GALOIS GROUPS OF $p$-EXTENSIONS OF HIGHER LOCAL FIELDS MODULO $p$-TH COMMUTATORS

VICTOR ABRASHKIN

Abstract. Suppose $K$ is $N$-dimensional local field of characteristic $p$, $\mathcal{G} = \text{Gal}(K_{\text{sep}}/K)$, $\mathcal{G}_{<p}$ is the maximal quotient of $\mathcal{G}$ of period $p$ and nilpotent class $< p$ and $K_{<p} \subset K_{\text{sep}}$ is such that $\text{Gal}(K_{<p}/K) = \mathcal{G}_{<p}$. We use nilpotent Artin-Schreier theory to identify $\mathcal{G}_{<p}$ with the group $G(L)$ obtained from a profinite Lie $\mathbb{F}_p$-algebra $L$ via the Campbell-Hausdorff composition law. The canonical $\mathcal{P}$-topology on $K$ is used to define a dense Lie subalgebra $L^\mathcal{P}$ in $L$. The algebra $L^\mathcal{P}$ can be provided with a system of $\mathcal{P}$-topological generators and its $\mathcal{P}$-open subalgebras correspond to all $N$-dimensional extensions of $K$ in $K_{<p}$. These results are applied to higher local fields $K$ of characteristic 0 containing primitive $p$-th root of unity. If $\Gamma = \text{Gal}(K_{\text{alg}}/K)$ we introduce similarly the quotient $\Gamma_{<p} = G(L)$, a dense $\mathbb{F}_p$-Lie algebra $L^\mathcal{P} \subset L$, and describe the structure of $L^\mathcal{P}$ in terms of generators and relations. The general result is illustrated by explicit presentation of $\Gamma_{<p}$ modulo third commutators.

Introduction

The concept of local field $K$ of dimension $N \geq 1$ was introduced as an essential ingredient of the theory of higher adeles in the study of arithmetic of algebraic varieties. One of first immense applications of this concept was the construction of higher dimensional generalization of local class field theory, cf. [16, 17, 18] and [22, 23, 24]. Higher local fields appear as a generalization of classical complete discrete valuation fields (finite extensions of $\mathbb{Q}_p$ and the fields of Laurent series with coefficients in finite fields). They can be introduced by induction: $N$-dimensional local field is a complete discrete valuation field with $(N-1)$-dimensional residue field (this residue field is called the first residue field $K^{(1)}$ of $K$). Similarly, we obtain next residue fields, the last (or $N$-th) residue field is necessarily finite and will be always denoted by $k \simeq \mathbb{F}_{p^{N_0}}$.

Basic definitions including highly important concept of special topology ($\mathcal{P}$-topology) for these fields are briefly explained in Sect. 2. Notice that the classification results and explicit aspects of class field theory can be found in [20, 30, 15, 27, 5].

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If a prime number \( p \) is the characteristic of the first residue field of \( K \) then the Galois group \( \Gamma(p) \) of the maximal \( p \)-extension of \( K \) contains the most important information about the whole Galois group \( \Gamma \) of \( K \). In the case of 1-dimensional fields the structure of \( \Gamma(p) \) is very well-known: if \( \text{char} \ K = p \) or \( \text{char} \ K = 0 \) but \( K \) contains no non-trivial \( p \)-th roots of unity the group \( \Gamma(p) \) is profinite free (I. Shafarevich), if \( K \) contains non-trivial \( p \)-th roots of unity then \( \Gamma(p) \) has a minimal system of generators containing \([K : \mathbb{Q}_p] + 2 \) elements and one (explicitly known) relation (S. Demushkin), cf. also [14]. As for the higher dimensional case, notice the paper [28] where the case of 2-dimensional fields with the first residue field of characteristic 0 was considered.

There is a standard way to use class field theory to study the structure of \( \Gamma \) modulo the third term of the \( p \)-central series of \( \Gamma(p) \), cf. [26] and various results about 1-dimensional local and global fields in [19] (an attempt to apply this method for higher local fields was also done in [9]). We can describe this strategy in terms of the central series of \( \Gamma \) as follows.

For \( s \geq 1 \), let \( C_s(\Gamma) \) be the closure of the subgroup of commutators of order \( \geq s \) in \( \Gamma \). Then class field theory used just to specify the elements of \( \Gamma \) modulo \( \Gamma^pC_2(\Gamma) \) and to study their action (by conjugation) on \( \Gamma^pC_2(\Gamma) \) mod \( \Gamma^pC_3(\Gamma) \) via explicit formulas for Hilbert symbol. The possibility of generalizing this method are quite limited:

a) The knowledge of action of elements of \( \Gamma^{ab} \) on \( C_2(\Gamma) \) mod \( C_3(\Gamma) \) is not sufficient to study \( \Gamma \) modulo \( \Gamma^MC_4(\Gamma) \), where, say, \( M = 1, s \geq 4 \) or \( M \geq 2, s \geq 3 \).

b) The most significant achievement of class field theory is its functoriality; this property plays no role under this approach.

c) Class field theory leads us from explicit constructions of abelian extensions (via Kummer or Artin-Schreier methods) to a functorial and invariant viewing of their theory. At the same time the explicit reciprocity laws (another great and non-trivial topic) bring us back to the original naive viewing of such field extensions. For this reason it is natural to expect more direct approach to the original problem.

In [1, 2] we initiated the study of \( \Gamma(p) \) modulo \( p \)-th commutators without any reference to class field theory. It was based on the combination of new techniques: nilpotent version of the Artin-Schreier theory developed by the author, and the Fontaine-Wintenberger theory of the field-of-norms functor. In the case of 1-dimensional local fields our method has been already applied in [6, 7, 8] to the study of the group \( \Gamma(p) \) modulo \( C_p(\Gamma) \) together with the induced ramification filtration in terms of generators and relations.

However, in the case of local fields of dimension \( N > 1 \) there is another serious obstacle: it is not possible to apply directly the techniques of generators and relations to study the group \( \Gamma(p) \). This could be
explained in the following way. For 1-dimensional $K = \mathbb{F}_p((t))$, any extension of $K$ of degree $p$ is contained in the composite of Artin-Schreier extensions $K(T_a)$, $T_a^p - T_a = t^{-a}$, where $a \in \{n \mid \gcd(a, p) = 1\} \cup \{0\}$. As a result, $\Gamma(p)$ is generated by the elements $\tau_a \in \Gamma(p)$ such that $\tau_a(T_a) = T_b + \delta_{ab}$. On the other hand, if $K = \mathbb{F}_p((t_2))((t_1))$ is 2-dimensional the situation is completely different. For example, the extension $K(T)$ such that $T^p - T = t^{-1}(1 + t_2 + \cdots + t_2^n + \ldots)$ is not contained in the compositum of all $K(T_n)$, where $T_0^p - T_0 = t_1^{-1}t_2^n, n \geq 0$.

As a result, the lifts of elements of the Galois groups of elementary $\mathbb{F}_p$-very small piece of $\Gamma$ contained in the compositum of all $N$ theory, where the abelian extensions of $K$ are described via quotients of the $K$-group $K_N(K)$. This group has no natural system of generators, but it contains a dense subgroup $K_N^{\text{top}}(K)$. This subgroup can be described via generators and can be taken instead of $K_N(K)$ when studying finite abelian extensions of $K$.

We describe briefly the content of the paper as follows.

a) For an $N$-dimensional local field $K$ of characteristic $p$ we apply nilpotent Artin-Schreier theory to fix an identification $\pi : G_{<p} \simeq G(L)$. Here $G_{<p}$ is the maximal quotient of $G = \text{Gal}(K_{\text{sep}}/K)$ of period $p$ and nilpotent class $< p$, $L$ is a profinite Lie $\mathbb{F}_p$-algebra and $G(L)$ is the profinite $p$-group obtained from the elements of $L$ via the Campbell-Hausdorff composition law. The identification $\pi$ is defined uniquely up to conjugation if we choose suitable element $e \in L \otimes \mathbb{K}$.

b) We use the $P$-topology on $K$ to define the Lie subalgebra $L^P$ in $L$. This algebra has $P$-topological structure and the appropriate $P$-topological system of generators. The algebra $L^P$ is dense in $L$, i.e. the profinite completion of $L^P$ coincides with $L$.

c) With respect to different identifications of nilpotent Artin-Schreier theory $\pi : G_{<p} \simeq G(L)$ the algebra $L^P$ gives rise to a class of conjugated subgroups $G^P_{<p} := \pi^{-1}(L^P) \in \text{cl}^P G_{<p}$; the profinite completion of $G^P_{<p}$ coincides with $G_{<p}$.

d) The subgroups $G^P_{<p}$ have $P$-topological systems of generators and could be used to study $N$-dimensional local field extensions $K'$ of $K$ in $K_{<p}$. More precisely, $H$ is an open subgroup in $G$ iff $H^P = G^P_{<p} \cap H$ is a $P$-open subgroup of finite index in $G^P_{<p}$. We have also $(G_{<p} : H) = (G^P_{<p} : H^P) = [K' : K]$, where $K' = K_{<p}^H$. In particular, $K'/K$ is Galois iff $H^P$ is normal in $G^P_{<p}$; in this case $\text{Gal}(K'/K) = G^P_{<p}/H^P$.

e) Suppose $t = (t_1, \ldots, t_N)$ is a system of local parameters in $K$, $m_K$ is the maximal ideal in the $N$-valuation ring $O_K$ of $K$, $\omega \in m_K$, and for $1 \leq m \leq N$, $h^{(m)}_\omega \in \text{Aut} K$ are such that $h^{(m)}_\omega(t_i) = t_i E(\omega^p)^{\delta_{m,i}}$, where $E(X)$ is the Artin-Hasse exponential. Then all lifts of $h^{(m)}_\omega$, $1 \leq m \leq N$, to $K_{<p}$ form a subgroup $G_\omega \subset \text{Aut} K_{<p}$ containing $G_{<p}$. 
Let \( \Gamma_\omega \) be the maximal quotient of \( G_\omega \) of period \( p \) and nilpotent class \(< p \). If \( \bar{G} \) is the image of \( G_{<p} \) in \( \Gamma_\omega \) we obtain the following short exact sequence of profinite \( p \)-groups

\[
1 \rightarrow \bar{G} \rightarrow \Gamma_\omega \rightarrow \langle h_\omega(1)^\mathbb{Z}/p \times \cdots \times h_\omega(N)^\mathbb{Z}/p \rightarrow 1, \]

the corresponding exact sequence of Lie \( \mathbb{F}_p \)-algebras (here \( \Gamma_\omega = G(L_\omega) \))

\[
0 \rightarrow \bar{L} \rightarrow L_\omega \rightarrow \prod_{1 \leq m \leq N} \mathbb{F}_p h_\omega(m) \rightarrow 0,
\]

and define the appropriate dense subalgebra \( L_\omega^p \) such that

\[
0 \rightarrow \bar{L}^p \rightarrow L_\omega^p \rightarrow \prod_{1 \leq m \leq N} \mathbb{F}_p h_\omega(m) \rightarrow 0.
\]

f) We apply methods from \([6, 7]\) to describe the structure of the Lie algebras \( \bar{L}^p \otimes k \) and \( L_\omega^p \otimes k \). In particular, we obtain recurrent procedure to recover the operators \( ad \bar{L}^p \), where \( \bar{L}^p \) are lifts of \( L_\omega^p \), and find explicit formula for \([\bar{L}^p(m), \bar{L}^p(n)] \in \bar{L} \). These results are illustrated via explicit description of the structure of the Lie algebra \( L_\omega^p \) modulo third commutators.

g) We apply the results from f) to the explicit description of \( \Gamma_{<p} = \Gamma/pC_p(\Gamma) \), where \( \Gamma \) is the Galois group of \( N \)-dimensional local field \( K \) containing non-trivial \( p \)-th root of unity \( \zeta_1 \). More precisely, we introduce a canonical class of conjugated dense subgroups \( \Gamma_{<p}^p \) in \( \Gamma_{<p} \) with \( \mathcal{P} \)-topological systems of generators. Then we apply Scholl’s construction of the field-of-norms functor to identify \( \Gamma_{<p}^p \) with \( \Gamma_{<p}^\omega \), where the \( \omega \in m_K \) is defined in terms related to the \( p \)-th root of unity \( \zeta_1 \). This result is illustrated in the case where \( K = \mathbb{Q}_p(\zeta_1)\{\{x\}\} \).

**Remark.**

a) The structure of \( \Gamma_{<p} \) depends only on a special power series constructed from \( \zeta_1 \in K \); this series appears in the \( p \)-adic Hodge theory as the period of \( G_m \).

b) There is still an open question in the description of \( \Gamma_{<p} \). We have not yet found explicitly the commutators \([\bar{L}^p(m), \bar{L}^p(n)] \). There is a strong evidence that there are lifts \( \bar{L}^p(m) \) which commute one with another: we verified this fact modulo \( C_4(\Gamma_\omega) \) by direct computation. The existence of commuting elements in sufficiently large Galois groups may have some relation to anabelian geometry, cf. \([11]\).

**Notation.** Let \( G \) be a topological group. For \( s \geq 1 \), denote by \( C_s(G) \) the closure of its subgroup of \( s \)-th commutators. Here \( C_1(G) = G \) and for \( s \geq 2 \), \( C_s(G) \) is the closure of the commutator subgroup \( (G, C_{s-1}(G)) \). Similarly, if \( L \) is a (topological) Lie algebra over some ring \( R \) then \( C_s(L) \) is the closure of its \( R \)-submodule of commutators of order \( \geq s \). If \( M \) and \( S \) are \( R \)-modules we denote very often by \( MS \) the extension of scalars \( M \otimes_R S \).
1. CONSTRUCTIONS OF NILPOTENT ARTIN-SCHREIER THEORY

In this section we review basic results of nilpotent Artin-Shreier theory, cf. [1, 2]. This theory allows us to work with $p$-extensions of fields of characteristic $p$ with Galois groups of nilpotent class $< p$. In these notes we use the simplest case of the theory involving Galois groups of period $p$. In other words, if char $K = p$ and $\Gamma = \text{Gal}(K_{\text{sep}}/K)$ our approach allows us to work efficiently with subfields of $K_{<p} := K_{\text{sep}}^{\Gamma_{<p}}$, where $\Gamma_{<p} = \Gamma/\Gamma^p C_p(\Gamma)$.

1.1. Groups and Lie algebras of nilpotent class $< p$. The basic ingredient of the nilpotent Artin-Shreier theory is the equivalence of the category of $p$-groups of nilpotent class $s_0 < p$ and the category of Lie $\mathbb{Z}/p$-algebras of the same nilpotent class. In the case of objects killed by $p$ this equivalence can be explained as follows.

Let $L$ be a Lie $\mathbb{F}_p$-algebra of nilpotent class $< p$, i.e. $C_p(L) = 0$.

Let $\mathfrak{A}$ be an enveloping algebra of $L$. Then there is a natural embedding $L \subset \mathfrak{A}$, the elements of $L$ generate the augmentation ideal $J$ of $\mathfrak{A}$ and we have a morphism of algebras $\Delta : \mathfrak{A} \to \mathfrak{A} \otimes \mathfrak{A}$ uniquely determined by the conditions $\Delta(l) = l \otimes 1 + 1 \otimes l$ for all $l \in L$. The Poincare-Birkhoff-Witt Theorem then implies:

- $L \cap J^p = 0$;
- $L \mod J^p = \{a \mod J^p \mid \Delta(a) \equiv a \otimes 1 + 1 \otimes a \mod (J \otimes 1 + 1 \otimes J)^p\}$;
- the set $\widehat{\exp}(L) \mod J^p$ is identified with the set of all ”diagonal elements“ mod deg $p$, i.e. with the set of all $a \in 1 + J \mod J^p$ such that $\Delta(a) \equiv a \otimes a \mod (J \otimes 1 + 1 \otimes J)^p$ (here $\widehat{\exp}(x) = \sum_{0 \leq i < p} x^i/i!$ is the truncated exponential).

In particular, there is a natural embedding $L \subset \mathfrak{A} \mod J^p$ and in terms of this embedding the Campbell-Hausdorff formula appears as

$$(l_1, l_2) \mapsto l_1 \circ l_2 = l_1 + l_2 + \frac{1}{2}[l_1, l_2] + \ldots, \ l_1, l_2 \in L,$$

where $\widehat{\exp}(l_1) \widehat{\exp}(l_2) \equiv \widehat{\exp}(l_1 \circ l_2) \mod J^p$. This composition law provides the set $L$ with a group structure and we denote this group by $G(L)$. The group $G(L)$ has period $p$ and nilpotent class $< p$. The correspondence $L \mapsto G(L)$ induces the equivalence of the category of $p$-groups of period $p$ and nilpotent class $s < p$ and the category of Lie $\mathbb{Z}/p$-algebras of the same nilpotent class $s$. This equivalence is naturally extended to the similar categories of pro-finite Lie algebras and pro-finite $p$-groups.

1.2. Nilpotent Artin-Schreier theory. Let $L$ be a finite Lie $\mathbb{F}_p$-algebra of nilpotent class $< p$. Consider the extensions of scalars $L_K$ and $L_{\text{sep}} := L_{K_{\text{sep}}}$. Then the elements of $\Gamma = \text{Gal}(K_{\text{sep}}/K)$ and the Frobenius $\sigma$ act on $L_{\text{sep}}$ through the second factor, $L_{\text{sep}}|_{\sigma=\text{id}} = L$ and
is not empty and for any fixed \( f \in \mathcal{F}(e) \), the map \( \tau \mapsto (-f) \circ \tau(f) \) is a continuous group homomorphism \( \pi_f(e) : \Gamma \to G(L) \). The correspondence \( e \mapsto \pi_f(e) \) has the following properties:

a) if \( f' \in \mathcal{F}(e) \) then \( f' = f \circ l \), where \( l \in G(L) \); in particular, \( \pi_f(e) \) and \( \pi_f(e) \) are conjugated via \( l \);

b) for any continuous group homomorphism \( \pi : \Gamma \to G(L) \), there are \( e \in G(L_K) \) and \( f \in \mathcal{F}(e) \) such that \( \pi_f(e) = \pi \);

c) for appropriate elements \( e, e' \in G(L_K) \), \( f \in \mathcal{F}(e) \) and \( f' \in \mathcal{F}(e') \), we have \( \pi_f(e) = \pi_{f'}(e') \) iff there is an \( x \in G(L_K) \) such that \( f' = x \circ f \) and, therefore, \( e' = \sigma(x) \circ e \circ (-x) \).

In the case of a profinite Lie algebra \( L = \varprojlim L_\alpha \), where all \( L_\alpha \) are finite Lie \( F_p \)-algebras, consider \( e = \varprojlim e_\alpha \in L_K \), where all \( e_\alpha \in L_\alpha K \).

Then there is \( f = \lim f_\alpha \in \varprojlim \mathcal{F}(e_\alpha) \subset L_{\text{sep}} \) (where all \( f_\alpha \in \mathcal{F}(e_\alpha) \)) and \( \pi_f(e) = \varprojlim \pi_{f_\alpha}(e_\alpha) \) maps \( \Gamma \) to \( G(L) = \varprojlim G(L_\alpha) \).

1.3. The diagonal element and abelian Artin-Schreier theory. Let \( \bar{K} = K/(\sigma - \text{id})K \) and \( M = \text{Hom}_{F_p\text{-lin}}(\bar{K}, F_p) \). If \( \bar{K} \) is provided with discrete topology (as an inductive limit of finite dimensional \( F_p \)-subspaces), its dual \( M \) has the pro-finite topology and

\[
M_K = \text{Hom}_{F_p\text{-lin}}(\bar{K}, \bar{K})
\]

Let \( \Pi : \Gamma \to \bar{K} \) be the natural projection and \( e \in M_K = \text{Hom}(\bar{K}, K) \) be such that \((\text{id}_M \otimes \Pi)e = \text{id}_{\bar{K}} \). Equivalently, let \( S \) be a section of \( \Pi \) and \( e := e_S = (\text{id}_M \otimes S)\text{id}_{\bar{K}} \).

In notation of Sect. [1.2] the identification of the abelian Artin-Schreier theory \( \pi^{ab} : \Gamma_{\leq 2} := \Gamma/\Gamma^p C_2(\Gamma) \cong M \) can be obtained as follows:

— choose \( f \in M_{\text{sep}} := M_{K_{\text{sep}}} \) such that \( \sigma f - f = e_S \) and for any \( \tau \in \Gamma_{\leq 2} \), let \( \tau f - f = \pi^{ab}(\tau) \in M_{\text{sep}}|_{\sigma = \text{id}} = M \).

Remark. In the above formulas (and in similar situations below if there is no risk of confusion) we use the simpler notation \( \sigma \) and \( \tau \) instead of \( \text{id}_M \otimes \sigma \) and \( \text{id}_{M_{\text{sep}}} \otimes \tau \).

The map \( \pi^{ab} \) does not depend on a choice of \( f \):

— if \( f_1 \in M_{\text{sep}} \) is such that \( \sigma f_1 - f_1 = e_S \) then \( f_1 - f \in M_{\text{sep}}|_{\sigma = \text{id}} = M \) and \( \tau f_1 - f_1 = \tau f - f \).

The map \( \pi^{ab} \) also does not depend on a choice of \( S \):
— if $S'$ is another section then there is $g \in M_K$ such that $e_{S'} - e_S = \sigma g - g$. Therefore, $f' := f + g$ satisfies the relation $\sigma f' - f' = e_{S'}$ and, $\tau f' - f' = \tau f - f$.

1.4. Identifications $\pi_f(e) : \Gamma_{<p} \simeq G(L)$.

Let $\tilde{L}$ be a free Lie $\mathbb{F}_p$-algebra with generating module $M$ and $L = \tilde{L}/C_p(\tilde{L})$. (Note that $M$ is a profinite limit of its finite quotients and $\tilde{L}$ is the corresponding profinite limit of finite Lie algebras.) Consider the natural projection

$$\text{pr} \otimes \Pi : L \otimes_{\mathbb{F}_p} K \longrightarrow L/C_2(L) \otimes_{\mathbb{F}_p} K = M_K$$

and set $\mathcal{E}(L_K) = \{ e \in L_K \mid (\text{pr} \otimes \Pi)e = \text{id}_K \}$. Agree to denote the image of $e$ in $M_K$ by $e_S$, where $S$ is the appropriate section of $\Pi$, cf. Sect. [1,3].

Choose $f \in \mathcal{F}(e)$ and consider the group homomorphism $\pi_f(e) : \Gamma_{<p} \longrightarrow G(L)$ such that for any $\tau \in \Gamma_{<p}$, $\pi_f(e)(\tau) = (-f) \circ \tau(f)$. Then $\pi_f(e)$ is a group isomorphism (use that $\Gamma$ is a free pro-$p$-group and $\pi_f(e) \mod p\mathcal{C}_2(\Gamma)$ is isomorphism by Sect. [1,3]).

If $f'$ is another element from $\mathcal{F}(e)$ then there is $l \in G(L)$ such that $f' = f \circ l$ and $\pi_f(e)(\tau) = (-f') \circ \tau(f') = (-l) \circ \pi_f(e)(\tau) \circ l$ is conjugated to $\pi_f(e)$. Study how $\pi_f(e)$ depends on a choice of $e \in \mathcal{E}(L_K)$.

**Proposition 1.1.** If $e, e' \in \mathcal{E}(L_K)$ then there is $x \in L_K$ and a section $A$ of the natural projection $\text{pr} : L \longrightarrow L/C_2(L) = M$ such that

$$e' = \sigma(x) \circ (A \otimes \text{id}_K)e \circ (-x),$$

where $A \in \text{Aut}_L eL$ is a unique extension of $A$.

**Proof.** Let $\{ l_\alpha \mid \alpha \in \mathcal{I} \}$ be an $\mathbb{F}_p$-basis of $K$. Let $\hat{l}_\alpha, \alpha \in \mathcal{I}$, be the dual (topological) basis for $M$, i.e. for any $\alpha_1, \alpha_2 \in \mathcal{I}$, $\hat{l}_\alpha(l_{\alpha_2}) = \delta_{\alpha_1\alpha_2}$.

Then we have the sections $S$ and $S'$ of $\Pi$ such that $e_S = \sum_\alpha \hat{l}_\alpha \otimes S(l_\alpha)$ and $e_{S'} = \sum_\alpha \hat{l}_\alpha \otimes S'(l_\alpha)$.

Apply induction on $r \geq 1$ to prove the existence of $x_r \in L_K$ and a section $A_r$ of the projection $L \longrightarrow M$ such that

$$e' \equiv \sigma(x_r) \circ (A_r \otimes \text{id}_K)e \circ (-x_r) \mod C_{r+1}(L_K),$$

where $A_r \in \text{Aut}_L eL$ is such that $A_r|_M = A_r$.

If $r = 1$ take $A_1 = \text{id}_M$ and $x_1 = \sum_\alpha \hat{l}_\alpha \otimes x_{1\alpha}$, where all $x_{1\alpha} \in K$ are such that $S'(l_\alpha) - S(l_\alpha) = \sigma(x_{1\alpha}) - x_{1\alpha}$.

If $r \geq 1$ and the required $x_r$ and $A_r$ exist then there is $l_{r+1} \in C_{r+1}(L_K)$ such that $e' \equiv \sigma x_r \circ (A_r \otimes \text{id}_K)e \circ (-x_r) \circ l_{r+1} \mod C_{r+2}(L_K)$.

Using that $K = \text{Im}(S) \oplus (\sigma - \text{id})K$ we can present $l_{r+1}$ as

$$l_{r+1} = l' + \sigma x' - x'$$
where \( l' = \sum_{\alpha} c_{\alpha} \otimes S(l_{\alpha}) \), all \( c_{\alpha} \in C_{r+1}(L) \) and \( x' \in C_{r+1}(L_K) \). It remains to set \( A_{r+1} (\tilde{l}_{\alpha}) = A_{r} (\tilde{l}_{\alpha}) + c_{\alpha} \) and \( x_{r+1} = x_{r} + x' \). The proposition is proved. \( \square \)

**Corollary 1.2.** With above notation there is \( f' \in F(e') \) such that for any \( \tau \in \Gamma_{<p} \), \( \pi_f'(e')(\tau) = A(\pi_f(e)(\tau)) \).

**Proof.** Let \( f' = x \circ (\mathcal{A} \otimes \text{id}_{sep}) f \), then \( f' \in F(e') \). Indeed,

\[
\sigma(f') = \sigma x \circ (\mathcal{A} \otimes \text{id}_{sep}) \sigma(f) = \sigma(x) \circ (\mathcal{A} \otimes \text{id}_K)e_S \circ (\mathcal{A} \otimes \text{id}_{sep})f = \sigma(x) \circ (\mathcal{A} \otimes \text{id}_K)e \circ (-x) \circ f' = e' \circ f'.
\]

Therefore, for any \( \tau \in \Gamma_{<p} \), \( \pi_f'(e')(\tau) \) is equal to

\[
(-f') \circ \tau(f') = (\mathcal{A} \otimes \text{id}_{sep})((-f) \circ \tau(f)) = A(\pi_f(e)(\tau)).
\]

\( \square \)

By the above corollary, a choice of \( e \in E(L_K) \) determines the class \( \pi_e \) of conjugated identifications \( \{ \pi_f(e) \mid f \in F(e) \} \) of \( \Gamma_{<p} \) with \( G(L) \).

When \( e \) is replaced by another \( e' \in E(L_K) \) the new class of conjugated identifications \( \pi_{e'} \) is obtained from \( \pi_e \) via the composition with some automorphism \( \mathcal{A} = \mathcal{A}(e, e') \in \text{Aut}_{L_{Lie}}(L) \) such that \( \mathcal{A} \equiv \text{id}_L \mod C_2(L) \).

### 1.5. Compatibility with field extensions.

Suppose \( K' \) is a field extension of \( K \) in \( K_{sep} \). Consider the above defined objects: \( M, L, e, e' \in E(L_K), f \in F(e) \) and \( \pi = \pi_f(e) : \Gamma_{<p} \simeq G(L) \) introduced in the context of the field \( K \). Let \( \Gamma_{<p}', M', L', e' \in E(L_{K'}) \), \( f' \in F(e') \) and \( \pi': \Gamma_{<p}' \simeq G(L') \) be the similar objects for the field \( K' \).

The embedding \( \text{Gal}(K_{sep}/K') \rightarrow \text{Gal}(K_{sep}/K) \) induces a natural group homomorphism \( \Theta : \Gamma_{<p}' \rightarrow \Gamma_{<p} \), which can be described in terms of the identifications \( \pi \) and \( \pi' \) as follows.

Consider \( e \otimes_K 1 \in L_K \otimes_K K' = L_{K'}, \supset M_{K'} = \text{Hom}(\bar{K}, K') \).

**Proposition 1.3.** There is a morphism of Lie algebras \( \mathcal{A} : L' \rightarrow L \) and \( x \in L_{K'} \) such that

a) \( e \otimes_K 1 = \sigma(x') \circ (\mathcal{A} \otimes \text{id}_K)e' \circ (-x') \);

b) for any \( \tau' \in \Gamma_{<p}', \pi(\Theta(\tau')) = \mathcal{A}(\pi'(\tau')) \);

c) if \( K' \subset K_{<p} \) then \( \pi(\text{Gal}(K_{<p}/K')) = \mathcal{A}(L') \).

**Proof.** Let \( \{ l'_\alpha \mid \alpha \in \mathcal{I}' \} \) be an \( \mathbb{F}_p \)-basis of \( K' = K'/(\sigma - \text{id}) K' \). Let \( \tilde{l}'_\alpha, \alpha \in \mathcal{I}' \), be the dual (topological) basis for \( M' \). Then for a suitable section \( S' \) of \( \Pi' : K' \rightarrow \bar{K}' \), we have \( e_{S'} = \sum_{\alpha} \tilde{l}'_\alpha \otimes S'(l'_\alpha) \) and \( \{ S'(l'_\alpha) \mid \alpha \in \mathcal{I}' \} \) is a basis of \( \text{Im}(S') \subset K' \). Proceeding similarly to the proof of Prop[1.1] prove the existence of \( x' \in L_{K'} \) and \( \tilde{l}_\alpha \in L \) such that

\[
e \otimes_K 1 = \sigma(x') \circ \left( \sum_{\alpha} \tilde{l}_\alpha \otimes S'(l'_\alpha) \right) \circ (-x').
\]
If $A' : M' \to L$ is a linear map such that for all $\alpha$, it holds $A'(\hat{\alpha}) = \tilde{\alpha}$, the above relation appears in the following form

$$e \otimes_K 1 = \sigma(x') \circ (A' \otimes \text{id}_{K'}) e' \circ (-x') ,$$

where $A'$ is a unique morphism of Lie algebras $L' \to L$ such that $A'|_{M'} = A'$. As a result, the both $(-x') \circ f$ and $(A' \otimes \text{id}_{sep}) f'$ belong to $F ((A' \otimes \text{id}_{K'}) e') \subset L_{sep}$. So, there is $l \in L$ such that

$$( -x') \circ (f \otimes_K 1) = (A' \otimes \text{id}_{sep}) f' \circ l .$$

If $x = x' \circ l$ and $A = \text{Ad} l \cdot A' \in \text{Hom}_{\text{Lie}} (L', L)$ then the above equality can be rewritten as

$$( -x) \circ (f \otimes_K 1) = (A \otimes \text{id}_{sep}) f' .$$

In particular, we have $e \otimes_K 1 = \sigma(x) \circ (A \otimes \text{id}_{K'}) e' \circ (-x)$ and for any $\tau' \in \Gamma_{<p}$, it holds $\pi (\Theta (\tau')) = (-f) \circ \tau'(f) = (A \otimes \text{id}_{K'}) ((-f') \circ \tau'(f')) = A (\pi (\tau'))$. The proposition is proved. \hfill \Box

1.6. Lifts of $\phi \in \text{Aut} K$ to $\phi_{<p} \in \text{Aut} K_{<p}$.

As earlier, $e \in \mathcal{E} (L_K)$, $f \in F (e)$, $\pi = \pi_f (e) : \Gamma_{<p} \simeq G (L)$.

Suppose $\phi \in \text{Aut} K$. We are going to describe (the lifts) $\phi_{<p} \in \text{Aut} K_{<p}$ such that $\phi_{<p|K} = \phi$.

Let $\phi_{<p} e := (\text{id} \otimes \phi) e \in L_K$. As earlier, for any given $\phi_{<p}$, establish the existence of $A = A (\phi_{<p}) \in \text{Aut}_{\text{Lie}} L$ and $C = C (\phi_{<p}) \in L_K$ such that

(1.1) $\phi_{<p} e = \sigma (C) \circ (A \otimes \text{id}_{K}) e \circ (-C)$.

Let $\mathcal{M} (\phi)$ be the set of all pairs $(C, A)$ satisfying (1.1). Let

$$\kappa : \mathcal{M} (\phi) \to \{ \phi_{<p} \in \text{Aut} K_{<p} \mid \phi_{<p|K} = \phi \}$$

be the map defined as follows.

If $(C, A) \in \mathcal{M} (\phi)$ then $g := C \circ (A \otimes \text{id}) f \in F (\phi_{<p} e)$. If $\phi_{<p}'$ is a lift of $\phi$ then $(\text{id} \otimes \phi_{<p}') f \in F (\phi_{<p} e)$. Then for some $l \in L$,

$$g = (\text{id} \otimes \phi_{<p}') (f \circ l) = (\text{id} \otimes \phi_{<p}') (\text{id} \otimes \pi^{-1}) f = (\text{id} \otimes (\phi_{<p}' \cdot \pi^{-1} l)) f$$

and the composition $\phi_{<p} := \phi_{<p}' \cdot \pi^{-1} l$ is a lift of $\phi$. It is easy to see that the lift $\phi_{<p}$ does not depend on the above choice of $\phi_{<p}'$. As a result, we can set $\kappa (C, A) = \phi_{<p}$.

Proposition 1.4.

a) If $\kappa (C, A) = \phi_{<p}$ then for any $\tau \in \Gamma_{<p}$, $\pi (\text{Ad}(\phi_{<p}) \tau) = A (\pi (\tau))$.

b) The map $\kappa$ is a bijection.

Proof. With above notation $\pi_g (\phi_{<p} e) (\tau) = (-g) \circ \tau (g) = A (\pi (\tau))$. On the other hand, $g = (\text{id}_{l} \otimes \phi_{<p}) f$ implies that

$$\pi_g (\phi_{<p} e) (\tau) = \phi_{<p} ( (-f) \circ \phi_{<p}^{-1} \tau \phi_{<p} (f) ) = \pi (\text{Ad}(\phi_{<p}) \tau) .$$

The part b) is implied by the following three facts:

b_{1}) the map $\kappa$ is injective.
Indeed, let \( \kappa(C_1, \mathcal{A}_1) = \kappa(C_2, \mathcal{A}_2) \). Then

\[
C_1 \circ (A_1 \otimes \text{id}) f = C_2 \circ (A_2 \otimes \text{id}) f .
\]

This implies that \( (A_{1}^{-1} \otimes \text{id})((-C_{2}) \circ C_{1}) \circ f = (A_{1}^{-1}A_{2} \otimes \text{id}) f \) and, therefore, for any \( \tau \in \Gamma_{<p} \), it holds \( \pi(\tau) = (A_{1}^{-1}A_{2})(\pi \tau) \) (use that \( (A_{1}^{-1} \otimes \text{id})((-C_{2}) \circ C_{1}) \in L_{K} \)). As a result, \( A_{1}^{-1}A_{2} = \text{id}_{L} \) and \( (C_{1}, \mathcal{A}_1) = (C_2, \mathcal{A}_2) \).

\[b_2\) \{\( \phi_{<p} \mid \phi_{<p}|_{K} = \phi \}\) is a principal homogeneous space over \( \Gamma_{<p} \)
with respect to the action \( \phi_{<p} \mapsto \phi_{<p} \cdot \tau, \tau \in \Gamma_{<p} \);

\[b_3\) the appropriate action of \( \tau \in \Gamma_{<p} \) on the pair \( (C, \mathcal{A}) \) appears in
the form \( (C, \mathcal{A}) \mapsto (C', \mathcal{A}') \), where for \( l_{\tau} := \pi(\epsilon) \tau, C' = C \circ l_{\tau} \) and for
any \( l \in L, \mathcal{A}'(l) = \mathcal{A}(l) \circ l_{\tau} = (\text{Ad}l_{\tau} : \mathcal{A})(l) \).

The proof of \( b_2 \) and \( b_3 \) is straightforward. For more details cf. [8].

The above formalism allows us to use the identification \( \pi = \pi_{f}(\epsilon) \)
to work with the group of all lifts \( \phi_{<p} \in \text{Aut} K_{<p} \) of automorphisms
\( \phi \in \text{Aut} K \). Note that:

– if \( (C', \mathcal{A}') \) and \( C'', \mathcal{A}'' \) correspond to the lifts \( \phi_{<p} \) and, \( \phi_{<p}' \) of, resp., \( \phi \) and \( \phi'' \), then the couple \( (C'' \circ (A' \otimes \text{id}_{K})C'', \mathcal{A}' \mathcal{A}'') \) corresponds
to the lift \( \phi_{<p} \phi'' \) of \( \phi \phi'' \);

– the elements \( \pi(\tau) = l_{\tau} \in G(L) \) appear as a special case of a
lift of \( \text{id}_{K} \) and correspond to the pairs \( (l_{\tau}, \text{Ad} l_{\tau}) \), where \( \text{Ad} l_{\tau} : l \mapsto
(-l_{\tau}) \circ l \circ l_{\tau} \).

2. Higher local fields and \( \mathcal{P} \)-topology

2.1. Higher local fields. Let \( K \) be an \( N \)-dimensional local field, i.e.

– if \( N = 0 \) then \( K \) is finite;

– if \( N \geq 1 \) then \( K \) is a complete discrete valuation field such that its
residue field is \((N - 1)\)-dimensional.

If \( N \geq 1 \) then the residue field of \( K \) is the first residue field of
\( K \). It will be usually denoted by \( K^{(1)} \). The corresponding valuation
ring \( O_{K}^{(1)} \) is the first valuation ring. We agree to set by induction
for all \( 1 < m \leq N, K^{(m)} = K^{(m-1)} \) — this is the \( m \)-th residue
field of \( K \). Note that \( K^{(N)} \) is 0-dimensional and, therefore, finite.

Define the \( N \)-valuation ring \( O_{K} \) of \( K \) by induction on \( N \) as follows.
If \( N = 0 \) set \( O_{K} = K \). If \( N \geq 1 \) and \( \text{pr} : O_{K}^{(1)} \rightarrow K^{(1)} \)
is the natural projection then set \( O_{K} = \text{pr}^{-1}O_{K}^{(1)} \).

If \( N \geq 1 \) then \( \pi := \{\pi_{1}, \ldots, \pi_{N}\} \) is a system of local
parameters in \( K \) if:

– \( \pi_{1} \) is (the first) uniformizer in \( K \);

– \( \pi_{2}, \ldots, \pi_{N} \in O_{K}^{(1)} \) and their projections \( \bar{\pi}_{2}, \ldots, \bar{\pi}_{N} \) to \( K^{(1)} \) form a
system of local parameters for \( K^{(1)} \).
If \([E : K] < \infty\) then the structure of \(N\)-dimensional field on \(K\) is uniquely extended to \(E\) and vice versa.

If \(\text{char } K = p \neq 0\) and \(\pi = \{\pi_1, \ldots, \pi_N\}\) is a system of local parameters in \(K\) then \(K\) appears as a field of iterated formal Laurent series \(K = k((\pi_N)) \cdots ((\pi_1))\), where \(k = K^{(N)} \simeq \mathbb{F}_q\) with \(q = p^{N_0}\). This is a part of the classification result, \([29]\). In this case \(K^{(1)}\) is identified with the subfield \(k((\pi_N)) \cdots ((\pi_2))\) of \(K\). More formally, there is a system of local parameters \(\bar{\pi} := \{\bar{\pi}_2, \ldots, \bar{\pi}_N\}\) in \(K^{(1)}\) such that their lifts to \(K\) form a subset \(\{\pi_2, \ldots, \pi_N\}\) of \(\pi\). We use the notation \(\iota_{\bar{\pi}}\) for the corresponding embedding of \(K^{(1)}\) into \(K\).

If \(\text{char } K = 0\) we always assume that \(\text{char } K^{(1)} = p > 0\) – such fields have most interesting arithmetical structure. The appropriate classification result for such fields \(K\) can be presented as follows.

Let \(K^{(1)} = k((\bar{\pi}_N)) \cdots ((\bar{\pi}_2))\) where \(\bar{\pi} := \{\bar{\pi}_2, \ldots, \bar{\pi}_N\}\) are local parameters for \(K^{(1)}\). The elements \(\bar{\pi}_2, \ldots, \bar{\pi}_N\) form a \(p\)-basis in \(K^{(1)}\), \([10]\). We can use this \(p\)-basis to construct an (absolutely unramified) lift \(K_{\bar{\pi}}^{(1)}\) of \(K^{(1)}\) to characteristic 0, \([10]\). Recall that \(K_{\bar{\pi}}^{(1)}\) is the fraction field of the ring \(\lim_{m \to \infty} O_m(K^{(1)})\), where

\[
O_m(K^{(1)}) = W_m(\sigma^{m-1} K^{(1)})[\pi_2, \ldots, \pi_N] \subset W_m(K^{(1)})
\]

are the lifts of \(K^{(1)}\) modulo \(p^m\). (Here \(\pi_2, \ldots, \pi_N\) are the Teichmuller representatives of \(\bar{\pi}_2, \ldots, \bar{\pi}_N\).) The field \(K_{\bar{\pi}}^{(1)}\) has a natural structure of \(N\)-dimensional local field of characteristic 0 with local parameters \(\{p, \pi_2, \ldots, \pi_N\}\). Now the classification result from \([29]\) states

\(K\) is a finite field extension of \(K_{\bar{\pi}}^{(1)}\).

In particular, we obtain an analogue \(\iota_{\bar{\pi}} : K_{\bar{\pi}}^{(1)} \to K\) of the above defined embedding \(K^{(1)} \to K\) in the characteristic \(p\) case.

Note also that,

— there is \(\pi_1 \in K\) such that \(\{\pi_1, \pi_2, \ldots, \pi_N\}\) is a system of local parameters in \(K\);

— the field \(K\) contains a (unramified 1-dimensional) local field \(F_{ur} = \text{Frac } W(k)\);

— the classification result from \([29]\) states also the existence of a finite totally ramified extension \(F'\) of \(F_{ur}\) such that \(K \subset F'K_{\bar{\pi}}^{(1)}\).

2.2. Definition and basic properties of \(\mathcal{P}\)-topology.

The topology on \(N\)-dimensional local field \(K\) (we refer to it as the \(\mathcal{P}\)-topology) can be introduced as follows, \([23, 29, 21]\).

If \(N = 0\) then the \(\mathcal{P}\)-topology on \(K\) is discrete.
Suppose \( N \geq 1 \) and \( \pi = \{ \pi_1, \ldots, \pi_N \} \) is a system of local parameters in \( K \). Then the \( \mathcal{P} \)-topology on \( K \) is introduced by induction on \( N \) via the following properties:

1. any \( \xi \in K \) can be uniquely presented as \( \mathcal{P} \)-convergent series
   \[
   \xi = \sum_{\alpha} [\alpha]_1^{\alpha_1} \cdots \pi_N^{\alpha_N}
   \]
   where the indices \( \alpha = (a_1, \ldots, a_N) \in \mathbb{Z}^N \), all \( \alpha_a \in K \), \( [\alpha] = \alpha_a \) if \( \text{char} K = p \) and \([\alpha]_1 = \alpha_a \) if \( \text{char} K = 0 \);

2. the \( \mathcal{P} \)-convergence property of \( \xi \) means the existence of integers \( A_i(a_1, \ldots, a_{i-1}) \in \mathbb{Z} \) with \( 1 \leq i \leq N \), such that: if \( \alpha_a \neq 0 \) then
   \[
   a_1 \geq A_1, a_2 \geq A_2(a_1), \ldots, a_N \geq A_N(a_1, \ldots, a_{N-1})
   \]

3. if \( \bar{\pi} := \{ \bar{\pi}_2, \ldots, \bar{\pi}_N \} \) is a system of local parameters for \( K^{(1)} \) used to define the \( \mathcal{P} \)-topology on \( K^{(1)} \) then \( \iota_{\bar{\pi}} \) induces the map
   \[
   s_{\bar{\pi}} : \sum_{\bar{a} = (a_2, \ldots, a_N)} [\alpha_1]_{\bar{\pi}_2}^{a_2} \cdots [\pi_N]_{\bar{\pi}_N}^{a_N} \mapsto \sum_{\bar{a}' = (0, a_2, \ldots, a_N)} [\alpha_1]_{\bar{\pi}_2}^{a_2} \cdots [\pi_N]_{\bar{\pi}_N}^{a_N}
   \]
   which is a \( \mathcal{P} \)-continuous (set-theoretic) section \( s : K^{(1)} \rightarrow O^{(1)}_K \) of the natural projection \( O^{(1)}_K \rightarrow K^{(1)} \).

The above properties imply that:

- a) a base of \( \mathcal{P} \)-open subsets \( \mathcal{U}_\pi(K) \) in \( K \) consists of the subsets \( \sum_{b \in \mathbb{Z}} \pi_1^b s_{\pi}(U_b) \), where all \( U_b \in \mathcal{U}_\pi(K^{(1)}) \) and for \( b \gg 0 \), \( U_b \equiv K^{(1)} \);

- b) a base \( \mathcal{C}_\pi(K) \) of sequentially compact (closed) subsets in \( K \) consists of \( \sum_{b \in \mathbb{Z}} \pi_1^b s_{\pi}(C_b) \) such that all \( C_b \in \mathcal{C}_\pi(K^{(1)}) \) and for \( b \ll 0 \), \( C_b \equiv 0 \);

- c) if \( \pi' = \{ \pi'_1, \ldots, \pi'_1 \} \) is another system of local parameters for \( K \) then the appropriate analogs of the above properties a)-c) also hold (i.e. the concept of \( \mathcal{P} \)-topology does not depend on the original choice of local parameters in \( K \));

- d) if \( [E : K] = n \) and the identification of \( K \)-vector spaces \( E = K^n \) is induced by a choice of some \( K \)-basis in \( E \) then \( \{ U^n \mid U \in \mathcal{U}_\pi(K) \} \) is a base of \( \mathcal{P} \)-open subsets in \( L \); similarly, \( \{ C^n \mid C \in \mathcal{C}_\pi(K) \} \) is a base of sequentially \( \mathcal{P} \)-compact subsets in \( E \);

- e) if \( C_1, C_2 \subseteq \mathcal{C}_\pi(K) \) then \( C_1 C_2 \) is also sequentially compact (i.e. there is \( C \subseteq \mathcal{C}_\pi(K) \) such that \( C_1 C_2 \subseteq C \));

- f) \( K \) is a \( \mathcal{P} \)-topological additive group but not a \( \mathcal{P} \)-topological field; however, \( K = \varinjlim \lim C \) and the multiplication \( C \times K \rightarrow K \) is
\( P \)-continuous (i.e. for any \( U \in \mathcal{U}_\pi(K) \) there is an \( U' \in \mathcal{U}_\pi(K) \) such that \( CU' \subset U \)).

Note that the subset of \( K \) consisting of the series \( \xi \) from above item (1) satisfying the condition:

if \( \alpha_a \neq 0 \) then \( a_1 > A_1, a_2 > A_2(a_1), \ldots, a_N > A_N(a_1, \ldots, a_{N-1}) \)
(with a fixed choice of \( A_i(a_1, \ldots, a_{i-1}) \), \( 1 \leq i \leq N \))
is sequentially compact. The family of all such subsets (with a fixed choice of a system of local parameters \( \pi = \{ \pi_1, \ldots, \pi_N \} \) forms the base \( \mathcal{C}_\pi(K) \).

We can similarly describe the base \( \mathcal{U}_\pi(K) \):

\( U \in \mathcal{U}_\pi(K) \) iff there are \( B_1, B_2(a_1), \ldots, B_N(a_1, \ldots, a_{N-1}) \in \mathbb{Z} \) (depending on \( U \)) such that \( \xi \in U \) are characterized by the condition:

if \( a_1 < B_1, a_2 < B_2(a_1), \ldots, a_N < B_N(a_1, \ldots, a_{N-1}) \) then \( \alpha_a = 0 \).

2.3. \( P \)-topology in characteristic \( p \).

Assume that \( K = \mathbb{K} \) has characteristic \( p \) and has a system of local parameters \( t = \{ t_1, \ldots, t_N \} \). We will use the simpler notation \( \mathcal{U}(K) \) and \( \mathcal{C}(K) \) instead of \( \mathcal{U}_t(K) \) and \( \mathcal{C}_t(K) \) when working with this fixed system of local parameters \( t \). Note that all \( C \in \mathcal{C}(K) \) and \( U \in \mathcal{U}(K) \) are \( k \)-linear vector spaces (where \( k = \mathbb{K}^{(N)} \)) and their elements appear as (some) formal \( k \)-linear combinations of the monomials \( t^a := t_1^{a_1} \cdots t_N^{a_N} \), where all \( a = (a_1, \ldots, a_N) \in \mathbb{Z}^N \).

Let \( \mathcal{I} = \mathcal{I}(K) \) and \( \mathcal{J} = \mathcal{J}(K) \) be the sets of indices such that \( \mathcal{C}(K) = \{ C_\alpha \mid \alpha \in \mathcal{I} \} \) and \( \mathcal{U}(K) = \{ U_\beta \mid \beta \in \mathcal{J} \} \).

It is easy to see by induction on \( N \) that for any \( \alpha \in \mathcal{I} \) and \( \beta \in \mathcal{J} \),

\[
\dim_k C_\alpha/C_\alpha \cap U_\beta < \infty.
\]

Note that \( C_\alpha/C_\alpha \cap U_\beta \) are provided with the \( k \)-bases consisting of the monomials \( t^a \mod C_\alpha \cap U_\beta \) such that \( t^a \in C_\alpha \setminus U_\beta \). These bases are compatible with respect to different choices of \( \alpha \) and \( \beta \). Therefore, \( \{ t^a \mid t^a \in C_\alpha \} \) is a \( P \)-topological \( k \)-basis in \( C_\alpha \), a base of \( P \)-open neighbourhoods in \( C_\alpha \) consists of \( k \)-vector subspaces containing almost all elements of this basis and \( C_\alpha = \varinjlim_{\beta} C_\alpha/C_\alpha \cap U_\beta \).

Let \( \mathcal{N}_{\alpha\beta} = (C_\alpha/C_\alpha \cap U_\beta)^D \) be the dual \( k \)-vector space for \( C_\alpha/C_\alpha \cap U_\beta \). Then \( \dim_k \mathcal{N}_{\alpha\beta} < \infty \), \( \mathcal{N}_{\alpha\beta}^D = C_\alpha/C_\alpha \cap U_\beta \) and the spaces \( \mathcal{N}_{\alpha\beta} \) are provided with compatible (dual) \( k \)-bases

\[
\{ T_a \mid t^{-a} \in C_\alpha \setminus U_\beta \},
\]

where for any \( t^{-b} \) with \( b \in \mathbb{Z}^N \), \( T_a(t^{-b}) = \delta_{ab} \).

Note that \( \mathcal{N}_\alpha^P := \text{Hom}_{\mathcal{P}-\text{cont}}(C_\alpha, k) = \varinjlim_{\beta} \text{Hom}(\mathcal{N}_{\alpha\beta}^D, k) = \lim_{\beta} \mathcal{N}_{\alpha\beta} \)
has a \( k \)-basis \( \{ T_a \mid t^{-a} \in C_\alpha \} \). Therefore, \( C_\alpha \) is the set of all formal
Consider the $\mathbb{F}_p$-vector spaces $\mathcal{E}_\alpha^p := \text{Hom}_{\mathbb{F}_p-\text{cont}}(C_\alpha, \mathbb{F}_p)$. Then

$$\mathcal{E}_\alpha^p \otimes k = \text{Hom}_{\mathbb{F}_p-\text{lin}, \mathbb{F}_p-\text{cont}}(C_\alpha, k) = \bigoplus_{n \in \mathbb{Z}/N_0} N_\alpha^p \otimes_k k^{(n)},$$

where $k^{(n)}$ is the (twisted) $k$-module $k \otimes_{\sigma^n} k$. In this identification the Frobenius $\sigma$ acts through the second factor on the left-hand side and shifts $\mathbb{Z}/N_0$-summands by +1 on the right-hand side.

In particular, the extensions of scalars $\mathcal{E}_\alpha^p \otimes k$ have the $k$-bases

$$\{ T_{an} = T_a \otimes 1^{(n)} \mid t^{-a} \in C_\alpha, n \in \mathbb{Z}/N_0 \}$$

which are compatible on $\alpha \in \mathcal{I}$. So, $\{ T_{an} \mid a \in \mathbb{Z}^N, n \in \mathbb{Z}/N_0 \}$ is a topological basis for $\mathcal{E}_\alpha^p \otimes k$, where $\mathcal{E}_\alpha^p := \text{Hom}_{\mathbb{F}_p-\text{cont}}(\mathcal{K}, \mathbb{F}_p) = \lim_{\leftarrow} \alpha \mathcal{E}_\alpha^p$.

**Proposition 2.1.** Let $\mathcal{E} := \text{Hom}(\mathcal{K}, \mathbb{F}_p)$. Then $\mathcal{E} = \mathcal{E}_\alpha^{p DD}$ (the double dual $\mathbb{F}_p$-vector space for $\mathcal{E}_\alpha^p$).

**Proof.** We verify this on the level of extensions of scalars as follows:

$$\text{Hom}(\mathcal{K}, \mathbb{F}_p) \otimes k = \lim_{\leftarrow} \alpha \text{Hom}(C_\alpha, \mathbb{F}_p) \otimes k =$$

$$\lim_{\leftarrow} \alpha \bigoplus_{n \in \mathbb{Z}/N_0} \text{Hom}_k(C_\alpha, k) \otimes_k k^{(n)} = \lim_{\leftarrow} \alpha \bigoplus_{n \in \mathbb{Z}/N_0} N_\alpha^{DD} \otimes_k k^{(n)}$$

$$= \lim_{\leftarrow} \alpha \left( \bigoplus_{n \in \mathbb{Z}/N_0} N_\alpha \otimes_k k^{(n)} \right)^{DD} = \lim_{\leftarrow} \alpha (\mathcal{E}_\alpha^p \otimes k)^{DD} = \mathcal{E}_\alpha^{p DD} \otimes k.$$

$\square$

**Corollary 2.2.** The vector space $\text{Hom}(\mathcal{K}, \mathbb{F}_p)$ is the profinite completion of its subspace $\text{Hom}_{\mathbb{F}_p-\text{cont}}(\mathcal{K}, \mathbb{F}_p)$.

**Proof.** Use that if $L$ is a $\mathbb{F}_p$-linear space then $L^{DD}$ is canonically isomorphic to the profinite completion of $L$. We sketch briefly the proof of this fact extracted from [13].

Suppose $L^D = \lim_{\leftarrow} Y_\alpha$, where all $Y_\alpha$ are finite dimensional vector subspaces in $L^D$. Then $L^{DD} = \lim_{\leftarrow} Y_\alpha^{DD}$. Note that $Y_\alpha \mapsto \text{Ann}(Y_\alpha) \subset L$ is a bijection between the set of finite dimensional subspaces in $L^D$ and the set of finite codimensional subspaces in $L$, and $Y_\alpha^{DD} \simeq L/\text{Ann}(Y_\alpha)$. Therefore, $L^{DD} \simeq \lim_{\leftarrow} L/Z_\alpha$ where $Z_\alpha = \text{Ann}Y_\alpha$ runs over the set of all finite codimensional subspaces in $L$. $\square$
Choose a system of local parameters $t = \{t_1, \ldots, t_N\}$ in $\mathcal{K}$ and let $\mathcal{C}(\mathcal{K}) = \{C_\alpha \mid \alpha \in \mathcal{T}\}$ and $\mathcal{U}(\mathcal{K}) = \{U_\beta \mid \beta \in \mathcal{J}\}$ be the corresponding bases of sequentially compact and open subsets in $\mathcal{K}$ from Sect. 2.3. Then the images $\bar{C}_\alpha = \Pi(C_\alpha)$ and $\bar{U}_\beta = \Pi(U_\beta)$ form the corresponding bases for $\bar{\mathcal{K}}$.

Choose $\alpha_0 \in \mathbb{K}$ with the absolute trace $\text{Tr}_{\mathbb{F}_p/\mathbb{F}_p} \alpha_0 = 1$.

Define $\Psi_p$-linear operators $S, R : \mathcal{K} \rightarrow \mathcal{K}$ as follows.

Suppose $\alpha \in \mathbb{K}^*$.

If $a \in \mathbb{Z}^N$, $a > \bar{0} = (0, \ldots, 0) \in \mathbb{Z}^N$ then set $S(t^a) = 0$ and $R(t^a) = -\sum_{i \geq 0} \sigma^i(t^a)$.

For $a = 0$, set $S(\alpha) = \alpha_0 \text{Tr}_{\mathbb{F}_p/\mathbb{F}_p} \alpha$ and $R(\alpha) = \sum_{0 \leq j < R} (\sigma^j \alpha_0) \sigma^j \alpha$.

If $a = -a_1 p^m < \bar{0}$ with $a_1 \in \mathbb{Z}_N^+(p)$ set $S(t^a) = t^{-a_1} \sigma^{-m} \alpha$ and $R(t^a) = \sum_{1 \leq i \leq m} \sigma^{-i}(t^a)$.

For $b = \sum_{a \in \mathbb{Z}^N} \alpha_a t^a \in \mathcal{K}$, set $S(b) = \sum_{a \in \mathbb{Z}^N} S(\alpha_a t^a)$, $R(b) = \sum_{a \in \mathbb{Z}^N} R(\alpha_a t^a)$.

The proof of the following proposition is straightforward. It uses just that $\sigma : \mathcal{K} \rightarrow \mathcal{K}$ is $\mathcal{P}$-continuous and $\mathcal{K}$ is a $\mathcal{P}$-topological group with respect to addition.

**Proposition 3.1.** a) $R$ and $S$ are $\mathcal{P}$-continuous.

b) For any $b \in \mathcal{K}$, $b = S(b) + (\sigma - \text{id}) R(b)$.

Notice that $S^2 = S$, $R^2 = R$ and $R S = S R = 0$. In particular, Prop. 3.1 implies that the elements $b \in \mathcal{K}$ can be uniquely presented modulo $(\sigma - \text{id}) \mathcal{K}$ in the following form

$$
\sum_{a \in \mathbb{Z}^N_+} \gamma_a t^{-a} + \gamma_0 \alpha_0
$$

where all $\gamma_a \in \mathbb{K}$ and $\gamma_0 \in \mathbb{F}_p$. We have also the following proposition.

**Proposition 3.2.** a) The morphism $\Pi(b) \mapsto S(b)$, where $b \in \mathcal{K}$, defines a $\mathcal{P}$-continuous section $S_{\text{loc}} : \bar{\mathcal{K}} \rightarrow \mathcal{K}$ of $\Pi$ such that $S_{\text{loc}} \Pi(\alpha t^{-a}) = \alpha t^{-a}$ if $a \in \mathbb{Z}_N^+(p)$, $\alpha \in \mathbb{K}$, and $S_{\text{loc}}(k/(\sigma - \text{id}) k) = \mathbb{F}_p \alpha_0 \subset \mathbb{K}$.

b) For a $\mathcal{P}$-continuous section $S$ of $\Pi$ there is a $\mathcal{P}$-continuous map $R_S : \mathcal{K} \rightarrow \mathcal{K}$ such that for any $b \in \mathcal{K}$, $b = S(\Pi(b)) + (\sigma - \text{id}) R_S(b)$.

**Proof.** Item a) follows from Prop. 3.1. For item b), just notice that

$$
b = S(b) + (\sigma - \text{id}) R(b) = S\Pi(b) + (S - S\Pi) b + (\sigma - \text{id}) R(b) = S\Pi(b) + (\sigma - \text{id})(R(S - S\Pi) + R)(b) = S\Pi(b) + (\sigma - \text{id}) R(b - S\Pi b)
$$

and $R_S := R(\text{id} - S\Pi)$ is $\mathcal{P}$-continuous. \qed
3.2. \(\mathcal{P}\)-topological module \(\tilde{\mathcal{K}}^{DP}\). Proceed similarly to Sect.2.3 by setting for all \(\alpha \in \mathcal{I}\) and \(\beta \in \mathcal{J}\):

1) \(\tilde{\mathcal{K}}^{D}_{\alpha\beta} := \text{Hom}(\tilde{C}/\bar{C}_\alpha \cap \bar{U}_\beta, \mathcal{F}_p) = \text{Hom}_{\mathcal{P}}(\mathcal{C}/\bar{C}_\alpha \cap \bar{U}_\beta, \mathcal{F}_p);\)

2) \(\tilde{\mathcal{K}}^{DP}_{\alpha} := \lim_{\beta} \tilde{\mathcal{K}}^{D}_{\alpha\beta} = \text{Hom}_{\mathcal{P}}(\mathcal{C}/\bar{C}_\alpha, \mathcal{F}_p);\)

3) \(\tilde{\mathcal{K}}^{DP} := \lim_{\alpha} \tilde{\mathcal{K}}^{DP}_{\alpha} = \text{Hom}_{\mathcal{P}}(\tilde{\mathcal{K}}, \mathcal{F}_p).\)

Remark. 1) For any \(\alpha\) and \(\beta\), \(\dim k\bar{C}_\alpha/(\bar{C}_\alpha \cap U_\beta) < \infty\).

2) \(\tilde{\mathcal{K}}^{D}_{\alpha} = \text{Hom}(\bar{C}_\alpha, \mathcal{F}_p) = (\tilde{\mathcal{K}}^{DP}_{\alpha})^{DD},\) in particular, \(\tilde{\mathcal{K}}^{D}_{\alpha}\) is the profinite completion of \(\tilde{\mathcal{K}}^{DP}_{\alpha}\).

3) It follows from 2) that \(\tilde{\mathcal{K}}^{DP} = \lim_{\alpha} (\tilde{\mathcal{K}}^{DP}_{\alpha})^{DD} = (\tilde{\mathcal{K}}^{DP})^{DD}\) is the profinite completion of \(\tilde{\mathcal{K}}^{DP}\).

Define the \(\mathcal{P}\)-topology on \(\tilde{\mathcal{K}}^{DP}\) as the projective limit of discrete topologies on all \(\tilde{\mathcal{K}}^{DP}_{\alpha}\).

Set for any \(\alpha \in \mathcal{I}\) and \(\beta \in \mathcal{J}\),

\(U^{D}_\alpha := \{u^D \in \tilde{\mathcal{K}}^{DP} \mid u^D(\bar{C}_\alpha) = 0\} = \text{Ann}(\bar{C}_\alpha).\)

\(C^{D}_\beta = \{c^D \in \tilde{\mathcal{K}}^{DP} \mid c^D(\bar{U}_\beta) = 0\} = \text{Ann}(\bar{U}_\beta).\)

Then in \(\tilde{\mathcal{K}}^{DP}\):

a) \(U(\tilde{\mathcal{K}}^{DP}) := \{U^{D}_\alpha \mid \alpha \in \mathcal{I}\}\) is a base of open neighborhoods;

b) \(C(\tilde{\mathcal{K}}^{DP}) = \{C^{D}_\beta \mid \beta \in \mathcal{J}\}\) is a base of sequentially compact subsets;

c) \(\tilde{\mathcal{K}}^{DP} = \lim_{\beta} C^{D}_\beta\) and for any \(\beta \in \mathcal{J}\), \(C^{D}_\beta = \lim_{\alpha} C^{D}_\beta/C^{D}_\beta \cap U^{D}_\alpha;\)

d) for any \(\alpha \in \mathcal{I}\), \(\tilde{\mathcal{K}}^{DP}_{\alpha} = \tilde{\mathcal{K}}^{DP}/U^{D}_\alpha.\)

These properties are implied easily via the following observations.

Let \(\{\alpha_i \mid 1 \leq i \leq N\}\) be a basis of \(k\) over \(\mathbb{F}_p\). Consider the set

\[(3.2) \quad \{\alpha_i t^{-a} \mid 1 \leq i \leq N, a \in \mathbb{Z}_N^+(p)\} \cup \{a_0\}.\]

Then:

— for any \(\alpha \in \mathcal{I}\), there is a subset of \(3.2\) which forms a \(\mathcal{P}\)-topological basis of \(\bar{C}_\alpha;\)

— for any \(\beta \in \mathcal{J}\), there is a subset of \(3.2\) which forms a \(\mathcal{P}\)-topological basis of \(\bar{U}_\beta.\)

Let

\[(3.3) \quad \{D^{(i)}_a \mid a \in \mathbb{Z}_N^+(p), 1 \leq i \leq N\} \cup \{D_0\}\]

be the dual system of elements of \(\tilde{\mathcal{K}}^{DP}\) for system \(3.2\).
Then for any $\alpha \in I$, there is a subset of (3.3) which forms a $P$-topological $\mathbb{F}_p$-basis of $U_\alpha^D$. Similarly, for any $\beta \in J$, there is a subset of (3.3) which forms a $P$-topological $\mathbb{F}_p$-basis of $C_\beta^D$.

As a result, the pairing $\bar{K} \times \bar{K}^D \to \mathbb{F}_p$ is a perfect pairing of $P$-topological modules. This pairing identifies $\bar{K}^D$ with $\text{Hom}_{P\text{-cont}}(\bar{K}, \mathbb{F}_p)$.

Consider the presentations of elements from $S(\bar{K}) \subset K$ in the form (3.1). For $a \in \mathbb{Z}_{N}^+(p)$ and $n \in \mathbb{Z}/N_0$, let $D_{an} \in \bar{K}^D \otimes_{\mathbb{F}_p} k$ be such that $D_{an}(\Pi(\gamma_a t^{-a})) = \sigma^n \gamma_a$ and $D_{an}(\Pi(\alpha_0)) = 0$. If $D_0 \in \bar{K}^D$ is the element appeared in (3.3), then $D_0(\gamma_a t^{-a}) = 0$ and $D_0(\alpha_0) = 1$.

The elements of the set (3.4)

$$D := \{D_{an} \mid a \in \mathbb{Z}_{N}^+(p), n \in \mathbb{Z}/N_0\} \cup \{D_0\}$$

form a $P$-topological basis for $\bar{K}_k^D$. In particular:

(1) the elements of $\bar{K}_k^D$ appear uniquely as $P$-convergent series

$$\sum_{a \in \mathbb{Z}_{N}^+(p)} \gamma_{an} D_{an} + \gamma_0 D_0,$$

where all $\gamma_{an}$ and $\gamma_0$ run over $k$;

(2) the appropriate subsets of (3.4) provide us with compatible $k$-bases for $\bar{K}_{\alpha\beta}^D$ and $\bar{K}_\alpha^D$;

(3) the elements of $\bar{K}_k^D$ can be presented uniquely as $P$-convergent series

$$\sum_{a \in \mathbb{Z}_{N}^+(p)} \sigma^n(\gamma_a) D_{an} + \gamma_0 D_0,$$

where $\gamma_0 \in \mathbb{F}_p$ and for $a \neq 0$, $\gamma_a \in k$.

Remark. The condition of $P$-convergence in (1) means that for any $\alpha \in I$, $\{\gamma_{an} \neq 0 \mid t^{-a} \in \tilde{C}_\alpha\}$ is finite. Similar condition holds in (3) (where $\gamma_{an}$ should be replaced by $\sigma^n \gamma_a$).

Let $\otimes^P$ be the $P$-topological tensor product. Consider

$$\bar{K}^D_k := \text{Hom}_{P\text{-cont}}(\bar{K}, K) = \bar{K}^D \otimes^P \mathbb{F}_p K = \bar{K}^D_k \otimes^P K.$$

The following property is straightforward.

Proposition 3.3. The elements of $\bar{K}^D_k$ can be presented uniquely as $P$-convergent sums $\sum_{a \in \mathbb{Z}^N} m_a t^{-a}$ with coefficients $m_a \in \bar{K}^D_k$.

Remark. The condition of $P$-convergence in Prop 3.3 means that for any $\alpha \in I$, $\beta \in J$, $\{m_a \neq 0 \mid t^{-a} \notin U_\beta, m_a \notin U_\alpha^D\}$ is finite.
3.3. Lie algebras \( \mathcal{L} \) and \( \mathcal{L}^p \).

Let \( \mathcal{\tilde{L}} \) be a free profinite Lie algebra over \( \mathbb{F}_p \) with the (profinite) module of free generators \( \mathcal{\tilde{K}}^{DP} = \text{Hom}(\mathcal{\tilde{K}}, \mathbb{F}_p) \). Let \( \mathcal{L} = \mathcal{\tilde{L}} / C_p(\mathcal{\tilde{L}}) \). Then \( \mathcal{L} \) is a projective limit of finite Lie \( \mathbb{F}_p \)-algebras generated by the finite quotients of \( \mathcal{\tilde{K}}^{DP} \).

Let \( \mathcal{\tilde{L}}^p \), resp., \( \mathcal{L}^p \), be the Lie subalgebra in \( \mathcal{\tilde{L}} \), resp. in \( \mathcal{L} \), generated by the elements of \( \mathcal{\tilde{K}}^{DP} \) and \( C_p(\mathcal{\tilde{L}}) \). Then \( \mathcal{L}_p(\mathcal{\tilde{L}}^p) = \mathcal{\tilde{L}}^p \cap C_p(\mathcal{\tilde{L}}) \) (use that the profinite completion of \( \mathcal{\tilde{K}}^{DP} \) is \( \mathcal{\tilde{K}}^D \)) and \( \mathcal{L}^p = \mathcal{\tilde{L}}^p / C_p(\mathcal{\tilde{L}}^p) \).

Note that \( \mathcal{L}^p \) inherits the \( \mathcal{P} \)-topological structure from \( \mathcal{\tilde{K}}^{DP} \) (use the topology of tensor product on \( \sum_{1 \leq i < p} (\mathcal{\tilde{K}}^{DP})^{\otimes i} \subset \mathcal{L}^p \)), and the profinite completion of \( \mathcal{L}^p \) coincides with \( \mathcal{L} \).

Introduce the Lie algebras \( \mathcal{L}_\alpha \) with generators \( \mathcal{K}_\alpha^{DP} = \text{Hom}(\mathcal{\tilde{C}}_\alpha, \mathbb{F}_p) \) and \( \mathcal{L}_\alpha^p \) with generators \( \mathcal{K}_\alpha^{DP} = \text{Hom}_{\mathcal{P},\text{cont}}(\mathcal{\tilde{C}}_\alpha, \mathbb{F}_p) \). Then \( \mathcal{L}_\alpha^p \) has discrete topology, its profinite completion coincides with \( \mathcal{L}_\alpha, \mathcal{\tilde{L}} = \varprojlim \mathcal{L}_\alpha \) and \( \mathcal{L}^p = \varprojlim \mathcal{L}_\alpha \). If \( \mathcal{L}_{\alpha\beta} \) is the subalgebra in \( \mathcal{L}_\alpha \) generated by \( \mathcal{K}_{\alpha\beta}^D \) then \( \mathcal{L}_{\alpha\beta} \) is finite and \( \mathcal{L}_\alpha = \varprojlim \mathcal{L}_{\alpha\beta} \).

The elements of the Lie algebra \( \mathcal{L}_k^p := \mathcal{L}^p \otimes_{\mathbb{F}_p} k \) appear as convergent \( k \)-linear combinations of the Lie monomials of the form

\[
\sum_{D_1, \ldots, D_r} \gamma_{D_1, \ldots, D_r} [\ldots [D_1, \ldots], D_r].
\]

where all \( D_1, \ldots, D_r \) belong to \( \mathcal{P} \). The condition of convergency means that for any \( \alpha \in \mathcal{I} \), all but finitely many of these monomials have the zero image in \( \mathcal{L}_\alpha^p \).

We can describe similarly the enveloping algebra of \( \mathcal{L}^p \). Namely, let \( \mathfrak{A}^p \) and \( \mathfrak{A}_\alpha^p \) be enveloping algebras for \( \mathcal{L}^p \) and, resp., \( \mathcal{L}_\alpha^p \) taken modulo \( p \)-th powers of the corresponding augmentation ideals. Then \( \mathfrak{A}^p = \varprojlim \mathfrak{A}_\alpha^p \) and \( \mathfrak{A}_{\alpha,k}^p \) consists of all polynomials of total degree \( < p \) in the subset of variables \( D_{an} \) from \( \mathfrak{A}_\alpha^p \), satisfying the condition \( t^{-a} \notin C_n \).

In other words, the elements of \( \mathfrak{A}^p \) are characterized in the algebra \( \mathfrak{A} \) as \( \mathcal{P} \)-continuous polynomials on \( \mathcal{\tilde{K}} \) with values in \( \mathbb{F}_p \) of total degree \( < p \). Of course, \( \mathfrak{A} \) and \( \mathcal{L} \) can be recovered as the profinite completion of \( \mathfrak{A}^p \) and, resp., \( \mathcal{L}^p \).

3.4. Class of conjugated subgroups \( \text{cl}(\mathcal{G}_{<p}) \).

Let \( \mathcal{G} = \text{Gal}(\mathcal{K}_{sep}/\mathcal{K}) \) be the absolute Galois group of the field \( \mathcal{K} \).

If \( \mathcal{G}_{<2} := \mathcal{G}/\mathcal{G}^p \mathcal{C}_2(\mathcal{G}) \) is the maximal abelian quotient of period \( p \) of \( \mathcal{G} \), then the classical Artin-Schreier duality \( \mathcal{\tilde{K}} \times \mathcal{G}_{<2} \rightarrow \mathbb{F}_p \) allows us to identify \( \mathcal{G}_{<2} \) with \( \mathcal{K}^{D} = (\mathcal{K}^{DP})^{DD} \) and to introduce a dense subgroup \( \mathcal{G}_{<2} := \mathcal{\tilde{K}}^{DP} \) in \( \mathcal{G}_{<2} \). Note that with respect to this identification, the
elements $D_a^{(i)} \in \bar{K}^{DP}$ from (3.3) appear as elements of $G_{<2}$ such that if $T_{bj} \in K_{sep}$ are such that $T_{bj} - T_{bj} = \alpha_j t^{-b}$ (in notation from Sect. 2.3) then $D_a^{(i)}(T_{bj}) = T_{bj} + \delta_{ab}\delta_{ij}$. Similarly, the elements $D_{mn} \in \bar{K}^{DP}$ from (3.4) act as follows: if $b \in \mathbb{N}$, $\gcd(b, p) = 1$ and $T_b \in K_{sep}$ is such that $T_b^n - T_b = t^{-b}$ then for $0 \leq m < N_0$, $D_{mn}(T_b^n) = T_b^{mn} + \delta_{ab}\delta_{nm}$.

The Artin-Schreier reciprocity law provides us with the identifications of local class field theory $G_{<2} \simeq K_N(K)/p$ and $G_{<2}^{op} \simeq K_N^{op}(K)/p$, where $K_N$ is $N$-th $K$-functor and $K_N^{op}$ is its $N$-th Milnor topological version, cf. e.g. [22, 23].

The subgroup $G_{<2}^{op}$ is considerably smaller than $G_{<2}$ but its profinite completion recovers the whole $G_{<2}$. In particular, $G_{<2}^{op}$ can be used instead of $G_{<2}$ when studying finite (abelian) extensions of $K$ inside $K_{<2} = K_{<2}^{op}(K_{<2})$. Our target is to introduce an analog of $G_{<2}^{op}$ in the case of $p$-extensions of nilpotent class $< p$.

From now on we will consider only $e \in \mathcal{E}(L_K^p) := L_K^p \cap \mathcal{E}(L_K)$. Under this assumption if $S$ is a section of the projection $\Pi : K \rightarrow K$ such that $e \mod C_2(L_K^p) = e_S$ then $S$ is $P$-continuous.

As earlier, choose $f \in F(e)$ and set $\pi = \pi_f(e) : G_{<p} \simeq G(L)$.\[
\textbf{Definition.} \quad \text{cl}^P(G_{<p}) \text{ is the class of conjugated subgroups of } G_{<p} \text{ containing } \pi_f(e)^{-1} G(L^p). \]

\textbf{Theorem 3.4.} The class $\text{cl}^P(G_{<p})$ does not depend on the choices of $e \in \mathcal{E}(L_K^p)$ and $f \in F(e)$.

\textbf{Proof.} Suppose $e' \in \mathcal{E}(L_K^p)$, $f' \in F(e')$ and set $\pi' = \pi_{f'}(e')$. We must prove that $\pi'^{-1} G(L^p)$ and $(\pi')^{-1} G(L^p)$ are conjugated in $G_{<p}$.

\textbf{Lemma 3.5.} There is an $x \in L_K^p$ and a $P$-continuous section $A : K^{DP} \rightarrow L^p$ of the natural projection $L^p \rightarrow L^p/C_2(L^p) = K^{DP}$ such that $e' = \sigma(x) \circ (A \otimes_{F_p} \text{id}_K)e \circ (-x)$, where $A \in \text{Aut}_L \mathbb{L}^P$ is such that $A|_{K^{DP}} = A.$

\textbf{Proof of lemma.} The proof appears as a $P$-topological version of the proof of Prop. 1.1, where we use the $(P$-continuous) operators from Prop. 3.1 and 3.2.

Let $\{l_{\alpha} \mid \alpha \in I\}$ be a $P$-topological $\mathbb{F}_p$-basis of $K$. Let $\hat{l}_\alpha, \alpha \in I$, be the dual ($P$-topological) $\mathbb{F}_p$-basis for $K^{DP}$, i.e. for any $\alpha_1, \alpha_2 \in I$, $\hat{l}_{\alpha_1}(l_{\alpha_2}) = \delta_{\alpha_1\alpha_2}$. Then for the corresponding sections $S$ and $S'$, we have the $P$-convergent series $e_S = \sum_{\alpha} \hat{l}_\alpha \otimes^P S(l_{\alpha})$ and $e_{S'} = \sum_{\alpha} \hat{l}_\alpha \otimes^P S'(l_{\alpha}).$

Apply induction on $r \geq 1$ to prove the existence of $x_r \in L_K^p$ and a section $A_r$ of the projection $L^p \rightarrow K^{DP}$ such that $e' \equiv \sigma(x_r) \circ (A_r \otimes \text{id}_K)e \circ (-x_r) \mod C_{r+1}(L_K^p)$, with the appropriate $P$-continuous automorphism $A_r$ of $L^p$.
If \( r = 1 \) take \( A_1 = \text{id}_M \) and \( x_1 = \sum \hat{t}_\alpha \otimes^P x_{1\alpha} \), where all \( x_{1\alpha} = \mathcal{R}(S'(l_\alpha) - S(l_\alpha)) \in \mathcal{K} \) and \( \mathcal{R} \) is the operator from Prop. 3.1. Note that \( x_1 \in \mathcal{L}_K^P \) because \( \mathcal{R} \) is \( \mathcal{P} \)-continuous.

If \( r \geq 1 \) and such \( x_r \) and \( A_r \) exist then there is \( l_{r+1} \in C_{r+1}(\mathcal{L}_K^P) \) such that \( e' = \sigma_x \circ (A_r \otimes^P \text{id}_K) e \circ (-x_r) \circ l_{r+1} \mod C_{r+2}(\mathcal{L}_K^P) \).

Let \( l_{r+1} = \sum \alpha c_\alpha \otimes^P b_\alpha \) with all \( c_\alpha \in C_{r+1}(\mathcal{L}_P^P) \) and \( b_\alpha \in \mathcal{K} \). Then
\[
l_{r+1} = l' + \sigma(x') - x',
\]
where \( l' = \sum \alpha c_\alpha \otimes^P (S\Pi)(b_\alpha) \) and \( x' = \sum \alpha c_\alpha \otimes^P R\alpha(b_\alpha) \) belong to \( C_{r+1}(\mathcal{L}_K^P) \subset \mathcal{L}_K^P \). It remains to set \( A_{r+1}(\hat{t}_\alpha) = A_r(\hat{t}_\alpha) + c_\alpha \) and \( x_{r+1} = x_r + x' \). The lemma is proved. \( \square \)

**Remark.** The main reason why the proof of Prop. 3.1 works in the \( \mathcal{P} \)-topological context is that \( \mathcal{L}_K^P = \mathcal{L}^P \otimes^P \mathcal{K} \) is stable with respect to the action of \( \mathcal{P} \)-continuous operators on the factor \( \mathcal{K} \).

Continue the proof of the theorem.

Denote by the same symbol \( \mathcal{A} \) the (unique) extension of \( \mathcal{A} \) to \( \mathcal{L} \) (use that \( \mathcal{L} \) is a profinite completion of \( \mathcal{L}^P \)). Then \( f'' = x \circ (\mathcal{A} \otimes \text{id}_{\text{sep}}) f \in \mathcal{F}(e') \) and for any \( \tau \in \mathcal{G}_{\text{sep}} \), \( \pi''(e') = (\mathcal{A} \cdot \pi) \tau \).

Then \( (\mathcal{A} \cdot \pi)(\mathcal{G}_{\text{sep}}^P) = \mathcal{A}(\mathcal{L}^P) = \mathcal{L}^P \). This implies that \( \pi^{-1}G(\mathcal{L}^P) = \pi''(e')^{-1}G(\mathcal{L}^P) \). But \( f', f'' \in \mathcal{F}(e') \) implies that \( (\pi')^{-1}G(\mathcal{L}^P) \) and \( \pi''(e')^{-1}G(\mathcal{L}^P) \) are conjugated in \( \mathcal{G}_{\text{sep}} \). \( \square \)

### 3.5. \( \mathcal{P} \)-Galois correspondence

As earlier, consider \( \mathcal{K}^P, \mathcal{K}_{DDP}, e \in \mathcal{E}(\mathcal{L}_K^P), f \in \mathcal{F}(e) \) and \( \pi := \pi_f(e) : \mathcal{G}_{\text{sep}}^P \simeq G(\mathcal{L}) \). Suppose \( \mathcal{K}' \) is a finite field extension of \( \mathcal{K} \) in \( \mathcal{K}_{\text{sep}} \). Let \( \mathcal{G}'_{\text{sep}}^P, \mathcal{K}_{DDP}' \), \( e' \in \mathcal{E}(\mathcal{L}_K^P)' \), \( f' \in \mathcal{F}(e') \) and \( \pi' = \pi_{f'}(e') : \mathcal{G}'_{\text{sep}}^P \simeq G(\mathcal{L}') \) be the similar objects for the field \( \mathcal{K}' \).

The natural morphism of profinite groups \( \Theta : \mathcal{G}_{\text{sep}}^P \rightarrow \mathcal{G}_{\text{sep}}^P \) can be described in the terms of identifications \( \pi \) and \( \pi' \) by Prop. 3.3. It admits the following \( \mathcal{P} \)-version.

**Proposition 3.6.** Suppose \( \mathcal{G}_{\text{sep}}^P \subset \mathcal{L}^P \mathcal{G}_{\text{sep}}^P \). Then:

a) there is \( \mathcal{G}_{\text{sep}}^P \subset \mathcal{L}^P \mathcal{G}_{\text{sep}}^P \), such that \( \Theta(\mathcal{G}_{\text{sep}}^P) \subset \mathcal{G}_{\text{sep}}^P \);

b) \( \Theta(\mathcal{G}_{\text{sep}}^P) = (\mathcal{G}_{\text{sep}}^P : \Theta(\mathcal{G}_{\text{sep}}^P)) = (\mathcal{G}_{\text{sep}} : \Theta(\mathcal{G}_{\text{sep}})) \), i.e. \( \Theta(\mathcal{G}_{\text{sep}}^P) = \Theta(\mathcal{G}_{\text{sep}}^P) \cap \mathcal{G}_{\text{sep}}^P \).

**Proof.** a) We can assume that \( f' \in \mathcal{F}(e') \) is such that \( \mathcal{G}_{\text{sep}}^P = \pi'^{-1}G(\mathcal{L}^P) \).

Then we can apply the \( \mathcal{P} \)-topological version of the proof of Prop. 3.3 to establish the existence of \( \mathcal{P} \)-continuous \( \mathcal{A} \in \text{Hom}_{\text{Lie}}(\mathcal{L}^P, \mathcal{L}^P) \) and \( x' \in \mathcal{L}_K^P \) such that
\[
e \otimes^P 1_{\mathcal{K}'} = \sigma(x') \circ (\mathcal{A} \otimes \text{id}_{\mathcal{K}'}) e'_l \circ (-x').
\]

From (3.5) it follows that both \( (-x') \circ f \) and \( (\mathcal{A} \otimes \text{id}_{\mathcal{K}'}) f' \) belong to \( \mathcal{F}((\mathcal{A} \otimes \text{id}) e'_{\text{sep}}) \subset \mathcal{L}_{\text{sep}} \). Therefore, there is \( l \in \mathcal{L} \) such that
\[
(-x') \circ f = (\mathcal{A} \otimes \text{id}_{\text{sep}}) f' \circ l.
\]
As a result, for any $\tau' \in G'_\leq p$, $\pi(\Theta(\tau')) = (-l) \circ A(\pi'(\tau')) \circ l$, and
\[
\pi(\Theta(G'_\leq p)) = (-l) \circ A(L^P) \circ (-l) \subset (-l) \circ L^P \circ l.
\]
Equivalently, for $g = \pi^{-1}(l) \in G_\leq p$, we have
\[
\Theta(G^P_\leq p) \subset (-g) \circ \pi^{-1}(L^P) \circ g \in cl^P G_\leq p.
\]
So, we can take $G^P_\leq p = \Ad(g)(\pi^{-1} L^P)$.

b) Assume that in the notation from a), $l = 0$. This guarantees $\Theta(G'_\leq p) \subset G^P_\leq p$ and $\pi = A \cdot \pi'$, where $\pi = \pi_f(e) : G_\leq p \simeq G(L)$, $\pi' = \pi_f(e') : G'_\leq p \simeq G(L')$ and $A : L' \rightarrow L$ is induced by $\Theta$.

Let $p^n = [K' : K] = (G_\leq p : \Theta(G'_\leq p))$.

- The case $[K' : K] = p$.

Here $K'/K$ is Galois of degree $p$, $(L : A(L')) = p$, $A(L')$ is an ideal in $L$, and $A(L') = C_2(L) + L$, where $L \subset K^D$ is of index $p$.

Let $A(L^P) = C_2(L^P) + L^0 \subset A(L')$, where in notation from Sect. 3.2:
\[
L^0 \subset K^{PD} = \lim_{\longrightarrow} Hom_{\mathrm{P-cont}}(C_\alpha, \mathbb{F}_p).
\]

Let $K' = K(T')$, where $T'^p - T' = c \in K$. Then $K' = K^H_\leq p$, with $H = \Theta(G'_\leq p)$ and $\pi(H) = G(C_2(L) + L)$. Therefore, $L \subset K^D$ is characterized by the trivial action on $T'$ or, equivalently, $L = \text{Ann}(\bar{c})$, where $\bar{c} \in \bar{K}$ is the image of $c$ under the natural projection $\Pi : K \rightarrow \bar{K}$.

We can assume that for some index $\alpha_0$, $\Pi(c) \in \bar{C}_{\alpha_0}$, because $\bar{K}$ is the union of all $\bar{C}_{\alpha} = \Pi(C_{\alpha})$. As a result:
- the $P$-subgroup $H^P$ appears in the form $\pi^{-1} G(C_2(L^P) + L^0)$;
- $L^0$ is the preimage of a subspace $L^0_{\alpha_0} \subset Hom_{\mathrm{P-cont}}(C_{\alpha_0}, \mathbb{F}_p)$;
- $L^0_{\alpha_0}$ consists of all finite $\mathbb{F}_p$-linear combinations of the elements $D^{(i)}_{a_0} \in C_{\alpha_0}$ from (3.3) which annihilate $\bar{c}$. This means that $L^0_{\alpha_0}$ is of index $p$ in $Hom_{\mathrm{P-cont}}(\bar{C}_{\alpha_0}, \mathbb{F}_p)$, $L^0$ is of index $p$ in $K^{PD}$, $A(L^P)$ is of index $p$ in $L^P$ and b) is proved in the case $n = 1$.

Inductive step.
Suppose $n \geq 2$ and b) is proved for field extensions of degree $p^{n-1}$.
Consider the tower $K \subset K_1 \subset K'$, $[K_1 : K] = p$, $[K' : K] = p^{n-1}$.
Using similar notation for $K'$ and $K_1$ we have:
(1) the fields tower $K \subset K_1 \subset K' \subset K_\leq p \subset K_{1,\leq p} \subset K'_{\leq p}$;
(2) the compatible identifications:
- $G_\leq p \simeq G(L) \subset G_{1,\leq p} \simeq G(L_1) \subset G'_\leq p \simeq G(L')$,
- $G^P_\leq p \simeq G(L^P) \subset G^P_{1,\leq p} \simeq G(L_1^P) \subset G^P_\leq p \simeq G(L^P)$
(3) the natural group homomorphisms:
- $\Theta_1 : G_{1,\leq p} \longrightarrow \Theta_1(G_{1,\leq p}) \subset G_\leq p$,
- $\Theta' : G_\leq p \longrightarrow \Theta'(G'_\leq p) \subset G_{1,\leq p}$
- $\Theta_1 : \Theta'(G_\leq p) \longrightarrow \Theta(G'_\leq p) \subset \Theta_1(G_{1,\leq p}) \subset G_\leq p$.
(4) the restrictions of the above $\Theta$, $\Theta'$, $\Theta_1$ to the corresponding $\mathcal{P}$-subgroups satisfy analogs of relations from c).

Note that $\text{Ker} \Theta_1 = \text{Gal}(\mathcal{K}_1/K_{<p}) = J$ is the profinite closure of $\mathcal{J}^p := \mathcal{G}_{1,<p}^p \cap J = \text{Ker} \Theta_1|_{\mathcal{G}_{1,<p}^p}$. Therefore, $\Theta_1(\mathcal{G}_{1,<p}) = \mathcal{G}_{1,<p}/\text{Ker} J^p$.

Similarly, $\Theta_1$ induces a group epimorphic map $\Theta'(\mathcal{G}_{<p}) \rightarrow \Theta(\mathcal{G}_{<p}^p)$ with the kernel $J$, the corresponding epimorphism $\Theta'(\mathcal{G}_{<p}^p) \rightarrow \Theta(\mathcal{G}_{<p}^p)$ has the kernel $J^p$ and $\Theta(\mathcal{G}_{<p}^p) = \Theta'(\mathcal{G}_{<p}^p)/J^p$.

Therefore, $(\mathcal{G}_{1,<p}^p : \Theta'(\mathcal{G}_{<p})) = (\Theta_1(\mathcal{G}_{1,<p}^p) : \Theta(\mathcal{G}_{<p}^p))$. By the inductive assumption, this index equals $p^{n-1}$. Finally, using the case $n = 1$ we obtain $(\mathcal{G}_{<p}^p : \Theta(\mathcal{G}_{<p}^p)) = p^n$. □

**Definition.** If $\mathcal{H} = \Theta(\mathcal{G}_{<p})$ then we set $\mathcal{H}^p = \mathcal{H} \cap \mathcal{G}_{<p}^p$.

Clearly, the conjugacy class of $\mathcal{H}^p$ in its profinite completion $\mathcal{H}$ is well defined.

**Corollary 3.7.** a) Any extension $\mathcal{K}'$ of $\mathcal{K}$ in $\mathcal{K}_{<p}$ (in the category of $N$-dimensional local fields) appears in the form $\mathcal{K}'_{<p}$, where $\mathcal{H}$ is the profinite completion of a $\mathcal{P}$-closed subgroup $\mathcal{H}^p$ of some $\mathcal{G}_{<p}^p \subset \mathcal{G}_{<p}$.

b) In the above notation, $\mathcal{K}'$ is Galois over $\mathcal{K}$ iff the subgroup $\mathcal{H}^p$ of $\mathcal{G}_{<p}^p$ is normal, and $\text{Gal}(\mathcal{K}'/\mathcal{K}) = \mathcal{G}_{<p}^p/\mathcal{H}^p$.

It remains to characterize the subgroups $\mathcal{H}^p \subset \mathcal{G}_{<p}^p$ such that $\mathcal{K}^p$ is an extension of $\mathcal{K}$ in $\mathcal{K}_{<p}$.

**Proposition 3.8.** Let $H \subset \mathcal{G}_{<p}^p$ be a subgroup. Then $H = \mathcal{H}^p$, where $\mathcal{K}^p = \mathcal{K}'$ is $N$-dimensional field extension of $\mathcal{K}$ iff

a) $(\mathcal{G}_{<p}^p : H) < \infty$;

b) $H$ is $\mathcal{P}$-open in $\mathcal{G}_{<p}^p$.

*Proof.* If $\mathcal{K}'$ is field extension of $\mathcal{K}$ in the category of $N$-dimensional fields then $[\mathcal{K}' : \mathcal{K}] = (\mathcal{G}_{<p}^p : \mathcal{H}) = (\mathcal{G}_{<p}^p : \mathcal{H}^p) < \infty$ and by Prop. 3.6 $\mathcal{H}^p$ is $\mathcal{P}$-closed. It is also $\mathcal{P}$-open as a closed subgroup of finite index in $\mathcal{G}_{<p}^p$.

To proceed in the opposite direction note that $H = G(L)$, where $L$ is a Lie subalgebra in $\mathcal{L}^p$ and the index $(\mathcal{L}^p : L)$ is a power of $p$. Choose an increasing sequence of Lie algebras $L = L_0 \subset L_1 \cdots \subset L_n = \mathcal{L}^p$ where each $L_{i-1}$ is ideal in $L_i$ and $(L_i : L_{i-1}) = p$. As a result, we can proceed by induction and it will be sufficient to consider the case $n = 1$.

Then $L \supset C_2(\mathcal{L}^p)$ and $L = C_2(\mathcal{L}^p) + L^0$, where

$$L^0 \subset \text{Hom}_{\mathcal{P}-\text{cont}}(\mathcal{K}, \mathcal{F}_p) = \lim_{\alpha} \text{Hom}_{\mathcal{P}-\text{cont}}(\mathcal{C}_\alpha, \mathcal{F}_p),$$

cf. notation from Sect. 3.2 Since $L^0$ is $\mathcal{P}$-open there is an index $\alpha_0$ such that $L^0$ is the preimage of a subgroup $L_{\alpha_0}$ of index $p$ in
the subgroup $H$ in $\text{Hom}(P)$. Therefore, there is $\bar{c} \in \tilde{C}_{\alpha_0}$ such that $L_{\alpha_0}^0 = \text{Ann} \bar{c}$ in $\text{Hom}_{\mathcal{P}-\text{cont}}(\tilde{C}_{\alpha_0}, \mathbb{F}_p)$.

Let $K' = \mathcal{K}(T')$, where $T' = T^\Pi = \Pi^{-1} \bar{c}$. Then $\mathcal{K}' = \mathcal{K}'_{<p}$, where the subgroup $\mathcal{H}$ of $\mathcal{G}_{<p}$ is such that $\mathcal{H} = G(C_2(\mathcal{L}) + L)$ and $L \subset \mathcal{K}'$ is characterized by the trivial action on $T'$. Therefore, the corresponding $\mathcal{P}$-subgroup $\mathcal{H}' = G(C_2(\mathcal{L}) + L')$, where $L'$ consists of finite $\mathbb{F}_p$-linear combinations of the elements $D_a$ from \([3,3]\) which annihilate $c$. Therefore, $L' = L^0$ and $\mathcal{H}' = H$.

3.6. **More general $\mathcal{P}$-groups.** Suppose $G' \subset \text{Aut} \mathcal{K}'$. For example, $\mathcal{K}'/\mathcal{K}$ is Galois and $G' = \text{Gal}(\mathcal{K}'/\mathcal{K})$. Consider the group $\Gamma' \subset \text{Aut} \mathcal{K}'_{<p}$ of all lifts of the elements of $G'$ to $\mathcal{K}'_{<p}$. These lifts can be treated in terms of couples $(C_{g'}^g, A_{g'}^g)$, where $g \in G'$, $C_{g'}^g \in \mathcal{L}_{\mathcal{K}'}$, and $A_{g'}^g \in \text{Aut}_{\mathcal{L}_{\mathcal{K}'}}$. This description uses the identification $\pi' = \pi_f(e') : G'_{<p} \simeq G(\mathcal{L}')$. After applying $\pi'^{-1}$ we obtain the exact sequence

$$1 \rightarrow G'_{<p} \rightarrow \Gamma' \rightarrow G' \rightarrow 1.$$  

Consider a subgroup $\Gamma'^{\mathcal{P}}$ of $\Gamma'$ coming from $C_{g'}^g \in \mathcal{L}_{\mathcal{K}'}^{\mathcal{P}} = \mathcal{L}^{\mathcal{P}} \otimes \mathcal{P} \mathcal{K}'$ and $\mathcal{P}$-continuous $A_{g'}^g$. (For example, use the $\mathcal{P}$-continuous operators $\mathcal{R}$ and $\mathcal{S}$ from Sect.3.1 to recover the corresponding pairs $(C_{g'}^g, A_{g'}^g)$, cf. e.g. Sect.1.2 below.) We obtain the following exact sequence

$$1 \rightarrow G'_{<p}^{\mathcal{P}} \rightarrow \Gamma'^{\mathcal{P}} \rightarrow G' \rightarrow 1.$$  

This construction of the subgroup $\Gamma'^{\mathcal{P}}$ of $\Gamma'$ does not depend on a choice of “$\mathcal{P}$-continuous” lifts of elements of $g \in G'$ (all such lifts differ by elements of $G'_{<p}^{\mathcal{P}}$).

The above construction in the case $G' = \text{Gal}(\mathcal{K}'/\mathcal{K})$ allows us to recover (uniquely up to isomorphism) the group $G'_{<p}$ from $\mathcal{H}' = \Theta(G'_{<p})$. Even more, if $\mathcal{K} \subset \mathcal{K}_1 \subset \mathcal{K}_2 \subset \mathcal{K}_{<p}$, $\mathcal{H}_i = \text{Gal}(\mathcal{K}_{<p}/\mathcal{K}_i)$, $\mathcal{H}'_i = \mathcal{H}_i \cap G'_{<p}$ with $i = 1, 2$, and $\mathcal{K}_2/\mathcal{K}_1$ is Galois then $\mathcal{H}'_2 = \mathcal{H}_2 \cap G'_{<p}$ is uniquely (up to isomorphism) recovered from $\mathcal{H}'_1$.

4. **The groups $\mathcal{G}'_{\omega}$ and $\Gamma'^{\mathcal{P}}$**

As earlier, $\mathcal{K}$ is $N$-dimensional local field of characteristic $p$ with fixed system of local parameters $t = \{t_1, \ldots, t_N\}$ and the last residue field $k \cong \mathbb{F}_p^{\tau_0}$. Fix $\alpha_0 \in k$ such that $\text{Tr}_{k/\mathbb{F}_p}(\alpha_0) = 1$. Let $S = S_{t, \alpha_0}$ be the section from Prop.3.2.

Take $e := e_S = \sum_{\alpha \in \mathbb{Z}_N^+} t^{-a}D_\alpha + \alpha_0 D_0$, choose $f \in \mathcal{F}(e)$ and consider $\pi = \pi_f(e) : G_{<p} \simeq G(\mathcal{L})$. 
Fix $\tilde{c}^0 = (c_1^0, \ldots, c_N^0) \in p\mathbb{Z}^N$ such that $c_0^0 > 0$, set $t^{c_0^0} := t_1^{c_1^0} \cdots t_N^{c_N^0}$. Choose $\omega = \sum_{i > 0} \beta_i t^{(i/p)+1} \in t^{c_0^0} \mathcal{O}_K^1$, where all $\beta_i = \beta_i(\omega) \in k; \beta_0 \neq 0$.

Let $E(X) = \exp \left( \sum_{i \geq 0} X^{i/p} / p^i \right)$ be the Artin-Hasse exponential.

4.1. Automorphisms $h_\omega^{(m)}$. For $1 \leq m \leq N$, let $h_\omega^{(m)}$ be the $\mathcal{P}$-continuous automorphism of $\mathcal{K}$ such that $h_\omega^{(m)}|_k = 1$, $h_\omega^{(m)}(t_m) = t_m E(\omega^p)$ and for all $j \neq m$, $h_\omega^{(m)}(t_j) = t_j$. Let $m_K$ be the maximal ideal in $\mathcal{O}_K$. Clearly, $m_K$ consists of all $\mathcal{P}$-convergent $k$-linear combinations of $t^p$, where $a \in \mathbb{Z}^N$.

For $n \in \mathbb{Z}$, let $h_\omega^{(m)n}$ be the $n$-th iteration of $h_\omega^{(m)}$ and, similarly, denote by $h_\omega^{(m_1)} h_\omega^{(m_2)}$ the composition of $h_\omega^{(m_1)}$ and $h_\omega^{(m_2)}$.

**Proposition 4.1.**

a) For any $n \geq 0$, $h_\omega^{(m)n}(t_m) \equiv t_m E(n \omega^p) \bmod t^{p^n} m_K$;

b) $h_\omega^{(m_1)} h_\omega^{(m_2)} \equiv h_\omega^{(m_2)} h_\omega^{(m_1)} \bmod t^{p^n} m_K$.

**Proof.** Note that $h_\omega^{(m)}(t_m) \equiv t_m \bmod t^{p^n} m_K$ and this implies for any $t \geq 0$, that $h_\omega^{(m)}(t^{e_p^m}) \equiv t^{e_p^m} \bmod t^{p^n} m_K$. As a result,

\[
(4.1) \quad h_\omega^{(m)}(\omega^p) \equiv \omega^p \bmod t^{p^n} m_K.
\]

Apply induction on $n \geq 0$ to prove part a) of the proposition.

If it is proved for some $n \geq 0$ then

\[
h_\omega^{(m)n+1}(t_m) \equiv h_\omega^{(m)}(t_m E(n \omega^p)) \equiv t_m E(\omega^p) E(n \omega^p) \equiv t_m E((n+1) \omega^p)
\]

modulo $t^{p^n} m_K$ (use that $E(X + Y) \equiv E(X)E(Y) \bmod \deg p$).

Similarly, relation (4.1) implies part b). \hfill \Box

**Remark.** The above proposition can be stated also for the truncated exponential $\exp(X) = 1 + X + \cdots + X^{p^{-1}/(p-1)!}$ instead of $E(X)$.

4.2. The groups $\mathcal{G}_\omega$ and $\mathcal{G}_\omega^\mathcal{P}$. Let $\hat{h}_\omega^{(m)} \in \mathrm{Aut}\mathcal{K}_{<p}$ be such that $\hat{h}_\omega^{(m)}|_K = h_\omega^{(m)}$. Denote by $\mathcal{G}_\omega$ the subgroup in $\mathrm{Aut}\mathcal{K}_{<p}$ generated by the elements of $\mathcal{G}_{<p}$ and the lifts $\hat{h}_\omega^{(m)}$ with $1 \leq m \leq N$. The elements $\hat{h}_\omega$ of $\mathcal{G}_\omega$ are characterised by the property $\hat{h}_\omega|_K \in \langle h_\omega^{(1)}, \ldots, h_\omega^{(N)} \rangle \subset \mathrm{Aut}\mathcal{K}$.

They can be uniquely specified by the couples $(C, \mathcal{A}) \in \mathcal{L}_\mathcal{K} \times \mathrm{Aut}\mathcal{L}$ such that

\[
\hat{h}_\omega(f) = C \circ (\mathcal{A} \otimes \text{id}) f
\]

or, equivalently, such that (where $h_\omega = \hat{h}_\omega|_K$)

\[
(4.2) \quad (\text{id}_\mathcal{L} \otimes h_\omega)e = \sigma C \circ (\mathcal{A} \otimes \text{id})e \circ (-C).
\]

If $\hat{h}_\omega'$ corresponds to $(C', \mathcal{A}')$ then the composition $\hat{h}_\omega' \cdot \hat{h}_\omega$ corresponds to the couple $(h_\omega'(C) \circ \mathcal{A}(C'), \mathcal{A} \mathcal{A}')$, where $h_\omega' = \hat{h}_\omega'|_K$. With this notation the subgroup $\mathcal{G}_{<p} \subset \mathcal{G}_\omega$ is identified with the subgroup of
couples \((l, \mathrm{Ad} l)\), where \(l \in \mathcal{L}\). Indeed, under the identification \(\pi = \pi_f(e) : G_{<p} \cong G(\mathcal{L})\) from Prop. 4.1 if \(\tau \in G_{<p}\) then \(\tau(f) = f \circ l = l \circ (\mathrm{Ad} l \otimes \text{id}) f\).

Suppose \(1 \leq m \leq N\) and \(\hat{h}_\omega^{(m)}\) is specified via the couple \((C^{(m)}, A^{(m)})\). Relation (4.2) can be treated via the following recurrent procedure.

Suppose \(s \geq 1\) and the couple \((C^{(m)}_s, A^{(m)}_s)\) satisfies relation (4.2) modulo \(s\)-th commutators \(C_s(\mathcal{L}_K)\). Use the operators \(\mathcal{R}\) and \(\mathcal{S}\) from Sect. 3.1 to obtain \(C_s' \in C_{s+1}(\mathcal{L}_K)\) and \(A_s' \in \text{Hom}_{\text{Lie}}(\mathcal{L}, C_{s+1}\mathcal{L})\) such that

\[
\sigma C_s' - C_s' + (A_s' \otimes \text{id}_K) e \equiv (\text{id}_\mathcal{L} \otimes h^{(m)}_\omega) e - \sigma C_s^{(m)} \circ (A^{(m)}_s \otimes \text{id}_K) e \circ (-C^{(m)}_s) \mod C_{s+1}(\mathcal{L}_K).
\]

Then the couple \((C_s + C_s', A_s + A_s')\) satisfies (4.2) modulo \(C_{s+2}(\mathcal{L}_K)\).

Denote by \(\hat{h}_\omega^{(m)}\) the lift of \(h^{(m)}_\omega\) which corresponds to the couple \((C^{(m)}_0, A^{(m)}_0) := (C^{(m)}_p, A^{(m)}_p)\). Note that \(C^{(m)}_0 \in \mathcal{L}_K^p\) and \(A^{(m)}_0 \mid_{\mathcal{L}_F}\) is a \(\mathcal{P}\)-continuous automorphism of the Lie algebra \(\mathcal{L}^p\).

Using that \((\text{id}_\mathcal{L} \otimes h^{(m)}_\omega) e \in \mathcal{L}^p_K\) and \(\mathcal{L} \cap \mathcal{L}^p = \mathcal{L}^p\) we obtain the following property:

**Proposition 4.2.** A lift \(\hat{h}_\omega^{(m)}\) corresponds to a couple \((C^{(m)}, A^{(m)})\) with \(C^{(m)} \in \mathcal{L}_K^p\) and \(A^{(m)} \in \text{Aut}_{\mathcal{P}\text{-cont}}(\mathcal{L}^p)\), if and only if there is \(l \in \mathcal{L}^p\) such that \(C^{(m)} = C^{(m)}_0 \circ l\) and \(A^{(m)} = \text{Ad} l \cdot A^{(m)}_0\).

**Definition.** \(\mathcal{G}^p_\omega \subset \mathcal{G}_\omega\) is a subgroup generated by \(\mathcal{G}^p_{<p} = \pi^{-1} \mathcal{L}^p\) and the lifts \(\hat{h}_\omega^{(m)}\), \(1 \leq m \leq N\).

**Remark.** (i) The elements of the group \(\mathcal{G}^p_\omega\) are specified via the couples \((C, A) \in \mathcal{L}_K^p \times \text{Aut}_{\mathcal{P}\text{-cont}}(\mathcal{L}^p)\) (which satisfy relation (4.2)).

(ii) The profinite completion of \(\mathcal{G}^p_\omega\) coincides with \(\mathcal{G}_\omega\).

Obviously, we have the following natural short exact sequences:

\[
(4.3) \quad 1 \rightarrow \mathcal{G}_{<p} \rightarrow \mathcal{G}_\omega \rightarrow \langle h^{(1)}_\omega, \ldots, h^{(N)}_\omega \rangle \rightarrow 1,
\]

\[
(4.4) \quad 1 \rightarrow \mathcal{G}^p_{<p} \rightarrow \mathcal{G}^p_\omega \rightarrow \langle h^{(1)}_\omega, \ldots, h^{(N)}_\omega \rangle \rightarrow 1.
\]

The structure of (4.3) can be uniquely recovered from (4.4) by going to profinite completions.

4.3. The commutator subgroups \(C_s(\mathcal{G}^p_\omega)\). Define the weight function in \(\mathcal{L}^p_K\) by setting for \(s \in \mathbb{N}\) and \((s-1)c^0 \leq a \lt sc^0\),

\[
\text{wt}(D_{an}) = s.
\]

Introduce the ideal \(\mathcal{L}^p(\omega)(s)\) of \(\mathcal{L}^p\) such that \(\mathcal{L}^p(\omega)(s)_k\) is generated by all \([\ldots[D_{a_{11}}, D_{a_{21}}], \ldots, D_{a_{nn}}]_k\) with \(\sum_i \text{wt}(D_{a_{ni}}) \geq s\). Clearly, for any \(s_1, s_2\), it holds \([\mathcal{L}^p(\omega)(s_1), \mathcal{L}^p(\omega)(s_2)] \subset \mathcal{L}^p(\omega)(s_1 + s_2)\).
Consider the lifts $h^{0(m)}_\omega \in G^P_\omega$ from Sect. 4.2. Denote by $\text{Ad}^{(m)}$ the automorphism of $G(\mathcal{L}^P)$ obtained from conjugation by $h^{0(m)}_\omega$ on $G^P_{<p}$ with respect to the identification $\pi(= \pi_f(e)) : G^P_{<p} \simeq G(\mathcal{L}^P)$.

Let for $a \in \mathbb{Z}_N^+(p)$, $\text{Ad}^{(m)}_k(D_{a0}) = D^{(m)}_a$ and $\text{Ad}^{(m)}_k(D_0) = D^{(m)}_0$.

**Lemma 4.3.** For any $1 \leq m, m' \leq N$,

a) $D^{(m)}_0 \equiv D_0 \mod \mathcal{L}^P_0 (3) + \mathcal{L}^P_0 (2) \cap C_2(\mathcal{L}^P)$;

b) if $a = (\tilde{a}^{(1)}, \ldots, \tilde{a}^{(N)}) \in \mathbb{Z}_N^+(p)$ and $\text{wt}(D_{a0}) = s$ then

$$D^{(m)}_{a0} \equiv D_{a0} - \sum_{i \geq 0} A_i a^{(m)} \bar{s} \in \mathcal{L}^P_0 (s + 2)_k + \mathcal{L}^P_0 (s + 1)_k \cap C_2(\mathcal{L}^P_k),$$

where the elements $A_i \in k$ are such that $E(\omega^P) = 1 + \sum A_i t^{p_\omega^P + p_0}$;

c) the commutator $(\tilde{h}^{0(m)}_\omega, \tilde{h}^{0(m')}_\omega) \in \pi^{-1} G(\mathcal{L}^P_0 (2))$.

We shall prove this lemma after finishing the proof of Prop. 4.4 below. Note that lemma 4.3 implies $\pi C_2(\mathcal{G}^P_\omega) \subset G(\mathcal{L}^P_0 (2))$.

Set $\mathcal{L}^P_0 (1) = \mathcal{L}^P$.

For $s \geq 2$, let $\mathcal{L}^P_\omega^P (s) \subset \mathcal{L}^P$ be such that $\pi C_s(\mathcal{G}^P_\omega) = G(\mathcal{L}^P_\omega^P (s))$.

**Proposition 4.4.** For $1 \leq s \leq p$, $\mathcal{L}^P_\omega^P (s) = \mathcal{L}^P_\omega^P (s)$.

**Proof.** Use induction on $s \geq 1$.

Clearly, $\mathcal{L}^P_\omega^P (1) = \mathcal{L}^P_\omega^P (1)$.

Suppose $s_0 \geq 2$ and for $1 \leq s \leq s_0$, $\mathcal{L}^P_\omega^P (s) = \mathcal{L}^P_\omega^P (s)$.

Let $\mathcal{L}^P_\omega^P \omega (s, m) = \sum_{a \in \mathbb{Z}_N^+(p)} k D_{a0} \cap \mathcal{L}^P$ be “the subspace of linear terms” in $\mathcal{L}^P$.

We have the following properties for all $s \leq s_0$:

- $\mathcal{L}^P_\omega^P (s + 1) = \mathcal{L}^P_\omega^P (s + 1) \cap \mathcal{L}^P_\omega^P + \mathcal{L}^P_\omega^P (s + 1) \cap C_2(\mathcal{L}^P)$;

- $\mathcal{L}^P_\omega^P (s + 1) \cap C_2(\mathcal{L}^P) = \sum_{s_1 + s_2 = s + 1} \mathcal{L}^P_\omega^P (s_1), \mathcal{L}^P_\omega^P (s_2)$;

- $\mathcal{L}^P_\omega^P (s + 1)$ is the ideal in $\mathcal{L}^P$ generated by $[\mathcal{L}^P_\omega^P (s), \mathcal{L}^P]$ and the elements $\text{Ad}^{(m)}_l (l) \circ (-l)$, where $l \in \mathcal{L}^P_\omega^P (s)$ and $1 \leq m \leq N$. (If $s_0 = 1$ we do need part c) of Lemma 4.3.)

Now statements a) and b) of Lemma 4.3 imply:

(c1) if $l \in \mathcal{L}^P_\omega^P (s)$ then $\text{Ad}^{(m)}_l (l) \circ (-l) \in \mathcal{L}^P_\omega^P (s + 1)$;

(c2) if $l \in \mathcal{L}^P_\omega^P \cap \mathcal{L}^P_\omega^P (s + 1)$ then there are $m$ and $l' \in \mathcal{L}^P_\omega^P \cap \mathcal{L} (s)$ such that

$$\text{Ad}^{(m)}_l (l') \circ (-l') \equiv l \mod \mathcal{L}^P_\omega^P (s + 1) \cap C_2(\mathcal{L}^P)$$

(use that $A_0 \neq 0$ and for any $a = (a^{(1)}, \ldots, a^{(N)}) \in \mathbb{Z}_N^+(p)$, there is $m$ such that $a^{(m)} \neq 0 \mod p$).
Then $[\mathcal{L}_\omega^P(s_0), \mathcal{L}^P] = [\mathcal{L}_\omega^P(s_0), \mathcal{L}^P(1)] \subset \mathcal{L}_\omega^P(s_0 + 1)$ and applying (c1) we obtain $\mathcal{L}_\omega^P(s_0 + 1) \subset \mathcal{L}_\omega^P(s_0 + 1)$.

For the opposite direction, note that by the inductive assumption,

$$\mathcal{L}_\omega^P(s_0 + 1) \cap C_2(\mathcal{L}^P) = \sum_{s_1 + s_2 = s_0 + 1} [\mathcal{L}_\omega^P(s_1), \mathcal{L}_\omega^P(s_2)] \subset \mathcal{L}_\omega^P(s_0 + 1)$$

and then (c2) implies that $\mathcal{L}_\omega^P \cap \mathcal{L}_\omega^P(s_0 + 1) \subset \mathcal{L}_\omega^P(s_0 + 1)$. As a result, $\mathcal{L}_\omega^P(s_0 + 1) \subset \mathcal{L}_\omega^P(s_0 + 1)$ and our proposition is proved. □

**Proof of Lemma 4.3.** Let

$$\mathcal{N} = \sum_{s \geq 1} t^{-\sigma^0 s} \mathcal{L}_\omega^P(s)_{m_K},$$

where $m_K$ is the maximal ideal of the $\mathcal{N}$-valuation ring $\mathcal{O}_K$ of $\mathcal{K}$. Clearly, $\mathcal{N}$ has an induced structure of a Lie algebra over $\mathbb{F}_p$ and $\mathcal{N} \subset \mathcal{L}_\omega^P$.

Let $e^{(m)} := (A_{\mathcal{K}}^{(m)} \otimes id_\mathcal{K}) e = \sum_{a \in Z_N(p)} t^{-a} D^{(m)}_{a0} + a_0 D^{(m)}_{00}.$

The recovering of $C^{(0)(m)} \in G(\mathcal{L}_\omega^P)$ and $e^{(m)}$ from relation

$$(4.5) \quad (id_{\mathcal{L}_\omega^P} \otimes h^{(m)}_\omega) e \circ C^{(0)(m)} = (\sigma C^{(0)(m)}) \circ e^{(m)},$$

is a part of the recurrent procedure from Sect. 4.2. Clearly, the operators $\mathcal{S}$ and $\mathcal{R}$ from Sect. 3.1 map $\mathcal{N}$ to itself. Therefore, when following the recurrent procedure we remain at each step in $\mathcal{N}$. As a result, all $e^{(m)}, C^{(0)(m)}, \sigma C^{(0)(m)} \in \mathcal{N}$.

For any $j \geq 0$, introduce the ideals $\mathcal{N}(j) := t^{\sigma^0 j} \mathcal{N}$ of $\mathcal{N}$. The operators $\mathcal{S}$ and $\mathcal{R}$ also map the ideals $\mathcal{N}(j)$ to itself.

The following properties are obtained by direct calculations:

(i) $(id_{\mathcal{L}_\omega^P} \otimes h^{(m)}_\omega) e = e + e^{(m)}_1 \bmod \mathcal{N}(2), \quad e^{(m)}_1 = e^{(m)}_1^{(m)+} + e^{(m)}_1^{(m)-} \in \mathcal{N}(1),$

$e^{(m)-}_1 = - \sum_{a \geq 0} t^{-a} a^{(m)} A_t D^{a+\sigma^0 + pt, 0}, \quad e^{(m)+}_1 = - \sum_{a \geq 0} a^{(m)} A_t t^{-a+\sigma^0 + pt} D_{a0}.$

(note that $e^{(m)+}_1 \in \mathcal{L}_{m_K}^P$ and, therefore, $\mathcal{S}(e^{(m)+}_1) = 0$);

(ii) the congruence $(id_{\mathcal{L}_\omega^P} \otimes h^{(m)}_\omega) e \equiv e \bmod \mathcal{N}(1)$ implies that $e^{(m)} \equiv e \bmod \mathcal{N}(1)$ and $C^{(0)(m)}, \sigma C^{(0)(m)} \in \mathcal{N}(1)$. Indeed, in the procedure of specification of $h^{(m)}_\omega$ it holds that for all $s$, $C^{(m)}_s, \sigma C^{(m)}_s \in \mathcal{N}(1)$ and $(A^{(m)}_t \otimes id_\mathcal{K}) e \equiv e \bmod \mathcal{N}(1);$ $(4.5)$

(iii) $e^{(m)} = (-\sigma C^{(0)(m)}) \circ (id_{\mathcal{L}_\omega^P} \otimes h^{(m)}_\omega) e \circ C^{(0)(m)} = (C^{(0)(m)} - \sigma C^{(0)(m)}) + e + e^{(m)}_1 \bmod \mathcal{N}(2) + t^{\sigma^0} \mathcal{N}(2)$, where $\mathcal{N}(2) := \sum_{s \geq 2} t^{-s\sigma^0} (\mathcal{L}_\omega^P(s) \cap C_2(\mathcal{L}^P))_{m_K}$

(use that $[\mathcal{N}(1), \mathcal{N}(1)] \subset \mathcal{N}(2)$ and $[\mathcal{N}(1), \mathcal{N}] \subset t^{\sigma^0} \mathcal{N}(2)$).
(iv) $S(N(2) + t^{\sigma_1} \tilde{N}(2)) \subset N(2) + t^{\sigma_1} \tilde{N}(2)$, $S(e^{(m)} - e - e_1^{(m)}) = e^{(m)} - e - e_1^{(m)}$, $S(C^{(m)} - \sigma C^{(m)} + e_1^{(m)}) = 0$. Therefore, item (iii) implies

$$e^{(m)} \equiv e + e_1^{(m)} \mod N(2) + t^{\sigma_1} \tilde{N}(2).$$

More explicitly,

$$e^{(m)} \equiv \sum_{a \in \mathbb{Z}_N(p)} t^{-a} \left(D_{a0} - a^{(m)} \sum_{i \geq 0} A_i D_{a + i, 0} \right) + \alpha_0 D_0$$

modulo $N(2) + t^{\sigma_1} \tilde{N}(2)$.

Deduce from this congruence statements a) and b) of our lemma.

Consider the presentation of an element $l \in L^P_K$ in the form of a $\mathcal{P}$-convergent series $l = \sum_{b \in \mathbb{Z}_N} t^b l_b$, with all $l_b \in L^P_K$.

Suppose $s \geq 1$ and $-(s - 1)\bar{c} > b > -s\bar{c}$.

It follows directly from definitions that:

— if $l \in N$ then $l_b \in L^P_{\mathcal{P}}(s)_K$;

— if $l \in N(2)$ then $l_b \in L^P_{\mathcal{P}}(s + 2)_K$;

— if $l \in t^{\sigma_1} \tilde{N}(2)$ then $l_b \in L^P_{\mathcal{P}}(s + 1)_K \cap C_2(L^P_K)$.

As a result, the parts a) and b) of lemma are obtained by comparing coefficients in (4.6).

Now note that for any $m_1$,

$$e^{(m_1)}(id_{L^P} \otimes h^{(m_1)}_{\omega}) \equiv e + e_1^{(m_1)} \mod N(2).$$

Let $N_{<p} = \sum_{s \geq 1} t^{-s_0} L^P_{\mathcal{P}}(s)_{m_{<p}}$, where $m_{<p}$ is the maximal ideal of the $N$-valuation ring $O_{N_{<p}}$. Again, $N_{<p}$ has the induced structure of a Lie $\mathbb{F}_p$-algebra and for any $j \geq 0$, $N_{<p}(j) = t^{j\bar{c}_0} N_{<p}$ is ideal in $N_{<p}$.

As earlier, $f, \sigma f \in N_{<p}$, and congruence (4.7) implies that

$$(id_{L^P} \otimes h^{(m_1)}_{\omega}) f \equiv e + e_1^{(m_1)} \mod N_{<p}(2).$$

where $f^{(m_1)} \in N_{<p}(1)$ is such that $\sigma f^{(m_1)} - f^{(m_1)} = e^{(m_1)}$.

This implies

$$(id_{L^P} \otimes h^{(m_1)}_{\omega}) f \equiv (id_{L^P} \otimes h^{(m_1)}_{\omega}) f \equiv f + f^{(m_1)} \mod N_{<p}(2)$$

and, therefore, $(id_{L^P} \otimes h^{(m_1)}_{\omega}) f \equiv f \mod N_{<p}(2)$.

On the other hand, the commutator $(h^{(m_1)}_{\omega}, h^{(m_1)}_{\omega})$ is a lift of $id_K$, i.e. it coincides with $\pi^{-1}(l_{mm_1}) \in G^P_{<p}$. Therefore, $l_{mm_1} \in L^P \cap N_{<p}(2) = L^P_{\mathcal{P}}(2)$. The part c) is proved.
4.4. The group $\Gamma^p$. Let $\hat{L} = L/L_{c_0}(p)$ and $\hat{L}^p = L^P/L^p_{c_0}(p)$. Then $\hat{L}^p$ is dense in $\hat{L}$. If $\mathcal{K}(p) = \mathcal{K}_c^{\hat{G}(L_c^0(p))}$ then $\hat{G} := \text{Gal}(\mathcal{K}(p)/\mathcal{K})$ and the identification $\pi_f(e)$ induces the identification $\bar{\pi} : \hat{G} \simeq G(\hat{L})$. This identification can be obtained via nilpotent Artin-Schreier theory: for $\bar{e} \in \hat{L}$ and $\bar{f} \in \hat{L}_{\mathcal{K}(p)}$, we have $\sigma \bar{f} = \bar{e} \circ \bar{f}$ and $\bar{\pi} = \pi_f(\bar{e})$. However, the algebra $\hat{L}_K$ is too big for the process of linearization, cf. below. This motivates the following definitions.

Let

$$M := \sum_{1 \leq s < p} t^{-s \hat{d}} L^0(s)_{m_K} + L^0(p)_{\mathcal{K}}$$

$$M^p := \sum_{1 \leq s < p} t^{-s \hat{d}} L^p_0(s)_{m_K} + L^p_0(p)_{\mathcal{K}}$$

$$M_{<p} := \sum_{1 \leq s < p} t^{-s \hat{d}} L^0(s)_{m_{<p}} + L^0(p)_{\mathcal{K}_{<p}}$$

where (as earlier) $m_{<p}$ is the maximal ideal of the $N$-valuation ring $\mathcal{O}_{\mathcal{K}_{<p}}$. Then $M$ and $M^p$ have the induced structure of Lie $\mathbb{F}_p$-algebras (use the Lie bracket from $\mathcal{L}_K$). For $s \geq 0$, $M(s) := t^{s \hat{d}} M$ and $M^p(s) := t^{s \hat{d}} M^p$ form a decreasing central filtrations of ideals in $\mathcal{M}$ and $\mathcal{M}^p$. Similarly, $M_{<p}$ is a Lie $\mathbb{F}_p$-algebra (containing $M$ as its subalgebra), for $s \geq 0$, $M_{<p}(s) := t^{s \hat{d}} M_{<p}$ is a decreasing central filtration of ideals in $\mathcal{M}_{<p}$ and $M_{<p}(s) \cap M^p = M^p(s)$. It can be easily seen that $e \in M^p$ and $f, \sigma f \in M_{<p}$.

There is a natural embedding

$$\tilde{M}^p := M^p/M^p(p-1) \subset \tilde{M}_{<p} := M_{<p}/M_{<p}(p-1),$$

and the induced decreasing filtrations of ideals $\tilde{M}^p(s)$ and $\tilde{M}_{<p}(s)$ (where $\tilde{M}^p(p-1) = M_{<p}(p-1) = 0$) are compatible with this embedding. For all $s \geq 0$, $(id_{\tilde{L}} \otimes h^0_{\omega}) - id_{\tilde{M}^p})^* \tilde{M}^p \subset \tilde{M}^p(s)$.

The algebras $\tilde{M}^p$ and $\tilde{M}_{<p}$ are naturally identified with the following subquotients of $\hat{L}^p_K$ and $\hat{L}_{<p}$:

$$\tilde{M}^p = \left( \sum_{1 \leq s < p} t^{-s \hat{d}} \tilde{L}^p_0(s)_{m} \right) \otimes \mathcal{O}_K/t^{(p-1)\hat{d}}$$

$$\tilde{M}_{<p} = \left( \sum_{1 \leq s < p} t^{-s \hat{d}} \tilde{L}^0_0(s)_{m_{<p}} \right) \otimes \mathcal{O}_{\mathcal{K}_{<p}}/t^{(p-1)\hat{d}}.$$ 

We can see easily that $\bar{e} \otimes 1 \in \tilde{M}^p$, $\bar{f} \otimes 1$, $\sigma \bar{f} \otimes 1 \in \tilde{M}_{<p}$ and $\sigma \bar{f} \otimes 1 = (\bar{e} \otimes 1) \circ (\bar{f} \otimes 1)$. The following property shows that we still have full control of the identification $\bar{\pi}$.

**Proposition 4.5.** The correspondence $\tau \mapsto (-\bar{f} \otimes 1) \circ \tau(\bar{f} \otimes 1)$ induces the natural projections $\mathcal{G}_{<p} \longrightarrow \bar{G} \simeq G(\hat{L})$ and $\mathcal{G}^p_{<p} \longrightarrow \bar{G}^p \simeq G(\hat{L}^p)$. 

Proof. $(-\bar{f} \otimes 1) \circ \tau \bar{f} \otimes 1$ comes from $(-f) \circ \tau f \in G(L)$. It remains to notice that $L \cap M(p-1) = \mathcal{L}_{c_0}(p)$ and $L \cap M^P(p-1) = \mathcal{L}_{c_0}^P(p)$. \hfill \Box

Remark. In the above setting we can replace $\mathcal{M}_{<\rho}$ by its analogue $\mathcal{M}_{K(p)}$, where the field $K(p)$ is used instead of $\mathcal{K}_{<\rho}$ (because $\bar{f} \in \mathcal{L}_{c(p)}$).

Let $\Gamma^P_\omega := \mathcal{G}_\omega^P/(\mathcal{G}_\omega^P)P_c(\mathcal{G}_\omega^P)$. Then $\Gamma_\omega := \mathcal{G}_\omega/P_c(\mathcal{G}_\omega)$ can be recovered as the pro-finite completion of $\Gamma^P_\omega$.

Proposition 4.6. Exact sequence (4.3) induces the following exact sequence of profinite p-groups

$$1 \rightarrow \mathcal{G}^P \rightarrow \Gamma^P_\omega \rightarrow \prod_{1 \leq m \leq N} \langle h^{(m)}_\omega \rangle / \langle h^{(m)p}_\omega \rangle \rightarrow 1.$$  

Proof. Consider the orbit of $\bar{f} \otimes 1$ with respect to the natural action of $\mathcal{G}_\omega \subset \text{Aut} \mathcal{K}_{<\rho}$ on $f$ (recall that all “values” of $f$ belong to $\mathcal{K}_{<\rho} \subset \mathcal{K}_{s\rho}$). Then the stabilizer $H$ of $\bar{f} \otimes 1$ equals $\mathcal{G}_\omega^P C_p(\mathcal{G}_\omega)$. This fact and the remaining part of the proof go along the lines of Prop. 3.5 from [6]. \hfill \Box

Suppose $\pi_\omega : \Gamma^P_\omega \simeq G(L^P_\omega)$ extends $\bar{\pi} = \pi_f(\bar{e})$ for a suitable Lie $F_p$-algebra $L^P_\omega$ containing $L^P$. Then the automorphisms $h^{(m)}_\omega$ give rise to Lie elements $l^{(m)}_\omega$ and we obtain the following property.

Corollary 4.7. There is a natural exact sequence of Lie $F_p$-algebras

$$0 \rightarrow \bar{L}^P \rightarrow L^P_\omega \rightarrow \prod_{1 \leq m \leq N} F_p l^{(m)}_\omega \rightarrow 0. \quad (4.8)$$

We recover the structure of $L^P_\omega$ below by analyzing the orbit of $\bar{f}$.

4.5. Filtered module $\mathcal{M}^f$ and the procedure of linearization.

Let $\bar{h}^{(m)}_\omega \in \Gamma^P_\omega$ be a lift of $h^{(m)}_\omega$ to $K(p)$. We use below the notation $\bar{h}^{(m)}_\omega$ for the corresponding element $\pi_\omega(\bar{h}^{(m)}_\omega) \in L^P_\omega$. For example, cf. Sect. 4.3 if $\bar{h}^{(m)}_\omega = \pi_\omega(\bar{h}^{(m)}_\omega)_{K(p)}$ then the notation $\text{Ad}^{(m)}(\bar{h}^{(m)}_\omega)$ appears as $\text{Ad}(\bar{h}^{(m)}_\omega)$. Let $\Gamma^{(m)}_\omega$ be a subgroup in $\Gamma^P_\omega$ generated by $\bar{h}^{(m)}_\omega$ and $\bar{G}^P = \bar{\pi}^{-1}G(\bar{L}^P)$ (clearly, it does not depend on the choice of $\bar{h}^{(m)}_\omega$). Then we have the following exact sequence

$$1 \rightarrow \mathcal{G}^P \rightarrow \Gamma^{(m)}_\omega \rightarrow \langle h^{(m)}_\omega \rangle / \langle h^{(m)p}_\omega \rangle \rightarrow 1.$$  

Let $L^{(m)}_\omega$ be a Lie subalgebra in $L^P_\omega$ such that $\pi_\omega(\Gamma^{(m)}_\omega) = G(L^{(m)}_\omega)$. We obtain the following exact sequence of $F_p$-Lie algebras

$$0 \rightarrow \bar{L}^P \rightarrow L^{(m)}_\omega \rightarrow F_p l^{(m)}_\omega \rightarrow 0,$$

obtained from (4.8) via the natural embedding $F_p l^{(m)}_\omega \rightarrow \prod_{1 \leq m \leq N} F_p l^{(m)}_\omega$.

The structure of the Lie algebras $L^{(m)}_\omega$ (as well as the groups $\Gamma^{(m)}_\omega$) can be studied via the "linearization techniques" from [8 7].
Namely, the action of \( \text{id}_L \otimes \bar{h}^{(m)}_\omega \) on \( \hat{L}^p_K \), which can be presented in the form \( \exp(\text{id}_L \otimes dh^{(m)}_\omega) \), where \( \text{id}_L \otimes dh^{(m)}_\omega \) is a derivation on \( \hat{M}^p \). Indeed, the elements of \( \hat{M}^p \) can be written uniquely as sums of elements of the form \( l \otimes t^{-a} \), where for some \( 1 \leq s < p \), \( (s-1)t^0 \leq a < st^0 \) and \( l \in \hat{L}^p(s)_k \). Then this derivation comes from the correspondences \( l \otimes t^{-a} \mapsto l \otimes (-a^{(m)})t^{-a\omega^p} \).

Let \( \hat{M}^f \) be the minimal Lie subalgebra in \( \hat{M}_{K(p)} \) obtained by joining to \( \hat{M}^p \) the element \( \bar{f} \) and its images under the action of the group generated by \( \text{id}_L \otimes \bar{h}^{(m)}_\omega \). The algebra \( \hat{M}^f \) still reflects all essential information about the structure of \( \Gamma_\omega \). Then

\[
(id_L \otimes \bar{h}^{(m)}_\omega) \bar{f} = \bar{C}^{(m)} \circ (\bar{A}^{(m)} \otimes \text{id}_{K(p)}) \bar{f},
\]

where \( \bar{C}^{(m)} \in \hat{M}^p \) and \( \bar{A}^{(m)} = \text{Ad} \bar{h}^{(m)}_\omega \in \text{Aut}\hat{L}^p \). So, this relation determines the action of the lift \( id_L \otimes \bar{h}^{(m)}_\omega \) on \( \hat{M}^f \).

For any \( n \geq 1 \), we have \( (id \otimes \bar{h}^{(m)n}) \bar{f} = \bar{C}^{(m,n)} \circ (\bar{A}^{(m)n} \otimes \text{id}) \bar{f} \), where the element \( \bar{C}^{(m,n)} \in \hat{M}^p \) can be presented in the following form

\[
(id \otimes \bar{h}^{(m)n-1}) \bar{C}^{(m)} \circ (\bar{A}^{(m)} \otimes \bar{h}^{(m)n-2}) \bar{C}^{(m)} \circ \cdots \circ (\bar{A}^{(m)n-1} \otimes \text{id}) \bar{C}^{(m)}.
\]

Let \( \bar{c}^{(m)}_i \in \hat{M}^p \) be such that for all \( 1 \leq i < p \), \( \bar{C}^{(m,n)} = \sum_{1 \leq i < p} n^i \bar{c}^{(m)}_i \).

(Such elements \( \bar{c}^{(m)}_i \) are unique because \( \det((n^i))_{1 \leq i, j < p} \neq 0 \mod p \).)

Summarizing our approach from \([5, 7]\) we obtain:

**Proposition 4.8.**

a) \( \bar{A}^{(m)} = \exp \bar{B}^{(m)} \), where \( \bar{B}^{(m)} \) is a derivation on \( \hat{L}^p \);

b) if \( \alpha_p := \text{Spec} \mathbb{F}_p[U] \) with \( U^p = 0 \), is a finite group scheme with coaditation \( \Delta U = U \otimes 1 + 1 \otimes U \) then the correspondence

\[
\bar{h}^{(m)U} : \bar{f} \mapsto (U \otimes \bar{c}^{(m)}_1) + \cdots + U^{p-1} \otimes \bar{c}^{(m)}_{p-1}) \circ (\bar{A}^{(m)U} \otimes \text{id}) \bar{f} =
\]

\[
= (U \otimes \bar{c}^{(m)}_1 + \cdots + U^{p-1} \otimes \bar{c}^{(m)}_{p-1}) \circ \left( \sum_{0 \leq n < p} U^n \otimes (\bar{B}^n / n! \otimes \text{id}) \right) \bar{f}
\]

induces a coaction of \( \alpha_p \) on \( \hat{M}^f \);

c) \( \bar{h}^{(m)n}_\omega(\bar{f}) = \bar{h}^{(m)U}_\omega|_{U=n} \);

d) if for all \( a \in \mathbb{Z}_N^0(p) \), \( V^{(m)}_a : = \text{ad} \bar{h}^{(m)}_\omega(D_a) \) then

\[
\sigma \bar{c}^{(m)}_1 - \bar{c}^{(m)}_1 + \sum_{a \in \mathbb{Z}_N^0(p)} t^{-a} V^{(m)}_a =
\]

\[
- \sum_{1 \leq k < p} \frac{1}{k!} t^{-(a_1 + \cdots + a_k)} \omega^{(m)}_{a_1} \cdots [D_{a_1 0}, D_{a_2 0}], \ldots, D_{a_k 0]}
\]

\[
- \sum_{2 \leq k < p} \frac{1}{k!} t^{-(a_1 + \cdots + a_k)} \cdots [V_{a_1 0}, D_{a_2 0}], \ldots, D_{a_k 0]}
\]
(4.9) $- \sum_{1 \leq k < p} \frac{1}{k!} t^{-(a_1 + \cdots + a_k)} [\ldots [\sigma \tilde{c}_1^{(m)}, D_{a_1}], \ldots, D_{a_k}]$

(the indices $a_1, \ldots, a_k$ in all above sums run over $\mathbb{Z}_N^0(p)$);

e) the solutions $\{\tilde{c}_1^{(m)}, V_{a_0}^{(m)} | a \in \mathbb{Z}_N^0(p)\}$ of (4.9) are in bijection with the lifts $\tilde{h}_\omega^{(m)}$ of $h_\omega^{(m)}$ to $K(p)$;

f) suppose $\tilde{c}_1^{(m)} = \sum_{i \in \mathbb{Z}_N} \tilde{c}_1^{(m)}(i)t^i$, where all $\tilde{c}_1^{(m)}(i) \in \mathcal{L}_k^p$; then different solutions of (4.9) have different $\tilde{c}_1^{(m)}(0)$, i.e. $\tilde{c}_1^{(m)}(0) \in \mathcal{L}_k^p$ are strict invariants of the lifts $\tilde{h}_\omega^{(m)}$.

Proof. – a) is just a general fact about the structure of unipotent automorphisms on modules with filtration of length $< p$;

– b) this is also a sufficiently general interpretation of unipotent additive action on modules with filtration of length $< p$ (Sect.3 of [8] contains necessary background of the specification of this situation to the Campbell-Hausdorff composition law.);

c) this follows obviously from b);

d) note that the relations

$$(\text{id} \otimes h_\omega^{(m)n})\tilde{e} = (\sigma \tilde{c}(m,n)) \circ (\text{Ad} n \tilde{l}_\omega^{(m)} \otimes \text{id}) \tilde{e} \circ (-\tilde{C}(m,n))$$

imply that

$$(\text{id} \otimes h_\omega^{(m)U})\tilde{e} = (\sigma \tilde{C}(m)U) \circ (\text{Ad} U \tilde{l}_\omega^{(m)} \otimes \text{id}) \tilde{e} \circ (-\tilde{C}(m)U),$$

where $\tilde{C}(m)U = U\tilde{c}_1^{(m)} + \cdots + U^{p-1}\tilde{c}_{p-1}^{(m)}$. This implies

$$\tilde{c}_1^{(m)}(U) \circ (\text{id} \otimes h_\omega^{(m)U})\tilde{e} \equiv (\sigma \tilde{c}_1^{(m)}(U)) \circ (\text{Ad} U \tilde{l}_\omega^{(m)} \otimes \text{id}) \tilde{e} \mod U^2,$$

and we need just to follow the coefficients for $U$;

Remark. Relation (4.9) can be uniquely lifted to $\tilde{\mathcal{L}}_K^p = \mathcal{L}_K^p \mod \mathcal{L}_k^p(p)_K$ by taking suitable unique lifts of $\tilde{c}_1^{(m)}$ (use that $\sigma$ is nilpotent on $\mathcal{M}(p - 1)_K \mod \mathcal{L}_k^p(p)_K$). In other words, we have unique lifts of $\tilde{c}_1^{(m)}$ to $\tilde{L}_K$ such that (4.9) is still an equality in $\tilde{L}_K^p$. We will use the same notation $\tilde{c}_1^{(m)}$ for these lifts.

– e) note, first, the corresponding data $\{\tilde{c}_1^{(m)}, V_{a_0}^{(m)} | \tilde{a} \in \mathbb{Z}_N^0(p)\}$ are in a bijection with the lifts $\tilde{h}_\omega^{(m)}$ of $h_\omega^{(m)}$ to $K(p)$, cf. Sect.1.6. Therefore, we should verify that $\tilde{c}_1^{(m)}$ determines uniquely the whole vector $\tilde{c}^{(m)}$. This follows formally from b) and can be verified as follows (we used a different approach in [7], cf. Remark in Sect.3.5).

Since $\tilde{h}_\omega^{(m)U} : \tilde{\mathcal{M}}^f \rightarrow \mathbb{F}_p[U] \otimes \tilde{\mathcal{M}}^f$ is the coaction of the group scheme $\alpha_p$, we have in $\mathbb{F}_p[U_1, U_2] \otimes \tilde{\mathcal{M}}^f$ that

$$(\text{id}_{U_1} \otimes \tilde{h}_\omega^{(m)U_2}) \cdot \tilde{h}_\omega^{(m)U_1} = \tilde{h}_\omega^{(m)U_1 + U_2}.$$
Then we obtain in $\mathcal{L}_{K(p)}$,
\[(\text{id} \otimes h^{(m)U_2}_\omega)\tilde{C}^{(m)U_1} \circ (\text{Ad}^{U_1} \tilde{f}^{(m)}_\omega \otimes \text{id})C^{(m)U_2} \circ (\text{Ad}^{U_1+U_2} \tilde{f}^{(m)}_\omega \otimes \text{id})\tilde{f}\]
implies
\[\left(\sum_{n \geq 1} U^n_1 (\text{id} \otimes h^{(m)U_2}_\omega)\tilde{c}^{(m)}_n\right) \circ \left(\sum_{n \geq 1} U^n_2 (\tilde{A}^{U_1} \otimes \text{id})\tilde{c}^{(m)}_n\right) = \sum_{n_1, n_2} \tilde{c}^{(m)}_{n_1+n_2} (U_1 + U_2)^{n_1+n_2}\]

For $n \geq 1$, the coefficient for $U_1 U_2^n$ in the RHS equals $(n+1)\tilde{c}^{(m)}_n$. The corresponding coefficient in the LHS coincides with the coefficient in
\[\left(U_1 (\text{id} \otimes h^{(m)U_2}_\omega)\tilde{c}^{(m)}_1\right) \circ \left((\text{id} + U_1 B^{(m)} \otimes \text{id}) \sum_{n \geq 0} U^n c^{(m)}_n\right)\]
and, therefore, equals $(B^{(m)} \otimes \text{id})c^{(m)}_n$ plus $\mathbb{F}_p$-linear combination of the elements of the following form (cf. Remark below)
\[\ldots (\text{id} \otimes d^{n_1} h^{(m)}_\omega / n_1!) c^{(m)}_1, c^{(m)}_{n_2}, \ldots, c^{(m)}_{n_s}\]
where $n_1 + \cdots + n_s = n$, $s \geq 1$ and $n_1 \geq 0$. As a result, $(n+1)\tilde{c}^{(m)}_{n+1}$ can be uniquely recovered from $\tilde{c}^{(m)}_1, \ldots, \tilde{c}^{(m)}_n$.

**Remark.** We used well-known relation,
\[X + UY \equiv X \circ \left(U \sum_{k \geq 1} \frac{1}{k!} [\ldots [Y, X], \ldots, X]\right) \mod U^2\]
with $U = U_1$ and $X = \sum_n U^n c^{(m)}_n$, cf. references in [8], Sect. 3.2.

f) follows by induction on mod $C_i(L^P_{\omega})$, $i \geq 1$, from relation (4.9). □

4.6. **The structure of $\Gamma^P$.** We are going to determine the structure of the Lie algebra $L^P_{\omega}$. In the above section we indicated the way how to specify the lifts $\tilde{h}^{(m)}_\omega = \pi^{-1}_\omega(\tilde{h}^{(m)}_\omega)$, $1 \leq m \leq N$. This can be done by applying recurrent procedure (4.9) to find the elements $\tilde{c}^{(m)}_1$ and $V^{(m)}_a = \text{ad}_k \tilde{h}^{(m)}_\omega(D_{a0})$, $a \in \mathbb{Z}_N^+(p)$. In addition, we should specify the commutators $[\tilde{h}^{(i)}_{\omega}, \tilde{h}^{(j)}_{\omega}] := \tilde{L}_{[i, j]}$ \in $\mathcal{L}^P$, $1 \leq i, j \leq N$.

**Remark.** From Lemma (4.3c) it follows that all $\tilde{L}_{[\omega, [i, j]]} \in C_2(L^P_{\omega}) = \mathcal{L}^P_{\omega}(2)$. It would be very interesting to find explicit expression for the elements $\tilde{L}_{[\omega, [i, j]]}$ in terms of the involved parameters $A_i = A_i(\omega)$ (recall that $E(\omega^p) = 1 + \sum_i A_i \omega^{i+pe}$). We verified by a direct calculation with relations (4.9) that for any lifts $\tilde{h}^{(i)}_{\omega}$ and $\tilde{h}^{(j)}_{\omega}$, the corresponding elements $\tilde{L}_{[\omega, [i, j]]} \in C_4(L^P_{\omega})$. 


The following property could be useful to study the properties of the elements $\tilde{L}_\omega[i, j]$. To simplify the notation set for all $m$, $\text{id} \otimes d_{h, \omega}^{(m)} = d^{(m)}$ and $\text{ad} \tilde{l}_\omega^{(m)} \otimes \text{id} = \text{ad}^{(m)}$.

**Proposition 4.9.** For all $1 \leq i, j \leq N$,

$$\tilde{L}_\omega[i, j] = (d^{(j)} - \text{ad}^{(j)})\tilde{c}_1^{(j)} - (d^{(i)} - \text{ad}^{(i)})\tilde{c}_1^{(i)} + [\tilde{c}_1^{(i)}, \tilde{c}_1^{(j)}].$$

**Proof.** With above assumption we have in $\tilde{\mathcal{L}}_K^\sigma[U]$ for all $m$, 

$$(\text{id} + Ud^{(m)})\tilde{e} \equiv (U\sigma \tilde{c}_1^{(m)}) \circ (\text{id} + U\text{ad}^{(m)})\tilde{e} \circ (-U\tilde{c}_1^{(m)}) \mod U^2.$$ 

Let $\tilde{E} = \exp\tilde{e}$ in the enveloping algebra $\tilde{\mathfrak{A}}_K^\sigma$ of $\tilde{\mathcal{L}}_K^\sigma$. Then we have the following congruence modulo $(U^2, (\tilde{\mathcal{J}}_K^\sigma)^p)$, where $\tilde{\mathcal{J}}_K^\sigma$ is the augmentation ideal in $\tilde{\mathfrak{A}}_K^\sigma$,

$$(\text{id} + Ud^{(m)})\tilde{E} \equiv (1 + \sigma\tilde{c}_1^{(m)} U) \cdot (\text{id} + U\text{ad}^{(m)})\tilde{E} \cdot (1 - \tilde{c}_1^{(m)} U).$$

Comparing the coefficients for $U$ we obtain 

$$(d^{(m)} - \text{ad}^{(m)})\tilde{E} \equiv \sigma\tilde{c}_1^{(m)} \cdot \tilde{E} - \tilde{E} \cdot \tilde{c}_1^{(m)} \mod (\tilde{\mathcal{J}}_K^\sigma)^p.$$ 

Note that $(d^{(i)}d^{(j)} - d^{(j)}d^{(i)})\tilde{E} \equiv 0$ because

$$d^{(m)}\tilde{E} = -\sum_{i \geq 0} A_i \sum_{s \geq 0} (1/s!)(a_1 + \cdots + a_s)^{(m)}t^{-(a_1 + \cdots + a_s) + \ell + p^s}D_{a_0}.$$ 

In addition, for any $i, j$, $\text{ad}^{(i)}$ and $d^{(j)}$ commute one with another.

Therefore, (4.10) implies

$$\text{ad} \tilde{L}_\omega[i, j](\tilde{E}) \equiv (d^{(i)} - \text{ad}^{(i)})(d^{(j)} - \text{ad}^{(j)})\tilde{E} - (d^{(j)} - \text{ad}^{(j)})(d^{(i)} - \text{ad}^{(i)})\tilde{E} \equiv \sigma(\mathcal{X})\tilde{E} - \tilde{E}\mathcal{X},$$

where $\mathcal{X} = (d^{(i)} - \text{ad}^{(i)})\tilde{c}_1^{(j)} - (d^{(j)} - \text{ad}^{(j)})\tilde{c}_1^{(i)} + [\tilde{c}_1^{(i)}, \tilde{c}_1^{(j)}].$

Let $\mathcal{X}_0 = \mathcal{X} - L_\omega[i, j]$, then

$$(4.11) \quad \sigma(\mathcal{X}_0)\tilde{E} \equiv \tilde{E}\mathcal{X}_0 \mod (\tilde{\mathcal{J}}_K^\sigma)^p.$$ 

It remains to prove that

$$\mathcal{X}_0 = 0.$$ 

**Lemma 4.10.** $\mathcal{X}_0 = 0$.

**Proof of Lemma.** As earlier, let $\tilde{\mathfrak{A}}_K^\sigma$ be the enveloping algebra for $\tilde{\mathcal{L}}_K^\sigma$. Let $\tilde{\mathfrak{A}}_K^\sigma(s), s \geq 1$, be the ideal in $\tilde{\mathfrak{A}}_K^\sigma$ generated by the monomials $D_{a_1n_1} \cdots D_{a_un_u}$ of weight $\geq s$, i.e. such that if $s_i \in \mathbb{N}$ for $1 \leq i \leq u$, are such that $(s_i - 1)c_0 \leq a_i < s_ic_0$ then $s_1 + \cdots + s_u \geq s$. For each $s$, $\tilde{\mathfrak{A}}_K^\sigma(s) \cap \tilde{\mathcal{L}}_K^\sigma = \tilde{\mathcal{L}}_K^\sigma(s)$, in particular, $\tilde{\mathfrak{A}}_K^\sigma(p) \cap \tilde{\mathcal{L}}_K^\sigma = 0$. 

Note that $\mathcal{X}_0 \in \tilde{\mathcal{L}}_K^\sigma$ and let $\mathcal{X}_0 = \mathcal{X}^{(1)} + \cdots + \mathcal{X}_0^{(p-1)}$, where each $\mathcal{X}_0^{(s)}$ is a $\mathcal{K}$-linear combination of the Lie monomials of weight $s$,

$$[\cdots [D_{a_1n_1}, D_{a_2n_2}], \cdots, D_{a_un_u}].$$

Clearly, $\mathcal{X}_0^{(s)} \in \tilde{\mathcal{L}}_K^\sigma(s)$ and if $\mathcal{X}_0^{(s)} \in \tilde{\mathfrak{A}}_K^\sigma(s + 1)$ then $\mathcal{X}_0^{(s)} = 0$. It will be enough to prove by induction on $1 \leq s \leq p$ that $\mathcal{X}_0 \in \tilde{\mathcal{L}}_K^\sigma(s) + \tilde{\mathfrak{A}}_K^\sigma(s + 1)\mathcal{K}$. 

Remark in the part d) of the proof of Prop. 4.8).

The structure of 4.7.

Suppose the lemma is proved for some \(1 \leq s \leq p - 1\).

Thus (compare with Prop. 3.1) all \([D_{a_0}, \chi_0^{(s)}]\) \(= \bar{\mathfrak{A}}^P_{\bar{\mathfrak{c}}} (s + 2)_K\) and

\[
\chi_0^{(s + 1)} \in \bar{\mathcal{L}}^P_{\bar{\mathfrak{c}}} (s + 1) + \bar{\mathfrak{A}}^P_{\bar{\mathfrak{c}}} (s + 2)_K.
\]

If weight of \(D_{a_0} = 1\) then \([D_{a_0}, \chi_0^{(s)}]\) has weight \(s + 1\), therefore, \([D_{a_0}, \chi_0^{(s)}] = 0\) and \(\chi_0^{(s)} = 0\). Similarly, \(\chi_0^{(s + 1)} \in \bar{\mathcal{L}}^P_{\bar{\mathfrak{c}}} (s + 1)\).

The lemma is proved.

Using the notation from item f) of Prop. 4.8 we obtain

**Corollary 4.11.** For \(1 \leq i, j \leq N\),

\[
\tilde{\epsilon}_{\omega, i, j} = \text{ad}^{(i)}(c_1^{(j)}(0)) - \text{ad}^{(j)}(c_1^{(i)}(0)) + \sum_{s \in \mathbb{Z}_N} [c_1^{(i)}(s), c_1^{(j)}(-s)] .
\]

4.7. The structure of \(\Gamma^P_\omega\) modulo third commutators. Consider the lift of relation (4.9) to \(\bar{\mathcal{L}}^P_K\) taken modulo \(C_2(L^P_\omega)_K = \bar{\mathcal{L}}^P_{\bar{\mathfrak{c}}} (2)_K\) (cf. Remark in the part d) of the proof of Prop. 4.8).

\[
(4.12) \quad \sigma \tilde{c}_1^{(m)} - \tilde{c}_1^{(m)} + \sum_{a \in \mathbb{Z}_N^0 (p)} t^{-a} V_{a_0}^{(m)} \equiv - \sum_{a \in \mathbb{Z}_N^0 (p), i \geq 0} A_i(\omega) t^{a + p - a} a^{(m)} D_{a_0} .
\]

Here \(V_{a_0}^{(m)} = \text{ad} \tilde{\epsilon}_{\omega, a_0} (D_{a_0})\) and \(a^{(m)}\) is the \(m\)-th component of \(a \in \mathbb{Z}_N^0 (p)\). Applying to (4.12) the operator \(\mathcal{R}\) from Sect. 3.1 we obtain:

1) \(V_0^{(m)} = \text{ad} \tilde{\epsilon}_{\omega, 0} (D_0) \in C_2(L^P_\omega)\);

2) for all \(a \in \mathbb{Z}_N^0 (p)\),

\[
V_{a_0}^{(m)} = \text{ad} \tilde{\epsilon}_{\omega, a_0} (D_{a_0}) \in \text{mod} C_2(L^P_{\omega, a_0}) .
\]

Property 2) means that all generators of \(L^{(m)}_{\omega, a_0}\) of the form \(D_{a_0}\) with \(a > c^0\) and \(a^{(m)} \neq 0 \mod p\), can be eliminated from the system

\[
\{D_{a_0} \mid a \in \mathbb{Z}_N^0 (p)\} \cup \{D_0\} \cup \{l_\omega^{(m)}\}
\]

of generators of \(L^{(m)}_{\omega, a_0}\). Indeed, because \(A_0 \neq 0\) and \(a^{(m)} \neq 0 \mod p\), all \(D_{a_0 + \omega_0, 0}\) belong to the ideal of second commutators

\[
C_2(L^{(m)}_{\omega, a_0}) = (\text{ad} \tilde{\epsilon}_{\omega, a_0}) L^P_k + C_2(L^P_k) ,
\]
and for any \( n \in \mathbb{Z}/N_0 \), all \( D_{a+e^n} = \sigma^n D_{a+e^0} \) also belong to \( C_2(L_{\omega,k}^{(m_\omega)P}) \). Property 1) means that \( L_{\omega,k}^{(m_\omega)P} \) has only one relation with respect to any minimal \( \mathcal{P} \)-topological set of generators. Therefore, \( \Gamma_{\omega,k}^{(m_\omega)P} \) can be treated as a \( \mathcal{P} \)-topological group with one relation.

On the other hand, the Lie algebra \( L_{\omega,k}^P \) has the system of generators

\[
\{D_{an} \mid a \in \mathbb{Z}_N^+(p) \} \cup \{D_0\} \cup \{\tilde{l}_\omega^{(m)} \mid 1 \leq m \leq N\}
\]

with the corresponding system of relations

\[
[D_{an}, \tilde{l}_\omega^{(m)}] = V_{an}^{(m)} \ , \ [D_0, \tilde{l}_\omega^{(m)}] = V_0^{(m)} \ , \ [[\tilde{l}_\omega^{(i)}, \tilde{l}_\omega^{(j)}]] = \tilde{l}_\omega[i, j],
\]

where \( a \in \mathbb{Z}_N^+(p) \) and \( 1 \leq m, i, j \leq N \).

Choose for every \( a \in \mathbb{Z}_N^+(p) \), \( m_a \in [1, N] \) such that \( a^{(m_a)} \neq 0 \mod p \). Then the relations \([D_{an}, \tilde{l}_\omega^{(m_a)}] = V_{an}^{(m_a)} \), \( a \in \mathbb{Z}_N^+(p) \), can be used to eliminate extra generators \( \{D_{an} \mid a > \bar{c}_0\} \) and to present the structure of \( L_{\omega,k}^P \) in terms of the corresponding minimal system of generators \( \{D_{an} \mid a \in \mathbb{Z}_N^+(p), a < \bar{c}_0\} \cup \{D_0\} \cup \{\tilde{l}_\omega^{(m)}\} \).

Consider second central step to obtain explicitly the above relations modulo \( C_3(L_{\omega,k}^P) \).

**Proposition 4.12.** For \( 1 \leq m \leq N \) and \( a \in \mathbb{Z}_N^+(p) \), there are the following congruences mod \( C_3(L_{\omega,k}^P) \):

\[
\begin{align*}
V_0^{(m)} & \equiv -\frac{1}{2} \sum_{\substack{i \geq 0 \atop 0 \leq n < N_0}} \sigma^n \left( A_i \sum_{a_1+a_2 = \bar{c}^n + p^n} a_1^{(m)} [D_{a_1,0}, D_{a_2,0}] \right), \\
V_{a_0}^{(m)} & \equiv -\sum_{n \geq 1} \sigma^n \left( A_i \sum_{a_1+a_2/p^n = \bar{c}^n + p^n + a/p^n} a_1^{(m)} [D_{a_1,0}, D_{a_2,-n}] \right), \\
- \sum_{i \geq 0} A_i a^{(m)} D_{\bar{c}^n + p^n + a, 0} & - \frac{1}{2} \sum_{n \geq 0} \sigma^{-n} \left( A_i \sum_{a_1+a_2 = \bar{c}^n + p^n} a_1^{(m)} [D_{a_1,0}, D_{a_2,0}] \right).
\end{align*}
\]

**Proof.** From (4.11) we obtain (apply the operator \( S \) from Sect. 3.1)

\[
\tilde{c}_i^{(m)} \equiv \sum_{\substack{a, i \geq 0 \atop n \geq 0}} \sigma^n (A_i) t^{\bar{c}^n + p^n - a} a^{(m)} D_{an} \mod C_3(L_{\omega,k}^P).
\]

Then the right-hand side of (4.9) modulo \( C_3(L_{\omega,k}^P) \) appears as
Corollary 4.13. $L_{\omega,k}^p \mod C_3(L_{\omega,k}^p)$ is the maximal $\sigma$-invariant quotient of nilpotent class 2 of a free Lie $k$-algebra with generators

\{D_{an} \mid a \in \mathbb{Z}_{N}^+(p), a < \bar{c}, n \in \mathbb{Z}/N_0 \} \cup \{D_{0} \} \cup \{\bar{r}^m \} \mid 1 \leq m \leq N \}

satisfying for $1 \leq m_1, m_2 \leq N$ and $a \in \mathbb{Z}_{N}^+(p)$, the relations:

$$\mathcal{R}(m_1, m_2) : [\bar{r}^m_1, \bar{r}^m_2] = 0 ,$$

$$\mathcal{R}_0(m) : [D_0, \bar{r}^m_0] + \frac{1}{2} \sum_{\substack{i \geq 0 \\ 0 \leq n < N_0}} \sigma^n \left( \sum_{a_{1+a_2/p^n} = \bar{c}^p/p^a + a/p^n} A_i \sum_{a_{1+a_2/p^n} = \bar{c}^p/p^a + a/p^n} a_{1(a_1)} a_{2(a_2)} - a_{1(a_1)} a_{2(a_2)} \right) [D_{a_1,0}, D_{a_2,0}] = 0 ,$$

$$\mathcal{R}_a(m_1, m_2) : \sum_{n \geq 1} \sigma^n \left( \sum_{\substack{a_{1+a_2/p^n} = \bar{c}^p/p^a + a/p^n} A_i \sum_{a_{1+a_2/p^n} = \bar{c}^p/p^a + a/p^n} a_{1(a_1)} a_{2(a_2)} - a_{1(a_1)} a_{2(a_2)} \right) [D_{a_1,0}, D_{a_2,0}] = 0 .$$

Remark. a) If $N = 1$ then there is only one (Demushkin) relation $\mathcal{R}_0(1)$, cf. [8], [7].

b) The simplest example can be obtained by choosing $\bar{c}^0 = (p, 0, \ldots, 0)$, $\omega^p = \bar{c}^0$ and $N_0 = 1$, cf. Sect. [5, 5].

c) The structure of $\Gamma_{\omega}^p$ depends only on $\omega$, more precisely, only on $\omega^p$ mod $(p-1)\bar{c}^0$, i.e. on the constants $A_i$ with $\iota < (p-2)\bar{c}^0/p$.

d) The structure of $\Gamma_{\omega}^p$ modulo $C_s(\Gamma_{\omega}^p)$, $s \leq p$, depends only on the constants $A_i$ with $\iota < (s-2)\bar{c}^0/p$.

4.8. The simplest example. $N = 2$, $N_0 = 1$, $\bar{c}^0 = (p, 0)$, $A_0 = 1$, all remaining $A_i = 0$.

The minimal generators:
\{ $D_{an} \mid a \in \mathbb{Z}_{N}^+(p), a < (p, 0) \} \cup \{D_{0} \} \cup \{\bar{r}^1, \bar{r}^2 \}.$

The relations:
$$\mathcal{R}(1, 2) : [\bar{r}^1, \bar{r}^2] ,$$
\[ \mathcal{R}_0(1) = [D_0, \overline{v}^{(1)}] + \sum_{1 \leq \alpha \leq \frac{L-1}{2}, \gamma \in \mathbb{Z}} \alpha [D_{(\alpha, \gamma)}, D_{(p-\alpha, -\gamma)}] \]

\[ \mathcal{R}_0(2) = [D_0, \overline{v}^{(2)}] + \sum_{0 \leq \alpha \leq \frac{L-1}{2}, \gamma \in \mathbb{Z}} \gamma [D_{(\alpha, \gamma)}, D_{(p-\alpha, -\gamma)}] \]

\[ \mathcal{R}_a(1; 2) : [D_a, a^{(2)} \overline{v}^{(1)} - a^{(1)} \overline{v}^{(2)}] - \delta_{a^{(1)}, 0} \sum_{\gamma} a^{(2)} [D_{(p-1, \gamma)}, D_{(p, a^{(2)}) - p\gamma}] + a^{(1)} \sum_{n \geq 1, \beta \in \mathbb{Z}^+ \backslash (p)} \beta [D_{(p-\beta)}, D_{a+(0, p^n \beta)}] + \frac{1}{2} \sum_{a_1 + a_2 = (p, 0) + a} \left( a_1^{(1)} a_2^{(2)} - a_1^{(2)} a_2^{(1)} \right) [D_{a_1}, D_{a_2}] \]

5. Characteristic 0 case

In this section \( K \) is an \( N \)-dimensional local field of characteristic 0. We assume that the first residue field \( K^{(1)} \) of \( K \) has characteristic \( p \). The last residue field \( k \) of \( K \) is isomorphic to \( \mathbb{F}_{p^{N_0}} \), \( N_0 \in \mathbb{N} \). We also fix a system of local parameters \( \pi = \{ \pi_1, \cdots, \pi_N \} \) of \( K \), denote by \( v^1 \) the first valuation of \( K \) such that \( v^1(p) = 1 \) and by \( O_K^1 \) the corresponding valuation ring. Starting Sect. 5.2, we assume that \( K \) contains a primitive \( p \)-th root of unity \( \zeta_1 \).

5.1. The field-of-norms functor. \( N \)-dimensional local fields are special cases of \((N - 1)\)-big fields used by Scholl [25] to construct a higher dimensional analogue of the Fontaine-Wintenberger field-of-norms functor, [7]. We can apply this construction to the case of higher local fields due to the fact that the structure of an \( N \)-dimensional field is uniquely extended to its finite field extensions. We don’t use here the construction of the field-of-norms functor from [4]: it is based on essentially close ideas but works in the category of higher local fields with additional structure given by “subfields of constants” (because the whole theory in [3] is based only on the concept of ramification for higher local fields).

Let \( K_{alg} \) be an algebraic closure of \( K \). Denote by the same symbol a unique extension of the valuation \( v^1 \) to \( K_{alg} \). For any \( 0 < c \leq 1 \), let \( \mathfrak{p}^c = \{ x \in K_{alg} \mid v^1(x) \geq c \} \). If \( L \) is a field extension of \( K \) in \( K_{alg} \), we use the simpler notation \( O_L^1/\mathfrak{p}^c \) instead of \( O_L^1/(\mathfrak{p}^c \cap O_L^1) \).

An increasing fields tower \( K_* = (K_n)_{n \geq 0} \), where \( K_0 = K \), is strictly deeply ramified (SDR) with parameters \((n_0, c)\), if for \( n \geq n_0 \), we have \( [K_{n+1} : K_n] = p^N \) and there is a surjective map \( \Omega_{O_{F_{n+1}}/p^n}^1 \rightarrow (O_{K_{n+1}}/\mathfrak{p}^c)^N \) or, equivalently, the \( p \)-th power map induces epimorphic maps

\[ i^1_n(K_*) : O_{K_{n+1}}^1/\mathfrak{p}^c \rightarrow O_{K_n}^1/\mathfrak{p}^c. \]
This implies that for all \( n \geq n_0 \), the last residue fields \( K_n^{(N)} \) are the same and there are systems of local parameters \( \{ \pi_{n1}, \ldots, \pi_{nN} \} \) in \( K_n \) such that for all \( 1 \leq m \leq N \), \( \pi_{n+1,m} \equiv \pi_{nm} \mod p^c \), where \( \pi_m = \pi_{0m} \).

Equivalently, on the level of the \( N \)-valuation rings \( \mathcal{O}_{K_n} \), the \( p \)-th power map induces epimorphic maps

\[
i_n(K_\bullet) : \mathcal{O}_{K_{n+1}} / p^c \longrightarrow \mathcal{O}_{K_n} / p^c.
\]

Let \( \mathcal{O} = \lim_{\rightarrow n} \mathcal{O}_{K_n} / p^c \). Then \( \mathcal{O} \) is an integral domain and we can introduce the field of fractions \( \mathcal{K} \) of \( \mathcal{O} \). The field-of-norms functor \( X \) associates to the SDR tower \( K_\bullet \) the field \( \mathcal{K} = X(K_\bullet) \). This field has characteristic \( p \), it inherits a structure of \( N \)-dimensional local field such that the elements \( t_m := \lim_{\leftarrow n} \pi_{nm}, 1 \leq m \leq N \), form a system of local parameters in \( \mathcal{K} \). Then the \( N \)-dimensional valuation ring \( \mathcal{O}_{\mathcal{K}} \) of \( \mathcal{K} \) coincides with \( \mathcal{O} \), and for \( n \geq n_0 \), the last residue fields of \( \mathcal{K} \) and \( K_n \) coincide. Since the identification \( \mathcal{O}_{\mathcal{K}} = \lim_{\rightarrow n} \mathcal{O}_{K_n} / p^c \) relates the appropriate power series in given systems of local parameters the field-of-norms functor is compatible with \( \mathcal{P} \)-topological structures on the fields \( K_n \) and \( \mathcal{K} \).

Suppose \( L \) is a finite extension of \( K \) in \( K_{alg} \). Then the tower \( L_\bullet = (LK_n)_{n \geq 0} \) is again SDR and \( X(L_\bullet) = \mathcal{L} \) is a separable extension of \( \mathcal{K} \) of degree \([LK_n : K_n]\), where \( n \gg 0 \). The extension \( \mathcal{L} / \mathcal{K} \) is Galois if and only if for \( n \gg 0 \), \( LK_n / K_n \) is Galois. From the definition of \( \mathcal{L} \) and \( \mathcal{K} \) it follows that we have a natural identification of groups \( \text{Gal}(\mathcal{L} / \mathcal{K}) = \text{Gal}(LK_n / K_n) \). As a result, \( X(K_{alg}) := \lim_{\rightarrow \mathcal{L}} X(L_\bullet) \) is a separable closure \( \mathcal{K}_{sep} \) of \( \mathcal{K} \) and the functor \( X \) identifies \( \text{Gal}(\mathcal{K}_{sep} / \mathcal{K}) \) with \( \text{Gal}(K_{alg} / K_{\infty}) \), where \( K_{\infty} = \lim_{\leftarrow n} K_n \).

Similarly to the classical 1-dimensional situation there is the following interpretation of the functor \( X \). Let \( \mathbb{C}_p(N) \) be the \( \nu^1 \)-adic completion of \( K_{alg} \) and let \( R_0(N) = \lim_{\leftarrow n \geq 0} \mathbb{C}_p(N) \) with respect to the \( p \)-th power maps on \( \mathbb{C}_p(N) \). The operations on \( R_0(N) \) are defined as follows: if \( a = \{ a_n \}_{n \geq 0} \) and \( b = \{ b_n \}_{n \geq 0} \) belong to \( R_0(N) \) then \( ab = \{ a_nb_n \}_{n \geq 0} \) and \( a + b = \{ c_n \}_{n \geq 0} \), where \( c_n = \lim_{m \to \infty} (a_{n+m} + b_{n+m})^{p^n} \). So, \( R_0(N) \) is a field of characteristic \( p \), there is a natural embedding \( \mathcal{K}_{sep} \subset R_0(N) \), and \( R_0(N) \) appears as the completion of \( \mathcal{K}_{sep} \) with respect to the first valuation.

We denote by \( R(N) \subset R_0(N) \) the completion of the \( N \)-valuation ring of \( \mathcal{K}_{sep} \), and let \( m_{R(N)} \) be the maximal ideal of \( R(N) \). Clearly,

\[
R(N) = \lim_{\leftarrow n \geq 0} \mathcal{O}_{\mathbb{C}_p(N)} = \lim_{\leftarrow n \geq 0} \mathcal{O}_{K_{alg}} / p .
\]
Note that the \( P \)-topology on \( K \) is uniquely extended to \( R_0(N) \) and this extension coincides with the extension of the \( P \)-topological structure of \( K \) to \( R_0(N) \) (as the completion of the \( P \)-topology on \( K_{\mathrm{sep}} \)).

5.2. Suppose \( u \in \mathbb{N} \) and \( \phi_1, \ldots, \phi_u \in \mathfrak{m}_K \setminus \{0\} \) are independent modulo \( p \)-th powers, i.e., for any \( n_1, \ldots, n_u \in \mathbb{Z} \), the product \( \phi_{1}^{n_1} \cdots \phi_{u}^{n_u} \in \mathcal{K}^{\star p} \) iff all \( n_i \equiv 0 \mod p \). Let \( K^\phi = \bigcup_{n \geq 0} K(\phi_{1n}, \ldots, \phi_{un}) \), where for \( 1 \leq i \leq u \), \( \phi_i = \phi_{i0} \) and for all \( n \in \mathbb{N} \), \( \phi_{i,n-1} = \phi_{in}^p \).

For \( n \in \mathbb{N} \), let \( \zeta_{n+1} \in \mathcal{K}_{\mathrm{alg}} \) be such that \( \zeta_{n+1}^p = \zeta_n \). Here \( \zeta_1 \in K \) is our given \( p \)-th primitive root of unity. Let \( \tilde{K}^\phi = K^\phi \{ \zeta_n | n \in \mathbb{N} \} \). Then \( \tilde{K}^\phi / K \) is normal and let \( \Gamma^\phi \) be its Galois group.

**Lemma 5.1.** If \( \Gamma^\phi_{<p} \) is the maximal quotient of \( \Gamma^\phi \) of period \( p \) and nilpotent class \( < p \) then there is a natural exact sequence of groups

\[
\text{Gal}(\tilde{K}^\phi / K) \longrightarrow \Gamma^\phi_{<p} \longrightarrow \text{Gal}(K(\phi_{11}, \ldots, \phi_{u1}) / K) \longrightarrow 1.
\]

**Proof.** Clearly, \( \Gamma^\phi_{K_{\mathrm{alg}}/K} = \langle \sigma, \tau_1, \ldots, \tau_u \rangle \), where for any \( 1 \leq i \leq m \leq u \) and some \( s_0 \in \mathbb{Z} \), \( \sigma \zeta_m = \zeta_{s_0}^{\sigma} \), \( \sigma \phi_{in} = \phi_{in}^{\sigma} \), \( \tau_m(\zeta_n) = \zeta_n, \tau_m(\phi_{in}) = \phi_{in} \zeta_{n+l_{mi}} \), and \( \sigma^{-1} \tau_m \sigma = \tau_{m}^{(1+ps_0)^{-1}} \).

Therefore, \( (\Gamma^\phi)^p = \langle \sigma^p, \tau_1^p, \ldots, \tau_m^p \rangle \) and for the subgroup of second commutators we have \( C_2(\Gamma^\phi) \subset \langle \tau_1^p, \ldots, \tau_m^p \rangle \subset (\Gamma^\phi)^p \). As a result, it holds \( (\Gamma^\phi)^p C_p(\Gamma^\phi) = \langle \sigma^p, \tau_1^p, \ldots, \tau_m^p \rangle \) and the lemma is proved. \( \square \)

We are going to apply the above lemma to our field \( K \) and the set of local parameters \( \pi = \{ \pi_1, \ldots, \pi_n \} \). The lemma provides us with the field extensions \( \tilde{K}^\pi \supset K^\pi \supset K \). Let \( \Gamma_{<p} := \Gamma / P C_p(\Gamma) \) and \( \Gamma_K^* = \text{Gal}(K_{\mathrm{alg}} / \tilde{K}^\pi) \). The embedding \( \Gamma_K^* \subset \Gamma \) induces a continuous homomorphism \( \iota^\pi : \Gamma_K^* \longrightarrow \Gamma_{<p} \). Denote by \( \kappa^\pi \) the natural surjection \( \Gamma_{<p} \longrightarrow \text{Gal}(K(\sqrt[p]{\pi_1}, \ldots, \sqrt[p]{\pi_N}) / K) = \prod_{1 \leq m \leq N} \langle \tau_m \rangle^{\mathbb{Z}/p} \), where

\[
\tau_m(\sqrt[p]{\pi_i}) = \sqrt[p]{\pi_i^{\kappa^\pi m}}.
\]

**Proposition 5.2.** The following sequence of profinite groups

\[
\Gamma_K^* \xrightarrow{\iota^\pi} \Gamma_{<p} \xrightarrow{\kappa^\pi} \prod_{1 \leq m \leq N} \langle \tau_m \rangle^{\mathbb{Z}/p} \longrightarrow 1
\]

is exact.

**Proof.** Note that the elements of the group \( \Gamma_{\tilde{K}^\pi} = \text{Gal}(K_{\mathrm{alg}} / \tilde{K}^\pi) \) together with a lift \( \tilde{\sigma} \in \Gamma_K^* \) of \( \sigma \) generate the group \( \Gamma_K^* \). Now the exact sequence from Lemma 5.1 implies that \( \text{Ker} \kappa^\pi \) is generated by \( \tilde{\sigma} \) and the image of \( \Gamma_{\tilde{K}^\pi} \subset \Gamma_K^* \). As a result, this kernel coincides with the image of \( \Gamma_K^* \) in \( \Gamma_{<p} \). \( \square \)

Let \( R(N) \) be the ring from Sect.5.1. Recall, there is a natural embedding \( k \subset R \) and for \( 1 \leq m \leq N \), \( t_n := \lim_{n \to \infty} \{ \pi_{mn} \}_{n,0} \in R(N) \), where \( \pi_{m0} = \pi_m \) and \( \pi_{m,n+1}^{pn} = \pi_mn \).
Following Sect. 2.3 set \( t = (t_1, \ldots, t_N) \) and \( K = k((t)) \). Then \( K \) is a closed subfield of \( R_0(N) = \text{Frac} \ R(N) \) with a system of local parameters \( t \). The tower \( K^\pi = \{K(\pi_{t1}, \ldots, \pi_{tN})\}_{n \geq 0} \) is SDR and \( K^\pi = K^\pi_{\sigma} \). Therefore, the field-of-norms functor \( X \) from Sect. 5.1 identifies \( X(K^\pi) \) with \( K \) and \( R_0(N) \) with the completion of \( K_{\text{sep}} \). In particular, there is a natural inclusion \( \Gamma \rightarrow \text{Aut} \ R_0(N) \) which induces the identification of \( \mathcal{G} = \text{Gal}(K_{\text{sep}}/K) \) and \( \Gamma_{K^\pi} \).

We are going to apply the results of the previous sections and will use the appropriate notation related to our field \( K \), e.g. \( \mathcal{G}_{<p} = \text{Gal}(K_{<p}/K) \), where \( K_{<p} = k^{\text{sep}_{C_p}(\mathcal{O})} \). The field-of-norms identification \( \mathcal{G} \simeq \Gamma_{K^\pi} \) composed with the morphism \( \tau \) from Prop. 5.2 induces a group homomorphism \( \iota_{<p} : \mathcal{G}_{<p} \rightarrow \Gamma_{<p} \) and we obtain the following property.

**Proposition 5.3.** The following sequence of profinite groups is exact

\[
\mathcal{G}_{<p} \xrightarrow{\iota_{<p}} \Gamma_{<p} \xrightarrow{\pi} \prod_{1 \leq m \leq N} \langle \tau_m \rangle_{\mathbb{Z}/p} \rightarrow 1.
\]

5.3. **Isomorphism \( \kappa_{<p} \).** Let \( \bar{c} \in \mathbb{Z}^N_{>0} \) be such that \( p = \pi^\nu u \), where \( u \in \mathcal{O}_K \) (as earlier, \( \pi^\nu = \pi^c_1 \cdots \pi^c_N \) with \( \bar{c} = (c_1, \ldots, c_N) \)).

Set \( c^0 = p^{\nu}/(p-1) \). Note that \( c^0 \in \mathbb{Z}^N_{>0} \). Consider a \( \eta \)-continuous embedding \( \eta \) of the field \( K \) into \( R_0(N) \) such that the image of \( (\text{id}_L \otimes \eta) \bar{c} \) in \( \mathcal{M}_{R_0(N)} \) coincides with \( \bar{c} \otimes 1 \).

Such field embedding satisfies \( \eta|_k = \text{id} \) and is uniquely determined by a choice of the elements \( \eta(t_m) \in R(N)_0 \), where \( 1 \leq m \leq N \), such that \( \eta(t_m) \equiv t_m \text{ mod } t^{(p-1)c^0} \).

**Proposition 5.4.** There is a unique lift \( \tilde{\eta} \) of \( \eta \) to \( K(p) \) such that

\[
(\text{id}_L \otimes \tilde{\eta}) \bar{f} \otimes 1 = \bar{f} \otimes 1.
\]

**Proof.** Let \( \bar{\eta} \) be an extension of \( \eta \) to \( K_{<p} \). Clearly, \( \sigma((\text{id}_L \otimes \bar{\eta}) \bar{f} \otimes 1) = (\bar{c} \otimes 1) \circ ((\text{id}_L \otimes \bar{\eta}) \bar{f} \otimes 1) \). So,

\[
(- (\bar{f} \otimes 1)) \circ ((\text{id}_L \otimes \bar{\eta}) \bar{f} \otimes 1) \in \mathcal{M}_{R_0(N)}|_{\sigma = \text{id}} = \bar{L}.
\]

In other words, there is \( l \in \bar{L} \) such that for \( g = \bar{\pi}^{-1}(l) \), it holds

\[
(\text{id}_L \otimes \bar{\eta}) \bar{f} \otimes 1 = (\bar{f} \otimes 1) \circ l = g(\bar{f}) \otimes 1.
\]
As a result, we can take \( \bar{\eta} = \eta \cdot g^{-1} \). The uniqueness of \( \bar{\eta} \) is obvious because any two such lifts differ by an element of \( \mathcal{G} \) but \( \mathcal{G} \) acts strictly on \( f \otimes 1 \).

Let \( \varepsilon = (\zeta_n \bmod p)_n \geq 0 \in R \subset R(N) \) be Fontaine’s element (here \( \zeta_1 \in K \) is our \( p \)-th root of unity and for all \( n, \zeta_n^p = \zeta_{n-1} \)).

Let \( \zeta_1 = 1 + \pi^{\varepsilon_2} \sum_{n \geq 0} [\beta_n] n^n \), where all \( [\beta_n] \) are the Teichmüller representatives of \( \beta_n \in k \) and \( \beta_0 \neq 0 \). Here \( \pi^{\varepsilon_2} O_K = (\zeta_1 - 1)O_K = p^{1/(p-1)}O_K \), i.e. \( p\varepsilon^2 = c_0 \).

Consider the identification of rings \( R(N)/\tau_{e_1} \cong O_{K_{alg}}/p \), coming from the projection \( R(N) = \lim_{\leftarrow n} \langle O_{K_{alg}} \rangle_n \) to \( \langle O_{K_{alg}} \rangle_0 \bmod p \). This implies

\[
\sigma^{-1} \varepsilon \equiv 1 + \sum_{\tau \geq 0} \beta_{\tau} t^{\tau_{e_1}} \bmod \tau_{e_1} R(N) \text{ and, therefore,}
\]

\[
\varepsilon \equiv 1 + \sum_{\tau \geq 0} \beta_{\tau} t^{\tau_{e_1}} \bmod \tau_{e_1} R(N).
\]

Assume the morphisms \( h^{(m)}_w \in \text{Aut} \mathcal{K} \) from Sect.4.1 are determined by \( \omega \) such that \( E(\omega^p) = 1 + \sum_{\tau \geq 0} \beta_{\tau} t^{\tau_{e_1}} \bmod \tau_{e_1} R(N) \).

As a result, for any \( m, \tau_m(t) \equiv h^{(m)}(t) \bmod \tau_{e_1} m_{R(N)} \).

Suppose \( \tau \in \Gamma \) (recall that \( \Gamma \subset \text{Aut} R_0(N) \)). Then for some integers \( m_1, \ldots, m_N \), we have the following congruence modulo \( \tau_{e_1} m_{R(N)} \)

\[
\tau(t) \equiv \{ t_1 e_{m_1}, \ldots, t_N e_{m_N} \} \equiv \{ h^{(m_1)}_w(t_1), \ldots, h^{(m_N)}_w(t_N) \}.
\]

Let \( h^\tau = h^{(m_1)}_w \cdots h^{(m_N)}_w \in \text{Aut} \mathcal{K} \). Then (use Prop.4.1)

\[
\tau|_\mathcal{K}(t) \equiv h^\tau(t) \bmod \tau_{e_1} m_{R(N)}.
\]

This means that \( \eta := \tau^{-1}|_\mathcal{K} \cdot h^\tau : \mathcal{K} \longrightarrow R_0(N) \) satisfies the assumption of Prop.5.4 and we can consider the corresponding lift \( \tilde{\eta} : \mathcal{K}(p) \longrightarrow R_0(N) \). Let \( \bar{\eta} \) be a lift of \( \tilde{\eta} \) to \( \mathcal{K}_{sep} \).

Set \( \tilde{h}^\tau := (\tau \cdot \tilde{\eta})|_{\mathcal{K}(p)} \).

Then \( \tilde{h}^\tau|_\mathcal{K} = (\tau \cdot \bar{\eta})|_{\mathcal{K}} = h^\tau \) and by Galois theory \( \tilde{h}^\tau \in \mathcal{K} \subset \text{Aut} \mathcal{K}(p) \).

As a result, we obtained the map (of sets) \( \kappa : \Gamma \longrightarrow \Gamma_\omega \) uniquely characterized by the property \( (\text{id}_L \otimes \tau) \tilde{f} = (\text{id}_L \otimes \kappa(\tau)) \tilde{f} \).

**Proposition 5.5.** \( \kappa \) induces a group isomorphism \( \kappa_{<p} : \Gamma_{<p} \longrightarrow \Gamma_\omega \).

**Proof.** Suppose \( \tau_1, \tau \in \Gamma \). Let \( \bar{C} \in \bar{L}_\mathcal{K} \) and \( A \in \text{Aut} \bar{L} \) be such that \( (\text{id}_L \otimes \kappa(\tau)) \bar{f} = A \circ (\bar{A} \otimes \text{id}_{\mathcal{K}(p)}) \bar{f} \). Then

\[
(\text{id}_L \otimes \kappa(\tau_1)) \bar{f} = (\text{id}_L \otimes \tau_1) \bar{f} = (\text{id}_L \otimes \tau_1)(\text{id}_L \otimes \tau) \bar{f} = (\text{id}_L \otimes \tau_1)(\text{id}_L \otimes \tau_1)(\text{id}_L \otimes \tau)(\bar{C} \circ (\bar{A} \otimes \text{id}_{\mathcal{K}(p)}) \bar{f}) = (\text{id}_L \otimes \kappa(\tau_1)) \bar{C} \circ (\bar{A} \otimes \tau)(\text{id}_L \otimes \kappa(\tau)) \bar{C} \circ (\bar{A} \otimes \text{id}_{\mathcal{K}(p)})(\text{id}_L \otimes \tau_1)(\text{id}_L \otimes \kappa(\tau))(\text{id}_L \otimes \kappa(\tau))(\text{id}_L \otimes \kappa(\tau)) \bar{f} = (\text{id}_L \otimes \kappa(\tau))(\text{id}_L \otimes \kappa(\tau))(\text{id}_L \otimes \kappa(\tau))(\text{id}_L \otimes \kappa(\tau)) \bar{f} = (\text{id}_L \otimes \kappa(\tau))(\text{id}_L \otimes \kappa(\tau))(\text{id}_L \otimes \kappa(\tau))(\text{id}_L \otimes \kappa(\tau)) \bar{f}.
\]
and, therefore, \( \kappa(\tau_1\tau) = \kappa(\tau_1)\kappa(\tau) \) (use that \( \Gamma_\omega \) acts strictly on the orbit of \( \bar{f} \)). In particular, \( \kappa \) factors through the natural projection \( \Gamma' \rightarrow \Gamma_{<p} \) and defines the group homomorphism \( \kappa_{<p} : \Gamma_{<p} \rightarrow \Gamma_\omega \).

Recall that we have the identification of \( \Gamma_{<p} \) with \( \Gamma_{<p} \) and, therefore, \( \kappa_{<p} \) identifies the groups \( \kappa(\Gamma_{<p}) \) and \( \Gamma_{<p} \). Besides, \( \kappa_{<p} \) induces a group isomorphism of the group \( \text{Gal}(K(t_1, \ldots, t_{1N})/K) \) and \( \text{Gal}(\bar{K}(\tau_1\omega, \ldots, \tau_m\omega)/\bar{K}(\tau_1\omega)) \) with the quotient \( \langle h_{1\omega}^{(1)}\rangle_{\bar{K}(\tau_1\omega)} \times \cdots \times \langle h_{1\omega}^{(N)}\rangle_{\bar{K}(\tau_1\omega)} \) of \( \Gamma_\omega \). Now Proposition 5.3 implies that \( \kappa_{<p} \) is a group isomorphism. \( \Box \)

5.4. Groups \( \Gamma^p_{<p} \). Consider the group isomorphism \( \kappa_{<p} : \Gamma_{<p} \rightarrow \Gamma_\omega \) from Sect. 5.3.

**Definition.** \( \text{cl}^p \Gamma_{<p} \) is the class of conjugated subgroups in \( \Gamma_{<p} \) containing \( \Gamma^p_{<p} := \kappa^{-1}(\Gamma^p_\omega) \).

This definition involves a choice of local parameters in \( K \).

**Proposition 5.6.** The class \( \text{cl}^p \Gamma_{<p} \) does not depend on a choice of a system of local parameters in \( K \).

**Proof.** Let \( \pi = \{\pi_1, \ldots, \pi_N\} \) and \( \pi' = \{\pi'_1, \ldots, \pi'_N\} \) be two systems of local parameters in \( K \). Let \( K^{(1)}_\pi = K(\pi_1, \ldots, \pi_N) \) and \( K^{(1)}_{\pi'} = K(\pi'_1, \ldots, \pi'_N) \), where (as earlier) for all \( m, \pi^p_1 = \pi_m \) and \( \pi^p_{1m} = \pi'_m \).

Denote by \( K^{(1)} \) the composite of \( K^{(1)}_\pi \) and \( K^{(1)}_{\pi'} \).

Consider the SDR-towers \( K^{(1)}_\pi \) and \( K^{(1)}_{\pi'} \), the fields-of-norms \( K = X(K^{(1)}_\pi) \) and \( K' = X(K^{(1)}_{\pi'}) \) with the corresponding systems of local parameters \( t = \{t_1, \ldots, t_N\} \) and \( t' = \{t'_1, \ldots, t'_N\} \).

Then we have the appropriate field extensions

\[
\mathcal{K} \subset K^{(1)} \subset \mathcal{K}(p) \subset \mathcal{K}_{<p} \subset R_0(N)
\]

\[
\mathcal{K}' \subset K^{(1)}(p) \subset \mathcal{K}'_{<p} \subset R_0(N)
\]

where \( \mathcal{K}(p) \) and \( \mathcal{K}(p) \) were defined earlier, \( \mathcal{K}_{<p} \) and \( \mathcal{K}(p) \) are their analogs if \( K \) is replaced by \( K' \). The exact sequence from Sect. 4.5

\[
1 \rightarrow \bar{G} \rightarrow \Gamma_\omega \rightarrow H_\omega \rightarrow 1,
\]

with \( H_\omega := \prod_{1 \leq m \leq N} \langle h_{1\omega}^{(m)} \rangle_{\bar{K}(\tau_1\omega)} / \langle h_{1\omega}^{(m)p} \rangle_{\bar{K}(\tau_1\omega)} \), gives rise to the exact sequence

\[
(5.2) \quad 1 \rightarrow \bar{G}^{(1)} \rightarrow \Gamma_\omega \rightarrow \text{Gal}(\mathcal{K}(1)/\mathcal{K}) \times H_\omega \rightarrow 1,
\]

where \( \bar{G}^{(1)} = \text{Gal}(\mathcal{K}(p)/\mathcal{K}(1)) \). We have a similar sequence for \( \mathcal{K}' \)

\[
(5.3) \quad 1 \rightarrow \bar{G}'^{(1)} \rightarrow \Gamma'_{\omega'} \rightarrow \text{Gal}(\mathcal{K'}(1)/\mathcal{K'}) \times H'_{\omega'} \rightarrow 1,
\]

where \( H'_{\omega'} \) is an analog of \( H_\omega \). The isomorphisms \( \kappa_{<p} : \Gamma_{\omega} \simeq \Gamma_{<p} \) and \( \kappa'_{<p} : \Gamma'_{\omega'} \simeq \Gamma_{<p} \) induce the isomorphisms

\[
\text{Gal}(\mathcal{K}(1)/\mathcal{K}) \simeq \text{Gal}(\mathcal{K}(1)/\mathcal{K}) \times H'_{\omega'} \simeq \text{Gal}(\mathcal{K}(1)/\mathcal{K}) \times H'_{\omega'},
\]
(5.4) \[ \text{Gal}(K_{<p}/K^{(1)}) \cong \tilde{G}^{(1)} \cong G^{p(1)}. \]

We want to study relation between the \( \mathcal{P} \)-structures on \( \tilde{G}^{(1)} \) and \( G^{p(1)} \) (induced from \( \tilde{G} \) and \( G^p \)).

Recall, cf. Sect. 5.4, there is a Lie algebra

\[ \tilde{M} = \left( \sum_s t^{-s\overline{c_0}} m_K \tilde{L}_{\overline{c_0}}(s) \right) \otimes \mathcal{O}_K/t^{(p-1)\overline{c_0}} \]

and its analogue \( \tilde{M}_{K(p)} \), where \( K \) is replaced by \( K(p) \). There are \( \overline{c} \otimes 1 \in \tilde{M} \) and \( \overline{f} \otimes 1 \in \tilde{M}_{K(p)} \) such that \( \sigma(\overline{f} \otimes 1) = (\overline{c} \otimes 1) \circ (\overline{f} \otimes 1) \) and the identification \( \pi_f(\overline{c}) : \tilde{G} \cong G(\tilde{L}) \) is given by \( \tau \mapsto (-\overline{f} \otimes 1) \circ (\tau \overline{f} \otimes 1) \).

The algebra \( \tilde{L}^P \) (as well as the group \( \tilde{G}^P \)) is defined in terms of the \( \mathcal{P} \)-topological structure on \( \tilde{M} \) coming from the corresponding structures on \( \mathcal{O}_K/t^{(p-1)\overline{c_0}} \) and \( \tilde{L} \). Note that \( \tilde{L} \) is the image of \( L \) and the \( \mathcal{P} \)-topology on \( L \) is induced from \( L/C_2(L) = \text{Hom}(K,F_p) \). Therefore, the \( \mathcal{P} \)-structure on \( \tilde{L} \) comes from

\[ \tilde{L}/C_2(\tilde{L}) = \text{Hom}(t^{-(p-1)\overline{c_0}} m_K/(\sigma-id)K,F_p). \]

As a result,

\[ \overline{c} \otimes 1 \in \tilde{M}^P = \left( \sum_s t^{-s\overline{c_0}} m_K \tilde{L}^P_{\overline{c_0}}(s) \right) \otimes \mathcal{O}_K/t^{(p-1)\overline{c_0}} \]

and \( \pi_f(\overline{c})^{-1}G(\tilde{L}^P) = \tilde{G}^P \).

Similar construction is used to obtain (in the context of \( K' \)) that

\[ \overline{c}' \otimes 1 \in \tilde{M}_{K(p)}^P = \left( \sum_s t^{-s\overline{c_0}} m_K \tilde{L}^P_{\overline{c_0}}(s) \right) \otimes \mathcal{O}_{K(p)}/t^{(p-1)\overline{c_0}} \]

and \( \pi_f(\overline{c})^{-1}G(\tilde{L}^P) = \tilde{G}^P \).

The \( \mathcal{P} \)-subgroups \( \tilde{G}^P \) and \( \tilde{G}^P \) “live in different worlds”, but the field-of-norms functor \( X \) identifies them with subgroups in \( \Gamma_{<p} = \text{Gal}(K_{<p}/K) \). This procedure can be specified as follows.

Let \( \Gamma_{<p} = G(L) \), where \( L \) is a suitable \( F_p \)-algebra. Introduce

\[ \tilde{M} = \left( \sum_{1 \leq s < p} (\zeta_1 - 1)^{-s} C_s(L)_{m_K} \right) \otimes \mathcal{O}_K/p. \]

\[ \tilde{M}_{<p} = \left( \sum_{1 \leq s < p} (\zeta_1 - 1)^{-s} C_s(L)_{m_K_{<p}} \right) \otimes \mathcal{O}_{K_{<p}}/p. \]

The projection \( \text{pr}_1 : R(N) = \lim_{n \to \infty}(\mathcal{O}_{K_{<p}}/p)_n \to (\mathcal{O}_{K_{<p}}/p)_{1} \) establishes the ring isomorphism \( R(N)/t^{(p-1)\overline{c_0}} = R(N)/t^{(p-1)\overline{c_0}} \cong \mathcal{O}_{K_{<p}}/p \), cf. Sect. 5.4 (note that \( \text{Ker} \text{pr}_1 \) is generated by \( t^{(p-1)\overline{c_0}} \)).

Consider the induced by \( \text{pr}_1 \) isomorphism \( \mathcal{O}_{K_{<p}}/t^{(p-1)\overline{c_0}} \cong \mathcal{O}_{K_{<p}}/p \). This gives for each \( 1 \leq s < p \), the compatible identifications of the
\[ \mathcal{O}_K(p)/t(p-1)e^0 \text{-module } t^{-k^0} \mathcal{O}_K(p) \otimes \mathcal{O}_K(p)/t(p-1)e^0 \text{ with } \mathcal{O}_{K,p}^p/p \text{-module } (\zeta_1 - 1)^{-k^0} \mathcal{O}_{K,p}^p \otimes \mathcal{O}_{K,p}^p/p. \]

In addition, the field-of-norms functor identifies the group \( \hat{\mathcal{G}} \) with \( \Gamma_1^p = \text{Gal}(K_{cp}/K_1^p) \subset \Gamma \) and the Lie subalgebra \( L_1^\pi \subset L \), where \( G(L_1^\pi) = \hat{\Gamma}_1^\pi \). As a result, we obtain the embedding \( F_{cp} : \hat{\mathcal{M}}_{K(p)} \rightarrow \hat{\mathcal{M}}_{cp} \) and the induced embedding \( F = F_{cp}|_{\hat{\mathcal{M}}_1} : \hat{\mathcal{M}} \rightarrow \hat{\mathcal{M}}_1 \).

Set \( e^\pi = F(e \otimes 1) \) and \( f^\pi = F_{cp}(f \otimes 1) \). Then \( \sigma f^\pi = e^\pi \circ f^\pi \) and the map \( \tau \mapsto (-f^\pi) \circ \tau(f^\pi) \) recovers the identification \( \hat{\Gamma}_1^\pi \simeq \hat{\mathcal{G}} \) or, equivalently, \( \kappa^\pi : L_1^\pi \simeq \hat{\mathcal{L}}. \) (We used it earlier when constructing the isomorphism \( \kappa_{cp} \).)

The identification \( \kappa^\pi \) is compatible with the \( \mathcal{P} \)-topology.

Indeed, the \( \mathcal{P} \)-topological structure on \( L_1^\pi \) comes (via tensor topology) from \( L_1^\pi/C_2(L_1^\pi) = \text{Hom}(p^{-1}m_K/(\sigma - \text{id})K, \mathbb{F}_p) \) and the field-of-norms identification of \( p^{-1}m_K/(\sigma - \text{id})K \) with \( t^{-(p-1)e^0}m_K/(\sigma - \text{id})K \).

Repeating the above arguments in the context of the system of parameters \( \pi' \) we obtain the \( \mathcal{P} \)-continuous identification \( \kappa^{\pi'} : L_1^{\pi'} \simeq \hat{\mathcal{L}}'. \)

Let \( \hat{\mathcal{G}}'(1) = G(\hat{\mathcal{L}}'(1)) \) and \( \hat{\mathcal{G}}'^{(1)} = G(\hat{\mathcal{L}}'^{(1)}) \). Then isomorphism \( \text{[5.4]} \) appears as compatible with \( \mathcal{P} \)-structures isomorphisms \( L_1^{(1)} \simeq \mathcal{L}_1^{(1)} \simeq \mathcal{L}^{(1)} \). Therefore, the conjugacy classes of \( (\kappa^\pi)^{-1}\mathcal{L}_1^{(1)} \) and \( (\kappa^{\pi'})^{-1}\mathcal{L}_1^{(1)} \) coincide.

Finally, applying the interpretation from Sect.3.6 we obtain from exact sequences \( \text{[5.2]} \) and \( \text{[5.3]} \) that the conjugate classes of \( \kappa_{cp}(\Gamma_1^P) \) and \( \kappa'_{cp}(\Gamma_1^{'P}) \) coincide.

As a result, \( \Gamma_{cp}^P \) is provided with the \( \mathcal{P} \)-topology induced from the subgroup \( \mathcal{G}_p^P \) and this topology does not depend on a choice of such subgroup (i.e. on the choice of local parameters in \( K \)).

For any (local \( N \)-dimensional) subfield \( K \subset K' \subset K_{cp} \), set \( H^P = H \cap \Gamma_{cp}^P \), where \( H = \text{Gal}(K_{cp}/K') \). As earlier (in the case of local fields of characteristic \( p \)), we easily obtain the following property.

**Corollary 5.7.** a) The profinite completion of \( H^P \) is \( H \);

b) \( (\Gamma_{cp} : H) = (\Gamma_{cp}^P : H^P) = [K' : K] \);

c) for all subfields \( K' \), the subgroups \( H^P \) of \( \Gamma_{cp}^P \) can be characterized as all \( \mathcal{P} \)-open subgroups of finite index in \( \Gamma_{cp}^P \).

5.5. **Explicit structure of \( \Gamma_{cp}^P \).** Recall that \( K \) is an \( N \)-dimensional local field, with the first residue field of characteristic \( p \) and the last residue field \( k \simeq \mathbb{F}_{p^{1/n_0}} \). Review the above results about \( \Gamma_{cp} \).

Suppose \( \pi = \{\pi_1, \ldots, \pi_N\} \) is a system of local parameters in \( K \) and \( \zeta_1 \in K \) is a primitive \( p \)-th root of unity. Then

\[
\zeta_1 = E \left( \sum_{\ell = 0}^{n_p - 1} \beta_\ell \pi^{2^{\ell+1}} \right),
\]
where $E(X) \in \mathbb{Z}_p[[X]]$ is the Artin-Hasse exponential, all $t \in \mathbb{Z}_p^N$, $[\beta]$ are the Teichmüller representatives of $\beta \in k$, $\bar{\beta} \neq 0$. (Recall that $\bar{\pi}^2 := \bar{\pi}_1^{c_1} \cdots \bar{\pi}_N^{c_N}$, where $\bar{c} = (c_1, \ldots, c_N)$.)

Associate with $K$ the topological $\mathbb{F}_p$-algebra $L^p$ as follows.

Consider an $N$-dimensional local field $K$ of characteristic $p$ with finite residue field $k \simeq \mathbb{F}_p^{N_0}$. Then we have the topological module $\overline{K}^p := \operatorname{Hom}_{\mathbb{F}_p\text{-cont}}(K/\sigma - \operatorname{id}K, \mathbb{F}_p)$, which generate the "maximal" $\mathbb{F}_p$-Lie algebra $L^p$ of nilpotent class $< p$, i.e. the quotient of free Lie algebra with module of generators $\overline{K}^p$ by the ideal of $p$-th commutators. (Note that $L^p$ appears as the projective limit of "maximal" Lie algebras $L_{C_\alpha}$ (of nilpotent class $< p$) generated by the elements of open subsets $\overline{C}^D \subset M$.)

Let $\sigma$ be the Frobenius automorphism on $k$ and denote by the same symbol the $\sigma$-linear automorphism of $L_k$ such that $D_{an} \mapsto D_{a,n+1}, D_0 \mapsto D_0$. (We also set $D_{bo} = \sigma^a(a_0)D_0$, where $a_0$ is a fixed element from $W(k) \subset K$ with absolute trace 1.)

Let $L^p = L^p_k/\mathcal{L}^0_k(p)|_{\sigma = \operatorname{id}}$ with induced $\mathcal{P}$-topological structure.

Introduce the Lie $\mathbb{F}_p$-algebra $L^p$ as the maximal algebra of nilpotent class $< p$ containing $\mathcal{L}$ and the generators $\{\bar{l}^{(m)} | 1 \leq m \leq N\}$. Introduce the ideal of relations $\mathcal{R}^k_{\bar{l}}$ in $L^p_k$ as follows.

Specify recurrent relation (1.9) to our situation:

\[
\sigma \bar{c}_1^{(m)} - \bar{c}_1^{(m)} + \sum_{a \in \mathbb{Z}_p^N(p)} t^{-a} V_{a_0}^{(m)} = 0
\]

\[
- \frac{1}{k!} t^{-(a_1 + \cdots + a_k)} \bar{c}_1^{(m)} \cdots \sum_{1 \leq k < p} \frac{1}{k!} t^{-(a_1 + \cdots + a_k)} \cdots [D_{a_10}, D_{a_20}, \ldots, D_{a_k0}]
\]

\[
- \sum_{2 \leq k < p} \frac{1}{k!} t^{-(a_1 + \cdots + a_k)} \cdots [V_{a_10}^{(m)}, D_{a_20}, \ldots, D_{a_k0}]
\]

\[
- \sum_{1 \leq k < p} \frac{1}{k!} t^{-(a_1 + \cdots + a_k)} \cdots [D_{a_10}, D_{a_20}, \ldots, D_{a_k0}]
\]

(5.5)

(5.5) \quad (the indices $a_1, \ldots, a_k$ in all above sums run over $\mathbb{Z}_p^0(p)$). Recall that $\bar{c}_{1}^{(m)} \in \mathcal{L}^p_k$ and $V_{a_0} \in \mathcal{L}^p_k$. Then the ideal $\mathcal{R}^k_{\bar{l}} \subset L^p_k$ is generated by the following elements:

1) $[D_{an}, \bar{l}^{(m)}] - \sigma^{n}(V_{a_0}^{(m)}), a \in \mathbb{Z}_p^0(p)$;
2) \([\bar{l}^{(i)}, \bar{l}^{(j)}] - \bar{l}[i, j]\), where \(\bar{l}[i, j] \in \bar{L}\) is given in notation of Cor. 4.11

\[
\bar{l}[i, j] = \text{ad}^{(i)}(\bar{c}^{(j)}(0)) - \text{ad}^{(j)}(\bar{c}^{(i)}(0)) + \sum_{\epsilon \in \mathbb{Z}^N} [\bar{e}^{(i)}(\epsilon), \bar{e}^{(j)}(-\epsilon)].
\]

Let \(L_k = L_k^p/R_k^p\) and \(L = L_k\mid_{\sigma = \text{id}}\).

**Remark.** Taking another solutions \(\{\bar{c}^{(m)}_1, V^{(m)}_a \mid a \in \mathbb{Z}_N^0(p), 1 \leq m \leq N\}\) of (5.3) is equivalent to replacing the generators \(\bar{l}^{(m)}\) modulo \(\bar{L}^p\).

Summarizing the above results we state the following theorem.

**Theorem 5.8.** a) \(\Gamma^{<p}_c \simeq G(L^p)\);

b) \(\Gamma_{<p}\) is the profinite completion of \(G(L^p)\).

5.6. **Simplest example.** The above description of \(\Gamma_{<p}\) essentially uses the equivalence of the category of \(p\)-groups and \(\mathbb{F}_p\)-Lie algebras (there is no operation of extension of scalars in the category of \(p\)-groups).

It could be also verified that the study of involved \(p\)-groups at the level of their Lie algebras gives much simpler form of the corresponding relations. At the same time the above presentation of \(L^p\) as the quotient of \(L^p\) by appropriate relations is not the simplest one. Some relations, e.g. \([D_{an}, \bar{l}^{(m)}] - V_{an}\) with \(a > \bar{c}^0\) can be used to exclude extra generators \(D_{an}\) with \(a > \bar{c}^0\). Ideally, the whole description should be done in terms of, say, the minimal system of generators \(\{D_{an} \mid a < \bar{c}^0\} \cup \{\bar{l}^{(m)} \mid 1 \leq m \leq N\}\). This was done in Sect. 4.1.7 where we presented our description modulo third commutators.

In the case of the last residue field \(k \simeq \mathbb{F}_p\) we do not need the operation of extension of scalars and can express the answer directly in terms of groups. This will not give any simplifications, but can be easily obtained when working modulo third commutators.

Namely, if \(k \simeq \mathbb{F}_p\) then \(\Gamma^{<p}_c \mod C_3(\Gamma^{<p}_c)\) appears as the group with \(\mathcal{P}\)-topological generators \(\{\tau_a \mid a \in \mathbb{Z}_N^0(p), a < \bar{c}^0\} \cup \{\bar{l}^{(m)} \mid 1 \leq m \leq N\}\) and the subgroup of relations generated (as normal subgroup) by following relations:

- \(R(i, j) = (\bar{l}^{(i)}, \bar{l}^{(j)}),\) here \(1 \leq i < j \leq N\);

- \(R_0(m) = (\tau_0, \bar{l}^{(m)}) \prod_{\epsilon > 0} \prod_{b+c = \bar{c}^0+pa} (\tau_b, \tau_c)^{b(m)/2} ;\)

- \(R_a(i, j) = (\tau_a, \bar{l}^{(i)a^{(j)}}/\bar{l}^{(j)a^{(i)}}) \prod_{n \geq 1} \prod_{\epsilon > 0} (\tau_b, \tau_c)^{(b(m) - b(j)c(i))\beta_i/2} \times \prod_{n \geq 1} \prod_{\epsilon > 0} (\tau_b, \tau_c)^{(b(m)c(j) - b(j)c(i))\beta_j/2} ;\)

where \(\mathcal{P}\) is the set of \(p\)-prime divisors of the residue field.\(\square\)
here $1 \leq m \leq N$, $1 \leq i < j \leq N$, $a \in \mathbb{Z}_N^+(p)$.

The above example could be simplified if we take 2-dimensional $K = \mathbb{Q}_p(\zeta_1)\{\pi_2\}$ with the system of local parameters $\pi = \{\pi_1, \pi_2\}$, where $E(\pi_1) = \zeta_1$. In this case $\widehat{c}^0 = (p, 0)$, $\beta_0 = 1$ and all remaining $\beta_i = 0$.

We have the system of minimal generators

$$\{\tau_a \mid a \in \mathbb{Z}_N^+(p), a < (p, 0)\} \cup \{\bar{h}^{(1)}, \bar{h}^{(2)}\}$$

and the following relations:

- $\mathcal{R}(1, 2) = (\bar{h}^{(1)}, \bar{h}^{(2)})$;

- $\mathcal{R}_0(1) = (\tau_0, \bar{h}^{(1)}) \prod_{1 \leq \alpha \leq \frac{p-1}{2}} \prod_{\gamma} (\tau_{(\alpha, \gamma)}, \tau_{(p-\alpha, -\gamma)})^a$

- $\mathcal{R}_0(2) = (\tau_0, \bar{h}^{(2)}) \prod_{1 \leq \alpha \leq \frac{p-1}{2}} \prod_{\gamma} (\tau_{(\alpha, \gamma)}, \tau_{(p-\alpha, -\gamma)})^\gamma$

- $\mathcal{R}_a(1, 2) : (\tau_a, \bar{h}^{(1)}a^{(2)}/\bar{h}^{(2)}a^{(1)}) \times \prod_{\gamma} (\tau_{(p-1, \gamma)}, \tau_{(p,a^{(2)}-p\gamma)})^{-a^{(2)}\delta_{0,a^{(1)}}}$

$$\times \prod_{n \geq 1, \beta} (\tau_{(p-n, -\beta)}, \tau_{a+(0,p^n\beta)})^{a^{(1)}} \times \prod_{b+c=(p,0)+a} (\tau_b, \tau_c)^{(b^{(1)}c^{(2)}-b^{(2)}c^{(1)})/2}$$

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