SAMPLING THEORY FOR STURM-LIOUVILLE PROBLEM WITH BOUNDARY AND TRANSMISSION CONDITIONS DEPENDENT ON EIGENPARAMETER

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Abstract. In this paper, we investigate the sampling analysis associated with discontinuous Sturm-Liouville problem which has transmission conditions at two points of discontinuity also contains an eigenparameter in a boundary condition and two transmission conditions. We establish briefly spectral properties of the problem and then we prove the sampling theorem associated with the problem.

1. Introduction

Let \( \sigma > 0 \), and denote by \( B^2_\sigma \), the Paley-Wiener space of all entire functions \( f \) of exponential type with band width at most \( \sigma \) which are \( L^2(\mathbb{R}) \) functions when restricted to \( \mathbb{R} \). This space is characterized by the following relation which is given by Paley and Wiener [1], known as the Paley-Wiener theorem:

\[
\text{f}(t) \in B^2_\sigma \iff f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\sigma}^{\sigma} e^{igt} g(x) \, dx, \quad g \in L^2(-\sigma, \sigma).
\]

The Whittaker-Kotel’nikov-Shannon (WKS) sampling theorem states that if \( f(t) \in B^2_\sigma \), then it is completely determined by its values at the points \( t_k = \frac{k\pi}{\sigma}, k \in \mathbb{Z} \) and can be reconstructed by means of the formula

\[
f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \sin c(\sigma t - k\pi), \quad t \in \mathbb{C},
\]

where

\[
\sin c(\sigma t - k\pi) = \begin{cases} \frac{\sin(\sigma t - k\pi)}{\sigma t - k\pi}, & t \neq \frac{k\pi}{\sigma} \\ 1, & t = \frac{k\pi}{\sigma} \end{cases}
\]

The series (1.2) is absolutely and uniformly convergent on compact subsets of \( \mathbb{C} \), uniformly convergent on \( \mathbb{R} \), (see [2, 3]).

One of the important generalizations of the WKS sampling theorem is the Paley-Wiener-Levinson theorem which can be stated as follows: Let \( \{t_k\}_{k \in \mathbb{Z}} \) be a sequence of real numbers satisfying

\[
D := \sup_{k \in \mathbb{Z}} \left| t_k - \frac{k\pi}{\sigma} \right| < \frac{\pi}{4\sigma}.
\]

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and let
\[ G(t) := (t - t_0) \prod_{k=1}^{\infty} \left(1 - \frac{t}{t_k}\right) \left(1 - \frac{t}{t_k-1}\right). \]

Then for any \( f(t) \in B^2_\sigma \),
\[ f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \frac{G(t)}{G'(t_k)(t - t_k)}, \quad t \in \mathbb{C}. \]

The series (1.6) converges uniformly an compact subsets of \( \mathbb{C} \), (see [2, 4]).

Series of the form (1.6) is called a Lagrange-type interpolation.

Another important generalization of the WKS sampling theorem is the theorem of Kramer [5], can be stated as follows: Let \( I \) be a finite closed interval, \( K(x, t) \) a function continuous in \( t \) such that \( K(x, t) \in L^2(I) \) for all \( t \in \mathbb{C} \), and let \( \{t_k\}_{k \in \mathbb{Z}} \) be a sequence of real numbers such that \( \{K(x, t_k)\}_{k \in \mathbb{Z}} \) is a complete orthogonal set in \( L^2(I) \). Suppose that
\[ f(t) = \int_I K(x, t) g(x) \, dx, \quad g \in L^2(I). \]

Then
\[ f(t) = \sum_{k=-\infty}^{\infty} f(t_k) \frac{\int K(x, t) K(x, t_k) \, dx}{\|K(x, t_k)\|^2_{L^2(I)}}. \]

Generalization of WKS sampling theorem has been investigated extensively, see [6 – 9]. Sampling theorems associated with Sturm-Liouville problems were investigated in [10 – 15]. Also [16, 17] and [18, 19] are the example works with an eigenparameter in the boundary conditions in direction of sampling analysis associated with continuous and discontinuous eigenproblems, respectively. In [20], the authors discussed the situation of deriving a sampling theorem of Kramer type when the kernels are discontinuous. They introduced the discontinuous Sturm-Liouville problems studied by Kobayashi [21] to Sturm-Liouville problems with eigenfunctions having two symmetrically located discontinuities and satisfying symmetric jump (or transmission) conditions. Their sampling result states that the transformation
\[ F(\lambda) = \int_0^{\pi} f(x) u(x, \lambda) \, dx, \quad f \in L^2(0, \pi), \]
can be reconstructed via the sampling form
\[ F(\lambda) = \sum_{n=0}^{\infty} F(\lambda_n) \frac{\omega(\lambda)}{(\lambda - \lambda_n) \omega'(\lambda_n)}, \]
where \( \{\lambda_n\}_{n=0}^{\infty} \) are the zeros of the function \( \omega(\lambda) \), and they are exactly the eigenvalues of the discontinuous Sturm-Liouville problem. In this problem, neither boundary conditions nor transmission conditions contain an eigenparameter. In [22], the author investigated the sampling analysis associated with discontinuous Sturm-Liouville problems which has one point of discontinuity and contains an eigenparameter in all boundary conditions and derived sampling representations for transforms whose kernels are either solutions or Green’s functions.
We consider the boundary value problem:

\[(1.9) \quad \tau(u) := -u''(x) + q(x)u(x) = \lambda u(x), \quad x \in I\]

with boundary conditions;

\[(1.10) \quad B_1(u) := \beta_1u(a) + \beta_2u'(a) = 0,\]

\[(1.11) \quad B_2(u) := \lambda (\alpha'_1u(b) - \alpha'_2u'(b)) - (\alpha_1u(b) - \alpha_2u'(b)) = 0,\]

and transmission conditions at two points of discontinuity \(c_1\) and \(c_2\);

\[(1.12) \quad T_1(u) := u(c^-_1) - \delta u(c^+_1) = 0,\]

\[(1.13) \quad T_2(u) := u'(c^-_1) - \delta u'(c^+_1) + \lambda u(c^-_1) = 0,\]

\[(1.14) \quad T_3(u) := \delta u(c^-_2) - \gamma u(c^+_2) = 0,\]

\[(1.15) \quad T_4(u) := \delta u'(c^-_2) - \gamma u'(c^+_2) + \lambda u(c^-_2) = 0,\]

where \(I = [a, c_1) \cup (c_1, c_2) \cup (c_2, b],\) \(a < c_1 < c_2 < b;\) \(\lambda\) is a complex spectral parameter; \(q(x)\) is a given real valued function which is continuous in \([a, c_1), (c_1, c_2)\) and \((c_2, b]\) and has finite limits \(q(c^+_1) = \lim_{x \to c_1^+} q(x)\) and \(q(c^-_2) = \lim_{x \to c_2^-} q(x)\); \(\alpha_i, \alpha'_i, \beta_i, \delta, \gamma \in \mathbb{R} (i = 1, 2), |\beta_1| + |\beta_2| \neq 0, \delta > 0, \gamma \neq 0; c_i := c_i \pm 0, (i = 1, 2)\) and \(\rho := (\alpha'_1\alpha_2 - \alpha_1\alpha'_2) > 0.\)

In the present work, we investigate the sampling analysis associated with discontinuous Sturm-Liouville problem \((1.9)-(1.15)\) which has transmission conditions at two points of discontinuity and contains an eigenparameter in a boundary condition and two transmission conditions. This is the difference between our problem and sampling theories associated with discontinuous eigenproblems studied in the literature. The problem of deriving a sampling theorem of Kramer type when the problem contains an eigenparameter in two transmission conditions besides in a boundary conditions does not exist as far as we know. To derive sampling theorem associated with the problem \((1.9)-(1.15)\), we study briefly the spectral properties of the problem \((1.9)-(1.15)\) and then we prove that integral transforms associated with the problem \((1.9)-(1.15)\) can also be reconstructed in a sampling form of Lagrange interpolation type.

2. **Spectral Properties**

To formulate a theoretic approach to the problem \((1.9)-(1.15)\) let \(L := L^2(a, c_1) \oplus L^2(c_1, c_2) \oplus L^2(c_2, b)\) and \(\mathbb{C}^3 := \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}\) and we define the Hilbert space \(H = L \oplus \mathbb{C}^3\) with an inner product

\[(2.1) \quad \langle F(.), G(.) \rangle_H := \int_a^b f(x)\overline{g}(x) \, dx + \delta^2 \int_{c_1}^{c_2} f(x)\overline{g}(x) \, dx + \gamma^2 \int_{c_1}^{b} f(x)\overline{g}(x) \, dx + \frac{\beta^2}{\rho} h_1 k_1 + h_2 k_2 + \delta h_3 k_3,\]

where
\[ F(x) = \begin{pmatrix} f(x) \\ h_1 \\ h_2 \\ h_3 \end{pmatrix}, \quad G(x) = \begin{pmatrix} g(x) \\ k_1 \\ k_2 \\ k_3 \end{pmatrix} \in H, \quad f(\cdot), g(\cdot) \in L^2(a,b) \text{ and } h_i, k_i \in \mathbb{C} \] 

\((i = 1, 2, 3)\).

For convenience we put
\[ R_b (u) := \alpha_1 u (b) - \alpha_2 u' (b), \quad R'_b (u) := \alpha'_1 u (b) - \alpha'_2 u' (b), \]
\[ R_{c_1} (u) := u (c_1^-), \quad R'_{c_1} (u) := u' (c_1^-) - \delta u' (c_1^+), \]
\[ R_{c_2} (u) := u (c_2^-), \quad R'_{c_2} (u) := \delta u' (c_2^-) - \gamma u' (c_2^+). \]

For functions \( f(x) \), which is defined on \( I \) and has finite limit \( f(c_i^+) = \lim_{x \to c_i^+} f(x) \), \( f(c_i^-) = \lim_{x \to c_i^-} f(x) \), by \( f_{(i)}(x) \) \((i = 1, 4)\) we denote the functions
\[
\begin{align*}
 f_{(1)}(x) := & \begin{cases} f(x), & x \in [a, c_1), \\ f(c_1^-), & x = c_1, \end{cases} & f_{(2)}(x) := & \begin{cases} f(x), & x \in (c_1, c_2), \\ f(c_1^+), & x = c_1, \end{cases} \\
 f_{(3)}(x) := & \begin{cases} f(x), & x \in (c_1, c_2), \\ f(c_2^-), & x = c_2, \end{cases} & f_{(4)}(x) := & \begin{cases} f(x), & x \in (c_2, b], \\ f(c_2^+), & x = c_2, \end{cases}
\end{align*}
\]

which are defined on \( I_1 := [a, c_1], I_2 := [c_1, c_2] \) and \( I_3 := [c_2, b] \), respectively.

In this space, we define a linear operator \( A \) by the domain of definition
\[ D(A) := \left\{ F(x) = \begin{pmatrix} f(x) \\ R_b (f) \\ R_{c_1} (f) \\ R_{c_2} (f) \end{pmatrix} \in H \mid f_{(i)}(\cdot), f'_{(i)}(\cdot) \text{ are absolutely continuous in } I_i \; (i = 1, 2, 3); \; \tau(f) \in L; \; B_1 (f) = 0, \; T_1 (f) = T_3 (f) = 0, \; h_1 = R_b (f), \; h_2 = R_{c_1} (f), \; h_3 = R_{c_2} (f) \right\} \]
and
\[ A \left( \begin{pmatrix} f(x) \\ R_b (f) \\ R_{c_1} (f) \\ R_{c_2} (f) \end{pmatrix} \right) = \begin{pmatrix} \tau(f) & R_b (f) & -R_{c_1}' (f) & -R_{c_2}' (f) \\ R_b (f) & R_{c_1} (f) & R_{c_1} (f) & R_{c_2} (f) \end{pmatrix} \in D(A). \]

Consequently, the problem (1.9)-(1.15) can be rewritten in operator form as \( AF = \lambda F \), i.e., the problem (1.9)-(1.15) can be considered as the eigenvalue problem for the operator \( A \).

The following theorem can be proven by the same methods in similar studies [21, 23–25].

**Theorem 1.** i) \( A \) is a symmetric operator in \( H \); the eigenvalue problem for the operator \( A \) and the problem (1.9)-(1.15) coincide. ii) All eigenvalues and eigenfunctions of the operator \( A \) (or the problem (1.9)-(1.15)) are real. iii) Two eigenfunctions \( u(x, \lambda) \) and \( v(x, \mu) \) corresponding to different eigenvalues \( \lambda \) and \( \mu \), are
orthogonal, i.e.,
\[ (2.8) \int_a^c u(x, \lambda) v(x, \mu) \, dx + \frac{\gamma^2}{\rho} \int_c^b u(x, \lambda) v(x, \mu) \, dx + \gamma^2 \int_a^b u(x, \lambda) v(x, \mu) \, dx + \int_a^c \frac{\gamma^2}{\rho} R'_b(u) R'_c(v) R_{c_1}(u) R_{c_1}(v) + \delta R_{c_2}(u) R_{c_2}(v) = 0. \]

Now, we will construct a special fundamental system of solutions of the equation (1.9). By virtue of theorem 1.5 in [26], we will define two solutions of the equation (1.9) as follows:

\[ (2.9) \phi_\lambda(x) = \begin{cases} \phi_{1\lambda}(x), & x \in [a, c_1], \\ \phi_{2\lambda}(x), & x \in (c_1, c_2), \\ \phi_{3\lambda}(x), & x \in (c_2, b], \end{cases}, \quad \chi_\lambda(x) = \begin{cases} \chi_{1\lambda}(x), & x \in [a, c_1], \\ \chi_{2\lambda}(x), & x \in (c_1, c_2), \\ \chi_{3\lambda}(x), & x \in (c_2, b]. \end{cases} \]

Let \( \phi_{1\lambda}(x) = \phi_1(x, \lambda) \) be the solution of the equation (1.9) on \([a, c_1]\), which satisfies the initial conditions
\[ (2.10) \quad u(a) = \beta_2, \quad u'(a) = -\beta_1. \]

By virtue of theorem 1.5 in [26], after defining this solution we may define the solution \( \phi_{2\lambda}(x) = \phi_2(x, \lambda) \) of the equation (1.9) on \([c_1, c_2]\) by means of the solution \( \phi_{1\lambda}(x) \) by the nonstandard initial conditions
\[ (2.11) \quad u(c_1) = \frac{1}{\delta} \phi_{1\lambda}(c_1^{-}), \quad u'(c_1) = \frac{1}{\delta} \{ \phi'_{1\lambda}(c_1^{-}) + \lambda \phi_{1\lambda}(c_1^{-}) \}. \]

After defining this solution, we may define the solution \( \phi_{3\lambda}(x) = \phi_3(x, \lambda) \) of the equation (1.9) on \([c_2, b]\) by means of the solution \( \phi_{2\lambda}(x) \) by the nonstandard initial conditions
\[ (2.12) \quad u(c_2) = \frac{\delta}{\gamma} \phi_{2\lambda}(c_2^{-}), \quad u'(c_2) = \frac{1}{\gamma} \{ \delta \phi_{2\lambda}(c_2^{-}) + \lambda \phi_{2\lambda}(c_2^{-}) \}. \]

Thus, \( \phi_\lambda(x) = \phi(x, \lambda) \) satisfies the equation (1.9) on \(I\), the boundary condition (1.10) and the transmission conditions (1.12)-(1.15).

Analogically, first we define the solution \( \chi_{3\lambda}(x) = \chi_3(x, \lambda) \) of the equation (1.9) on \([c_2, b]\) by the initial conditions
\[ (2.13) \quad u(b) = \lambda \alpha_2' - \alpha_2, \quad u'(b) = \lambda \alpha_1' - \alpha_1. \]

Again, after defining this solution, we define the solution \( \chi_{2\lambda}(x) = \chi_2(x, \lambda) \) of the equation (1.9) on \([c_1, c_2]\) by the initial conditions
\[ (2.14) \quad u(c_2) = \frac{\gamma}{\delta} \chi_{3\lambda}(c_2^+), \quad u'(c_2) = \frac{\gamma}{\delta} \{ \chi'_{3\lambda}(c_2^+) - \lambda \chi_{3\lambda}(c_2^+) \}. \]

After defining this solution, we define the solution \( \chi_{1\lambda}(x) = \chi_1(x, \lambda) \) of the equation (1.9) on \([a, c_1]\) by the initial conditions
\[ (2.15) \quad u(c_1) = \delta \chi_{2\lambda}(c_1^+), \quad u'(c_1) = \delta \{ \chi'_{2\lambda}(c_1^+) - \lambda \chi_{2\lambda}(c_1^+) \}. \]

Thus, \( \chi_\lambda(x) = \chi(x, \lambda) \) satisfies the equation (1.9) on \(I\), the boundary condition (1.11) and the transmission conditions (1.12)-(1.15).
Since the Wronskian $W(\phi_{i\lambda}, \chi_{i\lambda}; x)$ are independent on variable $x \in I_i$ ($i = 1, 2, 3$) and $\phi_{i\lambda}(x)$ and $\chi_{i\lambda}(x)$ are the entire functions of the parameter $\lambda$ for each $x \in I_i$, then the functions
\[
(2.16) \quad \omega_i(\lambda) := W(\phi_{i\lambda}, \chi_{i\lambda}; x) = \phi_{i\lambda}(x) \chi_{i\lambda}'(x) - \phi_{i\lambda}'(x) \chi_{i\lambda}(x), \quad (i = 1, 2, 3)
\]
are the entire functions of parameter $\lambda$. Taking into account (2.11), (2.12), (2.14) and (2.15), after a short calculation, we get
\[
\omega_1(\lambda) = \delta_2^2 \omega_2(\lambda) = \gamma_2^2 \omega_3(\lambda)
\]
for each $\lambda \in \mathbb{C}$.

**Corollary 1.** The zeros of the functions $\omega_1(\lambda)$, $\omega_2(\lambda)$ and $\omega_3(\lambda)$ coincide.

Then, we may introduce to the consideration the characteristic function $\omega(\lambda)$ as
\[
(2.17) \quad \omega(\lambda) := \omega_1(\lambda) = \delta_2^2 \omega_2(\lambda) = \gamma_2^2 \omega_3(\lambda).
\]

**Theorem 2.** The eigenvalues of the problem (1.9)-(1.15) are coincided zeros of the function $\omega(\lambda)$.

**Proof.** ...

**Lemma 1.** All eigenvalues $\lambda_n$ of the problem (1.9)-(1.15) are simple zeros of $\omega(\lambda)$.

**Proof.** ...

If $\lambda_n$ ($n = 0, 1, 2, \ldots$) denote the zeros of $\omega(\lambda)$, then
\[
(2.26) \quad \Phi_{\lambda_n}(x) := \begin{pmatrix} \phi_{\lambda_n}(x) \\ R'_b(\phi_{\lambda_n}) \\ R_{c_1}(\phi_{\lambda_n}) \\ R_{c_2}(\phi_{\lambda_n}) \end{pmatrix}
\]
are the corresponding eigenvectors of the operator $A$, satisfying the orthogonality relation
\[
(2.27) \quad \langle \Phi_{\lambda_n}(\cdot), \Phi_{\lambda_m}(\cdot) \rangle_H = 0 \quad \text{for} \quad n \neq m.
\]

Here $\{\phi_{\lambda_n}(\cdot)\}_{n=0}^\infty$ is a sequence of eigenfunctions of the problem (1.9)-(1.15) corresponding to the eigenvectors of $A$, i.e.,
\[
(2.28) \quad \Psi_{\lambda_n}(x) := \frac{\Phi_{\lambda_n}(x)}{\|\Phi_{\lambda_n}(x)\|_H} = \begin{pmatrix} \Psi_{\lambda_n}(x) \\ R'_b(\Psi_{\lambda_n}) \\ R_{c_1}(\Psi_{\lambda_n}) \\ R_{c_2}(\Psi_{\lambda_n}) \end{pmatrix}.
\]

Therefore $\{\chi_{\lambda_n}(\cdot)\}_{n=0}^\infty$ is another set of eigenfunctions which is related by $\{\phi_{\lambda_n}(\cdot)\}_{n=0}^\infty$ with
\[
(2.29) \quad \chi_{\lambda_n}(x) = k_n \phi_{\lambda_n}(x), \quad x \in I, \quad n \in \mathbb{Z}
\]
where $k_n \neq 0$ are non-zero constants, since all eigenvalues are simple.
3. Asymptotic Formulas for Eigenvalues and Eigenfunctions

The asymptotics formulas for eigenvalues and eigenfunctions can be derived similar to the classical techniques of [26], see also [23–25]. We state the results briefly.

Let \( \phi_\lambda (x) \) be the solutions of the equation (1.9) defined in section 2, and let \( \lambda = s^2 \). Then the following integral equations hold for \( k = 0 \) and \( k = 1 \):

\[
\frac{d^k}{dx^k}\phi_{1\lambda} (x) = \beta_2 \frac{d^k}{dx^k} (\cos s (x - a)) - \frac{\beta_1}{s} \frac{d^k}{dx^k} (\sin s (x - a)) + \frac{1}{s^2} \frac{d^k}{dx^k} (\sin s (x - y)) q (y) \phi_{1\lambda} (y) dy,
\]

(3.1)

\[
\frac{d^k}{dx^k}\phi_{2\lambda} (x) = \frac{1}{\delta} \frac{d^k}{dx^k} (\cos s (x - c_1)) + \frac{1}{s^2} \left\{ \phi'_{1\lambda} (c_1) + s^2 \phi_{1\lambda} (c_1) \right\} \times \frac{1}{s} \frac{d^k}{dx^k} (\sin s (x - y)) q (y) \phi_{2\lambda} (y) dy,
\]

(3.2)

\[
\frac{d^k}{dx^k}\phi_{3\lambda} (x) = \frac{\delta}{\gamma} \frac{d^k}{dx^k} (\cos s (x - c_2)) + \frac{1}{s^2} \left\{ \delta \phi'_{2\lambda} (c_2) + s^2 \phi_{2\lambda} (c_2) \right\} \times \frac{1}{s} \frac{d^k}{dx^k} (\sin s (x - y)) q (y) \phi_{3\lambda} (y) dy.
\]

(3.3)

Let \( \lambda = s^2 \) and \( |\text{Im} s| = t \). Then the functions \( \phi_{i\lambda} (x) \) have the following asymptotic representations for \( |\lambda| \to \infty \), which hold uniformly for \( x \in I_i \) (\( i = 1, 2, 3 \)):

\[
\frac{d^k}{dx^k}\phi_{1\lambda} (x) = \beta_2 \frac{d^k}{dx^k} (\cos s (x - a)) + O \left( \frac{1}{s} e^{t(x-a)} \right),
\]

(3.4)

\[
\frac{d^k}{dx^k}\phi_{2\lambda} (x) = \frac{s^2 \beta_2}{\delta} \cos (s (c_1 - a)) - \frac{d^k}{dx^k} (\sin s (x - c_1)) + O \left( e^{t(x-a)} \right),
\]

(3.5)

\[
\frac{d^k}{dx^k}\phi_{3\lambda} (x) = \frac{s^2 \beta_2}{\delta} \cos (s (c_1 - a)) \sin (s (c_2 - c_1)) - \frac{d^k}{dx^k} (\sin s (x - c_2)) + O \left( se^{t(x-a)} \right),
\]

(3.6)

By substituting (3.6) into the representation

\[
\omega (\lambda) = \gamma^2 \left\{ (s^2 \alpha_1 - \alpha_1) \phi_{3\lambda} (b) - (s^2 \alpha_2 - \alpha_2) \phi'_{3\lambda} (b) \right\},
\]

(3.7)

then the characteristic function \( \omega (\lambda) \) has the following asymptotic representation:

\[
\omega (\lambda) = \frac{s^5 \gamma \beta_2 \alpha_2^2}{\delta} \cos (s (c_1 - a)) \sin (s (c_2 - c_1)) \cos (s (b - c_2)) + O \left( s^4 e^{t(b-a)} \right).
\]

(3.8)

**Corollary 2.** The eigenvalues of the problem (1.9)-(1.15) are bounded below.

Now we can obtain the asymptotic approximation formula for the eigenvalues of the problem (1.9)-(1.15). Since the eigenvalues coincide with the zeros of the entire function \( \omega (\lambda) \), it follows that they have no finite limit. Moreover, we know that all eigenvalues are real and bounded below. Therefore, we may renumber them as \( \lambda_0 \leq \lambda_1 \leq ... \), listed according to their multiplicity.
The eigenvalues $\lambda_n = s_n^2$, $(n = 0, 1, \ldots)$ of the problem (1.9)-(1.15) have the following asymptotic representation for $n \to \infty$:

$$
(3.9) \quad \tilde{s}_n = \frac{(n + 1/2) \pi}{(c_1 - a)} + O \left( \frac{1}{n} \right),
$$

$$
(3.10) \quad \tilde{\tilde{s}}_n = \frac{n \pi}{(c_2 - c_1)} + O \left( \frac{1}{n} \right),
$$

$$
(3.11) \quad \tilde{\tilde{\tilde{s}}}_n = \frac{(n + 1/2) \pi}{(b - c_2)} + O \left( \frac{1}{n} \right).
$$

Proof. ...

Then from (3.4)-(3.6) (for $k = 0$) and the above theorem, the asymptotic behaviour of the eigenfunctions:

$$
(3.13) \quad \phi_{\lambda_n}(x) = \begin{cases} 
\phi_{1\lambda_n}(x), & x \in [a,c_1), \\
\phi_{2\lambda_n}(x), & x \in (c_1,c_2), \\
\phi_{3\lambda_n}(x), & x \in (c_2,b], 
\end{cases}
$$

of the problem (1.9)-(1.15) is given by

$$
\phi_{\tilde{\lambda}_n}(x) = \begin{cases} 
\beta_2 \cos \left( \frac{(n + 1/2) \pi}{(c_1 - a)} (x - a) \right) + O \left( \frac{1}{n} \right), & x \in [a,c_1), \\
O(1), & x \in (c_1,c_2), \\
O(1), & x \in (c_2,b],
\end{cases}
$$

$$
\phi_{\tilde{\tilde{\lambda}}_n}(x) = \begin{cases} 
\beta_2 \cos \left( \frac{n \pi}{(c_2 - c_1)} (x - a) \right) + O \left( \frac{1}{n} \right), & x \in [a,c_1), \\
\frac{n \pi}{(c_2 - c_1)} \beta_2 \cos \left( \frac{n \pi}{(c_2 - c_1)} (x - c_1) \right) + O(1), & x \in (c_1,c_2), \\
O(1), & x \in (c_2,b],
\end{cases}
$$

all these asymptotic formulas hold uniformly for $x$.

4. THE SAMPLING THEOREM

Now we can derive the sampling theorem associated with the problem (1.9)-(1.15).

Theorem 4. Let

$$
(4.1) \quad \phi_{\lambda}(x) = \begin{cases} 
\phi_{1\lambda}(x), & x \in [a,c_1), \\
\phi_{2\lambda}(x), & x \in (c_1,c_2), \\
\phi_{3\lambda}(x), & x \in (c_2,b],
\end{cases}
$$
be the solution of the equation (1.9) together with the conditions (2.10)-(2.12). Let $g(.) \in L^2(a,b)$ and 

(4.2)  \[ F(\lambda) = \int_a^b g(x) \phi_{1\lambda}(x) \, dx + \delta^2 \int_{c_1}^{c_2} g(x) \phi_{2\lambda}(x) \, dx + \gamma^2 \int_{c_2}^b g(x) \phi_{3\lambda}(x) \, dx. \]

Then $F(\lambda)$ is an entire function of exponential type that can be reconstructed from its values at the points $\{\lambda_n\}_{n=0}^{\infty}$ via the sampling formula

(4.3)  \[ F(\lambda) = \sum_{n=0}^{\infty} \omega(\lambda_n) \frac{\omega(\lambda)}{(\lambda - \lambda_n) \omega'(\lambda_n)}. \]

The series (4.3) converges absolutely on $\mathbb{C}$ and uniformly on compact subset of $\mathbb{C}$. Here $\omega(\lambda)$ is the entire function defined in (2.17).

Proof. ... \Box

Remark 1. To see that expansion (4.3) is a Lagrange type interpolation, we may replace $\omega(\lambda)$ by the canonical product

(4.26)  \[ \varpi(\lambda) = \begin{cases} \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right), & \text{if zero is not an eigenvalue,} \\ \lambda \prod_{n=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_n}\right), & \text{if } \lambda_0 = 0 \text{ is an eigenvalue.} \end{cases} \]

From Hadamard’s factorization theorem, see [6], $\omega(\lambda) = h(\lambda) \varpi(\lambda)$, where $h(\lambda)$ is an entire function with no zeros. Thus,

(4.27)  \[ \frac{\omega(\lambda)}{\omega'(\lambda_n)} = \frac{h(\lambda) \varpi(\lambda)}{h(\lambda_n) \varpi'(\lambda_n)} \]

and (4.2), (4.3) remain valid for the function $F(\lambda)/h(\lambda)$. Hence

(4.28)  \[ F(\lambda) = \sum_{n=0}^{\infty} F(\lambda_n) \frac{h(\lambda) \varpi(\lambda)}{(\lambda - \lambda_n) h(\lambda_n) \varpi'(\lambda_n)}. \]

We may redefine (4.2) by taking kernel $\phi_{\lambda}(.) / h(\lambda) = \phi_{\lambda}(.)$ to get

(4.29)  \[ F(\lambda) = \frac{F(\lambda)}{h(\lambda)} = \sum_{n=0}^{\infty} F(\lambda_n) \frac{\varpi(\lambda)}{(\lambda - \lambda_n) \varpi'(\lambda_n)}. \]

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