A DICHOTOMY OF SELF-CONFORMAL SUBSETS OF $\mathbb{R}$ WITH OVERLAPS

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Abstract. We show that self-conformal subsets of $\mathbb{R}$ that do not satisfy the weak separation condition have full Assouad dimension. Combining this with a recent results by Käenmäki and Rossi we conclude that an interesting dichotomy applies to self-conformal and not just self-similar sets: if $F \subset \mathbb{R}$ is self-conformal with Hausdorff dimension strictly less than 1, either the Hausdorff dimension and Assouad dimension agree or the Assouad dimension is 1. We conclude that the weak separation property is in this case equivalent to Assouad and Hausdorff dimension coinciding.

1. Introduction

In this article we study a family of subsets of the line $\mathbb{R}$, called self-conformal sets. We assume the reader is familiar with standard work on the subject of dimension theory and conformal attractors (see e.g. Falconer [4], [5], and Pesin [17]) but repeat some important results for completeness.

Definition 1.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^{1+\varepsilon}$ diffeomorphism such that there exists an open interval $J_f \subset \mathbb{R}$ such that $\inf_{x \in J_f} |Df(x)| > 0$ and $f(J_f) \subset J_f$. If, furthermore, there exists $0 < c < 1$ such that $\sup_{x \in J_f} |Df(x)| \leq c$ we call $f$ a conformal contraction (on $J_f$).

The self-conformal attractor is defined to be the unique set invariant under such maps that we call an iterated function system.

Definition 1.2. Let $\mathcal{I} = \{f_i\}_{i \in \Lambda}$ be a finite collection of conformal contractions on some open interval $J$, for some (finite) index set $\Lambda$. We call $\mathcal{I}$ the self-conformal iterated function system (IFS) and consider the unique, non-empty, and compact set $F$ satisfying

$$F = \bigcup_{i \in \Lambda} f_i(F).$$

This invariant set is known as the self-conformal attractor of $\mathcal{I}$.

The existence and uniqueness of $F$ follows from Hutchinson [9]. If there are no overlaps in the construction, the Hausdorff dimension can be found using thermodynamic formalism, see Pesin [17] and references contained therein. If there are exact overlaps this method can be adapted to find the Hausdorff dimension, see...
e.g. Peres et al. [16]. We are particularly interested in the case where the overlaps do not behave nicely, specifically where the weak separation condition does not hold. We show that the Assouad dimension of non-trivial $F \subset \mathbb{R}$, where the associated IFS does not satisfy the weak separation property, is 1. Combining this with the fact that the weak separation property implies coincidence of Assouad and Hausdorff dimension, see Käenmäki and Rossi [10], we conclude that either Hausdorff and Assouad dimension agree or the Assouad dimension is full; it equals 1. This means that the weak separation property is equivalent to Hausdorff and Assouad dimension being equal. This extends recent work by Fraser et al. [8], who considered the linear, self-similar, case.

The Assouad dimension was developed with embedding problems in mind (see Assouad [1] and [2], and Robinson [18]) and has become a tool in the study of strange attractors and fractals, see Fraser [7], Luukkainen [13], Mackay and Tyson [14], and references therein. In particular we note that the Assouad dimension behaves similarly to the Hausdorff and box-counting dimension in simple settings, for instance when the resulting attractor is Ahlfors regular, but can differ considerably once overlaps are allowed. The Assouad dimension can be heuristically described as a measurement of the maximal (relative) scaling exponent of the number of boxes one needs to cover (a neighbourhood of) a set.

**Definition 1.3.** Let $E \subseteq \mathbb{R}$ and let $N_\rho(E)$ be the least number of open sets of diameter at most $\rho$ that cover $E$. The Assouad dimension of $E$ is

$$\dim_A(E) = \inf \left\{ \alpha : \text{there exists } C > 0 \text{ such that} \right.$$ 

$$\text{for all } 0 < \rho < R \text{ we have } \sup_{x \in E} N_\rho(B(x, R) \cap E) \leq C \left( \frac{R}{\rho} \right)^\alpha \right\}.$$ 

We denote the Hausdorff and box-counting dimension by $\dim_H$ and $\dim_B$, respectively. For definitions and basic properties see Falconer [4]. At this point we also remark that for self-conformal attractors, irrespective of separation conditions, the upper box counting and Hausdorff dimension coincide (see Falconer [3]). This is, of course, in contrast with our main result Theorem 2.2, where we show that lack of the weak separation property implies $\dim_A F = 1$, irrespective of $\dim_H F$.

We use a finite alphabet $\Lambda$ to code points in the attractor. Let $\Lambda$ be the finite index set given by the IFS. We write $\Lambda^k$ to refer to the codings (or words) of length $k$ that have entries in $\Lambda$,

$$\Lambda^k = \{ w_1 w_2 \ldots w_k : w_i \in \Lambda \}.$$ 

Similarly, we define the set of all infinite words as $\Lambda^\infty$ and the set of all finite words as $\Lambda^* = \bigcup_{k \in \mathbb{N}_0} \Lambda^k$, where $\Lambda^0 = \{ \varepsilon_0 \}$ contains only the empty word $\varepsilon_0$. Equipped with these words we can describe composition of maps according to words. Let $w = w_1 w_2 \ldots w_{|w|} \in \Lambda^*$, we write

$$f_w = f_{w_1} \circ f_{w_2} \circ \cdots \circ f_{w_{|w|}},$$ 

where $|w|$ is the length of the word $w$. For $w = w_1 w_2 \cdots \in \Lambda^\infty$ we similarly write

$$f_w = \lim_{k \to \infty} f_{w_1} \circ f_{w_2} \circ \cdots \circ f_{w_k}.$$ 

Since all the $f_i$ are strict contractions, $f_w(x)$ maps all $x \in \mathcal{J}$ to the same point. Given two words $w \in \Lambda^*$ and $v \in \Lambda^* \cup \Lambda^\infty$ we let $vw$ be the concatenation of the two words. The attractor is then the projection of the codings under composition
of the mappings in $\mathcal{I}$. Let $x \in \mathbb{R}$; we have

$$F = \bigcup_{w \in \Lambda^n} f_w(x) = \lim_{k \to \infty} \bigcap_{i=1}^k \bigcup_{w \in \Lambda^i} f_w(\Delta),$$

for all compact sets $\Delta$ such that $f_i(\Delta) \subseteq \Delta$ for all $i \in \Lambda$. To ease notation we make the assumption, rescaling and translating if necessary, that the (compact) convex hull of $F$ is $[0, 1]$ and we set $\Delta = [0, 1]$. We can further assume, without loss of generality, that $\mathcal{J}$ is an $\varepsilon$-neighbourhood of $\Delta$ for some $\varepsilon > 0$.

We exclude the trivial case where all $f \in \mathcal{I}$ share the same fixed point. Here, trivially, $\dim_A = \dim_H = 0$ and we stipulate that the are at least two maps in $\mathcal{I}$ that do not share a common fixed point. We label the rightmost fixed point $x_1 \in \Delta$ and refer to the map with this fixed point as $f_1$, choosing arbitrarily if there is more than one. If $\inf_{i \in \Lambda, x \in \mathcal{J}} D f_i(x) > 0$, then $x_1 = 1$. This is not the case in general, but for non-trivial IFSs we must have $x_1 > 0$.

### 2. The Weak Separation Condition and Main Results

Many different separation conditions have been considered for IFS attractors, with the strong separation condition (SSC) and the open set condition (OSC) the most prominent. The strong separation condition is, as the name suggests, stronger than the open set condition but many results concerning the SSC can be extended, under suitable modification, to the OSC and the weak separation condition (WSP), which is a weaker condition still. It was introduced by Lau and Ngai [11] with an important alternative definition due to Zerner [19], which we refer to as the identity limit criterion (ILC). The ILC does not, in general, coincide with the WSP, but in the self-similar setting they do. While the weak separation property was introduced with self-similar sets in mind, several authors considered the self-conformal case, see [6], [10], [12], [15], [16]. In Lemma 3.3 we prove that for finite self-conformal iterated function systems of the real line the ILC is equivalent to the WSP.

The self-similar analogue of Theorem 2.2 proved in [8], makes heavy use of the ILC and in this paper we adopt their methods to the self-conformal case. Our proof sometimes closely follows, and sometimes differs from the approach in [8]. Lemma 3.6 for example, holds trivially in the self-similar setting, but needs some work for self-conformal attractors.

Let $0 < b \leq 1$ and set $\mathcal{A}_b$ to be the set of all codings such that the image under the associated mapping is of diameter comparable to $b$,

$$\mathcal{A}_b = \{ w \in \Lambda^* : \text{diam } f_{w_1 w_2 \ldots w_l}(\Delta) \leq b < \text{diam } f_{w_1 w_2 \ldots w_{|w|-1}}(\Delta) \},$$

and set $\mathcal{M}_b = \{ f_w : w \in \mathcal{A}_b \}$ to be the collection of (distinct) maps associated to those words.

**Definition 2.1.** Let $F$ be the attractor of the (self-conformal) iterated function system $\mathcal{I} = \{ f_i \}_{i \in \Lambda}$. The IFS $\mathcal{I}$ satisfies the weak separation property (WSP) if there exists $\gamma \in \mathbb{N}$ and a set $\mathcal{O}$ with non-empty interior such that, for all $b \in (0, 1]$ and all $x \in \mathbb{R}$,

$$\# \{ f \in \mathcal{M}_b : x \in f(\mathcal{O}) \} \leq \gamma.$$

We now state our main result.

**Theorem 2.2.** Let $F \subset \mathbb{R}$ be the attractor of the non-trivial iterated function system $\mathcal{I}$. If $\mathcal{I}$ does not satisfy the weak separation property, the Assouad dimension is full. That is

$$\neg \text{WSP} \Rightarrow \dim_A F = 1.$$
The following proposition follows from the work in Käenmäki and Rossi [10] for Moran constructions.

**Proposition 2.3.** Let $F \subseteq \mathbb{R}$ be the attractor of a non-trivial iterated function system $I$. If $I$ satisfies the WSP then $\dim_H F = \dim_A F$.

Combining Theorem 2.2 and Proposition 2.3 we obtain.

**Corollary 2.4.** Let $F \subseteq \mathbb{R}$ be the attractor of the non-trivial iterated function system $I$ and assume that $\dim_H F < 1$. The following are equivalent:

1. $I$ does not satisfy the weak separation property,
2. The Assouad dimension is full, i.e. $\dim_A F = 1$,
3. $\dim_H F = \dim_B F < \dim_A F = 1$.

### 3. Proofs

We now prove the results in the preceding section. First, we define the bounded distortion condition and the identity limit criterion of Zerner, and prove that this criterion is equivalent to the weak separation property for self-conformal attractors of $\mathbb{R}$. We finish this section by proving Theorem 2.2 using very weak pseudo tangents, to be defined.

It is well known that self-conformal attractors satisfy the bounded distortion condition, see [5].

**Definition 3.1.** The attractor $F$ of an iterated function system $I$ satisfies the bounded distortion condition if there exists a (uniform) distortion constant $K > 0$ such that for all $w \in \Lambda^*$ and all $x \in \mathcal{J}$,

$$K^{-1} \operatorname{diam}(f_w(\Delta)) \leq |Df_w(x)| \leq K \operatorname{diam}(f_w(\Delta)).$$

Since we will be dealing with the inverses $f_i^{-1}$, we have to be careful about the domains. Generally speaking, we are not interested in $f_i^{-1}(x)$ for $x \in \mathbb{R} \setminus \mathcal{J}$, and therefore define, for all $i \in \Lambda$,

$$f_i^{-1}(x) = \begin{cases} \lambda_r x + \lim_{x \to \sup \mathcal{J}} f_i(x) - \lambda_r \sup \mathcal{J}, & \text{for } x \geq \sup \mathcal{J}; \\
\lambda_r x \text{ such that } f_i(y) = x, & \text{for } x \in \mathcal{J}; \\
\lambda_l x + \lim_{x \to \inf \mathcal{J}} f_i(x) - \lambda_l \inf \mathcal{J}, & \text{for } x \leq \inf \mathcal{J}, \end{cases}$$

where $\lambda_r = \lim_{x \to \sup \mathcal{J}} Df_i(x)$ and $\lambda_l = \lim_{x \to \inf \mathcal{J}} Df_i(x)$. Note that this linearisation of $f_i$ beyond $\mathcal{J}$ is well defined as all the limits exist, and $|\lambda_i|, |\lambda_r| > 0$, such that $f_i^{-1}$ is repelling $x \in \mathbb{R} \setminus \mathcal{J}$. Further, we restrict the domains of $f_i$ to $\mathcal{J}$ and slightly abuse notation by writing $f_v$ instead of $f_v|\mathcal{J}$ for all $v \in \Lambda^*$. Let

$$\mathcal{E} = \{ f_v^{-1} \circ f_w : v, w \in \mathbb{I}^*, v \neq w \},$$

where $f_{Id}$ is the identity. We immediately see that $\mathcal{E}$ is a subset of the set of all bounded functions $\mathcal{C}(\mathcal{J}, \mathbb{R})$, which we equip with the supremum norm $\|\|_\infty$.

As noted above, the definition given by Zerner [19] is a very useful tool in the self-similar case, which is not equivalent to the WSP in general. Here we shall refer to it as the identity limit criterion (ILC).

**Definition 3.2** (Identity Limit Criterion – ILC). Let $\mathcal{E} \subseteq \mathcal{C}(\mathcal{J}, \mathbb{R})$ be as above. We say that $\mathbb{I}$ satisfies the identity limit criterion if the identity is not a limit point of $\mathcal{E}$, 

$$\text{Id } \notin \mathcal{E} \setminus \text{Id}.$$
Remark. The original definition did not restrict the functions to a bounded interval and instead considered \( \mathcal{E} \) as a subset of the space of all similarities \( S(\mathbb{R}^d, \mathbb{R}^d) \), endowed with the topology \( T \) of pointwise convergence. Convergence in the topological space \( (S(\mathbb{R}^d, \mathbb{R}^d), T) \) and convergence in the Banach space \( (C([0,1]^d, \mathbb{R}^d), \| \cdot \|_\infty) \) are equivalent for similarities and our definition of the ILC coincides with the original definition by Zerner for self-similar IFS.

It turns out that for self-conformal attractors with finite IFSs the notions of WSP and ILC coincide:

**Lemma 3.3.** Let \( \mathcal{I} \) be a self-conformal iterated function system of the unit line. The weak separation condition and the identity limit criterion are equivalent:

\[
\text{WSP} \iff \text{ILC}.
\]

**Proof.** (\( \neg \text{ILC} \Rightarrow \neg \text{WSP} \)) Assume for a contradiction that the WSP holds. Thus there exists \( \gamma \in \mathbb{N} \) and a set \( \mathcal{O} \) with non-empty interior such that for all \( 0 < b \leq 1 \) and \( x \in \mathbb{R} \) we have \#\{\( f \in \mathcal{M}_b : x \in f(\mathcal{O}) \} \leq \gamma \). It is easy to see that failure of the ILC implies that for every \( \epsilon > 0 \) we can find words \( v, w \in \Lambda^* \) such that \( d_H(\mathcal{O}, f_v^{-1} \circ f_w(\mathcal{O})) < \epsilon \) and \( 0 < \| f_v^{-1} \circ f_w - 1 \|_{\infty} < \epsilon \). Then

\[
\sup_{x \in \mathcal{J}} | f_v^{-1} \circ f_w(x) - x | < \epsilon,
\]

so \( f_v^{-1} \circ f_w(x) \in (x - \epsilon, x + \epsilon) \). Now \( f_v \) is a monotone continuous function and

\[
f_w(x) \in f_v((x - \epsilon, x + \epsilon)).
\]

Both \( f_v(x), f_w(x) \in \mathcal{J} \) and \( f_v(x) \in f_v((x - \epsilon, x + \epsilon)) \), hence

\[
| f_w(x) - f_v(x) | \leq \text{diam}(f_v((x - \epsilon, x + \epsilon)) \cap \mathcal{J}) \leq \text{diam}(f_v((x - \epsilon, x + \epsilon))).
\]

Using bounded distortion and the mean value theorem,

\[
\text{diam}(f_v((x - \epsilon, x + \epsilon))) \leq 2C K \epsilon \cdot \text{diam}(f_v(\Delta)).
\]

where \( K \) is the distortion constant and \( C = \text{diam} \mathcal{J} \). We will frequently use this estimate and from now on redefine \( K \) appropriately to include \( C \). This implies

\[
0 < \| f_v - f_w \|_{\infty} \leq 2K \epsilon \cdot \text{diam}(f_v(\Delta))
\]

But then for any \( N \in \mathbb{N} \) and \( z_1, z_2 \in \{v, w\}^N \),

\[
\| (f_{z_1} - f_{z_2}) \|_{\infty} \leq 2K \epsilon \text{diam}(f_v(\Delta)) K^{N-1} \text{diam}(f_v(\Delta))^{1-N} \leq 2K \epsilon K^{N} \text{diam}(f_v(\Delta))^{2-N}.
\]

Because \( \mathcal{O} \) has non-empty interior, there exists an open interval \( \Omega \subseteq \mathcal{O} \). Let \( \delta = \min_{\{v, w\}^N} \text{diam}(f_v(\Omega)) \) and set

\[
\epsilon < \min \left\{ \delta, \frac{\delta \text{diam}(f_v(\Delta))^{N-2}}{6K^N} \right\},
\]

where \( \delta' > 0 \) is small enough such that,

\[
\mathcal{T} := \inf_{z \in (v, w)^N} \text{diam}(f_v(\Delta)) \geq \inf_{z \in \Lambda} \{ Df_v(x) \} \sup_{z \in \mathcal{J}} \{ \text{diam}(f_v(\Delta)) \} =: \mathcal{T}.
\]

So there exists \( 0 < \kappa \leq 1 \) and \( b' \in [\mathcal{T}, \mathcal{T}] \) such that

\[
\# \{ f_z \in \mathcal{M}_{b'} : z \in \{v, w\}^N \} \geq \kappa \# \{ f_z : z \in \{v, w\}^N \}.
\]

For any \( N \in \mathbb{N} \), we can find \( \epsilon \) such that, for all \( z_1, z_2 \in \{v, w\}^N \),

\[
\| (f_{z_1} - f_{z_2}) \|_{\infty} \leq \delta \frac{\min_{\{v, w\}^N} \text{diam}(f_v(\Omega))}{3}.
\]
This implies that
\[ \bigcap_{z \in \{v, w\}^N} f_z(\Omega) \supseteq \bigcap_{z \in \{v, w\}^N} f_z(\hat{\Omega}) \neq \emptyset. \]

Let \( N > \gamma / \kappa \); by (3.3) there exists \( x \in \bigcap \{ f_z(\hat{\Omega}) : z \in \{v, w\}^N \} \), and by (3.4) there exists \( \varepsilon \) such that for all \( g \),
\[
\#\{ f \in \mathcal{M}_g : x \in f(\hat{\Omega}) \} \geq \kappa \#\{ f_z : z \in \{v, w\}^N \} \geq 2^N \geq \gamma,
\]
contradicting the WSP.

\[ \square \]

\((\neg \text{WSP} \Rightarrow \neg \text{ILC})\)

Let \( \mathcal{W} \subset \Lambda^* \) be such that for any distinct \( v, w \in \mathcal{W} \) the maps \( f_v, f_w \) satisfy \( f_v^{-1} \circ f_w \neq \text{Id} \). In particular \( \mathcal{E} \setminus \text{Id} \) satisfies this condition. Let \( j^+ = \sup \mathcal{J} \) and \( j^- = \inf \mathcal{J} \), and define, for all \( z \in \Lambda^* \),
\[ f_z(j^+) = \lim_{x \to j^+} f_z(x) \quad \text{and} \quad f_z(j^-) = \lim_{x \to j^-} f_z(x). \]
Clearly both limits exist. We let \( \mathcal{C} \) be the ‘normalised’ maps for words in \( \mathcal{W} \), that is
\[
\mathcal{C} = \{(L_v \circ f_v) : v \in \mathcal{W}_i\}, \quad \text{where} \quad L_v(x) = \frac{x - f_v(j^-)}{f_v(j^+) - f_v(j^-)}.
\]
Thus for any \( g \in \mathcal{C} \) we have, slightly abusing notation, \( g(j^-) = 0 \) and \( g(j^+) = 1 \).
Since the maps satisfy the bounded distortion condition, there exists \( 0 < K < \infty \), such that for all \( g \in \mathcal{C} \) and all \( x \in \mathcal{J} \),
\[ 0 < K^{-1} \leq Dg(x) \leq K < \infty. \]

Given any sequence of collections \( \{\mathcal{W}_i\}_{i \in \mathbb{N}} \) such that \( \#\mathcal{W}_i \geq i \) there exist two sequences of words \( (v_i) \) and \( (w_i) \) such that
\[ \left\| ((L_{v_i} \circ f_{v_i})^{-1} \circ L_{w_i} \circ f_{w_i}) - \text{Id} \right\|_\infty \to 0 \quad \text{as} \quad i \to \infty. \]

We will prove this statement by contradiction. Assume the contrary, that is there exists \( \varepsilon > 0 \) such that for all \( i \in \mathbb{N} \) and any distinct \( v, w \in \mathcal{W}_i \), the maps \( L_v \circ f_v \) and \( L_w \circ f_w \) are \( \varepsilon \)-separated, meaning there exists \( x \in \Delta \) such that \( |L_v \circ f_v(x) - L_w \circ f_w(x)| \geq \varepsilon \). Therefore there exists a neighbourhood around \( x \) of diameter at least \( r_\varepsilon = \varepsilon / (K' - K')^{-1} \), for \( K' = \max\{2, K\} \), where \( |L_v \circ f_v(x + r) - L_w \circ f_w(x + r)| \geq \varepsilon |r_\varepsilon - r| \) for \( r \in [-r_\varepsilon, r_\varepsilon] \). This applies for any distinct \( v, w \in \mathcal{W}_i \) and thus
\[ \#\mathcal{W}_i \leq n_\varepsilon := \left\{ \frac{2/r_\varepsilon}{1/r_\varepsilon} \right\}. \]

This bound can be obtained by considering the parallelogram with edges at slopes \( K \) and \( K^{-1} \) and opposing vertices \((j^-, 0)\) and \((j^+, 1)\). Let \( a_\varepsilon = [(\text{diam } \mathcal{J}) / r_\varepsilon] \) and subdivide this parallelogram equally into \( a_\varepsilon \) by \( a_\varepsilon \) smaller parallelograms. Each path along the edges of the smaller parallelograms, starting at \((j^-, 0)\), increasing in horizontal displacement, and ending at \((j^+, 1)\) is a continuous function with slopes \( K \) or \( K^{-1} \). Each function can be distinctly represented by a word in
\[ \mathcal{P}_\varepsilon = \{ e = e_1 e_2 \ldots e_{2a_\varepsilon} \in \{e^+, e^-\}^{2a_\varepsilon} : \#\{ i \in \{1, \ldots, 2a_\varepsilon\} : e_i = e^+ \} = a_\varepsilon \}, \]
where \( e^+ \) represents an edge with slope \( K \) of a small parallelogram, and \( e^- \) represents the corresponding edge with slope \( K^{-1} \). This is the most efficient way to distribute functions with slopes bounded by \( K \) and \( K^{-1} \) and any other function will be at most \( \varepsilon / 2 \) apart from one of the functions given by words in \( \mathcal{P}_\varepsilon \). It is elementary that \( \#\mathcal{P}_\varepsilon = \left(\frac{2a_\varepsilon}{a_\varepsilon}\right) \), giving the bound above.

We can now pick \( i > n_\varepsilon \), which implies that there are at least two maps in \( \mathcal{C}_i \) that are not \( \varepsilon \)-separated. Since \( \varepsilon \) was arbitrary we have a contradiction and (3.6) follows. Similarly, for \( \varepsilon_i \to 0 \) as \( i \to \infty \), replacing \( L_v \) by similarity \( L_v^\varepsilon \) that maps
\( f_\epsilon(j^-) \) into \((0 - \epsilon, 0 + \epsilon)\) and \( f_\epsilon(j^+) \) into \((1 - \epsilon, 1 + \epsilon)\), the convergence in \((3.6)\) holds.

We now prove \(-\)WSP \(\Rightarrow\) \(-\)ILC. Assume that \(I\) does not satisfy the weak separation property. Let \(A^+\gamma\) be the set of words (with minimal cardinality) such that there exists \(b \in (0, 1)\) and \(x_\gamma \in \Delta\) for which

\[
\# \{ f_w : w \in A^+\gamma \subseteq A_0, x_\gamma \in f_w(\Delta) \text{ and } Df_w(x_\gamma) > 0 \} > \gamma.
\]

Similarly define \(A^-\gamma\) for functions with negative derivative. The weak separation property implies that at least one of the sequences \(\{A^+\gamma\}_{\gamma \in \mathbb{N}}\) and \(\{A^-\gamma\}_{\gamma \in \mathbb{N}}\) exists. For definiteness assume \(\{A^+\gamma\}_{\gamma \in \mathbb{N}}\) is the infinite sequence, the other case follows by the same argument.

Fix \(\epsilon_\iota > 0\) such that \(\epsilon_\iota \to 0\) and choose \(\gamma_i\) large enough such that there are \(i\) maps whose images of \(\Delta\) are \(\epsilon_\iota\) close with respect to the Hausdorff metric \(d_H\), i.e. there are \(i\) distinct words \(v_1, \ldots, v_i \in A^+\gamma_i\) for which

\[
d_H(f_{v_j}(\Delta), f_{v_k}(\Delta)) < \epsilon_\iota \max\{ \text{diam}\(f_{v_j}(\Delta)\) \}.
\]

We refer to the collection of these words as \(B_i\). For each \(i\) choose \(w^i \in B_i\) arbitrarily and consider the set

\[
\mathcal{C}_i = \{ (L_{w^i} \circ f_{v_j}) : v \in B_i \}.
\]

By definition \(L_{w^i}\) maps \(f_{v_j}(j^-)\) into \((-\epsilon_\iota, \epsilon_\iota)\) and \(f_{v_j}(j^+)\) into \((1 - \epsilon_\iota, 1 + \epsilon_\iota)\) for all \(j\). The sequence of sets of words \(B_i\) satisfies all assumptions made for \((3.6)\), and there exist a sequence \((m_i)_{i \in \mathbb{N}}\) such that there are at least two words in \(B_{m_i}\), which have associated maps that are \(\varepsilon_{m_i}\)-close. Call these \(u_{m_i}^1\) and \(u_{m_i}^2\). We get

\[
\left\| L_{w^i} \circ f_{u_{m_i}^1} - L_{w^i} \circ f_{u_{m_i}^2} \right\|_\infty \leq \epsilon_{m_i},
\]

and so, for some \(c > 0\),

\[
\begin{align*}
\epsilon_{m_i} & \geq \left\| \left( L_{w^i} \circ f_{u_{m_i}^1} \right)^{-1} \circ L_{w^i} \circ f_{u_{m_i}^2} - \text{Id} \right\|_\infty \\
& \geq \left\| f_{u_{m_i}^1}^{-1} \circ f_{u_{m_i}^2} - \text{Id} \right\|_\infty.
\end{align*}
\]

The required conclusion follows on noting that \(\epsilon_{m_i} \to 0\). \(\square\)

3.1. Proof of Main Theorem. Throughout we will assume that the WSP does not hold and there exists at least one sequence \((v_i, w_i)_{i \in \mathbb{N}} \in (\Lambda^* \times \Lambda^*)^\mathbb{N}\) such that \(f_{v_i}^{-1} \circ f_{w_i} \to \text{Id}\) in \((\mathcal{C}(\mathcal{J}, \mathbb{R}), \|\cdot\|_\infty)\). Let \(x_i\) be the fixed point of \(f_i\), for \(i \in \Lambda\). Recall the \(x_1\) is the right-most fixed point. Fix \(x_0\) to be any other. It is evident that

\[
0 \leq f_0(x_0) = x_0 < x_1 = f_1(x_1) \leq 1.
\]

If there are multiple maps having fixed points \(x_0\) or \(x_1\), choose \(f_0\) and \(f_1\) arbitrarily amongst them.

Lemma 3.4. There exists \(\tilde{C} > 0\) and \(\beta > 0\) such that for all \(v \in \Lambda^*\) and \(x \in \mathcal{J}\),

\[
|Df_v(x) - Df_v(y)| \leq \tilde{C}|x - y|^\beta \text{diam}(f_v(\mathcal{J})).
\]

Proof. Write \(k = |v|\). Using the chain rule repeatedly we obtain

\[
Df_{v_1 \cdots v_k}(x) = Df_{v_1}(x) \cdot Df_{v_2}(f_{v_1}(x)) \cdot Df_{v_3}(f_{v_2v_1}(x)) \cdots Df_{v_k}(f_{v_{k-1}v_{k-2} \cdots v_1}(x)).
\]

It is a basic fact of maps satisfying the bounded distortion condition that \(|f_v(x) - f_v(y)| \leq C \text{diam}(f_v(\mathcal{J})|x - y|\) for all \(w \in \Lambda^*\) and \(x, y \in \mathcal{J}\), see e.g. Falconer [3] Corollary 4.4]. Let \(1 \leq j \leq k\). Fix \(x \in \mathcal{J}\), then for all \(y \in \mathcal{J}\),

\[
Df_v(f_{v_{j-1} \cdots v_1}(y)) = Df_v(f_{v_{j-1} \cdots v_1}(x) + r_j),
\]
where \( |r_j| \leq C \text{diam} f_{v_j-1...v_1}(J) |x-y| \). Now for all \( i \in \Lambda \), \( Df_i \) is \( \beta \)-Hölder for some \( \beta > 0 \) and so there exists \( C' > 0 \) such that
\[
|Df_{v_j}(f_{v_j-1...v_1}(y))| \leq |Df_{v_j}(f_{v_j-1...v_1}(x))| + C'|r_j|^{\beta}.
\]
Note that \( r_k \to 0 \) as \( k \to \infty \), whereas \( Df_i(x) \geq K^{-1} > 0 \). Therefore we can without loss of generality assume that
\[
Df_{v_j}(f_{v_j-1...v_1}(y)) = Df_{v_j}(f_{v_j-1...v_1}(x)) + C''|r_j|^{\beta},
\]
for some \( C'' \in [-C',C'] \). Thus,
\[
Df_{v_1...v_k}(y) = \prod_{j=1}^k Df_{v_j}(f_{v_j-1...v_1}(y))
\]
\[
= \prod_{j=1}^k (Df_{v_j}(f_{v_j-1...v_1}(x)) + C''|r_j|^{\beta})
\]
\[
= \left( \prod_{j=1}^k Df_{v_j}(f_{v_j-1...v_1}(x)) \right) \left( \prod_{j=1}^k \left( 1 + \frac{C''|r_j|^{\beta}}{Df_{v_j}(f_{v_j-1...v_1}(x))} \right) \right)
\]
\[
= Df_{v_1...v_k}(x) \exp \left( \sum_{j=1}^k \log \left( 1 + \frac{C''|r_j|^{\beta}}{Df_{v_j}(f_{v_j-1...v_1}(x))} \right) \right) .
\]
We conclude, for some \( C_1, C_2 \geq 1 \),
\[
|Df_{v_1...v_k}(x) - Df_{v_1...v_k}(y)|
\]
\[
\leq |Df_{v_1...v_k}(x)| \cdot \left| \exp \left( \sum_{j=1}^k \log \left( 1 + \frac{C''|r_j|^{\beta}}{Df_{v_j}(f_{v_j-1...v_1}(x))} \right) \right) - 1 \right|
\]
\[
\leq |Df_{v_1...v_k}(x)| \cdot \left( \exp \left( \sum_{j=1}^k \frac{C''(C \text{diam} f_{v_j-1...v_1}(J) |x-y|)^{\beta}}{Df_{v_j}(f_{v_j-1...v_1}(x))} \right) - 1 \right)
\]
\[
\leq |Df_{v_1...v_k}(x)| \cdot \left( \exp \left[ C^{\beta}C' |x-y|^{\beta} \left( \sum_{j=1}^k \text{diam}(f_{v_j-1...v_1}(J)) \right) \right] - 1 \right)
\]
\[
\leq |Df_{v_1...v_k}(x)| \left( e^{C_1|x-y|^\beta} - 1 \right)
\]
\[
\leq K \text{diam}(f_{v_1...v_k}(J)|x-y|^\beta.
\]
Setting \( \tilde{C} = KC_2 \) completes the proof. \( \square \)

Lemma 3.5. Let \( \varepsilon > 0 \) and choose \( v, w \in \Lambda^* \) such that \( \|f_v^{-1} \circ f_w - \text{Id}\|_{\infty} < \varepsilon \). Then there exist \( \tilde{C} > 0 \) and \( \beta > 0 \) such that for all \( x, y \in J \) that satisfy \( f_v^{-1} \circ f_w(x), f_v^{-1} \circ f_w(y) \in J \),
\[
|D(f_v^{-1} \circ f_w)(x) - D(f_v^{-1} \circ f_w)(y)| \leq \tilde{C}|x-y|^\beta.
\]
Proof. Using the chain rule for inverses, and writing \( g = f_v^{-1} \circ f_w \), we obtain,
\[
\frac{|D(f_v^{-1} \circ f_w)(x) - D(f_v^{-1} \circ f_w)(y)|}{|Df_v(g(x))|} - \frac{|Df_w(y)|}{|Df_v(g(y))|}
\]
\[
= \frac{|Df_v(x) \cdot (Df_v(g(x))) - Df_w(y) \cdot (Df_v(g(x)))|}{|Df_v(g(x))| \cdot (Df_v(g(y)))}
\]
Proof. from above. exists a subsequence such that \( f \) as required.

\[ \text{We now use the assumption that } \| f^{-1} \circ f_{1} - \text{Id} \|_{\infty} < \varepsilon \text{ and so } \text{diam}(f_{1}(J)) \leq \text{diam}(f_{1}(J)) \text{, and, redefining } \tilde{C} \text{ if necessary,} \]
\[
\left| D(f^{-1} \circ f_{1})(x) - D(f^{-1} \circ f_{1})(y) \right| \leq 2K^{2}\tilde{C}|x - y|^{\beta},
\]
as required. \( \square \)

Lemma 3.6. Given any sequences of words \((v_{i}), (w_{i})\) such that \( f_{v_{i}^{-1}} \circ f_{w_{i}} \to \text{Id} \) there exists a subsequence such that \( f_{v_{i}^{-1}} \circ f_{w_{i}}(x_{1}) \nearrow x_{1} \) from below or \( f_{v_{i}^{-1}} \circ f_{w_{i}}(x_{1}) \searrow x_{1} \) from above.

Proof. We first show that (i) has a subsequence \((k_{i})\) such that for every \( k_{i} \) in the sequence, \( f_{v_{k_{i}}} \circ f_{w_{k_{i}}}(x_{0}) \neq x_{0} \) or \( f_{v_{k_{i}}} \circ f_{w_{k_{i}}}(x_{1}) \neq x_{1} \). Assume for a contradiction there is no such subsequence. This means that there exists some \( \varepsilon > 0 \) and \( N \in \mathbb{N} \) such that for all \( v_{i}, w_{i} \in \bigcup_{k \geq N} A^{k}, \)
\[
\| f_{v_{i}^{-1}} \circ f_{w_{i}} - \text{Id} \|_{\infty} < \varepsilon \Rightarrow f_{v_{i}^{-1}} \circ f_{w_{i}}(x_{0}) = x_{0} \text{ and } f_{v_{i}^{-1}} \circ f_{w_{i}}(x_{1}) = x_{1}.
\]
Hence applying the map \( f_{1} \) with fixed point \( x_{1} \) we get \( f_{1}^{-1} \circ f_{v_{i}^{-1}} \circ f_{w_{i}}(x_{1}) = x_{1} \) and in general \( f_{1}^{-n} \circ f_{v_{i}^{-1}} \circ f_{w_{i}} \circ f_{1}^{n}(x_{1}) = x_{1} \), however, choosing \( N \) large enough such that \( \| f_{1}^{-1} \circ f_{v_{i}^{-1}} \circ f_{w_{i}} - f_{1} - \text{Id} \|_{\infty} < \varepsilon \) for \( v_{i}, w_{i} \in \bigcup_{k \geq N} A^{k} \) implies that \( x_{0} \) is also a fixed point of \( f_{1}^{-1} \circ f_{v_{i}^{-1}} \circ f_{w_{i}} \circ f_{1} \), and by induction,
\[
f_{1}^{-n} \circ f_{v_{i}^{-1}} \circ f_{w_{i}} \circ f_{1}^{n}(x_{0}) = x_{0} \text{ and } f_{1}^{-n} \circ f_{v_{i}^{-1}} \circ f_{w_{i}} \circ f_{1}^{n}(x_{1}) = x_{1}.
\]
But this implies that \( f_{v_{i}^{-1}} \circ f_{w_{i}} = \text{Id} \) as \( f_{v_{i}^{-1}} \circ f_{w_{i}} \in C^{1+\varepsilon} \) and since our choice was arbitrary we must have \( f_{v_{i}^{-1}} \circ f_{w_{i}} = \text{Id} \) for all but finitely many \( \nu \neq \nu \in \Lambda^{*} \), contradicting the WSP.

By redefining the maps, reflecting about 1/2 if necessary, we can assume that there exists a subsequence (\( k_{i} \)) satisfying \( f_{v_{k_{i}}} \circ f_{w_{k_{i}}}(x_{1}) \neq x_{1} \) and by choosing subsequences again we can assume \( f_{v_{k_{i}}} \circ f_{w_{k_{i}}}(x_{1}) < x_{1} \) or \( f_{v_{k_{i}}} \circ f_{w_{k_{i}}}(x_{1}) > x_{1} \) as required. \( \square \)

Using Lemma 3.6 we now assume \((v_{i}, w_{i}) \), \((\Lambda^{*} \times \Lambda^{*})^{N}\) satisfies \( g_{i} = (f_{v_{i}^{-1}} \circ f_{w_{i}} - \text{Id}) \to 0, \) and \( g_{i}(x_{1}) \nearrow 0 \). The latter claim can be made without loss of generality as the case when \( g_{i}(x_{1}) \searrow 0 \) can be proven by changing some of the signs in the work below; we omit details. Note that Lemma 3.5 gives us bounds on the derivative of \( g_{i} \). The most extremal case satisfies \( g_{i}(y) = -\|g_{i}\|_{\infty}, g_{i}(y + 2\delta) = \|g_{i}\|_{\infty}, \) and by symmetry, \( g_{i}(y + \delta) = 0 \) for some \( y \in J \) and \( \delta \geq 0 \). We must have \( Dg_{i}(y) = Dg_{i}(y + 2\delta) = 0 \) and therefore \( \sup_{x \in J} |g_{i}| \leq Dg_{i}(y + \delta) \). Then, \((\tilde{C}/(1+\beta))^{1+\beta} = \|g_{i}\|_{\infty} \) and so
\[
\delta \geq \left( \frac{\beta + 1}{C} \right)^{1/(1+\beta)} \cdot \|g_{i}\|_{\infty}^{1/(1+\beta)} = \tilde{K}_{1} \|g_{i}\|_{\infty}^{1/(1+\beta)}
\]
and
\[
Dg_{i}(y + \delta) \leq \tilde{C}\delta^{\beta} \leq \tilde{K}_{2} \|g_{i}\|_{\infty}^{\beta/(1+\beta)},
\]
for some \( \tilde{K}_{1}, \tilde{K}_{2} \to 0 \).
Lemma 3.7. Let \((v_i, w_i)\) be a pair as in Lemma 3.6. There exists a sequence of \((v_i, w_i)\) and \(\tilde{c} > 0\), independent of \(i\), such that,

\[
|g_i(x_i)| \geq \tilde{c} \|g_i\|_\infty.
\]

Proof. Assume that for a chosen \(i\) we have \(c_i := |g_i(x_i)|/\|g_i\|_\infty < \tilde{c}\). We can then replace the words \((v_i, w_i)\) by a different pair satisfying the conditions in Lemma 3.6, but also (3.8). Let \(m\) and \(c' \geq 1\) be such that \(c' \text{diam}(f^m(J)) < c_i\) but \(c' \text{diam}(f^{m-1}(J)) \geq c_i\) and

\[
\|f_1^{-m} \circ g_i \circ f_1^m - \text{Id}\|_\infty \leq \|g_i\|_\infty.
\]

Now,

\[
|f_1^{-m} \circ g_i \circ f_1^m - \text{Id})(x_i)| = |f_1^{-m} \circ g_i(x_i) - x_i|
\]

\[
= |f_1^{-m}(x_1 + c_i\|g_i\|_\infty) - x_i|
\]

\[
\geq |x_1 + K^{-1} \text{diam}(f_1^{m-1}(J))\|c_i\|_\infty - x_1|
\]

\[
\geq (c'K)^{-1}\|g_i\|_\infty,
\]

and so \(\tilde{c} \geq (c'K)^{-1} > 0\), and we can replace \((v_i, w_i)\) by \((v_i1^m, w_i1^m)\). \(\square\)

Let \(\rho_i = \sqrt[3]{\tilde{K}_1}\|g_i\|_{1+1/\beta}^{1/1+1/\beta}\) and write

\[
\xi_i(r) = \inf\{|g_i(x)| : x \in B_r(x_1)\} \text{ and } \Xi_i(r) = \sup\{|g_i(x)| : x \in B_r(x_1)\}.
\]

Note the following bounds, using Lemma 3.3,

\[
|g_i(x_1)| - 2\rho_i |Dg_i(x_1)| - \frac{\tilde{C}(2\rho_i)^{1+\beta}}{1+\beta} > 0,
\]

\[
|g_i(x_1)| - \rho_i |Dg_i(x_1)| - \frac{\tilde{C}(\rho_i)^{1+\beta}}{1+\beta} > 0.
\]

Now consider the derivative of \(g_i\) and let \(\tilde{z}_1\) and \(\tilde{z}_2\) be such that \(g_i(\tilde{z}_1) = \xi_i(\rho)\) and \(g_i(\tilde{z}_2) = \xi_i(2\rho)\). Therefore, \(Dg_i(\tilde{z}_1) \geq Dg_i(x_1) - 2\rho_i^\beta\) and so

\[
\xi_i(\rho_i) - \xi_i(2\rho_i) \geq \rho_i (|Dg_i(x_1)| - 2\tilde{C}\rho_i^\beta),
\]

and by (3.9),

\[
\xi_i(\rho_i) \geq \rho_i |Dg_i(x_1)| - 2\tilde{C}\rho_i^{1+\beta}.
\]

It follows that

\[
\Xi_i(\rho) \leq |g_i(x_1)| + \rho_i |Dg_i(x_1)| + \frac{\tilde{C}(\rho_i)^{1+\beta}}{1+\beta} \text{ by Lemma 3.3}
\]

\[
\leq \xi_i(\rho_i) + 2\rho_i |Dg_i(x_1)| + \frac{2\tilde{C}(\rho_i)^{1+\beta}}{1+\beta} \text{ by (3.10)}
\]

\[
\leq 3\xi_i(\rho_i) + \rho_i^{1+\beta} \left(4\tilde{C} + \frac{2\tilde{C}}{1+\beta}\right) \text{ by (3.11)}
\]

\[
\leq 3\xi_i(\rho_i) + \tilde{K}_3 \|g_i\|_\infty
\]

\[
(3.12) \leq \tilde{K} \xi_i(\rho_i) \text{ by Lemma 3.7}
\]

for some \(\tilde{K} > 0\) independent of \(i\).

Let \(\epsilon > 0\) be given, and choose \(k_0\) big enough such that

\[
\|g_{k_0}\|_\infty < \epsilon \tilde{K}_1 \|g_{k_0}\|_{1/(1+\beta)} = \epsilon \rho_i.
\]
Let $\mathcal{J}$ be a very weak pseudo tangent to $E$, then $\dim_A \hat{E} \leq \dim_A E$.

Our proof closely follows the argument in [5], and we only sketch some details.

### Definition 3.8
Let $K_1, K_2 \subseteq \mathbb{R}^d$ be compact. $[K]_\varepsilon$ denote the closed $\varepsilon$-neighbourhood of $K$, we write $d^e_H(K_1, K_2)$ for the left sided Hausdorff distance between two sets $K_1$ and $K_2$, given by

$$d^e_H(K_1, K_2) = \inf\{\varepsilon \geq 0 \mid K_1 \subseteq [K_2]_\varepsilon\}.$$
Proof. Let $E \subseteq \mathbb{R}^d$ be compact and set $s = \dim_A E$, then for all $\varepsilon > 0$ there exists constant $C_\varepsilon$ such that for all $0 < r < R$,

$$\sup_{x \in E} N_r(B(x, r) \cap E) \leq C_\varepsilon (R/r)^{s+\varepsilon}.$$ 

Now $T_i$ is bi-Lipschitz map such that $\alpha \leq \alpha_i \leq \overline{\alpha}$ for some $0 < \alpha \leq \overline{\alpha} < \infty$. Therefore

$$\sup_{x \in E} N_r(B(x, r) \cap T_i(E)) \leq C_\varepsilon \left( \frac{\overline{\alpha} R}{\alpha r} \right)^{s+\varepsilon} = C'_\varepsilon (R/r)^{s+\varepsilon},$$

for some $C'_\varepsilon > 0$, independent of $i$. Now choose $i$ large enough such that $d^*_H(\hat{E}, T_i(E)) < r$, thus a minimal cover for $T_i(E)$ can be extended to a cover of $\hat{E}$ by covering the $r$-neighbourhood of every $r$-ball with $c^0$ balls of radius $r$. We have

$$\sup_{x \in E} N_r(B(x, r) \cap \hat{E}) \leq c^0 \sup_{x \in E} N_r(B(x, r) \cap T_i(E)) \leq c^0 C'_\varepsilon (R/r)^{s+\varepsilon}.$$ 

So $\dim_A \hat{E} \leq s + \varepsilon$, and as $\varepsilon$ was arbitrary the required conclusion follows. \qed

We conclude the proof by showing that $[0, 1]$ is a very weak pseudo tangent of $F$, using the construction in [8]. Let $\varepsilon_i = 1/(ni)$ and define $F_i = \{ \phi_n \circ \psi_n(x_1) : 0 \leq n < i \} \cup \{ x_1 \}$. Consider

$$F_i^* = \{ \phi_{i-1} \circ f_\omega(x_1) : \omega \in \Lambda^* \}.$$

Clearly, for every $0 \leq l \leq i - 1$,

$$u_l = v_{k_{l-1}}^{-1} v_{k_{l-2}}^{-1} \ldots v_{k_{l+1}}^{-1} w_{k_l}^{-1} w_{k_{l-1}}^{-1} \ldots w_{k_0}^{-1} w_{k_{m_0}} \in \Lambda^N,$$

and $\phi_{i-1} = f_q$ for $q = 1^{-\alpha_0} v_{k_0}^{-1} \ldots 1^{-\alpha_{l-1}} v_{k_l}^{-1} 1$. We conclude

$$q u_l = 1^{-\alpha_0} v_{k_0}^{-1} \ldots 1^{-\alpha_{l-1}} v_{k_l}^{-1} w_{k_l}^{-1} w_{k_{l-1}}^{-1} \ldots w_{k_0}^{-1} w_{k_{m_0}},$$

and so $F_i \subseteq F_i^*$. We define $T_i = \phi_{i-1}$, then

$$T_i F \supseteq F_i^* \supseteq F_i = \{ \phi_n \circ \psi_n(x_1) : 0 \leq n < i \} \cup \{ x_1 \}.$$ 

Applying [9,15] gives us

$$-\kappa^{-1} \varepsilon_i \geq \phi_n \circ \psi_n - \phi_{n-1} \circ \psi_{n-1}(x_1) \geq -\kappa \varepsilon_i,$$

for all $0 \leq n < i$ and so $d^*_H([0, 1], T_i(F)) \leq 1/i \to 0$ as $i \to \infty$. Since $\phi_i$ is a bi-Lipschitz map with bounded distortion, $T_i$ is bi-Lipschitz and $\alpha_i$ is bounded. Therefore, applying Lemma [8,10] the required conclusion, $\dim_A F \geq \dim_A [0, 1] = 1$, follows. \qed

References

[1] P. Assouad. Espaces métriques, plongements, facteurs, *Thèse de doctorat d’État, Publ. Math. Orsay 229–766*, Univ. Paris XI, Orsay, (1977).
[2] P. Assouad. Étude d’une dimension métrique liée à la possibilité de plongements dans $\mathbb{R}^n$, *C. R. Acad. Sci. Paris Sér. A-B*, 288, (1979), 731–734.
[3] K. J. Falconer. Dimensions and measures of quasi self-similar sets, *Proc. Amer. Math. Soc.*, 106, (1989), 543–554.
[4] K. J. Falconer. *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley, 3rd Ed., 2014.
[5] K. J. Falconer. *Techniques in Fractal Geometry*, John Wiley, 1997.
[6] M. A. Ferrara and P. Panzone. Separation properties for iterated function systems of bounded distortion, *Fractals*, 19, (2011), 259–269.
[7] J. M. Fraser. Assouad type dimensions and homogeneity of fractals, *Trans. Amer. Math. Soc.*, 366, (2014), 6687–6733.
[8] J. M. Fraser, A. M. Henderson, E. J. Olson and J. C. Robinson. On the Assouad dimension of self-similar sets with overlaps, *Adv. Math.*, 273, (2015), 188–214.
A DICHOTOMY OF SELF-CONFORMAL SUBSETS OF $\mathbb{R}$ WITH OVERLAPS

[9] J. E. Hutchinson. Fractals and self-similarity, *Indiana Univ. Math. J.*, 30, (1981), 713–747.
[10] A. Käenmäki and E. Rossi. Weak separation condition, Assouad dimension, and Furstenberg homogeneity, *Ann. Acad. Sci. Fenn. Math.*, to appear. [arXiv:1506.07851v2].
[11] K.-S. Lau and S.-M. Ngai. Multifractal measures and a weak separation condition, *Adv. Math.*, 141, (1999), 45–96.
[12] K.-S. Lau, S.-M. Ngai and X.-Y. Wang. Separation conditions for conformal iterated function systems, *Monatsh. Math.*, 156, (2009), 325–355.
[13] J. Luukkainen. Assouad dimension: antifractal metrization, porous sets, and homogeneous measures, *J. Korean Math. Soc.*, 35, (1998), 23–76.
[14] J. M. Mackay and J. T. Tyson. *Conformal dimension. Theory and application*, University Lecture Series, 54. American Mathematical Society, Providence, RI, 2010.
[15] S.-M. Ngai and J.-X. Tong. Infinite iterated function systems with overlaps, *Ergodic Theory Dynam. Systems*, to appear.
[16] Y. Peres, M. Rams, K. Simon and B. Solomyak. Equivalence of positive Hausdorff measure and the open set condition for self-conformal sets, *Proc. Amer. Math. Soc.*, 129, (2001), 2689–2699.
[17] Y. Pesin. *Dimension theory in dynamical systems*, Chicago Lectures in Mathematics, University of Chicago Press, 1997.
[18] J. C. Robinson. *Dimensions, Embeddings, and Attractors*, Cambridge University Press, 2011.
[19] M. P. W. Zerner. Weak separation properties for self-similar sets, *Proc. Amer. Math. Soc.*, 124, (1996), 3529–3539.

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