Final spin of a coalescing black-hole binary: an Effective-One-Body approach

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We update the analytical estimate of the final spin of a coalescing black-hole binary derived within the Effective-One-Body (EOB) approach. We consider unequal-mass non-spinning black-hole binaries. It is found that a more complete account of relevant physical effects (higher post-Newtonian accuracy, ringdown losses) allows the analytical EOB estimate to “converge towards” the recently obtained numerical results within 2\%. This agreement illustrates the ability of the EOB approach to capture the essential physics of coalescing black-hole binaries. Our analytical approach allows one to estimate the final spin of the black hole formed by coalescing binaries in a mass range \((\nu = m_1 m_2 / (m_1 + m_2)^2 < 0.16)\) which is not presently covered by numerical simulations.

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I. INTRODUCTION

The dimensionless spin parameter \(\hat{a} = a / M = J / M^2\) of the black hole formed by the coalescence of a black-hole binary is a complicated function of the initial masses and spins of the two constituent black holes. [We consider inspiralling systems circularized by radiation reaction]. The function \(\hat{a}(m_1, m_2, S_1, S_2)\) is a useful diagnostic for comparing analytical approaches to the dynamics of coalescing binaries to the results of three-dimensional numerical simulations. Here we shall consider the simple case of non-spinning binaries, where \(\hat{a}\) becomes simply a function of the symmetric mass ratio \(\nu = m_1 m_2 / (m_1 + m_2)^2\) (which satisfies \(0 \leq \nu \leq 1/4\)).

As far as we know, the first estimate of \(\hat{a}\) was made in 2000\textsuperscript{1}, on the basis of a new analytical approach to the general relativistic two-body dynamics, the Effective-One-Body (EOB) approach\textsuperscript{2}, at a time where no reliable numerical simulations of coalescing black hole binaries were yet available. This estimate was \(\hat{a} = 0.795\) for \(\nu = 1/4\) (i.e. for the equal mass case), and was based on estimating the ratio \(J / M^2\) at a “matching radius” \(r_{\text{match}} \simeq 2.85 M\) where the two-black-hole system was replaced by a unique, ringing black hole. This estimate used a 2.5 post-Newtonian (PN) accurate description of the dynamics down to \(r_{\text{match}}\), and neglected the energy and angular momentum losses during the ringdown. In 2001, a combination of full numerical simulations with a “close-limit” approximation\textsuperscript{3} describing the ringing final black hole led to a similar estimate, namely \(\hat{a} \simeq 0.8\) for \(\nu = 1/4\)\textsuperscript{4}. This estimate was revised downwards in 2002, to \(\hat{a} \simeq 0.7\)\textsuperscript{5}, when a better (Lazarus-based) way of bridging the far- and close-limit approaches indicated that the angular momentum losses were larger than previously estimated. Indeed, Ref.\textsuperscript{5} had estimated the total angular momentum loss after crossing the Last Stable Orbit (LSO)\textsuperscript{1} to be around 2\%, and the corresponding energy loss to be around 3\%. [The EOB approach had estimated an energy loss beyond the LSO of about 1.4\%, half of it emitted during the plunge and the other half during the ringdown\textsuperscript{6}]. By contrast, Ref.\textsuperscript{5} estimated the angular momentum loss to be around 12\%. More recently, an update of the analytical EOB estimate\textsuperscript{7} using the now available 3.5PN accurate description of the two-body dynamics showed that increasing the PN accuracy of the dynamics (from 2.5PN to 3.5PN) had the effect of decreasing the final spin parameter from \(\hat{a} = 0.795\) to \(\hat{a} = 0.77\) (again for \(\nu = 1/4\), and again when neglecting the angular momentum loss below the matching radius \(r_{\text{match}}\)).

The estimates just recalled belong to a “prehistoric” era where, for a variety of reasons, numerical simulations did not exhibit a very convincing convergence among themselves, nor towards analytical results (see, however,\textsuperscript{5} and\textsuperscript{8}). This era has recently ended thanks to remarkable breakthroughs in numerical relativity. Different groups have finally succeeded in numerically simulating the merger of two black holes of comparable masses \(m_1\) and \(m_2\), possibly with spin, and their results exhibit convincing internal convergence, and a nice mutual consistency\textsuperscript{9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20}. In the particular case of non-spinning black holes with equal masses, \(m_1 = m_2\), different groups now agree on the value \(\hat{a} \simeq 0.69\) (which is within the span of the “prehistoric” estimates recalled above). The most extensive analysis of the final angular momentum of coalescing black holes (in the non-spinning case) to date has been carried out by Gonzalez et al.\textsuperscript{16}: they have considered a large sample of unequal-mass systems, corresponding to a symmetric mass ratio \(\nu\) varying in the range \(0.1613 \leq \nu \leq 1/4\), and have accurately determined the variation of \(\hat{a}\) with \(\nu\) within this range. The aim of the present paper is

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\textsuperscript{1} Also known as the innermost stable circular orbit (ISCO).
to generalize and update the analytical EOB estimates \[1, 7\] of \( \dot{a}(\nu) \) recalled above, both by explicitly considering general values of \( \nu \) and by improving the previous EOB treatments of the physical ingredients which are crucial in determining the value of \( \dot{a} \). Our motivation for this study is two-fold: on the one hand, we wish to see to what extent numerical results can be reproduced by analytical (EOB) estimates, and on the other hand we wish to understand, on this example, what are the physical ingredients which are crucial in accurately determining the plunge, merger and ringdown dynamics of coalescing black holes.

Let us recall that the EOB approach to the general relativistic two-body dynamics is a non-perturbatively resummed analytic technique which has been developed in Refs. \[1, 2, 5, 21, 22, 23\]. This technique uses, as basic input, the results of PN theory, such as: (i) PN-expanded equations of motion for two point-like bodies, (ii) PN-expanded radiative multipole moments, and (iii) PN-expanded energy and angular momentum fluxes at infinity. For the moment, the most accurate such results are the 3PN conservative dynamics \[24, 25\], and the 3.5PN energy flux \[26, 27, 28\]. Then the EOB approach “packages” this PN-expanded information in special resummed forms which extend the validity of the PN results beyond the expected weak-field-slow-velocity regime into (part of) the strong-field-fast-motion regime. The aim being to use the EOB approach for analytically describing the last inspiralling orbits, the transition from inspiral to plunge, and the plunge itself down to a “matching radius” \( r_{\text{match}} \) small enough to allow one to match there the plunge waveform to a ringdown one.

The basic new ingredient used below to improve the previous EOB estimates of \( \dot{a} \) is an approximate treatment of the energy and angular momentum losses during ringdown. These losses were neglected in \[1, 7\]. Our approximation will consist in estimating these losses non-perturbatively by plunging test-masses \[29, 30\] has shown that this modified radiation reaction stays closer to the “exact” gravitational wave angular momentum flux computed à la Regge-Wheeler-Zerilli.

The paper is organized as follows. In Sec. \[1\] we review the non-perturbative construction of the two-body dynamics incorporating radiation reaction effects while we devote Sec. \[11\] to the presentation of our results. Some conclusions are presented in Sec. \[14\]. We use geometric units \( G = c = 1 \).

II. EQUATIONS

In this section we recall the non-perturbative construction of the two-body dynamics including a radiation reaction force. We take advantage of the most complete PN results, i.e. we work at 3PN for the conservative part of the dynamics and at 3.5PN for the radiation damping. In the EOB framework, the complicated PN-expanded relative dynamics (in the center of mass frame) of the binary system of masses \( m_1 \) and \( m_2 \) is mapped into the simpler geodesic dynamics of a particle of mass \( \mu = m_1 m_2 / (m_1 + m_2) \) moving in some effective background geometry (in Schwarzschild gauge)

\[
ds^2 = -A(r) dt^2 + B(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) . \tag{1}
\]

Here, and below, we work with the dimensionless reduced variables \( r = R/M \) and \( t = T/M \), with \( M = m_1 + m_2 \); \( r, \theta, \varphi \) are polar coordinates in the effective problem that describe the relative motion. The coefficients of the effective metric at the 3PN approximation \[21\] read

\[
A^{3\text{PN}}(r) \equiv 1 - \frac{2}{r} + \frac{2\nu}{r^3} + \left( \frac{94}{3} - \frac{41}{32} \pi^2 \right) \frac{\nu}{r^4} , \tag{2}
\]

\[
(BA)^{3\text{PN}}(r) \equiv D^{3\text{PN}}(r) \equiv 1 - \frac{6\nu}{r^2} + 2(3\nu - 26) \frac{\nu}{r^4} . \tag{3}
\]

Note that we work with the PN expansion of the quantity \( D(r) \equiv B(r)/A(r) \), rather than with the \( g_{rr} \) metric coefficient \( B(r) = D(r)/A(r) \). [Recall that \( B(r) \) is equal to \( 1/A(r) \) in the Schwarzschild case (which corresponds to the test-mass limit \( \nu \to 0 \)).]

Though the EOB packaging of the complicated original PN dynamics into the much simpler metric coefficients \( A(r), D(r) \) already represents an efficient resummation of PN-expanded results, it is quite useful, especially at 3PN, to further resum the ‘packages’ \( A(r), D(r) \). Indeed, a general feature of the EOB philosophy is to smoothly connect the EOB structures to their \( \nu \to 0 \) limit \[21, 22\]. Here, this can be done by replacing the (Taylor expanded) metric coefficients given by Eqs. \(2\)-\(3\) with suitable Padé approximants. The simplest, and most robust, choices consist in using, as metric functions, the following definitions \( A(r) \equiv P_2^A[A^{3\text{PN}}] \) and \( D(r) \equiv P_2^D[D^{3\text{PN}}] \). These Padé approximants are used to ensure the following two facts: (i) the function \( A \) has a simple zero for a positive value of \( r \) (like \( A_{\text{Schw}} = 1 - 2/r \)) and (ii) the function \( D \) stays positive while \( r \) decreases (like \( D_{\text{Schw}} = 1 \)). The Padé resummation of \( A \) is useful for ensuring the existence and \( \nu \)-continuity of a last stable orbit (LSO), as well as the existence and \( \nu \)-continuity of a last unstable orbit, i.e. of a \( \nu \)-deformed analog of a
light ring (LR). We recall that the LR corresponds to the circular orbit of a massless particle, or of an extremely relativistic massive particle, and is technically defined by looking for the maximum of \( A(r)/r^2 \), i.e. by solving \( (d/dr)(A(r)/r^2) = 0 \). The Padé resummation of \( D \) is useful to ensure that the orbital frequency \( \Omega = d\varphi/dt \) has a clear maximum at (approximately) the EOB \( \nu \)-deformed light ring and then drops to zero (cf Fig. 1 below for \( \nu = 1/4 \)).

In the EOB approach one splits the general relativistic relative dynamics of a binary system into a conservative part, determined by the EOB Hamiltonian defined below, and a non-conservative part related to the loss of angular momentum through gravitational radiation. The EOB Hamiltonian (divided by \( \mu \)) is given by

\[
\hat{H} \equiv \frac{1}{\nu} \sqrt{1 + 2\nu \left( \hat{H}_{\mathrm{eff}} - 1 \right)} \tag{4}
\]

where \( \hat{H}_{\mathrm{eff}} \) denotes the so-called “effective Hamiltonian” (describing the geodesic dynamics of the “effective” test-mass \( \mu \)), originally written as \(^{21}\)

\[
\hat{H}_{\mathrm{eff}} \equiv \sqrt{A \left( 1 + \frac{p_x^2}{r^2} + \frac{p_r^2}{B} + z_3 \frac{p_r^4}{r^2} \right)} \tag{5}
\]

Here \( z_3 = 2\nu(4 - 3\nu) \), \( \hat{H} \equiv H/\mu \), \( \hat{H}_{\mathrm{eff}} \equiv H_{\mathrm{eff}}/\mu \), \( p_x \equiv P_x/(\mu M) \), \( r = R/M \) and \( p_r \) denotes the conjugate momentum to \( r \).

Following what we did in the test-mass limit case \(^{29, 30}\), the relative dynamics is somewhat more conveniently described by replacing the Schwarzschild-like radial variable \( r \) by the EOB generalization of the Regge-Wheeler tortoise coordinate \( r_* \), defined by integrating

\[
\frac{dr_*}{dr} = \left( \frac{B}{A} \right)^{1/2} \tag{6}
\]

where we recall that \( B = D/A \). One then needs to replace the \( r \)-conjugate momentum \( p_r \) by the \( r_* \)-conjugate momentum \( p_{r_*} \), such that \( p_{r_*} dr_* = p_r dr \), i.e. \( p_{r_*} = (A/B)^{1/2} p_r \). The reason for using this transformation is that \( p_{r_*} \) has a finite limit when \( r \) tends to the zero of \( A(r) \) (“\( \nu \)-deformed effective horizon”), while \( p_r \) diverges there. Actually, as we stop evolving the dynamics around the \( \nu \)-deformed light ring (i.e. before reaching the zero of \( A(r) \)), this change of variables is not really necessary. It is, however, convenient because it magnifies the radial axis in a crucial region, and prevents any excessive growth of the radial momentum during the plunge. [Let us mention in passing that the same kind of coordinate has been used in \(^{31}\) for studying gravitational perturbations of non-rotating relativistic stars.]

Neglecting (as it is consistent in a 3PN correction term) the square of the factor \( B/A \) entering the \( z_3 p_r^4/r^2 \) term, this leads to the following form for the effective Hamiltonian

\[
\hat{H}_{\mathrm{eff}} \equiv \sqrt{p_{r_*}^2 + A \left( 1 + \frac{p_x^2}{r^2} + z_3 \frac{p_r^4}{r^2} \right)} \tag{7}
\]

Hamilton’s equations for \((r, \varphi, p_r, p_\varphi)\) then read

\[
d\varphi = \frac{Ap_\varphi}{\nu r^2 H_{\mathrm{eff}}} \equiv \Omega \tag{8}
\]

\[
dr = \left( \frac{A}{B} \right)^{1/2} \frac{1}{\nu H_{\mathrm{eff}}} \left( p_{r_*} + z_3 \frac{2A}{r^2} p_r^3 \right) \tag{9}
\]

\[
dp_{r_*} = \hat{\mathcal{F}}_{\varphi} \tag{10}
\]

\[
dp_{r_*} = - \left( \frac{A}{B} \right)^{1/2} \frac{1}{2\nu H_{\mathrm{eff}}} \times \left\{ A' + \frac{p_x^2}{r^2} \left( A' - \frac{2A}{r} \right) + z_3 \left( \frac{A'}{r^2} - \frac{2A}{r^3} \right) p_r^4 \right\} \tag{11}
\]

where \( A' = dA/dr \). In these equations the extra term \( \hat{\mathcal{F}}_{\varphi} \) represents the non-conservative part of the dynamics, namely the radiation reaction force.

During the quasi-circular inspiral, a rather accurate expression for \( \hat{\mathcal{F}}_{\varphi} \) is the following Padé-resummed form \(^{32}\)

\[
\hat{\mathcal{F}}_{\varphi} \equiv \frac{\hat{\mathcal{F}}_{\varphi}^K}{\mu} = - \frac{32}{5} \nu\Omega^{7/3} \frac{\hat{f}_{\mathrm{DIS}}(v_\varphi; \nu)}{1 - v_\varphi/v_{\mathrm{DIS}}}, \tag{12}
\]

which is expressed in terms of the PN ordering parameter \( v_{\mathrm{DIS}} \equiv \Omega^{1/3} \). In this expression, the function \( \hat{f}_{\mathrm{DIS}} \) denotes the “factored flux function” of Ref. \(^{32}\), scaled to the Newtonian (quadrupole) flux [see Eq. (4.6)-(4.8) there].

Ref. \(^{11}\) assumed that the analytical continuation of the expression \(^{12}\) might still be a sufficiently accurate description of radiation reaction effects during the plunge. On the other hand, the authors of Ref. \(^{23}\) pointed out that the expression \(^{12}\) assumed the continued validity of the usual Kepler law \(^4\) \( \Omega^2 r^3 = 1 \) during the plunge. [This is why we label the expression \(^{12}\) with a superscript \( K \), for Kepler.] They, however, emphasized that the Kepler combination \( K = \Omega^2 r^3 \) significantly deviates from one after the crossing of the LSO, to become of order of 0.5 at the (effective) light ring. Ref. \(^{24}\) went on to argue for a different expression for the radiation reaction, say \( \hat{\mathcal{F}}_{\varphi} \) (without any superscript), that does not assume Kepler’s law. This new expression reads

\[
\hat{\mathcal{F}}_{\varphi} \equiv \frac{\hat{\mathcal{F}}_{\varphi}}{\mu} = - \frac{32}{5} \nu\Omega^{7/3} \frac{\hat{f}_{\mathrm{DIS}}(v_\varphi; \nu)}{1 - v_\varphi/v_{\mathrm{DIS}}}, \tag{13}
\]

\(^4\) modulo a factor \( \psi \) taken into account below.
FIG. 1: Time evolution of the orbital frequency $\Omega$ for $\nu = 1/4$: the maximum occurs near the $\nu$-deformed light ring at $r_{LR}(1/4) \approx 2.316$.

TABLE I: A sample of the numerical data of Fig. 2. From left to right the columns report: the symmetric mass ratio $\nu$, the final dimensionless angular momentum $\hat{a}^{\text{NR}}$ from Ref. [16], and our best estimates (with 3PN+3.5PN dynamics) with $\hat{F}_K$ ($\hat{a}^{\text{BH}}, M^{\text{BH}}/M$) and with $\hat{F}_R$ ($\hat{a}^{\text{BH}}, M^{\text{BH}}/M$).

| $\nu$ | $\hat{a}^{\text{NR}}$ | $\hat{a}^{K}_{\text{BH}}$ | $M^{K}_{\text{BH}}/M$ | $\hat{a}^{R}_{\text{BH}}$ | $M^{R}_{\text{BH}}/M$ |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 0.25  | 0.6871          | 0.6793          | 0.9555          | 0.7023          | 0.9589          |
| 0.2402| 0.6641          | 0.6575          | 0.9582          | 0.6792          | 0.9614          |
| 0.2227| 0.6248          | 0.6181          | 0.9631          | 0.6373          | 0.9659          |
| 0.2015| 0.5753          | 0.5687          | 0.9686          | 0.5850          | 0.9710          |
| 0.1825| 0.5281          | 0.5231          | 0.9732          | 0.5370          | 0.9751          |
| 0.1613| 0.4713          | 0.4706          | 0.9778          | 0.4819          | 0.9794          |
| 0.14  | /               | 0.4160          | 0.9821          | 0.4251          | 0.9834          |
| 0.12  | /               | 0.3631          | 0.9858          | 0.3700          | 0.9867          |
| 0.10  | /               | 0.3082          | 0.9891          | 0.3134          | 0.9898          |
| 0.08  | /               | 0.2514          | 0.9920          | 0.2549          | 0.9925          |
| 0.06  | /               | 0.1925          | 0.9946          | 0.1946          | 0.9949          |
| 0.04  | /               | 0.1312          | 0.9968          | 0.1322          | 0.9969          |
| 0.02  | /               | 0.0672          | 0.9986          | 0.0675          | 0.9986          |
| 0.01  | /               | 0.0341          | 0.9994          | 0.0341          | 0.9994          |

where $\nu = \Omega r_\nu$ and $r_\nu = r_0^{1/3}$ where the function $\psi$ is defined as in Eq. (22) of Ref. [22]. Note that the essential difference between the two possible expressions for the radiation reaction is that $\hat{F}_K \propto \Omega^{7/3}$, while $\hat{F}_R \propto \Omega^5 r^4$. See Ref. [30] (notably Fig. 2 there) for a detailed comparison of these two analytical representations of radiation reaction to “exact” numerical results during the plunge. In both possible expressions for the radiation reaction, our current “best estimate” of $\hat{f}$ is obtained by Padé approximating the currently most complete post-Newtonian results, namely the 3.5PN ones [26, 27, 28].

In the forthcoming analysis we shall compare (and contrast) the relative dynamics and the related final black hole spin using various PN-accuracies for the EOB dynamics (2PN Hamiltonian + 2.5PN radiation reaction, versus 3PN Hamiltonian + 3.5PN radiation reaction), as well as the two different expressions for radiation reaction briefly discussed above.

III. RESULTS

The computation of the relative dynamics needs two separate steps: (i) to initialize the system [38, 41] and (ii) to integrate it in time. The initial condition for the relative dynamics is given in a standard way, notably by specifying a non-zero initial value for $p_c$ according to the PN order that is being used. Our implementation follows Eqs. (4.16)-(4.21) of Ref. [1] and Eqs. (4.8) and (4.10) of Ref. [2] [see also Eqs. (9)-(13) of Ref. [29]] and does not need to be discussed explicitly here. Let us only mention that, for the initial relative separation that we shall take, namely $r_0 = 10$, the leading post-adiabatic approximation is sufficient for getting a smooth quasi-circular inspiral (without noticeable eccentricity).

The orbital frequency $\Omega$ develops a maximum at approximately the location of the last unstable EOB circular orbit defined by the condition $(A(r)/r^2)' = 0$. As already mentioned above, in the test-mass limit ($\nu \ll 1$), this condition defines the light ring $r = 3$. When $\nu \neq 0$, we shall refer to the solution of $(A(r)/r^2)' = 0$ as the $\nu$-deformed light ring (LR): $r_{LR}(\nu)$.

In the $\nu \ll 1$ limit, it was realized long ago [33] that the crossing of the light ring by a test particle corresponds to triggering the black hole quasi-normal modes. For related reasons (discussed in [30]), in the comparable-mass case, the crossing of the $\nu$-deformed light ring corresponds to an abrupt change of description: before this crossing one can still describe the two black holes as two point masses with EOB relative dynamics, while after this crossing one can replace the binary system by a single distorted black hole (as in the “close-limit” approximation of colliding black holes [3]). In other words, the EOB approach estimates the full waveform by matching at $r_{\text{match}} \approx r_{LR}$ the inspiral + plunge waveform computed from the EOB dynamics to a superposition of quasi-normal modes (QNMs) describing the ringdown of the final distorted black hole.

Refs. [1, 7] then estimated the mass and angular mo-

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5 The quantity $r_\nu$ is introduced to simplify some expressions, because it is such that the Kepler-looking law $\Omega^2 r_\nu^3 = 1$ holds, without correcting factor, during the inspiral (i.e. above the LSO).
momentum of the final black hole by the (EOB) energy and angular momentum of the binary system at the matching point\(^{6}\) \(r_{\text{match}}\). This procedure, with the choice \(r_{\text{match}} = r_{\text{LR}}(\nu)\), gives the following estimate of the mass and spin parameter of the final black hole

\[
M_{\text{BH}} \simeq H_{\text{LR}} \quad \hat{\Omega}_{\text{BH}} \simeq \frac{p_{\nu}}{H_{\text{LR}}^2} .
\]

This leading-order estimate of \(\hat{\Omega}_{\text{BH}}(\nu)\) is plotted for different post-Newtonian approximations in Fig. 2 and compared with the numerical relativity data of Ref. 16 (black solid line). The labelling of these leading-order "light ring" analytical estimates indicates both the choice of the PN accuracy for the dynamics ("2PN-LR" or "3PN-LR"), and the choice of a specific radiation reaction expression, namely Eq. 12 ("Rad Reac \(\propto \Omega^{7/3}\)\) or Eq. 13 ("Rad Reac \(\propto \Omega^{5/4}\)\).

It is evident from Fig. 2 that these "LR" approximations overestimate the actual result. It is also clear that using a 3PN-accurate dynamics, instead of a 2PN one, goes in the good direction.

As we were mentioning in the introduction, the basic physical ingredient which is lacking in these simple-minded LR estimates is the loss of angular momentum during the ringdown phase (the importance of this loss was first emphasized in \([5]\)). Here we shall use, as next approximation beyond the leading-order estimates \([11]\), an approximate treatment of the ringdown losses of angular momentum (and energy) which takes advantage of the corresponding results in the small mass case. Recently, by combining EOB and Regge-Wheeler-Zerilli techniques, Ref. 29 computed the gravitational wave signal (decomposed in multipoles) from the late inspiral, plunge and ringdown in the small \(\nu\) limit. In this calculation the dynamics of the source \((r(t), \varphi(t))\) is determined together with the waveform \(\Psi_{\ell m}(u)\), and this allows one to relate the retarded time \(u\), in terms of which the waveform is expressed, to the dynamical time \(t\), entering the equations of motion \([8]-[11]\). In particular, this allows one to define precisely the retarded time, say \(u_{\text{LR}}\), corresponding to the crossing of the light ring, and thereby the part of the waveform which corresponds to the ringdown (in the EOB sense). It was also verified in Ref. 29 that the ratio \(\Psi_{\ell m}(u)/\mu\) had a universal limit as \(\nu \to 0\). By integrating over \(u \geq u_{\text{LR}}\) the quadratic expressions in \(\Psi_{\ell m}(u)/\mu\) and its derivative \(29\), and summing over the various multipoles up to \(\ell = 4\), one obtains the following numbers for the energy and angular momentum losses during the ringdown, in the limiting case \(\nu \ll 1\):

\[
\begin{align*}
E_{\text{ringdown}}(\nu) &\simeq 0.2448 \nu^2 M , \\
J_{\text{ringdown}}(\nu) &\simeq 1.3890 \nu^2 M^2 .
\end{align*}
\]

so that we can define the following next-to-leading order approximations to the mass and spin of the final Kerr black hole:

\[
\begin{align*}
\hat{E}_{\text{ringdown}}(\nu) &\equiv E_{\text{ringdown}}(\nu) , \\
\hat{J}_{\text{ringdown}}(\nu) &\equiv J_{\text{ringdown}}(\nu) .
\end{align*}
\]

We plot in Fig. 2 the next-to-leading order estimate \([19]\) for both choices of radiation reaction and for different post-Newtonian approximations (they are labelled by, e.g., "2PN: Rad Reac \(\propto \Omega^{7/3}\)\). It is clear on Fig. 2 that these next-to-leading order estimates are closer to

\[
\begin{align*}
M_{\text{BH}}(\nu) &\equiv H_{\text{LR}} - \hat{E}_{\text{ringdown}}(\nu) , \\
J_{\text{BH}}(\nu) &\equiv p_{\nu} - \hat{J}_{\text{ringdown}}(\nu) , \\
\hat{\Omega}_{\text{BH}}(\nu) &\equiv \frac{\hat{J}_{\text{BH}}(\nu)}{M_{\text{BH}}^2(\nu)} .
\end{align*}
\]

\[\text{We use here the fact that such simple-minded} \nu \text{ rescalings have been found to be (surprisingly) rather accurate, see e.g. [33] for the energy loss in the head-on collision of two black holes, and its comparison to the test-mass limit [33].}\]
the numerical relativity ones. It is also clear that the use of the 3PN-accurate dynamics improves significantly the results, compared to the 2PN-accurate case. Actually, we see that the 3PN+3.5PN order next-to-leading estimates \(19\) closely “bracket” the numerical relativity result. Using the (originally proposed) “Kepler-type” radiation reaction \(12\) leads to a spin parameter which is slightly smaller than the numerical relativity one, while using the more recently proposed radiation reaction \(13\) leads to a spin parameter which is slightly larger than the numerical one. Note also that our analytical predictions are computed for all values of \(\nu\) within the full range \(0 < \nu < 0.25\). This allows us to predict (e.g. by using the 3PN: Rad Reac \(\propto \Omega^{7/3}\) curve, or some combination of the two 3PN curves) the angular momentum of the final black hole for the range of values \(\nu < 0.16\), which has not been explored (yet) by numerical simulations (but in which the analytical approximation should be rather reliable). For the sake of comparison, we list in Table I our best (bracketing) 3PN+3.5PN numbers for the dimensionless spin parameter, together with a selected sample of the numerical relativity data. We mention, in passing, that had we estimated the spin parameter as the ratio \(\mathcal{P}_{\nu}^2 = \mathcal{P}_{LR}^2 / H^2_{LR}\) (i.e. neglecting the energy loss due to ringdown), and worked at the 3PN level with our (a priori preferred) radiation reaction \(\mathcal{F}_{\nu}\), we would have obtained values even closer to the numerical relativity ones: e.g. \(\hat{a}(0.25) \simeq 0.6804\), \(\hat{a}(0.1825) \simeq 0.5281\), and \(\hat{a}(0.1613) \simeq 0.4757\). This (probably partly accidental) agreement illustrates the fact that our current (approximate) analytical framework has already captured most of the correct physics, and that small variations in the implementation of the EOB approach can probably lead to an excellent agreement with numerical relativity results.

IV. CONCLUSIONS

In conclusion, we have improved previous analytical estimates, derived within the Effective One Body (EOB) framework, of the final spin of an unequal-mass coalescing black-hole binary. Our improvements consist in taking a more complete account of the most relevant physics: we have used higher PN accuracy, we considered several ways of modelling the radiation reaction during the plunge, and, most crucially, we took into account the angular momentum and energy lost to gravitational radiation during the ringdown. Our final results differ by less than 2% from the recent numerical relativity estimates of \(10\) (see Table I). This nice agreement shows, in our opinion, the ability of the EOB approach to capture, qualitatively and quantitatively, the essential physics of the plunge and merger of black-hole binaries.

We did not try here to further reduce the small remaining difference between analytical and numerical results\(^8\). Our goal here was mainly to exhibit, in a simple case study, how the inclusion of more and more physical effects in the EOB approach led to a nice, monotonic convergence towards a numerical relativity result. In separate investigations \(30, 32\) we shall illustrate how the EOB framework can also nicely converge towards the gravitational waveform. The aim of these studies is to understand which parts of the physics included in the EOB method must be more precisely modelled to yield accurate representations of the various physical observables of merging binary black holes. Indeed, the general philosophy of the EOB approach is that this (resummed perturbative) analytical framework contains several flexibility parameters which can be determined by fitting EOB predictions to some non-perturbative data, such as numerical relativity simulations, or, possibly, actual observational data. An example of a flexibility parameter is the coefficient \(a_5(\nu)\) parametrizing presently uncalculable 4PN (or higher) additional contributions \(+a_5(\nu)/r^5 + \ldots\) to the crucial “radial potential” \(A(r)\) in the effective metric Eq.\(41\). Ref. \(8\) exemplified how the parameter \(a_5\) (such that \(a_5(\nu) = a_5\nu\)) could be fitted to numerical data (for initial configurations).

We are aware of the rather coarse nature of the approximation used above for estimating the ringdown losses. In fact, the EOB approach itself provides a better, and more consistent, way of estimating these losses. Indeed, by matching, at \(r_{\text{match}} \simeq r_{\text{LR}}\), a post-Newtonian improved plunge waveform (\(\ell = 2, m = \pm 2\)) (from the EOB 3PN+3.5PN dynamics) to a superposition of quasinormal modes (QNMs) of a Kerr black hole of mass and angular momentum given by the energy and angular momentum of the relative dynamics at the \(\nu\)-deformed EOB light ring, we can analytically determine the amplitude of the ringdown waveform. Then, from this waveform we can analytically estimate, within the EOB approach, the ringdown losses. In view of the many delicate issues connected with this matching procedure (see \(30, 36, 37\)), we leave to future work a detailed discussion. Let us, however, quote some preliminary results that we have obtained. In the \(\nu = 1/4\) case, and considering, for simplicity, only the contribution of the quadrupole (\(\ell = 2, m = \pm 2\)) part of the waveform, this matching procedure gives \(\mathcal{F}_{\nu=1/4}^{\text{EOBmatched}} \simeq 0.0899\) for the angular momentum carried away by \(\ell = 2, m = \pm 2\) gravitational waves (GW) during ringdown. This happens to be in good agreement (relative difference \(\lesssim 4\%\)) with our naively scaled estimate \(\mathcal{F}_{\nu=1/4}^{\text{ringdown}} \simeq 0.0868\). However, this good agreement is partly accidental. Several complicated effects go here in various (probably compensating) directions: (i) the matching can under- or

\(^8\) Nor do we wish to conclude from Table I that \(\mathcal{F}_{\nu}^{K}\) is a more accurate expression for the radiation reaction than \(\mathcal{F}_{\nu}\). One needs to consider the effect of higher PN contributions before reaching any conclusion.
over-estimate the amplitude of the ringdown signal, (ii) the exact scaling with $\nu$ is not exactly proportional to $\nu^2$, (iii) in the $\nu = 1/4$ case, the GW signal is dominated by the quadrupole $\ell = 2, m = \pm 2$, (iv) for smaller values of $\nu$ (when the reflection symmetry is lost) the higher multipoles provide significant contributions, and, indeed, our scaling estimate was based on the small $\nu$ limit and included all the multipoles up to $\ell = 4$. [As an example of the importance of higher multipoles for smaller $\nu$'s let us mention that we indeed found, for $\nu = 0.1$ a quadrupole-only matched value of $p_{\mathrm{ringdown}}^{\mathrm{EOBmatched}}(0.1) \approx 0.00631$ which is roughly half the scaled value $p_{\mathrm{ringdown}}^\text{scaled}(0.1) \approx 0.0139$.

As a further remark, we point out that our analytical results above suggest that a simple quadratic expression $a_{\text{fit}}(\nu) = a_1 \nu + a_2 \nu^2$ should provide a reasonable fit to the data. And indeed, fitting the data of [10] with this function, provides a good fit of $a_1 = 3.27690$ and $a_2 = -2.11405$. Moreover, this fit is found to stay close to our best analytical predictions for the range $\nu < 0.16$ not covered by numerical simulations. For instance, For $\nu = 0.1$, this gives $a_{\text{fit}}(0.1) \approx 0.3066$, to be compared with, say, $a_1^{\text{analytical}}(0.1) \approx 0.3082$, while for $\nu = 0.01$, this gives $a_{\text{fit}}(0.01) \approx 0.0325$ to be compared with, say, $a_1^{\text{analytical}}(0.01) \approx 0.0341$. Note also that the analytically expected value in the $\nu \to 0$ limit is given by $a_1 = 0 + O(\nu^2) = 3.4641 \nu + O(\nu^2)$, where the analytical value $\sqrt{12}$ for the $a_1$ coefficient derives from the well known specific angular momentum of a test particle at the LSO. The slight difference between $a_{\text{analytical}}^{\text{analytical}}(0.1) = 3.46410$ and $a_{\text{fit}}(0.1) = 3.27690$ is probably due to the deviations from analyticity in $\nu$, as $\nu \to 0$, implied by the appearance of strange fractional powers of $\nu$ (integer powers of $\nu^{1/5}$) during the transition between the LSO and the plunge, see Ref. [1].

Let us finally mention that, though we focussed here on nonspinning binaries, we intend to study, within the EOB approach, the dimensionless spin parameter of spinning black-hole binaries. Indeed, not only is the function $a(m_1, m_2, S_1, S_2)$ a useful diagnostics for comparing analytical and numerical results, but it has also an important physical meaning. If the cosmic censorship conjecture is correct, this function should always stay smaller than 1, even if the individual spins take their maximum Kerr values $S_1 = m_1^2, S_2 = m_2^2$. Both leading-order EOB analytical results [7, 21, 22], and recent full-scale numerical [12] results have indicated that this is indeed the case. It would be, however, quite interesting to improve the EOB estimates beyond the leading order, and to compare them in detail with numerical results.

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but during the entire plunge + merger phase (post-LSO). As explained above, it is our use of the analytical EOB description which allows us to define precisely the “ringdown phase”.

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