Free boson representation of $DY_h(gl_N)_k$

X.M. Ding $^{a,b}$, B. Y. Hou $^c$, B.Yuan Hou $^d$, L. Zhao $^c$

$^a$ CCAST, P.O. Box 8730, Beijing, 100080, China
$^b$ Institute of Theoretical Physics of Academia Sinica, P.O.Box 2735, 100080, China
$^c$ Institute of Modern Physics, Northwest University, Xian, 710069, China
$^d$ Physics Department of Graduate School of Academia Sinica, Beijing, 100039, China

Abstract

We construct a realization of the Yangian double $DY_h(gl_N)$ and $DY_h(sl_N)$ of an arbitrary level $k$ in terms of free boson fields with a continuous parameter. In the $\hbar \to 0$ limit this realization becomes the Wakimoto realization of $gl_N$ and $sl_N$, respectively. The vertex operators and the screening currents are also constructed with the same spirits; the screening currents commute with $DY_h(sl_N)$ modulo total difference.

1 Introduction

Quantum affine algebras and Yangian were proposed by Drinfeld as generalizations of classical universal Lie algebras and loop algebra with nontrivial Hopf algebra structures [8, 9, 10]. They play crucial role, which is similar the role of the Virasoro algebras and Kac-Moody algebras in the conformal field theories, in the perturbative integrable massive quantum field theories. Following the Faddeev-Reshetikhin-Takhtajan (FRT) formalism [11], both kinds of algebras can be considered as associative algebras defined through the Yang-Baxter relation (i.e. RLL-relations) with the structure constants determined by the solutions of the quantum Yang-Baxter equation (QYBE). Quantum affine algebra [11] and Yangian double with center (the central extension of the quantum double of Yangian) [21, 22, 23] are respectively affine extensions of classical universal Lie algebra and Yangian. From the RLL viewpoint, which corresponds to the trigonometric solution of QYBE and rational one, respectively. If one regards the Reshetikhin-Semenov-Tian-Shansky realization (RS) [22] as the affine analog of Faddeev-Reshetikhin-Takhtajan formalism. They both were proved to have important applications in certain physical problems, especially in
describing the dynamical symmetries of the perturbative integrable massive quantum field theories, calculating the correlation functions and form factors of some two-dimensional exactly solvable lattice statistical model and (1+1)-dimensional completely integrable quantum field theories \[4, 29, 34\]. In such physical applications, the infinite-dimensional representations of quantum affine algebra and Yangian double are required. Rephrasing in another words, doubled Yangian with central is needed, especially the representations with higher \((k > 1)\) level.

In practice, free field realization is proved to be quite effective and useful approach to study complicated algebras structure. In this aspect, the Heisenberg algebra (or free boson) representation has become a common method for obtaining representations of quantum affine algebras. For examples, the free boson representations of \(U_q(\hat{sl}_2)\) with an arbitrary level have been obtained in Refs.\[33, 30, 31, 25, 1\]. Free boson representation of \(U_q(\hat{sl}_N)\) with level 1 was constructed in \[14\]. Free boson representations of \(U_q(\hat{sl}_3)\) and \(U_q(\hat{sl}_N)\) with arbitrary level were constructed in \[2\] and \[3\] respectively. For the Yangian doubles, the free field representation of \(DY_{\bar{h}}(\hat{sl}_2)\) with level \(k\) was constructed in \[28\]. The level 1 free boson representation of \(DY_{\bar{h}}(\hat{sl}_N)\) was given in \[18\]. In Ref.\[16\] the free field representation of the Yangian double \(DY_{\bar{h}}(sl_N)\) with arbitrary level \(k\) are obtained by a simple correspondence between the quantum affine algebra \(U_q(\hat{sl}_N)\) and the Yangian double \(DY_{\bar{h}}(sl_N)\) \[16\]. This correspondence makes our derivation of free field representation for \(DY_{\bar{h}}(sl_N)\) greatly simplified. However, we have been unable to obtain screening currents for \(DY_{\bar{h}}(sl_N)\) following the same spirit, so the relations of quantum affine algebras and the Yangian double with center is nontrivial. So to construct a free fields realization of the screening current operator is the one of the motivations of this paper.

Another motivation is that, in the Ref.\[13\] Frenkel and Reshetikhin use the free field realization of the quantum affine algebra \(U_q(\hat{gl}_2)_k\) at critical level \(k = -2\), they investigate the structure of its infinite dimensional center of the quantum affine algebra \(U_q(\hat{gl}_2)_k\). Using the Wakimoto realization of the quantum affine algebra they define a new Poisson bracket algebra, which is nothing but the \(q\)-deformation of the classical Virasoro algebra ( in the general case are the \(q\)-deformations of the classical \(\mathcal{W}_q(sl_N)\) algebras ). Similarly as the quantum affine cases, infinite dimensional centers exist for the Yangian double with center at critical level, which are just the classical \(h - \mathcal{W}\) algebras \[17\]. In the simplest case, \(h\)-Virasoro is considered in Ref.\[1\] by using the free fields realization of the Yangian double with center \(DY_{\bar{h}}(gl_2)_k\) at level \(k = -2\). Similarly, one way to study the structure of the \(h - \mathcal{W}\) algebras is through the free field realization of the Yangian double with center \(DY_{\bar{h}}(g)_k\) at critical level, where the Lie algebra \(g\) has a higher rank. However, it is difficult to obtain the free fields realization of the currents for Yangian double \(DY_{\bar{h}}(gl_N)_c\) in the higher rank cases (rank greater than one) with the boson fields using in \[16\], either no consistent realization of quantum determinant, which decomposes the \(DY_{\bar{h}}(gl_N)_c\) to its subalgebra \(DY_{\bar{h}}(gl_N)_{c_1}\) could be given.

In this paper, we introduce free fields involved with a continuous parameter, we obtain
the $DY_h(gl_N)$ and $DY_h(sl_N)$ currents, and the screening currents and vertex operators.

The manuscript is arranged as follows. Section 2, at first, we briefly review the Drinfeld new realization of the Yangian $Y(g)$, then we give the defining relations of Drinfeld generators of the Yangian double with center $DY_h(gl_N)$; and the RS realization is also given, the isomorphism between these two realization can be established through Ding-Frenkel correspondence. In section 3, the three set of free boson fields with a continuous parameter are introduced; some relations, which will be used in the following sections are listed. With the help of the results obtained in the section 3, we give a free fields realization of $DY_h(gl_N)$ and $DY_h(sl_N)$ in the section 4. In section 5 is the free fields realization of the screening currents and the vertex operators, and brief discussions are given at the final section of the paper.

2 Drinfeld generators of Yangian double $DY_h(gl_N)$

At first, we introduce some notations, which will be used in the sequel. Let $\epsilon_\mu$ (1 $\leq \mu \leq N$) be the orthonormal basis in $\mathbb{R}^N$. We have the inner product $\langle \epsilon_\mu, \epsilon_\nu \rangle = \delta_{\mu,\nu}$. Define

$$\bar{\epsilon}_\mu = \epsilon_\mu - \epsilon, \quad \epsilon = \frac{1}{N} \sum_{\mu=1}^{N} \epsilon_\mu. \tag{2.1}$$

Obviously, $\sum_{\mu=1}^{N} \bar{\epsilon}_\mu = 0$. The simple roots, fundamental weights are,

$$\alpha_\mu = \epsilon_\mu - \epsilon_{\mu+1}, \quad \Lambda_\mu = \sum_{\nu=1}^{\mu} \bar{\epsilon}_\nu, \tag{2.2}$$

respectively. For simple, denote half of the Cartan matrix $a_{\mu,\nu}$ as $B_{\mu,\nu}$.

There are three realizations of Yangian $Y(g)$: Drinfeld realization $[8,9]$, Drinfeld new realization $[10]$, and $R$-matrix realization or FRT realization $[26,11]$. The isomorphism between the Drinfeld new realization and the $R$-matrix (or FRT) realization can be established by using the Ding and Frenkel correspondence $[3]$. Here, we only give the definition of the Drinfeld new realization of Yangian, for the Yangian double is defined in terms of quantum double of the Drinfeld new realization, while the Yangian double with center is the central extension of the Yangian double. There is the following theorem:

**Theorem 1** [10] *The Yangian $Y(g)$ is isomorphic to the associative algebra with generators, $x_{\mu,r}^{\pm}$, $h_{\mu,r}$, $\mu = 1, \ldots, \text{rank } g$, $r \in \mathbb{Z}_{\geq 0}$, and the following defining relations.*

\[
\begin{align*}
[h_{\mu,r}, h_{\nu,s}] &= 0, \\
[h_{\mu,0}, x_{\nu,s}^{\pm}] &= \pm 2d_\mu B_{\mu,\nu} x_{\mu,s}^{\pm}, \\
[h_{\mu,r+1}, x_{\nu,s}^{\pm}] - [h_{\mu,r}, x_{\nu,s+1}^{\pm}] &= \pm h d_\mu B_{\mu,\nu} (h_{\mu,r} x_{\nu,s}^{\pm} + x_{\nu,s}^{\pm} h_{\mu,r}), \\
[x_{\mu,r}^{+}, x_{\nu,s}^{-}] &= \delta_{\mu,\nu} h_{\mu,r+s},
\end{align*}
\]

3
The Drinfeld generators for all sequences of non-negative integers $r$, and they satisfy the following commutation relations:

$$[x_{\mu,r+1}^{\pm},x_{\nu,s}^{\pm}] - [x_{\mu,r}^{\pm},x_{\nu,s+1}^{\pm}] = \pm h d_{\mu} B_{\mu,\nu} (x_{\mu,r}^{\pm}, x_{\nu,s}^{\pm} + x_{\nu,s}^{\pm} x_{\mu,r}^{\pm}),$$

$$\sum_{\pi} [x_{\mu,r_{\pi(1)}}^{\pm}, [x_{\mu,r_{\pi(2)}}^{\pm}, \ldots, [x_{\mu,r_{\pi(m)}}^{\pm}, x_{\nu,s}^{\pm}] \ldots]] = 0,$$

for all sequences of non-negative integers $r_1, \ldots, r_\nu$, where $m = 1 - 2B_{\mu,\nu}$, $2d_{\mu} B_{\mu,\nu}$ is the symmetric Cartan matrix. And the sum is over all permutations $\pi$ of $1, \ldots, m$.

As an associative algebra, the Yangian double with central $DY_h(gl_N)_c$ is generated by the Drinfeld generators $\{k_{\mu,r}^{\pm}, k_{\mu,s}^{\pm}, x^{\pm}_{\nu,m} \mid \mu = 1, \ 2, \ \ldots, N; \nu = 1, \ 2, \ \ldots, N - 1; \ r \in Z_0, \ s \in Z_{<0}, \ m \in Z\}$, derivation operator $d$ and the center $c$.

The generating function of the Drinfeld generators of Yangian double with center $DY_h(gl_N)_c$ are $k_{\mu}^{\pm}(u)$ and $X_{\mu}^{\pm}(u)$,

$$k_{\mu}^{\pm}(u) = \sum_{r \geq 0} k_{\mu,r} u^{-r-1}, \ k_{\mu}^{\pm}(u) = \sum_{s < 0} h_{\nu,s} u^{-s-1},$$

$$X_{\mu}^{\pm}(u) = \sum_{m \in \mathbb{Z}} x_{\mu,m}^{\pm} u^{-m-1}, \ X_{\mu}^{\pm}(u) = \sum_{m \in \mathbb{Z}} x_{\mu,m}^{\pm} u^{-m-1},$$

and they satisfy the following commutation relations:

$$[d, \chi(u)] = \frac{dX}{du}, \ [c, \chi(u)] = 0, \ \chi(u) = k_{\mu}^{\pm}(u), \ X_{\mu}^{\pm}(u), \ X_{\mu}^{\pm}(u)$$

$$k_{\mu}^{\pm}(u)k_{\nu}^{\pm}(v) = k_{\nu}^{\pm}(v)k_{\mu}^{\pm}(u),$$

$$\rho(u - v - ch) k_{\mu}^{\pm}(u) k_{\nu}^{\pm}(v) = k_{\mu}^{\pm}(v) k_{\mu}^{\pm}(u) \rho(u - v),$$

$$\rho(u - v - ch) \frac{u - v - ch}{u - v + h - ch} k_{\mu}^{\pm}(u) k_{\nu}^{\pm}(v) = k_{\nu}^{\pm}(v) k_{\mu}^{\pm}(u) \frac{u - v}{u - v + h} \rho(u - v), \ \mu < \nu,$$

$$\rho(u - v - ch) \frac{u - v - h + ch}{u - v - ch} k_{\mu}^{\pm}(u) k_{\nu}^{\pm}(v) = k_{\nu}^{\pm}(v) k_{\mu}^{\pm}(u) \frac{u - v - h}{u - v} \rho(u - v), \ \mu > \nu,$$

$$k_{\mu}^{\pm}(u) X_{\mu}^{\pm}(v) k_{\mu}^{\pm}(u)^{-1} = \frac{u - v}{u - v + h} X_{\mu}^{\pm}(v),$$

$$k_{\mu+1}^{\pm}(u) X_{\mu}^{\pm}(v) k_{\mu+1}^{\pm}(u)^{-1} = \frac{u - v}{u - v - h} X_{\mu}^{\pm}(v),$$

$$k_{\mu}^{\pm}(u) X_{\mu}^{\pm}(v) k_{\mu}^{\pm}(u)^{-1} = \frac{u - v + h}{u - v} X_{\mu}^{\pm}(v),$$

$$k_{\mu+1}^{\pm}(u) X_{\mu}^{\pm}(v) k_{\mu+1}^{\pm}(u)^{-1} = \frac{u - v - h}{u - v} X_{\mu}^{\pm}(v),$$

$$k_{\mu}^{\pm}(u) X_{\mu}^{\pm}(v) k_{\mu}^{\pm}(u)^{-1} = \frac{u - v + h - ch}{u - v - ch} X_{\mu}^{\pm}(v),$$

$$k_{\mu+1}^{\pm}(u) X_{\mu}^{\pm}(v) k_{\mu+1}^{\pm}(u)^{-1} = \frac{u - v - h - ch}{u - v - ch} X_{\mu}^{\pm}(v),$$

$$k_{\mu}^{\pm}(u) X_{\mu}^{\pm}(v) k_{\mu}^{\pm}(u)^{-1} = X_{\mu}^{\pm}(v), \ \mu - \nu \geq 2, \ \text{or} \ \mu - \nu \leq -1,$$

$$k_{\mu}^{\pm}(u) X_{\mu}^{\pm}(v) k_{\mu}^{\pm}(u)^{-1} = X_{\mu}^{\pm}(v), \ \mu - \nu \geq 2, \ \text{or} \ \mu - \nu \leq -1,$$
(u - v \mp \hbar) X^\pm_\mu(u) X^\pm_\nu(v) = (u - v \pm \hbar) X^\pm_\mu(v) X^\pm_\mu(u),

(u - v + \hbar) X^+_\mu(u) X^+_{\mu+1}(v) = (u - v) X^+_{\mu+1}(v) X^+_\mu(u),

(u - v) X^-_\mu(u) X^-_{\mu+1}(v) = (u - v + \hbar) X^-_{\mu+1}(v) X^-_\mu(u),

X^\pm_\mu(u_1) X^\pm_\mu(u_2) X^\pm_\nu(v) - 2X^\pm_\mu(u_1) X^\pm_\nu(v) X^\pm_\mu(u_2)

\quad + X^\pm_\nu(v) X^\pm_\mu(u_1) X^\pm_\mu(u_2)

\quad + \{u_1 \leftrightarrow u_2\} = 0 \quad |\mu - \nu| = 1,

X^\pm_\mu(u) X^\pm_\nu(v) = X^\pm_\nu(v) X^\pm_\mu(u) \quad |\mu - \nu| \geq 2,

[X^\pm_\mu(u), X^-_\nu(v)] = \delta_{\mu,\nu} \left(\delta(u - (v + \hbar))k^+_{\mu+1}(u)k^+_\mu(u)^{-1} - \delta(u - v)k^-_{\mu+1}(v)k^-_\mu(v)^{-1}\right).

In which the scalar factor

$$\rho^+(u) = \frac{\Gamma\left(\frac{1}{N} - \frac{\mu}{N^2}\right) \Gamma\left(1 - \frac{1}{N} - \frac{\mu}{N^2}\right)}{\Gamma\left(-\frac{\mu}{N}\right) \Gamma\left(1 - \frac{1}{N} - \frac{\mu}{N^2}\right)},$$

(2.3)

while \(\delta(u - v) = \sum_{m+n=-1} u^m v^n\). These relations are understood as relations between formal series. The above relations can be derived through the RS realization, in the following we will turn to the RS realization.

The RS realization \([32]\) originated from the quantum inverse scattering method. The one of the advantages of the RS realization is that, the central elements can be written explicit by the quantum trace \([32]\). For this purpose, in the following we will drop the \(d\) operator in the Yangian double with center, and denote as \(DY'_h(gl_N)\). Introduce the following operators:

$$L^\pm(u) = (t^\pm_\mu(u))_{1 \leq \mu, \nu \leq N};$$

(2.4)

where the generating function,

$$t^\pm_\mu(u) = \delta_{\mu,\nu} + \hbar \sum_{m=0}^{\infty} t^\pm_{\mu,\nu}[m] u^{-m-1}; l^+_\mu(u) = \delta_{\mu,\nu} - \hbar \sum_{m=-1}^{\infty} l^-_{\mu,\nu}[m] u^{-m-1};$$

(2.5)

and the \(R\)-matrix \(R^\pm(u)\),

$$R^\pm(u) = \rho^+(u) \tilde{R}(u),$$

(2.6)

$$\rho^-(u) = \left(\frac{\Gamma\left(\frac{1}{N} + \frac{\mu}{N^2}\right) \Gamma\left(1 - \frac{1}{N} + \frac{\mu}{N^2}\right)}{\Gamma\left(-\frac{\mu}{N}\right) \Gamma\left(1 + \frac{\mu}{N^2}\right)}\right)^{-1} = \frac{1}{\rho^+(-u)},$$

(2.7)

and \(\tilde{R}(u) = \frac{u t^+ h P}{u + h}\) is the Yang’s solution of QYBE. The \(P\) is the flip operator of \(u \otimes v, Pu \otimes v = v \otimes u\), and the choice of the scalar function \(\rho^\pm(u)\) can be determined by the unitarity condition and crossing symmetry\([13]\). But here we adopt another approach but the crossing symmetry condition, which will be explained in the latter part of the paper.
From the universal R-matrix for the central extended Yangian double obtained in [21, 22, 23], $L^\pm(u)$ have the following unique decompositions:

\[
L^+(u) = 
\begin{pmatrix}
1 & 0 & \ldots & 0 \\
\h f_{2,1}^+(u - c) & 1 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
\h f_{N,1}^+(u - c) & \ldots & \h f_{N,N-1}^+(u - c) & 1
\end{pmatrix}
\begin{pmatrix}
\k_1^+(u) & 0 & \ldots & 0 \\
0 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & \k_N^+(u)
\end{pmatrix}
\times 
\begin{pmatrix}
1 & \h e_{1,2}^+(u) & \ldots & \h e_{1,N}^+(u) \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & 1
\end{pmatrix},
\]

\[
L^-(u) = 
\begin{pmatrix}
1 & 0 & \ldots & 0 \\
\h f_{2,1}^-(u) & 1 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
\h f_{N,1}^-(u) & \ldots & \h f_{N,N-1}^-(u) & 1
\end{pmatrix}
\begin{pmatrix}
\k_1^-(u) & 0 & \ldots & 0 \\
0 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & \k_N^-(u)
\end{pmatrix}
\times 
\begin{pmatrix}
1 & \h e_{1,2}^-(u) & \ldots & \h e_{1,N}^-(u) \\
0 & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \ldots & \ldots & 1
\end{pmatrix},
\]

the defining relations in $DY_h'(gl_N)$ are given as follows:

\[
R^\pm(u - v)L^\pm_1(u)L^\pm_2(v) = L^\pm_2(v)L^\pm_1(u)R^\pm(u - v),
\]

\[
R^+(u - v - c)\h L^\pm_1(u)L^\pm_2(v) = L^\pm_2(v)L^\pm_1(u)R^+(u - v),
\]

and the notation

\[
L_1(u) = L(u) \otimes 1, \quad L_2(u) = 1 \otimes L(u),
\]

is used. From the above expression, if we set

\[
X^-_\mu(u) = f^+_{\mu+1,\mu}(u) - f^-_{\mu+1,\mu}(u),
\]

\[
X^+_\mu(u) = e^+_{\mu,\mu+1}(u) - e^-_{\mu,\mu+1}(u),
\]

which define an isomorphism between the Drinfeld realization and the RS realization. The statement can be proved following the Ding-Frenkel equivalence [F].

Similar as $DY_h'(gl_N)_c$, an associative algebra, the Yangian double with central $DY_h(sl_N)'_c$ is a subalgebra of $DY_h'(gl_N)_c$, which is obtained with decomposing the $DY_h'(gl_N)_c$ by the quantum determinant. If we define the following currents:
Now we can write the generating relations for
\[ K^\pm(u) = \prod_{\mu=1}^N k^\pm_\mu (u + (\mu - 1)h), \]
\[ E_\mu(u) = \frac{1}{h} X^+_\mu(u + \frac{\mu}{2}h), \quad F_\mu(u) = \frac{1}{h} X^-_\mu(u + \frac{\mu}{2}h). \] (2.9)

and their modes expansion are,
\[ H^+_\mu(u) = 1 + h \sum_{m \geq 0} h_{\mu,m} u^{-m-1}, \quad H^-_\mu(u) = 1 - h \sum_{m < 0} h_{\mu,m} u^{-m-1}, \]
\[ E_\mu(u) = \sum_{m \in \mathbb{Z}} e_{\mu,m} u^{-m-1}, \quad F_\mu(u) = \sum_{m \in \mathbb{Z}} f_{\mu,m} u^{-m-1}. \]

The Drinfeld generators of \( DY'_h(sl_N)_c \) are \( \{h_{\nu,m}, e_{\nu,m}, f_{\nu,m}, c | \nu = 1, 2, ..., N-1; m \in \mathbb{Z} \} \).

Now we can write the generating relations for \( DY'_h(sl_N) \) as follows \[18].

\[ [H^\pm_\mu, H^\pm_\nu] = 0, \quad [c, \text{everything}] = 0, \] (2.10)
\[ (u - v + B_{\mu,\nu}h \mp ch)(u - v - B_{\mu,\nu}h)H^\pm_\mu(u)H^\pm_\nu(v) \]
\[ = (u - v - B_{\mu,\nu}h \mp ch)(u - v + B_{\mu,\nu}h)H^\pm_\mu(u)H^\pm_\nu(v), \] (2.11)
\[ (u - v - B_{\mu,\nu}h)H^\pm_\mu(u)E_\nu(v) = (u - v + B_{\mu,\nu}h)E_\nu(v)H^\pm_\nu(u), \] (2.12)
\[ (u - v + B_{\mu,\nu}h - ch)H^\pm_\mu(u)F_\nu(v) = (u - v - B_{\mu,\nu}h + ch)F_\nu(v)H^\pm_\nu(u), \] (2.13)
\[ (u - v + B_{\mu,\nu}h)H^-_\mu(u)F_\nu(v) = (u - v - B_{\mu,\nu}h)F_\nu(v)H^-_\nu(u), \] (2.14)
\[ (u - v - B_{\mu,\nu}h)E_\nu(u)E_\nu(v) = (u - v + B_{\mu,\nu}h)E_\nu(v)E_\nu(u), \] (2.15)
\[ (u - v + B_{\mu,\nu}h)F_\nu(u)F_\nu(v) = (u - v - B_{\mu,\nu}h)F_\nu(v)F_\nu(u), \] (2.16)
\[ [E_\mu(u), F_\nu(v)] = \frac{\delta_{\mu,\nu}}{h} \left( \delta(u - (v + ch))H^\pm_\mu(u) - \delta(u - v)H^\pm_\nu(v) \right), \] (2.17)
\[ E_\mu(u_1)E_\mu(u_2)E_\nu(v) - 2E_\mu(u_1)E_\nu(v)E_\mu(u_2) \]
\[ + E_\nu(v)E_\mu(u_1)E_\mu(u_2) + (\text{replacement: } u_1 \leftrightarrow u_2) = 0 \quad \text{for } |\mu - \nu| = 1, \] (2.18)
\[ F_\mu(u_1)F_\mu(u_2)F_\nu(v) - 2F_\mu(u_1)F_\nu(v)F_\mu(u_2) \]
\[ + F_\nu(v)F_\mu(u_1)F_\mu(u_2) + (\text{replacement: } u_1 \leftrightarrow u_2) = 0 \quad \text{for } |\mu - \nu| = 1, \] (2.19)
\[ E_\mu(u)E_\nu(v) = E_\nu(v)E_\mu(u) \quad \text{for } |\mu - \nu| > 1, \] (2.20)
\[ F_\mu(u)F_\nu(v) = F_\nu(v)F_\mu(u) \quad \text{for } |\mu - \nu| > 1, \] (2.21)
\[ [K^\pm(u), K^\pm(v)] = 0, \] (2.22)
\[ K^+(u)K^-(v) = K^-(v)K^+(u), \] (2.23)
\[ [K^\epsilon(u), H^\pm_\mu(v)] = [K^\epsilon(u), E_\mu(v)] \]
\[ = [K^\epsilon(u), F_\mu(v)] = 0, \quad \forall \epsilon = \pm, \forall 1 \leq \mu \leq N. \] (2.24)
In this section we shall construct a free boson representation of $DY_h(gl_N)_c$, so the Fourier coefficients of the power series $K(u) - 1$ are the central elements of the $DY_h(gl_N)_c$. It is obvious that the Heisenberg subalgebra of $DY_h(gl_N)_c$ is generated by the quotient of $DY_h(gl_N)_c$ by the ideal generated by this central elements. In fact we see that the formula $K^\pm(u) = q - det.L^\pm(u)$, and the scalar function is chosen so that the quantum determinant commuting with the all of the currents and with itself.

3 Free boson with continuous parameter

In this section we shall construct a free boson representation of $DY_h(sl_N)_k$ with arbitrary level $k$ ($c = k$ henceforth). For $N = 2$ this problem has already been solved in Ref. [28]. For generic $N$, the Yangian double currents are obtained using the quantum-Yangian double correspondence [16], but however, screening currents and vertex operators can not be get in the same way, at least, they can not be expressed as a nice form. So the correspondence between them are somewhat nontrivial. In the next section, we introduce another kinds Heisenberg algebras to overcome these problems.

$N^2 - 1$ free boson fields are needed to construct the $DY_h(sl_N)_k$ generators with arbitrary level. So we introduce three set boson fields $\hat{\lambda}_\mu$ ($1 \leq \mu \leq N$), $\hat{b}_{\mu,\nu}$ and $\hat{c}_{\mu,\nu}$ ($1 \leq \mu < \nu \leq N$), and while $\hat{b}_{\mu,\nu} = \hat{c}_{\mu,\nu} = 0$, for $\mu \geq \nu$. The quantum Heisenberg algebra $A_{h,k}(sl_N)$ generated by $\hat{\lambda}_\mu(t)$, $\hat{b}_{\mu,\nu}(t)$, $\hat{c}_{\mu,\nu}(t)$ is involved with continuous parameters $t$ ($t \in \mathbb{R} - 0$), which enjoy the the following commutation relations

\[ [\hat{\lambda}_\mu(t), \hat{\lambda}_\mu(t')] = \frac{\sinh^2(t + t')}{\sinh \frac{h}{2} t} \delta(t + t'), \quad (3.25) \]
\[ [\hat{\lambda}_\mu(t), \hat{\lambda}_\nu(t')] = -\frac{\sinh^2(t + t')}{\sinh \frac{h}{2} t} e^{\text{sign}(\mu - \nu) \frac{N}{2}} \delta(t + t'), \quad (3.26) \]
\[ \sum_{\mu=1}^{N} e^{(1-\mu)ht} \hat{\lambda}_\mu(t) = 0; \quad (3.27) \]
\[ [\hat{b}_{\mu,\nu}(t), \hat{b}_{\mu',\nu'}(t')] = -\frac{1}{h^2 t} \sinh^2 \frac{ht}{2} \delta_{\mu,\mu'} \delta_{\nu,\nu'} \delta(t + t'), \quad (3.28) \]
\[ [\hat{c}_{\mu,\nu}(t), \hat{c}_{\mu',\nu'}(t')] = \frac{1}{h^2 t} \sinh^2 \frac{ht}{2} \delta_{\mu,\mu'} \delta_{\nu,\nu'} \delta(t + t'), \quad (3.29) \]

and the other commutators vanish identical, where $g = N$ is the dual Coxeter number for the $sl_N$ algebra. Here $\hat{\lambda}_\mu(t)$ are the "fundamental weight type" fields, while $\hat{b}_{\mu,\nu}(t)$, $\hat{c}_{\mu,\nu}(t)$ are "ghost type" fields in conformal field theory alike. In fact, the free boson fields defined above can be viewed as the Yangian limit of the elliptic algebra $A_{h,\eta}(sl_N)$, with $\eta \to 0$ [24].

The Fock space corresponding to the above Heisenberg algebras can be specified as follows. Let $|0>$ be the vacuum state defined by
\[ \lambda_\mu(t)|0 >= b_{\mu,\nu}(t)|0 >= c_{\mu,\nu}(t)|0 >= 0, \quad (t > 0), \] 

(3.30)

The Fock space \( \mathcal{F}(l_\lambda, l_b, l_c) \) is then generated by the actions of the negative modes of \( \lambda_\mu, b_{\mu,\nu}, c_{\mu,\nu} \). We shall see later that this Fock space actually forms a (Wakimoto-like \cite{6, 12}) module for the Yangian double \( DY_h(sl_N) \) with level \( k \).

We define the free boson fields \( \chi_{\mu,\nu}(\beta|u; \alpha) \) and \( \chi_{\pm}(u) \) through

\[ \chi_{\mu,\nu}(\beta|u; \alpha) = \exp \left\{ -\hbar \int_{-\infty}^{0} \frac{dt}{2\pi i \sinh \frac{\beta}{2} h t} \dot{\chi}_{\mu,\nu}(t)e^{\alpha h t}e^{-iut} \right\} \]

\[ \exp \left\{ -\hbar \int_{0}^{\infty} \frac{dt}{2\pi i \sinh \frac{\beta}{2} h t} \dot{\chi}_{\mu,\nu}(t)e^{-\alpha h t}e^{iut} \right\}, \] 

(3.31)

\[ \chi_{\mu,\nu}^+(u) = \exp \{ 2\hbar \int_{0}^{\infty} \frac{dt}{2\pi i} \dot{\chi}_{\mu,\nu}(t)e^{-iut} \} \]

(3.32)

\[ \chi_{\mu,\nu}^-(u) = \exp \{ -2\hbar \int_{\infty}^{-\infty} \frac{dt}{2\pi i} \dot{\chi}_{\mu,\nu}(t)e^{-iut} \}, \] 

(3.33)

where \( \chi_{\mu,\nu}(u) (\chi_{\mu,\nu}^\pm(u)) \) stands for \( \lambda_\mu(u), b_{\mu,\nu}(u) \) or \( c_{\mu,\nu}(u) (\lambda_\mu^\pm(u), b_{\mu,\nu}^\pm(u) \) or \( c_{\mu,\nu}^\pm(u) \)). Then following the standard procedure we have

\[ b_{\mu,\nu}(u)b_{\mu',\nu'}^-(v) = \exp \left\{ -\int_{\tilde{c}} \frac{dt}{2i\pi} \frac{\ln(-t)}{t} \frac{\sinh \frac{\beta}{2} h t}{2\pi} \delta_{\mu,\nu'} \delta_{\nu,\nu'} e^{-i(u-v)t} \right\} :b_{\mu,\nu}(u)b_{\mu',\nu'}^-(v): \]

\[ = \left( \frac{\Gamma(\frac{i(u-v)}{h} + \frac{1}{2})}{\Gamma(\frac{i(u-v)}{h} + \frac{1}{2})} \right) \delta_{\mu,\nu'} \delta_{\nu,\nu'} 
\]

\[ b_{\mu',\nu'}^-(v)b_{\mu,\nu}(u) \]

\[ = \left( \frac{u-v+i\frac{h}{2}}{u-v-i\frac{h}{2}} \right) \delta_{\mu,\nu'} \delta_{\nu,\nu'} 
\]

(3.34)

where the \( \tilde{c} \) indicate the circle from \( +\infty \) to \( 0 \) in the upper half plain and \( 0 \) to \( -\infty \) in the lower half plain \cite{20, 24}. To derive the eq. (3.34), the identity

\[ \int_{\tilde{c}} \frac{dt}{2i\pi t} \frac{e^{-ut}}{1 - e^{-t/\eta}} = \ln \Gamma(\eta u) + (\eta u - 1/2)(\gamma - \ln \eta) - \frac{1}{2} \ln 2\pi, \] 

(3.35)

is used, in which the \( \gamma \) is the Euler number. In the same way, we can get the following identities.

\[ b_{\mu,\nu}(u)b_{\mu',\nu'}^+(v) = \left( \frac{u-v+i\frac{h}{2}}{u-v-i\frac{h}{2}} \right) \delta_{\mu,\nu'} \delta_{\nu,\nu'} 
\]

\[ b_{\mu',\nu'}^+(v)b_{\mu,\nu}^+(u), \] 

(3.36)

\[ ^1\text{Here and below, all OPE relations should be understood to hold in the analytic continuation sense.} \]
\[
\begin{align*}
\frac{c_{\mu,\nu}(u)c_{\mu',\nu'}(v)}{c_{\mu',\nu'}(v)c_{\mu,\nu}(u)} &= \left(\frac{u - v - i\frac{k}{2}}{u - v + i\frac{k}{2}}\right) \delta_{\mu,\mu'}\delta_{\nu,\nu'}, \\
\frac{c_{\mu,\nu}(u)c_{\mu',\nu'}(v)}{c_{\mu',\nu'}(v)c_{\mu,\nu}(u)} &= \left(\frac{u - v - i\frac{k}{2}}{u - v + i\frac{k}{2}}\right) \delta_{\mu,\mu'}\delta_{\nu,\nu'}, \\
\lambda_\mu^+(u)\lambda_\mu(k + N|v; \alpha) &= \lambda_\mu(k + N|u; \alpha)\lambda_\mu^-(v), \\
\lambda_\nu(k + N|v; \alpha)\lambda_\nu^+(u) &= \lambda_\nu^-(v)\lambda_\nu(k + N|u; \alpha), \\
\lambda_\mu^+(u)\lambda_\mu^-(v) &= \frac{\Gamma\left(\frac{i(u-v)}{N} + \frac{\alpha}{2}\right)\Gamma\left(\frac{i(u-v)}{N} + \frac{\alpha-1}{N} + 1\right)}{\Gamma\left(\frac{i(u-v)}{N} + \frac{\alpha}{2} + 1\right)\Gamma\left(\frac{i(u-v)}{N} + \frac{\alpha+1}{N} + 1\right)}, \\
\lambda_\nu^+(u)\lambda_\nu^-(v) &= \frac{\Gamma\left(\frac{i(u-v)}{N} + \frac{\alpha}{2}\right)\Gamma\left(\frac{i(u-v)}{N} + \frac{\alpha-1}{N} + 1\right)}{\Gamma\left(\frac{i(u-v)}{N} + \frac{\alpha}{2} + 1\right)\Gamma\left(\frac{i(u-v)}{N} + \frac{\alpha+1}{N} + 1\right)}, \\
\lambda_\mu^+(u)\lambda_\mu^-(v) &= \left(\frac{u - v - i\frac{k}{2}}{u - v + i\frac{k}{2}}\right) \delta_{\mu,\mu'}\delta_{\nu,\nu'}b^+_{\mu',\nu'}(v)b^+_{\mu,\nu}(u), \\
\lambda_\nu^+(u)\lambda_\nu^-(v) &= \left(\frac{u - v - i\frac{k}{2}}{u - v + i\frac{k}{2}}\right) \delta_{\mu,\mu'}\delta_{\nu,\nu'}b^+_{\mu',\nu'}(v)b^+_{\mu,\nu}(u).
\end{align*}
\]
In this section, by using the results obtained in the above sections, we can define a
identities are useful. In fact, the relations of the arguments in different terms are decided

\[
\begin{align*}
\text{Proposition 1} \\
\text{There is a homomorphism } \psi_{h,k} \text{ from } DY_h(sl_N)_k \text{ to } A_{h,k}(sl_N), \text{ which is} \\
\text{defined on the generators as follows:}
\end{align*}
\]

\[
\begin{align*}
k_\mu^+(u) & \mapsto : \lambda_\mu^+(u + \frac{k - N}{4} i) \prod_{\alpha = 1}^{\mu - 1} \frac{b_{\alpha+1}^+(u + \mu - \alpha i)}{b_{\mu+1}^+(u + \mu - \alpha i)}; \\
k_\mu^-(u) & \mapsto : \lambda_\mu^-(u - \frac{k - N}{4} i) \prod_{\rho = 1}^{\mu - 1} \frac{b_{\rho+1}^-(u + \rho + \gamma i)}{b_{\mu+1}^-(u + \rho + \gamma i)}; \\
X_\nu^+(u) & \mapsto \sum_{m=1}^\nu : (b+c)_{m,\nu}(u - \frac{m - \nu - 1}{2} i) \left( \frac{b_{m,\nu+1}^+(u - \frac{m - \nu - 1}{2} i)}{(c+b)_{m,\nu+1}^+(u - \frac{m - \nu - 1}{2} i)} \right).
\end{align*}
\]

In the process of constructing the boson realization of \( DY_h(sl_N)_k \), the following four
identities are useful. In fact, the relations of the arguments in different terms are decided
by these identities. There are

\[
\begin{align*}
b_{\mu,\nu}^+(u) : b_{\mu,\nu}^+(v) b_{\mu',\nu'}(v - i \frac{k}{2}) : b_{\mu,\nu}^+(u) b_{\mu',\nu'}(v + i \frac{k}{2}) &= 1, \\
b_{\mu,\nu}^+(u) : b_{\mu,\nu}^+(v + i \frac{k}{2}) b_{\mu',\nu'}(v) b_{\mu',\nu'}(v + i \hbar)^{-1} : b_{\mu,\nu}^+(u) b_{\mu',\nu'}(v + i \hbar)^{-1} &= 1, \\
\int b_{\mu,\nu}^+(u + i \frac{k}{2}) b_{\mu,\nu}^+(u) b_{\mu',\nu'}(v) b_{\mu',\nu'}(u + i \hbar) \quad 1, \\
\int b_{\mu,\nu}^+(u + i \frac{k}{2}) b_{\mu,\nu}^+(u) b_{\mu',\nu'}(v) b_{\mu',\nu'}(u + i \hbar) &= 1, \\
\int b_{\mu,\nu}^+(u) b_{\mu,\nu}^+(u + i \frac{k}{2}) b_{\mu,\nu}^+(v) b_{\mu',\nu'}(v) b_{\mu',\nu'}(v + i \hbar) &= 1, \\
\int b_{\mu,\nu}^+(u) b_{\mu,\nu}^+(u + i \frac{k}{2}) b_{\mu,\nu}^+(v) b_{\mu',\nu'}(v) b_{\mu',\nu'}(v + i \hbar) &= 1.
\end{align*}
\]

\section{Free boson representation of \( DY_h(gl_N)_k \) and \( DY_h(sl_N)_k \)}

In this section, by using the results obtained in the above sections, we can define a
\( DY_h(sl_N)_k \) module on the \( \mathcal{F}_r, b, c \). Then with the help of Eq.(2.4), we get a module of
\( DY_h(sl_N)_k \) on the \( \mathcal{F}_r, b, c \). The results can be expressed in terms as:

\end{proof}
is well defined

\[ (k + b)_{\nu+1}(u - \frac{\nu - 1}{2}i\hbar) \] 
\[ \prod_{n=1}^{\nu-1} b_{n,\nu}(u - \frac{n-\nu}{2}i\hbar) \] 
\[ - b_{\nu,\nu}(u + \frac{\nu + m}{2}i\hbar)^{-1}(b + c)_{\mu,\nu}(u + \frac{\nu + m + 1}{2}i\hbar)^{-1} \] 
\[ \times \left[ (1 + \delta_{\nu,m}) b_{\nu,\nu}(u + \frac{\nu + m}{2}i\hbar)^{-1}(b + c)_{\mu,\nu}(u + \frac{\nu + m + 1}{2}i\hbar)^{-1} \right] \]
\[ \times \prod_{n=m+1}^{\nu} b_{n,\nu}(u + \frac{\nu + n - 1}{2}i\hbar) b_{\nu,\nu+1}(u + \frac{\nu + \nu + 1}{2}i\hbar) \]
\[ - \sum_{m=\nu+1}^{N} (b + c)_{\nu,m}(u - \frac{2k + m - \nu - 1}{2}i\hbar) \]
\[ \times \lambda_\nu^+(u + \frac{3k + g_\nu + i\hbar}{4}) \lambda_{\nu+1}^+(u - \frac{3k + g_\nu + i\hbar}{4})^{-1} \]
\[ \times \left[ (1 + \delta_{\nu,m+1}) \frac{b_{\nu+1,m}(u - \frac{2k + m - \nu - 1}{2}i\hbar)}{(b + c)_{\nu+1,m}(u - \frac{2k + m - \nu - 2}{2}i\hbar)} \right] \]
\[ \times \prod_{n=m}^{\nu} \frac{b_{\nu,n}(u - \frac{2k + n - 1}{2}i\hbar)}{b_{\nu+1,n}(u - \frac{2k + n - \nu - 1}{2}i\hbar)} \] 

is well defined \( \mathcal{F}_{\lambda, b, c} \), and satisfies the commutation relations:

\[ k_{\nu}^\pm(v)k_{\mu}^\pm(u) = k_{\mu}^\mp(u)k_{\nu}^\pm(v), \]
\[ \rho(u - v + ik\hbar)k_{\mu}^-(u)k_{\nu}^+(v) = k_{\mu}^-(v)k_{\nu}^+(u)\rho(u - v), \]
\[ \rho(u - v + ik\hbar)\frac{u - v + ik\hbar}{u - v - ik\hbar}k_{\mu}^+(u)k_{\nu}^-(v) = k_{\nu}^-(v)k_{\mu}^+(u)\frac{u - v}{u - v - ik\hbar}\rho(u - v), \quad \mu < \nu, \]
\[ \rho(u - v + ik\hbar)\frac{u - v + ik\hbar}{u - v - ik\hbar}k_{\mu}^-(u)k_{\nu}^-(v) = k_{\nu}^-(v)k_{\mu}^+(u)\frac{u - v + ik\hbar}{u - v}\rho(u - v), \quad \mu > \nu, \]
\[ k_{\mu}^\pm(u)X_{\mu}^+(v)k_{\mu}^\pm(u)^{-1} = \frac{u - v}{u - v - ik\hbar}X_{\mu}^+(v), \]
\[ k_{\mu+1}^\pm(u)X_{\mu}^+(v)k_{\mu+1}^\pm(u)^{-1} = \frac{u - v}{u - v + ik\hbar}X_{\mu}^+(v), \]
\[ k_{\mu}^-(u)X_{\mu}^-(v)k_{\mu}^-(u)^{-1} = \frac{u - v - ik\hbar}{u - v}X_{\mu}^-(v), \]
\[ k_{\mu+1}^-(u)X_{\mu}^-(v)k_{\mu+1}^-(u)^{-1} = \frac{u - v + ik\hbar}{u - v}X_{\mu}^-(v), \]
\[ k_{\mu}^+(u)X_{\mu}^-(v)k_{\mu}^+(u)^{-1} = \frac{u - v - ik\hbar + ik\hbar}{u - v + ik\hbar}X_{\mu}^-(v), \]
\[ k_{\mu+1}(u)X^{-}_{\nu}(v)k_{\mu+1}(u)^{-1} = \frac{u-v+i\hbar + ikh}{u-v+ikh}X^{-}_{\nu}(v), \]
\[ k_{\mu}^{\pm}(u)X^{\pm}_{\nu}(v)k_{\mu}^{\pm}(u)^{-1} = X^{\pm}_{\nu}(v), \quad \mu - \nu \geq 2, \quad \text{or} \quad \mu - \nu \leq -1 \]
\[ k_{\mu}^{\pm}(u)X^{-}_{\nu}(v)k_{\mu}^{\pm}(u)^{-1} = X^{-}_{\nu}(v), \quad \mu - \nu \geq 2, \quad \text{or} \quad \mu - \nu \leq -1, \]
\[(u-v \pm i\hbar)X_{\mu}^{\pm}(u)X_{\mu}^{\pm}(v) \simeq (u-v \mp i\hbar)X_{\mu}^{\pm}(v)X_{\mu}^{\pm}(u) \sim \text{reg.}, \]
\[ (u-v-i\hbar)X_{\mu}^{\pm}(u)X_{\mu+1}^{\pm}(v) \simeq (u-v)X_{\mu+1}^{\pm}(v)X_{\mu}^{\pm}(u) \sim \text{reg.}, \]
\[ (u-v)X_{\mu}^{-}(u)X_{\mu+1}^{-}(v) \simeq (u-v)X_{\mu+1}^{-}(v)X_{\mu}^{-}(u) \sim \text{reg.}, \]
\[ X_{\mu}^{\pm}(u_1)X_{\nu}^{\pm}(u_2)X_{\nu}^{\pm}(v) - 2X_{\mu}^{\pm}(u_1)X_{\nu}^{\pm}(v)X_{\mu}^{\pm}(u_2) \]
\[ + X_{\mu}^{\pm}(v)X_{\mu}^{\pm}(u_1)X_{\mu}^{\pm}(u_2) \]
\[ + \{u_1 \leftrightarrow u_2\} \sim 0 \quad |\mu - \nu| = 1, \]
\[ X_{\mu}^{\pm}(u)X_{\nu}^{\pm}(v) \simeq X_{\nu}^{\pm}(v)X_{\mu}^{\pm}(u) \sim \text{reg.} \quad |\mu - \nu| \geq 2, \]
\[ [X_{\mu}^{\pm}(u), X_{\nu}^{-}(v)] \simeq \text{reg.} + \delta_{\mu,\nu} \left( \delta(u - (v - i\hbar))k_{\mu+1}^{\pm}(u)k_{\mu}^{\pm}(u)^{-1} \right. \]
\[ \left. - \delta(u - v)k_{\mu+1}^{-}(v)k_{\mu}^{-}(v)^{-1} \right). \]

where \text{reg.} means some regular terms and \(\simeq\) and \(\sim\) imply “equals up to’ with difference of such terms.

\textbf{Proof:} A somewhat long but straightforward calculation verifies this proposition. Using the Eq.\((2.3)\) and Eq.\((4.50)\) to Eq.\((4.53)\), we have the following expressions:

\[ H_{\mu}^{+}(u) \leftrightarrow \lambda_{\mu}^{+}(u + \frac{k - N}{4} - i\mu\frac{\hbar}{2}) \lambda_{\mu+1}^{+}(u + \frac{k - N}{4} - i\mu\frac{\hbar}{2})^{-1} \]
\[ \prod_{m=1}^{\mu} \prod_{n=\mu+1}^{N} \frac{b_{m,\mu+1}(u - \frac{m-1}{2}i\hbar)}{b_{m,\mu}(u + \frac{m}{2}i\hbar)} \frac{b_{n,\mu}(u - \frac{m-1}{2}i\hbar)}{b_{n+1,\mu}(u + \frac{m}{2}i\hbar)} \]  
\[ H_{\mu}^{-}(u) \leftrightarrow \lambda_{\mu}^{-}(u - \frac{k - N}{4} - i\mu\frac{\hbar}{2}) \lambda_{\mu+1}^{-}(u - \frac{k - N}{4} - i\mu\frac{\hbar}{2})^{-1} \]
\[ \prod_{m=1}^{\mu} \prod_{n=\mu+1}^{N} \frac{b_{m,\mu+1}(u + \frac{m-1}{2}i\hbar)}{b_{m,\mu}(u + \frac{m}{2}i\hbar)} \frac{b_{n,\mu}(u + \frac{m-1}{2}i\hbar)}{b_{n+1,\mu}(u + \frac{m}{2}i\hbar)} \]  
\[ E_{\mu}(u) \leftrightarrow \frac{1}{\hbar} \sum_{m=1}^{\mu} : (b+c)_{m,\mu}(u - \frac{m-1}{2}i\hbar) \left\{ \frac{b_{m,\mu+1}(u - \frac{m-1}{2}i\hbar)}{(c+b)_{m,\mu+1}(u - \frac{m}{2}i\hbar)} \right\} \right. \]
\[ \left. - \frac{b_{\mu+1}(u - \frac{m-1}{2}i\hbar)}{(c+b)_{\mu+1}(u - \frac{m-1}{2}i\hbar)} \right\} \prod_{n=1}^{m-1} \frac{b_{n,\mu+1}(u - \frac{n-1}{2}i\hbar)}{b_{n,\mu}(u + \frac{n}{2}i\hbar)} \]  
\[ F_{\mu}(u) \leftrightarrow \frac{1}{\hbar} : \left\{ \sum_{m=1}^{\mu} (b+c)_{m,\mu+1}(u + \frac{m}{2}i\hbar) \right\} \]
\[ \lambda_{\mu}^{-}(u - \frac{k - N}{4} - i\mu\frac{\hbar}{2}) \lambda_{\mu+1}^{-}(u - \frac{k - N}{4} - i\mu\frac{\hbar}{2})^{-1} \]
\[ \times \left( 1 + \delta_{\mu,\mu} \right) \frac{b_{\mu,\mu}(u + \frac{m}{2}i\hbar)}{b_{\mu,\mu}(u + \frac{m}{2}i\hbar)} \]
\[-b^{+}_{m,\mu}(u + \frac{m}{2} i\hbar)^{-1}(b + c)_{m,\mu}(u + \frac{m + 1}{2} i\hbar)^{-1}\]
\[
\times \prod_{n=m+1}^{\mu} \prod_{s=m+1}^{N} \frac{b^{-}_{n,\mu+1}(u + \frac{n-1}{2} i\hbar)}{b^{-}_{n,\mu}(u + \frac{n}{2} i\hbar)} \frac{b^{-}_{\mu+1,s}(u + \frac{s-1}{2} i\hbar)}{b^{-}_{\mu+1,n}(u + \frac{s}{2} i\hbar)}
\]
\[-\frac{(b+c)_{\mu,m}(u - \frac{2k + m - 1}{2} i\hbar)}{(b+c)_{\mu+1,m}(u - \frac{2k+m-1}{2} i\hbar)}\]
\[\lambda^{\pm}_{\mu}(u - \frac{3k + N}{4} i\hbar - i\mu \frac{\hbar}{2}) \lambda^{\pm}_{\mu+1}(u - \frac{3k + N}{4} i\hbar - i\mu \frac{\hbar}{2})^{-1}\]
\[\times \left[ (1 + \delta_{m,\mu+1}) \frac{b^{+}_{\mu+1,m}(u - \frac{2k+m-1}{2} i\hbar)}{(b+c)_{\mu+1,m}(u - \frac{2k+m-1}{2} i\hbar)} \right] \]
\[-\frac{b^{-}_{\mu+1,m}(u - \frac{2K+m-1}{2} i\hbar)}{(b+c)_{\mu+1,m}(u - \frac{2k+m-1}{2} i\hbar)}\]
\[\times \prod_{n=m}^{N} \frac{b^{+}_{\mu,n}(u - \frac{2k+n}{2} i\hbar)}{b^{-}_{\mu+1,n}(u - \frac{2k+n}{2} i\hbar)} \right] \right\} ^{i}, \quad (4.57)\]

\[(1 \leq \mu \leq N - 1) \text{ without confusion. The following corollary are the direct result of the proposition 1.} \]

**Corollary 1** The fields $H^{\pm}_{\mu}(u)$, $E^{\pm}_{\mu}(u)$ defined in equations Eq.(4.54), Eq.(4.56) and Eq.(4.57) are well-defined on the Fock space $\mathcal{F}(\lambda, b_{\nu}, l_{\nu})$ and satisfy

\[\quad [H^{\pm}_{\mu}(u), H^{\mp}_{\nu}(v)] = 0, \quad (4.58)\]
\[\quad H^{\pm}_{\mu}(u)H^{\mp}_{\nu}(v) = \frac{u - v \pm i\hbar + iB_{\mu,\nu}\hbar}{u - v} \frac{u - v - iB_{\mu,\nu}\hbar}{u - v + iB_{\mu,\nu}\hbar} H^{\pm}_{\mu}(v)H^{\pm}_{\mu}(u), \quad (4.59)\]
\[\quad (u - v + iB_{\mu,\nu}\hbar)H^{\pm}_{\mu}(u)E_{\nu}(v) = (u - v - iB_{\mu,\nu}\hbar)E_{\nu}(v)H^{\pm}_{\mu}(u), \quad (4.60)\]
\[\quad (u - v - iB_{\mu,\nu}\hbar)H^{\mp}_{\mu}(u)F_{\nu}(v) = (u - v + iB_{\mu,\nu}\hbar)F_{\nu}(v)H^{\mp}_{\mu}(u), \quad (4.61)\]
\[\quad (u - v - iB_{\mu,\nu}\hbar + i\hbar)H^{\pm}_{\mu}(u)F_{\nu}(v) = (u - v + iB_{\mu,\nu}\hbar + i\hbar)F_{\nu}(v)H^{\pm}_{\mu}(u), \quad (4.62)\]
\[\quad E_{\mu}(u)E_{\nu}(v) \simeq E_{\nu}(v)E_{\mu}(u) \sim \text{reg. for } B_{\mu,\nu} = 0, \quad (4.63)\]
\[\quad F_{\mu}(u)F_{\nu}(v) \simeq F_{\nu}(v)F_{\mu}(u) \sim \text{reg. for } B_{\mu,\nu} = 0, \quad (4.64)\]
\[\quad (u - v + iB_{\mu,\nu}\hbar)E_{\mu}(u)E_{\nu}(v) \simeq (u - v - iB_{\mu,\nu}\hbar)E_{\nu}(v)E_{\mu}(u) \sim \text{reg. for } B_{\mu,\nu} \neq 0, \quad (4.65)\]
\[\quad (u - v - iB_{\mu,\nu}\hbar)F_{\nu}(v)F_{\mu}(u) \simeq (u - v + iB_{\mu,\nu}\hbar)F_{\mu}(u)F_{\nu}(v) \sim \text{reg. for } B_{\mu,\nu} \neq 0, \quad (4.66)\]
\[\quad E_{\mu}(u)F_{\nu}(v) \sim F_{\nu}(v)E_{\mu}(u) \]
\[\quad \frac{\delta^{\mu,\nu}}{i\hbar} \left( \frac{1}{u - v + i\hbar}H^{\pm}_{\mu}(u) - \frac{1}{u - v}H^{\mp}_{\mu}(v) \right), \quad (4.67)\]
\[\quad E_{\mu}(u_{1})E_{\mu}(u_{2})E_{\nu}(v) - 2E_{\mu}(u_{1})E_{\nu}(v)E_{\mu}(u_{2}) + E_{\nu}(v)E_{\mu}(u_{1})E_{\mu}(u_{2}) \]
\[\quad \text{+(replacement: } u_{1} \leftrightarrow u_{2}) \text{ = 0 for } |\nu - \mu| = 1, \quad (4.68)\]
\[ F_\mu(u_1)F_\nu(u_2)F_\nu(v) - 2F_\mu(u_1)F_\nu(v)F_\mu(u_2) \\
+ F_\nu(v)F_\mu(u_1)F_\mu(u_2) + (\text{replacement: } u_1 \leftrightarrow u_2) = 0 \text{ for } |\nu - \mu| = 1, \tag{4.69} \]

The corollary also followed by a straightforward calculation. For the sake of illustration, here we give only an example only. First, considering the case

\[
H_\mu^+(u)H_\mu^-(v) = \frac{u - v + i\hbar + i\hbar u - v - iNh - i\hbar}{u - v + i\hbar - i\hbar u - v - iNh + i\hbar} \prod_{n=1}^{N} \prod_{m=\mu+1}^{N} \left( \frac{u - v - i(n-1)\hbar}{u - v - i(n-2)\hbar} \frac{u - v - i(n+1)\hbar}{u - v - i(n+1)\hbar} \right) \frac{u - v - i(m-1)\hbar}{u - v - i(m+1)\hbar} \frac{u - v - i(m-2)\hbar}{u - v - i(m+1)\hbar} \right) H_\mu^-(v)H_\mu^+(u)
\]

\[
= \frac{u - v + i\hbar + i\hbar u - v - iNh - i\hbar}{u - v + i\hbar - i\hbar u - v - iNh + i\hbar} \prod_{n=1}^{N} \left( \frac{u - v - i(n-1)\hbar}{u - v - i(n-2)\hbar} \frac{u - v - i(n+1)\hbar}{u - v - i(n+1)\hbar} \right) H_\mu^-(v)H_\mu^+(u), \tag{4.70} \]

and

\[
H_\mu^+(u)H_{\mu+1}^-(v) = \frac{u - v + i\hbar - i\frac{\hbar}{2} u - v - iNh + i\frac{\hbar}{2}}{u - v + i\hbar + i\frac{\hbar}{2} u - v - iNh - i\frac{\hbar}{2}} \prod_{n=1}^{N} \left( \frac{u - v - i(n-3/2)\hbar}{u - v - i(n+1/2)\hbar} \frac{u - v - i(n+1/2)\hbar}{u - v - i(n+1/2)\hbar} \right) H_{\mu+1}^-(v)H_\mu^+(u)
\]

\[
= \frac{u - v + i\hbar - i\frac{\hbar}{2} u - v + i\frac{\hbar}{2}}{u - v + i\hbar + i\frac{\hbar}{2} u - v - i\frac{\hbar}{2} H_{\mu+1}^-(v)H_\mu^+(u), \tag{4.71} \]

\[
H_\mu^+(u)H_\nu^-(v) = H_\nu^-(v)H_\mu^+(u), \text{ for } |\mu - \nu| \geq 2. \tag{4.72} \]

The above three equation can express as the form Eq. (4.59). In the above calculation the identities in the above sections are used without mention. The remaining calculation are the similar.

**Remark 1** Obviously, if we replace the \( \hbar \) by \(-i\hbar \), the Eq. (4.54) to Eq. (4.68) are the same as that of the Yangian double \( DY_h(sl_N) \), Eq. (4.10) to Eq. (2.20). But, the realization of the Eq. (4.54), Eq. (4.54), Eq. (4.57) should not be considered to be isomorphic with the Yangian double \( DY_h(sl_N) \), because the free fields realization carries a continuous parameter, while the Yangian double \( DY_h(sl_N) \) has discrete mode. For \( g = sl_2 \), see [23] for more information on this point.
So, we give the following proposition:

**Proposition 2** Under a homomorphic map, the free fields realization Eq.(4.50), Eq.(4.51), Eq.(4.52), Eq.(4.53) and Eq.(4.54), Eq.(4.56), Eq.(4.57) give a representation of the Yangian double $\text{DY}_h(\mathfrak{gl}_N)$ and $\text{DY}_h(\mathfrak{sl}_N)$.

5 Screening currents

In [16] the author give a boson field realization for the currents $\text{DY}_h(\mathfrak{sl}_N)$ by the so called $q$-affine-Yangian double correspondence. There we use ordinary oscillators with discrete mode, and the screening currents and the vertex operators are difficult to obtain by such boson fields. Because of the importance of the vertex operators and screening currents in the field theories, so construction of screening currents in terms of free fields are one of the main objects in the theories. Especially if we know these quantities we could have been able to calculate the cohomology of the action of our bosonization formulas on the Fock spaces $\mathcal{F}(\lambda, l_b, l_c)$. This problem can be overcome by using the boson fields defined in the Eq.(3.25),Eq.(3.28),Eq.(3.29). And the screening currents $S^l(u)$, $l = 1, \ldots, N - 1$ can be expressed as the below nice form (under the homomorphism of course),

\[
S_\mu(u) \mapsto: \lambda_{\mu+1}(k + N|u; (k + N)/4)\lambda_{\mu}(k + N|u; (k + N)/4)^{-1}\tilde{S}_\mu(u) : \quad (5.73)
\]

where,

\[
\tilde{S}_\mu(u) \mapsto -\frac{1}{\hbar} \sum_{m = \mu+1}^{N} (b + c)_{\mu+1,m}(u - \frac{N - m}{2}i\hbar)
\]

\[
\left\{ b_{\mu,m}^-(u - \frac{N - m - 1}{2}i\hbar)^{-1} (c + b)_{\mu,m}(u - \frac{N - m - 1}{2}i\hbar)^{-1} - b_{\mu,m}^+(u - \frac{N - m}{2}i\hbar)^{-1} (c + b)_{\mu,m}(u - \frac{N - m}{2}i\hbar)^{-1} \right\}
\]

\[
\prod_{n = m+1}^{N} b_{\mu+1,n}^-(u - \frac{N - n}{2}i\hbar) b_{\mu,n}^-(u - \frac{N - n - 1}{2}i\hbar), \quad (5.74)
\]

Then by a direct calculation, one can prove that, the currents have the following properties.

\[
H^\pm_\mu(u)S_\nu(v) \simeq S_\nu(v)H^\pm_\mu(u) \sim \text{reg.},
\]
\[
E^\mu(u)S_\nu(v) \simeq S_\nu(v)E^\mu(u) \sim \text{reg.},
\]
\[
(u - v - iB_{\mu,\nu})\tilde{S}_\mu(u)\tilde{S}_\nu(v) \simeq (u - v - iB_{\mu,\nu})\tilde{S}_\nu(v)\tilde{S}_\mu(u) \sim \text{reg.} \quad (5.75)
\]
\[ F_\mu(u)S_\nu(v) \simeq S_\nu(v)F_\mu(u) \]
\[ \sim \text{reg.} + \frac{\delta_{\mu,\nu}}{i\hbar} \left( \frac{1}{u-v+\frac{N}{2}i\hbar} - \frac{1}{u-v-i\hbar-\frac{N}{2}i\hbar} \right) \lambda_{\mu+1}(k+N|u;-(k+N)/4)\lambda_\mu(k+N|u;-(k+N)/4)^{-1} \]
\[ = \text{reg.} + \delta_{\mu,\nu} (k+N) \partial_\nu \left( \frac{1}{u-v} \lambda_{\mu+1}(k+N|u;-(k+N)/4) \right. \]
\[ \left. \times \lambda_\mu(k+N|u;-(k+N)/4)^{-1} \right), \quad (5.76) \]

where
\[ k+\alpha \partial_u f(u) = \frac{1}{i\hbar} \left( f(u-i\alpha\frac{\hbar}{2}) - f(u+i\hbar+i\alpha\frac{\hbar}{2}) \right), \quad (5.77) \]

For the purpose to define the vertices, one first introduce the state, \(|p_\Lambda, 0, 0\rangle\). It can be easy to show that this state is just the highest weight state with weight \(\Lambda = l_1\Lambda_1 + \ldots + l_{N-1}\Lambda_{N-1} = (l, \ldots, l)\). If the vertex operators with weight is given as
\[ \Phi_\Lambda(u; \alpha) \mapsto \sum_{\mu=1}^{N-1} \exp\left\{ -\hbar \int_0^0 \frac{dt}{2\pi i \sinh \frac{k+N}{2} \hbar \sinh \frac{\hbar}{2} t} \hat{\lambda}_\mu(t)e^{\alpha\hbar t}e^{-iut} \right\} \]
\[ \exp\left\{ -\hbar \int_0^{+\infty} \frac{dt}{2\pi i \sinh \frac{k+N}{2} \hbar \sinh \frac{\hbar}{2} t} \hat{\lambda}_\mu(t)e^{-\alpha\hbar t}e^{-iut} \right\} ; \quad (5.78) \]

and the highest weight state of \(DY_h(sl_N)\), can be obtained from the vacuum \(|0\rangle\) state by this operators, so the vertex operator is similar as the primary fields in the conformal field theory. The vertex related to the intertwining operators of type one and type two will be discussed elsewhere.

### 6 Discussions

In this paper we establish the free boson representation of the Yangian double \(DY_h(sl_N)\) with arbitrary level \(k\). Such representations of the Yangian double \(DY_h(sl_N)\) are expected to be useful in calculating the correlation functions of various quantum integrable systems in (1+1)-spacetime dimensions, e.g. the spin Calogero-Sutherland model [35], quantum nonlinear Schrödinger equation [27] and some field theoretic models such as Thirring model, Gross-Neveu model with \(U(N)\) gauge symmetries etc. Our representation of \(DY_h(sl_N)\) may also be used to analyze the behavior of Yangian double at the critical level \(k = -g\), a very fascinating area of great interest of study [13], which is just the \(h-W\) algebra. For the length of the manuscript, we will discuss this problem in a separated paper.
Acknowledgments: One of the authors (Ding) would like to thanks Prof. K. Wu and Prof. Z. Y. Zhu for fruitful discussion, and he was supported in part by the "China postdoctoral Science Foundation".

References

[1] Abada, A., Bougourzi, A.H., El Gradechi, M.A.: A deformation of the Wakimoto construction, *Preprint CRM-1829* (1992).

[2] Awata, H., Odake, S., Shiraishi, J.: *Lett. Math. Phys.* 30 (1994), 207.

[3] Awata, H., Odake, S., Shiraishi, J.: *Commun. Math. Phys.* 162 (1994), 61.

[4] Bernard, D., LeClair, A.: *Nucl. Phys.* B399 (1993), 709.

[5] Bougourzi, A.H., Sebbar, A., A Hopf algebra isomorphism between two realizations of the quantum affine algebra $U_q(gl_2)$, *Preprint q-alg/9701004*, ITP-SB-97-01 (1997).

[6] Ding, J., and Frenkel, I.B.: *Commun. Math. Phys.* 156 (1993), 277.

[7] Ding, X. M., Hou, B. Y., Zhao, L., $\hat{h}$ (Yangian) deformation of Miura map and the Virasoro algebra, *Preprint q-alg/9701014*.

[8] Drinfeld, V.G.: *Soviet Math. Dokl.* 283 (1985), 1060.

[9] Drinfeld, V.G.: In *Proceedings of the International Congress of Mathematicians*, p798, Berkeley, (1987).

[10] Drinfeld, V.G.: *Sov. Math. Dokl.* 36 (1988), 212.

[11] Faddeev, L.D., Reshetikhin, N.Yu., Takhtajan, L.A.: *Advanced Series in Mathematical Physics, Vol.10*, Singapore, World Scientific (1989), p299.

[12] Feigin, B.L., Frenkel, E.: *Russ. Math. Surv.* 43 (1989), 221.

[13] Frenkel, E., Reshetikhin, N.: *Commun. Math. Phys.* 178 (1996), 237.

[14] Frenkel, I.B., Jing, N.: *Proc. Natl. Acad. Sci. USA* 85 (1988), 9373.

[15] Frenkel, I.B., Reshetikhin, N.: *Commun. Math. Phys.* 146 (1992), 1.

[16] Hou, B.Y, Zhao, L., Ding, X.M.: $q$-affine-Yangian double correspondence and free boson representation of Yangian double with arbitrary level, *Preprint q-alg/9701023*.

[17] Hou, B.Y., Yang, W. L. : A $\hbar$-deformation of the $W_N$ algebra and its Vertex operators, *Preprint q-alg/9701025*.
[18] Iohara, K.: Bosonic representations of Yangian double $DY_h(g)$ with $g = gl_N, \ sl_N$, J. Phys.A.: Math. Gen. 29, 4593(1996) (Preprint q-alg/9603033 (1996)).

[19] Iohara, K., Konno, M.: Lett. Math. Phys. 37 (1996), 319.

[20] Jombo, M., Miwa, T.: J. Phys.A.: Math. Gen. 29, 2923(1996).

[21] Khoroshkin, S., Tolstoy, V.: Lett. Math. Phys. 36 (1996), 373.

[22] Khoroshkin, S.: Central Extension of the Yangian Double. In Collection SMF, Colloque “Septièmes Rencontres du Contact Franco-Belge en Algèbre”, June 1995, Reins; Preprint q-alg/9602031.

[23] Khoroshkin, S., Lebedev, D.: Intertwining operators for the central extension of the Yangian double, Preprint q-alg/9602030 (1996).

[24] Khoroshkin, S., Lebedev, D., Pakuliak, S.: Elliptic algebra $\mathcal{A}_{p,q}$ in the scaling limit, Preprint q-alg/9702002.

[25] Kimura, K.: On free field representation of the quantum affine algebra $U_q(\widehat{sl}_2)$, Preprint RIMS-910 (1992).

[26] Kirillov, A. N, Reshetikhin, N. Yu.: Lett. Math. Phys. 12 (1986), 199.

[27] Kojima, T., Korepin, V.E.: Determinant representations for dynamical correlation functions of the quantum nonlinear Schrodinger equation, Preprint RIMS-1115 (1996).

[28] Konno, H.: Free field representation of level-$k$ Yangian double $DY(sl_2)_k$ and deformation of Wakimoto Modules, Lett. Math. Phys. 40 (1997), 321. Preprint YITP-96-10 (1996).

[29] LeClair, A., Smirnov, F.: Int. J. Mod. Phys. A7 (1992), 2997.

[30] Matsuo, A.: Phys. Lett. B308 (1993), 260.

[31] Matsuo, A.: Commun. Math. Phys. 160 (1994), 33.

[32] Reshetikhin, N. Yu., Semenov-Tyan-Shansky, M. A.: Let. Math. Phys. 19 (1990), 133.

[33] Shiraishi, J.: Phys. Lett. A171 (1992), 243.

[34] Smirnov, F.A.: Int. J. Mod. Phys. A7 suppl. 1B (1992), 813, 839.

[35] Takimura, K., Uglov, D.: The orthogonal eigenbasis and norms of eigenvectors in the spin Calogero-Sutherland model, Preprint RIMS-1114 (1996).

[36] Wakimoto, M.: Commun. Math. Phys. 104 (1986), 605.