Algorithms for computer algebra calculations in spacetime

I. The calculation of curvature

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Abstract

We examine the relative performance of algorithms for the calculation of curvature in spacetime. The classical coordinate component method is compared to two distinct versions of the Newman-Penrose tetrad approach for a variety of spacetimes, and distinct coordinates and tetrads for a given spacetime. Within the system GRTensorII, we find that there is no single preferred approach on the basis of speed. Rather, we find that the fastest algorithm is the one that minimizes the amount of time spent on simplification. This means that arguments concerning the theoretical superiority of an algorithm need not translate into superior performance when applied to a specific spacetime calculation. In all cases it is the global simplification strategy which is of paramount importance. An appropriate simplification strategy can change an untractable problem into one which can be solved essentially instantaneously.

1 Introduction

It is fair to say that tetrad calculations are generally considered superior to classical coordinate methods for the calculation of curvature in spacetime. Experiments by Campbell and Wainwright now dating back many years showed that tetrad methods are faster than coordinate methods by factors of $2 \sim 4$. Even larger factors have been obtained by MacCallum in a Euclidean context. Within the well known system SHEEP, for example, the advice to the beginner is to always use frame versions of the metric (e.g. MacCallum and Skea p. 23). On the commercial side, within the system MACSYMA2 the demonstration CTENSOR4 begins with an explanation that “frame fields” (orthonormal bases) allow the computations to run much more quickly. The demonstration calculates the bases components of the Ricci tensor for the Kerr-Newman spacetime in Boyer-Lindquist coordinates, and is a good place to begin our discussion.

In Table 1 we have reproduced this demonstration within the system GRTensorII running under MapleV Release 3, and have included the calculation of the Weyl tensor. The Table demonstrates some interesting properties. The theoretical advantage of the frame approach is clearly demonstrated in the Boyer-Lindquist coordinates (Column BKN). However, under the elementary coordinate transformation $u = a \cos \theta$ this advantage fails to deliver superior performance (Column BKNU). The importance of strategic application of simplification at intermediate steps is illustrated in Column BKNS. For this test simplification of components has been carried out only after the components of the Ricci and Weyl tensors are calculated. It is worth noting that without some optimization in the simplification strategy (e.g. post-calculation simplification only) this calculation cannot be executed in MapleV on a 32 bit machine.

Clearly one could dismiss the findings in Table 1 if the implementation of the bases algorithms in GRTensorII were particularly inefficient compared to coordinate methods. We do not believe this to be the case.
Table 1: Average CPU time\(^1\) in seconds for the calculation and simplification of the bases components of the Ricci and Weyl tensors \((R_{(a)(b)} \text{ and } C_{(a)(b)(c)(d)})\) compared to the same for the coordinate components \((R_{ab} \text{ and } C_{abcd})\). BKN refers to the Kerr-Newman spacetime in Boyer-Lindquist coordinates, and BKNU the same but with the transformation \(u = a \cos \theta\). For both BKN and BKNU the simplification procedures have been structured for optimum performance\(^2\).

For the Column BKNS the tetrad components of BKN are used, but simplification procedures are not applied to intermediate calculation steps, only to the final results.

Rather, as we attempt to show in what follows, we believe that Table 1 reflects the fact that bases methods are not fundamentally superior to classical coordinate methods. We find that the most important criterion for speed is the minimization of the time spent on simplification. This underlines the importance of the choice of coordinates or tetrad in a computer algebra calculation and, more importantly, points out the fact that the user must be able to select the style of simplification which is most appropriate for the particular problem.

## 2 Protocol for comparisons

### 2.1 Choice of algorithms

Within the framework of tetrad methods, the formalism of Newman and Penrose\(^3\) has proven most useful for calculations in spacetime (see e.g. [9]). Some of the earliest applications of computer algebra to relativity stressed the efficiency of this formalism (e.g. [1]). McLenaghan [10] (see also Allen et al. [11]) has emphasized two distinct approaches within this formalism. These are distinguished as the methods of Cartan and Debever in [10], and as Methods A and B in [11], a notation which we adopt here. The methods are outlined in [10] and [11] with references and we do not repeat this material here.\(^4\) We simply note that Method A uses the definitions of Newman and Penrose explicitly, while Method B essentially uses definitions constructed so as to avoid inversion of coordinate indices.

In this paper we compare these two approaches to classical coordinate methods (suitably optimized).

### 2.2 Basis for comparison

The null tetrad formalism is sufficiently distinct from the classical coordinate approach that a basis for the comparison of the two methods is not clearly defined. In the classical approach the “curvature” of a spacetime

\(^1\)All times are in seconds as returned by the MapleV \texttt{status} function and are the average of four runs on a Sun Sparc 5 (see Section 2.4). The maximum deviation from the average is less than 5% for times exceeding 2 seconds and about 10% for shorter times.

\(^2\)We consider a worksheet (a sequence of calculation and simplification procedures) to be optimized when the execution time has reached a minimum.

\(^3\)Due to their length, the complete text of worksheets used for these tests have not been included in this report (except for an example in Appendix B), however they have been made publicly available [17].

\(^4\) The curvature component \(\Phi_{12}\) is consistently incorrect in [10] and [11]. In particular, the coefficients of \(\mu \tau\) and \(\nu \sigma\) are -1 and 1 respectively, not -2 and +2. This error is also present in the \texttt{Debever} package in MapleV Releases 2 and 3.
is usually considered evaluated when the coordinate components of the Ricci and Weyl tensors have been evaluated. In the Newman-Penrose (NP) formalism it is the tetrad components of these tensors (the $\Phi$s and $\Psi$s) that display the “curvature”. The complication that arises in a comparison of such different methods is the fact that the natural output of each method is distinct. Now given the coordinate components, and the null tetrad, the tetrad components follow in the usual way [9]. One could then form a basis for comparison by defining the “curvature” as the tetrad components of the Ricci and Weyl tensors. This is the comparison used in [9]. Naturally, this puts the classical component method at a disadvantage since the extra sums involved are not a natural part of the method. In this paper we have tried to cover all possibilities by having both NP approaches output the tetrad components of the Ricci and Weyl tensors, and the coordinate approach output both the coordinate and tetrad components of these tensors.

### 2.3 Choice of spacetimes

To compare the performance of algorithms for spacetime calculations it is essential that a variety of spacetimes be considered, and that within a given spacetime different tetrads (coordinates) be examined. Campbell and Wainwright [1] chose to examine the spacetimes of Griffiths [12], Lewis-Papapetrou [13], Bondi [14], and Debever [15]. More recently, the examination by Allen et al. [11] (which is a comparison of the NP approaches) included these spacetimes (with a more general form of the Debever metric, the Debever-McLenaghan-Tariq metric [15]) and also the plane wave, 2x2 decomposable, and static spherically symmetric spacetimes [16]. We have found these last three spacetimes to be too simple since the associated calculation times are too short to form a reliable basis for comparison. They also include one form of the Kerr-Newman metric and a general tetrad (their Case 9). Here we examine the Kerr-Newman metric in a variety of forms. We do not include the tetrad 9 given in [11] since it does not conform to the requirements of a null tetrad in the Newman-Penrose formalism.

### 2.4 Method of comparison

The comparisons were made by way of 26 re-executable Maple worksheets, and the calculations were performed with GRTensorII [5] under MapleV Release 3 [7] (in the X-Windows interface) with patchlevel 3 on a Sun SPARC5 running SunOS 4.1.4 and equipped with a 75MHz CPU and 64Mb of RAM.

The Maple worksheets and associated input files used in these tests are summarized in Appendix B. The associated tetrads, and line elements are shown in Appendix C. The reproduction of tetrads is prone to errors, and references [9], [10] and [11] all contain misprints in the tetrad components. Appendix C has been produced directly from the input files and so an error constitutes not a misprint, but an actual error in the input which would invalidate our conclusions in the case concerned. The worksheets are available for detailed examination and execution [17].

The worksheets are constructed in the following way. Either the contravariant or covariant tetrad is loaded into GRTensorII from an input file. The choice determines NP algorithm to be used for the tetrad part of the calculation. The tetrad components of the Ricci and Weyl tensors are then evaluated, usually in simplified form (see Section 3). The metric is then generated from the tetrad, and the covariant coordinate components of the Ricci and Weyl tensors are evaluated and simplified when necessary (again see Section 3). We have been careful to ensure that the metric, though generated from the tetrad, is presented in optimal form. Assuming that the tetrads are given, the worksheets then evaluate the tetrad components from the covariant coordinate components. These are then simplified to the exact form of the tetrad calculation.

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5 Campbell and Wainwright [1] also examined the Kerr-Newman metric. However, their description involves intermediate hand simplification in this case and does not include comparison to the component method. In this Section we take the position that a fair comparison of computer algorithm efficiency can only be carried out if no hand simplification of tensor components is permitted.

6 This configuration was chosen for its reliability and reproducibility of CPU times, not its speed. By way of a comparison, a Pentium 133 is about 1.5 times faster, however the DOS/Windows implementation of MapleV Release 3 reports only integer CPU times.

7 A tensor component is considered to be fully simplified when its size, as measured in Maple ‘words’, reaches a minimum.
We believe that each worksheet has been fully optimized\(^2\). That is, the procedures (e.g. precalculation of the spin coefficients) and simplification procedures (type and order of simplification) at each step of the calculation have been constructed so as to present each approach in its best performance mode. It should be pointed out that partial optimization of a worksheet is straightforward (see Appendix\(^3\)), but the full optimization of a worksheet is a somewhat involved task. Full optimization is necessary if a real comparison of approaches is to be given.

A sample worksheet is given in Appendix\(^4\) along with the associated input file. In GRTensorII the input files contain no information beyond the components of the basis or metric. In particular, no simplification information is contained in the input file. The simplification strategy used is read from the worksheet.

### 3 Comparisons

Summarized in Table 2 are the results of our tests\(^1\) and the results available from the previous tests in [1] and [11]. Before we discuss the comparisons it is appropriate to emphasize the importance of simplification procedures. The fastest calculation in the Kerr-Newman metric is the covariant NP tetrad calculation (Method B) in Row 9. If even only part of the simplification strategy is altered (e.g. the background simplification procedure used before the components are more fully simplified) the execution time can increase by a factor well over two orders of magnitude. The global simplification strategy is of paramount importance.

A number of interesting points emerge from Table 2.

**i)** It is appropriate to begin by comparing our results with previously published tests. Starting with the work of Campbell and Wainwright [1], although the exact form of their output is unavailable, we observe a notable agreement for the ratio \(A/C\) and their results for the Lewis-Papapetrou and Bondi metrics as shown in Table 2. For the Griffiths metric our component times are somewhat faster\(^5\). Quite naturally, a concern at the time was the storage requirements for the calculations. They report storage requirements for the component method a factor 2 \(\sim\) 5 times that of the tetrad approach. Whereas storage is no longer the concern that it once was, we note that we have observed storage requirements for the component calculations only about 1.1 \(\sim\) 1.5 times that of the tetrad method.

The paper by Allen et al. [11] is concerned with the ratio \(A/D\), that is, the relative performance of the two null tetrad methods. The exact form of their output is unavailable. In general we find that the performance of the Debever approach (Method B) is overestimated in [11]. Although the central thesis of [11] is the superiority of the Debever approach, this rests principally on their analysis of the Kerr-Newman metric. It is clear from Table 2 that this superior performance is tetrad (coordinate) dependent.

**ii)** For metrics of a general type (Rows 3 through 8 in Table 2) we find that the standard NP approach is faster than the alternative proposed in [10] and [11]. We find that this superior performance is not uniformly maintained if the general functions are replaced by specific ones. We have used the Kerr-Newman metric as an example, and as can be seen from Rows 9 through 14 of Table 2, the relative performance of the two tetrad approaches is highly dependent on the tetrads (coordinates). However, whereas the Debever approach can significantly outpace the NP approach (Rows 9 and 12), it is never far behind.

**iii)** For metrics of a general type (Rows 3 through 8 in Table 2) both tetrad approaches are superior to a calculation of the tetrad components from the coordinate components. This is exactly as one would expect. The extra sums involved slow the coordinate approach down (compare Columns \(B\) and \(C\) as well as \(E\) and \(F\)). Again, however, this superior performance is not uniformly maintained if the general functions are replaced by specific ones. Columns \(A/C\) and \(D/F\) for Rows 9 through 14 show that the classical approach can rival the tetrad approaches even for a calculation of the tetrad components.

**iv)** It could be argued that the natural output of the coordinate approach, as regards the calculation of curvature, is simply the coordinate components of the Ricci and Weyl tensors. For metrics of a general type the standard NP approach retains its superiority over the coordinate calculation (Column \(A/B\)). This is not

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\(^{8}\)The fourth component of \(m^a\) for this metric given in [1] is wrong. We believe this to be a misprint which would not alter the time reported.

\(^{9}\)The errors in \(\Phi_{12}\) within the MapleV \texttt{debever} package affects only their result for the Debever-McLenaghan-Tariq metric (See footnote 4).
| Column | A | B | C | D | E | F | A/D | A/B | A/C | D/E | D/F |
|-------|---|---|---|---|---|---|-----|-----|-----|-----|-----|
| 1. Coord’s | A | B | C | D | E | F | A/D | A/B | A/C | D/E | D/F |
| 2. Griff | 2.0 | 1.9 | 2.9 | 1.6 | 2.3 | 3.5 | 1.25 | 1.65 | 1.95 | 0.69 | 0.39 | 0.70 | 0.46 |
| 3. L-P | 1.9 | 2.8 | 4.9 | 2.6 | 2.9 | 6.1 | 0.73 | 1.44 | 0.68 | 0.39 | 0.40 | 0.90 | 0.43 |
| 4. Bondi1 | 3.5 | 6.2 | 10.5 | 4.4 | 6.2 | 10.4 | 0.80 | 0.83 | 0.56 | 0.33 | 0.27 | 0.71 | 0.42 |
| 5. Bondi2 | 1.5 | 1.8 | 4.7 | 2.6 | 1.8 | 4.6 | 0.58 | 0.83 | 0.32 | 0.71 | 0.57 | 1.44 | 0.57 |
| 6. Deb | 11.1 | 13.3 | 39.8 | 24.0 | 13.6 | 39.7 | 0.46 | 0.84 | 0.28 | 1.76 | 0.60 |
| 7. DMT1 | 10.9 | 13.5 | 40.6 | 22.3 | 13.5 | 40.7 | 0.49 | 0.81 | 0.27 | 1.65 | 0.55 |
| 8. DMT2 | 35.3 | 32.1 | 109.0 | 55.0 | 32.6 | 105.0 | 0.64 | 0.93 | 1.10 | 0.32 | 1.69 | 0.52 |
| 9. KN-Euc1 | 22.7 | 13.1 | 19.9 | 6.3 | 13.7 | 20.0 | 3.60 | 5.80 | 1.68 | 1.14 | 0.46 | 0.32 |
| 10. KN-Euc2 | 54.4 | 13.2 | 17.2 | 24.4 | 13.1 | 16.2 | 2.23 | 4.12 | 3.16 | 0.83 | 1.44 | 0.57 |
| 11. KN-BL1 | 26.6 | 27.5 | 38.4 | 30.0 | 30.0 | 34.6 | 0.89 | 0.97 | 0.69 | 1.00 | 0.87 |
| 12. KN-EF1 | 32.5 | 22.7 | 30.2 | 7.8 | 22.7 | 25.4 | 4.17 | 1.43 | 1.08 | 0.34 | 0.31 |
| 13. KN-BL2 | 12.6 | 6.7 | 14.9 | 18.2 | 9.1 | 15.8 | 0.69 | 1.88 | 0.84 | 2.00 | 1.15 |
| 14. KN-EF2 | 41.0 | 22.9 | 38.9 | 20.8 | 22.8 | 26.0 | 1.97 | 1.79 | 1.05 | 0.91 | 0.80 |

Table 2: CPU times and comparisons for optimized calculations. Columns A through F give CPU times in seconds.

Column
- **A**: total CPU time to generate and simplify the curvature components (Φs and Ψs) from a contravariant tetrad using the standard NP approach (‘Method A’).
- **B**: time for the simplified (covariant) coordinate components of the Ricci and Weyl tensors for the metric generated from the tetrad A.
- **C**: time (including B) to generate and simplify the tetrad curvature components from the coordinate components calculated in B, given the tetrad A.
- **D**: the time to generate and simplify the curvature components from a covariant tetrad using the modified NP approach of [11] (‘Method B’).
- **E**: time for the simplified (covariant) coordinate components of the Ricci and Weyl tensors for the metric generated from the covariant tetrad D.
- **F**: time (including E) to generate and simplify the tetrad curvature components from the coordinate components calculated in E given the tetrad D.

The differences in Columns B and E are due to differences in the exact form of the metric generated from the tetrad. The spacetimes and form of the output are distinguished by the following abbreviations:

- **Griff**: Griffiths metric [12]. Output in factored form.
- **L-P**: Lewis-Papapetrou metric [13]. Output in factored form.
- **Bondi1**: Bondi metric [14]. Output in factored form.
- **Bondi2**: Bondi metric. Output in expanded form.
- **Deb**: Debever metric [17]. Output in normal form (does not simplify further).
- **DMT1**: Debever-McLenaghan-Tariq metric [18]. Output in normal form (does not simplify further).
- **DMT2**: Debever-McLenaghan-Tariq metric in general form [15]. Output in normal form (does not simplify further).
- **KN-Euc1**: Kerr-Newman metric [11]. Output in factored form.
- **KN-Euc2**: Modified form of 9. Output in factored form.
- **KN-BL1**: Kerr-Newman metric in Boyer-Lindquist coordinates [18]. Output in factored form.
- **KN-BL2**: Kerr-Newman metric in modified Boyer-Lindquist coordinates using $u = a \cos \theta$. Output in factored form.
- **KN-EF1**: Kerr-Newman metric in advanced Eddington-Finkelstein coordinates [19]. Output in factored form.
- **KN-EF2**: Modified form of 12. Output in factored form.
Calculation | Mix | Mix1 | Mix2 | Mix3
---|---|---|---|---
$R_{(a)(b)}$ | 8.7 | 7.5 | 4.2 | 2.5
$C_{(a)(b)(c)(d)}$ | 8.5 | 1.1 | 7.4 | 1.1
Total | 17.2 | 8.6 | 11.6 | 3.6

| | $R_{ab}$ | 8.5 | 8.5 | 8.5 | 8.5 |
| | $C_{abcd}$ | 52.5 | 13.7 | 52.5 | 13.7 |
| | Total | 61.0 | 22.2 | 61.0 | 22.2 |

Table 3: Average CPU time in seconds for the calculation and simplification of the bases components of the Ricci and Weyl tensors ($R_{(a)(b)}$ and $C_{(a)(b)(c)(d)}$) compared to the same for the coordinate components ($R_{ab}$ and $C_{abcd}$). Mix refers to the standard 1-forms with trigonometric functions \[20\] and a time dependent basis inner product. The same inner product is used for Mix1 but the trigonometric functions have been transformed away. For Mix2 a constant basis inner product has been used, and in Mix3 the trigonometric functions have been transformed away. In all cases the simplification procedures have been structured for optimum performance.

true for the modified tetrad approach (Column D/E). Columns A/B and D/E indicate that at least in the Kerr-Newman metric the calculation of the coordinate components of the Ricci and Weyl tensors is usually faster than the calculation of the tetrad components.

4 Discussion

Our central conclusion is that the best algorithm, as regards speed, for the computer algebra calculation of curvature in spacetime is the one that minimizes the amount of time spent on simplification. This underlines the importance of the careful choice of coordinates or tetrad in a computer calculation and, more importantly, demonstrates that the user must be able to style the global simplification strategy in a manner most appropriate for the particular problem being studied. An appropriate simplification strategy can change an untractable problem into one which can be solved essentially instantaneously\[10\]. Our comparisons (Table 2) indicate that there is no uniformly superior algorithm. In the development of these comparisons we observed that the differences between procedures optimized with respect to the global simplification strategy for the procedure were less than the variations within a given procedure for different simplification strategies.

Although we have not found any algorithm to be uniformly superior, there certainly are cases where the appropriate algorithm stands out. A case in point is the mixmaster spacetime \[19\]. Here an appropriate choice of basis removes any angular dependence in the bases components of both the Ricci and Weyl tensors. The tetrad approach (not a null tetrad in this case) would certainly be expected to outperform a coordinate calculation in this case. This is confirmed in Table 3. We have considered both a constant basis inner product (for Mix1 and Mix3) and a time dependent one (for Mix and Mix2). The coordinate transformations are simply $\Theta = \cos \theta$ and $\Psi = \sin \psi$. It is clear that the classical coordinate calculation is no match for the basis approach exactly as one would guess. Interestingly, it is the Weyl tensor calculation that improves under coordinate transformation, and the Ricci tensor under change in the basis inner product.

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\[10\]Some simplification strategies appropriate to GRTensorII in MapleV are given in Appendix A.

\[11\]This example was suggested to us by Prof. C. W. Misner.
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[6] GRTensorII is a package which runs within MapleV. It is distinct from packages distributed with Maple and must be obtained independently. The GRTensorII software and documentation is distributed freely on the World-Wide-Web from the address [http://astro.queensu.ca/~grtensor/GRHome.html](http://astro.queensu.ca/~grtensor/GRHome.html). A Mathematica port of GRTensorII is expected to become available in Spring of 1996.

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[17] Complete worksheets and input files generating the entries of Tables 1–3 are available from [http://astro.queensu.ca/~grtensor/papers.html](http://astro.queensu.ca/~grtensor/papers.html). These worksheets are in the form of MapleV.3 input (‘.ms’ and ‘.mpl’) files.

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A Simplification strategies for GRTensorII in MapleV

We have found that the global simplification strategy is of central importance for the computer algebra calculation of curvature in spacetime. Summarized here are some general rules of simplification appropriate to GRTensorII running under MapleV. Although the general philosophy described here is more widely applicable, our comments are specific to the system used and so we have relegated this information to an Appendix.

(i) There should be some default simplification procedure applied to every step of a calculation. Failure to do this can make a simple calculation intractable. The Maple routine normal is a good starting point for the default. If the calculation involves exponentials (e.g. the Bondi metric) the routine expand may be more appropriate. Only if very general functions are involved is it appropriate to consider no default simplification. In general cases this may be the optimal choice.

(ii) For both coordinate and tetrad calculations the removal of trigonometric and like functions via elementary coordinate transformations will often improve performance.

(iii) Simplification of the metric tensor or tetrad components before further calculation will improve performance.

(iv) Precalculation and further simplification of the spin coefficients (and their complex conjugates) will improve performance only in more complicated cases. The same holds for the Christoffel symbols in the coordinate approach.

(v) For further simplification after an object has been calculated, the Maple routine simplify is seldom a good first choice. The routine expand followed by factor is often more appropriate. If the situation is sufficiently general (e.g. the Debever-McLenaghan-Tariq metric) there will be no further simplification if normal has been used as default.

(vi) If complicated functions are involved, it can be advantageous to substitute the explicit forms of the functions after a more general calculation is completed.

(vii) When a calculation is proceeding slowly, it should be halted, the simplification strategy altered, and the worksheet reexecuted.

For most situations these general rules will give adequate performance, and reduce the calculation of curvature for even complex spacetimes to an essentially trivial exercise. Usually, it is the answer that is of interest and not the fact that the simplification strategy is optimal. When optimal simplification strategies are the prime concern (as in this paper) the problem is more involved because of the large number of simplification procedures available and the size of the resultant parameter space to be explored.

12GRTensorII provides a menu of 12 distinct commonly used predefined simplification routines with the ability to introduce customized constraints and simplification routines. Any single parameter routine can be applied with gralter, and any multiple parameter routine can be applied with the command grmap.
## B Worksheets and input files

This Appendix is simply a bookkeeping operation which relates the input files necessary to run the worksheets. The first three worksheets are associated with Table 1, the last four with Table 3, and all the rest with Table 2. The worksheet `appc.ms` generates Appendix C from the input files. Abbreviations which appear twice in the last column of this Table refer to contravariant and covariant tetrad components, respectively.

Worksheets used to generate Table 1:

| Test | Worksheet | Input file |
|------|-----------|------------|
| BKN  | bkn1.ms   | bkn.mpl    |
| BNU  | bkn2.ms   | bknu.mpl   |
| BNNS | bkn3.ms   | bkn.mpl    |

Worksheets used to generate Table 2:

| Test    | Worksheet | Input file   | Worksheet | Input file   |
|---------|-----------|--------------|-----------|--------------|
| Grif    | griff1.ms | npupgrif.mpl | griff2.ms | npdngrif.mpl |
| L-P     | lewis1.ms | npuplew.mpl  | lewis2.ms | npdnlew.mpl  |
| Bondi1  | bondi1.ms | npupbon.mpl  | bondi2.ms | npdnbon.mpl  |
| Bondi2  | bondi1.ms | npupbon.mpl  | bondi2.ms | npdnbon.mpl  |
| Deb     | deb1.ms   | npupdeb.mpl  | deb2.ms   | npdndeb.mpl  |
| DMT1    | dmts1.ms  | npupsdmt.mpl | dmts2.ms  | npdnsdmt.mpl |
| DMT2    | dmt1.ms   | npupdm1.mpl  | dmt2.ms   | npndm2.mpl   |
| KN-Euc1 | kna1.ms   | npupkn1.mpl  | kna2.ms   | npdnkn1.mpl  |
| KN-Euc2 | knb1.ms   | npupkn2.mpl  | knb2.ms   | npdnkn2.mpl  |
| KN-BL1  | knc1.ms   | npupkn3.mpl  | knc2.ms   | npdnkn3.mpl  |
| KN-EF1  | knd1.ms   | npupkn4.mpl  | knd2.ms   | npdnkn4.mpl  |
| KN-BL2  | kne1.ms   | npupkn5.mpl  | kne2.ms   | npdnkn5.mpl  |
| KN-EF2  | knf1.ms   | npupkn6.mpl  | knf2.ms   | npdnkn6.mpl  |

Worksheets used to generate Table 3:

| Test | Worksheet | Input file |
|------|-----------|------------|
| Mix  | mixmr.ms  | mix.mpl    |
| Mix1 | mixm1.ms  | mix1.mpl   |
| Mix2 | mixm2.ms  | mix2.mpl   |
| Mix3 | mixm3.ms  | mix3.mpl   |
C  Tetrads and metrics produced from the input files

The following sections list tetrads and metrics used as inputs for the tests listed in Tables 1–3. This list has been produced directly from the input files which were used in the tests (listed in Appendix 3) and converted to \LaTeX using MapleV's \LaTeX output facility with only minor modifications to improve readability.

C.1  Kerr-Newman “frame field” (Table 1)

For this set of tests (whose output is given in Table 1), the Kerr-Newman spacetime is described by a frame consisting of four independent covariant vector fields whose inner product is the constant matrix $\eta_{(a)(b)} = \text{diag}(-1, -1, -1, 1)$. The basis vectors and corresponding line element for each case are given below.

**Spacetime: Kerr-Newman (Boyer-Lindquist coordinates) (BKN and BKNS)**

| Input file | Frame |
|------------|-------|
| bkn.mpl, $\omega_{1_2} = \begin{bmatrix} \sqrt{r^2 + a^2 \cos(\theta)^2} \\ \sqrt{r^2 - 2mr + a^2 + Q^2} \end{bmatrix}, 0, 0, 0$ | $C.1$ Kerr-Newman "frame field" (Table 1) |
| $\omega_{2_2} = \begin{bmatrix} 0, \sqrt{r^2 + a^2 \cos(\theta)^2} \end{bmatrix}, 0, 0$ | |
| $\omega_{3_2} = \begin{bmatrix} 0, 0, \frac{(r^2 + a^2) \sin(\theta)}{\sqrt{r^2 + a^2 \cos(\theta)^2}} - \frac{a \sin(\theta)}{\sqrt{r^2 + a^2 \cos(\theta)^2}} \end{bmatrix}$ | |
| $\omega_{4_2} = \begin{bmatrix} 0, 0, -a \frac{\sqrt{r^2 - 2mr + a^2 + Q^2} \sin(\theta)\sqrt{r^2 - 2mr + a^2 + Q^2}}{\sqrt{r^2 + a^2 \cos(\theta)^2}}, \sqrt{r^2 - 2mr + a^2 + Q^2} \end{bmatrix}$ | |
| Corresponding line element: $ds^2 = -\frac{(r^2 + a^2 \cos(\theta)^2)}{r^2 - 2mr + a^2 + Q^2} \frac{dt^2}{2} + (-r^2 - a^2 \cos(\theta)^2) \frac{d\theta^2}{2} + (1 + \cos(\theta))^2 \frac{dr^2}{2}$ | |
| $+ \frac{a^2 Q^2 \cos(\theta)^2 + 2a^2 mr + r^4 - a^2 Q^2 + r^2 a^2}{r^2 + a^2 \cos(\theta)^2} \frac{d\phi^2}{(r^2 + a^2 \cos(\theta)^2)}$ | |
| $+ 2a(1 + \cos(\theta))(\cos(\theta) + 1)(-2mr + Q^2) \frac{dt}{r^2 + a^2 \cos(\theta)^2}$ | |
| $+ \frac{(a^2 \cos(\theta)^2 + r^2 - 2mr + Q^2)}{r^2 + a^2 \cos(\theta)^2} \frac{d\phi^2}{2}$ | |

**Spacetime: Kerr-Newman (Boyer-Lindquist coordinates, $u = a\cos \theta$) (BKNU)**

| Input file | Frame |
|------------|-------|
| bkn.mpl, $\omega_{1_2} = \begin{bmatrix} \sqrt{r^2 + u^2} \\ \sqrt{r^2 - 2mr + a^2 + Q^2} \end{bmatrix}, 0, 0, 0$ | |
| $\omega_{2_2} = \begin{bmatrix} 0, -\sqrt{r^2 + u^2} \end{bmatrix}, 0, 0$ | |
| $\omega_{3_2} = \begin{bmatrix} 0, 0, \frac{(r^2 + a^2) \sqrt{a^2 - u^2}}{\sqrt{r^2 + u^2}} \end{bmatrix}, \frac{\sqrt{a^2 - u^2}}{\sqrt{r^2 + u^2}}$ | |
| $\omega_{4_2} = \begin{bmatrix} 0, 0, -\sqrt{r^2 - 2mr + a^2 + Q^2} \frac{(a^2 - u^2)}{r^2 + u^2}, \sqrt{r^2 - 2mr + a^2 + Q^2} \end{bmatrix}$ | |
Corresponding line element:
\[
d s^2 = -\frac{(r^2 + u^2)\, dt^2}{r^2 - 2m r + a^2 + Q^2} - \frac{(r^2 + u^2)\, d u^2}{(a - u)(a + u)} + \frac{(a - u)(a + u)}{(r^2 + u^2) a} + \frac{(a - u)(a + u)(-2m r + Q^2)\, d \phi \, d^t}{(r^2 + u^2) a} + \frac{(u^2 + r^2 - 2m r + Q^2)\, d t^2}{r^2 + u^2}
\]

C.2 Null tetrads (Table 2)

This section lists the set of null tetrads for the test cases used to generate Table 2. For each spacetime, four forms of input were used. The times listed in Column A of Table 2 are obtained using a contravariant null tetrad as input. In Columns B and C the metric is calculated from this tetrad and used for subsequent calculation. In Column D a covariant tetrad is loaded, and in E and F, its corresponding metric is used. Though the metric of column E is, of course, equivalent to that of B, there are often differences in representation which can in principle alter calculation times (though in practice we have found this effect to be minimal). Thus the line elements used as inputs for each tests are listed along with the tetrad used to calculate them.

| Spacetime: Griffiths (Grif) [12] |
|----------------------------------|
| **Input file** | **Contravariant tetrad** |
| npupgrif.mpl | \( l^a = [0, 1, 0, 0] \) |
|            | \( n^a = [1, 0, 0, 0] \) |
|            | \( m^a = \left[ \frac{1}{3} e^{(-2 I a (a + v))} \sqrt{3} (I + 2 a v + a y), \frac{1}{3} e^{(-2 I a (a + v))} \sqrt{3} (I + 2 a u + a y), \frac{1}{3} e^{(-2 I a (a + v))} \sqrt{3} (I - 2 a u - 2 a v), \frac{1}{3} e^{(-2 I a (a + v))} \sqrt{3} (I - 2 a u - 2 a v) \right] \) |
|            | \( \overline{m}^a = \left[ \frac{1}{3} e^{(2 I a (a + v))} \sqrt{3} (I + 2 a v + a y), \frac{1}{3} e^{(2 I a (a + v))} \sqrt{3} (I + 2 a u + a y), \frac{1}{3} e^{(2 I a (a + v))} \sqrt{3} (I - 2 a u - 2 a v), \frac{1}{3} e^{(2 I a (a + v))} \sqrt{3} (I - 2 a u - 2 a v) \right] \) |
| Corresponding line element: | \( d s^2 = 2\, d u \, d v - 2 a (y - 2 v) \, d u \, d x + 2 a (y - 2 v) \, d v \, d x + 2\, d v \, d y + \left( -4 a^2 (a - u)^2 + 2 a^2 y^2 - 2 a^2 v u - 4 a^2 v y - 6 a^2 u^2 - \frac{3}{2} - 6 a^2 v^2 \right) \, d x^2 - 2 a (u + 2 y + v) \, d x \, d y + \frac{1}{2} \, d y^2 \) |

| **Input file** | **Covariant tetrad** |
|-----------------|
| npdngrif.mpl | \( l_a = [1, 0, a (2 u - y), 1] \) |
|                | \( n_a = [0, 1, a (2 v - y), 1] \) |
|                | \( m_a = \left[ 0, 0, -\frac{1}{2} \sqrt{3} e^{(-2 I a (a + v))} (1 + 2 I a (u + v)), -\frac{1}{2} I \sqrt{3} e^{(-2 I a (a + v))} \right] \) |
|                | \( \overline{m}_a = \left[ 0, 0, -\frac{1}{2} \sqrt{3} e^{(2 I a (a + v))} (1 - 2 I a (u + v)), \frac{1}{2} I \sqrt{3} e^{(2 I a (a + v))} \right] \) |
| Corresponding line element: | \( d s^2 = 2\, d u \, d v - 2 a (y - 2 v) \, d u \, d x + 2 a (2 u - y) \, d v \, d x \) |
\[ + 2 \, dv \, dy + \left( -4a^2 y u + 2a^2 y^2 - 4a^2 v u - 4a^2 v y - 6a^2 u^2 - \frac{3}{2} - 6a^2 v^2 \right) \, dx^2 \]
\[ - 2a (u + 2y + v) \, dx \, dy + \frac{1}{2} \, dy^2 \]

**Spacetime: Lewis-Papapetrou (L-P)**  \[\text{[13]}\]

**Input file**  
**Contravariant tetrad**

\[ l^a = \begin{bmatrix} - \frac{1}{2} \sqrt{2} \left( e^{(-s(x,y))} + \frac{w(x,y) e^{s(x,y)}}{r(x,y)} \right) & 0 & 0 \end{bmatrix}, \quad 0, 0, \frac{1}{2} \sqrt{2} \]
\[ n^a = \begin{bmatrix} \frac{1}{2} \sqrt{2} \left( e^{(-s(x,y))} + \frac{w(x,y) e^{s(x,y)}}{r(x,y)} \right) & 0 & 0 \end{bmatrix}, \quad 0, 0, - \frac{1}{2} \sqrt{2} \]
\[ m^a = \begin{bmatrix} e^{(k(x,y)-s(x,y))} \end{bmatrix}, \quad 0, 0 \]
\[ \overline{m}^a = \begin{bmatrix} e^{(k(x,y)-s(x,y))} \end{bmatrix}, \quad 0, 0 \]

**Corresponding line element:**
\[
\begin{align*}
\text{ds}^2 &= \frac{dt^2}{e^{(-s(x,y))} + 2 \frac{w(x,y) e^{s(x,y)}}{r(x,y)}} + 2 \frac{w(x,y) e^{s(x,y)}}{r(x,y)} \\ &- \left( e^{(-s(x,y))} r(x,y) - w(x,y) e^{s(x,y)} \right) \left( e^{(-s(x,y))} r(x,y) + w(x,y) e^{s(x,y)} \right) \, dz^2
\end{align*}
\]

**Input file**  
**Covariant tetrad**

\[ l_a = \begin{bmatrix} \frac{1}{2} e^{s(x,y)} \sqrt{2}, 0, 0 \end{bmatrix}, \quad - \frac{1}{2} \frac{r(x,y) e^{(-s(x,y))} \sqrt{2} + \frac{1}{2} w(x,y) e^{s(x,y)} \sqrt{2}}{e^{(-s(x,y))} + 2 \frac{w(x,y) e^{s(x,y)}}{r(x,y)}} \]
\[ n_a = \begin{bmatrix} \frac{1}{2} e^{s(x,y)} \sqrt{2}, 0, 0 \end{bmatrix}, \quad 0, \frac{1}{2} \frac{r(x,y) e^{(-s(x,y))} \sqrt{2} + \frac{1}{2} w(x,y) e^{s(x,y)} \sqrt{2}}{e^{(-s(x,y))} + 2 \frac{w(x,y) e^{s(x,y)}}{r(x,y)}} \]
\[ m_a = \begin{bmatrix} 0, 0, - \frac{1}{2} e^{(k(x,y)-s(x,y))} \sqrt{2} \end{bmatrix}, \quad 0, 0 \]
\[ \overline{m}_a = \begin{bmatrix} 0, 0, - \frac{1}{2} e^{(k(x,y)-s(x,y))} \sqrt{2} \end{bmatrix}, \quad 0, 0 \]

**Corresponding line element:**
\[
\begin{align*}
\text{ds}^2 &= e^{2s(x,y)} \, dt^2 + 2 e^{(2s(x,y))} w(x,y) \, dt \, dz - e^{2k(x,y)-2s(x,y)} \, dx \, dy \\ &- \left( e^{(-s(x,y))} r(x,y) - w(x,y) e^{s(x,y)} \right) \left( e^{(-s(x,y))} r(x,y) + w(x,y) e^{s(x,y)} \right) \, dz^2
\end{align*}
\]

**Spacetime: Bondi (Bondi1, Bondi2)**  \[\text{[14]}\]

**Input file**  
**Contravariant tetrad**

\[ l^a = \begin{bmatrix} e^{(-Q(r,u,\theta))}, 0, 0, 0 \end{bmatrix}, \quad 0 \]
\[ n^a = \begin{bmatrix} - \frac{1}{2} e^{(-Q(r,u,\theta))} V(r, u, \theta), e^{(-Q(r,u,\theta))}, e^{(-Q(r,u,\theta))} U(r, u, \theta), 0 \end{bmatrix}, \quad 0 \]
\[ m^a = \begin{bmatrix} 0, 0, - \frac{1}{2} \frac{e^{(-\gamma(r,u,\theta))}}{r} \sqrt{2} \end{bmatrix}, \quad - \frac{1}{2} \frac{I e^{\gamma(r,u,\theta)} \sqrt{2}}{r \sin(\theta)} \]
\[ \overline{m}^a = \begin{bmatrix} 0, 0, - \frac{1}{2} \frac{e^{(-\gamma(r,u,\theta))}}{r} \sqrt{2} \end{bmatrix}, \quad - \frac{1}{2} \frac{I e^{\gamma(r,u,\theta)} \sqrt{2}}{r \sin(\theta)} \]

**Corresponding line element:**
\[ ds^2 = 2 (e^{Q(x,y)})^2 \frac{dr \, du + \left( \frac{(e^{Q(x,y)})^2 V(r,u,\theta)}{r} - r^2 (e^{\gamma(r,u,\theta)})^2 U(r,u,\theta)^2 \right)}{d\theta^2} = 2 U(r,u,\theta)^2 \, d\theta^2 - \frac{r^2 \sin(\theta)^2 \, d\phi^2}{(e^{\gamma(r,u,\theta)})^2} \]

**Corresponding line element:**

\[ ds^2 = 2 (e^{Q(x,y)})^2 \frac{dr \, du + \left( \frac{(e^{Q(x,y)})^2 V(r,u,\theta)}{r} - r^2 (e^{\gamma(r,u,\theta)})^2 U(r,u,\theta)^2 \right)}{d\theta^2} = 2 U(r,u,\theta)^2 \, d\theta^2 - \frac{r^2 \sin(\theta)^2 \, d\phi^2}{(e^{\gamma(r,u,\theta)})^2} \]

**Input file** | Covariant tetrad
---|---
**npdnbon.mpl**

- \( l_a = \left[ 0, e^{Q(x,y)}, 0, 0 \right] \)
- \( n_a = \left[ e^{Q(x,y)}, e^{Q(x,y)} \frac{V(r,u,\theta)}{r}, 0, 0 \right] \)
- \( m_a = \left[ 0, -\frac{1}{2} r U(r,u,\theta) e^{\gamma(r,u,\theta)} \sqrt{2}, \frac{1}{2} r e^{\gamma(r,u,\theta)} \sqrt{2}, \frac{1}{2} I r \sin(\theta) \theta \right] \)
- \( \overline{m_a} = \left[ 0, -\frac{1}{2} U(r,u,\theta) e^{\gamma(r,u,\theta)} \sqrt{2}, \frac{1}{2} r e^{\gamma(r,u,\theta)} \sqrt{2}, -\frac{1}{2} I r \sin(\theta) \theta \right] \)

**Space time: Debever (Deb)**

**Input file** | Contravariant tetrad
---|---
**npupdeb.mpl**

- \( l^a = \left[ -\frac{1}{2} L(x,y) P(x,y) \sqrt{2}, \right. \)
- \( \frac{N(x,y) \sqrt{2}}{2}, \frac{P(x,y) \sqrt{2}}{2}, 0 \]
- \( n^a = \left[ -\frac{1}{2} L(x,y) P(x,y) + M(x,y) N(x,y), \right. \)
- \( \frac{1}{2} Y(y) \sqrt{2}, \frac{1}{2} S(x,y), 0 \]
- \( m^a = \left[ \frac{1}{2} L(x,y) P(x,y) + M(x,y) N(x,y), \right. \)
- \( -\frac{1}{2} L(x,y) P(x,y) + M(x,y) N(x,y), 0, -\frac{1}{2} I X(x) \sqrt{2} \]
- \( \overline{m^a} = \left[ \frac{1}{2} L(x,y) P(x,y) + M(x,y) N(x,y), \right. \)
- \( -\frac{1}{2} L(x,y) P(x,y) + M(x,y) N(x,y), 0, \frac{1}{2} I X(x) \sqrt{2} \]

**Corresponding line element:**

\[ ds^2 = (L(x,y) - N(x,y)) (L(x,y) + N(x,y)) \, dt^2 + 2 (-P(x,y) N(x,y) + M(x,y) L(x,y)) \, dt \, dz \]
- \( (P(x,y) - M(x,y)) (P(x,y) + M(x,y)) \, dz^2 - \frac{S(x,y)^2 \, dy^2}{Y(y)^2} - \frac{S(x,y)^2 \, dx^2}{X(x)^2} \]

**Input file** | Covariant tetrad
---|---
**npndndeb.mpl**

- \( l_a = \left[ \frac{1}{2} L(x,y) \sqrt{2}, \frac{1}{2} M(x,y) \sqrt{2}, \frac{1}{2} S(x,y) \sqrt{2}, \frac{1}{2} Y(y), 0 \right] \)
\[ n_a = \begin{bmatrix} \frac{1}{2} L(x, y) \sqrt{2}, & \frac{1}{2} M(x, y) \sqrt{2}, & -\frac{1}{2} \frac{S(x, y) \sqrt{2}}{Y(y)}, & 0 \end{bmatrix} \]

\[ m_a = \begin{bmatrix} -\frac{1}{2} N(x, y) \sqrt{2}, & -\frac{1}{2} P(x, y) \sqrt{2}, & 0, & \frac{1}{2} \frac{IS(x, y) \sqrt{2}}{X(x)} \end{bmatrix} \]

\[ \overline{m}_a = \begin{bmatrix} -\frac{1}{2} N(x, y) \sqrt{2}, & -\frac{1}{2} P(x, y) \sqrt{2}, & 0, & -\frac{1}{2} \frac{IS(x, y) \sqrt{2}}{X(x)} \end{bmatrix} \]

Corresponding line element:
\[
ds^2 = (L(x, y) - N(x, y))(L(x, y) + N(x, y)) \, dt^2
\]
\[
+ 2 (-P(x, y) N(x, y) + M(x, y) L(x, y)) \, dt \, dz
\]
\[
+ (M(x, y) - P(x, y)) (P(x, y) + M(x, y)) \, dz^2
\]
\[- \frac{S(x, y)^2 \, dy^2}{Y(y)^2} - \frac{S(x, y)^2 \, dx^2}{X(x)^2} \]

Spacetime: Debever-McLenaghan-Tariq (DMT1)

Input file Contravariant tetrad

npupsdm.t  \[ l^a = \begin{bmatrix} \frac{1}{2} L(w, x) \sqrt{2}, & \frac{1}{2} M(w, x) \sqrt{2}, & 0, & -\frac{1}{2} \frac{S(w, x) \sqrt{2}}{w, x} \end{bmatrix} \]

\[ 0, & -\frac{1}{2} \frac{\sqrt{2}}{S(w, x)} \]

\[ n^a = \begin{bmatrix} \frac{1}{2} L(w, x) P(w, x) - M(w, x) N(w, x), & 0, & -\frac{1}{2} \frac{S(w, x) \sqrt{2}}{S(w, x)} \end{bmatrix} \]

\[ 0, & -\frac{1}{2} \frac{\sqrt{2}}{S(w, x)} \]

\[ m^a = \begin{bmatrix} -\frac{1}{2} L(w, x) P(w, x) - M(w, x) N(w, x), & 0, & -\frac{1}{2} \frac{S(w, x) \sqrt{2}}{S(w, x)} \end{bmatrix} \]

\[ 0, & -\frac{1}{2} \frac{\sqrt{2}}{S(w, x)} \]

\[ \overline{m}_a = \begin{bmatrix} -\frac{1}{2} L(w, x) P(w, x) - M(w, x) N(w, x), & 0, & -\frac{1}{2} \frac{S(w, x) \sqrt{2}}{S(w, x)} \end{bmatrix} \]

\[ 0, & -\frac{1}{2} \frac{\sqrt{2}}{S(w, x)} \]

Corresponding line element:
\[
ds^2 = (L(w, x) - N(w, x))(L(w, x) + N(w, x)) \, du^2
\]
\[
+ 2 (-P(w, x) N(w, x) + M(w, x) L(w, x)) \, du \, dv
\]
\[
+ (M(w, x) - P(w, x)) (P(w, x) + M(w, x)) \, dv^2
\]
\[- \frac{R(w, x)^2 \, dw^2}{S(w, x)^2} - \frac{S(w, x)^2 \, dw^2}{R(w, x)^2} \]

Input file Covariant tetrad

npdnsdm.t

\[ l_a = \begin{bmatrix} \frac{1}{2} \sqrt{2} L(w, x), & \frac{1}{2} \sqrt{2} M(w, x), & 0, & \frac{1}{2} \frac{S(w, x) \sqrt{2}}{w, x} \end{bmatrix} \]

\[ n_a = \begin{bmatrix} \frac{1}{2} \sqrt{2} L(w, x), & \frac{1}{2} \sqrt{2} M(w, x), & 0, & -\frac{1}{2} \frac{S(w, x) \sqrt{2}}{w, x} \end{bmatrix} \]

\[ m_a = \begin{bmatrix} -\frac{1}{2} \sqrt{2} L(w, x), & -\frac{1}{2} \sqrt{2} M(w, x), & \frac{1}{2} \frac{R(w, x) \sqrt{2}}{R(w, x)}, & 0 \end{bmatrix} \]

\[ \overline{m}_a = \begin{bmatrix} -\frac{1}{2} \sqrt{2} L(w, x), & -\frac{1}{2} \sqrt{2} M(w, x), & \frac{1}{2} \frac{R(w, x) \sqrt{2}}{R(w, x)}, & 0 \end{bmatrix} \]

Corresponding line element:
\[
ds^2 = (L(w, x) - N(w, x))(L(w, x) + N(w, x)) \, du^2
\]
\[ + 2 (-P(w,x)N(w,x) + M(w,x)L(w,x)) \, du \, dv \\
+ (M(w,x) - P(w,x))(M(w,x) + P(w,x)) \, dv^2 - R(w,x)^2 \, dw^2 - S(w,x)^2 \, dx^2 \]

Spacetime: Debever-McLenaghan-Tariq (modified) (DMT2) \[ \text{[DMT2]} \]

Input file Contravariant tetrad

\[ \text{npupdmt.mpl} \]

\[ l^a = \begin{bmatrix}
\frac{1}{2} L(u,v,w,x)P(u,v,w,x) - M(u,v,w,x)N(u,v,w,x) \\
- \frac{1}{2} L(u,v,w,x)P(u,v,w,x) - M(u,v,w,x)N(u,v,w,x)
\end{bmatrix}, 0, \frac{1}{2} S(w,x) \]

\[ n^a = \begin{bmatrix}
\frac{1}{2} L(u,v,w,x)P(u,v,w,x) - M(u,v,w,x)N(u,v,w,x) \\
- \frac{1}{2} L(u,v,w,x)P(u,v,w,x) - M(u,v,w,x)N(u,v,w,x)
\end{bmatrix}, 0, \frac{1}{2} S(w,x) \]

\[ m^a = \begin{bmatrix}
\frac{1}{2} L(u,v,w,x)P(u,v,w,x) - M(u,v,w,x)N(u,v,w,x) \\
- \frac{1}{2} L(u,v,w,x)P(u,v,w,x) - M(u,v,w,x)N(u,v,w,x)
\end{bmatrix}, 0, \frac{1}{2} I \sqrt{2} R(w,x) \]

\[ m^a = \begin{bmatrix}
\frac{1}{2} L(u,v,w,x)P(u,v,w,x) - M(u,v,w,x)N(u,v,w,x) \\
- \frac{1}{2} L(u,v,w,x)P(u,v,w,x) - M(u,v,w,x)N(u,v,w,x)
\end{bmatrix}, 0, \frac{1}{2} I \sqrt{2} R(w,x) \]

Corresponding line element:

\[ ds^2 = -(N(u,v,w,x) - L(u,v,w,x))(N(u,v,w,x) + L(u,v,w,x)) \, du \, dv \\
+ 2 (-P(u,v,w,x)N(u,v,w,x) + M(u,v,w,x)L(u,v,w,x)) \, du \, dv \\
- (P(u,v,w,x) - M(u,v,w,x))(P(u,v,w,x) + M(u,v,w,x)) \, dv^2 \\
- R(w,x)^2 \, dw^2 - S(w,x)^2 \, dx^2 \]

Input file Covariant tetrad

\[ \text{npndmt.mpl} \]

\[ l_a = \begin{bmatrix}
\frac{1}{2} \sqrt{2} L(u,v,w,x), \frac{1}{2} \sqrt{2} M(u,v,w,x), 0, \frac{1}{2} S(w,x) \sqrt{2}
\end{bmatrix} \]

\[ n_a = \begin{bmatrix}
\frac{1}{2} \sqrt{2} L(u,v,w,x), \frac{1}{2} \sqrt{2} M(u,v,w,x), 0, -\frac{1}{2} S(w,x) \sqrt{2}
\end{bmatrix} \]

\[ m_a = \begin{bmatrix}
-\frac{1}{2} \sqrt{2} N(u,v,w,x), -\frac{1}{2} \sqrt{2} P(u,v,w,x), \frac{1}{2} I R(w,x) \sqrt{2}, 0
\end{bmatrix} \]

\[ m_a = \begin{bmatrix}
-\frac{1}{2} \sqrt{2} N(u,v,w,x), -\frac{1}{2} \sqrt{2} P(u,v,w,x), -\frac{1}{2} I R(w,x) \sqrt{2}, 0
\end{bmatrix} \]

Corresponding line element:

\[ ds^2 = -(N(u,v,w,x) - L(u,v,w,x))(N(u,v,w,x) + L(u,v,w,x)) \, du \, dv \\
+ 2 (-P(u,v,w,x)N(u,v,w,x) + M(u,v,w,x)L(u,v,w,x)) \, du \, dv \\
+ (M(u,v,w,x) - P(u,v,w,x))(M(u,v,w,x) + P(u,v,w,x)) \, dv^2 \\
- R(w,x)^2 \, dw^2 - S(w,x)^2 \, dx^2 \]

Spacetime: Kerr-Newman \((x, y, z, t)\) (KN-Euc1)
\[ l^a = \left[ \frac{1}{2} \sqrt{x^2 - 2 M x + a^2 + Q^2} \right] \left\{ \begin{array}{l} \frac{1}{2} \sqrt{x^2 + a^2 y^2} \sqrt{x^2 - 2 M x + a^2 + Q^2} \\ \frac{1}{2} \sqrt{x^2 + a^2 y^2} \sqrt{x^2 - 2 M x + a^2 + Q^2} \end{array} \right. \]

\[ n^a = \left[ \frac{1}{2} \sqrt{x^2 - 2 M x + a^2 + Q^2} \right] \left\{ \begin{array}{l} \frac{1}{2} \sqrt{x^2 + a^2 y^2} \sqrt{x^2 - 2 M x + a^2 + Q^2} \\ \frac{1}{2} \sqrt{x^2 + a^2 y^2} \sqrt{x^2 - 2 M x + a^2 + Q^2} \end{array} \right. \]

\[ m^a = \left[ \frac{1}{2} \sqrt{\frac{1 - y^2}{x^2 + a^2 y^2}} \right] \left\{ \begin{array}{l} \frac{1}{2} \sqrt{\frac{1 - y^2}{x^2 + a^2 y^2}} \sqrt{\frac{x^2}{1 - y^2}} \\ \frac{1}{2} \sqrt{\frac{1 - y^2}{x^2 + a^2 y^2}} \frac{1}{2} \sqrt{\frac{1 - y^2}{x^2 + a^2 y^2}} \end{array} \right. \]

\[ \mathcal{m}^a = \left[ \frac{1}{2} \sqrt{\frac{1 - y^2}{x^2 + a^2 y^2}} \right] \left\{ \begin{array}{l} \frac{1}{2} \sqrt{\frac{1 - y^2}{x^2 + a^2 y^2}} \sqrt{\frac{x^2}{1 - y^2}} \\ \frac{1}{2} \sqrt{\frac{1 - y^2}{x^2 + a^2 y^2}} \frac{1}{2} \sqrt{\frac{1 - y^2}{x^2 + a^2 y^2}} \end{array} \right. \]

\[ \text{Corresponding line elements:} \]

\[ ds^2 = \frac{(a^2 y^2 + x^2 - 2 M x + Q^2) \, dt^2}{x^2 + a^2 y^2} + 2 \frac{1}{2} a ( - 2 M x + Q^2 ) ( - 1 + y ) ( y + 1 ) \, dt \, dz \]

\[ + (x^4 + x^2 a^2 + 2 a^2 M x - a^2 y^2)^2 x^2 - 2 a^2 y^2 M x + a^4 y^2 + a^2 y^2 Q^2) \]

\[ ( - 1 + y ) ( y + 1 ) \, ds^2 / x^2 + a^2 y^2) - \frac{(x^2 + a^2 y^2) \, dx^2}{x^2 - 2 M x + a^2 + Q^2} + (x^2 + a^2 y^2) \, dy^2 \]

\[ \text{Spacetime: Kerr-Newman } (x, y, z, t) \text{ (KN-Euc2)} \]

\[ l^a = \left[ \frac{1}{2} \sqrt{x^2 - 2 M x + a^2 + Q^2} \right] \left\{ \begin{array}{l} \frac{1}{2} \sqrt{x^2 + a^2 y^2} \sqrt{x^2 - 2 M x + a^2 + Q^2} \\ \frac{1}{2} \sqrt{x^2 + a^2 y^2} \sqrt{x^2 - 2 M x + a^2 + Q^2} \end{array} \right. \]

\[ \text{Input file Covariant tetrad} \]

\[ \text{Input file Contravariant tetrad} \]
Corresponding line element:
\[
ds^2 = \left( a^2 y^2 + x^2 - 2 M x + Q^2 \right) \frac{dt^2}{x^2 + a^2 y^2} + a \left( -2 M x + Q^2 \right) \left( -1 + y \right) \left( y + 1 \right) \frac{dz^2}{x^2 + a^2 y^2} + \left( x^4 + x^2 a^2 + 2 a^2 M x - a^2 Q^2 + a^2 y^2 x^2 - 2 a^2 y^2 M x + a^4 y^2 + a^2 y^2 Q^2 \right) \left( -1 + y \right) \left( y + 1 \right) \frac{dz^2}{x^2 + a^2 y^2}
\]

Input file Covariant tetrad

\[
l_a = \left[ \frac{1}{2} \left( \frac{\sqrt{2} \sqrt{x^2 - 2 M x + a^2 + Q^2}}{\sqrt{x^2 + a^2 y^2}} \right), - \frac{1}{2} \sqrt{2} a \frac{\sqrt{x^2 - 2 M x + a^2 + Q^2}}{\sqrt{x^2 + a^2 y^2}} \right] \\
n_a = \left[ \frac{1}{2} \sqrt{2} \frac{\sqrt{x^2 - 2 M x + a^2 + Q^2}}{\sqrt{x^2 + a^2 y^2}}, - \frac{1}{2} \sqrt{2} \frac{\sqrt{x^2 - 2 M x + a^2 + Q^2}}{\sqrt{x^2 + a^2 y^2}} \right] \\
m_a = \left[ \frac{1}{2} \sqrt{2} a \frac{1 - y^2}{\sqrt{x^2 + a^2 y^2}}, - \frac{1}{2} \sqrt{2} a \frac{1 - y^2}{\sqrt{x^2 + a^2 y^2}} \right] \\
m_a = \left[ \frac{1}{2} \sqrt{2} a \frac{1 - y^2}{\sqrt{x^2 + a^2 y^2}}, - \frac{1}{2} \sqrt{2} a \frac{1 - y^2}{\sqrt{x^2 + a^2 y^2}} \right]
\]

Corresponding line element:
\[
ds^2 = \left( a^2 y^2 + x^2 - 2 M x + Q^2 \right) \frac{dt^2}{x^2 + a^2 y^2} + a \left( -2 M x + Q^2 \right) \left( -1 + y \right) \left( y + 1 \right) \frac{dz^2}{x^2 + a^2 y^2} + \left( x^4 + x^2 a^2 + 2 a^2 M x - a^2 Q^2 + a^2 y^2 x^2 - 2 a^2 y^2 M x + a^4 y^2 + a^2 y^2 Q^2 \right) \left( -1 + y \right) \left( y + 1 \right) \frac{dz^2}{x^2 + a^2 y^2}
\]

Spacetime: Kerr-Newman (Boyer-Lindquist coordinates) (KN-BL1)

Input file Contravariant tetrad

\[
l^a = \left[ \frac{r^2 + a^2}{r^2 - 2 M r + a^2 + Q^2}, 1, 0, \frac{a}{r^2 - 2 M r + a^2 + Q^2} \right] \\
n^a = \left[ \frac{1}{2} \frac{r^2 + a^2}{r^2 + a^2 \cos(\theta)^2}, - \frac{1}{2} \frac{r^2 - 2 M r + a^2 + Q^2}{r^2 + a^2 \cos(\theta)^2}, 0, \frac{1}{2} \frac{r^2 + a^2 \cos(\theta)^2}{r^2 + a^2 \cos(\theta)^2} \right]
\]
\[ m^a = \begin{bmatrix} \frac{1}{2} (I \sin(\theta) r + \sin(\theta) a \cos(\theta)) a \sqrt{2} & 0 & \frac{1}{2} (r - I a \cos(\theta)) \sqrt{2} \\ \frac{1}{2} r^2 + a^2 \cos(\theta)^2 & 0 & \frac{1}{2} r^2 + a^2 \cos(\theta)^2 \end{bmatrix}, \]
\[ \overline{m}^a = \begin{bmatrix} \frac{1}{2} (I \sin(\theta) r + \sin(\theta) a \cos(\theta)) a \sqrt{2} & 0 & \frac{1}{2} (r + I a \cos(\theta)) \sqrt{2} \\ \frac{1}{2} r^2 + a^2 \cos(\theta)^2 & 0 & \frac{1}{2} r^2 + a^2 \cos(\theta)^2 \end{bmatrix}, \]

Corresponding line element:
\[ ds^2 = \frac{(2 \cos(\theta)^2 + r^2 + Q^2 - 2Mr) \ dt^2 + 2 (-1 + \cos(\theta)) (\cos(\theta) + 1) (Q^2 - 2Mr) a \ dt \ d\phi}{r^2 + a^2 \cos(\theta)^2} - \frac{(r^2 + a^2 \cos(\theta)^2) \ dr^2 + (-r^2 - a^2 \cos(\theta)^2) \ d\theta^2 + (-1 + \cos(\theta))}{(\cos(\theta) + 1) (-2a^2 \cos(\theta)^2 M r + a^4 \cos(\theta)^2 + r^2 a^2 \cos(\theta)^2 + a^2 Q^2 \cos(\theta)^2 - a^2 Q^2 + 2a^2 M r + r^4 + r^2 a^2)} \ d\phi^2 / (r^2 + a^2 \cos(\theta)^2) \]

Spacetime: Kerr-Newman (Eddington-Finklestein coordinates) (KN-EF1) [19]
\[
\mathbf{m}^a = \left[ -\frac{1}{2} I \sqrt{2} a \sin(\theta), 0, \frac{1}{2} I \sqrt{2} a \cos(\theta), 0, \frac{1}{2} \frac{\sqrt{2}}{r - I a \cos(\theta)} \right], \quad \mathbf{n}^a = \left[ \frac{\sqrt{2}}{r - I a \cos(\theta)} \right], \quad \mathbf{a}^a = \left[ -\frac{1}{2} I \sqrt{2} \left( r - I a \cos(\theta) \right) \sin(\theta) \right]
\]

Corresponding line element:
\[
ds^2 = \frac{\left( a^2 \cos(\theta)^2 + r^2 - 2 m r + Q^2 \right) d u^2}{r^2 + a^2 \cos(\theta)^2} + 2 d u d r
\]
\[
+ 2 \left( a \left( r^2 + a^2 \cos(\theta)^2 \right) \right) \left( \cos(\theta) + 1 \right) \left( r^2 - 2 m r + a^2 + Q^2 \right) d u d \phi
\]
\[
+ 2 \left( 0, \frac{\sqrt{2}}{r - I a \cos(\theta)}, 0, \frac{1}{2} \frac{\sqrt{2}}{r - I a \cos(\theta)} \right) \left( r^2 + a^2 \cos(\theta)^2 \right) \left( \cos(\theta) + 1 \right) \left( r^2 - 2 m r + a^2 + Q^2 \right) d \phi d \theta
\]
\[
\left( r^2 + a^2 \cos(\theta)^2 \right) \left( \cos(\theta) + 1 \right) \left( r^2 - 2 m r + a^2 + Q^2 \right) d \phi d \theta
\]

\[
\text{Input file Covariant tetrad}
\]

\[
l_a = \left[ 1, 0, 0, \left( -1 + \cos(\theta) \right) \right], \quad n_a = \left[ \frac{1}{2} \frac{1}{\sqrt{r^2 + a^2 \cos(\theta)^2}}, 1, 0, \sqrt{2} \right], \quad m_a = \left[ \frac{1}{2} \frac{I \sqrt{2} \sin(\theta)}{\sqrt{r^2 + a^2 \cos(\theta)^2}}, 0, \frac{1}{2} \frac{\sqrt{2}}{r + I a \cos(\theta)} \right]
\]

Corresponding line element:
\[
ds^2 = \frac{\left( a^2 \cos(\theta)^2 + r^2 - 2 m r + Q^2 \right) d u^2}{r^2 + a^2 \cos(\theta)^2} + 2 d u d r
\]
\[
+ 2 \left( a \left( r^2 + a^2 \cos(\theta)^2 \right) \right) \left( \cos(\theta) + 1 \right) \left( r^2 - 2 m r + a^2 + Q^2 \right) d u d \phi
\]
\[
+ 2 \left( 0, \frac{\sqrt{2}}{r + I a \cos(\theta)}, 0, \frac{1}{2} \frac{\sqrt{2}}{r + I a \cos(\theta)} \right) \left( r^2 + a^2 \cos(\theta)^2 \right) \left( \cos(\theta) + 1 \right) \left( r^2 - 2 m r + a^2 + Q^2 \right) d \phi d \theta
\]
\[
\left( r^2 + a^2 \cos(\theta)^2 \right) \left( \cos(\theta) + 1 \right) \left( r^2 - 2 m r + a^2 + Q^2 \right) d \phi d \theta
\]

\[
\text{Input file Contravariant tetrad}
\]

\[
l^a = \left[ r^2 + a^2, 1, 0, r^2 - 2 M r + a^2 + Q^2, 1 \right], \quad n^a = \left[ 1, r^2 + a^2, 1, \left( r^2 - 2 M r + a^2 + Q^2 \right), 0, \frac{1}{2} \frac{a}{\sqrt{r^2 + u^2}} \right], \quad m^a = \left[ \frac{1}{2} \frac{I \sqrt{2} \left( r - I u \right) \sqrt{2}}{\sqrt{u^2 - a^2 \left( r - I u \right) \sqrt{2}}}, 0, \frac{1}{2} \frac{\sqrt{2}}{\sqrt{u^2 - a^2 \left( r - I u \right) \sqrt{2}}} \right]
\]

\[
\text{Input file Boyer-Lindquist coordinates, } u = a \cos \theta \quad \text{(KN-BL2) [4]}
\]

\[
\text{Spacetime: Kerr-Newman}
\]

\[
pupkn5.mp1
\]

\[
l^a = \left[ \frac{r^2 + a^2}{2 \left( r^2 - 2 M r + a^2 + Q^2 \right)} \right], \quad n^a = \left[ 1, \frac{1}{2} \frac{a}{\sqrt{r^2 + u^2}}, \frac{1}{2} \frac{\left( r^2 + u^2 \right)^2}{r^2 - 2 M r + a^2 + Q^2}, 0, \frac{1}{2} \frac{a}{\sqrt{r^2 + u^2}} \right], \quad m^a = \left[ \frac{1}{2} \frac{\left( \sqrt{u^2 - a^2 \left( r - I u \right) \sqrt{2}} \right)}{\sqrt{r^2 + u^2}}, 0, \frac{1}{2} \frac{\sqrt{2}}{\sqrt{r^2 + u^2}} \right]
\]

\[
\text{Input file Contravariant tetrad}
\]

\[
l^a = \left[ \frac{r^2 + a^2}{2 \left( r^2 + a^2 \right)} \right], \quad n^a = \left[ 1, \frac{1}{2} \frac{a}{\sqrt{r^2 + u^2}}, \frac{1}{2} \frac{\left( r^2 + u^2 \right)^2}{r^2 - 2 M r + a^2 + Q^2}, 0, \frac{1}{2} \frac{a}{\sqrt{r^2 + u^2}} \right], \quad m^a = \left[ \frac{1}{2} \frac{\left( \sqrt{u^2 - a^2 \left( r - I u \right) \sqrt{2}} \right)}{\sqrt{r^2 + u^2}}, 0, \frac{1}{2} \frac{\sqrt{2}}{\sqrt{r^2 + u^2}} \right]
\]
Corresponding line element:

\[
\begin{align*}
\text{Corresponding line element:} \\
\mathrm{ds}^2 &= \left( u^2 - 2 M r + r^2 + Q^2 \right) \, dt^2 - 2 \left( Q^2 - 2 M r \right) (a - u) \, (a + u) \, dt \, d\phi \\
&\quad - \frac{u^2 - 2 M r + a^2 + Q^2}{r^2 + u^2} \, \left( I r + u \right) \left( I r - u \right) \, d u^2 \\
&\quad + \frac{2 u^2 M r - a^2 u^2 - r^2 a^2 - Q^2 u^2 - r^4 - r^2 a^2 + a^2 Q^2 - 2 a^2 M r \, (a - u)}{(a + u) \, \phi^2 / \left( (r^2 + u^2)^2 \right)}
\end{align*}
\]

\[
\text{Input file Covariant tetrad}
\]

\[
\begin{align*}
l_a &= \left[ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0 \right] \\
n_a &= \left[ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0 \right] \\
m_a &= \left[ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0 \right] \\
\overline{m}_a &= \left[ \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0 \right]
\end{align*}
\]

\[
\text{Corresponding line element:} \\
\mathrm{ds}^2 \left( u^2 - 2 M r + r^2 + Q^2 \right) \, dt^2 - 2 \left( Q^2 - 2 M r \right) (a - u) \, (a + u) \, dt \, d\phi \\
&\quad - \frac{u^2 - 2 M r + a^2 + Q^2}{r^2 + u^2} \, \left( I r + u \right) \left( I r - u \right) \, d u^2 \\
&\quad + \frac{2 u^2 M r - a^2 u^2 - r^2 a^2 - Q^2 u^2 - r^4 - r^2 a^2 + a^2 Q^2 - 2 a^2 M r \, (a - u)}{(a + u) \, \phi^2 / \left( (r^2 + u^2)^2 \right)}
\]

\[
\text{Spacetime: Kerr-Newman (Eddington-Finkelstein coordinates) (KN-EF2) [19]}
\]

\[
\text{Input file Contravariant tetrad}
\]

\[
\begin{align*}
l^a &= \left[ 0, 1, 0, 0 \right] \\
n^a &= \left[ \frac{r^2 + a^2}{r^2 + a^2 \cos(\theta)^2}, -\frac{1}{2} \left( r^2 - 2 m r + a^2 \right) \cos(\theta)^2, 0, -\frac{a}{r^2 + a^2 \cos(\theta)^2} \right] \\
m^a &= \left[ \frac{1}{2} \sqrt{2} a \sin(\theta), 0, \frac{1}{2} \sqrt{2} a \sin(\theta), 0 \right] \\
\overline{m}_a &= \left[ \frac{1}{2} \sqrt{2} a \sin(\theta), 0, \frac{1}{2} \sqrt{2} a \sin(\theta), 0 \right]
\end{align*}
\]

\[
\text{Corresponding line element:} \\
\mathrm{ds}^2 \left( a^2 \cos(\theta)^2 + r^2 - 2 m r + Q^2 \right) \, du^2 \\
&\quad + 2 \left( a \left( -1 + \cos(\theta) \right) \left( \cos(\theta) + 1 \right) \right) \left( -2 m r + Q^2 \right) \, du \, d\phi \\
&\quad + 2 \left( -1 + \cos(\theta) \right) \left( \cos(\theta) + 1 \right) \left( r^2 a^2 \cos(\theta)^2 - 2 a^2 m r \cos(\theta)^2 + a^4 \cos(\theta)^2 \right) \\
&\quad + 2 \left( -1 + \cos(\theta) \right) \left( \cos(\theta) + 1 \right) \left( r^2 a^2 \cos(\theta)^2 - 2 a^2 m r \cos(\theta)^2 + a^4 \cos(\theta)^2 \right)
\]
\[
+ a^2 Q^2 \cos(\theta)^2 + 2 a^2 m r + r^4 - a^2 Q^2 + r^2 a^2 \right) d \phi^2 / (r^2 + a^2 \cos(\theta)^2)
\]

Input file Covariant tetrad

\(\text{npdnkn6.mpl}\)

\[ l_a = \left[ 1, 0, 0, (-1 + \cos(\theta)) (\cos(\theta) + 1) a \right] \]

\[ n_a = \frac{1}{2} \left[ \frac{1}{r^2 + a^2 \cos(\theta)^2}, 1, 0, \right. \]

\[ \frac{1}{2} a (-1 + \cos(\theta)) (\cos(\theta) + 1) (r^2 - 2 m r + a^2 + Q^2) \]

\[ m_a = \left[ \frac{1}{2} I \sqrt{2} a \sin(\theta), 0, \frac{1}{2} (-r + I a \cos(\theta)) \sqrt{2}, \frac{1}{2} \sqrt{2} \sin(\theta) (r^2 + a^2) \right] \]

\[ m_a = \left[ \frac{1}{2} I \sqrt{2} a \sin(\theta), 0, \frac{1}{2} (r + I a \cos(\theta)) \sqrt{2}, \frac{1}{2} \sqrt{2} \sin(\theta) (r^2 + a^2) \right] \]

Corresponding line element:

\[
 ds^2 = \left( a^2 \cos(\theta)^2 + r^2 - 2 m r + Q^2 \right) du^2 + 2 du \, dr + 2 d u \, d \phi + 2 (\cos(\theta)^2 + r^2 - a^2 \cos(\theta)^2) \, d \phi^2 + \]

\[
 + a^2 Q^2 \cos(\theta)^2 \left( 2 a^2 m r + r^4 - a^2 Q^2 + r^2 a^2 \right) \, d \phi^2 / (r^2 + a^2 \cos(\theta)^2)
\]

C.3 Mixmaster spacetime (Table 3)

For this set of tests, not only are the basis vectors varied, but also their inner product. The input used for these tests is listed below.

Spacetime: Mixmaster (Mix) [20]

Input file Frame

\(\text{mix.mpl}\)

Inner product of basis vectors:

\[ \eta^{(a)(b)} = \text{diag} (e^{2a(t)}, e^{2b(t)}, e^{2c(t)}, -e^{2a(t)+2b(t)+2c(t)}) \]

Basis vectors:

\[ \omega_{1a} = [\cos(\psi), \sin(\psi) \sin(\theta), 0, 0] \]

\[ \omega_{2a} = [\sin(\psi), -\cos(\psi) \sin(\theta), 0, 0] \]

\[ \omega_{3a} = [0, \cos(\theta), 1, 0] \]

\[ \omega_{4a} = [0, 0, 0, 1] \]

Corresponding line element:

\[
 ds^2 = \left( \cos(\psi)^2 \left( e^{a(t)} \right)^2 + \sin(\psi)^2 \left( e^{b(t)} \right)^2 \right) d \theta^2 - 2 \sin(\psi) \cos(\psi) \sin(\theta) \left( e^{b(t)} - e^{a(t)} \right) \left( e^{b(t)} + e^{a(t)} \right) \, d \theta \, d \phi + \\
+ \sin(\psi)^2 \sin(\theta)^2 \left( e^{a(t)} \right)^2 + \cos(\psi)^2 \sin(\theta)^2 \left( e^{b(t)} \right)^2 + \cos(\theta)^2 \left( e^{c(t)} \right)^2 \, d \phi^2 + 2 \left( e^{c(t)} \right)^2 \cos(\theta) \, d \phi \, d \psi + \left( e^{c(t)} \right)^2 \, d \psi^2 - \left( e^{b(t)} \right)^2 \left( e^{b(t)} \right)^2 \, d t^2
\]

Spacetime: Mixmaster (Mix1) [21]
Inner product of basis vectors:

\[ \eta^{a(b)} = \text{diag}(e^{2a(T)}, e^{2b(T)}, e^{2c(T)}, -e^{2a(T)+2b(T)+2c(T)}) \]

Basis vectors:

\[
\begin{align*}
\omega_{1a} &= \left[ -\frac{\sqrt{1 - \Psi^2}}{\sqrt{1 - \Theta^2}}, \Psi \sqrt{1 - \Theta^2}, 0, 0 \right] \\
\omega_{2a} &= \left[ -\frac{\Psi}{\sqrt{1 - \Theta^2}}, \sqrt{1 - \Psi^2} \sqrt{1 - \Theta^2}, 0, 0 \right] \\
\omega_{3a} &= \left[ 0, \Theta, \frac{1}{\sqrt{1 - \Theta^2}}, 0 \right] \\
\omega_{4a} &= \left[ 0, 0, 0, 1 \right]
\end{align*}
\]

Corresponding line element:

\[
\begin{align*}
ds^2 &= -\frac{((e^{a(T)})^2 - (e^{a(T)})^2 \Psi^2 + \Psi^2 (e^{b(T)})^2)}{2 \Psi \sqrt{-(\Psi - 1)(\Psi + 1)}} \frac{d\Theta^2}{\Theta - 1} \frac{d\Phi}{\Phi + 1} \\
&\quad + 2 \Psi \sqrt{-(\Psi - 1)(\Psi + 1)}(e^{b(T)} - e^{a(T)})(e^{b(T)} + e^{a(T)}) d\Theta d\Phi + \\
&\quad + (e^{b(T)})^2 \Psi^2 - (e^{a(T)})^2 \Psi^2 \Theta^2 + (e^{b(T)})^2 - (e^{b(T)})^2 \Psi^2 (e^{b(T)})^2 + (e^{b(T)})^2 - (e^{b(T)})^2 \Theta^2 \\
&\quad + (e^{b(T)})^2 \Theta^2 \Psi^2 + \Theta^2 (e^{c(T)})^2 d\Phi^2 + 2 \frac{\Theta (e^{c(T)})^2 d\Phi d\Psi}{\sqrt{-(\Psi - 1)(\Psi + 1)}} \\
&\quad - \frac{(e^{c(T)})^2 d\Psi^2}{(\Psi - 1)(\Psi + 1)} - (e^{a(T)})^2 (e^{b(T)})^2 (e^{c(T)})^2 dT^2 \end{align*}
\]

Spacetime: Mixmaster (Mix2)

Inner product of basis vectors:

\[ \eta^{a(b)} = \text{diag}(1, 1, 1, -1) \]

Basis vectors:

\[
\begin{align*}
\omega_{1a} &= \left[ e^{a(t)} \cos(\psi), e^{a(t)} \sin(\psi) \sin(\theta), 0, 0 \right] \\
\omega_{2a} &= \left[ e^{b(t)} \sin(\psi), -e^{b(t)} \cos(\psi) \sin(\theta), 0, 0 \right] \\
\omega_{3a} &= \left[ 0, e^{c(t)} \cos(\theta), e^{c(t)}, 0 \right] \\
\omega_{4a} &= \left[ 0, 0, 0, e^{a(t)+b(t)+c(t)} \right]
\end{align*}
\]

Corresponding line element:

\[
\begin{align*}
ds^2 &= \left( \cos(\psi)^2 (e^{a(t)})^2 + \sin(\psi)^2 (e^{b(t)})^2 \right) d\theta^2 \\
&\quad - 2 \sin(\psi) \cos(\psi) \sin(\theta) \left( e^{b(t)} - e^{a(t)} \right) \left( e^{b(t)} + e^{a(t)} \right) d\theta d\phi + \\
&\quad (\sin(\psi)^2 \sin(\theta)^2) (e^{a(t)})^2 + \cos(\psi)^2 \sin(\theta)^2 (e^{b(t)})^2 + \cos(\theta)^2 (e^{c(t)})^2 d\phi^2 \\
&\quad + 2 (e^{c(t)})^2 \cos(\theta) d\phi d\psi + (e^{c(t)})^2 d\psi^2 - (e^{a(t)})^2 (e^{b(t)})^2 (e^{c(t)})^2 dT^2 \end{align*}
\]

Spacetime: Mixmaster (Mix3)
\[ \omega_{2a} = \left[ -\frac{e^b(T) \Psi}{\sqrt{1-\Theta^2}}, \frac{-e^b(T) \sqrt{1-\Theta^2} \sqrt{1-\Psi^2}}{\sqrt{1-\Psi^2}}, 0, 0 \right] \]

\[ \omega_{3a} = \left[ 0, \Theta e^c(T), \frac{e^c(T)}{\sqrt{1-\Psi^2}}, 0 \right] \]

\[ \omega_{4a} = [0, 0, 0, e^{(a(T)+b(T)+c(T))}] \]

Corresponding line element:

\[ ds^2 = \frac{-\left(\frac{e^a(T)}{\sqrt{1-\Psi^2}}\right)^2 + \left(\frac{e^a(T)}{\sqrt{1-\Psi^2}}\right)^2 \Psi^2 - \Psi^2 \left(\frac{e^b(T)}{\sqrt{1-\Psi^2}}\right)^2}{(\Theta - 1)(\Theta + 1)} d\Theta^2 \]

\[ - 2 \Psi \sqrt{-(\Psi - 1)(\Psi + 1)} \left(\frac{e^a(T)}{\sqrt{1-\Psi^2}} - e^b(T) \left(\frac{e^a(T)}{\sqrt{1-\Psi^2}} + e^a(T) \right) \right) d\Theta \ d\Phi + \]

\[ \left(\frac{e^a(T)}{\sqrt{1-\Psi^2}}\right)^2 \Psi^2 - \left(\frac{e^a(T)}{\sqrt{1-\Psi^2}}\right)^2 \Psi^2 \Theta^2 + \left(\frac{e^b(T)}{\sqrt{1-\Psi^2}}\right)^2 - \Psi^2 \left(\frac{e^b(T)}{\sqrt{1-\Psi^2}}\right)^2 - \left(\frac{e^b(T)}{\sqrt{1-\Psi^2}}\right)^2 \Theta^2 \]

\[ + \left(\frac{e^b(T)}{\sqrt{1-\Psi^2}}\right)^2 \Theta^2 \Psi^2 + \Theta^2 \left(\frac{e^c(T)}{\sqrt{1-\Psi^2}}\right)^2 \] d\Phi^2 + 2 \frac{\Theta \left(\frac{e^c(T)}{\sqrt{1-\Psi^2}}\right)^2}{\sqrt{-(\Psi - 1)(\Psi + 1)}} d\Phi \ d\Psi \]

\[ - \left(\frac{e^c(T)}{\sqrt{1-\Psi^2}}\right)^2 d\Psi^2 \left(\frac{e^a(T)}{\sqrt{1-\Psi^2}}\right)^2 \left(\frac{e^b(T)}{\sqrt{1-\Psi^2}}\right)^2 dT^2 \]
The following is the input file:

\[
\begin{align*}
\text{Ndim}_\_ & := 4 : \\
x1\_ & := t : \\
x2\_ & := r : \\
x3\_ & := u : \\
x4\_ & := \phi : \\
\eta_{12\_} & := 1 : \\
\eta_{34\_} & := -1 : \\
\beta_{11\_} & := \frac{(r^2+a^2)}{(r^2-2Mr+a^2+Q^2)} : \\
\beta_{12\_} & := 1 : \\
\beta_{14\_} & := \frac{a}{(r^2-2Mr+a^2+Q^2)} : \\
\beta_{21\_} & := \frac{1}{2}\frac{(r^2+a^2)}{(r^2+u^2)} : \\
\beta_{22\_} & := -\frac{1}{2}\frac{(r^2-2Mr+a^2+Q^2)}{(r^2+u^2)} : \\
\beta_{24\_} & := \frac{1}{2}\frac{a}{(r^2+u^2)} : \\
\beta_{31\_} & := \frac{1}{2}\left(I\left(a^2-u^2\right)^{1/2}r+(a^2-u^2)^{1/2}u\right)\frac{2^{1/2}}{(r^2+u^2)} : \\
\beta_{33\_} & := \frac{1}{2}\left(-(a^2-u^2)^{1/2}\right)\frac{(r-Iu)^{1/2}}{(r^2+u^2)} : \\
\beta_{34\_} & := \frac{1}{2}\frac{I}{(r^2+u^2)} : \\
\beta_{41\_} & := \frac{1}{2}\left(-I\left(a^2-u^2\right)^{1/2}r+(a^2-u^2)^{1/2}u\right)\frac{2^{1/2}}{(r^2+u^2)} : \\
\beta_{43\_} & := \frac{1}{2}\frac{I^2}{(r^2+u^2)} : \\
\beta_{44\_} & := -\frac{1}{2}\frac{I}{(r^2+u^2)} : \\
\text{Info}_\_ & := \text{Contravariant NPtetrad for Kerr-Newman metric} \\
& (u=a\cos(\theta) \text{ to Boyer-Lindquist coordinates}) \; : \\
\end{align*}
\]

The following is the annotated input/output of kne1.ms.

```
> restart:
> readlib(grii):
> grtensor();

GRTensorII Version 1.26a

December 1, 1995

Developed by Peter Musgrave, Denis Pollney and Kayll Lake

Copyright 1994 – 1995 by the authors.

Latest version available from: [http://astro.queensu.ca/GRHome.html]

To initiate help type ?grtensor

Defaults read from C:\MAPLEV3\lib/grtensor.ini
```

We now load the contravariant tetrad, calculate the scalars, and factor them.

```
> qload(npupkn5);

Default spacetime = npupkn5

For the npupkn5 spacetime :

Coordinates
\[
x^a = [t\; u\; \phi]
\]

Basis inner product

24
Null tetrad (contravariant components)

\[
\eta^{(a)}{}_{(b)} = \begin{bmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
\end{bmatrix}
\]

For the npupkn5 spacetime:

\[
\Phi_{00} = 0 \\
\Phi_{01} = 0 \\
\Phi_{02} = 0 \\
\Phi_{11} = \frac{1}{2} \frac{Q^2}{(r + I u)^2(r - I u)^2} \\
\Phi_{12} = 0 \\
\Phi_{22} = 0 \\
\Lambda = 0 \\
\Psi_{0} = 0 \\
\Psi_{1} = 0 \\
\Psi_{2} = \frac{Q^2 - M r - I u M}{(r + I u)(r - I u)^3} \\
\Psi_{3} = 0
\]
\( \Psi 4 = 0 \)

We now generate the metric and simplify it.

\[
\begin{align*}
&> \text{grcalc}(g(\text{dn},\text{dn})); \\
&> \text{gralter}(\_,\text{simplify},\text{factor}); \\
&> \text{grcalc}(\text{ds}); \\
&> \text{grdisplay}(\_);
\end{align*}
\]

For the npupkn5 spacetime:

Line element

\[
ds^2 = \left( \frac{r^2 - 2Mr + u^2 + Q^2}{r^2 + u^2} \right) dt^2 - 2 \left( \frac{-2Mr + Q^2}{a(r^2 + u^2)} \right) (a - u)(a + u) \text{ } dt \text{ } d\phi \\
- \left( \frac{(r^2 + u^2)}{r^2 - 2Mr + a^2 + Q^2} \right) \left( \frac{r - Iu}{a - u} \right) \text{ } du^2 \\
\left( \frac{(-r^4 - u^2r^2 - r^2a^2 + 2u^2Mr - 2rMa^2 + Q^2a^2 - u^2a^2 - u^2Q^2)(a - u)}{(a + u)} \right) \text{ } d\phi^2 / \left( \left( r^2 + u^2 \right) a^2 \right)
\]

The coordinate components of Ricci and Weyl are calculated and simplified.

(Display deleted for this appendix.)

\[
\begin{align*}
&> \text{grcalc}(g(\text{up},\text{up})); \\
&> \text{gralter}(\_,\text{radical},\text{expand},\text{factor}); \\
&> \text{grcalc}(R(\text{dn},\text{dn}),\text{Ricciscalar}); \\
&> \text{gralter}(\_,\text{expand},\text{factor}); \\
&> \text{grcalc}(C(\text{dn},\text{dn},\text{dn},\text{dn})); \\
&> \text{gralter}(\_,\text{expand},\text{factor});
\end{align*}
\]

The following lines define the Weyl Scalars.

\[
\begin{align*}
&> \text{grdefine}(\text{P0}',\_,'-\text{C}(a \ b \ c \ d)\text{NP1}('a)\text{NPm}('b)\text{NP1}('c)\text{NPm}('d)'); \\
&> \text{grdefine}(\text{P1}',\_,'-\text{C}(a \ b \ c \ d)\text{NP1}('a)\text{NPn}('b)\text{NP1}('c)\text{NPm}('d)'); \\
&> \text{grdefine}(\text{P2}',\_,'-\text{C}(a \ b \ c \ d)\text{NP1}('a)\text{NP1}('c)\text{NPn}('d)-\text{NP1}('a)\text{NPn}('b)\text{NPm}('c)\text{NPmbar}('d)/2'); \\
&> \text{grdefine}(\text{P3}',\_,'-\text{C}(a \ b \ c \ d)\text{NPn}('a)\text{NP1}('b)\text{NPn}('c)\text{NPmbar}('d)'); \\
&> \text{grdefine}(\text{P4}',\_,'-\text{C}(a \ b \ c \ d)\text{NPn}('a)\text{NPmbar}('b)\text{NPn}('c)\text{NPmbar}('d)');
\end{align*}
\]

The following lines define the Ricci Scalars.

\[
\begin{align*}
&> \text{grdefine}(\text{P00}',\_,\text{R}(a \ b)\text{NP1}('a)\text{NP1}('b)/2'); \\
&> \text{grdefine}(\text{P01}',\_,\text{R}(a \ b)\text{NP1}('a)\text{NPm}('b)/2'); \\
&> \text{grdefine}(\text{P02}',\_,\text{R}(a \ b)\text{NPm}('a)\text{NPm}('b)/2'); \\
&> \text{grdefine}(\text{P03}',\_,\text{R}(a \ b)\text{NPm}('a)\text{NPm}('b)\text{NPm}('a)\text{NPmbar}('b)/4');
\end{align*}
\]
> grdefine('P12',{},'R{a b}*NPn{^a}*NPm{^b}/2');
Created definition for P12

> grdefine('P22',{},'R{a b}*NPn{^a}*NPn{^b}/2');
Created definition for P22

We calculate and simplify the Weyl scalars.
> grcalc(P0,P1,P2,P3,P4);
> gralter(_,factor);
> grdisplay(_);

For the npupkn5 spacetime:
\[ P0 = 0 \]
\[ P1 = 0 \]
\[ P2 = \frac{Q^2 - M r - I u M}{(r + I u)(r - I u)^3} \]
\[ P3 = 0 \]
\[ P4 = 0 \]

We calculate and simplify the Ricci scalars.
> grcalc(P00,P01,P02,P11,P12,P22);
> gralter(_,factor);
> grdisplay(_);

For the npupkn5 spacetime:
\[ P00 = 0 \]
\[ P01 = 0 \]
\[ P02 = 0 \]
\[ P11 = \frac{1}{2} \frac{Q^2}{(r^2 + u^2)^2} \]
\[ P12 = 0 \]
\[ P22 = 0 \]

Unfortunately, the output obtained from each of these methods are not in exactly the same form. Since \( P2 \) is smaller as measured in Maple words, this must be considered the fully simplified form. In fact, by performing the MapleV operation \texttt{factor} on the denominators of \( \Psi_2 \) (an operation requiring less than .1 CPU seconds), it can be reduced to the required fully simplified form. In all tests listed in Appendix \[\textbf{B}\] the final results of calculation are either presented in exactly equivalent forms, or equivalent within some simplification operation requiring a negligible amount of time.