Approximating a Convex Body by a Polytope Using the Epsilon-Net Theorem

Márton Naszódi

Abstract. Giving a joint generalization of a result of Brazitikos, Chasapis and Hioni and results of Giannopoulos and Milman, we prove that roughly \(\left\lceil \frac{d}{\ln(1-\vartheta)} \right\rceil\) points chosen uniformly and independently from a centered convex body \(K\) in \(\mathbb{R}^d\) yield a polytope \(P\) for which \(\vartheta K \subseteq P \subseteq K\) holds with large probability. The proof is simple, and relies on a combinatorial tool, the \(\varepsilon\)-net theorem.

1. Introduction

A convex body (i.e., a compact convex set with non-empty interior) in \(\mathbb{R}^d\) is called centered, if its center of mass is the origin.

We study the following problem. Given a convex body \(K\) in \(\mathbb{R}^d\), a positive integer \(t \geq d + 1\), and \(\delta, \vartheta \in (0, 1)\). We want to show that under some assumptions on the parameters \(d, t, \delta, \vartheta\) (and without assumptions on \(K\)), the convex hull of \(t\) randomly, uniformly and independently chosen points of \(K\) contains \(\vartheta K\) with probability at least \(1 - \delta\).

The main result of [BCH16] concerns the case of very rough approximation, that is, where the number \(t\) of chosen points is linear in the dimension \(d\). It states that the convex hull of \(t = \alpha d\) random points in a centered convex body \(K\) is a polytope \(P\) which satisfies \(\frac{c_1}{d} K \subseteq P\), with probability \(1 - \delta = 1 - e^{-c_2 d}\), where \(c_1, c_2 > 0\) and \(\alpha > 1\) are absolute constants.

Our first result is a slightly stronger version of this statements, where the three constants are made explicit.

Theorem 1.1. Let \(K\) be a centered convex body in \(\mathbb{R}^d\). Choose \(t = 500d\) points \(X_1, \ldots, X_t\) of \(K\) randomly, independently and uniformly. Then

\[ \frac{1}{d} K \subseteq \text{conv}\{X_1, \ldots, X_t\} \subseteq K. \]

with probability at least \(1 - 1/e^d\).

Another instance of our general problem is Theorem 5.2 of [GM00], which concerns fine approximation, that is, where the number \(t\) of chosen points is exponential in the dimension \(d\). It states that for any \(\delta, \gamma \in (0, 1)\), if we choose \(t = e^{\gamma d}\) random points in any centered convex body \(K\) in \(\mathbb{R}^d\), then the polytope \(P\) thus obtained satisfies \(c(\delta) \gamma K \subseteq P\), with probability \(1 - \delta\).

The same argument yields Proposition 5.3 of [GM00], according to which for any \(\delta, \vartheta \in (0, 1)\), if we choose \(t = c(\delta) \left(\frac{e}{1-\vartheta}\right)^d\) random points in any centered convex body \(K\) in \(\mathbb{R}^d\), then the polytope \(P\) thus obtained satisfies \(\vartheta K \subseteq P\), with probability \(1 - \delta\).

Our main result is the following.

2010 Mathematics Subject Classification. 52A27, 52A20.

Key words and phrases. approximation by polytopes, convex body, epsilon-net theorem, Grünbaum’s theorem, VC-dimension.
Theorem 1.2. Let $\delta, \vartheta \in (0, 1)$, and let $K$ be a centered convex body in $\mathbb{R}^d$. Let

$$t := \left\lceil C \frac{(d + 1)e}{(1 - \vartheta)^d} \ln \frac{e}{(1 - \vartheta)^d} \right\rceil,$$

where $C \geq 2$ is such that

$$C^2 \left( \frac{(1 - \vartheta)^d}{e} \right)^{-2} \leq \frac{(\delta/4)^{1/(d+1)}}{e^3}.$$

Choose $t$ points $X_1, \ldots, X_t$ of $K$ randomly, independently and uniformly. Then

$$\vartheta K \subseteq \text{conv} \{X_1, \ldots, X_t\} \subseteq K$$

with probability at least $1 - \delta$.

By substituting $\vartheta = \frac{1}{d}, \delta = e^{-d-1}, C = 7$, we obtain Theorem 1.1.

By substituting $C = 3$, we obtain the two results of [GM00] mentioned above.

In Section 2, we present a generalization of a classical result of Grünbaum [GrÜ60], according to which any halfspace containing the center of mass of a convex body contains at least $1/e$ of its volume. In Section 3, we give a specific form of the $\varepsilon$-net theorem, a result from combinatorics obtained by Haussler and Welzl [HW87] building on works of Vapnik and Chervonenkis [VC68], and then refined by Komlós, Pach and Woeginger [KPW92]. Finally, in Section 4, we combine these two to obtain Theorem 1.2.

2. Convexity: A Stability Version of a Theorem of Grünbaum

Grünbaum’s theorem [GrÜ60] states that for any centered convex body $K$ in $\mathbb{R}^d$, and any half-space $F_0$ that contains the origin we have

$$\text{vol} (K) / e \leq \text{vol} (K \cap F_0),$$

where $\text{vol} (\cdot)$ denotes volume.

We say that a half-space $F$ supports $K$ from outside if the boundary of the half-space intersects $\text{bd} K$, but $F$ does not intersect the interior of $K$. Lemma 2.1 is a stability version of Grünbaum’s theorem.

**Lemma 2.1.** Let $K$ be a convex body in $\mathbb{R}^d$ with centroid at the origin. Let $0 < \vartheta < 1$, and $F$ be a half-space that supports $\vartheta K$ from outside. Then

$$\text{vol} (K) \frac{(1 - \vartheta)^d}{e} \leq \text{vol} (K \cap F).$$

**Proof of Lemma 2.1.** Let $F_0$ be a translate of $F$ containing $o$ on its boundary, and let $F_1$ be a translate of $F$ that supports $K$ from outside. Finally, let $p \in \text{bd} F_1 \cap K$. Then $\vartheta p + (1 - \vartheta)(K \cap F_0)$ (that is, the homothetic copy of $K \cap F_0$ with homothety center $p$ and ratio $1 - \vartheta$) is in $F$. Its volume is $(1 - \vartheta)^d \text{vol} (K \cap F_0)$, which by (1), is at least $(1 - \vartheta)^d \text{vol} (K) / e$, finishing the proof.

3. Combinatorics: The $\varepsilon$-net Theorem of Haussler and Welzl

**Definition 3.1.** Let $\mathcal{F}$ be a family of subsets of some set $U$. The Vapnik–Chervonenkis dimension (VC-dimension, in short) of $\mathcal{F}$ is the maximal cardinality of a subset $V$ of $U$ such that $V$ is shattered by $\mathcal{F}$, that is, $\{F \cap V : F \in \mathcal{F}\} = 2^V$.

A transversal of the set family $\mathcal{F}$ is a subset $Q$ of $U$ that intersects each member of $\mathcal{F}$.
It is well known that if $U$ is any subset of $\mathbb{R}^d$, and $\mathcal{F}$ is a family of half-spaces of $\mathbb{R}^d$, then the VC-dimension of $\mathcal{F}$ is at most $d + 1$.

The $\varepsilon$-net Theorem was first proved by Vapnik and Chervonenkis [VC68], then refined and applied in the geometric settings by Haussler and Welzl [HW87], and further improved by Komlós, Pach and Woeginger [KHW92]. We restate Theorem 3.1 of [KHW92].

**Lemma 3.2 ($\varepsilon$-net Theorem).** Let $0 < \varepsilon < 1/e$, and let $D$ be a positive integer. Let $\mathcal{F}$ be a family of some measurable subsets of a probability space $(U, \mu)$, where the probability of each member $F$ of $\mathcal{F}$ is $\mu(F) \geq \varepsilon$. Assume that the VC-dimension of $\mathcal{F}$ is at most $D$.

Let $t$ be

$$t := \left\lceil C \frac{D}{\varepsilon} \ln \frac{1}{\varepsilon} \right\rceil,$$

where $C \geq 2$ is such that

$$C^2 \varepsilon C^{-2} \leq \frac{(\delta/4)^{1/D}}{e^2}.$$

Choose $t$ elements $X_1, \ldots, X_t$ of $V$ randomly, independently according to $\mu$. Then $\{X_1, \ldots, X_t\}$ is a transversal of $\mathcal{F}$ with probability at least $1 - \delta$.

**Proof.** We give an outline of the last (routine, computational) part of the proof to obtain an explicit bound on the probability as stated in our lemma.

Let $E$ be the bad event, that is, when $\{X_1, \ldots, X_t\}$ is not a transversal of $\mathcal{F}$. At the end of the proof presented in [PA95] (Theorem 15.5 therein), it is obtained that for any integer $T$ which is larger than $t$, we have

$$\text{Prob}(E) < 2 \sum_{i=0}^{D} \left( \frac{T}{i} \right) \left( 1 - \frac{t}{T} \right)^{(T-t)\varepsilon^{-1}}.$$ 

With the choice of $T = \frac{4\varepsilon t}{C^2}$, using $\sum_{i=0}^{D} \left( \frac{T}{i} \right) \leq \left( \frac{eT}{T} \right)^D$, we obtain that if $C^2 \varepsilon C^{-2} \leq \frac{(\delta/4)^{1/D}}{e^2}$ holds, then $\text{Prob}(E) < \delta$, completing the proof of Lemma 3.2. □

For more on the theory of $\varepsilon$-nets, see [PA95, Mat02, AS16, MV17].

**4. Proof of Theorem 1.2**

**Proof of Theorem 1.2.** We consider the following set system on the base set $K$:

$$\mathcal{F} := \{K \cap F : F \text{ is a half space that supports } \partial K \text{ from outside}\}.$$ 

Clearly, the VC-dimension of $\mathcal{F}$ is at most $D := d + 1$. Let $\mu$ be the Lebesgue measure restricted to $K$, and assume that $\text{vol}(K) = 1$, that is, that $\mu$ is a probability measure. By (2), we have that each set in $\mathcal{F}$ is of measure at least $\varepsilon := \frac{1 - \theta^d}{e}$. Lemma 3.2 yields that if we choose $t$ points of $K$ independently with respect to $\mu$ (that is, uniformly), then with probability at least $1 - \delta$, we obtain a set $Q \subseteq K$ that intersects every member of $\mathcal{F}$. The latter is equivalent to $\partial K \subseteq \text{conv } Q$, completing the proof. □

**Acknowledgements**

The author thanks Nabil Mustafa for enlightening conversations on the $\varepsilon$-net theorem and topics around it.

The research was partially supported by the National Research, Development and Innovation Office (NKFIH) grant NKFI-K119670 and by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences. Part of the work was carried out during
a stay at EPFL, Lausanne at János Pach’s Chair of Discrete and Computational Geometry supported by the Swiss National Science Foundation Grants 200020-162884 and 200021-165977.

REFERENCES

[AS16] N. Alon and J. H. Spencer, The probabilistic method, Fourth, Wiley Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2016. MR3524748

[BCH16] S. Brazitikos, G. Chasapis, and L. Hioni, Random approximation and the vertex index of convex bodies, Archiv der Mathematik (2016), 1–13. http://dx.doi.org/10.1007/s00013-016-0975-2

[GM00] A. A. Giannopoulos and V. D. Milman, Concentration property on probability spaces, Adv. Math. 156 (2000), no. 1, 77–106. MR1800254

[Grü60] B. Grünbaum, Partitions of mass-distributions and of convex bodies by hyperplanes, Pacific J. Math. 10 (1960), 1257–1261. MR0124818

[HW87] D. Haussler and E. Welzl, $\varepsilon$-nets and simplex range queries, Discrete Comput. Geom. 2 (1987), no. 2, 127–151. MR884223

[KPW92] J. Komlós, J. Pach, and G. Woeginger, Almost tight bounds for $\varepsilon$-nets, Discrete Comput. Geom. 7 (1992), no. 2, 163–173. MR1139078

[Mat02] J. Matoušek, Lectures on discrete geometry, Graduate Texts in Mathematics, vol. 212, Springer-Verlag, New York, 2002.

[MV17] N. H. Mustafa and K. Varadarajan, Epsilon-approximations & epsilon-nets, ArXiv e-prints (2017Feb), available at [1702.03678]

[PA95] J. Pach and P. K. Agarwal, Combinatorial geometry., New York, NY: John Wiley & Sons, 1995 (English).

[VČ68] V. N. Vapnik and A. Ja. Červonenkis, On the uniform convergence of relative frequencies of events to their probabilities, Dokl. Akad. Nauk SSSR, 181 4 (1968), 781ff. in Russian; English translation in Theor. Probab. Appl., 16 (1971), 264–280.

MÁRTON NASZÓDI, ELTE, DEPT. OF GEOMETRY, LORAND EÖTVÖS UNIVERSITY, PÁZMÁNY PÉTER SÉTÁNY 1/C BUDAPEST, HUNGARY 1117

E-mail address: marton.naszodi@math.elte.hu