VARIATIONS OF 4-DIMENSIONAL TWISTS OBTAINED
BY AN INFINITE ORDER PLUG

MOTOOS TANGE

Abstract. In the previous paper the author defined an infinite order plug \((P, \varphi)\) which gives rise to infinite Fintushel-Stern’s knot-surgeries. Here, we give two 4-dimensional infinitely many exotic families \(Y_n, Z_n\) of exotic enlargements of the plug. The families \(Y_n, Z_n\) have \(b_2 = 3, 4\) and the boundaries are 3-manifolds with \(b_1 = 1, 0\) respectively. We give a plug (or \(g\)-cork) twist \((P, \varphi_{p,q})\) producing the 2-bridge knot or link surgery by combining the plug \((P, \varphi)\). As a further example, we describe a 4-dimensional twist \((M, \mu)\) between knot-surgeries for two mutant knots. The twisted double concerning \((M, \mu)\) gives a candidate of exotic \#\(^2\)\(S^2 \times S^2\).

1. Introduction

1.1. Corks and plugs. If two smooth manifolds \(X, X'\) are homeomorphic but non-diffeomorphic, then we say that \(X\) and \(X'\) are exotic (or exotic pair).

A cut-and-paste is a performance removing a submanifold \(Z\) from \(X\) and regluing \(Y\) via \(\phi : \partial Y \to \partial Z\). We use the notation \((X - Z) \cup_{\phi} Y\) for the cut-and-paste. We call a cut-and-paste a local move in this paper. Let \(Y\) be a (codimension 0) submanifold of a 4-manifold \(X\). Let \(\phi\) be a diffeomorphism \(\partial Y \to \partial Y\). We denote the local move with respect to \((Y, \phi)\) by

\[ X(Y, \phi) := [X - Y] \cup_{\phi} Y, \]

and call such a local move a twist \((Y, \phi)\).

For a pair of exotic 4-manifolds \(X, X'\), we call a compact contractible Stein manifold \(Cr\) a cork, if \(Cr\) is smoothly embedded in \(X\), and \(X'\) is obtained by a cut-and-paste of \(Cr \subset X\) according to a diffeomorphism \(\tau : \partial Cr \to \partial Cr\). Hence, the boundary diffeomorphism \(\tau\) cannot extend to inside \(Cr\) as a diffeomorphism. We also call the deformation a cork twist \((Cr, \tau)\). Suppose that \(X\) and \(X'\) are two exotic simply-connected closed oriented 4-manifolds. Then they are changed to each other by a cork twist \((Cr, \tau)\) with an order

Date: September 22, 2015.

1991 Mathematics Subject Classification. 57R55, 57R65.

Key words and phrases. 4-manifolds, exotic structure, cork, plug, Fintushel-Stern’s knot-surgery, rational tangle, knot mutation.

This work was supported by JSPS KAKENHI Grant Number 24-840006.
2 boundary diffeomorphism \( \tau \) (3, 7, 15). Namely, this means
\[
X' = X(Cr, \tau).
\]
Hence, in some sense, the existence of such a cork \((Cr, \tau)\) causes 4-dimensional differential structures. Akbulut and Yasui in [4] defined another kind of twists, which are called plug twists, and which change smooth structures. The definition is given in the later section. A study of cork and plug should play a key role in understanding differential structures of 4-manifolds.

In this paper, we produce two types of infinitely many exotic enlargements of \( P \). The meaning of studying enlargements is to investigate to what extent the ‘exotic producer’ like cork or plug can extend to a larger 4-manifold. Putting a plug \((P, \varphi)\) defined in [18] and other deformations together, we give infinite variations of 4-dimensional (plug or g-cork) twist for rational tangle replacement. In terms of local move of knot we give a 4-dimensional twist \((M, \mu)\) with respect to the knot mutation, however since \( M \) is not a Stein manifold, the twist is neither plug nor g-cork.

2. The definitions, results, and brief proofs.

In this section we give definitions appeared, and a sequence of results obtained in this paper. We also give several proofs proven immediately.

2.1. Infinite order cork and plug. Let \( P \) denote a 4-manifold described in Figure 1. A diffeomorphism \( \varphi : \partial P \to \partial P \) is defined to be Figure 2.

Remark 1. Throughout this paper, any unlabeled component in any diagrams of 4-manifolds or of 3-manifolds stands for a 0-framed 2-handle or a 0-surgery respectively.

The paper [18] shows that the twist \((P, \varphi)\) is an infinite order plug, and the square twist \((P, \varphi^2)\) is a generalized cork (a g-cork), as defined later. Namely, \((P, \varphi)\) and \((P, \varphi^2)\) satisfy the following:

Theorem 1 ([18]). \( P \) is a Stein 4-manifold. The map \( \varphi : \partial P \to \partial P \) has infinite order and \( \varphi \) cannot extend to a self-homeomorphism on inside \( P \).
There exists a 4-manifold $X$ such that $\{X(P, \varphi^k)\}$ is a family of mutually exotic 4-manifolds.

The map $\varphi^2$ can extend to a self-homeomorphism, but cannot extend to any self-diffeomorphism on $P$.

In general, we define an infinite order plug, cork and g-cork.

**Definition 1** (Infinite order plug). $(P, \varphi)$ is an infinite order plug if it satisfies the following conditions:

1. $P$ is a compact Stein 4-manifold.
2. $\varphi$ cannot extend to a self-homeomorphism on $P$.
3. There exists a 4-manifold $X$ and embedding $P \subset X$ such that $\{X(P, \varphi^k)\}$ is a family of mutually exotic 4-manifolds.

**Definition 2** (Infinite order cork). $(C, \varphi)$ is an infinite order cork if it satisfies the following conditions:

1. $C$ is a compact contractible Stein 4-manifold.
2. $\varphi^k$ cannot extend to any self-diffeomorphism on $C$ for any positive integer $k$.

If $X(C, \varphi^k)$ are mutually exotic 4-manifolds, then $(C, \varphi)$ is an infinite order cork for $\{X(C, \varphi^k)\}$. In the case where $C$ is not contractible, in place of being contractible in the (1) condition, we call $(C, \varphi)$ a generalized cork (or g-cork).

The order of each $\varphi$ in Definition 1 and 2 as a mapping class on the boundary 3-manifold is infinite. The twist $(P, \varphi)$ defined above is an infinite order plug and $(P, \varphi^2)$ is an infinite order g-cork. Furthermore, the plug twist $(P, \varphi)$ can make a Fintushel-Stern’s knot-surgery. Let $V$ and $C$ denote the neighborhoods of Kodaira’s singularity III and II. See Figure 3 for the diagrams. $C$ is called a cusp neighborhood. These diagrams can be also seen in [13].

**Theorem 2** ([18]). Let $X$ be a 4-manifold containing $V$ and let $K$ be a knot. Let $X_K$ be a knot-surgery of $X$ along the general fiber of $V$. For a
knot $K'$ obtained by changing a crossing of a diagram of $K$, there exists an embedding $i : P \hookrightarrow X_K$ such that for the embedding $i$ we have

$$X_{K'} = X_K(P, \varphi).$$

The $n$-th power $(P, \varphi^n)$ makes an $n$ times full-twist

$$X_{K_n} = X_K(P, \varphi^n),$$

where $K$ and $K_n$ are the two knots whose local diagrams are Figure 4 and whose remaining diagrams are the same thing.

**Figure 4.** $K_n$ is the $n$-full twist of $K$.

**Remark 2.** The crossing change is a local move for knots or links. Theorem 2 means that the plug twist $(P, \varphi)$ plays a role in 'the crossing change of 4-manifolds' obtained by knot-surgery in some sense. Similarly, for many other local moves of knots or links, one can construct a local move over 4-manifolds. We will give an example of a 4-dimensional local move (twist) coming from a local move (knot mutation) of knots and links at the later section.

Here we define the knot-surgery and (2-component) link-surgery according to [10]. Let $T \subset X$ be an embedded torus with trivial neighborhood



**Figure 3.** The neighborhoods of Kodaira's singularity III and II (cusp).
and $K$ a knot in $S^3$. The 0-surgery $M_K$ cross $S^1$ naturally contains an embedded torus $T_m = \{\text{meridian}\} \times S^1$ with the trivial neighborhood. Then, (Fintushel-Stern’s) knot-surgery $X_K$ is defined to be the fiber-sum

$$X_K = (M_K \times S^1)\#_{T_m=T}X.$$  

Let $U_1, U_2$ be two 4-manifolds containing an embedded tori $T_i \subset U_i$ with the trivial neighborhoods. Let $L = K_1 \cup K_2$ be a 2-component link. Let

$$\alpha_L : \pi_1(S^3 - L) \to \mathbb{Z}$$

be a homomorphism satisfying $\alpha_L(m_i) = 1$, where $m_i$ is the meridian curve of $K_i$. Let $M_L$ be the $\alpha(\ell_i)$-surgery of $L$, where $\ell_i$ is the longitude of $K_i$. Let $T_{m_i}$ be a torus $m_i \times S^1 \subset M_L \times S^1$. Then, we denote by $(U_1, U_2)_L$ the following double fiber-sum operation:

$$(U_1, U_2)_L = U_1\#_{T_1=T_m}(M_L \times S^1)\#_{T_{m_2}=T}U_2.$$  

In the case of $U = U_1 = U_2$, we write as $(U, U)_L = U_L$. We call $(U_1, U_2)_L$ the link-surgery by the link $L$.

2.2. Two kinds of enlargements $Y_n$ and $Z_n$. Akbulut-Yasui’s corks $(W_n, f_n)$ and plugs $(W_{m,n}, f_{m,n})$ in [4] can give exotic enlargements by attaching 2-handles. In this paper we consider two kinds of enlargements $Y_0 = P \cup h_1$ and $Z_0 = P \cup h_1 \cup h_2$, where $h_1$, and $h_2$ are two 2-handles on $P$ as indicated in Figure 5 and the framings are both $-1$. Hence, we have

$$Y_0 = \tilde{Y} \# \mathbb{CP}^2$$

and

$$Z_0 = \tilde{Z} \# \mathbb{CP}^2.$$  

$\tilde{Y}$ (and $\tilde{Z}$) are 4-manifolds presented by the left (and right) diagrams in Figure 6.

Let $Y_n$ and $Z_n$ define to be other enlargements obtained by twists

$$(1) \quad Y_n = Y_0(P, \varphi^n)$$
and

\[ Z_n = Z_0(P, \varphi^n) \]

with respect to the embeddings \( P \hookrightarrow Y_0 \) and \( Z_0 \). Since \( Y_n \) and \( Z_n \) are the 2-handle attachments of the simply-connected manifold \( P \), they are also simply-connected and the Betti numbers \( b_2 \) of them are 3 and 4 respectively.

The g-cork \((P, \varphi^2)\) in \cite15 gives the diffeomorphisms:

\[ Y_{n+2} \simeq Y_n \]

and

\[ Z_{n+2} \simeq Z_n. \]

In this paper we use notation \( \cong \) and \( \simeq \) as a diffeomorphism and a homeomorphism respectively. Hence, \( Y_{2n} \) (or \( Z_{2n} \)) is homeomorphic to \( Y_0 \) (or \( Z_0 \)) and \( Y_{2n+1} \) (or \( Z_{2n+1} \)) is homeomorphic to \( Y_1 \) (or \( Z_1 \)). Actually \( Y_n \) and \( Z_n \) give the four homeomorphism types.

**Proposition 1.** Let \( X \) be \( Y \) or \( Z \). In \( \{X_n\} \) there exist two homeomorphism types \( X_0 \) and \( X_1 \) and we have

\[
X_n \simeq \begin{cases} 
X_0 & n \equiv 0 \text{ mod } 2 \\
X_1 & n \equiv 1 \text{ mod } 2.
\end{cases}
\]

This proposition is proven by seeing intersection forms in later section.

The boundary \( \partial Y_n \) is diffeomorphic to the 3-manifold described by the left diagram in Figure 6. This is a 0-surgery on \(-\Sigma(2,3,5)\) as the left diagram in Figure 6. The boundary \( \partial Z_n \) is 1-surgery of the granny knot. The proof is in Figure 7.

![Figure 6. Diagrams of \( \tilde{Y} \) and \( \tilde{Z} \) (as 4-manifolds) and \( \partial \tilde{Y} \) and \( \partial \tilde{Z} \) (as 3-manifolds).](image)

From the viewpoint of geometry, \( Y_n \) and \( Z_n \) have the following property.

**Theorem 3.** Let \( n \) be a positive integer. \( Y_n \) and \( Z_n \) are submanifolds of irreducible symplectic manifolds.

For the differential structures, we get the following theorem.

**Theorem 4.** Let \( n \) be any positive integer \( n \). Then \( Y_{2n} \) and \( Y_0 \) are exotic.

Whether \( \{Y_n\} \) are mutually non-diffeomorphic manifolds is unknown, however, we can prove the following.
Theorem 5. Each of \( \{ Y_{2n} | n \in \mathbb{N} \} \) and \( \{ Y_{2n+1} | n \in \mathbb{N} \} \) contains infinitely many differential structures.

We will prove this theorem in Section 3.1. The differential structures \( \{ Z_n \} \) satisfy the following.

Theorem 6. \( \{ Z_{2n} | n \geq 0 \} \) and \( \{ Z_{2n+1} | n \geq 0 \} \) are two families of mutually exotic 4-manifolds.

2.3. A twist for a rational tangle replacement. Let \( K_i \) be a knot or link for \( i = 1, 2 \). \( K_2 \) is a tangle replacement of \( K_1 \), if the local move \( K_1 \leftrightsquigarrow K_2 \) satisfies the following:

- \( K_2 \) is a local move of \( K_1 \) with respect to a closed 3-ball \( B^3 \) that \( K_i \) and \( \partial B^3 \) transversely intersects at \( K_i \cap \partial B^3 \).
- \( K_1 \cap B^3 \) and \( K_2 \cap B^3 \) are proper embeddings of several arcs in \( B^3 \).
- the arcs are homotopic to each other by a homotopy that fixes the boundary.

The usual crossing change of knots and links is one example of tangle replacements. Figure 8 is a picture of tangle replacement which is described schematically.

In this paper we treat the tangle replacements satisfying the following conditions. Let \( T_i \) denote \( K_i \cap B^3 \).

- \( \partial T_i \subset \partial B^3 \) are four points
- \( B^3 \setminus K_1 \) is homeomorphic to \( B^3 \setminus K_2 \).

The first example is the case where \( B^3 \setminus K_1 \) is homeomorphic to the genus two handlebody. We call the replacement rational tangle replacement.

Let \( (p,q) \) be relatively prime integers with \( p \) even. We define a diffeomorphism \( \varphi_{p,q} : \partial P \to \partial P \) in Section 4. The pair \( (P, \varphi_{p,q}) \) satisfies the following:
Proposition 2. Let $p$ be an even integer with $p \neq 0$. The twist $(P, \varphi_{p,q})$ is an infinite order

$$\begin{cases} 
    \text{plug} & p \equiv 2 \text{ mod } 4 \text{ or } \\
    \text{g-cork} & p \equiv 0 \text{ mod } 4.
\end{cases}$$

Let $O_n$ denote the $n$-component unlink.

Theorem 7. Let $X$ be a 4-manifold containing $V$ and let $K_{p,q}$ be a non-trivial 2-bridge knot. Then there exists an embedding $i : P \hookleq V \subset X$ such that the twist $(P, \varphi_{p-1,q})$ with respect to $i$ gives the knot-surgery

$$X := X_{O_1} \sim X(P, \varphi_{p-1,q}) = X_{K_{p,q}}.$$ 

Let $X_1$ be a 4-manifold containing $C$ and let $K_{p,q}$ be a non-trivial 2-bridge link. Let $X$ be $X_1 \# X_2 \# S^2 \times S^2$. Then there exists an embedding $j : P \hookleq X$ such that the twist $(P, \varphi_{p,q})$ with respect to $j$ gives the link-surgery

$$X = (X_1, X_2)_{O_2} \sim X(P, \varphi_{p,q}) = (X_1, X_2)_{K_{p,q}}.$$ 

This is a generalization of the result (Theorem 1) that $(P, \varphi)$ is a plug and $(P, \varphi^2)$ is a g-cork. Namely, the case of $(p, q) = (2n, 1)$ corresponds to the equality $\varphi_{2n,1} = \varphi^n$.

By combining the twist and the inverse in Theorem 7 we also obtain a general rational tangle replacement

$$X_K \overset{\varphi_{p,q}^1}{\sim} X_{O_1} \overset{\varphi_{r,s}}{\sim} X_{K'}.$$ 

2.4. A twist for mutant knots. We call an involutive tangle replacement as in Figure 9 knot mutation and we call two knots $K, K'$ which are obtained by the knot mutation mutant knots. It is well-known that mutant knots have similar topological properties. Any two mutant knots have the same hyperbolic volume and HOMFLY polynomial, in particular, the same Alexander polynomial.
The next variation of \((P, \varphi)\) is a twist between knot-surgeries for any two mutant knots. The knot mutation is not a rational tangle replacement, because the local tangle complement is not homeomorphic to a handlebody. Indeed, compute the fundamental group of the local tangle complement. We found a twist \((M, \mu)\) of 4-manifold between the knot-surgeries for two mutant knots. Let \(M\) be a 4-manifold described by Figure 10. A map \(\mu : \partial M \to \partial M\) is defined in Section 4.3. From the diagram in Figure 10 we can prove that \(M\) is an oriented, simply-connected 4-manifold with \(\partial M = \partial P \# S^2 \times S^1\), \(H_*(M) \cong H_*(\vee^3 S^2)\), \(b_3(M) = 0\). \(\partial M \cong \partial P \# S^2 \times S^1\) is described in Figure 11.

**Theorem 8.** Let \(X\) be a 4-manifold containing \(V\). Let \(K, K'\) be any mutant knots. Then there exist a twist \((M, \mu)\) and an inclusion \(i : M \hookrightarrow X_K\) such that the square of the gluing map \(\mu : \partial M \to \partial M\) is homotopic to the trivial map on \(\partial M\) and changes the knot-surgeries as follows:

\[X_{K'} = X_K(M, \mu)\]

**Proposition 3.** The map \(\mu : \partial M \to \partial M\) extends to a self-homeomorphism on \(M\).

**Remark 3.** It are two subtle problems whether \(\mu\) extends to a self-diffeomorphism on \(M\). One reason is what \(M\) not be a Stein manifold. In fact any Stein filling of a reduced 3-manifold is a boundary-sum of Stein fillings of the connected-sum components of the 3-manifold due to [8]. It is well-known that the Stein filling of the \(S^2 \times S^1\) component must be diffeomorphic to \(D^3 \times S^1\) due to [9]. \(S^2 \times S^1\) is a connected-sum component of \(\partial M\). These facts and \(\pi_1(M) = e\) conclude that \(M\) never have any Stein structure. Therefore, if you are to interpret knot-surgeries for mutant knots as a plug or cork twist, then you must improve the construction of \(M\).

Another reason is what for mutant knots \(K\) and \(K'\), the Seiberg-Witten invariants are the same by Fintushel-Stern’s formula in [10].

**Remark 4.** If \(\mu\) can extend to \(M\) as a diffeomorphism, then two knot-surgeries of all pairs of mutant knots are diffeomorphic to each other. Unlike
Figure 11. A diffeomorphism $\partial M \cong \partial P \# S^2 \times S^1$

the examples by Akbulut [2], and Akaho [1], this diffeomorphism suggests a meaningful map coming from knot mutation.

If $\mu$ cannot extend to inside $M$ as any diffeomorphism, then $(M, \mu)$ would be a not-Stein $g$-cork giving a subtle effect.

The twisted double $D_{\mu}(M) := M \cup_\mu (-M)$ is homeomorphic to $\#^3 S^2 \times S^2$. Its diffeomorphism type is not-known. $D_{\mu}(M)$ has one connected-sum component of $S^2 \times S^2$, i.e. it is not irreducible.

**Proposition 4.** $M_0$ be a 4-manifold $M$ with a 2-handle deleted and let $\mu_0$ be a boundary diffeomorphism $\partial M_0 \to \partial M_0$ naturally induced from $\mu$. Then we have $D_{\mu}(M) = D_{\mu_0}(M_0) \# S^2 \times S^2$.

15756 Here we summarize several questions.

**Question 1.** Can the map $\mu$ extend to a self-diffeomorphism on $M$?

**Question 2.** Is $D_{\mu}(M)$ (or $D_{\mu_0}(M_0)$) an exotic $\#^3 S^2 \times S^2$ (or $\#^2 S^2 \times S^2$)?

**Question 3.** For mutant knots $K$ and $K'$, which twist $(M, \phi)$ can realize the deformation $X_K \leadsto X_{K'}$ as a cork or plug twist?

**Acknowledgements**

I thank Kouichi Yasui and Yuichi Yamada for giving me some useful suggestions and advice. Specially, Yasui pointed me out that $M$ is not Stein manifold.
3. Exotic enlargements.

3.1. The homeomorphism types. We consider four homeomorphism types $Y_0$, $Y_1$, $Z_0$ and $Z_1$ of the enlargement of $P$.

Lemma 1. The intersection forms of $Y_n$ and $Z_n$ are as follows:

\[ Q_{Y_n} = \begin{cases} \langle 0 \rangle \oplus H & n: \text{odd} \\ \langle 0 \rangle \oplus \langle 1 \rangle \oplus \langle -1 \rangle & n: \text{even}, \end{cases} \]

\[ Q_{Z_n} = \begin{cases} \oplus^2 H & n: \text{odd} \\ \oplus^2 \langle 1 \rangle \oplus^2 \langle -1 \rangle & n: \text{even}, \end{cases} \]

where $H$ is the quadratic form presented by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Proof of Proposition 11. Lemma 11 (3) and (4) imply the required assertion.

Proof of Lemma 11. From the homeomorphisms (3) and (4), we may consider homeomorphism types $Y_0$, $Y_1$ and $Z_0$, $Z_1$ respectively. From the picture in Figure 5 together with 2-handles, the intersection forms of $Y_0$ and $Z_0$ can be immediately seen $\langle 0 \rangle \oplus \langle 1 \rangle \oplus \langle -1 \rangle$ and $\oplus^2 \langle 1 \rangle \oplus^2 \langle -1 \rangle$ respectively. The diagram of $Z_1$ is the left of Figure 12 and the diagram of $Y_1$ is Figure 12 with the $-2$-framed component erased. Hence, the intersection forms of $Y_1$ and $Z_1$ are isomorphic to $\langle 0 \rangle \oplus H$ and $\oplus^2 H$ respectively.

Figure 12. $Z_1$ and $Z_2$.

3.2. Infinitely many exotic structures on $Y_0$ and $Y_1$. Fintushel and Stern in [10] computed the Seiberg-Witten invariant of the link-surgery. Let $L_n$ be the $(2,2n)$-torus link, in particular, the $(2,2)$-torus link is the Hopf link. Then the Seiberg-Witten invariant is as follows:

\[ SW_{E(1)L_n} = \Delta_{L_n}(t_1,t_2) = (t_1t_2)^{n-1} + (t_1t_2)^{n-3} + \cdots + (t_1t_2)^{-n+1}. \]

Thus, the basic classes are the following:

\[ B_{E(1)L_n} = \{ i(t_1 + t_2) | i = -n + 1, n + 3, \cdots, n - 1 \}. \]
Lemma 2. For any positive integer \( n \), \( E(1)_{L_0} \) is an irreducible symplectic manifold.

**Proof.** We assume that \( E(1)_{L_0} \) has an embedded sphere \( C \) with \( [C]^2 = -1 \). Since the intersection form is odd, \( n \) is even. We may assume \( C \) is a symplectic sphere. Let \( E' \) be the blow-downed manifold along \( C \). Then the Seiberg-Witten basic classes \( B_{E(1)_{L_0}} \) are of form \( \{k \pm PD(C) | k \in B_{E'} \} \). The basic classes \( k_+ = k \pm PD(C) \) satisfy \( (PD(k_+) - PD(k_-))^2 = 4C^2 = -4 \). However, from the basic classes \( E \), the self-intersection number of the difference of any two of the basic classes is zero. This is contradiction.

Since \( E(1)_{L_0}(n \neq 0) \) is a simply-connected, minimal symplectic manifold with \( b^2 > 1 \), it is irreducible due to [14]. Thus \( Z_n \) is also an irreducible symplectic manifold. \[ \Box \]

We prove Theorem 3.

**Proof of Theorem 3.** We will prove \( Y_n \subset E(1)_{L_0} \). The manifold \( P \subset Y_0 \) is embedded in \( E(1)_{L_0} \) by the definition. See [17] for the embedding. The twist of \( (P, \varphi^n) \) via the embedding \( P \hookrightarrow E(1)_{L_0} \) gets \( E(1)_{L_0} \). Then \( Y_0 \) changes to \( Y_n \) in \( E(1)_{L_0} \). This result is due to Theorem 2 or [18]. From Lemma 2, \( Y_n \) and \( Z_n \) are submanifolds of an irreducible symplectic 4-manifold. For \( n = 1 \), see Figure 13.

Applying the same twist for \( Z_0 \subset E(1)_{L_0} \), we can obtain an embedding \( Z_n \hookrightarrow E(1)_{L_0} \).

Notice that each of 2-handles \( h_1 \) or \( h_2 \) in \( Y_n \) and \( Z_n \) corresponds to the sections in \( E(1) - \nu(T^2) \).

**Proof of Theorem 4.** From Theorem 3 if \( n \) is positive, then \( Y_{2n} \) is irreducible, however \( Y_0 \) has a \((-1)\)-sphere. Thus \( Y_{2n} \) is not diffeomorphic to \( Y_0 \). From Proposition 1, \( Y_0 \) and \( Y_{2n} \) are homeomorphic.

Next, we will prove the existence of infinitely many mutually exotic differential structures in \( \{Y_n\} \). First, we prove the following lemmas:

Lemma 3. Let \( Q \) be a quadratic form \( \langle 0 \rangle \oplus \langle -1 \rangle \oplus \langle 1 \rangle \) on \( \mathbb{Z}^3 \). Any isomorphism \( (\mathbb{Z}^3, Q) \to (\mathbb{Z}^3, Q) \) preserving \( Q \) is presented by

\[
\begin{pmatrix}
\epsilon_1 & a & b \\
0 & \epsilon_2 & 0 \\
0 & 0 & \epsilon_3
\end{pmatrix},
\]

where each \( \epsilon_i \) is \( \pm 1 \) and \( a, b \) are any integers.

**Proof.** Let \( \phi \) be any isomorphism \( \phi : (\mathbb{Z}^3, Q) \to (\mathbb{Z}^3, Q) \) preserving the quadratic form \( Q = \langle 0 \rangle \oplus \langle -1 \rangle \oplus \langle 1 \rangle \). For the standard generator \( \{e_i\} \) in \( \mathbb{Z}^3 \), we denote the images by \( \phi(e_i) = a_i e_1 + b_i e_2 + c_i e_3 \). Since \( \phi \) preserves
Figure 13. $Y_2 \hookrightarrow [S^3 - \nu(L_n)] \times S^1 \cup 3$ vanishing cycles $\cup 2$-handle.

Q, we have

$$
\begin{align*}
-b_1^2 + c_1^2 &= 0, & -b_2^2 + c_2^2 &= -1, \\
-b_3^2 + c_3^2 &= 1, & -b_1b_2 + c_1c_2 &= 0, \\
-b_1b_3 + c_1c_3 &= 0, & -b_2b_3 + c_2c_3 &= 0.
\end{align*}
$$

Solving these equations, we have $c_2 = 0$, $b_3 = 0$, $b_2 = \pm 1$, and $c_3 = \pm 1$. Furthermore, we have $b_1 = c_1 = 0$. Here we put $b_2 =: \epsilon_2$, and $c_3 =: \epsilon_3$. Since the map $\phi$ is an automorphism on $\mathbb{Z}^3$, we have $a_1 =: \epsilon_1$, where $\epsilon_1 = \pm 1$. Hence, defining as $a = a_2$ and $b = a_3$, we get the presentation matrix of $\phi$.

\[\epsilon_1 \quad a \quad b \\
0 \quad \epsilon_2 \quad 0 \]

\[0 \quad 0 \quad \epsilon_2 \]

where each $\epsilon_i$ is $\pm 1$ and $a, b$ are any integers.

**Lemma 4.** Let $Q$ be a quadratic form $\langle 0 \rangle \oplus H$ on $\mathbb{Z}^3$. Any isomorphism $(\mathbb{Z}^3, Q) \to (\mathbb{Z}^3, Q)$ preserving $Q$ is presented by

where each $\epsilon_i$ is $\pm 1$ and $a, b$ are any integers.

**Proof.** Let $\phi$ be any isomorphism $(\mathbb{Z}^3, Q) \to (\mathbb{Z}^3, Q)$. For the standard generator $\{e_i\}$ in $\mathbb{Z}^3$, we denote the images by $\phi(e_i) = a_i e_1 + b_i e_2 + c_i e_3$. Since $\phi$ preserves $Q$, we have

$$
\begin{align*}
b_1c_1 &= 0, & b_2c_2 &= 0, & b_3c_3 &= 0, \\
b_1c_2 + c_1b_2 &= 0, & b_1c_3 + c_1b_3 &= 0, & b_2c_3 + c_2b_3 &= 1.
\end{align*}
$$

If $b_1 \neq 0$, then $c_1 = 0$ holds from the first equation. Then, from $c_2 = -\frac{a_1b_2}{b_1}$ and $c_3 = -\frac{a_1b_3}{b_1}$, we obtain $c_2 = c_3 = 0$. This is contradiction for the last equation. Thus $b_1 = 0$ holds. In the same way $c_1 = 0$ holds.
Since $b_2b_3c_2c_3 = 0$, we have $b_2c_3 = 0$ or $c_2b_3 = 0$. If $b_2c_3 = 0$, then $c_2b_3 = 1$, hence, $c_2 = b_3 = \pm 1$ and $b_2 = c_3 = 0$ (because $b_2c_2 = 0$ and $c_3b_3 = 0$). If $c_2b_3 = 0$, then $b_2c_3 = 1$, hence $c_2 = b_3 = \pm 1$ and $b_3 = c_2 = 0$.

Since the map $\phi$ is an isomorphism, we get $a_1 = \pm 1$. Therefore, we get the presented matrix of $\phi$ as above. □

Here we introduce the following result in [16]:

**Proposition 5 ([16])**. Suppose that $\Sigma$ is a smooth, embedded, closed 2-dimensional submanifold in a smooth 4-manifold $X$ with $b_2^+(X) > 1$ and for a basic class $K$ we have $\chi(\Sigma) - [\Sigma]^2 - K([\Sigma]) = 2n < 0$. Let $\epsilon$ denote the sign of $K([\Sigma])$. Then the cohomology class $K + 2\epsilon PD([\Sigma])$ is also a basic class.

**Lemma 5.** Let $m$ be a positive integer. There exists a generator $\{T_1, T_2, S_m\}$ in $H_2(P_m)$ such that $T_i$ ($i = 1, 2$) are realized by tori and the genus of the surface realizing $S_m$ is $m(m - 1)$. The presentation matrix with respect to this generator is

$$
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & -2m^2 - m - 1
\end{pmatrix}.
$$

**Proof.** Recall that $Y_0 = P \cup h_1$ and $Y_m = Y_0(P, \varphi^m)$. Here $h_1$ is the 2-handle in Figure [5]. We denote by $\alpha_1$ the attaching sphere of $h_1$ and by $\alpha_{1,m}$ the image $\varphi^m(\alpha_1)$. The attaching sphere $\alpha_{1,m}$ is the $(m, 2m + 1)$-torus knot on the boundary of the 0-handle (see the fourth pictures in Figure [14]).

Let $m_1, m_2$ be the meridians for the link $L_m$. Let $T_i$ be the embedded torus $T_{m_i} = m_i \times S^1$ in $Y_m$ corresponding to $m_i$. $T_i$ can be seen in Figure [15].

![Figure 14. The torus knot for S.](image-url)
Figure 15. These tori are embedded in $Y_m$ as $T_1, T_2$.

Let $S_m$ be an embedded surface made from the union of a slice surface in $P$ of $\alpha_{1,m}$ and the core disk of $h_1$. Hence, the pair $\{T_1, T_2, S_m\}$ is embedded surfaces generating $H_2(Y_m)$, because $Y_m$ consists of $\alpha_{1,m}$ and 0-framed 2-handles by canceling two 1-/2-handle canceling pairs. The latter 0-framed 2-handles correspond to the handle decomposition of $P$.

The genus is $g(S_m) = \frac{(m-1)2m}{2} = m(m-1)$ since $\alpha_{1,m}$ is the $(m, 2m + 1)$-torus knot. The self-intersection number of $S_m$ is $-2m^2 - m - 1$ by canceling other components by handle calculus. The intersection of $T_2$ and $S_m$ can be understood from what attaching sphere of $T_2$ is a meridian of $S_m$ homologically in the same way as Figure 13 in [15].

Thus, the presentation matrix for the generators $\{T_1, T_2, S_m\}$ becomes the claimed one.

**Proof of Theorem 5** Suppose that there exists a diffeomorphism $\delta : Y_m \cong Y_n$ for some $m, n$ with $0 \leq m < n$ and $n \equiv m \pmod{2}$. We denote by $\{T'_1, T'_2, S_n\}$ such a pair corresponding to $Y_n$. We get a smooth inclusion:

$$S_m \subset Y_m \overset{\delta}{\rightarrow} Y_n \hookrightarrow E(1)_{L_n}.$$ 

We denote $\delta(S_m)$ simply by $S_m$ in $E(1)_{L_n}$.

Suppose that $m$ is even. The isomorphism $f_\delta : (\mathbb{Z}^3, Q_{Y_m}) \rightarrow (\mathbb{Z}^3, Q_{Y_n})$ can be decomposed as follows:

$$(\mathbb{Z}^3, Q_{Y_m}) \rightarrow (\mathbb{Z}^3, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) \rightarrow (\mathbb{Z}^3, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) \rightarrow (\mathbb{Z}^3, Q_{Y_n}).$$

Using Lemma 3 we obtain the following presentation for $f_\delta$:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -n^2 - \frac{n}{2} & n^2 + \frac{n}{2} + 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} \epsilon_1 & a & b \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -m^2 - \frac{m}{2} - 1 \\ 0 & 1 & -m^2 - \frac{m}{2} \end{pmatrix}.$$
Hence, the class of $S_m$ in $Y_n$ via $\delta$ is presented as follows:

$$
[S_m] = \left( - (a + b) \left( m^2 + \frac{m}{2} \right) - a \right) [T'_1] + \left( (\epsilon_2 - \epsilon_3) \left( m^2 + \frac{m}{2} \right) \left( n^2 + \frac{n}{2} \right) + \left( n^2 + \frac{n}{2} \right) \epsilon_2 \right) [T'_2] + \left( \epsilon_2 - \epsilon_3 \right) \left( m^2 + \frac{m}{2} \right) \epsilon_2 [S_n].
$$

(7)

Thus, we have the following intersection number

$$
[S_m] \cdot ([T'_1] + [T'_2]) = (\epsilon_2 - \epsilon_3) \left( m^2 + \frac{m}{2} \right) + \epsilon_2.
$$

Here putting $k = PD(\epsilon_2(n - 1)([T'_1] + [T'_2]))$ and $\eta = \frac{1 - \epsilon_2 \epsilon_3}{2}$, we have $k([S_m]) = (n - 1)((2m^2 + m)\eta + 1) > 0$.

Here we have

$$\chi(S_m) - [S_m]^2 - k([S_m]) = 2 - 2m(m - 1) + (2m^2 + m + 1) - (n - 1)((2m^2 + m)\eta + 1) = 3m + 3 - (n - 1)((2m^2 + m)\eta + 1) = 3m - n + 4 - (n - 1)(2m^2 + m)\eta \leq 3m - n + 4.
$$

If $n$ satisfies $3m + 4 < n$, then $\chi(S_m) - [S_m]^2 - k([S_m]) = 2\ell < 0$ holds.

Using Proposition 5, we have a basic class $k + 2PD([S_m])$.

Here $S_n$ represents a section in $E(1)_{L_n}$ thus $[S_n]$ is a non-vanishing class in $H_2(E(1)_{L_n})$. From the basic classes (5) of $E(1)_{L_n}$, the coefficient of $[S_n]$ in $[S_m]$ must be 0. The coefficient of $[S_n]$ is an odd number. See the coefficient in (7). Thus, this has some contradiction. Therefore, if $3m + 4 < n$ is satisfied, then $Y_n$ is not diffeomorphic to $Y_m$.

Suppose that $m$ is odd. Any isomorphism $(\mathbb{Z}^3, Q_{Y_m}) \to (\mathbb{Z}^3, Q_{Y_n})$ can be decomposed as follows:

$$Q_{Y_m} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \to Q_{Y_n}.
$$

Using Lemma 4, we obtain the following presentation for $\varphi$:

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} \epsilon_1 & a & b \\ 0 & \epsilon_2 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -m^2 - \frac{m+1}{2} \end{pmatrix}
$$

or

$$
\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} \epsilon_1 & a & b \\ 0 & \epsilon_2 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -m^2 - \frac{m+1}{2} \end{pmatrix}
$$

In fact, any automorphism preserving $(0) \oplus H$ is the solution of

$$
[S_m] = \left( b - a \left( m^2 + \frac{m+1}{2} \right) \right) [T'_1] - \epsilon_2(m - n) \left( m + n + \frac{1}{2} \right) [T'_2] + \epsilon_2[S_n].
$$
VARIATIONS OF 4-DIMENSIONAL TWISTS OBTAINED BY AN INFINITE ORDER PLUG 17

\[ [S_m] = \left( b - a \left( m^2 + \frac{m + 1}{2} \right) \right) [T'_1] + \epsilon_2 \left( 1 - \left( m^2 + \frac{m + 1}{2} \right) \left( n^2 + \frac{n + 1}{2} \right) \right) [T'_2] \]

\[ -\epsilon_2 \left( m^2 + \frac{m + 1}{2} \right) [S_n] \]

Thus, we have

\[ [S_m] \cdot ([T'_1] + [T'_2]) = \epsilon_2 \text{ or } -\epsilon_2 \left( m^2 + \frac{m + 1}{2} \right). \]

Here putting \( k = PD(\epsilon_2(n - 1)([T'_1] + [T'_2])) \) or \( PD(-\epsilon_2(n - 1)([T'_1] + [T'_2])) \), we have \( k([S_m]) = n - 1 > 0 \) or \( (n - 1)(m^2 + \frac{m + 1}{2}) > 0 \) respectively. Thus, we have

\[
\chi(S_m) - [S_m]^2 - k([S_m]) = 2 - 2m(m - 1) + 2m^2 + m + 1 - \left\{ \begin{array}{ll}
3m + 4 - n \\
3m + 3 - (n - 1)(m^2 + \frac{m + 1}{2})
\end{array} \right.
\]

\[ \leq 3m + 4 - n. \]

If \( n \) satisfies \( 3m + 4 < n \), then \( \chi(S_m) - [S_m]^2 - k([S_m]) = 2\ell < 0 \) holds. Using Proposition 5 \( k + 2PD([S_m]) \) is also a basic class. In the same reason as the case where \( m \) is even, the coefficient of \([S_n] \) in \([S_m] \) must be 0, namely, we have

\[ 2m^2 + m + 1 = 0. \]

Since this equation does not have any integer solution, \( Y_m \) is non-diffeomorphic to \( Y_n \).

In both parities of \( m \) and \( n \), we can get an infinite subsequence \( \{m_i\} \) in \( \mathbb{N} \) such that \( Y_{m_i} \) are mutually non-diffeomorphic to each other. \( \square \)

3.3. 4-manifolds \( \{Z_{2n}\} \) and \( \{Z_{2n+1}\} \) obtained by a g-cork \((P, \varphi^{2n})\). In this section we show infinitely many non-diffeomorphic exotic enlargements \( Z_n \) of \( P \).

\[ Z_0 = P \cup h_1 \cup h_2 = \bar{Z}_0 \#^2 \mathbb{CP}^2. \]

Proof of Theorem 6 Let \( E_{D,i} \to D^2 \) \((i = 1, 2)\) be two copies of the fibration of the complement \( E(1) - \nu(T^2) \) of the neighborhood of a fiber \( T^2 \).

The definition of the link-surgery gives \( E(1)L_n = ([S^3 - \nu(L_n)] \times S^1) \cup_{\omega_1} E_{D,1} \cup_{\omega_2} E_{D,2} \) (see the first picture in Figure 18). Each gluing map \( \omega_i \) is a map from \( \partial E_{D,i} \) to one component of \( \partial \nu(L_n) \times S^1 \).

Here, \( Dv_1 \), and \( Ds_1 \) in \( E_{D,1} \) are the neighborhoods of the compressing disk for the vanishing cycle and a section of \( E_{D,1} \to D^2 \). \( Dv_2 \), \( Dv_3 \), and \( Ds_2 \) in the other component \( E_{D,2} \) are the neighborhoods of the compressing disks for the vanishing cycles and a section of \( E_{D,2} \to D^2 \).

We use the same notation \( Dv_i \) and \( Ds_j \) as the parts put on \([S^3 - \nu(L_n)] \times S^1 \) via gluing maps \( \omega_1 \) and \( \omega_2 \) (see the second picture in Figure 18). Since the following holds:

\[ E_{D,1} - Dv_1 - Ds_1 = M_c(2, 3, 6) \]
Figure 16. Milnor fiber attached one 2-handle ($M_c(2, 3, 5)$). The boundary is $\Sigma(2, 3, 6)$.

and

$$E_{D,2} - Dv_2 - Dv_3 - Ds_2 = M_c(2, 3, 5),$$

we get

$$E(1)_{L_n} = (([S^3 - \nu(L_n)] \times S^1) \cup_{i=1}^3 Dv_i \cup_{i=1}^2 Ds_i) \cup M_c(2, 3, 6) \cup M_c(2, 3, 5).$$

Here the Milnor fiber is defined to be the set

$$M_c(p, q, r) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 | z_1^p + z_2^q + z_3^r = \epsilon \text{ and } |z_1|^2 + |z_2|^2 + |z_3|^2 \leq 1\},$$

for a non-zero complex number $\epsilon$. The handle decomposition is seen in [13]. Figure 17 gives $Z_n \cup h_3 \cup h^3 = ([S^3 - \nu(L_n)] \times S^1) \cup_{i=1}^3 Dv_i \cup_{i=1}^2 Ds_i$. The

link-surgery is constructed as follows:

$$E(1)_{L_n} = Z_n \cup h_3 \cup h^3 \cup M_c(2, 3, 6) \cup M_c(2, 3, 5).$$

The handles $h_3$ and $h^3$ are the 2- and 3-handle indicated in Figure 17.

Let $R$ denote the union $h_3 \cup h^3$. The attaching region of $R$ is a thickened torus $T^2 \times D^1$ on $\partial Z_n$. The boundary $\partial(Z_n \cup R)$ is the disjoint union of $\Sigma(2, 3, 5)$ and $\Sigma(2, 3, 6)$. The isotopy class of the essential torus in $\partial Z_n$ is
uniquely determined from JSJ-theory. Thus the self-diffeomorphism on $Z_n$ can extend to $Z_n \cup R$ uniquely.

Next we attach the Milnor fibers on the boundaries $\Sigma(2, 3, 5)$ and $\Sigma(2, 3, 6)$. Here we claim the following lemma:

**Lemma 6** ([12], [17]). Any diffeomorphism on $\Sigma(2, 3, 5)$ or $\Sigma(2, 3, 6)$ extends to $M_c(2, 3, 5)$ or $M_c(2, 3, 6)$ respectively.

**Proof.** The proof is the same as Lemma 3.7 in [11]. We remark the case of $\Sigma(2, 3, 6)$ here. By the result in [6] the diffeotopy type of $\Sigma(2, 3, 6)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The non-trivial diffeomorphism on $\Sigma(2, 3, 6)$ is the restriction of rotating by the $180^\circ$ about the horizontal line in Figure 16. Thus the diffeomorphism extends to $M_c(2, 3, 6)$.

Thus, any self-diffeomorphism on $\Sigma(2, 3, 5)$ and $\Sigma(2, 3, 6)$ can extend to $M_c(2, 3, 5)$ or $M_c(2, 3, 6)$.

The diffeomorphism on $Z_n$ can extend to $Z_n \cup R \cup M_c(2, 3, 5) \cup M_c(2, 3, 6) = E(1)_{L_n}$. This means that the diffeomorphism type of $E(1)_{L_n}$ is determined by that of $Z_n$. Conversely, if $m \neq n$, then $Z_n$ and $Z_m$ are non-diffeomorphic.
Hence, we have the following corollary.

**Corollary 1.** Any diffeomorphism \( Z_n \to Z_m \) extends to a diffeomorphism \( E(1)_{L_n} \to E(1)_{L_m} \).

Hence, in this case \( Z \) has the same role as Gompf’s nuclei \( N \) in [11].

4. **Some variations of plug twists.**

In this section, combining the plug twist \((P, \varphi)\) and other twists, we show the 2-bridge knot-surgery and 2-bridge link-surgery (Theorem 7) are produced by the same \( P \).

4.1. **The 2-bridge knot-surgery.** For an irreducible fraction \( p/q \), take the continued fraction

\[
p/q = a_1 - \frac{1}{a_2 - \frac{1}{a_3 - \cdots - \frac{1}{a_n}}} = [a_1, a_2, a_3, \cdots, a_n].
\]

The continued fraction determines the 2-bridge knot or link diagram as **Figure 19** where \( k \) in the figure stands for the \( k \)-half twist. The isotopy type of \( K_{p,q} \) depends only on the relatively prime integers \((p, q)\) and does not depend on the way of the continued fraction.

**Figure 19.** An example of the 2-bridge knot or link \( K_{p,q} \).

The following deformations of coefficients do not change the isotopy class of \( K_{p,q} \) and the rational number \( p/q \):

(8) \((a_1, \cdots, a_i, a_{i+1}, \cdots, a_n) \leftrightarrow (a_1, \cdots, a_i \pm 1, a_{i+1} \pm 1, \cdots, a_n)\)

(9) \((a_1, \cdots, a_n) \leftrightarrow (\pm 1, a_1 \pm 1, \cdots, a_n), (a_1, \cdots, a_n \pm 1, \pm 1)\)

By using this deformation, for any irreducible fraction \( p/q \) we get the continued fraction

\[
p/q = [b_1, b_2, \cdots, b_N]
\]

such that \( N \) is an odd number and \( b_3, b_5, \cdots, b_N \) are all even. If \( b_1 \) is odd or even, then \( K_{p,q} \) is a knot or 2-component link respectively. We define the 3-braid indicating as in the right of **Figure 19** with respect to \((b_1, b_2, \cdots, b_N)\) to be \( B_{p,q} \).

Let \( p \) be an even integer. Then we take a continued fraction \( p/q = [b_1, \cdots, b_N] \) as above. Namely, \( N \) is an odd number and \( b_1, b_3, \cdots, b_N \) are
even integers. We denote the map \( \psi : \partial P \rightarrow \partial P \) as in Figure 20. We define \( \varphi_{p,q} : \partial P \rightarrow \partial P \) as follows:
\[
\varphi_{p,q} := \frac{b_p}{2} \circ \psi^{b_{p-1}} \circ \cdots \circ \frac{b_3}{2} \circ \psi^{b_2} \circ \frac{b_1}{2}.
\]
This definition may depend on the way of continued fraction of \( p/q \). We choose such a continued fraction for the fraction \( p/q \). Here we prove Proposition 2.

**Proof.** First, we show that \( \varphi_{p,q} \) is not a torsion element. Let \( B_3 \) be the 3-braid group with the following presentation:
\[
B_3 = \langle \sigma_1, \sigma_2 | \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle,
\]
and let \( B_3^0 \) be a subgroup generated by \( \sigma_1 \) and \( \sigma_2^2 \). The generators \( \sigma_1 \) and \( \sigma_2 \) are as in Figure 21. This group is a normal subgroup in \( B_3 \) and gives
\[
\sigma_1 \quad \quad \quad \sigma_2
\]

**Claim 1.** \( B_3^0 \cong F_2 \times Z \), where \( F_2 \) is the rank 2 free group.

**Proof.** We have the following short exact sequence:
\[
1 \rightarrow F_2 \xrightarrow{f_1} B_3^0 \xrightarrow{f_2} \mathbb{Z} \rightarrow 0,
\]
where \( f_2 \) is the number of half-twists between the first string and the second string, namely, it is the map \( B_3^0 \rightarrow B_2 \cong \mathbb{Z} \) obtained by forgetting the third string. The subgroup in \( B_3^0 \) satisfying \( f_2 = 0 \) is considered as the

---

**Figure 20.** The definition of \( \psi \).

**Figure 21.** The generators \( \sigma_1, \sigma_2 \) in \( B_3 \).
homotopy class of a path on the 2 holed disk with a base point. Thus we have \( \text{Ker}(f_2) \cong F_2 \). This exact sequence is splittable since \( B_2 \cong \langle \sigma_1 \rangle \) is the subgroup in \( B_0^3 \) as a lift of \( f_2 \).

Since \( F_2 \) and \( \mathbb{Z} \) are torsion-free, \( F_2 \times \mathbb{Z} \) is also torsion-free. This means that if \( \varphi_{p,q} \) is torsion, then \( \varphi_{p,q} = \text{id} \) holds. Since the twist \( (P, \varphi_{p,q}) \) of \( E(1)_{O_2} = 3C P^2 \# 19C P^2 \) is trivial, namely, \( \Delta_{K_{p,q}}(t_1, t_2) = 0 \). The 2-bridge knot with Alexander polynomial zero is the 2-component unlink only. Therefore if \( p \neq 0 \), then \( \varphi_{p,q} \) is not torsion.

We compute the intersection form of \( D_{\varphi_{p,q}}(P) \). The double is described in Figure 22 (the case of \( N = 3 \)). The two (0-framed) fine curves are the attaching spheres of the upper manifolds of the double. The curve is parallel to the thick curve in each box with \( \pm b_{2k+1} \)-half twist and is twisted in each box with \( \pm b_{2k} \)-half twist. The parallel and twisted diagram is described in Figure 23. The first deformation (homeomorphism) in Figure 23 is also seen in [18] and the second and fourth deformations (diffeomorphisms) are also seen in [18]. The third deformation in Figure 23 is an isotopy of the diagram. Hence, the intersection form is \( \oplus 2 \left( \begin{array}{cc} 0 & 1 \\ 1 & -\frac{1}{2}(b_1 + b_3 + \cdots + b_N) \end{array} \right) \).

We claim the following:

**Lemma 7.** Let \( [b_1, \cdots, b_N] \) be a continued fraction of \( p/q \) with \( N \) an odd natural number. If \( b_1, b_3, \cdots, b_N \) are all even integers, then \( p \equiv (-1)^{\frac{N-1}{2}} (b_1 + b_3 + \cdots + b_N) \) mod 4.

**Proof.** The integer \( p \) is equal to the (1,1)-component in the following matrix.

\[
\left( \begin{array}{cc} b_1 & -1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} b_2 & -1 \\ 1 & 0 \end{array} \right) \cdots \left( \begin{array}{cc} b_N & -1 \\ 1 & 0 \end{array} \right).
\]

Since we have

\[
\left( \begin{array}{cc} b_1 & -1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} b_2 & -1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{cc} b_3 & -1 \\ 1 & 0 \end{array} \right) = \left( \begin{array}{ccc} -b_1 - b_3 & 1 - b_1 b_2 \\ 1 - b_2 b_3 & -b_2 \end{array} \right) \mod 4.
\]

Suppose that

\[
\prod_{l=1}^{2k+1} \left( \begin{array}{cc} b_l & -1 \\ 1 & 0 \end{array} \right) \equiv (-1)^k \left( \frac{1}{1 + \sum_{s=1}^{k} c_s b_{2s+1}} -1 + \sum_{s=1}^{k} c_s b_{2s-1} \right) \mod 4,
\]

(11)
where \( c_i, d_i, e \) are some integers. Then we have
\[
\begin{pmatrix}
\sum_{s=0}^{k} b_{2s+1} \\
1 + \sum_{s=1}^{k} d_s b_{2s+1}
\end{pmatrix}
-1 + \frac{\sum_{s=1}^{k} c_s b_{2s-1}}{e}
\begin{pmatrix}
 b_{2k+2} \\
1 + \sum_{s=1}^{k} d_s b_{2s+1}
\end{pmatrix}
= \begin{pmatrix}
 -\sum_{s=0}^{k+1} b_{2s+1} \\
1 - b_{2k+2}b_{2k+3} - \sum_{s=1}^{k} d_s b_{2s+1} + eb_{2k+3}
\end{pmatrix}
\begin{pmatrix}
 b_{2k+2} \sum_{s=0}^{k} b_{2s+1} + 1 + \sum_{s=1}^{k} c_s b_{2s-1} \\
-1 + \sum_{s=1}^{k+1} d'_s b_{2s-1}
\end{pmatrix}
\equiv \begin{pmatrix}
 b_{2k+2} \sum_{s=0}^{k} b_{2s+1} + 1 + \sum_{s=1}^{k} c_s b_{2s-1} \\
-1 + \sum_{s=1}^{k+1} d'_s b_{2s-1}
\end{pmatrix}
\mod 4
\]
where \( c'_i, d'_i, e' \) are some integers. Thus (11) holds for \( k+1 \) instead of \( k \). The induction implies \( p \equiv (-1)^{\frac{N-1}{2}}(b_1 + b_3 + \cdots + b_N) \mod 4 \).

We go back to the proof of Proposition 2. The intersection form of \( D_\varphi_{p,q}(P) \) is
\[
\begin{pmatrix}
 0 & 1 \\
1 & (-1)^{\frac{N+1}{2}}p
\end{pmatrix}
\cong \begin{cases}
\oplus^2(1) \oplus^2(-1) & p \equiv 2 \mod 4 \\
\oplus^2H & p \equiv 0 \mod 4
\end{cases}
\]
The Boyer’s result means that if \( p \equiv 2 \mod 4 \), then \( (P, \varphi_{p,q}) \) is a plug and if \( p \equiv 0 \mod 4 \), then \( (P, \varphi_{p,q}) \) is a g-cork.

We decompose Theorem 7 into two propositions (Proposition 6 and 7).

**Proposition 6.** Let \( X \) be a 4-manifold containing \( V \) and \( K_{p,q} \) be a non-trivial 2-bridge knot (i.e., \( p \) is an odd number). Then there exists an embedding \( i : P \hookrightarrow V \subset X \) such that the twist \( (P, \varphi_{p-1,q}) \) gives the deformation:
\[
X_{K_{p,q}} = X(P, \varphi_{p-1,q}, i),
\]
where the embedding \( i \) is defined in Figure 24 and independent of \( K_{p,q} \).

**Proof.** The embedding \( i : P \hookrightarrow V \) is constructed in Figure 24. The twist \( (P, \varphi_{\frac{b_1-1}{2}}) \) is described in the first deformation in Figure 25. Consecutively, we do the twist \( (P, \psi_{b^2}) \) (the second deformation in Figure 25). Continuing the twists along (10), we totally obtain the twist \( (P, \varphi_{p,q}) \) in the last picture in Figure 25. Here \( -B_{p,q} \) is the mirror image of the braid \( B_{p,q} \).

We compute the intersection form of the twisted double \( D_\varphi_{p,q}(P) \).

**Remark 5.** In the similar way, we can also construct another embedding \( i' : P \hookrightarrow V \) by changing the crossings in the broken circles in Figure 24. This embedding is different from \( i \), because the twist \( (P, \varphi_{p-1,q}) \) gives \( V_{K_{p-2,q}} \). In general, the Alexander polynomials of \( K_{p-2,q} \) and \( K_{p,q} \) are different.

### 4.2. 2-bridge link-surgery.
We consider the case of link-surgery. Let \( C \) be a cusp neighborhood (i.e., Kodaira’s singular fibration II). The handle decomposition of \( C \) is described in Figure 3. We denote \( C \# C \# S^2 \times S^2 \) by \( W \).
Figure 22. The homeomorphism type of $D_{\varphi_{p,q}}(P)$.

Figure 23. The local pictures of the fine curves in the box $\pm b_{2k-1}$ and $\pm b_{2k}$ in the first picture in Figure 22 (the cases of $b_{2k-1} = 4$ or $b_{2k} = 4$).

Proposition 7. Let $X_i$ ($i = 1, 2$) be two 4-manifolds containing $C$ and let $X$ be $X_1 \# X_2 \# S^2 \times S^2$. If $K_{p,q}$ is a 2-bridge link (i.e. $p$ is an even number), then there exists an embedding $j : P \hookrightarrow W \subset X$ such that the twist $(P, \varphi_{p,q})$ gets

$$X(P, \varphi_{p,q}) = (X_1, X_2)_{K_{p,q}},$$

where the embedding $j : P \hookrightarrow X := X_1 \# X_2 \# (S^2 \times S^2)$ is the one obtained by the same way as indicated in Figure 22.
Proof. The application of $\varphi_{p,q}$ to Figure 26 in the same way as Figure 25 gives the twist $X(P, \varphi_{p,q}) = (X_1, X_2)_{K_{p,q}}$. □

Proof of Theorem 7. Let $K$ be a 2-bridge knot or link. Then Proposition 6 and 7, it follows the required assertion. □

4.3. A twist $(M, \mu)$. Let $M$ be the manifold described in Figure 10. We factorize the knot mutation into the three processes as in Figure 27. According to this process, we define $\mu$ to be the map obtained by the process as described in Figure 28. Here $\tilde{\varphi}_1, \tilde{\varphi}_2: \partial M \to \partial M$ are maps obtained by performing locally $\varphi, \varphi^{-1}$ on $\partial M$.

Proof of Theorem 8. Let $K, K'$ be a mutant pair. We find an embedding $M \hookrightarrow V_K$. Let $D$ be a knot diagram of the knot $K$ containing the local tangle of the right in Figure 19. For example, the first picture in Figure 29 is such a diagram. We move the local tangle surrounded by the broken line to a bottom position by some isotopy (the second picture). The resulting diagram gives a plate presentation with keeping the local picture in the bottom (the third picture).

We prove that $(M, \mu)$ is a twist between knot-surgeries for mutant pair $K$ and $K'$ by illustrating the case of $V_K \sim V_C$ in Figure 30 where $KT$ is the Kinoshita-Terasaka knot and $C$ is the Conway knot.
Figure 25. The construction of the twist by \((P, \varphi_{p-1,q})\) of \(V\). The box \(n\) stands for the \(n\)-half twist.

Figure 26. The embedding \(j : P \hookrightarrow \#C\#S^2 \times S^2 =: W\).
isotopy

\[ \sim \] crossing change \[ \sim \] crossing change

**Figure 27.** The factorization of the knot mutation.

**Figure 28.** The definition of \( \mu \).
By keeping track of the processes in Figure 28, the square $\mu^2$ is the two times of the last move in Figure 28. This means a 360$^\circ$ rotation of $\partial M$ along the torus. This is homotopic to the identity. □

Proof of Proposition 3. The first picture in Figure 31 presents the untwisted double $D(M) := M \cup_{\text{id}} (-M)$. We can easily check the diffeomorphism $D(M) \cong \#^3 S^2 \times S^2$ by handle calculus. Removing $M$ in $D(M)$, regluing by $\mu$, we get the next picture in Figure 31. The intersection form of the twisted double $D_\mu(M)$ is isomorphic to $\oplus^3 H$. Thus, by using Boyer’s result in [5], $\mu$ can extend to a self-homeomorphism $M \to M$. □

Here we define $M_0$ to be $M$ with a $-1$-framed 2-handle deleted (the left of Figure 32). The boundary map $\mu_0 : \partial M_0 \to \partial M_0$ is naturally induced from the map $\mu$, because the $-1$-framed 2-handle in $M$ is fixed via the map $\mu$. The diffeomorphism $D_{\text{id}}(M_0) \cong \#^2 S^2 \times S^2 \# S^3 \times S^1$ and the homeomorphism $D_{\mu}(M_0) \cong \#^2 S^2 \times S^2$ hold due to easy calculation.

Proof of Proposition 4. The outmost (Hopf-linked) pair of $-1$-framed 2-handle and 0-framed 2-handle in Figure 31 can be moved to the parallel position of the other Hopf-linked pair by several handle slides. Such handle slides are indicated in Figure 33. Hence, the pair can be removed as one Hopf link component with both framings 0. See the bottom row in Figure 33. The same deformation is seen in Fig.15 in [17]. The remaining part is $D_{\mu}(M_0)$. □

References

[1] M. Akaho, A connected sum of knots and Fintushel-Stern knot surgery on 4-manifolds, Turk. J. Math. 30(2006)87-93
[2] S. Akbulut, A fake compact contractible 4-manifold, J. Differential Geom. 33, (1991), 335-356.
[3] S. Akbulut and R. Matveyev, A convex decomposition theorem for 4-manifolds, Internat. Math. Res. Notices 1998, no. 7, 371-381.
[4] S. Akbulut and K. Yasui Corks, Plugs and exotic structures, Jour. of GGT, vol 2 (2008) 40-82.
[5] S. Boyer, Simply-Connected 4-Manifolds with a given Boundary, American Mathematical Society vol. 298 no.1 (1986) pp.331-357
[6] M. Boileau and J-P. Otal, Groupe des difféotopies de certaines variétés de Seifert, C.R. Acad. Sci. Paris Sér. Math. 303 (1986), no. 1, 19-22
Figure 30. The performance $V_{KT} \rightarrow V_{KT} - M \rightarrow (V_{KT} - M) \cup_{\mu} M = V_C$. The fine curve presents the removed handles for $V_{KT} - M$. 

$V_{KT}$

$V_{KT} - M$

$V_{KT} - M$

$V_C$
Figure 31. $D(M) \sim D_{\mu}(M)$ (via the local move $(M, \mu)$).

Figure 32. $M_0$ and $D(M_0) = \#^3S^2 \times S^2 \#^3 \times S^1$.

[7] C. L. Curtis, M. H. Freedman, W. C. Hsiang, and R. Stong, A decomposition theorem for h-cobordant smooth simply-connected compact 4-manifolds, Invent. Math. 123 (1996), no. 2, 343-348.

[8] Y. Eliashberg, Filling by holomorphic discs and its applications, Geometry of Low-Dimensional Manifolds: 2, Proc. Durham Symp. 1989, London Math. Soc. Lecture Notes, 151, 1990, 45-67.

[9] Y. Eliashberg, Topological characterization of Stein manifolds of dimension $> 2$, Internat. J. of Math. 1 (1990), 29-46.

[10] R. Fintushel and R. Stern, Knots, links, and 4-manifolds, Invent. Math. 134 2 (1998), pp. 363-400

[11] R. Gompf, Nuclei of Elliptic surfaces, Topology, 30 (1991) 479-512

[12] R. Gompf and A. Stipsicz, 4-manifolds and Kirby calculus, Graduate Studies in Mathematics, 20. American Mathematical Society, Providence, RI, 1999

[13] J. Harer, A. Kas, and R.C. Kirby, Handlebody decompositions of complex surfaces, Mem. Amer. Math. Soc. 62 (1986), no. 350.

[14] D. Kotschick, The Seiberg-Witten invariants of symplectic four-manifolds, Asterisque No. 241 (1997), Exp. No. 812, 4, 195-220

[15] R. Matveyev, A decomposition of smooth simply-connected h-cobordant 4-manifolds, J. Differential Geom. 44 (1996), no. 3, 571-582.
Figure 33. To move a pair of 2-handles to the position of the other pair.

[16] P. Ozsváth and Z. Szabó, Symplectic Thom conjecture, Ann. of Math. (2) 151 (2000), no. 1, 93-124

[17] M. Tange, The link surgery of $S^2 \times S^2$ and Scharlemann’s manifolds, Hiroshima Math Journal44-1(2014)35-62

[18] M. Tange, A plug with infinite order and some exotic 4-manifolds, Journal of GGT (2015) arXiv:1201.6000

Institute of Mathematics, University of Tsukuba, Ibaraki 305-8571 JAPAN
E-mail address: tange@math.tsukuba.ac.jp