Conservation laws and bosonization in integrable Luttinger liquids

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(Received )

We examine and explain the Luttinger-liquid character of models solvable by the Bethe ansatz by introducing a suitable bosonic operator algebra. In the case of the Hubbard chain, this involves two bosonic algebras which apply to all values of $U$, electronic density, and magnetization. Only at zero magnetization does this lead to the usual charge - spin separation. We show that our “pseudoparticle” operator approach clarifies, unifies, and extends several recent results, including the existence of independent right and left equations of motion and the concept of “pseudoparticle” (also known as “Bethe quasiparticle”).

PACS numbers: 05.30. Jp, 05.30 Fk, 67.40. Db, 72.15. Nj
Based on the similarities between the low-energy spectra of the Luttinger model and of single-component integrable models solvable by the Bethe ansatz (BA), Haldane [1] introduced the concept of “Luttinger liquid”. However, in contrast to the case of the Luttinger model [1], the study of the BA solvable models did not include an operator description of the low-energy physics: the Luttinger-liquid character of these models relied on the identification of a structure in the low-energy spectrum provided by the BA. Further, although there has been considerable progress in understanding the critical-energy spectrum of multicomponent integrable systems by combining BA and conformal-field theory [2], this approach also does not provide an operator description of the low-energy problem. Recent studies by Di Castro and Metzner [3] have revealed that instead of using bosonization [1], one can construct the Luttinger-liquid theory for a general “g-ology” model in terms of separate charge and spin conservation for states near the left and right “Fermi points”. Again, this construction was not extended to integrable models solvable by BA and its interpretation in an operator context was unclear for these models.

In the present letter we solve the problem of presenting a general operator approach that establishes the Luttinger-liquid character of integrable models solvable by BA. In particular, we use the pseudoparticle-operator algebra [4] to construct a general Luttinger-liquid theory, which extends the study of Ref. [3] to the case of BA integrable models by establishing explicitly the separate left-right conservation laws for these models. Moreover, we introduce the bosonic-operator algebra which describes the low-energy physics of integrable Luttinger liquids and justifies the similarities of the energy spectra of these systems to those of the Luttinger and g-ology models. We discuss the nature of the pseudoparticles, focusing on the transformation between electrons and pseudoparticles. This discussion confirms that the pseudoparticles cannot be “removed” from the many-body system [3] and allows us to write the pseudoparticle generators of the bosonic excitations explicitly in the electronic basis. More generally, our results establish a consistent relation between the Luttinger liquid introduced by Haldane [1] and the Landau-liquid character of the U(1) sectors of parameter space of integrable Luttinger liquids [4, 6]. Although our present analysis refers to systems
with local interactions, we will demonstrate explicitly elsewhere that it can be extended to the quantum-chaos related $1/r^2$ integrable models [7]. The “Bethe quasiparticles” of these models [7] are precisely the present pseudoparticles, which were first introduced in Refs. [5] (see also references therein) and refer to a large class of integrable quantum liquids [4].

We consider the particular case of the Hubbard chain [8] in an external magnetic field $H$ and chemical potential $\mu$ [4,5]. This describes $N_\uparrow$ up-spin and $N_\downarrow$ down-spin electrons in a lattice of $N_a$ sites ($N = N_\uparrow + N_\downarrow$, $n = N/N_a$, $n_\sigma = N_\sigma/N_a$, and $m = n_\uparrow - n_\downarrow$). We consider densities $0 < n < 1$ and spin densities $0 < m < n$. This corresponds to the sector of parameter space with $U(1) \times U(1)$ symmetry [4,5]. In this case the low-energy physics is dominated by a particular class of Hamiltonian eigenstates: these lowest-weight states (LWS) of both the “eta”-spin and spin algebras [8] which refer to real rapidities [4]. We call these states “LWS I” to distinguish them from the LWS associated with complex, non-real, rapidities, which we call “LWS II”. Importantly, in the $U(1) \times U(1)$ sector both the LWS II and the non-LWS have energy gaps relative to each canonical-ensemble ground state [10] and do not contribute to the low-energy physics [4]. In the pseudoparticle basis [4,10,11] one can define a many-pseudoparticle perturbation theory in which the non-interacting pseudoparticle ground state is the exact ground state of the many-electron problem [4,10]. All LWS I can be generated by acting on the vacuum $|V\rangle$ (zero-electron state) with pseudoparticle operators $b_\alpha^\dagger$ which obey the usual fermionic algebra [4], $\{b_\alpha^\dagger, b_\alpha\} = \{b_\alpha, b_\alpha^\dagger\} = 0$. Here $\alpha$ refers to the two pseudoparticle colors $c$ and $s$ [4,5]. The discrete pseudomomentum values are $q_j = \frac{2\pi}{N_a} I_j^\alpha$, where $I_j^\alpha$ are consecutive integers or half integers. There are $N_\alpha^* = 0, ..., N_a^*, \ldots, N_\alpha^*$ empty values, which we call $\alpha$ pseudoholes. In the case of the Hubbard chain, we have that $N_c^* = N_a$, $N_c = N$, $N_s^* = N_\uparrow$, and $N_s = N_\downarrow$, i.e., the pseudoparticle numbers are good quantum numbers. The boundary
conditions fix the numbers \( I_j^\alpha \): \( I_j^\alpha \) are integers (or half integers) for \( N_\uparrow \) even (or odd), and \( I_j^\alpha \) are integers (or half integers) for \( N_\downarrow \) odd (or even) \[\text{[4,8]}\]. The ground state associated with a canonical ensemble of \((N_\uparrow, N_\downarrow)\) values [and \((N_c = N_\uparrow + N_\downarrow, N_s = N_\downarrow)\)] has the form \[\text{[4,10]}\]

\[|0; N_\uparrow, N_\downarrow\rangle = \prod_{\alpha=c,s} \left[ \prod_{q=q_{F_\alpha}^{-}}^{q_{F_\alpha}^{+}} b_{q_\alpha}^\dagger \right]|V\rangle, \tag{1}\]

where when \( N_\alpha \) is odd (even) and \( I_j^\alpha \) are integers (half integers) the pseudo-Fermi points are symmetric and given by \( q_{F_\alpha}^{(+)} = -q_{F_\alpha}^{(-)} = \frac{\pi}{N_\alpha}[N_\alpha - 1] \). On the other hand, when \( N_\alpha \) is odd (even) and \( I_j^\alpha \) are half integers (integers) we have that \( q_{F_\alpha}^{(+)} = \frac{\pi}{N_\alpha}[N_\alpha - 2] \) and \( q_{F_\alpha}^{(-)} = \frac{\pi}{N_\alpha}[N_\alpha - 2] \). In the pseudoparticle basis the ground state of the many-electron problem is a “non-interacting” pseudoparticle ground state of simple Slater-determinant form, (1). When all \( q_{F_\alpha}^{(\pm)} \) are symmetric this state has zero momentum and is non-degenerate, whereas when any number of these points are nonsymmetric it has finite momentum and is degenerate \[\text{[10]}\]. Except for terms of \( 1/N_\alpha \) order, \( q_{F_\alpha}^{(\pm)} = \pm 2k_F = \pm \pi n \) and \( q_{F_\downarrow}^{(\pm)} = \pm k_{F_\downarrow} = \pm \pi n_\downarrow \).

Since the pseudoparticles dominate the physics at energy scales smaller than the gaps for the LWS II and non-LWS, it is clearly of interest to construct the electron-pseudoparticle operator transformation in this region. The explicit construction of this transformation for general BA solvable models is a very involved unsolved problem. For instance, we know that at low energies and finite momenta it maps one-pair electron operators onto multipair pseudoparticle operators \[\text{[4]}\]. However, although the one-pair electronic operator \( c_{k'k}^{\uparrow}c_{k'\sigma}^{\dagger} \) is, in general, of multipair pseudoparticle character (\( \sigma = \uparrow, \downarrow \) as an operator index and \( \sigma = \pm 1 \) otherwise), \( \hat{\rho}_\sigma(k) = \sum_{k'} c_{k'k}^{\dagger}c_{k'\sigma} \) is at \( k = 0 \) of one-pair pseudoparticle character. This result also holds for (nearly) vanishing momenta \( k = \pm \frac{2\pi}{N_\alpha} \), as we discuss later. We can write the pseudoparticle and electronic number operators \( \hat{N}_\alpha = \hat{\rho}_\alpha(0) \) (with \( \hat{\rho}_\sigma(k) = \sum_{q} b_{q_\alpha}^{\dagger}b_{q_\alpha} \) and \( \hat{N}_\sigma = \hat{\rho}_\sigma(0) \), respectively, in the electronic and pseudoparticle basis as \( \hat{N}_\alpha = \sum_{\sigma} G_{\alpha\sigma}\hat{N}_\sigma \) and \( \hat{N}_\sigma = \sum_{\sigma} G^{-1}_{\sigma\alpha}\hat{N}_\alpha \), respectively, where \( G \) and \( G^{-1} \) are the electron-pseudoparticle and pseudoparticle-electron charge matrices, respectively. Their entries are \( G_{c_\uparrow} = G_{c_\downarrow} = G_{s_\downarrow} = 1, G_{s_\uparrow} = 0 \) and \( G^{-1}_{c_\uparrow} = -G^{-1}_{c_\downarrow} = G^{-1}_{s_\downarrow} = 1, G^{-1}_{s_\uparrow} = 0 \). The simple form
of these expressions follows from the conservation of the number of \( \alpha \) pseudoparticles, \( N_\alpha \), so that \( N_\alpha = N_{\gamma(\alpha)} \), where \( \gamma(c) = \rho \) and \( \gamma(s) = \downarrow \). However, that \( \hat{\rho}_\alpha(0) = \hat{\rho}_{\gamma(\alpha)}(0) \) does not imply that \( \hat{\rho}_\alpha(k) = \hat{\rho}_{\gamma(\alpha)}(k) \) for \( k > 0 \); we shall return to this important point later.

The non-perturbative character of the usual electronic basis – e.g., the absence of a coherent \( \delta \)-function peak in the one-electron spectral function – is reflected in the properties of the electron - pseudoparticle transformation. Let us use the changes in the quantum numbers \( I_j^\alpha \) between ground states (1) differing in the electronic or pseudoparticle numbers by \( \pm 1 \) [10] to describe one electron of lowest-energy in terms of pseudoparticles and to show that the latter cannot be removed from the many-electron system [7]. For simplicity, we choose both numbers \( N_\sigma \) to be odd in the starting state (1) which we call \( \text{A} \) (our results are independent of this choice). This state has zero momentum and is non-degenerate. To define the up-spin electron in terms of pseudoparticles we compare the state \( \text{A} \) with the ground state \( |0; N_\uparrow \pm 1, N_\downarrow \rangle \). This has pseudoparticle numbers \( N_c = N \pm 1 \) and \( N_s = N_\downarrow \) (state \( \text{B} \)).

The changes in the electronic and pseudoparticle numbers seem to indicate that removing or adding one up-spin electron is equivalent to removing or adding one \( c \) pseudoparticle. However, this is not true. Looking carefully at the changes in the pseudoparticle quantum numbers \( I_j^\alpha \) from \( \text{A} \) to \( \text{B} \), we find that within the many-body system the up-spin electron consists of two excitations which cannot be decomposed: 1) an excitation removing or adding one \( c \) pseudoparticle from one of the pseudo-Fermi points, with momentum \( q_{Fc}^{(\pm)} = \pm 2k_F \); and 2) a collective excitation involving all \( s \) pseudoparticles which shifts their pseudomomenta by \( \mp \pi/Na \). This shift results from the change of the quantum numbers \( I_j^s \) from integers to half integers. Although each pseudomomentum shift, \( \mp \pi/Na \), is vanishing small, adding all contributions from the \( s \) pseudo-Fermi sea gives \( \mp k_{F_\downarrow} \). The total momentum of the up-spin electron is \( P = \pm [2k_F - k_{F_\downarrow}] = \pm k_{F_\uparrow} \) (here and below we omit \( 1/Na \) corrections to \( P \)). Since excitations 1) and 2) cannot be decomposed, we cannot remove or add one \( c \) pseudoparticle without making a collective excitation involving all \( s \) pseudoparticles. It is in this sense that we say that the \( c \) pseudoparticle cannot be removed from the many-body system and cannot exist as a free entity. A similar analysis shows that the down-spin electron is a
collective pseudoparticle object made out of one \( c \) pseudoparticle, one \( s \) pseudoparticle, and a pseudomomentum shift of all remaining \( c \) pseudoparticles. Its momentum is \( P = \pm 2k_F \pm 2k_F \pm k_{F \downarrow} = \pm k_{F \downarrow} \). Both the up- and down-spin electrons can be removed from and added to the system, but within the system they do not refer to Hamiltonian eigenstates and have zero lifetime: when added to the system they decay immediately into collective pseudoparticle excitations. Similarly, the \( s \) pseudoparticle cannot exist as a free entity out of the many-electron system: the ground state with numbers \( N_c = N \) and \( N_s = N_{\downarrow} \pm 1 \), \( |0; N_{\uparrow} \pm 1, N_{\downarrow} \pm 1 \rangle \), is, relative to the state \( \mathcal{A} \), nothing but an electronic spin flip. This is a collective pseudoparticle object made out of one \( s \) pseudoparticle, a pseudomomentum shift of all remaining \( s \) pseudoparticles, and a pseudomomentum shift of all \( c \) pseudoparticles. These three excitations cannot be decomposed.

Let \( \mathcal{H}_I \) be the Hilbert space spanned by the LWS I. In \( \mathcal{H}_I \) and normal-ordered relative to (1), the Hubbard Hamiltonian has an infinite number of terms which correspond to increasing “order” of pseudoparticle scattering [4]: formally, \( \hat{H} := \sum_{i=1}^{\infty} \hat{H}^{(i)} \), where the first two terms are \( \hat{H}^{(1)} = \sum_{q,\alpha} \epsilon_\alpha(q) : \hat{N}_\alpha(q) : \) and \( \hat{H}^{(2)} = \frac{1}{N_c} \sum_{q,\alpha} \sum_{q',\alpha'} \frac{1}{2} f_{\alpha\alpha'}(q, q') : \hat{N}_\alpha(q) \cdots \hat{N}_{\alpha'}(q') : \). The expressions for the pseudoparticle bands, \( \epsilon_\alpha(q) \), and of the “Landau” \( f \) functions, \( f_{\alpha\alpha'}(q, q') \), are given in Ref. [5]. The latter involve the velocities \( v_\alpha(q) = \frac{d\epsilon_\alpha(q)}{dq} \) and the two-pseudoparticle forward-scattering phase shifts \( \Phi_{\alpha\alpha'}(q, q') \). These are defined in the second paper of Ref. [5]. In particular, the velocities \( v_\alpha \equiv v_\alpha(q_{F\alpha}^{(+)}) \) and the parameters \( \xi_{\alpha\alpha'}^j = \delta_{\alpha\alpha'} + \Phi_{\alpha\alpha'}(q_{F\alpha}^{(+)} q_{F\alpha'}^{(+)})(-1)^j \Phi_{\alpha\alpha'}(q_{F\alpha}^{(+)} q_{F\alpha'}^{(-)}) \), with \( j = 0, 1 \), play a determining role at the critical point. \( \xi_{\alpha\alpha'}^1 \) are the entries of the transpose of the dressed-charge matrix [2]. An essential point is that at constant values of the electron and pseudoparticle numbers the electron-pseudoparticle transformation does not mix left and right electronic operators, \( i.e., \), \( \iota = \text{sgn}(k)1 = \pm 1 \) electronic operators are made out of \( \iota = \text{sgn}(q)1 = \pm 1 \) pseudoparticle operators only, \( \iota \) defining the right \( (\iota = 1) \) and left \( (\iota = -1) \) movers. Measuring the electronic momentum \( k \) and pseudomomentum \( q \) from the \( U = 0 \) Fermi points \( k_{F\sigma}^{(\pm)} = \pm \pi n_\sigma \) and pseudo-Fermi points \( q_{F\alpha}^{(\pm)} \), respectively, adds the index \( \iota \) to the electronic and pseudoparticle operators. The new momentum \( \tilde{k} \) and pseudomomentum \( \tilde{q} \) are such that
\( \tilde{k} = k - k_{F_\alpha}^{(\pm)} \) and \( \tilde{q} = q - q_{F_\alpha}^{(\pm)} \), respectively, for \( \iota = \pm 1 \). For instance, \( \hat{\rho}_{\alpha}(k) = \sum_{\iota} c_{k + \iota k \alpha}^\dagger c_{k \alpha} \) and \( \hat{\rho}_{\alpha}(k) = \sum_{\iota} b_{q + k \alpha}^\dagger b_{q \alpha} \). The perturbative character of the pseudoparticle basis \([4]\) implies that the above pseudoparticle Hamiltonian can be used as starting point for the construction of a critical-point Hamiltonian. This proceeds by linearizing the pseudoparticle bands \( \epsilon_{\alpha}(q) \) around the pseudo-Fermi points. In addition, one considers the terms associated with two-pseudoparticle scattering near the pseudo-Fermi surface. We emphasize that, in the case of a single-component model, as well as for the anisotropic Fermi points, \( \alpha, \iota \) are good quantum numbers, \( i.e. \) \( \alpha, \iota \) and \( \epsilon_{\alpha}(q) \) only \([4]\). The critical-point Hamiltonian can be written as : \( \hat{\mathcal{H}} := \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_2 + \hat{\mathcal{H}}_4 \), where \( \hat{\mathcal{H}}_0 = \sum_{\alpha, \iota, \tilde{q}} i v_{\alpha} \tilde{q} : \hat{N}_{\alpha, \iota}(\tilde{q}) : \) and \( \hat{\mathcal{H}}_2 + \hat{\mathcal{H}}_4 = \frac{2}{N_{\alpha}} \sum_{\alpha, \alpha', \iota, t, k} [g_2^{\alpha \alpha'}(k) \delta_{\iota, -t'} + g_4^{\alpha \alpha'}(k) \delta_{\iota, t'}] : \hat{\rho}_{\alpha, t}(k) :: \hat{\rho}_{\alpha', t'}(-k) :: \). Here \( \hat{N}_{\alpha}(\tilde{q}) = b_{q \alpha}^\dagger b_{q \alpha} \) and the couplings read \( g_2^{\alpha \alpha'}(k) = f_{\alpha \alpha'}^{-1}(k) \delta_{k, 0} \) and \( g_4^{\alpha \alpha'}(k) = f_{\alpha \alpha'}^{-1}(k) \delta_{k, 0} \). It follows that in the pseudoparticle basis the suitable critical-point Hamiltonian is a gology model of the type studied in Ref. \([3]\) with exotic \( k = 0 \) forward-scattering couplings, \( \sigma \) replaced by \( \alpha \), and the electronic operators by pseudoparticle operators. We emphasize that the existence of only \( k = 0 \) pseudoparticle scattering allows the ground-state pseudomomentum distribution, \( \langle \hat{N}_{\alpha}(q) \rangle \), to be of non-interacting character, \( i.e. \) equal to 1 inside and 0 outside the pseudo-Fermi surface, respectively \([4][5][10]\). The absence of the \( g_1 \) and \( g_3 \) terms implies that the \( \alpha, \iota \) pseudoparticle number operators, \( \hat{N}_{\alpha, \iota} = \sum_{\tilde{q}} \hat{N}_{\alpha, \iota}(\tilde{q}) \), such that \( \hat{N}_{\alpha, \iota} = \hat{N}_{\gamma(\alpha), \iota} \), are good quantum numbers, \( i.e. \) there are separate right and left conservation laws. This is a generalization of the results of Ref. \([3]\) with the Fermi points replaced by the pseudo-Fermi points. We emphasize that, in the case of a single-component model, as well as for the anisotropic \( S = 1/2 \) Heisenberg chain \([4]\) and the Hubbard chain at half filling \([4]\), we can omit the index \( \alpha \), so that the critical Hamiltonian can be rewritten as : \( \hat{\mathcal{H}} := \hat{\mathcal{H}}_0 + \hat{\mathcal{V}} \), with \( \hat{\mathcal{V}} = \frac{4}{N_{\alpha}} \sum_{\iota, t, k} [V_1(k) \delta_{t, t'} + V_2(k) \delta_{t, -t'}] : \hat{\rho}_{\iota}(k) :: \hat{\rho}_{t'}(-k) :: \). Here \( \hat{\rho}_{\iota}(k) = \sum_{\tilde{q}} b_{q + k \iota}^\dagger b_{q \iota} \), \( V_1(k) = \frac{f_{\iota}^1}{2\pi} \delta_{k, 0} \), and \( V_2(k) = \frac{f_{\iota}^{-1}}{2\pi} \delta_{k, 0} \). Therefore, in single-component integrable systems, \( \hat{\mathcal{H}} : \) is a Luttinger model with exotic \( k = 0 \) forward-scattering potentials. This universal form of the critical-point Hamiltonian in the pseudoparticle-operator basis justifies the
The separate right and left conservation laws provide the Luttinger-liquid parameters through equations of motions. Let $\vartheta$ be any conserved quantity such as charge ($\vartheta = \rho$), spin ($\vartheta = \sigma_z$), or spin projection ($\vartheta = \sigma$). Then $\dot{N}_{\vartheta\alpha}$ can be written as $\dot{N}_{\vartheta\alpha} = \sum_\alpha k_{\vartheta\alpha} \dot{N}_{\gamma(\alpha)\alpha}$, where the integers $k_{\vartheta\alpha}$ are $k_{\vartheta c} = k_{\sigma z c} = k_{\gamma c} = 1$, $k_{\varrho c} = 0$, $k_{\varrho s} = 0$, $k_{\sigma s} = -2$, $k_{\gamma s} = -1$, and $k_{\jmath s} = 1$ (note that $k_{\sigma\alpha} = G_{\sigma\alpha}^{-1}$ and $k_{\alpha\alpha'} = \delta_{\alpha\alpha'}$). The operator $\dot{\rho}_{\gamma(\alpha)\alpha}(k)$ can be written as $\dot{\rho}_{\gamma(\alpha)\alpha}(k) = \sum_\sigma G_{\sigma\alpha} \dot{\rho}_{\sigma(\alpha)\alpha}(k)$. (Although $\dot{\rho}_{\gamma(\alpha)\alpha}(0) = \dot{\rho}_{\alpha\alpha}(0)$, $\dot{\rho}_{\gamma(\alpha)\alpha}(k) \neq \dot{\rho}_{\alpha\alpha}(k)$ for $k \neq 0$, as we find below.) Let us consider the general operator $\dot{\rho}_{\vartheta\alpha}(k) = \sum_\alpha k_{\vartheta\alpha} \dot{\rho}_{\gamma(\alpha)\alpha}(k)$. It is useful to consider the combinations $\dot{\rho}_{\vartheta}(k) = \dot{\rho}_{\alpha\alpha}(k) \pm \dot{\rho}_{\alpha-1}(k)$ (note that $\dot{\rho}_{\vartheta}^{(+)}(k) = \dot{\rho}_{\vartheta}(k)$). Since the commutator $[\rho_{\vartheta}^{(+)}(k, t), \hat{\mathcal{H}}_2 + \hat{\mathcal{H}}_4] = 0$ for $k > 0$ and $[\rho_{\vartheta}^{(+)}(k, t), : \hat{\mathcal{H}} :]$ is proportional to $k$ at $k = 0$, the interesting quantity associated with the equation of motion for $\dot{\rho}_{\vartheta}(k, t)$ is the following ratio, which we evaluate using the forms of the coupling constants of $: \hat{\mathcal{H}} :$

$$i \frac{\partial \rho_{\vartheta}^{(+)}(k, t)}{\partial t} \bigg|_{k=0} = \frac{[\rho_{\vartheta}^{(+)}(k, t), : \hat{\mathcal{H}} :]}{k} \bigg|_{k=0} = \mathcal{V}_{\vartheta}^{(+)}(0, t) \rho_{\vartheta}^{(+)}(0, t) \tag{2}$$

where $\mathcal{V}_{\vartheta}^{(-)}(k, t) = \sum_{\alpha, \alpha'} k_{\vartheta\alpha} k_{\vartheta\alpha'} v_{\alpha} \delta_{\alpha\alpha'} + \left[ f_{\alpha\alpha'}^{-1} - f_{\alpha\alpha'}^{-1} \right] = \sum_\alpha v_\alpha \left[ \sum_{\alpha'} k_{\vartheta\alpha} \xi_{\alpha\alpha'}^1 \right]^2$ and $\mathcal{V}_{\vartheta}^{(+)} = 1/\left\{ \sum_{\alpha, \alpha'} k_{\vartheta\alpha} k_{\vartheta\alpha'} v_{\alpha} \delta_{\alpha\alpha'} - \left[ A_{\alpha\alpha'}^{(+)} + A_{\alpha\alpha'}^{-1} \right] \right\} = 1/\left\{ \sum_\alpha v_\alpha \left[ \sum_{\alpha'} k_{\vartheta\alpha} \xi_{\alpha\alpha'}^1 \right]^2 \right\}$. Here $A_{\alpha\alpha'}^{(+)} = A_{\alpha\alpha'}^{(+)}(q_{F\alpha}, q_{F\alpha'})$ and $A_{\alpha\alpha'}^{-1} = A_{\alpha\alpha'}^{(-)}(q_{F\alpha}, q_{F\alpha'})$, where $A_{\alpha\alpha'}(q, q')$ are the scattering amplitudes given by Eqs. (83) – (85) of the third paper of Ref. [5]. The velocities $\mathcal{V}_{\vartheta}^{(+)}$ and $\mathcal{V}_{\vartheta}^{(-)}$ determine the $\vartheta$ susceptibility, $K_{\vartheta} = 1/[\pi \mathcal{V}_{\vartheta}^{(+)}]$, and the coherent part of the $\vartheta$ conductivity spectrum, $\mathcal{V}_{\vartheta}^{(-)}(\delta(\omega))$, respectively [3]. This agrees with the studies of Ref. [4] for charge and spin and with the conductivity expressions of Ref. [4]. For single-component systems there is only one choice for $\vartheta$ and $\mathcal{V}^{(-)} = v[\xi^1]^2$ and $\mathcal{V}^{(+)} = v[\xi^0]^2 = v[\xi^1]^2$, in agreement with Ref. [1]. The $\mathcal{V}_{\vartheta}^{(+)}$ are the expressions for the $\vartheta$ conserved quantities of integrable multicomponent Luttinger liquids. Equation (2) involves the commutator of the pseudoparticle-Hamiltonian $: \hat{\mathcal{H}} :$ with an electronic operator and, therefore, the velocities $\mathcal{V}_{\vartheta}^{(+)}$ do not have the same simple form as for the g-ology model of Ref. [3]. Importantly, except for single-component integrable models, $\mathcal{V}_{\vartheta}^{(+)}$ does not equal the expression of $\mathcal{V}_{\vartheta}^{(-)}$ with $f_{\alpha\alpha'}^{-1} - f_{\alpha\alpha'}^{-1}$ replaced by $f_{\alpha\alpha'}^{-1} + f_{\alpha\alpha'}^{-1}$. 

Luttinger-liquid character of integrable models by BA.
The bosonization of $\hat{H}$ is straightforward and refers to the non-interacting pseudoparticle ground state (1). We find that 
\[ [\hat{\rho}_\alpha(k), \hat{\rho}_{\alpha'}(-k')] = \delta_{\alpha,\alpha'}\delta_{\lambda,\lambda'}\delta_{k,k'}(ik\frac{Na}{2\pi}) \] and the $\alpha$ bosonic operators are given by
\[ a^\dagger_{k\alpha} = \sqrt{\frac{2\pi}{Na|k|}} \sum_\lambda \theta(ik) \hat{\rho}_\alpha(k), \] (3)
for $k > 0$. Our bosonization reproduces the results of conformal-field theory [4]: the bosons (3) refer to the tower excitations of Ref. [4], whereas the HWS [4] of the Virasoro Algebras [2], which introduce the anomalous dimensions, are the current and “charge” excitations [1]. The low-energy separation refers to the colors $\alpha$ for all parameter space.

To complete our analysis we should show how to express these bosonic generators $\hat{\rho}_\alpha(k)$ in the electronic basis. This will allow a definition of the $\alpha$ low-energy separation in terms of usual electronic operators. Let us consider the reduced Hilbert space spanned by the Hamiltonian eigenstates of momentum $k = \frac{i2\pi}{Na}$ relative to the ground state (1), which we shall henceforth denote by $|0\rangle$, so that we can denote the Hamiltonian eigenstates by $|\alpha t\rangle = \hat{\rho}_{\lambda t}(k)|0\rangle$ with $k = \frac{i2\pi}{Na}$. These states are orthogonal to (1) and have a single pseudohole at one of the pseudo-Fermi points. In the reduced Hilbert space they constitute a complete orthonormal basis, so that $\langle \alpha t|\alpha' t'\rangle = \delta_{\alpha,\alpha'}\delta_{\lambda,\lambda'}$ and $\sum_{\alpha,\lambda} |\alpha t\rangle\langle \alpha t| = 1$. Based on the results of Ref. [4] we find that the one-pair electronic operator $\hat{\rho}_\alpha(k)$ at $k = \frac{i2\pi}{Na}$ is of one-pair pseudoparticle character and projects the ground state (1) in the above reduced Hilbert space, i.e., $\hat{\rho}_{\alpha t}(k)|0\rangle = \sum_{\alpha,\lambda} U_{\alpha\sigma}^{-1} |\alpha t\rangle = \hat{\rho}_{\lambda t}(k)|0\rangle$, where $U_{\alpha\sigma}^{-1} = \langle \alpha t|\hat{\rho}_\sigma(k)|0\rangle$: at $k = \frac{i2\pi}{Na}$ we have that $\hat{\rho}_\alpha(k) = \sum_{\alpha,\lambda} U_{\alpha\sigma}^{-1} \hat{\rho}_{\alpha t}(k)$ where we denote $U^{-1}$ by pseudoparticle - electron matrix. Introducing the “electronic” states $|\sigma t\rangle = \hat{\rho}_{\sigma t}(k)|0\rangle$ with $k = \frac{i2\pi}{Na}$, and applying the methods of the third paper of Ref. [3], we find $U_{\alpha\sigma}^{-1} = \langle \alpha t|\sigma t\rangle = \sum_{\alpha'} G_{\alpha\alpha'}^{-1} \xi_{\alpha\alpha'}^{1}$. However, here we are interested in the inverse problem, i.e., writing the operators $\hat{\rho}_\alpha(k)$ in the electronic basis. Fortunately, the $k = \frac{i2\pi}{Na}$ states $|\sigma t\rangle$ constitute a complete (but non-orthonormal) basis in the reduced Hilbert space. This follows from the fact that $\det U^{-1} > 0$. Therefore, we can invert the matrix $U^{-1}$ to find
\[ \hat{\rho}_{\alpha}(k) = \sum_{\sigma} U_{\sigma\alpha} \hat{\rho}_{\sigma}(k), \quad k = \frac{2\pi}{N_a}, \tag{4} \]

where the electron - pseudoparticle matrix \( U \) is such that \( U_{\sigma\alpha} = \sum_{\alpha'} G_{\alpha'\sigma} \xi_0^{\alpha\alpha'} \). In the electronic basis and at \( k = \frac{2\pi}{N_a} \) the operators (3) read \( a^\dagger_{k\alpha} = \sqrt{\frac{2\pi}{N_a|k|}} \sum_{\sigma,\iota} \theta(ik) U_{\sigma\alpha} \hat{\rho}_{\sigma\iota}(k) \). We emphasize that the one-pair electronic operator \( \hat{\rho}_{\sigma\iota}(k) \) does not change the numbers \( N_\sigma \) and, therefore, the pseudoparticle-collective part of the two associated one-electron excitations cancels. Equation (4) reveals that the generators of the \( \alpha \) bosonic (tower) excitations are an interaction-dependent mixture of up-spin and down-spin one-pair electronic operators: in the \( U(1) \otimes U(1) \) sector all the \( 2 \times 2 \) matrix elements \( U_{\sigma\alpha} \) are non-vanishing and interaction dependent. This shows the exotic character of the \( \alpha \) low-energy separation, which is different from the dynamical separation studied in Ref. [4]: this latter effect refers to the transport masses which, due to the pure \( k = 0 \) forward-scattering character of \( \hat{\mathcal{H}} \), only “see” the \( k = 0 \) \( \alpha \) operators \( \hat{\rho}_{\alpha\iota}(0) = \hat{\rho}_{\gamma(\alpha)\iota}(0) \). This leads to a dynamical charge - down-spin separation [4]. On the other hand, the \( k > 0 \) \( \alpha \) generators of (3) are not the operators \( \hat{\rho}_{\gamma(\alpha)\iota}(k) \). Instead, they are the exotic mixture of the electronic operators \( \hat{\rho}_{\sigma\iota}(k) \), (4). However, from the usual \( m = 0 \) and (or) \( n = 1 \) pictures [1,3,11], we expect \( c \) and \( s \) to become charge and spin, respectively, in the limit \( m \to 0 \), and \( c \) to become charge in the limit \( n \to 1 \). This is confirmed by Table 1 where we show some limiting forms of (4). A more detailed study of expression (4) will be presented elsewhere.

This work was supported by C.S.I.C. (Spain) and the University of Illinois. We thank F. Guinea for stimulating discussions. A. H. C. N. thanks CNPq (Brazil) for a scholarship.
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| Case | Description | Expression |
|------|-------------|------------|
| (a) | $U \to 0$ | $\rho_{c}(k)$ $\rho_{\uparrow}(k)$ |
| (b) | $U \to 0$ | $\hat{\rho}_{c}(k)$ $\hat{\rho}_\uparrow(k)$ |
| (c) | $n \to 0$ | $\xi_{c\sigma}^{0} \rho_{c}(k)$ $\sum_{\sigma} \mathcal{U}_{\sigma c}^{-1} \hat{\rho}_{\sigma}(k)$ $\hat{\rho}_{\rho}(k)$ |
| (d) | $m \to 0$ | $-\frac{\hat{\rho}_{c\sigma}(k)}{\sqrt{2}}$ $\hat{\rho}_{\downarrow}(k)$ $-\frac{\hat{\rho}_{c\sigma}(k)}{\sqrt{2}}$ $\hat{\rho}_{\downarrow}(k)$ $\sum_{\sigma} \mathcal{U}_{\sigma s}^{-1} \hat{\rho}_{\sigma}(k)$ |
| (e) | $n_{\downarrow} \to 0$ | $\sum_{\sigma} \mathcal{U}_{\sigma c}^{-1} \hat{\rho}_{\sigma}(k)$ $\hat{\rho}_{\rho}(k)$ |
| (f) | $n \to 1$ | $\hat{\rho}_{c}(k)$ $\hat{\rho}_{\uparrow}(k)$ |

**TABLE 1** – Limiting values of the $c$ and $s$ generators at $k = \frac{2\pi}{N_{a}}$ and for (a) $U \to 0$ when $0 < n < 1$ and $n_{\uparrow} > n_{\downarrow}$; (b) $U \to 0$ when $0 < n < 1$ and $n_{\uparrow} = n_{\downarrow}$; (c) $n \to 0$ when $n_{\uparrow} > n_{\downarrow}$ and $U > 0$; (d) $m \to 0$ when $0 < n < 1$ and $U > 0$; (e) $n_{\downarrow} \to 0$ when $n_{\downarrow} < n_{\uparrow}$ (here $\mathcal{U}_{\uparrow c}^{-1} = 1$); and (f) $n \to 1$ when $U > 0$. 

12