Dimensional reduction of the CPT-even electromagnetic sector of the Standard Model Extension

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The CPT-even abelian gauge sector of the Standard Model Extension is represented by the Maxwell term supplemented by \((K_F)_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma}\), where the Lorentz-violating background tensor, \((K_F)_{\mu\nu\rho\sigma}\), possesses the symmetries of the Riemann tensor. In the present work, we examine the planar version of this theory, obtained by means of a typical dimensional reduction procedure to \((1 + 2)\) dimensions. The resulting planar electrodynamics is composed of a gauge sector containing six Lorentz-violating coefficients, a scalar field endowed with a noncanonical kinetic term, and a coupling term that links the scalar and gauge sectors. The dispersion relation is exactly determined, revealing that the six parameters related to the pure electromagnetic sector do not yield birefringence at any order. In this model, the birefringence may appear only as a second order effect associated with the coupling tensor linking the gauge and scalar sectors. The equations of motion are written and solved in the stationary regime. The Lorentz-violating parameters do not alter the asymptotic behavior of the fields but induce an angular dependence not observed in the Maxwell planar theory.

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I. INTRODUCTION

Lorentz symmetry has been considered as a fundamental cornerstone of physics since the establishment of the special theory of relativity. The inquiry about to what extent this is an exact symmetry of nature constitutes the scope of most investigations in Lorentz violation nowadays. A motivation for such studies is that the observation of Lorentz symmetry small violations in current low-energy phenomena could indicate new routes for developing theories at the Planck-scale. The Standard Model Extension (SME) \([1], [2]\) is a theoretical framework that incorporates Lorentz-violating coefficients to the standard model and to general relativity, and has served as a suitable tool for constructing interesting approaches in this area.

The gauge sector of the SME embraces twenty three Lorentz-violating coefficients that yield some unconventional phenomena such as vacuum birefringence and Cherenkov radiation. The coefficients are usually classified in accordance with some criteria. One criterion is the possible violation of the CPT symmetry, being the parameters CPT-odd or CPT-even. The Carroll-Field-Jackiw (CFJ) term \([3]\) is the CPT-odd term of the SME, composed of four parameters, that engenders a parity-odd and birefringent electrodynamics whose properties were largely examined in connection with many diverse issues: consistency aspects and modifications induced in QED \([4, 6]\), supersymmetry \([7]\), generation by radiative correction \([8]\), vacuum Cherenkov radiation emission \([9]\), electromagnetic propagation in waveguides \([10]\), Casimir effect \([11]\), finite-temperature contributions and Planck distribution \([12, 13]\), anisotropies of the Cosmic Microwave Background Radiation \([14]\), classical electrodynamics solutions \([15]\), dimensional reduction \([16, 17, 18]\).

The CPT-even gauge sector of the SME has been studied since 2002, after the pioneering contributions by Kostelecky & Mewes \([19, 20]\). This sector is represented by the nineteen components of the fourth-rank tensor, \((K_F)_{\alpha\nu\rho\sigma}\), endowed with the same symmetries of the Riemann tensor, and a double null trace. The nineteen components are grouped in two subclasses: the ten birefringent ones, which are severely constrained by astrophysical tests of birefringence, and the nine nonbirefringent ones. This latter group can only be constrained by laboratory tests, which are continuously being proposed and realized. High-quality cosmological spectropolarimetry data \([21]\) have been employed to impose stringent upper bounds (as tight as \(10^{-37}\)) on the ten birefringent LIV parameters. On the other hand, the Cherenkov radiation \([22]\) and the absence of emission of Cherenkov radiation by UHECR (ultrahigh energy cosmic rays) \([23, 24]\) have been used to impose upper bounds on the nonbirefringent components. Photon-fermion vertex corrections induced by the LIV coefficients \([25, 27]\) have been employed to state upper bounds on these coefficients, as well.

The dimensional reduction of the CPT-odd gauge term of Standard Model Extension was performed in Ref. [16].
yielding a planar electrodynamics composed of the Maxwell-Chern-Simons electrodynamics coupled with a massless scalar field - the remanent of the third spacial component of the four-potential \( A^{(3)} = \phi \). It is interesting to note that the Chern-Simons terms appears naturally in such reduction. This planar model was studied in its consistency (stability, causality and unitarity) and had its classical solutions determined in Ref. [17]. The dimensional reduction of the Abelian-Higgs Maxwell-Carroll-Field-Jackiw model was performed in Ref. [18].

In the present work, we realize the dimensional reduction of the CPT-even gauge sector of the SME to \((1+2)\) -dimensions following the prescription adopted in Refs. [16], [18], that is, freezing the third spatial component in such a way the fields can not exhibit any dependence on it. The arising scalar field is the remanent of the third spacial component of the four-potential \( A^{(3)} = \phi \). We obtain a planar theory composed of the electromagnetic sector, a scalar massless field with noncanonical kinetic term, and a mixing term that couples the scalar and gauge sectors. In the gauge sector, Lorentz violation is induced by a fourth-rank tensor, \( Z_{\mu\nu\lambda\kappa} \), endowed with the symmetries of the Riemann tensor, which renders six independent components. The scalar sector presents an additional noncanonical kinetic term, \( C_{\mu\lambda} \partial^\mu \phi \partial^\lambda \phi \), where \( C_{\mu\lambda} \) is a Lorentz-violating symmetric second-rank tensor with six independent components. The scalar and gauge sectors are coupled by the third-rank tensor, \( T_{\mu\lambda\kappa} \), whose symmetries imply eight independent components. The traceless condition, coming from the \((1+3)\) dimensional model, reduced the number of independent parameters to nineteen. Once the planar model and its structural features are set up, we examine the effects of the Lorentz-violating parameters in the electromagnetic classical solutions (in the stationary regime), using the Green’s method. As in the original four-dimensional counterpart, stationary currents and static charge are able to create both magnetic and electric fields in this planar theory. The stationary solutions reveal that Lorentz-violating coefficients do not modify the asymptotic radial behavior of the Maxwell planar electrodynamics, but are able to generate terms with angular dependence. The dispersion relation stemming from the pure gauge sector allows to notice that the planar electrodynamics is free from birefringence, which is implied only at second order by the components of the tensor \( T_{\mu\lambda\kappa} \).

This work is organized as follows. In Sec. II, we briefly present general features of the CPT-even electrodynamics of the SME. In Sec. III, we perform the dimensional reduction procedure that leads to the planar theory of interest. In Sec. IV, we study this planar theory focusing on the equations of motion and the attainment of the dispersion relation. In Sec. V, we evaluate the classical stationary solutions for electric and magnetic field, remarking the deviations induced by the Lorentz-violating terms. In Sec. VI, we present our conclusions and final remarks.

II. THE PARAMETRIZATION OF THE CPT-EVEN GAUGE SECTOR OF SME IN (1+3) DIMENSIONS

The CPT-even sector of the Lorentz-violating electrodynamics of the SME photon sector is represented by the following Lagrangian:

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} (K_F)_{\mu\bar{\nu}\lambda\bar{\kappa}} F^{\mu\bar{\nu}} F^{\lambda\kappa} - J_\mu A^\mu,
\]

where the indices with hat, \( \hat{\mu}, \hat{\nu} \), run from 0 to 3, \( F_{\mu\bar{\nu}} \) is the usual electromagnetic field tensor, \( A^\mu \) is the four-potential, \( (K_F)_{\mu\bar{\nu}\lambda\bar{\kappa}} \) is a renormalizable, dimensionless coupling which has the same symmetries as the Riemann tensor

\[
(K_F)_{\mu\bar{\nu}\lambda\bar{\kappa}} = -(K_F)_{\nu\bar{\mu}\lambda\bar{\kappa}}, \quad (K_F)_{\mu\bar{\nu}\lambda\bar{\kappa}} = -(K_F)_{\bar{\mu}\nu\lambda\bar{\kappa}} ,\quad (K_F)_{\mu\bar{\nu}\lambda\bar{\kappa}} = (K_F)_{\lambda\bar{\kappa}\mu\bar{\nu}},
\]

\[
(K_F)^{\mu\bar{\nu}\lambda\bar{\kappa}} + (K_F)^{\mu\bar{\kappa}\lambda\bar{\nu}} + (K_F)^{\mu\bar{\kappa}\bar{\nu}\lambda} = 0.
\]

and a double null trace, \( (K_F)^{\mu\bar{\nu}} \mu\bar{\nu} = 0 \). The equation of motion is

\[
\partial_\nu F^{\nu\bar{\mu}} - (K_F)^{\mu\bar{\nu}\lambda\bar{\kappa}} \partial_\nu F_{\lambda\bar{\kappa}} = 0.
\]

The tensor \( (K_F)_{\alpha\nu\rho\sigma} \) has 19 independent components, from which nine do not yield birefringence. A very useful parametrization for addressing this theory is the one presented in Refs. [19, 20], in which these 19 components are
where \( \kappa \) is the planar version of the original \((K_F)\)-tensor, that is, \( Z_{\mu\nu\lambda\kappa} = [ (K_F)_{\mu\nu\lambda\kappa} ]_{1+2} \). It fulfills the following symmetry properties

\[
Z_{\mu\nu\lambda\kappa} = Z_{\lambda\kappa\mu\nu}, \quad Z_{\mu\nu\lambda\kappa} = -Z_{\nu\mu\lambda\kappa}, \quad Z_{\mu\nu\lambda\kappa} = -Z_{\mu\kappa\nu\lambda}, \quad Z_{\mu\kappa\nu\lambda} + Z_{\mu\lambda\kappa\nu} + Z_{\mu\nu\lambda\kappa} = 0.
\]

These symmetries imply \( Z_{\mu\nu\lambda\kappa} F^{\mu\nu} F^{\lambda\kappa} = Z_{\mu\nu\lambda\kappa} F^{\mu\nu} F^{\lambda\kappa} \), leading to

\[
(K_F)_{\mu\nu\lambda\kappa} F^{\mu\nu} F^{\lambda\kappa} = Z_{\mu\nu\lambda\kappa} F^{\mu\nu} F^{\lambda\kappa} + 4Z_{\mu\nu\lambda\kappa} F^{\mu\nu} F^{\lambda\kappa} + 4Z_{\mu\kappa\nu\lambda},
\]

where \( F^{\mu\nu} = \partial^{\mu} \phi \), and it were defined new second-rank and third-rank tensors

\[
T_{\mu\lambda\kappa} = (K_F)_{3\mu\lambda\kappa}, \quad C_{\mu\lambda} = (K_F)_{\mu\lambda\kappa}.
\]
With it, the dimensionally reduced Lagrangian is

\[
\mathcal{L}_{(1+2)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} Z_{\mu\nu} F^{\mu\nu} F^{\lambda\kappa} + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - C_{\mu\lambda} \partial^{\mu} \phi \partial^{\lambda} \phi + T_{\mu\lambda\kappa} \partial^{\mu} \phi F^{\lambda\kappa} - J_\mu A^\mu - J_\phi,
\]

(16)

The presence of the tensor \(C_{\mu\lambda}\) provides a noncanonical kinetic term for the scalar field. Some attempts of proposing Lorentz-violating constructions for topological defects with a term like this are already known in literature [20]. This term has recently been used to study acoustic black holes with Lorentz-violation in (1+2) dimensions [30] and also the Bose-Einstein condensation of a bosinic ideal gas [31]. The tensor \(T_{\mu\lambda\kappa}\), in turn, is responsible for the coupling between the scalar and gauge sectors in this planar theory. These two tensors satisfy the following symmetries:

\[
\begin{align*}
C_{\mu\lambda} &= C_{\lambda\mu}, \\
T_{\mu\lambda\kappa} &= -T_{\mu\kappa\lambda}, \\
T_{\mu\lambda\kappa} + T_{\lambda\kappa\mu} + T_{\kappa\mu\lambda} &= 0,
\end{align*}
\]

(17) (18) (19)

The (double) traceless property of the \(K_F\) tensor is now read as

\[
Z_{\mu\nu} \mu^\nu + 2C^\alpha_\alpha = 0.
\]

(20)

By the relations (17-19), we conclude that the tensors \(C_{\mu\lambda}, Z_{\mu\nu\lambda\kappa}, T_{\mu\lambda\kappa}\) contain six, six and eight independent components, respectively, comprising twenty parameters. The relation (20) states a constraint between them, remaining nineteen independent components, the same number of the tensor \(K_F\) (before the dimensional reduction).

The reduced model (16) is invariant under the following local gauge transformation:

\[
A_\mu \rightarrow A_\mu + \partial_\mu \Lambda, \quad \phi \rightarrow \phi,
\]

(21)

in such a way it preserves the U (1) local gauge symmetry of the 4-dimensional model. The full Lagrangian of this model is written as

\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - C_{00} (\partial_0 \phi)^2 + C_{0i} (\partial_0 \phi) (\partial_i \phi) - C_{ij} (\partial_i \phi) (\partial_j \phi) - (Z_{012} E^i B - \frac{1}{2} (k_{DE})_{ij} E^i E^j - \frac{1}{2} s B^2 - T_{00\kappa} \partial_0 \phi E^\kappa - \epsilon_{ij} T_{0ij} \partial_0 \phi E^j + \epsilon_{ij} T_{ilj} \partial_i \phi B).
\]

(22)

As in the original four-dimensional model, the Lorentz-violating coefficients have definite parity. In (1+2) dimensions, the parity operator acts doing \(r \rightarrow (-x, y)\), so that the fields go as

\[
A_0 \rightarrow A_0, \quad A \rightarrow (-A_x, A_y), \quad E \rightarrow (-E_x, E_y), \quad B \rightarrow -B.
\]

(23)

For more details, see Ref. [32]. We consider that the field \(\phi\) behaves as a scalar, \(\phi \rightarrow \phi\). This allows to conclude that this planar model possesses twelve parity-even components, and nine parity-odd ones, as shown in the Table I. Further, we see that the trace relation (20) involves only parity-even coefficients, whereas the relation (19) embraces only parity-odd parameters (when the indices of the tensor \(T_{\kappa\mu\lambda}\) assume three different values, \(T_{012} + T_{120} + T_{201} = 0\)). These two relations reduce the number of independent components from twenty one to nineteen, as it is expected. The fact that the components of the vectors \((Z_{012}, T_{00i})\) transform distinctly is a consequence of the way the vectors \(r, A, E\) behave under parity.\(^1\)

| Components | N  | N  |
|------------|----|----|
| Parity-even | \(C_{00}, C_{0i}, C_{1i}, C_{22}, L_1, (k_{DE})_{11}, (k_{DE})_{22}, s, T_{002}, T_{101}, T_{202}, T_{112}\) | 12 | 11 |
| Parity-odd | \(C_{0i}, C_{12}, L_2, (k_{DE})_{12}, T_{001}, T_{012}, T_{022}, T_{201}, T_{212}\) | 9  | 8  |
| Total      | 21 | 19 |

\(^1\) Note that in the case the field \(\phi\) behaves like a pseudoscalar \((\phi \rightarrow -\phi)\), the behavior of the components \(T_{00i}, T_{0ij}, T_{i0j}, T_{ilj}\) is reversed under parity.
If one neglects the coupling between the scalar and gauge sectors \((T_{\mu\lambda\kappa} = 0)\), one has a planar theory composed by the usual Maxwell electrodynamics modified by the term \(Z_{\mu\nu\lambda\kappa} F^{\mu\nu} F^{\lambda\kappa}\) and a scalar field endowed with a noncanonical kinetic term, whose properties will be examined. The planar Lagrangian density of the electromagnetic sector is

\[
\mathcal{L}_{EM(1+2)} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} Z_{\mu\nu\lambda\kappa} F^{\mu\nu} F^{\lambda\kappa} - J_\mu A^\mu,
\]

which represents a gauge-invariant theory (in the absence of external currents). We should note that the planar tensor \(Z_{\mu\nu\lambda\kappa}\) would possess \(3^4 = 81\) components in the absence of the symmetries. The symmetries properties, however, reduce them to only six independents components:

\[
Z_{0ilm} = [Z_{0112}, Z_{0212}], \quad Z_{0i0m} = [Z_{0101}, Z_{0202}, Z_{0102}], \quad Z_{ijlm} = [Z_{1212}],
\]

It is interesting to note that the permutation symmetry \((13)\) is now just a complementary relation, not implying a new constraint on the components of the tensor \(Z_{\mu\nu\lambda\kappa}\). This planar tensor does not share the double traceless condition of \(F_{\mu\nu}\) anymore. Instead of it holds Eq.\((20)\), that states a relation between its components and the ones of the tensor \(C_{\mu\lambda}\). For this reason, when the gauge and scalar sector are considered together, both ones only contribute with eleven components.

In order to propose an effective parametrization for gauge sector elements, it is helpful to write the Lagrangian element, \(Z_{\mu\nu\lambda\kappa} F^{\mu\nu} F^{\lambda\kappa}\), in terms of the electric and magnetic fields,

\[
Z_{\mu\nu\lambda\kappa} F^{\mu\nu} F^{\lambda\kappa} = 4Z_{0112} E^i B + 4Z_{0i0j} E^i E^j + 4Z_{1212} B^2,
\]

where it was used \(F^{0i} = -E^i\), \(F^{12} = F_{12} = -B\). We should remember that in \((1+2)\)–dimensions the magnetic field is a scalar. Thus, the two elements \(Z_{0ilm}\) can be read as elements of a two-vector,

\[
2Z_{0ilm} = 2Z_{0i12} = L_i, \quad L^i = 2Z^{0i12},
\]

with \(\mathbf{L} = (L_1, L_2)\). The three elements \(Z_{0i0j}\) are written as elements of a symmetric \(2 \times 2\) matrix

\[
2Z_{0i0j} = (k_{DE})_{ij} = (k_{DE})_{ji},
\]

whose components are

\[
k_{DE} = 2 \begin{bmatrix} Z_{0101} & Z_{0102} \\ Z_{0102} & Z_{0202} \end{bmatrix}.
\]

Finally, the single element \(Z_{ijlm}\) plays the role of a scalar,

\[
2Z_{ijlm} = 2Z_{1212} = s.
\]

Using the new definitions, the planar pure electromagnetic Lagrangian \((24)\) takes the form

\[
\mathcal{L}_{EM(1+2)} = \frac{1}{2} \mathbf{E}^2 - \frac{1}{2} (k_{DE})_{ij} E^i E^j - \frac{1}{2} (1 + s) B^2 + (\mathbf{L} \cdot \mathbf{E}) B,
\]

where it was used the contraction

\[
Z_{\mu\nu\lambda\kappa} F^{\mu\nu} F^{\lambda\kappa} = -4(\mathbf{L} \cdot \mathbf{E}) B + 2E^i (k_{DE})_{ij} E^j + 2sB^2.
\]

Another relevant aspect concerns the evaluation of the canonical energy-momentum tensor for the planar Lagrangian, \((24)\), carried out from the usual form \(\Theta^{\beta\rho} = [\partial / \partial (\partial_\beta A_\alpha)] \partial^\rho A_\alpha - g^{\beta\rho} \mathcal{L}\), and leading to the result,

\[
\Theta^{\beta\rho} = -(F^{\beta\alpha} + Z_{\beta\alpha\lambda\kappa} F_{\lambda\kappa}) \partial^\rho A_\alpha - g^{\beta\rho} \mathcal{L}.
\]
The energy density,
\[ \Theta^{00}_{EM} = \frac{1}{2}(E^2 + B^2) - \frac{1}{2} \left( k_{DE} \right)^{ij} E^i E^j + \frac{1}{2} s B^2, \]  
(34)
is obtained by using the Gauss’s law. It is interesting to mention that the same result is achieved via the construction of the density of Hamiltonian, \( H = \pi^\alpha A_\alpha - L \), where \( \pi^\alpha = \partial L / \partial (\partial_\alpha A_\alpha) \) is the conjugate momentum,
\[ \pi^\alpha = -F^{0\alpha} - Z^{0\alpha\lambda\kappa} F_{\lambda\kappa}. \]  
(35)

In components, we have \( \pi^0 = 0 \) and \( \pi^i = E^i - (k_{DE})^{ij} E^j + L^i B \). The pure gauge model has two first class constraints, \( \pi^0 \) and \( \partial_\mu \pi^\mu \), the latter one being the Gauss’s law. The Hamiltonian analysis implies the same energy density of Eq. (42). These outcomes show that the energy density can be regarded as positive definite, once the Lorentz-violating parameters are sufficiently small.

### IV. WAVE EQUATIONS FOR THE PLANAR ELECTRODYNAMICS

In order to obtain the classical solutions of the planar electrodynamics represented by Lagrangian (16), we should write the equations motion. In a general way, such equations are given by
\[ \partial_\alpha F^{\alpha\beta} - Z^{\beta\lambda\kappa} \partial_\alpha F_{\lambda\kappa} - 2T^{\mu\alpha\beta} \partial_\alpha \partial_\mu \phi = J^\beta, \]  
(36)

\[ \Box \phi + T^{\alpha\lambda\kappa} \partial_\alpha F_{\lambda\kappa} - 2C^{\alpha\lambda} \partial_\alpha \partial_\lambda \phi = -J. \]  
(37)

In the absence of the coupling term \( (T^{\mu\alpha\beta} = 0) \), the gauge and scalar sectors become decoupled and classically governed by the following equation:
\[ \partial_\alpha F^{\alpha\beta} - Z^{\beta\lambda\kappa} \partial_\alpha F_{\lambda\kappa} = J^\beta, \]  
(38)

\[ \Box \phi - 2C^{\alpha\lambda} \partial_\alpha \partial_\lambda \phi = -J, \]  
(39)

The main reason for neglecting the tensor \( T_{\mu\lambda\kappa} \) is that it appears as a second order contribution in the equations defined in terms only of the gauge field or the scalar field. In order to verify it, we isolate the scalar field in Eq. (37), in the absence of scalar sources, \( J = 0 \), writing
\[ \phi = -\frac{T^{\alpha\lambda\kappa} \partial_\alpha}{\Box - 2C^{\alpha\lambda} \partial_\alpha \partial_\lambda} F_{\lambda\kappa}. \]  
(40)

Replacing Eq. (16) in Eq. (38), there appears:
\[ \partial_\alpha F^{\alpha\beta} - Z^{\beta\lambda\kappa} \partial_\alpha F_{\lambda\kappa} + \frac{4T^{\mu\alpha\beta} T^{\phi\lambda\kappa} \partial_\alpha \partial_\mu \partial_\phi}{\Box - 2C^{\rho\tau} \partial_\rho \partial_\tau} F_{\lambda\kappa} = J^\beta. \]  
(41)

Such expression differs from the decoupled equation (38) by a second order term in the tensor \( T^{\mu\alpha\beta} \), justifying the vanishing choice \( (T^{\mu\alpha\beta} = 0) \) adopted. A similar procedure shows that the tensor \( T^{\mu\alpha\beta} \) contributes on the decoupled Eq. (39) only at second order, as well. Hence, the gauge and scalar sectors fulfill decoupled equations of motion at first order, confirming the validity of Eqs. (39) and (40).

In terms of the electric and magnetic fields, Eq. (38) yields
\[ \partial_i E^i - (k_{DE})_{ij} \partial_i E^j + L^i \partial_i B = \rho, \]  
(42)

\[ (1 + s) \epsilon^{ij} \partial_i B - \partial_ii E^i + (k_{DE})^{ij} \partial_i E^j - \epsilon^{ij} \partial_i (L^j E^i) - L^i \partial_i B = J^i, \]  
(43)
which correspond to modified forms for the Gauss’s law and Ampere’s law. Besides these equations, there is the Bianchi identity, \( \partial_\mu F^{\mu\nu} = 0 \), where \( F^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho} F_{\rho\alpha} \) is the dual of the electromagnetic field tensor in \((1 + 2)\)-dimensions, which is a three-vector, \( F^{\mu\nu} = (-\vec{B}, -\vec{E}) \). The symbol \((*)\) designates the dual of a 2-vector: \( (E^i)^* = \epsilon_{ij} E^j \), so that
The stationary version of this equation is
\[ F_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta} E^\alpha B^\beta. \]
Here, one has adopted the following convection: \( \epsilon_{012} = \epsilon^{012} = \epsilon_{12} = \epsilon^{12} = 1 \), \( F^{12} = F_{12} = -B \), \( F_{0i} = E^i \). As it is well-known, Bianchi identity corresponds to the Faraday’s law,
\[ \partial_t B + \nabla \times E = 0. \]  
(Eq. 44)

Eqs. (42), (43), (44) are the modified Maxwell equations corresponding to Lagrangian (24). Multiplying Eq. (43) by \( \epsilon_{ip} \), we have:
\[ (1 + s)\partial_p B - \epsilon_{ip}\partial_i E^i + \epsilon_{ip} (k_{DE})_{ij} \partial_j E^j - \partial_p (L^P E^i) - \epsilon_{ip} L^i \partial_i B = \epsilon_{ip} J^i, \]  
(Eq. 45)

The stationary version of this equation is
\[ n^i \partial_i B - 2n^i \partial_i (L^i E^i) + L^P \partial_p \partial_p E^i = -\epsilon_{ip} J^P, \]  
(Eq. 46)

where \( n = (1 + s) \). Applying the operator \( \partial_i \) on Eq. (46), it turns out:
\[ n\nabla^2 B - 2n\nabla^2 (L^i E^i) + L^P \partial_p \partial_p E^i = -\epsilon_{ip} \partial_i J^P, \]  
(Eq. 47)

\[ n\nabla^2 B + (L^j \partial_j) \nabla^2 A_0 = -\epsilon_{ip} \partial_i J^P. \]  
(Eq. 48)

Multiplying Eq. (42) by \( n \) and replacing Eq. (40) on it, it is possible to achieve a decoupled expression for the electric field,
\[ \partial_i E^i - (k_{DE})_{ij} \partial_j E^j + \frac{1}{n} L^i L^q \partial_i E^q = \rho + \frac{1}{n} \epsilon_{im} L^i J^m. \]  
(Eq. 49)

The dependence on the current in the nonhomogeneous part indicates that this planar model inherits a feature from the four-dimensional model: stationary currents may engender both magnetic and electric fields. These modified Maxwell equations exhibit an analogous form to the Maxwell ones of the four-dimensional theory, respecting the structure of differential operators in three and two spatial dimensions. A point of difference is that in the stationary original theory, the coupling between the scalar and magnetic sector is established only by the parity-odd coefficients. In this planar theory, the coupling is implemented by the parity-odd and parity-even coefficients \( L^i \).

In the Lorentz gauge, \( \partial_\mu A^\mu = 0 \), the wave equations for the 3-potential can be derived from
\[ \Box A^\beta - 2 Z^{\beta\alpha\lambda\kappa} \partial_\alpha \partial_\lambda A_\kappa = J^\beta. \]  
(Eq. 50)

For \( \beta = 0 \) and \( \beta = i \), we derive the equations for \( A^0 \) and \( A^i \), namely:
\[ \Box A^0 + (k_{DE})_{ij} \partial_i \partial_j A_0 + L^P \partial_i B = \rho, \]  
(Eq. 51)

\[ \Box A^i - \epsilon^{ij} L^j \partial_i \partial_0 A_j + (k_{DE})_{ij} \partial_0 ^2 A_j - (k_{DE})_{ij} \partial_0 \partial_j A_0 + \epsilon_{ij} \partial_i (L^P \partial_p) A_0 + s \epsilon^{ij} \partial_p B - L^j \partial_0 B = J^i, \]  
(Eq. 52)

whose stationary versions are
\[ \nabla^2 A^0 - (k_{DE})_{ij} \partial_i \partial_j A_0 - L^i \partial_i B = -\rho, \]  
(Eq. 53)

\[ \nabla^2 A^i - \epsilon_{ij} \partial_i (L^P \partial_p) A_0 - s \epsilon^{ij} \partial_p B = -J^i. \]  
(Eq. 54)

Using Eq. (40), the expression (51) for the scalar potential can be decoupled as:
\[ [\nabla^2 - (k_{DE})_{ij} \partial_i \partial_j + \frac{1}{n} (L^i L^j \partial_i \partial_j)] A_0 = -\rho - \frac{1}{n} \epsilon_{im} L^i J^m. \]  
(Eq. 55)

This expression can be also obtained starting from the differential equation for the electric field, Eq. (49), by replacing \( E^i = -\partial_j A_0 \). Considering that in the stationary regime it holds \( \partial_t A^i = 0 \), Eq. (55) is written as
\[ (1 + s)\nabla^2 A^i - \epsilon^{ij} \partial_i (L^P \partial_p) A_0 = -J^i. \]  
(Eq. 56)
This latter equation confirms that currents act as source for the both the electric and magnetic fields. On the other hand, it is possible to show that charges generate both electric and magnetic fields as well.

The magnetic field can be read from Eq. (48), \( \nabla^2 \left[ n B + (L^j \partial_j) A_0 \right] = - \epsilon_{ij} \partial_j J^i \), leading to

\[
B(r) = - \frac{1}{n} (L^j \partial_j) A_0(r) - \int G_B(r-r') \left[ \frac{1}{n} \epsilon_{ij} \partial_j J^m(r') \right] d^2 r',
\]

where the magnetic Green function satisfies

\[
\nabla^2 G_B(r-r') = \delta(r-r').
\]

In the momentum space, \( \hat{G}(p) = -1/p^2 \), implying \( G_B(r-r') = \frac{1}{2\pi} \ln |r-r'| \), so that the magnetic field is written as

\[
B(r) = \frac{1}{n} L^j E^j(r) - \frac{1}{2\pi n} \int \ln |r-r'| \left[ \epsilon_{im} \partial_i J^m(r') \right] d^2 r'.
\]

In the absence of currents, a simple relation holds between the magnetic and electric field:

\[
B(r) = \frac{1}{n} (L \cdot E(r)).
\]

An issue of interest is the complete wave equations which lead to the dispersion relations of this planar electrodynamics. In order to study it, we complete the wave equation for the electric field. We take the time derivative of Eq. (43), and replace the Bianchi identity, \( \partial_i B = - (\epsilon_{mn} \partial_m E^n) \) in it. After some manipulation, one obtains a wave equation for the electric field at the form \( M_{ij} E^j = 0 \), where

\[
M_{ij} = [-n \partial_j \partial_i + n \delta_{ij} \nabla^2 - \delta_{ij} \partial_i^2 - (k_{DE})^{ij} \partial_i^2] + L_j \partial_i \epsilon_{il} \partial_l - L_i \epsilon_{mj} \partial_i \partial_m.
\]

In the momentum space, it follows:

\[
M_{ij} = [np_j p_i - n \delta_{ij} p^2 + \delta_{ij} p_0^2 - (k_{DE})_{ij} p_0^2] + L_j \epsilon_{il} p_i - L_i \epsilon_{mj} p_m.
\]

The dispersion relation is achieved imposing \( \det M = 0 \). Evaluating the components,

\[
M_{11} = [np_1^2 - n p^2 + p_0^2 - (k_{DE})_{11} p_0^2 + 2L_1 p_0 p_2],
\]

\[
M_{22} = [np_2^2 - np^2 + p_0^2 - (k_{DE})_{22} p_0^2 - 2L_2 p_0 p_1],
\]

\[
M_{12} = M_{21} = np_1 p_2 - (k_{DE})_{12} p_0^2 + L_2 p_0 p_2 - L_1 p_0 p_1,
\]

one can write and factor the determinant, \( \det M = M_{11} M_{22} - (M_{12})^2 \), obtaining an exact dispersion relation:

\[
\det M = p_0^2 \left[ p_0^2 [1 - \text{tr}(k_{DE}) + \det(k_{DE})] + 2p_0 [L \times p + (k_{DE})_{ij} p_i L_j^*] - [n p^2 - n (k_{DE})_{ij} p_i p_j + (L \cdot p)^2] \right] = 0.
\]

This physical dispersion relation yields the solution

\[
p_0 = \frac{1}{D} \left[ L \times p + (k_{DE})_{ij} p_i L_j^* + \Omega \left( p^2 - (k_{DE})_{ij} p_i p_j \right)^{1/2} \right].
\]

where

\[
D = [1 - \text{tr}(k_{DE}) + \det(k_{DE})],
\]

\[
\Omega = \left[ (1+s) D + L^2 - (k_{DE})_{ij} L_i^* L_j^* \right]^{1/2}.
\]

From relation (67), we notice that both modes propagate with the same phase velocity, which implies absence of birefringence. To understand it, we should take the right \( (p_0+) \) and the left \( (p_0-) \) modes, corresponding to the \( \pm \)
signals in Eq. (67), propagating in the same sense. Hence, we should take the left ($p_{0-}$) mode with reversed momentum ($p \rightarrow -p$),

$$p_{0-}(-p) = \frac{1}{D} \left[ -(L \times p) - (k_{DE})_{ij} p_i L^*_j - \Omega \left( p^2 - (k_{DE})_{ij} p_i p_j \right)^{1/2} \right]. \quad (70)$$

meaning propagation to the right. We should note that it coincides with the right mode,

$$p_{0+}(p) = \frac{1}{D} \left[ (L \times p) + (k_{DE})_{ij} p_i L^*_j + \Omega \left( p^2 - (k_{DE})_{ij} p_i p_j \right)^{1/2} \right], \quad (71)$$

with a reversed global signal. These relations provide the same phase velocity. This situation is analogous to the one of the parity-odd dispersion relation of the CPT-even original model, discussed in Eqs. (36-44) of Ref. [33], which yields no birefringence. Such discussion reveals that the six Lorentz-violating parameters of the electromagnetic sector, $s$, $(k_{DE})_{ij}$, $L^i$, behave as nonbirefringent components (at any order). Hence, the birefringent components of this planar theory should be contained in the coupling tensor $T_{\mu \nu \kappa}$.

As this tensor modifies the equations of motion at second order, the birefringence is manifest only as a second order effect in the Lorentz-violating parameters. At first order, Eq. (67) implies the following physical dispersion relation:

$$p_0 = |p| \left[ 1 + \frac{s}{2} + \frac{1}{2} \text{tr}(k_{DE}) - (k_{DE})_{ij} \frac{p_i p_j}{2p^2} \pm \frac{1}{|p|} \right]. \quad (72)$$

From relation (72), we can also evaluate the group velocity,

$$u_g = \left[ 1 + \frac{s}{2} + \frac{1}{2} \text{tr}(k_{DE}) - (k_{DE})_{ij} \frac{\hat{p}_i \hat{p}_j}{2} \pm \epsilon_{ij} L^i \hat{p}^j \right], \quad (73)$$

showing that this electrodynamics could spoil causality. In order to perform a complete analysis on the consistency of this theory (stability, causality, and unitarity), one should carry out a detailed analysis via the Feynman gauge propagator, which is being regarded as a future perspective.

V. CLASSICAL STATIONARY SOLUTIONS

In this section, we should solve the equations for the electromagnetic and scalar sectors, obtaining stationary solutions at first order in the Lorentz-violating parameters.

A. The electrostatic and magnetostatic

A good starting point to study the stationary solutions for the pure electromagnetic sector is the differential equation for the scalar potential, Eq. (55), which at first order is read as:

$$[\nabla^2 - (k_{DE})^{ij} \partial_i \partial_j + \frac{1}{n} (L^i L^j \partial_i \partial_j)] A_0 = -\rho - \frac{1}{n} \epsilon_{im} L^i J^m. \quad (74)$$

The solution for this equation can be achieved by means of the Green method, which allows to write

$$A_0(\mathbf{r}) = -\int G(\mathbf{r} - \mathbf{r}') [\rho(\mathbf{r}') + \frac{1}{n} \epsilon_{im} L^i J^m(\mathbf{r}')] d^2 \mathbf{r}', \quad (75)$$

where $G(\mathbf{r} - \mathbf{r}')$ is the Green’s function, which fulfills the first order equation

$$[\nabla^2 - (k_{DE})^{ij} \partial_i \partial_j] G(\mathbf{r} - \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}'). \quad (76)$$

In Fourier space it holds

$$G(\mathbf{r} - \mathbf{r}') = \frac{1}{(2\pi)^2} \int d^2 p \, \tilde{G}(\mathbf{p}) \exp[-i \mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')], \quad (77)$$
whose replacement in Eq. \((73)\) leads to
\[
\tilde{G}(\mathbf{p}) = -\frac{1}{\mathbf{p}^2 - (k_{DE})^2} p^i p^j = -\frac{1}{\mathbf{p}^2} \left[ 1 + (k_{DE})^{ij} \frac{p^i p^j}{\mathbf{p}^2} + \ldots \right],
\]  
and we have evaluated \(\tilde{G}(\mathbf{p})\) at first order in the Lorentz-violating parameters, due to its usual smallness. Performing the Fourier integrations, we achieve the following Green function:
\[
G(\mathbf{R}) = \frac{1}{(2\pi)} \left[ \left( 1 + \frac{1}{2} (k_{DE})^{ij} \right) \ln R + \frac{1}{2} (k_{DE})^{ij} \frac{R_i R_j}{R^2} \right]
\]  
where \(\mathbf{R} = (\mathbf{r} - \mathbf{r}')\) and the terms involving the coefficients \((k_{DE})^{ij}\) are corrections to usual planar Green function, \(\ln R\). Here, it were used the following transforms:
\[
\int d^2 \mathbf{p} \frac{1}{\mathbf{p}^2} e^{-i\mathbf{p} \cdot \mathbf{R}} = -2\pi \ln R, \quad \int d^2 \mathbf{p} \frac{p^i p^j}{\mathbf{p}^4} e^{-i\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')} = -2\pi \left( \frac{\delta_{ij}}{2} \ln R + \frac{1}{2} \frac{R_i R_j}{R^2} \right).
\]  
The scalar potential is then written as
\[
A_0(\mathbf{r}) = -\frac{1}{2\pi} \int \left[ \left( 1 + \frac{1}{2} (k_{DE})^{ij} \right) \ln |\mathbf{r} - \mathbf{r}'| + \frac{1}{2} (k_{DE})^{ij} \frac{(\mathbf{r} - \mathbf{r}')_i (\mathbf{r} - \mathbf{r}')_j}{(\mathbf{r} - \mathbf{r}')^2} \right] \rho(\mathbf{r}') + \frac{1}{n} \epsilon_{im} L^i J^m(\mathbf{r}') d^2 \mathbf{r}',
\]  
which at first order in the Lorentz-violating parameters is
\[
A_0(\mathbf{r}) = -\frac{1}{2\pi} \int \left[ \left( 1 + \frac{1}{2} (k_{DE})^{ij} \right) \ln |\mathbf{r} - \mathbf{r}'| + \frac{1}{2} (k_{DE})^{ij} \frac{(\mathbf{r} - \mathbf{r}')_i (\mathbf{r} - \mathbf{r}')_j}{(\mathbf{r} - \mathbf{r}')^2} \right] \rho(\mathbf{r}') d^2 \mathbf{r}'
\]  
and
\[
-\frac{1}{2\pi} \epsilon_{im} L^i \int \ln |\mathbf{r} - \mathbf{r}'| J^m(\mathbf{r}') d^2 \mathbf{r}',
\]  
where \(n^{-1} \sim (1 - s)\). For a point-like charge distribution \((J^m(\mathbf{r}') = 0)\), \(\rho(\mathbf{r}') = q \delta(\mathbf{r}')\), one achieves the scalar potential
\[
A_0(\mathbf{r}) = -\frac{q}{2\pi} \left[ 1 + \frac{1}{2} (k_{DE})^{ij} \right] \ln r + \frac{1}{2} (k_{DE})^{ij} \frac{r^j r^j}{r^2} \right].
\]  
This scalar potential differs from the usual planar behavior by the term \((k_{DE})^{ij} r_i r_j / r^2\), which represents a directional factor whose magnitude remains constant with distance. In fact, supposing \(r_i = r \cos \theta_i, r_j = r \cos \theta_j\), one has \((k_{DE})^{ij} r_i r_j / r^2 = (k_{DE})^{ij} \cos \theta_i \cos \theta_j\). This shows that the Lorentz-violating corrections are unable to modify the asymptotic behavior of the electric field. Hence, the implied electric field,
\[
\mathbf{E}'(\mathbf{r}) = \frac{q}{2\pi} \left[ 1 + \frac{1}{2} (k_{DE})^{ij} \frac{r^j}{r^2} + \frac{1}{r^2} \left( (k_{DE})^{ij} r_j - \frac{(k_{DE})^{ij} r_i r_j}{r^2} \right) \right],
\]  

and
\[
\mathbf{E}'(\mathbf{r}) = \frac{q}{2\pi} \left[ 1 + \frac{1}{2} (k_{DE})^{ij} - \frac{(k_{DE})^{ij} \cos \theta_i \cos \theta_j}{r^2} + \frac{1}{r^2} (k_{DE})^{ij} r_j \right],
\]  
decays as \(1/r\), as it occurs in the Maxwell theory in \((1 + 2)\)–dimensions. This field has a radial behavior except for the Lorentz-violating contribution \((k_{DE})^{ij} r_i\), which constitutes the qualitative difference induced by Lorentz violation.

In this theory, a point-like charge, \((J^m(\mathbf{r}') = 0, \rho(\mathbf{r}') = q \delta(\mathbf{r}')\)), yields a nonnull magnetic field, that in accordance with Eq. \((57)\), at first order is \(B(\mathbf{r}) = (\mathbf{L} \cdot \mathbf{E}(\mathbf{r}))\). It then yields
\[
B(\mathbf{r}) = \frac{q}{2\pi} \frac{\mathbf{L} \cdot \mathbf{r}}{r^2}.
\]  
This field decays with \(1/r\) and does not present radial symmetry. It possesses an angular dependence that reflects the direction of the vector \(\mathbf{r}\) in relation to the background vector \(\mathbf{L}\). In this case, the modulation factor is \(|\mathbf{L}| \cos \beta\), being \(\beta\) the angle between \(\mathbf{r}\) and \(\mathbf{L}\).
A stationary current associated to a point-like charge with uniform velocity \( u \), \( J(r') = q u \delta(r') \), \( [\rho(r') = 0] \), yields the scalar potential:

\[
A_0 (r) = -\frac{q}{2\pi} (L \times u) \ln r,
\]  

(87)

whose electric field is

\[
E^i (r) = \frac{q}{2\pi} \left( (L \times u) \frac{r^i}{r^2} \right).
\]  

(88)

Once the vector product engenders a scalar in two dimensions, this electric field results aligned with the radial direction, without angular dependence. The scalar \((L \times u)\) acts as a modulation factor sensitive to the angle between the vectors \( L, u \), which can vary from zero (for \( L//u \)) to the maximum \(|L| |u|\) (for \( L \perp u \)).

The magnetic field associated with this current density, \( J(r') = q u \delta(r') \), \( [\rho(r') = 0] \), is attained from Eq. (59),

\[
B (r) = -\frac{q}{2\pi} (1 - s) \frac{\epsilon_{im} r^i u^m}{r^2} = -\frac{q}{2\pi} (1 - s) \frac{r \times u}{r^2},
\]  

(89)

where the Lorentz-violating contribution has the same form of Maxwell usual solution.

**B. The pure scalar sector**

A solution for the scalar field can be easily constructed. At first order, the scalar field evolution is governed by Eq. (39), which in the stationary limit is given by

\[
(\nabla^2 + 2C^{ij} \partial_i \partial_j) \phi = J.
\]  

(90)

The Green function for this equation satisfies \([\nabla^2 + 2C^{ij} \partial_i \partial_j]G(r - r') = \delta(r - r')\), while the solution is written as

\[
\phi (r) = \int G(r - r') J(r') d^2 r'.
\]  

(91)

Following the procedure developed for the scalar field, we obtain

\[
\tilde{G} (p) = -\frac{1}{p^2} \left[ 1 - 2C^{ij} \frac{p_i p_j}{p^2} \right],
\]  

(92)

the same structure Green’s function for the scalar potential, Eq. (78). So, we attain as Green’s function a result very similar to Eq. (76),

\[
G(R) = \frac{1}{(2\pi)^3} \left[ (1 - C^{ii}) \ln |R - C^{ij} R_j R_i | R^2 \right].
\]  

(93)

The scalar field is given as

\[
\phi (r) = \frac{1}{2\pi} \int \left[ (1 - C^{ii}) \ln |r - r'| - C^{ij} \frac{(r - r')_i (r - r')_j}{(r - r')^2} \right] J(r') d^2 r',
\]  

(94)

The scalar field generated by a point-like scalar source, \( J(r') = q \delta(r') \), is

\[
\phi (r) = \frac{q}{2\pi} \left[ (1 - C^{ii}) \ln r - C^{ij} \frac{r^i r^j}{r^2} \right].
\]  

(95)

We thus confirm that scalar field presents a very similar behavior to the one of the scalar potential, given by Eq. (83).
VI. CONCLUSIONS AND REMARKS

In this work, we have performed the dimensional reduction of the CPT-even gauge sector of SME, attaining a planar model enclosing both gauge and scalar sectors, coupled by a third-rank tensor stemming from the dimensional reduction. The symmetries of the planar Lorentz-violating tensors have been scrutinized, and the number of independent components were evaluated. The parity of these components was determined considering the field $\phi$ as a scalar, but it can be also examined supposing that $\phi$ behaves as a pseudoscalar (inherit the behavior of the component $A^{(3)}$). In the sequel, we have taken the coupling tensor as null, and examined the equations of motion for the electromagnetic and scalar sectors. These equations were solved by the Green’s method in the stationary regime.

One parallel should be made with the dimensional reduction of the Maxwell-Carroll-Field-Jackiw electrodynamics in Ref. [16]. In that case, it was obtained a planar model composed of the Maxwell-Chern-Simons electrodynamics, a scalar field and a coupling term, where the Lorentz violation was controlled by a 3-vector background, $v^\mu = (v_0, v^i)$. The stationary classical solutions of this model revealed that the background altered the asymptotic behavior of the fields. Indeed, while the Maxwell-Chern-Simons solutions decay exponentially for $r \to \infty$, the Lorentz-violating solutions exhibited a $1/r$ behavior for $r \to \infty$. In the dimension reduction of the CPT-even sector, the presence of Lorentz-violating terms do not alter the long distance profile of the solutions, keeping the asymptotic behavior of the pure Maxwell planar electrodynamics, $1/r$. It is noted, however, that the Lorentz-violating parameters induce an angular dependence in the field solutions. The canonical energy-momentum tensor was carried out, leading to an energy density which is positive definite for small Lorentz-violating parameters.

The dispersion relation of the planar abelian gauge model was exactly evaluated, revealing that the six Lorentz-violating parameters related to the electromagnetic sector do not yield birefringence. This means that the pure electrodynamics stemming from Lagrangian (61) presents no birefringence at any order. Such effect, however, may be engendered by the some components of the coupling tensor $T_{\mu\lambda\kappa}$, as a second order effect. Finally, the group velocity evaluation shows that this planar theory could be endowed with causality illness. A more careful analysis on the physical consistency of this model (stability, causality, unitarity) is under progress. Another point of interest is the investigation of topological defects, such stable vortex configurations, in this theoretical framework.

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