ON STABILITY OF THE THIRD ORDER PARTIAL DELAY 
DIFFERENTIAL EQUATION WITH NONLOCAL 
BOUNDARY CONDITIONS

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Abstract: In the present paper, the stability of the initial value problem for the third order partial delay differential equation with nonlocal boundary condition is studied. The first of accuracy absolute stable difference scheme for solution of this problem is presented. Stability estimates for solution of this difference scheme are proved. Numerical results are provided.

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1. Introduction

Classical and nonclassical problems for third order partial differential equations have been studied widely in the literature (for instance, see [1], [2], [3], [4], [5], [6], [7], [8], [9], [10], [11], [12], [13], [14]).

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Time delay is one of the most common phenomena occurring in many engineering applications. In control theory, the process of sampled-data control is a typical example where time delay happens in the transmission from measurement to controller.

Theory and applications of delay linear and nonlinear third order ordinary and partial differential and difference equations with the delay term were widely investigated (see, e.g., [15], [16], [17], [18], [19], [20], [21], [22] and the references given therein).

Our goal in this paper is to investigate the initial value problem for third order partial delay differential and difference equations with nonlocal boundary condition. The paper is organized as follows. Section 3 is Introduction. In Section 2, a theorem on stability of the initial value problem for the third order partial delay differential equation with nonlocal boundary condition is established. In Section 3, the first order of accuracy difference scheme for solution of this problem is studied. Stability estimates for solution of this difference scheme are proved. In Section 4, numerical results are provided. Finally, Section 5 is conclusion.

2. Stability of differential problem

In \([0, \infty) \times (0, l)\), the initial boundary value problem for the third order partial differential equation with time delay and nonlocal boundary conditions

\[
\begin{aligned}
\frac{\partial^3 u(t, x)}{\partial t^3} - (a(x)u_{tx}(t, x))_x + \delta u(t, x) \\
= -b ((-a(x)u_x(t - w, x))_x + \delta u(t, x)) \\
+ f(t, x), \quad 0 < t < \infty, \quad (0, l),
\end{aligned}
\]

\[
u(t, x) = g(t, x), \quad -w \leq t \leq 0, \quad x \in [0, l],
\]

\[
u(t, 0) = u(t, l), u_x(t, 0) = u_x(t, l), \quad 0 \leq t < \infty
\]

is considered. Throughout this paper, we will assume that

\(a(x) \geq a > 0, \ x \in (0, l)\) and \(a(l) = a(0)\).

We consider the Hilbert space \(L_2[0, l]\) of the all square integrable functions defined on \([0, l]\), equipped with the norm

\[\| f \|_{L_2[0, l]} = \left( \int_0^l |f(x)|^2 dx \right)^{\frac{1}{2}}.\]
Under compatibility conditions problem, (1) has a unique solution $u(t, x)$ for the smooth functions $a(x), x \in (0, \ell), g(t, x), -w \leq t \leq 0, x \in [0, l], f(t, x), 0 < t < \infty, x \in (0, l)$, and $\delta > 0, b \in R^1$.

Let us present a theorem on stability of problem (1).

**Theorem 1.** For the solutions of problem (1), we have following stability estimates:

\[
\max_{0 \leq t \leq nw} \| v_{tt}(t, \cdot) \|_{W_2^1[0, l]}, \max_{0 \leq t \leq nw} \| v_t(t, \cdot) \|_{W_2^2[0, l]}, \max_{0 \leq t \leq nw} \| v(t, \cdot) \|_{W_3^2[0, l]}
\]

\[
\leq M_2 \left[ (2 + |b| w)^n a_0 + \int_{j=1}^{n} (2 + |b| w)^{n-j} \left( \int_{(j-1)w}^{jw} \| f(s, \cdot) \|_{W_2^1[0, l]} \, ds \right) \right],
\]

\[
a_0 = \max \left\{ \max_{-w \leq t \leq 0} \| g_{tt}(t, \cdot) \|_{W_2^1[0, l]}, \max_{-w \leq t \leq 0} \| g(t, \cdot) \|_{W_2^3[0, l]} \right\},
\]

where $M_2$ does not depend on $g(t, x)$ and $f(t, x)$. Here, $W_2^k[0, l], k = 1, 2, 3$ are Sobolev spaces of all square integrable functions $\psi(x)$ defined on $[0, l]$ equipped with the norm

\[
\| \psi \|_{W_2^k[0, l]} = \left( \int_0^l \sum_{j=0}^{k} \left( \psi^{(j)}(x) \right)^2 \, dx \right)^{1/2}.
\]

**Proof.** This allows us to reduce the problem (1) to the initial value problem

\[
\begin{cases}
\frac{d^3 v(t)}{dt^3} + A \frac{dv(t)}{dt} = bAv(t-w) + f(t), & 0 < t < \infty, \\
v(t) = g(t), & -w \leq t \leq 0
\end{cases}
\]

in a Hilbert space $H = L_2[0, l]$ with a self-adjoint positive definite operator $A$ defined by formula

\[
Au(x) = -(a(x)u_x(x))_x + \delta u(x)
\]

with domain

\[
D(A) = \{ u(x) : u(x), u_x(x), (a(x)u_x)_x \in L_2[0, l], u(l) = u(0), u_x(l) = u_x(0) \}.
\]

The proof of Theorem 1 is based on the self-adjointness and positive definiteness of the space operator $A$ defined by formula (3) [25] and the following theorem on stability of the solution of the abstract problem (2).
Theorem 2. ([23]) For the solution of problem (2) the following estimate holds:

\[
\max_{0 \leq t \leq nw} \left\| A^2 \frac{d^2 v(t)}{dt^2} \right\|_H, \quad \max_{0 \leq t \leq nw} \left\| A v(t) \right\|_H, \quad \frac{1}{2} \max_{0 \leq t \leq nw} \left\| A^3 v(t) \right\|_H \\
\leq (2 + |b| w)^n a_0 + \int_0^{nw} \left\| A^{1/2} f(s) \right\|_H ds, \quad n = 1, 2, ..., (4)
\]

where

\[
a_0 = \max \left\{ \max_{-w \leq t \leq 0} \left\| A^2 \frac{d^2 g(t)}{dt^2} \right\|_H, \quad \max_{-w \leq t \leq 0} \left\| A g(t) \right\|_H \right\},
\]

\[
\max_{-w \leq t \leq 0} \left\| A^{3/2} g(t) \right\|_H. \]

3. Stability of the difference scheme

Now, we study the stable difference scheme for the approximate solution of problem (1). The discretization of problem (1) is carried out in two steps.

In the first step, the spatial discretization is carried out. We define the grid space

\[
[0, \ell]_h = \{ x = x_n \mid x_n = nh, \quad 0 \leq n \leq M, \quad Mh = \ell \}.
\]

We introduce the Hilbert space \( L_{2h} = L_2([0, \ell]_h) \) of the grid functions \( \varphi^h(x) = \{ \varphi^n \}_0^M \) defined on \([0, \ell]_h\), equipped with the norm

\[
\left\| \varphi^h \right\|_{L_{2h}} = \left( \sum_{x \in [0, \ell]_h} \left| \varphi^h(x) \right|^2 h \right)^{1/2}.
\]

To the differential operator \( A \) defined by the formula (3), we assign the difference operator \( A^x_h \) by the formula

\[
A^x_h \varphi^h(x) = \left\{ - \left( a(x) \frac{\varphi^h}{\varphi^h} \right)_x + \delta \varphi^n(x) \right\}^{M-1}_{-M+1}, \quad (5)
\]

acting in the space of grid functions \( \varphi^h(x) = \{ \varphi^n \}_0^M \) and satisfying the conditions \( \varphi^0 = \varphi^M, \varphi^1 - \varphi^0 = \varphi^M - \varphi^{M-1} \). Here

\[
\varphi^n_x = \frac{\varphi^n - \varphi^{n-1}}{h}, \quad 1 \leq n \leq M, \quad \varphi^n_x = \frac{\varphi^{n+1} - \varphi^n}{h}, \quad 0 \leq n \leq M - 1.
\]
It is well-known that $A_h^x$, defined by (5), is a self-adjoint positive definite operator in $L_{2h}$. With the help of $A_h^x$, the first discretization step results in the following problem

$$
\begin{align*}
\frac{\partial^3 u^h(t, x)}{\partial t^3} + A_h^x u^h(t, x) &= -bA_h^x u^h(t - w, x) \\
+ f^h(t, x), & x \in [0, \ell]_h, 0 < t < \infty, \\
u^h(t, x) &= g^h(t, x), -w \leq t \leq 0, x \in [0, \ell]_h, -w < t < 0.
\end{align*}
$$

(6)

In the second step, we replace the problem (6) with the following first order of accuracy difference scheme

$$
\begin{align*}
\frac{u^h_{k+2}(t) - 3u^h_{k+1}(t) + 3u^h_k(t) - u^h_{k-1}(t)}{\tau^3} + A_h^x u^h_{k+2}(t) &= A_h^x u^h_{k+1}(t) + A_h^x u^h_{k-1}(t) \\
= bA_h^x u^h_{k-N}(x) + f^h_k(x), & x \in [0, \ell]_h, \\
u^h_k(x) &= g^h(t_k, x), -N \leq k \leq 0, \\
(I_h + \tau^2 A_h^x) \frac{u^h_1(x) - u^h_0(x)}{\tau} &= g^h_t(0, x), \\
(I_h + \tau^2 A_h^x) \frac{u^h_2(x) - 2u^h_1(x) + u^h_0(x)}{\tau^2} &= g^h_{tt}(0, x), x \in [0, \ell]_h, \\
(I_h + \tau^2 A_h^x) \frac{u^h_{m+1}(x) - u^h_m(x)}{\tau} &= u^h_{m-N}(x) - u^h_{m-N-1}(x), x \in [0, \ell]_h, \\
(I_h + \tau^2 A_h^x) \frac{u^h_{m+2}(x) - 2u^h_{m+1}(x) + u^h_m(x)}{\tau^2} &= u^h_{m-N}(x) - 2u^h_{m-N-1}(x) + u^h_{m-N-2}(x), x \in [0, \ell]_h, m = 1, 2, \ldots,
\end{align*}
$$

(7)

where $\tau = 1/N$ and $t_k = k\tau$, $-N \leq k < \infty$.

**Theorem 3.** Let $\tau$ and $h$ be sufficiently small numbers. For the solution of difference scheme (7) the following estimates

$$
\begin{align*}
\max_{0 \leq k \leq (m+1)N-2} \left\| \frac{u^h_{k+2} - 2u^h_{k+1} + u^h_k}{\tau^2} \right\|_{W^1_{2h}} &\leq C_1 \left( 2 + \frac{\tau |b| (N - 2)}{m b_0} \right)^m, \\
\max_{1 \leq k \leq (m+1)N} \left\| \frac{u^h_k - u^h_{k-1}}{\tau} \right\|_{W^2_{2h}} &\leq C_1 \left( 2 + \frac{\tau |b| (N - 2)}{m b_0} \right)^m,
\end{align*}
$$

$$
\begin{align*}
\max_{0 \leq k \leq (m+1)N} \left\| u^h_k \right\|_{W^3_{2h}} &\leq C_1 \left( 2 + \frac{\tau |b| (N - 2)}{m b_0} \right)^m.
\end{align*}
$$
solution of the difference scheme (8).

The difference scheme (7) can be written in the abstract form

\begin{align*}
&+ \sum_{j=1}^{m} (2 + \tau |b| (N - 2))^m - j \tau \sum_{s=(j-1)N+1}^{jN} \|f(t_s)\|_{W_{2h}^{1/2}}, m = 0, 1, \ldots, \\
&b^h_0 = \max \left\{ \max_{-N \leq k \leq 0} \|g^h_{t_0}(t_k)\|_{W_{2h}^1}, \max_{-N \leq k \leq 0} \|g^h_0(t_k)\|_{W_{2h}^2}, \right. \\
&\left. \max_{-N \leq k \leq 0} \|g^h(t_k)\|_{W_{2h}^3} \right\}
\end{align*}

hold, where \( C_1 \) does not depend on \( \tau, h, g^h(t_k), \) and \( f^h_k(x) \). Here, \( W_{2h}^k, k = 1, 2, 3 \) are spaces of all mesh functions \( \psi^h(x) \) defined on \([0, l]_h\) equipped with the norm

\[ \| \psi^h \|_{W_{2h}^k} = \left( \sum_{x \in [0, l]_h} \sum_{j=0}^k \left( \psi^h_{j, x} \right)^2 \right)^{1/2}. \]

**Proof.** The difference scheme (7) can be written in the abstract form

\[
\begin{align*}
\frac{u^h_{k+2} - 3u^h_{k+1} + 3u^h_k - u^h_{k-1}}{\tau^3} + A_h \frac{u^h_{k+2} - u^h_{k+1}}{\tau} &= bA_h u^h_{k-N} + f^h_k, k \geq 1, \\
u^h_k &= g^h_k, -N \leq k \leq 0, \\
(I_h + \tau^2 A_h) \frac{u^h_{k+1} - u^h_k}{\tau} &= g^h(0), (I_h + \tau^2 A_h) \frac{u^h_{k+2} - 2u^h_{k+1} + u^h_k}{\tau^2} = g^h_0, \\
(I_h + \tau^2 A_h) \frac{u^h_{mN+2} - 2u^h_{mN+1} + u^h_{mN}}{\tau^2} &= u^h_{mN-2}u^h_{mN-1} + u^h_{mN-2}, \\
(I_h + \tau^2 A_h) \frac{u^h_{mN+1} - u^h_{mN}}{\tau} &= u^h_{mN-2}u^h_{mN-1}, m = 1, 2, \ldots
\end{align*}
\]

in a Hilbert space \( L_{2h} \) with self-adjoint positive definite operator \( A_h = A^*_h \) by formula (5).

Here, \( g^h_k = g^h_k(x), f^h_k = f^h_k(x) \) and \( u^h_k = u^h_k(x) \) are known and unknown abstract mesh functions defined on \([0, l]_h\) with the values in \( H = L_{2h} \). Therefore, the proof of Theorem 2 is based on the self-adjointness and positive definiteness of the space operator \( A_h \) (5) [26] and the following theorem on stability of the solution of the difference scheme (8). \( \square \)
Theorem 4. ([27]) For the solution of difference scheme (8) the following estimate holds:

\[ \frac{1}{2} \max_{0 \leq k \leq (m+1)N-2} \left\| \frac{1}{2} u_{k+2}^h - 2u_{k+1}^h + u_k^h \right\| + \max_{H} \left\| \frac{1}{\tau} A_h u_k^h - u_{k-1}^h \right\|, \]

\[ \max_{0 \leq k \leq (m+1)N} \left\| A_h^2 u_k^h \right\|_H \leq C_1 \left( (2 + \tau|b|(N - 2))^m b_0^h \right. \]

\[ \left. + \sum_{j=1}^{m} (2 + \tau|b|(N - 2))^{m-j} \tau \sum_{s=(j-1)N+1}^{jN} \left\| A_H^\frac{1}{2} f(t_s) \right\|_H \right], \quad m = 0, 1, \ldots, \]

where

\[ b_0 = \max \left\{ \max_{-N \leq k \leq 0} \left\| A_h^{\frac{1}{2}} g''(t_k) \right\|_H, \quad \max_{-N \leq k \leq 0} \left\| A_h g^h(t_k) \right\|_H, \right. \]

\[ \left. \max_{-N \leq k \leq 0} \left\| A_h^{\frac{3}{2}} g^h(t_k) \right\|_H \right\}. \]

4. Numerical results

When the analytical methods do not work properly, the numerical methods for obtaining the approximate solutions of partial differential equations play an important role in applied mathematics. In this section, we will use the first order of accuracy difference scheme to approximate the solution of a simple test problem

\[
\begin{cases}
\frac{\partial^3 u(t,x)}{\partial t^3} - \frac{\partial^3 u(t,x)}{\partial t \partial x^2} + 16 \frac{\partial u(t,x)}{\partial t} \\
= -0.1 \left( -\frac{\partial^2 u(t-1,x)}{\partial x^2} + 16u(t-1, x) \right) \\
-48e^{-2t} \sin 2x + 2e^{-2(t-1)} \sin 2x,
\end{cases}
\]

\[ 0 < t < \infty, \quad 0 < x < \pi, \]

\[ u(t, x) = e^{-2t} \sin 2x, \quad -1 \leq t \leq 0, \quad 0 \leq x \leq \pi, \]

\[ u(t, 0) = u(t, \pi), u_x(t, 0) = u_x(t, \pi), \quad 0 \leq t < \infty. \]

The exact solution of problem (9) is \( u(t, x) = e^{-2t} \sin 2x, 0 \leq x \leq \pi, -1 \leq t < \infty. \) For the approximate solutions of the problem (9), using the set of grid points

\[ [-1, \infty) \times [0, \pi]_h = \{(t_k, x_n) : t_k = k\tau, -N \leq k, N\tau = 1, \]
we get the first order of accuracy in $t$ difference scheme

\[
\begin{align*}
\frac{u_n^{k+2}-3u_n^{k+1}+3u_n^k-u_n^{k-1}}{\tau^2} - u_n^{k+2} & - u_n^{k+1} - 2(u_n^{k+2} - u_n^{k+1}) + u_n^{k+2} - u_n^{k+1} \\
+ & 16\frac{k_n^{k+2} - u_n^{k+1}}{\tau} = -(0.1) \left( -u_n^{k-N} - 2u_n^{k-N} + u_n^{k-N} \right) + 16u_n^{k-N} \\
-48e^{-2t_k} \sin 2x_n + 2e^{-2(t_k-N)} \sin 2x_n, \\
t_k = k\tau, \quad mN + 1 \leq k \leq (m + 1)N - 2, \\
m = 0, 1, \ldots, \quad 1 \leq n \leq M - 1, \\
N\tau = 1, \quad x_n = nh, \quad 1 \leq n \leq M - 1, \quad Mh = \pi, \\
u_n^0 = \sin(2nh), \\
u_n^1-u_n^0 & + \tau(-u_n^{n+1} - 2u_n^0 + u_n^{n-1} + 16u_n^1) + \\
\tau & \left( \frac{u_n^{n+1} - 2u_n^0 + u_n^{n-1}}{h^2} - 16u_n^0 \right) = -2\sin(2nh), \\
u_n^2-u_n^0 & + \tau\left( \frac{-2u_n^{n+1} + u_n^{n-1}}{h^2} + 16u_n^0 \right) \\
+ & 2\left( \frac{u_n^{n+1} - 2u_n^0 + u_n^{n-1}}{h^2} - 16u_n^1 \right) \\
+( & \frac{-u_n^{n+1} - 2u_n^0 + u_n^{n-1}}{h^2} + 16u_n^0 ) = 4\sin(2nh), \quad 0 \leq n \leq M, \\
\frac{u_n^{mN+1} - u_n^{mN}}{\tau} & + \tau\left( -\frac{u_n^{mN+1} - 2u_n^{mN} + u_n^{mN-1}}{h^2} + 16u_n^{mN+1} \right) \\
+ & \tau\left( \frac{u_n^{mN+2} - 2u_n^{mN} + u_n^{mN-1}}{h^2} - 16u_n^{mN} \right) = \frac{u_n^{mN} - u_n^{mN-1}}{\tau}, \\
\frac{u_n^{mN+2} - 2u_n^{mN+1} + u_n^{mN}}{h^2} & + 16u_n^{mN+2} \\
+ & \frac{u_n^{mN+1} - 2u_n^{mN+1} + u_n^{mN+1}}{h^2} - 16u_n^{mN+1} \\
+ & \frac{-u_n^{mN} - 2u_n^{mN} - u_n^{mN}}{h^2} + 16u_n^{mN} \\
= & \frac{u_n^{mN} - 2u_n^{mN-1} + u_n^{mN-2}}{\tau^2}, \quad 0 \leq n \leq M, \quad m = 1, 2, \ldots, \\
u_0^k = u_M^k, \quad u_1^k - u_0^k = u_M^k - u_M^{k-1}, \\
mN \leq k \leq (m + 1)N, \quad m = 1, 2, \ldots.
We can write (10) in the matrix form

\[
BU^{k+2} + CU^{k+1} + DU^k + EU^{k-1} = \varphi(U^{k-N}),
\]

\[
k = 1, 2, 3, \ldots, U^0 = \begin{bmatrix}
  0 \\
  \sin(2h) \\
  \vdots \\
  \sin(2(M-1)h) \\
  0
\end{bmatrix},
\]

\[
U^1 = (1 - 2\tau)U^0, U^2 = U^1 - (1 - 4\tau^2)U^0 \\
U^{mN+1} = F^{-1}HU^{mN} - F^{-1}U^{mN-1}, \\
U^{mN+2} = 2U^{mN+1} + F^{-1}PU^{mN} - 2F^{-1}U^{mN-1} + F^{-1}U^{mN-2}, m = 1, 2, \ldots,
\]

where \(B, C, D, E, F, H\) and \(P\) are \((M+1) \times (M+1)\) matrices, \(\varphi(U^{k-N})\), \(U^0\), \(U^1\) and \(U^r, r = k, k \pm 1, k + 2\) are \((M+1) \times 1\) column vectors defined by

\[
B = \begin{bmatrix}
  1 & 0 & 0 & 0 & \cdots & 0 & 0 & -1 \\
  a & b & a & \cdots & 0 & 0 & 0 \\
  0 & a & b & a & \cdots & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & \cdots & b & a & 0 \\
  0 & 0 & 0 & 0 & \cdots & a & b & a \\
  1 & -1 & 0 & 0 & \cdots & 0 & -1 & 1
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
  0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
  -a & c & -a & \cdots & 0 & 0 & 0 \\
  0 & -a & c & -a & \cdots & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & \cdots & c & -a & 0 \\
  0 & 0 & 0 & 0 & \cdots & -a & c & -a \\
  0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix},
\]

\[
DE(g) = \begin{bmatrix}
  0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\
  0 & g & 0 & \cdots & 0 & 0 & 0 \\
  0 & 0 & g & 0 & \cdots & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & \cdots & g & 0 & 0 \\
  0 & 0 & 0 & 0 & \cdots & 0 & g & 0 \\
  0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0
\end{bmatrix},
\]
\[
FH(z) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & z & -s & \\
0 & -s & 0 & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & z & -s & 0 \\
0 & 0 & 0 & 0 & -s & z & -s \\
1 & -1 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix},
\]

\[
P = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & -1 \\
s & p & s & \\
0 & s & p & s & 0 & 0 & 0 & 0 \\
. & . & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & p & s & 0 \\
0 & 0 & 0 & 0 & s & p & s \\
1 & -1 & 0 & 0 & 0 & 0 & -1 & 1
\end{bmatrix},
\]

\[
DE(g) = \begin{cases}
D, & g = d, \\
E, & g = e,
\end{cases}
\]

\[
FH(z) = \begin{cases}
F, & z = q, \\
H, & z = t,
\end{cases}
\]

\[
\varphi(U^{k-N}) = \begin{bmatrix}
0 \\
\varphi^k_1 \\
. \\
\varphi^k_{M-1} \\
0
\end{bmatrix}, \quad U^r = \begin{bmatrix}
U^r_0 \\
U^r_1 \\
. \\
U^r_{M-1} \\
U^r_M
\end{bmatrix}, \quad r = k, k \pm 1, k + 2,
\]

where

\[
\varphi^k_n = -(0.1) \left( -\frac{u^k_{n+1} - 2u^k_n + u^k_{n-1}}{h^2} + 16u^k_n \right)
- 48e^{-2tk} \sin 2x_n + 2e^{-2(tk-N)} \sin 2x_n,
\]

\[
t_k = k\tau, \quad mN + 1 \leq k \leq (m + 1)N - 2,
\]

\[
m = 0, 1, ..., 1 \leq n \leq M - 1.
\]

Here, we denote

\[
a = -\frac{1}{\tau h^2}, \quad b = \frac{1}{\tau h^2} + \frac{16}{\tau}, \quad c = -\frac{3}{\tau h^2} - \frac{2}{\tau h^2} - \frac{16}{\tau}, \quad d = \frac{3}{\tau h^2}, \quad e = -\frac{1}{\tau^3}, \quad t = 2 + \frac{2r^2}{h^2} + 16r^2, \quad p = -\frac{2r^2}{h^2} - 16r^2,
\]
\[ q = 1 + \frac{2\tau^2}{h^2} + 16\tau^2, \quad s = \frac{\tau^2}{h^2}. \]

The numerical solutions are recorded for different values of \( N \) and \( M \), and \( u_n^k \) represents the numerical solution of this difference scheme at \( u(t_k, x_n) \). Table 1 is constructed for \( N = M = 40, 80, 160 \) in \( t \in [0, 1] \), \( t \in [1, 2] \), \( t \in [2, 3] \) respectively and the errors are computed by

\[
mE_M^N = \max_{mN + 1 \leq k \leq (m+1)N, \ 0 \leq n \leq M} |u(t_k, x_n) - u_n^k|.\]

**Table 1.** Errors of difference scheme (10).

| \((N, M)\)       | \(N = M = 40\) | \(N = M = 80\) | \(N = M = 160\) |
|------------------|----------------|----------------|----------------|
| \( t \in [0, 1]\) | 0.0790         | 0.0400         | 0.0200         |
| \( t \in [1, 2]\) | 0.0823         | 0.0411         | 0.0205         |
| \( t \in [2, 3]\) | 0.0789         | 0.0402         | 0.0202         |

If \( N \) and \( M \) are doubled, the values of the errors are decreases by a factor of approximately \( 1/2 \) for the first order difference scheme (10). The errors presented in this table indicates the accuracy of difference scheme.

### 5. Conclusion

In the present paper, the stability of the initial boundary value problem for the third order partial delay differential equation with nonlocal conditions is investigated. The first order of accuracy difference scheme for solution of this problem is presented. Stability estimates for solution of this difference scheme are proved. Numerical results are provided. Some statements of the present paper were published, without proof, in [23], [24].

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References

[1] S. Amirov and A.I. Kozhanov, Mixed boundary value problem for a class of strongly nonlinear sobolev-type equations of higher order, *Doklady Mathematics*, 88 (2013), 446–448.

[2] J. Niu and P. Li, Numerical algorithm for the third-order partial differential equation with three-point boundary value problem, *Abstract and Applied Analysis*, 2014 (2014), # 630671.

[3] Y. Apakov, On the solution of a boundary-value problem for a third-order equation with multiple characteristics, *Ukrainian Mathematical Journal*, 64 (2012), 1–12.

[4] Y. Apakov and B. Irgashev, Boundary-value problem for a generate high-odd order equation, *Ukrainian Mathematical Journal*, 66 (2015), 1475–1490.

[5] Y. Apakov and S. Rutkauskas, On a boundary value problem to third order pde with multiple characteristics, *Nonlinear Analytical Modelling Control*, 16 (2011), 255–269.

[6] K. Belakroum, A. Ashyralyev and A. Guezane-Lakoud, A note on the non-local boundary value problem for a third order partial differential equation, *Filomat*, 32 (2018), 801–808.

[7] C. Latrous and A. Memou, A three-point boundary value problem with an integral condition for a third-order partial differential equation, *Abstract and Applied Analysis*, 2005 (2005), 33–43.

[8] A.I. Ahmed, On the solvability of higher-order operator-differential equations in a weighted Sobolev space, *International Journal of Applied Mathematics*, 34, No 1 (2021), 147–167; doi: 10.12732/ijam.v34i1.8.

[9] K. Belakroum, On the stability of nonlocal boundary value problem for a third order pde, *International Journal of Applied Mathematics*, 34, No 2 (2021), 391–400; doi: 10.12732/ijam.v34i2.14.

[10] B.I. Islamov and Y.K. Alikulov, Boundary value problem for loaded equation of parabolic-hyperbolic type of the third order in an infinite three-dimensional domain, *International Journal of Applied Mathematics*, 34, No 2 (2021), 377–389; doi: 10.12732/ijam.v34i2.13.
[11] M. Toyoda and T. Watanabe, Solutions for a fractional-order differential equation with boundary conditions of third order, *International Journal of Applied Mathematics*, 33, No 3 (2020), 403–412; doi: 10.12732/ijam.v33i3.4.

[12] M.M. Mutallimov, L.I. Amirova, F. A. Aliev, Sh. A. Faradjova and I.A. Maharramov, Remarks to the paper: sweep algorithm for solving optimal control problem with multi-point boundary conditions, *TWMS Journal of Pure and Applied Mathematics*, 9 (2018), 243–246.

[13] N. Sh. Iskenderov, S.I. Allahverdiyeva, An inverse boundary value problem for the boussinesq-love equation with nonlocal integral condition, *TWMS Journal of Pure and Applied Mathematics*, 11 (2020), 226–237.

[14] S.T. Demiray and H. Bulut, A new approach for nizhnik-novikov-veselov system, *Advanced Mathematical Models and Applications*, 4 (2019), 79–85.

[15] B. Cahlon and D. Schmidt, Stability criteria for certain third-order delay differential equations, *Journal of Computational and Applied Mathematics*, 188 (2006), 319–335.

[16] B. Baculíková, J. Dzurina, and Y.V. Rogovchenko, Oscillation of third order trinomial delay differential equations, *Applied Mathematics and Computation*, 218 (2012), 7023–7033.

[17] A.U. Afuwape and M.O. Omeike, Stability and boundedness of solutions of a kind of third-order delay differential equations, *Computational and Applied Mathematics*, 29 (2010), 329–342.

[18] S.R. Grace, Oscillation criteria for a third order nonlinear delay differential equations with time delay, *Opuscula Mathematica*, 35 (2015), 485–497.

[19] H. Xiang, Oscillation of the third-order nonlinear neutral differential equations with distributed time delay, *Italian Journal of Pure and Applied Mathematics*, 36 (2016), 769–782.

[20] G.A. Pikina, Predictive time optimal algorithm for a third-order dynamical system with delay, *Journal of Physics: Conference Series*, 891 (2017), #012278.

[21] M.J. Mardanov and S.T. Malik, Discrete maximum principle in system with a delay in control, In: *Proc. of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan*, 46 (2020), 284–293; doi: 10.29228/proc.33.
[22] A. Ashyralyev, D. Agirseven, and R.P. Agarwal, Stability estimates for delay parabolic differential and difference equations, *Applied and Computational Mathematics, 19* (2020), 175–204.

[23] A. Ashyralyev, E. Hincal, and S. Ibrahim, Stability of the third order partial differential equations with time delay, In: *AIP Conference Proceedings, 1997*, Int. Conf. on Analysis and Applied Mathematics (ICAAM), Mersin, Turkey, Sep 06-09, 2018 (Eds: by: A. Ashyralyev, A. Lukashov and S. Sadybekov), Amer. Inst. Physics, Melville, NY USA (2018), # 020086; doi: 10.1063/1.5049080.

[24] A. Ashyralyev, E. Hincal, and S. Ibrahim, A numerical algorithm for the third order partial differential equation with time delay, In: *AIP Conference Proceedings, 2183*, Int. Conf. on Analysis and Applied Mathematics (ICAAM), Istanbul, Turkey, Sep 04-08, 2019 (Ed: H. Cakalli), Institute of Science and Technology, Maltepe University, Istanbul, Turkey (2019), # 070014; doi: 10.1063/1.5136176.

[25] A. Ashyralyev, P.E. Sobolevskii, *New Difference Schemes for Partial Differential Equations*, Birkhäuser Verlag, Boston-Berlin (2004).

[26] A. Ashyralyev, E. Hincal, and S. Ibrahim, On the absolute stable difference scheme for the third order partial differential equation with time delay, *Symmetry, 12* (2020), Article # 1033; doi: 10.3390/sym12061033.

[27] A. Ashyralyev, A.S. Erdogan, and A.M. Sarsenbi, A note on the parabolic identification problem with involution and Dirichlet condition, *Bull. of the Karaganda University-Mathematics, 99* (2020), No 3, 130–139.