ON THE CLASSIFICATION OF FRACTAL SQUARES

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Abstract
In the previous paper [K. S. Lau, J. J. Luo and H. Rao, Topological structure of fractal squares, Math. Proc. Camb. Phil. Soc. 155 (2013) 73–86], Lau, Luo and Rao completely classified the topological structure of so called fractal square \( F \) defined by \( F = (F + D)/n \), where \( D \subseteq \{0, 1, \ldots, n-1\}^2 \), \( n \geq 2 \). In this paper, we further provide simple criteria for the \( F \) to be totally disconnected, then we discuss the Lipschitz classification of \( F \) in the case \( n = 3 \), which is an attempt to consider non-totally disconnected sets.

Keywords: Fractal Square; Totally Disconnected; Congruence; Lipschitz Equivalence.

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1. INTRODUCTION

For $n \geq 2$, let $D = \{d_1, \ldots, d_m\} \subseteq \{0, 1, \ldots, n-1\}$ be a digit set with cardinality $\#D = m$, and let $\{S_i\}_{i=1}^m$ be an iterated function system (IFS) on $\mathbb{R}^2$, where $S_i(x) = \frac{1}{n}(x + d_i)$ where $d_i \in D$. Then there exists a unique self-similar set $F \subset \mathbb{R}^2$ satisfying the set equation,

$$F = \bigcup_{i=1}^m S_i(F) = \frac{1}{n}(F + D)$$

(1.1)

which is called a fractal square. The geometric construction of a fractal square seems like that of middle third Cantor set; First we divide a unit square into $n^2$ small equal squares of which $m$ small squares are kept and the rest discarded, the positions of the $m$ chosen squares are parallel line segments.

Let $\mathcal{F}_n$ denote the collection of all fractal squares satisfying (1.1). It is easy to see that the fractal squares in $\mathcal{F}_n$ have the common Hausdorff dimension $\log m/\log n$ but distinct topological structures. In the above three types, the fractal squares of type (i) are called Cantor-type sets which play an important role in fractal geometry and dynamical systems, so we will give a further study on this case. Especially, we provide simple criteria for the existence of type (i) in $\mathcal{F}_{n,m}$.

Two sets $E$ and $F$ on $\mathbb{R}^d$ are said to be Lipschitz equivalent, and denoted by $E \approx F$, if there is a bi-Lipschitz map $g$ from $E$ onto $F$, i.e. $g$ is a bijection and there is a constant $C > 0$ such that

$$C^{-1}|x - y| \leq |g(x) - g(y)| \leq C|x - y|, \quad \forall x, y \in E.$$  

It is well-known that if $E \approx F$ then they have the same Hausdorff dimension, but the converse is not true in general. Lipschitz classification of sets has attracted a lot of interests in the literature. In fractal geometry, the fundamental works were due to Cooper and Pignatelli and Falconer and Marsh on Cantor sets. Recently, many generalizations on totally disconnected self-similar sets (Cantor-type sets) have been extensively studied. But there are few results on non-totally disconnected cases.

Motivated by that, our aim of the paper is to make an attempt in this direction.

For $\mathcal{F}_{n,m}$, the Lipschitz equivalence class is denoted by $\mathcal{F}_{n,m}/\approx$. When $n = 3, m = 2, 3, 4, 5$, we have

**Theorem 1.1.** $\#(\mathcal{F}_{3,2}/\approx) = 1$; $\#(\mathcal{F}_{3,3}/\approx) = \#(\mathcal{F}_{3,4}/\approx) = 2$; and $\#(\mathcal{F}_{3,5}/\approx) \leq 10$.

The first three classes are simple, while $\mathcal{F}_{3,5}$ is complicated, as it contains all the three types of fractal squares. The complete classification seems very difficult, but we conjecture that $\#(\mathcal{F}_{3,5}/\approx) = 10$ (see remarks in Sec. 4).

The paper is organized as follows: In Sec. 2, we discuss several criteria for a fractal square to be totally disconnected. We prove Theorem 1.1 by using various methods (see Theorems 3.3, 3.4, 3.6, and 3.10) in Sec. 3, and give some remarks on other cases in Sec. 4. Finally, we include all figures of fractal squares in $\mathcal{F}_{3,5}, \mathcal{F}_{3,6}, \mathcal{F}_{3,7}$ and $\mathcal{F}_{3,8}$ in an appendix.

2. CRITERIA FOR TOTAL DISCONNECTEDNESS

For fractal square $F$ as in (1.1), we define a set on $D$ by

$$E = \{(d_i, d_j) : (F + d_i) \cap (F + d_j) \neq \emptyset, \quad d_i, d_j \in D\}.$$  

We say that $d_i, d_j$ are $E$-connected if there exists a finite sequence $\{d_{i_1}, \ldots, d_{i_k}\} \subset D$ such that $d_{i_1} = d_i, d_{i_k} = d_j$, and $(d_{i_l}, d_{i_{l+1}}) \in E, 1 \leq l \leq k - 1$. The following criterion for connectedness was first proved by Hata and rediscovered by Kirat and Lau.

**Lemma 2.1.** A fractal square $F$ with a digit set $D$ is connected if and only if any two $d_i, d_j \in D$ are $E$-connected.

Let $B = [0,1]^2$ be the unit square, $\Sigma = \{1, \ldots, m\}$. Let $D_1 = D$ and $D_{k+1} = D + nD_k$, then

$$D_k = \{d_u := d_{u_1} + nd_{u_2} + \cdots + n^{k-1}d_{u_k} : u = j_1 \cdots j_k \in \Sigma^k\}, \quad k \geq 1.$$  

(2.1)

Denote by $S_u(B) = S_{j_1} \circ \cdots \circ S_{j_k}(B) = n^{-k}(B + d_u)$, we call such $S_u(B)$ (or any translation of $n^{-k}$
scaling of $B$) a $k$-square. Obviously, we have

$$F = \bigcap_{k=1}^{\infty} \bigcup_{u \in \Sigma^k} S_u(B).$$

(2.2)

By letting $F(k) = \bigcup_{u \in \Sigma^k} S_u(B)$, we call $F(k)$ a $k$th approximation of the fractal square $F$.

**Definition 2.2.** In $B$, a vertical path is a curve starting at point $(x,0)$ and ending at point $(x,1)$ for some $x \in [0,1]$; a horizontal path is a curve starting at point $(0,y)$ and ending at point $(1,y)$ for some $y \in [0,1]$; a cross path is the union of one vertical path and one horizontal path; a $\lambda$-path is the union of three arcs connecting an interior point of $B$ and three corners of $B$, respectively. (see Fig. 1.)

Obviously, a vertical path and a horizontal path meet each other, so a cross path is connected and reaches four points of the four sides of $B$, respectively. A $\lambda$-path is also connected. Intuitively, the shape of the $\lambda$-path looks like the letter "$\lambda$" or its rotations. The simplest $\lambda$-path may be the union of a diagonal and half of the other. From (2.2), it can be seen that $B \setminus F$ contains a vertical path if and only if there exist an integer $k \geq 1$ and a chain of edge-adjacent $k$-squares outside $F(k)$ which begins with $[\frac{x}{2^n}, \frac{x+1}{2^n}] \times [0, \frac{1}{2^n}]$ and ends with $[\frac{x}{2^n}, \frac{x+1}{2^n}] \times [1 - \frac{1}{2^n}, 1]$ for some $j \in \{0, 1, \ldots, n^k - 1\}$.

Similarly for the cross path and the $\lambda$-path. (see Fig. 2.)

The main use of the above four paths is to verify the total disconnectedness of $F$.

**Proposition 2.3 (Ref. 11).** A fractal square $F$ is totally disconnected if and only if $B \setminus F$ has a cross path.

The following criterion is more convenient for many cases in our consideration. Note that $F$ contains a vertical (horizontal) line segment if and only if $F^{(1)}$ does.

**Theorem 2.4.** A fractal square $F$ is totally disconnected if and only if $F$ contains no vertical line segments and $B \setminus F$ contains a vertical path.

**Proof.** If $F$ is totally disconnected, then the necessity is obvious since $B \setminus F$ is open and pathwise connected. For the converse part, let $C$ be a component of $F$, and $\text{Proj}_x C$ denote the orthogonal projection of $C$ onto the $x$-axis. If $C$ contains a vertical line segment, then $\text{Proj}_x C$ is connected and $C \setminus \text{Proj}_x C$ is empty. If $C$ contains no vertical line segments, then $\text{Proj}_x C$ is empty and $C \setminus \text{Proj}_x C$ is connected. Therefore, $C$ is a connected component of $F$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fractal_path.png}
\caption{From left to right: A vertical path, a cross path and a $\lambda$-path.}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{square_path.png}
\caption{Paths covered by squares.}
\end{figure}
projection of $C$ on the $x$-axis, then $\text{Proj}_x C$ is also pathwise connected. We claim $|\text{Proj}_x C| = 0$. Indeed, if otherwise, $|\text{Proj}_x C| > 0$. Choose an integer $k$ large enough such that

$$C \cap \left[ \frac{i}{n^k}, \frac{i+1}{n^k} \right) \times [0,1] \neq \emptyset,$$

$$C \cap \left[ \frac{i+1}{n^k}, \frac{i+2}{n^k} \right) \times [0,1] \neq \emptyset,$$

$$C \cap \left[ \frac{i+2}{n^k}, \frac{i+3}{n^k} \right) \times [0,1] \neq \emptyset$$

hold for some $i \in \{0,1, \ldots, n^k-3\}$. Let $I_j = \left[ \frac{2j+1}{2n^k}, \frac{2j+2}{2n^k} \right)$, $j = 0,1, \ldots, n^k-1$ be the $k$-squares in the rectangle $\left[ \frac{1}{2n^k}, \frac{1}{2n^k} \right) \times [0,1]$. Suppose $\alpha$ is a vertical path of $B \backslash F$. If $I_j$ belongs to the $k$th approximation of $F$, we denote it by $S_u(B)$ for some $u \in \Sigma^k$. Then $S_u(B;F)$ contains a path $S_u(\alpha) := \alpha_j$; if not, then $I_j \subset B \backslash F$. We can take a vertical line $\beta_j$ in $I_j$ with the same horizontal coordinate as the $\alpha_j$. Hence we construct a vertical path in $B \backslash F$ by joining the paths $\alpha_j, \beta_j$ which separate the component $C$. Thus, $C$ must lie in one vertical line. By the assumption, $C$ cannot be a vertical line segment, which implies $C$ is just a singleton. Therefore, $F$ is totally disconnected.

**Proposition 2.5.** Let $F$ be a fractal square. If $B \backslash F$ contains a $\lambda$-path, then $F$ is totally disconnected. Conversely, if $F$ is totally disconnected and at most one corner of $B$ is in $F$, then there exists a $\lambda$-path in $B \backslash F$.

**Proof.** The proof is essentially the same as above. We mention that if $F$ is totally disconnected and at most one corner of $B$ is in $F$, then $B \backslash F$ contains at least three corners of $B$. Hence we can construct a $\lambda$-path in $B \backslash F$ by using the pathwise connectedness of $B \backslash F$.

**Theorem 2.6.** If $m \leq n^2 - n - \left\lfloor \frac{n}{2} \right\rfloor$ then $F_{n,m}$ contains a totally disconnected fractal square.

**Proof.** Let $D_1 = \{(i, j) : i = 0, 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor, j = 1, 0, \ldots, n-1\}$, and a digit set $D = \{0,1, \ldots, n-1\} \setminus D_1$. Then $\#D = n^2 - n - \left\lceil \frac{n}{2} \right\rceil = m$, and $F = \frac{1}{n}(F + D)$ belongs to $F_{n,m}$. Since the set $D_1$ determines a $\lambda$-path in $B \backslash F^{[2]}$, so in $B \backslash F$, it implies that $F$ is totally disconnected by Proposition 2.5.

3. **CLASSIFICATION OF FRAC TAL SQUARES WHEN $n = 3$**

**Lemma 3.1 (Refs. 6 and 12).** Let $F, F' \in F_{n,m}$ be two fractal squares. If $F, F'$ are totally disconnected then $F \simeq F'$.

However, if two fractal squares are not totally disconnected, there are few results about their Lipschitz equivalence. In this section, we make an attempt on some special cases, such as connected fractal squares or fractal squares containing parallel line segments. We try to classify the Lipschitz equivalence classes of $F_{n,m}$ for $n = 3, m = 2, 3, 4, 5$.

For convenience, we use an $n \times n$ matrix $M = (m_{ij})_{1 \leq i,j \leq n}$ to represent a fractal square $F$ where

$$m_{ij} = \begin{cases} 1 & \text{if } (j-1, n-i) \in D \\ 0 & \text{otherwise} \end{cases}$$

We call $M$ the label matrix of $F$. It is easy to see that there is a one-to-one correspondence between $F$ and $M$. So we prefer to use the label matrix to depict the fractal square for simplicity.

Geometrically, two sets are called congruent if one can be transformed into the other by some rigid motions. From (2.2), it is seen that two fractal squares are congruent if their first approximations are congruent, which can be immediately observed from the label matrices.

**Lemma 3.2.** Let $g : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a linear transformation defined by $g(x) = Ax + v$ where $A$ is a $d \times d$ invertible matrix and $v \in \mathbb{R}^d$. Then $g$ is a bi-Lipschitz map.

**Proof.** Since $A$ is invertible, for any $x, y \in \mathbb{R}^d$, we have

$$|A(x - y)| \leq \|A\| |x - y|$$

and

$$|x - y| = |A^{-1}A(x - y)| \leq \|A^{-1}\| \|A(x - y)|,$$

where $\|A\|$ denotes the norm of matrix $A$. Hence

$$|A^{-1}|^{-1}|x - y| \leq |g(x) - g(y)| \leq \|A\||x - y|$$

proving that $g$ is a bi-Lipschitz map.

**Theorem 3.3.** $\#(F_{3,2}/\simeq) = 1$; $\#(F_{3,3}/\simeq) = \#(F_{3,4}/\simeq) = 2$.

**Proof.** Since $\log 2/\log 3 < 1$, all the fractal squares in $F_{3,2}$ are totally disconnected.\footnote{Remark} Hence $\#(F_{3,2}/\simeq) = 1$ by Lemma 3.1.
In $F_{3,3}$, every fractal square is either totally disconnected or connected (a line segment). Hence $\#(F_{3,3}/\sim) = 2$. In $F_{3,4}$, the totally disconnected fractal squares form one Lipschitz equivalence class by Lemma 3.1. Moreover, it can be easily checked that, up to congruence, there are only 6 different non-totally disconnected fractal squares, denoted by $F_i = \frac{1}{2}(F_1 + D_1)$ where $i = 1, \ldots, 6$. The corresponding label matrices are listed as follows:

We define a linear transformation $g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by $g(x) = Ax$ where $A = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $D_2 = AD_1$. Hence $AF_1 = \frac{1}{2}(AF_1 + AD_1) = \frac{1}{2}(AF_1 + D_2)$, implying $F_2 = g(F_1)$ by the uniqueness of attractor. So we get $F_1 \simeq F_2$ as $g$ is a bi-Lipschitz map by Lemma 3.2.

Similarly, it is easy to verify that $D_1 = A_1D_4$, $D_2 = A_2D_5$, $D_6 = A_3D_4$ and $D_3 = A_3D_5$, where

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$  

Therefore, $F_1 \simeq F_2 \simeq \cdots \simeq F_6$, proving $\#(F_{3,4}/\sim) = 2$.

In $F_{3,5}$, the total number of fractal squares is $C_5^3$. Up to congruence, there are 21 distinct fractal squares among them. However, in the rest of this section, we will show that there are at most 10 Lipschitz equivalence classes. First we use $\{F_i\}_{i=1}^{21}$ to denote 21 fractal squares, and each $F_i$ takes the following $M_i$ as its label matrix:

**Type (i):** totally disconnected fractal squares:

Type (ii): connected fractal squares:

$$M_5 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \quad M_4 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad M_6 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad M_7 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$  

Type (iii): fractal squares containing parallel line segments:

By the criteria in the last section, especially Theorem 2.4, fractal squares of type (i) are indeed totally disconnected. Hence $F_i, 1 \leq i \leq 5$ are Lipschitz equivalent by Lemma 3.1. For types (ii) and (iii), we have

$$M_{10} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad M_{11} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}. $$
Theorem 3.4. \( F_t \cong F_0; \quad F_0 \cong F_{t_0} \cong F_{t_1}; \quad F_{t_2} \cong F_{t_3}; \quad F_{t_4} \cong F_{t_5} \cong F_{t_6}; \quad F_{t_7} \cong F_{t_8}. \)

Proof. Let \( F_i = \frac{1}{3}(F_{i+1} + D_i), i = 1, \ldots, 21, \) and let

\[
A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1/2 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.
\]

Then \( D_1 = A_1D_7, D_1 = A_2D_5, D_11 = A_3D_9, D_15 = A_4D_4, D_19 = A_5D_8. \) By defining linear transformations \( g_i(x) = A_i x \) for \( i = 1, 2, 3, 4, \) we can obtain \( g_1(F_1) = F_2, g_2(F_2) = F_3, g_3(F_3) = F_4, \) and \( g_4(F_4) = F_5, \) where \( g_i \) are bi-Lipschitz maps.

Moreover, let \( g_v(x) = A_v x + v, \quad g_v(x) = A_v x + \frac{v}{3}, \) where \( v = \frac{1}{3}[1, 1, 1], \quad v' = \frac{1}{3}[1, 1, 0]. \) Then \( D_3 = A_2 D_{10} + 2v \) and \( D_{20} = A_0 D_{10} + v' \). Hence

\[
g_1(F_{t_0}) = g_5 \left( \frac{1}{3}(F_{t_0} + D_{t_0}) \right) = \frac{1}{3}(A_5 F_{t_0} + A_5 D_{t_0} + 3v)
\]

and

\[
g_1(F_{t_0}) = g_6 \left( \frac{1}{3}(F_{t_0} + D_{t_0}) \right) = \frac{1}{3}(A_6 F_{t_0} + A_6 D_{t_0} + \frac{3}{2}v')
\]

That implies \( g_1(F_{t_0}) = F_5 \) and \( g_6(F_{t_0}) = F_{t_0}, \) finishing the proof.

Lemma 3.5. \( F_t \) is a connected set which equals the closure of a union of infinitely countable circles, and so does \( F_0. \)

Proof. The connectedness can be obtained easily by Lemma 2.1. Let \( C = \{0, 1\} \times \{0, 1\} \cup \{0, 1\} \times \{0, 1\}, \) then \( C \) is a circle, and \( F_t = F_0, F_1 = C + D \) (see Fig. 3). By induction, we can get for any \( k \geq 1, \)

\[
\frac{C}{3} + D_{k+1} = \frac{C}{3} + \frac{D}{3} + \cdots + \frac{D}{3} \subset F_t.
\]

Hence

\[
\bigcup_{k=1}^{\infty} \left( \frac{C}{3} + D_{k+1} \right) \subset F_t.
\]

On the other hand, for any \( x \in F_t, \) there exists a sequence \( \{d_k\}, \) with \( d_k \in D \) such that \( x = \sum_{k=0}^{\infty} \frac{1}{3^k} d_k. \) By (2.1), for all \( k \geq 1, \) we have \( \sum_{k=1}^{\infty} \frac{1}{3^k} d_k \in \frac{1}{3} D_k \subset \bigcup_{k=1}^{\infty} \left( \frac{C}{3} + \frac{D_{k+1}}{3} \right) \), then \( x \in \bigcup_{k=1}^{\infty} \left( \frac{C}{3} + \frac{D_{k+1}}{3} \right). \)

We omit the proof for \( F_0 \) as it is the same as above.

A nonempty compact set \( T \subset \mathbb{R}^2 \) called a tree-like set if for any two distinct points \( x, y \in T, \) there is a unique path (or curve) in \( T \) connecting them.

Theorem 3.6. \( F_0 \) and \( F_t \) are not homeomorphic, hence are not Lipschitz equivalent.
Proof. Lemma 3.5 implies $F_0$ is not a tree-like set, so it suffices to show that $F_0$ is a tree-like set. By Lemma 2.1, $F_0$ is connected, hence is path-wise connected. Thus, for any two distinct points $x, y \in F_0$, there is a path $\pi(x,y)$ in $F_0$ connecting them. Next we show $F_0$ is a tree-like set by proving the uniqueness of the path $\pi(x,y)$.

Following notation of (2.2), let $u, v \in \Sigma^k$. It is known that if $S_u(B) \cap S_v(B)$ is singleton then $S_u(B) \cap S_v(B) = \emptyset$ for any $i, j \in \Sigma$; if $S_u(B) \cap S_v(B)$ is a line segment, say $L_{uv}$, then there exists a unique pair $(i,j) \in \Sigma \times \Sigma$ such that $S_u(B) \cap S_v(B)$ is a line segment with length $|L_{uv}| = \frac{|L_{uv}|}{3} = \frac{1}{3^k}$ (see Fig. 4). Define

$$E_k = \left\{ (d_u,d_v) : |S_u(B) \cap S_v(B)| = \frac{1}{3^k} \right\}$$

to be the set of edges for $D_k$. Then $(D_k,E_k)$ forms a tree by the argument above for any $k \geq 1$.

Assume $\pi'(x,y)$ is a path different from $\pi(x,y)$. Then there exists a point $z_0 \in \pi'(x,y) \setminus \{x,y\}$ such that $e_0 := \inf\{|z - z_0| : z \in \pi'(x,y) \setminus \{x,y\}\} > 0$.

Since $x, y \in F_0 \subset \bigcup_{i \in \Sigma} S_i(B)$ and $x \neq y$, there is a large enough $k_0 \geq \log_{1/2} \frac{3^k}{e_0}$ such that, for any $k \geq k_0$, there exist $u, v \in \Sigma^k$ such that $x \in S_u(B), y \in S_v(B)$ and $S_u(B) \cap S_v(B) = \emptyset$. By the tree structure of $(D_k,E_k)$, we can find a unique finite sequence $u_1, \ldots, u_k$ satisfying $u = u_1, \ldots, v = u_i$ and $(u_i, u_{i+1}) \in E_k$ for $i = 1, \ldots, k - 1$.

Thus $\pi(x,y), \pi'(x,y) \subset \bigcup_{i=1}^k S_{u_i}(B)$. Suppose $z_0 \in S_{u_0}(B)$, then let $z_1 \in \pi(x,y) \cap S_{u_0}(B)$, we get

$$|z_0 - z_1| \leq \text{diam}(S_{u_0}(B)) = \frac{\sqrt{2}}{3^k} < e_0.$$

That contradicts $|z_0 - z_1| \geq e_0$. \hfill \Box

Let $E \subset \mathbb{R}^2$ be a nonempty connected set. We say a point $a \in E$ is a $k$-branch point if $E \setminus \{a\}$ consists of $k$ components. It is known that $k$-branch points are topological invariants. A 1-branch point of $E$ is often called a top of $E$. The following lemma is obvious.

Lemma 3.7. Suppose that $x$ is a $k$-branch point in $E \subset \mathbb{R}^2$. Then for any $U \subset \mathbb{R}^2$, $E \setminus U$ has at least $k$ components provided that $U$ contains $x$ and the diameter $U$ is small enough.

Proof. Let $E_1, E_2, \ldots, E_k$ be the components of $E \setminus \{x\}$. Let $\delta$ be the minimum of the diameters diam$(E_j), 1 \leq j \leq k$. If diam$(U) < \delta/2$, then $(E \setminus U) \cap E_j$ is not empty and contributes at least one component to $E \setminus U$. \hfill \Box

Let $F \in F_{m,n}$ and $\Sigma = \{1, \ldots, m\}$. For any point $x \in F$, there exists an infinite word $i_1 i_2 \cdots$ such that

$$\{x\} = \bigcap_{k=1}^\infty S_{i_k-i_{k-1}}(F),$$

where $i_j \in \Sigma$ and $S_{i_k-i_{k-1}} = S_{i_k} \circ \cdots \circ S_{i_1}$. We call $i_1 i_2 \cdots$ a coding of $x$, and $(F)_{i_1-i_{k-1}} := S_{i_1-i_k}(F)$ a cylinder of $F$.

Lemma 3.8. Let $\{S_i\}_{i=1}^m$ be the IFS of $F_0$, which is depicted by Fig. 4a. Suppose $x$ belongs to $F_0$.

![Fractal Square](image)
Then

(i) If the coding of \( x \) is unique and contains finitely many symbols 2, 4, then \( x \) is a 1-branch point.

(ii) Suppose the coding of \( x \) is unique and contains infinitely many symbols 2, 4. If \( x \) is not eventually 2, then \( x \) is a 2-branch point; otherwise \( x \) is a 3-branch point.

(iii) If \( x \) has more than one coding, then \( x \) is either a 2-branch or a 4-branch point.

Proof. (i) Clearly if the coding of \( x \) does not contain the symbols 2, 4, then \( x \) is a 1-branch point, namely, \( x \) is a top of \( F_k \) (see Fig. 4c). Indeed, we can show by induction that if we delete the cylinder \( (F_k)_{i_1...i_k} \) from \( F_k \), then the resulting set is still connected. Hence, \( x \) is a 1-branch point by Lemma 3.7.

Now suppose that \( i_1i_2 \cdots \) contains symbols 2, 4, say \( i_k \) is the last symbol in 2, 4 and \( i_j \) belongs to \( \{1, 3, 5\} \) for all \( j > k \). Then \( x \) is a top of the cylinder \( (F_k)_{i_1...i_k} \). If \( x \) is not a top of \( F_k \), then \( x \) must belong to another cylinder, which means \( x \) has more than one coding.

(ii) Suppose \( i_1i_2 \cdots \) contains infinitely many symbols 2, 4, and it is not eventually 2. This means 4 will appear infinitely many times. Suppose \( i_k = 4 \). Let \( U = (F_k)_{i_{k+1}} \setminus \{ (F_k)_{i_1...i_k} \} \). Since \( (F_k)_{i_1...i_k} \setminus U \) consists of two components and \( U \) does not intersect other cylinders of \( F_k \), we conclude that \( F_k \setminus U \) has only two components. Therefore, \( x \) is a 2-branch point.

Now suppose that \( i_1i_2 \cdots \) is eventually 2. Suppose \( i_k = 2 \) for all \( k \geq k_0 \). Delete \( (F_k)_{i_1...i_k} \) but keep the three intersecting points with other cylinders, the resulting set consists of three components. Hence, \( x \) is a 3-branch point.

(iii) Now suppose \( x \) has more than one coding. If \( x \) has no coding of eventually 2, then \( x \) must be the common top of two cylinders and it is a 2-branch point. If \( x \) has a coding of eventually 2, say \( i_1i_2...i_k2^n \). If we delete \( x \), then \( (F_k)_{i_1...i_k} \) is partitioned into three pieces. The other part of \( F_k \) either connects to the top of \( (F_k)_{i_1...i_k} \) or connects to \( x \). Hence, \( x \) is a 4-branch point.

Lemma 3.9. Let \( \{ S_1 \}_{i=1}^r \) be the IFS of \( F_k \), which is depicted by Fig. 5a. Suppose \( x \) belongs to \( F_k \). Then

(i) If the coding of \( x \) is unique and contains finitely many symbols 5, then \( x \) is a 1-branch point.

(ii) If the coding of \( x \) is unique and contains infinitely many symbols 5, then \( x \) is a 4-branch point.

(iii) If \( x \) has more than one coding, then \( x \) is a 2-branch point.

Proof. (i) Clearly if the coding of \( x \) does not contain the symbol 5, then \( x \) is a 1-branch point, namely, \( x \) is a top corner of \( F_k \) (see Fig. 5c). Indeed, we can show by induction that if we delete the cylinder \( (F_k)_{i_1...i_k} \) from \( F_k \), then the resulting set is still connected. Hence \( x \) is a 1-branch point by Lemma 3.7.

Now suppose that \( i_1i_2 \cdots \) contains symbol 5, say \( i_k \) is the last 5 and \( i_j \in \{1, 2, 3, 4\} \) for all \( j > k \). Then \( x \) is a top of the cylinder \( (F_k)_{i_{k+1}} \). If \( x \) is not a top of \( F_k \), then \( x \) must belong to another cylinder, which means \( x \) has more than one coding.

(ii) If \( i_1i_2 \cdots \) contains infinitely many 5, suppose \( i_k = 5 \). Let \( U = (F_k)_{i_1...i_k} \setminus U \). Since \( (F_k)_{i_1...i_k} \setminus U \) consists of four components and \( U \) does not intersect other cylinders of \( F_k \), we conclude that \( F_k \setminus U \) has
only four components. Therefore, \( x \) is a 4-branch point.

(iii) If \( x \) has more than one coding, then \( x \) must be the common top of two cylinders, and it is a 2-branch point.

\[ \text{Theorem 3.10.} \text{ } F_6 \text{ and } F_9 \text{ are not homeomorphic, hence are not Lipschitz equivalent.} \]

\[ \text{Proof.} \text{ } \text{By Lemmas 3.8 and 3.9, we know that} \text{ } F_6 \text{ contains 3-branch points, while} \text{ } F_9 \text{ contains no 3-branch points. Therefore, they are not homeomorphic.} \]

4. REMARKS

Because of irregularity, it is difficult to study the remaining \( F_{11}, F_{14}, F_{17}, F_{18}, F_{20}, F_{21} \) of \( F_{3,5} \). We conjecture that they are not Lipschitz equivalent at all, and \( \#(F_{3,5}/\sim) = 10. \)

For the cases of \( F_{3,6}, F_{3,7} \) and \( F_{3,8} \), we summarize their topological classifications as follows: up to congruence, \( F_{3,6} \) only contains 16 fractal squares of which 6 are disconnected and 10 are connected; \( F_{3,7} \) only contains 8 connected fractal squares; and \( F_{3,8} \) only contains 3 connected fractal squares (please see their figures in the next Appendix section). Recently, Ruan and Wang proved that \( \#(F_{3,7}/\sim) = 8 \) and \( \#(F_{3,8}/\sim) = 3 \) by making use of an old result called Whyburn's theorem. However, it is still hopeless to handle the other cases completely.

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APPENDIX A. FIGURES OF FRACTAL SQUARES

Fig. A.1 Five totally disconnected fractal squares in $F_{3,5}$.

Fig. A.2 Six connected fractal squares in $F_{3,5}$.

Fig. A.3 10 fractal squares containing parallel line segments in $F_{3,5}$.

Fig. A.4 Six disconnected fractal squares in $F_{3,6}$.
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Fig. A.5  10 connected fractal squares in $F_3,6$.

Fig. A.6  Eight fractal squares in $F_3,7$.

Fig. A.7  Three fractal squares in $F_3,8$. 