Theoretical Description of Resistive Behavior near a Quantum Vortex-Glass Transition

Hideharu Ishida and Ryusuke Ikeda

Department of Physics, Kyoto University, Kyoto 606-8502

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Resistive behaviors at nonzero temperatures ($T > 0$) reflecting a quantum vortex-glass (VG) transition (the so-called field-tuned superconductor-insulator transition at $T = 0$) are studied based on a quantum Ginzburg-Landau (GL) action for a s-wave pairing case containing microscopic details. The ordinary dissipative dynamics of the pair-field is assumed on the basis of a consistency between the fluctuation conductance terms excluded from GL approach and an observed negative magnetoresistance. It is shown that the VG contribution, $G_{vg}$, to 2D conductance becomes insensitive to $T$ at an apparent VG transition field $B_{vg}^*$ defined at experimentally accessible temperatures but depends on the repulsive electron-electron interaction, and that, only in the dirty limit with no electron-repulsion, it takes a universal value at low $T$. Available resistivity data near $B_{vg}^*$ are explained based on our results, and an extension of the theory to 3D case is briefly discussed.

KEYWORDS: Superconductor-Insulator Transition, Vortex-Glass Transition, Vortex States

§1. Introduction

The issue of quantum superconductor-insulator (FSI) transition induced by an applied field in disordered superconductors has become one of long-standing problems in condensed matter physics. Most of resistivity data suggestive of an FSI transition have been discussed based on a scaling hypothesis postulated by Fisher on resistive behaviors at nonzero temperatures ($T > 0$) accompanying a 2D vortex-glass (VG) transition at $T = 0$. However, no theoretical calculation justifying his scenario on resistive behavior at $T > 0$ has been reported so far. In fact, recent experimental studies have often led even to an argument against the 2D VG ordering, i.e., superconducting ordering in $B \neq 0$, although theoretically the 2D quantum VG phase will exist more likely than the 3D VG phase does at $T > 0$ because the former is formally similar to the thermal 3D glass phase due to line defects which certainly exists at $T > 0$.

Fisher’s scenario is based on the phase-only approximation of a GL action. The phase fluctuation in a phase-only model for homogeneously disordered (i.e., nongranular) superconductors has no
dissipative dynamics, and hence, a *nondissipative* dynamics was assumed there.6 Further, since such a phase-only action is essentially equivalent to the quantum boson action at low energy, possible vortex phases were identified with the corresponding ones of the boson system. However, as in the description of thermal vortex phase diagram at nonzero temperatures,2 a more microscopic GL approach in which the pair-field yields a dissipative dynamics should be able to explain physical properties systematically even at low temperatures as far as a nontrivial intermediate phase3 does not occur as a consequence of a phase-only model valid for nongranular films. For instance, the so-called bose insulating state1,3 will correspond to the quantum vortex liquid regime14 with insulating GL fluctuation conductivity.

Actually, there are several reasons why the Fisher’s argument on the resistive behavior near a FSI transition should be reconsidered theoretically. First, the argument6 on a universal conductance value at the VG transition field, \( B = B_{\text{vg}} \), and on the scaling behavior at \( T > 0 \) of resistance is based on consideration and calculations of the fluctuation conductance \( G_s(T=0, \omega \to 0) \) in the disorder-tuned case at \( B = 0 \), where \( \omega \to 0 \) means taking the dc limit. However, at a \( T = 0 \) criticality, this conductance need not be equal to the quantity \( G_s(T \to 0, \omega = 0) \). It is the latter which is measurable in real experiments, while it is the former which is expected to take a universal value as a consequence of a quantum continuous transition.15 Further, the temperature range over which the presence of a \( T \)-independent conductance is suggested is often1,3 broad so that the dissipative dynamics, neglected in ref.6, of the pair-field does not seem to be negligible, and thus, it is unclear to what extent taking the \( T \to 0 \) limit is essential. In addition, the assumption6 that the derivation of \( G_s(T=0, \omega \to 0) \) at \( B = 0 \) is applicable to the nonzero field case is not justified, because the VG critical properties of response quantities are quite different from those for the \( B = 0 \) superconducting transition. For example, the 3D VG phase due to point-like quenched disorder has no static Meissner response to any disturbance of magnetic field, and hence the VG transition is not accompanied by a divergent diamagnetic susceptibility in contrast to the case of the normal-Meissner transition. It means that it is not justified at all to apply6 the scaling argument on transport quantities in the normal-Meissner transition to the FSI transition case.

Previously, one of the present authors has pointed out14 the possibility that the intervening metallic behavior at low \( T \) may imply the vortex flow conductance taking a nearly universal value along the 2D melting line \( B_m(T) \) of the disorder-free vortex solid, if \( B_m(T) \) is insensitive to \( T \) in the *quantum* regime. Such a quantum melting line insensitive to \( T \) is satisfied in the dirty limit\( \delta \) for a s-wave pairing and neglecting a repulsive interaction between electrons. Further, the \( B_m(0) \) is implicitly assumed in this proposal\( \delta \) to lie above a \( T = 0 \) VG transition field \( B_{\text{vg}} \). As demonstrated elsewhere\( \delta \) however, \( B_{\text{vg}} \) calculated in a mean field approximation in terms of microscopic parameter in the s-wave dirty limit\( \delta \) seems to, contrary to the observed trend\( \delta \), increase with increasing disorder measured by \( (E_F\tau)^{-1} \) due primarily to the \( \tau \)-dependence of the
corresponding \( H_{c2}(0) \equiv H_{c2}^0(0) = 0.56\phi_0 T_c \tau / l^2 \propto \tau^{-1} \), where \( E_F \) is the fermi energy, \( l = v_F \tau \) the mean free path, \( T_c \) the mean field transition temperature in zero field and in clean limit, and \( \phi_0 \) the flux quantum. This is not physically surprising, because not only the thermal fluctuation but also the vortex pinning are expected to be enhanced with increasing the microscopic disorder. Hence, an inclusion of the electronic interplay between the repulsive interaction and disorder seems to be necessary in order to get both \( B_{vg} \) and \( H_{c2}(0) \) decreasing with increasing disorder or decreasing the film thickness.\(^\text{19}\) Actually, we have judged\(^\text{19}\) that a situation in which the 2D melting field \( B_m(0) \) lies above \( B_{vg} \) will seldom happen and hence that a contribution to the conductivity arising from vortex pinning effects may not be negligible in explaining consistently the flat resistance curve (the intervening metallic behavior). In ref.\(^\text{11}\), the VG fluctuation term \( G_{vg}(T = 0, \omega \rightarrow 0) \) of conductance was considered as a byproduct of transport properties near the 3D thermal glass transition due to line disorder and was argued to be a nonuniversal quantity in general. However, there, no microscopic details were taken into account, and the measurable quantity \( G_{vg}(T \rightarrow 0, \omega = 0) \) was not examined.\(^\text{11}\)

In this paper, we examine the vortex-glass contribution \( G_{vg}(\omega = 0) \) to the dc conductance for a current perpendicular to an applied field at nonzero temperatures based on the microscopic study\(^\text{19}\) of quantum GL action for the s-wave pairing case. As demonstrated in ref.\(^\text{19}\), the coefficient of dissipative term has a remarkable \( T \)-dependence rather in the vicinity of \( T = 0 \), implying that the premise in previous works\(^\text{6}\) that one may start from a bosonic model at \( T = 0 \) is not justified in the present issue. In §2, \( G_{vg}(T > 0, \omega = 0) \) in the quantum critical region around and at the critical field \( B_{vg} \) is examined on a general ground. As in ref.\(^\text{19}\), the ordinary dirty limit neglecting an interplay between an electron-electron interaction and disorder will be called merely as the dirty limit, and the dynamics of the pair-field at low frequencies is assumed according to ref.\(^\text{19}\) to remain dissipative even in \( T \rightarrow 0 \) limit. Then, we find that \( G_{vg}(B_{vg}) \) becomes a nonuniversal constant depending on a strength of the electron-repulsion but that, in the dirty limit with no electron-repulsion, it unusually becomes a universal constant independent of material parameters. In §3, it is pointed out that, in \( B > B_{vg} \), the “bosonic” contribution including \( G_{vg} \), arising from the GL action, to the conductance approaches zero in \( T \rightarrow 0 \) limit and that a negative magnetoresponse of a superconducting origin\(^\text{4}\) at lower temperatures is provided by “fermionic” fluctuation contributions, such as the Maki-Thompson fluctuation term, excluded from the GL description. Based on these results, the resistive behavior near \( B_{vg} \) is discussed in §3 and compared with existing key resistive data in thin films. In §4 we comment on extensions of the present theory to 3D systems in low \( T \) limit.

§2. 2D VG Conductance in the Quantum Critical Regime

Since an FSI transition usually occurs far from the region near the zero-field superconducting transition where the vortex-pair excitations play essential roles, a high \( B \) approximation will be
invoked in which the pair-field in any static vortex state is described in terms of the lowest Landau level (LLL) modes which do not accommodate vortex-pair excitations. If first neglecting the random potential terms leading to the vortex pinning effects, the 2D quantum GL action on LLL fluctuations $\Psi$ of the pair-field takes the form

$$S_{\text{mfp}} = \int d^2r \left[ \beta \sum \mu(0) + \gamma |\omega| \right] |\Psi_0(r)|^2 + \frac{U_4}{2} \int_0^\beta du |\Psi(r, u)|^4, \quad (2.1)$$

where $\gamma$, $U_4 > 0$, $\beta = 1/k_B T$, $\Psi(r, u) = \sum_\omega \Psi_\omega(r)e^{-i\omega u}$, $\omega$ is a Matsubara frequency for bosons, and the fact that the squared gauge-invariant gradient $Q^2 = (-i\nabla + 2\pi/\phi_0 A)^2$ is replaced by the factor $r_B^{-2} = 2\pi B/\phi_0$ after operating any LLL eigenfunction was used. The mean field $H_{c2}(T)$-line is defined as $\mu(0) = 0$. In this paper, we focus on the temperature range defined by $T < T_{\text{cr}}^{\text{mf}}$, where

$$T_{\text{cr}}^{\text{mf}} \simeq 0.15 T_{c0} B/H_{c2}^{(0)}. \quad (2.2)$$

As is explained later, this temperature scale arises from the denominator of diffusion propagators, and in $T < T_{\text{cr}}^{\text{mf}}$, $\mu(0)$ and $U_4$ become $T$-independent on cooling, while the $T$-dependence of $\gamma$ depends remarkably on the presence of an electron-electron repulsive interaction.

The random potential terms of GL action were studied in ref.19. At high $T (< T_{c0})$ and low $B (< H_{c2}(0))$, they may be represented simply in terms of a single random $T_c$ term, i.e., as a local potential form, while, in low $T$ and high $B$ case of our interest, they become spatially nonlocal reflecting the fact that the only microscopic scale measuring the spatial variations of $\Psi$ in high $B$ and in 2D is the averaged vortex spacing $r_B$. The replicated GL action within LLL arising after the random-averaging is of the form

$$S_p^n = \sum_\alpha \left[ \sum_\omega \left( \mu(0) + \gamma |\omega| \right) |\varphi_0^{(\alpha)}(p, \omega)|^2 + \frac{U_4}{4\pi r_B^2} N_v^{-1} \sum_k \rho^{(\alpha)}(k, \omega) \rho^{(\alpha)}(-k, -\omega) \right]$$

$$- \sum_{\alpha'} \frac{U_4}{4\pi r_B^2} \sum_k f_{00}(k^2) \rho^{(\alpha)}(k, 0) \rho^{(\alpha')*(-k, 0)} \right], \quad (2.3)$$

where $\alpha$ and $\alpha'$ are replica indices, $N_v$ is the number of field-induced vortices, and $\rho^{(\alpha)}(k, \omega)$ is the Fourier transform of $|\Psi^{(\alpha)}(r, \tau)|^2$ and expressed by

$$\rho^{(\alpha)}(k, \omega) = \sum_{p, \omega_1} e^{ipk_3 - k^2/4} \varphi_0^{(\alpha)*}(p - k_3/2, \omega_1) \varphi_0^{(\alpha)}(p + k_3/2, \omega_1 + \omega), \quad (2.4)$$

where $\Psi(r) = \sum_p \varphi_0(p) u_{0,p}(r)$ with LLL eigenfunction $u_{0,p}(r)$ in a Landau gauge. Further, the length scales and $\Psi$ were rescaled, respectively, in the manners, $r/r_B \rightarrow r$ and $\beta^{1/2}\Psi \rightarrow \Psi$. The function $f_{00}(k^2)$ is positive and a regular function of $k^2$ (and also of $k/|k|$ when the Fermi surface is anisotropic) and, at least in the dirty limit, independent of material parameters such as $l$. Although its detailed functional form is not known even in the dirty limit, just the property
that the wavenumber $k$ in $f_{00}$ is entirely scaled by $r_B$ becomes essential in examining the critical conductance. The familiar random $T$ model corresponds to the specific case in which the $k^2$-dependences in $f_{00}(k^2)$ are neglected. Although this nonlocality in $f_{00}$ is safely negligible at high $T$ and low $B$ where the field is measured through the ratio $\xi_0^2/r_B^2$ with GL coherence length in dirty limit $\xi_0 \approx \sqrt{r_F l/T_{cr}}$, as already mentioned, it cannot be neglected in high $B$ and low $T$. As well as $U_4$, the bare pinning strength $U_p$ can be regarded as being $T$-independent in $T < T_{cr}^m$.

Until reaching eq.(2.27), we assume in this section the GL coefficients $\gamma$, $U_4$, and $U_p$ to be $T$-independent as if the GL action is the expression at $T = 0$. First, to illustrate properties of VG fluctuation, we invoke a systematic loop (or $1/M$) expansion and focus on its lowest order ($M = \infty$) terms by, as a mathematical tool, assuming that the complex scalar pair-field $\Psi$ has $M$-flavors. Up to the lowest order in $U_p$, the random-averaged propagator $G_0$ of LLL fluctuation $\phi^{(\alpha)}_0$ in this case satisfies

$$\left(G_0(\omega)\right)^{-1} = \mu(0) + \gamma|\omega| + \frac{U_4}{4\pi r_B^2} \sum_{\omega} G_0(\omega) - \Delta_0^{(R)} G_0(\omega).$$

Here the factor $\Delta_0^{(R)}$ is a coefficient of a renormalized pinning vertex off-diagonal in the replica indices and is given by

$$\Delta_0^{(R)} = \frac{U_p}{2\pi r_B^2} N_v^{-1} \sum_{k,\omega} e^{-k^2/2 f_{00}(k^2)} \left(1 + \sigma_{vg} e^{-k^2/2}\right)^{-2},$$

where $\sigma_{vg} = (U_4/2\pi r_B^2) \sum_{\omega} G_0^2(\omega)$. The solution of eq.(2.5) is easily found to have the form

$$\frac{G_0(0)}{G_0(\omega)} = 1 + G_0(0) \frac{\gamma|\omega|}{2} + \frac{2 t_{vg,0}^{-1} G_0(0) \gamma|\omega| s(|\omega|)}{1 + 4 t_{vg,0}^{-2} G_0(0) \gamma|\omega| s(|\omega|)},$$

where $s(|\omega|) = 1 - t_{vg,0}/2 + \gamma|\omega| G_0(0)/4$, and

$$t_{vg,0} = 1 - \Delta_0^{(R)} \left(G_0(0)\right)^2.$$
occurring in the thermal 2D LLL case, arises because the pinning strength does not carry nonzero frequency, while the relation \( x_4 \propto G_0(0) \) is due to the fact that the purely dissipative quantum fluctuation raises the dimensionality of fluctuation by two and behaves like the thermal 4D LLL fluctuation. Based on this general rule, it is easy to generalize the above expressions of the lowest order in \( U_p \) to the case with arbitrary pinning strength. To do this, we first note the definition of dynamical VG susceptibility

\[
\chi_{vg}(k; \omega_1, \omega_2) = N^{-1} \int_{\mathbf{r}, \mathbf{R}} e^{i\mathbf{k} \cdot \mathbf{R}} \, \langle \Psi_{\omega_1}^{*}(\mathbf{r}) \Psi_{\omega_1}^{*}(\mathbf{r} + \mathbf{R}) \rangle \langle \Psi_{\omega_2}(\mathbf{r} + \mathbf{R}) \Psi_{\omega_2}(\mathbf{r}) \rangle
\]

\[
= (\beta N_v)^{-1} e^{-k^2/2} \sum_{p,p'} e^{i(p-p')kz} \varphi_0(p, \omega_1) \varphi_0^{*}(p', \omega_1) \varphi_0(p + k_y, \omega_2) \varphi_0^{*}(p + k_y, \omega_2), \tag{2.10}
\]

which appears in the expression of \( G_{vg} \), where the overbar denotes the random average. By this definition, the irreducible vertex in a diagrammatic representation of \( \chi_{vg} \) is found to carry a quantity

\[
\Delta^{(R)} = \frac{U_p}{2\pi^2} \Delta(x_p; x_4) \tag{2.11}
\]

in the quantum regime \( T < T_{cr} \) and consequently, the quantity \( t_{vg,0} \) is given by replacing \( \Delta_0^{(R)} \) in eq.\( (2.8) \) by \( \Delta^{(R)}, \) where \( \Delta(x_p; x_4) \) is a unknown function of \( x_p \) and \( x_4. \) Using, for simplicity, eq.\( (2.5) \) (with replacement \( \Delta_0^{(R)} \to \Delta^{(R)}, \) \( t_{vg,0} \) near \( B_{vg,0} \) and in \( T \to 0 \) limit is independent of the details of \( \Delta^{(R)} \) and expressed in the form

\[
t_{vg,0} = \frac{\pi^2 \gamma B^2}{4} \frac{B - B_{vg,0}}{B_{vg,0}} \tag{2.12}
\]

up to \( O(B - B_{vg,0}) \). The \( B_{vg,0} \)-value was examined in details elsewhere.

In real disordered films, the critical fluctuation accompanying the second order VG transition at a critical field in \( T \to 0 \) limit is expected to be strong enough to change a temperature dependence near \( B_{vg,0} \). To find how behaviors near \( B_{vg,0} \) and the \( B_{vg} \)-value itself are affected by the VG critical fluctuation, let us consider an effective action for the VG fluctuation field, as in the context of spin-glass, expressed as a tensor \( Q_{\alpha,\alpha'}(\omega, \omega') \) with \( \alpha \neq \alpha' \). Here \( Q_{\alpha,\alpha'} \) depends on two frequency variables, reflecting that the pinning vertex carrying \( \Delta^{(R)} \) does not convey nonzero frequency. Although a detailed diagrammatic analysis is needed to construct the action consistently with the derivation of eq.\( (2.12) \), this procedure can be bypassed here. First, it will be easily seen that, when performing a Landau expansion of the effective action with respect to \( Q_{\alpha,\alpha'} \), a \( Q^n \)-order term is accompanied by \( n \) frequency-summations. Further, according to the diagrammatic rules mentioned above, \( \Delta^{(R)} \) appears everywhere as the combination \( x_p^{(R)} \equiv \Delta^{(R)}(G_0(0))^2 \), which is a constant at \( B_{vg,0} \) and in \( T \to 0 \) limit according to eq.\( (2.8) \) with \( \Delta_0^{(R)} \) replaced by \( \Delta^{(R)} \) and a modification to be done below. Then, the VG effective action should take the form

\[
S_{\text{eff}}(Q) = \int d^2r \left[ \sum_{\omega_1, \omega_2} \sum_{\alpha_1, \alpha_2} Q_{\alpha_2, \alpha_1}(\omega_2, \omega_1) \left( t_{vg,0} + x_p^{(R)}(-\nabla^2) \right) \right]
\]
\[ +g(\gamma^{(R)}|\omega_1, \gamma^{(R)}|\omega_2| x_p^{(R)})Q_{\alpha_1, \alpha_2}(\omega_1, \omega_2) + \sum_{m \geq 3} c_m(x_p^{(R)})^{m/2} \prod_{j=1}^{m} \sum_{\omega_j} Q_{\alpha_j, \alpha_{j+1}}(\omega_j, \omega_{j+1}) \right], \]

where the index \( j = m + 1 \) implies \( j = 1 \), \( c_m \)'s are constants, and the function \( g \) satisfies \( g(0, 0; x_p^{(R)}) = 0 \). First, it will be easily seen by considering this action at \( T = 0 \) that any renormalization of the Gaussian mass \( t_{vg,0} \) due to the interaction \((m \geq 3)\) terms is accompanied only by the parameter \( x_p^{(R)} \) and hence that the resulting renormalized one \( t_{vg} \) is given by

\[ t_{vg} \equiv t_{vg,0} + c_f l = 1 + c_f l - x_p^{(R)} \simeq \frac{\pi^2 \gamma^2}{U_4} \frac{B - B_{vg}}{B_{vg,0}}, \]

where \( B_{vg} \simeq B_{vg,0}(1 - c_f U_4/((\pi^2 r_B^2 \gamma))) \) is the fluctuation-corrected VG transition field. As usual, a nonzero coefficient \( c_f \) is estimated in terms of a UV-divergent self energy term which, at the one loop level in eq.(2.13), occurs from a positive Hartree-like contribution of the \( m = 4 \) term. Consequently, as expected, \( c_f \) becomes positive and thus, \( B_{vg} \) decreases with increasing the quantum fluctuation strength \( U_4/(r_B^2 \gamma) \) according to the above-mentioned expression. Next, let us consider the action at nonzero \( T << T_{cr} \) near \( B_{vg} \) where \( |t_{vg}| \ll 1 \). This situation is formally similar to that in 3D systems at nonzero (but low) fields near \( T_{c0} \) in which the so-called thermal XY scaling \( 1 \not\equiv T - T_{c0} \sim B^{3/4} \) is expected below a crossover field (a line defined from the relation \( \delta t_s = \delta t_d \) in ref.21) corresponding to \( T_{cr} \) in the present issue. In the case, the field or the vortex density measures the inverse-square of the microscopic length \( r_B \), and hence a scaling relation given above is expected by comparing the two lengths, \( r_B \) and the correlation length of the pair-field \( \sim (T - T_{c0})^{-\nu_{XY}} \) with \( \nu_{XY} \simeq 2/3 \), with each other. Similarly, in the present issue near \( B_{vg} \), we have only two variables \( t_{vg}(B) \) and \( \gamma^{(R)}/\beta \propto \gamma/((\beta \sqrt{\Delta^{(R)}}) \), and hence, the only \( T \)-dependent scaling variable is, in the present case, \( (\xi_{vg}(B, T \to 0))^{z \gamma^{(R)}/\beta} \) or equivalently, \( X_{vg} \equiv t_{vg}(B)(\beta/\gamma^{(R)})^{1/z \nu} \), where \( z \) is the dynamical exponent of the quantum 2D VG transition, and \( \xi_{vg}(B, T \to 0) = (t_{vg}(B))^{-\nu} \) with an exponent \( \nu > 0 \) is the VG correlation length in \( T \to 0 \) limit written in unit of \( r_B \). Thus, the scaling relation

\[ |B - B_{vg}| \sim B_{vg,0} \frac{U_4}{\pi^2 \gamma r_B^2} \left( \frac{\gamma}{\sqrt{\Delta^{(R)}}} T \right)^{1/z \nu}, \]

measuring the field range of quantum VG critical regime at nonzero \( T \), is expected near \( B_{vg} \), and the correlation length \( \xi_{vg} \) expressed in unit of \( r_B \) should have, at nonzero \( T \), the scaling form \( \xi_{vg}(B, T) = (t_{vg}(B))^{-\nu} S_{\xi}(X_{vg}) \), where the scaling function \( S_{\xi} \) has the limiting behaviors; \( S_{\xi}(X_{vg} \to 0) \to (X_{vg})^{\nu} \), \( S_{\xi}(X_{vg} \to +\infty) \) approaches a positive constant. For convenience, we will assume \( S_{\xi}(X_{vg}) = (X_{vg}/(c_{\xi} + X_{vg}))^{\nu} \) leading to the correlation length

\[ \xi_{vg}(B, T) = \left( t_{vg}(B) + c_{\xi} \left( \frac{\gamma}{\sqrt{\Delta^{(R)}}} T \right)^{1/z \nu} \right)^{-\nu} \]

and yielding the correct limiting behaviors, where \( c_{\xi} \) is a positive constant depending only on \( x_p^{(R)} \)
near $B_{vg}$. Note that eq.(2.16) implies $\xi_{vg}(B = B_{vg}, T) \sim T^{-1/2}$. Further, although a line at finite $T$, on which $\xi_{vg}$ apparently diverges, is defined in $B < B_{vg}$ as a consequence of the scaling (2.15), the appearance of such a line is not peculiar to the present issue: For instance, although the thermal 3D XY scaling in clean limit suggests a line at nonzero $B$ obeying this scaling, not a continuous transition but rather a first order (vortex lattice melting) transition occurs there. Similarly, the presence of such a line in the present 2D dirty case near $T = 0$ does not contradict the absence of a true 2D VG transition at $T > 0$. Rather, eq.(2.16) itself will fail with decreasing $B$ away from the quantum VG critical regime, since the absence of the 2D VG transition at $T > 0$ is usually guaranteed by a nonperturbative origin in a vortex solid.\(^{(23)}\)

Now, let us examine the VG fluctuation contribution $G_{vg}$ to the total conductance $G$. Near or below $H_{c2}(T)$-line at nonzero $T$, $G$ is generally expressed in the form\(^{(23)\,(24)}\)

$$G = G_n + G_s = G_n + \delta G_s + G_{fl} + G_{vg},$$

(2.17)

where $G_n$ and $G_s$ are, respectively, the quasiparticle contribution and the superconducting part of $G$, and $G_s$ consists of the Aslamasov-Larkin (AL) fluctuation term $G_{fl} + G_{vg}$ and of other fluctuation contributions $\delta G_s$ excluded from the GL description. The AL contribution is further divided into the part $G_{vg}$ due to VG fluctuation and the remaining one $G_{fl}$, which in clean limit becomes the vortex flow conductivity\(^{(23)}\) deep in the vortex liquid region at nonzero $T$. However, $G_{fl}$ in $B > B_{vg}$ must vanish\(^{(4)}\) in $T \to 0$ limit, although this vanishing may not be detected at low enough $B$ in systems with weaker quantum fluctuation. Nonvanishing contribution in $T \to 0$ limit may arise from $G_n$ and\(^{(23)\,(25)}\) $\delta G_s$, although discussing a rigorous $T = 0$ result is beyond the scope of this paper. In this section, we focus primarily on $G_{vg}$. Other fluctuation term $\delta G_{s}$ will be commented on in §3.

To study $G_{vg}$ (and $G_{fl}$) under a uniform current, we need the spatially averaged supercurrent $\langle j_{\Omega}(r) \rangle_r$ where $\Omega$ is the external frequency. In general, $\langle j_{\Omega}(r) \rangle_r$ will take the form

$$\langle j_{\Omega}(r_1) \rangle_r = \frac{2\pi l^2}{\phi_0} \beta^{-1} \sum_{\omega} \langle \overline{\mathcal{C}}(|\omega|, |\omega + \Omega|; Q_j) (Q_1 + Q_2^*) \tilde{\Psi}^*(r_2) \tilde{\Psi}_{\omega + \Omega}(r_1) |_{r_1 = r_2} > r_1$$

$$= \frac{2\sqrt{2\pi l^2} \beta^{-1}}{\phi_0 r_B} \sum_{\omega} C(|\omega|, |\omega + \Omega|) \sum_{p} \langle \varphi^*_1(p, \omega) \varphi_0(p, \omega + \Omega) (\hat{x} + i\hat{y}) + \varphi^*_0(p, \omega) \varphi_1(p, \omega + \Omega) (\hat{x} - i\hat{y}) \rangle,$$

(2.18)

where $Q_j = -i\partial / \partial r_j + 2\pi A(r_j) / \phi_0$, $\varphi_1(p, \omega)$ a fluctuation field in the next lowest Landau level (NLL), and $\tilde{\Psi}$ the full pair-field prior to decomposing into the Landau levels. The $\omega$ and $\Omega$ dependences in $C(|\omega|, |\omega + \Omega|)$ are negligible when considering $G_{vg}$ (and $G_{fl}$). Further, in obtaining the second line of eq.(2.18), terms consisting only of higher Landau levels were neglected by assuming a high field approximation\(^{(11,\,(12,\,(17)}\) in which a static vortex state is described within LLL. In particular, it is important to note that the NLL modes cannot participate in describing any static vortex
solid for a reason of symmetry. Since this massive NLL modes inevitably appear in considering a response to a uniform current, some terms associated with NLL modes have to be added in the GL action, which are expressed by

$$\delta S_p^\alpha = \sum_\alpha \sum_\omega \left[ G_1(\omega) \right]^{-1} |\varphi_1^\alpha(p, \omega)|^2 - \frac{U_p}{4\pi r_B^2 N_\rho} \sum_k \sum_{\alpha, \beta} \left[ \frac{k^2}{2} f_{11}(k^2) \rho_1^{(\alpha)}(k, 0) \rho_1^{(\beta)*}(k, 0) \right]$$

$$+ \left( \frac{k_+}{\sqrt{2}} f_{01}(k^2) \rho_1^{(\alpha)}(k, 0) \rho_1^{(\beta)*}(k, 0) + \text{c.c.} \right),$$

(2.19)

where $k_+ = k_y + ik_x$, and

$$\rho_1^{(\alpha)}(k, 0) = \sum_{\omega, \rho} e^{i p k_2 - \frac{k^2}{4}} \varphi_1^{(\alpha)*}(p - k_y/2, \omega) \varphi_0^{(\alpha)}(p + k_y/2, \omega).$$

(2.20)

For convenience of presentation, the NLL propagator $G_1(\omega) = |\varphi_1^\alpha(p, \omega)|^2 = (\mu_1(0) + \gamma(1)|\omega|)^{-1}$ is assumed here to have been renormalized in a proper way. The bare mass $\mu_1(0)$ is of order unity near $H_{c2}(0)$.

The terms $G_{fl} + G_{vg}$ of conductance are obtained in terms of Kubo formula

$$\left( \frac{\pi r_B^2}{M_1} \right)^2 R_Q(G_{fl} + G_{vg})$$

$$= \left( -\frac{\partial}{\partial \Omega} \right) (\beta N_\rho)^{-1} \sum_{p, p', \omega} \left| \frac{\varphi_0(p, \omega) \varphi_0^*(p', \omega) \varphi_1(p', \omega + \Omega) \varphi_1^*(p, \omega + \Omega)}{[G_1(0)]^2} \right|_{\Omega \to 0},$$

(2.21)

where $R_Q = \pi \hbar/2 e^2$ is the resistance quantum. In the present high field approximation, $G_1(0)$ appears in both $G_{vg}$ and $G_{fl}$ only as the combination $M_1 \equiv 2\pi l^2 C(0, 0) G_1(0)$. Below, it will be shown that, deep in the vortex liquid regime, this factor takes the universal value

$$M_1 = \pi r_B^2$$

(2.22)

independent of microscopic details as a consequence of gauge-invariance. Deep in the vortex liquid regime, the AL conductance $G_{fl}$ in the pinning-free case can be assumed to be given by the vortex flow expression. Since the vortex flow in a pinning-free system is not affected by the vortex-solidification at $T_m(B)$, we can focus on the mean field result of the vortex flow conductance, which is most easily derived in terms of a harmonic action

$$\delta S_{\text{har}} = \beta \left| \left| \Psi_{\text{MF}}(r) \right| \right|^2 r \sum_{\Omega} \left( \gamma/|\Omega| |s_L|^2 + \left| G_1(0) \right|^{-1} \left| s_L + \frac{2\pi}{\phi_0} (\delta A \times \hat{z}) r_B^2 \right|^2 \right),$$

(2.23)

where $s_L$ denotes a uniform displacement of vortices, and $\Psi_{\text{MF}}$ is the mean field solution of vortex solid. The combination $r_B^{-2}(\hat{z} \times s_L) + (2\pi/\phi_0) \delta A$ in eq. (2.23) is a consequence of the gauge-invariance and ensures the Josephson relation $E = -\partial \delta A / \partial t = -v_L \times B$, where $v_L = \partial s_L / \partial t$ is the vortex
velocity. By substituting $B s_L = \hat{z} \times \delta \mathbf{A}$ minimizing the above second term into the first term, the vortex flow conductivity is obtained as the coefficient of thus obtained first term, which is clearly independent of $G_1(0)$. On the other hand, using eq.(2.21) (with no pinning disorder effect) in the mean field approximation where $\varphi_0$’s $\omega$-dependence is absent, one obtains a conductance expression $\propto M_f^2$. Under the condition that these two expressions are identical with each other, eq.(2.22) follows. In fact, using the general fact already used in writing down eq.(2.23), that the uniform fluctuation belonging to NLL around the mean field solution $\Psi_{MF}$ is nothing but a uniform displacement solution with amplitude $s_{L,y} = i s_{L,x}$, the averaged supercurrent (2.18) is found to be proportional to $2\pi^2 C(0,0) (\mathbf{B} \times s_L + \delta \mathbf{A})$ up to $O(s_L)$ and $O(\delta \mathbf{A})$. By identifying this with that following from eq.(2.23), we again obtain eq.(2.22). Although, due to a pinning disorder effect on the static NLL mode, the pinning-free result (2.22) may be subject to a subtle change, we argue in terms of eq.(2.8) that, as far as $G_1(0) \ll G_0(0)$ is safely satisfied, eq.(2.22) is quantitatively valid near $B_{vg}$ in the present high field case.

Before examining $G_{vg}$ further, a form of $\chi_{vg}$ needs to be determined. In the Gaussian (or mean field) approximation, it is written as

$$\chi_{vg,0}(\mathbf{k}; \omega_1, \omega_2) = (G_0(0))^2 (\xi_{vg,0})^2$$

$$\times \left( 1 + \frac{(k \xi_{vg,0})^2}{2} + \frac{2(\xi_{vg,0})^2 \gamma(\omega_1)}{1 + \sqrt{1 + 4(\xi_{vg,0})^4 \gamma(\omega_1)}} + \frac{2(\xi_{vg,0})^2 \gamma(\omega_2)}{1 + \sqrt{1 + 4(\xi_{vg,0})^4 \gamma(\omega_2)}} \right)^{-1},$$

(2.24)

where $\xi_{vg,0} = t_{vg,0}^{-1/2}$ is the mean field VG correlation length in unit of $r_B$, and eq.(2.7) was used. To go beyond the mean field analysis for studying $G_{vg}$ and enter the quantum VG critical regime present in $T < T_{cr}$, we invoke the ordinary scaling hypothesis for the VG correlation function on the basis of the above mean field expression as follows:

$$\chi_{vg}(\mathbf{k}; \omega_1, \omega_2) = c_g (\xi_{vg} G_0(0))^2 S_x(k \xi_{vg}; \gamma(\omega_1) \xi_{vg} \gamma(\omega_2) \xi_{vg}),$$

(2.25)

where the correlation length $\xi_{vg}$ is given by eq.(2.16), and the scaling function $S_x$ and a positive coefficient $c_g$, as well as $c_{\xi}$ in eq.(2.16), may depend on $x_p(R)$ according to the action (2.13).

Now, the terms corresponding to $G_{vg}$ in r.h.s. of eq.(2.21) will be examined. First, closely following the analysis used for the thermal glass transitions, let us consider the contributions $G_{vg}^{(1a)}$ of the diagram Fig.1 (a) and $G_{vg}^{(1b)}$ of a sum of the family of Fig.1 (b) to eq.(2.21) with the coefficient (2.22), which are expressed as

$$R_Q G_{vg}^{(1a)} = \frac{U_p}{2\pi B} \int k^2 e^{-k^2/2} f_{11}(k^2) \left( -\frac{\partial}{\partial|\Omega|} \right) \beta^{-1} \sum_\omega \chi_{vg}(\mathbf{k}, \omega, \omega + \Omega)$$

$$= \pi x_p c_g(2 \pi \gamma(\omega_1)^2 \xi_{vg} \beta^{-1}) \int k^2 \sum_n S'_x(k; 2\pi |n| (\gamma(\omega_1) \xi_{vg} \beta^{-1})),$$

(2.26)
\[ R Q G^{(1b)}_{\nu g} = \frac{U_p^2}{4\pi r_B c_0} \left( -\frac{\partial}{\partial \Omega} \right) \beta^{-1} \sum_\omega (G_0(\omega) + G_0(\omega + \Omega))^2 \int_k \chi_{\nu g}(k; \omega, \omega + \Omega) \]

\[ = 4\pi c_0 \epsilon_p(x_p^{(R)} - \frac{\epsilon_p}{2} (\gamma^{(R)} / \nu g)^{\beta^{-1}}) \sum_n \int_k S'_X(k; 2\pi|n| \gamma^{(R)} / \nu g)^{\beta^{-1}}, \quad (2.27) \]

where \( c_0 = \int_k k^2 (f_0(k^2) f_1(k^2) - (f_0(k^2))^2) e^{-k^2/2} \), and \( \sum_\omega S'_X(k; |\omega|) = -(\partial/\partial |\Omega|) \sum_\omega S_X(k; |\omega|, |\omega + \Omega|)|_{\Omega = 0} \). Although we have assumed in eqs. (2.26) and (2.27) the "pinning lines" unrelated to the VG susceptibility to carry not \( \Delta^{(R)} \) but \( U_p/2\pi r_B^2 \), this simplification does not affect the conclusion given below, that \( R Q G^{(1b)}_{\nu g} \) in the dirty limit takes a universal value independent of \( T \) at \( B_{\nu g} \), because of the relations (2.11) and (2.34) (see below).

So far, no microscopic (electronic) model leading to the dissipative GL action has been specified, and a \( T = 0 \) limit of the resulting GL action has been simply assumed. Although we will consider below a simplified BCS Hamiltonian with a short-ranged repulsion and a nonmagnetic potential disorder in order to see how a \( T \)-insensitive GL action can be realized, it is instructive to first start from the dirty limit with no electron-repulsion. In this case, the expressions of coefficients \( U_4, \gamma, \mu(0), \mu_1(0) \), and \( C(0,0) \) are available in the literatures \[ 14, 18 \] and given by

\[ U_4 = 8\pi T^3 (\beta N(0))^{-1} \sum_{\epsilon > 0} (\Gamma(2\epsilon; B))^3, \quad (2.28) \]

\[ \gamma = 4\pi T^2 \beta^{-1} \sum_{\epsilon > 0} (\Gamma(2\epsilon; B))^2, \quad (2.29) \]

\[ \mu(0) = \ln \left( \frac{T}{T_{c0}} \right) + 4\pi T \beta^{-1} \sum_{\epsilon > 0} (\Gamma(2\epsilon; 0) - \Gamma(2\epsilon; B)), \quad (2.30) \]

\[ \mu_1(0) = \mu(0) + 4\pi T \beta^{-1} \sum_{\epsilon > 0} (\Gamma(2\epsilon; B) - \Gamma(2\epsilon; 3B)), \quad (2.31) \]

and

\[ C(0,0) = 2\pi T \beta^{-1} \sum_{\epsilon > 0} \Gamma(2\epsilon; B) \Gamma(2\epsilon; 3B), \quad (2.32) \]

where \( \Gamma(2\epsilon; B) = (2|\epsilon| T + \pi^2 B / \phi_0)^{-1} \), and \( \epsilon \) denotes a Matsubara frequency for fermions. Note that the relation (2.22) is satisfied just on the \( H_{c2}(T) \) line where \( \mu(0) = 0 \), implying that, in the dirty limit, the renormalized \( \mu_1(0) \) should approach \( 4\pi T \beta^{-1} \sum_{\epsilon > 0} (\Gamma(2\epsilon; B) - \Gamma(2\epsilon; 3B)) \) with decreasing \( B \). In \( T < T_{c1}^{mf} \), the \( T \)-dependence of \( \Gamma(2\epsilon; B) \) is cut off by the \( B \)-dependence, and the above coefficients become insensitive to \( T \) in the manner \( U_4 \rightarrow 4\pi T^2 r_B^2 / (N(0) l^4), \gamma \rightarrow 2\pi (r_B / l)^2 \equiv \gamma^{(0)}(T = 0), \mu(0) \rightarrow \ln(B/H_{c2}(0)), \) and \( \mu_1(0) \rightarrow \ln(3B/H_{c2}(0)) \). On the other hand, the pinning strength \( U_p \) is known only in \( T < T_{c1}^{mf} \) and given by \[ 14 \]

\[ U_p \simeq r_B^2 \left( \frac{\tau}{N(0)/l^2} \right)^2. \quad (2.33) \]
In terms of these $T$-independent GL coefficients, a "$T = 0$" critical field $B_{\text{vg}}$ can become well-defined within the dirty limit. Further, at lower temperatures than $T_{cr}$ which is estimated in dirty limit as $\sim 4\pi T_{cr}^\text{mf}/(E_F\tau)$ ($\propto B$), we have the relation

$$x_p = \frac{\pi^3}{2}a_x^2. \quad (2.34)$$

As a result of eq.(2.34), $x_p^{(R)}$ depends only on $x_p$. Just at $B_{\text{vg}}$, the combination $\gamma^{(R)}(\xi_{\text{vg}})^2/\beta$, as well as $x_p^{(R)}$ and $x_p$, becomes a constant independent of $T$ and of material parameters according to eqs.(2.11) and (2.14). Therefore, $G_{\text{vg}}^{(1b)}(B = B_{\text{vg}})$ is a universal constant divided by $R_Q$ in $T < \text{Min}(T_{cr}, T_{cr}^\text{mf})$. Similarly, $R_QG_{\text{vg}}^{(1a)}(B = B_{\text{vg}})$ becomes $(\gamma T/\sqrt{\Delta^{(R)}})^{2/\beta}$ multiplied by a universal positive constant.

It should be mentioned that we cannot verify directly whether the (universal) constant $c_{01}$ is positive or not, because the functional forms of $f_{00}$, $f_{11}$, and $f_{01}$ are not completely known, although, by definition, the VG contribution $G_{\text{vg}}$ to the conductance must be positive. On the other hand, we have a unlimitedly large number of diagrams contributing to $R_QG_{\text{vg}}(B = B_{\text{vg}})$ in the same way as Fig.1(b). All of them can be seen as such diagrams that, according to the already-mentioned LLL diagrammatic rule, the vertex correction unrelated to the VG susceptibility is of higher order in $x_p$ and $x_4$ (in $T < T_{cr}$) compared with those in Fig.1. In this sense, Fig.1(a) is the lowest order term, and the diagram in Fig.1(b) is the next lowest order term in $x_p$. However, just at $B = B_{\text{vg}}$ where $x_p$ and $x_4$ take constant values, the diagram of Fig.1(a) becomes of the same order as that of Fig.1(b) except the extra power in $\xi_{\text{vg}}^{-2}$. The same thing holds in the (formally) higher order diagrams in $x_p$ and $x_4$, and hence, they also contribute to a universal value of $R_QG_{\text{vg}}(B = B_{\text{vg}})$ together with a sum of the family of Fig.1(b). Since, unfortunately, we have no resummation scheme, useful at the critical point, for judging which of those diagrams should be adopted or may be neglected, it is difficult, as in the argument on a $T = 0$ critical conductance $R_QG_s(T = 0, \omega \to 0)$, to estimate here a concrete value of $R_QG_{\text{vg}}(B = B_{\text{vg}})$. We can just conclude that, in the present dirty limit, $R_QG_{\text{vg}}(T \to 0, B = B_{\text{vg}})$ is a universal positive number.

Of course, a universal critical $G_{\text{vg}}$ obtained above is not a consequence of the $T = 0$ scaling argument. Actually, the $\omega$-summation in eq.(2.27) was not changed above into a frequency-integral because $\gamma^{(R)}(\xi_{\text{vg}})^2/\beta$ is finite in $T \to 0$ limit and at $B_{\text{vg}}$, implying that $G_{\text{vg}}^{(1b)}(B = B_{\text{vg}})$ is not a dc limit of a $T = 0$ conductance but a dc conductance in the quantum regime $0 < T < T_{cr}$.

However, it will be difficult to explain available resistivity data in terms only of the results in the dirty limit with no electron-repulsion. As examined in ref.19, the mean field value $B_{\text{vg},0}$ calculated in the dirty limit seems to increase with increasing $(E_F\tau)^{-1}$ due to the corresponding increase of $H_{c2}^d(0)$, while the data in ref.4 have shown a trend opposite to this. Although the resulting fluctuation-corrected field $B_{\text{vg}} \sim B_{\text{vg},0}(1 - 2e_0/(\pi E_F\tau))$ (see the sentence below eq.(2.14)) may decrease with increasing $(E_F\tau)^{-1}$ depending on $E_F\tau$-values, it will be difficult to understand a
dependence of $B_{vg}$ on the film thickness $d \sim k_F^{-1} R_Q/(R_r E_F \tau)$ in the dirty limit with no electron-repulsion, where $R_r$ is the high temperature sheet resistance. Once the interplay between the electron-repulsion and disorder is taken into account, however, $H_{c2}(0)$ and hence, $B_{vg}$ decrease with increasing the electron-repulsion strength $\lambda_1 \simeq R_r/(8\pi R_Q)$. Further, a nonzero $\lambda_1$ results in a failure of the equality (2.34), and thus, the $G_{vg}(T \to 0)$ value at a critical field is not universal any longer but will depend on $\lambda_1$.

We argue that the properties in the close vicinity of $T = 0$ (corresponding to the region below $T_{rep}$) of disordered thin superconducting films have not been examined so far experimentally. According to the previous works, the interplay between an electron-repulsion and disorder appears in the GL action in two different ways: The GL coefficients $U_4$, $\mu(0)$, $\mu_1(0)$, $C(0,0)$, and $U_p$ are convergent in low $T$ limit at each order of the $\lambda_1$-perturbation series, and their $T$-dependences are controlled, through the denominator of Cooperons, by the $|\epsilon|$ (Matsubara frequency) value of the order $P^2/(8\pi \tau r_B^2)$. Namely, their $T$-dependences are lost, as well as those in the dirty limit, below $T_{cr}^{mf}$ independent of $\lambda_1$. In contrast, $\gamma$ at low enough $T$ is expanded in powers of $\lambda_1 \ln(T/T_{cr}^{mf})$ because this quantity is dominated by the lowest $|\epsilon|$ values, and consequently, a $T$-dependence of $\gamma$ induced by the electron-repulsion will become remarkable rather near $T = 0$ below $T_{rep} \simeq T_{cr}^{mf} \exp(-\lambda_1^{-1})$. Since, as far as we know, $R_r < R_Q$, or equivalently $8\pi \lambda_1 < 1$, is satisfied in real thin films with an FSI behavior, $T_{rep}$ will lie much below $T_{cr}^{mf}$ and seems to be inaccessibly low in real systems. Then, it is reasonable to assume the FSI behavior seen in real experiments to be a phenomenon in the intermediate region $T_{rep} \leq T \leq T_{cr}^{mf}$, where all GL coefficients are insensitive to $T$ so that an apparent VG critical field $B_{vg}^*$ and a nonuniversal constant $R_Q G_{vg}(B = B_{vg}^*)$ are well-defined.

In §4 of ref.19, a computation result on $\gamma$ was given supporting the argument on the presence of the intermediate temperature region, and it was suggested that the temperature scale corresponding to $T_{rep}$ will be below $0.1 T_{cr}^{mf}$ in the cases with realistic $R_r$-values. Unfortunately, it was difficult to judge whether $\gamma$ remains positive or vanishes in low $T$ limit, i.e., in $T < T_{rep}$. We simply expect here that $\gamma$ will significantly decrease on cooling below $T_{rep}$ and hence that, within the model of purely dissipative dynamics, the true $B_{vg}$ will lie at a much lower field than $B_{vg}^*$ (see Fig.3 below). What we wish to emphasize is that, as far as $8\pi \lambda_1 < 1$ is satisfied in real systems, the $T$-dependence of a microscopic origin is remarkable rather at extremely low temperatures below $T_{rep}$, and hence that any attempt, such as ref.6, to explain the FSI behaviors by assuming a $T = 0$ bosonic model is not justified.

§3. Description of 2D Resistive Behavior near $B_{vg}^*$

In this section, we discuss the resistivity curves around $B_{vg}^*$ and in $T < T_{cr}^{mf}$ on the basis of the results in §2 and give some results relevant to comparing with experimental data. Again, the GL coefficients will be assumed for a moment to be insensitive to $T$ so that a critical field $B_{vg}$ may be
well-defined.

First, let us start with the resistive behaviors far above \( B_{\text{vg}} (B > B_{\text{vg}}) \) where the Gaussian approximation, illustrated as eq.(2.24), for the VG fluctuation may be used. Assuming the low frequency behavior to be essential even in the Gaussian region, we will keep, for simplicity, only \( O(\omega_j) \) \((j = 1, 2)\) terms in eq.(2.24), and this \( \chi_{\text{vg},0} \) will be substituted into the first line of eq.(2.27). By arranging the \( \omega \)-summation and performing the \( |\Omega| \)-derivative, we obtain

\[
\frac{R_Q G_{\text{vg},0}^{(1b)}}{2\epsilon_{01}} = x^2 \beta^{-1} \left( \gamma^{(R)}\xi_{\text{vg},0}^4 + \frac{5}{2}\gamma^{(R)}\xi_{\text{vg},0}^2 \ln(1 + c_c^{-2}\xi_{\text{vg},0}^2) + (G_0(0))^{-2} \sum_{\omega>0} \frac{\partial}{\partial\omega} \left[ -(G_0(\omega))^2 \right] \right)
\times \ln \left( \frac{1 + c_c^{-2}\xi_{\text{vg},0}^2 + 2\omega\gamma^{(R)}\xi_{\text{vg},0}^4}{1 + 2\omega\gamma^{(R)}\xi_{\text{vg},0}^4} \right) + \ln(1 + c_c^{-2}\xi_{\text{vg},0}^2) \frac{G_0(\omega)}{2}(G_0(\omega) + G_0(0)) \right), \tag{3.1}
\]

which obviously vanishes in \( T \to 0 \) \((\beta^{-1}\gamma^{(R)}\xi_{\text{vg},0}^4 \to 0)\) limit. Similarly, one can verify that \( G_{\text{vg},0}^{(1a)} \) also vanishes at \( T = 0 \) if taking account of the frequency dependence of \( G_1 \). The additional \( \ln^2\xi_{\text{vg},0} \) dependences arise from an upper cutoff \((c_c^2\omega_B)^{-1}\) (with a constant \( c_c \) of order unity) of the \( |k| \)-integral. Judging from a similar situation one encounters in deriving 2D AL fluctuation conductance in \( T \to 0 \) limit and at \( B = 0 \), we believe that this divergence is specific to the present direct frequency-summation and may be avoided by the standard analytic continuation which we have not tried. However, this technical issue does not affect our conclusion that, as well as \( G_{\text{fl}}, G_{\text{vg}}(B > B_{\text{vg}}) \) vanishes in \( T \to 0 \) limit, because, as shown in ref.14, each term of perturbation series of fluctuation conductivity examined within a quantum GL action vanishes in \( T \to 0 \) irrespective of the presence or absence of a pinning-disorder term in the action. In \( B \ll B_{\text{vg}} \), such a perturbation should become more convergent with approaching \( T = 0 \), and hence, we can conclude that, as well as each term of the perturbation series, the resummation results of the perturbation series, i.e., \( G_{\text{fl}} \) and \( G_{\text{vg}} \) themselves in \( B > B_{\text{vg}} \), also vanish in \( T \to 0 \) limit.

On the other hand, in \( B < B_{\text{vg}} \) and out of the quantum critical regime defined by eq.(2.15), \( \xi_{\text{vg}} \) grows with decreasing \( T \) or \( B \) and hence, the VG fluctuation becomes "classical" even at low temperatures below \( T_{c\nu} \) in which the pair-field fluctuation, with higher energy than the VG fluctuation, is of a quantum character. In fact, as eq.(3.1) suggests, one may expect the 2D classical behavior \( G_{\text{vg}} \sim (\beta^{-1}\gamma^{(R)}\xi_{\text{vg}}^z) \) within the present analysis. However, the classical (i.e., thermal) 2D VG transition is washed out, e.g., by a nonperturbative effect such as the free vacancies or interstitials in the vortex solid. Therefore, the region in which the classical scaling behavior \( G_{\text{vg}} \sim \xi_{\text{vg}}^z \) is visible may be quite narrow.

According to eq.(2.27), the scaling behavior of \( R_Q G_{\text{vg}} \)

\[
R_Q G_{\text{vg}} = \mathcal{U} \left( c_u \frac{B - B_{\text{vg}}}{B_{\text{vg},0}} \left( \frac{T_{c\nu}}{T} \right)^{1/2\nu} \right) \tag{3.2}
\]
is expected in the quantum VG critical regime defined by the relation (2.15). Here, \( U(x = 0) \) is a positive and nonuniversal constant according to the result in §2, and

\[
c_u = \frac{\sqrt{\Delta(R)^2}}{U_4 T_{c0}} \left( \frac{\gamma T_{c0}}{\sqrt{\Delta(R)}} \right)^{(z\nu-1)/z\nu}.
\]

Further, if describing the "Gaussian" region, discussed above, in terms of eq.(3.2), the limiting behaviors \( U(x \to +\infty) \to 0 \) and \( U(x \to -1) \to (x + 1)^{-z\nu} \) have to be satisfied. Therefore, the resulting \((R_Q G_{vg})^{-1} v.s. T\) curves around \( B_{vg} \) in \( B-T \) plane are, just like the original speculation\[^{31}\] based on some \( B = 0 \) results, similar to that of a "renormalization-group flow" near a fixed point. Further, we note that, if \( z\nu > 1 \), the \( \gamma \)-dependence of \( c_u \) will be opposite to a naive expectation on a strength of quantum critical fluctuation. In the case with a quantum normal-Meissner (i.e., \( B = 0 \)) transition, for instance, one would expect the width of the quantum critical regime at a fixed \( T \) to scale like \( \gamma^{-1/z\nu} \) and hence to become narrower for a stronger quantum fluctuation. In contrast, in the present \( B > 0 \) problem, the quantum VG critical region becomes wider as the pair-field fluctuation is enhanced. Details of the low \( T \) regions in \( B-T \) phase diagram of thin films are sketched in Fig.2.

As shown previously,\[^{14}\] \( R_Q G_{fl} \) calculated in the pinning-free \((U_p = 0)\) case shows, around a field above \( B_m(0) \), "fan-shaped" resistivity curves similar to but more moderate than that of \( G_{vg} \) near \( B_{vg} \). This follows from the facts that, when \( \gamma \) and hence the melting line \( B_m(T) \) in the quantum regime are insensitive to \( T \), a "flat" \( G_{fl}^{-1} \) curve of the order of \( R_Q \) is realized at a field above \( B_m \) and that, in higher fields, \( G_{fl}^{-1} \) shows an insulating behavior reflecting \( G_{fl}(T = 0) = 0 \), while it yields, in lower fields closer to \( B_m \), the classical vortex flow behavior which is insensitive to \( T \) at such low temperatures (see, for instance, Fig.4 of ref.14). Although an inclusion of the vortex pinning effect will slightly change this behavior of \( G_{fl} \) particularly close to \( B_m \), it is generally questionable to interpret real resistance curves by neglecting the presence of \( G_{fl} \) and identifying the resistance only with \( G_{vg}^{-1} \) because, as mentioned above, \( G_{fl} \) can have a magnitude of the same order as \( G_{vg} \) near \( B_m \). The fluctuation corrections to \( G_n, \delta G_s \), will be taken into account later, and for a moment we identify the total conductance with the bosonic contribution \( G_{fl} + G_{vg} \) plus the normal contribution \( G_n \).

Now, we discuss typical resistance data under the assumptions that \( T_{rep} \) is inaccessibly low (see the end part of §2) and that an apparent critical field \( B_{vg}^* \) well-defined in \( T > T_{rep} \), will lie above \( B_m \). Due to the former assumption, all the GL coefficients and hence, \( B_{vg}^* \) are insensitive to \( T \) in \( T_{rep} < T < T_{cr}^{mfl} \). One will see soon that the latter assumption has already been verified\[^{4,7}\] in some data. First, let us start with the cases with weak disorder (small \( R_r \)), in which the quantum fluctuation is also weak simultaneously and \( B_{vg}^* \) will lie near but above \( B_m \) (see below). In this case, the contributions of \( G_n \) and/or \( G_{fl} \) occupy a large weight of the total conductance near \( B_{vg}^* \), and there may be no clear indication of quantum correction\[^{32}\] (\( \sim -\lambda_1 \ln(1/T\tau) \)) to the familiar residual
behavior (insensitive to \(T\)) of \(G_n\) at accessible temperatures above \(T_{\text{rep}}\). Since \(T_{\text{cr}} \leq B_l^2/\phi_0 < 1\) in fields of our interest, the negligible quantum correction to \(G_n\) implies that the neglect of the region below \(T_{\text{rep}}\) is justified. Further, since \(G_{fl}\) just above \(B_m\) in the present case is, as mentioned in the last paragraph, insensitive to \(T\), the background contribution \(G_n + G_{fl}\) is expected to be independent of \(T\) and to weakly depend on \(B\). On the other hand, according to eq.(3.3), the quantum \(VG\) critical region at a fixed \(T\) seems to be narrower at weaker disorder: As a rough estimation in this weak disorder case, if any \(\lambda_1\)-dependence is neglected in eq.(3.3), \(c_u\) certainly decreases with increasing disorder as far as \(z\nu > 1\), as seen in most of data. By taking account of these contributions to the conductance altogether, the FSI behavior (3.2) is expected to be visible in a narrow VG critical field range around \(B_{vg}^*\). We believe that this will be an appropriate explanation, for instance, to the data in MoSi films\(^4\) where the total resistance value near \(B_{vg}^*\) was remarkably suppressed due to a large weight of \(G_n + G_{fl}\) insensitive to \(T\). In fact, in ref.4, the field \(B_0\) below which the thermal activation barrier\(^3\) \(~\ln B^{-1}\), arising from the motions of vacancies and interstitials in a (short-range ordered) vortex solid, remains nonvanishing lies just below the critical field corresponding to \(B_{vg}^*\). Since \(B_0\) should be essentially the same as \(B_m\), it justifies the above assumption that \(B_{vg}^*\) lies just above \(B_m\).

It is possible that, in systems with still weaker disorder, even the quantum fluctuation behavior of \(G_{fl}\) is not seen. In such a case with inaccessibly low \(T_{\text{cr}}\), the flat behavior of \(G_{vg}\) should not be seen consistently. This situation corresponds to the data in ref.33 where the only quantum behavior was seen in a vortex flow behavior, which is itself the classical behavior of \(G_{fl}\), but suggestive of a quantum tunnelling effect. Clearly, the metallic behavior below \(H_{c2}\) there\(^2\) is not a reflection of a \(T = 0\) phase diagram: When the sample disorder is weaker, i.e., \(T_{\text{cr}}\) is low enough, one needs to enter a lower temperature region to find quantum VG behaviors. Theoretically, an intermediate vortex liquid phase with finite resistance much smaller than \(G_n^{-1}\) is absent\(^2\) in 2D homogeneously disordered films at \(T = 0\) (see also §4).

Returning again to systems with visible quantum critical behaviors of \(G_s\), in turn, let us consider situations with stronger disorder (larger \(R_r\)). In this case, a decrease of \(G_{fl}\) and \(G_n\) on cooling will affect the total conductance: The total conductance at \(B_{vg}^*\) will decrease on cooling until a constant limiting value \(G \simeq G_{vg}(B_{vg}^*)\) is reached. Such a behavior that the resistance just at \(B_{vg}^*\) is not flat but increases on cooling until the lowest (accessible) temperature is reached has been observed in various materials\(^6\). If such data are explained according to this scenario, the vanishing contribution \(G - G_{vg}\) in \(T \to 0\) will not be accompanied by the scaling behavior (2.15). However, an insulating behavior at \(B_{vg}^*\) may appear even if \(G \simeq G_{vg}\). As illustrated by eq.(2.26), a sub-leading term of \(G_{vg}(B_{vg}^*)\) will behave like \(~(\xi_{vg}(B = B_{vg}^*))^{-2} \propto T^2/z\). If the sum of such terms is positive just like eq.(2.26) itself, this also becomes an origin of an insulating behavior at \(B_{vg}^*\).

We note that, in this case, the sub-leading terms in the quantum VG critical region are multiplied
by a critical scaling function like \( U(x) \) in eq.(3.2) and hence that they can be discriminated from an insulating behavior of the noncritical term \( G - G_{vg} \) mentioned earlier.

In ref.7, nonmonotonic resistance curves (insulating at intermediate temperatures but superconducting on further cooling) were found even just below \( B_{vg}^* \) in Bi/Sb films. According to our theory, this nonmonotonic behavior below \( B_{vg}^* \) is also understood as arising from the sum of the above-mentioned insulating \( G_{fl}^{-1} \) above \( B_m(0) \) and the superconducting \( G_{vg}^{-1} \) below \( B_{vg}^* \). This non-monotonic behavior is visible because the window in \( B_m(0) < B < B_{vg}^* \) is moderately wide in contrast to that in MoSi case.\(^\text{13}\) Actually, the presence of a wide region \( B_0 < B < B_{vg}^* \), in which the resistance does not yield an activated behavior indicative of a vacancy- or interstitial-creep in a vortex solid, was pointed out in ref.7. As also mentioned in §3 of ref.19, this window should broaden with increasing disorder. However, our theoretical result shows that it is invalid to, based only on the nonactivated resistive behavior, identify the window in \( B_m(0) \) (or \( B_0 \)) with a putative quantum liquid phase with low but nonzero resistance at \( T = 0 \). Of course, in a system with still stronger disorder or at low enough \( T \) (but above \( T_{rep} \)), the contribution \( G_n + G_{fl} \) is already negligible, and the leading term of \( G_{vg}(B = B_{vg}^*) \), i.e., a (nonuniversal) constant \( G \simeq G_{vg}(B = B_{vg}^*) \), should be observed as the total conductance, as actually seen in the state 1 of ref.3 where a critical resistance value is much larger than \( R_r \). On the other hand, the apparent critical resistance values in ref.7 seem to correlate to the corresponding \( R_r \)-values, implying that the samples in ref.7 have intermediate strengths of disorder.

We emphasize again that, in the present theory, the intervening metallic behavior at a field is regarded not as a reflection of the true \( T = 0 \) phase diagram but as a phenomenon in the intermediate (accessible) temperature range \( T_{rep} < T < T_{mf}^{cr} \). This interpretation is never artificial. In fact, a 2D FSI behavior was also observed in 2D-like but bulk (underdoped) YBCO,\(^\text{5}\) in which the true \( T = 0 \) critical behavior must be of a 3D VG type (see §4), and the system at nonzero temperatures behaves as if it have a 2D VG transition at \( T = 0 \) (This situation is comparable with a dimensional crossover just above a thermal transition. Note that, in the present case, \( \xi_{vg}(B = B_{vg}^*) \) diverges in \( T \to 0 \) limit). Similarly to this, in the present case the temperature variation of resistance curves in \( B > B_{vg} \) (see Fig.2) should become insulating at inaccessibly low temperatures below \( T_{rep} \), although the resistance curves in \( B_{vg} < B \ll B_{vg}^* \) may decrease on cooling in \( T > T_{rep} \), more or less, as a result of classical (thermal) VG fluctuation.

Now, the fermionic fluctuation term \( \delta G_s \) of the conductance will be considered. Since discussing the normal part \( G_n \) in details is beyond the scope of this paper, let us assume here \( G_n(T > 0) \) to show an almost metallic behavior. Then, the dynamics of the pair-field should be dominated by the dissipative term, and the fermionic fluctuation conductance \( \delta G_s \), consisting of the Maki-Thompson terms and DOS terms, remains nonzero in low \( T \) limit and was previously regarded as a correction to \( G_n \) without examining in details (see §7 in ref.14). According to a recent systematic study in
ref.25, $\delta G_s$ at $T = 0$ is negative and given, on the Gaussian level, by

$$R_Q\delta G_s(T = 0) \simeq -\frac{\gamma^{(0)}(T = 0)}{3\pi\gamma} \ln \frac{1}{\mu(0)}, \quad (3.4)$$

which is valid above the mean field $H_{c2}(0)$ ($\mu(0) \simeq -1 + B/H_{c2}(0) > 0$). Note that, in the dirty limit, the r.h.s. of eq.(3.4) is independent of material parameters. If the renormalization of LLL fluctuation is performed in terms of eq.(2.5) in deriving the corresponding one to eq.(3.4) applicable in $B < H_{c2}(0)$, one finds eq.(3.4) is replaced by

$$R_Q\delta G_s^{(R)}(T = 0) \simeq -\frac{2\pi\gamma^{(0)}(T = 0) r_B^2}{3U_4}|\mu(0)|, \quad (3.5)$$

where effects of pinning disorder on the fluctuation renormalization were neglected for simplicity. By using eqs.(2.28) and (2.29), the r.h.s. of eq.(3.5) becomes of the order of $-E_F\tau|\mu(0)|$. Further, the increase of $|\delta G_s|$ on cooling remains valid after the fluctuation renormalization is performed. Since $|\mu(0)| < 1$ within the GL theory, while $R_QG_n \simeq E_F\tau$ in 2D and with no quantum correction, the fermionic conductance $G_n + \delta G_s$ can significantly reduce with decreasing $B$ below $H_{c2}(0)$ due to the $|\mu(0)|$-dependence in eq.(3.5). This seems to explain the negative magnetoresistance (MR) in MoSi thin films, which was not visible in 3D-like films and in highly disordered films nonsuperconducting even at $B = 0$. Further, a more remarkable negative MR had been also found in InO thin films34 showing the FSI behavior. In ref.34, the origin of this negative MR had been ascribed to a localization of bosons (i.e., pairs) which is also of a superconducting origin but, in contrast to our idea, does not seem to be supported through a microscopic calculation. We also note here that the data in ref.34 have suggested a metallic resistance in much higher fields than $B^*_v$ and even at low enough $T$. In our notation, this corresponds to a nonvanishing $G_n$ and is consistent with our assumption that, in contrast to ref.6, the FSI behavior should be explained based on a dissipative dynamics of the pair-field.

Our scenario on the FSI behaviors is conventional and precludes a possibility of an intermediate metallic vortex phase at $T = 0$. An argument favoring such an intermediate phase is based on the data suggestive of a quantum tunneling behavior in a $\ln G^{-1}$ v.s. $1/T$ plot and also on a computation of fluctuation conductivity applicable only to $B = 0$ and based on a neglect of dissipative dynamics. As verified in ref.4, however, such a metallic behavior over a broad field range was also seen in the case irrelevant to the vortex states, i.e., the case in a field parallel to the surface of thin film samples and with a current parallel to the field. This finding suggests that the quantum tunneling behavior may not be due to an intrinsic origin. Actually, an intermediate metallic behavior tends to be seen in rather weakly disordered films with $R_v \leq 1 \,k\Omega$. As mentioned earlier, one needs to enter lower temperatures in a weaker disorder case in order to search a true low $T$ behavior. To the best of our knowledge, there is no clear evidence of an intermediate metallic behavior in a stronger disorder case. Actually, an intervening metallic phase was argued based
only on models for the granular case in zero field which are inapplicable to the nonzero field case of homogeneous materials (see §1). Further, the argument\cite{35} favoring \( G(B = B_c) \approx G_n \) certainly contradicts the \( R_r \)-dependence of critical resistance values measured in ref.3.

§4. Discussions and Extension

In §3, we have discussed various resistive data suggestive of the FSI transition at \( T = 0 \) by taking account of three different superconducting terms \( G_{fl}, G_{vg}, \) and \( \delta G_s \) of conductance. Those data have been explained by noticing that the experimentally accessible temperatures in real thin films with \( R_r < R_Q \) will lie not in the vicinity of \( T = 0 \) but within an intermediate temperature range defined by us. The resistance data in various samples are apparently incompatible with one another and are not explained comprehensively within a scenario taking account only\cite{6} of \( G_{vg} \) without a detailed calculation. As in the thermal case\cite{24,12} a couple of contributions to conductance are necessary in order to explain available data in a unified manner. Although we have explained, by focusing primarily on systems with relatively weak disorder, why the apparent critical resistance value tends to increase with increasing \( R_r \), it has not been clarified whether the \( G_{vg}(B = B^*_{vg}) \)-value itself decreases or not with increasing \( R_r \). Further, attention was not paid much to the values of exponents \( z \) and \( \nu \). Although they are usually assumed to take universal values, this assumption is not necessarily valid for a quantum transition in random systems. These issues are left for future studies.

The extension of the present theory to 3D case will be explained here since, at a glance, recent data\cite{36} in thick (3D-like) MoSi films would seem to contradict the present theory. Similarly to the 2D case, the conductance \( G_\perp \) for a current perpendicular to \( B \) can be seen as consisting of the four terms;

\[
G_\perp = G_{n,\perp} + G_{fl,\perp} + \delta G_{s,\perp} + G_{vg,\perp},
\]

(4.1)

where the notation follows that in 2D case. Although the conductance defined in the applied field direction is not discussed here, our final conclusion remains valid for the parallel conductance. First, let us list the behaviors of each component in eq.(4.1) near \( T = 0 \). In 3D, the normal part \( G_{n,\perp} \) is safely assumed to be metallic, and the fluctuation correction \( \delta G_{s,\perp} \) is nondivergent in contrast to 2D case and merely a small correction, insensitive to \( B \) and \( T \), to \( G_{n,\perp} \), i.e., \( \delta G_{s,\perp} \sim O(1/E_F\tau) \).

According to ref.14, the AL fluctuation term \( G_{fl,\perp} \) (except \( G_{vg,\perp} \)), as in 2D case, approaches zero in \( T \to 0 \) limit above any transition field (see below). Further, it is easily found as a trivial extension of eq.(2.27) that the VG contribution \( G_{vg,\perp} \) has the form of a scaling function multiplied by \( \xi_{vg}^{-1} \), and thus,

\[
G_{vg}(B_{vg}) \sim \xi_{vg}^{-1}(T, B_{vg}) \sim T^{1/z},
\]

(4.2)

which implies \( G_{vg,\perp}(T \to 0, B_{vg}) = 0 \). Thus, above any transition field and at \( T = 0 \), the total conductance is essentially equivalent to \( G_n \). How about the \( B-D \) phase diagram (Fig.3) at \( T = 0 \),
where $D$ measures the strength of pinning disorder? In this case with dissipative dynamics, the quantum 3D GL model is equivalent to a classical 5D one, and the dimensionality of LLL fluctuation is three. Thus, the ordinary critical line $H_{c2}^*(T = 0)$, signalling the onset of the ordinary long-ranged phase coherence, lies in nonzero fields in this 3D case, while the corresponding field in 2D is zero. Just as in the $B = 0$ thermal transition in a bulk superconductor, the ordinary critical point $H_{c2}^*(0)$ (the left vertical line in Fig.3) is lowered from the mean field $H_{c2}(0)$ (the right vertical line in Fig.3) due to the 3D quantum fluctuation. When the pinning disorder is absent (i.e., $D = 0$), as mentioned previously in §5 of ref.14, a first order vortex solidification transition at $T = 0$ and in 3D should occur above $H_{c2}^*(T = 0)$. Its position, the end point of $B_m(0)$-line, was indicated as an open circle in Fig.3. On the other hand, as found in ref.37, the ordinary superconducting transition in classical 5D case with nonzero $D$ is expected to be of second order. This fact may be useful in understanding the present situation due to the above-mentioned correspondence between the classical 5D and the quantum-dissipated 3D cases. Since the first order solidification at the open circle is also accompanied by the ordinary superconducting ordering, however, both transitions must connect with each other in $B$-$D$ diagram, as described by the chain line. Then, it is straightforwardly concluded that the resistance for any current direction must be zero at least at and below $\text{Max}(B_m(0), H_{c2}^*(0))$: According to the definition of VG ordering (see eq.(2.10)), the presence of the ordinary phase coherence inevitably implies the presence of VG ordering, although a VG ordering can generally occur with no ordinary phase coherence. Namely, if an estimated $B_{vg}(D)$-line lies below $H_{c2}^*(0)$ for some $D$-values, then the resistance for any current direction vanishes not at $B_{vg}$ but already at $H_{c2}^*(0)$. Thus, since there is no region with finite resistance below $H_{c2}^*(0)$, no metallic intermediate phase with much smaller resistance is possible in 3D and at $T = 0$. Although, at large $D$, $B_{vg}$ may lie above $H_{c2}^*(0)$, it is obvious that this conclusion is not affected. The resistance curves apparently residual on cooling, observed in ref.36, are the reflection not of a true $T = 0$ metallic vortex phase but, just like the data in ref.33, merely of a pinning-induced enhancement of the vortex flow conductance in the vortex liquid region becoming narrower in $T \to 0$.

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1) A.F. Hebard and M.A. Paalanen, Phys. Rev. Lett. 65 (1990) 927; M.A. Paalanen, A.F. Hebard, and R.R. Ruel, Phys. Rev. Lett. 69 (1992) 1604.
2) S. Okuma and N. Kokubo, Phys. Rev. B 51 (1995) 15415.
3) V.F. Gantmakher et al., JETP Lett. 71 (2000) 160.
4) S. Okuma, S. Shinozaki, and M. Morita, Phys. Rev. B 63 (2001) 054523.
5) G.T. Seidler, T.F. Rosenbaum, and B.W. Veal, Phys. Rev. B 45 (1992) 10162.
6) M.P.A. Fisher, Phys. Re. Lett. 65 (1990) 923.
7) J.A. Chervenak and J.M. Valles Jr., Phys. Rev. B 61 (2000) R9245.
8) N. Mason and A. Kapitulnik, Phys. Rev. Lett. 82 (1999) 5341.
9) D.S. Fisher, M.P.A. Fisher, and D.A. Huse, Phys. Rev. B 43 (1991) 130.
10) D.R. Nelson and V.M. Vinokur, Phys. Rev. B 48 (1993) 13060.
11) R. Ikeda, J. Phys. Soc. Jpn. 68 (1999) 728.
12) R. Ikeda, J. Phys. Soc. Jpn. 65 (1996) 3998 and ibid 70 (2001) 219.
13) An argument favoring a metallic phase at $T = 0$ and $B = 0$ has recently been criticized, even for granular films, by T.K. Ng and D.K.K. Lee, Phys. Rev. B 63 (2001) 144509.
14) R. Ikeda, Int. J. Mod. Phys. B 10 (1996) 601.
15) M.P.A. Fisher, G. Grinstein, and S.M. Girvin, Phys. Rev. Lett. 64 (1990) 587.
16) K. Damle and S. Sachdev, Phys. Rev. B 56 (1997) 8714.
17) R. Ikeda, J. Phys. Soc. Jpn. 69 (2000) 559.
18) N.R. Werthamer, in Superconductivity, ed. by R.D. Parks (Dekker, NY, 1969).
19) H. Ishida, H. Adachi, and R. Ikeda, to appear in J.Phys.Soc.Jpn..
20) N. Markovic et al., Phys. Rev. B 60 (1999) 4320.
21) R. Ikeda, J. Phys. Soc. Jpn. 64 (1995) 1683.
22) I. D. Lawrie, Phys. Rev. Lett. 79 (1997) 131.
23) M.V. Feigel’man, V.B. Geshkenbein, and A.I. Larkin, Physica (Amsterdam) C 167 (1990) 177.
24) R. Ikeda, T. Ohmi, and T. Tsuneto, J. Phys. Soc. Jpn. 60 (1991) 1051.
25) V.M. Galitski and A.I. Larkin, Phys. Rev. B 63 (2001) 1174506, where no relevance to the FSI issue was mentioned.
26) R. Ikeda, J. Phys. Soc. Jpn. 64 (1995) 3925.
27) R. Ikeda, J. Phys. Soc. Jpn. 66 (1997) 1603.
28) Just like the pinning-free vortex solid, the static VG phase in high fields should be also described within LLL, and the NLL modes cannot participate in its description.
29) A.M. Finkel’stein, Physica (Amsterdam) B 197 (1994) 636.
30) H. Ishida and R. Ikeda, J. Phys. Soc. Jpn. 67 (1998) 983.
31) S.M. Girvin et al., Prog. Theor. Phys. Suppl. 107 (1992) 135.
32) H. Fukuyama, J. Phys. Soc. Jpn. 48 (1980) 2169.
33) P.H. Kes, M.H. Theumissen, and B. Becker, Physica (Amsterdam) C 282-287 (1997) 331.
34) V.F. Gantmakher et al., JETP Letters 68 (1998)363.
35) P. Phillips and D. Dalidovich, cond-mat/0104504.
36) S. Okuma, Y. Imamoto, and M. Morita, Phys. Rev. Lett. 86 (2001) 3136.
37) M.A. Moore and T.J. Newman, Phys. Rev. Lett. 75 (1995) 533.
38) N. Mason and A. Kapitulnik, Phys. Rev. B 64 (2001) 060504. If the first order transition reported in this paper is an intrinsic event, it should be a remnant not of the ordinary liquid-solid transition but of a zero temperature glass-glass transition between the 2D VG phase and the equivalent of Bragg-Bose glass (see T. Giamarchi and P. le Doussal, Phys. Rev. B 55 (1997) 6577 and also Fig.4 in R. Ikeda, J. Phys. Soc. Jpn. 70 (2001) 219 ) for the quantum 2D case, of which the latter is also presumably long range ordered only at $T = 0$. This 2D quantum first order melting may survive at nonzero temperatures, just like the vortex liquid to slush transition\cite{12}, with a critical (termination) point of the transition line in $B > 0$ and $T > 0$.\cite{21}
Figure Caption

Fig.1
Examples of diagrams contributing to $G_{vg}$. The solid curves denote the LLL fluctuation propagators, the chain curves are NLL propagators, the dashed lines are the pinning lines, and the hatched rectangle denotes $\chi_{vg}$.

Fig.2
Details of 2D phase diagram near zero temperature. Here $H_{c2}(0)$ is the mean field upper critical field, $B_m(0)$ the $T = 0$ vortex-solidification field estimated by neglecting the presence of the critical field $B_{vg}$, the chain line denotes $T_{rep}$, and the lower (upper) dashed straight line denotes $T_{cr}^m(B)$ ($T_{cr}(B)$). The quantum VG (QVG) critical regime is the $T$-dependent field range defined by eq.(2.15) around the apparent VG critical field $B_{vg}^*$. Details of the true zero temperature limit below $T_{rep}$ are not drawn.

Fig.3
Conjectured 3D $B$ v.s. $D$ (pinning disorder) phase diagram at $T = 0$. The chain curve $B_m(0)$ and the open circle imply the first order transition accompanied by a melting of a vortex solid, the left vertical line is the ordinary superconducting transition line $H_{c2}^*(0)$, the right vertical line the mean field $H_{c2}$ line, and the curved portion in large $D$ of the solid line indicates $B_{vg}$ occurring above $H_{c2}^*(0)$. The solid lines imply second order superconducting transition. As explained in the text, the VG susceptibility is divergent everywhere below $B_M \equiv \text{Max}(H_{c2}^*(0), B_{vg}, B_m(0))$ and hence, the resistance $R_\perp$ is zero in $B < B_M$, while $R_\perp \simeq G_n^{-1}$ in $B > B_M$. 
\[ \begin{align*}
H_{c2}(0) & \quad T_{\text{rep}}(B) \\
B_{v}^{*} & \quad \text{QVG critical regime} \\
B_{vg} & \quad T_{\text{cr}}(B) \\
B_{m}(0) & \quad T_{\text{cr}}^{\text{mf}}(B)
\end{align*} \]
