Estimation of RFID Tag Population Size by Gaussian Estimator

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Abstract—In this paper we propose a novel approach to estimating RFID tag population size. Unlike all previous \{0,1\} estimation schemes, we analysed our scheme under \{0,1,e\} and presented results under both \{0,1\} and \{0,1,e\} channel models. Under both the channel models we used a well justified Gaussian estimator for estimation. We have named our algorithm "Gaussian Estimation of RFID Tags," namely, GERT. The most prominent feature of GERT is the quality with which it estimates a tag population size. We supported all the required approximations with detailed analytical work and counted for all the approximation errors when we considered the overall quality of the estimation. GERT is shown to be more cost effective than the previous methods suggested so far, under the same performance constraints.

Index Terms—RFID, Collision Model, Gaussian Approximation

I. INTRODUCTION

A. The Problem Statement

Tag estimation is useful in many everyday applications including in tag identification, privacy-sensitive RFID systems and warehouse monitoring. In this paper, we propose a more efficient method to estimate the size of a tag population. There have been a good number of estimation schemes so far, and we have presented in the paper the comparative efficacy of our algorithm with some recently proposed approaches. The problem at hand can be formally stated as follows: For a given reliability requirement \(\alpha \in [0,1]\), a confidence interval \(\beta \in [0,1]\) a reader will have to estimate an unknown tag population size \(t\) in a particular area. The estimation has to maintain the minimum accuracy condition \(P[|\hat{t}-t| \leq \beta t] \geq \alpha\), where \(\hat{t}\) is the estimated value of the actual tag population size \(t\).

B. Proposed Scheme Overview

In this paper, we present a new algorithm named Gaussian Estimation of RFID Tags (GERT). At the beginning of the estimation process we run a probe using the Flazolet Martin algorithm [1] to get a rough upper bound \(t_m\) on the tag population size \(t\). Based on \(t_m\) and the accuracy requirements \(\alpha, \beta\) we determine the critical parameters \(p, f, n\), where \(p\) is the persistence probability (the probability that a tag remains active to respond to a forthcoming frame), \(f\) is the frame size and \(n\) is the required number of rounds. A standardized framed slotted Aloha protocol is what we used as communication protocol. The reader first broadcasts the frame size of the forthcoming frame to the tags in its vicinity. Each tag randomly picks a time-slot in the frame and replies during that slot. Previous \{0,1\} channel models do not differentiate between a singleton and a collision slot, where a 0 represents an empty slot and a 1 represents a non-empty slot. In this \{0,1,e\} model [2] [3], 0 represents ‘no reply’, 1 represents ‘exactly one reply’ and \(e\) stands for ‘any number of replies greater than 1’, to a slot in a frame. The reader receives a sequence of 0s, 1s and es. GERT calculates \(\frac{N_e-N_1}{f}\) for each round where, \(N_e\) represents the number of \(e\)s (i.e. collision slots) and \(N_1\) is the number of 1s in the reader sequence. After \(n\) rounds of these measurements we take the average of all these values. This average is finally substituted for the true mean in the expected value equation of the estimator to estimate the tag population size by an inverse function. Figure 1 demonstrates that \(E\left[\frac{N_e-N_1}{f}\right]\) is a monotonic function of \(f\) for given \(p\) and \(e\), except for a singularity at the start. We have analyzed in this paper the conditions under which \(E\left[\frac{N_e-N_1}{f}\right]\) is invertible and \(\frac{N_e-N_1}{f}\) is asymptotically Gaussian distributed for large \(f\), while meeting the imposed estimation accuracy requirement.

C. Related Work

The first work on tag estimation was that of Kodialam and Nandagopal, titled Unified Probabilistic Estimator (UPE) [4]. UPE was based on the number of empty slots or the number of collision slots in the frame. Kodialam et al. later proposed an improved framed slotted Aloha protocol-based estimation in [5] called Enhanced Zero Based (EZB) estimator. EZB makes its estimation based on the total number of empty slots in a frame. The difference between EZB and UPE is that UPE makes an estimation of the population size in each frame and at the end averages out all the estimation results whereas, EZB finds the average of the number of 0s in each frame and finally makes the estimation based on this average value. In this paper we compared the proposed method (GERT) with a relatively recent tag estimation scheme (ART) by Shahzad and Liu [6], which claims to be both faster and more cost effective in terms of the number of slots required for estimation than the prior algorithms. ART uses a \{0,1\} channel model and estimates the tag population size based on the average run size of 1s in a frame.
II. FORMULATION OF THE PROBLEM

We are considering \( \frac{N_e - N_1}{f} \) as the estimator in our proposed model. We define two important variables in (1),
\[
Z_f(t) \triangleq \frac{N_e - N_1}{f}, \quad g_f(t) \triangleq \text{E} \left[ \frac{N_e - N_1}{f} \right]
\] (1)
For a given persistence probability \( p \) and frame size \( f \), let \( X_{ij} \sim \text{Bernoulli}(\frac{p}{f}) \) be the variable that represents the probability that the \( i \)th tag replies to the \( j \)th slot or not. So,
\[
X_{ij} = \begin{cases} 
1, & \text{with probability } \frac{p}{f} \\
0, & \text{with probability } (1 - \frac{p}{f}) 
\end{cases} 
\] (2)
Let \( Y_j = \sum_{i=1}^{t} X_{ij}. \) It is straightforward to see that the random variable \( Y_j \) has the following probability distribution,
\[
p_{Y_j}(y) = \begin{cases} 
(1 - \frac{p}{f})^t = p_0, & y = 0 \\
t \left( \frac{p}{f} \right) (1 - \frac{p}{f})^{t-1} = p_1, & y = 1 \\
1 - \left( 1 - \frac{p}{f} \right)^t - t \left( \frac{p}{f} \right) (1 - \frac{p}{f})^{t-1} = p_e, & y = e
\end{cases}
\]
Introduction of the following two indicators namely \( Y_j^{(e)} \) and \( Y_j^{(c)} \) will make our analysis easier,
\[
Y_j^{(e)} = \begin{cases} 
1, & \text{when } Y_j = 1 \\
0, & \text{when } Y_j \neq 1
\end{cases}, \quad Y_j^{(c)} = \begin{cases} 
1, & \text{when } Y_j = e \\
0, & \text{when } Y_j \neq e
\end{cases}
\] (3)
Now using (1) we have,
\[
Z_f = \frac{1}{f} \sum_{j=1}^{f} (Y_j^{(e)} - Y_j^{(1)})
\] (4)
Next we define, \( Z_{j,f} \triangleq Y_j^{(e)} - Y_j^{(1)}. \) Then (4) gives,
\[
Z_f = \frac{1}{f} \sum_{j=1}^{f} Z_{j,f}
\] (5)
Since \( Y_j^{(1)} \in \{0,1\} \) and \( Y_j^{(c)} \in \{0,1\} \) it is easy to see that we have the following pdf for \( Z_{j,f} \),
\[
Z_{j,f} = \begin{cases} 
0, & \text{with probability } p_0 \\
-1, & \text{with probability } p_1 \\
1, & \text{with probability } p_e
\end{cases}
\] (6)
Let the mean and variance of \( Z_{j,f} \) be \( \mu_f \) and \( \sigma_f^2 \) respectively. Simple algebraic manipulations give us,
\[
E[Z_{j,f}] = \mu_f = p_e - p_1
\] (7)
\[
\sigma_f^2 = p_e + p_1 - (p_e - p_1)^2
\] (8)
Let, \( \mu_f \) and \( \sigma_f^2 \) be the mean and variance of \( Z_f \) respectively. Its easy to see \( \mu_f = \mu_f \). Using (1), (5), (7) and (8) we have,
\[
g_f(t) = 1 - \left( 1 - \frac{p}{f} \right)^t - 2t \left( \frac{p}{f} \right) (1 - \frac{p}{f})^{t-1}
\] (9)
\[
\sigma_f^2 = \frac{1}{f} [p_e + p_1 - (p_e - p_1)^2]
\] (10)
We notice in Figure 1 that, there is a dip at the beginning of the expectation curve which presents us with a case of singularity. We define, \( r \triangleq \frac{tp}{f} \). Proofs to the following two facts for \( \frac{f}{p} \gg 1 \), are given in Appendix A and Appendix B respectively,
1) at D in Figure 1 \( t_{LM} = \frac{f}{2p} \) or equivalently at \( r_{LM} = \frac{1}{2} \).
2) \( g_f(t) \) convex \( \frac{t}{2p} \) and concave for \( \frac{3t}{2p} \) for given \( f \) and \( p \).
At point B in Figure 1 the singularity ends. We numerically calculated the value of \( r \) at B to be \( r_{min} = 1.2564 \). Using the definition of \( r \) we get the corresponding frame size at B,
\[
f_{max} = \frac{tp}{r_{min}}
\] (11)
Which gives us the maximum allowable frame size for given \( t \) and \( p \) for which \( g_f(t) \) will avoid the singularity region and hence be invertible.

III. GAUSSIAN APPROXIMATION OF THE ESTIMATOR

Our estimation of the tag population size has to maintain the accuracy requirement given by the condition,
\[
P[|\hat{t} - t| \leq \beta t] \geq \alpha
\] (12)
Since we are using \( Z_f \) as our estimator to determine the value of \( \hat{t} \), using (1) the condition in (12) can be written as,
\[
P[g_f\{(1 - \beta)t\} \leq Z_f \leq g_f\{(1 + \beta)t\}] \geq \alpha
\] (13)
Now to perform our estimation of the tag population size maintaining the accuracy requirements given in (13), we need the following,
1) \( g_f(t) \) has to be an invertible function.
2) Analytical expression for the distribution of \( Z_f(t) \).

The previous section clearly analyzed the conditions under which \( g_f(t) \) is monotonic function and hence invertible. This section is particularly devoted to the analysis of the conditions under which \( Z_f(t) \) has a Gaussian distribution. We resort to
a triangular array version of Central Limit Theorem \cite{7}, i.e.,
Lindeberg Feller Theorem \cite{8} for that.

The statement of Lindeberg Feller Theorem says, when \( \{X_{n,i}\} \) is an independent array of random variables with \( E[X_{n,i}] = 0 \) and \( E[X_{n,i}^2] = \sigma^2_{n,i} \), \( Z_n = \sum_{i=1}^n X_{n,i} \) and \( B_n^2 = \text{Var}(Z_n) = \sum_{i=1}^n \sigma^2_{n,i} \), then \( Z_n \to N(0, B_n^2) \) if the condition below holds for every \( \epsilon > 0 \),

\[
\frac{1}{B_n^2} \sum_{i=1}^n E[X_{n,i}^2 | |X_{n,i}| > \epsilon B_n] \to 0 \tag{14}
\]

For our algorithm, we modify \( Z_{j,f} \) to get a new variable \( \tilde{Z}_{j,f} \), so that the new variable has zero mean and variance \( \sigma^2_f \). It is straightforward to show that, \( \tilde{Z}_{j,f} \) has the following probability distribution,

\[
\tilde{Z}_{j,f} = \begin{cases} 
-\mu_f, & \text{with probability } p_0 \\
1 - \mu_f, & \text{with probability } p_1 \\
1 - \mu_f, & \text{with probability } p_c 
\end{cases} \tag{15}
\]

Now in line with (14) and the statement of Lindeberg Feller theorem, simple algebraic manipulations using (3), (8), (10) and (15) we get, \( Z_{f} \sim N(\mu_f, \sigma^2_f) \) if the following holds,

\[
\frac{1}{f \sigma_f^2} \sum_{i=1}^f \left( \tilde{Z}_{j,f} A \left( |\tilde{Z}_{j,f}| > \epsilon \sqrt{f} \sigma_f \right) \right) \to 0 \tag{16}
\]

In the above condition given in (16), the indicator function \( A \left( |\tilde{Z}_{j,f}| > \epsilon \sqrt{f} \sigma_f \right) \) plays a pivotal role. For the variable \( Z_{i,f} \) we have the following 3 cases of the indicator function,

\[
\begin{align*}
|1 - \mu_f| > \epsilon \sqrt{f} \sigma_f & \implies (17) \\
|-1 - \mu_f| > \epsilon \sqrt{f} \sigma_f & \implies (18) \\
|\mu_f| > \epsilon \sqrt{f} \sigma_f & \implies (19)
\end{align*}
\]

It is easy to see that if none of (17), (18) and (19) holds, then (16) holds. It has been proved analytically in Appendix C that, if the following condition holds, then none of the (17), (18), (19) holds; that is, (16) holds,

\[
e^2 f \geq k(r) \tag{20}
\]

where \( k(r) \) is defined as \( k(r) \triangleq \max\{k_1(r), k_2(r), k_3(r)\} \), and the values for \( k_1, k_2 \) and \( k_3 \) are given by the following equations.

\[
k_1(r) = \frac{1}{\frac{e^{4r}(1+4r)^2}{(1+2r)^2} - 1} \tag{21}
\]

\[
k_2(r) = \frac{e^{2r} + (1 + 2r) \left( \frac{1}{2} - e^{r} \right)}{\frac{1}{2} \left( 1 + 2r \right)^2 - e^r} \tag{22}
\]

\[
k_3(r) = \frac{e^{2r} - 2e^r (1 + 2r) + (1 + 2r)^2}{(1 + 2r)^2 - e^r (1 + 4r)} \tag{23}
\]

The argument now reduces to, if (20) holds, then (16) holds. Solving (20) for frame size \( f \), we can write,

\[
f \geq \frac{k(r)}{e^r}. \tag{24}
\]

This serves as the lower bound on \( f \) for a given value of \( \epsilon \). If we select a frame size that satisfies (24), the distribution of the estimator will be \( Z_f \sim N(\mu_f, \sigma^2_f) \).

A. Quality Considerations of Gaussian Approximation

The quality of the above approximation depends on the value of the approximation error \( \epsilon \). Exactly speaking, satisfying (16) means,

\[
\begin{align*}
P [ \frac{Z_f - \mu_f}{\sigma_f} \leq u ] - P [ \frac{l - \theta}{\sigma_f} \leq u ] & \leq \epsilon \\
P [ l \leq \theta \leq u ] & \leq \epsilon \tag{25}
\end{align*}
\]

where, \( \theta \sim N(0,1) \). It is straightforward to see that, (25) implies \( P [ l \leq \theta \leq u ] \geq \alpha + \epsilon \). Which means that, If we approximate \( \frac{Z_f - \mu_f}{\sigma_f} \) as standard normal, to compensate for the approximation error we will have to maintain the actual reliability \( \alpha + \epsilon \) instead of the given reliability \( \alpha \). The fact that probability can not be greater than 1 renders \( \alpha + \epsilon \leq 1 \), giving the following upper bound on \( \epsilon \),

\[
\epsilon_{\text{max}} = 1 - \alpha \tag{26}
\]

Equation (26) gives the maximum value of \( \epsilon \) that we can operate on for a given reliability requirement \( \alpha \).

\begin{algorithm}
\caption{Estimate RFID Tag Population (\( \alpha, \beta, n_r \))}
\textbf{Input:}
\begin{enumerate}
  \item Required reliability \( \alpha \)
  \item Required confidence interval \( \beta \)
\end{enumerate}
\textbf{Output:} Estimated tag population size \( \hat{t} \)
1. Calculate \( t_{\text{min}} := \text{upper bound} \)
2. Calculate persistence probability \( p \) using (11)
3. Obtain \( f_{\text{max}} \) and \( f_{\text{min}} \) using (11) and (28) respectively.
4. Obtain \( \epsilon_{\text{max}} \) and \( f_{\text{min}} \) using (26) and (29) respectively.
5. Make the array, \( f_{\text{array}} = \{ f_{\text{min}}, f_{\text{max}} \} \)
6. For \( i := 1 : l_f \) do
    \begin{enumerate}
        \item Evaluate \( n_i \) for given \( p \) and \( f_{\text{array}}(i) \)
    \end{enumerate}
8. Obtain \( f_{\text{op}} \) and \( n_{\text{op}} \) such that \( (f_{\text{op}} + l) \times n_{\text{op}} := \text{min}\{ (f_{\text{array}}(i) + l) \times n_i \} \)
9. For \( j := 1 : n_{\text{op}} \) do
    \begin{enumerate}
        \item Provide the reader with frame size \( f_{\text{op}} \), persistence probability \( p \), and random seed \( R_j \).
        \item Run Aloha on the \( j \)th frame.
        \item Obtain \( Z_f(j) = \frac{N_j - N_i}{f_{\text{op}}} \) for the \( j \)th frame
    \end{enumerate}
10. \( \hat{Z}_f := \frac{1}{n_{\text{op}}} \sum_{j=1}^{n_{\text{op}}} Z_f(j) \)
11. Set \( g_f(l) := \hat{Z}_f \) and solve (9) to get the estimated value \( l \) for tag population size \( t \).
12. \textbf{return} \( l \)
\end{algorithm}
IV. SELECTION OF CRITICAL PARAMETERS

This section clarifies the steps to attain the optimum parameters mentioned in Algorithm 1.

A. Persistence Probability p

To decide the persistence probability p, we resort to the numerical evidences presented in Figure 2. Figure 2 shows that for different accuracy requirements the required number of slots for estimation \((f + l)\times n\) hits minimum for different values of \(t\). Since we will not aim at any reliability lower than 87\%, the minimum that applies to \(\alpha = 87\%\) curve will surely encompass the minimum of the higher accuracy requirements curves. So, we select \(t = 1500\) as the point where we bring down all the higher tag population sizes. Since at the time of estimation we do not know \(t\), we substitute \(t_m\) for \(t\) and get the following equation for persistence probability,

\[
p = \begin{cases} 
\frac{1500}{t_m} & \text{for } t_m > 1500 \\
1, & \text{otherwise}
\end{cases} \tag{27}
\]

B. Selection of \(\epsilon\)

We have already discussed the maximum value of \(\epsilon\) that we can operate on, given by the equation \(\text{(26)}\). When we are looking for the minimum value of \(\epsilon\) we are actually looking for the value of \(\epsilon\) corresponding to point B in Figure 1 in order to avoid singularity. We know from \(\text{(11)}\) at point B in Figure 1 \(f = f_{\text{max}}\) and \(r = r_{\text{min}}\). Now, using \(\text{(20)}\) we have,

\[
\epsilon_{\text{min}} = \sqrt{\frac{k(r_{\text{min}})}{f_{\text{max}}}} \tag{28}
\]

This would only work as long as \(\epsilon_{\text{min}} \leq \epsilon_{\text{max}}\). GERT performance shows that if we aim at 92\% reliability, the achieved reliability is around 97\% which is top notch. To aim at 92\% our \(\epsilon_{\text{min}}\) has to be less than 0.08. We numerically found that 0.08 is the value of epsilon for the number of tags as low as 200. To be on the safer side and appreciating the fact that \(t_m\) is a random quantity, we only estimate the tag population sizes greater than 300 under \(\{0,1,e\}\) channel model. That ensures \(\epsilon_{\text{min}} \leq \epsilon_{\text{max}}\) holds.

C. Frame size

It is apparent from \(\text{(20)}\) that, corresponding to \(\epsilon_{\text{max}}\) given in \(\text{(26)}\) we have the minimum frame size \(f_{\text{min}}\), as mentioned in the Algorithm 1 that will ensure that \(Z_f\) is Gaussian distributed. Using the equation \(\text{(20)}\) we get,

\[
f_{\text{min}} = \min_f \frac{k(f,p,t)}{\epsilon_{\text{max}}^2} \tag{29}
\]

D. Number of rounds \(n\)

For any \(f\) in \(f_{\text{array}}\) mentioned in Algorithm 1, \(g_f(t)\) is a monotonic function and \(Z_f \sim N(\mu_f, \sigma_f^2)\) with an approximation error \(\epsilon\) corresponding to a given \(f\), and we know that standard deviation gets scaled down \(\sqrt{n}\) times if we take \(n\) rounds of the measurements, using \(\text{(13)}\) we have,

\[
P \left[ \frac{g_f((1 - \beta)\mu_t - \mu_f)}{\sqrt{n}} \right] \leq \frac{Z_f - \mu_f}{\sqrt{n}} \leq \frac{g_f((1 + \beta)\mu_t - \mu_f)}{\sqrt{n}} \geq \alpha + \epsilon \tag{30}
\]

It is obvious that the left and right inequalities in \(\text{(30)}\) will give us two different values of \(n\), we name them \(n_{\text{left}}\) and \(n_{\text{right}}\) respectively. So, the required number of rounds for given frame size \(f\), can be given by

\[
n = \left\lceil \max\{n_{\text{left}}, n_{\text{right}}\} \right\rceil \tag{31}
\]

V. PERFORMANCE EVALUATION

We used MATLAB to get simulation results for GERT. Figures 3 and 4 illustrate the actual reliability of GERT for different reliability requirements under \(\{0,1\}\) and \(\{0,1,e\}\) channel models respectively. To get each point in Figures 3 and 4 we ran 300 to 600 trials. We see that the actual reliability of GERT is much greater than the required reliability under both channel models. This higher level of quality can be attributed to two distinct properties of GERT. Firstly, unlike ART, because of the restrictions imposed by the Gaussian approximation of \(Z_f\), for a given number of tags \(t\) GERT always maintains a commensurate frame size which is highly unlikely to get saturated. Secondly, we controlled quality by taking all the approximation errors into account when we calculated the overall estimation error. This gives us the advantage of being able to target a lower reliability than the required reliability. For example for \(\{0,1\}\) channel model we see that, when we have a required reliability of 91\%, we can actually aim at 87\%. Figure 5 and 6 present the corresponding number of slots required for the achieved reliability given in Figure 3 and 4 respectively. The required number of slots for ART for the reliability requirements 91\%, 95\% and 97\% are 1760, 2340 and 2880 respectively. As we see in Figure 5 except for the very small tag population sizes GERT under...
{0, 1} channel model takes 27%, 30% and 28% less number of slots than ART, to achieve 91%, 95% and 97% levels of reliability respectively. Comparing between Figures 5 and 6, we notice that GERT under {0, 1, e} requires fewer number of slots than GERT under {0, 1} to achieve similar level of accuracy. This shows the advantage of having more side information. This is because under {0, 1, e} each 1 is certain. Because of this added certainty, we need fewer number of rounds to meet the accuracy requirements under {0, 1, e} than under {0, 1} channel model.

VI. CONCLUSION

The key contribution of this paper is that it proposes a completely new and more effective technique for the estimation of an RFID tag population. The most prominent feature of GERT is the quality of estimation coming off rigorous analysis. All our analytical findings have been well supported by the simulation results. GERT is much more cost effective than any other proposed algorithm so far under the same accuracy constraints.

APPENDIX A

Lemma 1. The local minimum of \( g_f(t) \) curve occurs at a tag population size \( t_{LM} = \frac{1}{2p} \) or equivalently at \( r_{LM} = \frac{1}{2} \), given frame size \( f \) and persistence probability \( p \).

Proof: To find the local minimum we need to differentiate the expectation curve and set the derivative to 0. Solving that equation we will get the local minimum of the dip. Using (9),

\[
\frac{d}{dt} g_f(t) = \frac{d}{dt} \left[ 1 - \left( 1 - \frac{p}{f} \right)^t - 2t \left( \frac{p}{f} \right) \left( 1 - \frac{p}{f} \right)^{t-1} \right] = 0
\]

(32)

Simple algebraic calculations give us,
Here, \( t_{LM} \) stands for the \( t \) value where the local minimum of the dip occurs (i.e. at point \( D \) in Figure 1). Now, since the value of \( \frac{t}{f} << 1 \) we can approximate \( \ln \left(1 - \frac{p}{f}\right) \) as \( -\frac{p}{f} \). This will give us the following, \( t_{LM} \approx \frac{f}{2p} \). Which means the local minimum for the dip occurs at a value of \( t = t_{LM} \) which is supported by our simulation results.

As we defined earlier,

\[
r = \frac{tp}{f} \tag{34}
\]

Substituting, \( t_{LM} \) in (34) gives us \( r_{LM} = \frac{1}{2} \).

### APPENDIX B

**Lemma 2.** \( g_f(t) \) is a convex function of \( t \) for \( t < \frac{3f}{2p} \), and for the rest of the \( t \) values the function is concave, given frame size \( f \) and persistence probability \( p \).

**Proof:** To find an inverse we need \( g_f(t) \) to be a monotonic function of \( t \). To find which part of the \( g_f(t) \) demonstrates monotonic behavior we need the second derivative of \( g_f(t) \) and check for it’s convexity and concavity characteristics.

Again using (9)

\[
\frac{d^2}{dt^2} g_f(t) = \left(1 - \frac{p}{f}\right)^{t-1} \ln \left(1 - \frac{p}{f}\right) \left[ -2t \left(\frac{p}{f}\right) \ln \left(1 - \frac{p}{f}\right) \right. \\
\left. -4 \left(\frac{p}{f}\right) - \left(1 - \frac{p}{f}\right) \ln \left(1 - \frac{p}{f}\right) \right] 
\]

(35)

If we closely follow the equation we see that, the value of the common factor \( \left(1 - \frac{p}{f}\right)^{t-1} \ln \left(1 - \frac{p}{f}\right) \) is negative . So, for the total value to be positive the second derivative of the equation inside the third bracket will have to be negative. After algebraic manipulations and approximating \( \ln \left(1 - \frac{p}{f}\right) \) as \( -\frac{p}{f} \) we get for \( t < \frac{3f+p}{2p} \),

\[
\left[ 2t \left(\frac{p}{f}\right) - 4 \left(\frac{p}{f}\right) + \left(1 - \frac{p}{f}\right) \left(\frac{p}{f}\right) \right] < 0
\]

Or, equivalently, for \( t < \frac{3f+p}{2p} \approx \frac{3f}{2p} \), \( \frac{d^2}{dt^2} g_f(t) \) is positive, indicating \( g_f(t) \) is a convex function of \( t \) and for the rest of the \( t \) values the curve is concave. Substituting \( t = \frac{3f}{2p} \) in (34) gives us \( r = \frac{3}{2} \). Our simulation results strongly support that claim.

### APPENDIX C

**Derivation of Lindeberg Feller conditions for GERT under \([0, 1, e] \) channel model**

1) **First condition:** From equation (17) we have the following,

\[
\Rightarrow 1 - \mu_f > \epsilon \sqrt{f} \sigma_j, f
\]

\[
\Rightarrow 1 - (p_e - p_i) > \epsilon \sqrt{f} [p_e + p_i - (p_e - p_i)^2]
\]

\[
\Rightarrow 1 - 2(p_e - p_i) + (p_e - p_i)^2 > \epsilon^2 f [p_e + p_i - (p_e - p_i)^2]
\]

\[
\Rightarrow 1 - (2 + \epsilon^2 f) p_e + (2 - \epsilon^2 f) p_i + (1 + \epsilon^2 f) (p_e - p_i)^2 > 0
\]

(36)

letting \( \epsilon^2 f \) be represented by \( k \), and inserting the expression for \( p_0 \), \( p_1 \) and \( p_e \) we have,

\[
\Rightarrow 1 - (2 + k) \left[ 1 - \left(1 - \frac{p}{f}\right) e^{-t \left(1 - \frac{p}{f}\right) t^{-1}} \right] + (2 - k) \left[ t \left(\frac{p}{f}\right) \left(1 - \frac{p}{f}\right) e^{-t \left(1 - \frac{p}{f}\right) t^{-1}} \right] + (1 + k) \left[ 1 + \left(1 - \frac{p}{f}\right)^{2t} \right]
\]

\[
+ 4t^2 \left(\frac{p}{f}\right)^2 \left(1 - \frac{p}{f}\right)^{2(t-1)} - 2 \left(1 - \frac{p}{f}\right)^{t} + 4t \left(\frac{p}{f}\right)
\]

\[
\left(1 - \frac{p}{f}\right)^{2t-1} - 4t \left(\frac{p}{f}\right) \left(1 - \frac{p}{f}\right)^{t-1} \right] > 0
\]

(37)

We know, for \( x << 1 \) and \( y \gg 1 \), \( (1 - x)^y \) can be approximated as \( e^{y \ln[1-x]} \) and that in turn can be reduced to \( e^{y(1-x)} \) applying Taylor series. Applying this we get,

\[
\Rightarrow 1 - (2 + k) \left[ 1 - e^{-t \left(\frac{1}{2} + \frac{1}{2} \left(\frac{f}{p}\right)^2\right)} + t \left(\frac{p}{f}\right) e^{-t \left(1 - \frac{1}{2} + \frac{1}{2} \left(\frac{f}{p}\right)^2\right)} \right] + (2 - k) \left[ t \left(\frac{p}{f}\right) e^{-t \left(1 - \frac{1}{2} + \frac{1}{2} \left(\frac{f}{p}\right)^2\right)} \right] + (1 + k)
\]

\[
\left[ 1 + e^{-t \left(\frac{1}{2} + \frac{1}{2} \left(\frac{f}{p}\right)^2\right)} + 4t \left(\frac{p}{f}\right)^2 e^{-2(t-1) \left(\frac{1}{2} + \frac{1}{2} \left(\frac{f}{p}\right)^2\right)} - 2e^{-t \left(\frac{1}{2} + \frac{1}{2} \left(\frac{f}{p}\right)^2\right)} \right] + (1 + k) \left[ 4t \left(\frac{p}{f}\right) e^{-2(t-1) \left(\frac{1}{2} + \frac{1}{2} \left(\frac{f}{p}\right)^2\right)} \right]
\]

\[
- 4t \left(\frac{p}{f}\right) e^{-t \left(1 - \frac{1}{2} + \frac{1}{2} \left(\frac{f}{p}\right)^2\right)} \right] > 0
\]

(38)

Now we have a list of approximations to make. They are,

\[
e^{-t \left(\frac{1}{2} + \frac{1}{2} \left(\frac{f}{p}\right)^2\right)} \approx e^{-\frac{t}{2}} \tag{39}
\]

\[
e^{-2(t-1) \left(\frac{1}{2} + \frac{1}{2} \left(\frac{f}{p}\right)^2\right)} \approx e^{-2t \frac{f}{p}} \tag{40}
\]

\[
e^{-2(t-1) \left(\frac{1}{2} + \frac{1}{2} \left(\frac{f}{p}\right)^2\right)} \approx e^{-2t \frac{f}{p}} \tag{41}
\]

\[
e^{-t \left(\frac{1}{2} + \frac{1}{2} \left(\frac{f}{p}\right)^2\right)} \approx e^{-\frac{t}{2}} \tag{42}
\]

After all these approximations and using (34) we have,

\[
\Rightarrow 1 - (2 + k) \left(1 - e^{-r - re^{-r}} + (2 - k) re^{-r} + (1 + k) \left(1 + e^{-2r} + 4r e^{-2r} - 2e^{-r}\right) \right)
\]

\[
+ (1 + k) \left(4re^{-2r} - 4e^{-2r}\right) > 0
\]

(43)
For the equation (17) to not hold, (43) must not hold. Simple algebraic manipulations give us that for (43) to not hold the value of $k$ must be,

$$k \geq \frac{1}{-e^r(1+4t)} - 1 = k_1 = \frac{e^r}{e^r - 1}.$$ 

So, $k_1$ is the minimum value of $k$ for which (17) does not hold.

2) 2nd condition: From equation (18) we have the following,

$$\Rightarrow 1 + \mu_f > \epsilon \sqrt{f} \sigma_j, f \Rightarrow 1 + (p_e - p_t) > \epsilon \sqrt{f} [p_e + p_t - (p_e - p_t)^2]$$
$$\Rightarrow 1 + 2(p_e - p_t) + (p_e - p_t)^2 > \epsilon^2 f [p_e + p_t - (p_e - p_t)^2]$$
$$\Rightarrow 1 + (2 - \epsilon^2 f) p_e - (2 + \epsilon^2 f) p_t + (1 + \epsilon^2 f) (p_e - p_t)^2 > 0$$

(44)

Letting $\epsilon^2 f$ be represented by $k$, and inserting the expression for $p_0$, $p_1$ and $p_e$, we have,

$$\Rightarrow 1 + (2 - k) \left[ 1 - \left(1 - \frac{p}{f}\right)^t - t \left(\frac{p}{f}\right) \left(1 - \frac{p}{f}\right)^{t-1}\right]$$
$$- (2 + k) \left[ t \left(\frac{p}{f}\right) \left(1 - \frac{p}{f}\right)^{t-1}\right] + (1 + k) \left[ 1 + \left(1 - \frac{p}{f}\right)^t\right]$$
$$+ 4t^2 \left(\frac{p}{f}\right)^2 \left(1 - \frac{p}{f}\right)^{2(t-1)} - 2 \left(1 - \frac{p}{f}\right)^t \left(1 - \frac{p}{f}\right)^{t-1}\right] > 0$$

(45)

Like the first condition $(1 - x)^y$ can be approximated as $e_{[1-1-x]}$ and that in turn can be reduced to $e_{[1-1-x]}$ applying Taylor series. Applying this along with approximations made in (39), (40), (41), (42) and using (34) we have,

$$\Rightarrow -k \left(1 - e^{-r}\right) + (1 + k) \left(1 + e^{-2r} + 4r^2 e^{-2r} - 2e^{-r}\right)$$
$$+ (1 + k) \left(4r e^{-2r} - 4re^{-r}\right) > 0$$

(48)

For the equation (19) to not hold, (48) must not hold. Simple algebraic manipulations give us that for (48) to not hold the value of $k$ must be,

$$k \geq \frac{e^{2r} + (1 + 2r) (\frac{1}{4} - e^r)}{1} = k_2$$

(46)

So, $k_2$ is the minimum value of $k$ for which (18) does not hold.

3) 3rd condition: From equation (19) we have the following,

$$\Rightarrow \mu_f > \epsilon \sqrt{f} \sigma_j, f \Rightarrow (p_e - p_t) > \epsilon \sqrt{f} [p_e + p_t - (p_e - p_t)^2]$$
$$\Rightarrow (p_e - p_t)^2 > \epsilon^2 f [p_e + p_t - (p_e - p_t)^2]$$
$$\Rightarrow - \epsilon^2 f (p_e + p_t) + (1 + \epsilon^2 f) (p_e - p_t)^2 > 0$$
$$\Rightarrow -k \left[ 1 - \left(1 - \frac{p}{f}\right)^t\right] + (1 + k) \left[ 1 + \left(1 - \frac{p}{f}\right)^t + 4t^2\right]$$
$$\left(\frac{p}{f}\right)^2 \left(1 - \frac{p}{f}\right)^{2(t-1)} - 2 \left(1 - \frac{p}{f}\right)^t + 4t \left(\frac{p}{f}\right)^2 \left(1 - \frac{p}{f}\right)^{t-1}\right] > 0$$

(47)

Like we did in the previous two conditions, $(1 - x)^y$ can be approximated as $e_{[1-1-x]}$ and that in turn can be reduced to $e_{[1-1-x]}$ applying Taylor series. Applying this along with approximations made in (39), (40), (41), (42) and using (34) we have,

$$\Rightarrow -k \left(1 - e^{-r}\right) + (1 + k) \left(1 + e^{-2r} + 4r^2 e^{-2r} - 2e^{-r}\right)$$
$$+ (1 + k) \left(4r e^{-2r} - 4re^{-r}\right) > 0$$

(48)

For the equation (19) to not hold, (48) must not hold. Simple algebraic manipulations give us that for (48) to not hold the value of $k$ must be,

$$k \geq \frac{e^{2r} - 2e^r (1 + 2r) + (1 + 2r)^2}{(1 + 2r)^2 - e^{-r} (1 + 4r)} = k_3$$

So, $k_3$ is the minimum value of $k$ for which (19) does not hold.

From the above three conditions we see that if we select the value of $k$ such that $k = \max\{k_1, k_2, k_3\}$ all of (17), (18) and (19) will not hold or equivalently (16) will hold. That essentially means for $k = \max\{k_1, k_2, k_3\}$ the GERT estimator ZF under the $0, 1, e$ channel model is Gaussian distributed.

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