\textbf{$n$-Regular Functions in Quaternionic Analysis}

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Abstract

In this paper we study left and right $n$-regular functions that originally were introduced in [FL4]. When $n = 1$, these functions are the usual quaternionic left and right regular functions. We show that $n$-regular functions satisfy most of the properties of the usual regular functions, including the conformal invariance under the fractional linear transformations by the conformal group and the Cauchy-Fueter type reproducing formulas. Arguably, these Cauchy-Fueter type reproducing formulas for $n$-regular functions are quaternionic analogues of Cauchy's integral formula for the $n$-th order pole

$$f^{(n-1)}(w) = \frac{(n - 1)!}{2\pi i} \oint \frac{f(z) \, dz}{(z - w)^n}.$$ \hfill (1)

We also find two expansions of the Cauchy-Fueter kernel for $n$-regular functions in terms of certain basis functions, we give an analogue of Laurent series expansion for $n$-regular functions, we construct an invariant pairing between left and right $n$-regular functions and we describe the irreducible representations associated to the spaces of left and right $n$-regular functions of the conformal group and its Lie algebra.

1 Introduction

The foundational result in quaternionic analysis is the Cauchy-Fueter integral formulas for left and right regular functions. Thus it is natural to ask about quaternionic analogue of Cauchy's integral formula for the $n$-th order pole for all positive integers $n$

$$f^{(n-1)}(w) = \frac{(n - 1)!}{2\pi i} \oint \frac{f(z) \, dz}{(z - w)^n}.$$ \hfill (1)

For $n = 2$ we suggested an answer to this question in [FL1] with the first derivative replaced by the Maxwell equations in the quaternionic case. In a recent paper [FL4] we proposed a different quaternionic counterpart of (1), for general $n$, introducing left and right $n$-regular functions. For $n = 1$ we get the usual left and right regular functions; when $n = 2$ we call them doubly left and right regular functions, and in the doubly regular case the first derivative in (1) is replaced by the degree operator plus two.

In this paper we study left and right $n$-regular functions in more detail. Let $n$ be a positive integer and $V_2^n$ the irreducible representation of $SU(2)$ of dimension $n + 1$. Then $n$-regular functions are functions on the space of quaternions $\mathbb{H}$ or $\mathbb{H}^\times = \mathbb{H} \setminus \{0\}$ with values in $V_2^n$ and satisfying $n$ regularity conditions. The spaces of left and right $n$-regular functions form the most degenerate series of unitary representations of $SU(2, 2)$ that are often called the spin $\frac{n}{2}$ representations of positive and negative helicities and play important role in physics. In the context of quaternionic analysis, $n$-regular functions first appeared briefly in [FL4]. In the present paper we study these spaces in more detail as natural generalizations of quaternionic regular functions.
The spaces of $n$-regular functions provide a class of irreducible representations of the conformal group that were considered before, for example, by H. P. Jakobsen and M. Vergne in [JV] and in a more general case by S. T. Lee [Le].

We show that $n$-regular functions satisfy most of the properties of the usual regular functions and doubly regular functions, including the conformal invariance under the fractional linear transformations by the conformal group $SL(2, \mathbb{H}^C) \simeq SL(4, \mathbb{C})$ and the Cauchy-Fueter type reproducing formulas. Arguably, these Cauchy-Fueter type reproducing formulas for $n$-regular functions are quaternionic analogues of Cauchy’s integral formula for the $n$-th order pole [11]. We also study in detail the $K$-type bases of the spaces of $n$-regular functions, the duality between left and right regular functions and the $u(2, 2)$-invariant inner products on these spaces.

In [FL4] we constructed an algebra of quaternionic functions using the product of spaces of left and right regular functions. We expect this construction to have a straightforward generalization to $n$-regular functions, thus yielding an explicit realization of an infinite family of certain non-highest, non-lowest weight representations of the conformal group, parametrized by positive integers $n$ with an intrinsic algebra structure.

The conformal groups of the quaternions, Minkowski space and split quaternions are locally isomorphic to $SO(5, 1)$, $SO(4, 2)$ and $SO(3, 3)$ respectively. Thus our constructions complement a thorough study of minimal representations of the indefinite orthogonal groups $O(p, q)$ by T. Kobayashi and B. Ørsted [KØ] (see references therein for the previous work on this subject).

Their work also uses the space of solutions of the ultrahyperbolic wave equation on $\mathbb{R}^{p-1, q-1}$ to give concrete realizations of these minimal representations. Many results of quaternionic analysis extend to higher dimensions in the form of Clifford analysis (see, for example, [BDS, DSS, R]). It is interesting to see how results of this paper are generalized to Clifford analysis.

The paper is organized as follows. In Section 2 we define left and right $n$-regular functions (Definitions 3, 4) and prove conformal invariance under the fractional linear transformations by group $GL(2, \mathbb{H}^C)$ (Theorem 5). In Section 3 we derive the Cauchy-Fueter type formulas for $n$-regular functions (Theorem 9). In Section 4 we find two expansions of the Cauchy-Fueter kernel $k_{n/2}(Z - W)$ for $n$-regular functions in terms of certain basis functions $F_{l,\mu,\nu}^{(n)}(Z)$, $F_{l,\mu,\nu}^{(n)}(Z)$, $G_{l,\mu,\nu}^{(n)}(Z)$ and $G_{l,\mu,\nu}^{(n)}(Z)$ (Proposition 12). In Section 5 we prove a technical result that a certain differential operator $D_n$ that enters the Cauchy-Fueter formulas for $n$-regular functions can be inverted for (left or right) $n$-regular functions defined on all of $\mathbb{H}^\times$ (Proposition 15). We also give an analogue of Laurent series expansion for $n$-regular functions on $\mathbb{H}^\times$ (Corollary 16). In Section 6 we use the technical result from the previous section to construct a $\mathfrak{gl}(2, \mathbb{H}^C)$-invariant pairing (14) between left and right $n$-regular functions on $\mathbb{H}^\times$. We also prove orthogonality relations between the basis functions $F_{l,\mu,\nu}^{(n)}(Z)$, $F_{l,\mu,\nu}^{(n)}(Z)$, $G_{l,\mu,\nu}^{(n)}(Z)$ and $G_{l,\mu,\nu}^{(n)}(Z)$ (Proposition 20). In Section 7 we consider $\mathfrak{gl}(2, \mathbb{H}^C)$-modules

$$\mathcal{F}_n^+ = \mathbb{C}\text{-span of } \{F_{l,\mu,\nu}^{(n)}(Z)\}, \quad \mathcal{F}_n^- = \mathbb{C}\text{-span of } \{F_{l,\mu,\nu}^{(n)}(Z)\},$$

$$\mathcal{G}_n^+ = \mathbb{C}\text{-span of } \{G_{l,\mu,\nu}^{(n)}(Z)\}, \quad \mathcal{G}_n^- = \mathbb{C}\text{-span of } \{G_{l,\mu,\nu}^{(n)}(Z)\}$$

associated to left and right $n$-regular functions. We prove that these $\mathfrak{gl}(2, \mathbb{H}^C)$-modules are irreducible (Theorem 26) and identify their $K$-types, where $K$ is the maximal compact subgroup $U(2) \times U(2)$ of $U(2, 2)$ (Proposition 24). In Section 8 we give explicit descriptions of the $u(2, 2)$-invariant inner products on the $\mathfrak{gl}(2, \mathbb{H}^C)$-modules $\mathcal{F}_n^+$ and $\mathcal{G}_n^+$ (Theorem 27).

Since this paper is a continuation of [FL1, FL3, FL4], we follow the same notations and instead of introducing those notations again we direct the reader to Section 2 of [FL3].
2 Definitions and Conformal Invariance

We continue to use notations established in [FL1]. In particular, \(e_0, e_1, e_2, e_3\) denote the units of the classical quaternions \(\mathbb{H}\) corresponding to the more familiar 1, \(i, j, k\) (we reserve the symbol \(i\) for \(\sqrt{-1} \in \mathbb{C}\)). Thus \(\mathbb{H}\) is an algebra over \(\mathbb{R}\) generated by \(e_0, e_1, e_2, e_3\), and the multiplicative structure is determined by the rules

\[
e_0 e_i = e_i e_0 = e_i, \quad (e_i)^2 = e_1 e_2 e_3 = -e_0, \quad e_i e_j = -e_j e_i, \quad 1 \leq i < j \leq 3,
\]

and the fact that \(\mathbb{H}\) is a division ring. Next we consider the algebra of complexified quaternions (also known as biquaternions) \(\mathbb{H}_\mathbb{C} = \mathbb{C} \otimes \mathbb{R} \mathbb{H}\) and write elements of \(\mathbb{H}_\mathbb{C}\) as

\[
Z = z_0 e_0 + z_1 e_1 + z_2 e_2 + z_3 e_3, \quad z_0, z_1, z_2, z_3 \in \mathbb{C},
\]

so that \(Z \in \mathbb{H}\) if and only if \(z_0, z_1, z_2, z_3 \in \mathbb{R}\):

\[
\mathbb{H} = \{X = x^0 e_0 + x^1 e_1 + x^2 e_2 + x^3 e_3; x^0, x^1, x^2, x^3 \in \mathbb{R}\}.
\]

For \(Z = z^0 e_0 + z^1 e_1 + z^2 e_2 + z^3 e_3 \in \mathbb{H}_\mathbb{C}\), we use

\[
Z^+ = z^0 e_0 - z^1 e_1 - z^2 e_2 - z^3 e_3
\]

to denote the quaternionic conjugation and

\[
N(Z) = Z^+ Z = ZZ^+ = (z^0)^2 + (z^1)^2 + (z^2)^2 + (z^3)^2 \in \mathbb{C}
\]

to denote the quadratic norm. Let

\[
\mathbb{H}_\mathbb{C}^\times = \mathbb{H}_\mathbb{C} \setminus \{0\} \quad \text{and} \quad \mathbb{H}_\mathbb{C}^\times = \{Z \in \mathbb{H}_\mathbb{C}; N(Z) \neq 0\}
\]

be the sets of invertible elements in \(\mathbb{H}\) and \(\mathbb{H}_\mathbb{C}\) respectively. The algebra \(\mathbb{H}_\mathbb{C}\) can be naturally identified with the algebra of \(2 \times 2\) matrices with complex entries. Recall that we denote by \(S\) (respectively \(S'\)) the irreducible 2-dimensional left (respectively right) \(\mathbb{H}_\mathbb{C}\)-module, as described in Subsection 2.3 of [FL1]. The spaces \(S\) and \(S'\) can be realized as respectively columns and rows of complex numbers. Then

\[
S \otimes S' \simeq \mathbb{H}_\mathbb{C}.
\]

Note that \(S \otimes S\) and \(S' \otimes S'\) are respectively left and right modules over \(\mathbb{H}_\mathbb{C} \otimes \mathbb{H}_\mathbb{C}\).

Fix a positive integer \(n\). The \(n\)-fold tensor product

\[
\underbrace{S \otimes \cdots \otimes S}_{n \text{ times}}
\]

is a left module over

\[
\underbrace{\mathbb{H}_\mathbb{C} \otimes \cdots \otimes \mathbb{H}_\mathbb{C}}_{n \text{ times}}
\]

and contains the \(n\)-fold symmetric product which we denote

\[
\underbrace{S \otimes \cdots \otimes S}_{n \text{ times}}
\]

as a subspace. Similarly, the \(n\)-fold tensor product

\[
\underbrace{S' \otimes \cdots \otimes S'}_{n \text{ times}}
\]
is a right module over \( \mathfrak{g} \) and contains the subspace of \( n \)-fold symmetric tensors

\[
S' \odot \cdots \odot S'.
\]

We have a natural bilinear pairing between \( S' \) and \( S \):

\[
S' \times S \to \mathbb{C}, \quad (s'_1, s'_2) \times \left( \begin{array}{c} s_1 \vspace{0.5em} \\ s_2 \end{array} \right) \mapsto (s'_1, s'_2) \left( \begin{array}{c} s_1 \\ s_2 \end{array} \right) = s'_1 s_1 + s'_2 s_2.
\]

This pairing extends to a multilinear pairing

\[
(\underbrace{S' \times \cdots \times S'}_{n \text{ times}}) \times (\underbrace{S \times \cdots \times S}_{n \text{ times}}) \to \mathbb{C}
\]

by taking the product of pairings of the respective components, and then to

\[
(\underbrace{S' \otimes \cdots \otimes S'}_{n \text{ times}}) \times (\underbrace{S \otimes \cdots \otimes S}_{n \text{ times}}) \to \mathbb{C}
\]

(4)

by multilinearity. For convenience, we restate Lemma 5 from [FL4]:

**Lemma 1.** Let \( t \in \bigotimes_{n \text{ times}} S \) and \( t' \in \bigotimes_{n \text{ times}} S' \), then, for any \( 1 \leq j, k \leq n, j \neq k \),

\[
\sum_{i=0}^{3} (1 \otimes \cdots \otimes e_i \otimes \cdots \otimes e_j \otimes \cdots \otimes 1) t = 0 \quad \text{in } \bigotimes_{n \text{ times}} S
\]

and

\[
\sum_{i=0}^{3} t'(1 \otimes \cdots \otimes e_i \otimes \cdots \otimes e_j \otimes \cdots \otimes 1) = 0 \quad \text{in } \bigotimes_{n \text{ times}} S'
\]

We consider spaces of functions

\[
\hat{\mathcal{F}}_n = \{ f : \mathbb{H}_2 \to \bigotimes_{n \text{ times}} S \} \quad \text{and} \quad \hat{\mathcal{G}}_n = \{ g : \mathbb{H}_2 \to \bigotimes_{n \text{ times}} S' \}
\]

(possibly with singularities), and let the group \( GL(2, \mathbb{H}_2) \) act on these spaces as follows:

\[
\pi_{nl}(h) : f(Z) \mapsto (\pi_{nl}(h)f)(Z) = \frac{(cZ + d)^{-1} \otimes \cdots \otimes (cZ + d)^{-1}}{N(cZ + d)} \cdot f((aZ + b)(cZ + d)^{-1}), \quad (5)
\]

\[
\pi_{nr}(h) : g(Z) \mapsto (\pi_{nr}(h)g)(Z)
\]

\[= g((a' - Zd')^{-1}(-b' + Zd')) \cdot \frac{(a' - Zd')^{-1} \otimes \cdots \otimes (a' - Zd')^{-1}}{N(a' - Zd')}, \quad (6)\]

where \( f \in \hat{\mathcal{F}}_n, g \in \hat{\mathcal{G}}_n, h = (\begin{array}{cc} a' & b' \\ c' & d' \end{array}) \in GL(2, \mathbb{H}_2) \) with \( h^{-1} = (\begin{array}{cc} a & b \\ c & d \end{array}) \). Clearly, these two actions preserve the subspaces of functions with values in \( n \)-fold symmetric products \( S \otimes \cdots \otimes S \) and \( S' \otimes \cdots \otimes S' \) respectively.

Differentiating \( \pi_{nl} \) and \( \pi_{nr} \), we obtain actions of the Lie algebra \( \mathfrak{gl}(2, \mathbb{H}_2) \), which we still denote by \( \pi_{nl} \) and \( \pi_{nr} \) respectively. Using notations

\[
\partial = \begin{pmatrix} \partial_{11} & \partial_{21} \\ \partial_{12} & \partial_{22} \end{pmatrix} = \frac{1}{2} \nabla, \quad \partial^+ = \begin{pmatrix} \partial_{22} & -\partial_{21} \\ -\partial_{12} & \partial_{11} \end{pmatrix} = \frac{1}{2} \nabla^+, \quad \partial_{ij} = \frac{\partial}{\partial z_{ij}}
\]

we can describe these actions of the Lie algebra (cf. Lemma 4 in [FL4]).
Lemma 2. The Lie algebra action $\pi_{nl}$ of $gl(2, \mathbb{H}_C)$ on $\hat{F}_n$ is given by

$$\pi_{nl}(A 0 0) : f(Z) \mapsto - \text{Tr}(AZ \partial)f,$$
$$\pi_{nl}(0 B 0) : f(Z) \mapsto - \text{Tr}(B \partial)f,$$
$$\pi_{nl}(C 0 0) : f(Z) \mapsto \text{Tr}(ZCZ \partial + CZ)f$$

$$+ (CZ \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes CZ \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes CZ)f,$$

$$\pi_{nl}(0 0 D) : f(Z) \mapsto \text{Tr}(ZD \partial + D)f$$

$$+ (D \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes D \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes D)f.$$

Similarly, the Lie algebra action $\pi_{nr}$ of $gl(2, \mathbb{H}_C)$ on $\hat{G}_n$ is given by

$$\pi_{nr}(A 0 0) : g(Z) \mapsto - \text{Tr}(AZ \partial + A)g$$

$$- g(A \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes A \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes A),$$

$$\pi_{nr}(0 B 0) : g(Z) \mapsto - \text{Tr}(B \partial)g,$$

$$\pi_{nr}(C 0 0) : g(Z) \mapsto \text{Tr}(ZCZ \partial + ZC)g$$

$$+ g(ZC \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes ZC \otimes \cdots \otimes 1 + \cdots + 1 \otimes \cdots \otimes 1 \otimes ZC),$$

$$\pi_{nr}(0 0 D) : g(Z) \mapsto \text{Tr}(ZD \partial)g.$$

Proof. These formulas are obtained by differentiating (5) and (6). \qed

We introduce $2n$ first order differential operators

$$\nabla_k^+ = (1 \otimes \cdots \otimes e_0 \otimes \cdots \otimes 1) \frac{\partial}{\partial x^0} + (1 \otimes \cdots \otimes e_1 \otimes \cdots \otimes 1) \frac{\partial}{\partial x^1}$$

$$+ (1 \otimes \cdots \otimes e_2 \otimes \cdots \otimes 1) \frac{\partial}{\partial x^2} + (1 \otimes \cdots \otimes e_3 \otimes \cdots \otimes 1) \frac{\partial}{\partial x^3},$$

$$\nabla_k = \frac{\partial}{\partial x^0}(1 \otimes \cdots \otimes e_0 \otimes \cdots \otimes 1) - \frac{\partial}{\partial x^1}(1 \otimes \cdots \otimes e_1 \otimes \cdots \otimes 1)$$

$$- \frac{\partial}{\partial x^2}(1 \otimes \cdots \otimes e_2 \otimes \cdots \otimes 1) - \frac{\partial}{\partial x^3}(1 \otimes \cdots \otimes e_3 \otimes \cdots \otimes 1),$$

$k = 1, \ldots, n$, which can be applied to functions with values in $n$-fold tensor products $S \otimes \cdots \otimes S$ or $S' \otimes \cdots \otimes S'$ as follows. If $U$ is an open subset of $\mathbb{H}$ or $\mathbb{H}_C$ and $f : U \to S \otimes \cdots \otimes S$ is a differentiable function, then these operators can be applied to $f$ on the left. For example,

$$\nabla_1^+ f = (e_0 \otimes 1 \otimes \cdots \otimes 1) \frac{\partial f}{\partial x^0} + (e_1 \otimes 1 \otimes \cdots \otimes 1) \frac{\partial f}{\partial x^1}$$

$$+ (e_2 \otimes 1 \otimes \cdots \otimes 1) \frac{\partial f}{\partial x^2} + (e_3 \otimes 1 \otimes \cdots \otimes 1) \frac{\partial f}{\partial x^3}.$$

Similarly, these operators can be applied on the right to differentiable functions $g : U \to S' \otimes \cdots \otimes S'$; we often indicate this with an arrow above the operator. For example,

$$\nabla_1^- g = \frac{\partial g}{\partial x^0}(1 \otimes \cdots \otimes 1 \otimes e_0) + \frac{\partial g}{\partial x^1}(1 \otimes \cdots \otimes 1 \otimes e_1)$$

$$+ \frac{\partial g}{\partial x^2}(1 \otimes \cdots \otimes 1 \otimes e_2) + \frac{\partial g}{\partial x^3}(1 \otimes \cdots \otimes 1 \otimes e_3).$$
Definition 3. Let $U$ be an open subset of $\mathbb{H}$. A $C^1$-function $f : U \to S \odot \cdots \odot S$ is left $n$-regular if it satisfies $n$ differential equations

$$\nabla_k^+ f = 0, \quad k = 1, \ldots, n,$$

for all points in $U$. Similarly, a $C^1$-function $g : U \to S' \odot \cdots \odot S'$ is right $n$-regular if

$$g \nabla_k^+ = 0, \quad k = 1, \ldots, n,$$

for all points in $U$.

Since

$$\nabla_k^+ \nabla_k^+ = \nabla_k^+ \nabla_k = \Box, \quad k = 1, \ldots, n,$$

where

$$\Box = \frac{\partial^2}{(\partial x_0)^2} + \frac{\partial^2}{(\partial x_1)^2} + \frac{\partial^2}{(\partial x_2)^2} + \frac{\partial^2}{(\partial x_3)^2},$$

left and right $n$-regular functions are harmonic.

One way to construct $n$-regular functions is to start with a harmonic function $\varphi : \mathbb{H} \to S \odot \cdots \odot S$, then $(\nabla \odot \cdots \odot \nabla) \varphi$ is left $n$-regular. Similarly, if $\varphi : \mathbb{H} \to S' \odot \cdots \odot S'$ is harmonic, then $\varphi(\nabla \odot \cdots \odot \nabla)$ is right $n$-regular.

We also can talk about $n$-regular functions defined on open subsets of $\mathbb{H}_C$. In this case we require such functions to be holomorphic.

Definition 4. Let $U$ be an open subset of $\mathbb{H}_C$. A holomorphic function $f : U \to S \odot \cdots \odot S$ is left $n$-regular if it satisfies $n$ differential equations

$$\nabla_k^+ f = 0, \quad k = 1, \ldots, n,$$

for all points in $U$.

Similarly, a holomorphic function $g : U \to S' \odot \cdots \odot S'$ is right $n$-regular if

$$g \nabla_k^+ = 0, \quad k = 1, \ldots, n,$$

for all points in $U$.

Let $\mathcal{R}_n$ and $\mathcal{R}'_n$ denote respectively the spaces of (holomorphic) left and right $n$-regular functions on $\mathbb{H}_C$, possibly with singularities.

Theorem 5. 1. The space $\mathcal{R}_n$ of left $n$-regular functions $\mathbb{H}_C \to S \odot \cdots \odot S$ (possibly with singularities) is invariant under the $\pi_{nl}$ action (5) of $GL(2, \mathbb{H}_C)$.

2. The space $\mathcal{R}'_n$ of right $n$-regular functions $\mathbb{H}_C \to S' \odot \cdots \odot S'$ (possibly with singularities) is invariant under the $\pi_{nr}$ action (6) of $GL(2, \mathbb{H}_C)$.

Proof. Since the Lie group $GL(2, \mathbb{H}_C) \simeq GL(4, \mathbb{C})$ is connected, it is sufficient to show that, if $f \in \mathcal{R}_n$, $g \in \mathcal{R}'_n$ and $(A \ B \ C \ D) \in \mathfrak{gl}(2, \mathbb{H}_C)$, then $\pi_{nl}(A \ B \ C \ D) f \in \mathcal{R}_n$ and $\pi_{nr}(A \ B \ C \ D) g \in \mathcal{R}'_n$. Consider,
for example, the case of $\pi_{nl}(\frac{0}{C}0)f$, the other cases are similar. For $k = 1, \ldots, n$, we have:

$$
\nabla^+_k \pi_{nl}(\frac{0}{C}0)f = \nabla^+_k \left( \text{Tr}(CZ\partial + CZ)f + \sum_{j=1}^{n}(1 \otimes \cdots \otimes CZ \otimes \cdots \otimes 1)f \right)
$$

$$
= \nabla^+_k \left( \text{Tr}(CZ\partial + CZ)f + (1 \otimes \cdots \otimes CZ \otimes \cdots \otimes 1)f \right)
$$

$$
+ \sum_{j \neq k}(1 \otimes \cdots \otimes CZ \otimes \cdots \otimes 1)\nabla^+_k f
$$

$$
+ \sum_{j \neq k}(1 \otimes \cdots \otimes CZ \otimes \cdots \otimes 1)\sum_{i=0}^{3}(1 \otimes \cdots \otimes e_i \otimes \cdots \otimes e_i \otimes \cdots \otimes 1)f
$$

the first summand is zero essentially because the space of left regular functions is invariant under the action $\pi_l$ (equation (22) in [FL1]), the second summand is zero because $f$ satisfies $\nabla^+_k f = 0$, and the third summand is zero by Lemma 1. □

### 3 Cauchy-Fueter Formulas for $n$-Regular Functions

In this section we derive the Cauchy-Fueter type formulas for $n$-regular functions from the classical Cauchy-Fueter formulas for left and right regular functions.

**Lemma 6.** Let $f(Z)$ be a left $n$-regular function, then the $S \otimes \cdots \otimes S$-valued functions

$$(Z \otimes \cdots \otimes 1 \otimes \cdots \otimes Z)f(Z), \quad k = 1, \ldots, n,$$

are “left regular in $k$-th place” in the sense that they satisfy

$$
\nabla^+_k \left( (Z \otimes \cdots \otimes 1 \otimes \cdots \otimes Z)f(Z) \right) = 0.
$$

Similarly, if $g(Z)$ is a right $n$-regular function, then the $S' \otimes \cdots \otimes S'$-valued functions

$$(g(Z)(Z \otimes \cdots \otimes 1 \otimes \cdots \otimes Z), \quad k = 1, \ldots, n,$$

are “right regular in $k$-th place” in the sense that they satisfy

$$
\left[ (g(Z)(Z \otimes \cdots \otimes 1 \otimes \cdots \otimes Z) \right]^{\otimes -k} = 0.
$$

**Proof.** We have:

$$
\nabla^+_k \left( (Z \otimes \cdots \otimes 1 \otimes \cdots \otimes Z)f(Z) \right) = (Z \otimes \cdots \otimes 1 \otimes \cdots \otimes Z) \left[ \nabla^+_k f(Z) \right]
$$

$$
+ \sum_{j \neq k} \sum_{i=0}^{3}(Z \otimes \cdots \otimes e_i \otimes \cdots \otimes e_i \otimes \cdots \otimes Z)f(Z)
$$

the first term is zero because $f$ satisfies $\nabla^+_k f = 0$ and the second term is zero by Lemma 1. □

Proof of the second part of the lemma is similar.
Recall the quaternionic valued holomorphic 3-form $Dz$ on $\mathbb{H}_C$:
\[
Dz = e_0 dz^1 \wedge dz^2 \wedge dz^3 - e_1 dz^0 \wedge dz^2 \wedge dz^3 + e_2 dz^0 \wedge dz^1 \wedge dz^3 - e_3 dz^0 \wedge dz^1 \wedge dz^2.
\]
We also consider holomorphic 3-forms
\[
Z \otimes \cdots \otimes Dz \otimes \cdots \otimes Z, \quad k = 1, \ldots, n,
\]
on $\mathbb{H}_C$ with values in $\mathbb{H}_C \otimes \cdots \otimes \mathbb{H}_C$. We have an analogue of Cauchy’s integral theorem for $n$-regular functions.

**Lemma 7.** Let $U \subset \mathbb{H}$ be an open bounded subset with piecewise $C^1$ boundary $\partial U$. Suppose that $f(Z)$ is left $n$-regular and $g(Z)$ is right $n$-regular on a neighborhood of the closure $\overline{U}$. Then, for each $k = 1, \ldots, n$,
\[
\int_{\partial U} g(Z) \cdot (Z \otimes \cdots \otimes Dz \otimes \cdots \otimes Z) \cdot f(Z) = 0.
\]

Note that the expression inside the integral is a $\mathbb{C}$-valued function obtained by applying the pairing (4).

**Proof.** Essentially by the definition of $Dz$,
\[
d\left[ g(Z) \cdot (Z \otimes \cdots \otimes Dz \otimes \cdots \otimes Z) \cdot f(Z) \right] = g(Z) \cdot \nabla^+_{k} \left( (Z \otimes \cdots \otimes 1 \otimes \cdots \otimes Z) \cdot f(Z) \right)
+ g(Z) \cdot \nabla^+_{k} \left( (Z \otimes \cdots \otimes 1 \otimes \cdots \otimes Z) \cdot f(Z) \right) dz^0 \wedge dz^1 \wedge dz^2 \wedge dz^3.
\]
By Lemma 6 both summands are zero, and the result follows. \qed

Recall the degree operator $\deg$ acting on functions on $\mathbb{H}$:
\[
\deg f = x^0 \frac{\partial f}{\partial x^0} + x^1 \frac{\partial f}{\partial x^1} + x^2 \frac{\partial f}{\partial x^2} + x^3 \frac{\partial f}{\partial x^3},
\]
and let $(\deg + m), m \in \mathbb{Z}$, denote the degree operator plus $m$ times the identity:
\[
(\deg + m)f = \deg f + mf.
\]
Similarly, we can define operators $\deg$ and $(\deg + m)$ acting on functions on $\mathbb{H}_C$. For convenience we recall Lemma 8 from [FL2] (it applies to both cases).

**Lemma 8.**
\[
2(\deg + 2) = Z^+ \nabla^+ + \nabla Z = \nabla^+ Z^+ + Z\nabla.
\]  
(7)

We introduce an operator
\[
D_n = (\deg + n)(\deg + n - 1) \cdots (\deg + 2).
\]

Define a function of $Z$ and $W$ taking values in $\mathbb{H}_C \otimes \cdots \otimes \mathbb{H}_C$
\[
k_{n/2}(Z - W) = \frac{1}{2^n} \left( \nabla W \otimes \cdots \otimes \nabla W \right) \frac{1}{N(Z - W)} = \frac{(-1)^n}{2^n} \left( \nabla Z \otimes \cdots \otimes \nabla Z \right) \frac{1}{N(Z - W)}.
\]  
(8)

Observe that
\[
k_{n/2}(Z - W) = (-1)^n k_{n/2}(W - Z).
\]  
(9)

We have the following analogue of the Cauchy-Fueter formulas for $n$-regular functions.
Theorem 9. Let $U \subset \mathbb{H}$ be an open bounded subset with piecewise $C^1$ boundary $\partial U$. Suppose that $f(Z)$ is left $n$-regular on a neighborhood of the closure $\overline{U}$, then, for each $k = 1, \ldots, n$,

$$\frac{1}{2\pi^2} \int_{\partial U} k_{n/2}(Z - W) \cdot (Z \otimes \cdots \otimes Dz \otimes \cdots \otimes Z) \cdot f(Z) = \begin{cases} D_n f(W) & \text{if } W \in U; \\ 0 & \text{if } W \notin U. \end{cases}$$

If $g(Z)$ is right $n$-regular on a neighborhood of the closure $\overline{U}$, then, for each $k = 1, \ldots, n$,

$$\frac{1}{2\pi^2} \int_{\partial U} g(Z) \cdot (Z \otimes \cdots \otimes Dz \otimes \cdots \otimes Z) \cdot k_{n/2}(Z - W) = \begin{cases} D_n g(W) & \text{if } W \in U; \\ 0 & \text{if } W \notin U. \end{cases}$$

Proof. By Lemma 6 the $S \otimes \cdots \otimes S$-valued function

$$(Z \otimes \cdots \otimes 1 \otimes \cdots \otimes Z)f(Z)$$

is “left regular in the $k$-th place” in the sense that it is annihilated by $\nabla_k^+$. From the classical Cauchy-Fueter formula for left regular functions, we obtain:

$$\frac{1}{2\pi^2} \int_{\partial U} \tilde{k}_{1/2}(Z - W) \cdot (1 \otimes \cdots \otimes Dz \otimes \cdots \otimes 1) \cdot (Z \otimes \cdots \otimes 1 \otimes \cdots \otimes Z) f(Z)$$

$$= \begin{cases} (W \otimes \cdots \otimes 1 \otimes \cdots \otimes W) f(W) & \text{if } W \in U; \\ 0 & \text{if } W \notin U, \end{cases}$$

(10)

where

$$\tilde{k}_{1/2}(Z - W) = 1 \otimes \cdots \otimes (Z - W)^{-1} \otimes \cdots \otimes 1 = \frac{1}{2} \nabla_k^+ \frac{1}{N(Z - W)} = -\frac{1}{2} \nabla_k^+ \frac{1}{N(Z - W)}.$$

Applying $n - 1$ differential operators $\nabla_j$, $j = 1, \ldots, n$, $j \neq k$, to both sides of (10) (the derivative is taken with respect to $W$),

$$\frac{2^{n-1}}{2\pi^2} \int_{\partial U} k_{n/2}(Z - W) \cdot (Z \otimes \cdots \otimes Dz \otimes \cdots \otimes Z) \cdot f(Z)$$

$$= \begin{cases} (\nabla \otimes \cdots \otimes 1 \otimes \cdots \otimes \nabla)(W \otimes \cdots \otimes 1 \otimes \cdots \otimes W) f(W) & \text{if } W \in U; \\ 0 & \text{if } W \notin U \end{cases}$$

$$= \begin{cases} 2^{n-1} (\deg + n)(\deg + n - 1) \cdots (\deg + 2) f(W) & \text{if } W \in U; \\ 0 & \text{if } W \notin U, \end{cases}$$

where the last equality follows from (17) and Lemma 11 since $\nabla_j^+ f = 0$, for each $j = 1, \ldots, n$.

The case of right $n$-regular function is similar. \hfill \Box

We have an analogue of Liouville’s theorem for $n$-regular functions:

Corollary 10. Let $f : \mathbb{H} \rightarrow S \otimes \cdots \otimes S$ be a function that is left $n$-regular and bounded on $\mathbb{H}$, then $f$ is constant. Similarly, if $g : \mathbb{H} \rightarrow S' \otimes \cdots \otimes S'$ is a function that is right $n$-regular and bounded on $\mathbb{H}$, then $g$ is constant.
Proof. The proof is essentially the same as for the (classical) left and right regular functions on \( \mathbb{H} \), so we only give a sketch of the first part. From Theorem 3 we have:

\[
\frac{\partial}{\partial x^0} D_n f(X) = \frac{1}{2\pi^2} \int_{S_3^R} \frac{\partial k_{n/2}(Z-X)}{\partial x^0} \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot f(Z),
\]

where \( S_3^R \subset \mathbb{H} \) is the three-dimensional sphere of radius \( R \) centered at the origin

\[
S_3^R = \{ X \in \mathbb{H}; \ N(X) = R^2 \}
\]

with \( R^2 > N(X) \). If \( f \) is bounded, one easily shows that the integral on the right hand side tends to zero as \( R \to \infty \). Thus

\[
\frac{\partial}{\partial x^0} D_n f = 0.
\]

Similarly, the other partial derivatives

\[
\frac{\partial}{\partial x^1} D_n f = \frac{\partial}{\partial x^2} D_n f = \frac{\partial}{\partial x^3} D_n f = 0.
\]

It follows that \( D_n f \) and hence \( f \) are constant. \( \square \)

4 Expansion of the Cauchy-Fueter Kernel for \( n \)-Regular Functions

We often identify \( \mathbb{H}_C \) with \( 2 \times 2 \) matrices with complex entries. Similarly, it will be convenient to identify \( \mathbb{H}_C \otimes \cdots \otimes \mathbb{H}_C \) with complex \( 2^n \times 2^n \) matrices using the Kronecker product. Let \( C^{k \times k, n \text{ times}} \) denote the algebra of \( k \times k \) complex matrices. For example, if \( A = (a_{11} a_{12}), B = (b_{11} b_{12}) \in C^{2 \times 2} \), then their Kronecker product is

\[
A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{pmatrix} \in C^{4 \times 4}.
\]

It is easy to see that the Kronecker product satisfies

\[
(A \otimes B)(C \otimes D) = (AC) \otimes (BD).
\]

Next we recall the matrix coefficients \( t^l_{\nu \mu}(Z) \)'s of \( SU(2) \) described by equation (27) of [FL1] (cf. [V]):

\[
t^l_{\nu \mu}(Z) = \frac{1}{2\pi i} \int (sz_{11} + z_{21})^{-\mu} (sz_{12} + z_{22})^{l+\mu} s^{-l+\mu} \frac{ds}{s}, \quad l=0, 1, \frac{3}{2}, ..., \mu, \nu \in Z, l, \mu, \nu \leq l. \tag{11}
\]

\( Z = (z_{11}, z_{21}, z_{12}, z_{22}) \in \mathbb{H}_C \), the integral is taken over a loop in \( \mathbb{C} \) going once around the origin in the counterclockwise direction. We regard these functions as polynomials on \( \mathbb{H}_C \).

Since \( \Box t^l_{\nu \mu}(Z) = 0 \) and \( \Box (N(Z)^{-1} \cdot t^l_{\nu \mu}(Z^{-1})) = 0 \), by the observation made after Definition 3 the columns and rows of the two \( 2^n \times 2^n \) matrices

\[
(\partial \otimes \cdots \otimes \partial)^{n \text{ times}} t^l_{\nu \mu}(Z) \quad \text{and} \quad (\partial \otimes \cdots \otimes \partial)^{n \text{ times}} (N(Z)^{-1} \cdot t^l_{\nu \mu}(Z^{-1}))
\]

are constant.
are respectively left and right $n$-regular. We use this to construct bases of left and right $n$-regular functions. Let indices $l$, $\mu$ and $\nu$ range as follows:

$$l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \mu \in \mathbb{Z} + l + n/2, \ \nu \in \mathbb{Z} + l, \ -l - n/2 \leq \mu \leq l + n/2, \ -l \leq \nu \leq l.$$ 

Introduce left $n$-regular functions $\mathbb{H}_\mathbb{C} \to S \circ \cdots \circ S$

$$F_{l,\mu,\nu}^{(n)}(Z) = \left( \left( \frac{\partial_{11}}{\partial_{12}} \otimes \cdots \otimes \frac{\partial_{11}}{\partial_{12}} \right) \right)_{\nu-n/2}^{l+n/2} \left( Z \right) = \left( \left( \frac{\partial_{21}}{\partial_{22}} \otimes \cdots \otimes \frac{\partial_{21}}{\partial_{22}} \right) \right)_{\mu-n/2}^{l+n/2} \left( Z \right) \quad (12)$$

and right $n$-regular functions $\mathbb{H}_\mathbb{C} \to S' \circ \cdots \circ S'$

$$G_{l,\mu,\nu}^{(n)}(Z) = \frac{(l - \nu)!}{(l - \nu + n)!} \left( \left( \frac{\partial_{11}}{\partial_{12}} \otimes \cdots \otimes \frac{\partial_{11}}{\partial_{12}} \right) \right)_{\nu-n/2}^{n} \left( Z \right).$$

In order to construct the dual basis, we consider left $n$-regular functions $\mathbb{H}_\mathbb{C} \to S \circ \cdots \circ S$

$$\left( -1 \right)^n \left( \left( \frac{\partial_{11}}{\partial_{12}} \otimes \cdots \otimes \frac{\partial_{11}}{\partial_{12}} \right) \right) \left( N(Z)^{-1} \cdot t^l_{\nu,\mu}(Z^{-1}) \right);$$

using Lemmas 22 and 23 in [FL1], one can see that these are columns with entries being some scalar multiples of

$$N(Z)^{-1} \cdot t^{l+n/2}_{\nu-n/2,\mu-n/2}(Z^{-1}), \ldots, N(Z)^{-1} \cdot t^{l+n/2}_{\nu+n/2,\mu-n/2}(Z^{-1}).$$

Shift the index $\tilde{\mu}$ so that $\tilde{\mu} - n/2 = \mu$ and call the resulting function $F_{l,\mu,\nu}^{(n)}(Z)$. Note that this is not always the same as differentiating $N(Z)^{-1} \cdot t^{l}_{\nu,\mu+n/2}(Z^{-1})$, since $\mu + n/2$ may be bigger than $l$, which is outside of allowed range. Similarly, consider right $n$-regular functions $\mathbb{H}_\mathbb{C} \to S' \circ \cdots \circ S'$

$$\left( -1 \right)^n \left( \left( \frac{\partial_{11}}{\partial_{12}} \otimes \cdots \otimes \frac{\partial_{11}}{\partial_{12}} \right) \right) \left( N(Z)^{-1} \cdot t^l_{\nu,\mu}(Z^{-1}) \right);$$

using Lemmas 22 and 23 in [FL1], one can see that these are rows with entries being

$$N(Z)^{-1} \cdot t^{l+n/2}_{\nu-n/2,\mu-n/2}(Z^{-1}), \ldots, N(Z)^{-1} \cdot t^{l+n/2}_{\nu+n/2,\mu-n/2}(Z^{-1}).$$

Shift the index $\tilde{\mu}$ so that $\tilde{\mu} - n/2 = \mu$ and call the resulting function $G_{l,\mu,\nu}^{(n)}(Z)$. Again, this is not always the same as differentiating $N(Z)^{-1} \cdot t^{l}_{\nu,\mu+n/2}(Z^{-1})$, since $\mu + n/2$ may be bigger than $l$. Note that $F_{l,\mu,\nu}^{(1)}(Z)$, $F_{l,\mu,\nu}^{(1)}(Z)$, $G_{l,\mu,\nu}^{(1)}(Z)$ and $G_{l,\mu,\nu}^{(1)}(Z)$ are exactly the basis functions that appear in Proposition 24 in [FL1]:

$$F_{l,\mu,\nu}^{(1)}(Z) = \left( \left( l - \mu + 1/2 \right) t^l_{\nu,\mu+1/2}(Z) \right) \left( \left( l + \mu + 1/2 \right) t^l_{\nu-1/2,\mu}(Z) \right),$$

$$F_{l,\mu,\nu}^{(1)}(Z) = \left( \left( l - \nu + 1 \right) N(Z)^{-1} \cdot t^{l+1/2}_{\nu-1/2,\mu}(Z^{-1}) \right) \left( \left( l + \nu + 1 \right) N(Z)^{-1} \cdot t^{l+1/2}_{\nu+1/2,\mu}(Z^{-1}) \right),$$

$$G_{l,\mu,\nu}^{(1)}(Z) = \left( t^l_{\mu+1/2,\nu}(Z), t^l_{\mu-1/2,\nu}(Z) \right),$$

$$G_{l,\mu,\nu}^{(1)}(Z) = \left( N(Z)^{-1} \cdot t^{l+1/2}_{\nu-1/2,\mu}(Z^{-1}), N(Z)^{-1} \cdot t^{l+1/2}_{\nu+1/2,\mu}(Z^{-1}) \right).$$
Lemma 11. We have the following recursive relations between the functions $F^{(n)}_{l,\mu,\nu}(Z)$, $F^{(n)}_{l,\mu,\nu}(Z)$, $G^{(n)}_{l,\mu,\nu}(Z)$ and $G^{(n)}_{l,\mu,\nu}(Z)$.

\[
F^{(n+1)}_{l,\mu,\nu}(Z) = \left( \frac{\partial_{11} F^{(n)}_{l+1,2,\mu,\nu-1/2}(Z)}{\partial_{12} F^{(n)}_{l+1,2,\mu,\nu-1/2}(Z)} \right) = \left( \frac{\partial_{21} F^{(n)}_{l+1,2,\mu,\nu+1/2}(Z)}{\partial_{22} F^{(n)}_{l+1,2,\mu,\nu+1/2}(Z)} \right),
\]

\[
F^{(n+1)}_{l,\mu-1,2,\nu}(Z) = -\left( \frac{\partial_{11} F^{(n)}_{l,\mu,\nu}(Z)}{\partial_{12} F^{(n)}_{l,\mu,\nu}(Z)} \right), \quad F^{(n+1)}_{l,\mu,\nu}(Z) = -\left( \frac{\partial_{21} F^{(n)}_{l,\mu,\nu}(Z)}{\partial_{22} F^{(n)}_{l,\mu,\nu}(Z)} \right),
\]

\[
G^{(n+1)}_{l,\mu,\nu}(Z) = (G^{(n+1)}_{l,\mu+1,2,\nu}(Z), G^{(n+1)}_{l,\mu-1,2,\nu}(Z)),
\]

\[
G^{(n+1)}_{l,\mu,\nu}(Z) = (G^{(n+1)}_{l+1,2,\mu,\nu-1/2}(Z), G^{(n+1)}_{l+1,2,\mu,\nu+1/2}(Z)).
\]

Next, we derive two expansions of the Cauchy-Fueter kernel for $n$-regular functions in terms of these functions $F^{(n)}_{l,\mu,\nu}(Z)$, $F^{(n)}_{l,\mu,\nu}(Z)$, $G^{(n)}_{l,\mu,\nu}(Z)$ and $G^{(n)}_{l,\mu,\nu}(Z)$. This is an $n$-regular function analogue of Proposition 26 from [FL1] for the usual regular functions (see also Proposition 112 in [FL4]) and Proposition 12 in [FL4] for doubly regular functions.

Proposition 12. We have the following expansions

\[
k_{n/2}(Z - W) = \sum_{l,\mu,\nu} F^{(n)}_{l,\mu,\nu}(W) \cdot G^{(n)}_{l,\mu,\nu}(Z) = \sum_{l,\mu,\nu} F^{(n)}_{l,\mu,\nu}(Z) \cdot G^{(n)}_{l,\mu,\nu}(W),
\]

which converge uniformly on compact subsets in the region \{(Z, W) \in \mathbb{H}^n \times \mathbb{H}; W Z^{-1} \in \mathbb{D}^+\}. The sums are taken first over all $\mu = -l - n/2, -l, \ldots, l + n/2$, $\nu = -l, -l + 1, \ldots, l$, then over $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

Proof. We use induction on $n$. If $n = 1$, the result reduces to Proposition 26 in [FL1] (see also Proposition 112 in [FL4]). To prove the inductive step, we use the recursive relations from Lemma 11 to expand

\[
\sum_{l,\mu,\nu} F^{(n+1)}_{l,\mu,\nu}(W) \cdot G^{(n+1)}_{l,\mu,\nu}(Z) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},
\]

where

\[
A_{11} = \sum_{l,\mu,\nu} (\partial_{11})_W F^{(n)}_{l+1,2,\mu,\nu-1/2}(W) \cdot G^{(n)}_{l+1,2,\mu,\nu-1/2}(Z) = (\partial_{11})_W k_{n/2}(Z - W),
\]

\[
A_{12} = \sum_{l,\mu,\nu} (\partial_{21})_W F^{(n)}_{l+1,2,\mu,\nu+1/2}(W) \cdot G^{(n)}_{l+1,2,\mu,\nu+1/2}(Z) = (\partial_{21})_W k_{n/2}(Z - W),
\]

\[
A_{21} = \sum_{l,\mu,\nu} (\partial_{12})_W F^{(n)}_{l+1,2,\mu,\nu+1/2}(W) \cdot G^{(n)}_{l+1,2,\mu,\nu-1/2}(Z) = (\partial_{12})_W k_{n/2}(Z - W),
\]

\[
A_{22} = \sum_{l,\mu,\nu} (\partial_{22})_W F^{(n)}_{l+1,2,\mu,\nu-1/2}(W) \cdot G^{(n)}_{l+1,2,\mu,\nu-1/2}(Z) = (\partial_{22})_W k_{n/2}(Z - W)
\]

by induction hypothesis. This proves

\[
\sum_{l,\mu,\nu} F^{(n+1)}_{l,\mu,\nu}(W) \cdot G^{(n+1)}_{l,\mu,\nu}(Z) = \begin{pmatrix} (\partial_{11})_W k_{n/2}(Z - W) & (\partial_{21})_W k_{n/2}(Z - W) \\ (\partial_{12})_W k_{n/2}(Z - W) & (\partial_{22})_W k_{n/2}(Z - W) \end{pmatrix} = k_{(n+1)/2}(Z - W).
\]

The other expansion is proved similarly. □
5  *n*-Regular Functions on $\mathbb{H}^\times$

In this section we show that, if a (left or right) *n*-regular function is defined on all of $\mathbb{H}^\times$, then the operators $(\deg + m)$, $m = 2, \ldots, n$, can be inverted. Thus the operator $D_n^-(\deg + m)$ can be inverted as well. This will be needed, for example, when we define the invariant bilinear pairing for such functions.

We start with a left *n*-regular function $f : \mathbb{H}^\times \to \mathbb{S} \odot \cdots \odot \mathbb{S}$ and derive some properties of such functions. Of course, right *n*-regular functions $g : \mathbb{H}^\times \to \mathbb{S'} \odot \cdots \odot \mathbb{S'}$ have similar properties. Let $0 < r < R$, then, by the Cauchy-Fueter formula for *n*-regular functions (Theorem 9),

$$D_n f(W) = \frac{1}{2\pi^2} \int_{S^3_R} k_{n/2}(Z - W) \cdot (D_z \otimes Z \otimes \cdots \otimes Z) \cdot f(Z)$$

$$- \frac{1}{2\pi^2} \int_{S^3} k_{n/2}(Z - W) \cdot (D_z \otimes Z \otimes \cdots \otimes Z) \cdot f(Z),$$

for all $W \in \mathbb{H}$ such that $r^2 < N(W) < R^2$, where $S^3_R \subset \mathbb{H}$ is the sphere of radius $R$ centered at the origin

$$S^3_R = \{ X \in \mathbb{H}; N(X) = R^2 \}.$$

Define functions $\tilde{f}_+ : \mathbb{H} \to \mathbb{S} \odot \cdots \odot \mathbb{S}$ and $\tilde{f}_- : \mathbb{H}^\times \to \mathbb{S} \odot \cdots \odot \mathbb{S}$ by

$$\tilde{f}_+(W) = \frac{1}{2\pi^2} \int_{S^3_R} k_{n/2}(Z - W) \cdot (D_z \otimes Z \otimes \cdots \otimes Z) \cdot f(Z), \quad R^2 > N(W),$$

$$\tilde{f}_-(W) = -\frac{1}{2\pi^2} \int_{S^3} k_{n/2}(Z - W) \cdot (D_z \otimes Z \otimes \cdots \otimes Z) \cdot f(Z), \quad r^2 < N(W).$$

Note that $\tilde{f}_+$ and $\tilde{f}_-$ are left *n*-regular on their respective domains and that $\tilde{f}_-(W)$ decays at infinity at a rate $\sim N(W)^{-1-1/2}$.

For a function $\varphi$ defined on $\mathbb{H}$ or, slightly more generally, on a star-shaped open subset of $\mathbb{H}$ centered at the origin, let

$$((\deg + m)^{-1} \varphi)(Z) = \int_0^1 t^{m-1} \cdot \varphi(tZ) \, dt, \quad m \geq 1,$$

(cf. Subsection 2.4 of [FL4]). Similarly, for a function $\varphi$ defined on $\mathbb{H}^\times$ and decaying sufficiently fast at infinity, we can define $(\deg + m)^{-1} \varphi$ as

$$((\deg + m)^{-1} \varphi)(Z) = -\int_1^\infty t^{m-1} \cdot \varphi(tZ) \, dt.$$

Then

$$(\deg + m)((\deg + m)^{-1} \varphi) = (\deg + m)^{-1}((\deg + m)\varphi) = \varphi$$

for functions $\varphi$ that are either defined on star-shaped open subsets of $\mathbb{H}$ centered at the origin or on $\mathbb{H}^\times$ and decaying sufficiently fast at infinity. (In the same fashion one can also define $(\deg + m)^{-1} \varphi$ for functions defined on star-shaped open subsets of $\mathbb{H}_C$ centered at the origin or on $\mathbb{H}^\times_C$ and decaying sufficiently fast at infinity.)

We introduce functions

$$f_+ = D_n^{-1} \tilde{f}_+ \quad \text{and} \quad f_- = D_n^{-1} \tilde{f}_-. $$

**Proposition 13.** Let $f : \mathbb{H}^\times \to \mathbb{S} \odot \cdots \odot \mathbb{S}$ be a left *n*-regular function. Then $f(X) = f_+(X) + f_-(X)$, for all $X \in \mathbb{H}^\times$. 

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Proof. The proof is the same as that of Proposition 13 in [4]. Let \( f_\ast = f - f_+ - f_- \), we want to show that \( f_\ast \equiv 0 \). Note that \( f_\ast : \mathbb{H}^\times \to \mathbb{S} \circ \cdots \circ \mathbb{S} \) is a left \( n \)-regular function such that \( D_n f_\ast \equiv 0 \), hence \( f_\ast \) is a sum of homogeneous functions of degrees \(-2, -3, \ldots, -n:\)

\[
f_\ast = f_{-2} + f_{-3} + \cdots + f_{-n}, \quad (\text{deg } j)f_{-j} = 0.
\]

Since the operators \( \nabla^+ \), \( k = 1, \ldots, n \) lower the degree of homogeneity by one, each \( f_{-j} : \mathbb{H}^\times \to \mathbb{S} \circ \cdots \circ \mathbb{S} \) is left \( n \)-regular. We will show that each \( f_{-j} \), \( j = 2, 3, \ldots, n \), is identically zero.

Let

\[
f_{-n-1} = \frac{\partial}{\partial x^0} f_{-n}, \quad f_{-n-1} : \mathbb{H}^\times \to \mathbb{S} \circ \cdots \circ \mathbb{S},
\]

then \( f_{-n-1} \) is a left \( n \)-regular function that is homogeneous of degree \(-n+1\).

By the Cauchy-Fueter formulas for \( n \)-regular functions (Theorem 9),

\[
(-1)^{n-1}(n-1)!f_{-n-1}(X) = \frac{1}{2\pi^2} \int_{S^3}'' k_{n/2}(Z - X) \cdot (Dz \circ Z \circ \cdots \circ Z) \cdot f_{-n-1}(Z)
\]

\[- \frac{1}{2\pi^2} \int_{S^3}'' k_{n/2}(Z - X) \cdot (Dz \circ Z \circ \cdots \circ Z) \cdot f_{-n-1}(Z),
\]

where \( R, r > 0 \) are such that \( r^2 < N(X) < R^2 \). By Liouville’s theorem (Corollary 10), the first integral defines a left \( n \)-regular function on \( \mathbb{H} \) that is either constant or unbounded. On the other hand, the second integral defines a left \( n \)-regular function on \( \mathbb{H}^\times \) that decays at infinity at a rate \( \sim N(W)^{-1-n/2} \). We conclude that \( f_{-n-1} \equiv 0 \), hence \( f_{-n} \equiv 0 \) as well.

A similar argument combined with induction shows that \( f_{-n+1}, \ldots, f_{-2} \equiv 0 \). Hence \( f_\ast \equiv 0 \).

Definition 14. Let \( f : \mathbb{H}^\times \to \mathbb{S} \circ \cdots \circ \mathbb{S} \) be a left \( n \)-regular function. We define

\[
(\text{deg } m)^{-1}f = (\text{deg } m)^{-1}f_+ + (\text{deg } m)^{-1}f_-, \quad m = 1, 2, \ldots, n,
\]

then

\[
D_n^{-1} f = D_n^{-1} f_+ + D_n^{-1} f_-.
\]

Similarly, we can define \( (\text{deg } m)^{-1}g \) and \( D_n^{-1} g \) for right \( n \)-regular functions \( g : \mathbb{H}^\times \to \mathbb{S}' \circ \cdots \circ \mathbb{S}' \).

From the previous discussion we immediately obtain:

Proposition 15. Let \( f : \mathbb{H}^\times \to \mathbb{S} \circ \cdots \circ \mathbb{S} \) be a left \( n \)-regular function and \( g : \mathbb{H}^\times \to \mathbb{S}' \circ \cdots \circ \mathbb{S}' \) a right \( n \)-regular function. Then, for \( m = 1, 2, \ldots, n \),

\[
(\text{deg } m)( (\text{deg } m)^{-1}f ) = (\text{deg } m)^{-1} ((\text{deg } m)f) = f,
\]

\[
(\text{deg } m)( (\text{deg } m)^{-1}g ) = (\text{deg } m)^{-1} ((\text{deg } m)g) = g,
\]

\[
D_n \circ D_n^{-1} f = D_n^{-1} \circ D_n f = f,
\]

\[
D_n \circ D_n^{-1} g = D_n^{-1} \circ D_n g = g.
\]

From the expansions of the Cauchy-Fueter kernel [13] we immediately obtain an analogue of Laurent series expansion for \( n \)-regular functions.
Corollary 16. Let \( f : \mathbb{R}^k \to \mathbb{S} \oplus \cdots \oplus \mathbb{S} \) be a left \( n \)-regular function, write \( f = f_+ + f_- \) as in Proposition 15. Then the functions \( f_+ \) and \( f_- \) can be expanded as series

\[
f_+(X) = \sum_{l} \left( \sum_{\mu,\nu} a_{l,\mu,\nu} F_{l,\mu,\nu}^{(n)}(X) \right), \quad f_-(X) = \sum_{l} \left( \sum_{\mu,\nu} b_{l,\mu,\nu} F_{l,\mu,\nu}^{(n)}(X) \right).
\]

If \( g : \mathbb{R}^k \to \mathbb{S}' \oplus \cdots \oplus \mathbb{S}' \) is a right \( n \)-regular function, then it can be expressed as \( g = g_+ + g_- \) in a similar way, and the functions \( g_+ \) and \( g_- \) can be expanded as series

\[
g_+(X) = \sum_{l} \left( \sum_{\mu,\nu} c_{l,\mu,\nu} G_{l,\mu,\nu}^{(n)}(X) \right), \quad g_-(X) = \sum_{l} \left( \sum_{\mu,\nu} d_{l,\mu,\nu} G_{l,\mu,\nu}^{(n)}(X) \right).
\]

Formulas expressing the coefficients \( a_{l,\mu,\nu}, b_{l,\mu,\nu}, c_{l,\mu,\nu} \) and \( d_{l,\mu,\nu} \) will be given in Corollary 21.

6 Invariant Bilinear Pairing for \( n \)-Regular Functions

We define a pairing between left and right \( n \)-regular functions as follows. If \( f(Z) \) and \( g(Z) \) are left and right \( n \)-regular functions on \( \mathbb{R}^k \) respectively, then, by the results of the previous section, \( D_n^{-1} f \) and \( D_n^{-1} g \) are well defined, and we set

\[
\langle f, g \rangle_{\mathcal{R}_n} = \frac{1}{2\pi^2} \int_{Z \in S^3_{R}} g(Z) \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot (D_n^{-1} f)(Z), \quad k = 1, \ldots, n.
\]

where \( S^3_{R} \subset \mathbb{R}^k \) is the sphere of radius \( R \) centered at the origin

\[
S^3_{R} = \{ X \in \mathbb{R}^k; N(X) = R^2 \}.
\]

Note that we use the subscript \( \mathcal{R}_n \) in \( \langle f, g \rangle_{\mathcal{R}_n} \) to indicate that the pairing is between left and right \( n \)-regular functions; however, this is not a pairing between the spaces \( \mathcal{R}_n \) and \( \mathcal{R}'_n \), since the functions \( f \) and \( g \) are not allowed to have singularities away from the origin. Recall that by Lemma 6 in [FL1], the 3-form \( Dz \) restricted to \( S_{R}^3 \) becomes \( Z dS/R \), where \( dS \) is the usual Euclidean volume element on \( S^3_{R} \). Thus we can rewrite (14) as

\[
\langle f, g \rangle_{\mathcal{R}_n} = \frac{1}{2\pi^2} \int_{Z \in S^3_{R}} g(Z) \cdot (Z \otimes \cdots \otimes Z) \cdot (D_n^{-1} f)(Z) \frac{dS}{R} = \frac{1}{2\pi^2} \int_{Z \in S^3_{R}} g(Z) \cdot (Z \otimes \cdots \otimes Dz \otimes \cdots \otimes Z) \cdot (D_n^{-1} f)(Z), \quad k = 1, \ldots, n.
\]

Since \( \nabla^+_k D_n^{-1} f = 0 \) and, by Lemma 6

\[
\left[ g(Z)(Z \otimes \cdots \otimes 1 \otimes \cdots \otimes Z) \right]_{k-th \ place}^{\nabla^+_k} = 0,
\]

the integrand of (14) is a closed 3-form. Thus, by Lemma 7 the contour of integration can be continuously deformed. In particular, this pairing does not depend on the choice of \( R > 0 \).

Proposition 17. If \( f(Z) \) and \( g(Z) \) are left and right \( n \)-regular functions respectively on \( \mathbb{R}^k \), then

\[
\langle f, g \rangle_{\mathcal{R}_n} = \frac{1}{2\pi^2} \int_{Z \in S^3_{R}} g(Z) \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot (D_n^{-1} f)(Z) = \frac{(-1)^{n-1}}{2\pi^2} \int_{Z \in S^3_{R}} (D_n^{-1} g)(Z) \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot f(Z).
\]
Proof. Since the expression
\[
\int_{Z \in S^1_R} g(Z) \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot f(Z)
\]
is independent of the choice of \( R > 0 \), we have:
\[
0 = \frac{d}{dt} \bigg|_{t=1} \left( \int_{Z \in S^1_R} g(Z) \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot f(Z) \right)
= \int_{Z \in S^1_R} \left( ((\deg + m)g)(Z) \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot f(Z) \right.
\]
\[
\quad + g(Z) \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot ((\deg + 2 + n - m) f)(Z) \bigg),
\]
for all \( m \in \mathbb{Z} \). From this (15) follows.

Corollary 18. If \( f(Z) \) and \( g(Z) \) are left and right \( n \)-regular functions on \( \mathbb{H}_\mathbb{C} \) respectively and \( W \in \mathbb{D}^+_R \) (open domains \( \mathbb{D}^+_R \) were defined by equation (22) in [FL3]), the Cauchy-Fueter formulas for \( n \)-regular functions (Theorem 3) can be rewritten as
\[
f(W) = \langle k_{n/2}(Z - W), f(Z) \rangle_{\mathcal{R}_n} \quad \text{and} \quad g(W) = (-1)^{n-1} \langle g(Z), k_{n/2}(Z - W) \rangle_{\mathcal{R}_n}.
\]

We can rewrite the bilinear pairing (14) in a more symmetrical way. Let \( 0 < r < R \) and \( 0 < r_1 < R < r_2 \). Using the Cauchy-Fueter formulas for \( n \)-regular functions (Theorem 3), substituting
\[
g(Z) = \frac{1}{2\pi^2} \int_{W \in S^1_{r_2}} (D_{n-1}^- g)(W) \cdot (Dw \otimes W \otimes \cdots \otimes W) \cdot k_{n/2}(W - Z)
\]
\[
- \frac{1}{2\pi^2} \int_{W \in S^1_{r_1}} (D_{n-1}^- g)(W) \cdot (Dw \otimes W \otimes \cdots \otimes W) \cdot k_{n/2}(W - Z), \quad Z \in \mathbb{D}_{r_2} \cap \mathbb{D}_{r_1},
\]
into (14) and shifting contours of integration, we obtain:
\[
4\pi^4 \cdot \langle f, g \rangle_{\mathcal{R}_n} = \left( \int_{Z \in S^1_{r_2} \atop W \in S^1_R} (D_{n-1}^- g)(W) \cdot (Dw \otimes W \otimes \cdots \otimes W) \cdot k_{n/2}(W - Z) \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot (D_{n-1}^- f)(Z) \right.
\]
\[
\left. - \int_{Z \in S^1_{r_1} \atop W \in S^1_R} (D_{n-1}^- g)(W) \cdot (Dw \otimes W \otimes \cdots \otimes W) \cdot k_{n/2}(W - Z) \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot (D_{n-1}^- f)(Z) \right) \quad (16)
\]

As usual, we realize \( U(2) \times U(2) \) as a diagonal subgroup of \( GL(2, \mathbb{H}_\mathbb{C}) \):
\[
U(2) \times U(2) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in GL(2, \mathbb{H}_\mathbb{C}); \ a, d \in \mathbb{H}_\mathbb{C}, \ a^*a = 1, \ d^*d = 1 \right\}, \quad (17)
\]
where \( a^* \) and \( d^* \) denote the matrix adjoints of \( a \) and \( d \) in the standard realization of \( \mathbb{H}_\mathbb{C} \) as \( 2 \times 2 \) complex matrices. Then \( SU(2) \times SU(2) \) is realized as
\[
SU(2) \times SU(2) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in GL(2, \mathbb{H}_\mathbb{C}); \ a, d \in \mathbb{H}, \ N(a) = N(d) = 1 \right\}. \quad (18)
\]

Proposition 19. The bilinear pairing (14) is \( SU(2) \times SU(2) \) and \( \mathfrak{gl}(2, \mathbb{H}_\mathbb{C}) \)-invariant.
Proof. It is sufficient to show that the pairing is invariant under

\[ \mathbb{H}^\times \times \mathbb{H}^\times = \{ (a,0; 0,b) \mid a,b \in \mathbb{H}^\times \} \subset GL(2, \mathbb{H}_\mathbb{C}), \]

\[ \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{H}_\mathbb{C}), \quad B \in \mathbb{H}_\mathbb{C}, \text{ and inversion } \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{H}_\mathbb{C}). \]

First, let \( h = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in GL(2, \mathbb{H}_\mathbb{C}), a,d \in \mathbb{H}, \tilde{Z} = a^{-1}Zd. \) Recall that actions \( \pi_{nl} \) and \( \pi_{nr} \) are described by equations (3)-(6). Using Proposition 11 from [PL1], we obtain:

\[
2\pi^2 \cdot \langle \pi_{nl}(h)f, \pi_{nr}(h)g \rangle_{\mathcal{R}_n} = \int_{Z \in S^3_R} (\pi_{nr}(h)g)(Z) \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot (D_n^{-1}(\pi_{nl}(h)f))(Z) \\
= \int_{Z \in S^3_R} g(a^{-1}Zd) \cdot a^{-1} \otimes \cdots \otimes a^{-1} \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot D_n^{-1} \left( \frac{d \otimes \cdots \otimes d}{N(d)^{-1}} \cdot f(a^{-1}Zd) \right) \\
= \int_{\tilde{Z} \in S^3_{R'}} \tilde{g}((\tilde{Z}) \cdot (D\tilde{z} \otimes \tilde{Z} \otimes \cdots \otimes \tilde{Z}) \cdot (D_n^{-1}f)(\tilde{Z}) = 2\pi^2 \cdot \langle f, g \rangle_{\mathcal{R}_n},
\]

where \( R' = \sqrt{N(a)^{-1} \cdot N(d)} \cdot R. \)

Next, we recall that matrices \( \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{H}_\mathbb{C}), B \in \mathbb{H}_\mathbb{C}, \) act by differentiation (Lemma 2). For example, if \( B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathbb{H}_\mathbb{C}, \) using expressions (15)-(16) for the bilinear pairing and the symmetry relation (9), we obtain:

\[
4\pi^4 \cdot \left\langle \pi_{nl}(\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} f, g \right\rangle_{\mathcal{R}_n} = (-1)^{n-1}2\pi^2 \int_{W \in S^3_R} (D_n^{-1}g)(W) \cdot (Dw \otimes W \otimes \cdots \otimes W) \cdot \frac{\partial f}{\partial w_{11}}(W) \\
= \int_{Z \in S^3_R} g(D_n^{-1}g)(W) \cdot (Dw \otimes W \otimes \cdots \otimes W) \cdot \left( \frac{\partial}{\partial w_{11}} k_{n/2}(W-Z) \right) \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot (D_n^{-1}f)(Z) \\
- \int_{Z \in S^3_R} g(D_n^{-1}g)(W) \cdot (Dw \otimes W \otimes \cdots \otimes W) \cdot \left( \frac{\partial}{\partial z_{11}} k_{n/2}(W-Z) \right) \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot (D_n^{-1}f)(Z) \\
+ \int_{Z \in S^3_R} g(D_n^{-1}g)(W) \cdot (Dw \otimes W \otimes \cdots \otimes W) \cdot \left( \frac{\partial}{\partial z_{11}} k_{n/2}(W-Z) \right) \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot (D_n^{-1}f)(Z) \\
= -2\pi^2 \int_{Z \in S^3_R} \frac{\partial g}{\partial z_{11}}(Z) \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot (D_n^{-1}f)(Z) = -4\pi^4 \cdot \left\langle f, \pi_{nr}(\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} g \right\rangle_{\mathcal{R}_n}.
\]

Finally, if \( h = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \in GL(2, \mathbb{H}_\mathbb{C}), \) changing the variable to \( \tilde{Z} = Z^{-1} \) – which is an orientation reversing map \( S^3_R \rightarrow S^3_{1/R} \) – and using Proposition 11 from [PL1], we have:

\[
2\pi^2 \cdot \langle \pi_{nl}(h)f, \pi_{nr}(h)g \rangle_{\mathcal{R}_n} \\
= (-1)^{n} \int_{Z \in S^3_R} g(Z^{-1}) \cdot \frac{Z^{-1} \otimes \cdots \otimes Z^{-1}}{N(Z)} \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot D_n^{-1} \left( \frac{Z^{-1} \otimes \cdots \otimes Z^{-1}}{N(Z)} \cdot f(Z^{-1}) \right) \\
= -\int_{Z \in S^3_R} g(Z^{-1}) \cdot \frac{(Z^{-1} \cdot Dz \cdot Z^{-1}) \otimes Z^{-1} \otimes \cdots \otimes Z^{-1}}{N(Z)^2} \cdot (D_n^{-1}f)(Z^{-1}) \\
= -\int_{\tilde{Z} \in S^3_{1/R}} g(\tilde{Z}) \cdot (D\tilde{z} \otimes \tilde{Z} \otimes \cdots \otimes \tilde{Z}) \cdot (D_n^{-1}f)(\tilde{Z}) = -2\pi^2 \cdot \langle f, g \rangle_{\mathcal{R}_n},
\]
where for the second equality we used the property

\[(\deg + m)^{-1}\left(\frac{Z^{-1} \otimes \cdots \otimes Z^{-1}}{N(Z)} \cdot f(Z^{-1})\right) = -\frac{Z^{-1} \otimes \cdots \otimes Z^{-1}}{N(Z)} \cdot ((\deg + n + 2 - m)^{-1} f)(Z^{-1}).\]

(Note that the negative sign in \((\pi_n(h)f, \pi_m(h)g)R_n = -(f, g)R_n\) does not affect the invariance of the bilinear pairing under the Lie algebra \(\mathfrak{g}(2, \mathbb{H}_C)\).)

Next we describe orthogonality relations for \(n\)-regular functions. Recall functions \(F_{l,\mu,\nu}^{(n)}\), \(F_{l,\mu,\nu}^{(n)}\), \(G_{l,\mu,\nu}^{(n)}\) and \(G_{l,\mu,\nu}^{(n)}\) introduced in Section 4; these are the functions that appear in matrix coefficient expansions of the Cauchy-Fueter kernel (13).

**Proposition 20.** We have the following orthogonality relations:

\[
\langle F_{l,\mu,\nu}^{(n)}, G_{l',\mu',\nu'}^{(n)} \rangle_{R_n} = (-1)^{n-1} \langle F_{l,\mu,\nu}^{(n)}, \delta_{l',l} \cdot \delta_{\mu',\mu} \cdot \delta_{\nu',\nu} \rangle_{R_n},
\]

(19)

\[
\langle F_{l,\mu,\nu}^{(n)}, G_{l',\mu',\nu'}^{(n)} \rangle_{R_n} = \langle F_{l,\mu,\nu}^{(n)}, G_{l',\mu',\nu'}^{(n)} \rangle_{R_n} = 0.
\]

(20)

In particular, (14) is a non-degenerate bilinear pairing between

\[
\mathbb{C}\text{-span of } \{F_{l,\mu,\nu}^{(n)}(Z)\} \text{ and } \mathbb{C}\text{-span of } \{G_{l,\mu,\nu}^{(n)}(Z)\}
\]

and between

\[
\mathbb{C}\text{-span of } \{F_{l,\mu,\nu}^{(n)}(Z)\} \text{ and } \mathbb{C}\text{-span of } \{G_{l,\mu,\nu}^{(n)}(Z)\},
\]

where \(l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \mu \in \mathbb{Z} + l + n/2, \nu \in \mathbb{Z} + l, -l - n/2 \leq \mu \leq l + n/2, -l \leq \nu \leq l\).

**Proof.** Observe that the functions \(F_{l,\mu,\nu}^{(n)}\) and \(G_{l,\mu,\nu}^{(n)}\) are homogeneous of degree \(2l\), while the functions \(F_{l,\mu,\nu}^{(n)}\) and \(G_{l,\mu,\nu}^{(n)}\) are homogeneous of degree \(-2l - n - 2\). On the one hand, the bilinear pairing (14) is independent of the choice of radius \(R > 0\). Recall that, by Lemma 6 in [FL1], the 3-form \(Dz\) restricted to \(S^3_R\) equals \(Z \cdot dS/R\). It follows that if \(f\) is homogeneous of degree \(d_f\) and \(g\) is homogeneous of degree \(d_g\), then either \((f, g)R_n = 0\) or \(d_f + d_g = -(n+2)\). This proves (20) and

\[
\langle F_{l,\mu,\nu}^{(n)}, G_{l',\mu',\nu'}^{(n)} \rangle_{R_n} = \langle F_{l,\mu,\nu}^{(n)}, G_{l',\mu',\nu'}^{(n)} \rangle_{R_n} = 0 \text{ if } l \neq l'.
\]

Then (19) follows from the expansions (13), the Cauchy-Fueter formulas (Theorem 9) and equation (15).

**Corollary 21.** The coefficients \(a_{l,\mu,\nu}\), \(b_{l,\mu,\nu}\), \(c_{l,\mu,\nu}\) and \(d_{l,\mu,\nu}\) of Laurent expansions of \(n\)-regular functions given in Corollary (16) are given by the following expressions:

\[
a_{l,\mu,\nu} = \langle f, G_{l,\mu,\nu}^{(n)} \rangle_{R_n}, \quad b_{l,\mu,\nu} = (-1)^{n-1} \langle f, G_{l,\mu,\nu}^{(n)} \rangle_{R_n},
\]

\[
c_{l,\mu,\nu} = (-1)^{n-1} \langle F_{l,\mu,\nu}, g \rangle_{R_n}, \quad d_{l,\mu,\nu} = \langle F_{l,\mu,\nu}, g \rangle_{R_n}.
\]
7 Spaces of \( n \)-Regular Functions as Representations of the Conformal Lie Algebra

In this section we identify the irreducible components and the \( K \)-types of the spaces of left and right \( n \)-regular functions on \( \mathbb{H}^\times \) regarded as representations of the conformal Lie algebra \( \mathfrak{gl}(2, \mathbb{H}_\mathbb{C}) \). Let

\[
F_n^+ = \mathbb{C}\text{-span} \{ F_{l,\mu,\nu}^{(n)}(Z) \}, \quad F_n^- = \mathbb{C}\text{-span} \{ F_{l,\mu,\nu}^{(n)}(Z) \},
\]

\[
G_n^+ = \mathbb{C}\text{-span} \{ G_{l,\mu,\nu}^{(n)}(Z) \}, \quad G_n^- = \mathbb{C}\text{-span} \{ G_{l,\mu,\nu}^{(n)}(Z) \},
\]

where

\[
l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots, \quad \mu \in \mathbb{Z} + l + n/2, \quad \nu \in \mathbb{Z} + l, \quad -l - n/2 \leq \mu \leq l + n/2, \quad -l \leq \nu \leq l.
\]

We can think of \( \mathbb{C}[z^0, z^1, z^2, z^3] \otimes (S \cdots S) \) as the space of \( (S \cdots S) \)-valued polynomials on \( \mathbb{H} \) or \( \mathbb{H}_\mathbb{C} \). Similarly, \( \mathbb{C}[z^0, z^1, z^2, z^3, N(Z)^{-1}] \otimes (S' \cdots S') \) can be thought as \( (S' \cdots S') \)-valued Laurent polynomials on \( \mathbb{H}_\mathbb{C}^\times \). We can give basis-free descriptions of the spaces \( F_n^\pm \) and \( G_n^\pm \).

Proposition 22. We have:

\[
F_n^+ = \{ \text{left } n\text{-regular polynomials } f \in \mathbb{C}[z^0, z^1, z^2, z^3] \otimes (S \cdots S) \},
\]

\[
G_n^+ = \{ \text{right } n\text{-regular polynomials } g \in \mathbb{C}[z^0, z^1, z^2, z^3] \otimes (S' \cdots S') \},
\]

\[
F_n^+ \oplus F_n^- = \{ \text{left } n\text{-regular functions } f \in \mathbb{C}[z^0, z^1, z^2, z^3, N(Z)^{-1}] \otimes (S \cdots S) \},
\]

\[
G_n^+ \oplus G_n^- = \{ \text{right } n\text{-regular functions } g \in \mathbb{C}[z^0, z^1, z^2, z^3, N(Z)^{-1}] \otimes (S' \cdots S') \},
\]

\[
F_n^- = \{ \text{left } n\text{-regular functions } f \in \mathbb{C}[z^0, z^1, z^2, z^3, N(Z)^{-1}] \otimes (S \cdots S) ; \pi_{nl}(0 \ 1) f \in \mathbb{C}[z^0, z^1, z^2, z^3] \otimes (S \cdots S) \},
\]

\[
G_n^- = \{ \text{right } n\text{-regular functions } g \in \mathbb{C}[z^0, z^1, z^2, z^3, N(Z)^{-1}] \otimes (S' \cdots S') ; \pi_{nr}(0 \ 1) g \in \mathbb{C}[z^0, z^1, z^2, z^3] \otimes (S' \cdots S') \}.
\]

In particular, we have complex vector space isomorphisms

\[
\pi_{nl}(0 \ 1) : F_n^+ \leftrightarrow F_n^- \quad \text{and} \quad \pi_{nr}(0 \ 1) : G_n^+ \leftrightarrow G_n^-.
\]

Proof. The descriptions of \( F_n^+, G_n^+, F_n^+ \oplus F_n^- \) and \( G_n^+ \oplus G_n^- \) follow from Corollaries \[16\] and \[21\]. Note that the inversion \( \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \in GL(2, \mathbb{H}_\mathbb{C}) \) preserves \( \mathbb{C}[z^0, z^1, z^2, z^3, N(Z)^{-1}] \otimes (S \cdots S) \) as well as \( F_n^+ \oplus F_n^- \). Comparing the degrees of homogeneity, we see that \( \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \) switches \( F_n^+ \) and \( F_n^- \), and the description of \( F_n^- \) follows. Finally, the case of \( G_n^- \) is similar to that of \( F_n^- \). □

Recall that we realize the maximal compact subgroup \( U(2) \times U(2) \) of \( U(2, 2) \) as well as \( SU(2) \times SU(2) \) as diagonal subgroups of \( GL(2, \mathbb{H}_\mathbb{C}) \) using \[17\]-\[18\].

Proposition 23. The spaces \( F_n^+ \), \( F_n^- \) are invariant under the \( \pi_{nl} \)-actions of \( U(2) \times U(2) \) and \( \mathfrak{gl}(2, \mathbb{H}_\mathbb{C}) \). Similarly, the spaces \( G_n^+ \), \( G_n^- \) are invariant under the \( \pi_{nr} \)-actions of \( U(2) \times U(2) \) and \( \mathfrak{gl}(2, \mathbb{H}_\mathbb{C}) \).

Proof. To show that \( F_n^+ \) is invariant under the \( \pi_{nl} \)-action of \( \mathfrak{gl}(2, \mathbb{H}_\mathbb{C}) \), we need to show that if \( F_{l,\mu,\nu}^{(n)}(Z) \in F_n^+ \) and \( \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in \mathfrak{gl}(2, \mathbb{H}_\mathbb{C}) \), then \( \pi_{nl}(A \ B \ C \ D) F_{l,\mu,\nu}^{(n)}(Z) \in F_n^+ \). Indeed, consider first a special case of \( A = B = D = 0 \), then \( F_{l,\mu,\nu}^{(n)}(Z) \) is homogeneous of degree \( 2l \) and \( \pi_{nl}(0 \ 0) F_{l,\mu,\nu}^{(n)}(Z) \).
is homogeneous of degree $2l+1$. By the Cauchy-Fueter formula for $n$-regular functions (Theorem 9),

$$
\frac{(2l + n + 1)!}{(2l + 2)!} \pi_n l \left( \begin{array}{cc} 0 & 0 \\ C & 0 \end{array} \right) F_{l,\mu,\nu}^{(n)}(W) = \frac{1}{2\pi^2} \int_{S_R^3} k_{n/2}(Z - W) \cdot (Dz \otimes Z \otimes \cdots \otimes Z) \cdot \left( \pi_n l \left( \begin{array}{cc} 0 & 0 \\ C & 0 \end{array} \right) F_{l,\mu,\nu}^{(n)}(Z) \right), \tag{23}
$$

where $S_R^3 \subset \mathbb{H}$ is the sphere centered at the origin of radius $R$ large enough so that $W \in \mathbb{D}_R^+$. On the other hand, recall the expansion (13) of the Cauchy-Fueter kernel (2).

Proposition 24. Each $F_n^\pm(d)$ is homogeneous of degree $d$. Similarly, we define $F_n^\pm(d)$, $G_n^\pm(d)$ and $G_n^\pm(d)$. As in our previous papers, we denote by $(\tau_l, V_l)$ the irreducible $(2l + 1)$-dimensional representation of $SU(2)$ or $\mathfrak{sl}(2, \mathbb{C})$, with $l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$.

Proposition 24. Each $F_n^\pm(d)$ is invariant under the $\pi_n l$ action restricted to $SU(2) \times SU(2)$, and we have the following decomposition into irreducible components:

$$
F_n^+(2l) = V_l \otimes V_{l+\frac{1}{2}}, \quad F_n^-(d) = \{0\} \quad \text{if } d < 0, \quad F_n^+(d) = \{0\} \quad \text{if } d > -(n+2), \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots.
$$

Similarly, each $G_n^\pm(d)$ is invariant under the $\pi_{nr}$ action restricted to $SU(2) \times SU(2)$, and we have the following decomposition into irreducible components:

$$
G_n^+(2l) = V_{l+\frac{n}{2}} \otimes V_l, \quad G_n^+(d) = \{0\} \quad \text{if } d < 0, \quad G_n^-(d) = \{0\} \quad \text{if } d > -(n+2), \quad l = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots.
$$

Remark 25. When $n = 1$, this result becomes Proposition 21 in [FT1].

Proof. Recall that the actions of $SU(2) \times SU(2)$ on $F_n^\pm$ and $G_n^\pm$ are obtained by restricting the actions of $GL(2, \mathbb{H}_\mathbb{C})$ described by [3]-[7]. It follows that $SU(2) \times SU(2)$ preserves each $F_n^\pm(d)$ and $G_n^\pm(d)$.

Next we identify each $F_n^\pm(d)$ and $G_n^\pm(d)$ as a representation of $SU(2) \times SU(2)$. First, we consider the case of $F_n^+(d)$. It is clear from the definition of $F_n^+$ that $F_n^+(d) = \{0\}$ when $d < 0$. Consider now the case of

$$
F_n^+(2l) = \mathbb{C}\text{-span of } \left\{ F_{l,\mu,\nu}^{(n)}(Z) ; \mu \in \mathbb{Z} + l + n/2, -l + n/2 \leq \mu \leq l + n/2, -l \leq \nu \leq l \right\};
$$
this space has dimension \((2l + 1)(2l + n + 1)\). It is clear from (5) that the first copy of \(SU(2)\) acts via \(\gamma_l\) and that, as representations of \(SU(2) \times SU(2)\),

\[
\mathcal{F}_n^+ (2l) \simeq V \otimes V',
\]

where \(V'\) is a subrepresentation of \(V_{n-1} \otimes V_l\). Counting dimensions, we find that \(\dim V' = 2l + n + 1\). To show that \(V' \simeq V_{l + \frac{n}{2}}\), consider the action of an element \(d' = \left( \begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix} \right)\) in the second copy of \(SU(2)\). By the representation theory of \(SU(2)\), it is sufficient to show that this element acts on each \(F_{l,\mu,\nu}^{(n)}(Z)\) by multiplication by \(\lambda^{2l+n}\). Indeed, by the definition of \(F_{l,\mu,\nu}^{(n)}(Z)\) (equation (12)) and by Lemma 22 in [FL1], \(F_{l,\mu,\nu}^{(n)}(Z)\) is proportional to

\[
\left( \begin{array}{c} 0 \\ \vdots \\ 0 \\ 0 \end{array} \right) \otimes \cdots \otimes \left( \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right) t^l_{\nu - l}(Z),
\]

and the matrix coefficient \(t^l_{\nu - l}(Z)\) is in turn proportional to \(z^{l+\nu}_{11} z^{-l-\nu}_{21}\). Then it is easy to see from (5) that the element \(h = \left( \begin{smallmatrix} 1 & 0 \\ 0 & d' \end{smallmatrix} \right)\) is an element of \(SU(2) \times SU(2)\), where \(d' = \left( \begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix} \right)\), acts on

\[
\left( \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right) \otimes \cdots \otimes \left( \begin{array}{c} 0 \\ \vdots \\ 0 \end{array} \right) t^l_{\nu - l}(Z)
\]

by multiplication by \(\lambda^{2l+n}\).

To identify \(\mathcal{F}_n^-(2l-n-2)\) as a representation of \(SU(2) \times SU(2)\), observe that, by Proposition 22, the element \(\left( \begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)\) maps \(\mathcal{F}_n^+(2l)\) into \(\mathcal{F}_n^-(2l-n-2)\) (and vice versa) and switches the two factors in \(SU(2) \times SU(2)\). Finally, we can identify \(\mathcal{G}_n^\pm(d)\) using the bilinear pairing (13) which, by Propositions 19 and 20, restricts to non-degenerate \(SU(2) \times SU(2)\)-invariant pairings between

\[
\mathcal{F}_n^+(d) \quad \text{and} \quad \mathcal{G}_n^-(d-n-2)
\]

and between

\[
\mathcal{F}_n^-(d) \quad \text{and} \quad \mathcal{G}_n^+(d-n-2).
\]

We conclude our description of the representations \(\mathcal{F}_n^\pm\) and \(\mathcal{G}_n^\pm\) by showing their irreducibility.

**Theorem 26.** We have:

\[
(\pi_{nl}, \mathcal{F}_n^\pm), \quad (\pi_{nr}, \mathcal{F}_n^\pm), \quad (\pi_{nl}, \mathcal{G}_n^\pm), \quad (\pi_{nr}, \mathcal{G}_n^\pm)
\]

are irreducible representations of \(\mathfrak{sl}(2, \mathbb{H}_C)\) (as well as \(\mathfrak{gl}(2, \mathbb{H}_C)\)).

**Proof.** We will show that \(\mathcal{F}_n^\pm\) are irreducible. Then the irreducibility of \(\mathcal{G}_n^\pm\) is immediate from Propositions 19 and 20. It is easy to see from the description of the \(K\)-types of \(\mathcal{F}_n^\pm\) given by Proposition 24 that it is sufficient to show that any non-zero vector in \(\mathcal{F}_n^+(d)\) also generates non-zero vectors in

\[
\mathcal{F}_n^+(d-1) \quad \text{as long as} \quad \mathcal{F}_n^+(d-1) \neq \{0\} \quad \text{and}
\]

\[
\mathcal{F}_n^+(d+1) \quad \text{as long as} \quad \mathcal{F}_n^+(d+1) \neq \{0\}
\]

(and similarly for \(\mathcal{F}_n^-(d)\)). This follows from the observation that if \(v_d \in \mathcal{F}_n^+(d)\), then

\[
\pi_{nl} \left( \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \right) v_d \in \mathcal{F}_n^+(d-1), \quad \pi_{nl} \left( \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \right) v_d \in \mathcal{F}_n^+(d+1),
\]

for each \(\left( \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \right) \in \mathfrak{sl}(2, \mathbb{H}_C)\). Lemma 2 describing \(\pi_{nl} \left( \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \right)\) and the fact that conjugation by \(\left( \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \right) \in \mathfrak{gl}(2, \mathbb{H}_C)\) switches \(\left( \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \right)\) and \(\left( \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} \right)\). \(\square\)
Note that the results of this section show that \( \mathcal{F}_n^\pm \) and \( \mathcal{G}_n^\pm \) are irreducible Harish-Chandra modules of a very special kind – they have highest or lowest weights.

8 Unitary Structures on \( \mathcal{F}_n^\pm \) and \( \mathcal{G}_n^\pm \)

In this section we describe \( u(2,2) \)-invariant inner products on \((\pi_{nl}, \mathcal{F}_n^+)\), \((\pi_{nl}, \mathcal{F}_n^-)\), \((\pi_{nr}, \mathcal{G}_n^+)\) and \((\pi_{nr}, \mathcal{G}_n^-)\). We realize \( U(2,2) \) as the subgroup of elements of \( GL(2, \mathbb{H}_\mathbb{C}) \) preserving the Hermitian form on \( \mathbb{C}^4 \) given by the \( 4 \times 4 \) matrix \((\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})\). Explicitly,

\[
U(2,2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{H}_\mathbb{C}, \quad d^*d = 1 + b^*b, \quad a^*a = 1 + c^*c \right\}.
\]

(Recall that \( a^* \) and \( d^* \) denote the matrix adjoints of \( a \) and \( d \) in the standard realization of \( \mathbb{H}_\mathbb{C} \) as \( 2 \times 2 \) complex matrices.) Then the Lie algebra of \( U(2,2) \) is

\[
u(2,2) = \left\{ \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \mid A, B, D \in \mathbb{H}_\mathbb{C}, \quad A = -A^*, \quad D = -D^* \right\}. \tag{24}
\]

Recall from Subsection 2.3 of [FL1] the \( \mathbb{C} \)-antilinear maps \( S \to S' \) and \( S' \to S \) – matrix transposition followed by complex conjugation. Since they are similar to quaternionic conjugation, we use the same symbol for these maps:

\[
\begin{pmatrix} s_1 \\ s_2 \end{pmatrix}^+ = (\bar{s}_1, \bar{s}_2), \quad (s'_1, s'_2)^+ = \left( \frac{\bar{s}'_1}{s'_2} \right), \quad \left( \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}^+ \right) \in S, \ (s'_1, s'_2) \in S'.
\]

Note that

\[
(Zs)^+ = s^+\bar{Z}^+, \quad (s'Z)^+ = \bar{Z}^+s'^+, \quad Z \in \mathbb{H}_\mathbb{C}, \ s \in S, \ s' \in S'; \tag{25}
\]

where \( \bar{Z} \) denotes complex conjugation relative to \( \mathbb{H} \):

\[
\begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} = \begin{pmatrix} z_{12} & -\bar{z}_{21} \\ -\bar{z}_{12} & \bar{z}_{21} \end{pmatrix}. \tag{26}
\]

These conjugation maps extend to tensor products

\[
S \otimes \cdots \otimes S \quad \text{and} \quad S' \otimes \cdots \otimes S' \quad \text{\( n \) times}
\]

in the most obvious way. We introduce a map \( \sigma : \hat{\mathcal{F}}_n \to \hat{\mathcal{G}}_n \) defined by

\[
\sigma(f)(Z) = f^+(\bar{Z}^+) = f^+(\bar{Z}^*), \quad \text{then} \quad \sigma^{-1}(g)(Z) = g^+(\bar{Z}^+) = g^+(\bar{Z}^*).
\]

Then \( \sigma \) produces an isomorphism of real vector spaces \( \mathcal{R}_n \to \mathcal{R}'_n \), we call its inverse \( \sigma^{-1} \). Clearly, \( \sigma \) and \( \sigma^{-1} \) map functions that are polynomial on \( \mathbb{H}_\mathbb{C} \) (respectively \( \mathbb{H}_\mathbb{C}^\times \)) into functions that are polynomial on \( \mathbb{H}_\mathbb{C} \) (respectively \( \mathbb{H}_\mathbb{C}^\times \)). By Proposition 22 \( \sigma \) and \( \sigma^{-1} \) restrict to isomorphisms of real vector spaces

\[
\sigma : \mathcal{F}_n^+ \to \mathcal{G}_n^+, \quad \sigma^{-1} : \mathcal{G}_n^+ \to \mathcal{F}_n^+, \quad \sigma : \mathcal{F}_n^- \to \mathcal{G}_n^-, \quad \sigma^{-1} : \mathcal{G}_n^- \to \mathcal{F}_n^-.
\]

From (23)-(26) and Lemma 2 we obtain the following commutation relations between \( \sigma \) and the Lie algebra actions \( \pi_{nl}, \pi_{nr} \) of \( gl(2, \mathbb{H}_\mathbb{C}) \):

\[
\sigma \circ \pi_{nl} \left( \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right) f = \pi_{nr} \left( \begin{pmatrix} 0 & B^* \\ C^* & 0 \end{pmatrix} \right) \circ \sigma(f), \tag{27}
\]

\[
\sigma \circ \pi_{nl} \left( \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right) f = \pi_{nr} \left( \begin{pmatrix} 0 & B^* \\ C^* & 0 \end{pmatrix} \right) \circ \sigma(f), \quad B, C \in \mathbb{H}_\mathbb{C}. \tag{28}
\]
Let us consider pairings on $\mathcal{F}_n^\pm$ and $\mathcal{G}_n^\pm$:

\[
(f_1, f_2)_{\mathcal{R}_n} = \frac{1}{2\pi^2} \int_{Z \in S^3} f_2^+(Z) \cdot (D_n^{-1} f_1)(Z) dS, \quad f_1, f_2 \in \mathcal{F}_n^+, \tag{29}
\]

\[
(g_1, g_2)_{\mathcal{R}_n^+} = \frac{1}{2\pi^2} \int_{Z \in S^3} (D_n^{-1} g_1)(Z) \cdot g_2^+(Z) dS, \quad g_1, g_2 \in \mathcal{G}_n^+, \tag{30}
\]

where $S^3 \subset \mathbb{H} \subset \mathbb{H}_c$ is the unit sphere centered at the origin. Clearly, these pairings are complex anti-linear, positive definite on $\mathcal{F}_n^+$ and $\mathcal{G}_n^+$, and either positive or negative definite on $\mathcal{F}_n^-$ and $\mathcal{G}_n^-$, depending on the effect of the operator $D_n^{-1}$.

**Theorem 27.** The pairing (29) is invariant under $u(2, 2)$-invariant on $(\pi_{nl}, \mathcal{F}_n^+)$ and $(\pi_{nl}, \mathcal{F}_n^-)$. Similarly, the pairing (30) is invariant under $(\pi_{nr}, \mathcal{G}_n^+)$ and $(\pi_{nr}, \mathcal{G}_n^-)$. (The Lie algebra $u(2, 2)$ is realized as a real form of $\mathfrak{gl}(2, \mathbb{H}_c)$ as in (24).)

**Proof.** We will prove the invariance of (29) only, the other case is similar. First, we relate (29) to the bilinear pairing (14):

\[
(f_1, f_2)_{\mathcal{R}_n} = \langle f_1, \sigma \circ \pi_{nl} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} f_2 \rangle_{\mathcal{R}_n}.
\]

Using the $\mathfrak{gl}(2, \mathbb{H}_c)$-invariance of the bilinear pairing (14) (Proposition 19) and relations (27)-(28), we find:

\[
(\pi_{nl} \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} f_1, f_2)_{\mathcal{R}_n} = \langle \pi_{nl} \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} f_1, \sigma \circ \pi_{nl} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} f_2 \rangle_{\mathcal{R}_n} = -\langle f_1, \pi_{nr} \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} \circ \sigma \circ \pi_{nl} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} f_2 \rangle_{\mathcal{R}_n} = -\langle f_1, \sigma \circ \pi_{nl} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \circ \pi_{nl} \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} f_2 \rangle_{\mathcal{R}_n} = -\langle f_1, \pi_{nl} \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix} f_2 \rangle_{\mathcal{R}_n}
\]

and, similarly,

\[
(\pi_{nl} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} f_1, f_2)_{\mathcal{R}_n} = -\langle f_1, \pi_{nl} \begin{pmatrix} 0 & C^* \\ 0 & 0 \end{pmatrix} f_2 \rangle_{\mathcal{R}_n}.
\]

This proves the invariance of (29) under $\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in u(2, 2)$, $B \in \mathbb{H}_c$. To show that the pairing (29) is invariant under $\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \in u(2, 2)$, $A, D \in \mathbb{H}_c$, $A = -A^*$, $D = -D^*$, one checks directly that (29) is invariant under the group $U(2) \times U(2)$ realized as (17). \hfill \square

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