POINTWISE LINEABILITY IN SEQUENCE SPACES

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Abstract. We prove that several results of lineability/spaceability in the framework of sequence spaces are valid in a stricter sense.

1. Introduction

The notions of lineability and spaceability were introduced in the seminal paper [1] by Aron, Gurariy, and Seoane-Sepúlveda. If $V$ is a vector space and $\alpha$ is a cardinal number, a subset $A$ of $V$ is called $\alpha$-lineable in $V$ if $A \cup \{0\}$ contains an $\alpha$-dimensional linear subspace $W$ of $V$. When $V$ has a topology and the subspace $W$ can be chosen to be closed, we say that $A$ is $\alpha$-spaceable. For details on the theory of lineability/spaceability we refer to [2, 6] and for recent results we refer to [3, 4, 16, 13] and the references therein.

In this paper we shall be interested in the following stricter variant of lineability/spaceability, introduced in [14]. Let $\alpha$, $\beta$ and $\lambda$ be cardinal numbers and $V$ be a vector space, with $\dim V = \lambda$ and $\alpha < \beta \leq \lambda$. A set $A \subset V$ is $(\alpha, \beta)$-lineable if $A$ is $\alpha$-lineable and for every subspace $W_\alpha \subset V$ with $W_\alpha \subset A \cup \{0\}$ and $\dim W_\alpha = \alpha$, there is a subspace $W_\beta \subset V$ with $\dim W_\beta = \beta$ and $W_\alpha \subset W_\beta \subset A \cup \{0\}$. Furthermore, if $W_\beta$ can always be chosen as a closed subspace, we say that $A$ is $(\alpha, \beta)$-spaceable. Notice that the ordinary notions of lineability and spaceability are recovered when $\alpha = 0$.

A challenging fashion of the investigation of lineability/spaceability is that there are only few general techniques (see [5, 15]) and each problem seems to need ad hoc arguments to be solved. From now on all vector spaces are considered over a fixed scalar field $\mathbb{K}$ which can be either $\mathbb{R}$ or $\mathbb{C}$. For any set $A$ we shall denote by $\text{card} \, (A)$ the cardinality of $A$; in particular, we denote $c = \text{card} \, (\mathbb{R})$ and $\aleph_0 = \text{card} \, (\mathbb{N})$.

In some sense, the problems investigated in the framework of $(\alpha, \beta)$-lineability/spaceability can be divided into three categories:

- $(1, \beta)$-lineability/spaceability is obtained but the techniques seem to be not adapted to $(\alpha, \beta)$-lineability/spaceability for $\alpha > 1$;
- $(\alpha, \beta)$-lineability/spaceability is characterized for all $\alpha < \beta$;
- No technique is known to obtain anything else than lineability/spaceability.

Let us illustrate examples of all three situations:

In [14, Theorem 3.1] it is shown that if $p, q \geq 1$, the set

$$A := \{ u : \ell_p \to \ell_q : u \text{ is continuous and not injective} \}$$

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is \((1, \mathfrak{c})\)-lineable in the set of continuous linear operators from \(\ell_p\) to \(\ell_q\). Still in [14, Theorem 3.2], it was proved that

\[
L_p[0,1] \setminus \bigcup_{q>p} L_q[0,1]
\]

is \((1, \mathfrak{c})\)-spaceable in \(L_p[0,1]\) for every \(p > 0\). Both techniques seem to be not immediately adapted to \((n, \mathfrak{c})\)-lineability/spaceability for \(n > 1\). The second case is illustrated by the following result of [14, Theorem 3.2]: the set

\[
\ell_p \setminus \bigcup_{0 < q < p} \ell_q
\]

is \((\alpha, \mathfrak{c})\)-spaceable in \(\ell_p\) if, and only if, \(\alpha < \aleph_0\). Finally the third case can be illustrated by results related to norm attaining operators (see, for instance, [18, Proposition 6]).

In this paper we introduce the notion of pointwise lineability/spaceability. This new concept is closely linked to the notion of \((1, \beta)\)-lineability/spaceability but being of a stricter nature. Our main goal is to develop general techniques in the framework of pointwise lineability/spaceability in sequence spaces. Lineability and spaceability were exhaustively investigated in this framework (see [7, 8, 9, 11, 14] and the references therein) and in the present paper we shall show several of these results hold for the more involved notion of pointwise lineability/spaceability.

This paper is organized as follows. In Section 2 we introduce the notion of pointwise lineability and preliminary terminology that shall be used throughout the paper. In Section 3 we prove our main results, which show that the main results of [7, 8, 9, 17] are valid in the context of pointwise spaceability.

## 2. Pointwise Spaceability in Sequence Spaces

We start this section establishing the concept of pointwise lineability/spaceability.

**Definition 2.1.** Let \(V\) be an infinite-dimensional vector space over \(\mathbb{K}\) and let be \(A\) a non-empty subset of \(V\). We say that \(A\) is pointwise lineable if for each \(x \in A\) there is an infinite-dimensional subspace \(W_x\) such that

\[
x \in W_x \subset A \cup \{0\}.
\]

When \(\alpha \leq \dim (V)\) is a cardinal number, we will say that \(A\) is pointwise \(\alpha\)-lineable if for each \(x \in A\) there is a subspace \(W_{x,\alpha}\) such that

\[
\dim (W_{x,\alpha}) = \alpha \quad \text{and} \quad x \in W_{x,\alpha} \subset A \cup \{0\}.
\]

Similarly, we define pointwise spaceability.

It is obvious that pointwise lineability/spaceability implies \((1, \beta)\)-lineability/spaceability. It is also simple to show that the converse is not true. In fact, \(\mathbb{R}^2 \cup \{(1,1,1)\}\) is not pointwise lineable in \(\mathbb{R}^3\) but it is \((1, 2)\)-lineable in \(\mathbb{R}^3\). The following example is perhaps more interesting and less artificial.
Example 2.2. Let $X$ be an infinite-dimensional Banach space endowed with weak-topology and let $A \neq X$ be a basic neighborhood of 0. Thus, there exist $\varepsilon > 0$, $n \in \mathbb{N}$ and $\varphi_1, \ldots, \varphi_n \in X^* \setminus \{0\}$, where $X^*$ is the topological dual of $X$, such that

$$A = \{ x \in X : |\varphi_i(x)| < \varepsilon \text{ for all } i = 1, \ldots, n \}$$

Note that $\left( \bigcap_{i=1}^n \ker(\varphi_i) \right) \subset A$ and

$$\dim \left( \bigcap_{i=1}^n \ker(\varphi_i) \right) = \dim(X).$$

Let $W$ be a subspace of $X$ such that $W \subset A$ and $\dim(W) = \alpha < \dim(X)$ and let us see that $W \subset \left( \bigcap_{i=1}^n \ker(\varphi_i) \right)$. In fact, if there exists $x_0 \in W - \left( \bigcap_{i=1}^n \ker(\varphi_i) \right)$, then $\varphi_i(x_0) \neq 0$ for a certain $i$. Considering $\lambda := \frac{\varepsilon + 1}{|\varphi_i(x_0)|}$, since $\lambda x_0 \in W$ and $|\varphi_i(\lambda x_0)| = \varepsilon + 1 > \varepsilon$,

we have $\lambda x_0 \notin A$, a contradiction. Therefore, $W \subset \left( \bigcap_{i=1}^n \ker(\varphi_i) \right)$ and $A$ is $(\alpha, \dim(X))$-spaceable. In particular, $A$ is $(1, \dim(X))$-spaceable. Since $\bigcap_{i=1}^n \ker(\varphi_i) \subset X$; let $y_0 \in X \setminus \left( \bigcap_{i=1}^n \ker(\varphi_i) \right)$ and

$$\delta := \max_{1 \leq i \leq n} |\varphi_i(y_0)| > 0.$$

For $\alpha := (2\delta)^{-1} \varepsilon$, we have $\alpha y_0 \notin \bigcap_{i=1}^n \ker(\varphi_i)$ and

$$|\varphi_i(\alpha y_0)| = \alpha |\varphi_i(y_0)| < \varepsilon,$$

and this shows that $\alpha y_0 \notin A$. But it is obvious that there is no subspace of $X$ containing $\alpha y_0$ and contained in $A$. Therefore, $A$ is not pointwise spaceable.

Many results on lineability in sequence spaces invoke the “mother vector” technique, where a sequence $(x_k)_{k=1}^\infty$ (called mother vector) in a sequence space $E$ is used to generate an infinite-dimensional subspace of $E$ contained in a certain set of sequences. However, in general, the generated subspace does not contain the “mother vector” and it is by no means simple to construct the subspace containing the “mother vector”. Examples can be found in the papers [14, 18]. The concept of pointwise lineability/spaceability faces this technicality: to generate a subspace containing the mother vector. Thus, the notion of pointwise lineability/spaceability shall not be confused with the “mother vector” technique.

Recently, some results in this direction have been obtained covered under the veil of $(1, \beta)$-lineability/spaceability. Analyzing the proof of [12, Theorem 1.3] we conclude that, in fact, we have pointwise lineability/spaceability result.

In order to deal with a wide range of sequence spaces we shall use the notion of invariant sequence spaces (see Definition 2.3) that was introduced in [8]. For the sake
of illustration, in [9, Theorem 2.5] it is proved that if $E$ is an invariant sequence space over a Banach space $X$, then

(1) For every $\Gamma \subset (0, \infty]$, the set

\[
E \backslash \bigcup_{q \in \Gamma} \ell_q (X)
\]

is either empty or spaceable;

(2) For every $\Gamma \subset (0, \infty]$, the set

\[
E \backslash \bigcup_{q \in \Gamma} \ell^w_q (X)
\]

is either empty or spaceable, where $\ell^w_p (X)$ is the space of weakly $p$-summable sequences in $X$ (the exact definitions shall be presented throughout the paper).

Let us recall the notion of invariant sequence spaces.

**Definition 2.3.** (See [8, Definition 2.1]) Let $X \neq \{0\}$ be a Banach space over $\mathbb{K}$.

(a) Given $x = (x_j)_{j=1}^\infty \in X^\mathbb{N}$, we define the zero-free version of $x$, that we denote by $x^0$, as follows: if $x$ has only finitely many nonzero coordinates, then $x^0 = 0$, otherwise, $x^0 = (x_{j_k})_{k=1}^\infty$ where $x_{j_k}$ is the $k$-th nonzero coordinate of $x$.

(b) An invariant sequence space over $X$ is an infinite-dimensional Banach or quasi-Banach space $E$ whose elements are $X$-valued sequences satisfying the following conditions:

\begin{enumerate}
  
  \item[(b1)] For all $x \in X^\mathbb{N}$ such that $x^0 \neq 0$,
  
  \[ x \in E \Leftrightarrow x^0 \in E \]
  
  and
  
  \[ \|x\| \leq K \|x^0\| \]
  
  for some constant $K$ which depends only on $E$.

  \item[(b2)] $\|x_j\|_X \leq \|x\|_E$ for all $x = (x_j)_{j=1}^\infty \in E$ and all $j \in \mathbb{N}$.

\end{enumerate}

An invariant sequence space is an invariant sequence space over some Banach space $X$.

Usual sequence spaces are invariant sequence spaces. For instance, if $X$ is a Banach space, the classical sequence spaces $\ell_p (X)$, $\ell^w_p (X)$, $\ell^w_p (X)$, $c_0 (X)$, $c (X)$, $p \in (0, \infty)$ and the Lorentz space $\ell_{p,q}$ are invariant sequence spaces. For more details and examples, we refer to [8].

Let $X$ and $Y$ be Banach spaces, $\Gamma$ be an arbitrary set and $E$ be an invariant sequence space over $X$. If $E_\lambda$ is an invariant sequence space over $Y$ for each $\lambda \in \Gamma$, and $f : X \to Y$ is any function, following [17, Definition 1.3], we define

\[
G (E, f, (E_\lambda)_{\lambda \in \Gamma}) = \left\{ (x_j)_{j=1}^\infty \in E : (f (x_j))_{j=1}^\infty \notin \bigcup_{\lambda \in \Gamma} E_\lambda \right\}.
\]

According to [9, Definition 2.3], a function $f : X \to Y$ between normed spaces is:
(a) Non-contractive if \( f(0) = 0 \) and for all scalars \( \alpha \) there is some constant \( K(\alpha) > 0 \) such that
\[
\|f(\alpha x)\|_Y \geq K(\alpha) \|f(x)\|_Y,
\]
for all \( x \in X \).

(b) Strongly non-contractive if \( f(0) = 0 \) and, for all scalars \( \alpha \) there exists some constant \( K(\alpha) > 0 \) such that
\[
|\varphi(f(\alpha x))| \geq K(\alpha) |\varphi(f(x))|,
\]
for all \( x \in X \) and all continuous linear functionals \( \varphi \in Y' \). (From now on, \( Y' \) denotes the topological dual of \( Y \).)

An invariant sequence space \( E \) over a Banach space \( X \) is called strongly invariant sequence space when
\[
c_{00}(X) := \{ (x_j)_{j=1}^\infty \in c_0(X) : (x_j)_{j=1}^\infty \text{ is eventually null} \} \subset E
\]
and \( (x_j)_{j=1}^\infty \in E \) if, and only if, all the subsequences of \( (x_j)_{j=1}^\infty \) also belong to \( E \).

All the aforementioned examples of invariant sequence spaces are also strongly invariant sequence spaces. However, the notions are not exactly the same. The set
\[
E = \left\{ (x_j)_{j=1}^\infty \in \ell_1 : \sum_{j=1}^\infty x_j = 0 \right\},
\]
is the kernel of the continuous linear functional
\[
\varphi : \ell_1 \to \mathbb{K}
\]
\[
x = (x_j)_{j=1}^\infty \mapsto \varphi(x) = \sum_{j=1}^\infty x_j
\]
and thus \( E \) is a Banach space. It is plain that \( |x_j| \leq \|x\| \) for every \( j \) and, if \( x^0 \neq 0 \) then \( x \in E \iff x^0 \in E \); it is also plain that \( \|x^0\| = \|x\| \). Hence \( E \) is an invariant sequence space. However, \( E \) fails to be a strongly invariant sequence space. In fact, note that
\[
(1, -1, 2^{-1}, -2^{-1}, 2^{-2}, -2^{-2}, \ldots, 2^{-n}, -2^{-n}, \ldots)
\]
lies in \( E \) and, on the other hand, the subsequence
\[
(1, 2^{-1}, 2^{-2}, \ldots, 2^{-n}, \ldots)
\]
does not lie in \( E \) (it is also simple to check that \( c_{00} \) is not contained in \( E \)).

According to [17, Definition 2.3], if \( X \) and \( Y \) are Banach spaces and \( E \) is an invariant sequence space over \( Y \), a function \( f : X \to Y \) such that \( f(0) = 0 \) is called compatible with \( E \) if for any \( X \)-valued sequence \( (x_j)_{j=1}^\infty \) and all scalars \( a \neq 0 \), we have
\[
(f(x_j))_{j=1}^\infty \notin E \Rightarrow (f(ax_j))_{j=1}^\infty \notin E.
\]
For instance, it is easy to see that any non-contractive map \( f : X \to Y \) is compatible with \( \ell_q(Y) \) and \( c_0(Y) \).
3. Main results

In this section we shall show that, in general, the results from [8, 9, 17] also hold in the context of pointwise lineability/spaceability.

The following theorem can be found in [17, Theorem 2.5] and it generalizes [9, Theorem 2.5(a)]:

**Theorem 3.1.** Let $X$ and $Y$ be Banach spaces, $\Gamma$ be an arbitrary set, $E$ be an invariant sequence space over $X$ and $E_\lambda$ be a strongly invariant sequence space over $Y$ for all $\lambda \in \Gamma$. If $f : X \to Y$ is compatible with $E_\lambda$ for each $\lambda \in \Gamma$, then $G(\lambda, f, (E_\lambda)_{\lambda \in \Gamma})$ is either empty or $c$-spaceable.

The main result of this subsection shows that if $(E_\lambda)_{\lambda \in \Gamma}$ is “nested”, that is, given $\lambda_1, \lambda_2 \in \Gamma$, either $E_{\lambda_1} \subset E_{\lambda_2}$ or $E_{\lambda_1} \subset E_{\lambda_2}$, then we can assure pointwise spaceability in the above result. We need the following simple lemma for further reference.

**Lemma 3.2.** Let $(E_\lambda)_{\lambda \in \Gamma}$ be a family of nested strongly invariant sequence spaces and let $N' \subset N$ be such that

$$\text{card}(N') = \text{card}(N \setminus N') = \aleph_0.$$  

If $x = (x_j)_{j=1}^{\infty} \in X^N$ and $x \notin \bigcup_{\lambda \in \Gamma} E_\lambda$, then either $(x_j)_{j \in N'} \notin \bigcup_{\lambda \in \Gamma} E_\lambda$ or $(x_j)_{j \in N \setminus N'} \notin \bigcup_{\lambda \in \Gamma} E_\lambda$.

**Proof.** Considering the subsequences $(x_j)_{j \in N'}$, $(x_j)_{j \in N \setminus N'}$, let us assume that

$$(x_j)_{j \in N'} \in \bigcup_{\lambda \in \Gamma} E_\lambda \quad \text{and} \quad (x_j)_{j \in N \setminus N'} \in \bigcup_{\lambda \in \Gamma} E_\lambda.$$  

Thus, there are $\lambda_1, \lambda_2 \in \Gamma$ such that $(x_j)_{j \in N'} \in E_{\lambda_1}$ and $(x_j)_{j \in N \setminus N'} \in E_{\lambda_2}$ and, without loss of generality, assuming $E_{\lambda_1} \subset E_{\lambda_2}$, we have that both $(x_j)_{j \in N'}$ and $(x_j)_{j \in N \setminus N'}$ belong to $E_{\lambda_2}$. Let us take the sequences $u = (u_j)_{j=1}^{\infty}$ and $v = (v_j)_{j=1}^{\infty}$ given by

$$u_j = \begin{cases} 0, & \text{if } j \in N \setminus N', \\ x_j, & \text{if } j \in N', \end{cases} \quad \text{and} \quad v_j = \begin{cases} x_j, & \text{if } j \in N \setminus N', \\ 0, & \text{if } j \in N'. \end{cases}$$  

We clearly see that $u^0 = \left[(x_j)_{j \in N'}\right]^0$ and $v^0 = \left[(x_j)_{j \in N \setminus N'}\right]^0$. If $u^0 = 0$, then $u \in c_0(X) \subset E_{\lambda_2}$ because $E_{\lambda_2}$ is a strongly invariant sequence space. If $u^0 \neq 0$, then $u \in E_{\lambda_2}$ because, in particular, $E_{\lambda_2}$ is an invariant sequence space. By the same reason we have that $v \in E_{\lambda_2}$. Now, since

$$x = u + v,$$

and $E_{\lambda_2}$ is a vector space, we conclude that

$$x \in E_{\lambda_2} \subset \bigcup_{\lambda \in \Gamma} E_\lambda,$$

contradicting the fact that $x \notin \bigcup_{\lambda \in \Gamma} E_\lambda$.  \(\square\)
In order to fix some notation, if $X$ is a linear space, $x_k \in X$ for all $k \in \mathbb{N}$ and \{i_1 < i_2 < \cdots \} is a subset of $\mathbb{N}$, we denote by
\[
\sum_{k=1}^{\infty} x_k \otimes e_{i_k}
\]
the $X$-valued sequence having the $i_k$-th coordinate equal to $x_k$ and all the other coordinates are zero.

**Theorem 3.3.** Let $X$ and $Y$ be Banach spaces, $\Gamma$ be an arbitrary set, $E$ be an invariant sequence space over $X$ and $E_\lambda$ be a strongly invariant sequence space over $Y$ for each $\lambda \in \Gamma$. If $f: X \to Y$ is compatible with $E_\lambda$ for all $\lambda \in \Gamma$ and $(E_\lambda)_{\lambda \in \Gamma}$ is nested, then $G (E, f, (E_\lambda)_{\lambda \in \Gamma})$ is either empty or pointwise $c$-spaceable.

**Proof.** Let us assume that $G (E, f, (E_\lambda)_{\lambda \in \Gamma})$ is non-empty and consider
\[
x = (x_j)_{j=1}^{\infty} \in G (E, f, (E_\lambda)_{\lambda \in \Gamma}).
\]
Notice that if $\mathbb{N}_x = \{j \in \mathbb{N} : x_j \neq 0\}$, then $\text{card} (\mathbb{N}_x) = \aleph_0$ because $f (0) = 0$ and $c_{00} (Y) \subset E_\lambda$ for each $\lambda \in \Gamma$. So, we can infer that $x^0 \neq 0$. First, we have to show that
\[
x^0 \in G (E, f, (E_\lambda)_{\lambda \in \Gamma}).
\]
We know that $(f (x_j))_{j=1}^{\infty} \notin \bigcup_{\lambda \in \Gamma} E_\lambda$ and, thus, $[(f (x_j))_{j=1}^{\infty}]^0 \notin \bigcup_{\lambda \in \Gamma} E_\lambda$ since, for each $\lambda \in \Gamma$, $E_\lambda$ is an strongly invariant sequence space. Let us denote $x^0 = (x_{jk})_{k=1}^{\infty}$, where $x_{jk}$ is the $k$-th non-null coordinate of $x$. Hence, we have to show that $(f (x_{jk}))_{k=1}^{\infty} \notin \bigcup_{\lambda \in \Gamma} E_\lambda$.

Since $f (0) = 0$, it follows that
\[
[(f (x_{jk}))_{k=1}^{\infty}]^0 = [(f (x_j))_{j=1}^{\infty}]^0 \notin \bigcup_{\lambda \in \Gamma} E_\lambda.
\]
Therefore, $(f (x_{jk}))_{k=1}^{\infty} \notin \bigcup_{\lambda \in \Gamma} E_\lambda$ and so $x^0 \in G (E, f, (E_\lambda)_{\lambda \in \Gamma})$.

It follows from Lemma 3.2 that exists $\mathbb{N}_1 \subset \mathbb{N}$ such that
\[
\text{card} (\mathbb{N}_1) = \text{card} (\mathbb{N} \setminus \mathbb{N}_1) = \aleph_0
\]
and
\[
(f (x_j))_{j \in \mathbb{N}_1} \notin \bigcup_{\lambda \in \Gamma} E_\lambda.
\]
Let us consider some sequence $(\mathbb{N}_i)_{i=2}^{\infty}$ of countable and pairwise disjoint subsets of $\mathbb{N}$ such that
\[
\mathbb{N} \setminus \mathbb{N}_1 = \bigcup_{i=2}^{\infty} \mathbb{N}_i.
\]
Denoting
\[
\mathbb{N}_i = \{i_1 < i_2 < i_3 < \cdots \},
\]
for \(i = 1, 2 \ldots\), let us consider the sequence \((y_i)_{i=1}^{\infty}\) defined by
\[
y_1 = x = \sum_{k=1}^{\infty} x_k \otimes e_k
\]
and, whenever \(i \geq 2\),
\[
y_i = \sum_{k=1}^{\infty} x_k \otimes e_{i_k}.
\]
Notice that \(y_i^0 = x^0\) and hence \(0 \neq y_i^0 \in E\) for all \(i\). Since \(E\) is an invariant sequence space, it follows that \(y_i \in E\) for all \(i \in \mathbb{N}\). Also note that the set \(\{y_i : i \in \mathbb{N}\}\) is linearly independent. In fact, let \(\lambda_1, \ldots, \lambda_k \in \mathbb{K}\) be such that
\[
\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_k y_k = 0
\]
We can see that, for all \(m \in \mathbb{N}\), the \(1\)-th coordinate of
\[
\lambda_1 y_1 + \lambda_2 y_2 + \cdots + \lambda_k y_k
\]
is
\[
\lambda_1 x_{1_m} = 0.
\]
and, since \(m\) can be chosen satisfying \(x_{1_m} \neq 0\), we have \(\lambda_1 = 0\). It follows that
\[
\lambda_2 y_2 + \cdots + \lambda_k y_k = 0.
\]
Let \(m \in \mathbb{N}\) be such that \(x_m \neq 0\). For all \(i = 2, \ldots, k\), the \(i_m\)-th coordinate of
\[
\lambda_2 y_2 + \cdots + \lambda_k y_k
\]
is
\[
\lambda_i x_{i_m} = 0
\]
and so \(\lambda_2 = \cdots = \lambda_k = 0\). Furthermore, \(y_i \in G(E, f, (E_\lambda)_{\lambda \in \Gamma})\). Indeed, since \(y_i^0 = x^0\), denoting \(y_i = \left( y_{(m)}^{(i)} \right)_{m=1}^{\infty}\), we have
\[
\left[ \left( f \left( y_{(m)}^{(i)} \right) \right)_{m=1}^{\infty} \right]^0 = \left[ \left( f \left( x_j \right) \right)_{j=1}^{\infty} \right]^0 \notin E_\lambda,
\]
for all \(i \in \mathbb{N}\) and all \(\lambda \in \Gamma\). Let \(K\) be the constant of Definition 2.3(b1) and let us take \(\bar{s} = 1\) if \(E\) is a Banach space and \(\bar{s} = s\) if \(E\) is a quasi-Banach space, \(0 < s < 1\). For each \((a_i)_{i=1}^{\infty} \in \ell_{\bar{s}}\),
\[
\sum_{i=1}^{\infty} \|a_i y_i\|_{E}^{\bar{s}} = \sum_{i=1}^{\infty} |a_i|^\bar{s} \cdot \|y_i\|_{E}^{\bar{s}} \leq K^{\bar{s}} \sum_{i=1}^{\infty} |a_i|^\bar{s} \cdot \|y_i^0\|_{E}^{\bar{s}}
\]
\[
= K^{\bar{s}} \|x^0\|_{E}^{\bar{s}} \sum_{i=1}^{\infty} |a_i|^\bar{s} < \infty.
\]
Hence, \(\sum_{i=1}^{\infty} \|a_i y_i\|_{E}^{\bar{s}} < \infty\) and, in both cases, we conclude that \(\sum_{i=1}^{\infty} a_i y_i\) converges in \(E\).
Therefore, the operator
\[
T : \ell_{\bar{s}} \rightarrow E
\]
\[
(a_k)_{k=1}^{\infty} \mapsto T ((a_k)_{k=1}^{\infty}) = \sum_{k=1}^{\infty} a_k y_k
\]
Thus, $(\mu_k)_{k=1}^{\infty}$ is chosen satisfying $p_1 = 0$, that is, the condition (b2) of Definition 2.3 assures that convergence in $k \to \infty$. In order to do this, we will check that, if $x \in E$ and we have, for $i \geq 2$, that the $i_p$-th coordinate of $T ((\mu_k)_{k=1}^{\infty})$ is $\mu_i x_p$ and we conclude that $\mu_i = 0$ for all $i \geq 2$.

Recalling that $x = y_1 \in T (\ell_2)$, let us show that $\overline{T (\ell_2)} = \ell_2 \cup \{0\}$.

In order to do this, we will check that, if $z = (z_n)_{n=1}^{\infty} \in \overline{T (\ell_2)}$ is a non-zero sequence, then $(f (z_n))_{n=1}^{\infty}$ is well-defined and linear. Let us see that $f (x_k)_{k=1}^{\infty} \notin \bigcup_{\lambda \in \Gamma} E_\lambda$, there is $m \in \mathbb{N}$ such that $x_{1m} \neq 0$; but the 1_m-th coordinate of $T ((\mu_k)_{k=1}^{\infty})$ is $\mu_1 x_1$, and we conclude that $\mu_1 = 0$. Now, if we fix $p \in \mathbb{N}$ such that $x_p \neq 0$, we have, for $i \geq 2$, that the $i_p$-th coordinate of $T ((\mu_k)_{k=1}^{\infty})$ is $\mu_i x_p$ and we conclude that $\mu_i = 0$ for all $i \geq 2$.

Notice that, for each $k \in \mathbb{N}$,

$$T \left( \left( a^{(k)}_i \right)_{i=1}^{\infty} \right) = \sum_{i=1}^{\infty} a^{(k)}_i y_i$$

$$= a^{(k)}_1 \sum_{k=1}^{\infty} x_k \otimes e_k + \sum_{i=2}^{\infty} a^{(k)}_i \sum_{k=1}^{\infty} x_k \otimes e_{i_k}$$

$$= \sum_{k=1}^{\infty} a^{(k)}_1 x_k \otimes e_k + \sum_{i=2}^{\infty} \sum_{k=1}^{\infty} a^{(k)}_i x_k \otimes e_{i_k}$$

$$= \left( w^{(k)}_r \right)_{r=1}^{\infty},$$

where

$$w^{(k)}_r = \begin{cases} a^{(k)}_1 x_1, & \text{if } r = 1, \\
 a^{(k)}_1 x_{1m} + a^{(k)}_i x_{im}, & \text{if } r = i, \text{ } i \geq 2. \end{cases}$$

The condition (b2) of Definition 2.3 assures that convergence in $E$ implies coordinate-wise convergence. Fixed some $p \in \mathbb{N}$ such that $x_{1p} \neq 0$, we have

$$z_{1p} = \lim_{k \to \infty} a^{(k)}_1 x_{1p} = \left( \lim_{k \to \infty} a^{(k)}_1 \right) x_{1p},$$

that is, $\lim_{k \to \infty} a^{(k)}_1$ exists. Let $a_1 = \lim_{k \to \infty} a^{(k)}_1$. On the other hand, if $n = i_p$, $i \geq 2$, where $p$ is chosen satisfying $x_p \neq 0$, then

$$z_{i_p} = \lim_{k \to \infty} \left( a^{(k)}_1 x_{i_p} + a^{(k)}_i x_{ip} \right).$$

Thus,

$$\left( \lim_{k \to \infty} a^{(k)}_i \right) x_p = \lim_{k \to \infty} a^{(k)}_i x_p$$

$$= \lim_{k \to \infty} \left[ \left( a^{(k)}_1 x_{i_p} + a^{(k)}_i x_p \right) - a^{(k)}_1 x_{ip} \right]$$

$$= \lim_{k \to \infty} \left( a^{(k)}_1 x_{ip} + a^{(k)}_i x_p \right) - \lim_{k \to \infty} a^{(k)}_1 x_{ip}$$

$$= z_{ip} - a_1 x_{ip},$$
that is, \( \lim_{k \to \infty} a_i^{(k)} \) exists and it will be denoted by \( a_i \). Therefore, coordinatewise convergence yields

\[
z_n = \begin{cases} 
a_1 x_{1_m}, & \text{if } n = 1_m \in \mathbb{N}_1, \\
a_1 x_{i_m} + a_i x_m, & \text{if } n = i_m \in \mathbb{N}_i \text{ and } i \geq 2. 
\end{cases}
\]

Since \( (z_n)_{n=1}^{\infty} \) is non-zero is immediate that \( a_i \neq 0 \) for some \( i \in \mathbb{N} \).

If \( a_1 \neq 0 \), we have \( z_{1_m} = a_1 x_{1_m} \) for all \( m \in \mathbb{N} \). Note that

\[
(f (z_{1_m}))_{m=1}^{\infty} = (f (a_1 x_{1_m}))_{m=1}^{\infty} = (f (a_1 x_j))_{j \in \mathbb{N}_1}.
\]

Since, for all \( \lambda \in \Gamma \), \( f \) is compatible with \( E_\lambda \) and \( (f (x_j))_{j \in \mathbb{N}_1} \notin E_\lambda \), we have

\[
(f (z_{1_m}))_{m=1}^{\infty} \notin E_\lambda
\]

for all \( \lambda \). Hence, there exists a subsequence of \( (f (z_n))_{n=1}^{\infty} \) not belonging to \( E_\lambda \) for all \( \lambda \). Since \( E_\lambda \) is a strongly invariant sequence space for all \( \lambda \in \Gamma \), it follows that

\[
(f (z_n))_{n=1}^{\infty} \notin E_\lambda
\]

for all \( \lambda \notin \Gamma \).

If \( a_1 = 0 \) and \( a_p \neq 0 \) for some \( p > 1 \), then \( z_{p_m} = a_p x_m \) for all \( m \in \mathbb{N} \) and, in this case, \( (f (z_{p_m}))_{m=1}^{\infty} = (f (a_p x_m))_{m=1}^{\infty} \). An analogous argument allows us to conclude that

\[
(f (z_n))_{n=1}^{\infty} \notin E_\lambda
\]

This completes the proof that \( z \in G (E, f, (E_\lambda)_{\lambda \in \Gamma}) \) and therefore \( G (E, f, (E_\lambda)_{\lambda \in \Gamma}) \) is pointwise c-spaceable.

We stress that the previous theorem does not cover results of the type (2.2), but this can be done with similar arguments as follows.

If \( F \) is an invariant sequence space over the scalar field \( \mathbb{K} \) and \( Y \) is a Banach space, we define

\[
F^w (Y) := \\{(x_j)_{j=1}^{\infty} \in Y^\mathbb{N} : (\varphi (x_j))_{j=1}^{\infty} \in F \text{ for all } \varphi \in Y'\}
\]

and note that

\[
\sup_{\|\varphi\| \leq 1} \| (\varphi (x_j))_{j=1}^{\infty} \|_F < \infty
\]

for all \( (x_j)_{j=1}^{\infty} \in F^w (Y) \) (see [17, p. 178]).

If \( F = \ell_p \), the respective space \( F^w (Y) \) is the well-known invariant sequence space \( \ell^w_p (Y) \). According to [17, Definition 3.3], if \( X \) and \( Y \) are Banach spaces and \( F \) is an invariant sequence space over \( \mathbb{K} \), then a map \( f : X \to Y \) such that \( f (0) = 0 \) is called strongly compatible with \( F^w (Y) \) if \( \varphi \circ f \) is compatible with \( F \) for all continuous linear functionals \( \varphi : Y \to \mathbb{K} \). It is obvious that any strongly non-contractive map \( f : X \to Y \) is strongly compatible with \( \ell^w_p (Y) \) and with

\[
c^w_0 (Y) := \\{(y_j)_{j=1}^{\infty} \in \ell_\infty (Y) : \lim_{j \to \infty} \varphi (y_j) = 0, \text{ for all } \varphi \in Y'\}
\]

The following result was proved in [17, Theorem 2.5]:
Theorem 3.4. Let $X$ and $Y$ be Banach spaces, $\Gamma$ an arbitrary set, $E$ an invariant sequence space over $X$ and, for all $\lambda \in \Gamma$, let $F_\lambda$ be a strongly invariant sequence space over $K$. If $f : X \to Y$ is strongly compatible with $F_\lambda^w(Y)$ for all $\lambda \in \Gamma$, then

$$G^w(E, f, (F_\lambda)_{\lambda \in \Gamma}) := \left\{ (x_j)_{j=1}^\infty \in E : (f(x_j))_{j=1}^\infty \notin \bigcup_{\lambda \in \Gamma} F_\lambda^w(Y) \right\}$$

is either empty or spaceable.

The proof of Theorem 3.3 can be adapted, mutatis mutandis, to obtain the following result, which shows that Theorem 3.4 is valid for pointwise spaceability.

Theorem 3.5. Let $X$ and $Y$ be Banach spaces, $\Gamma$ an arbitrary set, $E$ an invariant sequence space over $X$ and, for all $\lambda \in \Gamma$, let $F_\lambda$ be a strongly invariant sequence space over $K$. If $f : X \to Y$ is strongly compatible with $F_\lambda^w(Y)$ for all $\lambda \in \Gamma$ and $(F_\lambda^w(Y))_{\lambda \in \Gamma}$ is nested, then $G^w(E, f, (F_\lambda)_{\lambda \in \Gamma})$ is either empty or pointwise c-spaceable.

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