A note on the Supergravity Description of Dielectric Branes

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Abstract

We comment on the recent papers by Costa et al and Emparan, which show how one might generate supergravity solutions describing certain dielectric branes in ten dimensions. The “basic” such solutions describe either $N$ fundamental strings or $N$ D4-branes expanding into a D6-brane, with topology $\mathbb{M}^2 \otimes S^5$ or $\mathbb{M}^5 \otimes S^2$ respectively.Treating these solutions in a unified way, we note that they allow for precisely two values of the radius of the relevant sphere, and that the solution with the smaller value of the radius has the lower energy. Moreover, the possible radii in both cases agree up to numerical factors with the corresponding solutions of the D6-brane worldvolume theory. We thus argue that these supergravity solutions are the correct gravitational description of the dielectric branes of Emparan and Myers.

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1 Introduction

The higher-dimensional counterparts of the Melvin universe [1], which describe gravitating magnetic fluxbranes, have appeared in various string- and M-theoretic guises over the last few years (see, e.g., [2]–[15]). Within the Kaluza-Klein context, the most natural way to generate such fluxbranes is via a twisted compactification of flat space [16, 17, 18]. Moreover, starting instead with the Euclidean Schwarzschild solution cross a trivial time direction, one can generate spherical branes via just such a twisted compactification [18]. In slightly different language, one can think of this as follows: take a $D$-dimensional uncharged black string; analytically continue the worldvolume coordinates; and perform a double dimensional reduction along a twisted direction. The resulting $(D-1)$-dimensional solution describes a spherical $(D-5)$-brane in the core of a magnetic fluxbrane. It has local, but zero net, magnetic charge $1$.

If one now adds $N$ units of 2-form charge to the $D$-dimensional black string, the $(D - 1)$-dimensional solution will exhibit $N$ units of 1-form charge, so is more rightly interpreted as a collection of $N$ particles “expanding” into a sphere under the influence of the background $(D-3)$-form magnetic field. Of course, these techniques need not only be applied to black strings and, in recent work, Costa et al [19] and Emparan [20] considered their application in an M-theoretic context. Starting with the basic branes of eleven-dimensional supergravity, then, these authors showed how to generate solutions of type IIA supergravity which describe various ten-dimensional branes expanding into a D6-brane under the influence of a background 8-form magnetic field. In particular, the reduction of certain M2- or M5-brane solutions generates ten-dimensional solutions describing F-strings or D4-branes expanding into a D6-brane with topology $\mathbb{M}^2 \otimes \mathbb{S}^5$ or $\mathbb{M}^5 \otimes \mathbb{S}^2$ respectively, where $\mathbb{M}^n$ denotes an $n$-dimensional Minkowski space. Applying T-duality generates more general configurations [20] – D$p$-branes expanding into a D$(p + 2)$-brane, and F-strings expanding into a D$p$-brane, for arbitrary values of $p$ – but these solutions will necessarily be smeared along certain directions.

One of the motivations behind this work was to try to get a handle on the possible supergravity description of the dielectric branes of Emparan [21] and Myers [22]. Imposing T-duality invariance on the non-abelian action relevant to the description of multiple D$p$-branes, gives rise to a whole host of worldvolume couplings which are not present in the abelian theory [22, 23]. In particular, the Chern-Simons piece of the action must include couplings to Ramond-Ramond (R-R) potentials of degree greater than $(p + 1)$, giving rise to possible couplings to higher-dimensional branes. The presence of such terms allows for the “dielectric effect”, the simplest example of which is that of $N$ D$p$-branes expanding into a D$(p + 2)$-brane under the influence of the $(p + 3)$-form R-R potential associated with the latter [22, 24]. The resulting dielectric brane, which is a minimum of the worldvolume energy functional, has worldvolume $\mathbb{M}^{p+1} \otimes S^2_{\text{NC}}$, where $S^2_{\text{NC}}$ is the non-commutative or fuzzy two-sphere.

Although this process is most properly described from within the non-abelian worldvolume theory of the D$p$-branes, one can alternatively consider the dual description from within the abelian theory of the D$(p + 2)$-brane. In the large $N$ limit, the non-commutative nature of the sphere is lost, and these two descriptions agree [22]. Since it is unlikely that such non-commutative structures will appear within a supergravity perspective, any supergravity solution purporting to describe this dielectric effect should be compared to solutions of the D$(p + 2)$-brane worldvolume theory. From this perspective, the standard coupling of a $(p + 1)$-form R-R potential to the D$(p + 2)$-brane

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1 More generally, spherical $(D-3-2n)$-branes in the core of $n$ intersecting fluxbranes can be generated by applying $n$ twists [15].

2 Further solutions, generated from certain intersecting configurations of M-branes, were considered by Emparan in [20] but we will say no more about these here.
gives rise to “dissolved” Dp-branes within the worldvolume of the D(p + 2)-brane \[25\]. And it is the presence of this dissolved Dp-brane charge which allows for spherical solutions of the D(p + 2)-brane theory.

The existence of such spherical D(p + 2)-branes with dissolved Dp-brane charge was anticipated by Emparan in \[21\]. The main point of this work, however, was to analyse spherical D(p + 2)-brane solutions with dissolved F-string charge. The interpretation of such solutions was that of N “Born-Infeld strings” \[26, 27\] expanding into a sphere under the influence of an external R-R potential. Various arguments can then be used to identify the string-like solution of Born-Infeld theory as a fundamental string, but it is clear that one lacks a proper description of this process from the point of view of the F-strings themselves.

Many features of the supergravity solutions discovered by Costa et al \[19\] and Emparan \[20\] suggest that they should, indeed, be thought of as the proper gravitational description of these dielectric branes for \( p = 4 \). We will provide new evidence which significantly strengthens the case for such a connection, and clarifies the relationship between these geometries and the unstable solutions of \[18\].

In sections 2 and 3 we review possible spherical solutions of the D6-brane worldvolume theory due to the presence of dissolved D4-brane and F-string charge respectively. We comment briefly on the more general case of the spherical Dp-brane with dissolved F-string charge, for arbitrary p, the \( p = 2 \) case being qualitatively different to the other examples. In section 4, we turn to the supergravity solutions of \[19, 20\]. To comment on them, a brief review of their construction is necessary. We treat both the F-string and D4-brane solutions in a unified way, since they have the same structure, and then compare directly with the worldvolume analysis of sections 2 and 3. In both cases, the solution is consistent for precisely two values of the radius of the five- or two-dimensional sphere, a point missed in the treatment of \[19\] because the limiting procedure used therein sends one of the radii to infinity. Moreover, the functional form of these radii as a function of \( N \), the number of D4-branes or F-strings, precisely matches that of the world volume calculation. That this should be the case is rather unexpected. Moreover, a brief consideration of the smeared solutions of Emparan \[20\] shows that the atypical structure of the D2-brane with dissolved F-string charge is also captured by the corresponding supergravity solution.

In subsection 4.3, we comment on the energetics of these two consistent solutions, and show that that with the smaller value of the radius has the lower energy, in agreement with the worldvolume picture. We then turn to a consideration of “off-shell” configurations, those with arbitrary values of the radius of the sphere. In general, such solutions exhibit conical deficits which can be viewed as “deficit branes” providing the tension necessary to hold the dielectric brane in equilibrium. We then argue that the tension of these deficit branes gives us a handle on the stability of the supergravity solutions, with the results that the solution at the smaller radius is in fact stable – at least to radial perturbations – and that the second solution is unstable\[4\]. These additional pieces of evidence exhibit the close relationship between the worldvolume and supergravity descriptions. We believe further study of this connection will greatly enhance our understanding of the dielectric effect.

## 2 Spherical D6-branes from D4-branes – the worldvolume theory

Consider, then, \( N \) flat D4-branes polarized by a 7-form R-R potential into a D6-brane. From the point of view of the D6-brane, such a configuration is described by a solution with topology

\[3\]

\[4\] Corresponding, unsmeared supergravity solutions for different values of \( p \) should exist, but they cannot be generated using the techniques of \[19, 20\].

\[4\] We are grateful to Roberto Emparan for pointing out these arguments to us.
$M^5 \otimes S^2$ and $N$ units of dissolved D4-brane flux. The flat space approximation\[5\] analogous to that considered by Myers [22] — a flat background geometry and a constant 8-form field strength — does indeed allow for such solutions. We will see that the supergravity solutions of Costa et al [19] and Emparan [20] correspond to this simple model with a surprising degree of accuracy.

The flat ten-dimensional metric is written as

$$ds^2 = ds^2(M^5) + dr^2 + r^2 d\Omega^2_2 + ds^2(E^2),$$

(2.1)

where $d\Omega^2_2$ denotes the round metric on a unit $n$-sphere, and the worldvolume of the D6-brane is taken to be the obvious $M^5 \otimes S^2$. We are interested in static solutions of the worldvolume theory for $r = R = \text{const}$. The constant background 8-form field strength and corresponding 7-form potential is chosen to be

$$F[8] = -2fr^2 \epsilon(M^5) \wedge dr \wedge \epsilon(S^2),$$

(2.2)

$$C[7] = \frac{2}{3}fr^3 \epsilon(M^5) \wedge \epsilon(S^2),$$

(2.3)

where $\epsilon(M)$ denotes the volume form on the space $M$, and where a convenient choice of gauge has been made.

The D6-brane action is a sum of Dirac-Born-Infeld [28] and Chern-Simons [25, 29, 30] terms:

$$S = S_{\text{BI}} + S_{\text{CS}},$$

(2.4)

$$S_{\text{BI}} = -\frac{T_6}{g_s} \int_{D6} e^{-\phi} \sqrt{-\det (P[G]_{ab} + \lambda F_{ab})},$$

(2.5)

$$S_{\text{CS}} = \frac{T_6}{g_s} \int_{D6} P \left[ \sum C[n] \right] \wedge e^{\lambda F},$$

(2.6)

where we have set the Kalb-Ramond 2-form to zero, $F$ is the abelian Born-Infeld 2-form field strength, $P[\ldots]$ denotes a pull-back to the worldvolume of the spacetime fields, $T_6 = \mu_6 = ((2\pi)^6 \alpha'^7/2 )^{-1}$ and $\lambda = 2\pi \alpha'$. The Chern-Simons action is a sum of terms containing all the R-R potentials, $C[n]$, for $1 \leq n \leq 7$:

$$S_{\text{CS}} = \frac{T_6}{g_s} \int_{D6} (P [C[7]] + \lambda P [C[5]] \wedge F + \ldots ),$$

(2.7)

which are the relevant couplings of these potentials to $D_p$-branes for $0 \leq p \leq 6$. The Born-Infeld field strength is then chosen such that the second term in (2.7) mimics the coupling of $N$ D4-branes to $C[5]$. With

$$F = \frac{N}{2} \epsilon(S^2),$$

(2.8)

we have

$$\lambda T_6 \int_{M^5 \otimes S^2} P [C[5]] \wedge F = NT_4 \int_{M^5} P [C[5]] ,$$

(2.9)

where we have used the relation $2\pi \lambda T_{p+2} = T_p$.

Substituting for this Born-Infeld field strength, and for the radius $r = R$, in the background metric (2.1), we find that the action (2.4) gives the potential [22]

$$V(R) = \frac{4\pi T_6 V_4}{g_s} \left( \sqrt{R^4 + \frac{1}{4} \lambda^2 N^2 - \frac{2}{3} \lambda R^3} \right),$$

(2.10)

\[5\]“Approximation” in the sense that it is not a consistent supergravity background.
where we have taken the dilaton to vanish, and where $V_n$ denotes the volume of $\mathbb{E}^n$. If we introduce the dimensionless worldvolume quantities

$$\rho_D = f R, \quad N_D = \frac{1}{2} f^2 \lambda N,$$

(2.11)

$$V_D = \frac{g_s}{4\pi T_6 V_4} f^2 \sqrt{V_n} = \sqrt{\rho_D^4 + N_D^2 - \frac{2}{3} \rho_D^3},$$

(2.12)

then the dimensionless potential, $V_D$, has extrema when the following condition holds:

$$N_D^2 = \rho_D^2 - \rho_D^3.$$  

(2.13)

For $N_D < \frac{1}{2}$ there are two non-trivial extrema, given by

$$\rho_D^2 = \frac{1}{2} \left( 1 \pm \sqrt{1 - 4N_D^2} \right),$$

(2.14)

as shown in figure 1. There are thus two solutions describing spherical D6-branes, one a maximum of the potential, the other a local minimum. In figure 2a we show how the stationary points of the potential change as the number of D4-branes change. Figure 2b shows the value of the potential at these extrema.

Myers \[22\] considers the case where $N_D < \frac{1}{2}$ in the limit $N_D \gg \rho_D^2$ in which case one may expand the potential (2.13) as $V_D \simeq N_D + \frac{\rho_D^3}{2N_D} - \frac{2}{3} \rho_D^3 + \ldots$. The single non-zero extremum is then at $\rho_D \simeq N_D$ so, in this limit, we lose the information about the maximum. A similar limit was taken in the supergravity solution of \[19\] which again obscured the fact that there is a second consistent solution. We shall see below that, prior to any limit being taken, the allowed radii of the supergravity solution have precisely the same functional form as (2.14). We believe this to be strong evidence that the worldvolume and supergravity pictures are describing the same phenomenon.

3 Spherical D6-branes from F-strings – the worldvolume theory

We now turn to the case considered by Emparan \[21\], in which $N$ F-strings expand into a D6-brane with topology $M^2 \otimes S^5$. Taking the same action as in (2.4), we place the D6-brane at constant $r = R$ in the flat spacetime given by

$$ds^2 = ds^2(M^2) + dr^2 + r^2 d\Omega_5^2 + ds^2(\mathbb{E}^2).$$

(3.16)

The Born-Infeld field strength is taken to be

$$F = \mathcal{E} \epsilon(M^2),$$

(3.17)

where $\mathcal{E}$ is a constant electric field, and we take the constant R-R 8-form field strength and corresponding 7-form potential to be

$$F_{[8]} = 5h \epsilon(M^2) \wedge dr \wedge r^5 \epsilon(\Omega_5),$$

(3.18)

$$C_{[7]} = \frac{5}{6} h \epsilon(M^2) \wedge r^6 \epsilon(\Omega_5).$$

(3.19)

The action (2.4) then reduces to

$$S = -\frac{T_6}{g_s} \int dt \left( R^2 \sqrt{1 - \lambda^2 \mathcal{E}^2} - \frac{5h}{6} R^6 \right) \equiv \int dt L,$$

(3.20)
Figure 1: This plot shows the potential energy, \( V_D(\rho_D) \), of the spherical D6-brane for different values of dissolved D4-brane charge, \( N_D \). The top curve is for \( N_D = 0 \) and the lower one for \( N_D = \frac{1}{2} \). The potential has been shifted by a constant such that the energy at \( \rho_D = 0 \) is zero.

Figure 2: Here we show how the charge, \( N_D \), and energy, \( V_D \), at the stationary points vary with the radius of the spherical D6-brane with dissolved D4-brane charge.
where \( l \) is the length of the string and \( \Omega_n \) denotes the volume of a unit \( n \)-sphere. Introducing the conjugate momentum, \( D = \frac{\delta L}{\delta \dot{E}} \), we find that the Hamiltonian, \((\mathcal{E}D - L)\), is given by \[21\]

\[
H = \frac{T_6 \Omega_5 l}{g_s} \left( \sqrt{R_{10}^2 + \left( \frac{g_s}{T_6 \Omega_5 \lambda} \right)^2 D^2 - \frac{5h^2}{6} R_{10}^6} \right). \tag{3.21}
\]

To relate \( D \) to \( N \), the number of dissolved F-strings, we take \( R \) to zero to obtain a string-like state with energy per unit length \( D/\lambda = DT_F \) (\( T_F \) being the F-string tension), so we simply associate \( D = N \) \[21\]. Now we use \( R_{11} = g_s \sqrt{\alpha'} \), \( \Omega_5 = \pi^3 \), and introduce the following dimensionless quantities

\[
\rho_F = hR, \\
N_F = 8h^5 R_{11} \lambda^2 N, \\
V_F = 8\lambda^3 R_{11} h^5 \frac{H}{\sqrt{V_1}} = \sqrt{\rho_F^6 + N_F^2} - \frac{5}{6} \rho_F^6. 
\]

(3.22, 3.23, 3.24)

The extrema of the potential \[3.24\] are given by solutions of

\[
N_F^2 = \rho_F^8 - \rho_F^{10}. \tag{3.25}
\]

Again there is a maximum charge, above which there are no stable solutions. To calculate it, we use the fact that at this maximum charge, the two extrema merge into a point of inflection, so \( V'_F = V''_F = 0 \) at the extrema. This gives \( N_F \leq \sqrt{\left(\frac{8}{5}\right)^{\frac{1}{3}} - \left(\frac{8}{5}\right)^{\frac{2}{3}}} \). In figure 3 we show how the potential changes with the number of F-strings dissolved in the D6-brane. How the location of the extrema changes with \( N_F \) is shown in figure 4a, there clearly being two extrema for \( N_F \) less than the maximum value. This system is seen to be qualitatively the same as the one considered in section 2, the D4-brane blowing up into a D6-brane.

### 3.1 Spherical Dp-branes from F-strings

The above analysis is not peculiar to D6-branes, in that spherical Dp-branes with dissolved F-string charge exist for arbitrary values of \( 2 \leq p \leq 8 \) \[21\]. It is easy to show that the dimensionless potential in the general case has the form \[21\]

\[
V_F = \sqrt{\rho_F^{2(p-1)} + N_F^2} - \frac{p-1}{p} \rho_F^p, 
\]

(3.26)

in terms of dimensionless quantities analogous to those defined in \[3.22\] and \[3.25\] above. The extrema of this potential are given by solutions of

\[
N_F^2 = \rho_F^{2(p-2)} - \rho_F^{2(p-1)}, \tag{3.27}
\]

so for \( p > 2 \) there are always two extrema. The \( p = 2 \) case is qualitatively different, however. There is no minimum in this case, the single non-trivial extremum being at

\[
\rho_F = \sqrt{1 - N_F^2}, \tag{3.28}
\]

and this is a maximum of the potential. We will see that the relevant (smeared) supergravity solution describing this configuration has a similar structure.
Figure 3: This plot shows the potential energy, $V(\rho_F)$, of the spherical D6 brane for different values of dissolved F-string charge, $N_F$ (3.23). The top curve is for $N_F = 0$ and the lower one for the maximum charge $N_F = \sqrt{\left(\frac{1}{5}\right)^4 - \left(\frac{1}{5}\right)^5}$. The potential has again been shifted by a constant to ensure that the zero of energy is at $\rho_F = 0$.

Figure 4: Here we show how the charge, $N_F$ (3.25), and energy, $V_F$ (3.24), at the stationary points vary with the radius of the spherical D6-brane with dissolved F-string charge.
4 The supergravity description

The eleven-dimensional starting point of the discussion of [19, 20] is the double analytic continuation of the rotating black M2- and M5-branes:

\[
\begin{align*}
\text{d}s^2 &= H^{-2/\tilde{d}} \left[ \text{d}s^2(M^{d-1}) + f \left( \text{d}\tau + \frac{k l \cosh \alpha}{\Delta r^d \tilde{f}} \sin^2 \theta \text{d}\tilde{\phi} \right)^2 \right] \\
&\quad + H^{2/\tilde{d}} \left[ \frac{\text{d}r^2}{f} + r^2 \left( \Delta \text{d}\theta^2 + \frac{\Delta \tilde{f}}{f} \sin^2 \theta \text{d}\tilde{\phi}^2 + \cos^2 \theta \text{d}\Omega_{\tilde{d}-1}^2 \right) \right],
\end{align*}
\]

where \((d, \tilde{d}) = (3, 6)\) and \((6, 3)\) for the M2- and M5-branes respectively, and where

\[
H = 1 + \frac{k \sinh^2 \alpha}{\Delta r^d}, \quad f = 1 - \frac{k}{\Delta r^d}, \quad \tilde{f} = r^{\tilde{d}-2} - \frac{k}{\Delta r^d}, \quad \Delta = 1 - \frac{l^2 \cos^2 \theta}{r^2}.
\]

For the M2-brane, the 3-form potential is given by

\[
A_{[3]} = \frac{k \sinh \alpha}{\Delta r^d H} \epsilon(M^2) \wedge \left[ \cosh \alpha \text{d}\tau - l \sin^2 \theta \text{d}\tilde{\phi} \right],
\]

whereas for the M5-brane it is

\[
A_{[3]} = \frac{k \sinh \alpha}{\Delta} \cos^3 \theta \left[ - \left( 1 - \frac{l^2}{r^2} \right) \cosh \alpha \text{d}\tilde{\phi} - \frac{l}{r^2} \text{d}\tau \right] \wedge \epsilon(S^2).
\]

The location, \(r_H\), of the “Euclidean horizon” or bolt is given by the zero of \(\tilde{f}\), so that

\[
k = r_H^{\tilde{d}-2} \left( r_H^2 - l^2 \right),
\]

and the “Euclidean angular velocity”, \(\Omega\), is

\[
\Omega = \frac{l}{\cosh \alpha \left( r_H^2 - l^2 \right)}.
\]

To avoid a conical singularity at \(r = r_H\), \(\tau\) must be periodic with period \(2\pi R_{11}\), where

\[
R_{11} = \frac{2k \cosh \alpha}{d r_{H}^{d-1} - (d-2) l^2 r_{H}^{d-3}} = g_s \sqrt{\alpha'}.
\]

Finally, we must also consider the quantization of M2- and M5-brane charge which gives

\[
k \cosh \alpha \sinh \alpha = c_{\tilde{d}} N(\alpha')^{\tilde{d}/3},
\]

where we have used the fact that the eleven-dimensional Planck length, \(l_P\), is given by \(l_P = g_s^{1/3} \sqrt{\alpha'}\). \(N\) denotes the number of M2- or M5-branes, and the constant \(c_{\tilde{d}}\) is

\[
c_{\tilde{d}} = \frac{(2\pi)^{2\tilde{d}/3}}{d \Omega_{d+1}}.
\]

so that \(c_6 = 8\) and \(c_3 = 1/2\).

\[\footnote{The standard versions of which are to be found in [31], [32].}\]
As explained in [19, 20], the ten-dimensional geometry is found by reducing along orbits of the Killing vector
\[ K = \frac{\partial}{\partial \tau} + B \frac{\partial}{\partial \phi}, \]
(4.38)
the fixed point set of the corresponding isometry being \( \{ r = r_H, \theta = 0 \} \). To dimensionally reduce, one identifies points separated by a distance of \( 2\pi R_{11} \) along integral curves of \( K \). Introducing a new angular coordinate \( \phi = \tilde{\phi} - B\tau \), which has standard \( 2\pi \) periodicity and is constant along orbits of \( K \), gives \( K = \partial/\partial \tau \). Then the ten-dimensional string frame metric is
\[
d s^2 = \Sigma^{1/2} H^{-3/d} d\mathbf{s}^2(\mathbb{M}^{d-1}) + \Sigma^{1/2} H^{1-3/d} \left[ \frac{dr^2}{f} + r^2 \left( \Delta d\theta^2 + \cos^2 \theta d\Omega^2_{d-1} \right) \right] \]
(4.39)
\[+ \Sigma^{-1/2} H^{1-3/d} \tilde{f}\Delta r^2 \sin^2 \theta d\phi^2,\]
where
\[
\Sigma = f \left( 1 + \frac{B kl \cosh \alpha \sin^2 \theta}{r \Delta f} \right)^2 + H \frac{\Delta \tilde{f}}{f} (Br \sin \theta)^2, \]
(4.40)
and the dilaton is given by
\[
e^{2\phi} = \Sigma^{3/2} H^{-3/d}. \]
(4.41)
In general, the metric (4.39) has a conical singularity in the \( r-\phi \) plane at \( r = r_H \). An analysis of the \( r \to r_H \) limit shows that the deficit angle is given by \( 2\pi (1 - a) \), where
\[
a = \frac{1}{2l + B \cosh \alpha (r_H^2 - l^2)}.
\]
(4.42)
Then by imposing
\[
B = \frac{1}{R_{11}} - \Omega = \frac{dr_H + (\tilde{d} - 2)l}{2r_H (r_H + l) \cosh \alpha},
\]
(4.43)
we have \( a = 1 \), and so no conical singularity. As described in [19, 20], the ten-dimensional metric has the correct form to describe an F-string or D4-brane expanding into a D6-brane. The fixed point set \( \{ r = r_H, \theta = 0 \} \) of \( K \) leads to a null singularity in the metric (4.39), the singular surface being identified with the spherical part of the worldvolume of the F-string or D4-brane: \( \mathbb{M}^2 \otimes S^5 \) or \( \mathbb{M}^5 \otimes S^2 \) respectively. Moreover, an analysis of the near-core, \( \{ r = r_H, \theta = 0 \} \), spacetime shows that, in both cases, the radius of the sphere into which the F-string or D4-brane expands is precisely equal to \( r_H \) [19, 20].

4.1 Smeared solutions: F-strings expanding into a Dp-brane

We have seen in section 3.1, from the worldvolume point of view, that the case of F-strings expanding into a D2-brane is qualitatively different to that of expansion into any other Dp-brane: there is only one static solution and it is unstable. Here we obtain a supergravity description, albeit a smeared one [20], describing \( N \) F-strings expanding into a Dp-brane, for arbitrary \( 2 \leq p < 6 \). We will see that the \( p = 2 \) is a special case in supergravity as well. The starting point in this case is a smeared

\footnote{We will not write down the dimensional reduction of the 3-form potentials (4.31) and (4.32) since they are superfluous to our discussion.}
version of the M2-brane metric (4.29):
\[ ds^2 = H^{-2/3} \left[ ds^2(\mathbb{M}^2) + f \left( dr + \frac{k l \cosh \alpha}{\Delta r_p f} \sin^2 \theta d\tilde{\phi} \right)^2 \right] \]
\[ + H^{1/3} \left[ ds^2(\mathbb{E}^{6-p}) + \frac{d^2}{\tilde{f}} + r^2 \left( \Delta d\theta^2 + \frac{\Delta \tilde{f}}{\tilde{f}} \sin^2 \theta d\tilde{\phi}^2 + \cos^2 \theta d\Omega^2_p \right) \right] \]
where the functions \( H, f, \tilde{f} \) and \( \Delta \) are given by (4.30), but with \( \tilde{d} \) replaced by \( p \). The solution is thus smeared over the obvious \( 6-p \) transverse directions. The relations (4.33) and (4.35) are unchanged up to the replacement of \( \tilde{d} \) with \( p \), as is the 3-form potential (4.31). The charge quantization condition is then
\[ k \cosh \alpha \sinh \alpha = c_p N \frac{(R_{11} \lambda)^2}{V_{6-p}}, \] where
\[ c_p = \frac{(2\pi)^4}{p \Omega_{1+p}}. \]
Dimensional reduction proceeds as above, and gives rise to a D6-brane with topology \( \mathbb{M}^2 \times \mathbb{E}^{6-p} \times S^{p-1} \), so that T-duality along the flat transverse directions gives the desired solution describing \( N \) F-strings expanding into a D\( p \)-brane with topology \( \mathbb{M}^2 \times S^{p-1} \) [24]. Again, the potential conical singularity is avoided if \( B \) is given by (4.43), up to the replacement of \( \tilde{d} \) with \( p \).

### 4.2 The radius of the sphere

The question with which we are concerned here is, for what values of \( r_H \) does the solution (4.39) make sense? That is, what values of \( r_H \) are allowed and, furthermore, do the allowed values of \( r_H \) match the radii of the polarized branes discussed in sections 2 and 3? To address this question, we need to think about which quantities to hold fixed as we vary the values of \( N \) and \( B \). At first sight, one might consider holding the rotation parameter, \( l \), fixed, but this is incorrect; we should rather be considering the set of possible solutions for constant \( 2\kappa_{10}^2 = (2\pi)^4 R_{11}^2 \lambda^3 \). In other words, we should fix \( R_{11} \), in order that our family of ten-dimensional solutions have the same Newton’s constant.

We thus use the four relations (4.33), (4.35), (4.36) and (4.43) to eliminate \( l, \alpha \) and \( k \), and to give a relation between \( r_H, N, B \) and \( R_{11} \). First rewrite (4.43) as
\[ l r_H^{d-1} - l^2 r_H^{d-2} + kB \cosh \alpha r_H - \frac{d}{2} k = 0, \] Now take (4.33) to eliminate \( l \) from (4.47) and (4.35). Then eliminate \( \alpha \) by using (4.35), giving
\[ \frac{1}{4} (\tilde{d} - 2)^2 (1 - BR_{11})^2 k^2 + [(\tilde{d} - 2)(1 - BR_{11})^2 + 1] r_H^d k + [(1 - BR_{11})^2 - 1] r_H^{2\tilde{d}} = 0. \]
We may now solve this quadratic for \( k \) and substitute into (4.36). We note that the requirement \( k \geq 0 \) gives the restriction
\[ 0 \leq BR_{11} \leq 1, \]
so that \( B = 1/R_{11} \) is the maximum magnetic field [18]. However, for the ten-dimensional Kaluza-Klein picture to be valid, we need all ten-dimensional length scales to be larger than the compactification scale, \( R_{11} \). The magnetic field gives just such a length scale, \( 1/B \), so the solution is truly
ten-dimensional only for $BR_{11} \ll 1$. At any rate, we now find

$$k = \eta^d_H,$$  \hspace{1cm} (4.50)

where we have defined

$$\eta = \frac{2}{(d - 2)(1 - BR_{11})^2} \left[ \sqrt{1 + \tilde{d}(d - 2)(1 - BR_{11})^2} - (\tilde{d} - 2)(1 - BR_{11})^2 - 1 \right].$$  \hspace{1cm} (4.51)

With

$$A = \frac{1}{\eta} \left( 1 + \frac{\tilde{d} - 2}{2} \eta \right),$$  \hspace{1cm} (4.52)

we have

$$\eta^2 c_{d}^2 \frac{1}{A^{2d}} \left( \frac{\lambda}{R_{11}^2} \right)^{\frac{2d}{d - 2}} N^2 + \left( \frac{r_H}{AR_{11}} \right)^{2d - 2} - \left( \frac{r_H}{AR_{11}} \right)^{2d - 4} = 0.$$  \hspace{1cm} (4.53)

The functions $A(BR_{11})$ and $\eta(BR_{11})$ which control this equation, are plotted in figure 5. By introducing the dimensionless quantities

$$\rho_S = \frac{r_H}{AR_{11}},$$  \hspace{1cm} (4.54)

$$N_S = \frac{c_{d}}{\eta A^d} \left( \frac{\lambda}{R_{11}^2} \right)^{\frac{2}{d - 2}} N,$$  \hspace{1cm} (4.55)

relevant for the supergravity solutions, we finally find

$$N_S^2 = \rho_S^{2(d-2)} - \rho_S^{2(d-1)},$$  \hspace{1cm} (4.56)

which is of precisely the same form as (2.14) for $\tilde{d} = 3$, and (3.25) for $\tilde{d} = 6$. Although we cannot reproduce the correct numerical factors for any value of $BR_{11}$, it is surprising enough that the form of the equations are the same when one considers that the worldvolume calculation did not make use of a consistent supergravity background.

In figure 6 we show how the radius, $\rho_S$, of the $(\tilde{d} - 1)$-sphere depends on $N_S$, the dimensionless quantity defining the $(d - 2)$-brane charge of the solution. There are various noteworthy points. For $N_S$ less than some critical value, there are always two values of $r_H$, which we denote by $r_+$ and $r_-$ (where $r_+ > r_-$$)$. As long as the F-string or D4-brane charge is not too large, then, there are two distinct supergravity solutions of the form (4.39), for which the spacetime has no conical singularities. In the following subsection, we will see that the solution with $r_H = r_-$ has the lower energy of the two. As we decrease the charge, $r_+$ increases, and $r_-$ decreases. For zero charge – essentially the situation considered by Dowker et al \[18\] – there is only a single non-trivial value of $r_H$, this solution being unstable \[18\]. It corresponds to $\alpha = 0$, whereas the trivial solution $r_H = 0$ corresponds to $k = 0$, which we shall use as a background to calculate the energy of the charged solutions.

We now have all the information we need to consider the $B \to 0$ limit for fixed $N$ and $R_{11}$ (we consider the case $\tilde{d} = 3$). For $BR_{11} \ll 1$, $\eta \propto BR_{11}$ and $A^{-1} \propto BR_{11}$ (see figure 5), so from (4.55) we have $N_S \to A^{-2} \to 0$. Figure 6 shows that there are two radii for which $N_S \to 0$, one at $\rho_S \to 0$ and the other at $\rho_S \to 1$. The former has $N_S \propto \rho_S$ (see figure 6) as $\rho_S \to 0$ so, from (4.54), we have $\frac{r_H}{R_{11}} \propto \frac{1}{A} \to 0$. The latter has $\rho_S \to 1$ so $\frac{r_H}{R_{11}} \propto A \to \infty$. The solution with $r_H = r_-$ was identified as the stable solution of the worldvolume theory, so as the magnetic field is reduced this radius
the dimensionless functions $\eta(BR_{11}), A(BR_{11})$

Figure 5: In order to find $r_H$ for a given set of parameters, $R_{11}$, $B$ and $N$ we need the dimensionless functions $A(BR_{11}), \eta(BR_{11})$. (We plot $A^{-1}(BR_{11})$ for clarity.) The scales on the left of the graphs, and the solid lines, correspond to $\tilde{d} = 3$ (D4→D6) and the right hand scales, and dashed lines, correspond to $\tilde{d} = 6$ (F1→D6).

It is not difficult to generalize the above analysis to include the smeared solution describing $N$ F-strings expanding into a D$p$-brane for arbitrary $2 \leq p < 6$, as discussed in section 4.1. For $p \neq 2$, we need only everywhere replace $\tilde{d}$ with $p$. The ten-dimensional solutions thus obey

$$N_S^2 = \rho_S^{2(p-2)} - \rho_S^{2(p-1)},$$

(4.57)

which is of the same form as (3.27). The $p = 2$ case is simpler since the equation analogous to (4.48) is then only linear in $k$. We have

$$k = [1 - (1 - BR_{11})^2] r_H^2,$$

(4.58)

so that, in terms of the relevant dimensionless quantities, we find

$$\rho_S = \sqrt{1 - N_S^2},$$

(4.59)

which is of the same form as (3.28). We note that this single consistent solution corresponds to the larger of the two radii, $r_H = r_+$, in the case of general $p$. As we will see below, it is only the smaller radius which is stable, so it would seem that F-strings cannot expand into a stable spherical D2-brane. At any rate, the atypical properties of the spherical D2-brane with dissolved F-string charge are precisely captured by the corresponding (smeared) supergravity solution. We expect that a localized solution should exist, and that it should also exhibit the same behaviour.
4.3 Energetics and stability

Following [19], to compute the energy of the ten-dimensional solutions we make use of the background subtraction method [33], where the relevant background is the \( k = 0 \) solution: the standard Melvin-like flux 7-brane solution [9] written in spherical oblate coordinates. That is to say we define the zero of energy to be the \( k = 0 \) solution. We also need to transform the metric (4.39) to the Einstein frame for this calculation, using \( ds^2_E = e^{-\phi/2} ds^2_S \). Specifically [33]

\[
\mathcal{E} = -\frac{1}{2\kappa_{10}^2} \left[ \int_\infty d^8x \sqrt{h} \mathcal{N} \mathcal{K} - \int_\infty d^8x \sqrt{h_0} \mathcal{N}_0 \mathcal{K}_0 \right],
\]

where \( h_{ij} \) is the metric induced on a constant \( r \) slice of the constant time slice, \( h \) is its determinant and the integration is performed as \( r \to \infty \). The lapse function, \( \mathcal{N} \), and extrinsic curvature, \( \mathcal{K} \), of the constant \( r \) slice, are given by

\[
\mathcal{N} = \sqrt{-g_{tt}}, \quad \mathcal{K} = \frac{1}{\sqrt{g_{rr}}} h^{ij} \partial_r h_{ij},
\]

and a “0” subscript denotes the corresponding quantities for the reference \( k = 0 \) background.

We find

\[
\mathcal{E} = \frac{\Omega_{d+1} V_{d-2}}{(2\pi)^4 R_{11}^2 \lambda^3} \left( \hat{d} \sinh^2 \alpha + \hat{d} + 2 \right) k.
\]

Defining the dimensionless energy, \( \mathcal{E}_S \), as

\[
\mathcal{E}_S = \frac{\epsilon_d (2\pi)^{(2-d)/3} \lambda^3}{A^d \eta V_{d-2} P_{11}^{d-2}} \mathcal{E},
\]

gives

\[
\mathcal{E}_S = \rho_S^{\hat{d}-2} \left( 1 + \frac{2}{d} \rho_S^2 \right),
\]

which is plotted in figure 6 for the two values of \( \hat{d} \). In both cases, the solution with \( r_H = r_- \) has lower energy than that with \( r_H = r_+ \), which agrees with the pictures presented in sections 2 and 3.

Although the solution with \( r_H = r_- \) has lower energy than that with \( r_H = r_+ \), this does not imply stability. We can argue, however, that it is in fact stable to radial perturbations by looking at “off–shell” configurations, that is by holding \( B \) and \( N \) constant and varying \( r_H \). This still yields a supergravity solution but in general there will be a conical singularity on the surface of, and inside, the dielectric sphere. The interpretation of this singularity is that it provides the tension necessary to hold the dielectric sphere at that radius. One can associate a “deficit brane” with this conical singularity and, just as for a cosmic string [35], its tension is proportional to the deficit angle, \( 2\pi (1 - a) \), with \( a \) as in (4.42). So if the deficit angle is positive, then the deficit brane has positive tension, and is having to pull on the dielectric sphere to keep it static; the dielectric sphere wants to expand. Similarly, when the tension is negative, the sphere wants to contract.

To compute the tension (mass per unit volume) of the deficit brane, we take the \( r \to r_H \) limit of the metric (4.39) and again make use of the background subtraction method [33], as applied to the resulting spacetime. As explained in [33], the relevant background in this case is a spacetime identical in all respects except that it is free of the conical singularity. Denoting the energy in
Figure 6: Analogous plots to figures 2 and 4 for the supergravity case without conical singularities. They show how the charge, $N_S$ (4.56), and energy, $\mathcal{E}_S$ (4.64), vary as the radii of the static solutions change. The solid line is for $\tilde{d} = 3$ (D4→D6) and the dashed line is for $\tilde{d} = 6$ (F1→D6).

which we are interested by $T$, and applying the formulae (4.60) and (4.61) to the case at hand, we find

$$T = \frac{\Omega_{\tilde{d}+1}V_{d-2}r_{H}^{\tilde{d}-1}(l + B \cosh \alpha(r_{H}^{2} - l^{2}))}{(2\pi)^{4}R_{11}^{2}\lambda^{3}}(1 - a) = \frac{\Omega_{\tilde{d}+1}V_{d-2}R_{11}^{\tilde{d}-2}}{(2\pi)^{4}\lambda^{3}}\mathcal{T}_S$$

(4.65)

which, as promised, is proportional to the deficit angle. $\mathcal{T}_S$ is the mass of the deficit brane in terms of the dimensionless quantities. It is interesting to note that we can derive this expression using different techniques. In particular, we have considered methods analogous to those used in [36] to compute the effective energy-momentum tensor of the conical singularity between two collinear black holes in four dimensions; and these methods give exactly the same result for the mass $T$.

To analyse the form of (4.65) further, we fix $B$ and $N$, and substitute for $l$, $\alpha$ and $k$ using the expressions (4.33), (4.35) and (4.36). Of course, we no longer have access to the expression (4.43) for $B$, since this was derived to ensure the absence of conical singularities. Indeed, we can no longer solve for $\alpha$ or $k$ analytically, and must proceed using numerical methods. The plot of the mass, $\mathcal{T}_S$, as a function of $\rho_S$ is shown in figure 7, for both the F-string and D4-brane solutions.

As argued previously, a deficit brane with positive tension indicates that the dielectric sphere wants to expand, and negative tension implies that it wants to contract. We see from figure 7, then, that at $r_H = r_+$ the sphere is unstable to both expansion and contraction, whereas at $r_H = r_-$ the sphere is stable. We should note that these arguments apply only to radial perturbations.

5 Discussion

We have provided strong evidence that the supergravity solutions of Costa et al [19] and Emparan [21] are describing the dielectric effect whereby F-strings or D4-branes expand into spherical
Figure 7: This plot shows the dimensionless mass (4.65) of the deficit brane for $\tilde{d} = 3$ (solid line, left hand scale), $\tilde{d} = 6$ (dashed line, right hand scale), and for some specific values of $B$ and $N$. The inset is a magnification of the region around $r_H = r_-$. The arrows show whether the sphere wants to expand or contract as one moves away from the zeros of $T_S$.

D6-branes. We have presented the two supergravity solutions in a unified manner showing that, in both cases, there are two distinct static solutions which are free of conical singularities. Moreover, the radius of these solutions has precisely the same functional form as that implied by the worldvolume analyses of Emparan [21] and Myers [22].

On the supergravity side we have seen that the energy of the $r_H = r_-$ solution was lower than that with $r_H = r_+$, although this alone does not guarantee stability. To address the latter issue, we considered the tension of the deficit brane formed by static solutions away from $r_H = r_\pm$, which indicated that the $r_H = r_-$ solution was stable and that with $r_H = r_\pm$ was unstable. Thus we see that the supergravity solutions capture the worldvolume properties with remarkable accuracy.

To further strengthen the interpretation of the supergravity solutions, we considered the special case of $N$ F-strings expanding into a D2-brane, for which the worldvolume analysis indicates that no stable dielectric solution should exist. Although the supergravity solution for this configuration was smeared in certain transverse directions, we have seen that indeed there was no stable solution.

A potentially interesting area of research would be to look at the possible tunnelling of these configurations. From the worldvolume perspective, it is clear that tunnelling should occur from the classically stable configuration at $r_H = r_-$ to some expanding solution. On the supergravity side, in the case with no dissolved charge, instantons describing such a process exist, and have been described in some detail by Dowker 	extit{et al} [18]. They correspond to the nucleation of spherical branes within a fluxbrane. It would be interesting to look for analogous instantons in the case with dissolved charge.

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