Yetter-Drinfelfd-Long bimodules are modules

Daowei Lu∗ Shuanhong Wang
Department of Mathematics, Southeast University
Nanjing, Jiangsu 210096, P. R. of China

Abstract. Let $H$ be a finite dimensional bialgebra. In this paper, we prove that the category of Yetter-Drinfelfd-Long bimodules is isomorphic to the Yetter-Drinfelfd category over the tensor product bialgebra $H \otimes H^*$ as monoidal category. Moreover if $H$ is a Hopf algebra with bijective antipode, the isomorphism is braided.

Keywords: Hopf algebra; Yetter-Drinfelfd-Long bimodule; Braided monoidal category.

Mathematics Subject Classification: 16W30.

Introduction

F. Panaite and F. V. Oystaeyen in [3] introduced the notion of $L$-$R$ smash biproduct, with the $L$-$R$ smash product and $L$-$R$ smash coproduct introduced in [2] as multiplication, respectively comultiplication. When an object $A$ which is both an algebra and a coalgebra and a bialgebra $H$ form a $L$-$R$-admissible pair $(H, A)$, $A \sharp H$ becomes a bialgebra with smash product and smash coproduct, and the Radford biproduct is a special case. It turns out that $A$ is in fact a bialgebra in the category $LR(H)$ of Yetter-Drinfelfd-Long bimodules (introduced in [3]) with some compatible condition.

The aim of this paper is to show that the category $LR(H)$ coincides with the Yetter-Drinfelfd category over the bialgebra $H \otimes H^*$, in the case when $H$ is finite dimensional. Hence any object $M \in LR(H)$ is just a module over the Drinfelfd double $D(H \otimes H^*)$.

The paper is organized as follows. In section 1, we recall the category $LR(H)$. In section 2, we give the main result of this paper.

Throughout this article, all the vector spaces, tensor product and homomorphisms are over a fixed field $k$. For a coalgebra $C$, we will use the Heyneman-Sweedler’s notation $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$ (summation omitted).

∗Corresponding author: ludaowei620@126.com
1 Preliminaries

Let $H$ be a bialgebra. The category $\mathcal{LR}(H)$ is defined as follows. The objects of $\mathcal{LR}(H)$ are vector spaces $M$ endowed with $H$-bimodule and $H$-bicomodule structures (denoted by $h \otimes m \mapsto h \cdot m, m \otimes h \mapsto m \cdot h, m \mapsto m_{(-1)} \otimes m_{(0)}, m \mapsto m_{<0>} \otimes m_{<1>}$, for all $h \in H, m \in M$), such that $M$ is a left-left Yetter-Drinfeld module, a left-right Long module, a right-right Yetter-Drinfeld module and a right-left Long module, i.e.

\begin{align}
(h_1 \cdot m)_{(-1)} h_2 \otimes (h_1 \cdot m)_{(0)} &= h_1 m_{(-1)} \otimes h_2 \cdot m_{(0)}, \\
(h \cdot m)_{<0>} \otimes (h \cdot m)_{<1>} &= h \cdot m_{<0>} \otimes m_{<1>}, \\
(m \cdot h_2)_{<0>} \otimes h_1 (m \cdot h_2)_{<1>} &= m_{<0>} \cdot h_1 \otimes m_{<1>} h_2, \\
(m \cdot h)_{(-1)} \otimes (m \cdot h)_{(0)} &= m_{(-1)} \otimes m_{(0)} \cdot h.
\end{align}

The morphisms in $\mathcal{LR}(H)$ are $H$-bilinear and $H$-bicolinear maps.

If $H$ has a bijective antipode $S$, $\mathcal{LR}(H)$ becomes a strict braided monoidal category with the following structures: for all $M, N \in \mathcal{LR}(H)$, and $m \in M, n \in N, h \in H$,

\begin{align}
h \cdot (m \otimes n) &= h_1 \cdot m \otimes h_2 \cdot n, \\
(m \otimes n)_{(-1)} \otimes (m \otimes n)_{(0)} &= m_{(-1)} n_{(-1)} \otimes m_{(0)} \otimes n_{(0)}, \\
(m \otimes n) \cdot h &= m \cdot h_1 \otimes n \cdot h_2, \\
(m \otimes n)_{<0>} \otimes (m \otimes n)_{<1>} &= m_{<0>} \otimes n_{<0>} \otimes m_{<1>} n_{<1>},
\end{align}

and the braiding

\[ c_{M,N} : M \otimes N \mapsto N \otimes M, \quad m \otimes n \mapsto m_{(-1)} \cdot n_{<0>} \otimes m_{(0)} \cdot n_{<1>}, \]

and the inverse

\[ c_{M,N}^{-1} : N \otimes M \mapsto M \otimes N, \quad n \otimes m \mapsto m_{(0)} \cdot S^{-1}(n_{<1>}) \otimes S^{-1}(m_{(-1)}) \cdot n_{<0>}. \]

2 Main result

In this section, we will give the main result of this paper.

Lemma 2.1. Let $H$ be a finite dimensional bialgebra. Then we have a functor $F : \mathcal{LR}(H) \longrightarrow H \otimes H^{*} \mathcal{YD}$ given for any object $M \in \mathcal{LR}(H)$ and any morphism $\vartheta$ by

\[ F(M) = M \quad \text{and} \quad F(\vartheta) = \vartheta, \]

where $H \otimes H^{*}$ is a bialgebra with tensor product and tensor coproduct.

Proof. For all $M \in \mathcal{LR}(H)$, first of all, define the left action of $H \otimes H^{*}$ on $M$ by

\[ (h \otimes f) \cdot m = \{f, m_{<1>}\} h \cdot m_{<0>}, \]

(2.1)
for all $h \in H, f \in H^*$ and $m \in M$. Then $M$ is a left $H \otimes H^*$-module. Indeed for all $h, h' \in H, f, f' \in H^*$ and $m \in M$,

\begin{align*}
(h \otimes f)(h' \otimes f') \cdot m &= (hh' \otimes ff') \cdot m \\
&= \langle ff', m_{<1>} \rangle hh' \cdot m_{<0>} \\
&= \langle f, m_{<1>} \rangle \langle f', m_{<1>2} \rangle h \cdot (h' \cdot m_{<0>}) \\
&= \langle f, m_{<0>_{<1>}} \rangle \langle f', m_{<1>} \rangle h \cdot (h' \cdot m_{<0>_{<0>}}) \\
&= \langle f', m_{<1>} \rangle (h \otimes f) \cdot (h' \cdot m_{<0>}) \\
&= (h \otimes f) \cdot ((h' \otimes f') \cdot m).
\end{align*}

And

\[(1 \otimes \varepsilon) \cdot m = \langle \varepsilon, m_{<1>} \rangle m_{<0>} = m,
\]
as claimed. Next for all $m \in M$, define the left coaction of $H \otimes H^*$ on $M$ by

\[
\rho(m) = m_{[-1]} \otimes m_{[0]} = \sum m_{(-1)} \otimes {h^i} \otimes m_{(0)} \cdot h_i,
\]

where $\{h_i\}_i$ and $\{h^i\}_i$ are dual bases in $H$ and $H^*$. Then on one hand,

\[(\Delta_{H \otimes H^*} \otimes \text{id})\rho(m) = \sum m_{(-1)1} \otimes h^i_1 \otimes m_{(-1)2} \otimes h^i_2 \otimes m_{(0)} \cdot h_i.
\]

Evaluating the right side of the equation on $id \otimes g \otimes id \otimes h \otimes id$, we obtain

\[m_{(-1)1} \otimes m_{(-1)2} \otimes m_{(0)} \cdot gh.
\]

On the other hand

\[(id \otimes \rho)\rho(m) = \sum m_{(-1)} \otimes h^i \otimes (m_{(0)} \cdot h_i)_{(-1)} \otimes h^j \otimes (m_{(0)} \cdot h_i)_{(0)} \cdot h_j \\
&= \sum m_{(-1)} \otimes h^i \otimes m_{(0)(-1)} \otimes h^j \otimes (m_{(0)(0)} \cdot h_i) \cdot h_j \\
&= \sum m_{(-1)1} \otimes h^i \otimes m_{(-1)2} \otimes h^j \otimes m_{(0)} \cdot h_i h_j.
\]

Evaluating the right side of the equation on $id \otimes g \otimes id \otimes h \otimes id$, we obtain

\[m_{(-1)1} \otimes m_{(-1)2} \otimes m_{(0)} \cdot gh.
\]

Since $g, h \in H$ were arbitrary, we have

\[(\Delta_{H \otimes H^*} \otimes \text{id})\rho = (id \otimes \rho)\rho.
\]

And since

\[(\varepsilon_{H \otimes H^*} \otimes \text{id})(\rho(m)) = \varepsilon(m_{(-1)})m_{(0)} = m,
\]

3
\( M \) is a left \( H \otimes H^* \)-comodule.

Finally

\[
\{(h \otimes f)_{1} \cdot m\}_{[-1]}(h \otimes f)_{2} \otimes [(h \otimes f)_{1} \cdot m]_{[0]} \\
= (h_{1} \cdot m_{<0>})_{[-1]}\langle f_{1}, m_{<1>} \rangle (h_{2} \otimes f_{2}) \otimes (h_{1} \cdot m_{<0>})_{[0]} \\
= \sum \langle f_{1}, m_{<1>} \rangle (h_{1} \cdot m_{<0>})(-1)h_{2} \otimes h_{1}f_{2} \otimes (h_{1} \cdot m_{<0>})_{(0)} \cdot h_{i} \\
(1.1) = \sum \langle f_{1}, m_{<1>} \rangle h_{1}m_{<0>}(\cdot) \otimes h_{1}f_{2} \otimes h_{2} \cdot m_{<0>}(\cdot) \cdot h_{i}.
\]

Evaluating the right side of the equation on \( id \otimes g \otimes id \), we obtain

\[
\langle f, m_{<1>}g_{2} \rangle h_{1}m_{<0>}(\cdot) \otimes h_{2} \cdot m_{<0>}(\cdot) \cdot g_{1}.
\]

And

\[
(h \otimes f)_{1}m_{[-1]} \otimes (h \otimes f)_{2} \cdot m_{[0]} \\
= \sum (h_{1} \otimes f_{1})(m_{[-1]} \otimes h') \otimes (h_{2} \otimes f_{2}) \cdot (m_{[0]} \cdot h_{i}) \\
= \sum h_{1}m_{[-1]} \otimes f_{1}h' \otimes (f_{2}, (m_{(0)} \cdot h_{i})_{<1}>h_{2} \cdot (m_{(0)} \cdot h_{i})_{<0>}.
\]

Evaluating the right side of the equation on \( id \otimes g \otimes id \), we obtain

\[
\begin{align*}
& h_{1}m_{[-1]} \otimes \langle f, g_{1}(m_{(0)} \cdot g_{2})_{<1>} \rangle h_{2} \cdot (m_{(0)} \cdot g_{2})_{<0>} \\
& (1.3) = h_{1}m_{[-1]} \otimes \langle f, m_{(0)}_{<1>}g_{2} \rangle h_{2} \cdot m_{<0>}(\cdot) \cdot g_{1} \\
& = \langle f, m_{<1>}g_{2} \rangle h_{1}m_{<0>}(\cdot) \otimes h_{2} \cdot m_{<0>}(\cdot) \cdot g_{1}.
\end{align*}
\]

Therefore \( M \) is a left-left Yetter-Drinfeld module over \( H \otimes H^* \). It is straightforward to verify that any morphism in \( LR(H) \) is also a morphism in \( H \otimes H^*,\mathcal{YD} \). The proof is completed. \( \square \)

**Lemma 2.2.** Let \( H \) be a finite dimensional bialgebra. Then we have a functor \( G : H \otimes H^*,\mathcal{YD} \rightarrow LR(H) \) given for any object \( M \in H \otimes H^*,\mathcal{YD} \) and any morphism \( \theta \) by

\[
G(M) = M \quad \text{and} \quad G(\theta) = \theta.
\]

**Proof.** We denote by \( \varepsilon^* \) the map \( \varepsilon_{H^*} \) defined by \( \varepsilon_{H^*}(f) = f(1) \) for all \( f \in H^* \). For any \( M \in H \otimes H^*,\mathcal{YD} \), denote the left \( H \otimes H^* \)-coaction on \( M \) by

\[
m \mapsto m_{[-1]} \otimes m_{[0]},
\]

for all \( m \in M \). Define the \( H \)-bimodule and \( H \)-bicomodule structures as follows:

\[
h \cdot m = (h \otimes \varepsilon) \cdot m, \quad \rho_L(m) = m_{[-1]} \otimes m_{[0]} = (id \otimes \varepsilon^*)(m_{[-1]}) \otimes m_{[0]},
\]

(2.3)
\[ m \cdot h = ((\varepsilon \otimes \text{id})m_{[-1]}, h)m_{[0]}, \quad \rho_R(m) = m_{<0>} \otimes m_{<1>} = \sum (1 \otimes h^i) \cdot m \otimes h_i. \quad (2.4) \]

for all \( h \in H \).

Obviously \( M \) is a left \( H \)-module. And

\[
(\Delta \otimes \text{id}) \rho_L(m) = \Delta((\text{id} \otimes \varepsilon^*)(m_{[-1]})) \otimes m_{[0]}
\]
\[
= (\text{id} \otimes \varepsilon^*)(m_{[-1]}1) (\text{id} \otimes \varepsilon^*)(m_{[-1]}2) \otimes m_{[0]}
\]
\[
= (\text{id} \otimes \varepsilon^*)(m_{[-1]}1) (\text{id} \otimes \varepsilon^*)(m_{[0][[-1]} \otimes m_{[0][0]}
\]
\[
= (\text{id} \otimes \rho_L) \rho_L(m).
\]

The counit is straightforward. Thus \( M \) is a left \( H \)-comodule. For all \( h, h' \in M \),

\[
m \cdot hh' = ((\varepsilon \otimes \text{id})m_{[-1]}, hh')m_{[0]}
\]
\[
= ((\varepsilon \otimes \text{id})m_{[-1]}1, h) ((\varepsilon \otimes \text{id})m_{[-1]}2, h') m_{[0]}
\]
\[
= ((\varepsilon \otimes \text{id})m_{[-1]}1, h) ((\varepsilon \otimes \text{id})m_{[0][-1]}, h') m_{[0][0]}
\]
\[
= ((\varepsilon \otimes \text{id})m_{[-1]}1, h) m \cdot h'
\]
\[
= (m \cdot h) \cdot h'.
\]

The unit is obvious. Thus \( M \) is a right \( H \)-module. Since

\[
(id \otimes \Delta) \rho_R(m) = \sum (1 \otimes h^i) \cdot m \otimes h_{i1} \otimes h_{i2}
\]
\[
= \sum (1 \otimes h^i h^j) \cdot m \otimes h^j \otimes h^i
\]
\[
= (\rho_R \otimes \text{id}) \rho_R(m),
\]

it follows that \( M \) is a right \( H \)-comodule. Moreover

\[
(h \cdot m) \cdot h' = ((h \otimes \varepsilon) \cdot m) \cdot h'
\]
\[
= ((\varepsilon \otimes \text{id})((h \otimes \varepsilon) \cdot m)_{[-1]}1, h')((h \otimes \varepsilon) \cdot m)_{[0]}
\]
\[
= ((\varepsilon \otimes \text{id})((h_1 \otimes \varepsilon) \cdot m)_{[-1]}1(h_2 \otimes \varepsilon)), h')((h_1 \otimes \varepsilon) \cdot m)_{[0]}
\]
\[
= ((\varepsilon \otimes \text{id})m_{[-1]}1, h')(h \otimes \varepsilon) \cdot m_{[0]}
\]
\[
= h \cdot (m \cdot h').
\]

Thus \( M \) is an \( H \)-bimodule. And

\[
(\rho_L \otimes \text{id}) \rho_R(m) = \sum (id \otimes \varepsilon^*)(1 \otimes h^i) \cdot m_{[-1]} \otimes ((1 \otimes h^i) \cdot m)_{[0]} \otimes h_i
\]
\[
= \sum (id \otimes \varepsilon^*)((1 \otimes h^i) \cdot m)_{[-1]}(1 \otimes h^j) \otimes ((1 \otimes h^j) \cdot m)_{[0]} \otimes h_i
\]
\[
= \sum (id \otimes \varepsilon^*)((1 \otimes h^i) m_{[-1]}1) \otimes (1 \otimes h^i) \cdot m_{[0]} \otimes h_i
\]
\[= (id \otimes \rho_R)\rho_L(m).\]

Thus \( M \) is an \( H \)-bicomodule.

We now prove (1.1). For all \( h \in H, m \in M \),

\[
(h_1 \cdot m)_{(-1)}h_2 \otimes (h_1 \cdot m)_{(0)}
= ((h_1 \otimes \varepsilon) \cdot m)_{(-1)}h_2 \otimes ((h_1 \otimes \varepsilon) \cdot m)_{(0)}
= (id \otimes \varepsilon^*)(((h_1 \otimes \varepsilon) \cdot m)[-1](h_2 \otimes \varepsilon)) \otimes ((h_1 \otimes \varepsilon) \cdot m)[0]
= (id \otimes \varepsilon^*)(h_1 \otimes \varepsilon)m[-1] \otimes (h_2 \otimes \varepsilon) \cdot m[0]
= h_1m[-1] \otimes h_2 \cdot m[0].
\]

We now prove (1.2):

\[
(h \cdot m)_{<0>} \otimes (h \cdot m)_{<1>}
= ((h \otimes \varepsilon) \cdot m)_{<0>} \otimes ((h \otimes \varepsilon) \cdot m)_{<1>}
= \sum (1 \otimes h^i)(h \otimes \varepsilon) \cdot m \otimes h_i
= \sum (h \otimes \varepsilon)(1 \otimes h^i) \cdot m \otimes h_i
= h \cdot m_{<0>} \otimes m_{<1>}
\]

We now prove (1.3): On one hand,

\[
(m \cdot h_2)_{<0>} \otimes h_1(m \cdot h_2)_{<1>}
= \langle (\varepsilon \otimes id)m[-1], h_2 \rangle m[0]_{<0>} \otimes h_1m[0]_{<1>}
= \sum \langle (\varepsilon \otimes id)m[-1], h_2 \rangle (1 \otimes h^i) \cdot m[0] \otimes h_1h_i.
\]

Evaluating the right side on \( id \otimes f \) for all \( f \in H^* \), we have

\[
\langle (\varepsilon \otimes id)m[-1], h_2 \rangle (1 \otimes f_2) \cdot m[0], f_1(h_1)
= \langle (\varepsilon \otimes id)(1 \otimes f_1)m[-1], h \rangle (1 \otimes f_2) \cdot m[0]
= \langle (\varepsilon \otimes id)((1 \otimes f_1) \cdot m)[-1](1 \otimes f_2), h \rangle (1 \otimes f_1) \cdot m[0].
\]

On the other hand,

\[
m_{<0>} \cdot h_1 \otimes m_{<1>}h_2 \cdot m_{<0>} \cdot h_1 \otimes h_ih_2
= \sum \langle (1 \otimes h^i) \cdot m \cdot h_1 \otimes h_i, h_2 \rangle
= \sum \langle (\varepsilon \otimes id)((1 \otimes h^i) \cdot m)[-1], h_1 \rangle ((1 \otimes h^i) \cdot m)[0] \otimes h_ih_2
\]

Evaluating the right side on \( id \otimes f \), we have

\[
\langle (\varepsilon \otimes id)((1 \otimes f_1) \cdot m)[-1], h_1 \rangle ((1 \otimes f_1) \cdot m)[0]f_2(h_2)
= \langle (\varepsilon \otimes id)(((1 \otimes f_1) \cdot m)[-1](1 \otimes f_2), h \rangle ((1 \otimes f_1) \cdot m)[0].
\]

Hence \((m \cdot h_2)_{<0>} \otimes h_1(m \cdot h_2)_{<1>} = m_{<0>} \cdot h_1 \otimes m_{<1>}h_2 \) since \( f \) was arbitrary.
We now prove (1.4):

\[(m \cdot h)_{(-1)} \otimes (m \cdot h)_{(0)}\]
\[= \langle (\varepsilon \otimes id)m_{[-1]}, h)(id \otimes \varepsilon^*)(m_{[0][-1]} \otimes m_{[0][0]}\]
\[= (id \otimes h)m_{[-1]} \otimes m_{[0]}\]
\[= \langle (\varepsilon \otimes id)m_{[-1]}2, h)(id \otimes \varepsilon^*)(m_{[-1]}1 \otimes m_{[0]}\]
\[= m_{(-1)} \otimes m_{(0)} \cdot h,\]
where in the third equality, \((id \otimes h)m_{[-1]}\) means the second factor of \(m_{[-1]}\) acts on \(h\).

Therefore \(M \in \mathcal{LR}(H)\). It is straightforward to verify that any morphism in \(\mathcal{H} \otimes \mathcal{H}^* \mathcal{YD}\) is also a morphism in \(\mathcal{LR}(H)\). The proof is completed.

**Theorem 2.3.** Let \(H\) be a finite dimensional bialgebra. Then we have a monoidal category isomorphism

\[
\mathcal{LR}(H) \cong \mathcal{H} \otimes \mathcal{H}^* \mathcal{YD}.
\]

Moreover if \(H\) is a Hopf algebra with bijective antipode \(S\), they are isomorphic as braided monoidal categories. Consequently

\[
\mathcal{LR}(H) \cong \mathcal{D}(H \otimes H^*) M,
\]
where \(\mathcal{D}(H \otimes H^*)\) is the Drinfeld double of \(H \otimes H^*\).

**Proof.** It is easy to see that the functor \(F : \mathcal{LR}(H) \longrightarrow \mathcal{H} \otimes \mathcal{H}^* \mathcal{YD}\) is monoidal and that \(F \circ G = id\) and \(G \circ F = id\). And for all \(M, N \in \mathcal{LR}(H)\), and \(m \in M, n \in N\),

\[
m_{[-1]} \cdot n \otimes m_{[0]} = \sum (m_{(-1)} \otimes h^i) \cdot n \otimes m_{[0]} \cdot h^i
\]
\[
= \sum m_{(-1)} \cdot n_{<0>} \otimes m_{[0]} \cdot n_{<1>}.
\]

The proof is completed.

**Corollary 2.4.** \((A, H)\) is an \(L-R\)-admissible pair if and only if \((A, H \otimes H^*)\) is an admissible pair.

By the isomorphism in Theorem 2.3, we can obtain the following result in [5] directly.

**Proposition 2.5.** Let \(H\) be a finite dimensional Hopf algebra. The canonical braiding of \(\mathcal{LR}(H)\) is pseudosymmetric if and only if \(H\) is commutative and cocommutative.

**Proof.** From [4], the canonical braiding of \(\mathcal{H} \otimes \mathcal{H}^* \mathcal{YD}\) is pseudosymmetric if and only if \(H \otimes H^*\) is commutative and cocommutative. By the bialgebra structure of \(H \otimes H^*\), the proof is completed.
Acknowledgements

This work was supported by the NSF of China (No. 11371088) and the Fundamental Research Funds for the Central Universities (No. KYLX15_0109).

References

[1] C. Kassel, Quantum groups, GTM 155, Springer, Berlin, 1995.

[2] F. Panaite, F. Van Oystaeyen, L-R-smash product for (quasi-) Hopf algebras, J. Algebra 309(2007): 168–191.

[3] F. Panaite, F. Van Oystaeyen, L-R-smash biproducts, double biproducts and a braided category of Yetter-Drinfeld-Long bimodules, Rocky Mountain Journal of Mathematics, 40(2008): 2013–2024.

[4] F. Panaite, M. Staic, F. Van Oystaeyen, Pseudosymmetric braidings, twines and twisted algebras, J. Pure Appl. Algebra, 214(2010): 867–884.

[5] F. Panaite, M. Staic, More examples of pseudosymmetric braided categories, J. Alg. Appl., 12(2011): 63–75.

[6] D. E. Radford, The structure of Hopf algebras with a projection, J. Algebra 92(1985): 322–347.

[7] L. Y. Zhang, L-R smash products for bimodule algebras, Progr. Nat. Science 16(2006): 580–587.