Oscillatory solutions for a diode with counter-streaming electron and ion beams

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Abstract. A plasma diode with counter-streaming electron and ion beams has as a rule more than one time-independent solution. Stability features of the steady state solution with ion reflection from a potential barrier are studied numerically. It was found that there is a threshold in the diode length beyond which nonlinear oscillations occur as an alternative instead of such solutions. Features of the oscillations are studied.

1. Introduction

Existence of the oppositely directed electron and ion beams is a frequently encountered situation in plasma diodes. Such beams form a feedback system for electric field and electron and ion densities where various complicated and strongly nonlinear processes may occur. It is known that there are the oscillations of the relaxation type in the Knudsen plasma diode [1]. Regions with the flows of particles of opposite signs are also inherent in some astrophysical objects. In particular, regions with counter-streaming electron and positron beams are supposed to exist in the vicinity of pulsars [2, 3]. These regions are called “pulsar diodes”. A rather strong electric field that accelerates particles occurs there. It is thought to be responsible for the powerful pulsar radiation. Sometimes the halting in the pulsar radiation is observed, after which the frequency and intensity change, but a trigger of such processes is not known. So, it is interesting to investigate dynamics of the beams in the plasma diode and to understand the mechanism of the oscillations genesis as well as that of the switch between different oscillation modes.

Previously, the monoenergetic particle flows of charge of opposite signs originated from opposite boundaries in the one-dimensional vacuum diode with a fixed potential difference between the electrodes have been studied in Ref. [4]. On examination are the particles moving collisionlessly and interacting through only the self-consisted electric field. All feasible types of potential distribution for this system have been analyzed. They differ in a number of potential maxima and minima (and in the ratio of the potential extrema values to the particle energy). It has been found that there can be several time-independent solutions corresponding to the same set of the system parameters (particle energies and currents). They vary in types of the potential distribution. However, the issue concerning with their stability features remains open. In the present paper we investigate the type of steady state solution with the potential distribution exhibiting the minimum near the electrode supplying electrons that are partially reflected from this minimum as also the maximum reflecting a portion of ions near the opposite electrode supplying ions. We study the perturbation evolution for the steady state solution of this type.
2. Statement of the problem

We consider one-dimensional diode with the interelectrode distance \( L \) formed with the two parallel infinite plane electrodes. Let \( z \) axis be normal to electrodes and \( U \) be the electrode potential difference. The left electrode \((z = 0)\) supplies a mono-energetic electron flow with velocity \( v_{e,0} \) and density \( n_{e,0} \) whereas the right one \((z = L)\) supplies an ion flow with average velocity \( v_{i,0} \) and density \( n_{i,0} \). We consider that both electron kinetic energy \( W_0 \) and electron density \( n_0 \) equals to ion ones. We introduce dimensionless variables choosing \( n_0 \) as a density unit, \( W_0 \) as an energy unit and the Debye length \( \lambda_D = [2\varepsilon_0 W_0/(e^2 n_0)]^{1/2} \) as a length unit with \( \varepsilon_0 \) being the free-space dielectric constant and \( e \) being the electron charge. Evidently, electron and ion velocities at electrodes \( v_{e,0} \) and \( v_{i,0} \) differ. We choose them as units of the electron and ion velocities, respectively. The time unit is \( \lambda_D/v_{i,0} \) whereas the potential one is \( 2W_0/e \).

Evolution of the potential distribution as well as the velocity distribution functions of electrons \( f_e \) and ions \( f_i \) is described by the system of equations including kinetic equation for electrons and ions and Poisson’s equation:

\[
\begin{align*}
\mu \frac{\partial f_e}{\partial \tau} + u_e \frac{\partial f_e}{\partial \zeta} - \varepsilon(\zeta) \frac{\partial f_e}{\partial u_e} &= 0, \\
\frac{\partial f_i}{\partial \tau} + u_i \frac{\partial f_i}{\partial \zeta} + \varepsilon(\zeta) \frac{\partial f_i}{\partial u_i} &= 0, \\
\frac{d^2 \eta}{d\zeta^2} &= n_e - n_i.
\end{align*}
\]

Here, \( \tau, \zeta, \text{ and } \eta \) are dimensionless time, coordinate and potential, \( n_e, u_e \) and \( f_e (n_i, u_i, f_i) \) are density, velocity and distribution function of electrons (ions), \( \varepsilon(\zeta) = -d\eta/d\zeta \) is electrical field strength, \( \mu = \sqrt{m/M} \) with \( m \) and \( M \) being electron and ion mass. We introduce \( \delta = L/\lambda_D \) and \( V = eU/2W_0 \) as dimensionless diode length and voltage between electrodes, while the left electrode potential is assumed to be equal to zero.

We suggest that ions cross the distance of the order of \( \lambda_D \) in more time than electrons are able to fly the electrode distance, i.e. the value \( \mu \delta \ll 1 \). For this reason, we can suggest that time needed for electron distribution to be formed is much less than ion time step i.e. we suppose that electron distribution instantly adjusts the potential change. We suppose electron velocity distribution function to be a delta-function. As mentioned above, we consider the situation when a portion of electrons is reflected from the potential barrier at the point \( \zeta_{\text{min}} \) located near the left electrode, \( r \) being the ratio of reflected electron density to emitted one. Since the problem for electrons is time-independent at each ion time step, the electron density is of the form [4]

\[
n_e(\eta, r; \zeta_{\text{min}}) = a(r)(1 + 2\eta)^{-1/2}, \quad a(r) = \begin{cases} 1 + r, & \zeta < \zeta_{\text{min}}^v, \\ 1 - r, & \zeta > \zeta_{\text{min}}^v. \end{cases}
\]

We suppose that emitted ions are uniformly distributed within the narrow velocity gap \( 2\Delta \ll 1 \), their distribution function at right boundary being

\[
f_i(\delta; u) = \frac{1}{2\Delta} \theta(\Delta^2 - (u + 1)^2), \quad u < 0.
\]

An initial problem is stated as follows. We suppose that for a time \( \tau^* \) exceeding several times of ion flying across the diode space there is time invariant potential distribution corresponding to the steady state ion density. Some details of calculated the steady state distributions are given below. At the time moment \( \tau^* \) we start a self-consistent simulation of potential and density distributions.

3. Simulation method

The simulation algorithm for the ion distribution function is very close to the \( E, K \)-code [5–7] modified for the case when ions enter from the right boundary. According to \( E, K \)-code calculation of the ion distribution function is based on the fact that this function is kept along each ion trajectory

\[
\frac{dc}{d\tau} = u, \quad \frac{du}{d\tau} = \frac{dn}{dc} = -\varepsilon(\zeta(\tau)).
\]

Starting with the values \( \zeta \) and \( u \) at the time \( \tau \) we can calculate the ion trajectory back in time till it crosses the boundary \( \zeta = 0 \) or \( \zeta = \delta \) at the time \( \tau = \tau_0 > 0 \) with velocity \( u_0 \) or till the moment \( \tau = 0 \)
if the trajectory has not crossed any boundary. Thus the ion distribution function is calculated for any velocity $u$: $f_i(\zeta, u(t), \tau) = f_i(\delta, u_0, \tau_0)$. More detailed description of the method for trajectory integration is given in [5–7].

It is evident that the ion distribution function at the point $\zeta$ and the moment $\tau$ is constant and equal to $1/2\Delta$ if the ion trajectory followed from $\zeta$ and $\tau$ crosses right boundary with the velocity value within $[-1 - \Delta, -1 + \Delta]$ and is equal to zero otherwise. The ion density is calculated as the sum over the all velocity intervals where the distribution function is nonzero.

In process of simulation at each time step $\tau^i$ we do not know the potential distribution (PD) change within the time interval $(\tau^{i-1}, \tau^i)$. For this reason at first we approximate the PD varying within this interval and calculate the ion density relating the time $\tau^i$ using PD at time $\tau^{i-1}$. This ion density is substituted to the right part of the Poisson's equation (3) to be as $n_i(\tau^i, \zeta)$. The electron density $n_e$ given by (4) with the PD approximated to time $\tau^i$ that is $\eta(\tau^{i-1}, \zeta)$ is substituted to equation (3), too. The obtained nonlinear ordinary second-order differential equation with the boundary conditions $\eta(0) = 0$, $\eta(\delta) = V$ is integrated numerically to calculate $\eta(\tau^i, \zeta)$. After that we calculated again the ion density relating the PD found and repeat this process. In order to ensure a self-consistency of the process at each time moment we carry out two iterations.

For numerical integration of the nonlinear equation we multiply both parts of it by $(d\eta/d\zeta)d\zeta$ and integrate with respect to $\zeta$. Thus we obtain an equation for electric field strength:

$$
\frac{\langle n_e^2 \rangle}{2} - \frac{\langle n_i^2 \rangle}{2} = \int_{\eta_1}^{\eta} n_e(\eta, r; \zeta_{\min}) d\eta - \int_{\zeta_1}^{\zeta} n_i(\zeta) \frac{d\eta}{d\zeta} d\zeta. \tag{7}
$$

Integral of electron density is readily found in analytical form. For the case of electron partial reflection from virtual electrode it includes the coordinate of potential minimum $\zeta_{\min}$ and the electron reflection coefficient $r$ as parameters. Then we integrate numerically the obtained first-order differential equation (7) for potential distribution and find $\zeta_{\min}$ and $r$ from the boundary conditions by shooting method.

### 4. Results and discussion

At the first step we obtain density and PDs that slightly deviate from the steady state distributions because of the finite precision of numerical simulation. This deviation may be considered as a small perturbation of the initial condition. It may to increase or decrease during the following time steps depending on whether the steady state solution we started from is stable or not. Observing the time evolution of the numerical solution we can judge whether the steady state solution is stable or not. As a result the perturbed solution may to approach to the steady state solution with a high precision (if it is stable) or to find its way into other type of solution that may be a time-dependent one.

The solution stability depends on the system parameters. In present paper we investigate the solution stability in relation to diode length $\delta$. We have performed calculation for the $\delta = 2.5$ and $3.5$ and the electrode potential difference $V = 0.1$. The semi-range of the ion velocity distribution function $\Delta$ has been chosen equal to 0.01.

We calculate the steady state solution for this case as follows. The PD has the minimum $\eta_{\min} = -0.5$ located near the left electrode as well as a maximum $\eta_{\max}$ located near the right electrode. The electron density is found by formulae (4). It depends on two parameters: $\zeta_{\min}$ and $r$. The ion density may be found by integrating the ion distribution function (5) with respect to $u$, what is carried out analytically. The ion density depends on two parameters: $\eta_{\max}$ and $\zeta_{\max}$. Considering that the electric field strength vanishes both in the position of potential minimum and that of its maximum we can find a relation between parameters $r$ and $\eta_{\max}$ by employing the Gauss theorem to the segment $(\zeta_{\min}, \zeta_{\max})$, i.e. integrating the difference $n_e - n_i$ by $\eta$ and equating the result to zero. Now we are able to calculate the PD in the diode. The parameters $\zeta_{\min}$ and $r$ are obtained by shooting method using the boundary conditions. The obtained steady state PDs are given in figure 1.
Figure 1. Steady state PDs for a diode with $\delta = 2.5$ (a) and $3.5$ (b); $V(\delta) = 0.1$.

We have carried out simulation of time-dependent process for two values of the diode length. In the case of $\delta = 2.5$ the initial perturbation was damped gradually and terminated in the state which does not vary with time and was very nearly the initial steady state solution. Thus we come to conclusion that for this diode length the solution is stable with respect to small perturbations. This process is illustrated by a plot of maximum potential against its position (figure 2). We can see that after some time interval the potential maximum moves along the spiral which terminates in the steady state solution.

Figure 2. Maximum potential against its position drawn for a diode with length $\delta = 2.5$

In the diode with $\delta = 3.5$, initial perturbation rises with time. Time dependences of the maximum potential and its position are presented in figure 3. It is seen that after some time the regular oscillation with increasing amplitude are formed. The relevant plot of the maximum potential against its position is demonstrated in figure 4. At first it is expanding spiral, but after several oscillation periods it transforms to the closed loop and the variation of potential maximum becomes periodical. Thus, in this case there is no steady state solution in the diode. A final state of the considered system is the periodical oscillating process.
Figure 3. Time dependencies of maximum potential position (a) and value (b) drawn for a diode with length $\delta = 3.5$.

Figure 4. Maximum potential against its position drawn for a diode with length $\delta = 3.5$; dotted line is the last closed loop.

5. Conclusions
In this work, we have examined stable features of steady-state solutions in the collisionless diode that is dynamically driven by counter-streaming electron and ion flows. The solutions characterized by ion reflection from a potential barrier were considered. Our study was carried out numerically by $E$,$K$-code. We have found that there is a threshold in the diode length beyond which the solutions turn out to be unstable. Nonlinear oscillations occur as an alternative instead of steady state solutions. The
result obtained may be useful in studying processes in diodes with the Knudsen discharge [1] and pulsar diodes [2,3].

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