The stability of the $b$-family of peakon equations

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Abstract

In the present work we revisit the $b$-family model of peakon equations, containing as special cases the $b = 2$ (Camassa–Holm) and $b = 3$ (Degasperis–Procesi) integrable examples. We establish information about the point spectrum of the peakon solutions and notably find that for suitably smooth perturbations there exists point spectrum in the right half plane rendering the peakons unstable for $b < 1$. We explore numerically these ideas in the realm of fixed-point iterations, spectral stability analysis and time-stepping of the model for the different parameter regimes. In particular, we identify exact, stationary (spectrally stable) lefton solutions for $b < -1$, and for $-1 < b < 1$, we dynamically identify ramp-cliff solutions as dominant states in this regime. We complement our analysis by examining the breakup of smooth initial data into stable peakons for $b > 1$. While many of the above dynamical features had been explored in

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earlier studies, in the present work, we supplement them, wherever possible, with spectral stability computations.

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(Some figures may appear in color only in the online journal)

1. Introduction

The family of partial differential equations:

$$u_t - u_{ttx} + (b + 1)uu_x = bu_xu_{xx} + uu_{xxx},$$

labelled by the parameter \( b \), is distinguished by the fact that it includes two completely integrable equations, namely the Camassa–Holm equation (the case \( b = 2 \) [1, 2]), and the Degasperis–Procesi equation (the case \( b = 3 \) [3, 4]). Each of the two integrable cases has a Lax pair (and is, thus, solvable via the inverse scattering transform), possesses multi-soliton solutions, and a bi-Hamiltonian structure [2, 5–7]. Furthermore, the cases \( b = 2, 3 \) have been singled out by various tests of integrability: The Wahlquist–Estabrook prolongation method, the Painlevé analysis, symmetry conditions, and a test for asymptotic integrability [3, 8–10].

The Camassa–Holm equation was originally proposed as a model for shallow water waves [1, 2]. The results of [11, 12] (see proposition 2 of [11] and equation (3.8) of [12]) show that, in a model of shallow water, the solution \( u \) of equation (1) corresponds to the horizontal component of velocity evaluated at some specific level in the cases \( b \geq 10/11 \) or \( b \leq -10 \). However, there is some debate about the precise range of validity of such models [13].

What makes the $b$-family particularly interesting to study from a mathematical physics viewpoint is that its members share the one-peakon solutions:

$$u = u_0 = c \exp(-|x - ct|),$$

that are admitted by the Camassa–Holm, and the Degasperis–Procesi equations. Indeed, the peakons solve the following weak formulation of equation (1):

$$u_t = \frac{1}{2} \left( \phi * \left[ \frac{b - 3}{2} u_x^2 - \frac{b}{2} u^2 \right] - u_t^2 \right)_x, \quad \phi = e^{-|x|},$$

where $*$ denotes convolution; the fact that $\phi/2$ is a Green’s function for the operator $1 - \partial_x^2$ was used in the reformulation. In effect, equation (3) is obtained from equation (1) by factoring out the operator $1 - \partial_x^2$.

Moreover, the whole $b$-family possesses $N$-peakon solutions [1, 2, 4, 14, 15] given by:

$$u(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x - q_j(t)|},$$

where the positions $q_j$ and amplitudes $p_j$ are the canonically conjugate coordinates and momenta in a finite-dimensional Hamiltonian system. In the cases $b = 2, 3$, this Hamiltonian system is completely integrable in the Liouville–Arnold sense. In the general case, the Hamiltonian system does not appear to be integrable [16]. Recently, the $b$-family was generalized to an equation containing two free functions with the property that it also admits multi-peakon solutions written as a linear combination of one-peakons [17].
Another interesting aspect of the $b$-family in the cases $b = 2, 3$ is that they admit smooth multi-soliton solutions on a nonzero background [1, 2, 5]. In the limit where the background goes to zero, the $N$-soliton solutions become the $N$-peakons solution as given in equation (4). For general $b$, only smooth one-solitons on nonzero background are known to exist [18].

The work of [19, 20] presented a numerical study of the solutions of equation (1) for different values of $b$. They observed that there are three distinct parameter regimes separated by bifurcations at $b = 1$ and $b = -1$, as follows:

- **Peakon regime**: For $b > 1$, arbitrary initial data asymptotically separates out into a number of peakons as $t \to \infty$.
- **Ramp-cliff regime**: For $-1 < b < 1$, solutions behave asymptotically like a combination of a ‘ramp’-like solution of Burgers equation (proportional to $x/t$), together with an exponentially-decaying tail (‘cliff’).
- **Lefton regime**: For $b < -1$, arbitrary initial data moves to the left and asymptotically separates out into a number of ‘leftons’ as $t \to \infty$, which are smooth, exponentially localized, stationary solitary waves.

The behaviour observed separately in each of the parameter ranges $b > 1$ and $b < -1$ can be understood as particular instances of the soliton resolution conjecture [21], a somewhat loosely defined conjecture which states that for suitable dispersive wave equations, solutions with ‘generic’ initial data will decompose into a finite number of solitary waves plus a radiation part which disperses away. Hone and Lafortune [22] provide a first step towards explaining this phenomenon analytically in the ‘lefton’ regime $b < -1$. Indeed, they show that in this parameter range a single lefton solution is orbitally stable, by applying the approach of Grillakis et al [23]. The main ingredients required for the stability analysis are the Hamiltonian structure and conservation laws for equation (1). The $b$-family is known to admit a Hamiltonian structure and two additional conservation laws [24]. In order to present the three conserved quantities, we rewrite the $b$-family (1) as a nonlocal evolution equation for the ‘momentum variable’ $m$ as:

$$m_t + um_x + bu_x m = 0, \quad m := u - u_{xx}. \quad (5)$$

Using equation (5), it is straightforward to verify that, for any value of $b \neq 0, 1$, there are at least three different functionals that are formally conserved by the time evolution of $m$ [24]. These are the Hamiltonian $E$ and two other functionals $C_1$ and $C_2$ given by:

$$E = \int m \, dx, \quad C_1 = \int m^{1/b} \, dx, \quad (6)$$

and,

$$C_2 = \int m^{-1/b} \left( \frac{m_x^2}{b^2 m^2} + 1 \right) \, dx. \quad (7)$$

Hone and Lafortune [22] use the fact that the lefton solutions are a critical point for a functional, which is a linear combination of the Hamiltonian $E$ and the conserved functional $C_2$.

In this article, our goal is to study the spectral stability of the peakon solutions (cf equation (2)). In particular, we are interested in the observation made numerically by Holm and Staley [19, 20] that the peakon solutions become unstable when $b < 1$. To do so, in section 2, we state the main analytical results concerning the spectrum associated to the eigenvalue problem arising from the linearization of equation (3) about the peakon solutions. These analytical
results are proven in section 3. The numerical results on the \( b \)-family (cf equation (1)) are presented in section 4. We explore both statically as appropriate, as well as dynamically, each of the classes of solutions therein. We examine their existence over parametric variations of \( b \), when possible/relevant (e.g. for the leftons) we consider their stability and we also explore their dynamics (especially for the ramp-cliff waveforms for which we cannot identify a reference frame in which they appear as steady). In section 5, we state our conclusions and present directions for future study.

2. Main results

The spectral stability of the peakon solution is explored by first considering equation (3) in the variables \( t \) and \( \xi = x - ct \):

\[
\begin{align*}
u_t - cu\xi &= \frac{1}{2} \left( \phi * \left[ \frac{b - 3}{2} u_{\xi}^2 - \frac{b}{2} u^2 \right] - u^2 \right)_{\xi}, \quad \phi = e^{-|\xi|}.
\end{align*}
\]

The equation above is invariant under the transformation:

\[
\begin{align*}
u \to -\nu, \quad \xi \to -\xi, \quad c \to -c.
\end{align*}
\]

Thus, as a consequence, any solution of (8) for negative \( c \) shares the stability properties of the corresponding (positive) solution with \( c > 0 \) obtained through the transformation (9). We can thus restrict ourselves to the case where \( c \) and the solution profile is sign-definite (here, assumed to be positive). This is an approach that is common in the class of unidirectional equations in the general family of Korteweg–de Vries models. In that context of unidirectional models for shallow water waves, the \( b \)-family describes the horizontal fluid velocity at a certain depth \([1, 2, 11, 12]\). Since \( c \) is positive, the scaling

\[
\begin{align*}
u \to cu, \quad t \to ct,
\end{align*}
\]

can be performed in order to set \( c = 1 \). Equation (8) now reads:

\[
\begin{align*}
u_t - u_{\xi} &= \frac{1}{2} \left( \phi * \left[ \frac{b - 3}{2} u_{\xi}^2 - \frac{b}{2} u^2 \right] - u^2 \right)_{\xi}, \quad \phi = e^{-|\xi|},
\end{align*}
\]

with one-peakon solution now given by:

\[
\begin{align*}
u = \phi = e^{-|\xi|}.
\end{align*}
\]

Now consider a small perturbation of the peakon of the form:

\[
\begin{align*}w(\xi, t) = v(\xi)e^{\lambda t},
\end{align*}
\]

where \( v(\xi) \) stands for the eigenvector associated with the eigenvalue \( \lambda \). Then, we substitute \( u = \phi + w \) into equation (3) and linearize by keeping only the first-order terms in \( v \). This way, we obtain the following eigenvalue problem associated to an integral operator \( L \):

\[
\begin{align*}
\lambda v = Lv \equiv \left( \phi * \left[ \frac{b - 3}{2} \phi' v' - \frac{b}{2} \phi v \right] + (1 - \phi) v \right)' - \phi e^{-|\xi|},
\end{align*}
\]

where the prime denotes derivative with respect to \( \xi \). For our analytical study, we are interested in the spectrum of \( L \) defined above.

To define an appropriate domain for the operator \( L \), we need to consider well-posedness of the \( b \)-family equation (3). The \( b \)-family is known to be well-posed for initial conditions in \( H^s(\mathbb{R}) \), \( s > 3/2 \) [25–31]. It is also known to be ill-posed for \( s < 3/2 \) when \( b \neq 1 \) [14, 15]. The peakons in the Camassa–Holm (\( b = 2 \)) are proven to be stable in \( H^1(\mathbb{R}) \) [32, 33] while the ones
in the Degasperis–Procesi \((b = 3)\) in \(L^2(\mathbb{R})\) \([34]\). However, due to the discussion above about well-posedness, Constantin and Strauss \([32]\) and Lin and Liu \([34]\) state that their stability results only apply to initial condition that are in the subsets \(H^s(\mathbb{R})\), \(s > 3/2\), of \(H^1(\mathbb{R})\) (for Camassa–Holm) or \(L^2(\mathbb{R})\) (for Degasperis–Procesi).

When considering the orbital stability of the peakon solutions \((12)\), we are thus interested in perturbed solutions \(u(\xi, t)\) with initial conditions of the form:

\[
u(\xi, t = 0) = \phi(\xi + \epsilon) + p_0(\xi) \in H^s(\mathbb{R}), s > 3/2,
\]

with \(p_0\) small enough in the chosen function space. In \((14)\), \(\epsilon\) is introduced to take into account a drift along the translation invariance symmetry direction. Since \(\phi\) itself is in \(H^s(\mathbb{R})\), for all \(s < 3/2\), we have that, from \((14)\),

\[
\phi(\xi + \epsilon) = u(\xi, t = 0) - p_0(\xi) \in H^s(\mathbb{R}), \text{for all } s < 3/2.
\]

Hence, a necessary condition for the initial conditions \(u(\xi, t = 0)\) \((14)\) to be in \(H^s(\mathbb{R})\), for some \(s > 3/2\) is that \(p_0\) be in \(H^s(\mathbb{R})\), for all \(s < 3/2\). Thus, at the linear level, we will look for eigenvectors of the form:

\[
v = \phi'(\xi) + p_1(\xi),
\]

for \(p_1 \in H^s(\mathbb{R})\), for all \(s < 3/2\). However, \(v = \phi'\) is discontinuous and thus not in the domain of \(L\) as defined in equation \((13)\) due to the term \(\phi'v'\). In the next section (see equation \((36)\)), we will define an extension \(\mathcal{L}_w\) of the operator \(L\) given in equation \((13)\) that admits discontinuous functions in its domain. For \(\mathcal{L}_w\), we will be interested in eigenfunctions in the set:

\[
A = \left\{ \phi'(\xi) + p_1(\xi) \left| p_1 \in H^s(\mathbb{R}), \text{for all } s < 3/2 \right. \right\}.
\]

The extension \(\mathcal{L}_w\) is not a closed operator on \(L^2(\mathbb{R})\) and thus its resolvent set is automatically empty (see for example \([35]\)). However, we show that \(\mathcal{L}_w\) is closed on the Banach space \(L^2(\mathbb{R}) \cap C_d(\mathbb{R})\) (see lemma \(3.4\)), where \(C_d(\mathbb{R})\) is the set of bounded functions that are continuous except at the origin, where the functions are allowed to have a finite jump discontinuity (see equation \((38)\)). In section \(3\), we prove the following theorem about the point spectrum of \(\mathcal{L}_w\):

**Theorem 2.1.** The linear operator \(\mathcal{L}_w\) defined in equation \((36)\) is closed on \(L^2(\mathbb{R}) \cap C_d(\mathbb{R})\) and its point spectrum consists of the origin \(\lambda = 0\) and, if \(b < 2\), of the two bands defined by \(0 < |\Re(\lambda)| < 2 - b\). If we restrict the eigenfunction to the set \(A\) defined in equation \((16)\), the band of point spectrum is reduced to \(0 < |\Re(\lambda)| \leq 1 - b\) when \(b < 1\).

The second statement within the theorem \(2.1\) above provides an explanation for the observation made numerically by Holm and Staley \([19, 20]\) that the peakon solutions are unstable when \(b < 1\). Remark \(3.6\) below illustrates the fact that if the chosen space is made of functions with more regularity than the ones in \(L^2(\mathbb{R}) \cap C_d(\mathbb{R})\), the width of the band obtained in the first part of theorem \(2.1\) decreases. Actually, equation \((47)\) shows that with enough regularity, the two bands in theorem \(2.1\) can be made as close as one wants to each other. A spectrum consisting of a strip about the imaginary axis also occurs in the study of the peaked periodic wave of both versions of the reduced Ostrovsky equations \([36]\). Although our solutions are not periodic, the nature of the result is similar.

In what follows, we also explore numerically the waveforms of the model for different values of \(b\). We identify the leftons as stationary solutions for \(b < -1\) and illustrate their potential spectral stability. We dynamically examine the ramp-cliff solutions for \(-1 < b < 1\) and show that progressively refined computations (involving more modes) suggest that the ramp-cliff
solutions deform into emitting peakons close to $b = 1$ (the more refined the computations, the closer to $b = 1$ this phenomenology arises). Beyond $b = 1$ in line with the theory above, we find that initial data breaks up spontaneously into arrays of peakons that appear to be dynamically robust. A complementary perspective that we provide to avoid issues with the discontinuity of the peakons involves the stability analysis of the solutions of non-vanishing background, as they approach the vanishing background (i.e. peakon) limit.

3. Computation of the point spectrum

In this section, we compute the point spectrum of equation (13) for values of $\lambda$ such that $\text{Re}(\lambda) \geq 0$. The case where $\text{Re}(\lambda) < 0$ can be obtained from the spectrum of the right side of the complex plane by making the observation that if $v(\xi)$ solves the eigenvalue problem of equation (13) for a given value of $\lambda = \lambda_0$, then $v(-\xi)$ solves that same eigenvalue problem with $\lambda = -\lambda_0$.

We first show that the operator $L$ defined in equation (13) does not have continuous eigenvectors.

**Proposition 3.1.** The eigenvalue problem of equation (13) does not have solutions in $H^1(\mathbb{R})$.

**Proof.** Consider the problem of equation (13) for $\xi < 0$. We apply the operator $1 - \partial^2_\xi$ to obtain the following differential equation:

$$\lambda(v - v'') + (v'' - v + b e^{\xi} v + (1 - b) e^{2\xi} v')' = 0,$$

where we have used the fact that $\phi/2$ is a Green’s function for the operator $1 - \partial^2_\xi$. It turns out there are two solutions to equation (17) converging as $\xi \to -\infty$, one as $e^{\xi}$ and one as $e^{N\xi}$. There is also a solution diverging as $e^{-2\xi}$. These decay and growth rates are found by solving the constant coefficient asymptotic system obtained by applying the limit $\xi \to -\infty$ to equation (17). Actually, equation (17) admits the two explicit solutions $v_1 = e^{-\xi}$ and $v_2 = e^{\xi}$. The third solution behaving as $e^{N\xi}$ as $\xi \to -\infty$ cannot be written in such a simple way and we will use its known properties rather than an explicit formula when dealing with it.

It should be noted that $\xi = 0$ in equation (17) is a regular singular point with exponents:

$$r_1 = 0, \quad r_2 = 1, \quad \text{and} \quad r_3 = -\lambda + 2 - b.$$  

A third solution $v_3$, linearly independent of $v_1$ and $v_2$, that is not singular at $\xi = 0$, can thus be found if we assume $\text{Re}(r_3) > 0$. It can be defined by its series expansion about $\xi = 0$:

$$v_3 = \begin{cases} \xi^{|r_3|} + O(\xi^{|r_3|+1}) & \text{if } r_3 \neq 1, \\ \xi \ln|\xi| + O(\xi^2 \ln|\xi|) & \text{if } r_3 = 1, \end{cases}$$

where $r_3$ is given in (18). Upon adding an appropriate multiple of $e^{\xi} - e^{-\xi}$ to $v_3$, the solution converges as $\xi \to -\infty$ and is zero at $\xi = 0$. Indeed, let $v = F$ be the solution of equation (17) defined as:

$$F = v_3 - C(e^{-\xi} - e^{\xi}), \quad \text{where } C \equiv \lim_{\xi \to -\infty} e^{\xi} v_3(\xi).$$

Then $F$ is such that $F(0) = 0$ and $F \to 0$ as $\xi \to -\infty$. The specific value of the constant $C$ is irrelevant in the rest of the discussion. Equation (20) is used to show that one can define a solution $F$ that has the properties that $F(0) = 0$ and $F \to 0$ as $\xi \to -\infty$, both of which are what
is needed for our purposes. Nevertheless, we can write $F$ as an integral and find a formula for $C$ in the following way. In the case where $b + \Re(\lambda) < 1$, the solution $\nu_3$ can be written as:

$$\nu_3 = \int_{0}^{\xi} (e^{\xi - y} - e^{\xi})e^{\lambda y}(1 - e^{y})^{-b - \lambda}dy.$$ 

Then we compute $C$ from (20) as:

$$C = -\int_{0}^{-\infty} e^{(\lambda + 1)y}(1 - e^{y})^{-b - \lambda}dy,$$

and get,

$$F = \nu_3 + \left(\int_{0}^{-\infty} e^{(\lambda + 1)y}(1 - e^{y})^{-b - \lambda}dy\right)(e^{-\xi} - e^{-\xi}).$$

(21)

It can be verified that indeed, $F(0) = \lim_{\xi \to -\infty} F(\xi) = 0$. The formula above only works for $b + \Re(\lambda) < 1$ because the integral in (21) does not converge otherwise due to the behaviour of the integrand at $y = 0$ for $\xi \neq 0$. To extend equation (21) to the weaker condition $b + \Re(\lambda) < 2$ (i.e. $r_3 > 0$), one splits the interval of integration in (21) into $(0, \xi)$ and $(\xi, -\infty)$. We combine the integral on the interval $(0, \xi)$ with the integral defining $\nu_3$ in (21) and obtain:

$$F = e^{\xi} \int_{0}^{\xi} (e^{\gamma} - e^{\xi})e^{\lambda y}(1 - e^{y})^{-b - \lambda}dy$$

$$+ \left(\int_{\xi}^{-\infty} e^{(\lambda + 1)y}(1 - e^{y})^{-b - \lambda}dy\right)(e^{-\xi} - e^{-\xi}).$$

(22)

The expression above provides a solution $v = F$ of equation (17) that is zero at $\xi = 0$ and $\xi \to -\infty$ valid for $b + \Re(\lambda) < 2$.

Applying the operator $1 - \partial_{\xi}^2$ to the eigenvalue problem of equation (13) for $\xi > 0$, one obtains a differential equation with only one converging solution as $\xi \to \infty$ given by $v = e^{-\xi}$. Hence, to look for a solution to equation (13) that is bounded, in the case where $\Re(r_3) > 0$, one considers:

$$v = \begin{cases} c_0 e^{-\xi} & \text{for } \xi > 0, \\ c_1 F(\xi) + c_2 e^{\xi} & \text{for } \xi < 0. \end{cases}$$

(23)

Since we look for a continuous solution, and because $F(0) = 0$, we need to take $c_0 = c_2$. The most general ansatz in this case is:

$$\nu_c = c_1 v_1 + c_2 v_2, \text{where } v_1 \equiv H(-\xi)F(\xi) \text{ and } v_2 \equiv e^{-|\xi|},$$

(24)

with $H$ being the Heaviside function.

**Lemma 3.2.** If we substitute $v = \nu_c$ into equation (13), one obtains:

$$\mathcal{L}\nu_c - \lambda \nu_c = \left(c_1 \frac{3(b - 2)}{2} \int_{-\infty}^{0} e^{2\xi} F(\xi')d\xi' + c_2 (\text{sgn}(\xi) - \lambda) e^{-|\xi|}\right) e^{-|\xi|}. \quad (25)$$

**Proof.** By substituting $v = v_2 = e^{-|\xi|}$ into equation (13), it is a straightforward computation to find that:

$$\mathcal{L}v_2 - \lambda v_2 = (\text{sgn}(\xi) - \lambda) e^{-|\xi|}. \quad (26)$$
When substituting $v = v_1 = H(-\xi)F(\xi)$, there are two cases to consider: $\xi > 0$ and $\xi < 0$. If $\xi > 0$, we substitute $v_1 = H(-\xi)F(\xi)$ into equation (13) and obtain:

\[
\mathcal{L}v_1 - \lambda v_1 = \left(\phi * \left[\frac{b-3}{2} \phi' v_1' - \frac{b}{2} \phi v_1\right] + (1 - \phi) v_1\right)' - \lambda v_1
\]

\[
= \left(\int_{-\infty}^{\xi} e^{-|\xi-\xi'|} e^{\xi'} \left[\frac{b-3}{2} F'(\xi') - \frac{b}{2} F(\xi')\right] d\xi'\right)'
\]

\[
= e^{-\xi} \int_{-\infty}^{\xi} e^{\xi'} \left[\frac{b-3}{2} F'(\xi') - \frac{b}{2} F(\xi')\right] d\xi' + (1 - e^{\xi})F'
\]

\[
= e^{-\xi} \left(\int_{-\infty}^{\xi} e^{2\xi'} F(\xi') d\xi'\right)
\]

For $\xi < 0$, one uses the fact that $F$ satisfies (17) itself obtained by the application of the operator $1 - \partial_{\xi}^2$ on the eigenvalue problem (13), that is:

\[
(1 - \partial_{\xi}^2) (\mathcal{L}v_1 - \lambda v_1) = 0 \text{ for } \xi < 0.
\]

This implies that $\mathcal{L}v_1 - \lambda v_1$ is a linear combination of $e^{\xi}$ and $e^{-\xi}$ for $\xi < 0$. Since $F$ converges to zero as $\xi \to -\infty$, we have that:

\[
\mathcal{L}v_1 - \lambda v_1 = B e^{\xi}, \text{ } B = \text{ const., for } \xi < 0.
\]

Furthermore, it can be checked that $\mathcal{L}v_1 - \lambda v_1$ is continuous at $\xi = 0$. This is done by computing explicitly $\mathcal{L}v_1 - \lambda v_1$ for $\xi < 0$ as:

\[
\mathcal{L}v_1 - \lambda v_1 = \left(\phi * \left[\frac{b-3}{2} \phi' v_1' - \frac{b}{2} \phi v_1\right] + (1 - \phi) v_1\right)' - \lambda v_1
\]

\[
= \left(\int_{-\infty}^{\xi} e^{-|\xi-\xi'|} e^{\xi'} \left[\frac{b-3}{2} F'(\xi') - \frac{b}{2} F(\xi')\right] d\xi'\right)'
\]

\[
= e^{-\xi} \int_{-\infty}^{\xi} e^{\xi'} \left[\frac{b-3}{2} F'(\xi') - \frac{b}{2} F(\xi')\right] d\xi' + (1 - e^{\xi})F'
\]

\[
= e^{-\xi} \left(\int_{-\infty}^{\xi} e^{2\xi'} F(\xi') d\xi'\right)
\]

We apply the limit as $\xi \to 0^+$ to the the last expression in (29) and the limit as $\xi \to 0^-$ to (27). Using the fact that $F(0) = 0$, we find that both limits are the same and thus $\mathcal{L}v_1 - \lambda v_1$ is continuous at $\xi = 0$. By continuity with equation (27), we establish that the constant $B$ in (28) is given by $B = \left(\frac{3(b-2)}{2} \int_{-\infty}^{0} e^{2\xi'} F(\xi') d\xi'\right)$. Therefore, we have that:

\[
\mathcal{L}v_1 - \lambda v_1 = e^{-|\xi|} \left(\frac{3(b-2)}{2} \int_{-\infty}^{0} e^{2\xi'} F(\xi') d\xi'\right).
\]

Together, (24), (26) and (30) give the results in the lemma.
To prove that equation (13) does not have continuous solutions, we need to prove that the right-hand-side (RHS) of equation (25) cannot be zero for any $c_1$ and $c_2$. For the RHS of equation (25) to be zero, $c_2$ must be zero since the expression it multiplies is discontinuous. The statement of the proposition then stems from the following lemma.

Lemma 3.3. Assume that $r_3 = 2 - b - \lambda$ has a positive real part. Let $F$ be the unique (up to multiplication by a scalar) solution to equation (17) such that $F(0) = 0$ and $F \to 0$ as $\xi \to -\infty$. Then,

$$\int_{-\infty}^{0} e^{2\xi} F(\xi) d\xi \neq 0.$$  

Proof. We make the substitution $v = e^\xi u$ into equation (17) to get a second-order equation for $u'$ since $v = e^\xi$ solves equation (17). The new equation admits $u' = e^{-2\xi}$ as a solution. We then make the substitution $u' = e^{-2\xi} w$ and get a first-order equation for $w'$ whose solution is:

$$w' = \tilde{B} \frac{e^{(\lambda + 1)\xi}}{(e^\xi - 1)^{\lambda - b}} \tilde{B} = \text{const.}$$  (31)

This way, we have that $e^{2\xi} v' = e^{2\xi} (e^\xi u)' = e^{2\xi} (e^\xi u + e^\xi u') = e^{2\xi} v + e^{2\xi} w$, thus implying:

$$e^{2\xi} v' - e^{2\xi} v = e^{2\xi} w.$$  (32)

We substitute $v = F$ in the equation above and integrate both sides from $-\infty$ to 0. We integrate by parts the first term of the left-hand-side, using the fact that $F(0) = 0$, and obtain:

$$-\frac{3}{2} \int_{-\infty}^{0} e^{2\xi} F(\xi) d\xi = \int_{-\infty}^{0} e^{2\xi} w(\xi) d\xi.$$

Since the sign of $w'$ never changes by equation (31) and $w \to 0$ as $\xi \to -\infty$ by equation (32), we have that $w'$ never changes sign for $\xi < 0$ and the integrals above are both nonzero. \[\Box\]

It follows from lemma 3.3 that the RHS of (25) cannot be zero unless $c_1 = c_2 = 0$ in the case where $r_3 = 2 - b - \lambda$ has a positive real part. The statement of the proposition that the eigenvalue problem of equation (13) does not have solutions in $H^1(\mathbb{R})$ thus follows. \[\Box\]

We now want to consider solutions to the eigenvalue problem (13) admitting a discontinuity at the origin such as in equation (15). To do so, we introduce an extension of $L$ as defined in equation (13). We first consider $L$ in the case $\xi < 0$, which we denote by $L_-:$

$$L_- v = \frac{d}{d\xi} \left( \int_{-\infty}^{\xi} e^{-|\xi-\xi'|} \left[ \frac{b - 3}{2} u_0(\xi') v' (\xi') \right] d\xi' - \frac{b}{2} \phi * \phi v + (1 - \phi) v \right)$$

$$= \frac{b - 3}{2} \frac{d}{d\xi} \left( e^{-\xi} \int_{-\infty}^{\xi} e^{2\xi'} v' (\xi') d\xi' + e^\xi \int_{\xi}^{0} v' (\xi') d\xi' \right)$$

$$- e^\xi \int_{0}^{\infty} e^{-2\xi'} v' (\xi') d\xi' + \frac{d}{d\xi} \left( -\frac{b}{2} \phi * (\phi v) + (1 - \phi) v \right),$$
and where we used the fact that $\phi' = -\text{sgn}(\xi)e^{-|\xi|}$. Then, we use integration by parts to eliminate $v'$ obtain:

\[
L_+ v = \frac{d}{d\xi} \left\{ (3 - b) \left[ e^\xi \int_0^\infty e^{-2\xi'} v(\xi') d\xi' + e^\xi \int_{-\infty}^\xi e^{2\xi'} v(\xi') d\xi' \right. \right. \\
\left. \left. \quad - e^\xi \left( v_0^+ + v_0^- \right) \right\} - \frac{b}{2} \phi \ast (\phi v) + (1 - \phi) v \right\},
\]

where

\[
v_0^\pm \equiv \lim_{\xi \to \pm \infty} v(\xi).
\]

In the case of $\xi > 0$, we get:

\[
L_+ v = \frac{d}{d\xi} \left\{ (3 - b) \left[ e^{-\xi} \int_0^0 e^{2\xi'} v(\xi') d\xi' + e^\xi \int_{-\infty}^\xi e^{-2\xi'} v(\xi') d\xi' \right. \right. \\
\left. \left. \quad - e^{-\xi} \left( v_0^+ + v_0^- \right) \right\} - \frac{b}{2} \phi \ast (\phi v) + (1 - \phi) v \right\},
\]

Thus the extension of the operator $L$ (from equation (13)) reads:

\[
L_w \equiv \begin{cases} 
L_+ & \text{for } \xi > 0, \\
L_- & \text{for } \xi < 0,
\end{cases}
\]

which has a larger domain, and $L_v = L_w v$ if $v \in \text{Dom}(L) \subseteq L^2(\mathbb{R})$. Indeed, the domain of $L$ (from the definition given in equation (13)), is restricted to $H^1(\mathbb{R})$, while the operator $L_w$ (from equations (33) and (34)) admits in its domain functions that have a finite-jump discontinuity at $\xi = 0$. Therefore, it is possible to verify that $\phi'$ is in the kernel of $L_w$ and that $\phi$ is a generalized eigenvector of the $L_w$, i.e.

\[
L_w \phi' = 0, \text{ and } L_w \phi = \phi'.
\]

Equation (35) is obtained by substituting $v = \phi' = -\text{sgn}(\xi)e^{-|\xi|}$ and $v = \phi$ into (33) and (34). The first equation in (35) is a consequence of the translation symmetry of the $b$-family, while the second is a consequence of the fact that the speed $c$ of the peakon solutions (2) is free.

We can group terms in equations (33) and (34) as:

\[
L_v v = \frac{d}{d\xi} \left\{ (3 - b) \left[ e^{-\xi} \int_{\min(\xi,0)}^{\max(\xi,0)} e^{2\xi'} v(\xi') d\xi' + e^\xi \int_{\max(\xi,0)}^\infty e^{-2\xi'} v(\xi') d\xi' \right. \right. \\
\left. \left. \quad - e^{-|\xi|} \left( v_0^+ + v_0^- \right) \right\} - \frac{b}{2} \phi \ast (\phi v) + (1 - \phi) v \right\},
\]

and rewrite $L_w$ (by applying the derivative operator in equation (36)) as:

\[
L_w v = (3 - b) \left( e^\xi \int_{\min(\xi,0)}^{\max(\xi,0)} e^{-2\xi'} v(\xi') d\xi' - e^{-\xi} \int_{-\infty}^{\min(\xi,0)} e^{2\xi'} v(\xi') d\xi' \right) \\
- \frac{b}{2} \phi' \ast (\phi v) + (3 - b) \text{sgn}(\xi)e^{-|\xi|} \left( \frac{v_0^+ + v_0^-}{2} - v \right) \\
+ ((1 - \phi)v)' \tag{37}
\]
In order to show that $\mathcal{L}_w$ is not closable on $L^2(\mathbb{R})$, it suffices to show that there is a sequence $v_n$ converging to zero, while $\mathcal{L}_w v_n$ does not [35, 37]. We choose the sequence of bump functions defined as:

$$v_n \equiv \begin{cases} 
\exp\left(\frac{1}{n^2 \xi^2 - 1}\right) & \text{for } |\xi| < 1/n, \\
0 & \text{otherwise.}
\end{cases}$$

Clearly, $v_n$ converges to 0 in $L^2(\mathbb{R})$, and all the terms in equation (36) do also except for the third one since $(v_{2n}^0 + v_{2n}^-)$ converges to $2e^{-1}$.

In order to define a space on which $\mathcal{L}_w$ is closed, we first introduce the following subspace of $L^\infty(\mathbb{R})$ made of functions that are continuous everywhere except at $\xi = 0$. More precisely:

$$C_d(\mathbb{R}) = \left\{ v \in L^\infty(\mathbb{R}) \mid v \in C_b(\mathbb{R} \setminus \{0\}) \text{ and } \lim_{\xi \rightarrow 0^\pm} v = v^\pm_0 \text{ exist} \right\}. \quad (38)$$

The set $C_d(\mathbb{R})$ with the $L^\infty(\mathbb{R})$ norm is a Banach space, since it is isomorphic to the direct sum $C_b((-\infty,0]) \oplus C_b([0,\infty))$ equipped with the norm $\left\{ \|v\|_{L^\infty((-\infty,0])}, \|v\|_{L^\infty([0,\infty))} \right\}$. The operator $\mathcal{L}_w$ is defined almost everywhere on $L^2(\mathbb{R}) \cap C_d(\mathbb{R})$, and thus we have the following lemma.

**Lemma 3.4.** The operator $\mathcal{L}_w$ is closed on $L^2(\mathbb{R}) \cap C_d(\mathbb{R})$.

**Proof.** We first consider the operator $\tilde{\mathcal{L}}_w$ defined by:

$$\tilde{\mathcal{L}}_w v = ((1 - \phi)v)' - (3 - b)\text{sgn}(\xi)e^{-|\xi|}v. \quad (39)$$

We prove that $\mathcal{L}_w - \tilde{\mathcal{L}}_w$ is compact on $L^2(\mathbb{R}) \cap C_d(\mathbb{R})$ followed by the use of theorem 1.11 of [38].

We first prove that each term on the first line of equation (37) is compact on $L^2(\mathbb{R}) \cap C_d(\mathbb{R})$ by proving they are compact on both $L^2(\mathbb{R})$ and $L^\infty(\mathbb{R})$. They are compact on $L^2(\mathbb{R})$ because each term on the first line of equation (37) can be written as an integral operator for some kernel $K \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$. As such, each of those terms defines a Hilbert-Schmidt integral operator, known to be compact (see [39], p 262). For example, the first term in parentheses in equation (37) corresponds to the kernel:

$$K_1 = \begin{cases} 
\xi^{-2}\xi' & \text{for } \xi' > \max(\xi, 0), \\
0 & \text{otherwise.}
\end{cases} \quad (40)$$

To prove the integral terms in equation (37) are compact on $L^\infty(\mathbb{R})$, we use the corollary 5.1 of [40], giving the conditions on the kernel of an integral operator for it to be compact on $L^\infty(\mathbb{R}^d)$. Those conditions reduce to the following in the case of $n = 1$ dimension.

Assume that there is a constant $M$ such that for almost all $\xi \in \mathbb{R}$, $K(\xi, \cdot) \in L^1(\mathbb{R})$ and $\|K(\xi, \cdot)\|_1 \leq M$. Then the operator is compact if and only if for any $\varepsilon > 0$ there exist $\delta > 0$ and $R > 0$ such that for almost all $\xi \in \mathbb{R}$ and all $h \in (-\delta, \delta)$ we have:

$$\int_{\mathbb{R} \setminus (-R,R)} |K(\xi, \xi')| d\xi' < \varepsilon \quad (41)$$

\(^5\) It states that if an operator is closed, then so is any relatively compact perturbation of that operator.
and
\[
\int_{\mathbb{R}} |K(\xi, \xi' + h) - K(\xi, \xi')|d\xi' < \varepsilon. \tag{42}
\]

To check those conditions on the kernel \(K_1\), defined in equation (40), we compute its \(L^1\) norm and find that it is bounded by \(M = 1/2\). We can also compute the integral in equation (41) and find that is is bounded by \(e^{-R}/2\). Finally, the integral in equation (42) is found to be bounded by \(1 - e^{-2|h|}\). The conditions of compactness on \(L^\infty(\mathbb{R})\) can also be verified straightforwardly for the two other terms of the first line of equation (37). For the second term, we have:

\[
K_2 = \begin{cases} 
  e^{-\xi + 2\xi'} & \text{for } \xi' < \min(\xi, 0), \\
  0 & \text{otherwise}.
\end{cases}
\]

The condition on the \(L^1\) norm, and conditions (41) and (42) are verified based on the fact that:

\[
K_2(\xi, \xi') = K(\xi, -\xi - \xi').
\]

For the third term in equation (37), we have the kernel:

\[
K_3 = -K_{3a} + K_{3b},
\]

where

\[
K_{3a} = \begin{cases} 
  e^{\xi - \xi' - |\xi'|} & \text{for } \xi > \xi', \\
  0 & \text{otherwise},
\end{cases} \quad K_{3b} = \begin{cases} 
  e^{\xi - \xi' - |\xi'|} & \text{for } \xi < \xi', \\
  0 & \text{otherwise}.
\end{cases}
\]

Since \(K_{3b}(\xi, \xi') = K_{3a}(\xi, -\xi - \xi')\), we only have to verify the conditions for \(K_{3a}\). An integral computation shows that the \(L^1\) norm of \(K_{3a}\) is bounded by \(1/2 + 1/e\). Furthermore, another integral computation shows that the integral in (41) is bounded by \(e^{-R}\). For the integral in (42), one has to consider several cases depending on the signs of \(\xi, h\), and \(\xi - h\). In each case, one finds that the integral is bounded by an expression that goes to zero as \(h \to 0\). Note that it would have been sufficient to show that the integral operators on the first line of equation (37) are continuous in order to prove the lemma. However, compactness is the stronger property that may be useful in the future to obtain the full spectrum.

We now prove that the remaining term of \(L_n - \hat{L}_n\) defined by:

\[
A_n \equiv \frac{(3 - b)}{2} \left( v_{0}^+ + v_{0}^- \right) \frac{\text{sgn}(\xi)e^{-|\xi|}}{2},
\]

is compact on \(L^2(\mathbb{R}) \cap C_{\mathbb{R}}(\mathbb{R})\). To prove compactness, we need to take a bounded sequence \(\{v_n\} \subset C_{\mathbb{R}}(\mathbb{R})\) and prove that \(\{A_n\}\) has a Cauchy subsequence. The boundedness of \(\{|v_n|\} \subset C_{\mathbb{R}}(\mathbb{R})\) implies the boundedness of \(\{v_{0}^+\}\), with \(v_{0}^+ \equiv \lim_{\pm \xi \to 0, \pm} v_n\). Thus, the sequence \(\{v_{n,0}^+ + v_{n,0}^-\} \subset C_{\mathbb{R}}(\mathbb{R})\) contains a Cauchy subsequence \(\{v_{n,0}^+ + v_{n,0}^-\}\). With the \(L^\infty(\mathbb{R})\) norm we have:

\[
\|A_{n,i} - A_{n,j}\|_{L^\infty(\mathbb{R})} = \frac{|3 - b|}{2} \left| (v_{0,0}^+ + v_{0,0}^-) - (v_{n,0}^+ + v_{n,0}^-) \right|,
\]

and with the \(L^2(\mathbb{R})\) norm:

\[
\|A_{n,i} - A_{n,j}\|_{L^2(\mathbb{R})} = \frac{|3 - b|}{2} \left| \int_{\mathbb{R}} \left( v_{n,0}^+ + v_{n,0}^- - v_{n,0}^+ - v_{n,0}^- \right) |e^{-|\xi|}| \right|_{L^2(\mathbb{R})}
\]

\[
= \frac{|3 - b|}{2} \left| \left( v_{n,0}^+ + v_{n,0}^- \right) - \left( v_{n,0}^+ + v_{n,0}^- \right) \right|.
\]
Thus, the sequence \( \{ A v_n \} \) is a Cauchy subsequence of \( \{ A v_n \} \) on both \( C_d(\mathbb{R}) \) and \( L^2(\mathbb{R}) \). We conclude that \( A \) is compact on \( L^2(\mathbb{R}) \cap C_d(\mathbb{R}) \).

It now suffices to prove that \( \hat{L}_w \) defined in equation (39) is closed. Assume we have a converging sequence in the domain of \( \hat{L}_w \). Let \( v_n \to v \) such that \( \hat{L}_w v_n \) also converging. We need to show that \( \hat{L}_w v_n \to \hat{L}_w v \). The convergence of the term \(-(3 - b)\text{sgn}(\xi)e^{-|\xi|}v_n\) to \(-(3 - b)\text{sgn}(\xi)e^{-|\xi|}v\) in \( L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}) \) is immediate. For the term \( \{(1 - \phi)v\}' \), the \( L^2(\mathbb{R}) \) convergence of \( v_n \) to \( v \) implies the \( L^2(\mathbb{R}) \) of \( (1 - \phi)v_n \) to \( (1 - \phi)v \). Furthermore, since \( (1 - \phi)v_n \)' itself is convergent in \( L^2(\mathbb{R}) \), it converges to \( (1 - \phi)v \) by definition of convergence on \( H^1(\mathbb{R}) \). The convergence of the term \( (1 - \phi)v \)' in the sup norm to \((1 - \phi)v \)' follows from the fact that \( \hat{L}_w v_n \) converges in both \( L^2(\mathbb{R}) \) and \( C_d(\mathbb{R}) \) to the same function, by the definition of the norm on \( L^2(\mathbb{R}) \cap C_d(\mathbb{R}) \) as being the maximum of the two norms.

We are now ready to prove theorem 2.1.

**Proof.** We first compute the point spectrum of \( \hat{L}_w \) associated with the eigenvalue problem of equation (36). The most general candidate for a discontinuous solution at \( \xi = 0 \) is given by equation (23). Without loss of generality, we re-parametrize the free parameters \( c_0 \) and \( c_2 \) with \( \tilde{c}_0 \) and \( \tilde{c}_2 \), which we define though the relation:

\[
\begin{align*}
c_0 &= -\tilde{c}_0 + \tilde{c}_2, \\
c_2 &= \tilde{c}_0 + \tilde{c}_2.
\end{align*}
\]

Using those in (23), we find that the most general ansatz for a solution in \( L^2(\mathbb{R}) \) in this case is:

\[
v_d = \tilde{c}_0 \phi' + c_1 v_1 + \tilde{c}_2 \phi, \text{ where } v_1 = H(-\xi)F(\xi),
\]

where we used the fact that \( \phi' = -\text{sgn}(\xi)e^{-|\xi|} \). Computing \( \hat{L}_w v_d \), using equations (25) and (35), we find:

\[
\begin{align*}
\hat{L}_w v_d - \lambda v_d &= e^{-|\xi|} \left( c_1 \frac{3(b - 2)}{2} \int_{-\infty}^{0} e^{2\xi} F(\xi) d\xi + \tilde{c}_2 \text{sgn}(\xi) - \lambda \right) + \tilde{c}_0 \lambda \text{sgn}(\xi) \\
&= e^{-|\xi|} \left( c_1 \frac{3(b - 2)}{2} \int_{-\infty}^{0} e^{2\xi} F(\xi) d\xi + \tilde{c}_2 (\text{sgn}(\xi) - \lambda) + \tilde{c}_0 \lambda \text{sgn}(\xi) \right).
\end{align*}
\]

Thus, \( v_d \) is a solution given that \( \tilde{c}_2 = -\tilde{c}_0 \lambda \) and \( c_1 \) is chosen such that

\[
c_1 = \frac{2\tilde{c}_2}{3(b - 2)} \lambda \int_{-\infty}^{0} e^{2\xi} F(\xi) d\xi.
\]

From the expansions given in equation (19), if we add the restriction that \( F \) be in \( C_d(\mathbb{R}) \), we have that:

\[
\text{Re}(r_3) = 2 - b - \text{Re}(\lambda) > 0,
\]

i.e. any \( \lambda \) satisfying \( 0 < \text{Re}(\lambda) < 2 - b \) is in the point spectrum.

The following lemma proves the second part of theorem 2.1.

**Lemma 3.5.** The function \( v_1 = H(-\xi)F(\xi) \), where \( F \) solves equation (17) such that \( F(0) = 0 \) and \( F(\xi) \to 0 \) as \( \xi \to -\infty \), is in \( H^s(\mathbb{R}) \), for all \( s < 3/2 \) if and only if \( r_3 \) given by equation (18) satisfies \( \text{Re}(r_3) = 2 - b - \text{Re}(\lambda) \geq 1 \).

**Proof.** Because the series expansion of equation (19) admits a different form for \( r_3 = 1 \) and \( r_3 \neq 1 \), we treat the two cases separately starting with \( r_3 \neq 1 \). In view of the definition of \( F \) from equations (19) and (20), if we require \( F \) to be in \( H^1(\mathbb{R}) \), it implies that \( \text{Re}(r_3) > 1/2 \). We
write $F$ as $F = \tilde{F} + H(-\xi) |\xi|^\rho e^{ix}$. Because $\Re(r_3) > 1/2$, we have that $\tilde{F}$ is in $H^2(\mathbb{R})$. It thus suffices to show that the function:

$$
T(\xi) \equiv H(-\xi) |\xi|^\rho e^{ix} = \begin{cases} 
|\xi|^\rho e^{ix}, & \xi < 0, \\
0, & \xi > 0,
\end{cases} \tag{44}
$$

is in $H^s(\mathbb{R})$ for all $s < 3/2$ if only if $\Re(r_3) \geq 1$. As an example, if we use $r_3 = 1$ in equation (44), then the Fourier transform of $T$ is given by:

$$
\hat{T}(w) = \frac{1}{(1 - iw)^{\rho + 1}},
$$

Recall that the condition for $T$ to be in $H^s(\mathbb{R})$ is that $(1 + w^2)^{s/2}T(w)$ be in $L^2(\mathbb{R})$ [41]. This condition is satisfied if and only if $s < 3/2$. The same condition on $s$ is obtained if we use $r_3$ such that $\Re(r_3) = 1$ and thus it is clear from $T$ given by equation (44) that it will be in $H^s(\mathbb{R})$ for all $s < 3/2$ if and only if $\Re(r_3) \geq 1$. It can also be checked directly by the following expression giving the Fourier transform of $T$ (cf equation (44)) for general values of $r_3$:

$$
\hat{T}(w) = \frac{i\Gamma(r_3 + 1)}{(1 - iw)^{\rho + 1}},
$$

where $\Gamma$ is the Gamma function. As per the case for $r_3 = 1$, from the second line of equation (19), we consider the function:

$$
T_1(\xi) \equiv H(-\xi) |\xi| \ln(|\xi|) e^{ix} = \begin{cases} 
|\xi| \ln(|\xi|) e^{ix}, & \xi < 0, \\
0, & \xi > 0,
\end{cases}
$$

which can be verified to be in $H^s(\mathbb{R})$ for all $s < 3/2$ by the expression of its Fourier transform:

$$
\hat{T}_1(w) = \frac{i(wv - 1)^2 \left( \ln(w^2 + 1) - 2 \arctan(w) - 2i(\gamma - 1) \right)}{2(w^2 + 1)^2}
$$

where $\gamma$ is Euler’s constant. □

Since lemma (3.5) proves the second part of the theorem, we have completed the proof. □

**Remark 3.6.** In theorem 2.1, we use the space $L^2(\mathbb{R}) \cap C_{\delta}(\mathbb{R})$ (with $C_{\delta}(\mathbb{R})$ defined in (38)). If more regularity is required by using the space:

$$
H^s_{\delta}(\mathbb{R}) \equiv \left\{ v \in L^2(\mathbb{R}) \middle| v|_{(-\infty,0)} \in H^1((-\infty,0)) \text{ and } v|_{(0,\infty)} \in H^1((0,\infty)) \right\} \tag{45}
$$

instead, one finds the point spectrum in the first part of theorem 2.1 to be $0 < |\Re(\lambda)| \leq 3/2 - b$. Indeed, the proof of theorem 2.1 goes through with the modification that the condition $\Re(r_3) \geq 1/2$ (instead of $\Re(r_3) > 0$ specified in equation (43)) must be satisfied in order for $F$ to be in $H^s_{\delta}(\mathbb{R})$. The closure of $\mathcal{L}_w$ holds because, for any interval $I \subset \mathbb{R}$, $\|v\|_{L^2(I)} \leq C_s \|v\|_{H^s(I)}$ for some constant $C_s$. Furthermore, if $H'^s$ is replaced by $H^s$, $1 \leq s < 3/2$, in the definition of $H^s_{\delta}(\mathbb{R})$ above, that is:

$$
H^s_{\delta}(\mathbb{R}) \equiv \left\{ v \in L^2(\mathbb{R}) \middle| v|_{(-\infty,0)} \in H^s((-\infty,0)) \text{ and } v|_{(0,\infty)} \in H^s((0,\infty)) \right\}, \tag{46}
$$

then the condition on $r_3$ becomes $\Re(r_3) > s - 1/2$. This follows from the Fourier transform computation done in the proof of lemma 3.5 and from lemma 5.2 of [41] giving a criterion for
a function to be in a fractional Sobolev space on a subset of $\mathbb{R}$. The band specified in the first part of theorem 2.1 is then found to be:

$$0 < |\text{Re}(\lambda)| < \frac{5}{2} - s - b,$$

which limits to the band specified in the second part of theorem 2.1 as $s \to 3/2$.

### 4. Numerical results

In this section, we present numerical results concerning the existence and spectral stability of standing and travelling wave solutions to the $b$-family of equations, i.e. equation (1). The discussion that follows next is complemented by systematically presenting results on spatio-temporal evolution of generic (Gaussian) and peakon initial data.

#### 4.1. Standing and travelling waves

First, we shall be interested in the ‘lefton’ solutions. A single lefton is a stationary solution of equation (1) given by the explicit formula [24]:

$$u = A \left( \cosh (\gamma (x - x_0)) \right)^{-\frac{1}{2}}, \quad \gamma = -\frac{b + 1}{2},$$

(48)

where $A$ and $x_0$ are its amplitude and centre, respectively. For a given $b$, this is a two-parameter family of solutions, given the arbitrary choice of $A$ and $x_0$. The form of equation (48) suggests that leftons exist only for the parameter regime $b < -1$, which we confirm numerically by parameter continuation in $b$. To investigate the spectral stability of a lefton solution $u = u_0(x)$ of the $b$-family, we linearize equation (1) about $u = u_0(x)$ by substituting the ansatz $u(x, t) = u_0(x) + v(x)e^{\lambda t}$ and keeping only terms that are linear in $v$ to obtain the following eigenvalue problem:

$$\lambda (v - v''') + ((b + 1)u_0v + (1 - b)u_0'v' - u_0v'' - u_0''v')' = 0,$$

(49)

where the $'$ denotes differentiation with respect to $x$.

We rearrange equation (49) to get $(I - \partial_x^3)^{-1} \mathcal{L}(u_0)v = \lambda v$, where $\mathcal{L}(u_0)$ is the linear operator:

$$\mathcal{L}(u_0) = -\partial_x \left( (b + 1)u_0I + (1 - b)u_0'\partial_x - u_0\partial_x^2 - u_0'' \right).$$

We can verify directly that $\mathcal{L}(u_0)u_0' = 0$, which results from translation invariance of the system. In addition, $\mathcal{L}(u_0)u_0 = 0$. This is a result of the additional degree of freedom in $A$ in equation (48), which generates a kernel eigenfunction $\frac{\partial v}{\partial x} = \frac{1}{2}u_0$.

To find the spectrum, we use Fourier spectral differentiation matrices for the differential operators and compute the eigenvalues using the built-in eigenvalue solver e.g in MATLAB. The maximum real part of the computed spectrum is of order $10^{-7}$, suggesting that the spectrum is purely imaginary. Due to the quartet symmetry of the spectrum of a Hamiltonian system (i.e. if $\lambda$ is in the spectrum, so is $-\lambda$, and $-\bar{\lambda}$), this is consistent with neutral (spectral) stability in a Hamiltonian system. In addition, we verify numerically that the kernel eigenfunctions are indeed $u_0'$ and $u_0$. These spectral results are obtained for a wide range of $A$ and $b < -1$.

For the peakon solutions, which are travelling waves, this method of computing the spectrum does not work since the peakon is not differentiable at its centre. As an alternative, we
will compute the spectrum of the family of smooth solitary waves on a nonzero background [18], which are solutions to the equation:

$$c(u'' - u) + (b + 1)u^2 + (1 - b)\left(\frac{u'}{2}\right)^2 - u'' = g, \quad \xi = x - ct, \quad (50)$$

obtained by integrating the co-travelling frame ODE obtained from equation (1). Notice that here the ‘ denotes differentiation with respect to the travelling frame variable $\xi$. We can compute these smooth solitons by numerical parameter continuation in $g$ using a secant based predictor-corrector method [51], starting with the peakon solution as an initial guess (figure 1, left panel). The limit of these smooth solitons is the peakon solution.

Using the same techniques as above, we can numerically compute the spectrum of these smooth solitons. Linearizing equation (1) about a solution $u(\xi)$ to equation (50), the eigenvalue problem becomes:

$$\lambda(v - v') + (c(v'' - v) + (b + 1)uv + (1 - b)u'v' - uv' - u''v')' = 0, \quad (51)$$

where we have shifted to a co-travelling frame with speed $c$. For $b = 1.5$, the maximum real part of the spectrum is of order $10^{-13}$ for sufficiently large $g$ (figure 1, bottom left), which suggests that the spectrum is purely imaginary for that parameter regime. When $g$ decreases below a threshold value, numerical spectral computation suggests the presence of an eigenvalue with positive real part. This eigenvalue depends smoothly on $g$. This threshold, however, is lower as the number of Fourier modes in the discretization is increased. Furthermore, the eigenfunction
associated with this eigenvalue resembles the derivative of the smooth soliton (figure 1, bottom right), and becomes increasingly singular as $g$ decreases. Qualitatively similar results are obtained for all values of $b \in [1, 2]$ that we tried. This suggests that while the smooth solitons are stable (see, for example, [42]), the peakon solution (which is in some sense a singular limit of these smooth solitons) is, in fact, unstable [43]. Indeed, these two results are both in line with the recent [42, 43], posted after our original submission and claiming, respectively, the stability of the smooth solitary waves in the model of interest, while the peaked solitary waves are shown to be spectrally unstable.

4.2. Numerical timestepping

We now turn our focus to spatio-temporal dynamics of the $b$-family of peakon equations (cf equation (1)). For our subsequent analysis, we will consider Gaussian initial data of the form of:

$$u(x, t = 0) := \frac{1}{\sigma \sqrt{\pi}} e^{-\frac{(x-x_0)^2}{2\sigma^2}},$$

(52)

where $\sigma$ and $x_0$ correspond to the width and centre of the Gaussian pulse, respectively. The previous works of [19, 20, 44] considered the so-called $m$-formulation:

$$m_t = -um_x - bu_m, \quad m := u - u_{xx},$$

(53)

which we adopt from now on, and the numerical scheme we employed in this work is discussed next.

We advance equation (53) forward in time with the initial data of equation (52) by using Fourier spectral collocation for the spatial discretization supplemented by periodic boundary conditions on $[0, 200]$, and the Runge–Kutta–Fehlberg method (RKF45) for the time marching. The latter additionally incorporates a time step-size adaptation strategy such that both absolute and relative tolerances are less than $10^{-8}$ per time step. Then, at each time step, the field $u$ is obtained from the field $m$ by inverting the Helmholtz operator $1 - \partial_x^2$ in Fourier space. We should mention in passing that the time integration is performed in Fourier space as well. We remove the aliasing errors by employing the so-called $3/2$-rule in order to ensure that the high wavenumber Fourier coefficients are well decayed (see, e.g. [45]). However, the numerical scheme we employ in this work is different than the one used in [19, 20, 44] since we do not introduce any artificial viscosity (as opposed to these previous yet notable works). This way, it is expected that the numerical results reported herein are closer and closer representations of the original physical system.

A series of benchmarks of the numerical scheme is discussed in the appendix in order to compare numerical results with respective ones that were obtained using artificial viscosity from previous works. In particular, using the initial data of equation (52), selected cases of spatio-temporal dynamics in $b$ are presented giving rise to peakons, leftons as well as ramp-cliffs, and the results discussed therein are well connected with the current literature. For example (see also figures 6–8), when $b < -1$, we observe the emergence of solitary pulses from Gaussian initial data (cf equation (52)) that move to the left, gradually asymptoting to a steady-state solution, i.e. leftons (cf equation (48)). It should be noted in passing that the number of leftons depends on how close or far away the selected value of $b$ is from $-1$, e.g. we observed the emergence of three and two leftons for $b = -2$ and $b = -1.5$, respectively (see the appendix for a detailed discussion on leftons).

We now turn our focus on the ramp-cliff regime corresponding to the case when $b \in (-1, 1)$. In the appendix, we present 4 cases of ramp-cliffs where the latter travel faster for gradually
increasing values of $b$. We can clearly observe the formation of these patterns and the self-similar expansion of their rear tails, while at the same time their front part steepens. It is worth noting here that we are not aware of a frame where such solutions can be considered as steady. However, we report at this point an artifact that was observed in our numerical simulations with $N = 16384$ collocation points and interval of time of integration $t \in [0, 3000]$. One would expect the emergence of ramp-cliffs propagating to the right of the computational domain. Nevertheless, for $b \geq 0.85$ we noticed that peakons were emitted from the ramp-cliffs, with the former emerging as robust travelling waves. We investigated this byproduct of the numerical scheme by considering the implications of theorem 3 in \cite{22}. In particular, it can be shown that if $m(x, t = 0) > 0$, then $m(x, t) > 0$, $\forall t > 0$ holds which in fact is the case as per the Gaussian initial data employed in this work.

Upon a careful inspection of the temporal evolution of the variable $m$, we noticed that it becomes negative past a time $t_0$, thus suggesting that one cannot continue the temporal integration beyond that time (due to the numerical scheme violating a theoretically established constraint). Moreover, we performed a spatial grid refinement by increasing the number of collocation points to $N = 32768$ in order to investigate further the dependence of $t_0$ on $N$. We still observed the emergence of such ‘spurious’ peakons but their appearance was delayed in time. This finding is somewhat expected: in this computation, we keep our spatial domain $[0, 200]$ fixed during the spatial grid refinement which implies that the wavenumbers are still multiples of $k = 2\pi/L$. Thus, when the number of collocation points is increased, the numerical scheme resolves progressively better the large wavenumbers which, in turn, results in the time delay of the emergence of those ‘spurious’ peakons. It is expected that if we increase the number of nodes to, e.g. $N = 65536$, this artifact will gradually disappear. As case examples of ramp-cliffs (in addition to the ones shown in figure 9 in the appendix), we demonstrate two cases with $b = 0.8$ and $b = 0.99$ in figure 2 where we stopped the integrator at $t \approx 290$ (past that time, we observed the non-positivity of the $m$ variable). It should be noted in passing that figure 2 depicts contour plots (see also the colourbar that is placed adjacent to the right of each panel) of the spatio-temporal evolution of $u(x, t)$.

We now investigate the peakon regime of the $b$-family, i.e., when $b > 1$. In particular, figure 3 presents selective cases of numerical simulations based on Gaussian initial data with $\sigma = 5$ and $x_0 = 50$, and $N = 8192$ Fourier modes. The top left and right panels correspond to the cases with $b = 1.5$ and $b = 2$ (CH) whereas the bottom left and right to values of $b = 2.5$ and $b = 3$ (DP), respectively. The emergence of sharply peaked waves can be discerned from these panels where the initial Gaussian pulse breaks into peakons as time progresses. Furthermore,
the time when the first peakon emerges in the simulations depends on the value of \( b \), that is, its emergence is ‘delayed’ when \( b \) is close to 1. However, when the value \( b \) is further away from that limit, the first peakon emerges at earlier times together with secondary peakons of smaller amplitude travelling across the computational grid. It should be noted also that the first peakon (having actually the largest amplitude) travels in the computational grid and undergoes nearly elastic collisions with other peakons of smaller amplitude. Such phenomenology is interesting in its own right and deserves further study, however it is beyond the scope of the present work.

We finally focus on theorem 2.1 (see section 2) which suggests that the point spectrum contains positive eigenvalues for \( b < 1 \), that is, the peakons are orbitally unstable for \( b < 1 \). We explore this theoretical finding numerically by considering a peakon centred at \( x_0 = 50 \) with speed (or amplitude) \( c \approx 0.031 \), and \( N = 32768 \) collocation points. The left and right panels of the top row of figure 4 present our numerical results for values of \( b = 0.88 \) (left panel) and \( b = 0.98 \) (right panel), respectively. It can be discerned from both panels that the peakons are orbitally unstable. The amplitude of the initial profile \( (t_0 = 0) \) gradually increases over time eventually leading to a collapse of the waveform (in particular, past \( t_0 \approx 130 \) for the spatial discretization employed herein).

On the other hand, i.e. when \( b > 1 \), we expect peakons to be orbitally stable. Indeed, this is the case as is shown in the middle and bottom panels of figure 4. In particular, the middle and bottom panels showcase profiles of peakons at \( t_0 = 0 \) and \( t_0 = 3000 \) (terminal time of integration) for \( b = 1.3 \) and \( b = 1.5 \), respectively (the same initial condition was used in both cases as in the top row of figure 4). It can be discerned from both panels that peakons appear to be robust over the time integration. However, a couple of remarks are in order at this point.
Figure 4. Top row: the emergence of the instability for peakon solutions to the $b$-family. Left and right panels present snapshots of peakon solutions at various times $t_0$ (see the legend therein) for $b = 0.88$ and $b = 0.98$, respectively. Middle and bottom rows: The stable regime $b > 1$ for $b = 1.3$ and $b = 1.5$. The panel in the middle and bottom rows showcases a peakon solution at $t_0 = 0$ and $t_0 = 3000$ with dashed and solid black lines, respectively. The insets therein, demonstrate the amplitude of the peakon as a function of time with a solid red line. A peakon centred at $x_0 = 50$ with speed $c \approx 0.031$ is employed as an initial condition. The number of collocation points that are employed in the above numerical simulations is $N = 32768$.

and in line with the middle and bottom panels of figure 4. We observe a small in-amplitude yet stationary localized error at the vicinity of the centre ($x_0 = 50$) of the initially placed peakon. It has been argued in [46] that when non-smooth initial data are considered in an evolution numerical experiment (such as peakons in the $b$-family), localized errors are expected to be formed in the vicinity of $x_0$ initially that remain stationary in time. This is the case in both panels of figure 4 and it is expected that this error gradually diminishes with grid refinement (see [46]). However, this error results in a slightly larger amplitude (and thus speed) of the pertinent peakon waveform but after a ‘transient’ period of time it remains constant over the time evolution, as this can be seen in the insets of the panels. Indicatively, the location of the peakon after 3000 time units in the bottom panel (i.e. for $b = 1.5$) is found to be at $x \approx 145.4$ whereas the theoretical expectation is $\approx 143.1$, thus suggesting a (relative) error of $\approx 1.6\%$. Despite this artifact, peakons for $b > 1$ appear to be highly robust and these findings are in accordance with theorem 2.1.
5. Conclusions and future directions

In the present work we have identified the solutions of the \( b \)-family of peakon equations. We have provided some analytical insight on the spectral problem, identifying the instability of the peakon waveforms via the consideration of their point spectrum of \( \mathcal{L}_b \). We should note in passing that in the recent work of [43], the linearization of the \( b \)-family is interpreted in a weak form, and is related to a closed operator on \( L^2(\mathbb{R}) \). The nonlocal portion of that operator therein is shown to be compact, thus enabling the computation of the entire spectrum which itself shows spectral instability on \( L^2(\mathbb{R}) \) for all \( b \neq 5/2 \). In our present case, we aim at obtaining the spectrum of \( \mathcal{L}_b \) on the spaces \( H^s_b(\mathbb{R}), s \geq 1 \) defined in equation (46). While we did not obtain (analytically) the spectral stability or instability of the peakons for \( b > 1 \), we did obtain the spectral and linear instability for \( b < 1 \) by computing the point spectrum of the operator. In the case \( b > 1 \), the orbital stability of peakons in the energy space \( H^1(\mathbb{R}) \) was shown for the Camassa–Holm equation \( (b = 2) \) in [32, 33] and in \( L^2(\mathbb{R}) \) for the Degasperis–Procesi equation \( (b = 3) \). It was furthermore shown in [47] for the Camassa–Holm equation that although the peakons are orbitally stable in \( H^1(\mathbb{R}) \), they are nonlinearly unstable with respect to perturbations in \( W^{1,\infty}(\mathbb{R}) \) and linearly unstable on the \( H^1_b(\mathbb{R}) \) space defined in (45). A similar situation between linear and nonlinearity stability exists for the integrable peakon Novikov equation. Indeed, the Novikov peakons are shown to be orbitally and asymptotically stable in \( H^1(\mathbb{R}) \) [48, 49], while linearly stable on \( H^1(\mathbb{R}) \) and linearly unstable on \( W^{1,\infty}(\mathbb{R}) \) [50]. In our case, for the \( b \)-family, we surmise that the peakons are orbitally stable on \( H^1(\mathbb{R}) \) for \( b > 1 \) (since they are for \( b = 2, 3 \)), while spectrally unstable, in line with the numerical observations herein and the recent work of [43]. We also surmise that the point spectrum we found in the second part of theorem 2.1 provides an analytical explanation to the instability of the peakons observed numerically in the case \( b < 1 \).

Our analytical insights have been corroborated by a diverse array of numerical computations. For structures that we could identify as steady, either in the original frame or in a co-travelling frame, we attempted to offer a complementary spectral picture. This was done in the case of the leftons for \( b < -1 \) which are stationary and were found to potentially be stable in this regime. On the other hand, in the regime \(-1 < b < 1 \), we could only perform dynamical simulations which illustrated the transient emergence and tendency towards breaking of ramp-cliff waveforms. The resulting formation of peakon structures (as \( b \to 1 \)) was identified as a feature that disappears as the high wavenumbers become better resolved. However, the peakon structures become indeed dominant for \( b > 1 \) where they spontaneously arise from smooth initial conditions and robustly persist for different values of \( b \), for integrable and non-integrable cases alike. Suggestive, although not definitive, towards their stability is the picture identified spectrally for the solutions on a finite background, tending towards these peakons as the background parameter \( g \) tends to 0.

While we believe that this study addresses some of the pending questions on this class of systems admittedly many more questions remain open and are worthwhile to explore in future studies. At the level of direct numerical simulations of the \( b \)-family of equation (1), is there a robust yet accurate spatial discretization scheme that can account for the non-smoothness of

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\( ^6 \) That computation could be done as in [43] by showing the compactness of the nonlocal part of \( \mathcal{L}_b \), which we were not able to do mainly because we are dealing with fractional Sobolev spaces. One could then use [36, theorem 1], to reduce the computation of the spectrum to that of the simpler operator \( \tilde{\mathcal{L}}_b \) given in (39). It would then follow from an extension of [43, theorem 3.1], which deals with the same operator, that, in addition to the point spectrum specified by (47), there is essential spectrum satisfying the reverse inequality, thus showing spectral instability except when \( b = 5/2 - \epsilon \).
the waveforms we are interested in without the use of any artificial viscosity (that itself perturbs the system)? Moreover, is there a meaningful (and consistent with our theoretical analysis) way in which the peakon spectral analysis can be numerically performed? Is there a frame (possibly a self-similarly evolving one) where the ramp-cliff structures can be considered as steady and thus be spectrally analysed? Are there higher-dimensional analogues of these different structures and, if so, which of the above properties persist or disappear even in the two-spatial-dimension case? These are only some among the numerous yet timely open questions. Work in these is currently underway and will be reported in future publications.

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Appendix. Spatio-temporal dynamics: from peakons to leftons and ramp-cliffs

We test our numerical scheme by reproducing a subset of the results of [4, 19] in the absence of artificial viscosity. In particular, the left \((b = 2)\) and right \((b = 3)\) panels of figure 5 correspond to the spatio-temporal evolution of \(u(x,t)\) by using Gaussian initial data (cf equation (52)) with \(\sigma = 5\) and \(x_0 = 100\), and \(\sigma = 5\) and \(x_0 = 33\) respectively. Those results compare well with figures 1 and 2 of [19] and [4], respectively.

Next, we focus on the regime \(b < -1\). In particular, figures 6–8 highlight numerical results on the lefton regime (cf equation (48)) by considering various values of \(b\) (with \(N = 8192\) Fourier modes). In particular, figure 6 presents the spatio-temporal evolution of \(u(x,t)\) for the cases with \(b = -3\) (top left panel), \(b = -2.5\) (top right panel), \(b = -2\) (bottom left panel), and \(b = -1.5\) (bottom right panel), respectively, when Gaussian initial data are employed with \(\sigma = 10\) and \(x_0 = 100\). The emergence of leftons is clearly evident in all those panels and we notice the appearance of more leftons when \(b(<-1)\) is larger in its absolute value (notice the appearance of four leftons in the top left and right panels whereas the bottom left and right ones contain three and two, respectively). We further investigated the emergence of leftons by considering different values of the Gaussian’s width and centre. Specifically, figure 7 presents results with \(\sigma = 7\) (and \(x_0 = 100\)) where the number of leftons decreases as \(b\) approaches \(-1\).

Figure 8 compares the numerically obtained (stationary) solution of the top right panel of figure 7 with equation (48). It should be noted that this result is the analogue of figure 6 in [19]. In the present case (with \(b = -2.5\)), three leftons appear at the terminal time of the evolution \((t = 2500)\) whose locations and amplitudes are computed. Then, those values are plugged into equation (48) and are plotted with stars, crosses and plus signs in figure 8. A perfect match can be clearly discerned, thus suggesting the accuracy and high-fidelity of the numerical scheme employed in this work, again, by using no artificial viscosity.
Figure 5. Left panel: Spatio-temporal evolution of a Gaussian profile with $\sigma = 5$ centred at $x_0 = 50$ and $b = 2$. Note that $N = 8192$ Fourier collocation points in space were used for this computation. Right panel: Same as the left one but for Gaussian initial data with $x_0 = 33$ and $b = 3$ (and same width, i.e. $\sigma = 5$). Here, $N = 4096$ collocation points were used.

Figure 6. Results on numerical simulations using $N = 8192$ Fourier collocation points. In particular, a Gaussian pulse centred at $x_0 = 100$ with $\sigma = 10$ was used as an initial condition to the $b$-family. Top left and right panels correspond to values of $b = -3$ and $b = -2.5$ whereas the bottom left and right ones to values of $b = -2$ and $b = -1.5$, respectively.

Next, we focus on the regime $b \in (-1, 1)$ in which ramp-cliff solutions were suggested to be observed from Gaussian initial data. Figure 9 corresponds to numerical results with $\sigma = 10$ and $x_0 = 100$ by employing $N = 16384$ Fourier modes. In particular, the top left and right panels correspond to the spatio-temporal evolution of $u(x, t)$ with $b = -1$ (i.e. at the bifurcation point) and $b = -0.5$, whereas the bottom left and right ones to $b = 0$ and $b = 0.5$, respectively. From the top left panel of figure 9 ($b = -1$), it can be discerned that the Gaussian pulse becomes
Figure 7. Same as figure 6 but for Gaussian initial data with $\sigma = 7$ (and $x_0 = 100$). Top left and right panels correspond to values of $b = -3$ and $b = -2.5$ whereas the bottom left and right ones to values of $b = -2$ and $b = -1.5$, respectively.

Figure 8. Spatial distribution of the solution of the top right panel of figure 7 at $t = 2500$ (i.e. $b = -2.5$, $\sigma = 7$ and $x_0 = 100$). The numerically obtained solution is shown with a solid black line whereas the exact lefton solutions (cf equation (48)) are shown with black stars, crosses and plus signs, respectively.

slightly wider but represents a nearly stationary solution (see, for example, figure 5 of [20]). On the other hand, the top right, bottom left and right panels corresponding to $b = -0.5$, $b = 0$ and $b = 0.5$, respectively, showcase examples of ramp-cliff solutions. It should be noted that their amplitude decreases over the time evolution although their velocity increases with $b$. 
Figure 9. Same as figure 7 but for Gaussian initial data with $\sigma = 10$ (and $x_0 = 100$) and $N = 16384$ Fourier modes. Top left and right panels correspond to values of $b = -1$ and $b = -0.5$ whereas the bottom left and right ones to values of $b = 0$ and $b = 0.5$, respectively.

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