Quantum corrections to the holomorphic structure of the mirror bundle along the caustic and the bifurcation locus

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Abstract. Given, in the Lagrangian torus fibration $R^4 \to R^2$, a Lagrangian submanifold $L$, endowed with a trivial flat connection, the corresponding mirror object is constructed on the dual fibration by means of a family of Morse homologies associated to the generating function of $L$, and it is provided with a holomorphic structure. Morse homology, however, is not defined along the caustic $C$ of $L$ or along the bifurcation locus $B$, where the family does not satisfy the Morse-Smale condition. The holomorphic structure is extended to the subset $C \cup B$, except cusps, yielding the so called quantum corrections to the mirror object.
1 Introduction

One of the reasons for which it may be desirable to embark on the study of mirror symmetry for dual torus fibrations is that Calabi-Yau threefolds represents, at least in String Theory, a case of remarkable interest. For the kind of problems this paper is concerned with, first steps in this direction were undertaken in [4], [2] and [3]: it was provided, for the torus fibration $T^{2n} \rightarrow T^n$, or, in general, for a smooth trivial family of Lagrangian tori, a correspondence between Lagrangian submanifolds endowed with a flat connection on one side and holomorphic bundles on the other. Some restrictions are necessary: the most substantial, besides the absence of singular fibres in the fibrations, is that Lagrangian submanifolds are assumed to exhibit no caustic. The constructions of the correspondence are different: by means of families of Floer homologies or by a kind of Fourier-Mukai transform; however, at least in this simple setting, they are equivalent. An attempt to allow for more general Lagrangian submanifolds, that is to include the caustic, is contained in [5]: some “quantum corrections” must be added to the construction and it is argued that these should be provided by pseudoholomorphic disks; it is also conjectured that Floer homology and pseudoholomorphic disks can be replaced, in an appropriate sense, by Morse homology and gradient lines. The analysis of caustic and bifurcation locus, particularly for the case of the perturbed elliptic umbilic, was carried on, in dimension 2, in [11] and [12], and an attempt of study of quantum corrections was developed in [13].

The present paper tries to generalize the content of the three previous works, by analysing the behaviour of gradient lines for given caustic and bifurcation locus and proposing quantum corrections to get a whole defined holomorphic mirror object: this is carried out by assigning, under suitable hypothesis, submanifolds $C$ and $B$, acting respectively as caustic and bifurcation locus, and a class of orbit equivalent gradient vector fields for each subset $U_i$ determined by $C$ and $B$, followed by a study of bifurcations relating the phase portraits in nearby $U_i$ and $U_j$. However, while in [13], for the specific case of the perturbed elliptic umbilic, the monodromy of the holomorphic structure of the mirror bundle around cusps was considered, this paper does not deal the generalization of this aspect. The fibration $T^4 \rightarrow T^2$ is considered, though, since quantum corrections are defined locally, the fibration $\mathbb{R}^4 \rightarrow \mathbb{R}^2$ is kept in mind. What follows is a summary of the content of this paper.
In chapter 2 caustic and bifurcation locus associated to a Lagrangian submanifold are introduced and their features are exposed: in particular, it is studied when a codimension 1 subset of $\mathbb{R}^2$ represents caustic and bifurcation locus of some Lagrangian submanifold.

Chapter 3 outlines the construction of the mirror object by Morse homology: it is not defined along caustic and bifurcation locus, which form a codimension 1 subset of $\mathbb{R}^2$.

In chapter 4 it is studied the behaviour of gradient lines and of phase portraits in a neighbourhood of folds of the caustic, not containing the bifurcation locus, and, as a consequence, quantum corrections are defined, in order to extend the holomorphic structure of the mirror object through such points of the caustic.

In chapter 5 gradient lines and phase portraits are analysed near codimension 1 points of the bifurcation locus, leading to the definition of quantum corrections for glueing the holomorphic structure of the mirror object along these points.

Chapter 6 is concerned with the relative position of caustic and bifurcation locus and with intersection of bifurcation lines.

Chapter 7 is devoted to show that the holomorphic mirror object has no monodromy around the codimension 2 subset of points for which quantum corrections, introduced in chapter 5 and 6, are not defined, and so it can be extended across such points. However, as already said, cusps are not considered. Theorem 7.6 sums up the achieved results.
2 \textit{CB}-diagrams

Consider $\mathbb{R}^4$, endowed with its canonical symplectic structure and standard Euclidean metric, and the natural projection $\mathbb{R}^4 \to \mathbb{R}^2$. To any smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ (see [1] for details) it is associated a 2-dimensional Lagrangian submanifold $L$, of which $f$ is its generating function; the set of critical values of the projection $L \hookrightarrow \mathbb{R}^4 \to \mathbb{R}^2$ is called caustic of $L$ (or of $f$) and denoted by $C$. Any Lagrangian submanifolds can be described locally by some generating functions. Generically, in neighbourhood of any point of $L$, but for a discrete set, $f$ can be assumed to be a function on the coordinates along the fibres.

**Proposition 2.1.** The caustic $C$ is a codimension 1 immersed submanifold of $\mathbb{R}^2$ with singularities.

$C$ is a stratified submanifold: generically, folds form the stratum of codimension 1 and cusps, the singularities of $C$, the stratum of codimension 2. Different branches of $C$ can intersect transversely one with another, generically at folds. In [II], the family of functions $f_x : \mathbb{R}^2 \to \mathbb{R}$, with $x \in \mathbb{R}^2$, $f_x(y) = f(y) - xy$, is associated to $L$. The solutions of the gradient system

$$\frac{dy}{dt} = \nabla f_x(y)$$

are named gradient lines. Observe that $f_x$ is a Morse function for $x \notin C$. The subset of $\mathbb{R}^2 \setminus C$, at whose points the gradient vector field $\nabla f_x$ is not Morse-Smale, is called bifurcation locus and denoted by $B$: at these points a saddle-to-saddle separatrix occurs in the phase portrait of $\nabla f_x$.

**Proposition 2.2.** The bifurcation locus $B$ is a codimension 1 immersed submanifold of $\mathbb{R}^2$.

$B$ is a stratified submanifold: generically, at codimension 1 points one saddle-to-saddle separatrix occurs, at codimension 2 points two saddle-to-saddle separatrixes occur. Codimension 2 points are the intersections of the codimension 1 stratum.

Thus $C$ and $B$ determine in $\mathbb{R}^2$ a diagram.

**Definition 2.3.** The $CB$-diagram associated to a Lagrangian submanifold $L$ of $\mathbb{R}^4$, is the partition of $\mathbb{R}^2$ determined by the caustic $C$ and by the bifurcation locus $B$ of $L$. 

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Definition 2.3 is extended as follows:

**Definition 2.4.** Given two codimension 1 submanifolds $C$ and $B$ of $\mathbb{R}^2$, holding the features outlined above of, respectively, the caustic and the bifurcation locus, a $CB$-diagram generated by $C$ and $B$ is the partition $U_i$, $i \in I$, $U_i$ connected, determined in $\mathbb{R}^2$ by $C$ and $B$, together with an assignment, for each $i \in I$, of a phase portrait $P_i$. It is denoted by $(C, B, (U_i, P_i)_{i \in I})$.

The question now is: when is a $CB$-diagram, generated by $C$ and $B$ and with family of phase portraits $P_i$, the $CB$-diagram of some Lagrangian submanifold $L$, whose caustic and bifurcation locus are respectively $C$ and $B$, and, for $x \in U_i$, $\nabla f_x$ is orbit isotopic (see the discussion preceding definition 2.8) to $P_i$, where $f$ is a generating function of $L$? The answer depends first of all on the family of phase portraits $P_i$: in fact, $P_i$ must be the phase portrait of some gradient vector field. Into this direction, a result is provided by the following lemma:

**Lemma 2.5.** A gradient vector field in $\mathbb{R}^2$ has no periodic orbits, moreover, if structurally stable, has only hyperbolic critical points and the intersection of the stable and unstable submanifolds $W^s(p)$ and $W^u(q)$ of any two critical points $p$ and $q$ is always transverse.

So to any $x \notin C \cup B$ it must be associated a vector field exhibiting only hyperbolic critical points, which in $\mathbb{R}^2$ turn out to be either stable nodes or unstable nodes or saddles, and with no saddle-to-saddle separatrix. In this case we can prove the following proposition:

**Proposition 2.6.** Given a phase portrait exhibiting a finite number of hyperbolic critical points, no periodic orbits and no saddle-to-saddle separatrixes, then there exists a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $\nabla f$ has the given phase portrait.

*Proof.* Let $p_j$, for $j \in J$, where is $J$ finite, be the critical points in the given phase portrait, and consider, for each $j$, a ball $B_j$ centred in $p_j$ and such that $B_j \cap B_k = \emptyset$ for $j \neq k$. As explained, each $p_j$ is expected to be either a node, stable or unstable, or a saddle: since there exist gradient vector field with such singular points, choose bounded functions $f_j$ such that $\nabla f_j$ exhibits in $B_j$ the critical point $p_j$ of the type as prescribed by the given phase portrait. Assume $B_j$ itself as domain of $f_j$. Choose now a point $p_m$ among the critical points $p_j$ and for each $j \neq m$ choose a path $\gamma_j$ from $p_m$ to $p_j$. Choose
also a tubular neighbourhood $N_j$ of $\gamma_j$ such that $(N_i \cap N_j) \setminus B_m = \emptyset$. To extend $f_m$ along $N_j$, assume $\partial B_j \cap N_j$ to be a level set of $f_j$ as well as a fibre of $N_j$, and take the fibres of $N_j$ as the level curves of the extended functions: since each $f_j$ is defined up to a constant, $f_m$ can be matched with $f_j$. Proceed till obtaining a bounded function $f$ defined on the contractible subset $B = (\cup_j B_j) \cup (\cup_j N_j)$. Now $f$ can be extended to a function defined on the whole $R^2$: indeed, as $B$ is contractible, the problem is equivalent to extending a bounded function from a ball $B$ to the whole $R^2$. Note that this can be carried out without introducing new critical points.

The proof relies on the fact that there exist gradient vector fields exhibiting a stable node or an unstable node or a saddle. Since there are also gradient vector fields, though non-structurally stable, exhibiting a saddle-node (over folds of the caustic, when referring to the setting considered in this paper), or a point, which can be called “saddle-node-saddle”, given by two saddles and a node glued together (over cusps), or saddle-to-saddle separatrixes (over points of the bifurcation locus), the following corollary generalizes proposition 2.6:

**Corollary 2.7.** Given a phase portrait exhibiting a finite number of critical points (hyperbolic or saddle-nodes or saddle-node-saddles) and no periodic orbits, then there exists a function $f : R^2 \to R$ such that $\nabla f$ has the given phase portrait.

Let $U_i$, with $i \in I$ and where $I$ is either finite or $N$, be the connected components of the $CB$-diagram associated to $L$. For every $i \in I$, $x_1, x_2 \in U_i$ and path $\gamma(t) \subset U_i$, with $t \in [0,1]$, $\gamma(0) = x_1$ and $\gamma(1) = x_2$, there exists an orbit isotopy from $x_1$ to $x_2$, that is a smooth family $\Phi_t$ of diffeomorphisms of $R^2$, with $t \in [0,1]$, such that $\Phi_0 = Id$ and $\Phi_t$ provides an orbit equivalence between $\nabla f_{x_1}$ and $\nabla f_{\gamma(t)}$ for every $t \in [0,1]$. Therefore, another property that the family $(P_i)$ of a $CB$-diagram generated by $C$ and $B$ must satisfy, in order to be the $CB$-diagram of some Lagrangian submanifold, is that, intuitively, if $U_k$ and $U_l$ are separated by $C$ or $B$, then it must be possible to switch from $P_k$ to $P_l$ by adding or removing a pair of critical points (forming at $C$ a degenerate critical point) or by exchanging the separatrices of two saddles (forming at $B$ a saddle-to-saddle separatrix). The last considerations can be resumed rigorously in the following definition:

**Definition 2.8.** A $CB$-diagram $(C, B, (U_i, P_i)_{i \in I})$ is admissible if and only if $I$: it is finite; $U_i$ is open for every $i \in I$; each $P_j$ exhibits a finite number of
only hyperbolic critical points and no closed orbits; if ∂U_i ∩ ∂U_j ≠ ∅ then for every path γ : [0, 1] → \overline{U}_i ∪ \overline{U}_j such that γ([−1, 0)) ⊂ U_i, γ((0, 1]) ⊂ U_j and γ(0) ∈ ∂U_i ∩ ∂U_j, there is a smooth family \{X_t\}_{t ∈ [−1, 1]}, such that \{X_t\} has phase portrait orbit isotopic to \{P_t\} for \( t ∈ [−1, 0), \) and to \{P_j\} for \( t ∈ (0, 1], \) and such that \( X_0 \) exhibits either a degenerate critical point or a saddle-to-saddle separatrix depending on whether \( γ(0) \) belongs respectively to \( C \) or \( B; \) moreover, any two family \( X_t \) and \( X'_t \) as above are orbit isotopic.

**Theorem 2.9.** Any admissible CB-diagram \((C, B, (U_i, P_i)_{i ∈ I})\) is the CB-diagram of some Lagrangian submanifold \( L \) at least on any compact subset of \( \mathbb{R}^2, \) in the sense that \( L \) has caustic and bifurcation locus diffeomorphic, respectively, to \( C \) and \( B, \) determining a partition \((W_i)_{i ∈ I}\) of \( \mathbb{R}^2, \) with \( W_i \) diffeomorphic to \( U_i, \) and, for each \( x ∈ W_i \) such that the projection of \( L \) over \( W_i \) is non-empty, the vector fields \( \nabla f_x \) has phase portrait orbit isotopic to \( P_i, \) where \( f \) is a local generating function of \( L. \)

**Proof.** Observe that, since the critical points of \( \nabla f_x \) correspond to the intersection points of \( L \) with the fibre over \( x, \) \( L \) will be defined only over those \( U_i \) endowed with a phase portrait \( P_i \) having at least a critical point. So, for every \( i ∈ I \) such that \( P_i \) is as described above, choose a point \( p_i ∈ U_i. \) By proposition 2.6 choose a gradient vector field \( X_i \) with phase portrait orbit equivalent to \( P_i \) and such that \( X_i(0) = 0. \) Let \( \tilde{f}_i \) be a function in the variable \( y = (y_1, y_2) \) such that \( \nabla \tilde{f}_i = X_i \) and let \( f_i = \tilde{f}_i + p_i \cdot y; \) observe that \( \nabla(f_i)_{p_i} = X_i, \) and, since \( p_i ∉ C ∪ B, \) there is a subset \( V'_i \) of \( U_i \) such that for all \( x ∈ V'_i \) the vector field \( \nabla(f_i)_x \) is orbit equivalent to \( X_i. \) Observe also that \( (∂f_i/∂y)(0) = (∂\tilde{f}_i/∂y)(0) + p_i = X_i(0) + p_i = p_i, \) so the equation \( x = ∂f/∂y \) defines a Lagrangian submanifold \( L_i \) over a neighbourhood \( V'_1 \) of \( p_i \) which can be assumed contained in \( V'_i. \)

\( L_i \) can be extended over an open subset \( V^2 \) of \( U_i \) diffeomorphic to \( U_i; \) clearly \( V^2 = V^1 \) when \( π_1(U_i) = 0; \) otherwise, consider for simplicity the case where \( π_1(U_i) = \mathbb{Z}: \) since \( p_i ∉ C \text{ and } V^1 \) is an open ball, \( L_i \) can be generated by a finite set of functions \( g^j_i, \) one for each sheet \( L^j_i \) of \( L_i \) over \( V^1 \) and defined on \( V^1 \) (this corresponds to the fact that each sheet \( L^j_i \) of \( L_i \) can be seen as the graph of a closed 1-form \( σ^j_i, \) which, being \( V^1 \) an open ball, is exact, that is, \( σ^j_i = dg^j_i \text{ for some function } g^j_i); \) it is enough now to extend each \( g^j_i \) to a function defined on an open subset \( V^2 \) diffeomorphic to \( U_i. \) This argument also shows how to extend a Lagrangian submanifolds, defined over two disjoint open subsets of the base of the fibration, where it has the same
number of sheets and no critical points, onto a new subset containing the two subsets.

The admissibility of the given $CB$-diagram implies that to each point of $C$ and $B$ it is associated a vector field, which by corollary 2.7, can be assumed to be a gradient vector field. Since $C$ and $B$ generically have two strata, choose,

Choose a point $q_{ik}$ on each connected component $C_{ik}, B_{ik}$ of the codimension 1 stratum of $C$, respectively $B$, where “intersection points” between $C$ and $C$ are removed (the issue of intersection points will be analyzed in chapter 6). Each $C_{ik}, B_{ik}$ will bound two subsets $U_i, U_k$ in the partition of the given $CB$-diagram. Let $\gamma_{ik}$ be a path from $p_i$ to $p_k$ as in definition 2.8, with $\gamma_{ik}(0) = q_{ik}$ and associated family of vector fields $X_{ik}^t$, and let $f_{ik}$ be a function such that $(\partial f_{ik}/\partial y)(0) = q_{ik}$ and $\nabla (f_{ik})_{\gamma(t)}$ is orbit equivalent to $X_{ik}^t$, for $t$ in a neighbourhood $N_{ik}$ of 0: this is possible because $X_{ik}^0$ exhibits a saddle-node or a saddle-to-saddle separatrix, particularly it is not stable, and after a small perturbation, that is for $t$ in a neighbourhood $N_{ik}$ of 0, it is orbit equivalent, by the admissibility of the given $CB$-diagram, to the vector fields $\nabla (f_{ik})_{p_i} = X_i$ and $\nabla (f_{ik})_{p_k} = X_k$ in, respectively, $\gamma_{ik}(N_{ik}) \cap U_i$ and $\gamma_{ik}(N_{ik}) \cap U_k$. The function $f_{ik}$ defines a Lagrangian submanifold $L_{ik}$ in a neighbourhood $V_{ik}$ of $q_{ik}$; $V_{ik}$ can be supposed to not intersect any of the subsets $V_j^2$ constructed above. For a generic choice of $f_{ik}$, $V_{ik}$ is a ball such that, along one of its diameters, the vector field $\nabla (f_{ik})_{x}$ is orbit equivalent to $X_{ik}^0$, while, in the two half-disks determined by such a diameter, it is orbit equivalent to respectively $X_i$ and $X_k$. This construction can be performed also for every point of the codimension 2 stratum of $C$ and $B$ and for the “intersection points” between $C$ and $B$.

The Lagrangian submanifolds defined above over the open sets $V_j^2, V_{ik}$ and in open neighbourhoods of cusps, of intersection points of bifurcation lines and of intersection points between $C$ and $B$, can be glued together, by extending them along every path $\gamma_{ik}$, in a new Lagrangian submanifold $L$: indeed, away from points of $C$ and $B$, which form a codimension 1 subset, generating functions, one for each sheets of the Lagrangian submanifolds, can be considered, and these functions, as already explained above, can be extended along every paths $\gamma_{ik}$.

The Lagrangian submanifold $L$ so obtained can be extended now along the remaining points of the caustic $C$. Suppose indeed to have two Lagrangian submanifolds $L_1$ and $L_2$ defined over two disjoint open balls $W_1$ and $W_2$ in the $(x_1, x_2)$-plane, such that they exhibits a caustic, formed only by folds,
along a diameter $C_i$ of $W_i$, for $i = 1, 2$. For simplicity, suppose that $L_i$ has two sheets over one of the two connected components determined by $C_i$ and no sheet over the other. Let $b_i$ one of the two points in $\partial W_i \cap \bar{C}_i$ and consider a path $C : [1, 2] \to \mathbb{R}^2$ such that $C((1, 2)) \cap W_i = \emptyset$, $C(i) = b_i$, for $i = 1, 2$ and such that it extends the paths $C_i$ (it can be $W_i = \emptyset$ for $i = 1$ or $i = 2$ or $W_1 = W_2$ but $b_1 \neq b_2$). By admissibility of the given $CB$-diagram, it can be assumed that $L_1$ and $L_2$ have the same number of sheets in the component of $W_i$ lying on the same side with respect to the path $C_1 \cup C \cup C_2$. Choose coordinates $t$ along $C_1 \cup C \cup C_2$ and $u$ such that, if $(t, u, y_t, y_u)$ are canonical coordinates, $C_1 \cup C \cup C_2$ lies on the $t$-axis and such that, since $C_i$ contains only folds, $L_i$ has equation

$$\begin{align*}
  u &= y_u^2 \\
  t &= y_t
\end{align*}$$

in $W_i$. This equation gives also the wanted extension along $C$, when the coordinate $t$ corresponds to points of $C$. Since the map $(x_1, x_2, y_1, y_2) \to (t, u, y_t, y_u)$ is a Lagrangian equivalence of the Lagrangian bundle $\mathbb{R}^4 \to \mathbb{R}^2$, the extension of $L$ along $C$ is obtained.

Finally, the extension of $L$ to the whole $\mathbb{R}^2$ is carried out as already done above, since the subset to which now $L$ is extended does not contain any point of $C$. Because the vector fields in $U_i$ and $U_k$, when these have $B$ has common boundary, are not orbit equivalent and since the given $CB$-diagram is admissible, it follows that for each path from $U_i$ to $U_k$ there is a point along this path where the corresponding vector field is equivalent to the one chosen over $q_{ik}$. In principle, this point is not unique, however if further bifurcation points appear, they must appear in pairs, that is, each pair will mark the apperance of the same saddle-to-saddle separatrix, so that the two bifurcations cancel each other and the admissibility of the given $CB$-diagram is preserved; after a perturbation, each pair of points can be removed, at least on a compact subset. That the bifurcation locus of $L$ is diffeomorphic to the given $B$ follows from the fact that $V_i^2$ has been constructed diffeomorphic to $U_i$. 

\[ \square \]

**Remark 2.10.** Note that if two Lagrangian submanifolds have diffeomorphic $CB$-diagrams, in the sense of theorem [2.9], this does not imply that they are Lagrangian equivalent: in fact, for example, two Lagrangian equivalent submanifolds have diffeomorphic caustic, however the converse is not true.
3 The mirror bundle

This chapter wants to be only a summary of the idea of the construction of the mirror object using families of Floer homologies, or, as in this paper, families of Morse homologies. Details are, in fact, already exposed in [4], [5] and [13].

In the Lagrangian torus fibration $T^4 \to T^2$, consider a 2-dimensional Lagrangian submanifold $L \hookrightarrow T^4$, endowed with a flat connection $\nabla$, and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a generating function of $L$. Floer homology for families of Lagrangian submanifolds is treated in [6] and its application to mirror symmetry in the construction of the mirror object on the dual fibration is in [4] and [5]: the fibre of the mirror object, an element of $D Coh(\hat{X})$, over a point $(x, w)$, where $x \in T^2$ and $w \in \hat{F}_x$, is given by the intersection Floer homology $HF((L, \nabla), (F_x, w))$, where $F_x$ is the fibre over $x$ and $\hat{F}_x$ the dual fibre (all the problems concerning the definition or the existence of $HF(L, F_x)$ are not discussed here, see rather the monograph [8]). A holomorphic frame is then defined (see [4] and [5]), gluing the fibres in a complex of holomorphic bundles. In particular, in [5] and [7] it is conjectured that near the caustic the moduli space of pseudoholomorphic disks is isotopic, after perturbation, to the moduli space of gradient lines of the generating function $f$. This conjecture is used in this paper: the fibre of the mirror object over $x$ is defined as the Morse homology $HM(f_x)$, when $x \notin C \cup B$: in fact in this case, $f_x$ is a Morse function and the Morse-Smale condition is satisfied (of course, all conditions on $f$ necessary to ensure the existence of Morse homology are assumed: to this purpose, for everything concerning Morse homology, see [10] and, above all, the monograph [14]). A holomorphic frame is then defined, yielding a complex of holomorphic bundles, away from $x \in C \cup B$: writing $\nabla = d + A$, a section $e(x)$ of the mirror object turns out to be holomorphic and descends on the torus fibres when multiplied by the weight

$$\exp \left[ 2\pi \left( \frac{h(x)}{2} - \frac{A(x)}{4\pi} + i \frac{\partial h}{\partial x} \cdot w \right) \right]$$

where $h$ is a multi-valued function on the base such that each sheet of $L$ is locally the graph of $dh$ (in other words, $h$ is a set of local generating functions, defined in the coordinates of the base, one for each sheet of $L$). The way to extend the holomorphic structure through the subset $C \cup B$ is provided by “quantum corrections”, that is morphisms gluing the mirror object along this subset. This is the purpose of the present paper. Since quantum cor-
rections are defined locally, it is enough to consider the Lagrangian fibration $\mathbb{R}^4 \to \mathbb{R}^2$.

**Remark 3.1.** Here is a kind of road map showing the way leading to the extension of the holomorphic structure of the mirror objet across $C \cup B$:

- in an admissible $CB$-diagram $(C, B, (U_i, P_i)_{i \in I})$, confront the phase portraits $P_i$ and $P_j$ for nearby $U_i$ and $U_j$;

- show that the Morse homologies $HM(f_x)$, for $x \in U_i$ are isomorphic to those for $x \in U_j$;

- pick up an isomorphism: the choice depends on which kind of points form the common boundary of $U_i$ and $U_j$, that is, folds not limit points of the bifurcation locus or codimension 1 bifurcation points; it is defined a map at the chains level, that is, on generators of the Morse complex, that is, on critical points of $\nabla f_x$, inducing an isomorphism (the quantum correction) in homology;

- with such a glueing, check that there is no monodromy when going around the set of the remaining points, that is, folds which are limit of the bifurcation locus and codimension 2 bifurcation points (cusps are not considered, as already said, in this paper), which form a codimension 2 subset; this means that the bundle whose fibres are $HM(f_x)$ is endowed now with non-vanishing sections which can be extended to any point of $C \cup B$ (but for cusps);

- observe that, since a holomorphic section is obtained from a section of $HM(f_x)$ multiplied by a weight, it follows that the glueing isomorphisms introduced in the previous steps induce a glueing at the level of holomorphic sections, allowing to extend them across $C \cup B$ (but for cusps);

- $HM(f_x)$ is now endowed with a holomorphic structure which can be extended to $C \cup B$ (but for cusps).

## 4 The caustic

Points of the caustic are characterized by a degenerate critical point of $\nabla f_x$, which, if $x$ is a fold, is related to a so-called birth-death pair.
**Definition 4.1.** A smooth 2-parameters family of vector fields, defined on an open subset $U$ of the plane, exhibits a birth-death pair if there exists a curve $C$, decomposing $U$ into two connected components $U_1$ and $U_2$, such that the family has $k + 2$ critical points $p_1(s)$, ... , $p_k(s)$, $p_{k+1}(s)$, $p_{k+2}(s)$ in $U_1$ and $k$ critical points $p_1(s)$, ... , $p_k(s)$ in $U_2$, where $s$ is the parameter, and the two critical points $p_{k+1}$, $p_{k+2}$ converge, for $s$ converging to $c \in C$, to a degenerate critical point $p(c)$, called birth-death point. The pair of critical points $(p_{k+1}(s), p_{k+2}(s))$ is called a birth-death pair.

**Proposition 4.2.** For a generic $f$, at any fold $x \in C$, the function $f_x$ exhibits a birth-death point.

**Proof.** For simplicity, the fold $x$ can be assumed to be the origin $(0,0)$. After a Lagrangian equivalence, since $f$ is generic, the local generating function $f$ of $L$ can be written, for example, as $f(y_1, y_2) = y_3$. This shows that $C$ determines in any neighbourhood of $(0,0)$ two subsets, characterized by the fact that the intersection of the fibre $F_{x'}$, for $x'$ in these two subsets, and $L$ is either empty or contains two points: these form a birth-death pair and, for $x' \rightarrow (0,0)$, glue together into a birth-death point. Equivalently, this means that the vector field $\nabla f_{x'}$ exhibits two critical points in one of these components, gluing together in a degenerate critical point at $(0,0)$, and no critical points in the other component. \qed

In the example considered in [11] and [12], concerned with the cusp and the elliptic umbilic, the birth-death pairs were formed by a saddle and an unstable node. Instead, for the hyperbolic umbilic, the birth-death pairs are given by a saddle and a stable node. These are also the only two cases that can be met.

Denote by $\mu(p_i)$ the Morse index of a critical point $p_i$.

**Proposition 4.3.** If $(p_1, p_2)$ is a birth-death pair corresponding to some fold, then $|\mu(p_1) - \mu(p_2)| = 1$.

**Proof.** At a fold $x$, by definition, $rk(H(f_x))(c(x)) = n - 1$, where $c(x)$ is the birth-death point: this means that both $p_1$ and $p_2$ has an eigenvalue $e(p_i)$ which vanishes when $p_1$ and $p_2$ glue together in $c(x)$; if $p_1$ and $p_2$ glue over a fold then $e(p_1)$ and $e(p_2)$ have opposite sign. The remaining eigenvalues of $p_1$ and $p_2$ can not vanish and so have the same signs. \qed

The quantity $\mu(p_1) - \mu(p_2)$ is named relative Morse index of $p_1$ and $p_2$. 

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Lemma 4.4. Given a birth-death pair \((p_1, p_2)\) with \(\mu(p_1) > \mu(p_2)\), then there exists a unique gradient line \(\gamma_{p_1, p_2}\).

Proof. The local phase portrait is topologically equivalent to that given by the generating function \(f(y_1, y_2) = y_1^2 \pm y_2^2\) (see [9]), for which the existence of a gradient line \(\gamma_{p_1, p_2}\) can be proved by direct computation. If \(W^u(p_1) \cap W^s(p_2)\) is not empty, then \(\mu(p_1) - \mu(p_2) = \dim(W^u(p_1) \cap W^s(p_2)) = 1\). Unicity follows now from the fact that \(W^u(p_1) \cap W^s(p_2)\) is connected: if not, the birth-death point would exhibit a homoclinic orbit, which can not occur for a gradient vector field. \(\square\)

So, as explained, the caustic divides a small open neighbourhood \(U\) of \(x\) into two open subsets \(U_1\) and \(U_2\): for example, for \(x' \in U_1\), \(\nabla f_{x'}\) has critical points \(p_1(x'), \ldots, p_k(x'), p_{k+1}(x'), p_{k+2}(x')\), where \((p_{k+1}(x'), p_{k+2}(x'))\) is the birth-death pair, and so \(p_1(x'), \ldots, p_k(x')\) are the critical points of \(\nabla f_{x'}\) for \(x' \in U_2\).

Definition 4.5. A square in a phase portrait is a set

\[(un, s_1, s_2, sn; \gamma_{un, s_1}, \gamma_{s_1, sn}, \gamma_{un, s_2}, \gamma_{s_2, sn})\]

whose elements are an unstable node \(un\), two saddles \(s_1\) and \(s_2\), a stable node \(sn\), and for each saddle a pair of separatrixes connecting them to the nodes. If \(s_2 = s_1\) and either \(\gamma_{un, s_1} = \gamma_{un, s_2}\) or \(\gamma_{s_1, sn} = \gamma_{s_2, sn}\) the square is said to be degenerate.

Theorem 4.7 is quoted from [10]: it is essential in defining the Morse complex, explaining the structure of the boundary of the moduli space \(\mathcal{M}(p, q) = (W^u(p) \cap W^s(q))/\mathbb{R}\) of gradient lines between two critical points \(p\) and \(q\) with relative Morse index \(\mu(p, q) = 2\). Given such points \(p\) and \(q\), if there exists a critical point \(r\), with \(\mu(p, r) = \mu(r, q) = 1\), and gradient lines \(\gamma_{p, r}\) and \(\gamma_{r, q}\), the triple \((\gamma_{p, r}, r, \gamma_{r, q})\) is called a broken gradient line from \(p\) to \(q\), and denoted also by \(\gamma_{p, r} \# \gamma_{r, q}\). Observe that the gradient lines of a square form two broken gradient lines \(\gamma_{un, s_1} \# \gamma_{s_i, sn}\) for \(i = 1, 2\). Finally, \(\gamma_1 \# \gamma_2 \neq \gamma_1' \# \gamma_2'\) if and only if \(\gamma_1 \neq \gamma_1'\) or \(\gamma_2 \neq \gamma_2'\).

Some hypothesis on the function \(f\) are needed in order to ensure a good behaviour of \(\mathcal{M}(p, q)\) and so to define a differential \(\partial\) such that \(\partial^2 = 0\).

Definition 4.6. A function \(f\) of class \(C^1\) satisfies the Palais-Smale condition if every sequence \((x_n)\), such that \(|f(x_n)|\) is bounded and \(|df(x_n)| \to 0\) for \(n \to \infty\), admits a convergent subsequence.
Theorem 4.7. Suppose \( f \) is a function of class \( C^3 \), having only non degenerate critical points, satisfying the Palais-Smale condition and the Morse-Smale condition. Let \( p \) and \( q \) be two critical points of \( f \), connected by the flow, and such that \( \mu(p) - \mu(q) = 2 \). Suppose that the space of gradient lines \( \mathcal{M}(p,q) \) from \( p \) to \( q \) is contained in a flow-invariant compact set. Then each connected component of \( \mathcal{M}(p,q) \) either is compact after including \( p \) and \( q \), and so diffeomorphic to the 2-sphere, or its boundary consists of two different broken gradient lines from \( p \) to \( q \).

Conversely each broken gradient line from \( p \) to \( q \) is contained in the boundary of precisely one component of \( \mathcal{M}(p,q) \).

In other words, each connected component of \( \mathcal{M}(p,q) \), if non-compact after including \( p \) and \( q \), determines by means of its boundary a square. Theorem 4.7, quoted from [10], is proved by assuming, for a matter of convergence, a compactness hypothesis: \( W^u(p) \cap W^s(q) \) is contained in a flow-invariant compact subset. In the case we are considering, that is \( \mathbb{R}^2 \), this could be a problem: there could be an unstable node \( un \) and stable node \( sn \) such that \( W^u(un) \cap W^s(sn) \) is not bounded and so the above compactness hypothesis cannot be applied. Actually, what is important for the purposes of Morse theory is to show, when \( un \) is connected to some saddle, that every connected component of \( W^u(un) \cap W^s(sn) \) is bounded by two different broken gradient lines (or, in other words, it forms a square): this allows to prove that \( \partial^2 = 0 \).

Suppose, for example, that in \( \mathbb{R}^2 \) \( W^u(un) \cap W^s(sn) \) is not bounded, consider the compactification \( S^2 \) of \( \mathbb{R}^2 \) and suppose also that the phase portrait can be extended to \( S^2 \) without adding new critical points: as \( \infty \) is not a critical point and because of unicity of solutions, the gradient line through \( \infty \), in the compactification \( S^2 \), either belongs to \( (W^u(un) \cap W^s(sn))/\mathbb{R}^2 \), if it connects \( un \) to \( sn \), or bounds \( (W^u(un) \cap W^s(sn))/\mathbb{R}^2 \), if it connects \( un \) to a saddle or a saddle to \( sn \). So, in the first case, except when compact after including \( un \) and \( sn \), \( W^u(un) \cap W^s(sn) \) has in \( \mathbb{R}^2 \) two non-bounded connected components, which, however, in \( S^2 \) form a unique connected bounded component of \( W^u(un) \cap W^s(sn) \) (see figure 4.1); in the second case, the gradient line through \( \infty \) is part of one of the two broken gradient lines bounding \( W^u(un) \cap W^s(sn) \) in \( S^2 \). So, even though the compactness hypothesis fails in \( \mathbb{R}^2 \), the equation \( \partial^2 \) still holds true.
Assumption 4.8. Each component of $W^u(un) \cap W^s(sn)$ satisfies the compactness hypothesis of theorem 4.7, that is, it is contained in a flow-invariant compact subset either of $\mathbb{R}^2$ or, provided the phase portrait can be extended to $S^2$ without the addition of further critical points, of the compactification $S^2$ of $\mathbb{R}^2$.

Lemma 4.9. Under the hypothesis of theorem 4.7 and assumption 4.8, given a saddle $s_1$, a stable node $sn$, an unstable nodes $un$ and gradient lines $\gamma_{un,s_1}$ and $\gamma_{s_1,sn}$ then there exists a saddle $s_2$ ($s_2 = s_1$ is a possibility), with separatrix $\gamma_{un,s_2}$ and $\gamma_{s_2,sn}$, thus forming a square (degenerate if $s_2 = s_1$), in $\mathbb{R}^2$ or eventually in its compactification $S^2$, together with $s_1$, $sn$ and $un$.

**Proof.** Theorem 4.7, eventually applied on $S^2$, implies that the boundary of $W^u(un) \cap W^s(sn)$ contains, besides the broken gradient line $\gamma_{un,s_1} \# \gamma_{s_1,sn}$, a second broken gradient line $(\gamma_{un,s_2}, s_2, \gamma_{s_2,sn})$ for some saddle $s_2$.

Here are few examples.

**Example 4.10.** Given a square $(un, s_1, s_2, sn)$, the two separatrixes of each saddle $s_i$ not forming the sides of the squares lie on the same side with respect to the broken gradient lines $\gamma_{un,s_i} \# \gamma_{s_i,sn}$. Suppose these separatrixes
are in the unbounded region $R_1$ determined by the square: then, if, in
the bounded region $R_2$, there are no other critical points, $R_2$ is a com-
ponent of $\mathcal{M}(un, sn)$, whose boundary is formed by the broken gradient lines
$\gamma_{un,s_1} \# \gamma_{s_1,sn}$ and $\gamma_{un,s_2} \# \gamma_{s_2,sn}$. Suppose instead that the two separatrixes lie
in $R_2$: then at least a stable and an unstable node, are contained in it (see
figure 4.1).

**Example 4.11.** If more than one square has $un$ and $sn$ among their vertexes,
then $\mathcal{M}(un, sn)$ has more than one connected component.

**Example 4.12.** A case where $\mathcal{M}(un, sn)$ is not bounded is shown in figure
4.1): in $\mathbb{R}^2$, $\mathcal{M}(un, sn)$ is the union of two unbounded connected sets $R_1$ and
$R_2$, such that $\partial R_i$ consists of a broken gradien line $\gamma_{un,s_i} \# \gamma_{s_i,sn}$ and of two
gradient lines $\gamma_{un}$ and $\gamma_{sn}$, which, in the compactification $S^2$ of $\mathbb{R}^2,$ connect,
respectively, $un$ and $sn$ to $\infty$; so, in $S^2$, the boundary of $\mathcal{M}(un, sn)$ consists
of the two broken gradient lines $\gamma_{un,s_1} \# \gamma_{s_1,sn}$ and $\gamma_{un,s_2} \# \gamma_{s_2,sn}$, while $\gamma_{un}$ and
$\gamma_{sn}$ are part of a single gradient line from $un$ to $sn$, belonging to $\mathcal{M}(un, sn)$.

**Example 4.13.** Figure 4.2 shows two types of degenerate squares:
$(un_1, s, s, sn)$ bounded by $\gamma_1 \# \gamma_3$ and $\gamma'_1 \# \gamma'_3$ with $\gamma_1 = \gamma'_1$, and $(un_2, s, s, sn)$
bounded by $\gamma_2 \# \gamma_3$ and $\gamma'_2 \# \gamma'_3$ with $\gamma_2 = \gamma'_2$. In the first case $W^u(un_1) \cap W^s(sn)$
is bounded and connected, in the second case $W^u(un_2) \cap W^s(sn)$ is not
bounded and has two connected component, but it is bounded and connected in the compactification $S^2$ of $\mathbb{R}^2$. Note that the square $(un_2, s, s, sn)$ requires the existence of the unstable node $un_1$ and so the existence of the square $(un_1, s, s, sn)$, but not the converse. This lacking of symmetry is understood in $S^2$: the square $(un_1, s, s, sn)$ requires the unstable node $un_2$. A similar example is obtained exchanging the roles of stable and unstable nodes.

![Degenerate squares](image_url)
Remark 4.14. If $f$ is bounded or in a bounded subset of $\mathbb{R}^2$, $f$ exhibits a finite number of critical points (in particular, this is true if $f$ is defined on a compact manifold). In the sequel, $f$ is assumed to have only a finite number of critical points.

Under the hypothesis of theorem 4.7, it can be constructed a complex, the Morse complex, whose homology groups are used to define the mirror object in Mirror Symmetry. Very briefly, given a Morse function satisfying the hypothesis of theorem 4.7 and a metric, the Morse complex is defined by the free module (in our case over $\mathbb{C}$) generated by critical points of the Morse function. The grading is given by the Morse index. The Morse differential is defined by counting gradient lines between critical points:

$$0 \to \mathbb{C}[un_i] \to^\partial \mathbb{C}[s_j] \to^\partial \mathbb{C}[sn_k] \to 0$$

Here $un_i$ denotes the unstable nodes, $s_j$ the saddles and $sn_k$ the stable nodes. The construction of the Morse differential, when we consider coefficients in $\mathbb{C}$, depends on the choice of an orientation of gradient lines between critical points. That this choice can be done in a compatible way, that is, the orientations of gradient lines between two critical points $p$ and $q$ such that $|\mu(p) - \mu(q)| = 2$ and the induced orientations on the gradient lines (between critical points of relative Morse index 1) in the boundary of $W^s(p) \cap W^u(q)$ are consistent, allowing to talk of a "coherent orientation", can be proved for finite dimensional oriented manifolds (see [14]). In this paper the sign of a gradient lines $\gamma$ is denoted by $n(\gamma)$.

In an admissible $CB$-diagram $(C, B, (U_i, P_i)_{i \in I})$, to each point $x \notin C \cup B$, it is associated a gradient vector field $\nabla f_x$ with phase portrait $P_i$: this enables to define a Morse complex

$$\mathbb{C}[p_1(x)] \oplus \ldots \oplus \mathbb{C}[p_k(x)]$$

where $p_i(x)$ are the critical points of $\nabla f_x$, and Morse homology groups. Since for all $x \in U_i$, the vector fields $\nabla f_x$ are orbit equivalent, it follows that the Morse complexes associated to them are isomorphic: this allows to write

$$\mathbb{C}[p_{U_i}^1] \oplus \ldots \oplus \mathbb{C}[p_{U_i}^k]$$

and to talk about the Morse complex and Morse homology over $U_i$.

The following lemma, besides stating the isomorphism between Morse homology over $U_1$ and $U_2$, when their boundaries are folds not limit points of $B$, suggests also how to pick up an isomorphism, leading to the definition of quantum corrections.
Lemma 4.15. Let $U_1$ and $U_2$ be open subsets such that $U_i \cap (C \cup B) = \emptyset$ and $\partial U_1 \cap \partial U_2 \subset C$ consists only of folds not limit points of $B$; let $p_1(x')$, $\ldots$, $p_k(x')$, $p_{k+1}(x')$, $p_{k+2}(x')$ be the critical points of $\nabla f_x'$ for $x' \in U_1$, and $p_1(x')$, $\ldots$, $p_k(x')$ be the critical points of $\nabla f_x'$ for $x' \in U_2$, so that $(p_{k+1}(x'), p_{k+2}(x'))$ is a birth-death pair. Then the homology groups of the Morse complexes

$$C[p_{U_2}^1] \oplus \ldots \oplus C[p_{U_2}^k]$$

$$C[p_{U_1}^1] \oplus \ldots \oplus C[p_{U_1}^k] \oplus C[p_{U_1}^{k+1}] \oplus C[p_{U_1}^{k+2}]$$

are isomorphic.

Proof. 1. We confront first $HM_1(U_1)$ with $HM_1(U_2)$, by computing $Im \partial_0$ and $Ker \partial_1$ in $U_1$ and $U_2$.

- Suppose, first, that the birth-death pair $(p_{k+1}, p_{k+2})$ is of type $(n, s)$, where $n$ is an unstable node and $s$ a saddle. It will be proved that $dim Im \partial_0^{U_1} = dim Im \partial_0^{U_2} + 1$ and $dim Ker \partial_1^{U_1} = dim Ker \partial_1^{U_2} + 1$.

Consider the phase portraits, looking at figure 4.3. For $x \in U_1$, by lemma 4.4 there exists a gradient line $\gamma_{n,s}$ from $n$ to $s$, which is, therefore, one of the two components of $W^s(s)$. This implies that $s$ may be connected, besides to $n$, to at most a second unstable node $\text{un}$, by means of the second component of $W^s(s)$. Moreover, it may also be connected to at most two unstable nodes, by means of its remaining two separatrices forming $W^u(s)$. As to $n$, the gradient lines forming $W^u(n)$ may connect $n$ to further saddles or stable nodes. For $x \in C$, $n$ and $s$ glue together in a birth-death point $d$: this exhibits an unstable manifold $W^u(d)$ and a center manifold $W^c(d)$ (see 4.3), corresponding to the zero eigenvalue; observe that $d$ is connected to $\text{un}$ by a gradient line in $W^c(d)$, to all saddles and stable nodes connected to $n$ in $U_1$, by means of other gradient lines in $W^c(d)$, and to the stable nodes connected to $s$ in $U_1$, by means of the two components of $W^u(d)$. In $U_2$, the center manifold $W^c(d)$ breaks, forming gradient lines from $\text{un}$ to all the stable nodes and saddles connected to $n$ and $s$. 
Observe, therefore, that if there are, in $U_1$, a second unstable node $un$ connected to $s$ and saddles $s_i$ connected to $n$, then in $U_2$ the new gradient lines from $un$ to $s_i$, not appearing in $U_1$, implies a change in $Im \partial_0 U_2$ with respect to $Im \partial_0 U_1$.

* So consider $Im \partial_0$. Denote by $un_1$, ..., $un_n$ the unstable nodes appearing in the phase portrait over $U_2$. We distinguish two cases:

** If no unstable node, except $n$, is connected to $s$ in $U_1$, then

$$Im \partial_0 U_1 = i(Im \partial_0 U_2) \oplus <\partial_0 U_1(n)>$$

where $i$ denotes the natural injection, defined by the continuity of the family $f_x$ in the parameter $x$,

$$\mathbb{C}[p_1 U_2] \oplus ... \oplus \mathbb{C}[p_k U_2] \hookrightarrow \mathbb{C}[p_1 U_1] \oplus ... \oplus \mathbb{C}[p_k U_1] \oplus \mathbb{C}[p_k^{U_1}] \oplus \mathbb{C}[p_k^{U_2}]$$

indeed, $\partial_0^{U_1}(n) \not\in i(Im \partial_0^{U_2})$ because writing $\partial_0^{U_1}(n) = s_i + ... + s_{i_1} + s$, for some saddles $s_i$, ..., $s_{i_1}$, by hypothesis, $s \not\in i(Im \partial_0^{U_2})$.

Note that it should rather be written $n(\gamma_{n,s_{i_1}}) s_{i_1} + ... + n(\gamma_{n,s_{i}}) s_i + n(\gamma_{n,s}) s$, where $n(\gamma_{n,s_{i}}), n(\gamma_{n,s}) \in \{1,-1\}$ depend on the choice of an orientation of gradient lines; however, for simplicity, it will often assumed, when possible, $n(\gamma_{n,s_{i}}) = n(\gamma_{n,s}) = 1$.

** Suppose now $un_k$ is a second unstable node (denoted by $un$ in figure[3]) connected to $s$ by the gradient line $\gamma_{un_k,s}$: if $\partial_0^{U_1}(un_k) = <s_{k_1} + ... + s_{k_m} - s>$ for some saddles $s_{k_1}, ..., s_{k_m}$ (note here the choice of signs: in fact, the two components $\gamma_{un_k,s}$ and $\gamma_{n,s}$ of $W^s(s)$ have opposite orientations), and, as above, $\partial_0^{U_1}(n) = s_{i_1} + ... + s_{i_1} + s$,
then $\partial U^2_0(un_k) = <s_{k_1} + ... + s_{k_m} + s_{i_1} + ... + s_{i_l}>$ and so

$$Im\partial U^1_0 = i(\partial U^2_0(<un_1, ..., un_{k-1}, un_{k+1}, ..., un_n>)) \oplus <\partial U^1_0(un_k) > \oplus <\partial U^1_0(n) > = i(\partial U^2_0(<un_1, ..., un_{k-1}, un_{k+1}, ..., un_n>)) \oplus <s_{k_1} + ... + s_{k_m} - s > \oplus <s_{i_1} + ... + s_{i_l} + s > = i(\partial U^2_0(<un_1, ..., un_{k-1}, un_{k+1}, ..., un_n>)) \oplus <s_{k_1} + ... + s_{k_m} + s_{i_1} + ... + s_{i_l} + s > = i(\partial U^2_0(<un_1, ..., un_{k-1}, un_{k+1}, ..., un_n>)) \oplus <\partial U^1_0(un_k) > \oplus <\partial U^1_0(n) > = i(Im\partial U^2_0) \oplus <\partial U^1_0(n) >$$

Note that if it were $\partial U^1_0(n) = <s_{i_1} + ... + s_{i_l} + s > = <s_{k_1} + ... + s_{k_m} - s > = \partial U^1_0(un_k)$, then $l = m$ and $s_{i_r} = s_{k_r}$ for all $1 \leq r \leq l = m$, thus $\partial U^2_0(un_k) = 0$: this means that $un_k$ is connected to all the saddles $s_{i_r} = s_{k_r}$ by both the separatrices forming $W^s(s_{i_r})$ (this is shown, reversing the roles of stable and unstable nodes, in figure 4.2 representing a degenerate square). Thus the relation $Im\partial U^1_0 = i(Im\partial U^2_0) \oplus <\partial U^1_0(n) >$ still holds. To better illustrate this case, consider only two saddles, $s$ and $s_g$, such that $\partial U^1_0(n) = <s_g + s > = <s_g - s > = \partial U^1_0(un_k)$, and let $\gamma_{n,s}$, $\gamma_{n,s_g}$ and $\gamma_{un_k,s}$, $\gamma_{un_k,s_g}$ be the gradient lines connecting $s$ and $s_g$, respectively, to $n$ and $un_k$, as in figure 4.4.

![Diagram](image)

Fig. 4.4: $\partial U^1_0(n) = \partial U^1_0(un_k)$

One of the two components of both $W^u(s)$ and $W^u(s_g)$ lies in the bounded region $R_2$ determined by $\gamma_{n,s}$, $\gamma_{n,s_g}$, $\gamma_{un_k,s}$ and $\gamma_{un_k,s_g}$,
so, since the vector fields are gradient, there must be at least another critical point in $R_2$: this must be a stable node $sn_g$, and, for simplicity, assume this is the only critical point, as shown in figure 4.5).

\[ \text{Fig. 4.5: } \partial U_1(n) = \partial U_1(un_k) : \text{ the phase portrait in } U_1 \]

in $U_2$, since no other saddles were assumed to be in $R^2$, by lemma a degenerate square with vertexes $un_k, s_g$ and $sn_g$ appears in the phase portrait, as shown in figure 4.6.

\[ \text{Fig. 4.6: } \partial U_1(n) = \partial U_1(un_k) : \text{ the phase portrait in } U_2 \]

* It remains to prove that $\dim \ker \partial U_1 = \dim \ker \partial U_2 + 1$.

** If $\partial U_1(s) = 0$, that is $s$ is not connected to any stable node, then

$$\ker \partial U_1 = i(\ker \partial U_2) \oplus <s>$$

since $s \notin i(\ker \partial U_2)$.

** If, instead, $\partial U_1(s)$ does not vanish, then either there exists a stable node $sn_g$ such that $\partial U_1(s) = sn_g$, or there exist two stable nodes $sn_g$ and $sn_h$ such that $\partial U_1(s) = sn_g + sn_k$ (as already explained, the signs in $\partial U_1$ depends on the orientation of gradient lines, however, it is done this choice to simplify the notation).
If \( \partial^U_2(s) = sn_g \) then, by lemma 4.9, there exists a saddle \( s_g \) forming a square together with \( n, s \) and \( sn_g \). Assume that the square is non-degenerate, that is \( s_g \neq s \). Suppose also, for the moment, that \( s_g \) is the only saddle, besides \( s \), connected to \( sn_g \).

Suppose that only an unstable node, that is, \( n \), is connected to \( s \).

If \( s_g \) is not connected to a second stable node, then

\[
\text{Ker}\partial^U_1 = i(\text{Ker}\partial^U_2) \oplus <s \pm s_g>
\]

where the sign depends on the orientation of gradient lines, particularly on the orientation of the square \((n, s, s_g, sn_g)\).

If, instead, \( s_g \) is connected to a second stable node \( sn_{g_1} \), as figure 4.7 shows, then by lemma 4.9 there exists a saddle \( s_{g_1} \), which suppose, for the moment, distinct from \( s \) and \( s_g \), forming a square together with \( n, s_g \) and \( sn_{g_1} \); repeating this argument, being finite the number of critical points, it follows that there is at most a finite number \( k \) of such squares having as vertexes the node \( n \), the saddles \( s_{g_i} \) and \( s_{g_{i+1}} \), and the stable node \( sn_{g_{i+1}} \), as shown in figure 4.7.
Fig. 4.7: $s_g$ is connected to a second stable node

From figure 4.7, it turns out now that

$$Ker\partial^{U_1}_1 = i(Ker\partial^{U_2}_1) \oplus <s \pm s_g \pm s_g \pm \ldots \pm s_g>$$

.**** Consider now the cases $s_g_1 = s$ and $s_g_1 = s_g$. If $s_g_1 = s$, there are two possible phase portraits (one of which, to be the phase portrait of a gradient vector field, requires in $U_1$ a further unstable node and yields in $U_2$ a degenerate square): however, in both cases, $Im\partial^{U_1}_0 = Ker\partial^{U_1}_1 = \mathbb{C}$ and $Im\partial^{U_2}_0 = Ker\partial^{U_2}_1 = \{0\}$. If $s_g_1 = s_g$ then $W^u(n) \cap W^s(sn)$ is not bounded and the phase portrait can not be extended to the compactification $S^2$ of $\mathbb{R}^2$ unless adding further critical points: such phase portrait contradicts the compactness assumption [4.8] and so is not considered.

**** If $s_{n_g}$ is connected to other saddles, $dim Ker\partial^{U_1}_1 = dim Ker\partial^{U_2}_1 + 1$ is still valid. Indeed, observe that two cases may occur: starting from $s_n_g$, consider chains formed by a finite alternate sequence of saddles and stable nodes, ending with a saddle, as $(s_{n_g}, s_{g_1}, s_{n_g_1}, s_{g_2}, s_{n_g_2}, \ldots, s_{n_g_{k-1}}, s_{g_k})$, or with a stable node, as $(s_{n_g}, s_{g_1}, s_{n_g_1}, s_{g_2}, s_{n_g_2}, \ldots, s_{g_j}, s_{n_g_j})$, for some finite $k$ or $j$ (two chains may have elements in common). In figure 4.8 the two cases are represented by the chain $(s_{n_g}, s_m)$ and by the chain $(s_{n_g}, s_{g_1}, s_{n_g_1})$. Referring to this situation and supposing for example that $\gamma_{s_{n_g}, s_{n_g}}$, $\gamma_{s_m, s_{n_g}}$ and $\gamma_{s_{g_1}, s_{n_g}}$ have positive orientation, and $\gamma_{s_g, s_{g_1}, s_{n_g}}$ has negative orientation, it follows that $s_m + s_g \in Ker\partial^{U_1}_1$, and that $s_m + s_g$, $s + s_g$, $s - s_m \in Ker\partial^{U_1}_1$, and thus $Ker\partial^{U_1}_1$ is generated by two of these elements. The same argument works when considering longer chains. Note also that when a chain ends with a stable node, a linear combination of the saddles of the chain with other saddles in the phase portrait never belongs to $Ker\partial^{U_1}_1$ or to $Ker\partial^{U_2}_1$.
It remains to consider the case where $s_g = s$. If $s$ has a double connection to $sn$ (as explained, it can occur into two ways, in one of which, a second unstable node is required in order to make the phase portrait that of a gradient vector field: this anticipates the situation, which will be considered soon, where $s$ is connected to two unstable nodes), then $\text{Ker} \partial_{U_1} = \langle s \rangle$ and $\text{Ker} \partial_{U_2} = \{0\}$. If, instead, $s$ has in $U_1$ a double connection to $n$, a homoclinic orbit appears in the phase portrait along $C$: this can not occur for a gradient vector field.

Suppose now that $s$ is connected to a second unstable node $un$: then by lemma 4.9 there exists a saddle $s_j$ forming a square together with $un$, $s$ and $sn_g$, as figure 4.9 shows (suppose for the moment $s_j \neq s_g$ and $s_j \neq s$).

Observe that if the orientations of gradient lines are chosen as in the case represented in figure 4.8, then the gradient line $\gamma_{s_j,sn_g}$ has
positive orientation. A computation proves that

$$\text{Ker} \partial^U_2 = \langle s_m + s_g, s_j + s_g, s_j - s_m \rangle = \langle s_m + s_g, s_j + s_g \rangle$$

and

$$\text{Ker} \partial^U_1 = \langle s_m + s_g, s_j + s_g, s_j - s_m, s - s_m, s + s_g, s - s_j \rangle = \langle s_m + s_g, s_j + s_g, s - s_g \rangle$$

thus

$$\text{Ker} \partial^U_1 = i(\text{Ker} \partial^U_2) \oplus \langle s - s_g \rangle$$

which implies $\text{dim} \text{Ker} \partial^U_1 = \text{dim} \text{Ker} \partial^U_2 + 1$.

It remains to consider the degenerate cases. If $s_j = s_g$, a degenerate square appears in $U_2$, but the relation just obtained above still holds. If $s_j = s$ and $s_g \neq s$, then $s$ has a double connection only with $sn_g$ (and not with $un$ because $s$ is already connected to $n$), which can be realized by two different phase portraits, depending on how $\gamma_{s,sn_g}$ winds in the phase portrait: in one case $W^u(n) \cap W^s(sn_g)$ is not bounded and the phase portrait cannot be extended to the compactification $S^2$ of $\mathbb{R}^2$ unless adding further critical points, but this is not compatible with the compactness hypothesis 4.8 in the other case, (note that the phase portrait, in order to be that of a gradient vector field, requires to be completed by the addition of further critical points) the compactness hypothesis is fulfilled and the expected relation between $\text{Ker} \partial^U_1$ and $\text{Ker} \partial^U_2$ is verified. Finally, the case $s_g = s = s_j$ was already examined in the degenerate case of $s$ connected to a unique unstable node $n$.

*** Suppose now that $\partial^U_1(s) = sn_g + sn_k$. Then, by lemma 4.9 there exist two saddles $s_g$ and $s_k$ forming two distinct squares together with $n$, $s$ and, respectively, $sn_g$ and $sn_k$ (note that the squares cannot be degenerate in $U_1$), and suppose, for the moment, $s_g \neq s_k$. In figure 4.10 it is presented the case where $s$ is not connected to any other unstable node, except $n$. 

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Fig. 4.10: $s$ is connected to two stable nodes

If instead $s$ is connected to a second unstable node $un_1$, then, as figure 4.11 shows, there exist two saddle $s_h$ and $s_j$ forming a square with $un_1$, $s$ and respectively $sn_k$ and $sn_g$. Suppose for the moment that all the saddles in the phase portrait are distinct.

Fig. 4.11: $s$ is connected to two stable nodes and to a second unstable node

In all the cases, it can be checked that a choice of a coherent orientation of squares allows to prove that $\text{dim Ker}\partial U_1 = \text{dim Ker}\partial U_2 + 1$. For instance, regarding the situation pictured in figure 4.9, choosing a coherent orientation, for example one such that $\gamma_{s,sn_g}$, $\gamma_{sn,sn_g}$, $\gamma_{s_j,sn_g}$ and $\gamma_{s_k,sn_k}$ have positive orientation while $\gamma_{s_g,sn_g}$, etc.
\( \gamma_{s,sn_k}, \gamma_{sh,sn_k} \) and \( \gamma_{sn,sn_k} \) have negative orientation, generators can be chosen such that

\[
\text{Ker} \partial U_1^1 = < s_g + s_j, s_g + s_n, s_k + s_h, s_k + s_m >
\]

and

\[
\text{Ker} \partial U_1^0 = i(\text{Ker} \partial U_2^1) \oplus < s + s_k + s_g >
\]

This result is also achieved when, as already proved, sequences of squares starting, as in figure 4.5, from \( n \) and \( s_g \) or from \( n \) and \( s_h \) are added in the phase portrait.

Finally, note that \( s_g \) may coincide with \( s_j \) or \( s_k \), and with \( s_h \) provided, in this case, both \( un_1 \) and \( sn_k \) are connected to \( s_g \), that is, \( s_g = s_h = s_j = s_k \). Depending on which direction the gradient lines between saddles and nodes wind in the phase portrait, further critical points must be included in order the vector field to be gradient. Moreover, when \( un_1 \) is in the phase portrait, it may happen that some moduli space of gradient lines from unstable nodes to stable nodes is not bounded and the phase portrait can not be extended to the compactification \( S^2 \) of \( \mathbb{R}^2 \) unless adding further critical points: for example, this happens with \( W^u(un_1) \cap W^s(sn_k) \) when \( s_k = s_g = s_g \) (in this case, the extension of the phase portrait to \( S^2 \) yields a homoclinic orbit through \( \infty \)). However, when the compactness hypothesis 4.8 is fulfilled, \( \dim \text{Ker} \partial U_1^1 = \dim \text{Ker} \partial U_2^1 + 1 \).

• Suppose now the birth-death pair \( (p_{k+1}, p_{k+2}) \) is represented by a stable node \( n \) and a saddle \( s \): we are going to prove that \( \dim \text{Im} \partial U_0^1 = \dim \text{Im} \partial U_0^2 \) and \( \dim \text{Ker} \partial U_1^0 = \dim \text{Ker} \partial U_1^2 \).

The existence in \( U_1 \), by lemma 4.4 of a gradient line \( \gamma_{n,s} \) from \( n \) to \( s \), implies, as already explained for a birth-death pair given by an unstable node and a saddle, that if, in \( U_1 \), \( n \) is connected to other unstable nodes \( un_j \) and saddles \( s_i \) and if \( s \) is connected at most to two unstable nodes and to a second stable node \( sn \), then in \( U_2 \) gradient lines from \( un_j \) and from \( s_i \) to \( sn \) appear in the phase portrait (see figure 4.1 after reverting all the arrows). Such gradient lines from \( s_i \) to \( sn \) implies a change in \( \text{Ker} \partial U_1^2 \) with respect to \( \text{Ker} \partial U_1^0 \).

The saddle \( s \) can be connected to at most two stable nodes, one of which is \( n \), and two unstable nodes.
* Consider first the simplest case: $s$ is connected only to $n$:

** clearly \( \dim \text{Im} \partial_0^{U_1} = \dim \text{Im} \partial_0^{U_2} \).

** as to kernels, in principle also other saddles $s_1, ..., s_i$ might be connected to $n$, of these, say $s_1, ..., s_j$, with $j \leq i$, might be each connected to a second stable node $s_{n_1}, ..., s_{n_j}$, as in figure 4.12 (more generally, finite chains of saddles and stable nodes, as $(n, s_{k_1}, s_{n_{k_1}}, s_{k_2}, s_{n_{k_2}}, ...)$, starting from $n$ and ending with a saddle or a stable node, should be considered, however the length of such chains does not affect the argument).

\[ s \rightarrow n \]

\[ s_2 \]

\[ s_1 \rightarrow s_{n_1} \]

Fig. 4.12: a saddle and a stable node forming a birth–death pair

Choosing the orientation for which the gradient lines $\gamma_{s,n_l}, \gamma_{s_1,n_l}$, for $1 \leq l \leq i$, are positive and, as a consequence, $\gamma_{s_{r,n_l}}$, for $1 \leq r \leq j$, are negative, it follows that

\[ \text{Ker} \partial_1^{U_2} = \langle s_{j+1}, ..., s_i \rangle \]

and

\[ \text{Ker} \partial_1^{U_1} = \langle s - s_{j+1}, ..., s - s_i \rangle \]

and so \( \dim \text{Ker} \partial_1^{U_1} = \dim \text{Ker} \partial_1^{U_2} \). Note that the saddles $s_1, ..., s_j$ were irrelevant in the computation of the kernel, so from now on, assume every chain ends with a saddle.

* Suppose also now that $s$ is connected to a second stable node $s_{n}$, but no unstable nodes are connected to $s$.

** Clearly \( \dim \text{Im} \partial_0^{U_1} = \dim \text{Im} \partial_0^{U_2} \).

** As explained, in $U_2$ gradient lines from the saddles $s_1, ..., s_i$ to $s_{n}$ appear in the phase portrait. Suppose that other saddles $s'_{1}, ..., s'_{r}$ are connected to $s_{n}$ (see figure 4.13).
Choose an orientation such that the gradient lines $\gamma_{s_i,sn}$, for $1 \leq l \leq r$, are positive. Observe that $\partial_1^{U_2}(s) = n - sn$. A computation shows that

$$Ker\partial_1^{U_1} = \langle s - s_1 + s'_1, ..., s - s_1 + s'_r, s - s_2 + s'_1, ..., s - s_i - s'_1 \rangle$$

and

$$Ker\partial_1^{U_2} = \langle -s_1 + s'_1, ..., -s_1 + s'_r, -s_2 + s'_1, ..., -s_i - s'_1 \rangle$$

and so $dimKer\partial_1^{U_1} = dimKer\partial_1^{U_2}$.

Note that if $s_l = s'_m$ for some $1 \leq l \leq i$ and $1 \leq m \leq r$, then, as shown in figure 4.14, there exists at least an unstable node to which $s$ is connected, anticipating the case, where $s$ is connected to an unstable node, which will be considered soon.

* Allow now $s$ to be connected also to one unstable node $un$. By lemma 4.9 there exist saddles $s_1$ and $s'_1$ forming squares with $un$, $sn$ and respectively $s_1$ and $s'_1$. Suppose, for the moment, the two...
squares non-degenerate and $s_1 \neq s'_1$. Suppose $un$ is connected, besides $s, s_1$ and $s'_1$, to other saddles $s_{g_1}, \ldots, s_{g_m}$, so that $\partial^{U_1}_0(un) = s + s_1 + s'_1 + \sum_{i=1}^{m} s_{g_i}$ (as usual, signs depend on the orientation, however fixing a choice does not modify the argument). Suppose that, both in $U_1$ and in $U_2$, the unstable nodes are $un$ and $un_{h_1}, \ldots, un_{h_n}$. Since the only unstable node connected to $s$ is $un$,

$$\text{Im} \partial^{U_1}_0 = i(<\sum_{j=1}^{n} \partial^{U_2}_0(un_{h_j})>) \oplus \partial^{U_1}_0(un)$$

moreover, being $\partial^{U_2}_0(un) = s_1 + s'_1 + \sum_{i=1}^{m} s_{g_i}$ and if

$s_1 + s'_1 + \sum_{i=1}^{m} s_{g_i} \notin <\sum_{j=1}^{n} \partial^{U_2}_0(un_{h_j})>$ then

$$\text{Im} \partial^{U_2}_0 = <\sum_{j=1}^{n} \partial^{U_2}_0(un_{h_j})> \oplus \partial^{U_2}_0(un)$$

and so $\text{dim} \text{Im} \partial^{U_1}_0 = \text{dim} \text{Im} \partial^{U_2}_0$. If instead $s_1 + s'_1 + \sum_{i=1}^{m} s_{g_i} \in <\sum_{j=1}^{n} \partial^{U_2}_0(un_{h_j})>$ then $\partial^{U_2}_0(un)$ is a linear combination of $\partial^{U_2}_0(un_{h_i})$: consider the situation represented in figure 4.15, where both $un$ and a second unstable node $un_{1}$ are connected to the same pair of saddles in such a way that $\partial^{U_i}_0(un) = \partial^{U_i}_0(un)$, for $i = 1, 2$,

\[\text{Fig. 4.15: } un \text{ and a second unstable node } un_{1} \text{ are connected to the same pair of saddles}\]
Observe that $s$ has a separatrix from $\infty$, thus, if the compactness hypothesis 4.8 is fulfilled, there exists an unstable node $un_{hp}$ with a gradient line from $un_{hp}$ to $\infty$; this gradient line, in the compactification $S^2$, becomes a gradient line from $un_{hp}$ to $s$ and $(un_{hp}, s, s_1, n)$ and $(un_{hp}, s, s_1', sn)$ are squares; thus $Im\partial_{U_0}^{U_1}(un) \in i(< \sum_{j=1}^{n} \partial_{U_2}^{U_2}(un_{hp}) >)$, and so it follows again $dimIm\partial_{U_0}^{U_1} = dimIm\partial_{U_0}^{U_2}$. Now, since an unstable node connected to $s$ does not affect the kernel of $\partial_1$, it also follows $dimKer\partial_{U_0}^{U_1} = dimKer\partial_{U_0}^{U_2}$.

It remains to consider few special cases.

** The situation where $s_1 = s_1'$ does not differ so much from that above: the same arguments can be suitably applied and the same conclusion achieved.

** If, instead one or both the squares formed by $un$ with $s$ and respectively $s_1$ and $s_1'$ are degenerate, the only possibility is that $W^u(s) \subset W^u(un)$, that is, $s$ has two separatrices connecting it to $un$ and having opposite orientations: this implies

$$\partial_{U_0}^{U_1}(n) = i(\partial_{U_0}^{U_2}(n))$$

and so again $dimIm\partial_{U_0}^{U_1} = dimIm\partial_{U_0}^{U_2}$.

* Suppose now that $s$ is connected to two unstable nodes $un_1$ and $un_2$. Then, by lemma 4.9, there are saddles $s_i$ and $s_i'$ forming squares (which can not be degenerate) with $un_i$, $s$ and respectively $n$ and $sn$, for $i = 1, 2$. The proof that $dimIm\partial_{U_0}^{U_1} = dimIm\partial_{U_0}^{U_2}$ and $dimKer\partial_{U_0}^{U_1} = dimKer\partial_{U_0}^{U_2}$ goes along the same lines of the case where $s$ is connected to a unique unstable node, however some attention must be paid to prove the relation between $Ker\partial_{U_2}^{U_1}$ and $Ker\partial_{U_2}^{U_2}$ when $\partial_{U_0}^{U_1}(un_i) = \partial_{U_0}^{U_1}(\tilde{un}_i)$ for some unstable node $\tilde{un}_i$ (see figure 4.16): in this case, in $U_1$ there are moduli spaces of gradient lines from unstable to stable nodes which are not bounded (namely, $\mathcal{M}(\tilde{un}_i, n)$ and $\mathcal{M}(\tilde{un}_i, sn)$) and the phase portrait cannot be extended to the compactification $S^2$ of $\mathbb{R}^2$ unless adding further critical points. This case is not consistent with the compactness hypothesis 4.8 (however see also remark 4.16).
Finally, if some of the saddles forming the square coincide, the situation can be analyzed in a way similar to the case, already examined, where \( n \) is an unstable node.

2. It remains now to confront \( HM_0(U_1) \) and \( HM_2(U_2) \), which amounts to compute respectively \( Ker \partial_0 \) and \( Im \partial_1 \).

- Suppose first that the birth-death pair \((n, s)\) is given by an unstable node \( n \) and a saddle \( s \).

  * Consider \( HM_0 \). In \( U_1 \), let \( un \) be an eventual second unstable node connected to \( s \). Consider chains \((n, s'_1, un'_1, ...)\) and \((un, s_{1j}, un_{1j}, ...)\) of saddles and unstable nodes starting respectively from \( n \) and \( un \). If at least one chain of each type ends with an unstable nodes, then they determine an element in \( Ker \partial_0^{U_1} \) which is not in \( Ker \partial_0^{U_2} \). For simplicity, suppose to have only one chain of each type and of shortest length: \((n, s'_1, un'_1)\) and \((n, s_1, un_1)\). Then
and observe that:

Remark 4.17. Let $\dim \ker \partial_0^{U_1} = \dim \ker \partial_0^{U_2}$, it follows that $\dim \ker \partial_0^{U_1} = \dim \ker \partial_0^{U_2}$ and so $HM_0(U_1) \cong HM_0(U_2)$.

* Consider now $HM_2$. Since $Im \partial_0^{U_1} = i(Im \partial_0^{U_2}) \oplus < \partial_1^{U_1}(s) >$ it remains to prove that $\partial_0^{U_1}(s) \in i(Im \partial_0^{U_2})$. Suppose $\partial_0^{U_1}(s) = sn_1 + sn_2$, then by lemma 4.9 there are saddles $s_i$, with $i = 1, 2$, forming squares together with $n$, $s$ and $sn_i$. Suppose also $\partial_1^{U_1}(s_i) = sn_i + sn_i'$ and note that $i(\partial_1^{U_1}(s_i)) = sn_i + sn_i'$ and that by lemma 4.9 there is a saddle $s'$ connected to $n$ and such that $\partial_1^{U_1}(s') = sn_1' + sn_2' = i(\partial_1^{U_1}(s'))$. It follows now that $\partial_1^{U_1}(s) = i(\partial_1^{U_1}(s_1 - s_2 + s'))$. This proves $HM_2(U_1) \cong HM_2(U_2)$.

• Suppose now that the birth-death pair $(n, s)$ is given by a stable node $n$ and a saddle $s$.

* Consider $HM_0$. Let $un_i$, for $i = 1, 2$, two unstable nodes eventually connected to $s$. By lemma 4.9 there are saddles $s_i$ forming squares respectively with $un_i$, $s$ and $n$. This implies that any linear combination of $un_1$ and $un_2$ never belongs to $\ker \partial_0^{U_1}$, thus $\dim \ker \partial_0^{U_1} = \dim \ker \partial_0^{U_2}$ and so $HM_0(U_1) \cong HM_0(U_2)$.

* Consider now $HM_2$. As $n \in \ker \partial_0^{U_1}$, it follows that $\dim \ker \partial_0^{U_1} = \dim \ker \partial_0^{U_2} + 1$. It remains to prove $\dim \ker \partial_0^{U_1} = \dim \ker \partial_0^{U_2} + 1$. Suppose $sn$ is an eventual second stable node connected to $s$ and consider chains $(n, s_1', sn_1', ...)$ and $(sn, s_1j, sn_1j, ...)$ of saddles and stable nodes starting respectively from $n$ and $sn$. Observe that $i(\partial_1^{U_2}(s_{ki})), i(\partial_1^{U_2}(s_{kj})) \in Im \partial_1^{U_1}$, and that $Im \partial_1^{U_1}$ contains also $\partial_1^{U_1}(s) = n - sn$ (for some choice of signs). Hence $\dim \ker \partial_0^{U_1} = \dim \ker \partial_0^{U_2} + 1$ and $HM_2(U_1) \cong HM_2(U_2)$.

\[\square\]

Remark 4.16. Observe that, in the last case shown, for instance, in figure 16, $\infty$ is the critical point to be added in order to extend the phase portrait to $S^2$, particularly, is a saddle: in this way on $S^2$ the relation $\dim \ker \partial_0^{U_1} = \dim \ker \partial_0^{U_2}$ still holds.

Remark 4.17. Let $i$ denote the natural injection

$$\mathbb{C}[p_1^{U_2}] \oplus ... \oplus \mathbb{C}[p_k^{U_2}] \hookrightarrow \mathbb{C}[p_1^{U_1}] \oplus ... \oplus \mathbb{C}[p_k^{U_1}] \oplus \mathbb{C}[p_{k+1}^{U_1}] \oplus \mathbb{C}[p_{k+2}^{U_1}]$$

and observe that:
1. if the birth-death pair is given by an unstable node $n$ and a saddle $s$ then

$$\dim \ker \partial_{U_1} = \dim \ker \partial_{U_2} + 1$$
$$\dim \im \partial_{U_1} = \dim \im \partial_{U_2} + 1$$

and

$$i(\ker \partial_{U_2}) \subseteq \ker \partial_{U_1}$$
$$\im \partial_{U_1} = i(\im \partial_{U_2}) \oplus <\partial_{U_1}(n)>$$

moreover, the 1-dimensional complement which added to $i(\ker \partial_{U_2})$ yields $\ker \partial_{U_1}$ contains the saddle $s$: a choice of a generator may be for example, referring for notations to figure 4.11

$$s \pm s_g \pm \sum_{i=1}^{n} s_{g_i} \pm s_k \pm \sum_{j=1}^{m} s_{k_j}$$

where the signs depend on the choice of a coherent orientation, $s_g$ is the saddle forming a square together with $n$, $s$ and $sn_i$, $s_{g_i}$, with $1 \leq i \leq N$ are the saddles in the chain of squares of length $N$ attached to $n$ and $s_g$ as in figure 4.17, $s_h$ is the saddle forming a square together with $n$, $s$ and $sn_2$, while $s_{h_j}$, for $1 \leq j \leq m$, are the saddles in the chain of squares of length $m$ attached to $n$ and $s_k$. The expression given for the generator includes all the cases: indeed, if $s$ is not connected to any stable node, there are no squares, so the generator is simply $s$; if $s$ is connected to a unique stable node $sn_1$, there is only the square with $s_g$, so the generator is $s \pm s_g$; if a chain of squares of length $N$ is attached to $n$ and $s_g$, then the generator is $s \pm s_g \pm \sum_{i=1}^{N} s_{g_i}$.

As to $HM_0$, observe that, considering only the unstable nodes $un$ and $n$, the element $un$ in $\ker \partial_{U_2}$ is replaced by $un + n$ in $\ker \partial_{U_2}$.

As to $HM_2$, observe simply that $\im \partial_{U_1} = i(\im \partial_{U_2})$.

2. if the birth-death pair is given by a stable node and a saddle, then

$$\dim \ker \partial_{U_1} = \dim \ker \partial_{U_2}$$
$$\dim \im \partial_{U_1} = \dim \im \partial_{U_2}$$
moreover, if $u_{n_1}$ and $u_{n_2}$ are unstable nodes connected to $s$ in $U_1$, writing $\text{Im} \partial^{U_2}_0 = I \oplus < \partial^{U_2}_0(u_{n_1}), \partial^{U_2}_0(u_{n_2}) >$ for some $I$, then

$$\text{Im} \partial^{U_1}_0 = i(I) \oplus < \partial^{U_1}_0(u_{n_1}), \partial^{U_1}_0(u_{n_2}) >$$

instead, as to kernel of $\partial_1$, suppose $s$ is connected, besides $n$, to a second stable node $sn$, assume that, starting from $n$, there are finite chains of the form $(n, s_{i_1}, s_{n_{i_1}}, s_{i_1 i_2}, s_{n_{i_1 i_2}}, ..., s_{i_1...i_p})$ ending with a saddle $s_{i_1...i_p}$, where $1 \leq i_r \leq i_{r_{\text{max}}}$ for $r = 1, ..., p$, and if from $sn$ there are finite chains of the form $(n, s_{j_1}, s_{n_{j_1}}, s_{j_1 j_2}, s_{n_{j_1 j_2}}, ..., s_{j_1...j_q})$ ending with a saddle $s_{j_1...j_p}$, where $1 \leq j_s \leq j_{s_{\text{max}}}$ for $s = 1, ..., q$, setting $\tilde{s}_{i_1...i_p} = \pm s_{i_1} \pm s_{i_1 i_2} \pm ... \pm s_{i_1...i_p}$ and $\tilde{s}_{j_1...j_q} = \pm s_{j_1} \pm s_{j_1 j_2} \pm ... \pm s_{j_1...j_q}$, and writing

$$\text{Ker} \partial^{U_2}_1 = K \oplus < \tilde{s}_{1...1} \pm \tilde{s}'_{1...1}, ..., \tilde{s}_{1...1} \pm \tilde{s}'_{j_{1_{\text{max}}}...j_{q_{\text{max}}}},$$

$$\tilde{s}_{21...1} \pm \tilde{s}'_{1...1}, ..., \tilde{s}_{21...1} \pm \tilde{s}'_{j_{1_{\text{max}}}...j_{q_{\text{max}}}}, ..., $$

$$\tilde{s}_{i_{1_{\text{max}}}...j_{p_{\text{max}}}} \pm \tilde{s}_{1...1}, ..., \tilde{s}_{i_{1_{\text{max}}}...j_{p_{\text{max}}}} \pm \tilde{s}_{j_{1_{\text{max}}}...j_{q_{\text{max}}}} >$$

for some $K$, then

$$\text{Ker} \partial^{U_1}_1 = i(K) \oplus < s \pm \tilde{s}_{1...1} \pm \tilde{s}'_{1...1}, ..., s \pm \tilde{s}_{1...1} \pm \tilde{s}'_{j_{1_{\text{max}}}...j_{q_{\text{max}}}}, $$

$$s \pm \tilde{s}_{21...1} \pm \tilde{s}'_{1...1}, ..., s \pm \tilde{s}_{21...1} \pm \tilde{s}'_{j_{1_{\text{max}}}...j_{q_{\text{max}}}}, ..., $$

$$s \pm \tilde{s}_{i_{1_{\text{max}}}...j_{p_{\text{max}}}} \pm \tilde{s}_{1...1}, ..., s \pm \tilde{s}_{i_{1_{\text{max}}}...j_{p_{\text{max}}}} \pm \tilde{s}_{j_{1_{\text{max}}}...j_{q_{\text{max}}}} >$$

where signs depend, as usual, on a choice of a coherent orientation. If, as in the proof of lemma 4.15, we assume that all chains from $n$ and $sn$ are of length 1, that is, $n$ is connected, besides $s$, to saddles $s_1, ..., s_n$ and these are not connected to any other stable node, and, similarly, $n$ is connected, besides $s$, to saddles $s_1', ..., s_m'$ and these too are not connected to any other stable node, then the above expression simplifies considerably

$$\text{Ker} \partial^{U_1}_1 = i(K) \oplus < s \pm s_1 \pm s_1', ..., s \pm s_1 \pm s_m', $$

$$s \pm s_2 \pm s_1', s \pm s_3 \pm s_1', ..., s \pm s_n \pm s_m' >$$

Particularly, if $s$ is connected to a unique stable node, that is, only to $n$, we have

$$\text{Ker} \partial^{U_1}_1 = i(K) \oplus < s \pm s_1, ..., s \pm s_n >$$
As to $HM_0$, observe simply that $\text{Ker} \partial_0^{U_1} = i(\text{Ker} \partial_0^{U_2})$.

As to $HM_2$, observe that, though $i(\partial_1^{U_2}(s'_{ki}))$, $i(\partial_1^{U_2}(sk_j)) \in \text{Im} \partial_1^{U_1}$, however $i(\partial_1^{U_2}(s'_{ki})) \neq \partial_1^{U_1}(s'_{ki})$ and $i(\partial_1^{U_2}(sk_j)) \neq \partial_1^{U_1}(sk_j)$.

Lemma 4.15 shows that $HM(U_1) \cong HM(U_2)$. The purpose now is to pick up an isomorphism which will be used to glue the holomorphic structure of the mirror object along the caustic $C$, providing the quantum corrections.

**Definition 4.18.** If $\partial U_1 \cap \partial U_2 \neq \emptyset$ contains only folds not limit points of the bifurcation locus $B$, define an isomorphism $HM(U_1) \cong HM(U_2)$ as follows:

1. if the birth-death point is represented by an unstable node $n$ and a saddle $s$, the isomorphism $M : HM(U_1) \cong HM(U_2)$ is the one induced in homology by the map

\[ \tilde{M} : \mathbb{C}[p_1^{U_1}] \oplus ... \oplus \mathbb{C}[p_k^{U_2}] \to \mathbb{C}[p_1^{U_1}] \oplus ... \oplus \mathbb{C}[p_k^{U_1}] \oplus \mathbb{C}[p_{k+1}^{U_2}] \oplus \mathbb{C}[p_{k+2}^{U_1}] \]

where $\tilde{M}$ is the natural injection $i$ on saddles and stable nodes and, if $un_i, un$ are the unstable nodes appearing in $U_2$ with $un$ the eventual second unstable node connected to $s$ in $U_1$ besides $n$, it is still the natural injection $i$ on $un_i$, while on $un$ it acts as

\[ \tilde{M}(un) = un + n \]

2. If the birth-death point is represented by a stable node $n$ and a saddle $s$, and if $s_i \neq s$, for $i = 1, ..., N$, are the saddles in the phase portraits over $U_1$ and $U_2$, such that, for $i = 1, ..., m \leq N$, $s_i$ are the saddles connected to $n$, define the isomorphism $M : HM(U_1) \cong HM(U_2)$ as the one induced in homology by the map

\[ \tilde{M} : \mathbb{C}[p_1^{U_2}] \oplus ... \oplus \mathbb{C}[p_k^{U_2}] \to \mathbb{C}[p_1^{U_1}] \oplus ... \oplus \mathbb{C}[p_k^{U_1}] \oplus \mathbb{C}[p_{k+1}^{U_1}] \oplus \mathbb{C}[p_{k+2}^{U_1}] \]

such that

\[ \tilde{M}(s_i) = \begin{cases} s_i \pm s, & 1 \leq i \leq m \\ s_i, & m < i \leq N \end{cases} \]

where the sign depends on orientation: + if $n(\gamma_{s_i,n}) = -n(\gamma_{s_i,n})$ and $-$ otherwise ($n(\gamma)$ denotes the sign of the gradient line $\gamma$), while it is the natural injection $i$ on stable and unstable nodes.

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To have a well-defined map $M$ in homology, it is necessary to check that $\tilde{M}$, defined on generators $p_i$, maps kernel and image of $\partial^{U_2}$ to, respectively, those of $\partial^{U_1}$.

**Lemma 4.19.** The map $\tilde{M}$ satisfies $\tilde{M}(\text{Im}\partial^{U_2}) \subseteq \text{Im}\partial^{U_1}$ and $\tilde{M}(\text{Ker}\partial^{U_2}) \subseteq \text{Ker}\partial^{U_1}$ and so it induces an isomorphism $M : HM(U_1) \cong HM(U_2)$.

**Proof.**  
1. When the birth-death pair is represented by an unstable node and a saddle, the lemma follows from formulas in part 1 of remark 4.17.

2. Suppose, instead, the birth-death pair is given by a stable node and a saddle.

- The relations $\tilde{M}(\text{Ker}\partial_0^{U_2}) \subseteq \text{Ker}\partial_1^{U_1}$ and $\tilde{M}(\text{Im}\partial_0^{U_2}) \subseteq \text{Im}\partial_1^{U_1}$ are easily verified.

- So consider the free $\mathbb{C}$-module generated by saddles. Since $\tilde{M}$ is non trivial on those saddles $s_i$ connected to $n$, we are going to compute $\tilde{M}(\partial_0^{U_2})$ on those unstable nodes connected to $s_i$. If there are unstable nodes $un_1$ and $un_2$ connected to $s$, let $s_i$ be the saddle forming squares with $s$, $n$ and $un_i$, for $i = 1, 2$. Writing $\partial_0^{U_1}(un_i) = s \pm s_i \pm ...$, where, as usual, signs depend on orientation, then $\partial_0^{U_2}(un_i) = s_i \pm ...$. So, in the special case where each node $un_i$ is connected, in $U_1$, to only the saddle $s$ and $s_i$, since in a square $n(\gamma_{s,n}) = n(\gamma_{s_i,n}) \iff n(\gamma_{un_i,s_i}) \not= n(\gamma_{un_i,s})$ it follows that $\tilde{M}(\partial_0^{U_2}(un_i)) = \tilde{M}(s_i) = s_i \pm s$, and so $\tilde{M}(\text{Im}\partial_0^{U_2}) \subseteq \text{Im}\partial_1^{U_1}$.

If, instead, $un_i$ is connected to other saddles besides $s$ and $s_i$ (suppose, for simplicity, only to the saddle $sk_i$), then the formula $\tilde{M}(\text{Im}\partial_0^{U_2}) \subseteq \text{Im}\partial_1^{U_1}$ is still easily verified provided that $sk_i$ is not connected to $n$.

In the opposite case, note, first of all, that the phase portrait must exhibit other critical points, at least a saddle, a stable node and an unstable node, in order to be the phase portrait of a gradient vector field: this is shown in figure 4.17, where it is represented only the unstable node $un_1$, and the critical points, added in the phase portrait, are denoted by $\tilde{s}$, $\tilde{sn}$, $\tilde{un}$.

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Fig. 4.17: \( u_1 \) and \( n \) are connected to the same saddle

Considering the squares \( S_1 \) and \( S_2 \), it follows, respectively, that

\[
n(\gamma_{s_1,n}) = n(\gamma_{s,n}) \iff n(\gamma_{u_1,s_1}) \neq n(\gamma_{u_1,s})
\]

\[
n(\gamma_{s_1,n}) = n(\gamma_{\tilde{s},n}) \iff n(\gamma_{u_1,s_{11}}) \neq n(\gamma_{u_1,\tilde{s}})
\]

so the quantum correction when crossing the caustic from \( U_2 \) to \( U_1 \) is given by

\[
\tilde{M}(s_i) = s_i - \epsilon(s_i) s
\]

for \( i = 1, 3, 4 \), where

\[
\epsilon(s_i) = \begin{cases} 
1 & , \ n(\gamma_{s_i,n}) = n(\gamma_{s,n}) \\
-1 & , \ n(\gamma_{s_i,n}) \neq n(\gamma_{s,n})
\end{cases}
\]

The Morse differential is defined as

\[
\partial^L_0(u_1) = n(\gamma_{u_1,s_1}) s_1 + n(\gamma_{u_1,s_{11}}) s_{11} + n(\gamma_{u_1,\tilde{s}}) \tilde{s}
\]

\[
\partial^L_0(u_{11}) = n(\gamma_{u_1,s}) s + n(\gamma_{u_1,s_1}) s_1 + n(\gamma_{u_1,s_{11}}) s_{11} + n(\gamma_{u_1,\tilde{s}}) \tilde{s}
\]

so it follows that

\[
\tilde{M}(\partial^L_0(u_1)) = n(\gamma_{u_1,s_1}) s_1 + n(\gamma_{u_1,s_{11}}) s_{11} + n(\gamma_{u_1,\tilde{s}}) \tilde{s} + \\
- [n(\gamma_{u_1,s_1}) \epsilon(s_1) + n(\gamma_{u_1,s_{11}}) \epsilon(s_{11}) + \\
+ n(\gamma_{u_1,\tilde{s}}) \epsilon(\tilde{s})] s
\]

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Since 
\[ n(\gamma_{un1,s1})\epsilon(s_1) = -n(\gamma_{un1,s}) \]
while
\[ n(\gamma_{un1,s1}) = n(\gamma_{un1,s}) \Leftrightarrow n(\gamma_{s1,n}) \neq n(\gamma_{s,n}) \Leftrightarrow \epsilon(s_1) \neq \epsilon(\bar{s}) \]
implies
\[ n(\gamma_{un1,s1})\epsilon(s_1) + n(\gamma_{un1,s})\epsilon(\bar{s}) = 0 \]
it follows that
\[ \tilde{M}(\partial^{U2}_{0}(un1)) = \partial^{U1}_{0}(un1) \]
Suppose now that another unstable node \( \tilde{u}n \) is connected to one of the saddle \( s_1 \), say, for example, \( s_1 \) (the case considered in figure 4.17 is a particular case, more generally see figure 4.18). Suppose also that \( \tilde{u}n \) is connected to further saddles \( \tilde{s}_h \) for some parameter \( h \). It follows that \( \tilde{M}(\partial^{U2}_{0}(\tilde{u}n)) \in Im\partial^{U1}_{0} \). By lemma 4.9 there exists a saddle \( s_q \) forming a square with \( \tilde{u}n, s_1 \) and \( n \), and so \( \tilde{M}(s_q) = s_q - \epsilon(s_q)s \)

\[ \tilde{M}(\partial^{U2}_{0}(\tilde{u}n)) = \tilde{M}(n(\gamma_{u\tilde{n},s_1})s_1 + n(\gamma_{u\tilde{n},s_q})s_q + \sum_h n(\gamma_{u\tilde{n},s_h})\bar{s}_h) = n(\gamma_{u\tilde{n},s_1})s_1 + n(\gamma_{u\tilde{n},s_q})s_q + \sum_h n(\gamma_{u\tilde{n},s_h})\bar{s}_h + -[n(\gamma_{u\tilde{n},s_1})\epsilon(s_1) + n(\gamma_{u\tilde{n},s_q})\epsilon(s_q)]s = n(\gamma_{u\tilde{n},s_1})s_1 + n(\gamma_{u\tilde{n},s_q})s_q + \sum_h n(\gamma_{u\tilde{n},s_h})\bar{s}_h = \partial^{U1}_{0}(\tilde{u}n) \]

Fig. 4.18: another unstable node is connected to \( s_1 \)
because, as seen above, for a square always
\[ n(\gamma_{\bar{u}n,s_1})\epsilon(s_1) + n(\gamma_{\bar{u}n,s_2})\epsilon(s_2) = 0 \]

As said at the beginning, the same conclusions are achieved if two unstable nodes \( u_{11} \) and \( u_{22} \) are connected to \( s \), and the proof does not change substantially.

It remains to check that \( \tilde{M}(\text{Ker}\partial^{U_2}_1) = \text{Ker}\partial^{U_1}_1 \). Observing that \( n(\gamma_{s_1,n})s_1 - n(\gamma_{s_j,n})s_j \) is an element of both \( \text{Ker}\partial^{U_1}_1 \) and \( \text{Ker}\partial^{U_2}_1 \) for every \( j = 2, \ldots, n \), and that \( n(\gamma_{s_1,n})s_1 - n(\gamma_{s_j,n})s_j = \epsilon(s_1)s_1 - \epsilon(s_j)s_j \), it follows that
\[
\text{Ker}\partial^{U_1}_1 = i(K)\oplus < s - \epsilon(s_1)s_1 - \epsilon(s'_1)s'_1, \ldots, s - \epsilon(s_1)s_1 + \epsilon(s'_1)s'_1, s_2 - \epsilon(s_1)s_2, \ldots, \epsilon(s_1)s_1 - \epsilon(s_n)s_n >
\]
and
\[
\text{Ker}\partial^{U_2}_1 = K\oplus < -\epsilon(s_1)s_1 - \epsilon(s'_1)s'_1, \ldots, -\epsilon(s_1)s_1 - \epsilon(s'_m)s'_m, \epsilon(s_1)s_1 - \epsilon(s_2)s_2, \ldots, \epsilon(s_1)s_1 - \epsilon(s_n)s_n >
\]

hence, since \( \tilde{M} = i \) on all saddles but \( s_j \),
\[
\tilde{M}(\text{Ker}\partial^{U_2}_1) = i(K)\oplus < -\epsilon(s_1)[s_1 - \epsilon(s_1)s_1 - \epsilon(s'_1)s'_1, \ldots, -\epsilon(s_1)[s_1 - \epsilon(s_1)s_1 - \epsilon(s'_m)s'_m, \epsilon(s_1)[s_1 - \epsilon(s_1)s_1 - \epsilon(s_2)s_2, \ldots, \epsilon(s_1)[s_1 - \epsilon(s_1)s_1 - \epsilon(s_n)s_n] >= \text{Ker}\partial^{U_1}_1
\]

\[ \square \]

5 The bifurcation locus

Points \( x \) of the bifurcation locus \( B \) are defined as those points at which a saddle-to-saddle separatrix appears in the phase portrait, so that to each bifurcation point \( x \) it is associated the pair of saddles \( s_1(x) \) and \( s_2(x) \) and the exceptional gradient line connecting them. Call them bifurcating saddles.

In dimension 2, \( B \) is an immersed submanifold of codimension 1. Bifurcation points characterized by the presence of two saddle-to-saddle sepratrixes form
a codimension 2 subset. This endows $B$ with a stratification, where strata are
given by codimension 1 and codimension 2 points (that is, points where one,
respectively, two saddle-to-saddle separatrixes appear in the phase portrait).

Consider two subset $U_1$ and $U_2$ such that $\partial U_1 \cap \partial U_2 \neq \emptyset$ is contained
in a bifurcation line. Note that if critical points are seen as sections of the
fibration $\mathbb{R}^4 \to \mathbb{R}^2$, these are smooth sections over $B$. The phase portraits
over points of, respectively, $U_1$ and $U_2$ differ, in general, because of gradient
lines appearing or disappearing between the bifurcating saddles and some
nodes (see figure 5.19). This changes the Morse differential $\partial$. However,
even when $\partial$ is unchanged by the bifurcation, generically the phase portraits
of $U_1$ and $U_2$ are not orbit isotopic. Consider, for simplicity, a 1-parameter
family $\{f_t\}$ of functions such that $t = 0$ is a bifurcation point. Suppose
there are two saddles $s_1$ and $s_2$ connected by a saddle-to-saddle separatrix
$\gamma_{s_1,s_2}$. Referring to figure 5.19, if $W^u_j(s_i(t))$ and $W^s_j(s_i(t))$ denote, for $j = 1, 2$
the two components of the unstable and respectively stable manifolds of $s_i$,
$i = 1, 2$, then the family $\{f_t\}$ provides isotopies, for $t \leq 0$, between $W^u_1(s_1(t))$
and $\gamma_{s_1,s_2} \cup s_1 \cup W^u_2(s_2(t))$ and between $W^s_1(s_1(t))$ and $W^s_2(s_1(t)) \cup s_1 \cup \gamma_{s_1,s_2}$, for
t $\geq 0$, between $W^u_1(s_1(t))$ and $\gamma_{s_1,s_2} \cup s_1 \cup W^u_2(s_2(t))$ and between $W^s_1(s_1(t))$
and $W^s_2(s_1(t)) \cup s_1 \cup \gamma_{s_1,s_2}$. Note that the above isotopy does not induce an
orbit isotopy.
This implies that if there is an unstable node $un_1$ such that $\gamma_{un_1,s_1} = W_1^s(s_1(t))$ for $t < 0$, then for $t > 0$, besides $\gamma_{un_1,s_1} = W_1^s(s_1(t))$, also the gradient line $\gamma_{un_1,s_2} = W_1^u(s_2(t))$ appears in the phase portrait; if there exists an unstable node $un_2$ such that $\gamma_{un_2,s_1} = W_2^s(s_1(t))$ for $t < 0$, then there exists also the gradient line $\gamma_{un_2,s_2} = W_1^s(s_2(t))$ for $t < 0$, which breaks for $t > 0$, while $\gamma_{un_2,s_1} = W_2^s(s_1(t))$ persists in the phase portrait; if there is a stable node $sn_1$ such that $\gamma_{s_2,sn_2} = W_1^u(s_2(t))$ for $t < 0$, then there is also the gradient line $\gamma_{s_1,sn_2} = W_1^u(s_1(t))$ for $t < 0$, which breaks for $t > 0$, while $\gamma_{s_2,sn_2} = W_1^u(s_2(t))$ persists in the phase portrait; if there exists a stable node $sn_2$ such that $\gamma_{s_2,sn_2} = W_2^s(s_2(t))$ for $t < 0$, then for $t > 0$ there is also, besides $\gamma_{s_2,sn_2} = W_2^s(s_2(t))$, the gradient line $\gamma_{s_1,sn_2} = W_1^u(s_1(t))$; instead, if there are a stable node $sn$ such that $\gamma_{s_1,sn} = W_1^u(s_1(t))$ or an unstable node $un$ such that $\gamma_{un,s_2} = W_2^s(s_2(t))$, these lines appear in both the phase portrait for $t < 0$ and $t > 0$. In this sense gradient lines between the bifurcating saddles and some nodes appear or break in the phase portraits over points of $U_1$ and $U_2$ when crossing the bifurcation locus, changing the Morse differential $\partial$. The purpose is to prove that the Morse complexes over $U_1$ and $U_2$ are isomorphic, and to pick up a suitable isomorphism, providing the quantum correction.

**Lemma 5.1.** In the situation described above of two bifurcating saddles, suppose there exists, for instance in $U_1$, an unstable node $un_1$ with gradient lines $\gamma_{un_1,s_1}$ and $\gamma_{un_1,s_2}$, and assume that the signs given to the separatrixes of the saddles $s_1$ and $s_2$ coincide in $U_1$ and $U_2$:
1. Suppose there is also a stable node $s_{n_2}$ with gradient lines $\gamma_{s_1,s_{n_2}}$ and $\gamma_{s_2,s_{n_2}}$, then

$$n(\gamma_{u_{n_1},s_1}) = n(\gamma_{u_{n_1},s_2}) \iff n(\gamma_{s_1,s_{n_2}}) = -n(\gamma_{s_2,s_{n_2}})$$

2. $n(\gamma_{u_{n_1},s_1}) = n(\gamma_{u_{n_1},s_2})$ in $U_1$ if and only if $n(\gamma_{u_{n_2},s_1}) = -n(\gamma_{u_{n_2},s_2})$ and/or $n(\gamma_{s_1,s_{n_1}}) = n(\gamma_{s_2,s_{n_1}})$ in $U_2$ (provided these gradient lines do exist).

Proof. 1. It is a consequence of a choice of an orientation for a square and already used along the proof of lemma 4.14.

2. Suppose there exists in $U_1$ a second unstable node $u_{n_2}$ with the gradient line $\gamma_{u_{n_2},s_1}$, then in $U_2$, besides $\gamma_{u_{n_1},s_1}$, both the lines $\gamma_{u_{n_2},s_1}$ and $\gamma_{u_{n_2},s_2}$ appear in the phase portrait. Observe now that since $n(W^u_{s_1}(s_1)) = -n(W^u_{s_2}(s_1))$ and, by assumption, $n(\gamma_{u_{n_1},s_1}) = n(\gamma_{u_{n_2},s_2})$, it follows that $n(\gamma_{u_{n_2},s_1}) = -n(\gamma_{u_{n_2},s_2})$. Suppose there exist in $U_2$ a stable node $s_{n_1}$ with the gradient line $\gamma_{s_2,s_{n_1}}$, then in $U_2$, besides $\gamma_{u_{n_1},s_1}$, also the lines $\gamma_{s_1,s_{n_1}}$ and $\gamma_{s_2,s_{n_1}}$ occur in the phase portrait. By lemma 4.19 there exist a saddle $s$ forming a square in $U_1$ with $u_{n_1}$, $s_2$ and $s_{n_1}$, and in $U_2$ with $u_{n_1}$, $s_1$ and $s_{n_1}$. If $n(\gamma_{u_{n_1},s_2}) = n(\gamma_{s_2,s_{n_1}})$ in $U_1$ then $n(\gamma_{u_{n_1},s}) = -n(\gamma_{s,s_{n_1}})$, which implies, assuming that signs of separatrixes coincide in $U_1$ and $U_2$, that $n(\gamma_{s_1,s_{n_1}}) = n(\gamma_{u_{n_1},s_1})$ in $U_2$. As, by hypothesis, $n(\gamma_{u_{n_1},s_1}) = n(\gamma_{u_{n_1},s_2})$, it follows that $n(\gamma_{s_1,s_{n_1}}) = n(\gamma_{s_2,s_{n_1}})$. If, instead, $n(\gamma_{u_{n_1},s_2}) = -n(\gamma_{s_2,s_{n_1}})$ in $U_1$ then $n(\gamma_{u_{n_1},s}) = n(\gamma_{s,s_{n_1}})$, which implies, that $n(\gamma_{s_1,s_{n_1}}) = n(\gamma_{u_{n_1},s_1})$ in $U_2$. As, by hypothesis, $n(\gamma_{u_{n_1},s_1}) = n(\gamma_{u_{n_1},s_2})$, it follows again that $n(\gamma_{s_1,s_{n_1}}) = n(\gamma_{s_2,s_{n_1}})$.

Let $s_1$, $s_2$, $s_3$, ..., $s_k$ be the saddles in the phase portrait, where $s_1$ and $s_2$ form a pair of bifurcating saddles. Assume to choose signs according to lemma 5.1 (this will be made more rigorous later in definition 5.7).

Lemma 5.2. Let $U_1$ and $U_2$ be open subsets such that $U_i \cap (B \cup C) = \emptyset$ and $\partial U_1 \cap \partial U_2 \subset B$ consists only of codimension 1 bifurcation points. Then the homology groups of the Morse complexes

$$\mathbb{C}[p^{U_1}] \oplus \ldots \oplus \mathbb{C}[p^{U_1}]$$

$$\mathbb{C}[p^{U_2}] \oplus \ldots \oplus \mathbb{C}[p^{U_2}]$$

are isomorphic.
Proof. We are going to prove that \( \dim \ker \partial U_1 = \dim \ker \partial U_2 \) and \( \dim \operatorname{Im} \partial U_1 = \dim \operatorname{Im} \partial U_2 \).

- It is easily verified that \( \dim \ker \partial_0 U_1 = \dim \ker \partial_0 U_2 \) and \( \dim \operatorname{Im} \partial U_1 = \dim \operatorname{Im} \partial U_2 \).

- It remains to check the above relations at the level of 1-chains, that is, on saddles. With the notation as in figure 5.19, what can change the Morse differential \( \partial \) in \( U_1 \) and \( U_2 \) are the gradient lines between the saddle \( s_1 \) and \( s_2 \) on one side and the nodes \( u_1, u_2, s_1n \) and \( s_1n \) on the other, thus, for simplicity, assume only these nodes in the phase portrait.

- Consider, first of all, how \( \operatorname{Im} \partial_0 \) can change when crossing the bifurcation line.

If neither \( u_1 \) nor \( u_2 \) are in the phase portrait then \( \operatorname{Im} \partial_0 U_1 = \operatorname{Im} \partial_0 U_2 = \{0\} \).

If just \( u_1 \) is in the phase portrait, then, as explained, \( U_1 \) exhibits the gradient lines \( \gamma_{u_1,s_1} \) and \( \gamma_{u_1,s_2} \) while \( U_2 \) only \( \gamma_{u_1,s_1} \). So, for a choice of signs as lemma 5.1 suggests (see further on definition 5.7), we have that
\[
\operatorname{Im} \partial_0 U_1 = < s_1 + s_2 + \sum_{j=1}^{m(u_1)} s_j >
\]
\[
\operatorname{Im} \partial_0 U_2 = < s_1 + \sum_{j=1}^{m(u_1)} r_j >
\]
where \( r_j \), for \( j = 1, \ldots, m(u_1) \), are further saddles connected to \( u_1 \), hence
\[
\dim \operatorname{Im} \partial_0 U_1 = \dim \operatorname{Im} \partial_0 U_2
\]

If just \( u_2 \) is in the phase portrait, then \( U_1 \) exhibits only the gradient line \( \gamma_{u_1,s_1} \) while \( U_2 \) both \( \gamma_{u_1,s_1} \) and \( \gamma_{u_1,s_2} \). So, according to lemma 5.1
\[
\operatorname{Im} \partial_0 U_1 = < s_1 + \sum_{l=1}^{m(u_2)} s_j >
\]
\[ Im\partial_0^U^2 = < s_1 - s_2 \pm \sum_{l=1}^{m(u_2)} t_l > \]

where \( t_l \), for \( l = 1, ..., m(u_2) \), are further saddles connected to \( u_2 \). Hence still \( \dim Im\partial_0^U^1 = \dim Im\partial_0^U^2 \).

If both \( u_1 \) and \( u_2 \) are in the phase portrait, then in \( U_1 \) we have the gradient lines \( \gamma_{u_1,s_1} \), \( \gamma_{u_1,s_1} \) and \( \gamma_{u_1,s_2} \) while in \( U_2 \) the lines \( \gamma_{u_1,s_1} \), \( \gamma_{u_2,s_1} \) and \( \gamma_{u_2,s_2} \). Lemma 5.1 implies that

\[
Im\partial_0^{U_1} = < s_1 + s_2 \pm \sum_{j=1}^{m(u_1)} r_j, s_1 \pm \sum_{l=1}^{m(u_2)} t_l >
\]

\[
Im\partial_0^{U_2} = < s_1 \pm \sum_{j=1}^{m(u_1)} r_j, s_1 - s_2 \pm \sum_{l=1}^{m(u_2)} t_l >
\]

so \( \dim Im\partial_0^{U_1} = \dim Im\partial_0^{U_2} \).

- Consider, now, how \( Ker\partial_1 \) can change when crossing the bifurcation line.

If neither \( s_n \) nor \( s_n \) are in the phase portrait then

\[ Ker\partial_1^{U_1} = Ker\partial_1^{U_2} = < s_1, s_2 > \]

so \( \dim Ker\partial_1^{U_1} = \dim Ker\partial_1^{U_2} \).

If just \( s_n \) is in the phase portrait, then in \( U_1 \) there is only the gradient line \( \gamma_{s_n,s_1} \) while in \( U_2 \) we have \( \gamma_{s_1,s_n} \) and \( \gamma_{s_2,s_n} \). Lemma 5.1 implies that

\[ Ker\partial_1^{U_1} = < s_1, s_2 \pm x_1, ..., s_2 \pm x_{m(s_1)} > \]

\[ Ker\partial_1^{U_2} = < s_1 - s_2, s_2 \pm x_1, ..., s_2 \pm x_{m(s_1)} > \]

where \( x_p \), for \( p = 1, ..., m(s_1) \), are further saddles connected to \( s_n \). Therefore \( \dim Ker\partial_1^{U_1} = \dim Ker\partial_1^{U_2} \).

If just \( s_n \) is in the phase portrait, then in \( U_1 \) there are the gradient lines \( \gamma_{s_n,s_2} \) and \( \gamma_{s_2,s_n} \), while in \( U_2 \) only \( \gamma_{s_2,s_n} \). Lemma 5.1 implies that

\[ Ker\partial_1^{U_1} = < s_1 + s_2, s_2 \pm y_1, ..., s_2 \pm y_{m(s_2)} > \]
Ker\(\partial_1^{U_2}\) = \(<s_1, s_2 \pm y_1, \ldots, s_2 \pm y_m(sn_2)>
\)

where \(y_q\), for \(q = 1, \ldots, m(sn_2)\), are further saddles connected to \(sn_2\), in particular, it follows that \(\dim \text{Ker} \partial_1^{U_1} = \dim \text{Ker} \partial_1^{U_2}\).

If both \(sn_1\) and \(sn_2\) are in the phase portrait, then this exhibits over \(U_1\) the gradient lines \(\gamma_{s_1,sn_2}\), \(\gamma_{s_2,sn_1}\) and \(\gamma_{s_2,sn_2}\), while over \(U_2\) the gradient lines \(\gamma_{s_1,sn_1}\), \(\gamma_{s_2,sn_1}\) and \(\gamma_{s_2,sn_2}\). Lemma 5.1 implies that

\[
\text{Ker} \partial_1^{U_1} =< s_1 + s_2 \pm x_1, s_2 \pm y_1 \pm x_1, \ldots, s_2 \pm y_1 \pm x_1, s_2 \pm y_1 \pm x_2, \ldots, s_2 \pm y_1 \pm x_1 + x_1 >
\]

\[
\text{Ker} \partial_1^{U_1} =< s_1 \pm x_1, s_2 \pm y_1 \pm x_1, \ldots, s_2 \pm y_1 \pm x_1, s_2 \pm y_1 \pm x_2, \ldots, s_2 \pm y_1 \pm x_1 + x_1 >
\]

so \(\dim \text{Ker} \partial_1^{U_1} = \dim \text{Ker} \partial_1^{U_2}\).

Remark 5.3. The presence of a second stable node \(sn\) connected to \(s_1\) or of a second unstable node \(un\) connected to \(s_2\) simply adds new terms in the expressions of \(\text{Im} \partial_0\) and \(\text{Ker} \partial_1\), which, however, are not modified by the bifurcation and so appears both in \(U_1\) and in \(U_2\).

We now pick up an isomorphism between the Morse homologies in \(U_1\) and \(U_2\).

Definition 5.4. For a bifurcation characterized by the appearance of the saddle-to-saddle separatrix \(\gamma_{s_1,s_2}\) and whose locus is a line \(B\), and, using notations as in figure 5.19, if orientation is chosen in such a way that in \(U_1\), as lemma 5.1 suggests, \(W^s_i(s_1)\) and \(W^u_i(s_2)\) have same sign and \(W^u_i(s_1)\) and \(W^u_i(s_2)\) have opposite sign, define an isomorphism \(\tilde{M} : HM(U_1) \rightarrow HM(U_2)\) as the one induced in homology by the map \(\tilde{M} : \oplus_{i=1}^k C[s_1^{U_1}] \rightarrow \oplus_{i=1}^k C[s_1^{U_2}]\) such that

\[
\tilde{M}(s_i) = \begin{cases} 
  s_1 - s_2, & i = 1 \\
  s_i, & i \neq 1
\end{cases}
\]

\(\tilde{M} = Id\) on nodes
Remark 5.5. Note that if in $U_1$ signs are not as definition 5.4 requires, by lemma 5.1 this condition is fulfilled in $U_2$, so that $\tilde{M}$ defines an isomorphism $\tilde{M} : HM(U_2) \to HM(U_1)$. The isomorphism $HM(U_2) \to HM(U_1)$ is provided by $\tilde{M}^{-1}$

$$\tilde{M}^{-1}(s_i) = \begin{cases} s_1 + s_2 & , \quad i = 1 \\ s_i & , \quad i \neq 1 \end{cases}$$

Lemma 5.6. The map $\tilde{M}$ induces a map in homology.

Proof. • Nothing to prove as to $HM_0$ and $HM_2$, being $\tilde{M} = Id$ on nodes.

• So consider $HM_1$.

- Lemma 5.2 implies that $\tilde{M}(Ker\partial_{U_1}) = Ker\partial_{U_2}$ and $\tilde{M}(Im\partial_{U_1}) = Im\partial_{U_2}$, when the only nodes in the phase portrait are those named $un_1$, $un_2$, $sn_1$ and $sn_2$. However, as noted in remark 5.3 there could be a further stable node $sn$ connected to $s_1$ and a further unstable node $un$ connected to $s_2$. In this case, it is necessary to check that $\partial_{U_1}(un)$ is mapped by $\tilde{M}$ into $Im\partial_{U_2}$: since $\partial_{U_1}(un) = \partial_{U_2}(un) = s_2 + \sum_{j=1}^m r_j$ for some saddles $r_j$ with $r_j \neq s_1$ for all $j = 1, \ldots, m$, and since $\tilde{M}$ acts as the identity on $s_2$ and $r_j$, it follows that $\tilde{M}(\partial_{U_1}(un)) = \partial_{U_2}(un)$.

- As to $Ker\partial_1$, consider, first, the following case: $s_2$ is not connected to any stable node, while to $sn$ it is connected, besides $s_1$, another saddle $s$; then $Ker\partial_{U_1} = Ker\partial_{U_2} = < s_1 \pm s, s_2 >$ and so $\tilde{M}(Ker\partial_{U_1}) = Ker\partial_{U_2}$. If, instead, $s_2$ is connected to two stable nodes $sn_1$ and $sn_2$, which, besides $s_2$, are connected, respectively, to further saddles $r_1$ and $r_2$, we have that $Ker\partial_{U_1} = < s_1 + s - r_1, s_2 + r_1 + r_2 >$ and $Ker\partial_{U_2} = < s_1 + s + r_2, s_2 + r_1 + r_2 >$; thus, being $\tilde{M}(s_1 + s + r_1) = s_1 + s + r_1 - s_2 = (s_1 + s + r_2) - (s_2 + r_1 + r_2)$, it follows that $\tilde{M}(Ker\partial_{U_1}) = Ker\partial_{U_2}$. The argument is independent from the chosen orientation. The same conclusion is achieved, modifying slightly the proof, if just one stable node is connected to $s_2$.

To apply definition 5.4 it is necessary that orientation is chosen in a proper way. The following definition selects the class of orientations for which quantum corrections can be constructed, in accordance with definition 5.4.
Definition 5.7. An orientation in a phase portrait is said to satisfy the “signs convention” if and only if it is chosen in such a way that, for any pair of bifurcating saddles \(s_1\) and \(s_2\) exhibiting the saddle-to-saddle separatrix \(\gamma_{s_1,s_2}\) along a certain bifurcation line, \(W^s_1(s_1)\) and \(W^s_1(s_2)\) are given the same sign, while \(W^u_1(s_1)\) and \(W^u_2(s_2)\) opposite sign (see figure 5.19 for notation).

The following proposition states that such class of orientations is not empty.

Proposition 5.8. For any given phase portrait, there exists a coherent orientation satisfying definition 5.7.

Proof. The coherent orientation of definition 5.7 is the orientation corresponding to a phase portrait where stable and unstable nodes \(sn\) and \(un\) are added in such a way that, for each pair of bifurcation saddles \(s_1\) and \(s_2\), \(W^s_1(s_1)\) and \(W^s_1(s_2)\) connect \(un\) with respectively \(s_1\) and \(s_2\), and \(W^u_1(s_1)\) and \(W^u_2(s_2)\) connect respectively \(s_1\) and \(s_2\) to \(sn\): in this case, indeed, \((un,s_1,s_2,sn)\), with these separatrixes, form a square, and this is just the orientation for a square. The existence of coherent orientations for any phase portrait proves now the proposition.

6 Intersection of caustic and bifurcation lines

As explained, the caustic \(C\) is an immersed submanifold of codimension 1 having, in dimension 2, two strata: the folds, forming the stratum of codimension 1, and the cusps, the stratum of codimension 2. Different branches of \(C\) can intersect transversely one with another, generically at folds, to which corresponds two birth-death pairs with no common points.

The bifurcation locus \(B\) is as well an immersed submanifold of codimension 2 with two strata: codimension 1 and codimension 2 bifurcations, where the corresponding phase portrait exhibits one or respectively two saddle-to-saddle separatrixes. This means that the intersection points of two bifurcation lines are codimension 2 bifurcations, each line representing one of the two saddle-to-saddle separatrixes exhibited by the codimension 2 bifurcation (see [11] and [12]).

A bifurcation line \(B\) can intersect the caustic \(C\), generically, at a fold: indeed, if the intersection were a cusp, the exceptional gradient line appearing at this point will break for any small perturbation, and, by a transversality
argument (see for example [12] for perturbations of the elliptic umbilic), will appear at a nearby point of the caustic, which, generically, is a fold. Actually, to be precise, having defined bifurcation points away from the caustic, it should be better to talk about points of the caustic being limit points of the bifurcation locus rather than intersection points of the caustic and bifurcation locus. If to $B$ it is associated the saddle-to-saddle separatrix $\gamma_{s_1,s_2}$ and to $C$ the pair of birth-death points $(s_i,n)$, where $i = 1$ or $i = 2$ and $n$ is a node, then, in a neighbourhood of the intersection point, $B$ is a half-line lying in one of the two subsets determined by $C$, precisely the one exhibiting $s_i$, and whose origin is the intersection point. As already explained above, generically this point is a fold. The two ways $B$ can meet $C$ are shown in figure 6.20: case (a) was described just above, case (b) occurs when $i \neq 1, 2$.

\[\begin{array}{c}
(a) \\
\hspace{1cm} C \\
\hspace{1.5cm} B \\
(b) \\
\hspace{1cm} C \\
\hspace{1.5cm} B
\end{array}\]

Fig. 6.20: Intersections between the caustic and the bifurcation locus

Consider now the intersection of two bifurcation lines $B_1$ and $B_2$ (see [12] for examples concerning the perturbed elliptic umbilic). The phase portrait corresponding to the intersection point $z$ of $B_1$ and $B_2$ contains two saddle-to-saddle separatrices $\gamma_1$ and $\gamma_2$, each appearing, respectively, along $B_1$ and $B_2$. The phase portrait associated to each subset determined by $B_1$ and $B_2$ is obtained from the phase portrait over $z$ by breaking $\gamma_1$ and $\gamma_2$: if $\gamma_1 \neq \gamma_2$ there are at least four of such phase portraits, which are not orbit isotopic.

To study when the assignement of $B_1$ and $B_2$, together with the exceptional gradient lines $\gamma_1$ and $\gamma_2$ which they represent, can give rise to an admissible $CB$-diagram, it is necessary to distinguish between two cases. In fact, some attention must be paid when $B_1$ and $B_2$ represent bifurcations with saddle-to-saddle separatrix $\gamma_{s_1,s_2}$ and $\gamma_{s_2,s_3}$ respectively.

**Lemma 6.1.** Let $B_1$ and $B_2$ be bifurcation lines representing bifurcations corresponding to saddle-to-saddle separatrices $\gamma_{s_1,s_2}$ and $\gamma_{s_3,s_4}$, with either
s_2 \neq s_3 \text{ or } s_4 \neq s_1\); if the phase portrait corresponding to the intersection point \(B_1 \cap B_2\) is the phase portrait of a gradient vector field, and if to each subset determined by \(B_1\) and \(B_2\) it is associated a phase portrait obtained, in the way described above, by breaking the exceptional gradient lines, then the resulting \(CB\)-diagram is admissible.

**Proof.** According to definition 2.8 it is enough to prove that there is a family of diffeomorphisms, providing orbit isotopies for \(t < 0\) and \(t > 0\), and such that at \(t = 0\), that is along \(B_1\) or \(B_2\), two separatrixes of the saddle \(s_1\) and \(s_2\), respectively of \(s_3\) and \(s_4\), form \(\gamma_{s_1,s_2}\) and \(\gamma_{s_3,s_4}\), in the way explained in section 5 and shown in figure 5.19. If the saddles \(s_i\) are all distinct, for \(i = 1, \ldots, 4\), there are disjoint neighbourhood \(U_{12}\) and \(U_{34}\) containing respectively \(\gamma_{s_1,s_2}\) and \(\gamma_{s_3,s_4}\) and such that \(U_{12} \cap (W^u(s_k) \cup W^s(s_k)) = \emptyset\) for \(k = 3, 4\) and \(U_{34} \cap (W^u(s_l) \cup W^s(s_l)) = \emptyset\) for \(l = 1, 2\). Because of the way the phase portraits in each subset determined by \(B_1\) and \(B_2\) are constructed, and since \(U_{12}\) and \(U_{34}\) are disjoint, it follows that it is possible to find a family of diffeomorphisms as above which is the identity on the complement of \(U_{34}\) or \(U_{12}\), and thus providing the required bifurcations along \(B_1\) and \(B_2\).

If \(s_1 = s_3\) (a similar argument if \(s_2 = s_4\)) then \(\gamma_{s_1,s_2} \cup \gamma_{s_1,s_4} = W^u(s_1)\), and so the two separatrixes of \(s_1\) forming \(W^s(s_1)\) lie on different sides with respect to \(\gamma_{s_1,s_2} \cup s_3 \cup \gamma_{s_3,s_4}\); this ensures that there are neighbourhood \(U_2\) and \(U_4\) of respectively \(\gamma_{s_1,s_2}\) and \(\gamma_{s_1,s_4}\), and containing the stable and unstable manifolds of respectively \(s_2\) and \(s_4\), intersecting each one at most in \(s_1 = s_3\), such that \(U_2 \cap (W^u(s_4) \cup W^s(s_4)) = \emptyset\) and \(U_4 \cap (W^u(s_2) \cup W^s(s_2)) = \emptyset\). Now the proof goes on as in the first part for distinct saddles.

The case of two intersecting bifurcation lines \(B_1\) and \(B_2\), corresponding respectively to saddle-to-saddle separatrixes \(\gamma_{s_1,s_2}\) and \(\gamma_{s_2,s_3}\), performs a different behaviour: the reason is that there are three different ways of breaking the two non-generic gradient line appearing at \(z\), giving rise to a codimension 1 bifurcation: breaking \(\gamma_{s_2,s_3}\) and leaving only \(\gamma_{s_1,s_2}\), as occurs along \(B_1\); or breaking \(\gamma_{s_1,s_2}\) and leaving \(\gamma_{s_2,s_3}\), as occurs along \(B_2\); or forming the saddle-to-saddle separatrix \(\gamma_{s_1,s_3}\). The bifurcation line \(B_3\) corresponding to \(\gamma_{s_1,s_3}\) appears in the \(CB\)-diagram as a half-line with origin in \(z\). A case of this kind is considered in [12] for the perturbed elliptic umbilic.

**Lemma 6.2.** If, in a \(CB\)-diagram, the bifurcation lines \(B_1\) and \(B_2\) corresponding to the saddle-to-saddle separatrixes \(\gamma_{s_1,s_2}\) and \(\gamma_{s_2,s_3}\) intersect each other in \(z\) and if the phase portrait at \(z\) is that of a gradient vector field,
then, to be admissible, the CB-diagram must contain also a bifurcation half-line $B_3$, corresponding to the saddle-to-saddle separatrix $\gamma_{s_1,s_3}$, and whose origin is $z$.

Proof. Consider the phase portrait over $z$, and observe that the two separatrices of $s_2$, not forming exceptional gradient lines, lie on the same side with respect to $\gamma_{s_1,s_2} \cup s_2 \cup \gamma_{s_2,s_3}$. There are two ways of breaking a saddle-to-saddle separatrix, yielding two phase portraits which are not orbit isotopic. So, while in the cases considered in lemma 6.1 the existence of disjoint neighbourhood, each one containing one of the saddle-to-saddle separatrices, provided four non-orbit equivalent phase portraits as a result of the breaking the two exceptional gradient lines, and corresponding to the four subsets determined by the intersection of $B_1$ and $B_2$, now, because of the relative position of $\gamma_{s_1,s_2}$ and $\gamma_{s_2,s_3}$, and of the remaining separatrices of $s_2$, there are five non-orbit isotopic phase portraits, shown in figure 6.23. Two of these differs by a bifurcation, associated to the saddle-to-saddle separatrix $\gamma_{s_1,s_3}$, and represented by a bifurcation line $B_3$ lying in one of the subsets, determined by $B_1$ and $B_2$, and dividing it into two disjoint subset. In other words, $B_3$ is an half-line with origin in $z$. In one of those two phase portraits $\gamma_{s_1,s_2}$ can occur but not $\gamma_{s_2,s_3}$, the opposite happens in the second. The bifurcation corresponding to $\gamma_{s_1,s_3}$ allows to switch from one to the other. This shows the admissibility of the CB-diagram containing $B_1$, $B_2$ and $B_3$. 

Figure 6.21 represents the two possibilities of intersection of bifurcation lines in an admissible CB-diagram.

![Diagram](image_url)

Fig. 6.21 : Intersections of bifurcation lines

The structure of the phase portraits in the subsets determined by intersection of bifurcation lines as in case (b) of figure 6.21 is as follows. The phase portrait at the intersection point contains the exceptional gradient lines $\gamma_{s_1,s_2}$
and $\gamma_{s_2, s_3}$; the remaining separatrices of $s_2$ lie on the same side with respect to $\gamma_{s_1, s_2} \cup s_2 \cup \gamma_{s_2, s_3}$ (see figure 6.22).

**Fig. 6.22:** the phase portrait over the intersection of $B_1$, $B_2$ and $B_3$

Breaking the two exceptional gradient lines provides five phase portrait, as explained in lemma 6.2 represented in figure 6.23.
Fig. 23: phase portraits in the subsets determined by $B_1$, $B_2$ and $B_3$

Note that the bifurcation line $B_3$, corresponding to the saddle-to-saddle separatrix $\gamma_{s_1,s_3}$, bounds $U_3$ and $U_4$. The bifurcation line $B_1$, corresponding to $\gamma_{s_1,s_2}$, separates $U_1$ from $U_5$ and $U_2$ from $U_3$, while $B_2$, corresponding to $\gamma_{s_2,s_3}$, separates $U_1$ from $U_2$ and $U_4$ from $U_5$.

7 Extension of quantum corrections

The quantum corrections in definitions 4.18 and 5.4 allow to glue the holomorphic objects, defined by means of Morse homology, on $U_1$ and $U_2$ along their common boundary, when this is either a subset of the caustic $C$, consisting of folds not limit points of the bifurcation locus $B$, or a subset of $B$ consisting of codimension 1 bifurcation points. This is a codimension 1 submanifold of $\mathbb{R}^2$. It remains to check that such holomorphic structure can be extended through the codimension 2 subset of $\mathbb{R}^2$ formed by folds which are limit points of $B$ (that is, the intersection points of $C$ and $B$), by codimension 2 bifurcation points (that is, the intersections of bifurcation lines) and cusps. To this purpose it will be computed the monodromy of the holomorphic structure given by quantum corrections and check that it is the identity. We are not going to analyze in this paper the cusps: this was considered, though only for the particular case of the perturbed elliptic umbilic, in [13].

Proposition 7.1. Suppose that the caustic $C$ and a bifurcation line $B$ intersect as shown in figure 7.24., then the holomorphic structure of the mirror object can be extended through the intersection point.
Proof. Suppose that \((n, s)\) is the birth-death pair associated to \(C\), appearing in \(U'_1\) and \(U'_2\), and \((s_1, s_2)\) the pair of bifurcating saddles associated to \(B\). Note that \(s \neq s_i\), for \(i = 1, 2\). From definition 5.4 it follows that: the quantum corrections, glueing the mirror object over \(U'_2\), \(U'_1\) along \(B\) and over \(U'_2\), \(U'_1\) along \(B\), and induced respectively by \(\tilde{M}^{U_2U_1}_B\), \(\tilde{M}^{U'_2U'_1}_B = (\tilde{M}^{U'_1U'_2}_B)^{-1}\), coincide on all saddles except \(s\), on which the former is not defined; \(\tilde{M}^{U'_2U'_1}_B\) is the identity on all saddles, except on \(s_1\): in fact, it acts as \(s_1 \rightarrow s_1 \pm s_2\), where the sign depends on orientation. On the other hand, by definition 4.18, the quantum corrections, glueing along \(C\) the mirror object over \(U'_1\), \(U'_2\), and denoted respectively by \(\tilde{M}^{U'_1U'_1}_C\) and \(\tilde{M}^{U'_2U'_2}_C\), are both given, if \(n\) is an unstable node, by the natural injection on saddles, if \(n\) is a stable node, by the natural injection on saddles not connected to \(n\), respectively, in \(U'_1\) and \(U'_2\), and by a shift by \(s\) on the remaining saddles. Hence, if \(n\) is unstable or if \(n\) is stable but \(s_1\) and \(s_2\) are not connected to \(n\) both in \(U'_1\) and \(U'_2\), then clearly

\[\tilde{M}^{U_2U_1}_B \tilde{M}^{U'_1U'_1}_C \tilde{M}^{U'_2U'_2}_B \tilde{M}^{U'_1U'_1}_C = Id\]

and so the holomorphic structure can be extended through the intersection point.

This equality, and the same conclusion, holds also in the remaining cases, though more care must be paid: indeed, if \(n\) is stable node and only \(s_1\) is connected to \(n\) both in \(U'_1\) and \(U'_2\), then \(\tilde{M}^{U'_1U'_1}_C\) and \(\tilde{M}^{U'_2U'_2}_C\) are the identity on \(s_2\) and a shift by \(s\) on \(s_1\), that is, \(s_1 \rightarrow s_1 \pm s\); if\(n\) is stable node, \(s_1\) is connected to \(n\) only in \(U'_1\) and \(s_2\) is connected to \(n\) both in \(U'_1\) and \(U'_2\), then, making for simplicity a choice of signs, although the argument is independent from it, \(\tilde{M}^{U'_1U'_1}_C\) is a shift by \(s\) both on \(s_1\) and \(s_2\), while \(\tilde{M}^{U'_2U'_2}_C\) is the identity.

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on $s_1$ and a shift by $s$ on $s_2$, therefore the composition $\tilde{M}^{U'_2U_2} \tilde{M}^{U'_1U'_2} \tilde{M}^{U_1U'_1}$ acts as follows

\[
\begin{align*}
  s_1 & \to s_1 + s \to s_1 + s - s_2 \to s_1 - s_2 \\
  s_2 & \to s_2 \to s_2 \to s_2
\end{align*}
\]

Finally, if $n$ is stable, $s_1$ is connected to $n$ only in $U_2'$, and $s_2$ is connected to $n$ both in $U'_1$ and $U'_2$, then $\tilde{M}^{U_2U_2}$ is the identity on $s_1$ and a shift by $s$ on $s_2$, while $\tilde{M}^{U'_1U'_1}$ is a shift by $s$ on both $s_1$ and $s_2$, therefore the composition $\tilde{M}^{U'_2U_2} \tilde{M}^{U'_1U'_2} \tilde{M}^{U_1U'_1}$ acts as follows

\[
\begin{align*}
  s_1 & \to s_1 \to s_1 - s_2 - s \to s_1 - s_2 \\
  s_2 & \to s_2 + s \to s_2 + s \to s_2
\end{align*}
\]

That the monodromy is the identity on nodes is easily verified, since $\tilde{M}^{U'_2U'_1}$ and $\tilde{M}^{U_2U_1}$ act as the identity on nodes. \hfill \Box

**Proposition 7.2.** Suppose that the caustic $C$ and a bifurcation line intersect as shown in figure 7.25, then the holomorphic structure of the mirror object can be extended through the intersection point.

Fig. 7.25: intersection of $C$ and $B$

*Proof.* Suppose that $(s_1, s_2)$ is the pair of bifurcating saddles at $B$ and $(n, s_i)$ is the birth-death pair associated to $C$, for $i = 1$ or $i = 2$. Let $s_3, s_4, \ldots, s_m$ be the saddles exhibited, besides $s_1$ and $s_2$, by the phase portrait in $U'_1$ and $U'_2$. Denote by $M^{U_1U'_1}$ and $M^{U'_2U_1}$ the quantum corrections glueing along $C$ the mirror object respectively over $U_1$, $U'_1$, and over $U'_2$, $U_1$, and by $M^{U'_2U'_1}$ the quantum correction glueing along $B$ the mirror object in $U'_1$ and $U_2$. 

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Note that, for $j \geq 3$, $M_{B}^{U_{1}'U_{2}'}(s_{j}) = s_{j}$ and since $s_{j}$ is connected to $n$ in $U_{1}'$ if and only if $s_{j}$ is connected to $n$ in $U_{2}'$, it follows $M_{C}^{U_{1}'U_{1}'}(s_{j}) = M_{C}^{U_{1}'U_{2}'}(s_{j})$; this implies that

$$M_{C}^{U_{1}'U_{1}'}M_{B}^{U_{1}'U_{2}'}M_{C}^{U_{1}'U_{1}'}(s_{j}) = s_{j}$$

It remains to check the action of the above composition of quantum corrections on $s_{1}$ and $s_{2}$.

Suppose first $n$ is an unstable node and $s_{i} = s_{1}$; by definition 4.18, $M_{C}^{U_{1}'U_{1}'}$ and $M_{C}^{U_{1}'U_{2}'}$ are the natural injection, in particular, they map $s_{2} \rightarrow s_{2}$ (on $s_{1}$ they are not defined); by definition 5.4, $M_{B}^{U_{1}'U_{2}'}(s_{1}) = s_{1} - s_{2}$ and $M_{B}^{U_{2}'U_{2}'}(s_{2}) = s_{2}$; hence

$$M_{C}^{U_{1}'U_{1}'}M_{B}^{U_{1}'U_{2}'}M_{C}^{U_{1}'U_{1}'} = Id$$

thus the mirror object can be extended through the intersection point.

If $n$ is an unstable node but $s_{i} = s_{2}$, the same conclusion is achieved, the only difference being that $M_{C}^{U_{1}'U_{1}'}$ and $M_{C}^{U_{1}'U_{2}'}$ are the natural injection on $s_{1}$ (and not defined on $s_{2}$).

Suppose now $n$ is a stable node and $s_{i} = s_{2}$. If $s_{1}$ is connected to $n$ in $U_{1}'$ (but the same result is achieved also if $s_{1}$ is connected to $n$ in $U_{2}'$), and assuming $n(\gamma_{s_{2},n}) = -n(\gamma_{s_{2},n})$ in $U_{1}'$ (the argument works as well, up to signs, for the opposite choice), then $M_{C}^{U_{1}'U_{1}'}(s_{1}) = s_{1} + s_{2}$, $M_{B}^{U_{1}'U_{2}'}(s_{1}) = s_{1} - s_{2}$ and $M_{B}^{U_{2}'U_{2}'}(s_{2}) = s_{2}$, while $M_{C}^{U_{1}'U_{2}'}(s_{1}) = s_{1}$ since $s_{1}$ is not connected to $n$ in $U_{2}'$. Hence

$$M_{C}^{U_{1}'U_{1}'}M_{B}^{U_{1}'U_{2}'}M_{C}^{U_{1}'U_{1}'} = Id$$

and the mirror object can be extended through the intersection point.

Also when $n$ is a stable node but $s_{i} = s_{1}$ there is no monodromy given by quantum corrections and the mirror object can be extended through the intersection point: indeed, the maps $M_{C}^{U_{1}'U_{1}'}$, $M_{B}^{U_{1}'U_{2}'}$ and $M_{C}^{U_{1}'U_{1}'}$ are the identity on $s_{2}$.

That the monodromy is the identity on nodes is easily verified, since $M_{B}^{U_{2}'U_{2}'}$ and $M_{B}^{U_{2}'U_{1}'}$ act as the identity on nodes.

Consider now intersection points of bifurcation lines: as seen, there are two cases.
Proposition 7.3. Suppose that two bifurcation lines $B_1$ and $B_2$ intersect as shown in figure 7.26, then the holomorphic structure of the mirror object can be extended through the intersection point.

\[ \text{Fig. 7.26: Intersection of } B_1 \text{ and } B_2 \]

Proof. Let $(s_1, s'_1)$ and $(s_2, s'_2)$ be the pair of bifurcating saddles corresponding respectively to $B_1$ and $B_2$. By lemma 6.1 at most either $s_1 = s_2$ or $s'_1 = s'_2$. Let $M^{U_1U_2}, \ldots, M^{U_4U_1}$ be the quantum corrections glueing the mirror object over $U_1$ and $U_2$, ..., $U_4$ and $U_1$, along the common bifurcation line bounding these domains. Then, by definition 5.4, $M^{U_3U_4} = (M^{U_1U_2})^{-1}$ and $M^{U_4U_1} = (M^{U_2U_3})^{-1}$ and the quantum corrections along $B_1$ commute with those along $B_2$. Therefore

\[ M^{U_4U_1}M^{U_3U_4}M^{U_2U_3}M^{U_1U_2} = Id \]

and so the mirror object can be extended through the intersection point. \qed

Proposition 7.4. Suppose that two bifurcation lines $B_1$ and $B_2$ intersect as shown in figure 7.27, then the holomorphic structure of the mirror object can be extended through the intersection point.

\[ \text{Fig. 7.27: Intersection of } B_1, B_2 \text{ and } B_3 \]
Proof. By lemma 6.2, assume that the phase portrait of $U_i$ is as represented in figure 6.23. We write the quantum corrections for each bifurcation line, showing their action on generators, when non-trivial, and their associated matrix, and then compute their composition:

1. from $U_1$ to $U_2$ the quantum correction $\Psi_{12}$ is non-trivial on $s_2$

$$s_2 \rightarrow s_2 + \psi_{12}(s_3)$$

where $\psi_{12}(s_3) \in \{-1, 1\}$, and its matrix is

$$\Psi_{12} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & \psi_{12}(s_3) & 1
\end{bmatrix}$$

2. from $U_2$ to $U_3$ the quantum correction $\Psi_{23}$ is non-trivial on $s_1$

$$s_1 \rightarrow s_1 + \psi_{23}(s_2)$$

where $\psi_{23}(s_2) \in \{-1, 1\}$, and its matrix is

$$\Psi_{12} = \begin{bmatrix}
1 & 0 & 0 \\
\psi_{23}(s_2) & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}$$

3. from $U_3$ to $U_4$ the quantum correction $\Psi_{34}$ is non-trivial on $s_1$

$$s_1 \rightarrow s_1 + \psi_{34}(s_3)$$

where $\psi_{34}(s_3) \in \{-1, 1\}$, and its matrix is

$$\Psi_{12} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
\psi_{34}(s_3) & 0 & 1
\end{bmatrix}$$

4. from $U_4$ to $U_5$ the quantum correction $\Psi_{45}$ is non-trivial on $s_2$

$$s_2 \rightarrow s_2 + \psi_{45}(s_3)$$

where $\psi_{45}(s_3) \in \{-1, 1\}$, and its matrix is

$$\Psi_{12} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & \psi_{45}(s_3) & 1
\end{bmatrix}$$
5. from $U_5$ to $U_1$ the quantum correction $\Psi_{51}$ is non-trivial on $s_1$

$$s_1 \rightarrow s_1 + \psi_{51}(s_2)$$

where $\psi_{51}(s_2) \in \{-1, 1\}$, and its matrix is

$$\Psi_{51} = \begin{bmatrix} 1 & 0 & 0 \\ \psi_{54}(s_2) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The composition $\Psi = \Psi_{51}\Psi_{45}\Psi_{34}\Psi_{23}\Psi_{12}$ has the following action on generators:

\[
\begin{align*}
\Psi(s_1) &= s_1 + (\psi_{23}(s_2) + \psi_{54}(s_2)) + \psi_{34}(s_3) + \psi_{45}(s_3)\psi_{23}(s_2) \\
\Psi(s_2) &= s_2 + \psi_{12}(s_3) + \psi_{45}(s_3) \\
\Psi(s_3) &= s_3
\end{align*}
\]

Note now that $\psi_{23}(s_2) = -\psi_{54}(s_2)$ and $\psi_{12}(s_3) = -\psi_{45}(s_3)$: in fact, observing the phase portraits in figure 6.23, once an orientation for the separatrixes of each saddle is chosen, the signs convention 4.8 is satisfied by $\psi_{23}(s_2)$ if and only if it is not satisfied by $\psi_{54}(s_2)$, and the same can be stated for $\psi_{12}(s_3)$ and $\psi_{45}(s_3)$. In this way $\Psi$ can be simplified as:

\[
\begin{align*}
\Psi(s_1) &= s_1 + \psi_{34}(s_3) + \psi_{45}(s_3)\psi_{23}(s_2) \\
\Psi(s_2) &= s_2 \\
\Psi(s_3) &= s_3
\end{align*}
\]

It remains to prove that $\psi_{34}(s_3) + \psi_{45}(s_3)\psi_{23}(s_2) = 0$. Consider the phase portrait over $U_1$, shown in figure 7.28: choose an orientation and compute the terms in the above equation.
Observe that $\psi_{45}(s_3)$ is determined by the signs of $W^s_1(s_2)$ and $W^s_1(s_3)$, $\psi_{23}(s_2)$ by the signs of $W^s_2(s_2)$ and $W^s_1(s_1)$, and $\psi_{34}(s_3)$ by $W^s_1(s_1)$ and $W^s_1(s_3)$. Suppose $n(W^s_1(s_2)) = n(W^s_1(s_3))$: since $\psi_{45}(s_3) = -\psi_{12}(s_3)$, then $\psi_{45}(s_3) = 1$, moreover $n(W^s_1(s_2)) = -n(W^s_1(s_2))$. Choose now $n(W^s_1(s_1))$: if $n(W^s_1(s_1)) = n(W^s_2(s_2))$ then $\psi_{23}(s_3) = -1$, on the other hand, being $n(W^s_1(s_1)) = -n(W^s_1(s_3))$, it also follows $\psi_{34}(s_3) = 1$; if $n(W^s_1(s_1)) = -n(W^s_2(s_2))$ then $\psi_{23}(s_3) = 1$ and $\psi_{34}(s_3) = -1$. In both cases $\psi_{34}(s_3) + \psi_{45}(s_3)\psi_{23}(s_2) = 0$. The same conclusion is achieved supposing $n(W^s_1(s_2)) = -n(W^s_1(s_3))$.

We can sum up all the results in the following theorem:

**Theorem 7.5.** For an admissible CB-diagram, such that for each $x \notin C \cup B$ Morse homology is defined, that is, both the hypothesis of theorem 4.7 and assumption 4.8 are fulfilled, and where the orientation of gradient lines satisfies the signs convention 5.7, the quantum corrections introduced in definitions 4.18 and 5.4 allow to extend the holomorphic structure of the mirror object to all folds of the caustic and to all bifurcation points.

Another step is necessary to provide a globally defined holomorphic object on the mirror fibration: to study its extensibility to cusps. This problem was analyzed in [13] (but see also [5]) for the perturbed elliptic umbilic: in that case, a quantum correction was defined, however, since it was related to the existence of a spin structure on the Lagrangian submanifold $L$ defined by the generating function $f$ (and to the orientation problem of a family in Morse
theory and Floer theory), rather than to the bifurcations of the family $f_x$, we prefer to postpone the analysis of cusps to another time.

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