THE SEMIGROUP STRUCTURE OF GAUSSIAN CHANNELS

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Abstract. We investigate the semigroup structure of bosonic Gaussian quantum channels. Particular focus lies on the sets of channels which are divisible, idempotent or Markovian (in the sense of either belonging to one-parameter semigroups or being infinitesimal divisible). We show that the non-compactness of the set of Gaussian channels allows for remarkable differences when comparing the semigroup structure with that of finite dimensional quantum channels. For instance, every irreversible Gaussian channel is shown to be divisible in spite of the existence of Gaussian channels which are not infinitesimal divisible. A simpler and known consequence of non-compactness is the lack of generators for certain reversible channels. Along the way we provide new representations for classes of Gaussian channels: as matrix semigroup, complex valued positive matrices or in terms of a simple form describing almost all one-parameter semigroups.

1. Introduction

A quantum channel describes the input-output relation of a quantum mechanical operation. Mathematically, it is described by a completely positive map which is trace-preserving (in the Schrödinger picture) or identity-preserving (in the Heisenberg picture). Considering equal input and output spaces we can concatenate quantum channels with themselves or other channels. Clearly, such a concatenation is again a valid quantum operation, so that the set of quantum channels forms a semigroup. Inverses only belong to this semigroup if the channels describe unitary evolution. Apart from this subgroup of reversible channels various subsets can be distinguished according to their semigroup properties; for instance (i) channels which are elements of one-parameter semigroups, i.e., physically speaking solutions of time-independent Markovian master equations, (ii) channels which are infinitesimal divisible, e.g., solutions of time-dependent Markovian master equations, (iii) channels which are divisible into others in a non-trivial way, (iv) channels which are indivisible, and (v) channels which are idempotent.
Whereas the characterization of one-parameter semigroups goes mainly back to the seventies [1, 2] the distinction of the above sets has been addressed more recently in [4] and methods to decide membership in (i) have been provided in [5]. The classical counterpart known as embedding problem for Markov chains was exhaustively studied in probability theory (see [3] and the references therein.)

The characterization of the above sets seems to become vastly more complex as the dimension $d$ of the considered system is increased: whereas for qubits ($d = 2$) essentially everything is known, the problem of deciding membership in (i) turns out to be NP-hard with increasing $d$ [6].

The present article is devoted to the study of the semigroup properties of *bosonic Gaussian channels* (i.e., ‘quasi-free’ maps)—a class where the underlying Hilbert space is infinite dimensional. The restriction to Gaussian channels is motivated by their physical relevance (they model, for instance, optical fibers and occur all along with quadratic interactions) and it is suggested by the mentioned complexity issue. Note that this restriction has two flavors: when we ask whether a channel is in one of the above sets we do not only restrict the channel under consideration, we also restrict the involved one-parameter semigroups or factorizations to within the Gaussian setting. In this way we can escape from infinite dimensional Hilbert space into finite dimensional phase space and formulate everything in terms of finite dimensional matrix analysis.

At this point one might expect that the basic picture of the set of finite dimensional quantum channels carries over to the Gaussian setting. There are, however, crucial differences, for instance: whereas the set of quantum channels in finite dimensions is compact, the set of Gaussian channels (even though having a finite dimensional parametrization) is not; every irreversible Gaussian channel turns out to be divisible in spite of the existence of Gaussian channels which are not infinitesimal divisible etc.

The article is organized as follows: Sec.2 introduces the basic notation and Sec.3 shows that the set of Gaussian channels is indeed a matrix semigroup. In Sec.4 which is mainly provided for completeness, we review the reversible case and emphasize the fact that not every canonical transformation has a generating quadratic Hamiltonian. Sec.5 deals with one-parameter semigroups, for which a simple representation is provided, and Sec.6 has a closer look at infinitesimal divisible channels. In Sec.7 we prove that all irreversible Gaussian channels are divisible by exploiting a simple mapping from the set of Gaussian channels to the cone of complex positive matrices. Finally,
in Secs. 8, 9 idempotent and gauge-covariant channels are investigated before Sec. 10 concludes with some open questions.

2. Gaussian channels

Let $Q_j, P_j, j = 1, \ldots, n$, be the canonical operators satisfying the canonical commutation relations ($\hbar = 1$)

$$[Q_j, P_k] = i\delta_{jk}, \quad [Q_j, Q_k] = [P_j, P_k] = 0.$$ 

We will use a notation $R = (Q_1, P_1, \ldots, Q_n, P_n)^T$ and for each $\xi \in \mathbb{R}^2n$, we define $W_\xi = e^{i\xi^T \sigma R}$. Here we have denoted

$$\sigma \equiv \sigma_n = \oplus_{i=1}^n \sigma_1, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$ 

The unitary operators $W_\xi$ are called Weyl operators and they correspond to displacements in phase space.

A Gaussian channel is a quantum channel which maps Gaussian states into Gaussian states [7, 8, 9]. The mathematical form of Gaussian channels is best described in the Heisenberg picture when we look at their action on the Weyl operators. A Gaussian channel then corresponds to a mapping

$$W_\xi \mapsto W_{X^T \xi} e^{-\frac{1}{2} \xi^T Y \xi},$$

where $X, Y$ are real $2n \times 2n$-matrices. Here and thereafter without loss of generality we restrict to channels which map zero-mean states into zero-mean states. Complete positivity (cp) imposes a constraint on the matrices $X, Y$, which can be written as

$$Y \geq i (\sigma - X \sigma X^T).$$

From now on, we will identify the Gaussian channels with the pairs $(X, Y)$ of real $2n \times 2n$-matrices satisfying (1). We denote by $\mathcal{G}$ the set of all Gaussian channels.

It is often useful to depict a Gaussian channel by its action on the first and second moments of a quantum state. We denote the first moments by a vector $d \in \mathbb{R}^{2n}$ whose components are expectation values $d_k \equiv \langle R_k \rangle$ and we define the covariance matrix as

$$\Gamma_{kl} = \langle \{R_k - d_k, R_l - d_l\}_+ \rangle.$$ 

A Gaussian channel $(X, Y)$ then acts as

$$d \mapsto Xd,$$

$$\Gamma \mapsto X \Gamma X^T + Y.$$
Example 1. The preparation of a Gaussian state is a simple instance of a Gaussian channel. A Gaussian channel with $X = 0$ has an input-independent output state with covariance matrix $Y$. The cp-condition (1) then reduces to $Y + i\sigma \geq 0$, which is nothing but the condition for $Y$ to be a valid covariance matrix.

Generally speaking, the $Y$ contribution in a Gaussian channel $(X, Y)$ can be regarded as noise term. It follows from (1) that $Y \geq 0$. If $Y = 0$, then the cp-condition (1) implies that $X\sigma X^T = \sigma$, meaning that $X$ is an element of the real symplectic group $Sp(2n)$. The group of unitary Gaussian channels is therefore identified with $Sp(2n)$. Let us notice, however, that generally $X$ can be any real matrix, as long as sufficient noise is added (i.e., $Y$ is large enough).

3. Semigroup Product

Concatenation of two Gaussian channels is again a Gaussian channel. Hence, the set of Gaussian channels forms a semigroup. The semigroup product is given by

$$(X_1, Y_1) \cdot (X_2, Y_2) = (X_1 X_2, Y_1 + X_1 Y_2 X_1^T)$$

(2)

and the identity element is $(I, 0)$.

Let us recall that we are using the Heisenberg picture and in the Schrödinger picture the order of the product is opposite to the order of application of channels. For instance, if $(X_1, Y_1)$ and $(X_2, Y_2)$ describe optical fibers, then the product $(X_1, Y_1) \cdot (X_2, Y_2)$ corresponds to the channel in which the signal first goes through $(X_2, Y_2)$ and then through $(X_1, Y_1)$.

The semigroup product (2) can also be written as an ordinary matrix product. For each $X, Y \in M_{2n}(\mathbb{R})$, we denote $x = X \otimes X$ and $y \in \mathbb{R}^{2n} \otimes \mathbb{R}^{2n}$ is the column vector defined through the condition

$$\langle e_i \otimes e_j | y \rangle = \langle e_i | Y e_j \rangle .$$

(3)

Then the mapping $\pi : \mathcal{G} \to M_{4n^2+2n+1}(\mathbb{R})$ with

$$\pi(X, Y) = \begin{pmatrix} x & y & 0 \\ 0 & 1 & 0 \\ 0 & 0 & X \end{pmatrix}$$

is an injective homomorphism. We conclude that the semigroup of Gaussian channels is a matrix semigroup—a useful property when it comes to the discussion of one-parameter semigroups.
4. Reversible channels

A channel is called reversible if it has an inverse which is also a channel (i.e., we mean physically reversible as opposed to mathematically invertible). Reversible channels are exactly the unitary channels, and a Gaussian channel \((X, Y)\) is reversible iff \(X \in Sp(2n)\) and \(Y = 0\).

For completeness we briefly discuss the group structure of reversible channels, which already exhibits some interesting features. The sympletic group \(Sp(2n)\) is a non-compact connected Lie group whose Lie algebra \(sp(2n)\) is given by all real matrices \(s\) such that \((s \sigma)^T = s \sigma\).

The exponential map
\[
\exp : sp(2n) \to Sp(2n), \quad s \mapsto e^s
\]
is not surjective - a common feature of non-compact Lie groups.

In physical terms, the lack of surjectivity of the exponential map means that there are canonical transformations \(S \in Sp(2n)\) for which there is no ‘Hamiltonian matrix’ \(s \in sp(2n)\) generating them via \(S = e^s\).

Clearly, if we consider the corresponding unitary \(U_S\) acting on Hilbert space (i.e., an element of the metaplectic representation of \(Sp(2n)\) \cite{10}), then there is always a Hamiltonian \(\tilde{H}\) such that \(U_S = \exp i \tilde{H}\). Such a Hamiltonian may be obtained from the spectral decomposition of \(U_S\).

The point is, however, that \(\tilde{H}\) cannot be a quadratic expression in the canonical operators if \(S \not\in \exp(sp(2n))\).

A necessary condition for \(S \in \exp(sp(2n))\) is that \(S\) has a real logarithm. We recall the following standard result from matrix analysis \cite{11}.

**Proposition 1.** A real matrix \(X \in M_n(\mathbb{R})\) has a real logarithm \(L \in M_n(\mathbb{R})\) (i.e. \(X = e^L\)) iff \(X\) is non-singular and the Jordan blocks of \(X\) corresponding to negative eigenvalues have even multiplicities.

The characterization of the set of \(\exp(sp(2n))\) goes back to Williamson \cite{12}. For our purposes, the following partial characterization should suffice.

**Proposition 2.** Let \(S \in Sp(2n)\). If \(-1\) is not an eigenvalue of \(S\), then \(S \in \exp(sp(2n))\) iff \(S\) has real logarithm.

A simple consequence of Prop. \ref{prop2} is that if \(S \in Sp(2n)\) is positive, then \(S \in \exp(sp(2n))\).

An important subgroup \(K(2n)\) of \(Sp(2n)\) is formed by those matrices which are in addition orthogonal, i.e., \(K(2n) = Sp(2n) \cap SO(2n)\). The corresponding maps are called passive transformations as they preserve the number of particles \cite{10}. The subgroup \(K(2n)\) is a (maximal) compact subgroup of \(Sp(2n)\) and it is isomorphic to \(U(n)\). Consequently,
the exponential map from the Lie algebra \( \mathfrak{k}(2n) \) to \( K(2n) \) is surjective. That is, every \( S \in K(2n) \) has the property that \( S \in \exp(\mathfrak{k}(2n)) \subset \exp(\mathfrak{sp}(2n)) \).

Using the Euler decomposition (cf. [10]) each \( S \in \mathfrak{sp}(2n) \) can be written as a product \( S = K_1DK_2 \), where \( K_1, K_2 \in K(2n) \) and \( D \in \mathfrak{sp}(2n) \) is a diagonal matrix of the form \( D = \text{diag}(d_1, 1/d_1, \ldots, d_n, 1/d_n) \) with \( d_1, \ldots, d_n > 0 \). The diagonal matrix \( D \) describes single-mode squeezings. Thus, every reversible Gaussian channel is a concatenation of passive transformations and single-mode squeezings.

From the previous discussion we arrive at the following conclusion.

**Proposition 3.** Let \( S \in \mathfrak{sp}(2n) \). There are \( S_1, S_2 \in \exp(\mathfrak{sp}(2n)) \) such that \( S = S_1S_2 \).

**Proof.** Since \( S \) has an Euler decomposition \( S = K_1DK_2 = K_1DK_1^TK_1^{-T}K_2 \), it is a product of a positive \( S_1 := K_1DK_1^T \) and an orthogonal \( S_2 = K_1^{-T}K_2 \) symplectic matrix. Both \( S_1, S_2 \in \exp(\mathfrak{sp}(2n)) \), as discussed earlier. \( \square \)

5. **One-parameter Gaussian semigroups**

5.1. **General form.** By one-parameter semigroup of Gaussian channels we mean a family of Gaussian channels, parametrized by \( \mathbb{R}^+ \), which satisfies the following conditions:

(i) **continuity:** \( X_t \) and \( Y_t \) depend on \( t \in \mathbb{R}^+ \) in a continuous way

(ii) **semigroup property:** \( X_tX_s = X_{t+s} \) and \( Y_{t+s} = Y_t + X_tY_sX_t^T \).

(iii) **connected to the identity:** \( X_0 = \mathbb{1} \) and \( Y_0 = 0 \)

As we have seen in Section 3, the semigroup of Gaussian channels is a matrix semigroup. It follows that every one-parameter semigroup of Gaussian channels has a generator [13]. In particular, the mappings \( t \mapsto X_t \) and \( t \mapsto Y_t \) are differentiable. The following characterization (although in a slightly different form) has been derived in [14].

**Proposition 4.** A family \((X_t, Y_t)_{t \geq 0}\) of Gaussian channels forms a one-parameter semigroup iff there exists real matrices \( A, B, H \) with \( iA + B \geq 0 \), \( A^T = -A \), \( H^T = H \) such that

\[
X_t = e^{t(A-H)\sigma}, \quad (5)
\]

\[
Y_t = 2 \int_0^t X_s^T \sigma B \sigma X_s \, ds. \quad (6)
\]
In the Heisenberg picture this corresponds to an evolution of any observable $O$ governed by a master equation $\partial_t O = \mathcal{L}(O)$ with a Liouvillian

\begin{align}
\mathcal{L}(O) &= i[\hat{H},O] + \sum_\alpha \hat{L}_\alpha^* O \hat{L}_\alpha - \frac{1}{2} \{ \hat{L}_\alpha^* \hat{L}_\alpha, O \}_+, \quad (7) \\
\hat{H} &= \frac{1}{2} \sum_{kl} H_{kl} R_k R_l, \quad (8) \\
\hat{L}_\alpha &= \sum_k L_{\alpha,k} R_k, \quad \text{with} \quad B + iA = L^* L. \quad (9)
\end{align}

A simple consequence of this characterization is the following.

**Corollary 1.** Let $X \in M_{2n}(\mathbb{R})$. The following conditions are equivalent:

(i) There exists a matrix $Y \in M_{2n}(\mathbb{R})$ such that $(X,Y)$ is an element of a one-parameter semigroup of Gaussian channels.

(ii) $X$ is non-singular and the Jordan blocks of $X$ corresponding to negative eigenvalues have even multiplicities.

**Proof.** By Prop. 1, the condition (ii) on $X$ is nothing but the existence of a real logarithm $L$ which is clearly necessary for $X$ to occur in an element of a one-parameter semigroup of Gaussian channels. Hence, (i) implies (ii).

In order to see the other direction, suppose that (ii) holds and let $L$ be a real logarithm of $X$. Let us decompose $L\sigma^T = A - H$ into a symmetric part $H = H^T$ and anti-symmetric part $A = -A^T$, respectively. Then there is always a $B \in M_{2n}(\mathbb{R})$ (e.g. $B = \|A\|_\infty \mathbb{1}$) such that $iA + B \geq 0$ and we can construct a one-parameter semigroup of Gaussian channels by following the characterization in Prop. 1. \qed

Prop. 1 gives a complete but rather cumbersome characterization of one-parameter semigroups of Gaussian channels. In particular, the appearing integral might complicate further use of the characterization. Fortunately, almost all generators of such semigroups allow for a simpler representation discussed in the next subsection.

**5.2. Simple form.** Suppose that $\{X_t\}_{t \geq 0}$ is a semigroup and fix a real symmetric matrix $\mathcal{Y}$. Then by setting

$$Y_t = \mathcal{Y} - X_t \mathcal{Y} X_t^T$$

the semigroup property for $(X_t,Y_t)_{t \geq 0}$ is satisfied. The cp-condition now reads

$$\mathcal{Y} - i\sigma \geq X_t (\mathcal{Y} - i\sigma) X_t^T. \quad (11)$$
Example 2. An amplification channel is of the form \( X = \sqrt{\eta} \mathbb{1}, \) \( Y = (\eta - 1) \mathbb{1} \) for some \( \eta \in (1, \infty) \). Amplification channels form a one-parameter semigroup. Namely,
\[
X_t = e^{t/BD}, \quad Y_t = (e^{2t} - 1)\mathbb{1}.
\]
This is of the simple form (10) with \( Y = -\mathbb{1} \).

Proposition 5. A one-parameter Gaussian semigroup \((X_t, Y_t)_{t \geq 0}\) is of the simple form (10) if the operator \((A - H)\sigma\) in Prop. 4 (i.e., the generator of \(X_t\)) does not have a pair of eigenvalues of the form \( \pm \lambda \).

Proof. The general form (6) gives \( \dot{Y}_0 = 2 \sigma^T B \sigma \equiv \tilde{B} \). On the other hand, the simple form (10) leads to
\[
\dot{Y}_0 = \tilde{A} Y + Y \tilde{A}^T,
\]
where \( \tilde{A} = (H - A)\sigma \). Hence, in order for \((X_t, Y_t)_{t \geq 0}\) to be of the form (10), we need to find \( \lambda \) such that
\[
\tilde{A} Y + Y \tilde{A}^T = \tilde{B}.
\]
The linear equation (12) can be written as
\[
(\mathbb{1} \otimes \tilde{A}^T + \tilde{A} \otimes \mathbb{1}) |\lambda\rangle = |\tilde{B}\rangle
\]
Hence, if \( \mathbb{1} \otimes \tilde{A}^T + \tilde{A} \otimes \mathbb{1} \) is invertible, then we have a solution for \( \lambda \).

The eigenvalues of \( \mathbb{1} \otimes \tilde{A}^T + \tilde{A} \otimes \mathbb{1} \) are of the form \( \lambda_i + \lambda_j \), where \( \lambda_i, \lambda_j \) are eigenvalues of \( \tilde{A} \) (this can be seen e.g. using Schur upper-triangular form for \( \tilde{A} \) and \( \tilde{A}^T \)). Therefore, if \( \tilde{A} \) has the property that the sum of any two of its eigenvalues is nonzero, the invertibility of \( \mathbb{1} \otimes \tilde{A}^T + \tilde{A} \otimes \mathbb{1} \) follows.

If \( \lambda \) is a solution then \( \lambda^T \) and hence \( \frac{1}{2} (\lambda + \lambda^T) \) are again solutions. Therefore, \( \lambda \) can be chosen symmetric. \( \square \)

Prop. 5 implies that almost all one-parameter semigroups admit a representation of the simple form in Eq. (10). Moreover, it shows that we can approximate any one-parameter Gaussian semigroup with a one-parameter semigroup of the simple form. Namely, assume \((X_t, Y_t)_{t \geq 0}\) is a one-parameter Gaussian semigroup which is not of the simple form (10). Suppose that \( A, B, H \) are as in Prop. 4. We can clearly make an arbitrarily small change in the matrices \( A \) and \( H \) (hence getting new matrices \( A' \) and \( H' \)) in a way that the set of eigenvalues of \((A' - H')\sigma \) do not contain pairs of the type \( \pm \lambda \). If necessary, we also make a small change to \( B \), obtaining \( B' \), to guarantee the condition \( iA' + B' \geq 0 \). Hence, there exists a one-parameter semigroup \((X'_t, Y'_t)_{t \geq 0}\) of the simple form which has generating matrices \( A', B', H' \) arbitrary close to \( A, B, H \).
Example 3. Not all one-parameter Gaussian semigroups are of the simple form (10). For instance, suppose \( n = 1 \) and choose \( A = 0 \) together with \( H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

In this case \( \tilde{A} = (H - A)\sigma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \).

The generated matrix \( X_t = e^{-t\tilde{A}} \) corresponds to squeezing.

With any \( Y \), the matrix \( \tilde{A}Y + Y\tilde{A} \) is diagonal. Therefore, the condition (12) cannot be satisfied whenever \( B \) (and hence \( \tilde{B} \)) is a non-diagonal matrix. On the other hand, since \( A = 0 \) any positive matrix is a possible choice for \( B \).

5.3. Bounded evolutions. We say that a one-parameter semigroup \((X_t, Y_t)_{t \geq 0}\) has bounded noise term if there is a constant \( c \) such that \( \|Y_t\| \leq c \) for all \( t \in \mathbb{R}^+ \).

Proposition 6. Let \((X_t, Y_t)_{t \geq 0}\) be a one-parameter semigroup of Gaussian channels. The following conditions are equivalent:

(i) There exists a positive matrix \( Y \) such that

\[
Y_t = Y - X_t Y X_t^T. \tag{13}
\]

(ii) \((X_t, Y_t)_{t \geq 0}\) has bounded noise term.

Proof. Suppose that (i) holds. For every \( t \in \mathbb{R}^+ \), we then have

\[
0 \leq Y_t = Y - X_t Y X_t^T \leq Y. \tag{14}
\]

Suppose that (ii) holds. Let \( \mu \) be an invariant mean of the semigroup \( \mathbb{R}^+ \) (see e.g. [15]). By the assumption, each matrix entry \( t \mapsto [Y_t]_{ij} \) is a bounded continuous function. We define the matrix \( Y \) by

\[
[Y]_{ij} = \mu([Y]_{ij}).
\]

As \( \mu \) is invariant, an application to the second semigroup condition \( Y_{t+s} = Y_t + X_t Y_s X_t^T \) gives the formula (13). Moreover, \( Y \geq 0 \) since \( Y_t \geq 0 \) for each \( t \in \mathbb{R}^+ \) and

\[
\langle v | Y v \rangle = \mu(\langle v | Y v \rangle) \geq 0.
\]

Let \((X_t, Y_t)_{t \geq 0}\) be a one-parameter semigroup and suppose there is a \( Y \) satisfying (10). The matrix \( Y \) is a valid covariance matrix (e.g., of a Gaussian state) iff

\[
Y - i\sigma \geq 0. \tag{15}
\]
Thus, in this situation the one-parameter semigroup \((X_t, Y_t)_{t \geq 0}\) has an invariant Gaussian state.

This is the case if the semigroup \((X_t)_{t \geq 0}\) is strictly contractive: 
\[ \|X_t\| < 1 \text{ for } t > 0. \]
Then we have \(\lim_{t \to \infty} X_t = 0\) so that \((11)\) becomes \((15)\).

**Example 4.** An *attenuation channel* is of the form 
\[
X_t = \sqrt{\eta} \mathbb{1}, \quad Y_t = (1 - \eta) \mathbb{1}
\]
for some \(\eta \in (0, 1)\). Attenuation channels form a bounded one-parameter semigroup. Namely, 
\[
X_t = e^{-t} \mathbb{1}, \quad Y_t = (1 - e^{-2t}) \mathbb{1}.
\]
For this one-parameter semigroup we have \(\mathcal{Y} = \mathbb{1}\). Therefore, the vacuum state (with covariance matrix \(\Gamma = \mathbb{1}\) and first moments \(d = 0\)) is an invariant state for the one-parameter semigroup of attenuation channels.

We notice that a one-parameter semigroup \((X_t, Y_t)_{t \geq 0}\) may have bounded noise term without having an invariant Gaussian state. For a simple example suppose that \((X_t, Y_t)_{t \geq 0}\) is a one-parameter semigroup of reversible channels. Then \(X_t = e^{-tH}\) for some symmetric matrix \(H\) and \(Y_t = 0\). If \(X_t\) is not orthogonal (in which case \(\mathcal{Y} \propto \mathbb{1}\)) we have \(\mathcal{Y} = 0\), which is clearly not a valid covariance matrix.

**6. Infinitesimal divisible channels**

We call a Gaussian channel parameterized by \((X, Y)\) *infinitesimal divisible* if either

(a) for every \(\varepsilon > 0\) there exists a finite set of Gaussian channels \((X_i, Y_i)\) such that:
   
   (i) \(\|(X_i, Y_i) - (\mathbb{1}, 0)\| < \varepsilon\)
   
   (ii) \(\prod_i (X_i, Y_i) = (X, Y)\)

or

(b) the channel can be approximated arbitrarily well with (a)-type of channels.

We note that in the classical case \([3]\) this is what is called limit of a null triangular array, and it is proved there that such limits are precisely solutions of time-dependent Kolmogorov equations. Also in the quantum case solutions of time-dependent Markovian master equations are clearly infinitesimal divisible. For finite dimensional quantum systems the close relation between the two sets has been studied in \([4]\).

The concatenation of two infinitesimal divisible channels is clearly infinitesimal divisible. The infinitesimal divisible channels thus form a subsemigroup of \(\mathcal{G}\).
It is clear that if \((X, Y)\) is an element of a one-parameter semigroup of Gaussian channels, then it is infinitesimal divisible. The converse is, however, not true. Namely, suppose that \(S\) is a symplectic matrix such that \(S \notin \exp(\mathfrak{sp}(2n))\). Then the channel \((S, 0)\) is not an element of a one-parameter Gaussian semigroup, but by Prop. \(8\) we have symplectic matrices \(S_1, S_2\) such that \(S = S_1 S_2\) and \(S_1, S_2 \in \exp(\mathfrak{sp}(2n))\). Since the channels \((S_1, 0)\) and \((S_2, 0)\) are infinitesimal divisible, so is also \((S, 0)\).

The following is a simple necessary condition for a channel to be infinitesimal divisible.

**Proposition 7.** If a Gaussian channel \((X, Y)\) is infinitesimal divisible, then \(\det X \geq 0\).

**Proof.** The claim follows from the continuity and multiplicativity of the determinant. \(\square\)

**Example 5.** The Gaussian channel describing a phase conjugating mirror (with minimal noise) corresponds to the choices

\[
X = \mathbb{1}_n \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = 2\mathbb{1}_{2n}.
\]

If \(n\) is odd, then \(\det(X) < 0\) and therefore \((X, Y)\) is not infinitesimal divisible.

In the spirit of Corollary \([1]\) we can formulate a converse of Prop. \([7]\).

**Proposition 8.** If \(X \in M_{2n}(\mathbb{R})\) satisfies \(\det(X) > 0\), then there exists a \(Y \in M_{2n}(\mathbb{R})\) such that \((X, Y)\) is an infinitesimal divisible Gaussian channel.

**Proof.** Consider the real Jordan decomposition \(X = MJM^{-1}\) and group the Jordan blocks in \(J\) such that \(J = J_- \oplus J_r\) with \(J_-\) being the collection of all Jordan blocks with negative eigenvalues and \(J_r\) containing all the others. By defining \(X_1 \equiv M(-I) \oplus \mathbb{1} M^{-1}\) and \(X_2 \equiv M(-J_-) \oplus J_r M^{-1}\) we get \(X = X_1 X_2\). Since \(\det(X) > 0\), the multiplicity of the eigenvalue \(-1\) of \(X_1\) is even. By Proposition \(1\), the \(X_i\)'s now have real logarithms which implies by Corollary \([1]\) that there are \(Y_i\)'s such that \((X_i, Y_i)\) are elements of a one-parameter semigroup of Gaussian channels. Consequently, \((X, Y)\) is infinitesimal divisible if we choose \(Y = Y_1 + X_1 Y_2 X_1^T\). \(\square\)

We also have the following simple observation.

**Proposition 9.** Let \((X, Y)\) be infinitesimal divisible Gaussian channel. Then every Gaussian channel \((X, \tilde{Y})\) with \(\tilde{Y} \geq Y\) is infinitesimal divisible.
Proof. We can split the additional noise $\tilde{Y} - Y$ into arbitrarily small pieces, $\prod_{j=1}^{m}(1, \frac{1}{m}(\tilde{Y} - Y)) = (1, \tilde{Y} - Y)$. □

7. Divisible channels

We call a Gaussian channel $(X, Y)$ divisible if there exist two non-reversible Gaussian channels $(X_1, Y_1)$ and $(X_2, Y_2)$ such that

$$(X_1, Y_1) \cdot (X_2, Y_2) = (X, Y).$$

In this section we show that actually all non-reversible Gaussian channels are divisible—in spite of the existence of Gaussian channels which are not infinitesimal divisible. This is in sharp contrast to the finite dimensional case, where one has also indivisible channels \[4\].

The main tool which we use to prove the result is a surjective mapping from $G$ onto the cone of positive matrices in $M_{2n}(\mathbb{C})$. Thus we define the mapping $p$ in the following way:

$$p : M_{2n}(\mathbb{R}) \times M_{2n}(\mathbb{R}) \rightarrow M_{2n}(\mathbb{C}), \ (X, Y) \mapsto i(X\sigma X^T - \sigma) + Y.$$ 

A comparison of this definition with the cp-condition \[1\] shows that $p(X, Y) \geq 0$ iff $(X, Y) \in G$. Moreover, $p(X, Y) = 0$ iff $X \in Sp(2n)$ and $Y = 0$. Thus, the kernel of $p$ is exactly the set of reversible elements of $G$.

An essential property of $p$ is the fact each positive matrix $P$ is an image of a Gaussian channel $(X, Y)$. This property is proved in the following two lemmas. This first lemma is a standard result in linear algebra, but we give a proof for the reader’s convenience.

**Lemma 1.** Suppose $M \in M_{2n}(\mathbb{R})$ is anti-symmetric. Then it can be written in the form $M = N\sigma N^T$ for some $N \in M_{2n}(\mathbb{R})$.

**Proof.** Since $M$ is normal, there is an orthogonal real matrix $R$ such that $R^TMR$ is a block diagonal matrix where each block is a 1-by-1 matrix or a 2-by-2 matrix of the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}, \quad a, b \in \mathbb{R}, b > 0. \quad (16)$$

As $M$ is anti-symmetric every 1-by-1 block has to be 0 and every 2-by-2 block of the form \[1\] has $a = 0$. For this kind of 2-by-2 matrix we can write

$$\begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} = \begin{pmatrix} \sqrt{b} & 0 \\ 0 & \sqrt{b} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{b} & 0 \\ 0 & \sqrt{b} \end{pmatrix}.$$

□
Lemma 2. Let $P \in M_{2n}(\mathbb{C})$ and $P \geq 0$. There is $(X,Y) \in \mathcal{G}$ such that $p(X,Y) = P$.

Proof. Let us write $P = P_R + iP_I$, where $P_R, P_I \in M_{2n}(\mathbb{R})$. Since $P \geq 0$, $P_R$ is symmetric positive and $P_I$ is anti-symmetric. Also the matrix $P_I + \sigma$ is anti-symmetric, and by Lemma 1 it can thus be written in the form

$$P_I + \sigma = X \sigma X^T,$$

where $X \in M_{2n}(\mathbb{R})$. Choosing $Y = P_R$ the claim is proven. □

Each positive matrix $P$ represents an equivalence class of Gaussian channels rather than a single channel. Namely, for two Gaussian channels $(X_1, Y_2)$ and $(X_2, Y_2)$ we have

$$p(X_1, Y_1) = p(X_2, Y_2) \text{ iff } Y_1 = Y_2 \text{ and } X_1\sigma X_1^T = X_2\sigma X_2^T.$$ 

If we concatenate two Gaussian channels corresponding to positive matrices $P_1$ and $P_2$ we obtain a Gaussian channel in the equivalence class

$$P = P_1 + X_1 P_2 X_1^T. \quad (17)$$

With this preparation we are now ready to prove that every non-reversible Gaussian channel is divisible.

Proposition 10. Let $(X,Y) \in \mathcal{G}$ be non-reversible. There exist non-reversible $(X_1, Y_1), (X_2, Y_2) \in \mathcal{G}$ such that $(X,Y) = (X_1, Y_1) \cdot (X_2, Y_2)$.

Proof. Let us first consider the case $\det X = 0$. Choose $X_1 = X$, $Y_1 = Y$, $X_2 = \mathbb{1}$ and $Y_2$ the projector onto the kernel of $X$. In this way we can write the channel $(X,Y)$ as a concatenation of two non-reversible channels.

Let us then suppose $\det X \neq 0$. Let $P = p(X,Y)$. We denote $P_1 = \varepsilon P$, and fix a pair $(X_1, Y_1)$ such that $p(X_1, Y_1) = P_1$. It is possible to choose $0 < \varepsilon < 1$ such that $\det X_1 \neq 0$. Indeed, by its definition $X_1$ satisfies

$$X_1\sigma X_1^T = \varepsilon X\sigma X^T + (1 - \varepsilon)\sigma, \quad (18)$$

hence we get

$$(\det X_1)^2 = \det(X_1\sigma X_1^T) = \det(\varepsilon X\sigma X^T + (1 - \varepsilon)\sigma).$$

For $\varepsilon = 0$ the right hand side is 1, and from the continuity of the determinant follows that $\det X_1 \neq 0$ for some $0 < \varepsilon < 1$.

We then define

$$P_2 := (1 - \varepsilon)X_1^{-1}PX_1^{-T}.$$ 

It follows that

$$X_1 P_2 X_1^T + P_1 = P.$$
This shows that in the equivalence class of Gaussian channels represented by $P$, there is at least one channel $(\tilde{X}, Y)$ which can be divided non-trivially. So it remains to prove that this holds then for all Gaussian channels in the equivalence class.

As we have noticed earlier, $(X, Y)$ and $(\tilde{X}, Y)$ are in the same equivalence class iff $X\sigma X^T = \tilde{X}\sigma \tilde{X}^T$. Thus, $S := X^{-1}\tilde{X} \in Sp(2n)$ and $X = \tilde{X}S^{-1}$. Therefore, a decomposition for $(\tilde{X}, Y)$ also leads to a decomposition of $(X, Y)$.

Since $Y_1 = \varepsilon Y \neq 0$, the channel $(X_1, Y_1)$ is non-reversible. Also, the channel $(X_2, Y_2)$ has to be non-reversible as $Y_1 \neq Y$ implies that $Y_2 \neq 0$. \hfill \Box

Note that $(X_1, Y_1)$ can be chosen arbitrary close to the ideal channel $(\mathbb{I}, 0)$. That is, form an arbitrary irreversible Gaussian channel we can ‘chop off’ an infinitesimal (irreversible) piece so that the remaining part is still a valid Gaussian quantum channel. The possibility of iterating this procedure, i.e., chopping off an infinitesimal pieces from the remainder and so further, might suggest that every Gaussian channel is infinitesimal divisible. This intuition, however, fails since the remaining channel (after having chopped off a piece) is, in the Gaussian context, not necessarily closer to the ideal channel (as it would be in the finite dimensional context). In fact, for Gaussian channel which are not infinitesimal divisible this procedure can bring us further and further away from the identity. This might be seen as a signature of the non-compactness of the set of Gaussian channels (as opposed to the compactness in the finite-dimensional context).

8. Idempotent channels

A Gaussian channel $(X, Y)$ is idempotent if

$$ (X, Y) \cdot (X, Y) = (X, Y). $$

This leads to the requirements $X^2 = X$ and $XYX^T = 0$. Since $Y$ is positive, the second condition can be written in the form $(X\sqrt{Y})(X\sqrt{Y})^T = 0$, which is equivalent to $XY = 0$. Therefore, we conclude that $(X, Y)$ is idempotent iff

$$ X^2 = X, \quad XY = 0. \quad (19) $$

In physical terms, idempotency means that a repeated use of the channel does not change the system any further.

Suppose that $X$ is invertible. Then the conditions (19) imply that $(X, Y) = (\mathbb{I}, 0)$, which is just the identity channel. However, there are also other idempotent channels as illustrated in the following example.
Example 6. Let $X$ and $Y$ be diagonal matrices of the form

$$X = \text{diag}(1, \ldots, 1, 0, \ldots, 0), \quad Y = \text{diag}(0, \ldots, 0, y_1, y_1, \ldots, y_{n-k}, y_{n-k}) ,$$

where $y_j \geq 1; j = 1, \ldots, n - k$. Then the pair $(X, Y)$ clearly satisfies (19). The cp-condition (1) breaks into conditions for $2 \times 2$-matrices,

$$i\sigma_1 \geq i\sigma_1, \quad y_j \geq 1, \quad y_j \geq 1,$$

which obviously hold. The channel $(X, Y)$ corresponds to a transformation where we do nothing for the first $2k$ modes but for the rest $2n - 2k$ modes we do a state preparation (see Example 1).

Let $S \in Sp(2n)$. If we concatenate a channel $(X, Y)$ with the reversible channels corresponding to $S$ and $S^{-1}$, we get

$$(S, 0) \cdot (X, Y) \cdot (S^{-1}, 0) = (SX S^{-1}, SY S^T). \quad (20)$$

It is easy to verify that if $(X, Y)$ is idempotent, then also $(SX S^{-1}, SY S^T)$ is idempotent. Therefore, Example 6 generates a full class of idempotent channels. In the following we show that actually all idempotent channels are of that form.

Proposition 11. A Gaussian channel $(X, Y)$ is idempotent iff there is a symplectic matrix $S$ such that

$$SX S^{-1} = \text{diag}(1, \ldots, 1, 0, \ldots, 0), \quad (21)$$

$$SY S^T = \text{diag}(0, \ldots, 0, y_1, y_1, \ldots, y_{n-k}, y_{n-k}) , \quad (22)$$

where $y_j \geq 1; j = 1, \ldots, n - k$.

Proof. Let us first show that $X^T$ is a symplectic projection, i.e. the symplectic space $(\mathbb{R}^{2n}, \sigma)$ is a direct sum of two subspaces $V_1, V_2$, mutually orthogonal with respect to the symplectic form. That is every vector $v$ is uniquely decomposed as $v_1 + v_2$ with $v_1 \in V_1, v_2 \in V_2$, and $v_1^T \sigma v_2 = 0$.

Indeed, put $v_1 = X^T v, v_2 = (\mathbb{1} - X^T)v$, then $v = v_1 + v_2$ and $Y v_1 = 0$. By condition (1) we get

$$v_2^T Y v_2 = (v_1 + iv_2)^* Y (v_1 + iv_2) \geq 2 v_1^T \sigma v_2$$

for all $v_1 \in V_1, v_2 \in V_2$, hence $v_1^T \sigma v_2 = 0$.

From (1) we also obtain

$$Y = (\mathbb{1} - X) Y (\mathbb{1} - X)^T \geq i(\mathbb{1} - X) \sigma (\mathbb{1} - X)^T .$$
Applying symplectic diagonalization of the positive symmetric matrix 
\((1 - X)Y(1 - X)^T\) in \(V_2\), one can always find a symplectic matrix \(S\) in 
\((R^{2n}, \sigma)\) satisfying \([22]\).

Since \(XY = 0\), we also have \(SXS^{-1}SYS^T = 0\) and \([21]\) follows. □

9. Gauge-covariant channels

We say that a Gaussian channel \((X, Y)\) is gauge-covariant if \([X, \sigma] = [Y, \sigma] = 0\). We denote by \(G_{\sigma}\) the set of all gauge-covariant Gaussian channels. It is clearly a subsemigroup of \(G\). Physically, gauge-covariant channels arise for instance from a number conserving (i.e., passive) coupling to an environment (cf. [17]).

Let us rearrange the matrix \(\sigma\) such that
\[
\sigma = \begin{pmatrix}
0 & 1_n \\
-1_n & 0
\end{pmatrix},
\]
then matrices \(M \in M_{2n}(\mathbb{C})\) commuting with \(\sigma\) are those of the form
\[
M = \begin{pmatrix}
A & B \\
-B & A
\end{pmatrix}, \tag{23}
\]
where \(A, B \in M_n(\mathbb{C})\). The maps \(M \mapsto A \pm iB\) are easily seen to be \(*\)-homomorphisms of \(M_{2n}(\mathbb{C})\) onto \(M_n(\mathbb{C})\), hence \(M \geq 0\) implies \(A \pm iB \geq 0\). Note that \(A^* = A, B^* = -B\) in this case. Let us show that conversely, \(M \geq 0\) if \(A \pm iB \geq 0\). Let \(A \pm iB \geq 0\), then \(A \geq 0\) and \(B^* = -B\). Assume first that \(A\) is nondegenerate, then \(1_n \pm iA^{-1/2}BA^{-1/2} \geq 0\), which implies \(1_n + A^{-1/2}BA^{-1/2} \geq 0\) and hence \(A + BA^{-1}B \geq 0\) which implies \(M \geq 0\). The case of degenerate \(A\) is obtained by approximation. Thus we have proved

**Lemma 3.** \(M \geq 0\) iff \(A \pm iB \geq 0\).

For a matrix \(M\) of the form \([23]\) we denote \(\hat{M} = A + iB\). Then \(\hat{\sigma} = i1_n\) and the cp-condition \([1]\) for gauge-covariant channels takes the form
\[
\hat{Y} \geq \pm \left(1_n - \hat{X}\hat{X}^*\right). \tag{24}
\]

Let \(\hat{X}^* = \hat{U}\hat{K}\) be the polar decomposition of the matrix \(\hat{X}^*\), where \(\hat{U}\) is unitary and \(\hat{K} = \sqrt{\hat{X}\hat{X}^*}\) is positive. Then the channel \((X, Y)\) is a concatenation of the reversible channel \((U, 0)\) and the channel \((K, Y)\). For the last channel the condition \([24]\) takes the form
\[
\hat{Y} \geq \pm \left(1_n - \hat{K}^2\right). \tag{25}
\]

There are several basic cases depending on the properties of \(\hat{K}\):
(i) $\hat{K} = 0$. The channel is idempotent for any $\hat{Y} \geq \mathbb{1}_n$. (State preparation with the covariance matrix $Y$).

(ii) $0 < \hat{K} < \mathbb{1}_n$, where by strict inequality we mean that the eigenvalues $k_j$ of $\hat{K}$ satisfy $0 < k_j < 1$. The channel is a member of a one-parameter semigroup of Gaussian channels with bounded noise term having an invariant state. The semigroup is defined via the relations $\hat{X}_t = \hat{K}^t, \hat{Y}_t = \hat{Y} - \hat{K}^t\hat{Y}\hat{K}^t$, where $\hat{Y}$ is the unique solution of the equation

$$\hat{Y} = \mathcal{Y} - \hat{K}\mathcal{Y}\hat{K}$$

corresponding to the covariance matrix of the invariant state. Indeed, the last equation written in the basis of eigenvectors of $\hat{K}$ has unique solution

$$\hat{Y} = (\nu_{ij}), \quad \nu_{ij} = (1 - k_i k_j)^{-1} y_{ij}, \quad \hat{Y} = (y_{ij}). \quad (26)$$

Let us prove that $\hat{Y} \geq \mathbb{1}_n$ and hence it corresponds to the covariance matrix of a Gaussian state. The matrix with the elements

$$(1 - k_i k_j)^{-1} = \int_0^\infty \exp \left( k_i k_j - 1 \right) dt \quad (27)$$

is positive since the matrix $(k_i k_j - 1)$ is conditionally positive (see Thm. 6.3.6. in [16]). By using the condition $\hat{Y} \geq \mathbb{1}_n - \hat{K}^2$ and a lemma concerning Schur products of positive matrices (Thm. 5.2.1. in [16]), we have

$$\hat{Y} = ((1 - k_i k_j)^{-1} y_{ij}) \geq ((1 - k_i k_j)^{-1} (1 - k_i^2) \delta_{ij}) = \mathbb{1}_n.$$

(iii) $\hat{K} = \mathbb{1}_n$. Then $\hat{Y} \geq 0$ and the channel is member of one-parameter semigroup of Gaussian channels $(1, tY)$ with unbounded noise term. In the case $\hat{Y} = 0$ this is identity channel.

(iv) $\hat{K} > \mathbb{1}_n$, that is $k_j > 1$. The channel is a member of one-parameter semigroup of Gaussian channels with $\hat{Y}$ defined as in (26). However in this case instead of (27) we must use

$$(1 - k_i k_j)^{-1} = -\int_0^\infty \exp \left( 1 - k_i k_j \right) dt, \quad (28)$$

therefore the matrix with the elements (28) is negative implying $\hat{Y} \leq -\mathbb{1}_n$. Thus the semigroup has unbounded noise term and there is no invariant state.

In general, one can decompose $\hat{K}$ into direct orthogonal sum of the matrices satisfying the conditions (i)-(iv). In case $\hat{Y}$ commutes with $\hat{K}$ one can further decompose the channel $(K, Y)$ into corresponding
channels. In particular, in the case \( n = 1 \) any gauge-invariant channel reduces to one of the cases (i)-(iv).

10. Conclusions and open questions

We conclude with some open questions. First of all, we lack a characterization of infinitesimal divisible Gaussian channels. Prop.8 provides a partial answer in terms of the determinant \( \det(X) \). A similar property, however, turned out to be necessary but not sufficient in the case of finite dimensional quantum channels [4]. Moreover, we left open the question which type of dynamical equations (e.g., time-dependent Markovian master equations) leads to solutions which coincide with the set of infinitesimal divisible channels.

For one-parameter semigroups our picture is more complete. Yet, there is no efficiently decidable criterion which enables us to say whether or not a given Gaussian channel is an element of such a one-parameter semigroup. The simple form [10] suggests to follow the lines of [5] [6] where an integer semi-definite program provided a solution for finite dimensional quantum channels. However, boundary cases (e.g., channels not admitting a simple form representation) will have to be treated with care.

Other questions arise when we slightly change the rules of the game. In the reversible case we saw that while a transformation might not be an element of a one-parameter semigroup within the Gaussian world, it can become one if we drop the restriction to the Gaussian world. So how does the general picture change if we allow for decompositions into arbitrary channels?

In a similar vein we may allow for tensor powers and thereby investigate the robustness of all the discussed properties w.r.t. taking several copies of a quantum channel. This might be interesting beyond Gaussian channels (e.g., for qubit maps) as well. In the Gaussian case we can easily find examples showing that things can change: take a reversible Gaussian channel with \( S \notin \exp(\mathfrak{sp}(2n)) \), then \( S \oplus S \) happens to have a Hamiltonian matrix as a generator.

Finally, it is desirable to relate semigroup properties of a quantum channel to other properties such as their capacities or to properties of quantum spin chains to which the channels can be assigned to via the finitely correlated state construction [18], [19].
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