KIRSZBRAUN-TYPE THEOREMS FOR GRAPHS

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ABSTRACT. A well-known theorem by Kirszbraun implies that all 1-Lipschitz functions \( f : A \subset \mathbb{R}^n \to \mathbb{R}^n \) with the Euclidean metric have a 1-Lipschitz extension to \( \mathbb{R}^n \). For metric spaces \( X, Y \) we say that \( Y \) is \( X \)-Kirszbraun if all 1-Lipschitz functions \( f : A \subset X \to Y \) have a 1-Lipschitz extension to \( X \). In this paper we focus on \( X \) and \( Y \) being graphs with the usual path metric; in particular, we characterize \( \mathbb{Z}^d \)-Kirszbraun graphs using some curious Helly-type properties.

1. Introduction

Let \( l_2 \) denote the usual Euclidean metric on \( \mathbb{R}^n \) for all \( n \). Given a metric space \( X \) and a subset \( A \) we will write \( A \subset X \) to mean the subset \( A \) endowed with the restricted metric from \( X \). Our story begins with two well-known theorems by Kirszbraun and Helly. Kirszbraun proved in [6] that for all Lipschitz functions \( f : A \subset (\mathbb{R}^n, l_2) \to (\mathbb{R}^n, l_2) \) there is an extension to a Lipschitz function on \( \mathbb{R}^n \) with the same Lipschitz constant. One of the ways to prove the Kirszbraun theorem (as in [11]) uses the result by Helly [5, 3]: Given a collection of convex sets \( B_1, B_2, \ldots, B_k \) if every \( n+1 \) subcollection has a non-empty intersection then \( \bigcap_{i=1}^k B_i \neq \emptyset \). The relationship between these two theorems is well-known; in this paper we bring it forth in the context of graphs.

Given metric spaces \( X \) and \( Y \), we say that \( Y \) is \( X \)-Kirszbraun if all 1-Lipschitz maps \( f : A \subset X \to Y \) have a 1-Lipschitz extension to \( X \). The Kirszbraun theorem says that \( \mathbb{R}^n \) is \( \mathbb{R}^n \)-Kirszbraun.

For \( m \in \mathbb{N} \) and \( n \in \mathbb{N} \cup \{ \infty \}; n > m \), a metric space \( X \) has the \((n, m)\)-Helly property if for a collection of closed balls \( B_1, B_2, \ldots, B_n \) (if \( n \neq \infty \) and any finite collection otherwise) of radius bigger than or equal to one we have that every subcollection of cardinality \( m \) has a non-empty intersection, then \( \bigcap_{i=1}^m B_i \neq \emptyset \). Since balls in \( \mathbb{R}^n \) with the Euclidean metric are convex, Helly’s theorem can be restated to say that \( \mathbb{R}^n \) is \((\infty, n + 1)\)-Helly.

Given a graph \( H \), we endow the set of vertices (also denoted by \( H \)) with the path metric. By \( \mathbb{Z}^d \) we will mean the Cayley graph of the group \( \mathbb{Z}^d \) with respect to standard generators. All graphs in this paper are non-empty, connected and undirected without multiple edges and self-loops. The following is the main theorem of this paper.

Theorem 1.1. A graph is \( \mathbb{Z}^d \)-Kirszbraun if and only if it is \((2d, 2)\)-Helly.

Let \( K_n \) denote the complete graph on \( n \)-vertices. \( K_n \) is \( G \)-Kirszbraun for all graphs \( G \) since all maps \( f : G \to K_n \) are 1-Lipschitz. On the other hand, \( \mathbb{Z}^2 \) is not \( \mathbb{Z}^2 \)-Kirszbraun: Let \( A := (0, 1), (0, -1), (1, 0), (-1, 0) \) and \( f : A \subset \mathbb{Z}^2 \to \mathbb{Z}^2 \) be given by

\[
\begin{align*}
f(0, 1) & := (0, 0), f(0, -1) := (0, 1), f(1, 0) := (1, 1), f(-1, 0) := (1, 0);
\end{align*}
\]

though \( f \) is 1-Lipschitz, it does not have a 1-Lipschitz extension \( \tilde{f} : A \cup \{(0, 0)\} \to \mathbb{Z}^2 \). This example can be modified to satisfy a certain extendability property; look at Corollary 4.3 and the example there after. It is well-known that trees are \( G \)-Kirszbraun for all graphs \( G \) [1, Section 3]. On

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We draw our motivation from two very distinct sources: the respective vertices; let $\text{Hom}$ Helly graphs can be found in the survey $k$ $v$ then $|A|$ For example, let $d$ proving that it is $(2-k)$ $K$ thus it is not $(2-d,2)$-Helly. An easy deduction (for instance following the discussion in $k$, $G$ $\text{Hel}$. A graph is called $H$ $d$ $G$ $A$ passes through a vertex in $k$ goes back to the original paper by Kirszbraun $d$, $G$ important properties is an old one and goes back to the original paper by Kirszbraun $d,2)$-Helly: Among graphs, research has focused mostly on a certain universality: A graph is called Helly if it is $(\infty,2)$-Helly. An easy deduction (for instance following the discussion in $d_H$ denote the path metric on the graph $H$. A walk in the graph $H$ of length $k$ is a sequence of $k+1$ vertices $p=(v_0,v_1,\ldots,v_k)$ such that $d_H(v_i,v_{i+1}) \leq 1$ for all $0 \leq i \leq k-1$; we say that $p$ starts at $v_0$ and ends at $v_k$. A geodesic from vertex $v$ to $w$ in a graph $G$ is a walk from $v$ to $w$ of the shortest length. Given a graph $H$, a subset $A \subseteq G$ and $b \in G \setminus A$, the geodesic culling of $A$ with respect to $b$ is $Cull(A,b) := \{a \in A : \text{there is no geodesic from } a \text{ to } b \text{ which passes through a vertex in } A \setminus \{a\}\}$ For example, let $A \subseteq \mathbb{Z}^2 \setminus \{(0,0)\}$. If $(i,j), (k,l) \in Cull(A,b)$ are elements of the same quadrant, then $|i| > |k|$ if and only if $|j| < |l|$.
Remark 3.1. Let $A \subset \mathbb{Z}^d$ be contained in the coordinate axes. Then $|\text{Cull}(A, \vec{0})| \leq 2d$.

Geodesic culling helps prove that certain 1-Lipschitz maps extend:

**Proposition 3.2.** Given a 1-Lipschitz map $f : A \subset G \rightarrow H$ and $b \in G \setminus A$, $f$ has a 1-Lipschitz extension to $A \cup \{b\}$ if and only if $f|_{\text{Cull}(A,b)}$ has a 1-Lipschitz extension to $\text{Cull}(A,b) \cup \{b\}$.

**Proof.** The forward direction of the proof is immediate because $\text{Cull}(A,b) \subset A$. For the backwards direction let $\tilde{f} : \text{Cull}(A,b) \cup \{b\} \subset G \rightarrow H$ be a 1-Lipschitz extension of $f|_{\text{Cull}(A,b)}$ and consider the map $\hat{f} : A \cup \{b\} \subset G \rightarrow H$ given by

$$\hat{f}(a) = \begin{cases} f(a) & \text{if } a \in A \\ \tilde{f}(b) & \text{if } a = b. \end{cases}$$

To prove that $\hat{f}$ is 1-Lipschitz we need to verify that for all $a \in A$, $d_H(\hat{f}(a), \tilde{f}(b)) \leq d_G(a, b)$. From the hypothesis it follows for $a \in \text{Cull}(A,b)$. Now suppose $a \in A \setminus \text{Cull}(A,b)$. Then there exists $a' \in \text{Cull}(A,b)$ such that there exists a geodesic from $a$ to $b$ passing through $a'$. This implies that $d_G(a, b) = d_G(a, a') + d_G(a', b)$. But

$$d_G(a, a') \geq d_H(\hat{f}(a), \tilde{f}(a')) = d_H(f(a), f(a'))$$

because $f$ is 1-Lipschitz

$$d_G(a', b) \geq d_H(\tilde{f}(a'), \tilde{f}(b)) = d_H(\tilde{f}(a'), \tilde{f}(b))$$

because $\tilde{f}$ is 1-Lipschitz.

By the triangle inequality, the proof is complete. \hfill $\square$

Given a graph $H$, a vertex $v \in H$ and $n \in \mathbb{N}$ denote by $B^H_n(v)$, the ball of radius $n$ in $H$ centered at $v$. We will now interpret the $(n, 2)$-Helly property in a different light.

**Proposition 3.3.** Let $H$ be a graph satisfying the $(n, 2)$-Helly property. For all 1-Lipschitz maps $f : A \subset G \rightarrow H$ and $b \in G \setminus A$ such that $|\text{Cull}(A,b)| \leq n$, there exists a 1-Lipschitz extension of $f$ to $A \cup \{b\}$.

**Proof.** Consider the extension of $f$ to $A \cup \{b\}$, $\tilde{f}$ where $\tilde{f}(b)$ is any vertex in

$$\bigcap_{b' \in \text{Cull}(A,b)} B^H_{d_G(a,b')}(\tilde{f}(b'));$$

the intersection is non-empty because $|\text{Cull}(A,b)| \leq n$ and for all $a, a' \in \text{Cull}(A,b)$,

$$d_H(f(a), f(a')) \leq d_G(a, a') \leq d_G(a, b) + d_G(b, a') \text{ implying } B^H_{d_G(a,b)}(f(a)) \cap B^H_{d_G(b,a')}(f(a')) \neq \emptyset.$$

The function $\tilde{f}|_{\text{Cull}(A,b) \cup \{b\}}$ is 1-Lipschitz; Proposition 3.2 completes the proof. \hfill $\square$

Let $C_n$ and $P_n$ denote the cycle graph and the path graph with $n$ vertices respectively.

**Corollary 3.4.** All graphs are $P_n$, $C_n$ and $\mathbb{Z}$-Kirszbraun.

In the case where $G = P_n, C_n$ or $\mathbb{Z}$ we have for all $A \subset G$ and $b \in G \setminus A$, $|\text{Cull}(A,b)| \leq 2$; the corollary follows from Proposition 3.3 and the fact that all graphs are $(2, 2)$-Helly.

Given $r_1, r_2, \ldots, r_n \in \mathbb{N}$, denote by $T_{(r_1, r_2, \ldots, r_n)}$ the star-shaped tree with a central vertex and $n$ disjoint walks of lengths $(r_i)_{1 \leq i \leq n}$ emanating from it.

**Corollary 3.5.** A graph $H$ has the $(n, 2)$-Helly’s property if and only if it is $T_{(r_1, r_2, r_3, \ldots, r_n)}$-Kirszbraun for all $r_1, r_2, \ldots, r_n$.

This follows from the fact that for all $A \subset T_{(r_1, r_2, r_3, \ldots, r_n)}$ and $b \in T_{(r_1, r_2, r_3, \ldots, r_n)} \setminus \{A\}$, $|\text{Cull}(A,b)| \leq n$. Now we are prepared to prove the main theorem of the paper.
Proof of Theorem 1.1. We will first prove the forward direction. Let \( H \) be a graph which is \( \mathbb{Z}^d \)-Kirszbraun. For all \( r \in \mathbb{N}^d \) there is an isometry from \( T_r \) to \( \mathbb{Z}^d \) mapping the walks emanating from the central vertex to the coordinate axes. Hence \( H \) is \( T_r \)-Kirszbraun for all \( r \in \mathbb{N}^d \). By Corollary 3.5, we have proved the \((2d,2)\)-Helly’s property for \( H \).

Now let us prove the backward direction. Let \( H \) have the \((2d,2)\)-Helly’s property. We want to prove that for all 1-Lipschitz maps \( f : A \subset \mathbb{Z}^d \rightarrow H \), there is a 1-Lipschitz extension. It is sufficient to prove this for finite subsets \( A \). We will proceed by induction on \( |A| \), viz., we will prove \( St(n) \):

Let \( f : A \subset \mathbb{Z}^d \rightarrow H \) be 1-Lipschitz with \( |A| = n \). Let \( b \in \mathbb{Z}^d \setminus A \). The function \( f \) has a 1-Lipschitz extension to \( A \cup \{b\} \).

We know \( St(n) \) for \( n \leq 2d \) by the \((2d,2)\)-Helly’s property. Let us assume \( St(n) \) for some \( n \geq 2d \); we want to prove \( St(n+1) \). Let \( f : A \subset \mathbb{Z}^d \rightarrow H \) be 1-Lipschitz with \( |A| = n+1 \) and \( b \in \mathbb{Z}^d \setminus A \).

Without loss of generality assume that \( b = \bar{0} \). Also assume that \( \text{Cull}(A, \bar{0}) = A \); otherwise we can use the induction hypothesis and Proposition 3.2 to obtain the required extension to \( A \cup \{\bar{0}\} \).

We will prove that there exists a set \( \bar{A} \subset \mathbb{Z}^d \) and a 1-Lipschitz function \( \bar{f} : \bar{A} \rightarrow H \) such that

1. If \( \bar{f} \) has an extension to \( \bar{A} \cup \{\bar{0}\} \) then \( f \) has an extension to \( A \cup \{\bar{0}\} \).
2. Either the set \( \bar{A} \) is contained in the coordinate axes of \( \mathbb{Z}^d \) or \( |\bar{A}| \leq 2d \).

By Remark 3.1 and the \((2d,2)\)-Helly’s property for \( H \) this is sufficient to complete the proof.

Since \( |A| \geq n + 1 > 2d \), there exists \( i, j \in A \) and a coordinate \( 1 \leq k \leq d \) such that \( i_k, j_k \) are non-zero and have the same sign. Suppose \( i_k \leq j_k \). There is a geodesic from \( j \) to \( i - i_k \bar{e}_k \) which passes through \( i \). Since \( A = \text{Cull}(A, \bar{0}) \) we have that

\[
i - i_k \bar{e}_k \notin \{\bar{0}\} \cup A.
\]

Thus \( j \notin \text{Cull}(A, i - i_k \bar{e}_k) \) and hence \( |\text{Cull}(A, i - i_k \bar{e}_k)| \leq n \). By \( St(n) \) there exists a 1-Lipschitz extension of \( f|_{\text{Cull}(A, i - i_k \bar{e}_k)} \) to \( \text{Cull}(A, i - i_k \bar{e}_k) \cup \{i - i_k \bar{e}_k\} \). By Proposition 3.2 there is a 1-Lipschitz extension of \( f \) to \( f' : A \cup \{i - i_k \bar{e}_k\} \rightarrow H \). But there is a geodesic from \( i \) to \( \bar{0} \) which passes through \( i - i_k \bar{e}_k \). Thus

\[
\text{Cull}(A \cup \{i - i_k \bar{e}_k\}, \bar{0}) \subset (A \setminus \{i\}) \cup \{i - i_k \bar{e}_k\}.
\]

Set \( A' := (A \setminus \{i\}) \cup \{i - i_k \bar{e}_k\} \). By Proposition 3.2, \( f' \) has a 1-Lipschitz extension to \( A' \cup \{i - i_k \bar{e}_k\} \cup \{\bar{0}\} \) if and only if \( f''|_{A'} \) has a 1-Lipschitz extension to \( A' \cup \{\bar{0}\} \).

Thus we have obtained a set \( A' \) and a 1-Lipschitz map \( f' : A' \subset \mathbb{Z}^d \rightarrow H \) for which

1. If \( f' \) has an extension to \( A' \cup \{\bar{0}\} \) then \( f \) has an extension to \( A \cup \{\bar{0}\} \).
2. The sum of the number of non-zero coordinates of elements of \( A' \) is less than the sum of the number of non-zero coordinates of elements of \( A \).

By repeating this procedure (formally this is another induction) we get the required set \( \bar{A} \subset \mathbb{Z}^d \) and 1-Lipschitz map \( \bar{f} : \bar{A} \rightarrow H \). This completes the proof. \( \square \)

4. Extensions of Theorem 1.1

There are two immediate extensions of the theorem; the proofs of these extensions are very similar and are left to the reader. The first extension deals with other Lipschitz constants; since we are interested in Lipschitz maps between graphs we restrict our attention to integral Lipschitz constants.

Corollary 4.1. Let \( t \in \mathbb{N} \) and \( H \) be a connected graph. Every \( t \)-Lipschitz map \( f : A \subset \mathbb{Z}^d \rightarrow H \) has a \( t \)-Lipschitz extension to \( \mathbb{Z}^d \) if and only if

for all balls \( B_1, B_2, \ldots B_{2d} \) of radii multiples of \( t \) mutually intersect \( \implies \cap B_i \neq \emptyset \).
In particular, if $H$ is a $\mathbb{Z}^d$-Kirszbraun graph then all $t$-Lipschitz maps $f : A \subset \mathbb{Z}^d \to H$ have a $t$-Lipschitz extension. However it is easy to construct graphs $G$ and $H$ for which $G$ is $H$-Kirszbraun but there exists a 2-Lipschitz map $f : A \subset G \to H$ which does not have a 2-Lipschitz extension. But before we state the example we need the following simple proposition.

**Proposition 4.2.** Let $G$ be a finite graph with diameter $n$ and $H$ be a connected graph such that $B_n^H(v)$ is $G$-Kirszbraun for all $v \in H$. Then $H$ is a $G$-Kirszbraun graph.

**Proof.** Let $f : A \subset G \to H$ be 1-Lipschitz and pick $a \in A$. Then $\text{Image}(f) \subset B_n^H(f(a))$. Since $B_n^H(f(a))$ is $G$-Kirszbraun the result follows. $\square$

Since trees are Helly graphs, we have as an immediate application of the above that $C_n$ is $G$-Kirszbraun if diameter$(G) \leq n - 1$. For instance $C_6$ is $T(1,1,1,1,1,1)$-Kirszbraun. Label the leaves of $T(1,1,1,1,1,1)$ as $a_i; 1 \leq i \leq 6$ respectively and consider the map

$$f : \{a_1, a_2, a_3, a_4, a_5, a_6\} \subset T(1,1,1,1,1,1) \to C_6$$

given by $f(a_i) := i$.

The function $f$ is 2-Lipschitz but it has no 2-Lipschitz extension to $T(1,1,1,1,1,1)$.

In the study of Helly graphs it is well-known (look for instance at [1, Section 3.2]) that results which are true with regard to 1-Lipschitz extensions usually carry forward to graph homomorphisms in the bipartite case after some small technical modifications. This is also true in our case.

A bipartite graph $H$ is called bipartite $(n,m)$-Helly if for balls $B_1, B_2, B_3, \ldots, B_n$ (if $n \neq \infty$ and any finite collection otherwise) and a partite class $H_1$, we have that any subcollection of size $m$ among $B_1 \cap H_1, B_2 \cap H_1, \ldots, B_n \cap H_1$ has a non-empty intersection implies

$$\bigcap_{i=1}^{n} B_i \cap H_1 \neq \emptyset.$$

Let $G, H$ be bipartite graphs with partite classes $G_1, G_2$ and $H_1, H_2$ respectively. The graph $H$ is called bipartite $G$-Kirszbraun if for all 1-Lipschitz maps $f : A \subset G \to H$ for which $f(A \cap G_1) \subset H_1$ and $f(A \cap G_2) \subset H_2$ there exists $\tilde{f} \in \text{Hom}(G, H)$ extending it.

**Corollary 4.3.** A graph is bipartite $\mathbb{Z}^d$-Kirszbraun if and only if it is bipartite $(2d,2)$-Helly.

As noted at the end of Section 1, as a consequence of Theorem 1.1, the graph $\mathbb{Z}^2$ is not $(4,2)$-Helly. However it is bipartite $(\infty,2)$-Helly; this will follow from the discussion after Proposition 4.4. Given a graph $H$ we say that $v \sim_H w$ to mean that $(v, w)$ form an edge in the graph. Let $H_1, H_2$ be graphs with vertex sets $V_1, V_2$ respectively, we denote:

1. Their **strong product** by $H_1 \boxtimes H_2$. It is the graph with the vertex set $V_1 \times V_2$ and edges given by

$$(v_1, v_2) \sim_{H_1 \boxtimes H_2} (w_1, w_2) \text{ if } v_1 = w_1, v_2 \sim_{H_2} w_2 \text{ or } v_1 \sim_{H_1} w_1, v_2 = w_2 \text{ or } v_1 \sim_{H_1} w_1, v_2 \sim_{H_2} w_2.$$

2. Their **tensor product** by $H_1 \times H_2$. It is the graph with the vertex set $V_1 \times V_2$ and edges given by

$$(v_1, v_2) \sim_{H_1 \times H_2} (w_1, w_2) \text{ if } v_1 \sim_{H_1} w_1 \text{ and } v_2 \sim_{H_2} w_2.$$

**Proposition 4.4.** If for a graph $G$, graphs $H_1$ and $H_2$ are $G$-Kirszbraun then $H_1 \boxtimes H_2$ is $G$-Kirszbraun. If for a bipartite graph $G$, bipartite graphs $H'_1$ and $H'_2$ are bipartite $G$-Kirszbraun then the connected components of $H'_1 \times H'_2$ are bipartite $G$-Kirszbraun.

**Proof.** We will prove this in the non-bipartite case; the bipartite case follows similarly. Let $f := (f_1, f_2) : A \subset G \to H_1 \boxtimes H_2$ be 1-Lipschitz. It follows that the functions $f_1$ and $f_2$ are 1-Lipschitz as well; hence they have 1-Lipschitz extensions $\tilde{f}_1 : G \to H_1$ and $\tilde{f}_2 : G \to H_2$. Thus $(\tilde{f}_1, \tilde{f}_2) : G \to H_1 \boxtimes H_2$ is 1-Lipschitz and extends $f$. $\square$
Trees are Helly graphs; thus it follows that $\mathbb{Z}$ is bipartite $(\infty, 2)$-Helly. By Proposition 4.4 so are the connected components of $\mathbb{Z} \times \mathbb{Z}$ which are graph isomorphic to $\mathbb{Z}^2$.

5. The recognition and the hole filling problem

In this section we will give a polynomial time algorithm to decide whether a given graph is $\mathbb{Z}^d$-Kirszhbraun and give a simple application. In the following by being given a graph we mean that we are given the adjacency matrix of the graph.

**Proposition 5.1.** There is a polynomial time algorithm for the recognition problem of $(n, m)$-Helly graphs and bipartite $(n, m)$-Helly graphs where $n, m \in \mathbb{N}$.

The case for $n = \infty, m = 2$ this has been proven in [2].

**Proof.** Let us seek the algorithm in the case of $(n, m)$-Helly graphs; as always, the bipartite case is similar. In the following, for a function $\theta : \mathbb{R} \rightarrow \mathbb{R}$ by $t = O(\theta(|H|))$ we mean $t \leq k\theta(|H|)$ where $k$ is independent of $|H|$ but might depend on $m, n$.

1. Determine the distance between the vertices of the graph. This takes time $O(|H|^2)$.
2. Now make a list of all the collections of balls; each collection being of cardinality $n$. Since the diameter of the graph $H$ is bounded by $|H|$; listing the centers and the radii of the balls takes time $O(|H|^{2n})$.
3. Find the collections for which all the subcollections of cardinality $m$ intersect. For each collection, this step takes time $O(|H|)$.
4. Check if the intersection of the balls in the collections found in the previous step is non-empty. This step again takes time $O(|H|)$.

The total time taken is $O(|H|^{2n+2})$. □

We will end this section with a simple application (also look at the motivations mentioned in Section 2). Fix $d \geq 2$. By a box $B_n$ in $\mathbb{Z}^d$ we mean a subgraph $\{0, 1, \ldots, n\}^d$ and by the boundary $\partial_n$ we mean the internal vertex boundary of $B_n$, that is, vertices of $B_n$ where at least one of the coordinates is either 0 or $n$. The hole-filling problem asks: Given a graph $H$ and a graph homomorphism $f \in Hom(\partial_n, H)$, does it extend to a graph homomorphism $\tilde{f} \in Hom(B_n, H)$?

**Proposition 5.2.** Let $H$ be a finite bipartite $(2d, 2)$-Helly graph. Then there is a polynomial (in the size of the box and $|H|$) time algorithm for the hole-filling problem.

The same holds true in the context of 1-Lipschitz maps for $(2d, 2)$-Helly graphs; the algorithm is similar. In general without the assumption that $H$ is $(2d, 2)$-Helly the crude upper bound for the problem is exponential.

**Proof.** In the following, for a function $\theta : \mathbb{R}^2 \rightarrow \mathbb{R}$, by $t = O(\theta(|H|, n))$ we mean $t \leq k\theta(|H|, n)$ where $k$ is independent of $|H|$ and $n$. Let $f \in Hom(\partial_n, H)$ be given. Since $H$ is bipartite $(2d, 2)$-Helly graph, by Theorem 1.1, $f$ extends to $B_n$ if and only if $f$ is 1-Lipschitz. Thus to decide the hole-filling problem we need to determine whether or not $f$ is 1-Lipschitz. This can be decided in polynomial time:

1. Determine the distance between the vertices of the graph. This takes time $O(|H|^2)$.
2. For each pair of vertices in the graph $\partial_n$, determine the distance between the pair and their image under $f$ and verify the Lipschitz condition. This takes $O(n^{2d-2})$.

The total time taken is $O(n^{2d-2} + |H|^2)$. □

For boxes in $\mathbb{Z}^2$ and $H = \mathbb{Z}$ this algorithm can be improved as in [9] to obtain the optimal complexity of $n \log n$.  

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6. FURTHER PROBLEMS

(1) In the view of our motivation, we focused on the $\mathbb{Z}^d$-Kirszbraun property. It will be interesting to find characterizations for other domain graphs like the triangular or the hexagonal lattice.

(2) Give a sharper time bound on the recognition problem as in Proposition 5.1.

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REFERENCES

[1] H.-J. Bandelt and V. Chepoi. Metric graph theory and geometry: a survey. In Surveys on discrete and computational geometry, volume 453 of Contemp. Math., pages 49–86. Amer. Math. Soc., Providence, RI, 2008.
[2] H.-J. Bandelt and E. Pesch. Dismantling absolute retracts of reflexive graphs. European J. Combin., 10(3):211–220, 1989.
[3] L. Danzer, B. Grünbaum, and V. Klee. Helly’s theorem and its relatives. In Proc. Sympos. Pure Math., Vol. VII, pages 101–180. Amer. Math. Soc., Providence, R.I., 1963.
[4] H. Federer. Geometric measure theory. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.
[5] E. Helly. Über mengen konvexer körper mit gemeinschaftlichen punkten. Jahresbericht der Deutschen Mathematiker-Vereinigung, 32:175–176, 1923.
[6] M. Kirszbraun. Über die zusammenziehende und lipschitzsche transformationen. Fundamenta Mathematicae, 22(1):77–108, 1934.
[7] U. Lang and V. Schroeder. Kirszbraun’s theorem and metric spaces of bounded curvature. Geom. Funct. Anal., 7(3):535–560, 1997.
[8] G. Menz and M. Tassy. A variational principle for a non-integrable model. http://arxiv.org/abs/1610.08103, 2016.
[9] I. Pak, A. Sheffer, and M. Tassy. Fast domino tileability. Discrete Comput. Geom., 56(2):377–394, 2016.
[10] S. Sheffield. Random surfaces. Astérisque, (304):vi+175, 2005.
[11] F. A. Valentine. A Lipschitz condition preserving extension for a vector function. Amer. J. Math., 67:83–93, 1945.

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