Commuting Involution Graphs for Certain Exceptional Groups of Lie Type

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Received: 3 September 2020 / Revised: 31 March 2021 / Accepted: 12 April 2021 / Published online: 16 May 2021 © The Author(s) 2021

Abstract
Suppose that $G$ is a finite group and $X$ is a $G$-conjugacy classes of involutions. The commuting involution graph $C(G, X)$ is the graph whose vertex set is $X$ with $x, y \in X$ being joined if $x \neq y$ and $xy = yx$. Here for various exceptional Lie type groups of characteristic two we investigate their commuting involution graphs.

Keywords Commuting involution graphs · Exceptional groups of Lie type · Disc structure

1 Introduction
Suppose that $G$ is a finite group and $X$ is a subset of $G$. The commuting graph, $C(G, X)$, has $X$ as its vertex set and two vertices $x, y \in X$ are joined by an edge if $x \neq y$ and $x$ and $y$ commute. The extensive bibliography in [9] points towards the many varied commuting graphs which have been studied. But here we shall be considering commuting involution graphs—these are commuting graphs $C(G, X)$ where $X$ is a $G$-conjugacy class of involutions. From now on $X$ is assumed to be a $G$-conjugacy class of involutions. Because involutions are often centre stage in the study of non-abelian simple groups, there is a large literature on their commuting involution graphs. Indeed, such graphs have been instrumental in the construction of some of the sporadic simple groups. For example, the three Fischer groups with the conjugacy class being the 3-transpositions were investigated by Fischer [11], resulting in the construction of these groups. Later, also prior to their construction, commuting involution graphs for the Baby Monster ($\{3, 4\}$-transpositions) and the
Table 1 Disc sizes for $\mathcal{C}(G,X)$, $G \cong 3D_4(2), E_6(2), 2F_4(2)'$ and $F_4(2)$

| $G$    | $X = tx$ | $|\Delta_1(t)|$ | $|\Delta_2(t)|$ | $|\Delta_3(t)|$ |
|--------|----------|-----------------|-----------------|-----------------|
| $3D_4(2)$ | 2A      | 18              | 288             | 512             |
|        | 2B      | 339             | 11112           | 57344           |
| $E_6(2)$ | 2A      | 127782          | 4954112         |                 |
|        | 2B      | 285311          | 8819313408      |                 |
|        | 2C      | 3384671         | 609992912640    | 977994252288    |
| $2F_4(2)'$ | 2A    | 90              | 1664            |                 |
|        | 2B      | 147             | 7712            | 3840            |
| $F_4(2)$ | 2A      | 2286            | 67328           |                 |
|        | 2B      | 2286            | 67328           |                 |
|        | 2C      | 20944           | 4364800         |                 |
|        | 2D      | 50511           | 113896448       | 236912640       |

Monster (6-transpositions) were analyzed. Recently the commuting involution graphs of the sporadic simple groups have received much attention, see [5, 12, 14, 15, 17]. For those simple groups of Lie type consult [1, 4, 8–10], while an analysis of the commuting involution graphs of finite Coxeter groups may be found in [2, 3].

The aim of this short note is to describe certain features of $\mathcal{C}(G,X)$ when $G$ is one of the exceptional Lie type groups of characteristic two. Specifically we consider $G$ being one of the simple groups $3D_4(2), E_6(2), 2F_4(2)'$ and $F_4(2)$.

For $x \in X$ we define the $i$th disc of $x$, $\Delta_i(x)$, $(i \in \mathbb{N})$ to be

$$\Delta_i(x) = \{ y \in X \mid d(x, y) = i \}$$

where $d(,) \text{ is the usual distance metric on the graph } \mathcal{C}(G,X)$. Of course, $G$ acting by conjugation on $X$ embeds $G$ in the group of graph automorphisms of $\mathcal{C}(G,X)$ and, evidently, $G$ is transitive on the vertices of $\mathcal{C}(G,X)$. We now choose $t \in X$ to be a fixed vertex of $\mathcal{C}(G,X)$—our main focus is the description of the discs of $t$ in $\mathcal{C}(G,X)$. The diameter of $\mathcal{C}(G,X)$ will be denoted by $\text{Diam } \mathcal{C}(G,X)$ and we shall rely upon the ATLAS [7] for the names of conjugacy classes of $G$. Our main result is as follows.

**Theorem 1** Let $G$ be isomorphic to one of $3D_4(2), E_6(2), 2F_4(2)'$ and $F_4(2)$.

(i) The sizes of the discs $\Delta_i(t)$ are listed in Table 1 and the $G$-conjugacy classes of $tx$ for $x \in \Delta_i(t), i \in \mathbb{N}$ are given in Table 2.

(ii) If $(G,X) = (E_6(2), 2A), (E_6(2), 2B), (2F_4(2)', 2A), (F_4(2), 2A), (F_4(2), 2B)$ or $(F_4(2), 2C)$, then $\text{Diam } \mathcal{C}(G,X) = 2$.

(iii) If $(G,X) = (3D_4(2), 2A), (3D_4(2), 2B), (E_6(2), 2C), (2F_4(2)', 2B)$ or $(F_4(2), 2D)$, then $\text{Diam } \mathcal{C}(G,X) = 3$. 

Table 1 Disc sizes for $\mathcal{C}(G,X), G \cong 3D_4(2), E_6(2), 2F_4(2)', F_4(2)$
These results were obtained computationally with the aid of MAGMA [6], GAP [16] and the ONLINE ATLAS [18]. In the course of these calculations we determined the $C_G(t)$-orbits on $X$ (where $C_G(t)$ is acting by conjugation). Representatives, in MAGMA format, for each of these orbits are to be found as downloadable files at [13], as they may be of value in other investigations of these groups. In Sect. 2 we also collate information on the action of $C_G(t)$ on $X$. In particular, we give the $C_G(t)$-orbit sizes on each (non-empty) $X_C$, $X_C$ being defined below.

We observe that some “obvious” groups are missing in this paper. First $G_2(2)_{0}^{'}$ being isomorphic to $PSU_3(3)$ means it is covered in [8]. As for $G \cong E_6(2)$, the cases $X = 2A$ and $X = 2B$ are done in [1], while there are partial results in the case $X = 2C$. Likewise [1] also has partial results for $E_7(2)$. While $E_8(2)$ is far and away beyond current computational capabilities.

We remark on the graphs studied here. First we note that as the outer automorphism of $F_4(2)$ interchanges the two classes $2A$ and $2B$, we have that $C(F_4(2), 2A)$ and $C(F_4(2), 2B)$ are isomorphic graphs. A very noteworthy consequence of the present work is that the distance between $t$ and $x$ in $C(G, X)$ is almost always determined by the $G$-class to which $tx$ belongs. The exceptions are $G \cong$.
is the union of eighteen $CG$-orbits of size 294,912 and one of size 1,179,648 with those of size 294,912 being in $\Delta_2(t)$ and the one of size 1,179,648 in $\Delta_3(t)$.

A word or two about the information in our tables is required. As mentioned we employ the class names given in the ATLAS though we make some modifications. First we suppress the “slave” notation. So, for example, the classes $7B \ast 2, 7C \ast 4$ of $3D_4(2)$ are just written as $7B, 7C$, respectively. Secondly we compress the letter part of a class name when we mean the union of these classes and their letters are in alphabetical sequence. As an example, in Table 2, for $G \cong F_4(2)$ and $X = 2D$, $8AF$ is short-hand for $8A \cup 8B \cup 8C \cup 8D \cup 8E \cup 8F$.

Let $C$ be a $G$-conjugacy class and define

$$X_C = \{ x \in X \mid tx \in C \}.$$ 

It is clear that $X_C$ will either be empty or be a union of certain $C_G(t)$-orbits of $X$ (where $G$ acts upon $X$ by conjugation). In locating which discs of $t$ contain the vertices in $X_C$ we sometimes need to determine how $X_C$ breaks into $C_G(t)$-orbits. Also of interest to us is the size of $X_C$ which leads us to class structure constants. Class structure constants are the sizes of sets

$$\{(g_1, g_2) \in C_1 \times C_2 \mid g_1 g_2 = g \}$$

where $C_1, C_2, C_3$ are $G$-conjugacy classes and $g$ is a fixed element of $C_3$. Now these constants can be calculated directly from the complex character table of $G$ which are recorded in the ATLAS and are available electronically in the standard libraries of the computer algebra package GAP [16]. If we take $C_1 = C$, $C_2 = X = C_3$ and $g = t$, then in this case

$$|X_C| = \frac{|G|}{|C_G(t)||C_G(h)|} \sum_{r=1}^{k} \chi_r(h) \overline{\chi_r(t)} \overline{\chi_r(1)},$$

where $h$ is a representative from $C$ and $\chi_1, \ldots, \chi_k$ the complex irreducible characters of $G$.

2 $C_G(t)$-Orbits on $X$

As promised, we tabulate the sizes of the $C_G(t)$-orbits in their action upon $X_C$ where $C$ is a $G$-conjugacy class for which $X_C$ is non-empty. In the ensuing tables we use an exponential notation to indicate the multiplicity of a particular size. Thus in the table for $G \cong 3D_4(2)$ with $X = 2B$ the entry $4^6, 24^{12}$ next to $2B$ is telling us that $X_{2B}$ is the union of eighteen $C_G(t)$-orbits, six of which have size 4 and twelve of which have size 24. Still looking at the same table, the entry $512, 1536$ next to $9AC$ indicates that each of $X_{9A}, X_{9B}$ and $X_{9C}$ is the union of two $C_G(t)$-orbits of sizes 512 and 1536. We give details of the permutation ranks in Table 3.
### 2.1 $G \cong \mathbb{Z}_3 D_4(2)$

**$X = 2A$**

|   | 2A | 18 | 3A | 512 | 4A | 288 |
|---|----|----|----|-----|----|-----|

**$X = 2B$**

|   | 2A | 3, 24 | 2B | $4^6, 24^{12}$ | 3A | 384 | 3B | 512 |
|---|----|-------|----|---------------|----|-----|----|-----|
| 4A | 24$^5$, 192 | 4B | $24^{10}$, 192 | 4C | 384$^6$ | 6A | 1536 |
| 6B | 384$^6$ | 7AC | 512 | 7D | 3072 | 8A | 384$^6$ |
| 8B | 384$^8$ | 9AC | 512, 1536 | 12A | 1536$^2$ | 13AC | 3072 |
| 14AC | 1536 | 18AC | 1536$^2$ | 21AC | 3072 | 28AC | 1536$^2$ |

### 2.2 $G \cong E_6(2)$

**$X = 2A$**

|   | 2A | 2790 | 2B | 124992 | 3A | 2097152 | 4B | 2856960 |
|---|----|------|----|--------|----|---------|----|---------|

**$X = 2B$**

|   | 2A | 63, 2160$^2$ | 2B | 56, 4320, 30240$^2$, 30720$^2$, 64512, 120960 | 2C | 60480$^2$, 725760$^2$, 967680 |
|---|----|-------------|----|------------------------|----|----------------------|
| 3A | 2359296 | 3B | 16777216 | 4A | 774144 |
| 4B | 725760, 967680$^2$, 2211840$^2$ | 4C | 1935360$^4$, 3870720$^4$, 4423680$^4$, 7741440$^2$ | 4D | 7864320$^2$, 8847360 |
| 4E | 46448640$^2$ | 4F | 2064384$^2$, 61931520$^4$ | 4J | 123863040$^2$ |
| 4K | 743178240 | 5A | 939524096 | 6A | 70778880$^2$ |
| 6D | 990904320 | 6F | 1056964608 | 8C | 990904320$^2$ |
| 12B | 1132462080$^2$ |
|    | X = 2C                                      |    |
|----|--------------------------------------------|----|
| 2A | 3, 84, 1536, 2016                          | 2B | 168, 224, 2016, 5376, 8064, 10752, 16128, 32256, 43008, 86016 |
|    |                                            |    | 2C | 96^2, 5376, 16128^3, 32256^4  |
| 3A | 917504, 1572864                            | 3B | 29360128 |
| 4A | 1536, 21504, 32256, 36864^3, 64512^3, 86016, 786432, 1032192 | 4B | 1536^2, 16128, 32256^4  |
|    |                                            |    | 4C | 64512^3, 129024^3, 258048^3 |
| 4D | 1032192, 1376256, 2752512^3, 11010048, 16515072 | 4E | 258048^4, 516096^10, 1032192^10, 2064384^12, 4128768^22, 8257536^2 |
|    |                                            |    | 4F | 1032192^2, 2064384^2, 275251^2, 4128768^8, 550502^4, 16515072^8 |
| 4G | 4128768^2, 8257536^2, 16515072^4, 33030144^3, 66060288^2 | 4H | 3748736^2, 66060288^2 |
|    |                                            |    | 4I | 11010048^2, 16515072^2, 33030144^0, 66060288^7, 88080384, 264241152 |
| 4J | 1376256^2, 2064384^2, 4128768^6, 8257536^16, 16515072^10, 33030144^22, 66060288^10 | 4K | 4128768, 8257536^6, 16515072^13, 33030144^12, 66060288^8, 264241152 |
|    |                                            |    | 5A | 234881024, 1409286144 |
| 6A | 2752512, 33030144^3, 44040192^2             | 6B | 402653184 |
|    |                                            |    | 6C | 528482304, 704643072 |
| 6D | 1835008, 66060288^4, 88080384^3, 132120576^4, 176160768, 264241152, 528482304 | 6E | 37748736^2, 66060288^2, 88080384^2, 132120576^2, 528482304 |
|    |                                            |    | 6F | 88080384, 352321536, 528482304, 704643072, 1056964608 |
| 6G | 2818572288 | 6H | 1056964608^2, 4227858432 | 6I | 8455716864 |
|----|------------|----|------------------------|----|-------------|
| 7C | 805306368  | 7D | 3221225472             | 8A | 1572864^2, 33030144^2, 37748736^2, 66060288^2, 88080384^2, 132120576^2, 528482304^2 |
| 8B | 37748736^2, 44040192^2, 66060288^2, 88080384^2, 132120576^2, 528482304^2 | 8C | 16515072^2, 33030144^2, 66060288^2, 88080384^2, 132120576^12, 264241152^10 | 8D | 132120576^2, 264241152^20, 1056964608^2 |
| 8E | 176160768^2, 528482304^4, 2113929216^2 | 8F | 2113929216^5 | 8G | 26441152^4, 528482304^4, 105664608^12 |
| 8H | 2113929216^3 | 8I | 427858432^6 | 8J | 1056964608^2, 2113929216^6, 4227858432^4 |
| 8K | 16515072 | 8L | 33030144 | 8M | 66060288 | 8N | 88080384 | 8O | 132120576 | 8P | 528482304 | 8Q | 12A | 402653184^2 | 12B | 132120576^2, 528482304^6 |
| 12C | 264241152^8, 528482304^4, 1056964608^16 | 12D | 1409286144^2, 2113929216^4, 4227858432 | 12E | 2818572288^2, 4227858432 |
| 12F | 4227858432, 8455716864 | 12G | 5637144576 | 12H | 352321536^2, 2113929216^6, 4227858432^4 | 12I | 8455716864^2 |
| 12J | 1056964608^2, 2113929216^6, 4227858432^8 | 12K | 1409286144^2, 4227858432 | 14A | 8455716864^4 |
| 12L | 16911433728 | 12M | 16911433728^2 | 12P | 16911433728^2 |
| 13A | 19327352832 | 14G | 16911433728 | 14H | 9663676416 |
| 15C | 22548578304 | 15D | 7516192768, 22548578304 | 16A | 8455716864^4 |
|   |   |   |   |   |
|---|---|---|---|---|
| 16C | 16911433728^4 | 17A | 45097156608 | 17B | 45097156608 |
| 18A | 9663676416^2 | 18B | 67645734912 | 20A | 16911433728^2 |
| 20B | 33822867456^4 | 21G | 19327352832 | 21H | 45097156608 |
| 24A | 8455716864^8 | 24B | 16911433728^4 | 24C | 33822867456^2 |
| 24D | 33822867456^2 | 28K | 9663676416^2 | 28L | 33822867456 |
| 30E | 22548578304^2 | 30F | 67645734912 |   |   |

continued
### 2.3 \( G \cong 2F_4(2)' \)

\[ X = 2A \]

\[
\begin{array}{ccccccc}
| G | & X = f^G | |X| | & \text{Permutation rank} \\
\hline
3D_4(2) & 2A & 819 & 4 \\
 & 2B & 68796 & 27 \\
E_6(2) & 2A & 5081895 & 5 \\
 & 2B & 8822169720 & 62 \\
 & 2C & 1587990549600 & 719 \\
2F_4(2)' & 2A & 1755 & 5 \\
 & 2B & 11700 & 30 \\
F_4(2) & 2A & 69615 & 5 \\
 & 2B & 69615 & 5 \\
 & 2C & 4385745 & 33 \\
 & 2D & 350859600 & 1002 \\
\end{array}
\]

### 2.4 \( G \cong F_4(2) \)

\[ X = 2A \]

\[
\begin{array}{ccccccc}
| G | & X = f^G | |X| | & \text{Permutation rank} \\
\hline
2A & 270 & 2C & 2016 & 3A & 32768 & 4C & 34560 \\
\end{array}
\]

\[ X = 2B \]

\[
\begin{array}{ccccccc}
| G | & X = f^G | |X| | & \text{Permutation rank} \\
\hline
2B & 270 & 2C & 2016 & 3A & 32768 & 4D & 34560 \\
\end{array}
\]
| $X = 2C$ |  |
|---|---|---|---|---|---|---|
| 2AB | 30 | 2C | $32^2, 180, 1920^2$ | 2D | $720^2, 960^4, 11520$ | 3AB | 32768 |
| 4AB | 15360 | 4CD | 11520 | 4F | 1024$^2$ | 4JK | 30720$^2$ |
| 4L | 737280 | 4M | $184320^2$ | 5A | $1048576$ | 6GH | 983040 |

| $X = 2D$ |  |
|---|---|---|---|---|---|---|
| 2AB | 3, 12, $72^2$, 192 | 2C | 9, $12^2$, 24$^2$, 72$^4$, 144$^2$, 192$^2$, 576$^4$ | 2D | 24$^4$, 144$^{29}$, 576$^{24}$ | 1152$^{16}$, 9216 |
| 3AB | 2048, 6144, 24576 | 3C | 262144 | 4AB | 192, 576$^8$, 1152$^4$, 9216, 12288 |
| 4CD | 144$^4$, 192$^3$, 288$^4$, 576$^{13}$, 1152$^2$, 2304$^4$, 4608$^4$, 12288 | 4EF | 576$^4$, 1536$^4$, 2304$^4$, 9216$^8$, 18432$^4$, 73728 |
| 4I | 9216$^{14}$, 18432$^8$, 36864$^3$, 73728$^2$ | 4JK | 1152$^4$, 1536$^4$, 2304$^4$, 4608$^{20}$, 9216$^{16}$, 18432$^{22}$ | 4L | 9216$^9$, 36864$^8$, 147456$^5$ |
| 4M | 2304$^2$, 4608$^12$, 9216$^{30}$, 18432$^{10}$, 36864$^2$ | 4N | 147456$^8$, 36864$^{12}$, 147456$^4$ | 4O | 36864$^{12}$, 147456$^4$ |
| 5A | 196608$^2$, 589824 | 6AB | 6144$^2$, 24576$^2$, 73728$^3$ | 6CD | 36864$^2$, 49152$^2$, 73728$^3$, 294912 |
| 6EF | 786432 | 6GH | 12288, 36864$^2$, 49152$^2$, 73728$^3$, 147456$^5$, 294912 | 6J | 73728$^8$, 147456$^4$, 294912$^4$ |
| 6K | 2359296 | 7AB | 1572864 | 8A | 294912$^4$ |
| 8B | 147456$^8$, 294912$^4$ | 8CF | 24576$^2$, 73728$^{10}$, 147456$^4$, 294912$^4$ | 8G | 589824$^2$ |
| 8HI | 294912$^6$ | 8J | 589824$^{16}$ | 8K | 589824$^6$ |
| 9AB | 1572864, 4718592 | 10AB | 589824$^2$, 1179648$^3$ | 10C | 589824$^2$, 1179648$^4$ |
| 12AB | 294912$^4$, 1179648 | 12CD | 786432$^2$ | 12EH | 98304$^2$, 294912$^4$, 589824$^4$ |
| 12IJ | 294912$^{12}$, 1179648 | 12KL | 2359296$^2$ | 12MN | 589824$^{14}$ |
| 12O | 2359296$^4$ | 13A | 9437184 | 14AB | 4718592 |
| 15AB | 1572864, 4718592 | 16AB | 2359296$^4$ | 17AB | 9437184 |
| 18AB | 4718592$^2$ | 20AB | 2359296$^4$ | 21AB | 9437184 |
| 24AD | 2359296$^4$ | 28AB | 4718592$^2$ | 30AB | 4718592$^2$ |

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