NEW BOUNDS ON THE SIZE OF PERMUTATION CODES WITH MINIMUM KENDALL $\tau$-DISTANCE OF THREE

A. ABDOLLAHI, J. BAGHERIAN, F. JAFARI, M. KHATAMI, F. PARVARESH, AND R. SOBHANI

Abstract. We study $P(n,3)$, the size of the largest subset of the set of all permutations $S_n$ with minimum Kendall $\tau$-distance 3. Using a combination of group theory and integer programming, we reduced the upper bound of $P(p,3)$ from $(p-1)!-1$ to $(p-1)!-\left\lceil \frac{p}{3} \right\rceil + 2 \leq (p-1)!-2$ for all primes $p \geq 11$. In special cases where $n$ is equal to 6, 7, 11, 13, 14, 15 and 17 we reduced the upper bound of $P(n,3)$ by 3, 3, 9, 11, 1 and 4, respectively.

1. Introduction

Rank modulation was proposed as a solution to the challenges posed by flash memory storages[8]. In the rank modulation framework, codes are permutation codes, where by a permutation code (PC) of length $n$ we simply mean a non-empty subset $C$ of $S_n$, the set of all permutations of $\mathbb{Z} := \{1, 2, \ldots, n\}$. Given a permutation $\pi := [\pi(1), \pi(2), \ldots, \pi(i), \pi(i+1), \ldots, \pi(n)] \in S_n$, an adjacent transposition, $(i, i+1)$, for some $1 \leq i \leq n-1$, applied to $\pi$ will result in the permutation $[\pi(1), \pi(2), \ldots, \pi(i+1), \pi(i), \ldots, \pi(n)]$. For two permutations $\rho, \pi \in S_n$, the Kendall $\tau$-distance between $\rho$ and $\pi$, $d_{K}(\rho, \pi)$, is defined as the minimum number of adjacent transpositions needed to transform $\rho$ into $\pi$. Under the Kendall $\tau$-distance a PC of length $n$ with minimum distance $d$ can correct up to $\frac{d-1}{2}$ errors caused by charge-constrained errors [8].

The maximum size of a PC of length $n$ and minimum Kendall $\tau$-distance $d$ is denoted by $P(n,d)$. Several researchers have presented bounds on $P(n,d)$ (see [1, 2, 8, 10, 11, 12]), some of these results are shown in Table 1. It is known that $P(n,1) = n!$ and $P(n,2) = \frac{n!}{2}$. Also it is known that if $\frac{\binom{n}{3}}{\frac{n!}{2}} < d \leq \binom{n}{2}$, then $P(n,d) = 2$ (see [2, Theorem 10]). However, determining $P(n,d)$ turns out to be difficult for $3 < d \leq \frac{\binom{n}{2}}{2}$. In this paper, we study the upper bound of $P(n,3)$. By sphere packing bound (see [8, Theorems 17 and 18]) $P(n,3) \leq (n-1)!$. It is proved that if $n > 4$ is a prime number or $4 \leq n \leq 10$, then $P(n,3) \neq (n-1)!$ (see [5, Corollary 2.5 and Theorem 2.6] or [2, Corollary 2]).

There are several works which uses optimization techniques to bound the size of permutation codes under various distance metrics (Hamming, Kendall, Ulam) (see [6, 9, 10]). In Section II of this paper, we show that for any non-trivial subgroup of $S_n$, we can derive an integer programming where the optimal value of the objective function gives an upper bound on $P(n,3)$. In Section III, by considering Young
subgroups (see Definition 3.1 below) of $S_n$, we can improve the upper bound of $P(n,3)$ as shown in Table II.

2. Preliminaries

Let $G$ be a finite group and denote by $\mathbb{C}[G]$ the complex group algebra of $G$. The elements of $\mathbb{C}[G]$ are of the formal sum

\[ \sum_{g \in G} a_g g, \]

where $a_g \in \mathbb{C}$. The complex group algebra is a $\mathbb{C}$-algebra with the following addition, multiplication and scalar product:

\[ \sum_{g \in G} a_g g + \sum_{g \in G} b_g g = \sum_{g \in G} (a_g + b_g) g, \]

\[ \sum_{g \in G} a_g g \cdot \sum_{g \in G} b_g g = \sum_{g \in G} \left( \sum_{g = g_1 g_2} a_{g_1} b_{g_2} \right) g, \]

\[ \lambda \sum_{g \in G} a_g g = \sum_{g \in G} (\lambda a_g) g, \]

where $\lambda \in \mathbb{C}$ and $\sum_{g \in G} b_g g \in \mathbb{C}[G]$. If $a_g = 0$ for some $g$, the term $a_g g$ will be neglected in $\mathbb{C}[G]$ and $\sum_{g \in G} a_g g$ is written as $a_1 g_1 + \cdots + a_k g_k$, where $\{ g \mid a_g \neq 0 \} = \{ g_1, \ldots, g_k \}$ is non-empty and otherwise $\sum_{g \in G} a_g g$ is denoted by 0.

For a non-empty subset $X$ of $G$, we denote by $\bar{X}$ the element $\sum_{x \in X} x$ of $\mathbb{C}[G]$.

Let $G$ be a finite group and $S$ be a non-empty inverse closed subset of $G$ not-containing the identity element 1 of $G$. Consider the Cayley graph $\Gamma := Cay(G, S)$ whose vertices are elements of $G$ and two vertices $g, h$ are adjacent if $gh^{-1} \in S$. Now we have a metric $d_{\Gamma}$ on $G$ defined by $\Gamma$ which is the length of a shortest path between two vertices in $Cay(G, S)$. For example if $G = S_n$ and $S = \{ (1,2), (2,3), \ldots, (n-1, n) \}$, the metric $d_{\Gamma}$ is the Kendall $\tau$-metric on $S_n$. Also if $G = S_n$ and $S = T \cup T^{-1}$, where $T := \{ (a, a+1, \ldots, b) \mid a < b, a, b \in [n] \}$, the metric $d_{\Gamma}$ is the Ulam metric on $S_n$.

**Definition 2.1.** For a positive ineteger $r$ and an element $g \in G$, the ball of radius $r$ in $G$ under the metric $d_{\Gamma}$ is denoted by $B_{\Gamma}^r(g)$ defined by $B_{\Gamma}^r(g) = \{ h \in G \mid d_{\Gamma}(g,h) \leq r \}$.

| $n$ | 6 | 7 | 11 | 13 |
|-----|---|---|----|----|
| LB  | 102 | 588 | 11!/20 | 13!/24 |
| UB  | $5! - 1^a$ | $6! - 1^a$ | $10! - 1^a$ | $12! - 1^a$ |
| UB  | $5! - 4$ | $6! - 4$ | $10! - 10$ | $12! - 12$ |

Table 1. Some results on bounds of $P(n,3)$. The superscripts shows the references from which the upper bound is taken, where “a” is 2 5, and gray color shows our main results.
Remark 2.2. Note that $B^r_r(g) = (S^r \cup \{1\})g$, where $S^r := \{s_1 \cdots s_r \mid s_1, \ldots, s_r \in S\}$. Also note that since $S$ is inverse closed, $B^r_r(g) = S^r g$ for all $r \geq 2$. It follows that $|B^r_r(g)| = |B^1_r(1)| = |S^r \cup \{1\}|$ for all $g \in G$.

Proposition 2.3. Let $G$ be a finite group and $d_T$ be the metric induced by the metric $Cay(G, S)$. Then a subset $C$ of $G$ is a code with $\min\{d_T(x, y) \mid x, y \in C\} \geq 3$ if and only if there exists $Y \subset G$ such that

\[ (S \cup \{1\})\overline{C} = \hat{G} - \hat{Y}, \]

Proof. Let $Y = G \setminus \cup_{c \in C} B^1_1(c)$. So $G = \cup_{c \in C} B^1_1(c) \cup Y$ and hence Remark 2.2 implies $\hat{G} = (S \cup \{1\})C + \hat{Y}$. Now the result follows from the fact that for any two distinct elements $c, c'$ in $C$, $(S \cup \{1\})c \cap (S \cup \{1\})c' = \emptyset$ since otherwise $d_T(c, c') \leq 2$ that is a contradiction. This completes the proof. □

Let $G$ be a finite group and $d_T$ be the metric induced by the metric $Cay(G, S)$. For a positive integer $r$, an $r$-perfect code or a perfect code of radius $r$ of $G$ under the metric $d_T$ is a subset $C$ of $G$ such that $G = \cup_{c \in C} B^1_1(c)$ and $B^1_1(c) \cap B^1_1(c') = \emptyset$ for any two distinct $c, c' \in C$. By a similar argument as the proof of Proposition 2.3 it can be seen that if $C$ is an $r$-perfect code, then $(S^r \cup \{1\})\overline{C} = \hat{G}$. We note that according to Remark 2.2 $C$ is an $r$-perfect code if and only if $|C||S^r \cup \{1\}| = |G|$. Notice that $\hat{G}$ can be extended to an algebra homomorphism $\hat{\rho}$ from $\mathbb{C}[G]$ to the algebra $Mat_k(\mathbb{C})$ of $k \times k$ matrices over $\mathbb{C}$ such that $g^\rho = g^\rho$ for all $g \in G$. Thus the image of $\hat{X}$ for any non-empty subset $X$ of $G$ under $\hat{\rho}$ is the element $\sum_{x \in X} x^\rho$ of $Mat_k(\mathbb{C})$. In particular by applying $\hat{\rho}$ on the equality $2.2$ we obtain

\[ \left( \sum_{s \in S^r \cup \{1\}} s^\rho \right) \left( \sum_{c \in C} c^\rho \right) = \sum_{g \in G} g^\rho - \sum_{y \in Y} y^\rho, \]

where the latter equality is between elements of $Mat_k(\mathbb{C})$.

Remark 2.4. Given a group $G$ and a non-empty set $X$, recall that we say $G$ acts on $X$ (from the right) if there exists a function $X \times G \to X$ denoted by $(x, g) \mapsto x^g$ for all $(x, g) \in X \times G$ if $(x^g)^h = x^{gh}$ and $x^1 = x$ for all $x \in X$ and all $g, h \in G$, where $1$ denotes the identity element of $G$. For any $x \in G$ the set $\text{Stab}_G(x) := \{g \in G \mid x^g = x\}$ is called the stabilizer of $x$ in $G$ which is a subgroup of $G$. If the action is transitive (i.e., for any two elements $x, y \in X$, there exists $g \in G$ such that $x^g = y$), all stabilizers are conjugate under the elements of $G$, more precisely $\text{Stab}_G(x)^g = \text{Stab}_G(y)$ whenever $x^g = y$, where $\text{Stab}_G(x)^g = g^{-1} \text{Stab}_G(x)g$.

Now suppose that $G$ acts on $X$ and $|X| = k$ is finite. Fix an arbitrary ordering the elements of $X$ so that $x_i < x_j$ whenever $i < j$ for distinct elements $x_i, x_j \in X$. Denote by $\rho^G_X$ the map from $G$ to $GL_k(\mathbb{Z})$ defined by $g \mapsto P_g$, where $P_g$ is the $|X| \times |X|$ matrix whose $(i, j)$ entry is $1$ if $x_i^g = x_j$ and $0$ otherwise. Note that the definitions of $\rho^G_X$ depends on the choice of the ordering on $X$, however any two such representations of $G$ are conjugate by a permutation matrix.

Let $H$ be a subgroup of a finite group $G$ and $X$ be the set of right cosets of $H$ in $G$, i.e., $X := \{Hg \mid g \in G\}$. Then $G$ acts transitively on $X$ via $(Hg, g_0) \mapsto Hg_0$. We note that, it is known that $X$ partitions $G$, i.e., $G = \cup_{x \in X} x$ and $x \cap x' = \emptyset$ for all distinct elements $x$ and $x'$ of $X$, and $|X| = |G|/|H|$.
Lemma 2.5. Let $H$ be a subgroup of a finite group $G$ and $X = \{Ha_1, \ldots, Ha_m\}$ be the set of right cosets of $H$ in $G$. If $Y \subseteq G$, then by fixing the ordering $Ha_i < Ha_j$ whenever $i < j$, the $(i, j)$ entry of $\sum_{y \in Y} y^{\rho} \in \mathbb{R}^{|Y|}$ is $|Y \cap a_i^{-1}Ha_j|$.

Proof. Clearly, for any $y \in Y$, the $(i, j)$ entry of $y^{\rho} = 1$ if $Ha_i y = Ha_j$ and 0 otherwise. So the $(i, j)$ entry of $y^{\rho} = 1$ if $a_i y a_j^{-1} \in H$ and therefore $y \in a_i^{-1}Ha_j$. Hence the $(i, j)$ entry of $\sum_{y \in Y} y^{\rho}$ is equal to $|\{y \in Y | y \in a_i^{-1}Ha_j\}|$. This completes the proof. □

Theorem 2.6. Let $G$ be a finite group and $d_T$ be the metric induced by the metric $Cay(G, S)$. Also Let $C$ be a code in $G$ with $\min\{|d_T(c, c')| c, c' \in C\} \geq 3$. If $H$ is a subgroup of $G$ and $Z$ is the set of right cosets of $H$ in $G$, then the optimal value of the objective function of the following integer programming gives an upper bound on $|C|$

\[
\text{Maximize } \sum_{i=1}^{|Z|} x_i,
\]

subject to \[(S \cup \{1\})^{\rho} (x_1, \ldots, x_{|Z|})^T \leq |H| \mathbf{1},
\]
\[x_i \in \mathbb{Z}, \; x_i \geq 0, \; i \in \{1, \ldots, |Z|\},
\]

where $\mathbf{1}$ is a column vector of order $|Z| \times 1$ whose entries are equal to 1.

Proof. By Proposition 2.3 there exists $Y \subseteq G$ such that

\[
(\sum_{s \in S \cup \{1\}} s^{\rho}) (\sum_{c \in C} c^{\rho}) = \sum_{g \in G} g^{\rho} - \sum_{y \in Y} y^{\rho},
\]

Suppose that $Z = \{Ha_1, \ldots, Ha_m\}$. Without loss of generality, we may assume that $a_1 = 1$. We fix the ordering $Ha_i < Ha_j$ whenever $i < j$. By Lemma 2.5 the $(i, j)$ entry of $\sum_{g \in G} g^{\rho}$ is equal to $|G \cap a_i^{-1}Ha_j|$ and since $a_i^{-1}Ha_j \subseteq G$, the $(i, j)$ entry of $\sum_{g \in G} g^{\rho}$ is equal to $|a_i^{-1}Ha_j| = |H|$, for all $i, j \in \{1, \ldots, |Z|\}$. So if $B$ is a column of $\sum_{g \in G} g^{\rho}$, then $B = |H| \mathbf{1}$. Let $C$ be the first column of $\sum_{c \in C} c^{\rho}$. Then Lemma 2.5 implies that for all $1 \leq i \leq |Z|$, $i$-th row of $C$, denoted by $c_i$, is equal to $|C \cap Ha_i|$. Since $C = C \cap G = \cup_{i=1}^{|Z|} (C \cap Ha_i)$ and $(C \cap Ha_i) \cap (C \cap Ha_j) = \emptyset$ for all $i \neq j$, $\sum_{i=1}^{\lfloor |Z|/2 \rfloor} c_i = |C|$. We note that by Lemma 2.5 all entries of matrix $F^{\rho}$, $F \in \{C, G, Y, (S \cup \{1\})\}$, are integer and non-negative. Therefore $C$ is an integer solution for the following system of inequalities

\[
(S \cup \{1\})^{\rho} (x_1, \ldots, x_{|Z|})^T \leq |H| \mathbf{1}
\]
such that $\sum_{i=1}^{|Z|} c_i = |C|$ and this completes the proof. □

3. Results

Let $G = S_n$ and $S = \{(i, i+1) | 1 \leq i \leq n - 1\}$, then the metric induced by $Cay(G, S)$ on $S_n$ is the Kendall $\tau$-metric. In this section, by using the result in Section II, we improve the upper bound of $P(n, 3)$ when $n \in \{6, 14, 15\}$ or $n \geq 7$ is a prime number.

Usual traditional with well-developed candidate for $B$ is the set of Young tableaux of a given shape which we are going to recall them \[7\].
Definition 3.1. By a number partition \( \lambda \) of \( n \) (with the length \( m \)) we mean an \( m \)-tuple \((\lambda_1, \ldots, \lambda_m)\) of positive integers such that \( \lambda_1 \geq \cdots \geq \lambda_m \) and \( n = \sum_{i=1}^{m} \lambda_i \).

Given a number partition \( \lambda \) of \( n \), by a Young tabloid of shape \( \lambda \) we mean an \( m \)-tuple \((n_1, \ldots, n_m)\) of non-empty subsets of \([n]\) consisting a set partition for \([n]\) with \( |n_i| = \lambda_i \) for all \( i = 1, \ldots, m \). We denote by \( \mathcal{YT}_n(\lambda) \) the set of all Young tabloids of shape \( \lambda \) of \( n \). With a Young tabloid \( n = (n_1, \ldots, n_m) \) of shape \( \lambda \), we associate a Young subgroup \( S_n \) of \( S_n \) by taking \( S_n = S_{n_1} \times \cdots \times S_{n_m} \).

Remark 3.2. The action of \( S_n \) on \( \mathcal{YT}_n(\lambda) \) is defined by \((n_1, \ldots, n_m)^\sigma = (n_1^\sigma, \ldots, n_m^\sigma) \) for all \( \sigma \in S_n \). Fix an arbitrary ordering of the elements of \( \mathcal{YT}_n(\lambda) \). The representation \( \rho_{\mathcal{YT}_n(\lambda)}^{S_n} \) is equivalent to the representation \( \rho_X^S \), where \( X \) is the set of right cosets of the Young subgroup \( S_n \) in \( S_n \) for some Young tabloid \( n = (n_1, \ldots, n_m) \). Note that if \( n \) and \( n' \) are two Young tabloids of the same shape \( \lambda \), the corresponding Young subgroups are conjugate in \( S_n \) and so the representations \( \rho_X^S \) and \( \rho_X^{S'} \), where \( X \) and \( X' \) are the set of right cosets of Young subgroups \( S_n \) and \( S' \) in \( S_n \), respectively, are equivalent (i.e., a matrix \( U \) exists such that \( U^{-1} \rho_X^{S'}(\sigma)U = \rho_X^{S_n}(\sigma) \) for all \( \sigma \in S_n \)) so that we use Young tabloid \( \{1, \ldots, \lambda_1\}, \{\lambda_1 + 1, \ldots, \lambda_1 + \lambda_2\}, \ldots, \{n - \lambda_m + 1, \ldots, n\} \) for considering Young subgroup corresponding to the partition \( \lambda = (\lambda_1, \ldots, \lambda_m) \), as we are studying these representations up to equivalence.

Lemma 3.3. Let \( H \) be Young subgroup of \( S_n \) corresponding to the partition \( \lambda := (n-1, 1) \) and \( X \) be the set of right cosets of \( H \) in \( S_n \). If \( S = \{(i, i+1) \mid 1 \leq i \leq n-1\} \), then \( (S \cup \{1\}) \rho_X^{S_n} \) is a conjugate by a permutation matrix of the following matrix

\[
\begin{pmatrix}
  n-1 & 1 & 0 & 0 & \cdots & 0 \\
  1 & n-2 & 1 & 0 & \cdots & 0 \\
  0 & 1 & n-2 & 1 & 0 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 1 & n-2 & 1 \\
  0 & 0 & \cdots & 0 & 1 & n-1
\end{pmatrix}
\]

Proof. Let \( T := S \cup \{1\} \). Without loss of generality we may assume that \( \lambda \) is the partition \( \{(1), \{2, \ldots, n\}\} \) of \( n \) and therefore \( H = Stab_{S_n}(1) \). If \( i \neq j \), then \( H(1, i) \cap H(1, j) = \emptyset \) since otherwise \( (1, i, j) \in H \) that is a contradiction. So we can let \( X = \{H(1, i) \mid 1 \leq i \leq n\} \), where we are using the convention \( H(1, 1) := H \).

We Fixing the ordering \( H(1, i) < H(1, j) \) if \( i < j \). By Lemma 2.3 the \((i, j)\) entry of \( T\rho_X^{S_n} \) is equal to \( |T \cap (1, i)H(1, j)| \). If \( i = j \), then Remark 2.4 imply \((1, i)H(1, i) = Stab_{S_n}(i) \) and hence \( T \cap (1, i)H(1, i) = T \setminus \{(i-1, i), (i, i+1)\} \) if \( 2 \leq i \leq n-1 \), \( T \cap (1, n)H(1, n) = T \setminus \{(n, n-1)\} \) and \( T \cap H = T \setminus \{(1, 2)\} \).

Now suppose that \( i \neq j \). Clearly \( (1, i)(i, j)(1, j) = (i, j) \). Let \( h \in H \). Then \( \sigma := (1, i)h(1, j) = \pi(i, j, i) \), where \( \pi := (1, i)h(1, i) \in Stab_{S_n}(i) \). Since \( \pi(i) = i \), \( \pi(j) = i \) and therefore \( \sigma \) is an transposition if and only if \( h = (i, j) \). Hence, if \( j = i + 1 \) and \( i - 1 \), then \( T \cap (1, i)H(1, j) \) is equal to \( \{(i, i+1)\} \) and \( \{(i-1, i)\} \), respectively, and otherwise \( T \cap (1, i)H(1, j) = \emptyset \). This completes the proof. \( \square \)
Theorem 3.4. Let $p \geq 7$ be a prime number and consider the $p \times p$ matrix
\[
M = \begin{pmatrix}
 p - 1 & 1 & 0 & 0 & \ldots & 0 \\
 1 & p - 2 & 1 & 0 & \ldots & 0 \\
 0 & 1 & p - 2 & 1 & 0 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\
 0 & 0 & \ldots & 1 & p - 2 & 1 \\
 0 & 0 & \ldots & 0 & 1 & p - 1
\end{pmatrix}.
\]
Consider the system of inequalities $Mx \leq (p-1)!1$ with $x \geq 0$ and $x_i$ are integers. Let $x_{\text{max}} := \max\{x_i \mid i = 1, \ldots, p\}$. Then

1. $|\{i \in [p] \mid x_i \leq \frac{(p-1)!}{p}\}| \geq \left\lceil \frac{p}{3} \right\rceil.$
2. If $\sum_{i=1}^{p} x_i = (p-1)! - k$, then $|\{i \mid x_i = x_{\text{max}}\}| \geq p - k - 2.$
3. $\sum_{i=1}^{p} x_i \leq (p-1)! - \left\lceil \frac{p}{3} \right\rceil + 2.$

Proof. Let $A := \{i \in [p] \mid x_i \leq \frac{(p-1)!}{p}\}$ and $B := \{i \mid x_i = x_{\text{max}}\}$. Consider the partition $\{\{1, 2\}, \{3, 4, 5\}, \{6, 7, 8\}, \ldots, \{p - 2, p - 1, p\}\}$ of $[p]$ if $p \equiv 2 \mod 3$ and the partition $\{\{1, 2\}, \{3, 4, 5\}, \{6, 7, 8\}, \ldots, \{p - 4, p - 3, p - 2\}, \{p - 1, p\}\}$ if $p \equiv 1 \mod 3$. Each member of partitions corresponds to an obvious inequality, e.g. $\{1, 2\}$ and $\{p - 2, p - 1, p\}$ are respectively corresponding to $(p-1)x_1 + x_2 \leq (p-1)!$ and $x_{p-2} + (p-2)x_{p-1} + x_p \leq (p-1)!$. Each inequality corresponding to a member $P$ of partitions forces $x_i \leq (p-1)!/p$ for some $i \in P$, where $x_i = \min\{x_j \mid j \in P\}$. Since the size of both partitions is the same $\left\lceil \frac{p}{3} \right\rceil$, $|A| \geq \left\lceil \frac{p}{3} \right\rceil$.

Let $i \in [p]$ be such that $x_i = x_{\text{max}}$. Thus $\sum_{i=1, i \neq i, k+1}^{p} x_i = x_{i-1} + (p - 2)x_i + x_{i+1} - \sum_{i=1}^{p} x_i \leq (p-1)! - (p-1)! - k.$ Thus $\sum_{i=1, i \neq i, k+1}^{p} x_i - x_i \in \{0, 1, \ldots, k\}$. It follows that $|\{i \mid x_i < x_{\text{max}}\}| \leq k + 2$ and so $|B| \geq p - k - 2.$

Let $\sum_{i=1}^{p} x_i = (p-1)! - k$ and suppose, for a contradiction, that $k < \left\lceil \frac{p}{3} \right\rceil - 2$. So $|B| \geq p - \left\lceil \frac{p}{3} \right\rceil + 1$ and therefore
\[
|A \cap B| \geq |A| + |B| - p \geq \frac{p}{3} + p - \frac{p}{3} + 1 - p \geq 1.
\]

Hence $A \cap B \neq \emptyset$ and $x_{\text{max}} \leq (p-1)!/p$. Since $p$ is prime, by Wilson theorem [4] P. 27 $(p-1)! \equiv -1 \mod p$. Since $x_{\text{max}}$ is integer, we have that $x_i \leq \frac{(p-1)!+1}{p} - 1$ for all $i \in [p]$. Therefore
\[
\sum_{i=1}^{p} x_i = (p-1)! - k \leq p \frac{(p-1)!+1}{p} - 1 = (p-1)! + 1 - p
\]
and so
\[
p \leq k + 1 < \left\lceil \frac{p}{3} \right\rceil - 1,
\]
which is a contradiction. So we must have $k \geq \left\lceil \frac{p}{3} \right\rceil - 2$. This completes the proof.

Corollary 3.5. For all primes $p \geq 11$, $P(p, 3) \leq (p-1)! - \left\lceil \frac{p}{3} \right\rceil + 2 \leq (p-1)! - 2.$

Proof. The result follows from Theorems 2.4, 3.4 and Lemma 3.3.

Theorem 3.6. If $n$ is equal to 6, 7, 11, 13 and 17, then $P(n, 3)$ is less than or equal to 116, 716, 10! – 10, 12! – 12 and 16! – 5, respectively.
Proof. Let $S := \{(i, i + 1) \mid 1 \leq i \leq n - 1\}$. In view of Theorem 2.6, we have used a CPLEX software \cite{3} to determine the upper bound for $P(n, 3)$ obtained from solving the integer programming corresponding to the subgroup $H$ of $S_n$, where $H$ is the Young subgroup corresponding to the partition $(2, 2, 2)$, when $n = 6, (5, 1, 1)$, when $n = 7, (9, 2)$, when $n = 11, (11, 2)$, when $n = 13$ and $(16, 1)$, when $n = 17$. 

**Theorem 3.7.** There are no 1-perfect codes under the Kendall $\tau$-distance in $S_n$ when $n \in \{14, 15\}$. 

Proof. Let $S = \{(i, i + 1) \mid 1 \leq i \leq n-1\}$ and $T := S \cup \{1\}$. We are using techniques in \cite{5} for proving this theorem. By \cite{5} Theorem 2.2, if $S_n$ contains a subgroup $H$ such that $n \nmid |H|$ and $(T)^{p_{|H|}}_{\rho_{\lambda}}$ is invertible, where $X$ is the set of right cosets of $H$ in $S_n$, then $S_n$ contains no 1-perfect codes under the Kendall $\tau$-distance. In the case $n = 14$, we consider Young subgroup $H$ corresponding to the partition $(6, 6, 2)$. By a software check the matrix $(T)^{p_{|H|}}_{\rho_{\lambda}}$ is invertible.

In the case $n = 15$, we consider Young subgroup $H$ corresponding to the partition $(4, 4, 4, 3)$ in $S_{15}$. By \cite{7} Corollary 2.2.22, if for all $\lambda \in \{(15, 14, 1), (13, 2), (13, 1, 2), (12, 2), (11, 4, 1), (10, 5, 1), (10, 2), (9, 6), (9, 1, 3), (9, 2, 2, 1), (10, 4, 1), (10, 3, 2), (9, 5, 1), (8, 6, 1), (9, 3, 2), (8, 2, 2), (6, 6, 3), (9, 4, 2), (4, 4, 4, 3), (7, 6, 2), (7, 4, 2), (6, 5, 4), (8, 5, 2), (8, 4, 3), (7, 5, 3), (6, 3, 3, 3), (8, 3, 2, 2), (5, 4, 4, 2), (7, 3, 3, 2), (5, 5, 3, 2), (6, 5, 2, 2), (7, 4, 2, 1), (6, 4, 3, 2)\}$, $T^{p_{|H|}}_{\rho_{\lambda}}$ is invertible, where $\rho_{\lambda}$ is the irreducible representation of $S_{15}$ corresponding to $\lambda$, then $T^{p_{|H|}}_{\rho_{\lambda}}$ is invertible. By software check all these matrices are invertible. This completes the proof. 

**Conjecture 3.8.** If $H$ is the Young subgroup corresponding to the partition $(p - 1, p - 1, 2)$ of $S_{2p}$, where $p \geq 3$ is a prime number, and $X$ is the set of right cosets of $H$ in $S_{2p}$, then $(S \cup \{1\})^{p_{|H|}}_{\rho_{\lambda}}$ is invertible. In particular, there is no 1-perfect permutation code of length $2p$ with respect to the Kendall $\tau$-distance.

4. Conclusion

Due to the applications of PCs under the Kendall $\tau$-distance in flash memories, they have attracted the attention of many researchers. In this paper, we consider the upper bound of the size of the largest PC with minimum Kendall $\tau$-distance $3$. Using Group theory, corresponding to any non-trivial subgroup of $S_n$, we formulate an integer programming where the optimal value of the objective function gives an upper bound on $P(n, 3)$. After that, by solving the integer programming corresponding to some subgroups of $S_n$, when $n \geq 7$ is a prime number or $n \in \{6, 14, 15\}$, we can improve the upper bound on $P(n, 3)$.

**References**

1. A. Barg and A. Mazumdar, Codes in permutations and error correction for rank modulation, IEEE Trans. Inform. Theory, 56 (2010), No. 7, 3158-3165.
2. S. Buzaglo and T. Etzion, Bounds on the size of permutation codes with the Kendall $\tau$-metric, IEEE Trans. Inform. Theory, 61 (2015), No. 6, 3241-3250.
3. The CPLEX Group, Version 12.10, (2019) www.ibm.com/products/ilog-cplex-optimization-studio
4. U. Dudley, A Guide to Elementary Number Theory, The Dolciani Mathematical Expositions, 41. MAA Guides, 5. Mathematical Association of America, Washington, DC, 2009.
[5] P. H. Edelman and D. White, Codes, transforms and the spectrum of the symmetric group, Pacific J. Math., 143 (1990), 47-67.

[6] F. Göloğlu, J. Lember, A.-E. Riet, and V. Skachek, New bounds for permutation codes in Ulam metric, in Proc. IEEE Int. Symp. Inf. Theory (ISIT), Jun. 2015, pp. 1726-1730.

[7] G. James and A. Kerber, The representation theory of the symmetric group, Encyclopedia of Mathematics and its Applications, 16, Addison-Wesley Publishing Co., Reading, Mass., 1981.

[8] A. Jiang, R. Mateescu, M. Schwartz, and J. Bruck, Rank modulation for flash memories, IEEE Trans. Inform. Theory, 55 (2009), 2659-2673.

[9] H. Tarnanen, Upper bounds on permutation codes via linear programming, Eur. J. Combinat., 20 (1999), 101-114.

[10] S. Vijayakumar, Largest permutation codes with the Kendall $\tau$-metric in $S_5$ and $S_6$, IEEE Comm. Letters, 20 (2016), No. 10, 1912-1915.

[11] X. Wang, Y. Zhang, Y. Yang and G. Ge, New bounds of permutation codes under Hamming metric and Kendall’s $\tau$-metric, Des. Codes Cryptogr., 85 (2017), No. 3, 533-545.

[12] X. Wang, Y. Wang, W. Yin and F-W. Fu, Nonexistence of perfect permutation codes under the Kendall $\tau$-metric, Des. Codes Cryptogr., 89 (2021), No. 11, 2511-2531.

DEPARTMENT OF PURE MATHEMATICS, FACULTY OF MATHEMATICS AND STATISTICS, UNIVERSITY OF ISFAHAN, ISFAHAN 81746-73441, IRAN.

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), 19395-5746 Tehran, Iran.

Email address: a.abdollahi@math.ui.ac.ir

DEPARTMENT OF PURE MATHEMATICS, FACULTY OF MATHEMATICS AND STATISTICS, UNIVERSITY OF ISFAHAN, ISFAHAN 81746-73441, IRAN.

Email address: bagherian@sci.ui.ac.ir

DEPARTMENT OF PURE MATHEMATICS, FACULTY OF MATHEMATICS AND STATISTICS, UNIVERSITY OF ISFAHAN, ISFAHAN 81746-73441, IRAN.

Email address: math_fatemeh@yahoo.com

DEPARTMENT OF PURE MATHEMATICS, FACULTY OF MATHEMATICS AND STATISTICS, UNIVERSITY OF ISFAHAN, ISFAHAN 81746-73441, IRAN.

Email address: m.khatami@sci.ui.ac.ir

DEPARTMENT OF ELECTRICAL ENGINEERING, UNIVERSITY OF ISFAHAN, ISFAHAN 81746-73441, IRAN.

School of Mathematics, Institute for Research in Fundamental Sciences (IPM), 19395-5746 Tehran, Iran.

Email address: f.parvash@eng.ui.ac.ir

DEPARTMENT OF APPLIED MATHEMATICS AND COMPUTER SCIENCE, FACULTY OF MATHEMATICS AND STATISTICS, UNIVERSITY OF ISFAHAN, ISFAHAN 81746-73441, IRAN.

Email address: r.sobhani@sci.ui.ac.ir