How to Wick rotate generic curved spacetime

Matt Visser

School of Mathematics and Statistics, Victoria University of Wellington;
PO Box 600, Wellington 6140, New Zealand.

E-mail: matt.visser@sms.vuw.ac.nz

Abstract:
It is an article of folklore that the collection of ideas identified as Euclidean quantum gravity may be derived from ordinary Lorentzian signature gravity by the procedure of Wick rotation. This note will attempt to shed some light on this relatively ill-understood procedure. I argue that it proves inappropriate and unhelpful to regard Wick rotation in terms of a complex deformation of the time coordinate. Rather, Wick rotation can more usefully be viewed as a complex deformation of the spacetime metric. This simple reformulation of the Wick rotation procedure, while it leaves flat space physics unaffected, has profound implications for quantum gravity.

Original date: March 1991. This old pre-arXiv essay was originally written for the 1991 Gravity Research Foundation essay contest, while I was affiliated with the Physics Department of Washington University in Saint Louis. For many years a PDF copy of the essay was available at the GRF website, but many of the older essays have now been removed from that website. Apart from purely cosmetic issues, this arXiv upload is an accurate reflection of that 1991 essay.

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1 Introduction

A considerable amount of interest is currently focussed on the complicated collection of ideas known as Euclidean quantum gravity. Now, it is an article of folklore that Euclidean quantum gravity may be derived from ordinary Lorentzian signature gravity by the procedure of Wick rotation. While Wick rotation is a well understood process in flat spacetime, its application to curved spacetimes is more problematic and less straightforward than one might have reason to hope. It is the purpose of this note to shed some light on this ill-understood procedure.

In a quantum field theory defined on flat Minkowski space, Wick rotation is physically justified by appealing to the causality constraints, as embodied in Feynman’s “iε” prescription [1, 2]. After Wick rotating from Minkowski space to Euclidean space these causality constraints survive in a modified form as “Osterwalder–Schrader positivity” (OS positivity) of the Euclidean version of the field theory [3]. On the other hand, in the usual formulation of the transition from Lorentzian signature gravity to Euclidean signature quantum gravity the effects of the causal structure of the Lorentzian signature theory seem to disappear completely as no analogue of OS positivity is manifest [4]. This circumstance is profoundly disturbing, and suggests that something crucial is missing in the naive analysis.
A related observation, indicating that the process of Wick rotating a general spacetime is less well-understood than one might suppose, is that the naive prescription \( t \to -it \) does not in general yield a real Euclidean signature metric. For example: when acting on a stationary spacetime, this prescription (interpreted as a “generalized coordinate transformation”) does succeed in making the time–time and space–space components of the metric both real and positive [5]. However, even for this relatively simple case the \( t \to -it \) prescription will render the time–space components of the metric pure imaginary [6]. The problem becomes even more acute in non–stationary spacetimes where even the time–time and space–space components of the metric may become complex. For instance, a particularly transparent example is provided by ordinary de Sitter space [7]. In the comoving coordinates corresponding to a spatially flat \( k = 0 \) slicing, de Sitter space may be described by the metric

\[
    ds^2 = -dt^2 + H^{-2} e^{Ht} \left( dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right).
\]

(1.1)

Naive analytic continuation yields

\[
    ds^2 = +dt^2 + H^{-2} e^{-iHt} \left( dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right).
\]

(1.2)

While certainly entertaining, this manifestly complex analytically continued metric is hardly the hoped for “Euclideanization” of de Sitter space. Worse, the act of analytic continuation depends on the coordinate system employed. The metric of de Sitter space may equally well be written down in terms of the comoving coordinates corresponding to a positive spatial curvature \( k = +1 \) slicing, so that

\[
    ds^2 = -dt^2 + H^{-2} \cosh^2 Ht \left( \frac{dr^2}{1 - r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right).
\]

(1.3)

Naive analytic continuation in this case yields a metric that is manifestly Euclidean:

\[
    ds^2 = +dt^2 + H^{-2} \cos^2 Ht \left( \frac{dr^2}{1 - r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right).
\]

(1.4)

In fact this analytically continued metric has a natural interpretation as the canonical metric on a \( S^4 \) hypersphere of radius \( H^{-1} \), and we see that the often quoted result that the Euclideanization of de Sitter space is the 4–sphere is a coordinate dependent result. For the comoving coordinates corresponding to a negative spatial curvature \( k = -1 \) slicing, the metric of de Sitter space is

\[
    ds^2 = -dt^2 + H^{-2} \sinh^2 Ht \left( \frac{dr^2}{1 + r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right).
\]

(1.5)
In this case, naive analytic continuation yields a metric that while real is certainly not Euclidean, the original Lorentzian metric of signature (1,3) being analytically continued to another Lorentzian metric of signature (3,1) given by

$$ds^2 = +dt^2 - H^{-2} \sin^2 Ht \left( \frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right).$$  \hspace{1cm} (1.6)

This explicit coordinate dependence of the naive Wick rotation procedure should inspire a deep and abiding feeling of unease — clearly something critical is amiss. The situation is further complicated by observing that a general curved spacetime may not even possess a global coordinate patch (indeed de Sitter space itself possesses this feature). In such a case, the differentiable structure of the manifold is given in terms of an atlas of coordinate charts together with a set of transition functions connecting overlapping charts \[8, 9\]. It is far from clear how to interpret a naive “t → −it” prescription on such a structure.

Such concerns may be dealt with by reinterpreting Wick rotation in terms of an analytic continuation of the metric, leaving the coordinate charts invariant. The (reinterpreted) Wick rotation is then a method for connecting a real Minkowski signature metric with an associated real Euclidean signature metric while leaving the differentiable structure of the manifold fixed. In the case of flat space this prescription will precisely reproduce the known results, but this viewpoint has the great advantage that it may meaningfully be extended to general curved spacetimes.

2 Wick rotation in flat spacetime

2.1 The “i\epsilon” prescription

Recall the elementary result that causality in Minkowski space is responsible for the usual “i\epsilon” prescription for quantum field propagators [1, 2]. For instance, the Feynman propagator for a scalar field is \[P = (E, p)\]:

$$\Delta_F(P) = \frac{i}{E^2 - p^2 - m^2 + i\epsilon}.$$  \hspace{1cm} (2.1)

Now, in the complex energy plane the poles in this propagator occur at energies \(E = \pm \sqrt{m^2 + p^2 - i\epsilon}\), so that when performing the usual Wick rotation, \(E \to iE\), the contour does not pass over the poles. [The contour \((-\infty, +\infty)\) is deformed to \((-i\infty, +i\infty)\).] In terms of the “rotated” energy variable the Euclidean propagator is:

$$\Delta_E(P) = \frac{-i}{E^2 + p^2 + m^2}.$$  \hspace{1cm} (2.2)
Since energy and time are Fourier conjugates of each other, it is usual to rephrase this deformation of the contour of integration over energy in terms of an analytic continuation of the time variable \( t \to -it \). Unfortunately, as we have just seen, this simple procedure does not have a consistent generalization when considering generic curved manifolds.

### 2.2 Deforming the Minkowski metric

To obtain an interpretation of Wick rotation that does generalize nicely to curved manifolds we note that the Minkowski space Feynman propagator may be recast as:

\[
\Delta_F(P) = \frac{i}{E^2(1 + i\epsilon) - p^2 - m^2}.
\]  

(2.3)

The poles in this propagator occur at

\[
E = \pm \sqrt{(m^2 + p^2)/(1 + i\epsilon)} = \pm \sqrt{m^2 + p^2 - i\epsilon}.
\]  

(2.4)

Here we have used the fact that \( \epsilon \) is infinitesimal and that \( m^2 + p^2 \) is guaranteed to be positive. When the propagator is written in this form the \( i\epsilon \) prescription has a natural interpretation in terms of a complex “not quite Minkowski” metric. Specifically, let us define

\[
\eta_{\epsilon} = \text{diag}(-1 - i\epsilon, +1, +1, +1).
\]  

(2.5)

In terms of the four momentum and this “not quite Minkowski” metric the Feynman propagator can be compactly written as:

\[
\Delta_F(P) = \frac{i}{-\eta_{\epsilon}(P, P) - m^2}.
\]  

(2.6)

The prescription for Wick rotation is now also clear. Keeping the real part of \( \epsilon \) positive, analytically continue this metric from \( \epsilon = 0 \) to \( \epsilon = +2i \). (We particularly wish to avoid passing through the point \( \epsilon = +i \) since the metric \( \eta_{\epsilon} \) is degenerate at that point.) Then \( \eta(\epsilon = 0) = \eta_L \), while \( \eta(\epsilon = +2i) = \eta_E \). We also point out that \( \sqrt{-\det(\eta_{\epsilon})} = \sqrt{1 + i\epsilon} \), so that \( \sqrt{-\det(\eta_{\epsilon})} = +1 \) and \( \sqrt{-\det(\eta_{E})} = +i = +i\sqrt{\det(\eta_{E})} \). Thus this prescription for continuing the metric — not the coordinates — is completely equivalent to the usual Wick rotation in flat spacetime. Moreover, this procedure will now be shown to generalize nicely to curved spacetime. As a preliminary step to this generalization, we introduce the constant time–like vector \( V = (1, 0, 0, 0) \). Then we may compactly write

\[
\eta = \eta_L + i\epsilon \frac{V \otimes V}{\eta_L(V, V)}.
\]  

(2.7)
3 Wick rotation in generic spacetimes

3.1 Deforming Lorentzian–signature metrics

To apply these ideas to generic curved spacetimes we shall need to invoke some additional technical machinery. It is well known that a manifold admits a time-orientable Lorentzian metric if and only if it also admits an everywhere non-vanishing timelike vector field [9].

We now construct the “not quite Lorentzian” metric:

\[ g_\epsilon = g_L + i\epsilon \frac{V \otimes V}{g_L(V,V)} \]  (3.1)

Note that in the flat space limit \( g_L \to \eta_L \), and \( V \to (1,0,0,0) \), so that one recovers the result of the previous section. Further, by construction, \( g_E = g(\epsilon = +2i) \) is a globally defined (positive definite) Euclidean metric on the original manifold. Neither \( V \) nor \( g_E \) are in any sense unique, a circumstance that is less than pleasing, but unavoidable. (In the case of flat space, \( V \) could be taken to be any constant timelike vector. Although \( V \) is not then unique, in the limit \( \epsilon \to +2i \), it proves to be the case that \( \eta_E \) is independent of \( V \).)

To see in what sense the metric (3.1) embodies the curved space version of the “i\epsilon” prescription we argue as follows: For the case of a fixed background geometry one is still provided with a fixed light–cone structure so that one may unambiguously impose the constraint that \( [\hat{\phi}(x), \hat{\phi}(y)] = 0 \) whenever the events \( x \) and \( y \) are spacelike separated. Now by invoking the (strong) equivalence principle we know that locally spacetime always “looks like” a piece of Minkowski space. More precisely, consider modes whose wavelengths \( \lambda \) are much smaller than the local “radius of curvature” \( \rho = (R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta})^{-1/4} \), and whose frequency \( \nu \) is much larger than \( c/\rho \). For such high frequency/short wavelength modes propagation through spacetime is in all important respects equivalent to propagation through flat space up to terms of order \( \lambda/\rho \) or \( \nu c/\rho \). It is in this sense that the “not quite Lorentzian” metric of equation (3.1) implies and is implied by curved space causality in the high frequency/short wavelength limit.

3.2 Quantum gravity: A functional integration over geometries

The situation is considerably more complex when one seeks to quantize the gravitational field itself. There is no generally agreed upon method for doing this. I have argued elsewhere [10] in favour of the primacy of the Lorentzian path integral approach. Adopting this approach for the time being we write the physical partition function as

\[ Z_L = \int \mathcal{D}g_L \exp \left( -i \int R(g_L) \sqrt{-g_L} \right) \]  (3.2)
Proceeding formally, using the definition \( g_\epsilon \equiv g_L + i\epsilon \{V \otimes V / g_L(V,V)\} \), we now extend this \( Z_L \) to an analytically continued object \( Z(\epsilon) \):

\[
Z(\epsilon) \equiv \int \mathcal{D}g_L \mathcal{D}V \exp\left(-i \int R(g_\epsilon) \sqrt{-g_\epsilon}\right) \equiv \int \mathcal{D}g_\epsilon \exp\left(-i \int R(g_\epsilon) \sqrt{-g_\epsilon}\right). \tag{3.3}
\]

Note that as \( \epsilon \to 0 \), \( Z(\epsilon) \to Z_L \) up to a multiplicative constant. On the other hand for \( \epsilon \to +2i \) we may define a Wick rotated Euclidean partition function by

\[
Z_E = \lim_{\epsilon \to +2i} Z(\epsilon) \propto \int \mathcal{D}g_E \exp\left(+ \int R(g_E) \sqrt{g_E}\right). \tag{3.4}
\]

The convergence of this Euclidean path integral is of course problematic due to the well known conformal instability — for a nice discussion see Mazur and Mottola [11].

The point I wish to emphasize in this note concerns the range of integration of the \( \mathcal{D}g_E \) functional integration over Euclidean metrics. Within the framework espoused in this note the situation is clear: The original functional integration was over Lorentzian metrics defined on Lorentzian manifolds — and the only Euclidean metrics that can be obtained by Wick rotation from a Lorentzian metric are defined on precisely those Euclidean manifolds that are compatible with the existence of a Lorentzian structure.

It is thus my claim that the \( \mathcal{D}g_E \) functional integral should not be taken to be a functional integral over all Euclidean manifolds — rather it is my claim that the range of integration is restricted for physical reasons to include only those Euclidean manifolds compatible with the existence of a Lorentzian structure.

4 Discussion

It is worthwhile to examine carefully the logic of the argument presented in this essay:

1: By specific example I have shown that a naive “\( t \to -it \)” prescription for defining Wick rotation is mathematically inconsistent and physically unsupportable.

2: The improved Wick rotation prescription that I advocate has the virtues of being mathematically well defined and physically reasonable in that it reproduces sensible flat space results. In particular, this version of Wick rotation does not disturb the topological structure of the manifold — if the manifold has a Lorentzian metric initially then the Wick rotated Euclidean metric will still be compatible with the existence of a Lorentzian metric.

3: If we treat the Lorentzian path integral as paramount we see that Wick rotation does not lead to arbitrary Euclidean metrics, and the associated Euclidean functional integral over Euclidean metrics is over a topologically restricted class of Euclidean metrics.
This restriction of the class of Euclidean manifolds to be included in the Euclidean functional integral is of great importance to understanding quantum gravity. One way of avoiding point (3) — though in no way diminishing the strength of points (1) and (2) — is to define the Euclidean path integral by fiat to include all Euclidean manifolds. If such a course is adopted it is extremely difficult to see how a Wick rotation back to Lorentzian signature might be meaningfully defined — If manifolds that do not admit a Lorentzian metric are included in the Euclidean path integral then they will not suddenly acquire a Lorentzian interpretation through Wick rotation.

At another level, I would argue that defining the Euclidean path integral by fiat is physically unreasonable in that it violates Ockham’s razor — we live in a Lorentzian signature universe — we believe that the path integral formalism gives a good description of quantum physics — arbitrarily extending the range of integration of the path integral to include manifolds that do not admit a Lorentzian structure is in no way supported by physical experiment, nor does it appear to add insight to the mathematical problems encountered.

In summary: The naive “t → −it” prescription for Wick rotation is physically and mathematically diseased when applied to questions of quantum gravity. There is a fundamental issue of paramount importance hiding in the seemingly innocuous question “What is Wick rotation?”.

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[5] We choose our metric conventions such that the flat space Minkowski metric is
   $\eta_L = \text{diag}(-1,+1,+1,+1)$. The flat space Euclidean metric is taken to be positive definite,
   $\eta_E = \text{diag}(+1,+1,+1,+1)$.

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Note added (2017):

Gary Gibbons has kindly pointed out to me the considerably earlier 1977 article by Candelas and Raine [12], which explored similar issues. Additionally, some early 1977 discussion of the conformal mode problem in Euclidean quantum gravity can be found in reference [13]. In 1987–1988 some related work by Ivashchuk appeared in the Russian-language literature [14, 15].
If I were writing this essay today, I would add a discussion of Jeff Greensite’s ideas regarding dynamical signature change [16, 17, 19, 20], the “complex lapse” approach of Sean Hayward [21], related work by Vladimir Ivashchuk [22], and the “real way” advocated by Fernando Barbero [23]. In the causal dynamical triangulation (CDT) formulation of quantum gravity [24], restricting the configuration space to Euclidean simplicial manifolds that are Wick rotations of Lorentzian simplicial manifolds seems essential to keeping the functional integral under control. Other related work over the past decade includes [25–29]. A recent MSc thesis related to this topic is that by Finnian Gray [30].

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