GEOMETRIC INTERPRETATION OF HALF-PLANE CAPACITY

STEVEN LALLEY
Department of Statistics
University of Chicago
Email: lalley@galton.uchicago.edu

GREGORY LAWLER
Department of Mathematics
University of Chicago
Email: lawler@math.uchicago.edu

HARIHARAN NARAYANAN
Laboratory for Information and Decision Systems
MIT
Email: har@mit.edu

Submitted August 30, 2009, accepted in final form October 8, 2009

AMS 2000 Subject classification: 60J67, 97I80
Keywords: Brownian motion, Conformal Invariance, Schramm-Loewner Evolution

Abstract

Schramm-Loewner Evolution describes the scaling limits of interfaces in certain statistical mechanical systems. These interfaces are geometric objects that are not equipped with a canonical parametrization. The standard parametrization of SLE is via half-plane capacity, which is a conformal measure of the size of a set in the reference upper half-plane. This has useful harmonic and complex analytic properties and makes SLE a time-homogeneous Markov process on conformal maps. In this note, we show that the half-plane capacity of a hull $A$ is comparable up to multiplicative constants to more geometric quantities, namely the area of the union of all balls centered in $A$ tangent to $\mathbb{R}$, and the (Euclidean) area of a $1$-neighborhood of $A$ with respect to the hyperbolic metric.

1 Introduction

Suppose $A$ is a bounded, relatively closed subset of the upper half plane $\mathbb{H}$. We call $A$ a compact $\mathbb{H}$-hull if $A$ is bounded and $\mathbb{H} \setminus A$ is simply connected. The half-plane capacity of $A$, $\text{hcap}(A)$, is defined in a number of equivalent ways (see [1], especially Chapter 3). If $g_A$ denotes the unique conformal transformation of $\mathbb{H} \setminus A$ onto $\mathbb{H}$ with $g_A(z) = z + o(1)$ as $z \to \infty$, then $g_A$ has the expansion

$$g_A(z) = z + \frac{\text{hcap}(A)}{z} + O(|z|^{-2}), \quad z \to \infty.$$ 

Equivalently, if $B_t$ is a standard complex Brownian motion and $\tau_A = \inf\{t \geq 0 : B_t \not\in \mathbb{H} \setminus A\}$,

$$\text{hcap}(A) = \lim_{y \to \infty} y \mathbb{E}^y [\text{Im}(B_{\tau_A})].$$

Let $\text{Im}[A] = \sup\{\text{Im}(z) : z \in A\}$. Then if $y \geq \text{Im}[A]$, we can also write

$$\text{hcap}(A) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathbb{E}^{x+iy} [\text{Im}(B_{\tau_A})] \, dx.$$
These last two definitions do not require $\mathbb{H} \setminus A$ to be simply connected, and the latter definition does not require $A$ to be bounded but only that $\text{Im}[A] < \infty$.

For $\mathbb{H}$-hulls (that is, for relatively closed $A$ for which $\mathbb{H} \setminus A$ is simply connected), the half-plane capacity is comparable to a more geometric quantity that we define. This is not new (the second author learned it from Oded Schramm in oral communication), but we do not know of a proof in the literature\textsuperscript{3}. In this note, we prove the fact giving (nonoptimal) bounds on the constant. We start with the definition of the geometric quantity.

**Definition 1.** For an $\mathbb{H}$-hull $A$, let $\text{hsiz}(A)$ be the 2-dimensional Lebesgue measure of the union of all balls centered at points in $A$ that are tangent to the real line. In other words
\[
\text{hsiz}(A) = \text{area} \left( \bigcup_{x+i y \in A} B(x+i y) \right),
\]
where $B(z, \epsilon)$ denotes the disk of radius $\epsilon$ about $z$.

In this paper, we prove the following.

**Theorem 1.** For every $\mathbb{H}$-hull $A$,
\[
\frac{1}{66} \text{hsiz}(A) < \text{hcap}(A) < \frac{7}{2\pi} \text{hsiz}(A).
\]

## 2 Proof of Theorem 1

It suffices to prove this for weakly bounded $\mathbb{H}$-hulls, by which we mean $\mathbb{H}$-hulls $A$ with $\text{Im}(A) < \infty$ and such that for each $\epsilon > 0$, the set $\{x+i y : y > \epsilon\}$ is bounded. Indeed, for $\mathbb{H}$-hulls that are not weakly bounded, it is easy to verify that $\text{hsiz}(A) = \text{hcap}(A) = \infty$.

We start with a simple inequality that is implied but not explicitly stated in [1]. Equality is achieved when $A$ is a vertical line segment.

**Lemma 1.** If $A$ is an $\mathbb{H}$-hull, then
\[
\text{hcap}(A) \geq \frac{\text{Im}[A]^2}{2}. \tag{1}
\]

**Proof.** Due to the continuity of hcap with respect to the Hausdorff metric on $\mathbb{H}$-hulls, it suffices to prove the result for $\mathbb{H}$-hulls that are path-connected. For two $\mathbb{H}$-hulls $A_1 \subseteq A_2$, it can be seen using the Optional stopping theorem that $\text{hcap}(A_1) \leq \text{hcap}(A_2)$. Therefore without loss of generality, $A$ can be assumed to be of the form $\eta(0,T]$ where $\eta$ is a simple curve with $\eta(0+) \in \mathbb{R}$, parameterized so that $\text{hcap}[\eta(0,t)] = 2t$. In particular, $T = \text{hcap}(A)/2$. If $g_t = g_{t(0,t]}$, then $g_t$ satisfies the Loewner equation
\[
\partial_t g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{2}
\]
where $U : [0,T] \to \mathbb{R}$ is continuous. Suppose $\text{Im}(z)^2 > 2 \text{hcap}(A)$ and let $Y_t = \text{Im}[g_t(z)]$. Then (2) gives
\[
-\partial_t Y_t^2 \leq \frac{4Y_t}{|g_t(z) - U_t|^2} \leq 4,
\]
which implies
\[
Y_t^2 \geq Y_0^2 - 4T > 0.
\]
This implies that $z \notin A$, and hence $\text{Im}[A]^2 \leq 2 \text{hcap}(A)$. \hfill $\Box$

The next lemma is a variant of the Vitali covering lemma. If $c > 0$ and $z = x + i y \in \mathbb{H}$, let
\[
\mathcal{I}(z, c) = (x - cy, x + cy),
\]
\[
\mathcal{R}(z, c) = \mathcal{I}(z, c) \times (0, y) = \{x' + iy' : |x' - x| < cy, 0 < y' \leq y\}.
\]

\textsuperscript{3}After submitting this article, we learned that a similar result was recently proved by Carto Wong as part of his Ph.D. research.
Lemma 2. Suppose $A$ is a weakly bounded $\mathbb{H}$-hull and $c > 0$. Then there exists a finite or countably infinite sequence of points \( \{z_1 = x_1 + iy_1, z_2 = x_2 + iy_2, \ldots \} \subset A \) such that:

- $y_1 \geq y_2 \geq y_3 \geq \cdots$;
- the intervals $I(x_1, c), I(x_2, c), \ldots$ are disjoint;

$$A \subset \bigcup_{j=1}^{\infty} \mathcal{R}(z_j, 2c). \quad (3)$$

Proof. We define the points recursively. Let $A_0 = A$ and given \( \{z_1, \ldots, z_j\} \), let

$$A_j = A \setminus \left[ \bigcup_{k=1}^{j} \mathcal{R}(z_j, 2c) \right].$$

If $A_j = \emptyset$ we stop, and if $A_j \neq \emptyset$, we choose $z_{j+1} = x_{j+1} + iy_{j+1} \in A$ with $y_{j+1} = \Im[z_j]$. Note that if $k \leq j$, then $|x_{j+1} - x_k| \geq 2c y_k \geq c (y_k + y_{j+1})$ and hence $I(z_{j+1}, c) \cap I(z_k, c) = \emptyset$. Using the weak boundedness of $A$, we can see that $y_j \to 0$ and hence $(3)$ holds.

\[\square\]

Lemma 3. For every $c > 0$, let

$$\rho_c := \frac{2\sqrt{2}}{\pi} \arctan(e^{-\theta}), \quad \theta = \theta_c = \frac{\pi}{4c}.$$ 

Then, for any $c > 0$, if $A$ is a weakly bounded $\mathbb{H}$-hull and $x_0 + iy_0 \in A$ with $y_0 = \Im(A)$, then

$$\hcap(A) \geq \rho_c^2 y_0^2 + \hcap[A \setminus \mathcal{R}(z, 2c)].$$

Proof. By scaling and invariance under real translation, we may assume that $\Im[A] = y_0 = 1$ and $x_0 = 0$. Let $S = S_c$ be defined to be the set of all points $z$ of the form $x + i uy$ where $x + iy \in A \setminus \mathcal{R}(i, 2c)$ and $0 < u \leq 1$.

Clearly, $S \cap A = A \setminus \mathcal{R}(i, 2c)$.

Using the capacity inequality [1, (3.10)]

$$\hcap(A_1 \cup A_2) - \hcap(A_2) \leq \hcap(A_1) - \hcap(A_1 \cap A_2), \quad (4)$$

we see that

$$\hcap(S \cup A) - \hcap(S) \leq \hcap(A) - \hcap(S \cap A).$$

Hence, it suffices to show that

$$\hcap(S \cup A) - \hcap(S) \geq \rho_c^2.$$

Let $f$ be the conformal map of $\mathbb{H} \setminus S$ onto $\mathbb{H}$ such that $z - f(z) = o(1)$ as $z \to \infty$. Let $S^* := S \cup A$. By properties of halfplane capacity [1, (3.8)] and (1),

$$\hcap(S^*) - \hcap(S) = \hcap[f(S^* \setminus S)] \geq \frac{\Im[f(i)]^2}{2}.$$

Hence, it suffices to prove that

$$\Im[f(i)] \geq \sqrt{2} \rho = \frac{4}{\pi} \arctan(e^{-\theta}). \quad (5)$$

By construction, $S \cap \mathcal{R}(z, 2c) = \emptyset$. Let $V = (-2c, 2c) \times \{0, \infty\} = \{x + iy : |x| < 2c, y > 0\}$ and let $\tau_V$ be the first time that a Brownian motion leaves the domain. Then [1, (3.5)],

$$\Im[f(i)] = 1 - E^i[\Im(B_{\tau_V})] \geq P\{B_{\tau_V} \in [-2c, 2c]\} \geq P\{B_{\tau_V} \in [-2c, 2c]\}.$$ 

The map $\Phi(z) = \sin(\theta z)$ maps $V$ onto $\mathbb{H}$ sending $[-2c, 2c]$ to $[-1, 1]$ and $\Phi(i) = i \sinh \theta$. Using conformal invariance of Brownian motion and the Poisson kernel in $\mathbb{H}$, we see that

$$P\{B_{\tau_V} \in [-2c, 2c]\} = \frac{2}{\pi} \arctan\left(\frac{1}{\sinh \theta}\right) = \frac{4}{\pi} \arctan(e^{-\theta}).$$

The second equality uses the double angle formula for the tangent.

\[\square\]
Lemma 4. Suppose $c > 0$ and $x_1 + iy_1, x_2 + iy_2, \ldots$ are as in Lemma 2. Then

$$\text{hsiz}(A) \leq [\pi + 8c] \sum_{j=1}^{\infty} y_j^2.$$  \hspace{1cm} (6)

If $c \geq 1$, then

$$\pi \sum_{j=1}^{\infty} y_j^2 \leq \text{hsiz}(A).$$  \hspace{1cm} (7)

Proof. A simple geometry exercise shows that

$$\text{area} \left[ \bigcup_{x+iy \in A} B(x+iy, y) \right] = [\pi + 8c] y_j^2.$$  \hspace{1cm} (6)

Since

$$A \subset \bigcup_{j=1}^{\infty} \mathcal{R}(z_j, 2c),$$

the upper bound in (6) follows. Since $c \geq 1$, and the intervals $I(z_j, c)$ are disjoint, so are the disks $B(z_j, y_j)$. Hence,

$$\text{area} \left[ \bigcup_{x+iy \in A} B(x+iy, y) \right] \geq \text{area} \left[ \bigcup_{j=1}^{\infty} B(z_j, y_j) \right] = \pi \sum_{j=1}^{\infty} y_j^2.$$  \hspace{1cm} (7)

Proof of Theorem 1. Let $V_j = A \cap \mathcal{R}(z_j, c)$. Lemma 3 tells us that

$$\text{hcap} \left[ \bigcup_{k=j}^{\infty} V_j \right] \geq \rho_c^2 y_j^2 + \text{hcap} \left[ \bigcup_{k=j+1}^{\infty} V_j \right],$$

and hence

$$\text{hcap}(A) \geq \rho_c^2 \sum_{j=1}^{\infty} y_j^2.$$  \hspace{1cm} (8)

Combining this with the upper bound in (6) with any $c > 0$ gives

$$\frac{\text{hcap}(A)}{\text{hsiz}(A)} \geq \frac{\rho_c^2}{\pi + 8c}.$$  \hspace{1cm} (9)

Choosing $c = \frac{8}{5}$ gives us

$$\frac{\text{hcap}(A)}{\text{hsiz}(A)} \geq \frac{1}{66}.$$  \hspace{1cm} (10)

For the upper bound, choose a covering as in Lemma 2. Subadditivity and scaling give

$$\text{hcap}(A) \leq \sum_{j=1}^{\infty} \text{hcap} [\mathcal{R}(z_j, 2cy_j)] = \text{hcap} [\mathcal{R}(i, 2c)] \sum_{j=1}^{\infty} y_j^2.$$  \hspace{1cm} (9)

Combining this with the lower bound in (6) with $c = 1$ gives

$$\frac{\text{hcap}(A)}{\text{hsiz}(A)} \leq \frac{\text{hcap} [\mathcal{R}(i, 2)]}{\pi}.$$  \hspace{1cm} (9)

Note that $\mathcal{R}(i, 2)$ is the union of two real translates of $\mathcal{R}(i, 1)$, $\text{hcap} [\mathcal{R}(i, 2)] \leq 2 \text{hcap} [\mathcal{R}(i, 1)]$ whose intersection is the interval $(0, i]$. Using (4), we see that

$$\text{hcap} [\mathcal{R}(i, 2)] \leq 2 \text{hcap} [\mathcal{R}(i, 1)] - \text{hcap} ((0, i]) = 2 \text{hcap} [\mathcal{R}(i, 1)] - \frac{1}{2}.$$  \hspace{1cm} (9)
But $\mathcal{R}(i, 1)$ is strictly contained in $A' := \{ z \in \mathbb{H} : |z| \leq \sqrt{2} \}$, and hence
\[
\text{hcap}[\mathcal{R}(i, 1)] < \text{hcap}(A') = 2.
\]
The last equality can be seen by considering $h(z) = z + 2z^{-1}$ which maps $\mathbb{H} \setminus A'$ onto $\mathbb{H}$. Therefore,
\[
\text{hcap}[\mathcal{R}(i, 2)] < \frac{7}{2},
\]
and hence
\[
\frac{\text{hcap}(A)}{\text{hsiz}(A)} < \frac{7}{2\pi}.
\]

An equivalent form of this result can be stated\(^4\) in terms of the area of the 1-neighborhood of $A$ (denoted $\text{hyp}(A)$) in the hyperbolic metric. The unit hyperbolic ball centered at a point $(x, y)$ is the Euclidean ball with respect to which $(x, ye^{-1})$ and $(x, ye)$ are diametrically opposite boundary points. For any $c$, choosing a covering as in Lemma 2,
\[
\text{hyp}(A) < \left( \frac{e}{2} \right)^2 \pi + 4ce \sum_{j=1}^{\infty} y_j^2.
\]

So by (8),
\[
\frac{\text{hcap}(A)}{\text{hyp}(A)} > \rho_c^2 \left( \frac{e}{2} \right)^2 \pi + 4ce^{-1}.
\]

Setting $c$ to $\frac{8}{5}$,
\[
\frac{\text{hcap}(A)}{\text{hyp}(A)} > \frac{1}{100}.
\]

For any $c > \frac{e-e^{-1}}{2}$,
\[
\text{hyp}(A) \geq \pi \left( \frac{e-e^{-1}}{2} \right)^2 \sum_{j=1}^{\infty} y_j^2.
\]

So by (9),
\[
\frac{\text{hcap}(A)}{\text{hyp}(A)} < \frac{\text{hcap}[\mathcal{R}(i, 3)]}{\pi \left( \frac{e-e^{-1}}{2} \right)^2}.
\]
\[
\text{hcap}(\mathcal{R}(i, 3)) \leq \text{hcap}(\mathcal{R}(i, 1)) + \text{hcap}(\mathcal{R}(i, 2)) - \text{hcap}((0, i)) \leq 5.
\]

Therefore,
\[
\frac{1}{100} < \frac{\text{hcap}(A)}{\text{hyp}(A)} < \frac{20}{\pi(e-e^{-1})^2}.
\]

References

[1] G. Lawler, *Conformally Invariant Processes in the Plane*, American Mathematical Society, 2005.

\(^4\)This answers a question by the anonymous referee