Perturbation calculation of the axial anomaly of Ginsparg-Wilson fermion

Ting-Wai Chiu and Tung-Han Hsieh

Department of Physics, National Taiwan University, Taipei, Taiwan 106, R.O.C

Abstract

We evaluate the axial anomaly for the general Ginsparg-Wilson fermion operator $D = D_c (1 + RD_c)^{-1}$ with $R = r 1$. For any chirally symmetric $D_c$ which in the free fermion limit is free of species doubling and behaves like $i \gamma_\mu p_\mu$ as $p \to 0$, the axial anomaly $\text{tr}[\gamma_5 (RD)(x,x)]$ for $U(1)$ lattice gauge theory with single fermion flavor is equal to $\frac{e^2}{32 \pi^2} \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x + \hat{\mu} + \hat{\nu})$ up to higher order terms and/or non-perturbative contributions. The $\bar{F} F$ term is $r$-invariant and has the correct continuum limit.

PACS #: 11.15.Ha, 11.30.Fs, 11.30.Rd

Key Words: Chiral Anomaly, Topological charge, Ginsparg-Wilson relation.

1 Introduction

In 1981, Ginsparg and Wilson [1] formulated a criterion for breaking the chiral symmetry of the massless Dirac operator on the lattice,

$$D\gamma_5 + \gamma_5 D = 2aD\gamma_5 R D$$

(1)

where $R$ is any invertible hermitian operator which is local in the position space and trivial in the Dirac space, and $a$ is the lattice spacing. The underlying reason for breaking the continuum chiral symmetry on the lattice is due to the Nielson-Ninomiya no-go theorem [2] which states that any Dirac operator on the lattice cannot simultaneously possess locality, free of species doubling and the chiral symmetry. However, one prefers to violate the chiral symmetry rather than the other two basic properties, is due to the fact that if the chiral
symmetry breaking is specified by the RHS of (1) having one \( \gamma_5 \) sandwiched by two Dirac operators, then not only the usual chiral symmetry can be recovered in the continuum limit \( a \to 0 \), but (1) also guarantees the remnant chiral symmetry on the lattice, that is, the action \( A = \bar{\psi} D \psi \) is invariant under the finite chiral transformation on the lattice

\[
\psi \to \exp[\theta \gamma_5 (1 - a R D)] \psi
\]

\[
\bar{\psi} \to \bar{\psi} \exp[\theta (1 - a D R) \gamma_5]
\]

where \( \theta \) is a global parameter. The infinitesimal form of (2) and (3) was first observed by Lüscher [3]. In fact, the GW relation (1) can be generalized to accommodate the asymmetric finite chiral transformations on the lattice [4] having two different hermitian operators, say \( S \) and \( T \) in (2) and (3) respectively, by replacing \( 2R \) in (1) by \( S + T \). Furthermore, the particular form of chiral symmetry breaking on the RHS of (1) also implies the chiral Ward identities, non-renormalization of vector and flavor non-singlet axial vector currents, and non-mixing of operators in different chiral representations [5].

In general, on the lattice, given any chirally symmetric Dirac operator \( D_c \)

\[
D_c \gamma_5 + \gamma_5 D_c = 0 ,
\]

which is free of species doubling but nonlocal as a consequence of the Nielson-Ninomiya no-go theorem, the general solution [6]

\[
D = D_c (1 + a R D_c)^{-1}
\]

is a chiral symmetry breaking transformation which gives a class of Dirac operators satisfying the GW relation (1). The general solution (5) also implies that any zero mode of \( D \) is also a zero mode of \( D_c \), and vice versa. That is, the zero modes of \( D \) are \( R \)-invariant. Then the index of \( D \) is also \( R \)-invariant and is equal to the index of \( D_c \). Thus the chiral symmetry breaking transformation (5) cannot generate a non-zero index for \( D \) if the index of \( D_c \) is zero [7]. It has become clear that a basic attribute of \( D(D_c) \) should be introduced, and it is called topological characteristics in ref. [7,8]. In general, the topological characteristics of a Dirac operator cannot be revealed by any perturbation calculations. Therefore, if we obtain non-zero axial anomaly for \( D \) in a perturbation calculation, it does not necessarily imply that the index of \( D \) must be non-zero for topologically non-trivial background gauge fields. The most reliable way to determine the topological characteristics of a Dirac operator is to perform the following numerical test [9,8]: turn on a topologically non-trivial smooth background gauge field with constant field tensors, then check whether there are any zero modes of \( D \), and measure the index of \( D \) versus the topological charge of the background gauge field. It should be emphasized
that the topological characteristics of $D$ is a basic attribute of $D$, which is not due to finite size effects, but persists for any lattice sizes, at any lattice spacings and in any background gauge configurations.

In Ginsparg and Wilson’s original paper [1], the axial anomaly for any $D$ satisfying Eq. (1) was derived, and it agrees with the continuum axial anomaly if $D$ is free of species doubling and in the free fermion limit behaves like $i\gamma_\mu p_\mu$ as $p \to 0$. However, in their derivation, Eq. (1) is used to eliminate $R$ in some of the intermediate expressions, then at the final steps of their computations, $D$ is replaced by $D_c$, which is equivalent to setting $R = 0$. Strictly speaking, their procedures are not completely self-consistent since $R$ should be kept nonzero throughout the entire computation, otherwise Eq. (1) cannot be used to eliminate $R$ from any expressions containing $D\gamma_5 RD$. In other words, setting $R = 0$ at their final steps of integrations actually invalidates their previous steps of using Eq. (1) to eliminate $R$. The proper procedure should keep $R \neq 0$ throughout the entire calculation, and then shows that the axial anomaly is independent of $R$, finally the limit $R = 0$ can be safely taken. This motivated us to re-derive the axial anomaly for a general $D$ satisfying the Ginsparg-Wilson relation, with the proper procedure, and in the context of recent developments. Furthermore, the realization of the Atiyah-Singer index theorem on a finite ( infinite ) lattice relies on the perturbation result of the axial anomaly as well as the topological characteristics of $D$. This provided us further impetus to carry out the tedious computations and present the details of our derivation in this paper. We note in passing that in ref. [10,11] the chiral anomaly for the overlap-Dirac operator [12,13] is calculated, and their results agree with the continuum axial anomaly. However, the overlap-Dirac operator is only one of the solutions satisfying the Ginsparg-Wilson relation, and of course their results do not imply that the same axial anomaly will be obtained for a general $D$ with another $D_c$.

If we also require $D$ to satisfy the hermiticity condition

$$D^\dagger = \gamma_5 D \gamma_5$$

(6)

then $D_c$ also satisfies this condition, and becomes an antihermitian operator ($D_c^\dagger = -D_c$) which has one to one correspondence to a unitary operator $V$,

$$D_c = \frac{\mathbb{I} + V}{\mathbb{I} - V}$$

(7)

where $V$ also satisfies the hermiticity condition $\gamma_5 V \gamma_5 = V^\dagger$. Thus the inverse operator in Eq. (5) must exist, and the general solution $D$ is well defined.

In this paper, we evaluate the axial anomaly $\text{tr}[\gamma_5 (RD) (x, x)]$ for $R$ diagonal in the position space, i.e., $R = r \mathbb{I}$ with parameter $r$. Then from Eq. (5), we
have

\[ D = D_c(\mathbb{I} + rD_c)^{-1} \]  \hspace{1cm} (8)

We shall restrict our discussions to the $U(1)$ gauge theory with single fermion flavor. However, it is straightforward to generalize our derivations to lattice QCD. For any $D_c$ which in free fermion limit is free of species doubling and behaves like $i\gamma_\mu p_\mu$ as $p \to 0$, our perturbation calculation shows that the $\tilde{F}\tilde{F}$ term of the axial anomaly $\text{tr}[\gamma_5(RD)(x, x)]$ is independent of $r$ and has the correct continuum limit, i.e.,

\[ r \text{ tr}[\gamma_5 D(x, x)] = \frac{e^2}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x + \hat{\mu} + \hat{\nu}) + \text{other terms} \]  \hspace{1cm} (9)

where the field tensor on the lattice is defined as

\[ F_{\mu\nu}(x) = \frac{1}{a}[A_\nu(x + \hat{\mu}) - A_\nu(x) - A_\mu(x + \hat{\nu}) + A_\mu(x)] \]  \hspace{1cm} (10)

The other terms in (9) in principle cannot be computed directly by any perturbation calculation to a finite order, however, their sum over the entire lattice can be determined and has significant impacts to the index theorem on the lattice, as we will show in section 2.

The outline of this paper is as follows. In section 2, we derive the divergence of the axial vector current for the Dirac operator satisfying the general Ginsparg-Wilson relation, and to set up the theoretical framework for the perturbation calculation in section 3. The topological characteristics of $D$ is shown to emerge naturally as an integer functional of $D$, after the axial anomaly is summed over the entire lattice. In section 3, we perform the derivation of the axial anomaly. In section 4, we conclude and discuss. In appendix A, we derive an identity for the $\tilde{F}\tilde{F}$ terms. In appendix B, we derive some useful properties of the kernel for the vector current which are used in the derivation of axial anomaly in section 3. In appendix C, we prove an identity for the Ginsparg-Wilson kernel of the vector current.

2 The axial vector current and its divergence

In this section we derive the divergence of the axial vector current $J_5^\mu(x; A, D)$ for the Dirac operator $D$ satisfying the Ginsparg-Wilson relation of the general form [4]
\[ D \gamma_5 (\mathbb{I} - S D) + (\mathbb{I} - D T) \gamma_5 D = 0 \quad (11) \]

where \( S \) and \( T \) are arbitrary invertible hermitian operators which are local in the position space and trivial in the Dirac space. The action for exactly massless fermion in a background gauge field is

\[ A = \sum_{x,y} \bar{\psi}_x D(x,y; A) \psi_y \quad (12) \]

where \( x \) and \( y \) are site indices, and the Dirac indices are suppressed. Then the action \( A \) is invariant under the chiral transformation

\[
\begin{align*}
\psi &\rightarrow \exp[\theta \gamma_5 (\mathbb{I} - SD)] \psi \\
\bar{\psi} &\rightarrow \bar{\psi} \exp[\theta (\mathbb{I} - DT) \gamma_5]
\end{align*}
\quad (13, 14)\]

where \( \theta \) is a global parameter. Hence, the divergence of the (associated Noether current) axial vector current, \( \partial^\mu J^5_\mu(x) \), can be extracted from the change of the action \( \delta A \) under the infinitesimal local chiral transformation at the site \( x \),

\[
\begin{align*}
\psi_x &\rightarrow \psi_x + \delta \theta_x \gamma_5 [(\mathbb{I} - SD) \psi]_x \\
\bar{\psi}_x &\rightarrow \bar{\psi}_x + \delta \theta_x [\bar{\psi} (\mathbb{I} - DT) \gamma_5]_x \\
\end{align*}
\quad (15, 16)\]

with the prescription

\[ A \rightarrow A + \delta \theta_x \partial^\mu J^5_\mu(x) \quad (17) \]

Then we obtain

\[
\partial^\mu J^5_\mu(x) = \bar{\psi}_x \gamma_5 (D \psi)_x + (\bar{\psi} D)_x \gamma_5 \psi_x \\
- (\bar{\psi} DT)_x \gamma_5 (D \psi)_x - (\bar{\psi} D)_x \gamma_5 (SD \psi)_x \\
\quad (18)\]

which satisfies the conservation law

\[ \sum_x \partial^\mu J^5_\mu(x) = 0 \quad (19) \]

due to the exact chiral symmetry (11) on the lattice. Now we take the lattice to be finite and with periodic boundary conditions, then we define \( \partial^\mu J^5_\mu(x) \) by the backward difference of the axial vector current

\[
\partial^\mu J^5_\mu(x) = \sum_\mu [J^5_\mu(x) - J^5_\mu(x - \hat{\mu})] \\
\quad (20)\]
such that it is parity even under the parity transformation, and the conservation law Eq. (19) is also satisfied.

To evaluate the fermionic average of the divergence of the axial vector current in a fixed background gauge field,

\[
\langle \partial^\mu J_5^\mu(x) \rangle = \frac{1}{Z} \int [d\psi][d\bar{\psi}] \partial^\mu J_5^\mu(x) \exp(-\bar{\psi}D\psi) \\
Z = \int [d\psi][d\bar{\psi}] \exp(-\bar{\psi}D\psi)
\]

one would encounter \( D^{-1} \) which is not well defined for the exactly massless fermion in topologically nontrivial gauge background. Thus, one needs to introduce an infinitesimal mass \( m \) which couples to the chirally symmetric Dirac operator \( D_c \) in the following way \([4]\)

\[
\hat{D} = (D_c + m) \left( \mathbb{1} - \frac{1}{2}(S + T)D_c \right)^{-1}
\]

and then evaluate the fermionic average (21) with \( D \) replaced by \( \hat{D} \), and finally take the limit ( \( m \to 0 \) ), i.e.,

\[
\langle \partial^\mu J_5^\mu(x) \rangle = \lim_{m \to 0} \frac{1}{Z} \int [d\psi][d\bar{\psi}] \partial^\mu J_5^\mu(x) \exp(-\bar{\psi}\hat{D}\psi) \\
Z = \int [d\psi][d\bar{\psi}] \exp(-\bar{\psi}\hat{D}\psi)
\]

Substituting \( \partial^\mu J_5^\mu(x) \) by Eq. (18) and using Eq. (23), we obtain

\[
\langle \partial^\mu J_5^\mu(x) \rangle = -\text{tr}[\left( \mathbb{1} - DT \right)_{xx}\gamma_5] - \text{tr}[\left( \mathbb{1} - SD \right)_{xx}\gamma_5] \\
+ m \text{ tr} \left( \left\{ \left[ \mathbb{1} - \frac{1}{2}(S + T)D \right] \hat{D}^{-1}(\mathbb{1} - DT) \right\}_{xx}\gamma_5 \right) \\
+ m \text{ tr} \left( \left\{ (\mathbb{1} - SD)\hat{D}^{-1}\left[ \mathbb{1} - \frac{1}{2}(S + T)D \right] \right\}_{xx}\gamma_5 \right)
\]

where \( \text{tr} \) denotes the trace in the Dirac space. The first two terms on the RHS of (26) is equal to

\[
\text{tr} \left\{ \gamma_5[(S + T)D]_{xx} \right\}
\]

while the last two terms in the limit ( \( m \to 0 \) ) give

\[
2m \text{ tr}[\hat{D}^{-1}_{xx}\gamma_5]
\]
which can be rewritten as

\[ 2m \sum_{\alpha,\beta} \sum_s \phi_s^\alpha(x) \gamma_5^{\alpha\beta} [\phi_s^\beta(x)]^* m + \lambda_s \]  

(29)

where \( \phi_s \) and \( \lambda_s \) are normalized eigenfunction and eigenvalue of \( D \),

\[ \sum_y \sum_\beta D_{xy}^{\alpha\beta} \phi_s^\beta(y) = \lambda_s \phi_s^\alpha(x) \]

\[ \sum_x [\phi_s^\alpha(x)]^* \phi_s^\alpha(x) = \delta_{ss'} \]  

(30)

In the limit \( m \to 0 \), only zero modes of \( D \) contribute to (29) and the result is

\[ 2 \sum_{s=1}^{N_+} [\phi_s^+(x)]^\dagger \phi_s^+(x) - 2 \sum_{t=1}^{N_-} [\phi_t^-(x)]^\dagger \phi_t^-(x) \]  

(31)

where \( \phi_s^+ \) and \( \phi_t^- \) are normalized eigenfunctions of \( D \) (or \( D_c \)) with eigenvalues \( \lambda_s = \lambda_t = 0 \) and chiralities +1 and −1 respectively. Therefore in the limit \( m \to 0 \), Eq. (26) becomes

\[ \langle \partial^\mu J_\mu^5(x) \rangle = \text{tr} \{ \gamma_5 [(S + T)D]_{xx} \} \]

\[ + 2 \sum_{s=1}^{N_+} [\phi_s^+(x)]^\dagger \phi_s^+(x) - 2 \sum_{t=1}^{N_-} [\phi_t^-(x)]^\dagger \phi_t^-(x) \]  

(32)

This is the anomaly equation for \( D \) satisfying the general Ginsparg-Wilson relation (11) which is the exact chiral symmetry on the lattice, where the axial vector current \( J_\mu^5(x) \) is the associated Noether current.

On the other hand, as usual, if one considers the action built from the chirally symmetric part of \( D \) [1],

\[ \mathcal{A}_s = \frac{1}{2} \bar{\psi} (D - \gamma_5 D \gamma_5) \psi \]  

(33)

then \( \mathcal{A}_s \) has the usual chiral symmetry, and the divergence of the associated Noether current can be extracted from the change of the action \( \delta \mathcal{A}_s \) under the infinitesimal local chiral transformation at the site \( x \)

\[ \psi_x \to \psi_x + \delta \theta_x \gamma_5 \psi_x \]

\[ \bar{\psi}_x \to \bar{\psi}_x + \delta \theta_x \bar{\psi}_x \gamma_5 \]
with the prescription (17), and one obtains

\[
\partial^\mu J_\mu^5(x) = \frac{1}{2} (\bar{\psi}[D, \gamma_5])_x \psi_x - \frac{1}{2} \bar{\psi}_x ([D, \gamma_5] \psi)_x \\
= \bar{\psi}_x \gamma_5 (D \psi)_x + (\bar{\psi} D)_x \gamma_5 \psi_x \\
- \frac{1}{2} \bar{\psi}_x [\bar{\psi} D (S + T) \gamma_5 D]_x \psi_x - \frac{1}{2} \bar{\psi}_x [\bar{\psi} \gamma_5 (S + T) D]_x \\
\] (34)

which also satisfies the conservation of total chiral charge, Eq. (19), due to the usual chiral symmetry. Although Eq. (34) looks fairly different from Eq. (18), however, the fermionic average of (34) in a background gauge field [14] is equal to that of (18), i.e., Eq. (32). It is evident that the difference between (18) and (34) has no physical consequences. In fact, it is possible to redefine \( \partial^\mu J_\mu^5(x) \) of Eq. (18) in many different ways provided that it satisfies the conservation law, Eq. (19), and its fermionic average agrees with Eq. (32), however, the corresponding \( J_\mu^5(x) \) is not equal to the Noether current associated to the exact chiral symmetry on the lattice (11). For example, if one redefines (18) as

\[
\partial^\mu J_\mu^5(x) = \bar{\psi}_x \gamma_5 (D \psi)_x + (\bar{\psi} D)_x \gamma_5 \psi_x \\
- \bar{\psi}_x (D \gamma_5 D \psi)_x - (\bar{\psi} D \gamma_5 SD)_x \psi_x \\
= [\bar{\psi} D \gamma_5 (\mathbb{I} - SD)]_x \psi_x + [\bar{\psi} x ([\mathbb{I} - DT])_x \gamma_5 D]_x \\
= [\bar{\psi} D \gamma_5 (\mathbb{I} - SD)]_x \psi_x - \bar{\psi}_x [D \gamma_5 (\mathbb{I} - SD)]_x \\
\] (35)

where Eq. (11) has been used in the last equality, then the fermionic average of Eq. (35) obviously agrees with Eq. (32), and the conservation law, Eq. (19) is also satisfied.

Likewise, the divergence of the vector current can be extracted from the change of the action \( \delta A \) under the infinitesimal local transformation

\[
\psi_x \rightarrow \psi_x + \delta \theta_x \psi_x \\
\bar{\psi}_x \rightarrow \bar{\psi}_x - \delta \theta_x \bar{\psi}_x
\]

with the prescription

\[
A \rightarrow A + \delta \theta_x \partial^\mu J_\mu^5(x),
\]

then we obtain

\[
\partial^\mu J_\mu^5(x) = (\bar{\psi} D)_x \psi_x - \bar{\psi}_x (D \psi)_x \\
\] (36)

where \( \partial^\mu J_\mu^5(x) \) is defined by the backward difference of the vector current.
\[ \partial_\mu J_\mu(x) = \sum_\mu [J_\mu(x) - J_\mu(x - \hat{\mu})] \]  

(37)

such that it is parity even under the parity transformation, and the conservation law due to the \( U_V(1) \) symmetry,

\[ \sum_x \partial_\mu J_\mu(x) = 0 \]  

(38)

is satisfied on a finite lattice with periodic boundary conditions. If the vector current is expressed in terms of the kernel \( K_\mu \) as

\[ J_\mu(x; A, D) = \sum_{y,z} \bar{\psi}_{x+y} K_\mu(x, y, z; A, D) \psi_{x+z} \]  

(39)

then by comparing Eqs. (34) and (36), the axial vector current satisfying (34) can be written as

\[ J_5^\mu(x; A, D) = \sum_{y,z} \bar{\psi}_{x+y} K_5^\mu(x, y, z; A, D) \psi_{x+z} \]  

(40)

where the kernel \( K_5^\mu \) is related to \( K_\mu \) by

\[ K_5^\mu = \frac{1}{2} (K_\mu \gamma_5 - \gamma_5 K_\mu) \]  

(41)

We note in passing that for the \( J_5^\mu \) defined in (35), one could not find a simple relationship between the kernel \( K_5^\mu \) of this axial vector current and the kernel \( K_\mu \) of the vector current, since (35) is non-linear in \( D \).

Now summing Eq. (32) over all sites of the lattice, the LHS is zero due to the conservation law (19), then the RHS gives the so-called index theorem on the lattice [14,3,12]

\[ N_- - N_+ = \frac{1}{2} \sum_x \text{tr} \{ \gamma_5 [(S + T)D]_{xx} \} \]  

(42)

where \( N_+ \) (\( N_- \)) denotes the number of zero modes of positive (negative) chirality. It has been shown in ref. [7] that the index is invariant with respect to \( S \) and \( T \) and is equal to that at the chiral limit,

\[ N_- - N_+ = \lim_{S,T \to 0} \frac{1}{2} \sum_x \text{tr} \{ \gamma_5 [(S + T)D]_{xx} \} = \frac{1}{2} \sum_x \text{tr} [\gamma_5 V_{x,x}] \]  

(43)
where Eq. (7) has been used. We note that \textit{a priori}, there is no compelling reasons to guarantee that \( D \) has zero modes in topologically non-trivial sectors. It could happen that \( D \) is local and free of species doubling in the free fermion limit, but turns out to have zero index in any background gauge fields \[8\]. In that case, \( D \) is called topologically trivial, and the index theorem \((42)\) is trivially satisfied with both sides equal to zero, however, it does not correspond to the Atiyah-Singer index theorem in continuum. Presumably the index of \( D \) is a topological and non-perturbative quantity, therefore the topological characteristics of \( D \) cannot be revealed by any perturbation calculations at finite orders. We refer to ref. \[7,8\] for further discussions on topological characteristics of \( D \). As we will show later in this section, the topological characteristics of \( D \) emerges naturally as an integer functional of \( D \), after the axial anomaly is summed over all lattice sites.

In the next section, we will show that the first term on the RHS of Eq. \((32)\),

\[
\text{tr}\{\gamma_5[(S + T)D]_{x,x}\}
\]

reproduces the topological charge density

\[
\rho(x) \equiv \frac{e^2}{16\pi^2} \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x + \hat{\mu} + \hat{\nu})
\]

up to higher order terms and/or non-perturbative contributions. The field tensor \( F_{\mu\nu}(x) \) on the lattice is defined in \((10)\) for QED. We note that in \((45)\) the positions of the field tensors \( F_{\mu\nu} \) and \( F_{\lambda\sigma} \) are chosen at \( x \) and \( x + \hat{\mu} + \hat{\nu} \) respectively such that \( \rho(x) \) satisfies

\[
\sum_x \delta_x \rho(x) = 0
\]

for any local deformations of the gauge field \[15\]. In the continuum limit, \( \rho(x) \) agrees with the Chern-Pontryagin density in continuum.

In order to extract the term which is quadratic in gauge fields from \((44)\), we consider the following operator \( \mathcal{H} \) \[14\] which is defined as

\[
\mathcal{H}[O] \equiv \sum_{x,y} x_\sigma y_\lambda \frac{\delta}{\delta A_\mu(0)} \frac{\delta}{\delta A_\mu(y)} \left[ O(x) \right]_{A=0}
\]

where \( O(x) \) is any observable. It is evident that the operator \( \mathcal{H} \) picks up the terms quadratic in gauge fields from the observable \( O \) and converts them into a constant.

The operator \( \mathcal{H} \) acting on \((45)\) gives
\[ \mathcal{H}[\rho] = \frac{e^2}{16\pi^2} \sum_{x,y} x_\sigma y_\lambda \frac{\delta}{\delta A_\nu(0)} \frac{\delta}{\delta A_\mu(y)} \left[ \epsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta}(x) F_{\gamma\delta}(x + \hat{\alpha} + \hat{\beta}) \right] \bigg|_{A=0} \quad (48) \]

\[ = \frac{e^2}{2\pi^2} \epsilon_{\mu\nu\lambda\sigma} \quad (49) \]

where Eq. (10) has been used. We note that it is not necessary to take the limit \( A_\mu = 0 \) after the differentiations since \( \rho \) is quadratic in gauge fields. The proof of Eq. (49) is given in appendix A. We note in passing that

\[ \mathcal{H}'[\epsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta}(x) F_{\gamma\delta}(x + \hat{\alpha} + \hat{\beta})] = \mathcal{H}'[\epsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta}(x) F_{\gamma\delta}(x)] = 8\epsilon_{\mu\nu\lambda\sigma} \]

where the operator \( \mathcal{H}' \) is similar to \( \mathcal{H} \) defined in (47) but without imposing \( A_\mu = 0 \) after differentiations with respect to the gauge fields. However we exclude the case of \( \rho \) having both field tensors located at \( x \) since it does not satisfy Eq. (46).

On the other hand, if we have \( \mathcal{H} \) act on (44) and obtain

\[ \mathcal{H} \left( \text{tr} \{ \gamma_5 [(S + T)D]_{xx} \} \right) = \frac{e^2}{2\pi^2} \epsilon_{\mu\nu\lambda\sigma} \quad (50) \]

then we can infer that

\[ \text{tr} \{ \gamma_5 [(S + T)D]_{xx} \} = \frac{e^2}{16\pi^2} \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x + \hat{\mu} + \hat{\nu}) + \text{other terms} \quad (51) \]

where "other terms" denotes those terms which cannot be determined by the second order perturbation calculation using \( \mathcal{H} \), which in general consists of higher order terms in \( A_\mu \) and/or derivatives, plus terms due to non-perturbative and/or topolgical effects (if any). The other terms in (51) in general cannot be computed directly by any perturbation calculation to a finite order, however, their sum over the entire lattice can be determined and might have significant impacts to the index theorem on the lattice, as we will show later.

Although Eq. (51) refers to the infinite lattice, however, we expect that it also holds for finite lattices. The argument [7] is as follows. If \( D \) is local, the boundary effects enter as \( \sim \exp(-m(R)L/a) \), where \( L \) is the lattice size, \( a \) is the lattice spacing, \( m(R) \) is a monotonic increasing function of \( R = rI \) ( take the simplest case ) with \( m(0) \) equal to zero ( this is equivalent to that \( D_c \) is nonlocal ) and \( m(\infty) \) a positive constant. As \( L \to \infty \), the finite size effects vanish and the axial anomaly is given by Eq. (51). For \( L \) is finite, one might naively expect that the \( L \) dependence would enter the \( F\tilde{F} \) term and "other terms", through the variable \( m(R)L/a \). However, the \( F\tilde{F} \) term cannot have \( R \) dependence, otherwise it would be in contrary to the fact that the index
of $D$ is $R$-invariant \cite{7}. Consequently, if $L$ is gradually decreased from infinity toward a finite value, all dependence of $L$ and $R$ only resides in "other terms". So, Eq. (51) also holds for finite lattices provided that $D$ is local.

After summing Eq. (51) over all sites of the lattice, we have

$$
2(N_- - N_+) = \sum_x \text{tr} \left\{ \gamma_5 [(S + T)D]_{x,x} \right\}
= \sum_x \frac{e^2}{16\pi^2} \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x + \hat{\mu} + \hat{\nu}) + \sum_x (\text{other terms})
$$

(52)

Now we consider the topologically nontrivial background $U(1)$ gauge field on a 4-dimensional torus ($x_\mu \in [0, L_\mu], \mu = 1, \cdots, 4$),

$$
e A_1(x) = \frac{2\pi h_1}{L_1} - \frac{2\pi q_1 x_2}{L_1 L_2} + A_1^{(0)} \sin \left( \frac{2\pi n_2}{L_2} x_2 \right)
$$

(53)

$$
e A_2(x) = \frac{2\pi h_2}{L_2} + A_2^{(0)} \sin \left( \frac{2\pi n_1}{L_1} x_1 \right)
$$

(54)

$$
e A_3(x) = \frac{2\pi h_3}{L_3} - \frac{2\pi q_2 x_4}{L_3 L_4} + A_3^{(0)} \sin \left( \frac{2\pi n_4}{L_4} x_4 \right)
$$

(55)

$$
e A_4(x) = \frac{2\pi h_4}{L_4} + A_4^{(0)} \sin \left( \frac{2\pi n_3}{L_3} x_3 \right)
$$

(56)

where $q_1, q_2, n_1, \cdots, n_4$ are integers. The global part is characterized by the topological charge

$$
Q = \frac{e^2}{32\pi^2} \int d^4x \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x) = q_1 q_2
$$

(57)

which must be an integer. The harmonic parts are parameterized by four real constants $h_1, h_2, h_3$ and $h_4$. The local parts are chosen to be sinusoidal fluctuations with amplitudes $A_1^{(0)}, A_2^{(0)}, A_3^{(0)}$ and $A_4^{(0)}$, and frequencies $\frac{2\pi n_2}{L_2}, \frac{2\pi n_1}{L_1}, \frac{2\pi n_3}{L_3}$ and $\frac{2\pi n_4}{L_4}$ respectively. The discontinuity of $A_1(x)$ at $x_2 = L_2$ due to the global part only amounts to a gauge transformation. The field tensors $F_{12}$ and $F_{34}$ are continuous on the torus, while other $F'$s are zero. To transcribe the continuum gauge fields to link variables on a finite lattice with periodic boundary conditions, we take the lattice sites at $x_\mu = 0, a, \cdots, (N_\mu - 1)a$, where $a$ is the lattice spacing and $L_\mu = N_\mu a$ is the lattice size. Then the link variables are

$$
U_1(x) = \exp \left[ i e A_1(x)a \right]
$$

(58)

$$
U_2(x) = \exp \left[ i e A_2(x)a + i \delta_{x_2, (N_2 - 1)a} \frac{2\pi q_1 x_1}{L_1} \right]
$$

(59)
\begin{align}
U_3(x) &= \exp \left[ ieA_3(x)a \right] \\
U_4(x) &= \exp \left[ ieA_4(x)a + i\delta_{x_4,(N_4-1)a} \frac{2\pi q_2 x_3}{L_3} \right]
\end{align}

The last term in the exponent of \( U_2(x) \) ( \( U_4(x) \) ) is included to ensure that the field tensor \( F_{12} \) ( \( F_{34} \) ) which is defined by the ordered product of link variables around a plaquette [ Eq. (A.8) ] is continuous on the torus. Then the topological charge of this gauge configuration on the finite lattice is

\[
Q = \frac{e^2}{32\pi^2} \sum_x \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x + \hat{\mu} + \hat{\nu}) = q_1 q_2
\]

which agrees with the topological charge [ Eq. (57) ] on the 4-dimensional torus. From Eq. (52), we obtain

\[
\sum_x (\text{other terms}) = 2(N_- - N_+ - q_1 q_2)
\]

which is also an integer. Since \( D \) does not have any genuine zeromodes in the topologically trivial gauge background, thus the integer (63) must be proportional to \( q_1 q_2 \) and can be represented as

\[
\sum_x (\text{other terms}) = (c[D] - 1) \sum_x \frac{e^2}{16\pi^2} \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x + \hat{\mu} + \hat{\nu})
\]

where \( c[D] \) is an integer functional of \( D \). Then Eq. (52) becomes

\[
N_- - N_+ = \frac{1}{2} \sum_x \text{tr} \{ \gamma_5[(S + T)D]_{xx} \}
= c[D] \sum_x \frac{e^2}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x + \hat{\mu} + \hat{\nu})
= c[D] Q
\]

where \( c[D] = c[D_c] \) due to the invariance of the zero modes and the index under the chiral symmetry breaking transformation (5) [7]. In general, Eq. (64) holds for any smooth background gauge configurations on a finite (infinite) lattice provided that the topological charge on the lattice (62) is an integer and the axial anomaly on the lattice satisfies Eq. (50). It is remarkable that, if any of these higher order, non-perturbative and topological contributions to the axial anomaly does exist, then their total effects to the index can only enter as an integer (\( c[D] - 1 \)) multiple of the topological charge of the background gauge...
field. We emphasize that the emergence of $c[D]$ is not due to the finite size of
the lattice but an intrinsic characteristics of $D$, in particular, when $c[D] = 0,$
then the axial anomaly vanishes at each site of the lattice, independent of the
size of the lattice. In general, the topological characteristics, $c[D]$, is an integer
functional of $D$, which in turn depends on some parameters of $D$ as well as the
gauge configuration. $c[D]$ could become chaotic when the background gauge
field is rough, as first demonstrated in ref. [8]. For smooth gauge configurations,
$D$ can be classified according to its topological characteristics as follows [7].
If $c[D] = 1$, then $D$ is called topologically proper; else if $c[D] = 0$, then $D$
is called topologically trivial; else $c[D] = \text{integer} \neq 0,1$, then $D$ is called
topologically improper. Only for $D$ is topologically proper, i.e., $c[D] = 1$, Eq.
(64) can realize the Atiyah-Singer index theorem on a finite ( infinite ) lattice.
[ Note the invariance of the topological charge in Eqs. (62) and (57) ]. Eq.
(64) provides the theoretical understanding of the numerical results [9,8] for
exact zero modes satisfying the Atiyah-Singer index theorem even on very
small lattices in two dimensions as well as in four dimensions.

We note that in ref. [8], using the exact reflection symmetry and the exact
solution of the free fermion propagator, the ( lowest order ) perturbation
theory is shown to break down at the topological phase boundaries in the $m_0$
parameter space of the overlap-Dirac operator. The zero index of $D$ at the
phase boundaries can be interpreted as $c[D] = 0$, due to non-perturbative
and/or topological contributions. In general, $c[D]$ incorporates all kinds of
contributions from all fermionic modes.

We also note that in the operator $\mathcal{H}$, the gauge fields are set to zero after the
differentiations with respect to the gauge fields. This implies that only free
fermion propagators are needed in the evaluation of the LHS of Eq. (50). In
the next section, we will show that Eq. (50) is indeed satisfied by the Ginsparg-
Wilson Dirac operator (5) with $R = r \mathbb{1}$, provided that $D_c$ in the free fermion
limit is free of species doubling and behaves like $i\gamma_\mu p_\mu$ as $p \to 0$.

Recently Lüscher [15] proved that for $U(1)$ lattice gauge theory, and for $S+T =
2$, if the axial anomaly $q(x; A, D) = \text{tr}[\gamma_5 D_{x,x}]$ satisfies

$$\sum_x \delta_x q(x; A, D) = 0 \quad (65)$$

for any local deformations of the gauge field, then

$$q(x; A, D) = \gamma \epsilon_{\mu \nu \lambda \sigma} F_{\mu \nu}(x) F_{\lambda \sigma}(x + \hat{\mu} + \hat{\nu}) + \partial_{\mu} G_{\mu}(x; A, D) \quad (66)$$

where $\gamma$ is a constant times an integer factor, and
\[ \partial_{\mu} G_{\mu}(x; A, D) = \sum_{\mu} [G_{\mu}(x; A, D) - G_{\mu}(x - \hat{\mu}; A, D)] . \] (67)

The explicit form of the current \( G_{\mu}(x; A, D) \) is supposed to be very complicated. As discussed in ref. [7], Eq. (66) can be generalized to any \( D \) satisfying (11) with all the \( S \) and \( T \) dependences residing in the term \( \partial_{\mu} G_{\mu}(x; A, D) \), while the \( F\tilde{F} \) term depends on the topological characteristics \( c[D] \) which is \((S,T)-\)invariant ( \( c[D] = c[D_c] \)), i.e.,

\[ q(x; A, D) = \frac{1}{2} \text{tr}\{\gamma_5[(S + T)D]_{x,x}\} = \gamma' c[D_c] \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x + \hat{\mu} + \hat{\nu}) + \partial_{\mu} G_{\mu}(x; A, D) \] (68)

\[ = \gamma' c[D_c] \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x + \hat{\mu} + \hat{\nu}) + \partial_{\mu} G_{\mu}(x; A, D) \] (69)

where the current \( G_{\mu}(x; A, D) \) in general is a functional of \( S \) and \( T \). Although Eq. (66) is proved for the infinite lattice in ref. [15], however, as discussed in ref. [7], if \( D \) is local, then the boundary effects which depend on \((S,T)\) and the lattice size \( L \), can only enter the term \( \partial_{\mu} G_{\mu}(x; A, D) \), thus Eqs. (66) and (69) are also true for finite lattices. The same argument has been presented in details to assert that Eq. (51) also holds for finite lattices. [See the second paragraph after Eq. (51).]

Now applying the operator \( \mathcal{H} \) to Eq. (69) and using Eq. (49), we obtain

\[ \mathcal{H}[q(x; A, D)] = \gamma' \mathcal{H}[c[D_c] \epsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta}(x) F_{\gamma\delta}(x + \hat{\alpha} + \hat{\beta})] + \mathcal{H}[\partial_{\alpha} G_{\alpha}(x; A, D)] = 8\gamma' \epsilon_{\mu\nu\lambda\sigma} + \mathcal{H}[\partial_{\alpha} G_{\alpha}(x; A, D)] \] (70)

where \( c[D] \) is presumably non-perturbative and thus cannot be determined by the second order operator \( \mathcal{H} \) acting on \( q(x; A, D) \). Hence \( D \) can only be assumed to be topologically proper and \( c[D] \) is set to unity throughout the entire perturbation calculation. Then the assertion of Eq. (50) implies that the LHS of Eq. (70) is \((S,T)-\)invariant. On the other hand, on the RHS of (70), the first term is \((S,T)-\)invariant, but the second term in general depends on \( S \) and \( T \). This implies that \( \mathcal{H}[\partial_{\alpha} G_{\alpha}(x; A, D)] \) must vanish identically, and Eq. (70) gives

\[ \gamma' = \frac{e^2}{32\pi^2} \] (71)

and (69) becomes

\[ q(x; A, D) = \frac{e^2}{32\pi^2} c[D_c] \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x + \hat{\mu} + \hat{\nu}) + \partial_{\mu} G_{\mu}(x; A, D) \] (72)
For the $U(1)$ gauge fields defined in Eqs. (58)-(61), summing Eq. (72) over all lattice sites yields

$$N_- - N_+ = c[D] Q$$

in agreement with Eq. (64).

It is interesting to note that the integer factor in the $\gamma$ of Eq. (66) turns out can be used to accomodate non-perturbative and/or topological effects, and it is identified to be the topological characteristics, $c[D]$ which is first introduced in ref. [7,8] in the study of the index of $D$ with repect to the background gauge fields. It also provides plausible explanations to some seemingly paradoxical situations that one obtains the correct axial anomaly in the perturbation calculation for a given $D$ but the numerical (nonperturbative) computations give exactly zero axial anomaly at each site, for any lattice sizes and for any gauge configurations.

3 The Axial Anomaly

In this section, we assert Eq. (50) for $S + T = 2r I$ by evaluating

$$I_2 \equiv \mathcal{H}[2 r \text{ tr}(\gamma_5 D_{x,x})]$$

$$= -2 r \sum_{m,\mu} n_{\sigma} m_{\lambda} \left\{ \frac{\delta}{\delta A_\nu(0)} \frac{\delta}{\delta A_\mu(m)} \langle (\bar{\psi} D \gamma_5 D)_{\nu} \psi \rangle \right\}_{A=0}$$

(73)

where $D$ satisfies the Ginsparg-Wilson relation (1) with $R = r I$, i.e.,

$$D = D_c (I + r D_c)^{-1}$$

(74)

for any chirally symmetric Dirac operator $D_c$ which in the free fermion limit, is free of species doubling and behaves like $i \gamma_\mu p_\mu$ as $p \to 0$.

In general, we can write $D_c$ in the momentum space as

$$D_c(p) = i \gamma_\mu C_\mu(p)$$

(75)

where $C_\mu(p)$ are arbitrary functions which satisfy the following properties in the free fermion limit:

(i) $C_\mu(p) \to p_\mu$ as $p \to 0$.

(ii) $C_\mu(p)$ has no zeros in the entire Brillouin zone except at the origin $p = 0$.

Then using Eqs. (74) and (75), we obtain
\[
D = \frac{i\mathcal{C} + rC^2}{1 + r^2C^2} \tag{76}
\]
\[
D^{-1} = -\frac{i\mathcal{C}'}{C^2} + r \tag{77}
\]

First we perform the differentiations with respect to the gauge field \( s \). The general formula is

\[
\frac{\delta}{\delta A_\nu(0)} \frac{\delta}{\delta A_\mu(m)} \langle T(U) \rangle \\
\equiv \frac{\delta}{\delta A_\nu(0)} \frac{\delta}{\delta A_\mu(m)} \left\{ \frac{1}{Z(U)} \int \bar{\psi} \psi T(U) e^{-\bar{\psi}D(U)\psi} \right\} \equiv \sum_{i=1}^{12} P_i \tag{78}
\]

where \( U \) denotes link variables and \( T(U) \) is an arbitrary functional of link variables and fermion fields, and

\[
P_1 = \left\langle T(U) \left( \bar{\psi} \frac{\delta}{\delta A_\mu(m)} D(U) \psi \right) \left( \bar{\psi} \frac{\delta}{\delta A_\nu(0)} D(U) \psi \right) \right\rangle \tag{79}
\]
\[
P_2 = - \left\langle T(U) \left( \bar{\psi} \frac{\delta}{\delta A_\nu(0)} \frac{\delta}{\delta A_\mu(m)} D(U) \psi \right) \right\rangle \tag{80}
\]
\[
P_3 = - \left\langle \left( \frac{\delta}{\delta A_\nu(0)} T(U) \right) \left( \bar{\psi} \frac{\delta}{\delta A_\mu(m)} D(U) \psi \right) \right\rangle \tag{81}
\]
\[
P_4 = - \left\langle \left( \frac{\delta}{\delta A_\mu(m)} T(U) \right) \left( \bar{\psi} \frac{\delta}{\delta A_\nu(0)} D(U) \psi \right) \right\rangle \tag{82}
\]
\[
P_5 = \left\langle \left( \frac{\delta}{\delta A_\mu(m)} \frac{\delta}{\delta A_\nu(0)} T(U) \right) \right\rangle \tag{83}
\]
\[
P_6 = - \left\langle T(U) \left( \bar{\psi} \frac{\delta}{\delta A_\mu(m)} D(U) \psi \right) \right\rangle \left\langle \left( \bar{\psi} \frac{\delta}{\delta A_\nu(0)} D(U) \psi \right) \right\rangle \tag{84}
\]
\[
P_7 = \left\langle \left( \frac{\delta}{\delta A_\mu(m)} T(U) \right) \right\rangle \left\langle \left( \bar{\psi} \frac{\delta}{\delta A_\nu(0)} D(U) \psi \right) \right\rangle \tag{85}
\]
\[
P_8 = - \left\langle T(U) \left( \bar{\psi} \frac{\delta}{\delta A_\nu(0)} D(U) \psi \right) \right\rangle \left\langle \left( \bar{\psi} \frac{\delta}{\delta A_\mu(m)} D(U) \psi \right) \right\rangle \tag{86}
\]
\[
P_9 = \left\langle \left( \frac{\delta}{\delta A_\nu(0)} T(U) \right) \right\rangle \left\langle \left( \bar{\psi} \frac{\delta}{\delta A_\mu(m)} D(U) \psi \right) \right\rangle \tag{87}
\]
\[
P_{10} = - \left\langle T(U) \right\rangle \left\langle \left( \bar{\psi} \frac{\delta}{\delta A_\nu(0)} D(U) \psi \right) \right\rangle \left\langle \left( \bar{\psi} \frac{\delta}{\delta A_\mu(m)} D(U) \psi \right) \right\rangle \tag{88}
\]
\[
P_{11} = \left\langle T(U) \right\rangle \left\langle \left( \bar{\psi} \frac{\delta}{\delta A_\nu(0)} \frac{\delta}{\delta A_\mu(m)} D(U) \psi \right) \right\rangle \tag{89}
\]
To evaluate $I_2$ in (73), we set

$$T(U) = (\bar{\psi} DR_5 D)n \psi_n = r(\bar{\psi} D_5 D)n \psi_n$$

Then $\langle T(U) \rangle = -r \text{tr}[\gamma_5 D_{n,n}]$ which vanishes in the free fermion limit. So, $P_{10}$, $P_{11}$ and $P_{12}$ do not contribute to $I_2$. Next we express the vector current in terms of the kernel $K_\mu$ as

$$J_\mu(k, U) = \sum_{i,j} \bar{\psi}_{k+i} K_\mu(k, i, j; U) \psi_{k+j}.$$  

However, the vector current $J_\mu$ satisfying the divergence condition Eq. (36) is not unique. A general and explicit realization is given by Ginsparg and Wilson [1] with $K_\mu(k, i, j; U)$ equal to $D_{k+i,k+j}(U)$ times the sign of $(i-j)_\mu$ and times the fraction of the shortest length paths from $k+j$ to $k+i$ which pass through the link from $k$ to $k+\hat{\mu}$. The Ginsparg-Wilson kernel can be expressed in terms of the derivative of the Dirac operator with respect to the gauge field,

$$J_\mu(k, U) = \sum_{i,j} \bar{\psi}_{k+i} K_\mu(k, i, j; U) \psi_{k+j} = i \sum_{m,n} \bar{\psi}_m \frac{\delta}{\delta A_\mu(k)} D_{mn}(U) \psi_n$$

where the second equality is proved in appendix C. In the free fermion limit, the action is translational invariant, $D_{mn} = D_{m-n}$, then Eq. (92) becomes

$$J_\mu(k) = \sum_{i,j} \bar{\psi}_{k+i} K_\mu(i, i-j) \psi_{k+i-j} = i \sum_{m,n} \bar{\psi}_m \frac{\delta}{\delta A_\mu(k)} D_{mn}(U) \psi_n \bigg|_{A=0}$$

with

$$K_\mu(i, i-j) = \text{sign}(j_\mu) f_\mu(i, j) D_j$$

where $f_\mu(i, j)$ is equal to the fraction of the shortest length paths from 0 to $j$ which pass through the link from $i - \hat{\mu}$ to $i$.

We note that Hasenfratz [16] also showed that

$$J_\mu(k, U) = i \sum_{m,n} \bar{\psi}_m \frac{\delta}{\delta A_\mu(k)} D_{mn}(U) \psi_n$$

satisfies the divergence condition, Eq. (36). However, an explicit realization of the kernel has not been prescribed. Furthermore, in the free fermion limit, the
kernel is shown [16] to satisfy the so called sum rules which are equivalent to Eqs. (B.8) and (B.9) proved in appendix B as well as in ref. [1].

According to the properties of \( K_k(i,j;\mu) \) discussed above and the following simple identity of the derivative of a link variable with respect to the gauge field,

\[
\frac{\delta}{\delta A_\nu(n)} U_\mu(k) = i\delta_{\mu \nu} \delta_{kn} U_\mu(k)
\]  

(95)

we obtain

\[
\frac{\delta}{\delta A_\nu(n)} K_k(i,j;\mu) \bigg|_{A=0} = c_{\mu \nu}(n) K_k(i,i-j)
\]  

(96)

where \( c_{\mu \nu}(n) \) are some constants independent of the gauge fields. Some useful properties of \( K_k(i,i-j) \) are derived in Appendix B and will be used in our evaluation of \( I_2 \).

Using Eqs. (91), (92), (93), and (96), the expressions of \( P_i's \) in Eqs. (79)-(87) become

\[
P_1 = -\langle (\bar{\psi} D \gamma_5 D)_{\nu} \psi_{\nu} J_{\mu}(m) J_\mu(0) \rangle
\]  

(97)

\[
P_2 = ic_{\mu \nu}(0) \langle (\bar{\psi} D \gamma_5 D)_{\nu} \psi_{\mu} J_\mu(m) \rangle
\]  

(98)

\[
P_3 = i \langle \left( \frac{\delta}{\delta A_\mu(0)} (\bar{\psi} D \gamma_5 D)_{\nu} \psi_{\nu} \bigg|_{A=0} \right) J_\mu(m) \rangle
\]  

(99)

\[
P_4 = i \langle \left( \frac{\delta}{\delta A_\mu(m)} (\bar{\psi} D \gamma_5 D)_{\nu} \psi_{\nu} \bigg|_{A=0} \right) J_\nu(0) \rangle
\]  

(100)

\[
P_5 = \langle \left( \frac{\delta}{\delta A_\mu(0)} \frac{\delta}{\delta A_\mu(m)} (\bar{\psi} D \gamma_5 D)_{\nu} \psi_{\nu} \bigg|_{A=0} \right) \rangle
\]  

(101)

\[
P_6 = \langle (\bar{\psi} D \gamma_5 D)_{\nu} \psi_{\nu} J_{\mu}(m) \rangle \langle J_\nu(0) \rangle
\]  

(102)

\[
P_7 = -i \langle \left( \frac{\delta}{\delta A_\mu(m)} (\bar{\psi} D \gamma_5 D)_{\nu} \psi_{\nu} \bigg|_{A=0} \right) \rangle \langle J_\nu(0) \rangle
\]  

(103)

\[
P_8 = \langle (\bar{\psi} D \gamma_5 D)_{\nu} \psi_{\nu} J_{\mu}(m) \rangle
\]  

(104)

\[
P_9 = -i \langle \left( \frac{\delta}{\delta A_\mu(0)} (\bar{\psi} D \gamma_5 D)_{\nu} \psi_{\nu} \bigg|_{A=0} \right) \rangle \langle J_\mu(m) \rangle
\]  

(105)

where the free fermion limit has been imposed for the RHS of Eq. (73), i.e.
\[ I_2 = -2 \sum_{i=1}^{9} \sum_{m,n} n_{\sigma} m_{\lambda} P_i \]  

(106)

In above expressions, only \( P_1 \) has non-vanishing contributions to \( I_2 \). This can be shown in the following. First consider the common factor \( \langle J_{\nu}(0) \rangle \) of \( P_6 \) and \( P_7 \). Using Eqs. (93) and (94), we obtain

\[ \langle J_{\nu}(0) \rangle = -\sum_{i,j} \text{tr} [D_{-j}^{-1} K_{\nu}(i, i - j)] \]

= \[ \sum_{i,j} \text{tr} [D_{-j}^{-1} \text{sign}(j_{\nu}) f(i, -j) D_{-j}] \]

= \[ -\text{tr}(\mathbb{1}) \sum_{i,j} \text{sign}(j_{\nu}) f(i, j) = 0 \]  

(107)

since \( \sum_{i} f(i, j) = \sum_{i} f(i, -j) \). Similarly, we can show that \( \langle J_{\mu}(m) \rangle = 0 \) since it is translational invariant. Then \( P_6, P_7, P_8 \) and \( P_9 \) in Eqs. (102)-(105) are zero.

For \( P_2 \), the factor \( \langle \bar{\psi} D R_{\gamma_5} D_n \psi_n J_{\mu}(m) \rangle \) enters (106) to give

\[ \sum_{m,n} n_{\sigma} m_{\lambda} \langle \bar{\psi} D R_{\gamma_5} D_n \psi_n J_{\mu}(m) \rangle \]

= \[ \sum_{m,n,i,j} n_{\sigma} m_{\lambda} \langle \bar{\psi} D R_{\gamma_5} D_n \psi_n \bar{\psi}_{m+i} K_{\mu}(i, i - j) \psi_{m+i-j} \rangle \]

= \[ -r \sum_{m,n,i,j} n_{\sigma} m_{\lambda} \{ \text{tr} [\gamma_5 D_{m+i-j-n} D^{-1}_{m-i} K_{\mu}(i, i - j)] \}

= 0 \]

where the identity

\[ \text{tr}(\gamma_5) = \text{tr}(\gamma_5 \gamma_{\mu}) = \text{tr}(\gamma_5 \gamma_{\mu} \gamma_{\nu} \gamma_{\sigma}) = 0 \]  

(108)

has been used, and the fact that each factor of \( D, D^{-1} \) or \( K_{\mu} \) can give at most one gamma matrix in the free fermion limit [ Eqs. (76)-(77) and (94) ].

For \( P_5 \), it enters (106) to give

\[ \sum_{m,n} n_{\sigma} m_{\lambda} \left\langle \frac{\delta}{\delta A_{\nu}(0)} \frac{\delta}{\delta A_{\mu}(m)} (\bar{\psi} D R_{\gamma_5} D)_{n} \psi_{n} \right|_{A=0} \right\rangle \]

= \[ \frac{1}{2} \sum_{m,n,i} n_{\sigma} m_{\lambda} \left\langle \frac{\delta}{\delta A_{\nu}(0)} \frac{\delta}{\delta A_{\mu}(m)} \bar{\psi}_i (D_{i,n} \gamma_5 + \gamma_5 D_{i,n}) \psi_{n} \right|_{A=0} \right\rangle \]  

(109)
where Eq. (1) has been used. Using Eq. (95), we immediately see that this expression only involves trace operations on terms containing one $\gamma_5$ and less than four $\gamma_\mu'$s matrices, thus it must be zero.

For $P_3$, it enters (106) to give

$$
\sum_{m,n} n_\sigma m_\lambda \left\langle \left( \frac{\delta}{\delta A_\nu(0)} (\bar{\psi} DR_5 D)_n \psi_n \right) \bigg| A_{=0} \right. \left. \right) J_\mu(m) \right\rangle
$$

$$
= \frac{1}{2} \sum_{m,n,k} n_\sigma m_\lambda \left\langle \left( \frac{\delta}{\delta A_\nu(0)} \bar{\psi}_k [D_{k,n} \gamma_5 + \gamma_5 D_{k,n}] \psi_n \right) \bigg| A_{=0} \right. \left. \right) J_\mu(m) \right\rangle
$$

$$
= -\frac{i}{2} \sum_{m,n,k,i,j} n_\sigma m_\lambda \left\langle \bar{\psi}_k [K_\nu(k, k - n) \gamma_5 + \gamma_5 K_\nu(k, k - n)] \psi_{k-n} \right. \\
\left. \bar{\psi}_{m+i} K_\mu(i, i - j) \psi_{m+i-j} \right\rangle
$$

$$
= \frac{i}{2} \sum_{m,n,k,i,j} n_\sigma m_\lambda \left\{ \\
\text{tr} \left( D^{-1}_{m+i-j-k}[K_\nu(k, k - n) \gamma_5 + \gamma_5 K_\nu(k, k - n)] D^{-1}_{k-n-m-i} K_\mu(i, i - j) \right) \right\}
$$

$$
-\text{tr} \left( D^{-1}_{k-n}[K_\nu(k, k - n) \gamma_5 + \gamma_5 K_\nu(k, k - n)] \right) \text{tr} \left( D^{-1}_{m+i-j} K_\mu(i, i - j) \right) \right\}
$$

$$
= 0
$$

where Eqs. (1) and (93) have been used in the first and the second equalities. From Eq. (94), we see that $[K_\nu(k, k - n) \gamma_5 + \gamma_5 K_\nu(k, k - n)]$ is proportional to $\gamma_5$, then using the trace identity Eq. (108) to give zero in the above expression.

Similarly, we show that $P_4$ also enters (106) to give zero.

Finally only $P_1$ remains in Eq. (106), i.e.,

$$
I_2 = -2 \sum_{m,n} n_\sigma m_\lambda P_1
$$

$$
= 2 \sum_{m,n} n_\sigma m_\lambda \langle (\bar{\psi} DR_5 D)_n \psi_n J_\mu(m) J_\nu(0) \rangle
$$

$$
= \sum_{m,n,k,i,s,t} n_\sigma m_\lambda \langle \bar{\psi}_k (D_{k-n} \gamma_5 + \gamma_5 D_{k-n}) \psi_n \cdot \\
\bar{\psi}_{m+i} K_\mu(i, i - j) \psi_{m+i-j} \bar{\psi}_s K_\nu(s, s - t) \psi_{s-t} \rangle
$$

where Eqs. (1) and (93) have been used. After the fermion fields are contracted, Eq. (110) becomes

$$
I_2 = I_a + I_b + I_c + I_d
$$

where
\[ I_a = - \sum_{m,n,i,j,s} n_\sigma m_\lambda \text{tr}[\gamma_5 D^{-1}_{n-m-i} K_\mu(i,i-j) D^{-1}_{m+i-j-s} K_\nu(s,n)] \]  
(112)

\[ I_b = - \sum_{m,n,i,s,t} n_\sigma m_\lambda \text{tr}[\gamma_5 D^{-1}_{n-s} K_\nu(s,s-t) D^{-1}_{s-t-m-i} K_\mu(i,n-m)] \]  
(113)

\[ I_c = - \sum_{m,n,k,i,j,s,t} n_\sigma m_\lambda \text{tr}[\gamma_5 D_{k-n}^{-1} D^{-1}_{n-m-i} K_\mu(i,i-j)] \]  
\[ D^{-1}_{m+i-j-s} K_\nu(s,s-t) D^{-1}_{s-t-k} \]  
(114)

\[ I_d = - \sum_{m,n,k,i,j,s,t} n_\sigma m_\lambda \text{tr}[\gamma_5 D_{k-n}^{-1} D^{-1}_{n-s} K_\nu(s,s-t)] \]  
\[ D^{-1}_{s-t-m-i} K_\mu(i,i-j) D^{-1}_{m+i-j-k} \]  
(115)

where those terms involving trace operation on products of one \( \gamma_5 \) and less than four \( \gamma_\mu \)s matrices are zero and have been dropped. Using the Fourier transforms

\[ D^{\pm 1}_n = \int_{-\pi}^\pi \frac{d^4 p}{(2\pi)^4} e^{ipn} D^{\pm 1}(p) \]  
(116)

\[ K_\mu(m,m-l) = \int_{-\pi}^\pi \frac{d^4 p}{(2\pi)^4} \int_{-\pi}^\pi \frac{d^4 p'}{(2\pi)^4} e^{i(p+p')n} e^{-i(p+p')(m-l)} K_\mu(p,p+p') \]  
(117)

and the identity

\[ \int_{-\pi}^\pi \frac{d^4 p}{(2\pi)^4} \sum_n n_\sigma e^{ipn} f(p) = \int_{-\pi}^\pi \frac{d^4 p}{(2\pi)^4} \delta^4(p) i \frac{\partial}{\partial p_\sigma} f(p) \]  
(118)

we obtain

\[ I_a = - \sum_{m,n,l,j,s} n_\sigma m_\lambda \int_{\{p_n\}} e^{i p_1 (n-m-l)} e^{i p_2 l - i p_3 j} e^{i p_4 (m+j-s)} e^{i p_5 s - i p_6 n} \]  
\[ \text{tr}[\gamma_5 D^{-1}(p_1) K_\mu(p_2,p_3) D^{-1}(p_4) K_\nu(p_5,p_6)] \]  

\[ = - \sum_{m,n,l,j,s} n_\sigma m_\lambda \int_{p_1 p_3 p_6} e^{i p_1 (n-m)} e^{i p_3 m} e^{-i p_6 n} \]  
\[ \text{tr}[\gamma_5 D^{-1}(p_1) K_\mu(p_1,p_3) D^{-1}(p_3) K_\nu(p_3,p_6)] \]  

\[ = \int_{p_a p_b} \delta(p_a) \delta(p_b) \delta_{\lambda}^{(a)} \delta_{\sigma}^{(b)} \int_{\{p\}} \text{tr}[\gamma_5 D^{-1}(p+p_b) K_\mu(p+p_b,p+a+p_b)] \]  
\[ D^{-1}(p+p_b) K_\nu(p+a+p_b,p) \]  
(119)

The differentiation with respect to \( p_a \) and \( p_b \) will generate many terms in the last expression. However, due to the identity (108), only those terms containing the product \( \gamma_5 \gamma_1 \gamma_2 \gamma_3 \gamma_4 \) and its permutations can have nonzero con-
tributions, thus the final result is proportional to \( \text{tr}(\gamma_5 \gamma_\mu \gamma_\nu \gamma_\lambda \gamma_\sigma) = 4 \epsilon_{\mu \nu \lambda \sigma} \). Then it is obvious to see that those terms containing \( \partial^{(a)}_\lambda \partial^{(b)}_\sigma D^{-1}(p + p_a + p_b) \), or \( \partial^{(a)}_\lambda \partial^{(b)}_\sigma K_\nu(p + p_a + p_b, p) \) must vanish since they are symmetric in \( \lambda \) and \( \sigma \). Furthermore, the terms containing \( \partial^{(a)}_\lambda K_{\mu}, \partial^{(b)}_\sigma K_{\mu}, \partial^{(a)}_\lambda K_{\nu}, \partial^{(b)}_\sigma K_{\nu} \), or \( \partial^{(a)}_\lambda \partial^{(b)}_\sigma K_{\mu}(p + p_a, p + p_a + p_b) \) would become zero after integrations over \( p_a \) and \( p_b \) with the delta functions in (119), due to the following identities (proved in Appendix B)

\[
\partial'_\lambda K_\mu(p, p + p')|_{p' = 0} = \left( \frac{i}{2} \partial_\mu \partial_\lambda - \frac{1}{2} \delta_{\mu \lambda} \partial_\mu \right) D(p) \tag{120}
\]

\[
\partial'_\lambda K_\mu(p + p', p)|_{p' = 0} = \left( \frac{i}{2} \partial_\mu \partial_\lambda + \frac{1}{2} \delta_{\mu \lambda} \partial_\mu \right) D(p) \tag{121}
\]

which are symmetric in \( \mu \) and \( \lambda \). Hence, the only nonzero term in (119) is

\[
I_a = \int_p \text{tr} \left( \gamma_5 [\partial_\sigma D^{-1}(p)] K_\mu(p, p) [\partial_\lambda D^{-1}(p)] K_\nu(p, p) \right) = - \int_p \text{tr} \left( \gamma_5 [\partial_\sigma D^{-1}(p)][\partial_\mu D(p)][\partial_\lambda D^{-1}(p)][\partial_\nu D(p)] \right) \tag{122}
\]

where the identity (proved in Appendix B)

\[
K_\mu(p, p) = i \partial_\mu D(p) \tag{123}
\]

has been used. By performing the same analysis on \( I_b, I_c \) and \( I_d \), we obtain

\[
I_b = - \sum_{m, n, l, s, t} n_\sigma m_\lambda \int_{(p_\mu)} e^{ip_1(n-s)} e^{ip_2 s - ip_3 d} e^{ip_4(t-m-l)} e^{ip_5 l - ip_6(n-m)} \text{tr}[\gamma_5 D^{-1}(p_1) K_\nu(p_2, p_3) D^{-1}(p_4) K_\mu(p_5, p_6)]
\]

\[
= - \sum_{m, n} n_\sigma m_\lambda \int_{p_1 p_3 p_6} e^{ip_1 n} e^{-ip_3 m} e^{-ip_6(n-m)} \text{tr}[\gamma_5 D^{-1}(p_1) K_\nu(p_1, p_3) D^{-1}(p_3) K_\mu(p_3, p_6)]
\]

\[
= \int_{p, p_a} \delta(p_a) \delta(p_b) \partial^{(a)}_\lambda \partial^{(b)}_\sigma \int_p \text{tr}[\gamma_5 D^{-1}(p + p_a + p_b) K_\nu(p + p_a + p_b, p)]
\]

\[
D^{-1}(p) K_\mu(p, p + p_a)]
\]

\[
= 0 \tag{124}
\]

where the zero is essentially due to the presence of \( D^{-1}(p) \) which does not depend on \( p_a \) or \( p_b \).
\[I_c = - \sum_{m,n,k,l,j,s,t} n_{\sigma} m_{\lambda} \int_{\{p_n\}} e^{ip_1(k-n)} e^{ip_2(n-m-l)} e^{ip_3l-ip_4j} e^{ip_5(m+j-s)} e^{ip_6-s-ip_7t} e^{ip_8(t-k)} \text{tr} [\gamma_5 D(p_1) D^{-1}(p_2) K_\mu(p_3, p_4) D^{-1}(p_5) K_\nu(p_6, p_7) D^{-1}(p_8)] \]
\[= - \sum_{m,n} n_{\sigma} m_{\lambda} \int_{p_1,p_2,p_4} e^{-ip_1n} e^{ip_2(n-m)} e^{ip_4m} \text{tr} [\gamma_5 D(p_1) D^{-1}(p_2) K_\mu(p_2, p_4) D^{-1}(p_4) K_\nu(p_4, p_1) D^{-1}(p_1)] \]
\[= \int_{p_a,p_b} \delta(p_a) \delta(p_b) \delta^{(a)} \delta^{(b)} \int_{p} \text{tr} [\gamma_5 D(p) D^{-1}(p + p_b) K_\mu(p + p_a + p_b) D^{-1}(p + p_a + p_b) K_\nu(p + p_a + p_b, p) D^{-1}(p)] \]
\[= - \int_{p} \text{tr} \left( \gamma_5 [\partial_\sigma D(p)] [\partial_\mu D^{-1}(p)] [\partial_\lambda D(p)] [\partial_\nu D^{-1}(p)] \right) \quad (125) \]

where the identity
\[D \partial_\mu D^{-1} + (\partial_\mu D) D^{-1} = 0 \quad (126)\]

and Eqs. (120), (121) and (123) have been used.

\[I_d = - \sum_{m,n,k,l,j,s,t} n_{\sigma} m_{\lambda} \int_{\{p_n\}} e^{ip_1(k-n)} e^{ip_2(n-s)} e^{ip_3l-ip_4t} e^{ip_5(t-m-l)} e^{ip_6l-ip_7j} e^{ip_8(m+j-k)} \text{tr} [\gamma_5 D(p_1) D^{-1}(p_2) K_\nu(p_3, p_4) D^{-1}(p_5) K_\mu(p_6, p_7) D^{-1}(p_8)] \]
\[= - \sum_{m,n} m_{\lambda} n_{\sigma} \int_{p_1,p_2,p_4} e^{-ip_1n} e^{ip_2n} e^{-ip_4m} e^{ip_1m} \text{tr} [\gamma_5 D(p_1) D^{-1}(p_2) K_\nu(p_2, p_4) D^{-1}(p_4) K_\mu(p_4, p_1) D^{-1}(p_1)] \]
\[= \int_{p_a,p_b} \delta(p_a) \delta(p_b) \delta^{(a)} \delta^{(b)} \int_{p} \text{tr} [\gamma_5 D(p + p_a) D^{-1}(p + p_a + p_b)] [\partial_\nu D(p)] [\partial_\mu D^{-1}(p)] K_\mu(p + p_a + p_b) D^{-1}(p + p_a)] \]
\[= \int_{p} \text{tr} \left( \gamma_5 [\partial_\lambda D(p)] [\partial_\sigma D^{-1}(p)] [\partial_\nu D(p)] [\partial_\mu D^{-1}(p)] \right) - \gamma_5 [\partial_\sigma D(p)] [\partial_\nu D^{-1}(p)] [\partial_\mu D(p)] [\partial_\lambda D^{-1}(p)] \right) \]
\[= -2 \int_{p} \text{tr} \left( \gamma_5 [\partial_\mu D^{-1}(p)] [\partial_\nu D(p)] [\partial_\lambda D^{-1}(p)] [\partial_\sigma D(p)] \right) \quad (127) \]

Therefore, summing \(I_a, I_b, I_c\) and \(I_d\) in (122), (124), (125) and (127), and using Eqs. (76), (77) and (108), we obtain

\[I_2 = -4 \int_{p} \text{tr} \left( \gamma_5 [\partial_\mu D^{-1}(p)] [\partial_\nu D(p)] [\partial_\lambda D^{-1}(p)] [\partial_\sigma D(p)] \right) \]

24
\[-4 \int \frac{\partial_\mu}{p} \text{tr} \left[ \gamma_5 \frac{C}{C^2} \partial_\nu \left( \frac{C}{1 + r^2 C^2} \right) \partial_\lambda \left( \frac{C}{C^2} \right) \partial_\sigma \left( \frac{C}{1 + r^2 C^2} \right) \right] \tag{128}\]

\[-4 \int \frac{\partial_\mu}{p} \text{tr} \left[ \gamma_5 \frac{C}{C^2} \partial_\nu \left( \frac{C}{1 + r^2 C^2} \right) \partial_\lambda \left( \frac{C}{C^2} \right) \partial_\sigma \left( \frac{C}{1 + r^2 C^2} \right) \right] \tag{129}\]

where the \( \partial_\mu \) operation in (129) produces (128), plus three terms which are symmetric in \( \mu \nu, \mu \lambda, \) and \( \mu \sigma \), respectively, hence neither of these three terms contributes to \( I_2 \). Now we perform the momentum integral in (129) by, first removing an infinitesimal ball \( B_\epsilon \) with center at the origin \( p = 0 \) and radius \( \epsilon \) from the Brillouin zone, then evaluating the integral, and finally taking the limit \( \epsilon \) to zero, i.e.,

\[ I_2 = -\frac{4}{(2\pi)^4} \lim_{\epsilon \to 0} \int_{\epsilon \leq |p_\mu| \leq \pi} d^4 p \frac{\partial_\mu}{p} \left\{ \text{tr} \left[ \gamma_5 \frac{C}{C^2} \partial_\nu \left( \frac{C}{1 + r^2 C^2} \right) \partial_\lambda \left( \frac{C}{C^2} \right) \partial_\sigma \left( \frac{C}{1 + r^2 C^2} \right) \right] \right\} \tag{130}\]

Then according to the Gauss theorem, the volume integral over the Brillouin zone (a four dimensional torus due to the periodic boundary conditions) with the ball \( B_\epsilon \) removed can be expressed as a surface integral on the surface \( S_\epsilon \) of the ball \( B_\epsilon \), provided that \( C_\mu(p) \) is nonzero for \( \epsilon \leq |p_\mu| \leq \pi \) (i.e., free of species doubling) such that the integrand in (130) is well defined. So, (130) becomes

\[ I_2 = -\frac{1}{4\pi^4} \lim_{\epsilon \to 0} \int_{S_\epsilon} d^3 s \frac{n_\mu}{p_\mu} \text{tr} \left[ \gamma_5 \frac{C}{C^2} \partial_\nu \left( \frac{C}{1 + r^2 C^2} \right) \partial_\lambda \left( \frac{C}{C^2} \right) \partial_\sigma \left( \frac{C}{1 + r^2 C^2} \right) \right] \tag{131}\]

where \( n_\mu \) is the \( \mu \)-th component of the outward normal vector on the surface \( S_\epsilon \). Since we have assumed that \( C_\mu(p) \to p_\mu \) as \( p \to 0 \), we can set \( C_\mu(p) = p_\mu \) on the surface \( S_\epsilon \) and obtain

\[ I_2 = -\frac{1}{4\pi^4} \lim_{\epsilon \to 0} \int_{S_\epsilon} d^3 s \frac{n_\mu}{p_\mu} \text{tr} \left[ \gamma_5 \frac{p}{p^2} \left( \frac{\gamma_\nu}{1 + r^2 p^2} \right) \left( \frac{\gamma_\lambda}{p^2} \right) \left( \frac{\gamma_\sigma}{1 + r^2 p^2} \right) \right] \]

\[ = -\frac{1}{\pi^4} \lim_{\epsilon \to 0} \int_{S_\epsilon} d^3 s \frac{n_\mu}{p^4(1 + r^2 p^2)^2} \tag{132}\]
where we have used the property
\[
\int_{\mathcal{S}_c} d^3s \, n_\mu p_\nu f(p^2) = \delta_{\mu\nu} \int_{\mathcal{S}_c} d^3s \, n_\mu p_\mu f(p^2)
\]
(133)

Finally, we have
\[
I_2 = -\frac{\epsilon_{\mu\nu\lambda\sigma}}{\pi^4} \lim_{\epsilon \to 0} \frac{1}{(1 + r^2 \epsilon^2)^2} \int_0^{2\pi} d\phi \int_0^\pi d\theta_2 \sin \theta_2 \int_0^\pi d\theta_1 (-\sin^2 \theta_1 \cos^2 \theta_1)
\]
\[
= \frac{1}{2\pi^2} \epsilon_{\mu\nu\lambda\sigma}
\]
(134)

This completes the task of evaluating \( I_2 = \mathcal{H}[ 2 \, \text{tr}(\gamma_5 RD) ] \) for \( R = r \mathbb{1} \), where (134) is one of the main results of this paper. Although (134) has been derived for the infinite lattice (a very large lattice with periodic boundary conditions), it is reasonable to expect that it also holds for finite lattices with periodic boundary conditions and with even number of sites in each direction, since the integral in (130) is essentially a topological invariant quantity. Implications of Eq. (134) have been discussed in section 2.

4 Conclusions and Discussions

In this paper, we have evaluated the axial anomaly for the Ginsparg-Wilson fermion operator \( D = D_c (\mathbb{1} + r D_c)^{-1} \). For any chirally symmetric \( D_c \) which in the free fermion limit is free of species doubling and behaves like \( i\gamma_\mu p_\mu \) as \( p \to 0 \), the axial anomaly for \( U(1) \) lattice gauge theory with single fermion flavor is
\[
r \, \text{tr}[\gamma_5 D(x, x)] = \frac{e^2}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x + \hat{\mu} + \hat{\nu})
\]
\[
+ \text{higher order and/or non-perturbative terms}
\]
(135)

where the field tensor \( F_{\mu\nu}(x) \) on the lattice is defined in Eq. (10). The \( F \bar{F} \) term is \( r \)-invariant and has the correct continuum limit. As shown in section 2, the higher order and/or non-perturbative terms might have significant impacts to the index of \( D \). For smooth background gauge configurations [e.g., Eq. (58)-(61)] with integer topological charge (62), if the axial anomaly satisfying Eq. (50), then the sum of these higher order and/or non-perturbative terms over all sites of the lattice is shown to be an integer multiple of the \( F \bar{F} \) term and this leads to the emergence of an integer functional, \( c[D] \), which is called

26
topological characteristics of $D$ in ref. [7,8]. In general, $c[D]$ incorporates all kinds of contributions from all fermionic modes. Due to the $(S,T)$-invariance of the zero modes [7], we have [ Eq. (64) in section 2 ]

$$N_- - N_+ = c[D] \frac{e^2}{32\pi^2} \sum_x \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x + \hat{\mu} + \hat{\nu})$$  \hspace{1cm} (136)$$

where $c[D] = c[D_c]$ is invariant for any $S$ and $T$ in the general GW relation, Eq. (11). Then for topologically proper $D(D_c)$, i.e., $c[D] = 1$, the Atiyah-Singer index theorem can be realized on a finite (infinite) lattice for smooth background gauge fields. Now it is obvious that the $F\tilde{F}$ term in Eq. (135) must be also $(S,T)$-invariant, otherwise it would be contrary to Eq. (136). Hence, we have

$$\frac{1}{2} \text{tr}\{\gamma_5[(S + T)D]_{x,x}\} = \frac{e^2}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x + \hat{\mu} + \hat{\nu})$$

+ higher order and/or nonperturbative terms.

(137)

Then one can deduce the following result in the continuum limit,

$$\frac{1}{2} \text{tr}\{\gamma_5[(S + T)D]_{x,x}\} = c[D] \frac{e^2}{32\pi^2} \epsilon_{\mu\nu\lambda\sigma} F_{\mu\nu}(x) F_{\lambda\sigma}(x).$$  \hspace{1cm} (138)$$

Now the limit $(S,T \to 0)$ in Eq. (138) can be safely taken. Since the limit $S,T \to 0$ is the chiral limit where $D = D_c$ and the GW chiral symmetry breaking [ the RHS of Eq. (1) ] is completely turned off, we conclude that the GW relation indeed does not play the crucial role to fix the axial anomaly of $D$ in the continuum limit. This is in agreement with the conclusion of ref. [7]. The crucial point for $D$ to have the correct axial anomaly and to realize the Atiyah-Singer index theorem in the continuum limit is the existence of a topologically proper $D_c$ which also satisfies the properties mentioned above, or in general, the constraints (a)-(e) given in ref. [7]. Then any GW fermion operator $D$ constructed by the general solution $D = D_c(\mathbb{1} + RD_c)^{-1}$ will have the desired topological properties. The role of the chiral symmetry breaking transformation (5) is to bypass the Nielson-Ninomiya no-go theorem such that $D$ can be constructed to be local, free of species doubling and well defined for any gauge configurations, while the essential chiral physics of $D_c$ is preserved under this transformation. Therefore, for practical computations on a finite lattice ( with finite lattice spacings ), one must keep $(S,T)$ finite as well as using a topologically proper $D(D_c)$ such that the axial anomaly could agree with the Chern-Pontryagin density in continuum, though the index is equal to the topological charge for any $(S,T)$. 

27
In this appendix, we explicitly show that

\[
\mathcal{H}'[\epsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta}(x) F_{\gamma\delta}(x + \hat{\alpha} + \hat{\beta})] = \mathcal{H}'[\epsilon_{\alpha\beta\gamma\delta} F_{\alpha\beta}(x) F_{\gamma\delta}(x)] = 8\epsilon_{\mu\nu\lambda\sigma} \tag{A.1}
\]

where \(\mathcal{H}'\) is defined as

\[
\mathcal{H}'[O] = \sum_{x,y} x_\sigma y_\lambda \frac{\delta}{\delta A_\nu(0)} \frac{\delta}{\delta A_\mu(y)} [O(x)] \tag{A.2}
\]

which is similar to \(\mathcal{H}\) defined in Eq. (47) but without imposing the gauge fields to zero after the differentiations with respect to the gauge fields.

First, we derive the field tensor for the \(U(1)\) gauge theory on the lattice. A plaquette on the \(\hat{\mu} - \hat{\nu}\) plane is defined as

\[
P_{\mu\nu}(x) = U_\mu(x) U_\nu(x + \hat{\mu}) U_\mu^\dagger(x + \hat{\nu}) U_\nu^\dagger(x) = \exp\left\{ ie \left[ A_\nu(x + \hat{\mu}) - A_\nu(x) - A_\mu(x + \hat{\nu}) + A_\mu(x) \right] \right\} \tag{A.3}
\]

and its expansion up to \(e^2\) is

\[
1 + ie \left[ A_\nu(x + \hat{\mu}) - A_\nu(x) - A_\mu(x + \hat{\nu}) + A_\mu(x) \right] - \frac{1}{2} e^2 a^2 \left[ A_\nu(x + \hat{\mu}) - A_\nu(x) - A_\mu(x + \hat{\nu}) + A_\mu(x) \right]^2 \nonumber \\
\simeq 1 + ie a^2 \left[ \partial_\mu A_\nu - \partial_\nu A_\mu \right] - \frac{1}{2} e^2 a^4 \left[ \partial_\mu A_\nu - \partial_\nu A_\mu \right]^2 + O(e^3, a^6) \tag{A.4}
\]

Then the real part of the sum of all plaquettes, i.e.,

\[
\frac{1}{e^2} \sum_x \sum_{\nu < \mu} \text{Re}[1 - P_{\mu\nu}(x)], \tag{A.5}
\]

goes to

\[
\frac{1}{4} \sum_{x,\mu,\nu} a^4 \left[ \partial_\mu A_\nu - \partial_\nu A_\mu \right]^2 \tag{A.6}
\]

in the continuum limit, thus agrees with the action of QED. Hence, we can identify the field tensor on the lattice to be
Using Eqs. (A.2) and (A.7), we obtain

\[
F_{\mu\nu}(x) = \frac{1}{a} [A_\nu(x + \hat{\mu}) - A_\nu(x) - A_\mu(x + \hat{\nu}) + A_\mu(x)]
\]  
(A.7)

\[
= \frac{1}{iea^2} \log[P_{\mu\nu}(x)]
\]  
(A.8)

We note that on a finite lattice with periodic boundary conditions, and for background gauge fields with nonzero topological charge, some of the link variables at the boundary need modifications [Eqs. (59) and (61)] such that the field tensors are continuous on the torus.

Using Eqs. (A.2) and (A.7), we obtain

\[
\mathcal{H}'[\epsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta}(x)F_{\gamma\delta}(x)] = \sum_{x,y} \sum_{\alpha,\beta,\gamma,\delta} x_\alpha y_\beta \frac{\delta}{\delta A_\alpha(0)} \frac{\delta}{\delta A_\beta(y)} \{ \epsilon_{\alpha\beta\gamma\delta} \\
\left[ A_\beta(x - \hat{\alpha})A_\delta(y - \hat{\gamma}) - A_\alpha(x - \hat{\beta})A_\delta(y - \hat{\gamma}) \\
- A_\beta(x - \hat{\alpha})A_\gamma(y - \hat{\delta}) + A_\alpha(x - \hat{\beta})A_\gamma(y - \hat{\delta}) \right] \}
\]  
\[
= \sum_{x,y} \sum_{\alpha,\beta,\gamma,\delta} x_\alpha y_\beta \frac{\delta}{\delta A_\alpha(0)} \frac{\delta}{\delta A_\beta(y)} \{ \\
(\epsilon_{\sigma\nu\lambda\mu} - \epsilon_{\sigma\nu\lambda\mu} - \epsilon_{\sigma\nu\mu\lambda} + \epsilon_{\nu\sigma\mu\lambda})A_\nu(x - \hat{\sigma})A_\mu(x - \hat{\lambda}) \\
+ (\epsilon_{\lambda\sigma\nu\mu} - \epsilon_{\mu\lambda\sigma\nu} - \epsilon_{\lambda\mu\nu\sigma} + \epsilon_{\mu\lambda\nu\sigma})A_\mu(x - \hat{\lambda})A_\nu(x - \hat{\sigma}) \}
\]  
(A.9)

where in the first and the second equalities, only those terms which have nonzero contributions are retained.

For Lüscher’s topological charge density having the second field tensor located at the site \(x + \hat{\mu} + \hat{\nu}\) rather than at \(x\), we obtain

\[
\mathcal{H}'[\epsilon_{\alpha\beta\gamma\delta}F_{\alpha\beta}(x)F_{\gamma\delta}(x + \hat{\alpha} + \hat{\beta})] = \sum_{x,y} \sum_{\alpha,\beta,\gamma,\delta} x_\alpha y_\beta \frac{\delta}{\delta A_\alpha(0)} \frac{\delta}{\delta A_\beta(y)} \{ \epsilon_{\alpha\beta\gamma\delta} \\
\left[ A_\beta(x + \hat{\alpha})A_\delta(x + \hat{\alpha} + \hat{\beta} + \hat{\gamma}) - A_\alpha(x)A_\delta(x + \hat{\alpha} + \hat{\beta} + \hat{\gamma}) \\
- A_\beta(x + \hat{\alpha})A_\gamma(x + \hat{\alpha} + \hat{\beta} + \hat{\delta}) + A_\alpha(x)A_\gamma(x + \hat{\alpha} + \hat{\beta} + \hat{\delta}) \\
+ A_\alpha(x + \hat{\beta})A_\gamma(x + \hat{\alpha} + \hat{\beta} + \hat{\delta}) - A_\alpha(x)A_\gamma(x + \hat{\alpha} + \hat{\beta} + \hat{\delta}) \right] \}
\]  
\[
= \sum_{x,y} \sum_{\alpha,\beta,\gamma,\delta} x_\alpha y_\beta \frac{\delta}{\delta A_\alpha(0)} \frac{\delta}{\delta A_\beta(y)} \{ \\
\epsilon_{\sigma\nu\lambda\mu}A_\nu(x + \hat{\sigma})A_\mu(x + \hat{\sigma} + \hat{\nu} + \hat{\lambda}) + \epsilon_{\sigma\mu\lambda\nu}A_\mu(x + \hat{\sigma})A_\nu(x + \hat{\sigma} + \hat{\mu} + \hat{\lambda})
\]
Among the eight lines of expressions at the second equality, the second line vanishes due to the cancellation of its two terms, and the same happens to the fourth, the sixth and the eighth lines. Then the remaining four lines add up to yield the final result. This completes the proof of the identity \((A.1)\).

\(\)\[A.10\]

B

In this appendix we derive some useful properties of the kernel \(K_\mu\) of the vector current

\[
J_\mu(n) = \sum_{m,l} \bar{\psi}_{n+m} K_\mu(n, m, l; U) \psi_{n+l}.
\] \hspace{1cm} (B.1)

These properties [Eqs. (B.10) - (B.12)] are given in the appendix of Ginsparg and Wilson’s original paper [1]. Here we present our derivation in details and correct a minor misprint in ref. [1]. As usual, the divergence of the vector current is extracted from the change of the action under an infinitesimal local transformation at the site \(n\),

\[
\psi_n \rightarrow \psi_n + \theta_n \psi_n \\
\bar{\psi}_n \rightarrow \bar{\psi}_n - \theta_n \bar{\psi}_n
\]

with the prescription

\[
A \rightarrow A + \theta_n \partial_\mu J_\mu(n)
\]

then we obtain

\[
\partial_\mu J_\mu(n) = \sum_m [\bar{\psi}_m D_{mn} \psi_n - \bar{\psi}_n D_{nm} \psi_m] 
\] \hspace{1cm} (B.2)
where $\partial_\mu J_\mu(n)$ is defined by the backward difference

$$
\partial_\mu J_\mu(n) = \sum_\mu [J_\mu(n) - J_\mu(n - \hat{\mu})].
$$

(B.3)

Since we will turn off the gauge field after performing the differentiations in Eq. (73), we only need to derive all properties of $K_\mu$ in the free field limit where the action is translational invariant $D_{mn} = D_{m-n}$. Then Eq. (B.2) can be rewritten as

$$
\partial_\mu J_\mu(n) = \sum_l (\bar{\psi}_{n+l}D_l\psi_n - \bar{\psi}_nD_l\psi_{n-l}).
$$

(B.4)

and Eq. (B.1) as

$$
J_\mu(n) = \sum_{m,l} \bar{\psi}_{n+m}K_\mu(m, m - l)\psi_{n+m-l}.
$$

(B.5)

To construct $K_\mu(m, m - l)$ such that Eq. (B.4) can be reproduced with Eqs. (B.3) and (B.5), the authors of ref. [1] set

$$
K_\mu(m, m - l) = L_\mu(m, l) D_t = \text{sign}(l_\mu) f_\mu(m, l) D_t.
$$

(B.6)

where $f_\mu(m, l)$ is equal to the fraction of the shortest length paths from 0 to $l$ which pass through the link from $m - \hat{\mu}$ to $m$. It is straightforward to verify that this definition of $K_\mu$ leads to Eq. (B.4). Using Eq. (B.5) and Eq. (B.3), we obtain

$$
\partial_\mu J_\mu(n) = \sum_{\mu, m, l} \left[ \bar{\psi}_{n+m}K_\mu(m, m - l)\psi_{n+m-l} - \bar{\psi}_{n-\mu+m}K_\mu(m, m - l)\psi_{n-\mu+m-l} \right]
$$

(B.7)

The number of shortest length paths from $n + m - l$ to $n + m$ is

$$
N_l = \frac{(|l_1| + |l_2| + |l_3| + |l_4|)!}{|l_1|!|l_2|!|l_3|!|l_4|!}.
$$

For a given set of positive integers $(s_1, s_2, s_3, s_4)$, and for all $l$ with $|l_\nu| = s_\nu, \nu = 1, \cdots, 4$, then for $l_\mu \geq 0$, each one of these paths passing through the link from $n + m - \hat{\mu}$ to $n + m$ contributes $\frac{1}{N_l} \bar{\psi}_{n+m}D_t\psi_{n+m-l}$ to $J_\mu(n)$; while for $l_\mu \leq 0$, a path passing through the link from $n + m$ to $n + m - \hat{\mu}$ contributes $-\frac{1}{N_l} \bar{\psi}_{n+m}D_t\psi_{n+m-l}$ to $J_\mu(n)$. Hence for each $l$, the contribution of each link to the RHS of Eq. (B.7) is
\[ \frac{1}{N_l} (\bar{\psi}_{n+m} D_l \psi_{n+m-l} - \bar{\psi}_{n-\hat{\mu}+m} D_l \psi_{n-\hat{\mu}+m-l}). \]

Adding their contributions along each shortest length path would cancel in pairs except the boundary terms

\[ \frac{1}{N_l} (\psi_{n+l} D_l \psi_n - \psi_n D_l \psi_{n-l}). \]

The sum over all shortest length paths then cancels the factor \( \frac{1}{N_l} \). Thus Eq. (B.4) is reproduced.

Next we prove two identities which are essential for deriving Eqs. (B.10) - (B.12),

\[ \sum_m \mu L(m,l) = l_\mu, \quad \text{(B.8)} \]

\[ \sum_m m\nu L_\mu(m,l) = \frac{l_\mu (l_\nu + \delta_{\mu\nu})}{2}. \quad \text{(B.9)} \]

The summation over sites \( m \) is defined as

\[ \sum_m \equiv \sum_{m_\mu} \sum_{m_\nu=0} \sum_{m_\sigma=0} \sum_{m_\lambda=0} \sum l_\mu \]

where the upper and lower limits of \( m_\mu \) depend on the sign of \( l_\mu \). For \( l_\mu > 0 \), the summation of \( m_\mu \) is from 1 to \( l_\mu \), while for \( l_\mu < 0 \), from 0 to \( l_\mu + 1 \).

\[ \sum_{m_\mu} \equiv \begin{cases} \sum_{m_\mu=1}^{l_\mu} & \text{if } l_\mu > 0 \\ \sum_{m_\mu=1}^{l_\mu+1} & \text{if } l_\mu < 0 \end{cases} \]

To prove the first identity, we observe that (see Fig. 1) all shortest length paths from 0 to \( l \) must go through one of the links pointing in the \( \hat{\mu} \) direction with a fixed \( m_\mu \), i.e., they are all perpendicular to the hyperplane with fixed \( m_\mu \). Therefore holding \( m_\mu \) fixed and summing over all other indices \(( m_\nu, m_\sigma, m_\lambda )\) of the fraction \( f_\mu(m, l) \) [defined in Eq. (B.6)] is equal to summing all probabilities for all shortest length paths going through the hyperplane and hence it must equal to one. Since there are \( l_\mu \) hyperplanes between 0 and \( l_\mu \), after summing over \( m_\mu \), we obtain Eq. (B.8),
Fig. B.1. The portion of the lattice containing the shortest length paths between 0 and \( l \). Only two directions (\( \hat{\mu} \) and \( \hat{\nu} \)) are shown on the plane while the other two directions (\( \hat{\sigma} \) and \( \hat{\lambda} \)) are orthogonal to the plane. The solid line with arrows denotes the projection of one of the shortest length paths onto the plane.

\[
\sum_{m_{\mu}} \sum_{m_{\nu}=0} \sum_{m_{\sigma}=0} \sum_{m_{\lambda}=0} \text{sign}(l_{\mu})f_{\mu}(m, l) = \sum_{m_{\mu}} \text{sign}(l_{\mu}) = l_{\mu}
\]

For the second identity, we first prove the case \( \mu = \nu \),

\[
\sum_{m_{\mu}} \sum_{m_{\nu}=0} \sum_{m_{\sigma}=0} \sum_{m_{\lambda}=0} m_{\mu}L_{\mu}(m, l) = \sum_{m_{\mu}} m_{\mu} \text{sign}(l_{\mu})
\]

\[
= \begin{cases} 
(1 + 2 + \cdots + l_{\mu}) & \text{if } l_{\mu} > 0 \\
(-1 - 2 - \cdots - |l_{\mu}| + 1) & \text{if } l_{\mu} < 0
\end{cases}
\]

\[
= \frac{l_{\mu}(l_{\mu} + 1)}{2}
\]

For \( \nu \neq \mu \), by symmetry, the fraction of shortest length paths going through those links with fixed \( m_{\mu} \) and \( m_{\nu} \) is (see Fig. 1),

\[
\sum_{m_{\sigma}=0} \sum_{m_{\lambda}=0} f_{\mu}(m, l) = \frac{1}{|l_{\nu}| + 1}
\]

Then

\[
\sum_{m_{\mu}} \sum_{m_{\nu}=0} \sum_{m_{\sigma}=0} \sum_{m_{\lambda}=0} m_{\nu}L_{\mu}(m, l) = \sum_{m_{\mu}} \sum_{m_{\nu}=0} m_{\nu} \text{sign}(l_{\mu}) \frac{1}{|l_{\nu}| + 1} = \frac{l_{\mu} l_{\nu}}{2}
\]

This completes the proof of the second identity.

With these two identities, we proceed to derive some useful properties of \( K_{\mu} \) in momentum space

\[
K_{\mu}(p, p + p') = \sum_{l,m} e^{-ipm} e^{i(p+p')(m-l)} K_{\mu}(m, m - l)
\]
\[= \sum_{l,m} e^{-ipl} e^{-ip'(m-l)} L_{\mu}(m, l) D_l.\]

Since we only need the derivatives of \(K_{\mu}\) with respect to \(p'_\nu\) evaluated at \(p' = 0\), we can expand the above equation up to first order of \(p'\)

\[K_{\mu}(p, p + p') \approx \sum_{l,m} e^{-ipl} \left[1 + ip'(m - l)\right] L_{\mu}(m, l) D_l\]

\[= \sum_{l} e^{-ipl} l_{\mu} D_l \left(1 + i \frac{p'_\nu l_{\nu} + p'_\mu}{2} - ip' l\right)\]

\[= i \partial_{\mu} \left(1 + \frac{p'_\nu \partial_{\nu}}{2} + \frac{ip'_l}{2}\right) D(p)\]

where \(D(p) = \sum_l e^{-ipl} D_l\) and Eqs. (B.8), (B.9) have been used. Therefore, we have

\[K_{\mu}(p, p) = i \partial_{\mu} D(p) \quad (B.10)\]

and

\[\partial'_\nu K_{\mu}(p, p + p')|_{p' = 0} = \left(\frac{i}{2} \partial_{\mu} \partial_{\nu} - \frac{1}{2} \delta_{\mu\nu} \partial_{\mu}\right) D(p) \quad (B.11)\]

Similarly, we obtain

\[K_{\mu}(p + p', p) = \sum_{l,m} e^{-i(p+p')m} e^{-ip(m-l)} K_{\mu}(m, m - l)\]

\[= \sum_{l,m} e^{-ipl} e^{-ip'm} K_{\mu}(m, m - l)\]

\[\approx \sum_{l,m} e^{-ipl} \left(1 - ip'm\right) K_{\mu}(m, m - l)\]

\[= \sum_{l} e^{-ipl} l_{\mu} D_l \left(1 - i \frac{p'_\nu l_{\nu} + p'_\mu}{2}\right)\]

\[= i \partial_{\mu} \left(1 + \frac{p'_\nu \partial_{\nu}}{2} - \frac{ip'_l}{2}\right) D(p)\]

and this leads to

\[\partial'_\nu K_{\mu}(p + p', p)|_{p' = 0} = \left(\frac{i}{2} \partial_{\mu} \partial_{\nu} + \frac{1}{2} \delta_{\mu\nu} \partial_{\mu}\right) D(p) \quad (B.12)\]

We note that the operator \(\partial_{\mu}\) in the second term of Eq. (B.11) is missed in Eqs. (37) and (A6) of Ginsparg and Wilson’s original paper [1]. Equations (B.10)-(B.12) are used in our derivation of chiral anomaly in section 3.
In this appendix we prove the second equality of Eq. (92) [ or Eq. (93) ] which has played an important role in our derivation of axial anomaly in section 3. The vector current in Eq. (92) is

\[
J_\mu(k, U) = \sum_{i,j} \bar{\psi}_{k+i} K_\mu(k, i, j; U) \psi_{k+j} = i \sum_{m,n} \bar{\psi}_m \frac{\delta}{\delta A_\mu(k)} D_{mn}(U) \psi_n \tag{C.1}
\]

An explicit realization of the kernel \( K_\mu(k, i, j; U) \) is given in ref. [1] as

\[
K_\mu(k, i, j; U) = \text{sign}((i - j) \mu) f_\mu(k + i, k + j) D_{k+i,k+j}(U) \tag{C.2}
\]

where \( f_\mu(k + i, k + j) \) is the fraction of the shortest length paths from \( k + j \) to \( k + i \) which pass through the link \((k, k + \mu)\). Note that the vector current \( J_\mu(k, U) \) satisfying Eq. (36) is not unique. But the kernel defined in Eq. (C.2) leads to the second equality of Eq. (C.1). First we note that the ordered product of link variables along one of the shortest length paths from \( n \) to \( m \), say, \( P \), is

\[
U_P(m, n) = \prod_P U(m, s) \cdots U(t, n) \tag{C.3}
\]

where \( U(m, s) \) denotes the link variable pointing from \( m \) to \( s \) with the usual convention

\[
U(m, m + \mu) = U_\mu(m) = \exp[iaeA_\mu(m)].
\]

Then \( U_P(m, n) \) enters the action \( \mathcal{A} = \sum_{m,n} \bar{\psi}_m D_{mn}(U) \psi_n \) via the following gauge invariant product

\[
\bar{\psi}_m \Gamma(m, n) U_P(m, n) \psi_n \tag{C.4}
\]

where \( \Gamma(m, n) \) is a matrix in the Dirac space and its explicit form is irrelevant to our present discussion. The normalized sum of (C.4) over all shortest length paths from \( n \) to \( m \) is equal to the term \( \bar{\psi}_m D_{mn}(U) \psi_n \) in the action, i.e.,

\[
D_{mn}(U) = \Gamma(m, n) \frac{1}{N_l} \sum_P U_P(m, n) \tag{C.5}
\]

where \( N_l \) is the total number of shortest length paths from \( n \) to \( m \),

\[
N_l = \frac{(|l_1| + |l_2| + |l_3| + |l_4|)!}{|l_1|! |l_2|! |l_3|! |l_4|!}, \quad l = m - n \tag{C.6}
\]
Then using the following simple identity
\[
\delta \frac{\delta}{\delta A_\nu(k)} U_\mu(n) = i \delta_{\mu\nu} \delta_{kn} U_\mu(n)
\] (C.7)

it is straightforward to obtain that, if \( U_P(m, n) \) contains the link \( U_\mu(k) \) ( or its hermitian conjugate ), then the derivative of \( D_{mn}(U) \) with respect to \( A_\mu(k) \) yields \( i \) \( U_P(m, n) \) times a sign factor which depends on the relative positions of \( m \) and \( n \) ( i.e. -1 for \( m_\mu > n_\mu \) but +1 for \( m_\mu < n_\mu \) [ see Eq. (C.4) ] ); and the derivative is zero if \( U_P(m, n) \) does not contain the link \( U_\mu(k) \). That is
\[
\delta \frac{\delta}{\delta A_\mu(k)} U_P(m, n) = \begin{cases} 
-i \ \text{sign}((m - n)_\mu) U_P(m, n), & \text{if } U_\mu(k) \in U_P(m, n) \\
0, & \text{otherwise}
\end{cases}
\] (C.8)

Now multiplying both sides of Eq. (C.8) by \( \Gamma(m, n) \), summing over all shortest length paths from \( n \) to \( m \), and dividing by \( N_l \), we obtain
\[
i \delta \frac{\delta}{\delta A_\mu(k)} D_{mn}(U) = \text{sign}((m - n)_\mu) \Gamma(m, n) \frac{1}{N_l} \sum_{P'} U_{P'}(m, n)
\]
\[
= \text{sign}((m - n)_\mu) f_\mu(m, n) \Gamma(m, n) \frac{1}{N_l} \sum_{P'} U_{P'}(m, n)
\] (C.9)

where Eq. (C.5) has been used on the LHS. The summation \( P' \) on the RHS denotes the sum over the shortest length paths between \( m \) and \( n \) which pass through the link \( (k, k + \hat{\mu}) \), and the total number of these paths is denoted by \( N' \), and \( f_\mu(m, n) = N'/N_l \) is the fraction of the shortest length paths from \( n \) to \( m \) which pass through the link \( (k, k + \hat{\mu}) \). Now we sandwich both sides of Eq. (C.9) by \( \bar{\psi}_m \) and \( \psi_n \) and sum over \( m \) and \( n \). Then on the RHS of the resulting equation, we can write \( m = k + i \) and \( n = k + j \) and replace summations over \( m \) and \( n \) by summations over \( i \) and \( j \), since only those shortest length paths from \( n \) to \( m \) which pass through the link \( (k, k + \hat{\mu}) \) can have nonzero contribution, finally we have
\[
i \sum_{m,n} \bar{\psi}_m \delta \frac{\delta}{\delta A_\mu(k)} D_{mn}(U) \psi_n
\]
\[
= \sum_{i,j} \bar{\psi}_{k+i} \text{sign}((i - j)_\mu) f_\mu(k + i, k + j) D_{k+i,k+j}(U) \psi_{k+j}
\] (C.10)
\[
= \sum_{i,j} \bar{\psi}_{k+i} K_\mu(k, i, j; U) \psi_{k+j}
\] (C.11)
where
\[ D_{k+i,k+j}(U) = \Gamma(k+i,k+j) \frac{1}{N_l} \sum_{P'} U_{P'}(k+i,k+j). \]  \hspace{1cm} (C.12)

This completes the proof of the second equality of Eq. (92).

**Acknowledgement**

This work was supported by the National Science Council, R.O.C. under the grant number NSC88-2112-M002-016.
References

[1] P. Ginsparg and K. Wilson, Phys. Rev. D25 (1982) 2649.
[2] H.B. Nielsen and N. Ninomiya, Nucl. Phys. B185 (1981) 20; B193 (1981) 173.
[3] M. Lüscher, Phys. Lett B428 (1998) 342.
[4] T.W. Chiu, ”GW fermion propagators and chiral condensate”, hep-lat/9810052.
[5] P. Hasenfratz, Nucl. Phys. B525 (1998) 401.
[6] T.W. Chiu and S.V. Zenkin, Phys. Rev. D59 (1999) 074501.
[7] T.W. Chiu, Phys. Lett. B445 (1999) 371.
[8] T.W. Chiu, ”Topological phases in Neuberger-Dirac operator”, hep-lat/9810002.
[9] T.W. Chiu, Phys. Rev. D58 (1998) 074511.
[10] Y. Kikukawa and A. Yamada, ”Weak coupling expansion of massless QCD with a Ginsparg-Wilson fermion and axial U(1) anomaly”, hep-lat/9806013.
[11] D. Adams, ”Axial anomaly and topological charge in lattice gauge theory with overlap-Dirac operator”, hep-lat/9812003.
[12] H. Neuberger, Phys. Lett. B417 (1998) 141.
[13] H. Neuberger, Phys. Lett. B427 (1998) 353.
[14] P. Hasenfratz, V. Laliena and F. Niedermayer, Phys. Lett. B427 (1998) 125.
[15] M. Lüscher, Nucl. Phys. B538 (1999) 515.
[16] P. Hasenfratz, ”The theoretical background and properties of perfect actions”, hep-lat/9803027.