The challenge of understanding the collective behaviors of social systems can benefit from methods and concepts from physics [1-6], not because humans are similar to electrons, but because certain large-scale behaviors can be understood without an understanding of the small-scale details [7], in much the same way that sound waves can be understood without an understanding of atoms. Democratic elections are one such behavior. Over the past few decades, physicists have explored scaling patterns in voting and the dynamics of political opinion formation, e.g. [8-13]. Here, we define the concepts of negative representation, in which a shift in electorate opinions produces a shift in the election outcome in the opposite direction, and electoral instability, in which an arbitrarily small change in electorate opinions can dramatically swing the election outcome, and prove that unstable elections necessarily contain negatively represented opinions. Furthermore, in the presence of low voter turnout, increasing polarization of the electorate can drive elections through a transition from a stable to an unstable regime, analogous to the phase transition by which some materials become ferromagnetic below their critical temperatures. Empirical data suggest that United States presidential elections underwent such a phase transition in the 1970s and have since become increasingly unstable.

Elections are, fundamentally, a means of aggregating many opinions into one—those of the citizens into that of the elected official. Here, an opinion refers to a person’s entire set of political beliefs; i.e. each citizen (and candidate) has one opinion. For simplicity, we focus on the case in which the set of all possible opinions can be embedded in a one-dimensional continuous space (e.g. a position on a left-right spectrum), as in many studies in the social choice literature (reviewed in section S2 C). However, our results can be extended to a multidimensional space (see section S2 A). The first part of this manuscript examines general properties of electoral representation (which we connect to the voting power literature in section S2 B) and instability, using a mathematical formalism that departs from the literature in that it makes no assumptions about how people vote or even the structure of the voting process. The second part of this manuscript presents a specific model that builds upon the social choice literature in order to demonstrate how, when the assumption of concave voter preferences is relaxed, instability and negative representation can arise. The specific model is shown to map onto the well-known mean-field Ising model [14] of magnetic materials. The emergence of instability can couple to geospatially varying local election outcomes and the potentially destabilizing effects of a two party system. Finally, the manuscript considers the implications of such a model for contemporary American elections.

We define an election by \( y[f(x)] \), a functional that maps the distribution of electorate opinions \( f(x) \)—defined so that for any interval \([a, b] \subset \mathbb{R}\), the number of citizens with opinions in that interval is \( \int_a^b f(x)dx \)—to the election outcome \( y \in \mathbb{R} \), i.e. the opinion of the elected official. Note that in this framework, candidacy is endogenous: candidate opinions—or equivalently, which candidates run—are themselves functions of the electorate opinions. Any electoral system, regardless of its detailed mechanisms (e.g. the number of candidates or parties, the existence of primary elections, restrictions on candidate entry, etc.) can be conceptualized as such a process that outputs the opinion of the winning candidate based on the electorate opinions. In order that no opinion be a priori privileged over others, the one restriction we place on this process is translational invariance, i.e.

\[
y[f(x + c)] + c = y[f(x)]
\]

for all \( c \) (section S1 A).

In order to measure the sensitivity of the election outcome to changes in electorate opinions [15], we define the representation of an opinion \( x \) by

\[
r_e(f, x) = \frac{\delta y}{c}
\]

where \( \delta y \) is the change in outcome that occurs if an individual’s opinion changes from \( x \) to \( x' = x + c \). (Representation should not be defined using only the distance between a citizen’s opinion and that of the elected candidate: opinions without causal influence on the election outcome are not represented, even if they happen to align with that outcome.) It is convenient to measure representation by

\[
r(f, x) = \lim_{c \to 0} r_e(f, x)
\]

when the limit exists, since \( r(f, x) \) does not depend on \( c \).

For a large population, a number of results hold if the election is differentiable (see section S1 B for details). First,

\[
r(f, x) = \frac{d}{dx} \delta f(x)
\]

Second, \( r_e(f, x) \) is the average of \( r(f, x) \) over the interval \([x, x + c]\), and thus representation of individual opinions can be measured by \( r(f, x) \) alone. Third, the total representation of the electorate’s opinions equals 1:

\[
\int_{-\infty}^{\infty} f(x)r(f, x)dx = 1
\]
model (see section S1 D), the possible election outcomes in the general election based on these considerations. Then, election (or, equivalently, choose to run/are selected to run candidates (or two major political parties) who choose their position into the unstable regime. Consider an election with two shows how polarization drives elections through a phase transition. For instance, this model cannot describe the deterministic voting underlying the Median Voter Theorem [17] but can nonetheless exactly capture the collective behavior to which such voting gives rise, namely the election of the median opinion (section S1 D).

If citizens with opinions that are far from both candidates are more likely to abstain from voting, a phenomenon known as alienation [18–22], then negative representation occurs. Since not voting for either candidate is equivalent to preferring them equally, a utility function \( u_x(y) \) that captures this behavior will be almost flat for large \( |y - x| \). One such utility function is

\[
\arg\max_{y \in \mathbb{R}} \int_{-\infty}^{\infty} u_x(y) f(x) \, dx
\]

where the utility function \( u_x \) denotes the political preferences of someone with opinion \( x \). We note that this model always admits at least one Nash equilibrium in candidate strategies (section S1 E); the instability we describe, which arises from multiple Nash equilibria, should be distinguished from scenarios in which no Nash equilibria exist (section S2 C). Although this model may not accurately describe individual and candidate behaviors, it can nonetheless be used to describe the properties of real-world elections, since representation and instability depend on electoral mechanisms only through the effects of these mechanisms on \( y[f] \)—the functional that characterizes how the election outcome varies with electorate opinions. For instance, this model cannot describe the deterministic voting underlying the Median Voter Theorem [17] but can nonetheless exactly capture the collective behavior to which such voting gives rise, namely the election of the median opinion (section S1 D).

We now show that all unstable elections contain negatively represented opinions. An election is unstable if an arbitrarily small change in opinion can cause a sizable change in the election outcome (fig. 1), i.e. for some \( f \) and \( x \),

\[
\lim_{\epsilon \to 0} cr_c(f, x) \neq 0 \tag{6}
\]

If an election \( y[f] \) is unstable for an opinion distribution \( f_0 \), then if some opinion \( x_0 \) changes by a small amount \( \epsilon \) (call the resulting opinion distribution \( f_1 \)), the election outcome changes by a larger amount \( C \), i.e. \( \delta y_1 \equiv y[f_1] - y[f_0] = C \) with \( |C| > |\epsilon| \). Now consider starting with \( f_0 \) and shifting all opinions except \( x_0 \) by \( -\epsilon \) (call the resulting opinion distribution \( f_2 \)). Since \( f_2(x) = f_1(x + \epsilon) \), \( \delta y_2 \equiv y[f_2] - y[f_0] = C - \epsilon \) by translational invariance (eq. (6)). Thus, \( \delta y_1 \) and \( \delta y_2 \) have the same sign (since \( |C| > |\epsilon| \)), despite being caused by changes of opinion in opposite directions, so one of the two changes in opinions must be negatively represented. This proof that unstable elections always contain negatively represented opinions relies only on translational invariance; therefore, we expect it to hold generally in real-world conditions, regardless of the size of the electorate, the number of candidates or parties, the existence of primaries, the effects of the Electoral College, etc.

Thus far we have presented general results that apply to all elections; we now examine a particular class of models in order to illustrate how negative representation can arise and to show how polarization drives elections through a phase transition into the unstable regime. Consider an election with two candidates (or two major political parties) who choose their positions in order to maximize their chances of winning the election (or, equivalently, choose to run/are selected to run in the general election based on these considerations). Then, under the assumptions of an affine linear utility difference model [16] (see section S1 D), the possible election outcomes are given by (Theorem 4 of [16]):

\[
\arg\max_{y \in \mathbb{R}} \int_{-\infty}^{\infty} u_x(y) f(x) \, dx \tag{7}
\]

where the utility function \( u_x \) denotes the political preferences of someone with opinion \( x \). We note that this model always admits at least one Nash equilibrium in candidate strategies (section S1 E); the instability we describe, which arises from multiple Nash equilibria, should be distinguished from scenarios in which no Nash equilibria exist (section S2 C). Although this model may not accurately describe individual and candidate behaviors, it can nonetheless be used to describe the properties of real-world elections, since representation and instability depend on electoral mechanisms only through the effects of these mechanisms on \( y[f] \)—the functional that characterizes how the election outcome varies with electorate opinions. For instance, this model cannot describe the deterministic voting underlying the Median Voter Theorem [17] but can nonetheless exactly capture the collective behavior to which such voting gives rise, namely the election of the median opinion (section S1 D).

If citizens with opinions that are far from both candidates are more likely to abstain from voting, a phenomenon known as alienation [18–22], then negative representation occurs. Since not voting for either candidate is equivalent to preferring them equally, a utility function \( u_x(y) \) that captures this behavior will be almost flat for large \( |y - x| \). One such utility function is

\[
u_x(y) = u(y - x) = e^{-\frac{(y - x)^2}{2a^2}} \tag{8}
\]

where \( a \) is a positive constant. Representation is then given by (section S1 E)

\[
r(f, x) \propto \frac{1}{N} u''(y^* - x) = \frac{1}{Na^4} (a^2 - (y^* - x)^2) e^{-\frac{(y^* - x)^2}{2a^2}} \tag{9}
\]

where \( N \) is the number of constituents. Opinions far from the election outcome \( (x - y^*) > a \) are negatively represented \( r(f, x) < 0 \), see fig. 2: the election outcome is inversely sensitive to changes in those opinions. For instance, given a center-left candidate and a candidate to the right, as left-wing individuals move farther left, they may become less likely to vote for the center-left candidate (choosing instead not to vote), which increases the probability that the candidate on the right will win. In response, the electoral equilibrium of future elections may shift rightward, as candidates no longer vie for these left-wing votes. (Eq. [8] is just an example; more generally, negative representation will occur if and only if \( u(y - x) \) is not concave.) Thus, individual choices to abstain when neither candidate is appealing lead to a system-level perversion in the aggregation of electorate opinions, in which the electorate becoming more left-wing can result in a more right-wing outcome, or vice versa.

When combined with a sufficiently polarized electorate, nonvoting (in particular, non-concave \( n \)) leads not only to negative representation but also to instability. (As proven earlier, instability cannot occur without negative representation; however, negative representation is possible without instability—

![FIG. 1. Election outcomes (vertical dashed lines) shift in response to a shift in the distribution of electorate opinions (f(x), denoted by the solid curves, where the horizontal axes (x) denote political opinion). a-b. When not everyone votes, an election can be unstable (eq. (11))—a small shift in opinions to the right causes a large swing in the election outcome. c-d. In a stable election, by contrast, a small shift in opinions causes a similarly small shift in outcome.](image)
see section S1F for further discussion.) For $u_x$ defined by eq. (8) and an opinion distribution $f(x)$ consisting of two normally distributed (potentially unequally sized) subpopulations centered at $\pm \Delta$ (without loss of generality, we define the origin as the (unweighted) average of the means of the subpopulations),

$$f(x) = w_1e^{-\frac{(x+\Delta)^2}{2\sigma^2}} + w_2e^{-\frac{(x-\Delta)^2}{2\sigma^2}}, \quad (10)$$

the outcome $y$ is given by the following condition:

$$y/\Delta = \tanh(Jy/\Delta + h) \quad (11)$$

where $J$, a dimensionless measure of the polarization of the electorate, is given by $J = \Delta^2/(\sigma^2 + \sigma^2)$ and $h$, a measure of the relative sizes of the two subpopulations of the electorate, is given by $h = \frac{1}{2}\ln \frac{w_2}{w_1}$. For $w_1 = w_2$ (i.e. for equally sized subpopulations), $h = 0$, and the election is stable for $J \leq 1$ and unstable for $J > 1$. In the stable regime, $y = 0$. In the unstable regime, there are two possible outcomes described by $\pm |y^*|$, and an arbitrarily small change in $h$ can cause $y$ to swing between its positive and negative values (fig. 3). Thus, small variations determine which subpopulation wins. The variations might include changes in population, nuances in the candidates’ personalities, changes in the rules (simple majority versus Electoral College system, for example), voting restrictions, and the effectiveness of turnout operations. From election cycle to election cycle, the outcome can swing between the two subpopulations, with the majority of the opinions in the losing subpopulation being negatively represented (section S1F). An intuitive explanation of how such negative representation and instability arises is that, due to low voter turnout, candidates are incentivized to focus on turning out their bases rather than winning over centrist voters.

This voting model (eq. (11)) is precisely equivalent to a mean-field Ising model of a ferromagnet [14], in which each spin (magnetic dipole) interacts with every other spin, highlighting the hidden dependencies that can arise from a system in which citizens’ behaviors are superficially independent (section S1G). This connection with the Ising model differs from other applications of the Ising model to complex systems [23–26] in two respects: first, our analysis takes the electorate opinions as given and therefore concerns the instability in the election dynamics rather than in the dynamics of the electorate opinions themselves, and second, we do not impose any of the assumptions of the Ising model, but rather show (section S1G) that such an equivalence naturally arises for certain classes of voting behavior. Because the citizens are effectively coupled through collectively choosing a candidate, the system exhibits emergent behavior in which there can be discontinuities in the election outcome despite the continuous voting behaviors of individuals. Such emergent discontinuities are a common feature of complex systems [23, 27, 31].

These emergent discontinuities can couple to geospatial pattern formation. Given the existence of local elections, we can consider a spatially varying election outcome $y(r_i)$, which serves as an order parameter for the 2D geographic system ($\vec{r}_i$ represents geographic location). Given local interactions, the 2D Ising model exhibits a universal behavior of spatial pattern formation [52]. Indeed, this is consistent with the observed geographic segregation associated with political polarization [33]. (As these dynamics are consistent with social influence [34] and homophily [35], we expect the order parameter $y(r_i)$ to couple to these forces.) One such model that exhibits this universal behavior can be constructed by coarse-graining the local electoral outcomes $y(r_i)$ into a statistical field $\bar{y}(\vec{r})$ with an effective local hamiltonian (normalized by temperature) that allows for geographic heterogeneity:
\[
\int d^2\vec{r}[t(\vec{r})\dot{y}(\vec{r})^2 + u(\vec{r})\dot{y}(\vec{r})^4 + K(\vec{r})(\nabla^2 y(\vec{r}))^2 - h(\vec{r})\dot{y}(\vec{r})]
\]
(12)

where \(u, K : \mathbb{R}^2 \rightarrow (0, \infty)\) and \(t, h : \mathbb{R}^2 \rightarrow \mathbb{R}\).

We can also consider a model in which the political parties are explicit. Generically, the memberships of the two parties can be expected to differ in opinion due to the forces described in the previous paragraph. A model that captures this breaking of symmetry between the two parties can be described by the following temperature-normalized hamiltonian for \(p : \mathbb{R} \rightarrow [0, 1]\):

\[
- \max_{y_1, y_2 \in \mathbb{R}} \int_{-\infty}^{\infty} \beta f(x)[p(x)u_x(y_1) + (1 - p(x))u_x(y_2)]dx
\]
(13)

where \(p(x)\) denotes the probability that a citizen with opinion \(x\) belongs to or leans towards one of the two parties (with \(1 - p(x)\) being the probability that they align with the other party), where \(\beta\) is a positive constant, and where the rest of the notation follows that of eq. (7). As long as at least one of these parties chooses its nominee so as to maximize the probability of winning the general election (e.g. if primary voters vote based on this consideration), then no changes need be made to the analysis in eqs. (7) to (11). But, if \(\text{both}\) parties choose their nominees primarily based upon the opinions of their members rather than upon electability, then differences in the parties’ members will generally cause the election to be unstable, regardless of general election voting behavior. For example, if each of the subpopulations in eq. (10) approximately corresponds to a political party, then nominees based on the median or mean opinions of the party members will be located at \(\pm \Delta\), thus rendering the election unstable for all \(\Delta > 0\) if the two parties have approximately equal support. Negative representation must then also be present; one such example would be if some opinions of the left-leaning party were to shift further to the left, resulting in a more left-leaning nominee that is less appealing to general election voters, increasing the likelihood that the right-leaning party’s nominee wins. Given this inherently polarizing effect of the two-party system, electoral reforms such as instant-runoff voting or approval voting that allow for third-party candidates to run without playing spoiler may reduce this instability.

Empirically, the opinions of the United States population, and in particular the opinions of those most likely to vote, have been polarizing over the past few decades \([37]\), while voter turnout has remained approximately constant \([38]\). Thus, we might expect that over time, election outcomes have undergone a phase transition from a stable to unstable regime, and indeed this appears to be the case (fig. 4). (While elections cannot be expected to follow the precise assumptions behind eq. (11), they nonetheless may fall into the same \textit{universality class} (section S1H), including in the case where the space of opinions is multidimensional (section S2A).) Changes to the electoral process, such as reforms in the early 1970s as to how presidential party nominees are chosen, together with increasing polarization, may have driven this phase transition into instability. That the policies of legislators in politically homogeneous districts are more strongly correlated with the preferences of the district’s median voter than the policies of legislators in heterogeneous districts \([39]\) lends further empirical support.

In addition to whatever other problems may arise from instability, unstable elections also necessarily contain negatively represented opinions, a result that, due to the generality of the assumptions used to prove it, is expected to hold for any real-world election, regardless of the structure of the electoral process, the number of candidates, or other details. Therefore, the impact of electoral reforms on this instability should be considered. For instance, when voter turnout is low, political polarization can fundamentally shift electoral dynamics, causing large swings from election to election and leaving many negatively represented, regardless of the outcome. These results suggest that policies that increase voter turnout will not only result in more voices being heard but will also stabilize elections and reduce negative representation.

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ever, it is often useful to choose

\[ f \]

number of citizens with opinions in the interval \([a, b]\) is no greater than 1 for all \(a, b\) (see fig. S1 for an example). For large enough \(N\), the error of up to 1 opinion will generally not be significant. Alternatively, a natural interpretation of a smooth \(f(x)\) is that the opinions are themselves probabilistic (for an example of explicitly probabilistic opinions, see section S2 B). Whether or not \(f(x)\) is chosen to be smooth does not matter for the results of the text, although for the results that rely on the assumption that the number of citizens is large, the mathematics are simpler if \(f(x)\) is assumed to be a function rather than a distribution. For instance, the expression for representation in the case of median voting involves evaluating \(f\) at its median, an operation which is not well-defined if \(f\) is a sum of Dirac delta functions.

In the limit of a large population (\(N \gg 1\)), the change \(\delta f\) in the opinion distribution arising from an individual opinion will be small compared to the opinion distribution as a whole, and so we expand \(\delta y\) to first order in \(\delta f\):

\[
\delta y = \int_{-\infty}^{\infty} \delta f(z) \frac{\delta y}{\delta f(z)} dz
\]

(14)

Note that eq. (14) does not apply to cases in which \(y[f]\) is not differentiable, e.g. when the election is unstable and small changes in the opinion distribution can have an outsized impact; thus we do not use results derived from these equations when analyzing instability.

We now derive eq. (4). Note that when an individual opinion changes from \(x\) to \(x' = x + c\), the opinion distribution changes by

\[
\delta f(z) = \delta(z - x - c) - \delta(z - x)
\]

(15)

where \(\delta(z)\) is the Dirac delta function. Representation (eq. (2)) is then obtained in terms of functional derivatives of the election by substituting eq. (15) into eq. (14):

\[
r_c(f, x) = \frac{1}{c} \frac{\delta y}{\delta f(x + c)} - \frac{\delta y}{\delta f(x)}
\]

(16)

which, combined with eq. (3), yields eq. (4) (reproduced below).

\[
r(f, x) = \frac{d}{dz} \frac{\delta y}{\delta f(x)}
\]

(17)

By the fundamental theorem of calculus, we see from eq. (16) that \(r_c(f, x)\) is the average of \(r(f, x)\) over \([x, x + c]\).

We now prove eq. (5) \((\int_{-\infty}^{\infty} r(f, x) dx = 1)\), which holds when the election (\(y[f]\)) is differentiable. For small \(\epsilon, f(z + \epsilon) = f(z) + \epsilon \frac{d}{dx} f(z) + O(\epsilon^2)\), and thus, using eq. (14), that approximates \(\sum_{i=1}^{N} \delta(x - x_i)\), in the sense that the difference between \(\int_{a}^{b} f(x) dx\) and the number of citizens with opinions in the interval \([a, b]\) is no greater than 1 for all \(a, b\) (see fig. S1 for an example). For large enough \(N\), the error of up to 1 opinion will generally not be significant. Alternatively, a natural interpretation of a smooth \(f(x)\) is that the opinions are themselves probabilistic (for an example of explicitly probabilistic opinions, see section S2 B). Whether or not \(f(x)\) is chosen to be smooth does not matter for the results of the text, although for the results that rely on the assumption that the number of citizens is large, the mathematics are simpler if \(f(x)\) is assumed to be a function rather than a distribution. For instance, the expression for representation in the case of median voting involves evaluating \(f\) at its median, an operation which is not well-defined if \(f\) is a sum of Dirac delta functions.

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\]

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\[
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\]

(17)

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We now prove eq. (5) \((\int_{-\infty}^{\infty} r(f, x) dx = 1)\), which holds when the election (\(y[f]\)) is differentiable. For small \(\epsilon, f(z + \epsilon) = f(z) + \epsilon \frac{d}{dx} f(z) + O(\epsilon^2)\), and thus, using eq. (14),
Since $f$ has compact support, which follows from $f$ being an approximation of the opinions of a finite number of citizens, integrating by parts yields

$$y[f(z + \epsilon)] = y[f(z)] - \epsilon \int_{-\infty}^{\infty} f(x)r(f, x)dx + O(\epsilon^2)$$  \hspace{1cm} (19)$$

which can be combined with eq. (11) in the limit $\epsilon \to 0$ to yield eq. (5). This proof assumes that $f$ is differentiable for illustrative purposes, but the result will more generally hold if $f$ and $r$ are treated as distributions (generalized functions).

### C. Nash equilibria of the electoral game

Consider a two-candidate election with endogenous candidacy: candidate positions (or, equivalently, candidates) are chosen in order to maximize the probability of victory. In this framework, the winner of the election will have adopted an unbeatable position $y^*$, provided such a position exists (i.e. a candidate with position $y^*$ will have at least a 50% chance of winning against a candidate with any other position). Formally, $(y^*, y^*)$ is a Nash equilibrium, since no candidate can improve her chances by changing her position. Because the voting game is symmetric, if $(y_1, y_2)$ is a Nash equilibrium, then so are $(y_1, y_1)$ and $(y_2, y_2)$; see, for instance, section 2.3 of [41]. Thus, if there is a unique Nash equilibrium, it must be of the form $(y^*, y^*)$.

### D. The utility difference model

Building upon the two-candidate election framework above (section S1 C), we describe the assumptions behind the utility difference model given by eq. (7) and give examples of its applications to median voting, mean voting, and an election difference model given by eq. (7). To do so, we must assume there is a single possible election outcome $y^*$ (see section S1 C). Then, from eq. (7),

$$y^* = \arg\max_{\tilde{y}} \int_{-\infty}^{\infty} u_x(\tilde{y})f(x)dx$$  \hspace{1cm} (22)$$

which implies

$$0 = \int_{-\infty}^{\infty} u_x'(y^*)f(x)dx$$  \hspace{1cm} (23)$$

Note that eq. (22) satisfies $y[f(x)] = y[\lambda f(x)]$ for any positive constant $\lambda$ (scale invariance), and let $\tilde{f}(x) = f(x)/N$ where $N$ is the size of the electorate, so that $\int_{-\infty}^{\infty} f(x)dx = 1$. Considering the change that arises from the addition of a single individual with opinion $x_0$ to the population, we define $\delta y^*$ by

$$y^* + \delta y^* = y[f(x) + \delta(x - x_0)] = y[\tilde{f}(x) + \epsilon \delta(x - x_0)]$$  \hspace{1cm} (24)$$

where $\epsilon = 1/N$. Thus, substituting $y^* + \delta y^*$ for $y^*$ and $\tilde{f}(x) + \epsilon \delta(x - x_0)$ for $f(x)$ into eq. (23),

$$0 = \int_{-\infty}^{\infty} u_x'(y^* + \delta y^*)[\tilde{f}(x) + \epsilon \delta(x - x_0)]dx$$  \hspace{1cm} (25)$$
Because $\int_{-\infty}^{\infty} u_x(x)f(x)dx$ is differentiable in $f$ and has a single maximum in $y$, expanding eq. (25) to lowest order in $\epsilon$ yields

$$\delta y^* = \epsilon \left[ -u'_{x_0}(y^*) \int_{-\infty}^{\infty} u_x'(y^*) f(x)dx \right] + O(\epsilon^2)$$  \hspace{1cm} (26)

Noting that the denominator is independent of $x_0$, of order 1 (i.e. independent of $N$), and negative (otherwise, $y^*$ would be a minimum rather than a maximum),

$$\frac{\delta y}{\delta f(x_0)} = \lim_{\epsilon \to 0} \frac{\delta y^*}{\epsilon} \propto u_x'(y^*)$$  \hspace{1cm} (27)

So, using eq. (4),

$$r(f, x) = \frac{d}{dx} \frac{\delta y}{\delta f(x)} = \frac{1}{N} \frac{d}{dx} \frac{\delta y}{\delta f(x)} \propto \frac{1}{N} \frac{d}{dx} u_x'(y^*)$$  \hspace{1cm} (28)

If $u_x(y) = u(y - x)$, as it must be for some function $u$ if the election is translationally invariant as in eq. (1), then

$$r(f, x) \propto -\frac{1}{N} u''(y^* - x)$$  \hspace{1cm} (29)

Eq. (29) provides a direct link between citizen preferences and the representation of opinions. (If needed, the constant of proportionality can be determined through eq. (5).) Consider the examples for $u$ given in section S1D. For $u(y - x) = -a^2(y - x)^2$ and $u(y - x) = -a(y - x)^2$, we can quickly derive the representation of opinions under mean voting ($r(f, x) = \frac{1}{N}$) and median voting ($r(f, x) = \frac{\delta y}{\delta f(x)} m$ where $m$ is the median of $f$), respectively. For $u(y - x) = -\sqrt{a^2(y - x)^2 + b^2}$, which yields an outcome between that of median and mean voting.

$$r(f, x) \propto (1 + \frac{a^2}{b^2}(x - y[f]))^{3/2}$$  \hspace{1cm} (30)

resulting in the representation of opinions being concentrated around the election outcome, but not infinitely concentrated as it is for median voting. For $u$ that are not concave, there will exist some $x$ such that $u''(x - y) > 0$, and representation will be negative for those opinions (see eq. (29)).

**F. Instability in the utility difference model**

Here, we explore the conditions under which instability can arise in the model described by eq. (7), and we elaborate on the concrete model of instability with outcomes given by eq. (11). For the model given by eq. (7), $r(f, x) = \frac{d}{dx} \frac{\delta y}{\delta f(x)}$ is shown to be well-defined as long as $y[f]$ is single-valued, i.e. eq. (7) has a single maximum (section S1E). Thus, instability can occur only when there are multiple maxima. (For a single maximum $y^*$ with $\int_{-\infty}^{\infty} u''(y - x)f(x)dx = 0$, the functional derivative of $y^*$ is not defined, but, as can be shown in a higher order analysis, there is no instability. In particular, for $\delta y^*$ defined by eq. (24), we can derive

$$\frac{1}{6} (\delta y^*)^3 = \int_{-\infty}^{\infty} f(x) u''(y^*) dx + O(\epsilon^3/3)$$  \hspace{1cm} (26)

in place of eq. (26), which yields $\delta y^* \propto 1/3 u_{x_0}(y^*)^{1/3} + O(\epsilon^2/3)$. Note that

$$r(f, x_0) = \lim_{\epsilon \to 0} \frac{\delta y}{\delta f(x_0)}$$

is well-defined for any given $\epsilon = 1/N$—although it is not given by eq. (4) since $\frac{\delta y}{\delta f(x)}$ does not exist—thus, there is no instability.)

For an opinion distribution in which eq. (7) has two maxima $y^*_1$ and $y^*_2$, if each position is taken by one candidate, then each candidate has a 50% chance of winning the election. However, an arbitrarily small change in the opinion distribution can favor one outcome over the other, giving either $y^*_1$ or $y^*_2$ a chance of winning that is arbitrarily close to 100% in the large-population limit. (For a finite population, the outcome of the election in this model is not deterministic, and the probability of a given candidate winning is continuous with respect to the opinion distribution. This discontinuity in $y[f]$ arising from multiple Nash equilibria in the large-population limit is analogous to a first-order phase transition.)

The existence of multiple maxima in $y$ implies that $\int_{-\infty}^{\infty} u(x - y)f(x)dx$ is not concave (ignoring the degenerate case in which $\int_{-\infty}^{\infty} u(x - y)f(x)dx$ is constant over some interval). Thus, instability can arise only in the case of non-concave $u$, which is precisely the same condition under which negative representation occurs. That instability can arise only in the presence of negative representation should not surprise us, since it was proven under more general conditions in the main text. For this class of models, we also find that negative representation implies that there exist distributions of opinions for which instability arises, i.e. if $u$ is not concave, then there exists an $f(x)$ such that eq. (7) has multiple maxima. To see why this is true, consider an opinion distribution $f$ such that $f(x) = f(-x)$. Then, if there is a single maximum of eq. (7), it must lie at $y^* = 0$. In order for $y^* = 0$ to be a maximum, we must have $\int_{-\infty}^{\infty} f(x) u''(x)dx = 2 \int_{0}^{\infty} f(x) u''(x)dx \leq 0$ (where the equality follows from the symmetry of $u$ and $f$). But, assuming $u$ is twice continuously differentiable and not concave, there exists an $f$ such that $f(x) = f(-x)$ and $\int_{0}^{\infty} f(x) u''(x)dx > 0$. For such an $f$, $y^* = 0$ is not a maximum, thus contradicting our assumption that there was a unique maximum.

To provide an example of how, for non-concave $u$, the election is unstable for certain opinion distributions, we consider the $u$ used for the example of negative representation: $u(y - x) = \exp\left[\frac{-(y-x)^2}{2\sigma^2}\right]$ for some positive constant $a$ (eq. (8)). We take the distribution of electorate opinions to be a sum of two (potentially unequally weighted) normal distributions of equal variance:

$$f(x) = \sum_{\alpha=1,2} \frac{u_{\alpha} e^{-\frac{(x-\mu_{\alpha})^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$  \hspace{1cm} (31)
Then, from eq. (7), the outcome of the election is then given by
\[
\arg\max_y \int_{-\infty}^{\infty} f(x)e^{-\frac{(y-x)^2}{2\sigma^2}} dx = \arg\max_y \sum_{\alpha=1,2} w_{\alpha} e^{-\frac{(y-\mu_{\alpha})^2}{2\sigma^2}}
\]
Without loss of generality, we can assume that \( \mu_2 = -\mu_1 \equiv \Delta \geq 0 \). Defining the normalized election outcome \( \hat{y} \equiv y/\Delta \), we solve eq. (32) to get the following condition:
\[
\hat{y} = \tanh(J\hat{y} + h) \tag{33}
\]
where \( J = \Delta^2/(\alpha^2 + \sigma^2) \) and \( h = \frac{1}{2} \ln \frac{w_2}{w_1} \). For \( w_1 = w_2 \) (i.e. for equally sized subpopulations), \( h = 0 \), and the election is stable with \( \hat{y} = 0 \) for \( J \leq 1 \) and unstable for \( J > 1 \). In the unstable regime, there are two possible outcomes described by \( \pm |\hat{y}| \), and an arbitrarily small change in \( h \) can cause \( y \) to swing between its positive and negative value. In this regime, the majority of one of the subpopulations will be negatively represented: for \( y^* < 0 \), over half of the subpopulation centered at \( \Delta \) will have opinions \( x \) with \( x - y^* > \Delta - y^* > a \) (\( \Delta > a \) in the unstable regime), and, from eq. (9), representation is negative for these opinions. Likewise, for \( y^* > 0 \) in the unstable regime, over half of the subpopulation centered at \( -\Delta \) will be negatively represented.

### G. Connection with the mean-field Ising model

We map the voting model that gives rise to eq. (33) (eq. (11) in the main text) onto a mean-field Ising model (14). To begin constructing this map, note that the left-hand side of eq. (32) gives the limiting value of \( y \) as \( N \to \infty \) when \( y \) is drawn from a probability distribution corresponding to the partition function
\[
Z = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} e^{-\frac{(y-x)^2}{2\sigma^2}} f(x) dx \right]^N dy \tag{34}
\]
Because \( y \) is a gaussian random variable, we can explicitly integrate over \( y \), which yields, up to a multiplicative constant,
\[
Z = \int_{-\infty}^{\infty} e^{-\frac{1}{N} \sum_i (y_i-x_i)^2} \Pi_{i=1}^N (f(x_i) dx_i) = \int_{-\infty}^{\infty} e^{-\frac{1}{N} \sum_i (y_i-x_i)^2} \Pi_{i=1}^N (f(x_i) dx_i) \tag{35}
\]
This equation describes \( N \) interacting probabilistic “spins,” with each spin weighted by the opinion distribution \( f(x) \), with an energy penalty proportional to its mean-square distance from all of the other spins. For \( f(x) \) given by eq. (33) (eq. (10) in the main text), the behavior is exactly that of a mean-field Ising model (with an external magnetic field for \( w_1 \neq w_2 \)); in general, for bimodal symmetric \( f(x) \) we expect a phase transition in which the system will spontaneously break the symmetry between the peaks as the peaks move farther apart. In the stable/disordered phase, both of the peaks of \( f(x) \) are sampled by the “spins;” in the unstable/ordered phase, however, only one of the two peaks is sampled, and therefore the other peak is not represented. Despite the fact that each individual votes independently from everyone else, citizens are coupled through their collectively choosing a candidate, which is reflected by the effective interactions between the “spins.” In the limit of weak interactions, i.e. \( a \to \infty \), we recover mean voting, since in this limit, \( u(y-x) = \exp \left[ -\frac{(y-x)^2}{2\sigma^2} \right] \approx 1 - \frac{1}{2\sigma^2} (y-x)^2 \). (Quadratic utility functions yield mean voting—see section [13].) Thus, mean voting is a way of “independently” aggregating opinions.

For a general \( u(y-x_i) = \exp[-V(y-x_i)] \) (we can always write \( u \) in this form with \( V(x) \geq c \) for some \( c \in \mathbb{R} \)), we note that eq. (7) yields an election outcome equivalent to the limit of \( y \) as \( N \to \infty \) with \( y \) drawn from
\[
Z = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} e^{-V(y-x)} f(x) dx \right]^N dy = \int_{-\infty}^{\infty} e^{-\sum_i V(y-x_i)} \Pi_{i=1}^N (f(x_i) dx_i) \tag{36}
\]
For quadratic \( V \), we saw above that we could exactly integrate over the election outcome \( y \) to yield pairwise quadratic interactions between the \( x_i \) variables, but for general \( V \), such an integration will yield many interaction terms of higher than quadratic order between these \( x_i \). Although such integration cannot be carried out precisely, we expect this interacting system to undergo a phase transition for bimodal \( f(x) \) if the expansion of \( V \) produces sufficiently strong interactions. Thus, the system’s behavior should be similar to the exactly solvable case in which \( V \) is quadratic.

### H. Empirical data

To determine if the stability of U.S. presidential elections has changed over time, we used data from Jordan et al. (40) on the polarization in the party platforms. Jordan et al. used a combination of machine learning and human judgment to de-
termine which of the frequently used words in the party platforms were polarizing and then determined the number of polarizing words (classified by political issue dimensions such as economic, foreign, etc.) in the Republican and Democratic platforms from 1944 to 2012. From this data, we calculated the total number of polarizing words as a percentage of all words in the platforms. We chose the percentage of polarizing words in the party platforms as a measure of political polarization over other measures of ideology—such as NOMINATE scores [42], which measure ideological purity based on agreement with other politicians—because we wanted an external, content-based measure of divergence in opinion rather than a measure of ideology that depends only on the positions that politicians take relative to one another. To construct our voting model (fig. 3), we plotted by year the fraction of polarizing words in the Democratic platforms and the negative of the fraction of polarizing words in the Republican platforms. To correct for any time-independent bias affecting the number of polarizing words in the party platforms—for instance, which words Jordan et al. designated as polarizing—we subtracted from the data for each party separately the fraction of polarizing words from the year of least polarization for that party (which was 1948 for both parties). Thus, the data shown are the changes in the fraction of polarizing words relative to their baseline value (0.0258 for Democratic platforms and 0.0693 for Republican platforms).

As was noted in the main text and explained in section S1C, our voting model (fig. 3) is equivalent to a mean-field Ising model. Real-world elections are unlikely to follow this model exactly, and even if they did, there is no reason to believe that there would be a simple relationship between polarization (for which \( J \) is a dimensionless measure) and time. Nonetheless, if U.S. presidential elections underwent a phase transition in the same universality class as the mean-field Ising model, then in the vicinity of the phase transition, polarization would increase in proportion to the square root of the time from the transition, regardless of the precise relationship between time and polarization. In much the same way, magnetization increases near a ferromagnetic phase transition in proportion to \((T - T_c)^\beta\), where \( T \) is temperature, \( T_c \) is the temperature at which the phase transition occurs, and \( \beta \) is known as a critical exponent, which depends only on the universality class to which the phase transition belongs [43]. Inspired by this universality, we fit the polarization of both parties to the piecewise function

\[
f(x) = \begin{cases} 
0 & x \leq x_0 \\
A\sqrt{x - x_0} & x > x_0 
\end{cases}
\]  

(37)

where \( x \) is the year and \( f(x) \) is the fraction of polarizing words in that year’s platform relative to the baseline value (see above), and \( A \) and \( x_0 \) are free parameters, corresponding to the amplitude of the polarization and the year that it begins, respectively. We found that \( x_0 = 1970.54 \) and \( A = 0.0079196 \) minimize the total sum of square errors, yielding \( R^2 \) values of 0.86 for the Democratic party and 0.89 for the Republican party. If the two parties are considered together, \( R^2 = 0.87 \).

### S2. Supplementary Text

#### A. Multidimensional opinion space

For the sake of simplicity, this paper focuses on systems with a one-dimensional opinion space, but the concepts developed in this paper can naturally be extended to a multidimensional opinion space, where the opinions of the electorate and candidates lie in \( \mathbb{R}^n \), as in [16, 41, 44, 45]. This extension will be briefly outlined here. The definition of representation is generalized by replacing eq. (3) with

\[
r_{c,\mu}(f, x) = \frac{\delta y_\mu c_\mu}{|c|^2} 
\]  

(38)

For a scalar measure, we use

\[
\text{tr}[r] = \bar{c} \cdot \delta \bar{y} 
\]  

(39)

When there exists an \( r_{\mu\nu}(f, x) \) such that

\[
\delta y_\mu = \sum_{\nu} r_{\mu\nu}(f, x) c_\nu + O(c^2) 
\]  

(40)

for all \( \bar{c} \), this \( r_{\mu\nu}(f, x) \) can be used in place of eq. (3) as a representation independent of \( \bar{c} \). In the large-population limit, eq. (16) is then replaced (using Einstein-summation notation) by the path-independent integral

\[
\text{tr}[r(x, x')] = \frac{c_\mu}{|c|^2} \left( \frac{\delta y_\mu}{\delta f(x + \bar{c})} - \frac{\delta y_\mu}{\delta f(x)} \right) = \frac{c_\mu}{|c|^2} \int_{x}^{x+\bar{c}} r_{\mu\nu}(f, x') dx' 
\]  

(41)

voters. The trace \( \text{tr}[r] \) gives a rotationally invariant scalar measure.

The representation normalization condition corresponding to eq. (5) is \( \int f(x) r_{\mu\nu}(f, x) dx = \delta_{\mu\nu} \) where the integral is taken over \( \mathbb{R}^n \).

In the multidimensional case, instability also implies a failure in representation. In a manner analogous to eq. (6), insta-
bility is characterized by
\[ \lim_{\varepsilon \to 0} \epsilon \partial_{\varepsilon} \varphi_{\nu} \neq 0 \] (43)

Generally, instability implies either that \( \lim_{\varepsilon \to 0} |c| \text{tr}[r_{ij}] \neq 0 \), in which case negative representation (defined by \( \text{tr}[r_{ij}] < 0 \)) follows in the same way as the one-dimensional case, or that an infinitesimal change in opinion causes a finite orthogonal change in the outcome of the election. In this case, by considering further infinitesimal changes in opinion parallel to the first change in election outcome, and assuming that the magnitude of the change in the outcome of the election cannot grow without bound, one either gets negative representation directly (\( \text{tr}[r_{ij}] < 0 \) for some \( c \))—or \( \text{tr}[r_{ij}] > 1 \), from which negative representation follows as it does in the one-dimensional case.

Just as in the one-dimensional case, increasing polarization can drive the election through a phase transition from a stable to unstable regime, as eqs. (35) and (36) apply equally well if the opinions \( x_i \) are vector quantities. The nature of the unstable regime will depend on the symmetries present in the problem. For example, consider a distribution of opinions given by
\[ f(\bar{x}) = \sum_{\alpha=1}^{N_\alpha} e^{-\frac{(\bar{x}-\bar{\mu}_\alpha)^2}{2\sigma^2}} \] (44)
where \( N_\alpha \) is the number of subpopulations. For \( N_\alpha = 2 \), the instability that occurs for sufficiently large \( |\bar{\mu}_1 - \bar{\mu}_2| \) will be very similar to what was described in the main text for the one-dimensional case. For \( N_\alpha > 2 \), if the set of points \( \{\bar{\mu}_\alpha\} \) possess permutation symmetry, the system will resemble that of a mean-field Potts model (assuming the \( V \) in eq. (36) is isotropic). However, unlike the case for \( N_\alpha = 2 \)—where regardless of the values of the \( \bar{\mu}_\alpha \), the system possesses the appropriate symmetry to belong to the Ising universality class (due to the overall translational invariance of the problem and thus the ability to redefine the origin)—for \( N_\alpha > 2 \), we should not expect the system to generally possess the permutation symmetry necessary for Potts universality. Thus, we expect the real-world phase transitions that we observe to be of the \( N_\alpha = 2 \) type, i.e. of the same type of instability observed for a one-dimensional opinion space. For \( N_\alpha > 2 \), the situation may be able to be approximated by \( N_\alpha = 2 \), especially if there are dynamical social or political forces that pull the distribution of opinions towards a situation in which competition is roughly balanced between two opposing (potentially multi-party) coalitions at any given point in time.

B. The Owen-Shapley index as a special case

In this section we show that there exist functionals \( y[f(x)] \) for which our representation measure (eq. (4)) reproduces the values of both the deterministic and probabilistic Owen-Shapley voting power indices. Thus, these voting power indices can be thought of as special cases of our measure. We give a brief background on the voting power literature and then consider the case of a one-dimensional opinion space, followed by a generalization to the case of a multidimensional opinion space for which the Owen-Shapley index was primarily designed.

When nothing is known about the preferences of voters, their political power has traditionally been measured by a priori voting power \([46]\), which reflects the probability that a given individual or entity will cast the deciding vote and is usually measured by either the Penrose index \([47]\) or the Shapley-Shubik index \([48]\). More precisely, to calculate a voter’s a priori voting power, consider a random division of the rest of the voters into two camps. Then the probability that the excluded voter will get his way regardless of which camp he joins is his voting power; the Penrose index and the Shapley-Shubik index differ only in the way in which they randomly choose a division. While the Penrose index assumes that each voter randomly chooses one side or the other, the Shapley-Shubik index re-weights the probabilities so that each ordering of voters is equally likely. These indices provide useful and counterintuitive results when the voters possess differing numbers of votes, as in the European Union. For instance, under the 1958 voting rules for the European Economic Community, Luxembourg, despite having had one vote, had no voting power, since there were no possible divisions of the other five countries such that Luxembourg’s vote would be decisive \([49]\). But in elections in which each voter has one vote, all measures of a priori voting power result in each voter having an equal amount of power. A measure of voting power that takes voter preferences into account is needed to determine how various opinions are differently represented. Many preference-based measures have been proposed—for instance, the spatial Shapley-Shubik index, also known as the Owen-Shapley index \([50]\)—but, as we will see, these measures implicitly assume that people vote in a particular way.

In one dimension, the deterministic Owen-Shapley index \([50]\) allows for only two possible orderings of the voters (left to right or right to left), for which the median voter is pivotal in both, thus yielding the same concentration of power that our representation measure (eq. (3)) yields in the case of median voting (\( y[f] = \text{median}[f] \equiv m \) yields \( r(f, x) = \frac{d}{dx} \frac{dy}{dx} = \delta(x-m) \)). Benati and Marzetti \([51]\) note that this extreme concentration of power is due to the deterministic nature of the Owen-Shapley model, which assigns zero probability to almost all orderings. They propose a generalized election model in which voters’ opinions have both a deterministic and a random component. In the one-dimensional case, their treatment is equivalent to denoting the probabilistic opinion \( X_i \) of voter \( i \) by
\[ X_i = x_i + \epsilon_i \] (45)

There are some fundamental differences in the motivation behind the indices \([50]\), but mathematically, they are rather similar, though neither is without drawbacks: while the assumptions behind the Penrose index are simpler, in general the sum of all voters’ Penrose indices will not equal 1, while the sum of all voters’ Shapley-Shubik indices will.
where the \( x_i \) are deterministic and the \( \epsilon_i \) are independent random variables with a continuous probability density function \( f_\epsilon(\epsilon_i) \). Denoting the distribution of the \( x_i \) over the population by \( f(x_i) \) (note that \( f \) will be a sum of delta functions for a finite population) and choosing an election in which people vote for the candidate closest to their probabilistic opinions \( X_i \), the Nash equilibrium for the two candidates’ opinions is the median of the distribution \( f_X(X) = (f \ast f_\epsilon)(x) = \int_{-\infty}^{\infty} f(x)f_\epsilon(x-x)dx \), i.e.,

\[
y[f] = \text{median}[f_X] = \text{median}[f \ast f_\epsilon] = m \quad (46)
\]

The continuity of \( f_X \) follows from that of \( f_\epsilon \), and we have made the additional assumption that \( f_X(m) \neq 0 \); otherwise, there is no unique Nash equilibrium.

We then calculate

\[
\frac{\partial y[f]}{\partial f(x)} = \int_{-\infty}^{\infty} \frac{\delta \text{median}[f_X]}{\delta f_X(x)} \frac{\delta f_X(x)}{\delta f(x)} dz = \int_{-\infty}^{\infty} \frac{\text{sign}(z - m)}{2f_X(m)} f_\epsilon(z) dz = \int_{-\infty}^{\infty} \frac{\text{sign}(z + x - m)}{2f_X(m)} f_\epsilon(z) dz
\]

which yields the representation measure (eq. (4)) for opinion \( i \):

\[
r(f, x_i) = \frac{d}{dx_i} \frac{\delta y[f]}{\delta f(x_i)} = \int_{-\infty}^{\infty} \frac{2\delta(z + x_i - m)}{2f_X(m)} f_\epsilon(z) dz = \frac{1}{f_X(m)} f_\epsilon(m - x_i)
\]

The rightmost side of eq. (48) is the probability that voter \( i \) is the median—i.e. pivotal—voter; thus, \( r(f, x_i) \) is equal to the generalized Owen-Shapley index for voter \( i \).

The Owen-Shapley index was developed primarily for multidimensional opinion spaces in \( \mathbb{R}^n \). Owen and Shapley [50] consider a randomly drawn unit vector \( \hat{\mathbf{v}} \in \mathbb{R}^n \) and then order individuals by defining \( i < j \) if \( \hat{\mathbf{X}}_i \cdot \hat{\mathbf{v}} < \hat{\mathbf{X}}_j \cdot \hat{\mathbf{v}} \). (The \( \hat{\mathbf{X}}_i \) are deterministic but can easily be modified to be partially probabilistic as in [51].) The Owen-Shapley index of \( i \) is then the probability that \( i \) is the median of the resulting ordering. To see how this power index is a special case of our multidimensional representation measure (eq. (42)), consider the following method of choosing a candidate: given a set of voters with (potentially probabilistic) opinions \( \hat{\mathbf{X}}_i \in \mathbb{R}^n \):

1) Randomly choose an orthonormal basis \( V = \{\hat{\mathbf{v}}_1, ..., \hat{\mathbf{v}}_n\} \) for \( \mathbb{R}^n \).

2) Conditioning on the orthonormal basis \( V \), let \( m_{\alpha}(V) \) be the median of the probability distribution function for \( \hat{\mathbf{v}}_\alpha \cdot \hat{\mathbf{X}} \), where \( \hat{\mathbf{X}} \) is randomly drawn from the voter opinions \( \hat{\mathbf{X}}_i \).

3) The election outcome is then given by \( y[f] = \sum_{\alpha = 1}^{n} m_{\alpha}(V)\hat{\mathbf{v}}_\alpha \).

Note that \( y \) is now a random variable (since it depends on the orthonormal basis \( V \)), and so the right-hand side of eq. (42) must be replaced by its expectation value, i.e. \( \tau_{\mu \nu}(f, \hat{\mathbf{x}}) = \mathbb{E}[\frac{\partial y[f]}{\partial f_{\mu}(\hat{\mathbf{x}})}]\). From eq. (48) (with \( \hat{\mathbf{v}} \cdot \hat{\mathbf{y}} \cdot \hat{\mathbf{x}} \), and \( \hat{\mathbf{v}} \cdot \hat{\mathbf{e}}_{\nu} \) substituting for \( y, x_i \), and \( \epsilon_i \)), \( \tau_{\mu \nu}(f, \hat{\mathbf{x}})\hat{\mathbf{v}}_\nu \) is equal to the probability that \( i \) will be the median voter along \( \hat{\mathbf{v}} \). Therefore, \( \tau r[\tau r(f, \hat{\mathbf{x}})] \) is equal to the expected number of basis vectors along which \( i \) will be the median, and so \( \tau r[\tau r(f, \hat{\mathbf{x}})] \) is equivalent to \( n \) times the Owen-Shapley index.

The agreement between \( r(f, \hat{\mathbf{x}}) \) and the Owen-Shapley index follows from the fact that \( \frac{d}{dx_i} \frac{\text{median}[f_\epsilon(\hat{\mathbf{x}})]}{\text{median}[f_X(\hat{\mathbf{x}})]} \) measures the probability that \( i \) is the median voter along \( \hat{\mathbf{v}} \) for this class of voting models. In this sense, the Owen-Shapley index of power implicitly assumes an election in which some sort of median is chosen. This model is appropriate when voters vote deterministically (although the options presented for them to vote on may be random). But such deterministic voting assumes that voters distinguish between very small differences in policy with 100% certainty, and it also assumes that there is no chance that a voter abstains. While these assumptions may hold for assemblies of elected officials (and in particular to the EU, where these measures are most commonly applied), they tend to fail for mass elections, in which a citizen may sometimes vote for the candidate farther from her opinion and sometimes may choose not to vote at all.

C. A brief review of social choice theory

Here, we briefly review the social choice literature on elections, focusing in particular on the aspects of spatial models of elections—i.e. models that denote the political preferences of a citizen by a point in some space that can be embedded in \( \mathbb{R}^n \) for some \( n \)—that are most relevant to our work. (These points are often referred to as ideal points—rather than opinions, as in this manuscript—in order to emphasize the simplification that takes place when opinions are placed in a low-dimensional space.) The social choice literature also considers more general questions concerning the aggregation of opinions, often with counterintuitive results [52–56]; see [58] for an overview. The simplest spatial models of elections take \( n = 1 \), i.e. they assume that political preferences can be determined by a citizen’s position on a line (often interpreted as the left-right political spectrum), e.g. [17] [59] [60]. These

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3 This election takes the form of the Random Utility Model mentioned in section 2.2 of [15].

4 For example, Arrow’s Impossibility Theorem [57] states, roughly speaking, that a dictatorship is the only system of aggregating preferences that 1) prefers A to B if the citizens unanimously prefer A to B, and 2) does not allow its preference between two alternatives A and B to be affected by citizens preferences concerning a third alternative C.
models assume that there are two candidates competing for a majority of the votes and that all citizens vote and that they vote for whichever candidate is closest to them. Under these conditions, there is a pure Nash equilibrium (see Methods section ‘Nash equilibria of the electoral game’) in the candidate strategies, which is for both candidates to adopt the position of the median voter. Downs [60] recognized the limitations of these assumptions and intuited qualitatively that due to the nonvoting that can arise from alienation, “democracy does not lead to effective, stable government when the electorate is polarized.”

Subsequent work has focused on the existence of Nash equilibria in candidate strategies under various assumptions concerning voter and candidate motivations. In a multidimensional opinion space ($n > 1$), the deterministic voting that gives rise to the Median Voter Theorem for $n = 1$ does not yield a Nash equilibrium without additional restrictions on the voters’ preferences (see e.g. [61–63]). Nash equilibria can often be restored by considering spatial models in which citizens cast votes probabilistically, since probabilistic voting allows for smoother changes in voting behavior with respect to changes in the candidate positions (e.g. [41,42,64–66]; see [16] for a review). Other assumptions include, for instance, voter abstention [18,67], policy-motivated candidates (e.g. [68,69]) and game-theoretic considerations on the part of the voters (e.g. [70,71]).

In our work, we are concerned with the the functional $y[f(x)]$, which maps the distribution of citizen opinions onto the electoral outcome, rather than with the mechanism that gives rise to this map. In the latter half of the main text, we build upon the probabilistic voting models developed in the literature, relaxing the assumption of concave voter preferences so as to provide a concrete demonstration of how negative representation and electoral instability can arise, the latter of which is reflected in the existence of multiple Nash equilibria.
warning signals for critical transitions. *Nature*, 461:53–59, 2009.

[29] Thierry Mora and William Bialek. Are biological systems poised at criticality? *Journal of Statistical Physics*, 144:268–302, 2011.

[30] Jean-Philippe Bouchaud. Crises and collective socio-economic phenomena: simple models and challenges. *Journal of Statistical Physics*, 151:567–606, 2013.

[31] Dion Harmon, Marco Lagi, Marcus AM de Aguia, David DCinellato, Dan Braha, Irving R Epstein, and Yaneer Bar-Yam. Anticipating economic market crises using measures of collective panic. *PloS one*, 10:e0131871, 2015.

[32] Alan J Bray. Theory of phase-ordering kinetics. *Advances in Physics*, 51:481–587, 2002.

[33] “Electorally competitive counties have grown scarcer in recent decades,” *Pew Research Center*, Washington, D.C.; pewresearch.org/fact-tank/2016/06/30/electorally-competitive-counties-have-grown-scarcer-in-recent-decades/, 2016.

[34] Bernard R Berelson, Paul F Lazarsfeld, William N McPhee, and William N McPhee. *Voting: A study of opinion formation in a presidential campaign*. University of Chicago Press, 1954.

[35] Miller McPherson, Lynn Smith-Lovin, and James M Cook. Birds of a feather: Homophily in social networks. *Annual review of sociology*, 27:415–444, 2001.

[36] Christian Borghesi, Jean-Claude Raynal, and Jean-Philippe Bouchaud. Election turnout statistics in many countries: similarities, differences, and a diffusive field model for decision-making. *Plos One*, 7:e36289, 2012.

[37] “Political polarization in the american public,” *Pew Research Center*, Washington, D.C.; peoplepress.org/2014/06/12/political-polarization-in-the-american-public/, 2014.

[38] Michael P. McDonald. “National general election VEP turnout rates, 1789-present.” *United States Election Project*, electoproject.org/national-1789-present. Accessed 2019-7-25.

[39] Elisabeth R Gerber and Jeffrey B Lewis. Beyond the median: Voter preferences, district heterogeneity, and political representation. *Journal of Political Economy*, 112:1364–1383, 2004.

[40] Soren Jordan, Clayton M Webb, and B. Dan Wood. The president, polarization and the party platforms. *The Forum*, 12:169–189, 2014.

[41] Peter Coughlin and Shmuel Nitzan. Electoral outcomes with probabilistic voting and Nash social welfare maxima. *Journal of Public Economics*, 15:113–121, 1981.

[42] Keith T Poole and Howard Rosenthal. A spatial model for legislative roll call analysis. *American Journal of Political Science*, 29:357–384, 1985.

[43] Mehran Kardar. *Statistical Physics of Fields*. Cambridge University Press, 2007.

[44] James M Enelow and Melvin J Hinich. A general probabilistic spatial theory of elections. *Public Choice*, 61:101–133, 1989.

[45] James M Enelow and Melvin J Hinich. *The Spatial Theory of Voting: An Introduction*. Cambridge University Press, 1984.

[46] Dan S Felsenthal and Moshé Machover. A priori voting power: what is it all about? *Political Studies Review*, 2:1–23, 2004.

[47] Lionel S Penrose. The elementary statistics of majority voting. *Journal of the Royal Statistical Society*, 109:53–57, 1946.

[48] Lloyd S Shapley and Martin Shubik. A method for evaluating the distribution of power in a committee system. *The American Political Science Review*, 48:787–792, 1954.

[49] Andrew Gelman, Jonathan N Katz, and Francis Tuerlinckx. The mathematics and statistics of voting power. *Statistical Science*, 17:420–435, 2002.

[50] Guillermo Owen and Lloyd S Shapley. Optimal location of candidates in ideological space. *International Journal of Game Theory*, 18:339–356, 1989.

[51] Stefano Benati and Giuseppe V Marzetti. Probabilistic spatial power indexes. *Social Choice and Welfare*, 40:391–410, 2013.

[52] Nicolas De Condorcet et al. *Essai sur l’application de l’analyse à la probabilité des décisions rendues à la pluralité des voix*. de l’imprimerie royale, Paris, 1785.

[53] Kenneth J Arrow. *Social Choice and Individual Values*. Yale University Press, 2012.

[54] Allan Gibbard. Manipulation of voting schemes: A general result. *Econometrica: Journal of the Econometric Society*, 41:587–601, 1973.

[55] Mark Allen Satterthwaite. Strategy-proofness and arrow’s conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10:187–217, 1975.

[56] Jon Elster and Aanund Hylland. *Foundations of Social Choice Theory*. Cambridge University Press, 1989.

[57] Kenneth J Arrow. A difficulty in the concept of social welfare. *Journal of Political Economy*, 58:328–346, 1950.

[58] Christian List. Social choice theory. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, 2013.

[59] Harold Hotelling. Stability in competition. *The Economic Journal*, 39:41–57, 1929.

[60] Anthony Downs. An economic theory of political action in a democracy. *Journal of Political Economy*, 65:135–150, 1957.

[61] Charles R Plott. A notion of equilibrium and its possibility under majority rule. *The American Economic Review*, 57:787–806, 1967.

[62] Otto A Davis, Melvin J Hinich, and Peter C Ordeshook. An expository development of a mathematical model of the electoral process. *American Political Science Review*, 64:426–448, 1970.

[63] Otto A Davis, Morris H DeGroot, and Melvin J Hinich. Social preference orderings and majority rule. *Econometrica: Journal of the Econometric Society*, 40:147–157, 1972.

[64] Michael D Intriligator. A probabilistic model of social choice. *The Review of Economic Studies*, 40:553–560, 1973.

[65] Tse-Min Lin, James M Enelow, and Han Dorussen. Equilibrium in multicandidate probabilistic spatial voting. *Public Choice*, 98:59–82, 1999.

[66] Peter J Coughlin. *Probabilistic Voting Theory*. Cambridge University Press, 1992.

[67] Melvin J Hinich, John O Ledyard, and Peter C Ordeshook. A theory of electoral equilibrium: A spatial analysis based on the theory of games. *The Journal of Politics*, 35:154–193, 1973.

[68] Randall L Calvert. Robustness of the multidimensional voting model: Candidate motivations, uncertainty, and convergence. *American Journal of Political Science*, 29:89–95, 1985.

[69] Assar Lindbeck and Jörgen W Weibull. A model of political equilibrium in a representative democracy. *Journal of Public Economics*, 51:195–209, 1993.

[70] John O Ledyard. The pure theory of large two-candidate elections. *Public Choice*, 44:7–41, 1984.

[71] Richard D McKelvey and John W Patty. A theory of voting in large elections. *Games and Economic Behavior*, 57:155–180, 2006.