Asymptotical photon distributions in the dissipative dynamical Casimir effect

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Abstract
Asymptotical formulae for the photon distribution function of a quantum oscillator with time-dependent frequency and damping coefficients, interacting with a thermal reservoir, are derived in the case of a large mean number of quanta. Different regimes of excitation of an initial thermal state with an arbitrary temperature are considered. New formulae are used to predict the statistical properties of the electromagnetic field created in the experiments on the dynamical Casimir effect that are now under preparation.

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1. Introduction
The so-called ‘dynamical Casimir effect’ (DCE), i.e. generation of photons from vacuum due to the motion of neutral boundaries, has been the subject of numerous theoretical studies for almost 40 years (for specific reviews on the DCE, see [1–3]; general reviews on the Casimir physics, including different manifestations of the static and dynamical Casimir effects, can be found, e.g., in [4, 5]).

One of the most important results obtained during a decade before and after 2000 was the prediction of a possibility of observation of the DCE in a laboratory, using high-\(Q\) cavities with dimensions of the order of a few centimetres and resonance frequencies of the order of a few GHz or higher. Namely, calculations performed by several groups of authors [6–12] (using quite different approaches) resulted in the same conclusion: if one could arrange periodic changes of parameters of some cavity (its dimensions or properties of the walls) for a sufficiently long time, then initial vacuum or thermal fluctuations of the electromagnetic field could be amplified to a detectable level due to the effect of parametric resonance.

This result stimulated the work of several experimental groups, from which the MIR group of the University of Padua [13] seems to be close to success (the name of the experiment MIR means ‘motion induced radiation’; this term was coined in [8]). Therefore, it seems interesting to calculate the statistical properties of quantum states that could be obtained under realistic experimental conditions.

The simplest model of the DCE in cavities is that of a quantum oscillator with a time-dependent frequency, describing the selected field mode that is in resonance with time variations of the cavity parameters. Such a model was first proposed in [14, 15] and later it was developed in [6, 7]. In the ideal case (without losses or interactions with other degrees of freedom), the oscillator goes from the initial ground state to the vacuum squeezed state [6], whose properties are well known. In particular, this state has a strongly oscillating photon distribution function (PDF) \(f(m) \equiv \langle m | \hat{\rho} | m \rangle\) (i.e. the probability to detect \(m\) quanta in the state described by the statistical operator \(\hat{\rho}\)):

\[
f(2m) = \frac{\langle n \rangle^m (2m)!}{(1 + \langle n \rangle)^{(m+1)/2} (2m)!^2}, \quad f(2m + 1) = 0,
\]

where \(\langle n \rangle\) is the mean number of quanta. Another important property of the vacuum squeezed state is a high degree of quadrature squeezing when \(\langle n \rangle \gg 1\).

Quite a different situation arises if the intermode interaction is essential (as happens in the effectively one-dimensional Fabry–Perot cavity [16, 17] or in the three- and two-dimensional cavities with accidental degeneracies of the eigenmode spectra [10, 18]) or coupling with a detector is strong enough [6]. Then the degree of squeezing in each mode becomes much smaller than in the vacuum squeezed state and the oscillations of the PDF disappear.

New features appear when dissipation becomes important. Just such a situation takes place in the MIR experiment. Its main idea is to simulate a motion of one of the cavity walls using an effective electron–hole ‘plasma mirror’, created periodically on the surface of a semiconductor slab (attached to the wall) by illuminating it with a sequence of...
short laser pulses. If the interval between pulses exceeds the recombination time of carriers in the semiconductor, a highly conducting layer will periodically appear and disappear on the surface of the semiconductor film, thus simulating periodical displacements of the boundary. In this way, rather big relative amplitudes of displacements and changes of the fundamental cavity eigenfrequency, of the order of 10^−3 or 10^−2, can be easily achieved using standard semiconductor plates having a thickness of the order of 1 mm. This is a great advantage over the schemes where real oscillations of the surface of the cavity walls are excited, because in the latter case the relative amplitude of displacements cannot exceed the value 10^−8 due to tremendous internal stresses arising inside the material for the frequencies of the order of a few GHz or higher.

Note that the thickness of the photo-excited conducting layer near the surface of the semiconductor slab is much smaller than the thickness of the slab itself. It is determined mainly by the absorption coefficient of the laser radiation, so it is about a few micrometers or less, depending on the laser wavelength. Therefore laser pulses with a surface energy density of about a few µJ cm^−2 can create a highly conducting layer with a carrier concentration exceeding 10^17 cm^−3, which gives rise to an almost maximal possible change of the cavity eigenfrequency for the given geometry [19]. It is worth noting that although the thickness of the conducting layer is less than the skin depth, it gives the same frequency shift as the conductor filling in all of the slab. This interesting fact was explained and verified experimentally in [20]. On the other hand, the conductivity of the layer is not extremely high due to a moderate value of the mobility in the available materials (such as highly doped GaAs), which is less than 1 m^2 V^−1 s^−1 [13]. For this reason, effects of dissipation during the excitation–recombination process inside the layer cannot be neglected, since they can change the picture drastically [2, 19]. A model taking into account dissipation was developed in [2], and some of its consequences with respect to photon statistics and PDF were considered recently in [3, 21]. In this paper, I analyse some interesting special cases that were not considered in previous papers.

2. Exact formulae for the PDF and statistical moments

I assume that the effects of dissipation can be taken into account by means of a model based on the Heisenberg–Langevin operator equations of the form (\( \hbar = 1 \))

\[
\frac{d\hat{x}}{dt} = \hat{p} - \gamma_x(t)\hat{x} + \hat{F}_x(t),
\]

\[
\frac{d\hat{p}}{dt} = -\gamma_p(t)\hat{p} - \alpha^2(t)\hat{x} + \hat{F}_p(t),
\]

where \( \hat{x} \) and \( \hat{p} \) are dimensionless quadrature operators of the selected mode of the EM field. These operators are normalized in such a way that the mean number of photons equals \( N = \frac{1}{2}(\hat{p}^2 + \hat{x}^2 - 1) \). Two noise operators \( \hat{F}_x(t) \) and \( \hat{F}_p(t) \) with zero mean values (commuting with \( \hat{x} \) and \( \hat{p} \)) are necessary to preserve the canonical commutator \( \{\hat{x}(t), \hat{p}(t')\} = i \). These operators give a simplified description of complicated processes inside a thin lossy dielectric (semiconductor) slab attached to one of the cavity walls. In the phenomenological model used in this paper, the net result of all those processes is encoded in the correlators of the noise operators (the Markov approximation is assumed)

\[
\langle \hat{F}_x(t)\hat{F}_x(t') \rangle = \delta(t - t')\chi_{xx}(t), \quad j, k = x, p.
\]

In principle, the function \( \chi_{jk}(t) \), as well as functions \( \gamma_x(t) \) and \( \gamma_p(t) \), should be derived from some ‘microscopical’ model of electron–photon and electron–phonon interactions inside the semiconductor slab. In the phenomenological model considered here, these coefficients can only be ‘guessed’. The simplest possibility consistent with the principles of quantum mechanics is as follows [2, 21]:

\[
\gamma_x(t) = \gamma_p(t) = \gamma(t),
\]

\[
\chi_x = -\chi_p = i\gamma(t), \quad \chi_{xx} = \chi_{pp} = \gamma(t)\Gamma,
\]

where \( \gamma(t) \) is identified with the imaginary part of the complex time-dependent eigenfrequency of the cavity \( \omega_c(t) = \omega(t) - i\gamma(t) \), which can be found from the solution of the classical electrodynamical problem by taking the instantaneous geometry and material properties (for example, by solving the Helmholtz equation with the complex dielectric function \( \varepsilon(t) \) inside the slab and the set of cavity dimensions \( \{L_j(t)\} \), where the time variable \( t \) is considered as a parameter). The coefficient \( G \) is related to the temperature of the reservoir \( \Theta \) as

\[
G = 1 + 2\langle n \rangle_{\text{th}} = \coth \left[ \frac{\hbar\omega_c}{2k_B\Theta} \right]
\]

(so that \( \langle n \rangle_{\text{th}} \) is the mean number of quanta for the given cavity mode in the thermodynamic equilibrium). The specific set of coefficients (5) and (6) has the following remarkable property: if the frequency \( \omega \) does not depend on time, then the second-order statistical moments \( \langle \hat{x}^2 \rangle \), \( \langle \hat{p}^2 \rangle \) and \( \langle \hat{x}\hat{p} + \hat{p}\hat{x} \rangle \) tend to the well-known equilibrium values for an arbitrary positive function \( \gamma(t) \) (without any corrections). Some arguments pro e contra the choice of damping coefficients in the form (5) can be found in [21].

Equations (2) and (3) can be solved explicitly for arbitrary time-dependent functions \( \gamma_x(t), \omega(t) \) and \( \tilde{F}_{x,p}(t) \):

\[
\hat{x}(t) = \hat{x}_0 + \hat{X}(t), \quad \hat{p}(t) = \hat{p}_0 + \hat{P}(t).
\]

The first terms are the solutions of homogeneous equations (without the noise operators)

\[
\hat{x}_0 = e^{-\Gamma(t)}[\hat{x}_0\Re\{\xi(t)\} - \hat{p}_0\Im\{\xi(t)\}],
\]

\[
\hat{p}_0 = e^{-\Gamma(t)}[\hat{x}_0\Re\{\xi(t)\} - \hat{p}_0\Im\{\xi(t)\}],
\]

where \( \hat{x}_0 \) and \( \hat{p}_0 \) are the values of operators at \( t = 0 \) (taken as the initial instant) and \( \Gamma(t) = \int_0^t \gamma(t')dt' \). The function \( \xi(t) \) is a special solution to the classical oscillator equation

\[
\ddot{\xi} + \alpha^2(t)\xi = 0,
\]

selected by the initial condition \( \xi(t) = \exp(-it) \) for \( t \to -\infty \), which is equivalent to fixing the value of the Wronskian

\[
\xi\dot{\xi}^* - \dot{\xi}\xi^* = 2i.
\]
The operators \( \hat{X}(t) \) and \( \hat{P}(t) \) represent the influence of the stochastic forces:
\[
\left( \frac{\hat{X}(t)}{\hat{P}(t)} \right) = e^{-i\tau} \int_0^t dt e^{i\tau} \mathcal{A}(t; \tau) \left( \hat{F}_x(t) \right),
\]
where the \( 2 \times 2 \) matrix
\[
\mathcal{A}(t; \tau) = \begin{pmatrix} a_x^F(t; \tau) & a_p^F(t; \tau) \\ a_x^p(t; \tau) & a_p^p(t; \tau) \end{pmatrix}
\]
consists of the following elements:
\[
a_x^F = \text{Im} \left[ \xi(\tau) \hat{\xi}(\tau) \right], \quad a_p^F = \text{Im} \left[ \hat{\xi}(\tau) \xi(\tau) \right],
\]
\[
a_x^p = \text{Im} \left[ \xi(\tau) \hat{\xi}(\tau) \right], \quad a_p^p = \text{Im} \left[ \hat{\xi}(\tau) \xi(\tau) \right].
\]

Combining equations (13)–(16) with (4)–(6) one can obtain the following expressions for the second-order moments of operators \( \hat{P}(t) \) and \( \hat{X}(t) \) (here \( f_i \equiv f(i) \)):
\[
\langle \hat{P}^2(t) \rangle = |\xi|^2 J_1 - \text{Re} \left[ \hat{\xi}^2 \hat{J}_1 \right],
\]
\[
\langle \hat{X}^2(t) \rangle = |\xi|^2 J_1 - \text{Re} \left[ \hat{\xi} \hat{\xi}^2 \hat{J}_1 \right],
\]
\[
\frac{1}{2} \langle \hat{X} \hat{P} + \hat{P} \hat{X} \rangle = \text{Re} \left[ \xi \hat{\xi}^4 J_1 - \hat{\xi} \hat{\xi}^2 \hat{J}_1 \right].
\]

The mean number of quanta can be written as \( \mathcal{N}(t) = \mathcal{N}_i(t) + \mathcal{N}_r(t) \), where the first term depends on the initial state (‘signal’), while the second term is determined by the interaction with the reservoir. From (17) and (18) one obtains
\[
\mathcal{N}_i(t) = E_i J_i - \text{Re} \left( \hat{E}_i^2 \hat{J}_i \right),
\]
\[
E_i = \frac{1}{2} \left( |\xi|^2 + |\hat{\xi}|^2 \right), \quad \hat{E}_i = \frac{1}{2} \left( \xi^2 + \hat{\xi}^2 \right).
\]

For the initial thermal state characterized by the parameter \( G_0 \) (which can be different from \( G \)), one has
\[
\mathcal{N}_r^{(0)}(t) = \frac{1}{2} \left\{ G_0 e^{-2\xi(t)} E(t) - 1 \right\}.
\]

One should remember that formulae (22) and (24) make sense for sufficiently big values of time \( t \), when the laser pulses have been switched off and the recombination processes have been over, so that the normalized frequency \( \omega(t) \) returns to its initial unit value (because the photon number operator \( \hat{N} = \frac{1}{2} \left( \hat{P}^2 + \hat{X}^2 - 1 \right) \) is defined with respect to the initial geometry of the cavity, coinciding with the final one).

I consider here only the special (although the most realistic) case of initial thermal states of the field. It is well known [22, 23] that the description of open quantum systems by means of the Heisenberg–Langevin equations with delta-correlated stochastic force operators is equivalent to the description in the Schrödinger picture by means of the master equation for the statistical operator. In the case of linear operator equations of motion, such as equations (2) and (3), the corresponding master equations contain only quadratic terms (various products of two operators \( \hat{P} \) and \( \hat{X} \) [22–25]). Consequently, any initial Gaussian state (whose special case is the thermal state) remains Gaussian in the process of evolution. For thermal states, mean values of the first-order moments are equal to zero, and this property is preserved in the process of evolution governed by equations (2) and (3). Then all statistical properties of the single mode are completely determined by the variances of the quadrature operators \( \sigma_{xx} \) and \( \sigma_{pp} \) and by their covariance \( \sigma_{xp} = \sigma_{px} \) (in the case involved, \( \sigma_{hh} = \frac{1}{6} \langle \hat{a} \hat{b} + \hat{b} \hat{a} \rangle \)). Using equations (9) and (10) one can verify that the time-dependent (co)variances can be obtained from formulae (17)–(19) by means of the replacement \( J \rightarrow J + \frac{1}{4} G_0 \exp(-2\tau) \).

The PDF \( f(m) \equiv |m| \rho |m| \) of the Gaussian states was found long ago [26–31]. For zero mean values \( \langle \hat{x} \rangle = \langle \hat{p} \rangle = 0 \), it can be expressed in terms of the Legendre polynomials
\[
f(m) = \frac{2D_{n+1/2}^{1/2}}{D_{n+1/2}} P_n \left( \frac{4\Delta - 1}{\sqrt{4\Delta - 1}} \right),
\]
where
\[
D_n = 1 + 4\Delta \pm 2r,
\]
\[
\tau = \sigma_{xx} + \sigma_{pp} \equiv 1 + 2\sqrt{\lambda},
\]
\[
\Delta = \sigma_{xx} \sigma_{pp} - \sigma_{xx}^2 \sigma_{pp}^2 \geq 1/4
\]
(last inequality is the Schrödinger–Robertson uncertainty relation). For the Gaussian quantum states, one can write \( \Delta = 1/(4\mu^2) \), where the quantity \( \mu \) is the quantum purity of the state: \( \mu = \text{Tr}(\hat{\sigma}^2) \). For the pure Gaussian quantum states (in the absence of dissipation) \( \Delta \equiv 1/4 \), and formula (25) goes directly to (1).

The explicit expression for the time-dependent coefficient \( \Delta \) is as follows:
\[
\Delta = \left( J + \frac{G_0}{2} e^{-2\tau} \right)^2 - |\hat{J}|^2.
\]

If the functions \( \omega(t) = \omega_0 [1 + \chi(t)] \) and \( \gamma(t) \) have the form of periodic pulses separated by intervals of time with \( \omega = \omega_0 = \text{const} \) and \( \gamma = 0 \) (this means that the quality factor of the cavity is supposed to be high enough), then the following formulae can be obtained for the quantities \( E_n = E(\text{nt}), \tilde{E}_n = \tilde{E}(\text{nt}), J_n = J(\text{nt}) \) and \( J_n = J(\text{nt}) \) after \( n \) periods (\( n \) pulses of laser irradiation in the case of DCE) under the realistic conditions \( |\chi(t)| \ll 1 \) and \( \gamma(t) \ll 1 \) [2, 21]:
\[
E_n = \cosh(2\nu n), \quad \tilde{E}_n = \sinh(2\nu n) e^{i\delta},
\]
\[
J_n = A_n^{(+)} + A_n^{(-)}, \quad \tilde{J}_n = e^{i\delta} \left( A_n^{(+)} - A_n^{(-)} \right),
\]
\[
A_n^{(\pm)} = \frac{G A}{4(\Lambda \pm i)} \left( e^{2\nu n} - e^{-2\nu n} \right),
\]
\[
\Lambda = \frac{\chi}{\nu} + \frac{\nu}{\chi} + 2\nu.
\]
where $\beta$ is some insignificant constant phase.

$$v = \int_{t_1}^{t_f} \omega_0 \chi(t) e^{-2i\omega_0 t} dt, \quad \Lambda = \int_{t_1}^{t_f} \gamma(\tau) d\tau.$$  \hspace{1cm} (33)

Here $t_1$ and $t_f$ are the initial and final moments of each pulse.

It is taken into account that $\Lambda, v \ll 1$. The mean number of photons grows exponentially under the conditions $2n v > 1$ and $v > \Lambda$:

$$N_n = \frac{1}{4} e^{2(n-\Lambda\Lambda)} \left( G_0 + \frac{G \Lambda}{v - \Lambda} \right) + O(1).$$  \hspace{1cm} (34)

The coefficient $G_0$ is given by formula (7), but with $\Theta$ replaced by the initial temperature of the field mode $\Theta_0$ (which can be made different from $\Theta$). Under the same conditions, formula (29) takes the form

$$\Delta_n = N_n \frac{G \Lambda}{v - \Lambda} + O(1).$$  \hspace{1cm} (35)

Note that the ratio $\Delta_n/N_n$ in this limit does not depend on the coefficient $G_0$ (the initial temperature of the field mode).

Formulas (30)–(35) hold provided the periodicity of pulse $T$ is adjusted to the initial period of oscillations of the field mode $T_0 = 2\pi/\omega_0$ as follows ($m = 1, 2, \ldots$):

$$T = \frac{1}{2} T_0 (m + \varphi/\pi), \quad \varphi = -\omega_0 \int_{t_1}^{t_f} \chi(t) dt.$$  \hspace{1cm} (36)

A small shift $\varphi$ of the resonance periodicity of pulses arises if the profile of pulses is asymmetrical (namely this situation takes place in reality).

3. Asymptotical formulae for photon statistics and squeezing

Formula (25) is exact. However, it is not very convenient for calculations in the case of the DCE if the number of created photons is big (say, $m \sim N > 1000$; otherwise the effect cannot be confirmed with certainty at the existing experimental level due to the noise in the measurement channel). Therefore asymptotical forms of exact formulae for $m \gg 1$ can be more useful. Note that the argument of the Legendre polynomial in (25) is always outside the interval $(-1, 1)$, being equal to unity only for thermal states with $\tau = 2\sqrt{\Delta}$. For this reason, it is convenient to use the asymptotical formula [32]

$$P_m(\cosh \xi) \approx \left( \frac{\xi}{\sinh \xi} \right)^{1/2} I_0 \left( \frac{m + 1/2}{\tau} \xi \right),$$  \hspace{1cm} (37)

where $I_0(\xi)$ is the modified Bessel function. Formula (25) shows that the behavior of the PDF depends on the sign of the coefficient $D_-$, i.e. on the ratio $2\Delta/\tau \approx \Delta_n/N_n$. If this ratio exceeds the unit value, then the argument of the Legendre polynomial is real (and bigger than unity). In this case, using the known asymptotical formula $I_0(x) \approx (2\pi x)^{-1/2} \exp(x)$ (if $x \gg 1$) and making some further simplifications, one can arrive (under the conditions $N \gg 1$ and $m \gg 1$) at the simple formula [3, 21]

$$f(m) \approx \frac{\exp[-(m + 1/2)/(2N)]}{\sqrt{2\pi N (m + 1/2)}}.$$  \hspace{1cm} (38)

Here, I consider in detail the case when $D_- < 0$ or $\Delta_n/N_n < 1$.

Then formula (37) is still valid, but it needs some transformations because the parameter $\xi$ becomes complex. It is clear from formula (25) that the real positive value $f(m)$ does not depend on the choice of sign of the square root function $\sqrt{D_-} = \pm i \sqrt{|D_-|}$, provided this sign is maintained, both inside the argument of the Legendre polynomial and in the coefficient in front of this polynomial. Choosing for definiteness the branch $\sqrt{D_-} = i \sqrt{|D_-|}$, it is convenient to write

$$\xi = \tilde{\xi} - i\pi/2 = -i\eta, \quad \sinh(\tilde{\xi}) = \frac{4\Delta - 1}{\sqrt{D_- |D_-|}},$$

so that $\tilde{\xi}$ is a real positive number. Using the relation $I_0(-i\eta) = J_0(\eta)$ in (37), one can represent (25) as

$$f(m) \approx \frac{\left( \frac{2\eta}{\tau} \right)^{1/2}}{\sqrt{\pi (m + 1/2)^2}} \frac{|D_-|}{D_+} \left( \frac{m + 1/2}{\tau} \right) J_0 \left( \frac{m + 1/2}{\tau} \right).$$

Replacing the cosine function by the sum of two exponentials, one obtains

$$f(m) \approx \frac{1}{\sqrt{\pi (m + 1/2)^2}} \left[ \frac{2r + \delta}{D_+} \right]^{m+1/2} + (-1)^m \left[ \frac{|D_-|}{2r + \delta} \right]^{m+1/2}.$$  \hspace{1cm} (39)

where $\delta = 4\Delta - 1$. The right-hand side of (39) is real and positive. Using the relation $2\Delta = \pi r + c$ for $\tau \gg 1$ (with $b = G \Lambda / (v + \Lambda)$ according to equation (35)), one can verify the relationship

$$\frac{2r + \delta}{D_+} = 1 - \frac{1}{\tau} + O \left( \tau^{-2} \right),$$  \hspace{1cm} (40)

$$\frac{|D_-|}{2r + \delta} = 1 - \frac{b^2 + 2c}{1 \tau} + \tau^{-1} + O \left( \tau^{-2} \right).$$  \hspace{1cm} (41)

Note that the coefficient at $\tau^{-1}$ in (40) does not contain the coefficients $b$ and $c$. Obviously, the expansion (41) is valid provided $1 - b$ is not too small. Formulae (39) and (41) clearly show that oscillations of the PDF are actually negligible (especially for $m \sim N$), unless $b \ll 1$ (i.e. in the case of extremely small dissipation). In the latter case, the coefficient $c$ can be expressed in terms of the initial purity, which, in turn, is related to the factor $G_0$ as $\mu = 1/G_0$, so that $c = G_0^2/2$. Using the approximate formula $1 - x \approx \exp(-mx)$, which holds for $x \ll 1$ and $m x^2 \ll 1$, one can rewrite (39) in the case of small dissipation as (replacing $r$ by $\tau$)

$$f(m) \approx \frac{\tau (m + 1/2)^{1/2}}{\sqrt{2\pi N (m + 1/2)}} \left[ \exp[-(m + 1/2)/\tau] + (-1)^m \exp[-(m + 1/2)G_0^2/\tau] \right].$$  \hspace{1cm} (42)
Formula (42) is valid under the conditions $\tau \approx 2N \gg 1$, $1 \ll m \ll \tau^2$ and $G \Lambda/\nu \ll 1$. Even in this case the oscillations of the PDF can be noticed only if $G_0 \sim 1$, i.e., for low-temperature initial thermal states. In particular, equation (42) coincides with the asymptomatic form of the ideal PDF of the vacuum squeezed state (1) if $G_0 = 1$ and $\tau \approx 2\langle n \rangle \gg 1$. In the conditions of the MIR experiment the ratio $\Lambda/\nu$ exceeds $1/2$ [3], which means that the oscillating term in (39) can be neglected even for $G = 1$ (zero temperature of the cavity walls), so that formula (38) can be used for any values of parameters $G_0$ and $G$ (under the conditions $N \gg 1$ and $m \gg 1$).

Using the Euler–MacLaurin summation formula, one can verify that the distribution function (38) has the correct normalization with an accuracy $O(\tau^{-1/2})$:

$$
\sum_{m=0}^{\infty} f(m) \approx \int_{0}^{\infty} f(m) dm + O[f(0)]
$$

$$
\approx \int_{0}^{\infty} \exp(-x/\tau) \frac{dx}{\sqrt{\pi \tau x}} + O(\tau^{-1/2}) = 1 + O(\tau^{-1/2}).
$$

The oscillating terms in (39) or (42) do not influence normalization, because they give corrections of a higher order. For example, combining the nearest positive and negative terms with $m = 2k$ and $m = 2k + 1$ and applying the Euler–MacLaurin summation (which contains now only slowly varying positive terms) formula to the sum over these positive pairs, one can see that the correction equals approximately $G_0/(2\tau)$.

With the same accuracy as in (44), the moments of the distribution function can be calculated as

$$
\langle m^k \rangle \approx \frac{1}{m^k} \int_{0}^{\infty} x^k \exp(-x/\tau) \frac{dx}{\sqrt{\pi \tau x}}
$$

$$
= \tau^{k} \frac{(2k-1)!!}{2^k} \approx N^k (2k-1)!!.
$$

For $k = 2$, formula (46) leads to the following formula for the variance of the number of created quanta: $\sigma_N = (m^2) - \langle m \rangle^2 \approx 2N^2$. It means that the field mode goes asymptotically to the so-called ‘superchaotic’ [33, 34] quantum state, whose statistics is essentially different from the statistics of the initial thermal state, characterized by the formula $\sigma_N = \sqrt{N(N+1)} \approx N^{3/2}$. The same result can be obtained from the general formula for the variance of the number of quanta in the Gaussian quantum states [30] $\sigma_N = \tau^2/2 - \Delta - 1/4$ (if $\langle \hat{x} \rangle = \langle \hat{p} \rangle = 0$).

It is interesting that the asymptotic value of the ratio $2\Delta/\tau = b$ in the case concerned coincides with the invariant squeezing coefficient. This coefficient can be introduced in the following way. Obviously, instantaneous values of variances $\sigma_{xx}, \sigma_{pp}$ and $\sigma_{xp}$ cannot serve as true measures of squeezing, since they depend on time during the course of free evolution of the oscillator. For example (in dimensionless units with $\omega_0 = 1$),

$$
\sigma_{xx}(t) = \sigma_{xx}^{(0)} \cos^2(t) + \sigma_{pp}^{(0)} \sin^2(t) + \sigma_{xp}^{(0)} \sin(2t),
$$

and it can happen that both variances $\sigma_{xx}$ and $\sigma_{pp}$ are large, but nonetheless the state is highly squeezed due to the large nonzero covariance $\sigma_{xp}$. It is reasonable to introduce some invariant characteristics that do not depend on time in the course of free evolution (or on phase angle in the definition of the field quadrature as $\hat{E}(\phi) = \hat{a} \exp(-i\phi) + \hat{a}^\dagger \exp(i\phi) / \sqrt{2}$). I define the invariant squeezing coefficient $S$ as the ratio of the minimal value of the variance $\sigma_{xx}(t)$ (as a function of time (47)) to the dimensionless variance $1/2$ in the vacuum state. Then straightforward calculations give the formula (similar results can be found in [31, 35–37])

$$
S = \frac{4\Delta}{\tau + \tau^2 - 4\Delta}.
$$

4. Conclusion
The main results of the paper are as follows. New formulae (39) and (42) show how the increase of dissipation and temperature transforms the strongly oscillating PDF (1) of the ideal squeezed vacuum state to the smooth asymptotical distribution (38). Namely this smooth distribution is expected to be observed in the MIR experiment due to strong dissipation. Another new result is the explicit demonstration of the correlation between the existence of squeezing (when the invariant squeezing coefficient $S$ is less than unity) and the oscillations of the PDF in the case concerned. In particular, no squeezing of the fundamental field mode is expected in the MIR experiment.

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