Numerical simulation of the hydro-mechanically coupled production process in fractured reservoirs: Using extended finite element method

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Abstract. Any analysis of hydro-mechanically coupled production processes needs to consider flow in fractures and matrix coupled simultaneously. In this paper, fractures were described explicitly using a discrete fracture model. The fluid exchange term in the matrix and fracture governing equations was used to couple the flow in the fracture and matrix. Based on poroelasticity, the momentum and mass coupling of the standard equation were established for fractured porous media. An improved extended finite element method (I-XFEM) was used, and a solver was developed to solve the fully coupled model efficiently. In this model, fractures are decoupled from the grids, and the calculation efficiency was improved greatly. The improved enrichment functions were used to characterize the physical field and guarantee the calculation accuracy. The accuracy of the model was verified using a single-fracture model. A multi-fracture model was designed. The results showed that the cumulative production is positively related to the Elastic modulus and Poisson’s ratio, indicating that solid deformation on reservoir development has a significant influence and that the Elastic modulus and Poisson’s ratio have a significant effect on the reservoir stress sensitivity.

1. Introduction

Many natural fractures often develop in fractured reservoirs, and there are obvious heterogeneity and multi-scale characteristics. Fractures are the main channel of fluid flow, which has a significant impact on the fluid pressure field and solid stress field. In addition, for low-permeability and ultra-low permeability reservoirs, hydraulic fracturing has been widely used. While improving the fluid flow conditions and production rate, it greatly strengthens the spatial multi-scale characteristics, anisotropy, and heterogeneity of reservoirs, making it very difficult to simulate the underground fluid flow and predict production in a dynamic production process in such reservoirs.

Terzaghi [1,2] was the first to examine the hydro-mechanically coupled problem and gave an empirical formula of effective stress for the first time. Biot [3,4,5] proposed the modified effective stress formula based on Terzaghi. Rice (1976) and Coussy (1995) then successively developed the related theory [6,7]. Hydro-mechanically coupled problems have always been a major focus [8,9,10,11]. In recent years, there has been increasing research on hydraulic fracturing [12,13,14]. In this paper, based on poroelasticity, the underground fluid pressure field was coupled with the stress field of solid rock. The fluid in the fracture was treated as a viscous fluid, and the simplified Navier-Stokes equations were used to describe the fluid flow in fractures. The flow equation in the fracture cross-section could be averaged because the fracture cross-section size was smaller than its length. The fluid flow of matrix and fracture was coupled with the fluid exchange terms on the fracture
surface. On this basis, a fully coupled production model was established.

This study used the discrete fracture model (DFM) to describe fractures, and each fracture is
described explicitly. For the complex fracture model, the extended finite element method was used to
discretize the fully coupled model to improve the computational efficiency. A set of enrichment
functions based on the properties of the pressure field and stress field or asymptotic analytical
solutions were used to capture the characteristics of the physical fields. An improved extended finite
element method (I-XFEM) for solving fully coupled equations was established. In the I-XFEM, grids
were decoupled from fractures, and fractures are allowed to pass through, or partially pass through the
elements, making the algorithm avoid using unstructured grids for local refinement. Thus, the
computational complexity is reduced greatly. In this paper, a simple single-fracture model was
established, and the fully coupled model solved by I-XFEM was verified by comparing with the
results of the finite element method based on COMSOL Multiphysics® version 5.6. The changes in
the production rate of the production wells were studied by changing the Elastic modulus and
Poisson's ratio of rock.

2. Extended finite element method

2.1. Level set
The level set functions were constructed based on signed distance functions, including the normal
level set function, \( f(x) = (x - x_f) \cdot n_f \) and tangential level set function \( g(x) = (x - x_{tip}) \cdot \tau_{tip} \). Where
\( x_f \) is a point on the fracture and \( x_{tip} \) is a tip point. \( n_f \) and \( \tau_{tip} \) is the unit normal and tangential
vector of the fracture surface, respectively.

2.2. Enrichments of standard physical field approximate space
The extended finite element method is based on the finite element approximation, which characterizes
the physical field by enrichment functions while introducing additional degrees of freedom. For
elements cut by fractures, the displacement and pressure derivatives on both sides of fractures are
discontinuous. The enrichments for two physical fields are as follows [16,17]:

\[
H(x) = H(f(x)) = \begin{cases} 
1, & f(x) > 0 \\
-1, & f(x) < 0 
\end{cases} \quad \text{\# MERGEFORMAT (1)}
\]

\[
\phi^{\text{out}}(x) = \sum_j [f(x_j)] H_j(x) - \sum_j f(x_j) H_j(x) \quad \text{\# MERGEFORMAT (2)}
\]

For the elements that contain tips, two enrichments were selected to characterize the singularity of
the stress and pressure field [18]. They are as follows:

\[
[F(r, \theta), d = 1, 2, 3, 4] = \left[ \sqrt{\sin \frac{\theta}{2}}, \sqrt{\cos \frac{\theta}{2}}, \sqrt{\sin \frac{\theta}{2}} \sin \theta, \sqrt{\cos \frac{\theta}{2}} \sin \theta \right] \quad \text{\# MERGEFORMAT (3)}
\]

\[
\phi^{\text{tip}}(x) = \sqrt{\cos \frac{\theta}{2}} \quad \text{\# MERGEFORMAT (4)}
\]

where \( r, \theta \) are the coordinates at the fracture tip in the local coordinate system. For the elements
where the fractures intersect, the junction function was used to enrich the local physical field space:

\[
J(x) = \begin{cases} 
H_p(x), & H_r(x) < 0 \\
-H_p(x), & H_r(x) < 0 
\end{cases} \quad \text{\# MERGEFORMAT (5)}
\]

The enriched displacement field approximation formula is expressed as follows:
\[ u = \sum_{i \in N} N_i(x)u_i + \sum_{n=1}^{n_f} \sum_{j \in N_n} N_j(x)(H_n(x) - H_n(x_j))u_{j,n}^n + \sum_{d=1}^{n_{nc}} N_{nc}(x)\sum_{w=1}^{d} (F_{d,w}^n(x) - F_{d,w}^n(x_j))b_{j,d}^w + \sum_{z=1}^{n_{zt}} N_{zt}(x)(J_z(x) - J_z(x_w))c_{z,w}^w \]

where \( N_i(x) \) are standard finite element shape functions supported by the set of nodes \( N_i \). \( u_i \) is the displacement of nodes. \( n_f, n_t, n_j \) are the number of fractures, number of fracture tips, and number of fracture junctions, respectively. \( N_n \) denotes the set of nodes in the support area completely cut by the \( n \)th fracture, and they hold additional degrees of freedom \( a_{j,n}^n \). \( N_{nc} \) denotes the set of nodes in the support area that contains the \( d \)th tip, and they hold additional degrees of freedom, \( b_{j,d}^w \). \( N_{zt} \) denotes the set of nodes in the support area that contains the \( z \)th intersection point of intersected fractures, and they hold additional degrees of freedom \( c_{z,w}^w \).

The enriched pressure field approximation formula is as follows:

\[ p = \sum_{i \in N} N_i p_i + \Psi(x) = \sum_{i \in N} N_i p_i + \sum_{j \in N_{nc}} N_j \phi_j(x) \bar{p}_j, \]

where \( \Psi(x) \) is the enrichment function corresponding to the additional degrees of freedom \( \bar{p}_j \).

3. Governing equations

3.1. Governing equation of displacement/stress field

For the quasi-static process, the stress balance equation can be expressed as

\[ \nabla \cdot \sigma + b = 0, \]

\[ \sigma = \sigma' - \alpha p I = C : \varepsilon - \alpha p I, \]

where \( \sigma \) is the total stress tensor and \( \sigma' \) is the effective stress, \( \sigma' = C : \varepsilon \). \( \alpha \) is the Biot coefficient. \( p \) is the pore pressure. \( I \) is the identity matrix. \( C \) is the constitutive tensor and \( \varepsilon \) is the strain tensor. Neglecting the body force, \( b \), the strong form of governing equation is multiplied by the test function \( \delta p \) and integrated to establish the equivalent integral weak form of the equations:

\[ \int_{\Omega} (\nabla \cdot \delta u) : (C : \varepsilon - \alpha p I) d\Omega + \int_{\Gamma_f} \delta u \cdot \tau d\Gamma - \int_{\Gamma_r} \delta u \cdot \tilde{t} d\Gamma = 0, \]

3.2. Governing equation of the pressure field

3.2.1. Fluid flow in the matrix

The state equation for the fluid pressure \( p \) is expressed as follows [20]:

\[ p = Q \xi - \alpha Q \varepsilon, \]

where \( \xi \) and \( \varepsilon \) are the increment of the pore fluid per unit volume and dilatation of the matrix, respectively. \( \alpha \) is the Biot coefficient, \( \alpha = 1 - \frac{K_h}{K_s} \), and \( Q \) is the Biot modulus, \( Q = \left[ \frac{\phi}{K_f} + \frac{\alpha - \phi}{K_s} \right]^{-1} \),

where \( K_f \), \( K_s \), and \( K_b \) are bulk modulus of the fluid, solid, and whole porous media, respectively.

The local balance of mass for the matrix can be expressed as

\[ \frac{\partial m}{\partial t} + \nabla \cdot \mathbf{w} = 0, \]
where \( m_f \) is the fluid mass and \( w_f \) is the mass flux. From Darcy’s formula, \( v_f = \frac{w_f}{\rho_f} = -\frac{k_m}{\mu} \nabla p \), where \( \rho_f \) is the fluid density and \( k_m \) and \( \mu \) are matrix permeability and fluid viscosity, respectively.

Substitute this into equation (12), gives the following:

\[
\frac{\partial}{\partial t} \left( \frac{m_f}{\rho_f} \right) = -\frac{k_m}{\mu} \nabla^2 p .
\]  
*MERGEFORMAT (13)

Substituting equation (11) into equation (13) gives

\[
\alpha \nabla \cdot \mathbf{u} + \frac{1}{Q} \frac{\partial p}{\partial t} - \frac{k_m}{\mu} \nabla^2 p = 0 .
\]  
*MERGEFORMAT (14)

The weak form of matrix flow governing equation can be written as

\[
\int_{\Omega} \alpha \delta p \nabla \cdot \mathbf{u} d\Omega + \int_{\Gamma} \delta p \nabla \cdot \mathbf{u} d\Gamma + \int_{\Omega} \frac{k_m}{\mu} \nabla (\delta p) \nabla p d\Omega = \int_{\Gamma} \left[ q_d \cdot \mathbf{n} \right] \delta p d\Gamma
\]

where \( q_d \) represents the fluid exchange between the fracture and the matrix.

### 3.2.2. Fluid flow in fractures

The mass balance equation for fluid flow in the fracture can be expressed as

\[
\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{v}_f = 0 .
\]  
*MERGEFORMAT (16)

Consider equation (16) and the equation of state of a fluid, \( C_v = \frac{1}{\rho} \frac{\partial \rho}{\partial p} \), equation (16) can be written as

\[
C_v \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{v}_f = 0 .
\]  
*MERGEFORMAT (17)

The weak form can be obtained by multiplying the test function \( \delta p \).

\[
\int_{\Omega} \delta p \nabla \cdot \mathbf{v}_f d\Omega - \int_{\Gamma} \delta p \mathbf{n} \cdot \mathbf{v}_f d\Gamma + \int_{\Omega} C_v \delta p d\Omega = 0 .
\]  
*MERGEFORMAT (18)

The fracture fluid is considered to be an incompressible Newtonian fluid. The balance of momentum inside the two-dimensional fracture is

\[
\mu \nabla \cdot \nabla \mathbf{v}_f = \nabla p .
\]  
*MERGEFORMAT (19)

Because the fracture width is much smaller than its length, equation (19) can result in the following:

\[
\frac{\partial p}{\partial y} = 0 \quad \text{along} \ n_f ,
\]  
*MERGEFORMAT (20)

\[
\mu \frac{\partial^2 \mathbf{v}_f}{\partial y^2} = \frac{\partial p}{\partial x} \quad \text{along} \ t_f .
\]  
*MERGEFORMAT (21)

Integrating equation (21) twice along the direction of the fracture width, the velocity profile can be obtained as follows:

\[
\mathbf{v}_f \cdot \mathbf{t}_f = v = \frac{1}{2\mu} \frac{\partial p}{\partial x} \left( y^2 - h^2 \right) + v_f ,
\]  
*MERGEFORMAT (22)

where \( 2h \) is the fracture aperture, \( v_f \), is the essential boundary condition on two faces of the fracture.

The second term in equation (18) can be expressed as follows:

\[
\int_{\Gamma} \delta \nabla \cdot \mathbf{v}_f d\Omega = \int_{\Gamma} \frac{\partial p}{\partial x} \mathbf{v}_f d\Gamma = \int_{\Gamma} \frac{\partial p}{\partial x} v d\Gamma = -\int_{\Gamma} \frac{2h}{3\mu} \frac{\partial p}{\partial x} d\Gamma
\]

\[
= \int_{\Gamma} \left( \frac{2h}{3\mu} \frac{\partial p}{\partial x} - 2hv_f \right) d\Gamma
\]

\*MERGEFORMAT (23)
where \( w \) and \( v \) denote the normal and tangential flux component. Substitute (23) into equation (18), the fluid exchange term can be written as
\[
\delta p \mathbf{v} \cdot n d \Gamma = - \frac{1}{r_f} \left( \frac{2h}{3 \mu} \frac{\partial \mathbf{v}}{\partial x} - 2h \nu \right) d \Gamma - \Delta \frac{C_G \delta p d \Omega}{r_f} = \int \delta p \mathbf{q}_f \cdot n_f d \Gamma \). \* MERGEFORMAT (24)

Substituting (24) into equation (15) and ignoring, \( v_f \), the weak form governing equation with fracture-matrix flow coupled, results in the following:
\[
\int_{\Omega} \alpha \delta p \nabla \cdot \mathbf{d} d \Omega + \int_{\Gamma_f} \mathbf{q}_f \delta \mathbf{p} d \Gamma + \int \mathbf{K}_p \mathbf{u}_f \delta \mathbf{p} d \Gamma + \int \mathbf{K}_p \mathbf{u}_f \delta \mathbf{p} d \Gamma + \int \left( \frac{2h}{3 \mu} \frac{\partial \mathbf{v}_f}{\partial x} - 2h \nu \right) \delta \mathbf{p} d \Gamma + \int C_G \delta \mathbf{p} d \Omega = 0 \]
\* MERGEFORMAT (25)

### 3.3. Numerical scheme of fully coupled governing equations

Symbolically, the displacement and pressure expression can be written as, \( \begin{cases} u = NU \\ p = HP \end{cases} \). Substituting it into equations (10) and (25) result in the following the numerical scheme for fully coupled governing equations:
\[
\begin{bmatrix} 0 & 0 \\ M_u & M_p \end{bmatrix} \begin{bmatrix} \dot{U} \\ \dot{P} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_u & \mathbf{K}_p \\ 0 & \mathbf{K}_{pp} \end{bmatrix} \begin{bmatrix} \mathbf{U} \\ \mathbf{P} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_u \\ \mathbf{F}_p \end{bmatrix} \] \* MERGEFORMAT (26)

where, \( \mathbf{q} = [1 \ 1 \ 0] \)
\[
\begin{align*}
\mathbf{K}_u &= \int_{\Omega} \mathbf{B}^T : \mathbf{C} : \mathbf{B} d \Omega \\
\mathbf{K}_p &= -\int_{\Gamma_f} \mathbf{B}^T \alpha \mathbf{q}_f d \Gamma + \int_{\Gamma} \mathbf{N}^T \mathbf{H} d \Gamma \ , \\
\mathbf{F}_u &= \int_{\Gamma} \mathbf{N}^T \cdot \mathbf{d} d \Gamma \\
\mathbf{F}_p &= \int_{\Gamma_f} \mathbf{B}^T \mathbf{q}_f d \Gamma + \int_{\Gamma} \frac{2h}{3 \mu} (\mathbf{D} : \mathbf{t}_f) \mathbf{t}_f d \Gamma \\
\mathbf{M}_u &= \int_{\Omega} \alpha \mathbf{H}^T \mathbf{q}_f \mathbf{B} d \Omega \\
\mathbf{M}_p &= \int_{\Theta} \mathbf{Q}^T \mathbf{H} d \Omega \ , \\
\mathbf{K}_{pp} &= \int_{\Gamma_f} \frac{k_n}{\mu} \mathbf{D}^T \mathbf{D} d \Omega \ , \\
\mathbf{F}_2 &= -\int_{\Gamma} \mathbf{H}^T \mathbf{q}_f d \Gamma \\
\end{align*}
\]

### 4. Numerical examples

#### 4.1. Validation model

A two-dimensional single-fracture model was established to verify the method in this paper. The model size was 100 x 100 m with a well at the center point. A fracture existed in the \( x \) direction. I-XFEM was used to solve the fully coupled model, and the results were compared with the results obtained using COMSOL 5.6. Figures 1 and 2 present the model and mesh diagrams of COMSOL and I-XFEM, respectively. Table 1 lists the parameters of the model. After production for 200 hours, Figures 3 and 4 show the stress field and pressure field of COMSOL and I-XFEM, respectively.
Figure 1. Model diagram of COMSOL

Figure 2. Model diagram of I-XFEM

Table 1. Parameters of the model

| Parameters                          | Value                |
|-------------------------------------|----------------------|
| Initial reservoir pressure          | 20 MPa               |
| Initial horizontal in-situ stress   | 30×25 MPa            |
| Bottom-hole pressure                | 10 MPa               |
| Biot coefficient                    | 0.8                  |
| Compressibility coefficient of fluid | 4×10^{-10} Pa^{-1}   |
| Young modulus $E$                   | 50×10^{9} Pa         |
| Poisson’s ratio $\nu$               | 0.25                 |
| Initial matrix porosity             | 0.1                  |
| Matrix permeability                 | 9.8692×10^{-17} m^2  |
| Fluid viscosity                     | 6×10^{-3} Pa·s       |
| Fluid density                       | 1000 kg/m^3          |
| Fracture width                      | 0.001 m              |
| Time                                | 200 h                |

Figure 3. Stress and pressure fields after 200 h, from the left to right: the x-direction effective stress, y-direction effective stress, and pressure (COMSOL)

Figure 4. Stress and pressure fields after 200 h, from left to right: the x-direction effective stress, y-direction effective stress, and pressure (I-XFEM)

The effectiveness of the model in this paper can be estimated by comparing the pressure field and stress field diagram. Figure 5 was obtained by calculating the cumulative production. The cumulative production of the two methods showed a good fit, indicating that the fully coupled model and solver in this paper have good calculation accuracy.

In addition, the number of grids and element nodes of the I-XFEM model in this paper was less. Thus, the calculation burden is lower, which quantitatively shows the high efficiency of the method.
4.2. Influence of rock mechanics parameters on production

A multi-fracture model (100 × 100 m) was designed to examine the influence of the rock mechanics parameters on production. Figure 6 presents a schematic diagram of the model. Five fractures were distributed in the model. Table 2 lists the relevant parameters. Figure 7 shows the stress field and pressure field after 100 hours of production.

Table 2. Parameters of the multi-fracture model

| Parameters                           | Value                      |
|--------------------------------------|----------------------------|
| Fracture width                       | 1×10⁻³ m                  |
| Initial reservoir pressure           | 20 MPa                     |
| Bottom-hole pressure                 | 10 MPa                     |
| fluid density                        | 1000 kg/m³                 |
| Compressibility coefficient of fluid | 4×10⁻¹⁰ Pa⁻¹               |
| Biot coefficient                     | 0.8                        |
| Matrix permeability                  | 9.8692×10⁻¹⁹ m²            |
| Young modulus $E$                    | 40×10⁹ Pa                  |
| Poisson’s ratio $\nu$                | 0.25                       |
| Compressibility coefficient of rock  | 3.75×10⁻¹¹ Pa⁻¹            |
| Initial matrix porosity              | 0.1                        |
| Initial horizontal in-situ stress    | 25×20 MPa                  |

Figure 7. Stress and pressure profile after 100 h production, from left to right: the x-direction effective stress, the y-direction effective stress, and the pressure.

The curve diagrams in Figures 8 and 9 were obtained by changing the Elastic modulus and Poisson’s ratio of the rock and studying the influence of these two rock mechanical parameters on the...
cumulative production. In this numerical model, the cumulative production is positively correlated with the Elastic modulus and Poisson’s ratio. The rock was approximated as a linear elastic material, and $K_s$ was positively correlated with $E$ and $\nu$, where $K_s$ is the bulk modulus of the rock. Thus, $K_s$ increases with increasing $E$ and $\nu$, which cause a decrease in $c_s = 1/K_s$. The decrease in $c_s$ makes the stress sensitivity weaker [21] and the cumulative production higher. The numerical example shows that the Elastic modulus and Poisson's ratio have an important influence on stress sensitivity and affect the underground flow and reservoir production.

5. Conclusion
A fully coupled reservoir production model was established based on poroelasticity, and a solver was developed based on I-XFEM. The DFM was used to describe the fractures. The numerical method, I-XFEM, based on the extended finite element, decoupled the fractures from grids, improving the calculation efficiency. In addition, the different physical field characteristics were characterized by different enrichment functions, and the convergence and accuracy of the calculation were improved. The efficiency and high accuracy of the method in this paper were verified by comparing the results of the single-fracture model. The influence of the Elastic modulus and Poisson's ratio of the rock on the cumulative production was studied using a numerical example. The results showed that the cumulative production was positively related to these two rock mechanics parameters and significantly affected the underground fluid flow and reservoir production by influencing the reservoir stress sensitivity.

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