Decentralized Time and Energy-Optimal Control of Connected and Automated Vehicles in a Roundabout

Kaiyuan Xu  
Christos G. Cassandras  
Wei Xiao

Abstract—The paper considers the problem of controlling Connected and Automated Vehicles (CAVs) traveling through a three-entry roundabout so as to jointly minimize both the travel time and the energy consumption while providing speed-dependent safety guarantees, as well as satisfying velocity and acceleration constraints. We first design a systematic approach to dynamically determine the safety constraints and derive the unconstrained optimal control solution. A joint optimal control and barrier function (OCBF) method is then applied to efficiently obtain a controller that optimally track the unconstrained optimal solution while guaranteeing all the constraints. Simulation experiments are performed to compare the optimal controller to a baseline of human-driven vehicles showing effectiveness under symmetric and asymmetric roundabout configurations, balanced and imbalanced traffic rates and different sequencing rules for CAVs.

I. INTRODUCTION

The performance of traffic networks critically depends on the control of conflict areas such as intersections, roundabouts and merging roadways which are the main bottlenecks in these networks. The economic loss caused by congestion in these areas has been well documented [1]. Coordinating and controlling vehicles in these conflict areas is a challenging problem in terms of safety, congestion, and energy consumption [2], [3]. The emergence of Connected and Automated Vehicles (CAVs) provides a promising solution to this problem through better information utilization and more precise trajectory design. The automated control of vehicles has gained increasing attention with the development of new traffic infrastructure technologies [4] and, more recently, CAVs [1].

Both centralized and decentralized methods have been studied to deal with the control and coordination of CAVs at conflict areas. Centralized mechanisms are often used in forming platoons in merging problems [5] and determining passing sequences at intersections [6]. These approaches tend to work better when the safety constraints are independent of speed and they generally require significant computation resources, especially when traffic is heavy. They are also not easily amenable to disturbances.

Decentralized mechanisms restrict all computation to be done on board each vehicle with information sharing limited to a small number of neighbor vehicles [7]–[10]. Optimal control problem formulations are used in some of these approaches, with Model Predictive Control (MPC) techniques employed as an alternative to account for additional constraints and to compensate for disturbances by re-evaluating optimal actions [11]–[13]. The objectives in such problem formulations typically target the minimization of acceleration or the maximization of passenger comfort (measured as the acceleration derivative or jerk). An alternative to MPC has recently been proposed through the use of Control Barrier Functions (CBFs) [14], [15] which provide provable guarantees that safety constraints are always satisfied.

In this paper, we build on the use of optimal control and CBF-based methods in unsignalized intersections [16] and merging [17] to study roundabouts with all traffic consisting of CAVs. There are several similarities between merging, intersections and roundabouts problems. The single-lane merging problem [17] contains a single Merging Point (MP) where safety constraints must be guaranteed, while CAVs follow the same moving direction in each lane. In intersection problems, CAVs have a number of possible paths which conflict at multiple MPs restricted to a small area. In a roundabout, CAVs have the same moving direction (either clockwise or counterclockwise) but multiple possible paths which cross at multiple MPs. A roundabout problem can be dealt with as either a whole system like an intersection or it can be decomposed into several coupled merging problems.

Roundabouts are important components of a traffic network because they usually perform better than typical intersections in terms of efficiency and safety [18]. However, they can become significant bottleneck points as the traffic rate increases due to an inappropriate priority system, resulting in significant delays when the circulating flow is heavy. Previous studies mainly focus on conventional vehicles and try to solve the problem through improved road design or traffic signal control [19]–[21]. More recently, however, researchers have proposed methods for decentralized optimal control of CAVs in a roundabout. The roundabout problem is formulated as...
an optimal control problem with an analytical solution provided in [22]. The problem is decomposed so that first the minimum travel time is solved under the assumption that all vehicles use the same maximum speed within the roundabout. Then, fixing this time, the control input that minimizes the energy consumption is derived analytically. The general framework for decentralized optimal control of CAVs used in urban intersections is implemented in a roundabout problem in [23]. The analysis is similar to [22] except that there is no circulating speed assumption.

In this paper, we formulate an optimal control problem for controlling CAVs traveling through a roundabout. Unlike [22, 23], we jointly minimize both the travel time and the energy consumption and also consider speed-dependent roadway utilization. In addition, to improve computational efficiency, we adopt the joint Optimal Control and Barrier Function (OCBF) introduced in [15]: we first derive the optimal solution when no constraints become active and subsequently optimally track this solution while also guaranteeing the satisfaction of all constraints through the use of CBFs. We first assume a First-In-First-Out (FIFO) sequencing policy over the entire system. We then divide the roundabout into separate merging problems so as to introduce different resequencing rules depending on the MP. We will show that the FIFO policy does not perform well in many “asymmetric” configurations and explore an alternative sequencing policy, termed Shortest Distance First (SDF), which our experimental results show to be superior to FIFO. However, a systematic study of the effect of the passing order is still the subject of ongoing research.

The paper is organized as follows. In Section II, the roundabout problem is formulated as an optimal control problem with safety constraints. In Section III, a decentralized framework is provided to determine the safety constraints related to a given CAV. An OCBF controller is designed in Section IV while simulation results are presented in Section V showing significant improvements in the performance of the OCBF controller compared to a baseline of human-driven vehicles. Finally, in Section VI we provide conclusions and future research.

II. PROBLEM FORMULATION

We initiate our study of roundabouts by considering a single-lane triangle-shaped roundabout with 3 entries and 3 exits as shown in Fig. 1. We consider the case where all traffic consists of CAVs which randomly enter the roundabout from three different origins $O_1$, $O_2$ and $O_3$ and have randomly assigned exit points $E_1$, $E_2$ and $E_3$. The gray road segments which include the triangle and three entry roads form the Control Zone (CZ) where CAVs can share information and thus be automatically controlled. We assume all CAVs move in a counterclockwise way in the CZ. The entry road segments are connected with the triangle at the three Merging Points (MPs) where CAVs from different road segments may potentially collide with each other. The MPs are labeled as $M_1$, $M_2$ and $M_3$. We assume that each road segment has one single lane (extensions to multiple lanes and MPs are possible following the analysis in [24]). The three entry road segments which are labeled as $l_1$, $l_2$ and $l_3$ have the same length $L$, while the road segments in the triangle which are labeled as $l_4$, $l_5$ and $l_6$ have the same length $L_0$ (extensions to different lengths are straightforward). In Fig. 1 a circle, square and triangle represent entering from $O_1$, $O_2$ and $O_3$ respectively. The color red, green and blue represents exiting from $E_1$, $E_2$ and $E_3$ respectively. The full trajectory of a CAV in terms of the MPs it must go through can be determined by its entry and exit points.

A coordinator, i.e., a Road Side Unit (RSU) associated with the roundabout, maintains a passing sequence for all CAVs and records the information of each CAV. The CAVs communicate with the coordinator but are not controlled by it. All control inputs are evaluated on board each CAV in a decentralized way. Each CAV is assigned a unique index upon arrival at the CZ according to the passing order. The most common scheme for maintaining a passing sequence is the First-In-First-Out (FIFO) policy according to each CAV’s arrival time at the CZ. The FIFO rule is one of the simplest schemes, yet works well in many occasions as also shown in [25]. For simplicity, in what follows we use the FIFO queue, but we point out that any passing order policy may be used, such as the Dynamic Resequencing (DR) method in [26]. We also introduce one such alternative in Section III.

Let $S(t)$ be the set of CAV indices in the coordinator queue table at time $t$. The cardinality of $S(t)$ is denoted as $N(t)$. When a new CAV arrives, it is allocated the index $N(t)+1$. Each time a CAV $i$ leaves the CZ, it is dropped and all CAV indices larger than $i$ decrease by one. When CAV $i\in S(t)$ is traveling in the roundabout, there are several important events whose times are used in our analysis: (i) CAV $i$ enters the CZ at time $t_i^e$, (ii) CAV $i$ arrives at MP $M_k$ at time $t_i^k$, $k\in\{1, 2, 3\}$, (iii) CAV $i$ leaves the CZ at time $t_i^l$. Based on this setting, we can formulate an optimal control problem as described next.

**Vehicle Dynamics** Denote the distance from the origin
where $O_j, j \in \{1, 2, 3\}$ to the current location of CAV $i$ along its trajectory as $x_j^i(t)$. Since the CAV’s unique identity $i$ contains the information about the CAV’s origin $O_j$, we can use $x_i(t)$ instead of $x_j^i(t)$ (without any loss of information) to describe the vehicle dynamics as

$$\begin{align*}
\dot{x}_i(t) &= \frac{v_i(t)}{u_i(t)} \\
\dot{v}_i(t) &= \frac{v_i(t)}{u_i(t)}
\end{align*}$$

(1)

where $v_i$ is the velocity CAV $i$ along its trajectory and $u_i$ is the acceleration (control input).

**Objective 1** Minimize the travel time $J_{i,1} = t_{i,1}^f - t_{i,1}^0$ where $t_{i,1}^0$ and $t_{i,1}^f$ are the times CAV $i$ enters and exits the CZ.

**Objective 2** Minimize energy consumption:

$$J_{i,2} = \int_{t_{i}^{0}}^{t_{i}^{f}} C_i(u_i(t)) dt$$

(2)

where $C_i(\cdot)$ is a strictly increasing function of its argument. Since the energy consumption rate is a monotonic function of the acceleration, we adopt this general form to achieve energy efficiency.

**Constraint 1** (Rear-end safety constraint) Let $i_p$ denote the index of the CAV which immediately precedes CAV $i$ on road segment $l_k$. The distance between $i_p$ and $i$ $z_{i,i_p}(t) = x_{i_p}(t) - x_i(t)$ should be constrained by a speed-dependent constraint:

$$z_{i,i_p}(t) \geq \varphi v_i(t) + \delta, \ \forall t \in [t_{i}^{0}, t_{i}^{f}], \ \forall i \in S(t)$$

(3)

where $\varphi$ denotes the reaction time (as a rule, $\varphi = 1.8$ is suggested, see [27]), $\delta$ denotes the minimum safety distance (in general, we may use $\delta_i$ to make this distance CAV-dependent but will use a fixed $\delta$ for simplicity). The index of the preceding CAVs index $i_p$ may change due to road segment changing events and is determined by the method described later in section III-B.

**Constraint 2** (Safe merging constraint) Let $t_{i,k}^b, k \in \{1, 2, 3\}$ be the arrival time of CAV $i$ at MP $M_k$. Let $i_m$ denote the index of the CAV that CAV $i$ may collide with when arriving at its next MP $M_k$. The distance between $i_m$ and $i$ $z_{i,i_m}(t) = x_{i_m}(t) - x_i(t)$ is constrained by:

$$z_{i,i_m}(t_k^b) \geq \varphi v_i(t_k^b) + \delta, \ \forall i \in S(t), \ k \in \{1, 2, 3\}$$

(4)

where $i_m$ can be determined and updated by the method described in section III-B.

**Constraint 3** (Vehicle limitations) The CAVs are also subject to velocity and acceleration constraints due to physical limitations or road rules:

$$v_{i,\text{min}} \leq v_i(t) \leq v_{i,\text{max}}, \forall t \in [t_{i}^{0}, t_{i}^{f}], \forall i \in S(t)$$

$$u_{i,\text{min}} \leq u_i(t) \leq u_{i,\text{max}}, \forall t \in [t_{i}^{0}, t_{i}^{f}], \forall i \in S(t)$$

(5)

where $v_{i,\text{max}} > 0$ and $v_{i,\text{min}} \geq 0$ denote the maximum and minimum speed for CAV $i$, $u_{i,\text{max}} > 0$ and $u_{i,\text{min}} < 0$ denote the maximum and minimum acceleration for CAV $i$. We further assume common speed limits dictated by the traffic rules at the roundabout, i.e. $v_{i,\text{min}} = v_{\text{min}}, v_{i,\text{max}} = v_{\text{max}}$.

Similar to previous work [17], we construct a convex combination of the two objectives above:

$$J_i = \int_{t_{i}^{0}}^{t_{i}^{f}} \left[ \alpha + (1 - \alpha) \frac{1}{2} \max\{u_{i,\text{max}}^2, u_{i,\text{min}}^2\} \right] dt$$

(6)

where $J_{i,1}$ and $J_{i,2}$ are combined with $\alpha \in [0, 1]$ after proper normalization. Here, we simply choose the quadratic function $C_i(u_i) = \frac{1}{2} u_i^2(t)$. If $\alpha = 1$, the problem degenerates into a minimum traveling time problem. If $\alpha = 0$, it degenerates into a minimum energy consumption problem.

By defining $\beta \equiv \frac{1}{2(1-\alpha)} \max\{u_{\text{max}}^2, u_{\text{min}}^2\}, \alpha \in [0, 1)$ and proper scaling, we can rewrite this minimization problem as

$$J_i(u_i) = \beta(t_{i}^{f} - t_{i}^{0}) + \int_{t_{i}^{0}}^{t_{i}^{f}} \frac{1}{2} u_i^2(t) dt$$

(7)

where $\beta$ is the weight factor derived from $\alpha$. Then, we can formulate the optimal control problem as follows:

**Problem 1**: For each CAV $i$ following the dynamics (1), find the optimal control input $u_i(t)$ that minimizes (7) subject to constraints (1), (5), (6), the initial condition $x_i(t_0) = 0$, and given $t_{i,0}^0$, $v_{i,0}$ and $x_{i,0}$.

### III. DECENTRALIZED CONTROL FRAMEWORK

Compared to the single-lane merging or intersection control problems where the constraints are determined and fixed immediately when CAV $i$ enters the CZ, the main difficulty in a roundabout is that the constraints generally change after every event (defined earlier). In particular, for each CAV $i$ at time $t$ only the merging constraint related to the next MP ahead is considered. In other words, we need to determine at most one $i_p$ to enforce (3) and one $i_m$ to enforce (4) at any instant time. There are two different ways to deal with this problem: (i) Treat the system as a single CZ with three MPs with knowledge of each CAV’s sequence of MPs, or (ii) Decompose the roundabout into three separate merging problems corresponding to the three MPs, each with a separate CZ. The first approach is more complex and heavily relies on the CZ sequencing rule used. If FIFO is applied, it is likely to perform poorly in a large roundabout, because a new CAV may experience a large delay in order to preserve the FIFO passing sequence. The second approach is easier to implement but may cause congestion in a small roundabout due to the lack of space for effective control at each separate CZ associated with each MP.

In order to solve **Problem 1** for each CAV $i$, we need to first determine the corresponding $i_p$ and $i_m$ (when they exist) required in the safety constraints (3) and (4). Once this task is complete and (3) and (4) are fully specified, then **Problem 1** can be solved. In what follows, this first task is accomplished through a method designed to determine the constraints in an event-driven manner which can be used in either of the two approaches above and for any desired sequencing policy. An extended queue table, an example of which is shown in Table 1, corresponding to Fig. 1, is used to record the essential state information and identify all conflicting CAVs. We specify the state-updating mechanism for this queue table so as to determine for each CAV $i$ the corresponding $i_p$ and $i_m$. Then,
we focus on the second approach and in Section IV develop a general algorithm for solving Problem 1 based on the OCBF method.

A. The Extended Coordinator Queue Table

Starting with the coordinator queue table shown in Fig. 1, we extend it to include 6 additional columns for each CAV i. The precise definitions are given below:

| Table I | THE EXTENDED COORDINATOR QUEUE TABLE $S(t)$ |
|---------|--------------------------------------------|
| idx     | state | curr. | ori. | 1st MP | 2nd MP | 3rd MP | $i_p$ | $i_m$ |
| 0       | $x_0$ | $l_0$ | $l_1$ | $M_1$, $M$ | $M_2$, $M$ | $M_3$, $M$ | 0     | 0     |
| 1       | $x_1$ | $l_0$ | $l_1$ | $M_1$, $M$ | $M_2$, $M$ | $M_3$, $M$ | 0     | 0     |
| 2       | $x_2$ | $l_2$ | $l_2$ | $M_2$, $M$ | 0     | 0     | 0     | 0     |
| 3       | $x_3$ | $l_2$ | $l_2$ | $M_2$, $M$ | $M_3$, $M$ | 0     | 0     | 0     |
| 4       | $x_4$ | $l_2$ | $l_2$ | $M_2$, $M$ | $M_3$, $M$ | 0     | 0     | 0     |
| 5       | $x_5$ | $l_1$ | $l_1$ | $M_1$, $M$ | 0     | 0     | 0     | 0     |
| 6       | $x_6$ | $l_1$ | $l_1$ | $M_1$, $M$ | 0     | 0     | 0     | 0     |
| 7       | $x_7$ | $l_1$ | $l_1$ | $M_1$, $M$ | $M_2$, $M$ | 0     | 0     | 0     |
| 8       | $x_8$ | $l_1$ | $l_1$ | $M_1$, $M$ | $M_2$, $M$ | 0     | 0     | 0     |
| 9       | $x_9$ | $l_1$ | $l_1$ | $M_1$, $M$ | $M_2$, $M$ | 0     | 0     | 0     |

- **idx**: Unique CAV index, which allows us to determine the order in which the CAV will leave the roundabout according to some policy (e.g., FIFO in Table I).
- **state**: CAV state $x_i = (x_i, v_i)$ where $x_i$ denotes the distance from the entry point to the location of CAV $i$ along its current road segment.
- **curr.**: Current CAV road segment, which allows us to determine the rear-end safety constraints.
- **ori.**: Original CAV road segment, which allows us to determine its relative position when in road segment curr.
- **1st-3rd MP**: These columns record all the MPs on the CAV trajectory. If a CAV has already passed an MP, this MP is marked with an $M$. Otherwise, it is unmarked. As a CAV may not pass all three MPs in the roundabout, some of these columns may be blank.
- **$i_p$**: Index of the CAV that immediately precedes CAV $i$ in the same road segment (if such a CAV exists).
- **$i_m$**: Index of the CAV that may conflict with CAV $i$ at the next MP. CAV $i_m$ is the last CAV that passes the MP ahead of CAV $i$. Note that if $i_m$ and $i$ are in the same road segment, then $i_m (= i_p)$ is the immediately preceding CAV. In this case, the safe merging constraint is redundant and need not be included.

Event-driven Update Process for $S(t)$: The extended coordinator queue table $S(t)$ is updated whenever an event (as defined earlier) occurs. Thus, there are three different update processes corresponding to each triggering event:

- **A new CAV enters the CZ**: The CAV is indexed and added to the bottom of the queue table.
- **CAV $i$ exits the CZ**: All information of CAV $i$ is removed. All rows with index larger than $i$ decrease their index values by 1.
- **CAV $i$ passes an MP**: Mark the MP with $M$ and update the current road segment value curr of CAV $i$ with the one it is entering.

B. Determination of Safety Constraints

Recall that for each CAV $i$ in the CZ, we need to consider two different safety constraints (3) and (4). First, by looking at each row $j < i$ and the corresponding current road segment value curr, CAV $i$ can determine its immediately preceding CAV $i_p$ if one exists. This fully specifies the rear-end safety constraint (3). Next, we determine the CAV which possibly conflicts with CAV $i$ at the next MP it will pass so as to specify the safe merging constraint (4). To do so, we find in the extended queue table the last CAV $j < i$ that will pass or has passed the same MP as CAV $i$. In addition, if the CAV is in the same road segment as CAV $i$, it coincides with the preceding CAV $i_p$. Otherwise, we find $i_m$, if it exists. As an example, in Table I (a snapshot of Fig. 1), CAV 8 has no immediate preceding CAV in $l_1$, but it needs to yield to CAV 7 (although CAV 7 has already passed $M_3$, when CAV 8 arrives at $M_3$ there needs to be adequate space between CAV 7 and 8 for CAV 8 to enter $l_4$). CAV 9 however, only needs to satisfy its rear-end safety constraint with CAV 8.

It is now clear that we can use the information in $S(t)$ in a systematic way to determine both $i_p$ in (3) and $i_m$ in (4).

Thus, there are two functions $i_p(e)$ and $i_m(e)$ which need to be updated after event $e$ if this event affects CAV $i$. The index $i_p$ can be easily determined by looking at rows $j < i$ in the extended queue table until the first one is found with the same value curr as CAV $i$. For example, CAV 9 searches for its $i_p$ from CAV 8 to the top and sets $i_p = 8$ as CAV 8 has the curr value $l_1$. Next, the index $i_m$ is determined. To do this, CAV $i$ compares its MP information to that of each CAV in rows $j < i$. The process terminates the first time that any one of the following two conditions is satisfied:

- The MP information of CAV $i_m$ matches CAV $i$. We define $i_m$ to “match” $i$ if and only if the last marked MP or the first unmarked MP of CAV $i_m$ is the same as the first unmarked MP of CAV $i$.
- All prior rows $j < i$ have been looked up and none of them matches the MP information of CAV $i$.

Combining the two updating processes for $i_p$ and $i_m$ together, there are four different cases as follows:

1. **Both $i_p$ and $i_m$ exist**. In this case, there are two possibilities: (i) $i_p \neq i_m$. CAV $i$ has to satisfy the safe merging constraint (4) with $i_p < i$ and also satisfy the rear-end safety constraint (3) with $i_m < i$. For example, for $i = 7$, we have $i_p = 6$ and $i_m = 4$ ($M_2$ is the first unmarked MP for CAV 7 and that matches the first unmarked MP for CAV 4). (ii) $i_p = i_m$. CAV $i$ only has to follow $i_p$ and satisfy the rear-end safety constraint (3) with respect to $i_p$. Thus, there is no safe merging constraint for CAV $i$ to satisfy. For example, $i = 4$ and $i_p = i_m = 3$.

2. **Only $i_p$ exists**. In this case, there is no safe merging constraint for CAV $i$ to satisfy. CAV $i$ only needs to follow the preceding CAV $i_p$ and satisfy the rear-end safety constraint (3) with respect to $i_p$. For example, $i = 1$ and $i_p = 0$. 

3. Only \( i_m \) exists. In this case, CAV \( i \) has to satisfy the safe merging constraint \([4]\) with the CAV \( i_m \) in \( S(t) \). There is no preceding CAV \( i_p \), thus there is no rear-end safety constraint. For example, \( i = 5, i_m = 1 (M_3 \) is the first unmarked MP for \( CAV \) 5 and that matches the last marked MP for \( CAV \) 1 with no other match for \( j = 4, 3, 2 \).

4. Neither \( i_p \) nor \( i_m \) exists. In this case, CAV \( i \) does not have to consider any safety constraints. For example, \( i = 2 \).

C. Sequencing Policies with Sub-coordinator Queue Tables

Thus far, we have assumed a FIFO sequencing policy in the whole roundabout and defined a systematic process for updating \( t_p(e) \) and \( t_m(e) \) after each event \( e \), hence, the entire extended queue table \( S(t) \). However, FIFO may not be a good sequencing policy if applied to the whole roundabout. In order to allow possible resequencing when a CAV passes an MP, we introduce next a sub-coordinator queue table \( S_k(t) \) associated with each \( M_k, k = 1, 2, 3 \). \( S_k(t) \) coordinates all the CAVs for which \( M_k \) is the next MP to pass or it is the last MP that they have passed. We define \( CZ_k \) as the CZ corresponding to \( M_k \) that consists of the three road segments directly connected to \( M_k \). A sub-coordinator queue table can be viewed as a subset of the extended coordinator queue table except that the CAVs are in different order in the two tables. As an example, Table \( \text{II} \) (a snapshot of Fig. \( \text{I} \)) is the sub-coordinator queue table corresponding to \( M_1 \) (in this case, still based on the FIFO policy).

| \( S_k(t) \) | \( \text{THE SUB-COORDINATOR QUEUE TABLE} \) | \( S_1(t) \) |
|---|---|---|
| idx | info | curr. | ori. | 1st MP | 2nd MP | 3rd MP | \( t_p \) | \( t_m \) |
| 6 | \( x_6 \) | \( l_4 \) | \( l_1 \) | \( M_1, M \) | | | | |
| 7 | \( x_7 \) | \( l_4 \) | \( l_1 \) | \( M_1, M \) | \( M_2 \) | \( M_3 \) | 6 | |
| 8 | \( x_8 \) | \( l_4 \) | \( l_1 \) | \( M_1, M \) | \( M_2 \) | | | |
| 0 | \( x_0 \) | \( l_4 \) | \( l_1 \) | \( M_1, M \) | \( M_2, M \) | \( M_3, M \) | | |
| 1 | \( x_1 \) | \( l_4 \) | \( l_1 \) | \( M_1, M \) | \( M_2, M \) | \( M_3, M \) | | |
| 9 | \( x_9 \) | \( l_4 \) | \( l_1 \) | \( M_1, M \) | \( M_2, M \) | \( M_3, M \) | 8 | |

**Event-driven Update Process for** \( S_k(t) \): The sub-coordinator queue table \( S_k(t) \) is updated as follows after each event that has caused an update of the extended coordinator queue table \( S(t) \) as follows:

- For each CAV \( i \) in a sub-coordinator queue table \( S_k(t) \), update its information depending on the event observed:
  - \( i \) A new CAV \( j \) enters \( CZ_k \) (either from an entry point to the roundabout CZ or a MP passing event): Add a new row to \( S_k(t) \) and resequence the sub-coordinator queue table according to the sequencing policy used.
  - \( ii \) CAV \( j \) exits \( CZ_k \): Remove all the information of CAV \( j \) from \( S_k(t) \).
- Determine \( t_p \) and \( t_m \) in each sub-coordinator queue table with the same method as described in section \( \text{III-B} \).
- Update CAV \( j \)'s \( i_p \) and \( i_m \) in the extended coordinator queue table with the corresponding information in \( S_k(t) \), while \( M_k \) is the next MP of CAV \( j \).

Note that CAV \( j \) may appear in multiple sub-coordinator queue tables with different \( i_p \) and \( i_m \) values. However, only the one in \( S_k(t) \) where \( M_k \) is the next MP CAV \( j \) will pass is used to update the extended coordinator queue table \( S(t) \). The information of CAV \( j \) in other sub-coordinator queue tables is necessary for determining the safety constraints as CAV \( j \) may become CAV \( i_p \) or \( i_m \) of other CAVs.

**Resequeencing rule:** The sub-coordinator queue table allows resequencing when a CAV passes a MP. A resequencing rule generally designs and calculates a criterion for each CAV and sorts the CAVs in the queue table when a new event happens. For example, FIFO takes the arrival time in the CZ as the criterion while the Dynamic Resequeencing (DR) policy \([26]\) uses the overall objective value in \( \text{VI} \) as the criterion.

We propose here a straightforward yet effective (see Section \( \text{V} \)) resequencing rule for the roundabout as follows. Let \( \tilde{x}_k^i(t) = x_i(t) - d_k^i \) be the position of CAV \( i \) relative to \( M_k \), where \( d_k^i \) denotes the fixed distance from the entry point (origin) \( O \) to merging point \( M_k \) along the trajectory of CAV \( i \). Then, consider

\[
y_i(t) = -\tilde{x}_k^i(t) - \varphi v_i(t)
\]

(8)

This resequencing criterion reflects the distance between the CAV and the next MP. The CAV which has the smallest \( y_i(t) \) value is allocated first, thus referring to this as the Shortest Distance First (SDF) policy. Note that \( \varphi v_i(t) \) introduces a speed-dependent term corresponding to the speed-dependent safety constraints. This simple resequencing rule is tested in Section \( \text{V} \). Other resequencing policies can also be easily implemented with the help of the sub-coordinator queue tables.

IV. JOINT OPTIMAL CONTROL AND CONTROL BARRIER FUNCTION CONTROLLER (OCBF)

We now return to the solution of Problem 1, i.e., the minimization of \( (\text{VI}) \) subject to constraints \( (\text{I}), (\text{III}), (\text{IV}), (\text{V}) \), the initial condition \( x_i(t_0^i) = 0 \), and given \( t_1^1, v_1^0 \) and \( x_1(t_1^1) \). The problem formulation is complete since we have used the extended coordinator table to determine \( t_p \) and \( t_m \) (needed for the safety constraints) associated with the closest MP to CAV \( i \) given the sequence of CAVs in the system. After introducing the sub-coordinator queue tables, we also allow some resequencing for CAVs passing each MP and focus on the CZ associated with that MP. Thus, each such problem resembles the merging control problem in \( [27]\), which can be analytically solved. However, as pointed out in \( [27]\), when one or more constraints become active, this solution becomes computationally intensive. The problem here is exacerbated by the fact that the values of \( t_p \) and \( t_m \) change due to different events in the roundabout system. Therefore, to ensure that a solution can be obtained in real time while also guaranteeing that all safety constraints are always satisfied, we adopt the OCBF approach which is obtained as follows: (i) an optimal control solution is first obtained for the unconstrained roundabout problem (as reported in \( [27]\), such solutions are computationally efficient obtain, typically requiring \( \ll 1 \) sec using MATLAB). (ii) This solution is used as a reference control which is optimally tracked subject to a set of CBFs, one for each of the constraints \( (\text{III}), (\text{IV}), (\text{V}) \). Using the forward invariance property of CBFs, this ensures that these
constraints are always satisfied. This whole process is carried out in a decentralized way.

**Unconstrained optimal control solution:** With all constraints inactive (including at \( t_0^f \)), the solution of Problem \( 1 \) has the same form as the unconstrained optimal control solution in the merging problem \( 17 \) so that the optimal control, velocity and position trajectories of CAV \( i \) have the form:

\[
\begin{align*}
  u_i^*(t) &= a_i t + b_i \\
  v_i^*(t) &= \frac{1}{2} a_i t^2 + b_i t + c_i \\
  x_i^*(t) &= \frac{1}{6} a_i t^3 + \frac{1}{2} b_i t^2 + c_i t + d_i
\end{align*}
\]

(9)-(11)

where the parameters \( a_i, b_i, c_i, d_i \) and \( t_0^f \) are obtained by solving a set of nonlinear algebraic equations:

\[
\begin{align*}
  \frac{1}{2} a_i \cdot (t_0^f)^2 + b_i \cdot t_0^f + c_i &= v_0^f, \\
  \frac{1}{6} a_i \cdot (t_0^f)^3 + \frac{1}{2} b_i \cdot (t_0^f)^2 + c_i t_0^f + d_i &= x_0^f, \\
  a_i t_0^f + b_i &= 0, \\
  \beta + \frac{1}{2} a_i^2 \cdot (t_0^f)^2 + a_i b_i t_0^f + a_i c_i &= 0.
\end{align*}
\]

This set of equations only needs to be solved when CAV \( i \) enters the CZ and, as already mentioned, it can be solved very efficiently.

**Optimal tracking controller with CBFs:** Once we obtain the unconstrained optimal control solutions \( 9-11 \), we use a function \( h(u_i^*(t), x_i^*(t), z_i(t)) \) as a control reference \( u_{ref}(t) = h(u_i^*(t), x_i^*(t), z_i(t)) \), where \( x_i(t) \) provides feedback from the actual observed CAV trajectory. We then design a controller that minimizes \( \int_{t_0^f}^{t_1} \frac{1}{2}(u_i(t) - u_{ref}(t))dt \) subject to all constraints \( 3, 4, 5 \) and \( 6 \). This is accomplished as reviewed next (see also \( 14, 15 \)).

First, let \( z_i(t) \equiv (x_i(t), v_i(t)) \). Due to the vehicle dynamics \( 1 \), define \( f(x_i(t)) = [v_i(t), 0]^T \) and \( g(x_i(t)) = [0, 1]^T \). The constraints \( 3, 4, 5 \) and \( 6 \) are expressed in the form \( b_k(x_i(t)) \geq 0, k \in \{1, ..., b\} \) where \( b \) is the number of constraints. The CBF method maps the constraint \( b_k(x_i(t)) \geq 0 \) to a new constraint which directly involves the control \( u_i(t) \) and takes the form

\[
L_f b_k(x_i(t)) + L_g b_k(x_i(t)) u_i(t) + \gamma(b_k(x_i(t))) \geq 0,
\]

(13)

where \( L_f, L_g \) denote the Lie derivatives of \( b_k(x_i(t)) \) along \( f \) and \( g \) respectively, \( \gamma(\cdot) \) denotes a class of \( K \) functions \( 28 \). The forward invariance property of CBFs guarantees that a control input that keeps \( 13 \) satisfied will also keep \( b_k(x_i(t)) \geq 0 \). In other words, the constraints \( 3, 4, 5 \) and \( 6 \) are never violated.

To optimally track the reference speed trajectory, a CLF function \( V(x_i(t)) \) is used. Letting \( V(x_i(t)) = (v_i(t) - v_{ref}(t))^2 \), the CLF constraint takes the form

\[
L_f V(x_i(t)) + L_g V(x_i(t)) u_i(t) + \epsilon V(x_i(t)) \leq e_i(t),
\]

(14)

where \( \epsilon > 0 \) and \( e_i(t) \) is a relaxation variable which makes the constraint soft. Then, the OCBF controller optimally tracks the reference trajectory by solving the optimization problem:

\[
\min_{u_i(t), e_i(t)} \int_{t_0}^{t_1} \left( \beta e_i^2(t) + \frac{1}{2} (u_i(t) - u_{ref}(t))^2 \right) dt
\]

(15)

subject to the vehicle dynamics \( 1 \), the CBF constraints \( 13 \) and the CLF constraints \( 14 \). There are several possible choices for \( u_{ref}(t) \) and \( v_{ref}(t) \). In the sequel, we choose the simplest and most straightforward ones:

\[
\begin{align*}
  v_{ref}(t) &= v_i^*(t), \\
  u_{ref}(t) &= u_i^*(t)
\end{align*}
\]

(16)

where \( v_i^*(t) \) and \( u_i^*(t) \) are obtained from \( 10 \) and \( 9 \).

Considering all constraints in Problem \( 1 \), the rear-end safety constraint \( 3 \) and the vehicle limitations \( 5 \) are straightforward to transform into a CBF form. As an example, consider \( 3 \) by setting \( b_1(x_i(t)) = z_{i,im}(t) - \varphi v_i(t) - \delta \). After calculating the Lie derivatives, the CBF constraint \( 13 \) can be obtained. The safe merging constraint \( 4 \) differs from the rest in that it only applies to specific time instants \( t_{im}^k \). This poses a technical complication due to the fact that a CBF must always be in a continuously differential form. We can convert \( 4 \) to such a form using the technique in \( 23 \) to obtain

\[
z_{i,im}(t) - \Phi(x_i(t)) v_i(t) - \delta \geq 0, \quad t \in [t_{im}^k, t_{im}^{k+1}]
\]

(17)

where \( t_{im}^{k,a} \) denotes the time CAV \( i \) enters the road segment connected to \( M_k \) and \( \Phi(\cdot) \) is any strictly increasing function as long as it satisfies the boundary constraints \( z_{i,im}(t_{im}^k) - \phi v_i(t_{im}^k) - \delta \geq 0 \) and \( z_{i,im}(t_{im}^{k+1}) - \phi v_i(t_{im}^{k+1}) - \delta \geq 0 \) (which is precisely \( 4 \)). Note that we need to satisfy \( 17 \) when a CAV changes road segments in the roundabout and the value of \( t_{im} \) changes. Since \( z_{i,im}(t_{im}^k) \geq -L_{im} + L_i \), where \( L_i \) is the length of the road segment CAV \( i \) is in, to guarantee the feasibility of \( 17 \), we set \( \Phi(x_i(t_{im}^k)) v_i(t_{im}^k) + \delta = -L_{im} + L_i \). Then, from \( 4 \), we get \( \Phi(x_i(t_k)) = \varphi \). Simply choosing a linear \( \Phi(\cdot) \) as follows:

\[
\Phi(x_i(t)) = \left( \varphi + \frac{L_{im} - L_i + \delta}{v_i(t_{im}^k)} \right) \frac{x_i(t)}{L_i} - \frac{L_{im} - L_i + \delta}{v_i(t_{im}^k)}
\]

(18)

it is easy to check that it satisfies the boundary requirements. Note that when implementing the OCBF controller, \( x_i(t) \) needs to be transformed into a relative position \( \tilde{x}_i^k + L_i \), which reflects the distance between CAV \( i \) and the origin of the current road segment. Then, \( z_{i,ip} \) and \( z_{i,im} \) are calculated after this transformation, where \( z_{i,ip} = \tilde{x}_i^k - \tilde{x}_i^k \). With all constraints converted to CBF constraints in \( 15 \), we can solve this problem by discretizing \( [t_0^f, t_1^f] \) into intervals of equal length \( \Delta \) and solving \( 15 \) over each time interval. The decision variables \( u_i(t) \) and \( e_i(t) \) are assumed to be constant on each such time interval and can be easily obtained by solving a Quadratic Program (QP) problem since all CBF constraints are linear in the decision variables \( u_i(t) \)
and \( e_i(t) \). By repeating this process until CAV \( i \) exit the CZ, the solution to (13) is obtained.

Algorithm 1 summarizes the overall process for solving the roundabout problem: each CAV \( i \) first determines the conflict CAVs \( i_p \) and \( i_m \) and then derives the OCBF controller that guides it through its trajectory.

Algorithm 1: A MP-based Algorithm for Roundabout Problems

```plaintext
1 for every T seconds do
2   if a new event occurs then
3     if new CAV entering event then
4       Determine the passing order for all CAVs
5       Plan an unconstrained optimal control
6       trajectory for the new CAV
7     end
8     Update the extended coordinator queue table
9       S(t)
10    for each CAV in S(t) do
11       Determine the safety contraints it needs to meet
12       Use the joint optimal control and barrier
13       function controller to obtain control for it
14     end
15 end
```

V. SIMULATION RESULTS

In this section, we use Vissim, a multi-model traffic flow simulation platform, as a baseline to compare a roundabout performance with huma-driven vehicles to our OCBF controller. We build the scenario shown in Fig. 1 in Vissim and use the same vehicle arrival patterns in the OCBF controller for consistent comparison purposes.

Simulation 1: The first simulation focuses on the performance of the OCBF controller. The basic parameter settings are as follows: \( L_o = 60m, L = 60m, \delta = 10m, \varphi = 1.88, \) \( v_{\text{max}} = 17m/s, v_{\text{min}} = 0, u_{\text{max}} = 5m/s^2, u_{\text{min}} = -5m/s^2 \). This scenario considers a symmetric configuration in the sense that \( L_o = L \). The traffic in the three incoming roads is generated through Poisson processes with all rates set to 360 CAV/s/h. Under these traffic rates, vehicles will sometimes line up waiting for other vehicles in the roundabout to pass. A total number of 200 CAVs are simulated. The simulation results of the performance of OCBF compared to that in Vissim are listed in Table III.

In this simulation, FIFO is chosen as the sequencing policy in the OCBF method. As seen in Table III, the travel time of CAVs in the roundabout improves about 34% using the OCBF method compared with that of Vissim when \( \alpha = 0.1 \) (with some additional improvement when \( \alpha = 0.2 \)). The CAVs using the OCBF method consume 52% and 26% less energy than that in Vissim with \( \alpha \) set to 0.1 and 0.2 respectively. A larger \( \alpha \) means more emphasis on the travel time than the energy consumption, which explains the shorter travel time and the larger energy consumption. When it comes to the total objective, the OCBF controller shows 44% and 32% improvement over the human-driven performance in Vissim when \( \alpha \) equals to 0.1 and 0.2 respectively. This improvement in both the travel time and the energy consumption is to be expected as the CAVs using the OCBF method never stop and wait for CAVs in another road segment to pass as Vissim do.

Simulation 2: The second simulation compares the performance of OCBF under different sequencing rules in an asymmetric configuration. The parameter settings are the same as the first case except that \( L = 100m \). The weight is set to \( \alpha = 0.2 \). The simulation results of the performance of OCBF with FIFO and OCBF with the SDF sequencing policy, as well that in Vissim, are shown in Table IV.

Table IV shows that a CAV using OCBF with FIFO spends around 33% less travel time but 84% more energy than that in Vissim. The average objective values of the two cases are almost the same, indicating that OCBF with FIFO works poorly in an asymmetric roundabout. For example, when a CAV enters segment \( l_4 \), it has to wait for another CAV that has entered \( l_2 \) just before it to run 40 more meters for safe merging. This is unreasonable and may also result in some extreme cases when the OCBF problem becomes infeasible. This problem can be resolved by choosing a better sequencing policy such as SDF. As shown in Table IV, OCBF+SDF outperforms OCBF+FIFO, achieving an improvement of 40% in travel time, 26% in energy consumption and 36% in the objective value compared to that in Vissim.

Simulation 3: The purpose of this experiment is to study the effect of traffic volume. The roundabout is set to be asymmetric with the same parameter settings as in Simulation 2. A total number of approximately 500 CAVs are simulated under two sets of incoming traffic rates: balanced incoming traffic (360 CAVs/h for each origin) and imbalanced incoming traffic (540 CAVs/h from \( O_1 \), 270 CAVs/h from...
CBFs, is designed and implemented to track the desired (unconstrained) trajectory while guaranteeing that all safety constraints and vehicle limitations are satisfied. Significant improvements are shown in the simulation experiments which compare the performance of the OCBF controller to a baseline of human-driven vehicles. Future research is directed at studying different sequencing policies (which our results show is important in asymmetric roundabout configurations), as well as considering the centrifugal discomfort caused when road segments are curved and extending the model to more complex roundabouts as well as to a multi-lane version which allows lane changing and overtaking.

VI. CONCLUSION

We have presented a decentralized optimal control framework for controlling CAVs traveling through a roundabout to jointly minimize both the travel time and the energy consumption while satisfying speed-dependent safety constraints, as well as velocity and acceleration constraints. A systematic method is designed to dynamically determine the safety constraints a CAV needs to meet. An OCBF controller, combining an unconstrained optimal control solution with O\textsubscript{2} and O\textsubscript{3}. The simulation results of the performance of OCBF+SDF compared to that in Vissim under both balanced and imbalanced incoming traffic are shown in Tables \textbf{V} and \textbf{VI}.

From Table \textbf{V} it is seen that the imbalanced traffic causes longer travel times (~2s) and more energy consumption (~13%) although the total traffic rates are the same. The imbalanced traffic results in an imbalanced performance of CAVs from different origins. The CAVs originated from \textit{O\textsubscript{1}} with heavy traffic perform worse than those from \textit{O\textsubscript{2}} and \textit{O\textsubscript{3}} with light traffic. However, when OCBF+SDF is applied to the system, the imbalanced traffic brings almost no performance loss and becomes somewhat more balanced after passing the roundabout. This result is interesting because the fact that traffic is imbalanced was not taken into consideration in our method. An explanation of this phenomenon is that SDF gives the CAVs from \textit{O\textsubscript{1}} a higher priority as they are more likely to be the closest to the MP while OCBF allows the CAVs to pass the roundabout quickly without stop; therefore, the CAVs from a heavy traffic flow are less likely to get congested.

### Table V

| Traffic Type | CAV Origin | Time  | Energy | Ave. Obj. |
|-------------|------------|-------|--------|-----------|
| Balanced    | All        | 24.7615 | 32.1295 | 109.5092  |
|             | From \textit{O\textsubscript{1}} | 24.7631 | 33.7724 | 111.1570  |
|             | From \textit{O\textsubscript{2}} | 23.6934 | 31.4432 | 105.4851  |
|             | From \textit{O\textsubscript{3}} | 25.8458 | 31.0375 | 111.8057  |
| Imbalanced  | Total      | 26.5170 | 36.0547 | 118.9203  |
|             | From \textit{O\textsubscript{1}} | 28.1765 | 39.5067 | 127.5582  |
|             | From \textit{O\textsubscript{2}} | 25.6977 | 35.5803 | 115.8857  |
|             | From \textit{O\textsubscript{3}} | 23.8207 | 29.2346 | 103.6744  |

### Table VI

| Traffic Type | CAV Origin | Time  | Energy | Ave. Obj. |
|-------------|------------|-------|--------|-----------|
| Balanced    | All        | 15.1650 | 24.8601 | 72.2507   |
|             | From \textit{O\textsubscript{1}} | 15.1613 | 24.8453 | 72.2245   |
|             | From \textit{O\textsubscript{2}} | 14.8221 | 25.3610 | 71.6801   |
|             | From \textit{O\textsubscript{3}} | 15.5156 | 24.3696 | 72.8560   |
| Imbalanced  | Total      | 15.2437 | 24.1693 | 71.8059   |
|             | From \textit{O\textsubscript{1}} | 15.4035 | 24.9519 | 73.0880   |
|             | From \textit{O\textsubscript{2}} | 14.6235 | 19.8901 | 65.5885   |
|             | From \textit{O\textsubscript{3}} | 15.5163 | 26.7188 | 75.2072   |

REFERENCES

[1] J. Ríos-Torres and A. A. Malikopoulos, “A survey on the coordination of connected and automated vehicles at intersections and merging at highway on-ramps,” IEEE Transactions on Intelligent Transportation Systems, vol. 18, no. 5, pp. 1066–1077, 2016.

[2] L. Chen and C. Englund, “Cooperative intersection management: A survey,” IEEE Transactions on Intelligent Transportation Systems, vol. 17, no. 2, pp. 570–586, 2015.

[3] M. Tideman, M. C. van der Voort, B. van Arem, and F. Tillema, “A review of lateral driver support systems,” in 2007 IEEE Intelligent Transportation Systems Conference, pp. 992–999, IEEE, 2007.

[4] L. Li, D. Wen, and D. Yao, “A survey of traffic control with vehicular communications,” IEEE Transactions on Intelligent Transportation Systems, vol. 15, no. 1, pp. 425–432, 2013.

[5] H. Xu, S. Feng, Y. Zhang, and L. Li, “A grouping-based cooperative driving strategy for cavs merging problems,” IEEE Transactions on Vehicular Technology, vol. 68, no. 6, pp. 6125–6136, 2019.

[6] H. Xu, Y. Zhang, C. G. Cassandras, L. Li, and S. Feng, “A bi-level cooperative driving strategy allowing lane changes,” Transportation research part C: emerging technologies, vol. 120, p. 102773, 2020.

[7] V. Milanes, J. Godoy, J. Villagrá, and J. Pérez, “Automated on-ramp merging system for congested traffic situations,” IEEE Transactions on Intelligent Transportation Systems, vol. 12, no. 2, pp. 500–508, 2010.

[8] J. Rios-Torres, A. Malikopoulos, and P. Psu, “Online optimal control of connected vehicles for efficient traffic flow at merging roads,” in 2015 IEEE 18th international conference on intelligent transportation systems, pp. 2432–2437, IEEE, 2015.

[9] Y. Bichiou and H. A. Rakha, “Developing an optimal intersection control system for automated connected vehicles,” IEEE Transactions on Intelligent Transportation Systems, vol. 20, no. 5, pp. 1908–1916, 2018.

[10] R. Hult, G. R. Campos, E. Steinmetz, L. Hammarstrand, P. Falcone, and H. Wyneers, “Coordination of cooperative autonomous vehicles: Toward safer and more efficient road transportation,” IEEE Signal Processing Magazine, vol. 33, no. 6, pp. 74–84, 2016.

[11] W. Cao, M. Mukai, T. Kawabe, H. Nishira, and N. Fujiki, “Cooperative vehicle path generation during merging using model predictive control with real-time optimization,” Control Engineering Practice, vol. 34, pp. 98–105, 2015.

[12] M. Mukai, H. Natori, and M. Fujita, “Model predictive control with a mixed integer programming for merging path generation on motor way,” in 2017 IEEE Conference on Control Technology and Applications (CCTA), pp. 2214–2219, IEEE, 2017.

[13] M. H. B. M. Nor and T. Namerikawa, “Merging of connected and automated vehicles at intersections and merging at highway on-ramps,” in 2018 57th Annual Conference of the Society of Instrument and Control Engineers of Japan (SICE), pp. 272–277, IEEE, 2018.

[14] W. Xiao, C. G. Cassandras, and C. Bela, “Bridging the gap between optimal trajectory planning and safety-critical control with applications to autonomous vehicles,” Automatica, 2021 (in print).

[15] W. Xiao, C. G. Cassandras, and C. Bela, “Decentralized merging control in traffic networks with noisy vehicle dynamics: A joint optimal control and barrier function approach,” in 2019 IEEE Intelligent Transportation Systems Conference (ITSC), pp. 3162–3167, IEEE, 2019.
[16] Y. Zhang and C. G. Cassandras, “Decentralized optimal control of connected automated vehicles at signal-free intersections including comfort-constrained turns and safety guarantees,” *Automatica*, vol. 109, p. 108563, 2019.

[17] W. Xiao and C. G. Cassandras, “Decentralized optimal merging control for connected and automated vehicles with safety constraint guarantees,” *Automatica*, vol. 123, p. 109333, 2021.

[18] A. Flannery and T. Datta, “Operational performance measures of american roundabouts,” *Transportation research record*, vol. 1572, no. 1, pp. 68–75, 1997.

[19] M. Martin-Gasulla, A. García, and A. T. Moreno, “Benefits of metering signals at roundabouts with unbalanced flow; Patterns in spain,” *Transportation Research Record*, vol. 2585, no. 1, pp. 20–28, 2016.

[20] X. Yang, X. Li, and K. Xue, “A new traffic-signal control for modern roundabouts: method and application,” *IEEE Transactions on Intelligent Transportation Systems*, vol. 5, no. 4, pp. 282–287, 2004.

[21] H. Xu, K. Zhang, and D. Zhang, “Multi-level traffic control at large four-leg roundabouts,” *Journal of Advanced Transportation*, vol. 50, no. 6, pp. 988–1007, 2016.

[22] L. Zhao, A. Malikopoulos, and J. Rios-Torres, “Optimal control of connected and automated vehicles at roundabouts: An investigation in a mixed-traffic environment,” *IFAC-PapersOnLine*, vol. 51, no. 9, pp. 73–78, 2018.

[23] B. Chalaki, L. E. Beaver, and A. A. Malikopoulos, “Experimental validation of a real-time optimal controller for coordination of cavs in a multi-lane roundabout,” in *2020 IEEE Intelligent Vehicles Symposium (IV)*, pp. 775–780, IEEE, 2020.

[24] W. Xiao, C. G. Cassandras, and C. Belta, “Decentralized optimal control in multi-lane merging for connected and automated vehicles,” in *2020 IEEE 23rd International Conference on Intelligent Transportation Systems (ITSC)*, pp. 1–6, 2020.

[25] H. Xu, C. G. Cassandras, L. Li, and Y. Zhang, “Comparison of cooperative driving strategies for cavs at signal-free intersections,” submitted to *IEEE Transactions on Intelligent Transportation Systems*, 2020.

[26] Y. Zhang and C. G. Cassandras, “A decentralized optimal control framework for connected automated vehicles at urban intersections with dynamic resequencing,” in *2018 IEEE Conference on Decision and Control (CDC)*, pp. 217–222, IEEE, 2018.

[27] K. Vogel, “A comparison of headway and time to collision as safety indicators,” *Accident analysis & prevention*, vol. 35, no. 3, pp. 427–433, 2003.

[28] W. Xiao, C. Belta, and C. G. Cassandras, “Decentralized merging control in traffic networks: A control barrier function approach,” in *Proceedings of the 10th ACM/IEEE International Conference on Cyber-Physical Systems*, pp. 270–279, 2019.