Banach spaces with the (strong) Gelfand–Phillips property

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Abstract
Several new characterizations of the Gelfand–Phillips property are given. We define a strong version of the Gelfand–Phillips property and prove that a Banach space has this stronger property iff it embeds into $c_0$. For an infinite compact space $K$, the Banach space $C(K)$ has the strong Gelfand–Phillips property iff $C(K)$ is isomorphic to $c_0$ iff $K$ is countable and has finite scattered height.

Keywords Banach space · Gelfand–Phillips property · Strong Gelfand–Phillips property

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1 Introduction
All topological spaces are assumed to be Tychonoff and infinite, all Banach spaces are infinite-dimensional over the field $\mathbb{F}$ of real or complex numbers, and all operators between Banach spaces are linear and continuous. For a Banach space $E$, we denote by $B_E$ the closed unit ball of $E$, and the dual space of $E$ is denoted by $E'$. For a bounded subset $B \subseteq E$ and a functional $\varphi \in E'$, we put $\|\varphi\|_B := \sup\{ |\varphi(x)| : x \in B\}$.

A bounded subset $B$ of a Banach space $E$ is called limited if every weak* null sequence $(\varphi_n)_{n \in \mathbb{N}}$ in $E'$ converges uniformly on $B$, that is $\lim_{n} \|\varphi_n\|_B = 0$. A Banach space $E$ is called Gelfand–Phillips if every limited set in $E$ is precompact. A subset $A$ of $E$ is precompact if its closure in $E$ is compact.

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In [16] Gelfand proved that every separable Banach space is Gelfand–Phillips. The condition of being separable is essential. Indeed, Phillips [27] showed that the space $\ell^\infty = C(\omega)$ is not Gelfand–Phillips, where $\omega$ is the Stone–Čech compactification of the discrete space $\omega$ of nonnegative integers.

Bourgain and Diestel [4] defined an operator $T : L \to E$ between Banach spaces to be limited if $T(B_1)$ is a limited subset of $E$, or equivalently, if the adjoint operator $T^*$ is weak*–norm sequentially continuous. In [12], Drewnowski observed the following characterization playing a considerable role in recognizing Gelfand–Phillips spaces.

**Theorem 1.1** For a Banach space $E$ the following assertions are equivalent:

(i) $E$ is Gelfand–Phillips;
(ii) every limited weakly null sequence in $E$ is norm null;
(iii) every limited operator with range in $E$ is compact.

It immediately follows from (ii) that every Schur space (in particular, $\ell^1(\Gamma)$ for some infinite set $\Gamma$) is Gelfand–Phillips.

Gelfand–Phillips spaces were intensively studied by many authors, see for example [12, 13, 23, 30–32] and more recent articles [6, 17, 21, 29]. Another direction for studying the Gelfand–Phillips property is to characterize Gelfand–Phillips spaces that belong to some important classes of Banach spaces. Since every Banach space is (isometrically) embedded in a $C(K)$-space, it is important to recognize Gelfand–Phillips spaces among Banach spaces of continuous functions in terms of the compact space $K$. Some sufficient conditions on compact spaces $K$ to have Gelfand–Phillips space $C(K)$ were obtained by Drewnowski [12], Drewnowski and Emmanuele [13], and by Schlumprecht in [30, 31].

In the first main result of the paper (Theorem 2.1), we obtain several new characterizations of Gelfand–Phillips spaces from which we deduce some sufficient conditions of being a Gelfand–Phillips space (Corollary 2.2). Our approach involves the family $\text{BNP}(E)$ of all bounded non-precompact subsets of a Banach space $E$, instead of limited sets in $E$. This approach leads us to the following characterization of the Gelfand–Phillips property: A Banach space $E$ is Gelfand–Phillips if and only if for every $B \in \text{BNP}(E)$, there is a weak* null sequence $(\chi_n)_{n \in \omega}$ in $E'$ such that $\|\chi_n\|_B \not\to 0$.

The importance of this reformulation is that it gives not only a new characterization of the Gelfand–Phillips property, but it also allows to introduce and study a strong version of that property, see Definition 1.2 below.

Analysing the aforementioned characterization of the Gelfand–Phillips property, one can naturally ask: When does there exist a fixed weak* null sequence $(\chi_n)_{n \in \omega}$ in the dual $E'$ such that $\|\chi_n\|_B \not\to 0$ for every $B \in \text{BNP}(E)$? This question motivates to introduce the following strong version of the Gelfand–Phillips property.

**Definition 1.2** A Banach space $E$ is defined to have the strong Gelfand–Phillips property if $E$ admits a weak* null-sequence $(\chi_n)_{n \in \omega}$ in $E'$ such that $\|\chi_n\|_B \not\to 0$ for every $B \in \text{BNP}(E)$. In this case we will say that $E$ is strongly Gelfand–Phillips.
Strongly Gelfand–Phillips spaces are studied in Sect. 3. It turns out that the class of such spaces is rather narrow: according to Theorem 3.1, a Banach space $E$ is strongly Gelfand–Phillips if and only if it embeds into $c_0$; by Theorem 3.4, for a compact space $K$ the Banach space $C(K)$ is strongly Gelfand–Phillips if and only if $C(K)$ is isomorphic to $c_0$ if and only if $K$ is a countable compact space of finite scattered height.

2 A characterization of Gelfand–Phillips spaces

A topological space $X$ is defined to be

- **sequentially compact** if every sequence in $X$ contains a convergent subsequence;
- **selectively sequentially pseudocompact** at a subset $A \subseteq X$ if for any open sets $U_n \subseteq X$, $n \in \omega$, intersecting the set $A$, there exists a sequence $(x_n)_{n \in \omega} \in \prod_{n \in \omega} U_n$ containing a convergent subsequence;
- **selectively sequentially pseudocompact** if $X$ is sequentially pseudocompact at $X$, see [11].

It is clear that every selectively sequentially pseudocompact space $X$ is pseudocompact, but the converse is not true in general since $X$ must contain non-trivial convergent sequences. In particular, the Stone–Čech compactification $\beta D$ of an infinite discrete space $D$ is not selectively sequentially pseudocompact. Compact selectively sequentially pseudocompact spaces form the class $K^{''}$ introduced by Drewnowski and Emmanuele [13].

A non-trivial class of selectively sequentially pseudocompact spaces is the class of Valdivia compact spaces widely studied in Functional Analysis. Let us recall that a compact space $K$ is **Valdivia compact** if $K$ is homeomorphic to a subspace $X$ of a Tychonoff cube $[-1,1]^\kappa$ such that for the set $\Sigma := \{x \in [-1,1]^\kappa : |\text{supp}(x)| \leq \omega\}$, the intersection $X \cap \Sigma$ is dense in $X$. Since the space $\Sigma$ is sequentially compact, every Valdivia compact space is selectively sequentially pseudocompact. It should be mentioned that Banach spaces whose dual unit ball is Valdivia compact in the weak* topology form an important and well-studied class of Banach spaces, see [22].

Let $E$ be a Banach space. For simplicity of notations, the space $E$ endowed with the weak topology is denoted by $E_w$, and $E'_w$ denotes the dual space $E'$ with the weak* topology. A bounded subset $S$ of $E'$ is called **norming** if the formula $\|x\| = \sup_{\chi \in S} |\chi(x)|$ determines an equivalent norm on $E$, see [15, p.160]. We generalize this classical notion as follows. Let $B$ be a subset of $E$. A bounded subset $S$ of $E'$ is defined to be **B-norming** if there exist a positive constant $\lambda$ such that

$$\frac{1}{\lambda} \|x\| \leq \|x\| \leq \lambda \|x\|$$

for every $x \in B$.

In some papers (e.g. [31]) $E$-norming sets are called $E$-norming up to a constant.

In the following theorem, which is the main result of this section, we give several new characterizations of Gelfand–Phillips spaces.
Theorem 2.1  For a Banach space $E$ the following assertions are equivalent:

(i) $E$ is Gelfand–Phillips.

(ii) For every $x \in \text{BNP}(E)$ there exists an operator $T : E \to c_0$ such that $T(X)$ is not precompact in the Banach space $c_0$.

(iii) For every $x \in \text{BNP}(E)$, there is a weak* null-sequence $(\chi_n)_{n \in \omega}$ in $E'$ such that $\|\chi_n\|_X \not\to 0$.

(iv) For every $B \in \text{BNP}(E)$ there exist an infinite subset $X \subseteq B$ such that the space $\mathcal{E}_X'$, is selectively sequentially pseudocompact at some $(X - X)$-norming set $S \subseteq E'$.

Proof  (i) $\Rightarrow$ (ii) Assume that $E$ is Gelfand–Phillips and take any set $X \in \text{BNP}(E)$. Since $E$ is Gelfand–Phillips, the non-precompact set $X$ is not limited, which means that $\|\chi_n\|_X \not\to 0$ for some weak* null-sequence $(\chi_n)_{n \in \omega}$ in $E'$. Since $(\chi_n)_{n \in \omega}$ is norm-bounded in $E'$, the linear map

$$T : E \to c_0, \quad T : x \mapsto (\chi_n(x))_{n \in \omega},$$

is bounded. It remains to show that $T(X)$ is not precompact in $c_0$. Assuming that $T(X)$ is precompact, for every $\epsilon > 0$ we can find a finite set $F \subseteq X$ such that $T(F)$ is an $\epsilon$-net in $T(X)$ (which means that for every $x \in T(X)$ there exists $y \in T(F)$ such that $\|x - y\| < \epsilon$). Since $(\chi_n)_{n \in \omega}$ is weak* null and $F$ is finite, there exists $m \in \omega$ such that $\sup_{n \geq m} \max_{e \in F} |\chi_n(y)| < \epsilon$. For every $x \in X$ choose $y \in F$ with $\|T(x) - T(y)\| < \epsilon$ and conclude that for every $n \geq m$, we have

$$|\chi_n(x)| \leq |\chi_n(y)| + |\chi_n(y) - \chi_n(x)| < \epsilon + \|T(y) - T(x)\| < 2\epsilon,$$

which implies that $\|\chi_n\|_X \not\to 0$ and contradicts the choice of the sequence $(\chi_n)_{n \in \omega}$.

(ii) $\Rightarrow$ (iii) Given any set $X \in \text{BNP}(E)$, apply (ii) and find an operator $T : E \to c_0$ such that $T(X)$ is not precompact in $c_0$. For every $n \in \omega$, let $\chi_n = e_n' T$, where $e_n'$ is the $n$th coordinate functional in $c_0$. It is clear that the sequence $(\chi_n)_{n \in \omega}$ is weak* null in $E'$. It remains to prove that $\|\chi_n\|_X \not\to 0$. To derive a contradiction, assume that $\|\chi_n\|_X \not\to 0$. Then for every $\epsilon > 0$ there exists $m \in \omega$ such that $\sup_{n \geq m} \max_{e \in F} |\chi_n(y)| < \epsilon$. Since $X$ and $T$ are bounded, the number $b = \sup \{\|T(x)\| : x \in X\}$ is finite. Since the bounded finite-dimensional set

$$K = \{y \in c_0 : \|y\| \leq b, \text{ and } e_n'(y) = 0 \text{ for all } n \geq m\}$$

is compact in $c_0$, there exists a finite set $F \subseteq K$ such that for every $x \in K$, there exists $y \in F$ such that $\|x - y\| < \epsilon$. Given any $x \in X$, let $y \in K$ be a unique element such that $e_n'(y) = \chi_n(x)$ for all $n < m$. The choice of the number $m$ ensures that $\|y - T(x)\| = \sup_{n \geq m} |\chi_n(x)| < \epsilon$. Since $y \in K$, there exists $z \in F$ such that $\|y - z\| < \epsilon$. Then, $\|x - z\| < 2\epsilon$ which means that $F$ is a $2\epsilon$-net for $T(X)$ and hence the set $T(X)$ is precompact in $c_0$, which contradicts the choice of the operator $T$.

(iii) $\Rightarrow$ (iv) Given any set $B \in \text{BNP}(E)$, we apply (iii) to find a weak* null sequence $(\chi_n)_{n \in \omega}$ in $E'$ such that $\inf_{n \in \omega} \|\chi_n\|_B > 0$. Multiplying the elements of the sequence $(\chi_n)_{n \in \omega}$ by a suitable constant, we can assume that $\inf_{n \in \omega} \|\chi_n\|_B > 2b$ where
\[ b := \sup\{||x - y|| : x, y \in B\} > 0. \]

Taking into account that \((\chi_n)_{n \in \omega}\) is weak* null, construct inductively a sequence of points \((x_n)_{k \in \omega}\) in \(B\) and an increasing sequence of numbers \((n_k)_{k \in \omega}\) such that for every \(k \in \omega\) the following two conditions are satisfied:

- \(|\chi_n(x_k) - x_i| > 2b\);
- \(|\chi_n(x_i)| < b\) for every \(i < k\).

The triangle inequality ensures that \(|\chi_n(x_k - x_i)| > b\) for every \(i < k\).

Let \(X = \{x_k : k \in \omega\}\). Since the sequence \((\chi_n)_{k \in \omega}\) is weak* null, the space \(E'_{w^*}\) is selectively sequentially pseudocompact at the set \(S = \{\chi_n : k \in \omega\}\). We claim that \(S\) is \((X - X)\)-norming. Let \(C := \sup\{||\chi|| : \chi \in S\}\). Then \(C < \infty\) and, for every \(z \in E\), we have \(\|z\|_S = \sup_{\chi \in S}\|\chi(z)\| \leq C \cdot \|z\|\). If \(z \in X - X\), then \(z = x_k - x_i\) for some \(k, i \in \omega\). If \(k = i\), then \(z = 0\) and \(\|z\|_S = \|x\|\). If \(k \neq i\), then

\[
\|z\|_S \geq \max\{||\chi_n(x_k - x_i)||, ||\chi_n(x_k - x_i)||\} > b \geq ||x_k - x_i|| = ||z||. 
\]

Therefore,

\[
\|z\| \leq \|z\|_S \leq C||z||
\]

for every \(z \in X - X\), which means that the set \(S\) is \((X - X)\)-norming.

(iv) \(\Rightarrow\) (i) Assuming that \(E\) is not Gelfand–Phillips, we can find a limited set \(B \in \text{BNP}(E)\). Since \(B\) is not precompact, there exists \(\varepsilon > 0\) such that for every finite subset \(F \subseteq B\) there exists \(x \in B\) such that \(||x - y|| > \varepsilon\) for all \(y \in F\). Using this property of \(\varepsilon\), we can inductively construct a sequence \(\{z_n\}_{n \in \omega} \subseteq B\) such that \(||z_n - z_i|| > \varepsilon\) for every \(i < n\). It is clear that the set \(\{z_n : n \in \omega\}\) is not precompact. By (iv), there exist an infinite subset \(X \subseteq \{z_n : n \in \omega\}\) such that the space \(E'_{w^*}\) is selectively sequentially pseudocompact at some \((X - X)\)-norming set \(S \subseteq E'_{w^*}\). Then, there exists a positive real constant \(c\) such that \(||z||_S \geq c||z||\) for every \(z \in X - X\). Consequently, for any distinct elements \(x, y \in X\), we have

\[
\|x - y\|_S = \sup_{f \in S} |f(x - y)| \geq c \|x - y\| > c\varepsilon. 
\]

Write the set \(X\) as \(\{x_n\}_{n \in \omega}\) for pairwise distinct points \(x_n\), and for every \(n < m\), choose a linear functional \(f_{n,m} \in S\) such that \(|f_{n,m}(x_m - x_n)| > c\varepsilon\).

The selective sequential pseudocompactness of \(E'_{w^*}\) at \(S\) implies that the set \(S\) is bounded in \(E'_{w^*}\), and hence bounded in \(E\) (by the Banach–Steinhaus Uniform Boundedness Principle). Therefore the set \(\{f(x) : f \in S, x \in X\}\) has compact closure \(K\) in the field \(\mathbb{F}\) (of real or complex numbers).

Using the sequential compactness of the compact metrizable space \(K^X\), we can construct a decreasing sequence \(\{\Omega_n\}_{n \in \omega}\) of infinite sets in \(\omega\) such that for every \(n \in \omega\), the sequence \(\{f_{n,m}(X) : m \in \Omega_n\}\) converges to some function \(f_n \in K^X\). Choose an infinite set \(\Omega \subseteq \omega\) such that \(\Omega \setminus \Omega_n\) is finite for every \(n \in \omega\). Since the compact metrizable space \(K^X\) is sequentially compact, we can replace \(\Omega\) by a smaller infinite set and additionally assume that the sequence \(\{f_n\}_{n \in \Omega}\) converges to some element \(f_\infty \in K^X\). Since the set \(\{f_\infty(x_n)\}_{n \in \Omega} \subseteq K\) admits a finite cover by sets.
of diameter $< \frac{1}{4} \varepsilon$, we can replace $\Omega$ by a suitable infinite subset and additionally assume that the set $\{ f_{n,m}(x_n) \}_{n \in \Omega}$ has diameter $< \frac{1}{4} \varepsilon$.

It follows that the function $f_\infty \in K^X$ belongs to the closure of the set $\{ f_{n,m} \}_{X} : n, m \in \Omega, n < m \}$. Since the space $K^X$ is first-countable, we can choose a sequence

$$\{(n_i, m_i) \}_{i \in \omega} \subseteq \{(n, m) \in \Omega \times \Omega : n < m \}$$

such that the sequence $\{ f_{n_i, m_i} \}_{i \in \omega}$ converges to $f_\infty$. Since $F^X$ is metrizable, for every $i \in \omega$ the element $f_{n_i, m_i} \mid X$ of $K^X \subseteq \|X\|$ has an open neighborhood $V_i \subseteq \|X\|$ such that the sequence $\{ V_i \}_{i \in \omega}$ converges to $f_\infty$ in the sense that every neighborhood of $f_\infty$ in $\|X\|$ contains all but finitely many sets $V_i$. Since $|f_{n_i, m_i}(x_{n_i} - x_{m_i})| > c\varepsilon$, we can replace each set $V_i$ by a smaller neighborhood of $f_{n_i, m_i} \mid X$ and additionally assume that $|g(x_{n_i}) - g(x_{m_i})| > c\varepsilon$ for every $g \in V_i$. For every $i \in \omega$, consider the open neighborhood $W_i := \{ f \in \|E'_w \| : f \mid X \in V_i \}$ of the functional $f_{n_i, m_i}$ in the space $E'_w$.

Since $E'_w$ is selectively sequentially pseudocompact at $S$, there exists a convergent sequence $(g_k)_{k \in \omega} \subseteq E'_w$ and an increasing number sequence $(i(k))_{k \in \omega}$ such that $g_k \in W_{i(k)}$ for every $k \in \omega$. Let $g_\infty \in E'_w$ be the limit of the sequence $(g_{i(k)})_{k \in \omega}$. The continuity of the restriction operator $E'_w \to \|X\|, f \mapsto f \mid X$, and the choice of the open sets $V_i$, $i \in \omega$, guarantee that $g_\infty \mid X = f_\infty$. Consequently, the sequence $(g_k \mid X)_{k \in \omega}$ converges to $g_\infty \mid X = f_\infty$ in $\|X\|$.

Then, for every $k \in \omega$, we can find a number $j_k > k$ such that

$$\max \{ |f_\infty(x_{n_{i(k)}}) - g_{j_k}(x_{n_{i(k)}})|, |f_\infty(x_{m_{i(k)}}) - g_{j_k}(x_{m_{i(k)}})| \} < \frac{1}{8} c\varepsilon.$$ 

For every $k \in \omega$, consider the functional $\mu_k := g_k - g_{j_k} \in E'$ and observe that the sequence $(\mu_k)_{k \in \omega}$ converges to zero in $E'_w$. On the other hand, for every $k \in \omega$, the choice of the sequence $(j_k)_{k \in \omega}$, the inequality $\operatorname{diam} \{ f_\infty(x_{n_i}) \}_{n \in \Omega} < \frac{1}{4} c\varepsilon$, and the inclusion $g_k \mid X \in V_{i(k)}$ imply

$$|g_{j_k}(x_{n_{i(k)}}) - g_{j_k}(x_{m_{i(k)}})| = |g_{j_k}(x_{n_{i(k)}}) - f_\infty(x_{n_{i(k)}}) + f_\infty(x_{m_{i(k)}}) - g_{j_k}(x_{m_{i(k)}}) + f_\infty(x_{m_{i(k)}}) - f_\infty(x_{n_{i(k)}})| \leq |g_{j_k}(x_{n_{i(k)}}) - f_\infty(x_{n_{i(k)}})| + |f_\infty(x_{m_{i(k)}}) - g_{j_k}(x_{m_{i(k)}})| + |f_\infty(x_{m_{i(k)}}) - f_\infty(x_{n_{i(k)}})| \leq \frac{1}{8} c\varepsilon + \frac{1}{8} c\varepsilon + \frac{1}{4} c\varepsilon = \frac{1}{2} c\varepsilon$$

and

$$|\mu_k(x_{n_{i(k)}}) - \mu_k(x_{m_{i(k)}})| = |g_k(x_{n_{i(k)}}) - g_{j_k}(x_{n_{i(k)}}) - g_k(x_{m_{i(k)}}) + g_{j_k}(x_{m_{i(k)}})| \geq |g_k(x_{n_{i(k)}}) - g_{j_k}(x_{m_{i(k)}})| - |g_{j_k}(x_{n_{i(k)}}) - g_{j_k}(x_{m_{i(k)}})| > c\varepsilon - \frac{1}{2} c\varepsilon = \frac{1}{2} c\varepsilon.$$

Then, for every $k \in \omega$,

$$\sup_{x \in \overline{B}} |\mu_k(x)| \geq \max \{ |\mu_k(x_{n_{i(k)}})|, |\mu_k(x_{m_{i(k)}})| \} \geq \frac{1}{4} c\varepsilon,$$
witnessing that the set $B$ is not limited. This contradiction shows that the Banach space $E$ is Gelfand–Phillips.

In Corollary 2.2 below, we apply Theorem 2.1 to obtain some new and several well-known sufficient conditions on Banach spaces to be a Gelfand–Phillips space. For this, we should recall some additional definitions.

Following Castillo et al. [6], we define a Banach space $E$ to be separably weak* -extensible if any weak* null sequence in the dual of any separable subspace of $E$ admits a subsequence which can be extended to a weak* null sequence in $E'$. This class is strictly wider than the following class of Banach spaces introduced by Correa and Tausk [7, 8]: A Banach space $E$ has the separable $c_0$-extension property if every operator $T : X \to c_0$ defined on a separable Banach subspace $X \subseteq E$ can be extended to an operator $\tilde{T} : E \to c_0$. By [7, 8], the class of Banach spaces with the separable $c_0$-extension property includes all weakly compactly generated Banach spaces and all Banach spaces with the separable complementation property (= every separable subspace is contained in a separable complemented subspace).

For a compact space $K$, let

$$P(K) := \{ \mu \in C(K)' : \|\mu\| = \mu(1_K) = 1 \}$$

be the space of probability measures on $K$, endowed with the weak* topology, inherited from $C(K)'_{w^*}$. It is well-known that the space $P(K)$ is compact. Identifying each $x \in K$ with the Dirac measure $\delta_x : C(K) \to \mathbb{F}$, $\delta_x : f \mapsto f(x)$, we identify $K$ with a closed subspace of $P(K) \subseteq C(K)'_{w^*}$. Observe that the set $\{ \delta_x : x \in K \}$ is $C(K)$-norming.

**Corollary 2.2** A Banach space $E$ is Gelfand–Phillips if one of the following conditions holds:

(i) the closed unit ball $B_E$ endowed with the weak* topology is selectively sequentially pseudocompact;

(ii) [16] $E$ is separable;

(iii) [6, Prop. 2] $E$ has the separable $c_0$-extension property;

(iv) (cf. [12, Th. 2.2] and [31, Prop. 2] the space $E'_{w^*}$ is selectively sequentially pseudocompact at some $E$-norming set $S \subseteq E'$;

(v) [13, Th. 4.1] $E = C(K)$ for some compact selectively sequentially pseudocompact space $K$;

(vi) $E = C(K)$ for some compact space $K$ such that $P(K)$ is selectively sequentially pseudocompact at some set $A \subseteq P(K)$ containing $K$.

**Proof** (i) If $B_E$ is selectively sequentially pseudocompact in the weak* topology, then by the equivalence (i) $\Leftrightarrow$ (iv) in Theorem 2.1, $E$ is Gelfand–Phillips because $B_E$ is $E$-norming.

(ii) If $E$ is separable, then $B_E$ is compact metrizable and hence selectively sequentially pseudocompact in the weak* topology. By (i), $E$ is Gelfand–Phillips.
(iii) Assume that $E$ has the separable $c_0$-extension property. In order to apply Theorem 2.1, it suffices for every $B \in \text{BNP}(E)$ to find an operator $T : E \to c_0$ such that $T(B)$ is not precompact in $c_0$. Given any $B \in \text{BNP}(E)$, find a separable Banach subspace $X \subseteq E$ such that the set $X \cap B$ is not precompact in $X$. By (ii), the separable Banach space $X$ is Gelfand–Phillips. By Theorem 2.1, there exists an operation $T : X \to c_0$ such that the set $T(X \cap B)$ is not precompact in $c_0$. Since $E$ has the separable $c_0$-extension property, the operator $T$ can be extended to an operator $\tilde{T} : E \to c_0$. It is clear that the set $\tilde{T}(B) \supseteq T(X \cap B)$ is not precompact in $c_0$.

(iv) Assume that the space $E'_w$ is selectively sequentially pseudocompact at some $E$-norming set $S \subseteq E'$. By (iv) of Theorem 2.1, the Banach space $E$ is Gelfand–Phillips.

(v) Assume that $E = C(K)$ for some compact selectively sequentially pseudocompact space $K$. Identify $K$ with the set of Dirac measures in $E'_w$. Then $K$ is an $E$-norming set and $E'_w$ is selectively sequentially pseudocompact at $K$. By (iv) of Theorem 2.1, the space $E$ is Gelfand–Phillips.

(vi) Assume that $E = C(K)$ for some compact space $K$ such that $P(K)$ is selectively sequentially pseudocompact at some set $A \subseteq P(K)$ containing $K$. Identify $P(K)$ with the subspace of positive functionals in $E'_w$ and observe that $E'_w$ is selectively pseudocompact at $A$ and $A \supseteq K$ is $E$-norming. By Theorem 2.1, the space $E$ is Gelfand–Phillips. □

Sinha and Arora [32, Corollary 2.4] showed that for any Valdivia compact space, the Banach space $C(K)$ is Gelfand–Phillips. As we mentioned above, every Valdivia compact is selectively sequentially pseudocompact, and hence their result follows from Corollary 2.2(v).

It was noticed in [6] that the condition to have the separable $c_0$-extension property in (iii) of Corollary 2.2 is only sufficient to have the Gelfand–Phillips property. Below, we give a concrete example. Let us recall that the split interval $\mathbb{I}$ is the space $[0, 1] \times \{0, 1\}$ endowed with the interval topology generated by the lexicographic order $\leq$ defined by $(x, i) \leq (y, j)$ if and only if either $x < y$ or $x = y$ and $i \leq j$. It is well known that the split interval is first-countable and separable but not metrizable.

**Example 2.3** For the split interval $\mathbb{I}$, the Banach space $C(\mathbb{I})$ is Gelfand–Phillips but fails to have the separable $c_0$-extension property.

**Proof** Being compact and first-countable, the split interval $\mathbb{I}$ is sequentially compact and hence selectively sequentially pseudocompact. By Corollary 2.2(vi), the Banach space $C(\mathbb{I})$ is Gelfand–Phillips. Since the space $\mathbb{I}$ is linearly ordered, separable and non-metrizable, we can apply Theorem 2.2 of [8] and conclude that the Banach space $C(\mathbb{I})$ does not have the separable $c_0$-extension property. □

By the Phillips result [27], the Banach space $C(\beta\omega)$ is not Gelfand–Phillips. Below, we generalize this result. Recall that a Tychonoff space $X$ is an $F$-space if every functionally open set $A$ in $X$ is $C^\omega$-embedded in the sense that every
bounded continuous function \( f : A \to \mathbb{R} \) has a continuous extension \( \tilde{f} : X \to \mathbb{R} \).

For numerous equivalent conditions for a Tychonoff space \( X \) to be an \( F \)-space, see [18, 14.25]. In particular, the Stone–Čech compactification \( \beta I \) of any discrete space \( I \) is a compact \( F \)-space.

**Example 2.4** For any infinite compact \( F \)-space \( K \), the Banach space \( C(K) \) is not Gelfand–Phillips.

**Proof** Being infinite, the Tychonoff space \( K \) contains a sequence \( \{V_n\}_{n \in \omega} \) of non-empty pairwise disjoint open sets. For every \( n \in \omega \), fix a point \( v_n \in V_n \) and a continuous function \( f_n : K \to [0, 1] \) such that \( f_n(v_n) = 1 \) and \( f_n(K \setminus V_n) = \{0\} \). Consider the operator \( T : c_0 \to C(K) \) assigning to each sequence \( x = (x_n)_{n \in \omega} \in c_0 \) the continuous function \( T(x) = \sum_{n \in \omega} x_n \cdot f_n \), and observe that \( T \) is an isometric embedding of \( c_0 \) into \( C(K) \). By Corollary 4.5.9 of [9], the Banach space \( C(K) \) has the Grothendieck property, which means that the identity map \( C(K)' \to C(K)' \) is sequentially continuous (where \( C(K)' \) denotes the dual space of \( C(K) \) endowed with the weak topology).

Since the operator \( T : c_0 \to C(K) \) is an embedding, the image \( B := T(B_{c_0}) \) is bounded and not precompact in \( C(K) \), i.e., \( B \in \text{BNP}(C(K)) \). Assuming that \( C(K) \) is Gelfand–Phillips, we can find a weak* null sequence \( S = \{\mu_n\} \) in \( E' \) such that \( \|\mu_n\|_B \to 0 \). Since the identity map \( C(K)' \to C(K)' \) is sequentially continuous, we obtain that the sequence \( S \) converges to zero in the weak topology of the dual Banach space \( C(K)' \). Then, for the adjoint operator \( T^* : C(K)' \to (c_0)' \to (\ell_1)' \), the sequence \( \{T^*(\mu_n)\}_{n \in \omega} \) converges to zero in the weak topology of the Banach space \( \ell_1 \). By the Schur Theorem [10, VII], this sequence converges to zero in norm. For every \( n \in \omega \) and \( x \in B_{c_0} \), we have

\[
\|\mu_n\|_B = \sup_{x \in B_{c_0}} |\mu_n(T(x))| = \sup_{x \in B_{c_0}} |T^*(\mu_n)(x)| = \|T^*(\mu_n)\| \to 0,
\]

which contradicts the choice of the sequence \( (\mu_n)_{n \in \omega} \). Thus the Banach space \( C(K) \) is not Gelfand–Phillips. \( \square \)

It is known that Gelfand–Phillips spaces are not preserved by taking quotients, see [30]. Below we present a simple example witnessing this fact.

**Example 2.5** There are compact Hausdorff spaces \( X \subseteq Y \) such that the Banach space \( C(Y) \) is Gelfand–Phillips but the Banach space \( C(X) \) is not Gelfand–Phillips. In particular, a quotient of a Gelfand–Phillips Banach space can fail to be Gelfand–Phillips.

**Proof** In the Cantor cube \( Y := \{0, 1\}^\mathfrak{c} \) of weight \( \mathfrak{c} \) take a subspace \( X \), homeomorphic to \( \beta \omega \). Being Valdivia compact, the Cantor cube \( \{0, 1\}^\mathfrak{c} \) is selectively sequentially pseudocompact and, by Corollary 2.2(v), the Banach \( C(Y) \) is Gelfand–Phillips. By Example 2.4, the Banach space \( C(X) \) is not Gelfand–Phillips. \( \square \)
It is worth mentioning that, by results of Schlumprecht [30, 31], the Gel-
fand–Phillips property is not a three space property (see also Theorem 6.8.h in [5]).

We finish this section with two questions. The first one is related to a known open
problem of characterizing Banach spaces $E$ for which the dual unit ball $B_{E'}$ is weak*
sequentially compact (for historical remarks and the latest results, see [25]).

**Problem 2.6** Characterize Banach spaces $E$ for which the dual unit ball $B_{E'}$ is weak* selectively sequentially pseudocompact.

The following problem is motivated by the conditions (v) and (vi) of Corollary 2.2.

**Problem 2.7** Is there an infinite compact space $K$ whose space of probability meas-
ures $P(K)$ is selectively sequentially pseudocompact but $K$ contains no non-trivial
convergent sequences?

**Remark 2.8** By [2], under Jensen’s Diamond Principle ♦ (which is stronger than
the Continuum Hypothesis), there exists an infinite compact space $K$ such that the
compact space $P(K)$ is selectively sequentially pseudocompact but $K$ contains no topological copies of the spaces $\beta\omega$ and $\omega + 1 = \omega \cup \{\omega\}$. By Corollary 2.2(vi),
the Banach space $C(K)$ is Gelfand–Phillips, yet $K$ contains no nontrivial conver-
gent sequences (a CH-example of a compact space with these two properties has
been constructed by Schlumprecht in [30, § 5.4]). The space $K$ shows that Problem 2.7 has an affirmative answer under ♦. So, this problem essentially asks about
the existence of a ZFC-example. It should be mentioned that infinite compact spaces
containing no topological copies of the spaces $\beta\omega$ and $\omega + 1$ are called Efimov. The
problem of the existence of Efimov compact spaces in ZFC is one of major unsolved
problems of Set-Theoretic Topology, see [19, 26].

\[\square\]

3 Banach spaces with the strong Gelfand–Phillips property

Below we characterize Banach spaces with the strong Gelfand–Phillips property.

**Theorem 3.1** A Banach space $E$ is strongly Gelfand–Phillips if and only if it
embeds into $c_0$.

**Proof** Assuming that $E$ is strongly Gelfand–Phillips, find a weak* null sequence
$(x_n)_{n \in \omega}$ in $E'$ such that $\|x_n\|_B \nrightarrow 0$ for any $B \in \text{BNP}(E)$. Therefore every bounded
subset of the Banach subspace $Z = \bigcap_{n \in \omega} x_n^{-1}(0)$ of $E$ is precompact, which implies
that the subspace $Z$ is finite-dimensional. Unifying the weak* null sequence $(x_n)_{n \in \omega}$
with a finite set of functionals separating points of the finite-dimensional space $Z$,
we can assume that $Z = \{0\}$. In this case the linear map
\[ T : E \to c_0, \quad T : x \mapsto (\chi_n(x))_{n \in \omega}, \]

is continuous and injective. Assuming that the operator \( T \) is not a topological embedding, we can find a sequence \( \{x_n\}_{n \in \omega} \subseteq E \) of elements of norm 1 such that \( T(x_n) \to 0 \).

We claim that the bounded set \( B = \{x_n\}_{n \in \omega} \) is not precompact in \( E \). Indeed, in the opposite case, by the completeness of \( E \), the sequence \( \{x_n\}_{n \in \omega} \) would contain a subsequence \( \{x_{n_k}\}_{k \in \omega} \) that converges in \( E \) to some element \( x_\infty \in E \) of norm \( ||x_\infty|| = 1 \). The continuity of the operator \( T \) ensures that \( T(x_\infty) = \lim_{n \to \infty} T(x_n) = 0 \), which contradicts the injectivity of \( T \). This contradiction shows that the set \( B \) is not precompact in \( E \).

Now the choice of the sequence \( \{x_n\}_{n \in \omega} \) ensures that the sequence \( \{||x_n||_B\}_{n \in \omega} \) does not converge to zero. Since \( \lim_{n \to \infty} ||T(x_n)|| = 0 \), for every \( \varepsilon > 0 \), we can find an \( n \in \omega \) such that \( ||T(x_i)|| = \sup_{k \in \omega} |\chi_k(x_i)| < \varepsilon \) for all \( i \geq n \). Since the sequence \( \{x_n\}_{n \in \omega} \) weak* null, there exists a natural number \( m \geq n \) such \( ||\chi_m(x_i)|| < \varepsilon \) for all \( i \leq n \) and \( k \geq m \). Then for every \( k \geq m \), we have

\[ ||\chi_k||_B = \sup_{i \in \omega} |\chi_k(x_i)| = \max \left\{ \max_{i \leq n} |\chi_k(x_i)|, \sup_{i > n} |\chi_k(x_i)| \right\} < \varepsilon, \]

which means that \( ||\chi_k||_B \to 0 \). This contradiction shows that the operator \( T : E \to c_0 \) is a topological embedding.

Conversely, assume now that \( E \) is a subspace of the Banach space \( c_0 \). For every \( n \in \omega \), let \( \chi_n = e'_n \upharpoonright E \) be the restriction of the coordinate functional \( e'_n \in c'_0 = \ell_1 \) to the subspace \( E \subseteq c_0 \). Clearly, \( \{\chi_n\}_{n \in \omega} \) is weak*-null in \( E' \). Repeating the argument of the proof of the implication \( (ii) \Rightarrow (iii) \) in Theorem 2.1, we can show that every bounded set \( B \subseteq E \subseteq c_0 \) with \( ||\chi_n||_B \to 0 \) is precompact. Thus the sequence \( \{\chi_n\}_{n \in \omega} \) witnesses that \( E \) has the strong Gelfand–Phillips property. \( \square \)

**Remark 3.2** It is well known (see, e.g. [24, 2.d.6]) that the Banach space \( c_0 \) contains closed infinite-dimensional subspaces which are not isomorphic to \( c_0 \). \( \square \)

As a corollary we obtain the following three space property for the class of strongly Gelfand–Phillips spaces.

**Corollary 3.3** Let \( E \) be a Banach space and \( H \subseteq E \) be a closed linear subspace. Then the Banach space \( E \) is strongly Gelfand–Phillips if and only if the Banach spaces \( H \) and \( E/H \) are strongly Gelfand–Phillips.

**Proof** If \( E \) is strongly Gelfand–Phillips, then by Theorem 3.1, \( E \) can be identified with a subspace of \( c_0 \). By Theorem 3.1, the Banach space \( H \subseteq E \subseteq c_0 \) is strongly Gelfand–Phillips. By a result of Johnson and Zippin [20], the quotient space \( c_0/H \) is isomorphic to a subspace of \( c_0 \), and so is the quotient space \( E/H \subseteq c_0/H \). By Theorem 3.1, the quotient space \( E/H \) is strongly Gelfand–Phillips.

Now assume that the Banach spaces \( H \) and \( E/H \) are strongly Gelfand–Phillips. By Theorem 3.1, these spaces are isomorphic to subspaces of \( c_0 \). Consequently, there are.
isomorphic embeddings \( f_1 : H \to c_0 \) and \( f_2 : E/H \to c_0 \). Being isomorphic to subspaces of \( c_0 \), the Banach spaces \( H \) and \( E/H \) are separable and so is the Banach space \( E \). By (an implication of) the Sobczyk Theorem [10, p.72], the linear embedding \( f_1 : H \to c_0 \) extends to an operator \( \tilde{f}_1 : E \to c_0 \). Let \( q : E \to E/H \) be the quotient operator. It can be shown that the operator \( f : E \to c_0 \times c_0, f : x \mapsto (\tilde{f}_1(x), f_2 \circ q(x)) \), is an isomorphic embedding of \( E \) into the Banach space \( c_0 \times c_0 \). By Theorem 3.1, the Banach space \( E \) has is strongly Gelfand–Phillips. \( \square \)

Let us recall that a Tychonoff space \( K \) is pseudocompact if each real-valued continuous function on \( K \) is bounded. Observe that for a pseudocompact space \( K \), the space \( C(K) \) of \( \mathbb{F} \)-valued continuous functions on \( K \) is a Banach space with respect to the norm \( \| f \| := \sup_{x \in K} |f(x)| \).

A topological space \( X \) is scattered if each nonempty subspace of \( X \) has an isolated point. For a topological space \( X \), let \( X^{(0)} := X \) and let \( X^{(1)} \) be the space of non-isolated points of \( X \). For a non-zero ordinal \( \alpha \), let \( X^{(\alpha)} := \bigcap_{\beta<\alpha} (X^{(\beta)})^{(1)} \). It is well known that a topological space \( X \) is scattered if and only if \( X^{(\alpha)} = \emptyset \) for some ordinal \( \alpha \). The smallest ordinal \( \alpha \) with \( X^{(\alpha)} = \emptyset \) is called the scattered height of \( X \).

By a classical result of Bessaga and Pełczyński [3] (see also [28, 2.14]), for a compact space \( K \), the Banach space \( C(K) \) is isomorphic to \( c_0 \) if and only if \( K \) is countable and has finite scattered height. Using this result we prove the second main result of this section.

**Theorem 3.4** For an infinite pseudocompact space \( K \) the following conditions are equivalent:

(i) the Banach space \( C(K) \) is strongly Gelfand–Phillips;

(ii) the Banach space \( C(K) \) is isomorphic to a subspace of \( c_0 \);

(iii) \( K \) is compact and countable, and the Banach space \( C(K) \) is isomorphic to \( c_0 \);

(iv) the space \( K \) is compact, countable and has finite scattered height.

**Proof** The equivalence (i) \( \Leftrightarrow \) (ii) follows from Theorem 3.1, and the implication (iii) \( \Rightarrow \) (ii) is trivial.

(ii) \( \Rightarrow \) (iii) Assume that the Banach space \( C(K) \) is isomorphic to a subspace of \( c_0 \). Then the Banach space \( C(K) \) has separable dual Banach space \( C(K)' \). Identifying each point \( x \in K \) with the Dirac measure supported at \( x \), we see that \( \| x - y \| = 2 \) for any distinct points \( x, y \in K \subseteq C(K)' \). Now the separability of the dual Banach space \( C(K)' \) implies that the Tychonoff space \( K \) is countable. Therefore, \( K \) is Lindelöf, and since \( K \) is also pseudocompact, it is compact by Theorems 3.10.21 and 3.10.1 of [14]. By the Bessaga–Pełczyński Theorem [28, 2.14], the Banach space \( C(K) \) is isomorphic to \( C[0, \omega^{\alpha}] \) for some ordinal \( \alpha \geq 0 \). Theorem 2.15 of [28] implies that the Banach space \( C[0, \omega^{\alpha}] \) has Szlenk index \( Sz(C[0, \omega^{\alpha}]) = \omega^{\alpha+1} \) and, by Proposition 2.27 in [28], \( Sz(c_0) = \omega \). By Corollary 2.19 in [28], the Banach space \( C(K) \), being isomorphic to a subspace of \( c_0 \), has Szlenk index \( Sz(C(K)) \leq Sz(c_0) \). Then

\[
\omega^{\alpha+1} = Sz(C[0, \omega^{\alpha}]) = Sz(C(K)) \leq Sz(c_0) = \omega
\]
implies that $\alpha = 0$. Therefore $C(K)$ is isomorphic to $C[0, \omega^\omega] = C[0, \omega]$, which is isomorphic to $c_0$.

(iii) $\Rightarrow$ (iv) Since $C(K)$ is isomorphic to $c_0$, the compact countable space $K$ has finite scattered height by Theorem 2 in [3] (see also [28, 2.14]).

(iv) $\Rightarrow$ (iii) If $K$ is compact, countable and has finite scattered height, then the Banach space $C(K)$ is isomorphic to $c_0$ by the Bessaga–Pełczyński theorem [3]. 

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