A Unified Approach to Construct Correlation Coefficient Between Random Variables

Majid Asadi* and Somayeh Zarezadeh†

Abstract

Measuring the correlation (association) between two random variables is one of the important goals in statistical applications. In the literature, the covariance between two random variables is a widely used criterion in measuring the linear association between two random variables. In this paper, first we propose a covariance based unified measure of variability for a continuous random variable $X$ and we show that several measures of variability and uncertainty, such as variance, Gini mean difference, cumulative residual entropy, etc., can be considered as special cases. Then, we propose a unified measure of correlation between two continuous random variables $X$ and $Y$, with distribution functions (DFs) $F$ and $G$, based on the covariance between $X$ and $H^{-1}G(Y)$ (known as the Q-transformation of $H$ on $G$) where $H$ is a continuous DF. We show that our proposed measure of association subsumes some of the existing measures of correlation. Under some mild condition on $H$, it is shown the suggested index ranges between $[-1, 1]$ where the extremes of the range, i.e., -1 and 1, are attainable by the Fréchet bivariate minimal and maximal DFs, respectively. A special case of the proposed correlation measure leads to a variant of Pearson correlation coefficient which, as a measure of strength and direction of the linear relationship between $X$ and $Y$, has absolute values greater than or equal to the Pearson correlation. The results are examined numerically for some well known bivariate DFs.

Keywords: Association; Correlation coefficient; Gini’s mean difference; Cumulative residual entropy; Fréchet bounds, Q-transformation; Bivariate copula.

1 Introduction

One of the fundamental issues in statistical theory and applications is to measure the correlation (association) between two random phenomena. The problem of assessing the correlation between two random variables (r.v.s) has a long history and because of importance...
of the subject, several criteria have been proposed in the statistical literature. Let \(X\) and \(Y\) be two continuous r.v.s with joint distribution function (D.F) \(F(x, y) = P(X \leq x, Y \leq y)\), \((x, y) \in \mathbb{R}^2\), and continuous marginal D.Fs \(F(x) = P(X \leq x)\) and \(G(y) = P(Y \leq y)\), respectively. In parametric framework, the Pearson correlation coefficient, which is the most commonly used type of correlation index, measures the strength and direction of the linear relationship between \(X\) and \(Y\). The Pearson correlation coefficient, denoted by \(\rho(X, Y)\), is defined as the ratio of the covariance between \(X\) and \(Y\), to the product of their standard deviations. That is

\[
\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{E(XY) - E(X)E(Y)}{\sigma_X \sigma_Y}, \tag{1}
\]

where \(\sigma_X > 0\) (\(\sigma_Y > 0\)) denotes the standard deviation of \(X\) (\(Y\)). An application of Cauchy-Schwarz inequality shows that \(\rho(X, Y)\) lies in interval \([-1, 1]\). In nonparametric framework, the widely used measures of association between two r.v.s are Kendall’s coefficient and Spearman’s coefficient. The Spearman correlation coefficient is defined as the Pearson correlation coefficient between the ranks of \(X\) and \(Y\) while the Kendall’s coefficient (of concordance) is expressed with respect to the probabilities of the concordant and discordant pairs of observations from \(X\) and \(Y\). For more information in properties and applications of these indexes of correlation we refer, among others, to Samuel et al. (2001); Shevlyakov and Oja (2016) and references therein. Although these correlation coefficients have been widely used in many disciplines, there have been also defined other indexes of associations which are particulary useful in certain areas of applications; see, for example, Yin (2004); Yitzhaki and Schechtman (2013); Nolde (2014); Grothe et al. (2014). In economic and financial studies a commonly used measure of association between r.v.s \(X\) and \(Y\) is defined based on Gini’s mean difference by Schezhtman and Yitzhaki (1987). The Gini’s mean difference corresponding to r.v. \(X\), denoted by \(\text{GMD}(X)\) (or alternatively with \(\text{GMD}(F)\)), is defined as

\[
\text{GMD}(X) = E(|X_1 - X_2|) = 2 \int F(x) \bar{F}(x) dx, \tag{2}
\]

where \(X_1\) and \(X_2\) are independent r.v.s distributed as \(X\) and \(\bar{F}(x) = 1 - F(x)\). The GMD(X) as a measure of variability, (which is also equal to \(4\text{Cov}(X, F(X))\)), shares many properties of the variance of \(X\) and is more informative than the variance for the distributions that are far from normality (see, Yitzhaki (2003)). Schezhtman and Yitzhaki (1987) defined the association between \(X\) and \(Y\) as the covariance between \(X\) and \(G(Y)\) divided by the covariance between \(X\) and \(F(X)\). In other words, they proposed the measure of association between \(X\) and \(Y\) as

\[
\Gamma(X, Y) = \frac{\text{Cov}(X, G(Y))}{\text{Cov}(X, F(X))}. \tag{3}
\]

As for a continuous r.v. \(Y\), \(G(Y)\) is distributed uniformly on \((0, 1)\), the index \(\Gamma(X, Y)\) measures the association between \(X\) and a uniform r.v. on the interval \((0, 1)\) which corresponds
to the rank of $Y$. The index $\Gamma(X,Y)$ has the requirements of a correlation coefficient and is well applied in a series of research works in economics and finance by Yitzhaki and his coauthors. We refer the readers, for more details on applications of $\Gamma(X,Y)$ and its extensions, to [Yitzhaki and Schechtman 2013] and references therein. Recently, [Asadi 2017] proposed a new measure of association between two continuous r.v.s $X$ and $Y$. This measure is defined on the basis of $\text{Cov}(X, \phi(X))$, where $\phi(x) = \log \frac{F(x)}{\bar{F}(x)}$, is the log-odds rate associated to $X$. The cited author provides some interpretations of this covariance and showed that it arises naturally as a measure of variability. For instance, it is shown that $\text{Cov}(X, \phi(X))$ can be expressed as a function of cumulative residual entropy (a measure of uncertainty defined in [Rao et al. 2004]). Then the measure of association between r.v.s $X$ and $Y$ is defined as the ratio of the covariance between $X$ and the log-odds rate of $Y$ divided by the covariance between $X$ and the log-odds rate of $X$. If we denote this measure by $\alpha(X,Y)$, then

$$\alpha(X,Y) = \frac{\text{Cov}(X, \phi_Y(Y))}{\text{Cov}(X, \phi_X(X))},$$

(4)

It should be noted that for a continuous r.v. $X$, $\phi_X(X)$ is distributed as standard Logistic distribution. Hence $\alpha(X,Y)$ measures the correlation between $X$ and a standard Logistic r.v., where the Logistic r.v. is the log-odds transformation of the r.v. $Y$.

The aim of the present paper is to give a unified approach to construct measures of association between two r.v.s. In this regard, we assume that $X$ and $Y$ have continuous DFs $F$ and $G$, respectively. First we consider the following covariance which we call it the $G$-covariance between $X$ and $Y$,

$$\mathcal{C}(X,Y) = \text{Cov}(X, G^{-1}F(X)),$$

(5)

where, for $p \in [0,1]$,

$$G^{-1}(p) = \inf\{x \in \mathbb{R} : G(x) \geq p\},$$

is the inverse function of DF $G$. The quantity $G^{-1}F(.)$ is known in the literature with different names. [Gilchrist 2000] called it $Q$-transformation (Q-T) and [Shaw and Buckley 2009] named it sample transmutation maps. Throughout the paper, we use the abbreviation Q-T for quantities of the form $G^{-1}F(.)$. Note that the covariance in (5) measures the linear dependency between $X$ and r.v. $G^{-1}F(X)$, where the latter one is a r.v. distributed as $Y$. Based on the covariance (5), we propose a unified index of correlation between $X$ and $Y$ which leads to new measures of correlations and subsumes some of the existing measures such as the Pearson correlation coefficient (in the case that the $X$ and $Y$ are identical) and Gini correlation coefficient (and its extensions). Then, we study several properties of our unified index of association.

The rest of the paper is organized as follows: In Section 2, first we give briefly some backgrounds and applications of quantity Q-T which have already presented in the
literature. Then, we give the motivations of using the covariance \(5\) by showing that some measures of variability such as variance, GMD (and its extensions) and cumulative residual entropy can be considered as special cases of \(5\). In Section 3, we propose our unified measure of association between the r.v.s \(X\) and \(Y\) based on the covariance between \(X\) and \(H^{-1}G(Y)\), where \(H\) is a continuous DF. We call this unified correlation as \(H\)-transformed correlation between \(X\) and \(Y\) and denote it by \(\beta_H(X,Y)\). It is shown that \(\beta_H(X,Y)\) has almost all requirements of a correlation index. For example, it is proved that for any continuous symmetric DF \(H\), \(-1 \leq \beta_H(X,Y) \leq 1\), where \(\beta_H(X,Y) = 0\) if \(X\) and \(Y\) are independent. When the joint distribution of \(X\) and \(Y\) is bivariate normal with Pearson correlation \(\rho(X,Y) = \rho\), we show that \(\beta_H(X,Y) = \rho\), for any \(H\).

We prove that for the association index \(\beta_H(X,Y)\) the lower and upper bounds of the interval \([-1, 1]\) are attainable. In fact, it is proved that \(\beta_H(X,Y) = -1 (1)\) if \(X\) and \(Y\) are jointly distributed as Fréchet bivariate minimal (maximal) distribution. A special case of \(\beta_H(X,Y)\), which we call it \(\rho\)-transformed correlation and denote it by \(\rho_t(X,Y)\), provides a variant of Pearson correlation coefficient \(\rho(X,Y)\), whose absolute value is always greater than or equal to the absolute value of Pearson correlation \(\rho(X,Y)\). That is, \(\rho_t(X,Y)\) provides a wider range than that of \(\rho(X,Y)\) for measuring the linear correlation between two r.v.s. The correlation \(\beta_H(X,Y)\) provides, in general, an asymmetric class of correlation measures in terms of \(X\) and \(Y\). We propose some symmetric versions of that in Section 3. The index \(\beta_H(X,Y)\) is computed for several bivariate distributions under different special cases for DF \(H\). In Section 4, a decomposition formula is given for G-covariance of sum of nonnegative r.v.s which yields to some applications for redundancy systems. The paper is finalized with some concluding remarks in Section 5.

### 2 Motivations

Let \(X\) and \(Y\) be two continuous r.v.s with joint DF \(F(x,y), (x,y) \in \mathbb{R}^2\), and marginal DFs \(F(x)\) and \(G(y)\), respectively. In developing our results the quantity Q-T, \(G^{-1}F(x)\), plays a central role. Balanda and MacGillivray (1990) showed that the behavior of Q-T can be used to assess the Kurtosis of two distributions (see, also, Groeneveld (1998)). They showed that for symmetric distributions the so called spread-spread function is essentially a function of Q-T. Shaw and Buckley (2009) mentioned that among the applications of Q-T is sampling from exotic distributions, e.g. t-Student. Authors have also used the plots of sample version of Q-T, in which the empirical distributions are replaced in \(G^{-1}(F(x))\), for assessing symmetry of the distributions; see Doksum et al. (1977) and references therein. Aly and Bleuer (1986) called the function Q-T as the Q-Q plot and obtained some confidence intervals for that. In comparing the probability distributions, the concept of dispersive (variability) ordering is used to measure variability of r.v.s (see,
Shaked and Shanthikumar (2007)). The concept of dispersive ordering relies mainly on quantity \( G^{-1}F(x) \). A DF \( F \) is said to be less than a DF \( G \) in dispersive ordering if \( G^{-1}F(x) - x \) is nondecreasing in \( x \). (The dispersive ordering had been already employed by Doksum (1975) in which he used the terminology “\( F \) is tail-ordered with respect to \( G \”).) Zwet (1964) used the quantity \( Q-T \) to compare the skewness of two probability density functions. The DF \( G \) is more right-skewed, respectively more left-skewed, than the DF \( F \) if \( G^{-1}F(x) - x \) is a nondecreasing convex, respectively concave, function (see also, Yeo and Johnson (2000)). In reliability theory, the convexity of the function \( Q-T \) is used, in a general setting, to study the aging properties of lifetime r.v.s with support \([0, \infty)\) (see, Barlow and Proschan (1981)). In particular case if \( G \) is exponential distribution, the convexity of \( Q-T \) is equivalent to the property that \( F \) has increasing failure rate. Also, according to the latter cited authors, a lifetime DF \( F \) is said to be less than a lifetime DF \( G \) in star-shaped order if \( G^{-1}F(x) - x \) is increasing in \( x \). In special case that \( G \) is exponential the star-shaped property of \( Q-T \) is equivalent to the property that \( F \) has increasing failure rate in average.

In the following, we use \( Q-T \) to define a variant of covariance between \( X \) and \( Y \) which we call it \( G \)-covariance. Throughout the paper, we assume that all the required expectations exist.

**Definition 1.** Let \( X \) and \( Y \) be two r.v.s with DFs \( F \) and \( G \), respectively. The \( G \)-covariance of \( X \) in terms of DF \( G \) is defined as

\[
\mathcal{C}(X, Y) = \text{Cov} \left(X, G^{-1}F(X)\right).
\]  

As \( G^{-1}F(x) \) is an increasing function of \( x \), we clearly have \( 0 \leq \text{Cov} \left(X, G^{-1}F(X)\right) \), where the equality holds if and only if \( F \) (or \( G \)) is degenerate. With \( \sigma_X^2 \) and \( \sigma_Y^2 \) as the variances of \( X \) and \( Y \), respectively, using Cauchy-Schwarz inequality, we have

\[
\text{Cov}^2(X, G^{-1}F(X)) \leq \text{Var}(X)\text{Var}(G^{-1}F(X)) = \sigma_X^2 \sigma_Y^2
\]

where the equality follows from the fact that \( G^{-1}F(X) \) is distributed as \( Y \). Hence, we get that

\[
0 \leq \mathcal{C}(X, Y) \leq \sigma_X \sigma_Y.
\]  

It can be easily shown that, in the right inequality of (8), we have the equality if and only if \( X \) and \( Y \) are distributed identically up to a location.

Note that \( \mathcal{C}(X, Y) \) can be represented as

\[
\mathcal{C}(X, Y) = \text{Cov}(X, G^{-1}F(X))
\]  


\[
= E(XG^{-1}F(X)) - E(G^{-1}F(X))E(X)
= E(XG^{-1}F(X)) - E(Y)E(X)
= \int xG^{-1}F(x)dF(x) - E(Y)E(X)
= \int yF^{-1}G(y)dG(y) - E(Y)E(X)
= \text{Cov}(Y, F^{-1}G(Y)) = C(Y, X). \tag{10}
\]

Also an alternative way to demonstrate \(C(X, Y)\) is

\[
C(X, Y) = \int xG^{-1}F(x)dF(x) - \int G^{-1}F(x)dF(x) \int xdF(x)
= \int_0^1 F^{-1}(u)G^{-1}(u)du - \int_0^1 G^{-1}(u)du \int_0^1 F^{-1}(u)du
= \text{Cov}(F^{-1}(U), G^{-1}(U)),
\]

where \(U\) is a uniform r.v. distributed on \((0, 1)\).

In the following we show that some well known measures of disparity and variability have a covariance representation and can be considered as special cases of the \(G\)-covariance \(C(X, Y)\).

(a) If \(G = F\), then we get

\[
C(X, Y) = C(X, X) = \text{Cov}(X, F^{-1}(F(X))) = \text{Cov}(X, X) = \text{Var}(X).
\]

In particular if the vector \((X, Y)\) has an exchangeable DF then

\[
C(X, Y) = \text{Var}(X) = \text{Var}(Y) = C(Y, X).
\]

(b) If \(G\) is uniform distribution on \((0, 1)\), then we get

\[
C(X, Y) = \text{Cov}(X, F(X)) = \frac{1}{4}\text{GMD}(X),
\]

where \(\text{GMD}(X)\) is the Gini’s mean difference in [2]. The Gini coefficient, which is a widely used measure in economical studies, is defined as the \(\text{GMD}(X)\) divided by twice the mean of the population. It should be also noted that the \(\text{GMD}(X)\) can be represented as the difference between the expected values of the maxima and the minima in a sample of two independent and identically distributed (i.i.d.) r.v.s \(X_1\) and \(X_2\). That is

\[
\text{GMD}(X) = 4\text{Cov}(X, F(X)) = E(\max(X_1, X_2) - \min(X_1, X_2));
\]

see, e.g., Yitzhaki and Schechtman (2013).

In reliability theory and survival analysis, the mean residual life (MRL) and mean inactivity time (MIT) are important concepts to assess the lifetime and aging
properties of devices and live organisms. These concepts, denoted respectively by $m(t)$ and $\tilde{m}(t)$, are defined at any time $t$ as $m(t) = E(X - t|X > t)$, and $\tilde{m}(t) = E(t - X|X < t)$. Recently, Asadi et al. (2016) have shown, in the case that $X$ is a nonnegative r.v., $\text{GMD}(X)$ (and hence $\mathcal{C}(X, Y)$) can also be expressed as the sum of expectations of MRL and MIT of the minimum of random sample of size 2.

(c) In the case that $G(y) = 1 - e^{-y}$, $y > 0$, the exponential distribution with mean 1, we obtain

$$\mathcal{C}(X, Y) = \text{Cov}(X, \Lambda(X)),$$

where $\Lambda(x) = G^{-1} F(x) = -\log \bar{F}(x)$, in which $\bar{F}(x) = 1 - F(x)$. The function $\Lambda(x)$, corresponding to a nonnegative r.v., is called in reliability theory as the cumulative failure rate and plays a crucial role in the study of aging properties of systems lifetime. Asadi (2017) has shown that the following equality holds for a nonnegative r.v.

$$\text{Cov}(X, \Lambda(X)) = -\int_0^\infty \bar{F}(x) \log \bar{F}(x) dx,$$

(11)

where the right hand side is known, in the literature, as the cumulative residual entropy (CRE) defined by Rao et al. (2004). As an alternative measure of Shannon entropy, the cited authors argued that CRE can be considered as a measure of uncertainty. They obtained several properties of CRE and illustrated that this measure is useful in computer vision and image processing. Asadi and Zohrevand (2007) showed that the CRE is closely related to the mean residual life, $m(t)$, of a nonnegative r.v. $X$. If fact, it is always true that the CRE can be represented as $\text{CRE} = E(m(X))$. Another interesting fact that can also be concluded from the discussion here is that the differential Shannon entropy of the equilibrium distribution (ED) corresponding to $F$ has a covariance representation. The density function of ED is given by

$$f_e(x) = \frac{\bar{F}(x)}{\mu},$$

where $0 < \mu < \infty$ is the mean of DF $F$. In a renewal process, the ED arises as the asymptotic distribution of the waiting time until the next renewal and the time since the last renewal at time $t$. Also a delayed renewal process has stationary increments if and only if the distribution of the actual remaining life is $f_e(x)$. Such process known in the literature as the stationary renewal process or equilibrium renewal process; see, Ross (1983). If $H(f_e)$ denotes the differential Shannon entropy of $f_e$, then

$$H(f_e) = -\int_0^\infty f_e(x) \log f_e(x) dx.$$
\[ = - \int_0^\infty \frac{\tilde{F}(x)}{\mu} \log \frac{\tilde{F}(x)}{\mu} \, dx \]
\[ = \frac{1}{\mu} \text{Cov}(X, \Lambda(X)) + \log \mu. \]

Finally, we should mention in this part, that the concept of generalized cumulative residual entropy (GCRE) which is introduced by Psarrakos and Navarro (2013) as

\[ \mathcal{E}_n(X) = \frac{1}{n!} \int_0^\infty \tilde{F}(x)[\Lambda(x)]^n \, dx. \] (12)

For \( n = 1 \), we get the CRE of \( X \). One can easily verify that, with \( G_n(y) = 1 - e^{-\sqrt{y}} \), \( \mathcal{E}_n(X) \) has the following covariance representation

\[ \mathcal{E}_n(X) = \frac{1}{n!} \text{Cov}(X, G_n^{-1}F(X)) - \frac{1}{(n-1)!} \text{Cov}(X, G_n^{-1}F(X)). \] (13)

(d) In the case that \( G \) is Logistic with DF \( G(y) = \frac{1}{1+e^{-y}}, y \in \mathbb{R} \), we obtain

\[ \mathcal{C}(X, Y) = \text{Cov}(X, \phi(X)), \]

where \( \phi(x) = \log \frac{F(x)}{F(x)} \) is the log-odds rate associated to r.v. \( X \). Log-odds rate is considered in the survival analysis to model the failure process of lifetime data to assess the survival function of observations (see, Wang et al. (2003)). It is easy to show that

\[ \mathcal{C}(X, Y) = \int_0^\infty \tilde{F}(x)[\Lambda(x)]^n \, dx. \]

Then it can be shown, in this case, that

\[ \mathcal{C}(X, Y) = \mathcal{C}(X, \phi(X)), \]

where \( \phi(x) = \log \frac{F(x)}{F(x)} \) is the log-odds rate associated to r.v. \( X \). Log-odds rate is considered in the survival analysis to model the failure process of lifetime data to assess the survival function of observations (see, Wang et al. (2003)). It is easy to show that

\[ \mathcal{C}(X, Y) = \int_0^\infty \tilde{F}(x)[\Lambda(x)]^n \, dx. \]

where the last term on the right hand side is called as the cumulative past entropy.

For some discussions and interpretations of \( \mathcal{C}(X, Y) \), presented in this part, see Asadi (2017).

(e) Let

\[ G(y) = \begin{cases} 
1 - \left(\frac{1}{y}\right)^{\frac{1}{\nu}}, & y > 1, \ 0 < \nu < 1; \ \text{Pareto distribution}, \\
1 - (1 - y)^{\frac{1}{\nu - 1}}, & 0 < y < 1, \ \nu > 1; \ \text{Power distribution}, \\
0, & \text{o.w.}
\end{cases} \] (14)

Then it can be shown, in this case, that

\[ \mathcal{C}(X, Y) = [I(0 < \nu < 1) - I(\nu > 1)] \text{Cov}(X, \tilde{F}^{\nu-1}(X)), \]

where \( I(A) \) is an indicator function which is equal to 1 when \( x \in A \) and otherwise is equal to zero. Hence, we get the extended Gini, \( \text{EGini}_\nu(X) \), defined as a parametric extension of \( \text{GMD}(X) \) of the form:

\[ \text{EGini}_\nu(X) = \nu [I(\nu > 1) - I(0 < \nu < 1)] \mathcal{C}(X, Y), \]
where \( \nu \) is a parameter ranges from 0 to infinity and determines the relative weight attributed to various portions of probability distribution. For \( \nu = 2 \), the extended Gini leads to GMD(X) (up to a constant). For more interpretations and applications of \( \text{EGini}_\nu(X) \) in economic studies based on different values of \( \nu \), we refer to Yitzhaki and Schechtman (2013).

(f) The upper and lower record values, in a sequence of i.i.d. r.v.s \( X_1, X_2, \ldots \), have applications in different areas of applied probability; see, Arnold et al. (1998). Let \( X_i \)'s have a common continuous DF \( F \) with survival function \( \bar{F} \). Define a sequence of upper record times \( U(n), n = 1, 2, \ldots \), as follows

\[
U(n + 1) = \min\{j : j > U(n), X_j > X_{U(n)}\}, \quad n \geq 1,
\]

with \( U(1) = 1 \). Then, the sequence of upper record values \( \{R_n, n \geq 1\} \) is defined by \( R_n = X_{U(n)}, n \geq 1 \), where \( R_1 = X_1 \). The survival function of \( R_n \) is given by

\[
\bar{F}^U_n(t) = \bar{F}(t) \sum_{x=0}^{n-1} \frac{(\Lambda(t))^x}{x!}, \quad t > 0, n = 1, 2, \ldots,
\]

where \( \Lambda(t) = -\log \bar{F}(t) \). If \( R_n \) denotes the \( n \)th upper record value, then it can be easily shown that, with \( G_n(y) = 1 - e^{-\sqrt{y}} \), the mean of difference between \( R_n \) and \( R_1 \) has the following covariance representation:

\[
E(R_n - R_1) = E(R_n - \mu) = \frac{1}{(n-1)!} \text{Cov}(X, G^{-1}_{n-1} F(X)), \quad n \geq 1,
\]

where \( \mu = E(R_1) = E(X_1) \).

The lower record values in a sequence of i.i.d. r.v.s \( X_1, X_2, \ldots \) can be defined in a similar manner. The sequence of record times \( L(n), n = 1, 2, \ldots \), is defined as \( L(1) = 1 \) and

\[
L(n + 1) = \min\{j : j > L(n), X_j < X_{L(n)}\}, \quad n \geq 1.
\]

Then the \( n \)th lower record value is defined by \( \tilde{R}_n = X_{L(n)} \). The DF of \( \tilde{R}_n \) is given by

\[
F^L_n(t) = F(t) \sum_{x=n}^{\infty} \frac{[\tilde{\Lambda}(t)]^x}{x!}, \quad t > 0, n = 1, 2, \ldots,
\]

in which \( \tilde{\Lambda}(t) = -\log F(t) \); see, Arnold et al. (1998).

Let \( \tilde{R}_n \) denote the \( n \)th lower record. Then, it can be shown that

\[
E(\tilde{R}_n - \tilde{R}_1) = E(\tilde{R}_n - \mu) = \frac{1}{(n-1)!} \text{Cov}(X, [\tilde{\Lambda}(X)]^{n-1}), \quad n \geq 1,
\]
where $\tilde{\Lambda}(t) = -\log F(t)$. Therefore the expectation of the difference between the $n$th upper and lower records has a covariance representation as follows

$$
E(R_n - \tilde{R}_n) = \frac{1}{(n-1)!} \text{Cov}(X, [\Lambda(X)]^{n-1}) - \frac{1}{(n-1)!} \text{Cov}(X, [\tilde{\Lambda}(X)]^{n-1})
$$

$$
= \frac{1}{(n-1)!} \text{Cov}(X, K_n^{-1}(F(X))),
$$

where $K_n(x)$ is a DF with inverse $K_n^{-1}(u) = (-\ln(1-u))^n - (-\ln(u))^n, 0 < u < 1$.

### 3 A Unified Measure of Correlation

We define our unified measure of correlation between $X$ and $Y$, as follows:

**Definition 2.** Let $X$ and $Y$ be two continuous r.v.s with joint DF $F(x, y), (x, y) \in \mathbb{R}^2$, and continuous marginal DFs $F(x)$ and $G(y)$, respectively. Let $H$ be a continuous DF. Then the $H$-transformed correlation between $X$ and $Y$, denoted by $\beta_H(X, Y)$, is defined as

$$
\beta_H(X, Y) = \frac{\text{Cov}(X, H^{-1}G(Y))}{\text{Cov}(X, H^{-1}F(X))},
$$

(15)

provided that all expectations exist and $\text{Cov}(X, H^{-1}F(X)) > 0$.

It is trivial that for continuous r.v. $Y$, the r.v. $H^{-1}G(Y)$ is distributed as r.v. $W$, where $W$ has DF $H$. Hence, $\beta_H(X, Y)$ measures the association between $X$ and a function of $Y$ where that function is the transformation $H^{-1}$ over $G(Y)$. The $H$-transformed correlation between $Y$ and $X$ can be defined similarly as

$$
\beta_H(Y, X) = \frac{\text{Cov}(Y, H^{-1}F(X))}{\text{Cov}(Y, H^{-1}G(Y))},
$$

provided that $\text{Cov}(Y, H^{-1}G(Y)) > 0$.

In what follows, we study the properties of $\beta_H(X, Y)$ and show that, under some mild condition on $H$, it has the necessary requirements of a correlation coefficient. Before that, we give the following corollary showing that $\beta_H(X, Y)$ subsumes some well known measures of association as special cases.

**Corollary 1.** The correlation index $\beta_H(X, Y)$ in (15) gives the following measures of association as special cases:

(a) If we assume that $H = G$ then we have

$$
\beta_H(X, Y) = \frac{\text{Cov}(X, Y)}{\text{Cov}(X, G^{-1}F(X))}
$$

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Cov(X, Y)
\frac{\text{Cov}(X, G^{-1}F(X)) \text{Cov}(Y, F^{-1}G(Y))}{\text{Cov}(X, G^{-1}F(X)) \text{Cov}(Y, F^{-1}G(Y))},

(16)

where the last equality follows from (10). In the following, we call (16) as \( \rho \)-transformed correlation between \( X \) and \( Y \) and denote it by \( \rho_t(X, Y) \). The measure \( \rho_t(X, Y) \) is a correlation index proportional to the Pearson correlation coefficient \( \rho(X, Y) \) in (1). In fact \( \rho_t(X, Y) = a \rho(X, Y) \), where

\[ a = \frac{\sigma_X \sigma_Y}{\text{Cov}(X, G^{-1}F(X)) \text{Cov}(Y, F^{-1}G(Y))}. \]

In particular, if the marginal DFs \( F \) and \( G \) are identical, then \( a = 1 \). (Note that, a sufficient condition to have \( F = G \) is that the joint DF of \( (X, Y) \) to be exchangeable. Recall that a random vector \( (X, Y) \) is said to have an exchangeable DF if the vectors \( (X, Y) \) and \( (Y, X) \) are identically distributed.) However, in general case based on (7), we always have

\[ \text{Cov}^2(X, G^{-1}F(X)) = \text{Cov}^2(Y, F^{-1}G(Y)) \leq \sigma_X^2 \sigma_Y^2. \]

Hence, we get that \( a^2 \geq 1 \). This, in turn, implies that the following interesting inequality holds between \( \rho(X, Y) \) and \( \rho_t(X, Y) \):

\[ 0 \leq |\rho(X, Y)| \leq |\rho_t(X, Y)|. \]

(17)

We will show in Theorem 1 that when \( X \) and \( Y \) are positively correlated then \( \rho(X, Y) \leq \rho_t(X, Y) \leq 1 \), and when \( X \) and \( Y \) are negatively correlated and \( G \) or \( F \) is a symmetric DF, then \(-1 \leq \rho_t(X, Y) \leq \rho(X, Y)\). These inequalities indicate that \( \rho_t(X, Y) \), as a measure of the strength and direction of the linear relationship between two r.v.s, in compare to the Pearson correlation \( \rho(X, Y) \), shows more intensity of correlation between the two r.v.s.. This may be due to the fact that in denominator of \( \rho(X, Y) \) the normalizing factor \( \sigma_X \) (\( \sigma_Y \)) depends only on the distribution of \( F \) (\( G \)) while in denominator of \( \rho_t(X, Y) \) the normalizing factor \( \text{Cov}^2(X, G^{-1}F(X)) \) (\( \text{Cov}^2(Y, F^{-1}G(Y)) \)) depends on both DFs \( F \) and \( G \).

(b) If \( H \) is uniform on interval \( (0, 1) \), i.e., \( H(x) = x \), \( 0 < x < 1 \), then \( \beta_H(X, Y) \) reduces to the Gini correlation in (3),

\[ \Gamma(X, Y) = \frac{\text{Cov}(X, G(Y))}{\text{Cov}(X, F(X))}. \]

(c) If \( H \) is Pareto distribution \( (0 < \nu < 1) \) or power distribution \( (\nu > 1) \), given in below

\[ H(x) = \begin{cases} 1 - \left(\frac{1}{x}\right)^{\frac{1}{1-\nu}}, & x > 1, \ 0 < \nu < 1; \\ 1 - (1-x)^{\frac{1}{\nu}}, & 0 < x < 1, \ \nu > 1; \\ 0, & \text{o.w.,} \end{cases} \]
we get the extended Gini (EGini\(_\nu\)) correlation defined as
\[
\Gamma(\nu, X, Y) = \frac{\text{Cov}(X, G_{\nu}^{-1}(Y))}{\text{Cov}(X, F_{\nu}^{-1}(X))}, \quad \nu > 0.
\]

Note that for \(\nu = 2\) we arrive at the Gini correlation.

(d) If \(H(x) = \frac{1}{1+e^{-x}}, x \in \mathbb{R}\), the standard Logistic distribution, then \(\beta_H(X, Y)\) becomes the association measure in (1), defined by Asadi (2017), which measures the correlation between \(X\) and the log-odds rate of \(Y\).

Before giving the main properties of the correlation in (15), we give the following expressions which indicate that the correlation coefficient \(\beta_H(X, Y)\) has representations in terms of joint DF \(F(x, y) = P(X \leq x, Y \leq y)\) and joint survival function \(\bar{F}(x, y) = P(X > x, Y > y)\). In the sequel, we assume that all the integrals are from \(-\infty\) to \(\infty\) unless stated otherwise. The correlation \(\beta_H(X, Y)\) can be expressed as
\[
\beta_H(X, Y) = \frac{1}{\text{Cov}(X, H^{-1}F(X))} \int \int (F(x, y) - F(x)G(y)) dxdH^{-1}G(y)
\]
\[
= \frac{1}{\text{Cov}(X, H^{-1}F(X))} \int \int (\bar{F}(x, y) - \bar{F}(x)\bar{G}(y)) dxdH^{-1}G(y).
\]

The validity of these expressions can be verified from Theorem 1 of Cuadras (2002) under the assumptions that the expectations exist and \(H^{-1}G(y)\) is a bounded variation function.

The following theorem gives some properties of \(\beta_H(X, Y)\).

**Theorem 1.** The correlation \(\beta_H(X, Y)\) satisfies in the following properties:

(a) For continuous r.v.s \(X\) and \(Y\), \(\beta_H(X, Y) \leq 1\) and when \(H\) is a symmetric DF, \(-1 \leq \beta_H(X, Y) \leq 1\).

(b) The maximum (minimum) value of \(\beta_H(X, Y)\) is achieved, if \(Y\) is a monotone increasing (decreasing) function of \(X\).

(c) For independent r.v.s \(X\) and \(Y\), \(\beta_H(X, Y) = \beta_H(Y, X) = 0\).

(d) \(\beta_H(X, Y) = -\beta_H(-X, Y) = -\beta_H(X, -Y) = \beta_H(-X, -Y)\).

(e) The correlation measure \(\beta_H(X, Y)\) is invariant under all strictly monotone functions of \(Y\).

(f) \(\beta_H(X, Y)\) is invariant under changing the location and scale of \(X\) and \(Y\).

(g) If the joint DF of \(X\) and \(Y\) is exchangeable, then \(\beta_H(X, Y) = \beta_H(Y, X)\).

**Proof.** We provide the proofs for parts (a) and (g). The proofs of other parts are straightforward (see, Yitzhaki and Schechtman (2013), p. 41, where the authors study the properties of Gini correlation \(\Gamma(X, Y)\)).
First, we show that $\beta_H(X, Y) \leq 1$ for any continuous DF $H$. To this, we need to show that $E(XH^{-1}G(Y)) \leq E(XH^{-1}F(X))$. Both functions $X$ and $H^{-1}F(X)$ are increasing functions. Then $E(XH^{-1}G(Y))$ achieves its maximum value when $H^{-1}G(Y)$ is an increasing function of $X$, (see Yitzhaki and Schechtman (2013), p. 41). This implies that $H^{-1}F(X) = H^{-1}G(Y)$ which, in turn, implies that the maximum value is achieved at $E(XH^{-1}F(X))$ and hence $\beta_H(X, Y) \leq 1$.

Now, let $H$ be a symmetric DF about constant $a$. To have $-1 \leq \beta_H(X, Y)$ it needs to show that $-Cov(X, H^{-1}F(X)) \leq Cov(X, H^{-1}G(Y))$. From Yitzhaki and Schechtman (2013), p. 41, $E(XH^{-1}G(Y))$ achieves its minimum value when $H^{-1}G(Y)$ is a decreasing function of $X$. This results in $H^{-1}G(Y) = H^{-1}(1 - F(X)) = 2a - H^{-1}(F(X))$ which, in turn, implies that $2a - E(XH^{-1}F(X)) \leq E(XH^{-1}G(Y))$ and hence $-Cov(X, H^{-1}F(X)) \leq Cov(X, H^{-1}G(Y))$. Hence, we have $-1 \leq \beta_H(X, Y)$.

As the random vector $(X, Y)$ has exchangeable distribution, $(X, Y)$ is identically distributed as $(Y, X)$ and hence the marginal distributions of $X$ and $Y$ are identical, i.e., $F = G$. Hence, we can write

$$\beta_H(X, Y) = \frac{Cov(X, H^{-1}G(Y))}{Cov(X, H^{-1}F(X))}$$

$$= \frac{Cov(X, H^{-1}F(Y))}{Cov(X, H^{-1}G(Y))}$$

$$= \frac{Cov(Y, H^{-1}F(X))}{Cov(Y, H^{-1}G(Y))} = \beta_H(Y, X).$$

The following theorem proves that in bivariate normal distribution, the correlation $\beta_H(X, Y)$ is equal to Pearson correlation $\rho(X, Y)$.

**Theorem 2.** Let $X$ and $Y$ have bivariate normal distribution with Pearson correlation coefficient $\rho(X, Y) = \rho$. Then, for any continuous DF $H$ with finite mean $\mu_H$,

$$\beta_H(X, Y) = \beta_H(Y, X) = \rho.$$

**Proof.** Assume that the marginal DFs of $X$ and $Y$ are $F$ and $G$, with means $\mu_F$ and $\mu_G$ and positive variances $\sigma^2_F$ and $\sigma^2_G$, respectively. Further let $Z$ denote the standard normal r.v. with DF $\Phi$. It is well known that for the bivariate normal distribution we have

$$E(X|Y) = \mu_F + \rho\sigma_F \frac{Y - \mu_G}{\sigma_G}.$$

Using this we can write

$$Cov(X, H^{-1}G(Y)) = E_Y \left[ (E(X|Y) - \mu_F) \left( H^{-1}G(Y) - \mu_H \right) \right]$$

Theorem 2. Let $X$ and $Y$ have bivariate normal distribution with Pearson correlation coefficient $\rho(X, Y) = \rho$. Then, for any continuous DF $H$ with finite mean $\mu_H$,

$$\beta_H(X, Y) = \beta_H(Y, X) = \rho.$$
\[ \rho \sigma_F E_Y \left[ \left( \frac{Y - \mu_G}{\sigma_G} \right) H^{-1}G(Y) \right] \]
\[ = \rho \sigma_F \int \left( \frac{y - \mu_G}{\sigma_G} \right) H^{-1}G(y) dG(y) \]
\[ = \rho \frac{\sigma_F}{\sigma_G} \int (G^{-1}(z) - \mu_G) H^{-1}\Phi(z) d\Phi(z) \]
\[ = \rho \frac{\sigma_F}{\sigma_G} \left( \int G^{-1}(z) H^{-1}\Phi(z) d\Phi(z) - \mu_G \mu_H \right) \]
\[ = \rho \frac{\sigma_F}{\sigma_G} \text{Cov}(G^{-1}\Phi(Z), H^{-1}\Phi(Z)) \]
\[ = \rho \sigma_F \text{Cov}(Z, H^{-1}\Phi(Z)), \]

where the last equality follows from the fact that \( G^{-1}(z) = \sigma_G z + \mu_G \). On the other hand, we can similarly show that \( \text{Cov}(X, H^{-1}F(X)) = \sigma_F \text{Cov}(Z, H^{-1}\Phi(Z)) \). Hence, we have \( \beta_H(X, Y) = \rho \).

Assuming that \( X \) and \( Y \) have joint bivariate DF \( F(x, y) \), with marginal DFs \( F(x) \) and \( G(y) \), then \( F(x, y) \) satisfies the Fréchet bounds inequality
\[ F_0(x, y) = \max\{F(x) + G(y) - 1, 0\} \leq F(x, y) \leq \min\{F(x), G(y)\} = F_1(x, y). \]

The Fréchet bounds \( F_0(x, y) \) and \( F_1(x, y) \) are themselves bivariate distributions known as the minimal and maximal distributions, respectively. These distributions show the perfect negative and positive dependence between the corresponding r.v.s \( X \) and \( Y \), respectively; in the sense that “the joint distribution of \( X \) and \( Y \) is \( F_0(x, y) \) (\( F_1(x, y) \)) if and only if \( Y \) is decreasing (increasing) function of \( X \)” (see, Nelsen (1998)). In the following theorem we prove that, under some conditions, the extremes of the range for \( \beta_H(X, Y) \) i.e., \(-1\) and \( 1 \), are attainable by the Fréchet bivariate minimal and maximal distributions, respectively. In other words, we show that for lower and upper bounds of Fréchet inequality we have \( \beta_H(X, Y) = -1 \) and \( \beta_H(X, Y) = 1 \), respectively.

**Theorem 3.** Let \( X \) and \( Y \) be two continuous r.v.s with DFs \( F(x) \) and \( G(y) \), respectively, and \( H \) be a continuous DF.

(a) If \( (X, Y) \) has joint DF \( F_1(x, y) \) then \( \beta_H(X, Y) = 1 \),

(b) If \( H \) is symmetric and \( (X, Y) \) has joint DF \( F_0(x, y) \) then \( \beta_H(X, Y) = -1 \).

**Proof.** (a) Let us define the sets \( A_x = \{ y | y \geq G^{-1}(F(x)) \} \) and \( A_x^c = \{ y | y < G^{-1}(F(x)) \} \).

Then, we have
\[ \text{Cov}(X, H^{-1}G(Y)) = \int \int (F(x, y) - F(x)G(y)) dH^{-1}G(y)dx \]
\[ = \int \int \left( \min\{F(x), G(y)\} - F(x)G(y) \right) dH^{-1}G(y)dx \]

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\[
= \int F(x) \int_{A_x} \bar{G}(y)dH^{-1}G(y)dx + \int \bar{F}(x) \int_{A_x^c} G(y)dH^{-1}G(y)dx.
\]

(18)

But, we have under the assumptions of the theorem
\[
\int_{A_x} \bar{G}(y)dH^{-1}G(y) = -\bar{F}(x)H^{-1}F(x) + \int_{F(x)}^{1} H^{-1}(u)du,
\]

(19)

and
\[
\int_{A_x^c} G(y)dH^{-1}G(y) = F(x)H^{-1}F(x) - \int_{0}^{F(x)} H^{-1}(u)du.
\]

(20)

From (18), (19) and (20), we get
\[
\text{Cov}(X, H^{-1}G(Y)) = \lim_{a \to -\infty} \left\{ \int_{a}^{\infty} F(x) \int_{F(x)}^{1} H^{-1}(u)dudx - \int_{a}^{\infty} \bar{F}(x) \int_{0}^{F(x)} H^{-1}(u)dudx \right\}
\]
\[
= \lim_{a \to -\infty} \left\{ \int_{a}^{\infty} F(x) \int_{F(x)}^{1} H^{-1}(u)dudx - \int_{a}^{\infty} \bar{F}(x) \int_{0}^{1} H^{-1}(u)dudx \right\}
\]
\[
= \lim_{a \to -\infty} \left\{ \int_{a}^{1} H^{-1}(u) \left( \int_{a}^{F^{-1}(u)} dH - \int_{a}^{\infty} \bar{F}(x)dx \right)du \right\}
\]
\[
= \lim_{a \to -\infty} \left\{ \int_{0}^{1} H^{-1}(u)(F^{-1}(u) - a - \mu_F + a) \right\}
\]
\[
= \int_{0}^{1} F^{-1}(u)H^{-1}(u)du - \mu_F \mu_H
\]

This shows that \( \beta_H(X, Y) = 1. \)

(b) In this case we define \( B_x = \{ y | y \geq G^{-1}(\bar{F}(x)) \} \) and \( B_x^c = \{ y | y < G^{-1}(\bar{F}(x)) \}. \) Then
\[
\text{Cov}(X, H^{-1}G(Y)) = \int_{a}^{\infty} \int \left( \max \{ F(x) + G(y) - 1, 0 \} - F(x)G(y) \right)dH^{-1}G(y)dx
\]
\[
= -\int_{a}^{\infty} \int_{B_x} \bar{F}(x)\bar{G}(y)dH^{-1}G(y)dx - \int \int_{B_x^c} F(x)G(y)dH^{-1}G(y)dx.
\]

Therefore, using the same procedure as part (a), it can be written
\[
\text{Cov}(X, H^{-1}G(Y)) = \lim_{a \to -\infty} \left\{ \int_{a}^{\infty} F(x) \int_{0}^{\bar{F}(x)} H^{-1}(u)dudx - \int_{a}^{\infty} \bar{F}(x) \int_{\bar{F}(x)}^{1} H^{-1}(u)dudx \right\}
\]
\[
= \lim_{a \to -\infty} \left\{ \int_{a}^{1} H^{-1}(u)dudx - \int_{a}^{\infty} \bar{F}(x)dx \int_{0}^{1} H^{-1}(u)dudx \right\}.
\]
$$\lim_{a \to -\infty} \left\{ \int_0^1 H^{-1}(1 - u)(F^{-1}(u) - a - \mu_F + a)du \right\}$$

$$= \int_0^1 (2\mu_H - H^{-1}(u))F^{-1}(u)du - \mu_F \mu_H$$

$$= - \int_0^1 H^{-1}(u)F^{-1}(u)du + \mu_F \mu_H$$

$$= - \text{Cov}(H^{-1}(U), F^{-1}(U))$$

$$= - \text{Cov}(X, H^{-1}(F(X)))$$

where the equality (c) follows from the assumption that $H$ is symmetric. Hence, we get that $\beta_H(X, Y) = -1$. This completes the proof of the theorem.

\[\square\]

**Remark 1.** It should be pointed out that, the symmetric condition imposed on $H$ in part (b) of Theorem 3 cannot be dropped in general case. As a counter example, it can be easily verify that if $H$ is exponential the upper bound 1 for $\beta_H(X, Y)$ is attainable by Fréchet bivariate maximal distribution, however, the lower bound -1 is not attainable by Fréchet bivariate minimal distribution.

A well known class of bivariate distributions, which is extensively studied in the statistical literature, is FGM family (see, Cambanis (1991)). The joint DF $F(x, y)$ of the r.v.s $X$ and $Y$ with, respectively, continuous marginal DFs $F(x)$ and $G(y)$, is said to be a member of FGM family if

$$F(x, y) = F(x)G(y) \left(1 + \gamma \hat{F}(x)\hat{G}(y)\right),$$

where $\gamma \in [-1, 1]$ shows the parameter of dependency between $X$ and $Y$. Clearly for $\gamma = 0$, $X$ and $Y$ are independent. It is well known that for FGM family the Pearson correlation coefficient $\rho(X, Y)$ lies in interval $[-1/3, 1/3]$ where the maximum is attained for the case when the marginal distributions are uniform (Johnson and Kotz (1977)). Schechtman and Yitzhaki (1999) proved that in FGM family, the Gini correlation $\Gamma(X, Y)$ lies between $[-1/3, 1/3]$, for any marginal DFs $F$ and $G$.

The following theorem gives an expression for $\beta_H(X, Y)$ in FGM family.

**Theorem 4.** Under the assumption that $F$, $G$ and $H$ have finite means, the association measure $\beta_H(X, Y)$, for the FGM class, is given by

$$\beta_H(X, Y) = \frac{\text{GMD}(F)\text{GMD}(H)}{4\text{Cov}(X, H^{-1}F(X))}. \tag{21}$$

**Proof.**

$$\text{Cov}(X, H^{-1}G(Y)) = \int \int (F(x, y) - F(x)G(y)) dH^{-1}G(y)dx$$

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\[
\begin{align*}
&= \gamma \int \int F(x) \bar{F}(x) G(y) \bar{G}(y) dH^{-1} G(y) dx \\
&= \gamma \int F(x) \bar{F}(x) dx \int G(y) \bar{G}(y) dH^{-1} G(y) \\
&= \gamma \int F(x) \bar{F}(x) dx \int H(u) \bar{H}(u) du \\
&= \frac{\gamma}{4} \text{GMD}(F) \text{GMD}(H),
\end{align*}
\]

where \( \bar{H} = 1 - H \). Hence, \( \beta_H(X, Y) \) can be represented as
\[
\beta_H(X, Y) = \gamma \frac{\text{GMD}(F) \text{GMD}(H)}{4 \text{Cov}(X, H^{-1} F(X))}.
\]

This completes the proof.

It should be pointed out that the correlation index \( \beta_H(X, Y) \) in FGM family does not depend on the DF \( G \) which is transmuted by \( H \). Also, it is trivial that, in the case where \( H \) is uniform DF on interval \((0, 1)\), \( \beta_H(X, Y) \) reduces to Gini correlation which is free of \( F \) and its values lies in \([-1/3, 1/3]\). If \( H = G \), we arrive at the following formula for \( \rho_t(X, Y) \):
\[
\rho_t(X, Y) = \frac{\gamma}{4} \frac{\text{GMD}(F) \text{GMD}(H) \text{GMD}(G)}{\text{Cov}(X, H^{-1} F(X)) \text{Cov}(Y, F^{-1} G(Y))}.
\]

Table 1 gives the range of possible values of \( \rho(X, Y) \) and \( \rho_t(X, Y) \), in FGM family, for different choices of DFs \( F \) and \( G \). When one of the two r.v.s is selected as uniform r.v. \( U \), then we get the Gini correlation and hence
\[
\rho_t(U, X) = \rho_t(X, U) = \frac{\text{GMD}(X) \text{GMD}(U)}{4 \text{Cov}(X, F(X))} = \frac{\gamma}{3}.
\]

This implies that the range of possible values of \( \rho_t(X, U) \) is \([-1/3, 1/3]\). As seen in the table, \( \rho_t(X, Y) \), in compare to the Pearson correlation \( \rho(X, Y) \), shows a wider range of correlation between the two r.v.s.

In the following, we give some examples in which \( \beta_H(X, Y) \) in (15) are computed for different transformation DFs \( H \). The following choices for \( H \) are considered:

- Exponential distribution \( H(x) = 1 - e^{-x} \), \( x > 0 \): Cumulative residual entropy based (CRE-Based) correlation.
- Logistic distribution, \( H(x) = \frac{1}{1+e^{-x}} \), \( x \in \mathbb{R} \): Odds ratio based (OR-Based) correlation.
- Pareto distribution, \( H(x) = 1 - \left(\frac{1}{x}\right)^2 \), \( x > 1 \): Extended Gini correlation with parameter \( \nu = 0.5 \) (EGini_{0.5}).
- Uniform distribution, \( H(x) = x \), \( 0 < x < 1 \): Gini correlation.
Example 1. Table 1 represents the values of $\beta_H(X,Y)$, in FGM family, for different choices of transformation DFs $H$ and different DFs $F$.

Example 2. In this example we consider two bivariate distributions and compute the correlation index $\beta_H(X,Y)$ for different choices of $H$:

(a) The first bivariate distribution which we consider is a special case of Gumbel-Barnett family of copulas, introduced by Rieghley (1980), given as

$$C_\theta(u,v) = u + v - 1 + (1 - u)(1 - v)e^{-\theta \log(1-u) \log(1-v)}, \quad 0 \leq \theta \leq 1. \quad (22)$$
In this copula if we take the standard exponential DFs as marginals of $X$ and $Y$, then we arrive at the Gumbel’s bivariate exponential DF (Gumbel 1960). The joint DF of Gumbel’s bivariate exponential distribution is written as

$$F_\theta(x, y) = 1 - e^{-x} - e^{-y} + e^{-x-y-\theta xy}, \quad x > 0, \ y > 0, \ 0 \leq \theta \leq 1.$$  \hfill (23)

For $\theta = 0$, $X$ and $Y$ are independent and $\rho(X, Y) = 0$. As $\theta$ increases, the absolute value of Pearson correlation, $|\rho(X, Y)|$, increases and takes value $\rho(X, Y) = -0.40365$ at $\theta = 1$. This distribution is applied for describing r.v.s with negative correlation. (Of course, positive correlation can be obtained by changing $X$ to $-X$ or $Y$ to $-Y$.) In Table 3, the range of Pearson correlation and the range of $H$-transformed correlation are given for Gumbel’s bivariate exponential distribution.

(b) The second bivariate distribution considered in Table 3 is bivariate Logistic distribution which is belong to Ali-Mikhail-Haq family of copulas (Hutchinson and Lai (1990)) with the following structure

$$C_\theta(u, v) = \frac{uv}{1 - \theta(1 - u)(1 - v)}, \quad -1 \leq \theta \leq 1.$$  \hfill (24)

With standard Logistic distributions as marginal DFs of $X$ and $Y$, we arrive at the joint DF of bivariate Logistic distribution as follows

$$F_\theta(x, y) = \frac{1 - e^{-x}}{1 + e^{-y - \theta e^{-y-x}}}, \quad x > 0, \ y \in \mathbb{R}, \ -1 \leq \theta \leq 1.$$  \hfill (25)

Note that Gumbel’s bivariate Logistic distribution is a special case of bivariate Logistic distribution when $\theta = 1$.

Both bivariate DFs in (23) and (25) are exchangeable. Hence for both cases, we obtain $\rho(X, Y) = \rho_t(X, Y)$. The range of possible values of $\beta_H(X, Y)$ is given on the basis of five different DFs $H$ introduced above. For each $H$, the values of lower bound and upper bound of $H$-transformed correlation for Gumbel’s bivariate exponential, which are attained in $\theta = 1$ and $\theta = 0$, respectively, are given in the first panel of Table 3. It is seen from the table that the widest range of correlation is achieved for $\text{EGini}_3$ among all other correlations. It is evident from the table that, the range of the values of Pearson correlation $\rho(X, Y)$ is even less than those of Gini and OR-based correlations. The minimum range of correlation corresponds to $\text{EGini}_0.5$. In the case that the DF $H$ is equal to the marginal DFs of the bivariate distribution, the associated correlation $\beta_H(X, Y)$ becomes the Pearson correlation, which in this case is the CRE-Based correlation. The second panel of Table 3 gives the correlation $\beta_H(X, Y)$, based on the above mentioned distributions $H$, in bivariate Logistic distribution. The lower bound and the upper bound of all correlations are attained for $\theta = -1$ and $\theta = 1$, respectively. In this case the maximum range of correlation is achieved for $\text{EGini}_3$ and the minimum range is achieved for $\text{EGini}_0.5$. 
Table 3: The ranges of $\rho$, and $\beta_H$ correlations for two exchangeable distributions.

**Gumbel’s Type I Bivariate Exponential Distribution**

$$F_\theta(x, y) = 1 - e^{-x} - e^{-y} + e^{-x-y-\theta xy}, \ x > 0, \ y > 0, \ 0 \leq \theta \leq 1.$$  

| Correlation index | Lower bound | Upper bound |
|-------------------|-------------|-------------|
| Pearson           | -0.40365    | 0           |
| CRE-Based         | -0.40365    | 0           |
| OR-Based          | -0.51267    | 0           |
| EGini$_{0.5}$     | -0.26927    | 0           |
| Gini              | -0.55469    | 0           |
| EGini$_{3}$       | -0.64125    | 0           |

**Bivariate Logistic Distribution**

$$F_\theta(x, y) = (1 + e^{-x} + e^{-y} + (1-\theta)e^{-x-y})^{-1}, \ x \in \mathbb{R}, \ y \in \mathbb{R}, \ -1 \leq \theta \leq 1.$$  

| Correlation index | Lower bound | Upper bound |
|-------------------|-------------|-------------|
| Pearson           | -0.25000    | 0.50000     |
| CRE-Based         | -0.26516    | 0.39207     |
| OR-Based          | -0.25000    | 0.50000     |
| EGini$_{0.5}$     | -0.22135    | 0.27865     |
| Gini              | -0.27259    | 0.50000     |
| EGini$_{3}$       | -0.26272    | 0.55556     |

**Example 3.** In this example, we consider again the copulas given in (22) and (24). However, here we assume that the marginal DFs are not the same (the bivariate distribution is not exchangeable). In the first bivariate distribution the marginals are two different Weibull DFs (with different shape parameters) and in the second case the marginals are two different power DFs (with different shape parameters), respectively. In Table 3, the ranges of possible values of $\rho(X, Y)$, $\rho_t(X, Y)$, and $\beta_H(X, Y)$ are presented for both bivariate DFs. The values of lower and upper bounds of $H$-transformed correlation for the two bivariate distributions which are attained in $\theta = 1$ and $\theta = 0$, and in $\theta = -1$ and $\theta = 1$, respectively, are numerically computed for different DFs $H$. In the first panel which corresponds to Gumbel-Barnett copula with Weibull-Weibull marginals, it is seen that the maximum range is attained for EGini$_{3}$ and the minimum range is achieved for Pearson correlation. Also as we showed in inequality [17], the results of the table show that the $\rho$-transformed correlation has a wider range than that of Pearson correlation.

The second panel of the table presents the correlations between $X$ and $Y$ for Ali-Mikhail-Haq copula with power-power marginal DFs. In this case, we see that the maximum range coincides with EGini$_{3}$, the next maximum ranges are related to OR-Based, and Gini correlations, respectively, and the minimum range is obtained in EGini$_{0.5}$. Also we see that $\rho_t(X, Y)$ indicates a wider range of correlation between $X$ and $Y$ comparing to
Pearson correlation \( \rho(X, Y) \).

Table 4: The ranges of \( \rho, \rho_t, \) and \( \beta_H \) correlations for two distributions with non-equal marginals.

| Gumbel-Barnett copula with Weibull-Weibull marginals |
|------------------------------------------|
| \( F_\theta(x, y) = 1 - e^{-x^2} - e^{-\sqrt{y}} + e^{-x^2 - \sqrt{y} - \theta x^2 \sqrt{y}}, \quad x > 0, \ y > 0, \ 0 \leq \theta \leq 1 \). |
| Correlation index | Lower bound | Upper bound |
|-------------------|-------------|-------------|
| Pearson           | -0.32420    | 0           |
| \( \rho \)-transformed | -0.43307    | 0           |
| CRE-Based         | -0.48426    | 0           |
| OR-Based          | -0.51759    | 0           |
| E\(G\)ini\(_0.5\) | -0.41563    | 0           |
| Gini              | -0.53692    | 0           |
| E\(G\)ini\(_3\)  | -0.55776    | 0           |

| Ali-Mikhail-Haq copula with power-power marginals |
|-----------------------------------------------|
| \( F_\theta(x, y) = \frac{x(2 - x)y(y^2 - 3y + 3)}{(1 + \theta(y - 1)^3(x - 1)^2)}, \quad 0 < x < 1, \ 0 < y < 1, \ -1 \leq \theta \leq 1 \) |
| Correlation index | Lower bound | Upper bound |
|-------------------|-------------|-------------|
| Pearson           | -0.27099    | 0.39668     |
| \( \rho \)-transformed | -0.27212    | 0.39833     |
| CRE-Based         | -0.26589    | 0.36447     |
| OR-Based          | -0.27387    | 0.45685     |
| E\(G\)ini\(_0.5\) | -0.24790    | 0.29890     |
| Gini              | -0.27887    | 0.45177     |
| E\(G\)ini\(_3\)  | -0.28324    | 0.51025     |

3.1 Some Symmetric Versions

We have to point out here that the Pearson’s and Spearman’s correlation coefficients are both symmetric measures of correlation. However the association measure \( \beta_H(X, Y) \) introduced in this paper is not generally a symmetric measure unless the two r.v.s are exchangeable. There are several ways that one can introduce a symmetric version of the correlation coefficient considered in this paper, i.e., to impose a correlation coefficient with the property \( \beta_H(X, Y) = \beta_H(Y, X) \). Motivated by the works of Yitzhaki and Wodon (2003); Yitzhaki and Olkin (1991); Yitzhaki (2003), in the following, we introduce three measures of correlation based on \( \beta_H(X, Y) \) which are symmetric in terms of \( F \) and \( G \).

(a) The first symmetric version of correlation can be considered as

\[
\tau_H(X, Y) = \frac{1}{2} (\beta_H(X, Y) + \beta_H(Y, X)). \tag{26}
\]
(b) The second symmetric version which can be constructed is based on the approach used by Yitzhaki (2003). Let \( \eta_X = \text{Cov}(X, H^{-1}F(X)) \) and \( \eta_Y = \text{Cov}(Y, H^{-1}G(Y)) \). Define \( \nu_H(X, Y) \) as follows

\[
\nu_H(X, Y) = \frac{\eta_X \beta_H(X, Y) + \eta_Y \beta_H(Y, X)}{\eta_X + \eta_Y}.
\]

Then \( \nu_H(X, Y) \), as a weighted function of \( \beta_H(X, Y) \) and \( \beta_H(Y, X) \), is a symmetric measure of correlation that lies between \([-1, 1]\) and have the requirements of a correlation coefficient described in Theorem 1.

(c) The third symmetric index which can be imposed based on \( \beta_H(X, Y) \) is as follows (see, Yitzhaki and Wodon (2003)). With \( \eta_X \), and \( \eta_Y \), as defined in (b), let \( \bar{\beta}_H(X, Y) = 1 - \beta_H(X, Y) \) and \( \bar{\beta}_H(Y, X) = 1 - \beta_H(Y, X) \). Consider \( \bar{\nu}_H(X, Y) \) as

\[
\bar{\nu}_H(X, Y) = \frac{\eta_X \bar{\beta}_H(X, Y) + \eta_Y \bar{\beta}_H(Y, X)}{\eta_X + \eta_Y} = 1 - \nu_H(X, Y).
\]

Then \( \bar{\nu}_H(X, Y) \) which is a weighted function of \( \bar{\beta}_H(X, Y) \) and \( \bar{\beta}_H(Y, X) \) is symmetric in \( F \) and \( G \) and ranges between \([0, 2]\). Yitzhaki and Wodon (2003) showed that \( \bar{\nu}_H(X, Y) \), in the case that \( H \) is uniform distribution gives a measure, called Gini index of mobility, that provides a consistent setting for analysis of mobility, inequality and horizontal equity. It can be easily shown that \( \bar{\nu}_H(X, Y) \) can be also presented as

\[
\bar{\nu}_H(X, Y) = \frac{\text{Cov} \left( X - Y, H^{-1}F(X) - H^{-1}G(Y) \right)}{\eta_X + \eta_Y}.
\]

In the following, we give an example that these symmetric measures are calculated.

**Example 4.** Consider the Gumbel-Barnett copula with two different Weibull distributions as marginals and the joint DF given in Example 3. Let \( \theta = 1 \) which corresponds to highest dependency between r.v.s \( X \), and \( Y \). Then the joint DF of \( X \) and \( Y \) is written as

\[
F(x, y) = 1 - e^{-x^2} - e^{-\sqrt{y}} + e^{-x^2 - \sqrt{y} - x^2 \sqrt{y}}, \quad x > 0, \ y > 0.
\]

Table 5 presents the values of correlations \( \beta_H(X, Y) \) and \( \beta_H(Y, X) \), symmetric correlations \( \tau_H(X, Y) \), \( \nu_H(X, Y) \) and \( \bar{\nu}_H(X, Y) \) for different distributions \( H \).

## 4 A Decomposition Formula

In this section we give a decomposition formula for \( C(T, Y) \), which provides some results on the connection between the variability of sum of a number of r.v.s in terms of sum of
| Table 5: The values of symmetric correlation coefficients for Example 4. |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| Index           | $\beta_H(X, Y)$ | $\beta_H(Y, X)$ | $\tau_H(X, Y)$ | $\nu_H(X, Y)$   |
| CRE-Based       | -0.48426        | -0.29817        | -0.39121       | -0.31673        |
| OR-Based        | -0.51759        | -0.47762        | -0.49761       | -0.48267        |
| EGI50.04       | -0.41563        | -0.12179        | -0.26871       | -0.13873        |
| Gini           | -0.53692        | -0.59375        | -0.56534       | -0.58537        |
| EGI3           | -0.55776        | -0.80720        | -0.68248       | -0.76379        |

Variabilities of each r.v. In a reliability engineering point of view, consider a system with standby components with the following structure. We assume that the system is built of $n$ units with lifetimes $X_1, \ldots, X_n$ which will be connected to each other sequentially as follows. Unit number 1 with lifetime $X_1$ starts operating and in the time of failure, the unit number 2 with lifetime $X_2$ starts working automatically, and so on until the $n$th unit, with lifetime $X_n$, fails. Hence, the lifetime of the system, denoted by $T$, would be $T = \sum_{i=1}^{n} X_i$. Assume that $\mu_i = E(X_i)$ denotes the mean time to failure of unit number $i$ and $\mu = E(T) = \sum_{i=1}^{k} \mu_i$ denotes the mean time to failure of the system.

Let again for any two r.v.s $X$ and $Y$ with DFs $F$ and $G$, respectively, we denote $\text{Cov}(X, G^{-1}F(X))$ by $\mathcal{C}(X, Y)$. Now we have the following result.

**Theorem 5.** For any r.v. $Y$ with DF $G$, we have the following decomposition for $\mathcal{C}(T, Y)$ in terms of $\mathcal{C}(X_i, Y)$, $i = 1, 2, \ldots, n$.

$$\mathcal{C}(T, Y) = \sum_{i=1}^{n} \beta_G(X_i, T) \mathcal{C}(X_i, Y),$$

where $\beta_G(X_i, T)$ is the $G$-transformed correlation between the system lifetime $T$ and component lifetime $X_i$ defined in [15].

**Proof.** Let $F_{X_i}$ and $F_T$ denote the DFs of component lifetime $X_i$ and the system lifetime $T$, respectively. From the covariance properties of sum of r.v.s, we can write

$$\mathcal{C}(T, Y) = \text{Cov}(T, G^{-1}F_T(T))$$

$$= \sum_{i=1}^{n} \text{Cov}(X_i, G^{-1}F_T(T))$$

$$= \sum_{i=1}^{n} \frac{\text{Cov}(X_i, G^{-1}F_T(T))}{\text{Cov}(X_i, G^{-1}F_{X_i}(X_i))} \text{Cov}(X_i, G^{-1}F_{X_i}(X_i))$$

$$= \sum_{i=1}^{n} \beta_G(X_i, T) \mathcal{C}(X_i, Y).$$

\qed
Corollary 2. It is interesting to note that the correlation between the system lifetime $T$ and its component lifetime $X_i$, i.e., $\beta_G(X_i, T)$ is always nonnegative. This is so because in $\beta_G(X_i, T), G^{-1}F_T(T)$ is trivially an increasing function of $X_i$, as $T$ is increasing function of $X_i$. Hence, Cov$(X_i, G^{-1}F_T(T))$ is nonnegative which, in turn, implies that $\beta_G(X_i, T)$ is nonnegative. Thus, we have

$$0 \leq \beta_G(X_i, T) \leq 1.$$  \hfill (27)

This result shows that the G-covariance between the system lifetime $T$ and r.v. $Y$ can be decomposed as a combination of the G-covariance between components lifetime and r.v. $Y$. From Theorem 5 and relation (27), we conclude that

$$C(T, Y) \leq \sum_{i=1}^{n} C(X_i, Y).$$  \hfill (28)

That is, the G-covariance between the system lifetime and r.v. $Y$ is less than the sum of G-covariance between its components and r.v. $Y$. In particular when the $X_i$’s are identical r.v.s, we have $C(T, Y) \leq nC(X_1, Y)$. In this situation, if we assume that $G = F_{X_1}$ then

$$C(T, X_i) \leq n\text{Var}(X_1), \quad i = 1, \ldots, n.$$  

Based on Corollary 2, the following inequalities are obtained for some well known measures of disparity as special cases:

(a) If $G = F_T$, then we get

$$\text{Var}(T) \leq \sum_{i=1}^{n} C(X_i, T).$$

(b) In the case that $G(x) = 1 - e^{-\frac{x}{k}},$ $x > 0$, $k > 0$, the Weibull distribution with shape parameter $1/k$, we obtain

$$\mathcal{E}_k(T) \leq \sum_{i=1}^{n} \mathcal{E}_k(X_i), \quad k = 1, 2, \ldots,$$

where the $\mathcal{E}_k(\cdot)$ is the GCRE defined in (12). In the special case where $k = 1$, we obtain the following inequality regarding CRE.

$$\mathcal{E}_1(T) \leq \sum_{i=1}^{n} \mathcal{E}_1(X_i).$$

Thus, it is concluded that the uncertainty of a stand by system lifetime, in the sense of CRE, is less than the sum of uncertainties of the its components lifetime. As a result we can also conclude equivalently that for the system described above

$$E(m_T(T)) \leq \sum_{i=1}^{n} E(m_{X_i}(X_i)),$$

where $m_T$ and $m_{X_i}$ are the MRL’s of the system and the components, respectively; see also, Nasr-Esfahani and Asadi (2018).
(c) Consider $G(\cdot)$ as the DF given in (14). Then, for $\nu > 0$,

$$\text{EGini}_\nu(T) \leq \sum_{i=1}^{n} \text{EGini}_\nu(X_i).$$

For $\nu = 2$, which corresponds to $G(x)$ as uniform distribution on $(0, 1)$, we get

$$\text{GMD}(T) \leq \sum_{i=1}^{n} \text{GMD}(X_i), \tag{29}$$

where $\text{GMD}(\cdot)$ is the Gini’s mean difference. This result was already obtained by Yitzhaki and Schechtman (2013).

5 Concluding Remarks

In the present article, we introduced a unified approach to construct a correlation coefficient between two continuous r.v.s. We assumed that the continuous r.v.s $X$ and $Y$ have a joint distribution function $F(x, y)$ with marginal distribution functions $F$ and $G$, respectively. We first considered the covariance between $X$ and transformation $G^{-1}F(X)$, i.e., $\text{Cov}(X, G^{-1}F(X))$. The function $G^{-1}(\cdot)$ is known in the literature as the $Q$-transformation (or sample transmutation maps). We showed that some well known measures of variability such as variance, Gini mean difference and its extended version, cumulative residual entropy and some other disparity measures can be considered as special cases of $\text{Cov}(X, G^{-1}F(X))$. Motivated by this, we proposed a unified measure of correlation between the r.v.s $X$ and $Y$ based on $\text{Cov}(X, H^{-1}G(Y))$, where $H$ is a continuous distribution function. We showed that the introduced measure, which subsumes some well known measures of associations such as Gini and Pearson correlations for special choices of $H$, has all requirements of a correlation index under some mild condition on DF $H$. For example it was shown that it lies between $[-1, 1]$. When the joint distribution of $X$ and $Y$ is bivariate normal, we showed that the proposed measure, for any choice of $H$, equals the Pearson correlation coefficient. We proved, under some conditions that for our unified association index, the lower and upper bounds of the interval $[-1, 1]$ are attainable by joint Fréchet bivariate minimal and maximal distribution functions, respectively. A special case of the introduced correlation in this paper, provided a variant of Pearson correlation coefficient $\rho(X, Y)$, which measures with the property that its absolute value is always greater than or equal to the absolute value of $\rho(X, Y)$. Since the proposed measure of correlation is asymmetric, some symmetric versions of that were also discussed. Several examples of bivariate DFs of $X$ and $Y$ were presented in which the correlation is computed for different choices of $H$. Finally, we presented a decomposition formula for $\text{Cov}(X, G^{-1}F(X))$ in which the r.v. $X$ was considered as the sum of $n$ r.v.s. As an application of the decomposition formula, some results were provided on the connection
between variability measures of a standby system in terms of the variability measures of its components.

The r.v.s that we considered in this article, were assumed to be continuous. One interesting problem which can be considered as a future study is to investigate the results for the case that the r.v.s are arbitrary (in particular discrete r.v.s). Another important problem which can be investigated is to propose some estimators for $\beta_H(X,Y)$ for different choices of $H$. In particular, we believe that providing estimators for $\rho_t(X,Y)$ and exploring their properties may be of special importance, for measuring the linear correlation between the real data collected in different disciplines and applications.

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