The Nearly Flat Universe

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Abstract

We study here what it means for the Universe to be nearly flat, as opposed to exactly flat. We give three definitions of nearly flat, based on density, geometry and dynamics; all three definitions are equivalent and depend on a single constant flatness parameter $\varepsilon$ that quantifies the notion of nearly flat. Observations can only place an upper limit on $\varepsilon$, and always allow the possibility that the Universe is infinite with $k = -1$ or finite with $k = 1$. We use current observational data to obtain a numerical upper limit on the flatness parameter and discuss its implications, in particular the “naturalness” of the nearly flat Universe.

Keywords: general relativity — cosmology

1 Introduction

Observational cosmologists tell us that, at present, the ratio of the sum of the densities of all forms of matter-energy in the Universe to the critical density required for spatial flatness is:

$$\Omega_{T,0} = 1.02 \pm 0.04.$$  \hfill (1)

We follow standard usage throughout, using the symbol $\Omega$ to denote densities relative to the critical density $\rho_{\text{crit}} \equiv (3/8\pi G)(\dot{a}/a)^2$, with $\rho_\Lambda = \Lambda/(8\pi G)$ representing the contribution from vacuum energy so that $\Omega_T = (\rho + \rho_\Lambda)/\rho_{\text{crit}}$. The subscript “0” indicates terms taken at the present time.

Eq. (1) is a 95\% confidence-level fit to combined data on fluctuations in the spectrum of cosmic microwave background (CMB) radiation from the WMAP satellite and observations of the magnitudes of distant Type Ia supernovae (SNIa), plus the assumption that the present value of Hubble’s parameter satisfies $H_0 = (\dot{a}/a)_0 > 50 \text{ km s}^{-1} \text{ Mpc}^{-1}$ \hfill (1). The data are consistent with the possibility that $\Omega_{T,0}$ is \textit{precisely} one, implying an infinite and spatially flat ($k = 0$) Universe. They are also marginally consistent with the idea, widely held during the 1990s, that $\Omega_{T,0} < 1$, implying a Universe which is infinite and negatively curved ($k = -1$). Most intriguingly, Eq. (1) suggests the possibility of a closed, finite and positively curved Universe with $\Omega_{T,0} > 1$ and $k = +1$ — a prospect which has received less theoretical attention over the years \hfill (1) but which

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Figure 1: Closed ($k = +1$), flat ($k = 0$) and open ($k = -1$) universes.

is now hinted at by both the location of the primary acoustic peak [6, 7] and the amplitude of low-order multipoles [8, 9] in the CMB power spectrum. These are three radically different models of the Universe (Fig. 1).

The flat $k = 0$ Universe is widely favored among cosmologists for two reasons. First, inflation dynamically causes a very small portion of the primordial Universe to grow so large as to encompass the entire observable Universe, so that the spatial structure of the Universe is greatly flattened. This, however, only forces the Universe to be nearly, rather than exactly flat [10, 11], and indeed there are examples of inflationary models with $k \neq 0$ [12, 13, 14, 15, 16, 17]. Secondly, it is felt by some that a nearly flat Universe involves a numerical coincidence or fine-tuning between $\Omega_T$ and unity, especially at early times. As we will discuss in §6, however, this apparent fine-tuning is a natural result of the definitions of the density ratio $\Omega_T$ and the critical density, combined with the cosmological equations. We must also mention that notions of naturalness and fine-tuning are subjective; for example, one might consider a flat Universe to be infinitely fine-tuned since it has $\Omega_T$ identically equal to one, thereby making it the most unnatural choice.

Our main purpose in this work is to define just what it means for the Universe to be nearly flat. We give three logically independent but physically equivalent definitions, based on density, geometry and dynamical behavior. All lead to the
same dimensionless constant, whose significance as a flatness parameter does not appear to have been widely appreciated. Explicitly, this parameter is the ratio of the de Sitter radius of the Universe to a constant of integration in the Friedmann-Lemaître equation.

We derive our flatness parameter on the basis of density in §3, geometry in §4, and dynamics in §5. The question of naturalness is addressed in §6, and in §7 we give our summary and conclusions.

Our discussion has features in common with that of Chernin [18]. Chernin discusses four parameters, which he calls Friedmann constants; these are $A_V$ associated with vacuum energy, $A_D$ associated with dark matter, $A_B$ associated with baryons, and $A_R$ associated with radiation. He emphasizes that all are very roughly equal (within a few orders of magnitude), and suggests that this may be evidence of a cosmic symmetry of unknown origin. On the other hand, we emphasize that for a nearly flat Universe, $A_V$ must be at least several orders of magnitude less than $A_D$, and that indeed this is responsible for the near flatness. Thus our work is consistent with and complementary to that of Chernin.

The issue of near-flatness is also raised in a recent article by Lake [19], who adopts a rather different approach from ours, based on the trajectories of model universes in the $\Omega_M - \Omega_\Lambda$ plane. Lake’s intent is to assess the likelihood that the Universe is exactly flat, whereas we would contend that observation cannot distinguish — even in principle — between a perfectly flat Universe and one that is sufficiently close to flat.

2 Cosmological Dynamics

We begin by briefly reviewing cosmological dynamics, focusing on the present epoch in which the Universe is dominated by a cosmological constant (or dark energy) and pressureless matter. The matter component could be further broken down into cold dark matter and baryonic matter, but these have the same effect on the dynamics and we treat them together. For early times, radiation is also important [18]. We assume a linear equation of state for the matter component, $p = w \rho$ with $w = \text{const}$. The basic equations of homogeneous and isotropic cosmology may be written (with $c = 1$):

$$8\pi G \rho = -\Lambda + 3 \left( \frac{k}{a^2} + \frac{\dot{a}^2}{a^2} \right),$$
$$8\pi G p = \Lambda - \left( \frac{k}{a^2} + \frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a} \right).$$

With the linear equation of state and some algebraic manipulation, we obtain a first integral of these (the Friedmann-Lemaître equation) as

$$\dot{a}^2 = \left( \frac{C}{a} \right)^{3w+1} + \frac{a^2}{R_e^2} - k,$$

\[3\]
Figure 2: Scale factor as a function of time in the standard concordance model.

where $C$ is a constant and $R_\Lambda$ is the de Sitter radius, defined by:

$$R_\Lambda^2 = \frac{3}{\Lambda} = \frac{3}{8\pi G \rho_\Lambda} = \frac{1}{H^2 \Omega_\Lambda} = \frac{1}{H_0^2 \Omega_{\Lambda, 0}}.$$  

Here $\Omega_{\Lambda, 0} = \rho_\Lambda/\rho_{\text{crit}, 0}$ is the present density ratio of vacuum energy; we note that the vacuum energy density is constant but the density ratio $\Omega_\Lambda$ is not. (The constant $C$ here is the same as in Lake’s paper [19], while $C$ and $R_\Lambda$ are referred to as $A_D$ and $A_V$ respectively in the paper by Chernin [18].) Most of the discussion in this paper will be based on Eq. (3). For the case of a flat ($k = 0$) Universe, it can be solved exactly when the matter component is dominated by, e.g., pressureless dust ($w = 0$) or radiation ($w = 1/3$):

$$a(t) = \begin{cases} (CR_\Lambda^2)^{1/3} \sinh^{2/3}(3t/2R_\Lambda) & (w = 0) \\ (CR_\Lambda)^{1/3} \sinh^{1/2}(2t/R_\Lambda) & (w = 1/3) \end{cases}.$$  

The first of these expressions governs the now-standard concordance model of present-era cosmology (see Fig. 2). It is found in surprisingly few of the standard texts [20, 21] and goes back to Heckmann in 1931 [22, 23].

Some special times are of particular interest in this model: the time when acceleration begins, the time of equality between energy densities of matter and dark energy, and the present time (when dark-energy density is approximately 2.8 times the matter density). These are conveniently characterized by:

$$\sinh(3t/2R_\Lambda) = \begin{cases} 1/\sqrt{2} & \text{(acceleration begins)} \\ 1 & \text{(equal matter/dark-energy density)} \\ 1/\sqrt{\Omega_{\Lambda, 0}/\Omega_{M, 0}} & \text{(at present, } \approx 1.67) \end{cases}.$$  

Best-fit values of the relevant cosmological parameters from the WMAP satellite
are $H_0 = 71 \text{ km s}^{-1} \text{ Mpc}^{-1}, \Omega_{M,0} = 0.27$ and $\Omega_{\Lambda,0} = 0.75 \ [1]$, so that the de Sitter radius works out to $R_{\Lambda} = 16 \text{ Gly}$ from Eq. (4).

Note that for the case $k = 0$ the equations are scale-invariant, so the scale function is arbitrary to within a multiplicative factor and is not a measurable quantity. For $k = \pm 1$ the scale function is measurable in terms of $H$ and $\Omega_T$, as we will show in the next section.

3 Density Definition of Nearly Flat

Since density can never be measured with perfect precision, it is clear that we cannot use the quantity $\Omega_{\tau,0}$ itself to verify whether or not the Universe is precisely flat. But the observational bound [1] certainly indicates that the Universe is either flat, or nearly so. In fact, we will use $\Omega_T$ to obtain our first definition of nearly flat.

The first of the cosmological field equations (2) together with the first integral (3) allows us to solve for $\Omega_T$ in terms of the scale factor as follows:

$$\Omega_T = \left[ 1 - \frac{k}{(C/a)^{3w+1} + a^2/R^2_{\Lambda}} \right]^{-1} = \left[ 1 - \frac{k}{f(a)} \right]^{-1},$$

(7)

where the function $f(a)$ reads

$$f(a) \equiv (C/a)^{3w+1} + a^2/R^2_{\Lambda} = C/a + a^2/R^2_{\Lambda}$$

(8)

for the present era with $w = 0$. We plot this function in Fig. 3 and note that its minimum value is $3(C/2R_{\Lambda})^{2/3}$. The total density parameter itself is plotted in

Figure 3: Behavior of the function $f(a)$ that determines the density ratio, with minimum $f_{\text{min}} = 3(C/2R_{\Lambda})^{2/3}$ at $a = (CR^2_{\Lambda}/2)^{1/3}$. We have used Eqs. (2) and (4) for $R_{\Lambda}$ and $C$ respectively, with WMAP values for $\Omega_{M,0}$ and $\Omega_{\Lambda,0}$.
Figure 4: Behavior of the density ratio versus scale factor, using Eqs. (4) and (12) for $R_\Lambda$ and $C$ respectively, and assuming WMAP density parameters at present ($a/a_0 = 1$): $\Omega_{T,0} = 1.02 \pm 0.04$. The flat case $\Omega_{T,0} = 1$ is also shown for comparison.

Fig. 4: note that $\Omega_T \to 1$ in the early Universe when $a \to 0$, and that $\Omega_T \to 1$ in the distant future when $a$ becomes very large. Since observations indicate that $\Omega_T$ is close to one, the second term in square brackets must be small, so we may expand as

$$\Omega_T \approx 1 + \frac{k}{C/a + a^2/R_\Lambda^2}.$$  \hspace{1cm} (9)

The parameter $\Omega_T$ will be close to unity if

$$f(a) = C/a + a^2/R_\Lambda^2 > f_{\text{min}} = 3(C/2R_\Lambda)^{2/3} \gg 1.$$  \hspace{1cm} (10)

A nearly flat Universe with a total density parameter $\Omega_T$ that is nearly equal to unity for all times is therefore equivalent to one with a small flatness parameter $\varepsilon$, defined as

$$\varepsilon \equiv R_\Lambda/C \ll 1.$$  \hspace{1cm} (11)

This parameter is the constant ratio of a fundamental constant in the theory ($R_\Lambda$) to a constant of integration ($C$), so it is allowed by the theory to have any value. (It is equal to Chernin’s $A_V/A_D$ \cite{18}. However, many theorists would consider it unnatural for $\varepsilon$ to differ from unity by many orders of magnitude, or for $C$ to differ by many orders of magnitude from the natural value $R_\Lambda$.

The constant $C$ is simply related to the matter density; this can be seen by comparing the first of Eqs. (2) with Eq. (3) divided by $a^2$ (for $w = 0$):

$$C = \frac{(8\pi G/3)(\rho a^3)}{\Omega_M H^2 a^3} = \Omega_{M,0} H_0^2 a_0^3,$$  \hspace{1cm} (12)
where the final steps follow from the definition of $\Omega_M$ and the fact that $C = \text{constant}$. This relation expresses the well-known way in which matter density decreases as the Universe expands.

Let us assume that the Universe is not exactly flat, but is nearly flat, and use the observational bound (11) to obtain an upper limit and rough estimate for the flatness parameter. If we divide the first of Eqs. (2) by $3H^2$ we obtain

$$\Omega_T - 1 = k/H^2a^2.$$ (13)

Combining our expressions (11) and (12) with this, we find that

$$\varepsilon = \left(\frac{H_0^2a_0^3\Omega_{\Lambda,0}\sqrt{\Omega_{T,0}}}{\Omega_{M,0}}\right)^{-1} = \frac{|\Omega_{T,0} - 1|^{3/2}}{\Omega_{M,0}\sqrt{\Omega_{\Lambda,0}}} \lesssim 0.012.$$ (14)

It then follows from the definitions (11) and (11) that the constant $C$ must be at least

$$C = \frac{1}{\varepsilon H_0\sqrt{\Omega_{\Lambda,0}}} \gtrsim 1300 \text{ Gly}.$$ (15)

Moreover the scale function can be determined explicitly if $k \neq 0$; from Eq. (13) it is given by

$$a = \left(\frac{k}{\Omega_T - 1}\right)^{1/2}\frac{1}{H},$$ (16)

which takes the value $a_0 \approx 99 \text{ Gly}$ at present. (This result nicely dislays how $a$ becomes indeterminate for $k = 0$ and $\Omega_T = 1$.) The value of the flatness parameter in Eq. (13) is small, but can hardly be considered unnatural. Similarly comments apply to the distances in Eq. (15) and (16).

It is amusing to note that an independent means of determining $\varepsilon$ is available in principle. From the first of Eqs. (2) we may obtain

$$\left(\frac{H}{H_0}\right)^2 = \Omega_{M,0}(a_0/a)^3 + \Omega_{\Lambda,0} - (\Omega_{T,0} - 1)(a_0/a)^2.$$ (17)

For $k = 1$, $(H/H_0)^2$ dips below its asymptotic value of $\Omega_{\Lambda,0}$, dropping to a minimum of $(H_*/H_0)^2 = \left[1 - (4/27)\varepsilon^2\right]\Omega_{\Lambda,0}$ at $(a_*/a_0) = \frac{3\Omega_{M,0}/2(\Omega_{T,0} - 1)}{\Omega_{\Lambda,0}}$. Thus observations of the expansion rate at this epoch would yield a definitive measurement of $\varepsilon$. Unfortunately the relevant time occurs so far in the future (of order 60 Gyr) that such a method is not of practical interest.

In summary, we may say that, in terms of density, the Universe is nearly flat if the flatness parameter defined in Eq. (11) is small, which means that the constant of integration $C$ is large in comparison to the de Sitter radius $R_{\Lambda}$. Future improvements in measurements of the density ratio $\Omega_{T,0}$ are obviously of great interest, particularly since its current value hints at a density higher than the critical one, implying that the Universe may have positive spatial curvature and be finite.
4 Geometrical Definition of Nearly Flat

Geometry provides the most intuitive method for defining near-flatness. A part of the surface of a sphere is nearly flat if its extent is small compared to the radius of the sphere. In the same way, we may say that the Universe is nearly flat if the region to which we have access is small compared to its characteristic radius. For a metric in the standard form,

$$ds^2 = dt^2 - a^2(t) \left( \frac{du^2}{1 - ku^2} + u^2 d\theta^2 + u^2 \sin^2 \theta \, d\phi^2 \right), \quad (18)$$

this may be expressed mathematically by saying that the Universe is flat within some region (i.e., within proper radius $\ell$) if its dimensionless comoving radial coordinate $u$ remains small throughout that region:

$$u^2 \ll 1. \quad (19)$$

Our region of interest is the accessible Universe, so we take $\ell$ to be the Hubble distance $\ell = c/H$.

Proper (or physical) distance $\ell$ is calculated in terms of $u$ by

$$\ell = \int_0^u \frac{a(t) \, du'}{\sqrt{1 - ku'^2}} = \begin{cases} a(t) \sin^{-1} u & (k = 1) \\
 a(t) u & (k = 0) \\
a(t) \sinh^{-1} u & (k = -1) \end{cases} = a(t) u \left[ 1 + \frac{k}{6} u^2 + \cdots \right]. \quad (20)$$

Eq. (20) is easily inverted for $u = u(\ell)$:

$$u = (\ell/a)[1 - \frac{k}{6} (\ell/a)^2 + \cdots]. \quad (21)$$

We will need only the lowest order in $(\ell/a)$, so that the value of $k$ is irrelevant.

Taking $\ell = 1/H = a/\dot{a}$ for the accessible Universe, we find for the square of the comoving Hubble distance:

$$u_{H}^2 = \frac{\Omega_T - 1}{k}. \quad (22)$$

The equivalence of the definitions in terms of density and geometry becomes clear if we use Eqs. (17) and (22) to relate $\Omega_T$ to $u_{H}$, as follows:

$$u_{H}^2 = \frac{\Omega_T - 1}{k}. \quad (23)$$

That is, the comoving Hubble distance and the deviation of the density ratio from unity are small together, and both vanish at very early and very late times. Using the WMAP value $\Omega_{T,0} \approx 1.02$, the present spatial extent of the Universe in terms of the comoving Hubble distance is $u_{H,0} \approx \sqrt{0.02} = 0.14$.

In brief summary, the geometric definition of near-flatness is completely equivalent to the density definition.
Figure 5: The mechanical analogy for the dynamics of the Universe for each of the three cases of spatial curvature. The maximum of the potential occurs at $x = 1/2^{1/3}$ where $V_{\text{min}} = -3/2^{2/3}$.

5 Dynamical Definition of Nearly Flat

The first integral (3) may be put into dimensionless form by the substitutions $\tau \equiv t/R_\Lambda$ and $x \equiv a/(CR_\Lambda^2)^{1/3}$ to obtain

$$\left(\frac{dx}{d\tau}\right)^2 - \left(1/x + x^2\right) = -k\varepsilon^{2/3}. \quad (24)$$

It is apparent that, for small $\varepsilon$, the dynamical behavior will be similar to that for $k = 0$. Thus, as in the previous sections, we are led to $\varepsilon$ as a measure of near-flatness. Eq. (24) can of course be solved exactly by quadratures, as we will do below, but for a perturbative solution the appropriate expansion parameter is clearly $k\varepsilon^{2/3}$.

It is worth noting the mechanical analogy for Eq. (24), which is a particle of mass 2 and total energy $E = -k\varepsilon^{2/3}$ in a potential $V(x) = -(1/x + x^2)$, as shown in Fig. 5. The particle flies rapidly outward from the origin, coasts over the top of the potential hill, and accelerates down the hill to infinity. Obviously, if the total energy $E$ is much smaller than the distance between the top of the hill to zero (which is $3/2^{2/3}$), then total energy is unimportant and the three cases $k = 0, \pm 1$ will exhibit the same qualitative behavior. The difference is a shift in the time required to reach a given value of $x$.

Let us illustrate the use of Eq. (24). Very early in the history of the Universe, its expansion is dominated by the $1/x$ term, while for its very late history it is dominated by the $x^2$ term. Only at intermediate times is the last perturbative term relevant. This is easily seen if we take the positive root of Eq. (24) and
solve exactly by quadratures, then expand in \( \delta \equiv k \varepsilon^{2/3} \) as follows:

\[
\tau = \int_0^x \frac{dx}{\sqrt{1/x + x^2 - \delta}} = \sum_{n=0}^{\infty} \delta^n c_n I_n .
\]  

The integrals \( I_n \) are given by

\[
I_n \equiv \int_0^x \frac{dy}{(1/y + y^2)^{n+1/2}} ,
\]  

with \( c_0 \equiv 1 \) for \( n = 0 \) and \( c_n \equiv \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \) for \( n > 0 \).

To zeroth order in \( \delta \) one has

\[
\tau = I_0 = (2/3) \ln(x^{3/2} + \sqrt{1 + x^4}) .
\]  

This is equivalent to the first of Eqs. (24), valid for \( k = 0 \). The higher-order terms in the expansion (25) represent the shift in time required for the Universe to attain a given value of \( a \) when \( k = \pm 1 \), relative to the \( k = 0 \) case. The integrals \( I_n \) cannot be expressed in terms of elementary functions for \( n > 0 \).

In particular, it is possible to solve exactly for the accumulated time shift very late in the history of the Universe using Eq. (26). In the limit as \( a \) goes to infinity, one has:

\[
\lim_{x \to \infty} I_n = \int_0^\infty \frac{dy}{(1/y + y^2)^{n+1/2}} = \frac{\Gamma(n + 1/2) \Gamma(2n)}{3 \Gamma(n + 3/2)} .
\]  

The leading contributions to \( \Delta \tau \equiv \tau - I_0 \) are then found (in the late-time limit) as

\[
\lim_{a \to \infty} \Delta \tau = \frac{1}{3}(k \varepsilon^{2/3}) \lim_{x \to \infty} I_1 + \frac{3}{8}(k \varepsilon^{2/3})^2 \lim_{x \to \infty} I_2 + \cdots = 0.29 k \varepsilon^{2/3} + 0.075 k^2 \varepsilon^{4/3} \cdots .
\]  

In conjunction with the upper limit (23) on \( \varepsilon \), the contributions of these two terms to the asymptotic shift in physical time \( \Delta t = R_\lambda \Delta \tau \) amount to no more than \( \pm 240 \) Myr and 3 Myr respectively.

The time shift \( \Delta \tau \) that has accumulated up to the present time can be calculated by evaluating Eq. (26) numerically with \( x = x_0 = (\Omega_\lambda,0/\Omega_{M,0})^{1/3} \); one
finds in this way that the first two terms in the expansion contribute no more than ±120 Myr and 2 Myr, respectively. These times are small on cosmological scales, but not negligible. For instance, they are much longer than the duration of the radiation-dominated era.

6 Flatness of the Early Universe

At present the Universe is either flat with \( k = 0 \), or nearly flat with flatness parameter \( \varepsilon \leq 10^{-2} \). But the early Universe was extremely close to density-flat, as emphasized by many authors, notably Dicke [25]. Some authors have suggested that \( k = 0 \) is the most natural choice, since otherwise the density ratio \( \Omega_T \) must have been fine-tuned to be very close to unity, to within about a part in \( 10^{17} \) at the time of nucleosynthesis. This is the “flatness problem” of standard cosmology, and one of the issues that the inflationary paradigm is claimed to address [10].

Here we advocate a rather different view, for two reasons. First, as mentioned in the introduction, the notion of naturalness is fundamentally subjective, and one is equally entitled to view the case \( k = 0 \) as infinitely fine-tuned, with \( \Omega_T \) identically equal to one. Second, as we will show below, the difference between the actual density and the critical density actually diverges like \( t^{-1} \) in the early radiation-dominated Universe. The density ratio \( \Omega_T \) approaches unity only because the critical density diverges even faster, like \( t^{-2} \). The apparent fine-tuning is thus inherent in the definition of the critical density and the equations of general relativity, in particular Eq. (7). As we will discuss in detail below, \( \Omega_T \) must approach very close to one at early times. Thus the quantity \( \Omega_T - 1 \) is not a useful measure of naturalness or fine-tuning, and we suggest the use of the constant flatness parameter \( \varepsilon \) instead. Since this is as large as \( 10^{-2} \) according to present observations, a nearly flat Universe cannot as yet be claimed to be excessively fine-tuned.

Let us first calculate the deviation of the total density from critical in the early Universe. From the first of the cosmological equations (2) and the definition of the critical density, we see that

\[
\rho_T - \rho_{\text{crit}} = \left( \frac{3}{8\pi G} \right) \left( k/a^2 \right). \tag{31}
\]

In the early radiation era, \( a \propto t^{1/2} \) so for \( k \neq 0 \) the difference in Eq. (31) diverges at early times like \( t^{-1} \). The quantity \( \Omega_T - 1 = (\rho_T - \rho_{\text{crit}})/\rho_{\text{crit}} \) is small only because the critical density diverges even more rapidly:

\[
\rho_{\text{crit}} = \left( \frac{3}{8\pi G} \right) (\dot{a}/a)^2 \propto t^{-2}. \tag{32}
\]

The smallness of the quantity \( \Omega_T - 1 \) at early times is thus an artifact of the definition of the critical density.
Let us next estimate the deviation of the density ratio from unity at the time of decoupling, which occurred at a redshift of about \( z_d \approx (a_0/a_d) \approx 10^3 \) at a time \( t \approx 10^{12} \) s; this is also roughly the time of matter and radiation density equality. From Eq. (7):

\[
\Omega_T - 1 \approx k/f(a) \approx ka/C .
\]  

This is a reasonable approximation in the matter-dominated era (i.e., until recently), during which \( a \propto t^{2/3} \). Thus the deviation of \( \Omega_T \) from unity at decoupling obeys

\[
(\Omega_{T,d} - 1)/(\Omega_{T,0} - 1) \approx a_d/a_0 \approx 1/z_d \approx 10^{-3} .
\]  

Since the present deviation is about \( \lesssim 10^{-2} \) we obtain a small deviation of about \( \lesssim 10^{-5} \) at decoupling.

Similarly, we may go back in time in the radiation era, during which \( w = 1/3 \) and \( a \propto t^{1/2} \), to a time of about \( t_n \approx 1 \) s, when nucleosynthesis occurred. As above, but with \( w = 1/3 \) in Eq. (7), we obtain

\[
(\Omega_{T,n} - 1)/(\Omega_{T,d} - 1) \approx (a_n/a_d)^2 \approx t_n/t_d \approx 10^{-12} .
\]  

Thus the deviation during nucleosynthesis was only about \( 10^{-17} \), an impressively small number. It follows merely from the present density deviation, which is not necessarily very small, plus the cosmological equations and the definition of the critical density.

A useful analogy from elementary physics might be the following: consider a test particle of mass \( m \) with total energy \( E \) falling into the Newtonian gravitational field of a mass \( M \). The ratio of this particle’s kinetic energy \( K = mv^2/2 \) to its potential energy \( |U| = GMm/r \) is \( K/|U| = (E/GMm)r + 1 \). Note that the difference \( K/|U| - 1 \) becomes arbitrarily small as one approaches \( r \to 0 \), in exactly the same way that \( \Omega_T - 1 \) does in cosmology as \( t \to 0 \). Yet one would hardly be justified in concluding from this that \( E \) “must be” zero on the grounds of naturalness.

In summary, the extremely small deviation of the density ratio from unity in the early Universe is a consequence of the definition of the critical density and the basic equations of relativistic cosmology for any value of \( k \). We therefore do not agree with the viewpoint that \( k = 0 \) is necessarily the most natural interpretation of current observational data. If future experiments produce a much smaller limit on the flatness parameter \( \varepsilon \) (say, \( 10^{-5} \)), then that might be a more convincing indication that the most natural value for \( k \) is zero.

Chernin has made similar observations, with which we concur [18], although our interpretation is somewhat different. Chernin emphasizes that the most notable feature of the Universe is not its near-flatness, nor the densities of its various components, but the closeness of the Friedmann constants \( \Lambda \) and \( \Omega_\Lambda \) to each other. He has subsequently argued that this apparent coincidence may in turn be understood most naturally in a closed universe whose radius of curvature is of the same order of magnitude as its present size [26].
7 Summary and Conclusion

We have presented a new quantitative definition of a nearly-flat Universe in terms of a flatness parameter $\varepsilon$, and shown that it follows uniquely from three independent lines of argument based on density, geometry and dynamical behavior. It is clear from our derivations that there is no way, even in principle, to distinguish between a precisely flat Universe and one whose flatness parameter is sufficiently small. Measurements of total density from CMB data currently set an upper limit on $\varepsilon$ of order $10^{-2}$, which is small but not extremely small or unnatural. By contrast, the variable quantity $\Omega_T - 1$ is not a useful flatness criterion, because $\Omega_T$ is necessarily driven toward unity at both early and late times (for any value of $k$) by the equations of relativistic cosmology and the definition of the critical density.

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