A general variational principle for spherically symmetric perturbations in diffeomorphism covariant theories

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We present a general method for the analysis of the stability of static, spherically symmetric solutions to spherically symmetric perturbations in an arbitrary diffeomorphism covariant Lagrangian field theory. Our method involves fixing the gauge and solving the linearized gravitational field equations to eliminate the metric perturbation variable in terms of the matter variables. In a wide class of cases—which include $f(R)$ gravity, the Einstein-æther theory of Jacobson and Mattingly, and Bekenstein’s TeVeS theory—the remaining perturbation equations for the matter fields are second order in time. We show how the symplectic current arising from the original Lagrangian gives rise to a symmetric bilinear form on the variables of the reduced theory. If this bilinear form is positive definite, it provides an inner product that puts the equations of motion of the reduced theory into a self-adjoint form. A variational principle can then be written down immediately, from which stability can be tested readily. We illustrate our method in the case of Einstein’s equation with perfect fluid matter, thereby re-deriving, in a systematic manner, Chandrasekhar’s variational principle for radial oscillations of spherically symmetric stars. In a subsequent paper, we will apply our analysis to $f(R)$ gravity, the Einstein-æther theory, and Bekenstein’s TeVeS theory.

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I. INTRODUCTION

In recent years, there have been a number of proposals for new theories which modify general relativity in an attempt to either explain outstanding experimental problems or provide a theoretical framework for the analysis of new phenomena. In the first category, we have theories such as Carroll et al.’s $f(R)$ gravity [1], which attempts to modify general relativity to explain the cosmic acceleration, and Bekenstein’s TeVeS theory [2], an attempt to formulate a covariant version of Milgrom’s MOND. In the second category, we have theories such as Jacobson and Mattingly’s “Einstein-æther theory” [3], a toy theory in which Lorentz symmetry is dynamically broken by a vector field which is constrained to be unit and timelike.

If such theories are to be phenomenologically viable, they must satisfy two criteria, among others. First, they must possess solutions which are quasi-Newtonian, i.e., they must reproduce the dynamics of Newtonian gravity up to some small relativistic corrections. Second, these quasi-Newtonian solutions must be stable, or at least must not be unstable on time scales sufficiently short to interfere with known physical phenomena in Newtonian gravity. In particular, there must exist static, spherically symmetric solutions of the theory, corresponding to the interior and exterior of a star, and moreover these solutions must not be unstable on a time scale significantly shorter than cosmological time scales.

It would therefore be useful to obtain a general method by which the stability of an arbitrary Lagrangian field theory might be analyzed. The perturbational equations of motion off of some background will be a set of linear partial differential equations in terms of some perturbational fields. We shall denote these fields $\psi^{\alpha}$, where $\alpha$ is an index running over the collection of perturbational fields. Suppose that it were the case that these equations take the form

$$-\frac{\partial^2}{\partial t^2} \psi^\alpha = T^\alpha_\beta \psi^\beta$$

(1)

where $T$, the time-evolution operator, is a linear operator containing spatial derivatives only. Suppose, further, that we can find an inner product $\langle \cdot, \cdot \rangle$ on the vector space of fields $\psi^\alpha$ in which the operator $T$ is self-adjoint, i.e., the domain of $T$ coincides with the domain of $T^\dagger$ and

$$\langle \psi; T \chi \rangle = (T \psi; \chi)$$

(2)

for all $\psi$ and $\chi$ in the domain of $T$. Let $\omega^2_{\psi}$ denote the greatest lower bound of the spectrum of $T$. Then it can be shown [4] that if $\omega^2_{\psi} > 0$ then the background solution is stable, whereas if $\omega^2_{\psi} < 0$ then perturbations exist that grow exponentially on a timescale $\tau = 1/|\omega_0|$. Furthermore, for all $\psi$ in the space of perturbational fields, we have (by the Rayleigh-Ritz principle)

$$\omega^2_{\psi} \leq \langle T \psi; \psi \rangle$$

(3)

Thus, in practice, then, we can obtain a bound on the spectrum of $T$ by plugging various “trial functions” $\psi^\alpha$, into (3). If we find a trial function which yields a negative result, we know that there exist solutions of (1) that grow exponentially in time, and we also obtain a upper bound on the the timescale on which the instability of fastest growth occurs.
Unfortunately, for a generally covariant theory, the perturbation equations can never directly arise in the simple form of (1). There will be gauge freedom, so the equations will not even be deterministic. There will be constraint equations, which typically will be of lower order in time derivatives than the other equations. Furthermore, even if one succeeds in suitably fixing a gauge and solving some subset of the equations so that the remaining equations take the form of (1), there may not exist an inner product that makes $T$ self-adjoint, and even if such an inner product exists, it may be highly nontrivial to find it.

In this paper, we will show that all of the above problems can be solved in a wide class of diffeomorphism covariant theories in the context of spherically symmetric perturbations of static, spherically symmetric solutions. The easiest task is to fix the gauge for such perturbations, and we make a choice of gauge in Section IV. In Section IV we analyze the linearized constraint equations. For spherically symmetric perturbations, there are two independent constraint equations, namely a “time-time” and a “time-radial” equation. One of the main results of this paper is our proof that these two equations can always be reduced to a single equation of lower differential order. Furthermore, if the original constraint equations are partial differential equations of second (or lower) differential order, we show that in a wide class of theories this resulting single equation can be solved algebraically for one of the metric perturbation variables, thus completely eliminating the constraints. In addition, we show that in a wide class of theories, the remaining independent component of the gravitational field equations can be used to eliminate the remaining metric perturbation variable. We thereby reduce the theory to solving a system of partial differential equations for just the matter perturbation variables.

In the theories that we shall consider here and in a subsequent paper, the equations of motion of this reduced theory take the form of (1). However, we still have the problem of determining whether there exists an inner product that makes $T$ self-adjoint, and finding this inner product if it does exist. In this regard, it might be expected that a Hamiltonian formulation of the theory would be useful, since for a Hamiltonian of a simple “kinetic plus potential” form, the kinetic energy expression will provide such an inner product. Now, by assumption, the original theory was obtained from a Lagrangian, and, from this Lagrangian, one can obtain a Hamiltonian [5]. Prior to gauge fixing and solving the constraints, a Lagrangian and Hamiltonian for perturbations can be obtained by expanding about the background solution and keeping the terms quadratic in the perturbation variables. However, it is well known that if one substitutes a choice of gauge into a Lagrangian or Hamiltonian, it no longer functions as a Lagrangian or Hamiltonian. Thus, it might appear that the existence of a Lagrangian or Hamiltonian formulation of the original theory will be useless for obtaining a Lagrangian or Hamiltonian formulation of the reduced theory, and also useless for finding an inner product in the reduced theory that makes $T$ self-adjoint.

In Section VII we make use of the fact that—although substitution of a gauge choice and/or a solution to some of the field equations into a Lagrangian or Hamiltonian does not yield a Lagrangian or Hamiltonian—substitution of a gauge choice and/or a solution to some of the field equations into the symplectic current still yields a symplectic current that is conserved when the remaining linearized equations of motion are satisfied. We show that—in a wide class of cases—the resulting symplectic current of the reduced theory gives rise to a bilinear form that makes $T$ symmetric. If this bilinear form is positive definite, it provides the desired inner product, thus enabling us to obtain a variational principle for determining stability.

In Section VII we shall illustrate our method by applying it to the case of the Einstein-fluid system. We thereby will re-derive in a systematic fashion the variational principle for analyzing radial oscillations of spherically symmetric stars in general relativity that was first obtained by Chandrasekhar [6, 7]. In a subsequent paper, the method will be applied to analyze $f(R)$ gravity, the Einstein-æther theory of Jacobson and Mattingly, and to Bekenstein’s TeVeS theory.

Our notation and conventions will generally follow [8]. Where relevant, we will use units in which $c = G = 1$. Boldface symbols will be used to denote differential forms. For definiteness, we will restrict consideration here to four-dimensional spacetimes that are spherically symmetric in the sense of having an $SO(3)$ isometry subgroup whose orbits are two-spheres. However, all of our analysis and results generalize straightforwardly to $n$-dimensional spacetimes that are spherically symmetric in the sense of having an $SO(n-1)$ isometry subgroup whose orbits are $(n-2)$-spheres.

## II. LAGRANGIAN FORMALISM

In this section, we shall briefly review some basic definitions and constructions in theories derived from a diffeomorphism covariant Lagrangian. We refer the reader to [9] and [5] for further details and discussion.

Consider a diffeomorphism covariant Lagrangian four-form $\mathcal{L}$, constructed from dynamical fields $\Psi$ that consist of the spacetime metric and perhaps additional matter fields, assumed to be described by tensor fields. For computational convenience, in this paper we will use the inverse metric $g^{ab}$ rather than the metric $g_{ab}$ as the independent dynamical variable describing the gravitational degrees of freedom. The first variation of $\mathcal{L}$ can be written in the form

$$\delta \mathcal{L} = \mathcal{E} \delta \Psi + d\theta,$$  \hspace{1cm} (4)
which defines not only the Euler-Lagrange equations of motion, \( \mathcal{E} = 0 \), but also the symplectic potential three-form, \( \theta(\Psi, \delta \Psi) \). In \( \theta \), any tensor indices of \( \delta \Psi \) are understood to be contracted with the corresponding dual tensor indices of \( \mathcal{E} \), and, of course, the sum over all fields comprising \( \Psi \) is understood. The antisymmetrized variation of the symplectic potential yields the symplectic current three-form \( \omega(\Psi; \delta_1 \Psi, \delta_2 \Psi) \), defined by

\[
\omega = \delta_1 \theta(\Psi, \delta_2 \Psi) - \delta_2 \theta(\Psi, \delta_1 \Psi) . \tag{5}
\]

(Here, \( \delta_1 \) and \( \delta_2 \) are taken to be variations along a two-parameter family, and hence they commute, \( \delta_1 \delta_2 = \delta_2 \delta_1 \).) Applying \( d \) to this equation and using (4), we obtain

\[
d\omega = - (\delta_1 \mathcal{E}) \delta_2 \Psi + (\delta_2 \mathcal{E}) \delta_1 \Psi . \tag{6}
\]

Thus, the symplectic current is conserved, \( d\omega = 0 \), whenever \( \delta_1 \Psi \) and \( \delta_2 \Psi \) satisfy the linearized equations of motion.

The Noether current three-form \( J_\xi \) associated with an arbitrary vector field \( \xi^a \) is defined by

\[
J_\xi = \theta(\Psi, \mathcal{L}_\xi \Psi) - \xi \cdot \mathcal{L}_\xi , \tag{7}
\]

where the “\( \cdot \)” denotes the contraction of \( \xi^a \) into the first index of \( \mathcal{L}_\xi \). Applying \( d \) to this equation, we obtain

\[
dJ_\xi = -\mathcal{E} L_\xi \Psi . \tag{8}
\]

Thus, \( J \) is conserved, \( dJ = 0 \), whenever \( \Psi \) satisfies the equations of motion (i.e., \( \mathcal{E} = 0 \)) or whenever \( \xi^a \) is a symmetry of \( \Psi \) (i.e., \( \mathcal{L}_\xi \Psi = 0 \)). Finally, we note that a simple calculation \( \check{\text{a}} \) shows that

\[
\delta J_\xi = - \xi \cdot \mathcal{E} + \omega(\Psi; \delta \Psi, \mathcal{L}_\xi \Psi) + d(\xi \cdot \theta) . \tag{9}
\]

### III. GAUGE FIXING

As noted in the Introduction, generally covariant theories are gauge theories; they possess “unphysical degrees of freedom”, corresponding to diffeomorphisms. In order to obtain deterministic equations of motion and obtain a variational principle of the sort we seek, we must eliminate the gauge degrees of freedom. There is no known algorithm for doing this in general spacetimes. However, in our work, we will be dealing with spherically symmetric perturbations of static, spherically symmetric spacetimes, and thus may restrict attention to spacetimes which are spherically symmetric but not necessarily static. For such spacetimes, it is always possible to put the metric in the following form:

\[
ds^2 = - \exp(2\Phi(r,t)) dt^2 + \exp(2\Lambda(r,t)) dr^2 + r^2 d\Omega^2 . \tag{10}
\]

(See, e.g., \( \check{\text{b}} \) for the details.) Apart from the rotational isometries, the only diffeomorphisms that preserve the metric form \( \check{\text{b}} \) are redefinitions of the time coordinate of the form \( t \rightarrow g(t) \), for an arbitrary monotonic function \( g \).

For the static background solution, \( \Phi \) and \( \Lambda \) can be chosen to be independent of \( t \), in which case the only remaining gauge freedom is \( t \rightarrow ct \) for some constant \( c \). Thus, for the background solution, \( \Phi \) is unique up to an additive constant and \( \Lambda \) is unique.

Now consider an arbitrary spherically symmetric (but not necessarily static) perturbation of the background metric. Let \( \phi(r,t) \) and \( \lambda(r,t) \) denote, respectively, the perturbations of \( \Phi \) and \( \Lambda \). In other words, \( 2e^{-2\Phi} \phi(r,t) \) is the perturbation of the \( g^{tt} \) metric component, and \( -2e^{-2\Lambda} \lambda(r,t) \) is the perturbation of the \( g^{rr} \) component. The gauge freedom in the perturbed metric is \( \delta g^{ab} \rightarrow \delta g^{ab} + \mathcal{L}_\xi (g^{(0)})^{ab} \), where \( (g^{(0)})^{ab} \) is the background metric and \( \nu^a \) is an arbitrary vector field that generates diffeomorphisms that preserve the metric form \( \check{\text{b}} \). Since the only such nontrivial diffeomorphisms are \( t \rightarrow g(t) \), the only such nontrivial \( \nu^a \) is \( \nu^a = h(t) t^a \), where \( h(t) \) is a positive function and \( t^a \) is the static Killing field of the background solution. Thus, we find that \( \lambda \) is gauge invariant and \( \phi \) has the gauge freedom

\[
\phi(r,t) \rightarrow \phi(r,t) + f(t) . \tag{11}
\]

where \( f = dh/dt \) is an arbitrary function of \( t \). Thus, our choice of the metric form \( \check{\text{b}} \) eliminates the gauge freedom in the metric perturbation variables except for the small residual freedom \( \check{\text{b}} \).

An important consequence of the existence of this residual gauge freedom is that if \( \lambda(r,t) \) and \( \phi(r,t) \) along with the appropriate perturbed matter variables solve the linearized equations of motion, then if we replace \( \phi(r,t) \) by \( \phi(r,t) + f(t) \), we must still obtain a solution to the linearized equations of motion. This implies that \( \phi \) can appear in the equations of motion only in the form \( \partial \phi / \partial r \).

### IV. SOLVING THE LINEARIZED CONSTRAINT EQUATIONS

The general analysis of \( \check{\text{c}} \) shows that in any generally covariant theory, there will be a “constraint” on the phase space of the theory associated with the infinitesimal diffeomorphism generated by an arbitrary vector field \( \xi^a \). In other words, if \( \xi^a \) is used to define the notion of “time translations”, there will be a corresponding restriction on the phase space of the theory imposed by some subset of the equations of motion. If \( \xi^a \) is used to define the notion of “time translations”, then, as we shall see below, the resulting constraint equations will typically be of lower differential order in time than the other (so-called “evolution”) equations of motion. Consequently, if we wish to get our equations of motion into the simple form \( \check{\text{d}} \), it normally will be necessary that we solve the constraint equations so that—after elimination of variables—the remaining variables are unconstrained. Remarkably, we now shall show that in a wide class of diffeomorphism covariant theories, for spherically symmetric perturbations of static spherically symmetric spacetimes, the linearized
constraint equations can be solved \textit{algebraically} for the metric perturbation $\lambda$ in terms of the matter perturbation variables. Thus, for this wide class of theories, the constraints can be easily eliminated.

Our analysis makes use of the fact proved in the Appendix of [11] that for an arbitrary diffeomorphism covariant theory, the Noether charge two-form $Q_\xi$ can always be defined so that that the Noether current $J_\xi$ (as defined in [17]), associated with an arbitrary vector field $\xi^a$ takes the form

$$J_\xi = \xi^a C_a + dQ_\xi$$

(12)

where, according to the analysis of [9], the equations $\xi^a C_a = 0$ are the constraints of the theory associated with the infinitesimal local symmetry $\xi^a$. (We will, in effect, rederive (12) by our calculation leading to (28) below.) Thus, if we perturb about a background solution, we obviously obtain

$$\delta J_\xi = \xi^a \delta C_a + d(\delta Q_\xi)$$

(13)

On the other hand, we previously noted that $\delta J_\xi$ satisfies (9). Now suppose that we are perturbing about a solution to the field equations, $E = 0$. Suppose further that this background solution possesses a Killing field $t^a$ that is also a symmetry of all of the background matter fields, so that $L_t \Psi = 0$. Then, choosing $\xi^a = t^a$, we find from (13) and (9) that

$$t^a \delta C_a = d[t \cdot \theta - \delta Q_t]$$

(14)

Thus, the linearized constraint equations, $t^a \delta C_a = 0$, associated with $t^a$ are equivalent to the equation

$$d\beta = 0$$

(15)

where

$$\beta = t \cdot \theta - \delta Q_t$$

(16)

The replacement of the equation $t^a \delta C_a = 0$ by the equation $d\beta = 0$ need not, in general, result in any simplification of the equations, i.e., there may be as many or more independent components of $\beta$ as $t^a \delta C_a$, and the equations $d\beta = 0$ may be of as high or higher differential order as the equations $t^a \delta C_a = 0$. However, in the case of spherically symmetric perturbations of static spherically symmetric solutions, the replacement of $t^a \delta C_a = 0$ by $d\beta = 0$ always results in a major simplification. The reason for this simplification can be seen as follows. By spherical symmetry, in the coordinates introduced in the previous section, the three-form $t^a \delta C_a$ must take the form

$$t^a \delta C_a = H_1(t,r) dt \wedge d\Omega + H_2(t,r) dr \wedge d\Omega$$

(17)

where $d\Omega = \sin \theta d\theta \wedge d\varphi$. Thus, the constraint equations give rise to two independent equations, namely $H_1 = 0$ and $H_2 = 0$. By contrast, by spherical symmetry, the two-form $\beta$ must take the form

$$\beta = F(t,r) d\Omega$$

(18)

Thus, $\beta$ has only one nonvanishing component, and the equation $d\beta = 0$ then reduces simply to $F = \text{const}$. For the situations we shall consider here and in the subsequent paper, an “origin”, $r = 0$, will be present in the spacetime. The two-form $\beta$ is locally constructed from the background and perturbed dynamical fields (see (10)), and thus must be smooth everywhere, including at $r = 0$. However, the spherical volume element $d\Omega = \sin \theta d\theta \wedge d\varphi$ is not smooth at $r = 0$. Consequently, we must have $F = 0$ at $r = 0$, and since $F$ is constant, we must have $F = 0$ everywhere. In summary, we have shown that the two independent components of the linearized constraints take the form

$$H_1(t,r) = \frac{\partial F}{\partial t}, \quad H_2(t,r) = \frac{\partial F}{\partial r}$$

(19)

and, thus, using the boundary conditions at $r = 0$, the two constraint equations $H_1 = 0$ and $H_2 = 0$ reduce to the single equation, $F = 0$. Furthermore, it is clear from (14) that the equation $F = 0$ will be of lower differential order (by one) in the dynamical variables than the equations $H_1 = 0$ and $H_2 = 0$. Thus, we obtain a major simplification by replacing the equations $H_1 = 0$ and $H_2 = 0$ by the equation $F = 0$.

It remains now only to get an explicit expression for $F$ in an arbitrary diffeomorphism covariant theory. The calculation of $\theta$ is straightforward, and the expression for the Noether charge $Q$ that satisfies (12) can be determined by following the procedures outlined in the Appendix of [11]. One may then obtain $\beta$—and, hence, $F$—from (10). However, instead of following this procedure, we shall give a very simple derivation of a general formula for the constraints $t^a \delta C_a$. By linearizing this formula, one can then immediately determine $H_1$ and $H_2$. The desired quantity $F$ can then be determined by inspection from (19).

As previously noted in Section II for an arbitrary vector field $\xi^a$, we have

$$dJ_\xi = -E L_\xi \theta .$$

(20)

Now, the right side of this equation is a four-form that depends linearly on $\xi^a$ and contains no higher than first derivatives of $\xi^a$, i.e., it is of the form

$$-E L_\xi \theta = (B_a \xi^a + C^a_b \nabla_a \xi^b) e ,$$

(21)

where $e$ is the volume element associated with the metric and $B_a$ and $C^a_b$ are locally constructed out of the dynamical fields. We can rewrite the right side as

$$(B_a \xi^a + C^a_b \nabla_a \xi^b) e = U_a \xi^a e + dV$$

(22)

where

$$U_a \equiv B_a - \nabla_b C^b_a$$

(23)

and

$$V_{cde} \equiv C^a_b \xi^b \epsilon_{acde}$$

(24)
Note that $\mathbf{V}$ does not depend upon derivatives of $\xi^a$.
Thus, we have proven that for all $\xi^a$ we have

$$d(\mathbf{J}_\xi - \mathbf{V}) = \xi^a U_a \epsilon. \tag{25}$$

However, the only way that this equation can hold for arbitrary $\xi^a$ is if both sides are zero. Namely, if $U_a \neq 0$ at some point $p$, then we could find a smooth $\xi^a$ of compact support such that $U_a \xi^a \geq 0$ everywhere and $U_a \xi^a > 0$ at $p$. However, the integral of the right side would then be positive, whereas the integral of the left side vanishes, thereby yielding a contradiction. Thus, we obtain

$$U_a = B_a - \nabla_b C^b_a = 0 \tag{26}$$

and

$$d(\mathbf{J}_\xi - \mathbf{V}) = 0. \tag{27}$$

Equation (26) is a generalized version of the Bianchi identity, applicable to an arbitrary diffeomorphism covariant theory, possibly containing matter fields. Equation (27) implies (12) that there exists a two-form $Q^l$ locally constructed out of $\xi^a$ and the dynamical fields such that

$$\mathbf{J}_\xi = \mathbf{V} + dQ^l. \tag{28}$$

Comparing with (12) and using the fact that $\mathbf{V}$ does not depend upon derivatives of $\xi^a$ (the same argument as used below (26)), we see that the constraints are given by

$$\xi^a C_a = \mathbf{V}. \tag{29}$$

It is worth noting that, taking into account (24) and (29), the constraints take the form

$$\xi^a C_a = C \cdot \epsilon, \tag{30}$$

where $C^a = C^a b^{\xi b}$, whereas the generalized Bianchi identity (26) takes the form

$$\nabla_b C^b_a = B_a. \tag{31}$$

We will obtain explicit formulas for $B_a$ and $C^a b$. It should be emphasized that (31) holds independently of whether the equations of motion are satisfied.

We wish now to explicitly calculate $\mathbf{V}$. For the purpose of this calculation, we assume that the Lagrangian $\mathcal{L}$ depends only on the inverse metric $g^{\alpha \beta}$ and a single matter field $A_1 a_2 ... a_n b_1 b_2 ... b_m$; the generalization to the case of more than one matter field is straightforward. We denote the gravitational equations of motion (obtained by varying $\mathcal{L}$ with respect to $g^{\alpha \beta}$) by $(\mathcal{E}_G)_{ab} \epsilon$, and we denote the matter equations of motion (obtained by varying $\mathcal{L}$ with respect to $A_1 a_2 ... a_n b_1 b_2 ... b_m$) by $(\mathcal{E}_M)_{ab}. \tag{32}$

Equation (32) then takes the more explicit form

$$d\mathbf{J}_\xi = - ((\mathcal{E}_G)_{ab} \mathcal{L} g^{\alpha \beta} + (\mathcal{E}_M)_{ab} \mathcal{L} A_1 a_2 ... a_n b_1 b_2 ... b_m) \epsilon \tag{33}$$

Using $\mathcal{L} g^{\alpha \beta} = - (\nabla^a \xi^b + \nabla^b \xi^a)$ and the standard formula for the Lie derivative of tensor field

$$L_{\xi} A_1 a_2 ... a_n b_1 b_2 ... b_m = \xi^c \nabla_c A_1 a_2 ... a_n b_1 b_2 ... b_m \tag{34}$$

we can simply read off the formulas

$$B_c = -(\mathcal{E}_M)_{ab} A_1 a_2 ... a_n b_1 b_2 ... b_m \nabla_c A_1 b_1 ... b_m. \tag{35}$$

In the above equations, the summations run over all possible substitutions of the indices $c$ and $d$ into the $i$th slot of $A_1 a_2 ... a_n b_1 b_2 ... b_m$ and $(\mathcal{E}_M)_{ab}. \tag{36}$

We then see that

$$U_c = -2 \nabla^a (\mathcal{E}_G)_{ac} - (\mathcal{E}_M)_{ab} A_1 a_2 ... a_n b_1 b_2 ... b_m \nabla_a A_1 a_2 ... a_n b_1 b_2 ... b_m \tag{37}$$

and

$$V_{def} = \epsilon_{def} \left( 2 \mathcal{E}_G^{abc} (\mathcal{E}_G)_{ab} + \sum_i \xi^b A_1 a_2 ... a_n b_1 b_2 ... b_m (\mathcal{E}_M)_{ab} A_1 a_2 ... a_n b_1 b_2 ... b_m \right) \tag{38}$$

Equation (37) is our desired formula for the constraints, which can be readily evaluated in cases of interest.

V. ELIMINATING THE METRIC PERTURBATION VARIABLES

We now wish to solve the linearized field equations

$$\delta \mathcal{E}_G = 0, \ \delta \mathcal{E}_M = 0 \tag{39}$$

for spherically symmetric perturbations of a static, spherically symmetric background solution. We shall show that, in a wide class of cases, it is possible to solve these
equations \textit{algebraically} for the metric perturbation variables, thereby reducing the problem to solving the linearized matter equations \( \delta \mathcal{E}_M = 0 \) for the matter variables alone.

To begin, we note that from the form of \( C_{ab} \) (see (35)), together with the fact that the background equations of motion \( (\mathcal{E}_G)_{ab} = 0 \), \( \mathcal{E}_M = 0 \) are satisfied, it follows immediately that (35) is equivalent to

\[
\delta C_{ab} = 0, \quad \delta \mathcal{E}_M = 0. \tag{39}
\]

Next, we note that by spherical symmetry, there are only four independent components of \( \delta C_{ab} \), which can be chosen to be the \( tt \), \( tr \), \( rr \), and \( \theta \theta \) components. However, we showed in the previous section that the \( tt \) and \( tr \) components of these equations can be replaced by a single equation of lower differential order, namely \( F = 0 \). Furthermore, the linearization of the generalized Bianchi identity \( \mathcal{E}_M \) off of a background solution (where \( C^{ab} = 0 \)) yields

\[
\nabla_a (\delta C^{a}_b) = \delta B_b. \tag{40}
\]

If we impose the linearized matter equations of motion, \( \delta \mathcal{E}_M = 0 \), then \( \delta B_b = 0 \). This implies that if the \( tt \), \( tr \), and \( rr \) components of \( \delta C_{ab} = 0 \) are satisfied, and if the linearized matter equations of motion, \( \delta \mathcal{E}_M = 0 \), hold, then the \( \theta \theta \) component of \( \delta C_{ab} = 0 \) is automatically satisfied. Thus, the full set of linearized equations (39) may be replaced by

\[
F = 0, \quad (\delta C)_{rr} = 0, \quad \delta \mathcal{E}_M = 0 \tag{41}
\]

We now restrict consideration to theories in which the field equations are at most second order in derivatives of the metric, and such that the second derivatives of the metric appear only in the form of curvature. This is the case for general relativity with typical forms of matter, as well as for Bekenstein’s TeVeS theory. It is also the case for \( f(R) \) gravity after a suitable redefinition of variables, since that theory can be recast as a scalar-tensor theory.

We now analyze the possible dependence of \( \delta C_{tt} \) on \( \lambda \) in order to determine the possible dependence of \( F \) on this quantity. By the Bianchi identity, the \( tt \)- and \( tr \)-components of the equations \( \delta C_{ab} = 0 \) cannot contain second time derivatives of the metric perturbation quantities \( \phi \) and \( \lambda \), since otherwise the left side of (41) would contain third time derivatives of the metric perturbation quantities, which could not be canceled by terms on the right side, which, by assumption, contain at most second derivatives (see (31)). (Recall that the generalized Bianchi identity (31) holds independently of the field equations, so its linearization (41) off of a solution must hold if \( \phi \) and \( \lambda \) are taken to be arbitrary functions of \( r \) and \( t \).) Thus, in particular, the expression \( \delta C_{tt} \) cannot contain a term in \( \partial^2 \lambda / \partial t^2 \). On the other hand, for a diagonal metric like (11), for each coordinate \( x^\mu \), the Riemann tensor cannot contain a term of the form \( \partial^2 g^{tt} / \partial x^\mu \partial x^\mu \), since the formula for the Riemann tensor involves antisymmetrizations over components. Choosing \( x^\mu = r \), we see that the Riemann tensor cannot depend on \( \partial^2 g^{rr} / \partial r^2 \). This implies that the expression \( \delta C_{tt} \) cannot contain a term of the form \( \partial^2 \lambda / \partial r^2 \). Finally, since the background spacetime is static and thus invariant under time reflection \( t \rightarrow -t \), it follows that the expression \( \delta C_{tt} \) must not change sign under time reversal and thus cannot contain a term of the form \( \partial \lambda / \partial t \) or \( \partial^2 \lambda / \partial t^2 \).

Thus, we conclude that for theories in which the field equations are at most second order in derivatives of the metric, and are such that the second derivatives of the metric appear only in the form of curvature, the quantity \( \lambda \) can appear in the expression \( \delta C_{tt} \) only in the form of \( \lambda \) and \( \partial \lambda / \partial r \). However, as noted previously, the expression \( F \) must be of lower differential order (by one) than \( \delta C_{tt} \) in all variables. Consequently, \( F \) can depend only algebraically on \( \lambda \).

Next, we analyze the possible dependence of \( \delta C_{tr} \) on \( \phi \) in order to determine the possible dependence of \( F \) on this quantity. As noted at the end of Section III, \( \phi \) can appear in the linearized field equations only in the form of \( \partial \phi / \partial r \) and its derivatives. However, since \( \delta C_{tr} \) must be odd under time reflection, \( \phi \) can appear in \( \delta C_{tr} \) only in the form of \( \partial \phi / \partial t \) and its \( r \)-derivatives. Since, by assumption, the field equations contain no higher than second derivatives of the metric perturbation quantities, it follows that \( \phi \) can appear in \( \delta C_{tr} \) only in the form of \( \partial^2 \phi / \partial r \partial t \). Suppose now that \( \delta C_{tr} \) contained a non-vanishing term proportional to \( \partial^3 \phi / \partial r \partial t^2 \). The only other term in the Bianchi identity (31) that can contain third derivatives of the metric perturbation quantities is \( \partial (\delta C^{rr}_{r}) / \partial r \). In order to cancel the term proportional to \( \partial^3 \phi / \partial r \partial t^2 \), it is necessary that \( \delta C_{tt} \) contain a term proportional to \( \partial^2 \phi / \partial r \partial t^2 \). However, this is impossible, since the field equations can contain \( \phi \) only in the form of \( \partial \phi / \partial r \) and its derivatives. Consequently, \( \delta C_{tr} \) cannot have any dependence whatsoever upon \( \phi \). It then follows immediately that \( F \) cannot depend upon \( \phi \).

Putting together the results of the previous two paragraphs, we conclude that for theories in which the field equations are at most second order in derivatives of the metric, and are such that the second derivatives of the metric appear only in the form of curvature, \textit{we can solve the equation} \( F = 0 \) \textit{algebraically for} \( \lambda \) \textit{in terms of the matter variables}.

Next, we consider the equation \( (\delta C)_{rr} = 0 \). For exactly the same reason as \( (\delta C)_{tt} \) cannot contain second time derivatives of the metric perturbation quantities, it follows that \( (\delta C)_{rr} \) cannot contain second derivatives of these quantities with respect to \( r \), and thus cannot contain a term of the form \( \partial^2 \phi / \partial r \partial t^2 \). By time reflection symmetry, it also cannot contain a term of the form \( \partial \phi / \partial t \) or \( \partial^2 \phi / \partial r \partial t \). However, as already noted above, \( \phi \) can appear in the linearized field equations only in the form of \( \partial \phi / \partial r \) and its derivatives. Thus, we also cannot have a term in \( (\delta C)_{rr} \) of the form \( \partial^2 \phi / \partial r \partial t^2 \) (as we already noted above) or \( \phi \). Consequently, \( (\delta C)_{rr} \) can depend only algebraically on \( \partial \phi / \partial r \). Thus, since we have...
already eliminated $\lambda$ in terms of the matter variables for the class of theories considered here, we can solve the equation $(\delta C)_{rr} = 0$ algebraically for $\partial \psi / \partial r$ in terms of the matter variables.

In summary, we have seen that we can eliminate the metric perturbation variables algebraically by solving the equation $F = 0$ for $\lambda$ and solving the equation $(\delta C)_{rr} = 0$ for $\partial \psi / \partial r$. We may then substitute these solutions into the linearized matter equations

$$\delta E_M = 0$$  \hspace{1cm} (42)

to obtain a system of equations involving only the unknown matter variables. The perturbation problem has thus been reduced to solving these equations.

VI. OBTAINING A VARIATIONAL PRINCIPLE

Having fixed the gauge and completely eliminated the remaining metric perturbation variables by the procedures of the previous three sections, we may find that the system of equations (12) for the matter variables can be put in the general form (11). Indeed, we shall see in the next section and in a subsequent paper that this is the case in a wide class of theories. Nevertheless, even if the reduced equations take the form (11), our work is not completed because it is not straightforward to determine if there exists an inner product that satisfies (2), and—even if such an inner product does exist—it is not straightforward to find it explicitly. This is particularly true if (11) is a complicated system of equations. In the absence of an inner product that makes $T^\alpha_\beta$ self-adjoint, we will not be able to formulate a variational principle, and it will not be straightforward to analyze stability.

We now shall show that the fact that the original theory was derived from a Lagrangian will enable us—at least in a very wide class of cases—to determine whether the desired inner product exists and to find it explicitly if it does. When the desired inner product does exist, we will thereby be able to immediately write down the variational principle (3) for determining stability.

At first sight, it might appear that the fact that the original theory was derived from a Lagrangian would be of little use. It is true that one can obtain a Lagrangian for the linearized theory by expanding the Lagrangian of the exact theory to quadratic order about the background solution. However, if one substitutes the gauge choice made in Section III into this Lagrangian, the resulting object no longer functions as a Lagrangian. Similarly, even if one found a Lagrangian for the gauge-fixed theory, when one substitutes the solution for the metric perturbation variables found in Section V into this Lagrangian, the resulting object would again fail to function as a Lagrangian. Thus, it is far from obvious how to obtain a Lagrangian for the reduced theory.

Nevertheless, the fact that the original theory was derived from a Lagrangian leaves an important imprint on the reduced theory. As discussed in Section III in the original, exact theory, one can define a symplectic current three-form $\omega(\Psi; \delta_1 \Psi, \delta_2 \Psi)$, which is constructed out of a background solution $\Psi$ and two linearized perturbations, $\delta_1 \Psi$ and $\delta_2 \Psi$, such that $\omega$ depends linearly on $\delta_1 \Psi$ and $\delta_2 \Psi$. Furthermore, the symplectic current satisfies the property that it is conserved, i.e.,

$$d \omega = 0$$  \hspace{1cm} (43)

whenever $\delta_1 \Psi$ and $\delta_2 \Psi$ satisfy the linearized equations of motion. Substitution of gauge choices for $\delta_1 \Psi$ and $\delta_2 \Psi$ and/or elimination of variables via some of the linearized field equations will not affect the conservation of $\omega$. Thus, we automatically obtain a conserved symplectic current for the reduced theory.

The conditions arising from the existence of a conserved symplectic current $\omega$ are most conveniently formulated in terms of the pullback, $\bar{\omega}$, of $\omega$ to the static hypersurfaces of the background solution, i.e., the hypersurfaces orthogonal to the static Killing field $t^\alpha$. Since $d \omega$ is a four-form in a four-dimensional space, the condition that $d \omega = 0$ is equivalent to $t \cdot d \omega = 0$. By a standard identity, we have

$$t \cdot d \omega = \mathcal{L}_t \omega - d(t \cdot \omega).$$  \hspace{1cm} (44)

Thus, when the equations of motion hold, we have

$$\mathcal{L}_t \omega = d(t \cdot \omega).$$  \hspace{1cm} (45)

Pulling this equation back to the static hypersurfaces, we obtain

$$\mathcal{L}_t \bar{\omega} = d(\bar{t} \cdot \bar{\omega}).$$  \hspace{1cm} (46)

In the theories we shall consider here and in a subsequent paper, the pullback of the symplectic current for the reduced theory takes the form

$$\bar{\omega} = W_{\alpha \beta} \left( \frac{\partial \psi_1^{\alpha}}{\partial t} \frac{\partial \psi_2^{\beta}}{\partial t} - \frac{\partial \psi_2^{\alpha}}{\partial t} \frac{\partial \psi_1^{\beta}}{\partial t} \right)$$  \hspace{1cm} (47)

where $\psi^{\alpha}$ denotes the dynamical variables for the reduced theory. Here, the three-form $W_{\alpha \beta}$ is constructed from the quantities appearing in the background solution, and thus is independent of $t$, i.e., $\mathcal{L}_t W_{\alpha \beta} = 0$. Thus, we obtain

$$\mathcal{L}_t \bar{\omega} = W_{\alpha \beta} \left( \frac{\partial \psi_1^{\alpha}}{\partial t} \frac{\partial \psi_2^{\beta}}{\partial t} - \frac{\partial \psi_2^{\alpha}}{\partial t} \frac{\partial \psi_1^{\beta}}{\partial t} + \psi_1^{\alpha} T^\beta \gamma_2 - \psi_2^{\alpha} T^\beta \gamma_1 \right)$$  \hspace{1cm} (48)

However, the linearized equations of motion (11) hold if and only if $\partial^2 \psi^{\alpha} / \partial t^2 = T^\alpha_\beta \psi^{\beta}$. Thus, we find that the quantity

$$W_{\alpha \beta} \left( \frac{\partial \psi_1^{\alpha}}{\partial t} \frac{\partial \psi_2^{\beta}}{\partial t} - \frac{\partial \psi_2^{\alpha}}{\partial t} \frac{\partial \psi_1^{\beta}}{\partial t} + \psi_1^{\alpha} T^\beta \gamma_2 - \psi_2^{\alpha} T^\beta \gamma_1 \right)$$  \hspace{1cm} (49)
must be an exact form on any static hypersurface, \( \Sigma \). Now, \( \psi_1^\alpha, \partial \psi_1^\alpha / \partial t, \psi_2^\alpha, \partial \psi_2^\alpha / \partial t \) are all freely specifiable initial data at \( t = 0 \) for the system of equations (11). Thus, if we choose \( \psi_1^\alpha, \partial \psi_1^\alpha / \partial t, \psi_2^\alpha, \partial \psi_2^\alpha / \partial t \) to be arbitrary smooth functions of compact support (or of sufficiently rapid decay), the integral of (49) over \( \Sigma \) must vanish. Inspecting the first two terms of (49), we see that if we choose \( \psi_1^\alpha, \psi_2^\alpha \) to vanish. Inspecting the first two terms of (49), we see that for all \( \psi_1^\alpha, \psi_2^\alpha \) of compact support (or of sufficiently rapid decay), the integral of (49) over \( \Sigma \) must vanish. Inspecting the first two terms of (49), we see that for all \( \psi_1^\alpha \) and \( \psi_2^\alpha \) of compact support (or of sufficiently rapid decay), we have

\[
\int_{\Sigma_0} W_{\alpha \beta} (\psi_1^\alpha T^\beta \gamma_1 - \psi_2^\alpha T^\beta \gamma_2) = 0.
\]

(51)

Now, suppose that \( W_{\alpha \beta} \) is positive definite in the sense that

\[
\int_{\Sigma_0} W_{\alpha \beta} \psi^\alpha \psi^\beta > 0
\]

(52)

for all \( \psi^\alpha \neq 0 \). Then

\[
(\psi_1, \psi_2) \equiv \int_{\Sigma_0} W_{\alpha \beta} \psi_1^\alpha \psi_2^\beta
\]

(53)

defines an inner product. Equation (51) is then precisely the statement that the operator \( T_{\alpha \beta} \) is symmetric in this inner product. We thereby obtain a variational principle [17], as desired.

On the other hand, if \( W_{\alpha \beta} \) fails to be positive definite (or negative definite), then there does not appear to be any reason to expect that an inner product exists that makes \( T_{\alpha \beta} \) self-adjoint [18]. In that case, we do not expect that there is a variational principle to determine stability; and one presumably must work directly with the equations of motion to analyze stability.

Finally, it is worth noting that if \( W_{\alpha \beta} \) is positive definite, then given that \( T_{\alpha \beta} \) must satisfy (51), it is easily verified that

\[
h = \frac{1}{2} \int_{\Sigma_0} W_{\alpha \beta} \left( \frac{\partial \psi^\alpha}{\partial t} \frac{\partial \psi^\beta}{\partial t} + \psi^\alpha T_{\beta} \gamma \psi^\gamma \right)
\]

(54)

defines a Hamiltonian for the dynamics given by (11) associated with the symplectic form

\[
\Omega(\psi_1, \psi_2) = \int_{\Sigma_0} \omega(\Psi; \psi_1, \psi_2).
\]

(55)

VII. ILLUSTRATION: CHANDRASEKHAR’S VARIATIONAL PRINCIPLE

As a concrete example of the procedure outlined here, let us consider Einstein gravity minimally coupled to an isentropic perfect fluid. A variational principle for the radial oscillations of static, spherically symmetric stars was derived by Chandrasekhar [6, 7] by a direct analysis of the equations of motion. We shall now re-derive this variational principle in a much more systematic and direct way using the method described in this paper.

In order to apply our method, it is essential that the equations of motion be derived from a Lagrangian. For general matter minimally coupled to \( g^{ab} \), the Lagrangian will be of the form

\[
\mathcal{L} = \frac{1}{16\pi} R + \mathcal{L}_{\text{mat}}[\Psi, g^{ab}]
\]

(56)

where \( A \) denotes the collection of matter fields, with tensor indices suppressed. For the case of a perfect fluid, there have been there are several approaches that have been used to provide a Lagrangian formulation; see [13] for an overview. We will use the “Lagrangian coordinate” method. In this formalism, we introduce an abstract three-dimensional manifold, \( \mathcal{M} \), of fluid worldlines, equipped with a volume three-form \( \mathbf{N} \). The fluid is then described by a map \( \chi : \mathcal{M} \to \mathcal{M} \), which assigns to each \( x \) in the spacetime manifold \( \mathcal{M} \) the fluid worldline that passes through \( x \). By introducing coordinates, \( X^A \), on \( \mathcal{M} \), where \( A = 1, 2, 3 \), we can represent \( \chi \) by the three scalar functions \( X^A(x) \) on \( \mathcal{M} \), which we view as the independent dynamical variables of the fluid.

We define a three-form \( N_{abc} \) on spacetime—representing the “density of fluid worldlines” or the “particle number density”—by

\[
N_{abc} = N_{ABC}(X) \nabla_a X^A \nabla_b X^B \nabla_c X^C.
\]

(57)

In terms of \( N_{abc} \) we define the scalar particle number density \( \nu \) by

\[
\nu^2 = \frac{1}{6} N_{abc} N^{abc},
\]

(58)

and we define the fluid four-velocity \( U^a \) by

\[
N_{abc} = \nu \epsilon_{abcd} U^d,
\]

(59)

where \( \epsilon_{abcd} \) is the spacetime volume four-form. The Lagrangian four-form for the perfect fluid is then given by

\[
\mathcal{L}_{\text{mat}} = -g(\nu) \epsilon,
\]

(60)

where \( g(\nu) \) is an arbitrary function of the comoving particle density \( \nu \). The choice of the function \( g(\nu) \) corresponds to the choice of equation of state of the fluid (see [69] below).

We will need both the equations of motion and the formula for \( \omega \) in our analysis, so we now proceed to derive these. As usual, the “gravitational part” of the Lagrangian, \( \mathcal{L}_G = (1/16\pi) R + \mathcal{L}_{\text{mat}} \), contributes \( (1/16\pi) G_{ab} \) to the gravitational equations of motion and contributes

\[
\omega^a_{\text{grav}} = S^a_{bc} d \epsilon_f (\delta g^{bc} \nabla_d \delta g^f - \delta g^{bc} \nabla_d \delta g^f),
\]

(61)
to the symplectic current, where
\[
S^a_{bc} \epsilon_{def} = \frac{1}{16\pi} \left( \delta^a_c \epsilon_{def} g_{bf} - \frac{1}{2} g^{ad} g_{bc} g_{ef} - \frac{1}{2} \delta^a_b \epsilon_{def} g_{cf} - \frac{1}{2} \delta^a_e \epsilon_{bdf} g_{cf} + \frac{1}{2} \delta^a_d \epsilon_{bef} g_{cf} \right)
\]
and we have defined \( \omega^a \), the dual to \( \omega \), such that
\[
\omega_{abc} = \omega^a e_{abc}
\]
with \( \omega_{bcd} \) defined by (59).

To obtain the matter equations of motion and the contribution of the matter fields to the symplectic current, where the fluid-space indices only, not the fluid-space indices. Note that the fluid-space coordinates \( X^A \) are simply scalars as far as the spacetime derivative operator \( \nabla_a \) is concerned, and thus no metric variations, \( \delta g^{ab} \), appear in this expression. The antisymmetrization in this expression (using (58) and \( \delta \nu = (\nu^2)/2\nu \), we obtain
\[
\delta(-\rho) = \left[ \frac{\rho'}{2\nu} \nabla_a (N_{ABC} \delta X^A) \nabla_b X^B \nabla_c X^C N_{abc} \right. \]
\[
\left. - \frac{1}{2} \left( \frac{\rho'}{2\nu} N_{abcd} \delta g^{cd} - \rho g_{ab} \right) \delta g^{ab} \right] \epsilon, \quad (65)
\]
where \( \rho' \) denotes the derivative of \( \rho(\nu) \) with respect to \( \nu \). The second term is a bracket of \( \mathcal{L}_{\text{mat}} \) with respect to \( \delta g^{ab} \) and thus is equal to \( \frac{1}{\nu} \) times the stress-energy tensor, \( T_{ab} \); this quantity provides the contribution of \( \mathcal{L}_{\text{mat}} \) to the gravitational field equations. Using (59) and the identity \( \epsilon_{abc} \epsilon^{cdef} = -4 \delta[\epsilon_a \delta d]b \), we obtain
\[
N_{abcd} \delta g^{cd} = 2\nu^2 (g_{ab} + U_a U_b), \quad (66)
\]
and thus the stress-energy tensor can be rewritten in more familiar terms as
\[
T_{ab} = \rho' U_a U_b + (\rho' \nu - \rho) g_{ab}. \quad (67)
\]
This can be recognized as the standard stress-energy for a perfect fluid
\[
T_{ab} = (\rho + P) U_a U_b + P g_{ab} \quad (68)
\]
under the identifications
\[
\rho \rightarrow \rho \quad \rho' \nu - \rho \rightarrow P. \quad (69)
\]
Integrating the first term in (65) by parts, we obtain
\[
- \frac{\rho'}{2\nu} \nabla_a (N_{ABC} \delta X^A) \nabla_b X^B \nabla_c X^C N_{abc} \]
\[
= \nabla_a \left( \frac{\rho'}{2\nu} N_{ABC} \delta X^A \nabla_b X^B \nabla_c X^C N_{abc} \right) \]
\[
+ N_{ABC} \delta X^A \nabla_a \left( \frac{\rho'}{2\nu} \nabla_b X^B \nabla_c X^C N_{abc} \right) \quad (70)
\]
From the first term on the right-hand side, we can read off the presymplectic current \( \theta^a_{\text{mat}} \);
\[
\theta^a_{\text{mat}} = - \frac{\rho'}{2\nu} N_{ABC} \delta X^A \nabla_b X^B \nabla_c X^C N_{abc} \quad (71)
\]
We can then take the antisymmetrized second variation of \( \theta^a_{\text{mat}} = \theta \cdot \epsilon \), as in (59), to obtain an expression for the symplectic current three-form \( \omega^a_{\text{mat}} \). By a straightforward calculation, we obtain
\[
\omega^a_{\text{mat}} = \left[ \delta_1 g^{bc} K^a_{bcA} + \delta_1 X^B L^a_{AB} \right. \]
\[
+ \nabla_b \delta_1 X^B M^{ab}_{AB} \bigg] \delta_2 X^A - \left. [1 \leftrightarrow 2], \quad (72) \right.
\]
with \( \omega^a_{\text{mat}} \) defined analogously to (53) and the tensors \( K^a_{bcA} \), \( L^a_{AB} \), and \( M^{ab}_{AB} \) defined by
\[
K^a_{bcA} = - \frac{\rho'}{2\nu} \left( \frac{1}{2} g_{bc} N_{ADE} \nabla_d X^D \nabla_c X^E N^{ade} + 2 N_{ABD} \nabla_b X^B \nabla_d X^D N^a c d + \delta^a_b N^d c N_{ADE} \nabla_d X^D \nabla_c X^E \right) \]
\[
+ \frac{1}{8\nu^3} \left( \rho'' - \rho' \right) N_{def} N_{AFG} \nabla_f X^F \nabla_g X^G N^{a fg}, \quad (73)
\]
\[
L^a_{AB} = - \frac{\rho'}{2\nu} \left( \partial_{B} N_{ACD} \nabla_c X^C \nabla_d X^D N^{acd} + 3 N_{ACD} \nabla_c X^C \nabla_d X^D \partial_{E} N_{BFG} \nabla_{[a} X^E \nabla_c X^F \nabla_{d]} X^G \right) \]
\[
+ \frac{1}{4\nu^3} \left( \rho'' - \rho' \right) N_{ACD} \nabla_b X^C \nabla_c X^D N^{abc} \partial_{E} N_{BFG} \nabla_{[a} X^E \nabla_b X^F \nabla_{d]} X^G N^{c fg}, \quad (74)
\]
and

\[ M^{ab}_{\ AB} = \frac{\varrho'}{2\nu} \left( 2N_{ABC} \nabla_c X^C N^{abc} + 3N_{ACD} \nabla_c X^C \nabla_d X^D N_{BEF} g^{[b(a} \nabla^c X^E \nabla^d] X^{F]} + \frac{1}{4\nu^3} (\varrho'' - \varrho') (N_{ACD} \nabla_c X^C \nabla_d X^D N^{aced}) (N_{BEF} \nabla_c X^E \nabla_f X^F N^{bef}) \right). \]  (75)

(The antisymmetrizations in (74) and (75) are again over tensor indices only.)

The second term in (70) yields the matter equations of motion. Since the Lagrangian coordinates \( X^A \) are scalars, we have \( \nabla_{(a} \nabla_{b)} X^A = 0 \), and since \( N^{abc} \) is completely antisymmetric, the vanishing of the second term for arbitrary \( \delta X^A \) is equivalent to

\[ 0 = \nabla_a \left( \frac{\varrho'}{2\nu} \nabla_b X^B \nabla_c X^C N^{abc} \right) = \frac{1}{2} \nabla_b X^B \nabla_c X^C \nabla_a \left( \frac{\varrho'}{\nu} N^{abc} \right). \]  (76)

This can be put in a more recognizable form by rewriting it in terms of the four-velocity \( U^a \)

\[ 0 = \frac{1}{2} \nabla_b X^B \nabla_c X^C \nabla_a (\varrho' U_d) \, c^{abcd}. \]  (77)

This, in turn, is equivalent to \( U^a \nabla_a (\varrho' U_d) = 0 \), which is just the relativistic Euler equation [13].

We now restrict our attention to spherically symmetric perturbations of static, spherically symmetric solutions. In order to obtain our desired variational principle, we must do the following:

(i) Define our choice of variable(s) to describe the fluid perturbations;

(ii) Obtain an expression for \( F \) and solve the equation \( F = 0 \) algebraically for \( \lambda \);

(iii) Write down the equation \( \delta C_{\tau \tau} = 0 \) and solve this equation algebraically for \( \partial \phi/\partial \tau \);

(iv) Substitute the solutions for \( \lambda \) and \( \partial \phi/\partial \tau \) into the linearization of the matter equations of motion (76), thereby rewriting this equation purely in terms of the perturbed fluid variable(s);

(v) Determine if this equation takes the form 11 and, if so, read off the operator \( T^{\alpha \beta} \);

(vi) Evaluate the pullback of \( \omega \)---or, equivalently, the time component of \( \omega^a \);

(vii) Determine if \( \omega \) takes the form (17), and, if so, read off \( W_{\alpha \beta} \); and

(viii) Determine if \( W_{\alpha \beta} \) defines a positive definite inner product and, if so, write down the variational principle 3.

Although each of these steps may require some algebra, we emphasize that none of the steps require any ingenuity, and we are guaranteed to succeed in obtaining a variational principle unless the matter equations of motion obtained in step (iv) fail to take the form 11, the pullback of \( \omega \) fails to take the form (17), or \( W_{\alpha \beta} \) fails to define a positive definite inner product.

To describe the fluid perturbations, we choose the coordinates \( X^A = \{ X^R, X^\Theta, X^\Phi \} \) on fluid space so that in the static background solution we have \( X^R = r, X^\Theta = \theta \), and \( X^\Phi = \varphi \). Since we consider only spherically symmetric perturbations, we have \( \delta X^\Theta = 0 \) and \( \delta X^\Phi = 0 \). Thus, the perturbation of the fluid is completely characterized by its radial “Lagrangian displacement”

\[ \xi(r, t) \equiv \delta X^R(r, t). \]  (78)

which describes the radial displacement of each fluid element from its “equilibrium position”.

Given our choice of \( X^A \), in order to be compatible with our assumption of spherical symmetry, the three-form \( N_{ABC} \) on “fluid space” must take the form

\[ N_{R\Theta\Phi} = q(X^R) \sin X^\Theta, \]  (79)

for some function \( q \). By (58), the number density, \( \nu \), of the fluid is then given by

\[ \nu = \frac{q(X^R)}{r^2} \sqrt{e^{-2\Delta} \left( \frac{\partial X^R}{\partial r} \right)^2 - e^{-2\Phi} \left( \frac{\partial X^R}{\partial t} \right)^2}. \]  (80)

The variation of this formula yields

\[ \delta \nu = \nu \left( \frac{\partial \xi}{\partial r} + \left( \frac{1}{\nu} \frac{\partial \nu}{\partial r} + \frac{\partial \Delta}{\partial r} + \frac{2}{r} \right) \xi - \lambda \right). \]  (81)

This formula will enable us to express all terms in the linearized equations of motion that arise from the variation of \( \nu \) in terms of our chosen dynamical variable \( \xi \).

Our next step is to obtain an expression for \( F \) and solve the equation \( F = 0 \). Since, in this case, our matter fields \( X^A \) are scalars, the terms in (76) that depend on the matter equations of motion vanish, and we simply have

\[ V_{abc} = 2\epsilon^{d}{}_{abc} t^d (E_G)_{de}. \]  (82)

From this, we can conclude that

\[ V_{\theta \varphi} = -2\sqrt{-g(E_G)_{tt} g^{tr}}, \quad V_{r \theta \varphi} = 2\sqrt{-g(E_G)_{tt} g^{tr}}. \]  (83)
Since $\delta V = t^a \delta C_a$, we can therefore identify the quantity $H_1$ defined in (17) as

$$H_1(t, r) = -2r^2 e^{\Phi - \Lambda} (\delta \xi_G)_{tr}, \quad (84)$$

Calculating the first-order equation of motion $(\delta \xi_G)_{tr}$, we find that

$$(\delta \xi_G)_{tr} = \frac{2}{r} \frac{\partial \lambda}{\partial t} - e^{2\lambda} \phi' \frac{\partial \xi}{\partial t}. \quad (85)$$

From (19), we can immediately deduce that

$$F(t, r) = 2r^2 e^{\Phi - \Lambda} \left( \phi' \nu - \frac{2}{r} e^{2\lambda} \lambda \right) = 0. \quad (86)$$

Thus, $\lambda$ can be eliminated in terms of the matter variable $\xi$ by

$$\lambda = \frac{r}{2} e^{2\lambda} \phi' \nu \xi. \quad (87)$$

The next step is to write down and solve the equation $(\delta C)_{rr} = 0$ for $\partial \phi/\partial r$. By direct calculation, we obtain

$$(\delta C)_{rr} = (\delta \xi_G)_{rrr} = \frac{2}{r} e^{-2\Lambda} \left( \frac{\partial \phi}{\partial r} - \left( \frac{2}{r} \frac{\partial \phi}{\partial t} + \frac{1}{r} \right) \lambda \right) - \phi'' \nu \delta \nu, \quad (88)$$

Substituting our solutions for $\lambda$ and $\partial \phi/\partial r$, we obtain

$$\phi' \left[ -e^{2\Lambda - 2\Phi} \frac{\partial^2 \xi}{\partial t^2} + \frac{\partial \phi}{\partial t} \right] + \left( \frac{\partial}{\partial t} + \frac{\partial}{\partial r} \right) \left[ \phi'' \nu \left( \frac{\partial}{\partial r} + \frac{1}{\nu \partial r} + \frac{1}{r} \right) \xi - \lambda \right] = 0. \quad (90)$$

Fortunately, this equation takes the form of (11), and it is straightforward to read off the operator $T$ from this equation.

The next step is to evaluate the pullback of the symplectic current—or, equivalently, $t_a \omega^a$—so as to obtain the inner product that will appear in the variational principle. As previously noted $\omega^a$ consists of two pieces, $\omega^a = \omega^a_{\text{grav}} + \omega^a_{\text{mat}}$, given by (61) and (72) respectively. By direct evaluation, we find that for spherically symmetric perturbations of a static, spherically symmetric background, we have $t_a \omega^a_{\text{grav}} = 0$, i.e., there is no “gravitational contribution” to the symplectic form (20). In a static background, the expression for $t_a \omega^a_{\text{mat}}$ simplifies considerably, since the fluid four-velocity $U^a$ will be parallel to $t^a$, so $t_a N^{abc} = \nu t_a U_{abc} = 0$. We obtain

$$t_a \omega^a_{\text{mat}} = -t_a \frac{\phi'}{2r^2} N_{ABC} \nabla_b X^A \nabla_c X^C \delta_{2X^A} \left[ \delta_1 g^{ad} N_{db} \right. \right. \left. + 3 \nabla^a (N_{DEF} \delta_1 X^D) \nabla_b X^E \nabla^c X^F \right] - \left[ 1 \leftrightarrow 2 \right]. \quad (92)$$

The first term in the square brackets also vanishes in the spherically symmetric case. Thus, we find that for spherically symmetric perturbations of static, spherically symmetric backgrounds, the symplectic form

$$\Omega = \int_{\Sigma} \Omega = -\frac{1}{\Sigma} d^3x \sqrt{h} \left( \omega^a_{\text{grav}} + \omega^a_{\text{mat}} \right)n_a, \quad (93)$$

where $\Omega$ is a static slice in the background spacetime, $n^a$ is its future-directed unit normal, and $\sqrt{h}$ is the volume element associated with the induced Riemannian metric $h_{ab}$ on $\Sigma$ is given by

$$\Omega = \frac{3}{2} \int d^3x \sqrt{h} \left( N_{ABC} \delta_2 X^A \nabla_b X^B \nabla_c X^C \right) \times \left( N_{DEF} \nabla^a \delta_1 X^D \nabla_b X^E \nabla^c X^F \right) - \left[ 1 \leftrightarrow 2 \right]. \quad (94)$$
In writing the above, we have used the fact that there is only one non-vanishing component of $\delta X^A$, so any term proportional to $\delta_1 X^A \delta_2 X^B$ (as opposed to terms depending on the derivatives of $\delta X^A$) will vanish under antisymmetrization. Writing this out in terms of our perturbational variables yields the simple expression

$$\Omega = 4\pi \int dr r^2 e^{3\Lambda - \Phi} \rho' \nu \left( \frac{\partial \xi_1}{\partial t} \xi_2 - \xi_1 \frac{\partial \xi_2}{\partial t} \right).$$  \hspace{1cm} (95)$$

The inner product to be used in our variational principle can now be read off by comparing (95) with (53). We obtain

$$(\xi_1, \xi_2) = 4\pi \int dr r^2 e^{3\Lambda - \Phi} \rho' \nu \xi_1 \xi_2$$ \hspace{1cm} (96)$$

Since $\rho' \nu = \rho + P$ under the substitutions in (69), this quadratic form will be positive for any fluid satisfying the null energy condition (i.e., $\rho + P > 0$).

Our variational principle is then of the form (43), with numerator

$$\int dr r^2 e^{3\Lambda - \Phi} \rho' \nu \xi^2$$ \hspace{1cm} (97)$$

VIII. CONCLUSION

We have presented a general procedure to analyse the stability of spherically symmetric perturbations about static spherically symmetric solutions of an arbitrary covariant field theory. This procedure involves solving the linearized constraints and eliminating the metric perturbation variables algebraically. The symplectic form is then used to define an inner product, from which a variational principle can be obtained.

It is important to emphasize that our procedure is entirely prescriptive, as illustrated by the outline in Section VII; while the method can fail at various points (e.g., the inner product obtained from the symplectic form could fail to be positive), there is in principle never any “art” involved in applying this procedure to a given field theory. This method is therefore a potentially powerful tool for analyzing the viability of alternative theories of gravity and other covariant field theories; in an upcoming work \cite{[15]}, we will apply this method to three alternative theories of gravity \cite{[1],[2],[3]}.

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[16] If any of the background matter fields are tensor fields that have a nonvanishing time component, then the perturbations of these components will transform nontrivially, in a manner similar to $\phi$, under a re-definition of the time coordinate. In that case, some linear combination(s) of $\phi$ and the matter variables can appear in un-
differentiated form, but, if one treats these combinations as independent variables, the remaining dependence on $\phi$ can appear only in the form $\partial \phi / \partial r$ and its derivatives.

[17] To obtain a variational principle, we actually require $T^{\alpha \beta}$ to be self-adjoint, not merely symmetric. However, since $T^{\alpha \beta}$ is real, it always admits self-adjoint extensions. If $T^{\alpha \beta}$ admits more than one self-adjoint extension (i.e., if $T^{\alpha \beta}$ is not essentially self-adjoint on the initial domain of smooth functions of compact support), this simply means that additional boundary conditions must be supplied in order to make the dynamics specified by (1) well defined.

[18] If $W_{\alpha \beta}$ fails to be positive definite but is non-degenerate, then we obtain a Krein space, in which $T^{\alpha \beta}$ is symmetric. However, this does not provide us with a variational principle to determine stability.

[19] In writing the equation in this form, we have used the background equation of motion $\frac{2}{r} \left( \frac{d \Phi}{dr} + \frac{d \Lambda}{dr} \right) = e^{2\Lambda} g' \nu$.

[20] This fact is undoubtedly directly related to the fact that there are no dynamical degrees of freedom of the gravitational field in the spherically symmetric case.

[21] Note that Chandrasekhar’s $\gamma$ can be obtained by the substitution $g'' \nu^2 \rightarrow \gamma P$ in the Lagrangian coordinate formalism.