Lifting degenerate simplices with a single volume constraint

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Abstract

Let $M^d$ be the spherical, Euclidean, or hyperbolic space of dimension $d \geq n + 1$. Given any degenerate $(n+1)$-simplex $A$ in $M^d$ with non-degenerate $n$-faces $F_i$, there is a natural partition of the set of $n$-faces into two subsets $X_1$ and $X_2$ such that $\sum_{X_1} V_n(F_i) = \sum_{X_2} V_n(F_i)$, except for a special spherical case where $X_2$ is the empty set and $\sum_{X_1} V_n(F_i) = V_n(S^n)$ instead. For all cases, if the vertices vary smoothly in $M^d$ with a single volume constraint that $\sum_{X_1} V_n(F_i) - \sum_{X_2} V_n(F_i)$ is preserved as a constant (0 or $V_n(S^n)$), we prove that if a stress invariant $c_{n-1}(\alpha_{n-1})$ of the degenerate simplex is non-zero, then the vertices will be confined to a lower dimensional $M^n$ for any sufficiently small motion. This answers a question of the author and we also show that in the Euclidean case, $c_{n-1}(\alpha_{n-1}) = 0$ is equivalent to the vertices of a dual degenerate $(n+1)$-simplex lying on an $(n-1)$-sphere in $\mathbb{R}^n$.

1 Introduction

Let $M^d$ of dimension $d \geq n + 1$ be the spherical, Euclidean, or hyperbolic space of constant curvature $\kappa$, and $A$ be a degenerate $(n+1)$-dimensional simplex in $M^d$ with non-degenerate $n$-faces. By degenerate we mean that the vertices $\{A_1, \ldots, A_{n+2}\}$ of $A$ are confined to a lower dimensional $M^n$. Due to the degeneracy, the convex hull of the vertices in $M^d$ is an $n$-dimensional region in $M^n$. The $n$-faces $F_i$ of $A$ form a double covering of this region with a natural partition of the set of $n$-faces into two subsets $X_1$ and $X_2$ such that $\sum_{X_1} V_n(F_i) = \sum_{X_2} V_n(F_i)$; except for a trivial exception in the spherical case when the vertices are not confined to any open half sphere, then in this case $X_2$ is the empty set and $\sum_{X_1} V_n(F_i) = V_n(S^n)$ instead. The partition can also be viewed as induced by Radon's theorem.

For all cases, if the vertices of $A$ vary smoothly and are confined to $M^n$, then obviously we have that $\sum_{X_1} V_n(F_i) - \sum_{X_2} V_n(F_i)$ (here $V_n(F_i)$ is short for $V_n(F_i(t))$ when the context is clear) is preserved as a constant (0 or $V_n(S^n)$) for any small motion. But what about the inverse? Inspired by earlier work of the author [14], we ask the following question:

**Question 1.1.** If the vertices of $A$ vary smoothly in $M^d$ with a single volume constraint that $\sum_{X_1} V_n(F_i) - \sum_{X_2} V_n(F_i)$ is preserved as a constant (0 or $V_n(S^n)$), then does this constraint confine the vertices of $A$ to a lower dimensional $M^n$ for any sufficiently small motion?

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In this paper we always require the $n$-faces to be non-degenerate during any motion. We provide two opposing views on the question. On the one hand, notice that up to congruence the degrees of freedom of $\mathbf{A}$ in $M^d$ is $(n+2)(n+1)/2$ (the number of edges), or subtract 1 if the motion of $\mathbf{A}$ is confined to $M^n$. So under a simple view of the excess degrees of freedom, with just a single volume constraint as that of Question 1.1 except for some extreme cases it is hardly expected that the answer might be affirmative. But on the other hand, if we treat all $(n+1)$-simplices up to congruence as points in an $(n+2)(n+1)/2$-dimensional manifold and the degenerate $(n+1)$-simplices as the boundary of the manifold, then in the manifold the region that satisfies $\sum F_i V_n(F_i) - \sum G_i V_n(G_i) = 0$ (or $V_n(S^n)$) is a codimension one region of the manifold and coincides with the boundary partially. Then under this view the answer to Question 1.1 is expected to be affirmative for “almost all” configurations, and the focus shifts to finding when $\mathbf{A}$ can be lifted to form a non-degenerate simplex.

1.1 Main results

In [14] we obtained a sequence of stress invariants $c_0(\alpha^0), \ldots, c_{n+1}(\alpha^{n+1})$ for $\mathbf{A}$, where $\alpha$ is a 1-stress on $\mathbf{A}$, and $\alpha^k$ is induced as a $k$-stress on $\mathbf{A}$. We denote the pair by $(\mathbf{A}, \alpha)$ and reserve the notation for the rest of this paper. The notion of $k$-stress on simplicial complexes was first introduced by Lee [5] (see also Rybnikov [9] and Tay et al. [13]). To answer Question 1.1 we have the following theorem where $c_{n-1}(\alpha^{n-1})$ plays a central role.

**Theorem 1.2.** (Main Theorem 1) If $\mathbf{A}$ varies smoothly in $M^d$ with a single volume constraint that $\sum_{F_i} V_n(F_i) - \sum_{G_i} V_n(G_i)$ is preserved as a constant ($0$ or $V_n(S^n)$) and $c_{n-1}(\alpha^{n-1}) \neq 0$, then the vertices are confined to a lower dimensional $M^n$ for any sufficiently small motion.

To provide a simple geometric interpretation of $c_{n-1}(\alpha^{n-1}) = 0$ for the Euclidean case (but not in general for the non-Euclidean case), we introduce the following notion. For $n \geq 2$, with a restriction of $\mathbf{A}$ to $\mathbb{R}^n$, a degenerate $(n+1)$-simplex $\mathbf{B}$ in $\mathbb{R}^n$ with vertices $\{B_1, \ldots, B_{n+2}\}$ is called a dual of $\mathbf{A}$, if it satisfies $A_i A_j \cdot B_k B_l = 0$ for all distinct $i, j, k, l$. We can show that such $\mathbf{B}$ always exists and is unique up to similarity.

Similarly to $\mathbf{A}$, we obtain a sequence of stress invariants $c_0(\beta^0), \ldots, c_{n+1}(\beta^{n+1})$ for $\mathbf{B}$, where $\beta$ is a 1-stress on $\mathbf{B}$, and $\beta^k$ is induced as a $k$-stress on $\mathbf{B}$. We will show that numerically we can set $\beta = \alpha$. Then we have the following result.

**Theorem 1.3.** (Main Theorem 2) If $\mathbf{B}$ is a dual of $\mathbf{A}$ in $\mathbb{R}^n$, then $c_{n-1}(\alpha^{n-1}) = 0$ if and only if $c_1(\beta^1) = 0$, which is also equivalent to the vertices of $\mathbf{B}$ lying on an $(n-1)$-dimensional sphere in $\mathbb{R}^n$.

The question remains as to what can be said about the non-Euclidean case. As a consequence of Theorem 1.3 here we give a quick example. For $n = 2$, $c_1(\alpha^1) = 0$ and $c_1(\beta^1) = 0$ coincide, so by Theorem 1.2 and 1.3 it means that in order for four points to be lifted from $\mathbb{R}^2$ to form a non-degenerate 3-simplex in $\mathbb{R}^3$ while preserving $\sum_{F_i} V_2(F_i) = 1$ One simple example is in the 2-dimensional plane: if $A_1, A_2, A_3$ are the vertices of a non-right triangle and $A_4$ is the orthocenter of the triangle, then $\mathbf{A}$ is dual to itself. This notion of dual is induced from a more conventional notion of dual of a convex polytope in $\mathbb{R}^n$, where we leave the details to Section 7.1.
\[ \sum_{X_2} V_2(F_i) \] during the motion, they have to move on to a common circle in \( \mathbb{R}^2 \) first before being lifted from \( \mathbb{R}^2 \). See also Example 7.8.

We introduce a notion of characteristic polynomial of \((A, \alpha)\) by defining

\[
f(x) = \sum_{i=0}^{n+1} (-1)^i c_i x^{n+1-i}.
\]

For the Euclidean case, by [14, Theorem 3.4] \( f(x) \) has one zero and \( n \) non-zero real roots. Similarly we let \( g(x) \) be the characteristic polynomial of \((B, \beta)\). The following result shows a duality between \( f(x) \) and \( g(x) \).

**Theorem 1.4.** (Main Theorem 3) If \( B \) is a dual of \( A \) in \( \mathbb{R}^n \) and the non-zero roots of \( f(x) \) are \( \{\lambda_1, \ldots, \lambda_n\} \), then the non-zero roots of \( g(x) \) are \( \{c/\lambda_1, \ldots, c/\lambda_n\} \) for some constant \( c \).

Theorem 1.4 proves a main part of Theorem 1.3, namely \( c_{n-1}(\alpha^{n-1}) = 0 \) if and only if \( c_1(\beta^i) = 0 \). But Theorem 1.4 is a more general result and is of interest in its own right. In fact, it shows that for any \( i \) with \( 0 \leq i \leq n \), \( c_{n-i}(\alpha^{n-i}) = 0 \) if and only if \( c_i(\beta^i) = 0 \).

For the Euclidean case, combining Theorem 1.2 and 1.3 together, we provide a different formulation to answer Question 1.1 with a slightly stronger statement that includes continuous motion as well. Notice that for the Euclidean case the initial value of \( \sum_{X_1} V_n(F_i) - \sum_{X_2} V_n(F_i) \) is always 0.

**Theorem 1.5.** Let \( B \) be a dual of \( A \) in \( \mathbb{R}^n \) and assume the vertices of \( B \) not lying on an \((n - 1)\)-dimensional sphere in \( \mathbb{R}^n \). If \( A \) varies continuously in \( \mathbb{R}^d \) with a single volume constraint that \( \sum_{X_1} V_n(F_i) - \sum_{X_2} V_n(F_i) \) is preserved as 0, then the vertices of \( A \) are confined to a lower dimensional \( \mathbb{R}^n \) for any sufficiently small motion.

While Theorem 1.5 is simply a statement that combines Theorem 1.2 and 1.3 together, somewhat surprisingly, an alternative elementary proof of Theorem 1.5 bypasses both Theorem 1.2 and 1.3 without using \( k \)-stress, and is also valid for continuous motion as well. Thus, combined with Theorem 1.3, it also makes Theorem 1.2 valid for continuous motion for the Euclidean case without using any advanced tools like algebraic geometry. The interested reader may directly read this proof in Section 8 without reading earlier sections. But for the non-Euclidean case we do not have a similarly simple answer, thus making \( k \)-stress still the main tool to solve Question 1.1.

In this paper we use many results developed in [14], e.g., the deriving of the stress invariant \( c_{n-1}(\alpha^{n-1}) \), or more generally \( c_k(\alpha^k) \) for \( 0 \leq k \leq n+1 \). While we shall not repeat all the proofs, we introduce the necessary notions and strive to make this note self-contained and readable independently of [14].

### 1.2 Background and motivations

A flexible polyhedron is a closed polyhedral surface in \( M^n \) that admits continuous non-rigid deformations such that all faces remain rigid. The first example of an embedded flexible polyhedron in \( \mathbb{R}^3 \) was discovered by Connelly [2], and the volume of all flexible

\(^2\)But in this paper we do not need to use the property that the roots are real.
polyhedra in \( \mathbb{R}^3 \) was shown to remain constant during the continuous deformation, proving the bellows conjecture formulated by Connelly and D. Sullivan (see [3, 10]). However, in the spherical case, Alexandrov [1] constructed a flexible polyhedron in an open half sphere \( S^3_+ \) that does not conserve the volume. Thus the validity of the bellows conjecture depends on the curvature value of the underlying space.

If we allow the faces of a polyhedron to be non-rigid but applying volume constraints on the faces instead, then in Question 1.1 the degenerate simplex \( A \) can be loosely treated as a variant of flexible degenerate simplex. But unlike the bellows conjecture, the volume rigidity result of Theorem 1.2 does not depend on the constant curvature value of the underlying space.

Under a similar setting as that of Question 1.1, we proved a weaker version of Theorem 1.2 in [14, Theorem 1.4] with a slight reformulation below.

**Theorem 1.6.** If \( A \) varies smoothly in \( M^d \) with \( n + 2 \) volume constraints that \( V_n(F_i) \) is preserved as a constant for all \( n \)-faces \( F_i \) and \( c_{n-1}(\alpha^{n-1}) \neq 0 \), then the vertices are confined to a lower dimensional \( M^n \) for any sufficiently small motion.

As a special case of Theorem 1.2, though Theorem 1.6 has \( n + 2 \) volume constraints and thus is weaker than Theorem 1.2, it is a somewhat surprising result itself, because \( n + 2 \) is still far less than the degrees of freedom of \( A \) in \( M^d \) up to congruence. In fact, after Theorem 1.6 was proved, it was gradually realized that the degrees of freedom of \( A \) may not be the main barrier for this particular setting. Combined with the easy fact of \( \sum X_1 V_n(F_i) = \sum X_2 V_n(F_i) \) (or \( \sum X_1 V_n(F_i) = V_n(S^n) \) for a special spherical case), this inspired us to move a step further and ask Question 1.1 which lead to the formulation of Theorem 1.2.

Our main tools to prove Theorem 1.2 are two results we developed in [14], Theorem 4.3 and Lemma 5.1. In Theorem 4.3 for any \( k \)-stress \( \omega \) on a cell complex in \( M^d \), we discovered a geometric invariant \( c_k(\omega) \) associated with \( \omega \), and \( c_{n-1}(\alpha^{n-1}) \) is obtained as a special case. Lemma 5.1 is a technical result that provides a crucial estimate of the volume differential of the \( n \)-faces.

To some extent, this seemingly simple volume constraint on \( A \) that \( \sum X_1 V_n(F_i) - \sum X_2 V_n(F_i) \) is preserved as a constant is a rigidity property under disguise.

## 2 Classification of degenerate simplices

We first classify the degenerate simplices based on the size of \( X_1 \) and \( X_2 \) (see Question 1.1).

**case 0.** One subset (say, \( X_2 \)) is the empty set, which can only happen in the spherical case when the vertices are not confined to any open half sphere. This is also the only case that \( \sum X_1 V_n(F_i) = V_n(S^n) \). All other cases have \( \sum X_1 V_n(F_i) = \sum X_2 V_n(F_i) \).

**case 1.** One subset (say, \( X_2 \)) contains exactly one \( n \)-face, which happens when one vertex \( A_i \) falls in the convex hull of the remaining vertices of \( A \) in \( M^d \).

**case 2.** For all the remaining cases, where \( X_1 \) and \( X_2 \) each contains at least two \( n \)-faces.
To prove Theorem 1.2, we provide some necessary background in Section 3–5. But for case 0 and case 1 above, those background is not needed and the proof is elementary. While our general proof of Theorem 1.2 covers all cases, to illustrate the theorem in a simple setting, we provide this elementary proof first.

The proof of Theorem 1.2 for case 1 is trivial: Given any non-degenerate \((n + 1)\)-simplex in \(M^d\) with \(n\)-faces \(F_i\), then for any \(n\)-face \(F_j\) we have \(\sum_{i\neq j} V_n(F_i) > V_n(F_j)\), which immediately proves case 1. Notice that this is a global property.

To prove case 0, it suffices to prove the following result.

**Theorem 2.1.** For any non-degenerate \((n + 1)\)-simplex \(B\) in \(S^d\) with \(n\)-faces \(F_i\), we have \(\sum V_n(F_i) < V_n(S^n)\).

**Proof.** Let the vertices be \(B_1, \ldots, B_{n+2}\), and \(F_i\) be the \(n\)-face that contains all the vertices except \(B_i\). Now let \(B'_{n+2}\) be the antipodal point of \(B_{n+2}\), and for \(i < n + 2\) denote by \(F'_i\) the \(n\)-face formed by \(F_i\) with vertex \(B_{n+2}\) replaced by \(B'_{n+2}\).

For \(i < n + 2\), let \(C_i\) be the midpoint of the half circle with end points \(B_{n+2}\) and \(B'_{n+2}\) and crossing \(B_i\). As \(B\) is non-degenerate, then \(C_1, \ldots, C_{n+1}\) form a non-degenerate \(n\)-dimensional simplex. Denote by \(G_i\) the \((n - 1)\)-face formed by all the vertices \(C_1, \ldots, C_{n+1}\) except \(C_i\).

If we treat \(B_{n+2}\) and \(B'_{n+2}\) as the north and south pole, then for \(i < n + 2\), the \(n\)-dimensional region formed by the union of \(F_i\) and \(F'_i\) can be cut into two regions by the upper and lower hemispheres, with the upper region as the join of \(G_i\) with \(B_{n+2}\), and the lower region as the join of \(G_i\) with \(B'_{n+2}\). Thus

\[
V_n(F_i) + V_n(F'_i) = c \cdot V_{n-1}(G_i),
\]

where the constant \(c\) is \(V_n(S^n) / V_{n-1}(S^{n-1})\). By an induction on \(n\) we have \(\sum V_{n-1}(G_i) \leq V_{n-1}(S^{n-1})\), where the equality holds only when \(n = 1\) (but we do not need to use the strict inequality here), thus

\[
\sum_{i < n+2} (V_n(F_i) + V_n(F'_i)) = c \cdot \sum V_{n-1}(G_i) \leq c \cdot V_{n-1}(S^{n-1}) = V_n(S^n). \tag{2.1}
\]

As \(B'_{n+2}\) and \(B_1, \ldots, B_{n+1}\) form a non-degenerate \((n + 1)\)-simplex, therefore

\[
\sum_{i < n+2} V_n(F'_i) > V_n(F_{n+2}).
\]

Plug it in (2.1), then

\[
V_n(S^n) > V_n(F_{n+2}) + \sum_{i < n+2} V_n(F_i) = \sum V_n(F_i),
\]

which finishes the proof. \(\Box\)

Notice that Theorem 2.1 is a global property as well. However both case 0 and case 1 seem more like isolated extreme cases, and the proof above does not indicate how to prove case 2. In fact we can show that, unlike case 0 and case 1 where \(c_{n-1}(\alpha^{n-1})\) is always non-zero, for case 2 \(c_{n-1}(\alpha^{n-1})\) can take values positive, negative or zero as well, and its sign depends not only on \(X_1\) and \(X_2\) but also on the geometric shape of \(A\). In other words, case 2 is more complicated.

We next provide the background that is needed to prove case 2.
3 Basic notions

As the linearity between points in the hyperbolic space $\mathbb{H}^d$ plays an important role in this paper, we use the hyperboloid model to describe $\mathbb{H}^d$ throughout the paper. Let $\mathbb{R}^{d,1}$ be a $(d + 1)$-dimensional vector space endowed with a metric $x \cdot x = -x_0^2 + x_1^2 + \cdots + x_d^2$, then $\mathbb{H}^d$ is defined by

$$\{x \in \mathbb{R}^{d,1} : x \cdot x = -1, \ x_0 > 0\},$$

which is the upper sheet of a two-sheeted hyperboloid. Also let the spherical space $\mathbb{S}^d$ be the standard unit sphere centered at the origin in $\mathbb{R}^{d+1}$.

As it is assumed that all $n$-faces of $A$ are non-degenerate, so up to a constant factor, there is an unique sequence of non-zero coefficients $\alpha_1, \ldots, \alpha_{n+2} \in \mathbb{R}$, such that

$$\sum \alpha_i A_i = 0 \quad \text{and} \quad \sum \alpha_i A_i = 0 \quad (\text{affine dependence for } \mathbb{R}^d),$$

$$\sum \alpha_i A_i = 0 \quad (\text{linear dependence for } \mathbb{S}^d \text{ or } \mathbb{H}^d). \quad (3.1)$$

We call $\alpha := \{\alpha_1, \ldots, \alpha_{n+2}\}$ a 1-stress on $A$, and denote the pair by $(A, \alpha)$ and reserve the notation for the rest of this paper.

3.1 $k$-stress on cell complex

The notion of $k$-stress plays an important role in our results. While in this paper we are only concerned with $k$-stresses on the boundary complex of a degenerate simplex in $M^d$, we introduce the notion in the general setting on cell complexes (not necessarily simplicial) in $M^d$.

By a $k$-dimensional convex polytope in $M^k$ we mean a compact subset which can be expressed as a finite intersection of closed half spaces. In the spherical case for convenience we also require the convex polytope to be confined to an open half sphere, so a half circle or $S^0$ is not considered as a convex polytope in this context. A cell complex in $M^d$ is a finite set of convex polytopes (called cells) in $M^d$, such that every face (empty set included) of a cell is also a cell in the set, and any two cells share a unique maximal common face. However we do not worry about overlapping or intersection between cells in $M^d$ caused by the embedding.

If $K$ is a cell complex in $M^d$, for convenience we denote by $K$ as well the set of all its cells, and by $K^r$ the subset of its $r$-cells.

**Definition 3.1.** Consider a cell complex $K$ (not necessarily of dimension $d - 1$ or $d$) in $M^d$. A $k$-stress ($2 \leq k \leq d + 1$) on $K$ is a real-valued function $\omega$ on the $(k - 1)$-cells of $K$, such that for each $(k - 2)$-cell $F$ of $K$,

$$\sum_{G \in K^{k-1}, F \subset G} \omega(G) u_{F,G} = 0,$$

where the sum is taken over all $(k - 1)$-cells $G$ of $K$ that contain $F$, and $u_{F,G}$ is the inward unit normal to $G$ at its facet $F$. For $k = 1$, a 1-stress is an affine dependence among the vertices for the Euclidean case, or a linear dependence for the non-Euclidean case.
The notion of (affine and linear) \(k\)-stresses was first introduced by Lee [5] on simplicial complexes with vertices chosen in the Euclidean space. The notion was introduced partly under the inspiration of Kalai’s proof [4] of the Lower Bound Theorem which used classical stress. McMullen [6] also considered weights on simple polytopes, a notion dual to \(k\)-stresses. Both \(k\)-stresses and weights were alternative approaches to proving the \(g\)-conjecture for simplicial convex polytopes, whose original proof of the necessity part by Stanley [11] used deep techniques from algebraic geometry. Lee [5] showed that the \(g\)-conjecture for simplicial spheres, which remains open, can be proved true if one can show that for a simplicial \((d-1)\)-sphere \(\Delta\) with vertices chosen generically in \(\mathbb{R}^d\), the dimension of the space of affine \(k\)-stresses on \(\Delta\) is \(g_k\) for \(k \leq \lfloor d/2 \rfloor\), where \((g_0, g_1, \ldots)\) is the \(g\)-vector of \(\Delta\). See, e.g., [11] for the definition of \(g_k\).

Rybnikov [9] provided a geometric variation of the notion of (affine) \(k\)-stress, extending it to cell-complexes in both Euclidean and spherical spaces. Our notion agrees with this notion.

### 3.2 \(k\)-stress on \(A\)

If \(F\) is a \(k\)-simplex in \(\mathbb{S}^d\) or \(\mathbb{H}^d\) (which is embedded in \(\mathbb{R}^{d+1}\) or \(\mathbb{R}^{d,1}\) respectively) and \(B_1, \ldots, B_{k+1}\) are the vertices, for convenience we introduce a new notation

\[ ||F|| := |\det(B_i \cdot B_j)_{1 \leq i, j \leq k+1}|^{1/2}. \]

For the spherical case it is \((k+1)!\) times the volume of the Euclidean \((k+1)\)-simplex whose vertices are \(O, B_1, \ldots, B_{k+1}\), and for the hyperbolic case the pseudo-volume.

**Definition 3.2.** Let \((A, \alpha)\) be as in (3.1) where \(\alpha\) is a 1-stress on \(A\). For a given \(k\) \((0 \leq k \leq n)\) and each simplicial \(k\)-face \(F\) of \(A\), define a \((k+1)\)-stress \(\alpha^{k+1}\) on \(A\) by \(\alpha^{k+1}(F) := \prod_{\alpha \in F} \alpha_k! V_k(F)\) for the Euclidean case, and \(\alpha^{k+1}(F) := (\prod_{\alpha \in F} \alpha_s)! ||F||\) for the non-Euclidean case.

**Remark 3.3.** For notational reasons that due to the slight difference between Lee’s and our notion of \((k+1)\)-stresses, we use \(\alpha^{k+1}\) to denote the \((k+1)\)-stress obtained by multiplying \(\alpha\) with itself for \(k+1\) times and then normalized by a volume factor, rather than taking the value of \(\prod_{\alpha \in F} \alpha_s\) directly. With the volume interpretation of \(||F||\) above, it is not hard to verify that \(\alpha^{k+1}\) is indeed a valid \((k+1)\)-stress on \(A\).

### 4 A geometric invariant of \(k\)-stress

As shown in Theorem [12] \(c_{n-1}(\alpha^{n-1})\) plays a central role in the answer to Question [14]. For completeness we provide the detail about how a geometric invariant \(c_k(\omega)\) is obtained for any \(k\)-stress \(\omega\) on a cell complex \(K\).

First consider a \(k\)-dimensional convex polytope \(F\) and any two points \(P\) and \(Q\) in \(M^d\) in general position with respect to \(F\), and denote by \(\hat{F}\) the \((k+2)\)-dimensional convex polytope in \(M^d\) which is the join of \(F\) with the segment \(PQ\) (e.g., if \(F\) is a \(k\)-simplex, then \(\hat{F}\) is a \((k+2)\)-simplex). Also let \(\theta_F\) be the dihedral angle of \(\hat{F}\) at face \(F\). If \(\hat{F}\) is non-degenerate, then \(\theta_F\) can vary in such a manner that the distances between any pair of vertices of \(\hat{F}\)
are fixed except between \(P\) and \(Q\). It follows that \(V_{k+2}(\hat{F})\) can be treated as a function of a single variable \(\theta_F\), and we write the differential as \(dV_{k+2}(\hat{F})/d\theta_F\). Some degeneracy is allowed and \(\hat{F}\) need not be a convex polytope in the strict sense, as long as \(V_{k+2}(\hat{F})\) and \(\theta_F\) can be properly defined.

We introduce the following definition.

**Definition 4.1.** Let \(F\) be a \(k\)-dimensional convex polytope in \(M^d\) and \(\hat{F}\), \(\theta_F\) be as above. If \(\theta_F\) varies while all edge lengths of \(\hat{F}\) are fixed except between \(P\) and \(Q\), then define \(g_F : M^d \times M^d \rightarrow \mathbb{R}\) by

\[
g_F(P, Q) := (k + 2)! \frac{dV_{k+2}(\hat{F})}{d\theta_F}.
\]

Also set \(g_B(P, Q) = 1\).

**Remark 4.2.** When \(F\) is a single point \(B\), it is not hard to verify that for the Euclidean case we have \(g_B(P, Q) = \|\hat{P}\hat{B}\| \cdot \|\hat{Q}\hat{B}\| \cdot \cos \theta_B = \hat{P}\hat{B} \cdot \hat{Q}\hat{B}\). For the non-Euclidean case by [14, Corollary 2.12]

\[
g_B(P, Q) = \frac{2}{1 + \kappa P} \cdot \hat{P}\hat{B} \cdot \hat{Q}\hat{B}.
\]

So \(g_B\) is also approximately the Riemannian metric at \(B\) as \(g_B \sim \frac{\hat{P}\hat{B} \cdot \hat{Q}\hat{B}}{\hat{P}\hat{B}}\) when \(P, Q \rightarrow B\), but \(g_B\) is defined globally on \(M^d\) instead of just locally on the tangent space at \(B\) as the standard Riemannian metric is. In fact \(g_B\) is also a positive definite kernel on \(\mathbb{H}^d\) for \(d = 1\) [14, Theorem 2.25], which to our knowledge is a new member to the family of known positive definite kernels, and we conjecture for \(d \geq 2\) as well.

Using the Schlafli differential formula as the main tool, we obtained a Schlafli differential formula for simplices based on edge lengths [14, Proposition 2.11] and used it to prove the following key result. See Milnor [7] for the description of the formula, see also Rivin and Schlenker [8] and Suárez-Peiró [12].

**Theorem 4.3.** (14 Theorem 2.13) Let \(K\) be a cell complex in \(M^d\) of constant curvature \(\kappa\) and \(\omega\) be a \((k+1)\)-stress on \(k\)-faces of \(K\) for \(k \geq 0\). Then as long as \(g_F(P, Q)\) is properly defined for each \(F \in K^k\), we have that

\[
c_{k+1}(\omega) := \sum_{F \in K^k} \omega(F) g_F(P, Q)
\]

is an invariant independent of the choice of points \(P, Q \in M^d\), and for the non-Euclidean case

\[
c_{k+1}(\omega) = \kappa (k + 2) k! \sum_{F \in K^k} \omega(F) V_k(F).
\]

Particularly for \((A, \alpha)\), we have the following definition.

**Definition 4.4.** Let \((A, \alpha)\) be as in \((3.1)\) where \(\alpha\) is a 1-stress on \(A\), and \(\alpha^{k+1}\) be the \((k+1)\)-stress on \(A\) as in Definition \((3.2)\). Then by Theorem \((4.3)\) we define a sequence of invariants \(c_1(\alpha^1), \ldots, c_{n+1}(\alpha^{n+1})\) for \((A, \alpha)\) (also set \(c_0(\alpha^0) = 1\)), and for the non-Euclidean case

\[
c_{k+1}(\alpha^{k+1}) = \kappa (k + 2) k! \sum_{F \subseteq \text{dim}(F) = k} \left( \prod_{A_s \in F} \alpha_s \right) \|F\| V_k(F).
\]
Remark 4.5. For the non-Euclidean case, by (4.4), \(c_{n+1}(\alpha^{n+1})\) vanishes unless \(A\) is not confined to any open half sphere in the spherical case (case 0), and \(c_{n+1}(\alpha^{n+1})\) also vanishes for the Euclidean case as a limit of the spherical case. However for case 0, we can set \(\alpha\) such that \(\alpha_i > 0\) for all \(i\), then all \(c_{k+1}(\alpha^{k+1})\) are positive, including \(c_{n+1}(\alpha^{n+1})\).

5 A differential formula

Here we provide a differential formula in Lemma 5.1, a crucial estimate of the volume differential of the \(n\)-faces and an important step for proving Theorem 1.2. Denote by \(A(t)\) the smooth motion of \(A\) in \(M^d\), and \(A(0) = A\) the initial position.

Let \(A_0(t)\) in \(M^d\) be the mirror reflection of \(A_1(t)\) through a lower dimensional \(M^n\) that contains points \(A_2(t), \ldots, A_{n+2}(t)\). If \(A_0(t) \neq A_1(t)\), then \(A_0A_1\) (short for \(A_0(t)A_1(t)\)) is twice the altitude vector for \(A_1(t)\) with respect to the linear (resp. affine) span of \(A_i(t)\) of \(i \geq 2\) for the non-Euclidean (resp. Euclidean) case. It is not hard to see that if \(A(t)\) varies smoothly over \(t\), then \(A_0(t)\) varies smoothly as well, thus \(A_0A_1^2\) also varies smoothly.

For \(t \geq 0\), \(\alpha_i\) can be extended to a continuous function \(\alpha_i(t)\) with \(\alpha_i(0) = \alpha_i\) (and additionally \(\sum_{i \geq 1} \alpha_i(t) = 0\) for the Euclidean case), such that \(\sum_{i \geq 1} \alpha_i(t)A_i(t)\) is a multiple of \(A_0A_1^2\). Denote \(\{\alpha_1(t), \ldots, \alpha_{n+2}(t)\}\) by \(\alpha_t\). For a fixed \(t\), \(\alpha_t\) is unique up to a constant factor. We have the following formula.

Lemma 5.1. ([14] Proposition 2.19) Let \(A(t), \alpha_t, \text{ and } A_0(t)\) be as above. Assume \(A(t)\) varies smoothly for \(t \geq 0\) in \(M^d\). If \(c_{k-1}(\alpha^{k-1}) \neq 0\) and both \(A_0A_1^2\) and \((A_0A_1^2)\)' are strictly increasing for small \(t \geq 0\), then for the non-Euclidean case

\[
2 \cdot k! \sum_{G \subseteq A(t)} \left( \prod_{A_s(t) \in G} \alpha_s(t) \right) \left(\prod_{A_s(t) \in G} \alpha_s(t) \right) \|G\| dV_k(G) \sim -\frac{1}{4} \alpha_1^2 c_{k-1}(\alpha^{k-1}) dA_0A_1^2, \quad (5.1)
\]

and for the Euclidean case

\[
2 \cdot (k!)^2 \sum_{G \subseteq A(t)} \left( \prod_{A_s(t) \in G} \alpha_s(t) \right) V_k(G) dV_k(G) \sim -\frac{1}{4} \alpha_1^2 c_{k-1}(\alpha^{k-1}) dA_0A_1^2.
\]

Remark 5.2. Here the notation “\(\sim\)” means that if the two sides of the formula above are written as \(f_1(t)dt\) and \(f_2(t)dt\) instead, then \(f_1(t) - f_2(t) = o(f_2(t))\) as \(t \to 0\).

For the purpose of this paper, we only need the formula for case \(k = n\), and the proof of Theorem 1.2 essentially follows from Lemma 5.1.

6 Proof of Theorem 1.2

Theorem 6.1. (Theorem 1.2) If \(A\) varies smoothly in \(M^d\) with a single volume constraint that \(\sum_{X_1} V_n(F_1) - \sum_{X_2} V_n(F_1)\) is preserved as a constant (0 or \(V_n(S^n)\)) and \(c_{n-1}(\alpha^{n-1}) \neq 0\), then the vertices are confined to a lower dimensional \(M^n\) for any sufficiently small motion.
We use the same notations as in Section 5 including \( A(t) \), \( \alpha_i \), and \( A_0(t) \). Here we only provide the proof for the non-Euclidean case, as the Euclidean case can be treated similarly, and both as a consequence of Lemma 5.1.

**Proof.** As \( A(t) \) is smooth, both \( \overrightarrow{A_0A_1^2} \) and \( (\overrightarrow{A_0A_1^2})' = 2\overrightarrow{A_0A_1} \cdot (\overrightarrow{A_0A_1})' \) are 0 at \( t = 0 \). If the vertices are not confined to a lower dimensional \( M^n \) for some small motion, without loss of generality we assume that both \( \overrightarrow{A_0A_1^2} \) and \( (\overrightarrow{A_0A_1^2})' \) are strictly increasing for small \( t \geq 0 \).

Let \( \alpha_1(t) := \|F_1(t)\| \) where \( F_i(t) \) are the \( n \)-faces of \( A(t) \). As \( \alpha_i \) is unique up to a constant factor and now \( \alpha_i(t) \) is fixed, so \( \alpha_i \) is also fixed for all small \( t \geq 0 \). In fact, for \( i \geq 2 \) if \( \theta_i \) (short for \( \theta_i(t) \)) is the dihedral angle between \( n \)-faces \( F_i(t) \) and \( F_1(t) \), then \( \alpha_i(t) = -\|F_i(t)\| \cos \theta_i \). Notice that \( \theta_i \) is in the neighborhood of either 0 or \( \pi \). As a convention also set \( \theta_1 = 0 \) in the following.

In Lemma 5.1 on the left side of (6.1) take \( k = n \), factoring out \( 2 \cdot n! \prod_{i \geq 1} \alpha_i(t) \) and replacing \( G \) with an \( n \)-face \( F_i(t) \) of \( A(t) \), we have the coefficient of \( dV_n(F_i) \) as \( \|F_i(t)\|/\alpha_i(t) \). Namely for small \( t \geq 0 \),

\[
\sum \frac{\|F_i(t)\|}{\alpha_i(t)} dV_n(F_i) \sim c \cdot c_{n-1} (\alpha^{n-1} \cdot dA_0A_1^2) \tag{6.1}
\]

for a non-zero constant \( c \).

As mentioned above we have \( \alpha_i(t) = -\|F_i(t)\| \cos \theta_i \) (including \( i = 1 \) where \( \theta_1 = \pi \) and \( \alpha_1(t) = \|F_1(t)\| \)), thus

\[
\sum \frac{1}{\cos \theta_i} dV_n(F_i) \sim c \cdot c_{n-1} (\alpha^{n-1} \cdot dA_0A_1^2). \tag{6.2}
\]

As \( \overrightarrow{A_0A_1} \) is twice the altitude vector for \( A_1(t) \) with respect to the linear span of \( F_1(t) \), we have \( \sin^2 \theta_i = O(\overrightarrow{A_0A_1^2}) \) for all \( i \geq 1 \), and thus

\[(1 + \cos \theta_i)(1 - \cos \theta_i) = \sin^2 \theta_i = O(\overrightarrow{A_0A_1^2}).\]

For each \( i \geq 1 \), with a properly chosen sign of \( \pm 1 \) depending only on whether \( F_i \) is in \( X_1 \) or \( X_2 \), we have

\[
\cos \theta_i \pm 1 = O(\overrightarrow{A_0A_1^2}). \tag{6.3}
\]

Recall that at the beginning of the proof, we assume that both \( \overrightarrow{A_0A_1^2} \) and \( (\overrightarrow{A_0A_1^2})' \) are strictly increasing for small \( t \geq 0 \). Denote \( \overrightarrow{A_0A_1^2} \) by \( f_0(t) \), then

\[
f_0(t) = \int_0^t f_0'(t) dt \leq t \cdot f_0'(t).\]

Thus \( f_0(t)/f_0'(t) \to 0 \) as \( t \to 0 \). As the right side of (6.2) is in the order of \( f_0'(t)dt \) for small \( t > 0 \), so on the left side any change in the coefficients in the order of \( O(f_0(t)) \) can be ignored. Then on the left side of (6.2) replacing \( \cos \theta_i \) with a proper \( \pm 1 \) from (6.3), for small \( t \geq 0 \)

\[
\sum_{X_1} dV_n(F_i) - \sum_{X_2} dV_n(F_i) \sim c \cdot c_{n-1} (\alpha^{n-1} \cdot dA_0A_1^2). \tag{6.4}
\]
This contradicts the assumption that \( \sum_{X_1} V_n(F_i) - \sum_{X_2} V_n(F_i) \) is preserved as a constant. Thus the vertices are confined to a lower dimensional \( M^n \) for small \( t \geq 0 \), and this completes the proof. \( \square \)

When \( c_{n-1}(\alpha^{n-1}) \neq 0 \), (6.4) implies that — if we ignore the smoothness requirement for a moment — for any non-degenerate \((n + 1)\)-simplex in a small neighborhood of \( A \), we always have \( \sum_{X_1} V_n(F_i) \neq \sum_{X_2} V_n(F_i) \) (or \( \sum_{X_1} V_n(F_i) \neq V_n(S^n) \) for case 0), and the strict inequality is fixed as either “\( > \)” or “\( < \)” that only depends on the sign of \( c_{n-1}(\alpha^{n-1}) \).

Remark 6.2. For case 2, further computation shows that even with fixed \( X_1 \) and \( X_2 \), \( c_{n-1}(\alpha^{n-1}) \) can take values positive, negative or zero as well; and starting with any configuration with a non-zero \( c_{n-1}(\alpha^{n-1}) \), through degenerate \((n + 1)\)-simplices only, up to congruence it can deform to any configuration with the same \( X_1 \) and \( X_2 \) and a zero \( c_{n-1}(\alpha^{n-1}) \), and from there can be lifted to form a non-degenerate simplex. Thus Theorem 1.2 for case 2 is a local property, and cannot be strengthened by replacing the statement “for any sufficiently small motion” with “for any motion”.

7 Geometric interpretations

As shown in Theorem 1.2, \( c_{n-1}(\alpha^{n-1}) = 0 \) is the critical position that \( A \) may be lifted from \( M^n \) to form a non-degenerate \((n + 1)\)-simplex. The main purpose of this section is to prove Theorem 1.3 and 1.4, which provide a simple geometric interpretation of \( c_{n-1}(\alpha^{n-1}) = 0 \) for the Euclidean case. The main idea is to use matrix theory to prove Theorem 1.2 first, and then prove Theorem 1.3 next. For the non-Euclidean case, while an explicit formula for \( c_{n-1}(\alpha^{n-1}) \) is provided in (4.4), we lack a nice geometric interpretation of \( c_{n-1}(\alpha^{n-1}) = 0 \) except for \( n = 2 \).

We first revisit some notions.

7.1 Dual of \( A \) in \( \mathbb{R}^n \)

For \( n \geq 2 \), with a restriction of \( A \) to \( \mathbb{R}^n \), a degenerate \((n + 1)\)-simplex \( B \) in \( \mathbb{R}^n \) with vertices \( \{B_1, \ldots, B_{n+2}\} \) is called a dual of \( A \), if it satisfies \( \overrightarrow{A_iA_j} \cdot \overrightarrow{B_kB_l} = 0 \) for all distinct \( i, j, k, l \).

In the following we show that such \( B \) always exists and is unique up to similarity.

Without loss of generality, let \( A_{n+2} \) be the origin \( O \) in \( \mathbb{R}^n \), and \( F_i \) be the \( n \)-face of \( A \) that does not contain the vertex \( A_i \). Fix a non-zero constant \( c \). For any \( i \leq n + 1 \), denote by \( G_i \) the \((n - 1)\)-face \( F_{n+2} \setminus \{A_i\} \). Then there is a unique point \( B_i \) in \( \mathbb{R}^n \), such that \( \overrightarrow{OB_i} \) is perpendicular to \( G_i \), and for any \( j \leq n + 1 \) with \( j \neq i \), we have \( \overrightarrow{OB_i} \cdot \overrightarrow{OA_j} = c \). Finally let \( B_{n+2} = O \) and \( B \) be a degenerate \((n + 1)\)-simplex in \( \mathbb{R}^n \) with vertices \( \{B_1, \ldots, B_{n+2}\} \). Then

\[
\overrightarrow{B_{n+2}B_i} \cdot \overrightarrow{A_{n+2}A_j} = c.
\]

(7.1)

If \( i, j, k \) and \( n + 2 \) are distinct, in (7.1) replace \( j \) with \( k \) and subtract from it, then \( \overrightarrow{B_{n+2}B_i} \cdot \overrightarrow{A_jA_k} = 0 \). Similarly for distinct \( i, j, k, l \), we have \( \overrightarrow{B_iB_l} \cdot \overrightarrow{A_jA_k} = 0 \), which verifies that \( B \) is a dual of \( A \). By the construction of \( B \), it is not hard to observe that \( B \) is also unique up to similarity.
If we denote by $E_i$ the $n$-face of $B$ that does not contain the vertex $B_i$, then as a non-degenerate simplex, $E_{n+2}$ is a dual\(^3\) of $F_{n+2}$ in $\mathbb{R}^n$ with respect to the origin $O$ (which is also $A_{n+2}$ and $B_{n+2}$ the same time), a notion that in fact induces the notion of dual of a degenerate simplex in $\mathbb{R}^n$.

### 7.2 Properties of the characteristic polynomial

Recall that the characteristic polynomial of $(A, \alpha)$ is defined by

$$f(x) = \sum_{i=0}^{n+1} (-1)^i c_i(\alpha^i)x^{n+1-i}.$$ 

For the Euclidean case, we showed in [14, Theorem 3.4] that $f(x)$ has one zero and $n$ non-zero real roots, thus $c_{n+1}(\alpha^{n+1})$ is always 0. But $c_{n+1}(\alpha^{n+1})$ is non-zero for a special spherical case (See Remark [4.5]), so for the generality of $f(x)$, we keep $(-1)^{n+1}c_{n+1}(\alpha^{n+1})$ as the constant term of $f(x)$.

For the rest of this section we consider the Euclidean case only.

For a $k$-simplex $F$ and two points $P$ and $Q$ in $\mathbb{R}^d$, for convenience of computation, we introduce a new notation $d_F(P, Q)$.

**Definition 7.1.** For a $k$-simplex $F$ in $\mathbb{R}^d$, define $d_F(P, Q)$ by $k!V_k(F)g_F(P, Q)$, where $g_F(P, Q)$ is defined in Definition 4.1. Also set $d_F(P, Q) = 1$.

**Remark 7.2.** Unlike the definition of $g_F$ where $F$ need be non-degenerate, here $d_F$ is well defined when $F$ is degenerate, and $P$ and $Q$ can be any points as well. See below.

If the vertices of $F$ are $P_1, \ldots, P_{k+1}$, then by [14, (3.2), (2.5)] we have

$$d_F(P, Q) = \det(PP_i \cdot QQ_j)_{1 \leq i, j \leq k+1}.$$ (7.2)

By Theorem 4.3 (4.2) and Definition 3.2, for any choice of $P$ and $Q$, we have

$$c_{k+1}(\alpha^{k+1}) = \sum_{F \subseteq A, \dim(F) = k} \alpha^{k+1}(F)g_F(P, Q)$$

$$= \sum_{F \subseteq A, \dim(F) = k} \left( \prod_{A_s \subseteq F} \alpha_s \right) k!V_k(F)g_F(P, Q)$$

$$= \sum_{F \subseteq A, \dim(F) = k} \left( \prod_{A_s \subseteq F} \alpha_s \right) d_F(P, Q),$$

and thus

$$c_{k+1}(\alpha^{k+1}) = \sum_{F \subseteq A, \dim(F) = k} \left( \prod_{A_s \subseteq F} \alpha_s \right) \det(\overrightarrow{PA_s} \cdot \overrightarrow{QA_t})_{A_s, A_t \subseteq F}. \quad (7.3)$$

---

\(^3\)This more conventional notion of dual is defined on any convex polytope $P$ in $\mathbb{R}^n$ by the following construction: $P^* = \{y \in \mathbb{R}^n : x \cdot y \leq c \text{ for all } x \in P\}$ with $c > 0$ and a requirement that $P$ contains the origin $O$ in its interior. But for a non-degenerate simplex in $\mathbb{R}^n$, if we are only concerned with the location of the vertices and not its interior, then it can get away with this requirement (that $P$ contains the origin $O$) as long as the origin $O$ is not on any hyperplane that contains an $(n-1)$-face.
Without loss of generality, we use the coordinate of $\mathbb{R}^n$ for $A$ in the following. Let $C_1$ be an $n \times n$ matrix whose $i$-th row is vector $\overrightarrow{A_{n+1}A_i}$ for $i \leq n$, $C_2$ be an $n \times n$ matrix whose $i$-th row is vector $A_{n+2}A_i$, and $D_1 = \text{diag}(\alpha_1, \ldots, \alpha_n)$ be a diagonal matrix.

**Lemma 7.3.** The characteristic polynomials of both matrix $C_1C_2^T D_1$ and $C_2^TD_1C_1$ are $f(x)/x$.

*Proof.* The coefficient of $x^{n-k}$ in the characteristic polynomial of $C_1C_2^T D_1$ is $(-1)^k$ times the sum of all principal minors of $C_1C_2^T D_1$ of order $k$, which can be shown to be $(-1)^k c_k(\alpha^k)$ by choosing $P = A_{n+1}$ and $Q = A_{n+2}$ in (7.3). Also using the fact that $c_{n+1}(\alpha^{n+1}) = 0$, thus $f(x)/x$ is the characteristic polynomial of $C_1C_2^T D_1$. As $C_1$ and $C_2^T D_1$ are two square matrices, then the characteristic polynomials of $C_1C_2^T D_1$ and $C_2^TD_1C_1$ coincide. Thus $f(x)/x$ is also the characteristic polynomial of $C_2^TD_1C_1$. \hfill $\square$

### 7.3 Proof of Theorem 1.4

We continue to use the same notations from Section 7.2.

Recall that if $B$ is a dual of $A$ in $\mathbb{R}^n$, then similarly to $A$, we obtain a sequence of invariants $c_0(\beta^0), \ldots, c_{n+1}(\beta^{n+1})$ for $B$, where $\beta$ is a 1-stress on $B$, and $\beta^k$ is induced as a $k$-stress on $B$. The characteristic polynomial of $(B, \beta)$ is similarly defined by

$$g(x) = \sum_{i=0}^{n+1} (-1)^i c_i(\beta^i)x^{n+1-i}.$$  

Notice that $c_{n+1}(\beta^{n+1}) = 0$ as well.

Let $E_1$ be an $n \times n$ matrix whose $i$-th row is vector $\overrightarrow{B_{n+1}B_i}$ for $i \leq n$, $E_2$ be an $n \times n$ matrix whose $i$-th row is vector $\overrightarrow{B_{n+2}B_i}$, and $D_2 = \text{diag}(\beta_1, \ldots, \beta_n)$ be a diagonal matrix. Then similarly to $f(x)$ (Lemma 7.3), for $g(x)$ we have

**Lemma 7.4.** The characteristic polynomials of both matrix $E_1E_2^T D_2$ and $E_2^TD_2E_1$ are $g(x)/x$.

For an $n \times n$ matrix $A$ with non-zero determinant, the eigenvalues of the inverse matrix $A^{-1}$ are the same as the inverse of the eigenvalues of $A$. So by Lemma 7.3 and 7.4, if we can show that the product of $C_2^TD_1C_1$ and $E_2^TD_2E_1$ is a multiple of the identity matrix $I_n$, then we prove Theorem 1.4. This is what we plan to do next.

As $B$ is a dual of $A$, then $A_iA_j \cdot \overrightarrow{B_kB_l} = 0$ for all distinct $i, j, k, l$. Then

$$A_iA_j \cdot \overrightarrow{B_kB_l} = A_i^T \overrightarrow{A_jB_k}.$$  

So for a fixed $i$, $A_iA_j \cdot \overrightarrow{B_kB_l}$ is independent of $j$ and $k$ as long as $i, j, k$ are distinct. We denote it by $r_i$. We will show that $\alpha_i r_i$ is independent of $i$.

Now consider the product of $\sum_{i=1}^{n+1} \frac{1}{r_i} A_{n+2}A_i$ and $\overrightarrow{B_{n+1}B_j}$ for a fixed $j$ with $j \leq n$. There are only two non-zero terms left, one is $\frac{1}{r_j} A_{n+2}A_j \cdot \overrightarrow{B_{n+1}B_j}$ that is equal to $\frac{1}{r_j} = 1$, and the other is $\frac{1}{r_{n+1}} A_{n+2}A_{n+1} \cdot \overrightarrow{B_{n+1}B_j}$ that is equal to $\frac{-1}{r_{n+1}} = -1$. So they cancel each other out. As $\overrightarrow{B_{n+1}B_j}$ with $j \leq n$ are $n$ linearly independent vectors in $\mathbb{R}^n$, so $\sum_{i=1}^{n+1} \frac{1}{r_i} A_{n+2}A_i$
must be 0. As \( \sum_{i=1}^{n+1} \alpha_i \cdot \overrightarrow{A_{n+i}A_i} = 0 \) and the coefficients are unique up to a constant factor, so \( 1/r_i \) is proportional to \( \alpha_i \) and thus \( \alpha_i r_i \) is independent of \( i \) for \( i \leq n + 1 \). As \( n \geq 1 \), by symmetry, \( \alpha_i r_i \) is independent of \( i \) for \( i \leq n + 2 \) as well.

Also by symmetry, \( \beta_i r_i \) is independent of \( i \) for \( i \leq n + 2 \). So \( \alpha \) and \( \beta \) only differ by a constant factor, and numerically we can set \( \beta = \alpha \).

Now consider the matrix \( C_1 E_2^T = (A_{n+1} A_i \cdot B_{n+2} B_j)_{1 \leq i,j \leq n} \), which by earlier argument is a diagonal matrix \( \text{diag}(r_1, \ldots, r_n) \). Similarly \( E_1 C_2^T \) is also \( \text{diag}(r_1, \ldots, r_n) \) as well. Since both \( \alpha_i r_i \) and \( \beta_i r_i \) are independent of \( i \) for \( i \leq n + 2 \), so by putting what know together, we have

\[
D_1 (C_1 E_2^T) D_2 (E_1 C_2^T) = c \cdot I_n
\]

for some constant \( c \), where \( I_n \) is the \( n \times n \) identity matrix. As the right side is \( c \cdot I_n \), so on the left side of the formula we can move \( C_2^T \) from the end to the front and regroup the matrices without changing the value.

**Lemma 7.5.** We have \( (C_2^T D_1 C_1) (E_2^T D_2 E_1) = c \cdot I_n \) for some constant \( c \).

By Lemma 7.3 and 7.4, \( f(x)/x \) and \( g(x)/x \) are the characteristic polynomials of \( C_2^T D_1 C_1 \) and \( E_2^T D_2 E_1 \) respectively. Then by Lemma 7.5, we prove Theorem 1.4.

**Theorem 7.6.** (Theorem 1.4) If \( \mathbf{B} \) is a dual of \( \mathbf{A} \) in \( \mathbb{R}^n \) and the non-zero roots of \( f(x) \) are \( \{\lambda_1, \ldots, \lambda_n\} \), then the non-zero roots of \( g(x) \) are \( \{c/\lambda_1, \ldots, c/\lambda_n\} \) for some constant \( c \).

### 7.4 Geometric interpretation of \( c_{n-1}(\alpha^{n-1}) = 0 \)

As Theorem 1.4 shows, if \( \mathbf{B} \) is a dual of \( \mathbf{A} \) in \( \mathbb{R}^n \), then for any \( i \) with \( 0 \leq i \leq n \), \( c_{n-i}(\alpha^{n-i}) = 0 \) if and only if \( c_1(\beta^i) = 0 \), where \( \beta \) is a 1-stress on \( \mathbf{B} \), and \( \beta^k \) is induced as a \( k \)-stress on \( \mathbf{B} \). Particularly \( c_{n-1}(\alpha^{n-1}) = 0 \) if and only if \( c_1(\beta^1) = 0 \), so for the Euclidean case interpreting \( c_{n-1}(\alpha^{n-1}) = 0 \) is the same as interpreting \( c_1(\beta^1) = 0 \).

With a switch of notation between \( c_1(\beta^1) \) and \( c_1(\alpha^1) \), here we provide a more general geometric interpretation of \( c_1(\alpha^1) = 0 \) for not only the Euclidean but also the non-Euclidean cases.

**Proposition 7.7.** (14 Proposition 2.21) For the spherical (resp. hyperbolic) case, \( c_1(\alpha^1) = 0 \) if and only if \( A_1, \ldots, A_{n+2} \) are affinely dependent in \( \mathbb{R}^{n+1} \) (resp. \( \mathbb{R}^{n,1} \)). For the Euclidean case, \( c_1(\alpha^1) = 0 \) if and only if \( A_1, \ldots, A_{n+2} \) are lying on an \( (n-1) \)-dimensional sphere in \( \mathbb{R}^n \).

As the proof is rather simple and provides some geometric intuition for the reader, we repeat it here.

**Proof.** For the non-Euclidean case, by 14.4 we have \( c_1(\alpha^1) = \kappa \cdot 2 \sum \alpha_i \). Since \( \sum \alpha_i A_i = 0 \), so \( c_1(\alpha^1) = 0 \) (now the same as \( \sum \alpha_i = 0 \)) if and only if \( A_1, \ldots, A_{n+2} \) are affinely dependent in \( \mathbb{R}^{n+1} \) or \( \mathbb{R}^{n,1} \).

For the Euclidean case, let \( O_1 \) be the center of the \((n-1)\)-dimensional sphere in \( \mathbb{R}^n \) that contains points \( A_2, \ldots, A_{n+2} \), and \( r \) be the radius. By 3.11 we have \( \sum \alpha_i = 0 \), then by choosing \( P = Q = O_1 \) in (7.3), we have

\[
c_1(\alpha^1) = \sum \alpha_i \overrightarrow{O_1 A_i}^2 = \alpha_1 (\overrightarrow{O_1 A_1}^2 - r^2).
\]
Therefore $c_1(\alpha^1) = 0$ if and only if $A_1$ is on the sphere as well. \hfill \Box

For $n = 2$, to illustrate Theorem 1.2 by using Proposition 7.7 we give a rather interesting example of “four points on a circle” below.

**Example 7.8.** In $\mathbb{R}^3$, given four points that are initially in convex position on a plane. If we allow the four points to vary smoothly in $\mathbb{R}^3$ but constrain them to preserve $\sum x_1 V_n(F_i) = \sum x_2 V_n(F_i)$ (see Question 1.1) during any motion, then the four points have to be confined to a plane first until they move on to a common circle where $c_1(\alpha^1) = 0$, and only from this circle they can be lifted from $\mathbb{R}^2$ to form a non-degenerate 3-simplex in $\mathbb{R}^3$. For the non-Euclidean case, the critical position when the four points can be lifted to form a non-degenerate 3-simplex is when $c_1(\alpha^1) = 0$ as well, namely when the points are affinely dependent in $\mathbb{R}^3$ or $\mathbb{R}^{2,1}$. Particularly for the spherical case, it is the same as the four points are on a small circle on $S^2$.\footnote{For the hyperbolic case, if we use the upper half-space model instead, then it is also the same as the four points are on a circle (or a straight line) in the upper half-plane $\mathbb{H}$, but the circle or line can be either fully or partially in $\mathbb{H}$.}

For $n = 2$, we give some simple examples to show that when $c_1(\alpha^1) = 0$, the four points can indeed be lifted from $\mathbb{R}^2$ to form a non-degenerate 3-simplex in $\mathbb{R}^3$ while preserving $\sum x_1 V_n(F_i) = \sum x_2 V_n(F_i)$ during the motion.

**Example 7.9.** In $\mathbb{R}^3$, given four points that are initially on the $xy$-plane with coordinates $(\pm a, \pm b, 0)$. As they form a rectangle, so they are on a circle and therefore $c_1(\alpha^1) = 0$. Now fix one pair of diagonal points during the motion, and for the other pair of diagonal points, let the coordinates $x$ and $y$ be constants and coordinate $z$ be $t$ for $t \geq 0$. So for any $t > 0$, the four points form a non-degenerate 3-simplex with four congruent faces, and thus $\sum x_1 V_n(F_i) = \sum x_2 V_n(F_i)$ is preserved during the motion.

**Example 7.10.** In $\mathbb{R}^3$, given four points that initially form an isosceles trapezoid on a plane. The lengths of its legs satisfy $l_1 = l_2$, and the diagonals satisfy $d_1 = d_2$. As the four points are on a circle and therefore $c_1(\alpha^1) = 0$. Now if we only require $l_1(t) = l_2(t)$ and $d_1(t) = d_2(t)$ during the motion in $\mathbb{R}^3$, with no requirements for the bases, then for $t > 0$ the four points form a non-degenerate 3-simplex whose two faces in $X_1$ are congruent to the two faces in $X_2$ respectively. Thus $\sum x_1 V_n(F_i) = \sum x_2 V_n(F_i)$ is preserved during the motion. This construction can also be similarly extended to the non-Euclidean case for a quadrilateral that satisfies $l_1 = l_2$ (legs) and $d_1 = d_2$ (diagonals); also notice that the notion of “parallel” no longer applies to the pair of bases, but it is also not needed for the construction.

In fact, Example 7.9 is a special case of Example 7.10.

### 7.5 Proof of Theorem 1.3

By Theorem 1.4 for the Euclidean case $c_{n-1}(\alpha^{n-1}) = 0$ if and only if $c_1(\beta^1) = 0$. Then by applying Proposition 7.7 to $B$, we prove Theorem 1.3.

**Theorem 7.11.** (Theorem 1.3) If $B$ is a dual of $A$ in $\mathbb{R}^n$, then $c_{n-1}(\alpha^{n-1}) = 0$ if and only if $c_1(\beta^1) = 0$, which is also equivalent to the vertices of $B$ lying on an $(n - 1)$-dimensional sphere in $\mathbb{R}^n$.\footnote{For the hyperbolic case, if we use the upper half-space model instead, then it is also the same as the four points are on a circle (or a straight line) in the upper half-plane $\mathbb{H}$, but the circle or line can be either fully or partially in $\mathbb{H}$.}
8 An alternative proof of Theorem 1.5

By combining Theorem 1.2 and 1.3 together, we provide a different formulation to answer Question 1.1 for the Euclidean case, with a slightly stronger statement that includes continuous motion as well.

**Theorem 8.1.** (Theorem 1.5) Let $\mathbf{B}$ be a dual of $\mathbf{A}$ in $\mathbb{R}^n$ and assume the vertices of $\mathbf{B}$ not lying on an $(n-1)$-dimensional sphere in $\mathbb{R}^n$. If $\mathbf{A}$ varies continuously in $\mathbb{R}^d$ with a single volume constraint that $\sum X_1 V_n(F_i) - \sum X_2 V_n(F_i)$ is preserved as 0, then the vertices of $\mathbf{A}$ are confined to a lower dimensional $\mathbb{R}^n$ for any sufficiently small motion.

Recall that we used $k$-stress to prove Theorem 1.2 and matrix theory (by proving Theorem 1.4 first) to prove Theorem 1.3. While Theorem 1.5 is simply a statement that combines Theorem 1.2 and 1.3 together, somewhat surprisingly, an alternative elementary proof of Theorem 1.5 below bypasses both Theorem 1.2 and 1.3 and uses neither $k$-stress nor matrix theory, and is valid for continuous motion as well. Again, for the non-Euclidean case, we lack a similarly simple statement/proof, thus making $k$-stress still the main tool to solve Question 1.1. Denote by $\mathbf{A}(t)$ the continuous motion (not necessarily smooth) of $\mathbf{A}$ and $\mathbf{A}(0) = \mathbf{A}$.

**Proof.** Assume the vertices of $\mathbf{A}(t)$ are lifted from $\mathbb{R}^n$ to form a non-degenerate $(n+1)$-simplex for small $t > 0$. Also assume $d = n + 1$, and $\mathbf{A}(0)$ is in a hyperplane in $\mathbb{R}^{n+1}$ whose $(n+1)$-th coordinate is 0. For an $n$-face $F_i(t)$ of $\mathbf{A}(t)$, let $u_i(t)$ be the outward unit normal to $\mathbf{A}(t)$ at $F_i(t)$ if $F_i(t)$ is in $X_1$, and be the inward unit normal at $F_i(t)$ if $F_i(t)$ is in $X_2$. If we treat $u_i(t)$ as a unit vector pointing from the origin $O$ to a point $B_i(t)$ in $\mathbb{R}^{n+1}$, then by the Minkowski relation for areas of facets of a Euclidean polytope, we have

$$\sum_{X_1} V_n(F_i(t)) B_i(t) - \sum_{X_2} V_n(F_i(t)) B_i(t) = 0. \quad (8.1)$$

Let $N$ be the north pole with coordinates $(0, \ldots, 0, 1)$, then without loss of generality we assume $B_i(t)$ is in the neighborhood of $N$ for every $i$. For a fixed $t > 0$, as $\sum_{X_1} V_n(F_i(t)) - \sum_{X_2} V_n(F_i(t)) = 0$, so by (8.1) all $B_i(t)$ are affinely dependent in $\mathbb{R}^{n+1}$. Then all $B_i(t)$ are lying on the intersection of the unit $n$-sphere and a hyperplane in $\mathbb{R}^{n+1}$, thus all $B_i(t)$ are lying on an $(n-1)$-sphere. Also notice that as $B_i(t) B_j(t) = u_j(t) - u_i(t)$, so for all distinct $i, j, k, l$,

$$B_i(t) B_j(t) \cdot A_k(t) A_l(t) = 0. \quad (8.2)$$

Denote by $\mathbf{B}(t)$ the collection of all points $B_i(t)$ for a fixed $t$. Notationally, $\mathbf{B}(0)$ is not $\mathbf{B}$. But is a set of $n + 2$ points that collapses to a single point $N$. When $t \to 0$ (but not including $t = 0$), applying the facts that (1) $\mathbf{B}(t)$ is approximately on the tangent space at $N$ (of the unit sphere) that is parallel to the hyperplane that contains $\mathbf{A}$, (2) formula (8.2), and (3) the uniqueness of $\mathbf{B}$ as a dual of $\mathbf{A}$ up to similarity, we show that up to a proper scaling of $\mathbf{B}(t)$ for all $t > 0$, $\mathbf{B}(t)$ converges to a shape that is similar to $\mathbf{B}$. Also notice that this property is independent of the path of $\mathbf{A}(t)$.

But as all $B_i(t)$ are lying on an $(n-1)$-sphere, this contradicts the assumption that the vertices of $\mathbf{B}$ not lying on a $(n-1)$-sphere. Thus the vertices of $\mathbf{A}(t)$ are confined to a lower dimensional $\mathbb{R}^n$ for small $t \geq 0$. \(\square\)
Remark 8.2. The reader may notice that the proof above does not require that $A(t)$ is smooth, and is valid for continuous motion as well. So combined with Theorem 1.3, it also makes Theorem 1.2 valid for continuous motion for the Euclidean case without using more advanced tools.

Remark 8.3. If the vertices of $B$ are lying on an $(n−1)$-sphere, then in the proof above by choosing a proper $B(t)$ such that $B(t)$ is similar to $B$ for all $t > 0$, we can, inversely, explicitly construct a non-degenerate $A(t)$ that preserves $\sum X_1 V_n(F_i) - \sum X_2 V_n(F_i) = 0$ during the motion.

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