Continuous-time State & Dynamics Estimation using a Pseudo-Spectral Parameterization

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Abstract—We present a novel continuous time trajectory representation based on a Chebyshev polynomial basis, which when governed by known dynamics models, allows for full trajectory and robot dynamics estimation, particularly useful for high-performance robotics applications such as unmanned aerial vehicles. We show that we can gracefully incorporate model dynamics to our trajectory representation, within a factor-graph based framework, and leverage ideas from pseudo-spectral optimal control to parameterize the state and the control trajectories as interpolating polynomials. This allows us to perform efficient optimization at specifically chosen points derived from the theory, while recovering full trajectory estimates. Through simulated experiments we demonstrate the applicability of our representation for accurate flight dynamics estimation for multirotor aerial vehicles. The representation framework is general and can thus be applied to a multitude of high-performance applications beyond multirotor platforms.

I. INTRODUCTION

High-performance autonomous robotic platforms are increasingly important in a variety of complex and dangerous tasks which normally require expert human piloting or intervention. Tasks such as exploration and reconnaissance, inspection, precision agriculture, search and rescue, and autonomous camera platforms are examples where autonomy is being successfully leveraged, with more applications being unlocked as the state of the art in robotics improves.

State estimation is essential for robotic applications\(^1\),\(^2\), but equally important is estimating the dynamics and control which induces the robot’s trajectory. In real-world scenarios, robots have to deal with highly dynamic conditions with noisy sensor information\(^3\). Under such conditions, constraining the state estimation with known robot dynamics allows for more robust state estimation. Conversely, the state estimates at any point on the trajectory can be used as a means to acquire knowledge about the dynamics at that time instance, such as estimating unmeasured force inputs. Some examples of unmeasured forces include rotor and actuator forces/torques, forces due to adversarial conditions (e.g. strong winds), and contact forces with the ground or objects. An additional application is performing system identification to better estimate intrinsic and inertial parameters to allow for more precise robot control\(^4\).

While state estimation has seen considerable research interest, optimizing for the control estimates has not been tackled before, to the best of our knowledge. Research on using robot kinematics to constrain state estimation has seen much interest in recent years. Hartley et al.\(^5\) and Wisth et al.\(^6\) leverage the use of the forward kinematics from encoder measurements to estimate contact with the ground, for biped and quadrupeds robots respectively. Nisar et al.\(^7\) do model the dynamics of the robot in the optimization process via preintegration, however, they only consider linear forces measured from an IMU and do not estimate the actual control inputs. Recent work has attempted to model the dynamical model parameters\(^8\),\(^9\) directly from sensor information using learning-based approaches. However, these tend to be brittle, poorly understood, and data and compute intensive, making general applicability difficult.

In this paper, we propose a novel mathematical framework for estimating both the state and control of a robot body, leveraging a pseudo-spectral parameterization based on Chebyshev polynomials. Our approach follows from the pseudo-spectral optimal control literature\(^10\),\(^11\) and utilizes a set of sparse collocation points along the trajectory within a factor graph based framework which allows for efficient optimization and inference, while providing significant accuracy and extensibility. The proposed framework only assumes knowledge of the robot’s dynamics model, and we demonstrate the applicability of our approach on a quadrotor model to estimate both the states over a trajectory and also the rotor speeds and forces, using only measurement information from a monocular camera. In contrast to prior work, we do not make use of other sensors (such as IMUs) in the estimation pipeline. While we illustrate results on a quadrotor platform, our framework can be generally applied to a variety of robot models, as well as tackle various problems such as wind estimation, contact force estimation, system identification, etc.

II. PRELIMINARIES

A. Chebyshev Polynomials and Chebyshev Points

In this section we briefly review Chebyshev polynomials and their use in approximating continuous functions. We largely follow the exposition in\(^12\), though the reader might also be interested in the hands-on development in\(^13\).

The Chebyshev series is a spectral decomposition composed of an orthogonal basis defined on a unit circle, analogous to the Fourier series. It is defined for functions \(f(\tau)\) on the interval \([-1, 1]\), but can be easily scaled to arbitrary bounds. Concretely, the Chebyshev series decomposition is defined as:

\[
f(\tau) = \sum_{k=0}^{\infty} a_k T_k(\tau) \quad \text{with} \quad a_k = \frac{2}{\pi} \int_{-1}^{1} \frac{f(\tau)T_k(\tau)}{\sqrt{1 - \tau^2}} \, d\tau \quad (1)
\]
with the factor $2/\pi$ changed to $1/\pi$ for $k = 0$.

Above, $T_k$ is the $k$th Chebyshev polynomial, defined as the projection of a cosine function to the midline $[-1, 1]$ of the unit circle, as shown in figure 1:

$$T_k(\tau) \triangleq \cos \left( k \arccos(\tau) \right), \quad -1 \leq \tau \leq 1 \quad (2)$$

Given an arbitrary real function $f$ on $[-1, 1]$, we can exploit the connection to the Fourier series by making use of the fast Fourier transform (FFT) to efficiently obtain the coefficients $a_k$ of the truncated Chebyshev series. To ensure convergence of the approximation and avoid issues such as Runge’s phenomenon [14], for a given degree of approximation $N$, we query $f$ at the following points:

$$\tau_j \triangleq \cos \left( j\pi/N \right), \quad 0 \leq j \leq N \quad (3)$$

The points defined by (3) are the Chebyshev-Gauss-Lobatto (CGL) points [10] or simply, Chebyshev points. Figure 2 shows that they arise simply from the projection of a regular grid on the unit circle.

The FFT of this function will only have $N$ non-zero values corresponding to cosines of increasing frequency, in one-to-one correspondence to the Chebyshev polynomials (2). Those non-zeroes are exactly the coefficients $a_k$, with $0 \leq k \leq N$.

B. Barycentric Interpolation

Given the samples of $f$ at the $N+1$ Chebyshev points $f_j$, we can evaluate any point in $f$ efficiently using the Barycentric Interpolation formula [14]. The general Lagrangian form is given by:

$$f(x) = \sum_{j=0}^{N} \frac{\lambda_j f_j}{x - x_j} \quad (4)$$

where

$$\lambda_j = \frac{1}{\prod_{k \neq j}(x_j - x_k)}$$

and $f(x) = f_j$ if $x = x_j$.

Computationally, the Chebyshev points provide an advantage over equidistant sampled points due to the simplicity of the resulting $\lambda$ values, giving us the following formula

$$f(x) = \sum_{j=0}^{N} (-1)^j f_j \prod_{j \neq k}^{N} \frac{(-1)^j}{x - x_j} \quad (5)$$

with the special case $f(x) = f_j$ if $x = x_j$, and the summation terms for $j = 0$ and $j = N$ being multiplied by $1/2$.

Since this is a linear operation, we can reparameterize (5) as an efficient inner product $f(x) = \mathbf{f \cdot w}$ where $\mathbf{f}$ is a vector of all the values of $f$ at the Chebyshev points, and $\mathbf{w}$ is an $(N + 1)$ vector of Barycentric weights.

C. Differentiation Matrix

Spectral collocation methods yield an efficient process for obtaining derivatives of the approximating polynomial via the differentiation matrix. Simply put, a polynomial of degree $N$ is determined by its values on the $(N+1)$ point grid, and its derivative, a polynomial of degree $N$, is determined by its values on the same grid.

Thus, the derivatives of the interpolating polynomial can be efficiently computed via a matrix-vector product. This is useful when performing optimization as we can compute the derivatives of arbitrary functions for (almost) free.

III. APPROACH

A. Problem Statement

We consider the following continuous-time optimization problem: determine the control function $u(t)$, and the corresponding state trajectory $x(t)$ at specific points, that jointly minimize a cost function of the form:

$$\sum_{i=1}^{m} \left\| z_i - h_i \left( \mathbf{x} \left( t_i \right), u \left( t_i \right), \Theta_i \right) \right\|^2_{R_i} \quad (6)$$

where $z_i$ is one of $m$ (vector-valued) measurements, $\mathbf{x} \left( t_i \right)$ and $u \left( t_i \right)$ are the $n$-dimensional vehicle state and the $p$-dimensional control function respectively at the corresponding measurement time $t_i$, $h_i$ is a nonlinear measurement model, and $\Theta_i$ is a set of parameters that $z_i$ also depends on, e.g., landmark coordinates, intrinsic parameters, etc.

Without loss of generality we assume Gaussian measurement noise, and denote the covariance matrix of the noise
on $z_i$ as $R_i$. However, the formulation is easily extended to use other noise models, e.g., robust error norms. Priors on both the known variables $\Theta$ and the unknown variables can be easily accommodated as well.

We further assume that the vehicle is subject to the following continuous dynamics model,

$$\dot{x}(t) = f(x(t), u(t)), \ t_0 \leq t \leq t_f,$$

with $t_0 \leq t_i \leq t_f$ for all discrete measurement times $t_i$. For the sake of simplicity, we omit (in-)equality constraints as are customarily stated in optimal control including possible boundary conditions, which are easily taken into account.

In this paper we propose the use of Chebyshev polynomials to parameterize the continuous state trajectory $x(t)$ and the continuous control input $u(t)$, following the lead from the optimal control literature\cite{10, 11], to render the variational optimization problem\cite{9} into a simple Non-Linear Programming (NLP) problem.

### B. Pseudo-spectral Parameterization

The pseudo-spectral parameterization consists of $N$ function values $f_j$ at the Chebyshev points $\tau_j$. This allows us to estimate an interpolating polynomial whose values at the Chebyshev points are exactly the function values, while being smooth and efficiently differentiable. Since we can easily switch between the function values $f_j$ and the series coefficients $a_k$ using the FFT, or indeed back again using the inverse FFT, the two representations are equivalent.

![Fig. 3. Least-squares fit of a degree 6 Chebyshev interpolant to 21 noisy samples of the function $\exp(2\sin(2\pi x) + \cos(2\pi x))$. The parameters we optimize for are the 7 values at the Chebyshev-Gauss-Lobatto points, indicated by the stem plots. Note they do not in general coincide with any of the samples.](image)

The idea is illustrated using a scalar example in Figure 3, where we fit a degree 6 polynomial to 21 noisy samples of a known function. The parameters are the 7 values of the sought function at the $6+1$ Chebyshev points. For every combination of these 7 values, there exists a unique polynomial interpolant $I(\tau)$, for $-1 \leq \tau \leq 1$. The optimal values are those that minimize the squared distance between $I(\tau)$ and the samples. We discuss below exactly how this minimization can be achieved in the general optimization setting.

### C. Approximating Trajectories and Control Functions

We borrow the main idea from pseudo-spectral optimal control by parameterizing the unknown, continuous state and control functions using their values at the Chebyshev points, i.e., using a pseudo-spectral parameterization.

Arbitrary time intervals $[t_0, t_f]$ can be accommodated using the affine transformation\cite{10}

$$t = [(t_f + t_0) + (t_f - t_0)\tau]/2.$$  

With this transformation, we define the pseudo-spectral state parameterization as the $N$-dimensional vectors $x_s$, one for each of the $m$ state variables $x_i(t)$. These represent the state values at the transformed Chebyshev points

$$t_j \triangleq \frac{t_f + t_0}{2} + \frac{t_f - t_0}{2} \cos (j\pi/N), \ 0 \leq j \leq N. \quad (7)$$

Similarly, we define the pseudo-spectral control parameterization as the $N$-dimensional vectors for each of the $p$ control variables. We collect all the parameters in the $m \times (N + 1)$ matrix $X$ and the $p \times (N + 1)$ matrix $U$, which constitute our unknowns in what follows.

### D. Pseudo-spectral Optimization: Minimizing Measurement Error

The key idea is to express the desired objective function in terms of the pseudo-spectral parameterization $\{X, U\}$. To this end, we replace the optimal control performance index with the sum of measurement-derived least-squares terms, while keeping the idea of enforcing the dynamic constraints through collocation. The latter will be explained in detail in the next section, while here we focus on expressing the measurement function as a function of the parameters $X$.

The main mechanism we will use is the barycentric interpolation formula, a very efficient interpolation method to predict the state $x(t)$ from $X$ at any arbitrary time $t$.

For a given $t$, trajectory interpolation can be written as

$$x(t) = Xw(t) \quad (8)$$

with $w(t)$ being an $(N + 1)$-dimensional weight vector as defined through $\{X, U\}$. Substituting (8) in the objective function $E_1(X)$, and writing $w_i = w(t_i)$ we obtain

$$E_1(X) = \sum_{i=1}^{m} \|z_i - h_i(Xw_i)\|_R_i^2 \quad (9)$$

which we can minimize using non-linear programming.

Of particular note is the ease by which measurements on derivatives of the state can be accommodated. We have

$$\dot{x}(t) = D_N Xw(t) \quad (10)$$

where $D_N$ is an $(N + 1) \times (N + 1)$ differentiation matrix, defined in [13, p. 53]. The equation above can be substituted in the measurement equation in an analogous manner. Similar matrices can be defined for the second time derivative, etc.
E. Enforcing Dynamics through Direct Collocation

The final piece in the puzzle is enforcing the vehicle dynamics, which we do through direct collocation, a well-known technique in optimal control [15], [16]. This method is the practice of optimizing over both state and controls while enforcing the dynamic constraints at a set of “collocation points”, and is not limited to pseudospectral methods. As in Chebyshev-based PSOC methods, we create dynamic defect terms at the Chebyshev points.

To this end, we similarly write the continuous control function $u(t)$ as a linear function of the parameters $U$,

$$u(t) = Uw(t)$$

and substituting this, (8) and (10) into the dynamics equation (3A), we obtain

$$D_N Xw_j = f(Xw_j, Uw_j, t_j), \quad 0 \leq j \leq N$$

which is a hard, possibly nonlinear, equality constraint at each of the (transformed) Chebyshev points $t_j$, with $w_j = w(t_j)$. A stochastic version of the same would amount to

$$E_2(X, U) \triangleq \frac{1}{N} \sum_{j=0}^{N-1} \| D_N Xw_j - f(Xw_j, Uw_j, t_j) \|^2_Q$$

with $Q$ an $m \times m$ covariance matrix, which besides stochasticity can also account for model error, and where we assume a Gaussian noise model for simplicity.

F. Nonlinear Programming

The final objective function to be minimized is thus

$$E(X, U) \triangleq E_1(X) + E_2(X, U)$$

which is the sum of measurement error, interpolated at $T$ arbitrary time instants $t_i$, $1 \leq i \leq T$, and the dynamic defects at the $N + 1$ transformed Chebyshev points $t_j$, $0 \leq j \leq N$.

IV. APPLICATION TO QUADROTORS

Quadrotors are an important class of autonomous agents used in a variety of applications, such surveillance, monitoring, and transport due to their speed and versatility. This has lead to a surge in research in recent years on various areas involving quadrotors, such as state estimation, parameter estimation, and quadrotor control [1]–[4], [7].

However, much of this research has been focused on online estimation rather than post-hoc analysis for evaluating performance, failures, and robustness. In this paper, we focus on the post-hoc analysis of quadrotor trajectories in order to evaluate the continuous-time states and controls.

A. Quadrotor Dynamics

We follow the exposition in Beard [17] and Altug et al. [18] to describe the quadrotor dynamics. We assume an X-configuration quadrotor as is standard in the literature, with the motors numbered counter-clockwise starting from the top-left when looking at the quadrotor from above. We denote the mass of the quadrotor as $m$, inertial tensor as $I$, and vector of motor speeds as $u$. We denote the world frame with $n$ (for navigation frame as is common in the aerospace literature) and the body-frame of the vehicle as $b$.

The Newton-Euler dynamics equations are given by

$$m\ddot{\mathbf{v}} = \mathbf{F}$$

$$\mathbf{L}_b = \tau^b + \omega^b \times \mathbf{L}_b$$

1) Force: The total force acting on the center of mass of the quadrotor is [18], [19]

$$\mathbf{F} = mg^n + k_b^u \sum k_f w_i^2 + \mathbf{f}_d$$

where the first term is due to the Earth’s gravitational force, $k_b^u$ is the body z-axis in the world frame, $w_i$ is the speed of the $i$th motor, $k_f$ is the thrust coefficient, and $\mathbf{f}_d = k_d \|v^n\| v^n$ is the force due to aerodynamic drag with $k_d$ being the drag constant assuming the density of air is constant.

2) Torque: We can represent the torque cross product as a matrix multiplication via a skew-symmetric matrix

$$\tau = \mathbf{L}_b \omega^b + \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \mathbf{L}_b \omega^b$$

$$\omega^b = \begin{bmatrix} p \\ q \\ r \end{bmatrix}, \mathbf{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

are the angular velocity and the inertial tensor respectively. Thus, we get the angular moments as,

$$\begin{bmatrix} I_{xx} \tau_x \\ I_{yy} \tau_y \\ I_{zz} \tau_z \end{bmatrix} = \begin{bmatrix} qr I_{zz} - qr I_{yy} \\ pr I_{xx} - pr I_{zz} \\ pq I_{yy} - pq I_{xx} \end{bmatrix}$$

where we defined $\delta_u = \text{vec}(\Sigma)$ as the $p(N+1)$-dimensional perturbation vector on the controls estimate $\hat{U}$. 

The total external torque (moments) applied in the body frame is given by,

\[ \tau^B = \tau^T - g_a + \tau_w \]

where \( \tau^T \) is the torque due to the motor speeds, \( g_a \) is the gyroscopic moments, and \( \tau_w \) is the torque due to drag. The gyroscopic moments are considered negligible [20].

Given the motor configuration for the quadrotor, and denoting the motor arm length as \( l \), we get \( \tau^T \) via a mixing matrix

\[
\begin{bmatrix}
\tau_x \\
\tau_y \\
\tau_z
\end{bmatrix} = \begin{bmatrix}
l & l & -l & -l \\
-l & l & l & -l \\
-C & C & -C & -C
\end{bmatrix} \begin{bmatrix}
f_1 \\
f_2 \\
f_3 \\
f_4
\end{bmatrix}
\]

(20)

Here, \( f_i \) is the thrust force of each motor, and \( C \) is the ratio of torque coefficient to thrust coefficient. The signs in the first 2 rows of the mixing matrix depend on the vehicle frame, and the signs in the third row depend on the direction of the motor rotation, with positive yaw direction being negative.

B. Time Derivatives

Given the dynamics equation (III-A), we can compute the Jacobian matrix \( J \) by computing the time derivative for each term of the 12 dimensional state vector.

1) Position: The time derivative of the position is simply the velocity, \( \dot{p} = v \).

2) Rotation: Our parameterization of rotation is based on the Lie Algebra so(3) to allow for optimization. By Euler’s Theorem, this implies that the time derivative of the rotation is the angular velocity in the body frame, \( \dot{R} = \omega \).

3) Velocity: The time derivative of the velocity is the normalized total force, \( \dot{v} = f/m \).

4) Angular Rate: For the angular rate, the time derivative is simply the inertial tensor normalized torque, \( \dot{\omega} = \tau/I \)

C. Experimental Setup

For collecting measurements and evaluating our proposed framework, we leverage the FlightGoggles simulator [21]. FlightGoggles provides us with both a photo-realistic simulation environment as well as easy querying of quadrotor dynamics in real-time for data collection with some minor modifications. We make use of the standard configuration for the simulated quadrotor and all the default parameters in the simulator.

Our proposed framework is based on the GTSAM factor graph optimization library [22]. To model the dynamics defects, we define a new type of factor which takes as input the state and control matrices \( \{X, U\} \) respectively. The measurement error is computed using the standard projection factors available in GTSAM. All our experiments use \( N = 128 \) for the polynomial degree. We leverage the provided machinery to compute all linear operations.

For measuring initial estimates of poses from images, we use a monocular visual odometry pipeline built upon GTSAM. We follow the standard procedure of performing feature matching and estimating the pose from the fundamental matrix. Note that we only use the image data for our experiments, however use of the IMUs can also be made to provide initial estimates for linear and angular velocities.

![Fig. 4. Example of estimated state trajectory using Chebyshev polynomials for a quadrotor platform. We can estimate the trajectory as well as the control inputs using information from only a single camera sensor.](image)

V. Results

We run \( R = 6 \) different runs of the simulator and estimate the state and control trajectories for each run. One of the estimated trajectories as well as the control values for a comparative analysis is shown in figure 5. While the exact resolutions are difficult to see, the general trends are clearly observable, demonstrating our framework’s capability.

The quantitative analysis of the estimated motorspeeds with respect to the ground truth provides more insight. We take the average error across the entire trajectory for each motor for each run of the simulator. The results are presented in table V with lower values being better.

| Motor 1     | Motor 2     | Motor 3     | Motor 4     |
|-------------|-------------|-------------|-------------|
| RPM | % err | RPM | % err | RPM | % err | RPM | % err |
| 1.7069 | 0.1441 | 1.9419 | 0.1641 | 1.9735 | 0.1667 | 1.6878 | 0.1425 |
| 4.1462 | 0.3616 | 4.3242 | 0.3772 | 4.3716 | 0.3814 | 4.1131 | 0.3588 |
| 4.6440 | 0.3995 | 4.5240 | 0.3892 | 4.5707 | 0.3932 | 4.5167 | 0.3885 |
| 6.3759 | 0.5961 | 5.8155 | 0.5072 | 6.2201 | 0.5424 | 5.8438 | 0.5096 |
| 7.0094 | 0.5919 | 7.3957 | 0.6240 | 7.0751 | 0.5977 | 7.3921 | 0.6240 |
| 6.9770 | 0.5989 | 7.0844 | 0.6081 | 6.8208 | 0.5861 | 6.9810 | 0.5998 |

TABLE I

AVERAGE ERROR BETWEEN TRUE AND ESTIMATED MOTORSPEEDS.

As can be seen from the quantitative results in table V our proposed framework is able to accurately estimate the control inputs with less than 1% relative error.

VI. Conclusion

In this paper, we have proposed a novel parameterization of the state and control trajectories based on Chebyshev polynomials in a pseudo-spectral optimization framework. Our approach is a general one, capable of being applied to various types of systems and robots without any major assumptions other than the dynamics model be known. Moreover, we are able to achieve the estimates using only a single monocular camera, allowing for applicability in a wide range of applications, while also allowing for further improvements by incorporating measurements from additional sensors such
as IMUs. Given the general nature of our approach, one line of future work would be to evaluate the use of our framework for other types of robots, such as mobile manipulators and soft robots. Another avenue of research would be to formulate our approach in incremental estimation settings, rather than the current global parameterization. This would allow our approach to be run in real-time and allow for online estimation of states and parameters, amongst other things. Finally, our framework could also be used in learning-based systems (e.g. imitation learning [23]) to allow for learning of complex dynamics which are hard to model manually.

Our hope is that this work leads to further generalization of robotic systems in real-world settings, as well as allow for easy and efficient estimation of robot parameters.

Fig. 5. Comparison of motorspeeds for the actuators for 2 simulation runs. The dotted lines represent the ground truth and the solid lines the estimates. Due to the global nature of the Chebyshev polynomials, we only see general trends rather than detailed resolution.
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