UNIQUE STRONG SOLUTIONS AND V-ATTRACTOR OF A
THREE DIMENSIONAL GLOBALLY MODIFIED
MAGNETOHYDRODYNAMIC EQUATIONS

G. DEUGOUÉ*
Department of Mathematics and Computer Science, University of Dschang
P. O. BOX 67, Dschang, Cameroon
Department of Mathematics and Statistics, Florida International University
MMC, Miami, FL 33199, USA

J. K. DJOKO
School of Computational and Applied Mathematics, University of the Witwatersrand
1 Jan Smuts Avenue, Braamfontein 2000, Johannesburg, South Africa

A. C. FOUAPE AND A. NDONGMO NGANA
Department of Mathematics and Computer Science, University of Dschang
P. O. BOX 67, Dschang, Cameroon
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Abstract. In this paper, we provide a detailed investigation of the problem of
existence and uniqueness of strong solutions of a three-dimensional system of
globally modified magnetohydrodynamic equations which describe the motion
of turbulent particles of fluids in a magnetic field. We use the flattening prop-
erty to establish the existence of the global V-attractor and a limit argument to
obtain the existence of bounded entire weak solutions of the three-dimensional
magnetohydrodynamic equations with time independent forcing.

1. Introduction. It is well known that the 3D magnetohydrodynamic (MHD)
equations describe the motion of electrically conducting fluids and one of these
models is given by

\[
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \frac{1}{R_e} \Delta \mathbf{u} - S \mathbf{u} \cdot \nabla \mathbf{B} + \nabla \left( p + \frac{|\mathbf{B}|^2}{2} \right) &= \mathbf{f}_1(t), \\
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{u} + \frac{1}{R_m} \text{curl(curlB)} &= \mathbf{f}_2(t), \\
\text{div} \mathbf{u} &= 0, \quad \text{div} \mathbf{B} = 0,
\end{aligned}
\]

where \( \mathbf{u}, \mathbf{B} \) and \( p \) represent respectively the fluid velocity, the magnetic field and
the pressure. \( \mathbf{f}_1 \) and \( \mathbf{f}_2 \) are given external forces field. \( R_e \) and \( R_m \) are the so-called
Reynolds and magnetic Reynolds numbers, respectively, and \( S = \frac{M^2}{R_e R_m} \) is a positive

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* Corresponding author.
constant, where $M$ is the Hartman number. $|B|^2 = B \cdot B$ and represents the length of the magnetic field. These equations take into account the coupling between Maxwell’s equations governing the magnetic field and the Navier-Stokes equations (NSE) governing the fluid motion [3]. They play a fundamental role in astrophysics, geophysics, plasma physics and in many other areas in applied sciences. They have been intensively investigated for many years (see [1, 2, 3, 22, 23] just to cite a few), but some very basic issues on their solvability remain unresolved. For example, the problem of uniqueness of weak solution and the global regularity of the solution remain one of the open problems in mathematical physics.

In this paper, we introduce the following system of 3D globally modified magnetohydrodynamic equations (GMMHDE) on a bounded domain $M \subset \mathbb{R}^3$ with smooth boundary $\partial M$

\[
\begin{aligned}
\frac{\partial u}{\partial t} + F_N(||u||_{V_1})((u \cdot \nabla)u) - \frac{1}{Re} \Delta u & = \nonumber \\
- SF_N (||u, B||_V) (|B \cdot \nabla)B| + \nabla\left(p + \frac{1}{2} |B|^2\right) & = f_1, \\
\frac{\partial B}{\partial t} + F_N(||u, B||_V) ((u \cdot \nabla)B - (B \cdot \nabla)u) + \frac{1}{Rm} \text{curl(curl}B) & = f_2, \\
\text{div}u = 0, \text{div}B = 0, 
\end{aligned}
\]

endowed with the following initial and boundary conditions

\[
\begin{aligned}
&\begin{cases}
  u(0, x) = u_0(x), & B(0, x) = B_0(x) \text{ for all } x \in M, \\
  u = 0 \text{ on } \partial M \text{ (non slip condition),} \\
  B \cdot n = 0 \text{ and curl } B \times n = 0 \text{ on } \partial M \text{ (perfectly conducting wall),}
\end{cases}
& (3)
\end{aligned}
\]

where $n$ is the unit outward normal on $\partial M$, $N \in (0, \infty)$ is fixed and $F_N : [0, \infty) \to (0, 1]$ is defined by

\[
F_N(r) = \min\{1, N/r\}, \quad r \in \mathbb{R}^+.
\]

The spaces $V_1$ and $V$ together with $||u||_{V_1}$ and $||u, B||_V$ will be defined later.

Note that when $B = 0$, system (2)-(3) becomes the 3D globally modified Navier-Stokes equations (GMNSE) which have been the object of intense investigations over the last years [4, 6, 7, 8, 10, 14, 15, 18, 19, 21, 31]. The 3D GMNSE were introduced and studied in [4]. Contrary to the original 3D NSE, this modified model of the 3D NSE has some good properties such as global existence, uniqueness and regularity. These results are interesting in their own right, but also the GMNSE are useful in obtaining new results about the 3D Navier-Stokes equations. For example, they were used in [4] to establish the existence of bounded entire weak solutions for them. Also in [15], the GMNSE were used to show that the attainability set of weak solutions of the three dimensional Navier-Stokes equations satisfying an energy inequality, are weakly compact and weakly connected. For convergence results of solutions of the GMNSE to solutions of the three dimensional Navier-Stokes equations, see [4, 19]. These results were extended in [24, 25] to the case of the 3D globally modified Cahn-Hilliard-Navier-Stokes equations and the 3D globally modified Allen-Cahn-Navier-Stokes equations.

The GMMHDE (2)-(3) is inspired from the globally modified Navier-Stokes equations (GMNSE) studied in [4]. As noted in [4] concerning the GMNSE, the GMMHDE are indeed globally modified. The factors $F_N(||u||_{V_1})$ and $F_N(||u, B||_V)$ which depend, respectively, on the norms $||u||_{V_1}$ and $||u, B||_V$ are used only to prevent large values of $||u||_{V_1}$ and $||u, B||_V$ dominating the dynamics. Just like the
GMNSE, the GMMHDE violate the basic laws of mechanics, but mathematically the model is well defined.

Motivated by the above references, we propose to analyze the globally modified magnetohydrodynamic equations (2)-(3). The following three points are our main contribution in this work:

(a) we prove the existence, uniqueness of the weak and strong solutions for the GMMHDE,
(b) the flattening property is used to establish the existence of the global V-attractor,
(c) using a limiting argument, we obtain the existence of bounded entire weak solutions of the three-dimensional MHD equations with time independent forcing.

The paper is structured as follows. In Section 2, we introduce the mathematical setting of our problem. In Section 3, we prove the existence, uniqueness of the weak and strong solutions for the 3D GMMHDE. In addition, we establish the continuous dependence of the solution on $N$ and on the initial value in the space $V$. We also investigate the relationship between the Galerkin approximations of the 3D GMMHDE and the 3D MHD for a fixed finite dimension. In Section 4, we study the asymptotic behavior of the strong solutions when the forcing terms are time independent and we prove the existence of the global attractor in $V$. In Section 5, we prove that solutions to the 3D GMMHDE converge to a weak solution of the 3D MHD equations. For a time-independent forcing, we also prove the existence of bounded entire weak solutions of the 3D MHD equations.

2. The mathematical setting. Let us now recall from [22, 26] the functional set up of the model (2)–(3) and its abstract formulation.

Bold notation will represent a vector or a tensor. We consider the well-known Hilbert spaces $L^2(\mathcal{M}), H^m(\mathcal{M}), H^m_0(\mathcal{M})$ and we use the notations

$$L^2(\mathcal{M}) := (L^2(\mathcal{M}))^3, \quad \mathbb{H}^m(\mathcal{M}) := (H^m(\mathcal{M}))^3, \quad \mathbb{H}^m_0(\mathcal{M}) := (H^m_0(\mathcal{M}))^3,$$

We also introduce the following spaces

$$V_1 = \{ u \in (C_c^\infty(\mathcal{M}))^3 : \text{div} u = 0 \},$$

$$V_1 = \text{the closure of } V_1 \text{ in } \mathbb{H}^1_0(\mathcal{M}),$$

$$H_1 = \{ u \in L^2(\mathcal{M}) : \text{div} u = 0 \text{ and } u \cdot n = 0 \text{ on } \partial \mathcal{M} \}$$

$$V_2 = \{ B \in (C_c^\infty(\mathcal{M}))^3 : \text{curl} B = 0, B \cdot n = 0 \text{ on } \partial \mathcal{M} \},$$

$$V_2 = \{ B \in \mathbb{H}^1(\mathcal{M}) : \text{curl} B = 0; B \cdot n = 0 \text{ on } \partial \mathcal{M} \},$$

$$H_2 = \text{the closure of } V_2 \text{ in } L^2(\mathcal{M}).$$

Thus $H_2 = H_1$. We endow $H_i, i = 1, 2$ with the inner product of $L^2(\mathcal{M})$ and the norm of $L^2(\mathcal{M})$ denote respectively by $(.,.)_{L^2}$ and $|.|_{L^2}$.

We equip $V_1$ with the following inner product

$$((u, v))_1 = \sum_{i=1}^3 \left( \frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2}.$$

We equip $V_2$ with the scalar product

$$((u, v))_2 = (\text{curl} u, \text{curl} v)_{L^2}.$$
We note that by Poincaré’s inequality, the scalar product \((.,.)_1\) defines in (7) coincides with the well known inner product in \(H_0^1(M)\). The norm generated by \((.,.)_2\) is equivalent to the norm induced by \(H^1(M)\) on \(V_2\) (see [11, Chapter VII]). Hereafter, we set

\[
H = H_1 \times H_2, \quad V = V_1 \times V_2. \tag{9}
\]

The dual space of \(V\) is denoted by \(V'\). We endow \(H\) with the inner products defined as: for all \(\phi = (u, B), \psi = (v, C) \in H\), \(\phi = (u, B), \psi = (v, C) \in H\).

\[
(\phi, \psi) = (u, v) + (B, C),
\]

\[
[\phi, \psi] = (u, v) + S(B, C). \tag{10}
\]

They generate equivalent norms (for \(0 < S < \infty\))

\[
|\phi|_H^2 = (\phi, \phi) = |u|^2_2 + |B|^2_2, \quad [\phi]_H^2 = [\phi, \phi] = |u|^2_2 + S|B|^2_2. \tag{11}
\]

We also endow \(V\) with the inner products

\[
((\phi, \psi)) = \frac{1}{R_e}((u, v))_1 + \frac{1}{R_m}((B, C))_2, \quad [[\phi, \psi]] = \frac{1}{R_e}((u, v))_1 + \frac{S}{R_m}((B, C))_2,
\]

which in turn generate the equivalent norms on \(V\)

\[
||\phi||_V^2 = ((\phi, \phi)), \quad ||\phi||_V^2 = [[\phi, \phi]]. \tag{13}
\]

In order to give an abstract formulation of problem (2)-(3), we introduce the operators \(A_1 \in \mathcal{L}(V_1, V_1')\), \(A_2 \in \mathcal{L}(V_2, V_2')\), and \(A \in \mathcal{L}(V, V')\) defined by

\[
\langle A_1 u, v \rangle = ((u, v))_1, \quad \text{for all } u, v \in V_1,
\]

\[
\langle A_2 B, C \rangle = ((B, C))_2, \quad \text{for all } B, C \in V_2,
\]

\[
\langle A \phi, \psi \rangle = ((\phi, \psi)), \quad \text{for all } \phi, \psi \in V. \tag{14}
\]

\(A_1\) can be defined as an unbounded operator generated by the Stokes problem

\[
\begin{cases}
-\Delta u + \nabla p = f \quad \text{in } M, \\
div u = 0 \quad \text{in } M, \\
u = 0 \quad \text{on } \partial M,
\end{cases}
\]

with domain \(D(A_1) = \{u \in V_1 : A_1 u \in H_1\}\). For more detail concerning the operator \(A_1\) we refer the reader to [9, 17, 26]. The operator \(A_2\) can also be defined as an unbounded operator generated by the boundary value problem

\[
\begin{cases}
\text{curl(curl}(B) = f \quad \text{in } M, \\
\text{div}B = 0 \quad \text{in } M, \\
B \cdot n = 0, \quad \text{curl}B \times n = 0 \quad \text{on } \partial M,
\end{cases} \tag{15}
\]

with domain \(D(A_2) = \{C \in V_2 : A_2 C \in H_2\}\). For more information about the operator \(A_2\), we refer the reader to [11]. \(A\) is then an unbounded operator with domain \(D(A) = D(A_1) \times D(A_2)\).

From the continuity of the injection of \(V_i\), in \(H_i, i = 1, 2\), there exist constant \(\kappa_i, i = 1, 2\) such that

\[
|u|_{L^2} \leq \kappa_1 |u|_{V_1} \quad \text{for all } u \in V_1, \quad |B|_{L^2} \leq \kappa_2 |B|_{V_2} \quad \text{for all } B \in V_2. \tag{16}
\]

The best constant \(\kappa_i\) is equal to \(\frac{1}{\sqrt{\lambda_i}}\), where \(\lambda_i^1\) is the first eigenvalue of the compact operator \(A_i^{-1}\) from \(H_i\) into itself.
As in [22], we introduce the trilinear form $\mathcal{B}_0$ on $V \times V \times V$ by

$$
\mathcal{B}_0(\phi_1, \phi_2, \phi_3) = b(u_1, u_2, u_3) - Sb(B_1, B_2, u_3) + b(u_1, B_2, B_3) - b(B_1, u_2, B_3),
$$

(17)

for all $\phi_i = (u_i, B_i) \in V (i = 1, 2, 3)$, where $b$ is a continuous trilinear form defined on $H^1(M) \times H^1(M) \times H^1(M)$ by

$$
b(u, v, w) = \sum_{i,j=1}^{3} \int_M u_i \frac{\partial v_j}{\partial x_i} w_j dx,
$$

which satisfies the following relations

$$
b(u, v, v) = 0, \forall u \in V_1, v \in H^1(M),
$$

$$
b(u, v, w) = -b(u, w, v), \forall u \in V_1, v, w \in H^1(M),
$$

$$
|b(u, v, w)| \leq c \|u\|_{V_1}^{1/2} \|v\|_{V_2}^{1/2} \|w\|_{L^2} \|v\|_{L^2}, \forall u \in D(A_1), v \in V_1, w \in H_1
$$

(18)

$$
|b(b_1, b_2, u)| \leq c \|b_1\|_{V_2} \|b_2\|_{V_2} \|u\|_{V_2}, \forall b_1, b_2 \in V_2, u \in V_1,
$$

$$
|b(b_1, b_2, u)| \leq c \|b_1\|_{V_2} \|b_2\|_{V_2}, \forall b_1 \in V_2, b_2 \in D(A_2), u \in H_1,
$$

$$
|b(b_1, u_1, b_2)| \leq c \|b_1\|_{V_2} \|u_1\|_{L^2} \|b_2\|_{L^2}, \forall b_1 \in V_2, u_1 \in D(A_1), b_2 \in H_2.
$$

The operator $b$ also satisfies the following estimate

$$
|b(u, v, w)| \leq \|u\|_{L^6} \|v\|_{L^2} \|w\|_{L^2}, \forall u, v, w \in H^1(M).
$$

(19)

**Proof.** To simplify the notations, we assume the summation over repeated indexes. Let $u, v, w \in H^1(M)$. We have, by using the generalized Hölder inequality with exponents $(6, 2, 4, 12)$

$$
|b(u, v, w)| \leq \int_M |u_i| \left| \frac{\partial v_j}{\partial x_i} \right| |w_j| dx
$$

(20)

$$
= \int_M |u_i| \left| \frac{\partial v_j}{\partial x_i} \right| |w_j|^{1/2} |w_j|^{1/2} dx
$$

(21)

$$
\leq |u_i|_{L^6} \left| \frac{\partial v_j}{\partial x_i} \right|_{L^2} |w_j|_{L^2}^{1/2} |w_j|_{L^6}^{1/2}
$$

(22)

$$
\leq |u|_{L^6} \|v\|_{L^2} \|w\|_{L^2}^{1/2} \|w\|_{L^6}^{1/2},
$$

(23)

which proves (19). 

From (18), we infer that

$$
\mathcal{B}_0(\phi_1, \phi_2, \phi_2) = 0, \forall \phi_1, \phi_2 \in V,
$$

$$
\mathcal{B}_0(\phi_1, \phi_2, \phi_3) = -\mathcal{B}_0(\phi_1, \phi_3, \phi_2), \forall \phi_i \in V, i = 1, 2, 3
$$

(24)

Now we introduce the continuous bilinear form $B : V \times V \to V'$ by

$$
\langle B(\phi_1, \phi_2), \phi_3 \rangle = \mathcal{B}_0(\phi_1, \phi_2, \phi_3).
$$

(25)

We introduce a diagonal matrix $M = (m_{ij})_{1 \leq i, j \leq 6}$ defined by:

$$
\begin{cases}
m_{ii} = 1 & \text{if } 1 \leq i \leq 3, \\
m_{ii} = S & \text{if } 4 \leq i \leq 6, \\
m_{ij} = 0 & \text{if } i \neq j.
\end{cases}
$$

(26)
Note that
\[ B_0(\phi_1, \phi_2, M\phi_2) = b(u_1, u_2, u_3) + Sb(B_1, B_2, B_3) - S[b(B_1, B_2, u_2) + b(B_1, u_2, B_2)]. \] (27)

It follows from (18) and (27) that
\[ B_0(\phi_1, \phi_2, M\phi_2) = 0 \quad \forall \phi_1, \phi_2 \in V, \]
\[ B_0(\phi_1, \phi_2, M\phi_3) = -B_0(\phi_1, \phi_3, M\phi_2), \quad \forall \phi_i \in V, i = 1, 2, 3. \] (28)

We recall that (see [22]) \( B_0, B \) and \( b \) satisfy the following estimates
\[ |B_0(\phi_1, \phi_2, \phi_3)| \leq c|\phi_1|_V^2|\phi_2|_H^{1/2}|\phi_3|_H, \quad \forall \phi_1 \in V, \phi_2 \in D(A), \phi_3 \in H, \]
\[ \|B(\phi, \phi)\|_V \leq c|\phi|_H^2|\phi|_V^{3/2}. \] (29)

Hereafter we set
\[ B^N_0(\phi_1, \phi_2, \phi_3) = F_N(\|u_2\|_V) b(u_1, u_2, u_3) - SF_N(\|u_2, B_2\|_V)b(B_1, B_2, u_3) \]
\[ + F_N(\|u_2, B_2\|_V)b(u_1, B_2, B_3) - F_N(\|u_2, B_2\|_V)b(B_1, u_2, B_3) \]
\[ \langle B^N(\phi_1, \phi_2), \phi_3 \rangle = B^N_0(\phi_1, \phi_2, \phi_3), \quad \forall \phi_i = (u_i, B_i) \in V, i = 1, 2, 3. \] (30)

Arguing similarly as in the proof of (29), we can check that the following inequalities hold
\[ |B^N_0(\phi_1, \phi_2, \phi_3)| \leq cN|\phi_1|_V^{1/2}|\phi_2|_H^{1/2}|\phi_3|_H^2 + cSN|\phi_1|_V^{1/2}|\phi_3|_H, \quad \forall \phi_1 \in V, \phi_2 \in D(A), \phi_3 \in H, \]
\[ |B^N_0(\phi_1, \phi_2)| \leq cN|\phi_1|_H^{1/4}|A\phi_1|_H^{3/4}|\phi_2|_H + cSN|\phi_1|_V^{1/2}|A\phi_1|_H^{1/2}|\phi_3|_H, \quad \forall \phi_1 \in D(A), \phi_2 \in H, \] (31)
\[ |B^N(\phi_1, \phi_2)|_V \leq cN|\phi_1|_H^{1/4}|A\phi_1|_H^{3/4}|\phi_2|_H, \quad \forall \phi_1 \in D(A), \phi_2 \in V, \]
\[ \|B^N(\phi_1, \phi_2)\|_V \leq cN|\phi_1|_V^2 + cNS|\phi_1|_V, \]
\[ B^N_0(\phi_1, \phi_2) = cN|\phi_1|_V^{1/2}|A\phi_1|_H^{1/2}|\phi_2|_H \]
\[ + c|\phi_1|_V^2|A\phi_1|_H^2|\phi_2|_H, \quad \forall \phi_1 \in D(A), \phi_2 \in H. \] (32)

We note that in (35) and (32) we have also used (18).

Using the operators \( A, B^N \) previously defined, we rewrite (2)-(3) in the form
\[ \begin{cases} \frac{dy}{dt} + Ay + B^N(y, y) = F, \\ y(0) = y_0 = (u_0, B_0), \end{cases} \] (36)

where \( y = (u, B) \) is the solution of (2)-(3) and \( F = (f_1, f_2) \).

We now define the concept of solution associated with (36).

**Definition 2.1.** Suppose that \( (u_0, B_0) \in H \) and \( f_i \in L^2(0, T; V_i'), i = 1, 2 \).

A weak solution to (36) is any pair \( y = (u, B) \in L^2(0, T; V) \) such that
\[ \begin{cases} \frac{dy}{dt} + Ay + B^N(y, y) = F \text{ in } D(0, +\infty; V'), \\ y(0) = y_0 = (u_0, B_0), \end{cases} \] (37)
or equivalently
\[
\begin{align*}
\left( \frac{d\mathbf{y}}{dt}, \psi \right) + \left( (\mathbf{y}, \psi) \right) + B_0^N(\mathbf{y}, \mathbf{y}, \psi) &= (f_1, \mathbf{v}) + (f_2, \mathbf{C}) \quad \text{for all } \psi = (\mathbf{v}, \mathbf{C}) \in V, \\
y(0) &= y_0.
\end{align*}
\tag{38}
\]

**Remark 1.** Note that if \( y \in L^2(0, T; V) \) for all \( T > 0 \) and satisfies \((37)_1\), it follows from \((33)-(34)\) that \( \frac{d\mathbf{y}}{dt} \in L^2(0, T; V') \), and consequently (see \([27]\)), \( y \in C([0, T); H) \).

**Remark 2.** By taking \( \psi = M\mathbf{y} \) in \((38)_1\) and using the fact that \( B_0^N(\mathbf{y}, \mathbf{y}, M\mathbf{y}) = 0 \) (see \((28)_1\)), we derive that \( y \) satisfies the energy equality
\[
\begin{align*}
|\mathbf{u}(t)|_{L^2}^2 + S|\mathbf{B}(t)|_{L^2}^2 + \frac{2}{R_e} \int_s^t \| \mathbf{u}(\xi) \|_{V_1}^2 d\xi + \frac{2S}{R_m} \int_s^t \| \mathbf{B}(\xi) \|_{V_2}^2 d\xi \\
= |\mathbf{u}_0|_{L^2}^2 + S|\mathbf{B}_0|_{L^2}^2 + 2 \int_s^t (f_1(\xi), \mathbf{u}(\xi)) d\xi \\
+ 2S \int_s^t (f_2(\xi), \mathbf{B}(\xi)) d\xi \quad \text{for all } 0 \leq s \leq t.
\end{align*}
\tag{39}
\]

Note that the weak formulation of \((2)\) with \( F_N \) replaced by 1 is studied in \([22]\), where the existence and uniqueness of solution was proved in the two-dimensional case.

Now we recall from \([4, 21, 24]\) the following properties of \( F_N \), where the proof can be found in \([4, 21]\).

\[
\begin{align*}
|F_N(p) - F_N(r)| &\leq \frac{|p - r|}{r}, \quad \forall p, r \in \mathbb{R}^+, \quad r \neq 0, \\
|F_N(\|\mathbf{u}\|_{V_1}) - F_N(\|\mathbf{v}\|_{V_1})| &\leq \frac{\|\mathbf{u} - \mathbf{v}\|_{V_1}}{\|\mathbf{v}\|_{V_1}}, \quad \mathbf{u}, \mathbf{v} \in V_1, \quad \mathbf{v} \neq 0, \\
|F_M(p) - F_M(r)| &\leq \frac{|M - N|}{r} + \frac{|p - r|}{r}, \quad \forall p, r, M, N \in \mathbb{R}^+, \quad r \neq 0,
\end{align*}
\tag{40}
\]

\[
|F_N(\|\mathbf{u}\|_{V_1}) - F_N(\|\mathbf{v}\|_{V_1})| \leq \frac{1}{N} F_N(\|\mathbf{u}\|_{V_1}) F_N(\|\mathbf{v}\|_{V_1}) \|\mathbf{u} - \mathbf{v}\|_{V_1}, \quad \mathbf{u}, \mathbf{v} \in V_1.
\]

In the rest of this paper we will denote by \( c \), a generic positive constant (possibly depending on \( S, R_e, R_m, \kappa_1, M, \kappa_2 \)), which can vary even within the same line. However, this constant is always independent of time and initial data.

3. **Existence and uniqueness of weak and strong solutions.** The following result deals with the uniqueness of weak solution to the problem \((36)\).

**Theorem 3.1.** There exists at most one weak solution \((\mathbf{u}, B)\) of \((36)\) in the sense of Definition 2.1.

**Proof.** Let \( y_i = (\mathbf{u}_i, \mathbf{B}_i), \ i = 1, 2 \) be weak solutions to \((36)\) that belong to \( L^2(0, T; V) \). We set \( \delta \mathbf{y} = (\delta \mathbf{u}, \delta \mathbf{B}) = y_1 - y_2 \). Then \( (\delta \mathbf{u}, \delta \mathbf{B}) \) satisfies
\[
\begin{align*}
\left\{ \frac{d\delta \mathbf{y}}{dt} + A\delta \mathbf{y} + B^N(\mathbf{y}_1, \mathbf{y}_1) - B^N(\mathbf{y}_2, \mathbf{y}_2) = 0, \\
\delta \mathbf{y}(0) = 0.
\end{align*}
\tag{41}
\]
We will now estimate each term in the right-hand side of (48).

\[
\begin{align*}
(B^N(y_1, y_1) - B^N(y_2, y_2), M\delta y) &= F_N(\|u_1\|_V) b(u_1, u_1, \delta u) - F_N(\|u_2\|_V) b(u_2, u_2, \delta u) \\
- S [F_N(\|u_1, B_1\|_V) b(B_1, B_1, \delta u) - F_N(\|u_2, B_2\|_V) b(B_2, B_2, \delta u)] + S [F_N(\|u_1, B_1\|_V) b(u_1, B_1, \delta B) - F_N(\|u_2, B_2\|_V) b(u_2, B_2, \delta B)] \\
- S [F_N(\|u_1, B_1\|_V) b(B_1, u_1, \delta B) - F_N(\|u_2, B_2\|_V) b(b_2, u_2, \delta B)].
\end{align*}
\]  

(42)

Now using the fact that \( b(u_2, \delta u, \delta u) = b(u_2, \delta B, \delta B) = 0 \), we have

\[
\begin{align*}
F_N(\|u_1\|_V) b(u_1, u_1, \delta u) - F_N(\|u_2\|_V) b(u_2, u_2, \delta u) &= F_N(\|u_1\|_V) b(u_1, u_1, \delta u) + F_N(\|u_1\|_V) - F_N(\|u_2\|_V) b(u_2, u_2, \delta u), \quad (43) \\
[F_N(\|u_1, B_1\|_V) b(B_1, B_1, \delta u) - F_N(\|u_2, B_2\|_V) b(B_2, B_2, \delta u)] &= F_N(\|u_1, B_1\|_V) b(B_1, B_1, \delta u) + F_N(\|u_2, B_2\|_V) b(B_2, B_2, \delta u) + F_N(\|u_1, B_1\|_V) b(B_2, B_2, \delta u) \delta u) \\
F_N(\|u_1, B_1\|_V) b(B_2, B_2, \delta B)] &= F_N(\|u_1, B_1\|_V) b(B_1, B_1, \delta B) + F_N(\|u_2, B_2\|_V) b(B_2, B_2, \delta B) + F_N(\|u_1, B_1\|_V) b(B_2, B_2, \delta B) + F_N(\|u_2, B_2\|_V) b(B_2, B_2, \delta B). \quad (45) \\
F_N(\|u_1, B_1\|_V) b(B_2, B_2, \delta B) + F_N(\|u_2, B_2\|_V) b(B_2, B_2, \delta B)] &= F_N(\|u_1, B_1\|_V) b(B_1, B_1, \delta B) + F_N(\|u_2, B_2\|_V) b(B_2, B_2, \delta B) + F_N(\|u_1, B_1\|_V) b(B_2, B_2, \delta B) + F_N(\|u_2, B_2\|_V) b(B_2, B_2, \delta B). \quad (46)
\end{align*}
\]

Inserting these estimates (43)-(46) in (42) and using also the fact that \( b(B_2, \delta u, \delta B) = -b(B_2, \delta B, \delta u) \), we obtain

\[
\begin{align*}
(B^N(y_1, y_1) - B^N(y_2, y_2), M\delta y) &= F_N(\|u_1\|_V) b(u_1, u_1, \delta u) + F_N(\|u_1\|_V) - F_N(\|u_2\|_V) b(u_2, u_2, \delta u) \\
- SF_N(\|u_1, B_1\|_V) b(B_1, B_1, \delta u) + SF_N(\|u_1, B_1\|_V) b(B_1, B_1, \delta B) + SF_N(\|u_2, B_2\|_V) b(u_2, B_2, \delta B) \\
- SF_N(\|u_1, B_1\|_V) b(u_2, B_2, \delta u) - SF_N(\|u_1, B_1\|_V) b(B_2, B_2, \delta B) \\
- SF_N(\|u_1, B_1\|_V) b(B_2, B_2, \delta B)] &= F_N(\|u_1, B_1\|_V) b(B_1, B_1, \delta B) + F_N(\|u_2, B_2\|_V) b(B_2, B_2, \delta B) + F_N(\|u_1, B_1\|_V) b(B_2, B_2, \delta B) + F_N(\|u_2, B_2\|_V) b(B_2, B_2, \delta B). \quad (47)
\end{align*}
\]

Taking the scalar product in \( H \) of (41) with \( M\delta y \), we obtain

\[
\frac{d\mathcal{Y}}{dt} + \frac{2}{R_e} \|\delta u\|_V^2 + \frac{2S}{R_m} \|\delta B\|_V^2 = -2(B^N(y_1, y_1) - B^N(y_2, y_2), M\delta y),
\]

(48)

with \( \mathcal{Y} = \|\delta u\|_V^2 + S\|\delta B\|_V^2 \) and \( (B^N(y_1, y_1) - B^N(y_2, y_2), M\delta y) \) given by (47).

We will now estimate each term in the right-hand side of (48).
Note that
\[ F_N(\|u_1\|_{V_1}) b(\delta u, u_1, \delta u) + [F_N(\|u_1\|_{V_1}) - F_N(\|u_2\|_{V_1})] b(u_2, u_2, \delta u) \]
\[ = F_N(\|u_1\|_{V_1}) b(\delta u, u_1, \delta u) + [F_N(\|u_1\|_{V_1}) - F_N(\|u_2\|_{V_1})] b(u_2 - u_1, u_2, \delta u) \]
\[ + [F_N(\|u_1\|_{V_1}) - F_N(\|u_2\|_{V_1})] b(u_1, u_2, \delta u) \]
\[ = F_N(\|u_1\|_{V_1}) b(\delta u, u_1, \delta u) + [F_N(\|u_1\|_{V_1}) - F_N(\|u_2\|_{V_1})] b(\delta u, u_2, \delta u) \]
\[ + [F_N(\|u_1\|_{V_1}) - F_N(\|u_2\|_{V_1})] b(u_1, u_2, \delta u) \]
\[ = F_N(\|u_1\|_{V_1}) b(\delta u, u_1, \delta u) + [F_N(\|u_1\|_{V_1}) - F_N(\|u_2\|_{V_1})] b(u_1, u_2, \delta u), \tag{49} \]

Since \( b(\delta u, \delta u, u) = 0 \).

From (19) and the Young inequality with exponents \((4/3, 4)\), we have
\[ 2F_N(\|u_2\|_{V_1}) b(\delta u, u_2, \delta u) \leq 2F_N(\|u_2\|_{V_1}) \|\delta u\|_{V_1} \|\nabla u_2\|_{L^2} \|\delta u\|_{L^2}^{1/2} \|\delta u\|_{L^2}^{1/2} \]
\[ = cN \|\delta u\|_{V_1}^{3/2} \|\delta u\|_{L^2}^{1/2} \]
\[ \leq \frac{1}{TR_e} \|\delta u\|_{V_1}^2 + cN^4 \|\delta u\|_{L^2}^2. \]

From (40), (19) and the Young inequality with exponents \((4/3, 4)\), we obtain
\[ 2|F_N(\|u_1\|_{V_1}) - F_N(\|u_2\|_{V_1})| |b(u_1, u_2, \delta u)| \]
\[ \leq \frac{2}{N} F_N(\|u_1\|_{V_1}) F_N(\|u_2\|_{V_1}) \|\delta u\|_{V_1} \|\nabla u_2\|_{L^2} \|\delta u\|_{L^2}^{1/2} \|\delta u\|_{L^2}^{1/2} \]
\[ \leq \frac{c}{N} F_N(\|u_1\|_{V_1}) F_N(\|u_2\|_{V_1}) \|\delta u\|_{V_1} \|u_1\|_{V_1} \|u_2\|_{V_1} \|\delta u\|_{L^2}^{1/2} \|\delta u\|_{L^2}^{1/2} \]
\[ \leq cN \|\delta u\|_{V_1}^{3/2} \|\delta u\|_{L^2}^{1/2} \]
\[ \leq \frac{1}{TR_e} \|\delta u\|_{V_1}^2 + cN^4 \|\delta u\|_{L^2}^2. \]

From (18)_4 and Young's inequality, we have
\[ 2S |F_N(\|u_1, B_1\|_{V}) b(\delta B, B_1, \delta u)| \]
\[ \leq \frac{cN}{\|(u_1, B_1)\|_{V}} \|\delta B\|_{L^2}^{1/4} \|\delta B\|_{V_2}^{3/4} \|\delta u\|_{V_1} \|B_1\|_{V_2} \]
\[ \leq cN \|\delta B\|_{L^2}^{1/4} \|\delta B\|_{V_2}^{3/4} \|\delta u\|_{V_1} \]
\[ \leq \frac{1}{TR_e} \|\delta u\|_{V_1}^2 + \frac{2}{7R_m} \|\delta B\|_{V_2}^2 + cN^8 \|\delta B\|_{L^2}^2. \]

From (40), (19) and the Young inequality with exponents \((4/3, 4)\), we obtain
\[ 2S |F_N(\|u_1, B_1\|_{V}) - F_N(\|u_2, B_2\|_{V}) b(B_2, B_1, \delta u)| \]
\[ \leq \frac{2S}{N} F_N(\|u_1, B_1\|_{V}) F_N(\|u_2, B_2\|_{V}) \|\delta u\|_{L^2}^{1/2} \|\delta u\|_{L^2}^{1/2} \]
\[ \leq \frac{c}{N} F_N(\|u_1, B_1\|_{V}) F_N(\|u_2, B_2\|_{V}) \|\delta u\|_{V_2} \|\delta u\|_{V_2} \|B_1\|_{V_2} \|\delta u\|_{L^2}^{1/2} \|\delta u\|_{L^2}^{1/2} \]
\[ \leq \frac{c}{N} F_N(\|u_1, B_1\|_{V}) F_N(\|u_2, B_2\|_{V}) \|\delta u\|_{V_2} \|\delta u\|_{V_2} \|B_1\|_{V_2} \|\delta u\|_{L^2}^{1/2} \]
and

\[ \begin{align*}
&\leq cN\|\delta u\|_{V}^{3/2}\delta u^{1/2}_{L^{2}} \\
&\leq \frac{1}{t}\|\delta u|_{V}^{2} + cN^{4}\delta u^{2}_{L^{2}}. 
\end{align*} \]

Arguing similarly as in (53), we infer that

\[ \begin{align*}
2S|F_{N}(\|u_{1}, B_{1}\|_{V}) - F_{N}(\|u_{2}, B_{2}\|_{V})b(B_{2}, u_{1}, \delta B)|
&\leq \frac{1}{t}\|\delta u, \delta B\|_{V}^{2} + cN^{4}\|\delta B\|_{L^{2}}^{2}, \\
&\leq \frac{1}{t}\|\delta u, \delta B\|_{V}^{2} + cN^{4}\|\delta B\|_{L^{2}}^{2}, \\
&\leq \frac{1}{t}\|\delta u, \delta B\|_{V}^{2} + cN^{4}\|\delta B\|_{L^{2}}^{2}.
\end{align*} \]

Using (19) and the Young inequality we have

\[ \begin{align*}
2S|F_{N}(\|u_{1}, B_{1}\|_{V})b(\delta u, B_{1}, \delta B)|
&\leq 2SF_{N}(\|u_{1}, B_{1}\|_{V})\|\delta u\|_{L^{2}}\|\nabla B_{1}\|_{L^{2}}\|\delta B\|_{L^{2}}^{1/2}\|\delta B\|_{L^{2}}^{1/2} \\
&\leq cF_{N}\|\|u_{1}, B_{1}\|_{V}\|\|\delta u\|_{V_{1}}\|B_{1}\|_{V_{2}}\|B_{1}\|_{L^{2}}^{1/2}\|\delta B\|_{V_{2}}^{1/2} \\
&\leq cN\|\|\delta u\|_{V_{1}}\|\|B_{1}\|_{L^{2}}^{1/2}\|\|\delta B\|_{V_{2}}^{1/2} \\
&\leq \frac{1}{R_{e}}\|\delta u\|_{V_{1}}^{2} + \frac{S}{R_{m}}\|\delta B\|_{V_{2}}^{2} + cN^{4}\|\delta B\|_{L^{2}}^{2}.
\end{align*} \]

and

\[ \begin{align*}
2S|F_{N}(\|u_{1}, B_{1}\|_{V})b(\delta B, u_{1}, \delta B)|
&\leq cF_{N}\|\|u_{1}, B_{1}\|_{V}\|\|\delta B\|_{V_{2}}\|u_{1}\|_{V_{1}}\|\delta B\|_{L^{2}}^{1/2}\|\delta B\|_{V_{2}}^{1/2} \\
&\leq cN\|\|\delta B\|_{V_{2}}^{1/2}\|\|\delta B\|_{L^{2}}^{1/2} \\
&\leq \frac{S}{R_{m}}\|\delta B\|_{V_{2}}^{2} + cN^{4}\|\delta B\|_{L^{2}}^{2}.
\end{align*} \]

Inserting these estimates (50)–(57) in (48), we infer that

\[ \begin{align*}
\frac{d\mathcal{Y}}{dt} + \frac{1}{R_{e}}\|\delta u\|_{V_{1}}^{2} + \frac{S}{R_{m}}\|\delta B\|_{V_{2}}^{2} \leq (cN^{4} + cN^{8})\mathcal{Y}.
\end{align*} \]

Momentarily dropping the term \[ \frac{1}{R_{e}}\|\delta u\|_{V_{1}}^{2} + \frac{S}{R_{m}}\|\delta B\|_{V_{2}}^{2}, \]
we have a differential inequality

\[ \frac{d\mathcal{Y}}{dt} \leq (cN^{4} + cN^{8})\mathcal{Y}, \]

from which we obtain by using the Gronwall lemma:

\[ \mathcal{Y} = |\delta u|_{L^{2}}^{2} + S|\delta B|_{L^{2}}^{2} = 0 (\text{since } \mathcal{Y}(0) = |\delta u(0)|_{L^{2}}^{2} + S|\delta B(0)|_{L^{2}}^{2} = 0), \]
i.e., \((u_{1}, B_{1}) = (u_{2}, B_{2})\).

\[ \square \]

The next result concerns the existence of weak and strong solutions to the problem (36).

**Theorem 3.2.** Suppose that \( f_{1} \in L^{2}(0, T; H_{1}), f_{2} \in L^{2}(0, T; H_{2}) \) for all \( T > 0 \) and \((u_{0}, B_{0}) \in H \) be given.

(i) If \((u_{0}, B_{0}) \in V, \) there exists a unique weak solution \( y = (u, B) \) of (36), which is in fact a strong solution in the sense that

\[ y \in C(0, T; V) \cap L^{2}(0, T; D(A_{1}) \times D(A_{2})). \]
(ii) If the initial condition \((u_0, B_0) \not\in V\) then every weak solution \(y = (u, B)\) of (36) is a strong solution, in the sense that

\[
(u, B) \in C(\delta, T; V) \cap L^2(\delta, T; D(A_1) \times D(A_2)) \quad \text{for all } T > \delta > 0.
\]

(61)

(iii) If \((u_0, B_0) \not\in V\) but \(f_1 \in L^\infty(0, \delta; H_1), f_2 \in L^\infty(0, \delta; H_2)\) for some \(\delta > 0\), then there exists at least one weak solution \((u, B)\) of (36).

Proof. (ii). The proof of (ii) is the consequence of assertion (i). In fact if \((u_0, B_0) \not\in V\) and \((u, B)\) is a weak solution of (36) then \((u, B) \in C(0, T; H) \cap L^2(0, T; V)\) for all \(T > 0\), therefore for any \(\epsilon > 0\), there exists \(0 < t_0 < \epsilon\) such that \((u(t_0), B(t_0)) \in V\). Then the function \((v(t), B_1(t)) := (u(t + t_0), B(t + t_0))\) is a weak solution of system (36) with \((v(0), B_1(0)) = (u(t_0), B(t_0)) \in V\) and forcing term \((f_1(t + t_0), f_2(t + t_0))\), and by assertion (i) \((v, B_1) \in C(0, T; V) \cap L^2(0, T; D(A_1) \times D(A_2))\) for all \(T > 0\), and in particular \((u, B)\) satisfies (36).

(i) and (iii). Since the injection \(V \subset H\) is compact, let \((w_1, \psi_i, i = 1, 2, \ldots) \subset V\) be a orthonormal basis of \(H\), where \((w_i, i = 1, 2, \ldots), \{\psi_i, i = 1, 2, \ldots\}\) are eigenvectors of \(A_1\) and \(A_2\), respectively. We set \(V_n = H_n = \text{span}\{(w_1, \psi_1), \ldots, (w_n, \psi_n)\}\) and denote by \(P_n = (P_{n1}, P_{n2})\), the orthogonal projector in \(H\) onto \(V_n\) for the scalar product \((., .)\) defines in (10). Note that \(P_n\) is also the orthogonal projector in \(D(A), V, V'\) onto \(V_n\).

We look for \(y_n = (u_n, B_n) \in H_n\) solution to the ordinary differential equations

\[
\begin{align*}
\frac{dy_n}{dt} + A y_n + P_n B^N(y_n, y_n) &= P_n F, \\
y_n(0) &= P_n(u_0, B_0) = P_n u_0 = (u_{0n}, B_{0n}).
\end{align*}
\]

(62)

Since \(P_n F = P_n(f_1, f_2)\) is a locally Lipschitz function in \((u, B)\), it follows from the theory of ordinary differential equation that the system (62) has a solution \(y_n\), (see also Theorem A1 of [5]). The function \(y_n\) exists on an interval \([0, T_n]\). It will follows from a priori estimates that \(y_n\) exists on the interval \([0, T]\).

We now derive needed a priori estimates.

Using the same techniques as in Remark 2, we see that \(y_n\) satisfies the following energy equality

\[
\frac{d}{dt} \left(\|u_n(t)\|_{L^2}^2 + S\|B_n(t)\|_{L^2}^2\right) + \frac{2}{R_e}\|u_n(t)\|_{V_1}^2 + \frac{2S}{R_m}\|B_n(t)\|_{V_2}^2
\]

\[
= 2(P_{n1} f_1(t), u_n(t)) + 2S(P_{n2} f_2(t), B_n(t)),
\]

(63)

for all \(t \geq 0\).

Note that

\[
2\|P_{n1} f_1, u_n\| \leq 2\|f_1\|_{V_1'} \|u_n\|_{V_1} \leq \frac{1}{R_e}\|u_n\|_{V_1}^2 + c\|f_1\|_{V_1'}^2,
\]

(64)

\[
2S\|P_{n2} f_2, B_n\| \leq 2S\|f_2\|_{V_2'} \|B_n\|_{V_2} \leq \frac{S}{R_m}\|B_n\|_{V_2}^2 + c\|f_2\|_{V_2'}^2,
\]

(65)

since \(\|P_n\|_{L(V, V')} \leq 1\). Inserting these estimates (64)-(65) in (63), we infer that

\[
\frac{d}{dt} \left(\|u_n(t)\|_{L^2}^2 + S\|B_n(t)\|_{L^2}^2\right) + \frac{1}{R_e}\|u_n(t)\|_{V_1}^2 + \frac{S}{R_m}\|B_n(t)\|_{V_2}^2
\]

\[
\leq c\|f_1(t)\|_{V_1'}^2 + c\|f_2(t)\|_{V_2'}^2,
\]

(66)
for all \( t \geq 0 \). Hence, we have

\[
|u_n(t)|^2_{L^2} + S|B_n(t)|^2_{L^2} + \frac{1}{R} \int_0^t \|u_n(s)\|_{V_q}^2 \, ds + \frac{S}{R_m} \int_0^t \|B_n(s)\|_{V_q}^2 \, ds
\leq |u_0|^2_{L^2} + S|B_0|^2_{L^2} + c \int_0^t \|f_1(s)\|_{V_q'}^2 \, ds + c \int_0^t \|f_2(s)\|_{V_q'}^2 \, ds,
\]

(67)

for all \( t \geq 0 \).

Also, from (63)-(65) and using also (16), we infer that

\[
\frac{d}{dt} \mathcal{Y}_n(t) + \beta \mathcal{Y}_n(t) \leq c\|f_1(t)\|_{V_q'}^2 + c\|f_2(t)\|_{V_q'}^2,
\]

(68)

\[
\mathcal{Y}_n(t) = |u_n(t)|^2_{L^2} + S|B_n(t)|^2_{L^2}, \quad \beta = \min \left( \frac{1}{\kappa_1^2 R_c}, \frac{1}{\kappa_2^2 R_m} \right),
\]

from which we obtain by the technique of Gronwall’s lemma

\[
\mathcal{Y}_n(t) \leq (|u_0|^2_{L^2} + S|B_0|^2_{L^2}) \exp^{-\beta t} + c \sum_{i=1}^2 \int_0^t \|f_i(s)\|_{V_q'}^2 \exp^{-\beta(t-s)} \, ds,
\]

(69)

for all \( t \geq 0 \).

This proves that the sequence \( y_n = (u_n, B_n) \) remains in a bounded set of \( L^\infty(0, T; H) \cap L^2(0, T; V) \). Hence, we can use a compactness argument (see [28]) to see that a subsequence \( y_{n'} = (u_{n'}, B_{n'}) \) satisfies

\[
y_{n'} \to y \quad \text{weak-star in } L^\infty(0, T; H),
\]

weakly in \( L^2(0, T; V) \),

\[
\text{strongly in } L^2(0, T; H),
\]

(70)

a.e., in \( (0, T) \times \mathcal{M} \),

with \( y = (u, B) \in L^\infty(0, T; H) \cap L^2(0, T; V) \).

As the weak convergence in \( L^2(0, T; V) \) is not enough to ensure that

\[
F_N(\|u_{n'}\|_{V_q}) \to F_N(\|u\|_{V_q}) \quad \text{as } n' \to \infty,
\]

(71)

\[
F_N(\|(u_{n'}, B_{n'})\|_{V}) \to F_N(\|(u, B)\|_{V}) \quad \text{as } n' \to \infty,
\]

we need to derive stronger a priori estimates. Hence, taking the inner product in \( H \) of (62) with \( Ay_{n'} \), we obtain

\[
\frac{d}{dt} \|y_{n'}\|_{V}^2 + 2|Ay_{n'}|_H^2 = 2(f_1, A_1 u_{n'}) + 2(f_2, A_2 B_{n'}) - 2B_0^N(y_{n'}, y_{n'}, Ay_{n'}). \quad (72)
\]

Now using (31) and Young’s inequality with the exponents \( (4, 4/3) \), we have

\[
2|B_0^N(y_{n'}, y_{n'}, Ay_{n'})| \leq cN \|y_{n'}\|_V^{1/2} \|Ay_{n'}\|_H^{3/2} \leq \frac{1}{2} |Ay_{n'}|_H^2 + cN^4 \|y_{n'}\|_V^2. \quad (73)
\]

In addition, by Young’s inequality, one obtains

\[
2|(f_1, A_1 u_{n'})| + 2|(f_2, A_2 B_{n'})| \leq \frac{1}{4} |A_1 u_{n'}|_{L^2}^2 + \frac{1}{4} |A_2 B_{n'}|_{L^2}^2 + c \sum_{i=1}^2 |f_i|_{L^2}^2.
\]

(74)

It follows from (72)-(74) that

\[
\frac{d}{dt} \|y_{n'}\|_{V}^2 + |Ay_{n'}|_H^2 \leq c|f_1|_{L^2}^2 + c|f_2|_{L^2}^2 + cN^4 \|y_{n'}\|_V^2. \quad (75)
\]
Now we distinguish two cases:

**Case 1:** \( y_0 = (u_0, B_0) \in V \).

We mention that \( \| (u_{0n'}, B_{0n'}) \|_V = \| P_n'(u_0, B_0) \|_V \leq \| y_0 \|_V \),

by the choice of the orthonormal basis \( \{(w_i, \psi_i)\} \) of \( H \).

Now, dropping the term \( |A| y_{n'} |^2_H \) in (75), we have the following differential inequality

\[
\frac{d}{dt} \| y_{n'} \|^2_V \leq c |f_1|^2_{L^2} + c |f_2|^2_{L^2} + c N^4 \| y_{n'} \|^2_V,
\]

(76)

from which we obtain by using Gronwall’s lemma

\[
\| y_{n'}(t) \|^2_V \leq \| y_0 \|^2 \exp \left[ c N^4 t \right] + c \int_0^t \left( |f_1(s)|^2_{L^2} + |f_2(s)|^2_{L^2} \right) \exp \left[ c N^4 (t - s) \right] ds.
\]

(77)

Hence, we derive from (67), (75), and (77) that \( (y_{n'}) = (u_{n'}, B_{n'}) \) satisfies

\[
\| (u_{n'}, B_{n'})(t) \|^2_V \leq K, \quad \int_0^T \left( |A_1 u_{n'}(s)|^2_{L^2} + |A_2 B_{n'}(s)|^2_{L^2} \right) ds \leq K_1,
\]

(78)

which proves that \( (u_{n'}, B_{n'}) \) is bounded in \( L^\infty(0, T; V) \cap L^2(0, T; D(A_1) \times D(A_2)) \).

Note that in (78), \( K \) and \( K_1 \) are positive constants independent of \( n' \) and depending only on data such as \( M, R_e, R_m, S, f_1, f_2, T, u_0 \) and \( B_0 \).

Note that from (31), we have for any \( \phi_3 \in H \),

\[
|\mathcal{B}_0^N(y_{n'}, y_{n'}, \phi_3)| \leq c N |A| y_{n'} | H | \phi_3 | H + c S N |A| y_{n'} | H | \phi_3 | H.
\]

Hence from this last inequality and from (78), we deduce that the sequence \( \{P_n B_N(y_{n'}, y_{n'})\} \) is bounded in \( L^2(0, T; H_1 \times H_2) \), for all \( T > 0 \).

Therefore, from (62), we infer that the sequence

\[
\frac{d}{dt}(u_{n'}, B_{n'}) \text{ is also bounded in } L^2(0, T; H).
\]

(79)

Since \( D(A) = D(A_1) \times D(A_2) \subset V \subset H \) with compact injection, we derive from [16, Theorem 5.1, Chapter 1] that there exists an element \( (u, B) \in L^\infty(0, T; V) \cap L^2(0, T; D(A)) \), and a subsequence of \( (u_{n'}, B_{n'}) \) (still) denoted \( (u_{n'}, B_{n'}) \) such that for all \( T > 0 \), we have

\[
(u_{n'}, B_{n'}) \to (u, B) \quad \text{weak-star in } L^\infty(0, T; V),
\]

\[
\text{weakly in } L^2(0, T; D(A)),
\]

\[
\text{strongly in } L^2(0, T; V),
\]

\[
\text{a.e., in } (0, T) \times \mathcal{M},
\]

\[
\frac{d}{dt}(u_{n'}, B_{n'}) \to \frac{d}{dt}(u, B) \text{ weakly in } L^2(0, T; H).
\]

(80)

Also, as \( (u_{n'}, B_{n'}) \) converges to \( (u, B) \) in \( L^2(0, T; V) \) for all \( T > 0 \), we can infer that there exists a subsequence (still) denoted \( (u_{n'}, B_{n'}) \) such that

\[
\| (u_{n'}, B_{n'}) \|_V \to \| (u, B) \|_V \text{ a.e., in } (0, +\infty),
\]

and therefore

\[
F_N(\| u_{n'} \|_{V_1}) \to F_N(\| u \|_{V_1}) \text{ a.e., in } (0, +\infty),
\]

\[
F_N(\| (u_{n'}, B_{n'}) \|_V) \to F_N(\| (u, B) \|_V) \text{ a.e., in } (0, +\infty).
\]
Hence, arguing as in [4], we can take the limit in (62) to infer that \((u, B)\) is a weak solution to (36) satisfying the energy equality (39).

**Case 2:** \((u_0, B_0) \in H\), but \((u_0, B_0) \notin V\), and \(f_i \in L^\infty(0, \infty; H_i), i = 1, 2\). To achieve our goal, we will reasoning similarly as in [4, 9, 29, 24, 25].

We set \(|f_i|_\infty = |f_i|_{L^\infty(0, \infty; H_i)}, i = 1, 2\).

From (75), we obtain
\[
\|y_n'(t)\|_V^2 \leq \left[\|y_n'(t_0)\|_V^2 + c \left(\|f_1\|_\infty^2 + \|f_2\|_\infty^2\right) (t - t_0)\right] \exp \left[cN^4(t - t_0)\right], \quad (81)
\]
for any \(t_0 \in [0, t]\).

From (69), we also have
\[
\|u_n'(t)\|_{L_2}^2 + S|B_n'(t)|_{L_2}^2 \leq \left(|u_0|_{L_2}^2 + S|B_0|_{L_2}^2\right) \exp^{-\beta t} + \frac{c}{\beta} \sum_{i=1}^{2} |f_i|_\infty^2 \quad = \mathbb{K}_1(t), \forall t \geq 0.
\]

By integration in \(t\) from \(t\) to \(t + \tau\) of (66), we obtain, after dropping unnecessary terms,
\[
\int_t^{t+\tau} \left[\frac{1}{R_c} \|u_n'(s)\|_{V_1}^2 + \frac{S}{R_m} \|B_n'(s)\|_{V_2}^2\right] ds \\
\leq \left(|u_n'(t)\|_{L_2}^2 + S|B_n'(t)|_{L_2}^2\right) + c\tau \sum_{i=1}^{2} |f_i|_\infty^2 \\
\leq \mathbb{K}_1(t) + c\tau \sum_{i=1}^{2} |f_i|_\infty^2,
\]
where we have also use (82).

Hence, from (83), we deduce that
\[
\int_t^{t+\tau} \left[\frac{1}{R_c} \|u_n'(s)\|_{V_1}^2 + \frac{1}{R_m} \|B_n'(s)\|_{V_2}^2\right] ds \leq \frac{\mathbb{K}_1(t)}{\min(1, S)} + \tau \mathbb{K}_2, \quad (84)
\]
where \(\mathbb{K}_2 = c \sum_{i=1}^{2} |f_i|_\infty^2\).

For \(t > 0\) and \(\tau > 0\) given, let us define \(\rho > 0\) by
\[
\rho^2 = \frac{2}{\tau} \left[\frac{|u_0|_{L_2}^2 + S|B_0|_{L_2}^2}{\min(1, S)} + \left(\frac{1}{\min(1, S)} + \frac{\mathbb{K}_2}{\tau}\right)\mathbb{K}_2\right], \quad (85)
\]
and consider the sets
\[
D_{n'} = \{s \in [t, t + \tau] : \|y_{n'}\|_V \geq \rho\},
\]
and let us denote \(|D_{n'}|\) the Lebesgue measure of \(D_{n'}\). Using (84), we infer that
\[
\rho^2|D_{n'}| \leq \int_{D_{n'+\tau}} \|y_{n'}(s)\|_V^2 ds \\
\leq \int_t^{t+\tau} \|y_{n'}(s)\|_V^2 ds \\
\leq \frac{1}{\min(1, S)} \left[|u_0|_{L_2}^2 + S|B_0|_{L_2}^2 + \mathbb{K}_2\right] + \tau \mathbb{K}_2 = \frac{\tau \rho^2}{2},
\]
therefore \(|D_{n'}| \leq \tau / 2\).
Hence we can ensure that for any \( t \geq 0, \tau > 0 \) and \( n' \geq 1 \) there exists a \( t_0 \in (t, t + \tau) \) such that
\[
\|y_{n'}(t_0)\|_V^2 \leq \frac{2}{\tau} \left[ \frac{|u_0|_{L^2}^2 + S|B_0|_{L^2}^2}{\min(1, S)} + \left( \frac{1}{\min(1, S)\beta} + \tau \right) K_2 \right].
\]
From this property, we have for any given \( \delta > 0 \) and any \( t \geq \delta \), there exists a \( t_0 \in (t - \delta, t) \) such that
\[
\|y_{n'}(t_0)\|_V^2 \leq \frac{2}{\delta} \left[ \frac{|u_0|_{L^2}^2 + S|B_0|_{L^2}^2}{\min(1, S)} + \left( \frac{1}{\min(1, S)\beta} + \delta \right) K_2 \right]. \tag{87}
\]
From (87) and (81), we have
\[
\|y_{n'}(t)\|_V^2 \leq \frac{2}{\delta} \left[ \frac{|u_0|_{L^2}^2 + S|B_0|_{L^2}^2}{\min(1, S)} + \left( \frac{1}{\min(1, S)\beta} + \delta \right) K_2 \right] \exp \left[ cN^4(t - t_0) \right]
\leq \left[ c \left( |f_1|_{L^\infty}^2 + |f_2|_{L^\infty}^2 \right) (t - t_0) \right] \exp \left[ cN^4(t - t_0) \right]
\leq \left\{ \frac{2}{\delta} \left[ \frac{|u_0|_{L^2}^2 + S|B_0|_{L^2}^2}{\min(1, S)} + \left( \frac{1}{\min(1, S)\beta} + \delta \right) K_2 \right] \right\} \exp \left[ cN^4\delta \right] \tag{88}
\]
for all \( t \geq \delta \).

From (67), (75) and (88), we deduce that the sequence \( y_{n'} = (u_{n'}, B_{n'}) \) is bounded in \( L^\infty(0, T; H) \), in \( L^2(0, T; V) \), in \( L^2(\delta, T; D(A)) \), and in \( L^\infty(\delta, T; V) \), for all \( T > \delta > 0 \). By making similar reasoning as in case 1, we can prove that the sequence \( \frac{d}{dt}(u_{n'}, B_{n'}) \) is also bounded in \( L^2(\delta, T; H) \) for all \( T > \delta > 0 \). Hence, there exists an element
\[
(u, B) \in L^\infty(0, T; H) \cap L^2(0, T; V) \cap L^\infty(\delta, T; V) \cap L^2(\delta, T; D(A))
\]
for all \( T > \delta > 0 \), and a subsequence (still denoted \( (u_{n'}, B_{n'}) \), such that
\[
(u_{n'}, B_{n'}) \to (u, B) \text{ weakly in } L^2(0, T; V), \\
(u_{n'}, B_{n'}) \to (u, B) \text{ a.e. in } (0, T) \times \mathcal{M}, \\
(u_{n'}, B_{n'}) \to (u, B) \text{ strongly in } L^2(0, T; H), \\
(u_{n'}, B_{n'}) \to (u, B) \text{ strongly in } L^2(\delta, T; V), \\
(u_{n'}, B_{n'}) \to (u, B) \text{ weakly-star in } L^\infty(0, T; H), \\
(u_{n'}, B_{n'}) \to (u, B) \text{ weakly in } L^2(\delta, T; D(A)), \\
(u_{n'}, B_{n'}) \to (u, B) \text{ weakly-star in } L^\infty(\delta, T; V), \tag{89}
\]
\[
\frac{d}{dt}(u_{n'}, B_{n'}) \to \frac{d}{dt}(u, B) \text{ weakly in } L^2(\delta, T; H).
\]
Since \( (u_{n'}, B_{n'}) \) converges strongly to \( (u, B) \) in \( L^2(\delta, T; V) \) for all \( T > \delta > 0 \), we can assume, eventually extracting a subsequence, that (71) is also satisfied in this case. Hence from (71) and (89) we can take the limit in (62) and prove that \( (u, B) \) is a solution to (36) satisfying (61). \( \square \)

### 3.1. Continuous dependence on initial values and \( N \)

Here, we prove that the semi flows generated by the solutions \( (u_N, B_N)(t, (u_0, B_0)) \) of the GMMHDE (36) with the parameter \( N \) depend continuously on the parameter \( N \) as well as on the initial value \( (u_0, B_0) \). We begin by given an important result.
Theorem 3.3. Assume that \( f_i \in L^2(0, T; H_i), i = 1, 2 \) for all \( T > 0 \), and let \( N_i > 0, y_{0i} = (u_{0i}, B_{0i}) \in V_i, i = 1, 2 \) be given. Let \( y_i = (u_i, B_i) \) be the solution to (36) corresponding to the parameter \( N_i \) and the initial value \( y_{0i} = (u_{0i}, B_{0i}) \), \( i = 1, 2 \). Then, there exists a positive constant \( c \) depending only on \( R_e, R_m, M \) and \( S \) such that

\[
\|y_1(t) - y_2(t)\|_V^2 \\
\leq \left\|y_{01} - y_{02}\right\|_V^2 + c(N_1 - N_2)^2 \int_0^t Z_1(s)ds \times \exp \left[ cN_1^4t + c \int_0^t Z_1(s)ds \right], \quad (90)
\]

\[
\int_0^t |Ay_1(s) - Ay_2(s)|_B^2 ds \\
\leq \left\|y_{01} - y_{02}\right\|_V^2 + c(N_1 - N_2)^2 \int_0^t Z_1(s)ds \quad (\times)
\]

\[
(\times) \left[1 + \left(cN_1^4t + c \int_0^t Z_1(s)ds \right) \exp \left[ cN_1^4t + c \int_0^t Z_1(s)ds \right] \right], \quad (91)
\]

for all \( t \geq 0 \), with \( Z_1 = |A_1u_2|_{L_2}^2 + |A_2B_2|_{L_2}^2 \).

Proof. Let us set \( y = y_1 - y_2 = (u_1, B_1) - (u_2, B_2) = (\delta u, \delta B) \). Then \( y = (\delta u, \delta B) \) satisfies

\[
\frac{dy}{dt} + Ay + B_{N_1}(y, y_1) - B_{N_2}(y, y_2) = 0. \quad (92)
\]

From [4], we have

\[
K_1 \equiv F_{N_1}(\|u_1\|_V)b(u_1, u_1, A_1\delta u) - F_{N_2}(\|u_2\|_V)b(u_2, u_2, A_1\delta u)
\]

\[
= F_{N_1}(\|u_1\|_V)b(\delta u, u_1, A_1\delta u) + F_{N_2}(\|u_2\|_V)b(u_2, \delta u, A_1\delta u)
\]

\[
+ [F_{N_1}(\|u_1\|_V) - F_{N_2}(\|u_2\|_V)]b(u_2, u_1, A_1\delta u). \quad (93)
\]

Making similar reasoning as in (93), we can also check that:

\[
K_2 \equiv F_{N_1}(\|u_1, B_1\|_V)b(B_1, B_1, A_1\delta u) - F_{N_2}(\|u_2, B_2\|_V)b(B_2, B_2, A_1\delta u)
\]

\[
= F_{N_1}(\|u_1, B_1\|_V)b(\delta B, B_1, A_1\delta u) + F_{N_2}(\|u_2, B_2\|_V)b(B_2, \delta B, A_1\delta u)
\]

\[
+ [F_{N_1}(\|u_1, B_1\|_V) - F_{N_2}(\|u_2, B_2\|_V)]b(B_2, B_1, A_1\delta u)
\]

\[
\equiv K_1 + K_2^2 + K_3^2, \quad (94)
\]

\[
K_3 \equiv F_{N_1}(\|u_1, B_1\|_V)b(u_1, B_1, A_2\delta B) - F_{N_2}(\|u_2, B_2\|_V)b(u_2, B_2, A_2\delta B)
\]

\[
= F_{N_1}(\|u_1, B_1\|_V)b(\delta u, B_1, A_2\delta B) + F_{N_2}(\|u_2, B_2\|_V)b(u_2, \delta B, A_2\delta B)
\]

\[
+ [F_{N_1}(\|u_1, B_1\|_V) - F_{N_2}(\|u_2, B_2\|_V)]b(u_2, B_1, A_2\delta B)
\]

\[
\equiv K_1 + K_2^2 + K_3^2, \quad (95)
\]

\[
K_4 \equiv F_{N_1}(\|u_1, B_1\|_V)b(B_1, u_1, A_2\delta B) - F_{N_2}(\|u_2, B_2\|_V)b(B_2, u_2, A_2\delta B)
\]

\[
= F_{N_1}(\|u_1, B_1\|_V)b(B, u_1, A_2\delta B) + F_{N_2}(\|u_2, B_2\|_V)b(B, \delta u, A_2\delta B)
\]

\[
+ [F_{N_1}(\|u_1, B_1\|_V) - F_{N_2}(\|u_2, B_2\|_V)]b(B, u_1, A_2\delta B)
\]

\[
\equiv K_1^4 + K_2^4 + K_3^4. \quad (96)
\]

Also, we can easily check that

\[
(B_{N_1}(y, y_1) - B_{N_2}(y, y_2), Ay) = K_1 - SK_2 + K_3 - K_4. \quad (97)
\]
Hence, taking the scalar product in $H$ of (92) with $A\mathbf{y}$, we obtain
\[
\frac{d}{dt}\|\mathbf{y}\|_V^2 + 2|A\mathbf{y}|_H^2 = -2K_1 + 2SK_2 - 2K_3 + 2K_4.
\] (98)

We can easily check that (see [4])
\[
2|K_1| \leq \frac{3}{12} |A\mathbf{y}|_H^2 + cN_1\|\mathbf{y}\|_V^2 + c|A_1\mathbf{u}_2|_{L^2}^2 \|\mathbf{y}\|_V^2 + c|A_1\mathbf{u}_2|_{L^2}^2 (\|\mathbf{y}\|_V^2 + (N_1 - N_1)^2).
\] (99)

We will now estimate $SK_1^2$, $K_3^2$ and $K_4^1, i = 1, 2, 3$ as follows:
\[
2S|K_2^1| = 2SF_N(\|\mathbf{y}_1, B_1\|_V) b(\delta B, B_1, A_1\delta u)
\leq cS\frac{|A_1\mathbf{u}_2|_{L^2}^2 \|\mathbf{y}\|_V}{\|\mathbf{u}_1, B_1\|_V^2} \|\mathbf{y}\|_V^{1/2} |A_2\delta B|_{L^2}^{1/2} \|B_1\|_{V_2} |A_1\delta u|_{L^2} \leq cN_1\|\mathbf{y}\|_V^{1/2} |A\mathbf{y}|_H^{3/2} \leq \frac{1}{12} |A\mathbf{y}|_H^2 + cN_1\|\mathbf{y}\|_V^2.
\] (100)

\[
2S|K_2^3| = 2SF_N(\|\mathbf{y}_2, B_2\|_V) b(\delta B, B_2, A_1\delta u)
\leq S\|\mathbf{y}_2, B_2, A_1\delta u\|_{V_2} \leq c|A_2\mathbf{u}_2|_{L^2}^2 \|\mathbf{y}\|_V |A\mathbf{y}|_H \leq \frac{1}{12} |A\mathbf{y}|_H^2 + c|A_2\mathbf{u}_2|_{L^2}^2 \|\mathbf{y}\|_V^2.
\] (101)

\[
2S|K_3^3| = 2S \left[ F_N(\|\mathbf{y}_1, B_1\|_V) - F_N(\|\mathbf{y}_2, B_2\|_V) \right] b(\delta B, B_1, A_1\delta u)
\leq c \left( \frac{\|\mathbf{y}\|_V}{\|\mathbf{y}_1, B_1\|_V^2} + \frac{|N_1 - N_2|}{\|\mathbf{u}_1, B_1\|_V} \right) |A_2\mathbf{u}_2|_{L^2}^2 \|B_1\|_{V_2} |A_1\delta u|_{L^2} \leq c\|\mathbf{y}\|_V |A\mathbf{y}|_H \leq \frac{1}{12} |A\mathbf{y}|_H^2 + cN_1\|\mathbf{y}\|_V^2.
\] (102)

\[
2|K_3^1| = 2F_N(\|\mathbf{y}_1, B_1\|_V) b(\delta B, B_1, A_2\delta B)
\leq c \left( \frac{\|\mathbf{y}\|_V}{\|\mathbf{y}_1, B_1\|_V^2} + \frac{|N_1 - N_2|}{\|\mathbf{u}_1, B_1\|_V} \right) |A_1\mathbf{u}_2|_{L^2}^2 \|B_1\|_{V_2} |A_2\delta B|_{L^2} \leq cN_1\|\mathbf{y}\|_V^{1/2} |A\mathbf{y}|_H^{3/2} \leq \frac{1}{12} |A\mathbf{y}|_H^2 + cN_1\|\mathbf{y}\|_V^2.
\] (103)

\[
2|K_3^3| = 2F_N(\|\mathbf{y}_2, B_2\|_V) b(\delta B, B_2, A_2\delta B)
\leq |b(\mathbf{u}_2, \delta B, A_2\delta B)| \leq c|A_1\mathbf{u}_2|_{L^2} \|\mathbf{y}\|_V |A\mathbf{y}|_H \leq \frac{1}{12} |A\mathbf{y}|_H^2 + c|A_1\mathbf{u}_2|_{L^2}^2 \|\mathbf{y}\|_V^2.
\] (104)

\[
2|K_3^3| = 2F_N(\|\mathbf{y}_1, B_1\|_V) b(\mathbf{u}_2, \delta B, A_2\delta B)
\leq c \left( \frac{\|\mathbf{y}\|_V}{\|\mathbf{y}_1, B_1\|_V^2} + \frac{|N_1 - N_2|}{\|\mathbf{u}_1, B_1\|_V} \right) |A_1\mathbf{u}_2|_{L^2}^2 \|B_1\|_{V_2} |A_2\delta B|_{L^2} \leq c\|\mathbf{y}\|_V |A\mathbf{y}|_H \leq \frac{1}{12} |A\mathbf{y}|_H^2 + cN_1\|\mathbf{y}\|_V^2.
\] (105)
Now inserting these estimates (99)-(108) in (98), we obtain

\[ 2|K_1^2| = 2F_{N_1}(\|(u_1, B_1)\|_V)|b(\delta B, u_1, A_2\delta B)| \]
\[
\leq c \frac{N_1}{\|(u_1, B_1)\|_V} \|\delta B\|_V^{1/2} \|u_1\|_V \|A_2\delta B\|_L^2^{3/2} \\
\leq cN_1 \|y\|_V^{1/2} \|A\|_H^{3/2} \\
\leq \frac{1}{12} \|A\|_H^2 + cN_1^2 \|y\|_V^2. 
\]

\[ 2|K_2^2| = 2F_{N_2}(\|(u_2, B_2)\|_V)|b(B_2, \delta u, A_2\delta B)| \]
\[
\leq c \|A_2B_2\|_L^2 \|y\|_V \|A\|_H \\
\leq \frac{1}{12} \|A\|_H^2 + c\|A_2B_2\|_L^2 \|y\|_V^2. 
\]

\[ 2|K_3^2| = 2 |F_{N_1}(\|(u_1, B_1)\|_V) - F_{N_2}(\|(u_2, B_2)\|_V)||b(B_2, u_1, A_2\delta B)| \]
\[
\leq c \left( \frac{\|y\|_V}{\|(u_1, B_1)\|_V} + \frac{|N_1 - N_2|}{\|(u_1, B_1)\|_V} \right) \|A_2B_2\|_L^2 \|u_1\|_V \|A_2\delta B\|_L^2 \\
\leq c (\|y\|_V + |N_1 - N_2|) \|A_2B_2\|_L^2 \|A\|_H \\
\leq \frac{1}{12} \|A\|_H^2 + c\|A_2B_2\|_L^2 \|y\|_V^2 + (N_1 - N_2)^2 \right]. 
\]

Now inserting these estimates (99)-(108) in (98), we obtain

\[
\frac{d}{dt} \|y\|_V^2 + \|A\|_H^2 \leq c(N_1^4 + |A_1u_2|_L^2 + |A_2B_2|_L^2) \|y\|_V^2 \\
+ c(|A_1u_2|_L^2 + |A_2B_2|_L^2)(N_1 - N_2)^2 \]
\[
\equiv c(N_1^4 + Z_1) \|y\|_V^2 + cZ_1(N_1 - N_2)^2, \forall t \geq 0. 
\]

It follows from the Gronwall lemma and (109) that

\[
\|y(t)\|_V^2 \leq \left[ \|y_{01} - y_{02}\|_V^2 + c(N_1 - N_2)^2 \int_0^t \|Z_1(s)\|ds \right] \exp \left[ cN_1^4 t + c \int_0^t \|Z_1(s)\|ds \right], 
\]

which proves (90) and where \(Z_1 = |A_1u_2|_L^2 + |A_2B_2|_L^2\).

Now using (109) and (110), we get

\[
\int_0^t \|A\|_H^2 \leq \left[ \|y_{01} - y_{02}\|_V^2 + c(N_1 - N_2)^2 \int_0^t \|Z_1(s)\|ds \right] (x) \\
(\times) \left[ 1 + \left( cN_1^4 t + c \int_0^t \|Z_1(s)\|ds \right) \exp \left[ cN_1^4 t + c \int_0^t \|Z_1(s)\|ds \right] \right], 
\]

which end the proof of Theorem 3.3. \(\square\)

As a consequence of Theorem 3.3, we have a continuous dependence on the initial value and \(N\). More precisely, if we denote by \((u_N, B_N)\) the solution to (36) corresponding to the parameter \(N\) and the initial value \((u_0, B_0)\), then we have the following result.

**Corollary 1.** We assume that \(f_i \in L^2(0, T; H_i), i = 1, 2\) for all \(T > 0\). Then for any \((u_0, B_0) \in V\) and \(N > 0\), we have

\[
(u_M, B_M) \rightarrow (u_N, B_N) \rightarrow (u_0, B_0) \text{ in } C(0, T; V) \cap L^2(0, T; D(A)) \\
as (M, (\bar{u}_0, \bar{B}_0)) \rightarrow (N, (u_0, B_0)) \text{ in } \mathbb{R}^+ \times V. 
\]
Proof. It follows from (90) and (91).

3.2. Comparison of Galerkin solutions of the 3D GMMHDE and 3D MHD. We introduce the following Galerkin ODE for the GMMHDE with parameter \( N \).

\[
\frac{d y_n^{(N)}}{dt} + A y_n^{(N)} + P_n B^N (y_n^{(N)}, y_n^{(N)}) = P_n F, \quad y_n^{(N)} = (u_n^{(N)}, B_n^{(N)}). \quad (113)
\]

In the following proposition, we check that the Galerkin ODE (113) is the same as the Galerkin ODE for the MHD (associated with the same initial value \((u_0, B_0)\) over the time interval \([0, T]\)) for some values of \( N \).

**Proposition 1.** We assume that \( f_i \in L^\infty(0, T; H_i), i = 1, 2 \) for all \( T > 0 \), and we consider the Galerkin approximations of the 3D GMMHDE and 3D MHD of fixed dimension \( n \) for the same initial value \((u_0, B_0)\) over the time interval \([0, T]\).

Then there exists a subsequence \( y_n^{(N)} = (u_n^{(N)}, B_n^{(N)}) \) of the sequence \( y_N \) which converges uniformly in \( C(0, T; \mathbb{R}^3) \) to a function \( y_\infty = (u_\infty, B_\infty) \) in \( C(0, T; \mathbb{R}^3) \) which is the corresponding solution of the \( n \)-dimensional Galerkin ODE for the 3D MHD. More precisely, there exists \( y_n^{(\infty)} \) and a constant \( \kappa_3 \) such that \( y_n^{(N)} \equiv y_n^{(\infty)} \) for all \( N \) satisfying \( N \geq \kappa_3^{1/2} \max \left( \frac{\lambda_1}{R_e}, \frac{\lambda_2^2}{R_m} \right), \lambda_3^{1/2} \),

\[
\kappa_3 = |u_0|_2^2 + S|B_0|_2^2 + \left[ \min \left( \frac{\lambda_1}{R_e}, \frac{\lambda_2^2}{R_m} \right) \right]^{-2} |f_1|_\infty^2 + S|f_2|_\infty^2 T.
\]

**Proof.** We first note that

\[
\begin{align*}
|A_1 u_n|_{L^2} &\leq \lambda_1 |u_n|_{L^2}, \quad |u_n|_{V_1} \leq (\lambda_1^2)^{1/2} |u_n|_{L^2}, \\
|A_2 B_n|_{L^2} &\leq \lambda_2 |B_n|_{L^2}, \quad |B_n|_{V_2} \leq (\lambda_2^2)^{1/2} |B_n|_{L^2}, \\
\lambda_1 |u_n|_{L^2}^2 &\leq |u_n|_{V_1}^2, \quad \lambda_2 |B_n|_{L^2}^2 \leq |B_n|_{V_2}^2,
\end{align*}
\]

where \( \lambda_1 \) and \( \lambda_2 \) are the corresponding eigenvalues of the operators \( A_1 \) and \( A_2 \).

We set \( |f_i|_\infty = |f_i|_{L^\infty(0, T; H_i)}, i = 1, 2 \).

The corresponding energy inequality for the ODE (113) reads

\[
\frac{d}{dt} \left( |u_n^{(N)}(t)|_{L^2}^2 + S|B_n^{(N)}(t)|_{L^2}^2 \right) + \min \left( \frac{\lambda_1}{R_e}, \frac{\lambda_2^2}{R_m} \right) \left( |u_n^{(N)}|_{L^2}^2 + S|B_n^{(N)}|_{L^2}^2 \right) \leq \frac{|f_1|_\infty^2 + S|f_2|_\infty^2}{\min \left( \frac{\lambda_1}{R_e}, \frac{\lambda_2^2}{R_m} \right)} T, \quad (115)
\]

where we have also used (114)_3 and the Young inequality.

Now from (115), we infer that

\[
|u_n^{(N)}(t)|_{L^2}^2 + S|B_n^{(N)}(t)|_{L^2}^2 \leq |u_0|_{L^2}^2 + S|B_0|_{L^2}^2 + \left[ \min \left( \frac{\lambda_1}{R_e}, \frac{\lambda_2^2}{R_m} \right) \right]^{-2} \left( |f_1|_\infty^2 + S|f_2|_\infty^2 \right) T \equiv \kappa_3,
\]

for all \( t \in [0, T] \).
From (113), using (29) and (114), we obtain that a.e. in $(0, T)$
\[
\frac{dy^{(N)}_n}{dt} \leq |Ay^{(N)}_n|_H + |P_n B^N (y^{(N)}_n, y^{(N)}_n)|_H + |P_n F|_H
\]
\[
\leq |A_1 u^{(N)}_n|_{L^2} + |A_2 B^{(N)}_n|_{L^2} + \left|B(y^{(N)}_n, y^{(N)}_n)\right|_H + \sum_{i=1}^2 |f_i|_{L^2}
\]
\[
\leq \lambda_1^n |y^{(N)}_n|_{L^2} + \lambda_2^n |B^{(N)}_n|_{L^2} + c \|y^{(N)}_n\|^3 \|A y^{(N)}_n\|_{H}^{1/2} + \sum_{i=1}^2 |f_i|_{\infty}
\]
\[
\leq (\lambda_1^n + \lambda_2^n) |y^{(N)}_n|_H + c \left(\frac{\lambda_1^n}{R_e} + \frac{\lambda_2^n}{R_m}\right)^{5/4} |y^{(N)}_n|_H^2 + \sum_{i=1}^2 |f_i|_{\infty}.
\]  
(117)

Hence from (117) and the uniform boundedness (in both $N$ and $n$) of $\left|y^{(N)}_n\right|_H = \left|\left(u^{(N)}_n, B^{(N)}_n\right)\right|_H$ on the interval $[0, T]$, we see that its derivative $\frac{dy^{(N)}_n}{dt}$ is also a.e. uniformly bounded in $N$ on the interval $(0, T)$. So, by using the Arzela Ascoli theorem there is a subsequence $y^{(N)}_n$ of $y^{(N)}_n$ which converges uniformly in $C(0, T; \mathbb{R}^3)$ to a function $y^{(\infty)} = (u^{(\infty)}, B^{(\infty)})$ in $C(0, T; \mathbb{R}^3)$.

$y^{(\infty)} = (u^{(\infty)}, B^{(\infty)})$ is in fact the solution of the $n$-dimensional Galerkin ODE for the 3D MHD. Indeed, this follows from the uniqueness of solutions of the Galerkin ODE for a given initial value (which is proved in [22, Theorem 3.1]) and the fact that

\[
1 \geq F_N \left(\left\|u^{(N)}_n, B^{(N)}_n\right\|_V\right) = \min \left(1, \frac{N}{\left\|u^{(N)}_n, B^{(N)}_n\right\|_V}\right)
\]
\[
\geq \min \left(1, \frac{N}{\left(\frac{\lambda_1^n}{R_e} + \frac{\lambda_2^n}{S R_m}\right)^{1/2} \kappa_3^{1/2}}\right),
\]  
(118)

\[
1 \geq F_N \left(\left\|u^{(N)}_n\right\|_V\right) = \min \left(1, \frac{N}{\left\|u^{(N)}_n\right\|_V}\right) \geq \min \left(1, \frac{N}{(\lambda_1^n)^{1/2} \kappa_3^{1/2}}\right),
\]  
(119)

so,

\[
F_N (\left\|u^{(N)}_n\right\|_V) = 1 \text{ and } F_N (\left\|u^{(N)}_n, B^{(N)}_n\right\|_V) = 1,
\]

for $N \geq \kappa_3^{1/2} \max \left(\left(\frac{\lambda_1^n}{R_e} + \frac{\lambda_2^n}{S R_m}\right), \kappa_3^{1/2}\right)$.

This means that for all $N$ such that $N \geq \max \left(\left(\frac{\lambda_1^n}{R_e} + \frac{\lambda_2^n}{S R_m}\right), \kappa_3^{1/2}\right)$, we have $(u^{(N)}_n, B^{(N)}_n) \equiv (u^{(\infty)}_n, B^{(\infty)}_n)$.

We note that in (118) and (119), we have the following estimates:

\[
\left\|(u^{(N)}_n, B^{(N)}_n)\right\|_V^2 = \frac{1}{R_e} \left\|u^{(N)}_n\right\|_{V_1}^2 + \frac{1}{R_m} \left\|B^{(N)}_n\right\|_{V_2}^2
\]
\[
\leq \frac{\lambda_1^n}{R_e} \left\|u^{(N)}_n\right\|_{L^2}^2 + \frac{\lambda_2^n}{S R_m} \left\|B^{(N)}_n\right\|_{L^2}^2
\]
\[
\leq \left(\frac{\lambda_1^n}{R_e} + \frac{\lambda_2^n}{S R_m}\right)^{\kappa_3},
\]
and
\[ \| \mathbf{u}_n^{(N)} \|_{V_1}^2 \leq \lambda_n^1 \| \mathbf{u}_n^{(N)} \|_{L^2}^2 \leq \lambda_n^1 \kappa_3. \]

\[ \square \]

4. **Existence of global attractor in V of the 3D GMMHDE.** We now assume that the forcing terms \( f_i \in H_i, i = 1, 2 \) do not depend on time. We fix \( N > 0 \) and we denote by \( (\mathbf{u}, \mathbf{B}) \in V \) the unique strong solution to (36). By setting \( S_N(t)(u_0, B_0) = (u, B)(t) \), it will follows from Theorems 3.1, 3.2 and 3.3 that \( \{S_N(t)\}_{t \geq 0} \) is a \( C^0 \) semigroup in \( V \).

4.1. **Absorbing set in H.** Here we prove that \( S_N(t) \) has an absorbing set \( \mathcal{B}_H \) which absorbs bounded sets of \( V \).

Making now similar reasoning as in (66), using also (16), we can check that
\[ \frac{d}{dt} \mathcal{Y}(t) + \beta \mathcal{Y}(t) \leq \frac{1}{\beta} (|f_1|_{L^2}^2 + S|f_2|_{L^2}^2), \tag{120} \]
where \( \mathcal{Y}(t) = |\mathbf{u}(t)|_{L^2}^2 + S|\mathbf{B}(t)|_{L^2}^2 \) and \( \beta = \min \left( \frac{1}{\kappa_1 R_e}, \frac{1}{\kappa_2 R_m} \right) \).

It follows from (120) that
\[ \mathcal{Y}(t) \leq \left[ |u_0|_{L^2}^2 + S|B_0|_{L^2}^2 \right] e^{-\beta t} + \frac{1}{\beta^2} (|f_1|_{L^2}^2 + S|f_2|_{L^2}^2) \left[ 1 - e^{-\beta t} \right]. \]

Hence from this last inequality, we infer that
\[ [(\mathbf{u}(t), \mathbf{B}(t))]_H^2 \leq [(u_0, B_0)]_H^2 e^{-\beta t} + \frac{1}{\beta^2} (|f_1|_{L^2}^2 + S|f_2|_{L^2}^2), \quad \forall t \geq 0. \tag{121} \]

We conclude that \( S_N(t) \) has an absorbing set \( \mathcal{B}_H \) in \( H \) given by
\[ \mathcal{B}_H = \left\{ (\mathbf{u}, \mathbf{B}) \in H : [(\mathbf{u}, \mathbf{B})]_H^2 \leq 1 + \frac{1}{\beta^2} (|f_1|_{L^2}^2 + S|f_2|_{L^2}^2) \right\}. \tag{122} \]

4.2. **Absorbing set in V.** Making similar reasoning as in the derivation of (75), taking into account the fact that \( \| \mathbf{u} \|_{V_1} \leq \kappa_1 |\mathbf{A}_1 \mathbf{u}|_{L^2}, \| \mathbf{B} \|_{V_2} \leq \kappa_2 |\mathbf{A}_2 \mathbf{B}|_{L^2} \) (with \( \kappa_i = \frac{1}{\sqrt{\lambda_i}}, i = 1, 2 \)) and using also (32) and the Young inequality with exponents \((8/7, 8)\), we have
\[ \frac{d}{dt} \|(\mathbf{u}, \mathbf{B})(t)\|_{V}^2 + \min \left( \frac{R_c}{\kappa_1^2}, \frac{R_m}{\kappa_2^2} \right) \|(\mathbf{u}, \mathbf{B})(t)\|_{V}^2 \]
\[ \leq 2|f_1|_{L^2}^2 + 2|f_2|_{L^2}^2 + C_N \|(\mathbf{u}, \mathbf{B})(t)\|_{H}^2, \tag{123} \]
where \( C_N = \left[ (cN)^8 + (cNS)^8 \right]^{\frac{1}{7}}. \)

From (123), we infer that
\[ \frac{d}{dt} \|(\mathbf{u}, \mathbf{B})(t)\|_{V}^2 + \min \left( \frac{R_c}{\kappa_1^2}, \frac{R_m}{\kappa_2^2} \right) \|(\mathbf{u}, \mathbf{B})(t)\|_{V}^2 \]
\[ \leq 2|f_1|_{L^2}^2 + 2|f_2|_{L^2}^2 + \tilde{C}_N \|(\mathbf{u}, \mathbf{B})(t)\|_{H}^2, \tag{124} \]
where \( \tilde{C}_N = \max (C_N, \frac{C_N}{\kappa^2}) \) and \( [(\cdot, \cdot)]_H \) defined as in (12).
Substituting now the bound (121) for \( [(u, B)(t)]_H^2 \) in (124), we obtain
\[
\frac{d}{dt} \| (u, B)(t) \|_V^2 + \min \left( \frac{R_c}{\kappa_1}, \frac{R_m}{\kappa_2} \right) \| (u, B)(t) \|_V^2
\leq \tilde{C}_N [(u_0, B_0)]_H^2 e^{-\beta t} + 2|f_1|_{L^2}^2 + 2|f_2|_{L^2}^2
\] (125)

By integration in \( t \) from 0 to \( t \) of (125), we deduce that

(i) if \( \min \left( \frac{R_c}{\kappa_1}, \frac{R_m}{\kappa_2} \right) \leq \beta = \min \left( \frac{1}{\kappa_1^2 R_c}, \frac{1}{\kappa_2^2 R_m} \right) \), then
\[
\| (u, B)(t) \|_V^2 \leq \left( \| (u_0, B_0) \|_V^2 + \tilde{C}_N t [(u_0, B_0)]_H^2 \right) e^{-\min \left( \frac{R_c}{\kappa_1^2 R_c}, \frac{R_m}{\kappa_2^2 R_m} \right) t}
\] (126)
\[+ 2|f_1|_{L^2}^2 + 2|f_2|_{L^2}^2 + \frac{\tilde{C}_N}{\beta^2} (|f_1|_{L^2}^2 + S|f_2|_{L^2}^2), \forall t \geq 0, \]

(ii) if \( \min \left( \frac{R_c}{\kappa_1^2}, \frac{R_m}{\kappa_2^2} \right) > \beta = \min \left( \frac{1}{\kappa_1^2 R_c}, \frac{1}{\kappa_2^2 R_m} \right) \), then
\[
\| (u, B)(t) \|_V^2 \leq \| (u_0, B_0) \|_V^2 e^{-\min \left( \frac{R_c}{\kappa_1^2 R_c}, \frac{R_m}{\kappa_2^2 R_m} \right) t}
\] (127)
\[+ \tilde{C}_N t [(u_0, B_0)]_H^2 e^{-\min \left( \frac{1}{\kappa_1^2 R_c}, \frac{1}{\kappa_2^2 R_m} \right) t}
\] \[+ 2|f_1|_{L^2}^2 + 2|f_2|_{L^2}^2 + \frac{\tilde{C}_N}{\beta^2} (|f_1|_{L^2}^2 + S|f_2|_{L^2}^2), \forall t \geq 0. \]

From (126) or (127), we infer that \( S_N(t) \) has an absorbing set \( B_V^{(N)} \) in \( V \) which is given by:
\[
B_V^{(N)} = \left\{ (u, B) \in V : \| (u, B) \|_V^2 \leq 1 + 2 \sum_{i=1}^2 |f_i|_{L^2}^2 + \frac{\tilde{C}_N}{\beta^2} (|f_1|_{L^2}^2 + S|f_2|_{L^2}^2) \right\}.
\] (128)

4.3. Asymptotic compactness of the semigroup in \( V \). Here, we prove the asymptotic compactness of the semigroup \( S_N(t) \). We recall that with the view to proof that the semigroup \( S_N(t) \) is asymptotically compact, it is enough to verify the following flattening property (see [12, 20, 13] for more details) of the semigroup \( S_N(t) \).

**Proposition 2** (Flattening property). For any bounded set \( B \) of \( V \) and any \( \sigma > 0 \), there exists \( T_\sigma(B) > 0 \) and a finite dimensional subspace \( V_\sigma \) of \( V \) such that \( \{ P_\sigma S_N(t) B, t \geq T_\sigma(B) \} \) is bounded and
\[
\| (I - P_\sigma) S_N(t)(u_0, B_0) \|_V \leq \sigma, \forall t \geq T_\sigma(B), \ (u_0, B_0) \in B,
\] (129)
where \( P_\sigma : V \to V_\sigma \) is the projection operator.

**Proof.** Without loss of generality, we can restrict ourselves to \( B = B_V^{(N)} \), the absorbing set of \( S_N(t) \) in \( V \) given by (128). Let \( \sigma > 0 \). We will find an integer \( N_\sigma > 0 \) such that the flattening property holds for the \( N_\sigma \)-dimensional subspace \( V_\sigma \) of \( V \) spanned by the first eigenfunctions \((w_k, \psi_k), k = 1, 2, ..., N_\sigma \), where \((w_k, \psi_k)\) are
the eigenfunctions used in the proof of Theorem 3.2. Let us denote by \( \beta_1, \beta_2 \) the eigenvalues defined by

\[
A_1 w_k = \beta_1 w_k, \quad A_2 \psi_k = \beta_2 \psi_k, \quad k = 1, 2, \ldots
\]

Let 

\[
\bar{\lambda} = \min \left( R_e \lambda_{N_e}, R_m \lambda_{N_m} \right).
\]

From the fact that \( B \) is a bounded absorbing set and \( \| P_\sigma (u, B) \|_V \leq \|(u, B)\|_V \) for all \( (u, B) \in V \), we infer that there exists \( T_\sigma (B) > 0 \) such that the set \( \{ P_\sigma S_N(t)B, t \geq T_\sigma (B) \} \) is bounded.

We now move to the proof of (129).

Let \((u_0, B_0) \in B_\nu^{(N)}, y(t) = (u, B)(t) = S_N(t)(u_0, B_0) \) and \( Z = \|(u, B)\|_V^2 \). For \( t \) large enough, we know that \((u, B)(t)\) is uniformly bounded in \( V \).

We now make the following observation: given \( \lambda \) and similar reasoning as in deriving (75), we obtain

\[
\frac{d}{dt}[e^{\bar{\lambda} t}Z(t)] + e^{\bar{\lambda} t}\left| A(u, B) \right|_H^2 \leq \left[ c |f_1|_{L^2}^2 + c |f_2|_{L^2}^2 \right] e^{\bar{\lambda} t} + (c N^4 + \bar{\lambda}) e^{\bar{\lambda} t}Z(t),
\]

consequently,

\[
e^{-\bar{\lambda} t} \int_0^t e^{\bar{\lambda} s} \left| A(u, B) (s) \right|_H^2 ds \leq \|(u_0, B_0)\|_V^2 + \left[ \frac{c |f_1|_{L^2}^2 + c |f_2|_{L^2}^2}{\bar{\lambda}} \right]
\]

\[
+ \frac{(c N^4 + \bar{\lambda})}{\bar{\lambda}} e^{-\bar{\lambda} t} \int_0^t e^{\bar{\lambda} s} Z(s) ds \leq \kappa_4 < \infty.
\]

Let \( y_\sigma(t) = (I - P_\sigma)(u, B)(t) = (I - P_\sigma)S_N(t)(u_0, B_0) = (w, \psi) \). Taking the inner product in \( H \) of (36) with \( A y_\sigma = A S_N(t)(u_0, B_0) - A P_\sigma S_N(t)(u_0, B_0) = A S_N(t)(u_0, B_0) - P_\sigma A S_N(t)(u_0, B_0) = (I - P_\sigma)A S_N(t)(u_0, B_0) \), we readily check that \( y_\sigma \) satisfies

\[
\frac{d}{dt} \| y_\sigma \|_V^2 + 2 \| A y_\sigma \|_H^2 = 2(f_1, A_1 w) + 2(f_2, A_2 \psi) - 2 \mathcal{R}_0^N(y, y, A y_\sigma).
\]

We note that (for \( t \) large enough)

\[
2 \| \mathcal{R}_0^N(y, y, A y_\sigma) \| \leq 2 c N \| y \|_V^{1/2} \| A y \|_H^{1/2} \| A y_\sigma \|_H + 2 c N \| y \|_V^{1/2} \| A y \|_H^{1/2} \| A y_\sigma \|_H
\]

\[
\leq \frac{1}{2} \| A y_\sigma \|_H^2 + 4 (c N)^2 \| y \|_V \| A y \|_H + 4 (c N)^2 \| y \|_V \| A y \|_H
\]

\[
\leq \frac{1}{2} \| A y_\sigma \|_H^2 + \kappa_N \| A y \|_H, \quad \text{since } y \|_V \text{ is bounded}
\]

and where we have use (31), the Young inequality and \( \kappa_N \) is a monotone non-decreasing function of the parameter \( N \).

Also, we note that

\[
2 |(f_1, A_1 w) + 2|(f_2, A_2 \psi)| \leq \frac{1}{4} |A_1 w|_{L^2}^2 + \frac{1}{4} |A_1 \psi|_{L^2}^2 + 4 |f_1|_{L^2}^2 + 4 |f_2|_{L^2}^2.
\]

Inserting these estimates (133)-(134) in (132), we obtain

\[
\frac{d}{dt} \| y_\sigma \|_V^2 + \| A y_\sigma \|_H^2 \leq 4 |f_1|_{L^2}^2 + 4 |f_2|_{L^2}^2 + \kappa_N \| A y \|_H
\]

and

\[
\frac{d}{dt} \| y_\sigma \|_V^2 + \bar{\lambda} \| y_\sigma \|_V^2 \leq 4 |f_1|_{L^2}^2 + 4 |f_2|_{L^2}^2 + \kappa_N \| A y \|_H.
\]
since \( \min (R_c \lambda_{N,N}, R_m \lambda_{N,N}^2) \| y_\sigma \|_{V}^2 \leq \| A y_\sigma \|_{H}^2, \) \( \bar{\lambda} = \min (R_c \lambda_{N,N}, R_m \lambda_{N,N}^2), \) which gives

\[
\| y_\sigma(t) \|_{V}^2 \leq \| y_\sigma(0) \|_{V}^2 e^{-\bar{\lambda} t} + \frac{1}{\bar{\lambda}} \left( 4 \| f_1 \|_{L^2}^2 + 4 \| f_2 \|_{L^2}^2 \right) + \kappa_N e^{-\bar{\lambda} t} \int_0^t e^{\bar{\lambda} s} \| A y(s) \|_{H}^2 ds
\]

\[
\leq \| y_\sigma(0) \|_{V}^2 e^{-\bar{\lambda} t} + \frac{1}{\bar{\lambda}} \left( 4 \| f_1 \|_{L^2}^2 + 4 \| f_2 \|_{L^2}^2 \right) + \kappa_N \left[ e^{-\bar{\lambda} t} \int_0^t e^{\bar{\lambda} s} \| A y(s) \|_{H}^2 ds \right]^{1/2} \]

\[
\leq \|(u_0, B_0)\|_{V}^2 e^{-\bar{\lambda} t} + \frac{1}{\bar{\lambda}} \left( 4 \| f_1 \|_{L^2}^2 + 4 \| f_2 \|_{L^2}^2 \right) + \frac{\kappa_N \lambda}{\lambda^{1/2}}. \tag{137}
\]

We note that in (137), we have also use (131).

Therefore, for \( t \) and \( N \) large enough in such a way that the initial condition term becomes smaller than \( \sigma/2 \) and so that the sum of the terms involving \( \bar{\lambda} \) in the denominator is smaller that \( \sigma/2 \), we derive that \( \| y_\sigma(t) \|_{V}^2 \leq \sigma \), which proves the flattening property of \( S_N(t) \).

**Theorem 4.1.** If \( f_i \in H_i, i = 1, 2 \), then the 3D GMMHDE (2) has a global attractor \( \wedge_{N_1} \) in \( V \) for each \( N_1 > 0 \). Moreover, the set-valued mapping \( N_1 \mapsto \wedge_{N_1} \) is upper semi-continuous, i.e.

\[
dist_V(\wedge_{N_1}, \wedge_{N_2}) \to 0 \text{ as } N_2 \to N_1, \tag{138}
\]

where \( \text{dist}_V \) is the Hausdorff semi-distance on \( V \).

**Proof.** The existence of the global attractor follows from the existence of the absorbing set in \( V \) as well as the flattening property proved in Proposition 2. Making similar reasoning as in [4], we can check the upper semi-continuity (138).

Note that for each \( N > 0 \), we have \( \wedge_N \subset B_V^{(N)} \) and from (128), we have \( B_V^{(N)} \subset B_V^{(M)} \) for \( N \leq M \).

5. Convergence to weak solution of the 3D MHD equations. We assume that \( f_i \in L^2(0, T; H_i), i = 1, 2 \) for all \( T > 0 \). Let \( (u_N, B_N(t)) \) be the weak solution to (36) with initial value \( (u_N^0, B_N^0) \in H \), where \( (u_N^0, B_N^0) \to (u_0, B_0) \) weakly in \( H \) as \( N \to \infty \). \( (u_N, B_N) \) satisfies the following

\[
\frac{d}{dt} \left( \| u_N(t) \|_{L^2}^2 + S \| B_N(t) \|_{L^2}^2 \right) + \frac{2}{R_c} \| u_N(t) \|_{V_1}^2 + \frac{2S}{R_m} \| B_N(t) \|_{V_2}^2
\]

\[
= 2(f_1(t), u_N(t)) + 2S(f_2(t), B_N(t)), \tag{139}
\]

from which, we deduce that

\[
\frac{d}{dt} \left( \| u_N(t) \|_{L^2}^2 + S \| B_N(t) \|_{L^2}^2 \right) + \frac{1}{R_c} \| u_N(t) \|_{V_1}^2 + \frac{S}{R_m} \| B_N(t) \|_{V_2}^2
\]

\[
\leq c \| f_1(t) \|_{V_1}^2 + c \| f_2(t) \|_{V_2}^2. \tag{140}
\]

From (140) we deduce that \( (u_N, B_N(t)) \in L^\infty(0, T; H) \cap L^2(0, T; V), \) for all \( T > 0, \) and then, by (33), the sequence \( \frac{d}{dt}(u_N, B_N) \) is bounded in \( L^{4/3}(0, T; V') \). Hence, by a diagonal argument, there exists a subsequence of \( (u_N, B_N) \) still denoted \( (u_N, B_N) \)
such that
\[
(u_N, B_N) \rightarrow (u, B) \quad \begin{cases} 
\text{weak-star in } L^\infty(0, T; H), \\
\text{weakly in } L^2(0, T; V), \\
\text{strongly in } L^2(0, T; H),
\end{cases}
\]
(141)
where \((u, B) \in L^\infty(0, T; H) \cap L^2(0, T; V)\), for all \(T > 0\).
As in [4], we will prove that \((u, B)\) is a weak solution to the 3D MHD (1).

**Lemma 5.1.** We have
\[
F_N(\|u_N\|_{V_1}) \rightarrow 1 \text{ in } L^p(0, T; \mathbb{R}),
F_N(\|(u_N, B_N)\|_V) \rightarrow 1 \text{ in } L^p(0, T; \mathbb{R}),
\]
(142)
as \(N \rightarrow +\infty\) for each \(p > 1\).

**Proof.** For the proof of (142)\(_1\) we refer the reader to [4], and arguing similarly as in [4], we can check (142)\(_2\). \(\square\)

Using Lemma 5.1, it is proved in [4] that as \(N \rightarrow +\infty\), we have
\[
\int_0^t F_N(\|u_N\|_{V_1}) b(u_N, u_N, w) ds \rightarrow \int_0^t b(u, u, w) ds,
\]
(143)
\(\forall t \in [0, T], w \in D(A_1)\).

Similarly, one can check that
\[
\int_0^t F_N(\|(u_N, B_N)\|_V) b(B_N, B_N, w) ds \rightarrow \int_0^t b(B, B, w) ds,
\]
\[
\int_0^t F_N(\|(u_N, B_N)\|_V) b(u_N, B_N, \psi) ds \rightarrow \int_0^t b(u, B, \psi) ds,
\]
\[
\int_0^t F_N(\|(u_N, B_N)\|_V) b(B_N, u_N, \psi) ds \rightarrow \int_0^t b(B, u, \psi) ds,
\]
(144)
as \(N \rightarrow +\infty\),
for all \(t \in [0, T], w \in D(A_1)\) and \(\psi \in D(A_2)\). This proves that the limit function \((u, B)\) is a weak solution of the three-dimensional MHD equations (1) for the given initial condition \((u_0, B_0)\), i.e. satisfies the variational equation
\[
(y(t), \psi_1) + \int_0^t ((y(s), \psi_1)) ds + \int_0^t B_0(y(s), y(s), \psi_1) ds \\
= ((u_0, B_0), \psi_1) + \int_0^t (f_1(s), w) ds + \int_0^t (f_2(s), \psi) ds t \geq 0,
\]
(145)
where \(y(t) = (u, B)(t)\) and \(\psi_1 = (w, \psi) \in D(A)\). Now by density, we infer that the equation (145) holds for all \(\psi_1 = (w, \psi) \in V\).

**Remark 3.** We note that the variational equation (145) differs from the variational equation (38) for the 3D GMMHDE by the absence of the \(F_N(\|u\|_{V_1})\) and \(F_N(\|(u, B)\|_V)\) factors multiplying the nonlinear term \(b\).

5.1. **Existence of bounded entire weak solutions of the 3D MHD equations.** In this part, we suppose that the forcing terms \(f_i \in H_i, i = 1, 2\). Following similar steps as in [4], we prove the existence of a bounded entire weak solution of the 3D MHD equations.
Theorem 5.2. There exists a bounded entire weak solution of the 3D MHD equations (1). More precisely, there exists a bounded entire weak solutions of (1) with the initial value \((u_0, B_0) \in U_0\), where \(U_0\) is a subset of \(H\) consisting of the weak \(H\)-cluster points of a sequence in \(\land N\).

Proof. For the proof, we will arguing similarly as to the proof of Theorem 14 in [4]. Thus, we omit the details and only give a sketch. We consider a sequence \((u^N_0, B^N_0) \in V\) with \((u^N_k, B^N_k) \in \land N\) for each \(N\). So \(S_N(t) \land N = \land N\) for all \(t \geq 0\). It then follows that there exists an entire strong solution of the 3D GMMHDE (36)

\[
\begin{align*}
(\tilde{u}_N, \tilde{B}_N) : \mathbb{R} \rightarrow V \text{ with } (\tilde{u}_N, \tilde{B}_N)(0) &= (u^N_0, B^N_0) \\
(\tilde{u}_N, \tilde{B}_N)(t) &\in \land N \text{ for all } t \in \mathbb{R} \text{ and each } N.
\end{align*}
\]

Note that \(\land N \subset \mathcal{B}_H\) for each \(N\), where \(\mathcal{B}_H\) is the absorbing set in \(H\) given by (122). As \(\mathcal{B}_H\) is independent of \(N\), it follows that the sequence \((\tilde{u}_N, \tilde{B}_N)\) is bounded in \(L^\infty(0, T; H) \cap L^2(0, T; V)\). Therefore, there exists a subsequence (still denoted \((\tilde{u}_N, \tilde{B}_N)\) which converges to a function \((\tilde{u}, \tilde{B}) \in L^\infty(0, T; H) \cap L^2(0, T; V)\) weakly-star in \(L^\infty(0, T; H)\), weakly in \(L^2(0, T; V)\) and strongly in \(L^2(0, T; H)\) for all \(T > 0\). Moreover, \((\tilde{u}, \tilde{B}) \in \mathcal{B}_H\) by the weak-star lower semi-continuity of the norm in \(L^\infty(0, T; H)\). As in [4], we can extend this weak solution backward in time and obtain an entire weak solution \((\tilde{u}, \tilde{B})\) of the 3D MHD equations (1) with values in \(\mathcal{B}_H\).

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E-mail address: agdeugoue@yahoo.fr
E-mail address: jules.djokokamdem@gmail.com
E-mail address: adeletsanou@yahoo.fr
E-mail address: ndongmoaristide@yahoo.fr