ON THE MINIMUM NUMBER OF FACETS OF A 2-NEIGHBORLY POLYTOPE

ALEKSEANDR MAKIMENKO

Abstract. Let \( \mu_{2n}(d,v) \) (respectively, \( \mu_{2n}^*(d,v) \)) be the minimal number of facets of a (simplicial) 2-neighborly \( d \)-polytope with \( v \) vertices, \( v > d \geq 4 \). It is known that \( \mu_{2n}(4,v) = v(v-3)/2 \), \( \mu_{2n}(d,d+2) = d+5 \), \( \mu_{2n}(d,d+3) = d+7 \) for \( d \geq 5 \), and \( \mu_{2n}(d,d+4) \in [d+5,d+8] \) for \( d \geq 6 \). We show that \( \mu_{2n}(5,v) = \Omega(v^{4/3}) \), \( \mu_{2n}(6,v) \geq v \), and the equality \( \mu_{2n}(6,v) = v \) holds only for a simplex and for a dual 2-neighborly 6-polytope (if it exists) with \( v \geq 27 \). By using g-theorem, we get \( \mu_{2n}(d,v) = \Delta(\Delta(d-3) + 3d-5)/2 + d + 1 \), where \( \Delta = v - d - 1 \). Also we show that \( \mu_{2n}(d,v) \geq d+7 \) for \( v \geq d+4 \).

1. Introduction

Let \( P \) be a \( d \)-polytope, i.e., a \( d \)-dimensional convex polytope. An \( i \)-dimensional face of \( P \) is called \( i \)-face, 0-faces are vertices, 1-faces are edges, \((d-1)\)-faces are facets, and \((d-2)\)-faces are ridges. A polytope \( P \) is simplicial if every facet is a simplex, and \( P \) is simple if it is dual to a simplicial polytope.

Let \( f_i(P) \) be the number of \( i \)-faces of \( P \), \( 0 \leq i \leq d - 1 \). The problem of estimating \( f_i(P) \) (where \( P \) belongs to some class of polytopes) in terms of \( f_0(P) \) is well known. For the class of simplicial polytopes, the problem is known as the upper bound and the lower bound theorems (see [9, Chap. 10] for details). In particular [2],

\[
\mu_{d-1}(P) \geq (d-1)(f_0(P) - d) + 2 \quad \text{for a simplicial } d\text{-polytope } P. \tag{1}
\]

In 1990, G. Blind and R. Blind [5] solved the lower bound problem for the class of polytopes without a triangle 2-face. We raise the question for the class of 2-neighborly polytopes.

A \( d \)-polytope \( P \) is called \( k \)-neighborly if every subset of \( k \) vertices forms the vertex set of some face of \( P \). Since every \( d \)-polytope is 1-neighborly, we will consider \( k \)-neighborly polytopes only for nontrivial cases \( k \geq 2 \). For \( k > d/2 \), there is only one combinatorial type of a \( k \)-neighborly \( d \)-polytope — a \( d \)-simplex [9, p. 123]. The same is true for \( f_0(P) = d + 1 \). Therefore, we suppose \( d \geq 2k \) and \( f_0(P) > d + 1 \).

A \( \lfloor d/2 \rfloor \)-neighborly polytope is called \( 2 \)-neighborly. In particular, every \( 2 \)-neighborly \( d \)-polytope is \( \lfloor d/2 \rfloor \)-neighborly for \( d \geq 4 \). The family of \( 2 \)-neighborly polytopes is constantly attracting the attention of researchers. For \( d = 2k \), these polytopes have the maximal number of facets over all \( d \)-polytopes with \( n \) vertices [13]:

\[
\mu_{d-1}(P_{\text{neighborly}}) = \frac{n}{n-k} \binom{n-k}{k}. \tag{2}
\]

There exists a widespread feeling that \( k \)-neighborly polytopes are very common among convex polytopes [9, p. 129–129a], [8, sec. 3.4], [16]. Moreover, combinatorial polytopes of NP-hard problems has \( k \)-neighborly faces with superpolynomial number of vertices [12, 13].

We will denote by \( \mu_{2n}(d,v) \) the minimal number of facets of a 2-neighborly \( d \)-polytope with \( v \) vertices. From (2), we get \( \mu_{2n}(4,v) = v(v-3)/2 \). In [11], there was posed the following question: \( \mu_{2n}(d,v) \geq v \)? This inequality was validated for two cases [11]: \( d \leq 6 \) and \( v \leq d + 5 \).

In Section 2, we list examples of 2-neighborly \( d \)-polytopes with small difference \( \mu_{d-1}(P) - f_0(P) \).
In Section 5 for 5- and 6-polytopes we show that
\[
\mu_{2n}(5,v) \geq \min_{n \in \mathbb{N}} \max \left\{ \frac{v(v-1)}{2}, \frac{n(n-3)}{2} + 1 \right\} = \Omega(v^{4/3})
\]
and \(\mu_{2n}(6,v) \geq v\) and the equality \(\mu_{2n}(6,v) = v\) holds only for a simplex and for a dual 2-neighborly polytope with \(v \geq 27\).

In Section 4 by using g-theorem, we get the lower bound for the number of \(i\)-faces of a simplicial 2-neighborly \(d\)-polytope. Let \(\mu_{2n}^s(d, v)\) be the minimal number of facets of a simplicial 2-neighborly \(d\)-polytope with \(v\) vertices. Obviously, \(\mu_{2n}(d, v) \leq \mu_{2n}^s(d, v)\). We prove that
\[
\mu_{2n}^s(d, v) = \frac{\Delta(\Delta(d-3) + 3d - 5)}{2} + d + 1, \quad \text{where} \quad \Delta = v - d - 1. \quad (3)
\]
Also we show that \(\mu_{2n}(d, v) \geq d + 7\) for \(v \geq d + 4\).

2. EXAMPLES OF 2-NEIGHBORLY POLYTOPES WITH SMALL NUMBER OF FACETS

In this section we list all known (to the author) from the literature examples of 2-neighborly polytopes with small number of facets.

First of all, it is well known that \(f_3(P) = f_0(P)(f_0(P) - 3)/2\) for a 2-neighborly 4-polytope \(P\). In particular, this is true for cyclic polytopes.

Let \(P\) be a 2-neighborly \(d\)-polytope. If \(Q\) is an \(r\)-fold pyramid with basis \(P\), then \(Q\) is a 2-neighborly \((d + r)\)-polytope with \(f_0(Q) = f_0(P) + r\) and \(f_{d+r-1}(Q) = f_{d-1}(P) + r\) [9, Sec. 4.2]. Hence, by using cyclic 4-polytopes, for every \(d > 4\) and \(v > d\) we can construct an example of a 2-neighborly \(d\)-polytope \(Q\) with \(v\) vertices and \((v + 4 - d)(v + 1 - d)/2 + d - 4\) facets, i.e.
\[
f_{d-1}(Q) - f_0(Q) = (f_0(Q) + 4 - d)(f_0(Q) - d - 1)/2. \quad (4)
\]

It is also well known that every \(d\)-polytope \(P\) with \(d + 2\) vertices is a simplicial one (and inequality \((\mathbb{1})\) holds for it) or a pyramid over a \((d - 1)\)-polytope with \(d + 1\) vertices. Consequently, if it is 2-neighborly, then \(f_{d-1}(P) \geq d + 5\) and the equality is attained only on a \((d - 4)\)-fold pyramid over a cyclic 4-polytope with 6 vertices. It is interesting to remark that the last one can be realized as a 0/1-polytope (see fig. \([\mathbb{1a}])

In [7], there are enumerated all combinatorial types of 5-polytopes with \(\leq 9\) vertices (all face lattices available at \text{http://www-imai.is.s.u-tokyo.ac.jp/~hmiyata/oriented_matroids/}). For 2-neighborly 5-polytopes with 8 vertices, the minimal number of facets is 12 and it is attained.
on the 0/1-polytope on the fig. [11]. The minimal number of facets of a 2-neighborly 5-polytope with 9 vertices is equal to 16 and the polytope also can be realized as a 0/1-polytope (see fig. [14]).

By enumerating minimal reduced Gale diagrams for 2-neighborly $d$-polytopes, one can conclude [11] that $\mu_{2n}(d, d + 3) = d + 3$ and the lower bound is attained on $P_{5, 8}$ (see fig. [1b]).

In [1], O. Aichholzer enumerated all 2-neighborly 0/1-polytopes of dimension $\leq 6$. For dimension 5, except the examples listed in fig. [1], there is a polytope with 10 vertices and 22 facets. For dimension 6, if the number of vertices is 10, then the minimum number of facets for such polytopes is equal to 14 (see fig. [1d]). For the other numbers of vertices 11, 12, and 13, minimum numbers of facets equal 17, 21, and 26.

All the mentioned results are summarized in Table 1. It is natural to try to construct examples of a 2-neighborly $d$-polytopes with as small as possible difference between facets and vertices. Some progress can be done with a join of two polytopes [10]:

$$P \ast P' := \text{conv} \left( \{ (x, 0, 0) \in \mathbb{R}^{d+d' + 1} \mid x \in P \} \cup \{ (0, y, 1) \in \mathbb{R}^{d+d' + 1} \mid y \in P' \} \right),$$

where $P$ is a $d$-polytope and $P'$ is a $d'$-polytope. The polytope $P \ast P'$ has dimension $d + d' + 1$, $f_0(P) + f_0(P')$ vertices, and $f_{d-1}(P) + f_{d-1}(P')$ facets [10] Sec. 15.1.3. Moreover, if $P$ and $P'$ are $k$-neighborly, then $P \ast P'$ is also $k$-neighborly.

Let $P_n^m$ be a 2-neighborly 4-polytope with $n$ vertices, $n \geq 5$, and let $P_n^1 = P_n^0 \ast P_n^0$. Hence $P_n^1$ is a 2-neighborly 9-polytope with $2n$ vertices and $n(n - 3)$ facets. Let us define $P_n^m$ recursively:

$$P_n^{m+1} = P_n^m \ast P_n^m, \quad m \in \mathbb{N}.$$

Thus $P_n^m$ is a 2-neighborly $d$-polytope with

$$d = 5 \cdot 2^m - 1, \quad f_0(P_n^m) = 2^m n, \quad f_{d-1}(P_n^m) = 2^{m-1} n(n - 3).$$

Therefore,

$$f_{d-1}(P_n^m) - f_0(P_n^m) = \frac{f_0(P_n^m)(f_0(P_n^m) - d - 1)}{2^{m+1}} < \frac{f_0(P_n^m)(f_0(P_n^m) - d - 1)}{0.4d}.$$

This difference has a bit better asymptotic than [11].

### 3. Small dimensions

A polytope $P$ is called $m$-simplicial if every $m$-face of $P$ is a simplex. A polytope is called $m$-simple if it is dual to an $m$-simplicial polytope. In this section we use the well known fact that a $k$-neighborly polytope is $(2k - 1)$-simplicial [9] Sec. 7.1]. The set of all $i$-faces of a $d$-polytope $P$ we denote by $F_i(P)$, $i = 0, 1, \ldots, d - 1$.

| $d$ | $v$ | $d + 2$ | $d + 3$ | $d + 4$ | $d + 5$ | $\ldots$ | $\nu(v - 3)/2$ |
|-----|-----|--------|--------|--------|--------|---------|-------------|
| 4   | 5   | $d + 5$| 16     | $\leq 22$|
| 6   |      | $d + 7$| $\leq 14$| $\leq 17$|
|      |      |        |         |         |         |          |              |

Table 1. The minimal number of facets of a 2-neighborly $d$-polytope with $v$ vertices.
The two statements below (Lemma 1 and Theorem 2) are generalized versions of the results from [11].

**Lemma 1.** Let $P$ be an $m$-simplicial $d$-polytope and $m \geq d/2$. Then $f_m(P) \geq f_{d-m-1}(P)$ and the equality is attained only if $P$ is $m$-simple.

**Proof.** Let us count incidences between $i$-faces and $(i-1)$-faces of $P$, $0 < i \leq m$. Note that every $(i-1)$-face of a $d$-polytope is incident with at least $(d-i+1)$ $i$-faces. Hence,

$$(d-i+1)f_{i-1}(P) \leq \sum_{p \in F_i(P)} f_{i-1}(p),$$

where $f_{i-1}(p)$ is the number of $(i-1)$-faces of $p$. Since $i$-faces are simplices,

$$(d-i+1)f_{i-1}(P) \leq (i+1)f_i(P) \quad \text{for } 0 < i \leq m. \quad (5)$$

Note that the equality here is fulfilled if and only if every $(i-1)$-face of $P$ is incident to exactly $(d-i+1)$ $i$-faces, i.e. $P$ is $(d-i)$-simple.

Let $d$ be even, $d = 2n$, $n \in \mathbb{N}$. Let $i = n$. From inequality (5), we get

$$(n+1)f_{n-1}(P) \leq (n+1)f_n(P). \quad (6)$$

Suppose that $m > n$. Thus, substituting $i \in \{n-1, n+1\}$ in (5), we obtain

$$(n+2)f_{n-2}(P) \leq nf_{n-1}(P) \quad \text{and} \quad nf_{n}(P) \leq (n+2)f_{n+1}(P).$$

Combining this with (6), we have

$$f_{n-2}(P) \leq f_{n+1}(P).$$

By repeating this procedure, it is easy to get

$$f_{d-m-1}(P) \leq f_m(P) \quad \text{for even } d.$$ Moreover, the equality is attained only on $m$-simple polytopes.

The case $d = 2n+1$, $n \in \mathbb{N}$, are proved by analogy. (By using the duality and the fact that a $d$-simplex has the minimal number of faces among all $d$-polytopes [9, p. 36, Ex. 8].)

**Theorem 2.** Let $P$ be a $k$-neighborly $d$-polytope and $2k < d \leq 3k - 1$. Then $f_{d-1}(P) \geq f_0(P)$ and the equality is attained only if $P$ is a simplex.

**Proof.** Recall that every $i$-face of a $k$-neighborly $d$-polytope $P$ is a simplex for $i < 2k$ [9 Sec. 7.1]. Using Lemma 1 we get

$$f_{d-k}(P) \geq f_{k-1}(P). \quad (7)$$

Note also that for a $k$-neighborly $d$-polytope $P$ we can use the implication

$$f_{k-1}(P) \leq f_{d-k}(P) \Rightarrow f_0(P) \leq f_{d-1}(P). \quad (8)$$

Indeed, $f_{k-1}(P) = \binom{f_0(P)}{k}$ for a $k$-neighborly polytope $P$. But $f_{d-k}(P) \leq \binom{f_{d-1}(P)}{k}$ for any $d$-polytope $P$ and the equality is attained only if $P$ is dual $k$-neighborly. Finally note that

$$\binom{f_0(P)}{k} \leq \binom{f_{d-1}(P)}{k} \Rightarrow f_0(P) \leq f_{d-1}(P).$$

Combining (7) and (8), we obtain

$$f_0(P) \leq f_{d-1}(P) \quad \text{for } d \leq 3k - 1.$$ Moreover, the equality is attained only if $P$ is dual $k$-neighborly. Hence, it is $(2k-1)$-simplicial and $(2k-1)$-simple, and must be a simplex, since $2k-1 + 2k-1 > d$ [9 Exercise 4.8.11].
3.1. Dimension 5.

**Lemma 3.** Let $P$ be a 2-neighborly 5-polytope. If $v \in \text{vert } P$ and $f_P(v)$ is the number of facets incident with $v$, then $f_P(v) \geq |\text{vert } P| - 1$. The equality $f_P(v) = |\text{vert } P| - 1$ is valid for all $v \in \text{vert } P$ if and only if $P$ is 2-simple.

**Proof.** Let $P$ be a 2-neighborly 5-polytope and $v \in \text{vert } P$. Let $Q$ be a vertex figure of $P$ at $v$, i.e. $Q$ is the intersection of $P$ by a hyperplane which strictly separates $v$ from $\text{vert } P \setminus \{v\}$. Thus, $Q$ is a 4-polytope and $f_P(v) = f_3(Q)$. On the other hand, $f_0(Q) = |\text{vert } P| - 1$, since $P$ is 2-neighborly.

All 3-faces of $P$ are simplices. Hence, all 2-faces of $Q$ are triangles. Note that every 1-face of a 4-polytope is incident with at least three 2-faces, and if there are exactly three 2-faces for every 1-face, then the polytope is 2-simple. By counting incidences between 1-faces and 2-faces of $Q$, we have

$$f_2(Q) \cdot 3 \geq f_1(Q) \cdot 3,$$

and the equality is attained only if $Q$ is 2-simple. From Euler’s equation \cite[Sec. 8.1]{9}, we get

$$f_3(Q) - f_0(Q) = f_2(Q) - f_1(Q) \geq 0.$$

□

Now we want to obtain some lower bound for $f_4(P)$. By counting incidences between vertices and facets of a 2-neighborly 5-polytope $P$ and using Lemma 3 we get:

$$\sum_{p \in F_4(P)} f_0(p) \geq f_0(P)(f_0(P) - 1).$$

Let $\hat{x}$ be the average number of vertices in a facet of $P$:

$$\hat{x} = \frac{1}{f_4(P)} \sum_{p \in F_4(P)} f_0(p).$$

Hence,

$$f_4(P) \geq \frac{f_0(P)(f_0(P) - 1)}{\hat{x}}. \tag{9}$$

On the other hand, the number of facets of $P$ must be greater than the number of facets of a facet of $P$. Since every facet of $P$ is a 2-neighborly 4-polytope,

$$f_4(P) \geq \frac{\lceil \hat{x} \rceil (\lceil \hat{x} \rceil - 3)}{2} + 1.$$

Hence,

$$f_4(P) \geq \max \left\{ \frac{f_0(P)(f_0(P) - 1)}{\hat{x}}, \frac{\lceil \hat{x} \rceil (\lceil \hat{x} \rceil - 3)}{2} + 1 \right\}.$$

Therefore, for large $f_0$ we have

$$f_4 = \Omega(f_0^{4/3}).$$

Another question raised by Lemma 3 is as follows. Is there a 2-neighborly 2-simple 5-polytope other than a simplex? Obviously, every vertex figure of such polytope is a 2-simplicial 2-simple 4-polytope. Recently, Brinkmann and Ziegler \cite{6} established that there are only seven 2-s2s 4-polytopes with at most 12 vertices. By using this list we can prove the following.

**Theorem 4.** If $P$ is a 2-neighborly 2-simple 5-polytope other than a simplex, then $f_0(P) \geq 14$. 
Proof. Let $Q$ be a vertex figure of $P$ at $v$, $v \in \text{vert } P$. Consequently, $Q$ is a 2-simple 2-simplicial 4-polytope with $f_0(Q) = f_0(P) - 1$. Below we use the list of all such polytopes with $f_0(Q) \leq 12$ from [6, Theorem 2.1].

If $f_0(Q) = 9$, then $f_1(Q) = 26$. Hence, every vertex in $P$ is incident to 26 2-faces of $P$. On the other hand, every 2-face of $P$ is a triangle. Therefore, $26f_0(P)$ must be equal to $3f_2(P)$. But $f_0(P) = 10$ and $f_2(P)$ is not integer.

If $f_0(Q) = 10$, then $f_2(Q) = 30$. Hence, every vertex in $P$ is incident to 30 3-faces of $P$. On the other hand, every 3-face of $P$ is a simplex. Therefore, $30f_0(P)$ must be equal to $4f_3(P)$. But $f_0(P) = 11$ and $f_3(P)$ is not integer.

If $f_0(Q) = 11$, then $Q$ has one facet with 7 vertices. Hence, every vertex in $P$ is incident to one facet with 8 vertices. But $P$ has 12 vertices.

If $f_0(Q) = 12$, then $f_2(Q) = 39$. Hence, every vertex in $P$ is incident to 39 3-faces of $P$. On the other hand, every 3-face of $P$ is a simplex. Therefore, $39f_0(P)$ must be equal to $4f_3(P)$. But $f_0(P) = 13$ and $f_3(P)$ is not integer. \hfill $\square$

3.2. Dimension 6. In [11], it was showed that $f_5(P) \geq f_0(P)$ for a 2-neighborly 6-polytope. Now we can update this result.

**Theorem 5.** Let $P$ be a 2-neighborly 6-polytope. Then $f_5(P) \geq f_0(P)$. The equality $f_5(P) = f_0(P)$ holds if and only if $P$ is a simplex or $P$ is a dual 2-neighborly polytope with $f_0(P) \geq 27$.

**Proof.** Let $f_i := f_i(P)$. From Euler’s equation [9], we get

$$f_0 - f_1 + f_4 - f_5 = f_3 - f_2.$$ 

Using Lemma [1], we have $f_3 - f_2 \geq 0$ and $f_1 - f_0 \leq f_4 - f_5$. Note also, that $f_1 = f_0(f_0 - 1)/2$ and $f_4 \leq f_5(f_5 - 1)/2$, and the equality is attained only for dual 2-neighborly polytopes. Therefore, $f_0(f_0 - 3)/2 \leq f_5(f_5 - 3)/2$ and $f_0 \leq f_5$.

Now, let 2-neighborly 6-polytope $P$ be dual 2-neighborly and $P$ is not a simplex. Let $T$ be a facet of $P$. Then $T$ is a 2-neighborly 2-simple 5-polytope other than a simplex. From Theorem [4], we know that every vertex figure of $T$ at $v \in \text{vert } T$ is a 2-simple 2-simplicial 4-polytope $Q = Q(v)$ with $f_3(Q) = f_0(Q) = f_0(T) - 1 \geq 13$. Every facet $q$ of $Q$ is a simplicial 3-polytope. Hence, $f_2(q) = 2f_0(q) - 4$ and

$$2f_2(Q) = \sum_{q \in F_3(Q)} f_2(q) = \sum_{q \in F_3(Q)} (2f_0(q) - 4) = 2f_3(Q)(y - 2),$$

where

$$y = y(v) = \frac{1}{f_3(Q)} \sum_{q \in F_3(Q)} f_0(q)$$

is the average number of vertices in a facet of $Q = Q(v)$. On the other hand [3, Theorem 2(prop. 3)],

$$f_2(Q) \leq f_3(Q)(f_3(Q) + 3)/4.$$ 

Therefore,

$$y(v) \leq \frac{f_3(Q) + 3}{4} + 2 = \frac{f_0(Q) + 3}{4} + 2 = \frac{f_0(T) + 2}{4} + 2.$$ 

Let $\hat{x}$ be the average number of vertices in a facet of the 5-polytope $T$:

$$\hat{x} = \frac{1}{f_4(T)} \sum_{t \in F_4(T)} f_0(t).$$
and
\[ \hat{y} = \frac{1}{f_0(T)} \sum_{v \in F_0(T)} y(v) \leq \frac{f_0(T) + 2}{4} + 2. \]

Now we are going to show that
\[ \hat{x} \leq \hat{y} + 1 \]
and then use inequality (9).

Let \( F_4(v) \) be the set of facets of \( T \) incident to the vertex \( v \):
\[ F_4(v) = \{ t \in F_4(T) \mid v \in t \}. \]

Note, that
\[ \sum_{v \in F_0(T)} \sum_{t \in F_4(v)} f_0(t) = \sum_{v \in F_0(T)} (f_0(q) + 1) = \sum_{v \in F_0(T)} \langle F_4(v) | (y(v) + 1) \rangle \]
and \( |F_4(v)| = f_0(T) - 1 \). Let \( n = f_4(T) \) and \( x_i \) be the number of vertices in \( i \)-th facet of \( T \), \( i \in [n] \). Thus,
\[ \sum_{v \in F_0(T)} \sum_{t \in F_4(v)} f_0(t) = \sum_{i=1}^{n} x_i^2 \]
and
\[ \sum_{v \in F_0(T)} |F_4(v)| (y(v) + 1) = (\hat{y} + 1) \sum_{i=1}^{n} x_i. \]

Since
\[ \left( \sum_{i=1}^{n} \frac{x_i}{n} \right)^2 \leq \sum_{i=1}^{n} \frac{x_i^2}{n}, \]
we get
\[ \sum_{i=1}^{n} \frac{x_i}{n} \leq \hat{y} + 1. \]

But \( \hat{x} = \sum_{i=1}^{n} \frac{x_i}{n} \) and \( \hat{y} \leq \frac{f_0(T) + 2}{4} + 2 \). Consequently,
\[ \hat{x} \leq \frac{f_0(T) + 2}{4} + 3. \]

From inequality (9), we have
\[ f_4(T) \geq \frac{4f_0(f_0 - 1)}{f_0 + 14}. \]

Thus, \( f_4(T) \geq 26 \) for \( f_0 \geq 14 \). Hence, \( f_5(P) \geq 27. \)

4. SIMPLICIAL POLYTOPES AND POLYTOPES WITH SMALL NUMBER OF VERTICES

The tight lower bound for the number of \( j \)-faces of a simplicial \( k \)-neighborly \( d \)-polytope can be found by using \( g \)-theorem \[4, 17\].

Let \( M_d = (m_{i,j}) \) be \((\lceil d/2 \rceil + 1) \times (d + 1)\)-matrix with entries
\[ m_{i,j} = \binom{d + 1 - i}{d + 1 - j} - \binom{i}{d + 1 - j}, \quad 0 \leq i \leq \lceil d/2 \rceil, \quad 0 \leq j \leq d. \]

The matrix \( M_d \) has nonnegative entries and the left \((\lceil d/2 \rceil + 1) \times (\lceil d/2 \rceil + 1)\)-submatrix is upper triangular with ones on the main diagonal. The \( g \)-theorem states that the \( f \)-vector \( f = (f_{-1}, f_0, f_1, \ldots, f_{d-1}) \) of a simplicial \( d \)-polytope is equal to \( gM_d \) where \( g \)-vector \( g = (g_0, g_1, \ldots, g_{\lceil d/2 \rceil}) \) is an \( M \)-sequence (see, e.g., \[18, Sec. 8.6\]).
Lemma 7. Let \( y \), \( z \), and \( y \) be three different vertices of \( P \). If \( y \), \( z \), and \( y \) are separated from the other vertices of \( P \), then \( \text{conv}\{x, y, z\} \) is a 2-face of \( P \).

Proof. Since \( x \), \( y \), and \( z \) are separated from the other vertices of \( P \), then
\[
\text{conv}\{x, y, z\} \cap \text{conv}(\text{vert} P \setminus \{x, y, z\}) = \emptyset.
\]
Let \( A \) be an affine hull of \( \{x, y, z\} \). Thus, \( \dim A = 2 \) and \( A \cap P = \text{conv}\{x, y, z\} \), since every pair of vertices of \( P \) form a 1-face of \( P \). Therefore, \( \text{conv}\{x, y, z\} \) is a face of \( P \). \( \square \)

Theorem 6. If \( P \) is a simplicial \( k \)-neighborly \( d \)-polytope \((k \leq d/2)\) with \( n \) vertices and \( n \geq d + 2 \), then
\[
f_j(P) \geq \sum_{i=0}^k \left( \binom{d + 1 - i}{d - j} - \binom{i}{d - j} \right) \binom{n - d - 2 + i}{i}.
\]
In particular,
\[
f_{d-1}(P) \geq \sum_{i=0}^k (d + 1 - 2i) \binom{n - d - 2 + i}{i}.
\]

Proof. From the equality \( f = gM_d \), we can evaluate the first \( k + 1 \) entries of \( g \):
\[
\begin{align*}
g_0 &= f_{-1}(P) = 1, \\
g_0m_{0,1} + g_1 &= f_0(P) = n, \\
g_0m_{0,2} + g_1m_{1,2} + g_2 &= f_1(P) = \binom{n}{2}, \\
&\quad \quad \quad \vdots \\
g_0m_{0,k} + g_1m_{1,k} + \cdots + g_k &= f_{k-1}(P) = \binom{n}{k}.
\end{align*}
\]

We suppose that \( k \leq d/2 \) and \( n \geq d + 2 \). By using induction on \( j \), we prove
\[
g_j = \binom{n - d - 2 + j}{j}, \quad 0 \leq j \leq k.
\]

Obviously, \( g_0 = 1 \) and \( g_1 = n - d - 1 \). Suppose that the equality (11) is true for \( j = l, 1 \leq l \leq k - 1 \). From (10), we have
\[
\sum_{i=0}^{l} \binom{n - d - 2 + i}{i} \binom{d + 1 - i}{d - l} + g_{l+1} = \binom{n}{l + 1}.
\]
Recall one of the formulation of Vandermonde’s convolution:
\[
\sum_i \binom{\alpha + i}{i} \binom{\beta - i}{\gamma - i} = \binom{\alpha + \beta + 1}{\gamma}.
\]
Hence, \( g_{l+1} = \binom{n - d - 1 + l}{l + 1} \).

By \( g \)-theorem,
\[
f_j = \sum_{i=0}^{[d/2]} g_{i}m_{i,j+1}.
\]
Since \( g \) is an \( M \)-sequence, we may assume \( g_i = 0 \) for \( i > k \). Hence,
\[
f_j \geq \sum_{i=0}^{k} g_{i}m_{i,j+1} = \sum_{i=0}^{k} \binom{n - d - 2 + i}{i} m_{i,j+1}.
\]

As a consequence of the theorem, we obtain (3). Note, that the right-hand side of (3) increases monotonically with respect to both \( d \) and \( \Delta \).

Below we need the following lemma.

Lemma 7. Let \( P \) be a \( 2 \)-neighborly \( d \)-polytope and \( x, y, z \) be three different vertices of \( P \). If \( x \), \( y \), and \( z \) are separated from the other vertices of \( P \), then \( \text{conv}\{x, y, z\} \) is a 2-face of \( P \).

Proof. Since \( x \), \( y \), and \( z \) are separated from the other vertices of \( P \), then
\[
\text{conv}\{x, y, z\} \cap \text{conv}(\text{vert} P \setminus \{x, y, z\}) = \emptyset.
\]
Let \( A \) be an affine hull of \( \{x, y, z\} \). Thus, \( \dim A = 2 \) and \( A \cap P = \text{conv}\{x, y, z\} \), since every pair of vertices of \( P \) form a 1-face of \( P \). Therefore, \( \text{conv}\{x, y, z\} \) is a face of \( P \). \( \square \)
Theorem 8. \( \mu_{2n}(d, v) \geq d + 7 \) for \( v \geq d + 4 \).

Proof. The proof is by induction over \( d \). For \( d = 4 \), the validity of the theorem immediately follows from (2).

Suppose that the theorem statement is true for \( d = m, m \geq 4 \). Let \( P \) be a 2-neighborly \((m + 1)\)-polytope with \( f_0(P) \geq m + 5 \). If \( P \) is simplicial, then the statement follows from (3).

Now suppose that \( P \) has a nonsimplicial facet \( Q \). Hence \( Q \) is a 2-neighborly \( m\)-polytope with \( f_0(Q) \geq m + 2 \). By the induction hypothesis, if \( f_0(Q) \geq m + 4 \), then \( f_{m-1}(Q) \geq m + 7 \). Therefore,

\[
f_m(P) \geq f_{m-1}(Q) + 1 \geq m + 8.
\]

The same is true for \( f_0(Q) = m + 3 \), since \( f_{m-1}(Q) \geq m + 7 \) in this case [14] (see Table 1).

Now it remains to verify the case, when \( f_0(Q) = m + 2 \) and \( P \) has no facets with \( \geq m + 3 \) vertices. Thus, \( f_{m-1}(Q) \geq m + 5 \) (see Table 1). Moreover, there exist \( x, y, z \in \text{vert} P \setminus \text{vert} Q \) that are separated from the other vertices of \( P \). By Lemma 7, \( f = \text{conv}\{x, y, z\} \) is a 2-face of \( P \). Hence, there are at least \( m - 1 \) facets of \( P \) that are incident to \( f \). Every such a facet has \( \leq m + 2 \) vertices, and three of them are \( x, y, z \). Consequently, it does not have a common ridge \(((m - 1)\)-face) with \( Q \). Therefore,

\[
f_m(P) \geq f_{m-1}(Q) + 1 + m - 1 \geq 2m + 5.
\]

\(\Box\)

Note, that by using \( P_{6,10} \) (see fig. 1d) as the basis of an \( m\)-fold pyramid, one can deduce \( \mu_{2n}(d, d + 4) \leq d + 8 \) for \( d \geq 6 \). Therefore, \( d + 7 \leq \mu_{2n}(d, d + 4) \leq d + 8 \) for \( d \geq 6 \).

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References

[1] O. Aichholzer. Extremal properties of 0/1-polytopes of dimension 5. In Polytopes – Combinatorics and Computation. Birkhauser, 2000, 111–130.
[2] D. Barnette. The minimum number of vertices of a simple polytope. Israel J. Math. 10, 1971, 121–125.
[3] M. Bayer. The extended f-vectors of 4-polytopes. J. Combin. Theory Ser. A, 44(1), 1987, 141–151.
[4] L.J. Billera and C.W. Lee. A proof of the sufficiency of McMullen’s conditions for f-vectors of simplicial polytopes. J. Combin. Theory Ser. A, 31(3), 1981, 237–255.
[5] G. Blind and R. Blind. Convex polytopes without triangular faces. Israel J. Math. 71, 1990, 129–134.
[6] P. Brinkmann, G.M. Ziegler. A flag vector of a 3-sphere that is not the flag vector of a 4-polytope. Mathematika, 63(1), 2017, 260–271.
[7] K. Fukuda, H. Miyata, S.Moriyama. Complete enumeration of small realizable oriented matroids. Discrete & Computational Geometry, 49(2), 2013, 359–381.
[8] R. Gillmann. 0/1-Polytopes: Typical and Extremal Properties. PhD Thesis, TU Berlin, 2006.
[9] B. Grünbaum. Convex polytopes, 2nd edition (V. Kabel, V. Klee and G.M. Ziegler, eds.), Springer, 2003.
[10] M. Henk, J. Richter-Gebert and G. Ziegler. Basic properties of convex polytopes. In J.E. Goodman and J. O’Rourke, editors, Handbook of Discrete and Computational Geometry. Chapman & Hall/CRC Press, Boca Raton, 2nd edition, 2004, 355–382.
[11] A.N. Maksimenko, On the number of facets of a 2-neighborly polytope, Model. Anal. Inform. Sist., 17(1) (2010), 76–82 (in Russian).
[12] A. Maksimenko. k-Neighborly Faces of the Boolean Quadratic Polytopes. Journal of Mathematical Sciences. 203(6), 2014, 816–822.
[13] A.N. Maksimenko, A special role of Boolean quadratic polytopes among other combinatorial polytopes, Model. Anal. Inform. Sist., 23(1) (2016), 23–40.
[14] A.N. Maksimenko. The lower bound for the number of facets of a k-neighborly d-polytope with d + 3 vertices. arxiv:1509.00362.
[15] P. McMullen. The maximum numbers of faces of a convex polytope. Mathematika, 17, 1970, 179–184.
[16] A. Padrol, *Many neighborly polytopes and oriented matroids*, Discrete & Computational Geometry, 50(4) (2013), 865–902.

[17] R.P. Stanley, *The number of faces of simplicial convex polytopes*. Advances in Math., 35(3), 1980, 236–238.

[18] G.M. Ziegler. *Lectures on Polytopes*, Springer, 1995.

Laboratory of Discrete and Computational Geometry, P.G. Demidov Yaroslavl State University, ul. Sovetskaya 14, Yaroslavl 150000, Russia

E-mail address: maximenko.a.n@gmail.com