Decoherence of Josephson charge qubit

Xian-Ting Liang

Department of Physics and Institute of Modern Physics, Ningbo University, Ningbo, 315211, China

Abstract

In this paper we investigate decoherence time of superconducting Josephson charge qubit (JCQ). Two kinds of methods, iterative tensor multiplication (ITM) method derived from the quasiadiabatic propagator path integral (QUAPI) and Bloch equations method are used. Using the non-Markovian ITM method we correct the decoherence time predicted by Bloch equations method. By comparing the exact theoretical result with the experimental data we suggest that the Ohmic noise plays the central role to the decoherence of the JCQ.

Keywords: Decoherence; Non-Markov approximation; Josephson charge qubit.

PACS numbers: 03.67. Hk, 03.65.Ta, 89.70.+c

I. INTRODUCTION

Solid state qubits are considered promising candidates for making processors of quantum computers because they can be scaled up to a large numbers. The qubits based on Josephson junction are these kinds of qubits. But how about their other qualities, in particular, how about their coherence? Many efforts not only theoretical [1, 2] but also experimental [14, 16, 17, 22] have been about their coherence? Many efforts not only theoretical [1, 2] but also experimental [14, 16, 17, 22] have been contributed to search decoherence time as well as decoherence mechanism of the qubit systems. The theoretical researches are in general based on the spin-boson model [3, 4]. By now, it is suggested that there are $1/f$ noise and Ohmic noise in the environment of the Josephson qubits. It is considered that the former is derived from the background charge fluctuations and the latter from the electromagnetic noise due to voltage fluctuations. But what is the most primary mechanism of the decoherence, or which is the main noise source in the environment of the qubits? By using the Bloch equations [5, 6] one can derive the relaxation time and dephasing time of the qubits. However, in the derivation of Bloch equations an approximation scheme in general the Markov approximation should be used. It has been pointed out recently, the Markov approximation is not a suitable approximation scheme in the investigations of the quantum system for quantum computing purposes because it is not usually valid at low temperatures and for short cycle time of quantum computation [7]. A similar viewpoint on a qubit of double quantum dots is also pointed out, see as [8]. Privman and his co-worker [9] investigated the decoherence of qubits with short-time approximation rather than the Markov one, and many interesting and novel results are obtained. However, it is not enough to only investigate the decoherence in a short time because the coherence in a longer time for some qubits (for example qubits for quantum registers) is also interesting. Fortunately, it is found out that by using short-time propagators one can construct a path integral [10] in a longer time. The well established iterative tensor multiplication (ITM) algorithm derived from the quasiadiabatic propagator path integral (QUAPI) [11] can be used to solve the evolutions of low-dimension open quantum systems in a moderate long time [12]. In this method the temporal non-local interactions is involved and it is non-Markovian. Thus, we expect that the non-Markovian method can be used to investigate the decoherence of Josephson qubits and then to detect the mainly mechanism of the decoherence in the qubits. There are many kinds of Josephson qubit models [13, 14, 15, 16, 17, 18, 19, 20], but in this paper we only investigate the Josephson charge qubit (JCQ). We shall obtain an accurate decoherence time of the JCQ by using the ITM algorithm without the Markovian approximation. Base on the accurate decoherence time we shall suggest a mainly mechanism of the decoherence in the JCQ.

II. MODELS

The elementary unit of the quantum computer is a qubit which is in fact a two-level quantum system [21]. There are many physical realizations for the system. However, any single qubit can be represented as a spin-1/2 particle, and its Hamiltonian can be written as $H(t) = -\frac{1}{2} \vec{B}(t) \cdot \vec{\sigma}$. Here $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrixes in a space of states $|\uparrow\rangle = (0 1)^T$ and $|\downarrow\rangle = (1 0)^T$ which form basis states of the Hamiltonian. The quantity $\vec{B}(t)$ has different physical meaning according to the difference of the physical realizations of the qubits. For example, if the qubit is realized by a spin of some particle the $\vec{B}(t)$ will be an effective magnetic field. But to the JCQ, the components of the “magnetic field” are set as $B_x = E_J$, where $E_J$ is the Josephson energy of the Josephson junction, $B_y = 0$, and $B_z = 4E_C (1 - 2n_g)$ [6]. The Hamiltonian of a general qubit can be represented as

$$H_s = -\frac{1}{2} B_z \sigma_z - \frac{1}{2} B_x \sigma_x.$$

If one modulates the gate voltage and makes $n_g = 1/2$, the JCQ system may be reduced to $H_s = -\frac{1}{2} B_x \sigma_x$. In this paper, we take $E_J = 51.8 \mu eV$, and $E_C = 122 \mu eV$ according to Ref. [22]. If one considers the interaction of the qubit and its environment, and takes the environment
as a bath, the whole Hamiltonian of the system-bath will be
\[ H = H_0 + H_{\text{env}}, \] (2)
where
\[ H_0 = H_s, \]
\[ H_{\text{env}} = \sum_k \left[ \frac{1}{2m_k} p_k^2 + \frac{1}{2} m_k \omega_k^2 \left( x_k - \frac{\lambda_k}{m_k \omega_k^2} \sigma_z \right)^2 \right]. \] (3)

In general, \( H_0 \) should plus counterterms \(- \sum_k \lambda_k^2/2m_k^2 \omega_k^4\) which can ensure that some important features of the qubit do not depend on the coupling strength. In our problem, the counterterms only contribute a global phase so we can ignore it. This is the well-known spin-boson model, a appropriately reduced open qubit model. In the following, we firstly analyze this model. On the one hand, for the bath, only the linearly coupling term is chosen in the coordinates \( x_k \), representing the lowest nontrivial term in the Taylor series expansion of the potential. It is accurate enough in the weak coupling case. On the other hand, for the qubit, only \( \sigma_z \) coupling term is included. The terms of \( \sigma_{x,y} \) coupling with the bath have not been included in the Hamiltonian. The reason is that \( \sigma_{x,y} \) have only off-diagonal matrix elements in the \( \sigma_z \) representation, i.e., they only change \( |↑⟩ \) to \( |↓⟩ \) and vice versa.

In order to obtain the reduced density matrix of the qubit in qubit-bath system, one should know the coupling coefficients \( \lambda_k \) in Eq. (2). However, we do not need to know their details because all characteristics of the bath pertaining to the dynamics of the observable system are captured in the spectral density function of noise
\[ J_X(\omega) = \frac{\pi}{2} \sum_k \frac{\lambda_k^2}{m_k \omega_k^2} \delta(\omega - \omega_k). \] (4)

In the case of truly macroscopic environment the spectral density is for all practical purposes a continuous function of frequency. In the following we shall see that the spectral density function \( J(\omega) \) instead of \( \lambda_k \) is directly used in obtaining the elements of the reduced density matrix. It is related to the power spectrum of the noise as
\[ S_X(\omega) = J_X(\omega) \hbar \coth(\omega \beta \hbar/2). \] (5)

Here, \( \beta = 1/k_B T \), where \( k_B \) is the Boltzmann constant and \( T \) is the temperature. Throughout the paper we take \( T = 30 \) mK according to Ref. 22. Due to very wide of the real and imaginary parts of the response function (see Eq. (14) in the following) in the time range for the 1/f bath, we in fact cannot investigate the evolutions of the reduced density matrix of qubit in the 1/f bath with the ITM. Thus, in this paper, we focus on the case that the environment is the Ohmic bath. The spectral density of the Ohmic bath can be expressed as
\[ J_X(\omega) = 2\pi\hbar \omega \exp \left( -\frac{\omega}{\omega_C} \right), \] (6)
where \( \omega_C \) is the cut-off frequency of the bath modes. The parameter \( \alpha \) is dimensionless strength of the dissipation which is determined by concrete qubit-bath system. For the JCQ model, Makhlin et al. proposed a numerical simulation value \( \alpha \approx 10^{-6} \). In this paper we suppose \( \alpha = 5 \times 10^{-6} \). If \( \alpha > 5 \times 10^{-6} \) the decoherence time will be shorter than the results obtained in this paper. On the other hand, if \( \alpha < 5 \times 10^{-6} \) the decoherence time will be longer than the results in this paper.

### III. QUAPI AND ITM

In the following, we firstly review the QUAPI method and then the ITM scheme. Suppose the initial state of the qubit-bath system has the form
\[ R(0) = \rho(0) \otimes \rho_{\text{bath}}(0), \] (7)
where \( \rho(0) \) and \( \rho_{\text{bath}}(0) \) are initial states of the qubit and bath. The evolution of its reduced density operator of the open qubit
\[ \tilde{\rho}(s', s'; t) = \text{Tr}_{\text{bath}}(s') e^{-iHt/\hbar} \rho(0) \otimes \rho_{\text{bath}}(0) e^{iHt/\hbar} |s'⟩⟨s'| \] (8)

is given by
\[ \tilde{\rho}(s', s'; t) = \sum s_{0}^{0} \cdot \cdots \cdot \sum s_{0}^{N} \cdot \sum s_{1}^{0} \cdot \cdots \cdot \sum s_{1}^{N} \cdot \sum \cdots \cdot \sum s_{i}^{0} \cdot \cdots \cdot \sum s_{i}^{N} \cdot \cdots \cdot \sum s_{N}^{0} \cdot s_{N}^{1} \cdots s_{N}^{N} \]
\[ \times \langle s_{0}^{0} | e^{-iH_{\text{env}}(s') \Delta t/2 \hbar} \rho_{\text{bath}}(0) e^{iH_{\text{env}}(s') \Delta t/2 \hbar} | s_{0}^{0} \rangle \]
\[ \times \langle s_{0}^{1} | e^{-iH_{\text{env}}(s') \Delta t/2 \hbar} \rho_{\text{bath}}(0) e^{iH_{\text{env}}(s') \Delta t/2 \hbar} | s_{0}^{1} \rangle \]
\[ \times \cdots \cdots \times \langle s_{0}^{N} | e^{-iH_{\text{env}}(s') \Delta t/2 \hbar} \rho_{\text{bath}}(0) e^{iH_{\text{env}}(s') \Delta t/2 \hbar} | s_{0}^{N} \rangle \]
\[ \times I(s_{0}^{N}, s_{1}^{N}, \cdots, s_{N}^{N}, s_{0}^{N}, s_{1}^{N}, \cdots, s_{N}^{N}, s_{0}^{N}, \Delta t), \] (9)
where the influence functional is
\[ I(s_{0}^{N}, s_{1}^{N}, \cdots, s_{N}^{N}, s_{0}^{N}, s_{1}^{N}, \cdots, s_{N}^{N}, s_{0}^{N}, \Delta t) = \text{Tr}_{\text{bath}} e^{-iH_{\text{env}}(s') \Delta t/2 \hbar} e^{-iH_{\text{env}}(s_{N}^{N}) \Delta t/2 \hbar} \]
\[ \times \cdots \cdots \times e^{-iH_{\text{env}}(s_{0}^{0}) \Delta t/2 \hbar} \rho_{\text{bath}}(0) e^{iH_{\text{env}}(s_{0}^{0}) \Delta t/2 \hbar} \]
\[ \times e^{iH_{\text{env}}(s_{N}^{N}) \Delta t/2 \hbar} e^{iH_{\text{env}}(s_{0}^{0}) \Delta t/2 \hbar} \].
(10)

The discrete path integral representation of the qubit density matrix contains temporal nonlocal terms \( I(s_{0}^{N}, s_{1}^{N}, \cdots, s_{N}^{N}, s_{0}^{N}, s_{1}^{N}, \cdots, s_{N}^{N}, s_{0}^{N}, \Delta t) \) which denotes the process being non-Markovian. With the quasiadiabatic discretization of the path integral, the influence functional, Eq. (10) takes the form
\[ I = \exp \left\{ i \sum_{k=0}^{N} \sum_{k'=0}^{k} \left( s_{k}^{+} - s_{k}^{-} \right) \left( \eta_{kk'} s_{k'}^{+} - \eta_{kk'} s_{k'}^{-} \right) \right\}, \] (11)
where \( s_{k}^{+} = s_{0}^{+} \) and \( s_{N}^{-} = s_{0}^{+} \). The coefficients \( \eta_{kk'} \) can be obtained by substituting the discrete path into
the Feynman-Vernon expression. Their expressions have been shown in [11]. Thus, the influence functional can be expressed with a product of terms corresponding to different $\Delta k$ as

$$I = \prod_{k=0}^{N} I_0 (s^+_k) \prod_{k=0}^{N-1} I_1 (s^+_k, s^+_{k+1}) \prod_{k=0}^{N-\Delta k} I_{\Delta k} (s^+_k, s^+_k + \Delta k)$$

$$\cdots \prod_{k=0}^{N-\Delta k_{max}} I_{\Delta k_{max}} (s^+_k, s^+_k + \Delta k_{max}).$$

(12)

Here, $\Delta k = k - k'$, where $k'$ and $k$ are points of discrete path integral expressions, see Ref.[11], and

$$I_0 (s^+_k) = \exp \left\{ -\frac{1}{\hbar} \left( s^+_i - s^-_i \right) (\eta_{ii} s^+_i - \eta^*_{ii} s^-_i) \right\},$$

$$I_{\Delta k} (s^+_i, s^+_i + \Delta k) = \exp \left\{ -\frac{1}{\hbar} \left( s^+_i + \Delta k - s^-_i - \Delta k \right) \right\} \times (\eta_{ii} s^+_i - \eta^*_{ii} s^-_i), \Delta k \geq 1.$$

(13)

The length of the memory of the time can be estimated by the following bath response function

$$\gamma (t) = \frac{1}{\pi} \int_{0}^{\infty} d\omega J(\omega) \left[ \coth \left( \frac{\beta \hbar \omega}{2} \right) \cos \omega t - i \sin \omega t \right].$$

(14)

It is shown that when the real and imaginary parts behave as the delta function $\delta (t)$ and its derivative $\delta' (t)$, the dynamics of the reduced density matrix is Markovian. However, if the real and imaginary parts are broader than the delta function the dynamics is non-Markovian. The broader of the Re$[\gamma (t)]$ and Im$[\gamma (t)]$ are, the longer of the memory time will be. The broader of the Re$[\gamma (t)]$ and Im$[\gamma (t)]$ are, the more serious the Markov approximation will distort the practical dynamics. In Fig.1 we plot Re$[\gamma (t)]$ and Im$[\gamma (t)]$ of the Ohmic bath.

**Fig.1**

From Fig.1 we see that the memory time is about $T_{mem} = 1 \times 10^{-11} \text{ s}$ for the Ohmic bath. Due to nonlocality of the influence functional, it is impossible to calculate the reduced density matrix by Eq. (14) in matrix multiplication scheme. However, the short range nonlocality of the influence functional Eq. (14) implies that the effects of the nonlocality should drop off rapidly as the “interaction distance” increases. In the Makri’s ITM scheme the interaction can be taken into account at each iteration step. The reduced density matrix at time $t = N \Delta t$ (N even) is given as

$$\tilde{\rho} (s^+_N, N \Delta t) = A^{(1)} (s^+_N; N \Delta t) I_0 (s^+_N),$$

(15)

where

$$A^{(1)} (s^+_k; (k + 1) \Delta t) = \int ds^+_k T^{(2)} (s^+_k, s^+_k; (k + 1) \Delta t) \times A^{(1)} (s^+_k; k \Delta t).$$

Here,

$$T^{(2\Delta k_{max})} (s^+_k, s^+_k + \Delta k_{max} - 1) = \prod_{n=k}^{k+\Delta k_{max}-1} K (s^+_n, s^+_n + \Delta k_{max} - 1) I_0 (s^+_n) I_1 (s^+_n, s^+_n + \Delta k_{max} - 1) \times I_2 (s^+_n, s^+_n + \Delta k_{max} - 1) \prod_{n=k}^{k+\Delta k_{max}} I_{\Delta k_{max}} (s^+_n, s^+_n + \Delta k_{max}),$$

(17)

and

$$A^{(\Delta k_{max})} (s^+_0, s^+_1, ..., s^+_\Delta k_{max}) = \langle s^+_0 | s^+_1 | ... | s^+_\Delta k_{max} | s^-_0 \rangle,$$

(18)

where

$$K (s^+_k, s^+_k) = \langle s^+_k | \exp (-i H_0 \Delta t / \hbar) | s^+_k \rangle \times \langle s^-_k | \exp (i H_0 \Delta t / \hbar) | s^-_{k+1} \rangle.$$

(19)

In the ITM scheme a short-time approximation instead of the Markovian approximation is used. The approximation makes a error of short-time propagator in order $(\Delta t)^3$ which is small enough as we set the time step $\Delta t$ very small. It is shown that when the time step $\Delta t$ is not larger than the characteristic time of the qubit system which can be calculated with $\hbar / E_f$ the calculation is accurate enough [5]. In particular, the scheme do not discard the memory of the temporal evolution, which may appropriate to solve the decoherence of qubits.

**IV. DECOHERENCE OF JOSEPHSON QUBITS**

To measure effects of decoherence one can use the entropy, the first entropy, and many other measures, such as maximal deviation norm etc., see [6]. However, essentially, the decoherence of a open quantum system is reflected through the decays of the off-diagonal coherent terms of its reduced density matrix. The decoherence is in general produced due to the interaction of the quantum system with other systems with a large number of degrees of freedom, for example the devices of measurement or environment. In this paper, we investigate the decoherence time of the JCQ via directly describing the evolution of the off-diagonal coherent terms, instead of using any measure of decoherence. In our following investigations, we suppose the cut-off frequency of the bath modes is $\omega_c = 5 \text{ (ps)}^{-1}$. We set the initial state of the qubit $\rho (0) = \frac{1}{2} (|0 \rangle + |1 \rangle)(\langle 0 | + \langle 1 |)$ which is a pure state and it has the maximum coherent terms, and the initial state of the environment $\rho_{bath} (0) = \prod_k e^{-\beta M_k} / Tr_b (e^{-\beta M_b})$.

**Decoherence time obtained from ITM scheme:** At first, we use the ITM scheme investigating the decoherence time of the Josephson qubits. The evolutions of the coherent elements of the reduced density matrix of the JCQ...
in the Ohmic bath is plotted in Fig. 2. Here, we simply choose $\Delta k_{\text{max}} = 1$ and $\Delta t = 1.27 \times 10^{-11}$ s in the ITM scheme. The choice of the time step is feasible as we consider that it should be not shorter than the memory time of the bath, because the latter is about $\tau_{\text{mem}} = 1 \times 10^{-11}$ s for the Ohmic bath, see Fig. 1. It is also appropriate as we consider that the time step should be not longer than the characteristic time of the qubit, where the latter is about $\tau = 1.3 \times 10^{-11}$ s.

**Fig 2**

It is shown that when we choose the parameter of the dimensionless strength of the dissipation $\alpha = 5 \times 10^{-6}$, the decoherence time of the JCQ is about $\tau_2 = 1.05299 \mu$s.

**Decoherence time calculated on Bloch equations:** It is well known that the decoherence time can be derived on Bloch equations. In this method, the relaxation and dephasing times $\tau_1, \tau_2$ can be calculated as \[ \tau_1^{-1} = 2\tau_2^{-1} = \frac{1}{2\hbar} J(\omega_0) \coth (\beta \hbar \omega_0/2), \]

where $\omega_0 = B_2/h$ is the natural frequency of the Josephson qubit. From Eq. (20) and using the same parameters of the qubit and the bath as above we can obtain that the decoherence time is $\tau_2 = 1.61966 \mu$s. It is shown that the time obtained from Eq. (20) is longer than that from the ITM scheme. We suggest that the difference is derived from the following two reasons. The first is that the Bloch equations are in general derived from the Markov approximation which discards the memory of the bath in the derivation of the dynamical evolution. The second is that the Eq. (20) is obtained from the second order approximation of perturbation series. The decoherence of the qubit described with this method is only the “resonant decoherence” \[ 24 \]. It is not equals to the actual decoherence accurately except for the “nonresonant decoherence” very small.

**Compared with the experimental results:** In our calculations, we use the parameters similar to Ref. [22], so we can compare our results to the experimental decoherence time. In [22] the decoherence time of a single-Cooper pair box, namely, the JCQ is estimated. The main decoherence source is thought to be spontaneous photon emission to the electromagnetic environment (which can just be described by the Ohmic bath). In [22] the authors pointed out that the experimental decoherence time of the JCQ could exceed $1 \mu$s. It is shown that by use of the Ohmic decoherence mechanism we can obtain a theoretical decoherence time of the JCQ not only by the ITM scheme but also through the Bloch equations method. Both of the theoretical results are agreement with the experimental one very well!

**V. CONCLUSIONS**

In this paper we investigated the decoherence time of the JCQ in the Ohmic bath with the ITM scheme based on the QUAPI and based on the Bloch equations. The results derived from the two kinds of methods are compared with each other. It is shown that the decoherence time obtained from the Bloch equations method is longer than that from the ITM scheme. We suggest that the difference is resulting from the different choices of the approximation scheme because the Markov approximation scheme used in the Bloch equations method discards the memory of the bath. It is also because the Bloch equations method discards the higher order decoherence, namely, only the “resonant decoherence” \[ 24 \] is left over. So the decoherence time obtained from this method is not equals to the actual decoherence time accurately. What is more important to us is that the experimental decoherence time of the JCQ due to spontaneous photon emission is well agreement with the ITM decoherence time of the JCQ because of the electromagnetic fluctuations. Both of the spontaneous photon emission and the electromagnetic fluctuations have the same decoherence mechanism and can be modeled by the Ohmic bath. These can lead to a conclusion that the Ohmic bath decoherence is a central mechanism in the JCQ and the decoherence time of the JCQ is about $1 \mu$s when the temperature is about $30 \text{ mK}$ and the Josephson energy is about $51.8 \mu\text{ev}$. The decoherence time is decided by the decoherence mechanism and affected by the experimental temperature. If the experimental temperature increase the dimensionless strength of the dissipation $\alpha$ will also increase and the decoherence time will be shorter, and vice versa.

**Acknowledgement 1** The project was supported by National Natural Science Foundation of China (Grant No. 10347133) and Ningbo Youth Foundation (Grant No. 2004A620003).

---

[1] Y. Makhlin, G. Schönh, and A. Shnirman, Rev. Mod. Phys. 73, 357 (2001).
[2] J. M. Martinis, S. Nam, J. Aumentado, and K. M. Lang, Phy. Rev. B 67, 094510 (2003).
[3] U. Weiss, Quantum Dissipative Systems, 2nd ed., (World Scientific Publishing, Singapore, 1999).
[4] A. J. Leggett, S. Chakravarty, A. T. Dorsey, M. P. A. Fisher, A. Garg and W. Zwerger, Rev. Mod. Phys. 59, 1 (1987).
[5] F. Bloch, Phys. Rev. 70, 460 (1964); R. K. Wangsness, and F. Bloch, Phys. Rev. 89, 728 (1953); F. Bloch, Phys. Rev. 105, 1206 (1957).
VI. CAPTIONS OF THE FIGURES

Fig.1: Real (line) and imaginary (short line) part of the response function of the Ohmic bath. Here, we set the temperature $T = 30 \text{ mK}$, $\alpha = 5 \times 10^{-6}$, and the unit of time is second (s).

Fig.2: The evolution of the off-diagonal elements of the reduced density matrix for the JCQ in the Ohmic bath. Here, we set $B_x = 51.8 \mu eV$, $B_z = 0$, $T = 30 \text{ mK}$, $\omega \text{C} = 5 (\text{ps})^{-1}$ Hz, $\alpha = 5 \times 10^{-6}$, and the unit of time is picosecond (ps). The initial state of the qubit and environment see the text.
\[ \gamma(\tau) \]
