KILLING MAGNETIC CURVES IN NON-FLAT
LORENTZIAN-HEISENBERG SPACES

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Abstract. We obtain some explicit formulas for Killing magnetic curves in non-flat Lorentzian-Heisenberg spaces.

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1. Introduction

Let \((M, g)\) be a three-dimensional (semi-)Riemannian manifold. The magnetic curves \(γ\) on \(M\) are generalizations of geodesics which satisfy the following differential equation:

\[
\nabla γ' \gamma' = ϕ(γ'),
\]

where \(\nabla\) is the Levi-Civita connection of \(g\) and \(ϕ\) is a skew-symmetric \((1,1)\)-tensor field. The tensor field \(ϕ\) is known as the Lorentz force and Equation (1.1) is said to be the Lorentz equation. Magnetic curves were investigated by several authors in Riemannian and semi-Riemannian manifolds (see [10], [11], [12], [13]).

Moreover, when the magnetic fields (which will be explained later) correspond to a Killing vector, the curves \(γ\) which fulfill Equation (1.1) are said to be Killing magnetic curves. Studying Killing magnetic curves is an actual topic of research in pure mathematics and theoretical physics. In [14], Romaniuc and Munteanu considered Killing magnetic curves in three-dimensional Euclidean space. In [15], the same authors studied these curves in three-dimensional Minkowski space. In [5], Erjavec gave some characterizations about Killing magnetic curves in \(SL(2, \mathbb{R})\). In [6] and [7], Killing magnetic curves were investigated in Sol space and almost cosymplectic Sol space, respectively. In [9], Munteanu and Nistor classified Killing magnetic curves in \(S^2 \times \mathbb{R}\). In [3], Calvaruso, Munteanu and Perrone obtained a complete classification for the Killing magnetic curves in three-dimensional almost paracontact manifolds. In [2], Bejan and Romaniuc proved that equipped with a Killing vector field \(V\), any arc length parameterized spacelike or timelike curve, normal to \(V\), is a magnetic trajectory associated to \(V\) in a Walker manifold. And finally, in [4], Derkaoui and Hathout occured explicit formulas for Killing magnetic curves in Heisenberg group.

In this paper, we determine the Killing magnetic curves in the three-dimensional Lorentzian-Heisenberg space. It is known that Lorentzian-Heisenberg space can be equipped with three non-isometric metrics. We will consider two of them which are non-flat.
2. Preliminaries

Let \((M, g)\) be a three-dimensional semi-Riemannian manifold. The magnetic curves on \((M, g)\) are trajectories of charged particles, moving on \(M\) under the action of a magnetic field \(F\). A magnetic field on \(M\) is a closed 2-form \(F\) on \(M\), to which one can associate a skew-symmetric \((1, 1)\)-tensor field \(\varphi\) on \(M\), uniquely determined by

\[
F(X, Y) = g(\varphi(X), Y),
\]

for all \(X, Y \in \chi(M)\). Here, the tensor field \(\varphi\) is called the Lorentz force.

The magnetic trajectories of \(F\) are regular curves \(\gamma\) in \(M\) which satisfy the Lorentz equation

\[
\nabla_t t = \varphi(t),
\]

where \(t = \gamma'\) is the speed vector of \(\gamma\).

Furthermore, to have a positively oriented orthonormal frame field \(\{e_1, e_2, e_3\}\) and represent the vectors \(X\) and \(Y\) as \(X = x_1 e_1 + x_2 e_2 + x_3 e_3\) and \(Y = y_1 e_1 + y_2 e_2 + y_3 e_3\), the vector product of two vector fields \(X = (x_1, x_2, x_3)\) and \(Y = (y_1, y_2, y_3)\) is given by

\[
X \wedge Y = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1).
\]

The mixed product of the vector fields \(X, Y, Z \in \chi(M)\) is then defined by

\[
g(X \wedge Y, Z) = dv_g(X, Y, Z),
\]

where \(dv_g\) denotes a volume on \(M\).

A vector field \(V\) is called a Killing vector field if it satisfy the Killing equation

\[
g(\nabla_X V, Y) + g(\nabla_Y V, X) = 0,
\]

for all \(X, Y \in \chi(M)\), where \(\nabla\) is the Levi-Civita connection of the metric \(g\).

Let \(F_V = i_V dv_g\) be the Killing magnetic force corresponding to the Killing magnetic vector field \(V\) on \(M\), where \(i\) denotes the inner product. The Lorentz force of \(F_V\) is described as

\[
\varphi(X) = V \wedge X,
\]

for all \(X \in \chi(M)\). Therefore, Equation (2.1) can be rewritten as

\[
\nabla_t t = V \wedge t,
\]

and solutions of above equation are called Killing magnetic curves corresponding to the Killing vector fields \(V\).

For shortness, we will call these curves as \(V\)-magnetic curves in this paper.

3. Metrics on Lorentzian-Heisenberg Spaces

Each left-invariant Lorentzian metric on the 3-dimensional Heisenberg group \(H_3\) is isometric to one of the following metrics:

\[
g_1 = -\frac{1}{\lambda^2} dx^2 + dy^2 + (xdy + dz)^2,
\]

\[
g_2 = \frac{1}{\lambda^2} dx^2 + dy^2 - (xdy + dz)^2, \quad \lambda > 0,
\]

\[
g_3 = dx^2 + (xdy + dz)^2 - ((1-x)dy - dz)^2.
\]

Furthermore, the Lorentzian metrics \(g_1, g_2, g_3\) are non-isometric and the Lorentzian metric \(g_3\) is flat [1]. We will deal with the metrics \(g_1\) and \(g_2\) (i.e. non-flat cases).
Remark 1. According to the coordinate change $u = \lambda^{-1}x$, $v = y$, $w = z + 2xy$, we rewrite the metrics as

\begin{align}
    g_1 &= -du^2 + dv^2 + \lambda^2(udv - u^2), \\
    g_2 &= du^2 + dv^2 - \lambda^2(udv - u^2), \quad \lambda > 0.
\end{align}

4. The metric $g_1$

An orthonormal basis on $(H_3, g_1)$ is given by

\begin{align}
    e_1 &= \frac{\partial}{\partial z}, \\
    e_2 &= \frac{\partial}{\partial y} - \frac{x}{\partial z}, \\
    e_3 &= \lambda \frac{\partial}{\partial x},
\end{align}

where the vector $e_3$ is timelike. The non-zero components of the Levi-Civita connection $\nabla$ of the metric $g_1$ are given by

\begin{align}
    \nabla_{e_1} e_2 &= \nabla_{e_2} e_1 = \frac{\lambda}{2} e_3, \\
    \nabla_{e_1} e_3 &= \nabla_{e_3} e_1 = \frac{\lambda}{2} e_2, \\
    \nabla_{e_2} e_3 &= -\nabla_{e_3} e_2 = \frac{\lambda}{2} e_1.
\end{align}

The Lie algebra of Killing vector fields of $(H_3, g_1)$ admits as basis

\begin{align}
    V_1 &= \frac{\partial}{\partial z}, V_2 = \frac{\partial}{\partial y}, V_3 = \lambda \frac{\partial}{\partial x} - \lambda y \frac{\partial}{\partial z}, \\
    V_4 &= \lambda^2 y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} - \frac{1}{2} \left( x^2 + \lambda^2 y^2 \right) \frac{\partial}{\partial z}.
\end{align}

Using [4.1], we rewrite equations [4.3] as follows:

\begin{align}
    V_1 &= e_1, \quad V_2 = xe_1 + e_2, \quad V_3 = -\lambda ye_1 + e_3, \\
    V_4 &= \frac{1}{2} \left( x^2 - \lambda^2 y^2 \right) e_1 + xe_2 + \lambda ye_3.
\end{align}

If $\gamma : I \to (H_3, g_1)$, $\gamma(t) = (x(t), y(t), z(t))$ is a regular curve, then its speed vector is described as

\begin{align}
    t &= \gamma'(t) = (x'(t), y'(t), z'(t))
\end{align}

and

\begin{align}
    t &= \gamma'(t) = (z' + xy')e_1 + y'e_2 + \frac{x'}{\lambda} e_3.
\end{align}

From equations [4.2], we have

\begin{align}
    \nabla_t t &= (z' + xy')e_1 + (y'' + x'z' + xy')e_2 + \frac{x''}{\lambda} + \lambda y'(z' + xy')e_3.
\end{align}

In the following subsections, we obtain some formulas for $V_i$ —magnetic curves ($i = 1, \ldots, 4$) in $(H_3, g_1)$. To solve the differential equations, we need help of Wolfram Mathematica.
4.1. $V_1$–magnetic curves. Using (4.3) and (4.4), we have

\begin{equation}
\nabla_t \mathbf{t} = -\frac{x'}{\lambda} \mathbf{e}_2 - y' \mathbf{e}_3.
\end{equation}

The equation $\nabla_t \mathbf{t} = V_1 \wedge \mathbf{t}$ gives us the following system:

\begin{equation}
S_1 : \begin{cases}
y'' + x'(z' + xy') + \frac{1}{\lambda} = 0, \\
x'' + y'\lambda(z' + xy') + \lambda = 0, \\
(z' + xy')' = 0.
\end{cases}
\end{equation}

By integrating $S_1$ and putting it in $S_1$, we obtain

\begin{equation}
S_1 : \begin{cases}
y'' + x'(c + \frac{1}{\lambda}) = 0, \\
x'' + y'\lambda(\lambda c + 1) = 0, \\
z' + xy' = c \text{ (constant)}. 
\end{cases}
\end{equation}

Solution of the system $S_1$ is

\begin{equation}
x(t) = -\frac{\lambda}{\lambda c + 1} [k_1 \cosh(\lambda c + 1)t + k_2 \sinh(\lambda c + 1)t] + k_3, \\
y(t) = \frac{\lambda}{\lambda c + 1} [k_1 \sinh(\lambda c + 1)t + k_2 \cosh(\lambda c + 1)t] + k_4,
\end{equation}

where $k_i$, $i = 1, ..., 4$ are constants. Setting equations in $S_1$ and by integration, we get

\begin{equation}
z(t) = \left(c + \frac{\lambda}{2(\lambda c + 1)} (k_1^2 - k_2^2) t + \frac{\lambda}{(\lambda c + 1)^2} \left(\frac{k_1^2 + k_2^2}{4} + \sinh(2\lambda c + 1)t\right)
\right.
\left. + \frac{k_1 k_2}{2} \cosh(2\lambda c + 1)t + \frac{1}{\lambda c + 1} [k_1 k_3 \sinh(\lambda c + 1)t]
\right.
\left. + k_2 k_3 \cosh(\lambda c + 1)t\right] + k_5,
\end{equation}

where $k_5$ is a constant.

If $c = -\frac{1}{\lambda}$, the system $S_1$ reduces to

\begin{equation}
S_1 : \begin{cases}
y'' = 0, \\
x'' = 0, \\
z' + xy' = -\frac{1}{\lambda}.
\end{cases}
\end{equation}

Its general solution

\begin{equation}
S_1 : \begin{cases}
x(t) = k_1 t + k_2, \\
y(t) = k_3 t + k_4, \\
z(t) = -\frac{k_1 k_2}{2} t^2 - \left(\frac{1}{\lambda} + k_2 k_3\right) t + k_5,
\end{cases}
\end{equation}

where $k_i$, $i = 1, ..., 5$ are constants. Therefore, we state the following theorem.

**Theorem 1.** All $V_1$–magnetic curves of $(H_3, g_1)$ satisfy the following equations:

(i) If $c = -\frac{1}{\lambda}$, then

\begin{equation}
\gamma(t) = \left(\begin{array}{c}
x(t) = k_1 t + k_2, \\
y(t) = k_3 t + k_4, \\
z(t) = -\frac{k_1 k_2}{2} t^2 - \left(\frac{1}{\lambda} + k_2 k_3\right) t + k_5.
\end{array}\right)
\end{equation}
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4.2. $V_2$-magnetic curves. According to (4.3) and (4.4), we have

\[(4.9)\quad V_2 \wedge \mathbf{t} = \frac{x'}{\lambda} \mathbf{e}_1 - \frac{x x'}{\lambda} \mathbf{e}_2 - (x y' - (z' + x y')) \mathbf{e}_3.\]

From the equation $\nabla_t V_2 = V_2$, we get

\[(4.10)\quad S_2 : \begin{cases} y'' + x'(z' + x y') &= -\frac{xx'}{\lambda}, \\ (\lambda y' - 1)(z' + x y') &= -x y' - \frac{x''}{\lambda}, \\ (z' + x y')' &= \frac{x'}{\lambda}. \end{cases}\]

By integrating $(S_2)_3$, we deduce

\[z' + x y' = \frac{x}{\lambda} + c,\]

where $c$ is a constant. Putting the last equation in $(S_2)_{1,2}$, we get

\[S_2 : \begin{cases} y'' &= -x'(\frac{2x}{\lambda} + c), \\ (\lambda y' - 1)(\frac{x}{\lambda} + c) &= -x y' - \frac{x''}{\lambda}, \end{cases}\]

and

\[(4.11)\quad \tilde{S}_2 : \begin{cases} y' &= -\frac{x^2}{\lambda} - xc, \\ x'' &= 2x^2 - x - \lambda c(3x^2 + c\lambda x + 1) = 0. \end{cases}\]

Without loss of generality, we can suppose that $c = 0$. In this case, the equation $x'' - 2x^2 - x = 0$ involves Jacobi elliptic functions as solutions. So, we can express the following proposition.

**Proposition 1.** The Killing magnetic curves in $(H_3, g_1)$ corresponding to the Killing vector field $V_2 = x \mathbf{e}_1 + \mathbf{e}_2$ are solutions of the system of differential equations (4.10).

4.3. $V_3$-magnetic curves. From (4.3) and (4.4), we have

\[(4.12)\quad V_3 \wedge \mathbf{t} = -y' \mathbf{e}_1 + (yx' + (z' + x y')) \mathbf{e}_2 + \lambda y y' \mathbf{e}_3.\]

Using the equation $\nabla_t V_3 = V_3 \wedge \mathbf{t}$, we obtain

\[(4.13)\quad S_3 : \begin{cases} y'' + x'(z' + x y') &= y x' + (z' + x y'), \\ \frac{x'}{\lambda} + \lambda y'(z' + x y') &= \lambda y y', \\ (z' + x y')' &= -y'. \end{cases}\]

By integrating $(S_3)_3$, we occur

\[z' + x y' = -y + c,\]

where $c$ is a constant. Putting the last equation in $(S_3)_{1,2}$, we get

\[(4.14)\quad \tilde{S}_3 : \begin{cases} x' &= \lambda^2 y^2 - \lambda^2 y c, \\ y'' - 2x'y + y + c(x' - 1) &= 0. \end{cases}\]
Without loss of generality, we can assume that \( c = 0 \). In this case, when we try to solve the system \( S_3 \), i.e., the equation \( y'' - 2\lambda^2 y^3 + y = 0 \), we encounter Jacobi elliptic functions. Therefore, we write the following proposition.

**Proposition 2.** The Killing magnetic curves in \((H_3, g_1)\) corresponding to the Killing vector field \( V_3 = -\lambda ye + e_3 \) are solutions of the system of differential equations \((4.13)\).

4.4. \( V_4 \)-magnetic curves. From \((4.3) \) and \((4.4) \), we write

\[
(4.15) \quad V_4 \land t = \left( \frac{xz'}{\lambda} - \lambda yy' \right)e_1 - \left( \frac{1}{2} (x^2 - \lambda^2 y^2) \right) e_2
- \left( \frac{1}{2} (x^2 - \lambda^2 y^2) \right) e_3.
\]

From the equation \( \nabla_t t = V_4 \land t \), we get

\[
(4.16) \quad S_4 : \begin{cases}
  y'' + x' (z' + xy') = -\frac{1}{2} (x^2 - \lambda^2 y^2) \frac{z'}{\lambda} + \lambda y (z' + xy'), \\
  y(x^2 - \lambda^2 y^2) (z' + xy') = -\frac{1}{2} (x^2 - \lambda^2 y^2) y' + x (z' + xy'), \\
  (z' + xy')' = \left( \frac{xz'}{\lambda} - \lambda yy' \right).
\end{cases}
\]

By integrating \((S_4)_3\), we obtain

\[
(4.17) \quad z' + xy' = \frac{x^2}{2\lambda} - \frac{\lambda y^2}{2} + c,
\]

where \( c \) is a constant. Putting the last equation in \((S_4)_{1,2}\), we get

\[
(4.18) \quad \tilde{S}_4 : \begin{cases}
  y'' + \frac{2c}{\lambda} (x^2 - \lambda^2 y^2) = \frac{1}{2} y (x^2 - \lambda^2 y^2) + c (\lambda y - x'), \\
  x'' + \lambda y' (x^2 - \lambda^2 y^2) = \frac{1}{2} x (x^2 - \lambda^2 y^2) + c (\lambda x - \lambda^2 y').
\end{cases}
\]

It seems very difficult to solve the system \( \tilde{S}_4 \) in general case. For a particular case \( x = \lambda y \), we deduce

\[
(4.19) \quad \tilde{S}_4 : \begin{cases}
  x'' + c \lambda x' - c \lambda x = 0, \\
  y'' + c \lambda y' - c \lambda y = 0.
\end{cases}
\]

By solving the second equation of the above system, we get

\[
y(t) = k_1 e^{-\frac{4}{3}(c \lambda + \sqrt{c \lambda (4 + c \lambda)})} + k_2 e^{\frac{4}{3}(-c \lambda + \sqrt{c \lambda (4 + c \lambda)})},
\]

where \( k_1 \) and \( k_2 \) are constants. From \((4.17)\), we obtain

\[
z(t) = ct - \frac{\lambda y^2}{2} = ct - \frac{\lambda}{2} \left( k_1 e^{-\frac{4}{3}(c \lambda + \sqrt{c \lambda (4 + c \lambda)})} + k_2 e^{\frac{4}{3}(-c \lambda + \sqrt{c \lambda (4 + c \lambda)})} \right)^2.
\]

Therefore, the solution of the system \( \tilde{S}_4 \) is given by

\[
(4.20) \quad \tilde{S}_4 : \begin{cases}
  x(t) = \lambda \left( k_1 e^{-\frac{4}{3}(c \lambda + \sqrt{c \lambda (4 + c \lambda)})} + k_2 e^{\frac{4}{3}(-c \lambda + \sqrt{c \lambda (4 + c \lambda)})} \right), \\
  y(t) = k_1 e^{-\frac{4}{3}(c \lambda + \sqrt{c \lambda (4 + c \lambda)})} + k_2 e^{\frac{4}{3}(-c \lambda + \sqrt{c \lambda (4 + c \lambda)})}, \\
  z(t) = ct - \frac{\lambda}{2} \left( k_1 e^{-\frac{4}{3}(c \lambda + \sqrt{c \lambda (4 + c \lambda)})} + k_2 e^{\frac{4}{3}(-c \lambda + \sqrt{c \lambda (4 + c \lambda)})} \right)^2.
\]

Hence, we write the following proposition.
Proposition 3. The Killing magnetic curves in \((H_3,g_1)\) corresponding to the Killing vector field 

\[ V_4 = \frac{1}{2}(x^2 - \lambda^2 y^2)e_1 + xe_2 + \lambda ye_3 \]

are solutions of the system of differential equations \((4.16)\). Moreover, the space curves given by parametric equations \((4.20)\) are \(V_4\)-magnetic curves in \((H_3,g_1)\).

In the last section, we follow the steps explained in the strategy mentioned in this section for the metric \(g_2\).

5. The metric \(g_2\)

We have an orthonormal basis on \((H_3,g_2)\)

\[
\begin{align*}
   e_1 &= \frac{\partial}{\partial y} - x \frac{\partial}{\partial z}, \\
   e_2 &= \lambda \frac{\partial}{\partial x}, \\
   e_3 &= \frac{\partial}{\partial z},
\end{align*}
\]

where the vector \(e_3\) is timelike. The non-zero components of the Levi-Civita connection \(\nabla\) of the metric \(g_2\) are given by

\[
\begin{align*}
   \nabla_{e_1}e_2 &= -\nabla_{e_2}e_1 = \frac{\lambda}{2} e_4, \\
   \nabla_{e_1}e_3 &= \nabla_{e_3}e_1 = \frac{\lambda}{2} e_2, \\
   \nabla_{e_2}e_3 &= \nabla_{e_3}e_2 = -\frac{\lambda}{2} e_1.
\end{align*}
\]

The Lie algebra of Killing vector fields of \((H_3,g_2)\) admits as basis

\[
\begin{align*}
   V_1 &= \frac{\partial}{\partial z}, \\
   V_2 &= \frac{\partial}{\partial y}, \\
   V_3 &= \lambda \frac{\partial}{\partial x} - \lambda y \frac{\partial}{\partial z}, \\
   V_4 &= -\lambda^2 y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + \frac{1}{2}(-x^2 + \lambda^2 y^2) \frac{\partial}{\partial z}.
\end{align*}
\]

Using \((5.1)\), we rewrite Equations \((5.3)\) as follows:

\[
\begin{align*}
   V_1 &= e_3, \\
   V_2 &= e_1 + xe_3, \\
   V_3 &= e_2 - \lambda ye_3, \\
   V_4 &= xe_1 - \lambda ye_2 + \frac{1}{2}(x^2 + \lambda^2 y^2)e_3.
\end{align*}
\]

If \(\gamma : I \rightarrow (H_3,g_2)\), \(\gamma(t) = (x(t),y(t),z(t))\) is a regular curve, then its speed vector is described as

\[ t = \gamma'(t) = (x'(t),y'(t),z'(t)) \]

and

\[
\nabla_t t = (y'' - x'(z' + xy'))e_1 + \left(\frac{x''}{\lambda} + \lambda y'(z' + xy')\right)e_2 + (z' + xy')'e_3.
\]
5.1. *V*$_1$–magnetic curves. We have

\begin{equation}
V_1 \land t = -\frac{x'}{\lambda} e_1 + y' e_2.
\end{equation}

From the equation $\nabla_4 t = V_1 \land t$, we get

\begin{align}
S_1 : \begin{cases}
y'' - x'(z' + xy') - \frac{1}{\lambda} = 0, \\
x'' + y'(z' + xy') - \lambda = 0, \\
(z' + xy')' = 0.
\end{cases}
\end{align}

By integrating $(S_1)_3$ and putting it in $(S_1)_{1,2}$, we obtain

\begin{align}
S_1 : \begin{cases}
y'' - x'(c - \frac{1}{\lambda}) = 0, \\
x'' + y'\lambda(\lambda c - 1) = 0, \\
z' + xy' = c \text{ (constant)}. 
\end{cases}
\end{align}

Solution of the system $(S_1)_{1,2}$ is

\begin{align}
V_1 \land t = V_1 \land t = \begin{cases}
x(t) = \frac{\lambda}{\lambda c - 1} [k_1 \cos((\lambda c - 1)t) + k_2 \sin((\lambda c - 1)t)] + k_3, \\
y(t) = \frac{1}{\lambda c - 1} [k_1 \sin((\lambda c - 1)t) - k_2 \cos((\lambda c - 1)t)] + k_4,
\end{cases}
\end{align}

where $k_i, \ i = 1,...,4$ are constants. Setting equations (5.7) in $(S_1)_3$ and by integration, we get

\begin{align}
z(t) &= (c - \frac{\lambda}{2(\lambda c - 1)} (k_1^2 - k_2^2))t - \frac{\lambda}{(\lambda c - 1)^2} (k_1^2 - k_2^2) \sin(2(\lambda c - 1)t) \\
&\quad - \frac{k_1 k_2}{2} \cos(2(\lambda c - 1)t) + \frac{1}{\lambda c - 1} (k_1 k_3 \sin((\lambda c - 1)t) \\
&\quad - k_2 k_3 \cos((\lambda c - 1)t)) + k_5,
\end{align}

where $k_5$ is a constant. If $c = \frac{1}{\lambda}$, the system $S_1$ reduces to

\begin{align}
S_1 : \begin{cases}
y'' = 0, \\
x'' = 0, \\
z' + xy' = \frac{1}{\lambda}.
\end{cases}
\end{align}

Its general solution

\begin{align}
S_1 : \begin{cases}
x(t) = k_1 t + k_2, \\
y(t) = k_3 t + k_4, \\
z(t) = -\frac{k_1 k_2}{2} t^2 + \left(\frac{1}{\lambda} - k_2 k_3\right) t + k_5,
\end{cases}
\end{align}

where $k_i, \ i = 1,...,5$ are constants. So, we have proved the theorem below.

**Theorem 2.** All $V_1$–magnetic curves of $(H_3, g_2)$ satisfy the following equations:

(i) If $c = \frac{1}{\lambda}$, then

\begin{align}
\gamma(t) = \begin{pmatrix}
x(t) = k_1 t + k_2, \\
y(t) = k_3 t + k_4, \\
z(t) = -\frac{k_1 k_2}{2} t^2 + \left(\frac{1}{\lambda} - k_2 k_3\right) t + k_5
\end{pmatrix}.
\end{align}

(ii) If $c \neq \frac{1}{\lambda}$, then

\begin{align}
\gamma(t) = \begin{pmatrix}
x(t) = \frac{\lambda}{\lambda c - 1} [k_1 \cos((\lambda c - 1)t) + k_2 \sin((\lambda c - 1)t)] + k_3, \\
y(t) = \frac{1}{\lambda c - 1} [k_1 \sin((\lambda c - 1)t) - k_2 \cos((\lambda c - 1)t)] + k_4,
\end{pmatrix}
\end{align}

\begin{align}
z(t) &= (c - \frac{\lambda}{2(\lambda c - 1)} (k_1^2 - k_2^2))t - \frac{\lambda}{(\lambda c - 1)^2} (k_1^2 - k_2^2) \sin(2(\lambda c - 1)t) \\
&\quad - \frac{k_1 k_2}{2} \cos(2(\lambda c - 1)t) + \frac{1}{\lambda c - 1} (k_1 k_3 \sin((\lambda c - 1)t) \\
&\quad - k_2 k_3 \cos((\lambda c - 1)t)) + k_5,
\end{align}
where \( k_i, \ i = 1, ..., 5 \) are constants.

**Remark 2.** These curves were considered by Lee in [8] according to corresponding metric \( g_2 \) in (3.1).

### 5.2. \( V_2 \)-magnetic curves.

Direct computations give

\[
V_2 \wedge t = -\frac{x'}{\lambda} e_1 + (xy' - (z' + xy'))e_2 - \frac{x'}{\lambda} e_3.
\]

The equation \( \nabla_t t = V_2 \wedge t \) concludes

\[
S_2: \begin{cases}
y'' - x'(z' + xy') = -\frac{x'}{\lambda}, \\
(\lambda y' + 1)(z' + xy') = xy' - \frac{x''}{\lambda}, \\
(z' + xy')' = -\frac{x'}{\lambda}.
\end{cases}
\]

By integrating \((S_2)_3\), we obtain

\[
z' + xy' = -\frac{x}{\lambda} + c,
\]

where \( c \) is a constant. Putting the last equation in \((S_2)_{1,2}\), we get

\[
\bar{S}_2: \begin{cases}
y' = -\frac{x^2}{\lambda} + xc, \\
x'' + 2x^3 - x - \lambda c(3x^2 - c\lambda x - 1) = 0.
\end{cases}
\]

Without loss of generality, we can assume that \( c = 0 \). This system \( \bar{S}_2 \), i.e., the equation \( x'' + 2x^3 - x = 0 \) involves Jacobi elliptic functions. So, we write the following proposition.

**Proposition 4.** The Killing magnetic curves in \((H_3, g_2)\) corresponding to the Killing vector field \( V_2 = e_1 + xe_3 \) are solutions of the system of differential equations (5.10).

### 5.3. \( V_3 \)-magnetic curves.

We have

\[
V_3 \wedge t = (xy' + (z' + xy'))e_1 - \lambda yy'e_2 + y' e_3.
\]

From the equation \( \nabla_t t = V_3 \wedge t \), we get

\[
S_3: \begin{cases}
y'' - x'(z' + xy') = xy' + (z' + xy'), \\
\frac{x''}{\lambda} + \lambda y'(z' + xy') = -\lambda yy', \\
(z' + xy')' = y'.
\end{cases}
\]

By integrating \((S_3)_3\), we deduce

\[
z' + xy' = y + c,
\]

where \( c \) is a constant. Putting the last equation in \((S_3)_{1,2}\), we have

\[
\bar{S}_3: \begin{cases}
x' = -\lambda^2 y^2 - \lambda^2 yc, \\
y'' - 2x'y - y - c(x' + 1) = 0.
\end{cases}
\]

Without loss of generality, we can suppose that \( c = 0 \). Then, the system \( \bar{S}_3 \), i.e., the equation \( y'' + 2\lambda^2 y^3 - y = 0 \) has solutions which include Jacobi elliptic functions. Thus, we give the proposition below.

**Proposition 5.** The Killing magnetic curves in \((H_3, g_2)\) corresponding to the Killing vector field \( V_3 = e_2 - \lambda ye_3 \) are solutions of the system of differential equations (5.13).
5.4. $V_4$—magnetic curves. We write

\[ V_4 \land t = \left( -\frac{1}{2} (x'^2 + \lambda^2 y^2) \frac{y'}{\lambda} - \lambda y(x' + xy') \right) e_1 + \left( (x'^2 + \lambda^2 y^2) \frac{y'}{2} - x(z' + xy') \right) e_2 - \left( \frac{xy'}{\lambda} + \lambda yy' \right) e_3. \]

The equation $\nabla_t = V_4 \land t$ gives us

\[ (5.16) \quad S_4 : \begin{cases}
\frac{x''}{\lambda} + \lambda y'(x' + xy') = \frac{1}{\lambda} (x'^2 + \lambda^2 y^2) y' - x(z' + xy'), \\
(z' + xy')' = -\frac{xy'}{\lambda} - \lambda yy'.
\end{cases} \]

By integrating $(S_4)_3$, we obtain

\[ (5.17) \quad z' + xy' = \frac{x^2}{2\lambda} - \frac{\lambda y^2}{2} + c, \]

where $c$ is a constant. Putting the last equation in $(S_4)_{1,2}$, we get

\[ (5.18) \quad \bar{S}_4 : \begin{cases}
y'' + \frac{y'}{\lambda} (x^2 + \lambda^2 y^2) = \frac{1}{\lambda} y(x'^2 + \lambda^2 y^2) + c(-\lambda y + x'), \\
x'' - \lambda y'(x^2 + \lambda^2 y^2) = \frac{1}{\lambda} x(x'^2 + \lambda^2 y^2) - c(\lambda x + \lambda^2 y').
\end{cases} \]

It seems a true challenge to solve the system $\bar{S}_4$ in general case. However, we can find a special solution by considering $c = \lambda = 1$. In this case,

\[ x(t) = \cos \frac{\sqrt{2}}{2} t, \quad y(t) = \sin \frac{\sqrt{2}}{2} t \]

will be a solution for the system $\bar{S}_4$. Using these relations in (5.17), we get

\[ z(t) = \frac{2 - \sqrt{2}}{4} t - \frac{1}{4} \sin \sqrt{2}t + k_1 \]

where $k_1$ is a constant. So, we find a solution as follows:

\[ (5.19) \begin{cases}
x(t) = \cos \frac{\sqrt{2}}{2} t, \\
y(t) = \sin \frac{\sqrt{2}}{2} t, \\
z(t) = \frac{2 - \sqrt{2}}{4} t - \frac{1}{4} \sin \sqrt{2}t + k_1.
\end{cases} \]

Therefore, we can state the last proposition of the paper.

**Proposition 6.** The space curves given by parametric equations

\[ \gamma(t) = \begin{pmatrix}
x(t) = \cos \frac{\sqrt{2}}{2} t, \\
y(t) = \sin \frac{\sqrt{2}}{2} t, \\
z(t) = \frac{2 - \sqrt{2}}{4} t - \frac{1}{4} \sin \sqrt{2}t + k_1
\end{pmatrix} \]

are $V_4$—magnetic curves in $(H_3, g_2)$, where $k_1$ is an arbitrary constants.

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