Constraints on heavy meson form factors derived from QCD analyticity, unitarity and heavy quark spin symmetry

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Using the analytic properties of two-point functions in QCD, as well as unitarity, bounds on the $B$ meson form factor $F(q^2)$ can be derived. Heavy quark spin symmetry, correctly taken into account, is shown to improve these bounds significantly.

1 Introduction

An evaluation of the Cabibo-Kobayashi-Maskawa matrix element $V_{cb}$ can be obtained by studying the weak decays of $B$ mesons to $D$ mesons: $B \rightarrow Dl\bar{\nu}$. The amplitude for this process is:

$$M_{fi} = \left[ \bar{\psi}_l \frac{-ig}{2\sqrt{2}} \gamma^\mu(1 - \gamma_5) \psi_l \right] \frac{i(\gamma_\mu + \gamma_\mu q/\sqrt{2}M_W)}{q^2 - m_W^2} \times V_{cb} < D(p') | \bar{c} \gamma_\mu b | B(p) >$$

The corresponding total cross-section can be taken from experiment. The leptonic vertex can be computed perturbatively; to find $V_{cb}$ all there is yet to do is to find a way to evaluate the hadronic matrix element.

We can do so by taking advantage of the fact that $B$ and $D$ are heavy mesons. It was shown that the form-factors which describe such transitions are, in the infinite mass limit, expressed through one universal function: the Isgur-Wise function. Using the heavy quark effective theory (HQET), finite mass corrections suited to each particular case can be computed.

In this paper, we will concern ourselves with the following matrix element:

$$< B(p') | \bar{b} \gamma^\mu b | B(p) > = (p + p')^\mu F(q^2) \quad q = p - p'$$

Specifically, we will try to obtain bounds on the slope (charge radius) of the relevant form factor $F$ near zero momentum transfer:

$$\rho^2 = \frac{t_0}{2} \left[ \frac{dF}{dq^2} \right]_{q^2 = 0} , t_0 = 4m_B^2$$

Thus, information on the Isgur-Wise function can be obtained; actually, to a good approximation $F$ coincides with this function.
The method applied in the following makes use of the analyticity and unitarity of the theory. A dispersion relation for the vacuum polarization function is written. A Lehmann spectral representation can be used for the imaginary part of this function. Keeping in the spectral sum only the contribution of certain states we get an integral inequality which relates the form-factors along the unitarity cut (physical region for pair production) and the value of the function far from this region. The spin symmetry results of HQET are used in order to get a better inequality. Finally, applying standard techniques related to vector-valued analytic functions the required constraints on \( \Pi \) are derived.

In the next section we outline the method, using as an example the simplest case (taking into account only the contribution of \( B - \bar{B} \) states in the spectral sum). In the following one, we address the problem of including the \( B - \bar{B}^* \) and \( B^* - \bar{B}^* \) states; for this, the results of HQET have to be correctly taken into account. Finally, we present some numerical results and a short discussion.

2 Description of the method

Let’s consider the vacuum polarization tensor:

\[
\Pi^{\mu\nu}(x) = i < 0 | T \{ V^\mu(x) V^\nu(0) \} | 0 > , \quad V^\mu = \bar{b} \gamma^\mu b
\]

Going into momentum space, we get the invariant amplitude \( \Pi(q^2) \):

\[
\Pi^{\mu\nu}(q) = (q^\mu q^\nu - q^2 g^{\mu\nu}) \Pi(q^2)
\]

The analytic properties of this function (in the complex \( q^2 \) plane) are easy to derive. In the spectral sum

\[
\text{Im} \Pi(q^2) = \frac{-1}{3q^2} \pi(2\pi)^3 \times \sum_{\Gamma} \delta(q - p_{\Gamma}) \delta(|q^0| - E_{\Gamma}) < 0 | V^\mu(0) | \Gamma > < \Gamma | V_\mu(0) | 0 >
\]

only states which contain a \( b \) and a \( \bar{b} \) quark will contribute.

Such states can be uniparticle states: \( \Upsilon_1, \Upsilon_2, \Upsilon_3, \Upsilon_4 \) (bound states of the \( b - \bar{b} \) pair) or multiparticle states: \( a B - \bar{B}, B - B^* \) or \( B^* - B^* \) pair (a \( B \) is a bound state of the heavy quark \( \bar{b} \) and one of the light quarks \( u, d \) or \( s \); the difference between \( B \) and \( B^* \) is in their internal quantum numbers; thus \( B \) is a pseudoscalar particle, while \( B^* \) is a vectorial one).

From the above relation, as well as the masses of these particles, it can be seen that the function \( \Pi \) will have three poles in \( q^2 = m_{\Upsilon_1}^2, m_{\Upsilon_2}^2 \) and \( m_{\Upsilon_3}^2 \)
and a cut starting from \( q^2 = t_0 = 4m_B^2 \) ( \( m_B^2 \gtrsim t_0 \), so this pole is covered by the cut). Therefore, we can write a dispersion relation for the derivative of \( \Pi \) :

\[
\frac{d\Pi(q^2)}{dq^2} = \Pi'(q^2) = \frac{1}{\pi} \int_{m_{\Upsilon}^2}^{\infty} \frac{Im \Pi(t)}{(t - q^2)^2} dt
\]

for \( q^2 \) on the real axis and \( q^2 < m_{\Upsilon}^2 \).

Furthermore, it can be shown that the contribution of a certain type of particles in the spectral sum is positive; so, keeping only the contribution of \( B - \bar{B} \) states in (4) we can write an inequality like:

\[
Im \Pi(q^2) \geq \frac{1}{3q^2} (2\pi)^3 \times 
\sum_{\Gamma(B\bar{B})} \delta(\mathbf{q} - \mathbf{p}_\Gamma)\delta(|q^0| - E_\Gamma) < 0|V|B\bar{B} < B\bar{B}|V|0 >
\]

The matrix elements appearing in this inequality can be expressed with the form factor defined in (2); thus we get:

\[
Im \Pi(q^2 = t) \geq \frac{n_f}{48\pi} \left( 1 - \frac{t_0}{t} \right)^{3/2} |F(t)|^2 \theta(t - t_0)
\]

where \( n_f = 3 \) comes from the fact that we have three different types of \( B \) mesons (we presume that the form factor is the same for \( B_u \), \( B_d \) and \( B_s \)); inserting in (3) : 

\[
\Pi'(q^2) \geq \frac{1}{16\pi^2} \int_{t_0}^{\infty} \frac{(t - t_0)^{3/2}}{t^{3/2}(t - q^2)^2} |F(t)|^2 dt
\]

In the above relation, there is only one unknown quantity: the \( F \) form factor (\( \Pi'(q^2) \) can be computed perturbatively for \( q^2 \ll m_{\Upsilon}^2 \)). To get information from (8) on the charge radius (3), we apply the following procedure:

Defining

\[
I = \frac{1}{16\pi^2\Pi'(q^2)} \int_{t_0}^{\infty} \frac{(t - t_0)^{3/2}}{t^{3/2}(t - q^2)^2} |F(t)|^2 dt
\]

we perform a conformal mapping:

\[
z = \frac{\sqrt{t_0 - t} - \sqrt{t_0}}{\sqrt{t_0 - t} + \sqrt{t_0}}
\]
which brings the $t = q^2$ plane inside the unit circle; the threshold $t_0$ goes into $z = -1$, the points on the physical cut go onto the circumference of the circle; \((9)\) becomes

$$I = \int_0^{2\pi} \frac{d\theta}{2\pi} |\Phi(z)F(z)|^2$$

(11)

with $z = e^{i\theta}$ and

$$\Phi(z) = \frac{1}{16} \sqrt{\frac{1}{2\pi t_0 \Pi'(0)}} (1 + z)^2 \sqrt{1 - z}$$

(12)

in the particular case when $q^2 = 0$.

Second, we get rid of the singularities of the integrand in the unit disk. It can be shown\(^4\) that the form factor $F$ also has poles at the square masses of the three $\Upsilon$ particles; in the $z$ plane, they will appear as simple poles $z_1, z_2, z_3$ somewhere on the real axis between $z = -1$ and $z = 0$. To make them disappear, multiply the integrand in (11) by the so-called Blaschke functions:

$$B(z) = \prod_{i=1}^3 \frac{z - z_i}{1 - z z_i^*}$$

(13)

which have zeroes in $z_1, z_2, z_3$, and, moreover, $|B(z)| = 1$ on the unit circle $|z| = 1$. \(^3\)

Then:

$$I = \int_0^{2\pi} \frac{d\theta}{2\pi} |\Phi(z)F(z)B(z)|^2$$

(14)

the integrand being an analytic function.

Finally, let’s consider the following expression:

$$I = \frac{1}{2\pi} \int_0^{2\pi} |G(z)|^2 d\theta , \ z = e^{i\theta} .$$

If $G$ is analytic inside the unit disk we can expand it in a power series:

$$G(z = e^{i\theta}) = c_0 + c_1 e^{i\theta} + c_2 e^{2i\theta} + \ldots$$

and actually perform the integration:

$$I = |c_0|^2 + |c_1|^2 + |c_2|^2 + \ldots$$

\(^a\) if we’d knew the residues of $F$ in its poles we would be able to use a better method - in essence, substract the singularities, instead of multiplying them out; see\(^3\)
so that $1 \geq I$ implies:

$$1 \geq |c_0|^2 + |c_1|^2 = |G(0)|^2 + |G'(0)|^2$$

This is a Schur-Caratheodory type inequality; in our case it reads:

$$1 \geq |B\Phi F|^2(0) + |(B\Phi F)'|^2(0)$$

(15)

Using the normalization $F(0) = 1$, this quadratic inequality will give a superior and an inferior bound on $\rho^2$.

3 Heavy quark spin symmetry

The next step is to try to improve these bounds. Obviously, one way of achieving this is to take into account as many terms as possible in the right hand side of (4). It is easily seen that, with some phenomenological input, the contribution due to uniparticle states can be computed; thus we get

$$\Pi'(q^2) \geq \frac{27}{4\pi^2} \sum_i \frac{M_{\Upsilon_i} \Gamma_{\Upsilon_i}}{(q^2 - M_{\Upsilon_i}^2)^2} + \frac{1}{16\pi^2} \int_{t_0}^{\infty} \frac{(t - t_0)^{3/2}}{t^{3/2}(t - q^2)^2} |F(t)|^2 dt$$

(16)

where the widths $\Gamma_{\Upsilon_i}$ are physically measurable quantities defined by

$$\sigma(e^+e^- \rightarrow \Upsilon_i) = 12\pi^2 \delta(t - M_{\Upsilon_i}^2) \frac{\Gamma_{\Upsilon_i} M_{\Upsilon_i}}{M_{\Upsilon_i}}$$

Further, we try to include contributions from the $B - \bar{B}^*$, $B^* - \bar{B}$ and $B^* - B^*$ states. The relevant matrix elements in (4) can be expressed through the following form-factors:

$$<B^*(p', \epsilon)|V^\mu|B(p)> = \frac{2ie_{\epsilon\nu\alpha\beta}}{m_B + m_B'} \epsilon_{\nu} p_{\mu} p_{\beta} V(q^2)$$

(17)

$$<B^*(p', \epsilon')|V^\mu|B^*(p, \epsilon)> = F_1(q^2)(\epsilon \cdot \epsilon') P_{\mu} + F_2(q^2)[\epsilon_{\mu}(\epsilon' \cdot P) + \epsilon'_{\mu}(\epsilon \cdot P)]$$

$$+ F_3(q^2)\frac{(\epsilon \cdot P)(\epsilon' \cdot P)}{m_B'} P_{\mu} + F_4(q^2)[\epsilon_{\alpha}(\epsilon' \cdot P) - \epsilon'_{\alpha}(\epsilon \cdot P)]\frac{g^\mu\alpha q^2 - q^\mu q^\alpha}{m_B'^2}$$

(18)

(\text{where } P = p + p', q = p - p'); \text{ thus, instead of } (8),(16) \text{ we have :}

$$\Pi'(q^2) \geq \frac{27}{4\pi^2} \sum_i \frac{M_{\Upsilon_i} \Gamma_{\Upsilon_i}}{(q^2 - M_{\Upsilon_i}^2)^2} +$$

(19)
\[
\frac{1}{16\pi^2} \left\{ \int_{t_0}^{\infty} \frac{(t - t_0)^{3/2}}{t^{3/2}(t - q^2)^2} |F(t)|^2 dt + \int_{t_0^*}^{\infty} \frac{4t}{t_0^* t^{3/2}(t - q^2)^2} |V(t)|^2 dt + \int_{t_0^*}^{\infty} \frac{(t - t_0^*)^{3/2}}{t^{3/2}(t - q^2)^2} \left[ 2|F_1(t)|^2 + 2\frac{t}{t_0^*} |F_2(t)|^2 + |\hat{F}_3(t)|^2 + \left( \frac{4t}{t_0^*} \right)^2 |F_4(t)|^2 \right] dt \right\}
\]

where \( t_0^* = (m_B + m_{B^*})^2, t_0^{**} = 4m_{B^*}^2 \); \( \hat{F}_3 \) is a linear combination of \( F_1, F_2 \) and \( F_3 \) (see (24) below).

Having six unknown functions in it, this inequality is, in this form, of no use. At this point, the results of HQET can be of help; it can be shown that the form-factors are related in the neighborhood of zero-recoil point as follows:

\[
V(t), F_2(t) \rightarrow F(t), F_1(t) \rightarrow -F(t)
\]

\[
F_3, F_4(t) \rightarrow 0, \quad \text{when} \quad t = q^2 \rightarrow 0
\]

If one assumes that these relations hold on the entire unitarity cut then the inequality (19) can be written in terms of a single form factor \( F \). But this assumption was shown not to hold, especially near thresholds.

Treating each term in (19) separately, we can still obtain an inequality like (21):

\[
1 \geq |B\Phi F|^2(0) + |(B\Phi F)'|^2(0) + |B\Phi V|^2(0) + |(B\Phi V)'|^2(0) + \ldots
\]

The full expressions for \( \Phi, \Phi V \ldots \) are given in 11. This relation contains only the values of the form factors and of their derivatives at \( q^2 = 0 \); therefore the relations (20) hold, and (19) will become:

\[
a(\rho^2)^2 - 2b\rho^2 + c \leq 1
\]

\[
a = 64 \sum (B\Phi)^2(0) \quad b = 8 \sum (B\Phi)(B\Phi)'(0)
\]

\[
c = \sum [(B\Phi)^2(0) + (B\Phi)^2(0)] - 1
\]

with solution:

\[
b - \sqrt{b^2 - ac} \leq \rho^2 \leq b + \sqrt{b^2 - ac}
\]

This is our final result.

In deriving (19), there is a technical point worth discussing. The contribution of \( B - \bar{B} \) and \( B - B^* \) in the spectral sum (10) are parametrized by single

\[\text{equation (10) here differs from eq. (13) in } 11 \text{ by a factor 2 in the term containing } V(t), \text{ because in } 11 \text{ the contribution of } B^* - B \text{ states was not included.}\]
form factors $F$ and $V$; which appear in the integral (19) as $|F|^2$ and $|V|^2$; but the matrix element of $B^* - \bar{B}^*$ is parametrized by four form factors, which appear in (19) as a positive definite quadratic form. To be able to apply the method described in section 2, we have to write this quadratic form as a sum of squares; it actually appears this way in (19), ${\hat{F}}(t)$ being defined as:

$$
{\hat{F}}(t) = \left( \frac{2t}{t_0} \right) \left( \frac{2t}{t_0} - 1 \right) F_1(t) + \frac{2t}{t_0} F_2(t) + \left( \frac{2t}{t_0} \right)^2 F_3(t) \tag{24}
$$

4 Results and discussion

To obtain numerical results for the bounds (23) there are two parameters whose value has to be chosen (between certain limits). The first one is $q^2$ in (19). Theoretically, we can choose any value for $q^2$ from $-\infty$ to $m_{\Upsilon_1}^2$; the best results are obtained with $q^2$ as big as possible. In practice, $\Pi'(q^2)$ is evaluated perturbatively; the reliability of this evaluation increases if $q^2$ is far from the physical region; it seems that $q^2 = 50$ GeV$^2$ is as close as we can get to $m_{\Upsilon_1}^2 = 89.5$ GeV$^2$ and still believe in the perturbation series. The second parameter is the mass of the $b$ quark, which appears in the evaluation of $\Pi$. As this quantity is not well defined, we choose to vary it from 4.7 to 5 GeV. Higher mass gives stronger limits.

In previous work, taking into account only the contribution of $\Upsilon$ and $B - \bar{B}$ states (not using spin symmetry at all) the following limits were obtained:

-5.0 $\leq \rho^2 \leq 4.5$. Taking into account the contribution of $B - B$ and $B - B^*$ the bounds become $-0.90 \leq \rho^2 \leq 2.60$. These results were obtained with $q^2 = 0$, $m_b = 5$ GeV and using one loop approximation to compute $\Pi'(0)$. Because the two and three loop contribution to $\Pi'(0)$ increase its value by approximately 40% (!), these limits are actually stronger than what they should be.

Using the full apparatus presented in this paper, we obtain:

-0.2 $\leq \rho^2 \leq 1.85$ for $m_b = 4.7$ GeV; $-0.1 \leq \rho^2 \leq 1.76$ for $m_b = 5.0$ GeV; at $q^2 = 50$ GeV$^2$:

-0.0 $\leq \rho^2 \leq 1.6$ for $m_b = 4.7$ GeV and 0.3 $\leq \rho^2 \leq 1.2$ for $m_b = 5.0$ GeV.

These results show that using only general properties of QCD (as analyticity and unitarity) we can derive nontrivial constraints on the behavior of heavy mesons form factors at transfer momentum close to zero. The use of heavy quark spin symmetry brings significant improvements.

Further results are expected from the application of this method to other transitions, like the physically interesting case of $B$ to $D^*$ transition.\[\text{[14, 15]}\]
5 Acknowledgements

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6 References

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