Spectral singularities for Non-Hermitian one-dimensional Hamiltonians: puzzles with resolution of identity

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Abstract
We examine the completeness of biorthogonal sets of eigenfunctions for non-Hermitian Hamiltonians possessing a spectral singularity. The correct resolutions of identity are constructed for delta like and smooth potentials. Their form and the contribution of a spectral singularity depend on the class of functions employed for physical states. With this specification there is no obstruction to completeness originating from a spectral singularity.

Preprint ICCUB-10-013

1. Introduction

Lately complex Hamiltonians with real spectrum \cite{1,2} attract more and more attention to describe the phenomena in complex crystals \cite{3}, in certain optical wave guides \cite{4} and in cosmology of dark energy \cite{5}. In the case of discrete spectrum the peculiarities of non-Hermitian Hamiltonians are related to the appearance of exceptional points due to coalescence of some energy levels \cite{6}. For such systems the complete biorthogonal set of eigen- and associated functions normally exist and is sufficient to characterize their physics. On the other hand, if a complex potential has bounded spatial asymptotics, in the spectrum one can find not only continuum eigenvalues related to scattering but also so called spectral singularities. The latter spectral points lead to poles in the resolvent of the Hamiltonian in the continuous part of the spectrum. This kind of spectral points are known for a long time for radial problem of three dimensional Schrödinger equation \cite{7}-\cite{12}. Recently the states corresponding to spectral singularities were discussed in one-dimensional Quantum Mechanics on the entire real axis \cite{13,14} and for periodic complex potentials \cite{15,16} as producing specific physical phenomena. However the observational relevance of such states strongly depend on whether they appear as independent building blocks in the complete set of biorthogonal eigenstates.

In \cite{13,16} the serious doubts were raised concerning the very existence of a complete resolution of identity in the case when spectral singularities arise in the energy spectrum.
In our work we thoroughly examine this issue and arrive rather at the opposite conclusion, namely, we build manifestly resolutions of identity for typical complex potentials and point out how wave functions related to spectral singularities are incorporated in them. Meanwhile we have found that the full contribution of eigenvectors of spectral singularities is provided by different mechanisms and depends on a class of test functions. In particular, for a narrower class of test functions one can reduce the contribution of a spectral singularity but at the expense of deletion of certain terms which are responsible for reproducing of some test functions from a wider class. The major part of this work including the Appendices is devoted to the rigorous justification of the completeness and the structure of resolutions of identities for different spaces of test functions. We exemplify this reduction with an instructive example to elucidate how different terms corresponding to the continuum and singular parts of the spectrum provide the identity.

The correct resolutions of identity are constructed not only for the delta-like but also for some smooth potentials forming the delta-like sequence.

2. Resolution of identity for imaginary delta-like potential

For continuous spectrum of the Hamiltonian

\[ h = -\partial^2 + z\delta(x), \quad \partial \equiv \frac{d}{dx}, \quad iz \in \mathbb{R} \]  

there are eigenfunctions

\[ \psi_+(x; k) = \frac{1}{\sqrt{2\pi}} \left( \frac{2k}{2k + iz} e^{ikx} - \frac{iz}{2k + iz} e^{-ikx} \right), \quad x \geq 0 \]
\[ \equiv \frac{1}{\sqrt{2\pi}} \left( e^{ikx} - \frac{iz}{2k + iz} e^{-ikx} + \frac{2z}{2k + iz} \theta(x) \sin kx \right) \]  

\[ \psi_-(x; k) = \frac{1}{\sqrt{2\pi}} \left( (1 + iz/2k) e^{-ikx} - \frac{iz}{2k} e^{ikx} \right), \quad x \geq 0 \]
\[ \equiv \frac{1}{\sqrt{2\pi}} \left( (1 + iz/2k) e^{-ikx} - \frac{iz}{2k} e^{ikx} \right) \]  

\[ h\psi_\pm = k^2\psi_\pm, \quad \theta(x) = \begin{cases} 1, & x \geq 0, \\ 0, & x < 0. \end{cases} \]

These functions satisfy to relations

\[ \psi_-(x; k) \equiv (1 + iz/2k)\psi_+(x; k) \]  

and

\[ W[\psi_+(x; k), \psi_-(x; k)] \equiv \psi'_+(x; k)\psi_-(x; k) - \psi_+(x; k)\psi'_-(x; k) = \frac{ik}{\pi}. \]

Let’s notice that the standard "normalization" for scattering is respected by \( \psi_+ \) where one can read off \( T \) and \( R \) (transmission and reflection coefficients) but not by \( \psi_- \) which in order to read off \( T \) and \( R \) would require to be divided by \( (1 + iz/2k) \).
Green function for $h$ takes the form,

$$G(x, x'; \lambda) = \frac{\pi i}{\sqrt{\lambda}} \psi_+(x_>; \sqrt{\lambda})\psi_-(x_<; \sqrt{\lambda}), \quad \text{Im} \sqrt{\lambda} \geq 0,$$

$$(h - \lambda)G = \delta(x - x'), \quad x_> = \max\{x, x'\}, \quad x_< = \min\{x, x'\}. \quad (7)$$

There is spectral singularity in the spectrum of $h$ for $\lambda = -z^2/4$, which is the only pole of Green function (7). The corresponding eigenfunctions of $h$ take the form,

$$\psi_0(x) \equiv e^{z|x|/2} = \sqrt{2\pi} \psi_-(x; -iz/2) = -\sqrt{2\pi} \lim_{k \to -iz/2} [(1 + \frac{2k}{iz})\psi_+(x; k)], \quad (8)$$

$$\psi_+(x; iz/2) \equiv \frac{1}{\sqrt{2\pi}} \left\{ \begin{array}{ll} \frac{1}{2} e^{-zx/2}, & x \geq 0 \\ \frac{1}{2} e^{zx/2} - \frac{1}{2} e^{zx/2}, & x < 0 \end{array} \right\} = \frac{1}{\sqrt{2\pi}} \psi_0(x) - \frac{1}{2} \psi_-(x; iz/2). \quad (9)$$

and

$$\psi_-(x; iz/2) \equiv \frac{1}{\sqrt{2\pi}} \left\{ \begin{array}{ll} \frac{2e^{zx/2} - e^{-zx/2}}{iz}, & x \geq 0 \\ e^{zx/2}, & x < 0 \end{array} \right\} \sqrt{2\pi} \psi_0(x) - 2\psi_+(x; iz/2). \quad (10)$$

The eigenfunctions $\psi_+(x; k)$ and $\psi_-(x; k)$ of $h$ satisfy (see Appendix 1) the biorthogonality relations,

$$\int_{-\infty}^{+\infty} [(1 + \frac{2k}{iz})\psi_+(x; k)]\psi_-(x; k') \, dx = (1 + \frac{2k}{iz}) \delta(k - k'), \quad (11)$$

where the eigenfunction $\psi_0(x)$ is included due to (8).

The resolution of identity constructed from $\psi_+(x; k)$ and $\psi_-(x; k)$ holds (see Appendix 2),

$$\delta(x - x') = \int_{\mathcal{L}} \psi_+(x; k)\psi_-(x'; k) \, dk, \quad (12)$$

where $\mathcal{L}$ is an integration path in complex $k$ plane, obtained from the real axis by its deformation near the point $k = -iz/2$ upwards and the direction of $\mathcal{L}$ is specified from $-\infty$ to $+\infty$. This resolution of identity is valid for test functions belonging to $C_\mathbb{R} \cap C_{(-\infty, 0]} \cap C_{[0, +\infty)} \cap L_2(\mathbb{R}; (1 + |x|)^\gamma), \gamma > -1$ as well as for some bounded and even slowly increasing test functions (more details are presented in Appendix 2) and, in particular, for eigenfunctions (2) and (3) of the Hamiltonian $h$.

One can rearrange the resolution of identity (12) for any $\varepsilon > 0$ (see Appendix 2) to the form

$$\delta(x - x') = \left( \int_{-\infty}^{-iz/2 - \varepsilon} + \int_{+\infty}^{-iz/2 + \varepsilon} \right) \psi_+(x; k)\psi_-(x'; k) \, dk +$$

$$+ \frac{1}{\pi} e^{z(x-x')/2} \frac{\sin \varepsilon(x - x')}{x - x'} + \frac{iz}{\pi} \theta(-x)\theta(x') \int_{-iz/2 - \varepsilon}^{-iz/2 + \varepsilon} \frac{1}{k} \sin kx \sin kx' \, dk.$$ 

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1 Alternatively one could shift the denominators in (2) to $2k + iz + i0$. Then in (12) one can keep integration along real axis.
- \frac{z}{4} \psi_0(x) \psi_0(x') \left[ 1 - \frac{2}{\pi} \int_0^{\epsilon(|x|+|x'|)} \sin \frac{t}{t} dt \right], \quad \varepsilon > 0 \tag{13}

and, consequently, to the form

\delta(x - x') = \lim_{\varepsilon \downarrow 0} \left\{ \left( \int_{-\infty}^{-i\varepsilon/2-\varepsilon} + \int_{-i\varepsilon/2+\varepsilon}^{+\infty} \right) \psi_+(x; k) \psi_-(x'; k) dk + \right.

+ \frac{1}{\pi} e^{\pm(x-x')/2} \frac{\sin \epsilon(x-x')}{x-x'} + \frac{i z}{\pi} \theta(-x) \theta(x') \int_{-i\varepsilon/2-\varepsilon}^{+\infty} \frac{1}{k} \sin kx \sin kx' \, dk -

\left. - \frac{z}{4} \psi_0(x) \psi_0(x') \left[ 1 - \frac{2}{\pi} \int_0^{\epsilon(|x|+|x'|)} \sin \frac{t}{t} dt \right] \right\}, \tag{14}

where the prime ' at the limit symbol emphasizes that this limit is regarded as a limit in the space of distributions.

We can reduce the resolution (14) (see Appendix 2) for test functions from \( C_\mathbb{R} \cap C_{(-\infty, 0]} \cap C_{[0, +\infty)} \cap L_2(\mathbb{R}; (1 + |x|)^\gamma), \gamma > -1 \) to the form

\delta(x - x') = \lim_{\varepsilon \downarrow 0} \left\{ \left( \int_{-\infty}^{-i\varepsilon/2-\varepsilon} + \int_{-i\varepsilon/2+\varepsilon}^{+\infty} \right) \psi_+(x; k) \psi_-(x'; k) dk - \right.

- \frac{z}{4} \psi_0(x) \psi_0(x') \left[ 1 - \frac{2}{\pi} \int_0^{\epsilon(|x|+|x'|)} \sin \frac{t}{t} dt \right] \right\} \equiv \tag{15}

\lim_{\varepsilon \downarrow 0} \left\{ \left( \int_{-\infty}^{-i\varepsilon/2-\varepsilon} + \int_{-i\varepsilon/2+\varepsilon}^{+\infty} \right) \psi_+(x; k) \psi_-(x'; k) dk - \frac{z}{2\pi} \psi_0(x) \psi_0(x') \int_{\epsilon(|x|+|x'|)}^{+\infty} \sin \frac{t}{t} dt \right\}

and for test functions from \( C_\mathbb{R} \cap C_{(-\infty, 0]} \cap C_{[0, +\infty)} \cap L_2(\mathbb{R}; (1 + |x|)^\gamma), \gamma > 1 \) to the form

\delta(x - x') = \text{p.v.'} \int_{-\infty}^{+\infty} \psi_+(x; k) \psi_-(x'; k) \, dk - \frac{z}{4} \psi_0(x) \psi_0(x') \equiv \tag{16}

\lim_{\varepsilon \downarrow 0} \left\{ \left( \int_{-\infty}^{-i\varepsilon/2-\varepsilon} + \int_{-i\varepsilon/2+\varepsilon}^{+\infty} \right) \psi_+(x; k) \psi_-(x'; k) dk - \frac{z}{4} \psi_0(x) \psi_0(x') \right\}.

The resolution of identity (16) seems to have a more natural form than (15) and especially (14), but formally, say, the right-hand part of the resolutions (15) and (16) reproduces a half only of the function \( \psi_0(x) \) in view of (11) and of the following,

\lim_{\varepsilon \downarrow 0} \left\{ - \frac{z}{4} \psi_0(x) \psi_0(x') \left[ 1 - \frac{2}{\pi} \int_0^{\epsilon(|x|+|x'|)} \sin \frac{t}{t} dt \right] \right\} \psi_0(x) \, dx = \tag{17}
\[
\lim_{\varepsilon \downarrow 0} \left\{ -\frac{z}{\pi} \psi_0(x') \int_0^{+\infty} dx \, e^{zx} \int_0^{+\infty} dt \, \frac{\sin t}{t} \right\} = \frac{1}{\pi} \psi_0(x') \lim_{\varepsilon \downarrow 0} \left\{ -\int_{\varepsilon [x']}^{+\infty} \frac{\sin t}{t} \, dt + \frac{\sin \varepsilon (x + |x'|)}{x + |x'|} e^{zx} \, dx \right\} = \psi_0(x') \left\{ \frac{1}{2} - \frac{1}{\pi} e^{-z|x'|} \lim_{\varepsilon \downarrow 0} \left[ \int_{\varepsilon [x']}^{+\infty} \frac{\sin \tau}{\tau} e^{z\tau/\varepsilon} \, d\tau - \int_0^{+\infty} \frac{\sin \tau}{\tau} e^{z\tau/\varepsilon} \, d\tau \right] \right\} = \psi_0(x') \left\{ \frac{1}{2} + \text{sign} (iz) \frac{i}{2\pi} e^{-z|x'|} \lim_{\varepsilon \downarrow 0} \ln \frac{|z| + \varepsilon}{|z| - \varepsilon} \right\} = \frac{1}{2} \psi_0(x'),
\]

where the formulae 2.5.3.12, 2.5.13.20 and 2.5.13.21 from [17] are taken into account. The missing one-half of the function \( \psi_0(x) \) is provided by the second and third terms of the right-hand part of the resolution of identity [14] due to the chain of equalities,

\[
\lim_{\varepsilon \downarrow 0} \int_{-\infty}^{+\infty} \frac{1}{\pi} e^{z(x-x')/2} \frac{\sin \varepsilon (x-x')}{x-x'} \, dx = \frac{1}{2\pi} \lim_{A \to +\infty} \int_{-A}^{A} dx \psi_0(x) \int_{-iz/2-\varepsilon}^{-iz/2+\varepsilon} dk \, e^{ik(x-x') + iz|x'|/2} = \frac{1}{2\pi} \lim_{A \to +\infty} \int_{-iz/2-\varepsilon}^{-iz/2+\varepsilon} e^{ikx'} \left[ \frac{e^{i(k-iz/2)A} - 1}{i(k-iz/2)} + \frac{1 - e^{-i(k+iz/2)A}}{i(k+iz/2)} \right] \, dk = \frac{1}{2\pi} \lim_{A \to +\infty} \int_{-iz-\varepsilon}^{-iz+\varepsilon} \left[ e^{izx'/2} \int_{-iz-\varepsilon}^{-iz+\varepsilon} e^{-ixx'} \frac{1 - e^{-ixA}}{i\tau} \, d\tau + \int_{-\varepsilon}^{\varepsilon} e^{-ixx'} \left( \frac{1 - e^{-ixA}}{i\tau} \right) \, d\tau \right] = \frac{1}{2\pi} \lim_{A \to +\infty} \int_{-iz-\varepsilon}^{-iz+\varepsilon} \left[ e^{-ixx'/2} \int_{-iz-\varepsilon}^{-iz+\varepsilon} e^{ixx'} \frac{1 - e^{-ixA}}{i\tau} \, d\tau + \int_{-\varepsilon}^{\varepsilon} e^{ixx'} \left( \frac{1 - e^{-ixA}}{i\tau} \right) \, d\tau \right] = \frac{1}{2} e^{-zx^2/2},
\]

(18)
where Riemann theorem and the formula 2.5.3.12 from [17] are used, as well as, the following relation is employed
\[
\lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} \left[ \frac{i}{\pi} \theta(-x) \theta(x') \right] \sin kx \sin kx' \frac{dk}{k} \psi_0(x) \, dx = \frac{1}{2} \theta(x')(e^{zx'/2} - e^{-zx'/2}),
\] (19)
which can be derived in the same way as (18) and, finally, due to the identities
\[
\frac{1}{2} e^{-zx/2} + \frac{1}{2} \theta(x)(e^{zx/2} - e^{-zx/2}) \equiv \frac{1}{2} e^{z|x|/2} \equiv \frac{1}{2} \psi_0(x).
\] (20)
Thus, the resolution of identity (14) maps the function \(\psi_0(x)\) entirely\(^2\) and there is no any paradox of a "defectiveness" of reduced resolutions of identity because the function \(\psi_0(x)\) does not belong to the reduced spaces of test functions \(C_0(\mathbb{R}; (1 + |x|)^\gamma), \gamma > -1\) or even \(C_0(\mathbb{R}; (1 + |x|)^\gamma), \gamma > 1\).

**Example 1.** In order to elucidate how the reduced resolution (16) provides identity let’s apply it to the function smoothing \(\psi_0(x)\), namely,
\[
\psi_0(x; \alpha) = \exp\left(\frac{1}{2}(z - \alpha)|x|\right); \quad \alpha > 0.
\] (21)
In the pointwise limit \(\alpha \to 0\) this function tends to \(\psi_0(x)\) but this limit is incompatible with the selected reductions of test function spaces. The binorm of this function is well defined,
\[
\int_{-\infty}^{+\infty} \left( \psi_0(x; \alpha)\right)^2 = -\frac{2}{z - \alpha},
\] (22)
and it might be taken as a possible definition for the binorm of \(\psi_0(x)\) in the limit \(\alpha \to 0\). However as this limit is pointwise in \(x\) and does not preserve test function spaces the question of what is a best definition for the binorm of \(\psi_0(x)\) remains open.

Now let’s apply the two components of the resolution (16) to \(\psi_0(x; \alpha)\). With a chain of lengthy but straightforward calculations based on Eqs. (2), (3), (8) one can show that,
\[
\lim_{\varepsilon \to 0, A \to +\infty} \left( \int_{-\infty}^{-iz/2 - \varepsilon} + \int_{-i\varepsilon + 2z}^{A} \right) \frac{dk}{k} \psi_-(x'; k) \int_{-\infty}^{+\infty} dx \psi_+(x; k) \exp\left(\frac{1}{2}(z - \alpha)|x|\right) -
\]
\[
\frac{z}{4} \psi_0(x') \int_{-\infty}^{+\infty} \exp\left(\frac{1}{2}(z - \alpha)|x|\right) dx = \left( \psi_0(x'; \alpha) - \frac{\psi_0(x')}{2 - (\alpha/z)} \right) + \frac{\psi_0(x')}{2 - (\alpha/z)} = \psi_0(x'; \alpha).
\] (23)
For small \(\alpha/|z| \ll 1\) the spectral singularity contributes almost as much as the continuum part of the spectrum but this contribution \(\sim \psi_0(x)\) does not belong to the reduced space of test functions and its role is solely to compensate a similar piece from resolution of the continuum spectrum. We notice also that when thinking about the operation \(\varepsilon \downarrow 0, A \to +\infty\) the contributions of the second and third terms of the right-hand part of (14) in the resolution of identity are (see Remark 3 of Appendix 2) singular discontinuous functionals whose supports consist of the only element which is the infinity.

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\(^2\)It is interesting that contributions of the second and third terms of the right-hand part of (14) in the resolution of identity are (see Remark 3 of Appendix 2) singular discontinuous functionals whose supports consist of the only element which is the infinity.
+\infty and the limit \( \alpha \downarrow 0 \) one finds different results depending on their order as it follows from previous discussion. In particular, one reproduces a half only of the function \( \psi_0(x) \) in full accordance with (16) if firstly the limit \( \alpha \downarrow 0 \) is performed.

Let us now comment some technical subtleties in the above relations and note that the integral from the right-hand part of (12) is understood (see Appendix 2) on its order as follows:

\[
\int_{\mathcal{L}} \psi_+(x; k) \psi_-(x'; k) \, dk = \lim_{A \to +\infty} \int_{\mathcal{L}(A)} \psi_+(x; k) \psi_-(x'; k) \, dk,
\]

where \( \mathcal{L}(A) \) is a path in complex \( k \) plane, made of the segment \([-A, A]\) by its deformation near the point \( k = -iz/2 \) upwards and the direction of \( \mathcal{L}(A) \) is specified from \(-A\) to \( A \). Since the integral from the right-hand part of (24) is a standard integral (not a distribution), in view of (13) the following relations take place,

\[
\int_{\mathcal{L}} \psi_+(x; k) \psi_-(x'; k) \, dk = \lim_{A \to +\infty} \lim_{\varepsilon \downarrow 0} \left[ \int_{-A}^{-i z/2 - \varepsilon} + \int_{-i z/2 + \varepsilon}^{A} \psi_+(x; k) \psi_-(x'; k) \, dk - \frac{z}{4} \psi_0(x) \psi_0(x') \right] = \lim_{A \to +\infty} \text{p.v.} \int_{-A}^{+\infty} \psi_+(x; k) \psi_-(x'; k) \, dk - \frac{z}{4} \psi_0(x) \psi_0(x'),
\]

where the limit for \( \varepsilon \downarrow 0 \) (and consequently ”p.v.”) is regarded as pointwise one (not as a limit in a function space). The latter equality in (25) is considered as a definition. Thus, the resolution of identity,

\[
\delta(x - x') = \text{p.v.} \int_{-\infty}^{+\infty} \psi_+(x; k) \psi_-(x'; k) \, dk - \frac{z}{4} \psi_0(x) \psi_0(x'),
\]

holds (cf. with (16)) and moreover this resolution is equivalent to (12), i.e. it is valid for all test functions for which (12) is valid.

The resolution of identity (15) contains both the eigenfunctions \( \psi_+(x; k) \) and \( \psi_-(x'; k) \) for positive \( k \) which describe scattering, and the eigenfunctions \( \psi_+(x; k) \) and \( \psi_-(x'; k) \) for negative \( k \), which are linear combinations of scattering state vectors. It follows from the identities:

\[
\psi_+(x; k) = -\frac{iz}{2k + iz} \psi_+(x; -k) + \frac{4k^2}{4k^2 + z^2} \psi_-(x; -k), \quad x \in \mathbb{R}, \ k \in \mathbb{C},
\]

\[
\psi_-(x; k) = \psi_+(x; -k) - \frac{iz}{2k - iz} \psi_-(x; -k), \quad x \in \mathbb{R}, \ k \in \mathbb{C}.
\]

We remind that the standard ”normalization” for scattering is respected by \( \psi_+ \) where one can read off \( T \) and \( R \) (transmission and reflection coefficients) but not by \( \psi_- \) which in order to read off \( T \) and \( R \) would require to be divided by \((1 + iz/2k)\).
With the help of (27) one can rearrange the resolution of identity (15) for test functions from $C \cap C^\infty_{(0,0]} \cap L_2(\mathbb{R}; (1 + |z|)^\gamma)$, $\gamma > -1$ (see Appendix 2) to the form

$$
\delta(x-x') = \lim_{\epsilon \downarrow 0} \left\{ \left( \int_0^{\epsilon |z|/2-\epsilon} + \int_{\epsilon |z|/2+\epsilon}^{+\infty} \right) \left[ \psi_+(x;k)\psi_+(x';k) + \frac{4k^2}{4k^2+z^2} \psi_-(x;k)\psi_-(x';k) \right] dk - \frac{z}{4} \psi_0(x)\psi_0(x') \left[ 1 - \frac{2}{\pi} \int_0^{\epsilon(|z|+|z'|)} \sin t \frac{dt}{t} \right] \right\},
$$

(28)

where the eigenfunctions $\psi_+(x,k)$ and $\psi_-(x,k)$ correspond to scattering states, or to the symmetric form

$$
\delta(x-x') = \lim_{\epsilon \downarrow 0} \left\{ \left( \int_0^{\epsilon |z|/2-\epsilon} + \int_{\epsilon |z|/2+\epsilon}^{+\infty} \right) \left[ \frac{i}{2}\frac{z}{k-i\epsilon} \psi_+(x;k)\psi_+(x';k) + \frac{4k^2}{4k^2+z^2} \psi_+(x;k)\psi_-(x';k) + \frac{4ik^2z}{(4k^2+z^2)(2k+i\epsilon)} \psi_-(x;k)\psi_-(x';k) \right] dk - \frac{z}{4} \psi_0(x)\psi_0(x') \left[ 1 - \frac{2}{\pi} \int_0^{\epsilon(|z|+|z'|)} \sin t \frac{dt}{t} \right] \right\},
$$

(29)

where all eigenfunctions in the integral over $k$ describe scattering.

Finally let us remark that resolutions of identity equivalent to (12) – (14) can be obtained from (28) – (29) by supplementing them with the two following terms:

$$
\frac{2}{\pi} \cos \frac{iz(x-x')}{2} \sin \frac{\epsilon(x-x')}{x-x'} - \frac{i}{4\pi} \int_{iz/2-\epsilon}^{iz/2+\epsilon} \frac{e^{ik(|z|+|z'|)}}{k+i\epsilon} dk.
$$

(30)

This fact can be easily checked with the help of the relation (70) from Appendix 2. As well resolutions of identity of the type (26) can be produced from (28) – (29) by the replacement

$$
\lim_{\epsilon \downarrow 0} \left\{ \left( \int_0^{\epsilon |z|/2-\epsilon} + \int_{\epsilon |z|/2+\epsilon}^{+\infty} \right) \rightarrow \text{p.v.} \int_0^{+\infty} \right\}
$$

and by neglecting the integral $\int_0^{\epsilon(|z|+|z'|)} \sin t \frac{dt}{t}$.

### 3. Smooth potentials with spectral singularity

In order to use the technique of Supersymmetric Quantum Mechanics let us consider the shifted Hamiltonian $h^+ = h + z^2/4$. With the help of the standard construction of linear SUSY one can transform $h^+$, using the function $\psi_0(x)$ as a transformation function, into the Hamiltonian

$$
h^- = q^- q^+ = -\partial^2 + \frac{z^2}{4} - z\delta(x) = (h^+)^t, \quad q^+ = \mp \partial - \chi(x), \quad \chi(x) = \frac{\psi_0^*(x)}{\psi_0(x)} = \frac{z}{2} \text{sign } x.
$$

(31)
It is easy to see that this SUSY construction is a limiting case of the linear SUSY construction with the smooth superpotential \( \chi_\alpha(x) = (z/2) \tanh \alpha x \) for \( \text{Re} \alpha \to +\infty \). The main elements of these constructions are presented in the following table.

| \( \alpha \in \mathbb{C} \), \( \text{Re} \alpha \neq 0 \) | \( \alpha = +\infty \) |
|---|---|
| \( q^\pm_\alpha = \mp \partial - \chi_\alpha(x) \) | \( \chi_\alpha(x) = \frac{z}{2} \tanh \alpha x \) | \( \chi_\infty(x) \equiv \chi(x) = \frac{z}{2} \text{sign } x \) |
| \( \varphi_\alpha^\pm = 0, \varphi_\alpha^\pm(x) = e^{\mp \int \chi_\alpha(x) \, dx} \) | \( \varphi_\infty^-(x) \equiv \psi_0(x) = e^{z|x|/2} \), \( \varphi_\infty^+(x) = e^{-z|x|/2} \) |
| \( h^\pm_\alpha = q^\pm_\alpha q^\pm_\alpha = -\partial^2 + V^\pm_\alpha(x), V^\pm_\alpha(x) = \frac{z^2}{4} - \frac{z/2(z + 2\alpha)}{\cosh^2 \alpha x} \) | \( V^\pm_\alpha(x) = \frac{z^2}{4} \pm z \delta(x) \) |

In both cases \( \alpha = +\infty \) and \( \alpha \in \mathbb{C}, \text{Re} \alpha \neq 0 \) the function \( \varphi^\pm_\alpha(x) \) is an eigenfunction of the Hamiltonian \( h^\pm_\alpha = q^\pm_\alpha q^\pm_\alpha \) for the eigenvalue \( E = 0 \) corresponding to the spectral singularity (see the table) in the spectrum of \( h^\pm_\alpha \).

An eigenfunction of the Hamiltonian

\[
h_\alpha \equiv -\partial^2 - \frac{z/2(z + 2 - \alpha)}{\cosh^2 \alpha x} = h^+_\alpha - \frac{z^2}{4}
\]

for an eigenvalue \( k^2 \) satisfies the differential equation

\[
- \psi'' - \frac{z/2(z + 2 - \alpha)}{\cosh^2 \alpha x} \psi = k^2 \psi.
\]

With the help of the change of variables

\[
\psi(x) = e^{ikx} \phi(\xi), \quad \xi = \frac{1}{e^{2\alpha x} + 1}
\]

one can reduce Eq. \( \text{[33]} \) to the Gauss hypergeometric equation

\[
\xi(\xi - 1) \phi'' + [(a + b + 1) \xi - c] \phi' + ab \phi = 0,
\]

\[
a = 1 - \frac{z}{2\alpha}, \quad b = \frac{z}{2\alpha}, \quad c = 1 - \frac{ik}{\alpha}.
\]

Using the properties of gamma-function and hypergeometric function of the first kind (Gauss series) \( F(a, b, c; \xi) \) \( \text{(see [18])} \), one can show that for the eigenvalue \( k^2 \) of the Hamiltonian \( h_\alpha \) there are two eigenfunctions:

\[
\psi_+(x; k, \alpha) = \frac{1}{\sqrt{2\pi}} \frac{\Gamma(1 + \frac{z}{2\alpha} - \frac{ik}{\alpha})}{\Gamma^2(1 - \frac{ik}{\alpha})} \frac{2k}{2k + iz} e^{ikx} F \left( 1 - \frac{z}{2\alpha}, \frac{z}{2\alpha}, 1 - \frac{ik}{\alpha}; \frac{1}{e^{2\alpha x} + 1} \right)
\]

The asymptotics in the second column of the table are valid in the case \( \text{Re} \alpha > 0 \) only.

The asymptotics in \( \text{[36]} \) and \( \text{[37]} \) are valid in the case \( \text{Re} \alpha > 0 \) only.
\[
\equiv \frac{1}{\sqrt{2\pi}} \left\{ e^{ikx} F\left(1 - \frac{z}{2\alpha}, \frac{z}{2\alpha}; 1 + \frac{ik}{\alpha}; e^{-2\alpha x} + 1\right) - \frac{\Gamma(1+\frac{ik}{\alpha})\Gamma(1+\frac{iz}{2\alpha})\Gamma(1-\frac{z}{2\alpha} - \frac{ik}{\alpha})}{\Gamma(1-\frac{ik}{\alpha})} \frac{2\pi iz}{2k} e^{-ikx} F\left(1 - \frac{z}{2\alpha}, \frac{z}{2\alpha}; 1 - \frac{ik}{\alpha}; e^{-2\alpha x} + 1\right) \right\} =
\]
\[
= \frac{1}{\sqrt{2\pi}} \left\{ \frac{\Gamma(1+\frac{ik}{\alpha})\Gamma(1+\frac{iz}{2\alpha})\Gamma(1-\frac{z}{2\alpha} - \frac{ik}{\alpha})}{\Gamma(1-\frac{ik}{\alpha})} \frac{2k}{2k+iz} e^{ikx}[1 + o(1)], \quad x \to +\infty, \right. 
\]
\[
\left. - \frac{\Gamma(1+\frac{ik}{\alpha})\Gamma(1+\frac{iz}{2\alpha})\Gamma(1-\frac{z}{2\alpha} - \frac{ik}{\alpha})}{\Gamma(1-\frac{ik}{\alpha})} \frac{2k}{2k+iz} e^{-ikx}[1 + o(1)], \quad x \to -\infty, \right.
\]

\[
\text{describing scattering for } k > 0, \text{ and}
\]
\[
\psi_-(x; k, \alpha) = \frac{1}{\sqrt{2\pi}} \left\{ - \frac{\sin \frac{\pi iz}{2\alpha}}{\sin \frac{\pi k}{\alpha}} e^{ikx} F\left(1 - \frac{z}{2\alpha}, \frac{z}{2\alpha}; 1 - \frac{ik}{\alpha}; e^{2\alpha x} + 1\right) + \right.
\]
\[
\left. + \frac{\Gamma^2(1 - \frac{ik}{\alpha})}{\Gamma(1 + \frac{z}{2\alpha} - \frac{ik}{\alpha})\Gamma(1 - \frac{z}{2\alpha} - \frac{ik}{\alpha})} \left(1 + \frac{iz}{2k}\right) e^{-ikx} F\left(1 - \frac{z}{2\alpha}, \frac{z}{2\alpha}; 1 + \frac{ik}{\alpha}; e^{2\alpha x} + 1\right) \right\} =
\]
\[
\equiv \frac{1}{\sqrt{2\pi}} e^{-ikx} F\left(1 - \frac{z}{2\alpha}, \frac{z}{2\alpha}; 1 - \frac{ik}{\alpha}; e^{-2\alpha x} + 1\right) =
\]
\[
= \frac{1}{\sqrt{2\pi}} \left\{ \frac{\Gamma^2(1 - \frac{ik}{\alpha})}{\Gamma(1 + \frac{z}{2\alpha} - \frac{ik}{\alpha})\Gamma(1 - \frac{z}{2\alpha} - \frac{ik}{\alpha})} \left(1 + \frac{iz}{2k}\right) e^{-ikx}[1 + o(1)] - \frac{\sin \frac{\pi iz}{2\alpha}}{\sin \frac{\pi k}{\alpha}} e^{ikx}[1 + o(1)], \quad x \to +\infty, \right.
\]
\[
\left. e^{-ikx}[1 + o(1)], \quad x \to -\infty, \right. 
\]

\[
\text{describing scattering in the opposite direction for } k > 0. \text{ These eigenfunctions are inter-}
\]
\[
\text{connected by the relation}\]
\[
\psi_-(x; k, \alpha) \equiv \frac{\Gamma^2(1 - \frac{ik}{\alpha})}{\Gamma(1 + \frac{z}{2\alpha} - \frac{ik}{\alpha})\Gamma(1 - \frac{z}{2\alpha} - \frac{ik}{\alpha})} \left(1 + \frac{iz}{2k}\right) \psi_+(x; k, \alpha) \quad (38)
\]

\[
\text{(cf. with (39)).}
\]

Taking into account properties of gamma-function and hypergeometric function, it is not hard to check that:

(1) the following expressions are valid for the Wronskian of the functions \(\psi_+(x; k, \alpha)\) and \(\psi_-(x; k, \alpha)\) and for Green function:

\[
W[\psi_+(x; k, \alpha), \psi_-(x; k, \alpha)] \equiv
\]
\[
\psi_+(x; k, \alpha)\psi_-(x; k, \alpha) - \psi_+(x; k, \alpha)\psi'_-(x; k, \alpha) = \frac{ik}{\pi}, \quad (39)
\]
\[
G(x, x'; \lambda, \alpha) = \frac{\pi i}{\sqrt{\lambda}} \psi_+(x >; \sqrt{\lambda}, \alpha)\psi_-(x <; \sqrt{\lambda}, \alpha), \quad \Im \sqrt{\lambda} \geq 0,
\]
\[
(h - \lambda)G = \delta(x - x'), \quad x > = \max\{x, x'\}, \quad x < = \min\{x, x'\}; \quad (40)
\]

\[
\text{We remind again that the standard "normalization" for scattering is respected by } \psi_+ \text{ where one can read off } T \text{ and } R (\text{transmission and reflection coefficients}) \text{ but not by } \psi_- \text{ which in order to read off } T \text{ and } R \text{ would require to be divided by } (1 + iz/2k).
\]
(2) in the limit for $\text{Re} \alpha \to +\infty$ the eigenfunctions $\psi_+(x; k, \alpha)$ and $\psi_-(x; k, \alpha)$ of the Hamiltonian $h_\alpha$ turn into the eigenfunctions $\psi_+(x; k)$ and $\psi_-(x; k)$ respectively (see (2) and (3)) of the Hamiltonian $h$:

$$\lim_{\text{Re} \alpha \to +\infty} \psi_+(x; k, \alpha) = \psi_+(x; k), \quad \lim_{\text{Re} \alpha \to +\infty} \psi_-(x; k, \alpha) = \psi_-(x; k);$$  

hence, the Green function (40) in the limit for $\text{Re} \alpha \to +\infty$ becomes the Green function (7):

(3) in the case $\text{Re} \alpha \neq 0$ there is a spectral singularity in the spectrum of $h_\alpha$ for the eigenvalue $E = -z^2/4$ which is the only pole of the Green function (40) inside of the continuous spectrum of $h_\alpha$ and the corresponding eigenfunction takes the form

$$\psi_0(x; \alpha) \equiv [2 \cosh \alpha x]^{z/(2\alpha)} = \varphi^-_\alpha(x) =$$

$$-\sqrt{2\pi} \Gamma^2(1 - \frac{z}{2\alpha}) \lim_{k \to -iz/2} \left[(1 + \frac{2k}{iz}) \psi_+(x; k, \alpha)\right] = \sqrt{2\pi} \psi_-(x; -iz/2, \alpha).$$  

With the help of the Green function method one can construct the resolution of identity from the eigenfunctions (36) and (37),

$$\delta(x - x') = \int_{\mathcal{L}} \psi_+(x; k, \alpha)\psi_-(x'; k, \alpha) dk,$$  

where $\mathcal{L}$ is the same path as in (12). Taking into account (12) it is not hard to rearrange (13) to the form

$$\delta(x - x') = \text{p.v.} \int_{-\infty}^{+\infty} \psi_+(x; k, \alpha)\psi_-(x'; k, \alpha) dk - \frac{z}{4 \Gamma^2(1 - \frac{2\alpha}{z})} \psi_0(x; \alpha)\psi_0(x'; \alpha),$$  

which is analogous to (26) and evidently converts into (26) in the limit for $\text{Re} \alpha \to +\infty$.

Let us note at last that after the change of variable $\tau = \tanh \alpha x$ Eq. (33) takes the form of the Legendre equation

$$(1 - \tau^2) \frac{d^2\psi}{d\tau^2} - 2\tau \frac{d\psi}{d\tau} + \left[\nu(\nu + 1) - \frac{\mu^2}{1 - \tau^2}\right] \psi = 0$$  

with

$$\mu = \frac{ik}{\alpha}, \quad \nu = -\frac{z}{2\alpha}. \tag{46}$$  

Hence, the eigenfunctions $\psi_+(x; k, \alpha)$ and $\psi_-(x; k, \alpha)$ can be expressed through modified Legendre functions [18]. Using (36), (37) and the relation 3.4.6 from [18] one can receive the following presentations for these eigenfunctions:

$$\psi_+(x; k, \alpha) = \frac{1}{\sqrt{2\pi}} \frac{\Gamma(1 + \frac{z}{2\alpha} - \frac{ik}{\alpha}) \Gamma(1 - \frac{z}{2\alpha} - \frac{ik}{\alpha})}{\Gamma(1 - \frac{k^2}{4\alpha})} \frac{2k}{2k + iz} P_{-z/(2\alpha)}^{ik/\alpha}(\tanh \alpha x),$$

$$\psi_-(x; k, \alpha) = \frac{1}{\sqrt{2\pi}} \Gamma(1 - \frac{ik}{\alpha}) P_{-z/(2\alpha)}^{ik/\alpha}(-\tanh \alpha x), \tag{47}$$

where $P_\nu^0(\tau)$ is modified associated Legendre function of the first kind.

After the change $z \to -z$ the formulae (22) - (47) are valid as well for the Hamiltonian $h^-_\alpha$. These formulae are valid as well, generally speaking, for complex $k$ and $z$ (in particular, for real $z$) and for purely imaginary $\alpha$. In the latter case it is better to shift $x \to x - x_0$, $\text{Im} x_0 \neq 0$ in order to have a nonsingular complex periodic potential. In the case $\text{Re} z \text{Re} \alpha < 0$ the function (42) is a wave function of the bound state of $h_\alpha$ for the energy level $E = -z^2/4$. It follows from (36) and (37) that the potential of $h_\alpha$ is reflectionless iff $\text{Re} \alpha \neq 0$, $z = 2\alpha n$, $n = 0, \pm 1, \pm 2, \ldots$. 

11
4. Conclusions

We have proven that wave functions related to spectral singularities are quite relevant to build the complete resolution of identity for non-Hermitian systems. Remarkably, depending on the class of test functions the structure of a resolution of identity is different in the sense of distribution theory. Nevertheless, there is no class of physically motivated test functions (wave packets) for which spectral singularities are negligible. Thus they are physical in the case when the system is characterized by a non-Hermitian potential. Namely, they contribute to transmission and reflection coefficients dramatically enhancing their values.

We have studied in detail the delta-like and a related smooth potential. But let us notice that one can exhibit additional smooth superpotentials, whose limiting cases coincide with \( \chi_{\varepsilon} (x) \) from (31), namely:

\[
\chi_{\varepsilon}(x) \equiv \frac{z x}{2 \sqrt{x^2 + \varepsilon^2}} \rightarrow \chi(x) \equiv \frac{z}{2} \text{sign} x, \quad \varepsilon \rightarrow 0,
\]

\[
\chi_{\varepsilon}(x) \equiv \frac{z}{\pi} \arctg \frac{x}{\varepsilon} \rightarrow \chi(x) \equiv \frac{z}{2} \text{sign} x, \quad \varepsilon \downarrow 0. \tag{48}
\]

Acknowledgments

The work of AA and AS was supported by grant RFBR 09-01-00145-a. AA was supported in part by CUR Generalitat de Catalunya under project 2009SGR502.

APPENDIX 1.

Proof of biorthogonality relations

In order to prove (11) we shall show that

\[
\lim_{A \to +\infty} \int_{-\infty}^{+\infty} \left\{ \int_{-A}^{A} \left[ (1 + 2 i k/z) \psi_+(x; k) \psi_-(x; k') \right] dx \right\} \varphi(k) dk = (1 + 2 k'/iz) \varphi(k') \tag{49}
\]

for any test function \( \varphi(k) \in C^\infty_{\mathbb{R}} \cap L_1(\mathbb{R}) \cap \{ f(k) : f'(k) \in L_1(\mathbb{R}) \} \). It follows from the condition \( \varphi(k) \in C^\infty_{\mathbb{R}} \cap L_1(\mathbb{R}) \cap \{ f(k) : f'(k) \in L_1(\mathbb{R}) \} \) and the relation

\[
\varphi(x) = \varphi(x_0) + \int_{x_0}^{x} \varphi'(t) dt
\]

that

\[
\lim_{x \to +\infty} \varphi(x) = \lim_{x \to -\infty} \varphi(x) = 0. \tag{50}
\]

With the help of a straightforward calculation one can transform the left-hand part of (49) to

\[
\frac{1}{\pi} \lim_{A \to +\infty} \int_{-\infty}^{+\infty} \left\{ \frac{\sin A(k - k')}{k - k'} \psi(k) - \frac{\sin k' A}{k'} e^{ikA} \varphi(k) \right\} dk, \quad \psi(k) = (1 + 2 k/iz) \varphi(k), \tag{51}
\]
Such a choice of the test functions space is motivated by properties of eigenfunctions (see (2) and (3)).

\[ \lim_{A \to +\infty} \left\{ \frac{\sin k' A}{k'} \int_{-\infty}^{+\infty} e^{ikA} \varphi(k) dk \right\} = \lim_{A \to +\infty} \left\{ i \frac{\sin k' A}{k'} \int_{-\infty}^{+\infty} e^{ikA} \varphi(k) dk \right\} = 0. \]  (52)

Thus in the limit \( A \to +\infty \) the expression (51) is reduced to the first term. By virtue of the Riemann theorem and due to the evident inclusions

\[ \frac{\psi(k)}{k - k'} \in L_1(\mathbb{R} \setminus [k' - \delta, k' + \delta]), \quad \frac{\psi(k) - \psi(k')}{k - k'} \in L_1([k' - \delta, k' + \delta]) \]

for any \( \delta > 0 \), the following relations are valid:

\[ \lim_{A \to +\infty} \left( \int_{-\infty}^{k' - \delta} + \int_{k' + \delta}^{+\infty} \right) \sin A(k - k') \frac{\psi(x)}{k - k'} dk = 0, \]

\[ \lim_{A \to +\infty} \int_{k' - \delta}^{k' + \delta} \sin A(k - k') \frac{\psi(k) - \psi(k')}{k - k'} dk = 0. \]  (53)

Hence,

\[ \frac{1}{\pi} \lim_{A \to +\infty} \int_{-\infty}^{+\infty} \frac{\sin A(k - k')}{k - k'} \psi(k) dk = \frac{\psi(k')}{\pi} \lim_{A \to +\infty} \int_{k' - \delta}^{k' + \delta} \frac{\sin A(k - k')}{k - k'} dk = \psi(k'). \]  (54)

Thus, (13) and, consequently, (11) are valid.

Let us notice that one can prove (11) also for test functions from a wider class with the help of the technique of Theorem 2 and Remark 1 of the next Appendix 2.

**APPENDIX 2.**

**Proofs of resolutions of identity**

Let \( CL_{\gamma} = C_{\mathbb{R}} \cap \{ f(x) : f(x)|_{(-\infty,0]} \in C_{(-\infty,0]}^\infty, f(x)|_{[0,\infty)} \in C_{[0,\infty)}^\infty \} \cap L_2(\mathbb{R}; (1 + |x|)^{\gamma}), \gamma \in \mathbb{R}, \) be the space of test functions\(^6\) The sequence \( \varphi_n(x) \in CL_{\gamma} \) is called convergent in \( CL_{\gamma} \) to \( \varphi(x) \in CL_{\gamma}, \)

\[ \lim_{n \to +\infty} \varphi_n(x) = \varphi(x) \]  (55)

if

\[ \lim_{n \to +\infty} \int_{-\infty}^{+\infty} |\varphi_n(x) - \varphi(x)|^2 (1 + |x|)^{\gamma} dx = 0, \]  (56)

and for any \( x_1, x_2 \in \mathbb{R}, x_1 < x_2, \)

\[ \lim_{n \to +\infty} \max_{[x_1,x_2]} |\varphi_n(x) - \varphi(x)| = 0. \]  (57)

\(^6\)Such a choice of the test functions space is motivated by properties of \( h \) eigenfunctions (see (2) and (3)).
We shall denote the value of a functional $f$ on $\varphi \in CL_\gamma$ conventionally as $(f, \varphi)$. A functional $f$ is called continuous if for any sequence $\varphi_n \in CL_\gamma$ convergent in $CL_\gamma$ to zero the equality
\[
\lim_{n \to +\infty} (f, \varphi_n) = 0
\]
is valid. The space of distributions over $CL_\gamma$, i.e. of linear continuous functionals over $CL_\gamma$ is denoted $CL'_\gamma$. The sequence $f_n \in CL'_\gamma$ is called convergent in $CL'_\gamma$ to $f \in CL'_\gamma$, if for any $\varphi \in CL_\gamma$ the relation takes place,
\[
\lim_{n \to +\infty} (f_n, \varphi) = (f, \varphi).
\]
A functional $f \in CL'_\gamma$ is called regular if there is $f(x) \in L_2(\mathbb{R}; (1 + |x|)^{-\gamma})$ such that for any $\varphi \in CL_\gamma$ the equality
\[
(f, \varphi) = \int_{-\infty}^{+\infty} f(x)\varphi(x) \, dx
\]
holds. In this case we shall identify the distribution $f \in CL'_\gamma$ with the function $f(x) \in L_2(\mathbb{R}; (1 + |x|)^{-\gamma})$. In virtue of Bunyakovskii inequality,
\[
\left| \int_{-\infty}^{+\infty} f(x)\varphi(x) \, dx \right|^2 \leq \int_{-\infty}^{+\infty} |f^2(x)| \, dx \int_{-\infty}^{+\infty} |\varphi^2(x)|(1 + |x|)^{2\gamma} \, dx,
\]
it is evident that $L_2(\mathbb{R}; (1 + |x|)^{-\gamma}) \subset CL'_\gamma$ and this inclusion is continuous.

For any $\gamma_1 < \gamma_2$ there is obviously continuous inclusion $CL_{\gamma_2} \subset CL_{\gamma_1}$. Let us also note that the Dirac delta function $\delta(x - x')$ belongs to $CL'_\gamma$ for any $\gamma \in \mathbb{R}$.

Proof of the resolution of identity (12) is based on the following lemma.

**Lemma 1.** Suppose that $\mathcal{L}(A)$ is a path in complex $k$ plane, made of the segment $[-A, A]$ by its deformation near the point $k = k_0$, $k_0 \in (-A, A) \subset \mathbb{R}$ upwards and the direction of $\mathcal{L}(A)$ is specified from $-A$ to $A$. Then for any $r > 0$, $k_0 \in \mathbb{R}$ and $A > |k_0|$ the inequalities hold
\[
\left| \int_{\mathcal{L}(A)} \frac{e^{ikr}}{k - k_0} \, dk \right| \leq \frac{AD}{(1 + r(A - |k_0|))(A - |k_0|)}
\]
and
\[
\left| \int_{\mathcal{L}(A)} \frac{e^{ikr} \, dk}{k - k_0} - \frac{1}{ir} \left( \frac{e^{iAr}}{A - k_0} + \frac{e^{-iAr}}{A + k_0} \right) \right| \leq \frac{4}{(A - |k_0|)^2r^2},
\]
where $D > 0$ is a constant independent of $r$, $k_0$ and $A$.

**Proof.** With the help of Jordan lemma one can easily check that
\[
\int_{\mathcal{L}(A)} \frac{e^{ikr}}{k - k_0} \, dk = -\left( \int_{-\infty}^{-A} + \int_{A}^{+\infty} \right) \frac{e^{ikr}}{k - k_0} \, dk = \int_{A + k_0}^{+\infty} e^{ik_0r - i\tau r} \frac{d\tau}{\tau} - \int_{A - k_0}^{-\infty} e^{ik_0r + i\tau r} \frac{d\tau}{\tau} =
\]
$e^{ikr} \left( \int_{A+k_0}^{A-k_0} \frac{\cos \tau r}{\tau} d\tau - i \int_{A+k_0}^{A-k_0} \frac{\sin \tau r}{\tau} d\tau - i \int_{A-k_0}^{A+k_0} \frac{\sin \tau r}{\tau^2} d\tau \right).$ \hfill (65)

For the first integral in the right-hand side of (65) the following estimation is valid:

$$\left| \int_{A+k_0}^{A-k_0} \frac{\cos \tau r}{\tau} d\tau \right| = \left| \frac{\sin r(A-k_0)}{r(A-k_0)} - \frac{\sin r(A+k_0)}{r(A+k_0)} + \int_{A+k_0}^{A-k_0} \frac{\sin \tau r}{\tau} d\tau \right| \leq$$

$$\frac{C/2}{1 + r(A-k_0)} + \frac{C/2}{1 + r(A+k_0)} + \int_{A-k_0}^{A+k_0} \frac{C/2}{\tau(1 + r\tau)} d\tau \leq \frac{C}{1 + r(A-k_0)} + \frac{C}{1 + r(A+k_0)}.$$

For the second and third integrals of the right-hand side of (65) we have:

$$\left| \int_{A-k_0}^{A+k_0} \frac{\sin \tau r}{\tau} d\tau \right| = 2 \int_{A+k_0}^{A-k_0} \frac{\sin^2(\tau r/2)}{\tau} d\tau = 2 \left| - \frac{\sin^2[r(A \pm k_0)/2]}{A \pm k_0} + \int_{A \pm k_0}^{A \pm k_0} \frac{\sin^2(\tau r/2)}{\tau^2} d\tau \right| \leq$$

$$\frac{(C/2)^2 r(A \pm k_0)/2}{(1 + r(A \pm k_0)/2)^2} + \int_{A \pm k_0}^{A \pm k_0} \frac{(C/2)^2 r(A \pm k_0)/2}{(1 + r(A \pm k_0)/2)^2} d\tau \leq \frac{C^2/2}{1 + r(A - |k_0|) / 2} \leq \frac{C^2}{1 + r(A - |k_0|)} \leq \frac{AC^2}{(1 + r(A - |k_0|))(A - |k_0|)},$$ \hfill (67)

The inequality (63) follows from (65) – (67) with $D = C + 2C^2$.

The inequality (64) is valid in view of the following chain of relations derived with the help of (65) and integration by parts:

$$\left| \int_{\mathcal{C}(A)} \frac{e^{ikr}}{k-k_0} dk - \frac{1}{ir} \left( \frac{e^{iAr}}{A-k_0} + \frac{e^{-iAr}}{A+k_0} \right) \right| =$$

$$\frac{1}{r} \left| \left( \int_{-\infty}^{-A} + \int_{A}^{+\infty} \right) \frac{e^{ikr}}{(k-k_0)^2} dk \right| = \frac{1}{r^2} \left| \frac{e^{-iAr}}{(A+k_0)^2} - \frac{e^{iAr}}{(A-k_0)^2} + 2 \left( \int_{-\infty}^{-A} + \int_{A}^{+\infty} \right) \frac{e^{ikr}}{(k-k_0)^3} dk \right| \leq$$

$$\frac{2}{(A-|k_0|)^2 r^2} + \frac{4}{r^2} \int_{-\infty}^{+\infty} \frac{dk}{(k-k_0)^3} = \frac{4}{(A-|k_0|)^2 r^2}.$$ \hfill (68)

Lemma 1 is proved.

Validity of the resolution of identity (12) in $CL_\gamma$ for any $\gamma > -1$ is a corollary of the following theorem.

**Theorem 1.** Suppose that
Proof. One can reduce the product \( \psi_+(x; k) \psi_-(x'; k) \) to the form
\[
\psi_+(x; k) \psi_-(x'; k) = \frac{1}{2\pi} \left\{ e^{ik(x-x')} + \frac{2iz}{k} \theta(-x)\theta(x') \sin kx \sin kx' - \frac{iz}{2k + iz} e^{ik(|x|+|x'|)} \right\},
\]
where notation (4) is used. Hence,
\[
\int_{\mathcal{L}(A)} \psi_+(x; k) \psi_-(x'; k) \, dk = \frac{1}{2\pi} \int \frac{\sin A(x-x')}{x-x'} \, dx - i\frac{z}{4\pi} \int_{\mathcal{L}(A)} \frac{e^{ik(|x|+|x'|)}}{k + iz/2} \, dk.
\]
(71)

In view of Lemma 1 the integral in the left-hand part of (71) belongs to \( L_2(\mathbb{R}; (1+|x|)^{-\gamma}) \) and therefore to \( CL_\gamma' \). Thus, to prove (69) it is sufficient to establish that for any \( \varphi(x) \in CL_\gamma \) the equality takes place,
\[
\lim_{A \to +\infty} \int_{-\infty}^{+\infty} \left[ \frac{\sin A(x-x')}{x-x'} - \frac{iz}{4\pi} \int_{\mathcal{L}(A)} \frac{e^{ik(|x|+|x'|)}}{k + iz/2} \, dk \right] \varphi(x) \, dx = \varphi(x').
\]
(72)

By virtue of Bunyakowski inequality, Lemma 1 and inequality
\[
(1 + r)^\beta \leq 1 + r^\beta, \quad r \geq 0, \quad 0 \leq \beta \leq 1
\]
(73)

we have:
\[
\left| \int_{-\infty}^{+\infty} \left[ \int_{\mathcal{L}(A)} \frac{e^{ik(|x|+|x'|)}}{k + iz/2} \, dk \right] \varphi(x) \, dx \right|^2 \leq
\]
\[
\int_{-\infty}^{+\infty} \left| \int_{\mathcal{L}(A)} \frac{e^{ik(|x|+|x'|)}}{k + iz/2} \, dk \right|^2 \frac{dx}{(1+|x|)^\gamma} \int_{-\infty}^{+\infty} \varphi^2(x)(1+|x|)^\gamma \, dx \leq
\]
\[
\frac{A^2D^2}{(A-|z|/2)^2} \int_{-\infty}^{+\infty} \frac{1 + \theta(-\gamma)|x|^{-\gamma}}{[1+([x]+|x'|)(A-|z|/2)]^2} \, dx \int_{-\infty}^{+\infty} \varphi^2(x)(1+|x|)^\gamma \, dx \leq
\]
\[
\frac{2A^2D^2}{(A-|z|/2)^2} \int_{0}^{+\infty} \frac{1 + \theta(-\gamma)x^{-\gamma}}{[1+x(A-|z|/2)]^2} \, dx \int_{-\infty}^{+\infty} \varphi^2(x)(1+|x|)^\gamma \, dx =
\]
\[
\frac{2A^2D^2}{(A-|z|/2)^3} \int_{0}^{+\infty} \frac{1 + \theta(-\gamma)(A-|z|/2)^\gamma \xi^{-\gamma}}{(1+\xi)^2} \, d\xi \int_{-\infty}^{+\infty} \varphi^2(x)(1+|x|)^\gamma \, dx \to 0, \quad A \to +\infty,
\]
(74)
wherefrom it follows that the second term in the left-hand part of (72) vanishes. Now the proof that the left-hand side of (72) is equal to \( \varphi(x') \) is analogous to the proof in Appendix 1. Thus, Theorem 1 is proved.

The applicability of the resolution of identity (12) for some bounded and slowly increasing test functions is based on the next theorem.

**Theorem 2.** Suppose that

1. \( \psi_+(x; k) \) and \( \psi_-(x; k) \) are defined by the formulae (2) and (3) respectively for any \( x \in \mathbb{R}, k \in \mathbb{C} \) and fixed purely imaginary \( z \neq 0 \);
2. \( \mathcal{L}(A) \) is a path in complex \( k \) plane, made of the segment \([-A, A]\) by its deformation near the point \( k = -iz/2 \) upwards and the direction of \( \mathcal{L}(A) \) is specified from \(-A\) to \(A\);
3. the function \( \eta(x) \in C^\infty_{\mathbb{R}}, \eta(x) \equiv 0 \) for any \( x \leq 1 \), \( \eta(x) \in [0, 1] \) for any \( x \in [1, 2] \) and \( \eta(x) \equiv 1 \) for any \( x \geq 2 \).

Then for any \( \kappa \in (0, 1), k_0 \in \mathbb{R} \) and \( x' \in \mathbb{R} \) the following relation holds:

\[
\lim_{A \to +\infty} \int_{-\infty}^{+\infty} \left[ \int_{\mathcal{L}(A)} \psi_+(x; k)\psi_-(x'; k) \, dk \right] \eta(\pm x)e^{ik_0x}|x|^{\kappa} \, dx = \eta(\pm x'e^{ik_0x'}|x'|^{\kappa}. \tag{75}
\]

**Proof.** We present the proof for the case with upper signs in (75) only because the proof for the case with lower signs is quite similar. By virtue of (71) and Lemma 1 the following asymptotics takes place for the integral over \( k \) in (75):

\[
\int_{\mathcal{L}(A)} \psi_+(x; k)\psi_-(x'; k) \, dk = \frac{1}{\pi} \sin A(x - x') - \frac{z}{4\pi} \left( \frac{e^{iA(|x|+|x'|)}}{A + iz/2} + \frac{e^{-iA(|x|+|x'|)}}{A - iz/2} \right) \frac{1}{|x| + |x'|} + O\left(\frac{1}{x^2}\right), \quad x \to \pm \infty. \tag{76}
\]

Hence, the integral over \( x \) in (75) converges for any \( A > |k_0| \). It follows from Theorem 1 that

\[
\lim_{A \to +\infty} \int_{-\infty}^{+\infty} \left[ \int_{\mathcal{L}(A)} \psi_+(x; k)\psi_-(x'; k) \, dk \right] \eta(x)\eta(3 + x' - x)e^{ik_0x}|x|^{\kappa} \, dx = \eta(x')\eta(3)e^{ik_0x'}|x'|^{\kappa} = \eta(x')e^{ik_0x'}|x'|^{\kappa}. \tag{77}
\]

Hence, to prove (75) it is sufficient to prove that

\[
\lim_{A \to +\infty} \int_{-\infty}^{+\infty} \left[ \int_{\mathcal{L}(A)} \psi_+(x; k)\psi_-(x'; k) \, dk \right] \left\{ \eta(x)[1 - \eta(3 + x' - x)]e^{ik_0x}|x|^{\kappa} \right\} \, dx = 0. \tag{78}
\]

The fact that contributions of the first and second terms of the right-hand side of (76) vanish in the limit (78) can be checked with the help of Riemann theorem and integration.
Remark 1. Theorem 2 is proved.

Theorem 2 is proved.

Remark 1. Theorems 1 and 2 provide the validity of resolution of identity (12) for test functions which are linear combinations of functions \( \eta(\pm x)e^{ik_0x}\abs{x}^\gamma \), in general, with different \( \gamma \in (0,1) \) and \( k_0 \in \mathbb{R} \) and functions from \( CL_\gamma \), in general, with different \( \gamma > -1 \). In particular, these theorems guarantee applicability of (12) for eigenfunctions (2) and (3) of the Hamiltonian \( h \).

Remark 2. One can rearrange the resolution of identity (12) to the form (13) with the help of (70) and the chain of relations,

\[
-\frac{iz}{2\pi} \int_{\ell(\varepsilon)} \frac{e^{ik(|x|+|x'|)}}{2k+iz} \, dk =
\]

\[
= -\frac{iz}{2\pi} e^{\varepsilon(|x|+|x'|)/2} \int_{\ell(\varepsilon)} \frac{dk}{2k+iz} + \int_{\ell(\varepsilon)} \frac{e^{ik(|x|+|x'|)} - e^{\varepsilon(|x|+|x'|)/2}}{2k+iz} \, dk
\]

\[
= -\frac{iz}{2\pi} \psi_0(x)\psi_0(x') \left[ -\frac{\pi i}{2} + \frac{i}{2} \int_{-i\varepsilon/2-\varepsilon}^{-iz/2+\varepsilon} \frac{\sin[(k+iz/2)(|x|+|x'|)/2]}{k+iz/2} \, dk \right]
\]

\[
= -\frac{z}{4} \psi_0(x)\psi_0(x') \left[ 1 - \frac{2}{\pi} \int_0^{\varepsilon(|x|+|x'|)} \frac{\sin t}{t} \, dt \right] \equiv -\frac{z}{2\pi} \psi_0(x)\psi_0(x') \int_{\varepsilon(|x|+|x'|)}^{+\infty} \frac{\sin t}{t} \, dt, \quad (80)
\]

where the relation 2.5.3.12 from [17] is taken into account and \( \ell(\varepsilon) \) is the integration path in complex plane of \( k \) defined by the equation \( k = -iz/2 + \varepsilon e^{i\vartheta} \), \( 0 \leq \vartheta \leq \pi \) with the direction corresponding to increasing \( \vartheta \).

The resolutions of identity (15) and (16) are based on the following lemmas.

Lemma 2. For any \( \gamma > -1 \) and \( x' \in \mathbb{R} \) the relation holds

\[
\lim_{\varepsilon \downarrow 0} \frac{\sin \varepsilon(x-x')}{x-x'} = 0. \quad (81)
\]
Proof. It is true that
\[
\frac{\sin \epsilon(x - x')}{x - x'} \in L_2(\mathbb{R}; (1 + |x|)^{-\gamma}) \subset CL_\gamma', \quad \gamma > -1. \quad (82)
\]
Thus, to prove the lemma it is sufficient to establish that for any \( \varphi(x) \in CL_\gamma', \gamma > -1 \), the relation
\[
\lim_{\epsilon \downarrow 0} \int_{-\infty}^{+\infty} \frac{\sin \epsilon(x - x')}{x - x'} \varphi(x) \, dx = 0 \quad (83)
\]
is valid. But its validity follows from Bunyakovskii inequality and (73):
\[
\left| \int_{-\infty}^{+\infty} \frac{\sin \epsilon(x - x')}{x - x'} \varphi(x) \, dx \right|^2 \leq \int_{-\infty}^{+\infty} \frac{\sin^2 \epsilon(x - x')}{(x - x')^2(1 + |x|)^\gamma} \int |\varphi^2(x)|(1 + |x|)^\gamma \, dx \leq
\]
\[
\int_{-\infty}^{+\infty} \frac{\sin^2 \epsilon(x - x')}{(x - x')^2} \left[ 1 + \theta(-\gamma)(|x'| + |x - x'|)^{-\gamma} \right] dx \int |\varphi^2(x)|(1 + |x|)^\gamma \, dx \leq
\]
\[
\int_{-\infty}^{+\infty} \frac{\sin^2 \epsilon(x - x')}{(x - x')^2} \left[ 1 + \theta(-\gamma)(|x'|^{-\gamma} + |x - x'|^{-\gamma}) \right] dx \int |\varphi^2(x)|(1 + |x|)^\gamma \, dx =
\]
\[
\epsilon \int_{-\infty}^{+\infty} \frac{\sin^2 \xi}{\xi^2} \left[ 1 + \theta(-\gamma)(|x'|^{-\gamma} + \epsilon^\gamma|\xi|^{-\gamma}) \right] \, d\xi \int |\varphi^2(x)|(1 + |x|)^\gamma \, dx \rightarrow 0, \quad \epsilon \downarrow 0, \quad (84)
\]
where we define that the value \(|x'|^{-\gamma} \) for \( x' = 0 \) and \( \gamma = 0 \) is equal to zero. Lemma 2 is proved.

Lemma 3. For any \( y \in \mathbb{R}, \epsilon > 0 \) and \( k_0 \in \mathbb{R}, |k_0| > \epsilon \) the inequality
\[
\left| \int_{k_0 + \epsilon}^{k_0 - \epsilon} \frac{e^{iky}}{k} \, dk \right| \leq \frac{\epsilon D}{(|k_0| - \epsilon)(2 + \epsilon|y|)} \quad (85)
\]
takes place, where \( D \) is a constant independent of \( y, \epsilon \) and \( k_0 \).

Proof. Lemma 3 is valid by virtue of the following chain of relations:
\[
\left| \int_{k_0 - \epsilon}^{k_0 + \epsilon} \frac{e^{iky}}{k} \, dk \right| = \left| e^{ik_0 y} \int_{-\epsilon}^{\epsilon} \frac{e^{i\tau y}}{k_0 + \tau} \, d\tau \right| = \left| \frac{1}{iy} \int_{-\epsilon}^{\epsilon} \frac{d(e^{i\tau y} - 1)}{k_0 + \tau} \right| =
\]
\[
\left| \frac{e^{iy(k_0 + \epsilon)} - e^{-iy(k_0 - \epsilon)}}{iy(k_0 + \epsilon)} - \frac{e^{-iy(k_0 + \epsilon)} - 1}{iy(k_0 - \epsilon)} + \frac{1}{iy} \int_{-\epsilon}^{\epsilon} \frac{e^{i\tau y} - 1}{(k_0 + \tau)^2} \, d\tau \right| \leq
\]
\[
2 \left| \frac{\sin(\epsilon y/2)}{|y(k_0 + \epsilon)|} \right| + 2 \left| \frac{\sin(\epsilon y/2)}{|y(k_0 - \epsilon)|} \right| + \frac{2}{|y|} \int_{-\epsilon}^{\epsilon} \frac{\sin(\epsilon y/2)}{(k_0 + \tau)^2} \, d\tau \leq
\]
Thus, to prove the lemma it is sufficient to establish that for any \( \gamma > -1 \), the relation

\[
\lim_{\varepsilon \to 0} \left[ \theta(-x) \theta(x') \int_{-iz/2-\varepsilon}^{iz/2+\varepsilon} \frac{1}{k} \sin kx \sin kx' \, dk \right] = 0
\]  

(89)
takes place.

**Proof.** In view of Corollary 1

\[
\frac{2\varepsilon C}{(|k_0| - \varepsilon)(2 + \varepsilon |y|)} + 2C \int_{0}^{\varepsilon} \frac{\tau \, d\tau}{(|k_0| - \tau)^2 (2 + \varepsilon |\tau|)} \leq \frac{2\varepsilon C}{(|k_0| - \varepsilon)(2 + \varepsilon |y|)} + \int_{0}^{\varepsilon} \frac{d\tau}{(|k_0| - \tau)^2} = \frac{2\varepsilon C}{2 + \varepsilon |y|} \left[ \frac{1}{|k_0| - \varepsilon} + \frac{1}{|k_0| - \varepsilon} - \frac{1}{|k_0|} \right] = \frac{2\varepsilon C(|k_0| + \varepsilon)}{|k_0||(|k_0| - \varepsilon)(2 + \varepsilon |y|)|} \leq \frac{\varepsilon D}{(|k_0| - \varepsilon)(2 + \varepsilon |y|)},
\]

(86)

where \( D = 4C, C = 2 \sup_{\varepsilon > 0} |(1 + 1/\xi) \sin \xi| \). Lemma 3 is proved.

**Corollary 1.** For any \( x \in \mathbb{R}, x' \in \mathbb{R} \) and purely imaginary \( z \neq 0 \) the inequality

\[
\left| \int_{-iz/2-\varepsilon}^{iz/2+\varepsilon} \frac{1}{k} \sin kx \sin kx' \, dk \right| \leq \frac{\varepsilon D}{(|z|/2 - \varepsilon)(2 + \varepsilon ||x| - |x'||)}
\]

(87)
is valid.

**Corollary 2.** In conditions of Lemma 3 the inequality

\[
\left| \int_{k_0-\varepsilon}^{k_0+\varepsilon} e^{iky} \frac{dk}{k + k_0} \right| = \left| e^{-iky} \int_{2k_0-\varepsilon}^{2k_0+\varepsilon} \frac{e^{iy\tau}}{\tau} \, d\tau \right| = \left| \int_{2k_0-\varepsilon}^{2k_0+\varepsilon} \frac{e^{iy\tau}}{\tau} \, d\tau \right| \leq \frac{\varepsilon D}{(2|k_0| - \varepsilon)(2 + \varepsilon |y|)}
\]

(88)
holds as well.

**Lemma 4.** For any \( \gamma > -1, x' \in \mathbb{R} \) and purely imaginary \( z \neq 0 \) the relation

\[
\lim_{\varepsilon \to 0} \left[ \theta(-x) \theta(x') \int_{-iz/2-\varepsilon}^{iz/2+\varepsilon} \frac{1}{k} \sin kx \sin kx' \, dk \right] = 0
\]  

(90)
takes place.

Thus, to prove the lemma it is sufficient to establish that for any \( \varphi(x) \in CL_{\gamma}, \gamma > -1 \), the relation

\[
\lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} \left[ \theta(-x) \theta(x') \int_{-iz/2-\varepsilon}^{iz/2+\varepsilon} \frac{1}{k} \sin kx \sin kx' \, dk \right] \varphi(x) \, dx = 0
\]

(91)
is valid. But its validity follows from Bunyakovskii inequality, Corollary 1 and (73):

\[
\left| \int_{-\infty}^{+\infty} \left[ \theta(-x) \theta(x') \int_{-iz/2-\varepsilon}^{iz/2+\varepsilon} \frac{1}{k} \sin kx \sin kx' \, dk \right] \varphi(x) \, dx \right|^2 \leq \ldots
\]
for all test functions
\[
\sum_{\text{two functions from } CL}\text{functionals (93) on the standard space}
\]
other hand, one can represent any test function which are defined by the expressions
\[
\lim_{\varepsilon \to 0} \sin \frac{\varepsilon}{x-x'} = 0, \
\lim_{\varepsilon \to 0} \left[ \theta(-x)\theta(x') \right] = 0
\]
\[
\lim_{\varepsilon \to 0} \left[ \theta(-x)\theta(x') \right] = 0
\]
\[
\lim_{\varepsilon \to 0} \left[ \theta(-x)\theta(x') \right] = 0
\]
\[
\lim_{\varepsilon \to 0} \left[ \theta(-x)\theta(x') \right] = 0
\]
where we define that the value \(|x'|^{-\gamma}\) for \(x' = 0\) and \(\gamma = 0\) is equal to zero. Lemma 4 is proved.

**Remark 3.** Let us consider the functionals
\[
\lim_{\varepsilon \to 0} \frac{\sin \varepsilon(x-x')}{x-x'} = \frac{\sin \frac{\varepsilon}{x-x'}}{x-x'}, \
\lim_{\varepsilon \to 0} \left[ \theta(-x)\theta(x') \right] = 0
\]
which are defined by the expressions
\[
\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} \frac{\sin \varepsilon(x-x')}{x-x'} \varphi(x) \, dx,
\]
for all test functions \(\varphi(x) \in CL, \gamma \in \mathbb{R}\), for which corresponding to (83) limits from (84) exist. It follows from Lemmas 2 and 4 that these functionals are trivial (equal to zero) for any \(\gamma > -1\), but at the same time in view of (18) and (19) these functionals are nontrivial (different from zero) for any \(\gamma < -1\). By virtue of Lemmas 2 and 4 the restrictions of the functionals (83) on the standard space \(D(\mathbb{R}) \subset CL, \gamma \in \mathbb{R}\) are equal to zero. Hence, the supports of these functionals for any \(\gamma \in \mathbb{R}\) do not contain any finite real number. On the other hand, one can represent any test function \(\varphi(x) \in CL, \gamma \in \mathbb{R}\) for any \(R > 0\) as a sum of two functions from \(CL\gamma\) in the form
\[
\varphi(x) = \eta(|x|-R)\varphi(x) + [1-\eta(|x|-R)]\varphi(x), \quad R > 0,
\]
where $\eta(x) \in C^\infty_\mathbb{R}$, $\eta(x) \equiv 1$ for any $x < 0$, $\eta(x) \in [0, 1]$ for any $x \in [0, 1]$ and $\eta(x) \equiv 0$ for any $x > 1$. In view of Lemmas 2 and 4 the values of the functionals (93) for $\varphi(x)$ are equal to their values for the second term of (95) for any arbitrarily large $R > 0$. Hence, the values of the functionals (93) for a test function depend only on the behavior of this function in any arbitrarily small (in the topological sense) vicinity of the infinity and are independent of values of the function in any finite interval of real axis. In this sense the supports of the functionals (93) for any $\gamma < -1$ consist of the unique element which is the infinity. At last, since (1) for any $\varphi(x) \in CL_\gamma$ and $\gamma \in \mathbb{R}$ the relation

$$
\lim_{R \to +\infty} \eta(|x| - R) \varphi(x) = \varphi(x)
$$

holds; (2) the restrictions of the functionals (93) on $D(\mathbb{R})$ are zero for any $\gamma \in \mathbb{R}$ and (3) the functionals (93) are nontrivial for any $\gamma < -1$, so the functionals (93) for any $\gamma < -1$ are discontinuous.

In the same way one can verify that the functional

$$
\lim_{\varepsilon \downarrow 0} \frac{\sin^2 \left( \frac{\varepsilon}{2} (x - x') \right)}{\varepsilon (x - z) (x' - z)}, \quad x' \in \mathbb{R}, \quad \text{Im} z \neq 0,
$$

considered actually in [19], is trivial for any $\gamma > 1$ (proof of this fact is analogous to one of Lemma 3 from Appendix of [19]) and nontrivial for any $\gamma < 1$ (see (71) in [19]), is discontinuous for any $\gamma < 1$ and its support for any $\gamma < 1$ consist of the unique element which is infinity.

**Lemma 5.** Suppose that the function $\psi_0(x) = e^{x|x|/2}$ is defined for any $x \in \mathbb{R}$ and some purely imaginary $z \neq 0$. Then for any $\gamma > 1$ and $x' \in \mathbb{R}$ the relation

$$
\lim_{\varepsilon \downarrow 0} \int_{-\infty}^{+\infty} \left[ \psi_0(x) \psi_0(x') \right] \frac{\sin t}{t} dt = 0
$$

takes place.

**Proof.** It is true that

$$
\psi_0(x) \psi_0(x') \int_{0}^{\varepsilon(|x|+|x'|)} \frac{\sin t}{t} dt \in L_2(\mathbb{R}; (1 + |x|)^{-\gamma}) \subset CL'_\gamma, \quad \gamma > 1.
$$

Thus, to prove the lemma it is sufficient to establish that for any $\varphi(x) \in CL_\gamma$, $\gamma > 1$, the relation

$$
\lim_{\varepsilon \downarrow 0} \int_{-\infty}^{+\infty} \left[ \psi_0(x) \psi_0(x') \right] \frac{\sin t}{t} dt \varphi(x) dx = 0
$$

is valid. By virtue of the inequality

$$
\left| \int_{0}^{r} \frac{\sin t}{t} dt \right| \leq \frac{Kr}{1 + r}, \quad r \geq 0
$$

It is more natural to use for the functionals (97) (see [19]) narrower spaces of test functions $C^\infty_\mathbb{R} \cap L_2(\mathbb{R}; (1 + |x|)^{\gamma})$, $\gamma \in \mathbb{R}$. All proofs are easily adaptable for these spaces.
Thus, to prove the lemma it is sufficient to establish that for any \( \varphi(x) \in CL_\gamma, \gamma > -1 \), the relation
\[
\lim_{\varepsilon \downarrow 0} \int_{\varepsilon i z / 2 - \varepsilon}^{i z / 2 + \varepsilon} \frac{e^{ik(|x|+|x'|)}}{2k + i z} \, dk = 0. \tag{103}
\]

**Proof.** In view of Corollary 2
\[
\int_{\varepsilon i z / 2 - \varepsilon}^{i z / 2 + \varepsilon} \frac{e^{ik(|x|+|x'|)}}{2k + i z} \, dk \in L_2(\mathbb{R}; (1 + |x|)^{-\gamma}) \subset CL_\gamma, \quad \gamma > -1. \tag{104}
\]

Thus, to prove the lemma it is sufficient to establish that for any \( \varphi(x) \in CL_\gamma, \gamma > -1 \), the relation
\[
\lim_{\varepsilon \downarrow 0} \int_{-\infty}^{+\infty} \int_{\varepsilon i z / 2 - \varepsilon}^{i z / 2 + \varepsilon} \frac{e^{ik(|x|+|x'|)}}{2k + i z} \, dk \varphi(x) \, dx = 0. \tag{105}
\]
is valid. But with the help of Corollary 2 and Bunyakovskii inequality the proof of its validity is quite analogous to the proof for Lemma 4. Lemma 6 is proved.

**Corollary 5.** The resolutions of identity (28) and (29) for test functions from $CL_\gamma$ with $\gamma > 1$ follows from (15), (27), (70) and Lemmas 2, 4 and 6.

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