SUPERSYMMETRIC TWO-DIMENSIONAL QCD AT FINITE TEMPERATURE

J.R. HILLER
Department of Physics
University of Minnesota-Duluth
Duluth, MN 55812 USA
E-mail: jhill@d.umn.edu

Light-cone coordinates and supersymmetric discrete light-cone quantization are used to analyze the thermodynamics of two-dimensional supersymmetric quantum chromodynamics with a Chern–Simons term in the large-$N_c$ approximation. This requires estimation of the entire spectrum of the theory, which is done with a new algorithm based on Lanczos iterations. Although this work is still in progress, some preliminary results are presented.

Keywords: supersymmetry, quantum chromodynamics, finite temperature, density of states

1. Introduction

Recent work\(^1\) has shown that thermodynamic properties can be computed for large-$N_c$ supersymmetric theories. The approach is based on light-cone coordinates\(^2\) and the numerical technique of supersymmetric discrete light-cone quantization (SDLCQ).\(^3,4\) Here we consider two-dimensional supersymmetric quantum chromodynamics with a Chern–Simons term (SQCD-CS),\(^5\) dimensionally reduced from three dimensions.

Light-cone coordinates\(^2\) are defined by the time variable, $x^+ = (t + z)/\sqrt{2}$, and spatial components, $\vec{x} = (x^-, \vec{x}_\perp)$, where $x^- = (t - z)/\sqrt{2}$ and $\vec{x}_\perp = (x, y)$. The light-cone energy and momentum are given by $p^- = (E - p_z)/\sqrt{2}$ and $\vec{p} = (p^+ = (E + p_z)/\sqrt{2}, \vec{p}_\perp = (p_x, p_y))$, respectively.

For field theories quantized in terms of these coordinates, the standard numerical technique is discrete light-cone quantization (DLCQ).\(^6,7\) Space is restricted to a light-cone box $-L < x^- < L, -L_\perp < x, y < L_\perp$ with

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periodic boundary conditions. Momentum is then discretized as \( p_i^+ \to \frac{2\pi}{L} n_i, \)
\( p_{i\perp} \to \left( \frac{2\pi}{L} n_{ix}, \frac{2\pi}{L} n_{iy} \right) \) with \( n_i, n_{ix} \) and \( n_{iy} \) all integers. The limit \( L \to \infty \) is exchanged for a limit in terms of the integer (harmonic) resolution \( K \equiv \frac{L}{2} P^+ \) for fixed total momentum \( P^+ \). Because the \( n_i \) are positive, the number of particles is limited to no more than \( K \). Integrals are replaced by discrete sums.

SDLCQ\(^3,4\) is a special form of DLCQ that preserves at least part of the supersymmetry algebra

\[
\{Q^+, Q^-\} = 2\sqrt{2} P^+, \quad \{Q^-, Q^-\} = 2\sqrt{2} P^-, \quad \{Q^+, Q^-\} = -4P_{\perp}.
\]

Instead of discretizing the Hamiltonian \( P^- \) directly, the supercharge \( Q^- \) is discretized, and \( P^- \) is computed from the algebra as

\[
P^-_{\text{SDLCQ}} = \frac{1}{2\sqrt{2}} \{Q^-, Q^-\} \neq P^-_{\text{DLCQ}}.
\]

For ordinary DLCQ, one recovers supersymmetry only in the infinite resolution limit.

After a brief summary of the SQCD-CS theory, we show how thermodynamic quantities can be constructed from the partition function. This requires knowledge of the spectrum of the theory, which we obtain numerically with an iterative Lanczos algorithm. Some preliminary results are presented and future work discussed.

2. Supersymmetric QCD

We consider a dimensional reduction from 2+1 to 1+1 dimensions of \( \mathcal{N} = 1 \) supersymmetric quantum chromodynamics with a Chern–Simons term (SQCD-CS).\(^5\) The action is

\[
S = \int d^3x \text{Tr} \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \xi^\dagger D^\mu \xi + i\bar{\Psi} D_\mu \Gamma^\mu \Psi \\
- g \left[ \bar{\Psi} \Lambda \xi + \xi^\dagger \Lambda \Psi \right] + i \bar{\Lambda} \Gamma^\mu D_\mu \Lambda \\
+ \frac{\kappa}{2} \epsilon^{\mu\nu\lambda} \left[ A_\mu \partial_\nu A_\lambda + \frac{2i}{3} g A_\mu A_\nu A_\lambda \right] + \kappa \bar{\Lambda} \Lambda \right\},
\]

where the adjoint fields are the gauge boson \( A_\mu \) (gluons) and a Majorana fermion \( \Lambda \) (gluinos) and the fundamental fields are a Dirac fermion \( \Psi \) (quarks) and a complex scalar \( \xi \) (squarks). The Chern–Simons coupling, \( \kappa \), has the effect of providing a mass for the adjoint fields. The covariant
derivatives are
\[ D_{\mu} \Lambda = \partial_{\mu} \Lambda + ig[A_{\mu}, \Lambda], \quad D_{\mu} \xi = \partial_{\mu} \xi + igA_{\mu} \xi, \]
\[ D_{\mu} \Psi = \partial_{\mu} \Psi + igA_{\mu} \Psi. \] (4)
The fields transform according to
\[ \delta A_{\mu} = \frac{i}{2} \bar{\varepsilon} \Gamma_{\mu} \Lambda, \quad \delta \Lambda = \frac{1}{4} F_{\mu \nu} \Gamma^{\mu \nu} \varepsilon, \]
\[ \delta \xi = \frac{i}{2} \bar{\varepsilon} \Psi, \quad \delta \Psi = -\frac{1}{2} \Gamma^{\mu} \varepsilon D_{\mu} \xi. \] (5)
We reduce to 1+1 dimensions by assuming the fields to be independent of the transverse coordinate \( x \). We define fermion components by
\[ \Lambda = \begin{pmatrix} \lambda, \tilde{\lambda} \end{pmatrix}^T, \quad \Psi = \begin{pmatrix} \psi, \tilde{\psi} \end{pmatrix}^T, \quad Q = \begin{pmatrix} Q^+, Q^- \end{pmatrix}^T. \] (6)
There are constraints, which in light-cone gauge \((A^+ = 0)\) are written
\[ \partial_- \tilde{\lambda} = -\frac{ig}{\sqrt{2}} (\lambda \tilde{\lambda}^\dagger + i\lambda \psi^\dagger - i\tilde{\lambda} \xi^\dagger), \] (7)
\[ \partial_- \tilde{\psi} = -\frac{ig}{\sqrt{2}} A^2 \psi + \frac{g}{\sqrt{2}} \lambda \xi - \kappa \lambda / \sqrt{2}, \quad \partial_- A^- = gJ, \] (8)
with
\[ J \equiv i[A^2, \partial_- A^2] + \frac{1}{\sqrt{2}} \{ \lambda, \lambda \} + \kappa \partial_- A^2 - i\hbar \partial_- \xi^\dagger + i\xi \partial_- \xi^\dagger + \sqrt{2} \psi \psi^\dagger. \] (9)
The reduced supercharge is
\[ Q^- = g \int dx_- \left\{ 2^{3/4} \left( i[A^2, \partial_- A^2] - \kappa \partial_- A^2 + \frac{1}{\sqrt{2}} \{ \lambda, \lambda \} \right) \frac{1}{\partial_- \lambda} \right. \\
\left. - \frac{1}{\sqrt{2}} \left( i\sqrt{2} \xi \partial_- \xi^\dagger - i\sqrt{2} \partial_- \xi \xi^\dagger + 2 \psi \psi^\dagger \right) \frac{1}{\partial_- \lambda} \\
- 2 \left( \xi^\dagger A^2 \psi + \psi^\dagger A^2 \xi \right) \right\}. \] (10)
In the large-\( N_c \) approximation, there are only single-trace Fock states, the mesons
\[ f_{11}^\dagger (k_1) a_{i_1 i_2}^\dagger (k_2) \cdots b_{i_{n-1} i_n}^\dagger (k_{n-1}) \cdots f_{n}^\dagger (k_n) |0\rangle \] (11)
and glueballs
\[ \text{Tr} [a_{i_1 i_2}^\dagger (k_1) \cdots b_{i_{n-1} i_n}^\dagger (k_{n-1}) |0\rangle], \] (12)
where \( f_i^\dagger \) and \( f_i^\dagger \) create fundamental partons and \( a_{ij}^\dagger \) and \( b_{ij}^\dagger \) create adjoint partons. Either type of state could be a boson or a fermion.
The theory possesses a useful $Z_2$ symmetry
\[ a_{ij}(k,n) \rightarrow -a_{ji}(k,n), \quad b_{ij}(k,n) \rightarrow -b_{ji}(k,n), \tag{13} \]
which further divides the Fock space between states with even and odd numbers of gluons. We then diagonalize in each sector separately.

3. Finite Temperature
From the partition function, $Z = e^{-p_0/T}$, we compute the bosonic free energy
\[ F_B = \frac{VT}{\pi} \sum_{n=1}^{\infty} \int_{M_n}^{\infty} dp_0 \frac{p_0}{\sqrt{p_0^2 - M_n^2}} \ln \left( 1 - e^{-p_0/T} \right). \tag{14} \]
and the fermionic free energy
\[ F_F = -\frac{VT}{\pi} \sum_{n=1}^{\infty} \int_{M_n}^{\infty} dp_0 \frac{p_0}{\sqrt{p_0^2 - M_n^2}} \ln \left( 1 + e^{-p_0/T} \right). \tag{15} \]
The total free energy, once expanded in logarithms and $p_0$ integrals are performed, is given by
\[ F(T,V) = -\frac{(K-1)\pi}{4} VT^2 - \frac{2VT}{\pi} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} M_n K_1 \left( (2l + 1) \frac{M_n}{T} \right). \tag{16} \]
The sum over $l$ is well approximated by the first few terms. We represent the sum over $n$ as an integral over a density of states $\rho$: $\sum_n \rightarrow \int \rho(M) dM$. The density is approximated by a continuous function, and the integral $\int dM$ is computed by standard numerical techniques.

4. Lanczos Algorithm for Density of States
The discrete density of states is $\rho(M^2) = \sum_n d_n \delta(M^2 - M_n^2)$, where $d_n$ is the degeneracy of the mass eigenvalue $M_n$. The density can be written in the form of a trace over the evolution operator $e^{-iP^- x^+}$:
\[ \rho(M^2) = \frac{1}{4\pi P^+} \int_{-\infty}^{\infty} e^{iM^2 x^+/2P^+} \text{Tr} e^{-iP^-x^+} dx^+. \tag{17} \]
We approximate the trace as an average over a random sample of vectors $\rho(M^2) \approx \frac{1}{S} \sum_{s=1}^{S} \rho_s(M^2), \tag{18}$
with $\rho_s$ a local density for a single vector $|s\rangle$, defined by
\begin{equation}
\rho_s(M^2) = \frac{1}{4\pi P^+} \int_{-\infty}^{\infty} e^{iM^2 x^+/2P^+} |s\rangle e^{-iP^- x^+} |s\rangle dx^+.
\end{equation}
(19)

The sample vectors $|s\rangle$ can be chosen as random phase vectors;\textsuperscript{10} the coefficient of each Fock state in the basis is a random number of modulus one.

We approximate the matrix element $\langle s|e^{-iP^- x^+}|s\rangle$ by Lanczos iterations.\textsuperscript{11} Let $D$ be the length of $|s\rangle$, and define $|u_1\rangle = \frac{1}{\sqrt{D}}|s\rangle$ as the initial Lanczos vector. The matrix element $\langle u_1|e^{-iP^- x^+}|u_1\rangle$ can be approximated by the $(1,1)$ element of the exponentiation of the Lanczos tridiagonalization of $P^-$. Let $P^-_s$ be the tridiagonal Lanczos matrix. It can be exponentiated by first diagonalizing it:
\begin{equation}
P^-_s \vec{c}_j^s = \frac{M^2_{sj}}{2P^+} \vec{c}_j^s,
\end{equation}
(20)
such that $P^-_s = U\Lambda U^{-1}$, with $U_{ij} = (c^s_j)_i$ and $\Lambda_{ij} \equiv \delta_{ij} \frac{M^2_{ij}}{2P^+}$. The $(1,1)$ element is given by
\begin{equation}
\left( e^{-iP^- x^+} \right)_{11} = \sum_n |(c^s_n)_1|^2 e^{-iM^2_{sn} x^+/2P^+}.
\end{equation}
(21)

The local density can now be estimated by
\begin{equation}
\rho_s(M^2) \simeq \sum_n w_{sn} \delta(M^2 - M^2_{sn}),
\end{equation}
(22)
where $w_{sn} \equiv D|(c^s_n)_1|^2$ is the weight of each Lanczos eigenvalue. Only the extreme Lanczos eigenvalues are good approximations to eigenvalues of the original $P^-$. The other Lanczos eigenvalues provide a smeared representation of the full spectrum.

From this density of states, we compute the cumulative distribution function (CDF), $N(M^2) = \int M^2 dM^2 \rho(M^2)$ as an average
\begin{equation}
N(M^2) \simeq \frac{1}{S} \sum_s N_s(M^2),
\end{equation}
(23)
of local CDFs
\begin{equation}
N_s(M^2) \equiv \int M^2 dM^2 \rho_s(M^2) \simeq \sum_n w_{sn} \theta(M^2 - M^2_{sn}).
\end{equation}
(24)

The convergence of the approximation is dependent on the number of Lanczos iterations per sample, as well as the number $S$ of samples. Test
runs indicate that taking 20 samples is sufficient. The number of Lanczos iterations needs to be on the order of 1000 per sample; using only 10 leaves errors on the order of 1-2%.

5. Preliminary Results

Some preliminary results are presented in the accompanying figures. Figure 1 compares the numerical results for the CDF to analytic results, which can be obtained when the Yang-Mills coupling is zero. The numerical results are quite good, with only one noticeable deviation, at large $M^2$, a region where the discrete spectrum is sparse. Figure 2 shows the free energy at particular values of the Yang–Mills coupling.

![Fig. 1. Cumulative distribution function for resolution $K = 13$ in the analytically solvable case of zero Yang–Mills coupling. The numerical and analytic solutions are compared.](image)

6. Future Work

Additional work is in progress to complete this study of finite temperature properties of two-dimensional SQCD. Beyond this particular effort, one can consider finite-$N_c$ effects, with baryons and mixing of mesons and glueballs, and the full three-dimensional theory. As these techniques mature, analysis of four-dimensional theories can be considered.
Fig. 2. Free energy at fixed Yang–Mills coupling $g$ as a function of temperature $T$ for resolution $K = 14$.

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