BEHAVIOR NEAR THE EXTINCTION TIME FOR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH SUBLINEAR DISSIPATION TERMS

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Abstract. This paper is focused on the behavior near the extinction time of solutions of systems of ordinary differential equations with a sublinear dissipation term. Suppose the dissipation term is a product of a linear operator $A$ and a positively homogeneous scalar function $H$ of a negative degree $-\alpha$. Then any solution with an extinction time $T^*$ behaves like $(T^* - t)^{1/\alpha} \xi^*$ as time $t \to T^-\ast$, where $\xi^*$ is an eigenvector of $A$. The proof first establishes the asymptotic behaviors of the “Dirichlet” quotient and the normalized solution. They are then combined with a perturbation technique that requires the function $H$ to satisfy some pointwise Hölder-like condition. The result allows the higher order terms to be general and the nonlinear function $H$ to take very complicated forms.

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1. Introduction

This paper continues our investigations of exact asymptotic behaviors of solutions of systems of nonlinear ordinary differential equations (ODE), see [11, 12, 21, 22], and the Navier–Stokes equations (NSE), see, e.g., [9, 23, 25, 26]. These papers, in turn, are inspired by the work by Foias and Saut in [13, 14]. However, in contrast to the cited work above and others in this introduction, the current paper studies the asymptotic behavior near a finite extinction time, instead of time infinity, for equations with sublinear dissipation terms, instead of superlinear in [22], or linear in the others.

We review the known results in literature first. Foias and Saut study the NSE with potential body forces in the following functional form

$$u' + Au + B(u, u) = 0,$$

(1.1)
where $A$ is the (linear) Stokes operator which has positive eigenvalues and $B(\cdot, \cdot)$ is a bilinear form. Equation (1.1) holds in a certain weak sense in a suitable functional space.

It is proved in [13] that any nontrivial solution $u(t)$ of (1.1) has the following asymptotic behavior

$$e^{\Lambda t}u(t) \to \xi_*$ in any $C^m$-norms as $t \to \infty,$

(1.2)

where $\Lambda$ an eigenvalue of $A$ and $\xi_*$ is an eigenfunction of $A$ associated with $\Lambda$.

The method and result by Foias and Saut are extended to abstract differential inequalities in [16]. In fact, one of the techniques in [13] with the use of the Dirichlet quotient was already utilized earlier in [17] to obtain some backward uniqueness results and lower bound estimates. For an alternative method to establish (1.2) for ODE, see [12].

Foias and Saut themselves follow [13] with a theory of asymptotic expansions for the solutions of the NSE (1.1) in [14, 15]. These asymptotic expansions can be obtained independently from the limit (1.2). This theory is developed further for both ODE and partial differential equations (PDE). See [12, 22, 24, 26, 29, 30] for systems without forcing functions, and [9, 11, 21, 23, 25] with forcing functions. It is even established for the Lagrangian trajectories for viscous incompressible fluids in [20]. In particular, for nonsmooth ODE systems, [12] combines both techniques in [13] and [14] to obtain the asymptotic expansions. The first asymptotic approximation (1.2) turns out to play an essential role in that work.

In all of the above papers, except for [22] which will be discussed below, the ODE or PDE have linear dissipation terms. This facts causes the solutions, in the case the forcing functions are not present, decay exponentially as time $t \to \infty$. On contrary, the author recently studied in [22] the following ODE system in $\mathbb{R}^n$

$$y' = -H(y)Ay + G(t, y),$$

(1.3)

where $A$ is a constant $n \times n$ matrix with positive eigenvalues, $H$ is a positive function at first, and $G$ is a higher order term.

In [22], $H$ is additionally assumed to be a positively homogeneous function of a positive degree $\alpha \in (0, \infty)$. It is proved that any nonzero, decaying solution of (1.3) behaves like $t^{-1/\alpha}\xi_*$ as $t \to \infty$, where $\xi_*$ is an eigenvector of $A$.

This paper considers the opposite scenario when the function $H$ in (1.3) has a negative degree $-\alpha$. In this case, many solutions start out with nonzero values and then become zero at a finite time. Such time is called the extinction time. Our goal is to describe the behavior of these solutions near this extinction time.

The main result can be briefly described as follows postponing the accurate assumptions on $A$, $H$ and $G$. Given a solution $y(t)$ of (1.3) with the extinction time $T_*$, that is, $y(t) \neq 0$ before $T_*$, and $y(t) \to 0$ as $t \to T_*^-$. Then $y(t)$ behaves exactly like $(T_* - t)^{1/\alpha}\xi_*$ as time $t \to T_*^-$, where $\xi_*$ is an eigenvector of $A$. Note that the existence of the extinction time is guaranteed under the small nonzero initial data condition, see Theorem 2.4 below. The proof will make use and adapt the techniques from [13, 22]. In particular, the recent perturbation method in [22] will be utilized. This method is needed to deal with the nonlinear dissipation in our problem. It is different from previous work for equations with linear dissipations. The mentioned perturbation method will be implemented successfully in this paper for the study of the asymptotic behavior near the finite extinction time, instead of at time infinity as in [22].

The paper is organized as follows. In Section 2 we prove in Theorem 2.4 that under appropriate conditions on $A$, $H$ and $G$, any solution of (1.3) with sufficiently small nonzero
initial condition will become zero at finite time. Solutions of this type are the objects of our investigation in this paper. The paper’s main result is formulated in Theorem 3.3 of Section 3. While the condition for $A$ is the natural Assumption 2.1, the more technical conditions for $H$ are specified in Assumption 3.2. A key requirement of $H$, namely, property (HC) is introduced in Definition 3.1. A number of examples are provided in Example 3.4. Section 4 prepares for the proof of Theorem 3.3. Properties of functions in the class $\mathcal{H}_{-\alpha}(\mathbb{R}^n, \mathbb{R})$ in relation with property (HC) are established in Lemma 4.1. Preliminary estimates for the solutions are obtained in Lemma 4.3. Although they provide only a rough description of $y(t)$, the upper and lower bounds with the same rate $1/\alpha$ obtained in (4.3) are important in our further analysis. Section 5 proves Theorem 3.3 in a special case in the form of equation (5.2). This will serve the proof for the general case in Section 7. In Section 6, we obtain essential properties of the solutions when the matrix $A$ is symmetric. Although it is for a particular case, these are the key steps of the main proof in the final section. For example, the quotient $\lambda(t)$ in (6.1) determines the eigenvalue $\Lambda$, see Proposition 6.1. Section 7 finally gives proof to the main result – Theorem 3.3. It combines all the previous preparations with the perturbation method mentioned earlier, see equation (7.2) for case of symmetric matrix $A$. This equation is a reduction of equation (1.3) of $y(t)$ to the simple form (5.2), but for the projection $R_{\Lambda}y(t)$ and with a frozen coefficient $\Lambda H(v_*)$. The unit vector $v_*$ is the limit of $R_{\Lambda}y(t)/|y(t)|$ in Proposition 6.3. The case of general matrix $A$ is converted to the symmetric case by the use of the standard equivalence (2.2) and linear transformation $z = Sy$.

Notation. Throughout the paper, $n \in \mathbb{N} = \{1, 2, 3, \ldots\}$ is the spatial dimension. For any vector $x \in \mathbb{R}^n$, we denote by $|x|$ its Euclidean norm. For an $n \times n$ real matrix $A = (a_{ij})_{1 \leq i, j \leq n}$, its Euclidean norm is

$$
\|A\| = \left( \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2}.
$$

The unit sphere in $\mathbb{R}^n$ is $S^{n-1} = \{ x \in \mathbb{R}^n : |x| = 1 \}$.

2. Existence of the extinction time

We consider the ODE system (1.3) in $\mathbb{R}^n$. We will show that the extinction time exists for, at least, certain small solutions of (1.3).

Assumption 2.1. Hereafter, $A$ is a (real) diagonalizable $n \times n$ matrix with positive eigenvalues.

Thanks to Assumption 2.1, the matrix $A$ has $n$ positive eigenvalues (counting their multiplicities)

$$
\Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \ldots \leq \Lambda_n,
$$

and there exists an invertible $n \times n$ (real) matrix $S$ such that

$$
A = S^{-1}A_0S, \text{ where } A_0 = \text{diag}[\Lambda_1, \Lambda_2, \ldots, \Lambda_n].
$$

In the case $A$ is symmetric, the matrix $S$ is orthogonal, i.e., $S^{-1} = S^T$, and

$$
\Lambda_1|x|^2 \leq x \cdot Ax \leq \Lambda_n|x|^2 \text{ for all } x \in \mathbb{R}^n.
$$

The function $H$ will be assumed to have some type of homogeneity which is specified in the next definition.
Definition 2.2. Let $X$ and $Y$ be two (real) linear spaces, and $\beta < 0$ be a given number.

A function $F : X \setminus \{0\} \rightarrow Y$ is positively homogeneous of degree $\beta$ if

$$F(tx) = t^\beta F(x) \text{ for any } x \in X \setminus \{0\} \text{ and } t > 0.$$ 

Define $\mathcal{H}_\beta(X,Y)$ to be the set of functions from $X \setminus \{0\}$ to $Y$ that are positively homogeneous of degree $\beta$.

If $F \in \mathcal{H}_\beta(X,Y)$ and $F$ is not the zero function, then the degree $\beta$ is unique.

Assumption 2.3. Let $t_0$ be any given number in $[0, \infty)$. We assume the followings.

(i) The function $H$ is in $\mathcal{H}_{-\alpha}(\mathbb{R}^n, \mathbb{R})$ for some $\alpha > 0$, and in $C(\mathbb{R}^n \setminus \{0\}, (0, \infty))$.

(ii) The function $G(t,x)$ is continuous on $[t_0, \infty) \times (\mathbb{R}^n \setminus \{0\})$, and there exist positive numbers $c_*, r_*, \delta$ such that

$$|G(t,x)| \leq c_* |x|^{1-\alpha+\delta} \text{ for all } t \geq t_0, \text{ and all } x \in \mathbb{R}^n \text{ with } 0 < |x| \leq r_*.$$ 

(2.4)

Because $H$ is positive and continuous on $\mathbb{S}^{n-1}$, we have

$$0 < c_1 = \min_{|x|=1} H(x) \leq \max_{|x|=1} H(x) = c_2 < \infty. \quad (2.5)$$

By writing $H(x) = |x|^{-\alpha}H(|x|/|x|)$ for any $x \in \mathbb{R}^n \setminus \{0\}$, we derive

$$c_1|x|^{-\alpha} \leq H(x) \leq c_2|x|^{-\alpha} \text{ for all } x \in \mathbb{R}^n \setminus \{0\}. \quad (2.6)$$

The following are standard facts in the theory of ODE, see, e.g., [18, 19]. Consider equation (1.3) on the set

$$D = \{(t,y) : t \geq t_0, y \in \mathbb{R}^n \setminus \{0\}\} \subset \mathbb{R}^{n+1}. \quad (2.7)$$

For any $y_0 \in \mathbb{R}^n \setminus \{0\}$, there exists an interval $[t_0, T_{\max})$, with $0 < T_{\max} \leq \infty$, and a solution $y \in C^1([t_0, T_{\max}), \mathbb{R}^n \setminus \{0\})$ that satisfies (1.3) on $(t_0, T_{\max})$, $y(t_0) = y_0$, and either

(a) $T_{\max} = \infty$, or

(b) $T_{\max} < \infty$, and for any $\varepsilon > 0$, $y$ cannot be extended to a function of class $C^1([t_0, T_{\max}+\varepsilon), \mathbb{R}^n \setminus \{0\})$ that satisfies (1.3) on the interval $(t_0, T_{\max}+\varepsilon)$.

It is well-known that, in the case (b), it holds, for any compact set $U \subset \mathbb{R}^n \setminus \{0\}$, that

$$y(t) \not\in U \text{ when } t \in [t_0, T_{\max}) \text{ is near } T_{\max}. \quad (2.8)$$

Note that such a solution $y(t)$ may not be unique.

Theorem 2.4. There exists a number $r_0 > 0$ such that for any $y_0 \in \mathbb{R}^n \setminus \{0\}$ with $|y_0| \leq r_0$ and any solution $y(t)$ on $[t_0, T_{\max})$ as described from (2.7) to (2.8) above, one has

$$T_{\max} < \infty \text{ and } \lim_{t \to T_{\max}} y(t) = 0. \quad (2.9)$$

In other words, $T_{\max}$ is the extinction time of the solution $y$.

Proof. On $(t_0, T_{\max})$, we have

$$\frac{d}{dt}(|y|^\alpha) = \alpha |y|^{\alpha-2}y' \cdot y = \alpha (-|y|^{\alpha-2}H(y)(Ay) \cdot y + |y|^{\alpha-2}G(t,y) \cdot y). \quad (2.10)$$

Step 1. Consider $A$ is symmetric first. Take $r_0 > 0$ such that $c_4(2r_0)^\delta = a_0 \overset{\text{def}}{=} c_1 \Lambda_1/2$.

For $t > t_0$ sufficiently close to $t_0$, we have $|y(t)| < 2r_0$. Let $[t_0, T)$ be the maximal interval in $[t_0, T_{\max})$ on which $|y(t)| < 2r_0$. 

Suppose $T < T_{\text{max}}$. On the one hand, it must hold that
\[
|y(T)| = 2r_0. \tag{2.11}
\]

On the other hand, we have from (2.3), (2.4), (2.6) and (2.10) that, for $t \in (t_0, T)$,
\[
\frac{d}{dt}(|y|^\alpha) \leq \alpha(-c_1A_1 + c_s|y|^{\delta}) \leq \alpha(-c_1A_1 + c_s(2r_0)^{\delta}) = -\alpha a_0 < 0. \tag{2.12}
\]

Thus, $|y(T)|^\alpha \leq |y_0|^\alpha$, which implies $|y(T)| \leq |y_0| \leq r_0$. This contradicts (2.11). Therefore, $T = T_{\text{max}}$. Integrating (2.12) gives
\[
|y(t)|^\alpha \leq |y_0|^\alpha - \alpha a_0 t \text{ for all } t \in [t_0, T_{\text{max}}). \tag{2.13}
\]

Step 2. Consider the general matrix $A$. Using the equivalence (2.2), we set $z(t) = Sy(t)$ and $z_0 = z(t_0) = Sy_0$. Then
\[
z' = -\tilde{H}(z)A_0z + \tilde{G}(t, z) \text{ for } t \in (t_0, T_s), \tag{2.14}
\]
where
\[
\tilde{H}(z) = H(S^{-1}z), \quad \tilde{G}(t, z) = SG(t, S^{-1}z) \text{ for } t \in [t_0, T_s) \text{ and } z \in \mathbb{R}^n \setminus \{0\}.
\]

Note that
\[
\|S^{-1}\|^{-1} \cdot |x| \leq |Sx| \leq \|S\| \cdot |x| \text{ for all } x \in \mathbb{R}^n. \tag{2.15}
\]

Clearly, $\tilde{H}$ satisfies the same condition as $H$ in Assumption (2.3). Moreover, $\tilde{G}(t, z)$ is continuous on $[t_0, \infty) \times (\mathbb{R}^n \setminus \{0\})$. For $t \in [t_0, T_s)$ and $0 < |z| \leq r_*/\|S^{-1}\|$, we have $0 < |S^{-1}z| \leq r_*$, and then, by (2.4) and (2.15),
\[
|\tilde{G}(t, z)| \leq \|S\| \cdot c_\ast |S^{-1}z|^{-\alpha + \delta} \leq c_\ast \|S\| \cdot \left\{ \begin{array}{ll}
\|S^{-1}\|^{1-\alpha+\delta} |z|^{1-\alpha+\delta}, & \text{if } 1-\alpha+\delta \geq 0, \\
\|S\|^{-(1-\alpha+\delta)} |z|^{1-\alpha+\delta}, & \text{otherwise}.
\end{array} \right.
\]

We apply the calculations in Step 1 to the solution $z(t)$ of (2.14). When $|y_0| > 0$ is sufficiently small, we have $|z_0| > 0$ is sufficiently small, and hence, similar to estimate (2.13),
\[
|z(t)|^\alpha \leq |z_0|^\alpha - \alpha \tilde{a}_0 t, \text{ for all } t \in [t_0, T_{\text{max}}) \text{ and some constant } \tilde{a}_0 > 0.
\]

Therefore,
\[
|y(t)|^\alpha \leq \|S^{-1}\|^\alpha |z(t)|^\alpha \leq \|S^{-1}\|^\alpha (|Sy_0|^\alpha - \alpha \tilde{a}_0 t) \text{ for all } t \in [t_0, T_{\text{max}}). \tag{2.16}
\]

Step 3. If $T_{\text{max}} = \infty$, then (2.16) implies that $|y(t)|^\alpha < 0$ for $t > |Sy_0|^\alpha/(\alpha \tilde{a}_0)$, which is an obvious contradiction. Therefore, $T_{\text{max}} < \infty$. As a consequence of (2.16),
\[
|y(t)| \leq R_0 \text{ on } [t_0, T_{\text{max}}), \text{ where } R_0 = \|S^{-1}\| \cdot |Sy_0| > 0. \tag{2.17}
\]

For any $\varepsilon > 0$, let $U = \{x \in \mathbb{R}^n : \varepsilon \leq |x| \leq 2R_0\}$ in (2.8). Taking into account (2.17), one must have $|y(t)| < \varepsilon$ when $t \in [t_0, T_{\text{max}}^{-})$ is near $T_{\text{max}}$. This proves the zero limit in (2.9). \(\Box\)

We remark that $y(t)$ may be zero for $t$ larger than the above $T_{\text{max}}$. However, this is excluded from our consideration of the set $D$ in (2.7). The reason is our sole focus on the finite extinction time and the solution before that time.
3. THE MAIN RESULT

We present the main result of the paper in this section. We first continue to describe the matrix $A$ after (2.1) and (2.2). Denote the distinct eigenvalues of $A$ by $\lambda_j$'s which are arranged to be (strictly) increasing in $j$, i.e.,

$0 < \lambda_1 = \Lambda_1 < \lambda_2 < \ldots < \lambda_d = \Lambda_n$ for some integer $d \in [1, n]$.

The spectrum of $A$ is $\sigma(A) = \{\Lambda_k : 1 \leq k \leq n\} = \{\lambda_j : 1 \leq j \leq d\}$.

For $1 \leq k, \ell \leq n$, let $E_{k\ell}$ be the elementary $n \times n$ matrix $(\delta_{ki}\delta_{\ell j})_{1\leq i,j\leq n}$, where $\delta_{ki}$ and $\delta_{\ell j}$ are the Kronecker delta symbols. For $\Lambda \in \sigma(A)$, define

$$\hat{R}_\Lambda = \sum_{1 \leq i \leq n, \lambda_i = \Lambda} E_{ii} \text{ and } R_\Lambda = S^{-1}\hat{R}_\Lambda S.$$ 

Then one immediately has

$$I_n = \sum_{j=1}^d R_{\lambda_j}, \quad R_{\lambda_j}R_{\lambda_j} = \delta_{ij}R_{\lambda_j}, \quad AR_{\lambda_j} = R_{\lambda_j}A = \lambda_jR_{\lambda_j}. \quad \text{(3.1)}$$

Thanks to (3.1), each $R_\Lambda$ is a projection, and $R_\Lambda(\mathbb{R}^n)$ is the eigenspace of $A$ associated with the eigenvalue $\Lambda$.

In the case $A$ is symmetric, $R_\Lambda$ is the orthogonal projection from $\mathbb{R}^n$ to the eigenspace of $A$ associated with $\Lambda$, and, hence,

$$|R_\Lambda x| \leq |x| \text{ for all } x \in \mathbb{R}^n. \quad \text{(3.2)}$$

For our next result, the function $H$ is required to have an extra property.

**Definition 3.1.** Let $E$ be a nonempty subset of $\mathbb{R}^n$ and $F$ be a function from $E$ to $\mathbb{R}$. We say $F$ has property (HC) on $E$ if, for any $x_0 \in E$, there exist numbers $r, C, \gamma > 0$ such that

$$|F(x) - F(x_0)| \leq C|x - x_0|^{\gamma} \quad \text{for any } x \in E \text{ with } |x - x_0| < r. \quad \text{(3.3)}$$

**Assumption 3.2.** The function $H$ belongs to $\mathcal{H}_{-\alpha}(\mathbb{R}^n, \mathbb{R})$ for some $\alpha > 0$, has property (HC) on the unit sphere $S^{n-1}$, and $H > 0$ on $S^{n-1}$.

It is obvious that the function $H$ in Assumption 3.2 is continuous on $S^{n-1}$. In fact, it is continuous and positive on $\mathbb{R}^n \setminus \{0\}$, see Lemma 4.1 below.

Let $t_0, T_\ast \in \mathbb{R}$ be two given numbers with $T_\ast > t_0 \geq 0$. Assume $y \in C^1([t_0, T_\ast), \mathbb{R}^n)$ satisfies $y(t) \neq 0$ for all $t \in [t_0, T_\ast)$,

$$\lim_{t \to T_\ast^-} y(t) = 0, \quad \text{and} \quad y' = -H(y)Ay + f(t) \text{ for } t \in (t_0, T_\ast), \quad \text{(3.5)}$$

where $f$ is a continuous function from $[t_0, T_\ast)$ to $\mathbb{R}^n$ such that

$$|f(t)| \leq M|y(t)|^{1-\alpha+\delta}, \text{ for all } t \in [t_0, T_\ast) \text{ and some constants } M, \delta > 0. \quad \text{(3.6)}$$

By defining $y(T_\ast) = 0$, we have $y \in C([t_0, T_\ast], \mathbb{R}^n)$.

The current setting of equation (3.5) with property (3.6) already covers the case of (1.3) and (2.4) considered in Section 2. Indeed, if (1.3) and (2.4) hold, then setting the function
Lemma 4.1. Let \( f(t) = G(t, y(t)) \) gives (3.5) and (3.6) on an interval \([t_0', T_*] \), for some number \( t_0' \in [t_0, T_*) \) sufficiently closed to \( T_* \).

The main result of this paper is the following.

**Theorem 3.3 (Main theorem).** Under Assumption 3.2, there exist an eigenvalue \( \Lambda \) of \( A \) and an eigenvector \( \xi_* \) of \( A \) associated with \( \Lambda \) such that

\[
|y(t) - (T_* - t)^{1/\alpha} \xi_*| = \mathcal{O}((T_* - t)^{1/\alpha + \varepsilon}) \text{ as } t \to T_*^- \text{ for some } \varepsilon > 0.
\]

More specifically,

\[
|(I_n - R_\Lambda)y(t)| = \mathcal{O}((T_* - t)^{1/\alpha + \varepsilon}) \text{ as } t \to T_*^- \text{ for some } \varepsilon > 0,
\]

\[
|R_\Lambda y(t) - (T_* - t)^{1/\alpha} \xi_*| = \mathcal{O}((T_* - t)^{1/\alpha + \varepsilon}) \text{ as } t \to T_*^- \text{ for some } \varepsilon > 0,
\]

and

\[
\alpha \Lambda H(\xi_*) = 1.
\]

Before proving Theorem 3.3, we give some examples of the function \( H \).

**Example 3.4.** For simplicity, we consider the case dimension \( n = 2 \). One can easily generalize them for any higher dimension \( n \).

(a) For \( x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\} \),

\[
H(x) = (x_1^4 + 5x_2^4)^{-3}, \quad H(x) = \left([3|x_1|^{3/2} + |x_2|^{3/2}]^{1/3} + (2|x_1|^{5/3} + 7|x_2|^{5/3})^{3/10}\right)^{-1/8}.
\]

The first function is in \( \mathcal{H}_{-3/4}(\mathbb{R}^2, \mathbb{R}) \) while the second one is in \( \mathcal{H}_{-1/16}(\mathbb{R}^2, \mathbb{R}) \). They both belong to \( C^1(\mathbb{R}^2 \setminus \{0\}) \). Hence, they have property (HC) on \( S^1 \) with the same power \( \gamma = 1 \) in (3.3).

(b) Another example is

\[
H(x) = \frac{\sqrt{|x_1|} + \sqrt{|x_2|}}{|x|}.
\]

Then \( H \) belongs to \( \mathcal{H}_{-1/2}(\mathbb{R}^2, \mathbb{R}) \), is positive on \( S^1 \) and has property (HC) on \( S^1 \) with the same power \( \gamma = 1/2 \) in (3.3). Unlike the previous two examples, this function \( H \) is not in \( C^1(\mathbb{R}^2 \setminus \{0\}) \).

In fact, the function \( H \) can be very complicated, see similar examples in [22] Example 5.7 and [12] Section 6.

4. Preliminary facts and estimates

In this section, we prepare for the proof of Theorem 3.3 with some elementary facts and estimates for the solution \( y(t) \).

**Lemma 4.1.** Let \( F \in \mathcal{H}_{-\alpha}(\mathbb{R}^n, \mathbb{R}) \) for some \( \alpha > 0 \).

(i) If \( F > 0 \) on \( S^{n-1} \), then \( F > 0 \) on \( \mathbb{R}^n \setminus \{0\} \).

(ii) If \( F \) is continuous on \( S^{n-1} \), then it is continuous on \( \mathbb{R}^n \setminus \{0\} \).

Assume \( F \) has property (HC) on \( S^{n-1} \) in (iii) (v) below.

(iii) Then \( F \) has property (HC) on \( \mathbb{R}^n \setminus \{0\} \).

(iv) If \( \varphi \) is a function from \( \mathbb{R}^n \setminus \{0\} \) to \( \mathbb{R}^n \setminus \{0\} \) that has property (HC) on \( \mathbb{R}^n \setminus \{0\} \), then \( F \circ \varphi \) has property (HC) on \( \mathbb{R}^n \setminus \{0\} \).

(v) If \( K \) is an invertible \( n \times n \) matrix, then the function \( x \in \mathbb{R}^n \setminus \{0\} \mapsto F(Kx) \) has property (HC) on \( \mathbb{R}^n \setminus \{0\} \).
Proof. Denote $E = \mathbb{R}^n \setminus \{0\}$. For any $x \in E$, we can write $F(x) = |x|^{-\alpha} F(x/|x|)$. Hence, part (i) is obvious. For parts (ii)–(iv), the proofs are similar to the proof of [22, Lemma 5.1], and “the verification of Assumption 5.2 for $\tilde{H}$” in the proof of [22, Theorem 5.3]. We present the key arguments here.

Let $x, \xi \in E$. Then
\[
|F(x) - F(\xi)| = |x|^{-\alpha} F(x/|x|) - |\xi|^{-\alpha} F(\xi/|\xi|) \\
\leq |x|^{-\alpha} |F(x/|x|) - H(\xi/|\xi|)| + |x|^{-\alpha} - |\xi|^{-\alpha}| F(\xi/|\xi|)
\] (4.1)

Using inequality (4.1) and the fact that functions $x \in E \mapsto x/|x|$ and $x \in E \mapsto |x|^{-\alpha}$ are $C^1$-functions, we can prove parts (ii) and (iii).

We prove part (iv) now. Suppose $F$ has property (HC) on $S^{n-1}$. By part (iii) $F$ has property (HC) on $E$. Clearly, $\varphi$ is a continuous function on $E$. Let $\xi$ be any vector in $E$. Consider $x \in E$ sufficiently close to $\xi$. As $x \to \xi$, we have $\varphi(x) \to \varphi(\xi)$. Using inequality (3.3) for function $F$ and $x_0 := \varphi(\xi) \in E$, $x := \varphi(x) \in E$ with constant $C$ and power $\gamma$, and then inequality (3.3) again for function $\varphi$ and $x_0 := \xi \in E$, $x \in E$ with constant $C'$ and power $\gamma'$, we have
\[
|F(\varphi(x)) - F(\varphi(\xi))| \leq C|\varphi(x) - \varphi(\xi)|^\gamma \leq CC'|x - \xi|^\gamma'.
\]

Therefore, the function $F \circ \varphi$ has property (HC) on $E$.

Part (v) is a direct consequence of part (iv) with $\varphi(x) = Kx$. We omit the details. $\square$

Let $f$ satisfy (3.6). For any number $\delta' \in (0, \delta)$, we have
\[
|f(t)| \leq M'|y(t)|^{1-\alpha+\delta'} \text{ for all } t \in [t_0, T_*),
\] (4.2)

where
\[
M' = M \max_{t \in [t_0, T_*)} |y(t)|^{\delta - \delta'} \in (0, \infty).
\]

Let $y(t)$ be a solution as in Theorem 3.3. We will obtain preliminary estimates for the solution $y(t)$. They even hold under a weaker condition than Assumption 3.2.

Condition 4.2. The function $H$ belongs to $\mathcal{H}_{-\alpha}(\mathbb{R}, \mathbb{R})$ for some $\alpha > 0$, and is positive, continuous on the unit sphere $S^{n-1}$.

Suppose $H$ satisfies Condition 4.2, then it satisfies (2.5) and, thus, inequalities in (2.6) hold true.

Lemma 4.3. Under Condition 4.2, there are positive constants $C_1$ and $C_2$ such that
\[
C_1(T_* - t)^{1/\alpha} \leq |y(t)| \leq C_2(T_* - t)^{1/\alpha} \text{ for all } t \in [t_0, T_*].
\] (4.3)

Proof. We prove for the case the matrix $A$ is symmetric first and then for $A$ not symmetric.

Case 1. Consider $A$ is symmetric. For $t \in (t_0, T_*)$, we calculate, similarly to (2.10),
\[
\frac{d}{dt}(|y|^\alpha) = \alpha (-|y|^{\alpha-2} H(y) \cdot y + |y|^{\alpha-2} f(t) \cdot y).
\] (4.4)

Utilizing (2.6), (2.3) and (3.6), one has
\[
\alpha (-c_2 \Lambda_n - M|y|^\delta) \leq \frac{d}{dt}(|y|^\alpha) \leq \alpha (-c_1 \Lambda_1 + M|y|^\delta).
\]

Let $\alpha_1 = c_1 \Lambda_1/2$ and $\alpha_2 = c_2 \Lambda_n + 1$. Let $r_0 > 0$ be such that $Mr_0^\delta = \min\{1, c_1 \Lambda_1/2\}$.
Thanks to (3.4), there is \( T \in (t_0, T_*) \) such that \( |y(t)| \leq r_0 \) on \((T, T_*)\). Hence,

\[
- \alpha a_2 \leq \frac{d}{dt}(|y|^{\alpha}) \leq -\alpha a_1 \text{ on } (T, T_*).
\]

(4.5)

For \( t \in [T, T_*] \), integrating (4.3) from \( t \) to \( T_* \) and recalling that \( y(T_*) = 0 \), we obtain

\[
\alpha a_1 (T_* - t) \leq |y(t)|^{\alpha} \leq \alpha a_2 (T_* - t) \text{ for all } t \in [T, T_*].
\]

(4.6)

Note also that

\[
0 < a_3 \overset{\text{def}}{=} \min_{t \in [t_0, T]} (T_* - t)^{-1/\alpha} |y(t)| \leq a_4 \overset{\text{def}}{=} \max_{t \in [t_0, T]} (T_* - t)^{-1/\alpha} |y(t)| < \infty.
\]

(4.7)

Combining (4.6) with (4.7), we obtain the desired estimates in (4.3) with

\[
C_1 = \min\{(\alpha a_1)^{1/\alpha}, a_3\} \text{ and } C_2 = \max\{(\alpha a_2)^{1/\alpha}, a_4\}.
\]

Case 2. Consider \( A \) is not symmetric. Let \( A = S^{-1}A_0S \) as in (2.2).

Set \( z(t) = S y(t) \) for \( t \in [t_0, T_*] \). Then \( z \) belongs to \( C^1([t_0, T_*], \mathbb{R}^n \setminus \{0\}) \cap C([t_0, T_*], \mathbb{R}^n) \), \( z(t) \neq 0 \) for all \( t \in [t_0, T_*] \), \( z(T_*) = 0 \), and

\[
z' = -\tilde{H}(z)A_0 z + \tilde{f}(t) \text{ for } t \in (t_0, T_*),
\]

(4.8)

where

\[
\tilde{H}(z) = H(S^{-1}z) \text{ for } z \in \mathbb{R}^n \setminus \{0\}, \text{ and } \tilde{f}(t) = S f(t) \text{ for } t \in [t_0, T_*].
\]

(4.9)

One can verify that \( \tilde{H} \) belongs to \( \mathcal{H}_{-\alpha}(\mathbb{R}^n, \mathbb{R}) \), and, thanks to Lemma 4.1, is positive and continuous on \( \mathbb{R}^n \setminus \{0\} \).

Clearly, the function \( \tilde{f} \) is continuous on \([t_0, T_*]\). Thanks to (3.6) and (2.15), it satisfies, for \( t \in [t_0, T_*] \),

\[
|\tilde{f}(t)| \leq ||S|| \cdot |f(t)| \leq ||S||M |y(t)|^{1-\alpha+\delta} \leq \widetilde{M} |z(t)|^{1-\alpha+\delta},
\]

(4.10)

where

\[
\widetilde{M} = \begin{cases} 
M ||S|| \cdot ||S^{-1}||^{1-\alpha+\delta}, & \text{if } 1 - \alpha + \delta \geq 0, \\
M ||S|| \cdot ||S||^{-(1-\alpha+\delta)} = M ||S||^{\alpha-\delta}, & \text{otherwise}.
\end{cases}
\]

Therefore, we can apply the result in Case 1 to the solution \( z(t) \) and equation (4.8). Then there exist two positive constants \( C'_1 \) and \( C'_2 \) such that

\[
C'_1(T_* - t)^{1/\alpha} \leq |z(t)| \leq C'_2(T_* - t)^{1/\alpha} \text{ for all } t \in [t_0, T_*].
\]

(4.11)

Combining (4.11) with the relations in (2.15), we obtain the estimates in (4.3) for \( y(t) \). \( \square \)

The following are two immediate consequences of Lemma 4.3 still under Condition 4.2.

By (2.6) and (4.3), we have, for all \( t \in [t_0, T_*] \),

\[
C_3 (T_* - t)^{-1} \leq H(y(t)) \leq C_4 (T_* - t)^{-1}, \text{ where } C_3 = c_1 C_2^{-\alpha} \text{ and } C_4 = c_2 C_1^{-\alpha}.
\]

(4.12)

We also observe from (3.6) and (4.3) that, for all \( t \in [t_0, T_*] \),

\[
|f(t)| \leq M (T_* - t)^{1/\alpha - 1+\delta/\alpha} \cdot \begin{cases} 
C_2^{1-\alpha+\delta}, & \text{if } 1 - \alpha + \delta \geq 0, \\
C_1^{1-\alpha+\delta}, & \text{otherwise}.
\end{cases}
\]

(4.13)
5. Proof for a Special Case

Let $a$ be an arbitrarily positive number. Consider the case

$$A = I_n \text{ and } H(x) = a|x|^{-\alpha},$$

that is, equation (1.3) becomes

$$y' = -a|y|^{-\alpha}y + f(t) \text{ for } t \in (t_0, T_*).$$

Theorem 3.3 for this particular case is simply the following.

**Theorem 5.1.** There exists a vector $\xi_* \in \mathbb{R}^n$ such that

$$|\xi_*| = (\alpha a)^{1/\alpha},$$

and, as $t \to T_*^-$,

$$|y(t) - (T_* - t)^{1/\alpha} \xi_*| = \mathcal{O}((T_* - t)^{1/\alpha + \epsilon}) \text{ for some } \epsilon > 0.$$  

**Proof.** Because of (4.2), we can assume that $\delta < \alpha$. With the matrix $A$ and function $H$ in (5.1), they certainly satisfy Assumption 2.1 and Condition 4.2. Then Lemma 4.3 applies and the estimates from above and below for $|y(t)|$ in (4.3), and estimate (4.13) for $|f(t)|$ hold true.

For $t \in (t_0, T_*)$, we have from (4.4) that

$$\frac{d}{dt}(|y|^\alpha) = -\alpha a + \alpha |y|^{\alpha - 2} f(t) \cdot y.$$  

Integrating equation (5.5) from $t$ to $T_*$ gives

$$|y(t)|^\alpha = \alpha a (T_* - t) + g(t), \text{ where } g(t) = -\alpha \int_t^{T_*} |y(\tau)|^{\alpha - 2} f(\tau) \cdot y(\tau) d\tau.$$  

Hence, for all $t \in [t_0, T_*]$, one has $\alpha a (T_* - t) + g(t) > 0$.

Using the Cauchy–Schwarz inequality, (5.6) and the upper bound of $|y(t)|$ in (4.3), we estimate

$$|g(t)| \leq \alpha \int_t^{T_*} |y(\tau)|^{\alpha - 1} |f(\tau)| d\tau \leq \alpha M \int_t^{T_*} |y(\tau)|^\delta d\tau$$

$$\leq \alpha MC^\delta_2 \int_t^{T_*} (T_* - \tau)^{\delta/\alpha} d\tau.$$  

We obtain

$$|g(t)| \leq C_3(T_* - t)^{1+\delta/\alpha} \text{ for all } t \in [t_0, T_*], \text{ where } C_3 = \frac{\alpha MC^\delta_2}{1 + \delta/\alpha}.$$  

We consider equation (5.2) as a linear equation of $y$ with time-dependent coefficient $-a|y(t)|^{-\alpha}$ and forcing function $f(t)$. By the variation of constants formula, we solve for $y(t)$ explicitly as

$$y(t) = e^{-J(t)} \left( y_0 + \int_{t_0}^t e^{J(\tau)} f(\tau) d\tau \right) \text{ for } t \in [t_0, T_*],$$

where

$$J(t) = a \int_{t_0}^t |y(\tau)|^{-\alpha} d\tau.$$  

(5.8)
Using (5.6) in (5.8), we rewrite \( J(t) \) as
\[
J(t) = \int_{t_0}^{t} \frac{a}{\alpha(T_\tau - t) + g(\tau)} d\tau = J_1(t) + J_2(t),
\]
where
\[
J_1(t) = \int_{t_0}^{t} \frac{1}{\alpha(T_\tau - t)} d\tau \quad \text{and} \quad J_2(t) = \int_{t_0}^{t} h(\tau) d\tau,
\]
with
\[
h(\tau) = \frac{-g(\tau)}{\alpha(T_\tau - t)(\alpha(T_\tau - t) + g(\tau))}.
\]
Clearly,
\[
J_1(t) = -\frac{1}{\alpha} \ln(T_\tau - t) + \frac{1}{\alpha} \ln(T_\tau - t_0).
\]
Therefore,
\[
y(t) = \frac{(T_\tau - t)^{1/\alpha}}{(T_\tau - t_0)^{1/\alpha}} e^{-J_2(t)} \left( y_0 + (T_\tau - t_0)^{1/\alpha} \int_{t_0}^{t} \frac{e^{J_2(\tau)}}{(T_\tau - \tau)^{1/\alpha}} f(\tau) d\tau \right).
\]
(5.9)
Consider the integrand \( h(\tau) \) of \( J_2(t) \). Taking into account the estimate of \( |g(\tau)| \) in (5.7), we assert that, as \( \tau \to T_\tau^- \),
\[
|h(\tau)| = O(|g(\tau)|(T_\tau - \tau)^{-2}) = O((T_\tau - \tau)^{-1+\delta/\alpha}).
\]
(5.10)
Therefore,
\[
\lim_{\tau \to T_\tau^-} J_2(t) = \int_{t_0}^{T_\tau} h(\tau) d\tau = J_\tau \in \mathbb{R},
\]
(5.11)
and
\[
J_2(t) = J_\tau - h_1(t), \quad \text{where} \quad h_1(t) = \int_{t}^{T_\tau} h(\tau) d\tau \in \mathbb{R}.
\]
(5.12)
It follows estimate (5.10) that
\[
|h_1(t)| = O((T_\tau - t)^{\delta/\alpha}) \quad \text{as} \quad t \to T_\tau^-.
\]
(5.13)
Regarding the integral in formula (5.9), we have, thanks to estimate (5.13) of \( |f(t)| \), that
\[
\frac{|f(t)|}{(T_\tau - t)^{1/\alpha}} = O((T_\tau - t)^{-1+\delta/\alpha}) \quad \text{as} \quad t \to T_\tau^-.
\]
(5.14)
Therefore,
\[
\lim_{t \to T_\tau^-} \int_{t_0}^{t} \frac{e^{J_2(\tau)}}{(T_\tau - \tau)^{1/\alpha}} f(\tau) d\tau = \int_{t_0}^{T_\tau} \frac{e^{J_2(\tau)}}{(T_\tau - \tau)^{1/\alpha}} f(\tau) d\tau = \eta_\tau \in \mathbb{R}^n,
\]
and
\[
\int_{t_0}^{t} \frac{e^{J_2(\tau)}}{(T_\tau - \tau)^{1/\alpha}} f(\tau) d\tau = \eta_\tau - \eta(t),
\]
(5.15)
where
\[
\eta(t) = \int_{t}^{T_\tau} \frac{e^{J_2(\tau)}}{(T_\tau - \tau)^{1/\alpha}} f(\tau) d\tau \in \mathbb{R}^n.
\]
It follows (5.11) and (5.14) that
\[ |\eta(t)| = \mathcal{O}((T^*_t - t)^{\delta/\alpha}) \text{ as } t \to T^*_t. \] (5.16)

Combining (5.9), (5.12) and (5.15) gives
\[ y(t) = \frac{(T^*_t - t)^{1/\alpha}}{(T^*_t - t_0)^{1/\alpha}} e^{-J_s} \left( y_0 + (T^*_t - t_0)^{1/\alpha}(\eta_0 - \eta(t)) \right) \text{ for } t \in [t_0, T^*_t]. \]

Then
\[ y(t) - (T^*_t - t)^{1/\alpha} e^{-J_s} \left( \frac{y_0}{(T^*_t - t_0)^{1/\alpha}} + \eta_0 \right) = (T^*_t - t)^{1/\alpha} e^{-J_s} (e^{h_1(t)} - 1) \left( \frac{y_0}{(T^*_t - t_0)^{1/\alpha}} + \eta_0 \right) \]
\[ - (T^*_t - t)^{1/\alpha} e^{-J_s+h_1(t)} \eta(t). \]

Let \( \xi_* = e^{-J_s}((T^*_t - t_0)^{-1/\alpha} y_0 + \eta_0) \in \mathbb{R}^n \). This expression and properties (5.13), (5.16) imply, as \( t \to T^*_t \),
\[ |y(t) - (T^*_t - t)^{1/\alpha} \xi_*| = \mathcal{O}((T^*_t - t)^{1/\alpha}(e^{h_1(t)} - 1) + |\eta(t)|)) \]
\[ = \mathcal{O}((T^*_t - t)^{1/\alpha}(h_1(t) + |\eta(t)|)), \]
thus,
\[ |y(t) - (T^*_t - t)^{1/\alpha} \xi_*| = \mathcal{O}((T^*_t - t)^{1/\alpha + \delta/\alpha}). \] (5.17)

Therefore, we obtain the desired estimate (5.4). Because of the lower bound of \( |\eta(t)| \) in (4.3), the vector \( \xi_* \) in (5.4) must be non-zero.

We prove property (5.5) now. By the triangle inequality and (5.17), one has
\[ |(T^*_t - t)^{-1/\alpha}|y(t)| - |\xi_*| = \mathcal{O}((T^*_t - t)^{\delta/\alpha}). \] (5.18)

From (5.19),
\[ (T^*_t - t)^{-1/\alpha}|y(t)| = \left( a \alpha + \frac{g(t)}{T^*_t - t} \right)^{1/\alpha}. \] (5.19)

Taking into account estimate (5.7) of \( |g(t)| \), we have from (5.19) that, as \( t \to T^*_t \),
\[ |(T^*_t - t)^{-1/\alpha}|y(t)| - (a \alpha)^{1/\alpha} = \mathcal{O} \left( \frac{|g(t)|}{T^*_t - t} \right) = \mathcal{O}((T^*_t - t)^{\delta/\alpha}). \] (5.20)

From the two asymptotic estimates (5.18) and (5.20), one must have \( |\xi_*| = (a \alpha)^{1/\alpha} \), which proves (5.3). The proof is complete. \( \square \)

**Remark 5.2.** In the case dimension \( n = 1 \), Theorem 5.1 already proves Theorem 3.3 for any positive constant \( A \) and positive function \( H \in \mathcal{H}_{-\alpha}(\mathbb{R}, \mathbb{R}) \). We justify this fact below.

Let \( y(t) \) be the solution of (3.5) as in Theorem 3.3. With \( H \in \mathcal{H}_{-\alpha}(\mathbb{R}, \mathbb{R}) \), we have
\[ H(x) = \begin{cases} |x|^{-\alpha} H(1), & \text{for } x > 0, \\ |x|^{-\alpha} H(-1), & \text{for } x < 0. \end{cases} \]

In general, \( H(1) \neq H(-1) \), hence it appears that we do not have equation (5.2) yet. However, for our continuous solution \( y(t) \neq 0 \) on \([t_0, T^*_t] \), we must have either \( y(t) > 0 \) on \([t_0, T^*_t] \) or \( y(t) < 0 \) on \([t_0, T^*_t] \). Therefore, \( y(t) \), in fact, satisfies (5.2) for all \( t \in (t_0, T^*_t) \), with \( a = AH(1) \) or \( a = AH(-1) \). Then Theorem 5.1 applies. (As a side note, because \( S^0 = \{-1, 1\} \), property (HC) on \( S^0 \) is automatically satisfied.)
6. Solutions when the matrix $A$ is symmetric

In this section, we assume, in addition to Assumption [2.1] the matrix $A$ is symmetric and the function $H$ satisfies Condition [4.2].

Let $y(t)$ be the solution as in Theorem [3.3]. For $t \in [t_0, T_*)$, define

$$\lambda(t) = \frac{y(t) \cdot Ay(t)}{|y(t)|^2} \text{ and } v(t) = \frac{y(t)}{|y(t)|}.$$  \hspace{1cm} (6.1)

(The $\lambda(t)$ imitates the Dirichlet quotient for the heat equations when $A$ is the negative Laplacian.)

Then $\lambda \in C^1([t_0, T_*), \mathbb{R})$ and $v \in C^1([t_0, T_*), \mathbb{R}^n)$. Moreover, one has $|v(t)| = 1$ and, thanks to $[2.3],$

$$\Lambda_1 \leq \lambda(t) \leq \Lambda_n \leq \|A\| \text{ for all } t \in [t_0, T_*).$$  \hspace{1cm} (6.2)

**Proposition 6.1.** One has

$$\lim_{t \to T_*} \lambda(t) = \Lambda \in \sigma(A).$$

**Proof.** For $t \in (t_0, T_*)$, we have

$$\lambda'(t) = \frac{2}{|y|^2}y' \cdot Ay - \frac{2(y \cdot Ay)}{|y|^4}y \cdot y = \frac{2}{|y|^2}y' \cdot (Ay - \lambda y).$$  \hspace{1cm} (6.3)

By equation [1.3], we write $y'$ as

$$y' = -H(y)(Ay - \lambda y) - \lambda H(y)y + f(t),$$

and use it in (6.3) to obtain

$$\lambda'(t) = -\frac{2H(y)}{|y|^2}|Ay - \lambda y|^2 - \frac{2\lambda H(y)}{|y|^2}y \cdot (Ay - \lambda y) + h(t),$$

where

$$h(t) = \frac{2}{|y(t)|^2}f(t) \cdot (Ay(t) - \lambda(t)y(t)).$$

Because $y \cdot (Ay - \lambda y) = 0$, it follows that

$$\lambda'(t) = -2H(y)|Av - \lambda v|^2 + h(t).$$  \hspace{1cm} (6.4)

Using [3.6], [6.2], the fact $|v(t)| = 1$, and, then, [4.3], we estimate

$$|h(t)| \leq 4M \|A\| \cdot |y(t)|^{-\alpha + \delta} \leq C_5(T_* - t)^{-1+\delta/\alpha} \text{ for all } t \in [t_0, T_*),$$  \hspace{1cm} (6.5)

where $C_5$ is $4M \|A\|C_2^{-\alpha + \delta}$ if $\delta \geq \alpha$, or $4M \|A\|C_1^{-\alpha + \delta}$ otherwise.

For $t, t' \in [t_0, T_*)$ with $t' > t$, integrating equation (6.4) from $t$ to $t'$ gives

$$\lambda(t') - \lambda(t) + 2 \int_t^{t'} H(y(\tau))(Av(\tau) - \lambda(\tau)v(\tau))^2 d\tau = \int_t^{t'} h(\tau) d\tau.$$  \hspace{1cm} (6.6)

Thanks to (6.5), the last integral can be estimated as

$$\int_t^{t'} h(\tau) d\tau \leq \frac{\alpha C_5}{\delta} (T_* - t)^{\delta/\alpha}.$$  \hspace{1cm} (6.7)
By taking the limit superior of (6.6), as \( t' \to T_*^- \), we derive
\[
\limsup_{t' \to T_*^-} \lambda(t') \leq \lambda(t) + \frac{\alpha C_5}{\delta} (T_* - t)^{\delta/\alpha} < \infty.
\] (6.8)

Then taking the limit inferior of (6.8), as \( t \to T_*^- \), yields
\[
\limsup_{t' \to T_*^-} \lambda(t') \leq \liminf_{t \to T_*^-} \lambda(t).
\]

This and (6.2) imply
\[
\lim_{t \to T_*^-} \lambda(t) = \Lambda \in [\Lambda_1, \Lambda_n].
\] (6.9)

It remains to be proved that \( \Lambda \) is an eigenvalue of \( A \). Using properties (6.7) and (6.9) in (6.6) and by the Cauchy criterion, as \( t, t' \to T_*^- \), we obtain
\[
\int_{t_0}^{T_*} H(y(\tau)) |Av(\tau) - \lambda(\tau)v(\tau)|^2 d\tau < \infty.
\] (6.10)

We claim that
\[
\forall \varepsilon \in (0, T_* - t_0), \exists t \in [T_* - \varepsilon, T_*) : |Av(t) - \lambda(t)v(t)| < \varepsilon.
\] (6.11)

Indeed, suppose the claim (6.11) is not true, then
\[
\exists \varepsilon_0 \in (0, T_* - t_0), \forall t \in [T_* - \varepsilon_0, T_*) : |Av(t) - \lambda(t)v(t)| \geq \varepsilon_0.
\] (6.12)

Combining (6.12) with property (4.12), we have
\[
\int_{T_* - \varepsilon_0}^{T_*} H(y(\tau)) |Av(\tau) - \lambda(\tau)v(\tau)|^2 d\tau \geq \int_{T_* - \varepsilon_0}^{T_*} C_3 (T_* - \tau)^{-1} \varepsilon_0^2 d\tau = \infty,
\]
which contradicts (6.10). Hence, the claim (6.11) is true.

Thanks to (6.11), there exists a sequence \( (t_j)_{j=1}^\infty \subset [t_0, T_*) \) such that
\[
\lim_{j \to \infty} t_j = T_* \quad \text{and} \quad \lim_{j \to \infty} |Av(t_j) - \lambda(t_j)v(t_j)| = 0.
\] (6.13)

The first equation in (6.13) implies \( \lambda(t_j) \to \Lambda \) as \( j \to \infty \). Because \( v(t_j) \in S^{n-1} \) for all \( j \), we can extract a subsequence \( (v(t_{j_k}))_{k=1}^\infty \), such that \( v(t_{j_k}) \to \bar{v} \in S^{n-1} \) as \( k \to \infty \).

Combining these limits with the second equation in (6.13) written with \( j = j_k \) and \( k \to \infty \) yields \( A\bar{v} = \Lambda \bar{v} \). Therefore, \( \Lambda \) is an eigenvalue of \( A \).

From here to the end of this section, \( \Lambda \) is the eigenvalue in Proposition 6.1.

**Proposition 6.2.** There exists \( \varepsilon > 0 \) such that
\[
|(I_n - R_A)v(t)| = O((T_* - t)^\varepsilon) \quad \text{as} \quad t \to T_*^-.
\] (6.14)

**Proof.** We calculate
\[
v' = \frac{1}{|y|} g' - \frac{1}{|y|^3} (g' \cdot y)y = -\frac{H(y)}{|y|} Ay + \frac{1}{|y|} f(t) + \frac{H(y)(Ay) \cdot y}{|y|^3} y - \frac{f(t) \cdot y}{|y|^3} y.
\]

Define the function \( g : [t_0, T_*) \to \mathbb{R}^n \) by
\[
g(t) = \frac{1}{|y(t)|} f(t) - \frac{f(t) \cdot y(t)}{|y(t)|^3} y(t).
\]

Then we have
\[
v' = -H(y)(Av - \lambda v) + g(t) \quad \text{for all} \quad t \in (t_0, T_*).
\] (6.15)
Using property (3.6) of \( f(t) \), one can estimate
\[
|g(t)| \leq 2M|y(t)|^{-\alpha + \delta} \text{ for all } t \in [t_0, T_*]. \tag{6.16}
\]

Let \( \lambda_j \in \sigma(A) \setminus \{\Lambda\} \). Applying \( R_{\lambda_j} \) to equation (6.15) and taking the dot product with \( R_{\lambda_j}v \) yield
\[
\frac{1}{2} \frac{d}{dt} |R_{\lambda_j}v|^2 = -H(y)(\lambda_j - \lambda)|R_{\lambda_j}v|^2 + R_{\lambda_j}g(t) \cdot R_{\lambda_j}v. \tag{6.17}
\]

Set
\[
\mu = \min\{|\lambda_j - \Lambda| : 1 \leq j \leq d, \lambda_j \neq \Lambda\} > 0. \tag{6.18}
\]

Applying Cauchy–Schwarz’s inequality, inequality (3.2) to \(|R_{\lambda_j}g(t)|\), estimate (6.16) for \(|g(t)|\), and then Cauchy’s inequality, we have
\[
|R_{\lambda_j}g(t) \cdot R_{\lambda_j}v| \leq 2M|y|^{-\alpha + \delta}|R_{\lambda_j}v| \leq \frac{\mu}{4} H(y)|R_{\lambda_j}v|^2 + \frac{4M^2|y|^{-2\alpha + 2\delta}}{\mu H(y)}. \tag{6.19}
\]

Using the first inequality of (2.6) to estimate the last \( H(y) \) gives
\[
|R_{\lambda_j}g(t) \cdot R_{\lambda_j}v| \leq \frac{\mu}{4} H(y)|R_{\lambda_j}v|^2 + \frac{4M^2|y|^{-\alpha + 2\delta}}{\mu c_1}.
\]

Utilizing the estimates in (1.3) for the norm \(|y(t)|\), we obtain, for \( t \in [t_0, T_*] \),
\[
|R_{\lambda_j}g(t) \cdot R_{\lambda_j}v| \leq \frac{\mu}{4} H(y)|R_{\lambda_j}v|^2 + \frac{C_6}{2} (T_* - t)^{-1 + 2\delta/\alpha}, \tag{6.19}
\]

where
\[
C_6 = \frac{8M^2}{\mu c_1} \cdot \begin{cases} 
C_2^{-\alpha + 2\delta}, & \text{if } \delta \geq \alpha/2, \\
C_1^{-\alpha + 2\delta}, & \text{otherwise}.
\end{cases}
\]

Below, \( T \in (t_0, T_*) \) is fixed and can be taken sufficiently close to \( T_* \) such that
\[
|\lambda(t) - \Lambda| \leq \frac{\mu}{4} \text{ for all } t \in [T, T_*]. \tag{6.20}
\]

**Case \( \lambda_j > \Lambda \).** In this case, combining (6.17) and (6.19) yields, for \( t \in (t_0, T_*) \),
\[
\frac{1}{2} \frac{d}{dt} |R_{\lambda_j}v|^2 \leq - (\lambda_j - \lambda - \frac{\mu}{4}) H(y)|R_{\lambda_j}v|^2 + \frac{C_6}{2} (T_* - t)^{-1 + 2\delta/\alpha}.
\]

By definition (6.18) of \( \mu \) and the choice (6.20), one has, for all \( t \in [T, T_*] \),
\[
\lambda_j - \lambda(t) - \frac{\mu}{4} = (\lambda_j - \Lambda) + (\Lambda - \lambda(t)) - \frac{\mu}{4} \geq \mu - \frac{\mu}{4} - \frac{\mu}{4} = \frac{\mu}{2}. \tag{6.21}
\]

Thus, for \( t \in [T, T_*] \),
\[
\frac{d}{dt} |R_{\lambda_j}v|^2 \leq -\mu H(y)|R_{\lambda_j}v|^2 + C_6 (T_* - t)^{-1 + 2\delta/\alpha}. \tag{6.22}
\]

Let \( t \) and \( \bar{t} \) be any numbers in \([T, T_*]\) with \( t > \bar{t} \). It follows (6.22) that
\[
|R_{\lambda_j}v(t)|^2 \leq e^{-\mu \int_{\bar{t}}^t H(y(\tau)) d\tau} |R_{\lambda_j}v(\bar{t})|^2 + C_6 \int_{\bar{t}}^t e^{-\mu \int_{\bar{t}}^{\tau} H(y(s)) ds} (T_* - \tau)^{-1 + 2\delta/\alpha} d\tau. \tag{6.23}
\]

With \( C_3 \) being the positive constant in (4.12), we fix a number \( \theta > 0 \) such that
\[
\theta \leq C_3 \text{ and } \theta \mu < 2\delta/\alpha.
\]

Then
\[
H(y(t)) \geq \theta(T_* - t)^{-1} \text{ for all } t \in [t_0, T_*]. \tag{6.24}
\]
Utilizing this estimate in (6.23) gives
\[
|R_{\lambda_j} v(t)|^2 \leq e^{-\mu \int_{\tau_1}^{\tau_2} dt} |R_{\lambda_j} v(t)|^2 + C_6 \int_{\tau_1}^{\tau_2} e^{-\theta \mu \int_{\tau_1}^{\tau_2} ds} (T_* - \tau)^{-1 + 2\delta/\alpha} d\tau
\]
\[
= \left( \frac{t}{T_5 - \bar{t}} \right)^{\theta \mu} |R_{\lambda_j} v(t)|^2 + C_6 (T_* - t)^{\theta \mu} \int_{\bar{t}}^{t} (T_* - \tau)^{-1 + 2\delta/\alpha - \theta \mu} d\tau
\]
\[
= \left( \frac{t}{T_5 - \bar{t}} \right)^{\theta \mu} |R_{\lambda_j} v(t)|^2 + \frac{C_6 (T_* - t)^{\theta \mu}}{2\delta/\alpha - \theta \mu} ((T_* - \bar{t})^{2\delta/\alpha - \theta \mu} - (T_* - t)^{2\delta/\alpha - \theta \mu}).
\]

Thus,\[
|R_{\lambda_j} v(t)|^2 \leq \left( \frac{|R_{\lambda_j} v(\bar{t})|^2}{(T_* - \bar{t})^{\theta \mu}} + \frac{C_6 (T_* - \bar{t})^{2\delta/\alpha - \theta \mu}}{2\delta/\alpha - \theta \mu} \right) (T_* - t)^{\theta \mu}.
\]

(6.25)

Having \( \bar{t} = T \) in (6.25), we obtain
\[
|R_{\lambda_j} v(t)| = \mathcal{O}((T_* - t)^{\theta \mu/2}) \text{ as } t \to T_*^{-}.
\]

(6.26)

Case \( \lambda_j < \Lambda \). Using (6.19) to have a lower bound for the last term in (6.17), we have
\[
\frac{1}{2} \frac{d}{dt} |R_{\lambda_j} v|^2 \geq (\lambda - \lambda_j - \frac{\mu}{4}) H(y) |R_{\lambda_j} v|^2 - \frac{C_6}{2} (T_* - t)^{-1 + 2\delta/\alpha}.
\]

Similar to (6.21), one has, for \( t \in [T, T_*] \),
\[
\lambda(t) \leq \lambda_j - \frac{\mu}{4} = (\lambda(t) - \Lambda) + (\Lambda - \lambda_j) - \frac{\mu}{4} \geq \frac{\mu}{4} + \mu - \frac{\mu}{4} = \frac{\mu}{2}.
\]

Hence,
\[
\frac{d}{dt} |R_{\lambda_j} v|^2 \geq \mu H(y) |R_{\lambda_j} v|^2 - C_6 (T_* - t)^{-1 + 2\delta/\alpha}.
\]

Then, for any \( t, \bar{t} \in [T, T_*] \) with \( t > \bar{t} \), one has
\[
e^{-\mu \int_{\bar{t}}^{T_*} H(y(\tau)) d\tau} |R_{\lambda_j} v(t)|^2 - |R_{\lambda_j} v(\bar{t})|^2 \geq -C_6 \int_{\bar{t}}^{T_*} e^{-\mu \int_{\bar{t}}^{\tau} H(y(s)) ds} (T_* - \tau)^{-1 + 2\delta/\alpha} d\tau.
\]

(6.27)

Note from (6.24) that \( \int_{\bar{t}}^{T_*} H(y(\tau)) d\tau = \infty \), and from (3.2) that \( |R_{\lambda_j} v(t)| \leq |v(t)| = 1 \). Then
\[
\lim_{t \to T_*^{-}} e^{-\mu \int_{\bar{t}}^{T_*} H(y(\tau)) d\tau} |R_{\lambda_j} v(t)|^2 = 0.
\]

Letting \( t \to T_*^{-} \) in (6.27) and using (6.24) yield
\[
|R_{\lambda_j} v(\bar{t})|^2 \leq C_6 \int_{\bar{t}}^{T_*} e^{-\mu \int_{\bar{t}}^{\tau} H(y(s)) ds} (T_* - \tau)^{-1 + 2\delta/\alpha} d\tau
\]
\[
= C_6 \int_{\bar{t}}^{T_*} \left( \frac{T_* - \tau}{T_* - \bar{t}} \right)^{\theta \mu} (T_* - \tau)^{-1 + 2\delta/\alpha} d\tau = \frac{C_6}{\theta \mu + 2\delta/\alpha} (T_* - \bar{t})^{2\delta/\alpha}.
\]

Therefore, we obtain
\[
|R_{\lambda_j} v(\bar{t})| = \mathcal{O}((T_* - \bar{t})^{\delta/\alpha}) \text{ as } \bar{t} \to T_*^{-}.
\]

(6.28)

We estimate \(|(I_n - R_{\lambda}) v(t)|\) now. We have
\[
|(I_n - R_{\lambda}) v(t)| = \sum_{1 \leq j \leq d, \lambda_j \neq \Lambda} R_{\lambda_j} v(t) \leq \sum_{1 \leq j \leq d, \lambda_j \neq \Lambda} |R_{\lambda_j} v(t)|.
\]

(6.29)
In the last sum in (6.29), we estimate $|R_{\lambda}v(t)|$ for all $\lambda_j > \Lambda$ by (6.26), and estimate $|R_{\lambda_j}v(t)|$ for all $\lambda_j < \Lambda$ by (6.28). This results in the desired estimate (6.14) for $|(I_n - R_{\lambda})v|$, with $\varepsilon = \min\{\theta\mu/2, \delta/\alpha\} = \theta\mu/2$.

We derive from Proposition 6.2 more specific estimates for $y(t)$. Let $\varepsilon > 0$ be as in Proposition 6.2. On the one hand, we have

$$|(I_n - R_{\lambda})y(t)| = |y(t)| \cdot |(I_n - R_{\lambda})v(t)|.$$  

Together with (4.3) and (6.14), it yields

$$|(I_n - R_{\lambda})y(t)| = O((T_* - t)^{1/\alpha + \varepsilon}) \text{ as } t \to T_*^-.$$  

(6.30)

On the other hand, by the triangle inequality and (4.3), one has

$$|R_{\lambda}y(t)| \leq |y(t)| + |(I_n - R_{\lambda})y(t)| \leq C_2(T_* - t)^{1/\alpha} + |(I_n - R_{\lambda})y(t)|,$$

$$|R_{\lambda}y(t)| \geq |y(t)| - |(I_n - R_{\lambda})y(t)| \geq C_1(T_* - t)^{1/\alpha} - |(I_n - R_{\lambda})y(t)|.$$  

Combining these inequalities with estimate (6.30) for $|(I_n - R_{\lambda})y(t)|$, we deduce that there exist numbers $T_0 \in [t_0, T_*)$ and $C_7, C_8 > 0$ such that

$$C_7(T_* - t)^{1/\alpha} \leq |R_{\lambda}y(t)| \leq C_8(T_* - t)^{1/\alpha} \text{ for all } t \in [T_0, T_*).$$  

(6.31)

**Proposition 6.3.** There exists a unit vector $v_* \in \mathbb{R}^n$ such that

$$|R_{\lambda}v(t) - v_*| = O((T_* - t)^{\varepsilon}) \text{ as } t \to T_*^- \text{ for some } \varepsilon > 0.$$  

(6.32)

**Proof.** Let $\varepsilon_0 > 0$ be such that (6.14) holds for $\varepsilon = \varepsilon_0$. Then one has

$$|1 - |R_{\lambda}v(t)|| = ||v(t)| - |R_{\lambda}v(t)|| \leq |v(t) - R_{\lambda}v(t)| = O((T_* - t)^{\varepsilon_0}).$$  

(6.33)

Let $T_0$ be as in (6.31). Recall that $C_4$ is the positive constant in (4.12). We fix a number $\varepsilon_1 > 0$ such that

$$C_4\varepsilon_1 < \delta/\alpha.$$  

(6.34)

Thanks to Proposition 6.1 there is $T \in [T_0, T_*)$ such that

$$|\lambda(t) - \Lambda| \leq \varepsilon_1 \text{ for all } t \in [T, T_*).$$

Note from (6.31) that $R_{\lambda}v(t) \neq 0$ for all $t \in [T, T_*)$. Applying $R_{\lambda}$ to equation (6.15) yields, for $t \in (t_0, T_*),$

$$\frac{d}{dt}R_{\lambda}v = -H(y)(\Lambda - \lambda)R_{\lambda}v + R_{\lambda}g(t).$$  

(6.35)

Then, for $t \in [T, T_*),$

$$\frac{d}{dt}|R_{\lambda}v| = \frac{1}{|R_{\lambda}v|} \left( \frac{d}{dt}R_{\lambda}v \right) \cdot R_{\lambda}v = -H(y)(\Lambda - \lambda)|R_{\lambda}v| + g_1(t),$$  

(6.36)

where

$$g_1(t) = \frac{R_{\lambda}g(t) \cdot R_{\lambda}v(t)}{|R_{\lambda}v(t)|}.$$  

Solving for solution $|R_{\lambda}v(t)|$ by the variation of constants formula from the differential equation (6.36) gives, for $\bar{t}, t \in [T, T_*)$ with $t > \bar{t},$

$$|R_{\lambda}v(t)| = e^{-\int_{\bar{t}}^{t} H(y(\tau))(\Lambda - \lambda(\tau))d\tau} \left(|R_{\lambda}v(\bar{t})| + \int_{\bar{t}}^{t} e^{\int_{\tau}^{t} H(y(s))(\Lambda - \lambda(s))ds} g_1(\tau)d\tau \right).$$
It yields
\[
\int_{\bar{t}}^{t} H(y(\tau)) (\Lambda - \lambda(\tau)) d\tau = \ln \left( |R_A v(\bar{t})| + \int_{\bar{t}}^{t} e^{\int_{s}^{\tau} H(y(s))(\Lambda - \lambda(s)) ds} g_1(\tau) d\tau \right) - \ln |R_A v(t)|.
\]
(6.37)

We have from (3.2), (6.16) and (4.3) that, for \( t \in [T, T_*) \),
\[
|g_1(t)| \leq |R_A g(t)| \leq |g(t)| \leq C_9 (T_* - t)^{-1 + \delta / \alpha},
\]
(6.38)
where \( C_9 \) is \( 2MC_2^{1-\alpha + \delta} \) if \( \delta \geq \alpha \), and is \( 2MC_1^{1-\alpha + \delta} \) otherwise.

By (6.41) and (4.12), we have, for \( t \in [\bar{t}, T_*) \),
\[
e^{\int_{\bar{t}}^{T_*} H(y(s))(\Lambda - \lambda(s)) ds} \left| g_1(\tau) \right| \leq e^{\int_{\bar{t}}^{T_*} C_{4\varepsilon_1} (T_* - s)^{-1 + \delta / \alpha} ds} C_9 (T_* - \tau)^{-1 + \delta / \alpha}
= C_9 (T_* - \bar{t})^{C_{4\varepsilon_1} (T_* - \bar{t})^{-1 + \delta / \alpha - C_{4\varepsilon_1}}.}
\]

Thanks to this and (6.34),
\[
\lim_{t \to T_*^-} \int_{\bar{t}}^{t} e^{\int_{s}^{\tau} H(y(s))(\Lambda - \lambda(s)) ds} g_1(\tau) d\tau = \int_{\bar{t}}^{T_*} e^{\int_{s}^{\tau} H(y(s))(\Lambda - \lambda(s)) ds} g_1(\tau) d\tau = \eta(\bar{t}) \in \mathbb{R}.
\]
(6.39)

Note that
\[
|\eta(\bar{t})| \leq C_9 (T_* - \bar{t})^{C_{4\varepsilon_1} \int_{\bar{t}}^{T_*} (T_* - \tau)^{-1 + \delta / \alpha - C_{4\varepsilon_1}} d\tau = \frac{C_9}{\delta / \alpha - C_{4\varepsilon_1}} (T_* - \bar{t})^{\delta / \alpha}.
\]
(6.40)

Passing to the limit as \( t \to T_*^- \) in (6.37), we have
\[
\int_{\bar{t}}^{T_*} H(y(\tau))(\Lambda - \lambda(\tau)) d\tau = \ln(|R_A v(\bar{t})| + \eta(\bar{t})) \in \mathbb{R}.
\]
(6.41)

By (6.41), we can define, for \( t \in [T, T_*) \),
\[
h(t) = \int_{\bar{t}}^{T_*} H(y(\tau))(\Lambda - \lambda(\tau)) d\tau \in \mathbb{R}.
\]

We rewrite (6.41) for \( \bar{t} = t \) as
\[
h(t) = \ln(|R_A v(t)| + \eta(t)) = \ln(1 + (|R_A v(t)| - 1) + \eta(t)).
\]

With this expression and properties (6.33) and (6.40), we have, as \( t \to T_*^- \),
\[
|h(t)| = O(|R_A v(t)| - 1) + |\eta(t)| = O((T_* - t)^{\varepsilon_0} + (T_* - t)^{\delta / \alpha}) = O((T_* - \bar{t})^{\varepsilon_2}),
\]
(6.42)
where \( \varepsilon_2 = \min\{\varepsilon_0, \delta / \alpha\} \).

Solving for \( R_A v(t) \) from (6.35) by the variation of constants formula, one has
\[
R_A v(t) = e^{-\int_{\bar{t}}^{t} H(y(\tau))(\Lambda - \lambda(\tau)) d\tau} \left( R_A v(\bar{t}) + \int_{\bar{t}}^{t} e^{\int_{s}^{\tau} H(y(s))(\Lambda - \lambda(s)) ds} R_A g(\tau) d\tau \right).
\]
(6.43)

Using the same arguments as those from (6.38) to (6.40) with \( R_A g(\tau) \) replacing \( g_1(\tau) \), we obtain, similar to (6.39) and (6.40), that
\[
\lim_{t \to T_*^-} \int_{\bar{t}}^{t} e^{\int_{s}^{\tau} H(y(s))(\Lambda - \lambda(s)) ds} R_A g(\tau) d\tau = \int_{\bar{t}}^{T_*} e^{\int_{s}^{\tau} H(y(s))(\Lambda - \lambda(s)) ds} R_A g(\tau) d\tau = X(\bar{t}) \in \mathbb{R}^n
\]
for all \( \bar{t} \in [T, T_*) \), and
\[
|X(\bar{t})| = O((T_* - \bar{t})^{\varepsilon_2}) \text{ as } \bar{t} \to T_*^-.
\]
(6.44)
Thus, we obtain the desired estimate (6.32). The proof is complete.

Case 1. Consider the case equation (3.5), we have that it is positive, continuous on $\mathbb{R}$ and (6.46) hold for where $f$ is symmetric first. We use the same notation as in Section 6.

Let $\epsilon > 0$ such that, as $t \to T_*$, we rewrite (6.43) as $R_\Lambda v(t) = e^{h(t)-h(\bar{t})} \left( R_\Lambda v(\bar{t}) + X(\bar{t}) - \int_t^{T_*} e^{h(t)-h(\tau)} R_\Lambda g(\tau) d\tau \right) = e^{h(t)} v_* - X(t)$.

Thus, $|R_\Lambda v(t) - v_*| \leq e^{h(t)} - 1 \cdot |v_*| + |X(t)|$.

Using (6.42) and (6.44), we deduce, as $t \to T_*$, $|R_\Lambda v(t) - v_*| = O(|h(t)|) + |X(t)| = O((T_* - t)\epsilon^2)$.

Therefore, we obtain the desired estimate (6.32). The proof is complete.

Some immediate consequences of (6.14) and (6.32) are

$$\lim_{t \to T_*^-} R_\Lambda v(t) = \lim_{t \to T_*^-} v(t) = v_* , \quad (6.45)$$

and

$$|v(t) - v_*| = O((T_* - t)\epsilon) \quad \text{as } t \to T_*^- \text{ for some } \epsilon > 0. \quad (6.46)$$

7. Proof of the main theorem

Proof of Theorem 3.3. The fact $H$ satisfies Assumption 3.2 implies, thanks to Lemma 4.1, that it is positive, continuous on $\mathbb{R}^n \setminus \{0\}$, and has property (HC) on $\mathbb{R}^n \setminus \{0\}$.

Case 1. Consider the case $A$ is symmetric first. We use the same notation as in Section 6.

The estimate (3.8) already comes from (6.30). We prove (3.9) next. Applying $R_\Lambda$ to equation (3.5), we have

$$\left( R_\Lambda y \right)' = -\Lambda H(y) R_\Lambda y + R_\Lambda f(t). \quad (7.1)$$

Let $v_*$ be the unit vector in Proposition 6.3 and $\epsilon_0 > 0$ be such that (6.14), (6.30), (6.32), and (6.46) hold for $\epsilon = \epsilon_0$.

We rewrite $H(y)$ on the right-hand side of (7.1) as

$$H(y) = |y|^{-\alpha} H(v) = |R_\Lambda y|^{-\alpha} H(v_*) + g_0(t),$$

where

$$g_0(t) = |y(t)|^{-\alpha} (H(v(t)) - H(v_*)) + (|y(t)|^{-\alpha} - |R_\Lambda y(t)|^{-\alpha}) H(v_*).$$

Then

$$\left( R_\Lambda y \right)' = -\Lambda H(v_*) |R_\Lambda y|^{-\alpha} R_\Lambda y + f_0(t), \quad (7.2)$$

where $f_0(t) = -\Lambda g_0(t) R_\Lambda y(t) + R_\Lambda f(t)$.

We estimate $|g_0(t)|$ first. We combine inequality (3.3) in Definition 8.1 applied to $F = H$, $E = \mathbb{S}^{n-1}$, $x_0 = v_*$ and $x = v(t)$ for $t$ sufficiently close to $T_*$, with estimate (6.46). Then there exists a number $\gamma > 0$ such that, as $t \to T_*^-$,

$$|H(v(t)) - H(v_*)| = O(|v(t) - v_*|\gamma) = O((T_* - t)^{\gamma\epsilon_0}). \quad (7.3)$$
It is elementary to see \(|s^{-\alpha} - 1| = O(|s - 1|)\) as \(s \to 1\). By taking \(s = |R_{\Lambda}v(t)|\), which goes to 1 as \(t \to T_{\star}^{-}\) thanks to (6.45), and using estimate (6.33), we derive
\[
|1 - |R_{\Lambda}v(t)|^{-\alpha}| = O\left(|1 - |R_{\Lambda}v(t)||\right) = O((T_{\star} - t)^{\varepsilon_{0}}) \text{ as } t \to T_{\star}^{-}.
\]

Combining (7.3), (7.4) with (4.3), we obtain
\[
|g_{0}(t)| = O(|y(t)|^{-\alpha}((T_{\star} - t)^{\varepsilon_{0}} + (T_{\star} - t)^{\varepsilon_{0}})) = O((T_{\star} - t)^{-1+\varepsilon_{1}})
\]
as \(t \to T_{\star}^{-}\), where \(\varepsilon_{1} = \varepsilon_{0}\min\{1, \gamma\}\).

We estimate \(|f_{0}(t)|\) now. As \(t \to T_{\star}^{-}\), we have from (4.3) and (4.13) that
\[
|R_{\Lambda}y(t)| = O((T_{\star} - t)^{1/\alpha}) \text{ and } |R_{\Lambda}f(t)| = O((T_{\star} - t)^{1/\alpha - 1+\delta/\alpha}).
\]

Combining (7.5) and (7.6) gives, as \(t \to T_{\star}^{-}\),
\[
|f_{0}(t)| = O((T_{\star} - t)^{-1+\varepsilon_{1}}(T_{\star} - t)^{1/\alpha} + (T_{\star} - t)^{1/\alpha - 1+\delta/\alpha}) = O((T_{\star} - t)^{1/\alpha - 1+\varepsilon_{2}/\alpha}),
\]
where \(\varepsilon_{2} = \min\{\varepsilon_{1}, \alpha, \delta\}\).

By the virtue of the lower bound of \(|R_{\Lambda}y(t)|\) in (6.31), we actually have
\[
|f_{0}(t)| = O(|R_{\Lambda}y(t)|^{1-\alpha+\varepsilon_{2}}).
\]

Fix a number \(t'_{0} \in [t_{0}, T_{\star}]\) such that \(R_{\Lambda}y(t_{0}) \neq 0\) and \(|f_{0}(t)| \leq M_{0}|R_{\Lambda}y(t)|^{1-\alpha+\varepsilon_{2}}\) for all \(t \in [t'_{0}, T_{\star}]\), where \(M_{0}\) is a positive constant. Of course, one already has \(R_{\Lambda}y(T_{\star}) = 0\).

We apply Theorem 5.1 to solution \(R_{\Lambda}y(t)\) of equation (7.2) on the interval \([t'_{0}, T_{\star}]\). Specifically, \(R_{\Lambda}y(t)\) satisfies equation (5.2) on \([t'_{0}, T_{\star}]\) with constant \(a = \Lambda H(v_{\star})\) and \(f = f_{0}\). Then there exists a nonzero vector \(\xi_{\star} \in \mathbb{R}^{n}\) such that
\[
|R_{\Lambda}y(t) - (T_{\star} - t)^{1/\alpha} \xi_{\star}| = O((T_{\star} - t)^{1/\alpha + \varepsilon_{3}}) \text{ for some } \varepsilon_{3} > 0.
\]
and
\[
|\xi_{\star}| = (\alpha \Lambda H(v_{\star}))^{1/\alpha}.
\]
The desired statement (3.9) immediately follows (7.7).

Because
\[
\xi_{\star} = \lim_{t \to T_{\star}^{-}} (T_{\star} - t)^{-1/\alpha} R_{\Lambda}y(t),
\]
by (3.9), and the fact \(\xi_{\star} \neq 0\), we have \(\xi_{\star} \in R_{\Lambda}(\mathbb{R}^{n}) \setminus \{0\}\). Hence, \(\xi_{\star}\) is an eigenvector of \(A\) associated with \(\Lambda\).

Next, we prove (3.10). Writing
\[
y(t) - (T_{\star} - t)^{1/\alpha} \xi_{\star} = (I_{n} - R_{\Lambda})y(t) + (R_{\Lambda}y(t) - (T_{\star} - t)^{1/\alpha} \xi_{\star}),
\]
and using the estimate (6.30) with \(\varepsilon = \varepsilon_{0}\), and estimate (7.7) yield
\[
|y(t) - (T_{\star} - t)^{1/\alpha} \xi_{\star}| = O((T_{\star} - t)^{1/\alpha + \varepsilon_{0}} + (T_{\star} - t)^{1/\alpha + \varepsilon_{3}}).
\]
This implies (3.7) with \(\varepsilon = \min\{\varepsilon_{0}, \varepsilon_{3}\}\).

Finally, we prove (3.10). Let \(w(t) = (T_{\star} - t)^{-1/\alpha} y(t)\) and write \(v(t) = w(t)/|w(t)|\). Passing \(t \to T_{\star}^{-}\) and noticing that \(v(t) \to v_{\star}\) and \(w(t) \to \xi_{\star}\), thanks to (6.45) and (7.7), we obtain
\[
v_{\star} = \xi_{\star}/|\xi_{\star}|
\]
Then it follows (7.8), the fact \(H\) is positively homogeneous of degree \(-\alpha\), and relation (7.9) that
\[
1 = \alpha \Lambda H(v_{\star})|\xi_{\star}|^{-\alpha} = \alpha \Lambda H(|\xi_{\star}|v_{\star}) = \alpha \Lambda H(\xi_{\star}).
\]
Hence, we obtain (3.10). This completes the proof for the case of symmetric matrix \(A\).
Case 2. Consider the case $A$ is not symmetric. Let $A_0$ and $S$ be as in (2.2).

Set $z(t) = Sy(t)$ on $[t_0, T_*]$. Then $z(t)$ satisfies equation (4.8) with $\tilde{H}$ and $\tilde{f}$ defined in (4.9). One can verify the following facts.

- $z(t) \neq 0$ for $t \in [t_0, T_*)$ and $z(T_*) = 0$.
- $\tilde{H} \in \mathcal{H}_{-\alpha}(\mathbb{R}^n)$ and, thanks to parts (i) and (v) of Lemma 4.1, $\tilde{H} > 0$ on $S^{n-1}$ and $\tilde{H}$ has property (HC) on $S^{n-1}$.
- Thanks to (4.10), $\tilde{f}(t)$ and $z(t)$ satisfy condition (3.6) with the same numbers $\alpha, \delta, t_0, T_*$, and constant $\tilde{M}$ in place of $M$.

We apply the results already established in Case 1 above to the solution $z(t)$ of equation (4.8). Note that $A_0$ replaces $A$ and $\tilde{R}_\Lambda$ replaces $R_\Lambda$. Then there exist an eigenvalue $\Lambda$ of $A_0$ and an eigenvector $\xi_0$ of $A_0$ associated with $\Lambda$ such that

$$|(I_n - \tilde{R}_\Lambda)z(t)|, |\tilde{R}_\Lambda z(t) - (T_* - t)^{1/\alpha} \xi_0| = O((T_* - t)^{1/\alpha + \varepsilon}),$$

for some number $\varepsilon > 0$, and

$$\alpha \Lambda \tilde{H}(\xi_0) = 1. \quad (7.11)$$

Let $\xi_* = S^{-1} \xi_0$. Then $\Lambda$ is an eigenvalue of $A$ and $\xi_*$ is an eigenvector of $A$ associated with $\Lambda$. We rewrite (7.10) as

$$|S(I_n - R_\Lambda) y(t)|, |S(R_\Lambda y(t) - (T_* - t)^{-1/\alpha} \xi_*)| = O((T_* - t)^{1/\alpha + \varepsilon}),$$

which implies (3.8) and (3.9). By (3.8), (3.9) and the triangle inequality, we obtain (3.7) in the same way as in Case 1.

Finally, (3.10) follows (7.11) and the relation $\tilde{H}(\xi_0) = H(\xi_*)$. The proof of Theorem 3.3 is complete.

There is another totally different approach to the local properties of solutions of ODE based on the Poincaré–Dulac normal form [1][2][28]. It has been generalized and developed by many and for so long, see the books [7][8][27], recent papers such as [3][6], and references therein. Our approach is relatively new and only recently used to explore different classes of equations and problems in ODE. For more comparisons between the other approach and ours, see [22] Remark 5.8 and [12] Remark 6.14.

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