Generalized Mittag-Leffler Type Function: Fractional Integrations and Application to Fractional Kinetic Equations

Kottakkaran Sooppy Nisar*

Department of Mathematics, College of Arts and Sciences, Prince Sattam Bin Abdulaziz University, Wadi Aldawaser, Saudi Arabia

The generalized fractional integrations of the generalized Mittag-Leffler type function (GMLTF) are established in this paper. The results derived in this paper generalize many results available in the literature and are capable of generating several applications in the theory of special functions. The solutions of a generalized fractional kinetic equation using the Sumudu transform is also derived and studied as an application of the GMLTF.

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1. INTRODUCTION

The Pochhammer symbol $\left(\varpi\right)_n$ is defined by (for $\varpi \in \mathbb{C}$) [see (1), p. 2 and p. 5]):

$$
\left(\varpi\right)_n := \begin{cases} 1 & (n = 0) \\ \varpi (\varpi + 1) \ldots (\varpi + n - 1) & (n \in \mathbb{N}) \\ \frac{\Gamma(\varpi + n)}{\Gamma(\varpi)} & (\varpi \in \mathbb{C} \setminus \mathbb{Z}_0^-). 
\end{cases}
$$

The familiar generalized hypergeometric function $pF_q$ is defined as follows (see [2]):

$$
pF_q \left[ \begin{array}{c} (\varpi_p) \\ (\chi_q) \end{array} ; x \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\varpi_j)_n}{\prod_{j=1}^{q} (\chi_j)_n} \frac{x^n}{n!},
$$

where $(\varpi_j)_n$ and $(\chi_j)_n$ given in (1) and $\chi_i$ can not be a negative integer or zero. Here $p$ or $q$ or both are permitted to be zero. For all finite $x$, the series (2) is absolutely convergent if $p \leq q$ and for $|x| < 1$ if $p = q + 1$. When $p > q + 1$, then the series diverge for $x \neq 0$ and the series does not terminate.

In particular, if $p = 2$ and $q = 1$, (2) reduces to the Gaussian hypergeometric function

$$
2F_1 (\varpi_1, \varpi_2; \varpi_3; x) = \sum_{k=0}^{\infty} \frac{(\varpi_1)_n (\varpi_2)_n}{(\varpi_3)_n} \frac{x^n}{n!}.
$$

The function $\Psi_z(z)$ is the generalized Wright hypergeometric series which is given by

$$r, \Psi_z(z) = \sum_{k=0}^{\infty} \left[ \frac{(a_i, \sigma_i)_{\infty}}{(b_j, \chi_j)_{\infty}} \right] z^k$$

where $a_i, b_j \in \mathbb{C}$, and real $\sigma_i, \chi_j \in \mathbb{R}$ ($i = 1, 2, \ldots, r; j = 1, 2, \ldots, s$). The asymptotic behavior of (4) for large values of argument $z \in \mathbb{C}$ were mentioned in [3, 4] (also see [5, 6]).

To proceed our study, we need the definitions of the Mittag-Leffler functions (MLF) denoted by $E_{\alpha, \gamma}(x)$ (see [7]) and $E_{\alpha, \gamma, \chi}(x)$ [8], respectively:

$$E_{\alpha, \gamma}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \chi)} (x, \sigma, \gamma \in \mathbb{C}; x < 0, \Re(\alpha) > 0).$$

(5)

$$E_{\alpha, \gamma, \chi}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \chi)} (x, \sigma, \gamma \in \mathbb{C}; \Re(\alpha) > 0, \Re(\chi) > 0).$$

(6)

Many more generalizations and extensions of MLF widely studied recently [9, 10]. Also, the MLF performs an important role in physics and engineering problems. The derivations of physical problems of exponential nature could be governed by the physical laws through the MLF (power-law) [11–13].

Very recently, Nisar [14] defined a generalized Mittag-Leffler type function which is defined as follows

For $p, \rho, \sigma, \zeta \in \mathbb{C}, \Re(\kappa) > 0, \delta \neq 0, -1, -2, \ldots, \kappa \zeta$, and $(\omega)\zeta$ denotes the Pochhammer symbol.

$$p^{\rho, \sigma, \zeta} (\omega) = p^{\rho, \sigma, \zeta} (k_1, k_2, \ldots, k_p; \omega_1, \omega_2, \ldots, \omega_q; z)$$

$$= \sum_{\omega_1, \ldots, \omega_q} \left( k_1, k_2, \ldots, k_p \right) \omega_1, \ldots, \omega_q \delta^{1} \Gamma(\rho \sigma)$$

(7)

By assuming particular values for various parameters in (7), we get many of the popular functions in the literature. For example, $p^{\rho, \sigma, \zeta}(\omega)$ gives the $K$– function [15] and $p^{\rho, \sigma, \zeta}(\omega)$ turns to $E_{p, \sigma}(z)$ [16]. Also, $p^{\rho, \sigma, \zeta}(\omega)$ reduces to $E_{p, \sigma}(\omega)$ [17] and $p^{\rho, \sigma, \zeta}(\omega)$ gives the Mittag-Leffler function $E_{p, \sigma}(\omega)$ [8]. Similar way, $p^{\rho, \sigma, \zeta}(\omega)$ turns to the Mittag-Leffler functions $E_{p, \sigma}(\omega)$ [7]. For more details one can be referred to Nisar [14].

### 2. Generalized Fractional Integration of GMLF

Fractional calculus is one of the prominent branch of applied mathematics that deals with non-integer order derivatives and integrals (including complex orders), and their applications in almost all disciplines of science and engineering [18–22]. In this line, the use of special functions in connection with fractional calculus also studied widely [23–27]. For the basics of fractional calculus and its related literature, interesting readers can be referred to as Kiryakova [28], Miller and Ross [29], and Srivastava et al. [30]. In this paper, we studied the generalized fractional calculus of more generalized function given in (7). The generalized fractional integral operators (FIOs) involving the Appell functions $F_3$ are given for $\sigma, \sigma', \tau, \tau', \epsilon \in \mathbb{C}$ with $\Re(\epsilon) > 0$ and $x \in \mathbb{R}^+$ as follows:

$$\left( t^{s, \sigma', \tau, \tau'} f \right)(x) = \frac{x^{-\sigma}}{\Gamma(\epsilon)} \int_0^x (x-t)^{\epsilon-1} t^{-\epsilon} f(t) dt$$

and

$$\left( t^{s, \sigma', \tau, \tau'} f \right)(x) = \frac{x^{-\sigma}}{\Gamma(\epsilon)} \int_x^\infty (t-x)^{\epsilon-1} t^{-\epsilon} f(t) dt.$$

The integral operators of the types (8) and (9) have been introduced by Marichev [31] and later extended and studied by Saigo and Maeda [32]. Recently, many researchers (see [33–35]) have studied the image formulas for MSM FIOs involving various special functions.

The corresponding fractional differential operators (FDOs) have their respective forms:

$$\left( D^{s, \sigma', \tau, \tau'} f \right)(x) = \frac{d}{dx} \left[ \left( \frac{8(\epsilon)}{\epsilon} \right) + 1 \right]$$

and

$$\left( D^{s, \sigma', \tau, \tau'} f \right)(x) = -\frac{d}{dx} \left[ \left( \frac{8(\epsilon)}{\epsilon} \right) + 1 \right]$$

Here, we recall the following results (see [32, 36]).

**LEMMA 2.1.** Let $\sigma, \sigma', \tau, \tau', \epsilon, \in \mathbb{C}$ be such that $\Re(\epsilon) > 0$ and

$$\Re(\sigma) > \max\{0, \Re(\sigma + \sigma' + \tau - \epsilon), \Re(\sigma' - \tau')\}.$$

Then there exists the relation

$$\left( t^{s, \sigma', \tau, \tau'} \right)(x) \Gamma(\sigma + \epsilon - \Re(\sigma - \sigma' - \tau))$$

and

$$\Gamma(\sigma + \tau - \Re(\sigma - \sigma' - \tau))$$

$$\Gamma(\sigma + \tau - \Re(\sigma - \sigma' - \tau))$$

**LEMMA 2.2.** Let $\sigma, \sigma', \tau, \tau', \epsilon, \in \mathbb{C}$ such that $\Re(\epsilon) > 0$ and

$$\Re(\sigma) > \max\{\Re(\tau), \Re(\sigma - \sigma' + \epsilon), \Re(-\sigma - \sigma' + \epsilon)\}.$$
Then
\[
(\Gamma(\sigma')\Gamma(\sigma + \sigma' - \epsilon + \sigma))^{-1} \chi - \epsilon + \sigma - \epsilon + \sigma = \chi^{-\sigma - \sigma' + \epsilon - \sigma}, \quad (13)
\]

The main aim of this paper is to apply the generalized operators of fractional calculus for the GMLTF in order to get certain new image formulas.

2.1. Sumudu Transform

The Sumudu transform is widely used to solve various types of problems in science and engineering and it is introduced by Watugala (see [17, 38]). The details of Sumudu transforms, properties, and its applications the interesting readers can be referred to Asiru [39], Belag扁 et al. [40], and Bulut et al. [41].

The Sumudu transform over the set functions
\[
A = \{f(t) \mid \exists M, r_1, r_2 > 0, |f(t)| < Me^{r_1t}, \text{ if } t \in (-1)^j \times [0, \infty),
\]

is defined by
\[
G(u) = S[f(t); u] = \int_{-\infty}^{\infty} f(ut)e^{-ut}dt, \quad u \in (-r_1, r_2).
\]

The main aim of this study is to establish the generalized fractional calculus operators and the generalized FKEs involving GMLTF.

**Theorem 2.1.** Let \(\eta, \eta', \chi, \chi', \epsilon, r, \sigma, \lambda, \gamma \in \mathbb{C}, \Re(\kappa) > 0, \delta \neq 0, -1, -2, \ldots, \) such that \(\Re(r) > \max(0, \Re(\eta + \eta') - \chi - \epsilon), \Re((\eta' - \chi')').\)

Applying the definition (7) on the left hand side (l.h.s) of Theorem 2.1,
\[
\mathcal{J}_1 = \sum_{r=0}^{\infty} \frac{\chi}{(\eta_1)r, (\eta_2)r, \ldots, (\eta_q)r, (\delta) \Gamma(\sigma r + \lambda)} \bigg|_{0}^{\eta_1' r, \chi', \chi', r, r, \ldots, \eta_q'} \Gamma(\sigma r + \lambda)(x)
\]

Changing the order of integration and summation gives
\[
\mathcal{J}_1 = \sum_{r=0}^{\infty} \frac{\chi}{(\eta_1)r, (\eta_2)r, \ldots, (\eta_q)r, (\delta) \Gamma(\sigma r + \lambda)} \bigg|_{0}^{\eta_1' r, \chi', \chi', r, r, \ldots, \eta_q'} \Gamma(\sigma r + \lambda)(x)
\]

Applying Lemma 2.1, we get
\[
\mathcal{J}_1 = \sum_{r=0}^{\infty} \frac{\chi}{(\eta_1)r, (\eta_2)r, \ldots, (\eta_q)r, (\delta) \Gamma(\sigma r + \lambda)} \bigg|_{0}^{\eta_1' r, \chi', \chi', r, r, \ldots, \eta_q'} \Gamma(\sigma r + \lambda)(x)
\]

Using \(\Gamma(x + \kappa) = (x)_k \Gamma(x),\) we have
\[
\mathcal{J}_1 = \chi^{t - \eta - \eta' + \epsilon - 1} \Gamma(\delta) \prod_{r=1}^{\eta_1'} \Gamma(\eta_r) \prod_{r=1}^{\eta_2'} \Gamma(\eta_r) \prod_{r=1}^{\eta_q'} \Gamma(\eta_r) \prod_{r=1}^{\eta_{q+5}} \Gamma(\eta_r) \prod_{r=1}^{\eta_{q+5}} \Gamma(\eta_r)
\]

In view of (4), we reached the required result. \(\square\)

**Theorem 2.2.** Let \(\eta, \eta', \chi, \chi', \epsilon, r, \sigma, \lambda, \gamma \in \mathbb{C}, \Re(\kappa) > 0, \delta \neq 0, -1, -2, \ldots, \) such that \(\Re(r) > \max(0, \Re(\eta + \eta') - \chi - \epsilon), \Re((\eta' - \chi')').\) Then

Applying the definition (7) on the left hand side (l.h.s) of Theorem 2.2,
\[
\mathcal{J}_2 = \left|_{0}^{\chi'} \Gamma(\delta) \prod_{r=1}^{\eta_1'} \Gamma(\eta_r) \prod_{r=1}^{\eta_2'} \Gamma(\eta_r) \prod_{r=1}^{\eta_q'} \Gamma(\eta_r) \prod_{r=1}^{\eta_{q+5}} \Gamma(\eta_r) \right|_{x}^{\chi'} \left(\frac{\kappa(1), (\eta_1), (\eta_2), \ldots, (\eta_q), (\delta) \Gamma(\sigma r + \lambda)}{x} \right)
\]

Changing the order of integration and summation gives
\[
\mathcal{J}_2 = \sum_{r=0}^{\infty} \frac{\chi}{(\eta_1)r, (\eta_2)r, \ldots, (\eta_q)r, (\delta) \Gamma(\sigma r + \lambda)} \bigg|_{0}^{\chi'} \Gamma(\sigma r + \lambda)(x)
\]
Applying Lemma 2.2, we get

\[
\mathcal{J}_2 = \sum_{r=0}^{\infty} \frac{(k_1, k_2, \cdots, k_p)_r}{(\omega_1)_r, (\omega_2)_r, \cdots, (\omega_q)_r} \frac{(\gamma)_r}{(\sigma r + \lambda)_r} (\gamma r + \lambda) \\
\Gamma(-\chi + \tau + r) \Gamma(\eta + \eta' - \varepsilon + \tau + r) \\
\times \frac{\Gamma(\tau + r) \Gamma(\eta - \chi + \tau + r) \Gamma(\eta + \eta' - \varepsilon + \tau + r) \Gamma(\eta + \eta' + \chi' - \varepsilon + \tau + r)}{\Gamma(\tau + r) \Gamma(\eta - \chi + \tau + r) \Gamma(\eta + \eta' + \chi' - \varepsilon + \tau + r)} x^{-\eta - \eta' - \varepsilon - \tau - r}.
\]

Using \( \Gamma(x + r) = (x)_r \Gamma(x) \), we have

\[
\mathcal{J}_2 = x^{-\eta - \eta' - \varepsilon - \tau} \Gamma(\delta) \prod_{r=1}^{\infty} \Gamma(\omega_j) \sum_{r=0}^{\infty} \prod_{r=1}^{\infty} \Gamma(\kappa_j + r) \Gamma(\gamma r + \lambda) \\
\Gamma(\tau + r) \Gamma(\eta - \chi + \tau + r) \Gamma(\eta + \eta' - \varepsilon + \tau + r) \\
\times \frac{\Gamma(\tau + r) \Gamma(\eta - \chi + \tau + r) \Gamma(\eta + \eta' - \varepsilon + \tau + r) \Gamma(\eta + \eta' + \chi' - \varepsilon + \tau + r)}{\Gamma(\tau + r) \Gamma(\eta - \chi + \tau + r) \Gamma(\eta + \eta' + \chi' - \varepsilon + \tau + r)} x^{-\eta - \eta' - \varepsilon - \tau - r}.
\]

In view of (4), we reached the required result.

The following corollaries can derive immediately from Theorems 2.1 and 2.2 with the help of Pochhammer symbol

**Corollary 2.1. Let \( \delta = \lambda = 1 \) in Theorem 2.1, we get**

\[
\begin{aligned}
&\left( t \int_0^t x^{t-1} e^{\omega_1 \cdot x^{t-1} e^{\omega_2 \cdot x^{t-1} e^{\omega_3 \cdot x^{t-1} e^{\omega_4 \cdot x^{t-1} \cdots e^{\omega_q \cdot x^{t-1} \cdots}}}} \right)(x) \\
= &\frac{\Gamma(t) \Gamma(t + \varepsilon - \eta - \eta' - \chi) \Gamma(t + \chi' - \eta' - \eta)}{\Gamma(t + \chi) \Gamma(t + \varepsilon - \eta - \eta' - \chi) \Gamma(t + \varepsilon - \eta - \eta' - \chi) \Gamma(t + \varepsilon - \eta - \eta' - \chi)} x^{-\eta - \eta' - \varepsilon - \tau - r} \\
&\times \prod_{r=1}^{\infty} (k_1, 1_{\omega_1}, 1_{\omega_2}, \cdots, 1_{\omega_q})_{\delta} \left( t \int_0^t x^{t-1} e^{\omega_1 \cdot x^{t-1} e^{\omega_2 \cdot x^{t-1} e^{\omega_3 \cdot x^{t-1} e^{\omega_4 \cdot x^{t-1} \cdots e^{\omega_q \cdot x^{t-1} \cdots}}}} \right)(x).
\end{aligned}
\]

**Corollary 2.2. If \( \delta = \lambda = 1 \) in Theorem 2.2, then**

\[
\begin{aligned}
&\left( t \int_0^t x^{t-1} e^{\omega_1 \cdot x^{t-1} e^{\omega_2 \cdot x^{t-1} e^{\omega_3 \cdot x^{t-1} e^{\omega_4 \cdot x^{t-1} \cdots e^{\omega_q \cdot x^{t-1} \cdots}}}} \right)(x) \\
= &\frac{\Gamma(t - \chi) \Gamma(t + \chi' - \eta + \eta' - \varepsilon + \tau)}{\Gamma(t) \Gamma(t + \chi) \Gamma(t + \varepsilon - \eta - \eta' - \chi) \Gamma(t + \varepsilon - \eta - \eta' - \chi) \Gamma(t + \varepsilon - \eta - \eta' - \chi)} x^{-\eta - \eta' - \varepsilon - \tau - r} \\
&\times \prod_{r=1}^{\infty} (k_1, 1_{\omega_1}, \gamma, \tau - \chi, \eta + \eta' - \varepsilon + \tau, \eta + \chi' - \varepsilon + \tau)_{\delta} \left( t \int_0^t x^{t-1} e^{\omega_1 \cdot x^{t-1} e^{\omega_2 \cdot x^{t-1} e^{\omega_3 \cdot x^{t-1} e^{\omega_4 \cdot x^{t-1} \cdots e^{\omega_q \cdot x^{t-1} \cdots}}}} \right)(x).
\end{aligned}
\]

In the next section, we derived the generalized FKEs and we consider the Sumudu transform methodology to achieve the results.

### 3. Generalized Fractional Kinetic Equations Involving GMLTF

The generalized fractional kinetic equations (FKEs) involving the GMLTF with the Sumudu transform is derived in this section. The FKEs are studied widely in many papers [42–45].

Let \( \mathcal{R} = (\mathcal{R}_i) \) be the arbitrary reaction defined by a time-dependent quantity. The destruction \( d \) and production \( p \) depend on the quantity \( \mathcal{R} \) itself: \( d = d(\mathcal{R}) \) or \( p = p(\mathcal{R}) \) [see [42]]. The fractional differential equation can be expressed by

\[
\frac{d\mathcal{R}_i}{dt} = -d(\mathcal{R}_i) + p(\mathcal{R}_i),
\]

where \( \mathcal{R}_i \) described by \( \mathcal{R}_i(t^*) = \mathcal{R}(t - t^*) \), \( t^* > 0 \) (see, [42]). A special case of (15) is

\[
\frac{d\mathcal{R}_i}{dt} = -c_i \mathcal{R}_i(t),
\]

with \( \mathcal{R}_i(t = 0) = \mathcal{R}_{i0}, c_i > 0 \) and the solution of (16) is

\[
\mathcal{R}_i(t) = \mathcal{R}_{i0} e^{-c_i t}.
\]

Performing the integration of (16) leads to

\[
\mathcal{R}(t) - \mathcal{R}_{i0} = -c \mathcal{O}^{-1}_t \mathcal{R}(t),
\]

where \( \mathcal{O}^{-1}_t \mathcal{R} \) is the particular case of Riemann–Liouville (R-L) integral operator and \( c > 0 \) is a constant. The fractional form of (18) is (see [42])

\[
\mathcal{R}(t) = \mathcal{R}_{i0} + \mathcal{O}^{-1}_t c \mathcal{R}_i(t),
\]

where \( \mathcal{O}^{-1}_t c \mathcal{R}_i(t) \) is given by

\[
\mathcal{O}^{-1}_t c \mathcal{R}_i(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-s)^{\mu-1} c s ds, \Gamma(\mu) > 0.
\]

**Theorem 3.1.** For \( \sigma, \lambda, \gamma \in \mathbb{C}, \delta \neq 0, -1, -2, \cdots, d > 0, \mu > 0 \) then the solution of

\[
\mathcal{R}(t) - \mathcal{R}_{i0} p \mathcal{E}^\mathcal{R}_q \mathcal{R}_i(t) = -a \mathcal{O}^{-1}_t \mathcal{R}(t)
\]

is given by

\[
\mathcal{R}(t) = \mathcal{R}_{i0} + \sum_{n=0}^{\infty} \frac{(\kappa_1, \kappa_2, \cdots, \kappa_p)_n}{(\omega_1)_n, (\omega_2)_n, \cdots, (\omega_q)_n} \frac{1}{\Delta(\sigma n + \lambda)} \mathcal{R}_{i0} \mathcal{E}_n \mathcal{R}(t) dt
\]

\[
\mathcal{E}_n \mathcal{R}(t) dt
\]

**Proof:** The Sumudu transform (ST) of the R-L fractional operator is

\[
\mathcal{S}[\mathcal{R}_i(t), u \rightarrow u^* \mathcal{G}(u)]
\]

where \( \mathcal{G}(u) \) is defined in (14). Now, applying the ST on both sides of (21) and using (7) and (23), we have

\[
\mathcal{S}[\mathcal{R}(t), u] - \mathcal{R}_{i0} \mathcal{S}[p \mathcal{E}^\mathcal{R}_q \mathcal{R}(t), u] = \mathcal{S}[\mathcal{R}(t), u] - \mathcal{R}_{i0} \mathcal{S}[\mathcal{R}(t), u],
\]

\[
\mathcal{S}[\mathcal{R}(t), u] = \mathcal{R}_{i0} \mathcal{S}[p \mathcal{E}^\mathcal{R}_q \mathcal{R}(t), u],
\]

\[
\mathcal{S}[\mathcal{R}(t), u] = \mathcal{R}_{i0} \mathcal{S}[\mathcal{R}_i(t), u].
\]
which gives
\[ R'(u) = \mathcal{S}_0 \left( \int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{(\kappa_n \kappa_n \cdots \kappa_p) \Delta_n (\gamma) \Delta_n (\sigma_n + \lambda)}{(\omega_n \omega_n \cdots \omega_p) \Delta_n (\delta_n + \lambda)} dt \right) \]
\[ - \delta^u u^\rho \mathcal{R}(u), \quad (25) \]
which implies that
\[ R'(u)[1 + \delta^u u^\rho] \]
\[ = \mathcal{S}_0 \sum_{n=0}^{\infty} \frac{(\kappa_n \kappa_n \cdots \kappa_p) \Delta_n (\gamma) \Delta_n (\sigma_n + \lambda)}{(\omega_n \omega_n \cdots \omega_p) \Delta_n (\delta_n + \lambda)} \int_0^\infty e^{-t} e^\rho dt. \]
\[ (26) \]
After some simple calculation, we get
\[ R'(u) = \mathcal{S}_0 \sum_{n=0}^{\infty} \frac{(\kappa_n \kappa_n \cdots \kappa_p) \Delta_n (\gamma) \Delta_n (\sigma_n + \lambda)}{(\omega_n \omega_n \cdots \omega_p) \Delta_n (\delta_n + \lambda)} \Gamma(n+1) \]
\[ \times \left[ \sum_{s=0}^{\infty} (-\delta u^\rho)^s \right]. \]
\[ (27) \]
The inverse ST of (27) and using the formula \[ S^{-1}[u^\rho; \tau] = \frac{\tau^{\rho-1}}{\Gamma(\rho)}, \]
\[ \mathcal{R}(\tau) > 0 \]
gives
\[ \mathcal{R}(\tau) = \mathcal{S}_0 \sum_{n=0}^{\infty} \frac{(\kappa_n \kappa_n \cdots \kappa_p) \Delta_n (\gamma) \Delta_n (\sigma_n + \lambda)}{(\omega_n \omega_n \cdots \omega_p) \Delta_n (\delta_n + \lambda)} \Gamma(n+1) \]
\[ \times \sum_{s=0}^{\infty} (-1)^s \delta^s \frac{\tau^{\rho-s-1}}{\Gamma(\rho+s+1)}. \]
\[ (28) \]
In view of the Mittag-Leffler function definition, we arrived the needful result.

\[ \textbf{Theorem 3.2.} \quad \text{For } \sigma, \lambda, \gamma \in \mathbb{C}, \delta \neq 0, -1, -2, \ldots, d > 0, \epsilon > 0 \]

\[ \text{then the equation} \]
\[ \mathcal{R}(\tau) - \mathcal{S}_0 P \mathcal{E}_{\rho,q}^{\lambda,\gamma} (\epsilon^t \tau^t) = -\delta^u \mathcal{D}_t^{-\epsilon} \mathcal{R}(\tau) \]
\[ (29) \]
\[ \text{have the following solution} \]
\[ \mathcal{R}(\tau) = \mathcal{S}_0 \sum_{n=0}^{\infty} \frac{(\kappa_n \kappa_n \cdots \kappa_p) \Delta_n (\gamma) \Delta_n (\sigma_n + \lambda)}{(\omega_n \omega_n \cdots \omega_p) \Delta_n (\delta_n + \lambda)} \tau^{\rho-n-1} E_{\epsilon,n} (-\delta^u t^t) \]
\[ (30) \]
\[ \text{Proof: Applying the Sumudu transform on both sides of (29)} \]
\[ \mathcal{S}[\mathcal{R}(\tau); u] - \mathcal{S}_0 \mathcal{S}[P \mathcal{E}_{\rho,q}^{\lambda,\gamma} (\epsilon^t \tau^t); u] = \mathcal{S}[-\delta^u \mathcal{D}_t^{-\epsilon} \mathcal{R}(\tau); u], \]
\[ (31) \]
\[ \text{and using (7) and (23), we get} \]
\[ \mathcal{R}'(u) = \mathcal{S}_0 \left( \int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{(\kappa_n \kappa_n \cdots \kappa_p) \Delta_n (\gamma) \Delta_n (\sigma_n + \lambda)}{(\omega_n \omega_n \cdots \omega_p) \Delta_n (\delta_n + \lambda)} dt \right) \]
\[ - \delta^u u^\rho \mathcal{R}(u), \quad (32) \]
which gives
\[ \mathcal{R}(u)[1 + \delta^u u^\rho] \]
\[ = \mathcal{S}_0 \sum_{n=0}^{\infty} \frac{(\kappa_n \kappa_n \cdots \kappa_p) \Delta_n (\gamma) \Delta_n (\sigma_n + \lambda)}{(\omega_n \omega_n \cdots \omega_p) \Delta_n (\delta_n + \lambda)} \Gamma(n+1) \]
\[ \int_0^\infty e^{-t} e^\rho dt, \]
\[ (33) \]
which can be simplified as
\[ \mathcal{R}(u) = \mathcal{S}_0 \sum_{n=0}^{\infty} \frac{(\kappa_n \kappa_n \cdots \kappa_p) \Delta_n (\gamma) \Delta_n (\sigma_n + \lambda)}{(\omega_n \omega_n \cdots \omega_p) \Delta_n (\delta_n + \lambda)} \Gamma(n+1) \]
\[ \times \left\{ \sum_{s=0}^{\infty} (-\epsilon^u)^s \right\}. \]
\[ (34) \]
Taking the Sumudu inverse of (34) and using \[ S^{-1}[u^\rho; \tau] = \frac{\tau^{\rho-1}}{\Gamma(\rho)}, \]
we get
\[ \mathcal{R}(t) = \mathcal{S}_0 \sum_{n=0}^{\infty} \frac{(\kappa_n \kappa_n \cdots \kappa_p) \Delta_n (\gamma) \Delta_n (\sigma_n + \lambda)}{(\omega_n \omega_n \cdots \omega_p) \Delta_n (\delta_n + \lambda)} \Gamma(n+1) \]
\[ \times \sum_{s=0}^{\infty} (-1)^s \epsilon^s \frac{\tau^{\rho-s-1}}{\Gamma(\rho+s+1)}. \]
\[ (35) \]
In view of the definition of the Mittag-Leffler function, we get the required result.

\[ \textbf{Theorem 3.3.} \quad \text{For } \sigma, \lambda, \gamma \in \mathbb{C}, \delta \neq 0, -1, -2, \ldots, d > 0, \epsilon > 0 \]

\[ \text{and } a \neq \delta \text{ then the equation} \]
\[ \mathcal{R}(t) - \mathcal{S}_0 P \mathcal{E}_{\rho,q}^{\lambda,\gamma} (\epsilon^t \tau^t) = -\delta^u \mathcal{D}_t^{-\epsilon} \mathcal{R}(t) \]
\[ (36) \]
\[ \text{have the following solution} \]
\[ \mathcal{R}(t) = \mathcal{S}_0 \sum_{n=0}^{\infty} \frac{(\kappa_n \kappa_n \cdots \kappa_p) \Delta_n (\gamma) \Delta_n (\sigma_n + \lambda)}{(\omega_n \omega_n \cdots \omega_p) \Delta_n (\delta_n + \lambda)} \tau^{\rho-n-1} E_{\epsilon,n} (-\delta^u t^t) \]
\[ (37) \]
\[ \text{Proof: Applying the Sumudu transform on both sides of (36)} \]
\[ \mathcal{S}[\mathcal{R}(t); u] - \mathcal{S}_0 \mathcal{S}[P \mathcal{E}_{\rho,q}^{\lambda,\gamma} (\epsilon^t \tau^t); u] = \mathcal{S}[-\delta^u \mathcal{D}_t^{-\epsilon} \mathcal{R}(t); u], \]
\[ (38) \]
\[ \text{and using (7) and (23), we get} \]
\[ \mathcal{R}^\rho(u) = \mathcal{S}_0 \left( \int_0^\infty e^{-t} \sum_{n=0}^{\infty} \frac{(\kappa_n \kappa_n \cdots \kappa_p) \Delta_n (\gamma) \Delta_n (\sigma_n + \lambda)}{(\omega_n \omega_n \cdots \omega_p) \Delta_n (\delta_n + \lambda)} dt \right) \]
\[ - \delta^u u^\rho \mathcal{R}(u), \quad (39) \]
which gives
\[ \mathcal{R}(u)[1 + \delta^u u^\rho] \]
\[ = \mathcal{S}_0 \sum_{n=0}^{\infty} \frac{(\kappa_n \kappa_n \cdots \kappa_p) \Delta_n (\gamma) \Delta_n (\sigma_n + \lambda)}{(\omega_n \omega_n \cdots \omega_p) \Delta_n (\delta_n + \lambda)} \Gamma(n+1) \]
\[ \int_0^\infty e^{-t} e^\rho dt, \]
\[ (40) \]
which can be simplified as
\[ \mathcal{R}(u) = \mathcal{S}_0 \sum_{n=0}^{\infty} \frac{(\kappa_n \kappa_n \cdots \kappa_p) \Delta_n (\gamma) \Delta_n (\sigma_n + \lambda)}{(\omega_n \omega_n \cdots \omega_p) \Delta_n (\delta_n + \lambda)} \Gamma(n+1) \]
\[ \times \left\{ \sum_{s=0}^{\infty} (-\epsilon^u)^s \right\} . \]
\[ (41) \]
which can be simplified as

$$J(u) = \mathcal{R}_0 \sum_{n=0}^{\infty} \left( \frac{1}{n!} \right)^n \left( \frac{1}{(\xi^2/2)^n} \right) \left( \frac{1}{(\xi^2/2)^{n+1}} \right) \Gamma(n+1)$$

Taking the Sumudu inverse of (41) and using $S^{-1}\{u^n; t\} = \frac{n^{n-1}}{\Gamma(n)}$, we get

$$J(t) = \mathcal{R}_0 \sum_{n=0}^{\infty} \left( \frac{1}{n!} \right)^n \left( \frac{1}{(\xi^2/2)^n} \right) \left( \frac{1}{(\xi^2/2)^{n+1}} \right) \Gamma(n+1)$$

In view of the definition of the Mittag-Leffler function, we get the required result.

If we take $\delta = 1$ in Theorem 3.1, we get the generalized FKE involving $\Gamma-$ function as follows:

**Corollary 3.1.** For $\sigma, \lambda, \gamma \in \mathbb{C}, \delta \neq 0, -1, -2, \cdots, d > 0$, $\epsilon > 0$ then

$$J(t) = \mathcal{R}_0 \int_0^\infty e^{-\xi \cdot \Gamma(n+1)}$$

is given by

$$J(t) = \mathcal{R}_0 \sum_{n=0}^{\infty} \left( \frac{1}{n!} \right)^n \left( \frac{1}{(\xi^2/2)^n} \right) \left( \frac{1}{(\xi^2/2)^{n+1}} \right) \Gamma(n+1)$$

If we take $\delta = 1, p = q = 0$ and $\gamma = 1$ in Theorem 3.1, then the generalized FKE involving the Wiman function:

**Corollary 3.3.** For $\sigma, \lambda, \gamma \in \mathbb{C}, \delta \neq 0, -1, -2, \cdots, d > 0$, $\epsilon > 0$ then

$$J(t) = \mathcal{R}_0 \int_0^\infty e^{-\xi \cdot \Gamma(n+1)}$$

is given by

$$J(t) = \mathcal{R}_0 \sum_{n=0}^{\infty} \left( \frac{1}{n!} \right)^n \left( \frac{1}{(\xi^2/2)^n} \right) \left( \frac{1}{(\xi^2/2)^{n+1}} \right) \Gamma(n+1)$$

**4. CONCLUSION**

The generalized fractional integrations of the generalized Mittag-Leffler type function is studied in this paper. The obtained results are expressed in terms of the generalized Wright hypergeometric function and generalized hypergeometric functions. To show the potential application of GMLTF, the solutions of fractional kinetic equations are derived with the help of Sumudu transform. The results obtained in this study have significant importance as the solution of the equations are general and can derive many new and known solutions of FKEs involving various type of special functions.

**DATA AVAILABILITY STATEMENT**

All datasets generated for this study are included in the article.

**AUTHOR CONTRIBUTIONS**

The author confirms being the sole contributor of this work and has approved it for publication.

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