BBGKY Hierarchy Underlying Many Particle Quantum Mechanics

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Recently, the one particle quantum mechanics has been obtained in the framework of an entirely classical subquantum kinetics. In the present paper we argue that, within the same scheme and without any additional assumption, it is possible to obtain also the n-particle non relativistic quantum mechanics. The main goal of the present effort is to show that the classical BBGKY hierarchical equation, for the n-particle reduced distribution function, is the ancestor of the n-particle Schrödinger equation. On the other hand we show that within the scenario of the subquantum structure of quantum particle, the Fisher information measure emerges naturally in quantum mechanics.

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In ref. [1], it is shown that quantum mechanics can be obtained in a self consistent scheme of an entirely classical many body physics. Within this framework the mean features of quantum mechanics (i.e. the probabilistic nature of the quantum description, the quantum potential, the Schrödinger equation, the quantum operators, the Heisenberg uncertainty principle) were obtained starting from a classical subquantum kinetics in the phase space and without invoking any additional principle. Furthermore the fundamental constant \( \hbar \) emerges naturally as an integration constant and represents a free parameter for the theory. Within this theory, the astonishing scenario of a subquantum structure for the quantum particle emerges spontaneously. The quantum particle turns out to have an internal structure and a spatial dispersion and appears to be composed by \( N \) identical point like and interacting subquantum objects, the monads, obeying to the laws of Newtonian physics. The theory can be viewed as describing a mechanism which permits to construct the quantum particle starting from its constituents and then quantum mechanics appears to describe a physical reality.

The statistical ensemble of these \( N \) monads is described in the phase space through the distribution function \( f = f(t, x, v) \) whose time evolution is governed by the kinetic equation \( \mathcal{L} f = C(f) \), being \( \mathcal{L} \) the Liouville operator and \( C(f) \) the collision integral that takes into account the intermonad interactions. We don’t make any assumption about the structure of the collision integral and the nature of the interaction between the monads except that, during the point-like collisions the monad number, momentum and energy are conserved. The projection of the above phase space kinetics into the physical space produces a hydrodynamics which naturally leads to the one particle spinless quantum mechanics when the external force field is conservative.

Recently in ref. [2], it is shown that the above kinetic approach can be easily adopted to treat the quantum particle with spin in an external electromagnetic field. Of course in this way the spinning quantum mechanics can be obtained.

The energy conservation during the collisions considered in the present scheme plays a very important role and could be also taken into account in the stochastic models of quantum mechanics as it has been recently observed in ref. [3].

A question which naturally arises at this point is if it is possible to extend the kinetic formalism of ref. [1] in order to treat non relativistic many particle quantum systems. This point is neglected in the literature where almost the totality of works devoted to the derivation and interpretation of quantum mechanics concern the single particle.

The main goal of the present paper is to show that it is possible to obtain, within the theory developed in ref. [1] and without any additional assumption, also the n-particle quantum mechanics starting from the classical kinetics governed in the phase space by the BBGKY hierarchy.

**BBGKY Hierarchy:** Let us consider a statistical ensemble constituted by \( N \) identical and indistinguished point-like particles (monads) obeying the laws of classical physics. It is well known that the n-particle reduced distribution function \( f_n = f_n(t, x_1, v_1, x_2, v_2, ..., x_n, v_n) \) obeys the following BBGKY hierarchical evolution equation

\[
\frac{\partial f_n}{\partial t} + \sum_{i=1}^{n} v_i \frac{\partial f_n}{\partial x_i} + \sum_{i=1}^{n} \sum_{\mu} F_i \frac{\partial f_n}{\partial v_i} = \sum_{i=1}^{n} C_i(f_{n+1}) ,
\]

being \( F_i = -\partial \mathcal{V}_i / \partial x_i \) and

\[
\mathcal{V}_i = \mathcal{V}^{(ex)}(x_i) + \frac{1}{2} \sum_{j \neq i}^{n} \mathcal{V}^{(in)}(|x_i - x_j|) .
\]

\( C_i(f_{n+1}) \) are the collision integrals which can be expressed in term of the function \( f_{n+1} \). The assumption that during the point-like collisions the monad number,
momentum and energy are conserved, implies that the three functions \( g_1(v_i) = 1, g_2(v_i) = v_i \) and \( g_3(v_i) = v_i^2 \) are the collision invariants of \( C_i(f_{n+1}) \)

\[
\int g_k(v_i) C_i(f_{n+1}) \, d^3v_i = 0 \quad ; \quad k = 1, 2, 3 .
\]

Let us consider the 6\( n \)-dimensional phase space \( (x, v) \) being \( x = (x_1, x_2, ..., x_n) \) and \( v = (v_1, v_2, ..., v_n) \). We introduce the two gradient operators \( \partial / \partial x = (\partial / \partial x_1, \partial / \partial x_2, ..., \partial / \partial x_n) \) and \( \partial / \partial v = (\partial / \partial v_1, \partial / \partial v_2, ..., \partial / \partial v_n) \). Then we define the 3\( n \)-dimensional force \( F = (F_1, F_2, ..., F_n) \) which can be derived from a potential

\[
F = -\frac{\partial V}{\partial x} ; \quad V = \sum_{i=1}^{n} V_i ,
\]

The BBGKY hierarchical equation \( [11] \), governing the evolution of the \( n \)-particle reduced distribution function \( f_n = f_n(t, x, v) \), can be written in the following compact form

\[
\frac{\partial f_n}{\partial t} + v \cdot \frac{\partial f_n}{\partial x} + \frac{F}{\mu} \cdot \frac{\partial f_n}{\partial v} = C(f_{n+1}) .
\]

The collision integral

\[
C(f_{n+1}) = \sum_{i=1}^{n} C_i(f_{n+1}) ,
\]

admits the three collision invariants \( g_1(v) = 1, g_2(v) = v \) and \( g_3(v) = v^2 \) satisfying the conditions

\[
\int g_k(v) C_i(f_{n+1}) \, d^3v_i = 0 \quad ; \quad k = 1, 2, 3 .
\]

as one can verify immediately starting from Eq. [4].

Hydrodynamics: We consider now the projection of the kinetics from the 6\( n \)-dimensional phase space into the 3\( n \)-dimensional space of the coordinates \( x \), where the system is described through the distribution function \( \rho_n(t, x) = \int f_n(t, x, v) \, d^3v \). After recalling that the total value of a given physical quantity with density \( A = A(t, x, v) \) can be calculated as \( \int A f_n \, d^3v \, d^3x \) we define the mean value of \( A \) in the point \( x \) through

\[
< A(t, x, v) >_v = \frac{\int A(t, x, v) f_n(t, x, v) \, d^3v}{\int f_n(t, x, v) \, d^3v} .
\]

This mean value represents the density of \( A \) in the space of the coordinates and permits to write the total value of \( A \) as \( \int \rho_n < A >_v \, d^3x \). We define the densities of current, of stress tensor and of heat flux vector are defined, respectively, by

\[
u = < v >_v ,
\]

\[
\sigma_{ij} = \mu < (v_i - u_i)(v_j - u_j) >_v ,
\]

\[
h_i = \frac{1}{2} \mu < |v - u|^2 (v_i - u_i) >_v .
\]

The density of energy is given by

\[
E = \frac{1}{2} \mu < v^2 >_v + V = \frac{1}{2} \mu u^2 + \varepsilon + V ,
\]

being \( \varepsilon > 0 \) the density of the internal energy.

\[
\varepsilon = \frac{1}{2} \sigma_{ii} = \frac{1}{2} \mu < v^2 >_v - < v >^2_v .
\]

Using the standard procedure consisting in multiplying Eq. [10] by the three collision invariants \( g_k(v) \) and integrating with respect to \( v \), the three following hydrodynamic equations can be obtained \([3, 4]\):

\[
\frac{\partial \rho_n}{\partial t} + \frac{\partial}{\partial x} \cdot (\rho_n u) = 0 ,
\]

\[
\mu \frac{D u}{D t} = \mathbf{F} + \mathbf{F} ,
\]

\[
\frac{\partial}{\partial t} \cdot (\rho_n s) = 0 .
\]

In the above equations \( D/Dt = \partial / \partial t + u \cdot \partial / \partial x \) is the total time or substantial or Lagrangian derivative, \( \mathbf{F} \) is the stress force

\[
\mathbf{F}_i = -\frac{1}{\rho_n} \frac{\partial}{\partial x_j} (\rho_n \sigma_{ij}) ,
\]

and \( s_i = Eu_i + \sigma_{ij} u_j + h_i \) is the energy flux density vector.

Eqs. [14]-[16] define the most general hydrodynamics of the particle system which behaves in the physical space as a fluid. In particular Eq. [16] imposes the conservation of the total energy \( H \) of the system

\[
H = \int E \rho_n \, d^3x .
\]

After recalling that the external force \( \mathbf{F} \) is conservative, from Eq. [15] it follows immediately that the irrerotationality of \( u \) implies the conservativity of \( \mathbf{F} \) and viceversa, namely

\[
u = \frac{1}{\mu} \frac{\partial S}{\partial x} ,
\]

\[
\mathbf{F} = -\frac{\partial W}{\partial x} .
\]

The condition [16] imposes that the system here considered is spinless and reduces the vector equation [15] into a scalar one

\[
\frac{\partial S}{\partial t} + \frac{1}{2\mu} \left( \frac{\partial S}{\partial x} \right)^2 + W + V = 0 .
\]

Quantum potential: From the two expressions of the force \( \mathbf{F} \) given by Eqs. [17] and [20] it follows immediately

\[
\frac{\partial W}{\partial x_j} = \frac{\partial \sigma_{jk}}{\partial x_k} + \sigma_{jk} \frac{\partial}{\partial x_k} ,
\]

with \( \xi = \ln \rho_n \). Note that Eq. [22] descends from the requirement that the force \( \mathbf{F} \) is conservative and can be
viewed as a condition constraining the forms of \( W \) and \( \sigma_{jk} \) which result to be two functionals of the field \( \xi \). The solutions of the latter equation are couples of \( W \) and \( \sigma_{jk} \) and in principle there can exist more than one solution. Even though it can appear that Eq. (22) contains considerable degrees of freedom, in the following we will show that the particular structure of this equation restricts strongly the number of its solutions.

We suppose now that just as the left hand side, both the first and the second term in the right hand side in Eq. (22) are curl free and will take the form \( \partial(\ldots)/\partial x_j \). This requirement for the first term can be satisfied in two different ways. The first and most natural way is related to the choice \( \sigma_{jk} = \sigma \delta_{jk} \). Alternatively we can pose \( \sigma_{jk} = \partial a_k/\partial x_j \) and after recalling the symmetry property \( \sigma_{jk} = \sigma_{kj} \) we have that \( a_k = \partial \alpha/\partial x_k \). Then the most general form of the density of stress tensor is

\[
\sigma_{jk} = \sigma \delta_{jk} + \frac{\partial^2 \alpha}{\partial x_j \partial x_k},
\]

being \( \sigma \) and \( \alpha \) two scalar functionals depending on \( \xi \). After taking into account the identity

\[
\frac{\partial^2 \alpha}{\partial x_j \partial x_k} \frac{\partial \xi}{\partial x_j} = \frac{\partial \alpha}{\partial x_k} \frac{\partial^2 \xi}{\partial x_j \partial x_k} - \frac{\partial \alpha}{\partial x_k} \frac{\partial^2 \xi}{\partial x_j \partial x_k}.
\]

Eq. (22) can be written as

\[
\frac{\partial}{\partial x_j} \left( W - \sigma - \frac{\partial^2 \alpha}{\partial x_k \partial x_k} - \frac{\partial \alpha}{\partial x_k} \frac{\partial \xi}{\partial x_k} \right) = \frac{\partial \alpha}{\partial x_k} \frac{\partial^3 \xi}{\partial x_k \partial x_k \partial x_j}. \tag{24}
\]

Clearly the two terms in the right hand side of Eq. (22) must be curl free. These requirements impose the two following conditions

\[
\frac{\partial \sigma}{\partial x_i} \frac{\partial \xi}{\partial x_m} = \frac{\partial \sigma}{\partial x_m} \frac{\partial \xi}{\partial x_i}, \tag{26}
\]

\[
\frac{\partial^2 \alpha}{\partial x_i \partial x_j} \frac{\partial \xi}{\partial x_m} = \frac{\partial \sigma}{\partial x_m} \frac{\partial^2 \xi}{\partial x_k \partial x_k} \frac{\partial \xi}{\partial x_i}. \tag{27}
\]

It is trivial to verify from Eq. (20) it results that \( \sigma = \sigma(\xi) \) is an arbitrary function of \( \xi \) while from Eq. (22) one can obtain that \( \alpha = c \xi \) with \( c \) a real arbitrary constant. With these positions Eq. (26) becomes

\[
\frac{\partial}{\partial x_j} \left( W - \sigma - \int \sigma \, d\xi - \frac{c}{2} \frac{\partial^2 \xi}{\partial x_k \partial x_k} - \frac{c}{2} \frac{\partial \xi}{\partial x_k} \frac{\partial \xi}{\partial x_k} \right) = 0. \tag{28}
\]

From this last equation we can obtain the expression of \( W \) while Eq. (22) gives the expression of \( \sigma_{jk} \). Finally, we can write the most general solution of Eq. (22) under the form

\[
\sigma_{jk} = c \frac{\partial^2 \ln \rho_n}{\partial x_j \partial x_k} + \delta_{jk} \frac{1}{\rho_n} \int \rho_n \frac{dU(\rho_n)}{d\rho_n} \, d\rho_n, \tag{29}
\]

\[
W = c \left[ \frac{\partial^2}{\partial x^2} \ln \rho_n + \frac{1}{2} \left( \frac{\partial}{\partial x} \ln \rho_n \right)^2 \right] + U(\rho_n). \tag{30}
\]

The above expressions of \( \sigma_{jk} \) and \( W \) define a family of solutions of Eq. (22) and represent the hydrodynamic constitutive equations for the system. Remark that these equations have been enforced exclusively from the fact that the internal stress forces are conservative.

In the above constitutive equations the terms containing \( U \) describe the standard classical Eulerian fluid. For instance the nonlinear potential \( U(\rho_n) = a \rho_n \) appearing in the so called cubic non-linear Schrödinger equation is originated from the stress tensor \( \sigma_{jk} = \frac{1}{2} a \rho_n \delta_{jk} \). Analogously the logarithmic non-linearity \( U(\rho_n) = b \ln \rho_n \) in the Schrödinger equation is originated from a constant stress tensor \( \sigma_{jk} = b \delta_{jk} \).

In the following, we pose \( U = 0 \) and focalize our attention to the term of Eqs. (22) and (24) proportional to the arbitrary constant \( c \), which is absent in classical kinetics. From the definition of the total internal energy, it results that \( H = \int d^3x \rho_n \varepsilon > 0 \). Starting from Eq. (24) and performing an integration by parts, we obtain \( H = -c \int d^3x \left( \frac{\partial \rho_n}{\partial \varepsilon} / (\partial x)^2 / \rho_n \right) \). Then we pose \( c = -\eta^2 / 4 \mu \) in order to put \( H > 0 \). The real positive constant \( \eta \) remains a free parameter of the theory.

We introduce the normalized \( n \)-particle reduced distribution function \( q_n(x) = \rho_n / N \) being \( N = \int \rho_n d^3x \) so that \( \int q_n d^3n = 1 \). Then we define the mass \( m = N \mu \) the internal potential \( W = NW \) and set \( h = N \eta \). From (30) follows the expression of \( W \)

\[
W = -\frac{\hbar^2}{2m} q_n^{1/2} \frac{\partial^2 q_n^{1/2}}{\partial x^2}. \tag{31}
\]

We recall that the potential \( W \) given by Eq. (31) is the quantum potential (Madelung 1926) and it is remarkable that here it has been obtained in the framework of an entirely classical subquantum kinetics. Note that the subquantum monad structure of the system generates the stress forces and then the potential \( W \). Clearly when this structure is suppressed the quantum potential vanishes. Furthermore the presence of the collision integrals \( C_i \) in the BBGKY kinetic Eq. (11) which take into account the monad interactions are necessary for the consistency of the theory.

**Many particle Schrödinger equation:*** We define the external potential \( V = NV \), and set \( S = NS \). Now the quantum fluid is described completely by the two scalar fields \( q_n \) and \( S \) whose evolution equations are (22) and (33), respectively.

\[
\frac{\partial q_n}{\partial t} + \frac{\partial}{\partial x} \left( q_n \frac{1}{m} \frac{\partial S}{\partial x} \right) = 0, \tag{32}
\]

\[
\frac{\partial S}{\partial t} + \frac{1}{2m} \left( \frac{\partial S}{\partial x} \right)^2 + W + V = 0. \tag{33}
\]

Alternatively, we can describe this fluid by means of the complex field \( \Psi_n = q_n^{1/2} \exp(iS/h) \) with \( \int |\Psi_n|^2 d^3n = 1 \), whose evolution equation

\[
\frac{i\hbar}{\partial t} \frac{\partial \Psi_n}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_n}{\partial x^2} + V \Psi_n. \tag{34}
\]
can be obtained directly by combining Eqs. (32) and (33). This later equation is the \( n \)-particle Schrödinger equation which, after recalling \( V = \sum_i V_i \) with \( V_i = N V_i \), assumes the form

\[
\frac{i\hbar}{\partial t} \Psi_n = \sum_{i=1}^{n} \left( -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi_n}{\partial x_i^2} + V_i \Psi_n \right) \quad . \tag{35}
\]

The total energy \( H \) given by Eq. (18) results to be the sum of two terms: \( H = H^{(cl)} + H \). The first term

\[
H^{(cl)} = \sum_{i=1}^{n} \int d^3 x \, \varrho_n \left( \frac{1}{2} m \bar{u}_i^2 + \sum_{i=1}^{n} \int d^3 x \, \varrho_n V_i(x) \right) \quad , \tag{36}
\]

corresponds to the energy of \( n \) interacting classical particles while the second term \( H = \int \varepsilon \varrho_n \, d^3 x \) is originated by the internal structure of the \( n \) bodies and is given by

\[
H = \frac{\hbar^2}{8m} \int d^3 x \, \varrho_n \left( \frac{1}{\varrho} \frac{\partial \varrho}{\partial x} \right)^2 = \frac{\hbar^2}{8m} I \quad . \tag{37}
\]

Eq. (37) hints on the existence of a tight link between quantum mechanics and Fisher information theory. It is remarkable that \( H \) results to be proportional to the Fisher information measure \( I \) (Fisher 1922) which describes an important feature of quantum mechanics [9, 10, 11, 12]. Once again, within the scenario of the subquantum structure of quantum particle, another basic physical concept, namely \( I \), emerges naturally. In the present context \( I \) has a very transparent meaning being simply a measure of the internal energy of quantum system. The fact that \( H \) is originated from the internal structure of the \( n \)-bodies becomes more transparent if we observe that \( H \) can be written also as

\[
H = \sum_{i=1}^{n} \frac{\hbar^2}{8m} \int d^3 x_i \, \varrho_i \left( \frac{1}{\varrho_i} \frac{\partial \varrho_i}{\partial x_i} \right)^2 = \frac{\hbar^2}{8m} \sum_{i=1}^{n} I_i \quad , \tag{38}
\]

from which it results \( I = \sum_i I_i \) and then \( H = \sum_i H_i \). An analogous property holds also for the quantum potential

\[
W_i = -\frac{\hbar^2}{2m} \varrho_i^{-1/2} \frac{\partial^2 \varrho_i^{1/2}}{\partial x_i^2} \quad . \tag{39}
\]

Of course starting from \( H = H^{(cl)} + H \) and taking into account Eqs. (35) and (36) one obtains immediately the well known expression of the Hamiltonian of the \( n \)-particle quantum mechanics

\[
H = \int d^3 x \left( \frac{\hbar^2}{2m} \left( \frac{\partial \Psi_n}{\partial x} \right)^2 + V |\Psi_n|^2 \right) = \sum_{i=1}^{n} H_i \quad (40)
\]

In the present picture the mass of the \( n \)-body quantum system is \( M = nm \). After recalling that \( m = N \mu \) follows immediately that the system is composed by \( N = nN \) monads.

**Conclusions:** The approach used here to obtain the Schrödinger equation for a system of \( n \) quantum particles starting from the BBGKY hierarchy describing a statistical system of \( N = nN \) subquantum monads, suggests that quantum mechanics can be viewed as a hidden variables theory. This approach can be used easily also to construct the non linear quantum mechanics [it is enough to consider \( \mathcal{U} \neq 0 \) in Eqs. (29), (30)] and then for study the non linear effects originated for instance from the particle bosonic or fermionic nature [13, 14].

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