On Generalized Derivations of some classes of finite dimensional algebras

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Abstract. In this paper we investigated the concept of generalized derivations of finite dimensional algebras and their properties. The definition of the generalized derivation depends on some parameters and in particular on values of the parameters, we obtain classical concept of derivation and its generalizations. We use an algorithm for calculating the generalized derivations of Lie and Leibniz algebras.

1. Introduction
Some papers appear in the study of generalized derivations of algebras. For example Leger and Luks [1] discusses the generalized derivations of Lie algebras. Also in the work of Hartwig et al. [2], the concept of generalized derivations of Lie algebras were discussed and they refer to it as $(\sigma, \tau)$-derivation. Another profound work on generalized derivations of Lie algebra is by Novotny and Hrivnak [3] where important results were found in the work which they call $(\alpha, \beta, \gamma)$-derivation of Lie algebras. In this paper the concept of the generalized derivations is extended to some classes of Lie and Leibniz algebras. A simple algorithmic approach is adopted in the algebra of low dimensional cases where possible subalgebras and subspaces were found. In the case of Lie algebras the results in this work conforms with that of [3]. It is noteworthy to mention here that at first, the definition is given on an arbitrary algebra and later extended to algebras of different identities. All algebras here are defined over the field of complex numbers $\mathbb{C}$.

2. Preliminary
This section we begin with simple definitions and facts needed later on in the course of our discussions.

Definition 2.1. [4] Let $(E_1, \cdot)$ and $(E_2, \ast)$ be two arbitrary algebras. A linear mapping $f : E_1 \to E_2$ is homomorphism if $f(x \cdot y) = f(x) \ast f(y)$ for all $x, y \in E_1$.

A bijective homomorphism is called an isomorphism and the algebra $E_1$ and $E_2$ are called isomorphic, it is denoted by $E_1 \cong E_2$. If $E_1 = E_2 = E$ then $f$ is called an endomorphism of the algebra $E$ denoted by $\text{End}E$. The $\text{End}E$ is an associative algebra with respect the composition operation. It is an easy exercise to show that $E_1 \cong E_2$ if and only if $\text{End}E_1 \cong \text{End}E_2$. These isomorphisms are related to each other as follows: Let $\sigma : E_1 \to E_2$ be an isomorphism and
Let \( d \in \text{End}_{E_1} \). The function \( \mu(d) : \text{End}_{E_1} \cong \text{End}_{E_2} \) defined by \( \mu(d) = \sigma d \sigma^{-1} \) is an isomorphism of algebras \( \text{End}_{E_1} \) and \( \text{End}_{E_2} \). The multiplicative group of \( \text{End} \) is denoted by \( \text{Aut}(E) \) i.e.,
\[
\text{Aut}(E) = \{ \sigma \in \text{End}(E) \mid \sigma \text{ is a bijection} \}.
\]
Here is the definition of \((\alpha, \beta, \gamma)\)-derivation of algebras from [3].

**Definition 2.2.** Let \((E, \cdot)\) be an algebra and \(\alpha, \beta, \gamma\) be elements of \(\mathbb{C}\). A linear operator \(d \in \text{End}_E\) is a \((\alpha, \beta, \gamma)\)-derivation of \(E\) if for all \(x, y \in E\) such that \(\alpha d(x \cdot y) = \beta d(x) \cdot y + \gamma x \cdot d(y)\).

The set of all \((\alpha, \beta, \gamma)\)-derivations of \(E\) is denoted by \(\text{Der}_{(\alpha, \beta, \gamma)} E\) i.e.,
\[
\text{Der}_{(\alpha, \beta, \gamma)} E = \{d \in \text{End}_E \mid \alpha d(x \cdot y) = \beta d(x) \cdot y + \gamma x \cdot d(y), \forall x, y \in E\}.
\]
Clearly \(\text{Der}_{(\alpha, \beta, \gamma)} E\) is a subspace of \(\text{End}_E\).

**Proposition 2.1.** Let \(\sigma : E_1 \rightarrow E_2\) be an isomorphism of arbitrary algebras \((E, \cdot)\) and \((E, \ast)\). The mapping \(\sigma : \text{End}_{E_1} \rightarrow \text{End}_{E_2}\) defined by \(\mu(d) = \sigma d \sigma^{-1}\) is an isomorphism of vector spaces \(\text{Der}_{(\alpha, \beta, \gamma)} E_1\) and \(\text{Der}_{(\alpha, \beta, \gamma)} E_2\).

**Proof.** Due to the isomorphism relation we have \(x \ast y = \sigma^{-1}(x) \cdot \sigma^{-1}(y)\). Let \(d \in \text{End}_{E_1}\) such that \(\alpha d(x \cdot y) = \beta d(x) \cdot y + \gamma x \cdot d(y)\). For all \(x, y \in E\) we show that \(d \in \text{Der}_{(\alpha, \beta, \gamma)} E_2\). Indeed,
\[
\alpha \mu(d)(x \ast y) = \alpha \mu(d)(\sigma^{-1}(x) \cdot \sigma^{-1}(y)) = \alpha(\sigma \circ d \circ \sigma^{-1})(\sigma \circ y \cdot \sigma^{-1}(x) \cdot \sigma^{-1}(y)) = \alpha(\sigma \circ d)(\sigma^{-1}(x) \cdot \sigma^{-1}(y)) = \sigma(\alpha d(\sigma^{-1}(x) \cdot \sigma^{-1}(y))) = \sigma(\beta d(\sigma^{-1}(x) \cdot \sigma^{-1}(y)) + \gamma(\sigma^{-1}(x) \cdot d(\sigma^{-1}(y)) = \beta \mu(d)(x) \ast y + \gamma x \ast d(y).
\]

Another important result is about the intersection of two different subspaces of \(\text{Der}_{(\alpha, \beta, \gamma)} E\). We now restrict the result on Leibniz algebras:

**Proposition 2.2.** Let \(f : L_1 \rightarrow L_2\) be an isomorphism of two Leibniz algebras. Then the mapping \(g : \text{End}_{L_1} \rightarrow \text{End}_{L_2}\) defined by \(g(d) = df^{-1}\) is an isomorphism of vector spaces \(\text{Der}_{(\alpha, \beta, \gamma)} L_1 \cap \text{Der}_{(\alpha', \beta', \gamma')} L_1\) and \(\text{Der}_{(\alpha, \beta, \gamma)} L_2 \cap \text{Der}_{(\alpha', \beta', \gamma')} L_2\), that is \(\forall \alpha, \beta, \gamma, \alpha', \beta', \gamma' \in \mathbb{C}\),
\[
g(\text{Der}_{(\alpha, \beta, \gamma)} L_1 \cap \text{Der}_{(\alpha', \beta', \gamma')} L_1) = \text{Der}_{(\alpha, \beta, \gamma)} L_2 \cap \text{Der}_{(\alpha', \beta', \gamma')} L_2.
\]

**Proof.** Let \((L_1, [\cdot, \cdot])\) and \((L_2, [\cdot, \cdot])\) be two Leibniz algebras. The relation \(g(d) = df^{-1}\) implies that for any \(x, y \in L_2\) there is \([x, y] = df^{-1}(x, f^{-1}(y))\) holds true. From Definition 2.2, we can show that \(d \in \text{Der}_{(\alpha, \beta, \gamma)} L_1 \cap \text{Der}_{(\alpha', \beta', \gamma')} L_1\) if the two following relations hold, i.e
\[
ad(f^{-1}(x), f^{-1}(y)) = \beta(df^{-1}(x), f^{-1}(y)) + \gamma[f^{-1}(x), df^{-1}(y)]
\]
and
\[
\alpha d'(f^{-1}(x), f^{-1}(y)) = \beta'[df^{-1}(x), f^{-1}(y)] + \gamma'[f^{-1}(x), df^{-1}(y)]
\]
for all \(x, y \in L_2\). Applying the map \(f\) in the above relations, we have
\[
\alpha f df^{-1}[(x), (y)] = \beta[df^{-1}(x), f^{-1}(y)] + \gamma[f^{-1}(x), f df^{-1}(y)]
\]
and
\[
\alpha' df^{-1}(x, f^{-1}(y)) = \beta'[df^{-1}(x), f^{-1}(y)] + \gamma'[f^{-1}(x), df^{-1}(y)].
\]
Hence,
\[
d \in \text{Der}_{(\alpha, \beta, \gamma)} L_2 \cap \text{Der}_{(\alpha', \beta', \gamma')} L_2.
\]
3. The algorithm
In this section we give the description of \((\alpha, \beta, \gamma)\)-derivations of Lie and Leibniz algebras. We begin by using an algorithm [5] to find \((\alpha, \beta, \gamma)\)-derivations. Let \(\{e_1, e_2, \ldots, e_n\}\) be a basis of an \(n\)-dimensional arbitrary algebra \(E\). Then
\[
e_i \cdot e_j = \sum_{k=1}^{n} \gamma_{ij}^k e_k.
\] (1)
The coefficients \(\{\gamma_{ij}^k\} \in \mathbb{C}^{n^3}\) of the above linear combinations are called the structure constants of \(E\) on the basis \(\{e_1, e_2, \ldots, e_n\}\). The element \(d\) of \(\text{Der}(\alpha, \beta, \gamma)\) being a linear transformation of the vector space \(E\) is represented in a matrix form \([a_{ij}]_{i,j=1,2,\ldots,n}\), i.e.,
\[
d(e_i) = \sum_{j=1}^{n} d_{ji} e_j, \quad i = 1, 2, \ldots, n.
\] (2)
According to the Definition 2.2 we have:
\[
\alpha d(e_i e_j) = \beta d(e_i) e_j + \gamma e_i d(e_j), \quad i, j = 1, 2, \ldots, n.
\] (3)
Therefore, using eqn(1) and eqn(2) in eqn(3), we get
\[
\alpha \left( \sum_{t=1}^{n} \gamma_{ij}^t e_t \right) = \beta \left( \sum_{t=1}^{n} d_{it} e_t \right) e_j + \gamma e_i \left( \sum_{t=1}^{n} d_{jt} e_t \right).
\]
Solving and equating the coefficients of the basis vectors give us the following system of equations:
\[
\alpha \left( \sum_{t=1}^{n} \gamma_{ij}^t d_{st} \right) = \beta \left( \sum_{t=1}^{n} d_{it} \gamma_{kj}^s \right) + \left( \gamma \sum_{t=1}^{n} d_{jt} \gamma_{ti}^s \right).
\]
Consequently,
\[
\sum_{t=1}^{n} \left( \alpha \gamma_{ij}^t d_{st} - \beta d_{it} \gamma_{kj}^s - \gamma d_{jt} \gamma_{ti}^s \right) = 0.
\]
The system includes \(n^2 + 3\) variables which are \(\{\alpha, \beta, \gamma, d_{ij}\}\) where \(i, j = 1, 2, \ldots, n\). The solution to the system gives the description of the generalized derivations of \(E\) on the basis \(\{e_1, e_2, \ldots, e_n\}\). Now in the next section we apply the algorithm to two-dimensional and three-dimensional cases for Lie and Leibniz algebras.

4. Application for low dimensional cases
In this section, we apply the algorithm to Lie and Leibniz algebras.

The following definition can be found in [3], [4] and [6].

**Definition 4.1.** A Lie algebra \(L\) is a vector space over a field \(\mathbb{C}\) with a bilinear mapping \(L \times L \to L\) satisfying the following conditions:
\[
\begin{align*}
(i) \quad [x, y] &= -[y, x] \text{ for all } x \in L; \\
(ii) \quad [[x, y], z] + [[y, z], x] + [[z, x], y] &= 0 \text{ for all } x, y, z \in L.
\end{align*}
\]
Proposition 4.1. Let $L$ be a complex Lie algebra and $\alpha, \beta, \gamma \in \mathbb{C}$. Then for $\text{Der}_{(\alpha,\beta,\gamma)}L$ the values of $\alpha, \beta, \gamma$ are distributed as follows:

\[
\text{Der}_{(1,1,1)}L = \text{Der}L
\]

\[
\text{Der}_{(1,1,0)}L = \{d \in \text{End}L| [d(xy)] = [d(x), y]\};
\]

\[
\text{Der}_{(1,1,-1)}L = \{d \in \text{End}L| [d(xy)] = [x, d(y)]\};
\]

\[
\text{Der}_{(1,0,0)}L = \{d \in \text{End}L| d([xy]) = 0\};
\]

\[
\text{Der}_{(0,1,1)}L = \{d \in \text{End}L| [d(x), y] = -[x, d(y)]\};
\]

\[
\text{Der}_{(0,1,-1)}L = \{d \in \text{End}L| [d(x), y] = [x, d(y)]\};
\]

\[
\text{Der}_{(0,1,0)}L = \{d \in \text{End}L| [d(x), y] = 0\};
\]

\[
\text{Der}_{(\delta,1,1)}L = \{d \in \text{End}L| \delta[d(xy)] = [d(x), y] + [x, d(y)]\} \delta \neq 0;
\]

\[
\text{Der}_{(\delta,1,0)}L = \{d \in \text{End}L| \delta[d(xy)] = [d(x), y]\} \delta \neq 1;
\]

The proof is provided in [3].

We apply this proposition to two and three dimensional Lie algebras over complex field. The list of isomorphism classes (IC) of two and three-dimensional complex Lie algebras can be found in [3].

As a results, we provide generalized derivations of Lie algebras in Table 1.
Table 1. Generalized Derivations of two and three dimensional Lie algebras

| IC     | $(\alpha, \beta, \gamma)$ | $\text{Der}_{(\alpha, \beta, \gamma)} L$ | Dim |
|--------|---------------------------|--------------------------------------|-----|
| $(1,1,1)$ | $(0 \ 0 \ 0)$ | $(d_{21} \ d_{22})$ | 2 |
| $(0,1,1)$ | $(d_{11} \ d_{12})$ | $(d_{21} \ -d_{11})$ | 3 |
| $(1,1,0)$ | $(d_{11} \ 0)$ | $(0 \ d_{11})$ | 1 |
| $(1,0,0)$ | $(d_{11} \ 0)$ | $(0 \ d_{21})$ | 2 |
| $L_2^1$ | $(0,1,1)$ | $(d_{11} \ 0 \ d_{22})$ | 3 |
| $(0,1,0)$ | trivial | 0 |
| $(1,1,-1)$ | trivial | 0 |
| $(\delta, 1, 0) \delta \neq 1$ | trivial | 0 |
| $(\delta, 1, 1) \delta \neq 0$ | $(0 \ 0 \ d_{22})$ | $(d_{21} \ d_{22})$ | 2 |
| $(1,1,1)$ | $(d_{11} \ d_{12})$ | $(d_{13} \ d_{22})$ | 6 |
| $(0,1,1)$ | $(d_{11} \ d_{12})$ | $(d_{22} \ -d_{11})$ | 6 |
| $(1,1,0)$ | $(d_{11} \ d_{12})$ | $(d_{13} \ d_{22})$ | 5 |
| $(1,0,0)$ | $(d_{11} \ d_{12})$ | $(d_{13} \ d_{23})$ | 6 |
| $L_3^2$ | $(0,1,0)$ | $(d_{11} \ d_{12})$ | 3 |
| $(1,1,-1)$ | $(d_{11} \ d_{12})$ | $(d_{13} \ d_{23})$ | 4 |
| $(0,1,-1)$ | $(d_{11} \ d_{12})$ | $(d_{13} \ d_{23})$ | 4 |
| $(\delta, 1, 0) \delta \neq 1$ | $(d_{11} \ d_{12})$ | $(d_{13} \ d_{23})$ | 3 |
| $(\delta, 1, 1) \delta \neq 0$ | $(d_{11} \ d_{12})$ | $(d_{13} \ d_{23})$ | 6 |

| IC     | $(\alpha, \beta, \gamma)$ | $\text{Der}_{(\alpha, \beta, \gamma)} L$ | Dim |
|--------|---------------------------|--------------------------------------|-----|
| $(1,1,1)$ | $(0 \ 0 \ 0)$ | $(d_{22} \ d_{23})$ | 4 |
| $(0,1,1)$ | $(d_{11} \ d_{12})$ | $(d_{21} \ -d_{11})$ | 6 |
| $(1,1,0)$ | $(d_{11} \ 0)$ | $(0 \ d_{11})$ | 3 |
| $(1,0,0)$ | $(d_{11} \ 0)$ | $(0 \ d_{11})$ | 5 |
| $L_3^1$ | $(0,1,1)$ | $(d_{11} \ d_{12})$ | 6 |
| $(0,1,0)$ | trivial | 0 |
| $(1,1,-1)$ | trivial | 0 |
| $(\delta, 1, 0) \delta \neq 1$ | trivial | 0 |
| $(\delta, 1, 1) \delta \neq 0$ | $(d_{11} \ d_{12})$ | $(d_{13} \ d_{23})$ | 3 |
| IC    | $\alpha,\beta,\gamma$ | $\text{Dim}$ | IC    | $\alpha,\beta,\gamma$ | $\text{Dim}$ |
|-------|-----------------------|--------------|-------|-----------------------|--------------|
| (1,1,1) | $d_{11} d_{12} d_{13} \begin{pmatrix} d_{21} & d_{22} & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$ | 5 | (1,1,1) | $d_{11} d_{12} d_{13} \begin{pmatrix} d_{12} & d_{12} & 0 \\ d_{31} & 0 & 0 \end{pmatrix}$ | 4 |
| (0,1,1) | $d_{11} 0 d_{13} \begin{pmatrix} 0 & 0 & d_{23} \\ -d_{11} & 0 & 0 \end{pmatrix}$ | 3 | (0,1,1) | $d_{11} 0 d_{13} \begin{pmatrix} 0 & d_{11} & 0 \\ 0 & 0 & -d_{11} \end{pmatrix}$ | 3 |
| (1,1,0) | $d_{11} 0 0 \begin{pmatrix} 0 & d_{11} & 0 \\ 0 & 0 & d_{11} \end{pmatrix}$ | 1 | (1,1,0) | $d_{11} 0 0 \begin{pmatrix} 0 & d_{11} & 0 \\ 0 & 0 & d_{11} \end{pmatrix}$ | 1 |
|       | trivial               | 0            | (1,0,0) | trivial               | 0 |
|       | (0,1,0)               | trivial      | (1,1,-1)| trivial               | 0 |
|       | (1,1,1)               | trivial      | (0,1,1) | trivial               | 0 |
|       | $d_{11} 0 0 \begin{pmatrix} 0 & d_{11} & 0 \\ 0 & 0 & d_{11} \end{pmatrix}$ | 1 | (0,1,1) | $d_{11} 0 0 \begin{pmatrix} 0 & d_{11} & 0 \\ 0 & 0 & d_{11} \end{pmatrix}$ | 1 |
| (1,0,-1)| trivial               | 0            | (0,1,-1)| trivial               | 0 |
| (1,1,1) | $d_{11} 0 d_{13} \begin{pmatrix} 0 & d_{21} & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$ | 4 | $\text{Dim}$ | (1,1,1) | $d_{11} 0 d_{13} \begin{pmatrix} 0 & d_{21} & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$ | 4 |
| (0,1,1) | $d_{11} 0 d_{13} \begin{pmatrix} 0 & d_{11} & d_{11} \\ 0 & d_{32} & -d_{11} \end{pmatrix}$ | 3 | (0,1,-1)| trivial               | 0 |
| (1,1,0) | $d_{11} 0 0 \begin{pmatrix} 0 & d_{11} & 0 \\ 0 & 0 & d_{11} \end{pmatrix}$ | 1 | (0,1,1) | $d_{11} 0 0 \begin{pmatrix} 0 & d_{11} & 0 \\ 0 & 0 & d_{11} \end{pmatrix}$ | 1 |
| (1,0,0) | $0 0 d_{13} \begin{pmatrix} 0 & d_{23} & d_{33} \\ 0 & 0 & 0 \end{pmatrix}$ | 3 | (1,0,0) | trivial               | 0 |
|       | trivial               | 0            | (0,1,0) | trivial               | 0 |
|       | (1,1,-1) | trivial      | (1,1,-1)| trivial               | 0 |
|       | $d_{11} 0 0 \begin{pmatrix} 0 & d_{23} & d_{33} \\ 0 & 0 & 0 \end{pmatrix}$ | 0 | (0,1,-1)| trivial               | 0 |
|       | (0,1,-1) | trivial      | (0,1,-1)| trivial               | 0 |
|       | $d_{11} 0 0 \begin{pmatrix} 0 & d_{11} & 0 \\ 0 & 0 & d_{11} \end{pmatrix}$ | 0 | (0,1,0) | trivial               | 0 |
|       | (d, 1, 0 $\neq 1$) | trivial      | (0,1,0) | trivial               | 0 |
|       | (d, 1, 1 $\neq 0$) | trivial      | (0,1,-1)| trivial               | 0 |
|       | (d, 1, 1 $\neq 0$) | trivial      | (1,0,-1)| trivial               | 0 |
|       | $a_{d_{11}} 0 d_{13} \begin{pmatrix} 0 & a_{d_{11}} & d_{23} \\ 0 & 0 & -a_{d_{11}} \end{pmatrix}$ | 3 | (2,1,1) | $d_{11} 0 d_{13} \begin{pmatrix} 0 & d_{11} & 0 \\ 0 & 0 & d_{11} \end{pmatrix}$ | 1 |
|       | (d, 1, 1) | trivial      | (d, 1, 1) | trivial               | 0 |
|       | $c_{d_{11}} 0 d_{13} \begin{pmatrix} d_{21} & c_{d_{11}} & d_{23} \\ 0 & 0 & -c_{d_{11}} \end{pmatrix}$ | 4 | $(-1,1,1) | d_{11} 0 d_{13} \begin{pmatrix} 2d_{21} & -2d_{21} & -2d_{21} \\ d_{11} & -2d_{32} & d_{33} \end{pmatrix}$ | 6 |
|       | (d, 1, 1) | trivial      | (d, 1, 1) | trivial               | 0 |
Proposition 4.2. Let hence Leibniz algebras are non commutative analogue of Lie algebras. Then for Der\(2\) 

Definition 4.2. A Leibniz algebra \(\alpha\) is a vector space \(V\) over a field \(\mathbb{C}\) equipped with a bilinear mapping \([\cdot,\cdot]: L \times L \rightarrow L\) which satisfies the following identity:

\[
[x, [y, z]] = [x, y], z] - [x, [y, z]], \forall x, y \in L.
\]

Note that for all \(x \in L\) if the identity \([x, x] = 0\) holds then the Leibniz identity becomes Lie, hence Leibniz algebras are non commutative analogue of Lie algebras.

Proposition 4.2. Let \(L\) be a Leibniz algebra over a field \(\mathbb{C}\), with \(\text{char} \neq 0\) and \(\alpha, \beta, \gamma, \in \mathbb{C}\). Then for \(\text{Der}_{(\alpha,\beta,\gamma)}L\) the values of \(\alpha, \beta, \gamma\) are given as follows;

\[
\begin{align*}
\text{Der}_{(1,1,1)}L &= \text{Der}L; \\
\text{Der}_{(1,1,0)}L &= \{d \in \text{End}L \mid [d(xy)] = [d(x), y]\}; \\
\text{Der}_{(1,1,-1)}L &= \{d \in \text{End}L \mid [d(xy)] = [d(x), y] - [x, d(y)]\}; \\
\text{Der}_{(1,0,0)}L &= \{d \in \text{End}L \mid [d(xy)] = 0\}; \\
\text{Der}_{(0,1,1)}L &= \{d \in \text{End}L \mid [d(xy)] = [x, d(y)]\}; \\
\text{Der}_{(0,1,-1)}L &= \{d \in \text{End}L \mid [d(xy)] = [x, d(y)]\}; \\
\text{Der}_{(0,0,1)}L &= \{d \in \text{End}L \mid [d(xy)] = 0\}; \\
\text{Der}_{(\delta,1,0)}L &= \{d \in \text{End}L \mid [d(xy)] = [d(x), y], \forall \delta \neq 0, 1\}.
\end{align*}
\]

Proof. Consider the Leibniz identity \([x, [y, z]] = [[x, y], z] + [y, [x, z]].\) Now applying the operator \(d\), where \(d\) is an \((\alpha, \beta, \gamma)\) derivation on the LHS and RHS of the identity, we obtain two cases, i.e \(\beta + \gamma = 0\) and \(\beta + \gamma \neq 0\).

1. Suppose \(\beta + \gamma = 0\). Here we consider two cases again. Either \(\beta = \gamma = 0\) or \(\beta = -\gamma \neq 0\).
   i). If \(\beta = \gamma = 0\), it is clear that

\[
\text{Der}_{(\alpha,\beta,\gamma)}L = \text{Der}_{(\alpha,0,0)}L.
\]

When \(\alpha = 1\), we get \(\text{Der}_{(1,0,0)}L\), and when \(\alpha = 0\) we have \(\text{Der}_{(0,0,0)}L\).

ii) If \(\beta = -\gamma \neq 0\), it is obvious that

\[
\text{Der}_{(\alpha,\beta,\gamma)}L = \text{Der}_{(0,\beta-\gamma,\beta)}L \cap \text{Der}_{(2\alpha,0,0)}L,
\]

so,

\[
\text{Der}_{(\alpha,\beta,\gamma)}L = \text{Der}_{(0,1,-1)}L \cap \text{Der}_{(\alpha,0,0)}L.
\]

In the same way,

\[
\text{Der}_{(\alpha,1,-1)}L = \text{Der}_{(0,1,-1)}L \cap \text{Der}_{(\alpha,0,0)}L.
\]

Hence we have

\[
\text{Der}_{(\alpha,\beta,\gamma)}L = \text{Der}_{(\alpha,1,-1)}L.
\]

When \(\alpha = 1\), we get \(\text{Der}_{(1,1,-1)}L\) but when \(\alpha = 0\), then we have \(\text{Der}_{(0,1,-1)}L\).

2. Suppose \(\beta + \gamma \neq 0\). We consider another two cases: Either \(\beta - \gamma \neq 0\) or \(\beta = \gamma \neq 0\).
   i). If \(\beta - \gamma \neq 0\), then

\[
\text{Der}_{(\alpha,\beta,\gamma)}L = \text{Der}_{(0,\beta-\gamma,\beta)}L \cap \text{Der}_{(2\alpha,\beta+\gamma,\beta+\gamma)}L.
\]
But since $\beta + \gamma = 1$, and $\beta - \gamma = 1$. We get $\beta = 1$, and $\gamma = 0$. So, $\alpha = \alpha/\beta + \gamma = \alpha/1 = \alpha$. Hence

$$\text{Der}_{(\alpha,\beta,\gamma)}L = \text{Der}_{(\alpha,\beta,0)}L.$$ 

When $\alpha = 0, \beta = 1, \gamma = 0$, we get $\text{Der}_{(0,1,0)}L$, and when $\alpha = 1, \beta = 1, \gamma = 0$, then, we obtain $\text{Der}_{(1,1,0)}L$.

ii). If $\beta = \gamma \neq 0$, we get

$$\text{Der}_{(\alpha,\beta,\gamma)}L = \text{Der}_{(\alpha,1,1)}L.$$ 

For $\alpha = 1$, we have $\text{Der}_{(1,1,1)}L$. However, when $\delta \neq 0, 1$ and $\gamma = 0$, we have $\text{Der}_{(\delta,1,0)}L$ but for $\gamma = 1$, we yield $\text{Der}_{(\delta,1,1)}L$. 

The dimension of the vector space $\text{Der}_{(\alpha,\beta,\gamma)}L$. For any $\alpha, \beta, \gamma \in \mathbb{C}$ is an isomorphism invariant of Leibniz algebras.

From [7] and [9], the list of IC of two and three-dimensional complex Leibniz algebras are used to obtain their generalized derivation.

The following table describes the $(\alpha, \beta, \gamma)$-derivations of Leibniz algebras.
Table 2. Generalized Derivations of two and three dimensional Leibniz algebras

| IC        | $(\alpha, \beta, \gamma)$ | $\text{Der}_{(\alpha, \beta, \gamma)}A$ | Dim | $(\alpha, \beta, \gamma)$ | $\text{Der}_{(\alpha, \beta, \gamma)}A$ | Dim |
|-----------|--------------------------|---------------------------------|-----|--------------------------|---------------------------------|-----|
| $L_2^1$   | $(1,1,1)$                | $(d_{11} 0 0)\begin{pmatrix} 2d_{11} & 0 \\ d_{21} & d_{21} \end{pmatrix}$ | 2   | $(1,1,0)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{21} \end{pmatrix}$ | 2   |
|           | $(1,1,-1)$               | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{21} \end{pmatrix}$ | 2   | $(1,0,0)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{21} \end{pmatrix}$ | 2   |
|           | $(0,1,1)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 3   | $(0,1,-1)$               | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 2   |
|           | $(0,1,0)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 2   | $(\delta,1,0)$           | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 3   |
|           | $(1,1,1)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 2   | $(1,1,0)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 1   |
| $L_2^2$   | $(1,1,-1)$               | trivial                          | 0   | $(1,0,0)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{21} \end{pmatrix}$ | 2   |
|           | $(0,1,1)$                | $(d_{11} 0 0)\begin{pmatrix} 2d_{11} & 0 \\ d_{21} & d_{21} \end{pmatrix}$ | 2   | $(0,1,-1)$               | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{21} \end{pmatrix}$ | 1   |
|           | $(0,1,0)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 2   | $(\delta,1,0)$           | trivial                          | 0   |
|           | $(1,1,1)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 2   | $(1,1,0)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 1   |
| $L_2^3$   | $(1,1,-1)$               | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{21} \end{pmatrix}$ | 1   | $(1,0,0)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 2   |
|           | $(0,1,1)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{21} \end{pmatrix}$ | 1   | $(0,1,-1)$               | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 1   |
|           | $(0,1,0)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 2   | $(\delta,1,0)$           | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 2   |
|           | $(1,1,1)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 2   | $(1,1,0)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 1   |
| $L_3^1$   | $(1,1,-1)$               | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{21} \end{pmatrix}$ | 2   | $(1,0,0)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 3   |
|           | $(0,1,1)$                | $(d_{11} 0 0)\begin{pmatrix} 0 & 0 \\ d_{21} & d_{22} \end{pmatrix}$ | 2   | $(1,0,-1)$               | trivial                          | 0   |
|           | $(0,1,0)$                | trivial                          | 0   | $(\delta,1,0)$           | trivial                          | 0   |
Continued from previous page

| IC     | $(\alpha,\beta,\gamma)$ | $\text{Der}_{(\alpha,\beta,\gamma)} A$ | Dim | $(\alpha,\beta,\gamma)$ | $\text{Der}_{(\alpha,\beta,\gamma)} A$ | Dim |
|--------|--------------------------|--------------------------------------|-----|--------------------------|--------------------------------------|-----|
| $(1,1,1)$ | $\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{22} & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$ | 3 | $(1,1,0)$ | $\begin{pmatrix} d_{11} & 0 & d_{13} \\ d_{21} & 0 & d_{21} \\ 0 & 0 & d_{33} \end{pmatrix}$ | 4 |
| $L_3^2$ | $(1,1,-1)$ | $\begin{pmatrix} d_{11} & 0 & d_{11} \\ 0 & 0 & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$ | 2 | $(1,0,0)$ | $\begin{pmatrix} 0 & 0 & d_{21} \\ 0 & 0 & d_{23} \\ 0 & 0 & d_{33} \end{pmatrix}$ | 3 |
| $(0,1,1)$ | $\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & 0 & -d_{11} \\ 0 & d_{11} & d_{23} \end{pmatrix}$ | 2 | $(0,1,-1)$ | $\begin{pmatrix} d_{11} & 0 & d_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | 2 |
| $(0,1,0)$ | trivial | 0 | $(0,1,\delta)$ | $\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{11} & \delta d_{23} \\ 0 & 0 & d_{11} \end{pmatrix}$ | 2 |
| $(1,1,1)$ | $\begin{pmatrix} 0 & 0 & d_{13} \\ 0 & d_{22} & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$ | 3 | $(1,1,0)$ | $\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{11} & 0 \\ 0 & 0 & d_{11} \end{pmatrix}$ | 1 |
| $L_3^3$ | $(1,1,-1)$ | $\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ 0 & d_{11} + d_{13} & d_{11} \\ 0 & d_{13} & d_{11} \end{pmatrix}$ | 3 | $(1,0,0)$ | $\begin{pmatrix} 0 & d_{12} & d_{13} \\ 0 & 0 & d_{24} \\ 0 & 0 & d_{34} \end{pmatrix}$ | 4 |
| $(0,1,1)$ | $\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | 4 | $(0,1,-1)$ | $\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ 0 & -d_{32} & -d_{32} \\ 0 & \delta d_{32} & \delta d_{32} \end{pmatrix}$ | 3 |
| $(0,1,0)$ | $\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | 3 | $(\delta,1,0)$ | $\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ 0 & d_{32} & d_{32} \\ 0 & -\delta d_{32} & -\delta d_{32} \end{pmatrix}$ | 4 |
| $(1,1,1)$ | $\begin{pmatrix} 0 & d_{12} & d_{13} \\ 0 & d_{22} & 0 \\ 0 & -2d_{22} & 0 \end{pmatrix}$ | 3 | $(1,1,0)$ | $\begin{pmatrix} 0 & d_{12} & d_{13} \\ 0 & d_{32} & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | 3 |
| $L_3^4$ | $(1,1,-1)$ | $\begin{pmatrix} 0 & d_{12} & d_{13} \\ 0 & 0 & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$ | 3 | $(1,0,0)$ | $\begin{pmatrix} 0 & d_{12} & d_{13} \\ d_{22} & 0 & 0 \\ d_{32} & 0 & 0 \end{pmatrix}$ | 4 |
| $(0,1,1)$ | $\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ 0 & d_{22} & d_{23} \\ 0 & -d_{22} & -d_{22} \end{pmatrix}$ | 5 | $(0,1,-1)$ | $\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ 0 & d_{22} & d_{23} \\ 0 & d_{33} & 0 \end{pmatrix}$ | 5 |
| $(0,1,0)$ | $\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | 3 | $(\delta,1,0)$ | $\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ 0 & \delta d_{22} & \delta d_{32} \\ 0 & d_{32} & d_{33} \end{pmatrix}$ | 6 |
| $(1,1,1)$ | $\begin{pmatrix} 0 & d_{12} & d_{13} \\ 0 & 0 & d_{23} \\ 0 & -d_{23} & 0 \end{pmatrix}$ | 3 | $(1,1,0)$ | $\begin{pmatrix} 0 & d_{12} & d_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | 2 |
| $L_3^5$ | $(1,1,-1)$ | $\begin{pmatrix} 2d_{22} & d_{12} & 0 \\ d_{21} & d_{22} & d_{23} \\ 0 & 0 & 0 \end{pmatrix}$ | 4 | $(1,0,0)$ | $\begin{pmatrix} 0 & d_{12} & d_{13} \\ 0 & d_{22} & d_{23} \\ 0 & d_{32} & d_{33} \end{pmatrix}$ | 6 |
| $(0,1,1)$ | $\begin{pmatrix} d_{11} & 0 & d_{13} \\ 0 & 0 & d_{23} \\ 0 & -d_{23} & 0 \end{pmatrix}$ | 3 | $(0,1,-1)$ | $\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{11} & 0 \\ 0 & 0 & d_{11} \end{pmatrix}$ | 1 |
| $(0,1,0)$ | $\begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ | 4 | $(\delta,1,0)$ | $\begin{pmatrix} d_{11} & 0 & 0 \\ 0 & d_{11} & 0 \\ 0 & 0 & d_{11} \end{pmatrix}$ | 1 |
Continued from previous page

| IC   | \((\alpha, \beta, \gamma)\) | \(\text{Dim} (\alpha, \beta, \gamma)\) | \(\text{Dim} (\alpha, \beta, \gamma)\) | \(\text{Dim} (\alpha, \beta, \gamma)\) | \(\text{Dim} (\alpha, \beta, \gamma)\) |
|------|-----------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| \((1,1,1)\) | \(d_{11} \quad d_{12} \quad 0\) | \(d_{12} \quad d_{22} \quad 0\) | \(d_{11} \quad d_{12} \quad 0\) | \(d_{12} \quad d_{11} \quad 0\) | \(d_{12} \quad d_{11} \quad 0\) |
| \((1,1,0)\) | \(d_{11} \quad 0 \quad d_{23}\) | \(0 \quad d_{22} \quad d_{23}\) | \(0 \quad d_{12} \quad d_{11}\) | \(0 \quad 0 \quad d_{33}\) | \(0 \quad 0 \quad d_{33}\) |
| \(L^0_{3} (1,1,-1)\) | \(d_{11} \quad 0 \quad 0\) | \(0 \quad d_{11} \quad 0\) | \(0 \quad 0 \quad -d_{11}\) | \(1 \quad (0,1,-1)\) | \(d_{11} \quad 0 \quad 0\) |
| \((0,1,0)\) | \(d_{31} \quad d_{32} \quad d_{33}\) | \(0 \quad 0 \quad 0\) | \(0 \quad 0 \quad 0\) | \(0 \quad 0 \quad 0\) | \(0 \quad 0 \quad 0\) |
| \(L^7_{3} (1,1,-1)\) | \(0 \quad 0 \quad 0\) | \(d_{11} \quad 0 \quad 0\) | \(d_{11} \quad 0 \quad 0\) | \(d_{11} \quad 0 \quad 0\) | \(d_{11} \quad 0 \quad 0\) |
| \((0,1,0)\) | \(d_{31} \quad d_{32} \quad d_{33}\) | \(0 \quad 0 \quad 0\) | \(0 \quad 0 \quad 0\) | \(0 \quad 0 \quad 0\) | \(0 \quad 0 \quad 0\) |
| \(L^8_{3} (1,1,-1)\) | \(0 \quad 0 \quad 0\) | \(d_{11} \quad 0 \quad 0\) | \(d_{11} \quad 0 \quad 0\) | \(d_{11} \quad 0 \quad 0\) | \(d_{11} \quad 0 \quad 0\) |
| \((1,1,1)\) | \(d_{11} \quad 0 \quad d_{13}\) | \(d_{13} \quad 2d_{23}\) | \(d_{13} \quad 0\) | \(d_{12} \quad d_{11} \quad d_{23}\) | \(d_{12} \quad d_{11} \quad d_{23}\) |
| \(L^9_{3} (1,1,-1)\) | \(0 \quad 0 \quad 0\) | \(0 \quad 0 \quad 0\) | \(0 \quad 0 \quad 0\) | \(0 \quad 0 \quad 0\) | \(0 \quad 0 \quad 0\) |
| \((0,1,0)\) | \(d_{11} \quad 0 \quad 0\) | \(d_{11} \quad 0 \quad 0\) | \(d_{11} \quad 0 \quad 0\) | \(d_{11} \quad 0 \quad 0\) | \(d_{11} \quad 0 \quad 0\) |
| \(L^9_{3} (1,1,-1)\) | \(0 \quad 0 \quad 0\) | \(0 \quad 0 \quad 0\) | \(0 \quad 0 \quad 0\) | \(0 \quad 0 \quad 0\) | \(0 \quad 0 \quad 0\) |
where
\[ d_{01} = d_{11} - d_{22}, \quad d_{02} = d_{11} + d_{33}, \quad d_{03} = d_{11} + d_{21}, \quad d_{04} = d_{12} + d_{22} - d_{21}, d_{05} = d_{12} + d_{22}, \]
\[ d_{06} = d_{11} + d_{21} - d_{22}, d_{07} = d_{21} + d_{33}, \quad d_{08} = d_{22} + d_{33}, \]
\[ d_{09} = d_{12} - d_{33}, \quad d_{01} = d_{11} - d_{22} - d_{21}, \quad g = d_{11} + 1/\alpha d_{11}, \]
\[ a = \sigma - 1, \quad c = a - 1, \quad e = 1/a - 1, \]
\[ f = 1/\alpha - 1. \]

5. Conclusion
The generalized derivations of Lie and Leibniz algebras of finite dimensional are studied. Two dimensional of Lie and Leibniz algebras have the dimension of the space of generalized derivations ranges between zero and three. While, ranges zero until six of the dimension of the space of generalized derivation of Leibniz algebras in dimension three are obtained.

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References
[1] Leger, G. and Luks, E. (2000) Generalized Derivations of Lie algebras, J. Algebra, 228, 165-203.
[2] Hartwig, J., Larsson D. and Silvestrov S. (2006) Deformation of Lie algebras using (\sigma, \tau)-derivation, Journal of algebra, 38(2), 109-138.
[3] Novotny, P. and Hrivnak, J. (2008) On (\alpha, \beta, \gamma)-derivation of Lie algebras and corresponding invariant functions, J. Geom. Phys., 58, 208-217.
[4] Hrivnak, J. (2007) Invariants of Lie algebras PhD Thesis, Faculty of Nuclear Science and Physical Engineering, Czech Technical University, Prague.
[5] Rakhimov, I. S., Said Husain, Sh. K. and Abdulhadi, A. (2016) On Generalized derivations of finite dimensional associative algebras, FEIIC International journal of Engineering and Technology, vol.13, No 2, pp 121-126.
[6] Sharifah Kartini Said Hussain, I. S. Rakhimov and W. Basri (2017) Algorithms for computations of Loday algebras invariants, AIP Conference Proceedings, vol 1830, 070028 pg 1-9.
[7] Gorbatsevich, V. V. (2013) On some basic properties of Leibniz algebras, arXiv preprint arXiv:1302.3345.
[8] Loday, J. L. and Prishvili, T. (1993) Universal enveloping algebras of Leibniz algebras and (co)homology, Mathematiche Annalen., 296(1), 139-155.
[9] Rikhsiboev, I. M. and Rakhimov, I. S. (2012) Classification of three dimensional complex Leibniz algebras, AIP Conference Proceedings, vol 1450, no. 1 pg 358-362.