Generic bi-Lyapunov stable homoclinic classes

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Abstract
We study, for $C^1$ generic diffeomorphisms, homoclinic classes which are Lyapunov stable both for backward and forward iterations. We prove that they must admit a dominated splitting and show that under some hypothesis they must be the whole manifold. As a consequence of our results we also prove that in dimension 2 the class must be the whole manifold and in dimension 3, these classes must have nonempty interior. Many results on Lyapunov stable homoclinic classes for $C^1$-generic diffeomorphisms are also deduced.

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1. Introduction

The study of the dynamics of chain recurrence classes and their interaction with other chain recurrence classes has become a major problem in $C^1$-generic dynamics since [BC], where generic dynamics and Conley’s theory (see section 10.1 of [R]) were unified. In particular, this has raised interest in the study of homoclinic classes, which are generically the chain recurrence classes containing periodic orbits.

So, an interesting problem is to study the possible structures and dynamics a homoclinic class may have, particularly those related to the global dynamics of the diffeomorphism. At the moment, very little is known related to this problem, especially when the class is wild, that is, accumulated by infinitely many distinct chain recurrence classes (isolated classes are now quite well understood, see [BDV, chapter 10]). In particular, very natural and simple questions remain wide open, such as if one of these homoclinic classes may have a nonempty interior and not be the whole manifold (the progress made so far has to do with [ABCD, ABD, PotS]).

On the other hand, a lot of work also has been done towards the understanding of generic dynamics far from homoclinic tangencies (see for example [PS, W1, C2, Y]), but also very few are known in their presence.
In this paper we deal with these kinds of difficulties in a simpler context than just wild homoclinic classes. We study homoclinic classes which are bi-Lyapunov stable, in particular, which are saturated by stable and unstable sets (for example, homoclinic classes with nonempty interior have this property). We are able, by using new techniques developed in [Gou3], to prove that these classes admit a dominated splitting, a weak form of hyperbolicity which helps to order the dynamics in a class (see [BDV, appendix B]). Also, we handle the case where the class is far from tangencies and we prove that if a generic diffeomorphism admits a homoclinic class which is bi-Lyapunov stable then the diffeomorphism must be transitive. Our results respond affirmatively to the second part of problem 5.1 in [ABD] and to the other part also in dimensions 2 and 3.

The main technical novelty of this paper relies on the use of Lyapunov stability together with a recent result from [Gou3] to control the bifurcation of periodic points.

Finally, we would like to add that many of our results treat the case of homoclinic classes which are only Lyapunov stable for \( f \) (and not for \( f^{-1} \)) and could have independent interest for other applications.

1.1. Context of the problems

Let \( M \) be a compact connected boundaryless manifold of dimension \( d \) and let \( \text{Diff}^1(M) \) be the set of diffeomorphisms of \( M \) endowed with the \( C^1 \) topology.

We shall say that a property is \( C^1 \)-generic, or just generic, if and only if there exists a residual set (\( G_\delta \)-dense) \( \mathcal{R} \) of \( \text{Diff}^1(M) \) such that every \( f \in \mathcal{R} \) satisfies that property. Some well-known results in the theory of generic dynamical systems are presented in appendix A and they will be referred to when they are used.

For a hyperbolic periodic point \( p \in M \) of some diffeomorphism \( f \) we denote its homoclinic class by \( H(p, f) \). It is defined as the closure of the transversal intersections between the stable and unstable manifolds of the orbit of \( p \). Homoclinic classes can also be defined in terms of homoclinically related periodic points by defining \( H(p, f) \) as the closure of the set of the homoclinically related periodic orbits. Both definitions coincide. We recall that two hyperbolic periodic orbits \( O, O' \) are said to be homoclinically related if \( W^s(O) \cap W^u(O') \neq \emptyset \) and \( W^s(O') \cap W^u(O) \neq \emptyset \).

It is a very important problem to study the structure of homoclinic classes since they are basic pieces of the dynamics (see [BDV, chapter 10]). In particular, a very natural question is whether they can admit nonempty interior. The only known examples are robustly transitive diffeomorphisms for which there is (generically) only one homoclinic class which coincides with the whole manifold ( [BDV, chapter 7]).

In [ABD] the following conjecture was posed (it also appeared as problem 1 in [BC]).

**Conjecture 1 (Abdenur et al [ABD]).** There exists a residual set \( \mathcal{R} \) of \( \text{Diff}^1(M) \) of diffeomorphisms such that if \( f \in \mathcal{R} \) admits a homoclinic class with nonempty interior, then the diffeomorphism is transitive.

Some progress has been made towards the proof of this conjecture (see [ABD, ABCD, PotS]); in particular, it has been proved in [ABD] that isolated homoclinic classes as well as homoclinic classes admitting a strong partially hyperbolic splitting verify the conjecture. Also, they proved that a homoclinic class with nonempty interior must admit a dominated splitting (see theorem 8 in [ABD]).

In [ABCD] the conjecture was proved for surface diffeomorphisms, other proofs for surfaces (which does not use the approximation by \( C^2 \) diffeomorphisms) can be found in [PotS] where the codimension one case is studied.
Also, from the work of Yang [Y] one can deduce the conjecture in the case \( f \) is \( C^1 \)-generic and far from homoclinic tangencies (we shall extend this remark in section 5).

When studying some facts about this conjecture, in [ABD] it was proved that if a homoclinic class of a \( C^1 \)-generic diffeomorphism has nonempty interior then this class should be bi-Lyapunov stable. In fact, in [ABD] they proved that isolated and strongly partially hyperbolic bi-Lyapunov stable homoclinic classes for generic diffeomorphisms are the whole manifold.

This concept is \textit{a priori} weaker than having a nonempty interior and it is natural to ask the following question.

**Question 1 (Problem 1 of [BC]).** Is a bi-Lyapunov stable homoclinic class of a generic diffeomorphism necessarily the whole manifold?

For generic diffeomorphisms, there exists a more general notion of basic pieces of the dynamics, namely, the chain recurrence classes \(^1\). It is not difficult to deduce from [BC] that, for generic diffeomorphisms, a chain recurrence class with a nonempty interior must be a homoclinic class; thus, the answer to conjecture 1 must be the same for chain recurrence classes and for homoclinic classes.

However, we know that question 1 admits a negative answer if posed for chain recurrence classes. Bonatti and Diaz constructed (see [BD]) open sets of diffeomorphisms in every manifold of dimension \( \geq 3 \) admitting, for generic diffeomorphisms there, uncountably many bi-Lyapunov stable chain recurrence classes which in turn have no periodic points.

Although this may suggest a negative answer for question 1 we present here some results suggesting an affirmative answer. In particular, we prove that the answer is affirmative for surface diffeomorphisms, and that in three-dimensional manifold diffeomorphisms the answer must be the same as for conjecture 1.

The main reason for which the techniques in [ABCD] (or in [PotS]) are not able to answer question 1 for surfaces is that differently from the case of homoclinic classes with interior, it is not so easy to prove that bi-Lyapunov stable classes admit a dominated splitting (in fact, the bi-Lyapunov stable chain recurrence classes constructed in [BD] do not admit any). Here we are able, by using new techniques introduced by Gourmelon in [Gou3], to show the existence of a dominated splitting for bi-Lyapunov stable homoclinic classes for generic diffeomorphisms. This will give us the result in dimension 2.

Also, using the same techniques, we are able to extend the results previously obtained in [PotS] to the context of question 1 which in turn allow us (combined with a new result of Yang [Y]) also to deduce an affirmative answer to the question in the far from tangencies context.

### 1.2. Definitions and statement of results

Let us first recall the definition of dominated splitting: a compact set \( H \), invariant under a diffeomorphism \( f \), admits a \textit{dominated splitting} if the tangent bundle over \( H \) splits into two \( Df \) invariant subbundles, \( T_H M = E \oplus F \), such that there exist \( C > 0 \) and \( 0 < \lambda < 1 \), satisfying that for all \( x \in H \):

\[
\|Df^n E(x)\|/\|Df^{-n} F(f^n(x))\| \leq C\lambda^n.
\]

\(^1\) The chain recurrent set is the set of points \( x \) satisfying that for every \( \epsilon > 0 \) there exists an \( \epsilon \)-pseudo orbit from \( x \) to \( x \), that is, there exist points \( x = x_0, x_1, \ldots, x_k = x \) with \( k \geq 1 \) such that \( d(f(x_i), x_{i+1}) < \epsilon \). Inside the chain recurrent set, the chain recurrence classes are the equivalence classes of the relation given by \( x \sim y \) when for every \( \epsilon > 0 \) there exists an \( \epsilon \)-pseudo orbit from \( x \) to \( y \) and one from \( y \) to \( x \) (see [R]). A compact invariant set \( K \) is chain transitive if \( K \) is a chain recurrence class for \( f|_K \).
In this case we say that the bundle $F$ dominates $E$. Let us note that Gourmelon ([Gou1]) proved that there always exists an adapted metric for which we can take $C = 1$ in the definition. Given a dominated splitting $T_H M = E \oplus F$ we say that $\dim E$ is the index of the dominated splitting. Similarly, for a periodic point $p$ of period $\pi(p)$, its index is the dimension of the eigenspace associated with the set of eigenvalues of modulus smaller than 1 for $Df^\pi(p)$. For a homoclinic class $H$, we define the minimal index of $H$ to the minimum of the indices of its periodic points (we define maximal index analogously).

One can have dominated splittings into more than two subbundles (see [BDV, appendix B]), in particular, a splitting of the form $T_H M = \bigoplus_{i=1}^n E^i$ is dominated if for every $j < k$, $E^k$ dominates $E^j$.

We shall say that a bundle $E$ is uniformly contracting (expanding) if there exists $n_0 > 0$ ($n_0 < 0$) such that $\|Df^n E(x)\| < 1/2 \forall x \in H$.

Recall that a chain recurrent class $H$ is Lyapunov stable\(^2\) for $f$ if for every neighbourhood $U$ of $H$ there is a $V$ neighbourhood of $H$ such that $f^n(V) \subset U \forall n \geq 0$; in particular, it is easy to see that Lyapunov stability implies that $W^s(x) \subset H$ for every $x \in H$. We shall say the class is bi-Lyapunov stable if it is Lyapunov stable for $f$ and $f^{-1}$.

Given a hyperbolic periodic point $p$ in a homoclinic class $H$ of a diffeomorphism $f$ we have that its continuation is well defined in a small neighbourhood $U$ of $f$ and we denote $p_g$ to the continuation for $g \in U$. Thus, we can also define the continuation of the homoclinic class $H$ for every $g \in U$ as $H_g = H(g, p_g)$ the homoclinic class for $g$ of $p_g$. It is well known that given a hyperbolic periodic point $p$ of a diffeomorphism $f$ and $U$ an open set containing $f$ where the continuation of $p$ is well defined, then, there exists a residual subset of $U$ where the map $g \mapsto H(g, p_g)$ is continuous in the Hausdorff topology. Also, we can assume that if $H$ is Lyapunov stable (or bi-Lyapunov stable) for $f$, then $H(g, p_g)$ will also be Lyapunov stable (respectively bi-Lyapunov stable) for every generic $g$ in $U$ (see appendix A).

**Theorem 1.1.** For every $f$ in a residual subset $R_1$ of $\Diff^1(M)$, if $H$ is a bi-Lyapunov stable homoclinic class for $f$, then, $H$ admits a dominated splitting. Moreover, it admits at least one dominated splitting with index equal to the index of some periodic point in the class.

This theorem solves affirmatively the second part of problem 5.1 in [ABD]. We note that theorem 1.1 does not imply that the class is not accumulated by sinks or sources. Also, we must note that the theorem is optimal in the following sense, in [BV] an example is constructed of a robustly transitive diffeomorphism (thus bi-Lyapunov stable) of $\mathbb{T}^3$ admitting only one dominated splitting (into two two-dimensional bundles) and with periodic points of all possible indices for saddles.

The theorem is in fact stronger, we in fact prove the following theorem which may be interesting for itself dealing with Lyapunov stable homoclinic classes.

**Theorem 1.2.** For every $f$ in a residual subset $R_1$ of $\Diff^1(M)$, if $H$ is a Lyapunov stable homoclinic class for $f$, and there is a periodic point $p \in H$ of period $\pi(p)$ such that $\det(Df^{\pi(p)}(p)) \leq 1$, then, $H$ admits a dominated splitting.

We recall now that a compact invariant set $H$ is strongly partially hyperbolic if it admits a three ways dominated splitting $T_H M = E^s \oplus E^c \oplus E^u$, where $E^s$ is nontrivial and uniformly contracting and $E^c$ is nontrivial and uniformly expanding.

In the context of question 1 it was shown in [ABD] that generic bi-Lyapunov stable homoclinic classes admitting a strongly partially hyperbolic splitting must be the whole

\(^2\) For $C^1$-generic diffeomorphisms, chain recurrence classes which are Lyapunov stable are also quasi-attractors. See [BC].
manifold. Thus, it is very important to study whether the extremal bundles of a dominated splitting must be uniform. We are able to prove this in the codimension one case, using only Lyapunov stability for future iterates.

**Theorem 1.3.** Let \( f \) be a diffeomorphism in a residual subset \( R_2 \) of \( \text{Diff}^1(M) \) with a homoclinic class \( H \) which is Lyapunov stable admitting a codimension one dominated splitting \( T_H M = E \oplus F \) where \( \dim(F) = 1 \). Then, the bundle \( F \) is uniformly expanding for \( f \).

This theorem is an extension of a related result from [PotS] (where conjecture 1 was studied) where the same result was proved in the case the class has a nonempty interior.

As a consequence of this theorem we get the following easy corollaries.

**Corollary 1.1.** Let \( H \) be a bi-Lyapunov stable homoclinic class for a \( C^1 \)-generic diffeomorphism \( f \) such that \( T_H M = E^1 \oplus E^2 \oplus E^3 \) is a dominated splitting for \( f \) and \( \dim(E^1) = \dim(E^3) = 1 \). Then, \( H \) is strongly partially hyperbolic and \( H = M \).

**Proof.** The class should be strongly partially hyperbolic because of the previous theorem. Corollary 1 of [ABD, p 185] implies that \( H = M \). \( \square \)

We say that a hyperbolic periodic point \( p \) is far from tangencies if there is a neighbourhood of \( f \) such that there are no homoclinic tangencies associated with the stable and unstable manifolds of the continuation of \( p \). The tangencies are of index \( i \) if they are associated with a periodic point of index \( i \), that is, its stable manifold has dimension \( i \). We get the following result following [ABCDW] (see also [W1] and [Gou2]):

**Corollary 1.2.** Let \( H \) be a bi-Lyapunov stable homoclinic class for a \( C^1 \)-generic diffeomorphism \( f \) which has a periodic point \( p \) of index 1 and a periodic point \( q \) of index \( d - 1 \) and such that \( p \) and \( q \) are far from tangencies. Then, \( H = M \).

**Proof.** Using corollary 3 of [ABCDW], we are in the hypothesis of corollary 1.1. \( \square \)

Using a result of Yang (theorem 3 of [Y]) and theorem 1.3 we are able to prove a similar result which is stronger than the previous corollary but which in turn has a hypothesis of a more global nature. We say that a diffeomorphism \( f \) is far from tangencies if it cannot be approximated by diffeomorphisms having homoclinic tangencies for some hyperbolic periodic point. Note that in the far from tangencies context, it is proved in [Y] that a Lyapunov stable chain recurrence class must be an homoclinic class.

**Proposition 1.1.** There exists a \( C^1 \)-residual subset of the open set of diffeomorphisms far from tangencies such that if \( H \) is a bi-Lyapunov stable chain recurrence class for such a diffeomorphism, then \( H = M \).

With our results we are able to deduce some stronger statements for bi-Lyapunov (and Lyapunov) stable homoclinic classes for generic diffeomorphisms in manifolds of low dimensions.

In [PotS], our results allowed us to deduce the conjecture in dimension 2 since a homoclinic class with a nonempty interior must admit a dominated splitting (this can be deduced from [BDP] in the case the homoclinic class has nonempty interior), hence, combining theorem 1.1 with theorem 1.3 we get the following theorem.

**Theorem 1.4.** Let \( f \) be a \( C^1 \)-generic surface diffeomorphism having a bi-Lyapunov stable homoclinic class. Then \( f \) is conjugated to a linear Anosov diffeomorphism in \( \mathbb{T}^2 \).
For the reader’s convenience, in section 3, we give a proof of theorem 1.4 which is independent of theorem 1.1. It contains the ideas of theorem 1.1 but fewer technicalities.

Theorem 1.2, together with the results of [PS] (see also [ABCD]), has the following interesting consequence in surface dynamics (we shall give more precise definitions, together with some direct corollaries in section 3):

**Theorem 1.5.** Let \( f \) be a \( C^1 \)-generic surface diffeomorphism having a Lyapunov stable homoclinic class \( H \) which has a dissipative periodic point. Then, \( H \) is a hyperbolic attractor.

Theorem 1.1 together with theorem 1.3 combine to give the following interesting result which supports the extension of the conjecture to bi-Lyapunov stable classes and solves completely problem 5.1 in [ABD] for three-dimensional manifolds.

**Proposition 1.2.** Let \( H \) be a bi-Lyapunov stable homoclinic class for a \( C^1 \)-generic diffeomorphism in dimension 3. Then, \( H \) has nonempty interior.

**Remark 1.1.** The proof of this proposition also gives us that if \( H \) is a bi-Lyapunov stable homoclinic class for a \( C^1 \)-generic diffeomorphism in any dimension, which admits a codimension one dominated splitting, \( T_H M = E \oplus F \) with \( \dim F = 1 \), and a periodic point \( p \) of index \( d − 1 \), then the class has a nonempty interior.

1.3. Idea of the proof

The idea of the proof of theorem 1.3 is the following.

First we prove that if the homoclinic class is bi-Lyapunov stable, the periodic points in the class (which are all saddles) should have eigenvalues (in the \( F \) direction) exponentially (with the period) far from 1. Otherwise we manage to obtain a sink or a source inside the class (by using an improved version of Franks’ lemma by Gourmelon [Gou3], which allows us to perturb the derivative controlling the invariant manifolds and also using Lyapunov stability) which is a contradiction. After this is done, one can conclude with the same arguments as in [PotS].

To get the dominated splitting in the class (theorem 1.1) we use a similar idea, if the class does not admit any dominated splitting, then, by [BDP] one can create a sink or a source by perturbation, so, using again Franks’ lemma with control of the invariant manifolds, we are able to ensure that the sink or source is contained in the class using also Lyapunov stability and thus reach a contradiction. A similar idea also gives information on the index of the dominated splitting.

To be able to use the improved version of Franks’ lemma given by Gourmelon, we shall use a recent result by Bochi and Bonatti [BB] which we present in section 4.

1.4. Organization of the paper

In section 2 we shall prove theorem 1.3 and in section 4 we shall prove theorem 1.1. The main argument used in the paper is presented with full details at the end of the proof of lemma 2.1. The other places where we use it will refer to it. In section 3 we prove the results in low dimensions, namely theorem 1.4 (with a proof independent of theorem 1.1), theorem 1.5 and proposition 1.2 using theorem 1.3.

We present the proof of proposition 1.1 based on a theorem of Yang [Y] in section 5.

In appendix A we recall some generic results which are very well known and shall be used in the course of the paper.
2. Proof of theorem 1.3

Let $\text{Per}(f)$ denote the set of periodic points of $f$. For $p \in \text{Per}(f)$, $\pi(p)$ denotes the period of $p$. We denote as $\text{Per}_\alpha(f)$ the set of index $\alpha$ periodic points. Let $\mathcal{O}$ be a periodic orbit and $E$ a $Df$ invariant subbundle of $T\mathcal{O}M$; $D\mathcal{O}f|E$ denotes the cocycle over the periodic orbit given by its derivative restricted to the invariant subbundle.

We shall make some definitions. Let $\mathcal{O}$ be a periodic orbit and $\mathcal{A}_O$ be a linear cocycle over $\mathcal{O}$. We say that $\mathcal{A}_O$ has a strong stable manifold of dimension $i$ if the eigenvalues $|\lambda_1| \leq |\lambda_2| \leq \cdots \leq |\lambda_d|$ of $\mathcal{A}_O$ satisfy that $|\lambda_i| < \min\{1, |\lambda_{i+1}|\}$. If the derivative of $\mathcal{O}$ has strong stable manifold of dimension $i$ then classical results ensure the existence of a local, invariant manifold tangent to the subspace generated by the eigenvectors of these $i$ eigenvalues and imitating the behaviour of the derivative (see [HPS]).

Let $\Gamma_i$ be the set of cocycles over $\mathcal{O}$ which have a strong stable manifold of dimension $i$.

We endow $\Gamma_i$ with the following distance, $d(\mathcal{A}_O, \mathcal{B}_O) = \max\{\|\mathcal{A}_O - \mathcal{B}_O\|, \|\mathcal{A}_O^{-1} - \mathcal{B}_O^{-1}\|\}$ where the norm is $\|\mathcal{A}_O\| = \sup_{p \in \mathcal{O}}\{\|\mathcal{A}_O(v)\| : v \in T_p\mathcal{O}\\setminus\{0\}\}$. Let $g$ be a perturbation of $f$ such that the cocycles $D\mathcal{O}f$ and $D\mathcal{O}g$ are both in $\Gamma_i$, and let $U$ be a neighbourhood of $\mathcal{O}$. We shall say that $g$ preserves locally the $i$-strong stable manifold of $f$ outside $U$, if the set of points of the $i$-strong stable manifold of $\mathcal{O}$ outside $U$ whose positive iterates do not leave $U$ once they entered it are the same for $f$ and for $g$.

We have the following theorem due to Gourmelon which will be the key tool for proving the results presented here.

**Theorem 2.1 (Gourmelon [Gou3]).** Let $f$ be a diffeomorphism, and $\mathcal{O}$ a periodic orbit of $f$ such that $D\mathcal{O}f \in \Gamma_i$, and let $\gamma : [0, 1] \rightarrow \Gamma_i$ be a path starting at $D\mathcal{O}f$. Then, given a neighbourhood $U$ of $\mathcal{O}$, there is a perturbation $g$ of $f$ such that $D\mathcal{O}g = \gamma(1)$, $g$ coincides with $f$ outside $U$ and preserves locally the $i$-strong stable manifold of $f$ outside $U$. Moreover, given $U$ a $C^1$ neighbourhood of $f$, there exists $\varepsilon > 0$ such that if $\text{diam}(\gamma) < \varepsilon$ one can choose $g \in U$.

We observe that Franks’ lemma (see [F2]) is the previous theorem with $i = 0$. Also, we note that Gourmelon’s result is more general since it allows us to preserve more than one stable and more than one unstable manifold (of different dimensions, see [Gou3]).

**Lemma 2.1.** Let $H$ be a homoclinic class which is Lyapunov stable of a $C^1$-generic diffeomorphism $f$ such that the class has only periodic orbits of index smaller or equal to $\alpha$. So, there exists $K_0 > 0$, $\lambda \in (0, 1)$ and $m_0 \in \mathbb{Z}$ such that for every $p \in \text{Per}_\alpha(f_M)$ of sufficiently large period one has

$$\prod_{j=0}^{k-1} \prod_{j=0}^{m_0-1} \|Df^{-1}_j| E^s(f^{-m_0-j}(p))\| < K_0 \lambda^k \quad \text{for } \left\lfloor \frac{\pi(p)}{m_0} \right\rfloor.$$

**Proof.** Let $\mathcal{R}$ be a residual subset of $\text{Diff}^1(M)$ such that if $f \in \mathcal{R}$ and $H$ is a Lyapunov stable homoclinic class of a periodic point $q$ of index $\alpha$, there exists a small neighbourhood $U$ of $f$ where the continuation $q_\theta$ of $q$ is well defined and such that for every $g \in U \cap \mathcal{R}$ one has that $H(g, q_\theta)$ is Lyapunov stable, and such that $g$ is a continuity point of the map $g \mapsto H(g, q_\theta)$. Also, being $f$ generic, we can assume that for every $g \in U \cap \mathcal{R}$ and every $p \in \text{Per}_\alpha(g) \cap H(g, q_\theta)$ we have that $H(g, q_\theta) = H(g, p)$, so, the orbits of $p$ and $q_\theta$ are

3 A linear cocycle $A$ of dimension $n$ over a transformation $f : \Sigma \rightarrow \Sigma$ is a map $A : \Sigma \rightarrow GL(n, \mathbb{R})$. When one point $p \in \Sigma$ is $f$-periodic, the eigenvalues of the cocycle at $p$ are the eigenvalues of the matrix given by $A_{f^n(p-1(p))} \cdots A_p$. See [BGV].
homoclinically related (see appendix A. We are using properties (a1), (a2), (a3), (a4) and (a5) of theorem A.1).

We can also assume that \( \mathcal{U} \) and \( \mathcal{R} \) were chosen so that for every \( g \in \mathcal{U} \cap \mathcal{R} \) every periodic point in \( H(g, q_g) \) has an index smaller or equal to \( \alpha \). We can also assume that \( q_g \) has index \( \alpha \) for every \( g \in \mathcal{U} \) (see property (a5) of theorem A.1).

Lemma II.5 of [M] asserts that to prove the thesis is enough to show that there exists \( \varepsilon > 0 \) such that the set of cocycles \( \Theta_a = \{ D_{\partial(p)} f I_{/\mathcal{E}} : p \in \text{Per}_a(f_{/\mathcal{H}}) \} \) which all have its eigenvalues of modulus bigger than one, verify that every \( \varepsilon \)-perturbation of them preserves this property. That is, given \( p \in \text{Per}_a(f_{/\mathcal{H}}) \) one has that every \( \varepsilon \)-perturbation \( \{ A_0, \ldots, A_{\pi(p)-1} \} \) of \( D_{\partial(p)} f \) verifies that \( A_{\pi(p)-1} \ldots A_0 \) has all its eigenvalues of modulus bigger than or equal to one.

Therefore, assuming by contradiction that the lemma is false, we get that \( \forall \varepsilon > 0 \) there exists a periodic point \( p \in \text{Per}_a(f_{/\mathcal{H}}) \) and a linear cocycle over \( p, \{ A_0, \ldots, A_{\pi(p)} \} \) satisfying that \( \| D_{f^{\pi(p)}} f I_{/\mathcal{E}} - A_i \| \leq \varepsilon \) and \( \| D_{f^{\pi(p)}} f I_{/\mathcal{E}} - A_{\pi(p)-1}^{-1} \| \leq \varepsilon \) and such that \( \prod_{i=0}^{\pi(p)-1} A_i \) has some eigenvalue of modulus smaller than or equal to 1.

In coordinates \( T_{\partial(p)} M = E^u \oplus (E^u)^\perp \), since \( E^u \) is invariant we have that the form of \( Df \) is given by

\[
D_{f^{\pi(p)}} f = \begin{pmatrix}
D_{f^{\pi(p)}} f I_{/\mathcal{E}} & K_1(f) \\
0 & K_2(f)
\end{pmatrix},
\]

Let \( \gamma : [0, 1] \to \Gamma_a \) given in coordinates \( T_{\partial(p)} M = E^u \oplus (E^u)^\perp \) by

\[
\gamma_t(t) = \begin{pmatrix}
(1-t)D_{f^{\pi(p)}} f I_{/\mathcal{E}} + t A_i & K_1(f) \\
0 & K_2(f)
\end{pmatrix}
\]

whose diameter is bounded by \( \varepsilon \) (see lemma 4.1 of [BDP]).

Now\(^4\), choose a point \( x \) of intersection between \( W^s(p, f) \) with \( W^u(q, f) \) and choose a neighbourhood \( U \) of the orbit of \( p \) such that:

(i) It does not intersect the orbit of \( q \).

(ii) It does not intersect the past orbit of \( x \).

(iii) It verifies that once the orbit of \( x \) enters \( U \) it stays there for all its future iterates by \( f \).

It is very easy to choose \( U \) satisfying (i) since both the orbit of \( p \) and the one from \( q \) are finite. Since the past orbit of \( x \) accumulates in \( q \) it is not difficult to choose \( U \) satisfying (ii). To satisfy (iii) one has only to use the fact that \( x \) belongs to the stable manifold of \( p \) so, after a finite number of iterates, \( x \) will stay in the local stable manifold of \( p \), and it is not difficult then to choose a neighbourhood \( U \) which satisfies (iii) also.

Applying theorem 2.1 we can perturb \( f \) to a new diffeomorphism \( \hat{g} \) so that the orbit of \( p \) has index greater than \( \alpha \) and so that it preserves locally its strong stable manifold. This allows us to ensure the intersection between \( W^u(q_g, \hat{g}) \) and \( W^s(p, \hat{g}) \).

This intersection is transversal so it persists by small perturbations. The same occurs with the index of \( p \) so we can assume that \( \hat{g} \) is in \( \mathcal{R} \cap \mathcal{U} \). Using Lyapunov stability of \( H(\hat{g}, q_{\hat{g}}) \) we obtain that \( p \in H(\hat{g}, q_{\hat{g}}) \). This is because a Lyapunov stable homoclinic class is saturated by unstable sets, so, since \( q_{\hat{g}} \subset H(\hat{g}, q_{\hat{g}}) \) we have that \( W^u(\hat{g}, q_{\hat{g}}) \subset H(\hat{g}, q_{\hat{g}}) \) and since \( W^s(p, \hat{g}) \cap W^u(\hat{g}, q_{\hat{g}}) \neq \emptyset \), we get that \( p \in W^u(\hat{g}, q_{\hat{g}}) \) and we get what we claimed.

This contradicts the choice of \( \mathcal{U} \) since we find a diffeomorphism in \( \mathcal{U} \cap \mathcal{R} \) with a periodic point with index bigger than \( \alpha \) in the continuation of \( H \), and so the lemma is proved. \( \square \)

\(^4\) The following argument will be used repeatedly along the paper.
Remark 2.1. One can recover lemma 2 of [PotS] in this context. In fact, if there is a codimension one dominated splitting of the form $T_H M = E \oplus F$ with $\dim F = 1$ then (using the adapted metric given by [Gou1]) for a periodic point of maximal index one has $\|Df^{-1}_{f(p)}\| \leq \|Df^{-1}_{E(p)}\|$ so,

$$\prod_{i=0}^{k} \prod_{j=0}^{m_{0-1}} Df^{-1}_{f^{-j}m_{0-1}(p)} = \lambda_k^{k} = \frac{\pi(p)}{m_0}.$$ 

And since $F$ is one-dimensional one has $\prod_{i=0}^{k} \|A_i\| = \|\prod_{i=0}^{k} A_i\|$ so $\|Df^{-\pi(p)}\| \leq K_0 \lambda^{\pi(p)}$ (maybe changing the constants $K_0$ and $\lambda$).

In fact, there is $\gamma \in (0, 1)$ such that for every periodic point of maximal index and big enough period one has $\|Df^{-\pi(p)}\| \leq \gamma \pi(p)$.

Also, it is not hard to see that if the class admits a dominated splitting of index bigger than or equal to the index of all the periodic points in the class, then, periodic points should be hyperbolic in the period along $F$ (for a precise definition and discussion on this topic one can read [BGY, W2]). ♦

Remark 2.2. As a consequence of the proof of the lemma we get that one can perturb the eigenvalues along an invariant subspace of a cocycle without altering the rest of the eigenvalues. The perturbation will be of similar size to the size of the perturbation in the invariant subspace. See lemma 4.1 of [BDP]. Note also that we could have perturbed the cocycle $\{K_2i (f)\}$ without altering the eigenvalues of the cocycle $D_{(p)} f_{E^*}$.

One can now conclude the proof of theorem 1.3 with the same techniques as in the proof of the main theorem of [PotS].

We have that $T_H M = E \oplus F$ with $\dim F = 1$. We first prove that the centre unstable curves tangent to $F$ should be unstable and with uniform size (this is lemma 3 of [PotS]). To do this, we first use lemma 2.1 to get this property in the periodic points and then use the results from [PS2, BC] to show that the property extends to the rest of the points. This dynamical properties imply also uniqueness of these central unstable curves.

Assuming the bundle $F$ is not uniformly expanded, one has two cases: one can apply Liao’s selecting lemma or not (see [W2]).

In the first case one gets weak periodic points inside the class, which contradicts the thesis of lemma 2.1. The second case is similar. If Liao’s selecting lemma does not apply, one gets a minimal set inside $H$ where $E$ is uniformly contracting and, using the dynamical properties of the centre unstable curves, classical arguments give a periodic point close to this minimal set. Since the stable manifold of this periodic point will be uniform, it will intersect the unstable manifold of a point in $H$, and then Lyapunov stability implies the point is inside the class and again contradicts lemma 2.1.

For more details see [PotS].

3. Low-dimensional consequences

We shall first prove theorem 1.4. We note that it is an easy corollary of theorems 1.1 and 1.3 since together they give hyperbolicity of the homoclinic class. However, this proof does not use theorem 1.1 and we believe it may illuminate the idea of the more general proof.

Theorem. Let $H$ be a bi-Lyapunov stable homoclinic class of a $C^1$-generic surface diffeomorphism $f$. Then, $H$ admits a dominated splitting and thus $H = T^2$ and $f$ is an Anosov diffeomorphism.
Proof. Consider a generic diffeomorphism \( f \) satisfying properties \((a1), (a2), (a3)\) and \((a4)\) of theorem A.1. Consider a periodic point \( q \in H \) fixed such that for a neighbourhood \( \mathcal{U} \) of \( f \) the class \( H(q, g) \) is bi-Lyapunov stable for every \( g \) in a residual subset of \( \mathcal{U} \).

If the class does not admit a dominated splitting, arguments from [M] allow us to deduce that after a small perturbation, for any \( \delta > 0 \) we have a periodic point \( p \) such that the angle between the stable and unstable bundle is smaller than \( \delta \). Let us assume, for example, that the determinant of \( Df^p\pi(p) \) is smaller than one.

By classical arguments (see for example lemma 7.7 of [BDV]) one knows that after composing \( D_O(p)f \) with a rotation of angle smaller than \( \delta \) one has that the resulting cocycle, \( A_O(p) \), has complex eigenvalues. These eigenvalues have modulus smaller than one (since the rotation has determinant equal to 1). Also, the same argument of the determinant implies that we can join \( D_O(p)f \) with \( A_O(p) \) by a curve with a small diameter and such that every cocycle in the curve has one eigenvalue with modulus smaller than one. In fact, by stopping the curve before getting both eigenvalues equal, one is assured to be in the hypothesis of theorem 2.1.

We can then apply theorem 2.1 to perturb the periodic point \( p \) preserving its strong stable manifold locally.

Arguing as in lemma 2.1 we can do the perturbation so that \( W^s(q, g) \) intersects \( W^u(p, g) \), and thus using Lyapunov stability, we obtain that \( p \in H(q, g) \), which is absurd since \( p \) is a sink.

The rest follows by applying theorem 1.3 which implies that \( H \) is hyperbolic, and thus, using local product structure and bi-Lyapunov stability, we get that \( H \) is open and closed, and thus \( H = M \). So, \( f \) is an Anosov diffeomorphism, and by Franks’ theorem [F1], it must be conjugated to a linear Anosov diffeomorphism of \( \mathbb{T}^2 \). □

Remark 3.1. Note that for theorem 1.4 we did not use the results of [PS] which involve \( C^2 \) approximations. In the above theorem, we have only used Lyapunov stability for \( f \), and so theorem 1.3 allows us only to prove hyperbolicity of one of the bundles. That is why we need to use this new generic property given by theorem 2 of [ABCD].

The last theorem has some immediate consequences which may have some interest on their own.

We say that an embedding \( f : \mathbb{D}^2 \to \mathbb{D}^2 \) is dissipative if for every \( x \in \mathbb{D}^2 \) we have that \(|\det(D_x f)| < b < 1\). Recall that for a dissipative embedding of the disc, the only hyperbolic attractors are the sinks [Ply].

Corollary 3.1. Let \( f : \mathbb{D}^2 \to \mathbb{D}^2 \) be a generic dissipative embedding. Then, every Lyapunov stable homoclinic class is a sink.
This gives, in particular, that for the well-known Henon map (see [BDV, chapter 4]), $C^1$-generic diffeomorphisms near, do not have strange attractors (note that they may have aperiodic quasi-attractors, which is not hard to show would be semiconjugated to adding machines, see [BDV, chapter 10]).

Now, we shall prove proposition 1.2. This gives that question 1 has the same answer as conjecture 1 for diffeomorphisms in three-dimensional manifolds. Compare with the results from [PotS].

**Proof of proposition 1.2.** Applying theorem 1.1 one can assume that the class $H$ admits a dominated splitting of the form $E \oplus F$, and without loss of generality one can assume that $\dim F = 1$.

Theorem 1.3 thus implies that $F$ is uniformly expanded so the splitting is $T_H M = E \oplus E^u$. Assume first that there exists a periodic point $p$ in $H$ of index 2. Thus, this periodic point has a local stable manifold of dimension 2 which is homeomorphic to a two-dimensional disc.

Since the class is Lyapunov stable for $f^{-1}$ the stable manifold of the periodic point is completely contained in the class.

Now, using Lyapunov stability for $f$ and the foliation by strong unstable manifolds given by [HPS] one gets (saturating by unstable sets the local stable manifold of $p$) that the homoclinic class contains an open set. This implies the thesis under this assumption.

So, we must show that if all the periodic points in the class have index 1 then the class is the whole manifold. As we have been doing, using the genericity of $f$ we can assume that there is a residual subset $R$ of $\text{Diff}^1(M)$ and an open set $U$ of $f$ such that for every $g \in U \cap R$ all the periodic points in the class have index 1.

We have two situations: on the one hand, we consider the case where $E$ admits two invariant subbundles, $E = E^1 \oplus E^2$, with a dominated splitting and thus, we get that $E^1$ should be uniformly contracting (using theorem 1.3) proving that the homoclinic class is the whole manifold (corollary 1.1).

If $E$ admits no invariant subbundles then, using theorem 4.1, we can perturb the derivative of a periodic point in the class, so that the cocycle over the periodic point restricted to $E$ has all its eigenvalues contracting. Hence, we can construct a periodic point of index 2 inside the class.

**Remark 3.2.** It is very easy to adapt the proof of this proposition to get that if a bi-Lyapunov stable homoclinic class of a generic diffeomorphism admits a codimension one dominated splitting, $T_H M = E \oplus F$ with $\dim F = 1$, and has a periodic point of index $d - 1$, then, the class has a nonempty interior.  

4. Existence of a dominated splitting

We prove here theorems 1.1 and 1.2 which state that a bi-Lyapunov stable homoclinic class of a generic diffeomorphism (or a Lyapunov stable homoclinic class with a dissipative periodic orbit) admits a dominated splitting.

The idea is the following: in case $H$ does not admit any dominated splitting we can perturb the derivative of some periodic point in order to convert it into a sink or a source with the techniques of [BDP, BGV]. We pretend to use theorem 2.1 to ensure that the stable or unstable manifold of a periodic point in the class intersects the unstable or stable set of the source or the sink, respectively, and reach a contradiction.

This technique generalizes to the case where the class does not admit a dominated splitting with index between the indices of the periodic points in the class since by using these kinds
of results we may construct either a periodic point of smaller index than those on the class or one of bigger index and manage to relate it to the points in the class using Lyapunov stability.

The section is divided into two: in the first part, we recall some of the notions defined in [BGV] and we state a recent result by Bochi and Bonatti (see\(^5\) [BB]) which will allow us to use theorem 2.1. We also give an idea of the proof since it is a key step in the proof.

In the second part, we prove theorems 1.1 and 1.2.

4.1. Perturbations of cocycles over small paths

The goal of this section is to prove theorem 4.1 which allows us to perturb linear cocycles without dominated splitting in order to use Gourmelon’s theorem 2.1.

Before we proceed with the statement of theorem 4.1 we shall give some definitions taken from [BGV] and others which we shall adapt to fit our needs.

Let

\[ A = (\Sigma, f, E, A) \]

be a large period linear cocycle\(^6\) of dimension \(d\) bounded by \(K\) over an infinite set \(\Sigma\), that is

- \(f : \Sigma \to \Sigma\) is a bijection such that all points in \(\Sigma\) are periodic and such that given \(n > 0\) there are only finitely many with period less than \(n\).
- \(E\) is a vector bundle over \(\Sigma\), that is, there is \(p : E \to \Sigma\) such that \(E_x = p^{-1}(x)\) is a vector space of dimension \(d\) endowed with a euclidian metric \(\langle \cdot, \cdot \rangle\).
- \(A : x \in \Sigma \mapsto A(x) \in GL(E_x, E_f(x))\) is such that \(\|A\| \leq K\) and \(\|A^{-1}\| \leq K\).

In general, we shall denote \(A^\ell x = A_{f^\ell-1}(x) \ldots A x\) where juxtaposition denotes the usual composition of linear transformations.

For every \(x \in \Sigma\) we denote by \(\pi(x)\) its period and \(M_x = A(x)\) which is a linear map in \(GL(E_x, E_x)\) (which allows us to study eigenvalues and eigenvectors).

For \(1 \leq j \leq d\)

\[ \sigma_j(x, A) = \frac{\log|\lambda_j|}{\pi(x)}, \]

where \(\lambda_1, \ldots, \lambda_d\) are the eigenvalues of \(M_x\) in increasing order of modulus. As usual, we call \(\sigma_j(x, A)\) the \(j\)th Lyapunov exponent of \(A\) at \(x\).

Given an \(f\)-invariant subset \(\Sigma' \subset \Sigma\), we can always restrict the cocycle to the invariant set defining the cocycle \(A|_{\Sigma'} = (f|_{\Sigma'}, \Sigma', E|_{\Sigma'}, A|_{\Sigma'})\).

We shall say that a subbundle \(F \subset E\) is invariant if \(\forall x \in \Sigma\) we have \(A_x(F_x) = F_{f(x)}\). When there is an invariant subbundle, we can write the cocycle in coordinates \(F \oplus F^\perp\) (note that \(F^\perp\) may not be invariant) as

\[
\begin{pmatrix}
A_F & C_F \\
0 & A[F]
\end{pmatrix},
\]

where \(C_F\) is uniformly bounded. This induces two new cocycles: \(A_F = (\Sigma, f, F, A|_F)\) on \(F\) (where \(A|_F\) is the restriction of \(A\) to \(F\)) and \(A[F] = (\Sigma, f, E[F], A[F])\) on \(E[F] \simeq F^\perp\) where \((A[F])_x \in GL((F_x)^\perp, (F_{f(x)})^\perp)\) is given by \(p_{f(x)}^2 \circ A_x\) where \(p_{f(x)}^2\) is the projection map from \(E\) to \(F^\perp\). Note that changing only \(A_F\) affects only the eigenvalues associated with \(F\) and changing only \(A[F]\) affects only the rest of the eigenvalues, recall remark 2.2. See section 4.1 of [BDP] for more discussions on this decomposition.

\(^5\) I would like to thank C Bonatti for providing me with a preliminary version of [BB].

\(^6\) Sometimes, we shall abuse notation and call it just cocycle.
If \( A \) has two invariant subbundles \( F \) and \( G \), we shall say that \( F \) is \( \ell \)-dominated by \( G \) (and denote it as \( F \prec_{\ell} G \)) on an invariant subset \( \Sigma' \subset \Sigma \) if for every \( x \in \Sigma' \) and for every vector \( v \in F_x \setminus \{0\}, w \in G_x \setminus \{0\} \) one has
\[
\frac{\|A^k_x(v)\|}{\|v\|} \leq \frac{1}{2} \frac{\|A^k_x(w)\|}{\|w\|}.
\]
We shall denote \( F \prec G \) when there exists \( \ell > 0 \) such that \( F \prec_{\ell} G \).

If there exists complementary invariant subbundles \( E = F \oplus G \) such that \( F \prec G \) on a subset \( \Sigma' \subset \Sigma \), we shall say that \( A \) admits a dominated splitting on \( \Sigma' \).

As in [BGV], we shall say that \( A \) is strictly without domination if it is satisfied that always that \( A \) admits a dominated splitting in a set \( \Sigma' \) it is satisfied that \( \Sigma' \) is finite.

Let \( \Gamma_\Sigma \) be the set of cocycles over the orbit of \( x \) with the distance (see also section 2)
\[
d(A_i, B_i) = \sup_{0 \leq i < \pi(x), x \in E \setminus \{0\}} \left\{ \frac{\|A^i_x(v) - B^i_x(v)\|}{\|v\|}, \frac{\|A^{-i}_x(v) - B^{-i}_x(v)\|}{\|v\|} \right\}
\]
and let \( \Gamma_\Sigma \) (or \( \Gamma_{\Sigma,0} \)) the set of bounded large period linear cocycles over \( \Sigma \). Given \( A \in \Gamma_\Sigma \) we denote as \( A_i \in \Gamma_i \) to the cocycle \( \{A_i, \ldots, A_{\pi^{-1}(x)}\} \).

We recall from section 2 that we say that the cocycle \( A_i \) has strong stable manifold of dimension \( i \) if \( \sigma^i(x, A_i) \subset \{0, \sigma^{i+1}(x, A_i)\} \).

For \( 0 \leq i \leq d \), let
\[
\Gamma_{\Sigma,i} = \{A \in \Gamma_\Sigma : \forall x \in \Sigma : A_i \text{ has strong stable manifold of dimension } i \}
\]
Following [BGV] we say that \( B \) is a perturbation of \( A \) (denoted by \( B \sim A \)) if for every \( \varepsilon > 0 \) the set of points \( x \in \Sigma \) such that \( B_i \) is not \( \varepsilon \)-close to the cocycle \( A_i \) is finite.

Similarly, we say that \( B \) is a path perturbation of \( A \) if for every \( \varepsilon > 0 \) one has that the set of points \( x \in \Sigma \) such that \( B_i \) is not a perturbation of \( A_i \) along a path of diameter \( \leq \varepsilon \) is finite. That is, there is a path \( \gamma' : [0, 1] \rightarrow \Gamma_\Sigma \) such that \( \gamma'(0) = A \) and \( \gamma'(1) = B \) such that \( \gamma'_i : [0, 1] \rightarrow \Gamma_i \) are continuous paths and given \( \varepsilon > 0 \) the set of \( x \) such that \( \gamma'_i([0, 1]) \) has diameter \( \geq \varepsilon \) is finite.

In general, we shall be concerned with path perturbations which preserve the dimension of the strong stable manifold, so we shall say that \( B \) is a path perturbation of index \( i \) of a cocycle \( A \in \Gamma_{\Sigma,i} \) iff \( B \) is a path perturbation of \( A \) and the whole path is contained in \( \Gamma_{\Sigma,i} \).

This induces a relation in \( \Gamma_{\Sigma,i} \) which we shall denote as \( \sim^*_i \).

We have that \( ~ \sim \sim^*_i \) are equivalence relations in \( \Gamma_\Sigma \) and \( \Gamma_{\Sigma,i} \), respectively, and clearly \( \sim_i \) is contained in \( ~ \).

The Lyapunov diameter of the cocycle \( A \) is defined as
\[
\delta(A) = \liminf_{\pi(x) \rightarrow \infty} [\sigma^d(x, A) - \sigma^1(x, A)].
\]
If \( A \in \Gamma_{\Sigma,i} \), we define \( \delta_{\min}(A) = \inf_{B \sim A} \delta(B) \). Similarly, we define \( \delta_{\min,i}(A) = \inf_{B \sim A} \delta(B) \) and it could be strictly bigger.

**Remark 4.1.** For any cocycle \( A \), it is easy to see that \( \delta_{\min,0}(A) = \delta_{\min}(A) \). It suffices to consider the path \( (1 - t)A + tB \) where \( B \) is a perturbation of \( A \) having the same determinant over any periodic orbit and verifying \( \delta(B) = \delta_{\min}(A) \) (see lemma 4.3 of [BGV] where it is shown that such a \( B \) exists). \( \diamond \)

We are now ready to state the following theorem which will allow us to use the results from [BGV] together with the improved version of Franks’ lemma given by Gourmelon. This theorem follows directly from the results of [BB] where a much more general result is proven. For completeness we shall make some comments on the proof which we shall omit in full generality.
Theorem 4.1 (Consequence of [BB]). Let \( A = (\Sigma, f, E, A) \) be a bounded large period linear cocycle of dimension \( d \). Assume that

- \( A \) is strictly without domination.
- \( A \in \Gamma_{\Sigma,i} \)
- For every \( x \in \Sigma \), we have \( |\det(M^A_t)| < 1 \) (that is, for all \( x \in \Sigma \) we have \( \sum_{j=1}^{d} \sigma_j(x, A) < 0 \)).

Then, \( \delta_{\min}(A) = 0 \). In particular, given \( \varepsilon > 0 \) there exists a point \( x \in \Sigma \) and a path \( \gamma_x \) of diameter smaller than \( \varepsilon \) such that \( \gamma_x(0) = A_x \), the matrix \( M^\gamma_x(1) \) has all its eigenvalues of modulus smaller than one, and such that \( \gamma(t) \in \Gamma_{\Sigma,i} \) for every \( t \in [0, 1] \).

It is well known since Mañé (see [M]) that this result holds in dimension 2, namely, a two-dimensional cocycle strictly without domination admits a path perturbation which decreases the Lyapunov diameter even without affecting at all the determinant. In particular, if the determinant starts being smaller than one, there is a path perturbation of index 1 which takes the Lyapunov diameter to zero (for any \( i = 0, 1, 2 \)).

In higher dimensions, a natural way to attack the problem is to use induction. A simple perturbation imitating proposition 3.7 of [BGV] allows us to obtain a diagonal cocycle (that is, for every \( x \in \Sigma \) all the Lyapunov exponents are different) which in turn gives a lot of one-dimensional invariant subbundles associated with each Lyapunov exponent. Let us call them \( E_1(x, A), \ldots, E_d(x, A) \).

This gives

Lemma 4.1. For every \( A \in \Gamma_{\Sigma,i} \), there exists \( B \sim^*_\gamma A \) such that for every \( x \in \Sigma \) the eigenvalues of \( M_B^x \) have all different modulus and their modulus is arbitrarily near the original one in \( M^A_x \), that is, \( |\sigma^B(x, A) - \sigma^A(x, B)| \to 0 \) as \( \pi(x) \to \infty \) (in particular, \( \delta(B) = \delta(A) \)).

When we wish to apply induction a problem appears; in general, an invariant subbundle of a strictly without domination cocycle need not be strictly without domination (which would allow us to use induction). However, with more care, one can make a more subtle perturbation (which will be necessarily global) in order to preserve the existence of this invariant subbundle as well as breaking the domination and then be able to use induction.

Let us sketch the proof of the theorem in the case of [BGV] (without need of path perturbations) for three-dimensional cocycles. First, we can assume that the cocycle is diagonal, that is, we have an invariant splitting of the form \( E_1 \oplus E_2 \oplus E_3 \) associated with the eigenvalues in increasing order of modulus. It is easy to show that we can consider the cocycle satisfying \( \delta(A) = \delta_{\min}(A) \). If \( E_1 \) is not dominated by \( E_2 \), with a small perturbation of \( A_{E_1 \oplus E_2} \) we can (using induction) reduce the Lyapunov diameter which contradicts \( \delta(A) = \delta_{\min}(A) \). Hence, we get that \( E_1 \) should be dominated by \( E_2 \). Since \( A \) is strictly without domination, we get that \( E_1 \| E_2 \) cannot be dominated by \( E_3 \| E_2 \), hence, again by a two-dimensional perturbation we can decrease the Lyapunov diameter and conclude.

Note that the last perturbation cannot be made if we wish to perform a small path perturbation of index 2, since the eigenvalues associated with \( E_3 \) and \( E_2 \) may cross in this process losing the dimension of the strong stable manifold.

To overcome this difficulty, we must first make a perturbation of \( A_{E_2 \oplus E_3} \) in order to break the domination between \( E_1 \) and \( E_2 \). This is achieved by the following proposition which we state without proof.

Proposition 4.1. Given \( K > 0, k > 0 \) and \( \varepsilon > 0 \), there exists \( N > 0 \) and \( \ell \) such that if

- There exists a diagonal cocycle \( A_{\ell} \) of dimension 2 and bounded by \( K \) over a periodic orbit of period \( \pi(x) > N \),


- There exists a unit vector $v \in E_1$ such that
\[
\frac{3}{2} \| A_{f^k(x)}^j E_1 \| \geq \frac{\| A_{f^k(x)}^j A_1^2 v \|}{\| A_1^2 v \|},
\]
then, there exists $B_n$ a path perturbation of $A_1$ of diameter less than $\varepsilon$ verifying that all along the path the cocycle has the same Lyapunov exponents and
\[
\| B_n^k \|_{E_1(x, [0, 1])} \leq 2 \| A_1^2 v \|.
\]

Note that the statement is quite general; in fact, some weaker statement could suffice. Roughly, the proposition states that if the direction $E_1$ associated with the smaller eigenvalue has some iterates where there is a greater contraction in some vector $v$, then, one can perturb the cocycle along the orbit in a path, without changing the eigenvalues and obtaining a contraction similar to that of $v$ in the new eigendirection for some iterates of the cocycle. Note that the important statement deals with the case where $v \notin E_1$ since if $v \notin E_1$ there is no need for making a perturbation. This allows us to break domination as we shall explain later.

First, we briefly explain the proof of this proposition: the hypothesis of the proposition tells us that we can send by a small perturbation along $\ell$ transformations that we can send the direction $v$ into the direction $E_1$. Since $E_2$ is more expanded than $E_1$ in the period, there will be some place where we can also send the direction of $E_1$ into the direction of $v$. This allows us to show that the perturbation makes the $E_1$ direction pass some iterates near the direction of $v$ and thus, inheriting its contraction.

To guarantee that the perturbation can be made in order to keep the eigenvalues one has to control the moment where the perturbation is made. Roughly, one has to show that one can take the direction $E_1$ into the $v$ direction sufficiently near the moment where the contraction takes place and that leaves time to ‘correct’ the Lyapunov exponents along the rest of the orbit.

Now, we shall explain how to break the domination by using this proposition. Consider a cocycle $A$ such that $F_j(A) = E_1(A) \oplus \cdots \oplus E_j(A)$ verifies that $F_j < E_{j+1}(A)$ but $F_j(A)$ is not dominated by $E_{j+1}(A) \oplus E_{j+2}(A)$. We shall indicate how one can use the previous proposition to find a new cocycle $B(A)$ which is a path perturbation of $A$ along a path that preserves all the Lyapunov exponents and verifies that $F_j(B)$ is not dominated by $E_{j+1}(B)$.

To perform this perturbation, we use the fact that since $F_j(A)$ is not dominated by $E_{j+1}(A) \oplus E_{j+2}(A)$ one can find points $x_0$ with periods converging to infinity such that there are vectors $v_0$ in $(E_{j+1} \oplus E_{j+2})_{x_0}$ which are more contracted along large periods of time than the vectors in $(F_j)_{x_0}$. This allows (by breaking this time of strong contraction into two) us to use proposition 4.1 to obtain a path perturbation $B$ which realizes this strong contraction along the bundle $(E_{j+1}(B))_{x_0}$ and thus breaking the domination as wanted.

After this is done, an induction argument permits to prove that if the subbundle $E_1(A) \oplus \cdots \oplus E_j(A)$ is strictly not dominated by the subbundle $E_{j+1}(A) \oplus \cdots \oplus E_{j}(A)$, then there exists a path perturbation $B$ of $A$ such that the exponents are constant along the path and such that for $B$ the bundle $E_j(B)$ is strictly nondominated by $E_{j+1}(B)$.

This argument allows us to translate the problem into a problem for two-dimensional cocycles which, as we said, is already known how to deal with. With these observations made, one can conclude the proof of theorem 4.1.

4.2. Proof of theorem 1.1 and 1.2

Let $H$ be a generic bi-Lyapunov stable homoclinic class. Let us assume that $H$ contains periodic points of index $\alpha$ and we consider $\Delta_0^\alpha \subset \{ p \in \text{Per}_\alpha(f/H) \}$ the set of index $\alpha$ and $\eta$-disipative periodic points in $H$ for some $\eta < 1$. 
It is enough to have one periodic point with determinant smaller than one to get that for some \( \eta \), the set \( \Delta_0^\eta \) will be dense in \( H \) (see lemma 1.10 of [BDP]). If no periodic point has a determinant smaller than one, we work with \( f^{-1} \) and use Lyapunov stability for \( f^{-1} \) (see property (a1) in appendix A).

Note that if \( H \) admits no dominated splitting, then neither does the cocycle of the derivatives over \( \Delta_0^\eta \). Then, we can assume that it is strictly without domination (maybe by considering an infinite subset). This implies that we can apply theorem 4.1 and there is a periodic point \( p \in \Delta_0^\eta \) which can be turned into a sink with a \( C^1 \) small perturbation done along a path contained in \( \Gamma_{\varepsilon,a} \) (which maintains or increases the index).

Now we are able to use theorem 2.1 and reach a contradiction. Consider a periodic point \( q \in \Delta_0^\eta \) fixed such that for a neighbourhood \( \mathcal{U} \) of \( f \) the class \( H(q_\varepsilon, g) \) is Lyapunov stable for every \( g \in \mathcal{U} \cap \mathcal{R} \) (recall the first paragraph of the proof of lemma 2.1 and appendix A).

Suppose the class does not admit any dominated splitting, then, we have a periodic point \( p \in \Delta_0^\eta \) such that \( f \) can be perturbed in an arbitrarily small neighbourhood of \( p \) to a sink for a diffeomorphism \( g \in \mathcal{U} \) (which we can assume is in \( \mathcal{R} \cap \mathcal{U} \) since sinks are persistent) and preserving locally the strong stable manifold of \( p \). So, we choose a neighbourhood of \( p \) such that it does not meet the orbit of \( q \) nor the past orbit of some intersection of its unstable manifold with the local stable manifold of \( p \) with the same argument as in lemma 2.1.

Thus, we get that \( W^u(q_\varepsilon, g) \cap W^s(p, g) \neq \emptyset \) and using Lyapunov stability we reach a contradiction since it implies that \( p \in H(q_\varepsilon) \) which is absurd since \( p \) is a sink.

The same argument would work if \( \Delta_0^\eta \) consisted of points which were \( \eta \)-dissipative for \( f^{-1} \), but we should have used Lyapunov stability for \( f^{-1} \) in that case.

Now we consider the finest dominated splitting of the class (that is, such that the subbundles admit no sub-dominated splitting, see [BDV, appendix B]) which we denote as \( T_H M = E_1 \oplus \cdots \oplus E_k \).

Assume that the indices of the homoclinic class form the segment \( [\alpha, \beta] \) (see property (a5) of theorem A.1) and that there exists \( l \) such that \( \sum_{j=1}^{l-1} \dim E_j < \alpha \) and that \( \sum_{j=1}^{l} \dim E_j > \beta \).

Then, theorem 4.1 allows us to conclude that there exists a periodic point of index \( \alpha \) such that an arbitrarily small perturbation of the cocycle \( Df|_{E_l} \) has all its eigenvalues of the same modulus (note that \( E_l \) varies continuously, hence, the cocycle \( Df|_{E_l} \) is continuous). Assume that, for example, the modulus is smaller than one. Then the previous argument also gives a contradiction by finding a periodic point in a perturbation of the class of index bigger or equal to \( \sum_{j=1}^{l} \dim E_j \). \( \square \)

**Remark 4.2.**

- Also the same ideas give that periodic points in the class must be volume hyperbolic in the period (not necessarily uniformly, see [BGY] for a discussion on the difference between hyperbolicity in the period and uniform hyperbolicity) for the extremal subbundles of the finest dominated splitting (see [BDV, appendix B] for definitions).

- Note that the same proof works for theorem 1.2. If a homoclinic class \( H \) of a generic diffeomorphism \( f \) is Lyapunov stable and the class has one periodic point \( p \) such that \( |\det(Df^p_\varepsilon(p))| \leq 1 \) then the class admits a dominated splitting. Moreover, this dominated splitting can be considered as having an index bigger than or equal to the minimal index of the class.

- In fact, we can assume that, if a Lyapunov stable homoclinic class admits no dominated splitting, then there exists \( \eta > 1 \) such that every periodic point \( p \), it has determinant bigger than \( \eta^{\varepsilon_1(p)} \). Otherwise, there would exist a subsequence \( p_n \) of periodic points with normalized determinant converging to 1. After composing with a small homothety, we are in the hypothesis of theorem 1.2. \( \diamond \)
5. Bi-Lyapunov stable homoclinic classes far from tangencies

We prove here that if \( f \) is a generic diffeomorphism far from homoclinic tangencies and admits a chain recurrence class which is bi-Lyapunov stable, then \( f \) must be transitive. First of all, one can reduce the study to homoclinic classes since in [Y] it is proved that Lyapunov stable chain recurrence classes are in fact homoclinic classes.

For this we shall use theorem 1.3 and a recent result of [Y]. See also chapter 9 of [C1].

First we state the following result due to Yang for generic Lyapunov stable homoclinic classes far from tangencies (we shall denote \( \text{Tang} \) as the set of diffeomorphisms which can be \( C^1 \) perturbed to get a homoclinic tangency between the stable and unstable manifold of a hyperbolic periodic point).

**Theorem 5.1 ([Y] Theorem 3).** Let \( f \in \mathcal{R} \), where \( \mathcal{R} \) is a residual subset of \( \text{Diff}^1(M) \setminus \text{Tang} \), and let \( H \) be a Lyapunov stable homoclinic class for \( f \) of minimal index \( \alpha \). Let \( T_H \mathcal{M} = E \oplus F \) be a dominated splitting for \( H \) with \( \dim E = \alpha \). Then, one of the following two options holds:

1. \( E \) is uniformly contracting.
2. \( E \) decomposes as \( E^s \oplus E^c \) where \( E^s \) is uniformly contracting and \( E^c \) is one dimensional and \( H \) is the Hausdorff limit of periodic orbits of index \( \alpha - 1 \).

With this theorem and theorem 1.3, proposition 1.1 follows easily. We must note that this theorem alone is enough to prove the same result for homoclinic classes with interior since they are not compatible with being accumulated in the Hausdorff topology with periodic points of index smaller than the class. Theorem 1.3 is necessary in the bi-Lyapunov stable case as will be clearly seen in the proof.

**Proof of proposition 1.1.** First of all, if the class has all its periodic points with index between \( \alpha \) and \( \beta \) we know that it admits a three ways dominated splitting of the form \( T_H \mathcal{M} = E \oplus G \oplus F \) where \( \dim E = \alpha \) and \( \dim F = d - \beta \). This is because we can apply the result of [W1] which says that far from homoclinic tangencies there is an index \( i \) dominated splitting over the closure of the index \( i \) periodic points together with the fact that index \( \alpha \) and \( \beta \) periodic points should be dense in the class since the diffeomorphism is generic (see theorem A.1).

Now we will show that \( H \) admits a strong partially hyperbolic splitting. If \( E \) is one dimensional, then it must be uniformly hyperbolic because of theorem 1.3. If not, suppose \( \dim E > 1 \). Then, if it is not uniform, theorem 5.1 implies that it can be decomposed as a uniform bundle together with a one-dimensional central bundle; since \( \dim E > 1 \) we get a uniform bundle of positive dimension.

The same argument applies for \( F \) using Lyapunov stability for \( f^{-1} \), so we get a strong partially hyperbolic splitting.

Corollary 1 of [ABD] completes the proof. \( \square \)

**Appendix A. Some generic properties**

The following theorem gives some well-known generic properties we shall be using in the proof of our results. The main reference is [C1] or [BDV] (and the references therein).

**Theorem A.1.** There exists a residual subset \( \mathcal{R} \) of \( \text{Diff}^1(M) \) such that if \( f \in \mathcal{R} \)

(a1) \( f \) is Kupka–Smale (that is, all its periodic points are hyperbolic and their invariant manifolds intersect transversally). Also, we can assume that the determinant of the derivative of every periodic point is different from 1.
(a2) The periodic points of \( f \) are dense in the chain recurrent set of \( f \). Moreover, if a chain recurrence class \( C \) contains a periodic point \( p \) then \( C = H(p) \).

(a3) Given \( p \in \text{Per}(f) \) there exists \( U_1 \) a neighbourhood of \( f \) such that for every \( g \in U_1 \cap \mathcal{R} \),
\[ g \text{ is a continuity point for the map } g \mapsto H(g, p_g) = H_g \text{ where } p_g \text{ is the continuation of } p \text{ for } g . \]
The continuity is with respect to the Hausdorff distance between compact subsets of \( M \).

(a4) If a homoclinic class \( H \) is Lyapunov stable for \( f \in \mathcal{R} \) then there exists \( U_2 \subset U_1 \) a neighbourhood of \( f \) such that \( H_g \) is Lyapunov stable for every \( g \in U_2 \cap \mathcal{R} \).

(a5) If \( f \in \mathcal{R} \) and \( H \) is a homoclinic class. There exists an interval \([\alpha, \beta]\) of natural numbers such that for every \( g \in U_3 \subset U_1 \), \( H_g \) has periodic points of every index in \([\alpha, \beta]\) and every periodic point in \( H_g \) has its index in that interval. Also, all periodic points of the same index are homoclinically related.

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Generic bi-Lyapunov stable homoclinic classes

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