Simultaneous amplitude and phase damping of a kind of Gaussian states and their separability

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Abstract
We give out the time evolution solution of simultaneous amplitude and phase damping for any continuous variable state. For the simultaneous amplitude and phase damping of a wide class of two-mode entangled Gaussian states, two analytical conditions of the separability are given. One is the sufficient condition of separability. The other is the condition of PPT separability where the Peres-Horodecki criterion is applied. Between the two conditions there may exist bound entanglement. The simplest example is the simultaneous amplitude and phase damping of a two-mode squeezed vacuum state. The damped state is non-Gaussian.
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1 Introduction
Quantum entanglement or inseparability plays a major role in all branches of quantum information and quantum computation. Peres\[1\] proposed a criterion for checking the inseparability of a state by introducing the partial transpose operation. This condition is necessary and sufficient for some lower dimensional discrete bipartite systems but is no longer sufficient for higher dimensions\[2\]. Despite many studies on the discrete states, much attention has been paid to the continuous variable states \[3\]. Recently, quantum teleportation of coherent states has been experimentally realized by exploiting a two-mode squeezed vacuum state as an entanglement resource\[4\]. Due to the decoherence of the environment, a pure entanglement state will become mixed. Thus it is important to know if a given bipartite continuous variable state is entangled or not. The decoherence may be caused by coupling to the thermal noise of the environment, amplitude damping, quantum dissipation and phase damping. Besides the phase damping, the other three types of decoherence preserve gaussian property of the state, and a two-mode squeezed vacuum state will evolve to a two mode gaussian mixed state. For the separability of two mode gaussian state, the positivity of the partially transposed state is necessary and sufficient \[5\]\[6\]\[7\]. However, a gaussian state will evolve to a non-gaussian one by phase damping, and the case of a two mode squeezed vacuum state under the only decoherence of phase damping was perfectly solved\[8\]. In real experiments, the general situation which should be taken into account is the coexistence of noise, amplitude and phase damping. Theoretically, the separability and entanglement of non-gaussian state are seldom investigated, here we provide an example.

2 Time Evolution of Characteristic Function
Considering the simultaneous damping, the density matrix obeys the following master equation in the interaction picture $\frac{d\rho}{dt} = (\mathcal{L}_1 + \mathcal{L}_2)\rho$. Where $\mathcal{L}_1$ is the amplitude damping part

$$\mathcal{L}_1\rho = \sum_i \frac{\Gamma_i}{2}[\{(\bar{n}_i + 1)(2a_i^\dagger a_i^\dagger \rho - a_i^\dagger a_i \rho - \rho a_i^\dagger a_i) + \bar{n}_i(2a_i^\dagger \rho a_i - a_i a_i^\dagger \rho - \rho a_i a_i^\dagger]\}],$$

with $\bar{n}$ the average photon number of the thermal environment. And $\mathcal{L}_2$ is the phase damping part (e.g. \[8\]),

$$\mathcal{L}_2 = \sum_i \frac{\gamma_i}{2}[2a_i^\dagger a_i \rho a_i^\dagger a_i - (a_i^\dagger a_i)^2 \rho - \rho (a_i^\dagger a_i)^2].$$
The state can be equivalently specified by its characteristic function. Every operator \( A \in \mathcal{B}(\mathcal{H}) \) is completely determined by its characteristic function \( \chi_A := \text{tr}[AD(\mu)] \), where \( D(\mu) = \exp(\mu^+ - \mu^* a) \) is the displacement operator, with \( \mu = [\mu_1, \mu_2, \cdots, \mu_s]^T \), \( a = [a_1, a_2, \cdots, a_s] \) and the total number of modes is \( s \). It follows that \( A \) may be written in terms of \( \chi_A \) as \( A = \int \prod_i \frac{d^2 \mu_i}{2 \pi} |\chi_A(\mu)|D(\mu) \). The density matrix \( \rho \) can be expressed with its characteristic function \( \chi \). The amplitude damping equation of \( \chi \) and its solution are well known, they are \( \frac{\partial \chi}{\partial t} = -\frac{1}{2} \sum_i \Gamma_i (|\mu_i|^2 - |\mu_i|^2) \), \( \chi(\mu, t) = \chi(\rho e^{-\Gamma_1 t/2}, 0) \exp[-\sum_i (|\mu_i|^2 + \frac{1}{2})(1 - e^{-\Gamma_i t}) |\mu_i|^2] \). We now give out the phase damping equation of \( \chi \), it will be

\[
\frac{\partial \chi}{\partial t} = \frac{1}{2} \sum_i \gamma_i \partial^2 \chi \partial \theta_i^2
\]

if we denote \( \mu_i \) as \( |\mu_i| e^{i \theta_i} \). We can see that with the characteristic function the amplitude damping equation is described by the amplitude of the parameter \( \mu_i \), the phase damping equation is described by the phase of the parameter \( \mu_i \). The solution to the phase equation of \( \chi \) then will be \( \chi(\mu, \mu^*, t) = \prod_i (2 \pi \gamma_i t)^{-1/2} \int \exp(-\sum_i \frac{x_i^2}{2 \gamma_i t}) \chi(\mu e^{ix_1}, \mu^* e^{-ix_1}, 0) dx \), where \( \mu e^{ix} \) is the abbreviation of \( [\mu_1 e^{ix_1}, \mu_2 e^{ix_2}, \cdots, \mu_s e^{ix_s}] \). The simultaneous amplitude and phase damping to any initial characteristic function then will be

\[
\chi(\mu, \mu^*, t) = \prod_i (2 \pi \gamma_i t)^{-1/2} \int \exp(-\sum_i \frac{x_i^2}{2 \gamma_i t} + (\pi_i + \frac{1}{2})(1 - e^{-\Gamma_i t}) |\mu_i|^2) \chi(\mu e^{-\Gamma_1 t/2 + ix_1}, \mu^* e^{-\Gamma_2 t/2 - ix_1}, 0) dx.
\]

The density matrix then can be obtained as well by making use of operator integral.

We will concentrate on the simultaneous amplitude and phase damping of two-mode \( x-p \) symmetric Gaussian state \( |\Psi \rangle \) whose characteristic function is \( \chi(\mu, 0) = \exp[-(A_{10} |\mu_1|^2 + A_{20} |\mu_2|^2) + B_0 \mu_1 \mu_2 + B'_0 \mu_1^* \mu_2^*] \). The time evolution of \( \chi \) is

\[
\chi(\mu, \mu^*, t) = \int_x \exp[-(A'_{10} |\mu_1|^2 + A'_{20} |\mu_2|^2) + B \mu_1 \mu_2 e^{it} + B' \mu_1^* \mu_2^* e^{-it}].
\]

Where \( A'_{10} = e^{-\Gamma_1 t} A_{10} + (\pi_i + \frac{1}{2})(1 - e^{-\Gamma_i t}), B_{10} = e^{-\frac{t}{2} (\Gamma_1 + \Gamma_2)} B_0, \pi = \frac{1}{2} (\gamma_1 + \gamma_2) \). The integral on \( x \) is one fold, for simplicity, we denote \( \int \exp(-\sum \frac{x_i^2}{2 \gamma_i}) f(x) dx = \int f(x) \). The \( x-p \) symmetric Gaussian state set is a quite large set. It contains two-mode squeezed vacuum state \( |\Psi \rangle = \frac{1}{\cosh r} \sum_n \mid n, n \rangle (r \) is the squeezing parameter) and two-mode squeezed thermal state \( \rho_{\text{PT}} \) as its special cases, with \( A_{10} = A_{20} = (n_0 + \frac{1}{2}) \cosh 2r, B_0 = (n_0 + \frac{1}{2}) \sinh 2r \) respectively.

3 PPT separability

For the sake of simplicity in description, let us firstly consider the situation of \( A'_{10} = A'_{20} = A', B = B' \). The general case will be obtained straightforward and be described at the end of this section. The first question is that if the state after damping is entangled or not. Then how much is the entanglement left? The necessary condition of a bipartite state being entangled is that the partial transpose of the density operator is not positive definite \( \rho_{\text{PT}} \). The partial transpose operation changes the characteristic function in the fashion of: \( \chi(\mu_1, \mu_2) \rightarrow \chi(\mu_1, -\mu_2^*) = \chi_{\text{PT}}(\mu_1, \mu_2) \). For the separability of a non-gaussian bipartite state, a necessary condition was proposed by Simon \( \rho_{\text{PT}} \) in terms of the second moment of the state. In the original literature canonical operators were used, here we use creation and annihilation operators instead. The necessary condition comes from the non-negativity of \( \rho_{\text{PT}} \) and the commutation relations. For any \( Q = \eta \eta^+ \) with \( \eta = c_1 a_1 + c_2 a_2 + c_3 a_1^+ + c_4 a_2^+ \) of every set of complex coefficients \( c_0 \), one has \( \langle Q \rangle = \text{tr}(Q \rho_{\text{PT}}) \geq 0 \). Hence the second moment matrix of \( \rho_{\text{PT}} \) should be semi-positive definite. The second moment such as \( \text{tr}(\rho_{\text{PT}} a_i^+ a_j) \) can be obtained from the second derivative of \( \chi \) with respect to \( \mu_i \) and \( \mu_j^* \), we have

\[
A - 1 \geq Be^{-\gamma t}
\]

Where \( A = A' + \frac{1}{2} \). Other necessary conditions may come from when \( \eta \) is the linear combination of higher power of the creation and annihilation operators, and they may be tighter than Ineq. (6). And this is really the case.
We will find a tighter condition by exploring the negative eigenvalues of the partial transpose of the density operator. The eigenvalue of \( \rho^{\text{PT}} \) can be simplified as the eigenvalues of a series of matrices (see Appendix).

\[
M^{(m)}_n = (A^2 - B^2)^{-1} C^m \sum_k \binom{m-n}{l-k} \binom{n}{k} (D/C)^l e^{-\gamma t(n-l)^2},
\]

where \( C = 1 - \frac{A}{B} \), \( D = \frac{B}{A} \). When \( m = 0 \), one has the first eigenvalue \( \lambda^{(0)} = (A^2 - B^2)^{-1} \) which is always positive. The matrix \( M^{(m)} \) possesses the symmetry of \( M^{(m)}_n = M^{(m)}_{m-n} \), so that it can be reduced, and we get more analytical solutions. The negative eigenvalues may appear at \( \lambda^{(1)} = (A^2 - B^2)^{-1}(C - De^{-\gamma t}) \), \( \lambda^{(2)} = (A^2 - B^2)^{-1}(C^2 - D^2e^{-2\gamma t}) \) and so on. Hence one of the necessary conditions of the non-negativity of \( \rho^{\text{PT}}(t) \), so that the necessary condition of a damped state \( \rho(t) \) is PPT separable is that

\[
C \geq De^{-\gamma t}.
\]

We will prove that this condition is also sufficient for PPT separability. We turn to the detailed properties of matrix \( M^{(m)} \). The necessary condition of separability comes from \( M^{(2)} \) is \( C \geq De^{-2d} \), this is a trivial result compared with Ineq. 5. We have checked other solvable eigenvalues for necessary condition of separability. They are also trivial compared with Ineq. 5. It may be anticipated that separable conditions come from all other \( M^{(m)} \) are weaker than Ineq. 5, that is Ineq. 5 is also a sufficient condition of PPT separability. To prove this, we just need to consider the PPT separability at the case of \( C = De^{-\gamma t} \). Because if a state corresponding to \( C = De^{-\gamma t} \) is PPT separable, then another state with stronger phase damping (with increasing \( m \) while preserving all other parameters unchanged) is definitely PPT separable, for this stronger phase damping state we have \( C > De^{-\gamma t} \) and it is PPT separable. Our proof of the PPT separability of the state at \( C = De^{-\gamma t} \) is not a most general proof. We can only prove the non-negativity of \( M^{(m)} \) by algebraic programming up to \( m = 17 \) at the case of \( C = De^{-\gamma t} \). Denote \( M^{(m)}_{in} = (A^2 - B^2)^{-1} C^m N^{(m)}_{in} \), and let \( N^{(m,j)} \) (with its elements \( N^{(m,j)}_{in} \), \( 0 \leq l, n \leq j \)) be the \( j \)-th main submatrix of \( N^{(m)} \), then to prove the non-negativity of \( M^{(m)} \) is to prove that the determinants of all \( N^{(m,j)} \) are non-negative. The algebraic programming gives \( \det N^{(m,j)} = d^{-p(d-1)^{j+1/2} P^{(m,j)}(d) \geq 0} \), where \( d = D/C = e^{\gamma t} \geq 1, \) and \( P^{(m,j)}(d) \) is a polynomial of \( d \) with all its coefficients being positive integers. \( p \) is some integer rely on \( j \). The algebraic programming runs for all \( m \leq 17 \) and proves the non-negativity of \( M^{(m)} \).

We suggest that \( M^{(m)} \) is also non-negative for \( m > 17 \), but this is not verified because of the computing time of the algebraic programming.

Another direct way of proving comes from perturbation theory. Firstly \( M^{(m)} \) can always be symmetrized. The zero order matrix is \( M^{(m)}_0 = M^{(m)}(\gamma t = 0) \) which is just the case of gaussian state, and all its eigenvalues and eigenvectors are well known. So that the first order and second order perturbation of the eigenvalues of \( M^{(m)} \) can be obtained. A more concise way to obtain the perturbation result is as follows: In the eigenvalue of characteristic function, if we use \( \langle \alpha | \Phi \rangle = \exp(-\frac{1}{2} |\alpha|^2) \sum_{m=0}^{\infty} \zeta_n^{(m)} \alpha_1^* \alpha_2^{m-n} \alpha_1^* - \alpha_2^* \) as a test wave function, we then get a matrix (see Appendix)

\[
M^{(m)}_{in} = \frac{1}{(A^2 - B^2) \sqrt{4 \pi \gamma t}} \int dx \exp[-\frac{x^2}{4 \gamma t}] \sum_k \binom{m-n}{l-k} \binom{n}{k} (C + D \cos x)^{m-n-l+k} (C - D \cos x)^{k} (1-\gamma t)(iD \sin x)^{l+n-2k}
\]

which is a linear transformation of \( M^{(m)} \) and has the same eigenvalues. The zero order of \( M^{(m)} \) is the matrix \( M^{(m)}_0 = M^{(m)}(\gamma t = 0) \) which is a diagonal matrix. Hence eigenvalues up to the first order perturbation of \( M^{(m)} \) are simply \( M^{(m)}_{in} \). When \( \gamma t \) is quite small, \( M^{(m)}_{in} \) can be approximated as \( M^{(m)}_{nn} \approx (A^2 - B^2)^{-1} (C + D)^{m-n} (D - C)^{n} f(m,n) \) with

\[
f(m,n) = 1 - \gamma t \left[ \frac{D}{D+C} (m-n) + \frac{D}{D-C} n + \frac{2D^2}{(D+C)(D-C)} (m-n)n \right].
\]

For odd \( n \), the \( n - th \) eigenvalue of the zero order approximation is negative. If \( f(m,n) \) is also negative, then the \( n - th \) eigenvalue of the first order approximation becomes positive. For given \( \gamma t \) and \( D/C \) we can always
The entropy of the state $\rho$ is its entropy $S(\rho) = -\text{Tr} \rho \log \rho$. The entropy of our damped state can be obtained by solving the characteristic function eigenequation which is $\int \frac{d\mu}{2\pi i} \frac{d\alpha}{2\pi} \chi(\mu, \mu^*, t) \langle \beta | D(-\mu) | \alpha \rangle | \Phi \rangle = \lambda(\beta | \Phi \rangle$. After integrals on $\mu$, $\alpha$ and $x$, then compare the coefficients of each $\beta^n$ item of the two side, one can get a series of matrices $L^{(m)}$ whose eigenvalues are that of the damped state $\rho$ and

$$L_{mn}^{(m)} = \frac{C^n}{A^2 - B^2} \sum_k \binom{m+n}{n-k} \binom{l}{k} C^{2k} (-D)^{l+n-2k} e^{-\gamma t(n-l)^2}.$$  

(12)

The entropy of the state $\rho$ will be $S(\rho) = -\text{Tr} L^{(0)} \log(L^{(0)}) - 2 \sum_{m=1}^{\infty} \text{Tr} L^{(m)} \log(L^{(m)})$. The reduced state of $\rho$ is $\rho_1 = \text{Tr}_2 \rho$ with its characteristic function $\chi_1(\mu_1, t) = \chi(\mu_1, 0, t) = \exp[-A' |\mu|^2]$, hence its entropy is $S(\rho_1) = A \log A - (A-1) \log(A-1)$. It has been proven that the coherent information $I^1 = \max(0, S(\rho_1) - S(\rho))$ provides lower bound of distillable entanglement of the state $[13]$. The coherent information is calculated and plotted in the figure. At time $t_0$ the coherent information turns to zero. In the figure we have $t_0 < t_1$, but for other parameters we may have $t_0 > t_1$.

We have investigated the symmetric damping setting of the two mode squeezed thermal state, that is, the two modes undergo the same damping and noise. The generalization to asymmetric damping setting and $x - \rho$
symmetric Gaussian state is straightforward with the method developed here. Denote $C_i = 1 - \frac{A_i}{A_1A_2-|B|^2}$, $D = \frac{B}{A_1A_2-|B|^2}$, the $M^{(m)}$ matrix will be
\[
M_{lm}^{(m)} = (A_1A_2 - B^2)^{-1}C_1^m \sum_k \binom{m-n}{l-k} \binom{n}{k} \left( \frac{|D|}{\sqrt{C_1C_2}} \right)^{l+n-2k} e^{-\gamma l(l-1)^2},
\]
(13)
The whole issue of the positivity of the asymmetric setting is equivalent that of symmetric setting and omitted here. The necessary and sufficient criterion of the PPT separability of the state will be
\[
\sqrt{C_1C_2} \geq |D| e^{-\gamma t},
\]
(14) and the PPT sufficient criterion is again obtained at the sense of algebraic programming and perturbation theory. Ineq. (14) will be generalized to
\[
\sqrt{(A_1 - 1)(A_2 - 1)} \geq |B| e^{-\gamma t},
\]
(15)

## 4 Separability

We will prove that
\[
\sqrt{(A_1 - 1)(A_2 - 1)} \geq |B| \Leftrightarrow \sqrt{C_1C_2} \geq |D|
\]
is the sufficient condition of the separability of the damped state $\rho$. Let us first consider a Gaussian density operator $\rho_G$ with its characteristic function $\chi_G = \exp[-A_1^\dagger |\mu|^2 - A_2^\dagger |\mu|^2 + B(\mu_1\mu_2 e^{ix} + B^*\mu_1^*\mu_2^* e^{-ix})]$. The Fourier transformation of $\chi_G \exp(\frac{1}{2} |\mu|^2)$ is a probability distribution function (pdf) if $\sqrt{(A_1 - 1)(A_2 - 1)} \geq |B|$, where $x$ is absorbed into $\mu$. This pdf enables the P-representation of $\rho_G$. Hence $\rho_G$ is separable when $\sqrt{C_1C_2} \geq |D|$. The P-representation of $\rho$ is a positive integral of the P-representation of $\rho_G$. Thus $\rho$ is separable. From physical consideration, we may think $\rho$ is the phase damping of $\rho_G$, thus when $\rho_G$ is separable, $\rho$ should be separable.

The problem left is that when $\sqrt{C_1C_2} < |D|$ and $\sqrt{C_1C_2} \geq |D| e^{-\gamma t}$, the state is separable or not. We have strong evidence to elucidate that the state is not separable, although we do not give a full proof. The evidence is like this: the state can not be expressed in P-representation for $\sqrt{C_1C_2} < |D|$, $\rho = \int P(\alpha_1, \alpha_2) |\alpha_1\alpha_2\rangle \langle \alpha_1\alpha_2| d^2\alpha_1d^2\alpha_2/\pi^2$ is only possible for $\sqrt{C_1C_2} \geq |D|$, where $P(\alpha_1, \alpha_2)$ is a pdf and $|\alpha_1\alpha_2\rangle$ denotes two-mode coherent state.

The Fourier transformation will be
\[
P(\alpha) = \frac{c}{(A_1 - 1)(A_2 - 1)-|B|^2} \int e^{-\gamma \chi} d\alpha, \quad \chi = \frac{1}{2} |\mu|^2 - \mu \alpha^* + \mu^* \alpha d^4 \mu / \pi^2.
\]
After the integral of $\mu$, we have
\[
P(\alpha) = \frac{c}{(A_1 - 1)(A_2 - 1)-|B|^2} \int e^{-\gamma \chi} d\alpha, \quad \chi = \frac{1}{2} |\mu|^2 - \mu \alpha^* + \mu^* \alpha e^{i\varphi},
\]
(17) and $\varphi = \theta_1 + \theta_2 + \phi$, then
\[
P(\alpha) = c \exp[-E_2|\alpha_1|^2 - E_1 |\alpha_2|^2 + Fe^{i\varphi} \alpha_1 \alpha_2 + F^* e^{-i\varphi} \alpha_1^* \alpha_2^*].
\]
(18) Clearly $P(\alpha)$ is real, and $P(\alpha)$ is positive. The positivity of $P(\alpha)$ is warranted by the fact that when $\sqrt{C_1C_2} \geq |D|$, the state is separable. $P(\alpha)$ is a pdf. If we fix $F$ while decreasing $E_i$ to reach a state with $\sqrt{C_1C_2} < |D|$, the sign of $P(\alpha)$ will not change by decreasing $E_i$. Thus $P(\alpha)$ is positive even when $\sqrt{C_1C_2} < |D|$.

What left is the singularity of $P(\alpha)$. We will prove that $P(\alpha)$ is singular if and only if $\sqrt{C_1C_2} < |D|$. The singularity may appear when $|\alpha_i| = r_i \to \infty$, $P(\alpha) \to \infty$. The maximum of $P(\alpha)$ reaches when $\varphi = 0$, thus in the following, we set $\varphi = 0$. Let $g(z) = \sum_{n=0}^\infty \binom{2/2}{2n} \sum_{l=0}^n \binom{n}{l} \exp[-\gamma l(2l-n)^2]$, then we have $g(z) \approx \sum_{n=0}^\infty \frac{z^n}{n!} \exp[-\frac{\pi^2}{2} n]$. Thus
\[
1 + \sum_{n=1}^\infty \frac{z^n}{n!} - 1 + \sum_{n=1}^\infty \frac{z^n}{n!} \geq 1 + \frac{1}{\sqrt{\gamma \pi}} \frac{1}{2\sqrt{2\pi n}} (e^2 - 1 - z), \quad \text{where we have used DeMoivre-Laplace theorem}
\]
\[
\sum_{n=1}^\infty \frac{z^n}{n!} \geq 1 + \frac{1}{\sqrt{\gamma \pi}} \frac{1}{2\sqrt{2\pi n}} (e^2 - 1 - z), \quad \text{where we have used DeMoivre-Laplace theorem}
\]
\[
1 + \frac{1}{\sqrt{\gamma \pi}} \frac{1}{2\sqrt{2\pi n}} (e^2 - 1 - z) \leq z \to \infty \frac{\ln g(z)}{z} = 1. \quad \text{We arrive at}
\]
\[ P(r_1, r_2) \to c \exp[-E_2 r_1^2 - E_1 r_2^2 + 2|F|r_1 r_2] \leq c \exp[-2(\sqrt{E_2 E_1} - |F|)r_1 r_2] \text{ when } r_1 r_2 \to \infty. \] 

The non-singularity condition of \( P(\alpha) \) is simply \( \sqrt{E_2 E_1} \geq |F| \), which is equivalent to Ineq. 10.

We now compare all three conditions of the separability of \( \rho \). If \( \sqrt{C_1 C_2} \geq |D| \), the state \( \rho \) is separable, needlessly to say we have \( \sqrt{C_1 C_2} \geq |D| e^{-\gamma^2} \), and \( \sqrt{(A_1 - 1)(A_2 - 1)} \geq |B| e^{-\gamma^2} \); if \( \sqrt{C_1 C_2} < |D| \) and \( \sqrt{C_1 C_2} \geq |D| e^{-\gamma^2} \), we have \( \sqrt{(A_1 - 1)(A_2 - 1)} \geq |B| e^{-\gamma^2} \), hence Ineq. 15 can be dropped as a necessary condition because it is weak than Ineq. 14; at this case we do not know the state \( \rho \) is separable or not, we suspect that the state is bound entangled; if \( \sqrt{C_1 C_2} < |D| e^{-\gamma^2} \), the state \( \rho \) is entangled. The conditions are expressed with the curves \( A - 1 - Be^{-\gamma^2}, C - De^{-\gamma^2} \) and \( A - B - 1 \) in the figure for the special case of \( A_1 = A_2, B = B^* \). The zero points of the curves are \( t_1, t_2, t_3 \), and \( t_1 \leq t_2 \leq t_3 \). For channel without phase damping, the state is a gaussian state. All three conditions will be the same, the zero points of the curves will coincide and \( t_1 = t_2 = t_3 \).

\section{5 Conclusions and Discussions}

In conclusion, the phase damping equation of a state is obtained in the form of characteristic function. It turns out to be a usual dissipation equation with respect to the phase angle of the complex variable of the characteristic function. The time evolution solution is given for any continuous variable state undergo simultaneous amplitude and phase damping and thermal noise. Two of the criteria are given for the amplitude and phase damping of a two mode \( x - p \) symmetric Gaussian state. One is the sufficient condition of the damped state. The other is Peres-Horodecki criterion which is not only necessary but also proved to be PPT sufficient. The proof is at the sense of algebraic programming and also perturbation theory. The logarithmic negativity and coherent information of the damped state are investigated.

The evolution of the state is like this: the entanglement of the state (if the state is prepared entangled initially) decreases with time, at some time it reaches 0, this time is determined by Peres-Horodecki criterion. Then the state may be bound entangled at the next time interval, we proved that the state has not a P- representation at this time interval. The end of this time interval is the time determined by the sufficient condition of the separability. After this time the state is separable.

For a channel without phase damping, the state remains a Gaussian state. The two criteria will coincide. For pure phase damping channel, \( \Gamma = 0 \), \( \pi = 0 \), we can see that the initially two-mode vacuum state will never evolve to a separable state. Hiroshima mentioned this result with numerical calculation.

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\section{6 Appendix}

From the eigenequation of partial transposed density matrix \( \rho^{PT}(t) |\Phi\rangle = \lambda |\Phi\rangle \), the eigenequation for characteristic function of it can be deduced as \( \int \frac{d^2 \mu}{\pi^2} \frac{d^2 \alpha}{\pi^2} \chi^{PT}(\mu, \mu^*, t) \langle \beta | D(-\mu) |\alpha\rangle \langle \alpha | \Phi \rangle = \lambda \langle \beta | \Phi \rangle \). Let \( \langle \alpha | \Phi \rangle = \exp(-\frac{1}{2} |\alpha|^2) \sum_{n=0}^{m} c_n^{(m)} \alpha_1^{m-n} \alpha_2^n \), then

\[ I_{mn} = \exp(\frac{1}{2} |\beta|^2) \int \frac{d^4 \mu}{\pi^2} \frac{d^4 \alpha}{\pi^2} \chi^{PT}(\mu, \mu^*, t) \langle \beta | D(-\mu) |\alpha\rangle \exp(-\frac{1}{2} |\alpha|^2) \alpha_1^{m-n} \alpha_2^n \]

\[ = \int_x \int \frac{d^4 \mu}{\pi^2} \exp[-A_1 |\mu_1|^2 - A_2 |\mu_2|^2 - B \mu_1 \mu_2^* e^{ix} - B^* \mu_1^* \mu_2 e^{-ix} + \mu |\beta|^2 | \beta_1^* - \mu_1 | \beta_2^* - \mu_2 |]^n. \]

Where the integral formula

\[ \int \frac{d^2 \tau}{\pi} \exp[-|\tau|^2 + \tau \sigma] f(\tau) = f(\sigma) \]
is used to integrate $\alpha$. This formula can further be used to integrate $\mu$. After the integral of $\mu_1$ we have

$$I_{mn} = \frac{1}{A_1} \int \int \frac{d^2 \mu_2}{\pi} \exp[-A_2 |\mu_2|^2 - \frac{1}{A_1} B^* \mu_2 e^{-ix}(\beta_1^* - B \mu_2^* e^{ix}) + \mu_2 \beta_2^*]$$

$$\times |\beta_1^* - \frac{1}{A_1}(\beta_1^* - B \mu_2^* e^{ix})|^{m-n}(\beta_2^* - \mu_2^n).$$

After the integral of $\mu_2$ we have

$$I_{mn} = \frac{1}{A_1 A_2 - |B|^2} \int \int \left(C\beta_1^* + D e^{-ix} \beta_2^* \right)^{m-n} \left(C\beta_2^* + D^* e^{-ix} \beta_1^* \right)^n,$$

with $C_i = 1 - \frac{A_i}{A_1 A_2 - |B|^2}$ and $D = \frac{B}{A_1 A_2 - |B|^2}$. For each $m$ we expand the binomial and complete the integral of $x$, $I_{mn}$ will be a polynomial of $\beta_1^*$ and $\beta_2^*$. The eigenequation will be

$$\sum_{n=0}^{m} c_n^{(m)} I_{mn}(\beta_1^*, \beta_2^*) = \lambda \sum_{k=0}^{m} c_k^{(m)} \beta_1^{*m-k} \beta_2^{*k}.$$

By comparing the power of $\beta^*$, and absorbing the phase of $D$ into the coefficient $c_n^{(m)}$, we at last get a matrix $M^{(m)}$ (as in Eq.13) whose eigenvalues are that of $\rho^{PT}(t)$. Eq.9 can be deduced in the same way.

References

[1] A. Peres, Phys. Rev. Lett. 77, 1413 (1996).
[2] P. Horodecki, Phys. Lett. A 232, 333(1997). M. Horodecki, P. Horodecki and R. Horodecki, Phys. Rev. Lett. 78, 574 (1997).
[3] S. L.Braunstein and A. K. Pati eds. Quantum Information Theory with Continuous Variables, (Kluwer, Dordrecht, 2002). S. L. Braunstein and H. J. Kimble, Phys. Rev. Lett. 80, 869 (1998).
[4] A. Furusawa, J. L. Sørensen, S. L. Braunstein, C. A. Fuchs, H. J. Kimble and E. S. Polzik, Science 282, 706 (1998).
[5] L. M. Duan, G. Giedke, J. I. Cirac and P. Zoller, Phys. Rev. Lett. 84, 2722 (2000).
[6] R. Simon, Phys. Rev. Lett. 84, 2726 (2000).
[7] X. B. Wang, Phys. Rev. Lett. 87, 137903 (2001).
[8] T. Hiroshima, Phys. Rev. A 63, 022305 (2001).
[9] D. Petz, An Invitation to the Algebra of Canonical Commutation Relations, Leuven University Press, Leuven (1990).
[10] A. Perelomov, Generalized Coherent states, Springer Verlag, Berlin (1986).
[11] L. Z. Jiang, Int. J. Quantum Inform. 2, 273 (2004).
[12] X.Y. Chen, P. L. Qiu, Phys. Lett. A, 314, 191(2003).
[13] G. Vidal and R. F. Werner, Phys. Rev. A 65, 032314 (2002).
[14] K. Audenaert, M. B. Plenio, and J. Eisert, Phys. Rev. Lett. 90,027901 (2003).
[15] I. Devetak and A. Winter, Proc. R. Soc. Lond. A, 461, 207(2005).