Packing spanning rigid subgraphs with restricted degrees

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Abstract

Let $G$ be a graph and let $l$ be an integer-valued function on subsets of $V(G)$. The graph $G$ is said to be $l$-partition-connected, if for every partition $P$ of $V(G)$, $e_G(P) \geq \sum_{A \in P} l(A) - l(V(G))$, where $e_G(P)$ denotes the number of edges of $G$ joining different parts of $P$. We say that $G$ is $l$-rigid, if it contains a spanning $l$-partition-connected subgraph $H$ with $|E(H)| = \sum_{v \in V(H)} l(v) - l(V(H))$. In this paper, we investigate decomposition of graphs into spanning partition-connected and spanning rigid subgraphs. As a consequence, we improve a recent result due to Gu (2017) by proving that every $(4kp - 2p + 2m)$-connected graph $G$ with $k \geq 2$ has a spanning subgraph $H$ containing a packing of $m$ spanning trees and $p$ spanning $(2k - 1)$-edge-connected subgraphs $H_1, \ldots, H_p$ such that for each vertex $v$, every $H_i - v$ remains $(k - 1)$-edge-connected and also $d_{H_i}(v) \leq \lceil \frac{d_G(v)}{2} \rceil + 2kp - p + m$. From this result, we refine a result on arc-connected orientations of graphs.

Keywords:
Partition-connected; rigid graph; sparse graph; supermodular; edge-decomposition; vertex degree.

1 Introduction

In this article, all graphs have no loop, but multiple edges are allowed and a simple graph is a graph without multiple edges. Let $G$ be a graph. The vertex set, the edge set, and the minimum degree of $G$ are denoted by $V(G)$, $E(G)$, and $\delta(G)$, respectively. The degree $d_G(v)$ of a vertex $v$ is the number of edges of $G$ incident to $v$. For a set $X \subseteq V(G)$, we denote by $G[X]$ the induced subgraph of $G$ with the vertex set $X$ containing precisely those edges of $G$ whose ends lie in $X$. Let $A$ and $B$ be two subsets of $V(G)$. This pair is said to be intersecting, if $A \cap B \neq \emptyset$. Let $l$ be a real function on subsets of $V(G)$ with $l(\emptyset) = 0$. For notational simplicity, we write $l(G)$ for $l(V(G))$ and write $l(v)$ for $l(\{v\})$. The function $l$ is said to be supermodular, if for all vertex sets $A$ and $B$, $l(A \cap B) + l(A \cup B) \geq l(A) + l(B)$. Likewise, $l$ is said to be $c$-intersecting supermodular, if for all vertex sets $A$ and $B$ with $|A \cap B| \geq c$, the above-mentioned inequality holds. When $c = 1$, the set function $l$ is said to be intersecting supermodular. The set function $l$ is called (i) nonincreasing, if $l(A) \geq l(B)$, for all nonempty vertex sets $A, B$ with $A \subseteq B$, (ii)
In 2014 Cheriyan, Durand de Gevigney, and Szigeti established the following generalized version.

Theorem 1.1.

is an integer and connected graph with no spanning minimally 2-rigid subgraphs. In 2005 Jordán [14] extended this result to

refers to a collection of pairwise edge-disjoint subgraphs. Throughout this article, all set functions valued function on subsets of \( V \)

denote by \( d \) \( \ell \) disjoint vertex sets \( A \) \( G \)

It was shown that the converse is true, when for all vertex sets \( A \) if for all vertex sets \( A \) if for any two disjoint vertex sets \( A \) and \( B \) with \( A \neq \emptyset \) and \( A \cup B \subset V(G) \), \( d_{G - B}(A) \geq l(A \cup B) - \sum_{v \in B} \ell(v) \), where \( \ell \) is a real function on subsets of \( V(G) \). When \( G \) is 1-weakly \( l \)-connected, \( G \) is said to be \( l \)-connected.

For every vertex set \( A \) of a directed graph \( G \), we denote by \( d_G^+(A) \) the number of edges entering \( A \) and denote by \( d_G^-(A) \) the number of edges leaving \( A \). An orientation of \( G \) is called \( l \)-arc-connected, if for every vertex set \( A \), \( d_G^+(A) \geq l(A) \). Likewise, an orientation of \( G \) is called \( r \)-rooted \( l \)-arc-connected, if for every vertex set \( A \), \( d_G^+(A) \geq l(A) - \sum_{v \in A} r(v) \), where \( r \) is a nonnegative integer-valued function on \( V(G) \) with \( l(G) = \sum_{v \in V(G)} r(v) \). An orientation of \( G \) is said to be smooth, if for each vertex \( v \), \( |d_G^+(v) - d_G^-(v)| \leq 1 \). A packing refers to a collection of pairwise edge-disjoint subgraphs. Throughout this article, all set functions are zero on the empty set, all variables \( k \), \( p \), and \( m \) are integer and nonnegative (\( k \) is positive), unless otherwise stated.

In 1961 Nash-Williams and Tutte obtained a necessary and sufficient condition for a graph to have \( m \) edge-disjoint spanning trees which contains the following result as a corollary.

Theorem 1.1. ([17, 18]) Every \( 2m \)-edge-connected graph contains \( m \) edge-disjoint spanning trees.

In 1982 Lovász and Yemini [16] showed that every 6-connected graph is 2-rigid and constructed a 5-connected graph with no spanning minimally 2-rigid subgraphs. In 2005 Jordán [14] extended this result to a packing version by proving that every \( 6p \)-connected graph has \( p \) edge-disjoint spanning 2-rigid subgraphs. In 2014 Cheriyan, Durand de Gevigney, and Szegiti established the following generalized version.
Theorem 1.2. ([1]) Every $(6p + 2m)$-connected graph has a packing of $m$ spanning trees and $p$ spanning 2-rigid subgraphs.

Recently, Gu (2017) formulated the following extension of Theorem 1.2 and used it to refine a result on arc-connected orientation of graphs.

Theorem 1.3. ([9]) Every $(4kp - 2p + 2m)$-connected graph with $k \geq 2$ has a packing of $m$ spanning trees and $p$ spanning $k$-rigid subgraphs.

In this paper, we generalize and improve the above-mentioned theorem to the following supermodular version. From this result, we improve Theorem 1.4 in [9] as mentioned in the abstract and also refine the result of Gu (2017) on arc-connected orientations of graphs. Moreover, we investigate spanning rigid subgraphs with small degrees on independent sets and derive that every $6k$-connected bipartite graph $G$ with one partite set $A$ and $k \geq 1$ has a spanning 2-connected subgraph $H$ such that for each $v \in A$, $d_H(v) \leq \lceil d_G(v)/k \rceil + 2$.

Theorem 1.4. Let $G$ be a simple graph, let $l$ be a nonincreasing intersecting supermodular nonnegative integer-valued function on subsets of $V(G)$, and let $p$ and $k$ be two positive integers with $k \geq 2$. If $G$ is $(4kp - 2p + 2l)$-connected, then it has a packing of a spanning $l$-partition-connected subgraph and $p$ spanning $k$-rigid subgraphs.

2 Basic tools

In this section, we present some basic tools for working with sparse and rigid graphs. The first one shows that minimal and maximal rigid subgraphs containing two given vertices are unique, when the original graph is sparse and $c \leq 2$. In particular, maximal rigid subgraphs are edge-disjoint.

Proposition 2.1. Let $F$ be an $\ell$-sparse graph, where $\ell$ is a $c$-intersecting supermodular weakly subadditive integer-valued function on subsets of $V(F)$. If $F[A]$ and $F[B]$ are two $\ell$-rigid subgraphs and $|A \cap B| \geq c$, then both of graphs $F[A \cup B]$ and $F[A \cap B]$ are $\ell$-rigid.

Proof. Since $F$ is $\ell$-sparse, we must have $e_F(A \cap B) \leq \sum_{v \in A \cap B} \ell(v) - \ell(A \cap B)$, which can conclude that
\[ e_F(A \cup B) \geq e_F(A) + e_F(B) - e_F(A \cap B) \geq \sum_{v \in A} \ell(v) - \ell(A) + \sum_{v \in B} \ell(v) - \ell(B) - \sum_{v \in A \cap B} \ell(v) + \ell(A \cap B). \]

According to the assumption, $\ell$ is supermodular on $A$ and $B$, and so
\[ e_F(A \cup B) \geq \sum_{v \in A \cup B} \ell(v) + \ell(A \cap B) - \ell(A) - \ell(B) \geq \sum_{v \in A \cup B} \ell(v) - \ell(A \cup B). \]

Therefore, the equalities must be hold, which can imply that both of graphs $F[A \cup B]$ and $F[A \cap B]$ are $\ell$-rigid. Hence the proposition holds. \qed
The next proposition is a useful tool for finding a pair of edges such that replacing them preserves sparse property of a given spanning sparse subgraph.

**Proposition 2.2.** Let $G$ be a graph and let $\ell$ be a 2-intersecting supermodular weakly subadditive integer-valued function on subsets of $V(G)$. If $F$ is a spanning $l$-sparse subgraph of $G$, $xy \in E(G) \setminus E(F)$, and $Q$ is an $\ell$-rigid subgraph of $F$ including $x$ and $y$ with the minimum number of vertices, then for every $e \in E(Q)$, the resulting graph $F - e + xy$ remains $\ell$-sparse.

**Proof.** Suppose, by way of contradiction, that there is an edge $uv$ such that $F' = F - uv + xy$ is not $\ell$-sparse so that there is a vertex set $A$ with $e_F(A) \geq \sum_{v \in A} \ell(v) - \ell(A)$. Since $e_F(A) \leq \sum_{v \in A} \ell(v) - \ell(A)$, we must have $x, y \in A$, and $A \setminus \{u, v\} \neq \emptyset$, and also $e_F(A) = \sum_{v \in A} \ell(v) - \ell(A)$. In other words, the graph $F[A]$ is $\ell$-rigid. Since $|V(Q)|$ is minimal and $A$ includes $x$ and $y$, one can conclude that $V(Q) \subseteq A$. This implies that $u, v \in A$, which is a contradiction. \[\square\]

**Proposition 2.3.** Let $F$ be an $\ell$-sparse graph with $x, y \in V(F)$, where $\ell$ is a 2-intersecting supermodular weakly subadditive integer-valued function on subsets of $V(F)$. Let $F[A]$ be an $\ell$-rigid subgraph with the minimum number of vertices including $x$ and $y$. If $xy \notin E(F)$ and $B$ is a vertex set including $x$ and $y$ such that $F[B] + xy$ is $\ell$-rigid, then the graph $F[A \cup B]$ must be $\ell$-rigid.

**Proof.** If $A$ is a subset of $B$, then since $F[B]/A$ is $\ell$-partition-connected, we have

$$e_F(B) \geq e_F(A) + e_{F[B]}(P) \geq \sum_{v \in A} \ell(v) - \ell(A) + \sum_{X \in P} \ell(X) - \ell(B) = \sum_{v \in B} \ell(v) - \ell(B),$$

where $P$ is the partition of $B$ with $P = \{A\} \cup \{v\} : v \in B \setminus A\}$. Now, assume that $|A \cap B| < |A|$. Since $|A|$ is minimal, $e_F(A \cap B) < \sum_{v \in A \cap B} \ell(v) - \ell(A \cap B)$. Since $F[B] + xy$ is $\ell$-rigid, we must have $e_F(B) = \sum_{v \in B} \ell(v) - \ell(B) - 1$, which can conclude that

$$e_F(A \cup B) \geq e_F(A) + e_F(B) - e_F(A \cap B) \geq \sum_{v \in A} \ell(v) - \ell(A) + \sum_{v \in B} \ell(v) - \ell(B) - \sum_{v \in A \cap B} \ell(v) + \ell(A \cap B).$$

According to the assumption, $\ell$ is supermodular on $A$ and $B$, and so

$$e_F(A \cup B) \geq \sum_{v \in A \cup B} \ell(v) + \ell(A \cap B) - \ell(A) - \ell(B) \geq \sum_{v \in A \cup B} \ell(v) - \ell(A \cup B).$$

Therefore, in both cases $F[A \cup B]$ must be $\ell$-rigid. Hence the proposition holds. \[\square\]

**Proposition 2.4.**[13] Let $F$ be a graph with $x, y \in V(F)$ and let $\ell$ be a subadditive integer-valued function on subsets of $V(F)$. If $F$ is $\ell$-sparse and $Q$ is an $\ell$-rigid subgraph of $F$ with the minimum number of vertices including $x$ and $y$, then for every vertex set $A$ with $\{x, y\} \subseteq A \subseteq V(Q)$, $d_Q(A) \geq 1$. 

4
3 A sufficient connectedness condition for a graph to be $\ell$-rigid

The following proposition establishes a necessary connectedness condition for a graph to be $\ell$-rigid.

**Proposition 3.1.** Let $G$ be a graph and let $\ell$ be a weakly subadditive real function on subsets of $V(G)$. If $G$ is $\ell$-rigid, then for any two disjoint vertex sets $A$ and $B$,

$$d_{G-B}(A) \geq \ell(A \cup B) - \sum_{v \in B} \ell(v) + (\ell(G \setminus A) - \ell(G)).$$

**Proof.** We may assume that $G$ is minimally $\ell$-rigid. Since $G$ is $\ell$-sparse, $e_G(A \cup B) \leq \sum_{v \in A \cup B} \ell(v) - \ell(A \cup B)$ and $e_G(A^c) \leq \sum_{v \in A^c} \ell(v) - \ell(A^c)$, where $A^c = V(G) \setminus A$. It is not hard to verify that $d_{G-B}(A) = |E(G)| - e_G(A \cup B) - e_G(A^c) + e_G(B)$. Therefore,

$$d_{G-B}(A) \geq \sum_{v \in V(G)} \ell(v) - \ell(G) - \sum_{v \in A \cup B} \ell(v) + \ell(A \cup B) - \sum_{v \in A^c} \ell(v) + \ell(A^c) + e_G(B),$$

which implies that

$$d_{G-B}(A) \geq \ell(A \cup B) - \sum_{v \in B} \ell(v) + \ell(A^c) - \ell(G) + e_G(B).$$

Hence the proposition is proved. $\Box$

**Corollary 3.2.** ([9]) Let $k$ be an integer with $k \geq 2$. If $G$ is a $k$-rigid graph of order at least three, then it must be $k$-edge-connected and essentially $(2k - 1)$-edge-connected, and also for each vertex $v$, the graph $G - v$ remains $(k - 1)$-edge-connected.

The following theorem gives a sufficient connectedness condition for a graph to be $\ell$-rigid.

**Theorem 3.3.** Let $G$ be a graph and let $\ell$ be a 2-intersecting supermodular weakly subadditive nonnegative integer-valued function on subsets of $V(G)$. If for each vertex $v$, $d_G(v) \geq 2\ell(v)$ and for any two disjoint vertex sets $A$ and $B$ with $A \cup B \subseteq V(G)$ and $e_G(A \cup B) > \sum_{v \in A \cup B} \ell(v) - \ell(A \cup B)$,

$$d_{G-B}(A) \geq 2\ell(A \cup B) - \sum_{v \in B} \ell(v),$$

then $G$ has a spanning $\ell$-rigid subgraph $H$ excluding a given arbitrary edge set of size at most $\ell(G)$.

**Proof.** Let $E$ be an edge set of size at most $\ell(G)$. Let $\mathcal{F}$ be a spanning $\ell$-sparse subgraph of $G \setminus E$ with the maximum size. Define $\mathcal{A}$ to be the collection of all vertex sets of the maximal $\ell$-rigid subgraphs of $\mathcal{F}$. Suppose, by way of contradiction, that $V(G) \notin \mathcal{A}$. Let $\mathcal{A}_0$ be the collection of all vertex sets $X$ in $\mathcal{A}$ with $e_G(X) = \sum_{v \in X} \ell(v) - \ell(X)$. Define $\mathcal{P}$ be the collection of all vertex sets in $\mathcal{A} \setminus \mathcal{A}_0$ along with the vertex sets $\{v\}$ with $v \in V(G) \setminus \cup_{X \in \mathcal{A} \setminus \mathcal{A}_0} X$. For any $X \in \mathcal{P}$, define $X_B$ to be the set of all vertices $v$ which appears in at least two vertex sets of $\mathcal{P}$, and set $X_A = X \setminus X_B$. It is easy to see that

$$\sum_{X \in \mathcal{P}} \sum_{v \in X} \ell(v) - \frac{1}{2} \sum_{X \in \mathcal{P}} \sum_{v \in X_B} \ell(v) \geq \sum_{v \in V(G)} \ell(v).$$
Since \(|E(F)|\) is maximal, for \(xy \in E(G) \setminus (E \cup E(F))\) there must be an \(\ell\)-rigid subgraph of \(F\) including \(x\) and \(y\) with vertex set in \(A\). For every \(X \in A_0\), we have \(e_G(X) = e_F(X)\), and so for every \(xy \in E(G) \setminus (E \cup E(F))\) there must be an \(\ell\)-rigid subgraph of \(F\) including \(x\) and \(y\) with vertex set in \(P\). Hence

\[
e_F(P) = e_G(P).
\]

By the assumption,

\[
e_{G \setminus E}(P) \geq \frac{1}{2} \sum_{X \in P} d_{G \setminus X_B}(X_A) - |E| \geq \sum_{X \in P} \left( \ell(X) - \frac{1}{2} \sum_{v \in X_B} \ell(v) \right) - \ell(G).
\]

Therefore,

\[
|E(F)| = e_F(P) + \sum_{X \in P} e_F(X) = e_{G \setminus E}(P) + \sum_{X \in P} \left( \sum_{v \in X_B} \ell(v) - \ell(X) \right) \geq \sum_{v \in V(G)} \ell(v) - \ell(G).
\]

Hence \(F\) must be \(\ell\)-rigid, a contradiction. \(\square\)

**Remark 3.4.** In the above-mentioned theorem we could reduce the needed lower bound by \(2\ell(G) - 2|E|\) for any two disjoint vertex sets \(A\) and \(B\) with \(|A \cup B| = |V(G)| - 1\), where \(E\) is the give edge set of size at most \(\ell(G)\). In fact, there are at most one vertex set \(X\) in \(A\) with \(|X| = |V(G)| - 1\), when \(|V(G)| \geq 4\). This refined version can imply Corollary 1.9 in [10].

4 A necessary and sufficient decomposition condition

By the result of Nash-Williams [17] and Tutte [18], a graph is \(m\)-partition-connected if and only if it can be decomposed into \(m\) edge-disjoint spanning trees. Recently, the present author generalized this result to the following supermodular version.

**Theorem 4.1.** ([13]) Let \(G\) be a graph and let \(l_1, l_2, \ldots, l_m\) be \(m\) intersecting supermodular subadditive integer-valued functions on subsets of \(V(G)\). Then \(G\) is \((l_1 + \cdots + l_m)\)-partition-connected if and only if it can be decomposed into \(m\) edge-disjoint spanning subgraphs \(H_1, \ldots, H_m\) such that every graph \(H_i\) is \(l_i\)-partition-connected.

For a special case, Theorem 4.1 can be developed to a rigid version as the following result which is a generalized version of Theorem 5.2 in [10]. However, an arbitrary \((\ell_1 + \ell_2)\)-rigid graph may have no spanning minimally \(\ell_1\)-rigid subgraphs, see Figure 1.

**Theorem 4.2.** Let \(G\) be a graph and \(\ell\) be a 2-intersecting supermodular weakly subadditive integer-valued function on subsets of \(V(G)\). Assume that for any two adjacent vertices \(u\) and \(v\), \(\ell(u) + \ell(v) = \ell(\{u, v\}) + 1\). Then \(G\) is \(p\ell\)-rigid if and only if it can be decomposed into \(p\) edge-disjoint spanning \(\ell\)-rigid subgraphs.
Proof. We may assume that $G$ is minimally $p\ell$-rigid so that $G$ is $p\ell$-sparse and $|E(G)| = \sum_{v \in V(G)} p\ell(v) - p\ell(G)$. The proof presented here is based on matroid theory. We use some properties of the matroid obtained from the union of $p$ copies of a matroid consists of all spanning $\ell$-sparse subgraphs of $G$. Let $F_1, \ldots, F_p$ be $p$ edge-disjoint spanning $\ell$-sparse subgraphs of $G$ with the maximum $|E(F)|$, where $F = F_1 \cup \cdots \cup F_p$. By a theorem of Edmonds on the rank of matroids [3], there is a spanning subgraph $H$ of $G$ with

$$|E(F)| = p \text{rank}_\ell(H) + |E(G) \setminus E(H)|,$$

where $\text{rank}_\ell(H)$ denotes the maximum of all $|E(F)|$ taken over all spanning $\ell$-sparse subgraphs $F$ of $H$. Take $F$ be a spanning $\ell$-sparse subgraph of $H$ with the maximum $|E(F)|$. Let $P$ be the collection of subsets of $V(F)$ obtained from the maximal $\ell$-rigid subgraphs of $F$. By the property of $\ell$, every edge $e \in E(F)$ itself is an $\ell$-rigid subgraph of $F$ and so lies in a maximal $\ell$-rigid subgraph of $F$. By the maximality of $F$, both ends of every edge $e \in E(H) \setminus E(F)$ must lie in an $\ell$-rigid subgraph of $F$; otherwise we can add it to $F$ to obtain a new spanning sparse subgraphs with larger size. Thus $e_X(P) \leq |E(G) \setminus E(H)|$, and also

$$\text{rank}_\ell(H) = |E(F)| = \sum_{X \in P} e_X(X) = \sum_{X \in P} \left( \sum_{v \in X} \ell(v) - \ell(X) \right),$$

which implies that

$$\sum_{X \in P} e_X(X) + e_X(P) = |E(F)| = p \sum_{X \in P} \left( \sum_{v \in X} \ell(v) - \ell(X) \right) + |E(G) \setminus E(H)|.$$

On the other hand,

$$|E(F)| = \sum_{X \in P} e_X(X) + e_X(P) \leq \sum_{X \in P} e_G(X) + |E(G) \setminus E(H)| \leq \sum_{X \in P} \left( \sum_{v \in X} p\ell(v) - p\ell(X) \right) + |E(G) \setminus E(H)|.$$

These inequalities can imply that for every $X \in P$, $e_X(X) = e_G(X)$ and also $e_X(X) = |E(G) \setminus E(H)|$. Therefore, $|E(F)| = |E(G)| = \sum_{v \in V(G)} p\ell(v) - p\ell(G)$, and so for every $F_i$, we must have $|E(F_i)| = \sum_{v \in V(G)} \ell(v) - \ell(G)$. Hence the theorem holds.

![Figure 1: An $\ell_{3,4}$-rigid graph with no spanning minimally $\ell_{2,3}$-rigid subgraphs.](image-url)
5 Structures of maximal packing spanning sparse subgraphs

Here, we state following fundamental theorem, which gives much information about maximal packing spanning sparse subgraphs. This result is a supplement of a recent result in [13] and provides another extension for Lemma 3.5.3 in [2].

Theorem 5.1. Let $G$ be a graph, let $l$ be an intersecting supermodular subadditive integer-valued function on subsets of $V(G)$, and let $\ell$ be a 2-intersecting supermodular subadditive integer-valued function on subsets of $V(G)$. If $F$ and $\mathcal{F}$ are two edge-disjoint spanning subgraphs of $G$ with the maximum $|E(F \cup \mathcal{F})|$ such that $F$ is $l$-sparse and $\mathcal{F}$ is $\ell$-sparse, then there is a partition $P$ of $V(G)$ with the following properties:

1. For every $A \in P$, the graph $F[A]$ is $l$-partition-connected.
2. There is no edges in $E(G) \setminus E(F \cup \mathcal{F})$ joining different parts of $P$.
3. For every $xy \in E(G) \setminus E(\mathcal{F})$ with $x, y \in A \in P$, there is a vertex set $X$ such that $\{x, y\} \subseteq X \subseteq A$ and $\mathcal{F}[X]$ is $\ell$-rigid.

Proof. Define $\ell_1 = l$, $\ell_2 = \ell$, $F_1 = F$, and $F_2 = \mathcal{F}$. Put $T = (F_1, F_2)$. Let $\mathcal{A}$ be the set of all 2-tuples $T = (\mathcal{F}_1, \mathcal{F}_2)$ with the maximum $|E(T)|$ such that $\mathcal{F}_1$ and $\mathcal{F}_2$ are edge-disjoint spanning subgraphs of $G$ and every $\mathcal{F}_i$ is $\ell_i$-sparse, where $E(T) = E(\mathcal{F}_1 \cup \mathcal{F}_2)$. Note that if $e \in E(G) \setminus E(T)$, then every graph $\mathcal{F}_i + e$ is not $\ell_i$-sparse; otherwise, we replace $\mathcal{F}_i$ by $\mathcal{F}_i' + e$ in $T$, which contradicts maximality of $|E(T)|$. Thus both ends of $e$ lie in an $\ell_i$-rigid subgraph of $\mathcal{F}_i$. Let $Q_i$ be the $\ell_i$-rigid subgraph of $\mathcal{F}_i$ including both ends of $e$ with minimum number of vertices. Let $e' \in Q_i$. Define $\mathcal{F}_i' = \mathcal{F}_i - e + e'$ and $\mathcal{F}_j' = \mathcal{F}_j$ for other $j$ with $j \neq i$. According to Proposition 2.2, the graph $\mathcal{F}_i'$ is again $\ell_i$-sparse and so $T' = (\mathcal{F}_1', \mathcal{F}_2') \in \mathcal{A}$. We say that $T'$ is obtained from $T$ by replacing a pair of edges. Let $\mathcal{A}_0$ be the set of all 2-tuples $T$ in $\mathcal{A}$ which can be obtained from $T$ by a series of edge replacements. Let $G_0$ be the spanning subgraph of $G$ with

$$E(G_0) = \bigcup_{T \in \mathcal{A}_0} (E(G) \setminus E(T)).$$

Now, we prove the following claim.

Claim. Let $T = (\mathcal{F}_1, \mathcal{F}_2) \in \mathcal{A}_0$ and assume that $T' = (\mathcal{F}_1', \mathcal{F}_2')$ is obtained from $T$ by replacing a pair of edges. If $x$ and $y$ are two vertices in an $\ell_i$-rigid subgraph of $\mathcal{F}_i' \cap G_0$, then $x$ and $y$ are also in an $\ell_i$-rigid subgraph of $\mathcal{F}_i \cap G_0$, where $i = 1, 2$.

Proof of Claim. Let $e'$ be the new edge in $E(T') \setminus E(T)$. Define $Q_i'$ to be the minimal $\ell_i$-rigid subgraph of $\mathcal{F}_i' \cap G_0$ including $x$ and $y$. We may assume that $e' \in E(Q_i')$; otherwise, $E(Q_i') \subseteq E(\mathcal{F}_i) \cap E(G_0)$ and the proof can easily be completed. Since $e' \in E(T') \setminus E(T)$, both ends of $e'$ must lie in an $\ell_i$-rigid subgraph of $\mathcal{F}_i$. Define $Q_i$ to be the minimal $\ell_i$-rigid subgraph of $\mathcal{F}_i$ including both ends of $e'$. By Proposition 2.2, for every edge $e \in E(Q_i)$, the graph $\mathcal{F}_i - e + e'$ remains $\ell_i$-sparse, which can imply that $E(Q_i) \subseteq E(G_0)$. Define $Q = (Q_i \cup Q_i') - e'$. Note that $Q$ includes $x$ and $y$, and also $E(Q) \subseteq E(G_0) \cap E(\mathcal{F}_i)$. By Proposition 2.3, the graph $Q$ must be $\ell_i$-rigid and so the claim holds.
Define $P$ to be the partition of $V(G)$ obtained from the components of $G_0$. Let $i \in \{1, 2\}$, let $C_0$ be a component of $G_0$, and let $xy \in E(C_0)$. By the definition of $G_0$, there is no edges in $E(G) \setminus E(F_1 \cup F_2)$ joining different parts of $P$, and also there are some 2-tuples $T^1, \ldots, T^n$ in $\mathcal{A}_0$ such that $xy \in E(G) \setminus E(T^n)$, $T = T^1$, and every $T^k$ can be obtained from $T^{k-1}$ by replacing a pair of edges, where $1 < k \leq n$. As we stated above, $x$ and $y$ must lie in an $\ell_i$-rigid subgraph of $F_i$. Let $Q'_i$ be the minimal $\ell_i$-rigid subgraph of $F_i$ including $x$ and $y$. By Proposition 2.2, for every edge $e \in E(Q'_i)$, the graph $F_i - e + xy$ remains $\ell_i$-sparse, which can imply $E(Q'_i) \subseteq E(G_0)$. Thus $x$ and $y$ must also lie in an $\ell_i$-rigid subgraph of $F_i \cap G_0$. Let $Q_i$ be the minimal $\ell_i$-rigid subgraph of $F_i$ including $x$ and $y$ so that $E(Q_i) \subseteq E(G_0)$. Since $\ell_i$ is subadditive, Proposition 2.4 implies that $d_{Q_i}(A) \geq 1$, for every vertex set $A$ with $\{x, y\} \subseteq A \subseteq V(Q_i)$. Thus $Q_i/\{x, y\}$ is connected and hence $V(Q_i) \subseteq V(C_0)$. In other words, for every $xy \in E(C_0)$, there is an $\ell_i$-rigid subgraph of $F_i \cap C_0$ including $x$ and $y$. Since $C_0$ is connected and $\ell_1$ is intersecting supermodular, all vertices of $C_0$ must lie in an $\ell_1$-partition-connected subgraph of $F_1 \cap C_0$. Thus $F[V(C_0)]$ itself must be $\ell_1$-partition-connected and also the edge set of $F[V(C_0)]$ is a subset of $E(C_0)$. For every edge $xy \in E(F)$ with $x, y \in V(C_0)$, by the above-mentioned claim, there is a minimal $\ell_2$-rigid subgraph $Q$ of $F_2 \cap G_0$ including $x$ and $y$. As we observed above, one can conclude that $E(Q) \subseteq E(C_0)$. Hence the proof is completed.

6 Packing spanning partition-connected and spanning rigid subgraphs

The following theorem presents a sufficient connectedness condition for the existence of a packing consists of a spanning $l$-partition-connected subgraph and a spanning $\ell$-rigid subgraph.

**Theorem 6.1.** Let $G$ be a graph, let $l$ be a nonincreasing intersecting supermodular nonnegative integer-valued function on subsets of $V(G)$, and let $\ell$ be a 2-intersecting supermodular subadditive nonnegative integer-valued function on subsets of $V(G)$. If for each vertex $v$, $d_G(v) \geq 2\ell(v) + 2l(v)$ and for any two disjoint vertex sets $A$ and $B$ with $A \cup B \subseteq V(G)$ and $e_G(A \cup B) > \sum_{v \in A \cup B} \ell(v) - \ell(A \cup B)$,

$$d_{G-B}(A) \geq 2\ell(A \cup B) - \sum_{v \in B} \ell(v) + \begin{cases} 0, & \text{when } A = \emptyset; \\ 2l(A \cup B), & \text{when } A \neq \emptyset, \end{cases}$$

then $G$ can be decomposed into a spanning $l$-partition-connected subgraph and a spanning $\ell$-rigid subgraph and also a given arbitrary edge set of size at most $l(G) + \ell(G)$.

**Proof.** Let $E$ be an edge set of size at most $l(G) + \ell(G)$. Let $F$ and $\mathcal{F}$ be two edge-disjoint spanning subgraphs of $G \setminus E$ with the maximum $|E(F)| + |E(\mathcal{F})|$ such that $F$ is $l$-sparse and $\mathcal{F}$ is $\ell$-sparse. Let $P$ be a partition of $V(G)$ with the properties described in Theorem 5.1. Define $\mathcal{A}$ to be the collection of all vertex sets of the maximal $\ell$-rigid subgraphs of all graphs $\mathcal{F}[X]$, where $X \in P$. We may assume that $V(G) \notin \mathcal{A}$. Let $\mathcal{A}_0$ be the collection of all vertex sets $X$ in $\mathcal{A}$ with $e_G(X) = \sum_{v \in X} \ell(v) - \ell(X)$. Define $\mathcal{P}$
be the collection of all vertex sets in $A \setminus A_0$ along with the vertex sets $\{v\}$ with $v \in V(G) \cup X_{v \in A \setminus A_0} X$. For any $X \in \mathcal{P}$, define $X_B$ to be the set of all vertices $v$ which appears in at least two vertex sets of $\mathcal{P}$, and set $X_A = X \setminus X_B$. It is easy to see that

$$
\sum_{X \in \mathcal{P}} \sum_{v \in X} \ell(v) - \frac{1}{2} \sum_{X \in \mathcal{P}} \sum_{v \in X_B} \ell(v) \geq \sum_{v \in V(G)} \ell(v). \quad (1)
$$

For every $X \in A_0$, we have $e_G(X) = e_{\mathcal{F}}(X)$, and so for every $xy \in E(F)$ with $x, y \in A \in \mathcal{P}$, there must be an $\ell$-rigid subgraph of $\mathcal{F}$ including $x$ and $y$ whose vertex set is a subset of $A$. Thus items (2) and (3) of Theorem 5.1 can imply that

$$
e_G(\mathcal{P}) = e_{\mathcal{F}}(\mathcal{P}) + e_F(\mathcal{P}). \quad (2)
$$

By the assumption,

$$e_{G \setminus E}(\mathcal{P}) \geq \frac{1}{2} \sum_{X \in \mathcal{P}} d_{G - X_B}(X_A) - |E| \geq \sum_{X \in \mathcal{P}} \left( \ell(X) - \frac{1}{2} \sum_{v \in X_B} \ell(v) \right) + \sum_{X \in \mathcal{P}, X_A \neq \emptyset} l(X) - l(G) - l(E).
$$

Now, we prove the following claim.

**Claim.** If $Q \subseteq \mathcal{P}$, then there is a vertex set $X \in \mathcal{P}$ with $X \subseteq Q$ and $X_A \neq \emptyset$.

**Proof of Claim.** Suppose, by way of contradiction, that every vertex $v$ of $Q$ appears in at least two vertex sets $X \in \mathcal{P}$ with $X \subseteq Q$. Note that $\mathcal{F}[Q]$ is not $\ell$-rigid and also the $\ell$-rigid subgraphs $\mathcal{F}[X]$ with $X \in \mathcal{P}$ and $X \subseteq Q$ are edge-disjoint. Thus

$$\sum_{v \in Q} \ell(v) - \ell(Q) > e_{\mathcal{F}}(Q) \geq \sum_{X \in \mathcal{P}, X \subseteq Q} \left( \sum_{v \in X} \ell(v) - \ell(X) \right) = \sum_{X \in \mathcal{P}, X \subseteq Q} \left( \frac{1}{2} \sum_{v \in X} \ell(v) + \frac{1}{2} \sum_{v \in X} \ell(v) - \ell(X) \right).
$$

Since $\sum_{v \in X} \ell(v) \geq 2\ell(X)$, one can conclude that

$$\sum_{v \in Q} \ell(v) - \ell(Q) > \sum_{v \in Q} \ell(v) + \sum_{X \in \mathcal{P}, X \subseteq Q} \left( \frac{1}{2} \sum_{v \in X} \ell(v) - \ell(X) \right) \geq \sum_{v \in Q} \ell(v),
$$

which implies $\ell(Q) < 0$. This is a contradiction. Hence the claim holds.

Since $l$ is nonincreasing and nonnegative, by the above-mentioned claim we must have

$$\sum_{X \in \mathcal{P}, X_A \neq \emptyset} l(X) \geq \sum_{Q \in \mathcal{P}} l(Q),$$

which implies that

$$e_{G \setminus E}(\mathcal{P}) \geq \sum_{X \in \mathcal{P}} \ell(X) - \frac{1}{2} \sum_{X \in \mathcal{P}} \sum_{v \in X_B} \ell(v) + \sum_{Q \in \mathcal{P}} l(Q) - l(G) - l(E). \quad (3)
$$

On the other hand,

$$|E(\mathcal{F})| = e_{\mathcal{F}}(\mathcal{P}) + \sum_{X \in \mathcal{P}} e_{\mathcal{F}}(X) = e_{G \setminus E}(\mathcal{P}) - e_F(\mathcal{P}) + \sum_{X \in \mathcal{P}} \left( \sum_{v \in X} \ell(v) - \ell(X) \right), \quad (4)$$
Remark 6.2. In the above-mentioned theorem we could reduce the needed lower bound by $2l(G) + 2\ell(G) - 2|E|$ for any two disjoint vertex sets $A$ and $B$ with $|A \cup B| = |V(G)| - 1$, where $E$ is the given edge set of size at most $l(G) + \ell(G)$. This refined version can imply Corollary 1.8 in [10].

The following corollary is an application of Theorem 6.1 which can help us to impose a bound on degrees.

**Corollary 6.3.** Let $G$ be a graph, let $l$ be a nonincreasing intersecting supermodular nonnegative integer-valued function on subsets of $V(G)$, and let $\ell$ be a 2-intersecting supermodular subadditive nonnegative integer-valued function on subsets of $V(G)$. If for each vertex $v$, $d_G(v) \geq 2l(v) + 2\ell(v)$ and for any two disjoint vertex sets $A$ and $B$ with $A \cup B \subseteq V(G)$ and $e_G(A \cup B) > \sum_{v \in A \cup B} \ell(v) - \ell(A \cup B)$,

$$d_{G-B}(A) \geq 2\ell(A \cup B) - \sum_{v \in B} \ell(v) + \begin{cases} 0, & \text{when } A = \emptyset; \\ 2\ell(A \cup B), & \text{when } A \neq \emptyset. \end{cases}$$

then $G$ has a spanning subgraph $H$ containing a packing of a spanning $l$-partition-connected subgraph and a spanning $\ell$-rigid subgraph such that for each vertex $v$,

$$d_H(v) \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor + l(v) + \ell(v).$$

**Proof.** Define $l'(v) = \left\lfloor \frac{d_G(v)}{2} \right\rfloor - l(v) - \ell(v)$, for each vertex $v$ so that $d_G(v) \geq 2l'(v) + 2l(v) + 2\ell(v)$. Define $l'(A) = 0$ for every vertex set $A$ with $|A| \geq 2$. By applying Theorems 6.1 and 4.1, the graph $G$ can be decomposed into a spanning $l'$-partition-connected subgraph $H'$, a spanning $l$-partition-connected subgraph $H_1$, and a spanning $\ell$-rigid subgraph $H_2$. Define $H = H_1 \cup H_2$. For each vertex $v$, we have $d_H(v) \geq l'(G - v) + l'(v) - l'(G) = l'(v)$. This implies that $d_H(v) = d_G(v) - d_{H'}(v) \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor + l(v) + \ell(v).$ 

6.1 Further improvements on connectivity requirements

In this subsection, we shall introduce another step toward improving Theorem 6.1 as the following stronger but more complicated version. This result improves the needed connectivity requirements a little.
Theorem 6.4. Let $G$ be a graph, let $l$ be a nonincreasing intersecting supermodular nonnegative integer-valued function on subsets of $V(G)$, and let $\ell$ be a 2-intersecting supermodular subadditive nonnegative integer-valued function on subsets of $V(G)$. Define $\lambda$ to be the minimum of all $|X|$ taken over all vertex sets $X$ with $e_G(X) > \sum_{v \in X} \ell(v) - \ell(X)$. Take $\phi$ to be a nonincreasing real function on subsets of $V(G)$ with $0 \leq \phi \leq 1$. If for each vertex $v$, $d_G(v) \geq 2\ell(v) + 2\ell(v)$ and for any two disjoint vertex sets $A$ and $B$ with $A \cup B \subseteq V(G)$ and $e_G(A \cup B) \geq \sum_{v \in A \cup B} \ell(v) - \ell(A \cup B)$,

$$d_{G-B}(A) + e(A \cup B) \geq 2\ell(A \cup B) - \sum_{v \in B} \ell(v) + l(A \cup B) \times \begin{cases} 2, & \text{when } B = \emptyset; \\ \frac{1}{\lambda} \phi(A \cup B), & \text{when } A = \emptyset; \\ 2 - \phi(A \cup B), & \text{when } A \neq \emptyset \text{ and } B \neq \emptyset, \end{cases}$$

then $G$ can be decomposed into a spanning $\ell$-partition-connected subgraph and a spanning $\ell$-rigid subgraph of size at most $l(G) + \ell(G)$, where $\epsilon(X) = 2l(G) + 2\ell(G) - 2|M|$ for every vertex set $X$ with $|X| = |V(G)| - 1$; and $\epsilon(X) = 0$ otherwise.

Proof. The proof follows with the same arguments of Theorem 6.1 with only minor modifications. In fact, if for a vertex set $Q \in P$, there is only one proper vertex subset $X$ of $Q$ with $X \in P$ and $X_A \neq \emptyset$, then there are at least $\lambda$ proper vertex subsets $X$ of $Q$ with $X \in P$ and $X_A = \emptyset$. \qed

Corollary 6.5. Let $G$ be a simple graph and let $k$ be an integer with $k \geq 2$. If $G$ is $4k$-edge-connected, and $G - B$ is $(4k - 1 - k|B|)$-edge-connected for every vertex set $B$, then $G$ has a spanning tree $T$ such that $G - E(T)$ is $k$-rigid.

Proof. Since $G$ has no multiple edges, it is not hard to verify that for every vertex set $X$ with $e_G(X) > k|X| - (2k - 1)$, we must have $|X| \geq 2k$ which implies that $k|X| - (4k - 2) \geq 2(k - 1)^2 \geq 1$. Now, it is enough to apply Theorem 6.4 with $\ell = \ell_{k,2k-1}$, $l = l_{1,1}$, $\lambda = 1$, and $\phi = 1$. \qed

Corollary 6.6. Let $G$ be a simple graph and let $k$ be an integer with $k \geq 2$. If $G$ is $(2k + 2)$-edge-connected and essentially $4k$-edge-connected, and $G - B$ is essentially $(4k - 1 - k|B|)$-edge-connected for every vertex set $B$, then $G$ has a spanning tree $T$ such that $G - E(T)$ is $k$-rigid.

The next corollary improves Corollary 1.11 in [10] a little.

Corollary 6.7. Every $6$-connected essentially $8$-edge-connected simple graph $G$ has a spanning tree $T$ such that $G - E(T)$ is $2$-connected.

7 A necessary and sufficient orientation condition for a graph to be $\ell$-rigid

In 1980 Frank formulated the following criterion for a graph to be $l$-partition-connected.
Theorem 7.1. ([5]) Let $G$ be a graph and let $l$ be an intersecting supermodular nonnegative integer-valued function on subsets of $V(G)$ with $l(\emptyset) = l(G) = 0$. Then $G$ is $l$-partition-connected if and only if it has an $l$-arc-connected orientation.

By applying a special case of the above-mentioned theorem due to Hakimi [11], we generalize Frank’s result to the following rigid version.

Theorem 7.2. Let $G$ be a graph and let $\ell$ be a weakly subadditive nonnegative integer-valued function on subsets of $V(G)$ with $\ell(\emptyset) = \ell(G) = 0$. Then $G$ is minimally $\ell$-rigid if and only if it has an $\ell$-arc-connected orientation such that for each vertex $v$, $d_G^-(v) = \ell(v)$.

Proof. First assume that $G$ has an $\ell$-arc-connected orientation such that for each vertex $v$, $d_G^-(v) = \ell(v)$. Obviously, $|E(G)| = \sum_{v \in V(G)} d_G^-(v) = \sum_{v \in V(G)} \ell(v)$. Furthermore, for every vertex set $A$, we have

$$e_G(A) = \sum_{v \in A} d_G^-(v) - \sum_{v \in A} \ell(v) - d_G^-(A) \leq \sum_{v \in A} \ell(v) - \ell(A).$$

Thus $G$ is $\ell$-sparse and hence minimally $\ell$-rigid. Now, assume that $G$ is minimally $\ell$-rigid. Since $G$ is $\ell$-sparse and $\ell$ is nonnegative, for every vertex set $A$, $e_G(A) \leq \sum_{v \in A} \ell(v) - \ell(A) \leq \sum_{v \in A} \ell(v)$. Since $|E(G)| = \sum_{v \in V(G)} \ell(v)$, the graph $G$ must have an orientation such that for each vertex $v$, $d_G^+(v) = \ell(v)$, see [11, Theorem 4]. Therefore, for every vertex set $A$, we must have

$$d_G^-(A) = \sum_{v \in A} d_G^-(v) - e_G(A) = \sum_{v \in A} \ell(v) - e_G(A) \geq \ell(A).$$

Thus the orientation of $G$ is $\ell$-arc-connected. Note that the equality holds only if $G[A]$ is $\ell$-rigid. $\square$

A combination of Theorem 6.1 and 7.2, can conclude the next result.

Theorem 7.3. Let $G$ be a graph, let $l$ be a nonincreasing intersecting supermodular nonnegative integer-valued function on subsets of $V(G)$, and let $\ell$ be a 2-intersecting supermodular subadditive nonnegative integer-valued function on subsets of $V(G)$. Let $r_1$ and $r_2$ be two nonnegative integer-valued functions on $V(G)$ which $l(G) = \sum_{v \in V(G)} r_1(v)$ and $\ell(G) = \sum_{v \in V(G)} r_2(v)$, and also $r_2 \leq \ell$. If for each vertex $v$, $d_G(v) \geq 2\ell(v) + 2l(v)$ and for any two disjoint vertex sets $A$ and $B$ with $A \cup B \subseteq V(G)$ and $e_G(A \cup B) > \sum_{v \in A \cup B} \ell(v) - \ell(A \cup B)$,

$$d_{G-B}(A) \geq 2\ell(A \cup B) - \sum_{v \in B} \ell(v) + \begin{cases} 0, & \text{when } A = \emptyset; \\ 2l(A \cup B), & \text{when } A \neq \emptyset. \end{cases}$$

then $G$ has an orientation along with two edge-disjoint spanning subdigraphs $H_1$ and $H_2$ such that $H_1$ is $r_1$-rooted $l$-arc-connected, $H_2$ is $r_2$-rooted $\ell$-arc-connected, and for each vertex $v$, $d_{H_1}^-(v) = l(v) - r_1(v)$, $d_{H_2}^-(v) = \ell(v) - r_2(v)$, and

$$d_{A}^+(v) \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor.$$

Furthermore, for a given arbitrary vertex $u$, the upper bound can be reduced to $\left\lfloor \frac{d_G(v)}{2} \right\rfloor$. 13
Proof. First assume that \( r_1 = r_2 = 0 \). For each vertex \( v \), define \( l_0(v) = \lfloor d_G(v)/2 \rfloor - \ell(v) - l(v) \) and define \( l_0(A) = 0 \) for every vertex set \( A \) with \( |A| \geq 2 \). By applying Theorems 6.1 and 4.1, the graph \( G \) can be decomposed into a spanning \( l_0 \)-partition-connected subgraph \( H_0 \), a spanning \( l \)-partition-connected subgraph \( H_1 \), and a spanning \( \ell \)-rigid subgraph \( H_2 \). By Theorem 7.2, every \( H_i \) has an \( l_i \)-arc-connected orientation, where \( l_1 = l \) and \( l_2 = \ell \). Consider the orientation of \( G \) obtained from these orientations. For each vertex \( v \), we must have \( d_{G}^{+}(v) \leq d_{G}(v) - \sum_{0 \leq i \leq 2} d_{H_i}^{-}(v) \leq \lfloor \frac{d_G(v)}{2} \rfloor \). In order to prove general case, one can apply the same arguments by replacing the set functions \( l - r_1 \) and \( \ell - r_2 \), where \( r_i(A) = \sum_{v \in V(G)} r_i(v) \) for every vertex set \( A \). Note that for reducing the upper bound for the vertex \( u \), the proof can be obtained by repeating the proof of Theorem 6.1 with minor modifications. \( \square \)

Corollary 7.4. Let \( G \) be a graph and let \( \ell \) be a \( 2 \)-intersecting supermodular subadditive nonnegative integer-valued function on subsets of \( V(G) \) and \( r \) be a nonnegative integer-valued function on \( V(G) \) with \( r \leq \ell \) and \( \ell(G) = \sum_{v \in V(G)} r(v) \). If \( G \) is \( \ell \)-weakly \( 2 \ell \)-connected, then it has an orientation along with a spanning \( \ell \)-rooted \( \ell \)-arc-connected subdigraph \( H \) such that for each vertex \( v \), \( d_{H}^{+}(v) = \ell(v) - r(v) \) and

\[
d_{G}^{+}(v) \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor.
\]

Furthermore, for a given arbitrary vertex \( u \) the upper bound can be reduced to \( \lfloor \frac{d_G(u)}{2} \rfloor \).

8 Spanning rigid subgraphs with small degrees on independent sets

In this section, we turn our attention to present the following strengthened version of Theorem 6.1 by restricting degrees. Note that this theorem can be refined to a more complicated version similar to Theorem 6.4.

Theorem 8.1. Let \( G \) be a graph, let \( l \) be a nonincreasing intersecting supermodular nonnegative integer-valued function on subsets of \( V(G) \), and let \( \ell \) be a \( 2 \)-intersecting supermodular subadditive nonnegative integer-valued function on subsets of \( V(G) \). Let \( k \) be a real number with \( k > 2 \) and let \( \rho \) be a nonnegative real function on \( V(G) \) with \( \rho \leq d_G \). If the following conditions hold:

1. For every \( S \subseteq V(G) \), \( e_G(S) \leq \sum_{v \in S} \rho(v) + \frac{k}{k-2} (l(G) + \ell(G)) \).

2. For each vertex \( v \), \( d_G(v) \geq kl(v) + kl(v) \) and for any two disjoint vertex sets \( A \) and \( B \) with \( A \cup B \subseteq V(G) \) and \( e_G(A \cup B) > \sum_{v \in A \cup B} \ell(v) - l(A \cup B) \),

\[
d_{G-B}^{+}(A) \geq kl(A \cup B) - \frac{1}{2} \sum_{v \in B} \ell(v) + \begin{cases} 0, & \text{when } A = \emptyset; \\ kl(A \cup B), & \text{when } A \neq \emptyset. \end{cases}
\]
then \( G \) has a spanning subgraph \( H \) containing a packing of a spanning \( l \)-partition-connected subgraph and a spanning \( \ell \)-rigid subgraph such that for each vertex \( v \),

\[
d_H(v) \leq \left\lfloor \frac{d_G(v) - 2\rho(v)}{k} \right\rfloor + \rho(v) + l(v) + \ell(v).
\]

**Proof.** We repeat the proof of Theorem 6.1 with some modifications. By an argument similar to the proof of Corollary 6.3, it is enough to show that \( G \) has a packing of a spanning \( l' \)-partition-connected subgraph and a spanning \( \ell \)-rigid subgraph, where \( l'(v) = l(v) + \left\lfloor \frac{k-2}{k}d_G(v) - \frac{k-2}{k}\rho(v) \right\rfloor - \ell(v) \) for each vertex \( v \), and \( l'(A) = l(A) \) for every vertex set \( A \) with \(|A| \geq 2\). Let \( F \) and \( \mathcal{F} \) be two edge-disjoint spanning subgraphs of \( G \) with the maximum \(|E(F)| + |E(\mathcal{F})|\) such that \( F \) is \( l' \)-sparse and \( \mathcal{F} \) is \( \ell \)-sparse.

Let \( P \) be a partition of \( V(G) \) with the properties described in Theorem 5.1. Define \( A \) to be the collection of all vertex sets of the maximal \( \ell \)-rigid subgraphs of \( \mathcal{F}[A] \), where \( A \in P \). We may assume that \( V(G) \notin A \). Let \( A_0 \) be the collection of all vertex sets \( X \) in \( A \) with \( e_G(X) = \sum_{v \in X} l(v) - \ell(X) \). Define \( P \) be the collection of all vertex sets in \( A \setminus A_0 \) along with the vertex sets \( \{v\} \) with \( v \in V(G) \setminus \cup_{X \in A_0} X \). For every \( X \in A_0 \), we have \( e_G(X) = e_F(X) \), and so for every \( xy \in E(F) \) with \( x, y \in A \), there must be an \( \ell \)-rigid subgraph of \( F \) including \( x \) and \( y \) whose vertex set lie in \( P \). Thus items (2) and (3) of Theorem 5.1 can imply that

\[
e_G(P) = e_F(P) + e_F(P).
\]

Take \( S \) to be the set of all vertices \( v \) such that \( \{v\} \in P \), and put \( P' = P \setminus \{\{v\} : v \in S\} \). For any \( X \in P \), define \( X_B \) to be the set of all vertices \( v \) which appears in at least two vertex sets of \( P \), and set \( X_A = X \setminus X_B \). It is not hard to check that

\[
\sum_{v \in S} d_G(v) - e_G(S) + e_G(S) \geq \sum_{X \in P'} d_{G - X_B}(X_A) - e_G(S) + e_G(S),
\]

which implies that

\[
\frac{2}{k}e_G(S) \geq \frac{1}{k} \sum_{X \in P'} d_{G - X_B}(X_A) - \frac{1}{k} \sum_{v \in S} d_G(v) + \frac{2}{k} e_G(S).
\]

Thus

\[
e_G(P) = \sum_{v \in S} d_G(v) - e_G(S) + e_G(S) \geq \frac{1}{k} \sum_{X \in P'} d_{G - X_B}(X_A) + \sum_{v \in S} \left( \frac{k-1}{k}d_G(v) - \frac{k-2}{k}\rho(v) \right) - \ell(G) - \ell(G).
\]

Since \( e_G(S) \leq \sum_{v \in S} \rho(v) + \frac{k-2}{k} (\ell(G) + \ell(G)) \), we must have

\[
e_G(P) \geq \frac{1}{k} \sum_{X \in P'} d_{G - X_B}(X_A) + \sum_{v \in S} \left( \frac{k-1}{k}d_G(v) - \frac{k-2}{k}\rho(v) \right) - l(G) - \ell(G).
\]

By the assumption,

\[
\frac{1}{k} \sum_{X \in P'} d_{G - X_B}(X_A) \geq \sum_{X \in P'} \left( \ell(X) - \frac{1}{2} \sum_{v \in X_B} \ell(v) \right) + \sum_{X \in P',X_A \neq \emptyset} l(X),
\]

which can imply that

\[
\frac{1}{k} \sum_{X \in P'} d_{G - X_B}(X_A) \geq \sum_{X \in P'} \ell(X) - \frac{1}{2} \sum_{X \in P} \sum_{v \in X_B} \ell(v) - \sum_{v \in S} \ell(v) + \sum_{X \in P',X_A \neq \emptyset} l(X).
\]
Hence Relations (7) and (8) can deduce that
\[ e_G(P) \geq \sum_{X \in P} \ell(X) - \frac{1}{2} \sum_{X \in P} \sum_{v \in X_B} \ell(v) + \sum_{v \in S} l'(v) + \sum_{X \in P', X_A \neq \emptyset} l'(X) - l'(G) - \ell(G). \]

Similar to the proof of Theorem 6.1, one can prove that for any \( Q \in P \), there is a vertex set \( X \) in \( P \) with \( X_A \neq \emptyset \). Since \( l' \) is nonincreasing and nonnegative, we must have
\[ \sum_{v \in S} l'(v) + \sum_{X \in P', X_A \neq \emptyset} l'(X) = \sum_{X \in P', X_A \neq \emptyset} l'(X) \geq \sum_{Q \in P} l'(Q), \]
which can imply that
\[ e_G(P) \geq \sum_{X \in P} \ell(X) - \frac{1}{2} \sum_{X \in P} \sum_{v \in X_B} \ell(v) + \sum_{Q \in P} (\sum l'(Q) - l'(G) - \ell(G)). \] \hspace{1cm} (9)

On the other hand,
\[ |E(F)| = e_F(P) + \sum_{X \in P} e_F(X) \geq e_G(P) - e_F(P) + \sum_{X \in P} (\sum (\ell(v) - \ell(X))). \] \hspace{1cm} (10)

Also,
\[ |E(F)| = e_F(P) + \sum_{Q \in P} e_F(Q) = e_F(P) + \sum_{Q \in P} (\sum (l'(v) - l'(Q))) = e_F(P) + \sum_{v \in V(G)} l'(v) - \sum_{v \in Q} l'(Q). \] \hspace{1cm} (11)

Therefore, Relations (9), (10), and (11) can conclude that
\[ |E(F)| + |E(F)| \geq \sum_{v \in V(G)} (l'(v) + \ell(v)) - l'(G) + \ell(G). \]

Thus we must have \( |E(F)| = \sum_{v \in V(G)} l'(v) - l'(G) \) and \( |E(F)| = \sum_{v \in V(G)} \ell(v) - \ell(G) \). Hence \( F \) is \( l' \)-partition-connected and \( F \) is \( \ell \)-rigid and the proof is completed. \( \square \)

**Corollary 8.2.** Let \( G \) be a bipartite graph with one partite set \( A \) and let \( k \) be a real number with \( k \geq 1 \). If \( G \) is \( 6k \)-connected, then it has a spanning \( 2 \)-rigid subgraph \( H \) such that for each \( v \in A \),
\[ d_H(v) \leq \left\lfloor \frac{d_G(v)}{k} \right\rfloor + 2. \]

**Proof.** For each \( v \in A \), define \( \rho(v) = 0 \), and for each \( v \in V(G) \setminus A \), define \( \rho(v) = d_G(v) \). Now, it is enough to apply Theorem 8.1 with \( \ell = \ell_{2,3} \) and use the fact that every \( 2 \)-rigid graph is \( 2 \)-connected. \( \square \)

**9 Hypergraph versions**

Let \( H \) be a hypergraph (possibly with repetition of hyperedges). The vertex set and the hyperedge set of \( H \) are denoted by \( V(H) \) and \( E(H) \), respectively. The (co-rank) rank of \( H \) is the (minimum) maximum
size of its hyperedges. The degree \(d_H(v)\) of a vertex \(v\) is the number of hyperedges of \(H\) including \(v\). For a set \(X \subseteq V(H)\), we denote by \(H[X]\) the induced sub-hypergraph of \(H\) with the vertex set \(X\) containing precisely those hyperedges \(Z\) of \(H\) with \(Z \subseteq X\). A spanning sub-hypergraph \(F\) is called \(l\)-sparse, if for all vertex sets \(A\), \(e_F(A) \leq \sum_{v \in A} l(v) - l(A)\), where \(e_F(A)\) denotes the number of hyperedges \(Z\) of \(F\) with \(Z \subseteq A\). Likewise, the hypergraph \(H\) is called \(l\)-partition-connected, if for every partition \(P\) of \(V(H)\), \(e_H(P) \geq \sum_{A \in P} l(A) - l(H)\), where \(e_H(P)\) denotes the number of hyperedges of \(H\) joining different parts of \(P\). We say that a hypergraph \(H\) is \(l\)-rigid, if it contains a spanning \(l\)-sparse sub-hypergraph \(F\) with \(|E(F)| = \sum_{v \in V(F)} l(v) - l(F)\). We call a hypergraph \(H\) directed, if for every hyperedge \(Z\), a head vertex \(u\) in \(Z\) is specified; other vertices of \(Z - u\) are called the tails of \(Z\). For a vertex \(v\), we denote by \(d_H^-(v)\) the number of hyperedges with head \(v\) and denote by \(d_H^+(v)\) and the number of hyperedges with tail \(v\).

We say that a directed hypergraph \(H\) is \(l\)-arc-connected, if for every vertex set \(A\), \(d_H^-(A) \geq l(A)\), where \(d_H^+(A)\) denotes the number of hyperedges \(Z\) with head vertex in \(A\) and \(Z \setminus A \neq \emptyset\). Likewise, \(H\) is called \(r\)-rooted \(l\)-arc-connected, if for every vertex set \(A\), \(d_H^-(A) \geq l(A) - \sum_{v \in A} r(v)\), where \(r\) is a nonnegative integer-valued function on \(V(H)\) with \(l(H) = \sum_{v \in V(H)} r(v)\). We denote by \(d_H \subseteq_B (A)\) the number of hyperedges \(Z\) with \(Z \cap B \neq \emptyset\) and \(Z \setminus (A \cup B) \neq \emptyset\), where \(A\) and \(B\) are two disjoint vertex sets.

### 9.1 A necessary and sufficient orientation condition for a hypergraph to be \(l\)-rigid

The following theorem is a hypergraph version of Theorem 7.1 which was proved by Frank, Király, and Király (2003).

**Theorem 9.1.** ([7]) Let \(H\) be a hypergraph and let \(l\) be an intersecting supermodular nonnegative integer-valued function on subsets of \(V(H)\) with \(l(\emptyset) = l(H) = 0\). Then \(H\) is \(l\)-partition-connected if and only if it has an \(l\)-arc-connected orientation.

Motivated by the above-mentioned theorem, we state the following hypergraph version of Theorem 7.2.

**Theorem 9.2.** Let \(H\) be a hypergraph and let \(\ell\) be a weakly subadditive nonnegative integer-valued function on subsets of \(V(H)\) with \(\ell(\emptyset) = \ell(H) = 0\). Then \(H\) is minimally \(\ell\)-rigid if and only if it has an \(\ell\)-arc-connected orientation such that for each vertex \(v\), \(d_H^+(v) = \ell(v)\).

**Proof.** First assume that \(H\) has an \(\ell\)-arc-connected orientation such that for each vertex \(v\), \(d_H^+(v) = \ell(v)\). Obviously, \(|E(H)| = \sum_{v \in V(H)} d_H^+(v) = \sum_{v \in V(H)} \ell(v)\). Furthermore, for every vertex set \(A\), we have

\[
e_H(A) = \sum_{v \in A} d_H^+(v) - d_H^-(A) = \sum_{v \in A} \ell(v) - \ell(A) \leq \sum_{v \in A} \ell(v) - \ell(A).
\]

Thus \(H\) is \(\ell\)-sparse and hence minimally \(\ell\)-rigid. Now, assume that \(H\) is minimally \(\ell\)-rigid. Since \(H\) is \(\ell\)-sparse and \(\ell\) is nonnegative, for every vertex set \(A\), \(e_H(A) \leq \sum_{v \in A} \ell(v) - \ell(A) \leq \sum_{v \in A} \ell(v)\). Since
\( |E(\mathcal{H})| = \sum_{v \in V(\mathcal{H})} \ell(v) \), the hypergraph \( \mathcal{H} \) must have an orientation such that for each vertex \( v \), \( d^-_{\mathcal{H}}(v) = \ell(v) \), see [7, Lemma 3.3]. Therefore, for every vertex set \( A \), we must have
\[
d^-_{\mathcal{H}}(A) = \sum_{v \in A} d^-_{\mathcal{H}}(v) - e_{\mathcal{H}}(A) = \sum_{v \in A} \ell(v) - e_{\mathcal{H}}(A) \geq \ell(A).
\]
Thus the orientation of \( \mathcal{H} \) is \( \ell \)-arc-connected. \( \square \)

9.2 Generalizations

In this subsection, we only state the hypergraphs versions of the main results of this paper, which their proofs follow with the same arguments that stated for whose graph versions.

**Proposition 9.3.** Let \( \mathcal{H} \) be a hypergraph and let \( \ell \) be a weakly subadditive real function on subsets of \( V(\mathcal{H}) \). If \( \mathcal{H} \) is \( \ell \)-rigid, then for any two disjoint vertex sets \( A \) and \( B \),
\[
d^-_{\mathcal{H} \ominus B}(A) \geq \ell(A \cup B) - \sum_{v \in A} \ell(v) + (\ell(\mathcal{H} \setminus A) - \ell(\mathcal{H})).
\]

**Theorem 9.4.** Let \( \mathcal{H} \) be a hypergraph with the co-rank \( c \), \( c \geq 2 \), let \( l \) be an intersecting supermodular subadditive integer-valued function on subsets of \( V(\mathcal{H}) \), and let \( \ell \) be a \( c \)-intersecting supermodular subadditive integer-valued function on subsets of \( V(\mathcal{H}) \). If \( F \) and \( F \) are two edge-disjoint spanning sub-hypergraphs of \( \mathcal{H} \) with the maximum \( |E(F \cup \mathcal{F})| \) such that \( F \) is \( l \)-sparse and \( F \) is \( \ell \)-sparse, then there is a partition \( P \) of \( V(\mathcal{H}) \) with the following properties:

1. For any \( A \in P \), the hypergraph \( F[A] \) is \( l \)-partition-connected.
2. There is no hyperedges in \( E(\mathcal{H}) \setminus E(F \cup \mathcal{F}) \) joining different parts of \( P \).
3. For every \( Z \in E(F) \) with \( Z \subseteq A \in P \), there is a vertex set \( X \) such that \( Z \subseteq X \subseteq A \) and \( F[X] \) is \( \ell \)-rigid.

**Theorem 9.5.** Let \( \mathcal{H} \) be a hypergraph with the rank \( r \) and co-rank \( c \), \( c \geq 2 \), let \( l \) be a nonincreasing intersecting supermodular nonnegative integer-valued function on subsets of \( V(\mathcal{H}) \), and let \( \ell \) be a \( c \)-intersecting supermodular subadditive nonnegative integer-valued function on subsets of \( V(\mathcal{H}) \). If for each vertex \( v \), \( d_{\mathcal{H}}(v) \geq r\ell(v) + rl(v) \) and for any two disjoint vertex sets \( A \) and \( B \) with \( A \cup B \subseteq V(\mathcal{H}) \) and \( e_{\mathcal{H}}(A \cup B) > \sum_{v \in A \cup B} \ell(v) - \ell(A \cup B) \),
\[
d^-_{\mathcal{H} \ominus B}(A) \geq r\ell(A \cup B) - \frac{r}{2} \sum_{v \in B} \ell(v) + \begin{cases} 0, & \text{when } A = \emptyset; \\
rl(A \cup B), & \text{when } A \neq \emptyset,
\end{cases}
\]
then \( \mathcal{H} \) can be decomposed into a spanning \( l \)-partition-connected sub-hypergraph and a spanning \( \ell \)-rigid sub-hypergraph, and also a given edge set of size at most \( l(\mathcal{H}) + \ell(\mathcal{H}) \).
Corollary 9.6. Let \( \mathcal{H} \) be a hypergraph with the rank \( r \) and co-rank \( c \), \( c \geq 2 \), let \( l \) be a nonincreasing intersecting supermodular nonnegative integer-valued function on subsets of \( V(\mathcal{H}) \), and let \( \ell \) be a \( c \)-intersecting supermodular subadditive nonnegative integer-valued function on subsets of \( V(\mathcal{H}) \) with \( l(\mathcal{H}) = \ell(\mathcal{H}) = 0 \). If for each vertex \( v \), \( d_\mathcal{H}(v) \geq r\ell(v) + r\ell(v) \) and for any two disjoint vertex sets \( A \) and \( B \) with \( A \cup B \subseteq V(\mathcal{H}) \) and \( e_\mathcal{H}(A \cup B) \geq \sum_{v \in A \cup B} \ell(v) - \ell(A \cup B) \),

\[
d_{\mathcal{H} \cup B}(A) \geq r\ell(A \cup B) - \frac{r}{2} \sum_{v \in B} \ell(v) + \begin{cases} 0, & \text{when } A = \emptyset; \\ r\ell(A \cup B), & \text{when } A \neq \emptyset, \end{cases}
\]

then \( \mathcal{H} \) has an orientation along with two edge-disjoint spanning sub-hypergraphs \( H_1 \) and \( H_2 \) such that \( H_1 \) is \( l \)-arc-connected, \( H_2 \) is \( \ell \)-arc-connected, and for each vertex \( v \), \( d_{H_1}^v = l(v) \), \( d_{H_2}^v = \ell(v) \), and

\[
d_{H_1}^v \leq \lfloor \frac{r-1}{r} d_\mathcal{H}(v) \rfloor.
\]

Furthermore, for a given arbitrary vertex \( u \) the upper bound can be reduced to \( \lfloor \frac{r-1}{r} d_\mathcal{H}(u) \rfloor \).

10 Applications

The following theorem improves Theorem 4.1 in [9] by imposing a bound on degrees.

Theorem 10.1. Every \( k \)-weakly \((4kp - 2p + 2m)\)-connected simple graph \( G \) with \( k \geq 2 \) has a spanning subgraph \( H \) containing a packing of \( m \) spanning trees and \( p \) spanning \( k \)-rigid subgraphs such that for each vertex \( v \),

\[
d_H(v) \leq \lfloor \frac{d_G(v)}{2} \rfloor + kp + m.
\]

Proof. Apply Corollary 6.3 with \( \ell = p\ell_{k,2k-1} \) and \( l = l_{m,m} \), and next apply Theorems 4.1 and 4.2. \( \square \)

The next result improves Theorem 1.4 in [9] by replacing essentially edge-connectivity by edge-connectivity.

Theorem 10.2. Every \( k \)-weakly \((4kp - 2p + 2m)\)-connected simple graph \( G \) with \( k \geq 2 \) has a spanning subgraph \( H \) containing a packing of \( m \) spanning trees and \( p \) spanning \( k \)-rigid \((2k - 1)\)-edge-connected subgraphs such that for each vertex \( v \),

\[
d_H(v) \leq \lfloor \frac{d_G(v)}{2} \rfloor + 2kp - p + m.
\]

Proof. By applying Corollary 6.3 with \( \ell = p\ell_{k,2k-1} \) and \( l = l_{k-p+p+m,m} \), and next applying Theorems 4.1 and 4.2, one can conclude that \( G \) has a spanning subgraph \( H \) containing a packing of \( p \) spanning \( k \)-rigid subgraphs \( G_1, \ldots, G_p \), and \( p \) spanning \( l_{k-1,0} \)-partition-connected subgraphs \( G'_1, \ldots, G'_p \), and also \( m \) spanning trees \( T_1, \ldots, T_m \) such that for each vertex \( v \), \( d_H(v) \leq \lfloor \frac{d_G(v)}{2} \rfloor + 2kp - p + m \). By Corollary 3.2, every graph \( G_i \) is \( k \)-edge-connected and essentially \((2k - 1)\)-edge-connected. Define \( H_i = G_i \cup G'_i \). Since \( \delta(H_i) \geq \delta(G_i) + \delta(G'_i) \geq k + (k - 1) = 2k - 1 \), the graph \( H_i \) must be \((2k - 1)\)-edge-connected. Now, it is enough to consider the graphs \( H_1, \ldots, H_p \) and \( T_1, \ldots, T_m \) as the packing of \( H \) with the desired properties. Note that \( G \) could have multiple edges with multiplicity at most \( p \). \( \square \)
10.1 2-connected \((2k - 1)\)-edge-connected \(\{r - 3, r - 1\}\)-factors

Recently, the present author [12] showed that every \((2\lceil r/6 \rceil + 2k)\)-edge-connected \(r\)-regular graph of even order with \(r \geq 4\) has a \(k\)-tree-connected \(\{r - 3, r - 1\}\)-factor. In the following, we improve this result for highly connected graphs. Before doing so, we recall the following lemma.

Lemma 10.3. ([12]) Every \(m\)-tree-connected graph \(G\) has a spanning forest \(F\) with odd degrees such that for each vertex \(v\), \(d_F(v) \leq \lceil \frac{d_G(v)}{m} \rceil\).

Theorem 10.4. Every \((2\lceil r/6 \rceil + 4k - 2)\)-connected \(r\)-regular graph of even order with \(r \geq 4\) has a \(k\)-rigid \(\{r - 3, r - 1\}\)-factor.

Proof. Put \(m = \lceil r/6 \rceil\). By Theorem 10.1, the graph \(G\) contains two edge-disjoint spanning subgraphs \(L'\) and \(L\) such that \(L'\) is \(k\)-rigid, \(L\) is \(m\)-tree-connected, and also for each vertex \(v\), \(d_{L'}(v) + d_L(v) \leq \lceil \frac{d_G(v)}{2} \rceil + k + m\). Note that for each vertex \(v\), we must have \(d_L(v) \leq \lceil \frac{d_G(v)}{2} \rceil + m\). By Lemma 10.3, the graph \(L\) has a spanning forest \(F\) with odd degrees such that for each vertex \(v\), \(d_{F}(v) \leq \lceil \frac{d_{L'}(v)}{m} \rceil \leq \lceil \frac{d_G(v)}{2m} \rceil + 1 = 4\).

It is not hard to check that \(G \setminus E(F)\) is the desired spanning subgraph we are looking for. \(\square\)

10.2 Arc-connected orientations of graphs

Recently, Gu [9] showed that every \((2k + 1)\)-weakly \((8k + 4)\)-connected simple graph has an orientation such that for each vertex \(v\), \(G - v\) remains \(k\)-arc-strong. In the following, we strengthen this result in the same way by replacing a special case of Theorem 10.2. For this purpose, we first recall the following lemma due to Király and Szigeti (2006).

Lemma 10.5. ([15]) An Eulerian graph \(G\) has a smooth orientation such that for each vertex \(v\), the resulting directed graph \(G - v\) is \(k\)-arc-strong, if and only if for each vertex \(v\), the graph \(G - v\) is \(2k\)-edge-connected.

The following theorem improves Theorem 1.7 in [9].

Theorem 10.6. Every \((2k + 1)\)-weakly \((8k + 4)\)-connected simple graph \(G\) has a \((2k + 1)\)-arc-strong smooth orientation such that for each vertex \(v\), the resulting directed graph \(G - v\) remains \(k\)-arc-strong.

Proof. By applying Theorem 10.2 with \(p = m = 1\) and replacing \(2k + 1\) instead of \(k\), the graph \(G\) can be decomposed into a spanning tree \(T\) and a spanning \((2k + 1)\)-rigid \((4k + 1)\)-edge-connected subgraph \(G'\). According to Corollary 3.2, for each vertex \(v\) the graph \(G' - v\) remains \(2k\)-edge-connected. It is not hard to check that there is a spanning forest \(F\) of \(T\) such that for each vertex \(v\), \(d_F(v)\) and \(d_{G'}(v)\) have the same parity. Define \(H\) to be the spanning Eulerian subgraph of \(G\) with \(E(H) = E(G') \cup E(F)\). Note that \(H\) must automatically be \((4k + 2)\)-edge-connected. By Lemma 10.5, the graph \(H\) has a smooth orientation.
such that for each vertex \( v \), the resulting directed graph \( H - v \) remains \( k \)-arc-strong. Since this orientation is Eulerian, it is also \((2k + 1)\)-arc-strong. Now, it enough to consider a smooth orientation for the spanning graph \( H' \) of \( G \) with \( E(H') = E(G) \setminus E(H) \) and induce whose orientation to \( G \). This can complete the proof.

\( \square \)

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