FUSION POTENTIALS FOR $G_k$
AND HANDLE SQUASHING*

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Submitted to: Nuclear Physics B

* This work is supported in part by funds provided by the U.S. Department of Energy (D.O.E.) under contract #DE-AC02-76ER03069, and by the Division of Applied Mathematics of the U.S. Department of Energy under contract #DE-FG02-88ER25066.
ABSTRACT

Using Chern–Simons gauge theory, we show that the fusion ring of the conformal field theory $G_k$ ($G$ any Lie algebra) is isomorphic to $\frac{P_{\text{res}}}{(\nabla V)}$ where $V$ is a polynomial in $u$ and $(\nabla V)$ is the ideal generated by conditions $\nabla V = 0$. We explicitly construct $V$ for all $G_k$. We also derive a residue-like formula for the correlation functions in the Chern–Simons theory thus providing an RCFT version of the residue formula for the topological Landau–Ginzburg model. An operator that acts like a measure in this residue formula has the interpretation of a handle-squashing operator and explicit formulae for this operator are given.
I. INTRODUCTION

In the study of rational conformal field theories (RCFT), the fusion algebra plays a central role. For example, Gepner et al.\cite{gepner1} characterized the fusion rules of $G_k$ Wess–Zumino–Witten (WZW) models and Verlinde\cite{verlinde} displayed the connection between modular transformations and the fusion coefficients. Over the last few years there has been much progress in better understanding the fusion rules of $G_k$ theory.\cite{3,4,5,6,7,8} A good introduction to RCFT and the fusion algebra may be found in Ref.\ [9].

Recently, Gepner\cite{10} has conjectured that the fusion ring $R$ of any RCFT is isomorphic to $P[u]((\nabla V))$ where $P[u]$ is a polynomial ring in $u_i$ over $\mathbb{Z}$, the components of some finite-dimensional vector $u$, and $(\nabla V)$ is the ideal generated by the $\nabla V = 0$ where $V$ is a polynomial in the $u_i$’s. $V$ is called the fusion potential of the ring $R$. Gepner was able to show that this conjecture was true for the RCFT $SU(N)_k$. In this note we show this conjecture is true for an arbitrary $G_k$. Furthermore, the extension of these techniques to arbitrary coset models appears promising.\cite{11} Since there is yet no classification of RCFT’s this seems to be about as far as one can come at the present time to verifying the conjecture.

In a related and somewhat parallel development, Witten\cite{12} has shown that Chern–Simons gauge theory in three dimensions is closely connected with conformal field theory in two dimensions. This connection has been explored in many ways. Understanding the holomorphic quantization and the connection to the KZ equation\cite{13} has been studied in Refs. [14 – 16] and understanding the modular transformations of the Hilbert space was detailed in Refs. [17,8]. Also, understanding how to explicitly compute fusion algebra via Chern–Simons theory was studied in Ref. [8].
In this paper we use Chern–Simons field theory to explicitly construct the fusion potentials of $G_k$. We then explore some of the ideas of Ref. [10] in the context of Chern–Simons theory to derive an interesting formula for the correlation functions of Chern–Simons gauge theory. Perhaps not surprisingly, the correlators are given by a residue-like formula, reminiscent of the correlators of topological models (see Ref. [18–20,33]). This formula is interesting because it relates the measure in Gepner’s approach to the $K$ matrix$^{2,21,22}$ of a RCFT.

This paper is organized as follows. Section II is a short review of the canonical quantizations of Chern–Simons gauge theory for the case of three-fold $T^2 \times \mathbb{R}$ ($T^2$ is the torus). This technique will be used in Section III where we first describe, in general, how to compute the fusion potential for $G_k$ and then simply write them all down. Finally, Section IV describes correlation functions in Chern–Simons theory from Gepner’s point of view and it is there we encounter a connection between the measure in Gepner’s paper and the $K$ matrix of RCFT. Section V is a short conclusion.

II. CANONICAL QUANTIZATION OF THE CHERN–SIMONS THEORY ON $T^2 \times \mathbb{R}$

In Ref. [12] Witten identified the Hilbert space of Chern–Simons theory as the space of conformal blocks of the associated RCFT. We will use this idea to reduce the computation of the fusion algebra of a $G_k$ conformal field theory to a quantum mechanical computation on the Hilbert space of the associated $G_k$ Chern–Simons theory. In what follows we assume familiarity with Refs. [12,14,15,16,17,8] and use the conventions of Ref. [8] throughout.

To begin, let us recall that we wish to canonically quantize the action

$$I_{CS} = \frac{k}{4\pi} \int_{\mathcal{N}} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right).$$  (2.1)
A is the \( g \)-valued one-form on \( \mathcal{N} \), \( g \) the Lie algebra of \( G \), where \( \mathcal{N} \) will be taken to be the three-fold \( \Sigma \times \mathbb{R} \) (\( \Sigma \) is a two-dimensional [Riemann] surface), \( k \) is, as usual by gauge invariance, an integer referred to as the level and \( \text{Tr} \) is the symmetric bilinear form of the Lie algebra \( g \) normalized here so that in terms of the generators \( \tau^a \) of the \( G \) action \( \text{Tr} (\tau^a \tau^b) = 2\delta^{ab} \). This is the Chern–Simons theory related to the \( G_k \) RCFT. To canonically quantize this action we must first fix a time direction and choose a gauge. We choose the time axis to be along the \( \mathbb{R} \) in \( \mathcal{N} \) and choose the axial gauge \( A_t = 0 \). The local coordinates on \( \Sigma \) are \( x_1 \) and \( x_2 \). Then, as discussed in the above references, we find that this gauge choice implies the superselection rule (in analogy with electromagnetism it is called the Gauss’ law constraint)

\[
F_{12} = \partial_1 A_2 - \partial_2 A_1 + [A_1, A_2] = 0
\]  

(2.2)

and also the action in this gauge (with \( \partial\Sigma = 0 \)) is

\[
I_{\text{CS}} = \frac{k}{2\pi} \int_{\mathcal{N}} d^3 x \text{Tr} (A_1 \partial_t A_2).
\]  

(2.3)

Finally the observables of the theory are simply Wilson lines around non-trivial one-cycles in \( \Sigma \),

\[
O_{\mu, c} = \text{Tr}_{\mu} \left( P e^{\int_c A} \right)
\]  

(2.4)

where \( c \) is a one-cycle corresponding to some element of \( \pi_1(\Sigma) \), \( \mu \) is a representation of \( g \) in which the \( \text{Tr} \) is to be taken.

We are not yet ready to quantize the action Eq. (2.3) because we need to include factors that come from the measure of the path integral. As shown for example in Ref. [12,15] one can very simply include these factors. They lead to an additional term in the Lagrangian and are proportional to the original action Eq. (2.1). Indeed when combined with Eq. (2.1) they
result in simply shifting \( k \) to \( k + c \) where \( c \) is the quadratic Casimir of \( g \). Thus again choosing gauge \( A_t = 0 \) and proceeding as before we see that the action we wish to quantize is simply that of Eq. (2.3) with \( k \) replaced with \( k + c \).

For the remainder of the paper we specialize to the case \( \Sigma = T^2 \). It is convenient to first satisfy the constraint Eq. (2.2) classically and quantize the remaining degrees of freedom. Note also that the gauge choice \( A_t = 0 \) does not fix the gauge completely: We may use a time independent and single-valued gauge transformation to make the gauge field on \( T^2 \) a constant vector field. Then, classically, the constraint Eq. (2.2) implies that the two components of the vector field must commute in \( g \), and so, in general, both are in the Cartan subalgebra of \( g \). As such define

\[
\int_{c_1} A = a_i \nu^i \\
\int_{c_2} A = b_i \nu^i
\]

where \( c_1, c_2 \) are the cycles associated to the two generators of \( \pi_1 (T^2) \), \( \nu^i \) are simple roots of \( g \). We may now plug Eq. (2.5) into Eq. (2.3) with \( k \) shifted to \( k + c \) and using \((c_1, c_1) = 0 = (c_2, c_2), (c_1, c_2) = 1\) we have

\[
I_{CS} = -\frac{k + c}{2\pi} \int dt a_i C^{ij} \partial_t b_j
\]

where \( C^{ij} = \text{Tr} (\nu^i \nu^j) \). We have now reduced Chern–Simons theory in \( N \) to just a quantum mechanics problem where the \( a_i \) and \( b_j \) are just the coordinates and the momenta. The commutator in terms of \( a_i, b_j \) is,

\[
[a_i, b_j] = -\frac{2\pi i}{k + c} (C^{-1})_{ij} \\
[a_i, a_j] = 0 = [b_i, b_j].
\]

Rather than represent these operators on a Hilbert space we find a more convenient set of operators is suggested by the observables of the theory Eq. (2.4). Whatever representation \( \mu \)
is chosen, $\mathcal{O}_{\mu c}$ will involve only exponentials of the $a_i$’s and $b_i$’s. It is thus natural to find a
Hilbert space realization of

$$A_j = e^{ia_j}, \quad B_j = e^{ib_j}$$

(2.8)

and so Eq. (2.7) implies

$$A_i B_j A_i^{-1} B_j^{-1} = e^{\frac{2\pi i}{C} (C^{-1})_{ij}}$$

(2.9)

$$A_i A_j = A_j A_i, \quad B_i B_j = B_j B_i.$$  

This is analogous to a Weyl basis for the CCR. We now simply define a vacuum $|0\rangle$ by

$$A_i |0\rangle = |0\rangle \forall i \quad \text{(note A, B are unitary operators),}$$

and from Eq. (2.9) use the $B_i$’s as raising operators to generate the Fock space of states. It is easy to see that due to the fact that the commutator in Eq. (2.9) is idempotent, the spectrum of eigenvalues of the $A_i$’s will repeat after some number of applications of the raising operators $B_i$’s. We may thus consistently truncate this Fock space to a finite-dimensional Hilbert space, which we call the “Hilbert space of the Gaussian model” because of the strong resemblance of Eq. (2.9) with those of a system of $n$ free bosons.

This resulting finite-dimensional Hilbert space is not yet to be identified with the space of conformal blocks: the operators $A_i$ are not invariant under the residual gauge invariance associated with Weyl transformations. As shown in Ref. [8], under the Weyl action the Gaussian Hilbert space described above breaks into Weyl covariant subspaces. Finally, implementing this remaining gauge invariance we project all the operators of the theory onto the completely Weyl-odd sector.* It is natural from the point of the characters$^{23,24,17}$ that one should identify this completely Weyl-odd sector with the conformal blocks. For more details see Ref. [8].

* We mean states $\psi$ s.t. $\forall \omega \in W$ ($W$ is the Weyl group) with $\omega^2 = 1$, that $\omega \psi = -\psi$.  

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In the basis where \( A_i |0\rangle = |0\rangle \) \( \forall i \) we have finally that the \( O_{\mu, c_1} \) are diagonal on these completely Weyl-odd states. Furthermore, because \( O_{\mu, c_2} \) is composed of raising operators and is, by definition, even under all \( \omega \in W \) (\( W \) is the Weyl group) it is easy to see that \( O_{\mu, c_2} \) is a map from completely Weyl-odd states to themselves. It is not hard to show that these maps are precisely the fusion matrices. In notation, let \( \mathcal{H} \) denote the Hilbert space of states corresponding to the conformal blocks. They are labelled by representations since

\[
\psi_0 \in \mathcal{H} \quad \text{and} \quad \psi_\mu = O_{\mu, c_2} \psi_0 .
\]

one has

\[
O_{\mu, c_2} \psi_\nu = N_{\mu\nu}^\tau \psi_\tau
\]

where \( N_{\mu\nu}^\tau \) are the fusion coefficients. The proof of Eq. (2.11) is found by comparing the manipulations above to Refs. [3,7]. Note the \( O_{\mu, c_1} \) are just the diagonalization of the fusion matrices \( O_{\mu, c_2} \) under \( S \);

\[
O_{\mu, c_1} = S O_{\mu, c_2} S^{-1} .
\]

This is the Verlinde theorem.\(^2\)\(^{,26}\) For more details on the method described here see Ref. [8].

III. FUSION POTENTIALS

Having described how one can compute the fusion rules of a \( G_k \) RCFT using Chern–Simons theory, we will now proceed to derive the fusion potential of any \( G_k \).

It is worth mentioning that there are other ways to characterize the fusion algebra, such as finding the generating functions first introduced in Refs. [27,28]. However, in this note we will follow the spirit of Ref. [10] where it is conjectured that the fusion ring of any RCFT is isomorphic to \( \frac{P(u)}{(\nabla V)} \) where \( V(u) \) a polynomial is called the fusion potential.
We will show that this is true for the conformal field theories $G_k$. Our strategy is as follows: We first show that there is a potential for the Gaussian model (described in Section II). We then demonstrate that the fusion rules of $G_k$ are realized on a subvariety of the variety $\mathcal{M}$ defined by the $\nabla_x V = 0$ conditions of the Gaussian model, and show how the Weyl action naturally removes from $\mathcal{M}$ all the points except those on the subvariety. Many of the ideas in this section come from Ref. [10]. Although the Gaussian model of Eq. (2.9) is something like a “free field” decomposition of the theory (in that it has such a strong resemblance to a system of $n$ free bosons) this author sees no firm connection between this approach and that of Ref. [29]. As the reader will see below, the idempotency and “free-field” Gestalt of Eq. (2.9) are the key notions that allow one to integrate the fusion rules to a single potential.

Before displaying the potentials for $G_k$, we pause to more explicitly describe the method. Imagine assigning to each $A_i$ operator of Eq. (2.8) a complex variable $x_i$. We will find a potential $V(x)$ such that the ring of fusions in the Gaussian model, given by Eq. (2.9), will be given by $\rho(x)$ with $\nabla_x V = 0$. In accordance with the general ideas of Gepner, the solutions $x^{(i)}$ of $\nabla_x V = 0$ will be an affine variety $\mathcal{M}$ whose points will be in a 1−1 correspondence with the states $|\vec{\ell}\rangle$ in the Hilbert space of the Gaussian model, the 1−1 map being given by

$$A_i |\vec{\ell}\rangle = x_i^{(i)} |\vec{\ell}\rangle .$$  

(3.1)

We thus see that the variety $\nabla_x V = 0$ is essentially isomorphic to $\Lambda_W/(k + c)\Lambda_R$. We wish to ultimately only discuss the ring of fusions of Weyl-even operators, like the Wilson line of Eq. (2.4). Thus consider the map

$$x_i \rightarrow u_i(x)$$  

(3.2)
where $u_i$ are invariant under the action of the Weyl group (under the Weyl group, take the $x_i$’s to transform as the $A_i$’s) and are taken to be the defining representations of $g$. Obviously, $P[u]$ contains all the operators of Eq. (2.4). We write the potential of the Gaussian model as a function of the $u_i$’s. Then the ring $R = \frac{P[u]}{(\nabla_u V)}$ is the fusion ring of $G_k$. We can show this as follows. Since

$$\nabla_x V = \left[ \frac{\partial u}{\partial x} \right] \nabla_u V = 0$$

on $\mathcal{M}$ we ask what subvariety is picked out by $\nabla_u V = 0$. It is simply $\mathcal{M}$ minus the points for which $\det \left[ \frac{\partial u}{\partial x} \right] = 0$. Now it is simple to show that $\det \left[ \frac{\partial u}{\partial x} \right]$ is completely Weyl-odd and that it is, up to some trivial factors, the vacuum state (as viewed as an operator) of the conformal field theory $G_k$

$$\psi_0 \propto \det \left[ \frac{\partial u}{\partial x} \right]_{x_i = B_i} |0\rangle \ , \quad (3.3)$$

Thus $\det \left[ \frac{\partial u}{\partial x} \right] = 0$ at precisely those points on $\mathcal{M}$ that correspond to Weyl orbits of length less than $|W|$, the order of the Weyl group. For those points of $\mathcal{M}$ on the Weyl orbits that have length $|W|$ we see that the map Eq. (3.2) maps all points in that orbit to the same point. Thus $\nabla_u V = 0$ corresponds to a subvariety with a point for each integral representation of $G_k$. This may be readily seen by comparing the above construction to the construction of the space of conformal blocks for $G_k$ described in Ref. [8]. Indeed, remembering that the $O_{\mu,c1}|_{A_i = x_i} \in P[u]$ and writing Eq. (2.11) on the variety $\mathcal{M}$:

$$O_{\mu,c1} O_{\nu,c1} S \psi_0 = N_{\mu\nu}^{\tau} O_{\tau,c1} S \psi_0$$

we see that this equation only gives a condition on the product of polynomials in $P[u]$ when $\psi_0 \neq 0$ which, as described above (see Eq. (3.3)), is precisely at the points $\nabla_u V = 0$. This shows that $R$, the fusion ring of $G_k$, is given by by

$$R = \frac{P[u]}{(\nabla_u V)} \ .$$

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We next construct the potentials $V$ for $G_k$. As described above it will be enough to compute the $V(x)$ of the Gaussian model. The $x_i$’s are the natural variables to write the potential in and are just the $q_i$’s of Gepner\textsuperscript{10} in the case $G = SU(N)$. For clarity of exposition we divide $G$ into two classes; $G$ simply-laced and $G$ non-simply-laced.

$G$ Simply-Laced

If $G$ is simply-laced then the matrix $C^{ij}$ of Eq. (2.6) is just the Cartan matrix. We recall that in a Gaussian fusion ring the “vanishing” conditions are just statements of the idempotency of the operators, for example in $U(1)_k$, $A^{2k} = \mathbb{1} = B^{2k}$. (Recall that we have convention that $\text{Tr}(\tau^a \tau^b) = 2 \delta^{ab}$.) It is not difficult to recognize that idempotency of the operators in Eq. (2.8) and Eq. (2.9) implies the “vanishing” conditions (and thereby the fusion rules) of the $G_k$ theory. We study idempotency in the Gaussian model of Eq. (2.9) in the following way: We find a list of $n(= \text{Rank } G)$ linearly independent vectors $\vec{r}_i$ of $\mathbb{Z}^n$ with smallest integer components such that for each $\vec{r}_i$, $\prod_j A^{r_{ij}} = \mathbb{1}$ ($\mathbb{1}$ in the Hilbert space of the Gaussian model). Since $A_\ell |0\rangle = |0\rangle \ \forall \ell = 1, \ldots , n$ it is enough (by Schur’s lemma) to require $\prod_j A^{r_{ij}}$ commutes with all other operators of the theory, namely we require

$$\left[ \prod_j A^{r_{ij}} , B_\ell \right] = 0 \ \forall \ell, i .$$

For simply-laced $G$ it is very easy to characterize this set of vectors. By virtue of Eq. (2.9) a possible set of the $\vec{r}_i$ is just given by the rows of the Cartan matrix $A^i_j$ multiplied by $k + c$,

$$r_{ji} = (k + c) A^i_j . \quad (3.4)$$
Now returning to the notion of the vanishing conditions as specifying a variety we wish to solve the $n$ simultaneous conditions

$$\prod_{j}^{n} x_{j}^{r_{ij}} = 1 \quad \forall i = 1, \ldots, n.$$  

This may be easily done and here we simply write down the potential whose gradient $\nabla_{x} V = 0$ are the conditions above.* Details of the particular case of $SU(N)$ is in Appendix A, added as an aid to the reader.

$(A_{N-1})_{k} = SU(N)_{k}$  

(Rank $= N - 1$). The fusion potential is

$$V = \frac{x_{1}^{N(k+N)+1}}{N(k+N)+1} - x_{1} + \sum_{i=2}^{N-1} \left( \frac{\alpha_{i}^{(k+N+1)}}{k+N+1} - \alpha_{i} \right)$$  

(3.5)

in which the $x_{j}$’s (that correspond to eigenvalues of the $A_{j}$’s of Eq. (2.8)) are given by $x_{j} = \alpha_{j} x_{1}^{i}$, $2 \leq j < N - 1$. (Note that the Jacobian in going from $x_{i}$’s to $\alpha_{i}$’s is always non-zero.)

$(D_{\ell})_{k}$  

Let $\kappa = k + c$. We find that one must distinguish the two cases $\ell =$even and $\ell =$odd. Thus we find,

$$V = \frac{x_{1}^{R\kappa+1}}{R\kappa+1} - x_{1} + \frac{x_{2}^{R\kappa+1}}{R\kappa+1} - x_{2} + \sum_{j=3}^{\ell-2} \left( \frac{\alpha_{j}^{\kappa+1}}{\kappa+1} - \alpha_{j} \right)$$  

(3.6)

where $R = 2$ is $\ell$ is even and $R = 4$ is $\ell$ is odd. The other $x_{j}$, $3 < j \leq \ell$ are:

$$x_{j} = \alpha_{j} x_{1}^{-(j-2)\text{mod } R} x_{2}^{j\text{mod } R}.$$

(3.7)

* Care must be taken when solving these equations not to remove points or introduce additional images of the variety. See Appendix A.
Again let $\kappa = k + c$. The potential is,

$$V = \frac{x_1^{\kappa+1}}{\kappa + 1} - x_1 + \frac{x_2^{2\kappa+1}}{3\kappa + 1} - x_2 + \sum_{j=3}^{6} \left( \frac{\alpha_j^{\kappa+1}}{\kappa + 1} - \alpha_j \right)$$

(3.8)

where the $x_j \ 3 \leq j \leq 6$ are

$$x_3 = \alpha_3 x_2^2 \quad x_5 = \alpha_5 x_2$$

$$x_4 = \alpha_4 \quad x_6 = \alpha_6 x_2^2$$

(3.9)

$(E_7)_K$

Let $\kappa = k + c$. The fusion potential for $(E_7)_k$ is

$$V = \frac{x_1^{2\kappa+1}}{2\kappa + 1} - x_1 + \sum_{j=2}^{7} \left( \frac{\alpha_j^{\kappa+1}}{\kappa + 1} - \alpha_j \right)$$

(3.10)

where the $x_j \ 2 \leq j \leq 7$ are

$$x_2 = \alpha_2 \quad x_5 = \alpha_5 x_1$$

$$x_3 = \alpha_3 \quad x_6 = \alpha_6$$

$$x_4 = \alpha_4 \quad x_7 = \alpha_7 x_1$$

(3.11)

$(E_8)_k$

We let $\kappa = k + c$. The fusion potential is,

$$V = \frac{x_1^{2\kappa+1}}{2\kappa + 1} - x_1 + \sum_{j=2}^{8} \left( \frac{\alpha_j^{\kappa+1}}{\kappa + 1} - \alpha_j \right)$$

(3.12)

where the $x_j \ 2 \leq j \leq 8$ are

$$x_2 = \alpha_2 \quad x_4 = \alpha_4 \quad x_6 = \alpha_6 \quad x_8 = \alpha_8$$

$$x_3 = \alpha_3 \quad x_5 = \alpha_5 x_1 \quad x_7 = \alpha_7 x_1$$

(3.13)
**G Non-Simply-Laced**

If $G$ is non-simply-laced then the matrix $C^{ij}$ of Eq. (2.6) will not be the Cartan matrix. However, the “vanishing” conditions are still a result of the idempotency of the $A_i$’s and $B_j$’s of Eq. (2.8) and Eq. (2.9) and one may modify the argument for the simply-laced case. Now the $\vec{r}_i$ vectors will correspond to rows in the matrix $C^{ij}$.

$$ r_{ji} = (k + c)C^{ji}. $$  \hspace{1cm} (3.14)

We will call $k + c = \kappa$ throughout. We will now write down the $C^{ij}$ matrix for each non-simply-laced group and also write down the corresponding potential for $G_k$.

**(B_\ell)_k \hspace{1cm} (\ell \geq 2)**

The $C^{ij}$ matrix is $(\ell \times \ell)$,

$$
\begin{bmatrix}
4 & -2 \\
-2 & 4 & -2 \\
& & -2 &\ddots \\
& & & 4 & -2 \\
& & & -2 & 2 \\
\end{bmatrix}
$$  \hspace{1cm} (3.15)

and the corresponding potential is,

$$ V = \frac{x_1^{2(\ell - 1)\kappa+1}}{2(\ell - 1)\kappa + 1} - x_1 + \sum_{i=2}^{\ell} \left( \frac{\alpha_j^{2\kappa+1}}{2\kappa + 1} - \alpha_j \right) $$  \hspace{1cm} (3.16)

where the $x_j$, $2 \leq j \leq \ell$ are given as $x_j = x_1^j$. 

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(\(C_k\)) \( (\ell \geq 3)\)

The \(C^{ij}\) matrix is \((\ell \times \ell)\),

\[
\begin{bmatrix}
  2 & -1 & & & \\
  -1 & 2 & -1 & & \\
  & -1 & \ddots & \ddots & \\
  & & 2 & -2 & \\
  & & -2 & 4 & \\
\end{bmatrix} .
\]  

(3.17)

In writing the potential down it is convenient to distinguish the cases \(\ell\) even and \(\ell\) odd. We find:

\(\ell\) even: \( V = x_1^{2\kappa+1} 2\kappa + 1 - x_1 + x_2^{2\kappa+1} 2\kappa + 1 - x_\ell + \sum_{j=2}^{\ell-1} \left( \frac{\alpha_j^{\kappa+1}}{\kappa + 1} - \alpha_j \right) \)  

where for \(2 \leq j \leq \ell - 1\), \(x_j = \alpha x_1^j \mod 2\) and

\(\ell\) odd: \( V = x_1^{4\kappa+1} 4\kappa + 1 - x_\ell + \sum_{i=1}^{\ell-1} \left( \frac{\alpha_i^{\kappa+1}}{\kappa + 1} - \alpha_j \right) \)  

where \(x_1 = \alpha x_1^2\) and \(x_j = \alpha x_2^j \mod 2\) for \(2 \leq j \leq \ell - 1\).

\((F_4)_k\)

The \(C^{ij}\) matrix is,

\[
\begin{bmatrix}
  2 & -1 & & & \\
  -1 & 2 & -2 & & \\
  & -2 & 4 & -2 & \\
  & & -2 & 4 & \\
\end{bmatrix} .
\]  

(3.20)

and the corresponding potential is

\[
V = \frac{x_1^{\kappa+1}}{\kappa + 1} - x_1 + \frac{x_2^{\kappa+1}}{2\kappa + 1} - x_2 + \frac{x_3^{\kappa+1}}{2\kappa + 1} - x_3 + \frac{x_4^{2\kappa+1}}{2\kappa + 1} - x_4 .
\]  

(3.21)
\((G_2)_k\)

The \(C_{ij}\) matrix is,

\[
\begin{bmatrix}
2 & -3 \\
-3 & 6
\end{bmatrix}
\]

and the potential is,

\[
V = \frac{x_1^{\kappa+1}}{\kappa + 1} - x_1 + \frac{x_2^{3\kappa+1}}{3\kappa + 1} - x_2.
\] (3.23)

IV. CORRELATION FUNCTIONS AND HANDLE-SQUASHING

In this section we use the ideas of Ref. [10] and some elementary facts about the Gaussian model to suggest an interesting connection between the measure (for inner products and correlators) used by Gepner\(^{10}\) and the \(K\) matrix of Verlinde\(^{3,34}\) (see Bott\(^{22}\)). The Gaussian model will suggest a simple formula for \(K^{-1}\).

We begin by noting that all the potentials of the last section look simply like the potential of a theory made by tensoring some number of “free fields.” The many Maxwell conditions one would have if one tried to combine the vanishing conditions of \(G_k\) into one potential might look very restrictive but in these “free field” variables the Maxwell conditions are trivial and have no content. Indeed, integrating the “vanishing” conditions seems artificial and only serves to make contact with what was done for Landau–Ginzburg models;\(^{19,20,30}\) integration and differentiation of a \(C\)-valued variable \(x_i\) with respect to \(x_i\) makes sense but (thinking of the map Eq. (3.2) above) such an integration or differentiation in the space of integrable representations does not seem to have a natural interpretation in the conformal field theory.* So, instead of proceeding as Gepner\(^{10}\) has by defining an inner product (and

* Note that the Jacobian of partials of the map Eq. (3.2) does have the loose interpretation as a map from the Wilson line representations to the space of states in \(G_k\).
thereby correlators) as integrations of polynomials with respect to some measure, we seek a way of defining the inner product that is more natural \textit{vis-a-vis} the conformal field theory.

To motivate our method recall that in the variety $\nabla_u V = 0$ each point corresponds to an integrable representation (and therefore to a state) of the rational conformal field theory. Furthermore, by Eq. (3.1) each point’s $x_i^{(\ell)}$-value is the entry of the matrix $A_i$ along the diagonal in the $(\ell,\ell)$th positions. Also recall that the inner product on the Hilbert space of $G_k$ really came from the inner product on the Hilbert space of the Gaussian model, the norm of which was set by $\langle 0|0 \rangle = 1$. Finally, we know that in terms of the raising operators (the $B_i$’s) of the Gaussian model there exists a distinguished polynomial $\Gamma(B)$ such that the vacuum state $\psi_0$ of the $G_k$ may be written as,

$$\psi_0 = \Gamma(B)|0\rangle \ .$$

(4.1)

Now combining all these ideas we have a description of the inner product in terms of the operators of the conformal field theory,

$$\delta_{ij} = (\psi_i, \psi_j) = \langle \psi_0, \mathcal{O}_{i,c1}\mathcal{O}_{j,c2}\psi_0 \rangle$$

$$\quad = \langle 0|\Gamma^+(B)\mathcal{O}_{i,c2}\mathcal{O}_{j,c2}\Gamma(B)|0\rangle \quad (4.2)$$

$$\quad = \langle 0|S^+\mathcal{O}_{i,c1}\mathcal{O}_{j,c1}\Gamma^+(A)\Gamma(A)S|0\rangle \ .$$

Now, for a Gaussian model $S|0\rangle = \frac{1}{\sqrt{R}} \sum_{\ell} |\ell\rangle$ (sum runs over all states of the Gaussian model) and thus

$$(\psi_i, \psi_j) = \delta_{ij} = \frac{|W|}{R} \text{Tr}_\mathcal{H} \left[ \mathcal{O}_{i,c1}\mathcal{O}_{j,c1}\Gamma^+(A)\Gamma(A) \right] \quad (4.3)$$

where the trace is taken over, $\mathcal{H}$, just the states in $G_k$ (the $\Gamma$’s project out everything in the Gaussian Hilbert space \textit{except} the states in $G_k$). $|W|$ is the order of the Weyl group and $R$ is the total number of states in the Gaussian Hilbert space, \textit{i.e.}

$$R = \left| \frac{\Lambda_W}{(k+e)\Lambda_R} \right| .$$
Further, we may write this as

\[ (\psi_i, \psi_j) = \delta_{ij} = \frac{|W|}{R} \sum_{\nabla_u V = 0} (\mathcal{O}_i \mathcal{O}_j \Gamma^+ \Gamma) (x) \quad (4.4) \]

where the sum is over the variety \( \nabla_u V = 0 \). Thus “integration” may be understood as a sum of the values of a function evaluated on the points of the variety. (The \( \mathcal{O}_i \) and \( \mathcal{O}_j \) in Eq. (4.4) are “polynomials” of Gepner as described in Section III). Equation (4.4) bears a striking resemblance to the residue formulae of Refs. [18 – 20] and is, in a sense, the RCFT version of those formulae. Note that this is a genus one formula and its generalization to the space of operators in higher genus is not immediately obvious.

Note that although \( \Gamma \) is completely Weyl-odd (see Eq. (4.1)) the operator \( \Gamma^+ \Gamma \) is Weyl-even. \( \Gamma^+ \Gamma \) is also positive and real. Furthermore, by construction

\[ [\Gamma^+ \Gamma (B), \mathcal{O}_{\mu,c2}] = 0 \quad \forall \mu \quad (4.5) \]

\( \Gamma^+ \Gamma \) is the conformal field theory analogue of Gepner’s measure.\(^{10}\) It may be expressed as a vector in the space of the operators \( \mathcal{O}_\mu \). This follows by virtue of Eq. (4.3).

It is not difficult to show that \( \frac{|W|}{R} \Gamma^+ \Gamma = K^{-1} \), the handle squashing operator.\(^{23,33,34}\) For example, Fig. 1 is a diagrammatic picture of Eq. (4.3). One may prove \( \frac{|W|}{R} \Gamma^+ \Gamma = K^{-1} \) directly by using the Verlinde formula Eq. (2.12). Indeed suppose there exists an \( M \) such that

\[ [\mathcal{O}_{\mu,c1}, M] = 0 \quad \forall \mu \quad (4.6) \]

and,

\[ \delta_{ij} = \text{Tr}_\mathcal{H} \left( \mathcal{O}_{i,c1} M \mathcal{O}_{j,c1} \right) \quad . \]

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By Eq. (4.6) such an $M$ defines a Hermitian bilinear form on the operators $O_{\mu,c1}$ Thus we have

$$\delta_{ij} = \sum_{\ell} \left( \psi_\ell \left| O_{\mu,c1} O_{k,c1} M \right| \psi_\ell \right)$$

$$= \sum_{\ell} N_{ij}^{\mu} \left( \psi_\ell \left| O_{m,c1} M \right| \psi_\ell \right)$$

and using

$$O_{\mu c1} \psi_j = \frac{S_{\mu}^{+j} - \rho_j}{S_0^+} \psi_j$$

and Eq. (2.12) we find that

$$(\psi_\ell | M | \psi_\ell) = M_{\ell \ell} = \left| \frac{S_0^\ell}{2} \right|^2 .$$

Thus $MK = \mathbf{1}.$

One final remark is in order. We can use the above descriptions of $G_k$ to give an expression for $K^{-1}.$ From the method described in Section II it is clear that $\Gamma(B)$ is simply characterizable as the operator in the Gaussian model that is associated to the completely Weyl-odd state $(\psi_0$ the vacuum of $G_k,$ see Eq. (4.1)) containing the vector $|\rho\rangle$ where $\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha.$ This gives one an explicit way of computing $K^{-1}$ in the Hilbert space of $G_k.$ As an example we give the following formulae for the case $G = SU(2)$ and $G = SU(3);$ 

$$G = SU(2) \quad \begin{cases} \Gamma(B) = \frac{B - B^{-1}}{\sqrt{2}} \\ K^{-1} = \frac{1}{2(k+2)} (3O_1 - O_3) \end{cases}$$

$$G = SU(3) \quad \begin{cases} \Gamma(B) = \frac{1}{\sqrt{6}} \left( B_1 B_2 - B_1^{-1} B_2^2 + B_1^{-2} B_2 - B_1^{-1} B_2^{-1} + B_1 B_2^{-2} - B_1^{-1} B_2^{-1} \right) \\ K^{-1} = \left| \frac{\Lambda_W}{(k+3)\Lambda_r} \right|^{-1} (9O_1 - 6O_8 + 3 (O_{10} + O_{10} - O_{27}) \end{cases}$$

These formulae are true for all $k.$ It is intriguing that $K^{-1}$ has this universal description.
V. CONCLUSIONS

In this note we have shown that there is a fusion potential for all $G_k$ and have used the ideas of Ref. [10] and Ref. [8] to motivate a rather explicit description of the handle squashing operator $K^{-1}$.

In closing we note that many of these notions seem to allow simple generalizations to coset models and that a more group-theoretic approach to the fusion potentials is being pursued by Schnitzer, and that recently there has been progress in studying the fusion rules from the $N = 2$ Landau–Ginzburg approach.
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APPENDIX A

In this Appendix we describe how one solves the conditions \( \prod_j x_j^{r_{ji}} = 1 \). The \( r_i \)'s correspond to rows in some matrix, \( C_{ij} \). A little thought indicates that the algebraic manipulations used in solving \( \prod_j x_j^{r_{ji}} = 1 \) correspond to finding a matrix \( a_{\ell m} \) with \( a_{\ell m} \in \mathbb{Z} \) and \( \det |a_{\ell m}| = 1 \) such that a new \( C_{ij} \) defined through

\[
C_{new}^{ij} = C^{\ell i} a_{\ell j}
\]

has a simpler form than \( C^{\ell i} \). That is, if \( r_i \)'s are taken to be the rows of \( C_{new}^{ij} \) then we wish to find \( a_{\ell j} \)'s such that in these new \( r_i \)'s the conditions \( \prod_j x_j^{r_{ji}} = 1 \) all involve at most two different \( x_i \)'s each. Note the conditions that \( a_{\ell m} \in \mathbb{Z} \) and \( \det |a_{\ell m}| = 1 \) are just the requirement that the new system (in the \( r_i \)'s taken as rows in \( C_{new}^{ij} \)) has neither added to nor removed points from the variety specified by the original set of conditions. An \( a_{\ell m} \) may very simply be found for all the \( C_{ij} \)'s in the text by careful row reductions \( C^{ij} \).

For example, for \( SU(N) \) it is not difficult to show that with an

\[
a_{\ell} = \begin{bmatrix}
1 & x & x' & \ldots \\
0 & 1 & x' & \ddots \\
0 & 0 & 1 & \ddots \\
& \ddots & \ddots & \ddots \\
& & & \ddots & \ddots \\
& & & & 1
\end{bmatrix}
\]

\( (N - 1) \times (N - 1) \) (A.2)

(where the \( x \)'s means some integers) that the \( C_{new}^{ij} (= \text{the Cartan matrix for the case of } SU(N)) \) may be put into the form

\[
C_{new}^{ij} = \begin{bmatrix}
2 & -1 & & & \\
& \ddots & \ddots & \ddots & \\
& & 0 & -1 & 0 \\
& & 0 & -1 & 0 \\
N-1 & 0 & 0 & \ldots & 0 \\
N & 0 & 0 & \ldots & 0
\end{bmatrix}
\]

from which follows the potential Eq. (3.5).
Fig. 1: The operator $\frac{|W|}{\Gamma^+} \Gamma^+ \Gamma$ is $K^{-1}$, the handle-squashing operator.