INCLUSION IDEALS ASSOCIATED TO UNIFORMLY INCREASING HYPERGRAPHS

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Abstract. In this paper, we introduce inclusion ideals $I(H)$ associated to a special class of non uniform hypergraphs $\mathcal{H}(X,E,d)$, namely, the uniformly increasing hypergraphs. We discuss some algebraic properties of the inclusion ideals. In particular, we give an upper bound of the Castelnuovo-Mumford regularity of the special dual ideal $I[^*(H)]$.

Key words : Hypergraph, Stable ideals, Castelnuovo-Mumford regularity, Primary decomposition, Alexander duality.

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1. INTRODUCTION

Let $E = \{E_1, ..., E_s\}$ be the collection of distinct subsets of the finite set $X = \{x_1, ..., x_n\}$. The pair $\mathcal{H}(X,E)$ is said to be a Hypergraph, if $E_i \neq \emptyset$ for each $i$, where $X$ and $E$ are the set of vertices and edges of $\mathcal{H}$, respectively. A hypergraph is said to be $d$-uniform hypergraph, if $|E_i| = d$ for each $i$. For example, all simple graphs are 2-uniform hypergraphs. It is worth mentioning here that hypergraphs are the generalized form of the simple graphs and the simplicial complexes.

Let $S = K[x_1, ..., x_n]$ be the polynomial ring over an infinite field $K$. If we associate each vertex $x_i$ to each variable $x_i$, then the edge ideal $I(\mathcal{H})$ associated to simple hypergraph (see [7]) $\mathcal{H}$ is:

$$I(\mathcal{H}) = (\{x^E = \prod_{x \in E} x \mid E \in E\}) \subseteq S$$

This edge ideal was introduced by Villarreal for a special case of hypergraphs, namely, the simple graphs. The edge ideal $I(\mathcal{H})$ of a hypergraph was firstly discussed by Faridi in [5] and [6], but in different context. The study of algebraic objects via combinatorial correspondence has fascinated many people, who have been working to build a dictionary between the algebraic properties of edge ideal and the combinatorial structure associated to it. For instance, the description about the edge ideals and the ideals associated to simplicial complexes can be found in [5], [6], [10], [11], [12] and [13].

In this paper, we introduce the monomial ideal $I(\mathcal{H})$ associated a special kind of
non uniform Hypergraphs namely \textit{uniformly increasing hypergraphs} $\mathcal{H}(\mathcal{X}, \mathcal{E}, d)$, see 2.1. We call $\mathcal{I}(\mathcal{H})$ as the \textit{inclusion ideal}, which is not a square-free ideal in contrast to the edge ideal, see 2.5. In 2.12, we prove that the \textit{special dual} of inclusion ideal $\mathcal{I}^*[\mathcal{H}]$ is the monomial ideal whose associated prime ideals are totally ordered by inclusion. As can be easily seen, by an appropriate change of variables, such ideals are the monomial ideals of Borel type introduced by Herzog, Popescu and Vladoiu in [9]. Moreover, Herzog and Popescu in [8] proved that such ideals are \textit{pretty clean}. The upper bound for the Castlenouvo-Mumford regularity of such ideals are discussed in [1] and [3]. The Castlenouvo-Mumford regularity of $I$ is given by
\[ \text{reg}(I) = \max_{\beta_{ij}(I) \neq 0} \{ j - i \} \]
where $\beta_{ij}(I)$ are the graded Betti numbers of $I$. In 3.6 we give the upper bound for the regularity of $I^*[\mathcal{H}]$, which is more finer than the bound already found by Ahmad, Anwar in [1] and Cimpoeas in [3].

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2. Inclusion Ideal and its special dual

This section is devoted for the introduction of \textit{uniformly increasing hypergraphs} $\mathcal{H}(\mathcal{X}, \mathcal{E}, d)$ and its inclusion ideal we associate to it.

\textbf{Definition 2.1.} Let $\mathcal{X} = \{x_1, ..., x_n\}$ be a finite set and $\mathcal{E} = \{E_1, ..., E_s\}$ be a collection of subsets of $\mathcal{X}$ such that $E_i \subseteq E_j$ for $1 \leq i < j \leq s$, where $E_i, E_j \in \mathcal{E}$, then the triplet $\mathcal{H}(\mathcal{X}, \mathcal{E}, d)$ is called \textit{uniformly increasing hypergraph}, if

(1) $|E_i| \geq 2$ and
(2) $|E_{i+1}| = |E_i| + d$, where $d \in \mathbb{Z}^+$ is the increment.

\textbf{Example 2.2.} Let $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$ be the set of vertices and $\mathcal{E} = \{E_1, E_2, E_3\}$ be the set of edges, where $E_1 = \{x_1, x_2\}$, $E_2 = \{x_1, x_2, x_3\}$, $E_3 = \{x_1, x_2, x_3, x_4\}$, where $|E_{i+1}| - |E_i| = 1$ for $1 \leq i \leq 3$, then the hypergraph $\mathcal{H}(\mathcal{X}, \mathcal{E}, 1)$ is:

\begin{center}
\includegraphics[width=0.5\textwidth]{hypergraph.png}
\end{center}

\textbf{Definition 2.3.} For a \textit{uniformly increasing hypergraph} $\mathcal{H}(\mathcal{X}, \mathcal{E}, d)$ over $n$-vertices, we define the \textit{containment vector} ‘a’ as $a = (a_1, a_2, ..., a_n)$ where each $a_i$ is the \textit{degree} of vertex $x_i$; that is

$$ a_i = | \{ E_j \mid x_i \in E_j \text{ with } j \in \{1, 2, ..., s\} \} | $$


where $| . |$ denotes the cardinality of the set.

\textbf{Example 2.4.} For the hypergraph in example 2.3, the \textit{containment vector} is $a = (3, 3, 2, 1)$. As $x_1$ belongs to $\{E_1, E_2, E_3\}$, so $a_1 = 3$. Similarly, we have $a_2 = 3$, $a_3 = 2$ and $a_4 = 1$.

\textbf{Definition 2.5.} Let $S = k[x_1, ..., x_n]$ be a polynomial ring over an infinite field $k$ and let $\mathcal{H}(\mathcal{X}, \mathcal{E}, d)$ be a \textit{uniformly increasing hypergraph} over $n$ vertices. Considering the
correspondence of each vertex \( x_i \) of \( \mathcal{H}(\mathcal{X}, \mathcal{E}, d) \) to each variable \( x_i \) in \( S \), the inclusion ideal \( \mathcal{I}(\mathcal{H}) \) of the uniformly increasing hypergraph is defined as,

\[
\mathcal{I}(\mathcal{H}) = (\{x^{E_i} = \prod_{x \in E_i} x^{s-i+1} | E_i \in \mathcal{E}, \ \forall, \ 1 \leq i \leq |\mathcal{E}|\}).
\]

It should be noted that \( \mathcal{I}(\mathcal{H}) \) is minimally generated by the above mentioned monomial generators and we will denote it with \( G(\mathcal{I}(\mathcal{H})) \).

**Example 2.6.** Let \( \mathcal{H}(\mathcal{X}, \mathcal{E}, d) \) be the hypergraph in above example\(^{2.2}\) Then by definition its inclusion ideal \( \mathcal{I}(\mathcal{H}) \) will be:

\[
\mathcal{I}(\mathcal{H}) = (x_1^3x_2^3, x_1^2x_2^2, x_1x_2x_3x_4).
\]

**Remark 2.7.** Note that, the inclusion ideal \( \mathcal{I}(\mathcal{H}) \) of a uniformly increasing hypergraph \( \mathcal{H}(\mathcal{X}, \mathcal{E}, d) \) on \( n \) vertices is a non square-free monomial ideal in \( S \) unlike the edge ideals associated to the uniform hypergraphs. That is why, we call the inclusion ideal instead of edge ideal.

Now we give the definition of the Alexander dual of an arbitrary monomial ideal \( I \) with respect to \( a \) given in (\([10]\), section 5.2).

**Definition 2.8.** Given two vectors \( c, d \in \mathbb{N}^n \) with \( d \preceq c \) (that is \( d_i \leq c_i \) for \( i = 1, \ldots, n \)), let \( c \setminus d \) denote the vector in \( \mathbb{N}^n \) whose \( i^{th} \) coordinate is

\[
c_i \setminus d_i = \begin{cases} c_i + 1 - d_i, & \text{if } d_i \geq 1; \\ 0, & \text{if } d_i = 0. \end{cases}
\]

If \( I \subset S \) is a monomial ideal whose minimal generators all divide \( x^c \), then the Alexander dual of \( I \) with respect to \( c \) is

\[
I^c = \cap (m_{c|b}^{c|b} \mid x^b \text{ is the minimal generator of } I),
\]

where each \( m_{c|b}^{c|b} \) is the irreducible ideal generated by powers of variables. That is for a vector \( c \setminus b \) in \( \mathbb{N}^n \), we have

\[
m_{c|b}^{c|b} = < x_{i|b_{i|b}}^c \mid c_i \setminus b_i \geq 1 >
\]

**Lemma 2.9.** If \( \mathcal{I}(\mathcal{H}) = (x^{b_1}, x^{b_2}, \ldots, x^{b_n}) \) with \( b_i \in \mathbb{N}^n \) is the inclusion ideal of a uniformly increasing hypergraph \( \mathcal{H}(\mathcal{X}, \mathcal{E}, d) \) on \( n \) vertices , then \( b_i \preceq a \) for all \( i \in \{1, \ldots, n\} \), where ‘a’ is the containment vector associated to \( \mathcal{H}(\mathcal{X}, \mathcal{E}, d) \).

**Proof.** Let \( a = (a_1, \ldots, a_n) \) be the containment vector associated to \( \mathcal{H}(\mathcal{X}, \mathcal{E}, d) \) such that each \( a_i = \text{deg}(x_i) \) follows from \(\([2.3]\)\). If \( x_k \in E_t \) such that \( x_k \not\in E_{t-1} \) for some \( 1 \leq t \leq s \), then \( x_k \in E_{t+1}, x_k \in E_{t+2}, \ldots, x_k \in E_s \), which implies \( a_k = \text{deg}(x_k) = s - t + 1 \). Let us take an arbitrary vector \( b_i = (b_{i1}, b_{i2}, \ldots, b_{in}) \in \mathbb{N}^n \) such that \( x^{b_i} \in G(\mathcal{I}(\mathcal{H})) \). Then from \(\([2.5]\)\) we have \( b_{ij} = 0 \) if \( x_j \not\in E_i \) or \( b_{ij} = s - i + 1 \) if \( x_j \in E_i \) for all \( j \in \{1, \ldots, n\} \). If \( b_{ij} = 0 \), then \( b_{ij} < a_j \) because \( a_j > 0 \). If \( b_{ij} = s - i + 1 \), then \( s - i + 1 \leq s - t + 1 = a_j \), for \( t \leq i \) such that \( x_j \in E_t \) with \( x_j \not\in E_{t-1} \).

Next we give the definition of the special dual of the inclusion ideal of a uniformly increasing hypergraph.
Definition 2.10. If $\mathcal{I}(\mathcal{H}) = (x^{b_1}, x^{b_2}, \ldots, x^{b_s})$ is the inclusion ideal of a uniformly increasing hypergraph $\mathcal{H}(X, E, d)$ on $n$ vertices, then the Alexander dual of $\mathcal{I}(\mathcal{H})$ with respect to the containment vector ‘$a$’ associated to the hypergraph $\mathcal{H}(X, E, d)$ is called the special dual of the inclusion ideal, and is denoted by $I^{[a]}(\mathcal{H})$. That is,

$$I^{[a]}(\mathcal{H}) = \bigcap_{i=1}^{s} (m^{a \setminus b_i} | x^{b_i} \text{ is the minimal generator of } I_i).$$

It should be noted that here $x^{b_i}$ is actually $x^{E_i}$.

Example 2.11. Let $\mathcal{I}(\mathcal{H})$ be the monomial ideal in above example 2.6, its special dual will be:

$$I^{[a]}(\mathcal{H}) = (x_1, x_2) \cap (x_1^2, x_2, x_3) \cap (x_1^3, x_2^2, x_3, x_4)$$

$$= (x_1^3, x_2^3, x_1x_2^2, x_2x_3^2, x_1^2x_4, x_2^2x_4, x_1x_3x_4, x_2x_3x_4)$$

here containment vector is $a = (3, 3, 2, 1)$.

For a monomial $m \in S = k[x_1, x_2, \ldots, x_n]$, we define $\text{supp}(m)$ as $\text{supp}(m) = \{x_i \mid x_i \mid m \text{ for } i \in \{1, 2, \ldots, n\}\}$. Also for a monomial ideal $I = (g_1, \ldots, g_m) \subset S$ with $\{g_1, \ldots, g_m\}$ is the minimal set of generators of $I$, we define $\text{supp}(I) = \cup_{i=1}^{m} \{\text{supp}(g_i)\}$.

Proposition 2.12. Let $\mathcal{I}(\mathcal{H})$ be an inclusion ideal in $S = k[x_1, \ldots, x_n]$ associated to a uniformly increasing hypergraph $\mathcal{H}(X, E, d)$ on $n$ vertices, then $\text{Ass}(S/I^{[a]}(\mathcal{H}))$ is totally ordered by inclusion.

Proof. Let $\mathcal{I}(\mathcal{H}) = (x^{b_1}, x^{b_2}, \ldots, x^{b_s}) \in S$ be the inclusion ideal associated to a uniformly increasing hypergraph $\mathcal{H}(X, E, d)$ on $n$ vertices. So, the irredundant irreducible primary decomposition of special dual $I^{[a]}(\mathcal{H})$ (from the definition 2.10) is as follows;

$$I^{[a]}(\mathcal{H}) = \bigcap_{i=1}^{s} (m^{a \setminus b_i} | x^{b_i} \text{ be the minimal generators of } \mathcal{I}).$$

For each $x^{b_i} \in G(\mathcal{I}(\mathcal{H}))$, we have an irreducible primary ideal $m^{a \setminus b_i}$ appearing in the above primary decomposition of $I^{[a]}(\mathcal{H})$ with $\text{supp}(x^{b_i}) = \text{supp}(m^{a \setminus b_i})$. As each generator $x^{b_i}$ of the inclusion ideal $\mathcal{I}(\mathcal{H})$ is associated to each edge $E_i$ of the increasing hypergraph. So, we have $\text{supp}(x^{b_1}) \subset \text{supp}(x^{b_2}) \subset \ldots \subset \text{supp}(x^{b_s})$ i.e, $\sqrt{m^{a \setminus b_i}} = \text{supp}(m^{a \setminus b_i}) = \text{supp}(x^{b_i})$. Hence the associated primes of the above primary decomposition of $I^{[a]}(\mathcal{H})$ are totally ordered under inclusion.

3. Regularity of the special dual $I^{[a]}(\mathcal{H})$

Let $k$ be an infinite field, $S = k[x_1, \ldots, x_n], n \geq 2$ the polynomial ring over $k$ and $I \subset S$ a monomial ideal. Let $G(I)$ be the minimal set of monomial generators of $I$ and $\text{deg}(I)$ the highest degree of a monomial of $G(I)$. Given a monomial $u \in S$ set $m(u) = \max\{i \mid x_i^{|u|}\}$ and $m(I) = \max_{u \in G(I)} m(u)$. Also, $I_{\geq t}$ be the ideal generated by the monomials of $I$ of degree $\geq t$. A monomial ideal $I$ is stable if for each monomial $u \in I$ and $1 \leq j < m(u)$ it follows $\frac{x_i^{|u|}}{x_i^{m(u)}} \in I$. 


**Proposition 3.1.** A monomial ideal \( I = (x_1^{a_1}, \ldots, x_n^{a_n}) \subset S \), with \( n - 1 \geq a_1 \geq a_2 \geq \ldots a_n \geq 1 \) has \( I_{\geq t(I)} \) stable, where \( t(I) = \sum_{i=2}^{n} a_i \).

**Proof.** If \( u \in I_{\geq t(I)} \), from above we get \( u = v \cdot x_j^{a_j} \) for some \( 1 \leq j \leq n \) and \( v \in (x_1, \ldots, x_n)^{t(I)-a_j} \), then \( u \) belongs to the stable ideal \( (x_1, \ldots, x_n)^{t(I)} \) and it is enough to show that \( (x_1, \ldots, x_n)^{t(I)} \subseteq I_{\geq t(I)} \). If \( w \in (x_1, \ldots, x_n)^{t(I)} \), then \( w = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n} \) with all \( a_i \geq 0 \) and \( \sum_{i=1}^{n} a_i \geq t(I) \). Now we will prove that there exists some \( k \) such that \( 1 \leq k \leq n \) with \( \alpha_k \geq a_k \). Suppose contrary that there does not exist such \( k \), that is \( a_i < a_i \) for all \( i \in \{1, \ldots, n\} \). Because \( \sum_{i=1}^{n} a_i \geq t(I) = \sum_{i=2}^{n} a_i \), therefore

\[
\alpha_1 \geq (a_2 - \alpha_2) + (a_3 - \alpha_3) + \ldots + (a_n - \alpha_n)
\]

So we have \( \alpha_1 \geq n - 1 \), hence \( a_1 > n - 1 \). which is a contradiction. So we can take above \( v = x_1^{a_1} \ldots x_k^{\alpha_k-a_k} \ldots x_n^{a_n} \) which shows that \( w \in I_{\geq t(I)} \).

**Remark 3.2.** In general one cannot get \( I_{\geq t(I)-1} \) stable, when \( I = (x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n}) \) with \( n - 1 \geq a_1 \geq a_2 \geq \ldots a_n \geq 1 \). For example, if \( n = 3 \) and \( I = (x_1^2, x_2^2, x_3) \) then \( t(I) = 3 \) and clearly \( I_{\geq 2} \) is not stable.

Now we recall the following result from [1]:

**Proposition 3.3.** [1] If \( I, J \) are monomial ideals such that \( I_{\geq q(I)} \) and \( J_{\geq q(J)} \) are stable ideals, then \( (I \cap J)_{\geq \max(q(I),q(J))} \) is stable.

**Lemma 3.4.** If \( \mathcal{I}(\mathcal{H}) \in S \) be the inclusion ideal associated to a uniformly increasing hypergraph \( \mathcal{H}(\mathcal{X}, \mathcal{E}, d) \) on \( n \) vertices, then \( \mathcal{I}_{\geq t(I)}^{[s]}(\mathcal{H}) \) is stable, where \( t(I) = \sum_{i=2}^{n} a_i \) with \( a = (a_1, a_2, \ldots, a_n) \) be the containment vector associated to \( \mathcal{H}(\mathcal{X}, \mathcal{E}, d) \).

**Proof.** Let \( \mathcal{I}(\mathcal{H}) = (x_1^{b_1}, x_2^{b_2}, \ldots, x_n^{b_n}) \subset S \) with \( b_i \in \mathbb{N}^n \) be the inclusion ideal associated to a uniformly increasing hypergraph \( \mathcal{H}(\mathcal{X}, \mathcal{E}, d) \) on \( n \) vertices. Then its special dual \( \mathcal{I}^{[s]}(\mathcal{H}) \) will be:

\[
\mathcal{I}^{[s]}(\mathcal{H}) = \bigcap_{i=1}^{s} (m^{a_i} x_i^{b_i} \text{ be the minimal generators of } \mathcal{I}(\mathcal{H}))
\]

or

\[
\mathcal{I}^{[s]}(\mathcal{H}) = \bigcap_{i=1}^{s} Q_i \text{ where } Q_i = (m^{a_i} x_i^{b_i}) \text{ for all } i \in \{1, \ldots, s\}
\]

Because \( \text{Ass}(S/\mathcal{I}^{[s]}(\mathcal{H})) \) totally ordered from [2,12] so we have

\[
\sqrt{Q_1} \subset \sqrt{Q_2} \subset \ldots \subset \sqrt{Q_s}.
\]

Also \( m^{a_i} x_i^{b_i} \) is an irreducible ideal with \( \text{supp}(m^{a_i} x_i^{b_i}) = \text{supp}(x_i^{b_i}) \) for each \( i \in \{1, \ldots, s\} \). So, we have \( Q_i = (x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n}) \) with \( a = (\alpha_1, \alpha_2, \ldots, \alpha_n) \) where \( \alpha_k = 0 \) for \( r_i + 1 \leq k \leq n \) and \( r_i - 1 \geq \alpha_1 = i \geq \ldots \geq \alpha_n = 1 \) follows from the definitions [2,1] and [2,10]. Therefore, \( Q_s = (x_1^{a_1}, x_2^{a_2}, \ldots, x_n^{a_n}) \) with \( n - 1 \geq \alpha_1 = s \geq \alpha_2 \geq \ldots \geq \alpha_n = 1 \). Hence from the above Propositions 3.1 and 3.3 it immediately follows that \( \mathcal{I}^{[s]}_{\geq t(I)}(\mathcal{H}) \) is stable.

Next we recall a Proposition from [4].
**Proposition 3.5.** Let $I$ be a monomial ideal and $e \geq \deg(I)$ an integer such that $I_{\geq e}$ is stable. Then $\text{reg}(I) \leq e$.

**Corollary 3.6.** Let $\mathcal{I}(\mathcal{H}) \in S$ be the inclusion ideal associated to a uniformly increasing hypergraph $\mathcal{H}(\mathcal{X}, \mathcal{E}, d)$ on $n$ vertices. Then for its special dual ideal $\text{reg}(\mathcal{I}^\ast(\mathcal{H})) \leq t(\mathcal{I}^\ast(\mathcal{H})) = \sum_{i=2}^{s} a_i$.

**Proof.** By the previous lemma 3.4 we have $\mathcal{I}^\ast(\mathcal{H})$ stable, as $\deg(\mathcal{I}^\ast(\mathcal{H})) = \max\{a_1, a_2, \ldots, a_n\}$ and clearly $t(\mathcal{I}^\ast(\mathcal{H})) \geq \deg(\mathcal{I}^\ast(\mathcal{H}))$. Hence we get $\text{reg}(\mathcal{I}^\ast(\mathcal{H})) \leq t(\mathcal{I}^\ast(\mathcal{H})) = \sum_{i=2}^{s} a_i$ by proposition 3.5. □

**Remark 3.7.** As $\mathcal{I}^\ast(\mathcal{H})$ is a monomial ideal whose associated prime ideals are totally ordered under inclusion. In [1] and [3], the authors have given the bound for the regularity of such ideals that is $\text{reg}(\mathcal{I}^\ast(\mathcal{H})) < q(\mathcal{I}^\ast(\mathcal{H})) = m(\mathcal{I}^\ast(\mathcal{H}))(\deg(\mathcal{I}^\ast(\mathcal{H}))-1) + 1$. Therefore, we have $\text{reg}(\mathcal{I}^\ast(\mathcal{H})) \leq q(\mathcal{I}^\ast(\mathcal{H}))$. But it is worth noting that $t(\mathcal{I}^\ast(\mathcal{H})) \leq q(\mathcal{I}^\ast(\mathcal{H}))$, for instance, for the ideal $\mathcal{I}^\ast(\mathcal{H}) = (x_1, x_2) \cap (x_1^2, x_2^2, x_3)$, $t(\mathcal{I}^\ast(\mathcal{H})) = 3 < q(\mathcal{I}^\ast(\mathcal{H})) = 4$. Moreover, it should be noted that the bound for the regularity of $\mathcal{I}^\ast(\mathcal{H})$ found in 3.6 can not be applied to any ideal whose associated prime ideals are totally ordered by inclusion. It is also important to mention here that $t(\mathcal{I}^\ast(\mathcal{H})) = \sum_{i=2}^{s} a_i$ is a combinatorial term depends on the containment vector ‘$a$’ associated to the uniformly increasing hypergraph $\mathcal{H}(\mathcal{X}, \mathcal{E}, d)$ on $n$ vertices.

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