Perturbative Wilson loop in two-dimensional non-commutative Yang-Mills theory

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Abstract

We perform a perturbative $O(g^4)$ Wilson loop calculation for the $U(N)$ Yang-Mills theory defined on non-commutative one space - one time dimensions. We choose the light-cone gauge and compare the results obtained when using the Wu-Mandelstam-Leibbrandt (WML) and the Cauchy principal value (PV) prescription for the vector propagator. In the WML case the $\theta$-dependent term is well-defined and regular in the limit $\theta \to 0$, where the commutative theory is recovered; it provides a non-trivial example of a consistent calculation when non-commutativity involves the time variable. In the PV case, unexpectedly, the result differs from the WML one only by the addition of two singular terms with a trivial $\theta$-dependence. We find this feature intriguing, when remembering that, in ordinary theories on compact manifolds, the difference between the two cases can be traced back to the contribution of topological excitations.

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I. INTRODUCTION

There has been recently a growing interest in field theories defined on non-commutative spaces. Although their ultimate “physical” motivation is provided, in our opinion, by their tight relation with some limiting cases of string theories [1–3], the very possibility of exploring some specific non-local field theories in a systematic way in search of unexpected properties is fascinating on its own. These field theories are indeed non-local and non-locality has dramatic consequences on their basic dynamical features [4,5].

A concrete way of turning ordinary theories into non-commutative ones is to replace the usual multiplication of fields in the Lagrangian with the $\star$-product. This product is constructed by means of a real antisymmetric matrix $\theta^{\mu
u}$ which parameterizes non-commutativity of Minkowski space-time:

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \quad \mu, \nu = 0, \ldots, D - 1.$$  

The $\star$-product of two fields $\phi_1(x)$ and $\phi_2(x)$ is defined as

$$\phi_1(x) \star \phi_2(x) = \exp \left[ \frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x_1^\mu} \frac{\partial}{\partial x_2^\nu} \right] \phi_1(x_1)\phi_2(x_2)|_{x_1=x_2=x}$$

and leads to terms in the action with an infinite number of derivatives of fields which makes the theory non-local. Then one may wonder whether the theory would still comply with unitarity requirements.

Unitarity of scalar field theories, in the presence of non-commutativity, has been discussed in Ref. [6]: the authors explicitly check that Cutkoski’s rules are correct when $\theta^{\mu\nu}$ is of the “spatial” type, i.e. $\theta^{0i} = 0$. This exactly corresponds to the case in which an elegant embedding into string theory is possible: low-energy excitations of a $D$-brane in a magnetic background are in fact described by field theories with space non-commutativity [3], [7].

On the other hand theories with $\theta^{0i} \neq 0$ have an infinite number of time derivatives and are non-local in time: in this situation unitarity may be in jeopardy, the concept of Hamiltonian losing, in some sense, its meaning (see however [8]).
Again, this fact is not surprising when observed from the string theory point of view: \( \theta^{0i} \neq 0 \) is obtained in the presence of an electric background. The truncation of such a string theory to its massless sector is not consistent in this case [9] (see also [11]). The breakdown of unitarity in time-like scalar non-commutative theories has been considered in Ref. [11].

In the context of non-commutative quantum Yang-Mills theories, unitarity has been recently discussed in Ref. [12].

In ordinary theories, a typical probe to check unitarity is provided by the exponentiation of a Wilson loop. A perturbative computation has been widely used to check unitarity, assuming gauge invariance, or vice versa [13].

To extend this test to non-commutative theories is highly problematic, even in the spatial case. As a matter of fact the definition of the loop via a non-commutative path-ordering [14-16], has so far received a physical interpretation in the presence of matter fields as a wave function of composite operators only in a lattice formulation [17].

Exponentiation itself has not been proven to our knowledge, even in the spatial case. Nevertheless in Ref. (12) a perturbative \( \mathcal{O}(g^4) \) generalization to the spatial non-commutative case of the familiar results in the usual theory has shown that exponentiation persists in spite of the phases which reflect non-commutativity.

A particularly interesting case in ordinary \( U(N) \) gauge theories occurs in one-space, one-time dimensions (\( YM_{1+1} \)): thanks to the invariance under area-preserving diffeomorphisms, the Wilson loop can be exactly computed [18].

If the calculation of the loop is performed at all orders in perturbation theory using the light-cone gauge \( A_- = 0 \), two different functions of the area are obtained, according to the two different prescriptions chosen for the vector propagator in momentum space, namely

\[
D_{++} = i \ [k_-^2]_{PV} \quad (3)
\]

and

\[
D_{++} = i \ [k_- + i\epsilon k_+]^{-2} \quad (4)
\]
PV denoting the Cauchy principal value. The two expressions above are usually referred in the literature as ’t Hooft [19] and Wu-Mandelstam-Leibbrandt (WML) [20] propagators. They correspond to two different ways of quantizing the theory, namely by means of a light-front or of an equal-time algebra [21], respectively.

The loop for $U(N)$, computed according to ’t Hooft, coincides with the exact one obtained on the basis of geometrical considerations [18]

$$W = \exp(i g^2 N A), \quad (5)$$

and exhibits the Abelian-like exponentiation one expects on the basis of unitarity arguments, whereas the WML propagator leads to a different, genuinely perturbative result in which topological effects are disregarded [22]

$$W_{WML} = \exp(i g^2 A) L_{N-1}^{(1)}(-ig^2 A), \quad (6)$$

$L_{N-1}^{(1)}$ being a Laguerre polynomial.

The WML propagator can be Wick-rotated, thereby allowing for an Euclidean treatment. This is indeed the normal procedure followed in order to obtain the solution above. The continuation of the propagator is instead impossible when using the PV prescription.

One should however bear in mind that in the Euclidean formulation the original Minkowski contour of integration $\Gamma$ is changed to a (complex) contour $\Gamma^*$, according to the rule $(x_0, x_1) \rightarrow (ix_2, x_1)$. The area $\mathcal{A}$ is thereby converted into $i\mathcal{A}$. From what we have said, the Euclidean formulation cannot be defined if we choose the ’t Hooft propagator.

One can now inquire to what extent these considerations can be generalized to a non-commutative $U(N)$ gauge theory, always remaining in 1+1 dimensions. This is just the subject of the present work.

In 1+1 dimensions, non-commutativity necessarily affects the time variable leading to a potentially pathological theory.

As the Euclidean formulation has better regularity properties, we shall first consider the Euclidean non-commutative YM. In so doing however, the non-commutative parameter has
to undergo a transition to imaginary values as well, so that the basic (deformed) algebra is preserved.

In the next Section the Euclidean Wilson loop will be computed at $O(g^4)$. We shall find that it exhibits all the expected features, namely pure area dependence and continuity in the limit of vanishing non-commutative parameter. The limiting case of large non-commutative parameter (maximal non-commutativity) will also be discussed as well as the problems one encounters when trying to go beyond the $O(g^4)$-result.

In Sect.3 we shall face the difficulties inherent to a Minkowski formulation. We find, by performing a direct calculation in Minkowski space with the WML propagator, that the Wilson loop is in agreement with the Euclidean one, after both the area $A$ and the parameter $\theta$ have undergone simultaneous analytic reflection $A \to iA, \theta \to i\theta$.

One obtains in this way a regular result, even if the theory has space-time non-commutativity. A similar behaviour occurs also in perturbative Green functions, which keep their functional dependence on momenta, in spite of different analyticity properties coming from the analytic continuation to Minkowski variables.

An Euclidean formulation is instead impossible if the ’t Hooft propagator is considered, as no Wick rotation is allowed for it. An $O(g^4)$ calculation performed directly for the Minkowski contour produces singular integrals. Those integrals can be regularized by introducing a cutoff, but the final expression will irretrievably diverge when the cutoff is removed.

Final comments and possible future developments are discussed in Sect.4.

II. THE EUCLIDEAN WILSON LOOP

In this section we analyze the $U(N)$ Yang-Mills theory on a two-dimensional non-commutative space. The classical Minkowski action reads

$$S = -\frac{1}{2} \int d^2x \text{Tr} \left( F_{\mu\nu} \ast F^{\mu\nu} \right)$$

(7)

where the field strength $F_{\mu\nu}$ is given by
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig(A_\mu \star A_\nu - A_\nu \star A_\mu) \]  

and \( A_\mu \) is a \( N \times N \) hermitian matrix. The \( \star \)-product was defined in Eq. (2). The action in Eq. (7) is invariant under \( U(N) \) non-commutative gauge transformations

\[ \delta_\lambda A_\mu = \partial_\mu \lambda - ig(A_\mu \star \lambda - \lambda \star A_\mu). \]  

We quantize the theory in the light-cone gauge \( n^\mu A_\mu \equiv A_- = 0 \), the vector \( n_\mu \) being light-like, \( n^\mu \equiv \frac{1}{\sqrt{2}}(1, -1) \). The gauge condition can be imposed either adding to the action the gauge-fixing term

\[ S_{g.f.} = \frac{1}{\alpha} \int d^2x \ Tr \left( (A_- \star A_-)(x) \right) = \frac{1}{\alpha} \int d^2x \ Tr \left( (A_-(x))^2 \right), \quad \alpha \to 0, \]  

or by means of a Lagrange multiplier.

It is known that Faddeev-Popov ghosts decouple even in non-commutative theories \[23\], whereas the field tensor has only one component \( F_{-+} = \partial_- A_+ \).

The free vector propagator coincides with the one of the ordinary theory, namely it has the expressions of Eqs. (3) or (4), according to the algebra used for quantizing the theory.

In two dimensions, when quantized in axial gauge, the theory looks indeed free; in particular the action remains trivially invariant under area preserving diffeomorphisms; yet it can give rise to non-trivial correlations by means of open Wilson lines and to non-trivial potentials between external sources starting from closed Wilson loops.

In the non-commutative case the Wilson loop can be defined by means of the Moyal product as \[14,15\]

\[ \mathcal{W}[C] = \frac{1}{N} \int \mathcal{D}A e^{iS[A]} \int d^2x \ Tr P_\star \exp \left( ig \int_C A_+(x + \xi(s)) d\xi^+(s) \right), \]  

where \( C \) is a closed contour in non-commutative space-time parameterized by \( \xi(s) \), with \( 0 \leq s \leq 1 \), and \( P_\star \) denotes non-commutative path ordering along \( x(s) \) from left to right with respect to increasing \( s \) of \( \star \)-products of functions. Gauge invariance requires integration over coordinates, which is trivially realized when considering vacuum averages \[16\].

The perturbative expansion of \( \mathcal{W}[C] \), expressed by Eq. (11), reads
\[ W[C] = \frac{1}{N} \sum_{n=0}^{\infty} (ig)^n \int_0^1 ds_1 \ldots \int_{s_{n-1}}^1 ds_n \dot{x}_-(s_1) \ldots \dot{x}_-(s_n) \langle 0 | \text{Tr} \mathcal{T} [A_+ (x(s_1)) \ast \ldots \ast A_+ (x(s_n))] | 0 \rangle, \tag{12} \]

and it is easily shown to be an even power series in \( g \), so that we can write

\[ W[C] = 1 + g^2 W_2 + g^4 W_4 + O(g^6). \tag{13} \]

Through an explicit evaluation one is convinced that the function \( W_2 \) in Eq. (13) is reproduced by the single-exchange diagram, which is exactly as in the ordinary \( U(N) \) theory.

Therefore the first interesting quantity is \( W_4 \). It is obtained by the exchange of two propagators which can either cross or uncross. In the pairings (12)(34) and (23)(41) they do not cross, whereas in the pairing (13)(24) they do. Let us denote by \( W_4^{(1)}, W_4^{(2)} \) and \( W_4^{(cr)} \) the contributions in the three cases.

It is not difficult to realize that \( W_4^{(1)} \) and \( W_4^{(2)} \) are the same as the corresponding ones in the ordinary theory, being unaffected by the Moyal phase. We therefore concentrate our attention on \( W_4^{(cr)} \) and consider for the time being an Euclidean formulation. By exploiting the invariance of \( W_4 \) under area-preserving diffeomorphisms, which holds also in this non-commutative context, we consider the simple choice of a circular contour\[x(s) \equiv (x_1(s), x_2(s)) = r(\cos(2\pi s), \sin(2\pi s)). \tag{14}\]

Were it not for the presence of the Moyal phase, a tremendous simplification would occur between the factor in the measure \( \dot{x}_-(s)\dot{x}_-(s') \) and the basic correlator \( < A_+(s)A_+(s') > \). The Moyal phase can be handled in an easier way if we perform a Fourier transform, namely if we work in the momentum space. The momenta are chosen to be Euclidean and the non-commutative parameter imaginary \( \theta \rightarrow \text{i} \theta \). In this way all the phase factors do not change their behaviour.

We start from Eq. (12) and single out its \( O(g^4) \) contribution; we use WML propagators in the Euclidean form \( (k_1 - ik_2)^{-2} \) and parameterize the vectors introducing polar variables. Then we are led to the expression
The commutative part of the wave function is given by

$$\mathcal{W}_4^{(cr)} = r^4 \int_0^1 ds_1 \int_1^1 ds_2 \int_1^1 ds_3 \int_1^1 ds_4 \times$$

$$\int_0^\infty \frac{dp \, dq}{p \, q} \int_0^{2\pi} d\psi \, d\chi \exp(-2i(\psi + \chi)) \exp(2ip \sin \psi \sin \pi(s_1 - s_3)) \times$$

$$\exp(2iq \sin \chi \sin (s_2 - s_4)) \exp(i\frac{\theta}{q} \sin[\psi - \chi + \pi(s_2 + s_4 - s_1 - s_3)])$$

$$= A^2 \pi^2 F(\frac{\theta}{A}). \tag{15}$$

Integrating over $\psi$ and $p$, we get

$$\mathcal{W}_4^{(cr)} = \pi r^4 \int [ds] \int_0^\infty dq \, q \oint_{|z|=1} \frac{dz}{iz^3} \left(e^{-q \sin[\pi(s_4 - s_2)](z-\frac{1}{2})} - \frac{2}{z} e^{-i\pi\sigma} \right) \frac{1 - \gamma z e^{-i\pi\sigma}}{1 - \gamma ze^{i\pi\sigma}}. \tag{16}$$

where $\sigma = s_1 + s_3 - s_2 - s_4$ and

$$\gamma = \frac{\theta q}{2r^2 \sin \pi(s_3 - s_1)}; \quad \int [ds] = \int_0^1 ds_1 \int_1^1 ds_2 \int_1^1 ds_3 \int_1^1 ds_4.$$

It is an easy calculation to check that the function $F$ is continuous at $\theta = 0$ with $F(0) = \frac{1}{27}$, exactly corresponding to the value of the commutative case obtained with the WML propagator (see the $O(g^4)$ term of Eq.(6)).

One can also evaluate the function $F$ in the large-$\theta$ limit. This is most easily done by considering the splitting

$$\mathcal{W}_4^{(cr)} = \pi r^4 \int [ds] \int_0^\infty dq \, q \oint_{|z|=1} \frac{dz}{iz^3} \left(e^{-q \sin[\pi(s_4 - s_2)](z-\frac{1}{2})} - 1 \right) \frac{1 - \gamma z e^{-i\pi\sigma}}{1 - \gamma ze^{i\pi\sigma}}$$

$$+ \pi r^4 \int [ds] \int_0^\infty dq \, q \oint_{|z|=1} \frac{dz}{iz^3} \left(1 - \frac{2}{z} e^{-i\pi\sigma} \right) \frac{1 - \gamma z e^{-i\pi\sigma}}{1 - \gamma ze^{i\pi\sigma}}. \tag{17}$$

In the first integral we are entitled to perform the large-$\theta$ limit, thereby obtaining

$$\mathcal{I}_1 \simeq \pi r^4 \int [ds] \int_0^\infty dq \, q \oint_{|z|=1} \frac{dz}{iz^3} \left(e^{-q \sin[\pi(s_4 - s_2)](z-\frac{1}{2})} - 1 \right) \frac{e^{-2i\pi\sigma}}{z^2}$$

$$= \frac{r^4 \pi^2}{2} \int [ds] e^{-2\pi i(s_1 + s_3 - s_2 - s_4)} = -\frac{r^4}{16} \left(1 - \frac{3i}{\pi} \right). \tag{18}$$

The second integral in turn gives
\[ I_2 = \pi r^4 \int [ds] \int_0^\infty \frac{dq}{q} \oint_{|z|=1} dz \frac{1 - \frac{2}{z} e^{-i\pi \sigma}}{1 - \gamma z e^{i\pi \sigma}} \]

\[ = 2\pi^2 r^4 \int [ds] \int_0^1 \gamma d\gamma e^{2i\pi \sigma} (1 - \gamma^2) \]

\[ = \frac{r^4 \pi^2}{2} \int [ds] e^{2\pi i (s_1 + s_3 - s_2 - s_4)} = -\frac{r^4}{16} (1 + \frac{3i}{\pi}). \tag{19} \]

Summing \( I_1 \) and \( I_2 \), we finally get

\[ \lim_{\theta \to \infty} \mathcal{W}_4^{(cr)} = -\frac{A^2}{8\pi^2}. \tag{20} \]

The conclusion is that a perturbative Euclidean calculation with the WML prescription is feasible and leads to a regular result, in spite of the fact that non-commutativity in the Minkowski formulation involves the time variable. Abelian-like exponentiation does not hold, not surprisingly for \( N > 1 \), as the same phenomenon occurs also in the ordinary commutative case (see Eq.(6)) \[21,22\]. However one should remark that the presence of the Moyal phase introduces a \( \theta \)-dependent term in \( \mathcal{W}_4 \) that changes sign from the two limiting values \( \theta = 0 \) and \( \theta \to \infty \). A destructive interference is likely to occur.

It would be very interesting in our opinion to find an algorithm for computing higher perturbative orders, generalizing to the non-commutative theory the matrix model occurring in the ordinary case \[24\]. However it would require a full control of the Moyal phases for diagrams of increasing complexities. Alternatively one could try to envisage an algorithm of a non-perturbative type. This is likely to be possible starting from a suitable brane formulation and will be explored in future investigations.

III. THE MINKOWSKI FORMULATION

In this section we face the problem of performing a perturbative \( O(g^4) \) calculation of the Wilson loop, without making use of an Euclidean formulation. This is necessary as eventually we want to consider ’t Hooft’s prescription for the vector propagator, which cannot be Wick-rotated.
We start considering again the WML case, but refrain from passing to Euclidean coordinates. The $\mathcal{O}(g^4)$ contribution from the crossed diagram reads

$$
\mathcal{W}_4^{(\text{cr})} = - \int \left[ ds \right] \hat{x}_-(s_1) \ldots \hat{x}_-(s_4) \int \frac{dp_+ dp_- dq_+ dq_-}{4\pi^2 [p^2] 4\pi^2 [q^2]} e^{i\theta \left[ p_- q_+ - q_- p_+ \right]}
\times \exp\left( i \left[ p_+ (x_- (s_1) - x_- (s_3)) + p_- (x_+ (s_1) - x_+ (s_3)) \right] \right)
\times \exp\left( i \left[ q_+ (x_- (s_2) - x_- (s_4)) + q_- (x_+ (s_2) - x_+ (s_4)) \right] \right)
\times \left( (x_+ (s_1) - x_+ (s_3) + \theta \ q_+) \left( (x_+ (s_2) - x_+ (s_4) - \theta \ p_+ \right) \right).
$$

(21)

where the denominators $[p_-^2]$ and $[q_-^2]$ have to be interpreted according to Eq.(4).

Following the theory of distributions, the equation above can be rewritten as

$$
\mathcal{W}_4^{(\text{cr})} = \int \left[ ds \right] \hat{x}_-(s_1) \ldots \hat{x}_-(s_4) \int \frac{dp_+ dp_- dq_+ dq_-}{4\pi^2 [p^2] 4\pi^2 [q^2]} e^{i\theta \left[ p_- q_+ - q_- p_+ \right]}
\times \exp\left( i \left[ p_+ (x_- (s_1) - x_- (s_3)) + p_- (x_+ (s_1) - x_+ (s_3)) \right] \right)
\times \exp\left( i \left[ q_+ (x_- (s_2) - x_- (s_4)) + q_- (x_+ (s_2) - x_+ (s_4)) \right] \right)
\times \left( (x_+ (s_1) - x_+ (s_3) + \theta \ q_+) \left( (x_+ (s_2) - x_+ (s_4) - \theta \ p_+ \right) \right).
$$

(22)

We now recall the well-known relation between the WML distribution and the Cauchy principal value

$$
[k_+ + i\epsilon k_+]^{-1} = [k_-]_{PV} - i\pi \delta(k_-) \ sign(k_+).
$$

(23)

Eq.(23) splits naturally into three contributions, the first one being nothing but $\mathcal{W}_4^{(\text{cr})}$ evaluated with 't Hooft’s prescription

$$
\mathcal{W}_4^{(\text{cr,1})} = \int \left[ ds \right] \hat{x}_-(s_1) \ldots \hat{x}_-(s_4) \int \frac{dp_+ dp_- dq_+ dq_-}{4\pi^2 [p^2]_{PV} 4\pi^2 [q^2]_{PV}} e^{i\theta \left[ p_- q_+ - q_- p_+ \right]}
\times \exp\left( i \left[ p_+ (x_- (s_1) - x_- (s_3)) + p_- (x_+ (s_1) - x_+ (s_3)) \right] \right)
\times \exp\left( i \left[ q_+ (x_- (s_2) - x_- (s_4)) + q_- (x_+ (s_2) - x_+ (s_4)) \right] \right)
\times \left( (x_+ (s_1) - x_+ (s_3) + \theta \ q_+) \left( (x_+ (s_2) - x_+ (s_4) - \theta \ p_+ \right) \right).
$$

(24)

It is easy to realize that the expression above vanishes at $\theta = 0$ owing to the measure $[ds]$. When $\theta \neq 0$ the integrations over the momenta can be performed, leading to the formal result.
\[ W_4^{(cr,1)} = -\frac{\theta^2}{4\pi^2} \int [ds] \frac{\dot{x}_-(s_1) \ldots \dot{x}_-(s_4)}{\left[ (x_-(s_1) - x_-(s_3))^2 \right]_{PV} \left[ (x_-(s_2) - x_-(s_4))^2 \right]_{PV}} \]

where the denominators have to be interpreted as derivatives of the Cauchy principal value distribution.

The second contribution comes from the product of the two \( \delta \)-distributions

\[ W_4^{(cr,2)} = -\frac{1}{16\pi^2} \int [ds] \hat{x}_-(s_1) \ldots \hat{x}_-(s_4) \int dp_+dq_+ \ sign(p_+) \ sign(q_+) \]

\[ \times \exp(i[p_+(x_-(s_1) - x_-(s_3))] \exp(i[q_+(x_-(s_2) - x_-(s_4))]) \]

\[ \times \left( (x_+(s_1) - x_+(s_3) + \theta q_+) (x_+(s_2) - x_+(s_4) - \theta p_+) = \right. \]

\[ = \frac{1}{4\pi^2} \int [ds] \hat{x}_-(s_1) \ldots \hat{x}_-(s_4) \left[ \left( \frac{x_+(s_1) - x_+(s_3)}{(x_-(s_1) - x_-(s_3))^2}_{PV} \right) \left( \frac{x_+(s_2) - x_+(s_4)}{(x_-(s_2) - x_-(s_4))^2}_{PV} \right) \right. \]

\[ - i\theta \left. \frac{x_+(s_1) - x_+(s_3)}{(x_-(s_1) - x_-(s_3))^2}_{PV} \right]_{PV} \]

\[ + i\theta \left. \frac{x_+(s_2) - x_+(s_4)}{(x_-(s_2) - x_-(s_4))^2}_{PV} \right]_{PV} \]

\[ + \theta^2 \left. \frac{1}{(x_-(s_1) - x_-(s_3))^2}_{PV} \left( x_-(s_2) - x_-(s_4) \right)^2 \right]_{PV} \].

Finally there is the term due to mixed products (see Eq.(25))

\[ W_4^{(cr,3)} = -i\pi \int [ds] \hat{x}_-(s_1) \ldots \hat{x}_-(s_4) \int dp_+dp_- dq_+dq_- \ e^{i[\theta(p_-q_+ - q_-p_+)]} \]

\[ \times \left[ \sign(p_+) \delta(p_-) + \sign(q_+) \delta(q_-) \right]_{PV} \]

\[ \times \exp(i[p_+(x_-(s_1) - x_-(s_3))] + p_-(x_+(s_1) - x_+(s_3))] \]

\[ \times \exp(i[q_+(x_-(s_2) - x_-(s_4))] + q_-(x_+(s_2) - x_+(s_4))] \]

\[ \times \left( (x_+(s_1) - x_+(s_3) + \theta q_+) (x_+(s_2) - x_+(s_4) - \theta p_+) \right). \]

This term vanishes for symmetry reasons.

Now the (singular) \( \theta \)-dependent terms in \( W_4^{(cr,2)} \) cancel against the analogous (singular) terms which would occur when expanding the exponential in \( W_4^{(cr,1)} \), namely
\[ W_4^{(cr,1)} |_I = -\frac{\theta^2}{4\pi^2} \int [ds] \frac{\dot{x}_-(s_1) \ldots \dot{x}_-(s_4)}{\left[ (x_-(s_1) - x_-(s_3))^2 \right]_{PV} \left[ (x_-(s_2) - x_-(s_4))^2 \right]_{PV}} \tag{28} \]

and

\[ W_4^{(cr,1)} |_{II} = -\frac{i\theta}{4\pi^2} \int [ds] \dot{x}_-(s_1) \ldots \dot{x}_-(s_4) \left( \frac{x_+(s_2) - x_+(s_4)}{\left[ (x_-(s_2) - x_-(s_4))^2 \right]_{PV} \left[ (x_-(s_1) - x_-(s_3))^2 \right]_{PV}} \right) - \left( \frac{x_+(s_1) - x_+(s_3)}{\left[ (x_-(s_1) - x_-(s_3))^2 \right]_{PV} \left[ (x_-(s_2) - x_-(s_4))^2 \right]_{PV}} \right) \tag{29} \]

In this way \( W_4^{(cr)} \) turns out to be regular, as expected.

At \( \theta = 0 \) it is represented by the first term of Eq.\( (26) \): the quantity

\[ W_4^{(cr)} (\theta = 0) = \frac{1}{4\pi^2} \int [ds] \dot{x}_-(s_1) \ldots \dot{x}_-(s_4) \left( \frac{x_+(s_1) - x_+(s_3)}{x_-(s_1) - x_-(s_3)} \frac{x_+(s_2) - x_+(s_4)}{x_-(s_2) - x_-(s_4)} \right) \tag{30} \]

is nothing but the usual WML contribution as it should, after the Minkowski replacement \( A \to iA \). From here on it is no longer necessary to prescribe denominators as the measure provides the necessary regularization.

In order to compute the large-\( \theta \) limit of Eq.\( (22) \), we remember that the first two terms if we expand the exponential in Eq.\( (25) \), namely Eqs.\( (28, 29) \), cancel against the corresponding ones in Eq.\( (26) \). The next (\( \theta \)-independent) term in such an expansion

\[ W_4^{(cr,1)} |_{III} = \frac{1}{8\pi^2} \int [ds] \frac{\dot{x}_-(s_1) \ldots \dot{x}_-(s_4)}{\left[ (x_-(s_1) - x_-(s_3))^2 \right] \left[ (x_-(s_2) - x_-(s_4))^2 \right]} \times \left[ \left( x_-(s_1) - x_-(s_3) \right) \left( x_+(s_2) - x_+(s_4) \right) - \left( x_+(s_1) - x_+(s_3) \right) \left( x_-(s_2) - x_-(s_4) \right) \right]^2 \tag{31} \]

has to be added to the surviving one in Eq.\( (26) \), to obtain

\[ W_4^{(cr)} |_{\theta \to \infty} = \frac{1}{8\pi^2} \int [ds] \frac{\dot{x}_-(s_1) \ldots \dot{x}_-(s_4)}{\left[ (x_-(s_1) - x_-(s_3))^2 \right] \left[ (x_-(s_2) - x_-(s_4))^2 \right]} \times \left( \left[ \left( x_-(s_1) - x_-(s_3) \right) \left( x_+(s_2) - x_+(s_4) \right) \right]^2 + \left[ \left( x_+(s_1) - x_+(s_3) \right) \left( x_-(s_2) - x_-(s_4) \right) \right]^2 \right) \tag{32} \]

all other terms in the expansion of Eq.\( (25) \) vanishing in the large-\( \theta \) limit.
The integrals in Eq. (32) can be fairly easily computed, using for instance a rectangular contour; the final outcome is

$$W_4^{(cr,1)}_{III} = \frac{\mathcal{A}^2}{8\pi^2},$$

the opposite sign with respect to the one in Eq. (20) being due to the Minkowski replacement $\mathcal{A} \rightarrow iA$.

We stress that here the calculation has been performed directly in a Minkowski context, without rotating the Moyal phase.

The quantities $W_4^{(cr,1)}_{I}$ and $W_4^{(cr,1)}_{II}$ occurring in the Wilson loop calculation with 't Hooft’s prescription, are genuinely singular: integration of the distributions in Eqs. (28, 29) cannot be defined with the measure $[ds] \dot{x}_-(s_1) \ldots \dot{x}_-(s_4)$. If a cutoff is introduced as a regulator, the expressions diverge unavoidably when the cutoff is removed. In particular Eq. (28) does not depend on the area.

Those quantities are therefore responsible of the failure of the Wilson loop calculation in Minkowski space if ‘t Hooft’s prescription is adopted for the propagator. However we find quite remarkable that singularities are precisely confined within these two terms. The rest of the expression is regular in spite of the singular nature of ‘t Hooft’s propagator and coincides with the result one obtains after analytic continuation starting from the WML propagator. We find this unexpected connection very intriguing when remembering that in ordinary theories the difference between the two formulations on a compact manifold concerns their topological structure: the WML formulation only captures the topologically trivial sector [22].

IV. CONCLUSIONS

In usual commutative two-dimensional Yang-Mills theories ($YM_{1+1}$), closed Wilson loops can be exactly computed, either using geometrical techniques or by summing a perturbative series in light-cone gauge, in which the propagator is handled as a Cauchy principal value
prescription. As a matter of fact in this case contributions from crossed diagrams vanish thanks to peculiar support conditions. Only planar diagrams survive, enforcing the picture of the loop as an exchange potential between two (static) colour sources. A simple area exponentiation is the final outcome.

If the propagator is instead prescribed in a causal way, crossed diagrams no longer vanish; the result one obtains by summing the series no longer exhibits Abelian-like exponentiation for $N > 1$ and only captures the zero instanton sector of the theory $[22]$.

When considering non-commutative $YM_{1+1}$, one may wonder whether such a theory should make sense at all, owing to the pathologies occurring when non-commutativity involves the time variable. Indeed even simple models exhibit either trivial or inconsistent solutions in such a case $[25]$.

Therefore we find quite remarkable that a sensible expression can be obtained for a closed Wilson loop as a fourth-order perturbative calculation using the $WML$ propagator. It exhibits a smooth limit when the non-commutative parameter $\theta$ tends to zero, thereby recovering the usual commutative result; in the large non-commutative limit $\theta \to \infty$, the crossed diagrams contribution changes sign, thus suggesting the possibility of a critical value of $\theta$. The loop does not obey the Abelian-like exponentiation constraint, but this is not surprising for $N > 1$, as it happens also in the usual commutative case. For $N = 1$ the commutative Wilson loop trivially exponentiates; this does not happen in the noncommutative theory, providing a further example of the deep difference between the two cases.

Of course it would be very interesting in our opinion to be able to generalize such a calculation beyond the $O(g^4)$-order and to sum the resulting perturbative series as it was done in ref. $[18]$ in the ordinary case. Here the difficulty lies in the fact that different patterns of crossing entail different Moyal phases; no combinatorial algorithm has been found so far to cope with them, to our knowledge. It remains an open problem.

More dramatic is the situation when considering ’t Hooft’s form of the free propagator, which does not allow a smooth transition to Euclidean variables. Here the presence of the Moyal phase produces singularities even at $O(g^4)$, which cannot be cured. Strictly speaking,
the Wilson loop diverges (at least perturbatively). Nevertheless we find extremely interesting
the circumstance that the difference between its Moyal dependent term and the analogous
one in the $WML$ case turns out to be confined only in the two singular terms $\mathcal{W}_4^{(cr,1)}|_I$
and $\mathcal{W}_4^{(cr,1)}|_{II}$, which in turn possess a trivial $\theta$-dependence. This might entail far-reaching
consequences on the topological structure of the theory, when considered on a compact
manifold.

These features are currently under investigation and the results will be reported else-
where.

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