Research Article

A Variational Principle for the Steady-State Heat Transfer Process in a Rigid Continuous Mixture

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The subject of this paper is the steady-state heat transfer process in a rigid mixture with $N$ continuous constituents, each of them representing a given continuous body. A continuous mixture consists of a convenient representation for bodies composed by several different materials or phases, in which the actual interfaces do not allow an adequate Classical Continuum Mechanics approach, once that the boundary conditions make the mathematical description of the problem unfeasible (as for instance in reinforced concrete, polymer strengthened concrete, and porous media). The phenomenon is mathematically described by a set of $N$ partial differential equations coupled by temperature-dependent terms that play the role of internal energy sources. These internal energy sources arise because, at each spatial point, there are different temperatures, each one associated with one constituent of the mixture. The coupling among the partial differential equations arises from the thermal interchange among continua in a thermal nonequilibrium context (different temperature levels). In this work, it is presented a functional whose minimization is equivalent to the solution of the original steady-state problem (variational principle). The features of this functional give rise to proofs of solution existence and solution uniqueness. It is remarkable that, with the functional to be proposed here, instead of solving a system of $N$ coupled partial differential equations, we need to look only for the minimum of a single functional.

1. Introduction

Based on an original idea pointed out by Fick (in 1855) [1] and Stefan (in 1872) [2] and developed later by Atkin and Craine [3, 4] and Bowen [5], the Continuum Theory of Mixtures represents a generalization of the Classical Continuum Mechanics [6–8] in which a composite body (or system) is regarded as a superposition of $N$ continua occupying, simultaneously, the whole volume of the body. In other words, the body is regarded as a mixture consisting of $N$ continuous constituents.

Each constituent possesses its own temperature field and exchanges energy with the other constituents of the mixture when there is no thermal equilibrium.

The main subject of the present work is the energy transfer phenomenon in a rigid continuous mixture. This phenomenon (for a mixture with $N$ constituents) will be described by a set of $N$ partial differential equations coupled by internal source terms, which takes into account the temperatures of all the constituents and provides the thermal interaction among them.

The study of this kind of phenomenon is motivated by the existence of multimaterial heterogeneous bodies (reinforced concrete bodies, carbon fiber materials bodies, porous bodies, and others), for which the usual heat transfer approach is not convenient.

The main objectives of this work are to present the following:

1. A minimum principle, suitable for the steady-state heat transfer process in a rigid continuous mixture.
2. A proof of solution existence.
(3) A proof of solution uniqueness.

A continuous mixture is defined as a superposition of $N$ continuous constituents, each one representing a given continuum (a given material), as suggested in Figure 1.

In fact, the continuous mixture viewpoint consists of assuming that a body composed by a set of $N$ continua, occupying, respectively, the spatial regions $\Omega_1, \Omega_2, \Omega_3, \ldots, \Omega_N$ with $\Omega_i \cap \Omega_j = \emptyset$ if $i \neq j$ and with $\cup_{i=1}^{N}\Omega_i = \Omega$, be represented by a set of $N$ continuous constituents, all of them occupying simultaneously the entire region $\Omega$ (the whole mixture), in such a way that the original interfaces vanish, as suggested in Figure 1.

This superposition allows the existence of $N$ distinct temperatures at each spatial point in the mixture.

It is to be noticed that these $N$ distinct temperature fields are not independent since the energy transfer process in each constituent is affected by the energy transfer process in all the others constituents of the mixture.

In order to take into account the thermal interaction among the constituents, some special fields must be considered in the balance equations. In other words, the energy transfer process in each constituent will be described by an equation which possesses a field (the internal source term) representing the amount of energy (per unit time and unit volume) supplied to it by the other ones of the mixture. This field will be denoted here by $\Psi_\alpha$ (for the constituent $C_\alpha$).

The field $\Psi_\alpha$ will depend on the difference between the temperature of the $\alpha$th constituent and each one of the other constituents (in other words, if two constituents have, at the same point, different temperatures, then there will exist a heat exchange between them).

### 2. Energy Balance

The energy balance for each constituent $C_\alpha$ of a given rigid continuous mixture (composed by $N$ constituents) is mathematically described as follows [4]:

$$\rho_\alpha c_\alpha \frac{\partial T_\alpha}{\partial t} = -\text{div} q_\alpha + q_\alpha + \Psi_\alpha, \quad \text{in } \Omega \quad \text{for } \alpha = 1, 2, \ldots, N,$$

(1)

in which $\Omega$ is a bounded open set representing the region occupied by the mixture, $\rho_\alpha$ represents the mass density of $C_\alpha$ (ratio between the mass of the constituent $C_\alpha$ to the corresponding volume of mixture—different from the classical mass density), $c_\alpha$ is the specific heat of $C_\alpha$, $T_\alpha$ is the temperature of $C_\alpha$, $q_\alpha$ is the partial heat flux associated with $C_\alpha$, $q_\alpha$ is the internal heat supply (per unit time and per unit volume) for $C_\alpha$, and $\Psi_\alpha$ is the energy supply arising from the interaction between $C_\alpha$ and the other constituents of the mixture.

The internal heat supply $\Psi_\alpha$ is an internal effect. It provides the coupling among the heat transfer processes in all the constituents and represents (for each $\alpha$) the amount of energy, per unit time and unit volume, supplied to the constituent $C_\alpha$ due to its thermal interaction with the other constituents of the continuous mixture. Hence, in addition to equation (1), the following must hold (energy balance for the mixture as a whole):

$$\sum_{\alpha=1}^{N} \Psi_\alpha = 0, \quad \text{in } \Omega. \quad (2)$$

### 3. Constitutive Relations

The partial heat flux (per unit time and unit area) associated with the constituent $C_\alpha$ is given by

$$q_\alpha = -\Lambda k_\alpha \phi_\alpha \nabla T_\alpha,$$

(3)

in which $\Lambda$ is a positive-definite second order tensor (usually constant), $k_\alpha$ is the thermal conductivity of the material represented by $C_\alpha$ and $\phi_\alpha$ is the ratio between $\rho_\alpha$ and the actual mass density of the material represented by $C_\alpha$. The fields $\phi_\alpha$ are such that

$$\phi_\alpha = 1, \quad \text{in } \Omega \quad \text{and } 0 < \phi_\alpha < 1 \quad \text{in } \Omega, \quad \text{for any } \alpha. \quad (4)$$

The fields $\phi_\alpha$ are always known, for any $\alpha$.

The internal heat supply $\Psi_\alpha$ is the sum of the heat supplies coming from all the constituents to the constituent $C_\alpha$. In other words, we may write

$$\Psi_\alpha = \sum_{\beta=1}^{N} \Gamma_{\alpha \rightarrow \beta},$$

(5)

in which $\Gamma_{\alpha \rightarrow \beta}$ is the heat supply from $C_\beta$ to $C_\alpha$. Hence,

$$\Gamma_{\alpha \rightarrow \beta} = -\Gamma_{\beta \rightarrow \alpha} \Rightarrow \Gamma_{\alpha \rightarrow \alpha} = 0. \quad (6)$$

As a consequence of the Second Law of Thermodynamics [9], the sign of $\Gamma_{\alpha \rightarrow \beta}$ at each point, depends only on the sign of the difference $(T_\beta - T_\alpha)$ at this point. If $T_\beta > T_\alpha$, $\Gamma_{\alpha \rightarrow \beta}$ must be positive (since $C_\alpha$ will receive energy from $C_\beta$). In other words, the sign of $\Gamma_{\alpha \rightarrow \beta}$ is the same sign of $(T_\beta - T_\alpha)$. This fact induces the following constitutive relationship:

$$\Gamma_{\alpha \rightarrow \beta} = R_{\alpha \beta} (T_\beta - T_\alpha), \quad R_{\alpha \beta} = R_{\beta \alpha} > 0, \quad \text{for any } \alpha \text{ and } \beta. \quad (7)$$

The simplest case is the one in which $R_{\alpha \beta}$ is assumed to be a constant for any $\alpha$ and $\beta$.

It is remarkable that since (7) holds, equation (2) is automatically satisfied.

### 4. Boundary Conditions

It will be assumed that each constituent of the mixture exchanges energy with the environment by convection [10] (Incropera and Dewitt, 1996). In this way, on $\partial \Omega$, we have the following boundary condition:

$$q_\alpha \cdot n = h_\alpha (T_\alpha - T_{\text{ref}}), \quad \text{on } \partial \Omega \quad \text{for } \alpha = 1, 2, \ldots, N,$$

(8)

in which $h_\alpha$ is a nonnegative constant (convection heat transfer coefficient), $T_{\text{ref}}$ is a temperature of reference, and $n$ is the unit outward normal defined on $\partial \Omega$. It will be assumed
that, for at least one constituent, \( h_a \) be positive on some nonempty subset of \( \partial \Omega \).

It is to be noticed that, when \( h_a \) is assumed zero for a given constituent, this constituent does not exchange energy with the environment (insulated boundary).

5. The Steady-State Mathematical Description

Inserting (3) in (1) and in (8) and considering (7) and (5), we have the following description for the steady-state heat transfer phenomenon in a rigid continuous mixture:

\[
\begin{align*}
\text{div}(Ak_{a} \phi_a \text{grad} T_a) + \sum_{\beta=1}^{N} R_{\alpha\beta}(T_{\beta} - T_{a}) + \dot{q}_{a} & = 0, \quad \text{in } \Omega, \\
- Ak_{a} \phi_a \text{grad} T_a \cdot n & = h_a(T_a - T_{\infty}) \quad \text{on } \partial \Omega \text{ for } a = 1, 2, \ldots, N,
\end{align*}
\]

in which the unknowns are the temperature fields \( T_1, T_2, \ldots, T_N \).

It will be shown now that these unknowns may be obtained from the minimization of a convex and coercive functional, ensuring the existence and the uniqueness of the solution.

6. The Minimum Principle

If we assume that \( \Lambda, k_a, \phi_a, R_{\alpha\beta} \) and \( h_a \) do not depend on the unknowns \( T_1, T_2, \ldots, T_N \), and the problem represented by the set of \( N \) partial differential equations and \( N \) boundary conditions is equivalent to the minimization of the following convex and coercive functional [11]:

\[
\begin{align*}
I[w_1, w_2, \ldots, w_N] = & \sum_{a=1}^{N} \int_{\Omega} \frac{1}{2} k_a \phi_a (\text{A grad } w_a \cdot \text{grad } w_a) dV \\
- & \sum_{a=1}^{N} \int_{\Omega} \dot{q}_a w_a dV \\
+ & \sum_{a=1}^{N} \sum_{\beta=a}^{N} \left[ \frac{1}{2} R_{\alpha\beta}(w_{\beta} - w_a)^2 dV \right] \\
+ & \sum_{a=1}^{N} \int_{\partial \Omega} \frac{1}{2} h_a (w_a - T_{\infty})^2 dA,
\end{align*}
\]

in which the admissible fields \( w_a \) must belong to the Sobolev space \( H^1(\Omega) \) [12].

Minimizing a functional, like the one presented in (10), instead of solving a system of partial differential equations, represents an enormous advantage since it provides less effort when compared with traditional methods (like finite difference). In addition, a minimum principle, like the one represented by (10), provides easy and powerful tools for demonstrating the uniqueness and the existence of the solution. Variational tools are always welcome [13, 14].

But, first of all, it is mandatory to show that the functional defined in (10) is the correct choice for the considered (original) problem. In other words, we must show the equivalence between (9) and the minimization of \( I[w_1, w_2, \ldots, w_N] \).

In order to show that the solution of problem (9) is equivalent to the minimization of the functional \( I[w_1, w_2, \ldots, w_N] \) defined in (10), we begin evaluating the first variation of \( I[w_1, w_2, \ldots, w_N] \), denoted here by \( \delta I \).

The first variation of \( I \) is given as follows [15]:

\[
\begin{align*}
\delta I = & \sum_{a=1}^{N} \int_{\Omega} Ak_{a} \phi_a (\text{grad } w_a \cdot \text{grad } \delta w_a) dV \\
- & \sum_{a=1}^{N} \int_{\Omega} \dot{q}_a \delta w_a dV + \\
+ & \sum_{a=1}^{N} \sum_{\beta=1}^{N} \int_{\Omega} R_{\alpha\beta}(w_{\beta} - w_a)(\delta w_{\beta} - \delta w_a) dV \\
+ & \sum_{a=1}^{N} \int_{\partial \Omega} h_a(\delta w_a)(\delta w_a - \delta w_a) dA,
\end{align*}
\]

or, employing Green’s identity [16],

\[
\begin{align*}
\delta I = & \sum_{a=1}^{N} \int_{\Omega} - \text{div}(k_a \phi_a (\text{A grad } w_a)) \delta w_a dV \\
+ & \sum_{a=1}^{N} \int_{\Omega} k_a \phi_a (\text{A grad } w_a) \cdot n \delta w_a dA + \\
+ & \sum_{a=1}^{N} \int_{\partial \Omega} h_a(\delta w_a)(\delta w_a - \delta w_a) dA + \sum_{a=1}^{N} \int_{\partial \Omega} \dot{q}_a \delta w_a dV \\
+ & \sum_{a=1}^{N} \sum_{\beta=1}^{N} \int_{\Omega} R_{\alpha\beta}(w_{\beta} - w_a)(\delta w_{\beta} - \delta w_a) dV.
\end{align*}
\]
Since
\[ \alpha \leq \beta \leq N, \quad 1 \leq \alpha \leq N \iff 1 \leq \alpha \leq \beta, \quad 1 \leq \beta \leq N. \] (13)

The last term of (12) may be rewritten as follows:
\[ \frac{N}{\alpha = 1} \sum_{\beta = 1}^{1} \int_{\Omega} R_{\alpha \beta}(w_{\beta} - u_{\alpha})(\delta w_{\beta} - \delta u_{\alpha})d\Omega \]
\[ = \frac{N}{\alpha = 1} \sum_{\beta = 1}^{1} \int_{\Omega} R_{\alpha \beta}(w_{\beta} - u_{\alpha})d\Omega \]
\[ + \frac{N}{\beta = 1} \sum_{\alpha = 1}^{1} \int_{\Omega} R_{\alpha \beta}(w_{\beta} - u_{\alpha})d\Omega, \]

or replacing \( \alpha \) by \( \beta \) in the last term and taking into account that \( R_{\alpha \beta} = R_{\beta \alpha} \) if \( \alpha = \beta \),
\[ \int_{\Omega} R_{\alpha \beta}(w_{\beta} - u_{\alpha})(\delta w_{\beta} - \delta u_{\alpha})d\Omega \]
\[ = \int_{\Omega} R_{\alpha \beta}(w_{\beta} - u_{\alpha})d\Omega, \]

So, (12) may be rewritten as
\[ \delta I = -\int_{\Omega} \sum_{\alpha = 1}^{N} \left\{ \text{div}(k_{\alpha} \phi_{\alpha}(\Lambda \text{ grad } u_{\alpha})) \right\} + \int_{\Omega} \sum_{\alpha = 1}^{N} \left\{ h_{\alpha}(w_{\alpha} - T_{\infty}) + k_{\alpha} \phi_{\alpha}(\Lambda \text{ grad } u_{\alpha}) \cdot n \right\} \delta u_{\alpha} d\Omega. \]

(16)

The extrema of the functional \( I \) are obtained for the fields \( u_{\alpha} \), \( \alpha = 1, 2, \ldots, N \) such that \( u_{\alpha} \equiv u_{\alpha} \) gives rise to \( \delta I = 0 \).

Impressing \( \delta I = 0 \) and taking into account that the fields \( \delta w_{1}, \delta w_{2}, \ldots, \delta w_{N} \) are arbitrary and independent, we have that the fields \( u_{\alpha} \) must satisfy (Euler–Lagrange equations and natural boundary conditions):
\[ \text{div}(\Lambda k_{\alpha} \phi_{\alpha} \text{ grad } u_{\alpha}) + \frac{N}{\alpha = 1} \sum_{\beta = 1}^{N} R_{\alpha \beta}(u_{\beta} - u_{\alpha}) + \delta u_{\alpha} = 0, \quad \text{in } \Omega, \]
\[ - \Lambda k_{\alpha} \phi_{\alpha} \text{ grad } u_{\alpha} \cdot n = h_{\alpha}(u_{\alpha} - T_{\infty}) \text{ on } \partial \Omega \quad \text{for } \alpha = 1, 2, \ldots, N. \]

(17)

It is easy to see that problems (17) and (9) are equivalent. In other words, the extrema of \( I \) are reached for the solutions of (9) (or (17)).

7. Uniqueness

It will be shown now that the solution of (17) (or (9)) is unique and corresponds to a minimum of the functional \( I \).

In other words, it will be shown that if \( T_{1}, T_{2}, \ldots, T_{N} \) satisfies (9), then
\[ I[T_{1}, T_{2}, \ldots, T_{N}] < I[w_{1}, w_{2}, \ldots, w_{N}], \]

provided \( T_{\alpha} \neq w_{\alpha} \) for some \( \alpha \).

(18)

Aiming to this, it is sufficient to prove that the functional \( I \) is strictly convex, which means it satisfies the following inequality [17]:
\[ I[\theta w + (1 - \theta) y] < \theta I[w] + (1 - \theta) I[y], \]

for any \( \theta \in (0, 1) \), for any \( y \neq w \),

in which \( w = (w_{1}, w_{2}, \ldots, w_{N}) \) and \( y = (y_{1}, y_{2}, \ldots, y_{N}) \).

The first step for proving (19) is to show that, for \( \theta \in (0, 1) \), the expression \( \theta I[w] + (1 - \theta) I[y] - I[\theta w + (1 - \theta) y] \) is always nonnegative.

Aiming to this, let us write the above expression as follows:
\[ \theta I[w] + (1 - \theta) I[y] - I[\theta w + (1 - \theta) y] = \]
\[ = \theta \int_{\Omega} \sum_{\alpha = 1}^{N} \frac{1}{2} k_{\alpha} \phi_{\alpha} (\Lambda \text{ grad } u_{\alpha}) \text{ grad } u_{\alpha} d\Omega + \]
\[ + \frac{N}{\alpha = 1} \sum_{\beta = 1}^{N} R_{\alpha \beta}(w_{\beta} - u_{\alpha})^{2} d\Omega + \]
\[ + \frac{N}{\alpha = 1} \sum_{\beta = 1}^{N} R_{\alpha \beta}(w_{\beta} - u_{\alpha})^{2} d\Omega + \]
\[ + (1 - \theta) \int_{\Omega} \sum_{\alpha = 1}^{N} \frac{1}{2} k_{\alpha} \phi_{\alpha} (\Lambda \text{ grad } u_{\alpha}) \text{ grad } u_{\alpha} d\Omega + \]
\[ - \frac{N}{\alpha = 1} \sum_{\beta = 1}^{N} \delta u_{\alpha} \delta u_{\alpha} d\Omega + \]
\[ + \frac{N}{\alpha = 1} \sum_{\beta = 1}^{N} \delta u_{\alpha} \delta u_{\alpha} d\Omega + \]
\[ + \frac{N}{\alpha = 1} \sum_{\beta = 1}^{N} \delta u_{\alpha} \delta u_{\alpha} d\Omega + \]
\[ + \frac{N}{\alpha = 1} \sum_{\beta = 1}^{N} \delta u_{\alpha} \delta u_{\alpha} d\Omega + \]
\[ + \frac{N}{\alpha = 1} \sum_{\beta = 1}^{N} \delta u_{\alpha} \delta u_{\alpha} d\Omega + \]

(20)

Reordering some terms and taking into account that
Now, we must prove that, for any uniqueness of the solution of (9) is proven.

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...positive valued for any difference $\alpha$ _herefore, the positiveness of (22) is proven. f_ this ensures

Aiming to this, let us suppose that, for some $\alpha$, the difference $w_a - v_a$ is not a constant. In this case, since $k_a\phi_a$ is positive valued for any $\alpha$ and $\Lambda$ is a positive-definite tensor, we have

$$\sum_{a=1}^{N} \int_{\Omega} \frac{1}{2} k_a\phi_a \Lambda \text{grad} (w_a - v_a) \cdot \text{grad} (w_a - v_a) d\Omega > 0.$$  

(23)

On the contrary, if the difference $w_a - v_a$ is a nonzero constant, we have that

$$\sum_{a=1}^{N} \int_{\Omega} \frac{1}{2} h_a (w_a - v_a)^2 d\Omega > 0.$$  

(24)

Therefore, the positiveness of (22) is proven. This ensures (19). Since (19) holds, the functional $I$ is strictly convex and, consequently, its minimum (if exists) is unique. Thus, the uniqueness of the solution of (9) is proven.

we may rewrite (20) as follows:

$$\theta \sum_{a=1}^{N} \int_{\Omega} \hat{q}_a w_a d\Omega + (1 - \theta) \sum_{a=1}^{N} \int_{\Omega} \hat{q}_a v_a d\Omega$$

$$= \sum_{a=1}^{N} \int_{\Omega} \hat{q}_a (\theta w_a + (1 - \theta) v_a) d\Omega,$$

(21)

$$\theta I [\nu] + (1 - \theta) I [\nu] - I [\theta \nu + (1 - \theta) \nu] = \theta (1 - \theta) \sum_{a=1}^{N} \int_{\Omega} \frac{1}{2} h_a (w_a - v_a)^2 d\Omega +$$

$$+ \theta (1 - \theta) \sum_{\beta=1}^{N} \int_{\Omega} \frac{1}{2} k_a\phi_a \Lambda \text{grad} (w_a - v_a) \cdot \text{grad} (w_a - v_a) d\Omega +$$

$$+ \theta (1 - \theta) \sum_{a=1}^{N} \sum_{\beta=1}^{N} \int_{\Omega} \frac{1}{2} R_{\phi \alpha} (w_\beta - v_\beta - (w_a - v_a))^2 d\Omega.$$  

(22)

The only question now is to ensure the existence of the solution.

8. Solution Existence

A sufficient condition for the existence of the minimum of $I$(and consequently for the existence of a solution to (9)) is the coerciveness of the functional. Since the functional $I$ is strictly convex, the coerciveness is ensured if the following holds [18]:

$$\lim_{y \to -\infty} \frac{1}{y} I [y \nu] = +\infty, \text{ for any } \nu \text{ such that } ||\nu|| \neq 0,$$

(25)

in which the above norm is defined as

$$||\nu|| = \left( \sum_{a=1}^{N} \int_{\Omega} \left( \text{grad } w_a \right)^2 + w_a^2 d\Omega \right)^{1/2}.$$  

(26)

The limit in (25) may be expressed as follows:

$$\lim_{y \to -\infty} \frac{1}{y} I [y \nu] = \lim_{y \to -\infty} \frac{1}{y} \left\{ \sum_{a=1}^{N} \int_{\Omega} \frac{1}{2} k_a\phi_a \Lambda \left( \text{grad } y w_a \right) \cdot \left( \text{grad } y w_a \right) d\Omega - \sum_{a=1}^{N} \int_{\Omega} \hat{q}_a y w_a d\Omega +$$

$$+ \sum_{a=1}^{N} \sum_{\beta=1}^{N} \int_{\Omega} \frac{1}{2} R_{\phi \alpha} (y w_\beta - y w_a)^2 d\Omega + \sum_{a=1}^{N} \int_{\Omega} \frac{1}{2} h_a \left( y w_a - T_{\infty} \right)^2 d\Omega \right\} =$$

$$= \lim_{y \to -\infty} \left\{ \gamma \sum_{a=1}^{N} \int_{\Omega} \frac{1}{2} \Lambda k_a\phi_a \Lambda \left( \text{grad } w_a \right) \cdot \left( \text{grad } w_a \right) d\Omega - \sum_{a=1}^{N} \int_{\Omega} \hat{q}_a w_a d\Omega +$$

$$+ \gamma \sum_{a=1}^{N} \sum_{\beta=1}^{N} \int_{\Omega} \frac{1}{2} R_{\phi \alpha} (w_\beta - w_a)^2 d\Omega + \gamma \sum_{a=1}^{N} \int_{\Omega} \frac{1}{2} h_a \left( w_a - \frac{T_{\infty}}{\gamma} \right)^2 d\Omega \right\}.$$  

(27)
In order to demonstrate the coerciveness, let us assume that some \( w_a \) is not a constant. In such case, we have that
\[
\int_{\Omega} \frac{1}{2} k_a \phi_a \Lambda \text{grad } w_a \cdot \text{grad } w_a \, dV > 0 \quad \text{for some } \alpha. \tag{28}
\]

Therefore, since \( \Lambda \) is positive-definite, (25) holds.

On the contrary, if all the \( w_a \) are equal to a given constant (that must be nonzero), we have that
\[
\sum_{a=1}^{N} \int_{\partial\Omega} \frac{1}{2} h_a (w_a)^2 \, dA > 0. \tag{29}
\]

As previously assumed, at least one \( h_a \) is nonzero. So, again, (25) holds.

The last possibility arises when we assume that all the \( w_a \) are constants, but at least one of them is different from the others. In such cases, we have, for a given \( \alpha \),
\[
\sum_{\beta \neq \alpha}^{N} \int_{\Omega} \frac{1}{2} R_{\alpha\beta} (w_{\beta} - w_{\alpha})^2 \, dV > 0,
\]
and, once more time, (25) holds.

In fact, in order to reach a limit different from \(+\infty\), we must have
\[
\sum_{a=1}^{N} \int_{\Omega} \frac{1}{2} k_a \phi_a \Lambda \text{grad } w_a \cdot \text{grad } w_a \, dV
\]
\[
+ \sum_{a=1}^{N} \sum_{\beta \neq a}^{N} \int_{\Omega} \frac{1}{2} R_{\alpha\beta} (w_{\beta} - w_{\alpha})^2 \, dV + \sum_{a=1}^{N} \int_{\partial\Omega} \frac{1}{2} h_a w_{\alpha}^2 \, dA = 0. \tag{31}
\]

Nevertheless, the above equation holds, if and only if \( w_a = C \equiv 0 \). But, in this case, we would have \( \| w \| = 0 \), and this contradicts (25).

Hence, the convex functional \( I \) is coercive [18]. In other words, it admits at least one minimum. Since the functional is strictly convex, this minimum is unique [19].

9. Final Remarks

The existence and the uniqueness of the solution of (9) has been proved. In addition, the equivalence between the variational principle (the minimum principle) and the original problem was proven too. This fact provides a useful tool for numerical simulations by means, for instance, of a finite element approximation.

It is to be noticed that, assuming that the internal supplies \( \dot{q}_a \) belong to \( L^2(\Omega) \) and that \( \Omega \) has the cone property [12], the fields \( T_{\alpha} \) which minimize the functional \( I \) are continuous and bounded. Such feature was expected for these quantities.

Nomenclature

\( c_a \): Specific heat of \( C_a \) (J kg\(^{-1}\) K\(^{-1}\))
\( C_a \): The \( a \)th constituent
\( \Omega \): Region occupied by the mixture
\( \Psi_a \): Internal source term (W m\(^{-3}\))
\( q_a \): Internal heat supply for \( C_a \) (W m\(^{-3}\))
\( \rho_a \): Mass density of \( C_a \) (kg m\(^{-3}\))
\( \Lambda \): Positive-definite second-order tensor
\( k_a \): Thermal conductivity of \( C_a \) (W m\(^{-1}\) K\(^{-1}\))
\( \Gamma_{\alpha \rightarrow \beta} \): Heat supply from \( C_\beta \) to \( C_\alpha \) (W m\(^{-3}\))
\( \phi_a \): Ratio between \( \rho_a \) and the actual mass density \( h_a \): Convection heat transfer coefficient (W m\(^{-2}\) K\(^{-1}\))
\( T_{\max} \): Temperature of \( C_a \) (K)
\( n \): Unit outward normal defined on \( \partial\Omega \)
\( R_{\alpha\beta} \): Internal heat transfer coefficient (W m\(^{-2}\) K\(^{-1}\))
\( I \): Functional
\( \delta I \): The first variation of \( I \)
\( N \): Number of constituents of the mixture.

Appendix

A. Numerical Example

In order to illustrate the use of the functional defined in (10) for obtaining numerical results, let us consider a particular isotropic rigid binary mixture; for instance, a reinforced concrete body or simply a clay body structured by steel strings.

In such case, problem (9) reduces to
\[
\text{div} (\Lambda k_1 \phi_1 \text{grad } T_1) + R_{12} (T_2 - T_1) + \dot{q}_1 = 0, \quad \text{in } \Omega,
\]
\[
\text{div} (\Lambda k_2 \phi_2 \text{grad } T_2) + R_{21} (T_1 - T_2) + \dot{q}_2 = 0, \quad \text{in } \Omega,
\]
\[
- \Lambda k_1 \phi_1 \text{grad } T_1 \cdot n = h_1 (T_1 - T_{\max}), \quad \text{on } \partial\Omega, \quad \text{on } \partial\Omega,
\]
\[
- \Lambda k_2 \phi_2 \text{grad } T_2 \cdot n = h_2 (T_2 - T_{\max}) \quad \text{on } \partial\Omega. \tag{A.1}
\]

Assuming that \( \Lambda \) is the identity tensor (isotropic mixture), that the body is represented by the interval \( x \in (-L, L) \), \( k_1, \phi_1, k_2, \phi_2, R_{12}, R_{21}, \dot{q}_1, \dot{q}_2, h_1, h_2 \) and \( T_{\max} \) are constants we have (dependence only on the rectangular Cartesian coordinate \( x \)), and we have (A.1) reduced to
\[ k_1 \phi_1 \frac{d^2 T_1}{dx^2} + R_{12} (T_2 - T_1) + \dot{q}_1 = 0, \quad -L < x < L, \]
\[ k_2 \phi_2 \frac{d^2 T_2}{dx^2} + R_{21} (T_1 - T_2) + \dot{q}_2 = 0, \quad -L < x < L, \]
\[ -k_1 \phi_1 \frac{d T_1}{dx} = h_1 (T_1 - T_{\infty}) \text{ at } x = L, \quad k_1 \phi_1 \frac{d T_1}{dx} = h_1 (T_1 - T_{\infty}) \text{ at } x = -L, \]
\[ -k_2 \phi_2 \frac{d T_2}{dx} = h_2 (T_2 - T_{\infty}) \text{ at } x = L, \quad k_2 \phi_2 \frac{d T_2}{dx} = h_2 (T_2 - T_{\infty}) \text{ at } x = -L. \] (A.2)

In this case, the functional defined in (10) reduces to

\[ I[w_1, w_2] = \int_{-L}^{L} \left( \frac{1}{2} k_1 \phi_1 \left( \frac{dw_1}{dx} \right)^2 + k_2 \phi_2 \left( \frac{dw_2}{dx} \right)^2 \right) \, dx - \left( \int_{-L}^{L} (\dot{q}_1 w_1 + \dot{q}_2 w_2) \, dx \right) + \int_{-L}^{L} \left( R_{12} (w_2 - w_1)^2 + R_{21} (w_1 - w_2)^2 \right) \, dx + \left[ \frac{1}{2} h_1 (w_1 - T_{\infty})^2 \right]_{x=L} + \left[ \frac{1}{2} h_2 (w_2 - T_{\infty})^2 \right]_{x=L}. \] (A.3)

Since there is a symmetry with respect to the origin, the above functional may be represented in the following way (taking into account that \( R_{12} = R_{21} \)):

\[ I[w_1, w_2] = \int_{0}^{L} \left( \frac{1}{2} k_1 \phi_1 \left( \frac{dw_1}{dx} \right)^2 + k_2 \phi_2 \left( \frac{dw_2}{dx} \right)^2 \right) \, dx - \int_{0}^{L} (\dot{q}_1 w_1 + \dot{q}_2 w_2) \, dx + \int_{0}^{L} R_{12} (w_2 - w_1)^2 \, dx + \left[ \frac{1}{2} h_1 (w_1 - T_{\infty})^2 \right]_{x=L} + \left[ \frac{1}{2} h_2 (w_2 - T_{\infty})^2 \right]_{x=L}. \] (A.4)

In order to present some numerical results, we shall assume a piecewise continuous linear approximation for both \( w_1 \) and \( w_2 \). The approximations are the following:

\[ w_1 = (w_1)_i + \frac{(w_1)_{i+1} - (w_1)_i}{x_{i+1} - x_i} (x - x_i), \quad x_i \leq x \leq x_{i+1}, \]
\[ w_2 = (w_2)_i + \frac{(w_2)_{i+1} - (w_2)_i}{x_{i+1} - x_i} (x - x_i), \quad x_i \leq x \leq x_{i+1}, \] (A.5)
\[ x_i = \left( \frac{i - 1}{M} \right) L = (i - 1) \Delta x, \quad i = 1, 2, 3, \ldots, M. \]

In this case, the functional defined in A.1–A.4 becomes a function of \( (w_1)_1, (w_1)_2, \ldots, (w_1)_M, (w_1)_{M+1}, (w_2)_1, (w_2)_2, \ldots, (w_2)_M \) and \( (w_2)_{M+1} \). These constants represent the approximation for the temperature at the considered points...
and $x_M$, $x_1, x_2, x_3, \ldots, x_{M+1}$, being obtained from the minimization of this function. For instance, $(w_i)$ is the approximation for the temperature $T_1$ at the point $x_i$.

Some selected results are presented in Figures 2–4, all of them obtained with $M = 500$.

With the objective of making the results more general, the following dimensionless quantities will be employed:

\[
X = \frac{x}{L} \quad \text{(dimensionless position).} \tag{A.6}
\]

\[
\Theta_1 = \frac{T_1 - T_\infty}{T_\infty} \quad \Theta_2 = \frac{T_2 - T_\infty}{T_\infty} \quad \text{(dimensionless temperatures).} \tag{A.7}
\]
Figure 4: The dimensionless temperatures $\Theta_1$ and $\Theta_2$, as function of the dimensionless position $X = (x/L)$, obtained with $\beta_1 = 100.0$, $\beta_2 = 0.0$, $\gamma_1 = 10.0$, and $\gamma_2 = 1.0$, considering $\alpha_1 = \alpha_2 = 1.0$ (a) and $\alpha_1 = \alpha_2 = 50.0$ (b).

Table 1: The dimensionless temperatures $\Theta_1$ and $\Theta_2$ at the dimensionless positions ($X = (x/L)$) $X = 0.0; X = 0.2, X = 0.4, X = 0.6, X = 0.8,$ and $X = 1.0$, obtained with $\beta_1 = 100.0, \beta_2 = 0.0,$ and $\gamma_1 = \gamma_2 = 0.5$, employing $M = 200$, for four values of $\alpha_1$ and $\alpha_2$ (with $\alpha_1 = \alpha_2$).

| $\alpha_1 = \alpha_2$ | $\Theta_1$ | $X = 0.0$ | $X = 0.2$ | $X = 0.4$ | $X = 0.6$ | $X = 0.8$ | $X = 1.0$ |
|-----------------------|-------------|------------|------------|------------|------------|------------|------------|
| 0.5                   | $\Theta_1$  | 161.47     | 160.24     | 156.48     | 150.17     | 141.25     | 129.63     |
|                       | $\Theta_2$  | 87.26      | 86.53      | 84.33      | 80.70      | 75.68      | 69.35      |
| 2.0                   | $\Theta_1$  | 136.18     | 135.14     | 131.99     | 126.69     | 119.19     | 109.41     |
|                       | $\Theta_2$  | 112.55     | 111.63     | 108.83     | 104.18     | 97.73      | 89.58      |
| 12.0                  | $\Theta_1$  | 126.45     | 125.46     | 122.48     | 117.49     | 110.47     | 101.38     |
|                       | $\Theta_2$  | 122.28     | 121.31     | 118.34     | 113.38     | 106.45     | 97.60      |
| 50.0                  | $\Theta_1$  | 124.86     | 123.89     | 120.91     | 115.93     | 108.96     | 99.97      |
|                       | $\Theta_2$  | 123.86     | 122.89     | 119.91     | 114.94     | 107.97     | 99.02      |

$$\alpha_1 = \frac{R_{12}L^2}{k_1\phi_1},$$  
$$\alpha_2 = \frac{R_{21}L^2}{k_2\phi_2},$$  
$$\beta_1 = \frac{q_1L^2}{T_{co}k_1\phi_1},$$  
$$\beta_2 = \frac{q_2L^2}{T_{co}k_2\phi_2},$$  
$$\gamma_1 = \frac{h_1L}{k_1\phi_1},$$  
$$\gamma_2 = \frac{h_2L}{k_2\phi_2}$$  

Table 1 presents a direct comparison among different values of $\alpha_1$ and $\alpha_2$ (with $\alpha_1 = \alpha_2$).

It is quite interesting to note that, as $\alpha_1$ and $\alpha_2$ increase, the temperature fields become more near. In fact, even with different values for $\beta_1, \beta_2, \gamma_1$, and $\gamma_2$, very large values for $\alpha_1$ and $\alpha_2$ give rise to a thermal equilibrium between the constituents of the mixture.

It is to be noticed that the dimensionless quantities defined in (A.6), (A.7), and (A.8) allow to rewrite (A.2) as follows:

$$\frac{d^2\Theta_1}{dx^2} + \alpha_1(\Theta_2 - \Theta_1) + \beta_1 = 0, \quad 0 < X < 1,$$

$$\frac{d^2\Theta_2}{dx^2} + \alpha_2(\Theta_1 - \Theta_2) + \beta_2 = 0, \quad 0 < X < 1,$$

while the functional reduces to
\[ I[w_1, w_2] = \int_0^1 \frac{1}{2} \left( \frac{d\omega_1}{dX}^2 + \frac{d\omega_2}{dX}^2 \right) dX - \int_0^1 (\beta_1 w_1 + \beta_2 w_2) dX + \int_0^1 \frac{1}{2}(\alpha_1 (w_2 - w_1)^2 + \alpha_2 (w_1 - w_2)^2) dX + \left[ \frac{1}{2} \gamma_1 (w_1)^2 \right]_{X=1} + \left[ \frac{1}{2} \gamma_2 (w_2)^2 \right]_{X=1}. \] (A.10)

Such problem could be numerically solved by means of the powerful Generalized Differential Quadrature Method [20, 21], besides other less sophisticated procedures like finite difference schemes.

**Data Availability**

The data used to support the findings of this study are available within the article.

**Conflicts of Interest**

The author declares that there are no conflicts of interest regarding the publication of this paper.

**References**

[1] A. Fick, “Ueber diffusion,” *Annalen der Physik und Chemie*, vol. 170, no. 1, pp. 59–86, 1855.
[2] J. Stefan, “Über das Gleichgewicht und die Bewegung insbesondere die Diffusion von Gasgemengen,” *Sitzungsber Akad Wiss Wien*, vol. 63, pp. 63–124, 1871.
[3] R. J. Atkin and R. E. Craine, “Continuum theories of Mixtures: basic theory and historical development,” *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 29, no. 2, pp. 209–244, 1976.
[4] R. J. Atkin and R. E. Craine, “Continuum theories of mixtures: applications,” *IMA Journal of Applied Mathematics*, vol. 17, no. 2, pp. 153–207, 1976.
[5] R. M. Bowen, *Theory of Mixtures in Continuum Physics III*, A. C. Eringen, Ed., Academic Press, Cambridge, MA, USA, 1976.
[6] K. R. Rajagopal and L. Tao, “Mechanics of mixtures,” *Series on Advances in Mathematics for Applied Sciences*, Vol. 35, World Scientific, Singapore, 1995.
[7] A. Bedford and D. S. Drumheller, “Theories of immiscible and structured mixtures,” *International Journal of Engineering Science*, vol. 21, no. 8, pp. 863–960, 1983.
[8] G. Ahmadi, “A generalized continuum theory for multiphase suspensions flows,” *International Journal of Engineering Science*, vol. 23, no. 1, pp. 1–25, 1985.
[9] M. L. M. Costa, R. Sampaio, and R. M. Saldanha Da Gama, “On the energy balance for continuous mixtures,” *Mechanics Research Communications*, vol. 20, pp. 53–58, 1993.
[10] F. Incropera and D. P. Dewitt, *Introduction to Heat Transfer*, John Wiley & Sons Inc, Hoboken, NJ, USA, 1996.
[11] J. N. Reddy, *Energy Principles and Variational Methods in Applied Mechanics*, John Wiley & Sons Inc., Hoboken, NJ, USA, 2002.
[12] A. Adams, *Sobolev Spaces*, Academic Press, Cambridge, MA, USA, 1975.
[13] M. Shamsuddin, S. R. Mishra, O. A. Bég, and A. Kadir, “Viscous dissipation and joule heating effects in non-fourier MHD squeezing flow, heat and mass transfer between rigid plates with thermal radiation: variational parameter method solutions,” *Arabian Journal for Science and Engineering*, vol. 44, no. 9, pp. 8053–8066, 2019.
[14] M. D. Shamshuddin, S. R. Sheri, and O. A. Bég, “Oscillatory dissipative conjugate heat and mass transfer in chemically reacting micropolar flow with wall couple stress: a finite element numerical study,” in *Proceedings of the Institution of Mechanical Engineers, Part E: Journal of Process Mechanical Engineering*, vol. 233, pp. 48–64, 2017.
[15] H. Sagan, *Introduction to the Calculus of Variations*, Dover Publications, Garden City, NY, USA, 1969.
[16] R. Courant and F. John, *Introduction to Calculus and Analysis*, Vol. 2, Springer, New York, NY, USA, 1989.
[17] A. E. Taylor, *Introduction to Functional Analysis*, Wiley Toppan, Tokyo, Japan, 1958.
[18] M. S. Berger, *Nonlinearity & Functional Analysis: Lectures on Nonlinear Problems in Mathematical Analysis*, Academic Press, Cambridge, MA, USA, 1977.
[19] R. R. M. Saldanha Da Gama, “Existence uniqueness and construction of the solution of the energy transfer problem in a rigid and nonconvex black body,” *ZAMP Zeitschrift fur angewandte Mathematik und Physik*, vol. 42, no. 3, pp. 334–347, 1991.
[20] M. U. Ashraf, M. Qasim, A. Wakif, M. I. Afridi, and I. L. Animasaun, “A generalized differential quadrature algorithm for simulating magnetohydrodynamic peristaltic flow of blood based nanofluid containing magnetite nanoparticles: a physiological application,” *Numerical Methods for Partial Differential Equations*, vol. 36, no. 1, pp. 1–27, 2020.
[21] A. Wakif and R. Sahaqui, “Generalized differential quadrature scrutinization of an advanced MHD stability problem concerned water based nanofluids with metal/metal oxide nanomaterials: a proper application of the revised two phase nanofluid model with convective heating and through flow boundary conditions,” *Numerical Methods for Partial Differential Equations*, vol. 36, no. 7, pp. 1–28, 2020.