A new operational matrix based on Boubaker polynomials for solving fractional Emden-Fowler problem

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Abstract: In this paper the singular Emden-Fowler equation of fractional order is introduced and a computational method is proposed for its numerical solution. For the approximation of the solutions we have used Boubaker polynomials and defined the formulation for its fractional derivative operational matrix. This tool was not used yet, however, this area has not found many practical applications yet, and here introduced for the first time. The operational matrix of the Caputo fractional derivative tool converts these problems to a system of algebraic equations whose solutions are simple and easy to compute. Numerical examples are examined to prove the validity and the effectiveness of the proposed method to find approximate and precise solutions.

Keywords: Boubaker Polynomials; Operational matrix of fractional derivatives; Collocation method; Fractional Emden-Fowler Type Equations.

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1 Introduction

In mathematical physics and nonlinear mechanics there exists sufficiently large number of particular basic singular fractional differential equations for which an exact analytic solution in terms of known functions did not exist (Podlubny I. (1999); Kilbas A et al.
One of these equations describing many phenomena in mathematical physics and astrophysics such as, the thermal behaviour of a spherical cloud of gas, isothermal gas sphere and theory of stellar structure, theory of thermionic currents among many others, is called the singular Emden-Fowler equations of fractional order formulated as: Syam, M. (2018); Syam et al. (2018); Huan et al. (2017); Rebenda and Smarda. (1978)

\[
D^{2\alpha} u(x) + \frac{\lambda}{x^\alpha} D^{\alpha} u(x) + s(x) g(u(x)) = h(x), \quad x \in (0, 1), \quad \lambda > 0, \quad \frac{1}{2} < \alpha \leq 1
\]  

subject to the conditions:

\[
u(0) = a, \quad D^{\alpha} u(0) = b,
\]

where \(a\) and \(b\) are constants. When \(\alpha = 1, \lambda = 2,\) and \(h(x) = 1,\) Eq.(1) becomes the Lane-Emden type equation.

\[D^{\alpha} \text{denote the Caputo fractional derivatives. It is generally defined as follows :}
\]

\[
D^{\alpha} u(x) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{x} \frac{u^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \quad n-1 < \alpha < n, n \in \mathbb{N}, \alpha > 0
\]

For the Caputo derivative we have \(D^{\alpha} C = 0,\) where \(C\) is a constant, and

\[
D^{\alpha} x^\beta = \begin{cases} 
0, & \text{for } \beta \in \mathbb{N} \cup \{0\} \text{ and } \beta < [\alpha] \\
\frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)} x^{\beta-\alpha}, & \text{for } \beta \in \mathbb{N} \cup \{0\} \text{ and } \beta \geq [\alpha] \text{ or } \beta \notin \mathbb{N} \text{ and } \beta > [\alpha]
\end{cases}
\]

where, \([\alpha]\) denotes the integer part of \(\alpha,\) that is the largest integer less or equal than \(\alpha,\) or the smallest integer greater than or equal to \(\alpha.\)

The problem (1) was studied by using the Residual Power Series Method by Syam, M. (2018), Homotopy analysis method (HAM) by Huan et al. (2017), Reproducing kernel Hilbert space method by Syam et al. (2018), The fractional differential transformation (FDT) Rebenda and Smarda. (1978), Polynomial Least Squares Method by Caruntu et al. (2019), Shifted Legendre Operational Matrix by Tripathi N. (2019), Chebyshev wavelets by Kazemi Nasab et al. (2018), Orthonormal Bernoulli’s polynomials by Sahu and Mallick. (2019), Orthonormal Bernstein polynomials by Abbas et al. (2019). For the solution of the classic Emden-Fowler equations (Case \(\alpha = 1\)), there are many Studies of analytical as well as numerical methods is provided in monographs by Chandrasekhar S. (1979); Chowdhury et al. (2009); Shang et al. (2009); Wazwaz A.M. (2005); Wazwaz Abdul-M. (2005); Yousefi S.A. (2007); Bencheikh et al. (2017).

The purpose of this paper is to use Boubaker operational matrix of fractional order for solving a singular initial value problems of fractional Emden-Fowler type equations (1). To the best of our knowledge this is the first time that the Boubaker operational matrices are used to obtain solutions of singular Emden-Fowler equations of fractional order. First we present a new theorem which can reduce the fractional Emden-Fowler problem to a system of algebraic equations. The Boubaker polynomials were established for the first time by Boubaker (2007), to solve heat equation inside a physical model. The first monomial definition of the Boubaker polynomials on interval \(x \in [0, 1],\) was introduced by Boubaker K. (2007, 2008); Labiadh and Boubaker. (2007); Bolandtalat et al. (2016); Rabiei et al. (2017); Davaeifar and Rashidinia. (2017):

\[
B_0(x) = 1, \quad B_n(x) = \sum_{p=0}^{\xi(n)} \binom{n-4p}{n-p} C_{n-p}^p (-1)^p x^{n-2p}, \quad n \geq 1,
\]
Operational matrix for fractional Emden-Fowler problem.

where \(\xi(n) = \left\lfloor \frac{2n + ((-1)^n - 1)}{4} \right\rfloor\) and \(C_{n-r}^r = \frac{(n-p)!}{r!(n-2p)!}\). The symbol \(\lfloor \cdot \rfloor\) denotes the floor function. The Boubaker polynomials could be calculated by following recursive formula:

\[
\mathbf{B}_m(x) = x\mathbf{B}_{m-1}(x) - \mathbf{B}_{m-2}(x), \quad m \geq 2. \tag{4}
\]

We will construct operational matrix of Caputo fractional derivatives \(D^{(\alpha)}\) for the Boubaker polynomials which are given by

\[
D^{(\alpha)} \mathbf{B}(x) \simeq D^{(\alpha)} \mathbf{B}(x), \tag{5}
\]

where \(\mathbf{B}(x) = [B_0(x), B_1(x), \ldots, B_N(x)]^T\) be Boubaker vector and the matrice \(D^{(\alpha)}\) are of order \((N + 1) \times (N + 1)\). In order to show the high performance of Boubaker operational matrix of fractional derivative, we apply it to solve equation (1).

The paper is organized as follows. In Section (2), we express Boubaker polynomials in terms of Taylor basis, and function approximation. In Section (3) The operational matrix of Caputo fractional derivatives is constructed. In Section (4), we use Boubaker polynomials method for solving fractional Emden-Fowler type equations. Section (5) illustrates some numerical examples to show the accuracy of this method. Finally, Section (6) concludes the paper.

2 Boubaker’s matrix and approximation of function

By using the expression (3) and taking \(n = 0, \ldots, N\), we can express Boubaker polynomials in terms of Taylor basis (Bolandtalat et al. (2016); Rabiei et al. (2017); Davaeifar and Rashidinia. (2017))

\[
\mathbf{B}(x) = \mathbf{M} \mathbf{T}(x), \quad x \in [0, 1], \tag{6}
\]

where

\[
\mathbf{T}(x) = [1, x, \ldots, x^N]^T, \tag{7}
\]

and if \(N\) is odd,

\[
\mathbf{M} =
\begin{bmatrix}
  m_{0,0} & 0 & 0 & 0 & \cdots & 0 & 0 \\
  0 & m_{1,0} & 0 & 0 & \cdots & 0 & 0 \\
  m_{2,1} & 0 & m_{2,0} & 0 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  m_{N-1, \frac{N-1}{2}} & 0 & m_{N-1, \frac{N-3}{2}} & 0 & \cdots & m_{N-1, 0} & 0 \\
  0 & m_{N, \frac{N-1}{2}} & 0 & m_{N, \frac{N-3}{2}} & \cdots & 0 & m_{N, 0}
\end{bmatrix}
\]

if \(N\) is even,

\[
\mathbf{M} =
\begin{bmatrix}
  m_{0,0} & 0 & 0 & 0 & \cdots & 0 & 0 \\
  0 & m_{1,0} & 0 & 0 & \cdots & 0 & 0 \\
  m_{2,1} & 0 & m_{2,0} & 0 & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & m_{N-1, \frac{N-1}{2}} & 0 & m_{N-1, \frac{N-3}{2}} & \cdots & m_{N-1, 0} & 0 \\
  m_{N, \frac{N}{2}} & 0 & m_{N, \frac{N-2}{2}} & \cdots & 0 & m_{N, 0}
\end{bmatrix}
\]
where
\[
B_n(x) = \sum_{p=0}^{\frac{n}{2}} m_{n,p} x^{n-2p}, \quad n = 0, 1, ..., N, \quad p = 0, 1, ..., \left\lfloor \frac{n}{2} \right\rfloor,
\]
(8)
\[
m_{n,p} = \frac{(n-4p)}{(n-p)} C_{n-p}^p (-1)^p.
\]
(9)

As can be observed, \( M \) denotes a lower triangular matrix and \( |M| = \prod_{i=0}^{N} m_{i,0} = 1 \) thus it is non-singular and \( M^{-1} \) exists.

We consider the set of Boubaker polynomials of \( N \)th degree as
\[
B(x) = [B_0(x), B_1(x), ..., B_N(x)]^T \subset L^2[0,1],
\]
(10)
and assume that \( S_N = \text{span} \{B_0(x), B_1(x), ..., B_N(x)\} \). Because \( S_N \) is a finite dimensional vector space, if \( u \) is an arbitrary function in \( L^2[0,1] \), then \( u \) has the best approximation out of \( S_N \) such as \( u_N \in S_N \), that is Kreyszig E. (1978)

\[
\forall y \in S_N, \|u - u_N\|_2 \leq \|u - y\|_2.
\]
(11)

Since \( u_N \in S_N \), there exist unique coefficients \( c_i, i = 0, 1, ..., N \) such that
\[
u (x) \simeq u_N (x) = \sum_{i=0}^{N} c_i B_n (x) = C^T B (x),
\]
(12)
where \( C \) is an \((N+1) \times 1\) vector given by \( C = [c_0, c_1, ..., c_N]^T \), \( B(x) \) is the vector function defined in Eq.(10), and coefficients vector \( C \) can be computed by \( C^T < B(x), B(x) >= < u(x), B(x) > \), such that
\[
< u(x), B(x) > = \int_0^1 u(x) B^T (x) dx,
\]
(13)
and \( < ..., > \) denotes the standard inner product on \( L^2[0,1] \). Thus, by definition \( Q = < B(x), B(x) > \), we get
\[
C^T = (\int_0^1 u(x) B^T (x) dx) Q^{-1},
\]
(14)
where \( Q \) is the following \((N+1) \times (N+1)\) matrix :
\[
Q = < B(x), B(x) > = \int_0^1 B(x) B^T (x) dx = M \left( \int_0^1 T(x) T^T (x) dx \right) M^T
\]
\[
= MHM^T.
\]
Where; \( H = [h_{ij}]_{(N+1) \times (N+1)} \) is the well-known Hilbert matrix, the components of which can be computed as follows:
\[
h_{ij} = \frac{1}{i+j+1}, \quad i, j = 1, ..., N.
\]
(15)
Operational matrix for fractional Emden-Fowler problem.

**Theorem 1:** (Kreyszig E. (1978); Davaefar and Rashidinia. (2017)) Elements $B_0, B_1, \ldots, B_N$ of a Hilbert space $L^2[0,1]$ constitute a linearly independent set in $L^2[0,1]$ if and only if

$$G(B_0, B_1, \ldots, B_N) = \begin{vmatrix} \langle B_0, B_0 \rangle & \langle B_0, B_1 \rangle & \cdots & \langle B_0, B_N \rangle \\ \langle B_1, B_0 \rangle & \langle B_1, B_1 \rangle & \cdots & \langle B_1, B_N \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle B_N, B_0 \rangle & \langle B_N, B_1 \rangle & \cdots & \langle B_N, B_N \rangle \end{vmatrix} \neq 0$$

produces that $Q$ is symmetric and also non-singular, so $Q^{-1}$ exists.

**Lemma 2:** (Kreyszig E. (1978); Rabiei et al. (2017)) Suppose that $u \in C^N[0,1]$ and $S_N = \text{span} \{ B_N(x), B_N(x), \ldots, B_N(x) \}$ Let $u_0$ be the best approximation for $u$ out of $S_N$ then

$$\| u(x) - u_0(x) \|_{L^2[0,1]} \leq \frac{\text{Max}_{x \in [0,1]}|u^{(N+1)}(x)|}{(N+1)\sqrt{2N+3}}$$

(16)

.: For proof, see Kreyszig E. (1978); Rabiei et al. (2017).

**Theorem 3:** (Kreyszig E. (1978); Rabiei et al. (2017)) Suppose that $u \in L^2[0,1]$ and $u(x)$ is approximated by $\sum_{i=0}^{N} c_i B_i(x)$, then we have

$$\lim_{N \to \infty} \left\| u(x) - \sum_{i=0}^{N} c_i B_i(x) \right\|_{L^2[0,1]} = 0$$

(17)

3 The Boubaker operational matrix of fractional derivative

The main objective of this section is to derive the operational matrix of Caputo fractional derivatives based on the Boubaker polynomials

For a vector $B(x)$, we can approximate the operational matrices of fractional order integration as Rabiei et al. (2017):

$$D^\alpha B(x) \simeq D^{(\alpha)}B(x)$$

(18)

where $D^{(\alpha)}$ is the $(N + 1) \times (N + 1)$ Caputo fractional operational matrix of integration for Boubaker polynomials. We compute $D^{(\alpha)}$ as follows:

$$D^\alpha B(x) \simeq MD^{(\alpha)}T(x) = MZ\bar{X}(x)$$

(19)

where the matrix $Z_{(N+1)\times(N+1)}$ is given by

$$Z = (Z_{i,j}) = \begin{cases} \frac{\Gamma(j+1)}{(j+1)^\alpha}, & i = j = \lceil \alpha \rceil, \ldots, N \\ 0, & \text{otherwise} \end{cases}$$

(20)
and \( \bar{X} = \bar{X}_{i+1}(N+1) \times (1) \), with
\[
\bar{X}_{i+1} = \begin{cases} 
  x^{i-\alpha}, & i = [\alpha],...,N \\
  0, & i = 0,1,...,[\alpha] - 1
\end{cases}
\] (21)

Now, \( \bar{X} \) is expanded in terms of Boubaker polynomials as
\[
\bar{X} = E^T B(x)
\] (22)

where \( E = [e_0, e_1, ..., e_m] \) and \( e_i = Q^{-1} \hat{E}_i \hat{E}_i = [\hat{e}_{i,0}, \hat{e}_{i,1}, ..., \hat{e}_{i,m}]^T \). The entries of the vector \( \hat{E}_i \) can be calculated as
\[
\hat{e}_{i,j} = \left( \int_0^1 x^{i-\alpha} B_j(x) dx \right) Q^{-1}
\] (23)

then we have
\[
D^{\alpha} B(x) \simeq D^{(\alpha)} B(x), D^{(\alpha)} = MZE^T
\] (24)

\( D^{(\alpha)} \) is the operational matrix of the Caputo fractional derivative.

### 4 Solution of singular Fractional Emden-Fowler problem

This section presents the derivation of the method for solving a singular initial value problems of fractional Emden Fowler type equations.

Let us consider the fractional Emden-Fowler equation of the form
\[
D^{2\alpha} u(x) + \frac{\lambda}{x^\alpha} D^{\alpha} u(x) + s(x) g(u(x)) = h(x), \quad x \in (0, 1), \quad \lambda > 0, \quad \frac{1}{2} < \alpha \leq 1
\] (25)

with initial conditions:
\[
u(0) = a, \quad D^{\alpha} u(0) = b
\] (26)

Approximating \( u(x), s(x) g(u(x)) \) by the Boubaker polynomials as
\[
u(x) = \sum_{i=0}^{m} c_i B_i(x) = C^T B(x), \quad s(x) g(u(x)) = s(x) g(C^T B(x))
\] (27)

where the unknowns are, \( C = [c_0, c_1, ..., c_m]^T \). Using operational matrix of fractional derivative, Eq. (24) can be written as
\[
C^T D^{(2\alpha)} B(x) + \frac{\lambda}{x^\alpha} C^T D^{(\alpha)} B(x) + s(x) g(C^T B(x)) = h(x)
\] (28)
Collocating Eq.(28) at \( m - 1 \) collocation points leads to

\[
C^T D^{(2\alpha)} B (x_i) + \frac{\lambda}{x^\alpha} C^T D^{(\alpha)} B (x_i) + s (x_i) g (C^T B (x_i)) = h (x_i). \tag{29}
\]

A set of suitable collocation points is defined as follows:

\[
x_i = \frac{1}{2} \left( \cos \left( \frac{i\pi}{n} \right) + 1 \right), i = 0, \ldots, m - 1 \tag{30}
\]

In addition, the initial conditions (26) provide two algebraic equations as

\[
C^T B (0) = a, \quad C^T D^{(\alpha)} B (0) = 0 \tag{31}
\]

Finally, we can compute the values for the components of \( C \) by solving the system of eq. (29) and (31). Hence, the approximate solution for \( u(x) \) can be computed by using Eq. (12).

5 Numerical examples

In this section, we applied the method presented in Section (4) to solve fractional Emden-Fowler Equation. We have done all the numerical computations with a computer program Matlab

**Example 1.** We consider the following fractional Emden-Fowler equations:

\[
D^{2\alpha} u(x) + \frac{2}{x^\alpha} D^{\alpha} u(x) + u(x)^n = 0, \tag{32}
\]

subject to the conditions: \( u(0) = 1, \quad D^{\alpha} u(0) = 0 \)

1. For \( \alpha = 1 \) and \( n = 0 \), Eq. (32) has as The exact solution for this problem is

\[
u(x) = 1 - \frac{1}{3!} x^2
\]

By applying this method, and taking \( m = 2 \), we find

\[
D^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \quad D^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{4}{3} \\ 0 \\ \frac{1}{6} \end{bmatrix}
\]

Hence, the solution is

\[
u(x) = C^T B (x) = \begin{bmatrix} \frac{4}{3} \\ 0, \frac{1}{6} \end{bmatrix} \left( \frac{1}{x^2 + 2} \right) = 1 - \frac{1}{3!} x^2
\]

which is the exact solution.
2. For $\alpha = 1$ and $n = 1$, Eq. (32) has as the exact solution

$$u(t) = \frac{\sin x}{x}$$

We solved the above problem, by applying the technique described in Section (4) with $m = 3$, we approximate solution as

$$u(x) = c_0B_0(x) + c_1B_1(x) + c_2B_2(x) + c_3B_3(x) = C^T B(x)$$

Here, we have

$$D^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \quad D^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

we find the following system

$$c_0 + 2c_2 = 1, \quad c_1 + c_3 = 0$$

$$c_0 + \frac{41}{12}c_1 + \frac{137}{16}c_2 + \frac{2465}{192}c_3 = 0, \quad c_0 + \frac{33}{4}c_1 + \frac{129}{16}c_2 + \frac{721}{64}c_3 = 0$$

which has the solution:

$$\{c_0 = \frac{25673}{19113}, \quad c_1 = -\frac{256}{19113}, \quad c_2 = -\frac{3280}{19113}, \quad c_3 = \frac{256}{19113}\}$$

So, in this case the approximate of $u_3(x)$ is

$$u_3(x) = \frac{25673}{19113}x - \frac{256}{19113}x^2 + \frac{3280}{19113}x^3 + 1$$

Table (1) shows the absolute errors between the approximate solutions obtained for values of $m = 3$, and $m = 6$ using the operational matrix of Boubaker polynomials and the exact solutions. It can be seen from table (1) that the solutions obtained by the present method is nearly identical with the exact solutions. Clearly, increasing more higher the values of $m$ leads to highly accurate results.

| Table 1 | Absolute error for different values of $m$ for $\alpha = 1$ |
|---------|-----------------------------------------------|
| $x$  | $0.1$ | $0.3$ | $0.5$ | $0.7$ | $0.9$ |
| $m = 3$ | 3.688E-5 | 1.507E-4 | 7.956E-5 | 1.938E-4 | 3.961E-4 |
| $m = 6$ | 3.389E-8 | 3.051E-7 | 8.479E-8 | 1.671E-7 | 2.961E-7 |
Operational matrix for fractional Emden-Fowler problem.

**Example 2.** We consider the following fractional Emden-Fowler equations (Syam, M. (2018)):

\[ D^{2\alpha}u(x) + \frac{1}{x^{\alpha}}D^{\alpha}u(x) + (1 + x^{\alpha})(u(x)) = h(x) \]

subject to the conditions:

\[ u(0) = 3, \quad D^{\alpha} u(0) = 0 \]

where

\[ h(x) = \Gamma (1 + 2\alpha) + \frac{\Gamma (1 + 2\alpha)}{\Gamma (1 + \alpha)} + (1 + x^{\alpha})(3 + x^{2\alpha}), \]

The exact solution is \( u(x) = 3 + x^{2\alpha} \)

Applying the method developed in Sections (3) and (4) for \( m = 2, \alpha = 1 \) we have:

\[ u(x) = c_0 B_0(x) + c_1 B_1(x) + c_2 B_2(x) = C^T B(x) \]

Therefore using Eq. (29) we obtain: \( 1.5c_0 + 2.75c_1 + 7.375c_2 = 8.875 \)

Now, by applying Eq.(29) we have: \( c_0 + 2c_2 = 3 \) and \( c_1 = 0 \)

Finally, we get \( c_0 = 1, c_1 = 0 \) and \( c_2 = 1 \)

Thus we can write

\[ u(x) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 + 2 \end{bmatrix} = 3 + x^2 \]

which is the exact solution.

Table (2) shows the absolute errors between the approximate solutions obtained for values of \( \alpha = 0.7, 0.8, \) and \( \alpha = 1 \) and the exact solutions.

**Table 2** Absolute error for different values of \( \alpha \) for \( m = 2 \)

| \( x \) | \( \alpha = 1 \) | \( \alpha = 0.85 \) | \( \alpha = 0.75 \) |
|-------|------------------|------------------|------------------|
|       | 0.1              | 0.3              | 0.5              | 0.7              | 0.9              |
| \( \alpha = 1 \) | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| \( \alpha = 0.85 \) | 6.760 7 \times 10^{-2} | 1.075 9 \times 10^{-2} | 8.371 2 \times 10^{-3} | 6.982 7 \times 10^{-3} | 1.187 0 \times 10^{-3} |
| \( \alpha = 0.75 \) | 9.703 2 \times 10^{-2} | 1.026 4 \times 10^{-2} | 5.164 3 \times 10^{-3} | 4.989 1 \times 10^{-3} | 1.992 4 \times 10^{-3} |

**Example 3.** We consider the following fractional Emden-Fowler equations:

\[ D^{2\alpha} u(x) + \frac{\lambda}{x^{\alpha}}D^{\alpha} u(x) - 2(2x^2 + 3) u(x) = h(x) \]

subject to the conditions: \( u(0) = 1, \quad D^{\alpha} u(0) = 0 \), with \( \alpha = 1, \lambda = 2 \) and \( h(x) = 0 \) . Eq (33) has as the exact solution Bencheikh et al. (2017) \( u(t) = \exp(x^2) \)

In fig(1), we plotted the exact solution and the approximate solutions of \( u(x) \) for \( m = 4 \) and 6. Definitely, by increasing the value of \( m \), the approximate value of \( u(x) \) will close to the exact values. and fig(2) present the absolute error in this case.

**Example 4.** We consider the following fractional Emden-Fowler equations (Syam, M. (2018)):

\[ D^{2\alpha} u(x) + \frac{1}{x^{\alpha}}D^{\alpha} u(x) - 9u(x) = h(x), \quad x \in (0, 1), \quad \frac{1}{2} < \alpha \leq 1 \]
subject to the boundary conditions

\[ u(0) = 2, \quad D^\alpha u(0) = 0 \]

where

\[ h(x) = -9 + \frac{\Gamma(1 + 2\alpha)}{\Gamma(1 + \alpha)} + \Gamma(1 + 2\alpha) + \left( \frac{\Gamma(1 + 3\alpha)}{\Gamma(1 + \alpha)} + \frac{\Gamma(1 + 3\alpha)}{\Gamma(1 + 2\alpha)} \right)x^{\alpha} - 9x^{2\alpha} - 9x^{3\alpha} \]

The exact solution is

\[ u(x) = 1 + x^{2\alpha} + x^{3\alpha} \]

For \( \alpha = 0.7, 0.8, 1 \) and \( m = 4 \). The operational matrix for \( D^{(\alpha)} \) for various \( \alpha \) are given as

\[
D^{(0.7)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
7.5067 & -5.1084 & -7.0969 & 8.2458 & -3.4888 & 0 \\
-0.7585 & -2.489 & -1.8287 & 4.1118 & 2.1815 & 0 \\
3.8518 & -5.5229 & -5.0826 & 8.4712 & -3.3057 & 0 \\
1.8646 & -2.3313 & -0.3187 & 2.3752 & 0.6146 & 0
\end{bmatrix}
\]

\[
D^{(1.4)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
4.7386 & 7.3096 & 0.5640 & -4.3120 & 2.8203 & 0 \\
-7.5585 & 0.9871 & 3.2963 & 0.2115 & -0.4759 & 0 \\
-3.8934 & -10.541 & -1.8169 & 11.281 & -3.7423 & 0
\end{bmatrix}
\]

\[
D^{(0.8)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
6.1986 & -3.3934 & -5.2443 & 5.8093 & -2.3809 & 0 \\
-2.6616 & 3.8043 & 2.6493 & -2.9733 & 1.3151 & 0 \\
1.7557 & -3.6652 & -2.8636 & 5.9132 & -2.2249 & 0 \\
2.3933 & -4.6647 & -1.7933 & 5.0768 & -0.5847 & 0
\end{bmatrix}
\]
Operational matrix for fractional Emden-Fowler problem.

\[
\mathbf{D}^{(1,6)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
13.216 & -6.4671 & -11.403 & 12.352 & -4.9845 & \\
-8.5829 & 6.6068 & 6.3194 & -5.0447 & 2.0389 & \\
-10.082 & -5.9541 & 3.6179 & 5.9058 & -1.4201 & \\
\end{bmatrix}
\]

\[
\mathbf{D}^{(1)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 \\
-5 & 0 & 3 & 0 & 0 & 0 \\
0 & -4 & 0 & 4 & 0 & 0 \\
\end{bmatrix}, \quad \mathbf{D}^{(2)} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 0 & 0 \\
0 & 6 & 0 & 0 & 0 & 0 \\
-24 & 0 & 12 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Eq. (5) with the initial condition has been solved with the proposed method and the values of the unknown matrix \( \mathbf{C}^T \) are obtained and listed in Table (3). The values of absolute errors obtained are shown in Table (4).

Table 3 Values of unknowns for \( m = 4 \) and for different values of \( \alpha \)

| Unknowns | \( c_0 \) | \( c_1 \) | \( c_2 \) | \( c_3 \) | \( c_4 \) |
|----------|--------|--------|--------|--------|--------|
| \( \alpha = 0.7 \) | -2.2706 | 0.5755 | 1.6168 | -7.5937 \times 10^{-2} | 1.8568 \times 10^{-2} |
| \( \alpha = 0.8 \) | -2.2895 | 7.9481 \times 10^{-2} | 1.6352 | 0.14592 | 9.5450 \times 10^{-3} |
| \( \alpha = 1 \) | -1.0004 | -0.99965 | 1.0002 | 0.99965 | 2.2216 \times 10^{-3} |

Table 4 Absolute errors for \( m = 5 \) and for different values of \( \alpha \)

| \( x \) | \( \alpha = 0.7 \) | \( \alpha = 0.8 \) | \( \alpha = 1 \) |
|-------|--------|--------|--------|
| 0.1   | 5.5839 \times 10^{-2} | 2.8251 \times 10^{-2} | 3.8348 \times 10^{-3} |
| 0.2   | 4.9238 \times 10^{-2} | 2.3678 \times 10^{-2} | 3.4764 \times 10^{-3} |
| 0.3   | 4.5776 \times 10^{-2} | 2.0665 \times 10^{-2} | 3.1270 \times 10^{-3} |
| 0.4   | 4.3242 \times 10^{-2} | 1.8550 \times 10^{-2} | 2.9831 \times 10^{-3} |
| 0.5   | 4.0654 \times 10^{-2} | 1.7195 \times 10^{-2} | 3.2362 \times 10^{-3} |
| 0.6   | 3.7510 \times 10^{-2} | 1.6605 \times 10^{-2} | 4.0721 \times 10^{-3} |
| 0.7   | 3.3528 \times 10^{-2} | 1.6810 \times 10^{-2} | 5.6716 \times 10^{-3} |
| 0.8   | 2.8539 \times 10^{-2} | 1.7824 \times 10^{-2} | 8.2100 \times 10^{-3} |
| 0.9   | 2.2428 \times 10^{-2} | 1.9619 \times 10^{-2} | 1.1857 \times 10^{-3} |
| 1.0   | 1.5099 \times 10^{-2} | 2.2125 \times 10^{-2} | 1.6778 \times 10^{-3} |

6 Conclusions

In this work, we get operational matrices of the fractional derivative by Boubaker polynomials. Then by using these matrices, we reduced the singular fractional Emden-Fowler type equations to a system of algebraic equations that can be solved easily. A
Figure 3 The graph of $u(x)$ with $m = 4$ and $\alpha = 0.7, 0.8, 1$.

Numerical example is included to demonstrate the validity and applicability of this method, and the results reveal that the method is very effective, straightforward, simple, and it can be applied by developing for the other related fractional problems, such as such partial fractional differential and integro-differential equations. This will be investigated in a future work.

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Constrained feedback RMPC for LPV systems with bounded rates of parameter variations and measurement errors

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Abstract: For Linear Parameter Varying (LPV) systems with bounded rates of parameter variations and bounded parameter measurement errors, a feedback Robust Model Predictive Control (RMPC) is designed by utilising the information on system parameters. A sequence of feedback control laws is designed based on the model with parameter-incremental uncertainty. Since the sequence of feedback control laws corresponds to the future variations of system parameters and introduces additional freedom, the control performance of RMPC can be improved. The recursive feasibility and closed-loop stability of the proposed RMPC are also proven.

Keywords: feedback RMPC; LPV systems; bounded rates; parameter variations; measurement errors.

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1 Introduction

Due to the capability of handling constraints explicitly, Model Predictive Control (MPC), also known as Receding Horizon Control (RHC), has become a popular technique for industrial process control and attracts much attention, especially robust MPC, such as Kothare et al. (1996) and Li et al. (2009). In some practical applications, the system parameters of LPV systems are often online measurable or vary with known bounded rates. For LPV systems with bounded rates of parameter variations, if the available information on the system parameters can be taken into account during controller design, the control performance is expected to be improved. Considering the measurable parameters of LPV systems, Lu and Arkun (2000) proposed a quasi-min-max MPC algorithm. For LPV systems with bounded rates of parameter variations and the parameters restricted into the unit simplex, Casavola et al. (2002) developed a feedback Min-Max MPC algorithm. But there is a major problem in its initialisation stage, which is pointed out by Ding and Huang (2007).

For LPV systems with independently varying parameters, Park and Jeong (2004) transformed the system into a system with ‘parameter-incremental’ uncertainties. Then, by applying the open-loop dual-mode control, i.e., some free control moves followed by a feedback control law, an RMPC algorithm is proposed. But due to the uncertainty of systems, the recursive feasibility of the controller proposed in Park and Jeong (2004) cannot be guaranteed, which directly results in that the closed-loop stability would not be guaranteed.

In practical applications, the parameter measurement error is another issue which must be considered. Therefore, this paper considers the RMPC of LPV systems with both independently varying parameters and parameter measurement errors. In terms of the measurement errors, the error bounds are used to calculate the possible areas where the parameters could belong to in the future, and these areas can be tackled with the parameter variations together. Then the dynamic system model is converted into a sequence of future models with parameter-incremental uncertainty by referring to Park and Jeong (2004), which includes not only the time-varying parameter variations but also the measurement errors. Corresponding to the model sequence, the proposed RMPC adopts a sequence of feedback control laws, instead of open-loop control strategy. The recursive feasibility and closed-loop stability can be guaranteed. Meanwhile, since the feedback control laws are designed according to the future parameter variations, the information on the parameter variations and measurement errors can be utilised in the MPC controller and then better control performance can be expected.

This paper is organised as follows: Section 2 introduces the problem and the issue about the recursive feasibility of RMPC. The feedback RMPC will be introduced with a modified model sequence with parameter-incremental uncertainties in detail in Section 3. Numerical example is given in Section 4 to verify the results proposed in this paper.
Constrained feedback RMPC for LPV systems

Notation: Denote \( u(k+i|k) \) and \( x(k+i|k) \) as the control input and system state of time \( k+i \), predicted at time \( k \). \( ||x||^2_Q = x^T Q x \), \( x(k|k) = x(k) \). The symbol * induces a symmetric structure, e.g., when \( L \) and \( R \) are symmetric matrices,

\[
\begin{bmatrix}
L & * \\
N & R
\end{bmatrix} =
\begin{bmatrix}
L & N^T \\
N & R
\end{bmatrix}.
\]

2 Background

Consider the discrete-time LPV system

\[
x(k+1) = A(\theta(k))x(k) + B(\theta(k))u(k)
\]

where \( x(k) \in \mathbb{R}^n \), \( u(k) \in \mathbb{R}^{n_u} \) and \( \theta(k) = \{ \theta_1(k), \theta_2(k), \ldots, \theta_L(k) \} \) are the system state, control input and parameter vector, respectively. The parameter vector \( \theta(k) \) is assumed measurable with measurement error \( \sigma \) at time \( k \). The measured values of \( \theta_i(k) \) is denoted as \( \hat{\theta}_i(k) \). Moreover, the real values, measured values and the changes of parameters satisfy the following constraints

\[
\theta_j \in \Sigma_j = \begin{bmatrix} \hat{\theta}_j & \hat{\theta}_j \end{bmatrix},
\]

\[
\Delta \theta_j(k) = \theta_j(k+1) - \theta_j(k) \in \delta_j = \begin{bmatrix} \delta_j & \hat{\delta}_j \end{bmatrix},
\]

\[
|\hat{\theta}_j(k) - \theta_j(k)| \leq \sigma_j.
\]

System (1) is subjected to the input constraints:

\[
|u_j(k)| \leq u_{j,\text{max}}, \quad j = 1, \ldots, m.
\]

At each time, the RMPC will calculate the control moves \( u(k+i|k) \) by optimising the following optimisation problem

\[
\min_{U(k)} \max_{\theta_j \in \Sigma_j, \Delta \theta_j \in \delta_j, j=1,\ldots,L} J_\infty(k) \text{ s.t. (1) - (5)}
\]

where \( J_\infty(k) = \sum_{i=0}^{\infty} \left[ ||x(k+i|k)||^2_{Q_1} + ||u(k+i|k)||^2_{R} \right] \), \( Q_1 \geq 0 \) and \( R \geq 0 \) are weighting matrices.

Remark 1: For LPV systems (1)--(5), although the approach in Kothare et al. (1996) can be directly used to design RMPC, the information on system parameters (2)--(3) is ignored, which may lead to poor performance. In order to improve the control performance, for LPV systems (1)--(5) without parameter measurement errors, Park and Jeong (2004) makes use of the information on parameters to design RMPC by tackling the uncertainty of LPV systems as parameter-incremental uncertainty. However, the open-loop strategy \( U(k) = \{ u(k|k), u(k+1|k), \ldots, u(k+N|k) \} \) is adopted by Park and Jeong (2004) which is similar to Wan and Kothare (2003). As pointed out by Pluemers et al. (2005), the proof about recursive feasibility of RMPC in Wan and Kothare (2003) is not correct due to the uncertainty of system. The same situation happens in Park and Jeong (2004) when \( N > 2 \).
For systems (1)–(3), Park and Jeong (2004) suggested the following method to form a system with parameter-incremental uncertainties to make use of the property of parameters. Since the parameter $\theta_j(k)$ can be measured at sample time and the bound rates of parameters variations are also available, the range of $\theta_j(k+i|k)$ can be computed and described as:

$$\theta_j(k+i|k) \in [\max(\theta_j(k) + i \times \delta_j, \bar{\theta}_j), \min(\bar{\theta}_j, \theta_j(k) + i \times \bar{\delta}_j)].$$

And then the following is defined in Park and Jeong (2004).

$$\mu_j(k+i|k) \triangleq \frac{1}{2} \left[ \min(\bar{\theta}_j, \theta_j(k) + i \times \bar{\delta}_j) + \max(\theta_j(k) + i \times \delta_j, \bar{\theta}_j) \right],$$

$$\rho_j(k+i|k) \triangleq \frac{1}{2} \left[ \min(\bar{\theta}_j, \theta_j(k) + i \times \bar{\delta}_j) - \max(\theta_j(k) + i \times \delta_j, \bar{\theta}_j) \right].$$

Park and Jeong (2004) modifies parameter uncertainties into parameter-incremental uncertainties as below,

$$A(\theta(k+i|k)) = A(\mu(k+i|k)) + B_p(k+i|k)\Delta C_q(k+i|k),$$

$$B(\theta(k+i|k)) = B(\mu(k+i|k)) + B_p(k+i|k)\Delta D_{qu}(k+i|k),$$

$$\Delta = \text{diag}(\eta_1 I, \eta_2 I, \ldots, \eta_L I)$$

where $\eta_i$ is a time-varying uncertain variable such that $\|\eta_i\| \leq 1, i = 1, 2, \ldots, p$. Thus, systems (1)–(3) can be transformed into the following structured uncertain system predicted at time $k$:

$$x(k+i+1|k) = A(\mu(k+i|k))x(k+i|k) + B(\mu(k+i|k))u(k+i|k) + B_p(k+i|k)p(k+i|k),$$

$$q(k+i|k) = C_q(k+i|k)x(k+i|k) + D_{qu}(k+i|k)u(k+i|k),$$

$$p(k+i|k) = \Delta q(k+i|k).$$

Since the recursive feasibility is the precondition of the closed-loop stability for systems with MPC, how to make good use of the information on system parameters (1)–(5) and to guarantee the recursive feasibility of RMPC become key issues to be studied. In the following, we will propose a feedback RMPC to achieve both of them for systems (1)–(5).

3 Feedback RMPC for system with parameter-incremental uncertainty

3.1 The modified model sequence with parameter-incremental uncertainties

For the parameters of systems (1)–(5), the measured values of parameters can be utilised at each time. With consideration of measurement errors (4), the following can be obtained.

$$\theta_j(k) \in \left[ \hat{\theta}_j(k) - \sigma_j, \hat{\theta}_j(k) + \sigma_j \right]$$
and for $\hat{\theta}_j(k+1)$, it must satisfied with

$$\hat{\theta}_j(k+1) \in \left[ \theta_j(k) + \delta_j - \sigma_j, \theta_j(k) + \bar{\delta}_j + \sigma_j \right].$$

Hence, for $\theta_j(k+1)$ we can get

$$\theta_j(k+1) \in \left[ \hat{\theta}_j(k+1) - \sigma_j, \hat{\theta}_j(k+1) + \sigma_j \right]$$

$$= \left[ \hat{\theta}_j(k) - \sigma_j + \bar{\delta}_j, \hat{\theta}_j(k) + \sigma_j + \bar{\delta}_j \right]$$

where $\bar{\delta}_j = \delta_j - 2\sigma_j$ and $\delta_j = \bar{\delta}_j + 2\sigma_j$. If $\hat{\theta}_j(k) > \hat{\theta}_j(k+1)$, $\hat{\theta}_j(k)$ is forced as $\hat{\theta}_j$ or $\hat{\theta}_j(k)$ due to (2).

In the same way, we can get that

$$\theta_j(k+i) \in \left[ \hat{\theta}_j(k+i) - \sigma_j, \hat{\theta}_j(k+i) + \sigma_j \right]$$

$$= \left[ \hat{\theta}_j(k) - \sigma_j + i \times \bar{\delta}_j, \hat{\theta}_j(k) + \sigma_j + i \times \bar{\delta}_j \right]$$

By referring to Park and Jeong (2004), we can revise the model with parameter-incremental uncertainties to include the measurement errors for systems (1)–(4). From the measured parameter $\hat{\theta}_j(k)$, the following is defined for $i \geq 0$.

$$\bar{\mu}_j(k+i|k) \triangleq \frac{1}{2} \left[ \min(\hat{\theta}_j, \hat{\theta}_j(k) + \sigma_j + i \times \bar{\delta}_j) + \max(\hat{\theta}_j(k) - \sigma_j + i \times \bar{\delta}_j, \hat{\theta}_j) \right],$$

$$\bar{\rho}_j(k+i|k) \triangleq \frac{1}{2} \left[ \min(\hat{\theta}_j, \hat{\theta}_j(k) + \sigma_j + i \times \bar{\delta}_j) - \max(\hat{\theta}_j(k) - \sigma_j + i \times \bar{\delta}_j, \hat{\theta}_j) \right].$$

Then similar to Park and Jeong (2004), systems (1)–(4) can be transformed into the following structured uncertain system predicted at time $k$:

$$x(k+i+1|k) = A(\bar{\mu}(k+i|k))x(k+i|k) + B(\bar{\mu}(k+i|k))u(k+i|k)$$

$$+ B_p(k+i|k)p(k+i|k),$$

$$q(k+i|k) = C_q(k+i|k)x(k+i|k) + D_{qu}(k+i|k)u(k+i|k),$$

$$p(k+i|k) = \Delta q(k+i|k).$$

It is worth to be pointed out that there are also uncertainties for the current system model due to measurement errors. Meanwhile, if there is no measurement error, the above model will be reduced to that in Park and Jeong (2004).

In addition, the original LPV system (1)–(2) can be converted into a structured feedback uncertain system as follows.

$$x(k+1) = \bar{\Lambda}(\alpha)x(k) + \bar{\tilde{B}}(\alpha)u(k) + \bar{\tilde{B}}_p(k),$$

$$q(k) = \bar{C}_q x(k) + \bar{D}_{qu} u(k),$$

$$p(k) = \Delta q(k).$$
Lemma 1: For the uncertain system $\Sigma(k)$ without input constraints, the policy $\pi = \{u(k), F_1, F_2, \ldots, F_{N-1}\}$, which guarantees (14) and $V(1,k) \leq \gamma$, is given by $u(k) = u_k$ and $F_i = Y_iX_i^{-1}$ with $P(i,k) = \gamma X_i^{-1}$, if there exists $\gamma > 0$, $X_i \in \mathbb{R}^{n \times n}$. 

3.2 The feedback robust MPC

From the analysis in the last section, the system model will vary along the sequence $\{\Sigma_{k,0}, \Sigma_{k,1}, \Sigma_{k,2}, \ldots, \Sigma_{k,N-1}\}$ and $\Sigma_{k,N-1} = \Psi$, which is denoted as $\Sigma(k)$. To avoid the difficulty to guarantee the recursive feasibility and make use of the information on system parameters, a closed-loop strategy should be adopted and the varying feedback control law $F_i$ at each time should correspond to $\Sigma_{k,i}$. Hence, the control strategy $\pi := \{u(k), F_1, F_2, \ldots, F_{N-1}\}$ is adopted, where $F_i$ is the feedback control gain at the $i$th step and after the $N$th step the feedback control gain is always $F_{N-1}$.

For $\Sigma(k)$ with $i > 0$, consider the following quadratic function:

$$V(i, k) = x(k + i|k)^T P(i, k)x(k + i|k),$$

where $P(i, k) = P(N-1, k)$ when $i > N - 1$.

From time $k + i$ to $k + i + 1$ ($i \geq 1$), the following robust stable condition is imposed on $V(i, k)$:

$$V(i + 1, k) - V(i, k) \leq -\|x(k + i|k)\|_{Q_1}^2 - \|u(k + i|k)\|_{R_1}^2,$$

which is equivalent to

$$\|(A(\bar{\mu}(k + i|k)) + B(\bar{\mu}(k + i|k))F_i)x(k + i|k)
+ B_p(\bar{\mu}(k + i|k))\|_{P(i+1,k)}^2 - \|x(k + i|k)\|_{P(i,k)}^2
\leq -\|x(k + i|k)\|_{Q_1}^2 - \|u(k + i|k)\|_{R_1}^2.$$

(14)

By summing (14) from $i = 1$ to $i = \infty$, it follows

$$\sum_{i=1}^{\infty} \left[\|x(k + i|k)\|_{Q_1}^2 + \|u(k + i|k)\|_{R_1}^2\right] \leq V(1,k).$$

(15)

Suppose there exists a non-negative parameter $\gamma$ such that $V(1,k) \leq \gamma$. Then, we can get

$$J_{\infty}(k) \leq \|x(k)\|_{Q_1}^2 + \|u(k)\|_{R_1}^2 + \gamma.$$

(16)

To guarantee (14) and $V(1,k) \leq \gamma$, the following lemma is given.
Constrained feedback RMPC for LPV systems

$X_i > 0, Y_i \in \mathbb{R}^{n_u \times n}$ ($i = 1, 2, \ldots, N - 1$) and positive-definite diagonal matrices $\Lambda_j \in \mathbb{R}^{n \times n}$, ($j = 0, 1, 2, \ldots, N - 1$), satisfying the following conditions.

$$
\begin{bmatrix}
1 & * & * \\
\Xi_1(k) & \Lambda_0 & * \\
\Xi_2(k) & 0 & \Xi_3(k)
\end{bmatrix} \geq 0
$$

(17)

$$
\begin{bmatrix}
X_i & * & * & * \\
R^{1/2} Y_i & \gamma I & * & * \\
Q^{1/2} X_i & 0 & \gamma I & * \\
\Xi_1(k+i) & 0 & 0 & \Lambda_i & * \\
\Xi_2(k+i) & 0 & 0 & 0 & \Xi_3(k+i)
\end{bmatrix} \geq 0,
$$

(18)

where $\Xi_1(k) = C_q(k)x(k) + D_q u_k$, $\Xi_2(k) = A(\hat{\mu}(k))x(k) + B(\hat{\mu}(k))u_k$, $\Xi_3(k) = X_1 - B_p(k)A^T B_p^2(k)$, $\Xi_1(k+i) = C_q(k+i)X_i + D_q u_k$, $\Xi_2(k+i) = A(\hat{\mu}(k+i))X_i + B(\hat{\mu}(k+i))Y_i$, $\Xi_3(k+i) = X_{i+1} - B_p(k+i)\Lambda_i B_p^2(k+i)$ and $X_N = X_{N-1}$.

Proof: From (8)–(10) (or (11)–(13)), we can get

$$
p(k+i)\{p(k+i)|^TP(k+i)\leq x(k+i)^T (C_q(k+i) + D_q u_k) + D_q u_k (F_{i+1})^T (C_q(k+i)) + D_q u_k (F_{i+1}) x(k+i)
$$

Condition (14) holds if the following condition can be guaranteed.

$$
\begin{bmatrix}
||F_i||_{R}^2 + Q \leq P(i,k) + ||A(\mu(k+i))|| \\
+B(\mu(k+i))F_i ||^2_{P(i+1,k)} \\
B_{p}^T (k+i) P(i+1,k) (A(\mu(k+i)) \times B)
\end{bmatrix} \leq 0,
$$

where $\chi = [x(k+i)]$, $\mathcal{B} = ||B_p(k+i)||_{P(i+1,k)}^2$. Then, by S-procedure, the above condition can be guaranteed if there exists $\Lambda'_i = \text{diag}(S_{1,i}, \ldots, S_{L,i})$, $S_{l,i} \geq 0$, $l = 1, 2, \ldots, L$ such that

$$
\begin{bmatrix}
\mathcal{A}_1 & * \\
\mathcal{A}_2 & \mathcal{B}_1
\end{bmatrix} \leq 0,
$$

where $\mathcal{A}_1 = ||A(\hat{\mu}(k+i)) + B(\hat{\mu}(k+i))F_i ||^2_{P(i+1,k)} - P(i,k) + ||F_i||_{R}^2 + Q + ||C_q(k+i) + D_q u_k ||^2_{\Lambda'_i}$, $\mathcal{A}_2 = B_{p}^T (k+i) P(i+1,k) (A(\mu(k+i)) + B(\mu(k+i)) F_i)$, $\mathcal{B}_1 = ||B_p(k+i)||_{P(i+1,k)}^2 - \Lambda'_i$.

Let $P(i,k) = \gamma X_i^{-1}$, $Y_i = F_{i}X_i$ and $\Lambda_i = \gamma (\Lambda'_i)^{-1}$. By using Schur complement, it can be concluded that the above condition is equivalent to (18).

In addition, by Schur complement, (17) is equivalent to

$$
\begin{bmatrix}
\mathcal{A}_1(0) & * \\
\mathcal{A}_2(0) & \mathcal{B}_1(0)
\end{bmatrix} \leq 0
$$
where \( \mathcal{A}_1(0) = ||A(\hat{\mu}(k))x(k) + B(\hat{\mu}(k))u_k||_{X_1^{-1}}^2 - 1 + ||C_q(k)x(k) + D_qu(k)) \)
\[ u_k \| \Lambda \|^{-1}, \mathcal{A}_2(0) = B_p^T(k)X_1^{-1}(A(\hat{\mu}(k))x(k) + B(\hat{\mu}(k))u_k) \]
and \( \mathcal{B}_1(0) = ||B_p(k)||_{X_1^{-1}}^2 - \Lambda_0^{-1}. \)

Left- and right-multiplying the above inequality by \([1 \ p(k)]\) and \([1 \ p(k)]^T\), respectively, and then by (8)–(10) and using S-procedure, we can get
\[ ||A(\hat{\mu}(k))x(k) + B(\hat{\mu}(k))u_k + B_p(k)p(k)||_{X_1^{-1}}^2 \leq 1. \]
That is, \( V(1, k) \leq \gamma \) holds if (17) is satisfied according to \( P(1, k) = \gamma X_1^{-1} \). Therefore, the lemma is proven.

From \( V(1, k) \leq \gamma \) and (14), it is obvious that \( V(i, k) \leq \gamma \). Then constraints (5) can be satisfied if the following lemma holds, whose proof can be easily obtained by the similar procedure in Kothare et al. (1996) or Li et al. (2009) and is omitted here.

**Lemma 2:** The input constraints (5) can be satisfied if there exists \( \gamma > 0, X_i \in \mathbb{R}^{n \times n}, \)
\( X_i > 0, Y_i \in \mathbb{R}^{n \times n}, Z_i \in \mathbb{R}^{n \times n} \) (\( i = 1, 2, \ldots, N - 1 \)) and positive-definite diagonal matrices \( \Lambda_j \in \mathbb{R}^{n \times n}, \) (\( j = 0, 1, 2, \ldots, N - 1 \)), satisfying conditions (17) and (18), and also satisfying the following conditions:
\[ ||(u_k)|| \leq u_t_{\max}, \quad l = 1, 2, \ldots, m \]
\[ Z_i \geq 0, (Z_i)_{ll} \leq u_i^2, \quad i = 1, 2, \ldots, N - 1, \quad l = 1, 2, \ldots, m. \] (20)

Lemma 2 can be proven in a similar way to the proof of the constraints in Kothare et al. (1996). Therefore, it is omitted here.

Based on Lemmas 1 and 2, the optimisation problem of feedback RMPC for \( \Sigma(k) \) can be formulated as below.

**Algorithm 1:** Let \( x(k) = x(k|k) \) be the state of the uncertain system \( \Sigma(k) \) measured at sampling time \( k \), and the input constraints are described as in (5). Then the policy \( \pi = \{u(k), F_1, F_2, \ldots, F_{N-1}\} \) that minimises the upper bound on the robust performance objective function at sampling time \( k \) is given by
\[ u(k) = u_{k}, F_i = Y_iX_i^{-1} \]
where \( \gamma > 0, X_i \in \mathbb{R}^{n \times n}, X_i > 0, Y_i \in \mathbb{R}^{n \times n}, Z_i \in \mathbb{R}^{n \times n} \) (\( i = 1, 2, \ldots, N - 1 \)) and positive-definite diagonal matrices \( \Lambda_j \in \mathbb{R}^{n \times n}, \) (\( j = 0, 1, 2, \ldots, N - 1 \)), are obtained from the solution (if it exists) of the following linear objective minimisation problem
\[ \min_{\gamma_0, \gamma, u_{k}, X_i, Y_i, Z_i, \Lambda_j} \gamma + \gamma_0 \] (21)
\[ \text{s.t.} \quad (17) - (20) \] (22)
\[ \begin{bmatrix} \gamma_0 x^T(k) u_k^T \\ x(k) Q_1^{-1} 0 \\ u_k 0 R^{-1} \end{bmatrix} \geq 0. \] (23)

The current control input is \( u(k) = u_k. \)
Remark 2: For RMPC based on Algorithm 1, if the control input $u(k)$ in control strategy $\pi$ is removed and $N$ is chosen as 1, the controller will be simplified to the design in Kothare et al. (1996). The added freedom in Algorithm 1 makes it possible to utilise the information on system parameters, which is helpful to improve the control performance.

To simplify the presentation, let $Q_i := (X_i, Y_i, Z_i, A_i)$. In terms of the recursive feasibility and closed-loop stability of Algorithm 1, the following theorem can be given.

Theorem 3: If there is a feasible solution of Algorithm 1 at time $k$ with system state $x(k)$, there will also exist a feasible solution for Algorithm 1 at next time, and the closed-loop system is asymptotically stable.

Proof: Since Algorithm 1 is feasible at time $k$, suppose $\Gamma^*(k) = \{\gamma^*_0(k), \gamma^*(k), u^*_k, A^*_k, Q^*_i, \ldots, Q^*_{N-1} \}$ as the optimal solution for the current state $x(k)$. That implies that $\Gamma^*(k)$ satisfies (17)–(20).

At time $k + 1$, for Algorithm 1, we construct a solution

$$\Gamma(k + 1) = \{\|x(k + 1)\|_Q^2 + \|Y_k^T(X_k^*)^{-1}x(k + 1)\|_R^2, a\gamma^*(k), Y_k^*(X_k^*)^{-1}x(k + 1), a\Lambda_k, aQ_k^*, \ldots, aQ_{N-1}^*, aQ_{N-1}^* \}$$

with definition $a := V(1, k + 1)/\gamma^*(k)$ where $V(1, k + 1) = x^T(k + 2)k + 1)\gamma^*(X_k^*)^{-1}x(k + 2k + 1), x(k + 2k + 1) = [A(\theta(k + 1)) + B(\theta(k + 1))Y_k^*(X_k^*)^{-1}]x(k + 1)$. The above definition means that $V(1, k + 1) = a\gamma^*(k)$.

From the model with parameter incremental uncertainty, $\Sigma_{k+1, i} \subseteq \Sigma_{k, i + 1}$ and $\Sigma_{k,i} \subseteq \Psi$. We observe that conditions (18) and (20) are affine in the matrices $(\gamma, Q^*_1, Q^*_2, \ldots, Q^*_{N-1})$. Multiplying them by parameter $a$ respectively, we can see that $\Gamma(k + 1)$ satisfies (20) and (18) when $i = 1, 2, \ldots, N - 1$.

Let $u_{k+1} = Y_k^*(X_k^*)^{-1}x(k + 1)$ and $\gamma_0(k + 1) = \|x(k + 1)\|_{Q_k^*}^2 + \|Y_k^*(X_k^*)^{-1}x(k + 1)\|_R^2$. It is obvious that (19) and (23) are satisfied by $\Gamma(k + 1)$. Furthermore, since $Q_k^*(k)$ satisfies (18) at time $k$, it implies that (17) holds at time $k + 1$ with the constructed solution $\Gamma(k + 1)$ due to the definition of $a$. That means, the constructed solution is a feasible solution of Algorithm 1 at time $k + 1$. Hence, the recursive feasibility of Algorithm 1 can be established.

In addition, from Lemma 1, it can be concluded that $\|x(k + 1)\|_Q^2 + \|Y_k^*(X_k^*)^{-1}x(k + 1)\|_R^2 + V(1, k + 1) \leq V(1, k) \leq \gamma^*(k)$, i.e., $\gamma_0(k + 1) + \gamma(k + 1) \leq \gamma^*(k)$. Therefore, it can be obtained that $\gamma_0^*(k + 1) + \gamma^*(k + 1) \leq \gamma_0^*(k + 1) + \gamma(k + 1) < \gamma_0^*(k) + \gamma^*(k)$ when $x(k) \neq 0$. That is, the closed-loop system is asymptotically stable. \hfill \square

4 Numerical example

Consider the following system:

$$x(k + 1) = [\theta(k)A_1 + (1 - \theta(k)) A_2] x(k) + [\theta(k)B_1 + (1 - \theta(k)) B_2] u(k)$$
where $A_1 = \begin{bmatrix} 1 & 0 \\ -0.3 & 1.4 \end{bmatrix}$, $A_2 = \begin{bmatrix} 1 & 0 \\ -0.1 & 1.1 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $|u| \leq 1$, $\theta(k) \in [0, 1]$, $|\Delta \theta| \leq \delta$, $|\hat{\theta}(k) - \theta(k)| \leq \sigma$ and

$$
\theta(k + 1) = \begin{cases} 
0, & \theta(k) + \delta \sin(k - 1) \leq 0 \\
\theta(k) + \delta \sin(k - 1) \\
1, & \theta(k) + \delta \sin(k - 1) \geq 1 
\end{cases}
$$

$$
\hat{\theta}(k) = \begin{cases} 
0, & \theta(k) + \sigma \leq 0 \\
\theta(k) + \sigma \\
1, & \theta(k) + \sigma \geq 1 
\end{cases}
$$

The initial state is chosen as $x(0) = [1, 1]^T$ and the weighting matrices are chosen as $Q_1 = \text{diag}(1, 0.1)$, $R = 0.001$. First, let us verify the recursive feasibility of Park and Jeong (2004). Investigate an extreme case for RHC in Park and Jeong (2004) with $N = 3$, i.e., the case with $\theta(k) = 1$, $\delta = 1$ and no measurement errors. For $x(k) = [1, 1]^T$ and $k = 1$, RHC in Park and Jeong (2004) optimizes the control inputs $u(k), u(k + 1), u(k + 2)$ to steer $x(k)$ to a terminal invariant set. The states $x(k), x(k + 1), x(k + 2), x(k + 3)$ are shown in Figure 1, which is a partial enlarged drawing. From Schuurmans and Rossiter (2000), $x(k + i), i > 1$ is a state constructed by the linear combination from the states $x(k + i)$ corresponding to the model vertices $\{A_1, B_1\}, \{A_2, B_2\}$. Thus, if Lemma 2 in Park and Jeong (2004) is correct, there must be a $u(k + 3|k + 1)$ with $u(k + 1|k + 1) = u(k + 1|k), u(k + 2|k + 1) = u(k + 2|k)$ steering $x(k + 1|k + 1)$ into the terminal set computed at time $k$. In Figure 1, states $x(k + 1|k + 1), x(k + 2|k + 1), x(k + 3|k + 1)$ are marked by a cycle. By computing, it is found that this $u(k + 3|k + 1)$ does not exist. That is, the RHC in Park and Jeong (2004) cannot guarantee the recursive feasibility.

Figures 2 and 3 show the state responses from $x(0) = [1, 1]^T$ with $\theta(0) = 0.6$, $\delta = 0.15$, $\sigma = 0$ and $\theta(0) = 0.6$, $\delta = 0.15$, $\sigma = 0.01$, respectively, where $N = 3$ for RHC in Park and Jeong (2004) and Algorithm 1. The results by using the techniques in Lu and Arkun (2000) and Kothare et al. (1996) are also included in Figures 2 and 3, respectively to make a comparison. From Figure 2, the performance of RMPC with Algorithm 1 is best, where the cost value of Algorithm 1 is 34.56, better than 42.69 of Lu and Arkun (2000) and 42.94 of Park and Jeong (2004). For the case with measurement errors $\sigma = 0.01$, the results are compared between the proposed Algorithm 1 and the technique in Kothare et al. (1996), which is the technique capable of dealing with LPV systems with measurement error in the previous literatures. The state response is shown in Figure 3 and the cost value of Algorithm 1 is 28.73 and that of Kothare et al. (1996) is 38.44. Therefore, it reflects that Algorithm 1 can achieve better control performance than the design in Kothare et al. (1996).

The above results verify the effectiveness of utilises the information of system parameters in Algorithm 1.

5 Conclusions

This paper presents a new approach to RMPC for LPV systems with bounded rates of parameter variations and bounded parameter measurement errors. By adopting a sequence of feedback control laws corresponding to the parameter variation of LPV systems, the information on system parameters can be made use of, which is helpful to reduce the design conservativeness and then improve the control performance of RMPC.
feasibility and closed-loop stability of the proposed MPC can be guaranteed by the proposed RMPC.

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