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Liouville Correspondence Between the Short-Pulse Hierarchy and the Sine-Gordon Hierarchy

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Abstract: The Liouville correspondence between the short-pulse integrable hierarchy and the sine-Gordon integrable hierarchy is studied. It is shown that the transformation relating the short-pulse equation with the sine-Gordon equation also establishes the correspondence between their flows and Hamiltonian conservation laws in respective hierarchy. This proposes an alternative approach to derive the Hamiltonian conservation laws of the short-pulse equation from the known ones of the classical sine-Gordon equation.

Keywords: Hamiltonian Conservation Law; Integrable System; Liouville Transformation; Short-Pulse Hierarchy; Sine-Gordon Hierarchy.

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1 Introduction

The purpose of this article is to study the correspondence between the short-pulse (SP) integrable hierarchy, which is associated with the SP equation [1]

\[ u_{xx} = u + \frac{1}{6} (u')_x, \]  
(1)

and the sine-Gordon (SG) integrable hierarchy, which is initiated with the classical SG equation [2, 3]

\[ Q_{yy} = \sin Q \]  
(2)

and contains the potential modified KdV (pmKdV) equation

\[ Q_t = Q_{yy} + \frac{1}{2} Q_y^3. \]

The SG equation (2), as a well-known integrable equation, has a wide range of applications in relativistic field theory, modern solid-state physics and differential geometry, etc., which can also be used to describe propagation of crystal defects and splay waves on a liquid membrane. Recently, much attention has been paid to the SP equation (1) and its several novel generalisations. Equation (1) was proposed as a model equation in an appropriate dimensionless form to describe the evolution of ultra-short pulses in nonlinear medium of the silica optical fibre, where \( u = u(t, x) \) represents the magnitude of the electric field [1]. Due to the numerical analysis presented in [4], the SP equation (1) has been recognised as a better alternative of the cubic nonlinear Schrödinger equation to approximate the solution of electronic Maxwell system in the case when the pulse length shortens. Apart from the significant importance in physics, the SP equation illustrates the feature of nice mathematical structure. Geometrically, (1) appeared as one of Rabelo’s equations associated with pseudo-spherical surfaces [5–7]. In addition, the integrability of the SP equation has also been investigated from various points of view. It possesses a zero-curvature representation with a parameter in the context of differential geometry [5], admits infinite number of conservation laws [8] as well as a Lax pair with the linear spectral problem of the Wadati–Konno–Ichikawa (WKI) type [9], and supports the bi-Hamiltonian structure [10, 11]. In [12], the interrelation between mKdV hierarchy and short-pulse hierarchy following from the more general gauge transformation was proposed and investigated. Other multicomponent generalisations and complex versions of (1) were also introduced and studied in the literatures [8, 13–19].

It is worth mentioning that a novel connection that relates the SP equation (1) with the SG equation (2) was found and the associated transformation was used to derive exact soliton solutions of the SP equation from the known ones of the SG equation. Some of these resulting solitons of the SP equation are regarded as suitable solutions representing the propagation of ultra-short pulses. Specifically, Sakovich et al. constructed the chain of transformations between (1) and (2) [7, 9] and obtained the loop-soliton solutions as well as smooth-soliton solutions of the SP equation [20]. Matsuno obtained the multisoliton solutions including multiloop and multibreather ones, as well as periodic soliton solutions of the SP equation...
Many other results were established with respect to the soliton solutions of the SP equation (see [23, 24] and references therein).

Since the SP equation (1) and the SG equation (2) both present the bi-Hamiltonian feature, each of them is associated with the corresponding infinite bi-Hamiltonian integrable hierarchy [25, 26]. In view of the correspondence between the SP equation and the SG equation, it is natural to anticipate that the respective hierarchy should be related in a certain manner. Recently, similar investigation about the correspondence between two integrable hierarchies has been considered comprehensively. For example, the correspondence between the Camassa–Holm (CH) integrable hierarchy and the KdV integrable hierarchy was established by the Liouville transformation in [27] and [28]; see also [29, 30]. More explicitly, through the Liouville transformation connecting the respective iso-spectral problem, the positive and negative flows of the CH hierarchy are generated by the negative and positive flows of the KdV hierarchy, respectively. The correspondence between the Hamiltonian conservation laws of the CH hierarchy and the KdV hierarchy is also derived in [27]. In a recent paper [31], it is proved that the integrable modified Camassa-Holm (mCH) hierarchy and the integrable modified KdV (mKdV) hierarchy (both integrable hierarchies in the negative direction begin with the corresponding Casimir flows) admit the Liouville correspondence. The integrable mCH hierarchy is initiated with the following nonlinear evolution equation

\[ m_t + ((u^2 - u_x^2)m)_x = 0, \quad m = u - u_{xx}, \quad (3) \]

known as the mCH equation, which has been studied extensively in recent years (see [32–37] and references therein). It is worth noting that the SP equation (1) is regarded as the scaling limit equation of the mCH equation (3) with the first-order term \( u_{xx} \).

In this article, we firstly establish the Liouville correspondence between the integrable SP and SG hierarchies. The corresponding Liouville transformation that takes the following form [9, 21]

\[ \cos Q(t, y) = \frac{1}{\sqrt{1 + u_x^2(t, x)}}, \quad y = \int_t^x \sqrt{1 + u_x^2} \, dx, \quad \tau = t, \quad (4) \]

not only converts the SP equation (1) into the SG equation (2) but also relates the iso-spectral problems of the SP hierarchy and the SG hierarchy. Based on the Liouville transformation (4), we are able to construct the following nontrivial identity

\[ \mathcal{R}^n = \partial_{\tau}^{-n}(\mathcal{R}^{-1})^n \partial_{\tau}, \quad n \in \mathbb{Z}^+, \]

where \( \mathcal{R} \) and \( \mathcal{R}^{-1} \) are the respective recursion operators of the SP hierarchy and SG hierarchy. Adhering to this crucial relation and the particular structures of the corresponding flows, we bear out such one-to-one Liouville correspondence between the two integrable hierarchies.

Another main result of this article is to establish the relationship between the Hamiltonian conservation laws for the SP hierarchy and those for the SG hierarchy through the transformation (4). The associated conservation laws play a crucial role in the investigation of the qualitative properties of the SP equation (1); see for example [38]. Therefore, the induced relationship between the Hamiltonian conservation laws for the SP equation and the known ones of the SG equation turns out to be of value for studying the SP equation. In [8], the integrals of the SP equation were derived from the Lax pair. Here, we propose an alternative approach to obtain the Hamiltonian conservation laws of the SP equation.

The remainder of this article is organised as follows. In Section 2, we particularise some results on integrability of the SP equation, the SG equation and their corresponding hierarchies. The main results in this article are also presented. Next in Section 3, based on the particular structures of flows in the two hierarchies combined with the relationship between the respective recursion operators, we exploit the Liouville transformation to establish the one-to-one correspondence between the flows in the SP and SG hierarchies. Finally, in Section 4, we establish the relationship between the Hamiltonian conservation laws of the SP and SG hierarchies.

## 2 Preliminaries and Main Results

We begin with the bi-Hamiltonian form of the mKdV equation

\[ U_t = U_{yy} + \frac{3}{2} U^2 U_y = \mathcal{L}_E \frac{\delta E_1}{\delta U} = \mathcal{L}_{E_2} \frac{\delta E_2}{\delta U}, \quad (5) \]

where

\[ \mathcal{L}_E = \partial_y^3 + \partial_y U \partial_y U \partial_y \]

are the compatible Hamiltonian operators and

\[ E_1(U) = \int_0^1 U^2 \, dy \quad \text{and} \quad E_2(U) = \int_0^1 \left( -\frac{1}{2} U_y^2 + \frac{1}{8} U^4 \right) \, dy \]

are the corresponding Hamiltonian functionals. In terms of the Hamiltonian pair (6), the mKdV integrable hierarchy, both in positive and negative directions, takes the following form
with the higher-order Hamiltonian functionals \( E_n, n \in \mathbb{Z} \).

Each equation in the hierarchy (7) is constructed by applying successively the recursion operator \( E = \mathcal{L}_1\mathcal{L}_1^{-1} \) to the seed symmetry \( G_1[U] = U_y \). The mKdV equation (5) appears to be \( U_{\tau} = \mathcal{E} U_y \) and is exactly the second member of the positive flows in this hierarchy. However, in view of the bi-Hamiltonian formulation of the seed equation:

\[
\mathcal{E} U_y = U_{\tau} \quad \mathcal{L}_1 \mathcal{L}_1^{-1} \mathcal{L}_1 \mathcal{L}_1^{-1} U = U_y,
\]

we obtain different structure of negative flows in the mKdV hierarchy (7) by choosing different Hamiltonian functional \( E_0(U) \).

If we take

\[
E_0(U) = \int (\cos(\partial_y U) - 1) dy \quad \text{with} \quad \partial_y \mathcal{E} = \partial_y \sin(\partial_y U) - \int \sin(\partial_y U) dy,
\]

then

\[
\mathcal{L}_1 \frac{\partial E_0}{\partial U} = \partial_y \sin(\partial_y U) - \int \sin(\partial_y U) dy,
\]

Hence, in this case, the negative flows of mKdV hierarchy (7) begin with

\[
U_i = G_i[U] = \mathcal{L}_1 \frac{\delta E_0}{\delta U} = \sin(\partial_y U),
\]

which is just the SG equation

\[
Q_{y} = \sin Q
\]

for the potential function \( Q = \partial_y U \).

Therefore, in such case, the negative flows of the mKdV hierarchy (7) have the form

\[
U_i = G_i[U] = (\mathcal{L}_1 \mathcal{L}_1^{-1})^{i-1} \sin(\partial_y U), \quad n = 1, 2, \ldots
\]

At the \( n \)th stage, the associated potential function \( Q = \partial_y U \) satisfies

\[
Q_i = (\mathcal{K}^{-1})^{i-1} \mathcal{J} \sin Q,
\]

where

\[
\mathcal{K} = \partial_y + Q \partial_y Q_y \quad \text{and} \quad \mathcal{J} = \partial_y^{-1}
\]

are the compatible Hamiltonian operators admitted by the SG equation (9). In view of the Hamiltonian pair (11), the SG equation (9) also admits a hierarchy consisting of an infinite number of integrable bi-Hamiltonian equations in both the positive and negative directions:

\[
Q_i = \mathcal{K}_i \frac{\delta E_0}{\delta Q} = \mathcal{J}_i \frac{\delta E_0}{\delta Q}, \quad n \in \mathbb{Z}.
\]

These integrable flows in (12) can be obtained by applying successively the recursion operator \( \mathcal{R}_i = \mathcal{J}_i \mathcal{K}_i^{-1} \) to the corresponding seed symmetry

\[
Q_i = \mathcal{K}_i [Q] = \mathcal{J}_i \frac{\delta E_0}{\delta Q} = \mathcal{J}_i \frac{\delta E_0}{\delta Q} = Q_y
\]

with

\[
\mathcal{R}_0(Q) = \int (-\cos Q + 1) dy \quad \text{and} \quad \mathcal{R}_1(Q) = -\frac{1}{2} Q_y dy.
\]

Observe that the SG equation (9) can be written as

\[
Q_i = \mathcal{R}_i [Q] = \mathcal{R}_i^{-1} Q_y = \mathcal{K}_i \frac{\delta E_0}{\delta Q} = \mathcal{J}_i \frac{\delta E_0}{\delta Q} = \partial_y^{-1} \sin Q
\]

with the associated Hamiltonian functional

\[
\mathcal{R}_i(Q) = -\frac{1}{2} \int \cos Q (\partial_y \sin Q)^2 dy.
\]

Furthermore, in the positive direction, the corresponding flows are

\[
Q_i = \mathcal{K}_i [Q] = \mathcal{J}_i Q_y = Q_{y+1} + \frac{1}{2} Q_y^3,
\]

which includes the potential mKdV (pmKdV) equation

\[
Q_i = \mathcal{J}_i [Q] = \mathcal{J}_i Q_y = Q_{y} + \frac{1}{2} Q_y^3,
\]

Whereas, if we take the Hamiltonian functional \( E_0(U) \) for the seed equation (8) by

\[
E_0(U) = \int U dy \quad \text{with} \quad \frac{\delta E_0}{\delta U} = 1,
\]

then the negative flows of the mKdV hierarchy (7) are generated from the Casimir equation

\[
U_i = G_i[U] = \mathcal{L}_1 \frac{\delta E_0}{\delta U} \quad \text{with} \quad \mathcal{L}_1 \frac{\delta E_0}{\delta U} = \mathcal{L}_1 \frac{\delta E_0}{\delta U} = 0.
\]

The corresponding negative flows \( U_i = G_i[U], n = 1, 2, \ldots \), are given by
\[ (\partial_y + U \partial_x^{-1} U)(L_x L_y^{-1})^{n-1} U_y = C_n, \]  

(15)

with \( C_n \) being the corresponding constants of integration for the flows \( U = G_n[U] \), \( n = 1, 2, \ldots \). It was shown in [31] that, in such case, the Liouville transformation that relates the corresponding iso-spectral problems establishes the one-to-one correspondence between the flows in the integrable mCH hierarchy initiated with the mCH equation (3) and the mKdV hierarchy with negative flows given by (15).

However, this brings out a natural question, whether there exists the correspondence in a certain manner which can relate the mKdV hierarchy with negative flows given by (10) or more suitably the integrable SG hierarchy (12) to some other integrable hierarchy? It was proved in [7] and [9] (see also [21]) that the SG equation (9) and the SP equation also holds for their respective integrable hierarchies. In view of these results, the first main result of this article is to verify rigorously that such a relationship between the SG hierarchy and the SP hierarchy initiated with the SP equation (14) which belongs to the SG hierarchy (12).

In this article, with the aim to investigate the correspondence between the SG hierarchy and the SP hierarchy, we start from the perspective of the iso-spectral problems for the SG hierarchy

\[ \Phi_y = \begin{pmatrix} \lambda \cos Q & \lambda \sin Q \\ \lambda \sin Q & -\lambda \cos Q \end{pmatrix} \Phi, \]  

(23)

and the SP hierarchy

\[ \Psi_x = \begin{pmatrix} \mu & \mu u_x \\ u_x & -\mu \end{pmatrix} \Psi, \]  

(24)

where \( \lambda \) and \( \mu \) are the respective spectral parameters. It is straightforward to verify that the transformation

\[ Q = \arccos \frac{1}{\sqrt{1 + u_y^2}}, \quad y = \int \sqrt{1 + u_x^2} \text{d}z \]  

(25)

relates the isospectral problem (23) with (24). Note further that transformation (25) together with \( \tau = t \) are actually equivalent to the chain of transformations (16) and the hodograph-type transformation introduced in [21]. The main result on the Liouville correspondence between two hierarchies is the following.

**Theorem 1.** For each \( n \in \mathbb{Z} \), the \((n+1)^{th}\) equation \( u_i = K_n[u] \) in the SP hierarchy (20) begins with the following equation

\[ u_i = K_n[u] = \kappa \frac{\partial H}{\partial u} = \mathcal{J} \frac{\partial H}{\partial u} = u_x \]  

(22)

which is known as the WKI equation describing the nonlinear transverse oscillations of elastic beams under tension [39]. Actually, it is implied in [9] that the WKI equation (22) is connected by the transformation (16) with the pmKdV equation (14) which belongs to the SG hierarchy (12).

The SP equation (17) in this hierarchy is the second member \( u_i = K_1[u] = \mathcal{R} u_y \) of the positive flows. While, it follows from

\[ u_i = K_0[u] = \kappa \frac{\partial H}{\partial u} = \mathcal{J} \frac{\partial H}{\partial u} = u_x \]  

(17)

with the associated Hamiltonian functional

\[ \mathcal{H}_n(u) = \int \left( -\sqrt{1 + u_x^2} + 1 \right) \text{d}x \quad \text{and} \quad \frac{\partial \mathcal{H}}{\partial u} = \left( \frac{u_x}{\sqrt{1 + u_x^2}} \right)_x, \]  

(21)

that the negative flows in the SP hierarchy (20) begin with the following equation

\[ u_i = K_n[u] = \mathcal{R}^{-1} u_x = \mathcal{J} \frac{\partial H}{\partial u} = \left( \frac{u_x}{\sqrt{1 + u_x^2}} \right)_x, \]  

(22)

can be constructed by applying successively the recursion operator \( \mathcal{R} = \kappa \mathcal{J}^{n-1} \) to the seed symmetry \( K_n[u] = u_x \). The
It follows from the bi-Hamiltonian structure (20), one can recursively construct an infinite hierarchy of Hamiltonian functionals

\[ \ldots \mathcal{H}_2, \mathcal{H}_1, \mathcal{H}_0, \mathcal{H}_f, \ldots \]  

which are Hamiltonian conservation laws admitted by the SP equation (17). On the other hand, the recursive equation (12) leads to an infinite hierarchy of Hamiltonian functionals

\[ \ldots \mathcal{R}_2, \mathcal{R}_1, \mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_2, \ldots \]  

which are conserved under the SG flow (9) [25, 26]. We will prove that the Liouville transformation (25) will also link the corresponding Hamiltonian conservation laws (26) and (27) admitted by the two integrable equations. More precisely, we establish the following theorem.

**Theorem 2.** Under the Liouville transformation (25), each Hamiltonian conservation law \( \mathcal{H}_n(Q) \) admitted by the SG equation in (27) and the Hamiltonian conservation law \( \mathcal{H}_n(u) \) of the SP equation in (26) satisfy the following relations

\[ \mathcal{H}_n(u) = -\mathcal{R}_n(Q), \quad n \in \mathbb{Z}. \]

### 3 The Correspondence between the Short Pulse and Sine-Gordon Hierarchies

We now focus our attention on the SP and SG hierarchies. First of all, in light of the recursion operator

\[ \mathcal{R} = \mathcal{K} \mathcal{J}^{-1} = (\partial_s^{-1} v_s + v_s \partial_s^{-1} u_s) \partial_s^{-1}, \]

the SP hierarchy (20) admits the following positive flows

\[ u_l = \mathcal{K}_s[u] = \mathcal{R}^l u_s, \quad n = 0, 1, \ldots \]  

On the other hand, starting with

\[ u_l = \mathcal{K}_o[u] = \partial_s^{-1} \frac{u_s}{\sqrt{1 + u_s^2}}, \]

the \((n-1)\text{th}\) member in the negative flows of the SP hierarchy (20) can be written as

\[ u_l = \mathcal{K}_{-[n-1]}[u] = \mathcal{R}^{-(n-1)} \partial_s^{-1} \frac{u_s}{\sqrt{1 + u_s^2}}, \quad n = 1, 2, \ldots \]  

Similarly, the positive flows of the SG hierarchy (12) take the following form

\[ Q_l = \mathcal{K}_{n+1}[Q] = \mathcal{R}^n Q_s, \quad n = 0, 1, \ldots \]  

where

\[ \mathcal{R} = \mathcal{K} \mathcal{J}^{-1} = v_s \partial_s^{-1} Q_s \partial_s^{-1} \]

is the corresponding recursion operator. While, in the negative direction, since \( \partial_s \mathcal{H}_n / \partial Q = \sin Q \), the \((n-1)\text{th}\) negative flow \( Q_l = \mathcal{R}^{-(n-1)} Q_s, n = 1, 2, \ldots \), can be reexpressed by

\[ \mathcal{J}^{-1} \mathcal{R}^{-(n-1)} Q_s = \sin Q, \quad n = 1, 2, \ldots \]

Hereafter, for the sake of convenience, for each positive integer \( n \geq 1 \), we write the \( n\text{th} \) equation in the positive direction of the SP and SG hierarchies by \((\text{SP})_n\) and \((\text{SG})_n\), respectively. While for each non-negative integer \( n \geq 0 \), the \( n\text{th} \) negative flow in the SP and SG hierarchies are denoted by \((\text{SP})_n\) and \((\text{SG})_n\), respectively. With these notations, we now restate Theorem 1 and present the explicit description of the correspondence between the two hierarchies.

**Theorem 3.** Under the transformations

\[ y = \int^t \frac{1}{\sqrt{1 + u_x^2(t, z) dz}} \]

and

\[ Q(r, y) = \arccos \frac{1}{\sqrt{1 + u_x^2(t, x)}}, \]

for each \( l \in \mathbb{Z} \), the \((\text{SP})_{n+1}\) equation is mapped into the \((\text{SG})_{l+1}\) equation, and conversely. More precisely,

(i) for each integer \( n \geq 0 \), \( u \) is a solution of the \((\text{SP})_{n+1}\) equation (29) if and only if \( Q \) satisfies \( Q_l = Q_s \) for \( n = 0 \) or the \((\text{SG})_{n+1}\) equation (33) for \( n \geq 1 \);

(ii) for each integer \( n \geq 1 \), \( u \) satisfies the \((\text{SP})_{n+1}\) equation (30) if and only if \( Q \) solves the \((\text{SG})_{n+1}\) equation (31).

The proof of Theorem 3 relies on the following lemma, which sets up the relationship between the recursion operators \( \mathcal{R} \) (28) and \( \mathcal{R} \) (32) admitted by SP and SG hierarchies, respectively, with \( u(t, x) \) and \( Q(r, y) \) connected by the transformations (34) and (35).

**Lemma 1.** Let \( \mathcal{R} \) be the recursion operator (28) for the SP hierarchy and \( \mathcal{R} \) be the recursion operator (32) for the SG hierarchy. Then, under the transformations (34) and (35),

\[ \mathcal{R}^n \partial_s^{-1} (\mathcal{R}^{-1})^n \partial_s \]

holds for each integer \( n \geq 1 \).
Proof. We prove (36) by induction. First, in the case of \( n = 1 \), based on the form of the operator \( \mathcal{R} \) (28), it suffices to prove the following operator identity
\[
\mathcal{R} \partial_y \mathcal{R} = (\partial_y^{-1} + u \partial_y^{-1} u_y) \mathcal{R} = \partial_y.
\]
In view of the transformations (34) and (35),
\[
\partial_y = \sqrt{1 + u^2} \partial_y, \quad \partial_y^{-1} = \sqrt{\frac{1}{1 + u^2}} = \partial_y^{-1} \cos Q
\]
and
\[
\cos Q = \frac{1}{\sqrt{1 + u^2}}, \quad \sin Q = \frac{u}{\sqrt{1 + u^2}}.
\]
Hence, for a test function \( \rho \in C_0^\infty(\mathbb{R}) \), using (38), (39) and integration by parts, we have
\begin{multline}
\partial_y^{-1} \mathcal{R} \rho = \partial_y^{-1} (\cos Q \cdot \mathcal{R} \rho) = \partial_y^{-1} (\cos Q \cdot \rho_{yy} + \cos Q \cdot Q \partial_y^{-1} Q \rho_y) \\
= \int \cos Q \cdot \rho_{yy} \, dx + \int \cos Q \cdot Q \partial_y^{-1} Q \rho_y \, dx \\
= \cos Q \cdot \rho_y + \int \sin Q \cdot Q \rho_y \, dx + \int (\sin Q) \partial_y^{-1} Q \rho_y \, dx \\
= \cos Q \cdot \rho_y + \sin Q \cdot \partial_y^{-1} Q \rho_y
\end{multline}
and
\[
u = \partial_y^{-1} u \mathcal{R} \rho = u \partial_y^{-1} \mathcal{R} \rho
\]
\[
= \tan Q \left( \int \sin Q \left( \rho_{yy} + Q \partial_y^{-1} Q \rho_y \right) \, dx \right)
\]
\[
= \tan Q \left( \sin Q \partial_y^{-1} Q \rho_y \right)
\]
\[
= \tan Q \left( \sin Q \cdot \rho_y - \int \cos Q \cdot Q \rho_y \, dx - \int (\sin Q) \partial_y^{-1} Q \rho_y \, dx \right).
\]
Combining (40) with (41) yields
\[
(\partial_y^{-1} + u \partial_y^{-1} u_y) \mathcal{R} \rho = \cos Q \cdot \rho_y + \frac{\sin^2 Q}{\cos Q} \rho_y = \frac{1}{\cos Q} \rho_y = \rho_y
\]
and verifies the identity (37).

Next, we assume (36) holds for \( n = k \), namely
\[
\mathcal{R}^k = \partial_y^{-1} (\mathcal{R}^{-1})^k \partial_y.
\]
Then, for \( n = k + 1 \), thanks to the result of \( n = 1 \), one has
\[
(\mathcal{R})^{k+1} = \partial_y^{-1} (\mathcal{R}^{-1})^k \partial_y \mathcal{R} \mathcal{R} = \partial_y^{-1} (\mathcal{R}^{-1})^k \partial_y \mathcal{R}^{-1} \partial_y \mathcal{R} = \partial_y^{-1} (\mathcal{R}^{-1})^{k+1} \partial_y,
\]
which implies that (36) holds for each \( n \geq 1 \). Therefore, the lemma is proved.

Proof of Theorem 3. (i). We start with the following \((SP)_n\), equation for \( n \geq 1 \)
\[
u = \mathcal{R}^n \nu = (\partial_y^{-1} + u \partial_y^{-1} u_y) \mathcal{R}^{-1} \partial_y, \quad n = 1, 2, \ldots
\]
Suppose that \( u = u(t, x) \) is the solution of (42). We first calculate the \( t \)-derivative of the new variable \( y \) defined in (34) for the solution \( u(t, x) \). More precisely, using (42), we deduce that
\[
\frac{\partial}{\partial t} y = \frac{\partial y}{\partial t} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial t} = \cos Q \partial_y^{-1} \partial_y \mathcal{R}^{-1} \partial_y - \frac{1}{\cos Q} (\cos Q \cdot \partial_y^{-1} Q \partial_y^{-1} Q \rho_y)
\]
\[
= \frac{\partial}{\partial t} \mathcal{R}^n \nu = \frac{\partial}{\partial t} \mathcal{R}^n \left( \partial_y^{-1} (\mathcal{R}^{-1})^{n-1} \partial_y \mathcal{R}^{-1} \partial_y \mathcal{R}^{-1} \partial_y \mathcal{R}^{-1} \partial_y \mathcal{R}^{-1} \partial_y \right).
\]
On the other hand, by (35), the corresponding new function \( \tau(t, y) \) satisfies
\[
\cos Q(t, y) = \frac{1}{\sqrt{1 + u^2(t, x)}},
\]
Differentiating the above expression with respect to \( t \) and using the transformation (34), we have
\[
\sin Q(\tau, \partial_y) = \frac{u}{\sqrt{1 + u^2}},
\]
which together with the relationship (38), (39), (43), and (44) gives rise to
\[
\sin Q(\tau, \partial_y) \frac{1}{\sqrt{1 + u^2}} = \frac{u}{\sqrt{1 + u^2}},
\]
Hence, we derive the following equation
\[
\frac{\partial}{\partial t} y = \frac{\partial y}{\partial t} = \cos Q \partial_y^{-1} \partial_y \mathcal{R}^{-1} \partial_y - \frac{1}{\cos Q} (\cos Q \cdot \partial_y^{-1} Q \partial_y^{-1} Q \rho_y)
\]
and
\[
\frac{\partial}{\partial t} \mathcal{R}^n \nu = \frac{\partial}{\partial t} \mathcal{R}^n \left( \partial_y^{-1} (\mathcal{R}^{-1})^{n-1} \partial_y \mathcal{R}^{-1} \partial_y \mathcal{R}^{-1} \partial_y \mathcal{R}^{-1} \partial_y \mathcal{R}^{-1} \partial_y \right).
\]
Thanks to Lemma 1, we deduce by (38) that \( Q(t, y) \) satisfies
\[ R^{n-1}Q = \frac{\partial}{\partial x} \sin Q, \]

which is exactly the \((SG)_{n=1}\) equation (33) for \(n = 1, 2, \ldots\).

For the case of \(n=0\), plugging \(u_i = K_i[u] = u_i\) into (43) and (44) yields

\[
\sin Q \left( Q_x + Q_y \int_t^x \frac{u_x u_y}{1+u_y^2} \, dz \right) = \frac{u_x u_y}{1+u_y^2}.2
\]

Combining integration by parts and the sufficiently fast decay property of \(u(t, x)\) as \(|x| \to +\infty\), using the relationship (38) and (39), we derive from the above identity

\[
\sin Q \left( Q_x + Q_y \int_t^x \frac{\partial_x}{\partial (1+u_y^2)} \, dz \right) = -\left( \frac{1}{1+u_y^2} \right).
\]

and then

\[
\sin Q \left( Q_x + Q_y \int_t^x \frac{\partial_x}{\partial (1+u_y^2)} \, dz \right) = -(\cos Q) \frac{1}{1+u_y^2} \sin Q \cdot Q_x \frac{1}{1+u_y^2}.
\]

We deduce that \(Q(t, y)\) satisfies the \((SG)\) equation \(Q = Q_x\).

Conversely, if \(Q(t, y)\) is a solution of the \((SG)_{n=1}\) equation for integers \(n \geq 0\), using the fact that the transformations (34) and (35) are the bijections, and tracing the previous steps backwards, one can sufficiently to verify that the reverse argument is also true. Therefore, part (i) is proved.

(ii). Now, we investigate the \((SP)_{n=1}\) equation (30) for \(n \geq 1\). Suppose that \(u = u(t, x)\) solves (30). The \(t\)-derivative of the corresponding new variable \(y\) defined by (34) satisfies

\[
y = \int_t^y \frac{u_x u_y}{\sqrt{1+u_y^2}} \, dz = \int_t^y \frac{u_x}{\sqrt{1+u_y^2}} \partial_y R^{(n-1)}(y) \, dz
\]

where the identities (38), (39) and integration by parts are used.

Then, for the corresponding new function \(Q(t, y)\) that is related with the solution \(u(t, x)\) through (35), differentiating the equation

\[
\cos Q(t, x) = \frac{1}{\sqrt{1+u_y^2(t, x)}}
\]

with respect to \(t\) leads to

\[
\sin Q \cdot (Q_x + Q_y) = \frac{u_x}{(1+u_y^2)^2}.
\]

Using (38), (39), and (45), together with the operator identity (36), we have

\[
\sin Q \left( Q_x + \tan Q \cdot Q_y \partial^{-1}_x R^{(n-1)} Q_x - Q_x \partial^{-1}_x Q_y \partial^{-1}_x R^{(n-1)} Q_y \right)
\]

\[
= \frac{u_x}{(1+u_y^2)^2} \partial_x R^{(n-1)}(y) \partial_x \sin Q
\]

\[
= \sin Q \cdot \cos Q \cdot \partial_x \left( \frac{1}{\cos Q} \partial_x R^{(n-1)} Q_y \right).
\]

Hence,

\[
Q_x + \tan Q \cdot Q_y \partial^{-1}_x R^{(n-1)} Q_x - Q_x \partial^{-1}_x Q_y \partial^{-1}_x R^{(n-1)} Q_x
\]

\[
= \partial_x R^{(n-1)} Q_y + \cos Q \left( \frac{1}{\cos Q} \partial_x R^{(n-1)} Q_y \right),
\]

which implies

\[
Q_x = (\partial_x + Q_x \partial_x R^{(n-1)} \partial_x Q_x) = R^{(n-1)} Q_x,
\]

and verifies that the corresponding \(Q(t, y)\) is the solution of the \((SG)\) equation (31) for each integer \(n \geq 1\). The converse argument is a direct result of the fact that (34) and (35) are the bijections. Hence, Theorem 3 is proved in general.

4 The Correspondence between the Hamiltonian Conservation Laws of the Short Pulse and sine-Gordon Equations

According to the Magri's theorem, one can recursively construct the infinite hierarchy of Hamiltonian conservation laws for the bi-Hamiltonian integrable systems. In particular, for the SP equation (17), the corresponding recursive formula

\[
\frac{\delta H_{n-1}}{\delta u} = J \frac{\delta H_n}{\delta u}, \quad n \in \mathbb{Z},
\]

formally provide an infinite collection of the Hamiltonian conservation laws, where \(K\) and \(J\) are the two compatible Hamiltonian operators (18) admitted by the SP equation.
While, for the SG equation (9), we determine the involved Hamiltonian conservation laws \( \mathcal{H}_n \) by

\[
\mathcal{H}_n = \int \mathcal{L}_n \, dt = \int \mathcal{L}_n \, dt, \quad n \in \mathbb{Z},
\]

(47)

using the Hamiltonian pair \( \mathcal{K} \) and \( \mathcal{J} \) defined in (11).

In this section, we establish the correspondence between the two hierarchies of Hamiltonian conservation laws \( \{ \mathcal{H}_n \} \) and \( \{ \mathcal{H}_n \} \) subject to the transformations (34) and (35) and prove Theorem 2. Let us begin with the following two facts.

**Lemma 2.** Let \( \{ \mathcal{H}_n \} \) and \( \{ \mathcal{H}_n \} \) be the sequences of Hamiltonian functionals satisfying the (46) and (47), respectively. Then their corresponding variational derivatives are determined by the following relationship

\[
\frac{\delta \mathcal{H}_n}{\delta u} = \partial_x f_{n}^{\mathcal{H}_n}, \quad n \in \mathbb{Z},
\]

under the transformations (34) and (35).

**Proof.** To prove this lemma, we use the induction argument. First of all, we consider the case of \( n \geq 0 \). Using (13) and (21), we find

\[
\frac{\delta \mathcal{H}_n}{\delta u} = \frac{u_{x}}{(1 + u_{x}^2)^{1/2}} \quad \text{and} \quad \frac{\delta \mathcal{H}_n}{\delta Q} = Q_{y},
\]

(49)

In view of the relationship (38) and (39), performing the \( x \)-derivative for

\[
\cos Q(t, x) = \frac{1}{\sqrt{1 + u_{x}^2}}
\]

leads to

\[
\sin Q(t, x) = \frac{u_{x}}{(1 + u_{x}^2)^{1/2}}
\]

which together with (49) verifies that (48) holds for \( n = 0 \).

Now, suppose, by induction, that (48) holds for \( n = k \) with \( k \geq 0 \), in other words

\[
\frac{\delta \mathcal{H}_n}{\delta u} = \partial_x f_{n}^{\mathcal{H}_n}.
\]

Then, for \( n = k + 1 \), in view of the recursive formulae (46) and (47),

\[
\frac{\delta \mathcal{H}_{k+1}}{\delta u} = \partial_x f_{k+1}^{\mathcal{H}_n}.
\]

Next, we deal with the case for the integers \( n \leq -1 \). Using (13) and (21), we find

\[
\frac{\delta \mathcal{H}_n}{\delta u} = \frac{u_{x}}{(1 + u_{x}^2)^{1/2}} \quad \text{and} \quad \frac{\delta \mathcal{H}_n}{\delta Q} = Q_{y},
\]

(49)

In view of the relationship (38) and (39), performing the \( x \)-derivative for

\[
\cos Q(t, y) = \frac{1}{\sqrt{1 + u_{y}^2}}
\]

leads to

\[
\sin Q(t, y) = \frac{u_{y}}{(1 + u_{y}^2)^{1/2}}
\]

which together with (49) verifies that (48) holds for \( n = 0 \).

Now, suppose, by induction, that (48) holds for \( n = k \) with \( k \geq 0 \), in other words

\[
\frac{\delta \mathcal{H}_n}{\delta u} = \partial_x f_{n}^{\mathcal{H}_n}.
\]

Then, for \( n = k + 1 \), in view of the recursive formulae (46) and (47),

\[
\frac{\delta \mathcal{H}_{k+1}}{\delta u} = \partial_x f_{k+1}^{\mathcal{H}_n}.
\]

Therefore, a straightforward induction verifies (48) for \( n \leq -1 \). This completes the proof of the lemma.

The next lemma provides a formula for the change of the variational derivatives with respect to \( u \) and \( Q \), respectively.

**Lemma 3.** Let \( u(t, x) \) and \( Q(r, y) \) be linked by the transformations (34) and (35). Assume further that \( u(t, x) \) satisfies the equation \( u_{t} = K_{\tau} reverse of Q \) in the SP hierarchy and \( Q(r, y) \) is the solution of the \( (\mathbb{G}_{Q}) \) equation \( Q_{\eta} = K_{\eta} \) under the transformations (34) and (35), then

\[
\frac{\delta \mathcal{H}(u)}{\delta u} = -K^{-1} \frac{\delta \mathcal{F}(Q)}{\delta Q},
\]

(51)

where \( K \) is the Hamiltonian operator given by (11).

**Proof.** First of all, in view of (35), we denote

\[
Q(r, y) = F[u(t, x)] = \arccos \frac{1}{\sqrt{1 + u_{x}^2}},
\]

According to (34), (38), and (39), we deduce with a test function \( \rho \in \mathcal{C}_c^\infty \) that
\[
\frac{d}{dt} y(u+\varphi) = \int_0^1 \frac{u_{t}}{\sqrt{1+u_{t}^2}} \rho(x) dx = \sin(\varphi) \rho(x) \int_0^1 \cos(\varphi) \rho(x) dx.
\]

Hence,
\[
\frac{d}{dt} F[u+\varphi] = Q_y \left( \frac{d}{dt} y(u+\varphi) + \frac{d}{dt} F[u+\varphi] \right)
= Q_y \left( \sin(\varphi) \rho - \partial_\varphi \cos(\varphi) \rho \right) + \frac{d}{dt} F[u+\varphi].
\]

Moreover, in terms of the Fréchet derivative
\[
\frac{d}{dt} F[u+\varphi] = \mathcal{D}_{\varphi}[\rho] = \frac{1}{1+u_{t}^2} \partial_\varphi \rho = \cos(\varphi) \partial_\varphi \rho.
\]

We thus discover
\[
\frac{d}{dt} F[u+\varphi] = (\cos(\varphi) \partial_\varphi - Q_y \sin(\varphi) + Q_y \partial_\varphi \cos(\varphi)) \rho.
\]

Next, it follows from the assumption that
\[
\frac{d}{dt} \mathcal{H}(u+\varphi) = \frac{d}{dt} \mathcal{F}_h(F[u+\varphi]).
\]

Then, the usual definition of variational derivative leads to
\[
\frac{d}{dt} \mathcal{H}(u+\varphi) = \left( \int_0^1 \frac{\partial \mathcal{H}}{\partial u} \rho(x) dx \right)
\]

Using (52) and integration by parts, we obtain
\[
\frac{d}{dt} \mathcal{F}_h(F[u+\varphi]) = \left( \int_0^1 \frac{\partial \mathcal{F}_h}{\partial y} \frac{\partial y}{\partial u} F[u+\varphi] dy \right)
\]

Indeed, since \( \mathcal{F}_h(Q) = -\int Q^2 / 2 dy \) is one of the Hamiltonian functionals for the equation \( Q_y = K_{\varphi,\rho} \) which belongs to the SG hierarchy (12), we then deduce from the convolution property for the hierarchy of the Hamiltonian functionals that
\[
\int \frac{\partial \mathcal{F}_h}{\partial y} \frac{\partial y}{\partial u} F[u+\varphi] dy = \int \frac{\partial \mathcal{F}_h}{\partial Q} \frac{\partial Q}{\partial y} dy = \int \frac{\partial \mathcal{F}_h}{\partial y} \frac{\partial y}{\partial u} dy = 0,
\]

where \( \mathcal{F} = \partial_\varphi \) is one of the Hamiltonian operators admitted by the SG hierarchy. Therefore, we conclude from (53) that (51) holds and prove the lemma.

Finally, under the hypothesis of Lemma 3, we define the functional
\[
\mathcal{H}(u) = \mathcal{F}_h(Q)
\]

for each \( n \in \mathbb{Z} \). From Lemma 3, we see that
\[
\frac{\delta \mathcal{H}(u)}{\delta y} = -K \frac{\delta \mathcal{F}_h}{\delta Q}.
\]

Meanwhile, Lemma 2 and the recursive formula (46) together with the relationship (50) readily lead to
\[
\frac{\delta \mathcal{H}(u)}{\delta y} = -K \mathcal{F} \frac{\delta \mathcal{H}(u)}{\delta y} = -K \mathcal{F} \frac{\delta \mathcal{H}(u)}{\delta y} = -\frac{\delta \mathcal{H}_n(u)}{\delta y},
\]

which immediately implies that
\[
\mathcal{H}(u) = -\mathcal{H}_n(u),
\]

from which
\[
\mathcal{H}_n(u) = -\mathcal{F}_h(Q), \quad n \in \mathbb{Z},
\]

follows. Assembling the previous two lemmas, we conclude that there exists an one-to-one correspondence between the two sequences of the Hamiltonian conservation laws \( \{ \mathcal{H}_n \} \) and \( \{ \mathcal{F}_h \} \). The proof to Theorem 2 is thereby completed.

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