CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS DEFINED BY NEW MULTIPLIER TRANSFORMATIONS

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Abstract. In the present paper the new multiplier transformations $J^\delta_p(\lambda, \mu, l)$ ($\delta, l \geq 0$, $\lambda \geq \mu \geq 0$; $p \in \mathbb{N}$) of multivalent functions is defined. Making use of the operator $J^\delta_p(\lambda, \mu, l)$, two new subclasses $\mathcal{P}_p(\lambda, \mu, l)$ and $\mathcal{F}_p(\lambda, \mu, l)$ of multivalent analytic functions are introduced and investigated in the open unit disk. Some interesting relations and characteristics such as inclusion relationships, neighborhoods, partial sums, some applications of fractional calculus and quasi-convolution properties of functions belonging to each of these subclasses $\mathcal{P}_p(\lambda, \mu, l)$ and $\mathcal{F}_p(\lambda, \mu, l)$ are investigated. Relevant connections of the definitions and results presented in this paper with those obtained in several earlier works on the subject are also pointed out.

1. INTRODUCTION AND DEFINITIONS

Let $A(n, p)$ denote the class of functions normalized by

\[ f(z) = z^p + \sum_{k=n+p}^{\infty} a_k z^k \quad (p, n \in \mathbb{N} : = \{1, 2, 3, \ldots\}) \] (1.1)

which are analytic and $p$-valent in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

Let $f(z)$ and $g(z)$ be analytic in $U$. Then, we say that the function $f$ is subordinate to $g$ if there exists a Schwarz function $w(z)$, analytic in $U$ with $w(0) = 0$, $|w(z)| < 1$ such that $f(z) = g(w(z))$ ($z \in U$). We denote this subordination $f \prec g$ or $f(z) \prec g(z)$ ($z \in U$). In particular, if the function $g$ is univalent in $U$, the above subordination is equivalent to $f(0) = g(0)$, $f(U) \subset g(U)$.

For $f \in A(n, p)$ given by (1.1) and $g(z)$ given by

\[ g(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k \quad (p, n \in \mathbb{N} : = \{1, 2, 3, \ldots\}) \] (1.2)

their convolution (or Hadamard product), denoted by $(f \ast g)$, is defined as

\[ (f \ast g)(z) := z^p + \sum_{k=n+p}^{\infty} a_k b_k z^k =: (g \ast f)(z) \quad (z \in U). \] (1.3)

Note that $f \ast g \in A(n, p)$. In particular, we set

\[ A(1, 0) := A_p, \quad A(1, n) := A_n, \quad A(1, 1) := A_{11} = A. \]

For a function $f$ in $A(n, p)$, we define the multiplier transformations $J^\delta_p(\lambda, \mu, l)$ as follows:

**Definition 1.1.** Let $f \in A(n, p)$. For the parameters $\delta, \lambda, \mu, l \in \mathbb{R}$; $\lambda \geq \mu \geq 0$ and $\delta, l \geq 0$ define the multiplier transformations $J^\delta_p(\lambda, \mu, l)$ on $A(n, p)$ by the following

\[ J^\delta_p(\lambda, \mu, l)f(z) = f(z) \] (p + l)J^\delta_p(\lambda, \mu, l)f(z) = \lambda \mu z^2 f''(z) + (\lambda - \mu + (1 - p)\lambda \mu) z f'(z) + (p(1 - \lambda + \mu) + l) f(z)

(1.4)

for $z \in U$ and $p, n \in \mathbb{N} : = \{1, 2, \ldots\}$.

If $f$ is given by (1.1) then from the definition of the multiplier transformations $J^\delta_p(\lambda, \mu, l)$, we can easily see that

\[ J^\delta_p(\lambda, \mu, l)f(z) = z^p + \sum_{k=n+p}^{\infty} \Phi^k_p(\delta, \lambda, \mu, l) a_k z^k \] (1.5)
where
\[
(1.6) \quad \Phi^k_p(\delta, \lambda, \mu, l) = \left[ \frac{(k-p)(\lambda \mu k + \lambda + \mu) + p + l}{p + l} \right]^\delta.
\]

Remark 1.1. It should be remarked that the operator \(J_p^\delta(\lambda, \mu, l)\) is a generalization of many other operators considered earlier. In particular, for \(f \in A(n,p)\) we have the following:

1. \(J_p^\delta(1, 0, 0, f(z)) \equiv D^\delta f(z), \quad (\delta \in N_0 := N \cup \{0\}\) the Salagean differential operator [42].
2. \(J_p^\delta(\lambda, 0, 0, f(z)) \equiv D^{\delta_0} f(z), \quad (\delta \in N_0)\) the generalized Salagean differential operator introduced by Al-Oboudi [2].

3. \(J_p^\delta(\lambda, \mu, 0, f(z)) \equiv D_{\lambda, \mu} f(z), \quad (\delta \in N_0)\) the operator studied by Deniz and Orhan [18], in special case \(0 \leq \mu \leq \lambda \leq 1\) the operator was studied firstly Raducanu and Orhan [39].

4. \(J_p^\delta(1, 0, 1, f(z)) \equiv I_p^\delta f(z), \quad (\delta \in N_0)\) the operator considered by Cho and Srivastava [15] and Cho and Kim [16].

5. \(J_p^\delta(1, 0, 1, f(z)) \equiv D_{\lambda, \mu} f(z), \quad (\delta \in N_0)\) the operator investigated by Uralegaddi and Sonomonatha [54].

6. \(J_p^\delta(\lambda, 0, 0, f(z)) \equiv D_{\lambda, \mu} f(z), \quad (\delta \in R^+ \cup \{0\}\) the operator studied by Acu and Owa [1].

7. \(J_p^\delta(\lambda, 0, 1, f(z)) \equiv I(\delta, \lambda, \mu) f(z), \quad (\delta \in R^+ \cup \{0\}\) the operator introduced by Catas [11].

8. \(J_p^\delta(1, 0, 0, f(z)) \equiv D_{\lambda, \mu} f(z), \quad (\delta \in N_0)\) the operator considered by Shan et al. [45].

9. \(J_p^\delta(\lambda, 0, 0, f(z)) \equiv D_{\lambda, \mu} f(z), \quad (\delta \in N_0)\) the operator investigated by Kwon [25].

10. \(J_p^\delta(1, 0, 1, f(z)) \equiv I_{\rho}(\delta, \lambda, \mu) f(z), \quad (\rho \in N_0)\) the operator considered by Kumar et al. [48].

11. \(J_p^\delta(\lambda, 0, 1, f(z)) \equiv I_{\rho}(\delta, \lambda, \mu) f(z), \quad (\rho \in N_0)\) the operator studied recently by Catas et al. [12].

For special values of parameters \(\lambda, \mu, l\) and \(p\), from the operator \(J_p^\delta(\lambda, \mu, l)\) the following new operators can be obtained:

- \(J_p^\delta(\lambda, \mu, 1) \equiv J_p^\delta(\lambda, \mu)\)
- \(J_p^\delta(\lambda, \mu, l) \equiv J_p^\delta(\lambda, \mu, l)\).

Now, by making use of the operator \(J_p^\delta(\lambda, \mu, l)\), we define a new subclass of functions belonging to the class \(A(n,p)\).

Definition 1.2. Let \(\lambda \geq \mu \geq 0\); \(l, \delta \geq 0; \quad p \in N\) and for the parameters \(\sigma, A, B\) such that
\[
(1.7) \quad -1 \leq A < B \leq 1, \quad 0 < B \leq 1 \quad \text{and} \quad 0 \leq \sigma < p,
\]
we say that a function \(f(z) \in A(n,p)\) is in the class \(P_{\lambda, \mu, l}(A, B; \sigma, p)\) if it satisfies the following subordination condition:
\[
(1.8) \quad \frac{1}{p - \sigma} \left[ J_p^\delta(\lambda, \mu, l) f(z) \right]' - \sigma < \frac{1 + Az}{1 + Bz} \quad (z \in U).
\]

If the following inequality holds true,
\[
(1.9) \quad \left| \frac{[J_p^\delta(\lambda, \mu, l) f(z)']'(z \neq 0)}{B^2 [J_p^\delta(\lambda, \mu, l) f(z)']'(z \neq 0)} - p \right| < 1 \quad (z \in U)
\]
the inequality (1.9) is equivalent the subordination condition (1.8).

We note that by specializing the parameters \(\lambda, \mu, l, \delta, \sigma, A, B\) and \(p\), the subclass \(P_{\lambda, \mu, l}(A, B; \sigma, p)\) reduces to several well-known subclasses of analytic functions. These subclasses are:

1. \(P_{\lambda, \mu, l}^0(1, 0, 1, 0, 1) \equiv P_{0,0,l}^0(1, 0, 1, 0, 1) \equiv R\) (see Mac-Grigor [31]);
2. \(P_{\lambda, \mu, l}(A, B; \sigma, p) \equiv P_{0,0,l}^0(A, B; \sigma, p) \equiv S_{\rho}(A, B, \sigma)\) (see Aouf [3]);
3. \(P_{\lambda, \mu, l}^0(-1, 1, -1; 1, 0, 1) \equiv S(\rho, \alpha > 1/2)\) (see Cho [15]);
4. \(P_{\lambda, \mu, l}^0(-1, 1, -1; 0, 1, 0) \equiv S(\alpha > 1/2)\) (see Sohi [49]);
5. \(P_{\lambda, \mu, l}^0(\lambda, 0, 0, 0, 0, 0) \equiv S(\rho)(\alpha > 1/2)\) (see Chen [13]);
6. \(P_{\lambda, \mu, l}^0(1, 0, 1, 0, 0, 0) \equiv R(\alpha, \beta, \gamma)\) (see Kim and Lee [24]).
Thus, by specializing the parameters \(\lambda, \mu, l, \delta, \sigma, A, B\) and \(p\), we obtain the following familiar subclasses of analytic functions in \(\mathcal{U}\) with negative coefficients:

1. \(\mathcal{P}_{\lambda,\mu,l}(1, -1, 1, 1, \alpha, 1) \equiv \mathcal{P}^*(1, (0 \leq \alpha < 1))\) (see for \(f \in A\) Sarangi and Uralegaddi [43] and for \(f \in \mathcal{A}(n)\) Sekine and Owa [44]);
2. \(\mathcal{P}_{\lambda,\mu,l}(1, 1, 1, 1, \alpha, 1) \equiv \mathcal{L}^*(\alpha, \beta, \gamma)\) (0 \(\leq \alpha \leq 1\), 0 \(\leq \beta, 0 \leq \gamma < 1\) (see Kim and Lee [24]);
3. \(\mathcal{P}_{\lambda,\mu,l}(A, B; \sigma, p) \equiv \mathcal{P}^*(p, A, B, \sigma)\) (see Aouf [4]);
4. \(\mathcal{P}_{\lambda,\mu,l}(1, 1, 1, 1, \alpha, 1) \equiv \mathcal{D}^*(\beta)\) (0 \(\leq \beta < 1\) (see Kim and Lee [24]);
5. \(\mathcal{P}_{\lambda,\mu,l}(1, 1, 1, 1, \alpha, 1) \equiv \mathcal{F}_p(1, \beta)\) (0 \(\leq \beta < 1\) (see Aouf [6]);
6. \(\mathcal{P}_{\lambda,\mu,l}(1, 1, 1, 1, \alpha, 1) \equiv \mathcal{P}^*(1, 1)\) (see Shakla and Dardonath [46]);
7. \(\mathcal{P}_{\lambda,\mu,l}(1, 1, 1, 1, \alpha, 1) \equiv \mathcal{P}^*(1, 1)\) (see Gupta and Jain [22]);
8. \(\mathcal{P}_{\lambda,\mu,l}(1, 1, 1, 1, \alpha, 1) \equiv \mathcal{P}^*(1, 1)\) (see Owa [36]);

In our present paper, we shall make use of the familiar integral operator \(\mathcal{I}_{\theta,p}\) defined by (see, for details, [9, 27, 30]; see also [54])

\[
(\mathcal{I}_{\theta,p})(z) := \frac{\theta + p}{z^p} \int_0^z t^{\rho-1} f(t) dt \quad f \in \mathcal{A}(n,p); \quad \theta + p > 0; \quad p \in \mathbb{N}
\]

as well as the fractional calculus operator \(\mathcal{D}^\rho\) for which it is well known that (see, for details, [37, 50] and [53]; see also Section 7)

\[
\mathcal{D}^\rho\{z^p\} = \frac{\Gamma(\rho + 1)}{\Gamma(\rho + 1 - \nu)} z^{\rho - \nu} \quad (\rho > 1; \nu \in \mathbb{R})
\]

in terms of Gamma function.

The main object of the present paper is to investigate the various important properties and characteristics of two subclasses of \(\mathcal{A}(n, p)\) of normalized analytic functions in \(\mathcal{U}\) with negative and positive coefficients, which are introduced here by making use of the multiplier transformations \(\mathcal{M}^\rho(\lambda, \mu, l)\) defined by (1.4). Inclusion relationships for the class \(\mathcal{P}_{\lambda,\mu,l}(A, B; \sigma, p)\) are investigated by applying the techniques of convolution. Furthermore, several properties involving generalized neighborhoods and partial sums for functions belonging to these subclasses are investigated. We also derive many results for the Quasi-convolution of functions belonging to the class \(\mathcal{P}_{\lambda,\mu,l}(A, B; \sigma, p)\). Finally, some applications of fractional calculus operators are considered. Relevant connections of the definitions and results presented here with those obtained in several earlier works are also pointed out.

**Remark 1.2.** Throughout our present investigation, we tacitly assume that the parametric constraints listed in (1.6), (1.7) and Definition 1.1 are satisfied.

## 2. Inclusion Properties of the Function Class \(\mathcal{P}_{\lambda,\mu,l}(A, B; \sigma, p)\)

For proving our first inclusion result, we shall need the following lemmas.

**Lemma 2.1** (See Fejer [19] or Ruscheweyh [41]). Assume \(a_1 = 1\) and \(a_m \geq 0\) for \(m \geq 2\), such that \(\{a_m\}\) is a convex decreasing sequence, i.e., \(a_m - 2a_{m+1} + a_{m+2} \geq 0\) and \(a_{m+1} - a_{m+2} \geq 0\) for \(m \in \mathbb{N}\). Then

\[
\Re \left\{ \sum_{m=1}^{\infty} a_m z^{m-1} \right\} \geq \frac{1}{2}
\]

for all \(z \in \mathcal{U}\).

**Lemma 2.2** (See Liu [28]). Let \(-1 \leq A_2 < A_1 < B_1 < B_2 \leq 1\). Then, we can write the following subordination result:

\[
\frac{1 + B_1 z}{1 + A_1 z} \geq \frac{1 + B_2 z}{1 + A_2 z}.
\]

**Lemma 2.3.** If \(\lambda - \mu \geq \frac{l+p}{2p}\) or \(\lambda = \mu = 0\), then \(\Re \left\{ 1 + \sum_{k=n+p}^{\infty} \Phi^{-1}(\lambda, \mu, l) \right\} \geq \frac{1}{2}\) for all \(z \in \mathcal{U}\).

**Proof.** Define:

\[
q(z) = 1 + \sum_{k=2}^{\infty} \frac{1}{\Phi^{-1}(\lambda, \mu, l) z^{k-n-2}} = 1 + \sum_{k=2}^{\infty} B_k z^{k-n-2}
\]

where

\[
B_k = \frac{1}{\Phi^{-1}(\lambda, \mu, l) z^{k-n-2}} = \frac{p + l}{(k + n - 2)(\lambda \mu (k + n + p - 2) + \lambda - \mu) + p + l}
\]

for all \(n, p \in \mathbb{N}, k \geq 2\).
Since the values \( k, l, p, \lambda \) and \( \mu \) are positive, we have \( B_k > 0 \) for all \( k \in \mathbb{N} \). We can easily find that

\[
B_{k+1} = \frac{p + l}{(k + n - 1)(\lambda \mu (k + n + p - 1) + \lambda - \mu) + p + l}
\]

(2.3)

\[
B_{k+2} = \frac{p + l}{(k + n)(\lambda \mu (k + n + p) + \lambda - \mu) + p + l}
\]

(2.4)

and thus from (2.3) and (2.4), we can see that

\[
B_{k+1} - B_{k+2} \geq 0
\]

for all \( k \in \mathbb{N} \). Next, we show that the inequality

\[
B_k - 2B_{k+1} + B_{k+2} \geq 0
\]

(2.5)

holds for all \( k \in \mathbb{N} \). Using (2.2), (2.3) and (2.4) we find that

\[
B_k - 2B_{k+1} + B_{k+2} = \frac{p + l}{(k + n - 2)(\lambda \mu (k + n + p - 2) + \lambda - \mu) + p + l}
\]

\[
- \frac{2}{(k + n - 1)(\lambda \mu (k + n + p - 1) + \lambda - \mu) + p + l}
\]

\[
+ \frac{p + l}{(k + n)(\lambda \mu (k + n + p) + \lambda - \mu) + p + l}
\]

\[
= \frac{2(\lambda \mu)^2 [3(k + n)(k + n - 1) + p(2(\lambda - \mu) - 1) - l] + (\lambda - \mu)^2}{C_2 C_1 C_0}
\]

where \( C_i = [(k + n - i)(\lambda \mu (k + n + p - i) + \lambda - \mu) + p + l] \) and from the hypothesis of Lemma 2.3, we deduce that (2.5) holds for all \( k \in \mathbb{N} \). Thus the sequence \( \{B_k\} \) is convex decreasing and by Lemma 2.1 we obtain that

\[
\mathbb{R}\{q(z)\} = \mathbb{R}\left\{ 1 + \sum_{k=2}^{\infty} B_k z^{k+n-2} \right\} = \mathbb{R}\left\{ 1 + \sum_{k=n+p}^{\infty} \frac{1}{\Phi_p(1, \lambda, \mu, l)} \right\} > \frac{1}{2}
\]

for all \( z \in \mathcal{U} \). The proof of Lemma 2.3 is completed.

Theorem 2.1. If \( \lambda - \mu \geq \frac{1 + \mu}{2p} \) or \( \lambda = \mu = 0 \) and \( \delta \geq 0 \), then

\[
\mathcal{P}_{\lambda, \mu, l}^{\delta+1}(A, B; \sigma, p) \subseteq \mathcal{P}_{\lambda, \mu, l}^{\delta}(A, B; \sigma, p).
\]

Proof. Let \( f \in \mathcal{P}_{\lambda, \mu, l}^{\delta+1}(A, B; \sigma, p) \). Using the definition of \( \mathcal{P}_{\lambda, \mu, l}^{\delta+1}(A, B; \sigma, p) \) we obtain that

\[
1 - \frac{1}{p - \sigma} \left( \frac{z[\mathcal{J}_{p}^{\delta+1}(\lambda, \mu, l)f(z)]'}{zp} - \sigma \right) < \frac{1 + Az}{1 + Bz} \quad (z \in \mathcal{U}).
\]

(2.6)

Applying definition of \( \mathcal{J}_{p}^{\delta}(\lambda, \mu, l)f(z) \) and the properties of convolution we find that

\[
\frac{z[\mathcal{J}_{p}^{\delta}(\lambda, \mu, l)f(z)]'}{zp} = \left( 1 + \sum_{k=n+p}^{\infty} \frac{1}{\Phi_p(1, \lambda, \mu, l)} z^{k-p} \right) \ast \left( \frac{1}{p} \sum_{k=n+p}^{\infty} \frac{k}{p} \Phi_p(\delta + 1, \lambda, \mu, l) a_k z^{k-p} \right)
\]

\[
= \left( 1 + \sum_{k=n+p}^{\infty} \frac{1}{\Phi_p(1, \lambda, \mu, l)} z^{k-p} \right) \ast \left( \frac{z[\mathcal{J}_{p}^{\delta+1}(\lambda, \mu, l)f(z)]'}{zp} \right).
\]
Therefore from the last equalities and (1.8) we get

$$\frac{1}{p - \sigma} \left( \frac{[J^\delta_p(\lambda, \mu, l)f(z)]'}{z^{p-1}} - \sigma \right)$$

(2.8)

$$= \frac{1}{p - \sigma} \left\{ \left( 1 + \sum_{k=n+p}^{\infty} \frac{1}{p \Phi_p(1, \lambda, \mu, l)} z^{k-p} \right) \ast \left( \frac{z[J^\delta_{p+1}(\lambda, \mu, l)f(z)]'}{pz^p} \right) - \sigma \right\}$$

$$= q(z) \cdot \frac{1 + Aw(z)}{1 + Bw(z)}$$

where \(|w(z)| < 1\) and \(w(0) = 0\). From the Herglotz theorem and Lemma 2.3 we thus obtain

$$q(z) = \int_{|x| = 1} \frac{d\varpi(x)}{1 - xz} \quad (z \in \mathcal{U}),$$

when \(\varpi(x)\) is a probability measure on the unit circle \(|x| = 1\), that is

$$\int_{|x| = 1} d\varpi(x) = 1.$$  

It follows from (2.8) that

$$\frac{1}{p - \sigma} \left( \frac{z[J^\delta_p(\lambda, \mu, l)f(z)]'}{z^p} - \sigma \right) = \int_{|x| = 1} \frac{1 + Axz}{1 + Bxz} d\varpi(x) < \frac{1 + Az}{1 + Bz}$$

because \(\frac{1 + Az}{1 + Bz}\) is convex univalent in \(\mathcal{U}\). Hence we conclude that

$$\mathcal{P}^{\delta+1}_{\lambda,\mu,l}(A, B; \sigma, p) \subseteq \mathcal{P}^\delta_{\lambda,\mu,l}(A, B; \sigma, p),$$

which completes the proof of Theorem 2.1.

**Theorem 2.2.** If \(\delta \geq 0\) and \(-1 \leq A_2 \leq A_1 < B_1 \leq B_2 \leq 1\), then

(2.9)

$$\mathcal{P}^{\delta+1}_{\lambda,\mu,l}(B_1, A_1; \sigma, p) \subseteq \mathcal{P}^\delta_{\lambda,\mu,l}(B_2, A_2; \sigma, p).$$

**Proof.** Making use of Lemma 2.2, we can write

$$\mathcal{P}^\delta_{\lambda,\mu,l}(B_1, A_1; \sigma, p) \subseteq \mathcal{P}^\delta_{\lambda,\mu,l}(B_2, A_2; \sigma, p).$$

Using (2.6) and (2.9), we have

$$\mathcal{P}^{\delta+1}_{\lambda,\mu,l}(B_1, A_1; \sigma, p) \subseteq \mathcal{P}^\delta_{\lambda,\mu,l}(B_1, A_1; \sigma, p) \subseteq \mathcal{P}^\delta_{\lambda,\mu,l}(B_2, A_1; \sigma, p) \subseteq \mathcal{P}^{\delta+1}_{\lambda,\mu,l}(B_2, A_2; \sigma, p)$$

so, we obtain

$$\mathcal{P}^{\delta+1}_{\lambda,\mu,l}(B_1, A_1; \sigma, p) \subseteq \mathcal{P}^\delta_{\lambda,\mu,l}(B_2, A_2; \sigma, p).$$

Thus, the proof is complete.

### 3. Basic Properties of the Function Class \(\mathcal{P}^\delta_{\lambda,\mu,l}(A, B; \sigma, p)\)

We first determine a necessary and sufficient condition for a function \(f(z) \in \mathcal{A}(n, p)\) of the form (1.10) to be in the class \(\mathcal{P}^\delta_{\lambda,\mu,l}(A, B; \sigma, p)\).

**Theorem 3.1.** Let the function \(f(z) \in \mathcal{A}(n, p)\) be defined by (1.10). Then the function \(f(z)\) is in the class \(\mathcal{P}^\delta_{\lambda,\mu,l}(A, B; \sigma, p)\) if and only if

(3.1)

$$\sum_{k=n+p}^{\infty} k(1 + B)\Phi_p^k(\delta, \lambda, \mu, l) |a_k| \leq (B - A)(p - \sigma)$$

where \(\Phi_p^k(\delta, \lambda, \mu, l)\) is given by (1.6).
**Proof.** If the condition (3.1) hold true, we find from (1.10) and (3.1) that
\[
\left| [J_p^\delta(\lambda, \mu, l)f(z)]' - pz^{p-1} \right| - |B[J_p^\delta(\lambda, \mu, l)f(z)]' - z^{p-1}[pB + (A - B)(p - \sigma)]|,
\]
\[
= \left| - \sum_{k=n+p} k\Phi_k^p(\delta, \lambda, \mu, l) |a_k| z^{k-1} \right| - \left| (B - A)(p - \sigma)z^{p-1} - B \sum_{k=n+p} k\Phi_k^p(\delta, \lambda, \mu, l) |a_k| z^{k-1} \right|
\]
\[
\leq \sum_{k=n+p} k(1 + B)\Phi_k^p(\delta, \lambda, \mu, l) |a_k| - (B - A)(p - \sigma) \leq 0 \quad (z \in \partial \mathcal{U} = \{ z : z \in \mathbb{C} \text{ and } |z| = 1 \}).
\]
Hence, by the **Maximum Modulus Theorem**, we have
\[
f(z) \in \mathcal{P}^\delta_{\lambda, \mu, l}(A, B; \sigma, p).
\]
Conversely, assume that the function \( f(z) \) defined by (1.10) is in the class \( \mathcal{P}^\delta_{\lambda, \mu, l}(A, B; \sigma, p) \). Then we have
\[
\left| \frac{[J_p^\delta(\lambda, \mu, l)f(z)]' - p}{B[J_p^\delta(\lambda, \mu, l)f(z)]' - [pB + (A - B)(p - \sigma)]} \right| = \left| \frac{\sum_{k=n+p} k\Phi_k^p(\delta, \lambda, \mu, l) |a_k| z^{k-p}}{(B - A)(p - \sigma)z^{p-1} - B \sum_{k=n+p} k\Phi_k^p(\delta, \lambda, \mu, l) |a_k| z^{k-p}} \right| < 1 \quad (z \in \mathcal{U}).
\]
Now, since \( |\Re(z)| \leq |z| \) for all \( z \), we have
\[
\Re \left( \frac{\sum_{k=n+p} k\Phi_k^p(\delta, \lambda, \mu, l) |a_k| z^{k-p}}{(B - A)(p - \sigma)z^{p-1} - B \sum_{k=n+p} k\Phi_k^p(\delta, \lambda, \mu, l) |a_k| z^{k-p}} \right) < 1.
\]
We choose values of \( z \) on the real axis so that the following expression:
\[
\left| [J_p^\delta(\lambda, \mu, l)f(z)]' \right|_{z=p-1}^{z=p}
\]
is real. Then, upon clearing the denominator in (3.3) and letting \( z \to 1^- \) though real values, we get the following inequality
\[
\sum_{k=n+p} k(1 + B)\Phi_k^p(\delta, \lambda, \mu, l) |a_k| \leq (B - A)(p - \sigma).
\]
This completes the proof of Theorem 3.1.

**Remark 3.1.** Since \( \mathcal{P}^\delta_{\lambda, \mu, l}(A, B; \sigma, p) \) is contained in the function class \( \mathcal{P}^\delta_{\lambda, \mu, l}(A, B; \sigma, p) \), a sufficient condition for \( f(z) \) defined by (1.1) to be in the class \( \mathcal{P}^\delta_{\lambda, \mu, l}(A, B; \sigma, p) \) is that it satisfies the condition (3.1) of Theorem 3.1.

**Corollary 3.1.** Let the function \( f(z) \in \mathcal{A}(n, p) \) be defined by (1.10). If the function \( f(z) \in \mathcal{P}^\delta_{\lambda, \mu, l}(A, B; \sigma, p) \), then
\[
|a_k| \leq \frac{(B - A)(p - \sigma)}{k(1 + B)\Phi_k^p(\delta, \lambda, \mu, l)} \quad (k, p \in \mathbb{N}).
\]
The result is sharp for the function \( f(z) \) given by
\[
f(z) = z^p - \frac{(B - A)(p - \sigma)}{k(1 + B)\Phi_k^p(\delta, \lambda, \mu, l)} z^k \quad (k, p \in \mathbb{N}).
\]
We next prove the following growth and distortion properties for the class \( \mathcal{P}^\delta_{\lambda, \mu, l}(A, B; \sigma, p) \).

**Remark 3.2.**

1. Putting \( A = -\beta, \ B = \beta, \ \delta = 0 \) and \( \sigma = \alpha \) in Theorem 3.1, we obtain the corresponding result given earlier by Aouf [6].
2. Putting \( A = (\gamma - 1)\beta, \ B = \alpha \beta, \ \delta = 0, \ p = n = 1 \) and \( \sigma = 0 \) in Theorem 3.1, we obtain result of Kim and Lee [24].
3. Putting \( \delta = 0 \) in Theorem 3.1, we obtain Theorem 1 in [4].
4. Putting \( A = (2a - 1)b, \ B = b, \ \delta = 0, \ n = 1 \) and \( \sigma = 0 \) in Theorem 3.1, we arrive at the Theorem of Owa [36].
5. Putting \( \delta = 0 \) and \( \sigma = 0 \) in Theorem 3.1, we obtain the corresponding result due to Shukla and Dashrath [46].

**Theorem 3.2.** If a function \( f(z) \) be defined by (1.10) is in the class \( \mathcal{P}^\delta_{\lambda, \mu, l}(A, B; \sigma, p) \), then
\[
\left( \frac{p!}{(p - q)!} \right) - \frac{(B - A)(p - \sigma)(n + p - 1)!}{(1 + B)\Phi_n^{n+p}(\delta, \lambda, \mu, l)(n + p - q)!} |z|^n |z|^{p-q}
\]
\[
\leq |f^{(q)}(z)| \leq \left( \frac{p!}{(p - q)!} + \frac{(B - A)(p - \sigma)(n + p - 1)!}{(1 + B)\Phi_n^{n+p}(\delta, \lambda, \mu, l)(n + p - q)!} |z|^n |z|^{p-q}
\]
for $q \in \mathbb{N}_0$, $p > q$ and all $z \in \mathcal{U}$. The result is sharp for the function $f(z)$ given by

$$f(z) = z^p + \frac{(B - A)(p - \sigma)}{(n + p)(1 + B)\Phi^p_n(\delta, \lambda, \mu, l)}z^{n+p} \ (p \in \mathbb{N}).$$

**Proof.** In view of Theorem 3.1, we have

$$\frac{(n + p)(1 + B)\Phi^p_n(\delta, \lambda, \mu, l)}{(B - A)(p - \sigma)(n + p)!} \sum_{k=n+p}^{\infty} k! |a_k| \leq \frac{1}{(1 + B)\Phi^p_n(\delta, \lambda, \mu, l)} \sum_{k=n+p}^{\infty} k! (1 + B)\Phi^p_n(\delta, \lambda, \mu, l) \ |a_k| \leq 1,$$

which readily yields

$$\sum_{k=n+p}^{\infty} k! |a_k| \leq \frac{(B - A)(p - \sigma)(n + p - 1)!}{(1 + B)\Phi^p_n(\delta, \lambda, \mu, l)} \ (k, p \in \mathbb{N}).$$

Now, by differentiating both sides of $(1.10)$ $q$—times with respect to $z$, we obtain

$$f^{(q)}(z) = \frac{p!}{(p - q)!}z^{p-q} - \sum_{k=n+p}^{\infty} \frac{k!}{(k - q)!} a_k z^{k-q} \ (q \in \mathbb{N}_0; \ p > q).$$

Theorem 3.2 follows readily from $(3.8)$ and $(3.9)$.

Finally, it is easy to see that the bounds in $(3.6)$ are attained for the function $f(z)$ given by $(3.7)$.

### 4. INCLUSION RELATIONS INVOLVING NEIGHBORHOODS

Following the earlier investigations (based upon the familiar concept of neighborhoods of analytic functions) by Goodman [21], Ruscheweyh [40] and others including Srivastava et al. [50, 52], Orhan [33, 34], Deniz et al. [17], Aouf et al. [8] (see also [11]).

Firstly, we define the $(n, \eta)$—neighborhood of function $f(z) \in \mathcal{A}(n, p)$ of the form $(1.1)$ by means of Definition 4.1 below.

**Definition 4.1.** For $\eta > 0$ and a non-negative sequence $S = \{s_k\}_{k=1}^{\infty}$, where

$$s_k := \frac{k(1 + B)\Phi^k_p(\delta, \lambda, \mu, l)}{(B - A)(p - \sigma)} \ (k \in \mathbb{N}).$$

The $(n, \eta)$—neighborhood of a function $f(z) \in \mathcal{A}(n, p)$ of the form $(1.1)$ is defined as follows:

$$\mathcal{N}^\eta_{n,p}(f) := \left\{ g : g(z) = z^p + \sum_{k=n+p}^{\infty} b_k z^k \in \mathcal{A}(n, p) \text{ and } \sum_{k=n+p}^{\infty} s_k |b_k - a_k| \leq \eta \ (\eta > 0) \right\}.$$

For $s_k = k$, Definition 4.1 would correspond to the $\mathcal{N}_\eta$—neighborhood considered by Ruscheweyh [40].

Our first result based upon the familiar concept of neighborhood defined by $(4.2)$.

**Theorem 4.1.** Let $f(z) \in \mathcal{P}^k_{\lambda, \mu, l}(A, B; \sigma, p)$ be given by $(1.1)$. If $f$ satisfies the inclusion condition:

$$f(z) + \varepsilon z^p (1 + \varepsilon)^{-1} \in \mathcal{P}^k_{\lambda, \mu, l}(A, B; \sigma, p) \ \ (\varepsilon \in \mathbb{C}; \ |\varepsilon| < 1; \ |\sigma| > 0),$$

then

$$\mathcal{N}^\eta_{n,p}(f) \subset \mathcal{P}^k_{\lambda, \mu, l}(A, B; \sigma, p).$$

**Proof.** It is not difficult to see that a function $f$ belongs to $\mathcal{P}^k_{\lambda, \mu, l}(A, B; \sigma, p)$ if and only if

$$\frac{[J_p^k(\lambda, \mu, l)f(z)]' - p^{p-1}}{B[J_p^k(\lambda, \mu, l)f(z)]' - z^{p-1}[pB + (A - B)(p - \sigma)]} \neq \tau \ (z \in \mathcal{U}; \ \tau \in \mathbb{C}; \ |\tau| = 1),$$

which is equivalent to

$$(f * h)(z) / z^p \neq 0 \ (z \in \mathcal{U}),$$

where for convenience,

$$h(z) := z^p + \sum_{k=n+p}^{\infty} c_k z^k = z^p + \sum_{k=n+p}^{\infty} \frac{k(1 + B)\Phi^k_p(\delta, \lambda, \mu, l)}{(B - A)(p - \sigma)} z^k.$$

We easily find from $(4.7)$ that

$$|c_k| \leq \frac{k(1 + B)\Phi^k_p(\delta, \lambda, \mu, l)}{(B - A)(p - \sigma)} \ (k \in \mathbb{N}).$$
Furthermore, under the hypotheses of theorem, (4.3) and (4.6) yields the following inequalities:

$$ \frac{(f(z) + \varepsilon z^p)(1 + \varepsilon)^{-1} * h(z)}{z^p} \neq 0 \quad (z \in \mathcal{U}) $$

or

$$ \frac{f(z) * h(z)}{z^p} \neq \varepsilon \quad (z \in \mathcal{U}), $$

which is equivalent to the following:

$$(4.9) \quad \frac{f(z) * h(z)}{z^p} \geq \eta \quad (z \in \mathcal{U}; \eta > 0).$$

Now, if we let

$$ g(z) := z^p + \sum_{k=n+p}^{\infty} b_k z^k \in \mathcal{N}_{n,p}^\eta(f), $$

then we have

$$ \left| \frac{(f(z) - g(z)) * h(z)}{z^p} \right| = \left| \sum_{k=n+p}^{\infty} (a_k - b_k) c_k z^{k-p} \right| \leq \sum_{k=n+p}^{\infty} \frac{k(1 + B)\Phi_k^p(\delta, \lambda, \mu, l)}{(B - A)(p - \sigma)} |a_k - b_k| |z|^{k-p} < \eta \quad (z \in \mathcal{U}; \eta > 0). $$

Thus, for any complex number $\tau$ such that $|\tau| = 1$, we have

$$ (g * h)(z) / z^p \neq 0 \quad (z \in \mathcal{U}), $$

which implies that $g \in \mathcal{P}_{\lambda, \mu, \eta}(A, B; \sigma, p)$. The proof is complete.

We now define the $(n, \eta)$–neighborhood of a function $f(z) \in \mathcal{A}(n, p)$ of the form (1.10) as follows:

**Definition 4.2.** For $\eta > 0$, the $(n, \eta)$–neighborhood of a function $f(z) \in \mathcal{A}(n, p)$ of the form (1.10) is given by

$$ \mathcal{N}_{n,p}^\eta(f) := \left\{ g : g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \in \mathcal{A}(n, p) \text{ and } \sum_{k=n+p}^{\infty} \frac{k(1 + B)\Phi_k^p(\delta, \lambda, \mu, l)}{(B - A)(p - \sigma)} ||b_k|| - |a_k| \leq \eta \quad (\eta > 0) \right\}. $$

Next, we prove

**Theorem 4.2.** If the function $f(z)$ defined by (1.10) is in the class $\mathcal{P}_{\lambda, \mu, \eta}(A, B; \sigma, p)$, then

$$(4.11) \quad \mathcal{N}_{n,p}^\eta(f) \subset \mathcal{P}_{\lambda, \mu, \eta}(A, B; \sigma, p) $$

where

$$ \eta := \frac{n[\lambda \mu(n + p) + \lambda - \mu]}{n[\lambda \mu(n + p) + \lambda - \mu] + p + l}. $$

The result is the best possible in the sense that $\eta$ cannot be increased.

**Proof.** For a function $f(z) \in \mathcal{P}_{\lambda, \mu, \eta}(A, B; \sigma, p)$ of the form (1.10) Theorem 3.1 immediately yields

$$(4.12) \quad \sum_{k=n+p}^{\infty} \frac{k(1 + B)\Phi_k^p(\delta, \lambda, \mu, l)}{(B - A)(p - \sigma)} |a_k| \leq \frac{p + l}{n[\lambda \mu(n + p) + \lambda - \mu] + p + l}.$$ 

Similarly, by taking

$$ g(z) := z^p - \sum_{k=n+p}^{\infty} |b_k| z^k \in \mathcal{N}_{n,p}^\eta(f) \quad \left( \eta = \frac{n[\lambda \mu(n + p) + \lambda - \mu]}{n[\lambda \mu(n + p) + \lambda - \mu] + p + l} \right), $$

we find from the definition (4.10) that

$$(4.13) \quad \sum_{k=n+p}^{\infty} \frac{k(1 + B)\Phi_k^p(\delta, \lambda, \mu, l)}{(B - A)(p - \sigma)} ||b_k|| - |a_k| \leq \eta \quad (\eta > 0). $$
With the help of (4.12) and (4.13), we have
\[
\sum_{k=n+p}^{\infty} \frac{k(1+B)\Phi_p^k(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma)} |b_k| \leq \sum_{k=n+p}^{\infty} \frac{k(1+B)\Phi_p^k(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma)} |b_k| + \sum_{k=n+p}^{\infty} \frac{k(1+B)\Phi_p^k(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma)} |b_k| - |a_k| + \sum_{k=n+p}^{\infty} \frac{k(1+B)\Phi_p^k(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma)} |b_k| - |a_k| \\
\leq \frac{p+l}{n[\lambda \mu(n+p) + \lambda - \mu] + p+l} + \eta = 1.
\]

Hence, in view of Theorem 3.1 again, we see that \( g(z) \in \mathcal{P}^{\delta+1}_{\lambda, \mu, l}(A, B; \sigma, p) \).

To show the sharpness of the assertion of Theorem 4.2, we consider the functions \( f(z) \) and \( g(z) \) given by
\[
f(z) = z^p - \left[ \frac{(B-A)(p-\sigma)}{(n+p)(1+B)\Phi_p^{n+p}(\delta+1, \lambda, \mu, l)} \right] z^{n+p} \in \mathcal{P}^{\delta+1}_{\lambda, \mu, l}(A, B; \sigma, p)
\]
and
\[
g(z) = z^p - \left[ \frac{(B-A)(p-\sigma)}{(n+p)(1+B)\Phi_p^{n+p}(\delta+1, \lambda, \mu, l)} + \frac{(B-A)(p-\sigma)}{(n+p)(1+B)\Phi_p^{n+p}(\delta, \lambda, \mu, l)} \eta^* \right] z^{n+p}
\]
where \( \eta^* > \eta \).

Clearly, the function \( g(z) \) belong to \( \mathcal{N}^{\eta^*}_{n,p}(f) \). On the other hand, we find from Theorem 3.1 that \( g(z) \notin \mathcal{P}^{\delta}_{\lambda, \mu, l}(A, B; \sigma, p) \). This evidently completes the proof of Theorem 4.2.

5. PARTIAL SUMS OF THE FUNCTION CLASS \( \mathcal{P}^{\delta}_{\lambda, \mu, l}(A, B; \sigma, p) \)

Following the earlier work by Silverman [47] and recently Liu [29] and Deniz et al. [17], in this section we investigate the ratio of real parts of functions involving (1.10) and its sequence of partial sums defined by
\[
\kappa_m(z) = \begin{cases} 
  z^p, & m = 1, 2, \ldots, n + p - 1; \\
  z^p - \sum_{k=n+p}^{m} |a_k| z^k, & m = n + p, n + p + 1, \ldots \quad (k \geq n + p; \ n, p \in \mathbb{N})
\end{cases}
\]
and determine sharp lower bounds for \( \Re \left\{ \frac{f(z)}{\kappa_m(z)} \right\}, \Re \left\{ \frac{\kappa_m(z)}{f(z)} \right\} \).

**Theorem 5.1.** Let \( f \in \mathcal{A}(n, p) \) and \( \kappa_m(z) \) be given by (1.10) and (5.1), respectively. Suppose also that
\[
\sum_{k=n+p}^{\infty} \theta_k |a_k| \leq 1 \quad \text{(where } \theta_k = \frac{k(1+B)\Phi_p^k(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma)} \).
\]
Then for \( m \geq k + p \), we have
\[
\Re \left( \frac{f(z)}{\kappa_m(z)} \right) > 1 - \frac{1}{\theta_{m+1}}
\]
and
\[
\Re \left( \frac{\kappa_m(z)}{f(z)} \right) > \frac{\theta_{m+1}}{1 + \theta_{m+1}}.
\]
The results are sharp for every \( m \) with the extremal functions given by
\[
f(z) = z^p - \frac{1}{\theta_{m+1}} z^{m+1}.
\]

**Proof.** Under the hypothesis of the theorem, we can see from (5.2) that \( \theta_{k+1} > \theta_k > 1 \quad (k \geq n + p) \). Therefore, we have
\[
\sum_{k=n+p}^{m} |a_k| + \theta_{m+1} \sum_{k=m+1}^{\infty} |a_k| \leq \sum_{k=n+p}^{\infty} \theta_k |a_k| \leq 1
\]
by using hypothesis (5.2) again.
Upon setting
\[
\omega(z) = \theta_{m+1} \left[ \frac{f(z)}{\kappa_m(z)} - \left(1 + \frac{1}{\theta_{m+1}} \right) \right]
\]  \hspace{1cm} (5.7)
\[
= 1 - \frac{\theta_{m+1} \sum_{k=m+1}^{\infty} |a_k| z^{k-p}}{1 - \sum_{k=n+p}^{m} |a_k| z^{k-p}}.
\]
By applying (5.6) and (5.7), we find that
\[
\frac{|\omega(z) - 1|}{|\omega(z) + 1|} = \frac{-\theta_{m+1} \sum_{k=m+1}^{\infty} |a_k| z^{k-p}}{2 - 2 \sum_{k=n+p}^{m} |a_k| z^{k-p} - \theta_{m+1} \sum_{k=m+1}^{\infty} |a_k| z^{k-p}} \leq 1 \quad (z \in \mathcal{U}; \; k \geq n + p),
\]  \hspace{1cm} (5.8)
which shows that \( \Re(\omega(z)) > 0 \) (\( z \in \mathcal{U} \)). From (5.7), we immediately obtain the inequality (5.3).

To see that the function \( f \) given by (5.5) gives the sharp result, we observe for \( z \to 1^- \) that
\[
\frac{f(z)}{\kappa_m(z)} = 1 - \frac{1}{\theta_{m+1}} z^{m-p+1} \to 1 - \frac{1}{\theta_{m+1}},
\]
which shows that the bound in (5.3) is the best possible.

Similarly, if we put
\[
\phi(z) = (1 + \theta_{m+1}) \left[ \frac{\kappa_m(z)}{f(z)} - \frac{\theta_{m+1}}{1 + \theta_{m+1}} \right]
\]  \hspace{1cm} (5.9)
\[
= 1 + \frac{(1 + \theta_{m+1}) \sum_{k=m+1}^{\infty} |a_k| z^{k-p}}{1 - \sum_{k=n+p}^{m} |a_k| z^{k-p}},
\]
and make use of (5.6), we can deduce that
\[
\frac{\phi(z) - 1}{\phi(z) + 1} = \frac{(1 + \theta_{m+1}) \sum_{k=m+1}^{\infty} |a_k| z^{k-p}}{2 - 2 \sum_{k=n+p}^{m} |a_k| z^{k-p} + \theta_{m+1} - 1) \sum_{k=m+1}^{\infty} |a_k| z^{k-p}} \leq 1 \quad (z \in \mathcal{U}; \; k \geq n + p),
\]  \hspace{1cm} (5.10)
which leads us immediately to assertion (5.4) of the theorem.

The bound in (5.4) is sharp with the extremal function given by (5.5). The proof of theorem is thus completed.

6. PROPERTIES ASSOCIATED WITH QUASI-COVOLUTION

In this part, we establish certain results concerning the Quasi-covolution of function is in the class \( \overline{\mathcal{P}}_{\lambda, \mu, l}^\delta (A, B; \sigma, p) \). For the functions \( f_j(z) \in \mathcal{A}(n, p) \) given by
\[
f_j(z) = z^p - \sum_{k=n+p}^{\infty} |a_{k,j}| z^k \quad (j = 1, m, \; p \in \mathbb{N}),
\]  \hspace{1cm} (6.1)
we denote by \( (f_1 \ast f_2)(z) \) the Quasi-covolution of functions \( f_1(z) \) and \( f_2(z) \), that is,
\[
(f_1 \ast f_2)(z) = z^p - \sum_{k=n+p}^{\infty} |a_{k,1}| |a_{k,2}| z^k.
\]  \hspace{1cm} (6.2)

Theorem 6.1. If \( f_j(z) \in \overline{\mathcal{P}}_{\lambda, \mu, l}^\delta (A, B; \sigma_j, p) \) (\( j = 1, m \)), then
\[
(f_1 \ast f_2 \ast \cdots \ast f_m)(z) \in \overline{\mathcal{P}}_{\lambda, \mu, l}^\delta (A, B; \Upsilon, p),
\]  \hspace{1cm} (6.3)
where
\[
\Upsilon := p - \frac{\prod_{j=1}^{m} (B - A)(p - \sigma_j)}{(B - A)((n + p)(1 + B)\Phi_p^{n+p}(\delta, \lambda, \mu, l))^{m-1}}.
\]  \hspace{1cm} (6.4)
The result is sharp for the functions \( f_j(z) \) given by
\[
f_j(z) = z^p - \frac{(B - A)(p - \sigma_j)}{(n + p)(1 + B)\Phi_p^{n+p}(\delta, \lambda, \mu, l)} z^{p+n} \quad (j = 1, m),
\]  \hspace{1cm} (6.5)
Proof. For \( m = 1 \), we see that \( \Upsilon = \sigma_1 \). For \( m = 2 \), Theorem 3.1 gives

\[
(6.6) \quad \sum_{k=n+p}^{\infty} \frac{k(1+B)\Phi^k_p(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma_j)} |a_{k,j}| \leq 1 \quad (j = 1, 2).
\]

Therefore, by the Cauchy-Schwarz inequality, we obtain

\[
(6.7) \quad \sum_{k=n+p}^{\infty} \frac{k(1+B)\Phi^k_p(\delta, \lambda, \mu, l)}{\sqrt{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}} |a_{k,1}| |a_{k,2}| \leq 1.
\]

To prove the case when \( m = 2 \), we have to find the largest \( \Upsilon \) such that

\[
(6.8) \quad \sum_{k=n+p}^{\infty} \frac{k(1+B)\Phi^k_p(\delta, \lambda, \mu, l)}{(B-A)(p-\Upsilon)} |a_{k,1}| |a_{k,2}| \leq 1,
\]
or such that

\[
(6.9) \quad \frac{|a_{k,1}| |a_{k,2}|}{(B-A)(p-\Upsilon)} \leq \frac{\sqrt{|a_{k,1}| |a_{k,2}|}}{\sqrt{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}}.
\]

this, equivalently, that

\[
(6.10) \quad \sqrt{|a_{k,1}| |a_{k,2}|} \leq \frac{(B-A)(p-\Upsilon)}{\sqrt{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}}.
\]

Further, by using (6.7), we need to find the largest \( \Upsilon \) such that

\[
\frac{\sqrt{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}}{k(1+B)\Phi^k_p(\delta, \lambda, \mu, l)} \leq \frac{(B-A)(p-\Upsilon)}{\sqrt{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}}
\]
or, equivalently, that

\[
(6.11) \quad \frac{1}{(B-A)(p-\Upsilon)} \leq \frac{k(1+B)\Phi^k_p(\delta, \lambda, \mu, l)}{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}.
\]

It follows from (6.9) that

\[
(6.12) \quad \Upsilon \leq p - \frac{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}{(B-A)k(1+B)\Phi^k_p(\delta, \lambda, \mu, l)}.
\]

Now, defining the function \( \psi(k) \) by

\[
(6.13) \quad \psi(k) = p - \frac{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}{(B-A)k(1+B)\Phi^k_p(\delta, \lambda, \mu, l)},
\]

we see that \( \psi'(k) \geq 0 \) for \( k \geq p + n \). This implies that

\[
\Upsilon \leq \psi(n+p) = p - \frac{\prod_{j=1}^{2}(B-A)(p-\sigma_j)}{(B-A)(n+p)(1+B)\Phi^{n+p}_p(\delta, \lambda, \mu, l)}.
\]

Therefore, the result is true for \( m = 2 \).

Suppose that the result is true for any positive integer \( m \). Then we have \((f_1 \bullet f_2 \bullet \cdots \bullet f_m \bullet f_{m+1})(z) \in \mathcal{F}^k_{\delta, \lambda, \mu, l}(A, B; \gamma, p)\), when

\[
\gamma = p - \frac{(B-A)(p-\Upsilon)(B-A)(p-\sigma_{m+1})}{(B-A)(n+p)(1+B)\Phi^{n+p}_p(\delta, \lambda, \mu, l)}
\]

where \( \Upsilon \) is given by (6.4). After a simple calculation, we have

\[
\gamma \leq p - \frac{\prod_{j=1}^{m+1}(B-A)(p-\sigma_j)}{(B-A)(n+p)(1+B)\Phi^{n+p}_p(\delta, \lambda, \mu, l)^m}.
\]

Thus, the result is true for \( m + 1 \). Therefore, by using the mathematical induction, we conclude that the result is true for any positive integer \( m \).

Finally, taking the functions \( f_j(z) \) defined by (6.5), we have

\[
(f_1 \bullet f_2 \bullet \cdots \bullet f_m)(z) = z^p - \left\{ \prod_{j=1}^{m} \frac{(B-A)(p-\sigma_j)}{(p+n)(1+B)\Phi^{p+n}_p(\delta, \lambda, \mu, l)} \right\} z^{p+n}
\]

\[
= z^p - \Lambda_{p+n} z^{p+n},
\]
which shows that
\[
\sum_{k=p+n}^{\infty} \frac{k(1+B)\Phi^k_p(\delta, \lambda, \mu, l)}{(B-A)(p-\bar{\chi})} A_k = \frac{(n+p)(1+B)\Phi^{n+p}_p(\delta, \lambda, \mu, l)}{(B-A)(p-\bar{\chi})} A_{p+n}
\]
\[
= \frac{(n+p)(1+B)\Phi^{n+p}_p(\delta, \lambda, \mu, l)}{(B-A)(p-\bar{\chi})} \times \left\{ \prod_{j=1}^{2} \frac{(B-A)(p-\sigma_j)}{(p+n)(1+B)\Phi^{p+n}_p(\delta, \lambda, \mu, l)} \right\}.
\]

Consequently, the result is sharp.

Putting \( \sigma_j = \sigma \) (\( j = \overline{1,m} \)) in Theorem 6.1, we have;

**Corollary 6.2.** If \( f_j(z) \in \overline{P}_{\lambda,\mu,l}(A, B; \sigma, p) \) (\( j = \overline{1,m} \)), then
\[
(f_1 \cdot f_2 \cdot \cdots \cdot f_m)(z) \in \overline{P}_{\lambda,\mu,l}(A, B; \bar{\chi}, p),
\]
where
\[
\bar{\chi} := p - \frac{[(B-A)(p-\sigma)]^m}{(B-A)[(n+p)(1+B)\Phi^{n+p}_p(\delta, \lambda, \mu, l)]^{m-1}}.
\]
The result is sharp for the functions \( f_j(z) \) given by
\[
f_j(z) = \frac{(B-A)(p-\sigma)}{(n+p)(1+B)\Phi^{p+n}_p(\delta, \lambda, \mu, l)} z^{p+n} \quad (j = \overline{1,m}).
\]

**Remark 6.1.** For special values of parameters \( \lambda, \mu, l, \delta, \sigma, A, B, n \) and \( p \), our results reduce to several well-known results as follows:

1. Putting \( A = -1, B = 1, \delta = 0 \) and \( m = 2 \) in Theorem 6.1, we obtain the corresponding results of Yaguchi et al. [55] and Aouf and Darwish [7] for \( n = 1 \).
2. Putting \( A = -1, B = 1, \delta = 0 \) and \( m = 2 \) in Corollary 6.2, we obtain the corresponding results of Lee et al. [26] and for \( n = 1 \) and Sekine and Owa [44] for \( p = 1 \).
3. Putting \( A = -1, B = 1, \delta = 0 \) and \( m = 3 \) in Corollary 6.2, we obtain the corresponding result due to Aouf and Darwish [7] for \( n = 1 \).
4. Putting \( A = -\beta, B = \beta, \delta = 0 \) and \( \sigma = \alpha \) in Theorem 6.1, we obtain the corresponding result due to Aouf [5].

**Theorem 6.2.** Let the function \( f_j(z) \) (\( j = \overline{1,m} \)) given by (6.1) be in the class \( \overline{P}_{\lambda,\mu,l}(A, B; \sigma_j, p) \). Then the function
\[
h(z) = z^p - \sum_{k=n+p}^{\infty} \left( \sum_{j=1}^{m} |a_{k,j}|^2 \right) z^k
\]
belongs to the class \( \overline{P}_{\lambda,\mu,l}(A, B; \chi, p) \), where
\[
\chi := p - \frac{m(B-A)(p-\sigma^*)^2}{(n+p)(1+B)\Phi^{n+p}_p(\delta, \lambda, \mu, l)} \quad (\sigma^* := \min\{\sigma_1, \sigma_2, \ldots, \sigma_m\}).
\]
The result is sharp for the functions \( f_j(z) \) (\( j = \overline{1,m} \)) given by (6.5).

**Proof.** By virtue of Theorem 3.1 we have
\[
\sum_{k=p+n}^{\infty} \left\{ \frac{k(1+B)\Phi^k_p(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma_j)} \right\}^2 |a_{k,j}|^2 \leq \left\{ \sum_{k=p+n}^{\infty} \frac{k(1+B)\Phi^k_p(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma_j)} |a_{k,j}| \right\}^2 \leq 1.
\]

Then it follows that for \( j = \overline{1,m} \),
\[
\frac{1}{m} \sum_{k=p+n}^{\infty} \left\{ \frac{k(1+B)\Phi^k_p(\delta, \lambda, \mu, l)}{(B-A)(p-\sigma_j)} \right\}^2 \left( \sum_{j=1}^{m} |a_{k,j}|^2 \right) \leq 1.
\]

Therefore, we need to find the largest \( \chi \) such that
\[
\frac{1}{m} \sum_{k=p+n}^{\infty} \left\{ \frac{k(1+B)\Phi^k_p(\delta, \lambda, \mu, l)}{(B-A)(p-\chi)} \right\}^2 \left( \sum_{j=1}^{m} |a_{k,j}|^2 \right) \leq 1.
\]
This implies that
\[(6.19) \quad \chi \leq p - \frac{m(B - A)(p - \sigma^*)^2}{k(1 + B)\Phi_p^k(\delta, \lambda, \mu, l)} \quad (\sigma^* := \min\{\sigma_1, \sigma_2, \ldots, \sigma_m\}, \ k \geq p + n).\]
Now, defining the function \(\Im(k)\) by
\[(6.20) \quad \Im(k) := p - \frac{m(B - A)(p - \sigma^*)^2}{k(1 + B)\Phi_p^k(\delta, \lambda, \mu, l)},\]
we see that \(\Im(k)\) is an increasing function of \(k, \ k \geq p + n\). Setting \(k = p + n\) in (6.19) we have
\[\chi \leq \Im(n + p) := p - \frac{m(B - A)(p - \sigma^*)^2}{(n + p)(1 + B)\Phi_p^{n+p}(\delta, \lambda, \mu, l)}\]
which completes the proof of Theorem 6.2.

Setting \(\sigma_j = \sigma (j = 1, m)\), in Theorem 6.2, we arrive at the following result.

Corollary 6.3. Let the functions \(f_j(z) (j = 1, m)\) given by (6.1) be in the class \(\widetilde{P}_{\lambda, \mu, l}^\delta(A, B; \sigma, p)\). Then the function
\[h(z) = z^\rho - \sum_{p = n + p}^\infty \left( \sum_{j=1}^m |a_{k,j}|^2 \right) z^k\]
belongs to the class \(\widetilde{P}_{\lambda, \mu, l}^\delta(A, B; \chi, p)\), where
\[\chi := p - \frac{m(B - A)(p - \sigma)^2}{(n + p)(1 + B)\Phi_p^{n+p}(\delta, \lambda, \mu, l)}\]

The result is sharp for the functions \(f_j(z) (j = 1, m)\) given by (6.5).

Remark 6.2.

(1) Putting \(A = -1, B = 1, \delta = 0\) and \(m = 2\) in Theorem 6.2, we obtain the corresponding results of Yaguchi et al. [55].

(2) Putting \(A = -1, B = 1, \delta = 0\) and \(m = 2\) in Corollary 6.3, we obtain the corresponding results of Aouf and Darwish [7] for \(n = 1\), Sekine and Owa [44] for \(p = 1\).

(3) Putting \(A = -\beta, B = \beta, \delta = 0\) and \(\sigma = \alpha\) in Theorem 6.2, we obtain the corresponding result due to Aouf [5].

7. APPLICATIONS OF FRACTIONAL CALCULUS OPERATORS

Various operators of fractional calculus (that is, fractional integral and fractional derivatives) have been studied in the literature rather extensively (cf., e.g., [37, 50, 53]; see also [14, 51] the various references cited therein). For our present investigation, we recall the following definitions.

Definition 7.1. Let \(f(z)\) be analytic in a simply connected region of the \(z\)-plane containing the origin. The fractional integral of \(f\) of order \(\nu\) is defined by
\[(7.1) \quad D_z^{-\nu}f(z) = \frac{1}{\Gamma(\nu)} \int_0^z f(\zeta) (z - \zeta)^{1-\nu} d\zeta \quad (\nu > 0),\]
where the multiplicity of \((z - \zeta)^{-\nu}\) is removed by requiring that \(\log(z - \zeta)\) is real for \(z - \zeta > 0\).

Definition 7.2. Let \(f(z)\) be analytic in a simply connected region of the \(z\)-plane containing the origin. The fractional derivative of \(f\) of order \(\nu\) is defined by
\[(7.2) \quad D_z^\nu f(z) = \frac{1}{\Gamma(1-\nu)} \int_0^z f(\zeta) (z - \zeta)^{-\nu} d\zeta \quad (0 \leq \nu < 1),\]
where the multiplicity of \((z - \zeta)^{-\nu}\) is removed by requiring that \(\log(z - \zeta)\) is real for \(z - \zeta > 0\).

Definition 7.3. Under the hypotheses of Definition 7.2, the fractional derivative of order \(n + \nu\) is defined, for a function \(f(z)\), by
\[(7.3) \quad D_z^{n+\nu} f(z) = \frac{d^n}{dz^n} (D_z^\nu f(z)) \quad (0 \leq \nu < 1; \ n \in \mathbb{N}_0).\]

In this section, we shall investigate the growth and distortion properties of functions in the class \(\widetilde{P}_{\lambda, \mu, l}^\delta(A, B; \sigma, p)\), which involving the operators \(I_{\sigma, p}\) and \(D_z^\nu\). In order to derive our results, we need the following lemma given by Chen et al. [14].

Lemma 7.1 (see [14]). Let the function \(f(z)\) defined by (1.10). Then
\[(7.4) \quad D_z^\nu (I_{\sigma, p} f)(z) = \frac{\Gamma(p + 1)}{\Gamma(p + 1 - \nu)} z^{p-\nu} - \sum_{k=n+p}^\infty \frac{(\sigma + p)\Gamma(k + 1)}{(\sigma + k)\Gamma(1 - \nu)} a_k z^{k-\nu}\]
Thus, by using (7.9) and (7.11), for all $z \in \mathbb{C}$ such that $\Re(z) > -p$, we have
\begin{equation}
\mathcal{I}_{\vartheta,p}\{(D^\nu_z f)(z)\} = \frac{(\vartheta + p)\Gamma(p + 1)}{(\vartheta + p - \nu)\Gamma(p + 1 - \nu)z^{p-\nu}} - \sum_{k=n+p}^{\infty} \frac{(\vartheta + p)\Gamma(k + 1)}{(\vartheta + k - \nu)\Gamma(k + 1 - \nu)}a_k z^{k-\nu}
\end{equation}
(\nu \in \mathbb{R}; \vartheta > -p; p,n \in \mathbb{N})

provided that no zeros appear in the denominators in (7.4) and (7.5).

**Theorem 7.1.** Let the functions $f(z)$ defined by (1.10) be in the class $\overline{P}_{\lambda,\mu,l}(A,B;\sigma,p)$. Then
\begin{equation}
|D^\nu_z\{(I_{\vartheta,p} f)(z)\}| \geq \left\{ \Gamma(p + 1) \frac{(\vartheta + p)\Gamma(n + p + 1)(B - A)(p - \sigma)}{(\vartheta + n + p\Gamma(n + p + 1 + \nu)(n + p)(1 + B)\Phi^{n+p}_{p}(\delta,\lambda,\mu,l)} |z|^{n} \right\} |z|^{\nu + p}
\end{equation}
(\nu \in \mathbb{C}; \nu > 0; \vartheta > -p; p,n \in \mathbb{N})

and
\begin{equation}
|D^\nu_z\{(I_{\vartheta,p} f)(z)\}| \leq \left\{ \Gamma(p + 1) \frac{(\vartheta + p)\Gamma(n + p + 1)(B - A)(p - \sigma)}{(\vartheta + n + p\Gamma(n + p + 1 + \nu)(n + p)(1 + B)\Phi^{n+p}_{p}(\delta,\lambda,\mu,l)} |z|^{n} \right\} |z|^{\nu + p}
\end{equation}
(\nu \in \mathbb{C}; \nu > 0; \vartheta > -p; p,n \in \mathbb{N}).

Each of the assertions (7.6) and (7.7) is sharp.

**Proof.** In view of Theorem 3.1, we have
\begin{equation}
\frac{(n + p)(1 + B)\Phi^{n+p}_{p}(\delta,\lambda,\mu,l)}{(B - A)(p - \sigma)} \sum_{k=n+p}^{\infty} |a_k| \leq \sum_{k=n+p}^{\infty} \frac{k(1 + B)\Phi^{k}_{p}(\delta,\lambda,\mu,l)}{(B - A)(p - \sigma)} |a_k| \leq 1,
\end{equation}
which readily yields
\begin{equation}
\sum_{k=n+p}^{\infty} |a_k| \leq \frac{(B - A)(p - \sigma)}{(n + p)(1 + B)\Phi^{n+p}_{p}(\delta,\lambda,\mu,l)}.
\end{equation}

Consider the function $F(z)$ defined in $U$ by
\begin{equation}
F(z) = \frac{\Gamma(p + 1 - \nu)}{\Gamma(p + 1)}z^{-\nu}D^\nu_z\{(I_{\vartheta,p} f)(z)\} = z^p - \sum_{k=n+p}^{\infty} \frac{(\vartheta + p)\Gamma(k + 1)\Gamma(p + 1 + \nu)}{(\vartheta + k - \nu)\Gamma(k + 1 + \nu)} |a_k| z^{k}
\end{equation}
\begin{equation}
= z^p - \sum_{k=n+p}^{\infty} \Theta(k) |a_k| z^{k} \quad (z \in U)
\end{equation}
where
\begin{equation}
\Theta(k) := \frac{(\vartheta + p)\Gamma(k + 1)\Gamma(p + 1 + \nu)}{(\vartheta + k - \nu)\Gamma(k + 1 + \nu)} \quad (k \geq p + n; p,n \in \mathbb{N}; \nu > 0).
\end{equation}

Since $\Theta(k)$ is a decreasing function of $k$ when $\nu > 0$, we get
\begin{equation}
0 < \Theta(k) \leq \Theta(n + p) = \frac{(\vartheta + p)\Gamma(n + p + 1)\Gamma(p + 1 + \nu)}{(\vartheta + n + p\Gamma(n + p + 1 + \nu)}\Gamma(p + 1)} \quad (\nu > 0; \vartheta > -p; p,n \in \mathbb{N})
\end{equation}

Thus, by using (7.9) and (7.11), for all $z \in U$, we deduce that
\begin{equation}
|F(z)| \geq |z|^p - \Theta(n + p) |z|^{n+p} \sum_{k=n+p}^{\infty} |a_k|
\end{equation}
\begin{equation}
\geq |z|^p - \frac{(\vartheta + p)\Gamma(n + p + 1)\Gamma(p + 1 + \nu)(B - A)(p - \sigma)}{(\vartheta + n + p\Gamma(n + p + 1 + \nu)}\Gamma(p + 1)(n + p)(1 + B)\Phi^{n+p}_{p}(\delta,\lambda,\mu,l)} |z|^{n+p}
\end{equation}
and

\[ |\mathcal{F}(z)| \leq |z|^p + \Theta(n + p)|z|^{n+p} \sum_{k=n+p}^{\infty} |a_k| \leq |z|^p + \frac{(\vartheta + p)\Gamma(n + p + 1)\Gamma(p + 1 + \nu)(B - A)(p - \sigma)}{(\vartheta + n + p)\Gamma(n + p + 1 + \nu)\Gamma(p + 1)(n + p)(1 + B)\Phi^{n+p}_p(\delta, \lambda, \mu, l)}|z|^{n+p} \]

which yield the inequalities (7.6) and (7.7) of Theorem 7.1. Equalities in (7.6) and (7.7) are attained for the function \(f(z)\) given by

\[
\mathcal{D}_z^{-\nu}\{\mathcal{I}_{\varphi,p}f(z)\} = \left\{ \frac{\Gamma(p + 1)}{\Gamma(p + 1 - \nu)} - \frac{(\vartheta + p)\Gamma(n + p + 1)(B - A)(p - \sigma)}{(\vartheta + n + p)\Gamma(n + p + 1 + \nu)(n + p)(1 + B)\Phi^{n+p}_p(\delta, \lambda, \mu, l)}z^n \right\}z^{p+\nu} \]

or, equivalently,

\[
\mathcal{I}_{\varphi,p}f(z) = z^p - \frac{(\vartheta + p)(B - A)(p - \sigma)}{(\vartheta + n + p)(n + p)(1 + B)\Phi^{n+p}_p(\delta, \lambda, \mu, l)}z^{n+p}. \]

Thus, we complete the proof of Theorem 7.1.

**Theorem 7.2.** Let the functions \(f(z)\) defined by (1.10) be in the class \(\mathcal{P}^\delta_{\lambda,\mu,l}(A, B; \sigma, p)\). Then

\[
|\mathcal{D}_z^{\nu}\{\mathcal{I}_{\varphi,p}f(z)\}| \geq \left\{ \frac{\Gamma(p + 1)}{\Gamma(p + 1 - \nu)}z^{p+\nu} - \frac{(\vartheta + p)\Gamma(n + p)(B - A)(p - \sigma)}{(\vartheta + n + p)\Gamma(n + p + 1 + \nu)(n + p)(1 + B)\Phi^{n+p}_p(\delta, \lambda, \mu, l)}|z|^n \right\}z^{p+\nu} \tag{7.12}
\]

and

\[
|\mathcal{D}_z^{\nu}\{\mathcal{I}_{\varphi,p}f(z)\}| \leq \left\{ \frac{\Gamma(p + 1)}{\Gamma(p + 1 - \nu)}z^{p+\nu} + \frac{(\vartheta + p)\Gamma(n + p)(B - A)(p - \sigma)}{(\vartheta + n + p)\Gamma(n + p + 1 + \nu)(n + p)(1 + B)\Phi^{n+p}_p(\delta, \lambda, \mu, l)}|z|^n \right\}z^{p+\nu} \tag{7.13}
\]

Each of the assertions (7.12) and (7.13) is sharp.

**Proof.** It follows from Theorem 3.1 that

\[
\sum_{k=n+p}^{\infty} k |a_k| \leq \frac{(B - A)(p - \sigma)}{(1 + B)\Phi^{n+p}_p(\delta, \lambda, \mu, l)}. \tag{7.14}
\]

Consider the function \(Q(z)\) defined in \(\mathcal{U}\) by

\[
Q(z) = \frac{\Gamma(p + 1 - \nu)}{\Gamma(p + 1)} z^\nu \mathcal{D}_z^{\nu}\{\mathcal{I}_{\varphi,p}f(z)\} = z^p - \sum_{k=n+p}^{\infty} \frac{(\vartheta + p)\Gamma(k)(p + 1 - \nu)}{(\vartheta + k)\Gamma(k + 1 - \nu)\Gamma(p + 1)} k |a_k| z^k = z^p - \sum_{k=n+p}^{\infty} \varphi(k)k |a_k| z^k \quad (z \in \mathcal{U})
\]

where, for convenience,

\[
\varphi(k) := \frac{(\vartheta + p)\Gamma(k)(p + 1 - \nu)}{(\vartheta + k)\Gamma(k + 1 - \nu)\Gamma(p + 1)} \quad (k \geq p + n; p, n \in \mathbb{N}; 0 \leq \nu < 1). \tag{7.15}
\]

Since \(\varphi(k)\) is a decreasing function of \(k\) when \(0 \leq \nu < 1\), we find that

\[
0 < \varphi(k) \leq \varphi(n + p) = \frac{(\vartheta + p)\Gamma(n + p)(p + 1 - \nu)}{(\vartheta + n + p)\Gamma(n + p + 1 - \nu)\Gamma(p + 1)} \quad (0 \leq \nu < 1; \vartheta > -p; p, n \in \mathbb{N}). \tag{7.16}
\]
Hence, with the aid of (7.14) and (7.16), for all \( z \in \mathcal{U} \), we have

\[
|Q(z)| \geq |z|^p - \varphi(n + p) |z|^{n+p} \sum_{k=n+p}^{\infty} k |a_k| \\
\geq |z|^p - \frac{(\vartheta + p) \Gamma(n + p) \Gamma(p + 1 - \nu)(B - A)(p - \sigma)}{(\vartheta + n + p) \Gamma(n + p + 1 - \nu)(1 + B) \Phi_{\nu + \varphi}^{n+p}(\delta, \lambda, \mu, l)} \left|z\right|^{n+p}
\]

and

\[
|Q(z)| \leq |z|^p + \varphi(n + p) |z|^{n+p} \sum_{k=n+p}^{\infty} k |a_k| \\
\leq |z|^p + \frac{(\vartheta + p) \Gamma(n + p) \Gamma(p + 1 - \nu)(B - A)(p - \sigma)}{(\vartheta + n + p) \Gamma(n + p + 1 - \nu)(1 + B) \Phi_{\nu + \varphi}^{n+p}(\delta, \lambda, \mu, l)} \left|z\right|^{n+p}
\]

which yield the inequalities (7.15) and (7.16) of Theorem 7.2. Equalities in (7.15) and (7.16) are attained for the function \( f(z) \) given by

\[
D_\nu^\nu \{(I_{\vartheta,p}f)(z)\} = \left\{ \frac{\Gamma(p + 1)}{\Gamma(p + 1 - \nu)} \left( (\vartheta + p) \Gamma(n + p + 1)(B - A)(p - \sigma) \right) \right\} \left( (\vartheta + n + p)(n + p + 1 - \nu)(1 + B) \Phi_{\nu + \varphi}^{n+p}(\delta, \lambda, \mu, l) \right) \left|z\right|^{n+p}
\]

or, equivalently, by

\[
(I_{\vartheta,p}f)(z) = z^p - \frac{(\vartheta + p)(B - A)(p - \sigma)}{(\vartheta + n + p)(n + p + 1 - \nu)(1 + B) \Phi_{\nu + \varphi}^{n+p}(\delta, \lambda, \mu, l)} \left|z\right|^{n+p}.
\]

Consequently, we complete the proof of Theorem 7.2.

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