GEOMETRY OF CENTROAFFINE SURFACES IN $\mathbb{R}^5$

NATHANIEL BUSHEK AND JEANNE N. CLELLAND

Abstract. We use Cartan’s method of moving frames to compute a complete set of local invariants for nondegenerate, 2-dimensional centroaffine surfaces in $\mathbb{R}^5 \setminus \{0\}$ with nondegenerate centroaffine metric. We then give a complete classification of all homogeneous centroaffine surfaces in this class.

1. Introduction

An immersion $\tilde{f} : M \to \mathbb{R}^n \setminus \{0\}$ is called a centroaffine immersion (cf. Definition 2.1) if the position vector $\tilde{f}(x)$ is transversal to the tangent space $\tilde{f}_*(T_x M)$ for all $x \in M$. Centroaffine geometry is the study of those properties of centroaffine immersions that are invariant under the action of the centroaffine group $GL(n, \mathbb{R})$ on $\mathbb{R}^n \setminus \{0\}$. Much attention has been given to the study of centroaffine curves and hypersurfaces ([11], [8], [9], [10], [13], [17], [5]), and more recently to the study of centroaffine immersions of codimension 2 ([13], [14], [15], [23], [21], [22]). In [20], applications of centroaffine geometry to the problem of feedback equivalence in control theory are discussed.

In this paper, we consider the case of a 2-dimensional centroaffine surface in $\mathbb{R}^5 \setminus \{0\}$. We will use Cartan’s method of moving frames to construct a complete set of local invariants for a large class of such surfaces under certain nondegeneracy assumptions (cf. Definition 3.1, Assumption 3.2). In addition, we give a complete classification of the homogeneous centroaffine surfaces in this class—i.e., those that admit a 3-dimensional Lie group of symmetries that acts transitively on an adapted frame bundle canonically associated to the surface (cf. Definition 6.1). Our primary results are Theorem 4.3 and Theorem 5.3, which describe local invariants for centroaffine surfaces with definite and indefinite centroaffine metrics, respectively, and Theorem 6.6, which describes the homogeneous examples.

The paper is organized as follows. In §2, we introduce the basic concepts of centroaffine geometry and centroaffine surfaces in $\mathbb{R}^5 \setminus \{0\}$, including the centroaffine frame bundle and the Maurer-Cartan forms. In §3, we begin the method of moving frames and identify first-order invariants for centroaffine surfaces. Based on these invariants, surfaces may be locally classified as “spacelike,” “timelike,” or “null.” In §4 and §5, we continue the method of moving frames for the spacelike and timelike cases, respectively. (We do not consider the null case here; it may be explored in a future paper.) Finally, in §6, we classify the homogeneous examples in both the spacelike and timelike cases.

2. Centroaffine surfaces in $\mathbb{R}^5$, adapted frames, and Maurer-Cartan forms

Five-dimensional centroaffine space is the manifold $\mathbb{R}^5 \setminus \{0\}$, equipped with a natural $GL(5, \mathbb{R})$-action. Specifically, $GL(5, \mathbb{R})$ acts on $\mathbb{R}^5 \setminus \{0\}$ by left multiplication: for $x \in \mathbb{R}^5 \setminus \{0\}$ and $g \in GL(5, \mathbb{R})$,
The group $GL(5, \mathbb{R})$ may be regarded as a principal bundle over $\mathbb{R}^5 \setminus \{0\}$: write an arbitrary element $g \in GL(5, \mathbb{R})$ as

$$g = [e_0 \ e_1 \ e_2 \ e_3 \ e_4],$$

where $e_0, \ldots, e_4 \in \mathbb{R}^5 \setminus \{0\}$ are linearly independent column vectors. Then define the bundle map $\pi : GL(5, \mathbb{R}) \to \mathbb{R}^5 \setminus \{0\}$ by

$$\pi([e_0 \ e_1 \ e_2 \ e_3]) = e_0.$$  

The fiber group $H$ is isomorphic to the stabilizer of the point

$$[1 \ 0 \ 0 \ 0 \ 0]^T \in \mathbb{R}^5 \setminus \{0\},$$

and this construction endows the manifold $\mathbb{R}^5 \setminus \{0\}$ with the structure of the homogeneous space $GL(5, \mathbb{R})/H$.

We also think of the bundle $\pi : GL(5, \mathbb{R}) \to \mathbb{R}^5 \setminus \{0\}$ as the centroaffine frame bundle $\mathcal{F}$ over $\mathbb{R}^5 \setminus \{0\}$. For each point $e_0 \in \mathbb{R}^5 \setminus \{0\}$, the fiber over $e_0$ consists of all frames $(e_0, e_1, e_2, e_3, e_4)$ for the tangent space $T_{e_0}(\mathbb{R}^5 \setminus \{0\})$—i.e., all frames for which the first vector in the frame is equal to the position vector.

The Maurer-Cartan forms $\omega^i_j$ on $\mathcal{F}$ are defined by the equations

$$de_i = e_j \omega^j_i, \quad 0 \leq i, j \leq 4,$$

and they satisfy the Cartan structure equations

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j.$$  

(For details, see [7] or [1].)

We are interested in the geometry of 2-dimensional immersions $\bar{f} : M^2 \to \mathbb{R}^5 \setminus \{0\}$; we will use Cartan’s method of moving frames to compute local invariants for such immersions under the action of $GL(5, \mathbb{R})$.

**Definition 2.1.** An immersion $\bar{f}$ of a 2-dimensional manifold $M$ into $\mathbb{R}^5 \setminus \{0\}$ is called a centroaffine immersion if the position vector $\bar{f}(x)$ is transversal to the tangent space $T_{\bar{f}(x)}(\mathbb{R}^5 \setminus \{0\})$ for all $x \in M$. The image $\Sigma = \bar{f}(M)$ is called a centroaffine surface in $\mathbb{R}^5 \setminus \{0\}$.

In order to begin the method of moving frames, consider the induced bundle of centroaffine frames along $\Sigma = \bar{f}(M)$; this is simply the pullback bundle $\mathcal{F}_0 = \bar{f}^* \mathcal{F}$ over $M$. A centroaffine frame field along $\Sigma$ is a section of $\mathcal{F}_0$—i.e., a smooth map $f : M \to GL(5, \mathbb{R})$ such that $\pi \circ f = \bar{f}$. Throughout the remainder of this paper, we will consider the pullbacks of the Maurer-Cartan forms on $\mathcal{F}$ to $M$ via such sections $f$, and we will suppress the pullback notation.

We will gradually adapt our choice of centroaffine frame fields based on the geometry of $\Sigma$. For our first adaptation, consider the subbundle $\mathcal{F}_1 \subset \mathcal{F}_0$ consisting of all frames for which $(e_1(x), e_2(x))$ span the tangent space $T_{\bar{f}(x)}(\mathbb{R}^5 \setminus \{0\})$ for each $x \in M$. A section $f : M \to \mathcal{F}_1$ will be called a 1-adapted frame field along $\Sigma$. Any two 1-adapted frames $(e_0, \ldots, e_4)$, $(\tilde{e}_0, \ldots, \tilde{e}_4)$...
based at the same point $x \in M$ are related by a transformation of the form

$$\text{(2.4)} \quad \begin{bmatrix} \tilde{e}_0 & \tilde{e}_1 & \tilde{e}_2 & \tilde{e}_3 \end{bmatrix} = \begin{bmatrix} e_0 & e_1 & e_2 & e_3 & e_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & r_{03} & r_{04} \\ 0 & a_{11} & a_{12} & r_{13} & r_{14} \\ 0 & a_{21} & a_{22} & r_{23} & r_{24} \\ 0 & 0 & 0 & b_{33} & b_{34} \\ 0 & 0 & 0 & b_{43} & b_{44} \end{bmatrix},$$

where the $2 \times 2$ submatrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $B = \begin{bmatrix} b_{33} & b_{34} \\ b_{43} & b_{44} \end{bmatrix}$ are elements of $GL(2, \mathbb{R})$. We will denote the group of all matrices of the form in (2.4) by $G_1$; then the bundle $\mathcal{F}_1$ is a principal bundle over $M$ with fiber group $G_1$.

If $f, \tilde{f} : M \to \mathcal{F}_1$ are two 1-adapted frame fields along $\Sigma$, then $f, \tilde{f}$ are related by the equation

$$\tilde{f}(x) = f(x) \cdot g(x)$$

for some smooth function $g : M \to G_1$, as in equation (2.4). Then the corresponding $\mathfrak{gl}(5, \mathbb{R})$-valued Maurer-Cartan forms $\Omega = [\omega^i_j], \tilde{\Omega} = [\tilde{\omega}^i_j]$ on $M$ are related as follows:

$$\text{(2.5)} \quad \tilde{\Omega} = g^{-1} dg + g^{-1} \Omega.$$

3. Reduction of the structure group and first-order invariants

Now consider the pullbacks of the Maurer-Cartan forms to $M$ via a 1-adapted frame field $f$. From equation (2.2) for $de_0$ and the fact that the image of $de_0$ is spanned by $e_1$ and $e_2$, we have

$$\omega_0^0 = \omega_0^3 = \omega_0^4 = 0. \quad \text{(3.1)}$$

Moreover, the 1-forms $\omega_0^1, \omega_0^2$ are semi-basic for the projection $\pi : \mathcal{F}_1 \to M$; in fact, they form a basis for the semi-basic 1-forms on $\mathcal{F}_1$.

Differentiating equations (3.1) yields:

$$0 = d\omega_0^0 = -(\omega_1^0 \wedge \omega_0^1 + \omega_2^0 \wedge \omega_0^2),$$

$$0 = d\omega_0^3 = -(\omega_1^3 \wedge \omega_0^1 + \omega_2^3 \wedge \omega_0^2),$$

$$0 = d\omega_0^4 = -(\omega_1^4 \wedge \omega_0^1 + \omega_2^4 \wedge \omega_0^2). \quad \text{(3.2)}$$

Cartan’s Lemma (see, e.g., [7] or [1]) then implies that there exist functions $h_i^j = h_j^i$, $k = 0, 3, 4$, on $M$ such that

$$\begin{bmatrix} \omega_1^i \\ \omega_2^i \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} \\ h_{12} & h_{22} \end{bmatrix} \begin{bmatrix} \omega_0^i \\ \omega_0^i \end{bmatrix}. \quad \text{(3.3)}$$

For simplicity of notation, let $h^k$ denote the matrix

$$h^k = \begin{bmatrix} h_{11}^k & h_{12}^k \\ h_{12}^k & h_{22}^k \end{bmatrix}, \quad k = 0, 3, 4.$$
If \( f, \tilde{f} : M \rightarrow F_1 \) are two 1-adapted frame fields related by a transformation of the form (2.4), then we can use equation (2.5) to determine how the corresponding matrices \( h^k, \tilde{h}^k \) are related. First, it follows from the fact that \( \tilde{e}_0 = e_0 \) that

\[
d\tilde{e}_0 = [\tilde{e}_1 \quad \tilde{e}_2] \begin{bmatrix} \tilde{\omega}^1_0 \\ \tilde{\omega}^2_0 \end{bmatrix} = [e_1 \quad e_2] \begin{bmatrix} \omega^1_0 \\ \omega^2_0 \end{bmatrix} = de_0.
\]

Then, since we have

\[
[e_1 \quad e_2] = [\tilde{e}_1 \quad \tilde{e}_2] A,
\]

we must have

\[
(3.4) \quad \begin{bmatrix} \tilde{\omega}^1_0 \\ \tilde{\omega}^2_0 \end{bmatrix} = A^{-1} \begin{bmatrix} \omega^1_0 \\ \omega^2_0 \end{bmatrix}.
\]

Similar considerations show that

\[
(3.5) \quad \begin{bmatrix} \hat{\omega}^1_4 \\ \hat{\omega}^2_4 \end{bmatrix} = \frac{1}{(\det B)} A^T \begin{bmatrix} \omega^3_1 \\ \omega^3_2 \end{bmatrix} - b_{34} \begin{bmatrix} \omega^4_1 \\ \omega^4_2 \end{bmatrix},
\]

\[
\begin{bmatrix} \hat{\omega}^0_1 \\ \hat{\omega}^0_2 \end{bmatrix} = A^T \begin{bmatrix} \omega^0_1 \\ \omega^0_2 \end{bmatrix} - r_{03} \begin{bmatrix} \omega^3_1 \\ \omega^3_2 \end{bmatrix} - r_{04} \begin{bmatrix} \omega^4_1 \\ \omega^4_2 \end{bmatrix}.
\]

Together, equations (3.4), (3.5) imply that

\[
(3.6) \quad \begin{align*}
\tilde{h}^3 &= \frac{1}{(\det B)} A^T (b_{44} h^3 - b_{34} h^4) A, \\
\tilde{h}^4 &= \frac{1}{(\det B)} A^T (-b_{43} h^3 + b_{33} h^4) A, \\
\tilde{h}^0 &= A^T h^0 A - r_{03} \tilde{h}^3 - r_{04} \tilde{h}^4.
\end{align*}
\]

**Definition 3.1.** A centroaffine surface \( \Sigma = \tilde{f}(M) \) will be called nondegenerate if the matrices \( h^0, h^3, h^4 \) are linearly independent in \( \text{Sym}^2(\mathbb{R}) \) at every point of \( M \).

Henceforth, we assume that \( \Sigma \) is nondegenerate; from the group action (3.6) it is clear that this definition is independent of the choice of 1-adapted frame field \( f : M \rightarrow F_1 \) along \( \Sigma \).

The next step is to use the group action (3.6) to find normal forms for the matrices \( h^0, h^3, h^4 \). First consider the action on \( h^3, h^4 \): it can be written as the composition of two separate actions by the matrices \( A, B \in GL(2, \mathbb{R}) \):

\[
A \cdot (h^3, h^4) = (A^T h^3 A, A^T h^4 A),
\]

\[
B \cdot (h^3, h^4) = \left( \frac{1}{(\det B)} (b_{44} h^3 - b_{34} h^4), \frac{1}{(\det B)} (-b_{43} h^3 + b_{33} h^4) \right).
\]
If we let $P$ denote the 2-dimensional subspace of $\text{Sym}^2(\mathbb{R})$ spanned by $(h^3, h^4)$, then we see that the action by $B$ preserves $P$, while $A$ acts on $P$ via

\begin{equation}
A \cdot P = A^T P A.
\end{equation}

It is well-known (see, e.g., [12]) that the action

\[ A \cdot h = A^T h A \]

on $\text{Sym}^2(\mathbb{R})$ preserves the indefinite quadratic form

\begin{equation}
Q(h) = -\det(h)
\end{equation}

up to a scale factor, and that this action has precisely 6 orbits, represented by the matrices

\[
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 \\
0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
-1 & 0 \\
0 & 0
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}, \quad
\begin{bmatrix}
-1 & 0 \\
0 & -1
\end{bmatrix}.
\]

It follows that the induced action (3.7) on the Grassmanian of 2-planes in $\text{Sym}^2(\mathbb{R})$ has precisely 3 orbits, depending on whether the plane $P$ is spacelike, timelike, or null with respect to the quadratic form (3.8). These orbits are represented by the 2-planes

\begin{equation}
P_1 = \text{span} \left( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \quad \text{(spacelike)},
\end{equation}

\begin{equation}
P_2 = \text{span} \left( \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \quad \text{(timelike)},
\end{equation}

\begin{equation}
P_3 = \text{span} \left( \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \quad \text{(null)}.
\end{equation}

The type of the plane $P$ spanned by $(h^3, h^4)$ (spacelike, timelike, or null) is preserved by the group action (3.6); thus the type of $P$ at any point $x \in M$ is well-defined, independent of the choice of 1-adapted frame field $f : M \to \mathcal{F}_1$.

At this point, the method of moving frames dictates that we divide into cases based on the type of $P$. In order to proceed, we make the following assumption:

**Assumption 3.2.** Assume that $\Sigma$ has constant type—i.e., that the type of $P$ is the same at every point $x \in M$.

In this paper we will consider only the spacelike and timelike cases; the null case is considerably more complicated and may be explored in a future paper.

4. The Spacelike Case

First, suppose that the plane $P$ spanned by $(h^3, h^4)$ is spacelike at every point of $M$. According to the group action (3.7), we can find a 1-adapted frame field along $\Sigma$ for which $P = P_1$, as in equation (3.9). Furthermore, we can then use the action by $B$ to find a 1-adapted frame field along $\Sigma$ for which

\[
h^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad h^4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]
It is straightforward to show that these conditions are preserved by transformations of the form \((3.6)\) with
\[
A = \lambda A_0, \quad B = \lambda^2 (A_0)^2, 
\]
where \(\lambda \in \mathbb{R}^*, A_0 \in O(2, \mathbb{R})\). For simplicity, we will restrict to transformations with \(A_0 \in SO(2, \mathbb{R}), \lambda > 0\); this has the advantage of producing a frame bundle whose fiber is a connected Lie group. Thus we will assume that
\[
(4.1) \quad A = \begin{bmatrix} \lambda \cos(\theta) & -\lambda \sin(\theta) \\ \lambda \sin(\theta) & \lambda \cos(\theta) \end{bmatrix}, \quad B = \begin{bmatrix} \lambda^2 \cos(2\theta) & -\lambda^2 \sin(2\theta) \\ \lambda^2 \sin(2\theta) & \lambda^2 \cos(2\theta) \end{bmatrix},
\]
where \(\lambda > 0, \theta \in \mathbb{R}\).

Next, consider the effect of the action \((3.6)\) on \(h^0\). With \(A = I_2\) and \(r_{03}, r_{04}\) chosen appropriately, we can add any linear combination of \(h^3, h^4\) to \(h^0\). Thus we can find a 1-adapted frame field for which \(h^0\) is a multiple (nonzero by the nondegeneracy assumption) of the identity matrix \(I_2\). Then under the action \((3.6)\) with \(A, B\) as in \((4.1)\), we have
\[
\tilde{h}^0 = \lambda^2 h^0 + \begin{bmatrix} -r_{03} & -r_{04} \\ -r_{04} & r_{03} \end{bmatrix}.
\]

Therefore, we can find a 1-adapted frame field along \(\Sigma\) satisfying the additional condition that \(h^0 = \pm I_2\), and this condition is preserved by transformations of the form \((3.6)\) with \(A, B\) as in \((4.1)\), \(\lambda = 1\), and and \(r_{03} = r_{04} = 0\).

**Definition 4.1.** Let \(\Sigma = \bar{f}(M)\) be a nondegenerate, spacelike centroaffine surface in \(\mathbb{R}^5 \setminus \{0\}\). A 1-adapted frame field \(f : M \to \mathcal{F}_1\) will be called 2-adapted if it satisfies the conditions
\[
(4.2) \quad h^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad h^4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad h^0 = \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix},
\]
with \(\epsilon = \pm 1\), at every point of \(M\).

Any two 2-adapted frames \((e_0, \ldots, e_4), (\tilde{e}_0, \ldots, \tilde{e}_4)\) based at the same point \(x \in M\) are related by a transformation of the form
\[
(4.3) \quad \begin{bmatrix} \tilde{e}_0 & \tilde{e}_1 & \tilde{e}_2 & \tilde{e}_3 & \tilde{e}_4 \end{bmatrix} = \begin{bmatrix} e_0 & e_1 & e_2 & e_3 & e_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & r_{13} & r_{14} \\ 0 & \sin(\theta) & \cos(\theta) & r_{23} & r_{24} \\ 0 & 0 & 0 & \cos(2\theta) & -\sin(2\theta) \\ 0 & 0 & 0 & \sin(2\theta) & \cos(2\theta) \end{bmatrix}.
\]

We will denote the group of all matrices of the form in \((4.3)\) by \(G_2\); then the 2-adapted frame fields along \(\Sigma\) are the smooth sections of a principal bundle \(\mathcal{F}_2 \subset \mathcal{F}_1\) over \(M\) with fiber group \(G_2\).

The equations \((4.2)\) are equivalent to the condition that the Maurer-Cartan forms associated to a 2-adapted frame field satisfy the conditions
\[
(4.4) \quad \omega^0_1 = \omega^1_0, \quad \omega^2_0 = -\omega^0_0, \\
\omega^1_1 = \omega^0_0, \quad \omega^2_1 = \omega^1_0, \\
\omega^0_0 = \epsilon \omega^1_0, \quad \omega^0_2 = \epsilon \omega^2_0.
\]
Differentiating equations (4.4) yields
\[ (2\omega_1^4 - \omega_3^3) \land \omega_1^4 + (\omega_2^4 - \omega_1^2 - \omega_3^3) \land \omega_0^2 = 0, \]
\[ (\omega_2^4 - \omega_1^2 - \omega_3^3) \land \omega_0^4 + (\omega_3^3 - 2\omega_2^2) \land \omega_0^2 = 0, \]
\[ (2\omega_2^4 - \omega_3^3) \land \omega_0^4 + (\omega_1^2 + \omega_2^2 - \omega_3^3) \land \omega_0^2 = 0, \]
\[ (\omega_1^3 + \omega_2^3 - \omega_3^3) \land \omega_0^4 + (2\omega_1^2 + \omega_3^3) \land \omega_0^2 = 0, \]
\[ (2\epsilon \omega_1^3 - \omega_0^3) \land \omega_0^4 + (\epsilon \omega_1^3 - \omega_0^3) \land \omega^0 = 0, \]
\[ (\epsilon \omega_1^3 + \epsilon \omega_2^3 - \omega_4^3) \land \omega_0^4 + (2\epsilon \omega_2^3 + \omega_3^3) \land \omega_0^2 = 0. \]

Applying Cartan’s Lemma to these equations shows that there exists a 1-form \( \alpha \) and functions \( h^i_{jk} \) on \( \mathcal{F}_2 \) such that
\[
\omega_0^0 = h^0_{31}\omega_1^1 + h^0_{32}\omega_2^2; \quad \omega_4^0 = h^0_{41}\omega_0^1 + h^0_{42}\omega_0^2; \quad \omega_1^4 = \alpha + h^1_{21}\omega_1^1 + h^1_{22}\omega_2^2; \\
\omega_1^1 = h^1_{11}\omega_0^1 + h^1_{12}\omega_2^2; \quad \omega_2^2 = h^2_{21}\omega_0^1 + h^2_{22}\omega_2^2; \\
\omega_2^1 = -\alpha + h^1_{31}\omega_0^1 + h^1_{32}\omega_2^2; \quad \omega_2^2 = h^2_{31}\omega_0^1 + h^2_{32}\omega_2^2; \\
\omega_3^3 = h^3_{31}\omega_0^1 + h^3_{32}\omega_2^2; \quad \omega_3^3 = 2\alpha + h^3_{41}\omega_0^1 + h^3_{42}\omega_2^2; \\
\omega_3^4 = -2\alpha + h^4_{31}\omega_0^1 + h^4_{32}\omega_2^2, \quad \omega_4^4 = h^4_{11}\omega_0^1 + h^4_{12}\omega_2^2.
\]

Moreover, the functions \( h^i_{jk} \) satisfy the relations
\[
2h^1_{12} - h^3_{32} + h^3_{41} = 0, \\
2h^2_{21} - h^3_{31} - h^3_{42} = 0, \\
h^1_{11} + 2h^1_{22} + h^2_{21} + h^4_{32} - 4h^1_{41} = 0, \\
h^1_{12} + 2h^1_{21} + h^2_{22} - h^4_{31} - h^4_{42} = 0, \\
h^0_{32} - h^0_{41} + 2\epsilon (h^1_{21} - h^1_{12}) = 0, \\
h^0_{31} + h^0_{42} + 2\epsilon (h^2_{21} - h^2_{22}) = 0.
\]

If \( f, \tilde{f} : M \to \mathcal{F}_2 \) are two 2-adapted frame fields related by a transformation of the form (4.3), then we can once again use equation (2.5) to determine how the corresponding functions \( h^i_{jk}, \tilde{h}^i_{jk} \) are related. Some of these relationships are more complicated than others; the most straightforward to compute are those corresponding to the forms \( \omega^0_3, \tilde{\omega}_3^0 \). These forms appear as the coefficients of \( \epsilon_0 = \epsilon_0 \) in the equations (2.2) for \( d\tilde{e}_3, d\tilde{e}_4 \). By applying equations (4.3) and (4.6), one can show that
\[
\begin{bmatrix}
\tilde{h}^0_{31}
\tilde{h}^0_{41}
\end{bmatrix}
= 
\begin{bmatrix}
\cos(2\theta) & \sin(2\theta) \\
-\sin(2\theta) & \cos(2\theta)
\end{bmatrix}
\begin{bmatrix}
\tilde{h}^0_{31}
\tilde{h}^0_{41}
\end{bmatrix}
+ \epsilon
\begin{bmatrix}
r_{13}
\epsilon
\end{bmatrix},
\]
\[
\begin{bmatrix}
\tilde{h}^0_{32}
\tilde{h}^0_{42}
\end{bmatrix}
= 
\begin{bmatrix}
\cos(2\theta) & \sin(2\theta) \\
-\sin(2\theta) & \cos(2\theta)
\end{bmatrix}
\begin{bmatrix}
\tilde{h}^0_{32}
\tilde{h}^0_{42}
\end{bmatrix}
+ \epsilon
\begin{bmatrix}
r_{23}
\epsilon
\end{bmatrix}.
\]

Thus we can find a 2-adapted frame field along \( \Sigma \) satisfying the conditions that
\[
h_{31}^0 = h_{32}^0 = h_{41}^0 = h_{42}^0 = 0,
\]
\[
\begin{bmatrix}
\tilde{h}^0_{31}
\tilde{h}^0_{41}
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{h}^0_{32}
\tilde{h}^0_{42}
\end{bmatrix}
+ \epsilon
\begin{bmatrix}
r_{13}
\epsilon
\end{bmatrix},
\]
\[
\begin{bmatrix}
\tilde{h}^0_{32}
\tilde{h}^0_{42}
\end{bmatrix}
= 
\begin{bmatrix}
\tilde{h}^0_{32}
\tilde{h}^0_{42}
\end{bmatrix}
+ \epsilon
\begin{bmatrix}
r_{23}
\epsilon
\end{bmatrix}.
\]
and these conditions are preserved by transformations of the form (4.3) with \( r_{13} = r_{14} = r_{23} = r_{24} = 0 \).

**Definition 4.2.** Let \( \Sigma = \tilde{f}(M) \) be a nondegenerate, spacelike centroaffine surface in \( \mathbb{R}^5 \setminus \{0\} \). A 2-adapted frame field \( f : M \to F_2 \) will be called \( 3\)-adapted if it satisfies the conditions
\[
(4.9)\quad h_{31}^0 = h_{32}^0 = h_{41}^0 = h_{42}^0 = 0
\]
at every point of \( M \).

Any two 3-adapted frames \( (e_0, \ldots, e_4), \ (\tilde{e}_0, \ldots, \tilde{e}_4) \) based at the same point \( x \in M \) are related by a transformation of the form
\[
(4.10)\quad [\tilde{e}_0 \ \tilde{e}_1 \ \tilde{e}_2 \ \tilde{e}_3 \ \tilde{e}_3] = [e_0 \ e_1 \ e_2 \ e_3 \ e_4] \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) & 0 & 0 \\ 0 & \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 0 & \cos(2\theta) & -\sin(2\theta) \\ 0 & 0 & 0 & \sin(2\theta) & \cos(2\theta) \end{bmatrix}.
\]

We will denote the group of all matrices of the form in (4.10) by \( G_3 \); note that \( G_3 \cong SO(2, \mathbb{R}) \). Then the 3-adapted frame fields are the smooth sections of a principal bundle \( F_3 \subset F_2 \) over \( M \) with fiber group \( G_3 \).

The equations (4.9) are equivalent to the condition that the Maurer-Cartan forms associated to a 3-adapted frame field satisfy the conditions
\[
(4.11)\quad \omega_3^0 = \omega_4^0 = 0.
\]

Differentiating equations (4.11) yields
\[
(4.12)\quad \omega_3^1 \wedge \omega_3^0 + \omega_3^2 \wedge \omega_0^0 = \omega_4^1 \wedge \omega_1^0 + \omega_4^2 \wedge \omega_0^0 = 0,
\]
and applying Cartan’s Lemma shows that there exist functions \( h_{jk}^i \) on \( F_3 \) such that
\[
(4.13)\quad \omega_3^1 = h_{31}^1 \omega_0^1 + h_{32}^1 \omega_0^2, \quad \omega_3^2 = h_{31}^2 \omega_0^1 + h_{32}^2 \omega_0^2,
\]
\[
\omega_4^1 = h_{41}^1 \omega_0^1 + h_{42}^1 \omega_0^2, \quad \omega_4^2 = h_{41}^2 \omega_0^1 + h_{42}^2 \omega_0^2.
\]
Moreover, on \( F_3 \), the last two relations in equations (4.7) simplify to
\[
(4.14)\quad h_{21}^1 - h_{12}^2 = h_{21}^2 - h_{12}^1 = 0.
\]

At this point, we have canonically associated to any nondegenerate, spacelike centroaffine surface in \( \mathbb{R}^5 \setminus \{0\} \) a frame bundle \( F_3 \) over \( M \) with fiber group isomorphic to \( SO(2, \mathbb{R}) \). Thus we have the following theorem:

**Theorem 4.3.** Let \( \tilde{f} : M \to \mathbb{R}^5 \setminus \{0\} \) be a centroaffine immersion whose image \( \Sigma = \tilde{f}(M) \) is a nondegenerate, spacelike centroaffine surface. Then the pullbacks of the Maurer-Cartan forms on \( GL(5, \mathbb{R}) \) to the bundle \( F_3 \) of 3-adapted frames on \( \Sigma \) determine a well-defined Riemannian metric
\[
I = (\omega_3^0)^2 + (\omega_4^0)^2
\]
on \( \Sigma \), called the centroaffine metric. Moreover, there is a well-defined “centroaffine normal bundle” \( N\Sigma \) whose fiber \( N_x \Sigma \) at each point \( x \in M \) is spanned by the vectors \( (e_3(x), e_4(x)) \) of any 3-adapted frame at \( x \), together with a well-defined Riemannian metric
\[
I_{\text{normal}} = (\omega_0^3)^2 + (\omega_0^4)^2
\]
In order to obtain more information about the centroaffine metric, consider the structure equations (2.3) for the semi-basic forms \( \omega_0^1, \omega_0^2 \) on \( M \). Based on our adaptations, it is straightforward to compute that
\[
d\omega_0^1 = -\alpha \wedge \omega_0^2, \quad d\omega_0^2 = \alpha \wedge \omega_0^1.
\]
Therefore, \( \alpha \) is the Levi-Civita connection form associated to the centroaffine metric on \( \Sigma \), and the Gauss curvature \( K \) of this metric is determined by the equation
\[
d\alpha = K \omega_0^1 \wedge \omega_0^2.
\]
The remaining structure equations (2.3) determine relations between the functions \( h_{ij} \) on \( F_3 \) and their covariant derivatives with respect to \( \omega_0^1, \omega_0^2 \). These relations may be viewed as analogs of the Gauss and Codazzi equations for Riemannian surfaces in Euclidean space. In particular, the analog of the Gauss equation is
\[
K = \frac{1}{2} \left( -h_{32}^3 h_{31}^4 - h_{41}^3 h_{42}^3 + h_{41}^3 h_{42}^3 h_{31}^3 - h_{32}^3 h_{42}^3 h_{31}^3 + h_{41}^3 h_{31}^3 + h_{31}^1 - h_{32}^4 + 2 h_{42}^1 \right) - \epsilon,
\]
while the remainder of the relations are partial differential equations involving the functions \( h_{ij} \). An analog of Bonnet’s Theorem (see [6]) guarantees that, at least locally, any solution of this PDE system gives rise to a nondegenerate, spacelike centroaffine surface, and that this surface is unique up to the action of \( GL(5, \mathbb{R}) \) on \( \mathbb{R}^5 \setminus \{0\} \). In particular, the functions \( h_{ij} \) on \( F_3 \) form a complete set of local invariants for such surfaces.

5. The timelike case

Now, suppose that the plane \( P \) spanned by \( (h^3, h^4) \) is timelike at every point of \( M \). According to the group action (3.7), we can find a 1-adapted frame field along \( \Sigma \) for which \( P = P_2 \), as in equation (3.9). Furthermore, we can then use the action by \( B \) to find a 1-adapted frame field along \( \Sigma \) for which
\[
h^3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad h^4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.
\]
(This choice of \( h^3, h^4 \) represents a null basis for \( P_2 \) with respect to the indefinite quadratic form (3.8) on \( P_2 \).) It is straightforward to show that these conditions are preserved by transformations of the form (3.6) with either
\[
A = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}, \quad B = \begin{bmatrix} a_{11}^2 & 0 \\ 0 & a_{22}^2 \end{bmatrix}, \quad a_{11}, a_{22} \neq 0,
\]
or
\[
A = \begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & a_{12}^2 \\ a_{21}^2 & 0 \end{bmatrix}, \quad a_{12}, a_{21} \neq 0.
\]
Since the latter transformation may be obtained from the former simply by interchanging \((e_1, e_2)\) and \((e_3, e_4)\), we will restrict our attention to transformations of the form (5.1), where \( A, B \) are diagonal matrices and \( B = A^2 \).
Next, consider the effect of the action (3.6) on $h^0$. With $A = I_2$ and $r_{03}, r_{04}$ chosen appropriately, we can add any linear combination of $h^3, h^4$ to $h^0$. Thus we can find a 1-adapted frame field for which $h^0$ is a multiple (nonzero by the nondegeneracy assumption) of the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then under the action (3.6) with $A, B$ as in (5.1), we have

$$\tilde{h}^0 = a_{11}a_{22}h^0 + \begin{bmatrix} -r_{03} & 0 \\ 0 & -r_{04} \end{bmatrix}.$$ 

Thus we can find a 1-adapted frame field along $\Sigma$ satisfying the additional condition that $h^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and this condition is preserved by transformations of the form (3.6) with $A, B$ as in (5.1), such that $a_{11}a_{22} = 1$ and $r_{03} = r_{04} = 0$. For simplicity, we will assume that $a_{11} > 0$; then we can set

$$A = \begin{bmatrix} e^\lambda & 0 \\ 0 & e^{-\lambda} \end{bmatrix}, \quad B = \begin{bmatrix} e^{2\lambda} & 0 \\ 0 & e^{-2\lambda} \end{bmatrix}, \quad \lambda \in \mathbb{R}. \quad (5.2)$$

**Definition 5.1.** Let $\Sigma = \bar{f}(M)$ be a nondegenerate, timelike centroaffine surface in $\mathbb{R}^5 \setminus \{0\}$. A 1-adapted frame field $f : M \to \mathcal{F}_1$ will be called $2$-adapted if it satisfies the conditions

$$h^3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad h^4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad h^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad (5.3)$$

at every point of $M$.

Any two 2-adapted frames $(e_0, \ldots, e_4), (\tilde{e}_0, \ldots, \tilde{e}_4)$ based at the same point $x \in M$ are related by a transformation of the form

$$\begin{bmatrix} \tilde{e}_0 & \tilde{e}_1 & \tilde{e}_2 & \tilde{e}_3 & \tilde{e}_4 \end{bmatrix} = \begin{bmatrix} e_0 & e_1 & e_2 & e_3 & e_4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^\lambda & 0 & r_{13} & r_{14} \\ 0 & 0 & e^{-\lambda} & r_{23} & r_{24} \\ 0 & 0 & 0 & e^{2\lambda} & 0 \\ 0 & 0 & 0 & 0 & e^{-2\lambda} \end{bmatrix}. \quad (5.4)$$

We will denote the group of all matrices of the form in (5.4) by $G_2$; then the 2-adapted frame fields along $\Sigma$ are the smooth sections of a principal bundle $\mathcal{F}_2 \subset \mathcal{F}_1$ over $M$ with fiber group $G_2$.

The equations (5.3) are equivalent to the condition that the Maurer-Cartan forms associated to a 2-adapted frame field satisfy the conditions

$$\omega_1^3 = \omega_0^1, \quad \omega_2^3 = 0, \quad \omega_4^4 = 0, \quad \omega_2^4 = \omega_0^2, \quad \omega_0^0 = \omega_0^1, \quad \omega_2^0 = \omega_0^1. \quad (5.5)$$
Differentiating equations (5.5) yields
\[
(2\omega_1^1 - \omega_3^3) \wedge \omega_1^0 + \omega_2^1 \wedge \omega_0^0 = 0, \\
\omega_2^1 \wedge \omega_0^0 - \omega_3^3 \wedge \omega_0^0 = 0, \\
-\omega_3^3 \wedge \omega_1^0 + \omega_1^2 \wedge \omega_0^0 = 0, \\
\omega_1^2 \wedge \omega_0^0 + (2\omega_2^2 - \omega_4^4) \wedge \omega_0^0 = 0, \\
(2\omega_1^1 - \omega_3^3) \wedge \omega_1^0 + (\omega_1^1 + \omega_2^2) \wedge \omega_0^0 = 0, \\
(\omega_1^1 + \omega_2^2) \wedge \omega_0^0 + (2\omega_1^1 - \omega_3^3) \wedge \omega_0^0 = 0.
\]
(5.6)

Applying Cartan’s Lemma to these equations shows that there exists a 1-form \( \alpha \) and functions \( h^i_{jk} \) on \( \mathcal{F}_2 \) such that
\[
\omega_3^0 = h_{31}^0 \omega_1^0 + h_{32}^0 \omega_0^2, \\
\omega_1^0 = \alpha + h_{11}^1 \omega_1^1 + h_{12}^2 \omega_0^2, \\
\omega_2^1 = h_{11}^2 \omega_1^0 + h_{12}^2 \omega_0^2, \\
\omega_3^3 = 2\alpha + h_{31}^3 \omega_1^0 + h_{32}^3 \omega_0^2, \\
\omega_3^4 = h_{31}^4 \omega_1^0 + h_{32}^4 \omega_0^2.
\]
(5.7)

Moreover, the functions \( h^i_{jk} \) satisfy the relations
\[
2h_{22}^2 - h_{21}^1 - h_{32}^3 = 0, \\
h_{22}^1 + h_{31}^3 = 0, \\
h_{11}^2 + h_{32}^3 = 0, \\
2h_{11}^1 - h_{12}^2 - h_{41}^4 = 0, \\
2h_{11}^1 - 2h_{12}^2 + h_{32}^3 = 0, \\
2h_{22}^2 - 2h_{21}^1 + h_{41}^4 = 0.
\]
(5.8)

If \( f, \bar{f} : M \to \mathcal{F}_2 \) are two 2-adapted frame fields related by a transformation of the form (5.4), then we can once again use equation (2.5) to determine how the corresponding functions \( h^i_{jk}, \tilde{h}^i_{jk} \) are related. As in the spacelike case, the most straightforward to compute are those corresponding to the forms \( \tilde{\omega}_3^0, \tilde{\omega}_4^0 \). These forms appear as the coefficients of \( \tilde{e}_0 = e_0 \) in the equations (2.2) for \( d\tilde{e}_3, d\tilde{e}_4 \). By applying equations (5.4) and (5.7), one can show that
\[
\begin{bmatrix}
\tilde{h}_{31}^0 \\
\tilde{h}_{41}^0
\end{bmatrix} = \begin{bmatrix}
e^{2\lambda} & 0 \\
0 & e^{-2\lambda}
\end{bmatrix} \begin{bmatrix}
h_{31}^0 \\
h_{41}^0
\end{bmatrix} + \begin{bmatrix}r_{23} \\
r_{24}\end{bmatrix},
\]
(5.9)
\[
\begin{bmatrix}
\tilde{h}_{32}^0 \\
\tilde{h}_{12}^0
\end{bmatrix} = \begin{bmatrix}
e^{2\lambda} & 0 \\
0 & e^{-2\lambda}
\end{bmatrix} \begin{bmatrix}
h_{32}^0 \\
h_{12}^0
\end{bmatrix} + \begin{bmatrix}r_{13} \\
r_{14}\end{bmatrix}.
\]

Thus we can find a 2-adapted frame field along \( \Sigma \) satisfying the conditions that
\[
h_{31}^0 = h_{32}^0 = h_{41}^0 = h_{42}^0 = 0,
\]
and these conditions are preserved by transformations of the form (5.4) with \( r_{13} = r_{14} = r_{23} = r_{24} = 0 \).
Definition 5.2. Let $\Sigma = \bar{f}(M)$ be a nondegenerate, timelike centroaffine surface in $\mathbb{R}^5 \setminus \{0\}$. A 2-adapted frame field $f : M \to \mathcal{F}_2$ will be called $3$-adapted if it satisfies the conditions
\begin{equation}
(5.10) \quad h^0_{31} = h^0_{32} = h^0_{41} = h^0_{42} = 0
\end{equation}
at every point of $M$.

Any two 3-adapted frames $(e_0, \ldots, e_4)$, $(\tilde{e}_0, \ldots, \tilde{e}_4)$ based at the same point $x \in M$ are related by a transformation of the form
\begin{equation}
(5.11) \quad [\tilde{e}_0 \ \tilde{e}_1 \ \tilde{e}_2 \ \tilde{e}_3 \ \tilde{e}_4] = [e_0 \ e_1 \ e_2 \ e_3 \ e_4] \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & e^\lambda & 0 & 0 & 0 \\
0 & 0 & e^{-\lambda} & 0 & 0 \\
0 & 0 & 0 & e^{2\lambda} & 0 \\
0 & 0 & 0 & 0 & e^{-2\lambda}
\end{bmatrix}.
\end{equation}

We will denote the group of all matrices of the form in (5.11) by $G_3$; note that $G_3 \cong SO^+(1,1)$. Then the 3-adapted frame fields are the smooth sections of a principal bundle $\mathcal{F}_3 \subset \mathcal{F}_2$ over $M$ with fiber group $G_3$.

The equations (5.10) are equivalent to the condition that the Maurer-Cartan forms associated to a 3-adapted frame field satisfy the conditions
\begin{equation}
(5.12) \quad \omega^0_3 = \omega^0_4 = 0.
\end{equation}
Differentiating equations (5.12) yields
\begin{equation}
(5.13) \quad \omega^2_3 \wedge \omega^1_0 + \omega^1_3 \wedge \omega^2_0 = \omega^2_4 \wedge \omega^1_0 + \omega^1_4 \wedge \omega^2_0 = 0,
\end{equation}
and applying Cartan’s Lemma shows that there exist functions $h^i_{jk}$ on $\mathcal{F}_3$ such that
\begin{equation}
(5.14) \quad \omega^1_3 = h^1_{31} \omega^1_0 + h^1_{32} \omega^2_0, \quad \omega^2_3 = h^2_{31} \omega^1_0 + h^1_{31} \omega^2_0,
\end{equation}
\begin{equation}
\omega^1_4 = h^1_{41} \omega^1_0 + h^1_{42} \omega^2_0, \quad \omega^2_4 = h^1_{41} \omega^1_0 + h^1_{41} \omega^2_0.
\end{equation}
Moreover, on $\mathcal{F}_3$, the last two relations in equations (5.8) simplify to
\begin{equation}
(5.15) \quad h^1_{11} - h^2_{12} = h^2_{22} - h^1_{21} = 0.
\end{equation}

At this point, we have canonically associated to any nondegenerate, timelike centroaffine surface in $\mathbb{R}^5 \setminus \{0\}$ a frame bundle $\mathcal{F}_3$ over $M$ with fiber group isomorphic to $SO^+(1,1)$. Thus we have the following theorem:

**Theorem 5.3.** Let $\bar{f} : M \to \mathbb{R}^5 \setminus \{0\}$ be a centroaffine immersion whose image $\Sigma = \bar{f}(M)$ is a nondegenerate, timelike centroaffine surface. Then the pullbacks of the Maurer-Cartan forms on $GL(5, \mathbb{R})$ to the bundle $\mathcal{F}_3$ of 3-adapted frames on $\Sigma$ determine a well-defined Lorentzian metric
\begin{equation}
I = 2 \omega^1_0 \omega^2_0
\end{equation}
on $\Sigma$, called the centroaffine metric. Moreover, there is a well-defined “centroaffine normal bundle” $N\Sigma$ whose fiber $N_x \Sigma$ at each point $x \in M$ is spanned by the vectors $(e_3(x), e_4(x))$ of any 3-adapted frame at $x$, together with a well-defined Lorentzian metric
\begin{equation}
I_{\text{normal}} = 2 \omega^3_0 \omega^4_0
\end{equation}
on $N\Sigma$. 

12
In order to obtain more information about the centroaffine metric, consider the structure equations (2.3) for the semi-basic forms $\omega^1_0, \omega^2_0$ on $M$. Based on our adaptations, it is straightforward to compute that

\[(5.16)\]
\[d\omega^1_0 = -\alpha \wedge \omega^1_0, \quad d\omega^2_0 = \alpha \wedge \omega^2_0.\]

Therefore, $\alpha$ is the Levi-Civita connection form associated to the centroaffine metric on $\Sigma$, and the Gauss curvature $K$ of this metric is determined by the equation

\[d\alpha = K \omega^1_0 \wedge \omega^2_0.\]

As in the spacelike case, the remaining structure equations (2.3) determine relations between the functions $h^i_{jk}$ on $F_3$ and their covariant derivatives with respect to $\omega^1_0, \omega^2_0$. The analog of the Gauss equation is

\[(5.17)\]
\[K = h^3_{41} h^4_{32} - h^3_{32} h^4_{41} + \frac{1}{2} (h^1_{32} + h^2_{41}) - 1,\]

while the remainder of the relations are partial differential equations involving the functions $h^i_{jk}$. As in the spacelike case, any solution of this PDE system locally gives rise to a non-degenerate, timelike centroaffine surface; this surface is unique up to the action of $GL(5, \mathbb{R})$ on $\mathbb{R}^5 \setminus \{0\}$, and the functions $h^i_{jk}$ on $F_3$ form a complete set of local invariants for such surfaces.

### 6. Homogeneous examples

The goal of this section is to give a complete classification (up to the $GL(5, \mathbb{R})$-action on $\mathbb{R}^5 \setminus \{0\}$) of the homogeneous examples of spacelike and timelike nondegenerate centroaffine surfaces in $\mathbb{R}^5 \setminus \{0\}$. First we must define precisely what we mean by the term “homogeneous.” Because any such surface $\Sigma = \tilde{f}(M)$ has a well-defined Riemannian or Lorentzian metric, any symmetry of $\Sigma$ must preserve the centroaffine metric on $\Sigma$ and hence must in fact be an isometry of $\Sigma$ with its centroaffine metric. Furthermore, if the group of symmetries of $\Sigma$ acts transitively, then the centroaffine metric must have constant Gauss curvature $K$. As is well-known, the maximal isometry group of any Riemannian or Lorentzian surface has dimension less than or equal to three, and in the maximal case the isometry group acts transitively on the orthonormal frame bundle. Thus we will use the following definition:

**Definition 6.1.** Let $F_3$ be the bundle of 3-adapted frames along a nondegenerate, spacelike or timelike centroaffine surface $\Sigma = \tilde{f}(M)$ in $\mathbb{R}^5 \setminus \{0\}$. A diffeomorphism $\phi : F_3 \to F_3$ is called a symmetry of $\Sigma$ if $\phi^*\Omega = \Omega$; i.e., if $\phi$ preserves the Maurer-Cartan forms on $F_3$. A nondegenerate spacelike or timelike centroaffine surface $\Sigma = \tilde{f}(M)$ will be called homogeneous if the group of symmetries of $\Sigma$ is a 3-dimensional Lie group that acts transitively on $F_3$.

**Remark 6.2.** It might also be of interest to consider the slightly less restrictive assumption that $\Sigma$ has a 2-dimensional group of symmetries that acts transitively on the base manifold $M$, but we will not consider this scenario here.

Our procedure for classifying the homogeneous examples is as follows. Observe that if $\Sigma$ is homogeneous, then all the structure functions $h^i_{jk}$ must be constant on the bundle $F_3$ of 3-adapted frames on $\Sigma$. When this condition is imposed, the structure equations (2.3) become algebraic relations among the constants $h^i_{jk}$. Given any solution to these relations, the structure equations (2.3) imply that the corresponding Maurer-Cartan form $\Omega = [\omega^1_j]$
takes values in a 3-dimensional Lie algebra $\mathfrak{g}$ which is realized explicitly as a Lie subalgebra of $\mathfrak{gl}(5, \mathbb{R})$. Thus $\Omega$ is also the Maurer-Cartan form of the connected Lie group $G \subset \text{GL}(5, \mathbb{R})$ generated by exponentiating $\mathfrak{g}$, and this equivalence of the Maurer-Cartan forms on $F_3$ with those on $G$ implies that $F_3$ is a homogeneous space for $G$; indeed, $G$ must be precisely the symmetry group that was assumed to act transitively on $F_3$.

Now, choose any point $f_0 = (e_0, e_1, e_2, e_3, e_4) \in F_3$. Recall that we can view $f_0$ as an element of $\text{GL}(5, \mathbb{R})$. The centroaffine surface $\Sigma$ is equivalent via the $\text{GL}(5, \mathbb{R})$-action to the surface $\tilde{\Sigma} = f_0^{-1} \cdot \Sigma$, and the bundle $\tilde{F}_3$ of 3-adapted frames over $\tilde{\Sigma}$ is given by $\tilde{F}_3 = f_0^{-1} \cdot F_3$.

So without loss of generality, we may assume that $f_0$ is the identity matrix $I_5$. With this assumption, the tangent space $T_{f_0}F_3$ is equal to the Lie algebra $\mathfrak{g}$, and $F_3$ must in fact be equal to $G$. Finally, $\Sigma$ is given by the image of $G$ under the projection \((2.1)\).

In order to carry out this procedure, we consider the spacelike and timelike cases separately.

### 6.1. Spacelike homogeneous examples.

From the adaptations of §4, the matrix $\Omega = [\omega^i_j]$ of Maurer-Cartan forms on $F_3$ may be written as

\[(6.1) \quad \Omega = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & -2 & 0
\end{pmatrix} \alpha +
\begin{pmatrix}
0 & \epsilon & 0 & 0 & 0 \\
1 & \frac{1}{2}(h_{31}^3 + h_{32}^3) - h_{32}^4 + h_{41}^4 & \frac{1}{2}(h_{32}^3 - h_{41}^3) & h_{31}^1 & h_{31}^1 \\
0 & \frac{1}{2}(h_{32}^3 - h_{41}^3) & \frac{1}{2}(h_{31}^3 + h_{42}^3) & h_{32}^1 & h_{42}^1 \\
0 & 1 & 0 & h_{31}^3 & h_{41}^3 \\
0 & 0 & 1 & h_{31}^4 & h_{41}^4
\end{pmatrix} \omega^1_0 +\]

while the structure equations (2.3) may be written as

\[(6.2) \quad d\Omega = -\Omega \wedge \Omega.\]

Substituting (6.1) into equation (6.2) and imposing the condition that all the functions $h^i_{jk}$ are constant leads to a system of algebraic equations for the $h^i_{jk}$. A somewhat tedious, but straightforward, computation shows that this system has precisely two solutions, one for $\epsilon = 1$ and one for $\epsilon = -1$. These are described in the following two examples.
Example 6.3. When $\epsilon = 1$, the unique solution to equation (6.2) with all $h^i_{jk}$ constant is

$$\Omega = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 2
\end{bmatrix} \alpha + \begin{bmatrix}
0 & 0 & 0 & 0 & 1/3 \\
0 & 0 & 0 & 0 & 1/3 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix} \omega_0^0 + \begin{bmatrix}
1 & 0 & 0 & -1/3 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{bmatrix} \omega_0^2.
$$

Furthermore, the Gauss equation (4.16) implies that the centroaffine metric has Gauss curvature $K = -\frac{1}{3}$.

Denote the matrices in equation (6.3) by $M_0, M_1, M_2$, respectively, so that

$$\Omega = M_0 \alpha + M_1 \omega_0^0 + M_2 \omega_0^2.$$

It is straightforward to compute that

$$[M_0, M_1] = -M_2, \quad [M_1, M_2] = \frac{1}{3} M_0, \quad [M_2, M_0] = -M_1.
$$

These bracket relations imply that the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(5, \mathbb{R})$ spanned by $(M_0, M_1, M_2)$ is isomorphic to $\mathfrak{so}(1, 2)$. (They also suggest that a more natural basis might be obtained by multiplying each of $M_1, M_2$ by $\sqrt{3}$.) Furthermore, it is straightforward to check that $\mathfrak{g}$ acts irreducibly on $\mathbb{R}^5 \setminus \{0\}$. It is well-known (see, e.g., [2]) that $\mathfrak{so}(1, 2)$ has a unique irreducible 5-dimensional representation and that this representation arises from a (unique) irreducible representation of $SO^+(1, 2)$; it follows that the Lie group $G \subset GL(5, \mathbb{R})$ corresponding to the Lie algebra $\mathfrak{g}$ is isomorphic to $SO^+(1, 2)$.

The easiest way to compute a local parametrization for $G$—and hence for $\Sigma$—is to compute the 1-parameter subgroups generated by $M_0, M_1, M_2$ and take products of the resulting group elements.

**Warning:** This must be done carefully in order to ensure that the resulting products cover the entire group $G$, which in turn guarantees that the resulting parametrization is surjective onto $\Sigma$. The subtlety of this issue can already be seen in the standard representation for $\mathfrak{so}(1, 2)$: the basis

$$\bar{M}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad \bar{M}_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{M}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

15
has the same bracket relations as \((M_0, \sqrt{3}M_1, \sqrt{3}M_2)\), and exponentiating this basis yields the 1-parameter subgroups

\[
\bar{g}_0(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{bmatrix},
\]

\[
\bar{g}_1(u) = \begin{bmatrix} \cosh(u) & \sinh(u) & 0 \\ \sinh(u) & \cosh(u) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{g}_2(v) = \begin{bmatrix} \cosh(v) & 0 & \sinh(v) \\ 0 & 1 & 0 \\ \sinh(v) & 0 & \cosh(v) \end{bmatrix}.
\]

Now consider the following two maps \(f_1, f_2 : \mathbb{R}^3 \to SO^+(1, 2)\), which are obtained by multiplying the elements \(\bar{g}_0(t), \bar{g}_1(u), \bar{g}_2(v)\) in different orders:

\[
f_1(u, v, t) = \bar{g}_1(u)\bar{g}_2(v)\bar{g}_0(t)
\]

\[
= \begin{bmatrix} \cosh(u) \cosh(v) & \sinh(u) \cos(t) - \cosh(u) \sinh(v) \sin(t) & \sinh(u) \sin(t) + \cosh(u) \sinh(v) \cos(t) \\ \sinh(u) \cosh(v) & \cosh(u) \cos(t) - \sinh(u) \sinh(v) \sin(t) & \cosh(u) \sin(t) + \sinh(u) \sinh(v) \cos(t) \\ \sinh(v) & -\cosh(v) \sin(t) & \cosh(v) \cos(t) \end{bmatrix},
\]

\[
f_2(u, v, t) = \bar{g}_1(u)\bar{g}_0(t)\bar{g}_2(v)
\]

\[
= \begin{bmatrix} \cosh(u) \cosh(v) + \sinh(u) \sinh(v) \sin(t) & \sinh(u) \cos(t) & \cosh(u) \sinh(v) + \sinh(u) \cosh(v) \sin(t) \\ \sinh(u) \cosh(v) + \cosh(u) \sinh(v) \sin(t) & \cosh(u) \cos(t) & \sinh(u) \sinh(v) + \cosh(u) \cosh(v) \sin(t) \\ \sinh(v) \cos(t) & -\sin(t) & \cosh(v) \cos(t) \end{bmatrix}.
\]

It is not difficult to show that \(f_1\) is surjective onto \(SO^+(1, 2)\), whereas we can see from the middle column that \(f_2\) is not. Thus we must perform this construction with care.

Now, the obvious correspondence

\[
\bar{M}_0 \leftrightarrow M_0, \quad \bar{M}_1 \leftrightarrow \sqrt{3}M_1, \quad \bar{M}_2 \leftrightarrow \sqrt{3}M_2,
\]

defines a Lie algebra isomorphism between the standard representation of \(\mathfrak{so}(1, 2)\) and our Lie algebra \(\mathfrak{g} \subset \mathfrak{gl}(5, \mathbb{R})\). Therefore, the surjectivity of the map \(f_1\) above implies that the analogous map \(\bar{f} : \mathbb{R}^3 \to G\) will also be surjective onto \(G\), and it follows that the map \(\bar{f} = \pi \circ f\) will be surjective onto \(\Sigma\). The map \(\bar{f}\) will also turn out to be independent of \(t\), and when regarded as a function of the two variables \((u, v)\) it will define a surjective parametrization \(\bar{f} : \mathbb{R}^2 \to \Sigma\).

With these considerations in mind, define the 1-parameter subgroups

\[
g_0(t) = \exp(tM_0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos(t) & \sin(t) & 0 & 0 \\ 0 & -\sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & 0 & \cos(2t) & \sin(2t) \\ 0 & 0 & 0 & -\sin(2t) & \cos(2t) \end{bmatrix},
\]
\[ g_1(u) = \exp(u \sqrt{3} M_1) = \begin{bmatrix}
\frac{1}{4} (3 \cosh(2u) + 1) & \frac{\sqrt{3}}{2} \sinh(2u) & 0 & \frac{1}{4} (\cosh(2u) - 1) & 0 \\
\frac{\sqrt{3}}{2} \sinh(2u) & \cosh(2u) & 0 & \frac{1}{2\sqrt{3}} \sinh(2u) & 0 \\
0 & 0 & \cosh(u) & 0 & \frac{1}{\sqrt{3}} \sinh(u) \\
\frac{3}{4} (\cosh(2u) - 1) & \frac{\sqrt{3}}{2} \sinh(2u) & 0 & \frac{1}{4} (\cosh(2u) + 3) & 0 \\
0 & 0 & \sqrt{3} \sinh(u) & 0 & \cosh(u)
\end{bmatrix}, \]

\[ g_2(v) = \exp(v \sqrt{3} M_2) = \begin{bmatrix}
\frac{1}{4} (3 \cosh(2v) + 1) & 0 & \frac{\sqrt{3}}{2} \sinh(2v) & \frac{1}{4} (1 - \cosh(2v)) & 0 \\
0 & \cosh(v) & 0 & 0 & \frac{1}{\sqrt{3}} \sinh(v) \\
\frac{\sqrt{3}}{2} \sinh(2v) & 0 & \cosh(2v) & -\frac{1}{2\sqrt{3}} \sinh(2v) & 0 \\
\frac{3}{4} (1 - \cosh(2v)) & 0 & -\frac{\sqrt{3}}{2} \sinh(2v) & \frac{1}{4} (\cosh(2v) + 3) & 0 \\
0 & \sqrt{3} \sinh(v) & 0 & 0 & \cosh(v)
\end{bmatrix}. \]

Then set
\[ \bar{f}(u, v, t) = \pi (g_1(u) \cdot g_2(v) \cdot g_0(t)) = \begin{bmatrix}
\frac{1}{2} \left( 3 \cosh^2(u) \cosh^2(v) - 1 \right) \\
\sqrt{3} \sinh(u) \cosh(u) \cosh^2(v) \\
\sqrt{3} \cosh(u) \sinh(v) \cosh(v) \\
\frac{3}{2} \left( \cosh^2(v) (\cosh^2(u) - 2) + 1 \right) \\
3 \sinh(u) \sinh(v) \cosh(v)
\end{bmatrix}. \]

It follows from the discussion above that \( \bar{f} \) is a surjective map onto \( \Sigma \). Moreover, it is straightforward to check that the tangent vectors \( \bar{f}_u, \bar{f}_v \) are linearly independent for all \((u, v) \in \mathbb{R}^2\); therefore \( \bar{f} \) parametrizes a smooth surface \( \Sigma \subset \mathbb{R}^5 \setminus \{0\} \), as expected. Topologically, \( \Sigma \) is diffeomorphic to a plane; this is to be expected, as \( F_3 \) is isomorphic to the orthonormal frame bundle of the hyperbolic plane \( \mathbb{H}^2 \), with \( \pi \) as the projection map.

In fact, we can describe \( \Sigma \) more intrinsically: if we denote the coordinates of a point \( x \in \mathbb{R}^5 \setminus \{0\} \) as \((x_0, \ldots, x_4)\), then the coordinates of \( \bar{f}(u, v) \) satisfy the quadratic equations
\[ x_1(x_0 - x_3 - 1) - x_2 x_4 = 0, \]
\[ x_2(x_0 + x_3 - 1) - x_1 x_4 = 0, \]
\[ (4x_0 - 1)^2 - 12 x_1^2 - 12 x_2^2 - 9 = 0. \]

Therefore, \( \Sigma \) is contained in the (real) algebraic variety \( X \subset \mathbb{R}^5 \setminus \{0\} \) defined by equations \(6.6\). \( \Sigma \) is not, however, equal to all of \( X \); for instance, \( X \) contains an affine plane consisting of all points of the form \((1, 0, 0, x_3, x_4)\), and this plane intersects \( \Sigma \) only when \( x_3 = x_4 = 0 \). Projections of \( \Sigma \) to the \((x_1, x_2, x_0), (x_1, x_2, x_3), \) and \((x_1, x_2, x_4)\) coordinate 3-planes are shown in Figure 1. 

17
Example 6.4. When $\epsilon = -1$, the unique solution to equation (6.2) with all $h_{ij}$ constant is

$$\Omega = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & -2
\end{bmatrix} \alpha + \begin{bmatrix}
0 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & -\frac{1}{3} & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{3} \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix} \omega_0^1 + \begin{bmatrix}
1 & 0 & 0 & \frac{1}{3} & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0
\end{bmatrix} \omega_0^2.$$

Furthermore, the Gauss equation (4.16) implies that the centroaffine metric has Gauss curvature $K = \frac{1}{3}$.

As in the previous example, denote the matrices in equation (6.7) by $M_0, M_1, M_2$, respectively. Then we have

$$[M_0, M_1] = -M_2, \quad [M_1, M_2] = -\frac{1}{3} M_0, \quad [M_2, M_0] = -M_1.$$  

These bracket relations imply that the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(5, \mathbb{R})$ spanned by $(M_0, M_1, M_2)$ is isomorphic to $\mathfrak{so}(3, \mathbb{R})$. Furthermore, it is straightforward to check that $\mathfrak{g}$ acts irreducibly on $\mathbb{R}^5 \setminus \{0\}$. Similarly to the previous example, $\mathfrak{so}(3, \mathbb{R})$ has a unique irreducible 5-dimensional representation, and this representation arises from a (unique) irreducible representation of $SO(3, \mathbb{R})$; it follows that the Lie group $G \subset GL(5, \mathbb{R})$ corresponding to the Lie algebra $\mathfrak{g}$ is isomorphic to $SO(3, \mathbb{R})$.

We will compute a local parametrization for $\Sigma$ as in the previous example: compute the 1-parameter subgroups of $G$ generated by $M_0, M_1, M_2$ and take products of the resulting group elements. Ensuring the surjectivity of the resulting parametrization is easier than in the previous example. First, observe that a basis $(\bar{M}_0, \bar{M}_1, \bar{M}_2)$ for the standard representation of $\mathfrak{so}(3, \mathbb{R})$ with the same bracket relations as $(M_0, \sqrt{3} M_1, \sqrt{3} M_2)$ is given by

$$\bar{M}_0 = \begin{bmatrix}0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0\end{bmatrix}, \quad \bar{M}_1 = \begin{bmatrix}0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0\end{bmatrix}, \quad \bar{M}_2 = \begin{bmatrix}0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0\end{bmatrix}.$$
Exponentiating this basis yields the 1-parameter subgroups

$$\bar{g}_0(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{bmatrix},$$

$$\bar{g}_1(u) = \begin{bmatrix} \cos(u) & -\sin(u) & 0 \\ \sin(u) & \cos(u) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{g}_2(v) = \begin{bmatrix} \cos(v) & 0 & -\sin(v) \\ 0 & 1 & 0 \\ \sin(v) & 0 & \cos(v) \end{bmatrix}.$$  

Then the map \(f : \mathbb{R}^3 \to SO(3, \mathbb{R})\) defined by

$$f(u, v, t) = \bar{g}_1(u)\bar{g}_2(v)\bar{g}_0(t) = \begin{bmatrix} \cos(u)\cos(v) - \sin(u)\cos(t) + \cos(u)\sin(v)\sin(t) & -\sin(u)\sin(t) - \cos(u)\sin(v)\cos(t) \\ \sin(u)\cos(v) & \cos(u)\cos(t) + \sin(u)\sin(v)\sin(t) & \cos(u)\sin(t) - \sin(u)\sin(v)\cos(t) \\ \sin(v) & -\cos(v)\sin(t) & \cos(v)\cos(t) \end{bmatrix}$$

is easily seen to be surjective onto \(SO(3, \mathbb{R})\). Thus the analogous map \(\bar{f} : \mathbb{R}^3 \to G\) will be surjective onto \(G\), and the map \(\bar{f} = \pi \circ f\) will be surjective onto \(\Sigma\).

So, define the 1-parameter subgroups

$$g_0(t) = \exp(tM_0) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \cos(t) & \sin(t) & 0 & 0 \\ 0 & -\sin(t) & \cos(t) & 0 & 0 \\ 0 & 0 & 0 & \cos(2t) & \sin(2t) \\ 0 & 0 & 0 & -\sin(2t) & \cos(2t) \end{bmatrix},$$

$$g_1(u) = \exp(u\sqrt{3}M_1) = \begin{bmatrix} \frac{1}{4}(3\cos(2u) + 1) & -\frac{\sqrt{3}}{2}\sin(2u) & 0 & \frac{1}{4}(1 - \cos(2u)) & 0 \\ \frac{\sqrt{3}}{2}\sin(2u) & \cos(2u) & 0 & -\frac{1}{2\sqrt{3}}\sin(2u) & 0 \\ 0 & 0 & \cos(u) & 0 & -\frac{1}{\sqrt{3}}\sin(u) \\ \frac{3}{4}(1 - \cos(2u)) & \frac{\sqrt{3}}{2}\sin(2u) & 0 & \frac{1}{4}(\cos(2u) + 3) & 0 \\ 0 & 0 & \sqrt{3}\sin(u) & 0 & \cos(u) \end{bmatrix},$$
\[ g_2(v) = \exp(v\sqrt{3}M_2) = \begin{bmatrix}
\frac{1}{4}(3\cos(2v) + 1) & 0 & -\frac{\sqrt{3}}{2}\sin(2v) & \frac{1}{4}(\cos(2v) - 1) & 0 \\
0 & \cos(v) & 0 & 0 & -\frac{1}{\sqrt{3}}\sin(v) \\
\frac{\sqrt{3}}{2}\sin(2v) & 0 & \cos(2v) & \frac{1}{2\sqrt{3}}\sin(2v) & 0 \\
\frac{3}{4}(\cos(2v) - 1) & 0 & -\frac{\sqrt{3}}{2}\sin(2v) & \frac{1}{4}(\cos(2v) + 3) & 0 \\
0 & \sqrt{3}\sin(v) & 0 & 0 & \cos(v)
\end{bmatrix}. \]

Then set
\[
(6.9) \quad \bar{f}(u, v, t) = \pi (g_1(u) \cdot g_2(v) \cdot g_0(t)) = \begin{bmatrix}
\frac{1}{2} (3\cos^2(u) \cos^2(v) - 1) \\
\sqrt{3}\sin(u) \cos(u) \cos^2(v) \\
\frac{3}{2} (\cos^2(v)(2 - \cos^2(u)) - 1) \\
3\sin(u) \sin(v) \cos(v)
\end{bmatrix}.
\]

It follows from the discussion above that \( \bar{f} \) is a surjective map onto \( \Sigma \), and we see that \( \bar{f} \) is also independent of \( t \). Unlike in the previous example, the tangent vectors \( \bar{f}_u, \bar{f}_v \) are not linearly independent for all \( (u, v) \in \mathbb{R}^2 \); indeed, \( \bar{f}_u = 0 \) whenever \( v \) is an odd multiple of \( \frac{\pi}{2} \). Nevertheless, the restriction of \( \bar{f} \) to some neighborhood of the point \( (u, v) = (0, 0) \) is a smooth embedding, and then homogeneity implies that \( \Sigma \) is smooth everywhere. Topologically, \( \Sigma \) is diffeomorphic to a sphere; this is to be expected, as \( \mathcal{F}_3 \) is isomorphic to the orthonormal frame bundle of the unit sphere \( S^2 \), with \( \pi \) as the projection map.

As in the previous example, we can show that \( \Sigma \) is contained in the intersection of three quadric hypersurfaces in \( \mathbb{R}^5 \setminus \{0\} \). The coordinates of \( \bar{f}(u, v) \) satisfy the quadratic equations
\[
\begin{align*}
x_1(x_0 + x_3 - 1) + x_2x_4 &= 0, \\
x_2(x_0 - x_3 - 1) + x_1x_4 &= 0, \\
(4x_0 - 1)^2 + 12x_2^2 + 12x_3^2 - 9 &= 0.
\end{align*}
\]

Therefore, \( \Sigma \) is contained in the (real) algebraic variety \( X \subset \mathbb{R}^5 \setminus \{0\} \) defined by equations \((6.10)\). Once again, \( \Sigma \) is not equal to all of \( X \): the variety \( X \) contains affine planes consisting of all points of the form \((1, 0, 0, x_3, x_4)\) or \((-\frac{1}{2}, 0, 0, x_3, x_4)\); the former intersects \( \Sigma \) only when \( x_3 = x_4 = 0 \), and the latter intersects \( \Sigma \) only when \( x_3^2 + x_4^2 = \frac{9}{4} \). Projections of \( \Sigma \) to the \((x_1, x_2, x_0), (x_1, x_2, x_3), \) and \((x_1, x_2, x_4)\) coordinate 3-planes are shown in Figure 2.
6.2. **Timelike homogeneous examples.** From the adaptations of §5, the matrix \( \Omega = [\omega^i_j] \) of Maurer-Cartan forms on \( \mathcal{F}_3 \) may be written as

\[
\Omega = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 \\
\end{pmatrix} \begin{pmatrix} \alpha \\
\omega_1 \\
\omega_2 \\
\end{pmatrix}
\]  

while the structure equations (2.3) may be written as

\[
d\Omega = -\Omega \wedge \Omega.
\]  

Substituting (6.11) into equation (6.12) and imposing the condition that all the functions \( h^i_{jk} \) are constant leads to a system of algebraic equations for the \( h^i_{jk} \). This system has precisely one solution, which is described in the following example.

**Example 6.5.** The unique solution to equation (6.12) with all \( h^i_{jk} \) constant is

\[
\Omega = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & -2 \\
\end{pmatrix} \begin{pmatrix} \alpha \\
\omega_1 \\
\omega_2 \\
\end{pmatrix}
\]  

Furthermore, the Gauss equation (5.17) implies that the centroaffine metric has Gauss curvature

\[ K = -\frac{1}{3}. \]

As in the spacelike examples, denote the matrices in equation (6.13) by \( M_0, M_1, M_2 \), respectively, so that

\[
\Omega = M_0 \alpha + M_1 \omega_1^0 + M_2 \omega_0^2.
\]

Then we have

\[
[M_0, M_1] = M_1, \quad [M_1, M_2] = \frac{1}{3} M_0, \quad [M_2, M_0] = M_2.
\]
These bracket relations imply that the Lie algebra $\mathfrak{g} \subset \mathfrak{gl}(5, \mathbb{R})$ spanned by $(M_0, M_1, M_2)$ is isomorphic to $\mathfrak{so}(2, 1)$. Furthermore, it is straightforward to check that $\mathfrak{g}$ acts irreducibly on $\mathbb{R}^5 \setminus \{0\}$. Similarly to the previous examples, $\mathfrak{so}(2, 1)$ has a unique irreducible 5-dimensional representation, and this representation arises from a (unique) irreducible representation of $SO^+(2, 1)$; it follows that the Lie group $G \subset GL(5, \mathbb{R})$ corresponding to the Lie algebra $\mathfrak{g}$ is isomorphic to $SO^+(2, 1)$.

We will compute a local parametrization for $\Sigma$ more or less as in the spacelike examples, by computing 1-parameter subgroups of $G$ and taking products of the resulting group elements. Unfortunately, the basis $(M_0, M_1, M_2)$ is not well-suited to generating a surjective parametrization, so first we need to modify it slightly. To this end, observe that a basis $(\bar{M}_0, \bar{M}_1, \bar{M}_2)$ for the standard representation of $\mathfrak{so}(2, 1)$ with the same bracket relations as $(M_0, \sqrt{3}M_1, \sqrt{3}M_2)$ is given by

$$
\bar{M}_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \bar{M}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \bar{M}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.
$$

Now consider the modified basis

$$
\bar{M}_0' = \bar{M}_0, \quad \bar{M}_1' = \frac{1}{\sqrt{2}}(\bar{M}_1 - \bar{M}_2), \quad \bar{M}_2' = \frac{1}{\sqrt{2}}(\bar{M}_1 + \bar{M}_2).
$$

Exponentiating this modified basis yields the 1-parameter subgroups

$$
\bar{g}_0(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-t} \end{bmatrix},
$$

$$
\bar{g}_1(u) = \begin{bmatrix} \cos(u) & -\frac{1}{\sqrt{2}}\sin(u) & \frac{1}{\sqrt{2}}\sin(u) \\ \frac{1}{\sqrt{2}}\sin(u) & \frac{1}{2}(1 + \cos(u)) & \frac{1}{2}(1 - \cos(u)) \\ -\frac{1}{\sqrt{2}}\sin(u) & \frac{1}{2}(1 - \cos(u)) & \frac{1}{2}(1 + \cos(u)) \end{bmatrix},
$$

$$
\bar{g}_2(v) = \begin{bmatrix} \cosh(v) & \frac{1}{\sqrt{2}}\sinh(v) & \frac{1}{\sqrt{2}}\sinh(v) \\ \frac{1}{\sqrt{2}}\sinh(v) & \frac{1}{2}(\cosh(v) + 1) & \frac{1}{2}(\cosh(v) - 1) \\ \frac{1}{\sqrt{2}}\sinh(v) & \frac{1}{2}(\cosh(v) - 1) & \frac{1}{2}(\cosh(v) + 1) \end{bmatrix}.
$$

Then the map $f : \mathbb{R}^3 \rightarrow SO^+(2, 1)$ defined by

$$
f(u, v, t) = \bar{g}_1(u)\bar{g}_2(v)\bar{g}_0(t) =
\begin{bmatrix}
\cos(u)\cosh(v) & \frac{1}{\sqrt{2}}e^t(\cos(u)\sinh(v) - \sin(u)) & \frac{1}{\sqrt{2}}e^{-t}(\cos(u)\sinh(v) + \sin(u)) \\
\frac{1}{\sqrt{2}}(\sin(u)\cosh(v) + \sinh(v)) & \frac{1}{2}e^t(\sin(u)\sinh(v) + \cosh(v) + \cos(u)) & \frac{1}{2}e^{-t}(\sin(u)\sinh(v) + \cosh(v) - \cos(u)) \\
-\frac{1}{\sqrt{2}}(\sin(u)\cosh(v) - \sinh(v)) & -\frac{1}{2}e^t(\sin(u)\sinh(v) - \cosh(v) + \cos(u)) & -\frac{1}{2}e^{-t}(\sin(u)\sinh(v) - \cosh(v) - \cos(u))
\end{bmatrix}
$$

is surjective onto $SO^+(2, 1)$. Thus the analogous map $\bar{f} : \mathbb{R}^3 \rightarrow G$ will be surjective onto $G$, and the map $\bar{f} = \pi \circ f$ will be surjective onto $\Sigma$. 

22
So, define the 1-parameter subgroups

\( g_0(t) = \exp(tM_0), \quad g_1(u) = \exp \left( u \sqrt{\frac{2}{3}} (M_1 - M_2) \right), \quad g_2(v) = \exp \left( v \sqrt{\frac{2}{3}} (M_1 + M_2) \right). \)

(The explicit expressions for these group elements are each too large to fit on one line and are not particularly enlightening.) Then set

\[
\tilde{f}(u, v, t) = \pi \left( g_1(u) \cdot g_2(v) \cdot g_0(t) \right)
\]

\( (6.16) \)

\[
= \begin{bmatrix}
\frac{1}{4} [3 \cos^2(u) (\cosh(2v) + 1) - 2] \\
\frac{\sqrt{3}}{2} \cos(u) [\sin(u) (\cosh(2v) + 1) + \sinh(2v)] \\
- \frac{\sqrt{3}}{2} \cos(u) [\sin(u) (\cosh(2v) + 1) - \sinh(2v)] \\
- \frac{3}{8} [\cos^2(u) (\cosh(2v) + 1) - 2 (\cosh(2v) + \sin(u) \sinh(2v))] \\
- \frac{3}{8} [\cos^2(u) (\cosh(2v) + 1) - 2 (\cosh(2v) - \sin(u) \sinh(2v))] \\
\end{bmatrix}.
\]

It follows from the discussion above that \( \tilde{f} \) is a surjective map onto \( \Sigma \), and we see that \( \tilde{f} \) is also independent of \( t \). Moreover, it is straightforward to check that the tangent vectors \( \tilde{f}_u, \tilde{f}_v \) are linearly independent for all \( (u, v) \in \mathbb{R}^2 \); therefore \( \tilde{f} \) parametrizes a smooth surface \( \Sigma \subset \mathbb{R}^5 \setminus \{0\} \), as expected.

In this case, \( \mathcal{F}_3 \) is isomorphic to the orthonormal frame bundle of the timelike surface \( S^{2,1} \) consisting of all spacelike unit vectors in \( \mathbb{R}^{2,1} \). This surface is a hyperboloid of one sheet, and so we might expect that \( \Sigma \) would be diffeomorphic to a cylinder. However, if we regard the domain of the parametrization \( (6.16) \) as \( S^1 \times \mathbb{R} \), we see that the map \( (6.16) \) is invariant under the transformation

\( (u, v) \to (u + \pi, -v). \)

Therefore, \( \Sigma \) is diffeomorphic to a Möbius band, which is precisely the quotient of the cylinder by this map. (We note, however, that the map \( f : S^1 \times \mathbb{R} \to G \) is one-to-one, as the frame vectors \( e_1(u, v), \ldots, e_4(u, v) \) are not preserved by the transformation \( (6.17) \).)

As in the spacelike examples, we can show that \( \Sigma \) is contained in the intersection of three quadric hypersurfaces in \( \mathbb{R}^5 \setminus \{0\} \). The coordinates of \( \tilde{f}(u, v) \) satisfy the quadratic equations

\[
3x_2^2 - x_4(4x_0 + 2) = 0,
\]

\[
3x_1^2 - x_3(4x_0 + 2) = 0,
\]

\[
2x_0^2 - x_0 - 3x_1x_2 - 1 = 0.
\]

Therefore, \( \Sigma \) is contained in the (real) algebraic variety \( X \subset \mathbb{R}^5 \setminus \{0\} \) defined by equations \( (6.18) \). \( \Sigma \) is an interesting and somewhat complicated subset of \( X \): first observe that the projection of \( X \) to the \( (x_0, x_1, x_2) \) coordinate 3-plane consists of the hyperboloid of one sheet defined by the third equation in \( (6.18) \), minus all points of the form \( (-\frac{1}{2}, x_1, x_2) \) except for \( (-\frac{1}{2}, 0, 0) \). The projection of \( X \) to this punctured hyperboloid is one-to-one, except over the point \( (-\frac{1}{2}, 0, 0) \), where the inverse image consists of all points of the form \( (-\frac{1}{2}, 0, 0, x_3, x_4) \). Meanwhile, \( \Sigma \) consists of that portion of \( X \) that projects to the portion of the hyperboloid with \( x_0 > -\frac{1}{2} \), together with the curve

\[
\left\{ (-\frac{1}{2}, 0, 0, x_3, x_4) \mid x_3x_4 = \frac{9}{4}, x_3, x_4 > 0 \right\}.
\]
Projections of $\Sigma$ to the $(x_1, x_2, x_0)$, $(x_1, x_2, x_3)$, $(x_1, x_2, x_4)$, $(x_1, x_3, x_0)$, and $(x_1, x_4, x_0)$ coordinate 3-planes are shown in Figure 3.

![Figure 3](image)

**Figure 3.** Projections of surface from Example 6.5 to 3-D subspaces

We collect the results of this section in the following theorem:

**Theorem 6.6.** Let $\tilde{f} : M \to \mathbb{R}^5 \setminus \{0\}$ be a centroaffine immersion whose image $\Sigma = \tilde{f}(M)$ is a homogeneous, nondegenerate, spacelike or timelike centroaffine surface. Then $\Sigma$ is equivalent via the $\text{GL}(5, \mathbb{R})$-action on $\mathbb{R}^5 \setminus \{0\}$ to one of the following:

- the immersion of the hyperbolic plane $\mathbb{H}^2$ of Example 6.3;
- the immersion of the unit sphere $S^2$ of Example 6.4;
- the immersion of the Lorentzian surface $S^{2,1}$ of Example 6.5.

**References**

1. Jeanne Clelland, *From Frenet to Cartan: The Method of Moving Frames*, to appear.
2. William Fulton and Joe Harris, *Representation theory*, Graduate Texts in Mathematics, vol. 129, Springer-Verlag, New York, 1991, A first course, Readings in Mathematics.
3. Hitoshi Furuhata, *Minimal centroaffine immersions of codimension two*, Bull. Belg. Math. Soc. Simon Stevin 7 (2000), no. 1, 125–134.
4. Hitoshi Furuhata and Takashi Kurose, *Self-dual centroaffine surfaces of codimension two with constant affine mean curvature*, Bull. Belg. Math. Soc. Simon Stevin 9 (2002), no. 4, 573–587.
5. Robert B. Gardner and George R. Wilkens, *The fundamental theorems of curves and hypersurfaces in centro-affine geometry*, Bull. Belg. Math. Soc. Simon Stevin 4 (1997), no. 3, 379–401.
6. P. Griffiths, *On Cartan’s method of Lie groups and moving frames as applied to uniqueness and existence questions in differential geometry*, Duke Math. J. 41 (1974), 775–814.
7. Thomas A. Ivey and J. M. Landsberg, *Cartan for beginners: differential geometry via moving frames and exterior differential systems*, Graduate Studies in Mathematics, vol. 61, American Mathematical Society, Providence, RI, 2003.

24
8. Detlef Laugwitz, *Differentialgeometrie in Vektorräumen, unter besonderer Berücksichtigung der unendlichdimensionalen Räume*, Friedr. Vieweg & Sohn, Braunschweig, 1965.

9. An Min Li and Chang Ping Wang, *Canonical centroaffine hypersurfaces in \( \mathbb{R}^{n+1} \)*, Results Math. **20** (1991), no. 3-4, 660–681, Affine differential geometry (Oberwolfach, 1991).

10. Hui Li Liu and Chang Ping Wang, *The centroaffine Tchebychev operator*, Results Math. **27** (1995), no. 1-2, 77–92, Festschrift dedicated to Katsumi Nomizu on his 70th birthday (Leuven, 1994; Brussels, 1994).

11. O. Mayer and A. Myller, *La géométrie centroaffine des courbes planes*, Annales Scientifiques de l’Université de Jassy **18** (1933), 234–280.

12. John Milnor and Dale Husemoller, *Symmetric bilinear forms*, Springer-Verlag, New York-Heidelberg, 1973, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73.

13. Katsumi Nomizu and Takeshi Sasaki, *Centroaffine immersions of codimension two and projective hypersurface theory*, Nagoya Math. J. **132** (1993), 63–90.

14. Christine Scharlach, *Centroaffine first order invariants of surfaces in \( \mathbb{R}^4 \)*, Results Math. **27** (1995), no. 1-2, 141–159, Festschrift dedicated to Katsumi Nomizu on his 70th birthday (Leuven, 1994; Brussels, 1994).

15. _____, *Centroaffine differential geometry of (positive) definite oriented surfaces in \( \mathbb{R}^4 \)*, New developments in differential geometry, Budapest 1996, Kluwer Acad. Publ., Dordrecht, 1999, pp. 411–428.

16. Christine Scharlach, Udo Simon, Leopold Verstraelen, and Luc Vrancken, *A new intrinsic curvature invariant for centroaffine hypersurfaces*, Beiträge Algebra Geom. **38** (1997), no. 2, 437–458.

17. Christine Scharlach and Luc Vrancken, *A curvature invariant for centroaffine hypersurfaces. II*, Geometry and topology of submanifolds, VIII (Brussels, 1995/Nordfjordeid, 1995), World Sci. Publ., River Edge, NJ, 1996, pp. 341–350.

18. _____, *Centroaffine surfaces in \( \mathbb{R}^4 \) with planar \( \nabla \)-geodesics*, Proc. Amer. Math. Soc. **126** (1998), no. 1, 213–219. MR 1452827 (98f:53010)

19. Chang Ping Wang, *Centroaffine minimal hypersurfaces in \( \mathbb{R}^{n+1} \)*, Geom. Dedicata **51** (1994), no. 1, 63–74.

20. George R. Wilkens, *Centro-affine geometry in the plane and feedback invariants of two-state scalar control systems*, Differential geometry and control (Boulder, CO, 1997), Proc. Sympos. Pure Math., vol. 64, Amer. Math. Soc., Providence, RI, 1999, pp. 319–333.

21. Yun Yang and Huili Liu, *Minimal centroaffine immersions of codimension two*, Results Math. **52** (2008), no. 3-4, 423–437.

22. Yun Yang, Yanhua Yu, and Huili Liu, *Flat centroaffine surfaces with the degenerate second fundamental form and vanishing Pick invariant in \( \mathbb{R}^4 \)*, J. Math. Anal. Appl. **397** (2013), no. 1, 161–171.

Department of Mathematics, UNC - Chapel Hill, CB #3250, Phillips Hall, Chapel Hill, NC 27599

E-mail address: bushek@unc.edu

Department of Mathematics, 395 UCB, University of Colorado, Boulder, CO 80309-0395

E-mail address: Jeanne.Clelland@colorado.edu

25