Spherically symmetric static black holes in Einstein-aether theory

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In this paper, we systematically study spherically symmetric static spacetimes in the framework of Einstein-aether theory, and pay particular attention to the existence of black holes (BHs). In the theory, two additional gravitational modes (one scalar and one vector) appear, due to the presence of a timelike aether field. To avoid the vacuum gravit-Cerenkov radiation, they must all propagate with speeds greater than or at least equal to the speed of light. In the spherical case, only the scalar mode is relevant, so BH horizons are defined by this mode, which are always inside or at most coincide with the metric (Killing) horizons. In the present studies we first clarify several subtle issues: In particular, we find that, out of the five non-trivial field equations, only three are independent, so the problem is well-posed, as now generically there are only three unknown functions, two, $F(r)$ and $B(r)$, describe the spacetime and one, $A(r)$, describes the aether field. In addition, the two second-order differential equations for $A$ and $F$ are independent of $B$, and once they are found, $B$ is given simply by an algebraic expression of $A$, $F$ and their derivatives. In the case $c_{13} = c_{14} = 0$, we are able to solve these equations analytically, and obtained a three-parameter family of exact solutions, where $c_i$'s are the four coupling constants of the theory with $c_{ij} \equiv c_i + c_j$. To simplify the problem further, we explore the symmetry of field redefinitions, and work first with the redefined metric and aether field, and then obtain the physical ones by the inverse transformations. These clarifications significantly simplify the computational labor, which is important, as the problem is highly involved mathematically. In fact, it is exactly because of these, we find various numerical BH solutions with an accuracy that is at least two orders higher than previous ones. More important, these BH solutions are the only ones that satisfy the self-consistent conditions and meantime are consistent with all the observational constraints obtained so far. The locations of universal horizons are also identified, together with several other observationally interesting quantities, such as the innermost stable circular orbits (ISCO), the ISCO frequency, and the maximum redshift $z_{\text{max}}$ of a photon emitted by a source orbiting the ISCO. All of these quantities are found to be quite closed to their relativistic limits.

I. INTRODUCTION

The detection of the first gravitational wave (GW) from the coalescence of two massive black holes (BHs) by advanced LIGO marked the beginning of a new era, the GW astronomy \cite{Abbott:2016blz}. Following this observation, soon more than ten GWs were detected by the LIGO/Virgo scientific collaboration \cite{LIGOScientific:2017aiz,LIGOScientific:2018mvr,LIGOScientific:2018dkp}. More recently, about 50 GW candidates have been identified after LIGO/Virgo resumed operations on April 1, 2019, possibly including the coalescence of a neutron-star (NS)/BH binary. However, the details of these detections have not yet been released \cite{TheLIGOScientific:2020stg}. The outbreak of interest on GWs and BHs has further gained momentum after the detection of the shadow of the M87 BH \cite{EventHorizonTelescope:2019dse,EventHorizonTelescope:2019ivm,EventHorizonTelescope:2021sag,EventHorizonTelescope:2022xvh}.

One of the remarkable observational results is the discovery that the mass of an individual BH in these binary systems can be much larger than what was previously expected, both theoretically and observationally \cite{Abbott:2016blz,Abbott:2017vtc,Abbott:2017xyo}, leading to the proposal and refinement of various formation scenarios \cite{Spalding:2016sno,Cardoso:2018lia}. A consequence of this discovery is that the early inspiral phase may also be detectable by space-based observatories, such as LISA \cite{Bender:2018cya}, TianQin \cite{Li:2018wci}, Taiji \cite{Huang:2018hjr}, and DECIGO \cite{Kawamura:2011fd}, for several years prior to their coalescence \cite{Cardoso:2015dvg,Gair:2017yad}. Such space-based detectors may be able to see many such systems, which will result in a variety of profound scientific consequences. In particular, multiple observations with different detectors at different frequencies of signals from the same source can

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provide excellent opportunities to study the evolution of the binary in detail. Since different detectors observe at disjoint frequency bands, together they cover different evolutionary stages of the same binary system. Each stage of the evolution carries information about different physical aspects of the source.

As a result, multi-band GW detections will provide an unprecedented opportunity to test different theories of gravity in the strong field regime [23–28]. Massive systems will be observed by ground-based detectors with high signal-to-noise ratios, after being tracked for years by space-based detectors in their inspiral phase. The two portions of signals can be combined to make precise tests for different theories of gravity. In particular, joint observations of binary black holes (BBHs) with a total mass larger than about 60 solar masses by LIGO/Virgo and space-based detectors can potentially improve current bounds on dipole emission from BBHs by more than six orders of magnitude [23], which will impose severe constraints on various theories of gravity [29].

In recent works, some of the present authors generalized the post-Newtonian (PN) formalism to certain modified theories of gravity and applied it to the quasi-circular inspiral of compact binaries. In particular, we calculated in detail the waveforms, GW polarizations, response functions and energy losses due to gravitational radiation in Brans-Dicke (BD) theory [30], and screened modified gravity (SMG) [31–33] to the leading PN order, with which we then considered projected constraints from the third-generation detectors. Such studies have been further generalized to triple systems in Einstein-aether (Æ-) theory [34, 35]. When applying such formulas to the first relativistic triple system discovered in 2014 [36], we studied the radiation power, and found that quadrupole emission has almost the same amplitude as that in general relativity (GR), but the dipole emission can be as large as the quadrupole emission. This can provide a promising window to place severe constraints on Æ-theory with multi-band GW observations [23, 26].

More recently, we revisited the problem of a binary system of non-spinning bodies in a quasi-circular inspiral within the framework of Æ-theory [37–42], and provided the explicit expressions for the time-domain and frequency-domain waveforms, GW polarizations, and response functions for both ground- and space-based detectors in the PN approximation [43]. In particular, we found that, when going beyond leading-order in the PN approximation, the non-Einsteinian polarization modes contain terms that depend on both the first and second harmonics of the orbital phase. With this in mind, we calculated analytically the corresponding parameterized post-Einsteinian parameters, generalizing the existing framework to allow for different propagation speeds among scalar, vector and tensor modes, without assuming the magnitude of its coupling parameters, and meanwhile allowing the binary system to have relative motions with respect to the aether field. Such results will particularly allow for the easy construction of Einstein-aether templates that could be used in Bayesian tests of GR in the future.

In this paper, we shall continuously work on GWs and BHs in the framework of Æ-theory, but move to the ringdown phase, which consists of the relaxation of the highly perturbed, newly formed merger remnant to its equilibrium state through the shedding of any perturbations in GWs as well as in matter waves. Such a remnant will typically be a Kerr BH, provided that the binary system is massive enough and GR provides the correct description. This phase can be well described as a sum of damped exponentials with unique frequencies and damping times - quasi-normal modes (QNMs) [44].

The information contained in QNMs provide the keys in revealing whether BHs are ubiquitous in our Universe, and more important whether GR is the correct theory to describe the event even in the strong field regime. In fact, in GR according to the no-hair theorem [45], an isolated and stationary BH is completely characterized by only three quantities, mass, spin angular momentum and electric charge. Astrophysically, we expect BHs to be neutral, so it must be described by the Kerr solution. Then, the quasi-normal frequencies and damping times will depend only on the mass and angular momentum of the final BH. Therefore, to extract the physics from the ringdown phase, at least two QNMs are needed. This will require the signal-to-noise ratio (SNR) to be of the order 100 [46]. Although such high SNRs are not achievable right now, it was shown that [47] they may be achievable once the advanced LIGO and Virgo reach their design sensitivities. In any case, it is certain that they will be detected by the ground-based third-generation detectors, such as Cosmic Explorer [48, 49] or the Einstein Telescope [50], as well as the space-based detectors, including LISA [17], TianQin [18], Taiji [19], and DECIGO [20], as just mentioned above.

In the framework of Æ-theory, BHs with rotations have not been found yet, while spherically symmetric BHs have been extensively studied in the past couple of years both analytically [51–62] and numerically [63–69]. It was shown that they can be also formed from gravitational collapse [70]. Unfortunately, in these studies, the parameter space has all been ruled out by current observations [71]. Therefore, as a first step to the study of the ringdown phase of a coalescing massive binary system, in this paper we shall focus ourselves mainly on spherically symmetric static BHs in the parameter space that satisfies the self-consistent conditions and the current observations [71]. As shown explicitly in [72], spherically symmetric BHs in the new physically viable phase space can be still formed from the gravitational collapse of realistic matter.

It should be noted that the definition of BHs in Æ-theory is different from that given in GR. In particular, in Æ-theory there are three gravitational modes, the scalar, vector and tensor, which will be referred to as the spin-0, spin-1 and spin-2 gravitons, respectively. Each of them moves in principle with a different speed, given,
where \(c_i\)'s are the four dimensionless coupling constants of the theory, and \(c_{ij} \equiv c_i + c_j\), \(c_{ijk} \equiv c_i + c_j + c_k\). The constants \(c_S\), \(c_V\) and \(c_T\) represent the speeds of the spin-0, spin-1 and spin-2 gravitons, respectively. In order to avoid the existence of the vacuum gravi-Cerenkov radiation by matter such as cosmic rays [74], we must require
\[
c_S, \quad c_V, \quad c_T \geq c, \tag{1.2}
\]
where \(c\) denotes the speed of light. Therefore, as far as the gravitational sector is concerned, the horizon of a BH should be defined by the largest speed of the three different species of gravitons. However, in the spherically symmetric spacetimes, the spin-1 and spin-2 gravitons are not excited, and only the spin-0 graviton is relevant. Thus, the BH horizons in spherically symmetric spacetimes are defined by the metric [75],
\[
g^{(S)}_{\mu\nu} = g_{\mu\nu} - \left( c^2_S - 1 \right) u_\mu u_\nu, \tag{1.3}
\]
where \(u_\mu\) denotes the four-velocity of the aether field, which is always timelike and unity, \(u_\mu u^\mu = -1\).

Because of the presence of the aether in the whole spacetime, it uniquely determines a preferred direction at each point of the spacetime. As a result, the Lorentz symmetry is locally violated in \(a\)-theory [76] \(^1\). It must be emphasized that the breaking of Lorentz symmetry can have significant effects on the low-energy physics through the interactions between gravity and matter, no matter how high the scale of symmetry breaking is [81], unless supersymmetry is invoked [82]. In this paper, we shall not be concerned with this question. First, we consider \(a\)-theory as a low-energy effective theory, and second the constraints on the breaking of the Lorentz symmetry in the gravitational sector is much weaker than that in the matter sector [76]. So, to avoid this problem, in this paper we simply assume that the matter sector still satisfies the Lorentz symmetry. Then, all the particles from the matter sector will travel with speeds less or equal to the speed of light. Therefore, for these particles, the Killing (or metric) horizons still serve as the boundaries. Once inside them, they will be trapped inside the metric horizons (MHs) forever, and never be able to escape to spatial infinities.

With the above in mind, in this paper we shall carry out a systematic study of spherically symmetric spacetimes in \(a\)-theory, clarify several subtle points, and then present numerically new BH solutions that satisfy all the current observational constraints [71]. In particular, we shall show that, among the five non-trivial field equations (three evolution equations and two constraints), only three of them are independent. As a result, the system is well defined, since in the current case there are only three unknown functions: two describe the spacetime, denoted by \(F(r)\) and \(B(r)\) in Eq.(3.1), and one describes the aether field, denoted by \(A(r)\) in Eq.(3.2).

An important result, born out of the above observations, is that the three independent equations can be divided into two groups, which decouple one from the other, that is, the equations for the two functions \(A(r)\) and \(F(r)\) [cf. Eqs.(3.5) and (3.6)] are independent of the function \(B(r)\). Therefore, to solve these three field equations, one can first solve Eqs.(3.5) and (3.6) for \(A(r)\) and \(F(r)\). Once they are found, one can obtain \(B(r)\) from the third equation. It is even more remarkable, if the third equation is chosen to be the constraint \(C^u = 0\), given by Eq.(3.10), from which one finds that \(B(r)\) is then directly given by the algebraic equation (3.11) without the need of any further integration. Considering the fact that the field equations are in general highly involved mathematically, as it can be seen from Eqs.(3.5)-(3.10) and Eqs.(A.1)-(A.4), this is important, as it shall significantly simplify the computational labor, when we try to solve these field equations.

Another important step of solving the field equations is Foster’s discovery of the symmetry of the action, the so-called field redefinitions [83]: the action remains invariant under the replacements,
\[
(g_{\mu\nu}, u^\mu, c_i) \rightarrow (\hat{g}_{\mu\nu}, \hat{u}^\mu, \hat{c}_i), \tag{1.4}
\]
where \(\hat{g}_{\mu\nu}, \hat{u}^\mu\) and \(\hat{c}_i\) are given by Eqs.(2.23) and (2.24) through the introduction of a free parameter \(\sigma\). Taking the advantage of the arbitrariness of \(\sigma\), we can choose it as \(\sigma = c^2_S\), where \(c^2_S\) is given by Eq.(1.1). Then, the spin-0 and metric horizons for the metric \(\hat{g}_{\mu\nu}\) coincide [63, 65, 67]. Thus, instead of solving the field equations for \((g_{\mu\nu}, u^\mu)\), we first solve the ones for \((\hat{g}_{\mu\nu}, \hat{u}^\mu)\), as in the latter the corresponding initial value problem can be easily imposed at horizons. Once \((\hat{g}_{\mu\nu}, \hat{u}^\mu)\) is found,
using the inverse transformations, we can easily obtain \((g_{\mu\nu}, u^\mu)\).

With the above observations, we are able to solve numerically the field equations with very high accuracy, as to be shown below [cf. Table I]. In fact, the accuracy is significantly improved and in general at least two orders higher than the previous works.

In theories with breaking Lorentz symmetry, another important quantity is the universal horizon (UH) \([66, 67]\), which is the causal boundary even for particles with infinitely large speeds. The thermodynamics of UHs and relevant physics have been extensively studied since then (see, for example, Section III of the review article \([80]\), and references therein). In particular, it was shown that such horizons can be formed from gravitational collapse of a massless scalar field \([72]\). In this paper, we shall also identify the locations of the UHs of our numerical new BH solutions.

The rest of the paper is organized as follows: Sec. II provides a brief review to \(\ae\)-theory, in which the introduction of the field redefinitions, the current observational constraints on the four dimensionless coupling constants \(c_i\)'s of the theory, and the definition of the spin-0 horizons (SOHs) are given.

In Sec. III, we systematically study spherically symmetric static spacetimes, and show explicitly that among the five non-trivial field equations, only three of them are independent, so the corresponding problem is well defined: three independent equations for three unknown functions. Then, from these three independent equations we are able to obtain a three-parameter family of exact solutions for the special case \(c_{13} = c_{14} = 0\), which depends in general on the coupling constant \(c_2\). However, requiring that the solutions be asymptotically flat makes the solutions independent of \(c_2\), and the metric reduces precisely to the Schwarzschild BH solution with a non-trivially coupling aether field [cf. Eq.(3.34)], which is timelike over the whole spacetime, including the region inside the BH. To further simplify the problem, in this section we also explore the advantage of the field redefinitions [83]. In particular, we show step by step how to choose the initial values of the differential equations Eqs.(3.63) and (3.64) on SOHs, and how to reduce the phase space from four dimensions, spanned by \((F_H, \tilde{F}_H, \tilde{A}_H, A_H)\), to one dimension, spanned only by \(A_H\). So, finally the problem reduces to finding the values of \(A_H\) that lead to asymptotically flat solutions of the form \((3.79)\) \([63, 67]\).

In Sec. IV, we spell out in detail the steps to carry out our numerical analysis. In particular, as we show explicitly, Eq.(3.65) is not independent from other three differential equations. Taking this advantage, we use it to monitor our numerical errors [cf. Eq.(4.7)]. To check our numerical code further, we reproduce the BH solutions obtained in \([63, 67]\), but with an accuracy two orders higher than those obtained in \([67]\) [cf. Table I]. Unfortunately, all these BH solutions have been ruled out by the current observations \([71]\). So, in Sec. IV.B we consider cases that satisfy all the observational constraints, and obtain various new static BH solutions.

Then, in Sec. V, we present the physical metric and \(\ae\)-field for these viable new BH solutions, by using the inverse transformations from the effective fields to the physical ones. In this section, we also show explicitly that the physical fields, \(g_{\mu\nu}\) and \(u^\mu\), are also asymptotically flat, provided that the effective fields \(\tilde{g}_{\mu\nu}\) and \(\tilde{u}^\mu\) are, which are related to \(g_{\mu\nu}\) and \(u^\mu\) via the coordinate transformations given by Eq.(3.42). Then, we calculate explicitly the locations of the metric, spin-0 and universal horizons, as well as the locations of the innermost stable circular orbits (ISCO), the Lorentz gamma factor, the gravitational radius, the orbital frequency of the ISCO, the maximum redshift of a photon emitted by a source orbiting the ISCO (measured at the infinity), the radii of the circular photon orbit, and the impact parameter of the circular photon orbit. All of them are given in Table IV-V. In Table VI we also calculate the differences of these quantities obtained in \(\ae\)-theory and GR. From these results, we find that the differences are very small, and it is very hard to distinguish GR and \(\ae\)-theory through these quantities, as far as the cases considered in this paper are concerned.

Finally, in Sec. VI we summarize our main results and present some concluding remarks. There is also an appendix, in which the coefficients of the field equations for both \((g_{\mu\nu}, u^\mu)\) and \((\tilde{g}_{\mu\nu}, \tilde{u}^\mu)\) are given.

II. \(\ae\)-THEORY

In \(\ae\)-theory, the fundamental variables of the gravitational sector are \([84]\),

\[
(g_{\mu\nu}, u^\mu, \lambda),
\]

with the Greek indices \(\mu, \nu = 0, 1, 2, 3\), and \(g_{\mu\nu}\) is the four-dimensional metric of the spacetime with the signature \((-+, +, +, +)\) \([37, 70]\), \(u^\mu\) is the aether four-velocity, as mentioned above, and \(\lambda\) is a Lagrangian multiplier, which guarantees that the aether four-velocity is always timelike and unity. In this paper, we also adopt units so that the speed of light is one \((c = 1)\). Then, the general action of the theory is given by \([75]\),

\[
S = S_m + S_\ae,
\]

where \(S_m\) denotes the action of matter, and \(S_\ae\) the gravitational action of the \(\ae\)-theory, given, respectively, by

\[
S_\ae = \frac{1}{16\pi G_\ae} \int \sqrt{-\tilde{g}} \, d^4x \left[ \mathcal{L}_\ae (g_{\mu\nu}, u^\alpha, c_i) + \mathcal{L}_\lambda (g_{\mu\nu}, u^\alpha, \lambda) \right],
\]

\[
S_m = \int \sqrt{-g} \, d^4x \left[ \mathcal{L}_m (g_{\mu\nu}, u^\alpha; \psi) \right].
\]

with the Greek indices \(\mu, \nu = 0, 1, 2, 3\), and \(g_{\mu\nu}\) is the
Here \( \psi \) collectively denotes the matter fields, \( R \) and \( g \) are, respectively, the Ricci scalar and determinant of \( g_{\mu \nu} \), and
\[
\mathcal{L}_{\Lambda} = \lambda \left( g_{\alpha \beta} u^\alpha u^\beta + 1 \right), \\
\mathcal{L}_{\varphi} = R(g_{\mu \nu}) - M^\alpha_{\beta \nu \mu} (D_{\alpha} u^\mu) (D_{\beta} u^\nu),
\]
where \( D_{\mu} \) denotes the covariant derivative with respect to \( g_{\mu \nu} \), and \( M^\alpha_{\beta \nu \mu} \) is defined as
\[
M^\alpha_{\beta \nu \mu} \equiv c_1 g^{\alpha \beta} \mu \nu + c_2 \delta^\alpha_\mu \delta^\beta_\nu + c_3 \delta^\alpha_\nu \delta^\beta_\mu - c_4 u^\alpha u^\beta \mu \nu.
\]
(2.5)
Note that here we assume that matter fields couple not only to \( g_{\mu \nu} \) but also to the aether field \( u^\mu \). However, in order to satisfy the severe observational constraints, such a coupling in general is assumed to be absent [75].

The four coupling constants \( c_i \)'s are all dimensionless, and \( G_{\varphi} \) is related to the Newtonian constant \( G_N \) via the relation [85],
\[
G_N = \frac{G_{\varphi}}{1 - \frac{2}{5} c_{14}}.
\]
(2.6)
The variations of the total action with respect to \( g_{\mu \nu} \) \( u^\mu \) and \( \lambda \) yield, respectively, the field equations,
\[
R^{\mu \nu} - \frac{1}{2} g^{\mu \nu} R - S^{\mu \nu} = 8 \pi G_N T^{\mu \nu},
\]
(2.7)
\[
\mathcal{E}_{\mu \nu} = 8 \pi G_N T_{\mu \nu},
\]
(2.8)
\[
g_{\alpha \beta} u^\alpha u^\beta = -1,
\]
(2.9)
where \( R^{\mu \nu} \) denotes the Ricci tensor, and
\[
S_{\alpha \beta} \equiv D_{\mu} \left[ J_{\mu}^{\alpha \beta} u_{\beta} + J_{\alpha \beta} u_{\mu} - u_{\beta} (J_{\alpha}^{\nu \mu}) \right]
+ c_1 \left[ (D_{\alpha} u_{\mu}) (D_{\beta} u^\mu) - (D_{\mu} u_{\alpha}) (D^\mu u_{\beta}) \right]
+ c_4 a_{\alpha \beta} + \lambda u_{\alpha} u_{\beta} - \frac{1}{2} g_{\alpha \beta} J_{\delta}^\mu D_{\delta} u^\nu,
\]
(2.10)
with
\[
J_{\mu}^{\alpha \beta} \equiv M_{\mu \beta \nu \alpha}, \quad a^{\alpha} \equiv u^\alpha D_{\alpha} u^\mu n^\mu.
\]
(2.11)
From Eq.(2.8), we find that
\[
\lambda = u_{\beta} D_{\alpha} J_{\alpha}^{\beta \nu} + c_4 a^2 - 8 \pi G_{\varphi} T_{\alpha} u^\alpha,
\]
(2.12)
where \( a^2 \equiv a_{\alpha \beta} a^{\alpha \beta} \).

It is easy to show that the Minkowski spacetime is a solution of \( \varphi \)-theory, in which the aether is aligned along the time direction, \( a_{\mu} = \delta_{\mu}^\alpha \). Then, the linear perturbations around the Minkowski background show that the theory in general possess three types of excitations, scalar (spin-0), vector (spin-1) and tensor (spin-2) modes [73], with their squared speeds given by Eq.(1.1).

In addition, among the 10 parameterized post-Newtonian (PPN) parameters [86, 87], in \( \varphi \)-theory the only two parameters that deviate from GR are \( \alpha_1 \) and \( \alpha_2 \), which measure the preferred frame effects. In terms of the four dimensionless coupling constants \( c_i \)'s of the \( \varphi \)-theory, they are given by [86],
\[
\alpha_1 = - \frac{8 (c_1 c_{14} - c_- c_{13})}{2c_1 - c_- c_{13}},
\]
\[
\alpha_2 = \frac{1}{2} \alpha_1 + \frac{(c_1 - 2c_{13}) (3c_2 + c_{13} + c_4)}{c_{123} (2 - c_{14})},
\]
(2.13)
where \( c_- = c_1 - c_3 \). In the weak-field regime, using lunar laser ranging and solar alignment with the ecliptic, Solar System observations constrain these parameters to very small values [86],
\[
|\alpha_1| \leq 10^{-4}, \quad |\alpha_2| \leq 10^{-7}.
\]
(2.14)
Recently, the combination of the GW event GW170817 [89], observed by the LIGO/Virgo collaboration, and the event of the gamma-ray burst GRB 170817A [90] provides a remarkably stringent constraint on the speed of the spin-2 mode, \( -3 \times 10^{-15} < c_7 - 1 < 7 \times 10^{-16} \), which, together with Eq.(1.1), implies that
\[
|c_{13}| < 10^{-15}.
\]
(2.15)
Requiring that the theory: (a) be self-consistent, such as free of ghosts and instability; and (b) satisfy all the observational constraints obtained so far, it was found that the parameter space of the theory is considerably restricted [71]. In particular, \( c_{14} \) and \( c_2 \) are restricted to
\[
0 \leq c_{14} \leq 2.5 \times 10^{-5}, \quad 0 \leq c_{14} \leq c_2 \leq 0.095.
\]
(2.16)
(2.17)
The constraints on other parameters depend on the values of \( c_{14} \). If dividing the above range into three intervals: (i) \( 0 \leq c_{14} \leq 2 \times 10^{-7} \); (ii) \( 2 \times 10^{-7} < c_{14} \leq 2 \times 10^{-6} \); and (iii) \( 2 \times 10^{-6} \leq c_{14} \leq 2.5 \times 10^{-5} \), in the first and last intervals, one finds [71],
\[
(i) \quad 0 \leq c_{14} \leq 2 \times 10^{-7}, \quad c_{14} \leq c_2 \leq 0.095,
\]
(2.18)
\[
(ii) \quad 2 \times 10^{-6} \leq c_{14} \leq 2.5 \times 10^{-5}, \quad 0 \leq c_2 - c_{14} \leq 2 \times 10^{-7}.
\]
(2.19)
In the intermediate regime (ii) \( 2 \times 10^{-7} < c_{14} \leq 2 \times 10^{-6} \), in addition to the ones given by Eqs.(2.16) and (2.17), the following constraints must be also satisfied,
\[
-10^{-7} \leq \frac{c_{14} (c_{14} + 2c_2 c_{14} - c_2)}{c_2 (2 - c_{14})} \leq 10^{-7}.
\]
(2.20)
Note that in writing Eq.(2.20), we had set \( c_{13} = 0 \), for which the errors are of the order \( \mathcal{O}(c_{13}) \approx 10^{-15} \), which
can be safely neglected for the current and forthcoming experiments. The results in this intermediate interval of $c_{14}$ were shown explicitly by Fig. 1 in [71]. Note that in this figure, the physically valid region is restricted only to the half plane $c_{14} \geq 0$, as shown by Eq.(2.16).

Since the theory possesses three different modes, and all of them are moving in different speeds, in general these different modes define different horizons [75]. These horizons are the null surfaces of the effective metrics,

$$g_{\alpha\beta}^{(A)} \equiv g_{\alpha\beta} - (c_A^2 - 1)u_\alpha u_\beta,$$  \hspace{1cm} (2.21)

where $A = S, V, T$. If a BH is defined to be a region that traps all possible causal influences, it must be bounded by a horizon corresponding to the fastest speed. Assuming that the matter sector always satisfies the Lorentz symmetry, we can see that in the matter sector the fastest speed will be the speed of light. Then, overall, the fastest speed must be one of the three gravitational modes.

However, in the spherically symmetric case, the spin-1 and spin-2 modes are not excited, so only the spin-0 gravitons are relevant. Therefore, in the present paper the relevant horizons for the gravitational sector are the 50Hs 2. In order to avoid the existence of the vacuum gravi-Cerenkov radiation by matter such as cosmic rays [74], we assume that $c_S \geq 1$, so that 50Hs are always inside or at most coincide with the metric horizons, the null surfaces defined by the metric $g_{\alpha\beta}$. The equality happens only when $c_S = 1$.

\section{Field Redefinitions}

Due to the specific symmetry of the theory, Foster found that the action $S_\omega (g_{\alpha\beta}, u^\alpha, c_i)$ given by Eqs.(2.3)-(2.5) does not change under the following field redefinitions [83],

$$(g_{\alpha\beta}, u^\alpha, c_i) \rightarrow (\hat{g}_{\alpha\beta}, \hat{u}^\alpha, \hat{c}_i),$$ \hspace{1cm} (2.22)

where

$$\hat{g}_{\alpha\beta} = g_{\alpha\beta} - (\sigma - 1)u_\alpha u_\beta, \quad \hat{u}^\alpha = \frac{1}{\sqrt{\sigma}} u^\alpha,$$

$$\hat{g}^{\alpha\beta} = g^{\alpha\beta} - (\sigma^{-1} - 1)u^\alpha u^\beta, \quad \hat{u}_\alpha = \sqrt{\sigma} u_\alpha,$$ \hspace{1cm} (2.23)

and

$$\hat{c}_1 = \frac{\sigma}{2} \left[ (1 + \sigma^{-2}) c_1 + (1 - \sigma^{-2}) c_3 - (1 - \sigma^{-1})^2 \right],$$

$$\hat{c}_2 = \sigma (c_2 + 1 - \sigma^{-1}),$$

$$\hat{c}_3 = \frac{\sigma}{2} \left[ (1 - \sigma^{-2}) c_1 + (1 + \sigma^{-2}) c_3 - (1 - \sigma^{-2}) \right],$$

$$\hat{c}_4 = c_4 - \frac{\sigma}{2} \left[ \left(1 - \frac{1}{\sigma}\right) c_1 + \left(1 - \frac{1}{\sigma^2}\right) c_3 \right] - \left(1 - \frac{1}{\sigma}\right)^2,$$ \hspace{1cm} (2.24)

with $\sigma$ being a positive otherwise arbitrary constant. Then, the following useful relations between $c_i$ and $\hat{c}_i$ hold,

$$\hat{c}_2 = c_2 + 1 - \sigma, \quad \hat{c}_{14} = c_{14},$$

$$\hat{c}_{13} = \sigma (c_{13} - 1) + 1, \quad \hat{c}_{123} = c_{123},$$

$$\hat{c}_{-} = \sigma^{-1} (c_{-} + 1).$$ \hspace{1cm} (2.25)

Note that $\hat{g}^{\alpha\beta} \hat{g}_{\alpha\beta} = \delta^\alpha_\alpha$ and $\hat{u}_\alpha \equiv \hat{g}_{\alpha\beta} \hat{u}^\beta$. Then, from Eq.(2.23), we find that

$$\hat{g}_{\alpha\beta} \hat{u}^\alpha \hat{u}^\beta = -1, \quad \hat{g} = \sigma g,$$ \hspace{1cm} (2.26)

where $\hat{g}$ is the determinant of $\hat{g}_{\alpha\beta}$. Thus, replacing $G_\omega$ and $L_\lambda$ by $\hat{G}_\omega$ and $\hat{L}_\lambda$ in Eq.(2.3), where

$$\hat{G}_\omega \equiv \sqrt{\sigma} G_\omega, \quad \hat{L}_\lambda \equiv \lambda (\hat{g}_{\alpha\beta} \hat{u}^\alpha \hat{u}^\beta + 1),$$ \hspace{1cm} (2.27)

we find that

$$S_\omega (g_{\alpha\beta}, u^\alpha, c_i, G_\omega, \lambda) = \hat{S}_\omega \left(\hat{g}_{\alpha\beta}, \hat{u}^\alpha, \hat{c}_i, \hat{G}_\omega, \lambda \right).$$ \hspace{1cm} (2.28)

As a result, when the matter field is absent, that is, $L_m = 0$, the Einstein-aether vacuum field equations take the same forms for the fields $(\hat{g}_{\alpha\beta}, \hat{u}^\alpha, \hat{c}_i, \lambda)$,

$$\hat{R}^{\mu\nu} - \frac{1}{2} \hat{g}^{\mu\nu} \hat{R} = \hat{S}^{\mu\nu},$$ \hspace{1cm} (2.29)

$$\hat{E}^{\mu}_\mu = 0,$$ \hspace{1cm} (2.30)

$$\hat{g}_{\alpha\beta} \hat{u}^\alpha \hat{u}^\beta = -1,$$ \hspace{1cm} (2.31)

where $\hat{R}^{\mu\nu}$ and $\hat{R}$ are the Ricci tensor and scalar made of $\hat{g}_{\alpha\beta}$. $S^{\mu\nu}$ and $\hat{E}^{\mu}_\mu$ are given by Eq.(2.10) simply by replacing $(g_{\mu\nu}, u^\mu, c_i)$ by $(\hat{g}_{\mu\nu}, \hat{u}^\mu, \hat{c}_i)$.

Therefore, for any given vacuum solution of the Einstein-aether field equations $(g_{\mu\nu}, u^\mu, c_i, \lambda)$, using the above field redefinitions, we can obtain a class of the vacuum solutions of the Einstein-aether field equations, given by $(\hat{g}_{\mu\nu}, \hat{u}^\mu, \hat{c}_i, \lambda)$.

Certainly, such obtained solutions may not always satisfy the physical and observational constraints found so far [71].

\footnote{If we consider Hořava gravity [79] as the UV complete theory of the hypersurface-orthogonal æ-theory (the khronometric theory) [91–94], even in the gravitational sector, the relevant boundaries will be the UHs, once such a UV complete theory is taken into account [80].}

\footnote{It should be noted that this holds in general only for the vacuum case. In particular, when matter presence, the aether field will be directly coupled with matter through the metric redefinitions.}
In this paper, we shall take advantage of such field redefinitions to simplify the corresponding mathematical problems by assuming that the fields described by \((g_{\mu\nu}, u^{\mu}, c, \lambda)\) are the physical ones, while the ones described by \((\hat{g}_{\mu\nu}, \hat{u}^{\mu}, \hat{c}, \lambda)\) as the “effective” ones, although both of the two metrics are the vacuum solutions of the Einstein-aether field equations, and can be physical, provided that the constraints recently given in [71] are satisfied.

The gravitational sector described by \((\hat{g}_{\mu\nu}, \hat{u}^{\mu}, \hat{c}, \lambda)\) has also three different propagation modes, with their speeds \(c_A\) given by Eq.(1.1) with the replacement \(c_i\) by \(\hat{c}_i\). Each of these modes defines a horizon, which is now a null surface of the metric,

\[
\hat{g}^{(A)}_{\alpha\beta} \equiv \hat{g}_{\alpha\beta} - (\hat{c}_A^2 - 1) \hat{u}_\alpha \hat{u}_\beta, \quad (2.32)
\]

where \(A = S, V, T\). It is interesting to note that

\[
\hat{c}_A^2 = \frac{c^2}{\sigma}, \quad (2.33)
\]

Thus, choosing \(\sigma = c^2\), we have \(\hat{c}_S = 1\), and from Eq.(2.32) we find that

\[
\hat{g}^{(S)}_{\alpha\beta} = \hat{g}_{\alpha\beta}, \quad (\sigma = c^2\), \quad (2.34)
\]

that is, the S0H of the metric \(\hat{g}_{\alpha\beta}\) coincides with its MH. Moreover, from Eqs.(2.21) and (2.23) we also find that

\[
\hat{g}^{(S)}_{\alpha\beta} = \hat{g}_{\alpha\beta}, \quad (\sigma = c^2\). \quad (2.35)
\]

Therefore, with the choice \(\sigma = c^2\), the MH of \(\hat{g}_{\alpha\beta}\) is also the S0H of the metric \(g_{\alpha\beta}\).

### B. Hypersurface-Orthogonal Aether Fields

When the aether field \(u_\mu\) is hypersurface-orthogonal (HO), the Einstein-aether field equations depend only on three combinations of the four coupling constants \(c_i\)’s. To see this clearly, let us first notice that, if the aether is HO, the twist \(\omega^\mu\) vanishes [75], where \(\omega^\mu\) is defined as \(\omega^\mu \equiv \epsilon^{\mu\nu\alpha\beta} u_\nu D_\alpha u_\beta\). Since

\[
\omega_\mu \omega^\mu = (D_\mu u_\nu) (D^\nu u^\mu) - (D_\mu u_\nu) (D^\mu u^\nu) - (u^\mu D_\mu u_\nu) (u^\nu D_\nu u^\nu), \quad (2.36)
\]

we can see that the addition of the term

\[
\Delta \mathcal{L}_x \equiv c_0 \omega_\mu \omega^\mu, \quad (2.37)
\]

to \(\mathcal{L}_x\) will not change the action, where \(c_0\) is an arbitrary real constant. However, this is equivalent to replacing \(c_i\) by \(\hat{c}_i\) in \(\mathcal{L}_x\), where

\[
\hat{c}_1 \equiv c_1 + c_0, \quad \hat{c}_2 \equiv c_2, \quad \hat{c}_3 \equiv c_3 - c_0, \quad \hat{c}_4 \equiv c_4 - c_0, \quad (2.38)
\]

Thus, by properly choosing \(c_0\), we can always eliminate one of the three parameters, \(c_1\), \(c_3\) and \(c_4\), or one of their combinations. Therefore, in this case only three combinations of \(c_i\)’s appear in the field equations. Since

\[
\hat{c}_{13} = c_{13}, \quad \hat{c}_{14} = c_{14}, \quad \hat{c}_2 = c_2, \quad (2.39)
\]

without loss of the generality, we can always choose these three combinations as \(c_{13}, c_{14}\) and \(c_2\).

To understand the above further, and also see the physical meaning of these combinations, following Jacobson [94], we first decompose \(D_\beta u_\alpha\) into the form

\[
D_\beta u_\alpha = \frac{1}{3} \theta h_{\alpha\beta} + \sigma_{\alpha\beta} + \omega_{\alpha\beta} - a_\alpha u_\beta, \quad (2.40)
\]

where \(\theta\) denotes the expansion of the aether field, \(h_{\alpha\beta}\) the spatial projection operator, \(\sigma_{\alpha\beta}\) the shear, which is the symmetric trace-free part of the spatial projection of \(D_\beta u_\alpha\), while \(\omega_{\alpha\beta}\) denotes the antisymmetric part of the spatial projection of \(D_\beta u_\alpha\), defined, respectively, by

\[
h_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta, \quad (A) \equiv D_\lambda u^\lambda, \quad (2.11)
\]

\[
\sigma_{\alpha\beta} \equiv D_\beta u_\alpha + a_\alpha u_\beta - \frac{1}{3} \theta h_{\alpha\beta}, \quad (2.41)
\]

\[
\omega_{\alpha\beta} \equiv D_\beta u_\alpha + a_\alpha u_\beta, \quad (2.42)
\]

with \((A, B) \equiv (AB + BA)/2\) and \([A, B] \equiv (AB - BA)/2\). Recall that \(a_\mu\) is the acceleration of the aether field, given by Eq.(2.11).

In terms of these quantities, Jacobson found

\[
\int d^4x \sqrt{-g} \mathcal{L}_x = \int d^4x \sqrt{-g} \left[ R - \frac{1}{3} c_\theta \theta^2 + c_\sigma a^2 - c_\sigma a^2 - c_\omega \omega^2 \right], \quad (2.43)
\]

where

\[
c_\theta \equiv c_{13} + 3c_2, \quad c_\sigma \equiv c_{13}, \quad c_\omega \equiv c_1 - c_3, \quad c_0 \equiv c_{14}, \quad (2.43)
\]

and

\[
\sigma^2 = -\frac{1}{3} \theta^2 + (D_\mu u_\nu) (D^\mu u^\nu) \alpha^2. \quad (2.44)
\]

Note that in the above action, there are no crossing terms of \((\theta, \sigma_{\alpha\beta}, \omega_{\alpha\beta}, a_\alpha)\). This is because the four terms on the right-hand side of Eq.(2.40) are orthogonal to each other, and when forming quadratic combinations of these quantities, only their “squares” contribute [94].

From Eq.(2.43) we can see clearly that \(c_{14}\) is related to the acceleration of the aether field, \(c_{13}\) to its shear, while its expansion is related to both \(c_2\) and \(c_{13}\). More interesting, the coefficient of the twist is proportional to \(c_1 - c_3\). When \(u_\mu\) is hypersurface-orthogonal, we have \(\omega^2 = 0\), so the last term in the above action vanishes identically, and only the three free parameters \(c_\theta, c_\sigma\) and \(c_\omega\) remain.
It is also interesting to note that the twist vanishes if and only if the four-velocity of the aether satisfies the conditions [95],

\[ u_{[\mu}D_{\nu]u_{\alpha]} = 0. \]  

(2.45)

When the aether is HO, it can be shown that Eq. (2.45) is satisfied. In addition, in the spherically symmetric case, Eq. (2.45) holds identically.

Moreover, it can be also shown [95] that Eq. (2.45) is the necessary and sufficient condition to write the four-velocity \( u_{\mu} \) in terms the gradient of a timelike scalar field \( \phi \),

\[ u_{\mu} = \frac{\phi_{,\mu}}{\sqrt{-\phi_{,\alpha}\phi^{\alpha}}}. \]  

(2.46)

Substituting it into the action (2.42), one obtains the action of the infrared limit of the healthy extension [91, 92] of the Hořava theory [79], which is often referred to as the khronometric theory \(^4\), where \( \phi \) is called the khronon field.

It should be noted that the khronometric theory and the HO æ-theory are equivalent only in the action level. In particular, in addition to the scalar mode, the khronometric theory has also an instantaneous mode \([66, 96]\), a mode that propagates with an infinitely large speed. This is mainly due to the fact that the field equations of the khronometric theory are the four-order differential equations of \( \phi \). It is the presence of those high-order terms that lead to the existence of the instantaneous mode \(^5\). On the other hand, in æ-theory, including the case with the HO symmetry, the field equations are of second-order for both the metric \( g_{\mu\nu} \) and the aether field \( u_{\mu} \). As a result, this instantaneous mode is absent. For more details, we refer readers to [80] and references therein.

### III. SPHERICALLY SYMMETRIC VACUUM SPACETIMES

**A. Field Equations for \( g_{\mu\nu} \) and \( u^\alpha \)**

As shown in the last section, to be consistent with observations, we must assume \( c_S \geq 1 \). As a result, S0Hs must be inside MHs. Since now S0Hs define the boundaries of spherically symmetric BHs, in order to cover spacetimes both inside and outside the MHs, one way is to adopt the Eddington-Finkelstein (EF) coordinates,

\[ ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = -F(r)dv^2 + 2B(r)dvdr + r^2d\Omega^2, \]  

(3.1)

where \( d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2 \) and \( x^\mu = (v, r, \theta, \phi) \), while the aether field takes the general form,

\[ u^\alpha \partial_\alpha = A(r)\partial_v - \frac{1 - F(r)A^2(r)}{2B(r)A(r)} \partial_r, \]  

(3.2)

which is respect to the spherical symmetry, and satisfies the constraint \( u_\mu u^\mu = -1 \). Therefore, in the current case, we have three unknown functions, \( F(r), A(r) \) and \( B(r) \).

Then, the vacuum field equations \( E_{\mu\nu} \equiv G_{\mu\nu} - \Lambda g_{\mu\nu} = 0 \) and \( \Lambda g^{\mu\nu} \) can be divided into two groups \([63, 67]\): one represents the evolution equations, given by

\[ E_{\theta\theta} = E_{\phi\phi} = \Lambda v^\theta = 0, \]  

(3.3)

and the other represents the constraint equation, given by

\[ C^v = 0, \]  

(3.4)

where \( C^\alpha \equiv E^{\alpha\alpha} + u^\alpha \Lambda \). Note that in Eq. (35) of \([67]\) two constraint equations \( C^v = C^r = 0 \) were considered.

However, \( C^r \) and \( C^v \) are not independent. Instead, they are related to each other by the relation \( C^v = (F/B)C^r \). Thus, \( C^v = 0 \) implies \( C^r = 0 \), so there is only one independent constraint. On the other hand, the three evolution equations can be cast in the forms \(^6\),

\[ F'' = F(A, A', F, F', r, c_i) = \frac{1}{2F^2A^4D} \left( f_0 + f_1 F + f_2 F^2 + f_3 F^3 \right) + f_4 F^4, \]  

(3.5)

\[ A'' = A(A, A', F, F', r, c_i) = \frac{1}{2F^2A^2D} \left( a_0 + a_1 F + a_2 F^2 + a_3 F^3 \right), \]  

(3.6)

\[ \frac{B'}{B} = B(A, A', F, F', r, c_i) = \frac{1}{2rA^2D} \left( b_0 + b_1 F + b_2 F^2 \right), \]  

(3.7)

where a prime stands for the derivative with respect to \( r \), and

\[ D \equiv d_-(J^2 + 1) + 2d_+J, \]  

(3.8)

with \( J \equiv FA^2 \) and

\[ d_+ \equiv (c_3^2 \pm 1)c_{14}(1 - c_{13})(2 + c_{13} + 3c_2). \]  

(3.9)

---

\(^4\) In \([93, 94]\), it was also referred to as T-theory.

\(^5\) In the Degenerate Higher-Order Scalar-Tensor (DHOST) theories, this mode is also referred to as the “shadowy” mode \([97]\).

\(^6\) It should be noted that in \([67]\) the second-order differential equation for \( F \) [cf. Eq. (36) given there] also depends on \( B \). But, since from the constraint \( C^v = 0 \), given by Eq. (3.10), one can express \( B \) in terms of \( A, F \) and their derivatives, as shown explicitly by Eq. (3.11), so there are no essential differences here, and it should only reflect the facts that different combinations of the field equations are used.
The coefficients \( f_n \), \( a_n \) and \( b_n \) are independent of \( F(r) \) and \( B(r) \) but depend on \( F'(r) \), \( A(r) \) and \( A'(r) \), and are given explicitly by Eqs.(A.1), (A.2) and (A.3) in Appendix A. The constraint equation (3.4) now can be cast in the form,

\[
n_0 + n_1 F + n_2 F^2 = 0, \tag{3.10}
\]

where \( n_n \)'s are given explicitly by Eq.(A.4) in Appendix A.

Thus, we have three dynamical equations and one constraint for the three unknown functions, \( F, A \) and \( B \). As a result, the system seems over determined. However, a closer examination shows that not all of them are independent. For example, Eq.(3.7) can be obtained from Eqs.(3.5), (3.6), and (3.10). In fact, from Eq.(3.10), we find that the function \( B \) can be written in the form

\[
B(r) = \pm \frac{1}{2\sqrt{2A^2}} \left[ 2A^2 \left[ 4J(1 + 2c_2 + c_{13}) 
- (2c_2 + c_{13})(J + 1)^2 
+ 4rA \left[ 2AJ' - 4JA' 
+ c_2(J - 1)(JA' - A' - AJ') \right] 
+ r^2 \left[ c_{14}(JA' + A' - AJ')^2 
- (c_2 + c_{13})(JA' - A' - AJ')^2 \right] \right]^{1/2}. \tag{3.11}
\]

Recall that \( J = FA^2 \). Note that there are two branches of solutions for \( B(r) \) with opposite signs, since Eq.(3.10) is a quadratic equation of \( B \). However, only the “+” sign will give us \( B = 1 \) at the spatial infinity, while the “-” sign will yield \( B(r \to \infty) = -1 \). Therefore, in the rest of the paper, we shall choose the “+” sign in Eq.(3.11). Then, first taking the derivative of Eq.(3.11) with respective to \( r \), and then combining the obtained result with Eqs.(3.5) and (3.6), one can obtain Eq.(3.7) \(^7\).

To solve these equations, in this paper we shall adopt the following strategy: choosing Eqs.(3.5), (3.6) and (3.11) as the three independent equations for the three unknown functions, \( F, A, \) and \( B \). The advantage of this choice is that Eqs.(3.5) and (3.6) are independent of the function \( B \). Therefore, we can first solve these two equations to find \( F \) and \( A \), and then obtain the function \( B \) directly from Eq.(3.11). In this approach, we only need to solve two equations, which will significantly save the computation labor, although we do use Eq.(3.7) to monitor our numerical errors.

To solve Eqs.(3.5) and (3.6), we can consider them as the “initial” value problem at a given “moment”, say, \( r = r_0 \) \([63, 67]\). Since they are second-order differential equations, the initial data will consist of the four initial values,

\[
\left\{ A(r_0), A'(r_0), F(r_0), F'(r_0) \right\}. \tag{3.12}
\]

In principle, \( r_0 \) can be chosen as any given (finite) moment. However, in the following we shall show that the most convenient choice will be the locations of the SOHs. It should be noted that a SOH does not always exist for any given initial data. However, since in this paper we are mainly interested in the case in which a SOH exists, so whenever we choose \( r_0 = r_{S0H} \), it always means that we only consider the case in which such a SOH is present.

To determine the location of the SOH for a given spherical solution of the metric (3.1), let us first consider the out-pointing normal vector, \( N_\mu \), of a hypersurface \( r = \text{Constant} \), say, \( r_0 \), which is given by \( N_\mu = \partial(r - r_0) / \partial x^\nu = \delta_0^\nu \). Then, the metric and spin-0 horizons of \( g_{\mu\nu} \) are given, respectively, by

\[
g_{\alpha\beta}N^\alpha N^\beta = 0, \tag{3.13}
\]

\[
g_{\alpha\beta}^{(S)}N^\alpha N^\beta = 0, \tag{3.14}
\]

where \( N^\mu \equiv g^{\mu\nu}N_\nu \), and \( g_{\alpha\beta}^{(S)} \) is defined by Eq.(1.3). For the metric and aether given in the form of Eqs.(3.1) and (3.2), they become

\[
F(r_{MH}) = 0,
\]

\[
(c_S^2 - 1) (J(r_{S0H})^2 + 1) + 2 (c_S^2 + 1) J(r_{S0H}) = 0,
\]

where \( r = r_{MH} \) and \( r = r_{S0H} \) are the locations of the metric and spin-0 horizons, respectively. Note that Eqs.(3.15) and (3.16) may have multiple roots, say, \( r_{MH}^1 \) and \( r_{S0H}^1 \). In these cases, the location of the metric (spin-0) horizon is always taken to be the largest root of \( r_{MH}^1 \) (or \( r_{S0H}^1 \)).

Depending on the value of \( c_S \), the solutions of Eq.(3.14) are given, respectively, by

\[
J(r_{S0H}^\pm) = \frac{1 \pm c_S}{1 \pm c_S} \equiv J^\pm, \quad c_S \neq 1, \tag{3.17}
\]

and

\[
J(r_{S0H}) = 0, \quad c_S = 1. \tag{3.18}
\]

It is interesting to note that on SOHs, we have

\[
\mathcal{D}(r_{S0H}) = 0, \tag{3.19}
\]

as it can be seen from Eqs.(3.8), (3.9) and (3.16).

As mentioned above, for some choices of \( c_i \), Eq.(3.14) does not always admit a solution, hence a SOH does not exist in this case. A particular choice was considered in

\(^7\) From this proof it can be seen that obtaining Eq.(3.7) from Eq.(3.11) the operation of taking the first-order derivatives was involved. Therefore, in principle these two equations are equivalent modulated an integration constant.
in which we have \(c_1 = 0.051\), \(c_2 = 0.116\), \(c_3 = -c_1\) and \(c_4 = 0\). For this choice, we find that \(c_S \simeq 1.37404\), \(J^+ \simeq -0.157556\) and \(J^- \simeq -6.34696\). As shown in Fig. 1, the function \(J(r)\) is always greater than \(J^\pm\), so no S0H is formed, as first noticed in [63]. Up to the numerical errors, Fig. 1 is the same as that given in [63], which provides another way to check our general expressions of the field equations given above.

In addition, we also find that the two exact solutions obtained in [53] satisfy these equations identically, as it is expected.

![FIG. 1: The solution for \(c_1 = 0.051\), \(c_2 = 0.116\), \(c_3 = -c_1\) and \(c_4 = 0\), first considered numerically in [63]. There are an outer and inner MHs, at which \(J\) vanishes. But, \(J\) does not cross the constant line of \(J^+\), so that a S0H is absent. This graph is the same as the one given in [63] (up to the numerical errors).](image)

Then, we find that Eqs.(3.5)-(3.6) now reduce to

\[
F'' = -\frac{2}{r} F' + \frac{c_2 \bar{F}(r)}{4r^2 A^4},
\]

\[
A'' = \frac{2}{r^2(A + A^3 F)} \left[ r^2(A')^2 - rAA' - A^2 - rA^3 A'(F + rF') + A^4 F \right] - \frac{c_2 \bar{F}(r)}{4r^2(A + A^3 F)},
\]

where

\[
\bar{F}(r) \equiv \left[ rA' - 2A + rA^2 A'F + A^3 (2F + rF') \right]^2.
\]

Combining Eqs.(3.21) and (3.22), we find the following equation,

\[
W'' + W'^2 + \frac{2}{r} W' - \frac{2}{r} = 0,
\]

where

\[
W = \ln \left( \frac{1 - FA^2}{A} \right).
\]

Eq.(3.24) has the general solution,

\[
W = \ln \left( w_2 + \ln \left( 1 + \frac{w_1 r^3}{\rho^2} \right) \right),
\]

where \(w_1\) and \(w_2\) are two integration constants. Then, the combination of Eqs.(3.25) and (3.26) yields,

\[
F(r) = \frac{1}{A^2} - \frac{w_2}{A} \left( \frac{1}{r^2} + w_1 r \right).
\]

Substituting Eq.(3.27) into Eq.(3.21), we find

\[
F'' = -\frac{2}{r} F' + F_0,
\]

where \(F_0 \equiv 9c_2 w_1^2 w_2^2/4\). Integrating Eq.(3.28), we find

\[
F(r) = F_2 \left( 1 - \frac{2m}{r} \right) + \frac{F_0}{6} r^2,
\]

where \(m\) and \(F_2\) are two other integration constants. On the other hand, from Eq.(3.27), we find that

\[
A(r) = -\frac{w_2}{2F} \left[ \left( \frac{1}{r^2} + w_1 r \right) \pm \sqrt{\frac{4F}{w_2^2} + \left( \frac{1}{r^2} + w_1 r \right)^2} \right].
\]

Substituting the above expressions for \(A\) and \(F\) into the constraint (3.11), we find that

\[
B = \sqrt{F_2}.
\]

---

\(^8\) When \(c_{14} = c_{13} = 0\), the speeds of the spin-0 and spin-1 modes can be infinitely large, as it can be seen from Eq.(1.1). Then, cautions must be taken, including the calculations of the PPN parameters [88].
Note that the above solution is asymptotically flat only when \( w_1 = 0 \), for which we have
\[
F(r) = F_2 \left( 1 - \frac{2m}{r} \right), \quad B(r) = \sqrt{F_2},
\]
\[
A(r) = -\frac{w_2}{2F} \left( 1 - \frac{4F}{w_2^2} \right) + \frac{1}{r^2}.
\] (3.32)

Using the gauge residual \( \psi' = C_0 \psi + C_1 \) of the metric (3.1), without loss of the generality, we can always set \( F_2 = 1 \), so the corresponding metric takes the precise form of the Schwarzschild solution,
\[
ds^2 = -\left( 1 - \frac{2m}{r} \right) dt^2 + 2dtdr + r^2d\Omega^2,
\] (3.33)
while the aether field is given by
\[
A(r) = -\frac{w_2}{2F} \left( 1 - \frac{4F}{w_2^2} \right) + \frac{1}{r^2}.
\] (3.34)

It is remarkable to note that now the aether field has no contribution to the spacetime geometry, although it does feel the gravitational field, as it can be seen from Eq.(3.34).

It should be also noted that Eqs.(3.33) and (3.34) were a particular case of the solutions first found in \([53]\) for the case \( c_{14} = 0 \) by further setting \( c_{13} = 0 \). But, the general solutions given by Eqs.(3.29) - (3.31) are new, as far as we know.

C. Field Equations for \( \tilde{g}_{\mu\nu} \) and \( \tilde{u}^\mu \)

Note that, instead of solving the three independent equations directly for \( A, B \) and \( F \), we shall first solve the corresponding three equations for \( \tilde{A}, \tilde{B} \) and \( \tilde{F} \), by taking the advantage of the field redefinitions introduced in the last section, and then obtain the functions \( A, B \) and \( F \) by the inverse transformations of Eqs.(3.39) and (3.41) to be given below. This will considerably simplify mathematically the problem of solving such complicated equations.

To this goal, let us first note that, with the field redefinitions (2.23), the line element corresponding to \( g_{\mu\nu} \) in the coordinates \((v, r, \theta, \phi)\), takes the form,
\[
ds^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu
\]
\[
= -\left[ F + \frac{(\sigma - 1)}{4A^2} \right] dv^2
\]
\[
+ 2 \left[ B + \frac{1}{2} \sigma A^2 F + 1 \right] dvd\Omega
\]
\[-(\sigma - 1)A^2 B^2 d\Omega^2. \] (3.35)
To bring the above expression into the standard EF form, we first make the coordinate transformation,
\[
\tilde{v} = C_0 v - C(r), \] (3.36)
where \( C_0 \) is an arbitrary real constant, and \( C(r) \) is a function of \( r \). Then, choosing \( C(r) \) so that
\[
\frac{dC(r)}{dr} = \frac{2C_0 A^2 B}{J(\sqrt{\sigma} - 1) + (\sqrt{\sigma} + 1)}, \] (3.37)
we find that in the coordinates \( \tilde{x}^\mu = (\tilde{v}, r, \theta, \phi) \), the line element (3.35) takes the form,
\[
d\tilde{s}^2 \equiv \tilde{g}_{\mu\nu}dx^\mu dx^\nu = \tilde{g}_{\mu\nu}\tilde{d}x^\mu \tilde{d}x^\nu
\]
\[= -\tilde{F}(r)dv^2 + 2\tilde{B}(r)vd\Omega + r^2d\Omega^2, \] (3.38)
where
\[
\tilde{F} = J^2(\sigma - 1) + 2J(\sigma + 1) + (\sigma - 1),
\]
\[
\tilde{B} = \frac{\sqrt{\sigma} B}{C_0}. \] (3.39)

On the other hand, in terms of the coordinates \( \tilde{x}^\mu \), the aether four-velocity is given by
\[
\tilde{u}^\alpha \frac{\partial}{\partial x^\alpha} = \tilde{u}^\alpha \frac{\partial}{\partial \tilde{x}^\alpha} = \tilde{A}(r)\partial_{\tilde{v}} - \frac{1 - \tilde{F}(r)\tilde{A}^2(r)}{2\tilde{B}(r)\tilde{A}(r)} \partial_{\tilde{r}}, \] (3.40)
where
\[
\tilde{A} = \frac{2C_0 A}{J(\sqrt{\sigma} - 1) + (\sqrt{\sigma} + 1)}, \] (3.41)
which satisfies the constraint \( \tilde{u}^\alpha \tilde{u}^\beta \tilde{g}_{\alpha\beta} = -1 \), with
\[
\tilde{g}_{\mu\nu} \equiv \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \frac{\partial x^\beta}{\partial \tilde{x}^\nu} \tilde{g}_{\alpha\beta}, \quad \tilde{u}_\mu \equiv \frac{\partial x^\alpha}{\partial \tilde{x}^\mu} \tilde{u}_\alpha. \] (3.42)

It should be noted that the metric (3.38) still has the gauge residual,
\[
\tilde{\psi} = C_1 \tilde{v} + C_2, \] (3.43)
where \( C_1 \) and \( C_2 \) are two arbitrary constants, which will keep the line element in the same form, after the rescaling,
\[
\tilde{F} = \frac{\tilde{F}}{C_1}, \quad \tilde{B} = \frac{\tilde{B}}{C_1}. \] (3.44)

Later we shall use this gauge freedom to fix one of the initial conditions.

In the rest of this paper, we always refer \((\tilde{g}_{\mu\nu}, \tilde{u}^\alpha)\) as the field obtained by the field redefinitions. The latter is related to \((\tilde{g}_{\mu\nu}, \tilde{u}^\alpha)\) via the inverse coordinate transformations of Eq.(3.42). Then, the Einstein-aether field equations for \((\tilde{g}_{\mu\nu}, \tilde{u}^\alpha)\) will take the same forms as those given by Eqs.(2.29) - (2.31), but now in terms of \((\tilde{g}_{\mu\nu}, \tilde{u}^\alpha, \tilde{c}_i)\) in the coordinates \( \tilde{x}^\mu \), where \( \tilde{c}_i \equiv \tilde{c}_i \). On the other hand, since the metric (3.38) for \( \tilde{g}_{\mu\nu} \) takes the same form as the metric (3.1) for \( g_{\mu\nu} \), and so does the aether field (3.40) for \( \tilde{u}^\mu \) as the one (3.2) for \( u^\mu \), it is not difficult to see that the field equations for \( F(r), \tilde{A}(r) \)
and $\tilde{B}(r)$ will be given precisely by Eqs.(3.5) - (3.10), if we simply make the following replacement,

$$ (F, A, B, c_i) \rightarrow (\tilde{F}, \tilde{A}, \tilde{B}, \tilde{c}_i) .$$  

(3.45)

As a result, we have

$$ \tilde{F}'' = \tilde{F} \left( \tilde{A}, \tilde{A}', \tilde{F}, \tilde{F}', r, \tilde{c}_i \right) 
= \frac{1}{2r^2 A^2 D} \left[ \tilde{f}_0 + \tilde{f}_1 \tilde{F} + \tilde{f}_2 \tilde{F}^2 + \tilde{f}_3 \tilde{F}^3 
+ \tilde{f}_4 \tilde{F}^4 \right],$$  

(3.46)

$$ \tilde{A}'' = \tilde{A} \left( \tilde{A}, \tilde{A}', \tilde{F}, \tilde{F}', r, \tilde{c}_i \right) 
= \frac{1}{2r^2 A^2 D} \left[ \tilde{a}_0 + \tilde{a}_1 \tilde{F} + \tilde{a}_2 \tilde{F}^2 + \tilde{a}_3 \tilde{F}^3 \right],$$  

(3.47)

$$ \frac{\tilde{B}'}{\tilde{B}} = \frac{b_0 + \tilde{b}_1 \tilde{F} + \tilde{b}_2 \tilde{F}^2}{b_0 + \tilde{b}_1 \tilde{F} + \tilde{b}_2 \tilde{F}^2},$$  

(3.48)

and

$$ \tilde{C}^\alpha = \tilde{n}_0 + \tilde{n}_1 \tilde{F} + \tilde{n}_2 \tilde{F}^2 = 0,$$  

(3.49)

where

$$ \tilde{D}(r) = \tilde{d}_- \left( \tilde{J}^2(r) + 1 \right) + 2 \tilde{d}_+ \tilde{J}(r), $$  

$$ \tilde{J}(r) = \tilde{F}(r) \tilde{A}'(r), $$  

$$ \tilde{d}_\pm = (\tilde{c}_S^2 \pm 1) \tilde{c}_4 (1 - \tilde{c}_3) (2 + \tilde{c}_3 + 3 \tilde{c}_2).$$  

(3.50)

The coefficients $\tilde{f}_n$, $\tilde{a}_n$, $\tilde{b}_n$ and $\tilde{n}_n$ are given by $f_n$, $a_n$, $b_n$ and $n_n$ after the replacement (3.45) is carried out.

Then, the metric and spin-0 horizons for $\tilde{g}_{\mu\nu}$ are given, respectively, by

$$ \tilde{g}_{\alpha\beta} N^\alpha N^\beta = 0,$$  

(3.51)

$$ \tilde{g}^{(S)}_{\alpha\beta} \bar{N}^\alpha \bar{N}^\beta = 0,$$  

(3.52)

where $\bar{N}_\alpha = \partial x^\mu / \partial \tilde{x}^\alpha N_\mu = \delta^\sigma_\alpha = \delta^\sigma_\alpha$ and

$$ \tilde{g}^{(s)}_{\alpha\beta} \equiv \tilde{g}_{\alpha\beta} - (\tilde{c}_S^2 - 1) \tilde{a}_\alpha \tilde{a}_\beta.$$  

(3.53)

In terms of $\tilde{F}$ and $\tilde{A}$, Eqs.(3.51) and (3.52) becomes

$$ \tilde{F}(\tilde{r}_{MH}) = 0,$$  

(3.54)

$$ (\tilde{c}_S^2 - 1) \left( \tilde{J}(\tilde{r}_{SOH})^2 + 1 \right) + 2 (\tilde{c}_S^2 + 1) \tilde{J}(\tilde{r}_{SOH}) = 0,$$  

(3.55)

where $r = \tilde{r}_{MH}$ and $r = \tilde{r}_{SOH}$ are respectively the locations of the metric and spin-0 horizons for the metric $\tilde{g}_{\mu\nu}$. Similarly, at $r = \tilde{r}_{SOH}$ we have

$$ \tilde{D}(\tilde{r}_{SOH}) = 0.$$  

(3.56)

Comparing the field equations given in this subsection with the corresponding ones given in the last subsection, we see that we can get one set from the other simply by the replacement (3.45).

In addition, in terms of $\tilde{g}_{\alpha\beta}$ and $N_\alpha$, Eqs.(3.51) and (3.52) reduce, respectively, to

$$ \tilde{g}_{\alpha\beta} N^\alpha N^\beta = 0,$$  

(3.57)

$$ \tilde{g}^{(s)}_{\alpha\beta} \bar{N}^\alpha \bar{N}^\beta = 0.$$  

(3.58)

Since $\tilde{r} = r$, we find that

$$ \tilde{r}_{MH} = \tilde{r}_{SOH}, \quad \tilde{r}_{SOH} = \tilde{r}_{SOH},$$  

(3.59)

where $\tilde{r}_{MH}$ and $\tilde{r}_{SOH}$ ($\tilde{r}_{MH}, \tilde{r}_{SOH}$) are the locations of the metric and spin-0 horizons of the metric $\tilde{g}_{\alpha\beta}$ ($\tilde{g}_{\alpha\beta}$). The above analysis shows that these horizons determined by $\tilde{g}_{\alpha\beta}$ are precisely equal to those determined by $\tilde{g}_{\alpha\beta}$.

D. $\sigma = c_s^2$

To solve Eqs.(3.46) - (3.49), we take the advantage of the choice $\sigma = c_s^2$, so that the speed of the spin-0 mode of the metric $\tilde{g}_{\mu\nu}$ becomes unity, i.e., $\tilde{c}_s = 1$. Since $\tilde{c}_i = \tilde{c}_i$, we also have $\tilde{c}_S = \tilde{c}_S = 1$. Then, from Eq.(1.1) we find that this leads to,

$$ \tilde{c}_2 = \frac{2\tilde{c}_{14} - 2\tilde{c}_{13} - \tilde{c}_{13}^2 \tilde{c}_{14}}{2 - 4\tilde{c}_{14} + 3\tilde{c}_{13} \tilde{c}_{14}}.$$  

(3.60)

For such a choice, from Eq.(3.50) we find that $\tilde{d}_- = 0$, and

$$ \tilde{D}(r) = 2 \tilde{d}_+ \tilde{J}(r) = 2 \tilde{d}_+ \tilde{A}'(r) \tilde{F}(r).$$  

(3.61)

Then, Eq.(3.56) yields $\tilde{F}(\tilde{r}_{SOH}) = 0$, since $\tilde{A} \neq 0$, which also represents the location of the MH, defined by Eq.(3.54). Therefore, for the choice $\sigma = c_s^2$ the MH coincides with the SOH for the effective metric $\tilde{g}_{\mu\nu}$, that is,

$$ \tilde{r}_{MH} = \tilde{r}_{SOH}, \quad (\sigma = c_s^2).$$  

(3.62)

As shown below, this will significantly simplify our computational labor. In particular, if we choose this surface as our initial moment, it will reduce the phase space of initial data from 4 dimensions to one dimension only.

For $\tilde{c}_S = 1$, Eqs.(3.46)-(3.49) reduce to,

$$ \tilde{F}'' = \frac{1}{4 \tilde{d}_+ r^2 A^6} \left( \tilde{f}_0 + \tilde{f}_1 + \tilde{f}_2 \tilde{F} + \tilde{f}_3 \tilde{F}^2 \right),$$  

(3.63)

$$ \tilde{A}'' = \frac{1}{4 \tilde{d}_+ r^2 A^4} \left( \tilde{a}_0 + \tilde{a}_1 + \tilde{a}_2 \tilde{F} + \tilde{a}_3 \tilde{F}^2 \right),$$  

(3.64)

$$ \frac{\tilde{B}'}{\tilde{B}} = \frac{1}{4 \tilde{d}_+ r A^4} \left( \tilde{b}_0 + \tilde{b}_1 + \tilde{b}_2 \tilde{F} \right),$$  

(3.65)

$$ \tilde{n}_0 + \tilde{n}_1 \tilde{F} + \tilde{n}_2 \tilde{F}^2 = 0.$$  

(3.66)
As shown previously, among these four equations, only three of them are independent, and our strategy in this paper is to take Eqs. (3.63), (3.64) and (3.66) as the three independent equations. The advantage of this approach is that Eqs. (3.63), (3.64) are independent of \( B(r) \), and Eq. (3.66) is a quadratic polynomial of \( B(r) \). So, we can solve Eqs. (3.63), (3.64) as the initial value problem first to find \( \tilde{F}(r) \) and \( \tilde{A}(r) \), and then insert them into Eq. (3.66) to obtain directly \( \tilde{B}(r) \), as explicitly given by Eq. (3.11), after taking the replacement (3.45) and the choice of \( \tilde{c}_2 \) of Eq. (3.60) into account.

From Eqs. (3.63) and (3.64) we can see that they become singular at \( r = \tilde{r}_{SOH} \) (Recall \( \tilde{F}(\tilde{r}_{SOH}) = 0 \), unless \( \tilde{f}_0(\tilde{r}_{SOH}) = \tilde{a}_0(\tilde{r}_{SOH}) = 0 \). As can be seen from the expressions of \( f_0(r) \), \( a_0(r) \) given in Appendix A, \( \tilde{f}_0(\tilde{r}_{SOH}) = \tilde{a}_0(\tilde{r}_{SOH}) = 0 \) imply \( \tilde{b}_0(\tilde{r}_{SOH}) = 0 \). Therefore, to have the field equations regular across the SOH, we must require \( \tilde{b}_0(\tilde{r}_{SOH}) = 0 \). It is interesting that this is also the condition for Eq. (3.65) to be non-singular across the SOH. In addition, using the gauge residual (3.43), we shall set \( \tilde{B}_H = 1 \), so Eq. (3.66) [which can be written in the form of Eq. (3.11), after the replacement (3.45)] will provide a constraint among the initial values of \( \tilde{F}_H', \tilde{A}_H' \), and \( \tilde{A}_H' \), where \( \tilde{F}_H' \equiv \tilde{F}'(\tilde{r}_{SOH}) \) and so on. In summary, on the SOH we have the following

\[
\begin{align*}
\tilde{F}_H &= 0, \quad (3.67) \\
\tilde{b}_0 \left( \tilde{A}_H, \tilde{A}_H', \tilde{F}_H, \tilde{r}_{SOH} \right) &= 0, \quad (3.68) \\
\tilde{B}_H &= 1. \quad (3.69)
\end{align*}
\]

From the expression for \( \tilde{b}_0 \) given in Appendix A, we can see that Eq. (3.68) is quadratic in \( \tilde{A}_H' \), and solving it on the SOH, in general we obtain two solutions,

\[
\tilde{A}_H' = \tilde{A}_H'_{\pm} \left( \tilde{A}_H, \tilde{F}_H, \tilde{r}_{SOH} \right). \quad (3.70)
\]

Then, inserting it, together with Eqs. (3.67) and (3.69), into Eq. (3.10), we get

\[
\tilde{n}_0_{\pm} \left( \tilde{A}_H, \tilde{F}_H, \tilde{r}_{SOH} \right) = 0, \quad (3.71)
\]

where the “\( \pm \)” signs correspond to the choices of \( \tilde{A}_H' = \tilde{A}_H'_{\pm} \). In general, Eq. (3.71) is a fourth-order polynomial of \( \tilde{F}_H' \), so it normally has four roots, denoted as

\[
\tilde{F}_H' = \tilde{F}_H'_{\pm}(\tilde{A}_H'_{\pm}, \tilde{r}_{SOH}), \quad (3.72)
\]

where \( n = 1, 2, 3, 4 \). For each given \( \tilde{F}_H'_{\pm} \), substituting it into Eq. (3.70) we find a corresponding \( \tilde{A}_H'_{\pm} \), given by

\[
\tilde{A}_H'_{\pm} = \tilde{A}_H'_{\pm}(\tilde{A}_H, \tilde{r}_{SOH}). \quad (3.73)
\]

Thus, once \( \tilde{A}_H \) and \( \tilde{r}_{SOH} \) are given, the quantities \( \tilde{F}_H'_{\pm} \) and \( \tilde{A}_H'_{\pm} \) are uniquely determined from Eqs. (3.72) and (3.73). For each set of \( (\tilde{A}_H, \tilde{r}_{SOH}) \), in general there are eight sets of \( (\tilde{A}_H', \tilde{F}_H') \).

If we choose \( r = \tilde{r}_{SOH} \) as the initial moment, such obtained \( (\tilde{A}_H', \tilde{F}_H') \), together with \( \tilde{F}_H = 0 \), and a proper choice of \( \tilde{A}_H \), can be considered as the initial conditions for the differential equations (3.63) and (3.64).

However, it is unclear which one(s) of these eight sets of initial conditions will lead to asymptotically flat solutions, except that the one with \( \tilde{F}_H' < 0 \), which can be discarded immediately, as it would lead to \( \tilde{F} = 0 \) at some radius \( r > \tilde{r}_{SOH} \), which is inconsistent with our assumption that \( r = \tilde{r}_{SOH} \) is the location of the SOH [67]. So, in general what one needs to do is to try all the possibilities.

Therefore, if we choose \( r = \tilde{r}_{SOH} \) as the initial moment, the four-dimensional phase space of the initial conditions, \( (\tilde{F}_H, \tilde{F}_H', \tilde{A}_H, \tilde{A}_H') \), reduces to one-dimensional, spanned by \( \tilde{A}_H \) only.

In the following, we shall show further that \( \tilde{r}_{SOH} \) can be chosen arbitrarily. In fact, introducing the dimensionless quantity, \( \xi \equiv \tilde{r}_{SOH}/r \), we find that Eqs. (3.63) - (3.65) and (3.49) can be written in the forms,

\[
\begin{align*}
\frac{d^2 \tilde{F}(\xi)}{d\xi^2} &= G_1(\xi, \tilde{c}_i), \quad (3.74) \\
\frac{d^2 \tilde{A}(\xi)}{d\xi^2} &= G_2(\xi, \tilde{c}_i), \quad (3.75) \\
\frac{1}{\tilde{B}(\xi)} \frac{d\tilde{B}(\xi)}{d\xi} &= G_3(\xi, \tilde{c}_i), \quad (3.76) \\
C^{\tilde{c}_i} \left( \tilde{A}(\xi), \tilde{A}'(\xi), \tilde{F}(\xi), \tilde{F}'(\xi), \tilde{B}(\xi), \xi, \tilde{c}_i \right) &= 0, \quad (3.77)
\end{align*}
\]

where \( G_i' \)s are all independent of \( \tilde{r}_{SOH} \), \( C^{\tilde{c}_i} \equiv \tilde{r}_{SOH}^{\tilde{c}_i} \tilde{C}^{\tilde{c}_i} \), and the primes in the last equation stand for the derivatives respect to \( \xi \). Therefore, Eqs. (3.74)-(3.77), or equivalently, Eqs. (3.46)-(3.49), are scaling-invariant and independent of \( \tilde{r}_{SOH} \). Thus, without loss of the generality, we can always set

\[
\tilde{r}_{SOH} = 1, \quad (3.78)
\]

which does not affect Eqs. (3.74) - (3.77), and also explains the reason why in [63, 67] the authors set \( \tilde{r}_{SOH} = 1 \) directly. At the same time, it should be noted that once \( \tilde{r}_{SOH} = 1 \) is taken, it implies that the unit of length is fixed. For instance, if we have a BH with \( \tilde{r}_{SOH} = 1 \) km, then setting \( \tilde{r}_{SOH} = 1 \) means the unit of length is in km.

Once \( \tilde{A}_H \) is chosen, we can integrate Eqs. (3.74) and (3.75) in both directions to find \( \tilde{F}(\xi) \) and \( \tilde{A}(\xi) \), one is toward the center, \( \xi = \tilde{r}_{SOH}/r = \infty \), in which we have \( \xi \in [1, \infty) \), and the other is toward infinity, \( \xi = \tilde{r}_{SOH}/r = 0 \), in which we have \( \xi \in (0, 1) \). Then, from Eq. (3.11) we can find \( \tilde{B}(\xi) \) uniquely, after the replacement of Eq. (3.45). Again, to have a proper asymptotical behavior of \( \tilde{B}(r) \), the “\( + \)” sign will be chosen.
At the spatial infinity $\xi = \tilde{r}_{S0H}/r \to 0$, we require that the spacetime be asymptotically flat, that is \cite{63, 67} \(^9\),

\[
\tilde{F}(\xi) = 1 + \tilde{F}_1 \xi + \frac{1}{48} \tilde{c}_1 \tilde{F}_1^3 \xi^3 + \cdots ,
\]

\[
\tilde{A}(\xi) = 1 - \frac{1}{2} \tilde{F}_1 \xi + \frac{1}{2} \tilde{A}_2 \xi^2 - \left( \frac{1}{96} \tilde{c}_1 \tilde{F}_1^3 - \frac{1}{16} \tilde{c}_1 \tilde{F}_1^2 \xi - \frac{1}{12} \tilde{c}_1 \tilde{F}_1 \xi^2 + \cdots \right) \xi^3 + \cdots ,
\]

\[
\tilde{B}(\xi) = 1 + \frac{1}{16} \tilde{c}_1 \tilde{F}_1^2 \xi^2 - \frac{1}{12} \tilde{c}_1 \tilde{F}_1 \xi^3 + \cdots , \tag{3.79}
\]

where $\tilde{F}_1 \equiv \tilde{F}'(\xi = 0)$ and $\tilde{A}_2 \equiv \tilde{A}''(\xi = 0)$. It should be noted that the Minkowski spacetime is obtained from its MH, defined by

\[
F(\xi) = \frac{1}{\sqrt{F_M}}, \quad A = \frac{1}{\sqrt{F_M}}, \quad B = \sqrt{F_M}, \tag{3.80}
\]

where $F_M$ is a positive otherwise arbitrary constant. Therefore, in the asymptotic expansions of Eq.(3.79), we had set $F_M = 1$ at the zeroth order of $\xi$. However, the initial conditions imposed at $r = \tilde{r}_{S0H}$ give above usually leads to $F_M \neq 1$, even for spacetimes that are asymptotically flat. Therefore, we need first to use the gauge residual \eqref{3.43} to bring $\tilde{F}(\xi = 0) = \tilde{A}(\xi = 0) = \tilde{B}(\xi = 0) = 1$, before using Eq.(3.79) to calculate the constants $\tilde{A}_2$ and $\tilde{F}_1$.

From the above analysis we can see that finding spherically symmetric solutions of the $\alpha$-theory now reduces to finding the initial condition $\tilde{A}_H$ that leads to the asymptotic behavior \eqref{3.79}, for a given set of $c_i$’s.

Before proceeding to the next section, we would like to recall that when $\sigma = c_2^2$, we have $\tilde{g}_{a\beta} = \tilde{g}_{a\beta}$, as shown by Eq.(2.35). That is, the SOH for the metric $\tilde{g}_{a\beta}$ now coincides with the MH of $\tilde{g}_{a\beta}$. With this same very choice, $\sigma = c_2^2$, the MH for $\tilde{g}_{a\beta}$ also coincides with its SOH. Thus, we have

\[
\tilde{r}_{S0H} = \tilde{r}_{S0H} = \tilde{r}_{MH} = \tilde{r}_{S0H} = \tilde{r}_{MH} \equiv r_H, \quad (\sigma = c_2^2).
\tag{3.81}
\]

It must be noted that $r_H$ defined in the last step denotes the location of the SOH of $\tilde{g}_{a\beta}$, which is usually different from its MH, defined by

\[
g_{a\beta} N^a N^\beta |_{r = r_{MH}} = 0, \tag{3.82}
\]

since in general we have $c_S \neq 1$, so $g^{(S)}_{a\beta} \equiv g_{a\beta} - (c^2_2 - 1) u_a u_\beta \neq g_{a\beta}$. As a result, we have $r_{MH} \neq r_{S0H}$ for $c_S \neq 1$.

\(^9\) Note that in \cite{63, 67} a factor $1/2$ is missing in front of $A_2$ in the expression of $A(x)$.

However, it is worth emphasizing again that, for the choice $\sigma = c_2^2$ we have $\tilde{c}_S = \tilde{c}_S = 1$, so the metric and spin-0 horizons of both $\tilde{g}_{a\beta}$ and $\tilde{g}_{a\beta}$ all coincide, and are given by the same $r_H$, as explicitly shown by Eq.(3.81). More importantly, it is also the location of the SOHs of the metric $g_{a\beta}$.

### IV. NUMERICAL SETUP AND RESULTS

#### A. General Steps

It is difficult to find analytical solutions to Eqs.(3.74)-(3.77). Thus, in this paper we are going to solve them numerically, using the shooting method, with the asymptotical conditions \eqref{3.79}. In particular, our strategy is the following:

(i) Choose a set of physical $c_i$’s satisfying the constraints \eqref{2.16}-\eqref{2.20}, and then calculate the corresponding $\tilde{c}_i$’s with $\sigma = c_2^2$.

(ii) Assume that for such chosen $c_i$’s the corresponding solution possesses a SOH located at $r = r_H$, and then follow the analysis given in the last section to impose the conditions $\tilde{F}_H = 0$ and $\tilde{B}_H = 1$.

(iii) Choose a test value for $\tilde{A}_H$, and then solve Eq.(3.68) for $\tilde{A}'_H$ in terms of $\tilde{F}'_H$ and $\tilde{A}_H$, i.e., $\tilde{A}'_H = \tilde{A}'_H(\tilde{F}'_H, \tilde{A}_H)$.

(iv) Substitute $\tilde{A}'_H$ into Eq.(3.71) to obtain a quartic equation for $\tilde{F}'_H$ and then solve it to find $\tilde{F}'_H$.

(v) With the initial conditions \{ $\tilde{F}_H, \tilde{A}_H, \tilde{F}'_H, \tilde{A}'_H$ \}, integrate Eqs.(3.74) and \eqref{3.75} from $\xi = 1$ to $\xi = 0$.

However, since the field equations are singular at $\xi = 1$, we will actually integrate these equations from $\xi = 1 - \epsilon$ to $\xi \approx 0$, where $\epsilon$ is a very small quantity. To obtain the values of $F(\xi)$ at $\xi = 1 - \epsilon$, we first Taylor expand them in the form,

\[
F(1 - \epsilon) = \sum_{k=0}^{2} \frac{F^{(k)}|_{\xi=1}}{k!} (-1)^k \epsilon^k + \mathcal{O}(\epsilon^3), \tag{4.1}
\]

where $F \equiv \{ \tilde{A}, \tilde{A}', \tilde{F}, \tilde{F}' \}$ and $F^{(k)} \equiv d^k F/d\xi^k$. For each $F$, we shall expand it to the second order of $\epsilon$, so the errors are of the order $\epsilon^3$. Thus, if we choose $\epsilon = 10^{-14}$, the errors in the initial conditions $F(1-\epsilon)$ is of the order $10^{-42}$. For $F = \tilde{A}$, $\tilde{F}$, we already obtained $F(1)$ and $F'(1)$ from the initial conditions. In these cases, to get $\tilde{A}'(1)$ and $\tilde{F}'(1)$, we use the field equations \eqref{3.74} and \eqref{3.75} and L’Hospital’s rule. On the other hand, for $F = \tilde{A}'$, expanding it to the second-order of $\epsilon$, we have

\[
\tilde{A}'(1 - \epsilon) = \tilde{A}'(1) - \tilde{A}''(1) \epsilon + \frac{1}{2} \tilde{A}^{(3)}(1) \epsilon^2 + \mathcal{O}(\epsilon^3), \tag{4.2}
\]

where $\tilde{A}^{(3)}(1) \equiv d^3 \tilde{A}(\xi)/d\xi^3 \bigg|_{\xi=1}$ can be obtained by first taking the derivative of Eq.(3.75) and then taking the limit $\xi \to 1$, as now we have already known
\(A(1), \hat{A}'(1), \hat{A}''(1), \hat{F}(1), \hat{F}'(1)\) and \(\hat{F}''(1)\). Similarly, for \(F = \hat{F}\), from Eq.\((3.74)\) we can find \(\hat{F}^{(3)}(1)\).

(vi) Repeat (iii)-(v) until a numerical solution matched to Eq.\((3.79)\) is obtained, by choosing different values of \(A_H\) with a bisectional search. Clearly, once such a value of \(A_H\) is found, it means that we can numerically find an asymptotically flat solution of the Einstein-aether field equations outside the S0H. Note that, to guarantee that Eq.\((3.79)\) is satisfied, the normalization of \{\(F, A, B\)\} need to be done according to Eq.\((3.80)\), by using the remaining gauge residual of Eq.\((3.43)\).

(vii) To obtain the solution in the internal region \(\xi \in (1, \infty)\), we simply integrate Eqs.\((3.74)\) and \((3.75)\) from \(\xi = 1\) to \(\xi \to \infty\) with the same value of \(\hat{A}_H\) found in the last step. As in the region \(\xi \in (0, 1)\), we can’t really set the “initial” conditions precisely at \(\xi = 1\). Instead, we will integrate them from \(\xi = 1 + \epsilon\) to \(\xi = \xi_{\infty} \gg 1\). The initial values at \(\xi = 1 + \epsilon\) can be obtained by following what we did in Step (v), that is, Taylor expand \(F(\xi)\) at \(\xi = 1 + \epsilon\), and then use the field equations to get all the quantities up to the third-order of \(\epsilon\).

(viii) Matching the results obtained from steps (vi) and (vii) together, we finally obtain a solution of \((\hat{F}(\xi), \hat{A}(\xi))\) on the whole spacetime \(\xi \in (0, \infty)\) (or \(r \in (0, \infty)\)).

(ix) Once \(\hat{F}\) and \(\hat{A}\) are known, from Eq.\((3.11)\), we can calculate \(\hat{B}\), so that an asymptotically flat black hole solution for \(\{\hat{A}, \hat{B}, \hat{F}\}\) is finally obtained over the whole space \(r \in (0, \infty)\).

Before proceeding to the next subsection to consider the physically allowed region of the parameter space of \(c_i\)'s, let us first reproduce the results presented in Table I of [67], in order to check our numerical code, although all these choices have been ruled out currently by observations [71]. To see this explicitly, let us first note that the parameters chosen in [63, 67] correspond to

\[
\begin{align*}
\hat{c}_2 &= -\frac{\hat{c}_1^3}{3\hat{c}_1^2 - 4\hat{c}_1 + 2}, \\
\hat{c}_3 &= 0, \\
\hat{c}_4 &= 0,
\end{align*}
\]

(4.3)

so that now only \(\hat{c}_1\) is a free parameter. With this choice of \(\hat{c}_1\)'s, the corresponding \(c_i\)'s can be obtained from Eqs.\((2.25)\) with \(\sigma = c_5^2\), which are given by,

\[
\begin{align*}
c_{14} &= \hat{c}_1, \\
c_2 &= \frac{-2c_{13} + 2\hat{c}_1 + 2c_{13}\hat{c}_1 - 2\hat{c}_1^2 - c_{13}\hat{c}_1^2}{2 - 4\hat{c}_1 + 3\hat{c}_1^2},
\end{align*}
\]

(4.4)

where \(c_{13}\) is arbitrary. This implies that Eqs.\((2.25)\) are degenerate for the choices of Eqs.\((4.3)\). It can be seen from Eq.\((4.4)\), in all the cases considered in [67], we have \(c_{14} > 2.5 \times 10^{-5}\). Hence all the cases considered in [63, 67] do not satisfy the current constraints [71].

With the above in mind, we reproduce all the cases considered in [63, 67], including the ones with \(\hat{c}_1 > 0.8\).

| \(\hat{c}_1\) | \(\hat{r}_g/r_H\) | \(\hat{F}_H\hat{A}_H^2\) | \(\hat{\gamma}_{ff}\) |
|---|---|---|---|
| 0.1 | 0.98948936 | 2.0961175 | 1.6028048 |
| 0.2 | 0.97802140 | 2.0716798 | 1.5769479 |
| 0.3 | 0.96522924 | 2.0391972 | 1.5476848 |
| 0.4 | 0.95054650 | 1.9965155 | 1.5140905 |
| 0.5 | 0.93304411 | 1.9405758 | 1.4748439 |
| 0.6 | 0.91106847 | 1.8666845 | 1.4279611 |
| 0.7 | 0.88131278 | 1.7673168 | 1.3702427 |
| 0.8 | 0.85830290 | 1.6283536 | 1.2959142 |
| 0.9 | 0.74751927 | 1.4155736 | 1.1921231 |
| 0.91 | 0.73301185 | 1.387021 | 1.1790400 |
| 0.92 | 0.71650458 | 1.3563710 | 1.1652344 |
| 0.93 | 0.69745439 | 1.3232418 | 1.1506047 |
| 0.94 | 0.67507450 | 1.2871125 | 1.1350208 |
| 0.95 | 0.64816499 | 1.2472379 | 1.1183101 |
| 0.96 | 0.61476429 | 1.2024805 | 1.1002331 |
| 0.97 | 0.57133058 | 1.1509356 | 1.0804355 |
| 0.98 | 0.51038168 | 1.0889067 | 1.0583387 |
| 0.99 | 0.41063001 | 1.0068873 | 1.0328120 |

Our results are presented in Table I, where

\[
\begin{align*}
\hat{\gamma}_{ff} &\equiv \hat{u}_a^\alpha u^\alpha_{ob}, \\
\hat{r}_g &\equiv -r_H \times \lim_{\xi \to 0} \frac{d\hat{F}(\xi)}{d\xi} = 2G_{\omega}M_{\text{ADM}},
\end{align*}
\]

(4.5)

(4.6)

where \(u^\alpha_{ob}\) is the tangent (unit) vector to a radial freefall trajectory that starts at rest at spatial infinity, and \(M_{\text{ADM}}\) denotes the Komar mass, which is equal to the ADM mass in the spherically symmetric case for the metric \(g_{\alpha\beta}\) [53].

From Table I we can see that our results are exactly the same as those given in [67] up to the same accuracy. But, due to the improved accuracy of our numerical code, for each of the physical quantity, we provided two more digits.

Additionally, in Fig. 2 we plotted the functions \(\hat{F}, \hat{B}, \hat{A}\) and \(C\) for four representative cases listed in Table I (\(\hat{c}_1 = 0.1, 0.3, 0.6, 0.99\)). Here, the quantity \(\bar{C}\) is defined as

\[
\bar{C} = \left| \frac{d\ln \hat{B}}{d\xi} - G_3 \right|,
\]

(4.7)

which vanishes identically for the solutions of the field equations, as it can be seen from Eq.\((3.76)\). In the rest of this paper, we shall use it to check the accuracy of our numerical code.
During the numerical calculations, we find that the asymptotical method, denoted by ˜, in which we also plot ˜. In this subsection we only consider the cases with constraints (2.18). Fig. 3 shows the functions ˜, since ˜ is very small, without loss of the generality, in plotting Fig. 3, we chose ˜, ˜, and ˜, as listed in Table I. The horizontal axis is ˜.

From Fig. 2, we note that the properties of { ˜, ˜, } depend on the choice of ˜. The quantity ˜ is approximately zero within the whole integration range, which means that our numerical solutions are quite reliable.

B. Physically Viable Solutions with S0Hs

With the above verification of our numerical code, we turn to the physically viable solutions of the Einstein-aether field equations, in which a S0H always exists. Since c13 is very small, without loss of the generality, in this subsection we only consider the cases with c13 = 0.

As the first example, let us consider the case c14 = 2 × 10⁻⁷, c2 = 9 × 10⁻⁷, and c3 = −c1, which satisfies the constraints (2.18). Fig. 3 shows the functions ˜, ˜, ˜, in which we also plot ˜ = ˜ ˜ and the GR limit of ˜, denoted by ˜ with ˜ = 1 − ˜/˜.

In plotting Fig. 3, we chose ˜ = 10⁻¹⁴. With the shooting method, ˜ is determined to be ˜ ≃ 2.4558992.¹⁰

In our calculations, we stop repeating the bisection search for ˜, when the value ˜ giving an asymptotically flat solution is determined to within 10⁻²³. Technically, these accuracies could be further improved. However, for our current purposes, they are already sufficient.

As we have already mentioned, theoretically Eq. (3.76) will be automatically satisfied once Eqs. (3.74), (3.75) and (3.77) hold. However, due to numerical errors, in practice, it can never be zero numerically. Thus, to monitor our numerical errors, we always plot out the quantity ˜ defined by Eq.(4.7), from which we can see clearly the numerical errors in our calculations. So, in the right-hand panels of Fig. 4, we plot out the curves of ˜, denoted by ˜, in each case.

Clearly, outside the S0H, ˜ ≲ 10⁻¹⁷, while inside the S0H we have ˜ ≲ 10⁻¹⁰. Thus, the solutions inside the horizon are not as accurate as the ones given outside of the horizon. However, since in this paper we are mainly concerned with the spacetime outside of the S0H, we shall

TABLE II: c², ˜ and ˜/rH calculated from different {c₂,c₁₄} with c₁₃ = 0 and a fixed ratio of c₂/c₁₄.

| c₂     | c₁₄     | c²   | ˜     | ˜/rH   |
|--------|---------|------|-------|--------|
| 9 × 10⁻⁷ | 2 × 10⁻⁷ | 4.4999935 | 2.4558992 | 1.1450729 |
| 9 × 10⁻⁸ | 2 × 10⁻⁸ | 4.4999994 | 2.4559003 | 1.1450730 |
| 9 × 10⁻⁹ | 2 × 10⁻⁹ | 4.4999999 | 2.4559004 | 1.1450730 |

¹⁰ During the numerical calculations, we find that the asymptotical behavior (3.79) of the metric coefficients at ξ ≈ rH/r ∼ 0 sensitively depends on the value of ˜. To make our results reliable, among all the steps in our codes, the precision is chosen to be not less than 37.
TABLE III: \( c_s^2, \, \tilde{A}_H \) and \( \tilde{r}_g/r_H \) calculated from different \( \{c_2, c_{14}\} \) with \( c_{13} = 0 \) and changing \( c_2/c_{14} \).

| \( c_2 \)       | \( c_{14} \)       | \( c_s^2 \)       | \( \tilde{A}_H \)       | \( \tilde{r}_g/r_H \) |
|-----------------|-------------------|-----------------|----------------|------------------|
| \( 2.01 \times 10^{-5} \) | \( 2 \times 10^{-5} \) | \( 1.0049596 \) | \( 1.4562430 \) | \( 1.0005850 \) |
| \( 7 \times 10^{-7} \) | \( 5 \times 10^{-7} \) | \( 1.3999982 \) | \( 1.6196457 \) | \( 1.0381205 \) |
| \( 9 \times 10^{-7} \) | \( 2 \times 10^{-8} \) | \( 44.9999393 \) | \( 6.4676346 \) | \( 1.2629671 \) |
| \( 9 \times 10^{-5} \) | \( 2 \times 10^{-7} \) | \( 449.93921 \) | \( 19.053220 \) | \( 1.3091657 \) |

not consider further improvements of our numerical code inside the horizon. The other quantities, such as \( c_s^2 \) and \( \tilde{r}_g \), are all given by the first row of Table II.

Following the same steps, we also consider other cases, and some of them are presented in Tables II-III. In particular, in Table II, we fix the ratio of \( c_2/c_{14} \) to be 9/2. In addition, the values of \( \{c_2, c_{14}\} \) are chosen so that they satisfy the constraints of Eq. (2.18). In Table III, the ratio \( c_2/c_{14} \) is changing and the values of \( \{c_2, c_{14}\} \) are chosen so that they are spreading over the whole viable range of \( c_{14} \), given by Eqs. (2.16)-(2.20).

From these tables we can see that quantities like \( \tilde{A}_H \) and \( \tilde{r}_g \) are sensitive only to the value of \( c_2/c_{14} \), instead of their individual values. This is understandable, as for \( c_{13} = 0 \) and \( c_{14} \lesssim 2.5 \times 10^{-5} \), Eq. (1.1) shows that \( c_S \approx c_S(c_2/c_{14}) \). Therefore, the same ratio of \( c_2/c_{14} \) implies the same velocity of the spin-0 graviton. Since S0H is defined by the speed of this massless particle, it is quite reasonable to expect that the related quantities are sensitive only to the value of \( c_S \).

The resulting \( \tilde{F}, \, \tilde{A}, \, \tilde{B} \) and \( \tilde{C} \) for the cases listed in Tables II and III are plotted in Figs. 4 and 5, respectively.

V. PHYSICAL SOLUTIONS (\( g_{\alpha \beta}, \, u^\mu \))

The above steps reveal how we find the solutions of the effective metric \( g_{\mu \nu} \) and aether field \( \tilde{u}^\mu \). To find the corresponding physical quantities \( g_{\mu \nu}, u^\mu \), we shall follow two steps: (a) Reverse Eqs. (3.39) and (3.41) to find a set of the physical quantities \{\( F(\xi), A(\xi), B(\xi) \}\} (Note that we have \( \xi = \tilde{r}_{S0H}/r = r_{S0H}/r \)). (b) Apply the rescaling \( \nu \rightarrow C_0 \nu \) to make the set of \{\( F(\xi), A(\xi), B(\xi) \}\} take the standard form at spatial infinity \( r = \infty \).

To these purposes, let us first note that, near the spatial infinity, Eqs. (3.79), (3.39) and (3.41) lead to

\[
F(\xi) = \frac{C_0^2}{\sigma} \left( 1 + F_1 \xi + \frac{1}{16} c_{14} F_1^2 \xi^2 + O(\xi^2) \right),
\]

\[
B(\xi) = \frac{C_0}{\sqrt{\sigma}} \left( 1 + \frac{1}{16} c_{14} F_1^2 \xi^2 - \frac{1}{12} c_{14} F_1^3 \xi^3 \right) + O(\xi^4),
\]

\[
A(\xi) = \frac{\sqrt{\sigma}}{C_0} \left[ 1 - \frac{1}{2} F_1 \xi + \frac{1}{2} A_2 \xi^2 - \frac{1}{2} A_2 F_1 - \frac{1}{16} F_1^3 \xi^3 + \frac{1}{96} c_{14} F_1^3 \xi^3 \right] + O(\xi^4),
\]

where

\[
F_1 = \tilde{F_1}, \quad c_{14} = \tilde{c}_{14}, \quad A_2 = \sqrt{\tilde{A}_2} = \frac{3}{4} (\sqrt{\sigma} - 1) \tilde{F}_1^2.
\]
the four-order of $\xi^4$.

A. Metric and Spin-0 Horizons

Again, we take the case of $c_{14} = 2 \times 10^{-7}$, $c_2 = 9 \times 10^{-7}$, and $c_3 = -c_1$ as the first example. The results for the normalized $F$, $A$, $B$ and $J$ in this case are plotted in Fig. 6. To see the whole picture of these functions on $r \in (0, \infty)$, they are plotted as functions of $r/r_H$ inside the horizon, while outside the horizon they are plotted as functions of $(r/r_H)^{-1}$. This explains why in the left-hand panel of Fig. 6, the MH ($r = r_{MH}$) stays in the left-hand side of the SOH, while in the right-hand panel, they just reverse the order. In this figure, we didn’t plot the GR limits for $B$ and $F$ since they are almost overlapped with their counterparts. From the analysis of this case, we find the following:

(a) The values of $F$ and $B$ are almost equal to their GR limits all the time. This is true even when $r$ is approaching the center $r = 0$, at which a spacetime curvature singularity is expected to be located.

(b) Inside the SOH, the oscillations of $A$ and $J$ become visible, which was also noted in [63]. Such oscillations continue, and become more violent as the curvature singularity at the center is approaching.

The functions of $\{F, A, B, J\}$ for the other cases listed in Tables II-III are plotted in Fig. 7. In this figure, the plots are ordered according to the magnitude of $c_2$. Besides, some amplified figures are inserted in (a)-(d) near the region around the point of $F = 0$. Similarly, in (e)-(f), some amplified figures are inserted near the region around the point of $J = J^+$.

The position of $r = r_{MH}$, at which we have $F(r_{MH}) = 0$, is marked by a full solid circle, while the position of $r = r_{SOH}$, at which we have $J(r_{SOH}) = J^+$ [cf. Eq. (3.17)], is marked by a pentagram, and in all these cases we always have $r_{MH} > r_{SOH}$. The values of $J^+$ and $J^-$ are given by the brown and purple solid lines, respectively. Note we always have $J^+ > J^-$ for $c_S > 1$. By using these two lines, we can easily find that there is only one $r_{SOH}$ in each case, i.e., $r_{SOH}^+$ in Eq. (3.17).

From the studies of these representative cases, we find the following: (i) As we have already mentioned, in all these cases the functions $B$ and $F$ are very close to their GR limits. (ii) Changing $c_S$ won’t influence the maximum of $A$ much. In contrast, the maximum of $|J|$ inside the SOH is sensitive to $c_S^2$. (iii) The oscillation of $A(r)$ gets more violent as $c_S^2$ is increasing. (iv) The value of $|r_{MH} - r_{SOH}|$ is getting bigger as $c_S^2$ deviating from 1. (v) In all these cases, we have only one $r_{SOH}$, i.e., only one intersection between $J(r)$ and $J^+$, in each case. (vi) Just like what we saw in Tables II-III, in the cases

\[11\] In [63], the author just considered the oscillational behavior of $A$. The physical quantities $F$, $A$, and $B$ were not considered.
with the same $c_s$ (but different values of $c_{14}$ and $c_2$), the corresponding functions $\{F, A, B, J\}$ are quite similar.

From Tables II-III and Fig. 7, we would like also to note that the value of $r_{MH}$ is always close to the corresponding $r_g$. To understand this, let us consider Eq. (5.1), from which we find that

$$ F(\xi) = 1 + F_1 \xi + \frac{1}{48} c_{14} F_1^3 \xi^3 + \mathcal{O} \left( \xi^4, c_{14}, c_2 \right), \quad (5.3) $$

after normalization. Recall $\xi \equiv r_H/r$ and $r_H \equiv r_{S0H}$. Then, from Eqs. (4.6), (3.79), (5.2) and (5.3), we also find that

$$ \frac{r_g}{r_{S0H}} = - \tilde{F}_1 = - F_1. \quad (5.4) $$

On the other hand, from Eq. (3.15), we have

$$ F(\xi) \bigg|_{r=r_{MH}} = 1 + F_1 \frac{r_{S0H}}{r_{MH}} + \frac{1}{48} c_{14} F_1^3 \left( \frac{r_{S0H}}{r_{MH}} \right)^3 + \mathcal{O} \left( \xi^4, c_{14}, c_2 \right) = 0, \quad (5.5) $$

from which we obtain,

$$ \frac{r_{MH}}{r_{S0H}} = - F_1 - \frac{1}{48} c_{14} F_1^3 \left( \frac{r_{S0H}}{r_{MH}} \right)^2 + \mathcal{O} \left( \frac{r_{S0H}}{r_{MH}} \right)^3 $$

$$ = \frac{r_g}{r_{S0H}} + \frac{1}{48} c_{14} \left( \frac{r_g}{r_{S0H}} \right)^2 \left( \frac{r_{S0H}}{r_{MH}} \right)^2 + \mathcal{O} \left( \xi^3, c_{14}, c_2 \right), \quad (5.6) $$

where Eq. (5.4) was used. For the expansion of $F$ to be finite, we must assume

$$ \mathcal{O} \left( \xi^3, c_{14}, c_2 \right) \lesssim \mathcal{O} \left[ \frac{1}{48} c_{14} \left( \frac{r_g}{r_{S0H}} \right)^3 \left( \frac{r_{S0H}}{r_{MH}} \right)^2 \right]. \quad (5.7) $$

At the same time, recall that we have $c_{14} \lesssim 2.5 \times 10^{-5}$ and $r_{S0H} \lesssim r_{MH}$. Besides, we also have $r_g/r_{S0H} \simeq \mathcal{O}(1)$. Thus, from Eq. (5.6) we find

$$ \left| \frac{r_{MH}}{r_{S0H}} - \frac{r_g}{r_{S0H}} \right| \lesssim \mathcal{O}(c_{14}). \quad (5.8) $$

This result reveals why the values of $r_{MH}/r_{S0H}$ and $r_g/r_{S0H}$ are very close to each other, although not necessarily the same exactly.

Finally, let us take a closer look at the difference between GR and $\alpha$-theory, although in the above we al-
FIG. 7: Solutions for different combinations of \( \{c_2, c_{14}\} \) listed in Tables II-III. Here, A, B, J and F are represented by the red solid line, green dotted line, orange dashed line, and blue dash-dotted line, respectively. These figures are ordered according to the magnitude of \( c_2^2 \). In each of the figure, the values \( J^+ \) and \( J^- \) are given respectively by the brown and purple solid lines with \( J^+ > J^- \), while the positions of \( r = r_{MH} \) and \( r = r_{SOH} \) are marked by a small full solid circle and a pentagram, respectively. Additionally, the value of \( r_{MH}/r_{SOH} \) is also given in each case.
FIG. 8: $\Delta F$ and $\Delta B$ for $c_2 = 9 \times 10^{-7}$, $c_{14} = 2 \times 10^{-7}$ and $c_{13} = 0$. The panels (a) and (c) show the region outside the S0H, while the panels (b) and (d) show the region inside the S0H.

ready mentioned that the results from these two theories are quite similar. To see these more clearly, we first note that the GR counterparts of $F$ and $B$ are given by

$$F_{\text{GR}} = 1 - \frac{r_{\text{MH}}}{r}, \quad B_{\text{GR}} = 1.$$  \hfill (5.9)

Thus, the relative differences can be defined as

$$\Delta F = \frac{F - F_{\text{GR}}}{F_{\text{GR}}}, \quad \Delta B = \frac{B - B_{\text{GR}}}{B_{\text{GR}}}. \hfill (5.10)$$

Again, considering the representative case $c_2 = 9 \times 10^{-7}$, $c_{14} = 2 \times 10^{-7}$ and $c_{13} = 0$, we plot out the differences $\Delta F$ and $\Delta B$ in Fig. 8, from which we find that in the range $\xi \in (10^{-12}, 1)$ we have $O(\Delta F) \lesssim 10^{-9}$. On the other hand, in the range $\xi \in (1, 10^{12})$, we have $O(\Delta F) \lesssim 10^{-5}$. Similarly, in the range $\xi \in (10^{-12}, 1)$, we have $O(\Delta B) \lesssim 10^{-8}$. In addition, in the range $\xi \in (1, 10^{7})$ we have $O(\Delta B) \lesssim 10^{-3}$. Thus, we confirm that $F$ and $B$ are indeed quite closed to their GR limits.

B. Universal Horizons

In theories with the broken LI, the dispersion relation of a massive particle contains generically high-order momentum terms [80],

$$E^2 = m^2 + c_k^2 k^2 \left(1 + \sum_{n=1}^{2(z-1)} a_n \left(\frac{k}{M_*}\right)^n\right), \hfill (5.11)$$

from which we can see that both of the group and phase velocities become unbounded as $k \to \infty$, where $E$ and $k$ are the energy and momentum of the particle considered, and $c_k$ and $a_n$’s are coefficients, depending on the species of the particle, while $M_*$ is the suppression energy scale of the higher-dimensional operators. Note that there must be no confusion between $c_k$ here and the four coupling constants $c_i$’s of the theory. As an immediate result, the causal structure of the spacetimes in such theories is quite different from that given in GR, where the light cone at a given point $p$ plays a fundamental role in determining the causal relationship of $p$ to other events [99]. In a UV complete theory, the above relationship is expected even in the gravitational sector. One of such examples is the
particles even with infinitely large speeds cannot escape from it, once they are inside it [cf. Fig. 9]. So, it acts as an absolute horizon to all particles (with any speed), which is often called the UH [66, 67, 80]. At the horizon, as can be seen from Fig. 9, we have [102],

\[
\zeta \cdot u \bigg|_{r=r_{UH}} = -\frac{1}{2A} (1 + J) \bigg|_{r=r_{UH}} = 0, \quad (5.12)
\]

where \( J \equiv F A^2 \). Therefore, the location of an UH is exactly the crossing point between the curve of \( J(r) \) and the horizontal constant line \( J = -1 \), as one can see from Figs. 6 and 7. From these figures we can also see that they are always located inside S0Hs, as expected. In addition, the curve \( J(r) \) is oscillating rapidly, and crosses the horizontal line \( J = -1 \) back and forth infinite times. Therefore, in each case we have infinite number of \( r_{UH-i} \) (\( i = 1, 2, ... \)). In this case, the UH is defined as the largest value of \( r_{UH-i} \) (\( i = 1, 2, ... \)). In Table IV, we show the locations of the first eight UHs for each case, listed in Tables II and III. It is interesting to note that the formation of multi-roots of UHs was first noticed in [67], and later observed in gravitational collapse [72].

C. Other Observational Quantities

Another observationally interesting quantity is the ISCO, which is the root of the equation,

\[
2rF'(r)^2 - F [3F'(r) + rF''(r)] = 0. \quad (5.13)
\]

Note that in GR we have \( r_{ISCO}/r_H = 3 \) [98]. Due to the tiny differences between the Schwarzschild solutions and the ones considered here, as shown in Fig. 8, it is expected that \( r_{ISCO} \)'s in these cases are also quite closed to its GR limit. As a matter of fact, we find that this is indeed the case, and the differences in all the cases considered above appear only after six digits, that is, \( |r_{ISCO} - r_{ISCO}^{GR}| \leq 10^{-6} \), as shown explicitly in Tables V and VI.

In Table V, we also show several other physical quantities. These include the Lorentz gamma factor \( \gamma_f \), the gravitational radius \( r_g \), the orbital frequency of the ISCO \( \omega_{ISCO} \), the maximum redshift \( z_{max} \) of a photon emitted by a source orbiting the ISCO (measured at the infinity), and the impact parameter \( b_{ph} \) of the circular photon orbit...
(CPO), which are defined, respectively, by [67],
\[ \gamma_{ff} = \left( A + \frac{1}{4A} \right)_{r=r_{MH}}, \]
\[ r_g = -r_{SOH} \frac{dF}{d\xi} \bigg|_{\xi=0}, \]
\[ \omega_{ISCO} = \frac{\sqrt{dF/dr}}{2r} \bigg|_{r=r_{ISCO}}, \]
\[ z_{max} = 1 + \frac{\omega_{ISCO} F^{-1/2}}{\sqrt{F - \omega_{ISCO}^2}} - 1, \]
\[ b_{ph} = \frac{r}{\sqrt{F}} \bigg|_{r=r_{ph}}, \]
where the radius \( r_{ph} \) of the CPO is defined as
\[ \left( 2F - r \frac{dF}{dr} \right)_{r=r_{ph}} = 0. \]

As pointed previously, these quantities are quite closed to their relativistic limits, since they depend only on the spacetimes described by \( F \) and \( B \). As shown in Fig. 8, the differences of these spacetimes between \( \alpha \)-theory and GR are very small. To see this more clearly, let us introduce the quantities,
\[ \Delta r_{ISCO} \equiv \frac{r_{ISCO}}{r_{MH}} - \left( \frac{r_{ISCO}}{r_{MH}} \right)_{GR}, \]
\[ \Delta \omega_{ISCO} \equiv \frac{r_g \omega_{ISCO}}{r_g^{(S)}} - \left( \frac{r_g \omega_{ISCO}}{r_g^{(S)}} \right)_{GR}, \]
\[ \Delta z_{max} = z_{max} - \left( z_{max} \right)_{GR}, \]
\[ \Delta b_{ph} = \frac{b_{ph}}{b_{ph}} - \left( \frac{b_{ph}}{b_{ph}} \right)_{GR}, \]
where the GR limits of \( r_{ISCO}/r_{MH}, r_g \omega_{ISCO}, z_{max} \) and \( b_{ph}/r_g \) are, respectively, 3, \( 2 \times 6^{-3/2}, 3/\sqrt{2} - 1 \) and \( 3 \sqrt{3}/2 \). As can be seen from Table VI, all of these quantities are fairly closed to their GR limits.

Therefore, we conclude that it is quite difficult to distinguish GR and \( \alpha \)-theory through the considerations of the physical quantities \( r_{ISCO}, \omega_{ISCO}, z_{max} \) or \( b_{ph} \), as far as the cases considered in this paper are concerned. Thus, it would be very interesting to look for other choices of \( \{ c_2, c_{13}, c_{14} \} \) (if there exist), which could result in distinguishable values in these observational quantities.

VI. CONCLUSIONS

In this paper, we have systematically studied static spherically symmetric spacetimes in the framework of Einstein-aether theory, by paying particular attention to black holes that have regular SOHs. In \( \alpha \)-theory, a time-like vector - the aether, exists over the whole spacetime. As a result, in contrast to GR, now there are three gravitational modes, referred to as, respectively, the spin-0, spin-1 and spin-2 gravitons.

To avoid the vacuum gravi-Čerenkov radiation, all these modes must propagate with speeds greater than or at least equal to the speed of light [74]. However, in the spherically symmetric spacetimes, only the spin-0 mode is relevant in the gravitational sector [75], and the boundaries of BHs are defined by this mode, which are the null surfaces with respect to the metric \( g^{(S)}_{\mu\nu} \) defined in Eq.(1.3), the so-called SOHs. Since now \( c_S \geq c \), where \( c_S \) is the speed of the spin-0 mode, the SOHs are always inside or at most coincide with the metric (Killing) horizons. Then, in order to cover spacetimes both inside and outside the BHs, working in the Eddington-Finkelstein coordinates (3.1) is one of the natural choices.

In the process of gravitational radiations of compact objects, all of these three fundamental modes will be emitted, and the GW forms and energy loss rate should be different from that of GR. In particular, to the leading order, both monopole and dipole emissions will co-exist with the quadrupole emission [34, 35, 38–41, 43]. Despite of all these, it is remarkable that the theory still remains as a viable theory, and satisfies all the constraints, both theoretical and observational [71], including the recent detection of the GW, GW170817, observed by the LIGO/Virgo collaboration [89], which imposed the severe constraint on the speed of the spin-2 gravitational mode, \( -3 \times 10^{-15} < c_T - 1 < 7 \times 10^{-16} \). Consequently, it is one of few theories that violate Lorentz symmetry and meantime is still consistent with all the observations carried out so far [71, 103].

Spherically symmetric static BHs in \( \alpha \)-theory have been extensively studied both analytically [51–61] and numerically [63–69], and various solutions have been obtained. Unfortunately, all these solutions have been ruled out by current observations [71].

Therefore, as a first step, in this paper we have investigated spherically symmetric static BHs in \( \alpha \)-theory that satisfy all the observational constraints found lately in [71] in detail, and presented various numerical new BH solutions. In particular, we have first shown explicitly that among the five non-trivial field equations, only three of them are independent. More important, the two second-order differential equations given by Eqs.(3.5) and (3.6) for the two functions \( F(r) \) and \( A(r) \) are independent of the function \( B(r) \), where \( F(r) \) and \( B(r) \) are the metric coefficients of the Eddington-Finkelstein metric (3.1), and \( A(r) \) describes the aether field, as shown by Eq.(3.2). Thus, one can first solve Eqs.(3.5) and (3.6) to find \( F(r) \) and \( A(r) \), and then from the third independent equation to find \( B(r) \). Another remarkable feature is that the function \( B(r) \) can be obtained from the constraint (3.10), and is given simply by the algebraic expression of \( F, A \) and their derivatives, as shown explicitly by Eq.(3.11). This not only saves the computational labor, but also makes the calculations more accurate, as pointed out explicitly in [67], solving the first-order differential equation (3.7) for \( B(r) \) can “potentially be affected by numerical inaccuracies when evaluated very close to the horizon”.

Then, now solving the (vacuum) field equations of
spherically symmetric static spacetimes in æ-theory simply reduces to solve the two second-order differential equations (3.5) and (3.6). This will considerably simplify the mathematical computations, which is very important, especially considering the fact that the field equations involved are extremely complicated, as one can see from Eqs.(3.5)-(3.10) and (A.1) - (A.4). Then, in the case involved are extremely complicated, as one can see from Eqs.(3.5)-(3.10) and (A.1) - (A.4). Then, in the case $c_{13} = c_{14} = 0$ we have been able to solve these equations explicitly, and obtained a three-parameter family of exact solutions, which in general depends on the coupling constant $c_2$. However, requiring that the solutions be asymptotically flat, we have found that the solutions become independent of $c_2$, and the corresponding metric reduces precisely to the Schwarzschild BH solution with a non-trivially coupling aether field given by Eq.(3.34), which is always timelike even in the region inside the BH.

To simplify the problem further, we have also taken the advantage of the field redefinitions that are allowed by the internal symmetry of æ-theory, first discovered by Foster in [83], and later were used frequently, including the works of [63, 65, 67]. The advantage of the field redefinitions is that it allows us to choose the free parameter $\sigma$ involved in the field redefinitions, so that the SOH of the redefined metric $\hat{g}_{\mu\nu}$ will coincide with its MH. This will reduce the four-dimensional space of the initial conditions, spanned by $\hat{F}_H, \hat{F}_H', \hat{A}_H, \hat{A}_H'$, to one-dimension, spanned only by $\hat{A}_H$, if the initial conditions are imposed on the SOH. In Sec. III.D, we have shown step by step how one can do it. In addition, in this same subsection we have also shown that the field equations are invariant under the rescaling $r \rightarrow Cr$. In fact, introducing the dimensionless coordinate $\xi \equiv r_{SOH}/r$, the relevant four field equations take the scaling-invariant forms of Eqs.(3.74) - (3.77), which are all independent of $r_{SOH}$. Thus, when integrating these equations, without loss of generality, one can assign any value to $r_{SOH}$.

We would like also to note that in Section III.C we worked out the relations in detail among the fields $(g_{\mu\nu}, u^i, c_i)$, $(\hat{g}_{\mu\nu}, \hat{u}^i, \hat{c}_i)$ and $(\hat{g}_{\mu\nu}, \hat{u}^i, \hat{c}_i)$, and clarified several subtle points. In particular, the redefined metric $\hat{g}_{\mu\nu}$ through Eqs.(2.23) and (2.24) does not take the standard form in the Eddington-Finkelstein coordinates.

### TABLE IV: $r_{UH-i}$’s for different cases listed in Tables II and III. Note that here we just show first eight UHs of Eq.(5.12) for each case.

| $c_2^2$ | $r_{MH}/r_{UH-1}$ | $r_{MH}/r_{UH-2}$ | $r_{MH}/r_{UH-3}$ | $r_{MH}/r_{UH-4}$ | $r_{MH}/r_{UH-5}$ | $r_{MH}/r_{UH-6}$ | $r_{MH}/r_{UH-7}$ | $r_{MH}/r_{UH-8}$ |
|---------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| 1.0049596 | 1.40913534 | 9.12519836 | 68.6766490 | 524.111256 | 4006.80012 | 30638.7274 | 234291.582 | 1791613.54 |
| 1.3999982 | 1.39634652 | 6.27835216 | 33.1700700 | 178.825436 | 967.454326 | 5237.31538 | 234291.582 | 1791613.54 |
| 4.4999935 | 1.36429738 | 2.74101697 | 6.42094860 | 38.6164383 | 95.7784255 | 28355.5481 | 153524.182 | 591850964 |
| 4.4999994 | 1.36429738 | 2.74101595 | 6.42094387 | 38.6164383 | 95.7784255 | 28355.5481 | 153524.182 | 591850964 |
| 4.4999999 | 1.36429738 | 2.74101584 | 6.42094340 | 38.6164754 | 95.7782326 | 28355.5481 | 153524.182 | 591850964 |
| 4.4999999 | 1.36429738 | 2.74101595 | 6.42094387 | 38.6164383 | 95.7784255 | 28355.5481 | 153524.182 | 591850964 |
| 449.93921 | 1.33429146 | 1.39226716 | 1.46010811 | 1.5385402 | 1.62835485 | 1.73026559 | 1.84507183 | 1.97362062 |

### TABLE V: The quantities $r_{SOH}, \gamma_{ff}, r_{ISCO}, \omega_{ISCO}, z_{max}$ and $b_{ph}$ for different cases listed in Tables II and III.

| $c_2^2$ | $r_{MH}/r_{SOH}$ | $\gamma_{ff}$ | $r_{ISCO}/r_{MH}$ | $r_{g}\omega_{ISCO}$ | $z_{max}$ | $b_{ph}/r_{g}$ |
|---------|-------------------|-------------|-------------------|-------------------|-------|--------------|
| 1.0049596 | 1.00058469 | 1.62614814 | 3.00000083 | 0.13608278 | 1.12132046 | 2.59807604 |
| 1.3999982 | 1.03812045 | 1.63971715 | 3.00000002 | 0.13608276 | 1.12132035 | 2.59807621 |
| 4.4999935 | 1.14507287 | 1.67376648 | 3.00000000 | 0.13608276 | 1.12132034 | 2.59807621 |
| 4.4999994 | 1.14507298 | 1.67376647 | 3.00000000 | 0.13608276 | 1.12132034 | 2.59807621 |
| 4.4999999 | 1.14507299 | 1.67376647 | 3.00000000 | 0.13608276 | 1.12132034 | 2.59807621 |
| 449.93939 | 1.26296693 | 1.69775757 | 3.00000000 | 0.13608276 | 1.12132034 | 2.59807621 |
| 449.93921 | 1.30916545 | 1.70149318 | 3.00000000 | 0.13608276 | 1.12132034 | 2.59807621 |
TABLE VI: \( \Delta r_{\text{ISCO}}, \Delta \omega_{\text{ISCO}}, \Delta z_{\text{max}} \) and \( \Delta b_{\text{ph}} \) for different cases listed in Tables II and III.

| \( c_s^2 \) | \( \Delta r_{\text{ISCO}} \) | \( \Delta \omega_{\text{ISCO}} \) | \( \Delta z_{\text{max}} \) | \( \Delta b_{\text{ph}} \) |
|-------|----------------|----------------|----------------|---------------|
| 1.0049596 | \( 8.3 \times 10^{-7} \) | \( 1.3 \times 10^{-8} \) | \( 1.2 \times 10^{-7} \) | \( -1.7 \times 10^{-7} \) |
| 1.3999982 | \( 1.8 \times 10^{-8} \) | \( 2.2 \times 10^{-10} \) | \( 2.0 \times 10^{-9} \) | \( -3.2 \times 10^{-9} \) |
| 4.4999935 | \( 4.0 \times 10^{-9} \) | \( 1.5 \times 10^{-12} \) | \( 4.9 \times 10^{-11} \) | \( -4.2 \times 10^{-10} \) |
| 4.4999994 | \( 4.0 \times 10^{-10} \) | \( 1.5 \times 10^{-11} \) | \( -7.2 \times 10^{-11} \) | \( -3.2 \times 10^{-10} \) |
| 4.4999999 | \( 4.0 \times 10^{-11} \) | \( 2.3 \times 10^{-11} \) | \( -1.2 \times 10^{-10} \) | \( -4.5 \times 10^{-10} \) |
| 44.999939 | \( 1.5 \times 10^{-10} \) | \( 9.6 \times 10^{-11} \) | \( -5.6 \times 10^{-10} \) | \( -1.9 \times 10^{-9} \) |
| 449.93921 | \( 1.1 \times 10^{-9} \) | \( 1.1 \times 10^{-11} \) | \( -4.5 \times 10^{-10} \) | \( -1.1 \times 10^{-9} \) |

as shown explicitly by Eq. (3.35). Instead, only after a proper coordinate transformation given by Eqs. (3.36) and (3.37), the resulting metric \( g_{\mu\nu} \) takes the standard form, as given by Eq. (3.38). Then, the field equations for \((g_{\mu\nu}, u^\mu, c_i)\) take the same forms as the ones for \((g_{\mu\nu}, \tilde{u}^\mu, \tilde{c}_i)\). Therefore, when we solved the field equations in terms of the redefined fields, they are the ones of \((g_{\mu\nu}, \tilde{u}^\mu)\), not the ones for \((g_{\mu\nu}, u^\mu)\).

After clarifying all these subtle points, in Sec. IV, we have worked out the detail on how to carry out explicitly our numerical analysis. In particular, to monitor the numerical errors of our code, we have introduced the quantity \( \tilde{C} \) through Eq. (4.7), which is essentially Eq. (3.65). Theoretically, it vanishes identically. But, due to numerical errors, it is expected that \( \tilde{C} \) has non-zero values, and the amplitude of it will provide a good indication on the numerical errors that our numerical code could produce.

To show further the accuracy of our numerical code, we have first reproduced the BH solutions obtained in \([63, 67]\), but with an accuracy that are at least two orders higher [cf. Table I]. It should be noted that all these BH solutions have been ruled out by the current observations \([71]\). So, after checking our numerical code, in Sec. IV.B, we considered various new BH solutions that satisfy all the observational constraints \([71]\), and presented them in Tables II and III, as well as in Figs. 3-5.

Then, in Sec. V, we have presented the physical metric \( g_{\mu\nu} \) and \( \varphi \)-field \( u^\mu \) for these viable new BH solutions obtained in Section IV. Before presenting the results, we have first shown that the physical fields, \( g_{\mu\nu} \) and \( u^\mu \), are also asymptotically flat, provided that the effective fields \( \tilde{g}_{\mu\nu} \) and \( \tilde{u}^\mu \) are [cf. Eqs. (5.1) and (5.2)]. Then, the physical BH solutions were plotted out in Figs. 6 and 7. Among several interesting features, we would like to point out the different locations of the metric and spin-0 horizons for the physical metric \( g_{\mu\nu} \), denoted by full solid circles and pentagrams, respectively.

Another interesting point is that all these physical BH solutions are quite similar to the Schwarzschild one. In Fig. 8 we have shown the differences for the case \( c_2 = 9 \times 10^{-7} \), \( c_{14} = 2 \times 10^{-7} \) and \( c_{13} = 0 \), but similar results also hold for the other cases, listed in Tables II and III.

In this section, we have also identified the locations of the UHs of these solutions and several other observationally interesting quantities, which include the ISCO \( r_{\text{ISCO}} \), the Lorentz gamma factor \( \gamma_{ff} \), the gravitational radius \( r_g \), the orbital frequency \( \omega_{\text{ISCO}} \) of the ISCO, and the maximum redshift \( z_{\text{max}} \) of a photon emitted by a source orbiting the ISCO (measured at the infinity), the radii \( r_{\text{ph}} \) of the CPO, and the impact parameter \( b_{\text{ph}} \) of the CPO. All of them are given in Table IV-V. In Table VI we also calculated the differences of these quantities obtained in \( \varphi \)-theory and GR. Looking at these results, we conclude that it’s very hard to distinguish GR and \( \varphi \)-theory through these quantities, as far as the cases considered in this paper are concerned. We would also like to note that for each BH solution, there are infinite number of UHs, \( r = r_{UH-i} \), \( (i = 1, 2, 3, \ldots) \), which was also observed in \([67]\). In Table IV we have listed the first eight of them, and the largest one is usually defined as the UH of the BH. In contrast, there are only one S0H and one MH for each solution. These features are also found in the gravitational collapse of a massless scalar field in \( \varphi \)-theory \([72]\).

An immediate implication of the above results is that the QNMs of these BHs for a test field, scalar, vector or tensor \([104]\), will be quite similar to these given in GR. Our preliminary results on such studies indicate that this is indeed the case. However, we expect that there should be significant differences from GR, when we consider the metric perturbations of these BH solutions - the gravitational spectra of perturbations \([105]\), as now the BH boundaries are the locations of the S0Hs, not the locations of the MHs. This should be specially true for the cases with large speeds \( c_s \) of the spin-0 modes, as in these cases the S0Hs are significantly different from the MHs, and located deeply inside them. Thus, imposing the non-out-going radiation on the S0Hs will be quite different from imposing the non-out-going radiation on the corresponding MHs. We wish to report our results
along this direction soon in another occasion.

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Appendix A: The coefficients of \( f_n, a_n, b_n \) and \( n_n \)

In this appendix, we shall provide the explicit expressions of the coefficients of \( f_n, a_n, b_n \) and \( n_n \), encountered in the Einstein-aether field equations in the spherically symmetric spacetimes, for which the metric is written in the Eddington-Finkelstein coordinates (3.1), with the aether field taking the form of Eq.(3.2). In particular, the coefficients of \( f_n, a_n \) and \( b_n \) appearing in Eqs.(3.5) - (3.7) are given by

\[
\begin{align*}
\mathcal{A}_0 &= -4 (c_2 + c_{13}) (c_2 + c_{13} - (c_2 + 1)) r A(r) A'(r) \\
&= -((c_1 c_2 - c_2^2 + c_{13}^2 + c_2 + 1) c_{14}^2 - (c_2 + 2) c_{13} c_{14}) r^2 A'(r)^2 \\
&- 4 (c_{14} + 1) c_{13}^2 + 2 (c_2 + 1) c_{13} + (c_2 - 2) c_{14} c_{13} + c_2 (c_2 + 2) - 2 (c_2^2 + 3 c_2 + 1) c_{14}) r A(r)^4 F'(r) \\
&+ 2 (c_2^2 + (2 - 3 c_2) c_{14} - 9 c_{13} c_{14} + (c_2 + 1) c_{14}^2 - c_{13}^2 (4 c_{14} + 1)) r^2 A(r)^3 A'(r) F'(r) \\
&- (5 c_{14} c_2^2 - c_2^2 + (c_2 + 1) c_{14}^2 + (7 c_2 - 2) c_{13} c_{14} + c_{13}^2 (4 c_{14} + 1)) r^2 A(r)^6 F'(r)^2 \\
&- 2 (c_2 + c_{13}) c_{14} (c_2-c_{13} + c_{14}) r^2 A(r)^5 A'(r) F'(r)^2 + (c_2 + c_{13}) c_{14} (c_2-c_{13} + c_{14}) r^3 A(r)^8 F'(r)^3 \\
&- (c_2 + c_{13}) A(r)^2 ((c_2 - c_{13} + c_{14}) r^3 A'(r)^2 F'(r)^2 + 4 c_2 (c_{14} - 1) + 2 c_{13} c_{14}) \\
\mathcal{A}_1 &= -8 (c_2 + c_{13}) (c_4 c_2 + c_2 + c_{13} + c_{14}) r A(r)^3 A'(r) \\
&+ 4 (c_2^2 - 3 c_{13} c_{14} c_2 + 2 c_{14} c_2 + (c_2 + 1) c_{14}^2 - c_{13} (c_2 + 1)) r^2 A(r)^2 A'(r)^2 \\
&- 4 ((1 - 2 c_{14}) c_2^2 + (c_{14} - c_2 + 2 c_2 + c_{14} + 4) c_{13} + c_2 (c_2 + 4) + (5 c_2^2 + 9 c_2 + 4) c_{14}) r A(r)^6 F'(r) \\
&+ 2 (c_{14} c_2^2 + 3 c_2^2 - 3 (c_2 + 1) c_{14}^2 + (11 c_2 + 6) c_{13} c_{14} + c_{13}^2 (4 c_{14} - 3)) r^2 A(r)^5 A'(r) F'(r) \\
&+ (c_2^2 + (3 c_2 + 2) c_{14} c_2 + 3 c_{13} c_{14} c_2 + (c_2 + 1) c_{14}^2 + c_{13}^2 (2 c_{14} - 1)) r^2 A(r)^8 F'(r)^2 \\
&+ 2 (c_2 + c_{13}) c_{14} (c_2 - c_{13} + c_{14}) r^3 A(r)^7 A'(r) F'(r)^2 \\
&- 2 (c_2 + c_{13}) A(r)^4 ((-c_2 + c_{13} + c_{14} - 4) r^3 A'(r)^2 F'(r)^2 - 4 (2 c_2 + c_{13} + 1)) \\
\mathcal{A}_2 &= 2 (c_{14} c_2^2 + 3 c_2^2 - 3 (c_2 + 1) c_{14}^2 + (11 c_2 + 6) c_{13} c_{14} + c_{13}^2 (4 c_{14} - 3)) r^2 A(r)^4 A'(r)^2 \\
&+ 4 ((-c_{14} - 1) c_2^2 + 2 (c_2 - 1) c_{13} + (c_2 + 4) c_{14} c_{13} + (c_2 - 2) c_2 + (4 c_2^2 + 8 c_2 + 2) c_{14}) r A(r)^8 F'(r) \\
&+ 4 ((c_2 + c_{14}) c_2 + (c_2 + 1) c_{14} - c_{13} + c_{14} + 2) c_2 - (c_{13}^2 + c_{14}^2) r^2 A(r)^7 A'(r) F'(r) \\
&+ (-c_2 + c_{13}) A(r)^6 ((c_2 - c_{13} + c_{14}) r^3 A'(r)^2 F'(r) + 8 c_2 + 4 (6 c_2 + 3 c_{13} + 4) c_{14}) \\
\mathcal{A}_3 &= 8 (c_2 + c_{13}) (c_4 c_2 + c_2 + c_{13} + c_{14}) r A(r)^7 A'(r) + 8 (2 c_2^2 + 3 c_{13} c_2 + c_2 + c_{13}^2 + c_{13}) c_{14} A(r)^8 \\
&+ 4 (c_2^2 - 3 c_{13} c_4 c_2 + 2 c_{14} c_2 + (c_2 + 1) c_{14}^2 - c_{13}^2 (c_{14} + 1)) r^2 A(r)^6 A'(r)^2 \\
&+ 4 (c_2 + c_{13}) (c_2 + c_{14} - (c_2 + 1) c_{14}) r A(r)^{10} F'(r) \\
&+ 2 (c_2 + c_{13} + c_{14}) (c_2 - c_{13} - (c_2 + 1) c_{14}) r^2 A(r)^9 A'(r) F'(r), \\
\mathcal{A}_4 &= 4 (c_2 + c_{13}) (c_2 + c_{13} - (c_2 + 1) c_{14}) r A(r)^9 A'(r) - 2 (c_2 + c_{13}) (2 c_2 (c_{14} - 1) + c_{13} c_{14}) A(r)^{10} \\
&+ (-c_{14} c_2^2 + c_2^2 - c_{13}^2 - (c_2 + 1) c_{14}^2 + (c_2 + 2) c_{13} c_{14}) r^2 A(r)^8 A'(r)^2, \\
\end{align*}
\]
\[ a_0 = 4 \left( -c_{14}^{2} + (-c_{14} c_{2} + 2c_{2} + 2c_{14} - 2) c_{13} + (c_{2} - 2) c_{2} + 2 \left( c_{2}^{2} + 3c_{2} + 1 \right) c_{14} \right) r A(r)^{2} A'(r) \\
+ (c_{14}^{2} + (5c_{2} c_{14} + 8) c_{13} - (c_{2} + 1) c_{2}^{2} - (c_{2} - 8) c_{2} - (5c_{2}^{2} + 18c_{2} + 8) c_{14}) r^2 A(r) A'(r)^2 \\
+ (c_{2} + c_{14}) c_{14} (c_{2} - c_{13} + c_{14}) r^3 A'(r)^3 + 4 (c_{2} + c_{13}) (c_{2} c_{4} + c_{2} + c_{13} + c_{14}) r A(r)^3 F'(r) \\
- 2 (2c_{14} - 1) c_{14}^2 + ((3c_{2} - 2) c_{14} + 4) c_{13} - (c_{2} + 1) c_{2}^2 + c_{2} (c_{2} + 2) + (3c_{2} + 4c_{2} + 4) c_{14}) r^2 A(r)^4 A'(r) F'(r) \\
+ (c_{2}^{2} - (c_{2} + 2) c_{14} c_{2} + c_{13} c_{14} c_{2} + c_{13}^{2} - (c_{2} + 1) c_{2}^2) r^2 A(r)^{5} F'(r)^2 \\
+ (c_{2} + c_{13}) c_{14} (c_{2} - c_{13} + c_{14}) r^3 A(r)^{6} A'(r) F'(r)^2 \\
- 2 (c_{2} + c_{13}) A(r)^{6} (c_{14} (c_{2} - c_{13} + c_{14} - 4) r^3 A'(r)^2 F'(r) + 2c_{2} + c_{2} c_{14} + c_{13} c_{14} + 4), \\
a_1 = 4 ((2c_{14} + 1) c_{14}^{2} + (3c_{14} c_{2} + 2c_{2} - 4c_{2} - 4c_{2} + 4c_{14}) r A(r)^{4} A'(r) \\
- (8c_{14} - 3) c_{14}^{2} - (23c_{2} + 4) c_{14} - 8) c_{14} + 3 (c_{2} + 1) c_{2}^2 - c_{2} (3c_{2} - 8) - (c_{2}^2 - 8c_{2} - 8) c_{14}) r^2 A(r)^{3} A'(r)^2 \\
- 2 (c_{2} + c_{13}) c_{14} (c_{2} - c_{13} + c_{14} - 4) r^3 A(r)^{2} A'(r)^3 - 8 (c_{2} + 1) (c_{2} + c_{13}) c_{14} r A(r)^{7} F'(r) \\
+ 4 ((c_{14} + 1) c_{14}^2 + 2 (c_{2} c_{14} - 1) c_{13} - (c_{2} + 1) c_{2}^2 - c_{2} (c_{2} + 2) + (c_{2} + 2c_{2} + 2) c_{14}) r^2 A(r)^{6} A'(r) F'(r) \\
+ (c_{14} c_{2}^{2} - c_{2}^2 + c_{13}^2 + (c_{2} + 1) c_{2}^2 - c_{2} (c_{2} + 2) c_{13} c_{14}) r^2 A(r)^{7} A'(r)^2 \\
- 2 (c_{2} + c_{13}) A(r)^{9} (c_{14} (c_{2} - c_{13} + c_{14}) r^3 A'(r)^2 F'(r) - 2c_{2} - (6c_{2} + 3c_{13} + 4) c_{14}), \\
a_2 = - 2 (c_{2} + c_{13}) (6c_{2} + 3c_{13} + 4) c_{14} - 2 (c_{2} + 2) A(r)^{7} \\
- 4 ((c_{14} + 1) c_{14}^2 + (c_{2} (3c_{14} + 2) + 2) c_{13} + c_{2} (c_{2} + 2) - 2 (c_{2} + c_{14}) r A(r)^{6} A'(r) \\
+ (3c_{2}^2 + (5c_{2} - 6) c_{14} c_{2} + 19c_{14} c_{12} - 3 (c_{2} + 1) c_{2}^2 + c_{2} (8c_{14} + 3)) r^2 A(r)^{5} A'(r)^2 \\
+ (c_{2} + c_{13}) c_{14} (c_{2} - c_{13} + c_{14}) r^3 A(r)^{4} A'(r)^3 - 4 (c_{2} + c_{13}) (c_{2} + c_{13} - (c_{2} + 1) c_{14}) r A(r)^{9} F'(r) \\
- 2 (c_{2} - c_{13} + c_{14}) (c_{2} - c_{13} - (c_{2} + 1) c_{14}) r^2 A(r)^{8} A'(r) F'(r), \\
a_3 = 2 (c_{2} + c_{13}) (2c_{2} (c_{14} - 1) + c_{13} c_{14}) A(r)^{9} - 4 (c_{2} + c_{13}) (c_{2} + c_{13} - (c_{2} + 1) c_{14}) r A(r)^{9} A'(r) \\
+ (c_{14} c_{2}^{2} - c_{2}^2 + c_{13}^2 + (c_{2} + 1) c_{2}^2 - c_{2} (c_{2} + 2) c_{13} c_{14}) r^2 A(r)^{7} A'(r)^2, \\
(A.2) \]

and

\[ b_0 = 4 (c_{2} + 1) (c_{2} + c_{13}) c_{14} A(r)^{2} - 4 (c_{2} + c_{13})^2 c_{14} r A(r)^{4} A'(r) \\
+ (c_{2} + c_{13}) c_{14} (c_{2} - c_{13} + c_{14}) r^2 A'(r)^2 - 4 (c_{2} + c_{13})^2 c_{14} r A(r)^{4} F'(r) \\
- 2 (c_{2} + c_{13}) c_{14} (-c_{2} + c_{13} + c_{14} - 4) r^2 A(r)^{3} A'(r) F'(r) \\
+ (c_{2} + c_{13}) c_{14} (c_{2} - c_{13} + c_{14}) r^2 A(r)^{6} F'(r)^2, \]

\[ b_1 = -(c_{2} - c_{13} - c_{14}) A'(r)^2 - c_{2} A(r)^2 F'(r) + 2c_{2} + c_{13} + 2 \\
+ (c_{2} - c_{13} + c_{14}) r A(r)^{2} A'(r) F'(r) + 4 (c_{2} + c_{13})^2 c_{14} r A(r)^{9} F'(r), \\
b_2 = 4 (c_{2} + c_{13})^2 c_{14} r A(r)^{5} A'(r)^2 - 4 (c_{2} + c_{13}) (c_{2} + c_{13} - (c_{2} + 1) c_{14}) r A(r)^{9} A'(r) \\
+ (c_{2} + c_{13}) c_{14} (c_{2} - c_{13} + c_{14}) r^2 A(r)^{4} A'(r)^2, \]  

(A.3)

On the other hand, the coefficients \( n_{\alpha} \)’s appearing in Eq.(3.10) are given by

\[ n_0 = \frac{c_{2} A'(r)}{2 r A(r)^{3} B(r)^{3}} - \frac{c_{2}}{2 r A(r)^{2} B(r)^{3}} - \frac{c_{13}}{4 r^2 A(r)^{2} B(r)^{3}} - \frac{1}{r^2 B(r)^{3}} \\
- \frac{(c_{2} + c_{13} + c_{14}) A'(r) F'(r)}{4 A(r)^{2} B(r)^{3}} - \frac{(c_{2} + c_{13} - c_{14}) A'(r)^2}{8 A(r)^{4} B(r)^{3}} + \frac{(c_{2} + 2) F'(r)}{2 r B(r)^{3}} \\
- \frac{(c_{2} + c_{13} - c_{14}) A'(r)^2 F'(r)^2}{8 B(r)^{3}} , \]

\[ n_1 = \frac{(-c_{2} - c_{13} - c_{14}) A'(r)^2}{4 A(r)^{2} B(r)^{3}} - \frac{c_{2} A(r)^{2} F'(r)}{2 r B(r)^{3}} + \frac{2c_{2} + c_{13} + 2}{2 r^2 B(r)^{3}} \\
+ \frac{(-c_{2} - c_{13} + c_{14}) A(r) A'(r) F'(r)}{4 B(r)^{3}} , \]

\[ n_2 = \frac{c_{2} A(r) A'(r)}{2 r B(r)^{3}} + \frac{(-c_{2} - c_{13} + c_{14}) A'(r)^2}{8 B(r)^{3}} + \frac{(-2c_{2} - c_{13}) A(r)^{2}}{4 r^2 B(r)^{3}} . \]  

(A.4)
When $c_{S} = 1$, i.e. $c_{2} = (−2c_{13} + 2c_{14} − c_{13}^{2}c_{14}) / (2 − 4c_{14} + 3c_{13}c_{14})$, the coefficients $f_{0}, a_{0}, b_{0}$ and $n_{0}$ reduce to

\[
f_{0} = \left[ rA(r)^{2}F'(r) + \frac{c_{14}c_{13} + 2c_{13} − 2c_{14}}{2(c_{13} − 1)c_{14}} \right] b_{0},
\]
\[
a_{0} = \frac{2(c_{13} − 1)c_{14}A'(r) + (c_{13} − 2)(c_{14} − 2)A(r)}{c_{13}(2rA(r)^{2}F'(r) + 1) + 2 − 2c_{14}(rA(r)^{2}F'(r) + 1)} f_{0},
\]
\[
b_{0} = \frac{1}{((3c_{13} − 4)c_{14} + 2)^{2}} \left\{ −2(c_{13} − 1)^{2}c_{14}^{2}c_{13}^{2} + (−3c_{13}^{2} − 4c_{14} + 4)c_{13} + 4(c_{14} − 1)c_{14} \right\} r^{2}A'(r)^{2} \nonumber \\
−4(c_{13} − 1)^{2}c_{14}^{2}A'(r)A'(r) \left[ (4c_{14}c_{13}^{2} + (3c_{14}^{2} − 16c_{14} + 4)c_{13} − 4(c_{14}^{2} − 4c_{14} + 2))rA(r)^{2}F'(r) \right. \nonumber \\
+4(c_{13} − 1)^{2}c_{14}^{2}A'(r)A'(r) \left[ (4c_{14}c_{13}^{2} + (−3c_{14}^{2} − 4c_{14} + 4)c_{13} + 4(c_{14} − 1)c_{14} \right. \nonumber \\
\left. +4(c_{14} − 1)c_{14}r^{2}A'(r)^{2}F'(r)^{2} + 8(c_{13} − 1)^{2}c_{14}rA(r)^{2}F'(r) + 4(c_{13} − 1)((c_{13} − 2)c_{14} + 2)) \right\}, \quad (A.5)
\]

and

\[
n_{0} = \frac{c_{14}(-2c_{13}^{2} + (3c_{14} + 4)c_{13} − 4c_{14})}{8((3c_{13} − 4)c_{14} + 2)B(r)^{3}} \times \nonumber \\
f'(r) \left[ c_{14}(-2c_{13}^{2} − 3c_{14}c_{13} + 4c_{13} + 4c_{14} − 4)A'(r) − 2(c_{14}c_{13}^{2} + (2 − 6c_{14})c_{13} + 6c_{14} − 4)A(r) \right] \nonumber \\
× \left[ 4(13c_{13} − 4)c_{14} + 2)A(r)(rA(r)^{2}B(r)^{3})^{-1} + 8((3c_{13} − 4)c_{14} + 2)rA(r)^{4}B(r)^{3} \right]^{-1} \nonumber \\
× \left[ c_{14}(-2c_{13}^{2} + (3c_{14} + 4)c_{13} − 4c_{14})r^{2}A'(r)^{2} − 4(c_{14}c_{13}^{2} + 2c_{13} − 2c_{14})rA(r)A'(r) \right. \nonumber \\
−8((3c_{13} − 4)c_{14} + 2)A(r)^{4}B(r)^{2} + (−2c_{14}c_{13}^{2} + (8c_{14} + 4)c_{13} − 8c_{14})A(r)^{2} \right]. \quad (A.6)
\]

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