_{\text{spin}} \left[ \langle p' | \hat{\rho}(t) | p' \rangle \langle p | \hat{j} | p \rangle \right],$$

$$= \delta(p - p') e \frac{\partial \hat{H}_0(p)}{\partial p}, \quad \hat{H} = \hat{H}_0 + \delta \hat{V},$$

(24)

where $\hat{H}_0$ is the unperturbed Hamiltonian, and the perturbation term $\delta \hat{V}$ describes the applied electric field, see (27) below. It is convenient to express the DM in terms of GFs (see §2.1 from Ref. 21).

$$\langle \lambda | \hat{\rho}(t) | \lambda' \rangle = \langle \Psi_t^T(\lambda; t) \Psi_t(\lambda'; t) \rangle = -i \lim_{\tau \to t} \langle \lambda | \hat{G}_\tau^c(t', t) | \lambda' \rangle =$$

$$= -i \lim_{\tau \to t} \langle \lambda' | \left( \hat{G}_\tau - \hat{G}_\tau \right) | (t', t) \rangle = - \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t}$$

$$\times \left[ \frac{\hbar}{2} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \langle \lambda' | \left( \hat{G}_E - \hat{G}_E \right) | (E, E - \omega) \rangle | \lambda \rangle, \right.$$  

(25)

where $\lambda \equiv (p, s)$. We assume that the unperturbed DM $\hat{\rho}^{(0)}$ is stationary (though not necessary equilibrium) and is characterized by energy distribution function $f_\beta$. Thus, the zero-order DM is time-independent, and the zero-order GFs are homogeneous in time.

The perturbation [see (27) below] affects both $\hat{\rho}(t)$ and $\hat{j}$ in (23). We call the correction to $\hat{j}$ “diamagnetic part of the current operator” $\hat{j}_D$; the unperturbed part of $\hat{j}$ we call “normal part of the current operator” $\hat{j}_N$:

$$\langle p' | \hat{j}_D | p \rangle = \delta(p - p') e \frac{\partial V}{\partial p},$$

$$\langle p' | \hat{j}_N | p \rangle = \delta(p - p') e \frac{\partial \hat{H}_0}{\partial p}.$$

(26)

When the system is perturbed by external electric field $E = -\frac{1}{c} \frac{\partial A}{\partial t}$, the perturbation operator is given by

$$\langle p' | \delta \hat{V} | p \rangle = \delta(p - p') \frac{1}{2m} \left\{ \left[ p - e c (A + \tilde{A}) \right]^2 - \left[ p - e c \tilde{A} \right] \right\} = \frac{e}{mc} \left[ p - e c \tilde{A} \right] \delta(p - p')$$

(27)

$$= - \frac{1}{c} A \langle p' | \hat{j}_N | p \rangle, \quad \langle p' | \hat{j}_D | p \rangle = - \frac{e}{mc} \delta(p - p') A.$$

Let us denote the Keldysh-contour time ordered GF as $\hat{\mathcal{G}}$. The applied electric field affects $\hat{\mathcal{G}}$; the first-order correction is given by

$$\delta^{(1)} \mathcal{G}(E, E - \omega) = \mathcal{G}^E \left[ - \frac{e}{c h} \lambda A_\omega \right] \mathcal{G}^{E - \omega},$$

(28)

Expressing (28) in a usual $2 \times 2$-matrix form\(^\text{22}\), we get perturbation expressions for GFs in (25).

$$\delta^{(1)} (\hat{G}_R - \hat{G}_A)(E, E - \omega) =$$

$$= \left[ \frac{e}{c h} \frac{e}{c} \mathcal{G}^{E - \omega} - \mathcal{G}^E \lambda A \mathcal{G}^{E - \omega} \right],$$

(29)

$$\delta^{(1)} G_K(E, E - \omega) = - \sum_{\beta = 1}^2 \frac{e A_\beta}{c h} \left[ (h_E - h_{E - \omega}) \hat{G}_R^E \lambda \hat{G}_A^{E - \omega} + h_{E - \omega} \hat{G}_R^E \lambda \hat{G}_A^{E - \omega} - h_E \hat{G}_A^E \lambda \hat{G}_R^{E - \omega} \right],$$

(30)

In equilibrium

$$h_E = \begin{cases} 0, & \text{tanh} \frac{E h}{2T}, \quad E h < -\mu, \\ \tanh \frac{E h}{2T}, & \text{tanh} \frac{E h}{2T}, \quad E h \geq -\mu. \end{cases}$$

(31)

where $T$ is the temperature (measured in energy units). We split the DM into “normal” and “diamagnetic” parts [like we did with the current operator in (23)]:

$$\langle \lambda | \delta \hat{\rho}_N(\omega) | \lambda' \rangle =$$

$$i \sum_{\beta = 1}^2 \frac{e A_\beta}{c h} \left[ \int_{-\infty}^{\infty} \frac{dE}{2\pi} \langle \lambda' | \left( h_E - h_{E - \omega} \right) \hat{G}_R^E \lambda \hat{G}_A^{E - \omega} \rangle | \lambda \rangle, \right.$$  

(32)

$$\langle \lambda | \delta \hat{\rho}_D(\omega) | \lambda' \rangle =$$

$$i \sum_{\beta = 1}^2 \frac{e A_\beta}{c h} \left[ \int_{-\infty}^{\infty} \frac{dE}{2\pi} \langle \lambda' | \left( h_{E - \omega} - 1 \right) \hat{G}_R^E \lambda \hat{G}_A^{E - \omega} \rangle | \lambda \rangle,$$

(33)
Then we rewrite (23) in the frequency space, substituting the SOI-dependent part of (38) is non-universal (the momentum correction). This complicates the calculation of (38). The quantities except for the Drude conductivity, see Appendix A, we can calculate in the momentum representation. From

\[
\text{Tr} \left[ v_R \hat{C}_R^E \hat{G}_R^E \right] = \frac{i}{\hbar} \text{Tr} \left[ \left( \alpha \beta \right) \hat{G}_R^E - \hat{G}_R^E \hat{G}_R^E \right] = \frac{i}{\hbar} \text{Tr} \left[ \left( \alpha \beta \right) \hat{G}_R^E \right] = -\delta_{ij} \text{Tr} \hat{E}^E
\]

we conclude that

\[
\text{at } \omega = 0 \quad \text{Tr} \left[ \delta_{ij} \right] = 0.
\]

Both equilibrium and non-equilibrium contributions to the persistent current in a mesoscopic ring are given by the trace of \(\left[ \delta \hat{\rho}_0 \right] \), while the trace of \(\left[ \delta \hat{\rho}_N (\omega) \right] \) is the linear response to the applied electric field.

In the rest of this section we assume that \( \alpha \) is directed along the \( \beta \)-axis, and we measure the charge current in the \( \alpha \)-direction. Then (for arbitrary energy distribution \( f_e \))

\[
\sigma_{\alpha \beta} (\omega) = \sigma_{\alpha \beta}^N (\omega) + \sigma_{\alpha \beta}^D (\omega),
\]

\[
\sigma_{\alpha \beta}^N (\omega) = \frac{e^2}{2\omega A_\alpha} \text{Tr} \left[ \delta \hat{G}_N (\omega) \right] = \frac{e^2}{\hbar^2} \text{Tr} \left[ \delta \hat{G}_R^E \hat{G}_R^E \right],
\]

where we assumed that the (momentum) trace is \( E \)-independent. (This assumption is valid for all universal quantities except for the Drude conductivity, see Appendix A.) The diamagnetic correction to the conductivity is given by

\[
\sigma_{\alpha \beta}^D (\omega) = \frac{e^2}{2\omega A_\alpha} \text{Tr} \left[ \delta \hat{G}_D (\omega) \right] = \frac{e^2}{\hbar^2} \text{Tr} \left[ \delta \hat{G}_R^E + \hat{G}_R^E \right] - \hat{G}_R^E \hat{G}_R^E
\]

In the frequency integral \( \int_{-\infty}^{\infty} \frac{dE}{2\pi} \) in (38), large negative frequencies \( E \sim -\mu / \hbar \) give important contribution to the result, so that the \( E \)-dependence of \( \tau \) in (18) cannot be neglected; this complicates the calculation of (38). The SOI-dependent part of (38) is non-universal (the momenta of GFs are not bounded in the vicinity of \( p_f \)). Since (38) does not contain products of different (retarded and advanced) GFs, it cannot contain diffusons or cooperons; hence it is incoherent and cannot produce corrections to the conductivity tensor having the same order in SOI, as our results below [see Eqs. (39), (40), and (41)] show. In what follows, we study the universal contribution \( \sigma_0 \) and do not calculate \( \sigma_1 \).

An attempt to use the Kubo formula \( \sigma_0 \) for calculating the leading (Drude) conductivity contribution leads to divergences. In fact, \( \sigma_0 \) is valid only for calculating corrections (due to \( \omega \neq 0 \), SOI interaction, etc.) to the main (Drude) conductivity value. See Appendix A for details of calculating the Drude conductivity.

We derived the universal contribution to the conductivity tensor \( \sigma_0 \) for a general case of non-equilibrium distribution function. The result \( \sigma_0 \) is the same as the one derived for the equilibrium case. Thus, we see that corrections to the conductivity are independent of the distribution function \( f_e \). Note that this is not true for the leading (Drude) conductivity \( \sigma_1 \), which does depend on \( f_e \).

In what follows, we always perform calculations in the rotated coordinate system, where the spin-orbit part of the Hamiltonian is given by \( \hat{A} \). In this coordinate system, the conductivity tensor is diagonal in all considered geometries; its anisotropic part is proportional to \( \sigma_3 \). See the discussion after (29). We denote the isotropic and anisotropic parts of the conductivity tensor \( \sigma \) with symbols \( \sigma_{\alpha \beta} \) and \( \sigma_{\alpha \beta} \), respectively.

\[
\sigma = \sigma_{\alpha \beta} \sigma_{\alpha \beta} + \sigma_{\alpha \beta} \sigma_{\alpha \beta}.
\]

In an arbitrary coordinate system, the (an)isotropic properties of a 2D symmetric tensor can be characterized by two non-negative scalars \( \sigma_{\alpha \beta} \) and \( |\sigma_{\alpha \beta}| \) — isotropic and anisotropic amplitudes defined by

\[
\sigma_{\alpha \beta} = \frac{1}{2} \text{Tr} \sigma, \quad |\sigma_{\alpha \beta}| = \sqrt{\text{Tr} \left[ \sigma - \sigma_{\alpha \beta} \sigma_{\alpha \beta} \right]^2 / 2}
\]

It is easy to check that both \( \sigma_{\alpha \beta} \) and \( |\sigma_{\alpha \beta}| \) are independent of the choice of the coordinate system. Finally, we give explicit expressions for the anisotropic part \( \sigma_{\alpha \beta} \) of the conductivity in the original and rotated by \( \pi/4 \) coordinate systems:

\[
\text{in the original CS } \sigma_{\alpha \beta} = \sigma_{\alpha \beta} + \sigma_{\alpha \beta}, \quad \text{and } \sigma_{\alpha \beta} + \sigma_{\alpha \beta} / 2
\]

VI. THE LOOP EXPANSION

It is convenient to represent the different contributions to the averaged conductivity in graphical form as diagrams. The simplest (bubble) diagram [see Fig. 2(a)] is produced by the Kubo formula \( \sigma_{\alpha \beta} \) by substituting \( \hat{G}_R \) and \( \hat{C}_A \) with the averaged GFs \( G_R \) [given by (19)] and \( G_A = [G_R]^{\dagger} \). The bubble has neither diffuson nor cooperon lines. One can proceed by connecting the retarded GF \( G_R \) of a bubble with an advanced one \( G_A \) by a cooperon or diffuson ladder in all possible (two)
ways. Doing so we obtain diagrams depicted in Figures 1 and 2 containing one cooperon and one diffuson. Adding more and more diffuson or cooperon lines in all possible ways, one obtains an infinite number of diagrams. In this section we describe how the most important diagrams can be selected out of this infinity for further calculation.

A. Two ways of drawing diagrams

In Fig. 1 the same (weak localization) diagram is drawn in two equivalent representations: on the lhs, the cooperon is drawn in the “ladder” form (see the lhs of Fig. 3(b)); while on the rhs the “coordinate” form [wavy line with two ends, see the rhs of Fig. 3(b)] is used.

The weak localization diagram is usually drawn in the lhs (ladder)-form (or as topologically-equivalent “bubble with maximally anti-crossing disorder-averaging lines”, cf. Fig. 4.8 from Ref. 10). Below we use the rhs (coordinate)-form [its advantages are discussed in Sec. VII B below]; Fig. 3 gives a recipe how a diagram can be transformed from one form into another and back.

A diagram in the coordinate representation consists of Green function boxes (bubbles, triangles, squares, pentagons, etc.) connected by wavy lines and/or diffuson lines. A vertex of a Green function box (GFB) may be occupied by a (i) observable operator, (ii) external field operator, or (iii) end of a cooperon and/or diffuson line.

B. Loops formed by cooperons and diffusons

There are two important momentum scales in the disorder averaging technique: (i) the (characteristic) absolute value of the momentum argument in averaged GFs \( p \sim p_F \), and (ii) \( h/l \ll p_F \). \( (l \) is the mean free path of an electron between two subsequent elastic scatterings off impurities.) Momentum integrals from products of GFs of the form

\[
\int \frac{d^d p}{(2\pi)^d} \prod_{i=1}^r G_R(p - q_i) \prod_{i=1}^d G_A(p - q_i) \quad (42)
\]

usually converge within the interval \( p_F - h/l \leq p \leq p_F + h/l \); hence \( h/l \) characterizes the deviation of momentum argument of an averaged GF from \( p_F \). The assumption that “large” momentum \( p_F \) is much larger than “small” momentum \( h/l \) is crucial for the disorder averaging technique, since \((p_F h/l)^{-1} \ll 1\) is its main expansion parameter (see Sec. VII C below).

The mean scattering free path \( l \) is also a scale on which averaged GFs \((18)\) and \((19)\) decay, e.g., in 3D \( G_{R/A}(r - r') \propto \exp[-|r - r'|/l] \). We can interpret this saying that the length of a Green function line in our diagrams is \( l \). Within the disorder averaging technique we cannot observe effects on scales shorter than \( l \), i.e., to say that the length of a GF-line is \( l \), is almost the same, as to say that this length is zero; thus we can consider a GF-line not as a line, but as a point.

Now if we draw some diagram in its “coordinate representation” (see Sec. VII A) and squeeze all Green function lines into points, the result will contain only cooperon or diffuson (CD) lines forming a certain number of loops. For example, a bubble in Fig. 2(a) has no loops (since it has no CD lines which could form a loop); a weak localization (WL) diagram in Fig. 1(b) has one loop, and all diagrams in Figs. 4,5 have two loops.

The number of CD-loops is equal to the number of independent “small” momentum variables (which are \( \sim h/l \), or, in other words – to the number of integrals over the CD-momentum variables.

C. Comparing two arbitrary diagrams

Let us estimate two arbitrary diagrams for the same physical quantity. An estimate for a GFB is \( \propto \nu_\nu'^{-1} \), where \( \nu \) is the number of Green function lines composing the GFB. Every CD line has a prefactor \((4\pi\nu')^{-1}\) [see Sec. VII B below]. We estimate the DoS as \( \nu \sim m/(2\pi\hbar^d) \), where \( \lambda \sim \hbar/p_F \), and \( d \) is the spatial dimension \((d = 2 \text{ or } d = 3) \). Let us denote \( L_{1,2}, H_{1,2}, C_{1,2} \) the corresponding number of loops, GFs, and CD-lines in the two considered diagrams; the quantities \( h_1 \), denote number of GF lines in the \( j \)th GFB of the first diagram, and \( h_{2n} \) do the same for the second diagram.

The calculation of diagrams is often much simpler in the diffusion approximation – i.e., assumption that \( q^2 l \ll h \), where \( q^2 \) is the characteristic momentum of a CD line (i.e., “small” momentum variable). [The validity
of the diffusion approximation in our calculation arises from the assumption that \( q^2 l \sim x h \ll h \). Sometimes a GFB gains additional smallness of the order of \( q^2 l / h \ll 1 \). One has to calculate a GFB in order to reveal how much of these “extra” \( q^2 l / h \) it has – this is not uniquely defined by the number of loops. In the following estimates we assume \( q^2 l \sim h \) in order not to mix up expansions in two different small parameters: \( q^2 l / h \ll 1 \) and \( (p_1 l / h)^{-1} \ll 1 \). Then the relation between two different arbitrary diagrams is estimated as

\[
\begin{align*}
\text{1st diagram} & \sim \int \prod_{i=1}^{L_1} \frac{d^d q_i}{(2\pi)^d} \left[ \prod_{j=1}^{H_1} 2\pi \nu_{\tau_{\nu}} h_{1j}^{-1} \right] \omega_{\nu \tau_{\nu}}^{C_1}, \\
\text{2nd diagram} & \sim \int \prod_{i=1}^{L_2} \frac{d^d q_i}{(2\pi)^d} \left[ \prod_{j=1}^{H_2} 2\pi \nu_{\tau_{\nu}} h_{2j}^{-1} \right] \omega_{\nu \tau_{\nu}}^{C_2}. 
\end{align*}
\]

We use the fact that \( L_i = C_i - H_i + 1 \) for \( i = 1, 2 \), and \( \sum_{j=1}^{H_1} h_{1j} - \sum_{j=1}^{H_2} h_{2n} = 2(C_1 - C_2) \), so that

\[
\frac{1}{(2\pi)^d} \left( \frac{\hbar}{p_{1l}} \right)^{d-1} L_1 L_2, \quad d > 1. \tag{44}
\]

As we discussed above, apart from \((p_1 l / h)^{-1} \gg 1\), there is an additional small expansion parameter \( q^2 l / h \ll 1\); so the total number of expansion parameters is two. (Later, in Sec. XI the number of small parameters is three.) The loop expansion predicts how large (small) an arbitrary diagram is only in powers of \((p_1 l / h)^{-1}\).

To conclude, the statement that “every loop brings a smallness \((p_1 l / h)^{-1}\) is known in mesoscopics; for the diagrams produced by the non-linear \( \sigma \)-mode\(^{28}\) it is explained in Sec. III.3.c of Ref. 28. However, we are not aware of earlier articles, where this statement is justified for diagrams within the usual disorder-averaging diagrammatic technique; this was the reason to include Sec. VII in this text.

### VII. ZERO-FREQUENCY COOPERON AND DIFFUSON IN THE PRESENCE OF SOI

Both cooperon and diffuson\(^{10}\) can be graphically represented as a sum of “ladders”; each “ladder” is given by retarded Green function \( G_R \) (bold line in our drawings) connected to the advanced one \( G_A \) (bold line) with some number of disorder-averaging (dashed) lines. An elementary building block of every such ladder is made of one dashed line connecting \( G_R \) with \( G_A \). Below we discuss a convenient way of rearranging spin indices in every building block of the ladder.

#### A. Separating spin indices

One can write the expression for two GF-lines connected with a disorder-averaging line in different ways; we will write it either as

\[
\begin{align*}
\text{p}_1, \alpha & \rightarrow_{G_R} \text{p}_2, \beta \rightarrow_{G_A} \text{p}_2', \alpha' \\
\text{p}_1', \alpha' & \rightarrow_{G_A} \text{p}_2', \beta' \rightarrow_{G_R} \text{p}_2, \beta
\end{align*}
\]

with

\[
\frac{1}{2\pi \nu_{\tau_{\nu}}} \left[ G_R(p_1) G_R(p_2) \right]_{\alpha \beta} \left[ G_A(p_1') G_A(p_2') \right]_{\alpha' \beta'}, \tag{45}
\]

or as

\[
\begin{align*}
\text{p}_1, \alpha & \rightarrow_{G_R} \text{p}_2, \beta \rightarrow_{G_A} \text{p}_2', \alpha' \\
\text{p}_1', \alpha' & \rightarrow_{G_A} \text{p}_2', \beta' \rightarrow_{G_R} \text{p}_2, \beta
\end{align*}
\]

with

\[
\frac{1}{2\pi \nu_{\tau_{\nu}}} \left[ G_R(p_1) G_R(p_2) \right]_{\alpha \beta} \left[ G_A(p_1') G_A(p_2') \right]_{\alpha' \beta'}, \tag{46}
\]

where the identity is used:

\[
2\delta_{\alpha \beta_1} \delta_{\beta \alpha_1} = 3 \sum_{\alpha=0}^{\sum} \sigma_\alpha \sigma_{\alpha_1} = \sum_{\alpha=0}^{\sum} \sigma_\alpha \left( \sigma_\alpha \right)^T, \quad \sigma_\alpha \equiv \sigma_{2 \alpha} - \sigma_{1 \alpha}.
\]
Every diagram with cooperons or diffusons contains an infinite number of elementary blocks\(^{(45)}\) and/or \(^{(46)}\). Below we will always separate spin variables in them according to \(^{(45)}\) or \(^{(46)}\).

### B. Defining cooperon and diffuson

Now let us transform every elementary building block in the diffuson series according to \(^{(45)}\):

\[
\begin{align*}
D_{\mathbf{q}}^{\alpha\beta} &= \frac{1}{4\pi\mathbf{v}_T} \left[ \sum_{n \geq 0} X_{\mathbf{q}}^\alpha \right]_{\alpha\beta} = \frac{1}{4\pi\mathbf{v}_T} \left[ 1 - X_D(q) \right]_{\alpha\beta}, \quad X_D^\alpha(q) = \frac{1}{4\pi\mathbf{v}_T} \int \frac{d^2p}{(2\pi\hbar)^2} \text{Tr}_{\text{spin}} \left[ \sigma_\alpha \mathbf{G}^p_E(\mathbf{p}) \sigma_\beta \mathbf{G}^p_A(\mathbf{p} - \mathbf{q}) \right],
\end{align*}
\]

where \(\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}_3\) and

\[
D_{\mathbf{q}}^{\alpha\beta} = \frac{1}{4\pi\mathbf{v}_T} \left[ \sum_{n \geq 0} X_{\mathbf{q}}^\alpha \right]_{\alpha\beta} = \frac{1}{4\pi\mathbf{v}_T} \left[ 1 - X_D(q) \right]_{\alpha\beta}, \quad X_D^\alpha(q) = \frac{1}{4\pi\mathbf{v}_T} \int \frac{d^2p}{(2\pi\hbar)^2} \text{Tr}_{\text{spin}} \left[ \sigma_\alpha \mathbf{G}^p_E(\mathbf{p}) \sigma_\beta \mathbf{G}^p_A(\mathbf{p} - \mathbf{q}) \right].
\]

where \(\mathbf{Tr}_{\text{spin}}\) stands for the trace only in spin indices. So \(^{(45)}\) helped us to convert the diffuson series into geometric series that we could sum. Analogously we use \(^{(46)}\) to transform the cooperon series:

\[
\begin{align*}
C_{\mathbf{q}}^{\alpha\beta} &= \frac{1}{4\pi\mathbf{v}_T} \left[ \sum_{n \geq 0} X_{\mathbf{q}}^\alpha \right]_{\alpha\beta} = \frac{1}{4\pi\mathbf{v}_T} \left[ 1 - X_C(q) \right]_{\alpha\beta}, \quad X_C^\alpha(q) = \frac{1}{4\pi\mathbf{v}_T} \int \frac{d^2p}{(2\pi\hbar)^2} \text{Tr}_{\text{spin}} \left[ \sigma_\alpha \mathbf{G}^p_E(\mathbf{p}) \sigma_\beta \mathbf{G}^p_A(\mathbf{p} - \mathbf{q}) \right].
\end{align*}
\]

From \(^{(11)}\) it follows that in a system with time reversal symmetry

\[
\sigma_2 \mathbf{G}^R_A(-\mathbf{p}) \sigma_2 = \mathbf{G}^R_A(\mathbf{p}),
\]

so that

\[
X_C^\alpha(q) = X_D^\alpha(q) \text{ and } C_{\mathbf{q}}^{\alpha\beta} = D_{\mathbf{q}}^{\alpha\beta}.
\]

The series \(^{(45)}\) or \(^{(50)}\) depend on four momentum and four spin variables, without external four GF lines only \(D_{\mathbf{p}_1 - \mathbf{p}_2}^{\alpha\beta} \) or \(C_{\mathbf{p}_1 + \mathbf{p}_2}^{\alpha\beta} \) is left, which is a function of two momentum and two spin variables.

We call the quantities \(D_{\mathbf{q}}^{\alpha\beta}\) and \(C_{\mathbf{q}}^{\alpha\beta}\) the diffuson and the cooperon, respectively.

Diagrammatically, \(D_{\mathbf{q}}^{\alpha\beta}\) can be drawn in two ways. The lhs-diagram in Fig. 3(a) is more similar to the series in \(^{(48)}\), while the rhs-diagram in Fig. 3(a) stresses the fact that the diffuson without external four GF lines has only two ends.

This rhs-diagram in Fig. 3(a) reflects better the spatial structure of the diffuson in the coordinate space. From Eq. \(^{(48)}\) one can see that the distance between points 1 and 2 in Fig. 3(a) is zero, and the same is true for the points 3 and 4. We merged these identical points in the rhs-
diagram in Fig. 3(a), so that 1=2 and 3=4. A diffusion at frequency \( \omega \) in coordinate space decays on the scale \( \min(|L_\omega|, L_\nu) \gg l \):

\[
L_\omega = l / \sqrt{2 i \omega t}, \quad L_\nu = l / \sqrt{2 x}, \quad L_\sigma = l / \sqrt{2 x}, \quad L_\nu \ll \hbar, \quad x \ll 1 \implies \min(|L_\omega|, L_\sigma) \gg l.
\] (54)

Thus, we see that the distance (in coordinate space) between points 1 and 2 (or 3 and 4) in Fig. 3(a) is (within our accuracy) infinitesimal, and this fact is graphically reflected by merging points 1 and 2 together (as well as points 3 and 4) in the rhs of Fig. 3(a). A similar reasoning is valid for the cooperon as well, see Fig. 3(b).

C. Explicit 2D-expressions for \( q = 0 \)

The diffusion at zero momentum, \( q = 0 \), can be calculated without assuming that the SOI amplitudes \( x_{\alpha, b} \) are small. Using the fact that

\[
G^R_0(p) + G^R_0(-p) = G_R(p) + G_R(-p),
\]
ones one obtains a sum rule:

\[
X_D^{22} = X_D^{00} - X_D^{11} + X_D^{33}.
\] (55)

For \( q = 0 \), \( X_D \) is a diagonal 4 \times 4 matrix with elements

\[
X_D^{00} = 1, \quad X_D^{11} = \frac{1 + K}{1 + (x_a + x_b)^2} K, \quad X_D^{33} = \frac{1}{K}, \quad K = \sqrt{[1 + (x_a + x_b)^2][1 + (x_a - x_b)^2]}.
\] (56)

The components of the diffusion at zero momentum are given by the diagonal matrix

\[
4 \pi i Q \nu D_0 = \frac{2 m \nu t}{\hbar} D_0 = \text{diag}\left[\frac{L_\omega}{\nu} 1 + \frac{1 + K}{x_a + x_b} 1 + \frac{1 + K}{x_a - x_b} K - 1\right],
\] (57)

where the electron (orbital) dephasing length \( L_\omega \) (due to inelastic scattering) serves as a cut-off for the infinity, and the first element \( L_\omega / \nu \) does not contribute to physical quantities when the Vollhardt-Wolff theorem holds (see Appendix D). The sum of the two diagrams in Fig. 2 is equal to the diagram in Fig. 2(a) with one velocity vertex renormalized:

![Diagram](image)

which corresponds to the expression

\[
\delta_\alpha = v_\alpha + \sum_{\gamma=1}^3 \sigma_{\gamma} D^{\gamma R}_{0} \text{Tr} \left[ \sigma_{\gamma} G_R(p) v_\alpha G_A(p) \right] = \frac{p_\alpha}{m},
\] (59)

where \( D^{\gamma R}_{0} \) are components of the zero-momentum diffusion given by (57,55).

Since the vertex renormalization of this type occurs in every diagram for the conductivity, we take it everywhere into account by substituting the velocity operator \( v_\alpha \) with its renormalized value \( \delta_\alpha = p_\alpha / m \); the only exception is the zero-loop contribution (calculated in Sec. VIII), which is represented by the diagram in Fig. 2(a) with only one velocity vertex being renormalized.

D. Explicit 2D-expressions for \( q \neq 0 \)

Because of the SOI, most components of the diffusion do not have a pole at zero momentum and frequency even if the dephasing effects are neglected; the diffusion gains a non-zero “mass”, see (57). In case of pure-Rashba or pure-Dresselhaus SOI, this “mass” is quadratic in the SOI amplitude. This still remains true in case when one (Rashba or Dresselhaus) SOI amplitude is much smaller than the other one, so that the SOI-induced anisotropy of the energy spectrum (17) \( \delta \) is a small parameter. Consequently, the integrals in diffusion momenta \( k \) and \( q \) converge on the scale of \( q \leq x/l \), and it is convenient to introduce dimensionless variables

\[
K \equiv k / x \hbar, \quad Q \equiv q / x \hbar,
\] (60)

so that \( K \leq 1 \) and \( Q \leq 1 \). Like in Sec. VIII the calculation shows that only the \((1,1)\)-minor (i.e., \(3 \times 3\) matrix block) of the \(4 \times 4\) diffusion matrix is affected by the SOI. In other words, the upper line and the left column of the diffusion matrix are independent of SOI:

\[
D_{00} = \frac{1}{2m \nu t^2 \hbar^2 / 2 \hbar - i \omega t},
\]

\[
D_{q0} = D_{aq} = 0, \quad a = 1 \ldots 3.
\] (61)

The element \( D_{00} \) gives no contribution to the conductivity in systems with time-reversal symmetry (see Appendix D); it becomes important, when this symmetry is broken, see Sec. XII. In the rest of this Section we reduce \(4 \times 4\) matrices of \( X_D \) and \( D_q \) to corresponding \((1,1)\)-minors.

To simplify the calculation, we further assume that \( x \ll 1 \) (diffusion approximation). Then the diffusion is obtained using (49) with \( X_D \) given by

\[
X_D \approx 1 - x^2 \left[ Y_Q^{(0)} - \delta Y_Q^{(1,0)} \right],
\] (62)

where \( \delta \) is defined in (17), and

\[
Y_Q^{(0)} = \frac{Q^2}{2} \left[ \begin{array}{cccc} 0 & -2iQ_x & 0 & -2iQ_y \\ 2iQ_x & 0 & 2iQ_y & 0 \\ 0 & 2iQ_y & 0 & -2iQ_x \\ -2iQ_x & 0 & 2iQ_y & 0 \end{array} \right],
\] (63)

\[
Y_Q^{(1,0)} = \frac{1}{2} \left[ \begin{array}{cccc} 1 & 1 & 0 & -iQ_y \\ 0 & 0 & -iQ_y & 0 \\ -iQ_x & -iQ_y & 0 & -iQ_x \\ 0 & iQ_y & iQ_x & 0 \end{array} \right].
\] (64)
Note that the above expression for $X_D$ is Hermitian and obeys the sum rule (55).

The resulting diffuson matrix $D_Q$ has a denominator $\propto (\det Y_Q)^n$, where $n > 0$ is an integer, and

$$8 \det Y_Q = 2 + Q^2 + Q^6$$

$$= (Q^2 + 1) \left( Q^2 - \frac{1 - i \sqrt{7}}{2} \right) \left( Q^2 - \frac{1 + i \sqrt{7}}{2} \right). \quad (65)$$

The expression (65) is independent of the direction of $Q$, so the same is true for the denominators of all diffuson components. Consequently, an arbitrary diagram containing a diffuson line with a non-zero momentum (e.g., rhs of Fig. 4) has denominators [consisting of powers of $\det Y_Q^0$, $\det Y_{K+Q'}$, and $\det Y_Q^0$], which are invariant with respect to two “mirror reflections”: (i) $(K_x \rightarrow -K_x, Q_x \rightarrow -Q_x)$, (ii) $(K_y \rightarrow -K_y, Q_y \rightarrow -Q_y)$, and (iii) $(K_x \leftrightarrow K_y, Q_x \leftrightarrow Q_y)$. [Note that the original Hamiltonian $H$ does not possess any of these symmetries.] These symmetries are used in the program[4] for reducing the size of the integrands.

Finally, we note that the easiest way to obtain the results of this section is to utilize the computer program[4].

VIII. THE ZERO-LOOP CONTRIBUTION

In the zero-loop approximation (ZLA), the calculation can be performed without assuming that SOI amplitudes are small, $x_{ab} \ll 1$ (i.e., without assuming the validity of the diffusion approximation). Only two diagrams (see Fig. 2) having zero loops contribute to the ZLA. As we discussed in Sec. VII.C, their sum is equal to the diagram in Fig. 2(a) with one velocity vertex substituted by its renormalized value (59).

$$\sigma_{q\bar{q}}^{(0)} - \delta x_{q\bar{q}} \sigma_D = \frac{e^2}{\hbar} \frac{1}{2} \text{Tr} \left[ \Delta \Delta A - \sigma_0 \frac{p_{a\bar{b}} p_{a\bar{b}}}{m^2} \tilde{g}^{E} \tilde{g}^{E} \right]$$

$$= \frac{e^2}{\hbar} \int \frac{d^2p}{(2\pi\hbar)^2} \frac{p_{a\bar{b}}}{m^2} \left( \tilde{g}^{E-} \tilde{g}^{E-} + \tilde{g}^{E+} \tilde{g}^{E+} - 2\tilde{g}^{E} \tilde{g}^{E} \right)$$

$$- \frac{e^2}{\hbar} \frac{1}{2} \text{Tr} \left[ \left( \tilde{g}^{E} \tilde{g}^{E} \right) \left( \tilde{g}^{E+} \tilde{g}^{E+} - \tilde{g}^{E-} \tilde{g}^{E-} \right) \right], \quad (66)$$

where we used (21). The rest of (66) equals

$$\frac{1}{2} \text{Tr} \left[ \left( \tilde{g}^{E-} \tilde{g}^{E-} \right) \left( \tilde{g}^{E+} \tilde{g}^{E+} \right) \right]$$

$$= \frac{1}{2} \sum_{i=1}^{2} \left( \frac{e}{mc} \right)^2 \frac{1}{2} \text{Tr} \left[ \tilde{g}^{E-} \tilde{g}^{E-} \right]$$

$$= \frac{1}{2} \text{Tr} \left[ \left( \tilde{g}^{E-} \tilde{g}^{E-} \right) \left( \tilde{g}^{E+} \tilde{g}^{E+} \right) \right]$$

$$= \frac{1}{2} \text{Tr} \left[ \left( \tilde{g}^{E} \tilde{g}^{E} \right) \left( \tilde{g}^{E+} \tilde{g}^{E+} - \tilde{g}^{E-} \tilde{g}^{E-} \right) \right]. \quad (68)$$

We see that the anisotropic terms in (67) and (68) cancel each other so that the charge conductivity (66) is proportional to the unity tensor $I$. Thus, within the ZLA $\sigma_{ab} = 0$ and

$$\sigma_{q\bar{q}}^{(0)} = \frac{e^2}{\hbar} \frac{x_a^2 + x_b^2}{4\mu_T/\hbar} + \frac{e^2}{\hbar} \frac{\sigma_0 - x_a x_b}{x_c} \right), \quad (69)$$

which is confirmed by the computer algebra calculation[4] for the limiting case when $2x_a x_b \ll x_a^2 + x_b^2 \ll 1$.

The absence of CD-loops in the ZLA-diagrams in Fig. 2 means that the ZLA-contribution neglects interference between electrons. Thus, the result (69) is valid also for the phase-coherent system (e.g., at high temperatures). The ZLA is local – that is, independent of the macroscopic (on scales $\gg l$) geometrical details of the sample, being the same in 2D and quasi-1D cases. Since the ZLA-diagrams contain no crossings of disorder-averaging lines, their contribution coincides with the results of the Boltzmann equation approach, see the discussion in §9.6 from Ref. 50.

Note that at finite frequency $\omega$ there are non-zero anisotropic corrections to the conductivity tensor. We do not present them in the main text; see[4] for details.

According to the loop expansion (see Sec. VII.D), diagrams having one (weak localization) and two loops may contribute to the conductivity of the same order, or even larger, than (69). We calculate contributions coming from these diagrams in the following sections.

IX. THE WEAK LOCALIZATION CONTRIBUTION

The weak localization contribution is provided by the diagram in Fig. 1(b) with the renormalized (dashed) Hikami box

$$\sigma^{(1)} = \frac{\hbar}{2\mu_T} \left[ (x_a^2 + x_b^2) + x_c x_d \right] \quad (70)$$

where both vertices are renormalized according to (58). The contribution of the diagrams in (70) can be
written in the form

\[
\sigma^{(1)}_{\alpha \beta} = \int \frac{d^2 k}{(2\pi \hbar)^2} \sum_{i,j=0}^{3} \mathbf{H}^{a\beta}_{ij}(k) D^{i,j}_{k}
\]  

(71)

where we used \( \mathbf{H}_{ij}^{a\beta} = A_{ij}^{a\beta} + B_{ij}^{a\beta} + C_{ij}^{a\beta} \),

\[
2-1.\text{max} \rightarrow A_{ij}^{a\beta}(k) = \frac{\hbar}{2m\tau} \sum_{i=0}^{3} \int \frac{d^2 p_1}{(2\pi \hbar)^2} \text{Tr} \left[ \sigma \mathbf{G}_R^T(-p_1) \left( -\frac{p_{1n}}{m} \right) \mathbf{G}_A^T(-p_1) \delta_j G_R(p + k) \right] \times \int \frac{d^2 p_2}{(2\pi \hbar)^2} \text{Tr} \left[ \sigma \mathbf{G}_R(k - p_2) \frac{2\pi}{m} G_A(k - p_2) \delta_i G_R(p_2) \right],
\]

(72)

\[
2-2.\text{max} \rightarrow B_{ij}^{a\beta}(k) = \int \frac{d^2 p}{(2\pi \hbar)^2} \text{Tr} \left[ \sigma \mathbf{G}_R(p) \delta_j G_A^T(k - p) \frac{2\pi}{m} G_R(k - p) \right] \times \int \frac{d^2 p_2}{(2\pi \hbar)^2} \text{Tr} \left[ \sigma \mathbf{G}_R(k - p_2) \frac{2\pi}{m} G_R(k - p_2) \delta_i G_A(p) \right],
\]

(73)

\[
2-3.\text{max} \rightarrow C_{ij}^{a\beta}(k) = \int \frac{d^2 p_1}{(2\pi \hbar)^2} \text{Tr} \left[ \sigma \mathbf{G}_A^T(k - p_1) \frac{2\pi}{m} G_R^T(k - p_1) \delta_j G_A(p) \right] \times \int \frac{d^2 p_2}{(2\pi \hbar)^2} \text{Tr} \left[ \sigma \mathbf{G}_A(-p_2) \frac{2\pi}{m} G_R(-p_2) \delta_i G_A^T(p + k) \right].
\]

(74)

The expressions (72), (73), (74) are generated by our program\(^4\) and taken from the (automatically created) files 2-1.max, 2-2.max, and 2-3.max.

In the absence of orbital dephasing and at zero frequency, the isotropic part of (70) diverges reproducing the well-known result for the weak antilocalization correction\(^3^1\),\(^3^2\).

The anisotropic part of (70), (71) converges. Its leading (in the SOI) contribution is given by

\[
\sigma^{(1)}_{an} = 2x_a x_b S_{20}^0 \frac{e^2}{\hbar} + \frac{e^2}{\hbar} \times O(\mu \tau / \hbar)^{-2},
\]

(75)

where we assumed that \( 2x_a x_b \ll x_a^2 + x_b^2 \ll 1 \), and

\[
S_{20}^0 = \frac{(131\pi + 262 \arctan \sqrt{7})/ \sqrt{7} - 88 - 7 \log 2}{224\pi} \approx 0.14
\]

(76)

These results are obtained in\(^4\), where the expression for the renormalized Hikami box (together with other details of calculation) can be found.

The results (69), (75) manifest the general rule (39): the disorder-averaged conductivity tensor is diagonal in the considered (rotated by \( \pi/4 \)) basis, where the SOI is given by (6). To demonstrate this rule, consider an arbitrary diagram produced by averaging the Kubo formula (57) for the off-diagonal conductivity element \( \sigma_{xy} \). Let us change the sign of \( p_{1z} \) and of \( c_2 \) everywhere in the expression for the diagram. The identity (47) remains valid if the sign of any Pauli matrix is changed, so that expressions for diffusons and cooperons will not change, as well as Hikami boxes, except for the Hikami box with the vertex \( v_\tau \), which will change sign. Thus the total expression will change its sign; on the other hand, since our transformation is only the change of variables (over which the Tr is taken), the expression must remain invariant. So we conclude that every diagram for \( \sigma_{xy} \) is zero, and the disorder-averaged conductivity tensor is diagonal.

X. TWO-LOOP CONTRIBUTION

A. Expansion in SOI

In the remainder of the paper we assume that the SOI parameters (17) are small, \( x \ll 1, |\delta| \ll 1 \) (we also assumed this in Sec. IX), and expand contributions to the conductivity in powers of \( x \) and \( \delta \). According to Ref. [\( 4 \)] in an infinite 2DEG,

\[
\sigma_{yy}(-a, b) = \sigma_{xx}(a, b) = \sigma_{xx}(-a, -b)
\]

or

\[
\sigma_{yy}(x, -\delta) = \sigma_{xx}(x, \delta).
\]

(77)

Then the anisotropic part of the conductivity tensor is given by the following expansion:

\[
\sigma_{an} = \frac{e^2}{\hbar} \sum_{m,n,r \geq 0} S^m_n \chi^m \delta^{2m+1} \frac{1}{(pyl/\hbar)^{r}}.
\]

(78)
FIG. 4: Three relevant two-loop diagrams, which contribute to $S_{10}^1$. See Ref. 8 for more details. Each diagram contains small-momentum singularities, which mutually cancel each other in accordance with the theorem from Appendix A.

FIG. 5: Six irrelevant two-loop diagrams, which do not contribute to $S_{10}^1$.

Physical quantities should depend only on even powers of SOI amplitudes; thus, we expect that $\forall n, r S_{nm}^r = 0$ for arbitrary odd $r$; our calculations confirm this statement for several values of $n$ and $r$. The two loop diagrams can only affect terms with $r \geq 1$ in (78); the calculation shows that $S_{10}^1 \neq 0$. Below we calculate $\sigma_{an}(i)$ for a 2D case (an infinite film – see Sec. X B), (ii) in quasi-1D case (infinite wire – see Sec. X C), and (iii) for the quasi-1D ring pierced by magnetic flux (see Sec. XII).

From Eqs. (77), (78), and (41) we conclude that the anisotropic contribution $\sigma_{an}$ to the conductivity tensor [see the definition in (10)] can be extracted from the odd in $\delta$ part of $\sigma_{xx}$:

$$\sigma_{an}(x, \delta) = \frac{\sigma_{xx}(x, \delta) - \sigma_{xx}(x, -\delta)}{2}. \quad (79)$$

B. The 2D case

In total, there are nine two-loop diagrams: three are shown in Fig. 4 and the other six in Fig. 5. The calculation in (14) shows that the leading contribution $S_{10}^1$ is produced by three diagrams in Fig. 4. Each relevant diagram consists of two Hikami boxes (dashed) and three diffuson and/or cooperon lines. For Ref. [2], we calculated the Hikami boxes manually (that is, we programmed expressions for them ourselves and then computer evaluated them). In order to facilitate the calculation, we made a variable change, which allowed us to express the sum of three diagrams in Fig. 4 as the diagram in Fig. 4(a) with the renormalized upper Hikami box (HB). We do not use this trick in this paper because (i) it works only for systems with time-reversal invariance and (ii) such tricks became useless after we modified the program.

Let us now describe how Hikami boxes (HBs) for diagrams in Fig. 4 are calculated. Every diagram in Fig. 4 contains two dashed HBs, and every dashed HB is given by the sum of three non-dashed HBs, like in Eq. (70). Thus, every diagram in Fig. 4 is a sum of nine diagrams with non-dashed HBs. For example, the diagram 4(c) can be expanded into a sum of nine diagrams in Fig. 6. Let us take the diagram 4(d) as an example. It is equal to

$$\begin{align*}
\sigma &= \int \frac{d^2 k_1}{(2\pi\hbar)^2} \int \frac{d^2 k_2}{(2\pi\hbar)^2} \mathcal{H}^{s_1 s_2 s_3}_{f_1 f_2 f_3}(k_1, k_2) D^{s_1 f_1}_{k_1} D^{s_2 f_2}_{k_2} D^{s_3 f_3}_{k_1 - k_1} \\
&= \int \frac{d^2 k}{(2\pi\hbar)^2} \int \frac{d^2 q}{(2\pi\hbar)^2} \mathcal{H}^{s_1 s_2 s_3}_{f_1 f_2 f_3}(k, k) D^{s_1 f_1}_{k + q} D^{s_2 f_2}_{k + q} D^{s_3 f_3}_{-k},
\end{align*} \quad (80)$$

where we used (53). The quantity $\mathcal{H}^{s_1 s_2 s_3}_{f_1 f_2 f_3}$ is the product of two HBs; it is calculated in the (automatically generated)
it to the antilocalization regime. Similarly to the weak-antilocalization problem, the non-analyticity in (83) can be smeared by introducing the finite dephasing rate $\gamma^{-1}_\phi > 0$. A convenient way to do this is to consider the response for the finite-frequency electric field; see Sec. [X].

C. The quasi-1D case

Consider a long wire whose cross section $L_\perp$ is much larger than the mean scattering length $l$, but with the corresponding Thouless energy $E_{c,\perp} = \hbar v_F/2L_\perp^2$ being large compared with the SOI-induced spectrum splitting (7). When (85) holds, Hikami boxes remain two-dimensional, while diffusons and cooperons become one-dimensional (cf. §7.4.1 from Ref. 10). In other words, in (81) still $\text{Tr}_{\text{p}} \equiv \int d^2 p/(2\pi)^2$, but in (80) the momentum integration becomes one-dimensional,

$$\int \frac{d^2 k}{(2\pi)^2} \int \frac{d^2 q}{(2\pi)^2} \rightarrow \frac{1}{L_\perp} \int_{-\infty}^{\infty} \frac{dk_y}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{dq_y}{(2\pi)^2},$$

and one should set $k_y = q_y = 0$ in the integrand. From (86) one can estimate that the quasi-1D SOI-induced contribution to the conductivity is $\tilde{l}^2/(\chi L_\perp)^2 \gg 1$ times larger than the 2D one.

A quasi-1D sample is macroscopically anisotropic (i.e., does not possess rotational symmetry), but this anisotropy becomes relevant only on the scales much larger than $l$. Thus, the Eq. (77), as well as the claim in Ref. 6 that “the conductivity tensor is isotropic when $\tilde{\sigma} = 0$” are valid in the ZLA and for the WL-diagram (since the distance between all vertices in corresponding diagrams is of the order of $l$), but may not be valid for the contribution of the two-loop diagrams in Fig. 4. In fact, we obtain the following anisotropic contribution for the conductivity tensor in the quasi-1D case (in the rotated CS):

$$\sigma^{(2)}_{an} = \frac{e^2}{\hbar} \frac{\tilde{l}^2}{p_{\perp} l^2 L_\perp^2} \begin{bmatrix} -0.39 & 0 \\ 0 & 6.7 \end{bmatrix} + \delta \begin{bmatrix} -852 & 0 \\ 0 & 13 \end{bmatrix},$$

which leads to

$$\sigma^{(2)}_{an} = \frac{e^2}{\hbar} \frac{\tilde{l}^2}{p_{\perp} l^2 L_\perp^2} \begin{bmatrix} 3.5 \ & 4335 \end{bmatrix}.$$

Thus, in the quasi-1D case the singularity in the SOI-correction to the conductivity tensor is more pronounced: (i) it occurs even when the (averaged) energy spectrum is isotropic and (ii) it diverges at vanishing SOI when orbital dephasing effects are neglected.
XI. THE FINITE FREQUENCY CASE

In this Section we will see that the anisotropic part of the conductivity tensor becomes an analytic function of the SOI amplitudes $(x_a, x_b)$ at finite frequency $\omega \neq 0$; we assume that the frequency is large enough compared to the strength of the SOI so that $|\omega r| \gg x^2$.

Here we have to expand corrections to the conductivity tensor not only in SOI, but in powers of $|\omega r|$ as well. Similarly to the zero-frequency case we choose three expansion parameters, $x_c, x_1$, and $W$, defined in the following way:

$$
x_c = \sqrt{2i\omega r}, \quad x_1 = \sqrt{x^2 + x_c^2} \ll 1,
W = 2x_c/x_1^2, \quad |x_1|, |x_1| |W| \ll 1.
$$

(89)

Differently from the zero-frequency case (see Sec. VII), the effect of the SOI on the diffusion can be calculated perturbatively in case when $|\omega r| \gg x^2$. The leading contribution is equal to the diffusion in the absence of SOI:

$$
D_{q}^{\alpha \beta} = \frac{\hbar}{m} \frac{1}{q^2/\hbar^2 - 2i\omega r \delta_{\alpha \beta}}, \quad \alpha, \beta = 0 \ldots 3.
$$

(90)

The derivation of SOI-corrections to (90) is straightforward, but lengthy, so we do not present it in the text of the article; see the program for more details.

Comparing (90) with expressions for the diffusion at $\omega = 0$ (see Secs. VII C and VII D) we see that the diffusions for $|\omega r| \ll x^2$ and $|\omega r| \gg x^2$ are very different. Consequently, while at $\omega = 0$ the vertex renormalization cancels the anomalous part of the velocity operator, this is no longer the case for $\omega \neq 0$ and at large frequencies the effect of the vertex renormalization is negligible.

Despite that, the calculation of the finite frequency case is similar to the one for $\omega = 0$; differences come from the fact that now we have three expansion parameters instead of two (17) for $\omega = 0$. We obtain

$$
a_{an}^{(2)} = -2 \cdot 0.25 \cdot \frac{-2i\omega r \cdot 2x_0 x_b}{(x_0^2 + x_b^2 - 2i\omega r)^2} \frac{e^2}{2\pi p y l}.
$$

(91)

This finite-frequency result obtained first in [17] can be interpreted in terms of dephasing; substituting $-i\omega r \rightarrow \tau/\tau_\phi$, we obtain

$$
a_{an}^{(2)} = \begin{cases} 5.6 \times 10^{-3} \cdot \frac{\tau_\phi l}{\tau_\phi + \tau_\perp}, & \tau_\phi \ll \tau_\perp, \\
0.13 \cdot \frac{\tau_\phi l}{\tau_\phi + \tau_\perp}, & \tau_\phi \ll \tau_\perp,
\end{cases}
$$

(92)

where the Dyakonov-Perel’ relaxation times are defined as $\tau_\phi = (x_a \mp x_b)^2$.

XII. THE QUASI-1D RING PIERCED BY MAGNETIC FLUX

The simplest way of breaking the time reversal invariance of the Hamiltonian is considering a constant vector potential field $A$. It arises in a quasi-one-dimensional ring pierced by a magnetic flux. (The magnetic flux can be described by subtracting $eA/c$ from momentum arguments of GFs.) In a ring geometry, the SOI-interaction of the type (4) cannot be provided by usual Rashba and Dresselhaus mechanisms. However, such SOI is forbidden and thus can occur due to different reasons, e.g., like in InAs nanowires.

Let us assume that $A$ is directed along the ring’s circumference so that $A \equiv A_\parallel \equiv A_x$. The presence of non-zero vector potential breaks the identities (52) and (53) together with the generalized Vollhardt-Wölfle theorem from Appendix D: thus making the anisotropy effect more pronounced because of the small-momentum diffusion-singularities, which now remain uncompensated.

Eq. (53) now changes into

$$
C_{q}^{\alpha \beta} = D_{q}^{\alpha \beta} + A_\parallel \frac{e}{c}.
$$

(93)

The calculation of the HBs is the same as in Sec. X B and Sec. X C: Like in Sec. X C we have to perform summation over two diffusion variables, see (86). For the diagrams which contain no cooperons, this summation is performed in the same way, like for the infinite quasi-1D wire, see Eq. (86). The diagrams with cooperons become different: in every such diagram, $eA_\parallel/c$ is always subtracted from one (out of two) cooperon momentum. The summation rule becomes then different from (86):

$$
\int \frac{d^2 k}{(2\pi)^2} \int \frac{d^2 q}{(2\pi)^2} \int_0^\infty \frac{d k ||}{2\pi L_||} \int_0^\infty \frac{d k \perp}{2\pi L_\perp} \sum_{q_i \parallel} (94)
$$

with $L_\parallel$ being the circumference of the ring, and $L_\perp$ its cross section. The summation is performed over all integer $n$ and it cannot be approximated with integration; one can use the Poisson summation formula instead:

$$
\frac{1}{L_\parallel} \sum_{n \in \mathbb{Z}} f \left( q_i - \frac{2\pi}{c} A \right) = \sum_{n \in \mathbb{Z}} \exp \left[ 2\pi i n \frac{\Phi}{\Phi_0} \frac{c}{|e|} \right] C_n,
$$

(95)

where the leading ($A$-dependent) contribution comes from the terms with $n = \pm 1$, and we will neglect contribution of terms with $|n| > 1$. Keeping only divergent terms (i.e., terms, containing massless cooperon/diffusion matrix elements), and assuming that $xL_\parallel \gg l$ we obtain the flux-dependent correction to the conductivity tensor:

$$
\sigma_{an}^{(2)} = \frac{\epsilon^2}{\hbar} \left[ \cos \left( 2\pi \frac{\phi}{\phi_0} \right) - 1 \right] \frac{l L_\parallel}{x L_\perp} \frac{\hbar}{p y l} (\Sigma_0 + \Sigma_1 \delta),
$$

(96)

with the coefficients
\[ \Sigma_i = a_i e^{-\frac{i \omega}{2}} + \exp \left( -\frac{L_i}{2} \sqrt{2 \sqrt{2} - 1} \right) \left( b_i \cos \left( \frac{L_i}{2} \sqrt{2 \sqrt{2} + 1} \right) + c_i \sin \left( \frac{L_i}{2} \sqrt{2 \sqrt{2} + 1} \right) \right), \]

\[ a_0 = -\frac{2 L_i^2 + 274 L_i - 219}{128}, \quad a_1 = -2 L_i^2 - L_i - 1, \]

\[ b_0 = -\frac{7124 \sqrt{7} L_i + 4513 \sqrt{2 \sqrt{2} - 1} \sqrt{7} - 2965 \sqrt{2 \sqrt{2} + 1}}{1792 \sqrt{7}}, \]

\[ c_0 = \frac{28 L_i - 4513 \sqrt{2 \sqrt{2} + 1} \sqrt{7} - 2965 \sqrt{2 \sqrt{2} - 1}}{1792 \sqrt{7}} = \frac{6 \cdot 10^{-3} L_i}{4}, \quad c_1 = -8.6 \cdot 10^{-5} L_i^3 + 4.5 L_i^2 - 0.46 L_i - 0.87, \]

where \( L_i = xL_i/l \). Like in Sec. [X], we assumed that the divergence at small SOI \((x \to 0)\) is regularized by the orbital dephasing. In Eqs. (97) we wrote numerical values of \( b_1 \) and \( c_1 \) instead of their analytic expressions in order to save space. The analytic expressions for \( b_1 \) and \( c_1 \) can be found in [14].

The result (96) has the same order of magnitude in the loop expansion [in powers of \((p_\| / \hbar)^{-1}\)], as (i) the infinite-plane result (82) and (83), as well as the (ii) quasi-1D result (84). However, out of all three considered geometries it is the most sensitive one with respect to orbital dephasing. In fact, in the coherent limit, \( L_\phi \to \infty \), it diverges as \( \propto L_\phi \) for finite SOI-amplitude \( x \), and as \( \propto L_\phi^2 \) in the limit \( x \ll 1/L_\phi \to 0 \).

**XIII. CONCLUSIONS**

We presented symbolic program [14] for generating, sorting, and calculating diagrams in the disorder-averaging diagrammatic technique. This program strongly facilitates analytical calculations, allowing one to calculate subtle effects due to spin-orbit interaction which were virtually inaccessible before due to the large number of integrals to be calculated. The possibility to automatize the calculation improves the usefulness of the diagrammatic approach, especially also in comparison to the non-linear \( \sigma \)-model [20], as a tool for studying disordered systems.

Using this program, we studied anisotropic corrections to the conductivity tensor due to the spin-orbit interaction (SOI). The arising anisotropy is a phase-coherence effect; therefore it strongly depends on the geometry of the sample. In the quasi-1D case the anisotropic correction is larger than in an infinite 2D-plane. Moreover, while in 2D-case the effect arises due to the anisotropy of the energy spectrum induced by the interference between Rashba and Dresselhaus types of SOI, in the quasi-1D case the conductivity is anisotropic even in the presence of only one type of SOI (Rashba or Dresselhaus), that is, when the energy spectrum is isotropic. The (microscopic) anisotropy of the conductivity tensor arises due to the macroscopic (shape) anisotropy of the sample (on the scale much larger than the mean free path \( l \)).

We also studied the case when the time-reversal symmetry of the system is broken (a ring pierced by a magnetic flux); then the anisotropy of the conductivity tensor becomes more sensitive to orbital dephasing effects due to the uncompensated small-momentum divergences in the integration over the diffusion momentum.

In all the considered geometries, the effect is non-analytical in the amplitude of the spin-orbit interaction, if the orbital dephasing effects are not taken into account. Once the dephasing effects are considered, the conductivity becomes an analytical function of the spin-orbit amplitudes.

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**APPENDIX A: INHOMOGENEOUS DISTRIBUTION OF ELECTRIC FIELD IN A HOMOGENEOUS WIRE**

The Kubo formula (37) is not valid for the calculation of the leading (Drude) contribution to the conductivity. From Sec. [X] one obtains:

\[ \text{Tr} \left[ \delta \hat{N}_{\alpha \beta} \right] = -\frac{i e^2 A_\alpha \omega}{\hbar m^2 \omega} \int_{-\infty}^{\infty} \frac{dE}{2\pi} \text{Tr} \left[ \hat{p}_{\alpha \beta}^{\mathrm{F}} \delta \hat{E} \right], \quad \omega \to 0, \]

(A1)

where the main contribution to \( \text{Tr} [\hat{p}_{\alpha \beta}^{\mathrm{F}} \delta \hat{E}] \) is given by \( \text{Tr} [\hat{p}_{\alpha \beta}^{\mathrm{F}} \delta \hat{E}] \). (Note that we do not take into account spin degree of freedom in this section and use expressions [18] for Green functions; consequently, \( \text{Tr} \) operators here do
not contain trace over spin.)

Only values of \( E \) close to the Fermi level (\( E = 0 \)) contribute to the integral in (A1), so that

\[
\text{Tr} \left[ \frac{\hat{p}_n^2}{2m} (\hat{g}_r^E - \hat{g}_a^E) \right] = 2im \tau \text{ Tr} \left[ \frac{\hat{p}_n^2}{2m} (\hat{g}_r^E - \hat{g}_a^E) \right] = 2im \tau \text{ Tr} \left[ \frac{\hat{p}_n^2}{2m} (\hat{g}_r^E - \hat{g}_a^E) \right].
\]

(A2)

From the Lehmann representation we obtain the matrix element of \( \hat{g}_r^E - \hat{g}_a^E \) in some (arbitrary) \( \lambda \)-representation:

\[
\langle \lambda | \hat{g}_r^E - \hat{g}_a^E | \lambda' \rangle = -2\pi i \sum_n \delta(E - E_n) \langle \lambda | n \rangle \langle n | \lambda' \rangle, \quad E_n = \langle n | \hat{H} | n \rangle,
\]

(A3)

where \( | n \rangle \) are the exact eigenstates of the unaveraged Hamiltonian. In a (spinless) disordered system, the states \( | n \rangle \) are non-degenerate due to the disorder; thus one can rewrite (A3) in the \( | e \rangle \)-basis with \( | n \rangle \equiv | e \rangle:

\[
\text{Tr} \left[ \frac{\hat{p}_n^2}{2m} (\hat{g}_r^E - \hat{g}_a^E) \right] = \frac{1}{d} \sum_e \langle e | \hat{p}_n^2 | e \rangle \langle e | \hat{g}_r^E - \hat{g}_a^E | e \rangle = -\frac{2\pi i}{d} \sum_{e,e'} \delta(E - E') \langle e | \hat{p}_n^2 | e \rangle \langle e | \hat{g}_r^E | e \rangle \langle e' | \hat{g}_r^E | e' \rangle = -\frac{2\pi i}{d} \sum_e \delta(E - E') \langle e' | \hat{p}_n^2 | e' \rangle,
\]

(A4)

where \( d \) is the dimension (\( d = 2 \) in 2D). Substituting (A4) into (A1) we obtain

\[
f^\lambda(\omega) = \sigma(\omega) \frac{i\omega A^\lambda_n}{\epsilon} = \frac{2\pi i A^\lambda_n}{mc d} \int dE d
\]

\[
\nu T r_f E | E \rangle = \hat{p}_n^2 | E \rangle, \quad \omega \rightarrow 0.
\]

(A5)

Thus the main (Drude) contribution to the conductivity per spin projection is given by

\[
\sigma_D = \frac{2e^2}{dmh} \tau \int dE d\nu T r_f E | E \rangle = \hat{p}_n^2 | E \rangle.
\]

(A6)

The easiest case is equilibrium at \( T = 0 \): then \( f^\nu \rightarrow -\delta(E) \) and the integral is equal to \( \nu | \hat{p}_n^2 | 0 \rangle \rangle \equiv \nu \mu \), so that \( \sigma_D = \mu \tau \cdot E^2 / h \cdot 4\pi \nu / d \).

Consider a wire of length \( L \) between two leads under the voltage \( V \), so that the energy distribution in the leads is given by \( f_E^{RL} \). If the effect of the interaction between electrons in the wire is weak, its distribution function linearly depends on the coordinate:

\[
f_E(r) = \frac{r}{L} f_E^{RL} + \left( 1 - \frac{r}{L} \right) f_E^{LR}.
\]

(A7)

From (A6) we conclude that also the leading (Drude) contribution to the conductivity is slightly inhomogeneous. Consequently, the stationary distribution of the electric field in the wire will also be inhomogeneous, so that the wire will be homogeneously charged. Thus the charge-neutrality of the current-currying wire is slightly violated.

\section*{APPENDIX B: GENERATING DIAGRAMS ON COMPUTER}

The diagrams are generated in the program \cite{14} from the Kubo formula \cite{15} according to the following algorithm.

1. The simplest diagram is obtained from \cite{15} by substituting GF-operators \( \hat{G}_R^F \) and \( \hat{G}_A^{E-W} \) with their averaged values. The result corresponds to the diagram in Fig. 2(a).

2. Add one cooperon or one diffusion line to the diagram(s) obtained on the previous step in all possible ways.

3. Leave only diagrams having no more than two loops (to calculate the number of loops, the diagram has to be redrawn in the “coordinate representation”, see Sec. VI A).

4. Recursively perform two previous steps, until the last step produces no more new diagrams.

5. Redraw all diagrams in the “coordinate representation”, see Sec. VI A.

6. E.g., consider the diagram in Fig. 4(a) without lonely disorder lines inserted in the Hikami boxes; each of its three diffusion lines is given by the infinite series \cite{16}. Leaving only the first terms in these series, we note that the resulting diagram is exactly the one depicted in Fig. 1(a) with three disorder (dashed) lines left. We see that the diagram in
In principle, the above steps can be done manually, but it is better to make use of the computer program. On the first step we started with one diagram; in the end we get 215 ones.

APPENDIX C: $S_{01}^{\alpha \omega}$ AT $\omega = 0$: CALCULATING
\[ \int \frac{d^2k}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2} \text{ ON COMPUTER: ANALYTICS AND NUMERICS} \]

Once all diagrams with no more than two CD-loops are generated, they are automatically divided into different groups according to the number of HBs and diffusion lines (in total we get eight groups). The Hikami boxes of all diagrams are automatically calculated. We have checked that the diagrams of only one (the fifth) group contribute to $\sigma_{an}$. The diagrams of the fifth group are calculated in the directory $2D.5/.$

Within the same group, the Hikami boxes can be summed up; after that the expression for the sum of all diagrams from the group can be written in the form (60), with $H_{\alpha b, \gamma a}^f$ being the sum of all HBs.

Then we integrate over the diffusion momenta. We use dimensionless momentum variables ($K$ and $Q$) defined in (60). We checked that only the diagrams in Fig. 4 contribute to $S_{01}^{\alpha \omega}$. Each of these diagrams has three diffusion/cooperon lines; we label their momenta as $K, Q$, and $K + Q$. In case of small anisotropy of the energy spectrum, $\delta \ll 1$, the denominator of the diffusion/cooperon depends only on the modulus of its momentum. Thus the angular dependence of the denominator comes only from $(K+Q)^2 = K^2+Q^2+2KQ \cos \psi$, where $\psi$ is the angle between $K$ and $Q$.

We calculate $\int \frac{d^2k}{(2\pi)^2} \int \frac{d^2q}{(2\pi)^2}$ in polar coordinates. In total there are two angular integrations; since the denominator depends only on one angle, another angular integration can be easily (and analytically) performed. The second angular integration is more complicated: the integrand is given by the sum of rational functions which have the form
\[ \int_0^{2\pi} \frac{d\psi}{2\pi} \frac{P_1(\sin \psi, \cos \psi)}{P_2(\cos \psi)}, \]
where $P_{1,2}$ are polynomials. The denominator is even in $\psi$, so we can leave only even (in $\psi$) part of the numerator. Then numerator thus can be expressed as another polynomial:
\[ \int_0^{2\pi} \frac{d\psi}{2\pi} \frac{P_1(\sin \psi, \cos \psi)}{P_2(\cos \psi)} \quad P_2(\cos \psi) = \sum_{n=1} a_n \cos^n \psi. \]

Next, we perform the analytical integration over $\psi$. Because of the large number of terms to integrate and large size of the expressions, this analytical integration (basically, calculation of residues) can only be done on computer. We use the fact that the denominators of all mass-full elements of the diffusion $D_{kq}$ can be factorized into two expressions. In the program they are denoted as $uno = 1 + (K + Q)^2$ and $due = 2 - (K + Q)^2 + (K + Q)^4$. The size of the integration result grows rapidly with powers of $uno$ and $due$ in the denominator, so it is necessary to split integrands into elementary fractions.

We assume that integrals over $(k, q)$ converge on the scale of $k \ll x/\hbar$ and $q \ll x/\hbar$, so that the integrals in dimensionless variables ($K, Q$) converge on the scale of $K \ll 1, Q \ll 1$. Together with the assumption $x \ll 1$, this permits us to use Taylor expansions (e.g., for Hikami boxes) in powers of $k/\hbar \ll 1$ and $q/\hbar \ll 1$. The usage of Taylor expansion here corresponds to the diffusive limit and is justified by the fact that we do not get any large-momenta divergences.

Next, the integrand is symmetric with respect to $K \leftrightarrow Q$, and the integration operator has this symmetry too. So we symmetrize every term of the integrand, and express it in terms of new variables $P = K^2 + Q^2$ and $A = 2KQ/P$. Accordingly, our integration operator is changed:
\[ \int_0^{\infty} dK \int_0^{\infty} dQ KQ = 2 \int_0^{\infty} dK \int_0^{K} dQ KQ = 2 \int_0^{\infty} dP \int_0^{P} dA \frac{PA}{2J}, \quad A = \frac{2KQ}{K^2 + Q^2}; \]

and $J$ is the Jacobian:
\[ J = \left| \frac{\partial(P, A)}{\partial(K, Q)} \right| = \left| 4 \frac{K^2 - Q^2}{K^2 + Q^2} \right| = 4 \sqrt{1 - A^2}. \]

The integration results produced by integrate.max for $A \rightarrow 0$ have insufficient precision (since the difference of large numbers has low numerical accuracy), so we integrate by $P$ and $A$ not from zero but from $0.001$. Since $VP$ the integrand $= 0$ at $A = 0$, and maximal contribution to the result occurs for $A \rightarrow 1$, this adjustment of the lower limit introduces only a negligible error into the result.

We treat separately (see files $KpQ$) the terms containing massless elements of $D_{kq}$, and see that no divergences at small $(P, A)$ occur, as it is predicted according to the VW-theorem in Appendix D.

The $\int_0^{\infty} dP$ together with the subsequent $\int_0^{1} dA$ is performed in directory $2D.5/A/$. (As a check, we performed the numerical integration using alternative variables in...
the directory 2D_5/B/, which led to the same result.) We use functions qagi for \(\int_0^1 dP\) and qaws for \(\int_0^1 dA\), which are part of the quadpack package. In order to minimize possible rounding corrections, all calculations are done with high precision (35 digits).

The integrand for \(\int_0^1 \frac{dA}{\sqrt{1-A}}\) is plotted in Fig. 7. The final result \(83\) has been calculated using different lisp realizations (clisp, gc1 and sbc1), and for different numbers of digits in the numerical integration \((20, 25, \text{and } 35)\).

In conclusion, the steps of the diffusion momentum integration are:

1. Perform the first (easy) angular integration.
2. Select terms that contain massless component \(D_{k+q}^{00}\). Integrate them by \(\psi\) (second angular integration variable).
3. Split other terms into elementary fractions, in order to decrease powers in their denominators. Integrate them by \(\psi\), and combine them with the terms obtained in the previous step.
4. Calculate the integrand in many points within the interval \(A \in [0, 1]\). Interpolate (linearly) the integrand in the integration interval \(A \in [0, 1]\), and integrate the interpolation result by \(A\).

**APPENDIX D: GENERALIZATION OF THE VOLLHARDT-WÖLFLE THEOREM FOR HAMILTONIANS WITH SPIN-ORBIT INTERACTION**

We assume that the following statements (demonstrated above for the case of Rashba and Dresselhaus SOI) are valid in general for a system with time reversal symmetry:

- The anomalous part of the velocity operator vanishes at \(\omega = 0\) due to the vertex renormalization \(85\).
- The only divergent (massless) element of the diffusion matrix \(85\) is \(D^{00}\).

Then, the Vollhardt-Wölffe theorem can be generalized to spin-dependent Hamiltonians:

**Theorem 1** In the the calculation of linear response coefficients, no diffusion-type divergences occur if the unperturbed system possesses time reversal symmetry.

In particular, diffusion singularities cannot occur in the calculation of the conductivity or spin susceptibility, as well as in the calculation of corresponding cumulants.

We consider an arbitrarily complicated diagram with one or more diffusions. We choose any of them and prove that the coefficient in front of \(D^{00}\) vanishes when the momentum flowing through this diffusion approaches zero. Let us draw only the selected diffusion in the “coordinate representation” (see Sec. VI.A); other diffusions and all cooperons (if present) remain in the “ladder representation”; e.g., Fig. 8. Drawn in this way, a diagram consists of two bubbles with a wavy diffusion line between them [e.g., Fig. 8(a)]. In a diagram for linear response to applied electric field, (at least) one of two vertices (we assume that it is the rhs-one) is proportional to the renormalized velocity operator \(85\). Let us downgrade all diffusions/cooperons to crosses [removing disorder-averaging (dashed) lines connecting them]. After that the rhs-bubble of an arbitrary diagram can be written as

\[
\int \frac{d^2p}{(2\pi \hbar)^2} pA \text{ Tr } \left[ \partial_0 G_R^{(m)}(p-q,p)G_A^{(n)}(p,p-q) \right],
\]

where \(G_R^{(m)}(p-q,p)\) and \(G_A^{(n)}(p,p-q)\) are unaveraged Green functions in the \(m\)th and \(n\)th order of the perturbation theory in the disorder potential. [Note that we did not draw the crosses on the GFs-lines in (D1).] Since the disorder cannot break the time reversal symmetry of the unperturbed Hamiltonian,

\[
\forall m, n, p_1, p_2 \left\{ \begin{array}{l}
\sigma_2 \left[ G_R^{(m)}(-p_2, -p_1) \right]^T \sigma_2 = G_R^{(m)}(p_1, p_2), \\
\sigma_2 \left[ G_A^{(n)}(-p_2, -p_1) \right]^T \sigma_2 = G_A^{(n)}(p_1, p_2),
\end{array} \right.
\]

which is a generalization of (22). From our assumptions it follows that \(G_{R/A}\) commutes with the renormalized velocity vertex \(vA\). Applying the transformation (22) to GFs in (D1), transposing matrices under the trace, and substituting \(p \rightarrow -p\) in the integral we obtain

\[
- \int \frac{d^2p}{(2\pi \hbar)^2} pA \text{ Tr } \left[ \partial_0 G_{R}^{(m)}(p-q,p)G_{A}^{(n)}(p,p-q) \right] = \int \frac{d^2p}{(2\pi \hbar)^2} pA \text{ Tr } \left[ \partial_0 G_{R}^{(m)}(p+p.q)G_{A}^{(n)}(p+q,p) \right].
\]
so that \( |D_1| \) vanishes at \( q = 0 \), and the diffuson divergence is regularized. The proof can be performed for every diffuson in the diagram, so that all diffusons are regularized. An example of mutually cancelling diffuson-
divergences is depicted in Fig. 8: both diagrams \( 8(a) \) and \( 8(b) \) diverge at \( q \to 0 \), but their sum is regular. A similar cancellation takes place for the diagrams in Fig. 5: the singularity in \( D_0^0 \) in the diagram \( 4(a) \) is cancelled (i.e., regularized) by the diagram \( 4(b) \) while the singularity in \( D_0^0 \) in the diagram \( 4(a) \) is cancelled by the diagram \( 4(c) \).

Such cancellation is provided by the fact that in a system with time-reversal invariance cooperon components are exactly the same as components of a diffuson. (This remains true also in the presence of interaction between electrons.)

Thus we demonstrated that (in the absence of interaction) there can be no diffuson type singularities in the presence of the time-reversal symmetry, thus generalizing the theorem proved in Ref. 9 for the spinless case without SOI.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig8}
\caption{Illustration to the proof of the Theorem. Only one diffuson (with momentum \( q \)) is depicted in the "coordinate representation"; this diffuson \( D_q \) divides both diagrams in two parts (bubbles). The vertex on the rhs of both diagrams is proportional to the renormalized velocity operator \( \bar{v} \). The rhs-bubble of \( b \) is the mirrored (and then rotated by \( 180^\circ \)) rhs-bubble of \( a \). The sum of two diagrams is regular at \( q \to 0 \).}
\end{figure}

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\item If the splitting of the integration result into elementary fractions is not performed, the subsequent numerical integration is problematic. This happens because huge integration results usually contain the difference of large numbers, which is calculated numerically with low accuracy, and result in various numerical errors, like, e.g., "false" divergences.
\item The asymptotic for \( \nu \ll 1 \) can be easily calculated: one ex-
pands 1/uno and 1/due in small $A$; then $\int_0^{2\pi} d\psi/(2\pi)$ becomes trivial, since $\psi$ appears now only in the numerator, and we get a convenient integrand for $\int_0^\pi dA \int_0^\infty dP$.

http://www.netlib.org/quadpack.

This condition is not fulfilled for a spin-independent Hamiltonian: $D^{\alpha\beta} \propto D^{00}\delta_{\alpha\beta}$, so that all four non-zero matrix elements diverge at zero frequency and momentum. However, $(D^{11}, D^{22}, D^{33})$ come with zero coefficient so that the theorem remains valid.

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| Symbol | Its definition |
|--------|----------------|
| $d$    | spatial dimension |
| $\sqrt{i}$ | $(1 + i)/\sqrt{2}$ |
| $C$ and $D$ | see (51) and (49) |
| $G_R$ and $G_A$ | see (19); $G_A = G_R^\dagger$ |
| $l$    | mean free path of an electron between subsequent elastic scattering off impurities |
| $L_\phi$ | electron (orbital) dephasing length due to inelastic scattering |
| $\tau^{-1}_\phi = v_F/L_\phi$ | dephasing rate due to inelastic scattering |
| $\mu$  | (temperature-dependent) chemical potential |
| $p_F = \sqrt{2m\mu}$ | Fermi momentum |
| $v_F = p_F/m$ | Fermi velocity |
| $\nu$, DoS | density of states |
| $\Re$ and $\Im$ | real and imaginary part |
| $\sigma_{\mu}$ and $\sigma_{\alpha\nu}$ | see (39) and (40) |
| $S$    | see (73) |
| $\sigma_0$ | $2 \times 2$ unity matrix |
| $\sigma_1$, $\sigma_2$, and $\sigma_3$ | Pauli matrices |
| $\xi_\mu$ | see (18) |
| $p_F$, $x_\alpha$, and $x_\beta$ | see (16) |
| $\delta$ and $\delta$ | see (17) |
| $\mathrm{CD}$ | cooperon or diffuson |
| $\mathrm{CS}$ | coordinate system |
| $\mathrm{DM}$ | density matrix |
| $\mathrm{GF}$ | Green function |
| $\mathrm{GFB}$ | Green functions box |
| $\mathrm{HB}$ | Hikami box |
| $\mathrm{lhs}$ | left hand side |
| $\mathrm{rhs}$ | right hand side |
| $\mathrm{SOI}$ | spin-orbit interaction |
| $\mathrm{VW}$ | Vollhardt-Wölfle |
| $\mathrm{WL}$ | weak localization |
| $\mathrm{ZLA}$ | zero-loop approximation |

**TABLE I:** Some often used notations and abbreviations.