Applications of Partial Differential Equations

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Abstract. There are many engineering, physical and other applications that need special mathematical equations to solve them. One of the most important equations that have a large role in the applications of science is partial differential equations. The partial differential equations are two types: linear and nonlinear. There is a wide range of functional equations that highlight the importance of PDE for example: electrostatics, heat conduction, transmission line, quantum mechanics and wave theory. In this study we will discuss the theoretical part of those applications that are used PDEs, trying to clarify more than order of the partial differential equations in one, two, three dimensions.

Keywords: PDE, linear, nonlinear, application, dimension, partial, differential, equation.

Introduction:

The partial differential equations are one of the most important mathematical equations used in many applications and they have an important role in the interpretation of many phenomena. Partial differential equations are used in the description of many sciences such as Laplace’s equation, transmission line equations, heat conduction equation, Poisson’s equation (electrostatics, elasticity theory and elsewhere), Helmholtz’s equation (wave theory), Schrodinger’s equation (quantum mechanics), Transverse vibrations equation and wave equation.

There are many different types of partial differential equations (PDEs) that differ according to order (first order, second order and so on) each of them has its characteristics. For example, the heat conduction equation is used to describe the spread of heat in a medium where the spatial and temporal distribution of the diffusion and evolution of temperature is homogeneous over time; in addition to it also describes the spread of solute in the solvent where the concentration changes with time and space.

Wave equation describing the average time with dependent displacements through elastic materials and in light of this, we find solutions to the propagation of disturbances. The wave equation is used to describe the waves that arise from the center of the earthquake such as P-wave (light shaking); it is one of the waves of elastic materials and it called seismic waves. It is considered the fastest wave in seismic waves, so when an earthquake is done, it is considered the first signal in the earthquake monitoring device. S-wave (strong shaking), it is called shear wave; it is a seismic wave which is shakes the Earth in a vertical direction on the propagation of the wave. The wave equation is also used to describe vibrations of musical instruments such as guitar and description of sound waves, which are used to describe ripples that appear on the water surface as a result of stone throwing.

One of the important equations in partial differential equations is the Laplace equation, which is used in many applications such as applications of static electricity, geometry, fluid flow, potential theory.
and harmonic functions theory. Partial differential equations help to find solutions of many complex theories and contribute to the development of the theoretical side of the equation as well as to the application side.

In this paper we present an introduction to partial differential equations, a useful and complex branch in mathematics and it is indispensable. This is in addition to standing on the applications of partial differential equations that explain how they work. This study is a theoretical aspect in the analysis of the real and functional power of these equations. The reference to applications of partial differential equations in different sciences has been used to arrive at a complete and real modeling of the importance of these equations in applied fields.

1. Linear Partial Differential Equations:

In order to enter into the study of partial differential equations, we must first examine four types of linear equations that are considered as the key to entering into PDEs. These equations are fundamental in the study of Partial Differential Equations. Each of the four linear equations has its characteristics and is considered as the primary models of partial differential equations. The methods used to solve partial differential equations are an initial image of linear equations.

In linear equations the function \( u \) is expressed as unknown function on open set \( U \) where \( U \subset \mathbb{R}^n \).

We express the dependent variable here by \( u \) and the independent variable is \( x \) where:

\[
x = (x_1, x_2, x_3, \ldots, x_n)
\]

and

\[
x \in U
\]

where \( U \) is considered a vector for independent variables. This applies to all four linear equations.

It is noteworthy that \( x \) sometimes expresses the spatial variables only so we need a variable which expresses time and here we find \( t \). Thus we find that we can define partial differential equations as an equation containing both \( x, u, \) partial derivatives of \( u \) or \( x, u, t, \) partial derivatives of \( t \).

We also mentioned that the partial differential equations have an order. We recognize the partial differential equation order by a higher order for derivatives, for example: if the higher derivative in the equation is 2, the equation here is called a second order partial differential equation.

1.1. The Linear Transport Equation:

\[
u_t + cu_x = 0
\]

The linear transport equation is the first order partial differential equation. It expresses the amount of motion through a constant speed \( c \) of quantity \( u \) in the presence of two variables that are spatial variable \( x \) and temporal variable \( t \). The solution here is a wave moving according to the value of \( c \):

- When \( c > 0 \) the wave moves right.
- When \( c < 0 \) the wave moves left.

When solving this equation, the solution formula will be:

\[
u(x, t) = f(x - ct)
\]
1.2. The Heat Equation

\[ u_t = k \Delta u \text{ in one dimension} \]

In this equation heat is expressed in a homogeneous medium by \( u(x, t) \) as a function in \( x \) and \( t \) and \( k \) is a constant where \( k > 0 \). Here we can define the Laplacian (\( \Delta \)) in Cartesian coordinates as:

\[ \Delta f(x) = \nabla \cdot \nabla f(x) = \sum_{i=1}^{n} \frac{\partial^2 f(x)}{\partial x_i^2} \]

It is worth mentioning that the temperature equation is also called the diffusion equation is used to express the spread of something in the medium and \( u \) is expresses of the concentration of the amount of diffusing \(^{10}\).

1.3. The Wave Equation

\[ u_{tt} = c^2 \Delta u \]

This equation that expresses the propagation of the wave where \( c \) expresses the wave speed and \( u \) is a displacement as \( u = u(x, t) \). This equation explains the transmission of guitar waves when vibrating. The displacement varies according to the medium in which it is located where:

In the case of the guitar \( x \in \mathbb{R} \)

In the case of the drum membrane \( x \in \mathbb{R}^2 \)

And \( u_{tt} \) is acceleration; it derives for the second time from speed equation. This gives the wave equation characteristics different from those shown in the heat equation \(^{11}\).

1.4. Laplace’s Equation

\[ \Delta u = 0 \text{ as a first order of PDE} \]

\[ \Delta^2 u = 0 \text{ as a second order of PDE} \]

In Laplace's equations \( u(x, t) \) is independent of time \( t \). They appear in many applications such as:

Fluid mechanics (steady state flow of inviscid fluids).
Heat conduction (steady state).

Electrostatics.

Gravitation.

Note: steady state is the state of the medium or system that does not change as a result of the addition of an external factor.}

In two dimensional heat conduction equation where $t$ is independent variable and $\frac{\partial u}{\partial t} = 0$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

When in three dimensional:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

When starting to solve any equation of Laplace Equation in any of the applications used them, they solve in region R for specific boundary conditions where:

Dirichlet boundary conditions where $u$ specified on curve $C$ in two dimensions or surface $S$ in three dimensions.

Neumann boundary conditions where the derivate of $u$ specified on curve $C$ in two dimensions or surface $S$ in three dimensions.

In this condition, we will mix between Dirichlet boundary conditions and Neumann boundary conditions.}

1.5. Other important applications of Partial Differential Equations:

a) Helmholtz’ equation (in wave theory, it is two dimensional form in $(x, y)$ dimensions) / second order PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$$

b) Poisson’s equation (in elasticity theory “electrostatics”, it is two dimensional form) / second order PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$$

c) Schrodinger’s equation (in quantum mechanics, it is two dimensional form in $(x, y, z)$ dimensions) / second order PDE:
\[- \frac{h^2}{8\pi^2 m} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \right) = E\psi \]

*where h is Planck's constant*

d) Transverse vibrations equation (in homogeneous rod) / forth order PDE:

\[ a^2 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0 \]

2. Nonlinear Partial Differential Equations:

We have already mentioned some of the applications of linear partial differential equations, and although applications to use linear equations are wide-ranging but there is a set of applications that does not apply linear partial differential equations. Here came the need to Nonlinear Partial Differential Equations. Now we will review some applications of these equations in the scientific fields which have contributed greatly to solve complex engineering and physical problems.

2.1. The Inviscid Burgers Equation:

This equation is used in many mathematics applications such as gas dynamics, fluid mechanics and nonlinear acoustics. The equation is based on the following formula:

\[ u_t + uu_x = 0 \]

\[ \frac{\partial u}{\partial t} + uu_x = 0 \]

This equation is most famous example of first order nonlinear equation. It looks like transport equation (linear PDE) but appearance of term \( uu_x \) made it nonlinear.

We can find other forms of Burgers Equation for kinematic viscosity we get viscous Burgers Equation:

\[ \frac{\partial u}{\partial t} + uu_x = \nu \frac{\partial^2 u}{\partial x^2} \]

2.2. Fisher’s Equation:

\[ u_t = \Delta u + f(u), \]

\[ f(u) = u(1-u) \]

This equation is used as a model of spatial distribution of population dynamics. It is worth mentioning that the equation \( f(u) = u(1-u) \) is an ordinary differential equation “ODP”.

This ODP is used to describe limited population growth. So we can say that Fisher’s Equation describes the balance between nonlinear reaction and linear diffusion. In the solution model for this problem, derivatives and integrations are replaced by algebraic matrices that give an approximate solution. This method is called Sinc collocation method.
2.3. The Porous Medium Equation:

\[ u_t = \Delta(u^m) \]

where, \( m > 0 \) is a constant

The heat equation is one of the most important fundamental equations in PDE. From its initial version is derived many formulas that are used in other applications. So the Porous Medium Equation (PME) is called the nonlinear heat equation.

This equation is used as a model for compacted soil or porous rock. The variable \( u \) is depends on \( x \) and \( t \), \( u(x,t) \geq 0 \). This variable is used to measure the density of the compressible gas in a specific location (\( x \)) at specific time (\( t \)). For \( m \) (constant), it depends on the equation of the gas pressure related to its density. When \( m = 1 \) this equation will be heat equation (linear), but for \( m \neq 1 \) the equation will be nonlinear. In this case \( m \geq 2 \) for gas flow.

This is additionally used in many physical applications such as:

- Boundary layer theory.
- Fluid flow
- Diffusion or heat transfer
- Lubrication
- Mathematical biology

2.4. The Korteweg-deVries (KdV) Equation:

\[ u_t + uu_x + u_{xxx} = 0 \]

This equation of the third order is used to describe the waves of water when there is a height of the water wave expressed by \( u = (x,t) \). It is also used to solve traveling wave equations (solitary waves), in this case the height should be symmetric about the crest. In addition to, it also offers an approximate solution model for fluid mechanics.

2.5. The Shallow Water Equations:

\[ h_t + (hv)_x = 0 \]

\[ v_t + vv_x - gh_x = 0. \]

Where:

- \( (g) \) is gravitational acceleration.
- \( (h) \) is the height of a shallow layer of water.
- \( (v) \) is the velocity of a shallow layer of water.
• \((x)\) is the horizontal spatial variable.

• \((t)\) is variable which represents time.

This equation (SWE) is used to describe the flow under the pressure surface of the liquid (free surface). It belongs to hyperbolic partial differential equations set 19. In other words, this equation can be considered to be used to describe a thin layer of liquids that have a fixed density as it is bounded by a free surface and the bottom in hydrostatic 20.

2.6. The Navier-Stokes Equations:

It can say that of Navier Stock equations are sum of the gravitational forces, pressure forces and viscous forces are equal to multiply mass by acceleration 22.

\[
\mathbf{F}_g + \mathbf{F}_p + \mathbf{F}_v = m\mathbf{a}
\]

Gravitational forces:

\[
\text{force of gravity (} \mathbf{F}_g \text{)} = m \mathbf{g} = \rho \mathbf{g} dxdydz
\]

\[
= \rho \left( \frac{\partial |v|}{\partial t} + \left[ v_x \frac{\partial |v|}{\partial x} \right] \hat{i} + \left[ v_y \frac{\partial |v|}{\partial y} \right] \hat{j} + \left[ v_z \frac{\partial |v|}{\partial z} \right] \hat{k} \right) dxdydz
\]

\[
\rho g_x + \sum F_x = \rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right)
\]

\[
\rho g_y + \sum F_y = \rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right)
\]

\[
\rho g_z + \sum F_z = \rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right)
\]

Pressure forces

\[
\mathbf{F}_p = -\nabla P * dV = -\nabla P dxdydz
\]

\[
\rho g_x \frac{\partial P}{\partial x} + \sum F_x = \rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right)
\]

\[
\rho g_y \frac{\partial P}{\partial y} + \sum F_y = \rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right)
\]

\[
\rho g_z \frac{\partial P}{\partial z} + \sum F_z = \rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right)
\]

Viscous forces
\[ \vec{F}_v = \nabla \tau \ast dV = -\nabla \tau dx dy dz \]

\[ \rho \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \sum F_x = \rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) \]

\[ \rho \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \sum F_y = \rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) \]

\[ \rho \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \sum F_z = \rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) \]

- Stress strain equation:

\[ \tau_{xy} = \tau_{yx} = \mu (\epsilon_{xy} + \epsilon_{yx}) = \mu \left( \frac{\partial \epsilon_{xy}}{\partial t} + \frac{\partial \epsilon_{yx}}{\partial t} \right) = \mu \left( \frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) \]

\[ \tau_{yz} = \tau_{zy} = \mu (\epsilon_{yz} + \epsilon_{zy}) = \mu \left( \frac{\partial \epsilon_{yz}}{\partial t} + \frac{\partial \epsilon_{zy}}{\partial t} \right) = \mu \left( \frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right) \]

\[ \tau_{xz} = \tau_{zx} = \mu (\epsilon_{xz} + \epsilon_{zx}) = \mu \left( \frac{\partial \epsilon_{xz}}{\partial t} + \frac{\partial \epsilon_{zx}}{\partial t} \right) = \mu \left( \frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z} \right) \]

\[ \tau_{xx} = -\frac{2}{3} \mu \nabla \ast \vec{v} + 2 \mu \frac{\partial v_x}{\partial x} \]

\[ \tau_{yy} = -\frac{2}{3} \mu \nabla \ast \vec{v} + 2 \mu \frac{\partial v_y}{\partial y} \]

\[ \tau_{zz} = -\frac{2}{3} \mu \nabla \ast \vec{v} + 2 \mu \frac{\partial v_z}{\partial z} \]

So by substituting with stress strain equation into momentum equation, we will find:

\[ \rho \frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) = \rho \left( \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) \]

\[ \rho \frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) = \rho \left( \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) \]

\[ \rho \frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) = \rho \left( \frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) \]
3. Conclusion

This research was a review of some of the models of linear and nonlinear partial differential equations. The actual applications using these equations were very important in science and engineering. The existence of such equations added much to the possibility of finding suitable solutions for many complex problems.

The mathematical part is now inseparable from the sciences such as physics, engineering and others. There is a strong correlation that occurred in the presence of those equations, which facilitated a lot for scientists to reach important explanations that benefit science and humanity.

It is worth mentioning that differential equations have a great influence on solving many physical problems and other sciences. Before the existence of differential equations, it was the case between scientists to reach the governing equation of the subject of research without reaching a solution.

Therefore, the laws of mathematics in general, linear differential equations and non-linear differential equations should be pursued to keep up with the analysis and solutions of many natural phenomena that are the subject of discussions of the scientific community.

Scientists should continue to solve mathematical problems and include equations necessary to help other sciences. This is in addition to resorting to the technical part and software to facilitate the introduction of equations and solve them and get more approximate results. The presence of technology has an active and strong role in scientific research and scientific community and easy access to many solutions for scientific problems, so we must include all the developments of science to the technological side to save time and effort.

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