Renormalization group trajectories from resonance factorized S-matrices

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Abstract

We propose and investigate a large class of models possessing resonance factorized S-matrices. The associated Casimir energy describes a rich pattern of renormalization group trajectories related to flows in the coset models based on the simply laced Lie Algebras. From a simplest resonance S-matrix, satisfying the \( \phi^3 \)-property, we predict new flows in non-unitary minimal models.

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Recently Al. Zamolodchikov has introduced the “staircase model” [1], which is defined by a S-matrix of a single massive particle $m$ with amplitude

$$S(\theta, \theta_0) = \frac{(\sinh(\theta) - i \cosh(\theta_0))}{(\sinh(\theta) + i \cosh(\theta_0))}.$$ 

The important characteristic of this S-matrix is to possess two resonance poles at $\theta = -\frac{i\pi}{2} \pm \theta_0$. The ultraviolet behaviour is governed by a theory of central charge $c = 1$. At intermediate distances, however, this model exhibits a rather rich renormalization group (RG) trajectories by varying the real parameter $\theta_0$. For $\theta_0$ large enough, Al. Zamolodchikov [1] has found that the associated Casimir energy (defined in a torus of radius $R$) interpolates between the central charges $c_p = 1 - 6/p(p+1), p = 3, 4, ...$ of the minimal models $M_p$. Roughly, the Casimir energy $E_0(R, \theta_0)$ forms plateaux of approximately length $\theta_0^2$ in each value of $c_p$ before smoothly crossing over to the next lower fixed point $c_{p-1}$. In the literature the flow, $M_p \rightarrow M_{p-1}$, is also known [2, 3] as the effect of the $\phi_{1,3}$ perturbation to the critical point $M_p$.

The purpose of this Letter is to extend these ideas to more general classes of flows. We propose a resonant $Z(N)$-factorizable scattering theory in which the behaviour of its associated RG trajectories will be related to a certain deformation of the minimal $W(A_{N-1})$ conformal models. It turns out that this theory is related to the scattering of the $A_{N-1}$ Toda model [10] for complex values of its coupling constant. This relation allows us to easily conjecture the resonance scattering of the D and E Lie algebras. This last relation is only formal, due to the fact that the Toda Lagrangian needs an extra meaning for complex values of its coupling constant. However, the associated scattering theory is perfectly well defined, producing typical RG trajectories of flows in the deformed coset models based on simply laced Lie groups. Finally, new flows are predicted in non-unitary minimal models from the simplest resonance scattering theories satisfying the “$\phi_3$-property”.

We first start by describing the resonance $Z(N)$ scattering. The spectrum consists of a set of particles and antiparticles with masses $m_i = \sin\left(\frac{i\pi}{N}\right)/\sin\left(\frac{\pi}{N}\right), i = 1, 2, ..., N - 1[4]$. The antiparticle appears in the particle-particle amplitude, and the factorizability implies [5] that the only constrains are the crossing and the unitarity conditions,

$$S_{i,j}(\theta)S_{i,j}(-\theta) = 1, \quad S_{i,j}(\theta) = S_{N-j,i}(i\pi - \theta) \quad (1)$$

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where \( N - j \) is the antiparticle of the \( j \)-particle.

There is a family of solutions for Eq. (1) as a function of a real parameter \( \theta_0 \), which is responsible for the resonance poles. The minimal solution reads,

\[
S_{1,1}(\theta, \theta_0) = \frac{\sinh \frac{1}{2}(\theta + i\frac{2\pi}{N}) \sinh \frac{1}{2}(\theta - \theta_0 - i\frac{2\pi}{N}) \sinh \frac{1}{2}(\theta + \theta_0 - i\frac{2\pi}{N})}{\sinh \frac{1}{2}(\theta - i\frac{2\pi}{N}) \sinh \frac{1}{2}(\theta - \theta_0 + i\frac{2\pi}{N}) \sinh \frac{1}{2}(\theta + \theta_0 + i\frac{2\pi}{N})} \tag{2}
\]

The physical pole is at \( \theta = \frac{2\pi i}{N} \), while the resonance poles appear in the unphysical sheet at \( \theta = -\frac{i\pi}{N} \pm \theta_0 \). The other amplitudes \( S_{i,j} \) are obtained from \( S_{1,1} \) by applying the bootstrap approach at \( \theta = \frac{i\pi}{N}(i - j + 2(a - b)) \); \( a=1,2,...,j-1, \ b=1,2,...,i-1 \). For \( N = 2 \), we obtain Al.Zamolodchikov’s model \( [1] \).

Here our interest is to study this theory at intermediate distances, by analyzing the finite volume effects to the Casimir energy. An effective way to study the Casimir energy \( E(R, \theta_0) \) in a geometry of finite volume \( R \) is via the thermodynamic Bethe ansatz (TBA) approach \( [6, 7] \) at temperature \( T = \frac{1}{R} \). For the sake of simplicity let us first concentrate in the case \( N = 3 \). In this case the Casimir energy \( E(R, \theta_0) \) is given by,

\[
E(R, \theta_0) = \frac{-m}{\pi} \int_{-\infty}^{\infty} d\theta \cosh(\theta)L(\epsilon) \tag{3}
\]

where \( L(\epsilon) = \ln(1 + e^{-\epsilon(\theta)}) \), \( m \) is the mass of the particle and its antiparticle, and the pseudoenergy \( \epsilon(\theta) \) satisfies the following integral equation,

\[
\epsilon(\theta) + \frac{1}{2\pi} \int_{-\infty}^{\infty} d\theta' \psi(\theta - \theta', \theta_0)L(\epsilon) = mR \cosh(\theta) \tag{4}
\]

where \( \psi(\theta, \theta_0) = -i \frac{d}{d\theta} \log [(S_{1,1}(\theta, \theta_0)S_{1,1}(i\pi - \theta, \theta_0))] \)

The ultraviolet limit of the Casimir energy \( , R \rightarrow 0, \) is independent of \( \theta_0 \) and we find the behaviour \( E(R, \theta_0) \approx -\frac{2\pi}{6R} \), which implies that the background conformal theory has central charge \( c = 2 \) \( [8] \). In order to analyze the behaviour of the function \( c(R, \theta_0) = -\frac{6RE(R, \theta_0)}{\pi} \) for finite values of \( R \) we numerically solved Eq. (2), in the convenient variable \( X = \log(mR/2) \), by standard interactive procedure. For \( \theta_0 = 0 \), \( c(R,0) \) behaves as a smooth function between the ultraviolet \( (c=2) \) and the infrared \( (c=0) \) regimes. By increasing \( \theta_0 \), however, we observe that certain plateaux start to form precisely around the values
that parametrize the central charge of the minimal model of the $W(A_2)$ algebra, namely $c = 2(1 - 12/p(p + 1))$, $p=4,5,\ldots$. For example at $\theta_0 = 40$, we notice at least 8 plateaux starting at $p=12$ and subsequently visiting the other fixed points $p=11,\ldots,4$, until finally reaching the infrared region. In Fig.1(a,b) we show this behaviour for $\theta_0 = 20, 40$. The same pattern can be viewed from the beta function along the RG trajectories. Following Al.Zamolodchikov’s notation [1], one can define the beta function as,

$$\beta(g) = -\frac{d}{dX}c(R, \theta_0), \quad g = 2 - c(R, \theta_0)$$

In Fig.2(a,b), we show $\beta(g)$ for $\theta_0 = 20, 40$. The zeros of $\beta(g)$ are formed precisely at $g = 24/p(p+1), p=4,5,\ldots$, in accordance with the plateaux mentioned above.

In the case of general $N$ we should expect a similar behaviour. From Al.Zamolodchikov’s discussion of $N=2$ and our present results we conclude that each time that $X \approx -(p - N)\frac{h}{2}$ the function $c(R, \theta_0)$ will crossover its value of $c_p = (N - 1) (1 - N(N + 1)/p(p + 1))$, $p=N+1,N+2,\ldots$, to the next fixed point with central charge $c_{p+1}$. Indeed, by linearizing the TBA equations around $X \approx -(p - N)\frac{h}{2}$ one remains with the same equation that describes the flow in the $W(A_{N-1})$ minimal models perturbed by the least $Z(N)$-invariant operator [12]. However we stress that the bulk of each plateau has the approximately length of $\frac{\theta_0}{N}$, in agreement with the fact that the finite-size corrections are $N$-dependent.

As an important remark we mention that our proposed resonance $Z(N)$-factorized model is easily connected to the one of the $A_{N-1}$ Toda field theory [11], by making an analytical continuation to the complex values of the Toda coupling constant. The coupling constant $\alpha$ enters in the S-matrices through a function $b_{Toda}(\alpha)$ [11]. By setting $b_{Toda}(\alpha) = \frac{\pi}{N} \pm i\theta_0$ in Eq.(2) we recover the minimal S-matrix of the $A_{N-1}$ Toda model [11]. This lead us to conjecture that the resonance scattering theories, based on the simple laced Lie algebras ADE, can be obtained from the corresponding Toda S-matrices [11]. The resonance parameter $\theta_0$ is introduced through the simple relation $b_{Toda}(\alpha) = \frac{\pi}{h} \pm i\theta_0$, where $h$ is the dual Coxeter number of the respective ADE Lie algebra. Here we have also analyzed the TBA equations around $X \approx -(p - h)\frac{\theta_0}{N}$ and performed numerical checks. Our conclusion is that
the ADE resonance scattering will produce RG trajectories associated with the flows in the coset model $G_{p-h} \otimes G_{1}/G_{p-h+1}$ ($G=A,D,E$) [12] perturbed by the field $\Phi$ with conformal dimension $\Delta_\Phi = 1 - h/(p+1)$ [9]. The scaling corrections in the infrared regime is made by the “dual” operator $\tilde{\Phi}$ with conformal dimension $\Delta_{\tilde{\Phi}} = 1 + h/(p-1)$ (for $p=h+1$, this field is replaced by the spinless combination $TT$ of the stress energy tensor $T$). The field $\Phi(\tilde{\Phi})$ is the analogue of the operators $\phi_{1,3}(\phi_{3,1})$ of the minimal models. It has been argued [1, 13] that the combination $\lambda \phi_{1,3} + \tilde{\lambda} \phi_{3,1}$ plays a fundamental role in the description of Al. Zamolodchikov’s staircase model as a perturbed conformal field theory. In our case, the straightforward generalization will consider the combination $\lambda \Phi + \tilde{\lambda} \tilde{\Phi}$. We remark, however, that this picture has to be checked by a careful analysis of the finite size corrections to the fixed point [16].

Let’s us now introduce a resonance scattering model possessing the “$\phi^3$-property” that will be connected with new flows in non-unitary minimal models with $c < 1$. The model consists of a single particle $a$ and its two-body S-matrix is given by,

$$S_{a,a}(\theta, \theta_0) = \frac{\tanh \frac{1}{2}(\theta + i \frac{2\pi}{3}) \tanh \frac{1}{2}(\theta - \theta_0 - i \frac{\pi}{3}) \tanh \frac{1}{2}(\theta + \theta_0 - i \frac{\pi}{3})}{\tanh \frac{1}{2}(\theta - i \frac{2\pi}{3}) \tanh \frac{1}{2}(\theta - \theta_0 + i \frac{\pi}{3}) \tanh \frac{1}{2}(\theta + \theta_0 + i \frac{\pi}{3})}$$ (6)

The pole at $\theta = \frac{2\pi i}{3}$ produces the particle itself ($\phi^3$-property) and the resonance poles are located at $\theta_0 = \frac{4\pi}{3} \pm \theta_0$. It turns out that the amplitude $S_{a,a}(\theta, \theta_0)$ satisfies the relation $S_{a,a}(\theta, \theta_0) = S_{1,1}(\theta, \theta_0)S_{1,2}(\theta, \theta_0)$, where $S_{i,j}(\theta, \theta_0)$ are the S-matrices of the resonance $Z(3)$-model. The equivalent Toda theory [10] is one proposed by Mikhailov [14] as a particular reduction of the $Z(3)$ Toda model. From the TBA point of view, this implies that the Casimir energy associated to $S_{a,a}(\theta, \theta_0)$ is precisely half of that of the $Z(3)$-model. Hence, now the plateaux will be formed around the values $c_p = 1 - 12/p(p+1)$, $p = 4, 5, \ldots$. This result suggests that we are dealing with RG trajectories of the non-unitary minimal models. We recall that in this case the Casimir energy is identified with the effective central charge $c_{eff} = (24\Delta - c)$, where $\Delta$ is the lowest conformal dimension [15]. Indeed, this is satisfied by the following classes of non-unitary minimal models,

$$M_{\frac{q}{2q+1}}, M_{\frac{q+1}{2q+1}}, \quad q = p - 2 = 2, 3, \ldots$$ (7)
The flow pattern is well showed in the Fig.(3). We see that while in the $M_{\frac{4}{2y+1}}$ model the relevant(irrelevant) operator associated with the perturbation(infrared corrections) is $\phi_{1,5}(\phi_{2,1})$, in the $M_{\frac{2y+1}{2}}$ theory the situation is replaced by $\phi_{2,1}(\phi_{1,5})$. In the RG trajectories the field $\phi_{1,5}$ and $\phi_{2,1}$ interchange their roles of relevant and irrelevant operators, building an extremely interesting pattern of flow. For an extra support we also have numerically studied the spectrum of the simplest case, namely $M_{3/5} + \phi_{2,1} \rightarrow M_{2/5}$. Our results are compatible with the typical behaviour expected of the RG flows. Also, in the case of $M_{3/5}$, the field $\phi_{1,5}$ (not present on its Kac-table) is substituted by the level 2 descendent of the $\phi_{1,5}$ operator.

In summary we have discussed rich classes of renormalization RG trajectories obtained from resonance scattering models based on the ADE Lie algebras. New flows in non-unitary minimal models have also been predicted. In our conclusions the main bulk of technical details were omitted, and they will be presented in a forthcoming publication [16].

Acknowledgements

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Figure Captions

Fig. 1(a,b) The scaling function $c(R, \theta_0)$ for (a) $\theta_0 = 20$, and for (b) $\theta_0 = 40$

Fig. 2(a,b) The beta function $\beta(g)$ for (a) $\theta_0 = 20$ and for (b) $\theta_0 = 40$. For $\theta_0 = 40$ we have omitted the first zero at $p = 5$ in order to better show the remaining zeros of $\beta(g)$

Fig. 3 The flow pattern in the non-unitary minimal models $M_{\frac{2q}{2q+1}}; M_{\frac{2q+1}{2q+2}}, q = 2, 3, \ldots$. The horizontal(vertical) arrows represent the relevant(irrelevant) operators defining the ultraviolet(infrared) corrections to the fixed point.