Finite Markov Chain in Inventory Control

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Abstract. In this paper we present an introduction to the finite markov chain. Transition probabilities are calculated, as well as a transition probability matrix. We present the necessary fundamentals of probability theory on markov chains for stochastic processes and stochastic modelling in inventory control.

1. Introduction
Inventory is an important factor but unfortunately it often fails to be monitored. Companies should have a database of how many requests must be fulfilled for each period in the future, with the right inventory level so that it is not too much or too little. Thus certainly needed an inventory management to analyze the optimum level of inventory. Need planning requires demand forecasting which is calculated from past requests, to estimate future needs.

This study discusses the uncertainty of planning and control of production which becomes more complicated in connection with the fluctuating demand [1]. In addition, there are several costs that must be taken into account including production costs, storage costs, and the cost of shortages of inventory. The stochastic nature of the probability of the Markov chain is one of the right choices to overcome this uncertainty. The Finite Markov chain method will be used in optimizing profits in this study.

Markov Chain as a Stochastic process where current events depend on previous events and only depend on that moment [2]. On stochastic processes provide an interesting properties of finite Markov chains. Where the number of steps in which a state is reachable or accessible from another state in a finite Markov chain with M (≥ 2) states. Markov Chain are widely used models in a variety of areas of theoretical and applied mathematics and science, including statistics, operations research, industrial engineering, linguistics, artificial intelligence, demographics, genomics [3].

Some previous study [4] a novel dividend valuation model is put forward by using a Markov Chain.Barbuet all propose further advancements in the Markov chain stock model [5]. Markov decision process approach, the states of a Markov chain represent possible states of demand for the inventory item. Meanwhile, the decision problem using markov chain combined with demand and inventory positionsand some cost such as the production cost, holding cost, shortage cost and sales price are to generate the profit [6].

In this work, we applied finite markov chain model as a stochastic process in inventory control[5-7]. This paper is organized into four parts, the main idea on the finite markov chain is described in Section 2. Than, Section 3 presents the results of inventory control obtained by application of finite
markov chain as a process stochastic and model stochastic problem. Last, Section 4 deals with the conclusion.

2. Inventory control
The goal of inventory control procedures is to maximize profits with minimum inventory investment, without impacting customer satisfaction levels. How much stock should I order? A balance between inventory and level of service to customers must be achieved. Special cases occur where customer requests are stochastic in nature. This research is focused on the inventory model where demand is not constant but has a non-linear form that is dependent on the available stock. The optimal solution to add back to the supply model with stochastic demand has been described by Benkherouf [7].

Inventory management, on the other hand, is a broader term that covers how you obtain, store, and profit from raw materials and finished goods alike. The right stock, at the right levels, in the right place, at the right time, at the right cost. Inventory level control is an activity related to planning, implementing, and controlling the determination of material needs in such a way that on one hand the operational needs can be met on time and on the other hand the investment of material inventory can be optimally suppressed. Inventory costs are all expenses and losses incurred as due to inventory. These costs are the purchase price, ordering costs, setup costs, storage costs, and inventory shortages.

This study addresses the problem of supply due to fluctuating demand, then using the Markov Chain method can link current demand with previous demand, so the results of this study can optimize the optimal amount of inventory, as well as various inventory costs that follow.

Figure 1. Simulation of inventory rate for 1 year

Fig. 1. Show a simulation of stochastic inventory levels so that demand at any time varies and is random, with the type of variation that can change or remain. The simulation period is for one year with a decrease in supply and demand of 20 units, parameter \( \beta \) is 0.7, while voting \( \sigma \) is 5.

2.1 Stochastic processes
Stochastic processes are defined as the process of compiling and indexing a set of random variables \( \{x_t\} \), with the time index \( t \) being in a set of \( T \). Sometimes \( T \) is considered as a set of non-negative integers, and \( X_t \) represents the measured characteristics that are considered at time \( t \). \( X_t \) presented the level of inventory of certain products at the end of the month [7]. One structure of stochastic processes is when the current status of the system can be one of \( M + 1 \) categories separated from each other called states labeled 0, 1, ..., \( M \). The random variable \( X_t \) represents the state system at time \( t \), so the only possible values are 0, 1, ..., \( M \). The system is observed at a certain point in time, labeled \( t = 0, 1, 2, ..., \). Therefore, the stochastic process \( X_t = \{X_0, X_1, X_2, ...\} \) provides a mathematical representation
of the state of the physical status that changes with time. Such a process is called a discrete time stochastic process with a limited state [5,7].

However, the fluctuating stochastic decision process can be explained by a limited number of states. The probability of transition between states is explained by a chain called the Markov Chain. The stochastic process is interesting to describe the nature of the system that operates over several time periods in the stochastic model.

2.2 Stochastic modelling

Here is the model of stochastic differential equation [7]

\[
dx_t = -\left(g + ax(t^p)I(x(t) > 0)\right) dt - \sigma dw_t + \sum_{t \geq 0} Q_i \delta(t - t_i)
\]

where:

- \(x_t\) = rate inventory
- \(I\) = the indicator function 0, 1
- \(w_t\) = standard Brownian motion
- \(\delta\) = dirac function
- \(Q_i\) = Quantity ordered at time \(t\), \(t_1 < t_2 < \ldots < t_n\)

with conditions:

- \(g > 0, \sigma > 0, a > 0, 0 < \beta < 1\)

3. Finite Markov Chain

The Markov process is a stochastic process in which the emergence of a state in the future depends on the state that immediately precedes it and depends only on it. [2] So if \(t_0 < t_1 < \ldots < t_n\) (\(n = 0,1,2,\ldots\)) represents a certain time \(t\), the random variable group \(\{X_t\}\) is a Markov chain if it has the following Markov properties:

\[
P\{X_{t+1} = j \mid X_0 = k_0, X_1 = k_1, \ldots, X_{t-1} = k_{t-1}, X_t = i\} = P\{X_{t+1} = j \mid X_t = i\}\]

for \(t = 0,1,\ldots\) dan setiaputani, \(j, k_0, k_1, \ldots\) \(k_{t-1}\)

A finite Markov chain is a memoryless homogeneous discrete stochastic process with a finite number of states. The transition probability of an \(x\), does not depend on the previous states, so this process is memoryless. Homogeneous process occurs if the transition probability \(p_{ij} = P(X_{t+1} = j \mid X_t = i)\) does not depend on time \(t\). A finite markov chain \(L\) with \(m\) state and \(p_{ij}\) probability of transition state \(i\) to \(j\). Which is \(V = \{1, 2, \ldots, m\}\), by the \(m \times m\) transition matrix \(M = (p_{ij})\)

3.1. Calculated the transition probability

Markov process will become Steady State after several iterations where the probability will be fixed. This probability is called the Steady State Probability. Steady State Probabilities of a Transition Matrix are obtained using formulas:

\[
(N^n(i+1) M^n(i+1)) = (N^n(i) M^n(i)) \times \text{transition probability matrix}
\]

The Steady State condition will produce the same probability in the next iteration, such that:

\[
(N^n(i) M^n(i)) = (N^n(i) M^n(i)) \times \text{transition probability matrix}
\]
The specific purpose of this study is to determine supply by estimating demand using the Markov Chain method. Suppose a decision problem has several S stationary policies, and it is assumed that P and R are a transition matrix one step related to the k-th policy, \( S = 1, 2, \ldots, s \), with a number of steps the enumeration.

Because the alternatives that may be taken are so many it will be difficult to enumerate all policies explicitly. Calculations for evaluating these various policies will also be very large. The iteration policy expectation of total income at stage \( n \) is expressed by the recursive equation:

\[
\hat{f}_n(i) = v_i + \sum_{j=1}^{m} P_{ij} \hat{f}_{n+1}(j) \quad i = 1, 2, \ldots, m
\]

The policy iteration steps are modified as follows:
1. Determination of value
   For any policy \( s \) with \( P_s \) and \( R_s \) matrices, With \( m \) unknown factors, namely \( f_s(1) \), \( f_s(2) \), \ldots, \( f_s(m) \)
2. Policy improvements
   If the resulting policy \( t \) equals \( s \), stop, \( t \) is the optimum policy. If not, set \( s = t \), and return to the value determination step.

3.2. Transition probability matrix
The transition probabilities of \( n \)-steps can be denoted as the form of a matrix.

\[
P^{(n)} = \begin{bmatrix}
(p_{00}^{(n)}) & (p_{01}^{(n)}) & \cdots & (p_{0M}^{(n)}) \\
(p_{10}^{(n)}) & (p_{11}^{(n)}) & \cdots & (p_{1M}^{(n)}) \\
\vdots & \vdots & \ddots & \vdots \\
(p_{M0}^{(n)}) & (p_{M1}^{(n)}) & \cdots & (p_{MM}^{(n)})
\end{bmatrix}
\]

if \( P_{ij} \geq 0 \) and \( \sum_{j=0}^{\infty} P_{ij}^{(n)} = 1; i, j = 0, 1, 2, \ldots, n \).

The elements of the higher transition period \( P_{ij}^{(n)} \) are obtained by matrix multiplication, directly.

\[
P_{ij}^{(2)} = P_{ij} P_{ij} = P_{ij}^2 \quad P_{ij}^{(3)} = P_{ij}^2 P_{ij} = P_{ij}^3 \quad P_{ij}^{(n)} = P_{ij}^{n-1} P = P_n
\]

The total inventory cost for each state \( i \) and decision \( x \) is the sum of the ordering costs, storage costs for state \( i \) and shortage costs.

\[
C_i(x) = a + b_i + E
\]

4. The Case Study
The government’s BPJS program causes the need for generic drugs to be very high at this time. Therefore, the supply of generic drugs must always be maintained so that the health care process is smooth. Problem modeling using the Markov Chain, which is by collecting data on the demand, inventory data, storage costs in warehouses, ordering costs and depreciation costs of goods. Table 1 is the frequency distribution of demand that have been processed from initial data after calculating class length.
Table 1. The frequency distribution of demand of generic drugs

| Demand            | Frekuensi | Prob. |
|-------------------|-----------|-------|
| 26456825 - 41231071 | 6         | 0.125 |
| 41231072 - 56005318 | 5         | 0.104 |
| 56005319 - 70779565 | 14        | 0.292 |
| 70779566 - 85553832 | 4         | 0.083 |
| 85553833 - 100328079 | 12        | 0.25  |
| 100328080 - 115102326 | 2        | 0.042 |
| 115102327 - 129876573 | 5        | 0.104 |
| Total             | 48        | 1     |

Based on the Markov Chain stipulation, that the decision x, a state of the system experiences a transition from state i to state j = I + x - d with probability \( P_{ij}(x) = P(d) \).

State i : initial inventory

\( a = \) order cost

\( b = \) storage cost

\( E = \) Shortage Cost

\( C_i(x) = \) Total Cost

The Markov Chain calculation for each state from table 1, i.e.:

For \( i = 0 ; x = 600 ; \) if \( j = 0 \)

We found \( d = 600 \) with Probability \( P_0(600) = P(d \geq 100) = 1 \), else = 0.

Shortage Cost = \[ \sum (d - i - x) P(d) \] = Rp 430.000 + Rp 2.750.000 [0.42 + 0.17 + 0.13 + 0.08 + 0.04] = Rp 2.755.930.000.

Matrix transition are obtained:

\[
\begin{bmatrix}
0.104 & 0.042 & 0.25 & 0.083 & 0.292 & 0.104 & 0.125 \\
0.104 & 0.042 & 0.25 & 0.083 & 0.292 & 0.104 & 0.125 \\
0.104 & 0.042 & 0.25 & 0.083 & 0.292 & 0.104 & 0.125 \\
0.104 & 0.042 & 0.25 & 0.083 & 0.292 & 0.104 & 0.125 \\
0.104 & 0.042 & 0.25 & 0.083 & 0.292 & 0.104 & 0.125 \\
0.104 & 0.042 & 0.25 & 0.083 & 0.292 & 0.104 & 0.125 \\
0.104 & 0.042 & 0.25 & 0.083 & 0.292 & 0.104 & 0.125 \\
\end{bmatrix}
\]

5. Result

\[ f_i^{(k)} = C_i(x_i^{(k)}) + \sum_j p_{ij}(x_i^{(k)}) f_j^{(k)} \]
7 linear equations are obtained:

\[
\begin{align*}
    f_0^{(0)} &= 2526.000 + (0.104)f_0^{(0)} + (0.042)f_{14774246}^{(0)} + (0.25)f_{29548492}^{(0)} + (0.083)f_{44322737}^{(0)} + (0.22)f_{5906983}^{(0)} + (0.104)f_{73871229}^{(0)} + (0.125)f_{88645474}^{(0)}
    \\
    f_{14774246}^{(0)} &= 13566.069,958,328 + (0.104)f_0^{(0)} + (0.042)f_{14774246}^{(0)} + (0.25)f_{29548492}^{(0)} + (0.183)f_{44322737}^{(0)} + (0.292)f_{5906983}^{(0)} + (0.104)f_{73871229}^{(0)} + (0.125)f_{88645474}^{(0)}
    \\
    f_{29548492}^{(0)} &= 28,237,29,893,750 + (0.104)f_0^{(0)} + (0.042)f_{14774246}^{(0)} + (0.25)f_{29548492}^{(0)} + (0.083)f_{44322737}^{(0)} + (0.292)f_{5906983}^{(0)} + (0.104)f_{73871229}^{(0)} + (0.125)f_{88645474}^{(0)}
    \\
    f_{44322737}^{(0)} &= 49,486,72,285,922 + (0.104)f_0^{(0)} + (0.042)f_{14774246}^{(0)} + (0.25)f_{29548492}^{(0)} + (0.083)f_{44322737}^{(0)} + (0.292)f_{5906983}^{(0)} + (0.104)f_{73871229}^{(0)} + (0.125)f_{88645474}^{(0)}
    \\
    f_{5906983}^{(0)} &= 72,920,58,760,355 + (0.104)f_0^{(0)} + (0.042)f_{14774246}^{(0)} + (0.25)f_{29548492}^{(0)} + (0.083)f_{44322737}^{(0)} + (0.292)f_{5906983}^{(0)} + (0.104)f_{73871229}^{(0)} + (0.125)f_{88645474}^{(0)}
    \\
    f_{3871229}^{(0)} &= 3,207,430,802,652 + (0.104)f_0^{(0)} + (0.042)f_{14774246}^{(0)} + (0.25)f_{29548492}^{(0)} + (0.083)f_{44322737}^{(0)} + (0.292)f_{5906983}^{(0)} + (0.104)f_{73871229}^{(0)} + (0.125)f_{88645474}^{(0)}
    \\
    f_{88645474}^{(0)} &= 137,890,291,874,277 + (0.104)f_0^{(0)} + (0.042)f_{14774246}^{(0)} + (0.25)f_{29548492}^{(0)} + (0.083)f_{44322737}^{(0)} + (0.292)f_{5906983}^{(0)} + (0.104)f_{73871229}^{(0)} + (0.125)f_{88645474}^{(0)}
\end{align*}
\]

To solve the linear equation system, the Lingo 10 program is used. The results are:

\[
\begin{align*}
    f_0^{(0)} &= 0.000000 \\
    f_{14774246}^{(0)} &= 0.5900953 \times 10^{15} = 590.095.300.000.000 \\
    f_{29548492}^{(0)} &= 0.6047665 \times 10^{15} = 604.765.500.000.000 \\
    f_{44322737}^{(0)} &= 0.6260159 \times 10^{15} = 626.015.900.000.000 \\
    f_{73871229}^{(0)} &= 0.6494494 \times 10^{15} = 649.449.400.000.000 \\
    f_{88645474}^{(0)} &= 0.7144195 \times 10^{15} = 714.419.500.000.000
\end{align*}
\]

6. Conclusion
The decision on ordering was initially based on estimation needs, without specific methods to help solve it. In order for inventory costs to be optimal, the analysis must be carried out using certain methods in accordance with the objectives to be achieved. The Markov Chain method is one of the right methods because the inventory calculation according to the Markov Chain is more accurate than other inventory calculations, where the calculation results already reflect the number of inventory units and costs that must be incurred.

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