Minimality, distality and equicontinuity for semigroup actions on compact Hausdorff spaces

(Dedicated to the memory of Professor Isaac Namioka)

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Abstract

Let $T \times X \to X$, $(t,x) \mapsto tx$, denoted $(T,X)$, be a semiflow on a compact Hausdorff space $X$ with phase semigroup $T$. If each $t \in T$ is surjective, $(T,X)$ is called surjective; and if each $t \in T$ is bijective $(T,X)$ is called invertible and in this case it induces $X \times T \to X$ by $(x,t) \mapsto xt := t^{-1}x$, denoted $(X,T)$. In this paper, we show that $(T,X)$ is equicontinuous surjective iff it is uniformly distal iff $(X,T)$ is equicontinuous surjective. As applications of this theorem, we also consider the minimality and distality of $(X,T)$ if $(T,X)$ is invertible with these dynamics.

Keywords: Equicontinuity; distality; minimality; almost periodicity; reflection principle

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1. Introduction

Let $T$ be a topological semigroup with a neutral element $e$; that is, $T$ is a $T_2$-space and meanwhile it is a multiplicative semigroup with $te = et = t$ for all $t \in T$ such that $(s, t) \mapsto st$ of $T \times T$ to $T$ is continuous.

Let $X$ be a non-singleton compact $T_2$-space, unless otherwise stated, in this paper. Given any $A \subset X$, by $\text{Int}X$ and $\text{cls}X$ we will denote respectively the interior and closure of $A$ relative to $X$. We will write $\Delta X = \{(x, x) \mid x \in X\}$ for the diagonal set of $X \times X$. We say that $T \times X \twoheadrightarrow X, (t, x) \mapsto tx$ is a semiflow with phase space $X$ and with phase semigroup $T$, denoted $(T, X)$, if $(t, x) \mapsto tx$ is jointly continuous from $T \times X$ to $X$ such that $ex = x \forall x \in X$ and $(ts)x = t(s)x \forall s, t \in T, x \in X$. When $T$ is a topological group, i.e., $(s, t) \mapsto st^{-1}$ of $T \times T$ onto $T$ is continuous, then we shall call $(T, X)$ a flow with the phase group $T$ (cf. [35, 26, 30, 4, 20]).

Given any integer $k \geq 2$, write $X^k = X \times \cdots \times X$ ($k$-times) and $(T, X^k)$ stands for the product semiflow defined by $(t, (x_1, \ldots, x_k)) \mapsto (tx_1, \ldots, tx_k)$.

**Standing notation 1.1.** Given $(T, X), x \in X$ and subsets $A, U, V$ of $X$, by $Tx$ we denote the orbit $\{tx \mid t \in T\}$ of $x$, write $TA = \bigcup_{t \in T} tA = \bigcup_{x \in A} Tx$, and set

$$N_T(x, U) = \{t \in T \mid tx \in U\} \quad \text{and} \quad N_T(U, V) = \{t \in T \mid U \cap r^{-1}V \neq \emptyset\}.$$ 

In addition, $r^{-1}x = \{y \in X \mid ty = x\}$ for all $x \in X$ and $t \in T$.

**Standing notation 1.2.** Let $\mathcal{US}_X$ be the compatible symmetric uniform structure of the compact $T_2$-space $X$; then for all $e \in \mathcal{US}_X, x \in X$, and $B \subset X$, we will write

1) $e[x] = \{y \in X \mid (x, y) \in e\}, e[B] = \bigcup_{x \in B} e[x]$;
2) $T(e[x], x) = \bigcup_{i \in T} (e[x] \times \{x\})$; and
3) $\varepsilon_n$ stands for an entourage in $\mathcal{U}_X$ with $(\varepsilon_n)^n \subset \varepsilon$ for all $n \geq 2$. In other words, if $x, z \in X$ are such that

$$(x, y_1) \in \varepsilon_n, (y_1, y_2) \in \varepsilon_n, \ldots, (y_{n-2}, y_{n-1}) \in \varepsilon_n, \text{ and } (y_{n-1}, z) \in \varepsilon_n,$$

then $(x, z) \in \varepsilon$.

See, e.g., [39, 35] and [4, Appendix II].

**Standing notation 1.3.** 1. $(T, X)$ is called a surjective semiflow if each $t \in T$ is a surjective self-map of $X$, i.e., $tX = X$ for all $t \in T$;
2. $(T, X)$ is said to be invertible if each $t \in T$ is bijective. In this case, by $\langle T \rangle$ we will denote the smallest group of self-homeomorphisms of $X$ containing $T$, and then $\langle T \rangle, X$ is a flow.

However, it should be noted that since $T$ is in general neither a syndetic nor a normal sub-semigroup of $(T, X)$ and $(\langle T \rangle, X)$ do not possess the same dynamics in general. In fact, contrary to what one might hope or expect, the passage from group to the semigroup case is not straightforward in many important cases; cf., e.g., [21]. First let us see an explicit example for this as follows.

**Examples 1.4.** 1. There exists an invertible semiflow $(T, X)$ such that there are points of $X$ which are almost periodic for $(T, X)$ but not for $(\langle T \rangle, X)$.

**Proof.** Indeed, let $X = [-1, 2]$ with the usual topology and for every $\alpha$, $0 < \alpha < 1$, define two self homeomorphisms of $X$ as follows:

$$f_\alpha : X \to X, \begin{cases} x \mapsto x & \text{if } -1 \leq x \leq 0, \\ x \mapsto \alpha x & \text{if } 0 \leq x \leq 1, \\ x \mapsto (2 - \alpha)x + 2(\alpha - 1) & \text{if } 1 \leq x \leq 2; \end{cases}$$

and

$$g_\alpha : X \to X, \begin{cases} x \mapsto (2 - \alpha)x + (1 - \alpha) & \text{if } -1 \leq x \leq 0, \\ x \mapsto 1 - \alpha(1 - x) & \text{if } 0 \leq x \leq 1, \\ x \mapsto x & \text{if } 1 \leq x \leq 2. \end{cases}$$

Now let $T = \langle f_\alpha, g_\alpha \mid 0 < \alpha < 1 \rangle^*$ be the discrete semigroup generated by $f_\alpha, g_\alpha$, $0 < \alpha < 1$. It is easy to see that each $t \in T$ is bijective so that $(T, X)$ is invertible. And $\Lambda = \{0, 1\}$ is minimal for $(T, X)$ so that every point of $\Lambda$ is almost periodic for $(T, X)$ but not for $(\langle T \rangle, X)$. In fact, since for every $x \in \Lambda$ we have $\text{cls}_X(T)x = X$, and $-1$ and $2$ are fixed points, which are the only almost periodic points of $(\langle T \rangle, X)$, thus $x \in \Lambda$ is not almost periodic for $(\langle T \rangle, X)$. Here each $t \in T \setminus \{e\}$ restricted to $\Lambda$ is not surjective.

2. There is an invertible semiflow $(T, X)$ on a compact metric space $X$ such that each orbit $Tx$ is not dense in $X$ but $(\langle T \rangle, X)$ has a residual set of points that have dense orbits.

**Proof.** In fact, $(T, X)$ above is such a semiflow. The orbit of every point $x \in (-1, 2)$ is not dense in $X$ for $(T, X)$ but dense for $(\langle T \rangle, X)$. \[3\]
Standing assumption 1.5. 1. The phase semigroup of any semiflow is a discrete infinite semigroup with a neutral element $e$. In this case, every compact subset of $T$ is finite.

2. The phase space of any semiflow is always assumed to be a compact $T_2$-space with the compatible uniform structure $\mathcal{U}$.

1.1. Basic notions

Let $(T, X)$ be a semiflow with phase semigroup $T$. We then first introduce or unify the most basic and important dynamics notions needed throughout this paper.

1.1.1. Equicontinuity by $\epsilon$-$\delta$

(a) $(T, X)$ is called equicontinuous in case given $\epsilon \in \mathcal{U}_X$ there exists a $\delta \in \mathcal{U}_X$ such that for all $t \in T$, $t(x, y) \in \epsilon$ if $x, y \in X$ with $(x, y) \in \delta$.

(b) We say $(T, X)$ is equicontinuous at $x \in X$, denoted $x \in \text{Equi} (T, X)$, if for all $\epsilon \in \mathcal{U}_X$ there is a $\delta \in \mathcal{U}_X$ such that $t(\delta[x]) \subseteq \epsilon(x)$ for all $t \in T$; or equivalently, $T(\delta[x], x) \subseteq \epsilon$.

By Definition (a), the equicontinuity of $(T, X)$ is independent of the topology of the phase semigroup $T$. In addition, since here $X$ is a compact $T_2$-space, thus it holds that:

Lebesgue’s covering lemma (cf. [39, 38]). If $\mathcal{V}$ is an open cover of $X$, then there exists a ‘Lebesgue number’ $\delta \in \mathcal{U}_X$ such that given $x \in X$, $\delta[x] \subseteq V$ for some $V \in \mathcal{V}$.

We can then easily obtain the following basic uniformity fact:

**Lemma 1.6.** If $\text{Equi} (T, X) = X$, then $(T, X)$ is equicontinuous.

**Proof.** Let $\text{Equi} (T, X) = X$. For $\epsilon \in \mathcal{U}_X$ and $x \in X$, there is a $\delta_x \in \mathcal{U}_X$ with $T(\delta_x[x], x) \subseteq \epsilon/2$.

Since $X$ is compact, by the Lebesgue covering lemma there exists a $\delta \in \mathcal{U}_X$ such that for all $y \in X$, $\delta[y] \subseteq \delta_x[x]$ for some $x \in X$. So by triangle inequality, $T(\delta[y], y) \subseteq \epsilon$. Since $\epsilon$ and $y$ are arbitrary, $(T, X)$ is equicontinuous.

1.1.2. Minimality

(c) A subset $A$ of $X$ is invariant if $T x \subseteq A$ for all $x \in X$, or equivalently, $T A \subseteq A$. It is negatively invariant if $t^{-1} A \subseteq A$ for all $x \in A$ and $t \in T$. 

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A subset \( \Lambda \) of \( X \) is referred to as minimal if \( \Lambda \) is a non-empty, closed, and invariant set containing no proper subsets with those properties. If \( X \) itself is minimal, then we call \((T, X)\) a minimal semiflow.

An \( x \in X \) is called a minimal point if \( \text{cls}_T X \) is a minimal set of \((T, X)\). If every point of \( X \) is minimal, then \((T, X)\) is called pointwise minimal.

Clearly, the minimality is also independent of the topology of \( T \); and \( \Lambda \) is minimal if and only if it is exactly the orbit closure of each of its points.

Moreover, if \((T, X)\) is minimal, then \( TX = X \) but there is no the property that \( tX = X \forall t \in T \) in general. Let’s see such a counterexample, which is motivated by Furstenberg’s [30, p. 40] for the case that \( \alpha = 1/2 \).

**Example 1.7.** Let \( X = [0, 1] \) be the unit interval with the usual topology and for each \( \alpha \) with \( 0 < \alpha < 1 \), define two injective mappings of \( X \) into itself as follows:

\[
f_\alpha : X \rightarrow X, \quad x \mapsto \alpha x
\]

and

\[
g_\alpha : X \rightarrow X, \quad x \mapsto 1 - \alpha(1 - x).
\]

Now let \( T = \langle f_\alpha, g_\alpha \mid 0 < \alpha < 1 \rangle \) be the discrete free semigroup generated by \( f_\alpha, g_\alpha, 0 < \alpha < 1 \).

It is easy to see that each \( t \in T \) is injective and \((T, X)\) is equicontinuous minimal, but each \( t \in T, t \neq e \), is not surjective.

However, we will show that this is actually in the affirmative for some special phase semigroups; see Propositions 3.7 and 3.16 and Corollary 3.18. In addition minimality will be equivalently described by ‘almost periodicity’; see Lemma 2.6 in §2.2.

### 1.1.3. Distality, proximity and regional proximity

The concept of “distality” has been proved to be a very fruitful one for topological dynamics of flows, giving rise to a rather extensive theory; see [26, 31, 4]. We will discuss this for semiflows in this paper.

We say that \( x \in X \) is proximal to \( y \in X \), write \((x, y) \in P(T, X)\) or \( P(X) \) or \( y \in P[x] \), if there exist a net \( \{t_n\} \) in \( T \) and \( z \in X \) with \( t_n(x, y) \rightarrow (z, z) \). By definition,

\[
P(X) = \bigcap_{\alpha \in \mathcal{Y}_X} \bigcup_{t \in T} t^{-1} \alpha.
\]

\((T, X)\) is called distal if for all \( x, y \in X \) with \( x \neq y \), one can find some \( \alpha \in \mathcal{Y}_X \) with \( \tau(x, y) \neq \alpha \) for every \( t \in T \).

An \( x \in X \) is called a distal point of \((T, X)\) if there exists no point other than itself in \( \text{cls}_T X \) to be proximal to it under \((T, X)\).

\((T, X)\) is called a point-distal semiflow if there exists a point \( x \in X \) such that \( x \) is a distal point of \((T, X)\) and \( TX \) is dense in \( X \) (cf. Veech [53]).
If \((T, X)\) is a point-distal flow with \(x\) a distal point, then each of \(Tx\) is a distal point. This is also true in surjective semiflows as follows:

**Lemma 1.8.** Let \((T, X)\) be a point-distal semiflow. Then \((T, X)\) is surjective if and only if the set of distal points is dense in \(X\).

**Proof.** The “only if” part. Let \(x\) be a distal point of \((T, X)\) with \(X = \text{cls}_XTx\). Then \(s^{-1}(sx) = x\) for all \(s \in T\). This implies that \(sx\) is distal for \((T, X)\), for all \(s \in T\). Indeed, otherwise, there is some \(y \in \text{cls}_XTsx\) with \(y \neq sx\) such that \((y, sx) \in P(X)\); so \((z, x) \in P(X)\) and \(z \neq x\), for all \(z \in s^{-1}y \neq 0\); this is a contradiction. We will postpone the proof of the “if” part of Lemma 1.8 in §6.1 using Ellis’ semigroup.

It should be mentioned that a distal point \(x\) does not satisfy that every \(y \in X\setminus \text{cls}_XTx\) is not proximal to \(x\) under \((T, X)\) unless \((T, X)\) is pointwise minimal. For example, let \(f : I \to I, x \mapsto x^2\) where \(I = [0, 1]\); then \(x = 0\) is a distal point but \((x, y) \in P(X)\) for all \(0 < y < 1\). Moreover, it is evident that

\[
(T, X) \text{ is distal iff } P(X) = A_X \text{ (cf. [26, Lemma 5.12] for } T \text{ in groups).}
\]

Thus a minimal semiflow is distal at a point \(x\) if and only if the proximal cell \(P[x]\) is pointwise minimal. For example, let \(f : I \to I, x \mapsto x^2\) where \(I = [0, 1]\); then \(x = 0\) is a distal point but \((x, y) \in P(X)\) for all \(0 < y < 1\). Moreover, it is evident that

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\[
(T, X) \text{ is distal iff } P(X) = A_X \text{ (cf. [26, Lemma 5.12] for } T \text{ in groups).}
\]
A flow \((T, X)\) is equicontinuous iff it is uniformly distal.

**Proof.** Let \((T, X)\) be an equicontinuous flow with \(\varepsilon - \delta\) as in §1.1.1 (a). Then if \((x, y) \notin \varepsilon\), then \((t^{-1}x, t^{-1}y) \notin \delta \forall t \in T\). Since \(T^{-1} = T\) for \(T\) is a group, so \((T, X)\) is uniformly distal. Conversely, assume \((T, X)\) is uniformly distal with \(\varepsilon - \delta\) as in Definition 1.9. It is obvious that \((x, y) \in \delta\) implies that \((tx, ty) \in \delta\) for all \(t \in T\) by \(T^{-1} = T\) again. □

Notice that the group structure of \(T\) plays a role in both of the “if” and “only if” parts. However, since there is no \(T = T^{-1}\) for a general semiflow with \(T\) not a group, hence according to Example 1.7 “Equicontinuous \(\Leftrightarrow\) Uniformly distal” is not obvious for semiflows with which we will be mainly concerned. See Theorem 1.14 below.

Let \((T, X)\) be an arbitrary semiflow. Next we will introduce another important relation which is weaker than proximity on \(X\).

(j) We say that \(x \in X\) is **regionally proximal** to \(y \in X\), denoted \((x, y) \in Q(T, X)\) or \(Q(X)\), if there are nets \([x_n], [y_n]\) in \(X\) and \([t_n]\) in \(T\) such that \(t_n(x_n, y_n) \to (z, z)\) for some \(z \in X\).

Clearly,

\[
Q(X) = \bigcap_{x \in X} \text{cls}_{X \times X} \bigcup_{t \in T} t^{-1} \alpha
\]

is a closed symmetric reflexive relation on \(X\). Then:

**Lemma 1.10.** Let \((T, X)\) be a semiflow with \(Q(X) = \Delta_X\). Then \(X \times T \to X\), which is defined by \((x, t) \mapsto xt := t^{-1}x\), is equicontinuous.

**Proof.** Given \(\varepsilon \in \mathcal{B}_X\), by \(Q(X) \subset \varepsilon\) and by the finite intersection property, there is some \(\delta \in \mathcal{B}_X\) such that \(\text{cls}_{X \times X} \bigcup_{t \in T} t^{-1} \delta \subset \varepsilon\). Thus \(\delta t \subset \varepsilon\) for all \(t \in T\). This shows that \(X \times T \to X\) is equicontinuous. □

It should be noticed that although \(P(X)\) and \(Q(X)\) both are reflexive symmetric relations on \(X\), yet if \(T\) is non-abelian they need not be invariant in our semigroup setting. In view of this, even if \(P(X)\) and \(Q(X)\) are closed equivalence relations on \(X\), \((T, X/P)\) and \((T, X/Q)\) do not need to make sense in general semiflows.

1.1.4. Amenability and C-semigroup

It is known that the structure of a topological semigroup is closely related to some dynamics of its actions; see e.g. [14]. We will consider here two kinds of phase semigroups as follows.

(k) A discrete semigroup \(T\) is called **amenable** if every semiflow \((T, Y)\) with the phase semigroup \(T\) permits an invariant Borel probability measure, i.e., there is a Borel probability measure \(\mu\) on \(Y\) such that \(\mu(B) = \mu(t^{-1}B) \forall t \in T\) for all Borel subset \(B \subseteq Y\) (cf. [19, 18]).

(l) Let \(T\) be a topological semigroup, not necessarily discrete; then \(T\) is called a **C-semigroup** if \(T \setminus sT\) and \(T \setminus Ts\) are relatively compact in \(T\) for all \(s \in T\) (cf. [41]).

In particular each abelian semigroup is amenable by the classical Markov-Kakutani fixed-point theorem. If \(T\) is a topological group, then \(sT = Ts = T\) for all \(s \in T\) so it is a C-semigroup. Clearly, \(T = (\mathbb{Z}, +)\) is a C-semigroup. In addition, under the usual non-discrete topology, \(T = (\mathbb{R}, +)\) is a C-semigroup, but not under the discrete topology.
1.1.5. Ellis enveloping semigroups

Let \( X^\times \) be the set of all functions from \( X \) to itself, continuous or not. The topology of pointwise convergence for \( X^\times \) is defined as follows: A net \( \{ f_n \} \) in \( X^\times \) converges to \( f \) if and only if \( f_n(x) \to f(x) \) for each \( x \in X \) (cf. [39]). A subbase of this topology is the family of all subsets of the form \( \{ f \mid f(x) \in U \} \), where \( x \) is a point of \( X \) and \( U \) is open in \( X \).

Then we recall several notions based on the semiflow \((T, X)\) as follows:

(m) By \( E(T, X) \) or simply \( E(X) \), we denote the Ellis semigroup of \((T, X)\); that is, \( E(X) \) is the closure of \( T \) in \( X^\times \) in the sense of the pointwise topology (cf., e.g., [26, 30, 4]).

(n) An element \( u \in E(X) \) is called an idempotent in \( E(X) \), denoted \( u \in J(E(X)) \), if \( u^2 = u \).

(o) \( I \neq \emptyset \) is called a minimal left ideal in \( E(X) \) if \( E(X)I \subseteq I \) and no proper non-empty subset of \( I \) has this property.

- Since \( E(X) \) is a compact right-topological semigroup (i.e., \( E(X) \) is a semigroup and a compact \( T_2 \)-space with \( R_q: p \mapsto pq \) continuous, for all \( q \in E(X) \)), there always exists an idempotent in each minimal left ideal in \( E(X) \) (cf. [26, 4]).

- Moreover, \( (x, x') \in P(X) \) iff \( \exists p \in E(X) \) with \( p(x) = p(x') \) iff there is a minimal left ideal \( I \) in \( E(X) \) such that \( p(x) = p(y) \forall p \in I \).

Clearly \( E(X) \) associated to \((T, X)\) is independent of the topology of the phase semigroup \( T \). For a flow \((T, X)\) we will consider whether or not \( T \) is a topological group under the pointwise topology in §8.

The proof of the following basic lemma is taken nearly word-for-word from [26, 2 of Proposition 5.16]. We will postpone the details in §6.1 following Lemma 6.3.

**Lemma 1.11** (cf. [26, 4] for \( T \) in groups). *Given any semiflow \((T, X)\), \( P(X) \) is an equivalence relation on \( X \) iff there is only one minimal left ideal in \( E(X) \).*

1.2. Main theorems

Although its proof is very easy (cf., e.g., [26, 4, 29]), yet it is a very useful important fact in topological dynamics that

\[ \text{If } (T, X) \text{ is an equicontinuous flow, then it is distal} \text{ (cf. [26, Proposition 4.4 and Corollary 5.4]).} \]

In fact, an equicontinuous flow is uniformly distal by §1.1.3(●). We note that the group structure of \( T \) plays a role in its various proofs available in the literature (cf. [26, 4, 29]). Moreover, if \( T \) is only a semigroup, the above important result need not be true. For instance, Example 1.7 is equicontinuous but not distal with \( P(X) = X \times X \neq \Delta_X \).

Let us see a more simple counterexample with abelian phase semigroup, which together with Examples 1.4 shows that some dynamics in semiflows are very different with the flow case.

**Example 1.12.** Let \( X = \{a, b, c\} \) be a discrete space and we let \( f: X \to X \) by \( a \mapsto b \mapsto c \mapsto c \). Then the cascade \((f, X)\) with phase semigroup \( \mathbb{Z}_+ \) is equicontinuous but it is not distal.

Here \( T \) is abelian and \((T, X)\) is not minimal. Of course, if \((T, X)\) is minimal equicontinuous with \( T \) an abelian (or more general, amenable) semigroup, then we shall show it is uniformly distal (cf. Corollary 3.3).
In both of Examples 1.7 and 1.12, each \( t \in T \) is not surjective. In view of this, recently Ethan Akin and Xiangdong Ye have independently suggested (in personal communications) the following assertion:

\[
\text{If } (T, X) \text{ is an equicontinuous surjective semiflow on a compact metric space with } T \text{ an abelian semigroup, then } (T, X) \text{ is distal.}
\]

In fact, by constructing an equivalent isometric metric \( d_T \) on \( X \), Akin’s [1, (d) of Proposition 2.4] implies that \( (T, X) \) is distal if \( X \) is a compact metric space with phase semigroup \( T = \mathbb{Z}_+ \).

In §2, using several different approaches, we shall be able to prove the Akin–Ye assertion without the metric condition on the phase space \( X \) and with no the abelian hypothesis on the phase semigroup \( T \).

Precisely speaking we shall prove the following theorem, which consists of Theorem 2.1, Lemma 6.7, (1) of Proposition 6.10, and Theorem 6.12.

**Theorem 1.13.** Let \( (T, X) \) be a semiflow with phase semigroup \( T \). Then the following statements hold:

1. If \( (T, X) \) is equicontinuous surjective, then it is distal.
2. If \( (T, X) \) is distal, then it is invertible.
3. If \( (T, X) \) is distal, then so is \( \langle (T), X \rangle \).
4. If \( (T, X) \) is invertible equicontinuous, then \( \langle (T), X \rangle \) is an equicontinuous flow.

Since \( (T) \) is in general much more bigger than \( T \), (3) and (4) of Theorem 1.13 are non-trivial. They are useful in our later applications.

By (1) and (2) of Theorem 1.13, an equicontinuous semiflow is surjective iff it is distal (cf. Corollary 6.8). However, ‘surjective’ is naturally satisfied in many interesting cases such as ‘homogeneous’ semiflow (cf. Proposition 3.2), minimal semiflow with ‘amenable’ phase semigroup (cf. Proposition 3.7), and \( \ell \)-recurrent semiflow with \( C \)-semigroup \( T \) (cf. Corollary 3.17).

Recall that ‘equicontinuity’ asserts that if two initial points \( x, y \) are sufficiently close, then their orbits \( T x \) and \( T y \) are synchronously close. And ‘uniform distality’ asserts that if two initial points \( x, y \) are sufficiently far away, then their orbits \( T x \) and \( T y \) are synchronously far away.

Thus, “equicontinuity” looks very different with “uniform distality”. However, as a result of Theorem 1.13 we can obtain that they are in fact equivalent to each other.

**Theorem 1.14.** A semiflow \( (T, X) \) is uniformly distal if and only if it is equicontinuous surjective.

**Proof.** (1) The “if” part: By Theorem 1.13, \( (T, X) \) is invertible and further \( \langle (T), X \rangle \) is an equicontinuous flow. Then by §1.1.3(三星), it follows that \( (T, X) \) is uniformly distal.

(2) The “only if” part: First by Theorem 1.13, \( (T, X) \) is invertible. Then by Definition 1.9, it follows that given \( \varepsilon \in \mathbb{H}_X \), there is a \( \delta \in \mathbb{H}_X \) such that if \( (x, y) \in \delta \) then \( (t^{-1} x, t^{-1} y) \in \varepsilon \) for all \( t \in T \) and \( x, y \in X \). Thus \( T^{-1} \) acts equicontinuously on \( X \). Whence by (4) of Theorem 1.13, \( \langle (T^{-1}), X \rangle \) is equicontinuous so that \( (T, X) \) is invertible equicontinuous. This proves Theorem 1.14.

Let \( (T, X) \) be a semiflow and \( M \) a closed invariant subset of \( X \). If \( (T, X) \) is equicontinuous, then so is \( (T, M) \). However, if \( (T, X) \) is surjective (even though invertible), \( (T, M) \) need not be surjective. For instance, 1 of Examples 1.4. We can here construct a more simple example as follows. Let \( f : [0, 1] \to [0, 1], x \mapsto x^2 \) and \( M = [0, 1/2] \); then \( f(M) \subseteq M \).
As a consequence of Theorem 1.14, we can easily obtain the following, which is evident if $T$ is a group because $tM \subseteq M \forall t \in T$ with $T = T^{-1}$ implies that $M \subseteq t^{-1}M \subseteq M \forall t \in T$ so that $tM = M \forall t \in T$. However, for a semigroup $T$, it becomes non-trivial.

**Corollary 1.15.** Let $M$ be a closed invariant subset of a semiflow $(T, X)$. If $(T, X)$ is equicontinuous surjective, then $(T, M)$ is also equicontinuous surjective.

**Proof.** By Theorem 1.14, $(T, X)$ is uniformly distal so that $(T, M)$ is also uniformly distal. Then $(T, M)$ is equicontinuous surjective by Theorem 1.14 again.

If $(T, X)$ is a flow with $Q(X) = \Delta X$, then by Lemma 1.10 it is equicontinuous. By using (1) of Theorem 1.13, ‘pointwise almost automorphy’ and ‘Veech’s relation’ $V(X)$, Dai and Xiao in [17] have proved the following fact (Corollary 1.16). However, we now can simply prove it by only using Lemma 1.10 and Theorem 1.13.

**Corollary 1.16** (cf. [17, Theorem 5.4]). A semiflow $(T, X)$ is equicontinuous surjective if and only if $Q(X) = \Delta X$.

**Proof.** Let $(T, X)$ be equicontinuous surjective. Then by equicontinuity, $P(X) = Q(X)$. Now by (1) of Theorem 1.13, we see $Q(X) = \Delta X$. Conversely, assume $Q(X) = \Delta X$. Since $P(X) = \Delta X$, $(T, X)$ is distal and so invertible by (2) of Theorem 1.13. Then by Lemma 1.10 and (4) of Theorem 1.13, $(T, X)$ is equicontinuous surjective. This proves Corollary 1.16.

As be mentioned before, if using no (2) and (4) of Theorem 1.13, then we need a long zigzag proof for this result as in [17].

In 1963 Furstenberg proved that “If $X$ is simply connected non-trivial, then $X$ does not admit of a minimal distal flow for any locally compact abelian group $T$” (cf. [29, Theorem 11.1]). It is natural to ask if there admits of a minimal distal semiflow or not. In fact, by (3) of Theorem 1.13 there exists no minimal distal semiflow too.

**Theorem 1.17.** If $X$ is simply connected non-trivial, then $X$ does not admit of a minimal distal semiflow for any locally compact abelian semigroup $T$.

**Proof.** Assume $(T, X)$ is a minimal distal semiflow with $T$ a locally compact abelian semigroup. Then under the discrete topology of $(T)$, $(T, X)$ is a minimal distal flow by (3) of Theorem 1.13. This then contradicts Furstenberg’s theorem.

In particular, the $n$-sphere, $n \geq 2$, cannot support a minimal distal semiflow for any abelian semigroup $T$.

### 1.3. Applications

There has already been some applications of Theorem 1.13 in the recent work [17] and in the proofs of Theorems 1.14 and 1.17 and Corollary 1.16. Next we shall give some other applications here.

First, let $\sigma: \Sigma_k^+ \to \Sigma_k^+$ be the shift map of the symbolic space $\Sigma_k^+ = [0, 1, \ldots, k-1]^\mathbb{Z}_+$, where $k \geq 2$. Suppose $(\sigma, X)$ is a subsystem of $(\sigma, \Sigma_k^+)$ such that $X$ is any infinite, closed, $\sigma$-invariant subset of $\Sigma_k^+$. Then by Theorem 1.13, it follows that:
(9) If \((\sigma, X)\) is equicontinuous, then it is not surjective.

\textbf{Proof.} Indeed, if \((\sigma, X)\) is surjective, then by Theorem 1.13 it is distal. However, there necessarily exists a pair of proximal points for \((\sigma, X)\) (cf. \cite[p. 158]{30}).

\textbf{Definition 1.18.} An invertible \((T, X)\) defines a semiflow \(X \times T \to X\) by \((x, t) \mapsto xt = r^{-1}x\). Here \((X, T)\) will be called the reflection of \((T, X)\), which is also thought of as the ‘history’ of \((T, X)\).

It should be noted that the phase semigroup of the reflection \((X, T)\) is \(T\), not \(T^{-1}\), with the \textit{discrete} topology in general.

\textbf{Remarks 1.19.} 1. Let \(G\) be a non-discrete topological group. If \((G, X)\) is a flow and if \(T\) is a subsemigroup of \(G\), then \((X, T)\) is of course a semiflow where \(T\) with the non-discrete topology inherited from \(G\).

2. If \((T, X)\) is invertible with \(T\) a locally compact \(C\)-semigroup and if \(t_0 \to t\) in \(T\) implies that \(r^{-1}x \to r^{-1}x\) for all \(x \in X\), then \((x, t) \mapsto xt\) is jointly continuous by Corollary 8.17 in Appendix so \((X, T)\) is a semiflow where \(T\) with the non-discrete topology.

If \((T, X)\) is a minimal cascade corresponding to a \(\mathbb{Z}_+\)-action, then for every minimal set \(X_0\) of its reflection \((X, T)\) we have \(X_0T = X_0\) and furthermore \(X_0 = TX_0\); so \(X_0 = X\). This indicates that \((X, T)\) is also minimal. However, for an invertible semiflow \((T, X)\) with \(T \neq \mathbb{Z}_+, \ 'X_0T = X_0'\) need not imply \(X_0 = TX_0\). Moreover, \((T, X)\) and \((X, T)\) do not share the same dynamics in general. For example, a recurrent/transitive point of a cascade \((T, X)\) need not be a recurrent/transitive point of its reflection \((X, T)\). In addition, in 1 of Examples 1.4, every point \(x\) of \(\Lambda\) is minimal for \((T, X)\) but not minimal for its reflection \((X, T)\); for otherwise, \(x\) is minimal for \((\langle T \rangle, X)\).

Nevertheless, as applications of Theorem 1.13, we will consider in §6 the minimality, distality, and equicontinuity dynamics of \((X, T)\) as \((T, X)\) itself possesses these dynamics. We will mainly show the following three reflection principles.

\textbf{Reflection principle I} (cf. Propositions 6.1 and 6.10). Let \(T \times X \to X\) be an invertible semiflow with the reflection \(X \times T \to X\). Then:

1. \((T, X)\) is equicontinuous if and only if \((X, T)\).
2. \((T, X)\) is minimal distal if and only if \((X, T)\).
3. \((T, X)\) is distal if and only if \((X, T)\).

Reflection principle I may be utilized for proving the “only if” part of Theorem 1.14 above. Moreover, it will be useful for us to show Furstenberg’s structure theorem of distal minimal semiflows (Theorem 6.14) in §6.2.

Here we are going to give another application. Let \((T, X)\) and \((T, Y)\) be two semiflows and \(\pi: X \to Y\) a continuous surjective map. If \(\pi(tx) = t\pi(x)\) for all \(t \in T\) and \(x \in X\), then \((T, Y)\) is called a factor of \((T, X)\) and \(\pi\) an epimorphism of \((T, X)\) and \((T, Y)\), denoted \(\pi: (T, X) \to (T, Y)\).

\textbf{Definition 1.20.} We will say that \((T, X)\) is a relatively equicontinuous (or an almost periodic) extension of \((T, Y)\) in case there is an epimorphism \(\pi: (T, X) \to (T, Y)\) such that for every \(\varepsilon \in \mathcal{H}_X\) there exists \(\delta \in \mathcal{H}_X\) satisfying that if \((x, x') \in \delta\) with \(\pi(x) = \pi(x')\) then \((tx, tx') \in \varepsilon \forall t \in T\) (cf. \cite[p. 95]{4}). If \(Y\) is one-pointed, then this reduces to Definition (a) in §1.1.1.

Theorem 1.21 below is actually a relativized version of (1) of Theorem 1.13 before. Its special case that \((T, X)\) is a skew-product semiflow driven by \((T, Y)\) with \(T = \mathbb{R}_+\) has been proven in \cite{46, 47} and \cite{48} by using Ellis’ enveloping semigroup.
Theorem 1.21 (cf. [29, Proposition 2.1] for $T$ in groups). If an invertible semiflow $(T, X)$ is a relatively equicontinuous extension of a distal semiflow $(T, Y)$, then $(T, X)$ is distal.

Proof. Let $\pi : (T, X) \to (T, Y)$ be a relatively equicontinuous epimorphism. First by Theorem 1.13, $(T, Y)$ is invertible and then $(Y, T)$ is distal by Reflection principle I. Note that $\pi : X \to Y$ is also a homomorphism from $(X, T)$ onto $(Y, T)$. We will show that $(X, T)$ is distal.

For that, let $x, x' \in X$ with $(x, x') \in P(X, T)$. Then by distality of $(Y, T)$, $\pi(x) = \pi(x')$. Taking a net $\{t_n\}$ in $T$ with $t^{-1}_n(x, x') \to (z, z)$ for some $z \in X$, by the relative equicontinuity of $(T, X)$ and $\pi(t^{-1}_n x) = \pi(t^{-1}_n x')$, it follows that $x = x'$ and so $(X, T)$ is distal. Again using Reflection principle I, $(T, X)$ is distal. This proves Theorem 1.21.

The above proof implies that if $\pi : (T, X) \to (T, Y)$ is a relatively equicontinuous epimorphism of flows $\pi$ is of distal type (cf., e.g., [4, p. 95]). It would be interesting to know if this holds for any invertible semiflow or not. Next for invertible semiflows with amenable semigroups we can obtain the following Reflection principle II.

Reflection principle II (cf. Theorem 6.19 and Proposition 6.20). Let $(T, X)$ be invertible with $T$ an amenable semigroup and $x \in X$. Then:

1. $x$ is a minimal point for $(T, X)$ if and only if it is for $(X, T)$. Moreover, if $x$ is a minimal point of $(T, X)$, then $\text{cls}_x TX = \text{cls}_x xT$.
2. $x$ is a distal point of $(T, X)$ if and only if $x$ is a distal point of $(X, T)$.

It should be mentioned that in light of Examples 1.4 the amenability of the phase semigroup $T$ is very important for the statement of Reflection principle II. As a result of our Reflection principle II, we can generalize Theorem 1.21 in amenable semigroups as follows:

Theorem 1.22. Let $T$ be an amenable semigroup and $(T, X), (T, Y)$ two minimal invertible semiflows. If $(T, X)$ is a relatively equicontinuous extension of $(T, Y)$ and $(T, Y)$ is point-distal, then $(T, X)$ is a point-distal semiflow.

Proof. Let $\pi : X \to Y$ be a relatively equicontinuous extension. Let $y \in Y$, with $\pi(x) = y$ for some $x \in X$, be a distal point of $(T, Y)$, then $y$ is also a distal point of $(Y, T)$ by Reflection principle II so that if $(x, x') \in P(X, T)$ then $\pi(x') = y$, i.e., $x' \in \pi^{-1}(y)$. Hence as in the proof of Theorem 1.21, we can easily show that $x' = x$. Thus $x$ is a distal point of $(X, T)$ and then of $(T, X)$.

In addition, using Reflection principle II in the classical case that $T = \mathbb{R}_+$ and $(T) = \mathbb{R}$ we can easily obtain the following:

If $\mathbb{R} \times X \to X$, $(t, x) \mapsto tx$ is a flow such that $\mathbb{R}_+ \times X \to X$, $(t, x) \mapsto tx$ has a distal point $x \in X$, then $x$ is a distal point of $\mathbb{R} \times X \to X$.

In particular, if $\mathbb{R}_+ \times X \to X$ is a distal semiflow, then $\mathbb{R} \times X \to X$ is a distal flow (cf. Sacker-Sell [47]).

The following is another consequence of Reflection principle II.

Theorem 1.23 (cf. [49] for $T = \mathbb{R}_+$). Let $(T, X)$ be invertible with $T$ an amenable semigroup such that $P((T), X)$ is an equivalence relation on $X$. Then $P((T), X) = P(T, X) = P(X, T)$.
Proof. We only prove \( P(T, X) = P(T, X) \). Clearly, \( P(T, X) \subseteq P(T, X) \). To show the converse inclusion, let \( (x, x') \in P(T, X) \). Let \( \mathbb{I} \) be the unique minimal left ideal in \( E((T), X) \) by Lemma 1.11. Then there exists some \( u \in \mathbb{I} \) with \( u(x) = u(x') \). We will show that \( u \in E(T, X) \).

For that, we need to consider the natural flow \( \pi_*: (T) \times E((T), X) \rightarrow E((T), X) \), which has only one minimal subset \( \mathbb{I} \). Clearly \( E(T, X) \subseteq E((T), X) \) and \( \pi_*: T \times E(T, X) \rightarrow E(T, X) \) is compatible with \( (T), E((T), X) \). We can choose a minimal subset \( K \subseteq E(T, X) \) with respect to \( (T, E(T, X)) \). Then by 1 of Reflection principle II, \( K \) is also \( T^{-1} \)-invariant so that \( K \) is \( (T) \)-invariant and then \( (T) \)-minimal. Thus \( K = \mathbb{I} \). This implies \( u \in E(T, X) \) and thus \( (x, x') \in P(T, X) \).

The proof of Theorem 1.23 is therefore complete.

Notice that in general, \( T \cup T^{-1} \not\subseteq (T) \) for an invertible semiflow \( (T, X) \). 2 of Reflection principle II says that an \( x \in X \) is a distal point of \( (T, X) \) iff \( x \) is a distal point of \( (X, T) \). However,

\[
\text{Is } x \text{ a distal point of the induced flow } (\langle T \rangle, X) ?
\]

It is exactly the localization problem of (3) of Theorem 1.13. As a consequence of Theorem 1.23, the answer to this question is YES in the setting of Theorem 1.23.

In the same situation of Theorem 1.23, an \( x \in X \) is a distal point of \( (T, X) \) iff \( x \) is a distal point of \( (\langle T \rangle, X) \).

In fact, this still holds without the condition that \( P((T), X) \) is an equivalence relation on \( X \); see 1 of Theorem 5.4 by purely topological methods.

We note that if \( T = \mathbb{R} \), in the proof of Theorem 1.23 and so \( (T) = \mathbb{R} \), then \( \mathbb{I} \) is contained in the \( \omega \)-limit set \( \omega(e) \) of \( e \in T \) under \( (\langle T \rangle, E((T), X)) \). Since \( e = id_x \in E(T, X) \) and \( \omega(e) \subseteq E(T, X) \) by \( \langle T \rangle = \mathbb{R} \), then \( \mathbb{I} \subseteq E(T, X) \). This is actually the idea of Yī’s proof in the \( \mathbb{R} \)-action case in [49, p. 7]. Clearly Yī’s idea does not work for our Theorem 1.23 here.

In addition, when \( (T) \) is abelian and if \( E(T, X) \subset C(X, X) \), then by Theorem 6.6 it follows that \( P((T), X) \) is an equivalence relation on \( X \). Moreover, if \( P((T), X) \) is closed in \( X \times X \), then it is an equivalence relation (cf. [4, Corollary 6.11]).

Next we consider invertible semiflows with \( C \)-semigroups as our phase semigroups instead of amenable semigroups.

Reflection principle III (cf. Theorem 6.31). Let \( (T, X) \) be invertible with \( T \) a \( C \)-semigroup not necessarily discrete. Then \( (T, X) \) is minimal iff so is \( (X, T) \).

Comparing with Reflection principle II, we pose the following

Question 1.24. Let \( (T, X) \) be invertible with \( T \) a \( C \)-semigroup and \( x \in X \). Then, does it hold that \( x \) is minimal for \( (T, X) \) iff so is \( x \) for \( (X, T) \)?

As a simple application of our reflection principles, we will study the sensitivity of semiflows and their reflections in §7.

2. Distality of equicontinuous surjective semiflows

Recall that a semiflow \( (T, X) \) is distal if and only if no diagonal pair is proximal (cf. §1.1.3 (g)). This section will be mainly devoted to proving (1) of Theorem 1.13 using three different approaches, which asserts that every equicontinuous surjective semiflow is distal. That is the following Theorem 2.1.
Theorem 2.1. If \((T, X)\) is an equicontinuous surjective semiflow, then \((T, X)\) is distal.

Our new approaches (Proofs (I), (II) and (III) below) introduced in proving Theorem 2.1 are all certainly valid for flows with phase groups.

2.1. Using pointwise recurrence of transition maps

For Proof (I) of Theorem 2.1, in preparation we first recall a classical notion. Let \(f\) be a continuous self-map of \(X\). A point \(x \in X\) is said to be (forwardly) recurrent if there is a net \(\{n_e\}\) in \(\mathbb{Z}_+\) with \(n_e \to +\infty\) such that \(f^{n_e}(x) \to x\). Further \((f, X)\) is called pointwise recurrent if each point of \(X\) is recurrent for \((f, X)\). Then the following is easily seen by definition.

Lemma 2.2. If \(x \in X\) is a recurrent point of \((f, X)\), then \(x \in f(X)\).

Proof. Let \(x \in X\) be recurrent for \((f, X)\). Then there is a net \(\{n_e\}\) in \(\mathbb{Z}_+\) with \(n_e \to +\infty\) and 
\[ f^{n_e}(x) \to x \] 
so that \(f(f^{n_e-1}(x))) \to x\). Since \(X\) is compact \(T_2\), there is a subnet \(\{n_j\}\) of \(\{n_i - 1\}\) such that \(f^{n_j}(x) \to y \in X\) and thus \(f(y) = x\). Thus \(x \in f(X)\). \(\square\)

Thus if \((f, X)\) is pointwise recurrent, then \(f\) is surjective (cf. [2, Lemma 3.1]). The following simple observation is very useful for Theorem 2.1.

Lemma 2.3. Suppose that \(f: X \to X\) is equicontinuous surjective. Then every point of \(X\) is recurrent for \((f, X)\).

Proof. Let \(x \in X\) and define \(\{x_n\}\) inductively by \(f(x_1) = x\), \(f(x_2) = x_1\), \ldots, \(f(x_n) = x_{n-1}\), \ldots. Let \(\varepsilon \in \mathcal{H}_X\) and let \(\delta\) correspond to \(\varepsilon\) in the definition of equicontinuity. Let \(n > 0\) and \(s > 0\) be integers such that \((x_n, x_{n+s}) \in \delta\), so \((f^{n+s}(x_n), f^{n+s}(x_{n+s})) \in \varepsilon\). Then \((x, f^i(x)) \in \varepsilon\) and thus \(x\) is (forwardly) recurrent for \((f, X)\). \(\square\)

Now we can prove Theorem 2.1 by using the pointwise recurrence as follows:

Proof (I) of Theorem 2.1. For \(t \in T\), \((t, X \times X)\) is equicontinuous surjective and thus by Lemma 2.3 it is pointwise recurrent. Suppose \((y, y') \in P(T, X)\) with \(y \neq y'\). Let \(\varepsilon \in \mathcal{H}_X\) such that \((y, y') \notin \varepsilon\). Let \(\delta \in \mathcal{H}_X\) correspond to \(\varepsilon/3\) as in §1.1.1 (a). Since \(y\) is proximal to \(y'\), we now can take \(\tau \in T\) such that \((\tau y, \tau y') \in \delta\), so \((\tau^n y, \tau^n y') \in \varepsilon/3\) for all \(n > 0\). Then there cannot be \(n_j\) with \(\tau^n(y, y') \to (y, y')\). But this contradicts the pointwise recurrence. The proof of Theorem 2.1 is thus completed. \(\square\)

2.2. Using almost periodicity

Let \((T, X)\) be a semiflow with \(T\) a topological semigroup not necessarily discrete here. We will first recall the concept of “almost periodicity”.

Definition 2.4 ([33, 30]).

(i) A subset \(A\) of \(T\) is said to be (left-)\(\text{thick}\) in \(T\) if for all compact subset \(K\) of \(T\) one can find some \(s \in T\) such that \(K \cap s \subseteq A\).

(ii) A subset \(A\) of \(T\) is called (left-)\(\text{syndetic}\) in \(T\) if there is a compact subset \(K\) of \(T\) with \(K \cap s \neq \emptyset\) for every \(t \in T\). (Here “left-” corresponds to the left-action of \(T\) on \(X\).)

(iii) A point \(x \in X\) is called \(\text{almost periodic}\) if \(N_T(x, U)\) is syndetic in \(T\) for all neighborhood \(U\) of \(x\) in \(X\). If every point of \(X\) is almost periodic, then \((T, X)\) is called \(\text{pointwise almost periodic}\).
(iv) \((T, X)\) is called uniformly almost periodic (u.a.p.) if given \(\varepsilon \in \mathcal{B}_X\), there exists a syndetic set \(A\) in \(T\) such that \(A x \subseteq s[x]\) for all \(x \in X\).

Here “thick set” is weaker than the notion—replete set [35, Definition 3.37] that requires containing some bilateral translate \(K \cup s K\) of each compact subset \(K\) of \(T\).

Given \(k \in T\), let \(L_k: T \rightarrow T, t \mapsto kt\) be the left translation mapping of \(T\). Then for subsets \(K, A\) of \(T\), we simply write
\[
K^{-1}A = \bigcup_{k \in K} L_k^{-1}[A], \quad \text{where } L_k^{-1}[A] = \{t \in T | kt \in A\}.
\]

Since here \(T\) is only a semigroup, \(K^{-1}A\) is possibly empty. If \(e \in K\) then \(A \subseteq K^{-1}A\). Thus a subset \(A\) of \(T\) being syndetic in \(T\) can be equivalently described as follows:

**A is syndetic in \(T\) if and only if there exists a compact subset \(K\) of \(T\) with \(T = K^{-1}A\).**

Note that in some literature, an “almost periodic” point is defined as \(N_T(x, U)\) is “syndetic” in the sense that there is a compact subset \(K\) of \(T\) with \(T = KN_T(x, U)\).

It should be mentioned that “\(T = K^{-1}N_T(x, U)\)” in (iii) of Definition 2.4 is not permitted to be replaced by “\(T = KN_T(x, U)\)” in semiflows; see [9, Proposition 4.8] for a counterexample which says that there is a semiflow on a compact metric space such that it has an almost periodic point in the sense of (iii) of Definition 2.4 but has no “almost periodic” points in the latter sense.

The following two equivalent conditions will be very useful for our later arguments involving almost periodicity.

**Lemma 2.5** (cf. [30] for \(T = \mathbb{Z}_+\)). A subset \(S\) of \(T\) is syndetic in \(T\) if and only if \(S \cap R \neq \emptyset\) for each thick set \(R\) in \(T\).

**Proof.** Let \(S\) be syndetic in \(T\) and let \(K\) a compact subset of \(T\) defined by syndeticity of \(S\). Then for each thick set \(R\) in \(T\), there is some \(t_0 \in T\) with \(Kt_0 \subseteq R\). Since \((Kt_0) \cap S \neq \emptyset\), hence \(R \cap S \neq \emptyset\).

Conversely, let \(S \cap R \neq \emptyset\) for all thick set \(R\) in \(T\). If \(S\) is not syndetic, then for each compact subset \(K\) of \(T\) there is \(t_K \in T\) such that \(Kt_K \cap S = \emptyset\). Set \(R = \bigcup_{t \in X} Kt_K\) where \(X\) is the set of all non-empty compact subsets of \(T\). Clearly \(R\) is thick in \(T\), but \(S \cap R = \emptyset\), a contradiction. This proves Lemma 2.5.

Since our phase space \(X\) is a compact \(T_2\)-space, every orbit closure contains a minimal set by Zorn’s lemma. So it contains an almost periodic point by the following basic result (cf. [33, 30, 9]). We will present a proof here for self-closeness.

**Lemma 2.6** (cf. [35] and [26, Proposition 2.5] for \(T\) in groups). Let \((T, X)\) be a semiflow where \(T\) not necessarily discrete; then a point \(x\) of \(X\) is almost periodic iff it is a minimal point.

**Notes.**
1. Instead of ‘compact \(X\)’ ‘regular \(X\)’ is enough for the necessity.
2. Moreover, this lemma shows that the almost periodicity of any point of \(X\) is independent of the topology of the phase semigroup.

**Proof.** Let \(x\) be almost periodic for \((T, X)\); and if \(\Lambda\) is a minimal subset of \(\text{cls}_x T x\) with \(x \notin \Lambda\), there are neighborhoods \(U\) of \(x\) and \(V\) of \(\Lambda\) such that \(U \cap V = \emptyset\). For every compact subset \(K\) of \(T\) and \(y_0 \in \Lambda\), there is a \(\delta \in \mathcal{B}_X\) so small that \(K(\delta[y_0]) \subset V\). Since \(y_0 \in \delta[y_0]\) for some \(y_0 \in T\), then \(Kt_0 \times V\) so \(Kt_0 \subset N_T(x, V)\). Thus \(N_T(x, V)\) is thick in \(T\). But \(N_T(x, U)\) is syndetic in \(T\), we conclude a contradiction \(N_T(x, V) \cap N_T(x, U) \neq \emptyset\). (Instead of compact, regular phase space is enough for this.)
Conversely, let $x$ be a minimal point and let $U$ be an open neighborhood of $x$. Since $Ty$ is dense in $\text{cls}_xTx$ for all $y \in \text{cls}_xTx$, hence $\{t^{-1}U \mid t \in T\}$ is an open cover of the compact subspace $\text{cls}_xTx$. Thus one can find a finite subset $\{k_1, \ldots, k_n\}$ of $T$ such that $\text{cls}_xTx \subseteq k_1^{-1}U \cup \cdots \cup k_n^{-1}U$; hence for any $t \in T$, $tx \in k_i^{-1}U$ and so $k_i tx \in U$ for some $1 \leq i \leq n$; this implies that $N_f(x,U)$ is syndetic in $T$; therefore $x$ is almost periodic for $(T,X)$.

**Lemma 2.7.** If $(x,y) \in P(T,X)$, then $(x,y)$ is not almost periodic for $(T,X \times X)$.

*Proof.* By the joint continuity of $tx$, if $(x,y) \in P(X)$, then for every $\epsilon \in \mathcal{B}_X$, $\{t \in T \mid t(x,y) \in \epsilon\}$ is a thick subset of $T$. Thus we can obtain the conclusion. \hfill $\square$

The following lemma is a generalization as well as strengthening of Lemma 2.3. See [4, Lemma 2.3] for $T$ in groups.

**Lemma 2.8.** If $(T,X)$ is equicontinuous surjective, then every point of $X$ is an almost periodic point of $(T,X)$.

*Proof.* Let $x \in X$ and let $M$ be a minimal subset of $\text{cls}_xTx$. If $x \notin M$, then there is an $\epsilon \in \mathcal{B}_X$ with $x \notin \epsilon[M]$. Let $tx$ be arbitrarily close to some $y \in M$. Since $x$ is a recurrent point for $(t,X)$ by Lemma 2.3, there is a net $(n_k)$ in $\mathbb{N}$ with $n_k \rightarrow \infty$ and $t^{n_k}x \rightarrow x$. Then by equicontinuity, it follows that $t^{n_k}x$ is arbitrarily close to $t^{n_k-1}y \in M$ and so $x$ is arbitrarily close to $M$, contradicting $x \notin \epsilon[M]$. Thus $x \in M$ and so every point of $X$ is almost periodic by Lemma 2.6. \hfill $\square$

Note that in view of Example 1.12 the ‘surjective’ condition is important for Lemma 2.8. However, it is not a necessary condition for almost periodicity; for instance, Example 1.7.

Now, based on Lemma 2.7 and Lemma 2.8, we can present another concise proof of Theorem 2.1 as follows.

*Proof (II) of Theorem 2.1.* Since $(T,X \times X)$ is equicontinuous surjective, then by Lemma 2.8, $(T,X \times X)$ is pointwise almost periodic. Thus $(T,X)$ is distal by Lemma 2.7. \hfill $\square$

It is interesting to notice that “equicontinuous + surjective $\Rightarrow$ distal” (Theorem 2.1) and “equicontinuous + surjective $\Rightarrow$ almost periodic” (Lemma 2.8) can not be localized. In fact we can easily construct a counterexample on the interval $I = [0, 1]$ with the usual topology. However, “distality $\Rightarrow$ almost periodicity” may be localized (cf. Theorem 5.1 in §5).

**Example 2.9.** Let $f : I \rightarrow I$ be defined by $x \mapsto x^2$. Then 0 and 1 are the only recurrent (fixed) points of $(f,I)$. Moreover, $(f,I)$, as a flow with phase group $\mathbb{Z}$, is equicontinuous at each $x \in (0, 1)$ but $x \in (0, 1)$ is neither an almost periodic point and nor a distal point of $(f,I)$.

Using our Theorem 2.1 as an important tool, Dai and Xiao in [17] have proved the following equivalence of ‘uniformly almost periodic’ and ‘equicontinuous surjective’ (see 1 and 2 of [26, Proposition 4.4] for $T$ in groups).

**Theorem 2.10 ([17]).** Let $(T,X)$ be a semiflow with phase semigroup $T$ not necessarily discrete. Then $(T,X)$ is u.a.p. if and only if it is equicontinuous surjective.

It should be noted that the compactness of $X$ is essential for Theorem 2.10. For example, the $C^0$-flow $\mathbb{R} \times X \rightarrow X$, $(t,x) \mapsto t + x$, where $X = \mathbb{R}$ with the usual topology, is equicontinuous but it is not u.a.p. with no almost periodic point in fact.

Theorem 2.10 implies the following important fact (cf. [4, Corollary 2.6] for $T$ a group and [17, Proposition 3.6] for the general case by different approaches).
Corollary 2.11. Every factor of an equicontinuous surjective semiflow is always equicontinuous surjective.

Proof. Let \((T, Y)\) be a factor of an equicontinuous surjective semiflow \((T, X)\) via an epimorphism \(\pi: X \to Y\). We proceed to show \((T, Y)\) is u.a.p.. Given \(\varepsilon \in \mathcal{B}_Y\), there is \(\delta \in \mathcal{B}_X\) with \(\pi \delta \subset \varepsilon\). Since \((T, X)\) is u.a.p. by Theorem 2.10, there is a syndetic subset \(A\) of \(T\) such that \(Ax \subseteq \delta(x)\) for all \(x \in X\). Now for each \(y \in Y\), having chosen \(x \in \pi^{-1}(y)\), \(Ay = A\pi x = \pi Ax \subseteq \pi \delta(x) \subseteq \varepsilon(y)\). Thus \((T, Y)\) is u.a.p.. 

Although the notion ‘syndetic’ is closely related to the topology of the phase semigroup \(T\), yet Theorem 2.10 shows that ‘u.a.p.’ does not depend on it, since distality and equicontinuity do not depend on it.

Let \((T, X)\) be a semiflow with phase semigroup \(T\). Then:

1. \((T, X)\) is u.a.p. if and only if \((T, X)\) is surjective and for every \(f \in C(X)\) the uniform closure of \(fT\) is a compact subset of \(C(X)\).
2. \((T, X)\) is equicontinuous if and only if for every \(f \in C(X)\) the uniform closure of \(fT\) is a compact subset of \(C(X)\).

Proof. Let \((T, X)\) be u.a.p. and \(f \in C(X)\). Then by Theorem 2.10, \(fT\) and its uniform closure are equicontinuous and \((T, X)\) is surjective. Hence \(fT\) is relatively compact in \(C(X)\) under the supremum norm by Ascoli’s theorem (cf. \([39, \text{Theorem 7.18 in p. 234}]\)).

Now assume \(fT\) is relatively compact in \(C(X)\) under the supremum norm, for all \(f \in C(X)\); and let \(\alpha \in \mathcal{B}_X\) be any given. Then there exist \(\varepsilon > 0\) and \(F\) a finite subset of \(C(X)\) such that if \(|f(x) - f(y)| < \varepsilon\) for all \(f \in F\), then \((x, y) \in \alpha\) (cf. \([4, \text{p. 47}]\)). By Ascoli’s theorem \(fT\), for each \(f \in F\), is equicontinuous. Thus since \(F\) is finite, there exists an index \(\delta \in \mathcal{B}_X\) with

\[|f_t(x) - f_t(y)| < \varepsilon \quad \forall t \in T, \quad f \in F \quad \text{and} \quad (x, y) \in \delta.\]

Hence \(T \delta \subseteq \alpha\) and \((T, X)\) is equicontinuous and so u.a.p. by Theorem 2.10.

The same argument implies the second assertion of Proposition 2.12 and we thus omit the details here. 

2.3. Using Ellis’ enveloping semigroup

Based on Ellis’ semigroup (cf. §1.1.5), the following another short proof of Theorem 2.1 without using the pointwise recurrence of an equicontinuous surjection is the other important idea of this paper.

Proof (III) of Theorem 2.1. Since \((T, X)\) is equicontinuous, then \((p, x) \mapsto p(x)\) of \(E(X) \times X\) to \(X\) is jointly continuous and hence the topology of pointwise convergence coincides with the compact-open topology for \(E(X)\) (cf. \([39, \text{Theorem 7.15}]\)). It follows easily from equicontinuity and surjectivity of each \(t \in T\) that all \(p \in E(X)\) are surjective. Indeed, let \(T \ni t_n \to p \in E(X)\) and \(p(X) \neq X\); then there is an \(\varepsilon \in \mathcal{B}_X\) so small that \(U = \varepsilon[p(X)] \neq X\). Since \(p(X) \subset U\) and \(t_n \to p\) in the sense of compact-open topology, \(t_n(X) \subseteq U\) as \(n\) sufficiently large. This contradicts
that $tX = X$ for all $t \in T$. Now for every idempotent $u$ in $E(X)$, since $u(u(x)) = u(x)$ for all $x \in X$ and $u(X) = X$, thus $u = id_X$. So if $(x, y) \in P(X)$, then $\{p \mid p(x) = p(y)\}$ is a non-empty closed subsemigroup of $E(X)$ and so there is an idempotent $u$ in $E(X)$ with $u(x) = u(y)$ and so $x = y$. This proves Theorem 2.1.

\[ \square \]

3. When is a semiflow surjective?

In light of Examples 1.7 and 1.12, the “surjective” condition is essential for our assertion of Theorem 2.1. In this section we will now introduce some sufficient conditions for that ‘each $t \in T$ is surjective’ for a semiflow $(T, X)$ with some special phase semigroups $T$.

3.1. Homogeneity condition

Let $(T, X)$ be a semiflow. Since $X$ is compact by our convention, each $(t, X)$ must have almost periodic points by Lemma 2.6 and so it has (forwardly) recurrent points. This point is very useful for us to justify the surjectiveness of a semiflow by the so-called “homogeneity” condition as follows.

**Definition 3.1** ([30]). We say that $(T, X)$ is homogeneous if there exists a minimal semiflow $(G, X)$ such that $tgx = gtx$ for all $t \in T, g \in G$ and $x \in X$.

**Proposition 3.2.** Let $(T, X)$ be a homogeneous semiflow. Then $(T, X)$ is surjective. Hence if $(T, X)$ is in addition equicontinuous it is distal.

**Proof.** Let $t \in T$. Since $(T, X)$ is homogeneous, then the (forwardly) recurrent points are dense in $X$ for $(t, X)$. Because if $x$ is recurrent for $(t, X)$ it is such that $x \in \text{cls}_{X^s}(tx, x \mid n \geq 1) \subseteq tX$ and $tX$ is closed, it follows that $t$ is a self-surjection of $X$ for each $t \in T$. Thus $(T, X)$ surjective, and then it is distal by Theorem 2.1 if it is equicontinuous.

Particularly, if $(T, X)$ is minimal with $T$ abelian, then it is homogeneous and thus $(T, X)$ is surjective by Proposition 3.2. Here we will present a more simple independent proof for this as follows.

**Corollary 3.3.** Let $(T, X)$ be a minimal semiflow with $T$ abelian. Then $(T, X)$ is surjective and hence if $(T, X)$ is in addition equicontinuous it is (uniformly) distal.

**Proof.** Let $Z = tX$ for all $t \in T$. Then $Z$ is closed and since $T$ abelian $Z$ is $T$-invariant. Thus $Z = X$. This completes the proof by Theorem 2.1 (and Theorem 1.14).

It should be noticed here that in view of Example 1.7 the abelian condition in Corollary 3.3, which guarantees the homogeneity, is essential. This result will be generalized to amenable semigroups by Proposition 3.7 in §3.2, using ergodic theory.

Given any integer $d \geq 1$, as a consequence of Proposition 3.2 and Theorem 2.1, the following corollary seems to be non-trivial because it is beyond Ellis’ joint continuity theorem.

**Corollary 3.4.** Let $\mathbb{R}_d^d \times X \rightarrow X$ be a separately continuous semiflow, where $(\mathbb{R}_d^d, +)$ is under the usual Euclidean topology. If $(\mathbb{R}_d^d, X)$ is minimal equicontinuous, then it is distal.

**Proof.** Write $T = \mathbb{R}_d^d$, which is an additive abelian semigroup. First, under the discrete topology of $T$, $(T, X)$ is a minimal semiflow. Then by Corollary 3.3, it follows that for each $t \in T$, $x \mapsto tx$ is a continuous surjection of $X$. Therefore, $(\mathbb{R}_d^d, X)$ is distal by Theorem 2.1.
3.2. Amenable semigroup condition

More general than the case of abelian phase semigroup, now we will consider amenable one (cf. §1.1.4 (k)).

Notation 3.5. Let $\mu$ be a Borel probability measure on the compact $T_2$-space $X$. Then:

1. $\mu$ is called quasi-regular if it is “outer-regular” for all Borel subsets of $X$ (i.e. for all Borel set $B$ and $\epsilon > 0$ one can find an open set $U$ with $B \subseteq U$ and $\mu(U \setminus B) < \epsilon$) and each open subset of $X$ is “inner regular” for $\mu$ (i.e. for any open set $U$ and $\epsilon > 0$ one can find a compact set $K$ with $K \subseteq U$ and $\mu(U \setminus K) < \epsilon$).

2. By $\text{supp}(\mu)$ we mean the support of the Borel probability measure $\mu$ in $X$; i.e., $x \in \text{supp}(\mu)$ iff every open neighborhood of $x$ has positive $\mu$-measure. Every point of $\text{supp}(\mu)$ is also called a density point of $\mu$.

Lemma 3.6. Let $\mu$ be an invariant quasi-regular Borel probability measure of $(T, X)$; then $\text{supp}(\mu)$ is a closed set of $\mu$-measure 1 such that $t[\text{supp}(\mu)] = \text{supp}(\mu)$ for all $t \in T$.

Proof. Set $S = \text{supp}(\mu)$. By definition, it easily follows that $S$ is closed; and moreover, $S$ is of $\mu$-measure 1. Otherwise, by the quasi-regularity of $\mu$ there exists a compact subset $K$ of $X$ with $K \cap S = \emptyset$ such that $\mu(K) > 0$; then $K$ contains at least one point of $S$. For, if not, then there is an open neighborhood $V_x$ of any $x \in K$ with $\mu(V_x) = 0$ and so by the compactness of $K$, $\mu(K) = 0$ contradicting $\mu(K) > 0$.

Now given $t \in T$, since $tS$ is a Borel set and $\mu(tS) = 1$, we can easily obtain that $tS = S$. Indeed, $tS \supseteq S$ is obvious. (If $S \setminus tS \neq \emptyset$, then $X \setminus tS$ is an open set containing points of $S$ so that $\mu(X \setminus tS) > 0$, a contradiction to $\mu(tS) = 1$.) Next assume $tS \nsubseteq S$ and then we can choose some $y \in tS - S$ and $x \in tS$ such that $tx = y$. Now we can pick an open set $U$ with $y \in U \subseteq X - S$. Hence $0 = \mu(U) = \mu(t^{-1}U)$. This contradicts that $x \in t^{-1}U, x \in S$ and $t^{-1}U$ is an open set.

This thus completes the proof of Lemma 3.6. □

Now we can easily conclude the following by Theorem 2.1 together with Lemma 3.6, which generalizes Corollary 3.3.

Proposition 3.7. Let $(T, X)$ be a semiflow with $T$ an amenable semigroup and with a dense set of almost periodic points. Then $(T, X)$ is surjective; and hence if $(T, X)$ is in addition equicontinuous it is distal.

Note. In view of Lemma 3.6, the statement of Proposition 3.7 is still true if $(T, X)$ is only a general minimal semiflow admitting an invariant Borel probability measure.

Proof. Let $x \in X$ be an almost periodic of $(T, X)$ and write $X_x = \text{cls}_X T x$. Then $(T, X_x)$ is a minimal subsemiflow of $(T, X)$. Since $T$ is amenable by hypothesis, hence by amenability and Riesz’s theorem there exists an invariant quasi-regular Borel probability measure $\mu$ for $(T, X_x)$. Moreover, since $(T, X_x)$ is minimal and $\text{supp}(\mu) \subseteq X_x$ is $T$-invariant, thus $\text{supp}(\mu) = X_x$. Then by Lemma 3.6, it follows that each $t \in T$ restricted to $X_x$ is a surjection of $X_x$. Thus $x \in tX$ for all $t \in T$. This shows that $tX = X$ for all $t \in T$, because almost periodic points are dense in $X$ and $tX$ is closed. Finally by Theorem 2.1, it follows that $(T, X)$ is distal, if it is equicontinuous. This therefore proves Proposition 3.7. □
3.3. C-semigroup condition and ℓ-recurrence

It was already known that if x is a recurrent point of a continuous self-map f of X then x ∈ f(X) (by Lemma 2.2). However, even for a minimal semiflow (T, X), X ≠ tX in general; see Example 1.7. Now we will generalize Lemma 2.2 to semiflows with a kind of special phase semigroups.

Definition 3.8 ([41]). Let T be a topological semigroup, which is not necessarily discrete. Then:

1. T is called a right C-semigroup if Ts is relatively co-compact in T, i.e., clS(T \ Ts) is compact, for each s ∈ T.
2. We could define left C-semigroup in a similar way.

When T is right and left C-semigroup, it is called a C-semigroup as in Definition (l) in §1.1.4. For example, let T = [1, ∞) with e = 1; then T is a multiplicative C-semigroup under the usual topology.

Next we need the notion—recurrent point—for a semiflow with general phase semigroup beyond T = Z⁺.

Definition 3.9. Let (T, X) be a semiflow, where T is a non-compact topological semigroup, not necessarily discrete. By ℳ we will denote the family of all compact neighborhoods of e. Then:

1. T is called locally compact if e has a compact neighborhood in T, i.e., ℳ ≠ ∅.
2. Given x ∈ X, y ∈ X is called a limit point of x, denoted by y ∈ ℓT(x), if y ∈ \∪_{K∈ℳ} clXK\^c x, where K\^c is the complement of K in T.
3. If x ∈ ℓT(x), then x is called an ℓ-recurrent point of (T, X); see [16, Definition 2.11].

Of course, even if T = Z, an ℓ-recurrent point need not be an almost periodic point for a general semiflow. For instance, every point of X is ℓ-recurrent for ((T), X) in 1 of Examples 1.4, but it is not almost periodic except the two ends −1 and 2 of X.

Lemma 3.10. Let (T, X) be a semiflow with a locally compact phase semigroup T and x ∈ X. Then x is an ℓ-recurrent point of (T, X) if and only if there is a net \{t_n\} in T such that

1. t_n x → x,
2. for every K ∈ ℳ, there is some n_K with t_n ∈ K\^c ∀n ≥ n_K.

Note. If a net \{t_n\} in T satisfies condition (2), then we shall say t_n → ∞.

Proof. The sufficiency is obvious; so we only need to prove the necessity. For this, assume x is an ℓ-recurrent point of (T, X).

Let ℳ be the neighborhoods filter of x. Define a binary relation ≥ on ℳ × ℳ as follows:

(U, K) ≥ (U', K') ⇔ U ⊆ U' and K ⊆ K'.

Then,

(a) if (U, K) ≥ (U', K') and (U'', K'') ≥ (U''', K'''), then (U, K) ≥ (U'', K''');
(b) if (U, K) ∈ ℳ × ℳ, then (U, K) ≥ (U, K);
(c) if (U, K), (U', K') ∈ ℳ × ℳ, then there is (U'', K'') ∈ ℳ × ℳ such that (U'', K'') ≥ (U, K) and (U'', K'') ≥ (U', K').

Thus (ℳ × ℳ, ≥) is a directed set. Now for every (U, K) ∈ ℳ × ℳ, we can take some t_{U,K} ∈ T such that t_{U,K} x ∈ U and t_{U,K} ∈ K'. It is easy to see that \{t_{U,K}\} is a net in T satisfies conditions (1) and (2). This proves Lemma 3.10. □
Remarks 3.11. Suppose \((T, X)\) is a semiflow where \(T\) is a locally compact non-compact topological semigroup.

(a) An \(x\) of \(X\) is not necessarily an \(\ell\)-recurrent point if there is only an infinite sequence \(\{t_n\}\) in \(T\) with \(t_n x \to x\).

For example, let \(X = \mathbb{R} \cup \{\infty\}\) be the one-point compactification of the 1-dimensional Euclidean space \(\mathbb{R}\) (so \(X\) is homeomorphic with the unit circle) and let \(T = (\mathbb{R}, +)\) with the usual topology. Define a flow on \(X\) with the phase group \(T\) as follows:

\[
T \times X \to X, \quad (t, x) \mapsto t + x.
\]

If \(t_n \to 0\) in \(T\), then \(t_n x \to x\) for each \(x \in X\). But \(\ell_T(x) = \{\infty\}\) for all \(x \in X\).

(b) When \(T\) is a group, an almost periodic point is always an \(\ell\)-recurrent point.

Proof. If \(A\) is a syndetic subset of \(T\), then \(A\) is never contained in any \(K \in \mathcal{K}_c\) for \(T\) is non-compact.

(c) More generally than the above (b), let \(T\) be such that each syndetic set is not relatively compact in \(T\). Then every almost periodic point is \(\ell\)-recurrent for \((T, X)\).

Remarks 3.12. The almost periodicity is a strong form of recurrence, yet (b) of Remark 3.11 is false in general if \(T\) is not a group, even for semiflows on compact metric spaces with no isolated points. Let’s construct such an example as follows.

(1) Let \(Y\) be a locally compact, non-compact, Polish space with no isolated points like \(Y = \mathbb{R}^d\), and let \(T = \{e\} \cup Y\), where \(e = id_Y : y \mapsto y\) is the identity self-map of \(Y\) and for every \(t \in T\) with \(t \neq e\) let \(t : y \mapsto t y\) be the constant map. Then \(T\) is a locally compact, non-compact, \(\sigma\)-compact (in fact separable), and non-abelian topological subsemigroup of \(C(Y, Y)\) under the topology defined by the way: for every net \(\{t_n\}\) in \(T\),

\[
t_n \to t \mbox{ in } T \iff t_n y \to t y \mbox{ } \forall y \in Y.
\]

In this case, \(e\) is an isolated point of \(T\) and \(T \setminus \{e\}\) is homeomorphic with \(Y\), i.e., \(t_n \to t\) in \(T\) iff \(t_n \to t\) in \(Y\).

(2) We now consider the naturally induced semiflow on \(Y\) with the phase semigroup \(T\) as follows:

\[
T \times Y \to Y, \quad (t, y) \mapsto t y \mbox{ where } t y = \begin{cases} y & \mbox{if } t = e, \\ t & \mbox{if } t \neq e. \end{cases}
\]

Given \(y_0 \in Y\) set \(S_{y_0} = \{t \in T | t y_0 = y_0\} = \{e, t_{y_0}\}\) where \(t_{y_0} y = y_0 \forall y \in Y\). Clearly \(S_{y_0}\) is a syndetic subsemigroup of \(T\) by (ii) of Definition 2.4 so every point of \(Y\) is a periodic point of \((T, Y)\). However \(T y = \text{cls}_T t y = Y, \forall y \in Y\), is not compact.

(Notice that it is a well-known fact that

Let \((G, X)\) be a flow with a locally compact separable phase group \(G\) on a locally compact \(T_2\)-space \(X\). Then for every \(x \in X\),

(a) if \(x\) is an almost periodic point, \(\text{cls}_G x\) is compact (cf. [26, Proposition 2.5] and [4, Lemma 1.6]);

(b) \(x\) is periodic if and only if \(Gx\) is compact (cf. [4, Theorem 1.5]).

But \((T, Y)\) shows that these statements need not be true in general semiflows.)
Remarks 3.13. Let \( T \) be a locally compact, \( \sigma \)-compact, and non-compact topological semigroup with an increasing sequence \( \{ K_n \} \) of compact neighborhoods of \( e \) such that \( T = \bigcup_n K_n \) and let \((T, X)\) be a semiflow. Then:

1. \( \ell_T(x) = \bigcap_n \text{cls}_X K_n^\circ x \) for all \( x \in X \). Thus, if \( X \) is a metric space, then \( y \in \ell_T(x) \) if and only if \( \exists t_n \in K_n^\circ \) with \( t_n x \to y \) as \( n \to \infty \).

2. If \( s^{-1} K \) is relatively compact in \( T \) for all \( s \in T \) and \( K \in \mathcal{K}_e \), then \( \ell_T(x) \) is invariant for \((T, X)\) with \( X \) a metric space. Thus \( \ell_T(x) \), for \( x \in X \), is an invariant closed non-empty set if \( X \) is a compact metric space.

Proof. Indeed, for all \( y \in \ell_T(x) \) and \( s \in T \), let \( t_n \in K_n^\circ \) with \( t_n x \to y \). Then \( s t_n x \to s y \). For every compact subset \( K \) of \( T \), there is some \( n_0 > 0 \) such that \( s t_n \notin K \) as \( n > n_0 \). This shows that we can select out a subsequence \( \{ t_n \} \) from \( \{ s t_n \} \) with \( t_n \in K_n^\circ \) such that \( t_n x \to s y \). Thus \( \ell_T(x) \) is invariant for all \( x \in X \).

We notice that the classical topological semigroups \( T = \mathbb{R}_+^d \) and \( \mathbb{Z}_1^d \) both are locally compact non-compact \( \sigma \)-compact.

Remarks 3.14. Let \((T, X)\) be a semiflow on a uniform \( T \)-space \((X, \mathcal{K}_X)\) not necessarily compact with phase semigroup \( T \). Then:

(a) A point \( x \in X \) is called Birkhoff recurrent if for every \( \varepsilon \in \mathcal{K}_X \) one can find a compact subset \( K \) of \( T \) such that \( T x \subseteq \varepsilon K \) \( \forall t \in T \); see [45, Definition V7.05] for \( T = \mathbb{R} \) and [9, Definition 3.1] for general topological semigroup.

(b) It has been proved that

\[
\text{If } (T, X) \text{ is a semiflow with } X \text{ a compact metric space, then a point } x \text{ of } X \text{ is almost periodic if and only if it is Birkhoff recurrent ([9, Theorem 1.3]).}
\]
Whenever $T$ is a topological group and $X$ is a locally compact metric space instead of a compact metric space, this statement still holds (cf. [9, Corollary 4.2]). In view of this, the following question is natural:

Does the statement of [9, Theorem 1.3] still hold if $(T, X)$ is a semiflow on a locally compact metric space $X$? (cf. [9, Question 4.9])

(c) Now in the same situation of (2) of Remark 3.12, $Y$ is a locally compact, non-compact, Polish space. If $y \in Y$ were Birkhoff recurrent for $(T, Y)$, then $\text{cls}_Y Ty = Y$ would be compact by [9, Lemma 3.4]. Therefore, every point of $Y$ is almost periodic but not Birkhoff recurrent. This thus gives us a negative solution to [9, Question 4.9].

**Remark 3.15.** Let $(T, X)$ be a semiflow with $T$ a locally compact non-compact semigroup and $x \in X$. If $Tx$ is dense in $X$ such that $\text{Int}_X Tx = \emptyset$, then $X = \ell_T(x)$; particularly, $x$ is $\ell$-recurrent.

**Proof.** Given $y \in X$, let $U$ be an arbitrary neighborhood of $y$ and $K \in \mathcal{X}_e$. Then $U \nsubseteq Kx$; otherwise, $\text{Int}_X Tx \neq \emptyset$. Then $tx \in U$ for some $t \in K^e$. Thus $y \in \ell_T(x)$.

Now we can generalize Lemma 2.2 from the special case $T = \mathbb{Z}_+$ to every left $C$-semigroup (cf. 2 of Definition 3.8) as follows:

**Proposition 3.16.** Let $(T, X)$ be a semiflow with $T$ a locally compact, non-compact, left $C$-semigroup and $x \in X$. If $y \in \ell_T(x)$, then $y \in \text{cls}_T X$ for every $t \in T$. Hence $\ell_T(x) \subseteq tX$ for all $t \in T$.

**Proof.** Let $t \in T$. Since $T \setminus \{t\}$ is relatively compact in $T$ and $y$ is a limit point of $x$ (cf. 2 of Definition 3.9), there is a net $(t_n)$ in $T$ with $t_n x \to y$. Take $t_n x \to z \in \text{cls}_T X$ (passing to a subnet if necessary). Thus $tz = y$ so $y \in \text{cls}_T X$. This proves Proposition 3.16.

The following is a simple consequence of Proposition 3.16, which generalizes [2, Lemma 3.1] from $T = \mathbb{Z}_+$ to a general left $C$-semigroup.

**Corollary 3.17.** Let $(T, X)$ be a semiflow with $T$ a locally compact, non-compact, left $C$-semigroup. If $(T, X)$ is pointwise $\ell$-recurrent, i.e., $x \in \ell_T(x)$ for all $x \in X$, then $(T, X)$ is surjective.

Note that an $\ell$-recurrent point need not be a minimal point. So Corollary 3.17 is comparable with Proposition 3.7. Moreover, (3) of Remark 3.12 shows that the left $C$-semigroup condition is essential for Corollary 3.17, since $\emptyset \not\subseteq tX$ for all $t \in T, t \neq e$.

**Corollary 3.18.** Let $(T, X)$ be a semiflow with $T$ a locally compact, non-compact, left $C$-semigroup and $x \in X$. Then:

1. If $Tx$ is dense in $X$ with $\text{Int}_X Tx = \emptyset$, then $X = tX$ for all $t \in T$.
2. If $(T, X)$ is equicontinuous and $\text{cls}_T X = X$ with $\text{Int}_T X = \emptyset$, then $(T, X)$ is a minimal u.a.p. semiflow.

**Proof.** (1) Let $Tx$ be dense in $X$ with $\text{Int}_X Tx = \emptyset$. By Remark 3.15, $\ell_T(x) = X$. Then the assertion (1) follows at once from Proposition 3.16.

(2) Based on (1) it follows that $(T, X)$ is surjective. Then by Theorem 2.10, $(T, X)$ is u.a.p. and so it is minimal. This proves Corollary 3.18.

In view of Example 1.12, the condition "$\text{Int}_X Tx = \emptyset$" is important for the assertions of Corollary 3.18.
4. Inheritance theorems

It is a well-known fact that for every flow \((T, X)\) and for all syndetic subgroup \(S\) of \(T\), \((T, X)\) is distal if and only if \((S, X)\) is distal (cf. [26, Proposition 5.14]). In fact, this kind of inheritance theorem also holds for semiflows with phase semigroups as follows:

**Proposition 4.1** (Inheritance theorem). Let \((T, X)\) be a semiflow with phase semigroup \(T\) not necessarily discrete, and let \(S\) be a syndetic subsemigroup of \(T\). Then:

1. \(P(T, X) = P(S, X)\);
2. \((T, X)\) is distal if and only if \((S, X)\) is distal;
3. \((T, X)\) is u.a.p. if and only if \((S, X)\) is u.a.p.;
4. \((T, X)\) is equicontinuous surjective if and only if so is \((S, X)\);
5. \((T, X)\) is invertible if and only if so is \((S, X)\);
6. If \((T, X)\) is invertible, then \(Q(T, X) = Q(S, X)\).

**Note.** When \(T\) is a topological group, see [26, Lemma 5.13] for (1) of Proposition 4.1, [26, Proposition 5.14] for (2) of Proposition 4.1, [26, Proposition 4.17] for (3) of Proposition 4.1, [4, Exercise 2.3] for (4) of Proposition 4.1, and [26, Lemma 4.16] for (6) of Proposition 4.1.

**Proof.** (1) Evidently \(P(S, X) \subseteq P(T, X)\). On the other hand, let \((x, y) \in P(T, X)\) and let \(\alpha \in \mathscr{U}_X\), then \(A_\alpha := \{t \in T \mid t(x, y) \in \alpha\}\) is a thick set of \(T\). Since \(S\) is syndetic in \(T\), thus \(S \cap A_\alpha \neq \emptyset\). This shows \((x, y) \in P(S, X)\). Thus \(P(S, X) = P(T, X)\).

(2) Since “distal \(\Rightarrow P = \Delta_X\)” for every semiflow on \(X\), then (2) follows at once from (1).

(3) Let \((T, X)\) be u.a.p.; then \((T, X)\) is distal and equicontinuous by Theorem 2.10. Thus \((S, X)\) is distal and equicontinuous by (2) so that \((S, X)\) is u.a.p. by Theorem 2.10 again.

Conversely, if \((S, X)\) be u.a.p., then \((T, X)\) is u.a.p. because every syndetic subset of \(S\) is also syndetic in \(T\) by (ii) of Definition 2.4.

(4) This follows evidently from Theorem 2.1, (3) and Theorem 2.10.

(5) Let \((S, X)\) be invertible. Since \(S\) is syndetic in \(T\), there is a compact subset \(K\) of \(T\) such that for any \(t \in \bar{K}\), there are \(k \in K\) and \(s \in S\) with \(kt = s\). Let \(K' = \{k \in K \mid \exists t \in T \text{ s.t. } kt \in S\}\); then for any \(t \in T\), there is some \(k' \in K'\) with \(kt = s \in S\). This implies that every \(t \in T\) is an injection of \(X\) and each \(k' \in K'\) is a surjection of \(X\). Thus each \(k' \in K'\) is a homeomorphism of \(X\) so that each \(t \in T\) is a homeomorphism of \(X\).

(6) Clearly \(Q(T, X) \supseteq Q(S, X)\). Let \(K\) be a compact subset of \(T\) such that for any \(t \in T\), there are \(k_t \in K\) and \(s_t \in S\) such that \(k_t t = s_t\). Given any \(\alpha \in \mathscr{U}_X\), there is some \(\beta \in \mathscr{U}_X\) with \(K\beta \subseteq \alpha\). Then \(t^{-1}\beta = s_t^{-1}k_t\beta \subseteq s_t^{-1}\alpha\) so that \(t^{-1}\beta \subseteq S^{-1}\alpha\). This shows that \(Q(T, X) \subseteq Q(S, X)\).

Note that in (4) of Proposition 4.1, since the syndetic subsemigroup \(S\) need not be dense in \(T\), this statement is thus non-trivial. Moreover according to Theorem 1.14, it can be equivalently illustrated as follows:

(4) \((T, X)\) is uniformly distal if and only if so is \((S, X)\).

Next we can obtain a simple consequence of Proposition 4.1. The following is, more or less, motivated by Clay’s [12, Theorem 9].

**Proposition 4.2.** Let \((T, X)\) be a semiflow with \(T\) an abelian semigroup not necessarily discrete, and let \(S\) be a syndetic subsemigroup of \(T\). If there are a point \(x\) such that \(Tx\) is dense in \(X\) and a fixed point \(p\) (i.e. \(Tp = \{p\}\)), then \(Q(T, X) = Q(S, X) = X \times X\).
Proof. We first show that $Tx \times Tx \subset P(T, X)$. In fact, for all $t, s \in T$ and $\alpha \in \mathcal{W}_T$, we can find some $\tau \in T$ such that $[t, s]\tau \subset \alpha[p]$. Then $\tau(tx, sx) \subset \alpha$. This implies that $(tx, sx) \in P(T, X)$. Thus $Tx \times Tx \subseteq P(T, X) = P(S, X)$ by (1) of Proposition 4.1. Further by $\text{cls}_{X \times X}P \subseteq Q$, it follows that $Q(T, X) = Q(S, X) = X \times X$. This proves Proposition 4.2.

Now, in Theorem 2.1, the condition that $(T, X)$ is surjective may be superficially relaxed by using Proposition 4.1 as follows:

Corollary 4.3. Let $(T, X)$ be a semiflow such that $S = \{t \in T | t \text{ is a self-surjection of } X\}$ is syndetic in $T$. If $(T, X)$ is equicontinuous, then it is distal and hence it is invertible.

Proof. Clearly $S$ is a syndetic subsemigroup of $T$. Thus by Theorem 2.1, $(S, X)$ is distal. So $(T, X)$ is distal by (2) of Proposition 4.1.

Moreover, if starting from Theorem 2.10 as Proof (II) of Theorem 2.1, we can easily obtain Theorem 2.1. But the proof of Theorem 2.10 is itself based on Theorem 2.1 in [17].

Finally, we note that the compactness of the phase space $X$ is important for Theorem 2.1. Otherwise, the statement is false; see [17, Example 3.7].

5. Distality of points by product almost periodicity

It is well known that $(T, X)$ is distal iff $(T, X \times X)$ is pointwise almost periodic (cf. [17, Proposition 2.5]); also see 1 and 3 of [26, Proposition 5.9] for flows). In fact, by a purely topological proof, we can obtain the following characterization of distal points, which implies that every distal point is an almost periodic point.

Theorem 5.1. Let $(T, X)$ be a semiflow and $x \in X$. Then $x$ is a distal point of $(T, X)$ if and only if $(x, y)$ is an almost periodic point of $(T, X \times X)$ for all almost periodic point $y$ of $(T, X)$.

Proof. (1) Necessity: Let $y \in X$ be any almost periodic point of $(T, X)$. By Zorn’s lemma, there exists a maximal subset $A$ of $X$ with $y \in A$ such that for all $a_1, \ldots, a_k \in A$, $(a_1, \ldots, a_k)$ is almost periodic for $(T, X^k)$, for all $k \geq 1$. Now for $z = (z_\alpha)_{\alpha \in A} \in X^A$ with $z_\alpha = a \forall a \in A$, we can take an almost periodic point $(z', x')$ in $\text{cls}_{X^A \times X}T(z, x)$ for $(T, X^A \times X)$. Since $z$ is almost periodic for $(T, X^k)$, then there is a net $(t_\alpha)$ in $T$ with $t_\alpha(z', x') \rightarrow (z, x')$ and $(z, x')$ is also almost periodic for $(T, X^A \times X)$ and so $x' \in A$ by maximality of $A$. Further we can select a net $(s_\alpha)$ in $T$ such that $s_\alpha(z, x) \rightarrow (z, x')$ and then $s_\alpha(x', x) \rightarrow (x', x')$ with $x' \in \text{cls}_{X}T x$. Thus $x = x' \in A$ by distality of $(T, X)$ at $x$. Then $x, y \in A$. Therefore by definition of $A$, $(x, y)$ is almost periodic for $(T, X \times X)$.

(2) Sufficiency: Since $X$ is compact, by Zorn’s lemma we can choose a $y_0 \in X$ which is almost periodic for $(T, X)$. So $x$ is almost periodic for $(T, X)$ and further every $y \in \text{cls}_{X}T x$ is almost periodic. Thus, for all $y \in \text{cls}_{X}T x$, $(x, y)$ is almost periodic. This implies that $x$ must be distal (by Lemma 2.7).

The proof of Theorem 5.1 is thus completed.

It should be noticed that by using IP*-recurrence of a distal point and his central sets of $\mathbb{Z}_+$, Furstenberg’s [30, (i) $\Leftrightarrow$ (iv) in Theorem 9.11] says that $x \in X$ is distal for $(T, X)$ iff $(x, z)$ is almost periodic for $(T, X \times Z)$ for all almost periodic point $z \in Z$, for the special case $T = \mathbb{Z}_+$ with $X$ a compact metric space (cf. [15, Theorem 4] for general semiflows on compact $T_2$-spaces).

Definition 5.2. We say that $(T, X)$ satisfies the Bronstein condition if the set of almost periodic points of $(T, X \times X)$ is dense in $X \times X$. 25
The Bronstein condition is a very important one in topological dynamics; see, e.g., [54]. Then as a consequence of Theorem 5.1 and Lemma 1.8, we can easily obtain the following result, which says that the point-distal (cf. §1.1.3 (i)) implies the Bronstein condition.

Corollary 5.3. If \((T, X)\) is a point-distal surjective semiflow, then \((T, X)\) satisfies the Bronstein condition.

Proof. Since \((T, X)\) is surjective point-distal, then by Lemma 1.8 it follows that the distal points are dense in \(X\). Then by Theorem 5.1, for all distal point \(x \in X\) and every \(y \in X\), \((x, y)\) is almost periodic for \((T, X \times X)\). This proves Corollary 5.3.

If \((T, X)\) is invertible point-distal with \(T\) an amenable semigroup, then we shall show later on that its reflection \((X, T)\) is point-distal (cf. Proposition 6.20). Here, based on Theorem 5.1, we can first prove that \((T, X)\) is point-distal.

Theorem 5.4. Let \((T, X)\) be invertible with \(T\) an amenable semigroup and \(x \in X\). Then:

1. \(x\) is a distal point of \((T, X)\) iff \(x\) is a distal point of \(((T), X)\).
2. \((T, X)\) is point-distal iff \(((T), X)\) is a point-distal flow.

Proof. (1). Clearly if \(x\) is a distal point of \(((T), X)\), then it is a distal point of \((T, X)\). Conversely, let \(x\) be a distal point of \((T, X)\); we will show \(x\) is also a distal point of \(((T), X)\). Given \(x \in \text{cls} T x\), by Theorem 5.1, \((y, x)\) is an almost periodic point of \((T, X \times X)\). Then \(W = \text{cls} T y x\) is a minimal subset of \((T, X \times X)\) by Lemma 2.6. Since \(T\) is amenable, by Proposition 3.7 or by 1 of Reflection principle II, it follows that \(((T), W)\) is a minimal subflow of \(((T), X \times X)\) and so \(\text{cls} T x = \text{cls} T y\). Thus by Lemma 2.6 again, \((y, x)\) is an almost periodic point of \(((T), X \times X)\). Using Theorem 5.1 again, \(x\) is a distal point of \(((T), X)\).

(2). In view of 1 of Theorem 5.4, we only need to show that if \(((T), X)\) is minimal, then \((T, X)\) is minimal. In fact, when \(\Lambda\) is a minimal subset of \((T, X)\), by 1 of Reflection principle II we can see \(\Lambda = X\). This proves Theorem 5.4.

Next we will present an application of Theorem 5.4. In 1970 [53] Veech proved the following theorem:

If \((T, X)\) is a point-distal flow on a non-trivial compact metric space \(X\), then \((T, X)\) has a non-trivial equicontinuous factor (cf. [53, Theorem 6.1])

Next by 2 of Theorem 5.4 and Veech’s theorem we can easily obtain an invertible semiflow version of Veech’s theorem as follows:

Corollary 5.5. Let \((T, X)\) be point-distal invertible with \(T\) an amenable semigroup and with \(X\) a non-trivial compact metric space. Then \((T, X)\) has a non-trivial equicontinuous (invertible) factor.

Question 5.6. Is Corollary 5.5 true with no separability hypothesis on \(X\)? This is also open in point-distal flows (cf. Veech [54, p. 802]).

Question 5.7. Let \(T\) be a locally compact non-compact topological semigroup, \((T, X)\) a semiflow and \(x \in X\). If \((x, y)\) is an \(\ell\)-recurrent point of \((T, X \times Y)\) in the sense of Definition 3.9.3 for every \(\ell\)-recurrent point \(y\) of any \((T, Y)\), is \(x\) a distal point of \((T, X)\)? (See [30, (i) \(\Leftrightarrow\) (iii) of Theorem 9.11] for \(T = \mathbb{Z}_n\).

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6. Dynamics of reflections of invertible semiflows

This section will be mainly devoted to proving (2), (3) and (4) of Theorem 1.13 and our Reflection principles I, II and III using Theorem 2.1. As applications of our reflection principles, we will prove Furstenberg’s structure theorem of minimal distal semiflows in §6.2 and we shall consider minimal non-sensitive invertible semiflows in §7.

Recall that a semiflow \((T, X)\) is invertible iff each \(t \in T\) is bijective; and in this case, \(\langle T \rangle\) denotes the group generated by \(T\). Then \(\langle (T), X \rangle\) is a flow on \(X\). However since \(T\) is neither a syndetic nor a normal subsemigroup of \(T\) in general, the dynamics properties of \(\langle (T), X \rangle\) can not be naturally inherited to \((T, X)\) in many cases.

When \((T, X)\) is invertible, \((X, T)\) denotes its reflection or ‘history’ defined as in Definition 1.18. If \((T, X)\) had certain dynamical property \(\Psi\) in the past, i.e., \((X, T)\) has \(\Psi\), then does \((T, X)\) have \(\Psi\)? This kind of dynamics is called satisfying “reflection principle” here.

6.1. Distality and equicontinuity of reflections

Theorem 2.1 implies the following, for which we will present a direct proof with no uses of Ellis’ joint continuity theorem (Theorem 8.8 in §8) and Ellis’s semigroup.

**Proposition 6.1.** Let \((T, X)\) be an invertible semiflow; then \((T, X)\) is equicontinuous if and only if so is \((X, T)\).

**Proof.** By symmetry we only prove the “only if” part and so assume \((T, X)\) is equicontinuous. To be contrary, suppose that \((X, T)\) is not equicontinuous at some point \(x \in X\). Then there are \(x_i, x'_i\) with \(x_i \to x\) and \(x'_i \to x\) in \(X\) and \(t_i \in T\) such that \((x_i, t_i, x'_i) = t_i^{-1}(x_i, x'_i) \to (z, z')\) where \(z \neq z'\). This shows that \((z, z') \in Q(T, X)\); i.e., \(z\) is regionally proximal to \(z'\) for \((T, X)\) (cf. Definition (j) in §1.1.3). Then it follows easily from the equicontinuity of \((T, X \times X)\) that \((z, z')\) is a proximal pair of \((T, X)\), contradicting \((T, X)\) distal by Theorem 2.1. Thus \((X, T)\) must be equicontinuous. This proves Proposition 6.1. \(\square\)

**Definition 6.2.** Let \(E\) be a multiplicative semigroup. Then:

1. A left ideal in \(E\) is a non-empty subset \(I\) such that \(EI \subseteq I\).
2. A minimal left ideal in \(E\) is one which does not properly contain a left ideal.
3. Let \(J(I)\) denote the set of idempotents in a left ideal \(I\); i.e., \(u \in J(I)\) iff \(u^2 = u\) and \(u \in I\).

This is more general than §1.1.5 (o); yet we will be mainly interested to the special case \(E = E(X)\) associated to a semiflow \((T, X)\). Since \(E\) is a compact \(T_2\) right-topological semigroup in this case, hence \(J(I) \neq \emptyset\) for all minimal left ideal in \(E\) (by [4, Lemma 6.6]).

We will need a purely algebraic lemma for us to characterize the distality of any semiflows (Lemma 6.7 and Theorem 6.22).

**Lemma 6.3** (cf. [4, Lemmas 6.1, 6.2 and 6.3]). Let \(E\) be any semigroup and let \(I, I'\) be two minimal left ideals in \(E\) with \(J(I) \neq \emptyset\). Then:

1. \(Ip = I\) for all \(p \in I\).
2. \(pu = p\) for all \(u \in J(I), p \in I\).
3. If \(u \in J(I)\) and \(p \in I\) with \(up = u\), then \(p \in J(I)\).
4. If \(u \in J(I)\) then \(uI\) is a group with the neutral element \(u\).
5. If \(p \in I\) then there is a unique \(u \in J(I)\) with \(up = p\).
(6) Let \( u, v \in J(I) \) and let \( p \in uI \). Then there is an \( r \in I \) with \( rp = v \) and \( pr = u \).

(7) \( I = \bigcup_{\pi \in J(I)} uI \).

(8) If \( u, v \in J(I) \) with \( u \neq v \), then \( uI \cap vI = \emptyset \).

(9) Suppose \( p, q, r \in I \) satisfy \( qp = rp \). Then \( q = r \).

(10) If \( u \in J(I) \), then there is a unique \( v \in J(I') \) such that \( uv = v \) and \( vu = u \), denoted \( u \sim v \).

**Note.** (1) of Lemma 6.3 implies that each minimal left ideal \( I \) of \( E(T, X) \) is a closed subset of \( E(T, X) \), since \( I = Ip = E(T, X)p \) for any \( p \in I \) and \( E(T, X) \) is compact and \( q \mapsto qp \) is continuous. Thus \( I \) is a minimal left ideal of \( E(T, X) \) if it is a minimal subset of the induced semiflow

\[
T \times E(T, X) \to E(T, X), \quad (t, p) \mapsto t \circ p.
\]

Here we will mainly need (2), (4), (7), (9), and (10) of Lemma 6.3 in our later arguments.

**Proof of Lemma 1.11.** Let \( I \) be the only minimal left ideal in \( E(X) \) and \( (x, y), (y, z) \in P(X) \). Then \( p(x) = p(y) \) and \( p(y) = p(z) \) for all \( p \in I \). So \((x, z) \in \mathcal{P}(X)\).

For the "only if" part, let \( P(X) \) be transitive, \( I, K \) minimal left ideals in \( E(X) \), and \( u \in J(I) \) and \( v \in J(K) \) with \( uv = v, vu = u \) by (10) of Lemma 6.3. Let \( x \in X \). Then \((x, ux) \in P(X) \) and \((x, vx) \in P(X) \) implies \((ux, vx) \in P(X) \). But \((ux, vx) = (ux, vx) \) implies that \((ux, vx) \) is an almost periodic point of \((T, X \times X) \). Hence \(ux = vx \) by Lemma 2.7 and \( u = v \) so that \( I = K \).

Following §1.1.3 (h), an \( x \in X \) is a distal point of \((T, X) \) if and only if it is proximal only to itself in \( \text{cls}_X Tx \).

**Lemma 6.4 (cf. Veech [53] for \( T \) in groups).** Let \((T, X) \) be a semiflow and \( x \in X \). Then \( x \) is a distal point of \((T, X) \) iff \( x = u(x) \) for all \( u \in J(E(X)) \).

**Proof.** Let \( x \) be a distal point of \((T, X) \) and \( u \in J(E(X)) \). Since \((x, u(x)) \in P(E(X)) \) is almost periodic (by Theorem 5.1), hence \( x = u(x) \).

Conversely, assume \( x = u(x) \) for all \( u \in J(E(X)) \) and let \( y \in \text{cls}_X Tx \) such that \((x, y) \in P(X) \). There is a minimal left ideal \( I \) such that \( p(x) = p(y) \) \( \forall p \in I \). Since \( x \in Ix \) and so \( y \in Iy \), it follows that \( Iy = Ix \) so \( y \in Iy \). Then there is \( u \in J(I) \) with \( y = u(y) = u(x) = x \). This shows that \( x \) is a distal point of \((T, X) \).

As a consequence of Lemmas 6.3 and 6.4, the following (2) of Theorem 6.5 is more or less motivated by [53, Proposition 2.1], which is useful for proving the "if" part of Lemma 1.8.

**Theorem 6.5.** Let \((T, X) \) be a semiflow with Ellis’ semigroup \( E(X) \) and \( x \in X \). Then:

1. For every minimal left ideal \( I \) in \( E(X) \), \( pI \cap J(I) \neq \emptyset \) for all \( p \in E(X) \).
2. \( x \) is a distal point of \((T, X) \) iff \( x \in p(X) \) for all \( p \in E(X) \) iff \( x \in u(X) \) for all \( u \in J(E(X)) \).

**Proof.** (1) Let \( p \in E(X) \) and \( I \) a minimal left ideal in \( E(X) \). Then \( pI \subseteq I \) and further by (7) and (4) of Lemma 6.3 there are \( q \in I, \delta \in I \), and \( v \in J(I) \) such that \( pq\delta = v \). Since \( q\delta \in I \), hence \( pI \cap J(I) \neq \emptyset \).

(2) Suppose \( x \) is a distal point; then \( x = v(x) \) for every \( v \in J(E(X)) \) by Lemma 6.4. Thus \( x \in p(X) \) for all \( p \in E(X) \) by (1). Conversely, suppose that \( x \in v(X) \) for all \( v \in J(E(X)) \). Let \( u \in J(E(X)) \) be arbitrary. Then there exists \( y \in X \) such that \( u(y) = x \). So \( u(x) = u^r(y) = u(y) = x \). Thus by Lemma 6.4, \( x \) is a distal point of \((T, X) \).

The proof of Theorem 6.5 is thus complete.  

\[ \square \]
Proof of the "if" part of Lemma 1.8. Let the set of distal points of $(T, X)$ be dense in $X$ and $t \in T$. Since every distal point belongs to $tX$ by Theorem 6.5 and $tX$ is a closed set, hence $tX = X$. The proof of Lemma 1.8 is thus complete.

Recall that if $E(T, X) \subset C(X, X)$ then $(T, X)$ is called "weakly almost periodic" in some literature, for example, [27]. So the following says that the proximal relation is an equivalence relation for every weakly almost periodic semiflow with abelian phase semigroup.

Theorem 6.6. Let $(T, X)$ be a semiflow with $T$ an abelian semigroup and $J(E(X)) \subset C(X, X)$. Then the following two statements hold:

1. There exists a unique minimal left ideal $I$ in $E(X)$ and moreover $I$ contains a unique idempotent $u$. Hence $P(X)$ is an equivalence relation on $X$.

2. If $x \in X$ is an almost periodic point, then it is a distal point. Hence if there is a dense set of almost periodic points, then $(T, X)$ is distal.

Note. In fact, if $E(X) \subset C(X, X)$ and there is a dense set of almost periodic points, then $(T, X)$ is not only distal but also equicontinuous by (c) of Lemma 6.7 and Ellis's joint continuity theorem (cf. Theorem 8.8 in §8).

Proof. (1) Let $I_1$ and $I_2$ be two minimal left ideals in $E(X)$. Then by (10) of Lemma 6.3, it follows that there are idempotents $u \in I_1$ and $v \in I_2$ such that $uv = v$. Thus there is a net $(t_n)$ in $T$ with $t_n \to v$ in $E(X)$ such that $I_1 \ni \lim t_n u = \lim ut_n = uv = v$. Then $I_1 \cap I_2 \neq \emptyset$ and thus $I_1 = I_2$. This shows that there is only one minimal left ideal $I$ in $E(X)$. Thus $P(X)$ is an equivalence relation on $X$ by Lemma 1.11.

Let $u, v \in J(I)$. Then by (2) of Lemma 6.3, $uv = u = uu$. By the above argument, we can see $vu = uv = uu$ and so by (9) of Lemma 6.3 $u = v$.

(2) Let $x$ be an almost periodic point of $(T, X)$. Then by $x \in Ix$ where $I$ is as in (1), $x = ux$. Thus by Lemma 6.4, $x$ is a distal point of $(T, X)$. Because $u \in C(X, X)$ by weak almost periodicity, if the set of almost periodic points of $(T, X)$ is dense in $X$ then $u = id_x$. Thus $(T, X)$ is pointwise distal by Lemma 6.4, and so it is distal.

Ellis’ classical characterization of distality states that $(T, X)$ is a distal flow if and only if $E(T, X)$ is a group (cf. [24, Theorem 1], [26, Proposition 5.3] and [4, Theorem 5.6]). Another important consequence of Lemma 6.3 is the following semiflow version of Ellis’ characterization, which has already played an important role in [17].

Lemma 6.7. Let $(T, X)$ be a semiflow, where $T$ is a discrete semigroup (but not necessarily $e \in T$). Then the following statements are pairwise equivalent:

(a) $(T, X)$ is a distal semiflow.

(b) $E(X)$ is a minimal left ideal in itself with $id_x \in E(X)$.

(c) $E(X)$ is a group with the neutral element $id_x$.

Notes. 1. Condition (b) implies that $(T, X)$ is pointwise minimal, because $E(X)x$ is a minimal set of $(T, X)$ and $x \in E(X)x$ for $id_x \in E(X)$.

2. In particular, if $f : X \to X$ is a distal continuous map, then $\exists n_x \to \infty$ such that $f^{n_x}x \to x$ for all $x \in X$ by (c). Thus, if $(f, X)$ is distal it is rigid. Here $(f, X)$ is called rigid if there is a sequence $n_x \to +\infty$ such that $f^{n_x}(x) \to x$ for all $x \in X$ (cf. [2]).
Proof. Condition (a) \(\Rightarrow\) (b): Let \(I\) be a minimal left ideal in \(E(X)\) and \(u \in J(I)\). Then by Lemma 6.4, \(x = u(x)\) for all \(x \in X\). Thus \(u = id_X\) and further \(E(X)\) is a minimal left ideal with the unique idempotent \(id_X \in E(X)\).

Condition (b) \(\Rightarrow\) (c): \(E(X)\) is a group by (4) of Lemma 6.3 with \(u = id_X \in E(X)\).

Condition (c) \(\Rightarrow\) (a): Suppose \((x, y) \in P(X)\). Then \(p(x) = p(y)\) for some \(p \in E(X)\) so \(x = y\) by \(p^{-1}p = id_X\), since \(E(X)\) is a group with \(e = id_X \in E(X)\). Thus (a) holds.

The proof of Lemma 6.7 is thus completed.

The most important part of Lemma 6.7 is “(a) \(\Rightarrow\) (c)” which implies (2) of Theorem 1.13. Now we will present an independent direct proof for it without using Lemma 6.3.

Another proof of “(a) \(\Rightarrow\) (c)” of Lemma 6.7. Note that ‘distal’ implies ‘pointwise almost periodic’ (by Theorem 5.1). Since \((T, X^T)\) is distal, \(E := E(T, X)\) which is the orbit closure of \(id_X\) is minimal. Now for every \(p \in E\), since \(Ep\) is closed \(T\)-invariant, \(Ep = E\). This easily follows that every \(p \in E\) has an inverse.

This algebraic characterization of distality is very useful. Notice that if \(e \notin T\) and \((T, X)\) is distal, then either \(id_X\) is a pointwise limit point of \(T\) in \(E(X)\) or \(tx = x \forall x \in X\) for some \(t \in T\) by (b) of Lemma 6.7.

Now by Lemma 6.7 or by the fact that every distal map is pointwise recurrent, we can obtain the following, (2) of which is just (2) of Theorem 1.13.

**Corollary 6.8.** Let \((T, X)\) be a semiflow with phase semigroup \(T\). Then:

1. If \((T, X)\) is distal, then it is invertible and admits an invariant Borel probability measure.
2. If \((T, X)\) is equicontinuous, then it is distal if and only if it is surjective.
3. If \((T, X)\) is point-distal surjective with \(E(X) \subset C(X, X)\), then \((T, X)\) is an u.a.p. semiflow.

Proof. (1) Let \((T, X)\) be distal; then by Lemma 6.7, \(E(X)\) is a group with \(e = id_X\). Let \(Homeo(X)\) be the group of all self-homeomorphisms of \(X\). Then \(T \subset Homeo(X)\) and \(E(X) = cl_{S_X}(T)\). Thus by Furstenberg’s structure theorem of distal minimal flows [29] (cf. Theorem 6.14 below), it follows that \((T, X)\) and so \((T, X)\) admit invariant Borel probability measures.

(2) This follows easily from Lemma 6.7 and Theorem 2.1.

(3) Let \(x\) be a distal point with \(cl_{S_X}(Tx) = X\). By Lemma 1.8, each point of \(Tx\) is distal for \((T, X)\). Given any \(u \in J(E(X))\), \(sx = usx\) for all \(s \in T\). Since \(u \in C(X, X)\) and \(Tx\) is dense, so \(u = id_X\). Thus \(E(X)\) is a group by (4) of Lemma 6.3 and so \((T, X)\) is minimal distal by Lemma 6.7. This implies that \((E(X), X)\) is equicontinuous. Thus \((T, X)\) is u.a.p.

The proof of Corollary 6.8 is thus completed.

It is interesting that a distal map is always surjective, while an equicontinuous map is not by Examples 1.7 and 1.12 in §1. Also this indicates that distal is the more natural concept. However, under locally (weakly) almost periodic condition, the equicontinuous is equivalent to the distal in flows (cf. [26, 6]).

**Lemma 6.9.** If \((T, X)\) is minimal invertible such that for each \(t \in T, (t^{-1}, X)\) is rigid, that is, \(id_X \in cl_{S_X}(r^n|n = 1, 2, \ldots, \infty)\), then \((X, T)\) is minimal.

Proof. Let \(X_0\) be a minimal set of \((X, T)\) by Zorn’s lemma, and let \(t \in T, t \neq e\) be any given. Then there exists a net \(\{n_k\}\) in \(\mathbb{N}\) with \(r^{n_k} \to id_X\) in \(X^T\) under the pointwise topology. Thus for every point \(x_0 \in X_0, r^{n_k}x_0 \to x_0\) and so \(t(t^{n_k}x_0) = t^{n_k+1}x_0 \to tx_0\). Since \(n_k + 1 \geq 0\), then \(r^{n_k+1}x_0 \in X_0\) and so \(tx_0 \in X_0\). Hence \(TX_0 \subseteq X_0\) and then \(X_0 = X\) for \((T, X)\) is minimal.
As another result of Lemma 6.7, we can then obtain using algebraic approaches the following simple observation for distal semiflows, which are (3) of Theorem 1.13 and 2 and 3 of Reflection principle 1.

**Proposition 6.10.** Let \((T, X)\) be a semiflow with phase semigroup \(T\). Then:

1. If \((T, X)\) is distal, then so is \(((T), X)\).
2. If \((T, X)\) is minimal distal, then \((X, T)\) and \(((T), X)\) both are minimal distal.

**Proof.** (1) Since \(E(T, X)\) is a group with \(e = id_X\) by Lemma 6.7, then \((T, X)\) is invertible and \(T^{-1} \subseteq E(X)\). So \(E(X, T) \subseteq E(X)\). If \(p(x) = p(y)\) for some \(p \in E(X, T)\) then by distality of \((T, X)\) we see \(x = y\). Therefore, \((X, T)\) is distal. Moreover, since \(\langle T \rangle \subseteq E(X)\), thus \(((T), X)\) is distal by Lemma 6.7 again.

(2) By (1), we only need prove the minimality of \((X, T)\). To this end, let \(t \in T\). Since the distal cascade \((t^{-1}, X)\) induces a distal semiflow \(f : (n, x) \mapsto t^{-n}x\) of \(\mathbb{N} \times X\) to \(X\) where \(\mathbb{N}\) is discrete additive, then by Lemma 6.7 the Ellis semigroup of \((f, \mathbb{N}, X)\) contains \(id_X\); i.e., \((t^{-1}, X)\) is rigid. Then (2) follows from Lemma 6.9.

The proof of Proposition 6.10 is therefore completed.

We note that using Ellis’ semigroup (Lemma 6.7) we have easily concluded Proposition 6.10. However, if we make no use of this and the \(\beta\)-compactification of \(T\), based on Theorem 5.1 in §5 and using only topological approaches we can prove it as follows.

**Proof II of Proposition 6.10.** Let \((T, X)\) be distal. We will divide our non-enveloping semigroup proof into relatively independent four steps.

**Step 1.** Every point of \(X\) is almost periodic for \((T, X)\). Moreover, \((T, X)\) is invertible.

**Proof.** The first part of Step 1 follows at once from Theorem 5.1. Now given \(t \in T\), since \((t, X)\) is pointwise almost periodic, then \(tX = X\). This shows that \((T, X)\) is invertible.

Although \((T, X)\) is pointwise almost periodic by Step 1, yet because \(T\) need not be syndetic in \((T)\) and \((T, X)\) need not be minimal the following Step 2 is non-trivial.

**Step 2.** \(((T), X)\) is pointwise almost periodic.

**Proof.** Let \(x \in X\) be any given and write \(Y_x = \text{cls}_XY_x\). Clearly by Step 1, \((T, Y_x)\) is minimal distal so that \(\text{cls}_XY = Y_x\) for all \(y \in Y_x\). Given \(y \in Y_x\), and \(t \in T\), since \(y\) is a (forwardly) minimal point for \((\pi, Y_x)\) by Step 1, there is a net \(\{n_k\}\) in \(\mathbb{N}\) with \(r^{n_k}y \rightarrow y\). So \(r^{n_k-1}y \rightarrow r^{-1}y \in Y_x\), for \(r^{n_k-1}y \in Y_x\), and \(Y_x\) is closed. This shows \(Y_xT \subseteq Y_x\). Thus \(Y_x = \text{cls}_XY_x \subseteq \text{cls}_XY \subseteq Y_x\) for all \(y \in Y_x\). This shows that each \(y \in Y_x\) and so \(x\) are almost periodic for \(((T), X)\).

**Step 3.** \((T \times X)\) is distal and so \(((T), X \times X)\) is pointwise almost periodic.

**Proof.** It follows easily from definition that \((T \times X)\) is distal. Then \(((T), X \times X)\) is pointwise almost periodic by Steps 1 and 2.

**Step 4.** Let \((T, Z)\) be a semiflow with any phase semigroup \(T\). If \((T, Z \times Z)\) is pointwise almost periodic, then \((T, Z)\) is distal.

**Proof.** This follows at once from Lemma 2.7.
Now, since \((T, X \times X)\) is pointwise almost periodic by Step 3, \((T, X)\), which is minimal if so is \((T, X)\), is distal by Step 4. Thus \((X, T)\) is distal.

Next, assume \((T, X)\) is minimal distal. Then \((r^{-1}, X)\) is pointwise almost periodic (forwardly) and so every negatively-invariant closed subset of \(X\) is also \(\pi\)-invariant. This implies the minimality of \((X, T)\). The proof II of Proposition 6.10 is therefore complete.

We will continue to consider the minimality of the reflection \((X, T)\) under much more weaker conditions in §6.3. Moreover, for an amenable phase semigroup, we will show in §6.3 that \((T, X)\) is distal at some point \(x \in X\) if and only if so is \((X, T)\) at the same point \(x\) (see Corollary 6.27).

The following result is originally due to Ellis [23, Theorem 3] (also see [4, Theorem 3.3]) in the case that \((T, X)\) is a flow.

**Corollary 6.11.** Let \((T, X)\) be a surjective semiflow. Then \((T, X)\) is equicontinuous if and only if \(E(X)\) is a group of self-homeomorphisms of \(X\).

**Proof.** First from equicontinuity, all \(p \in E(X)\) are continuous. Then the necessity follows at once from Theorem 2.1 and Lemma 6.7. Conversely, if \(E(X)\) is a group of homeomorphisms of \(X\), then by Ellis’ joint continuity theorem (cf. [4, Theorem 4.3] and also Theorem 8.8 in Appendix), it follows that \(E(X)\) and so \(T\) acts equicontinuously on \(X\). This proves Corollary 6.11.

In Corollary 6.11, it is essential that \(T\) consists of surjections, and not merely a semigroup of continuous maps. Corollary 6.11 may follows from (3) of Corollary 6.8.

Given any semigroup \(T\) of bijections of \(X\), \(T \cup T^{-1}\) is not necessarily equal to the group \((T)\). In addition, if \(T\) acts equicontinuously on \(X\), then so does \(T \cup T^{-1}\) by Proposition 6.1. However, since \(T\) need not be abelian, the equicontinuity of \((T)\) cannot be trivially obtained.

Nevertheless Theorem 2.1 together with Lemma 6.7 implies the following important fact, which is just (4) of Theorem 1.13.

**Theorem 6.12.** Let \(G\) be a semigroup of self-homeomorphisms of \(X\). Then \(G\) is equicontinuous on \(X\) if and only if \(E(G)\) is a group of self-homeomorphisms of \(X\).

**Proof.** It suffices to show the “only if” part. Let \(G\) is equicontinuous on \(X\). By Corollary 6.11, \(E(G, X)\) is a group consist of self-homeomorphisms of \(X\). Further \(E(G, X)\) acts equicontinuously on \(X\). Since \((G) \subseteq E(G, X)\), thus \((G)\) is equicontinuous on \(X\).

Motivated by Proof (III) of Theorem 2.1, we can present another self-contained topological proof of Theorem 6.12 without using Lemmas 6.3 and 6.7.

**Proof II of Theorem 6.12.** We only need to show the “only if” part; and then assume \(G\) is equicontinuous on \(X\). By \(C_{\text{cpt-op}}(X, X)\) we denote the space \(C(X, X)\) of all continuous self-maps of \(X\) equipped with the compact-open topology, and let \(E\) be the closure of \(G\) in \(C_{\text{cpt-op}}(X, X)\). Then by Ascoli’s theorem \(E\) is compact and moreover, each \(p \in E\) is a surjection of \(X\). We will show that \(E\) is a group.

First we note that \((f, g) \mapsto fg := f \circ g\) of \(C_{\text{cpt-op}}(X, X) \times C_{\text{cpt-op}}(X, X)\) to \(C_{\text{cpt-op}}(X, X)\) is separately continuous. This implies that \(EE \subseteq E\) and thus \(E\) is a compact semi-topological semigroup. Since each \(p \in E\) is surjective, \(E\) has the unique idempotent \(id_E\).

Given any \(p \in E\), since \(Ep\) is a closed subsemigroup of \(E\) so that it contains an idempotent, hence \(id_E \in Ep\) and so there is some \(q \in E\) such that \(qp = id_E\). This shows that \(E\) is a group of self-homeomorphisms of \(X\). Finally, since \(G\) and then \(E\) acts equicontinuously on \(X\), so does \((G)\) because of \((G) \subseteq E\).
Finally we notice that whereas Proposition 6.1 may be a consequence of Theorem 6.12, its direct proof is of independent interest.

6.2. Furstenberg’s structure theorem of distal minimal semiflows

Let $T$ be any discrete semigroup with neutral element $e$ and let $\theta$ be some ordinal. Following Furstenberg [29] we introduce a basic notion.

**Definition 6.13.** A projective system of minimal semiflows with phase semigroup $T$ is a collection of minimal semiflows $(T, X_\lambda)$ with compact $T_2$ phase spaces $X_\lambda$ indexed by ordinal numbers $\lambda \leq \theta$, and a family of epimorphisms, $\pi_\lambda^0: (T, X_\lambda) \to (T, X_\theta)$, for $0 \leq \nu \leq \lambda \leq \theta$, satisfying:

1. If $0 \leq \nu < \lambda < \eta \leq \theta$, then $\pi_\lambda^0 = \pi_\eta^0 \circ \pi_\nu^0$.
2. If $\mu \leq \theta$ is a limit ordinal, then $X_\mu$ is the minimal subset of the Cartesian product semiflow $(T, \prod_{\lambda \leq \mu} X_\lambda)$ consisting of all $x = (x_\lambda)_{\lambda \leq \mu} \in \prod_{\lambda \leq \mu} X_\lambda$ with $x_\nu = \pi_\nu^0(x_\lambda)$ for all $\nu < \lambda < \mu$ and then for $\lambda < \mu$, $\pi_\lambda^\mu: X_\mu \to X_\lambda$ is just the projection map. In this case, we say that $(T, X_\mu)$ is the projective limit of the family of minimal semiflows $\{(T, X_\lambda) | \lambda < \mu\}$.

Let $(T, X)$ be an invertible semiflow and let $G = \langle T \rangle$ be the discrete group of self homeomorphisms of $X$ generated by $T$ associated to $(T, X)$.

If $(T, Y)$ is another invertible semiflow and if $\pi: (T, X) \to (T, Y)$ is an epimorphism, then there is a natural extension

$\pi: (G, X) \to (G, Y)$

where for all $g = t_1 t_2 \cdots t_n \in G$, $t_i \in T \cup T^{-1}$,

$gx = t_1 t_2 \cdots t_n x \ \forall x \in X \ \text{and} \ \ gY = t_1 t_2 \cdots t_n Y \ \forall y \in Y.$

Since $t = t_1 t_2$ relative to $(T, X)$ implies that $t = t_1 t_2$ relative to $(T, Y)$, thus $\pi: (G, X) \to (G, Y)$ is well defined. However, it should be noticed that $G$ is defined by $(T, X)$, not by the factor $(T, Y)$.

Recall that $\pi: (T, X) \to (T, Y)$ is a relatively equicontinuous extension iif for all $x \in \mathcal{B}_X$ there is $\delta \in \mathcal{B}_X$ such that whenever $(x, x') \in \delta$ with $\pi(x) = \pi(x')$, then $(tx, tx') \in \varepsilon$ for all $t \in T$ (cf. Definition 1.20).

Now based on Definitions 1.20 and 6.13, we are ready to state the Furstenberg structure theorem for minimal distal semiflows as follows:

**Theorem 6.14** (Furstenberg’s structure theorem). Let $(T, X)$ and $(T, Y)$ be distal minimal semiflows and let $\pi: (T, X) \to (T, Y)$ be an epimorphism. Then there is a projective system of minimal semiflows $\{(T, X_\lambda) | \lambda \leq \theta\}$, for some ordinal $\theta \geq 1$, with $X_0 = X$, $X_\theta = Y$ such that if $0 \leq \lambda < \theta$, then $\pi_1^{\lambda+1}: (T, X_{\lambda+1}) \to (T, X_\lambda)$ is a relatively equicontinuous extension.

**Proof.** According to Proposition 6.10, $((T, X))$ and $((T, Y))$ are distal minimal flows. We now write $G = \langle T \rangle$ associated to $(T, X)$.

Then $\pi: (G, X) \to (G, Y)$ is an epimorphism of distal minimal flows with phase group $G$.

Thus by Furstenberg’s structure theorem of distal minimal flows (cf. [29] or [4, Theorem 7.1]), it follows that there is a projective system of minimal flows $\{(G, X_\lambda) | \lambda \leq \theta\}$ with $X_0 = X$, $X_\theta = Y$ such that if $0 \leq \lambda < \theta$, then $\pi_1^{\lambda+1}: (G, X_{\lambda+1}) \to (G, X_\lambda)$ is a relatively equicontinuous extension. In order to show that

$$(T, X) = (T, X_0) \to \cdots \to (T, X_{n+1}) \xrightarrow{\pi_1^{n+1}} (T, X_n) \to \cdots \to (T, X_1) \xrightarrow{\pi_1^2} (T, X_0) = (T, Y)$$
is actually the desired projective system of minimal semiflows with phase semigroup $T$, it is sufficient to prove that $(T, X_\lambda)$, $0 < \lambda < \theta$, is a minimal semiflow.

Indeed, given $0 < \lambda < \theta$, since $\pi^\theta_\lambda: (G, X_\lambda) \to (G, X_\lambda)$ is an epimorphism, it follows that $\pi^\theta_\lambda: (T, X) \to (T, X_\lambda)$ is also an epimorphism so $(T, X_\lambda)$ is a minimal semiflow for all $\lambda < \theta$. This proves Theorem 6.14. \hfill \Box

**Corollary 6.15.** If $(T, X)$ is a minimal distal semiflow, then it has a non-trivial equicontinuous factor, i.e., there is an epimorphism $\pi: (T, X) \to (T, Y)$ such that $(T, Y)$ is a non-trivial equicontinuous surjective semiflow.

Of course if $T$ is amenable and $X$ is metric, then the statement of Corollary 6.15 can follow from Corollary 5.5.

### 6.3. Minimality of reflections

If $(T, X)$ is a flow, $x \in X$, and $U$ a neighborhood of $x$, then $(N_T(x, U))^{-1} = N_{(T, X)}(x, U)$ is right-syndetic in $T$ by $T = T^{-1}$. So if $x$ is almost periodic for $(T, X)$, then it is also almost periodic for the reflection $(X, T)$. But if $(T, X)$ is only an invertible semiflow, then $(N_T(x, U))^{-1}$ need not be a right-syndetic subset of $T$ so that $x$ need not be almost periodic for $(X, T)$; see 1 of Examples 1.4. However, we will be concerned with a question or reflection principle as follows:

**If $(T, X)$ is minimal invertible, is $(X, T)$ minimal too?**

In this subsection, we shall show that this question is in the affirmative if $T$ is an amenable semigroup (cf. §1.1.4(k)) or if $T$ is a right $C$-semigroup (cf. Definition 3.8).

#### 6.3.1. Abelian phase semigroup

First of all, whereas the following observation is simple, it might be useful for our later proof of Proposition 6.17.

**Lemma 6.16.** Let $f: X \to X$ be a homeomorphism and let $x \in X$ be a (forwardly) recurrent point for $f^{-1}$. Then $f(x)$ belongs to $\text{cls}_X(f^{-n}(x) | n = 0, 1, 2, \ldots)$.

Motivated by Proposition 6.10 stated in §6.1, we can easily obtain the following result using Lemma 6.16.

**Proposition 6.17.** If $(T, X)$ is minimal invertible with $T$ an abelian semigroup, then $(X, T)$ is minimal.

**Proof.** Let $X_0$ be a minimal set of $(X, T)$ and $t \in T$. Let $x_0 \in X_0$ be a minimal point for $(t^{-1}, X_0)$ with phase semigroup $\mathbb{Z}_\pm$. As $x_0$ is recurrent for $t^{-1}$, it follows from Lemma 6.16 that $tx_0 \in X_0$. Then by commutativity of $T$, $tT^{-1}x_0 = T^{-1}tx_0 \subseteq X_0$ so $tX_0 \subseteq X_0$. Whence $X_0$ is invariant for $(T, X)$. This proves Proposition 6.17. \hfill \Box

In fact, Proposition 6.17 can be differently proved as follows:

**Proof II of Proposition 6.17.** Let $X_0$ be a minimal set of $(X, T)$. Then if $t \in T$, then $X_0 \cap tX_0 \neq \emptyset$ (since $t^{-1}x = x$ by Corollary 3.3). But since $T$ is abelian, then $tX_0$ is minimal for $(X, T)$ so $tX_0 = X_0$. This shows that $X_0 = X$. \hfill \Box
The lighting point of Proposition 6.17 is that \( T \) is not necessarily a syndetic subsemigroup of the group \( (T) \) of homeomorphisms of \( X \) generated by \( T \).

Let \( N \) be a non-empty closed invariant set of \( (X,T) \) and \( t \in T \); then for every \( x \in N \), its the \( \alpha \)-limit points set \( \alpha_t(x) \) under \((t, X)\) is such that \( \alpha_t^n(x) \subseteq N \) for all \( n \in \mathbb{Z}_+ \). More generally, we can obtain the following.

**Corollary 6.18.** Let \((T, X)\) be an invertible semiflow and \( N \) an invariant closed non-empty subset of its reflection \((X, T)\). Then for every abelian subsemigroup \( S \subseteq T \), there exists some point \( x \in N \) such that \( S \cdot x \in N \).

**Proof.** Let \( S \) be an abelian subsemigroup of \( T \). Since \( N \) is invariant for \((X,T)\), it is invariant for \((X,S)\). Then there is a minimal set \( N_0 \) for \((X,S)\) with \( N_0 \subseteq N \). By Proposition 6.17, \( N_0 \) is a minimal set for \((S,X)\), so \( S \cdot N_0 = N_0 \subseteq N \). This proves Corollary 6.18.

6.3.2. **Amenable phase semigroup**

Recall that as in §1.1.4(k) a semigroup \( T \) is said to be amenable iff every semiflow on a compact \( T \)-space with the phase semigroup \( T \) admits an invariant Borel probability measure.

Since each abelian semigroup is an amenable semigroup, then the following theorem covers Proposition 6.17 by different ergodic approaches.

**Theorem 6.19.** Let \((T, X)\) be an invertible semiflow with \( T \) an amenable semigroup and \( x \in X \). Then \( x \) is an almost periodic point of \((T, X)\) if and only if \( x \) is an almost periodic point of \((X, T)\). Moreover, if \( x \) is an almost periodic point of \((T, X)\), then \( \text{cls}_X T x = \text{cls}_X T X \).

**Note.** If “with \( T \) an amenable semigroup” is replaced by “admitting an invariant Borel probability measure”, then the statement still holds.

**Proof.** Let \( X_0 \) be a minimal subset of \((T, X)\). Since \( T \) is amenable, there is an invariant quasi-regular Borel probability measure \( \mu \) for \((T, X_0)\) such that \( \text{supp}(\mu) = X_0 \). Then by Lemma 3.6, it follows that for each \( t \in T \) is a surjection of \( X_0 \) and so is \( t^{-1} \) and then all \( t \) restricted to \( X_0 \) are self-homeomorphisms of \( X_0 \). This shows that \( X_0 \) is also a closed invariant subset of \((X, T)\). We will show that \( X_0 \) is also minimal for \((X, T)\).

To be contrary assume that \( X_0 \) is not minimal for \((X, T)\); then by Zorn’s lemma, there exists a proper non-empty closed subset \( Y \) of \( X_0 \) such that \((Y, X)\) is a minimal semiflow. Since \( T \) is amenable, there is an invariant quasi-regular Borel probability measure \( \nu \) for \((Y, T)\) such that \( \text{supp}(\nu) = Y \). Then by Lemma 3.6 again, it follows that for each \( t \in T \), \( t^{-1} : Y \to Y \) is surjective and so is \( t^{-1} \) and then \( t \) restricted to \( Y \) is a self-homeomorphism of \( Y \). This shows that \( Y \) is also a closed invariant subset of \((T, X_0)\). But this contradicts that \((T, X_0)\) is minimal.

By symmetry, we can show that every minimal set \((X, T)\) is a minimal set \((T, X)\). The proof of Theorem 6.19 is therefore complete.

In view of 1 of Examples 1.4, the condition that \( T \) is amenable is essential for the above proof of Theorem 6.19. In fact, the key idea is that each \( t \in T \) is surjective restricted to every minimal subset. Amenability just guarantees this condition.

Recall that Proposition 6.10 claims that if \((T, X)\) is distal, then so is \((X, T)\). However, from Theorem 6.19 we can obtain the following “reflection principle of distality” which asserts that if \( x \in X \) is a distal point of \((T, X)\) and if the phase semigroup \( T \) is amenable, then \( x \) is also a distal point for \((X, T)\). So if \( f : X \to X \) is a homeomorphism such that it is forwardly distal at a point \( x \), then it is backwardly distal at \( x \).
**Proposition 6.20.** Let $(T, X)$ be invertible with $T$ an amenable semigroup and $x \in X$. If $x$ is a distal point of $(T, X)$, then $x$ is a distal point of $(X, T)$.

**Notes.** 1. If “with $T$ an amenable semigroup” is replaced by “admitting an invariant Borel probability measure”, then the statement still holds. 2. Proposition 6.20 is in fact a corollary of Theorem 5.4. But we will present an independent proof here.

**Proof.** Let $x \in X$ be distal for $(T, X)$. Then by Theorem 5.1, $x$ is minimal for $(T, X)$. By Theorem 6.19, $x$ is a minimal point for $(X, T)$. Let $Z = \text{cls}_{X}T$ corresponding to $(X, T)$. Clearly $Z = \text{cls}_{X}T$ by Theorem 6.19 again. We will show that $x$ is not proximal to any $x' \neq x$ in $Z$ in the sense of $(X, T)$. In fact, if $x'$ is in $Z$, then $x'$ is a minimal point of $(X, T)$. Whence $x'$ is also a minimal point of $(T, X)$ by Theorem 6.19 once more. Then by Theorem 5.1, $(x, x')$ is a minimal point for $(T, X \times X)$. This implies by Theorem 6.19 that $(x, x')$ is a minimal point of $(X \times X, T)$. Thus, if $x$ is proximal to $x'$ for $(X, T)$, then $\text{cls}_{X \times X}(x, x')T$ is contained in the diagonal of $X \times X$ by minimality of $(x, x')$ under $(X \times X, T)$. Thus $x = x'$. The proof of Proposition 6.20 is thus complete.

In preparation for our next equicontinuity consequence of Proposition 3.7, we need to recall a notion for our convenience.

**Definition 6.21.** A subsemigroup $S$ of $X^X$ is called a semi-topological semigroup if under the topology $p$ of pointwise convergence, $(f, g) \mapsto f \circ g$ is separately continuous.

If $E(X)$ is a topological group with the pointwise topology and if $(T, X)$ is minimal, then $(T, X)$ is equicontinuous (cf. [17, Proposition 5.5]). However, if $E(X)$ is only a topological semigroup but $E(X) \subset C(X, X)$, then $(T, X)$ is still equicontinuous by the following.

**Theorem 6.22.** Let $(T, X)$ be a semiflow with $T$ an amenable semigroup and with a dense set of almost periodic points. Then, $(T, X)$ is u.a.p. iff $(T, X)$ is equicontinuous iff $E(T, X)$ is a topological semigroup with $E(T, X) \subset C(X, X)$.

**Proof.** The “only if” parts are obvious. Next we show the “if” parts of Theorem 6.22. In fact, we only need prove that if $E(X) \subset C(X, X)$ is a topological semigroup, then $(T, X)$ is equicontinuous and surjective by Theorem 2.1 and Theorem 2.10. For this, we now assume that $E(X) \subset C(X, X)$ is a topological semigroup in the sense of the pointwise topology $p$.

Let $\mathcal{I}$ be a minimal left ideal in $E(X)$. Then we can first show that

(i) Given any $p \in \mathcal{I}$, $p\mathcal{I} = \mathcal{I}$.

**Proof.** Indeed, applying Proposition 3.7 with $T \times \mathcal{I} \to \mathcal{I}$, $(t, p) \mapsto tp$, it follows that for every $t \in T$, $tl = l$. Then if $T \ni t_{a} \overset{r}{\rightarrow} p$ and $q \in \mathcal{I}$, there are $q_{n} \in \mathcal{I}$ with $t_{a}q_{n} = q$ and $q_{n} \overset{2}{\rightarrow} r$ for some $r \in \mathcal{I}$ so that $pr = q$ by the joint continuity of $(f, g) \mapsto f \circ g$. Thus, $p\mathcal{I} = \mathcal{I}$.  

Then by (i) there follows that

(ii) $up = p$, for $u \in J(\mathcal{I})$ and $p \in \mathcal{I}$.

**Proof.** By (i), $ull = \mathcal{I}$, so $p = uq$, for some $q \in \mathcal{I}$. Then $up = uuq = uq = p$.  

Next, if $u, v$ are idempotents in $\mathcal{I}$, then by (ii), it follows that $(u, v)u = (u, u)$, (9) of Lemma 6.3 implies that

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(iii) \( u = v. \)

Therefore, \( \mathbb{I} \) has a unique idempotent \( u \) in \( \mathbb{I} \). Of course \( u \in C(X,X) \). Since \( (T,X) \) has a dense set of almost periodic points, hence \( ux = x \) for each \( x \in X \). This implies by (4) of Lemma 6.3 that \( \mathbb{I} = E(X) \subset C(X,X) \) is a group. Thus \( (T,X) \) is equicontinuous by Corollary 6.11.

If \( T \) is a topological group, then we can improve the statement of Theorem 6.22 by a completely different proof as follows:

- A flow \( (T,X) \) is equicontinuous if and only if \( E(T,X) \) is a topological semigroup with \( E(T,X) \subset C(X,X) \). (See Theorem 8.14 in Appendix §8.)

It should be noticed that the ‘topological semigroup’ condition is essential for the above theorem (cf. Theorem 8.14 in §8) as shown by the following example.

Example 6.23. Let \( X \) be the one-point compactification of the reals and define a homeomorphism \( f: X \to X \) by \( x \mapsto x + 1 \) for all \( x \in X \). Then \( \infty \) is the unique almost periodic point and \( (f,X) \) with phase group \( \mathbb{Z} \) is not equicontinuous; but \( E(f,X) \subset C(X,X) \) consists of the powers of \( f \) together with the constant map \( c: x \mapsto \infty \). By Theorem 6.6, \( I = [c] \) is the unique minimal left ideal in \( E(f,X) \). Moreover, it is easy to see that \( E(f,X) \) is a semi-topological semigroup but not a topological semigroup. Indeed, let \( t_n = f^n \) and \( s_n = f^{1-n} \) for any \( n \geq 1 \). Clearly, \( t_n \to c \) and \( s_n \to c \) but \( f = \lim t_n s_n \neq (\lim t_n)(\lim s_n) = c \).

On the other hand, let us consider \( (f,X) \) in Example 6.23 from the viewpoint of semiflow. It shows that the condition ‘with a dense set of almost periodic points’ is essential in Theorem 6.22.

Example 6.24. Let \( f: X \to X \) be the same as in Example 6.23. But here we now consider \( (f,X) \) with phase semigroup \( \mathbb{Z} \). Clearly \( \infty \) is also the unique almost periodic point and \( (f,X) \) is not equicontinuous; but \( E(f,X) \subset C(X,X) \) consists of the powers \( f^n, n \geq 0 \), together with the constant map \( c: x \mapsto \infty \) of \( X \) into itself. By Theorem 6.6, \( I = [c] \) is the unique minimal left ideal in \( E(f,X) \). Moreover, it is easy to see that \( E(f,X) \) is a topological semigroup but not a topological group.

Recall that any subset \( A \) of \( X \) is called non-trivial if \( A \neq \emptyset \) and moreover \( A \neq X \). Then it is easy to verify that

- If \( T \) is a group, then \( (T,X) \) is minimal if and only if \( X \) does not contain a non-trivial invariant open subset.

However, in our semigroup situation, this becomes a non-trivial case. First of all, we can easily get the following simple fact for an invertible semiflow \((\pi,T,X)\).

Lemma 6.25. Let \((T,X)\) be an invertible semiflow; then the following two statements hold:

1. \( W \subset X \) is an invariant open set of \((T,X)\) iff \( X \setminus W \) is an invariant closed set of \((T,X)\).
2. \((X,T)\) is minimal iff \( TU = X \) for every non-empty open set \( U \).

The following seems to be helpful for considering the minimality of the reflection \((X,T)\) with \( T \) a non-abelian semigroup. See [2, Theorem 1.1(2)-b] for cascades on compact metric spaces.

Theorem 6.26. Let \((T,X)\) be invertible. Then \((X,T)\) is minimal if and only if \((T,X)\) does not have a non-trivial invariant open subset of \( X \). Hence, \((T,X)\) is minimal if and only if there is no non-trivial open invariant set of \((X,T)\).
Proof. Let $(X, T)$ be minimal and assume $U$ is a non-trivial open invariant subset of $(T, X)$. Then $X \setminus U$ is invariant non-empty closed for $(X, T)$ by Lemma 6.25 and so $X \setminus U = X$ contradicting $U$ non-trivial. Thus $X$ does not contain a non-trivial open invariant subset for $(T, X)$.

Conversely, let $X$ have no non-trivial open invariant subset for $(T, X)$ and assume $(X, T)$ is not minimal. Then we can find a non-trivial closed invariant subset $\Theta$ of $(X, T)$. Then $X \setminus \Theta$ is a non-trivial open invariant subset of $(T, X)$ by Lemma 6.25 again. Thus this concludes that $(X, T)$ is a minimal semiflow.

It is clear that every minimal flow admits no non-trivial open invariant set. Now, by Theorem 6.19 and Theorem 6.26, we can easily obtain the following semigroup-action result.

**Corollary 6.27.** If $(T, X)$ is minimal invertible with $T$ amenable, then there exists no non-trivial, open, and invariant set for $(T, X)$.

Proof. If this were false, then $(X, T)$ would not be minimal by Theorem 6.26. But this contradicts Theorem 6.19. This completes the proof of Corollary 6.27.

Another result of Theorem 6.19 is the following theorem, which is a generalization of a classical theorem of Tumarkin [45, Theorem V7.13] from the important case of $T = (\mathbb{R}, +)$ to the case of general amenable semigroups.

**Theorem 6.28.** Let $T$ be an amenable semigroup. If $\Lambda$ is a minimal subset of $(T, X)$ such that $\text{Int}_X \Lambda \neq \emptyset$, then $\Lambda$ is clopen in $X$.

Proof. Let $y \in \Lambda$ be an interior point of $X$. Then we can pick some index $s \in \mathcal{S}_X$ such that $s[y] \subseteq \Lambda$. Then $U := \bigcup_{t \in T} te[y]$ is an open, invariant, and non-empty subset of $X$ such that $U \subseteq \Lambda$. Thus by Theorem 6.19 (more precisely by Corollary 6.27), it follows that $U = \Lambda$. This proves Theorem 6.28.

Let $n$ be a positive integer. From Urysohn’s theorem the dimension of a compact subset of an $n$-dimensional manifold which has no interior points does not exceed $n-1$ (cf. [35, Lemma 2.14]). Hence we have the following

**Corollary 6.29** (Hilmy [45, Theorem 7.16] for $T = \mathbb{R}$ and [35, Theorem 2.15] for groups). Let $(T, M^n)$ be an invertible semiflow on an $n$-dimensional manifold $M^n, n \geq 1$, such that $T$ is amenable. If $\Lambda$ is a compact minimal subset with $\text{Int}_{M^n} \Lambda \neq \emptyset$, then $\text{Int}_{M^n} \Lambda \neq \emptyset$ and dim $\Lambda \leq n - 1$.

Proof. If $\text{Int}_{M^n} \Lambda = \emptyset$, then by Urysohn’s theorem dim $A \leq n - 1$. Now assume Int$_{M^n} \Lambda \neq \emptyset$; then by Theorem 6.28, it follows that $\Lambda$ is clopen non-trivial in $M^n$. This is a contradiction.

Let $(G, X)$ be a flow with phase group $G$ and $T$ a normal syndetic subgroup of $G$. Then it is a well-known fact that

- An $x \in X$ is an almost periodic point of $(G, X)$ if and only if $x$ is an almost periodic point of $(T, X)$ (cf., e.g., [26, Proposition 2.8] and [4, Theorem 1.13]).

By Theorem 6.28 we can obtain the following same flavor result using amenability instead of the normality of $T$.

**Corollary 6.30.** Let $(G, X)$ be an invertible semiflow on a compact connected $T_2$-space $X$ and $T$ a discrete syndetic amenable subsemigroup of $G$. Then $(G, X)$ is minimal iff so is $(T, X)$.
Proof. We only show the “only if” part. Let \( Y \) be a minimal subset of \((T, X)\). Let \( G = K^{-1} T \) for some subset \( K = \{k_1, \ldots, k_n\} \) of \( G \). Then
\[
X = \text{cl}_{\text{sX}} G Y = \text{cl}_{\text{sX}} K^{-1} T Y = \bigcup_{k \in K} k^{-1} \text{cl}_{\text{sX}} T Y = \bigcup_{k \in K} k^{-1} Y.
\]
Thus \( Y \) has non-empty interior. This implies by Theorem 6.28 that \( Y \) is clopen so that \( Y = X \). \( \square \)

It should be noted that if \( T \) is not discrete syndetic, then the statement of Corollary 6.30 need not be correct. For example, let \( \pi : \mathbb{R} \times \mathbb{T} \to \mathbb{T} \) be periodic of period 1 on the unit circle; then \( \mathbb{Z} \) is syndetic in \( \mathbb{R} \) under the usual topology but \((\pi, \mathbb{Z}, \mathbb{T})\) is not minimal.

6.3.3. C-semigroup and an open question

We do not know if the amenability condition in Theorem 6.19 may be replaced by the one that \((X, T)\) is homogeneous; that is, there is a group \( G \) of homeomorphisms of \( X \) such that \((G, X)\) is minimal with \( r^t g = g r^t \) for all \( t \in T \) and \( g \in G \). More generally, the following questions would be interesting:

1. If \( x \in X \) is a minimal point of \((T, X)\), whether or not \( x \) is a minimal point of \((X, T)\), where \( T \) is a non-amenable semigroup.
2. When \((T, X)\) is minimal invertible with \( T \) non-amenable, is \((X, T)\) minimal?

In view of Examples 1.4, the general solution to Question 1 above is NO. However the answer to Question 2 above is in the affirmative if the phase semigroup \( T \) is a C-semigroup (cf. Definition 3.8), as we proceed to show.

Theorem 6.31. Let \((T, X)\) be an invertible semiflow where \( T \) is not necessarily discrete. Then:

1. If \( T \) is a left C-semigroup and \((X, T)\) is minimal, then \((T, X)\) is minimal.
2. If \( T \) is a right C-semigroup and \((X, T)\) is minimal, then \((T, X)\) is minimal.

Proof. First of all, note that if \( T \) is a compact topological semigroup, then the statements are evidently true. Indeed, let \((T, X)\) be minimal and then for all \( x, y \in X \), \( T y = \text{cl}_{\text{sX}} T y = X \) and so \( t y = x \) for some \( t \in T \). This implies that for all \( x, y \in X \), \( y = r^t x = x t \) for some \( t \in T \) and thus \( \text{cl}_{\text{sX}} T x = T X \) for every \( x \in X \). Hence \((X, T)\) is minimal. Analogously, \((T, X)\) is minimal if so is \((X, T)\).

We now then suppose that \( T \) is a non-compact semigroup. Since minimality is independent of the topology of the phase semigroup \( T \), we assume \( T \) is an infinite discrete semigroup without loss of generality. (Note that C-semigroup relies on topology of \( T \), but the general case can be analogously proved.)

1. Let \((X, T)\) be minimal with \( T \) a left C-semigroup. We now proceed to show that \((T, X)\) is minimal. To this end, for every \( x \in X \), define the \( \omega \)-limit set of \( x \) with respect to \((T, X)\) as follows:
\[
\omega_T(x) = \bigcap_{F \in \mathcal{F}} \text{cl}_{\text{sX}} F^c x
\]
where \( \mathcal{F} \) is the collection of finite subsets of \( T \) and \( F^c \) is the complement of \( F \) in \( T \). Clearly, \( \omega_T(x) \) is closed non-empty by the “finite intersection property” (noting that \( T \) is non-compact and \( X \) is compact by hypothesis) and \( \omega_T(x) \subseteq \text{cl}_{\text{sX}} T x \).

We will show that \( \omega_T(x) \) is an invariant set of \((X, T)\). For this, let \( y \in \omega_T(x) \) and \( s \in T \) be arbitrarily given. Let \( F \in \mathcal{F} \) be arbitrary. Since \( K := s F \cup \text{cl}_{T}(T \setminus s T) \) is finite and \( s F^c \supseteq K^c \)
Lemma 7.2. Let \((T, X)\) be a semiflow with each \(t \in T\) an open self-map of \(X\). If \((T, X)\) is non-sensitive and \(\text{Tran}^-(T, X) \neq \emptyset\), then \(\text{Tran}^- (T, X) \subseteq \text{Equi}(T, X)\).
Proof. Let \( x_0 \in \text{Tran}^-(T, X) \) and \( x \in \text{Equi}_s(T, X) \) both be any given points. We then need to verify \( x_0 \in \text{Equi}_s(T, X) \). For this, let \( \eta \in \mathcal{H}_X \) such that if \( y, z \in \eta[x] \) then \( t(y, z) \in \varepsilon \) for all \( t \in T \). Since \( x_0 \) is negatively transitive, there is an \( s \in T \) such that \( s^{-1}x_0 \cap \eta[x] \neq \emptyset \). Then by the openness of \( s \), there exists an \( \delta \in \mathcal{H}_X \) such that \( \delta[x_0] \subseteq s(\eta[x]) \). Now for \( y, z \in \delta[x_0] \) and \( t \in T \), \( t(y, z) = st(y', z') \in \varepsilon \) for some \( y', z' \in \eta[x] \) with \( sy' = y, sz' = z \). This shows that \( x_0 \in \text{Equi}_s(T, X) \) for all \( \varepsilon \in \mathcal{H}_X \).

**Proposition 7.3.** Let \((T, X)\) be a minimal non-sensitive semiflow with \( T \) not necessarily discrete. Then the following two statements hold:

1. If \((T, X)\) is invertible with \( T \) an amenable semigroup, then \((T, X)\) is equicontinuous.
2. If \( T \) is a right \( C \)-semigroup, then \((T, X)\) is equicontinuous.

**Proof.**

1. By Theorem 6.19, \( \text{cls}_{X}XT = X \) for all \( x \in X \). Thus, \( \text{Tran}^-(T, X) = X \). Then by Lemma 7.2, \( \text{Equi}_s(T, X) = X \) and so \((T, X)\) is equicontinuous by Lemma 1.6.

2. Given \( \varepsilon \in \mathcal{H}_X \), there are \( x_0 \in X \) and \( \delta' \in \mathcal{H}_X \) such that \( T(\delta'[x_0], x_0) \subseteq \frac{\delta'}{\varepsilon} \) by non-sensitivity. Now since \((T, X)\) is minimal, for every \( x \in X \) there are \( s \in T \) and \( \delta \subseteq \delta' \) such that \( s(\delta[x]) \subseteq \delta'[x_0] \). In addition, since \( T \setminus Ts \) is relatively compact in \( T \), we can take an \( \eta \in \mathcal{H}_X \) with \( \eta \subseteq \delta \) so small that \( t(\eta[x], x) \subset \frac{\delta'}{\varepsilon} \) for all \( t \in T \setminus Ts \). Thus \( T(\eta[x], x) \subseteq \varepsilon \). Since \( \varepsilon \) is arbitrary, \( \text{Equi}(T, X) = X \) and so \((T, X)\) is equicontinuous by Lemma 1.6.

**Corollary 7.4.** Let \((T, X)\) be a minimal semiflow with \( T \) not necessarily discrete. Suppose that

1. \((T, X)\) is a right \( C \)-semigroup or
2. \((T, X)\) is invertible with \( T \) amenable.

Then \((T, X)\) is either sensitive or equicontinuous.

**Proof.** If \((T, X)\) is sensitive, then it evidently not equicontinuous. Now if \((T, X)\) is non-sensitive, then it is equicontinuous by Proposition 7.3.

Since \( \mathbb{Z}_+ \) is a right \( C \)-semigroup, hence the case (1) of Corollary 7.4 is a generalization of the Auslander-Yorke dichotomy theorem [7].

**Corollary 7.5.** Let \((T, X)\) be a minimal semiflow with \( T \) not necessarily discrete such that

1. \((T, X)\) is a right \( C \)-semigroup or
2. \((T, X)\) is invertible with \( T \) amenable.

Then \((T, X)\) is equicontinuous if and only if \( \text{Equi}(T, X) \neq \emptyset \).

**Proof.** If \( \text{Equi}(T, X) \neq \emptyset \), then \((T, X)\) is non-sensitive and so it is equicontinuous by Proposition 7.3. This proves Corollary 7.5.

The following corollary is a reflection principle on sensitivity of invertible semiflows in amenable semigroups or \( C \)-semigroups.

**Corollary 7.6.** Let \((T, X)\) be a minimal invertible semiflow with \( T \) not necessarily discrete such that \( T \) is either a \( C \)-semigroup or an amenable semigroup. Then \((T, X)\) is sensitive if and only if so is \((X, T)\).

**Proof.** Assume \((T, X)\) is sensitive. If \((X, T)\) were not sensitive, then it would be non-sensitive minimal by Theorems 6.19 and 6.31 and so equicontinuous by Proposition 7.3. Moreover, by Reflection principle I (Proposition 6.1), it follows that \((T, X)\) would be equicontinuous. This is a contradiction. Conversely, if \((X, T)\) is sensitive, then we could similarly prove that \((T, X)\) is sensitive.
Then by Theorem 2.1 and Corollary 7.4, we can easily obtain the following.

**Corollary 7.7.** Let \((T, X)\) be minimal surjective with \(T\) a right \(C\)-semigroup not necessarily discrete. If there is some \(t \in T\) non-invertible, then \((T, X)\) is sensitive.

**Proof.** If \((T, X)\) were non-sensitive, then by Corollary 7.4 \((T, X)\) would be equicontinuous and so distal by Theorem 2.1. This is a contradiction.

Now by using distality instead of amenability and \(C\)-semigroup, we can obtain the following dichotomy.

**Proposition 7.8.** Let \((T, X)\) be a minimal distal semiflow. Then \((T, X)\) is equicontinuous iff \((T, X)\) is non-sensitive.

**Proof.** Let \((T, X)\) be non-sensitive. By Lemma 7.2, \(\text{Tran}^{-}(T, X) \subseteq \text{Equi}(T, X)\). Further by Reflection principle I, \(\text{Tran}^{-}(T, X) = X\) so \((T, X)\) is equicontinuous. The other side implication is evident. Thus the proof is complete.

This shows that the dynamics of any non-equicontinuous minimal distal semiflow could not be predictable.

### 8. Appendix: revisit to Robert Ellis’ joint continuity theorems

Let \(T\) be a multiplicative topological group or semigroup and \(X\) a compact \(T_2\)-space, and let \(T \times X \to X, (t, x) \mapsto tx\) be a separately continuous flow or semiflow. Then the problem to find conditions on \(T\) so that \((t, x) \mapsto tx\) jointly continuous goes back, at least, to Baire (1899); see, e.g., [8, 28, 23, 52, 44, 4]. In this Appendix, we will revisit Robert Ellis’ joint continuity theorems and generalize some of them to locally compact Hausdorff semi-topological semigroups based on Isaac Namioka’s theorem.

**Standing notation.** In this appendix “locally compact Hausdorff space” will be abbreviated as “l.c.\(T_2\)-space”.

Every l.c.\(T_2\)-space is of the second category and every closed or open subset of an l.c.\(T_2\)-space as a subspace itself is an l.c.\(T_2\)-space.

Differently with Ellis’ original proof [23] in which the group structure of \(T\) plays an essential role, we will employ mainly the following basic joint continuity theorem due to Isaac Namioka 1974 [44, Theorem 1.2] as our tool.

**Lemma 8.1** (Namioka [44]). Let \(G\) be an l.c.\(T_2\)-space or a separable Baire space and \(X\) a compact \(T_2\)-space, and let \((Z, d)\) be a metric space. If a map \(f: G \times X \to Z\) is unilaterally continuous, then there exists a dense \(G_δ\)-set \(R\) in \(G\), such that \(f\) is jointly continuous at each point of \(R \times X\).

Some alternate proofs and generalizations of Namioka’s theorem have been given by, for examples, [50, 40, 37, 10] and [32, Lemma 1.36].

Since \(f: G \times X \to Z\) is unilaterally continuous, then \(E_f: G \to C_p(X, Z)\) given by \(g \mapsto f(g, .)\) is continuous, where \(C_p(X, Z)\) is the space of all continuous functions from \(X\) to \(Z\) with the pointwise topology \(p\). Thus Lemma 8.1 is a corollary of [4, Lemma 4.2] due to Troallic [50].
Under the setup of Lemma 8.1, \( C_\alpha(X,Z) \) denotes the uniform space \( C(X,Z) \) where the topology of uniform convergence on \( C(X,Z) \) is induced by the standard supremum norm:

\[
\|\phi - \psi\| = \sup_{x \in X} |\phi(x) - \psi(x)| \quad \forall \phi, \psi \in C(X,Z).
\]

It is very convenient to reformulate Lemma 8.1 in terms of functions spaces as follows. In our later application of this lemma, \( Z \) will be the unit interval \( I = [0,1] \) with the usual Euclidean metric.

**Lemma 8.2.** Let \( G \) be an l.c.T.2-space and \( X \) a compact T.2-space, and let \((Z,d)\) be a metric space. If \( f : G \times X \to Z \) is unilaterally continuous, then there exists a dense \( G_\delta \)-set \( R \) in \( G \), such that the induced map \( F : G \to C_\alpha(X,Z) \), \( t \mapsto f(t,\cdot) \) \( \forall t \in G \), is continuous at each point of \( R \).

**Proof.** This is just a consequence of [44, Theorem 2.2]; however we present its proof here for reader’s convenience. Let \( R \) be a dense \( G_\delta \)-set in \( G \) given by Lemma 8.1. Then for any \( \tau \in R \), \( f : G \times X \to Z \) is continuous at each point of \( \{\tau\} \times X \). Since \( X \) is compact, we see that \( F \) is continuous at \( \tau \) in the sense of the topology of uniform convergence on \( C(X,Z) \). Indeed, given any \( \varepsilon > 0 \), for any \( x \in X \), there are open neighborhoods \( U_x \) of \( \tau \) in \( G \) and \( V_x \) of \( x \) in \( X \) such that

\[
d(f(\tau,y), f(t,y)) < \varepsilon \quad \forall t \in U_x \text{ and } y \in V_x.
\]

Choosing \( x_1, \ldots, x_n \in X \) so that \( X = V_{x_1} \cup \cdots \cup V_{x_n} \) and letting \( U = \bigcap_{i=1}^n U_{x_i} \), it follows that

\[
\|F(\tau) - F(t)\| = \sup_{x \in X} d(f(\tau,x), f(t,x)) < \varepsilon \quad \forall t \in U.
\]

This concludes the desired.

A topological space \( Y \) is called completely regular iff for each member \( y \) of \( Y \) and each neighborhood \( U \) of \( y \) there is a continuous function \( \alpha \) on \( Y \) to the closed unit interval \( I \) such that \( \alpha(y) = 0 \) and \( \alpha \) is identically \( 1 \) on \( Y \setminus U \). It is clear that the family \( C(Y,I) \) of all continuous functions on a completely regular space \( Y \) to the unit interval \( I \) distinguishes points and closed sets in the sense that for closed subset \( A \) of \( Y \) and each point \( y \in Y \setminus A \) there is an \( \alpha \in C(Y,I) \) such that \( \alpha(y) \) does not belong to the closure of \( \alpha(A) \).

If \( X \) is a completely regular T.1-space, then by the classical Embedding Lemma (cf. [39, Chapter 4]) \( X \) is homeomorphic to a subspace of the cube \( Q = l^\infty(\mathbb{N}) \). Therefore we can easily obtain the following

**Lemma 8.3.** Let \( X \) be a completely regular T.1-space and \( W \) a topological space. Then a map \( f : W \to X \) is continuous at a point \( w_0 \in W \) if and only if \( \alpha \circ f : W \to I \) is continuous at the point \( w_0 \) for each \( \alpha \in C(X,I) \).

With this lemma at hands, we do not need here to strengthen Lemma 8.2 by uniform space instead of a metric space \( Z \) as [50, 42] there.

Recall that \((T,X)\) is weakly almost periodic iff \( E(X) \subseteq C(X,X) \). Comparing with [3, Corollary 6], an interesting point of the following is that \( X \) is not necessarily to be a metric space.

**Theorem 8.4.** Let \((T,X)\) be a weakly almost periodic flow and \( \text{Tran}(T,X) = \{x | \text{cl}_{\mathcal{S}Y} T x = x\} \). Then \( \text{Tran}(T,X) \subseteq \text{Equi}(T,X) \).

**Note.** Therefore by Lemma 1.6, if \((T,X)\) is a minimal flow, then \((T,X)\) is equicontinuous if and only if it is weakly almost periodic (cf. [4, Theorem 4.6]).
Lemma \( \Theta \): \( T \to C(X, I) \); \( t \mapsto \theta(t, \cdot) \forall t \in T \).
is continuous under the topology of uniform convergence on $C(X,I)$. Next, we will prove that $\Theta$ is continuous at $\tau$ under the topology of uniform convergence on $C(X,I)$.

Indeed, let $\tau$ be an arbitrary admissible element of $T$ and let $\{t_\gamma \mid \gamma \in \Gamma\}$ be a net in $T$ with $t_\gamma \to \tau$ under the topology of $\Xi$. We need to show that $||\Theta(t_\gamma) - \Theta(\tau)|| \to 0$.

By condition (c), $R \cap \text{cl}_{\Xi} M \neq \emptyset$; and so it follows that we can choose an $a \in R$ with $ta_j \to a$ for some net $\{a_j \mid j \in J\} \in M$. Then $t_\gamma a_j \to ta_j$ for any $j \in J$ by condition (b). Now given any $\epsilon > 0$, there exists a neighborhood $U$ of $a$ in $T$ such that $||\Theta(a) - \Theta(t)|| < \epsilon$ for each $t \in U$, because $\Theta$ is continuous at the point $a \in R$.

Therefore, there exist two indices $j_0 \in J$ and $\gamma_0 \in \Gamma$ such that $||\Theta(t_\gamma a_j) - \Theta(ta_j)|| < 2\epsilon$ if $j > j_0$ and $\gamma > \gamma_0$. Since $a : x \mapsto a_j, j \in J$, is a surjection of $X$, then

$$||\Theta(t_\gamma) - \Theta(\tau)|| = \sup_{x \in X}||\Theta(t_\gamma x) - \Theta(\tau x)|| = \sup_{x \in X}||\Theta(t_\gamma(a_j x)) - \Theta(\tau(a_j x))||$$

as $j \to j_0$ in the directed index set $J$. Thus $||\Theta(t_\gamma) - \Theta(\tau)|| \to 0$ for $\epsilon > 0$ is arbitrary; and so $\Theta$ is continuous at the point $\tau$ from $(T, \Xi)$ to $(C(X,I), \|\|)$.

This, of course, implies that $\theta : (t, x) \mapsto \theta(tx)$ of $T \times X$ to $I$ is jointly continuous at each point of $\{\tau\} \times X$. The proof of Theorem 8.8 is thus completed.

Note that the group structure of $T$ plays a role in Namioka’s proof of Ellis’ joint continuity theorem ([23, Theorem 1] and [44, Theorem 3.1]). From Theorem 8.8, we can easily obtain the following four corollaries and Ellis’ joint continuity theorem.

As the first simple application of Theorem 8.8, we can obtain an affirmative answer to the following open question:

*Let $S$ be a compact $T_2$ semi-topological semigroup with a dense algebraic subgroup $G$. Suppose a net $g_\alpha \to g$ in $G$. Does $g_\alpha^{-1}$ converge to $g^{-1}$ in $G$? (See [42, Question 10.3].)*

**Corollary 8.9.** Let $S$ be a compact $T_2$ semi-topological semigroup with a dense algebraic subgroup $G$. Then $G$ is a topological subgroup of $S$.

**Proof.** Let $T = S, X = S$ and define $T \times X \to X$ by $(t, x) \mapsto tx$ and $X \times T \to X$ by $(x, t) \mapsto xt$. Since $G$ is a subgroup and dense in $S$, it follows that $\text{cl}_{\Xi} gG = T = \text{cl}_{T} Gg$ for all $g \in G$. Thus $g : x \mapsto gx$ and $g : x \mapsto xg$ are surjections of $X$ for each $g \in G$ and further $T$ is admissible at each element $g \in G$. Then by Theorem 8.8, $(t, x) \mapsto tx$ is continuous on $G \times X$ and $(x, t) \mapsto xt$ is continuous on $X \times G$. Now let $g_\alpha \to x$ in $G$ and let $g_\alpha^{-1} \to y$ in $S$; then by the continuity, $xy = e = yx$. Whence $y = x^{-1}$. This concludes the proof of Corollary 8.9.

The interesting point of Corollary 8.9 is that $G$ as a subspace of $S$ is not necessarily locally compact so Ellis’ theorem (cf. Theorem 8.15 below) plays no role here.

**Corollary 8.10.** Let $T$ be a semigroup of continuous self-surjections of a compact $T_2$-space $X$; and let $\Xi$ be a topology on $T$ such that $(T, X)$ is admissible. Then $(t, x) \mapsto tx$ of $T \times X$ into $X$ is jointly continuous.

Given any integer $d \geq 1$, the following corollary seems to be non-trivial because it is beyond Ellis’ joint continuity theorem.
Corollary 8.11. Let \( \mathbb{R}^d \times X \to X, (t, x) \mapsto tx \) be a separately continuous semiflow, where \( (\mathbb{R}^d, +) \) is under the usual Euclidean topology. If \( X \) is minimal, then \( (t, x) \mapsto tx \) is jointly continuous on \( \mathbb{R}^d \times X \).

Proof. Write \( T = \mathbb{R}^d \), which is an additive abelian semigroup. First, under the discrete topology of \( T \), \((T, X)\) becomes a minimal semiflow. Then by Corollary 3.3, it follows that for each \( t \in T \), \( x \mapsto tx \) is a continuous surjection of \( X \). Therefore, under the Euclidean topology of \( \mathbb{R}^d \), the following conditions are satisfied:

(a) \( T \) is a locally compact \( T_2 \)-space; and \((t, x) \mapsto tx \) is separately continuous of \( T \times X \) to \( X \).

(b) The right translation \( R_t: t \mapsto t + s \) of \( T \) to itself is continuous, for each \( s \in T \).

(c) \( \text{Int}_T \text{cls}_T(\tau + \{t\} \pi, \text{is a surjection of } X)) \neq 0 \), for each \( \tau \in T \).

Then by Lawson’s theorem (cf. [42, Theorem 5.2] and also see Theorem 8.8), \((t, x) \mapsto tx \) is jointly continuous on \( T \times X \). This completes the proof of Corollary 8.11.

This corollary may be applied to two interesting cases. First, let \( \mathbb{R}^d \times X \to X \) be a semiflow; then it is well known that the induced Ellis semiflow \( \mathbb{R}^d \times E(X) \to E(X) \) is only separately continuous, not necessarily jointly continuous. However, for any minimal left ideal \( \mathbb{I} \) of \( E(X) \), \( \mathbb{R}^d \times \mathbb{I} \to \mathbb{I} \) is a jointly continuous semiflow by Corollary 8.11. Particularly, if \((\pi, \mathbb{R}^d, X)\) is distal, then \( E(X) \) itself is a minimal left ideal in \( E(X) \) (by Lemma 6.7) so that \((\pi, \mathbb{R}^d, E(X))\) is a semiflow with the phase semigroup \( \mathbb{R}^d \) under the usual topology.

Secondly, let \( \beta \mathbb{R}^d \) be the Stone-Čech compactification of \( \mathbb{R}^d \). Then \( \beta \mathbb{R}^d \) is a compact Hausdorff right-topological semigroup in a natural manner and there is a natural separately continuous semiflow \( \mathbb{R}^d \times \beta \mathbb{R}^d \to \beta \mathbb{R}^d \). Therefore, for any minimal left ideal \( \mathbb{I} \) of \( \beta \mathbb{R}^d \), \( \mathbb{R}^d \times \mathbb{I} \to \mathbb{I} \) is a jointly continuous semiflow by Corollary 8.11.

Let \( C_p(X, X) \) denote the Hausdorff space \( C(X, X) \) equipped with the topology \( p \) of pointwise convergence. Clearly, \( C_p(X, X) \) is a semi-topological semigroup, since the maps \( R_f: f \mapsto f \circ g \) and \( L_g: f \mapsto g \circ f \) of \( C_p(X, X) \) to itself are continuous for each \( g \in C_p(X, X) \). Then for any subgroup \( G \) of homeomorphisms on \( X \), by an argument similar to the proof of [30, Proposition 8.3], we can see that the closure \( \text{cls}_{C_p(X, X)} G \) of \( G \) in \( C_p(X, X) \) is a subsemigroup of \( C_p(X, X) \).

The following corollary is a generalization of [23, Lemma 3] using different approach. There Ellis is for compact metric phase space \( X \).

Corollary 8.12. Let \( G \) be a group of self-homeomorphisms of a compact \( T_2 \)-space \( X \); and let \( T = \text{cls}_{C_p(X, X)} G \). If \( T \) is an l.c. subset of \( C_p(X, X) \), then \( (g, x) \mapsto gx \) of \( G \times X \) to \( X \) is jointly continuous, where \( G \) is regarded as a subspace of \( C_p(X, X) \).

Note. If \( G \) itself is a compact subset of \( C_p(X, X) \), then \( (G, X, \pi) \) is equicontinuous (cf. [4, Theorem 4.3] and Theorem 6.11 before).

Proof. We consider \( T \times X \to X \) defined by the evaluation map \((t, x) \mapsto tx \). As \( \text{cls}_T g G = T \) for each \( g \in G \), \( T \) is admissible at each element of \( G \). Thus Corollary 8.12 follows at once from Theorem 8.8.

We shall say that for a group \( G \), an action \( G \times X \to X \) is effective if whenever \( g \neq e \) for \( g \in G \) then \( gx \neq x \) for some \( x \in X \). This is only a minor technical condition. If the action is not effective, let \( F = \{ t \in G \mid tx = x \ \forall x \in X \} \). Then \( F \) is a closed (since \( X \) is \( T_2 \)) normal subgroup of \( T \). The
quotient group $G/F$ acts on $X$ by $(Ft)x = tx$, and this action is clearly effective. Therefore, we can assume that the action of $G$ on $X$ is effective.

Another consequence of Theorem 8.8 is the following

**Corollary 8.13.** Let $G \times X \to X$ be an effective flow with compact $T_2$ phase space $X$ and discrete phase group $G$. If $G$ is abelian, then $G$ is a topological subgroup of the enveloping semigroup $E(X)$ in the pointwise topology.

**Proof.** Since $G$ effectively acts on $X$, we may see $G \subseteq E(X)$. Let $\Pi: E(X) \times G \to E(X)$ be defined by $\Pi: (f, g) \mapsto f \circ g$, which is separately continuous in the pointwise topology by noting that $f \circ g = g \circ f$ for any $f \in E(X)$ and $g \in G$ (cf. [4, (1) of Lemma 3.4]). Clearly, $\Pi$ is effective. Write $E = E(X)$. Let $T = \text{cls}_{C_\alpha(E, E)} G$ where we have identified $G$ with $[\Pi_g | g \in G]$ such that $G \subseteq C_\alpha(E, E)$.

On the other hand, it is well-known fact that the Ellis semigroup of $(E, G)$ is such that $E(E, G) \approx E$ (cf. [4, p. 55]). Thus, for any $\xi \in E(E, G) \subseteq E^E$, $\xi: f \mapsto f \circ \xi$ of $E$ to $E$ is continuous in the pointwise topology, i.e., $\xi \in C_\alpha(E, E)$. So, $T = E(E, G)$ is a compact Hausdorff subset of $C_\alpha(E, E)$.

Therefore by Corollary 8.12, it follows that $E \times G \to E$ is jointly continuous in the pointwise topologies and so $G$ is a paratopological group in the pointwise topology. Moreover, if $g_n \to g$ in $G$ and $g_n^{-1} \to f \in E$ with the involved pointwise topologies, then $g_n^{-1} \circ g_n = e$ and $g_n^{-1} \circ g_n \to f \circ g$ for $\Pi$ is continuous. Whence $f \circ g = e$ and then $f = g^{-1}$ since $G$ is a group. This implies that $G$ is a topological subgroup of $E$ in the pointwise topology.

The proof of Corollary 8.13 is thus completed. \qed

Notice that under the situation of Corollary 8.13, while $G$ is abelian, $E(X)$ is not necessarily abelian (cf. [4, p. 55]); otherwise, $E(X)$ becomes a compact $T_2$ semi-topological semigroup and then the conclusion of Corollary 8.13 follows at once from Corollary 8.9.

The following is a slight generalization of a theorem of Ellis, in which the only new ingredient is condition (1) $\implies$ (3).

**Theorem 8.14.** Let $G$ be a group of self-homeomorphisms of a compact $T_2$-space $X$; and let $T = \text{cls}_{C_\alpha(X, X)} G$. Then the following conditions are pairwise equivalent.

1. $T$ is a compact $T_2$-topological subsemigroup of $C_\alpha(X, X)$.
2. $T$ is a compact $T_2$-topological subsemigroup of $C_\alpha(X, X)$.
3. $G$ is equicontinuous on $X$.

**Note 1.** Example 6.24 shows that the statement of Theorem 8.14 is not true if $G$ is a semigroup of homeomorphisms of $X$ in place of $G$ being a group.

**Note 2.** It is comparable with [4, Theorems 3.3 and 4.4]. Condition (2) $\iff$ (3) is just Ellis’ [23, Theorem 3]. Here our proof is completely independent of Ellis [23] and it is more concise than his one.

**Proof.** Condition (1) $\implies$ (3). Let $T$ be a compact $T_2$-topological subsemigroup of $C_\alpha(X, X)$. Then $(f, g) \mapsto f \circ g$ of $T \times T$ to $T$ is continuous in the topology $\Xi$ of pointwise convergence on $T$ inherited from $C_\alpha(X, X)$. We will prove that $T \times X \to X, (t, x) \mapsto tx$ is jointly continuous. According to Theorem 8.8, it suffices to show that $(T, \Xi)$ is admissible. Obviously we only need to check condition (c). Indeed, since $G$ is a group consisting of homeomorphisms on $X$, hence $\text{cls}_{T} G = T$ for all $g \in G$. Now for any $\tau \in T \setminus G$ and $t \in T$, take nets $\{\tau_i\} \subseteq G, \{t_i\} \subseteq T$
with \(\tau_i \to \tau\) and \(\pi_i \circ \tau_i = t\). By choosing a subnet of \(\{t_i\}\) in the compact \(T\) if necessary, we may assume \(t_i \to f \in T\). Thus, \(\pi \circ f = t\) and then \(\tau T = T\) for each \(\tau \in T\). Thus \((T, \Xi)\) is admissible. Furthermore, \(\pi\) is continuous on \(T \times X\) and so \(T\) is equicontinuous on \(X\) since \(T \times X\) is compact.

Condition (3) \(\Rightarrow\) (2). Since \(G\) is equicontinuous, hence \(G\) is distal on \(X\) and further \(T\) is a compact \(T_2\)-space with a group structure ([24, Theorem 1]).\(^2\) Thus by Theorem 8.8, it follows that the map \((u, v) \mapsto u \circ v\) of \(T \times T\) to \(T\) is continuous. Now let \(\{t_i\} \subset T\) be a net with \(t_i \to t\). If \(t_i^{-1} \to r\), then \(t_i t_i^{-1} = e\) implies that \(r = r^{-1}\). Thus \(t_i^{-1} \to r^{-1}\). Therefore \(T\) is a compact group relative to the space \(C_T(X, X)\).

Condition (2) \(\Rightarrow\) (1). This is trivial by definitions.

The proof of Theorem 8.14 is thus completed. \(\Box\)

Finally we will simply reprove another classical theorem of Ellis using our Theorem 8.8 above as follows.

**Theorem 8.15** ([23, Theorem 2]). Let \(G\) be an \(l.c.T_2\)-space with a group structure such that \((x, y) \mapsto xy\) of \(G \times G\) to \(G\) is separately continuous. Then \(G\) is a topological group.

**Proof.** Let \(X\) be the one-point compactification of \(G\) with point at infinity \(\infty\). Then \(G\) may be thought of as a subset of \(C_\infty(X, X)\) by setting \(g \infty = \infty\) and \(\infty g = \infty\) for all \(g \in G\). By Theorem 8.8, it follows that

\[
G \times X \to X, \quad (g, x) \mapsto gx \quad \text{and} \quad X \times G \to X, \quad (x, g) \mapsto xg
\]

are jointly continuous. Thus, \((x, y) \mapsto xy\) of \(G \times G\) to \(G\) is continuous.

Now let \(g \in G\) and \([g_y]\) a net in \(G\) with \(g_{y} \to g\) in \(G\). Since \(X\) is compact, we may assume \(g_{y}^{-1} \to h\) in \(X\). Thus by \(g_{y} g_{y}^{-1} = e = g_{y}^{-1} g_{y}\), we see that \(gh = e = hg\) and \(h = g^{-1} \in G\). Therefore, the map \(g \mapsto g^{-1}\) of \(G\) to \(G\) is continuous. The proof is completed. \(\Box\)

Comparing our independent self-closed proof of Theorem 8.15 with Ellis’ presented in [23], here we need not use [8, Exercise 17] which is not accessible for many readers. The proof that inversion is continuous is somewhat more involved in the available literature (see, e.g., [22, 23] and [4, p. 63]). Theorem 8.15 is comparable with [43, Theorem 2] and [34, Lemma, p. 982] where \(G\) is a Polish space.

Following Definition 3.8 a topological semigroup \(T\) is a left \(C\)-semigroup if and only if \(T \setminus sT\) is relatively compact in \(T\) for every \(s \in T\).

**Theorem 8.16.** Let \(X\) be a compact \(T_2\)-space and let \(T\) be a non-compact \(l.c.T_2\)-topological semigroup consisting of self-surjections of \(X\). If \(T\) is a left \(C\)-semigroup and \((t, x) \mapsto tx\) of \(T \times X\) onto \(X\) is separately continuous, then \((T, X)\) is a semiflow (i.e. \((t, x) \mapsto tx\) is jointly continuous).

**Proof.** The conditions (a) and (b) of Definition 8.6 evidently hold. Since \(T\) is not compact, hence \(T \setminus \text{cl}(sT)\) is a non-empty open set. Thus \(\tau T\) has a non-empty interior. This implies condition (c) of Definition 8.7. Then our statement follows at once from Theorem 8.8. \(\Box\)
Note that under the usual topology, \((\mathbb{R}^d_+ , +)\), for \(d \geq 2\), is not a left C-semigroup. Thus Corollary 8.11 has different flavor with Theorem 8.16.

**Corollary 8.17.** Let \(T\) be a non-compact l.c. C-semigroup and \(X\) a compact \(T_2\)-space. Suppose that \((T, X)\) is an invertible semiflow. If \(t_n \to t\) in \(T\) implies that \(t_n^{-1}x \to t^{-1}x\) for all \(x \in X\), then the reflection \((X, T)\) is a semiflow.

**Proof.** By hypothesis, \((x, t) \mapsto xt = t^{-1}x\) is separately continuous. Then \((X, T)\) is a semiflow by Theorem 8.16.

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