Hölderian convergence of fractional extended nabla operator to fractional derivative

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Abstract
In this paper, we construct the fractional extended nabla operator as fractional power of linear spline of backward difference operator. Then we prove the strong convergence of this operator to fractional derivative in a Hölder space setting. Finally numerical examples are presented.

Keywords. fractional power, fractional derivatives, Hölderian convergence, fractional differential equation,

MSC (2010). 26A33, 47B38, 46E15, 39A70, 97N50, 47A60

1 Introduction
It is well-known that fractional calculus is a developing field both from the theoretical and applied point of view. The fractional differential equations turned out to be the best tool for modeling memory-dependent processes [5]. We refer to the monograph [22], which contains almost complete qualitative fractional differential equation theory, and to the monograph [7] for an application oriented exposition.

Besides this rapid development, the notion of difference operators has been extended to fractional calculus in different ways [11], [15], [17]. The discrete calculus provides a natural setting to define such operators. However, in literature there is no single definition of fractional difference operators and this situation can be confusing (see for example [1], [2] and [14]).

Another way to define this operators is to consider the fractional power of positive discrete operators see [4].

Effectively, functional calculus is a consistent way to define operators of the form $A^\alpha$ for a given linear operator $A$ in a Banach space. The fundamental aspects of the theory of fractional powers of non-negative operators are given in [13]. Sectorial operators satisfy a resolvent condition that leads to define the fractional power of such operators. The functional calculus for sectorial operators has been developed by M. Haase in the book [10].

Apart from [4], we do not know about any other work done on fractional difference derivative in terms of spectral operator theory.

In this paper, we define the fractional difference as fractional power of the nabla operator in a Hölder space. The Hölder spaces offer an interesting point of view in the analysis of fractional integrals and derivatives. This framework

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was developed by Samko et al. for fractional operators in the sense of Marchaud [17], [18]. In this functional framework we study the strong convergence of the extended backward differences to the derivative. We construct the fractional operators associated and the strong convergence result is proved. Lastly, some examples are provided to show the effectiveness of the approach.

This paper is organized as follows: The section 2 is devoted to preliminaries and some Hölderians tools. Then, the operators in this context are defined. In section 3, we give the basic definitions and results for the fractional power of sectorial operator and we construct the different fractional operators as fractional power of sectorial operator in Hölder spaces setting. In section 4, we discuss the strong convergence of the operators involved. Some examples are given in section 5.

2 Preliminaries on operators in Hölder spaces

Without loss of generality, we assume that the functions are defined on the interval $[0, 1]$. Let $H^\beta$ be the Banach space of Hölderian function on $[0, 1]$ with exponent $\beta$, where $0 < \beta < 1$ and such that $f(0) = 0$, endowed with the norm $\|f\|_\beta = \omega_\beta(f, 1)$, where

$$\omega_\beta(f, \delta) = \sup_{0 < |s-t| \leq \delta} \frac{|f(t) - f(s)|}{|t-s|^\beta}$$

Let $H^\beta_0$ be the subspace defined by

$$H^\beta_0 = \left\{ f \in H^\beta, \lim_{\delta \to 0} \omega_\beta(f, \delta) = 0 \right\}$$

**Remark 1** If $f \in H^\beta_0$ then $|f(t) - f(t-h)| = o(h^\beta)$ uniformly in $t$, for $t = h$ we get $|f(h)| = o(h^\beta)$.

**Remark 2** If $f \in H^\beta$ then $\|f\|_\infty \leq \|f\|_\beta$. Indeed, for every $x \in [0, 1]$, $|f(x)| = \frac{|f(x) - f(0)|}{x^\beta} x^\beta \leq \|f\|_\beta$.

**Remark 3** If $f' \in H^\beta$ then $\omega_\beta(f, h)$ tends to 0 as $h$ tends to 0.

Indeed, for every $x, y \in [0, 1]$, $x \neq y$ there exists some $\xi \in [x, y]$ such that

$$\frac{|f(x) - f(y)|}{|x-y|^\beta} = |x - y|^{1-\beta} |f'(\xi)| \leq |x - y|^{1-\beta} \|f'\|_\beta$$

which leads to

$$\omega_\beta(f, h) \leq h^{1-\beta} \|f'\|_\beta$$

The Hölder norm of the piecewise linear interpolation is given by the next lemma. As far as we know, this result was first proved by H. E. White, Jr, in a general setting see [21, 3.2 Corollary p 106] but we follow [16] in the presentation.
Lemma 2.1 (see lemma 3.1 [16]) Let \( t_0 = 0 < t_1 < \cdots < t_n = 1 \) be a partition of \([0,1]\) and \( f \) be a real valued polygonal line function on \([0,1]\) with vertices at \( t_i \)'s, i.e. \( f \) is continuous on \([0,1]\) and its restriction to each interval \([t_i, t_{i+1}]\) is an affine function. Then for any \( 0 \leq \beta < 1 \),
\[
\sup_{0 \leq s < t \leq 1} \frac{|f(t) - f(s)|}{|t-s|^\beta} = \max_{0 \leq i < j \leq 1} \frac{|f(t_j) - f(t_i)|}{(t_j - t_i)^\beta}
\]

Definition 1 For \( 0 < h < 1 \) fixed, let \( \Delta_h \) be the subdivision of \([0,1]\) in \( n \) subintervals with \( n = \lfloor 1/h \rfloor \) and \( t_k = kh, \) for each \( k = 0,1,\ldots,n \) where \( \lfloor a \rfloor \) means the integer part of \( a \). We denote by \( \mathcal{I}_h \in \mathcal{L}(H^\beta) \) the piecewise linear interpolation operator defined by
\[
(\mathcal{I}_h f)(x) := \sum_{k=1}^{n} \left( \frac{x - t_{k-1}}{h} f(t_k) + \frac{t_k - x}{h} f(t_{k-1}) \right) \mathbb{1}_{[t_{k-1}, t_k]}(x)
\]

In the following lemma the remainder of piecewise linear interpolation is expressed in Hölder norm.

Lemma 2.2 Let \((r_h f)(x) = (I - \mathcal{I}_h) f(x)\) then
\[
\| (r_h f) \|_{\beta} \leq 4 \omega_\beta(f, h)
\]

Proof First let us suppose that \( x, y \in [t_{k-1}, t_k] \) then
\[
(r_h f)(x) - (r_h f)(y) = f(x) - f(y) - \frac{x - y}{h} (f(t_k) - f(t_{k-1}))
\]

It follows, from \( |x - y| < h \) that
\[
\frac{|(r_h f)(x) - (r_h f)(y)|}{|x - y|^\beta} \leq \frac{|f(x) - f(y)|}{|x - y|^\beta} + \frac{|f(t_k) - f(t_{k-1})|}{h^\beta} \leq 2 \omega_\beta(f, h)
\]

Second, suppose that \( x \in [t_{k-1}, t_k] \) and \( y \in [t_k, t_{k+1}] \) then from the first case
\[
\frac{|(r_h f)(x) - (r_h f)(y)|}{|x - y|^\beta} \leq \frac{|(r_h f)(x) - (r_h f)(t_k)|}{|x - t_k|^\beta} + \frac{|(r_h f)(t_k) - (r_h f)(y)|}{|t_k - y|^\beta} \leq 4 \omega_\beta(f, h)
\]

Third, suppose that \( x \in [t_{k-1}, t_k] \) and \( y \in [t_{m-1}, t_m] \) with \( |x - y| > h \) then
\[
\frac{|(r_h f)(x) - (r_h f)(y)|}{|x - y|^\beta} \leq \frac{|(r_h f)(x) - (r_h f)(t_k)|}{|x - y|^\beta} + \frac{|(r_h f)(t_k) - (r_h f)(t_{m-1})|}{|x - y|^\beta} + \frac{|(r_h f)(t_{m-1}) - (r_h f)(y)|}{|x - y|^\beta}
\]

Knowing that \((r_h f)(t_k) = (r_h f)(t_{m-1}) = 0\) then
\[
\frac{|(r_h f)(x) - (r_h f)(y)|}{|x - y|^\beta} \leq 4 \omega_\beta(f, h)
\]
We denote by $A = \frac{d}{dx}$ the differential operator acting on $H^\beta$ with domain,

$$D(A) = \{ f \in H^\beta, f' \in H^\beta \},$$

and for $0 < h < 1$, let $\nabla_h \in \mathcal{L}(H^\beta)$ the nabla operator defined by

$$(\nabla_h f)(x) := \frac{f(x) - f(x - h)}{h}, \quad \text{for } x \in [h, 1]$$

We set $(\nabla_h f)(x) = f(x)$ for all $0 < x < h$.

Lastly, we introduce the extended nabla operator as the polygonal line with vertices $(t_k, \nabla f(t_k))$, $k = 0, 1, \ldots, n$, in the following definition.

**Definition 2** We define the operator $A_h$ for any $f \in H^\beta$ by

$$A_h f(x) = (I_h \nabla_h f)(x)$$

Obviously $A_h$ is a linear bounded operator with $\|A_h\|_\beta \leq \frac{2}{h}$.

In the next proposition, it can be pointed out that the sequence $(A_h)_h$ has no uniform limit as $h$ tends to 0.

**Proposition 2.3** The sequence $(A_h)_h$ of extended nabla operator is not a convergent sequence in $\mathcal{L}(H^\beta)$ as $h$ tends to 0.

**Proof** We need only to prove that $(A_h)_h$ is not a Cauchy sequence.

Let $\Delta_h, \Delta_{h/2}$ be two subdivisions of $[0, 1]$ and $f(x) = x^\beta$. We then have for $t = h/2$

$$|(A_h - A_{h/2}) f(t)| = \left| \frac{h/2 f(h) - f(h/2)}{h} \right| \geq \left| \left( \frac{h}{2} \right)^{\beta - 1} - \left( \frac{1}{2} \right)^{\beta - 1} \right|$$

Since

$$\| (A_h - A_{2h}) f \|_\beta \geq \| (A_h - A_{2h}) f(h/2) \|$$

then

$$\|A_h - A_{2h}\|_\beta \|f\|_\beta \geq \| (A_h - A_{2h}) f \|_\beta \geq \left( \frac{1}{2} \right)^{\beta - 1} - \frac{1}{2}$$

We have thus seen that $(A_h)_h$ is not a convergent sequence in $\mathcal{L}(H^\beta)$.

In the next proposition the strong convergence of extended nabla operator to the derivative operator is proved.

**Proposition 2.4** For every $f \in D(A)$, such that $f' \in H^\beta_0$ the sequence $(A_h)_h$ converges strongly to $A$ as $h$ tends to 0.
Proof Note that

\((A - A_h) f(x) = (I - I_h) Af(x) + I_h (A - \nabla_h) f(x) = (r_h f') (x) + I_h (A - \nabla_h) f(x)\).

Then

\[
\frac{|(A - A_h) f(x) - (A - A_h) f(y)|}{|x - y|^3} \leq \|r_h f'\|_\beta + \|I_h (A - \nabla_h) f(x) - I_h (A - \nabla_h) f(y)|}{|x - y|^3}
\]

From lemma 2.3

\[
\|r_h f'\|_\beta \leq 4\omega_\beta (f', h)
\]

For some \(i, j, \xi_1 \in ]t_{i-1}, t_i[\) and \(\xi_2 \in ]t_{j-1}, t_j[\) by lemma 2.1 we have

\[
\frac{|I_h (A - \nabla_h) f(x) - I_h (A - \nabla_h) f(y)|}{|x - y|^3} \leq \frac{|f'(t_i) - \nabla_h f(t_i)|}{|t_i - t_j|^3} + \frac{|f'(t_j) - \nabla_h f(t_j)|}{|t_i - t_j|^3}
\]

\[
\leq \frac{|f'(t_i) - f'(\xi_1)|}{|t_i - t_j|^3} + \frac{|f'(t_j) - f'(\xi_2)|}{|t_i - t_j|^3},
\]

\[
\leq 2\omega_\beta (f', h)
\]

Hence

\[
\frac{|(A - A_h) f(x) - (A - A_h) f(y)|}{|x - y|^3} \leq 6\omega_\beta (f', h)
\]

Therefore \(\lim_{h \to 0} \|A f - A_h f\|_\beta = 0\).

3 Fractional power of sectorial operator

3.1 Sectorial property

We first, recall the Haase concept of sectorial operators [10] Section 2.1, p19]. In the following \(R(\lambda, B) = (\lambda I - B)^{-1}, \rho(B)\) and \(\sigma(B) = \mathbb{C} \setminus \rho(B)\) denote respectively the resolvent, the resolvent set and the spectrum of a linear operator \(B\) on a Banach space \(Z\). Let \(S_\omega\) denote the open sector

\[
\{ z \in \mathbb{C}, z \neq 0 \text{ and } |\arg z| < \omega \}, 0 < \omega \leq \pi.
\]

Definition 3 The operator \(B\) is sectorial of angle \(\omega < \pi\) (in short: \(B \in \text{Sect}(\omega)\)) if:
1) \(\sigma(B) \subset S_\omega\) and
2) \(M(B, \omega') := \sup \{ \| R(\lambda, B) \|, \lambda \notin \overline{S_{\omega'}} \} < \infty\) for all \(\omega < \omega' < \pi\).

A family of operators \(\{B_i\}_i\) is uniformly sectorial of angle \(\omega\) if \(B_i \in \text{Sect}(\omega)\) for each \(i\) and \(\sup \{ M(B_i, \omega') \} < \infty\) for all \(\omega < \omega' < \pi\).

Remark 4 Sectorial operator in Haase definition don’t have to be densely defined see [13] Definition 3.8, p 97].
We are now able to define the fractional power of a sectorial operator with help of the Balakrishnan representation (see [10, Proposition 3.1.12])

**Proposition 3.1** Let $B$ an operator with domain $\mathcal{D}(B), B \in \text{Sect}(\omega)$, and let $0 < \alpha < 1$. Then for all $f \in \mathcal{D}(B)$

$$B^\alpha f(x) = -\frac{\sin \alpha \pi}{\pi} \int_0^\infty \lambda^{\alpha-1} R(-\lambda, B)f(x) d\lambda$$

(3)

### 3.2 Fractional power of the derivative

We define the fractional derivative as fractional power of sectorial operator in Hölder space. To do so, we examine sectoriality of $A$.

**Proposition 3.2** The operator $A$ on $H^\beta$ is sectorial of angle $\frac{\pi}{2}$.

**Proof** For all $\lambda \in \mathbb{C}$, the resolvent of the operator $A$ on $H^\beta$ is given by

$$R(\lambda, A)f(x) = -\int_0^x e^{\lambda(x-t)} f(t) dt$$

Let’s take $\lambda \in \mathbb{C}$ with $\text{Re}(\lambda) < 0$ and let $x, h$ be such that $0 \leq x - h < x \leq 1$. We have

$$|R(\lambda, A)f(x) - R(\lambda, A)f(x-h)|
= \left| \int_0^{x-h} -e^{\lambda t} [f(x-t) - f(x-h-t)] dt + \int_{x-h}^x -e^{\lambda t} f(x-t) dt \right|$$

If $p$ and $q$ are two real positive conjugates, the Hölder inequality implies:

$$|\lambda| |R(\lambda, A)f(x) - R(\lambda, A)f(x-h)|
\leq |\lambda| \left( \int_0^{x-h} e^{p \text{Re}(\lambda)t} dt \right)^{\frac{1}{p}} \left( \int_0^{x-h} |f(x-t) - f(x-h-t)|^q dt \right)^{\frac{1}{q}}
+ |\lambda| \left( \int_{x-h}^x e^{p \text{Re}(\lambda)t} dt \right)^{\frac{1}{p}} \left( \int_{x-h}^x |f(x-t)|^q dt \right)^{\frac{1}{q}}
\leq \sup_{0 \leq t \leq x-h} |f(t) - f(t-h)| (x-h)^\frac{q}{p} |\lambda| \left( e^{p \text{Re}(\lambda)(x-h)} - 1 \right)^{\frac{1}{q}}
+ \sup_{x-h \leq t \leq x} |f(x-t)| (h)^\frac{q}{p} |\lambda| \left( e^{p \text{Re}(\lambda)x} - e^{p \text{Re}(\lambda)(x-h)} \right)^{\frac{1}{q}}
\leq \omega_\beta(f, h) h^\frac{q}{p} 2 |\lambda| \left( \frac{-1}{p \text{Re}(\lambda)} \right)^{\frac{1}{q}}$$

Knowing that for every parameter $\text{Re}(\lambda) < 0$ the infimum, on $]1, \infty[$, of function

$$\left( \frac{-1}{p \text{Re}(\lambda)} \right)^{\frac{1}{q}}$$
is given by

$$\inf_{p>1} \left( \frac{-1}{p \Re(\lambda)} \right)^{1/p} = \begin{cases} -\frac{1}{e^{\frac{1}{p} \Re(\lambda)}} & \text{if } -\frac{\pi}{\Re(\lambda)} \leq 1 \\ e^{\frac{1}{p} \Re(\lambda)} & \text{if } -\frac{\pi}{\Re(\lambda)} > 1 \end{cases}$$

Consequently, by $\Re(\lambda) = |\lambda| \cos \omega$ we have

$$|\lambda| \left( -\frac{1}{\Re(\lambda)} \right) = |\lambda| \left( -\frac{1}{|\lambda| \cos \omega} \right) = -\frac{1}{\cos \omega}$$

and

$$|\lambda| e^{\frac{1}{p} \Re(\lambda)} = |\lambda| e^{\frac{1}{p} |\lambda| \cos \omega}.$$

In addition, the function defined on $[0, \infty[$ by $xe^{x \cos \omega}$ admits for maximum value $\frac{1}{\cos \omega}$, which gives the estimate

$$|\lambda| \inf_{p>1} \left( \frac{-1}{p \Re(\lambda)} \right)^{1/p} \leq -\frac{1}{\cos \omega}.$$

According to the previous arguments we get

$$|\lambda| |R(\lambda, A)f(x) - R(\lambda, A)f(x - h)| \leq \frac{2}{\cos \omega} \omega_\beta (f, h) h^\beta$$

which implies

$$|\lambda| \omega_\beta (R(\lambda, A)f, h) \leq \frac{2}{\cos \omega} \omega_\beta (f, h) h^\beta$$

and

$$|\lambda| \|R(\lambda, A)f\|_\beta \leq \frac{2}{\cos \omega} \|f\|_\beta$$

Therefore, for every $\lambda \in \mathbb{C} \setminus \mathcal{S}_\omega$, $\frac{\pi}{2} < \omega \leq \pi$,

$$\|\lambda R(\lambda, A)\| \leq -\frac{2}{\cos \omega}$$

**Corollary 3.3** Let $0 < \alpha < 1$ and $f \in D(A)$. Then

$$A^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} f'(t)dt$$

**Proof** Using the Balakrishnan representation of fractional power of sectorial operator, the previous representation follows.

In the next subsection the fractional power of operator $\nabla_h$ and $A_h$ are constructed.

### 3.3 Fractional nabla operators

Before studying the sectoriality of $\nabla_h$ and $A_h$ we begin by a surprising and useful result. In fact, elementary calculations show that the operator $I_h$ commutes with $\nabla_h$. This property has an interesting consequence for the resolvent operator given in the next lemma and stated in general framework.
Lemma 3.4 Let $X$ be a Banach space, $B,T \in \mathcal{L}(X)$ such that $T$ is idempotent and $T$ commute with $B$ then, for every $\lambda \in \rho(B)$, $\lambda \neq 0$

$$R(\lambda, TB) = TR(\lambda, B) + \frac{1}{\lambda} (I - T)$$

Proof To obtain the resolvent operator for $TB$ we consider the equation, for $f, g \in X$,

$$f = (\lambda I - TB) g$$

then by idempotence of the operator $T$ and commutative property we get

$$Tf = (\lambda I - B)Tg$$

combining the above two equations we have

$$f - Tf = \lambda (I - T) g$$

using the fact that

$$Tg = (\lambda I - B)^{-1} Tf$$

then

$$g = R(\lambda, B)Tf + \frac{1}{\lambda} (f - Tf)$$

Proposition 3.5 The family $(\nabla h)_h$ is uniformly sectorial of angle $\pi/2$ on $H^\beta$.

Proof It can be easily proved using Laplace and inverse Laplace transforms that

$$R(\lambda, \nabla h)f(x) = -h \sum_{j=0}^{n} \frac{1}{(1 - \lambda h)^{j+1}} f(x - t_j)$$

First, we check the boundedness of $R(\lambda, \nabla h)$ in $\mathcal{L}(H^\beta)$. For every $0 \leq x < y \leq 1$,

$$\left| \frac{R(\lambda, \nabla h)f(x) - R(\lambda, \nabla h)f(y)}{|x - y|^\beta} \right|$$

$$\leq \frac{1}{|x - y|^\beta} \left\{ -h \sum_{j=0}^{n} \frac{1}{(1 - \lambda h)^{j+1}} [f(x - t_j) - f(y - t_j)] + \left| -h \sum_{[y/h] + 1}^{[x/h]} \frac{1}{(1 - \lambda h)^{j+1}} f(x - t_j) \right| \right\}$$

$$\leq \|f\|_\beta \sum_{j=1}^{n} \frac{h}{|1 - \lambda h|^2}$$

Using the sum of a geometric series we have

$$\left| \frac{R(\lambda, \nabla h)f(x) - R(\lambda, \nabla h)f(y)}{|x - y|^\beta} \right| \leq \frac{h}{|1 - \lambda h| - 1} \|f\|_\beta \quad (4)$$
Now, observe that for any \( \lambda \in \mathbb{C} \setminus \overline{\mathbb{D}} \), we have \( |\lambda h - 1| > 1 \) and

\[
\frac{|\lambda| |R(\lambda, \nabla_h)f(x) - R(\lambda, \nabla_h)f(y)|}{|x - y|^\beta} \leq \frac{|\lambda| h}{|1 - \lambda h| - 1} \|f\|_\beta
\]

Knowing that

\[
|1 - \lambda h|^2 = h^2 |\lambda|^2 + 1 - 2 |\lambda| \cos(\arg \lambda) \geq h^2 |\lambda|^2 + 1 - 2h |\lambda| \cos \omega
\]

Then

\[
\frac{|\lambda| h}{|1 - \lambda h| - 1} \leq \frac{|\lambda| h}{\sqrt{h^2 |\lambda|^2 + 1 - 2h |\lambda| \cos \omega - 1}}
\]

Put \( \varphi(z) = \frac{z}{\sqrt{z^2 + 1 - 2z \cos \omega - 1}} \) for \( z > 0 \). It is easy to see that \( \varphi(+\infty) = 1 \),

\[
\varphi(0^+) = -\frac{1}{\cos \omega}
\]

and the derivative satisfies

\[
\varphi'(z) = \frac{(-1 + \cos^2 \omega) z^2}{(\sqrt{z^2 + 1 - 2z \cos \omega - 1})^2 \sqrt{z^2 + 1 - 2z \cos \omega} (1 - z \cos \omega + \sqrt{z^2 + 1 - 2z \cos \omega})}
\]

\[
< 0
\]

Consequently, for every \( z \in [0, +\infty] \), \( 1 < \varphi(z) \leq \frac{-1}{\cos \omega} \), which implies that

\[
\frac{|\lambda| |R(\lambda, \nabla_h)f(x) - R(\lambda, \nabla_h)f(y)|}{|x - y|^\beta} \leq \frac{-1}{\cos \omega} \|f\|_\beta \quad (5)
\]

We conclude that the family \( (\nabla_h)_{h} \) is uniformly sectorial of angle \( \frac{\pi}{2} \).

Consequently the extended nabla operator \( A_h \) is also sectorial as shown in the next corollary.

**Corollary 3.6** The family \( (A_h)_{h} \) is uniformly sectorial of angle \( \frac{\pi}{2} \) on \( H^\beta \).

**Proof** From lemma 3.4 we have

\[
R(\lambda, A_h)f(x) = \mathcal{I}_h (R(\lambda, \nabla_h)f)(x) + \frac{1}{\lambda} (f - \mathcal{I}_h f)(x)
\]

Then

\[
\frac{|\lambda| |R(\lambda, A_h)f(x) - R(\lambda, A_h)f(y)|}{|x - y|^\beta} \leq \frac{|\lambda| |\mathcal{I}_h R(\lambda, A_h)f(x) - \mathcal{I}_h R(\lambda, A_h)f(y)|}{|x - y|^\beta} + \frac{|(I - \mathcal{I}_h)f(x) - (I - \mathcal{I}_h)f(y)|}{|x - y|^\beta}
\]

from lemmas 2.4 and 2.8, there exist \( 0 \leq m, l \leq n \) such that
\[
\frac{|\lambda| |R(\lambda, A_h) f(x) - R(\lambda, A_h) f(y)|}{|x - y|^\beta} \leq \frac{|\lambda| |R(\lambda, A_h) f(t_m) - R(\lambda, A_h) f(t_i)|}{|t_m - t_i|^\beta} + 4\omega_\beta(f, h)
\]

From proposition 3.5 we get the estimate
\[
\frac{|\lambda| |R(\lambda, A_h) f(x) - R(\lambda, A_h) f(y)|}{|x - y|^\beta} \leq \left( \frac{-1}{\cos \omega} + 4 \right) \|f\|_\beta
\]

As a result we are able to define the fractional power of \(\nabla_h\) and \(A_h\). This is the purpose of the following theorem.

**Theorem 3.7** Let \(0 < \alpha < 1\), then fractional nabla operator is

\[
\nabla_h^\alpha f(x) = \frac{h^{1-\alpha}}{\Gamma(1 - \alpha)} \sum_{j=0}^{x/h} \Gamma(j + 1 - \alpha) \sum_{i=1}^{1} \frac{\Gamma(j + 1 - \alpha)}{\Gamma(j + 1)} \nabla_h f(x - t_j)
\]

and the fractional operator \(A_h^\alpha\) is

\[
A_h^\alpha f(x) = \sum_{k=0}^{n} \left( \frac{x - t_{k-1}}{h} \sum_{j=1}^{k} \Gamma(j + 1 - \alpha) \sum_{i=1}^{1} \Gamma(j + 1 - \alpha) \sum_{i=1}^{1} \frac{\Gamma(j + 1 - \alpha)}{\Gamma(j + 1)} \nabla_h f(t_{k-1} - t_j) \right) \mathbb{1}_{[t_{k-1}, t_k]}(x)
\]

We call \(A_h^\alpha\) the fractional extended nabla operator.

**Proof** Using Balakrishnan representation of fractional power of sectorial operator (3), we get when \(0 < \alpha < 1\)

\[
\nabla_h^\alpha f(x) = \frac{\sin \alpha \pi}{\pi} \int_{0}^{+\infty} \lambda^{\alpha - 1} (\lambda + \nabla_h)^{-1} \nabla_h f(x) d\lambda
\]

\[
= -\frac{\sin \alpha \pi}{\pi} \int_{0}^{+\infty} \lambda^{\alpha - 1} R(-\lambda, \nabla_h) \nabla_h f(x) d\lambda
\]

Then

\[
\nabla_h f(x) = \frac{\sin \alpha \pi}{\pi} \int_{0}^{+\infty} \lambda^{\alpha - 1} \frac{1}{\Gamma(j+1)} \nabla_h f(x - t_j) d\lambda
\]

\[
= h \sum_{j=0}^{x/h} \left( \frac{\sin \alpha \pi}{\pi} \int_{0}^{+\infty} \lambda^{\alpha - 1} \frac{1}{(1 + \lambda h)^{j+1}} d\lambda \right) \nabla_h f(x - t_j)
\]

Similar calculations to those in [4] Theorem 3.1] give

\[
\int_{0}^{+\infty} \lambda^{\alpha - 1} \frac{1}{(1 + \lambda h)^{j+1}} d\lambda = h^{-\alpha} \frac{\Gamma(j + 1 - \alpha) \Gamma(\alpha)}{\Gamma(j + 1)}
\]
Therefore

\[ \nabla_h^\alpha f(x) = \frac{h^{1-\alpha}}{\Gamma(1-\alpha)} \sum_{j=0}^{[x/h]} \frac{\Gamma(j+1-\alpha)}{\Gamma(j+1)} \nabla_h f(x-t_j) \]

We now turn to the evaluation of \( A_h^\alpha f \). From lemma 3.4 we get

\[ A_h^\alpha f(x) = -\frac{\sin \alpha \pi}{\pi} \int_0^{+\infty} \lambda^{\alpha-1} R(-\lambda, A_h) A_h f(x) d\lambda \]

\[ = -\frac{\sin \alpha \pi}{\pi} \int_0^{+\infty} \lambda^{\alpha-1} \mathcal{I}_h R(-\lambda, \nabla_h) \mathcal{I}_h \nabla_h f(x) d\lambda \]

the required evaluation of \( A_h^\alpha f \) then follows.

**Remark 5** The operator \( \nabla_h^\alpha \) is nothing but the Grünwald-Letnikov operator

\[ \nabla_h^\alpha f(x) = h^{-\alpha} \sum_{j=0}^{[x/h]} (-1)^j \binom{\alpha}{j} f(x-jh) \]

Let us mention that this operator was defined in a formal way as a generalization of difference formulas of integer order by replacing the integer order by a real number.

It is worth to underscore that in [4] discrete Grünwald-Letnikov approximations was called the Riemann-Liouville fractional derivative of order \( \alpha \).

The remaining problem is to study if the strong convergence \( (A_h f)_h \to Af \) can gives rise to the convergence of power operators; this is the aim of the next section.

### 4 Hölderian convergence of fractional extended nabla operator to fractional derivative

The following result, known elsewhere, is given in a suitable form for later uses.

**Lemma 4.1** There exists a function \( \Phi_\alpha \), such that

\[ \forall m \geq 1 \quad \frac{\Gamma(m+\alpha)}{\Gamma(m+1)} = m^{\alpha-1} + \frac{1}{\Gamma(1-\alpha)} \Phi_\alpha(m), \]

with \( |\Phi_\alpha(m)| \leq \frac{\Gamma(2-\alpha)}{2} m^{\alpha-2} \).

**Proof** From the definition of the beta function we have

\[ \frac{\Gamma(m+\alpha)}{\Gamma(m+1)} = \frac{1}{\Gamma(1-\alpha)} \int_0^1 t^{(m+\alpha-1)}(1-t)^{-\alpha} dt \]

(6)
Let $t = e^{-u}$ then the equality (3) becomes

$$\frac{\Gamma(m+\alpha)}{\Gamma(m+1)} \cdot \frac{1}{\Gamma(1-\alpha)} \int_0^{+\infty} e^{-mu}(u-1)^{-\alpha} du = \frac{1}{\Gamma(1-\alpha)} \int_0^{+\infty} e^{-mu}u^{-\alpha} \left( \frac{u}{e^u} - 1 \right)^{\alpha} du$$

Using the generating function of the Bernoulli numbers

$$G(u) = \frac{u}{e^u - 1} = \sum_{k=0}^{+\infty} B_k \frac{u^k}{k!} = 1 + \varphi(u) > 0$$

where $\varphi : [0, +\infty[ \to ]-1, 0]$ is a continuous function. We have

$$\frac{\Gamma(m+\alpha)}{\Gamma(m+1)} \cdot \frac{1}{\Gamma(1-\alpha)} \int_0^{+\infty} e^{-mu}u^{-\alpha} (1 + \varphi(u))^\alpha du$$

Now, Taylor’s formula with integral remainder applied to the function $(1 + \varphi(u))^\alpha$ gives

$$(1 + \varphi(u))^\alpha = 1 + \alpha \varphi(u) \int_0^{1} (1 + \xi \varphi(u))^{\alpha-1} d\xi$$

Therefore

$$\frac{\Gamma(m+\alpha)}{\Gamma(m+1)} \cdot \frac{1}{\Gamma(1-\alpha)} \int_0^{+\infty} e^{-mu}u^{-\alpha} du + \frac{\alpha}{\Gamma(1-\alpha)} \int_0^{+\infty} e^{-mu}u^{-\alpha} \varphi(u) \int_0^{1} (1 + \xi \varphi(u))^{\alpha-1} d\xi du$$

and then

$$\frac{\Gamma(m+\alpha)}{\Gamma(m+1)} = m^{\alpha-1} + \frac{1}{\Gamma(1-\alpha)} \Phi_\alpha(m)$$

where

$$\Phi_\alpha(m) = \alpha \int_0^{+\infty} e^{-mu}u^{-\alpha} \varphi(u) \int_0^{1} (1 + \xi \varphi(u))^{\alpha-1} d\xi du$$

From the identity $1 + \xi \varphi(u) = 1 - \xi + \xi (1 + \varphi(u))$ and the fact that $1 + \varphi(u) > 0$ for every $u \geq 0$, we have

$$1 + \xi \varphi(u) \geq 1 - \xi \quad \text{and} \quad (1 + \xi \varphi(u))^{\alpha-1} \leq (1 - \xi)^{\alpha-1}$$

Consequently

$$\int_0^{1} (1 + \xi \varphi(u))^{\alpha-1} d\xi \leq \int_0^{1} (1 - \xi)^{\alpha-1} d\xi \leq \frac{1}{\alpha}$$

Hence

$$\alpha \left| \int_0^{+\infty} e^{-mu}u^{-\alpha} \varphi(u) \int_0^{1} (1 + \xi \varphi(u))^{\alpha-1} d\xi du \right| \leq \int_0^{+\infty} e^{-mu}u^{-\alpha} |\varphi(u)| du$$

The function $\frac{\varphi(u)}{u}$ is strictly increasing on $[0, +\infty]$,

$$\lim_{u \to +\infty} \frac{\varphi(u)}{u} = -\frac{1}{2} \quad \text{and} \quad \lim_{u \to 0^+} \frac{\varphi(u)}{u} = 0$$

So, $\frac{\varphi(u)}{u} \leq \frac{1}{2}$, and the inequality (7) becomes

$$|\Phi_\alpha(m)| \leq \frac{1}{2} \int_0^{+\infty} e^{-mu}u^{1-\alpha} du$$
\[
|\Phi_\alpha(m)| \leq \frac{\Gamma(2-\alpha)m^{\alpha-2}}{2}
\]

Before stating the convergence theorem, we define for all \( f \) in \( D(A) \), the function \( \varphi \) by
\[
\varphi(x) := \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} A_h f(t) \, dt
\]

For the construction of the convergence result proof we need the following lemmas

**Lemma 4.2** For all \( 0 < \beta < 1 \) such that \( 1-\alpha-\beta > 0 \) we have
\[
\omega_\beta(\varphi, h) \leq \frac{8}{\Gamma(2-\alpha)} \|f\|_{\beta} h^{1-\alpha-\beta}
\]

**Proof** Remark that the function \( \varphi \) satisfies the following estimation
\[
\frac{|\varphi(x) - \varphi(y)|}{|x-y|^\beta} \leq \frac{1}{\Gamma(2-\alpha)} \|A_h f\|_{\beta} (\max(x,y))^{1-\alpha} \leq \frac{1}{\Gamma(2-\alpha)} \|A_h f\|_{\beta}
\]

then \( \varphi \in H^\beta[0,1] \).

Let us now estimate \( \omega_\beta(\varphi, h) \), we distinguish two cases,

First \( t_{k-1} < x < y \leq t_k \):

Notice that
\[
\Gamma(1-\alpha) \varphi(x) = \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} (x-t)^{-\alpha} \left( \frac{t-t_{i-1}}{h} \nabla_h f(t_i) + \frac{t_i-t}{h} \nabla_h f(t_{i-1}) \right) \, dt + \int_{t_{k-1}}^{x} (x-t)^{-\alpha} \left( \frac{t-t_{k-1}}{h} \nabla_h f(t_k) + \frac{t_k-t}{h} \nabla_h f(t_{k-1}) \right) \, dt
\]

Hence
\[
\Gamma(1-\alpha)(\varphi(x) - \varphi(y))
\]
\[
= \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} ((x-t)^{-\alpha} - (y-t)^{-\alpha}) \left( \frac{t-t_{i-1}}{h} \nabla_h f(t_i) + \frac{t_i-t}{h} \nabla_h f(t_{i-1}) \right) \, dt \\
+ \int_{t_{k-1}}^{x} ((x-t)^{-\alpha} - (y-t)^{-\alpha}) \left( \frac{t-t_{k-1}}{h} \nabla_h f(t_k) + \frac{t_k-t}{h} \nabla_h f(t_{k-1}) \right) \, dt + \int_{x}^{y} (y-t)^{-\alpha} \left( \frac{t-t_{k-1}}{h} \nabla_h f(t_k) + \frac{t_k-t}{h} \nabla_h f(t_{k-1}) \right) \, dt
\]

and
\[
\Gamma(1-\alpha) |\varphi(x) - \varphi(y)|
\]
\[
\leq 2 \|f\|_{\beta} \left\{ \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} ((x-t)^{-\alpha} - (y-t)^{-\alpha}) \, dt + \int_{t_{k-1}}^{x} ((x-t)^{-\alpha} - (y-t)^{-\alpha}) \, dt + \int_{x}^{y} (y-t)^{-\alpha} \right\}
\]

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which leads to

\[ |\varphi(x) - \varphi(y)| \]
\[ \leq \frac{2}{\Gamma(2 - \alpha)} \|f'\|_\beta \sum_{i=1}^{k-1} ((x - t_{i-1})^{1-\alpha} - (y - t_{i-1})^{1-\alpha} - (x - t_i)^{1-\alpha} + (y - t_i)^{1-\alpha}) \]
\[ + \frac{2}{\Gamma(2 - \alpha)} \|f'\|_\beta [(x - t_{k-1})^{1-\alpha} - (y - t_{k-1})^{1-\alpha} + 2(y - x)^{1-\alpha}] \]

\[ \text{finally} \]
\[ \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\beta} \leq \frac{4}{\Gamma(2 - \alpha)} \|f'\|_\beta |y - x|^{1-\alpha - \beta} \]
\[ \leq \frac{4}{\Gamma(2 - \alpha)} \|f'\|_\beta h^{1-\alpha - \beta} \]

Second \( t_{k-1} < x \leq t_k < y \leq t_{k+1} : \)
\[ \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\beta} \leq \frac{|\varphi(x) - \varphi(t_k)|}{|x - t_k|^\beta} + \frac{|\varphi(t_k) - \varphi(y)|}{|t_k - y|^\beta} \]
\[ \leq \frac{8}{\Gamma(2 - \alpha)} \|f'\|_\beta h^{1-\alpha - \beta} \]

Therefore
\[ \omega_\beta(\varphi, h) \leq \frac{8}{\Gamma(2 - \alpha)} \|f'\|_\beta h^{1-\alpha - \beta} \]

**Lemma 4.3** There exists \( C > 0 \) such that for every \( 0 \leq k \leq n \)

\[ \frac{|\varphi(t_k) - \nabla^\alpha f(t_k)|}{h^\beta} \leq \left( 1 + \frac{C}{\Gamma(1 - \alpha)} \right) \|f'\|_\beta h^{1-\alpha - \beta} + \frac{2^\beta}{\Gamma(2 - \alpha)} \omega_\beta(f', 2h) \]

**Proof** Obviously if \( k = 0 \) the lemma holds for every \( C > 0 \). Assume now that \( k > 0 \), then

\[ \varphi(t_k) = \frac{1}{\Gamma(1 - \alpha)} \sum_{i=1}^{k} \int_{t_{j-1}}^{t_j} (t_k - t)^{-\alpha} \left[ \frac{t - t_{j-1}}{h} \nabla_h f(t_j) + \frac{t_j - t}{h} \nabla_h f(t_{j-1}) \right] dt \]

A simple integration leads to

\[ \varphi(t_k) \]
\[ = \frac{1}{\Gamma(2 - \alpha)} \sum_{i=1}^{k} (t_k - t_{j-1})^{1-\alpha} \nabla_h f(t_{j-1}) - (t_k - t_j)^{1-\alpha} \nabla_h f(t_j) \]
\[ + \frac{1}{\Gamma(2 - \alpha)} \sum_{i=1}^{k} \frac{(t_k - t_{j-1})^{2-\alpha} - (t_k - t_j)^{2-\alpha}}{(2 - \alpha)h} [\nabla_h f(t_j) - \nabla_h f(t_{j-1})] \]

which can be arranged as follows
\[ \phi(t_k) \]

\[ = \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^{k-1} [(t_k - t_j)^{1-\alpha} - (t_k - t_{j+1})^{1-\alpha}] \nabla_h f(t_j) \]

\[ + \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^{k} \left( \frac{(t_k - t_{j-1})^{2-\alpha} - (t_k - t_j)^{2-\alpha}}{2-\alpha} - (t_k - t_j)^{1-\alpha} \right) (\nabla_h f(t_j) - \nabla_h f(t_{j-1})) \]

Then

\[ \phi(t_k) - \nabla_h^\alpha f(t_k) = S_1 + S_2 - h^{1-\alpha} \nabla_h f(t_k) \]

where

\[ S_1 = \frac{h^{1-\alpha}}{\Gamma(1-\alpha)} \sum_{i=1}^{k-1} \left[ \frac{j^{1-\alpha} - (j-1)^{1-\alpha}}{1-\alpha} - \frac{\Gamma(j+1-\alpha)}{\Gamma(j+1)} \right] \nabla_h f(t_k - t_j) \]

and

\[ S_2 = \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^{k} \left( \frac{(t_k - t_{j-1})^{2-\alpha} - (t_k - t_j)^{2-\alpha}}{2-\alpha} - (t_k - t_j)^{1-\alpha} \right) (\nabla_h f(t_j) - \nabla_h f(t_{j-1})) \]

By using the fact that

\[ 0 \leq \frac{(t_k - t_{j-1})^{2-\alpha} - (t_k - t_j)^{2-\alpha}}{(2-\alpha)h} - (t_k - t_j)^{1-\alpha} = \frac{1}{h} \int_{t_{j-1}}^{t_j} ((t_k - t)^{1-\alpha} - (t_k - t_j)^{1-\alpha}) dt \]

\[ \leq (t_k - t_{j-1})^{1-\alpha} - (t_k - t_j)^{1-\alpha} \]

and

\[ |\nabla_h f(t_j) - \nabla_h f(t_{j-1})| \leq (2h)^{\beta} \omega_\beta(f', 2h) \]

\[ |S_2| \text{ can be estimated by} \]

\[ |S_2| \leq \frac{(2h)^{\beta} \omega_\beta(f', 2h)}{\Gamma(2-\alpha)} \]

It remain to estimate \(|S_1|\). To doing so, we use the lemma \[4, 1\]

\[ \frac{\Gamma(j+1-\alpha)}{\Gamma(j+1)} = j^{-\alpha} + \frac{1}{\Gamma(\alpha)} \Phi_{1-\alpha}(j) \]

with

\[ |\Phi_{1-\alpha}(j)| \leq \frac{\Gamma(1+\alpha)}{2} j^{-\alpha-1} \]

Therefore

\[ \frac{j^{1-\alpha} - (j-1)^{1-\alpha}}{1-\alpha} - \frac{\Gamma(j+1-\alpha)}{\Gamma(j+1)} = \int_{t_{j-1}}^{t_j} (s^{-\alpha} - j^{-\alpha}) ds - \frac{1}{\Gamma(\alpha)} \Phi_{1-\alpha}(j) \]
and for every \( j \geq 2 \)

\[
\left| \frac{j^{1-\alpha} - (j-1)^{1-\alpha}}{1-\alpha} - \frac{\Gamma(j+1-\alpha)}{\Gamma(j+1)} \right| \leq (j-1)^{-\alpha} - j^{-\alpha} + \frac{1}{\Gamma(\alpha)} |\Phi_1 - (j)|
\]

This leads to

\[
\sum_{j=1}^{k-1} \left| \frac{j^{1-\alpha} - (j-1)^{1-\alpha}}{1-\alpha} - \frac{\Gamma(j+1-\alpha)}{\Gamma(j+1)} \right| \\
\leq \frac{1}{1-\alpha} - \Gamma(2-\alpha) + 1 - (k-1)^{-\alpha} + \alpha \sum_{j=1}^{k-1} j^{-\alpha-1} \\
\leq C
\]

with

\[
C = \frac{1}{1-\alpha} - \Gamma(2-\alpha) + 1 + \frac{\alpha}{2} \zeta(1+\alpha) > 0
\]

where \( \zeta(\cdot) \) is the Riemann zeta function.

Finally

\[
|S_1| \leq \frac{h^{1-\alpha}}{1-\alpha} C \|f'\|_\beta
\]

Now we can put the pieces together to get

\[
\frac{|\varphi(t_k) - \nabla_h^\alpha f(t_k)|}{h^\beta} \leq \left( 1 + \frac{C}{\Gamma(1-\alpha)} \right) \|f'\|_\beta h^{1-\alpha-\beta} + \frac{\alpha^2}{\Gamma(2-\alpha)} \omega(\beta; f', 2h)
\]

The following theorem shows that the sequence \((A_h^\alpha)_h\) converges strongly to \(A^\alpha\).

**Theorem 4.4** Let \( X_\beta \) be the space \( X_\beta = \{ f \in H^\beta \text{ such that } f' \in H_0^\beta \} \).

Then for all \( \beta \) such that \( 1 - \alpha - \beta > 0 \) the sequence \((A_h^\alpha)_h\) converges strongly to the fractional derivative \(A^\alpha\) on \( X_\beta\) as \( h \) tends to 0.

**Proof** For every \( 0 \leq x < y \leq 1 \),

\[
(A^\alpha - A_h^\alpha) (f)(x) - (A^\alpha - A_h^\alpha) (f)(y) \\
= -\frac{\sin \pi \alpha}{\pi} \int_0^{+\infty} \lambda^{\alpha-1} (R(-\lambda, A)(Af) - R(-\lambda, A_h)(A_hf))(x) d\lambda \\
+ \frac{\sin \pi \alpha}{\pi} \int_0^{+\infty} \lambda^{\alpha-1} (R(-\lambda, A)(Af) - R(-\lambda, A_h)(A_hf))(y) d\lambda
\]

by introducing a mixed term we get

\[
(A^\alpha - A_h^\alpha) (f)(x) - (A^\alpha - A_h^\alpha) (f)(y) \\
= -\frac{\sin \pi \alpha}{\pi} \int_0^{+\infty} \lambda^{\alpha-1} R(-\lambda, A)(Af - A_hf)(x) - (Af - A_hf)(y)) d\lambda \\
+ \frac{\sin \pi \alpha}{\pi} \int_0^{+\infty} \lambda^{\alpha-1} R(-\lambda, A) - R(-\lambda, A_h))(A_hf - (A_h f)(y)) d\lambda
\]
Denote by $I_1(x,y)$ and $I_2(x,y)$ respectively the first and the second integral in the equality above.

We begin by the first integral

$$|I_1(x,y)| \leq \int_0^{+\infty} \lambda^{\alpha - 1} |R(-\lambda, A)((Af-A_hf)(x) - (Af-A_hf)(y))| \, d\lambda$$

$$\leq \int_0^{+\infty} \lambda^{\alpha - 1} \int_0^x e^{-\lambda t} |(Af-A_hf)(y-t) - (Af-A_hf)(x-t)| \, dt \, d\lambda$$

$$+ \int_0^{+\infty} \lambda^{\alpha - 1} \int_x^y e^{-\lambda t} |(Af-A_hf)(y-t) - (Af-A_hf)(x-t)| \, dt \, d\lambda$$

which leads to

$$\frac{I_1(x,y)}{|x-y|^\beta} \leq \|Af-A_hf\|_\beta \int_0^{+\infty} \lambda^{\alpha - 1} \left( \int_0^y e^{-\lambda t} \, dt \right) \, d\lambda$$

The estimate

$$\frac{I_1(x,y)}{|x-y|^\beta} \leq \frac{6\Gamma(\alpha)}{1-\alpha} \omega_\beta(f', h)$$

follows from Fubini’s theorem and inequality 2.

Consider now the second integral

First notice that

$$\sin \frac{\pi \alpha}{\pi} \int_0^{+\infty} \lambda^{\alpha - 1} R(-\lambda, A) A_h f(x) \, d\lambda$$

$$= \sin \frac{\pi \alpha}{\pi} \int_0^{+\infty} \lambda^{\alpha - 1} \left( \int_0^x e^{-\lambda(x-t)} A_h f(t) \, dt \right) \, d\lambda$$

$$= \sin \frac{\pi \alpha}{\pi} \int_0^x \left( \int_0^{+\infty} \lambda^{\alpha - 1} e^{-\lambda(x-t)} \, d\lambda \right) A_h f(t) \, dt$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} A_h f(t) \, dt$$

Then

$$\sin \frac{\pi \alpha}{\pi} \int_0^{+\infty} \lambda^{\alpha - 1} (R(-\lambda, A) - R(-\lambda, A_h)) A_h f(x) \, d\lambda$$

$$= \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} A_h f(t) \, dt - \mathcal{I}_h \nabla^\alpha_h f(x)$$

$$= (r_h \varphi)(x) + \mathcal{I}_h (\varphi(x) - \nabla^\alpha_h f(x))$$

From lemmas 2.2 and 2.3 we have for some $k$ and $m$

$$\frac{\sin \pi \alpha}{\pi} |I_2(x,y)| \leq 4\omega_\beta(\varphi, h) + \frac{|\varphi(t_k) - \nabla^\alpha_h f(t_k) - \varphi(t_m) + \nabla^\alpha_h f(t_m)|}{|t_k - t_m|^\beta}$$

From lemmas 2.2 and 2.3 we deduce

$$\frac{\sin \pi \alpha}{\pi} |I_2(x,y)| \leq 2 \left( \frac{16}{\Gamma(2-\alpha)} + \frac{C}{\Gamma(1-\alpha)} + 1 \right) \|f'\|_\beta h^{1-\alpha-\beta} + \frac{2^\beta+1}{\Gamma(2-\alpha)} \omega_\beta(f', 2h)$$
\[ \left\| (A^\alpha - \nabla_h^\alpha f) \right\|_\beta \leq 2 \left( \frac{16}{\Gamma(2 - \alpha)} + \frac{C}{\Gamma(1 - \alpha)} + 1 \right) \left\| f' \right\|_\beta h^{1 - \alpha - \beta} + \frac{2^{3 + 1} + 6}{\Gamma(2 - \alpha)} \omega_\beta(f', 2h) \]

and the conclusion of the theorem holds.

5 Numerical examples

In this section two examples are discussed.

5.1 Exemple 1

Consider the fractional derivative of \( f(x) = x^\mu \ln x \). The analytical expression of the fractional derivative of \( f \) is

\[ A^\alpha f(x) = \frac{\Gamma(\mu + 1)}{\Gamma(\mu + 1 - \alpha)} x^{\mu - \alpha} [\ln x + \psi(\mu + 1) - \psi(\mu + 1 - \alpha)] \]

Where \( \psi(\cdot) \) denote the digamma function see [19, Formula (103)].

In the next tables error at the step size \( h \) is the Hölderian error defined by

\[ \max_{0 \leq i < j \leq 1/h} \left| \frac{([A^\alpha f - \nabla_h^\alpha f](t_i) - (A^\alpha f - \nabla_h^\alpha f)(t_j)]}{|t_i - t_j|^\beta} \right| \]

(8)

According to our theoretical consideration the convergence is ensured by \( h^{1 - \alpha - \beta} \) and \( \omega_\beta(f', 2h) \).

Let us establish an estimation of \( \omega_\beta(f', h) \)

For every \( 0 \leq x < y \leq 1 \)

\[ f'(y) - f'(x) = \mu(y^{\mu - 1} \ln y - x^{\mu - 1} \ln x) + y^{\mu - 1} - x^{\mu - 1} \]

Using the fact that

\[ y^{\mu - 1} \ln y - x^{\mu - 1} \ln x = \int_x^y \frac{d}{dt}(t^{\mu - 1} \ln t) dt = (\mu - 1) \int_x^y t^{\mu - 2} \ln t dt + \frac{1}{\mu - 1} (y^{\mu - 1} - x^{\mu - 1}) \]

Then for every \( 1 + \beta < \beta' < \mu \) we have

\[ \int_x^y t^{\mu - 2} \ln t dt = \int_x^y t^{\mu - \beta' + \beta' - 2} \ln t dt \]

Setting \( M = \max_{t \in [0,1]} |t^{\mu - \beta'} \ln t| \)

we have

\[ \left| \int_x^y t^{\mu - 2} \ln t dt \right| \leq M \int_x^y t^{\beta' - 2} dt = \frac{M}{\beta' - 1} \left( y^{\beta' - 1} - x^{\beta' - 1} \right) \]

Therefore
\[ |f'(y) - f'(x)| \leq \frac{M\mu(\mu - 1)}{\beta' - 1} \left( y^{\beta' - 1} - x^{\beta' - 1} \right) + \left( \frac{\mu}{\mu - 1} + 1 \right) \left( y^{\mu - 1} - x^{\mu - 1} \right) \]

It follows that
\[ \frac{|f'(y) - f'(x)|}{|y - x|^{\beta}} \leq \frac{M\mu(\mu - 1)}{\beta' - 1} |y - x|^{\beta' - 1 - \beta} + \left( \frac{\mu}{\mu - 1} + 1 \right) |y - x|^{\mu - 1 - \beta} \]
\[ \leq \left( \frac{M\mu(\mu - 1)}{\beta' - 1} + \left( \frac{\mu}{\mu - 1} + 1 \right) |y - x|^{\mu - \beta'} \right) |y - x|^{\beta' - 1 - \beta} \]

If \(|y - x| \leq h\) then
\[ \frac{|f'(y) - f'(x)|}{|y - x|^{\beta}} \leq \left( \frac{M\mu(\mu - 1)}{\beta' - 1} + \left( \frac{\mu}{\mu - 1} + 1 \right) h^{\mu - \beta'} \right) h^{\beta' - 1 - \beta} \]

and
\[ \omega_{\beta}(f', h) \leq \left( \frac{M\mu(\mu - 1)}{\beta' - 1} + \left( \frac{\mu}{\mu - 1} + 1 \right) h^{\mu - \beta'} \right) h^{\mu - 1 - \beta} \]

In case \(\mu = 3/2, \alpha = 0.3, \beta = 0.1\) the convergence is ensured since
\[ \mu - 1 - \beta = 0.4 > 0. \]

The results concerning errors are presented in Table 1 for \(\mu = 3/2, \alpha = 0.3, \beta = 0.1\)

| \(h\) | Error     |
|------|-----------|
| \(2^{-6}\) | 0.0079082 |
| \(2^{-7}\) | 0.0040833 |
| \(2^{-8}\) | 0.0021392 |
| \(2^{-9}\) | 0.0011478 |
| \(2^{-10}\) | 0.0006054 |
| \(2^{-11}\) | 0.0003150 |
| \(2^{-12}\) | 0.0001622 |

Table 1: Error defined by (8) when \(f(x) = x^\mu \ln x\) for \(\mu = 3/2, \alpha = 0.3\) and \(\beta = 0.1\).

In figure 1 on the left the graphs of \(A^\alpha f\) and \(A_h^\alpha f\) are shown. On the right we give the Hölderian errors.
\[ D^\alpha y(t) = \frac{40320}{\Gamma(9 - \alpha)} t^{8 - \alpha} - 3 \frac{\Gamma(5 + \alpha/2)}{\Gamma(5 - \alpha/2)} t^{4 - \frac{3\alpha}{2}} + \frac{9}{4} \Gamma(\alpha + 1) + \left(\frac{3}{2} t^{\frac{3}{2}} - t^4\right)^3 - [y(t)]^\frac{3}{2} \]

The initial condition is \( y(0) = 0 \). The exact solution of this problem is

\[ y(t) = t^8 - 3t^{4 + \frac{\alpha}{2}} + \frac{9}{4} t^\alpha. \]

For \( \alpha = 0.5 \), we display the results in table 2 for \( \beta = 0.1 \) and \( \beta = 0.01 \) respectively.

| \( h \) | Errors for \( \beta = 0.1 \) | Errors for \( \beta = 0.01 \) |
|------|-----------------|-----------------|
| 2^{-7} | 0.0347581 | 0.0224598 |
| 2^{-8} | 0.0269360 | 0.0163528 |
| 2^{-9} | 0.0206910 | 0.0118018 |
| 2^{-10} | 0.0158085 | 0.0084716 |
| 2^{-11} | 0.0120388 | 0.0060613 |
| 2^{-12} | 0.0091502 | 0.0043283 |
| 2^{-13} | 0.0069465 | 0.0030872 |

Table 2: Hölderian errors for problem (9) with \( \alpha = 0.5 \)

6 Conclusion

In this work, we defined a fractional operator as a fractional power of a piecewise linear interpolation of a backward difference on a Hölder space. We
proved the strong convergence of this operator to fractional derivative, and we supported our results with examples. We think that we have now a kind of process to define Euler-like formulas which contribute to solve numerically fractional differential equations in Hölder spaces. However, several questions can be the subject of further works, in particular, the analysis of the order of approximation, and the results that can be expected if one replace the linear spline $I_h$ by a spline of degree $n > 1$, or if one replaces the operator $\nabla_h$ by another more accurate approximation of the derivative.

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