ON A GROUP ASSOCIATED TO $z^2 - 1$

LAURENT BARTHOLDI AND ROSTISLAV I. GRIGORCHUK

Abstract. We construct a group $G$ acting on a binary rooted tree; this discrete group mimics the monodromy action of iterates of $f(z) = z^2 - 1$ on associated coverings of the Riemann sphere.

We then derive some algebraic properties of $G$, and describe for that specific example the connection between group theory, geometry and dynamics.

The most striking is probably that the quotient Cayley graphs of $G$ (aka “Schreier graphs”) converge to the Julia set of $f$.

1. Introduction

The purpose of this paper is to introduce, in a very concrete context, the construction of the iterated monodromy group of a branched covering of a Riemann surface. Our main focus will be on the covering of the Riemann sphere given by the map $z \mapsto z^2 - 1$, whose group $G$ possesses many interesting algebraic and analytic properties. Our main guideline is that these properties derive from dynamic properties of the covering.

We will show:

Theorem 1.1. Let $G$ be the iterated monodromy group (see Definition 2.4) of $z^2 - 1$. Then $G$ acts on the binary rooted tree, and is generated by

$$a = \langle b, 1 \rangle(1, 2), \quad b = \langle a, 1 \rangle.$$

where $\sigma$ permutes the top two branches of $G$, and $\langle x, y \rangle$ describes the automorphism acting respectively as $x$ and $y$ on the left and right subtrees.

1. $G$ is weakly branch (Proposition 3.1);
2. $G$ is a torsion-free group (Proposition 3.4);
3. $G$ contains a free monoid of rank 2, and hence has exponential growth (Proposition 3.5);
4. $G$ does not contain any non-abelian free group (Proposition 3.7);
5. $G$ is not solvable, but every proper quotient of $G$ is (free abelian)-by-(finite 2-group) (Proposition 3.10);
6. $G$ has a presentation (Proposition 3.6)

$$G = \langle a, b \mid [[a^p, b^p], [b^p, a^2], [a^2, a] \rangle \rangle$$

for all $p$ a power of 2).

The Schur multiplier $H_2(G, \mathbb{Z})$ is free abelian of infinite rank.

7. $G$ has solvable word problem (Proposition 3.8);
8. $G$’s spectrum is a Cantor set, and is the intersection of a line and the Julia set of a degree-2 polynomial mapping in $\mathbb{R}^3$ (Proposition 4.1)

$$F(\lambda, \mu, \nu) = (\lambda^2 + 2\lambda
\nu - 2\mu^2, \lambda\nu + 2\nu^2, -\mu^2).$$

Date: October 31, 2018.
9. G's limit space is the Julia set \( \mathfrak{J} \) of \( z^2 - 1 \), and the Schreier graphs \( G/P_n \) converge to \( \mathfrak{J} \) (Proposition I.4).

The group \( G \) has many properties in common with the group \( H \) introduced by Brunner, Sidki and Vieira [BSV99]; the main difference is the existence of free monoids in \( G \), which do not seem to exist in \( H \). We also believe that most proofs for \( G \) are simpler than the analogous statements for \( H \); in particular, the spectrum on \( H \) has not been computed.

2. COVERING MAPS AND GROUPS

This section describes the general construction of a group from a rational map of the Riemann sphere \( \mathbb{C} \).

Let \( f \in \mathbb{C}(z) \) be a branched self-covering of \( \mathbb{C} \). A point \( z \in \mathbb{C} \) is critical if \( f'(z) = 0 \), and is a ramification point if it is the \( f \)-image of a critical point. The postcritical set of \( f \) is \( \{ f^n(z) : n \geq 1, \, z \text{ critical} \} \).

Assume \( P \) is finite, and write \( M = \mathbb{C} \setminus P \). Then \( f \) induces by restriction a self-covering of \( M \).

Let \(*\) be a generic point in \( M \), i.e. be such that the iterated inverse \( f \)-images of \(*\) are all distinct. If \( f \) has degree \( d \), then for all \( n \in \mathbb{N} \) there are \( d^n \) points in \( f^{-n}(*\)\), and these form naturally the \( n \)th layer \( T_n \) of a \( d \)-regular rooted tree \( T = \bigcup_{n \geq 0} T_n \). The root of \( T \) is \(*\), and there is an edge between \( z \) and \( f(z) \) for all \( z \in T \setminus \{ *\} \). In particular, write \( T_1 = f^{-1}(*) \) the first layer of \( T \).

Let \( \gamma \) be a loop at \(*\) in \( M \). Then for all \( v \in T_n \) there is a unique lift \( \gamma_v \) of \( \gamma \) starting at \( v \) such that \( f^n(\gamma_v) = \gamma \); and furthermore the endpoint \( v^\gamma \) of \( \gamma_v \) also belongs to \( T_n \).

**Proposition 2.1.** For any such \( \gamma \) the map \( v \mapsto v^\gamma \) is tree automorphism of \( T \), and depends only on the homotopy class of \( \gamma \) in \( \pi_1(M, *) \).

**Definition 2.2.** The iterated monodromy group \( G_T(f) \) of \( f \) is the subgroup of \( \text{Aut}(T) \) generated by all maps \( v \mapsto v^\gamma \), as \( \gamma \) ranges over \( \pi_1(M, *) \).

This definition is actually independent of the choice of \(*\):

**Proposition 2.3.** Let \(*'\) be another generic basepoint, with tree \( T' \), and choose a path \( p \) from \(*\) to \(*'\). Then there is an isomorphism \( \phi : T \to T' \) such that

\[
\begin{array}{ccc}
\pi_1(M, *) & \overset{p_*}{\longrightarrow} & \pi_1(M, *) \\
\text{act} & & \text{act} \\
G_T(f) & \overset{\phi_*}{\longrightarrow} & G_{T'}(f)
\end{array}
\]

commutes.

We write \( G(f) \) for \( G_T(f) \), defined up to conjugation in \( T \). Abstractly, \( G(f) \) is a quotient of \( \pi_1(M, *) \), a free group of rank \(|P| - 1 |.

2.1. Recurrence. Choose now for each \( v \in T_1 \) a path \( \ell_v \) from \(*\) to \( v \) in \( M \). Consider a loop \( \gamma \) at \(*\); then it induces a permutation of \( T_1 \) given by \( v \mapsto v^\gamma \), and for each \( v \in T_1 \) its lift \( \gamma_v \) at \( v \) yields a loop \( \ell_v \gamma \ell_v^{-1} \) at \(*\), again written \( \gamma_v \).

**Proposition 2.4.** \( \gamma_v \) depends only on the class of \( \gamma \) in \( G(f) \).
We therefore have a natural wreath product embedding
\[ \phi : G(f) \to G(f) \wr \mathfrak{S}_{T_1} \]
mapping \( \gamma \) to \( (v \to \gamma_v, v \to v') \). Enumerating \( T_1 = \{ v_1, v_2, \ldots, v_d \} \), we write
\[ \phi(g) = \langle g_{v_1}, \ldots, g_{v_d} \rangle \pi_g, \]
where \( \pi_g \) is the permutation of \( T_1 \) corresponding to \( g \), in disjoint cycle notation.

Since \( G(f) \) is finitely generated (by at most \( |P| - 1 \) elements), it can be completely described by a permutation of \( d \) points and a sequence of \( d \) elements of \( G(f) \), written as products of generators. We will call this description a wreath presentation.

2.2. Main example. We study now the main example \( f(z) = z^2 - 1 \), and derive an explicit action of \( G = G(f) \) on a “standard” binary rooted tree \( \{ x, y \}^* \).

The postcritical set \( P \) is \( \{ 0, 1, \infty \} \), so \( M \) is a thrice-punctured sphere and \( G \) is 2-generated.

For convenience, pick as base point \( * \) a point close to \((1 - \sqrt{5})/2 \); then \( T_1 = \{ x, y \} \) with \( x \) close to \( * \) and \( y \) close to \(-*\).

Consider the following representatives of \( \pi_1(M, \ast) \)'s generators: \( a \) is a straight path approaching \(-1\), turning a small loop in the positive orientation around \(-1\), and returning to \(*\). Similarly, \( b \) is a straight path approaching \(-1\), turning around \(-1\) in the positive orientation, and returning to \(*\).

Let \( \ell_x \) be a short arc from \(*\) to \( x \), and let \( \ell_y \) be a half-circle above the origin from \(*\) to \( y \).

Let us compute first \( f^{-1}(a) \), i.e. the path traced by \( \pm \sqrt{z + 1} \) as \( z \) moves along \( a \). Its lift at \( x \) moves towards \( 0 \), passes below it, and continues towards \( y \). Its lift at \( y \) moves towards \( 0 \), passes above it, and continues towards \( x \). We have \( a_x = b \) and \( a_y = 1 \), so the wreath presentation of \( a \) is \( \phi(a) = \langle b, 1 \rangle \langle x, y \rangle \).

Consider next \( f^{-1}(b) \). Its lift at \( x \) moves towards \(-1\), loops around \(-1\), and returns to \( x \). Its lift at \( y \) moves towards \( 1 \), loops and returns to \( y \). We have \( b_x = a \) and \( b_y = 1 \), so the wreath presentation of \( b \) is \( \phi(b) = \langle a, 1 \rangle \).

3. Algebraic Properties

We will exploit as much as possible the wreath presentation of \( G = G(z^2 - 1) \) to obtain algebraic information on \( G \). Recall that \( G \) is given with an action on the tree \( T = \{ x, y \}^* \). We write \( xT, yT \) the subtrees below \( x, y \) respectively.

Most of our proofs are close adaptations of \[DSV99\].

Proposition 3.1. \( G \) is weakly branch, i.e. in \( G \)'s action on \( T \)

1. \( G \) acts transitively on each layer \( T_n \);
2. at every vertex \( v \), there is a non-trivial \( g \in G \) acting only on the subtree below \( g \), and fixing its complement.

Proof. \( G \) acts transitively on \( T_1 \), since \( a \) permutes its two elements. Now \( \text{Stab}(x) = \langle a^2, b, b' \rangle \) maps onto \( G \) by restriction of the action to \( xT \); hence by induction acts transitively on each of its layers. Combining with the transitive action on \( T_1 \), we get a transitive action on all layers of \( T \).

Consider next \( G' \). From \([a, b] = \langle a^{-1}b, b^{-1}a \rangle \) we get \( G' \neq 1 \). From the relation \([b, a^2] = \langle [a, b], 1 \rangle \), we see that \( G' \) contains \( G' \times 1 \) as a subgroup acting only on \( xT \). This can be iterated, so \( G \) contains a copy of \( G' \) acting only below the vertex \( v \). \( \square \)
Consider again the embedding $\phi : G \to G/\mathfrak{S}_{x,y}$; define a norm on the generators of $G$ by $\|a\| = \|a^{-1}\| = 1$ and $\|b\| = \|b^{-1}\| = \sqrt{2}$, and extend it by the triangle inequality:

$$\|g\| = \min\{\|s_1\| + \cdots + \|s_n\| : g = s_1 \cdots s_n, s_i \in \{a^{\pm 1}, b^{\pm 1}\}\}.$$ 

**Lemma 3.2.** If $\phi(g) = \langle g_x, g_y \rangle \pi$, then $\|g_i\| \leq \frac{\|g\|+1}{\sqrt{2}}$ for all $i \in \{x,y\}$.

**Proof.** Write in a minimal form $g$ as a product of $N$ letters $a$ (with signs) and $M$ letters $b$ (with signs). Then $g_i$ contains at most $(N + 1)/2$ letters $b$ and $M$ letters $a$.

From now on write $c = [a, b]$.

**Proposition 3.3.** $G' = \langle c, e^a, e^{a^2} \rangle$; we have $G/G' = \mathbb{Z}^2$ generated by $a$ and $b$ and $G'/\langle G' \times G' \rangle = \mathbb{Z}$ generated by $c$.

**Proof.** The first claim follows from computations: $e^b = c$, and $e^{a^3} = (e^{a^2})^{-1} ce^a$.

Next, we have $G' = \langle G' \times G' \rangle(c)$; indeed $e^a = \langle [b, a], 1 \rangle e^{-1}$, and $e^{a^2} = \langle [b, a^{-1}], [b, a] \rangle c$.

Now assume for contradiction that $a^m b^n \in G'$ with $|m| + |n|$ minimal. Then clearly $m$ is even, say $m = 2p$. We have, for some $k \in \mathbb{Z}$ with $|k| \leq |m| + |n|$,

$$a^m b^n = \langle b^p a^n, b^n \rangle = \langle g, h \rangle c^k = \langle g a^k, h a^{-bk} \rangle$$

and therefore $a^{n-k} b^p$ and $a^k b^p$ both belong to $G'$. This contradicts our assumption on minimality, so we have proven the second claim.

Assume finally that $c^k \in G' \times G'$ with $|k|$ minimal. Then $a^k \in G'$ which contradicts the second claim.

**Proposition 3.4.** $G$ is a torsion-free group.

**Proof.** Since $G$ acts on the binary tree, it is residually a 2-group, and its only torsion must be 2-torsion. Assume for contraction that $G$ contains an element $g$ of order 2, of minimal norm.

By Proposition 3.3, $a$ and $b$ are of infinite order. We may therefore assume $\|g\| \geq 2$. If $g$ fixes $x$, then its restrictions $g_x$ and $g_y$ are shorter by Lemma 3.2, and one of them has order 2, contradicting $g$’s minimality.

If $g$ does not fix $x$, then we may write $g = \langle g_x, g_y \rangle a$ for some $g_x, g_y \in G$. We then have $h = g^2 = \langle g_x b g_y, g_y g_x b \rangle = 1$ and therefore $g_y g_x b = 1$. Now for any element $h$ fixing $x$ we have $h_x h_y \in \langle a, b^2, G' \rangle$; this last subgroup does not contain $b$ by Proposition 3.3, so we have a contradiction.

**Proposition 3.5.** The subsemigroup $\{a, b\}^*$ of $G$ is free; therefore $G$ has exponential growth.

**Proof.** Consider two words $u, v$ in $\{a, b\}^*$ that are equal in $G$, and assume $|u| + |v|$ is minimal. We have $u_x = v_x$ and $u_y = v_y$ in $G$, which are shorter relations, so we may assume these words are equal by induction.

Now if $u_x$ and $v_x$ start with the same letter, this implies that $u$ and $v$ also start with the same letter, and cancelling these letters would give a shorter pair of words $u, v$ equal in $G$. \[\square\]
Proposition 3.6. $G$ has finite $L$-presentation

$$G = \langle a, b \mid [a^p, b^p], [b^p, a^{2p}], [a^2, a^{2p}] \rangle \text{ for all } p \text{ a power of } 2.$$  

The Schur multiplier $H_2(G, \mathbb{Z})$ is free abelian of infinite rank.

Proof. This follows from the standard method in [Bar02b], since $G$ is contracting and $G'$ is finitely generated.

We first obtain all relations $[[a^i, b], b]$ for odd $i$, and their iterates under the substitution $a \mapsto b, b \mapsto a^2$ induced by the inclusion of $G' \times 1$ in $G'$.

Then we note that $[[a^i, b], b]$ is a consequence of $[[a, b], b]$ and $[[b, a^2], a^2]$.

The Schur multiplier computation follows from Hopf’s formula $H_2(G, \mathbb{Z}) = (R \cap [F, F])/[R, F]$ for a group $G = F/R$ presented as quotient of a free group. The homomorphism $\sigma$ induces the standard shift on $\mathbb{Z}\infty$.

The method of the following proof is inspired by the PhD thesis of Edmeia Da Silva [Sil01].

Proposition 3.7. $G$ does not contain any non-abelian free subgroup.

Proof. Take $g, h \in G$; we seek a relation $w(g, h)$ between them. Consider the standard metric $|g|$ on words, given by assigning length 1 to each generator $a^\pm 1, b^\pm 1$.

Given any $g \in G$, we have $|g_x| \leq |g|$ and $|g_y| \leq |g|$; these inequalities are strict, unless $g \in B = \{a^{-i}b^na^j \mid n \in \mathbb{N}, i, j \in \{0, 1\}\}$. We also have $|g_xg_y| \leq |g|$.

We define the following ordering on pairs $(g, h)$ of group elements: Write $H = \text{Stab}_G(x)$ the stabilizer of $x$. The height of $(g, h)$ is $L(g, h) = |g| + |h| - \frac{1}{2}\#(g, h) \cap H$; that is, we order first by total length, and then by number of elements in $(g, h)$ fixing $x$.

The basis of the induction is that for $(g, h)$ of height at most 2 these elements must be generators, and we have the relation, say, $[[a, b], b]$ among them.

If $g, h \in H$, they may be written as $g = \ll g_x, g_y \gg$ and $h = \ll h_x, h_y \gg$. Then either one of $g_x, h_x$ is not in $B$, so by induction there is a relation $u(g_x, h_x)$; or we consider the cases $g_x = b^i, h_x = b^j$ and $g_x = a^{-1}b^na^j, h_x = a^{-1}b^ja$ which have a common power; and up to symmetry $g_x = a^{-1}b^ja, h_x = b^j$ when we have the relation $[g_x, h_x]$. By the same argument, there is a relation $v(g_y, h_y)$. Consider now relation $w = [u, v]$; we have $w(g, h) = [1, u(g_y, h_y), v(g_x, h_x), 1] = 1$.

If at least one, say $g$, fixes $x$, we may write $h = \ll h_x, h_y \gg(x, y)$, and may proceed as above with $g$ and $h^2$, since $L(g_x, h_xh_y) < L(g, h)$ and $L(g_y, h_xh_y) < L(g, h)$.

If none of $g, h$ fixes $x$, we write $g = \ll g_x, g_y \gg(x, y)$ and $h = \ll h_x, h_y \gg(x, y)$, and consider the elements in the following table:

| $g_x \in H$ | $g_y \in H$ | $g_y \notin H$ | $g_x \notin H$ | $g_y \in H$ | $g_y \notin H$ |
|-------------|-------------|-------------|-------------|-------------|-------------|
| $h_x \in H$ | $g^2, h^2$   | $g^2, h^2$   | $g^2, h^2$   | $g^2, h^2$   | $g^2, h^2$   |
| $h_y \notin H$ | $g^2, h^2$   | $gh^{-1}, h^{-1}$ | $gh, h^2$   | $g^2, h^2$   |
| $h_x \notin H$ | $g^2, h^2$   | $gh, h^2$   | $gh^{-1}, h^{-1}$ | $g^2, h^2$   |

By inspection, in the pair of elements $e, f$ corresponding to $g, h$ at both fix $x$, and their respective projections satisfy $L(e_x, f_x) < L(g, h)$ and $L(e_y, f_y) < L(g, h)$; therefore they satisfy relations $u, v$, and $e, f$ satisfy the relation $[u, v]$. 

Proposition 3.8. The word problem is solvable in $G$.

Proof. The solution to the word problem follows from Lemma 3.2: indeed take a word $w$ of length $n$. If it does not fix $x$, then it is not trivial. Otherwise its projections $w_x, w_y$ are shorter words, so can inductively be checked for triviality in $G$. Then $w$ is trivial in $G$ if and only if $w_x$ and $w_y$ are both trivial in $G$. 

Write now $d = [c, a]$ and $e = [d, a]$.

Proposition 3.9. We have $G'' = (\gamma_3 \times \gamma_3)$, and $\gamma_3 = (\gamma_3 \times \gamma_3)/(d, e)$.\rangle

In the lower central series of $G$ we have $\gamma_1/\gamma_2 = \mathbb{Z}^2$, $\gamma_2/\gamma_3 = \mathbb{Z}$, and $\gamma_3/\gamma_4 = \mathbb{Z}/4$. Therefore all quotients except the first two in the lower central series are finite.

In the lower 2-central series defined by $G_1 = G$ and $G_{n+1} = [G, G_n]G_{n/2}^2$, we have

$$\dim_{\mathbb{F}_2}\Gamma_n/\Gamma_{n+1} = \begin{cases} i + 2 & \text{if } n = 2^i \text{ for some } i; \\ \max\{i + 1 | 2^i \text{ divides } n\} & \text{otherwise}. \end{cases}$$

Proof. We moreover claim we have the following generating sets:

- $\gamma_1 = (a, b)$;
- $\gamma_2 = (c = [a, b] = (a, a^{-b}), c^{-1}a = (c, 1), c^{-a^{-1}-1} = (1, c))$;
- $\gamma_3 = (d = [c, a], e = [d, a], [e^{-1}, b] = (d, 1), (e, 1), (1, d), (1, e))$;
- $\gamma_4 = (d^4, e, (d, 1), (e, 1), (1, d), (1, e))$.

Most follow from simple computations; let us justify $d^4 \in \gamma_4$. Since it is clear that $\gamma_4$ contains $e$, we denote by $\equiv$ congruence modulo $e$, which is weaker than congruence modulo $\gamma_4$. We have

$$d^2 \equiv d^2e = b^{-1}a^{-1}ba^{-2}b^{-1}aba^2 = b^{-1}a^{-b}b^{-1}a^{-1}ba^2 \text{ modulo } [a^2b, a^2] = 1$$

$$= (d^2e)^{-a^{-b}} \equiv d^{-2}.$$

We have already computed $G/G' = \mathbb{Z}^2$ in Proposition 3.3 and $G'/\gamma_3 = \mathbb{Z}$ is generated by $c$. This last computation gives $\gamma_3/\gamma_4 = \mathbb{Z}/4$.

Next, $G''$ is generated as a normal subgroup by $[c, (e, 1)] = (d, 1)$, so $G'' = \gamma_3 \times \gamma_3$.

For the 2-central series see [Bar02a], where the argument is developed for the BSV group.

Proposition 3.10. $G$ is not solvable, but all its quotients are (free abelian)-by-(finite 2-group).

Proof. By Proposition 3.9, we have $G'' > \gamma_3 \times \gamma_3$, so $G'' > G'' \times G''$, and hence $G^{(n)} > G^{(n-1)} \times G^{(n-1)}$ for all $n$. Assume for contradiction that $G$ is solvable; this means $G^{(n)} = 1$ for some minimal $n$, a contradiction with the above statement.

Now consider a non-trivial normal subgroup $N$ of $G$. By [Gri00, Theorem 4], we have $(\gamma_3)^{2^n} < N$ for some $n$. On the other hand, $G$ is an abelian-by-(finite 2) extension of $(\gamma_2)^{2^n}$. The result follows.
4. Geometric and Analytic Properties

In this part we isolate results on the geometry of the action of $G$ on the boundary of the tree. The inspiration for this part is the paper [BG00] where computations of spectra were performed, and [BGN02] where the relation between the Schreier graphs of an i.m.g. group and the Julia set of its polynomial were explicit.

Consider first the spectrum computation. Given a group $G$ with generating set $S$ and a unitary representation $\pi$ on a Hilbert space $\mathcal{H}$, its spectrum is

$$ \text{spec}(\pi) = \frac{1}{|S|} \text{spec}_{B(\mathcal{H})}\left(\sum_{s \in S} \pi(s)\right). $$

It is a closed subset of $\mathbb{C}$; if $S$ is symmetric, then the spectrum is always contained in $\mathbb{R}$.

The main representation of interest is the left-regular representation $\rho$ on $\ell^2(G)$. A result of Kesten [Kes59] shows that 1 is in $\text{spec}(\rho)$ if and only if $G$ is amenable.

Here we concentrate on the “natural” representation $\pi$ of $G$ on $L^2(\partial T, \mu)$, where $\partial T$ is the boundary of the tree $T$. We denote by $\pi_n$ the representation of $G$ on the $n$th layer $T_n$.

Introduce for $n \geq 0$ the following homogeneous polynomials:

$$ Q_n(\lambda, \mu, \nu) = \det \left( \lambda + \mu (\pi_n a + \pi_n a^{-1}) + \nu (\pi_n b + \pi_n b^{-1}) \right). $$

Then the spectrum of $\pi_n$ is obtained by solving $Q_n(\lambda, -\frac{\mu}{4}, -\frac{\nu}{4}) = 0$.

We are not able to derive a closed form for $Q_n$; however we have the

**Proposition 4.1.** Define the polynomial mapping $F : \mathbb{R}^3 \to \mathbb{R}^3$ by

$$ (\lambda, \mu, \nu) \mapsto (\lambda^2 + 2\lambda\nu - 2\mu^2, \lambda\nu + 2\nu^2, -\mu^2). $$

Then $Q_n$ is given by

$$ Q_0(\lambda, \mu, \nu) = \lambda + 2\mu + 2\nu; $$

$$ Q_1(\lambda, \mu, \nu) = Q_0(\lambda, \mu, \nu) \cdot (\lambda - 2\mu + 2\nu); $$

and for $n \geq 1$,

$$ Q_{n+1}(\lambda, \mu, \nu) = Q_n(F(\lambda, \mu, \nu)). $$

Define $K$ as the closure of the set of all backwards $F$-iterates of $\{Q_1 = 0\}$. Then the spectrum of $\pi$ is the intersection of $\{\mu = \nu = -\frac{\mu}{4}\}$ with $K$.

**Proof.** The first two follow from direct computation, using $\pi_0(a) = \pi_0(b) = (1)$, and

$$ \pi_{n+1}(a) = \begin{pmatrix} 0 & \pi_n(b) \\ 1 & 0 \end{pmatrix}, \quad \pi_{n+1}(b) = \begin{pmatrix} \pi_n(a) & 0 \\ 0 & 1 \end{pmatrix}. $$

Next, we compute, writing $a, b$ for $\pi_n(a), \pi_n(b)$ respectively,

$$ Q_{n+1}(\lambda, \mu, \nu) = \det \begin{pmatrix} \lambda + \mu (a + a^{-1}) & \mu(1 + b) \\ \mu(b^{-1} + 1) & \lambda + 2\nu \end{pmatrix} $$

$$ = \det ((\lambda + \nu(a + a^{-1})(\lambda + 2\nu) - \mu^2(1 + b)(b^{-1} + 1)) $$

$$ = Q_n(\lambda^2 + 2\lambda\nu - 2\mu^2, \lambda\nu + 2\nu^2, -\mu^2). $$

The claim follows.
It seems difficult to obtain a more explicit description of \( \text{spec}(\pi) \), since the polynomials \( Q_n \) do not factor well. For instance, \( Q_5 \) has an irreducible factor of degree 5, and \( Q_5(\lambda, -\frac{1}{4}, -\frac{1}{4}) \) has Galois group \( S_5 \). Figure 4 displays the spectrum of \( \pi_6 \) and its associated spectral measure; from there we see that the spectrum of \( \pi \) is a Cantor set.

Now we turn our attention to the Schreier graphs of \( G \). These are graphs \( \mathcal{G}_n \) with \( 2^n \) points corresponding to the vertices in \( T_n \), with edges labelled \( a \) and \( b \) connecting \( v \) to \( v^a, v^b \) respectively for all \( v \in T_n \).

These graphs are therefore 4-regular graphs; the Green function of the random walk on \( \mathcal{G}_n \) is given by the integral of \( \frac{1}{1-t} \) with respect to the spectral measure given above. The striking fact is the following:

**Proposition 4.2.** The graphs \( \mathcal{G}_n \) are planar, and can be drawn in \( \mathbb{C} \) in such a way that they converge in the Hausdorff metric to the Julia set \( J \) of \( z^2 - 1 \).

**Proof.** This follows from [BGN02], since \( z^2 - 1 \) is hyperbolic.

\( \mathcal{G}_n \) is constructed as follows: it is built of two parts \( A_n, B_n \) connected at a distinguished vertex. Each of these is 4-regular, except at the connection vertex where each is 2-regular, and \( A_n \) contains only the \( a^{\pm 1} \)-edges while \( B_n \) contains only the \( b^{\pm 1} \)-edges.

\( A_0 \) and \( B_0 \) are the graphs on 1 vertex with a single loop of the appropriate label.
If $n = 2k$ is even, then $B_{2k+1} = B_{2k}$, and $A_{2k+1}$ is obtained by taking an $a$-labelled $2^{k+1}$-gon $v_0, \ldots, v_{2k+1-1}$, and attaching to each $v_i$ with $i \neq 0$ a copy of $B_{2j}$ where $j$ is the largest power of 2 dividing $i$. Its distinguished vertex is $v_0$.

If $n = 2k - 1$ is odd, then $A_{2k} = A_{2k-1}$, and $B_{2k}$ is obtained by taking an $b$-labelled $2^k$-gon $v_0, \ldots, v_{2^k-1}$, and attaching to each $v_i$ with $i \neq 0$ a copy of $A_{2j+1}$ where $j$ is the largest power of 2 dividing $i$. Its distinguished vertex is $v_0$.

The first Schreier graphs $\mathfrak{G}_n$ of $G$ are drawn in Figure 4. Compare with the Julia set in Figure 5.

References

[Bar02a] Laurent Bartholdi, The 2-dimension series of the just-nonsolvable bsv group, preprint, 2002.
[Bar02b] Laurent Bartholdi, L-presentations and branch groups, to appear in J. Algebra, 2002.
[BG00] Laurent Bartholdi and Rostislav I. Grigorchuk, On the spectrum of Hecke type operators related to some fractal groups, Trudy Mat. Inst. Steklov. 231 (2000), 5–45, math.GR/9910102.
[BGN02] Laurent Bartholdi, Rostislav I. Grigorchuk, and Volodymyr V. Nekrashevych, *From fractal groups to fractal sets*, submitted, 2002.

[BSV99] Andrew M. Brunner, Said N. Sidki, and Ana Cristina Vieira, *A just nonsolvable torsion-free group defined on the binary tree*, J. Algebra **211** (1999), no. 1, 99–114.

[Gri00] Rostislav I. Grigorchuk, *Just infinite branch groups*, New horizons in pro-p groups (Markus P. F. du Sautoy Dan Segal and Aner Shalev, eds.), Birkhäuser Boston, Boston, MA, 2000, pp. 121–179.

[Kes59] Harry Kesten, *Full Banach mean values on countable groups*, Math. Scand. **7** (1959), 146–156.

[Sil01] Edmídia Fernandes da Silva, *Uma família de grupos quase não-solúveis definida sobre árvores n-árias, n ≥ 2*, Ph.D. thesis, Universidade de Brasília, 2001.