NON-PERTURBATIVE ASPECTS OF QUANTUM ELECTRODYNAMICS ON CURVED SPACE AND INVESTIGATIONS IN MATRIX GRAVITY

by

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Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in Physics with Dissertation in Mathematical Physics

New Mexico Institute of Mining and Technology
Socorro, New Mexico
April, 2009
ABSTRACT

This Dissertation is devoted to the detailed study of two major subjects. In the first part, we study non-perturbative aspects of quantum electrodynamics on Riemannian manifolds by using heat kernel asymptotic expansion techniques. Here, we established the existence of a new non-perturbative heat kernel asymptotic expansion for a Laplace type operator on homogeneous Abelian bundles with parallel curvature, and we evaluated explicitly the first three coefficients of the expansion. As an application of this important result, we computed the imaginary part of the non-perturbative effective action in quantum electrodynamics and derived a generalization of the classical Schwinger’s result for the creation of scalar and spinor particles in an electromagnetic field induced by the gravitational field. We also discovered new infrared divergences due to the gravitational corrections, which represents a completely new physical effect.

In the second part of the Dissertation, we studied some aspects of a newly developed non-commutative theory of the gravitational field called Matrix Gravity. There are two versions of Matrix Gravity, in the first one, called Matrix General Relativity, the action functional is obtained by generalizing the Hilbert-Einstein functional to matrix-valued quantities. In the second one, called Spectral Matrix Gravity, the action is constructed from the first two spectral invariants of a non-Laplace type
second order partial differential operator. For the first version, we found the dynamical equations of the theory, while, for the second version, we computed the first non-commutative corrections to Einstein equations in the weak deformation limit. For Spectral Matrix Gravity we analyzed the spectrum of the theory on DeSitter space and found that the dynamical degrees of freedom are represented by a number of massive spin-2 and massive scalar particles. Furthermore, we developed the kinematics of test particles in Matrix Gravity and found the first and second order corrections to the usual Riemannian geodesic flow. We evaluated the anomalous non-geodesic acceleration in a particular case of static spherically symmetric background. We applied this result to study the problem of the Pioneer anomaly.
ACKNOWLEDGMENT

There are many people who deserve to be thanked for my personal and professional growth here at New Mexico Tech.

I would like to thank my advisor Prof. Ivan G. Avramidi for all he has taught me during these years and for being such an inspiring researcher. Thank you, Ivan, for all the time spent with me talking about Mathematics and Physics and for always suggesting new and interesting directions of research. You have been a great mentor and collaborator helping me to become a better scientist and to be prepared for my future career.

I would like to thank Dr. Giampiero Esposito for introducing me to the exciting field of Mathematical Physics and for his genuine support over the years. Thank you, Giampiero, for always believing that I can succeed.

I would like to thank Prof. Klaus Kirsten for working with me on previous papers and for his valuable suggestions on how to improve my techniques. I am sure that my time at Baylor working with you will be a productive one.

I thank the members of my Ph.D. committee for valuable suggestions on how to improve the present manuscript.

A special thank you goes to Megan, her love and support over the years that we have been together helped me to overcome difficult times and to focus on my studies. Thank you, Megan, for being such a wonderful and loving person.

A well deserved thank you goes to my family and friends back in Italy. I
thank them for believing in me and for respecting the choice that I made to leave the
country. I greatly appreciated their encouragement to pursue the path I choose.

I would also like to thank the many friends here at New Mexico Tech for
making life easy and pleasant in Socorro and for all the "good times". I thank Gina
for her help with my bureaucratic issues and for the many laughs.

I would like to conclude by citing some verses form Dante Alighieri’s
Divine Comedy, Canto XXVI; They inspired my life.

“Consider well the seed that gave you birth:
you were not made to live your lives as brutes,
but to be followers of worth and knowledge.”

This dissertation was typeset with \LaTeX\ by the author.

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1 \LaTeX\ document preparation system was developed by Leslie Lamport as a special version of
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matical Society. The \LaTeX\ macro package for the New Mexico Institute of Mining and Technology
dissertation format was adapted from Gerald Arnold’s modification of the \LaTeX\ macro package for
The University of Texas at Austin by Khe-Sing The.
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CHAPTER 1

INTRODUCTION

One of the most important achievements in theoretical and mathematical physics in the 20th century was the development of a quantized theory of fields. Today, this area of research is still thriving and presenting researchers with complex mathematical and experimental challenges. Quantum field theory begun when physicists tried to unify the newly discovered special theory of relativity with quantum mechanics. The seminal paper by P. A. M Dirac in 1927 [45], *The Quantum Theory of the Emission and Absorption of Radiation*, represents the first attempt to create a quantized theory of the electromagnetic field and is generally recognized as the beginning of quantum field theory.

After the work of Dirac an extraordinary amount of work in this field, both theoretical and experimental, was carried out by many physicists leading to striking discoveries. Amongst the most important are the prediction of the existence of antiparticles, relativistic effects on the spectra of atoms like Lamb shift, the discovery of a number of elementary particles, the formulation of the scattering matrix which describes the interaction and decay of particles, and the development of gauge theories which lead to a unified formulation of three of the fundamental forces of Nature.

Perhaps one of the most fundamental contributions to the development of the quantum theory of fields was given by Feynman in his dissertation [48], *The Principle of Least Action in Quantum Mechanics*, where he developed the formalism
that is known as path integral. This method is based on the description of quantum systems by using the Lagrangian formalism instead of the Hamiltonian one. This method leads to a covariant formulation of quantum mechanics and quantum field theory, namely a formulation in which the form of the fundamental relations does not depend on the system of reference used. The extreme importance of this formalism is promptly recognized when one attempts to unify quantum mechanics with Relativity which is, essentially, a covariant theory.

The path integral gives the transition amplitudes of quantum processes as an integral (sum) over all the possible paths in the configuration space joining the initial and final states of the process. Since the integral, in Euclidean formulation, contains the exponential of the (negative) classical action, it is straightforward to realize that only the paths that are close to the one that extremises the classical action contributes the most to the sum. The paths that are far from the classical path become exponentially small and are negligibly small in the sum. Unfortunately, in many cases of physical interest, the path integral cannot be evaluated exactly, and therefore some approximations need to be used. The features of the path integral make this formalism particularly suitable for a semiclassical (or WKB) approximation.

It is well known that this is not the only case in which one uses approximation techniques. Several approximation methods have been the most important tools for studying phenomena in quantum field theory. The reason for that is very simple: as any other field of study in Physics, quantum field theory presents very complicated non-linear problems for which a closed analytic solution cannot be found explicitly. Therefore various approximation schemes have been developed in order to obtain observable predictions at different regimes.
One of the most important approximation schemes, especially for its manifestly covariant form, is the background field method which was developed in major part by DeWitt [40, 41]. This method is, actually, a generalization of the method of generating functionals developed by Schwinger [72, 73]. The most important object in the background field method is the so called effective action. The effective action is a functional of the background field and, in principle, contains all the information about the quantum theory. In order to yield observable predictions, quantum field theory needs to give the probability amplitudes of a variety of scattering processes in which the initial interacting states are known and the final products can be measured by using particle detectors. Theoretically, this interaction is studied by means of the so called $S$-matrix (or scattering matrix). In the Feynman diagrammatic technique the $S$-matrix is described in terms of the propagator and vertex functions. The effective action determines the propagator and the vertex functions with regard to all quantum corrections. Of course, once these basic ingredients are known, the complete $S$-matrix is determined and, hence, measurable predictions can be made [39]. Moreover the effective action yields, upon variation, the effective equations of motion which describe the back-reaction of quantum processes on the classical background field. Another important feature of the effective action is its low-energy limit (also called effective potential), which is the natural tool for studying the vacuum structure of the quantum theory under consideration.

The effective action is one of the most powerful tools in quantum field theory and quantum gravity (see [72, 43, 14, 17, 28]). The effective action is a functional of the background fields that encodes, in principle, all the information of quantum field theory. It determines the full one-point propagator and the full vertex functions and, hence, the whole $S$-matrix. Moreover, the variation of the effective
action gives the effective equations for the background fields, which makes it possible to study the back-reaction of quantum processes on the classical background. In particular, the low energy effective action (or the effective potential) is the most appropriate tool for investigating the structure of the physical vacuum in quantum field theory.

The effective action is expressed in terms of the propagators and the vertex functions. One of the most powerful methods to study the propagators is the heat kernel method, which was originally proposed by Fock [52] and later generalized by Schwinger [72] who also applied it to the calculation of the one-loop effective action in quantum electrodynamics. Finally, DeWitt reformulated it in the geometrical language and applied it to the case of gravitational field (see his latest book [43]).

Unfortunately, in most interesting physical cases the effective action cannot be computed exactly, and therefore approximation methods need to be developed. Since the effective action can be written in terms of a path integral, one of the most effective approximation methods is the semiclassical perturbative expansion of the path integral in the number of loops, also known as the loop expansion. The basic idea is the following: all the fields are decomposed in a classical background part \( \phi \) and a quantum disturbance \( h \), like \( \varphi = \phi + \sqrt{\hbar} h \), where \( \hbar \) is the Planck constant. By substituting this decomposition in the classical action, one can expand the action in the quantum fields. The quadratic part of the expansion in the quantum fields gives the propagator and the higher order terms give the various vertex functions. This information is basically all we need in order to obtain the effective action, because, as we mentioned before, it is constructed in terms of the propagator and the vertex functions [39, 43]. The number of loops in the perturbative expansion...
sion corresponds to the power of the Planck’s constant: $\hbar$ corresponds to one-loop expansion, $\hbar^2$ represents two-loop expansion, and so on. One of the most effective mathematical tools to study the propagators in quantum field theory is the proper time method or heat kernel method which was developed by Schwinger in [72, 73] and then generalized to include curved spacetime by DeWitt in [40, 42]. Some nice reviews on this subject can be found in [33, 79].

It is clear that the various approximation methods only give correct results within their own specific regime. In other words each approximation method has its limits of validity. There are, mainly, three types of regimes that have been extensively studied in the literature (a good review can be found in [7]),

- **Semiclassical Approximation:** This approximation is used in the case in which the fields of interest have a large mass. The main idea of the method is to expand all the relevant quantities (like effective action) in a series of inverse powers of the mass. In this way, since the mass is large, the higher order terms become smaller and smaller and can be treated as perturbations. Of course this approximation fails completely if one considers small or vanishing masses because the higher order terms, in the expansion in inverse powers of the mass, would become larger and larger posing problems for the convergence of the expansion.

- **High Energy Approximation:** This approximation is particularly useful in the case in which one is interested in weak, rapidly varying background fields. This approximation analyzes the short-wave, and hence the high energy, part of the spectrum of the background field. The idea is to construct an expansion
in powers of the field strength. Obviously this expansion will not produce correct results when the field strength becomes large.

- **Low Energy Approximation**: This approximation is the opposite of the previous one, and it is used when one is interested in strong and slowly varying background fields. This approximation probes the long-wave, and therefore the low energy, part of the spectrum of the background field. In this approximation the idea is to construct an expansion in the derivatives of the fields. Exactly as in the other cases described above, this approximation fails if the field strength becomes small and the derivatives become large.

In the first part of this Dissertation we will mainly study quantum electrodynamics on curved spacetime. However, before describing in details the work done here, it is important to briefly review the past literature and results about this subject. One of the first attempts to utilize the heat kernel method for studying the effective action in quantum electrodynamics was carried out by Schwinger in [72]. In his paper he was able to successfully derive the effective action for quantum electrodynamics and evaluate its imaginary part which, in turn, gives the probability amplitude for the creation of pairs of particles in an electromagnetic field. This result, although of fundamental importance, was obtained on a flat (Minkowski) spacetime. The generalization of the results obtained by Schwinger in curved spacetime represent a much more complicated task. As mentioned before, in general cases the heat kernel and, therefore, the effective action cannot be computed exactly: the curved spacetime case is one of them. For this reason the research was focused on trying to obtain some results in particular regimes by using suitable approximation methods.
In order to study such case one of the oldest methods used was the Minackshisundaram-Pleijel short-time asymptotic expansion of the heat kernel. This method is essentially perturbative, in fact in this approach the expansion is written in terms of powers of the curvature of the spacetime and its derivatives. Of course this method is inadequate when the curvature becomes large (strongly curved spacetimes). If some of the geometric invariants of the curvature are large, the method of partial summation has been developed in which one sums the terms in curvature that are large. This is still a perturbative approach because the expansion is written in terms of powers of the geometric invariants of the curvature that are not large.

In the low-energy approximation, when the curvature, but not their derivatives, is large, an effective manifestly covariant method for the evaluation of the heat kernel asymptotics has been developed \[6, 7, 10\]. In the case the derivatives of the curvature are small, this method effectively sums the contribution from the curvature of the spacetime in the heat kernel expansion. However this study does not contain the electromagnetic field. A recent paper \[24\] overcomes this difficulty by treating covariantly constant gauge fields over symmetric spaces which are manifolds with covariantly constant Riemann tensor.

The first part of this Dissertation deals with the evaluation of the heat kernel asymptotic expansion for covariantly constant electromagnetic field on arbitrary Riemannian manifolds. We obtain, for the first time in literature, an expansion, for the heat kernel, in powers of the geometric invariants of the curvature of the spacetime but to all orders of the electromagnetic field. This represents a completely new, non-perturbative, heat kernel asymptotic expansion. Before this important result, the heat kernel expansion in presence of an electromagnetic field in curved spacetime,
and hence the effective action, were only known as power series in the electromagnetic field strength and its derivatives. Our result, instead, presents analytic functions of the electromagnetic field which means that we have effectively summed all the infinite contributions coming from the electromagnetic field to the effective action in curved spacetime. In this sense our results are non-perturbative in the electromagnetic field. In physical language this means that we have a tool, which was missing before, for studying the behavior of matter in a gravitational field under the influence of a very strong electromagnetic field. One can promptly realize that these results can be applied, for instance, to the study of matter close to any astrophysical object which possesses a strong electromagnetic field (like magnetars, pulsars, etc.). Moreover our results can be used as a tool in order to study charged black holes, in particular the important subject of creation of pairs of particles near a black hole possessing a strong electromagnetic field. In other words, we can apply our non-perturbative method to any physical system under the influence of the gravitational field and a strong electromagnetic field. Since this result is completely new, it is possible that new, non-perturbative, physical phenomena, unpredictable if one uses perturbative methods, might be found in the near future. The results obtained in the first part of this Dissertation have a profound impact also in mathematics especially in the area of study of Kähler manifolds which are, basically, complex manifolds with some specific algebraic conditions imposed on them. The complex structure on Kähler manifolds is a covariantly constant antisymmetric two-tensor which plays the role of our covariantly constant electromagnetic field. Our result, applied to this case, would give a new heat kernel asymptotic expansion of Kähler manifolds which can be used to obtain specific geometric information about these manifolds. This is an important topic that would be interesting to study in the near future.
As an application of the new non-perturbative heat kernel asymptotic expansion analyzed in this Dissertation, we studied the effective action for scalar and spinor fields under the influence of the gravitational and the electromagnetic field. In particular we considered the imaginary part of the effective action. The imaginary part actually measures the probability amplitude for the creation of pairs of particles, in particular if it vanishes no particles are created. In this Dissertation we applied the heat kernel asymptotic expansion in order to find the imaginary part of the effective action for particles in a gravitational field under the influence of an electromagnetic field. This important study generalizes the result obtained by Schwinger in [72] to curved spacetime and effectively yields the probability amplitude of creation of pairs in an electromagnetic field induced by the gravitational field. Our result is of particular importance because, in this effect, we take into account, in a non-perturbative fashion, the effect of the electromagnetic field in particle creation in curved spacetime. This is a completely new result that was absent in the literature.

Undeniably the Standard Model is the most successful achievement of quantum field theory, unifying strong, weak and electromagnetic interactions. However a quantized theory of the remaining fundamental force of Nature, namely gravitational interaction, remains still elusive. Researchers have put a great deal of effort in order to find a consistent theory of quantum gravity. This work, over the years, produced a number of different theories trying to reconcile quantum theory and gravitation but none of them has been proved to be the correct one yet. A review of the present status of quantum gravity can be found in [44] and references therein.

For this, and other reasons, recent research has been focused on the development of alternative theories of gravity. The aim is to find a new theory which
would address the problem of quantization of the gravitational field and also some recent discrepancies between the predictions of General Relativity and the observations in particular physical systems.

In the second part of this Dissertation we will present various results concerning a newly developed theory of the gravitational field called Matrix Gravity. Gravity is a universal physical phenomenon. It is this universality that leads to a successful geometric interpretation of gravity in terms of Riemannian geometry in General Relativity. General Relativity is widely accepted as a good approximation to the physical reality at large range of scales.

However some important open issues in General Relativity are still under debate. First of all, the experimental evidence points to the fact that all matter exhibits quantum behavior at microscopic scales. Thus, it is generally believed that the classical general relativistic description of gravity is inadequate at short distances due to quantum fluctuations. However, despite the enormous efforts to unify gravity and quantum mechanics during the last several decades we still do not have a consistent theory of quantum gravity. There are, of course, some promising approaches, like string theory, loop gravity and non-commutative geometry. But, at the time, none of them provides a complete consistent theory that can be verified by existing or realistic future experiments.

Secondly, in the last decade or so it became more and more evident that there might be a few problems in the classical domain as well. In addition to the old problem of gravitational singularities in General Relativity these gravitational anomalies include such effects as dark matter, dark energy, Pioneer anomaly, flyby anomaly, and others [60]. They might signal to new physics not only at the Planckian
scales but at very large (galactic) scales as well.

This suggests that General Relativity, that works perfectly well at macroscopic scales, should be modified (or deformed) both at microscopic and at galactic (or cosmological) scales (or, in the language of high energy physics, both in the ultraviolet and the infrared). It is very intriguing to imagine that these effects (that is, the quantum origin of gravity and gravitational anomalies at large scales) could be somehow related. Of course, this modification should be done in such a way that at the usual distances the usual General Relativity is recovered. This condition puts some constraints (experimental bounds) on the deformation parameters; in the case of non-commutative field theory such bounds on the non-commutativity parameter were obtained in [36].

The main ideas of General Relativity are closely related to the geometric interpretation of linear second-order partial differential operators which describe the propagation of waves, in particular light, in the spacetime [74]. In fact, in Einstein’s General Relativity, light is used to measure distances and to synchronize clocks in different points of the space. At the time of discovery of General Relativity the electromagnetic radiation (light) was the only known field that could serve the purpose. Today it is known that there exist different kinds of fields that can transmit information in the spacetime in particular fields with some internal structure (like gauge fields). Matrix Gravity is based on the idea that the structure of spacetime can be analyzed with fields possessing an internal structure rather than light. The role of the electromagnetic field in General Relativity is played, in Matrix Gravity, by some other gauge field (e.g. gluons or other vector bosons) [20].

This generalization to gauge fields completely changes the structure of the
spacetime. In General Relativity one is concerned with propagation of light which is described by a hyperbolic partial differential operator. The matrix of the second derivatives is smooth symmetric and non-degenerate, moreover it transforms like a contravariant two-tensor of type \((2,0)\). These properties allow us to interpret the matrix of the second derivatives as a Riemannian metric. Obviously, once the metric on the manifold is known, one can construct all the relevant geometric quantities that are used in General Relativity, like the Christoffel symbols, Riemann tensor and the invariants coming from it. In Matrix Gravity the picture is different. The propagation of gauge fields is determined by a system of hyperbolic partial differential operators. In this case the matrix of the second derivatives becomes endomorphism-valued, or, in other words, it becomes a “matrix of matrices”. The latter object does not describe a Riemannian metric but rather a more general collection of Finsler metrics. This metric is a generalization of the Riemann metric in which the distance between neighboring points is an homogeneous function of the point and the tangent vectors. The spacetime manifold is, now, equipped with a matrix-valued metric which describes a collection of Finsler metrics. At this point the construction of all the geometric quantities we need is the same as in General Relativity, however, in Matrix Gravity, the abovementioned quantities are matrix-valued.

Once the geometric framework is set, we need to focus on the dynamics of the gravitational field. General Relativity is nothing but the dynamical theory of the metric tensor defined on the spacetime manifold. Analogously, Matrix Gravity is the dynamical theory of the matrix-valued metric. However, because of the matrix-valued nature of the geometrical quantities, namely their non-commutative nature, in this theory, the definition of an action which yields the dynamics is not unique. If fact, if we try to generalize the Hilbert-Einstein action to matrix-valued quantities,
we are soon faced with the problem of ordering the matrix-valued measure on the spacetime manifold with the matrix-valued scalar curvature. In order to avoid this problem we can define the action for Matrix Gravity via spectral invariants of the partial differential operator which describes the dynamics of the theory. In general Relativity the action can be written as a specific combination of the first two global spectral invariants of a Laplace type operator (which in Euclidean formulation describes the propagation of light). In Matrix Gravity we can write the action as same combination of the first two global spectral invariants of a more general non-Laplace type partial differential operator. In this way the action is uniquely defined and does not depend on the order in which we write the geometric quantities.

The evaluation of the dynamical equations of the theory is an important step because we can use them in order to study some particular case of physical interest. In this Dissertation we analyzed the kinematics of test particles in the ambit of Matrix Gravity. We found that the motion of test particles in the spacetime is quite different from the predictions of General Relativity. Since Matrix Gravity is basically a dynamical theory of a collection of Finsler metrics, a particle is described by $N$ different mass parameters instead of one mass parameter $m$: For each Finsler metric we have a mass parameter. Every single Finsler metric in the collection determines a particular geodesic which is followed by the corresponding mass parameter. The sum of all the different mass parameters is the usual mass. The idea here is similar to the concept of colors in Quantum Chromodynamics (QCD): In Matrix Gravity the mass consists of different mass parameters as in QCD; the proton, for example, consists of three quarks of different color. The different mass parameters describe the tendency for a particle to move along a particular geodesic.

We would like to stress, at this point, that in this picture the trajectory of a parti-
cle “splits” in a system of trajectories (Finsler geodesics) close to the Riemannian geodesic. Obviously when everything commutes the bundle of trajectories collapses in one trajectory which coincides with the one predicted by General Relativity. As a result, the test particles exhibit a non-geodesic motion in the sense that they do not follow any geodesic derived from any Riemannian metric. This non-geodesic motion can be interpreted in terms of an anomalous acceleration affecting the test particles. Driven by this interesting result, we applied the kinematics of test particle in Matrix Gravity to the Pioneer spacecrafts which present an unexplained (in the ambit of General Relativity) acceleration.

The outline of the Dissertation is as follows. In the next chapter we will review some basic technical material and concepts that have been used throughout the Dissertation. In the third Chapter we will derive the first three heat kernel asymptotic coefficients for a covariant Laplace type operator in powers of the Riemannian curvature but to all orders of the electromagnetic field. In the fourth Chapter we will use the results obtained in Chapter three in order to study the effective action in non-perturbative quantum electrodynamics and compute, in particular, its imaginary part which gives the probability amplitude for the creation of pairs of particles in the electromagnetic field induced by the gravitational field. In the fifth Chapter we will be mainly concerned with the computation of the non-commutative Einstein equations derived from the action in Matrix Gravity constructed by generalizing the usual Hilbert-Einstein action to matrix-valued quantities. In the sixth Chapter we will focus on the action for Matrix Gravity derived from spectral invariants of a non-Laplace type operator. We utilize the heat kernel asymptotic expansion technique to compute the spectral invariants that form the action. In the second part of the
Chapter we use the spectral action to find the non-commutative corrections to Einstein equations in the low-energy limit, and discuss the spectrum of the theory. In the seventh Chapter we analyze the kinematics of test particles in Matrix Gravity, especially the new phenomenon of non-geodesic motion which is related to some anomalous acceleration. In order to obtain a more specific expression we study the motion of test particles in a static and spherically symmetric spacetime by using an algebra of $2 \times 2$ commuting matrices. As an application of Matrix Gravity, in the eighth Chapter, we describe the Pioneer anomaly in the ambit of the anomalous acceleration of test particles in Matrix Gravity. At the end of the Chapter we give an estimate of the free parameters of the theory to match the value of the observed anomalous acceleration of the Pioneer spacecrafts. We conclude, then, the Dissertation with a summary of the most important results obtained in this work and some ideas for future directions of research.

We would like, at this point, to fix the units which will be used throughout this Dissertation. In quantum field theory it is convention to set

$$\hbar = c = e = 1.$$ 

According to these units, all the relevant quantities can be expressed in terms of length $l$. More precisely we obtain

$$[x] = [x^0] = l, \quad [m] = [\text{Energy}] = l^{-1}, \quad [F] = [R] = l^{-2},$$

where $m$ is the mass, $F$ is the electromagnetic field strength and $R$ is the scalar curvature. Moreover, for the gravitational constant $G$ and the cosmological constant $\Lambda$ we have

$$[G] = l^2, \quad [\Lambda] = l^{-2}.$$
CHAPTER 2

MATHEMATICAL BACKGROUND IN QUANTUM FIELD THEORY AND GRAVITY

In this chapter we review the basic ideas and techniques that have been used over the years in the literature in order to develop a covariant formalism in quantum field theory. We will present the main ideas of the formal development of covariant methods in quantum field theory and the derivation and use of the effective action. Moreover, we will discuss the heat kernel method as a way (mostly used in this Dissertation) to deal with the effective action calculations in quantum field theory. In the second part of this chapter we will describe the basic mathematical tools that are used in Matrix Gravity.

2.1 Introduction to Quantum Field Theory

Classical mechanics is one of the milestones of human understanding of the physical world. Newton’s dynamical equations describe, by means of a set of second order differential equations, the motion of a system of point masses. Given the initial position and the initial velocity for each mass, the subsequent motion is completely determined. This description of nature was the paradigm for many years. With the discovery of electric and magnetic phenomena, Maxwell realized that these entities were better described by utilizing the concept of field.

The solution of Newton’s equations is a dynamical trajectory, which is an
object that associates to every instant in time a vector in the space. A field, instead, is a relation that associates to every point in space and every instant in time an object in a particular space. Maxwell described the electric and magnetic fields as vector fields. One of the most important achievements of Maxwell was the formulation of a dynamical theory in which the electric and magnetic fields were described as two manifestations of one single entity: the electromagnetic field.

One of the most striking prediction of the theory of electromagnetism was the finiteness of the speed of propagation of the electromagnetic radiation in any inertial system of reference. This was a completely new feature which was not present in classical mechanics in which information could propagate instantly. Moreover, the constancy of the speed of light in any reference frame was in direct contrast with the well established classical Galilean transformations. In order to overcome these difficulties, Einstein developed a theory, special relativity, in which the constancy of speed of light in any inertial reference frame was the starting point. In this theory, time and space are treated on an equal footing forming a single 4-dimensional entity called spacetime. In special relativity the electromagnetic field is described by a $4 \times 4$ antisymmetric matrix $F_{\mu\nu}$ which transforms as a tensor (a 2-form) under Lorentz transformations. It is soon realized, in the framework of special relativity, that the usual electric and magnetic fields do not have an absolute meaning. In fact, in different reference frames, the electric field can become a magnetic field and vice versa. This means that observers in different inertial systems of reference would not agree on what to call electric and magnetic field. In these circumstances, only certain invariant combinations of the electric and magnetic fields are physical observables. For this reason it is often convenient to utilize the spectral decomposition of the 2-form $F_{\mu\nu}$. In this way the electromagnetic field is represented in terms of
invariants and projection on invariant subspaces. This means that there is a reference frame in which $F_{\mu\nu}$ can be written as a block diagonal matrix where the entries are the field invariants.

In the period in which Einstein developed the theory of special relativity, another important theory, which would change the vision of the world, was just at its early stages. Quantum mechanics was developed in order to explain certain experimental observations, especially in atomic physics, which could not be predicted in the ambit of classical mechanics and electromagnetism. A few years later, quantum mechanics and special relativity became accepted in the scientific community as the theories best suited to describe the physical world. However, it was soon realized that the two theories were not completely compatible with each other. The reason is the following: in special relativity, space and time play the same role and are treated equally in the formulation of the fundamental equations. In quantum mechanics, instead, time still plays a privileged role, as one can see by just analyzing the Schrödinger equation.

Quantum field theory was developed in order to reconcile quantum mechanics and special relativity in a quantum theory that would be relativistically invariant. In this theory, the microscopic world is described in terms of fields which transform, in a certain specific way, under the Poincaré group. The Poincaré group contains two subgroups, the Abelian group of translations along the four coordinates and the non-Abelian Lorentz group of rotations in Minkowski space. The fields are characterized by their transformation properties under the Lorentz group. More specifically, the Lorentz group has, in general, two types of representations in terms of matrices of the corresponding algebra. The first type representations are called
single-valued and the second type representations are double-valued. Fields that transform under the single-valued representations of the Lorentz group are tensor fields, such as scalar and vector fields. The fields that transform under the double-valued representation of the Lorentz group are called spinor fields. These representations are also called spinor representations of the Lorentz group. The more familiar concept of spin of a field is related to the particular dimension of the representation of the Lorentz group under which the field transforms.

The dynamics of a field is described by an action functional. From the action one derives the dynamical equations for the fields by utilizing the stationary action principle. For instance, the dynamics of scalar fields is described by the Klein-Gordon equation and the dynamics of fields of spin 1/2 is described by the Dirac equation. In general the fields satisfy hyperbolic second order partial differential equations together with some suitable boundary conditions which are usually determined from the particular physical situation. The canonical quantization procedure for field theory is the following: one solves the dynamical equations by using the Fourier transform method. In this way the field will be described by a Fourier integral containing a combination of positive and negative frequencies. The field is quantized in the canonical quantization scheme by treating the field as an operator and by imposing specific equal time commutation relations. These relations induce similar commutation relations to the coefficients of the negative and positive frequencies in the solution for the field. These coefficients, which are now operators as well, are interpreted as creation and annihilation operators.

Let us, now, describe in more detail the Klein-Gordon and the Dirac equations in the light of their use in Chapter 4. It is well known that the Klein-Gordon
equation can be derived by varying the following action functional with respect to \( \varphi \)

\[
S_{\text{KG}} = \int_M dx \, g^{1/2} \left( g^{\mu\nu} \nabla_\mu \varphi^* \nabla_\nu \varphi + m^2 \varphi^* \varphi \right), \tag{2.1}
\]

where \( \varphi \) represents a complex field, \( m \) its mass, \( M \) the spacetime manifold, \( dx \, g^{1/2} \) the invariant measure on \( M \) and \( \nabla_\mu \) is the covariant derivative. Since in this Dissertation we are primarily interested in curved spacetimes, we consider the following generalization to curved spacetime of the above action functional

\[
S_{\text{KG}} = \int_M dx \, g^{1/2} \left( g^{\mu\nu} \nabla_\mu \varphi^* \nabla_\nu \varphi + \xi R \varphi^* \varphi + m^2 \varphi^* \varphi \right), \tag{2.2}
\]

where \( R \) is the scalar curvature, and \( \xi \) is a dimensionless coupling constant which represents the interaction of \( \varphi \) with the gravitational field. In general, one tries to find the simplest generalization from flat to curved spacetime. However, more complicated choices can be made if there are important, physical or technical, reasons to do so [51].

We would like to stress, at this point, that the action (2.2) describes the dynamics of a free scalar field in curved space. In order to describe a self-interacting scalar field, one can add a self-interaction term to the action (2.2); The most common term is \( \lambda (\varphi^* \varphi)^2 \), where \( \lambda \) is the coupling constant for self-interaction. In this case, by varying the action functional containing \( \lambda (\varphi^* \varphi)^2 \), one would get the following dynamical equation

\[
L_{\text{scalar}} \varphi = 0, \tag{2.3}
\]

where

\[
L_{\text{scalar}} = -\Delta + \xi R + m^2 + Q_{\text{scalar}}, \tag{2.4}
\]
where $\Delta$ is the Laplacian containing covariant derivatives and $Q_{\text{scalar}} = 2\lambda \varphi^*(\varphi^* \varphi)$.

In Chapter 4 we consider charged free scalar fields. In this case no self-interaction is present and the term $Q_{\text{scalar}}$ is set to zero.

The dynamics of fermions, such as electrons, in flat spacetime is described by the following action

$$S_D = \int_M dx \, g^{1/2} \bar{\psi} (i\gamma^\mu \partial_\mu + m) \psi ,$$

(2.5)

where $\psi$ represents the spinor field, $\bar{\psi}$ is the Dirac conjugate and $\gamma^\mu$ are the gamma matrices. The variation of this action with respect to the independent field leads to the Dirac equation. The generalization of the above action to curved spacetime is obtained by replacing the ordinary derivative with covariant derivative. No scalar analytic term proportional to the curvature of the spacetime can be added to the above action because any invariant of the spacetime curvature does not have the right dimensions. In order to obtain a wave equation for spinors, one considers the square of the Dirac operator $D = i\gamma^\mu \nabla_\mu + m$. More specifically, one has

$$L_{\text{spinor}} = \bar{\psi} D D \psi = (\gamma^\mu \gamma^\nu \nabla_\mu \nabla_\nu + m^2) \psi ,$$

(2.6)

By writing the coefficient of the second derivatives as sum of the commutator and anticommutator of the gamma matrices and by taking into account that the commutator of the covariant derivatives introduces the Riemann curvature tensor $R^{ab\mu\nu}$, as follows

$$[\nabla_\mu, \nabla_\nu] \psi = \frac{1}{4} R^{ab\mu\nu} \gamma_{[a} \gamma_{b]} \gamma_\mu \gamma_\nu \psi ,$$

(2.7)

it is not difficult to prove that one obtains

$$L_{\text{spinor}} = -\Delta + \frac{1}{4} R + m^2 .$$

(2.8)
As we can notice, from the last expression, there is a very specific value, \( \xi = 1/4 \), of the coupling constant for spinors in the gravitational field which is the same in any dimension.

Let us consider, now, the case in which a background electromagnetic field \( F_{\mu\nu} \) is present, which will be studied in Chapters 3 and 4. As it is explained in detail in the next section, introducing a background electromagnetic field in the formalism is equivalent to replacing the covariant derivatives in the action \( \nabla_\mu \) with \( \mathcal{D}_\mu = \nabla_\mu + iA_\mu \), where \( A_\mu \) represents the vector potential. For charged scalar fields, the introduction of the electromagnetic field is treated in detail in the next section. It is interesting to consider spinor fields in curved spacetime under the influence of a background electromagnetic field. By using the exact same argument that lead us to equation (2.8), and by noticing that, in this case,

\[
[\nabla_\mu, \nabla_\nu]\psi = \left( \frac{1}{4} R^{ab}_{\mu\nu} \gamma_{[a} \gamma_{b]} + iF_{\mu\nu} \right) \psi, \tag{2.9}
\]

one gets

\[
L_{\text{spinor}} = -\Delta + \frac{1}{4} R + m^2 + Q_{\text{spinor}}, \tag{2.10}
\]

where in this case it is not difficult to show that

\[
Q_{\text{spinor}} = -\frac{i}{2} F_{\mu\nu} \gamma^\mu \gamma^\nu. \tag{2.11}
\]

Obviously, in absence of the underlying electromagnetic field, the term \( Q_{\text{spinor}} \) vanishes. The presence of the term \( Q_{\text{spinor}} \) in the previous equation represents the reason why spinors and scalars behave differently in an electromagnetic field. It is the spin, represented by the antisymmetric product of gamma matrices, that directly couples with the electromagnetic field.
In fact, in the semiclassical approximation, charged scalar fields are described by just their electric charge. Spinor fields, instead, are described by the electric charge and the so-called magnetic moment

$$\mu = \frac{1}{m} s ,$$  \hspace{1cm} (2.12)

where $s$ represents the spin of the particle. For instance, the intrinsic magnetic moment of the electron is the negative of the Bohr magneton $\mu_B$ which, in the usual units, has the value

$$\mu_B = \frac{e \hbar}{2m_e} = 9.27 \cdot 10^{-24} \ J \ T^{-1} .$$  \hspace{1cm} (2.13)

The magnetic moment couples to the electromagnetic field making spinors behave differently from scalars in an electromagnetic field. This different behavior is the reason why we obtain different results for the creation of scalars and spinors in curved spacetime under the influence of a strong electromagnetic field in Chapter 4.

An important difference between quantum mechanics and quantum field theory is in the description of particles. In quantum mechanics, a certain wave function is an element of the Hilbert space and describes a system with a fixed number of quantum particles. Moreover, it is well known that the number of particles is conserved. In quantum field theory, a field is an element of a more general Fock space which is a direct sum of Hilbert spaces; this field describes quantum states with variable number of particles. This means that in quantum field theory the number of particles is not constant leading to the interesting phenomenon of creation of particles. In this framework, the operators introduced in the fields above are interpreted as creation and annihilation of particles.

Quadratic actions containing single fields describe the propagation of free fields without interactions. However, interesting physical processes arise from the
interaction of two or more fields. These processes are described by adding a Lagrangian of interaction to the free Lagrangian of the field. In particle physics one tries to predict the outcome of a process of interaction of two or more fields in a finite region of the spacetime. In particular, one is interested in the probability that specific “in” quantum states become, after interaction, some “out” quantum states, in other words scattering of particles. This probability amplitude is given in terms of the so called $S$-matrix. As explained later, these fundamental quantities are expressed in terms of the so called effective action.

One of the most interesting features of quantum field theory is the structure of the vacuum. The vacuum is defined as the quantum state with no particles. However, in quantum field theory the vacuum is not really “empty”. In fact, there are continuous processes of creation and annihilation of pairs of particles. This means that the vacuum is a very dynamical entity. In the mathematical framework of the effective action, these processes of creation and annihilation of pairs in the vacuum are described by the imaginary part of the effective action. One interesting process related to the properties of the vacuum is the Schwinger mechanism. Suppose that we introduce a constant electric field in a region where there are no particles (vacuum). As we already mentioned above, pairs of particles and antiparticles are created and annihilated. Now if the electric field is not strong enough (of energy less than the rest mass of the pairs created), all the created pairs will behave like small electric dipoles and will align with the field generating a “dielectric effect”. This dielectric effect of the vacuum is also known as vacuum polarization. Now let us suppose that the energy density of the electric field is much greater than the rest mass density of the pairs produced in the vacuum. At this point once a pair (of opposite charges) is created the positive one tends to move parallel to the electric field lines and the
other, charged negatively, tends to move in the opposite direction. In the assumption of strong electric field, the two particles that are created drift apart to the point in which they cannot annihilate each other. This means that particles have been created because of the presence of a strong electric field. Schwinger found that the rate of particle production of mass \( m \) and charge \( e \) per unit volume and per unit time in a constant electric field \( E \) is given by [72]

\[
R = \frac{1}{16\pi^4} E^2 \sum_{n=1}^{\infty} n^{-2} \exp\left(-\frac{n\pi m^2}{E}\right).
\]

Formally, the presence of the electric field leads to an imaginary contribution in the effective action which is interpreted as probability of creation of particles. This process of particle creation in an electric field was studied by Schwinger in a Minkowski (flat) space and for strong constant electric fields. In this Dissertation, we generalize his result to Riemannian (curved) spaces and strong covariantly constant electric fields.

In all the discussion above there is an important element missing; namely the above theories do not take into account General Relativity (GR). GR is a theory of the gravitational field in which the spacetime is described by a manifold and the gravitational field is described by a symmetric non-degenerate 2-tensor field \( g^{\mu\nu} \). GR successfully describes a wide range of physical phenomena at large scales. Despite this, it is believed that at a more fundamental level any theory describing the physical world should be quantized. Great efforts have been made in order to find a consistent theory of quantum gravity and yet this is still an open problem. Many theories have been conceived in order to unify quantum mechanics and General Relativity, but none of them seems to be correct. One of the most difficult problems to overcome when trying to construct a quantum theory of gravity is that General
Relativity falls in a class of theories that are non-renormalizable. This means that the infinities that appear in the quantized theory cannot be consistently removed, unlike renormalizable theories in which this procedure is well defined. One of the most important achievements in the direction of a full quantized theory of gravity, was the development of quantum field theory on curved space. In this theory, the gravitational background is treated classically and the fields defined in the spacetime are quantized. Even though quantum field theory in curved space is just an effective theory (it should be the result of a certain limiting case of the full and still unknown quantum theory of gravity), it has predicted some important phenomena. The most famous one is the Hawking radiation. Hawking discovered that a black hole emits a thermal radiation with a black body spectrum due to quantum effects. The Hawking radiation process reduces the mass of the black hole and is also known as black hole evaporation. Besides the problem of quantization, it has been recently discovered that the predictions of General Relativity are not in full agreement with the observations. Because of the open issues in General Relativity and its problems with quantization, it is generally believed that a new theory of gravity needs to be found. For this reason, in this Dissertation we study a modified theory of gravity called Matrix Gravity in which the gravitational field is described by a matrix-valued metric tensor. The idea is to understand whether or not this modified theory is able to address and solve the open issues that are present in General Relativity.

2.2 The Electromagnetic Field as \( U(1) \) Gauge Theory

In this section we will explicitly show that the electromagnetic field can be described as the gauge theory of the \( U(1) \) group following the discussion in [69]. We will see that the electromagnetic field arises naturally by requiring the invari-
ance under local $U(1)$ transformations of the Lagrangian for a charged scalar field. A charged scalar field is described by two real functions or, equivalently, by one complex function. In what follows we will restrict ourselves to the case of a (flat) Minkowski spacetime. The Lagrangian for a complex field $\phi$ can be written as follows

$$\mathcal{L} = (\partial_\mu \phi^*) (\partial^\mu \phi) - m^2 \phi^* \phi.$$  

(2.15)

Obviously, this Lagrangian is invariant under internal rotations of the field; namely under the following transformation

$$\phi \rightarrow e^{-i\Lambda} \phi,$$  

(2.16)

with $\Lambda$ being an arbitrary constant. This transformation implies that the field is rotated at every point in the space at the same time. This is in contrast with the principles of special relativity in which information propagates at finite speed. For this reason a more appropriate transformation (which respects the principles of Relativity), is the following

$$\phi \rightarrow e^{-i\Lambda(x)} \phi, \quad \phi^* \rightarrow e^{i\Lambda(x)} \phi^*.$$  

(2.17)

These transformations are called local $U(1)$ transformations. At this point one can easily see that the original Lagrangian is no longer invariant under the transformations (2.17). More precisely, under the transformations (2.17) the field and its derivative vary as follows

$$\delta \phi = -i\Lambda(x) \phi, \quad \delta (\partial_\mu \phi) = -i\Lambda(x)(\partial_\mu \phi) - i[\partial_\mu \Lambda(x)] \phi.$$  

(2.18)

By using the above variations, one can show that the variation of the Lagrangian (2.15) is not vanishing, but acquires an additional term

$$\delta \mathcal{L} = j^\mu \partial_\mu \Lambda(x),$$  

(2.19)
where
\[ j_\mu = i \left[ -\phi \partial_\mu \phi^* + \phi^* \partial_\mu \phi \right] . \] (2.20)

Now, in order to make the original Lagrangian invariant under the transformations (2.17), we need to introduce an additional term for which the variation cancels exactly the variation of the original Lagrangian. This means that we add to (2.15) the following term
\[ \mathcal{L}_1 = -j^\mu A_\mu , \] (2.21)
where \( A_\mu \) is a vector field which varies, under the transformations (2.17) as follows
\[ A_\mu \rightarrow A_\mu + \partial_\mu \Lambda(x) . \] (2.22)

We now proceed with the variation of \( \mathcal{L} + \mathcal{L}_1 \), and check if this combination is invariant under local \( U(1) \) transformations. By taking the variation of both terms we obtain
\[ \delta \mathcal{L} + \delta \mathcal{L}_1 = -(\delta j^\mu)A_\mu = -2\phi^* \phi A_\mu \partial_\mu \Lambda(x) . \] (2.23)
As we can see, the last combination is not yet invariant. In order to cancel the above term, we introduce, in the same spirit as earlier, the following term
\[ \mathcal{L}_2 = A_\mu A^\mu \phi^* \phi . \] (2.24)

Now, by taking the variation of the combination \( \mathcal{L} + \mathcal{L}_1 + \mathcal{L}_2 \), we finally obtain
\[ \delta \mathcal{L} + \delta \mathcal{L}_1 + \delta \mathcal{L}_2 = 0 . \] (2.25)
This means that the correct Lagrangian for a charged field which is invariant under local \( U(1) \) transformations is the following
\[ \mathcal{L}_{CF} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi - i \left[ -\phi \partial^\mu \phi^* + \phi^* \partial^\mu \phi \right] A_\mu + A_\mu A^\mu \phi^* \phi . \] (2.26)
At this point we see that the vector field $A_\mu$ enters the Lagrangian and interacts with the field $\phi$. One, therefore, expects that this field would be dynamical. Let us then, write a Lagrangian that describes the dynamics of the new field $A_\mu$. The Lagrangian for $A_\mu$, in general, must be a scalar quadratic in the derivatives of the field and invariant under local gauge transformations. The simplest choice would be $\mathcal{L}_A = (\partial_\mu A_\nu)(\partial^\mu A^\nu)$. However, this choice is not gauge invariant. A gauge invariant quantity which can be constructed from the derivatives of $A_\mu$ is the following 2-form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \quad (2.27)$$

Therefore, the Lagrangian describing the dynamics of the field $A_\mu$ invariant under the local gauge transformations (2.17) is

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (2.28)$$

The dynamics of the complex scalar field $\phi$ coupled with the $A_\mu$ field is described by the Lagrangian

$$\mathcal{L}_{\text{Total}} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m^2 \phi^* \phi - i \left[ -\phi \partial^\mu \phi^* + \phi^* \partial^\mu \phi \right] A_\mu + A_\mu A^\mu \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (2.29)$$

Let us, now, see what the dynamical equations for the field $A_\mu$ are. It is not difficult to prove, by varying the action constructed from (2.28), that the dynamical equations for $A_\mu$ are

$$\partial_\mu F^{\mu\nu} = - \mathcal{J}^\nu, \quad (2.30)$$

where

$$\mathcal{J}_\nu = i(\phi^* \partial_\nu \phi - \phi \partial_\nu \phi^*) - 2A_\nu \phi^* \phi. \quad (2.31)$$

These are nothing but the covariant version of Maxwell’s equations for the electromagnetic field, where $A_\mu$ is the vector potential.
We would like to mention, here, that this discussion has a nice geometrical interpretation. The final Lagrangian (2.29) can be written, by rearranging the terms, as follows

\[ \mathcal{L}_{CF} = (\nabla_{\mu} \phi^*) (\nabla_{\mu} \phi) - m^2 \phi^* \phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \]  

(2.32)

where we have defined a covariant derivative as

\[ \nabla_{\mu} = \partial_{\mu} + iA_{\mu}. \]  

(2.33)

This derivative is nothing but the covariant derivative defined on the $U(1)$ bundle which transforms covariantly under $U(1)$ transformations. In this interpretation, the vector potential $A_{\mu}$ is nothing but the connection coefficient. Since, in general, the covariant derivatives do not commute, their commutator is called curvature. The commutator of the covariant derivative on the $U(1)$ bundle is

\[ [\nabla_{\mu}, \nabla_{\nu}] \phi = iF_{\mu\nu} \phi. \]  

(2.34)

Therefore, the electromagnetic 2-form $F_{\mu\nu}$ is nothing but the curvature of the $U(1)$ bundle. This discussion shows that there exists a nice geometrical interpretation of the origin of the electromagnetic field as a gauge theory.

2.3 Effective Action in Quantum Field Theory

In this section we will describe the role of the effective action within the framework of quantum field theory. We will be mainly concerned, here, with the description of the quantization of non-gauge field theories (for a detailed description of the quantization of gauge field theories see [43]).

The basic object of any physical theory is the spacetime which is described
as a $n$-dimensional manifold, say $M$, with the following topological structure

$$M = I \times \Sigma,$$

(2.35)

where $I$ is a one-dimensional manifold diffeomorphic either to part of or to the whole real line, and $\Sigma$ is an $(n-1)$-dimensional manifold. The manifold $M$ is assumed to be globally hyperbolic and equipped with a pseudo-Riemannian metric. These conditions are sufficient for saying that the spacetime manifold $M$ possesses a foliation into spacelike sections diffeomorphic to $\Sigma$. This topology is necessary in order to have the correct causal structure of the spacetime. The additional structure of a vector bundle $\mathcal{V}$ can be defined over the spacetime manifold $M$, where each fiber is isomorphic to a vector space $V$. The sections of the vector bundle $\mathcal{V}$ over the manifold $M$, which we can denote by $\varphi^i$, are the fields. The fields describe different kinds of particles (depending on the structure of the vector bundle) in quantum field theory [28]. In what follows we will consider bosonic fields. The label “$i$” attached to the field represents a compact notation introduced by DeWitt and denotes not only its components but also the point in spacetime where the field is defined. This label can be considered as a set of two labels $i \equiv (A, x)$ where $A$ is a discrete index taking values from 1 to some $D$ associated with the field, and $x$ is the spacetime point [16], for instance we can write

$$\varphi^i \equiv \varphi^A(x) \quad \text{and} \quad \varphi^i' \equiv \varphi^A(x').$$

(2.36)

The set of all possible fields $\varphi^i$ on every point of the spacetime manifold is a manifold itself, $\mathcal{M}$, and is called configuration space.

The first and most important assumption in quantum field theory is that every isolated dynamical system is describable in terms of a characteristic action
functional, $S$, defined on the configuration space $\mathcal{M}$ with values on the real line,

$$S : \mathcal{M} \rightarrow \mathbb{R}.$$  \hfill (2.37)

The dynamics of the isolated system is described by the least action principle; In other words, the dynamics is determined by setting the first functional derivative of the action with respect to the independent fields to zero,

$$\frac{\delta S}{\delta \phi^i} = 0.$$  \hfill (2.38)

The fields that satisfy the above equation, and suitable boundary and initial conditions, are called dynamical fields. They form a subspace of the configuration space $\mathcal{M}$ called dynamical subspace which is often called, in quantum field theory, the *mass shell* [1].

Most of the problems in quantum field theory deal with processes of scattering of particles. In more details, in the remote past we have well defined measurable field states (or particles) which are described by the linearized equations of motion. As the system evolves in time, the field states interact in a specific finite region in the spacetime. The equations describing this interaction are highly nonlinear and cannot be solved exactly. After the interaction, in the remote future, we have again well defined measurable field states which, in general, are different from the initial states. We will call the initial state $|\text{in}\rangle$ and the final states $|\text{out}\rangle$ which are defined, respectively, in the remote past and in the remote future. The scattering process is, then, essentially described by the transition amplitude $\langle \text{in}|\text{out}\rangle$.

A powerful method for studying the transition amplitudes is given by the Schwinger variational principle which gives a relation between the variation of the
transition amplitude $\langle \text{in}|\text{out} \rangle$ and the variation of the action describing the dynamical system in the region of interaction. In more detail, the principle states that

$$
\delta \langle \text{in}|\text{out} \rangle = \frac{i}{\hbar} \langle \text{in}|\delta S|\text{out} \rangle .
$$

(2.39)

This principle is generally recognized as the principle of quantization because all the information about the quantum system can be derived from the above equation [16, 39, 43]. In general the action is replaced with a functional obtained by adding, to the previous action, a linear interaction with an external classical source $J_i$, namely

$$
S(\varphi) \rightarrow S(\varphi) + J_i \varphi_i ,
$$

(2.40)

where repeated indices mean a summation over the discrete labels and an integration over the continuous ones. Under this variation of the action functional the transition amplitude $\langle \text{in}|\text{out} \rangle$ becomes a functional of the external source $J_i$, in other words

$$
Z(J) = \langle \text{in}|\text{out} \rangle_{S(\varphi) \rightarrow S(\varphi) + J_i \varphi_i} .
$$

(2.41)

Some of the most important objects in quantum field theory are the chronological mean values of the quantum fields defined as

$$
\frac{\langle \text{in}|T(\varphi_i \cdots \varphi_i)|\text{out} \rangle}{\langle \text{in}|\text{out} \rangle} ,
$$

(2.42)

where $T$ represents the chronological ordering operator which orders the noncommuting quantum fields with respect to their time label from right to left.

By considering the following specific variation of the action

$$
\delta S = \delta J_i \varphi_i ,
$$

(2.43)
it can be shown \[16, 28, 39, 43\] that \(Z(J)\) is the generating functional for all the chronological amplitudes, namely

\[
Z(J + \eta) = \sum_{n \geq 0} \frac{i^n}{n!} \eta_i \cdots \eta_n \langle \text{in} | T(\varphi^i \cdots \varphi^i) | \text{out} \rangle .
\] (2.44)

The chronological amplitudes are easily obtained by repeatedly differentiating the above functional with respect to the auxiliary fields \(\eta\) and then setting \(\eta\) equal to zero.

From the functional \(Z(J)\) one can construct a new functional \(W(J)\) defined as follows

\[
Z(J) = e^{iW(J)} .
\] (2.45)

The utility of this newly introduced functional \(W(J)\) is soon recognized by taking its functional derivatives with respect to the external sources; explicitly, one obtains \[16, 28, 39, 43\]

\[
\langle \varphi^i \cdots \varphi^i \rangle = (-i)^n e^{-iW(J)} \frac{\delta^n}{\delta J_i \cdots \delta J_i} e^{iW(J)} ,
\] (2.46)

where \(\langle \varphi^i \cdots \varphi^i \rangle\) is the mean value of the quantum fields. In particular one has \[16\]

\[
\langle \varphi^i \rangle = \phi^i ,
\] (2.47)

\[
\langle \varphi^i \varphi^j \rangle = \phi^i \phi^j + \frac{1}{i} G^{ij} ,
\] (2.48)

\[
\langle \varphi^i \varphi^j \varphi^k \rangle = \phi^i \phi^j \phi^k + \frac{3}{i} \phi^i (G^{jk}) - G^{ijk} ,
\] (2.49)

where the parentheses () denote symmetrization over the included indices, \(\phi^i\) represents the background (or mean) field, \(G^{ij}\) is the one-point Green function (or propagator) and \(G^{ijk}\) is called multi-point Green function. To summarize, \(Z(J)\) is
the generating functional for the chronological products and \( W(J) \) is the generating functional for the Green functions.

It is clear that the mean field \( \phi \) is a functional of the external source \( J \). It is not difficult to show that the functional derivative of the mean field with respect to the external source is the propagator \( G^{ij} \). Therefore, if the matrix (propagator) \( G^{ij} \) is non-degenerate we can write the external source \( J \) as a functional of \( \phi \). By using this property, it can be shown [39, 43] that there exists a functional \( \Gamma(\phi) \), the \textit{effective action} which depends on the mean (background) field \( \phi \) and is the functional Legendre transform of \( W(J) \), namely

\[
\Gamma(\phi) = W(J) - J_i \phi^i \quad \text{(2.50)}
\]

In terms of the effective action the dynamical equations of the theory take the form

\[
\frac{\delta \Gamma}{\delta \phi^i} = -J_i \quad \text{(2.51)}
\]

\[
\frac{\delta^2 \Gamma}{\delta J_i \delta J_k} G^{km} = -\delta^m_i \quad \text{(2.52)}
\]

where the first equation represents the effective equations of motion determining the dynamics of the background field, and the second relation defines the full propagator of the theory, namely the propagator of the background field with regards to all quantum corrections. The higher functional derivatives of the effective action determine the full vertex functions. We can say, then, that the effective action is the generating functional of the full vertex functions. The vertex functions and the full propagator determine the full Green functions and, hence, the chronological amplitudes which, in turn, give the complete matrix of scattering processes. It is clear, at this point of the discussion, that the effective action is the most important object in the theory because it encodes all the information about the quantum fields.
An advantage of basing the theory on the effective action is that the external sources no longer appear. Moreover, as we will see later in this section, the functional integral representation of the effective action has a particularly suitable form for a perturbative analysis.

A very useful representation of the effective action is via the Feynman path integral. By integrating the Schwinger variational principle, one obtains the following expression for the (in|out) amplitudes, namely

$$
\langle\text{in}|\text{out}\rangle = \int_M D\varphi \exp\left\{ \frac{i}{\hbar} [S(\varphi) + J_k \varphi^k] \right\},
$$

(2.53)

where $D\varphi$ represents the functional measure defined on the configuration space. From this last expression it is possible to get a useful representation of the effective action, more precisely

$$
\exp\left\{ \frac{i}{\hbar} \Gamma(\phi) \right\} = \int_M D\varphi \exp\left\{ \frac{i}{\hbar} \left[ S(\varphi) - \frac{\delta \Gamma(\phi)}{\delta \phi^k}(\varphi^k - \phi^k) \right] \right\}.
$$

(2.54)

Strictly speaking this expression is purely formal, however meaningful results can be obtained in the framework of perturbation theory. For this reason it is convenient to use the semi-classical approximation of the effective action. One decomposes the effective action according to:

$$
\Gamma = S + \Sigma,
$$

(2.55)

where $\Sigma$ is called the self-energy functional which describes all the radiative corrections to the classical theory. The self-energy functional is computed in terms of an asymptotic expansion in powers of $\hbar$ as follows

$$
\Sigma \sim \sum_{k \geq 1} \hbar^k \Gamma_{(k)}.
$$

(2.56)
The next step is to substitute the above expansion for $\Gamma$ in the functional integral representation (2.54), and to make a change of variables in the functional integral

$$\varphi = \phi + \sqrt{\hbar} \, h,$$  \hspace{1cm} (2.57)

where $\phi$ is the background field and $h$ represents a small quantum perturbation. Because of the above change of variables, the measure in the functional integral (2.54) transforms as $D\varphi = Dh$ and the classical action in the functional integral is expanded in terms of $h$ as follows [28],

$$S(\phi + \sqrt{\hbar} \, h) = S(\phi) + \sqrt{\hbar} \frac{\delta S(\phi)}{\delta \phi^i} h^i - \frac{\hbar}{2} h^i \mathcal{L}_{ij}(\phi) h^j + \sum_{n=3}^{\infty} \frac{\hbar^{n/2}}{n!} \frac{\delta^{(n)} S(\phi)}{\delta \phi^{i_1} \cdots \delta \phi^{i_n}} h^{i_1} \cdots h^{i_n},$$ \hspace{1cm} (2.58)

where $\mathcal{L}$ is a partial differential operator, which is also called the operator of small disturbances, defined as the second variation of the classical action

$$\mathcal{L}_{ij} = -\frac{\delta^2 S}{\delta \varphi^i \delta \varphi^j}. \hspace{1cm} (2.59)$$

We would like to stress that in a non-gauge field theory this operator is non-degenerate, and therefore has a well defined Green function $G = \mathcal{L}^{-1}$ and a well defined functional determinant. In what follows we will consider non-gauge field theories. At this point all the needed quantities are expanded in powers of the small disturbances $\hbar$. The final step is to expand both sides of the functional integral relation (2.54) in powers of $\hbar$ and equate the terms of equal powers in $\hbar$. This expansion is called the loop expansion where the number of loops is given by the power of $\hbar$. The terms in the final expansion in $\hbar$ are functional integrals as well. However, since we previously expanded in powers of the small disturbances, the functional
integrals are of Gaussian form and can actually be computed. These Gaussian integrals contain the quadratic form \( h^i \mathcal{L}_{ij}(\phi) h^j \) and the result of the integration is written in terms of the functional determinant of the operator \( \mathcal{L} \), namely \( \text{Det} \mathcal{L} \) and also in terms of the bare propagator \( G = \mathcal{L}^{-1} \). In general one finds the following integrals

\[
\int_M \mathcal{D}h \exp \left\{ -\frac{i}{2} h^i \mathcal{L}_{ij} h^j \right\} = (\text{Det} \mathcal{L})^{-\frac{1}{2}}, \tag{2.60}
\]

\[
\int_M \mathcal{D}h \exp \left\{ -\frac{i}{2} h^i \mathcal{L}_{ij} h^j \right\} h^{k_1} \cdots h^{k_{2n+1}} = 0, \tag{2.61}
\]

\[
\int_M \mathcal{D}h \exp \left\{ -\frac{i}{2} h^i \mathcal{L}_{ij} h^j \right\} h^{k_1} \cdots h^{k_{2n}} = \frac{(2n)!}{2^n n!} \text{Det} \mathcal{L}^{-\frac{1}{2}} G^{(k_1 k_2 \cdots G^{(k_{2n-1} k_{2n})}}. \tag{2.62}
\]

The above expansion and the Gaussian integrals give a method to evaluate recursively all the terms \( \Gamma^{(k)} \) of the expansion of the effective action. Since we are particularly interested in the one-loop effective action, we will explicitly compute \( \Gamma^{(1)} \). By substituting the expansions (2.58) and (2.56) in the expression (2.54), one obtains

\[
\exp \left\{ i \Gamma^{(1)}(\phi) \right\} \exp \left\{ i \sum_{k=2}^{\infty} h^{k-1} \Gamma^{(k)}(\phi) \right\} = \int_M \mathcal{D}h \exp \left\{ -\frac{i}{2} h^i \mathcal{L}_{ij} h^j \right\} \exp \left\{ i \sum_{n=3}^{\infty} \frac{h^{n-1}}{n!} \frac{\delta^n S(\phi)}{\delta \phi^{i_1} \cdots \delta \phi^{i_n}} h^{i_1} \cdots h^{i_n} \right\}
\]

\[ - i \sum_{k=1}^{\infty} h^{k-\frac{1}{2}} \frac{\delta \Gamma^{(k)}}{\delta \phi^j} h^j \right\}. \tag{2.63}
\]

By equating the same power of \( h \) on both sides of this equation we obtain, in particular for the one-loop effective action, the following expression

\[
\exp \{ i \Gamma^{(1)}(\phi) \} = \int_M \mathcal{D}h \exp \left\{ -\frac{i}{2} h^i \mathcal{L}_{ij} h^j \right\}. \tag{2.64}
\]

By using the integral in (2.60) it is easy to show that \[16\ 43\]

\[
\Gamma^{(1)} = \frac{i}{2} \log \text{Det} \mathcal{L}. \tag{2.65}
\]
It is clear now that the effective action is a fundamental object in quantum field theory. All the information about quantum theory is encoded in the functional structure of the effective action, its functional derivatives give the full vertex functions and therefore the full propagators of the theory with are used to build the scattering matrix. We would like to stress that neither the classical action nor the self-energy functional are physical objects by themselves, only the effective action, \( \Gamma = S + \Sigma \), describes physical and measurable processes. If the self-energy functional has some divergent terms one can add equal and opposite counter-terms to the classical action. The coupling constants of these terms are the observable ones. The classical action with the addition of these counter-terms is called the renormalized classical action. This is, in a nutshell, the main idea of renormalization theory [16].

In a physical theory the effective action describes the in-out vacuum transition amplitude via

\[
\langle \text{out}|\text{in} \rangle = \exp[i\Gamma_{(1)}] .
\]

(2.66)

The real part of the effective action describes the polarization of the vacuum of quantum fields by the background fields and the imaginary part describes the creation of particles. Namely, the probability of production of particles (in the whole spacetime) is given by

\[
P = 1 - |\langle \text{out}|\text{in} \rangle|^2 = 1 - \exp[-2 \text{Im} \Gamma_{(1)}] .
\]

(2.67)

Unitarity requires that the imaginary part of the effective action should be non-negative

\[
\text{Im} \Gamma_{(1)} \geq 0 .
\]

(2.68)

Notice that when the imaginary part of the effective action is small, one has

\[
P \approx 2 \text{Im} \Gamma_{(1)} .
\]

(2.69)
The one-loop effective Lagrangian is defined by

\[ \Gamma_{(1)} = \int_M dx \, g^{1/2} \mathcal{L}. \]  

(2.70)

Therefore, the rate of particle production per unit volume per unit time is given by the imaginary part of the effective Lagrangian

\[ R = \frac{P}{VT} \approx 2 \text{ Im } \mathcal{L}. \]  

(2.71)

We would like to make a final remark here. In quantum field theory, the operator \( \mathcal{L} \) describes the propagation of small disturbances in the spacetime and is a hyperbolic operator. By performing a Wick rotation, \( t \rightarrow it \), Minkowski space gets mapped into the Euclidean space. In particular, the hyperbolic operator \( \mathcal{L} \) becomes an elliptic operator. The Euclidean formulation has some advantages: Elliptic operators have been intensively studied and important information is known about their spectrum. Moreover, it is easier to study the convergence of the integrals presented above resulting from the path integral formulation. Of course once a solution is found in Euclidean formulation, one can rotate back to Minkowski formulation and obtain the solution for Minkowski space. After a Wick rotation one obtains the Euclidean effective action which is defined by

\[ \Gamma_{(1)} = \frac{1}{2} \log \text{ Det } \mathcal{L}, \]  

(2.72)

where \( \varphi \) is the fermionic number of the field, (+1) for bosons and (−1) for fermions. This particular form of the one-loop effective action will be used in Chapter 4.

In this section we exposed the very basic and main ideas of the effective action approach to quantum field theory. A more detailed description of the subject (including gauge theories) can be found in the references that have been cited throughout the section.
2.4 The Heat Kernel Method

In the previous section we saw that the one-loop effective action is written in terms of the functional determinant of the operator of small disturbances which, in general, is a second order partial differential operator. The main challenge, at this point, is to find a formal way to deal with the determinant of an operator. The functional determinant is defined as a formal expression, and therefore needs to be regularized. In renormalizable field theories this procedure can be carried out in a consistent way. However, many field theories of physical interest, including General Relativity, are non-renormalizable. The short time asymptotic expansion of the trace of the heat kernel is constructed in terms of spectral invariants which determine the spectral asymptotics of the operator. Before starting to analyze the heat kernel methods, we will introduce the Laplace type operator on manifolds and present its main features.

2.4.1 Laplace Type Operators

We consider an $n$-dimensional manifold $M$ which is smooth, compact and without boundary equipped with a positive definite Riemannian metric $g$. The couple $(M, g)$ will denote a Riemannian manifold with the properties described above. The coordinates on $M$ will be denoted by $x^\mu$ where $\mu$ ranges over $\{1, \cdots n\}$. For any point $p$ on the manifold we can locally define the tangent space $T_p M$ to the manifold $M$ at the point $p$. The space $T_p M$ is the vector space of tangent vectors to $M$ at the point $p$. Moreover, the space $T^*_p M$ is the cotangent space to the manifold $M$ at the point $p$ and represents the space of the linear functionals, also called forms, acting on the tangent vectors.
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The tangent bundle $T_M$ is the disjoint union of the tangent spaces at each point of the manifold $M$. The notion of bundle can be easily generalized, in particular, to vector spaces $V$. A vector bundle $\mathcal{V}$ is the disjoint union of the vector spaces $V$ at each point of the manifold $M$, moreover the dual bundle $\mathcal{V}^*$ is the vector space of all the linear functionals defined on $\mathcal{V}$. A section of the vector bundle is a smooth map

$$\varphi : M \longrightarrow \mathcal{V},$$

such that at each point of the manifold $M$ it associates a vector in the vector bundle $\mathcal{V}$. It is easily recognized that this map represents a vector field defined on the manifold $M$ which we will denote by $\varphi^A$ where $A$ ranges over $\{1, \cdots, \dim V\}$. Of course this idea can be generalized to functions, tensors, etc. giving functions, tensor fields etc. defined on a manifold. The vector bundle $\mathcal{V}$ itself has the structure of a manifold. We equip the vector bundle with a non degenerate, Hermitian positive definite metric

$$E : \mathcal{V} \times \mathcal{V} \longrightarrow \mathbb{R},$$

which we will call the fiber metric on the vector bundle $\mathcal{V}$ and which can be naturally identified with the map

$$E : \mathcal{V} \longrightarrow \mathcal{V}^*.$$

Since we aim at introducing operators on manifolds, particularly the Laplacian, we need to construct a suitable space where the operator can be defined, namely a functional space. For $M$ we introduce the natural Riemannian volume element defined by the Riemannian metric $g_{\mu\nu}$ on the manifold $M$ as follows: $d\text{vol}(x) = dx\; g^{1/2}$, where $g = |\det g_{\mu\nu}|$. We introduce the set $C^\infty(\mathcal{V})$ of all smooth sections of the bundle
\( \mathcal{V} \), which is a vector space. Let \( \varphi \) and \( \psi \) be in \( C^\infty(\mathcal{V}) \). By using the fiber metric, in the above functional space we introduce the following inner product

\[
(\varphi, \psi) = \int_M d\text{vol}(x) \varphi^A(x) E_{AB}(x) \psi^B(x).
\]  

(2.76)

Therefore the space \( C^\infty(\mathcal{V}) \), equipped with this inner product, is an inner product space. The inner product space can be made into a normed space by defining the following norm

\[
\|\varphi\|^2 = (\varphi, \varphi) = \int_M d\text{vol}(x) \varphi^A(x) E_{AB}(x) \varphi^B(x).
\]  

(2.77)

By completing the space \( C^\infty(\mathcal{V}) \) with respect to this norm one obtains the Hilbert space \( \mathcal{L}^2(\mathcal{V}) \) of square integrable sections of the vector bundle \( \mathcal{V} \).

A connection on the vector bundle \( \mathcal{V} \) is a linear map

\[
\nabla : C^\infty(\mathcal{V}) \longrightarrow C^\infty(T^*M \otimes \mathcal{V}),
\]  

(2.78)

from the smooth sections of the bundle \( \mathcal{V} \) to 1-form valued sections of the bundle \( \mathcal{V} \) obeying the Leibnitz rule. We assume that the connection is compatible with the Hermitian metric on the vector bundle \( \mathcal{V} \). In a more explicit form we write

\[
\nabla_\mu \varphi = (\mathbb{I} \partial_\mu + \mathcal{A}_\mu) \varphi,
\]  

(2.79)

where \( \mathbb{I} \) represents the identity on the vector bundle \( \mathcal{V} \) and \( \mathcal{A}_\mu \) are the connection coefficients which bear a mixture of fiber and manifold indices. We would like to stress here that the operator defined above is a derivative operator. By explicitly writing all the indices we have

\[
\nabla_\mu \varphi^A = (\delta^A_B \otimes \partial_\mu + \mathcal{A}_\mu^A_B) \varphi^B.
\]  

(2.80)
We can also define the formal adjoint $\nabla^*$ of the derivative operator $\nabla$ with the help of the Riemannian metric on $M$ and the Hermitian structure on $\mathcal{V}$. Finally, let $Q \in C^\infty(\text{End}(\mathcal{V}))$ be a smooth section of the bundle of the endomorphisms of the vector bundle $\mathcal{V}$.

A Laplace type operator is a linear partial differential operator

$$\mathcal{L} : C^\infty(\mathcal{V}) \longrightarrow C^\infty(\mathcal{V}) ,$$

of the following form

$$\mathcal{L} = \nabla^* \nabla + Q = -g^{\mu\nu} \nabla_\mu \nabla_\nu + Q .$$

In local coordinates the Laplacian operator can be written, in a manifestly self-adjoint form, as

$$g^{\mu\nu} \nabla_\mu \nabla_\nu = \Delta = g^{-1/2}(\partial_\mu + A_\mu)g^{1/2}g^{\mu\nu}(\partial_\mu + A_\mu) .$$

One can write the above operator by explicitly separating the terms with second, first and zeroth order in the derivatives, namely it is possible to show that

$$\mathcal{L} = -g^{\mu\nu} \partial_\mu \partial_\nu + b^\mu \partial_\mu + c ,$$

where

$$b^\mu = -2g^{\mu\nu}A_\mu - g^{-1/2}\partial_\nu(g^{1/2}g^{\mu\nu}) ,$$

$$c = Q - g^{\mu\nu}A_\mu A_\nu - g^{-1/2}\partial_\mu(g^{1/2}g^{\mu\nu}A_\nu) .$$

Associated to any partial differential operator there is a function $\sigma(x, \xi)$ called the symbol,

$$\sigma : M \times T^*_xM \longrightarrow C^\infty(\text{End}(\mathcal{V})) ,$$
which is obtained from the operator by replacing the derivatives with a covector $i\xi_\mu$ (momentum). For the operator $\mathcal{L}$ considered above, it reads

$$\sigma(x, \xi) = \tilde{\gamma}^{\mu\nu}(x)\xi_\mu \xi_\nu + ib^\mu(x)\xi_\mu + c(x) .$$  \hspace{1cm} (2.88)

The leading symbol $\sigma_L$ of $\mathcal{L}$ is the part of the symbol with the highest power of the covector $\xi$. In the case we are analyzing, it has the form

$$\sigma_L(x, \xi) = \tilde{\gamma}^{\mu\nu}(x)\xi_\mu \xi_\nu .$$  \hspace{1cm} (2.89)

From the form of the Laplace type operator we can say that the second order part of $\mathcal{L}$ is determined by the metric $g^{\mu\nu}$ on the manifold $M$, the first order part of $\mathcal{L}$ is determined by the connection $\mathcal{A}_\mu$ on the vector bundle $\mathcal{V}$ and the zeroth order part of $\mathcal{L}$ is given by the endomorphism $Q$. It is important to say that any second order partial differential operator with a scalar leading symbol given by the metric is of Laplace type and can be put in the above form by a suitable choice of the connection and the endomorphism.

Second order partial differential operators can be classified by utilizing their leading symbol. For Laplace type operators this classification is equivalent to the following:

- The operator $\mathcal{L}$ is \textit{elliptic} if the eigenvalues of $g^{\mu\nu}$ are all different from zero and have the same sign.
- The operator $\mathcal{L}$ is \textit{hyperbolic} if the eigenvalues of $g^{\mu\nu}$ are all non-vanishing and all have the same sign except one that has the opposite sign.
- The operator $\mathcal{L}$ is \textit{parabolic} if the eigenvalues of $g^{\mu\nu}$ have all the same sign except one which is zero.
It can be proved that the Laplacian, and therefore the operator $\mathcal{L}$, is an elliptic and symmetric partial differential operator. An important property that we will assume for the second order partial differential operators in the rest of this work is for them to be self-adjoint. An operator is called essentially self-adjoint if for any $\varphi, \phi \in C^\infty(V)$,

$$(\mathcal{L} \varphi, \psi) = (\varphi, \mathcal{L} \psi).$$

(2.90)

It can be proved that we can always find a unique self-adjoint extension. Moreover, we will assume, from now on, that the operators have a positive definite leading symbol.

It is worth mentioning, here, that for a Laplace type operator the leading symbol is scalar. However, in Matrix Gravity we consider more general partial differential operators of non-Laplace type with non-scalar leading symbol. In fact, the coefficient of the second derivative of the operator we consider in Matrix Gravity bears two spacetime indexes and two fiber indexes making it a matrix-valued symmetric, non-degenerate tensor of type $(0, 2)$. A general second order partial differential operator acting on a vector bundle has the general form

$$L = -a^{\mu\nu}(x)\partial_\mu \partial_\nu + b^\mu(x)\partial_\mu + c(x),$$

(2.91)

with coefficients $(a^{\mu\nu}(x))_A^B$, $(b^\mu(x))_A^B$ and $(c(x))_A^B$. The operator $L$ is called of non-Laplace type if it is self-adjoint and if the leading symbol is not scalar, which means that it cannot be written as product of the identity $I$ on the bundle and a 2-times contravariant tensor. More explicitly, the leading symbol of a non-Laplace type operator is

$$\sigma_L(x, \xi) = a^{\mu\nu}(x)\xi_\mu \xi_\nu.$$
We will study non-Laplace type operators and their heat kernel asymptotic expansion in Chapter 6.

For an elliptic self-adjoint second order partial differential operator with positive definite leading symbol on a compact manifold, the following properties are well known [53]:

- The spectrum, \( \{ \lambda_n \}_{n=1}^{\infty} \) is real discrete and bounded from below

\[
\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots
\]

with some real constant \( \lambda_0 \).

- The eigenvalues have the following asymptotic behavior \( \lambda_k \sim Ck^{\frac{2}{n}} \) as \( k \to \infty \), where \( n = \dim(M) \).

- The eigenspaces are finite-dimensional.

- The eigenvectors, \( \{ \varphi_n \}_{n=1}^{\infty} \), are smooth sections of the vector bundle \( \mathcal{V} \) and form a complete orthonormal basis for the functional space \( L^2(\mathcal{V}) \).

### 2.4.2 Spectral Functions

In this section we will present some basic material regarding spectral functions, in particular the spectral zeta function. This particular object is of primary interest in quantum field theory because it gives a way to define the regularized functional determinant of an operator, which appears in the one-loop effective action. The spectrum of an operator contains important information both from the physical and mathematical points of view. That is why many efforts have been put
in order to find objects that would give knowledge about the spectrum. Of fundamental importance to the study of the spectrum of an operator are particular spectral invariants called spectral functions. Here, we will be mainly interested in two particular spectral functions, namely the heat trace and the spectral zeta function. The heat trace is the function $\Theta$ defined as

$$\Theta(t) = \sum_{n=1}^{\infty} e^{-t\lambda_n},$$

(2.93)

where $\lambda_n$ are the eigenvalues of the operator counted with their multiplicity. The heat trace is related, as we will show later in this Chapter, to the spectral zeta function and, therefore, to the one-loop effective action. The heat trace is a well defined function for positive $t$, moreover it can be analytically continued to the plane $\text{Re } t > 0$. We would like to mention that if the spectral functions are known exactly, then the spectrum is completely determined. Unfortunately, in general, the spectral functions are not known exactly. However, their asymptotic expansions are known, and they give important information about certain parts of the spectrum.

The spectral zeta function $\zeta$ is a generalization of the Riemann zeta function, defined as follows:

$$\zeta(s) = \sum_{n=1, \lambda_n \neq 0}^{\infty} \lambda_n^{-s},$$

(2.94)

where $\lambda_n$ are the non-vanishing eigenvalues of the operator counted with their multiplicity. The spectral zeta function can be analytically continued on the whole complex $s$-plane yielding a meromorphic function with only simple poles and regular at the origin. There is a nice representation of the spectral zeta function in terms of the heat trace. One can utilize the well known integral representation of the Gamma function

$$\Gamma(s) = \int_0^{\infty} dt \ t^{s-1} e^{-t}.$$
By performing the following change of variables \( t \rightarrow \lambda_n t \), it is not difficult to show that
\[
\lambda_n^{-s} = \frac{1}{\Gamma(s)} \int_{0}^{\infty} dt \; t^{s-1} e^{-\lambda_n t}.
\]
(2.96)

By taking the sum of this last expression one obtains
\[
\zeta(s) = \sum_{n=1, \lambda_n \neq 0}^{\infty} \lambda_n^{-s} = \frac{1}{\Gamma(s)} \int_{0}^{\infty} dt \; t^{s-1} \sum_{n=1, \lambda_n \neq 0}^{\infty} e^{-\lambda_n t}.
\]
(2.97)

Alternatively, by recalling the expression for the heat trace, one can write
\[
\zeta(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} dt \; t^{s-1} \left[ \Theta(t) - N \right],
\]
(2.98)

where \( N = \dim \ker \mathcal{L} \) is the number of zero eigenvalues of \( \mathcal{L} \). From this expression we clearly notice that the heat trace and the spectral zeta function are related to one another by a Mellin transformation.

In particular the spectral zeta function is used in order to define complex powers of an elliptic non-degenerate self-adjoint operator [53]. One can use this general result to evaluate the determinant of a suitable operator which appears in the semiclassical approximation of the path integral in quantum mechanics and quantum field theory. The relation between the spectral zeta function and the determinant of an operator can be formally shown as follows. Let \( \mathcal{L} \) be a Laplace type operator. Because of its discrete spectrum one defines the logarithm of the determinant of \( \mathcal{L} \) in complete analogy to the finite dimensional case, namely
\[
\log \Det_N(\mathcal{L}) = \log \prod_{n=1, \lambda_n \neq 0}^{N} \lambda_n = \sum_{n=1, \lambda_n \neq 0}^{N} \log(\lambda_n).
\]
(2.99)

Now, by recalling the expression for the spectral zeta function, and taking the derivative with respect to the parameter \( s \), one gets
\[
\zeta_N'(s) = - \sum_{n=1, \lambda_n \neq 0}^{N} \lambda_n^{-s} \log(\lambda_n).
\]
(2.100)
By setting $s = 0$ in the last expression, one obtains a formula for $\log \text{Det}_N(\mathcal{L})$, more explicitly

$$\log \text{Det}_N(\mathcal{L}) = -\zeta''(0).$$  \hfill (2.101)

In particular the determinant of the operator $\mathcal{L}$ can be obtained by taking the limit as $N \to \infty$ and by exponentiating the last expression, more explicitly

$$\text{Det}(\mathcal{L}) = e^{-\zeta'(0)}. \hfill (2.102)$$

In order to relate the spectral zeta function to the one-loop effective action in quantum field theory, we recall equation (2.65). The operator of small disturbances on the right hand side of equation (2.65) is an elliptic, non-degenerate and self-adjoint second order partial differential operator. By using the relation obtained above, one can write an expression for the one-loop effective action in the zeta function regularization as follows

$$\Gamma_{(1)} = -\frac{i}{2} \zeta'(0).$$  \hfill (2.103)

Since the spectral zeta function and the heat trace are related by the Mellin transform, we can find an expression for the one-loop effective action and the heat trace. By taking the derivative with respect to the parameter $s$ in (2.98) it is not difficult to show that

$$\Gamma_{(1)} = -\frac{i}{2} \int_0^\infty dt \frac{1}{t^4} \Theta(t). \hfill (2.104)$$

The one-loop effective action will be of primary interest in Chapter 5 where we will evaluate its imaginary part. This integral needs to be regularized because it diverges at $t = 0$ (ultraviolet divergences), moreover it could also diverge at $t = \infty$ (infrared divergence). The integral can be regularized by means, for example, of a cutoff regularization. It is clear, now, from the last expression, that the knowledge of the
heat trace is of fundamental importance in order to obtain information about the one-loop effective action. Unfortunately the heat trace cannot be computed exactly in many interesting cases. However, one can find suitable asymptotic expansions to get some information on \( \Gamma_{(1)} \). In the next section we will analyze the heat kernel (and in particular the heat trace) and some methods for the computation of its asymptotic expansion.

### 2.4.3 The Heat Kernel

Let \( \mathcal{L} \) be the operator that we utilized so far. For \( t > 0 \) the one-parameter family of operators

\[
U(t) = \exp(-t\mathcal{L}) ,
\]

forms a semigroup of bounded operators on \( L^2(V) \) which is called the heat semigroup. Associated with the heat semigroup one can define the heat kernel as follows

\[
U(t|x, x') = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \otimes \varphi_n^*(x') ,
\]

where \( \lambda_n \) are the eigenvalues of the operator \( \mathcal{L} \) counted with their multiplicity, and \( \varphi_n \) are the corresponding eigenfunctions. The heat kernel satisfies the following partial differential equation

\[
(\partial_t + \mathcal{L}) U(t|x, x') = 0 ,
\]

with the initial condition

\[
U(0|x, x') = \delta(x, x') ,
\]
where $\delta(x, x')$ represents the covariant Dirac delta function. It can be shown [53] that the heat semigroup is a trace-class operator, namely its $L^2$-trace is well defined

$$\text{Tr} \exp(-tL) = \int_M d\text{vol} \text{tr}_V U(t|x, x) ,$$  \hspace{1cm} (2.109)

where $\text{tr}_V$ represents the trace over the vector bundle indexes. By using the definition of the $L^2$-trace and the explicit expression (2.106) for $U(t|x, x')$, it is easy to realize that the heat trace, in (2.93), is equal to the trace of the heat semigroup

$$\Theta(t) = \text{Tr} \exp(-tL) .$$  \hspace{1cm} (2.110)

It is completely clear, now, that the knowledge of the trace of the heat kernel is equivalent to the knowledge of the one-loop effective action.

### 2.4.4 Asymptotic Expansion of the Heat Kernel

As we mentioned earlier, the heat kernel, and therefore its trace, cannot be evaluated explicitly in many cases of interest. For this reason some approximation schemes have been developed. It is important, at this point, to briefly introduce some two-point geometric quantities that will be used throughout the Dissertation. A more complete introduction to this subject can be found in [43, 74].

Let us fix a point, say $x'$, on the manifold $M$ and consider a sufficiently small neighborhood of $x'$, say a geodesic ball with a radius smaller than the injectivity radius of the manifold. Then, it can be proved that there exists a unique geodesic that connects every point $x$ to the point $x'$. In what follows we will restrict ourselves to this neighborhood. In order to avoid a cumbersome notation, we will denote by *Latin letters* tensor indices associated to the point $x$ and by *Greek letters* tensor indices associated to the point $x'$. Of course, the indices associated with the point $x$
(resp. \(x'\)) are raised and lowered with the metric at \(x\) (resp. \(x'\)). Also, we will denote by \(\nabla_a\) (resp. \(\nabla'_\mu\)) the covariant derivative with respect to \(x\) (resp. \(x'\)). We will use the standard notation of square brackets to denote the coincidence limit of two-point functions, more precisely, for any functions of \(x\) and \(x'\) we define

\[
[f](x) \equiv \lim_{x \to x'} f(x, x') .
\]

The world function \(\sigma(x, x')\) is defined as one half of the square of the length of the geodesic between the points \(x\) and \(x'\). It satisfies the equation \([5, 14, 43, 39]\)

\[
\sigma = \frac{1}{2} u^a u_a = \frac{1}{2} u_\mu u^\mu ,
\]

where

\[
u_a = \nabla_a \sigma, \quad \nu_\mu = \nabla'_\mu \sigma .
\]

The variables \(u^\mu\) are nothing but the normal coordinates with the origin at the point \(x'\). Next, one defines the tensors which are the second derivative of the world function \([5, 14]\)

\[
\eta^a_b = \nabla_b \nabla'^a \sigma , \quad \xi^a_b = \nabla^a \nabla_b \sigma ,
\]

and the tensor

\[
X^{\mu\nu} = \eta^\mu_a \eta^\nu_a .
\]

In particular, one needs the tensor \(\gamma^a_\mu\) inverse to \(\eta^a_\mu\) defined by

\[
\gamma^a_\mu \eta^\mu_b = \delta^a_b , \quad \eta^\mu_a \gamma^b_\nu = \delta^a_\nu .
\]

The two-point quantities defined above satisfy the following equations \([5, 14, 43, 39]\)

\[
\xi^a_b u^b = u^a , \quad \eta^\mu_a u^\mu = u^\mu , \quad \eta^\mu_a u_\mu = u_a .
\]
with the boundary conditions

\[ [\sigma] = [u^\mu] = [u^\rho] = 0 , \]  
\[ (2.118) \]

\[ [\xi^\mu_a] = [\delta^\mu_b] , \quad [\eta^\mu_a] = -[\delta^\mu_a] . \]  
\[ (2.119) \]

Another useful two-point quantity is the Van Vleck-Morette determinant which is defined as

\[ \Delta(x, x') = g^{-1/2}(x) \det[-\nabla_a \nabla'_a \sigma(x, x')] g^{-1/2}(x') . \]  
\[ (2.120) \]

This quantity should not be confused with the Laplacian \( \Delta = g^{ab} \nabla_a \nabla_b \). Usually, the meaning of \( \Delta \) will be clear from the context. Following [5, 14], we find it convenient to parameterize it by

\[ \Delta(x, x') = \exp[2\zeta(x, x')] . \]  
\[ (2.121) \]

Next, one defines new derivative operators by [5, 14]

\[ \bar{\nabla}_\mu = \gamma^a_\mu \nabla_a . \]  
\[ (2.122) \]

These operators commute when acting on objects that have been parallel transported to the point \( x' \) (in other words the objects that do not have Latin indices). In fact, when acting on such objects these operators are just partial derivatives with respect to normal coordinate \( u \)

\[ \bar{\nabla}_\mu = \frac{\partial}{\partial u^\mu} . \]  
\[ (2.123) \]

Next, the parallel displacement operator \( \mathcal{P}(x, x') \) of sections of the vector bundle \( \mathcal{V} \) along the geodesic from the point \( x' \) to the point \( x \) is defined as the solution of the equation [5, 14, 43, 39]

\[ u^{\rho} \nabla_{\rho} \mathcal{P}(x, x') = 0 , \]  
\[ (2.124) \]
with the initial condition

\[ [\mathcal{P}] = 1 \]  
(2.125)

It is not difficult to show that the parallel displacement operator satisfies the equation

\[ [\nabla_a, \nabla_b] \mathcal{P} = \mathcal{R}_{ab} \mathcal{P}, \]  
(2.126)

where \( \mathcal{R}_{ab} \) represents the curvature of the connection on the vector bundle \( \mathcal{V} \). Finally, one defines the two-point quantity which is the derivative of the parallel transport operator \([5, 14]\)

\[ \mathcal{A}_\mu = \mathcal{P}^{-1} \bar{\nabla}_\mu \mathcal{P}. \]  
(2.127)

It is important, for future reference, to present here the coincidence limits of higher derivatives of the two-point functions introduced above. They are expressed in terms of the curvature and, in particular, the ones that we will need are found to be \([14, 5, 43, 39]\)

\[ [\zeta, a] = 0 \]  
(2.128)
\[ [\zeta, bc] = \frac{1}{6} R_{bc} \]  
(2.129)
\[ [\eta^{\mu}_{a; b}] = 0 \]  
(2.130)
\[ [\eta^{\mu}_{c, ab}] = -\frac{1}{3} (R^{\mu}_{\nu \alpha \beta} + R^{\mu}_{\alpha \nu \beta}) \]  
(2.131)
\[ [\mathcal{A}_\nu] = 0 \]  
(2.132)
\[ [\mathcal{A}_c; b] = -\frac{1}{2} R_{vb} \]  
(2.133)

where \( R^{\mu}_{\nu \alpha \beta} \) is the Riemann tensor and \( R_{\mu \nu} = R^{\alpha}_{\mu \alpha \nu} \) is the Ricci tensor.

We will briefly present, here, the role of the previous geometric quantities in the evaluation of the heat kernel and, therefore, of its trace. We consider a second
order partial differential operator of Laplace type, \( \mathcal{L} = -\Delta + Q \). The heat kernel cannot be computed exactly in this case, however, one can find an asymptotic expansion for small time following \([5, 14, 39]\). We know that the heat kernel \( U(t|x, x') \) satisfies the equation (2.107) with the initial condition (2.108). To start, we consider the following ansatz,

\[
U(t|x, x') = A(t; x, x') \exp \left\{ -\frac{S(x, x')}{t} \right\},
\]

where \( S \) represents the action (2.37). By substituting the ansatz in the heat equation it is not difficult to show that

\[
\partial_t A + \frac{1}{t^2} \left[ S - g^{ab}(\nabla_a S)(\nabla_b S) \right] A + \frac{1}{t} \left[ 2g^{ab}(\nabla_a S)\nabla_b - \Delta S \right] A + (-\Delta + Q) A = 0.
\]

Since we are considering the asymptotic expansion for small \( t \), we set the coefficient of \( t^{-2} \) in the previous equation to zero, by doing so we obtain

\[
g^{ab}(\nabla_a S)(\nabla_b S) = S.
\]

These are nothing but the Hamilton-Jacobi equations for the action \( S \). By using (2.112), it can be proved that the solution of this equation is

\[
S(x, x') = \frac{\sigma(x, x')}{2}.
\]

By substituting the explicit solution for \( S \) in (2.135), we obtain

\[
\partial_t A + \frac{1}{t} \left[ u^a \nabla_a + \frac{1}{2} \xi_a^a \right] A = -(-\Delta + Q) A.
\]

At this point we introduce an ansatz for the function \( A(t; x, x') \). By careful inspection of the previous equation it is useful to write

\[
A(t; x, x') = (4\pi t)^{-n/2} \Delta^{\frac{1}{2}}(x, x') \mathcal{P}(x, x') \Omega(t; x, x'),
\]

where \( x \) and \( x' \) are the space-time points, and \( \Delta \) is the Laplacian operator.
where $\mathcal{P}$ is the parallel transport operator defined in (2.124) and the choice of the factor $(4\pi t)^{-n/2}$ will assure that the solution satisfies the initial condition (2.108). By utilizing this last ansatz we obtain

$$\partial_t \Omega + \frac{1}{2t} \left[ \Delta^{-1} u^a (\nabla_a \Delta) + \xi_a^a - n \right] \Omega + \frac{1}{t} u^a \nabla_a \Omega = -\Delta^{-1/2} \mathcal{P}^{-1} (-\Delta + Q) \mathcal{P} \Delta^{1/2} \Omega .$$

(2.140)

At this point, it can be proved [5, 14, 39] that the Van Vleck-Morette determinant (2.120) satisfies the equation

$$\Delta^{-1} u^a \nabla_a \Delta + \xi_a^a - n = 0 ,$$

(2.141)

By using this remark the equation for $\Omega$ becomes

$$\left[ \partial_t + \frac{1}{t} u^a \nabla_a + \Delta^{-1/2} \mathcal{P}^{-1} (-\Delta + Q) \mathcal{P} \Delta^{1/2} \right] \Omega = 0 .$$

(2.142)

Since we want an asymptotic expansion as $t \to 0$ we write $\Omega$ as asymptotic series

$$\Omega(t; x, x') \sim \sum_{k=0}^{\infty} a_k(x, x') t^k .$$

(2.143)

By substituting this expression in the equation above we finally obtain an expression for the asymptotic expansion of the heat kernel

$$U(t|x, x') \sim (4\pi t)^{-n/2} \Delta^{1/2} \mathcal{P} \mathcal{P} (x, x') \exp \left\{ -\frac{\sigma(x, x')}{2t} \right\} \sum_{k=0}^{\infty} a_k(x, x') t^k ,$$

(2.144)

where the heat kernel coefficients $a_k$ satisfy the DeWitt recurrence relation

$$[(k + 1) + u^a \nabla_a] a_{k+1}(x, x') = \Delta^{-1/2} \mathcal{P}^{-1} (-\Delta + Q) \mathcal{P} \Delta^{1/2} a_k(x, x') ,$$

(2.145)

with the initial condition

$$a_0(x, x') = 1 .$$

(2.146)
This expansion will be generalized in Chapter 3 in order to include, in a non-perturbative way, the electromagnetic field.

The lower order diagonal heat kernel coefficients are well known and have the form \[53, 5, 14\]

\[a_{\text{diag}}^0 = 1, \quad (2.147)\]

\[a_{\text{diag}}^1 = \frac{1}{6} R, \quad (2.148)\]

\[a_{\text{diag}}^2 = \frac{1}{30} \Delta R + \frac{1}{72} R^2 - \frac{1}{180} R_{\mu\nu} R^{\mu\nu} + \frac{1}{180} R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu} + \frac{1}{12} R_{\mu\nu} R^{\mu\nu}, \quad (2.149)\]

where \(a_{\text{diag}}^k = a_k(x, x)\). To avoid confusion we should stress that the normalization of the coefficients \(a_k\) differs from the papers \[5, 13, 14\].

It can be easily proven, by taking the trace of both sides of (2.144), that the heat trace, which is of primary interest here, has the following asymptotic expansion

\[\text{Tr} \exp\{-t\mathcal{L}\} \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k A_k, \quad (2.150)\]

where \(A_k\) are global heat kernel coefficients

\[A_k = \int_M d\text{vol} \text{tr}_V a_{\text{diag}}^k. \quad (2.151)\]

One should point out, here, that the heat trace for a non-Laplace type operator has the same asymptotic expansion as \(t \to 0\) \[53\].

In the next section we will describe two methods for the evaluation of the heat kernel asymptotic expansion: the covariant Taylor expansion method which we
use in the computation of the non-perturbative heat kernel asymptotics on homogeneous Abelian bundles in Chapters 3 and 4, and the covariant Fourier transform method that we use for the evaluation of the action in Spectral Matrix Gravity in Chapter 6.

2.4.5 Covariant Taylor Expansion and Covariant Fourier Transform

In this section we will briefly discuss two methods for evaluating the coefficients of the heat kernel asymptotic expansion which are different from the DeWitt method presented in the previous section. In the previous section we described some two-point geometric functions that are widely used for heat kernel calculations on Riemannian manifolds. Since we ultimately want to find an expansion for the heat kernel and its trace, we need to develop an expansion for the two-point quantities. In the first part of this section we will briefly describe the Taylor expansion of the two-point quantities following [5, 14]. A more detailed discussion on this subject can be found in [5, 14].

Let us consider, as before, two neighboring points $x$ and $x'$ connected by a unique geodesic. The derivatives, $u^\mu$, of the world function $\sigma(x, x')$ form a set of normal coordinates for the neighborhood, $\mathcal{U}$, under consideration. Our aim is to find an expansion in terms of the normal coordinates for functions defined on $\mathcal{U}$. Since the language that we use is covariant, we need an expansion which is independent on the system of coordinates that we choose. It is evident, from the discussion above, that the quantity $u^\mu$ is a vector at the point $x'$ and a scalar at the point $x$. Since scalars are the simplest invariant quantities, one can develop the Taylor expansion for a scalar function $f$ at the point $x$ by using the coordinate $u^\mu$. By parameterizing the geodesic between $x$ and $x'$ with an affine parameter one can show that for a scalar
function we obtain the expansion \([14, 5]\)

\[
f(x) = \sum_{n=0}^{\infty} \left( -1 \right)^n \frac{n!}{n!} u_0^a \cdots u_0^n \left[ g_{\mu_1}^{a_1} \cdots g_{\mu_n}^{a_n} \nabla_{a_1} \cdots \nabla_{a_n} f \right],
\]

(2.152)

where \(g_{\mu}^a\) is the parallel transport of covectors from the point \(x\) to the point \(x'\). By multiplying this expression by the parallel transport operators (2.124), as many as needed, we can find an expression for the covariant Taylor expansion of arbitrary tensors. In particular, we can find the covariant Taylor expansion for the two-point quantities that we need. It can be shown that [14, 5]

\[
\eta^\mu_b = g_b^{\nu} \left[ -\frac{1}{3} R^\mu_{\alpha \beta} u^\alpha u^\beta - \frac{1}{2} \nabla_\alpha R^\mu_{\beta \gamma} u^\beta u^\gamma + \frac{3}{5} \nabla_\alpha \nabla_\beta R^\mu_{\gamma \delta} u^\gamma u^\delta \right. \\
- \left. \frac{7}{15} R^\mu_{\alpha \beta \gamma} R^\nu_{\gamma \delta} u^\alpha u^\beta u^\gamma u^\delta + O(u^5) \right],
\]

(2.153)

\[
\zeta = \frac{1}{12} R_{\alpha \beta} u^\alpha u^\beta - \frac{1}{24} \nabla_\alpha R_{\beta \gamma} u^\beta u^\gamma + \frac{1}{80} \nabla_\alpha \nabla_\beta R_{\gamma \delta} u^\gamma u^\delta \\
+ \frac{1}{360} R_{\mu \nu \alpha \beta} R^\mu_{\alpha \beta \gamma \delta} u^\alpha u^\beta u^\gamma u^\delta + O(u^5),
\]

(2.154)

\[
\Delta^{1/2} = 1 + \frac{1}{12} R_{\alpha \beta} u^\alpha u^\beta - \frac{1}{24} \nabla_\alpha R_{\beta \gamma} u^\beta u^\gamma + \frac{1}{80} \nabla_\alpha \nabla_\beta R_{\gamma \delta} u^\gamma u^\delta \\
+ \frac{1}{288} R_{\alpha \beta} R_{\gamma \delta} u^\alpha u^\beta u^\gamma u^\delta + \frac{1}{360} R_{\mu \nu \alpha \beta} R^\mu_{\alpha \beta \gamma \delta} u^\alpha u^\beta u^\gamma u^\delta + O(u^5),
\]

(2.155)

\[
X^{\mu \nu} = g^{\mu \nu} + \frac{1}{3} R^\mu_{\alpha \beta} u^\alpha u^\beta - \frac{1}{6} \nabla_\alpha R^\mu_{\beta \gamma} u^\beta u^\gamma + \frac{1}{20} \nabla_\alpha \nabla_\beta R^\mu_{\gamma \delta} u^\gamma u^\delta \\
+ \frac{1}{15} R^\mu_{\alpha \beta \gamma} R^\nu_{\alpha \beta \gamma} u^\alpha u^\beta u^\gamma u^\delta + O(u^5),
\]

(2.156)

\[
\mathcal{A}_\mu = -\frac{1}{2} R_{\mu \alpha} u^\alpha + \frac{1}{3} \nabla_\alpha R_{\mu \beta} u^\alpha u^\beta + \frac{1}{24} R_{\mu \alpha \beta \gamma} R^\mu_{\beta \alpha \gamma} u^\beta u^\gamma + \frac{1}{8} \nabla_\alpha \nabla_\beta R_{\mu \gamma} u^\beta u^\gamma + O(u^5).
\]

(2.157)
where the above quantities have been defined, respectively, in (2.114), (2.121), (2.120), (2.115) and (2.127). We would like to stress that all coefficients of such expansions are evaluated at the point \( x' \).

Here, we will briefly describe the covariant Fourier transform which we will use in Chapter 6 to evaluate the heat kernel asymptotic coefficients for a non-Laplace type operator. The Fourier integral can be defined by using the two-point functions that we introduced earlier. Let \( f \) be a function defined on \( \mathcal{U} \). Its covariant Fourier transform is defined as follows \[5, 14\]

\[
f(x) = \int_{\mathbb{R}^n} \frac{dk}{(2\pi)^n} g^{-1/2}(x') e^{-ik_{\mu}u_{\mu}} \hat{f}(k; x'),
\]

(2.158)

where \( k_{\mu} \) represents the coordinate in the momentum space. The inverse Fourier transform is written as

\[
\hat{f}(k; x') = \int_M du g^{1/2}(x') e^{ik_{\mu}u_{\mu}} f(x).
\]

(2.159)

We can transform the last integral from normal coordinates to local coordinates. It is not difficult to show, by making a change of variables, that

\[
du = g^{1/2}(x)g^{-1/2}(x')\Delta(x, x')dx.
\]

(2.160)

By using this formula one gets \[5, 14\]

\[
\hat{f}(k; x') = \int_M dx \Delta(x, x') g^{1/2}(x) e^{ik_{\mu}u_{\mu}} f(x).
\]

(2.161)

We would like to mention, here, that by multiplying this expression by the parallel transport operators, as many as needed, we can find an expression for the covariant Fourier transform of arbitrary tensors. At this point it is convenient to derive a
representation of the covariant Dirac delta function in terms of the Fourier integral.
It is not difficult to show, by substituting (2.161) into (2.158), that one obtains
\[ \delta(x, x') = \Delta^{1/2}(x, x') \int_{\mathbb{R}^n} \frac{dk}{(2\pi)^n} \exp\{i k \mu d\mu\}. \] (2.162)
The covariant Fourier transform represents another important tool for the evaluation of the heat kernel asymptotic expansion which we will mainly use in Chapter 6.

2.4.6 Perturbation Theory for the Heat Semigroup

The operators for which we want to evaluate the heat kernel asymptotic expansion are known in terms of power series in a small formal parameter \(\varepsilon\), namely
\[ \mathcal{L} = \mathcal{L}_0 + \sum_{k=1}^{\infty} \mathcal{L}_k, \] (2.163)
where \(\mathcal{L}_0\) represents the unperturbed part and \(\mathcal{L}_k\) is of order \(\varepsilon^k\).

In order to evaluate the heat kernel for the operator \(\mathcal{L}\) we need to compute, first, the heat semigroup for \(\mathcal{L}\). Since \(\mathcal{L}\) is given in terms of a perturbative series, we are faced with the problem of finding an expansion for the exponent of two non-commuting operators. The expansion for the exponent of two non-commuting operators, known in the literature, is called Volterra series.

Let \(A\) and \(B\) be arbitrary, non-commuting operators, then the following can be proved \([22, 76]\)
\[ e^{A+B} = e^A + \sum_{k=1}^{\infty} \int_0^1 d\tau_k \int_0^{\tau_k} d\tau_{k-1} \cdots \int_0^{\tau_2} d\tau_1 e^{(1-\tau_k)A}Be^{(\tau_k-\tau_{k-1})A} \cdots e^{(2-\tau_1)A}Be^{\tau_1A}. \] (2.164)
By considering only few of the low order terms (the ones we will use in this work) we write

\[ e^{A+B} = e^A + \int_0^1 d\tau_1 e^{(1-\tau_1)A} Be^{\tau_1 A} \]

\[ + \int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 e^{(1-\tau_2)A} Be^{(\tau_2-\tau_1)A} Be^{\tau_1 A} + \cdots. \]  

(2.165)

This series can be also written in another way. Let us suppose that the operator \( A \) is of zeroth order (unperturbed) and the operator \( B \) represents higher orders in the perturbation theory. The Volterra series can be written in terms of a specific operator \( T \) acting on the unperturbed semigroup as follows \[22, 76\]

\[ \exp(A + B) = T \exp A, \]  

(2.166)

where

\[ T = I + \sum_{k=1}^\infty \int_0^{\tau_k} d\tau_k \int_{\tau_{k-1}}^{\tau_2} d\tau_{k-1} \cdots \int_0^{\tau_1} d\tau_1 \tilde{B}(\tau_1)\tilde{B}(\tau_2) \cdots \tilde{B}(\tau_k) \]  

(2.167)

and

\[ \tilde{B}(\tau) = e^{\tau A}Be^{-\tau A}. \]  

(2.168)

This particular form of the Volterra series will be utilized in Chapter 3.

### 2.5 Mathematical Framework in Matrix Gravity

#### 2.5.1 Motivation and Discussion

In the rest of the present Chapter, we will describe the motivations and the mathematical framework of the second major topic in this Dissertation, namely Matrix Gravity. The main idea in Matrix Gravity is to describe the gravitational field as a matrix-valued symmetric two-tensor field. It is well known that General
Relativity is nothing but the dynamical theory of the metric 2-tensor field which is, basically, an isomorphism between tangent and cotangent bundles. The dynamics of the metric is described by the Hilbert-Einstein action,

\[ S_{\text{HE}} = \frac{1}{16\pi G} \int_M dx \, g^{1/2} (R - 2\Lambda) , \]

where \( G \) is the Newtonian gravitational constant, \( \Lambda \) is the cosmological constant and \( R \) is the scalar curvature.

In Matrix Gravity, the metric 2-tensor field \( g^{\mu\nu} \), is replaced by an endomorphism-valued 2-tensor field \( a^{\mu\nu} \) which represents an isomorphism of more general bundles over the manifold \( M \). The main idea here, similar to General Relativity, is to develop a dynamical theory of this endomorphism-valued 2-tensor field \( a^{\mu\nu} \). This generality brings a much richer structure and content to the model.

We would like to stress, at this point, that the dynamical equations that we will derive in this Dissertation for Matrix Gravity, are classical and therefore they should be studied from the classical point of view.

The motivation for such a deformation of General Relativity is explained in detail in [20]. The very basic physical concepts are the notions of event and the spacetime. An event is a collection of variables that specifies the location of a point in space at a certain time. To assign a time to each point in space one needs to place clocks at every point (say on a lattice in space) and to synchronize these clocks. Once the position of the clocks is fixed the only way to synchronize the clocks is by transmitting the information from a fixed point (say, the origin of the coordinate system in space) to all other points. This can be done by sending a signal through space from one point to another. Therefore, the synchronization procedure depends
on the propagation of the signal through space, and, as a result, on the properties of the space it propagates through, in particular, on the presence of any physical background fields in space. The propagation of signals is described by a wave equation (a hyperbolic partial differential equation of second order). Therefore, the propagation of a signal depends on the matrix of the coefficients (a symmetric 2-tensor) $g^{\mu\nu}(x)$ of the second derivatives in the wave equation which must be non-degenerate and have the signature $(- + \cdots +)$. This matrix can be interpreted as a pseudo-Riemannian metric, which defines the geodesic flow, the curvature and the Einstein equations of General Relativity (for more details, see [20]).

The picture described above applies to the propagation of light, which is described by a single wave equation. However, now we know that at microscopic scales there are other fields that could be used to transmit a signal. In particular, the propagation of a multiplet of $N$ gauge fields is described not by a single wave equation but by a hyperbolic system of second order partial differential equations. We would like to stress, here, that the number $N$ of gauge fields that one should use in order to describe the gravitational field is not known at this time. It is possible that a quantized version of this theory could shed light on the precise number of gauge fields to use, however this problem needs further research. For simplicity of calculations, in Chapter 7 we analyze a model of Matrix Gravity where two gauge fields are taken into account. For now, we will leave $N$ arbitrary.

The coefficients at the second derivatives of such a system are not given by just a 2-tensor like $g^{\mu\nu}(x)$ but by a $N \times N$ matrix-valued symmetric 2-tensor $a^{\mu\nu}(x)$ as in (2.91). If $a^{\mu\nu}$ does not factorize as $a^{\mu\nu} \neq \Xi g^{\mu\nu}$, where $\Xi$ is some non-degenerate matrix, then there is no geometric interpretation of this hyperbolic system in terms
of a single Riemannian metric. Instead, one obtains a new kind of geometry that is called *Matrix Geometry*, which is equivalent to a collection of Finsler geometries. In this theory, instead of a single Riemannian geodesic flow, there is a system of $N$ Finsler geodesic flows. Moreover, a gravitating particle is described not by one mass parameter but by $N$ mass parameters (which could be different). The general idea is similar to the concept of colors in quantum chromodynamics. In Matrix Gravity each particle is considered to be composed of $N$ different “colors” each of them described by a different mass parameter. Each of these colors follows its own Finsler geodesic.

In this sense, the introduction of a matrix-valued metric $a^{\mu\nu}$ leads to the splitting of a single Riemannian geodesic to a system of $N$ close Finsler geodesics. We argue that at microscopic distances, and high energies, a single Riemannian geodesic is described by a system of $N$ Finsler geodesics. As we will see in Chapter 7 the equations of the geodesics in Matrix Gravity are non-linear with respect to the mass parameters $\mu_i$. This makes it not possible to write the geodesics in Matrix Gravity as a weighted sum (with weight $\mu_i$) of all the Finsler geodesics for each mass parameter. However, if one consider the following splitting $\mu_i = 1/N + \alpha_i$ where $\alpha_i$ are assumed to be small, one could linearize the equations for the geodesics and obtain, in the first order, a description of the geodesics in Matrix Gravity as weighted sum (with weights $\alpha_i$) of the single Finsler geodesics.

Notice that because the tensor $a^{\mu\nu}$ is matrix-valued, various components of this tensor do not commute, that is, $[a^{\mu\nu}, a^{\alpha\beta}] \neq 0$. In this sense, such geometry may be also called *non-commutative Riemannian geometry*. In the commutative limit, $a^{\mu\nu} \rightarrow g^{\mu\nu}$, and the standard Riemannian geometry with all its ingredients is recovered. Only the total mass of a gravitating particle is observed. For more details and discussions see [20][21].
We would like to mention, at this point, that Matrix Gravity contains two important physical consequences. The first is the presence, in the theory, of a new non-geodesic acceleration. In General Relativity, the motion of a test particle in the gravitational field is described by the equation

\[ \frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = 0, \]  

(2.170)

where \( \dot{x} \) represents the tangent vector to the geodesic and \( \Gamma^\mu_{\alpha\beta} \) are the Christoffel symbols. This equation describes the motion of a test particle free from external forces. As we will see in Chapter 7, in Matrix Gravity the equation for the geodesics becomes the following

\[ \frac{d^2 x^\mu}{dt^2} + \Gamma^\mu_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta = A^\mu_{\text{anom}}(x, \dot{x}). \]  

(2.171)

In this equation a non-geodesic acceleration is present which is written only in terms of the non-commutative part of the metric. In other words, the test particles in Matrix Gravity, do not follow any Riemannian geodesic. The second important physical consequence of this model is the violation of the equivalence principle. In General Relativity, test particles move along specific geodesics of a Riemannian metric independently of their masses. In Matrix Gravity, instead, test particles exhibit a non-geodesic motion \( A^\mu_{\text{anom}} \) which depends on the different mass parameters. Therefore, test particles that are described by different mass parameters will follow different trajectories in the spacetime. This is the origin of the violation of the equivalence principle.

**Main Differences Between Matrix Gravity and Non-commutative Gravity**

It is important to stress that this approach for deforming General Relativity is different from the ones proposed in the framework of non-commutative geometry. 
where the coordinates do not commute and the standard product between functions is replaced by the Moyal product \([62]\). In flat space one usually introduces non-commutative coordinates satisfying the commutation relations

\[
[x^\mu, x^\nu] = \theta^{\mu\nu}.
\]

(2.172)

Here, \(\theta^{\mu\nu}\) is a real constant anti-symmetric matrix, and one replaces the standard algebra of functions with the non-commutative algebra with the Moyal star product

\[
f(x) \star g(x) = \exp\left(\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial z^\nu}\right) f(x + y) g(x + z) \bigg|_{y = z = 0}.
\]

(2.173)

An extensive review of different realizations of gravity in the framework of non-commutative geometry, especially in connection with string theory, can be found in \([75]\). An extension of the star product and noncommutativity from flat to curved spacetime can be found in \([31, 55]\).

We list below the most relevant differences between the two approaches; a more detailed and extensive discussion can be found in \([20, 21, 26, 49]\). The biggest problem with the curved manifolds is the nature of the object \(\theta^{\mu\nu}\). All these models are defined, strictly speaking, only in perturbation theory in the deformation parameter. That is, one takes \(\theta^{\mu\nu}\) as a formal parameter and considers formal power series in \(\theta^{\mu\nu}\). In the approach of \([20, 21]\) to Matrix Gravity the deformation parameters are not formal and the theory is defined for all finite values of the deformation parameter.

In the standard non-commutative approach the coordinates themselves are non-commutative. This condition raises the questions of whether the spacetime has the structure of a manifold, and how one can define analysis on such spaces. Moreover, one needs a way to relate the non-commutative coordinates with the usual (commutative) coordinates. In Matrix Gravity one does not have non-commutative
coordinates. The spacetime, here, is a proper smooth manifold with the standard analysis defined on it.

In non-commutative geometry approach the deformation parameter $\theta^{\mu\nu}$ is non-dynamical, therefore there are no dynamical equations for it. This poses the question of what kind of physical, or mathematical, conditions can be used in order to determine it. In addition, in many models of non-commutative gravity (as in [31]), $\theta^{\mu\nu}$ is a non-tensorial object which makes it dependent on the choice of the system of coordinates. This feature leads the theory to be not invariant under the usual group of diffeomorphisms. In [31] the authors construct all the relevant geometric quantities (such as connection, curvature, etc.) in terms of the non-commutative deformation parameter. In this framework, they obtain an expansion of these quantities and of the action up to second order. In their approach the object $\theta^{\mu\nu}$ is a constant anti-symmetric matrix (not a tensor). This violates the usual diffeomorphism invariance, Lorentz invariance, etc. for which there exist very strict experimental bounds. The main result in [31] is the derivation of the deformed Einstein equations. The zero-order part (Einstein) is diffeomorphism-invariant, and the corrections (quadratic in theta) are not. Therefore the theory contains some preferred system of coordinates and its whole content depends on it. Of course, the theory needs to justify the choice of such system of coordinates.

In the approach of [20, 21] there is no need to introduce any non-tensorial objects. As a result the theory is diffeomorphism-invariant. So, there are no problems related to the violation of Lorenz invariance, etc. and there are no preferred systems of coordinates. Moreover, the non-commutative part of the metric in this approach is dynamical. There are non-commutative Einstein equations for it. The
goal of this chapter is, in particular, to derive these dynamical equations in the perturbation theory.

In [55], the authors assume that $\theta^{\mu\nu}$ is a covariantly constant tensor. But then, there are strict algebraic constraints on the Riemann curvature tensor of the commutative metric (obtained by a commutator of second covariant derivatives). In Matrix Gravity such algebraic constraints are absent— the commutative metric is arbitrary.

In the usual approach of non-commutative geometry (as in, for example, [31]), when one defines the affine connection, the covariant derivative, the curvature and the torsion the ordering of factors is not unique. There is no natural reason why one should prefer one ordering over the other. That is, the connection coefficients can be placed on the left, or on the right, (or one could symmetrize over these two possibilities) from the object of differentiation. Another aspect of the ordering problem is the fact that there is no unique way to raise and lower indices. One can act with the metric from the left or from the right. The approach in [21], instead, is pretty much unique. There is no need to define the affine connection, the covariant derivative, the curvature and the torsion. There is no ordering problem.

The definition of a “measure”, in standard non-commutative geometry, as a star determinant (as in [31]) does not guarantee its positivity. It only guarantees the positivity in the zero order of the perturbation theory. In the definition of [20, 21], the measure is positive even in strongly non-commutative regime.

Moreover, the Moyal star product is non-local which makes the whole theory non-local with possible unitarity problems. In the approach [21] the action functional is a usual local functional of sigma-model type (like General Relativity,
but with additional non-commutative degrees of freedom). There may be problems with the renormalizability (which requires further study) but not with unitarity.

One should also mention the relation of Matrix Gravity to so called “analog models of gravity”. In particular, the analysis in [32] is surprisingly similar to the analysis of the papers [19, 20]. The authors of [32] consider a hyperbolic system of second order partial differential equations, the corresponding Hamilton-Jacobi equations and the Hamiltonian system as in [19, 20]. In fact, their $f^{\mu\nu}$ is equivalent to the matrix-valued tensor $a^{\mu\nu}$, However, their goal was very different—they impose the commutativity conditions on $f^{\mu\nu}$ (eq. (44) in [32]) to enforce a unique effective metric for the compatibility with the Equivalence Principle. They barely mention the general geometric interpretation in terms of Finsler geometries as it “does not seem to be immediately relevant for either particle physics or gravitation”. The motivation of the authors of [32] is also very different from the approach of [20, 21]. Their idea is that gravity is not fundamental so that the effective metric simply reflects the properties of an underlying physics (such as fluid mechanics and condensed matter theory). They just need to have enough fields to be able to parameterize an arbitrary effective metric. In the approach of Matrix Gravity, the matrix-valued field $a^{\mu\nu}$ is fundamental; it is: i) non-commutative and ii) dynamical.

The action of Matrix Gravity can be constructed in two different ways. One approach, developed in [19, 20], called Matrix General Relativity, is to try to extend all standard concepts of differential geometry to the non-commutative setting and to construct a matrix-valued connection and a matrix-valued curvature. We will be mainly interested in this approach in Chapter 5.

The second approach, developed in [21], called Spectral Matrix Gravity, is
based on constructing the action form the coefficients of the spectral asymptotics of a non-Laplace type self-adjoint elliptic partial differential operator $L$ of second order with a positive definite leading symbol. We will analyze this particular approach in Chapter 6.

### 2.5.2 Matrix General Relativity

We will describe, in this section, the construction of the action for Matrix General Relativity following [19, 20]. The formalism that we are going to describe is related to the algebra-valued formulation of Mann [64] and Wald [81]. In these papers the authors introduce algebra-valued tensor fields and generalize the formalism of differential geometry to the algebra-valued case. More precisely they were studying a consistent theory to describe the interaction of a collection of massless spin-2 fields. The authors found that in order to have a consistent theory, the algebra to consider must be associative and commutative. In this case the theory simply becomes a sum of usual Hilbert-Einstein actions for the fields without cross-interaction terms. In the approach of [20, 21], the algebra is associative but non-commutative where the gauge group is simply the product of the group of diffeomorphism of a real manifold with the internal group. Because of this form of the gauge transformations one can allow the algebra to be non-commutative which leads to a different dynamics from the one described in [64] and [81].

Let $\mathcal{V}$ be an $N$-dimensional Hermitian vector bundle over $M$, let $\mathcal{S} = \mathcal{T}M \otimes \mathcal{V}$ be the bundle constructed by taking the tensor product of the tangent bundle to the manifold $M$ with the vector bundle $\mathcal{V}$, and let $\mathcal{S}^* = T^*M \otimes \mathcal{V}$, where $T^*M$ is the cotangent bundle to $M$. Let $a$ be a symmetric self-adjoint element of $TM \otimes$
\[ TM \otimes \text{End}(\mathcal{V}), \] that is,
\[ a^{\mu\nu} = a^{\nu\mu}, \quad (a^{\mu\nu})^* = a^{\nu\mu}. \tag{2.174} \]

Suppose that \( a^{\mu\nu} \) is an isomorphism between \( \mathcal{T} \) and \( \mathcal{T}^* \), then the inverse isomorphism \( b_{\mu\nu} \) satisfies the equation
\[ a^{\rho\nu} b_{\nu\mu} = b_{\mu\nu} a^{\rho\nu} = \delta^\rho_\mu \cdot \mathbb{I}. \tag{2.175} \]

There are some properties of the matrix \( b_{\mu\nu} \) that need attention. The first property is the following: the matrix \( b_{\mu\nu} \) satisfies the equation
\[ b^{\mu\nu} = b_{\nu\mu}, \tag{2.176} \]
but it is not necessarily a self-adjoint matrix symmetric in its tensor indices. Moreover, one can use \( a^{\mu\nu} \) and \( b_{\mu\nu} \) to lower and raise indices, although particular care is required in these operations because, in general, \( a^{\mu\nu} \) and \( b_{\mu\nu} \) do not commute and \( b_{\mu\nu} \) is not symmetric in its tensorial indices [20].

Let \( \mathcal{A}^{\alpha}_{\lambda\mu} \) be the matrix-valued Christoffel symbol defined as [20]
\[ \mathcal{A}^{\alpha}_{\lambda\mu} = \frac{1}{2} b_{\lambda\sigma} (a^{\rho\gamma} \partial_\gamma a^{\sigma\mu} - a^{\rho\gamma} \partial_\gamma a^{\sigma\alpha} - a^{\rho\gamma} \partial_\gamma a^{\sigma\nu}) b_{\rho\mu}, \tag{2.177} \]

it is not difficult to prove that this quantity transforms as a connection coefficient. It is important to notice, at this point, that in matrix geometry the connection (2.177) is not symmetric in the two lower indices.

In complete analogy with the ordinary Riemannian geometry, by using the matrix-valued Christoffel symbol, one can define the matrix-valued Riemann tensor as follows [20]
\[ \mathcal{R}^{1}_{\alpha\mu} = \partial_\mu \mathcal{A}^{1}_{\alpha\nu} - \partial_\nu \mathcal{A}^{1}_{\alpha\mu} + \mathcal{A}^{1}_{\beta\mu} \mathcal{A}^{1}_{\alpha\nu} - \mathcal{A}^{1}_{\beta\nu} \mathcal{A}^{1}_{\alpha\mu}. \tag{2.178} \]
Once the matrix curvature (Riemann) tensor is defined one can construct the matrix Ricci tensor, namely

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}.$$  \hspace{1cm} (2.179)

In order to write the action for Matrix Gravity, one needs to introduce the matrix scalar curvature $R$. Since the metric $a^{\mu\nu}$ and the Ricci tensor $R_{\mu\nu}$ are matrices, they do not commute in general and the definition of the scalar curvature, obtained by contracting the metric tensor with the Ricci tensor from the left, would be different if the contraction would be performed with the metric tensor on the right. In order to avoid this choice, we use a symmetrized definition of the matrix-valued scalar curvature as follows

$$R = \frac{1}{2} (a^{\mu\nu} R_{\mu\nu} + R_{\mu\nu} a^{\mu\nu}).$$  \hspace{1cm} (2.180)

In order to write an action for the model under consideration a generalization of the concept of measure is needed. As a guiding principle, any generalization of the measure $\mu$ has to lead, in the commutative limit, to the ordinary Riemannian measure $\sqrt{\det g_{\mu\nu}}$. Let $\rho$ be a matrix-valued scalar density, which can be defined, for example, as follows

$$\rho = \int_{\mathbb{R}^n} \frac{d\xi}{\pi^2} \exp(-a^{\mu\nu} \xi_\mu \xi_\nu).$$  \hspace{1cm} (2.181)

Then $\rho$ only depends on the metric $a$ and transforms in the correct way under diffeomorphisms of $M$. We would like to stress, here, that the choice of the measure is not unique. However, the definition (2.181) seems to be the most natural because it represents the quantity that appears as the $A_0$ coefficient in the heat kernel asymptotic of a generalized Laplace operator with matrix-valued symbol defined on the manifold $M$ under consideration. Of course different choices of the measure would lead
to different non-commutative limits of the theory. More precisely, the zeroth order of the expansion in the deformation parameter of the action always gives the usual General Relativity. The second order term, which gives the dynamical equations for the non-commutative part of the metric, instead, changes if the definition of measure is different. Further studies are required in order to fully understand how the choice of the measure affects the dynamics of the non-commutative part of the metric.

Now that all the relevant geometric quantities have been described, one can construct the action functional for the field $a^{\mu\nu}$ following [20]. This functional has to be invariant under both diffeomorphisms of $M$ and gauge transformations. By using the matrix-valued scalar curvature, defined in (2.180), and the matrix-valued scalar density (2.181), one obtains, by analogy to (2.169), [20]

$$S_{\text{MGR}}(a) = \frac{1}{16\pi G} \int_M dx \frac{1}{N} \text{Tr} [\rho (R - 2\Lambda)].$$

(2.182)

It is worth noticing that because of the cyclic property of the trace, the relative position of $\rho$ and the scalar curvature is irrelevant, moreover it is easily shown that the action functional (2.182) is invariant under the diffeomorphisms of $M$ and under the gauge transformations. Of course, as $a^{\mu\nu} \rightarrow g^{\mu\nu}$ this action reproduces the Hilbert-Einstein action (2.169) of General Relativity.

The field equations for the tensor $a^{\mu\nu}$, that we call non-commutative Einstein equations are obtained by varying the action with respect to $a^{\mu\nu}$. In the vacuum we have,

$$\frac{\partial L}{\partial a^{\alpha\beta}} - \partial_{\mu} \frac{\partial L}{\partial a^{\alpha\beta}{}_{\mu}} = 0,$$

(2.183)

where $a^{\alpha\beta}{}_{\mu} = \partial_{\mu} a^{\alpha\beta}$ and $L$ is the Lagrangian density.
The action has an additional new *global* gauge symmetry

\[ \sigma^{\mu\nu}(x) \mapsto U \sigma^{\mu\nu}(x) U^{-1}, \quad (2.184) \]

where \( U \) is a constant unitary matrix (for more details, see the papers cited above). By the Noether theorem this symmetry leads to the conserved currents (vector densities)

\[ \mathcal{J}^\mu = \left[ \sigma^{\alpha\beta}, \frac{\partial \mathcal{L}}{\partial (\sigma^{\alpha\beta})} \right], \quad \partial_\mu \mathcal{J}^\mu = 0. \]

In other words, this suggests the existence of *new physical charges*

\[ Q = \int d\hat{x} \mathcal{J}^0, \]

where \( d\hat{x} \) denotes the integration over the space coordinates only. These charges have purely non-commutative origin and vanish in the commutative limit.

This model may be viewed as a “non-commutative deformation” of Einstein gravity, which describes, in the weak deformation limit, General Relativity, and a multiplet of self-interacting massive two-tensor fields of spin 2 that interact also with gravity.

### 2.5.3 Spectral Matrix Gravity

By using the equations (2.147), (2.148) and (2.151), it is easy to see that the Hilbert-Einstein action (2.169) is nothing but a linear combination of \( A_0 \) and \( A_1 \) for a Laplace type operator. In full analogy, the action of Spectral Matrix Gravity proposed in [21] is a linear combination of the global heat kernel coefficients \( A_0 \) and \( A_1 \) for a general second order non-Laplace type operator, more precisely

\[ S = \frac{1}{16\pi G N} [6A_1 - 2\Lambda A_0]. \quad (2.185) \]
We would like to point out here that the above action can be also thought of as a particular case of the Spectral Action Principle introduced in the framework of non-commutative geometry in [38] and [37]. For the Laplace operator, \( L = -\Delta \), the heat kernel coefficients are (2.147) and (2.148) and, therefore, the action of Spectral Matrix Gravity reduces to the standard Hilbert-Einstein action (2.169).

We would like to stress, here, that we are interested in a much more complicated general case of an arbitrary non-Laplace type operator (with a non-scalar leading symbol). In this case there is no preferred Riemannian metric and the whole language of Riemannian geometry is not very helpful in computing the heat kernel asymptotics. That is why, until now, there are no explicit general formulas for the coefficient \( A_1 \). A class of so-called natural non-Laplace type operators was studied in [15, 18] where this coefficient was computed explicitly.

We would like to mention, here, that similar calculations have been performed in non-commutative geometry regarding heat kernel asymptotics expansion. In [70, 71], the author evaluates the relevant geometric quantities from an approximate power expansion of the trace of the heat kernel for a Laplace operator on a compact fuzzy space. In [80], the author studies the quantization of non-commutative gravity in two dimensions by considering a non-commutative deformation (using the Moyal product) of the Jackiw-Teitelboim model for gravity. In this case the path integral can be evaluated exactly and the operator for the quantum fluctuations can be found. Once the operator is known one can study the first two heat kernel asymptotic coefficients and obtain information about the conformal anomaly and the Polyakov action.
CHAPTER 3

NON-PERTURBATIVE HEAT KERNEL ASYMPTOTICS ON HOMOGENEOUS ABELIAN BUNDLES

Abstract

We study the heat kernel for a Laplace type partial differential operator acting on smooth sections of a complex vector bundle with the structure group $G \times U(1)$ over a Riemannian manifold $M$ without boundary. The total connection on the vector bundle naturally splits into a $G$-connection and a $U(1)$-connection, which is assumed to have a parallel curvature $F$. We find a new local short time asymptotic expansion of the off-diagonal heat kernel $U(t|x, x')$ close to the diagonal of $M \times M$ assuming the curvature $F$ to be of order $t^{-1}$. The coefficients of this expansion are polynomial functions in the Riemann curvature tensor (and the curvature of the $G$-connection) and its derivatives with universal coefficients depending in a non-polynomial but analytic way on the curvature $F$, more precisely, on $tF$. These functions generate all terms quadratic and linear in the Riemann curvature and of arbitrary order in $F$ in the usual heat kernel coefficients. In that sense, we effectively sum up the usual short time heat kernel asymptotic expansion to all orders of the curvature $F$. We compute the first three coefficients (both diagonal and off-diagonal) of this new asymptotic

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1The material in this chapter has been published in *Communications in Mathematical Physics*: I. G. Avramidi and G. Fucci, Non-Perturbative Heat Kernel Asymptotics on Homogeneous Abelian Bundles, *Comm. Math. Phys.* (2009) doi: 10.1007/s00220-009-0804-6
expansion.

3.1 Introduction

The heat kernel is one of the most powerful tools in quantum field theory and quantum gravity as well as mathematical physics and differential geometry (see for example [53, 79, 14, 17, 13, 58, 78, 56] and further references therein). It is of particular importance because the heat kernel methods give a framework for manifestly covariant calculation of a wide range of relevant quantities in quantum field theory like one-loop effective action, Green’s functions, effective potential etc.

Unfortunately the exact computation of the heat kernel can be carried out only for exceptional highly symmetric cases when the spectrum of the operator is known exactly, (see [35, 56, 58] and the references in [11, 24, 23, 25]). Although these special cases are very important, in quantum field theory we need the effective action, and, therefore, the heat kernel for general background fields. For this reason various approximation schemes have been developed. One of the oldest methods is the Minackshisundaram-Pleijel short-time asymptotic expansion (2.144), (2.150) of the heat kernel as \( t \to 0 \) (see the references in [53, 5, 79]).

Despite its enormous importance, this method is essentially perturbative. It is an expansion in powers of the curvatures \( R \) and their derivatives and, hence, is inadequate for large curvatures when \( tR \sim 1 \). To be able to describe the situation when at least some of the curvatures are large one needs an essentially non-perturbative approach, which effectively sums up in the short time asymptotic expansion of the heat kernel an infinite series of terms of certain structure that contain large curvatures (for a detailed analysis see [7, 12] and reviews [13, 17]). For example, the
partial summation of higher derivatives enables one to obtain a non-local expansion of the heat kernel in powers of curvatures (high-energy approximation in physical terminology). This is still an essentially perturbative approach since the curvatures (but not their derivatives) are assumed to be small and one expands in powers of curvatures.

On another hand to study the situation when curvatures (but not their derivatives) are large (low energy approximation) one needs an essentially non-perturbative approach. A promising approach to the calculation of the low-energy heat kernel expansion was developed in non-Abelian gauge theories and quantum gravity in \([6, 7, 8, 9, 10, 11, 25, 23, 24]\). While the papers \([6, 7, 9, 10]\) dealt with the parallel \(U(1)\)-curvature (that is, constant electromagnetic field) in flat space, the papers \([8, 11, 25]\) dealt with symmetric spaces (pure gravitational field in absence of an electromagnetic field). The difficulty of combining the gauge fields and gravity was finally overcome in the papers \([23, 24]\), where homogeneous bundles with parallel curvature on symmetric spaces were studied.

In this chapter we compute the heat kernel for the covariant Laplacian with a large parallel \(U(1)\) curvature \(F\) in a Riemannian manifold (that is, strong covariantly constant electromagnetic field in an arbitrary gravitational field). Our aim is to evaluate the first three coefficients of the heat kernel asymptotic expansion in powers of Riemann curvature \(R\) but in all orders of the \(U(1)\) curvature \(F\). This is equivalent to a partial summation in the heat kernel asymptotic expansion as \(t \to 0\) of all powers of \(F\) in terms which are linear and quadratic in Riemann curvature \(R\).
3.2 Setup of the Problem

Let $M$ be an $n$-dimensional compact Riemannian manifold without boundary and $S$ be a complex vector bundle over $M$ realizing a representation of the group $G \otimes U(1)$. Let $\varphi$ be a section of the bundle $S$ and $\nabla$ be the total connection on the bundle $S$ (including the $G$-connection as well as the $U(1)$-connection). Then the commutator of covariant derivatives defines the curvatures

$$[\nabla_\mu, \nabla_\nu] \varphi = (R_{\mu\nu} + i F_{\mu\nu}) \varphi,$$  \hspace{1cm} (3.1)

where $R_{\mu\nu}$ is the curvature of the $G$-connection and $F_{\mu\nu}$ is the curvature of the $U(1)$-connection (which will be also called the electromagnetic field).

In the present chapter we consider the Laplacian

$$\mathcal{L} = -\Delta.$$  \hspace{1cm} (3.2)

The asymptotic expansion of the heat kernel $U(t|x, x')$ for the Laplacian has the form (2.150) and its coefficients are (2.151). The diagonal heat kernel coefficients $d_k^{\text{diag}}$ are polynomials in the jets of the metric, the $G$-connection and the $U(1)$-connection; in other words, in the curvature tensors and their derivatives. Let us symbolically denote the jets of the metric and the $G$-connection by

$$R_{(n)} = \left\{ \nabla_{(\mu_1} \cdots \nabla_{\mu_n} R^a_{\mu_{n+1} \mu_{n+2}} \right\}, \hspace{1cm} \nabla_{(\mu_1} \cdots \nabla_{\mu_n} R^a_{\mu_{n+1} \mu_{n+2}} \},$$  \hspace{1cm} (3.3)

and the jets of the $U(1)$ connection by

$$F_{(n)} = \nabla_{(\mu_1} \cdots \nabla_{\mu_n} F^a_{\mu_{n+1}} \right\},$$  \hspace{1cm} (3.4)

Here and everywhere below the parenthesis indicate complete symmetrization over all indices included.
By counting the dimensions it is easy to describe the general structure of the coefficients $a_k^{\text{diag}}$. Let us introduce the multi-indices of nonnegative integers

$$i = (i_1, \ldots, i_m), \quad j = (j_1, \ldots, j_l). \quad (3.5)$$

Let us also denote

$$|i| = i_1 + \cdots + i_m, \quad |j| = j_1 + \cdots + j_l. \quad (3.6)$$

Then symbolically

$$a_k^{\text{diag}} = \sum_{N=1}^{k} \sum_{l=0}^{N} \sum_{m=0}^{N-l} \sum_{\|i\|+\|j\|+2N=2k} C_{(k,l,m),i,j} F_{(j_1)} \cdots F_{(j_l)} R_{(i_1)} \cdots R_{(i_m)}, \quad (3.7)$$

where $C_{(k,l,m),i,j}$ are some universal constants. The lower order diagonal heat kernel asymptotic coefficients are $(2.147)-(2.149)$. In the present chapter we study the case of a parallel $U(1)$ curvature (covariantly constant electromagnetic field), i.e.

$$\nabla_\mu F_{\alpha\beta} = 0. \quad (3.8)$$

That is, all jets $F_{(n)}$ are set to zero except the one of order zero, which is $F$ itself. In this case eq. (3.7) takes the form

$$a_k^{\text{diag}} = \sum_{N=1}^{k} \sum_{l=0}^{N} \sum_{m=0}^{N-l} \sum_{\|i\|+\|j\|+2N=2k} C_{(k,l,m),i} F^l R_{(i_1)} \cdots R_{(i_m)}, \quad (3.9)$$

where $C_{(k,l,m),i}$ are now some (other) numerical coefficients.

Thus, by summing up all powers of $F$ in the asymptotic expansion of the heat kernel diagonal we obtain a new (non-perturbative) asymptotic expansion

$$U^{\text{diag}}(t) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k a_k^{\text{diag}}(t), \quad (3.10)$$
where the coefficients $\tilde{a}_k^{\text{diag}}(t)$ are polynomials in the jets $R_{(n)}$

$$\tilde{a}_k^{\text{diag}}(t) = \sum_{N=1}^{k} \sum_{m=0}^{N} \sum_{k+2N=2k} f^{(k)}_{(m,i)}(t) R_{(i_1)} \cdots R_{(i_m)}, \quad (3.11)$$

and $f^{(k)}_{(m,i)}(t)$ are some universal dimensionless tensor-valued analytic functions that depend on $F$ only in the dimensionless combination $tF$.

For the heat trace we obtain then a new asymptotic expansion of the form

$$\text{Tr } \exp(-t\mathcal{L}) \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k \tilde{A}_k(t), \quad (3.12)$$

where

$$\tilde{A}_k(t) = \int_M d\text{vol} \text{ tr} \tilde{a}_k^{\text{diag}}(t). \quad (3.13)$$

This expansion can be described more rigorously as follows. We rescale the $U(1)$-curvature $F$ by

$$F \mapsto F(t) = t^{-1} \tilde{F}, \quad (3.14)$$

so that $tF(t) = \tilde{F}$ is independent of $t$. Then the operator $\mathcal{L}(t)$ becomes dependent on $t$ (in a singular way!). However, the heat trace still has a nice asymptotic expansion as $t \to 0$

$$\text{Tr } \exp[-t\mathcal{L}(t)] \sim (4\pi t)^{-n/2} \sum_{k=0}^{\infty} t^k \tilde{A}_k, \quad (3.15)$$

where the coefficients $\tilde{A}_k$ are expressed in terms of $\tilde{F} = tF(t)$, and, therefore, are independent of $t$. Thus, what we are doing is the asymptotic expansion of the heat trace for a particular case of a singular (as $t \to 0$) time-dependent operator $\mathcal{L}(t)$.

Let us stress once again that the eq. (3.11) should not be taken literally; it only represents the general structure of the coefficients $\tilde{a}_k^{\text{diag}}(t)$. To avoid confusion
we list below the general structure of the low-order coefficients in more detail

\[
\tilde{a}_0^{\text{diag}}(t) = f^{(0)}(t),
\]

\[
\tilde{a}_1^{\text{diag}}(t) = f^{(1)}_{(1,1)} \alpha^\mu \nu \rho \sigma(t) \nabla_a \nabla_b R^{\mu \nu \rho \sigma} + f^{(1)}_{(1,2)} \alpha^\mu \nu \rho \sigma(t) \nabla^a \nabla^b R^{\mu \nu}
\]

\[
\tilde{a}_2^{\text{diag}}(t) = f^{(2)}_{(1,1)} \alpha^\mu \nu \rho \sigma \tau \lambda(t) \nabla_a \nabla_b R^{\mu \nu \rho \sigma \tau \lambda} + f^{(2)}_{(1,2)} \alpha^\mu \nu \rho \sigma(t) \nabla^a \nabla^b R^{\mu \nu}
\]

\[
\tilde{a}_3^{\text{diag}}(t) = f^{(2)}_{(2,1)} \alpha^\mu \nu \rho \sigma \tau \lambda(t) \nabla_a \nabla_b R^{\mu \nu \rho \sigma \tau \lambda} + f^{(2)}_{(2,2)} \alpha^\mu \nu \rho \sigma(t) \nabla^a \nabla^b R^{\mu \nu}
\]

\[
\tilde{a}_4^{\text{diag}}(t) = f^{(2)}_{(2,3)} \alpha^\mu \nu \rho \sigma \tau \lambda(t) \nabla_a \nabla_b R^{\mu \nu \rho \sigma \tau \lambda},
\]

with obvious enumeration of the functions. It is the universal tensor functions \(f_{(1,m)}^{(i)}(t)\) that are of prime interest in this chapter. Our main goal is to compute the functions \(f_{(1,m)}^{(i)}(t)\) for the coefficients \(\tilde{a}_0^{\text{diag}}(t), \tilde{a}_1^{\text{diag}}(t)\) and \(\tilde{a}_2^{\text{diag}}(t)\).

Of course, for \(t = 0\) (or \(F = 0\)) the coefficients \(\tilde{a}_k(t)\) are equal to the usual diagonal heat kernel coefficients

\[
\tilde{a}_k^{\text{diag}}(0) = a_k^{\text{diag}}.
\]

Therefore, by using the explicit form of the coefficients \(a_k^{\text{diag}}\) given by (2.149) we obtain the initial values for the functions \(f_{(1,m)}^{(i)}\). Moreover, by analyzing the corresponding terms in the coefficients \(a_3^{\text{diag}}\) and \(a_4^{\text{diag}}\) (which are known, [53, 5, 78]), one can obtain partial information about some lower order Taylor coefficients of the functions \(f_{(j,k)}^{(i)}(t)\):

\[
f^{(0)}(t) = 1 - \frac{1}{12} t^2 F_{\mu \nu} F^{\mu \nu} + O(t^3),
\]

\[
f^{(1)}_{(1,1)} \alpha^\mu \nu \rho \sigma(t) = \frac{1}{6} \delta^\alpha_{[\mu} \delta^\beta_{\nu]} + O(t),
\]

\[
f^{(1)}_{(1,2)} \alpha^\mu \nu \rho \sigma(t) = \frac{1}{6} ti F^{\mu \nu} + O(t^2),
\]
\[ f_{(1,1)}^{(2) \, \rho \sigma \nu \sigma \rho} (t) = \frac{1}{30} g^{\rho \sigma} \delta_{\nu \sigma} + O(t), \quad (3.23) \]

\[ f_{(1,2)}^{(2) \, \alpha \beta \mu \nu} (t) = -\frac{1}{15} t i F^{[\alpha} [\nu \delta^{\beta]} + O(t^2), \quad (3.24) \]

\[ f_{(2,1)}^{(2) \, \gamma \delta \mu \nu \sigma \rho} (t) = \frac{1}{180} g_{\mu \nu} (\sigma g_{\gamma \delta} \rho) - \frac{1}{180} g_{\rho \mu} (\sigma g_{\gamma \delta}) + O(t), \quad (3.25) \]

\[ f_{(2,2)}^{(2) \, \alpha \beta \mu \nu} (t) = \frac{1}{12} \delta_{\mu \nu} + O(t), \quad (3.26) \]

\[ f_{(2,3)}^{(2) \, \alpha \beta \mu \nu \sigma \rho} (0) = -\frac{1}{36} t i F^{\alpha \beta} \delta_{[\sigma \nu]} ^{\rho} - \frac{1}{30} t i F^{\mu \nu} \delta_{[\sigma \rho]} ^{\delta} + \frac{1}{9} \delta_{[\sigma \rho]} ^{\delta} i F^{\nu} \delta_{[\sigma \rho]} ^{\delta} + O(t^2). \quad (3.27) \]

This information can be used to check our final results.

Notice that the global coefficients \( \tilde{A}_k(t) \) have exactly the same form as the local ones; the only difference is that the terms with the derivatives of the Riemann curvature do not contribute to the integrated coefficients since they can be eliminated by integrating by parts and taking into account that \( F \) is covariantly constant.

Moreover, we study even more general non-perturbative asymptotic expansion for the off-diagonal heat kernel and compute the coefficients of zero, first and second order in the Riemann curvature. We will show that there is a new non-perturbative asymptotic expansion of the off-diagonal heat kernel as \( t \to 0 \) (and \( F = t^{-1} \tilde{F} \), so that \( tF \) is fixed) of the form

\[ U(t|x, x') \sim \mathcal{P}(x, x') \Delta^{1/2}(x, x') U_0(t|x, x') \sum_{k=0}^{\infty} (t / 2)^k b_k(t|x, x'), \quad (3.28) \]
where $U_0$ is an analytic function of $F$ such that for $F = 0$
\begin{align}
U_0(t|x, x') \bigg|_{F=0} &= (4\pi t)^{-n/2} \exp \left[ -\frac{\sigma(x, x')}{2t} \right]. \tag{3.29}
\end{align}
Here $b_k(t|x, x')$ are analytic functions of $t$ that depend on $F$ only in the dimensionless combination $tF$. Of course, for $t = 0$ they are equal to the usual heat kernel coefficients, that is,
\begin{align}
b_{2k}(0|x, x') &= a_k(x, x'), \quad b_{2k+1}(0|x, x') = 0. \tag{3.30}
\end{align}
Moreover, we will show below that the odd-order coefficients vanish not only for $t = 0$ and any $x \neq x'$ but also for any $t$ and $x = x'$, that is, on the diagonal,
\begin{align}
b^{\text{diag}}_{2k+1}(t) = 0. \tag{3.31}
\end{align}
Thus, the heat kernel diagonal has the asymptotic expansion (3.10) as $t \to 0$ with
\begin{align}
\tilde{a}^{\text{diag}}_k(t) = (4\pi t)^{n/2} U^{\text{diag}}_0(t) b^{\text{diag}}_{2k}(t). \tag{3.32}
\end{align}

In what follows we will consider the operators
\begin{align}
D_\mu = \tilde{\nabla}_\mu - \frac{1}{2} i F_{\mu\alpha} u^\alpha. \tag{3.33}
\end{align}
Obviously, they form the algebra
\begin{align}
[D_\mu, D_\nu] = i F_{\mu\nu}, \quad [D_\mu, u^\nu] = \delta_{\mu}^\nu. \tag{3.34}
\end{align}
For a covariantly constant electromagnetic field, considered in this work, the following relation holds [14, 5]
\begin{align}
\nabla_\mu F_{\alpha\beta} = 0. \tag{3.35}
\end{align}
In this case we find it useful to decompose the quantity $\mathcal{A}_\mu$ as

$$\mathcal{A}_\mu = -\frac{1}{2}iF_{\mu\nu}u^\nu + \tilde{\mathcal{A}}_\mu .$$

(3.36)

It can be easily shown that $\tilde{\mathcal{A}}_\mu$ has the following Taylor expansion

$$\tilde{\mathcal{A}}_\mu = -\frac{1}{2}R_{\mu\alpha}u^\alpha + \frac{1}{24}R_{\mu\nu\rho\sigma}iF^\rho_{\gamma}u^\gamma u^\delta u^\epsilon + \frac{1}{3}\nabla_\alpha R_{\mu\beta}u^\alpha u^\beta$$

$$+ \frac{1}{24}R_{\mu\nu\rho\sigma}R^\sigma_{\gamma\alpha}u^\alpha u^\beta u^\gamma - \frac{1}{8}\nabla_\alpha R_{\mu\beta}u^\alpha u^\beta u^\gamma$$

$$- \frac{1}{720}R_{\mu\nu\rho\sigma}R_{\gamma\alpha\beta\delta}iF^\alpha_{\epsilon}u^\gamma u^\delta u^\epsilon u^\nu u^\mu + O(u^6) .$$

(3.37)

We would like to stress that, the expansion for $\tilde{\mathcal{A}}_\mu$ is valid in the case of a covariantly constant electromagnetic field.

By utilizing the Taylor expansion for all the relevant quantities we are able to find an expansion for the heat kernel. First of all, the heat kernel can be presented in the form

$$U(t|x, x') = \exp (-t\mathcal{L}) \mathcal{P}(x, x')\delta(x, x') ,$$

(3.38)

which can also be written as

$$U(t|x, x') = \mathcal{P}(x, x')\Delta^\frac{1}{2}(x, x')\exp(-t\tilde{\mathcal{L}})\delta(u) ,$$

(3.39)

where $\delta(u)$ is the usual delta-function in the normal coordinates $u^\mu$ (recall that $u^\mu$ depends on $x$ and $x'$ and $u = 0$ when $x = x'$) and $\tilde{\mathcal{L}}$ is an operator defined by

$$\tilde{\mathcal{L}} = \mathcal{P}^{-1}(x, x')\Delta^{-\frac{1}{2}}(x, x')\mathcal{L}\Delta^{\frac{1}{2}}(x, x')\mathcal{P}(x, x') .$$

(3.40)

As is shown in [5, 14] the operator $\tilde{\mathcal{L}}$ can be written in the form

$$\tilde{\mathcal{L}} = -(D_\mu + \mathcal{A}_\mu - \xi_\mu)X^{\mu\nu}(D_\nu + \mathcal{A}_\nu + \xi_\nu) ,$$

(3.41)
where $\zeta_\mu = \bar{\nabla}_\mu \zeta$.

Now, by using these equations and by recalling the formula in (3.36), one can rewrite the operator in (3.41) in another way as follows

$$\tilde{\mathcal{L}} = -(X_{\mu\nu} D_\mu D_\nu + Y^\mu D_\mu + Z),$$  \hspace{1cm} (3.42)

where $X^{\mu\nu}$ is defined in (2.115) and

$$Y^\mu = (\bar{\nabla}_\mu X^{\mu\nu}) + 2X^{\mu\nu} \bar{\gamma}_\mu,$$  \hspace{1cm} (3.43)

$$Z = \bar{\gamma}_\mu X^{\mu\nu} \bar{\gamma}_\nu - \zeta_\mu X^{\mu\nu} \zeta_\nu + (\bar{\nabla}_\mu X^{\mu\nu}) \bar{\gamma}_\nu + (\bar{\nabla}_\mu X^{\mu\nu}) \zeta_\nu + X^{\mu\nu} \bar{\nabla}_\mu \bar{\gamma}_\nu + X^{\mu\nu} \bar{\nabla}_\mu \zeta_\nu.$$  \hspace{1cm} (3.44)

By using the covariant Taylor expansion of the two-point quantities that we described in Chapter 2, we obtain an expansion for the coefficients $X^{\mu\nu}$, $Y^\mu$ and $Z$ of the operator $\mathcal{L}$ up to the fifth order.

### 3.3 Perturbation Theory

Our goal is now to develop the perturbation theory for the heat kernel. We need to identify a small expansion parameter $\epsilon$ in which the perturbation theory will be organized as $\epsilon \to 0$. First of all, we assume that $t$ is small, more precisely, we require $t \sim \epsilon^2$. Also, since we will work close to the diagonal, that is, $x$ is close to $x'$, we require that $u^\mu \sim \epsilon$. This will also mean that $\bar{\nabla} \sim \epsilon^{-1}$ and $\partial_t \sim \epsilon^{-2}$. Finally, we assume that $F$ is large, that is, of order $F \sim \epsilon^{-2}$. To summarize,

$$t \sim \epsilon^2, \quad u^\mu \sim \epsilon, \quad F \sim \epsilon^{-2}. \hspace{1cm} (3.45)$$
3.3.1 Perturbation Theory for the Operator $\mathcal{L}$

Now, we expand the operator $\mathcal{L}$ in a formal power series in $\varepsilon$ (recall that $D \sim \varepsilon^{-1}$ and $u \sim \varepsilon$) to obtain

$$\mathcal{L} \sim - \sum_{k=0}^{\infty} \mathcal{L}_k,$$

(3.46)

where $\mathcal{L}_k$ are operators of order $\varepsilon^{k-2}$. In particular,

$$\mathcal{L}_0 = D^2,$$

(3.47)

$$\mathcal{L}_1 = 0,$$

(3.48)

$$\mathcal{L}_k = X^{\mu \nu}_k D_\mu D_\nu + Y^\mu_k D_\mu + Z_k, \quad k \geq 2.$$

(3.49)

where

$$D^2 = g^{\mu \nu} D_\mu D_\nu,$$

(3.50)

and $X^{\mu \nu}_k$, $Y^\mu_k$ and $Z_k$ are some tensor-valued polynomials in normal coordinates $u^\mu$.

Note that $X^{\mu \nu}_k$ are homogeneous polynomials in normal coordinates $u^\mu$ and $F$ of order $\varepsilon^k$. Similarly, $Y^\mu_k \sim \varepsilon^{k-1}$ and $Z_k \sim \varepsilon^{k-2}$. Of course, here the terms $Fuu$ are counted as of order zero. That is, they have the form

$$X^{\mu \nu}_k = P^{\mu \nu}_{(1), k}(u),$$

(3.51)

$$Y^\mu_k = P^\mu_{(2), k-1} + F_{\alpha \beta} P^{\mu \alpha \beta}_{(3), k+1}(u),$$

(3.52)

$$Z_k = P_{(4), k-2} + F_{\alpha \beta} P^{\rho \theta}_{(5), k}(u) + F_{\alpha \beta} F_{\rho \sigma} P^{\rho \theta \sigma}_{(6), k+2}(u),$$

(3.53)

where $P_{(j), k}(u)$ are homogeneous tensor valued polynomials of degree $k$.

By using the covariant Taylor expansions in (2.156), (3.37) and (2.154)
we find the explicit expression of the coefficients

\[ X^{\mu\nu}_2 = C^{\mu\nu}_{2\, \alpha\beta} u^\alpha u^\beta, \]  
(3.54)

\[ Y^\mu_2 = E^{\mu}_{2\alpha} u^\alpha + G^{\mu}_{2\alpha\beta} u^\alpha u^\beta, \]  
(3.55)

\[ Z_2 = H_{2\alpha\beta} u^\alpha u^\beta + L_2, \]  
(3.56)

\[ X^{\mu\nu}_3 = C^{\mu\nu}_{3\, \alpha\beta\gamma} u^\alpha u^\beta u^\gamma, \]  
(3.57)

\[ Y^\mu_3 = E^{\mu}_{3\alpha\beta} u^\alpha u^\beta, \]  
(3.58)

\[ Z_3 = H_{3\alpha} u^\alpha, \]  
(3.59)

\[ X^{\mu\nu}_4 = C^{\mu\nu}_{4\, \alpha\beta\gamma\delta} u^\alpha u^\beta u^\gamma u^\delta, \]  
(3.60)

\[ Y^\mu_4 = E^{\mu}_{4\alpha\beta\gamma} u^\alpha u^\beta u^\gamma + G^{\mu}_{4\alpha\beta\gamma\delta} u^\alpha u^\beta u^\gamma u^\delta, \]  
(3.61)

\[ Z_4 = H_{4\alpha\beta} u^\alpha u^\beta + L_4_{\alpha\beta\gamma\delta} u^\alpha u^\beta u^\gamma u^\delta, \]  
(3.62)

where

\[ C^{\mu\nu}_{2\, \alpha\beta} = \frac{1}{3} R^{\mu\nu}_{(\alpha \beta)}, \]  

\[ E^{\mu}_{2\alpha} = -\frac{1}{3} R^\mu_{\alpha} - R^\mu_{\alpha}, \]  

\[ G^{\mu}_{2\alpha\beta\gamma} = -\frac{1}{12} R^{\mu}_{(\alpha \beta iF_{\gamma})}, \]  

\[ H_{2\alpha\beta} = -\frac{1}{24} R_{\mu(\alpha iF^\mu_{\beta})}, \]  

\[ L_2 = \frac{1}{6} R, \]  
(3.63)

\[ C^{\mu\nu}_{3\, \alpha\beta\gamma} = -\frac{1}{6} \nabla_{(\alpha} R^{\mu}_{\beta \gamma)}, \]  

\[ E^{\mu}_{3\alpha\beta\gamma} = \frac{1}{3} \nabla_{(\alpha} R^{\mu}_{\beta)} - \frac{1}{6} \nabla^\mu R_{\alpha\beta} + \frac{2}{3} \nabla^{(\alpha} R^{\mu}_{\beta)}, \]  

\[ H_{3\alpha} = \frac{1}{3} \nabla_{\mu} R^\mu_{\alpha} - \frac{1}{6} \nabla_{\alpha} R, \]  
(3.64)
\[ C_{\mu\nu\alpha\beta} = \frac{1}{15} R^\mu_{\nu\alpha\beta} \nabla_{\gamma} R_{\beta\gamma} - \frac{1}{20} \nabla_{(\alpha} \nabla_\beta R^\mu_{\nu)\gamma} , \]
\[ E_{\mu\nu\alpha\beta} = -\frac{1}{15} R^\mu_{\nu\alpha\beta} \nabla_{\gamma} R_{\nu\gamma} - \frac{1}{60} R^\mu_{\nu\alpha\beta} R_{\nu\gamma} - \frac{1}{4} R^\mu_{\nu\alpha\beta} R_{\nu\gamma} , \]
\[ + \frac{1}{10} \nabla_{(\alpha} \nabla_\beta R^\mu_{\nu\gamma}) - \frac{3}{20} \nabla_{(\alpha} \nabla_\beta R^\mu_{\nu\gamma} - \frac{1}{4} \nabla_{(\alpha} \nabla_\beta R^\mu_{\nu\gamma} , \]
\[ G_{\mu4\alpha\beta\gamma} = \frac{1}{40} R^\mu_{\alpha\beta\nu} R_{\nu\gamma} i F_{(\alpha\beta) ,} \]
\[ H_{\mu\nu\alpha\beta} = \frac{1}{4} R_{\mu\nu\alpha\beta} - \frac{1}{30} R_{\mu\alpha\nu\beta} - \frac{1}{4} \nabla_{(\alpha} \nabla_\beta R^\mu_{\nu\gamma}) + \frac{1}{60} R_{\mu\nu\alpha\beta} R^\mu_{\nu\gamma} , \]
\[ + \frac{1}{60} R_{\mu\nu\alpha\beta} R^\mu_{\nu\gamma} + \frac{1}{40} \Delta R_{\mu\alpha\beta} + \frac{3}{40} \nabla_{\alpha} \nabla_\beta R , \]
\[ L_{\mu\nu\alpha\beta} = -\frac{1}{80} R_{\mu\nu\alpha\beta} R^\mu_{\nu\gamma} i F_{(\alpha\beta)} - \frac{1}{80} R_{\mu\nu\alpha\beta} R^\mu_{\nu\gamma} i F_{(\alpha\beta)} - \frac{1}{24} \nabla_{(\alpha} \nabla_\beta R^\mu_{\nu)\gamma} i F_{(\alpha\beta) ,} \]
\[ O_{\mu4\alpha\beta\gamma\delta\epsilon} = \frac{1}{576} R^\mu_{\alpha\beta\gamma\delta\epsilon} R_{\nu\gamma} i F_{(\alpha\beta\gamma)} \tag{3.65} \]

Here and everywhere below the parenthesis denote the complete symmetrization over all indices enclosed; the vertical lines indicate the indices excluded from the symmetrization.

### 3.3.2 Perturbation Theory for the Heat Semigroup

Now, by using the perturbative expansion (3.46) of the operator \( \tilde{L} \) and recalling that \( D^2 \sim \varepsilon^{-2} \) and \( t \sim \varepsilon^2 \), we see that the operator \( tD^2 \) is of zero order and the operator \( tL_k \), \( k \geq 2 \), is of (higher) order \( \varepsilon^k \). Therefore, we can consider the terms \( tL_k \) with \( k \geq 2 \) as a perturbation.

By using the Volterra series for the operator in (3.46) we obtain
\[
\exp(-t\tilde{L}) = T(t) \exp(tD^2) , \tag{3.66}
\]
where \( T(t) \) is an operator defined by a formal perturbative expansion
\[
T(t) \sim \sum_{k=0}^{\infty} T_k(t) , \tag{3.67}
\]
with $T_k(t)$ being of order $\varepsilon^k$. Explicitly, up to terms of fifth order we obtain

$$T_0(t) = I, \quad (3.68)$$

$$T_1(t) = 0, \quad (3.69)$$

$$T_2(t) = t \int_0^1 d\tau_1 V_2(t\tau_1), \quad (3.70)$$

$$T_3(t) = t \int_0^1 d\tau_1 V_3(t\tau_1) \quad (3.71)$$

$$T_4(t) = t \int_0^1 d\tau_1 V_4(t\tau_1) + t^2 \int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 V_2(t\tau_1)V_2(t\tau_2) \quad (3.72)$$

and

$$V_k(s) = e^{sD^2} \mathcal{L}_k e^{-sD^2}. \quad (3.73)$$

3.3.3 Perturbation Theory for the Heat Kernel

As we already mentioned above the heat kernel can be computed from the heat semigroup by using the equation (3.39). By using the heat semigroup expansion from the previous section we now obtain the heat kernel in the form

$$U(t|x,x') \sim \mathcal{P}(x,x')\Delta^{1/2}(x,x')U_0(t|x,x') \sum_{k=0}^{\infty} t^{k/2} b_k(t|x,x'), \quad (3.74)$$

where

$$U_0(t|x,x') = \exp(tD^2)\delta(u), \quad (3.75)$$

and

$$b_k(t|x,x') = t^{k/2}U_0^{-1}(t|x,x')T_k(t)U_0(t|x,x'). \quad (3.76)$$
Thus, the calculation of the heat kernel coefficients reduces to the evaluation of the zero-order heat kernel $U_0(t|x, x')$ and to the action of the differential operators $T_k(t)$ on it.

The zero order heat kernel $U_0(t|x, x')$ can be evaluated by using the algebraic method developed in [6, 7]. First, the heat semigroup $\exp(tD^2)$ can be represented as an average over the (nilpotent) Lie group (3.34) with a Gaussian measure

$$\exp(tD^2) = (4\pi t)^{-n/2} J(t) \int_{\mathbb{R}^n} dk \exp \left\{ -\frac{1}{4} k^\mu M_{\mu\nu}(t) k^\nu + k^\mu D_\mu \right\},$$

where

$$J(t) = \det \left( \frac{t F}{\sinh(t F)} \right)^{1/2},$$

and $M(t)$ is a symmetric matrix defined by

$$M(t) = i F \coth(t F).$$

We would like to stress, at this point, that here and everywhere below all the functions of the 2-form $F$ are analytic and should be understood in terms of a power series in $F$.

Then by using the relation

$$\exp(k^\mu D_\mu)\delta(u) = \delta(u + k),$$

one obtains

$$U_0(t|x, x') = (4\pi t)^{-n/2} J(t) \exp \left\{ -\frac{1}{4} u^\mu M_{\mu\nu}(t) u^\nu \right\},$$

which is nothing but the Schwinger kernel for an electromagnetic field on $\mathbb{R}^n$ [72].

To obtain the asymptotic expansion of the heat kernel diagonal we just need to set $x = x'$ (or $u = 0$). At this point, we notice the following interesting fact.
The operators $tL_k$, $tV_k(t\tau)$ and $T_k(t)$ are differential operators with homogeneous polynomial coefficients (in $u^\mu$) of order $\varepsilon^k$. Recall that $u \sim \varepsilon$, $t \sim \varepsilon^2$ and $F \sim \varepsilon^{-2}$, so that $tF$ and $Fuu$ are counted as of order zero. Since the zero order heat kernel $U_0$ is Gaussian, then the off-diagonal coefficients $b_k(t|x,x')$ are polynomials in $u$. The point we want to make now is the following.

**Lemma 1.** The off-diagonal odd-order coefficients $b_{2k+1}$ are odd order polynomials in $u^\mu$, that is, they satisfy

$$b_{2k+1}(t|x,x') \bigg|_{u \to -u} = -b_{2k+1}(t|x,x'), \quad (3.82)$$

and, therefore, vanish on the diagonal,

$$b_{2k+1}^{\text{diag}}(t) = 0. \quad (3.83)$$

**Proof.** We discuss the transformation properties of various quantities under the reflection of the coordinates, $u \mapsto -u$. First, we note that the operator $D$ changes sign, and, therefore, the operator $L_0 = -D^2$ is invariant. Next, from the general form of the operator $L_k$ discussed above we see that $L_k \mapsto (-1)^k L_k$. Therefore, the same is true for the operator $V_k(t\tau)$, that is, $V_k \mapsto (-1)^k V_k$.

Now, the operator $T_k(t)$ has the following general form

$$T_k = t^k \sum_{m=1}^{[k/2]} \int_0^{\tau_1} \cdots \int_0^{\tau_m} \sum_{|j|=k} C_{m,j} V_{j_1}(t\tau_1) \cdots V_{j_m}(t\tau_m), \quad (3.84)$$

where the summation goes over multiindex $j = (j_1, \ldots, j_m)$ of integers $j_1, \ldots, j_m \geq 2$ such that $|j| = j_1 + \cdots + j_m = k$, and $C_{m,j}$ are some numerical coefficients. Therefore, the operator $T_k$ transforms as $T_k \mapsto (-1)^k T_k$. 
Since the zero-order heat kernel $U_0$ is invariant under the reflection of coordinates $u \mapsto -u$, we finally find that the coefficients $b_k$ transform according to $b_k \mapsto (-1)^k b_k$. Thus, $b_{2k}$ are even polynomials and $b_{2k+1}$ are odd-order polynomials.

\[ \square \]

By using this lemma and by setting $x = x'$ we obtain the asymptotic expansion of the heat kernel diagonal

\[
U^\text{diag}(t) \sim (4\pi t)^{-n/2} J(t) \sum_{k=0}^{\infty} t^k b^\text{diag}_{2k}(t),
\]

where the function $J(t)$ is defined in (3.78). Thus, we obtain

\[
\tilde{a}^\text{diag}_k(t) = J(t)b^\text{diag}_{2k}(t).
\]

### 3.3.4 Algebraic Framework

As we have shown above the evaluation of the heat semigroup is reduced to the calculation of the operators $V_k(s)$ defined by (3.73), which reduces, in turn, to the computation of general expressions

\[
e^{s\mathcal{D}^2} u^{\nu_1} \cdots u^{\nu_n} \mathcal{D}_{\mu_1} \cdots \mathcal{D}_{\mu_m} e^{-s\mathcal{D}^2} = Z^{\nu_1}(s) \cdots Z^{\nu_n}(s) A_{\mu_1}(s) \cdots A_{\mu_m}(s),
\]

where

\[
Z^{\nu}(s) = e^{s\mathcal{D}^2} u^{\nu} e^{-s\mathcal{D}^2},
\]

\[
A_{\mu}(s) = e^{s\mathcal{D}^2} \mathcal{D}_{\mu} e^{-s\mathcal{D}^2}.
\]

Obviously, the operators $A_{\mu}$ and $Z_{\nu}$ form the algebra

\[
[A_{\mu}(s), Z^{\nu}(s)] = \delta_{\mu}^{\nu}, \quad [A_{\mu}(s), A_{\nu}(s)] = iF_{\mu\nu}, \quad [Z^{\mu}(s), Z^{\nu}(s)] = 0.
\]
The operators $A_\mu(s)$ and $Z^\nu(s)$ can be computed as follows. First, we notice that $A_\mu(s)$ satisfies the differential equation

$$\partial_s A_\mu(s) = \text{Ad}_{D^2} A_\mu(s),$$

(3.91)

with the initial condition

$$A_\mu(0) = D_{\mu}.$$

Hereafter $\text{Ad}_{D^2}$ is an operator acting as a commutator, that is,

$$\text{Ad}_{D^2} A_\mu(s) \equiv [D^2, A_\mu(s)].$$

(3.92)

The solution of eq. (3.91) is

$$A_\mu(s) = \exp(s \text{Ad}_{D^2}) D_{\mu},$$

(3.93)

which can be written in terms of series as

$$A_\mu(s) = \sum_{k=0}^{\infty} \frac{s^k}{k!} (\text{Ad}_{D^2})^k D_{\mu}.$$  

(3.94)

Now, by using the algebra (3.34) we first obtain the commutator

$$[D^2, D_\mu] = -2i F_{\mu\alpha} D^\alpha,$$

(3.95)

and then, by induction,

$$(\text{Ad}_{D^2})^k D_\mu = (-2i)^k F_{\mu\alpha_1} F_{\alpha_2} \cdots F_{\alpha_{k-1}\alpha_k} D_{\alpha_k} = [(-2i)^k]_{\mu\alpha} D^\alpha.$$  

(3.96)

By substituting this result in the series (3.94) we finally find that

$$A_\mu(s) = \Psi_{\mu}^{\alpha}(s) D_{\alpha},$$

(3.97)
where
\[
\Psi(s) = \exp(-2siF) .
\] (3.98)

Similarly, for the operators \(Z'(s)\) we find
\[
Z'(s) = \exp(s \text{Ad}_{D^2})u^\mu = \sum_{k=0}^{\infty} \frac{s^k}{k!} (\text{Ad}_{D^2})^k u^\mu .
\] (3.99)

Now, by using the commutators in (3.34), we find
\[
\text{Ad}_{D^2} u^\mu = \left[D^2, u^\mu\right] = 2D^\mu ,
\] (3.100)
and then, by induction, we obtain, for \(k \geq 2\),
\[
(\text{Ad}_{D^2})^k u^\mu = 2((-2iF)^{k-1} \mu^\alpha D_\alpha .
\] (3.101)

Thus the operator \(Z'(s)\) in (3.89) takes the form
\[
Z'(s) = u^\mu - 2sD^\mu + 2 \sum_{k=2}^{\infty} \frac{s^k}{k!}((-2iF)^{k-1} \mu^\alpha D_\alpha .
\] (3.102)

This series can be easily summed up to give
\[
Z'(s) = u^\mu + \Omega^\mu_\alpha (s)D_\alpha ,
\] (3.103)

where
\[
\Omega(s) = \frac{1 - \exp(-2siF)}{iF} = 2 \exp(-siF) \frac{\sinh(siF)}{iF} .
\] (3.104)

Now, by using (3.98) and (3.104) we obtain
\[
\Omega^{-1}(s) = \frac{1}{2}iF \left[\coth(siF) + 1\right] = \frac{1}{2} \left[M(s) + iF\right] .
\] (3.105)

We will need the symmetric and the antisymmetric parts of \(\Omega^{-1}(s)\). By recalling that the matrix \(F\) is anti-symmetric it is easy to show
\[
\Omega_{(\mu\nu)}^{-1}(s) = \frac{1}{2} M_{\mu\nu}(s) .
\] (3.106)
\[ \Omega_{\nu}^{-1}(s) = \frac{1}{2} i F_{\mu \nu} , \quad (3.107) \]

Here and everywhere below the square brackets denote the complete antisymmetrization over all indices included.

For the future reference we also notice that
\[ \Omega^{-1}(s)\Omega^T(s) = \Psi^{-1}(s) = \exp(2siF) , \quad (3.108) \]

Finally, we define another function
\[ \Phi(s) = \Psi(s)\Omega^{-1}(s) = \left(\Omega^{-1}(s)\right)^T = \frac{1}{2} [M(s) - iF] . \quad (3.109) \]

It is useful to remember that the functions \( \Psi, F\Omega \) and \( \Phi\Omega \) are dimensionless.

### 3.3.5 Flat Connection

Next, we transform the operators \( Z^{\mu} \) to define new (time-dependent) derivative operators by
\[ D_{\mu}(s) = \Omega^{-1}_{\mu \nu}(s)Z^{\nu}(s) . \quad (3.110) \]

By using the explicit form of the operators \( Z^{\mu} \) and \( D_{\mu} \) we have
\[ D_{\mu}(s) = D_{\mu} + \Omega^{-1}_{\mu \nu}(s)u^{\nu} \]
\[ = \bar{\nabla}_{\mu} + \frac{1}{2} M_{\mu \nu}(s)u^{\nu} . \quad (3.111) \]

Since the operators \( Z^{\mu} \) commute, the operators \( D_{\mu}(s) \) obviously commute as well. In other words, the connection \( D_{\mu} \) is flat. Therefore, it can also be written as
\[ D_{\mu}(s) = e^{-\Theta(s)\bar{\nabla}_{\mu}}e^{\Theta(s)} , \quad (3.112) \]
where,
\[ \Theta(s) = \frac{1}{4} u^\mu M_{\mu\nu}(s) u^\nu . \] (3.113)

Now, we can rewrite the operators \( A_\mu(s) \) and \( Z^\mu(s) \) in (3.97) and (3.103) in terms of the operators \( D_\mu(s) \)

\[ A_\mu(s) = \Psi_\mu(s) \left( D_\alpha(s) - \Omega^{-1}_{\alpha\beta}(s) u^\beta \right) , \]
\[ Z^\mu(s) = \Omega^\alpha_\mu(s) D_\alpha(s) . \] (3.114)

It is useful, for future calculations, to prove the following

**Lemma 2.** Let \( D_\mu \) and \( u^\nu \) be operators satisfying the algebra

\[ [D_\mu, u^\nu] = \delta_\mu^\nu , \quad [D_\mu, D_\nu] = [u^\mu, u^\nu] = 0 . \] (3.115)

Then

\[ D_{\mu_1} \cdots D_{\mu_n} u^\rho = n \delta^\rho_{(\mu_1} D_{\mu_2} \cdots D_{\mu_n)} , \]
\[ D_{\mu_1} \cdots D_{\mu_n} u^\rho u^\sigma = n(n - 1) \delta^\rho_{(\mu_1} \delta^\sigma_{\mu_2} D_{\mu_3} \cdots D_{\mu_n)} + 2n u^\rho \delta^\sigma_{(\mu_1} D_{\mu_2} \cdots D_{\mu_n)} . \] (3.117)

**Proof.** Let \( \chi(\xi) = \xi^\mu D_\mu \) and

\[ \varphi^\mu(t) = \left[ e^{i\chi(\xi)}, u^\rho \right] = \left( e^{i\chi(\xi)} u^\rho e^{-i\chi(\xi)} - u^\rho \right) e^{i\chi(\xi)} , \] (3.118)

Then

\[ e^{i\chi(\xi)} u^\rho e^{-i\chi(\xi)} = \sum_{k=0}^{\infty} \frac{t^k}{k!} \left( \text{Ad}_{\chi(\xi)} \right)^k u^\rho . \] (3.119)

By using the commutation relation in (3.115) we have

\[ [\chi(\xi), u^\rho] = \xi^\rho , \] (3.120)
and, therefore,
\[ e^{tX(\xi)} u^\rho e^{-tX(\xi)} = u^\rho + t\xi^\rho. \]  
(3.121)

Thus
\[ \phi^\rho(t) = t\xi^\rho e^{tX(\xi)}. \]  
(3.122)

By expanding in Taylor series both sides of the last equation we obtain
\[ \sum_{k=0}^{\infty} \frac{t^{k+1}}{(k+1)!} \xi^{\mu_1} \cdots \xi^{\mu_{k+1}} = \sum_{k=0}^{\infty} \frac{t^{k+1}}{k!} \xi^{\mu_1} \cdots \xi^{\mu_{k+1}} D_{\mu_2} \cdots D_{\mu_{k+1}}. \]  
(3.123)

Now by equating the same powers of \( t \) in both series we obtain the claim (3.116).

The second relation can be proved in a similar manner. We introduce, in this case, the following generating function
\[ \phi^{\rho\sigma}(t) = \left[ e^{tX(\xi)}, u^\rho u^\sigma \right]. \]  
(3.124)

By the same argument used in the proof of the first relation we obtain that
\[ \phi^{\rho\sigma}(t) = \left[ e^{tX(\xi)}, u^\rho u^\sigma \right] = 2t\xi^{(\rho} u^{\sigma)} e^{tX(\xi)} + t^2 \xi^\rho \xi^\sigma. \]  
(3.125)

Now, as before, by expanding the last equation in Taylor series and equating the same powers of \( t \) we obtain the claim (3.117).

\[ \blacksquare \]

3.4 Evaluation of the Operator \( T \)

The perturbative expansion of the operator \( T \) is given by the eq. (3.67), with the operators \( T_k \) being integrals of the operators \( V_4(s) \) and their product. Thus, according to (3.70)-(3.72), to compute the operator \( T \) up to the fourth order we need to compute the operators \( V_2(s), V_3(s), V_4(s) \) and \( V_2(s_1)V_2(s_2) \).
3.4.1 Second Order

Now, by using the explicit expression for $\mathcal{L}_2$ given by eqs. (3.49), (3.56) and (3.63), utilizing the results of the Section 3, exploiting eqs. (3.114), (3.116) and (3.117), using eqs. (3.98), (3.104), (3.108) and (3.109) after some straightforward but cumbersome calculations we obtain

$$V_2(s) = \frac{1}{6} R + N^{\sigma}_{(2)} D_{\sigma} + P^{\rho\delta}_{(2)} D_{\rho} D_{\delta} + W^{\sigma\delta\gamma}_{(2)} D_{\sigma} D_{\delta} D_{\gamma} + Q^{\rho\sigma\delta\gamma}_{(2)} D_{\rho} D_{\sigma} D_{\delta} D_{\gamma},$$

where

$$N^{\sigma}_{(2)} = \left( R_{\alpha} - \frac{1}{3} R_{\alpha} \right) \Omega^{\alpha\sigma} \Phi_{\mu\eta} t_{\eta},$$

$$P^{\rho\delta}_{(2)} = \frac{1}{3} R_{\alpha} \delta^{\rho}_{\delta} \Omega^{(\gamma} \Omega^{\beta)} \Phi_{\mu\nu} t_{\mu} t_{\nu} - \frac{1}{2} M_{\mu\nu},$$

$$W^{\sigma\delta\gamma}_{(2)} = -\frac{1}{12} R_{\alpha} \delta^{\sigma}_{\gamma} \Omega^{(\rho} \Omega^{\beta)\sigma} \Phi_{\mu\nu} t_{\mu} + 3 \Psi_{\nu},$$

$$Q^{\rho\sigma\delta\gamma}_{(2)} = \frac{1}{12} R_{\alpha} \delta^{\sigma}_{\gamma} \Omega^{(\rho} \Phi_{\mu\nu} \delta^{\beta)} \Phi_{\mu\nu} + 3 \Psi_{\nu}.$$

Note that all these coefficients as well as the operators $D_{\mu}$ depend on the time variable $s$. We will indicate explicitly the dependence of various quantities on the time parameter only in the cases when it causes confusion, in particular, when there are two time parameters.

3.4.2 Third Order

Similarly, by using the explicit expression for $\mathcal{L}_3$ given by (3.49), (3.59) and (3.64), utilizing the results of the Section 3, exploiting eqs. (3.114), (3.116) and
(3.117), using eqs. (3.98), (3.104), (3.108) and (3.109) after some straightforward but cumbersome calculations we obtain

\[ V_3(s) = N^s_{(3)} D_\sigma D_\rho D_\epsilon \]

\[ + Q^s_{(3)} D_\sigma D_\rho D_\epsilon D_\delta + Y^s_{(3)} D_\sigma D_\rho D_\epsilon D_\delta D_\kappa, \]  

where

\[ N^s_{(3)} = -\frac{1}{6} \left( \nabla_\alpha R + 2 \nabla_\mu R^\mu_{\alpha} \right) \Omega^\alpha, \]  

\[ (3.132) \]

\[ P^s_{(3)} = -\frac{1}{6} \left( \nabla_\mu R_{\alpha \beta} - 2 \nabla_\alpha R^\mu_{\beta} + 4 \nabla_\alpha R^\mu_{\beta} \right) \Omega^\alpha \Omega^{[\beta} \Omega^{\gamma]} \Phi_{\mu \kappa} u^\kappa, \]  

\[ (3.133) \]

\[ W^s_{(3)} = -\frac{1}{3} \nabla_\alpha R^\mu_{\beta \gamma} \Omega^\alpha \Omega^{[\beta} \Omega^{\gamma] \Omega_{\mu \nu} \Phi_{\nu \kappa} u^\kappa \]  

\[ + \frac{1}{6} \left( \nabla_\mu R_{\alpha \beta} - 2 \nabla_\alpha R^\mu_{\beta} + 4 \nabla_\alpha R^\mu_{\beta} \right) \Omega^\alpha \Omega^{[\beta} \Omega^{\gamma] \Omega_{\mu \nu} \Psi_{\nu \kappa} u^\kappa, \]  

\[ (3.134) \]

\[ Q^s_{(3)} = \frac{1}{3} \nabla_\alpha R^\mu_{\beta \gamma} \Omega^\alpha \Omega^{[\beta} \Omega^{\gamma] \Omega_{\mu \nu} \Psi_{\nu \kappa} u^\kappa, \]  

\[ (3.135) \]

\[ Y^s_{(3)} = -\frac{1}{6} \nabla_\alpha R^\mu_{\beta \gamma} \Omega^\alpha \Omega^{[\beta} \Omega^{\gamma] \Omega_{\mu \nu} \Psi_{\nu \kappa} u^\kappa \]  

\[ (3.136) \]

Here again, for simplicity, we omitted the dependence of the coefficient functions and the derivatives on the time variable \( s \).

### 3.4.3 Fourth Order

**Operator** \( V_4(s) \)

By taking into account the definition of \( L_4 \) in (3.49) by using eqs. (3.60)-(3.62), (3.114), (3.116) and (3.117), and the explicit form of the functions \( \Psi \) and \( \Omega \), we obtain

\[ V_4(s) = P^s_{(4)} D_\sigma D_\rho D_\epsilon \]

\[ + Q^s_{(4)} D_\sigma D_\rho D_\epsilon D_\delta + Y^s_{(4)} D_\sigma D_\rho D_\epsilon D_\delta D_\kappa + S^s_{(4)} D_\sigma D_\rho D_\epsilon D_\delta D_\kappa D_\lambda, \]  

\[ (3.137) \]
where

\[ P^{\sigma \rho}_{(4)} = \frac{1}{60} \left[ R_{\mu \nu} R^\mu_{\alpha \beta} + R_{\mu \nu \lambda \alpha} R^\mu_{\lambda \beta} \right] \Omega^{\alpha \nu} \Omega^{\beta \rho}, \]

\[ + \frac{1}{40} \left[ \Delta R_{\alpha \beta} + 3 \nabla_\alpha \nabla_\beta R \right] \Omega^{\sigma \alpha} \Omega^{\beta \rho}, \]

\[ + \frac{1}{4} \left[ R_{\mu \alpha} R^\mu_{\beta} + \nabla_\alpha \nabla_\beta R^{\rho \sigma} \right] \Omega^{\alpha \rho} \Omega^{\beta \sigma}, \]  

(3.138)

\[ W^{\sigma \rho \mu}_{(4)} = \frac{1}{60} \left[ 6 \nabla_\alpha \nabla_\beta R^\mu_{\gamma \delta} + 15 \nabla_\alpha \nabla_\beta R^\mu_{\gamma} + 15 R^\mu_{\alpha \beta} R^\gamma_{\gamma} - 9 \nabla_\alpha \nabla_\beta R^\mu_{\gamma} \right] \Omega^{\alpha \sigma} \Omega^{\beta \rho} \Omega^{\gamma \nu} \Phi_{\mu \xi} u^\xi, \]

(3.139)

\[ Q^{\sigma \rho \mu \nu}_{(4)} = \frac{1}{300} \left[ 20 R^\mu_{\alpha \beta \gamma} R^\nu_{\gamma \delta} + 15 \nabla_\alpha \nabla_\beta R^\mu_{\gamma} \right] \Omega^{\alpha \sigma} \Omega^{\beta \rho} \Omega^{\gamma \nu} \Omega^{\delta \epsilon} \]

\[ \times \left[ \Phi_{\mu \xi} \Phi_{\nu \xi} u^\xi u^\epsilon - \frac{1}{2} M_{\mu \nu} \right] \]

\[ + \frac{1}{240} R^\mu_{\alpha \beta \gamma} \Omega^{\alpha \sigma} \Omega^{\beta \rho} \Omega^{\gamma \nu} \Gamma^{\delta \epsilon}_{\mu \nu} \left[ 3 \delta^\mu_{\epsilon} + \Psi^\mu_{\epsilon} \right] \]

\[ + \frac{1}{240} R^\mu_{\alpha \beta \gamma} \Omega^{\alpha \sigma} \Omega^{\beta \rho} \Omega^{\gamma \nu} \Gamma^{\delta \epsilon}_{\mu \nu} \left[ 3 \delta^\mu_{\epsilon} + 13 \Psi^\mu_{\epsilon} \right] \]

\[ - \frac{1}{24} R^\mu_{\alpha \beta \gamma} \Omega^{\alpha \sigma} \Omega^{\beta \rho} \Omega^{\gamma \nu} \Gamma^{\delta \epsilon}_{\mu \nu} \left[ \delta^\mu_{\epsilon} + 5 \Psi^\mu_{\epsilon} \right] \]

\[ + \frac{1}{20} \left[ 3 \nabla^\alpha \nabla^\beta \nabla^\mu \nabla^\nu - 2 \nabla^\alpha \nabla^\mu \nabla^\nu \right] \Omega^{\alpha \sigma} \Omega^{\beta \rho} \Omega^{\gamma \nu} \mu^\epsilon, \]

(3.140)

\[ Y^{\sigma \rho \alpha \beta}_{(4)} = -\frac{1}{10} \nabla^\alpha \nabla^\beta R^{\mu \nu \gamma \delta} \Omega^{\alpha \sigma} \Omega^{\beta \rho} \Omega^{\gamma \nu} \Omega^{\delta \epsilon} \Phi_{\mu \xi} u^\xi \]

\[ - \frac{1}{120} R^\alpha_{\gamma \delta} \Omega^{\gamma \delta} \Omega^{\rho \sigma} \Omega^{\omega \nu} \Omega^{\kappa \xi} \left[ 3 \delta^\nu_{\kappa} + 13 \Psi^\nu_{\kappa} \right] \Phi_{\mu \xi} u^\epsilon, \]  

(3.141)
\[
S^{(4)}_{\text{pre-excl}} = \frac{1}{20} \nabla^\alpha \nabla^\beta R^\gamma\psi^\delta \Omega_\alpha (\sigma \Omega^\nu_\beta \Omega^\nu_\delta \Omega^\nu_\gamma \Omega^\nu_\lambda \psi^{\kappa \lambda}) \\
+ \frac{1}{2880} R^\mu\eta\gamma\psi^\delta \Omega_\alpha (\sigma \Omega^\nu_\beta \Omega^\nu_\delta \Omega^\nu_\gamma \Omega^\nu_\lambda \psi^{\kappa \lambda} ) \left[ 62 \psi^{\kappa \lambda} \delta^\nu_\delta \right] \\
+ 125 \psi^{\kappa \lambda} \delta^\nu_\delta + 5 \delta^{\kappa \lambda} \delta^\nu_\delta .
\] (3.142)

**Operator \(V_2(s_1)V_2(s_2)\)**

Next, we need to compute the product of two operators \(V_2(s)\) depending on different times \(s_1\) and \(s_2\) by using the eq. (3.126). To simplify the notation we denote the derivatives \(D_{\mu}(s_k)\) depending on different times \(s_k\) simply by \(D_\mu^{(k)}\). To present the product \(V_2(s_1)V_2(s_2)\) in the “normal” form we need to move all derivative operators \(D_\mu^{(1)}\) to the right and all coordinates \(u^\nu\) to the left. In order to perform this task we need the commutator of the derivative operator \(D_\mu^{(1)}\) with the coefficients of the operator \(V_2(s_2)\). First, by using the commutators found earlier we obtain the relevant commutators

\[
\begin{align*}
\left[ D_{\mu_1}^{(1)} \ldots D_{\mu_n}^{(1)}, N^\nu_\eta(s_2) \right] &= nf^{\rho}_{(\mu_1}(s_2)D_{\rho\mu_2}^{(1)} \ldots D_{\mu_n)}^{(1)}, \\
\left[ D_{\mu_1}^{(1)} \ldots D_{\mu_n}^{(1)}, P^\eta_\nu(s_2) \right] &= n(n-1)g^\eta_{(\mu_1 \mu_2}(s_2)D_{\rho3}^{(1)} \ldots D_{\rho_n)}^{(1)} \\
&\quad + nh^\eta_{(\mu_1}(s_2)D_{\rho_2}^{(1)} \ldots D_{\rho_n)}^{(1)}, \\
\left[ D_{\mu_1}^{(1)} \ldots D_{\mu_n}^{(1)}, W^\eta_\nu(s_2) \right] &= np^{\eta}_{(\mu_1}(s_2)D_{\rho_2}^{(1)} \ldots D_{\rho_n)}^{(1)},
\end{align*}
\] (3.143) (3.144) (3.145)

where

\[
\begin{align*}
f^{\rho}_{\lambda} &= \left( R_{\beta}^\beta - \frac{1}{3} R_{\beta}^\beta \right) \Omega^\beta_\alpha \Phi_{\mu\lambda}, \\
g^\eta_{\nu, \kappa} &= \frac{1}{3} R^\mu_{(\alpha \beta)} \Omega^\alpha_\mu \Omega^{\beta \eta} \Phi_{\mu \delta} \Phi_{\nu \kappa}, \\
h^\eta_{\lambda} &= \frac{2}{3} R^\mu_{(\alpha \beta)} \Omega^\alpha_\mu \Omega^{\beta \eta} \Phi_{\mu \kappa} \Phi_{\nu \lambda}, \\
p^{\eta}_{\nu, \kappa} &= -\frac{1}{12} R^\mu_{\alpha \beta} \Omega^{\mu \nu} \Omega^{\beta \eta} \left[ \delta_{\nu}^{\kappa} + 7 \Psi_{\nu}^{\kappa} \right] \Phi_{\mu \lambda}.
\end{align*}
\] (3.146) (3.147) (3.148)
where

\[ (0, \ldots, 2) \]

and

\[ (3,1) \]

and

Next, by using the expression for the operator \( V_2(s) \) in (3.126) and the non-vanishing commutators in (3.143)-(3.144) we obtain

\[
V_2(s_1)V_2(s_2) = \frac{1}{36} R^2 + \frac{1}{6} R \left[ V_2(s_1) + V_2(s_2) \right] + L(s_1, s_2), \tag{3.149}
\]

where

\[
L(s_1, s_2) = \sum_{k=1}^{4} \sum_{n=0}^{4} C_{\mu_1\ldots\mu_k\nu_1\ldots\nu_2}^{\alpha\beta\gamma\delta}(s_1, s_2) D_{\mu_1}^{(1)} \cdots D_{\mu_k}^{(1)} D_{\nu_1}^{(2)} \cdots D_{\nu_2}^{(2)}, \tag{3.150}
\]

and

\[
C^\rho_{(0,1)} = N^\alpha_2(s_1) f^\rho_\alpha(s_2),
\]

\[
C^\alpha\beta\rho_{(1,1)} = 2 N^\alpha_2(s_1) N^\rho_2(s_2) + 2 P^\alpha_{(2)}(s_1) f^\rho_\alpha(s_2),
\]

\[
C^\alpha\beta\rho_{(2,1)} = 2 P^\alpha_{(2)}(s_1) N^\rho_2(s_2) + 3 W^\alpha\beta_{(2)}(s_1) f^\rho_\alpha(s_2),
\]

\[
C^{\alpha\beta\gamma\rho}_{(3,1)} = 2 W^{\alpha\beta\gamma}_{(2)}(s_1) N^\rho_2(s_2) + 4 Q^{\alpha\beta\gamma}_{(2)}(s_1) f^\rho_\alpha(s_2),
\]

\[
C^{\alpha\beta\gamma\rho}_{(4,1)} = 2 Q^{\alpha\beta\gamma\rho}_{(2)}(s_1) N^\rho_2(s_2), \tag{3.151}
\]

\[
C^{\alpha\sigma\rho}_{(0,2)} = N^\alpha_2(s_1) h^{\alpha\sigma}_{\rho}(s_2) + 2 P^\alpha_{(2)}(s_1) g^{\alpha\rho}_{\sigma}(s_2),
\]

\[
C^{\alpha\sigma\rho}_{(1,2)} = 2 N^\alpha_2(s_1) P^{\alpha\rho}_{(2)}(s_2) + 2 P^\alpha_{(2)}(s_1) h^{\alpha\sigma}_{\rho}(s_2) + 6 W^{\alpha\beta\gamma}_{(2)}(s_1) g^{\rho\gamma}_{\alpha\beta}(s_2),
\]

\[
C^{\alpha\beta\sigma\rho}_{(2,2)} = 2 P^\alpha_{(2)}(s_1) P^{\beta\rho}_{(2)}(s_2) + 3 W^{\alpha\beta\gamma}_{(2)}(s_1) h^{\alpha\sigma}_{\rho}(s_2) + 12 Q^{\alpha\beta\gamma\rho}_{(2)}(s_1) g^{\rho\sigma}_{\alpha\gamma}(s_2),
\]

\[
C^{\alpha\beta\gamma\rho\sigma}_{(3,2)} = 2 W^{\alpha\beta\gamma}_{(2)}(s_1) P^{\rho\sigma}_{(2)}(s_2) + 4 Q^{\alpha\beta\gamma\rho}_{(2)}(s_1) h^{\rho\sigma}_{\alpha\beta}(s_2),
\]

\[
C^{\alpha\beta\gamma\rho\sigma}_{(4,2)} = 2 Q^{\alpha\beta\gamma\rho\sigma}_{(2)}(s_1) P^{\rho\sigma}_{(2)}(s_2), \tag{3.152}
\]

\[
C^{\alpha\beta\rho\sigma\tau\upsilon}_{(0,3)} = N^\alpha_2(s_1) p^{\alpha\rho\sigma\tau\upsilon}_{\xi}(s_2),
\]

\[
C^{\alpha\beta\rho\sigma\tau\upsilon}_{(1,3)} = 2 N^\alpha_2(s_1) W^{\alpha\beta\rho\sigma\tau\upsilon}_{(2)}(s_2) + 2 P^\alpha_{(2)}(s_1) p^{\alpha\rho\sigma\tau\upsilon}_{\mu}(s_2),
\]

\[
C^{\alpha\beta\rho\sigma\tau\upsilon}_{(2,3)} = 2 P^\alpha_{(2)}(s_1) W^{\alpha\beta\rho\sigma\tau\upsilon}_{(2)}(s_2) + 3 W^{\rho\sigma\tau\upsilon}_{(2)}(s_1) p^{\rho\sigma\tau\upsilon}_{\mu}(s_2),
\]

\[
C^{\alpha\beta\gamma\rho\sigma\tau\upsilon}_{(3,3)} = 2 W^{\alpha\beta\gamma}_{(2)}(s_1) W^{\rho\sigma\tau\upsilon}_{(2)}(s_2) + 4 Q^{\alpha\beta\gamma\rho\sigma\tau\upsilon}_{(2)}(s_1) p^{\rho\sigma\tau\upsilon}_{\mu}(s_2),
\]

\[
C^{\alpha\beta\gamma\rho\sigma\tau\upsilon}_{(4,3)} = 2 Q^{\alpha\beta\gamma\rho\sigma\tau\upsilon}_{(2)}(s_1) W^{\rho\sigma\tau\upsilon}_{(2)}(s_2), \tag{3.153}
\]
\[ C^{\alpha\rho\sigma\chi}_{(1,4)} = N^\alpha_{(2)}(s_1)Q^{\rho\sigma\chi}_{(2)}(s_2) , \]
\[ C^{\alpha\beta\rho\sigma\chi}_{(2,4)} = P^{\alpha\beta}_{(2)}(s_1)Q^{\rho\sigma\chi}_{(2)}(s_2) , \]
\[ C^{\alpha\beta\gamma\rho\sigma\chi}_{(3,4)} = W^{\alpha\beta\gamma}_{(2)}(s_1)Q^{\rho\sigma\chi}_{(2)}(s_2) , \]
\[ C^{\alpha\beta\gamma\delta\rho\sigma\chi}_{(4,4)} = Q^{\alpha\beta\gamma\delta}_{(2)}(s_1)Q^{\rho\sigma\chi}_{(2)}(s_2) . \] (3.154)

### 3.5 Generalized Hermite Polynomials

Thus, we reduced the calculation of the asymptotic expansion of the heat kernel to the calculation of the derivatives \( D_{\mu}(s) \) of the zero order heat kernel \( U_0(t|x,x') \) given by (3.81). The needed derivatives of the zero order heat kernel can be expressed in terms of the following symmetric tensors

\[ \mathcal{H}_{\mu_1\ldots\mu_n}(s) = U_0^{-1}(t|x,x')D_{\mu_1}(s)\cdots D_{\mu_n}(s)U_0(t|x,x') , \] (3.155)

and

\[ \Xi_{\nu_1\ldots\nu_m\mu_1\ldots\mu_n}(s_1, s_2) = U_0^{-1}(t|x,x')D^{(1)}_{\nu_1}(s_1)\cdots D^{(1)}_{\nu_m}(s_1)D^{(2)}_{\mu_1}(s_2)\cdots D^{(2)}_{\mu_n}(s_2)U_0(t|x,x') , \] (3.156)

where we denoted as before \( D^{(k)}_{\mu} = D_{\mu}(s_k) \).

We recall that the derivatives \( D^{(1)}_{\mu} \) and \( D^{(2)}_{\nu} \) do not commute! Also, \( U_0 \) is a scalar function that depends on \( x \) and \( x' \) only through the normal coordinates \( u^\mu \). The derivative operator \( D_{\mu}(s) \) is defined by (3.111), and, when acting on a scalar function is equal to

\[ D_{\mu}(s) = \frac{\partial}{\partial u^\mu} + \frac{1}{2}M_{\mu\nu}(s)u^\nu \]
\[ = e^{-\Theta(s)} \frac{\partial}{\partial u^\mu} e^{\Theta(s)} , \] (3.157)

where the tensor \( M_{\mu\nu}(s) \) is defined by (3.79) and the function \( \Theta(s) \) is a quadratic form defined by (3.113).
Therefore, by using the explicit form of the zero order heat kernel (3.81) we see that the tensors $H_{\mu_1 \cdots \mu_n}(s)$ can be written in the form

$$H_{\mu_1 \cdots \mu_n}(s) = \exp[\Theta(t) - \Theta(s)] \frac{\partial}{\partial u^{\mu_1}} \cdots \frac{\partial}{\partial u^{\mu_n}} \exp[\Theta(s) - \Theta(t)] ,$$

(3.158)

The tensors $H_{\mu_1 \cdots \mu_n}(s)$ are polynomials in $u^\mu$. They differ from the usual Hermite polynomials of several variables (see, for example, [34]) by some normalization. That is why, we call them just Hermite polynomials. The generating function for Hermite polynomials

$$H(\xi, s) = \sum_{n=0}^{\infty} \frac{1}{n!} \xi_{\mu_1} \cdots \xi_{\mu_n} H_{\mu_1 \cdots \mu_n}(s) ,$$

(3.159)

can be computed as follows

$$H(\xi, s) = \exp\{\Theta(t) - \Theta(s)\} \exp\left(\xi^\mu \frac{\partial}{\partial u^\mu}\right) \exp\{\Theta(s) - \Theta(t)\}$$

$$= \exp\left\{\frac{1}{2} \xi^\alpha \Lambda_{\alpha\beta}(s) \left[\xi^\beta + 2u^\beta\right]\right\} ,$$

(3.160)

where

$$\Lambda(s) = \frac{1}{2} \left[M(s) - M(t)\right]$$

$$= \frac{1}{2} \left[\frac{iF}{\sinh(tiF)} \frac{\sinh((t-s)iF)}{\sinh(siF)}\right] .$$

(3.161)

By expanding the exponent in $\xi$ we obtain the Hermite polynomials explicitly. They can be read off from the expression

$$\xi^{\mu_1} \cdots \xi^{\mu_n} H_{\mu_1 \cdots \mu_n}(s) = \sum_{k=0}^{[\frac{n}{2}]} \frac{(2k)!}{2^k k!} \binom{n}{2k} \left(\xi^\alpha \Lambda_{\alpha\beta}(s) \xi^\beta\right)^k \left(\xi^\nu \Lambda_{\nu\sigma}(s) u^\sigma\right)^{n-2k} .$$

(3.162)

For convenience some low-order Hermite polynomials are given explicitly in tensorial form in the next section.
Similarly, the tensors $\Xi_{\nu_1 \cdots \nu_m \mu_1 \cdots \mu_n}(s_1, s_2)$ can be written in the form

$$\Xi_{\nu_1 \cdots \nu_m \mu_1 \cdots \mu_n}(s_1, s_2) = \exp \left[ \Theta(t) - \Theta(s_1) \right]$$

$$\times \frac{\partial}{\partial t^{\nu_1}} \cdots \frac{\partial}{\partial t^{\nu_m}} \exp \left[ \Theta(s_1) - \Theta(s_2) \right] \frac{\partial}{\partial u^{\mu_1}} \cdots \frac{\partial}{\partial u^{\mu_n}} \exp \left[ \Theta(s_2) - \Theta(t) \right].$$

They are obviously polynomial in $u^{\mu}$ as well. We call them Hermite polynomials of second kind. The generating function for these polynomials is defined by

$$\Xi(\xi, \eta, s_1, s_2) = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \xi^{\nu_1} \cdots \xi^{\nu_m} \eta^{\mu_1} \cdots \eta^{\mu_n} \Xi_{\nu_1 \cdots \nu_m \mu_1 \cdots \mu_n}(s_1, s_2),$$

and can be computed as follows

$$\Xi(\xi, \eta, s_1, s_2) = \exp \left[ \Theta(t) - \Theta(s_1) \right] \exp \left( \xi^\rho \Lambda_{\rho \sigma}(s_1) \eta^\sigma \right) \exp \left( \eta^\rho \Lambda_{\rho \sigma}(s_2) \xi^\sigma \right).$$

Notice that

$$\Xi(\xi, \eta, s_1, s_2) = \mathcal{H}(\xi, s_1) \mathcal{H}(\eta, s_2) \exp \left( \xi^\rho \Lambda_{\rho \sigma}(s_2) \eta^\sigma \right).$$

This enables one to express all Hermite polynomials of second kind $\Xi_{(n)}(s_1, s_2)$ in terms of the Hermite polynomials $\mathcal{H}_{(m)}(s_1), \mathcal{H}_{(l)}(s_2),$ and the matrix $\Lambda(s_2)$. Namely, they can be read off from the expression

$$\xi^{\nu_1} \cdots \xi^{\nu_m} \eta^{\mu_1} \cdots \eta^{\mu_n} \Xi_{\nu_1 \cdots \nu_m \mu_1 \cdots \mu_n}(s_1, s_2) = \sum_{k=0}^{\min(m,n)} k! \binom{m}{k} \binom{n}{k} \xi^{\nu_{m-k}} \mathcal{H}_{\nu_1 \cdots \nu_{m-k}}(s_1) \eta^{\mu_{n-k}} \mathcal{H}_{\mu_1 \cdots \mu_{n-k}}(s_2) \left( \xi^\rho \Lambda_{\rho \sigma}(s_2) \eta^\sigma \right)^k.$$
3.5.1 Calculation of Hermite Polynomials

The Hermite polynomials are defined by

\[ H_{\mu_1 \cdots \mu_n} = \exp \left\{ -\frac{1}{2} u^\alpha \Lambda_{\alpha \beta} u^\beta \right\} \frac{\partial}{\partial u^{\mu_1}} \cdots \frac{\partial}{\partial u^{\mu_n}} \exp \left\{ \frac{1}{2} u^\alpha \Lambda_{\alpha \beta} u^\beta \right\} \]

\[ = \left( \frac{\partial}{\partial u^{\mu_1}} + \Lambda_{\mu_1 \nu_1} u^{\nu_1} \right) \cdots \left( \frac{\partial}{\partial u^{\mu_n}} + \Lambda_{\mu_n \nu_n} u^{\nu_n} \right) \cdot 1. \quad (3.168) \]

They can be computed explicitly as follows. First, let

\[ H_n(\xi) = \xi^{\mu_1} \cdots \xi^{\mu_n} H_{\mu_1 \cdots \mu_n}, \quad (3.169) \]

and

\[ B = \xi^{\mu} \frac{\partial}{\partial u^{\mu}}, \quad A = \xi^{\mu} \Lambda_{\mu \nu} u^{\nu}. \quad (3.170) \]

Then

\[ H_n(\xi) = (A + B)^n \cdot 1. \quad (3.171) \]

Finally, let

\[ C = [B, A] = \xi^{\mu} \Lambda_{\mu \nu} \xi^{\nu}. \quad (3.172) \]

Obviously, the operators \( A, B, C \) form the Heisenberg algebra

\[ [B, A] = C, \quad [A, C] = [B, C] = 0. \]

**Lemma 3.** There holds,

\[ (A + B)^n = \sum_{k=0}^{[n/2]} \sum_{m=0}^{n-2k} \frac{(2k)!}{2^k k!} \binom{n}{2k} \binom{n-2k}{m} C^k A^{n-2k-m} B^m. \quad (3.173) \]

**Proof.** Notice that \( e^{t(A+B)} \) is the generating functional for \((A + B)^n\). Now, by using the Baker-Hausdorff-Campbell formula

\[ e^{t(A+B)} = e^{tC} e^{tA} e^{tB}, \]
expanding both sides in $t$ and computing the Taylor coefficients of the right hand side we obtain the eq. (3.173).

By using this result we obtain an explicit expression for (3.171)

$$\mathcal{H}_{(n)}(\xi) = \xi^{\mu_1} \cdots \xi^{\mu_n} \mathcal{H}_{\mu_1 \cdots \mu_n} = \sum_{k=0}^{[n]} \frac{n!}{2^k k! (n-2k)!} C^k A^{n-2k}. \quad (3.174)$$

By setting $A = 0$ we immediately obtain the (diagonal) values of Hermite polynomials at $u = 0$

$$\left[ \mathcal{H}_{\mu_1 \cdots \mu_{2n+1}} \right]_{\text{diag}} = 0, \quad (3.175)$$

$$\left[ \mathcal{H}_{\mu_1 \cdots \mu_{2n}} \right]_{\text{diag}} = \frac{(2n)!}{2^n n!} \Lambda_{\mu_1 \mu_2} \cdots \Lambda_{\mu_{2n-1} \mu_{2n}}. \quad (3.176)$$

We list below a few low order Hermite polynomials needed for our calculation

$$\mathcal{H}_{(0)} = 1, \quad (3.177)$$

$$\mathcal{H}_{\mu_1} = \Lambda_{\mu_1 \alpha} u^\alpha, \quad (3.178)$$

$$\mathcal{H}_{\mu_1 \mu_2} = \Lambda_{\mu_1 \mu_2} + \Lambda_{\mu_1 \alpha} \Lambda_{\mu_2 \beta} u^\alpha u^\beta, \quad (3.179)$$

$$\mathcal{H}_{\mu_1 \mu_2 \mu_3} = 3 \Lambda_{\mu_1 \mu_2} \Lambda_{\mu_3 \alpha} u^\alpha + \Lambda_{\mu_1 \alpha} \Lambda_{\mu_2 \beta} \Lambda_{\mu_3 \gamma} u^\alpha u^\beta u^\gamma, \quad (3.180)$$

$$\mathcal{H}_{\mu_1 \mu_2 \mu_3 \mu_4} = 3 \Lambda_{\mu_1 \mu_2} \Lambda_{\mu_3 \mu_4} + 3 \Lambda_{\mu_1 \mu_2} \Lambda_{\mu_3 \mu_4} \Lambda_{\mu_5 \mu_6} u^\alpha u^\beta u^\gamma u^\delta + \Lambda_{\mu_1 \alpha} \Lambda_{\mu_2 \beta} \Lambda_{\mu_3 \gamma} \Lambda_{\mu_4 \delta} u^\alpha u^\beta u^\gamma u^\delta. \quad (3.181)$$
\begin{align*}
\mathcal{H}_{\mu_1\mu_2\mu_3\mu_4} &= 15\Lambda_{(\mu_1\mu_2\Lambda_{\mu_3\mu_4}\Lambda_{\mu_5})\alpha}u^\alpha + 5\Lambda_{(\mu_1\mu_2\Lambda_{\mu_3\alpha}\Lambda_{\mu_4\beta}\Lambda_{\mu_5\gamma})u^\alpha u^\beta u^\gamma,} \\
&+ \Lambda_{(\mu_1\alpha\Lambda_{\mu_2\beta}\Lambda_{\mu_3\gamma\Lambda_{\mu_4}}u^\alpha)u^\beta u^\gamma u^\delta,} 
\end{align*}
(3.182)

\begin{align*}
\mathcal{H}_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} &= 15\Lambda_{(\mu_1\mu_2\Lambda_{\mu_3\mu_4}\Lambda_{\mu_5\mu_6})} + 45\Lambda_{(\mu_1\mu_2\Lambda_{\mu_3\mu_4}\Lambda_{\mu_5\alpha}\Lambda_{\mu_6\beta})u^\alpha u^\beta} \\
&+ 15\Lambda_{(\mu_1\mu_2\Lambda_{\mu_3\alpha\Lambda_{\mu_4\beta}\Lambda_{\mu_5\mu_6})u^\alpha u^\beta u^\gamma} \\
&+ \Lambda_{(\mu_1\alpha\Lambda_{\mu_2\beta\Lambda_{\mu_3\gamma\Lambda_{\mu_4}}u^\alpha\Lambda_{\mu_5\mu_6})u^\alpha u^\beta u^\gamma u^\delta,} 
\end{align*}
(3.183)

We list below some of the generalized Hermite polynomials of second kind. Now we have two sets of Hermite polynomials that depend on the quadratic forms \( \Lambda \) at two different times, \( s_1 \) and \( s_2 \). Let us define

\begin{align*}
\mathcal{H}_{(\alpha)}(s_1) &= \xi^\mu_1 \cdots \xi^\mu_{\alpha} H_{\mu_1 \cdots \mu_\alpha}(s_1), \\
\mathcal{H}_{(\alpha)}(s_2) &= \eta^\mu_1 \cdots \eta^\mu_{\alpha} H_{\mu_1 \cdots \mu_\alpha}(s_2),
\end{align*}
(3.184)

and

\( \Lambda(s_2) = \xi^\alpha \Lambda_{\alpha\beta}(s_2) \eta^\beta. \) (3.186)

Then from eq. (3.167) we obtain the quantities \( \Xi_{(m,n)} \) that we need in our calculations

\begin{align*}
\Xi_{(0,1)}(s_1, s_2) &= \mathcal{H}_{(1)}(s_2), \\
\Xi_{(1,1)}(s_1, s_2) &= \Lambda(s_2) + \mathcal{H}_{(1)}(s_1) \mathcal{H}_{(1)}(s_2), \\
\Xi_{(2,1)}(s_1, s_2) &= 2\Lambda(s_2) \mathcal{H}_{(1)}(s_1) + \mathcal{H}_{(1)}(s_2) \mathcal{H}_{(2)}(s_1), \\
\Xi_{(3,1)}(s_1, s_2) &= 3\Lambda(s_2) \mathcal{H}_{(2)}(s_1) + \mathcal{H}_{(1)}(s_2) \mathcal{H}_{(3)}(s_1), \\
\Xi_{(4,1)}(s_1, s_2) &= 4\Lambda(s_2) \mathcal{H}_{(3)}(s_1) + \mathcal{H}_{(1)}(s_2) \mathcal{H}_{(4)}(s_1),
\end{align*}
(3.187)
\[ \Xi_{(0,2)}(s_1, s_2) = \mathcal{H}_2(s_2) \] (3.192)

\[ \Xi_{(1,2)}(s_1, s_2) = 2\Lambda_2(s_2)\mathcal{H}_1(s_2) + \mathcal{H}_2(s_2)\mathcal{H}_1(s_1) , \] (3.193)

\[ \Xi_{(2,2)}(s_1, s_2) = 2\Lambda_2^2(s_2) + 4\Lambda_2(s_2)\mathcal{H}_1(s_2)\mathcal{H}_1(s_1) + \mathcal{H}_2(s_2)\mathcal{H}_2(s_1) \] (3.194)

\[ \Xi_{(3,2)}(s_1, s_2) = 6\Lambda_2^2(s_2)\mathcal{H}_1(s_1) + 6\Lambda_2(s_2)\mathcal{H}_1(s_2)\mathcal{H}_2(s_1) 
+ \mathcal{H}_2(s_2)\mathcal{H}_3(s_1) , \] (3.195)

\[ \Xi_{(4,2)}(s_1, s_2) = 12\Lambda_2^2(s_2)\mathcal{H}_2(s_1) + 8\Lambda_2(s_2)\mathcal{H}_1(s_2)\mathcal{H}_3(s_1) 
+ \mathcal{H}_2(s_2)\mathcal{H}_4(s_1) , \] (3.196)

\[ \Xi_{(0,3)}(s_1, s_2) = \mathcal{H}_3(s_2) \] (3.197)

\[ \Xi_{(1,3)}(s_1, s_2) = 3\Lambda_2(s_2)\mathcal{H}_2(s_2) + \mathcal{H}_3(s_2)\mathcal{H}_1(s_1) , \] (3.198)

\[ \Xi_{(2,3)}(s_1, s_2) = 6\Lambda_2^2(s_2)\mathcal{H}_1(s_2) + 6\Lambda_2(s_2)\mathcal{H}_2(s_2)\mathcal{H}_1(s_1) 
+ \mathcal{H}_3(s_2)\mathcal{H}_2(s_1) , \] (3.199)

\[ \Xi_{(3,3)}(s_1, s_2) = 6\Lambda_2^3(s_2) + 18\Lambda_2^2(s_2)\mathcal{H}_1(s_2)\mathcal{H}_1(s_1) 
+ 9\Lambda_2(s_2)\mathcal{H}_2(s_2)\mathcal{H}_2(s_1) + \mathcal{H}_3(s_2)\mathcal{H}_3(s_1) , \] (3.200)

\[ \Xi_{(4,3)}(s_1, s_2) = 24\Lambda_2^3(s_2)\mathcal{H}_1(s_1) + 36\Lambda_2^2(s_2)\mathcal{H}_1(s_2)\mathcal{H}_2(s_1) 
+ 12\Lambda_2(s_2)\mathcal{H}_2(s_2)\mathcal{H}_3(s_1) + \mathcal{H}_3(s_2)\mathcal{H}_4(s_1) , \] (3.201)
\[ \Xi_{(0,4)}(s_1, s_2) = \mathcal{H}_{(4)}(s_2) , \]  
\[ \Xi_{(1,4)}(s_1, s_2) = 4\Lambda(s_2)\mathcal{H}_{(3)}(s_2) + \mathcal{H}_{(4)}(s_2)\mathcal{H}_{(1)}(s_1) , \]  
\[ \Xi_{(2,4)}(s_1, s_2) = 12\Lambda^2(s_2)\mathcal{H}_{(2)}(s_2) + 8\Lambda(s_2)\mathcal{H}_{(3)}(s_2)\mathcal{H}_{(1)}(s_1) + \mathcal{H}_{(4)}(s_2)\mathcal{H}_{(2)}(s_1) , \]  
\[ \Xi_{(3,4)}(s_1, s_2) = 24\Lambda^3(s_2)\mathcal{H}_{(1)}(s_2) + 36\Lambda^2(s_2)\mathcal{H}_{(2)}(s_2)\mathcal{H}_{(1)}(s_1) + 12\Lambda(s_2)\mathcal{H}_{(3)}(s_2)\mathcal{H}_{(2)}(s_1) + \mathcal{H}_{(4)}(s_2)\mathcal{H}_{(4)}(s_1) . \]  

The coincidence limit of the quantities \( \Xi_{(m,n)} \), with \( m + n \) odd, vanishes identically

\[ [\Xi_{(m,n)}(s_1, s_2)]^{\text{diag}} = 0 , \quad \text{if } (m + n) \text{ is odd} . \]  

By recalling the coincidence limits of the Hermite polynomials we obtain the following

\[ [\Xi_{(1,1)}(s_1, s_2)]^{\text{diag}} = \Lambda(s_2) , \]  
\[ [\Xi_{(3,1)}(s_1, s_2)]^{\text{diag}} = 3\Lambda(s_1)\Lambda(s_2) , \]  
\[ [\Xi_{(0,2)}(s_1, s_2)]^{\text{diag}} = \Lambda(s_2) , \]  
\[ [\Xi_{(2,2)}(s_1, s_2)]^{\text{diag}} = \Lambda(s_1)\Lambda(s_2) + 2\Lambda^2(s_2) , \]  
\[ [\Xi_{(4,2)}(s_1, s_2)]^{\text{diag}} = 3\Lambda^2(s_1)\Lambda(s_2) + 12\Lambda(s_1)\Lambda^2(s_2) , \]  
\[ [\Xi_{(1,3)}(s_1, s_2)]^{\text{diag}} = 3\Lambda^2(s_2) , \]  
\[ [\Xi_{(3,3)}(s_1, s_2)]^{\text{diag}} = 9\Lambda(s_1)\Lambda^2(s_2) + 6\Lambda^3(s_2) , \]  
\[ [\Xi_{(2,4)}(s_1, s_2)]^{\text{diag}} = 3\Lambda(s_1)\Lambda^2(s_2) + 12\Lambda^3(s_2) , \]  
\[ [\Xi_{(4,4)}(s_1, s_2)]^{\text{diag}} = 9\Lambda^2(s_1)\Lambda^2(s_2) + 72\Lambda(s_1)\Lambda^3(s_2) + 24\Lambda^4(s_2) . \]
3.6 Off-diagonal Coefficients $b_k$

By using the machinery developed above, we can now write the coefficients of the asymptotic expansion of the heat kernel in terms of generalized Hermite polynomials. We define the following quantity

$$b_{2,(1)}(t|x,x') = \int_0^1 d\tau \left[ N^e_{(2)}(\tau\tau)\mathcal{H}_{e}(\tau\tau) + P^\varphi_{(2)}(\tau\tau)\mathcal{H}_{\varphi}(\tau\tau) + W^{\rho\varphi}_{(2)}(\tau\tau)\mathcal{H}_{\rho\varphi}(\tau\tau) ight. \right. \nonumber
\left. \left. + Q^{\rho\varphi}_{(2)}(\tau\tau)\mathcal{H}_{\rho\varphi}(\tau\tau) \right] . \tag{3.217}$$

Then, by referring to the formulas (3.126), (3.131), (3.137) and (3.149) and by using the following formula for multiple integrals

$$\int_a^b d\tau_n \int_a^{\tau_n} d\tau_{n-1} \cdots \int_a^{\tau_2} d\tau_1 f(\tau_1) = \frac{1}{(n-1)!} \int_a^b \tau^{n-1} f(\tau) , \tag{3.218}$$

we obtain

$$b_2(t|x,x') = \frac{1}{6} R + b_{2,(1)}(t|x,x') , \tag{3.219}$$

$$b_3(t|x,x') = t^{-1/2} \int_0^1 d\tau \left[ N^e_{(3)}(\tau\tau\tau)\mathcal{H}_{e}(\tau\tau\tau) + P^\varphi_{(3)}(\tau\tau\tau)\mathcal{H}_{\varphi}(\tau\tau\tau) + W^{\rho\varphi}_{(3)}(\tau\tau\tau)\mathcal{H}_{\rho\varphi}(\tau\tau\tau) \right. \right. \nonumber
\left. \left. + Q^{\rho\varphi}_{(3)}(\tau\tau\tau)\mathcal{H}_{\rho\varphi}(\tau\tau\tau) + Y^{\mu\rho\varphi}_{(3)}(\tau\tau\tau)\mathcal{H}_{\mu\rho\varphi}(\tau\tau\tau) \right] , \tag{3.220}$$

$$b_4(t|x,x') = \frac{1}{72} R^2 + \frac{1}{6} R b_{2,(1)}(t|x,x') \nonumber
+ t^{-1} \int_0^1 d\tau \left[ P^e_{(4)}(\tau\tau\tau\tau)\mathcal{H}_{e}(\tau\tau\tau\tau) + W^{ee}_{(4)}(\tau\tau\tau\tau)\mathcal{H}_{ee}(\tau\tau\tau\tau) \right. \right. \nonumber
\left. \left. + Q^{ee}_{(4)}(\tau\tau\tau\tau)\mathcal{H}_{ee}(\tau\tau\tau\tau) + Y^{ee\varphi}_{(4)}(\tau\tau\tau\tau)\mathcal{H}_{ee\varphi}(\tau\tau\tau\tau) + S^{ee\varphi\varphi}_{(4)}(\tau\tau\tau\tau)\mathcal{H}_{ee\varphi\varphi}(\tau\tau\tau\tau) \right] \right. \nonumber
\left. + \sum_{k=1}^{4} \sum_{n=0}^{4} \int_0^{\tau_2} d\tau_1 \int_0^{\tau_2} d\tau_1 \sum_{n=0}^{\tau_2} C^{\mu_1\cdots\mu_n\nu_1\cdots\nu_k}_{(n,k)}(\tau_1,\tau_2) \right) . \tag{3.221}$$
3.7 Diagonal Coefficients $b_k$

In order to obtain the diagonal values $b_k^{\text{diag}}(t)$ of the coefficients $b_k(t|x, x')$ we just need to set $u = 0$ in eqs. (3.219), (3.220) and (3.221). For the rest of this section we will employ the usual convention of denoting the coincidence limit by square brackets, that is,

$$[f(u)]^{\text{diag}} = f(0).$$  \hspace{1cm} (3.222)

By inspection of the equation defining the generalized Hermite polynomials, one can easily notice that, in the coincidence limit, all the ones with an odd number of indices vanish identically, namely

$$[\mathcal{H}_{\mu_1...\mu_{2n+1}}]^{\text{diag}} = 0. \hspace{1cm} (3.223)$$

By using the last remark we have the following expression for the coincidence limit of (3.219), i.e.

$$b_2^{\text{diag}}(t) = \frac{1}{6} R + b_2^{\text{diag}}(1), \hspace{1cm} (3.224)$$

where

$$b_2^{\text{diag}}(1) = \int_0^1 d\tau [P_{(2)}(\tau) \mathcal{H}_{\gamma\delta}(\tau) + Q_{(2)}(\tau) \mathcal{H}_{\rho\gamma\delta}(\tau)]^{\text{diag}}. \hspace{1cm} (3.225)$$

By using the explicit form of the coefficients $P_{(2)}, Q_{(2)}$ and the generalized Hermite polynomials, we obtain

$$b_2^{\text{diag}}(1) = J_{(1)}^{\alpha\beta}(t) R^{\alpha\beta}_{\gamma\delta}(t) + J_{(2)}^{\mu\nu}(t) R_{\mu\nu}(t) + J_{(3)}^{\mu\nu}(t) R_{\mu\nu}, \hspace{1cm} (3.226)$$
where

\[
J_{(1)}^{\mu\nu}(t) = \int_0^1 d\tau \left\{ -\frac{1}{6} \Omega^{\alpha\gamma} \Omega^{\delta\chi} M_{\mu\nu} \Lambda_{\gamma\delta} + \frac{1}{4} (\delta_{\nu}^{\gamma} + 3\Psi_{\nu}^{\gamma}) \Omega^{\alpha\rho} \Omega^{\beta\chi} \Psi_{\mu}^{\delta} \Lambda_{\rho\sigma} \Lambda_{\delta\gamma} \right\}, \quad (3.227)
\]

\[
J_{(2)}^{\mu\nu}(t) = \frac{1}{24} \int_0^1 d\tau \left( \delta_{\nu}^{\gamma} + 7\Psi_{\nu}^{\gamma} \right) \Omega^{\alpha\gamma} \Lambda_{\gamma\delta}, \quad (3.228)
\]

\[
J_{(3)}^{\mu\gamma}(t) = \int_0^1 d\tau \Omega^{\mu\gamma} \Psi^{\rho\delta} \Lambda_{\rho\delta}. \quad (3.229)
\]

Here all functions in the integrals depend on \( t\tau \).

Next, we introduce the following matrices

\[
\mathcal{A}(s) = \Omega(s)\Lambda(s) = \frac{1}{2} \frac{\exp[(t - 2s)IF] - \exp(-tIF)}{\sinh(tIF)}, \quad (3.230)
\]

\[
\mathcal{B}(s) = \Omega(s)\Lambda(s)\Omega(s)^T = \frac{\coth(tIF)}{IF} - \frac{\cosh[(t - 2s)IF]}{IF \sinh(tIF)}, \quad (3.231)
\]

\[
\Gamma(s) = \Omega^{-1}(s) - \frac{1}{4} \Psi(s)\Lambda(s) - \frac{3}{4} \Lambda(s)
= \frac{1}{8} \left( 3IF \coth(tIF) + \frac{IF}{\sinh(tIF)} \cosh[(t - 2s)IF] \right). \quad (3.232)
\]

Then, by using the relation

\[
\Omega(s)\Lambda(s)\Psi(s)^T = \Omega^T(s)\Lambda(s) = \mathcal{A}^T(s), \quad (3.233)
\]
we obtain

\[ J_{(1)}^{\alpha\beta\mu\nu}(t) = \int_0^1 d\tau \left\{ -\frac{1}{3} B^{\alpha\beta}(\tau) \Gamma_{\mu\nu}(\tau) \right. \]
\[ \left. + \frac{1}{6} \left( \mathcal{A}_{\mu}^{(a}(\tau) \mathcal{A}_{\nu)}(\tau) + 3 \mathcal{A}_{\mu}^{(a}(\tau) \mathcal{A}_{\nu}(\tau) \right) \right\} , \quad (3.234) \]

\[ J_{(2)}^{\mu\nu}(t) = \frac{1}{3} \int_0^1 d\tau \mathcal{A}^{(\mu\nu)}(\tau) = \frac{1}{6} \delta^{\mu\nu} , \quad (3.235) \]

\[ J_{(3)}^{\mu\nu}(t) = -\int_0^1 d\tau \mathcal{A}^{(\mu\nu)}(\tau) = -\frac{1}{2} \left( \frac{1}{tiF} - \coth(tiF) \right)^{[\mu\nu]} . \quad (3.236) \]

Unfortunately the integral \( J_{(1)}^{\alpha\beta\mu\nu} \) cannot be computed explicitly, in general.

As we already mentioned above all odd order coefficients \( b_{2k+1} \) have zero diagonal values. We see this directly for the coefficient \( b_3 \), which is given by (3.220). That is, by recalling the formulas in (3.132) through (3.136) and the remark (3.223) we have

\[ b_3^{\text{diag}}(t) = 0 . \quad (3.237) \]

Finally, we evaluate the diagonal values of fourth order coefficient \( b_4 \) given by (3.221). It can be written as follows

\[ b_4^{\text{diag}}(t) = \frac{1}{72} R^2 + \frac{1}{6} R^{2,1} \text{Re}^{\text{diag}}(t) + b_4^{\text{diag}}(t) + b_4^{\text{diag}}(t) . \quad (3.238) \]

By noticing that for odd \( n + k \), the diagonal values of the coefficients \( C_{(n,k)} \) vanish,

\[ \left[ C_{(n,k)}^{\mu_1\cdots\mu_n\nu_1\cdots\nu_k} \right]^{\text{diag}} = 0 , \quad (3.239) \]

and by using the explicit form of Hermite polynomials and the generating function
we obtain

\[
\begin{align*}
b_{4,1(2)}^{\text{diag}}(t) &= t^{-1} \int_0^1 d\tau \left\{ P_{(4)}^{\beta\gamma}(t\tau) \Lambda_{\alpha\beta}(t\tau) + 3 \left[ Q_{(4)}^{\kappa\lambda}(t\tau) \right]^{\text{diag}} \Lambda_{(\alpha\kappa)}(t\tau) \Lambda_{(\beta\lambda)}(t\tau) + 15 S_{(4)}^{\text{sym}}(t\tau) \Lambda_{(\alpha\beta\kappa\lambda)}(t\tau) \Lambda_{(\gamma\eta)}(t\tau) \right\} , \\
b_{4,3(2)}^{\text{diag}}(t) &= \int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 \left\{ [P_{(2)}^{\beta\delta}(\tau_1)]^{\text{diag}} f^\rho_\delta(\tau_2) \Lambda^{(1)}_{\alpha\rho} + 2 \left[ P_{(2)}^{\alpha\beta}(\tau_1) \right]^{\text{diag}} g^{(\rho\sigma)}_{\alpha\beta}(\tau_2) \Lambda^{(2)}_{\rho\sigma} + 12 Q^{\text{sym}}_{(2)}(\tau_1) f^\rho_\delta(\tau_2) \Lambda^{(1)}_{\alpha\rho} \Lambda^{(2)}_{\gamma\delta} + \left( 2 \left[ P_{(2)}^{\alpha\beta}(\tau_1) \right]^{\text{diag}} \left[ P_{(2)}^{\alpha\beta}(\tau_2) \right]^{\text{diag}} \right) \Lambda^{(1)}_{\alpha\rho} \Lambda^{(2)}_{\rho\sigma} + 6 \left[ P_{(2)}^{\alpha\beta}(\tau_1) \right]^{\text{diag}} \left[ P_{(2)}^{\rho\sigma}(\tau_2) \right]^{\text{diag}} \Lambda^{(2)}_{\rho\sigma} \Lambda^{(2)}_{\alpha\beta} \right.
\end{align*}
\]

where the superscript on the matrix \( \Lambda \) denotes its dependence on either \( t\tau_1 \) or \( t\tau_2 \).

We see that the scalar curvature appears only in the term \( b_{4,1(2)}^{\text{diag}}(t) \). Now, the term \( b_{4,3(2)}^{\text{diag}}(t) \) only contains derivatives of the curvature and quantities which are quadratic in the curvature with some of their indices contracted. It has the following
form

\[ b_{4,(2)}^{\text{diag}}(t) = \frac{1}{60} B_{\alpha\beta}(t) R_{\mu\nu\lambda}^{\alpha} R_{\nu\lambda}^{\beta} + A_{\mu
u}^{(1)}(t) R_{\mu}^{\lambda} R_{\nu}^{\gamma} + A_{\mu
u}^{(2)}(t) R_{\mu}^{\alpha} R_{\nu}^{\beta} + \frac{1}{60} B_{\alpha\beta}(t) R_{\gamma}^{\alpha} R_{\gamma}^{\beta} + A_{\mu
u}^{(3)}(t) R_{\mu}^{\gamma} R_{\nu}^{\gamma} - \frac{1}{30} B_{\alpha\beta}(t) R_{\nu}^{\alpha} R_{\mu}^{\beta} + A_{\mu
u}^{(4)}(t) R_{\gamma}^{\alpha} R_{\gamma}^{\beta} + \frac{1}{4} B_{\alpha\beta}(t) R_{\gamma}^{\alpha} R_{\gamma}^{\beta} + A_{\mu
u}^{(5)}(t) \nabla_{\gamma}^{\alpha} \nabla_{\mu}^{\beta} R_{\gamma}^{\gamma} + A_{\mu
u}^{(6)}(t) \nabla_{\gamma}^{\alpha} \nabla_{\mu}^{\beta} R_{\gamma}^{\gamma} + \frac{1}{40} B_{\mu
u}(t) \Delta R_{\gamma}^{\gamma} + \frac{3}{40} B_{\mu
u}(t) \nabla_{\gamma}^{\alpha} R_{\mu}^{\beta} + \frac{1}{4} B_{\mu
u}(t) \nabla_{\mu}^{\gamma} R_{\beta}^{\beta} \]
The term \( b_{4,(3)}^{\text{diag}}(t) \) only contains quantities which are quadratic in the curvature with none of their indices contracted. It has the form

\[
b_{4,(3)}^{\text{diag}}(t) = D_{\alpha\beta\mu\nu\gamma\delta\rho\sigma}^{(1)}(t) R_{\alpha\beta\mu\nu} R_{\gamma\delta\rho\sigma} + D_{\mu\nu\rho\sigma}^{(2)}(t) R_{\mu\nu} R_{\rho\sigma} + D_{\mu\nu\beta}^{(3)}(t) R_{\mu\nu} R_{\beta} + D_{\mu\nu\beta}^{(4)}(t) R_{\mu\nu} R_{\gamma\delta}, \tag{3.250}
\]

where \( D_{\mu_1\cdots\mu_n}^{(i)}(t) \) are some tensor-valued functions that depend on \( tF \). They have the form

\[
D_{\mu_1\cdots\mu_n}^{(i)}(t) = \int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 \, d_{\mu_1\cdots\mu_n}^{(i)}(t_1, t_2). \tag{3.251}
\]

To describe our results for the tensors \( d^{(k)} \) we define new tensors

\[
\mathcal{E}_{(r)\mu}^{\nu} = \delta_\mu^{\nu} + p \, \Psi_\mu^{\nu}, \tag{3.252}
\]

\[
S_{\alpha\beta\rho\sigma\nu\kappa} = \mathcal{B}_{\beta\gamma\rho\sigma}, \Phi_{\alpha\kappa}, - \mathcal{A}_{\beta\gamma\rho\sigma}, M_{\alpha\nu} - \frac{3}{4} \Omega_\mu^{(\eta}, \Omega_\nu^{\gamma)} \mathcal{E}_{(1),\rho \sigma}, \Phi_{\alpha\kappa}, \Lambda_{\gamma\epsilon} \Lambda_{\chi\delta} + \frac{3}{2} \Omega_\mu^{(\eta}, \Omega_\nu^{\gamma)} \Psi_\sigma \mathcal{E}_{(3),\rho \sigma}, \Lambda_{\gamma\epsilon} \Lambda_{\chi\delta} \Lambda_{\eta\kappa} \tag{3.253}
\]

\[
\mathcal{V}_{\gamma\delta\rho\sigma\nu\kappa}(t_1, t_2) = \Lambda_{\gamma\delta}(t_1) \left( \mathcal{B}_{\beta\gamma\rho\sigma}, \Phi_{\alpha\kappa}(t_1) \right) + 2 \left( \mathcal{A}_{\beta\gamma\rho\sigma}, \mathcal{A}_{\alpha\kappa}, \Phi_{\gamma\delta}(t_1) \right)(t_2)
- \frac{1}{4} \left( \Lambda_{\gamma\delta}(t_1) \right) \left( \mathcal{B}_{\beta\gamma\rho\sigma}, M_{\gamma\delta} \right)(t_2)
- \Lambda_{\gamma\delta}(t_1) \left( \mathcal{A}_{\beta\gamma\rho\sigma}, \mathcal{A}_{\alpha\kappa}, M_{\gamma\delta} \right)(t_2)
- \frac{3}{4} \left( \Lambda_{\gamma\delta}(t_1) \right) \left( \Lambda_{\gamma\delta}(t_1) \right) \left( \Lambda_{\gamma\delta}(t_1) \right) \left( \Lambda_{\gamma\delta}(t_1) \right)
\times \left( \Omega_\mu^{(\eta}, \Omega_\nu^{\gamma)} \mathcal{E}_{(1),\rho \sigma}, \Phi_{\alpha\kappa}(t_2) \right)
+ \frac{3}{16} \left( \Lambda_{\gamma\delta}(t_1) \right) \left( \Lambda_{\gamma\delta}(t_1) \right) \left( \Lambda_{\gamma\delta}(t_1) \right) \left( \Lambda_{\gamma\delta}(t_1) \right) \left( \Lambda_{\gamma\delta}(t_1) \right)
+ \frac{8}{3} \left( \Lambda_{\gamma\delta}(t_1) \right) \left( \Lambda_{\gamma\delta}(t_1) \right) \left( \Lambda_{\gamma\delta}(t_1) \right) \left( \Omega_\mu^{(\eta}, \Omega_\nu^{\gamma)} \mathcal{E}_{(3),\rho \sigma}, \Lambda_{\chi\delta} \Lambda_{\eta\kappa} \right) \left( \Omega_\mu^{(\eta}, \Omega_\nu^{\gamma)} \mathcal{E}_{(3),\rho \sigma}, \Lambda_{\chi\delta} \Lambda_{\eta\kappa} \right)
\tag{3.254}
\]
Then the tensors \( d^{(k)} \) have the form

\[
d_{αβγδρστ}^{(1)}(τ_1, τ_2) = \frac{-1}{9} (Ω^{ρ}λΩ^κM_{αρ})(τ_1) (B_{βσρ}Φ_{γτ}Φ_{μκ})(τ_2) \\
+ \frac{1}{9} (B_{βσρ}M_{αρ})(τ_1) (B_{δστ}Ω^{-1}_{(γρ)})(τ_2) \\
+ \frac{1}{9} (Ω_{β}^{λ}Ω^κM_{αρ})(τ_1) (A_{βστ}A_{στ}M_{τσ})(τ_2) \\
+ \frac{1}{12} (Ω_{β}^{λ}Ω^κM_{αρ})(τ_1) (Ω_κ^{λ}Ω^κΩ^γE_{(1)γ})(τ_2) \\
− \frac{1}{24} (B_{δστ}M_{τσ})(τ_1) (Λ_σM_{τσ})(τ_2) \\
+ 4Ω^{(λ}Ω^{κ}M_{αρ})(τ_1) (Λ_σΛ_τM_{τσ})(τ_2) (Ω_κ^{λ}Ω^κE_{(3)μ})(τ_2) \\
+ \frac{1}{3} (Ω_{β}^{λ}Ω^κΩ^κE_{(3)μ})(τ_1) V_{γδστμτ}(τ_1, τ_2), \tag{3.255}
\]

\[
d_{μναβτω}(τ_1, τ_2) = \frac{1}{9} (Ω^{λ}Ω^κM_{αρ})(τ_1) (Φ_{μτ}A_{τκ})(τ_2) \\
− \frac{1}{9} (B_{δστ}M_{αρ})(τ_1) (A_{μτ})(τ_2) \\
− \frac{1}{9} (A_{μτ})(τ_1) (B_{δστ}M_{αρ})(τ_2) \\
+ \frac{1}{12} (Ω^{λ}Ω^κΩ_{a}^{η}E_{(3)μ})(τ_1) (A_{μτ}A_{τκ})(τ_2) \\
− \frac{1}{36} (Ω^{λ}Ω^κM_{αρ})(τ_1) (A_{γτ}E_{(γτ)})(τ_2) \\
+ \frac{1}{3} (Ω^{λ}Ω^κΩ_{a}^{η}E_{(3)μ})(τ_1) (A_{μτ}Φ_{μτ})(τ_2) \\
+ \frac{1}{2} (A_{μτ}A_{τκ})(τ_1) + \frac{1}{4} (A_{μτ}A_{τκ})(τ_1) (A_{μστ}E_{(μστ)})(τ_2) \\
+ \frac{1}{36} (Ω^{λ}Ω_{a}^{η}E_{(7)μ})(τ_1) S_{αβτωρστ}(τ_2), \tag{3.256}
\]

\[
d_{μνρστ}^{(3)}(τ_1, τ_2) = \frac{-1}{36} (Ω^{λ}Ω_{a}^{η}E_{(7)μ})(τ_1) (Φ_{ατ}A_{τκ})(τ_2) + \frac{2}{9} (A_{μτ})(τ_1) (A_{αστ})(τ_2) \\
+ \frac{1}{144} Ω_{a}^{λ}Ω_{α}^{η}E_{(7)μ}(τ_1) (A_{βκ}A_{τκ}E_{(7)μ})(τ_2), \tag{3.257}
\]
\[ d_{\mu\nu\alpha\beta}(t_1, t_2) = \frac{1}{3} \left( \Omega_\beta^{(\eta \Psi_\alpha')}, M_{\alpha\eta} \right)(t_1) \left( \Phi_{\mu\nu}, \mathcal{A}_{\nu\mu}(t_2) \right) \] 

\[ \quad - 2\frac{2}{3} \left( \Omega_\nu^{(\eta \Psi_\mu')}(t_1) \left( B_{\beta\sigma}^{(\mu \Phi_{\alpha\nu})}, \Phi_{\rho\kappa}(t_2) \right) + \frac{1}{3} \left( B_{\beta\sigma}^{(\mu M_{\alpha\eta})}(t_1) \mathcal{A}_{\nu\mu}(t_2) \right) \right) \] 

\[ \quad + \frac{1}{3} \mathcal{A}_{\nu\mu}(t_1) \left( B_{\beta\sigma}^{(\mu M_{\alpha\eta})}(t_2) \right) \] 

\[ \quad + \frac{2}{3} \left( \Omega_\nu^{(\eta \Psi_\mu')}(t_1) \left( \mathcal{A}_{\nu\alpha}^{(\mu}, \mathcal{A}_{\nu\beta \kappa}(t_2) \right) \right) \] 

\[ \quad + \frac{2}{3} \left( \Omega_\nu^{(\eta \Psi_\mu')}(t_1) \left( \mathcal{A}_{\nu\alpha}^{(\mu}, \mathcal{A}_{\nu\beta \kappa}(t_2) \right) \right) \] 

\[ \quad + \frac{1}{2} \left( \Omega_\nu^{(\epsilon \Psi_\mu')}(t_1) \left( \mathcal{A}_{\nu\alpha}^{(\mu}, \mathcal{A}_{\nu\beta \kappa}(t_2) \right) \right) \] 

\[ \quad + \frac{1}{2} \left( \Omega_\nu^{(\epsilon \Psi_\mu')}(t_1) \left( \mathcal{A}_{\nu\alpha}^{(\mu}, \mathcal{A}_{\nu\beta \kappa}(t_2) \right) \right) \] 

\[ \quad + \frac{1}{2} \left( \Omega_\nu^{(\epsilon \Psi_\mu')}(t_1) \left( \mathcal{A}_{\nu\alpha}^{(\mu}, \mathcal{A}_{\nu\beta \kappa}(t_2) \right) \right) \] 

\[ \quad - \frac{1}{4} \left( \left( \mathcal{A}_{\nu\alpha}^{(\mu}, \mathcal{A}_{\nu\beta \kappa}(t_2) \right) + \frac{1}{4} \left( \Omega_\nu^{(\epsilon \Psi_\mu')}(t_1) \left( \mathcal{A}_{\nu\alpha}^{(\mu}, \mathcal{A}_{\nu\beta \kappa}(t_2) \right) \right) \right) \times \left( \Omega_\nu^{(\epsilon \Psi_\mu')}(t_1) \left( \mathcal{A}_{\nu\alpha}^{(\mu}, \mathcal{A}_{\nu\beta \kappa}(t_2) \right) \right), \] 

\[ \tag{3.258} \]

\[ d_{\mu\nu\alpha\beta}(t_1, t_2) = \frac{1}{12} \left( \Omega_\beta^{(\epsilon \mathcal{E}_{(7\eta)})}, \Omega_\eta \right)(t_1) \left( \Phi_{\mu\nu\gamma}, \mathcal{A}_{\nu\gamma}(t_2) \right) \] 

\[ \quad + \frac{1}{2} \left( \Omega_\nu^{(\epsilon \Psi_\mu')}(t_1) \left( \Phi_{\alpha\gamma}, \mathcal{A}_{\beta\gamma}(t_2) \right) \right) \] 

\[ \quad - \frac{1}{3} \mathcal{A}_{\alpha\beta\gamma}(t_1) \mathcal{A}_{\mu\nu}(t_2) - \frac{2}{3} \mathcal{A}_{\alpha\beta\gamma}(t_2) \mathcal{A}_{\mu\nu}(t_1) \] 

\[ \quad - \frac{1}{6} \left( \Omega_\beta^{(\epsilon \mathcal{E}_{(7\eta)})}, \Omega_\nu \right)(t_1) \left( \mathcal{A}_{\nu\gamma}, \mathcal{A}_{\nu\gamma}(t_2) \right) \] 

\[ \quad - \frac{1}{6} \left( \Omega_\nu^{(\epsilon \Psi_\mu')}(t_1) \left( \mathcal{A}_{\alpha\beta\gamma}, \Lambda_{\mu\nu}(t_2) \right) \right), \] 

\[ \tag{3.259} \]

\[ d_{\mu\nu\alpha\beta}(t_1, t_2) = -2 \left( \Omega_\nu^{(\epsilon \Psi_\mu')}, \Phi_{\alpha\gamma}, \mathcal{A}_{\beta\gamma}(t_2) \right) + 2 \mathcal{A}_{\nu\alpha}(t_1) \mathcal{A}_{\alpha\beta}(t_2) \] 

\[ \quad + 4 \left( \Omega_\nu^{(\epsilon \Psi_\mu')}(t_1) \left( \mathcal{A}_{\alpha}^{(\gamma}, \mathcal{A}_{\beta\gamma}(t_2) \right) \right) \right) \times \left( \Omega_\nu^{(\epsilon \Psi_\mu')}(t_1) \left( \mathcal{A}_{\alpha}^{(\gamma}, \mathcal{A}_{\beta\gamma}(t_2) \right) \right), \] 

\[ \tag{3.260} \]
3.8 Conclusions

In this chapter we studied the heat kernel expansion for a Laplace operator acting on sections of a complex vector bundle over a smooth compact Riemannian manifold without boundary. We assumed that the curvature $F$ of the $U(1)$ part of the total connection (the electromagnetic field) is covariantly constant and large, so that $tF \sim 1$, that is, $F$ is of order $t^{-1}$. In this situation the standard asymptotic expansion of the heat kernel as $t \to 0$ does not apply since the electromagnetic field cannot be treated as a perturbation.

In order to calculate the heat kernel asymptotic expansion we use an algebraic approach in which the nilpotent algebra of the operators $D_\mu$ plays a major role. In this approach the calculation of the asymptotic expansion of the heat kernel is reduced to the calculation of the asymptotic expansion of the heat semigroup and, then, to the action of differential operators on the zero-order heat kernel. Since the zero-order heat kernel has the Gaussian form the heat kernel asymptotics are expressed in terms of generalized Hermite polynomials.

The main result of this work is establishing the existence of a new non-perturbative asymptotic expansion of the heat kernel and the explicit calculation of the first three coefficients of this expansion (both off-diagonal and the diagonal ones). As far as we know, such an asymptotic expansion and the explicit form of these modified heat kernel coefficients are new.

We presented our result as explicitly as possible. Unfortunately, some of the integrals of the tensor-valued functions cannot be evaluated explicitly in full generality. They can be evaluated, in principle, by using the spectral decomposition
of the two-form $F$,

$$ F = \sum_{k=1}^{[n/2]} B_k E_k, \quad F^2 = - \sum_{k=1}^{[n/2]} B_k^2 \Pi_k, \quad \text{(3.261)} $$

where $B_k$ are the eigenvalues, $E_k$ are the (2-dimensional) eigen-two-forms, and $\Pi_k = -E_k^2$ are the corresponding eigen-projections onto 2-dimensional eigenspaces. Then for any analytic function of $tiF$ we have

$$ f(tiF) = \sum_{k=1}^{[n/2]} f(tB_k) \frac{1}{2} (\Pi_k + iE_k) + \sum_{k=1}^{[n/2]} f(-tB_k) \frac{1}{2} (\Pi_k - iE_k). \quad \text{(3.262)} $$

However, this seems impractical in general case in $n$ dimensions. It would simplify substantially in the following cases: i) there is only one eigenvalue (one magnetic field) in a corresponding two-dimensional subspace, that is, $F = B_1E_1$ (which is essentially 2-dimensional), and ii) all eigenvalues are equal so that $F^2 = -I$ (which is only possible in even dimensions).
CHAPTER 4

LOW-ENERGY EFFECTIVE ACTION IN
NON-PERTURBATIVE
ELECTRODYNAMICS IN CURVED SPACETIME

Abstract

We study the heat kernel for the Laplace type partial differential operator acting on smooth sections of a complex spin-tensor bundle over a generic $n$-dimensional Riemannian manifold. Assuming that the curvature of the $U(1)$ connection (that we call the electromagnetic field) is constant we compute the first two coefficients of the non-perturbative asymptotic expansion of the heat kernel which are of zero and the first order in Riemannian curvature and of arbitrary order in the electromagnetic field. We apply these results to the study of the effective action in non-perturbative electrodynamics in four dimensions and derive a generalization of the Schwinger’s result for the creation of scalar and spinor particles in electromagnetic field induced by the gravitational field. We discover a new infrared divergence in the imaginary part of the effective action due to the gravitational corrections, which seems to be a new physical effect.

$^2$The material in this chapter has been submitted for peer review to Journal of Mathematical Physics: I.G. Avramidi and G. Fucci, Low-Energy Effective Action in Non-Perturbative Electrodynamics in Curved Spacetime, arXiv: 0902.1541 [hep-th]
4.1 Introduction

Schwinger used, in [72], the heat kernel asymptotic expansion technique to evaluate the one-loop effective action in quantum electrodynamics. In particular he solved, exactly, the case of a constant electromagnetic field and derived an heat kernel integral representation for the effective action. He showed that the heat kernel becomes a meromorphic function and a careful evaluation of the integral leads to an imaginary part of the effective action. Schwinger computed the imaginary part of the effective action and showed that it describes the effect of creation of electron-positron pairs by the electric field. This effect is now called the Schwinger mechanism. This is an essentially non-perturbative effect (non-analytic in electric field) that vanishes exponentially for weak electric fields.

Therefore its evaluation requires non-perturbative techniques for the calculation of the heat kernel in the situation when curvatures (but not their derivatives) are large (low energy approximation).

In [27] we computed the heat kernel for the covariant Laplacian with a strong covariantly constant electromagnetic field in an arbitrary gravitational field. We evaluated the first three coefficients of the heat kernel asymptotic expansion in powers of Riemann curvature $R$ but in all orders of the electromagnetic field $F$. In the present chapter we use those results to compute explicitly the terms linear in the Riemann curvature in the non-perturbative heat kernel expansion for the scalar and the spinor fields and compute their contribution to the imaginary part of the effective action. In other words, we generalize the Schwinger mechanism to the case of a strong electromagnetic field in a gravitational field and compute the gravitational corrections to the original Schwinger result.
4.2 Setup of the Problem

Let $M$ be an $n$-dimensional compact Riemannian manifold (with positive-definite metric $g_{\mu\nu}$) without boundary and $S$ be a complex spin-tensor vector bundle over $M$ realizing a representation of the group $\text{Spin}(n) \otimes U(1)$. Let $\varphi$ be a section of the bundle $S$ and $\nabla$ be the total connection on the bundle $S$ (including the spin connection as well as the $U(1)$-connection). Then the commutator of covariant derivatives defines the curvatures

$$[\nabla_\mu, \nabla_\nu] \varphi = (R_{\mu\nu} + iF_{\mu\nu}) \varphi ,$$  \hspace{1cm} (4.1)$$

where $F_{\mu\nu}$ is the curvature of the $U(1)$-connection (which will be also called the electromagnetic field) and $R_{\mu\nu}$ is the curvature of the spin connection defined by

$$R_{\mu\nu} = \frac{1}{2} R^{ab}_{\mu\nu} \Sigma_{ab} ,$$  \hspace{1cm} (4.2)$$

with $\Sigma_{ab}$ being the generators of the spin group $\text{Spin}(n)$ satisfying the commutation relations

$$[\Sigma_{ab}, \Sigma^{cd}] = 4 \delta^{[c}_{[a} \Sigma^{d]}_{b]} .$$  \hspace{1cm} (4.3)$$

Note that for the scalar fields $R_{\mu\nu} = 0$ and for the spinor fields

$$\Sigma_{ab} = \frac{1}{2} \gamma_{ab} ,$$  \hspace{1cm} (4.4)$$

where $\gamma_{ab} = \gamma_{[a} \gamma_{b]}$ (more generally, we define $\gamma_{a_1 \ldots a_m} = \gamma_{[a_1} \ldots \gamma_{a_m]}$) and $\gamma_a$ are the Dirac matrices generating the Clifford algebra

$$\gamma_a \gamma_b + \gamma_b \gamma_a = 2 g_{ab} I .$$  \hspace{1cm} (4.5)$$
4.2.1 Differential Operators

In the present chapter we consider a second-order Laplace type partial differential operator,

\[ L = -\Delta + \xi R + Q, \quad (4.6) \]

where \( \Delta = g^{\mu\nu}\nabla_\mu \nabla_\nu \) is the Laplacian, \( \xi \) is a constant parameter, and \( Q \) is a smooth endomorphism of the bundle \( S \). This operator is elliptic and self-adjoint and has a positive-definite leading symbol. Usually, for scalar fields we set

\[ Q^{\text{scalar}} = 0. \quad (4.7) \]

Moreover, for canonical scalar fields the coupling

\[ \xi^{\text{scalar}} = \begin{cases} 0 & \text{for canonical scalar fields}, \\ \frac{(n-2)}{4(n-1)} & \text{for conformal scalar fields}. \end{cases} \quad (4.8) \]

Another important case is the square of the Dirac operator acting on spinor fields

\[ L = D^2, \quad (4.9) \]

where

\[ D = i\gamma^\mu \nabla_\mu. \quad (4.10) \]

It is easy to see that in this case we have

\[ \xi^{\text{spinor}} = \frac{1}{4}, \quad (4.11) \]

and

\[ Q^{\text{spinor}} = -\frac{1}{2} i F_{\mu\nu} \gamma^{\mu\nu}. \quad (4.12) \]
The object of primary interest in quantum field theory is the (Euclidean) one-loop effective action determined by the functional determinant

$$\Gamma_{(1)} = \mathcal{g} \log \text{Det} (L + m^2),$$

(4.13)

where $\mathcal{g}$ is the fermion number of the field equal to (+1) for boson fields and (−1) for fermion fields, $m$ is a mass parameter, which is assumed to be sufficiently large so that the operator $(L+m^2)$ is positive. Notice that the usual factor $\frac{1}{2}$ is missing because the field is complex, which is equivalent to the contribution of two real fields. Of course, this formal expression is divergent. To rigorously define the determinant of a differential operator one needs to introduce some regularization and then to renormalize it. One of the best ways to do it is via the heat kernel method.

### 4.2.2 Spectral Functions

The determinant of the operator $L + m^2$, considered above, can be defined within the so-called zeta-function regularization as follows. First, one defines the zeta function by

$$\zeta(s) = \mu^{2s} \text{Tr} (L + m^2)^{-s} = \int_M dx \, g^{1/2} Z(s),$$

(4.14)

where

$$Z(s) = \frac{\mu^{2s}}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} e^{-tm^2} \text{tr} \, U^{\text{diag}}(t),$$

(4.15)

$\text{tr}$ denotes the fiber trace over the bundle $\mathcal{S}$ and $\mu$ is a renormalization parameter introduced to preserve dimensions. Therefore, the zeta-regularized one-loop effective action is simply

$$\Gamma_{(1)} = -\mathcal{g} \zeta'(0),$$

(4.16)
and the one-loop effective Lagrangian is given by

$$\mathcal{L} = -\varrho Z'(0).$$  \hspace{1cm} (4.17)

The effective Lagrangian can be also defined simply in the cut-off regularization by

$$\mathcal{L} = -\varrho \int_0^\infty \frac{dt}{\varepsilon} e^{-tm^2} \text{tr} \, U^\text{diag}(t),$$  \hspace{1cm} (4.18)

where $\varepsilon$ is a regularization parameter, which should be set to zero after subtracting the divergent terms. Another regularization is the dimensional regularization, in which one simply defines the effective action by the formal integral

$$\mathcal{L} = -\varrho e^{2\varepsilon} \int_0^\infty \frac{dt}{t} e^{-tm^2} \text{tr} \, U^\text{diag}(t),$$  \hspace{1cm} (4.19)

where the heat trace is formally computed in complex dimension $(n - 2\varepsilon)$ with sufficiently large real part of $\varepsilon$ so that the integral is finite. The renormalized effective action is obtained then by subtracting the simple pole in $\varepsilon$.

For elliptic operators (in the Euclidean setup) the heat trace is a smooth function of $t$; in many cases it is even an analytic function of $t$ in the neighborhood of the positive real axis. However, in the physical case for hyperbolic operators (in the Lorentzian setup) the heat trace can have singularities even on the positive real axis of $t$. As we will show later in the approximation under consideration (for constant electromagnetic field) it becomes a meromorphic function of $t$ with an essential singularity at $t = 0$ and some poles $t_k, k = 1, 2, \ldots$, on the positive real axis. It turns out that the imaginary part of the effective action does not depend on the regularization method and is uniquely defined by the contribution of these poles.
These poles should be avoided from above, which gives

$$\text{Im } \mathcal{L} = -\Im \sum_{k=1}^{\infty} \text{Res} \left\{ t^{-1} e^{-im^2 t} \text{ tr } U^{\text{diag}}(t); t_k \right\} .$$  \hspace{1cm} (4.20)$$

This method was first elaborated and used by Schwinger [72] in quantum electrodynamics to calculate the electron-positron pair production by a constant electric field. One of the goal of our work is to generalize the Schwinger results for the case of constant electromagnetic field in a gravitational field. We will compute the extra contribution to the particle production by a constant electromagnetic field induced by the gravitational field.

### 4.2.3 Heat Kernel Asymptotic Expansion

In the previous Chapter we studied the case of a parallel $U(1)$ curvature (covariantly constant electromagnetic field), i.e. [25]

$$\nabla_{\mu} F_{\alpha\beta} = 0 . \hspace{1cm} (4.21)$$

In the present chapter we will also assume that the potential term $Q$ is covariantly constant

$$\nabla_{\mu} Q = 0 . \hspace{1cm} (4.22)$$

By summing up all powers of $F$ in the asymptotic expansion of the heat kernel diagonal we obtained a new (non-perturbative) asymptotic expansion

$$U^{\text{diag}}(t) \sim (4\pi t)^{-n/2} \exp (-tQ) J(t) \sum_{k=0}^{\infty} t^k b_2(t) , \hspace{1cm} (4.23)$$

where

$$J(t) = \det \left( \frac{tiF}{\sinh(tiF)} \right)^{1/2} , \hspace{1cm} (4.24)$$
and $b_2k(t)$ are the modified heat kernel coefficients which are analytic functions of $t$ at $t = 0$ which depend on $F$ only in the dimensionless combination $tF$. Here and everywhere below all functions of the 2-form $F$ are analytic at 0 and should be understood in terms of a power series in the matrix $F = (F^\mu_\nu)$. Notice the position of indices here, it is important! There is a difference here between Euclidean case and the Lorentzian one since the raising of indices by a Minkowski metric does change the properties of the matrix $F$. Also, here det denotes the determinant with respect to the tangent space indices.

The fiber trace of the heat kernel diagonal has then the asymptotic expansion

$$\text{tr } U\text{diag}(t) \sim (4\pi t)^{-n/2} \Phi(t) \sum_{k=0}^\infty t^k B_{2k}(t),$$

where

$$\Phi(t) = J(t)\text{tr } \exp(-tQ) ,$$

$$B_{2k}(t) = \frac{\text{tr } \exp(-tQ) b_{2k}(t)}{\text{tr } \exp(-tQ)},$$

are new (non-perturbative) heat kernel coefficients of the operator $L$. The integrals

$$\int_M dx g^{1/2} B_{2k}(t)$$

are then the spectral invariants of the operator $L$.

### 4.3 Calculation of the Coefficient $B_2(t)$

In Chapter 3 we obtained, in particular, the first three coefficients of the heat kernel asymptotic expansion, namely, [25]

$$b_0(t) = 1,$$

$$b_1(t) = 0,$$

$$b_2(t) = \{\Sigma_{\mu\alpha} W_{\nu\beta}(t) + V_{\mu\alpha\nu\beta}(t)\} R^{\mu\alpha\nu\beta},$$
where

\[ W(t) = \frac{1}{2} \left( \coth(tF) - \frac{1}{tF} \right), \quad (4.31) \]

\[ V_{\mu\nu}^{\alpha}(t) = \left( \frac{1}{3} - \xi \right) \delta^\mu_\nu \delta^\alpha_\beta + \int_0^1 d\tau \left\{ -\frac{1}{24} \mathcal{O}_{\mu\nu}(\tau) \mathcal{Z}_{\beta}(\tau) + \frac{1}{6} \mathcal{A}[\mu\alpha](\tau) \mathcal{A}[\nu\beta](\tau) \right. \]

\[ -\frac{1}{12} \mathcal{A}[\mu\nu](\tau) \mathcal{A}^{\alpha\beta}(\tau) - \frac{1}{4} \mathcal{A}[\mu\nu](\tau) \mathcal{A}^{\alpha\beta}(\tau) \right\}, \quad (4.32) \]

and

\[ A(\tau) = \frac{1}{2} \frac{\exp[(1 - 2\tau)tF] - \exp(-tF)}{\sinh(tF)} , \quad (4.33) \]

\[ B(\tau) = \frac{\coth(tF)}{tF} - \frac{1}{tF \sinh(tF)} \cosh[(1 - 2\tau)tF] , \quad (4.34) \]

\[ Z(\tau) = 3tF \coth(tF) + \frac{tF}{\sinh(tF)} \cosh[(1 - 2\tau)tF] . \quad (4.35) \]

The trace coefficients are then given by

\[ B_0(t) = 1 , \quad (4.36) \]

\[ B_2(t) = \left\{ \Psi_{\mu\alpha}(t) W_{\nu\beta}(t) + V_{\mu\nu\alpha\beta}(t) \right\} R^{\mu\nu\alpha\beta} , \quad (4.37) \]

where

\[ \Psi(t)_{\mu\alpha} = \frac{\text{tr} \exp(-tQ) \Sigma_{\mu\alpha}}{\text{tr} \exp(-tQ)} . \quad (4.38) \]

4.3.1 Spectral Decomposition

To evaluate it we use the spectral decomposition of the matrix \( F = (F^\mu_\nu) \),

\[ F = \sum_{k=1}^N B_k E_k , \quad (4.39) \]
where $B_k$ are some real invariants and $E_k = (E_k^{\mu\nu})$ are some matrices satisfying the equations

\begin{align}
E_{k\mu\nu} &= -E_{k\nu\mu}, \\
E_k^k E_k^k &= 0,
\end{align}

(4.40)

and for $k \neq m$

\begin{align}
E_k E_m &= 0.
\end{align}

(4.42)

Here, of course, $N \leq [n/2]$. The invariants $B_k$ (that we call “magnetic fields”) should not be confused with the heat trace coefficients $B_0$ and $B_1$.

Next, we define the matrices $\Pi_k = (\Pi_k^{\mu\nu})$ by

\begin{align}
\Pi_k &= -E_k^2.
\end{align}

(4.43)

They satisfy the equations

\begin{align}
\Pi_{k\mu\nu} &= \Pi_{k\nu\mu}, \\
E_k \Pi_k &= \Pi_k E_k = E_k,
\end{align}

(4.44)

(4.45)

and for $k \neq m$

\begin{align}
E_k \Pi_m &= \Pi_m E_k = 0, \quad \Pi_k \Pi_m = 0.
\end{align}

(4.46)

To compute functions of the matrix $F$ we need to know its eigenvalues. We distinguish two different cases.

**Euclidean Case.** In this case the metric has Euclidean signature $(++\cdots +)$ and the non-zero eigenvalues of the matrix $F$ are $\pm iB_1, \ldots, \pm iB_N$, (which are all imaginary).
Of course, it may also have a number of zero eigenvalues. In this case the matrices \( \Pi_k \) are nothing but the projections on 2-dimensional eigenspaces satisfying

\[
\Pi_k^2 = \Pi_k, \quad \Pi_k^{\mu\nu} = 2.
\] (4.47)

In this case we also have

\[
B_k = \frac{1}{2} E_k^{\mu\nu} F_{\mu\nu}.
\] (4.48)

Then we have

\[
(iF)^{2m} = \sum_{k=1}^{N} B_k^{2m} \Pi_k, \quad (m \geq 1)
\] (4.49)

\[
(iF)^{2m+1} = \sum_{k=1}^{N} B_k^{2m+1} iE_k, \quad (m \geq 0),
\] (4.50)

and, therefore, for any analytic function of \( tiF \) at \( t = 0 \) we have

\[
f(tiF) = f(0) + \sum_{k=1}^{N} \left\{ \frac{1}{2} \left[ f(tB_k) + f(-tB_k) - 2f(0) \right] \Pi_k + \frac{1}{2} \left[ f(tB_k) - f(-tB_k) \right] iE_k \right\}.
\] (4.51)

**Pseudo-Euclidean Case.** This is the physically relevant case of pseudo-Euclidean (Lorentzian) metric with the signature \((- + \cdots +)\). Then the non-zero eigenvalues of the matrix \( F \) are \( \pm B_1 \) (which are real) and \( \pm iB_2, \ldots, \pm iB_N \), (which are imaginary). We will call the invariant \( B_1 \), determining the real eigenvalue, the “electric field” and denote it by \( B_1 = E \), and the invariants \( B_k, k = 2, \ldots, N \), determining the imaginary eigenvalues, the “magnetic fields”. So, in general, there is one electric field and \((N - 1)\) magnetic fields. Again, there may be some zero eigenvalues as well.

In this case the matrices \( \Pi_2, \ldots, \Pi_N \) are the orthogonal eigen-projections as before, but the matrix \( \Pi_1 \) is equal to the negative of the corresponding projection, in
particular,
\[ \Pi_1^2 = -\Pi_1, \quad \Pi_1 E_1 = -E_1, \quad \Pi_1^{\mu\nu} = -2. \] (4.52)

Now, we have
\[ (iF)^{2m} = -(iE)^{2m} \Pi_1 + \sum_{k=2}^{N} B_k^{2m} \Pi_k, \quad (m \geq 1) \] (4.53)

\[ (iF)^{2m+1} = (iE)^{2m+1} E_1 + \sum_{k=2}^{N} B_k^{2m+1} iE_k, \quad (m \geq 0). \] (4.54)

Thus, to obtain the results for the pseudo-Euclidean case from the result for the Euclidean case we should just substitute formally
\[ B_1 \mapsto iE, \quad iE_1 \mapsto E_1, \quad \Pi_1 \mapsto -\Pi_1. \] (4.55)

In this way, we obtain for an analytic function of \( itF \),
\[
f(tiF) = f(0)I - \frac{1}{2}[f(itE) + f(-itE) - 2f(0)]\Pi_1 + \frac{1}{2}[f(itE) - f(-itE)]E_1 \\
+ \sum_{k=2}^{N} \left\{ \frac{1}{2}[f(tB_k) + f(-tB_k) - 2f(0)]\Pi_k + \frac{1}{2}[f(tB_k) - f(-tB_k)]iE_k \right\}. \] (4.56)

### 4.3.2 Scalar and Spinor Fields

First of all, we note that for scalar fields
\[ \Phi^{\text{scalar}}(t) = J(t), \quad \Psi^{\text{scalar}}_{\mu\nu}(t) = 0. \] (4.57)

For the spinor fields we have
\[ \Phi^{\text{spinor}}(t) = J(t) \text{ tr } \exp \left( \frac{1}{2} iF_{\mu\nu} \gamma^{\mu\nu} \right), \] (4.58)
\[\psi_{\alpha\beta}(t) = \frac{1}{2} \frac{\text{tr} \gamma_{\alpha\beta} \exp\left(\frac{1}{2} i F_{\mu\nu} \gamma^{\mu\nu}\right)}{\text{tr} \exp\left(\frac{1}{2} i F_{\rho\sigma} \gamma^{\rho\sigma}\right)}.\] (4.59)

Here \(\text{tr}\) denotes the trace with respect to the spinor indices.

We will compute these functions as follows. We define the matrices

\[T_k = \frac{1}{2} i E_k^{\mu\nu} \gamma_{\mu\nu}.\] (4.60)

Then by using the properties of the matrices \(E_k\) and the product of the matrices \(\gamma_{\mu\nu}\)

\[\gamma^{\mu\nu} \gamma_{\alpha\beta} = \gamma^{\mu\nu} \gamma_{\alpha\beta} - 4 \delta_{[\alpha}^{[\mu} \gamma_{\beta]}^{\nu]} - 2 \delta_{[\alpha}^{[\mu} \delta_{\beta]}^{\nu]} I,\] (4.61)

(and some other properties of Dirac matrices in \(n\) dimensions) one can show that these matrices are mutually commuting involutions, that is,

\[T_k^2 = I,\] (4.62)

and

\[[T_k, T_m] = 0.\] (4.63)

Also, the product of two different matrices is (for \(k \neq m\))

\[T_k T_m = -\frac{1}{4} E_k^{\mu\nu} E_m^{\rho\sigma} \gamma_{\mu\nu\rho\sigma}.\] (4.64)

More generally, the product of \(m > 1\) different matrices is

\[T_{k_1} \cdots T_{k_m} = \left(\frac{i}{2}\right)^m E_{k_1}^{\mu_1\mu_2} \cdots E_{k_m}^{\mu_{2m-1}\mu_{2m}} \gamma_{\mu_1 \cdots \mu_{2m}}.\] (4.65)

It is well known that the matrices \(\gamma_{\mu_1 \cdots \mu_k}\) are traceless for any \(k\) and the trace of the product of two matrices \(\gamma_{\mu_1 \cdots \mu_k}\) and \(\gamma_{\nu_1 \cdots \nu_m}\) is non-zero only for \(k = m\). By using these properties we obtain the traces

\[\text{tr} T_k = 0,\] (4.66)
\[ \text{tr} \gamma^{\alpha\beta} T_k = -2^{[n/2]} i E_k^{\alpha\beta}, \quad (4.67) \]

and for \( m > 1 \):

\[ \text{tr} T_{k_1} \cdots T_{k_m} = 0, \quad (4.68) \]

\[ \text{tr} \gamma^{\alpha\beta} T_{k_1} \cdots T_{k_m} = 0, \quad (4.69) \]

when all indices \( k_1, \ldots k_m \) are different.

Now, by using the spectral decomposition of the matrix \( F \) we easily obtain first

\[ J(t) = \prod_{k=1}^{N} \frac{t B_k}{\sinh(t B_k)}, \quad (4.70) \]

and

\[ \text{tr} \exp \left( \frac{1}{2} i t F_{\mu\nu} \gamma^{\mu\nu} \right) = \text{tr} \prod_{k=1}^{N} \exp \left( t T_k B_k \right), \quad (4.71) \]

\[ \text{tr} \gamma^{\alpha\beta} \exp \left( \frac{1}{2} i t F_{\mu\nu} \gamma^{\mu\nu} \right) = \text{tr} \gamma^{\alpha\beta} \prod_{k=1}^{N} \exp \left( t T_k B_k \right). \quad (4.72) \]

By using the properties of the matrices \( T_k \) we get

\[ \exp \left( t T_k B_k \right) = \cosh(t B_k) + T_k \sinh(t B_k). \quad (4.73) \]

Therefore

\[ \text{tr} \exp \left( \frac{1}{2} i t F_{\mu\nu} \gamma^{\mu\nu} \right) = 2^{[n/2]} \prod_{k=1}^{N} \cosh(t B_k), \quad (4.74) \]

and

\[ \text{tr} \gamma^{\alpha\beta} \prod_{k=1}^{N} \exp(t T_k B_k) = \prod_{j=1}^{N} \cosh(t B_j) \sum_{k=1}^{N} \tanh(t B_k) \text{tr} \gamma^{\alpha\beta} T_k \]

\[ = -2^{[n/2]} \prod_{j=1}^{N} \cosh(t B_j) \sum_{k=1}^{N} \tanh(t B_k) i E_k^{\alpha\beta}. \quad (4.75) \]
Thus for the spinor fields

$$\Phi^{\text{spinor}}(t) = 2^{n/2} \prod_{k=1}^{N} tB_k \coth(tB_k), \quad (4.76)$$

and

$$\Psi^{\text{spinor}}_{\alpha\beta}(t) = -\frac{1}{2} \sum_{k=1}^{N} \tanh(tB_k) i E_{\alpha\beta}. \quad (4.77)$$

By the way, this simply means that

$$\Psi^{\text{spinor}}(t) = -\frac{1}{2} \tanh(tiF). \quad (4.78)$$

### 4.3.3 Calculation of the Tensor $V_{\mu\nu\beta}(t)$

Next, we compute the tensor $V_{\mu\nu\beta}(t)$. First, we rewrite in the form

$$V_{\mu\nu\beta}(t) = \left( \frac{1}{3} - \xi \right) \delta^\delta_{v,\alpha} + \int_0^1 d\tau \left\{ -\frac{1}{24} B_{[\mu,\nu]}(\tau) Z^{[\alpha]}(\tau) \ight.$$

$$\left. + \frac{1}{16} \chi_{\mu\nu}(\tau) \chi_{\alpha\beta}(\tau) - \frac{1}{12} Y_{\mu\nu}(\tau) Y_{\alpha\beta}(\tau) \right\}, \quad (4.79)$$

where

$$\chi(\tau) = -\coth(tiF) + \frac{\cosh[(1 - 2\tau tiF)]}{\sinh(tiF)}, \quad (4.80)$$

$$Y(\tau) = 1 + \frac{\sinh[(1 - 2\tau tB_k)]}{\sinh(tB_k)}. \quad (4.81)$$
Next, we parameterize these matrices as follows

\[
\mathcal{B}(\tau) = 2\tau(1 - \tau)\mathbb{I} + \sum_{k=1}^{N} f_{1,k}(\tau)\Pi_{k},
\]

\[
\mathcal{Z}(\tau) = 4\mathbb{I} + \sum_{k=1}^{N} f_{2,k}(\tau)\Pi_{k},
\]

\[
\mathcal{Y}(\tau) = 2(1 - \tau)\mathbb{I} + \sum_{k=1}^{N} f_{3,k}(\tau)\Pi_{k},
\]

\[
\mathcal{X}(\tau) = \sum_{k=1}^{N} f_{4,k}(\tau)iE_{k},
\]

\[
W(t) = \sum_{k=1}^{N} f_{5,k}(t)iE_{k},
\]

where

\[
f_{1,k}(\tau) = \frac{\coth(tB_{k})}{tB_{k}} - \frac{1}{tB_{k}\sinh(tB_{k})}\cosh[(1 - 2\tau)tB_{k}] - 2\tau(1 - \tau),
\]

\[
f_{2,k}(\tau) = 3tB_{k}\coth(tB_{k}) + \frac{tB_{k}}{\sinh(tB_{k})}\cosh[(1 - 2\tau)tB_{k}] - 4,
\]

\[
f_{3,k}(\tau) = \frac{\sinh[(1 - 2\tau)tB_{k}]}{\sinh(tB_{k})} - (1 - 2\tau),
\]

\[
f_{4,k}(\tau) = -\coth(tB_{k}) + \frac{\cosh[(1 - 2\tau)tB_{k}]}{\sinh(tB_{k})},
\]

\[
f_{5,k}(t) = \frac{1}{2} \left( \coth(tB_{k}) - \frac{1}{tB_{k}} \right).
\]

This parametrization is convenient because all functions \(f_{m,k}(\tau)\) are analytic functions of \(t\) at \(t = 0\) and \(f_{m,k}(\tau)\bigg|_{t=0} = 0\).

Then we obtain

\[
V_{\nu}^{\mu\alpha\beta}(t) = \left( \frac{1}{6} - \xi \right)\delta_{\nu}^{(\alpha}\delta_{\beta)} + \sum_{k=1}^{N} \varphi_{k}(t)\Pi_{k}^{(\mu}(\nu)^{\alpha]_{\beta]}}
\]

\[
+ \sum_{k=1}^{N} \sum_{m=1}^{N} \left[ \rho_{km}(t)\Pi_{k}^{(\mu}(\nu)^{\alpha]_{\beta]} - \sigma_{km}(t)E_{k}^{\mu\alpha}E_{m}^{\nu\beta} \right],
\]
where

$$\varphi_k(t) = -\frac{1}{12} \int_0^1 d\tau [2 f_{1,k}(\tau) + \tau (1 - \tau) f_{2,k}(\tau) + 4(1 - \tau) f_{3,k}(\tau)] ,$$  \hspace{1cm} (4.93)

$$\rho_{km}(t) = -\frac{1}{48} \int_0^1 d\tau [f_{1,k}(\tau) f_{2,m}(\tau) + f_{2,k}(\tau) f_{1,m}(\tau) + 4 f_{3,k}(\tau) f_{3,m}(\tau)] ,$$  \hspace{1cm} (4.94)

$$\sigma_{km}(t) = \frac{1}{16} \int_0^1 d\tau f_{4,k}(\tau) f_{4,m}(\tau) .$$  \hspace{1cm} (4.95)

### 4.3.4 Calculation of the Coefficient Functions

The remaining coefficient functions $\varphi_k(t)$, $\rho_{km}(t)$ and $\sigma_{km}(t)$ are analytic functions of $t$ at $t = 0$. Here we give the solution the integrals above which have the following general form

$$A(\alpha, x) = \int_0^1 d\tau \tau^\alpha \cosh[(1 - 2\tau)x] ,$$  \hspace{1cm} (4.96)

and

$$B(\alpha, x) = \int_0^1 d\tau \tau^\alpha \sinh[(1 - 2\tau)x] .$$  \hspace{1cm} (4.97)

After a change of variables, it is not difficult to prove that for $\text{Re}(\alpha) > -1$ we get

$$A(\alpha, x) = \frac{\sinh(x)}{2x} + \frac{\alpha}{(2x)^{\alpha+1}} \left[ e^{x} \gamma(\alpha, 2y) + (-1)^{\alpha+1} e^{-x} \gamma(\alpha, -2y) \right] ,$$  \hspace{1cm} (4.98)

and

$$B(\alpha, x) = -\frac{\cosh(x)}{2x} + \frac{\alpha}{(2x)^{\alpha+1}} \left[ e^{x} \gamma(\alpha, 2y) + (-1)^{\alpha} e^{-x} \gamma(\alpha, -2y) \right] ,$$  \hspace{1cm} (4.99)

where $\gamma(\alpha, x)$ is the lower incomplete gamma function.
Moreover, if the coefficient \( \alpha \) is an integer, as in our case, we can write the formulas above as

\[
A(\alpha, x) = \frac{\sinh(x)}{2x} - \frac{\alpha!}{(2x)^{\alpha+1}} \sum_{k=1}^{\alpha-1} \frac{(2x)^k}{k!} \left[ \frac{e^{-x} + (-1)^{\alpha+k+1}e^x}{2} \right], \quad (4.100)
\]

and

\[
B(\alpha, x) = -\frac{\cosh(x)}{2x} - \frac{\alpha!}{(2x)^{\alpha+1}} \sum_{k=1}^{\alpha-1} \frac{(2x)^k}{k!} \left[ \frac{e^{-x} + (-1)^{\alpha+k}e^x}{2} \right]. \quad (4.101)
\]

For even \( \alpha = 2m \) we have

\[
A(2m, x) = \frac{\sinh(x)}{2x} - \frac{(2m)!}{(2x)^{2m+1}} \left\{ \sinh x \sum_{k=1}^{m} \frac{(2x)^{2k}}{(2k)!} + \cosh x \sum_{k=0}^{m-1} \frac{(2x)^{2k+1}}{(2k+1)!} \right\}, \quad (4.102)
\]

\[
B(2m, x) = -\frac{\cosh(x)}{2x} - \frac{(2m)!}{(2x)^{2m+1}} \left\{ \cosh x \sum_{k=1}^{m} \frac{(2x)^{2k}}{(2k)!} + \sinh x \sum_{k=0}^{m-1} \frac{(2x)^{2k+1}}{(2k+1)!} \right\}. \quad (4.103)
\]

For odd \( \alpha = 2m + 1 \) we have

\[
A(2m+1, x) = \frac{\sinh(x)}{2x} - \frac{(2m+1)!}{(2x)^{2m+2}} \left\{ \cosh x \sum_{k=1}^{m} \frac{(2x)^{2k}}{(2k)!} + \sinh x \sum_{k=0}^{m-1} \frac{(2x)^{2k+1}}{(2k+1)!} \right\}, \quad (4.104)
\]

\[
B(2m+1, x) = -\frac{\cosh(x)}{2x} - \frac{(2m+1)!}{(2x)^{2m+2}} \left\{ \sinh x \sum_{k=1}^{m} \frac{(2x)^{2k}}{(2k)!} + \cosh x \sum_{k=0}^{m-1} \frac{(2x)^{2k+1}}{(2k+1)!} \right\}. \quad (4.105)
\]

From these last formulas we can compute \( \varphi_k(t) \), \( \rho_{km}(t) \) and \( \sigma_{km}(t) \) by using following particular case of the above integrals

\[
\int_0^1 d\tau \cosh[(1 - 2\tau)x] = \frac{\sinh x}{x}, \quad (4.106)
\]

\[
\int_0^1 d\tau \sinh[(1 - 2\tau)x] = 0. \quad (4.107)
\]
By differentiating these integrals with respect to $x$ we obtain all other integrals we need

\[
\int_0^1 d\tau \tau \cosh[(1 - 2\tau)x] = \frac{1}{2} \sinh x, \quad (4.108)
\]

\[
\int_0^1 d\tau \tau^2 \cosh[(1 - 2\tau)x] = \frac{1}{2} \left( \frac{1}{x} + \frac{1}{x^3} \right) \sinh x - \frac{1}{2} \frac{1}{x^2} \cosh x, \quad (4.109)
\]

\[
\int_0^1 d\tau \tau \sinh[(1 - 2\tau)x] = -\frac{1}{2} \frac{1}{x} \cosh x + \frac{1}{2} \frac{1}{x^2} \sinh x. \quad (4.110)
\]

We also have the integrals

\[
\int_0^1 d\tau \cosh[(1 - 2\tau)x] \cosh[(1 - 2\tau)y] = \frac{1}{2} \left\{ \frac{\sinh(x + y)}{x + y} + \frac{\sinh(x - y)}{x - y} \right\}, \quad (4.111)
\]

\[
\int_0^1 d\tau \cosh[(1 - 2\tau)x] \sinh[(1 - 2\tau)y] = 0, \quad (4.112)
\]

\[
\int_0^1 d\tau \sinh[(1 - 2\tau)x] \sinh[(1 - 2\tau)y] = \frac{1}{2} \left\{ \frac{\sinh(x + y)}{x + y} - \frac{\sinh(x - y)}{x - y} \right\}. \quad (4.113)
\]

By using these integrals we obtain

\[
\varphi_k(t) = \frac{1}{6} + \frac{3}{8} \frac{1}{(tB_k)^2} - \frac{1}{24} \coth(tB_k) \left( tB_k + 9 \frac{1}{tB_k} \right), \quad (4.114)
\]

\[
\sigma_{km}(t) = \frac{1}{16} \coth(tB_k) \coth(tB_m) - \frac{1}{16} \frac{\coth(tB_k)}{tB_m} - \frac{1}{16} \frac{\coth(tB_m)}{tB_k} \left( \frac{1}{32} \frac{\coth(tB_m) + \coth(tB_k)}{t(B_k + B_m)} + \frac{1}{32} \frac{\coth(tB_m) - \coth(tB_k)}{t(B_k - B_m)} \right), \quad (4.115)
\]
\[
\rho_{km}(t) = -\frac{1}{48} \left( 4 + 9 \frac{1}{(tB_k)^2} + 9 \frac{1}{(tB_m)^2} - 8 \frac{1}{tB_k} \coth(tB_k) - 8 \frac{1}{tB_m} \coth(tB_m) 
\right. \\
\left. -(tB_k) \coth(tB_k) - (tB_m) \coth(tB_m) - 3 \frac{B_k}{tB_k^2} \coth(tB_k) 
\right.
\]
\[
= -3 \frac{B_m}{tB_m^2} \coth(tB_m) + 3 \left( \frac{B_k}{B_m} + \frac{B_m}{B_k} \right) \coth(tB_m) \coth(tB_k) \\
\left. - \frac{1}{2} \left[ \frac{B_m}{B_k} + \frac{B_k}{B_m} - 4 \right] \frac{\coth(tB_m) + \coth(tB_k)}{t(B_k + B_m)} 
\right.
\]
\[
\left. - \frac{1}{2} \left[ \frac{B_k}{B_m} + \frac{B_m}{B_k} + 4 \right] \frac{\coth(tB_m) - \coth(tB_k)}{t(B_k - B_m)} \right). 
\] (4.116)

### 4.3.5 Trace of the Heat Kernel Diagonal

The trace of the heat kernel diagonal in the general case within the considered approximation is given by

\[
\text{tr} \ U^\text{diag}(t) \sim (4\pi t)^{-n/2} \Phi(t) \{ 1 + tB_2(t) + \cdots \} , 
\] (4.117)

where the function \( \Phi(t) \) was computed above and the coefficient \( B_2 \) is given by

\[
B_2(t) = \left( \frac{1}{6} - \xi \right) R + \sum_{k=1}^{N} \left\{ \Psi^{\mu\alpha}(t) f_{5,k}(t) i E_k^{\nu\beta} R_{\mu\alpha\nu\beta} + \varphi_k(t) \Pi_k^{\mu\nu} R_{\mu\nu} \right\} \\
+ \sum_{k=1}^{N} \sum_{m=1}^{N} \left\{ \rho_{km}(t) \Pi_k^{\mu\nu} \Pi_m^{\alpha\beta} R_{\mu\alpha\nu\beta} - \sigma_{km}(t) E_k^{\mu\alpha} E_m^{\nu\beta} R_{\mu\alpha\nu\beta} \right\} . 
\] (4.118)

Let us specify it for the two cases of interest.
Scalar Fields

For scalar fields we have

\[ \Phi^{\text{scalar}}(t) = \prod_{k=1}^{N} \frac{tB_k}{\sinh(tB_k)}, \quad (4.119) \]

\[ B^{2,\text{scalar}}(t) = \left( \frac{1}{6} - \xi \right)R + \sum_{k=1}^{N} \varphi_k(t) \Pi^\mu{}_{\rho}^\nu R_{\mu\nu} \]

\[ + \sum_{k=1}^{N} \sum_{m=1}^{N} \left\{ \rho_{km}(t) \Pi^\mu{}_{k}^\nu \Pi^\alpha{}_{m}^\beta R_{\mu\alpha\nu\beta} - \sigma_{km}(t) E_{k}^\mu E_{m}^\nu R_{\mu\alpha\nu\beta} \right\}, \quad (4.120) \]

Spinor Fields

For the spinor fields we obtain

\[ \Phi^{\text{spinor}}(t) = 2^{[n/2]} \prod_{k=1}^{N} tB_k \coth(tB_k), \quad (4.121) \]

\[ B^{2,\text{spinor}}(t) = -\frac{1}{12}R + \sum_{k=1}^{N} \varphi_k(t) \Pi^\mu{}_{\rho}^\nu R_{\mu\nu} \]

\[ + \sum_{k=1}^{N} \sum_{m=1}^{N} \left\{ \rho_{km}(t) \Pi^\mu{}_{k}^\nu \Pi^\alpha{}_{m}^\beta R_{\mu\alpha\nu\beta} - \lambda_{km}(t) E_{k}^\mu E_{m}^\nu R_{\mu\alpha\nu\beta} \right\}, \]

where

\[ \lambda_{km}(t) = \sigma_{km}(t) + \frac{1}{8} \frac{\tanh(tB_m)}{B_k} + \frac{1}{8} \frac{\tanh(tB_k)}{B_m} \]

\[ - \frac{1}{8} \frac{\tanh(tB_m) \coth(tB_k)}{B_m} - \frac{1}{8} \frac{\tanh(tB_k) \coth(tB_m)}{B_m}. \quad (4.123) \]
4.3.6 Equal Magnetic Fields

We will specify the obtained result for the case when all magnetic invariants are equal to each other, that is,

\[ B_1 = \cdots = B_N = B. \quad (4.124) \]

Scalar Fields

For scalar fields it takes the form

\[ \Phi^{\text{scalar}}(t) = \left( \frac{tB}{\sinh(tB)} \right)^N, \quad (4.125) \]

\[ B_2^{\text{scalar}}(t) = \left( \frac{1}{6} - \xi \right) R + \varphi(t) H_1^{\mu\nu} R_{\mu\nu} \]

\[ + \rho(t) H_1^{\mu\nu} H_1^{\mu\nu} R_{\mu\nu\rho\sigma} - \sigma(t) X_1^{\mu\nu} X_1^{\mu\nu} R_{\mu\nu\rho\sigma}, \quad (4.126) \]

where

\[ H_1^{\mu\nu} = \sum_{k=1}^{N} \Pi_k^{\mu\nu}, \quad X_1^{\mu\nu} = \sum_{k=1}^{N} E_k^{\mu\nu}, \quad (4.127) \]

\[ \varphi(t) = \frac{1}{6} + \frac{3}{8} \frac{1}{(tB)^2} - \frac{1}{24} tB \coth(tB) - \frac{3}{8} \frac{\coth(tB)}{tB}, \quad (4.128) \]

\[ \sigma(t) = \frac{1}{16} - \frac{3}{32} \frac{\coth(tB)}{tB} + \frac{3}{32} \frac{1}{\sinh^2(tB)}, \quad (4.129) \]

\[ \rho(t) = -\frac{5}{24} - \frac{3}{8} \frac{1}{(tB)^2} + \frac{1}{24} tB \coth(tB) + \frac{7}{16} \frac{\coth(tB)}{tB} - \frac{1}{16} \frac{1}{\sinh^2(tB)}. \quad (4.130) \]
Spinor Fields

For the spinor fields we obtain

\[
\Phi^{\text{spinor}}(t) = 2^{[n/2]} [tB \coth(tB)]^N , \tag{4.131}
\]

\[
B_2^{\text{spinor}}(t) = - \frac{1}{12} R + \varphi(t) H_1^{\mu\nu} R_{\mu\nu}
\]
\[+ \rho(t) H_1^{\mu\nu} H_1^{\alpha\beta} R_{\mu\nu\alpha\beta} - \lambda(t) X_1^{\mu\nu} X_1^{\alpha\beta} R_{\mu\nu\alpha\beta} , \tag{4.132}\]

where

\[
\lambda(t) = - \frac{3}{16} + \frac{1}{32} \frac{1}{\sinh^2(tB)} + \frac{1}{4} \frac{\tanh(tB)}{tB} - \frac{3}{32} \frac{\coth(tB)}{tB} . \tag{4.133}\]

4.3.7 Electric and Magnetic Fields

Now we specify the above results for the pseudo-Euclidean case when there is one electric field and \((N - 1)\) equal magnetic fields. By using the recipe (4.55) we obtain the following results.

Scalar Fields

For scalar fields we have

\[
\Phi^{\text{scalar}}(t) = \frac{tE}{\sin(tE)} \left( \frac{tB}{\sinh(tB)} \right)^{N-1} , \tag{4.134}\]

\[
B_2^{\text{scalar}}(t) = \left( \frac{1}{6} - \xi \right) R - \bar{\varphi}(t) \Pi_1^{\mu\nu} R_{\mu\nu} + \varphi(t) H_2^{\mu\nu} R_{\mu\nu} + \bar{\rho}(t) \Pi_1^{\mu\nu} \Pi_1^{\alpha\beta} R_{\mu\nu\alpha\beta}
\]
\[+ \bar{\sigma}(t) E_1^{\mu\nu} E_1^{\alpha\beta} R_{\mu\nu\alpha\beta} - 2 \rho_1(t) H_2^{\mu\nu} \Pi_1^{\alpha\beta} R_{\mu\nu\alpha\beta} + 2 \sigma_1(t) X_2^{\mu\nu} E_1^{\alpha\beta} R_{\mu\nu\alpha\beta}
\]
\[+ \rho(t) H_2^{\mu\nu} H_2^{\alpha\beta} R_{\mu\nu\alpha\beta} - \sigma(t) X_2^{\mu\nu} X_2^{\alpha\beta} R_{\mu\nu\alpha\beta} . \tag{4.135}\]
where

\[ H_2^{\mu \nu} = \sum_{k=2}^{N} \Pi_k^{\mu \nu}, \quad X_2^{\mu \nu} = \sum_{k=2}^{N} E_k^{\mu \nu}, \quad (4.136) \]

\[ \tilde{\phi}(t) = \frac{1}{6} - \frac{3}{8} \left( \frac{1}{(tE)^2} - \frac{1}{24} tE \cot(tE) + \frac{3}{8} \frac{\cot(tE)}{tE} \right), \quad (4.137) \]

\[ \tilde{\rho}(t) = -\frac{5}{24} + \frac{3}{8} \left( \frac{1}{(tE)^2} + \frac{1}{24} tE \cot(tE) - \frac{7}{16} \frac{\cot(tE)}{tE} + \frac{1}{16} \frac{1}{\sin^2(tE)} \right), \quad (4.138) \]

\[ \tilde{\sigma}(t) = \frac{1}{16} + \frac{3}{32} \frac{\cot(tE)}{tE} - \frac{3}{32} \frac{1}{\sin^2(tE)}, \quad (4.139) \]

\[ \sigma_1(t) = \frac{1}{16} \cot(tE) \coth(tB) - \frac{1}{16} \frac{\cot(tE)}{tB} - \frac{1}{16} \frac{\coth(tB)}{tE} \]

\[ + \frac{1}{16} \frac{B \cot(tE) + E \coth(tB)}{t(B^2 + E^2)}, \quad (4.140) \]

\[ \rho_1(t) = -\frac{1}{48} \left\{ 4 - \frac{9}{(tE)^2} + \frac{9}{(tB)^2} + \frac{8}{tE} \cot(tE) - \frac{8}{tB} \coth(tB) \right\} \]

\[ - (tE) \cot(tE) - (tB) \coth(tB) \]

\[ - 3 \frac{E}{tB^2} \cot(tE) + 3 \frac{B}{tE^2} \coth(tB) + 3 \left( \frac{E}{B} - \frac{B}{E} \right) \coth(tB) \cot(tE) \]

\[ + \frac{5B^2 - E^2}{tB(B^2 + E^2)} \coth(tB) - \frac{5E^2 - B^2}{tE(B^2 + E^2)} \cot(tE) \right\}. \quad (4.141) \]
Spinor Fields

For the spinor fields we obtain

\[ \Phi^{\text{spinor}}(t) = 2^{[n/2]} t E \cot(tE) [tB \coth(tB)]^{N-1}, \quad (4.142) \]

\[ B_2^{\text{spinor}}(t) = -\frac{1}{12} R - \tilde{\phi}(t) \Pi^\mu_1^\alpha R_{\mu\nu} + \varphi(t) H_2^\mu_1^\alpha R_{\mu\nu} + \tilde{\rho}(t) \Pi^\mu_1^\alpha \Pi_1^\beta R_{\mu\alpha\nu\beta} \]

\[ + \tilde{\lambda}(t) E_1^\mu E_1^\alpha R_{\mu\alpha\nu\beta} - 2\tilde{\rho}_1(t) H_2^\mu_1^\alpha \Pi_1^\beta R_{\mu\alpha\nu\beta} + 2\lambda_1(t) X_2^\mu_1^\alpha E_1^\alpha R_{\mu\alpha\nu\beta} \]

\[ + \rho(t) H_2^\mu_2^\alpha R_{\mu\alpha\nu\beta} = \lambda(t) X_2^\mu_2^\alpha X_2^\alpha R_{\mu\alpha\nu\beta}, \quad (4.143) \]

where

\[ \tilde{\lambda}(t) = -\frac{3}{16} \frac{1}{32 \sin^2(tE)} + \frac{1}{4} \frac{\tan(tE)}{tE} + \frac{3}{32} \frac{\cot(tE)}{tE}, \quad (4.144) \]

\[ \lambda_1(t) = \frac{1}{16} \cot(tE) \coth(tB) - \frac{1}{16} \frac{\cot(tE)}{tB} - \frac{1}{16} \frac{\coth(tB)}{tE} \]

\[ + \frac{1}{16} \frac{B \cot(tE) + E \coth(tB)}{t(B^2 + E^2)} + \frac{1}{8} \frac{\tanh(tB)}{tE} - \frac{1}{8} \frac{\tan(tE)}{tB} \]

\[ - \frac{1}{8} \frac{\tanh(tB) \cot(tE)}{tE} + \frac{1}{8} \frac{\tan(tE) \coth(tB)}{tB}. \quad (4.145) \]

4.4 Imaginary Part of the Effective Lagrangian

Now, we can compute the imaginary part of the effective Lagrangian in the same approximation taking into account linear terms in the curvature. The effective action is given by the integral over \( t \) of the trace of the heat kernel diagonal. Of course, it should be properly regularized as discussed above. The most important point we want to make is that in the presence of the electric field the heat kernel is no longer a nice analytic function of \( t \) but it becomes a meromorphic function of \( t \) in the complex plane of \( t \) with poles on the real axis determined by the trigonomet-
ric functions in the coefficient functions computed above. As was pointed out first by Schwinger these poles should be carefully avoided by deforming the contour of integration which leads to an imaginary part of the effective action determined by the contribution of the residues of the poles. This imaginary part is always finite and does not depend on the regularization. We compute below the imaginary part of the effective Lagrangian for the scalar and the spinor fields.

The trace of the heat kernel, $\text{tr} \ U^\text{diag}(t)$, was computed above and is given by (4.117). Now, by using (4.20) the calculation of the imaginary part of the effective Lagrangian is reduced to the calculation of the residues of the functions $t^{-n/2-1}e^{-tm^2}\Phi(t)$ and $t^{-n/2}e^{-tm^2}\Phi(t)B_2(t)$ at the poles on the real line. By using the result (4.134) and (4.142) for the function $\Phi$ it is not difficult to see that the function $t^{-n/2-1}e^{-tm^2}\Phi(t)$ is a meromorphic function with isolated simple poles at $t_k = k\pi/E$ with $k = 1, 2, \ldots$. The function $t^{-n/2}e^{-tm^2}\Phi(t)B_2(t)$ is also a meromorphic function with the same poles but the poles could be double or even triple. The imaginary part is, then, simply evaluated by summing the residues of the integrand at the poles. It has the following form

$$\text{Im} \ L = \pi(4\pi)^{-n/2}E^{n/2}G_0(x, y) + \pi(4\pi)^{-n/2}E^{n/2-1}
\left[ G_1(x, y)R 
+ \ G_2(x, y)\Pi_1^{\mu \nu}R_{\mu \nu}
+ \ G_3(x, y)H_2^{\mu \nu}R_{\mu \nu}
+ \ G_4(x, y)\Pi_1^{\mu \nu}\Pi_1^{\alpha \beta}R_{\mu \nu \alpha \beta} \\
+ \ G_5(x, y)E_1^{\mu \alpha}E_1^{\nu \beta}R_{\mu \nu \alpha \beta}
+ \ G_6(x, y)H_2^{\mu \nu}\Pi_1^{\alpha \beta}R_{\mu \nu \alpha \beta} \\
+ \ G_7(x, y)X_2^{\mu \alpha}X_2^{\nu \beta}R_{\mu \nu \alpha \beta} + \ G_8(x, y)H_2^{\mu \nu}H_2^{\alpha \beta}R_{\mu \nu \alpha \beta} \\
+ \ G_9(x, y)X_2^{\mu \alpha}X_2^{\nu \beta}R_{\mu \nu \alpha \beta} \right],$$

(4.146)

where

$$x = \frac{B}{E}, \quad y = \frac{m^2}{E},$$

(4.147)
and $G_i(x,y)$ are some functions computed below.

### 4.4.1 Scalar Fields

At this point it is useful to introduce some auxiliary functions so that the final result for the quantities $G_i^{\text{scalar}}(x,y)$ can be written in a somewhat compact form, namely

$$f_k(x,y) = \left[ \frac{k\pi x}{\sinh (k\pi x)} \right]^{N-1} \exp (-k\pi y) , \tag{4.148}$$

$$g_k(x,y) = (N-1)(k\pi x) \coth(k\pi x) + k\pi y , \tag{4.149}$$

$$h_k(x,y) = \frac{1}{2} N(N-1)(k\pi x)^2 \coth^2(k\pi x) + \left( \frac{n}{2} - N \right) k\pi y$$

$$+ \frac{1}{2}(N-1)[(n-2N)+2k\pi y](k\pi x) \coth(k\pi x)$$

$$+ \frac{1}{2}(k\pi)^2 [1 - (N-1)x^2 + y^2] , \tag{4.150}$$

$$l_k(x,y) = -k\pi x + \left[ \left( \frac{n}{2} - N \right) + k\pi y \right] \coth(k\pi x) + N(k\pi x) \coth^2(k\pi x) , \tag{4.151}$$

$$\Omega_{1,k}(x,y) = \frac{1}{8} + \frac{n-2N}{48} - \frac{3}{8(k\pi)^2} \left( \frac{n}{2} - N + 2 \right)$$

$$+ \frac{1}{24} \left( 1 - \frac{9}{(k\pi)^2} \right) g_k(x,y) , \tag{4.152}$$

$$\Omega_{2,k}(x,y) = -\frac{1}{6} - \frac{n-2N}{48} + \frac{1}{32(k\pi)^2} \left( \frac{n}{2} - N + 2 \right) \left( \frac{n}{2} - N + 13 \right)$$

$$- \frac{1}{24} \left( 1 - \frac{21}{2(k\pi)^2} \right) g_k(x,y) + \frac{1}{16} \frac{h_k(x,y)}{(k\pi)^2} , \tag{4.153}$$

$$\Omega_{3,k}(x,y) = \frac{1}{16} - \frac{3}{64(k\pi)^2} \left( \frac{n}{2} - N + 1 \right) \left( \frac{n}{2} - N + 2 \right)$$

$$- \frac{3}{32(k\pi)^2} \left[ h_k(x,y) + g_k(x,y) \right] , \tag{4.154}$$
\[ \Omega_{4,k}(x, y) = - \frac{1}{8} - \frac{n - 2N}{48} + \frac{3}{8(k\pi)^2} \left( 1 - \frac{1}{x^2} \right) + \frac{1}{8} \left( \frac{1}{x - x} \right) f_k(x, y) \]
\[ \frac{k\pi}{k\pi} \]
\[ + \frac{1}{8(k\pi)^2} \left( \left( \frac{n}{2} - N + 1 \right) + g_k(x, y) \right) \left[ \frac{3x^4 - 1}{x^2(x^2 + 1)} \right] \]
\[ - \frac{1}{8(k\pi^2)} \left( \frac{x^4 - 3 - (k\pi x)^2}{x^2 + 1} - \frac{1}{24} g_k(x, y) \right), \quad (4.155) \]
\[ \Omega_{5,k}(x, y) = \frac{1}{8(k\pi)^2} \left( \left( \frac{n}{2} - N + 1 \right) + g_k(x, y) \right) \frac{1}{x(x^2 + 1)} - \frac{1}{8} \frac{l_k(x, y)}{(k\pi)} \]
\[ - \frac{1}{8(k\pi x)} \left( \frac{x^3}{x^2 + 1} \right). \quad (4.156) \]

By using these quantities we obtain the functions \( G_1^{\text{scalar}}(x, y) \) in the form of the following series

\[ G_0^{\text{scalar}}(x, y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2}} f_k(x, y), \]  
\[ (4.157) \]
\[ G_1^{\text{scalar}}(x, y) = \left( \frac{1}{6} - \xi \right) \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2-1}} f_k(x, y), \]  
\[ (4.158) \]
\[ G_2^{\text{scalar}}(x, y) = - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2-1}} f_k(x, y) \Omega_{1,k}(x, y), \]  
\[ (4.159) \]
\[ G_3^{\text{scalar}}(x, y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2-1}} f_k(x, y) \phi(k\pi x), \]  
\[ (4.160) \]
\[ G_4^{\text{scalar}}(x, y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2-1}} f_k(x, y) \Omega_{2,k}(x, y), \]  
\[ (4.161) \]
\[ G_5^{\text{scalar}}(x, y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2-1}} f_k(x, y) \Omega_{3,k}(x, y), \]  
\[ (4.162) \]
\[ G_6^{\text{scalar}}(x, y) = - \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{(k\pi)^{n/2-1}} f_k(x, y) \Omega_{4,k}(x, y), \]  
\[ (4.163) \]
Scalar fields:

\[
G^\text{scalar}_7(x, y) = \sum_{k=1}^{\infty} \left( \frac{-1}{k\pi} \right)^{k+1} f_k(x, y) \Omega_{5,k}(x, y),
\]

\[
G^\text{scalar}_8(x, y) = \sum_{k=1}^{\infty} \left( \frac{-1}{k\pi} \right)^{k+1} f_k(x, y) \rho(k\pi x),
\]

\[
G^\text{scalar}_9(x, y) = \sum_{k=1}^{\infty} \left( \frac{-1}{k\pi} \right)^{k+1} f_k(x, y) \sigma(k\pi x).
\]

### 4.4.2 Spinor Fields

Exactly as we did in the previous section, we introduce, now, some auxiliary functions that will be useful in the presentation of the final result, namely

\[
f_{S,k}(x, y) = \left[ (k\pi x) \coth(k\pi x) \right]^{N-1} \exp(-k\pi y),
\]

\[
g_{S,k}(x, y) = (N-1)(k\pi x) \coth(k\pi x) - (N-1)(k\pi x) \tanh(k\pi x) + k\pi y,
\]

\[
h_{S,k}(x, y) = \frac{1}{2} (k\pi y)^2 - (N-1)^2 (k\pi x)^2 + \left( \frac{n}{2} - N \right) k\pi y
\]

\[+ \frac{1}{2} (N-1) (n-2N+2k\pi y) (k\pi x) \left[ \coth(k\pi x) - \tanh(k\pi x) \right]
\]

\[+ \frac{1}{2} N(N-1)(k\pi x)^2 \coth^2(k\pi x) + \frac{1}{2} (N-1)(N-2)(k\pi x)^2 \tanh^2(k\pi x),
\]

\[
l_{S,k}(x, y) = -(N\pi x + \left( \frac{n}{2} - N + k\pi y \right) \coth(k\pi x) + N(k\pi x) \coth^2(k\pi x),
\]

\[
p_{S,k}(x, y) = (N-2)(k\pi x + \left( \frac{n}{2} - N + k\pi y \right) \tanh(k\pi x) - (N-2)(k\pi x) \tanh^2(k\pi x),
\]

\[
\Lambda_{1,k}(x, y) = \frac{1}{8} + \frac{n-2N}{48} - \frac{3}{8(k\pi)^2} \left( \frac{n}{2} - N + 2 \right) + \frac{1}{24} \left( 1 - \frac{9}{(k\pi)^2} \right) g_{S,k}(x, y),
\]

\[
(4.164)
\]

\[
(4.165)
\]

\[
(4.166)
\]

\[
(4.167)
\]

\[
(4.168)
\]

\[
(4.169)
\]

\[
(4.170)
\]

\[
(4.171)
\]

\[
(4.172)
\]
\begin{align*}
\Lambda_{2,k}(x,y) &= -\frac{1}{6} - \frac{n - 2N}{48} + \frac{1}{32(k\pi)^2} \left( \frac{n}{2} - N + 2 \right) \left( \frac{n}{2} - N + 13 \right) \\
&\quad - \frac{1}{24} \left( 1 - \frac{21}{2(k\pi)^2} \right) g_{S,k}(x,y) + \frac{1}{16} \left( \frac{2}{k\pi} \right)^2 h_{S,k}(x,y), \\
\Lambda_{3,k}(x,y) &= -\frac{3}{16} - \frac{3}{64(k\pi)^2} \left( \frac{n}{2} - N + 1 \right) \left( \frac{n}{2} - N + 2 \right) \\
&\quad - \frac{3}{32(k\pi)^2} \left[ g_{S,k}(x,y) + h_{S,k}(x,y) \right],
\tag{4.173}
\end{align*}

\begin{align*}
\Lambda_{4,k}(x,y) &= -\frac{1}{8} - \frac{n - 2N}{48} + \frac{3}{8(k\pi)^2} \left( 1 - \frac{1}{x^2} \right) + \frac{1}{8} \left( \frac{1 - x}{x} \right) \frac{l_{S,k}(x,y)}{k\pi} \\
&\quad + \frac{1}{8(k\pi)^2} \left( \frac{n}{2} - N + 1 + g_{S,k}(x,y) \right) \frac{3x^4 - 1}{x^2(x^2 + 1)} \\
&\quad - \frac{1}{8} \coth(k\pi x) \left[ \frac{x^4 - 3}{x^2 + 1} - \frac{(k\pi x)^2}{3} \right] - \frac{1}{24} g_{S,k}(x,y), \\
\Lambda_{5,k}(x,y) &= \frac{1}{8(k\pi)^2} \left( \frac{n}{2} - N + 1 + g_{S,k}(x,y) \right) \frac{1}{x(x^2 + 1)} - \frac{1}{8} \frac{l_{S,k}(x,y)}{k\pi} \\
&\quad - \frac{1}{8} \coth(k\pi x) \left( \frac{x^3}{x^2 + 1} \right) + \frac{1}{4(k\pi)} \left[ \tanh(k\pi x) + p_{S,k}(x,y) \right].
\tag{4.174}
\end{align*}

By using the above functions we can write the explicit expression for the quantities $G^\text{spinor}_i(x,y)$

\begin{align*}
G^\text{spinor}_0(x,y) &= 2[\frac{1}{2}] \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2}} f_{S,k}(x,y), \\
G^\text{spinor}_1(x,y) &= -\frac{2[\frac{1}{2}]}{12} \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x,y), \\
G^\text{spinor}_2(x,y) &= -2[\frac{1}{2}] \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x,y) \Lambda_{1,k}(x,y),
\tag{4.175}
\end{align*}
\[ G_{3}^{\text{spinor}}(x, y) = 2^\left[\frac{3}{2}\right] \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x, y) \varphi(k\pi x), \quad (4.180) \]

\[ G_{4}^{\text{spinor}}(x, y) = 2^\left[\frac{4}{2}\right] \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x, y) \Lambda_{2,k}(x, y), \quad (4.181) \]

\[ G_{5}^{\text{spinor}}(x, y) = 2^\left[\frac{5}{2}\right] \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x, y) \Lambda_{3,k}(x, y), \quad (4.182) \]

\[ G_{6}^{\text{spinor}}(x, y) = -2^\left[\frac{6}{2}\right] \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x, y) \Lambda_{4,k}(x, y), \quad (4.183) \]

\[ G_{7}^{\text{spinor}}(x, y) = 2^\left[\frac{7}{2}\right] \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x, y) \Lambda_{5,k}(x, y), \quad (4.184) \]

\[ G_{8}^{\text{spinor}}(x, y) = 2^\left[\frac{8}{2}\right] \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x, y) \varphi(k\pi x), \quad (4.185) \]

\[ G_{9}^{\text{spinor}}(x, y) = -2^\left[\frac{9}{2}\right] \sum_{k=1}^{\infty} \frac{1}{(k\pi)^{n/2-1}} f_{S,k}(x, y) \lambda(k\pi x). \quad (4.186) \]

Notice that because of the infrared cutoff factor \( e^{-k\pi y} \) the functions \( G_i(x, y) \) are exponentially small for massive fields in weak electric fields when the parameter is large, \( y \gg 1 \) (that is, \( m^2 \gg E \)), independently on \( x \). In this case, all these functions are approximated by just the first term of the series corresponding to \( k = 1 \).

### 4.5 Strong Electric Field in Four Dimensions

The formulas obtained in the previous section are very general and are valid in any dimensions. In this section we will present some particular cases of major interest.
4.5.1 Four Dimensions

In this section we will consider the physical case when \( n = 4 \). Obviously in four dimensions we only have two invariants, and, therefore, \( N = 2 \). The imaginary part of the effective Lagrangian reads now

\[
\text{Im } \mathcal{L} = \pi(4\pi)^{-2}E^2 G_0(x, y) + \pi(4\pi)^{-2}E \left[ G_1(x, y)R + G_2(x, y)\Pi_1^{\mu\nu}R_{\mu\nu} + G_3(x, y)\Pi_1^{\mu\nu}R_{\mu\nu}\right] + \ldots \]

(4.187)

For scalar fields in four dimensions the functions \( G_i^{\text{scalar}}(x, y) \) take the form

\[
G_0^{\text{scalar}}(x, y) = \frac{x}{\pi} \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{k \sinh(k\pi x)},
\]

(4.188)

\[
G_1^{\text{scalar}}(x, y) = \left( \frac{1}{6} - \xi \right) x \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)},
\]

(4.189)

\[
G_2^{\text{scalar}}(x, y) = -x \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)} \left\{ \frac{1}{8} - \frac{3}{4(4\pi)^2} + \frac{1}{24} \left( k\pi - \frac{9}{k\pi} \right) \left[ y + x \coth(k\pi x) \right] \right\},
\]

(4.190)

\[
G_3^{\text{scalar}}(x, y) = x \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)} \left\{ \frac{1}{6} + \frac{3}{8(k\pi)^2} - \frac{1}{24} k\pi x \coth(k\pi x) - \frac{3}{8} \coth(k\pi x) \right\},
\]

(4.191)
\[ G^{\text{scalar}}_4(x, y) = x \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)} \left\{ -\frac{13}{96} + \frac{13}{16(k\pi)^2} - \frac{x^2}{32} + \frac{y^2}{32} + \left( \frac{7}{16(k\pi)} - \frac{k\pi}{24} \right) y \right\} + \left( \frac{y}{16} + \frac{7}{16k\pi} - \frac{k\pi}{24} \right) x \coth(k\pi x) + \frac{x^2}{16} \coth^2(k\pi x) \right\}, \quad (4.192) \]

\[ G^{\text{scalar}}_5(x, y) = x \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)} \left\{ -\frac{1}{64} - \frac{3}{32(k\pi)^2} + \frac{3x^2}{64} - \frac{3y^2}{64} - \frac{3}{32k\pi} y \right\} - \frac{3}{32} \left( \frac{y}{x} + \frac{1}{k\pi} \right) \coth(k\pi x) - \frac{3}{32} x^2 \coth^2(k\pi x) \right\}, \quad (4.193) \]

\[ G^{\text{scalar}}_6(x, y) = -\frac{x}{x^2 + 1} \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)} \left\{ -\frac{1}{4} - \frac{1}{2(k\pi)^2} + \frac{3x^2}{4(k\pi)^2} + \frac{x^2}{8} (x^2 - 1) \right\} - \frac{k\pi}{24} y(x^2 + 1) - \frac{1}{8k\pi} y \left( \frac{1}{x^2} - 3x^2 \right) - \frac{1}{4} (x^4 - 1) \coth^2(k\pi x) \]
\[ + \left[ \frac{1}{4k\pi} \left( x^3 + \frac{1}{x} \right) + \frac{y}{8} \left( \frac{1}{x} - x^3 \right) \right] \coth(k\pi x) \right\}, \quad (4.194) \]

\[ G^{\text{scalar}}_7(x, y) = \frac{1}{x^2 + 1} \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)} \left\{ \frac{1}{8(k\pi)^2} + \frac{x^2}{8}(x^2 + 1) + \frac{y}{8k\pi} \right\} - \frac{x}{8} \left[ y(1 + x^2) - \frac{1}{k\pi} (1 - x^2) \right] \coth(k\pi x) \]
\[ - \frac{1}{4} x^2 (x^2 + 1) \coth^2(k\pi x) \right\}, \quad (4.195) \]

\[ G^{\text{scalar}}_8(x, y) = x \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)} \left\{ -\frac{7}{48} - \frac{3}{8}(k\pi x)^2 - \frac{1}{16} \coth^2(k\pi x) \right\} + \left( \frac{k\pi}{24} x + \frac{7}{16k\pi x} \right) \coth(k\pi x) \right\}, \quad (4.196) \]

\[ G^{\text{scalar}}_9(x, y) = x \sum_{k=1}^{\infty} \frac{e^{-k\pi y}}{\sinh(k\pi x)} \left\{ -\frac{1}{32} - \frac{3}{32k\pi x} \coth(k\pi x) + \frac{3}{32} \coth^2(k\pi x) \right\}. \quad (4.197) \]
For spinor fields in four dimensions the functions $G_{i}^{\text{spinor}}(x, y)$ take the form

\[
\begin{align*}
G_{0}^{\text{spinor}}(x, y) &= \frac{4x}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} \coth(k\pi x) e^{-k\pi y}, \\
G_{1}^{\text{spinor}}(x, y) &= -\frac{x}{3} \sum_{k=1}^{\infty} \coth(k\pi x) e^{-k\pi y}, \\
G_{2}^{\text{spinor}}(x, y) &= -4x \sum_{k=1}^{\infty} \coth(k\pi x) e^{-k\pi y} \left\{ \frac{1}{8} - \frac{3}{4(k\pi)^2} \right\}, \\
G_{3}^{\text{spinor}}(x, y) &= \frac{\pi}{16} \sum_{k=1}^{\infty} \coth(k\pi x) e^{-k\pi y} \left\{ \frac{3}{8} + \frac{1}{8 (k\pi)^{2}} - \frac{1}{24}k\pi x \coth(k\pi x) + \frac{1}{8} \coth^{2}(k\pi x) - \frac{x}{16} \right\}, \\
G_{4}^{\text{spinor}}(x, y) &= 4x \sum_{k=1}^{\infty} \coth(k\pi x) e^{-k\pi y} \left\{ \frac{1}{6} + \frac{13}{16(k\pi)^{2}} - \frac{x^{2}}{16} + \frac{y^{2}}{32} \right\}, \\
G_{5}^{\text{spinor}}(x, y) &= 4x \sum_{k=1}^{\infty} \coth(k\pi x) e^{-k\pi y} \left\{ -\frac{3}{16} + \frac{3}{32(k\pi)^{2}} + \frac{3x^{2}}{32} \right\}.
\end{align*}
\]
\[ G_{6}^{\text{spinor}}(x, y) = -\frac{4x}{x^2 + 1} \sum_{k=1}^{\infty} \coth(k\pi x)e^{-k\pi y} \left\{ -\frac{3}{8} + \frac{3}{4(k\pi)^2} \left( x^2 - \frac{2}{3x^2} \right) \right\} \]
\[ + \frac{x^2}{8}(2x^2 - 1) - \frac{x^2 y}{24} \left( k\pi - \frac{9}{k\pi} \right) - \frac{y}{24} \left( k\pi + \frac{3}{k\pi x^2} \right) \]
\[ + \left[ \frac{1}{4k\pi x} + \frac{y}{8x} - \frac{x^3}{8} \left( y - \frac{2}{k\pi} \right) \right] \coth(k\pi x) \]
\[ + \left[ \frac{1}{8k\pi x} + \frac{k\pi x}{24} + \frac{x^3}{24} \left( k\pi - \frac{9}{k\pi} \right) \right] \tanh(k\pi x) \]
\[ - \frac{1}{4}(x^4 - 1) \coth^2(k\pi x) \left\}, \right. \tag{4.204} \]

\[ G_{7}^{\text{spinor}}(x, y) = \frac{4}{x^2 + 1} \sum_{k=1}^{\infty} \coth(k\pi x)e^{-k\pi y} \left\{ \frac{1}{8(k\pi)^2} + \frac{x^2}{4} \left( x^2 + 1 \right) \right\} \]
\[ + \frac{y}{8k\pi} - \frac{x^2}{4}(x^2 + 1) \coth^2(k\pi x) \]
\[ + \frac{x}{8} \left[ \frac{1}{k\pi} (1 - x^2) - y(1 + x^2) \right] \coth(k\pi x) \]
\[ + \frac{x}{8} \left[ \frac{1}{k\pi} (1 + 2x^2) + 2y(1 + x^2) \right] \tanh(k\pi x) \left\}, \right. \tag{4.205} \]

\[ G_{8}^{\text{spinor}}(x, y) = 4x \sum_{k=1}^{\infty} \coth(k\pi x)e^{-k\pi y} \left\{ -\frac{7}{48} - \frac{3}{8(k\pi x)^2} + \frac{1}{24}k\pi x \coth(k\pi x) \right\} \]
\[ + \frac{7 \coth(k\pi x)}{16k\pi x} - \frac{1}{16} \coth^2(k\pi x) \left\}, \right. \tag{4.206} \]

\[ G_{9}^{\text{spinor}}(x, y) = -4x \sum_{k=1}^{\infty} \coth(k\pi x)e^{-k\pi y} \left\{ -\frac{9}{32} + \frac{3}{32} \coth^2(k\pi x) + \frac{1}{4} \frac{\tanh(k\pi x)}{k\pi x} \right\} \]
\[ - \frac{3 \coth(k\pi x)}{32k\pi x} \right\}. \tag{4.207} \]
4.5.2 Supercritical Electric Field

As we already mentioned above, the functions $G_i(x, y)$ are exponentially small for massive fields in weak electric fields for large $y = m^2/E$, as $y \to \infty$. Now we are considering the opposite case of light (or massless) fields in strong (supercritical) electric fields, when $y \to 0$ with a fixed $x$. This corresponds to the regime

$$m^2 << B, E. \quad (4.208)$$

Scalar Fields

The infrared (massless) limit for scalar fields is regular—there are no infrared divergences. This is due to the presence of the hyperbolic sine $\sinh(k \pi x)$ in the denominator, which gives a cut-off for large $k$ in the series, and therefore, assures its convergence. The result for the massless limit in the scalar case can be simply obtained by setting $y = 0$ in the above formulas for the functions $G_i(x, y)$.

Spinor Fields

The spinor case is quite different. The presence of the hyperbolic cotangent $\coth(k \pi x)$ does not provide a cut-off for the convergence of the series as $k \to \infty$. This leads, in the spinor case in four dimensions, to the presence of infrared divergences as $y = m^2/E \to 0$. By carefully studying the behavior of the series as $k \to \infty$ for a finite $y$ and then letting $y \to 0$ we compute the asymptotic expansion of the functions $G_i^{\text{spinor}}(x, y)$ as $y \to 0$. 
We obtain

\[ G_{0}^{\text{spinor}}(x, y) = \frac{2x}{3} + O(y), \quad (4.209) \]

\[ G_{1}^{\text{spinor}}(x, y) = -\frac{1}{3\pi y} + \frac{x}{8} + O(y), \quad (4.210) \]

\[ G_{2}^{\text{spinor}}(x, y) = -\frac{2}{3\pi y} + \frac{3x}{4} + O(y), \quad (4.211) \]

\[ G_{3}^{\text{spinor}}(x, y) = -\frac{1}{6\pi y^2} + \frac{2}{3\pi y} + \frac{3}{2\pi} \log(\pi y) + \frac{x^2(\pi x - 24) + 18}{72x} + O(y), \quad (4.212) \]

\[ G_{4}^{\text{spinor}}(x, y) = -\frac{5}{6\pi y} + \frac{7x}{8} + O(y), \quad (4.213) \]

\[ G_{5}^{\text{spinor}}(x, y) = -\frac{3}{4\pi y} + \frac{5x}{16} + O(y), \quad (4.214) \]

\[ G_{6}^{\text{spinor}}(x, y) = \frac{x^2 - 1}{6\pi(x^2 + 1)^2} + \frac{2}{3\pi y} - \frac{x^4 - 3}{2\pi(x^2 + 1)} \log(\pi y) \]

\[ -\frac{2\pi(x^2 + 1)}{6\pi(x^2 + 1)^2} - \frac{6\pi(9x^4 + 3x^2 - 4) - 36\pi(x^4 - 1) + \pi^2 x^3(x^2 - 1)}{72\pi x(x^2 + 1)} + O(y), \quad (4.215) \]

\[ G_{7}^{\text{spinor}}(x, y) = -\frac{x(x^2 + 2)}{2\pi(x^2 + 1)} \log(\pi y) + \frac{6\pi(x^2 + 1) + \pi}{12\pi(x^2 + 1)} + O(y), \quad (4.216) \]

\[ G_{8}^{\text{spinor}}(x, y) = \frac{1}{6\pi y^2} - \frac{5}{6\pi y} - \frac{7}{4\pi} \log(\pi y) - \frac{x^2(\pi x - 30) + 18}{72x} + O(y), \quad (4.217) \]

\[ G_{9}^{\text{spinor}}(x, y) = \frac{3}{4\pi y} + \frac{5}{8\pi} \log(\pi y) - \frac{3x}{8} + O(y). \quad (4.218) \]

Thus, we clearly see the infrared divergences of order \( x^2/y^2 = B^2/m^4 \), \( x/y = B/m^2 \) and \( \log y = \log(m^2/E) \).
4.5.3 Pure Electric Field

We analyze now the case of pure electric field without a magnetic field, that is, $B = 0$, which corresponds to the limit $x \to 0$ with fixed $y$. This corresponds to the physical regime when

$$B \ll m^2, E.$$  \hfill (4.219)

In this discussion we present the results in arbitrary dimension first and then we specialize them to the physical dimension $n = 4$.

Scalar Fields

We now evaluate the functions $G_i(x, y)$ for $x = 0$ and a finite $y$. In this limit we are presented with series of the following general form

$$\chi_{n}^{\text{scalar}}(y) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} e^{-ky}}{k^{n/2}}.$$ \hfill (4.220)

This series can be expressed in terms of the polylogarithmic function defined by

$$\text{Li}_j(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^j},$$ \hfill (4.221)

so that, we have

$$\chi_{n}^{\text{scalar}}(y) = -\text{Li}_{\frac{n}{2}}(-e^{-\pi y}).$$ \hfill (4.222)

It is not difficult to notice that the limit as $x \to 0$ of the functions $G_3^{\text{scalar}}$, $G_6^{\text{scalar}}$, $G_7^{\text{scalar}}$, $G_8^{\text{scalar}}$ and $G_9^{\text{scalar}}$ vanish identically, that is,

$$G_3^{\text{scalar}}(0, y) = G_6^{\text{scalar}}(0, y) = G_7^{\text{scalar}}(0, y) = G_8^{\text{scalar}}(0, y) = G_9^{\text{scalar}}(0, y) = 0.$$ \hfill (4.223)

The explicit expression for the remaining non-vanishing $G_i^{\text{scalar}}$ for pure electric field
in $n$ dimensions is

$$G_0^{\text{scalar}}(0, y) = -\pi^{-n/2} \text{Li}_2(-e^{-\pi y}) ,$$ (4.224)

$$G_1^{\text{scalar}}(0, y) = -\left(\frac{1}{6} - \xi\right) \frac{1}{\pi^{n/2-1}} \text{Li}_{n-1}(-e^{-\pi y}) ,$$ (4.225)

$$G_2^{\text{scalar}}(0, y) = -\frac{1}{48\pi^{n/2+1}} \left\{ 2\pi^3 y \text{Li}_{n-2}(-e^{-\pi y}) + (n + 4)\pi^2 \text{Li}_{n-1}(-e^{-\pi y}) 
- 18\pi y \text{Li}_{n-1}(-e^{-\pi y}) - 9(n + 2) \text{Li}_{n+1}(-e^{-\pi y}) \right\} ,$$ (4.226)

$$G_4^{\text{scalar}}(0, y) = \frac{1}{384\pi^{n/2+1}} \left\{ -16\pi^3 y \text{Li}_{n-1}(-e^{-\pi y}) - 4\pi^2 (2n + 9 - 3y^2) \text{Li}_{n-1}(-e^{-\pi y}) 
+ 12(n + 12)\pi y \text{Li}_{n-1}(-e^{-\pi y}) + 3(n + 2)(n + 24) \text{Li}_{n+1}(-e^{-\pi y}) \right\} ,$$ (4.227)

$$G_5^{\text{scalar}}(0, y) = \frac{1}{256\pi^{n/2+1}} \left\{ 4\pi^2 (1 - 3y^2) \text{Li}_{n-1}(-e^{-\pi y}) - 12n\pi y \text{Li}_{n}(-e^{-\pi y}) 
- 3n(n + 2) \text{Li}_{n+1}(-e^{-\pi y}) \right\} .$$ (4.228)

In the physical case of $n = 4$ some of the polylogarithmic functions can be expressed in terms of elementary functions. In this case we have

$$G_0^{\text{scalar}}(0, y) = -\frac{1}{\pi^2} \text{Li}_2(-e^{-\pi y}) ,$$ (4.229)

$$G_1^{\text{scalar}}(0, y) = -\left(\frac{1}{6} - \xi\right) \frac{1}{\pi} \ln(1 + e^{-\pi y}) ,$$ (4.230)

$$G_2^{\text{scalar}}(0, y) = \frac{1}{48\pi^3} \left\{ \frac{2\pi^3 y e^{-\pi y}}{1 + e^{-\pi y}} + 8\pi^2 \ln(1 + e^{-\pi y}) 
+ 18\pi y \text{Li}_2(-e^{-\pi y}) + 54\text{Li}_3(-e^{-\pi y}) \right\} ,$$ (4.231)
\[ G^{\text{scalar}}_4(0, y) = \frac{1}{384\pi^3} \left( \frac{16\pi^3 ye^{-\pi y}}{1 + e^{-\pi y}} + 4\pi^2(17 - 3y^2) \ln(1 + e^{-\pi y}) \right) + 192\pi y \text{Li}_2(-e^{-\pi y}) + 504\pi^3 \left( -e^{-\pi y} \right), \quad (4.232) \]

\[ G^{\text{scalar}}_5(0, y) = -\frac{1}{256\pi^3} \left( 4\pi^2(1 - 3y^2) \ln(1 + e^{-\pi y}) + 48\pi y \text{Li}_2(-e^{-\pi y}) + 72\pi^3 \left( -e^{-\pi y} \right) \right), \quad (4.233) \]

We study now the behavior of these functions as \( y \to 0 \), which corresponds to the limit

\[ B = 0, \quad m^2 << E. \quad (4.234) \]

By taking the limit as \( y \to 0 \) of the expression (4.222) and by noticing that

\[ \text{Li}_n(-1) = -(1 - 2^{1-n})\zeta(n), \quad (4.235) \]

where \( \zeta(x) \) denotes the Riemann zeta function, we obtain

\[ G^{\text{scalar}}_0(0, 0) = \frac{(1 - 2^{1-n/2})\zeta(n)}{\pi^{n/2}}. \quad (4.236) \]

Next, by taking the limit as \( y \to 0 \) and by using the formula (4.235), it is not difficult to obtain

\[ G^{\text{scalar}}_1(0, 0) = -\left( \frac{1}{6} - \zeta \right)\pi^{1-n/2}(1 - 2^{2-n/2})\zeta\left( \frac{n}{2} - 1 \right), \quad (4.237) \]

\[ G^{\text{scalar}}_2(0, 0) = -\frac{1}{48\pi^{n/2+1}} \left( -(n + 4)\pi^2(1 - 2^{2-n/2})\zeta\left( \frac{n}{2} - 1 \right) + 9(n + 2)(1 - 2^{-n/2})\zeta\left( \frac{n}{2} + 1 \right) \right), \quad (4.238) \]
\[ G_4^{\text{scalar}}(0, 0) = \frac{1}{384\pi^{n/2}+1} \left( 4\pi^2(2n+9)(1-2^{-n/2})\zeta\left( \frac{n}{2} - 1 \right) \right. \\
- \left. 3(n+2)(n+24)(1-2^{-n/2})\zeta\left( \frac{n}{2} + 1 \right) \right) , \quad (4.239) \]

\[ G_5^{\text{scalar}}(0, 0) = \frac{1}{256\pi^{n/2}+1} \left( -4\pi^2(1-2^{-n/2})\zeta\left( \frac{n}{2} - 1 \right) \right. \\
+ \left. 3n(n+2)(1-2^{-n/2})\zeta\left( \frac{n}{2} + 1 \right) \right) . \quad (4.240) \]

We consider, at this point, the physical case of four dimensions. By setting \( n = 4 \) in (4.236) we obtain

\[ G_0^{\text{scalar}}(0, 0) = \frac{1}{12} . \quad (4.241) \]

Now, we notice the following relation

\[ (1-2^{-n/2})\zeta\left( \frac{n}{2} - 1 \right) = \eta\left( \frac{n}{2} - 1 \right) , \quad (4.242) \]

where \( \eta(x) \) is the Dirichlet eta function. In the particular case of four dimensions we have that

\[ \lim_{n \to 4}(1-2^{-n/2})\zeta\left( \frac{n}{2} - 1 \right) = \eta(1) = \ln 2 . \quad (4.243) \]

By using the last remark we obtain the values of the functions \( G_i(n, y) \) in four dimensions

\[ G_1^{\text{scalar}}(0, 0) = -\left( \frac{1}{6} - \xi \right) \frac{1}{\pi} \ln 2 , \quad (4.244) \]

\[ G_2^{\text{scalar}}(0, 0) = \frac{1}{6\pi} \ln 2 - \frac{27}{32\pi^3} \zeta(3) , \quad (4.245) \]

\[ G_4^{\text{scalar}}(0, 0) = \frac{17}{96\pi} \ln 2 - \frac{63}{64\pi^3} \zeta(3) , \quad (4.246) \]

\[ G_5^{\text{scalar}}(0, 0) = -\frac{1}{64\pi} \ln 2 + \frac{27}{128\pi^3} \zeta(3) . \quad (4.247) \]
Spinor Fields

For spinor fields the expressions for the non-vanishing $G_i^{\text{spinor}}$ in the limit $x \to 0$ are

\[
G_0^{\text{spinor}}(0, y) = 2^{[n/2]} \pi^{-n/2} \text{Li}_{n/2}(e^{-\pi y}) ,
\]

\[
G_1^{\text{spinor}}(0, y) = -\frac{2^{[n/2]}}{12} \frac{1}{\pi^{n/2-1}} \text{Li}_{-1}^{n/2}(e^{-\pi y}) ,
\]

\[
G_2^{\text{spinor}}(0, y) = -\frac{2^{[n/2]}}{48 \pi^{n/2-1}} \left\{ 2 \pi^3 y \text{Li}_2^{n/2}(e^{-\pi y}) + (n + 4) \pi^2 \text{Li}_{-1}^{n/2}(e^{-\pi y}) \
- 18 \pi y \text{Li}_2^{n/2}(e^{-\pi y}) - 9(n + 2) \text{Li}_{-1}^{n/2}(e^{-\pi y}) \right\} ,
\]

\[
G_4^{\text{spinor}}(0, y) = \frac{2^{[n/2]}}{384 \pi^{n/2+1}} \left\{ -16 \pi^3 y \text{Li}_{-2}^{n/2}(e^{-\pi y}) - 4 \pi^2 (2n + 12 - 3y^2) \text{Li}_{-1}^{n/2}(e^{-\pi y}) \
+ 12(n + 12) \pi y \text{Li}_2^{n/2}(e^{-\pi y}) + 3(n + 2)(n + 24) \text{Li}_{-1}^{n/2}(e^{-\pi y}) \right\} ,
\]

\[
G_5^{\text{spinor}}(0, y) = -2^{[n/2]} \frac{3}{256 \pi^{n/2+1}} \left\{ 4 \pi^2 (4 + y^2) \text{Li}_{-1}^{n/2}(e^{-\pi y}) + 4n \pi y \text{Li}_2^{n/2}(e^{-\pi y}) \
+ n(n + 2) \text{Li}_{-1}^{n/2}(e^{-\pi y}) \right\} .
\]

In the particular case of $n = 4$ the above results read

\[
G_0^{\text{spinor}}(0, y) = \frac{4}{\pi^2} \text{Li}_2(e^{-\pi y}) ,
\]

\[
G_1^{\text{spinor}}(0, y) = \frac{1}{3\pi} \ln(1 - e^{-\pi y}) ,
\]

\[
G_2^{\text{spinor}}(0, y) = -\frac{1}{12 \pi^3} \left\{ \frac{2 \pi^3 y e^{-\pi y}}{1 - e^{-\pi y}} - 8 \pi^2 \ln(1 - e^{-\pi y}) \
- 18 \pi y \text{Li}_2(e^{-\pi y}) - 54 \text{Li}_3(e^{-\pi y}) \right\} ,
\]
\[
G_4^{\text{spinor}}(0, y) = \frac{1}{96\pi^3} \left\{ \frac{16\pi^3 y e^{-\pi y}}{1 - e^{-\pi y}} - 4\pi^2 (20 - 3y^2) \ln(1 - e^{-\pi y})
- 192\pi y \text{Li}_2(e^{-\pi y}) - 504\text{Li}_3(e^{-\pi y}) \right\},
\]
\[
G_5^{\text{spinor}}(0, y) = \frac{3}{16\pi^3} \left\{ \pi^2 (4 + y^2) \ln(1 - e^{-\pi y}) - 4\pi y \text{Li}_2(e^{-\pi y}) - 6\text{Li}_3(e^{-\pi y}) \right\}.
\]

In the case of spinor fields, for \( n > 4 \), there is a well-defined limit as \( y \to 0 \). In fact, by taking the massless limit, \( y \to 0 \), of the expression (4.248) and noticing that

\[
\text{Li}_n(1) = \zeta(n),
\]

we obtain

\[
G_0^{\text{spinor}}(0, 0) = \frac{2^{(n/2)}\pi^{n/2}}{\Gamma(n/2)} \zeta\left(\frac{n}{2}\right).
\]

Analogously, in the limit as \( y \to 0 \) the result for the remaining \( G_i^{\text{spinor}} \) can be written as follows

\[
G_1^{\text{spinor}}(0, 0) = -\frac{2^{[n/2]}\pi^{1-n/2}}{12} \zeta\left(\frac{n}{2} - 1\right),
\]
\[
G_2^{\text{spinor}}(0, 0) = -\frac{2^{[n/2]}\pi^{2}}{48\pi^{n/2+1}} \left\{ (n + 4)\pi^2 \zeta\left(\frac{n}{2} - 1\right) - 9(n + 2)\zeta\left(\frac{n}{2} + 1\right) \right\},
\]
\[
G_4^{\text{spinor}}(0, 0) = \frac{2^{[n/2]}\pi^{2}}{384\pi^{n/2+1}} \left\{ - 4\pi^2 (2n + 12)\zeta\left(\frac{n}{2} - 1\right)
+ 3(n + 2)(n + 24)\zeta\left(\frac{n}{2} + 1\right) \right\},
\]
\[
G_5^{\text{spinor}}(0, 0) = -2^{[n/2]}\pi^{2}\zeta\left(\frac{n}{2} - 1\right) + n(n + 2)\zeta\left(\frac{n}{2} + 1\right).
\]
We turn our attention, now, to the physical case of \( n = 4 \). From the expression in (4.259) we obtain the following result

\[
G_{\text{spinor}}^0(0,0) = \frac{2}{3}. \tag{4.264}
\]

It is evident, from the expressions in (4.254)-(4.257), that the functions \( G_{\text{spinor}}^i(0,y) \) in four dimensions represent a special case since there is an infrared divergence as \( m \to 0 \) (or \( y \to 0 \)). This means that there is no well-defined value for the massless limit \( y \to 0 \). Instead, we find a logarithmic divergence, \( \log(\pi y) \). In order to analyze this case we set \( n = 4 \) from the beginning in the expressions for finite \( y \), and then we examine the asymptotics as \( y \to 0 \). By using the equations (4.254)-(4.257) we obtain

\[
G_{\text{spinor}}^1(0,y) = \frac{1}{3\pi} \log(\pi y) + O(y), \tag{4.265}
\]

\[
G_{\text{spinor}}^2(0,y) = \frac{2}{3\pi} \log(\pi y) - \frac{1}{6\pi} + \frac{9}{2\pi^3} \zeta(3) + O(y), \tag{4.266}
\]

\[
G_{\text{spinor}}^4(0,y) = \frac{5}{6\pi} \log(\pi y) - \frac{1}{6\pi} + \frac{21}{4\pi^3} \zeta(3) + O(y), \tag{4.267}
\]

\[
G_{\text{spinor}}^5(0,y) = \frac{3}{4\pi} \log(\pi y) + \frac{9}{8\pi^3} \zeta(3) + O(y). \tag{4.268}
\]

Notice that, in four dimensions the functions \( G_{\text{spinor}}^i(x,y) \) are singular at the point \( x = y = 0 \). In particular, the limits \( x \to 0 \) and \( y \to 0 \) are not commutative, that is, the limits as \( x \to 0 \) of the eqs. (4.209)-(4.218) (obtained as \( y \to 0 \) for a finite \( x \)) are different from the eqs. (4.265)-(4.268) (obtained as \( y \to 0 \) for \( x = 0 \)).
4.6 Concluding Remarks

In this chapter we have continued the study of the heat kernel and the effective action for complex (scalar and spinor) quantum fields in a strong constant electromagnetic field and a gravitational field initiated in \[27\]. We study here an essentially non-perturbative regime when the electromagnetic field is so strong that one has to take into account all its orders. In this situation the standard asymptotic expansion of the heat kernel does not apply since the electromagnetic field cannot be treated as a perturbation. In \[27\] we established the existence of a new non-perturbative asymptotic expansion of the heat kernel and computed explicitly the first three coefficients of this expansion.

We computed the first two coefficients (of zero and the first order in the Riemann curvature) explicitly in \(n\)-dimensions by using the spectral decomposition of the electromagnetic field tensor. We applied this result for the calculation of the effective action in the physical pseudo-Euclidean (Lorentzian) case and computed explicitly the imaginary part of the effective action both in the general case and in the cases of physical interest. We also computed the asymptotics of the obtained results for supercritical electric fields.

We have discovered a new infrared divergence in the imaginary part of the effective action for massless spinor fields in four dimensions (or supercritical electric field), which is induced purely by the gravitational corrections. This means physically that the creation of massless spinor particles (or massive particles in supercritical electric field) is magnified substantially by the presence of the gravitational field. Further analysis shows that a similar effect occurs for any massless fields (also scalar fields) in the second order in the Riemann curvature. This effect could have
important consequences for theories with spontaneous symmetry breakdown when the mass of charged particles is generated by a Higgs field. Such theories would exhibit a significant amount of created particles (in the massless limit an infinite amount) at the phase transition point when the symmetry is restored and the massive charged particles become massless. That is why this seems to be an interesting new physical effect that deserves further investigation.
CHAPTER 5

NONCOMMUTATIVE EINSTEIN EQUATIONS IN MATRIX GENERAL RELATIVITY

Abstract

We study a non-commutative deformation of General Relativity where the gravitational field is described by a matrix-valued symmetric two-tensor field. The equations of motion are derived in the framework of this new theory by varying a diffeomorphisms and gauge invariant action constructed by using a matrix-valued scalar curvature. Interestingly the genuine non-commutative part of the dynamical equations is described only in terms of a particular tensor density that vanishes identically in the commutative limit. A non-commutative generalization of the energy-momentum tensor for the matter field is studied as well.

5.1 Introduction

The purpose of this chapter is to derive the equations of motion for the field $a^{\mu\nu}$ that generalizes the role played by $g^{\mu\nu}$ in the general theory of relativity. Since this model is a non-commutative extension of Einstein’s General Relativity we will call the corresponding equations of motions non-commutative Einstein’s

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3The material in this chapter has been published in Classical and Quantum Gravity: G. Fucci and I. G. Avramidi, Noncommutative Einstein Equations, Class. Quant. Grav. 25 (2008) 025005
5.2 Variation of the Action

The action functional for Matrix Gravity has been introduced in (2.182). By varying the action functional, (2.182), we can derive the equations of motion for the field \( a^{\mu\nu} \), which is the main goal and the main result of the present chapter. These equations will be matrix-valued and they will constitute a generalization of the ordinary Einstein’s equations that we will call non-commutative Einstein equations. In order to find the dynamics of the model we vary the action (2.182) with respect to the field \( a^{\mu\nu} \) considered as independent variable, namely

\[
 a^{\mu\nu} \rightarrow a^{\mu\nu} + \delta a^{\mu\nu} .
\]

By doing so we obtain, for the variation of the action, the following

\[
 \delta S = S (a^{\mu\nu} + \delta a^{\mu\nu}) - S (a^{\mu\nu}) = \frac{1}{16\pi G} \int_M d^4x \frac{1}{N} \text{Tr} \left[ G_{\mu\nu} \delta a^{\mu\nu} \right] ,
\]

where \( G_{\mu\nu} \) is some matrix valued symmetric tensor density. Then, of course, the desired equations of motion are

\[
 G_{\mu\nu} = 0 .
\]

It is important to notice that the matrix-valued tensor density (5.2) has to coincide with the Einstein tensor in the commutative limit, more precisely we need that, in the commutative limit, the following relation holds

\[
 \frac{1}{N} \text{Tr} \ G_{\mu\nu} = \sqrt{g} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) .
\]

Our main task, then, is to find the explicit form of the equations of motion that result from the variation of the action (2.182). In all the calculations that will
follow the order of the terms is important, unless explicitly stated, due to the matrix nature of them.

First of all, we rewrite the action in a more explicit form which is more suitable for the subsequent variation, namely

$$S_{\text{MGR}}(a) = \frac{1}{16\pi G} \int_M dx \frac{1}{N} \text{Tr} \left[ \rho \frac{1}{2} (\sigma^{\mu\nu} R_{\mu\nu} + R_{\mu\nu} \sigma^{\mu\nu}) \right].$$  \hspace{1cm} (5.4)

By varying the terms in (5.4) with respect to the independent field $\sigma^{\mu\nu}$, and by using the cyclic property of the trace we get

$$\delta S_{\text{MGR}}(a) = \frac{1}{16\pi G} \int_M dx \frac{1}{N} \text{Tr} \left[ \delta \rho R + \frac{1}{2} (R_{\mu\nu}, \rho) \delta \sigma^{\mu\nu} + \frac{1}{2} (\rho, \sigma^{\mu\nu}) \delta R_{\mu\nu} \right],$$  \hspace{1cm} (5.5)

where the curly brackets $\{, \}$ denote anti-commutation, namely $[A, B] = AB + BA$.

From the expressions (2.178) and (2.179), we can evaluate the variation of the matrix-valued Ricci tensor, more precisely we have

$$\delta R_{\mu\nu} = \partial_\nu (\delta \mathcal{A}^{\alpha}_{\mu\alpha}) - \partial_\nu (\delta \mathcal{A}^{\alpha}_{\mu\alpha}) + \delta \mathcal{A}^{\alpha}_{\lambda\alpha} \mathcal{A}^\lambda_{\mu\nu} + \mathcal{A}^\lambda_{\alpha\lambda} \delta \mathcal{A}^{\lambda}_{\mu\nu} +$$

$$- \delta \mathcal{A}^{\alpha}_{\lambda\mu} \mathcal{A}^\lambda_{\mu\nu} - \mathcal{A}^\alpha_{\lambda\nu} \delta \mathcal{A}^{\lambda}_{\mu\alpha}. \hspace{1cm} (5.6)$$

From now on, for simplicity of notation, we set

$$B^{\mu\nu} \equiv \{\rho, \sigma^{\mu\nu}\}. \hspace{1cm} (5.7)$$

By substituting (5.6) in (5.5), and by using the cyclic property of the trace we obtain

$$\delta S_{\text{MGR}}(a) = \frac{1}{16\pi G} \int_M dx \frac{1}{N} \text{Tr} \left[ \delta \rho R + \frac{1}{2} (R_{\mu\nu}, \rho) \delta \sigma^{\mu\nu} + \frac{1}{2} B^{\mu\nu} \partial_\nu (\delta \mathcal{A}^{\alpha}_{\mu\alpha}) +$$

$$- \frac{1}{2} B^{\mu\nu} \partial_\nu (\delta \mathcal{A}^{\alpha}_{\mu\alpha}) + \frac{1}{2} \mathcal{A}^\alpha_{\mu\alpha} B^{\mu\nu} \delta \mathcal{A}^{\alpha}_{\lambda\nu} + \frac{1}{2} B^{\mu\nu} \mathcal{A}^\alpha_{\lambda\nu} \delta \mathcal{A}^{\alpha}_{\mu\nu} +$$

$$- \frac{1}{2} \mathcal{A}^\alpha_{\mu\alpha} B^{\mu\nu} \delta \mathcal{A}^{\alpha}_{\lambda\nu} - \frac{1}{2} B^{\mu\nu} \mathcal{A}^\alpha_{\lambda\mu} \delta \mathcal{A}^{\alpha}_{\mu\alpha} \right]. \hspace{1cm} (5.8)$$
By integrating by parts and by collecting similar terms we get
\[
\delta S_{MGR}(a) = \frac{1}{16\pi G} \int_M dx \frac{1}{N} \text{Tr} \left[ \delta \rho R + \frac{1}{2} \{R_{\mu\nu}, \rho\} \delta a^{\mu\nu} - \frac{1}{2} \left( B_{\mu\nu,\alpha} - B^{\mu\nu, \mathcal{A}_{\alpha\lambda}} + \mathcal{A}_{\mu\nu} B_{\alpha\lambda} + B_{\mu\nu} \mathcal{A}_{\alpha\lambda} \right) \delta a^{\mu\nu} \right].
\]
(5.9)

We can rewrite the last expression in a more compact form, namely
\[
\delta S_{MGR}(a) = \frac{1}{16\pi G} \int_M dx \frac{1}{N} \text{Tr} \left[ \delta \rho R + \frac{1}{2} \{R_{\mu\nu}, \rho\} \delta a^{\mu\nu} - \frac{1}{2} C_{\mu\nu} \delta a^{\mu\nu} + \frac{1}{2} D^\mu \delta \mathcal{A}_{\mu\alpha} \right],
\]
(5.10)

where the matrix-valued tensor densities \(C_{\mu\nu}\) and \(D^\mu\) have the explicit expression
\[
C_{\mu\nu} = \{a^{\mu\nu}, \rho, \mathcal{A}_{\alpha\lambda}\} - \rho \{a^{\mu\nu}, \mathcal{A}_{\alpha\lambda}\} - \rho \{a^{\mu\nu}, \mathcal{A}_{\mu\alpha}\} + \rho \{a^{\mu\nu}, \mathcal{A}_{\nu\beta}\} + \rho \{a^{\mu\nu}, \mathcal{A}_{\alpha\beta}\} - \rho \{a^{\mu\nu}, \mathcal{A}_{\nu\alpha}\} - \rho \{a^{\mu\nu}, \mathcal{A}_{\mu\beta}\} + \rho \{a^{\mu\nu}, \mathcal{A}_{\alpha\beta}\},
\]
(5.11)

and
\[
D^\mu = \{a^{\mu\nu}, \rho, \mathcal{A}_{\nu\alpha}\} - \rho \{a^{\mu\nu}, \mathcal{A}_{\nu\alpha}\} - \rho \{a^{\mu\nu}, \mathcal{A}_{\mu\alpha}\} + \rho \{a^{\mu\nu}, \mathcal{A}_{\mu\nu}\} + \rho \{a^{\mu\nu}, \mathcal{A}_{\nu\nu}\} - \rho \{a^{\mu\nu}, \mathcal{A}_{\mu\nu}\} - \rho \{a^{\mu\nu}, \mathcal{A}_{\nu\mu}\} + \rho \{a^{\mu\nu}, \mathcal{A}_{\mu\nu}\}.
\]
(5.12)

It is worth noticing that in the commutative limit, or, in other words, when all the matrices commute, the tensor densities \(C_{\mu\nu}\) and \(D^\mu\) are identically zero, and the variation of the action \(\delta S_{MGR}\) simply reduces to the standard result of the general theory of relativity.

We can write, now, the variation of the connection coefficients. By using the expression (2.177), and by noticing that
\[
\delta b_{\mu\nu} = -b_{\mu\rho} \{\delta a^{\rho\sigma}, b_{\sigma\nu}\},
\]
we obtain the following
\[
\delta \mathcal{A}_{\mu\nu} = -b_{\lambda\mu} \{\delta a^{\lambda\rho}, \mathcal{A}_{\rho\nu}\} - \mathcal{A}_{\mu\nu} \delta a^{\nu\rho} b_{\rho\mu} + \frac{1}{2} b_{\lambda\rho} \delta a^{\rho\gamma} (\partial_\gamma a^{\alpha\nu}) b_{\rho\mu} + \frac{1}{2} b_{\lambda\rho} \delta a^{\rho\gamma} (\partial_\gamma a^{\alpha\mu}) b_{\rho\mu} + \frac{1}{2} b_{\lambda\rho} \delta a^{\rho\gamma} (\partial_\gamma a^{\alpha\sigma}) b_{\rho\mu} + \frac{1}{2} b_{\lambda\rho} \delta a^{\rho\gamma} (\partial_\gamma a^{\alpha\sigma}) b_{\rho\mu} + \frac{1}{2} b_{\lambda\rho} \delta a^{\rho\gamma} (\partial_\gamma a^{\alpha\sigma}) b_{\rho\mu}.
\]
(5.13)
Once we have the explicit expression for the variation of the connection coefficients, we can evaluate the last two terms that appear in the variation of the action (5.10). We start with the first of the two

\[
- \frac{1}{2} \int_M dx \, \text{Tr} V(C^\mu{}^\nu{}^\alpha \delta \mathcal{A}^\alpha_{\mu\nu}) =
\]

\[
= \frac{1}{2} \int_M dx \, \text{Tr} \left[ \mathcal{A}^{\alpha}{}_{\beta} C^\mu{}^\nu{}^\alpha b_{\mu\lambda} \delta a^{\lambda\beta} + b_{\beta\nu} C^\mu{}^\nu{}^\alpha \mathcal{A}^{\alpha}_{\mu\lambda} \delta a^{\lambda\beta} +
\right.
\]

\[
- \frac{1}{2} (\partial_\gamma a^{\alpha\sigma}) b_{\nu\rho} C^\mu{}^\nu{}^\alpha b_{\mu\sigma} \delta a^{\rho\gamma} + \frac{1}{2} (\partial_\gamma a^{\alpha\sigma}) b_{\rho\nu} C^\mu{}^\nu{}^\alpha b_{\mu\sigma} \delta a^{\sigma\gamma} +
\]

\[
+ \frac{1}{2} (\partial_\gamma a^{\alpha\sigma}) b_{\rho\nu} C^\mu{}^\nu{}^\alpha b_{\mu\sigma} \delta a^{\sigma\gamma} (\partial_\gamma \delta a^{\alpha\sigma}) +
\]

\[
+ \frac{1}{2} b_{\rho\nu} C^\mu{}^\nu{}^\alpha b_{\mu\sigma} \delta a^{\gamma\rho} (\partial_\gamma \delta a^{\alpha\sigma}) + \frac{1}{2} b_{\rho\nu} C^\mu{}^\nu{}^\alpha b_{\mu\sigma} a^{\rho\gamma} (\partial_\gamma \delta a^{\alpha\sigma}) \right],
\]

(5.14)

where in this last expression we used the cyclic property of the trace.

We introduce the following definition, which will be useful in order to simplify the notation,

\[
F_{\mu\lambda\rho} = b_{\rho\nu} C^\mu{}^\nu{}^\alpha b_{\mu\lambda}.
\]

(5.15)

By using the above definition, the expression in (5.14) can be rewritten as follows

\[
- \frac{1}{2} \int_M dx \, \text{Tr} V(C^\mu{}^\nu{}^\alpha \delta \mathcal{A}^\alpha_{\mu\nu}) =
\]

\[
= \frac{1}{2} \int_M dx \, \text{Tr} \left[ \mathcal{A}^{\alpha}{}_{\beta} C^\mu{}^\nu{}^\alpha \mathcal{A}^{\alpha}_{\mu\lambda} \delta a^{\lambda\beta} + F_{\mu\lambda\rho} d^{\alpha\gamma} C^\mu{}^\nu{}^\alpha b_{\mu\sigma} \delta a^{\rho\gamma} +
\right.
\]

\[
- \frac{1}{2} (\partial_\gamma a^{\alpha\sigma}) F_{\rho\sigma\rho} \delta a^{\alpha\gamma} + \frac{1}{2} (\partial_\gamma a^{\alpha\sigma}) F_{\rho\sigma\rho} \delta a^{\sigma\gamma} + \frac{1}{2} (\partial_\gamma a^{\alpha\sigma}) F_{\rho\sigma\rho} \delta a^{\rho\gamma} +
\]

\[
- \frac{1}{2} F_{\rho\sigma\rho} a^{\alpha\gamma} (\partial_\gamma \delta a^{\alpha\sigma}) + \frac{1}{2} F_{\rho\sigma\rho} d^{\alpha\gamma} (\partial_\gamma \delta a^{\alpha\sigma}) + \frac{1}{2} F_{\rho\sigma\rho} a^{\rho\gamma} (\partial_\gamma \delta a^{\alpha\sigma}) \right],
\]

(5.16)

where the first two terms in the last expression has been derived by using the relation

\[
\mathcal{A}^{\alpha}{}_{\beta} C^\mu{}^\nu{}^\alpha b_{\mu\lambda} \delta a^{\lambda\beta} = \mathcal{A}^{\alpha}{}_{\beta} a^{\gamma\rho} b_{\rho\nu} C^\mu{}^\nu{}^\alpha b_{\mu\lambda} \delta a^{\lambda\beta} = \mathcal{A}^{\alpha}{}_{\beta} a^{\gamma\rho} F_{\rho\alpha\lambda} \delta a^{\lambda\beta}.
\]

(5.17)
By integrating by parts and by relabeling dummy indices we find the final expression for (5.16), namely

\[ -\frac{1}{2} \int_M dx \text{ Tr} (C^{\mu \nu}_\alpha \delta \mathcal{F}^{\alpha}_{\mu \nu}) = \frac{1}{2} \int_M dx \text{ Tr} \left[ \mathcal{F}^{\alpha}_{\beta \gamma} \delta \mathcal{F}^{\beta \gamma}_\alpha + F_{\beta \alpha \rho} \mathcal{F}^{\beta \gamma} \mathcal{F}^{\gamma \alpha}_\rho + \right. \]

\[ - \frac{1}{2} \left( \partial_\beta \mathcal{F}^{\beta \gamma} \right) \left( F_{\rho \sigma \lambda} - F_{\lambda \sigma \rho} - F_{\sigma \rho \lambda} \right) \delta a^{\rho \gamma} + \]

\[ + \frac{1}{4} \left( \partial_\gamma \left[ \left( F_{\rho \sigma \lambda} - F_{\lambda \sigma \rho} - F_{\sigma \rho \lambda} \right) a^{\rho \gamma} \right] \right) \delta a^{\rho \sigma} . \]  

(5.18)

For the last term in the variation of the action (5.10), we use similar arguments which lead us to the expression (5.18). In this case we introduce the following definition:

\[ G_{\beta \alpha \rho} = b_{\beta \alpha} D^\mu b_{\mu \rho} . \]  

(5.19)

By using the definition above and the cyclic property of the trace we obtain

\[ \frac{1}{2} \int_M dx \text{ Tr} (D^\mu \delta \mathcal{F}^{\alpha}_{\mu \alpha}) = \]

\[ - \frac{1}{2} \int_M dx \text{ Tr} \left[ \mathcal{F}^{\alpha}_{\beta \gamma} \delta \mathcal{F}^{\beta \gamma}_\alpha + G_{\beta \gamma \alpha} \mathcal{F}^{\beta \gamma} \mathcal{F}^{\gamma \alpha}_\rho \delta a^{\rho \beta} + \right. \]

\[ - \frac{1}{2} \left( \partial_\beta \mathcal{F}^{\beta \gamma} \right) G_{\rho \sigma \lambda} \delta a^{\rho \gamma} \delta a^{\rho \sigma} + \frac{1}{2} \left( \partial_\gamma \mathcal{F}^{\gamma \alpha} \right) G_{\rho \sigma \lambda} \delta a^{\rho \gamma} \delta a^{\rho \sigma} + \]

\[ - \frac{1}{2} \left( \partial_\gamma \mathcal{F}^{\gamma \alpha} \right) G_{\rho \sigma \lambda} \delta a^{\rho \gamma} \delta a^{\rho \sigma} . \]  

(5.20)

By integrating by parts and relabeling dummy indices we get

\[ \frac{1}{2} \int_M dx \text{ Tr} (D^\mu \delta \mathcal{F}^{\alpha}_{\mu \alpha}) = \]

\[ - \frac{1}{2} \int_M dx \text{ Tr} \left[ \mathcal{F}^{\alpha}_{\beta \gamma} \delta \mathcal{F}^{\beta \gamma}_\alpha + G_{\beta \gamma \alpha} \mathcal{F}^{\beta \gamma} \mathcal{F}^{\gamma \alpha}_\rho \delta a^{\rho \beta} + \right. \]

\[ - \frac{1}{2} \left( \partial_\beta \mathcal{F}^{\beta \gamma} \right) G_{\rho \sigma \lambda} \delta a^{\rho \gamma} \delta a^{\rho \sigma} + \]

\[ - \frac{1}{2} \left( \partial_\gamma \mathcal{F}^{\gamma \alpha} \right) G_{\rho \sigma \lambda} \delta a^{\rho \gamma} \delta a^{\rho \sigma} + \frac{1}{4} \left( \partial_\gamma \left[ \left( G_{\rho \sigma \lambda} - F_{\lambda \sigma \rho} - F_{\sigma \rho \lambda} \right) a^{\rho \gamma} \right] \right) \delta a^{\rho \sigma} . \]  

(5.21)

It is worth noticing that in the above expressions, (5.18) and (5.21), the tensor densities $F$ and $G$ always appear in the same combination. This observation
justifies the following definitions

\[ X_{\rho\lambda\sigma} = F_{\rho\lambda\sigma} - F_{\lambda\sigma\rho} - F_{\sigma\rho\lambda}, \quad (5.22) \]

and

\[ Y_{\rho\lambda\sigma} = G_{\rho\lambda\sigma} - G_{\lambda\sigma\rho} - G_{\sigma\rho\lambda}. \quad (5.23) \]

By using the two definitions above we can rewrite the arguments of the traces in (5.18) and in (5.21) respectively as

\[ -\frac{1}{2} C_{\mu\nu}^{\alpha\delta} \mathcal{A}^{\alpha\mu} = \frac{1}{2} \left[ \mathcal{A}^{\alpha\beta} \partial_{\gamma} d^{\gamma} F_{\rho\alpha\lambda} + F_{\beta\alpha\gamma} d^{\gamma} \mathcal{A}^{\alpha\beta} \right. \]

\[ + \left. \frac{1}{2} \partial_{\gamma} (X_{\lambdaho\beta}) \right] \delta a^{\beta}, \quad (5.24) \]

and

\[ \frac{1}{2} D_{\mu}^{\delta} \mathcal{A}^{\alpha\mu} = \frac{1}{2} \left[ \mathcal{A}^{\alpha\beta} \partial_{\gamma} d^{\gamma} G_{\gamma\alpha\lambda} + G_{\beta\gamma\alpha} d^{\gamma} \mathcal{A}^{\alpha\beta} \right. \]

\[ + \left. \frac{1}{2} \partial_{\gamma} (Y_{\lambda\rho\beta}) \right] \delta a^{\beta}. \quad (5.25) \]

By combining the results (5.24) and (5.25) we obtain the expression for the last two terms in the variation of the action, namely

\[ -\frac{1}{2} C_{\alpha\mu}^{\mu\alpha} + \frac{1}{2} D_{\delta\beta}^{\alpha\mu} \]

\[ = \frac{1}{2} \left( \mathcal{A}^{\alpha\beta} \partial_{\gamma} d^{\gamma} (F_{\gamma\alpha\lambda} - G_{\gamma\alpha\lambda}) + \right. \]

\[ + (F_{\beta\alpha\gamma} - G_{\beta\alpha\gamma}) d^{\gamma} \mathcal{A}^{\alpha\beta} \right. \]

\[ + \left. \frac{1}{2} \partial_{\gamma} (X_{\lambda\rho\beta}) \right] \delta a^{\beta}. \quad (5.26) \]

5.3 Noncommutative Einstein Equations

With the expression (5.26) for the last two terms in (5.10), the variation of the action has the form (5.1), which is suitable for the derivation of the dynamical
equations of the model. Before writing the complete dynamical equations, we will simplify further the expression (5.26).

The definition (5.22) gives a linear relation between the matrix-valued tensor density $X$ and a particular combination of matrix-valued tensor density $F$, a similar linear relation between $Y$ and $G$ is given in (5.23). By using simple tensor algebra, it can be easily shown that those relations can be inverted, namely we can write

$$ F_{\rho\lambda\sigma} = -\frac{1}{2}(X_{\lambda\sigma\rho} + X_{\sigma\rho\lambda}), \quad (5.27) $$

and

$$ G_{\rho\lambda\sigma} = -\frac{1}{2}(Y_{\lambda\sigma\rho} - Y_{\rho\lambda\sigma}). \quad (5.28) $$

By substituting the equations (5.27) and (5.28) in the expression (5.26) we obtain the following

$$ -\frac{1}{2}C_{\alpha\beta\gamma}^\mu \delta_{\beta\gamma}^{\mathcal{Y}} \delta_{\mu\nu}^{\mathcal{X}} + \frac{1}{2}D^{\mu} \delta_{\lambda\sigma}^{\mathcal{X}} D^{\nu} = \frac{1}{4} \left[ \delta_{\mu\nu}^{\mathcal{Y}} \delta_{\alpha\beta}^{\mathcal{X}} (Y_{\alpha\lambda\gamma} - X_{\alpha\lambda\gamma}) + (Y_{\lambda\gamma\alpha} - X_{\lambda\gamma\alpha}) \right] + 
+ \left[ (Y_{\alpha\gamma\lambda} - X_{\alpha\gamma\lambda}) + (Y_{\gamma\beta\lambda} - X_{\gamma\beta\lambda}) \right] \delta_{\alpha\beta}^{\mathcal{X}} \delta_{\mu\nu}^{\mathcal{Y}} + 
+ (\partial_{\beta\gamma} \delta_{\mu\nu}^{\mathcal{X}}) (Y_{\rho\kappa\sigma} - X_{\rho\kappa\sigma}) - \partial_{\gamma} (H_{\rho\kappa\beta} - X_{\rho\kappa\beta} \delta_{\alpha\beta}^{\mathcal{X}}) \delta_{\alpha\beta}^{\mathcal{X}}. \quad (5.29) $$

We can see, in the last formula, that the tensor densities $X$ and $Y$ enter always in the same combination. It is useful, therefore, to define the following tensor density

$$ H_{\mu\nu\rho} = Y_{\mu\nu\rho} - X_{\mu\nu\rho}. \quad (5.30) $$

With this last definition we can rewrite (5.29) as

$$ -\frac{1}{2}C_{\alpha\beta\gamma}^{\mu \delta_{\beta\gamma}^{\mathcal{X}} \delta_{\mu\nu}^{\mathcal{Y}}} + \frac{1}{2}D^{\mu} \delta_{\lambda\sigma}^{\mathcal{X}} D^{\nu} = \frac{1}{4} \left[ \delta_{\mu\nu}^{\mathcal{Y}} \delta_{\alpha\beta}^{\mathcal{X}} H_{\alpha\beta\gamma\lambda} + \delta_{\mu\nu}^{\mathcal{Y}} \delta_{\alpha\beta}^{\mathcal{X}} H_{\alpha\beta\gamma\lambda} + 
+ H_{\alpha\gamma\beta} \delta_{\mu\nu}^{\mathcal{X}} \delta_{\rho\lambda}^{\mathcal{X}} + H_{\gamma\beta\alpha} \delta_{\mu\nu}^{\mathcal{X}} \delta_{\rho\lambda}^{\mathcal{X}} \right] \delta_{\alpha\beta}^{\mathcal{X}} + 
+ (\partial_{\beta\gamma} \delta_{\mu\nu}^{\mathcal{X}}) H_{\rho\kappa\sigma} - \partial_{\gamma} (H_{\rho\kappa\beta} \delta_{\alpha\beta}^{\mathcal{X}}) \delta_{\alpha\beta}^{\mathcal{X}} \delta_{\alpha\beta}^{\mathcal{X}}. \quad (5.31) $$
By using the compatibility condition of the metric tensor $\gamma^{\mu\nu}$ with the connection coefficients $\gamma^\alpha_{\mu\nu}$, we can write that

$$\partial_\beta \gamma^{\alpha\sigma} = -\gamma^\rho_{\gamma\beta} \gamma^{\alpha\sigma} - \gamma^\sigma_{\gamma\beta} \gamma^{\alpha\rho} ,$$

(5.32)

moreover we obtain that

$$- \partial_\gamma (H_{\delta\rho\gamma} \gamma^{\alpha\rho}) = -(\partial_\gamma H_{\delta\rho\gamma}) \gamma^{\alpha\rho} + H_{\delta\rho\gamma} \gamma^{\rho\gamma} \gamma^{\alpha\rho} + H_{\delta\rho\gamma} \gamma^{\rho\gamma} \gamma^{\alpha\rho} .$$

(5.33)

Since $H_{\mu\nu\rho}$ is a tensor density, we can write

$$\mathcal{D}_\gamma H_{\delta\rho\gamma} = \partial_\gamma H_{\delta\rho\gamma} - \gamma^{\alpha\delta}_\gamma H_{\delta\rho\gamma} - \gamma^{\alpha\rho}_\gamma H_{\delta\rho\gamma} - \gamma^{\alpha\sigma}_\rho H_{\lambda\rho\sigma} - \gamma^{\alpha\gamma}_\lambda H_{\delta\rho\gamma} .$$

(5.34)

By using the results obtained in (5.32), (5.33) and (5.34) we can express (5.31) as follows

$$- \frac{1}{2} C^{\alpha\nu}_{\delta\gamma} \gamma^\alpha_{\mu\nu} + \frac{1}{2} D^{\nu}_\delta \gamma^\alpha_{\mu\nu} = \frac{1}{4} \left\{ \frac{1}{2} \gamma^{\rho\gamma}_\alpha \gamma^{\alpha\rho}_{\gamma\beta} H_{\alpha\lambda\nu} + 2H_{\alpha\lambda\beta}\gamma^{\nu\gamma} \gamma^{\rho\gamma}_{\alpha\beta} \right\} +$$

$$-(\mathcal{D}_\gamma H_{\delta\rho\gamma}) \gamma^{\alpha\rho} - \left[ \gamma^{\alpha\delta}_\gamma, H_{\delta\rho\gamma} \right] \gamma^{\rho\gamma} - H_{\lambda\rho\gamma} \left[ \gamma^{\alpha\rho}_{\alpha\beta}, \gamma^{\alpha\rho} \right] - \left[ \gamma^{\alpha\rho}_{\beta\gamma}, H_{\lambda\rho\gamma} \right] \gamma^{\rho\gamma} +$$

$$-\left[ \gamma^{\alpha\rho}_{\gamma\sigma}, H_{\delta\rho\gamma} \right] \gamma^{\rho\gamma} - \gamma^{\rho\gamma}_{\beta\gamma} [H_{\lambda\rho\gamma}, \gamma^{\alpha\rho}] \right) \delta \alpha^{\beta} .$$

(5.35)

At this point we introduce the operator $P$ defined as

$$P \gamma H_{\delta\rho\gamma} = \mathcal{D} H_{\delta\rho\gamma} + \left[ \gamma^{\alpha\delta}_\gamma, H_{\delta\rho\gamma} \right] + \left[ \gamma^{\alpha\rho}_\gamma, H_{\delta\rho\gamma} \right] + \left[ \gamma^{\alpha\sigma}_\rho, H_{\lambda\rho\gamma} \right] + \left[ \gamma^{\alpha\gamma}_\lambda, H_{\delta\rho\gamma} \right] .$$

(5.36)

By using the last definition in (5.35) one obtains

$$- \frac{1}{2} C^{\alpha\nu}_{\delta\gamma} \gamma^\alpha_{\mu\nu} + \frac{1}{2} D^{\nu}_\delta \gamma^\alpha_{\mu\nu} = \frac{1}{4} \left\{ \frac{1}{2} \gamma^{\rho\gamma}_\alpha \gamma^{\alpha\rho}_{\gamma\beta} H_{\alpha\lambda\nu} + 2H_{\alpha\lambda\beta}\gamma^{\nu\gamma} \gamma^{\rho\gamma}_{\alpha\beta} \right\} +$$

$$-(P \gamma H_{\delta\rho\gamma}) \gamma^{\alpha\rho} + \left[ \gamma^{\alpha\rho}_\beta, H_{\lambda\rho\gamma} \right] \gamma^{\rho\gamma} - H_{\lambda\rho\gamma} \left[ \gamma^{\alpha\rho}_{\alpha\beta}, \gamma^{\rho\gamma} \right] - \gamma^{\rho\gamma}_{\beta\gamma} [H_{\lambda\rho\gamma}, \gamma^{\alpha\rho}] \delta \alpha^{\beta} .$$

(5.37)
We finally have all the ingredients that we need in order to write the dynamical equations of the theory. Now we only have to find an expression for the variation $\delta \rho$. The definition of $\rho$ is given in (2.181), and its variation can be straightforwardly evaluated as follows

$$
\delta \rho = -\int_{\mathbb{R}^n} \frac{d\xi}{\pi^2} \int_0^1 ds \, e^{-(1-s)A(\xi)} \delta a^{\mu\nu} \xi_{\mu} \xi_{\nu},
$$

(5.38)

where

$$
A(\xi) = a^{\mu\nu} \xi_{\mu} \xi_{\nu}.
$$

(5.39)

Once we have the expression (5.38) for the variation, we can use the cyclic property of the trace to write that

$$
\text{Tr}_V(\delta \rho \, \mathcal{R}) = \text{Tr}_V \left[ -\int_{\mathbb{R}^n} \frac{d\xi}{\pi^2} \int_0^1 ds \, e^{-(1-s)A(\xi)} \mathcal{R} e^{-(1-s)A(\xi)} \delta a^{\mu\nu} \right].
$$

(5.40)

By combining (5.40), (5.37) and (5.10) we obtain the non-commutative Einstein equations in absence of matter, namely

$$
\mathcal{G}_{\mu\nu} = 0,
$$

(5.41)

where

$$
\mathcal{G}_{\mu\nu} = \frac{1}{2} [\rho, \mathcal{R}_{\mu\nu}] + \mathcal{F}_{\mu\nu} + \frac{1}{2} \delta^{a}{}_{[\mu\nu]} a^{a}{}_{\rho} H_{\alpha\gamma\rho} + \frac{1}{2} H_{\mu\nu} a^{\rho} \delta^{a}{}_{[\mu\nu]} - \frac{1}{4} (P, H_{\mu\nu}) a^{\rho} + \frac{1}{4} [H_{\mu\nu} \alpha, a^{\rho}] - \frac{1}{4} H_{\mu\nu} [H_{\mu\nu}, a^{\rho}].
$$

(5.42)

is the non-commutative Einstein tensor, $\mathcal{F}_{\mu\nu}$ is defined by

$$
\mathcal{F}_{\mu\nu} = -\int_{\mathbb{R}^n} \frac{d\xi}{\pi^2} \int_0^1 ds \, e^{-(1-s)A(\xi)} \mathcal{R} e^{-(1-s)A(\xi)} \delta_{\mu\nu},
$$

(5.43)
and the tensor density $H$ has the explicit form

$$H_{\alpha\lambda\gamma} = b_{\alpha\nu}(\delta_{\gamma}^{\nu} D_{\mu} - C_{\gamma\mu}) b_{\mu\lambda} - b_{\lambda\nu}(\delta_{\nu}^{\gamma} D_{\mu} - C_{\nu\mu}) b_{\mu\alpha} - b_{\gamma\nu}(\delta_{\nu}^{\alpha} D_{\mu} - C_{\nu\mu}) b_{\mu\lambda} . \quad (5.44)$$

These equations are the main result of the present chapter. One can show that the first two terms in the equations (5.42) represent a straightforward generalization of Einstein’s equation to endomorphism-valued objects and the rest of the terms can be considered as a genuine non-commutative part which is not present in Einstein’s equation. It is interesting to note that the pure non-commutative part is completely described by the tensor density $H_{\mu\nu\rho}$ defined in (5.44).

Moreover the equation (5.41) satisfies the requirement (5.3), which, in words, expresses the necessity that our model reduces, in the commutative limit, to the standard theory of General Relativity. In fact, the trace of the pure non-commutative terms vanishes, because of the presence of the commutators, and the first two terms just give

$$\frac{1}{N} \text{Tr}_V \left( \frac{1}{2} \{ \rho, R_{\mu\nu} \} + F_{\mu\nu} \right) = \sqrt{g} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) . \quad (5.45)$$

For an arbitrary matrix algebra the equation (5.41) becomes more complicated than the ordinary Einstein’s equation due to presence of the new tensor density $H_{\mu\nu\rho}$. We mention, now, a particular case in which (5.41) simplifies. The formalism used so far deals with geometric quantities which are endomorphism-valued, namely they take values in $\text{End}(V)$. By choosing a basis in the vector space $V$ we can represent $\text{End}(V)$ by means of matrices. Let us suppose that the algebra under consideration is Abelian, in this case all the elements commute with each other and the tensor density $H_{\mu\nu\rho}$ vanishes identically and the equation (5.41) becomes

$$R_{\mu\nu} - \frac{1}{2} b_{\mu\nu} R = 0 . \quad (5.46)$$
Therefore, in case of a commutative matrix algebra, the equation of motion of our model have the same form as Einstein’s equation, with the only difference that (5.46) is matrix-valued.

5.4 The Action for the Matter Field

In order to have a complete theory for the gravitational field we need to describe the dynamics of the matter field in the framework of matrix general relativity. The main idea is to extend the general results of classical field theory. We will consider, in the following, the dynamics of a multiplet of free scalar fields propagating on a manifold $M$. We can construct an invariant action by using the matrix valued metric $a^{\mu\nu}$ and the measure $\rho$. A typical action is

$$S_{\text{matter}}(a, \phi) = \frac{1}{4} \int_M dx \left\{ -\partial_\mu \rho, \partial_\nu a^{\mu\nu} \phi \right\} - \left\langle \phi, \rho \{ Q \phi \} \right\rangle,$$

(5.47)

where $\langle \ , \ \rangle$ denotes the fiber inner product on the vector bundle $V$, and $Q$ is a constant mass matrix determining the masses of the scalar fields. The equations of motion of the scalar fields are then obviously

$$\left[ -\partial_\mu \rho, a^{\mu\nu} \partial_\nu + \{ Q \} \right] \phi = 0.$$

(5.48)

The complete action of the gravity and matter is described then by

$$S(a, \phi) = S_{\text{MGR}}(a) + S_{\text{matter}}(a, \phi).$$

(5.49)

By varying the above action with respect to $a^{\mu\nu}$ one obtains the non-commutative Einstein equation in presence of matter

$$G_{\mu\nu} = 8\pi G N T_{\mu\nu}.$$  

(5.50)
where $T_{\mu\nu}$ is the matrix energy-momentum tensor defined by

$$T_{\mu\nu} = \frac{-1}{2} \frac{\delta S_{\text{matter}}}{\delta a^{\mu\nu}}. \quad (5.51)$$

By using the explicit lagrangian (5.47) for the matter field, we obtain the expression for the energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{8} \left[ (\rho, \partial_\mu \varphi \otimes \partial_\nu \varphi) + M_{\mu\nu} + N_{\mu\nu} \right] + (\mu \leftrightarrow \nu), \quad (5.52)$$

where the explicit form of $M_{\mu\nu}$ and $N_{\mu\nu}$ is obtained by using the variation of the scalar density $\rho$ in (5.38), namely

$$M_{\mu\nu} = - \int_{\mathbb{R}^n} \frac{d\xi}{\pi^2} \int_0^1 ds e^{-sA(\xi)} \{ a^{\alpha\beta}, \partial_\alpha \varphi \otimes \partial_\beta \varphi \} e^{-sA(\xi)} \xi_\mu \xi_\nu, \quad (5.53)$$

and

$$N_{\mu\nu} = - \int_{\mathbb{R}^n} \frac{d\xi}{\pi^2} \int_0^1 ds e^{-sA(\xi)} \{ Q, \varphi \otimes \varphi \} e^{-sA(\xi)} \xi_\mu \xi_\nu. \quad (5.54)$$

It is worth remarking, here, that the above formula (5.52) for the energy-momentum tensor $T_{\mu\nu}$ reduces, in the commutative limit, to the standard result, e.g. [51].

### 5.5 Conclusions

The main idea of this new model is to describe the gravitational field by a multiplet of gauge fields with some internal structure. For this purpose the metric field $g^{\mu\nu}$, which describes gravity in General Relativity, is replaced by a matrix-valued 2-tensor field $a^{\mu\nu}$. This allows the model to have a much richer content in describing gravitational phenomena. A more general geometric picture is developed by allowing the metric to be matrix-valued. Most of the geometric quantities,
used in describing gravity, can be generalized to be endomorphism-valued. In this framework it is possible to introduce an action for the gravitational field which is diffeomorphisms and gauge invariant, that leads, after performing the variation with respect to $\widetilde{d}^{\mu \nu}$, to the modified (non-commutative) Einstein equation. It is interesting that the non-commutative part of the modified equations only depends on a specific tensor density $H_{\mu \nu \rho}$ and on a linear combination of its commutators.

An important question is related to the quantization of the present model. The analysis developed in this chapter is purely classical and the theory is represented by nothing but a generalized sigma model. The problems of quantization of the present theory, then, are the same that we encounter in performing the quantization of a sigma model.

We would like to make a final remark. In our model all the geometric quantities that we need to develop the formalism are endomorphism-valued. Once a basis for the vector bundle $V$ has been fixed, we can represent elements of $\text{End}(V)$ by matrices. Of course the description of physical phenomena has to be independent from the particular realization of the representation. This is, ultimately, related to the gauge invariance of the theory. We believe that by an opportune choice of gauge, namely an opportune representation of $\text{End}(V)$ by matrices, the dynamical equation (5.41) could be simplified further. The search for such particular gauge, if it exists, requires further studies in matrix differential geometry and matrix General Relativity.
CHAPTER 6

NONCOMMUTATIVE CORRECTIONS IN SPECTRAL MATRIX GRAVITY

Abstract

We study a non-commutative deformation of General Relativity based on spectral invariants of a partial differential operator acting on sections of a vector bundle over a smooth manifold. We compute the first non-commutative corrections to Einstein equations in the weak deformation limit and analyze the spectrum of the theory. Related topics are discussed as well.

6.1 Introduction

The main goal of this chapter is to study the action of Spectral Matrix Gravity in the weak deformation limit and to describe the corresponding corrections to Einstein equations.

We will describe a method for the calculation of the heat kernel developed in [7, 15], which is based on the covariant Fourier transform proposed in [5, 14]. In what follows we specialize the discussion to second order partial differential op-

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4The material in this chapter has been published in *Classical and Quantum Gravity*: G. Fucci and I. G. Avramidi, Non-Commutative Corrections in Spectral Matrix Gravity, *Class. Quant. Grav.* (2009) 26 045019 (24pp)
operators with non-scalar leading symbol which naturally arise in the framework of Matrix Gravity. These operators can be written in a manifestly self-adjoint form as follows \[21\]

\[ L = -\rho^{-1} \nabla_\mu \rho a^{\mu \nu} \rho \nabla_\nu \rho^{-1} + \bar{Q} , \quad (6.1) \]

where \( a^{\mu \nu} \) is a matrix-valued symmetric tensor of type \((2, 0)\), \( \rho \) is a matrix-valued density of weight \(1/2\) and \( \bar{Q} \) is a matrix-valued function.

The heat kernel for a general non-Laplace type second order partial differential operator \( L \) is the kernel of the heat semigroup, that is,

\[ U(t|x, x') = \exp(-tL) \delta(x, x') , \quad (6.2) \]

where \( \delta(x, x') \) is the delta-function (in the density form). By utilizing the Fourier integral representation for the covariant delta function \(2.162\), we obtain

\[ U(t|x, x') = \Delta^{1/2}(x, x') \Phi(t|k, x, x') \int_{\mathbb{R}^n} d\xi (2\pi)^n \exp\{i\xi \rho a^{\mu \nu} (x, x') - 1/2 \rho^{-1} \Delta - 1/2 L \Delta^{1/2} \rho e^{i\xi \sigma^{\mu \nu} a^{\mu \nu}} ) . \quad (6.3) \]

where

\[ \Phi(t|k, x, x') = \exp(-tA) \cdot 1 , \quad (6.4) \]

\[ A = e^{-i\xi \sigma^{\mu \nu} a^{\mu \nu}} \rho^{-1} \Delta^{-1} L \Delta^{1/2} \rho e^{i\xi \sigma^{\mu \nu} a^{\mu \nu} } . \quad (6.5) \]

By using the coincidence limits of the two-point functions, in section 2.2.4, we obtain the heat kernel diagonal

\[ U(t|x, x) = \int_{\mathbb{R}^n} d\xi (2\pi)^n \Phi(t|k, x, x) . \quad (6.6) \]

By substituting the operator \(6.1\) in equation \(6.5\), we get

\[ A = -e^{-i\xi \sigma^{\mu \nu} a^{\mu \nu}} \rho^{-1} \Delta^{-1} \rho^{-1} \nabla_\mu \rho a^{\mu \nu} \rho \nabla_\nu \rho^{-1} \Delta^{1/2} \rho e^{i\xi \sigma^{\mu \nu} a^{\mu \nu} } + \bar{Q} , \quad (6.7) \]
where \( Q = \mathcal{P}^{-1} \tilde{Q} \mathcal{P} \). We rewrite this operator in a more convenient form

\[
A = -\tilde{X}_\mu \tilde{d}^{\mu \nu} X_\nu + Q ,
\]

(6.8)

where

\[
X_\nu = e^{-i\xi_{\rho'} \sigma_{\rho'}} \mathcal{P}^{-1} \Delta^{-\frac{1}{2}} \rho \nabla_\rho \mathcal{P}^{-1} \Delta^{-\frac{1}{2}} \mathcal{P} e^{i\xi_{\rho'} \sigma_{\rho'}},
\]

(6.9)

\[
\bar{X}_\mu = e^{i\xi_{\rho} \sigma_{\rho}} \mathcal{P} \Delta^{-\frac{1}{2}} \rho^{-1} \nabla_\rho \Delta^{-\frac{1}{2}} \mathcal{P} e^{-i\xi_{\rho} \sigma_{\rho}}.
\]

(6.9)

It is useful to introduce, now, two quantities

\[
C_\nu = -\rho \rho^{-1} \quad \text{and} \quad \bar{C}_\nu = -\rho^{-1} \rho^{-1} .
\]

(6.10)

Then we get

\[
X_\nu = \nabla_\nu + C_\nu + \xi_{\nu} + E_\nu + i \xi_{\rho} \eta^\rho_{\nu},
\]

(6.11)

\[
\bar{X}_\mu = \nabla_\mu - \bar{C}_\mu + \xi_{\mu} + E_\mu + i \xi_{\rho} \eta^\rho_{\mu},
\]

(6.12)

and

\[
A = -(\nabla_\mu - \bar{C}_\mu + \xi_{\mu} + E_\mu + i \xi_{\rho} \eta^\rho_{\mu}) \tilde{d}^{\mu \nu}(\nabla_\nu + C_\nu + \xi_{\nu} + E_\nu + i \xi_{\rho} \eta^\rho_{\nu}) + Q .
\]

(6.13)

Finally, a straightforward calculation gives

\[
A = H + K + \mathcal{L} .
\]

(6.14)

Here

\[
H = \xi_{\rho' \sigma} \xi_{\rho} \eta^\rho_{\rho'} \eta^\rho_{\sigma} \tilde{d}^{\rho \sigma},
\]

(6.15)

\[
K = -i \xi_{\rho'} (\mathcal{B}^{\rho'} \nabla_\rho + \mathcal{G}^\rho),
\]

(6.16)

\[
\mathcal{L} = -\tilde{\mathcal{D}}_\mu \tilde{d}^{\mu \nu} \mathcal{D}_\nu + Q ,
\]

(6.17)
where

\[ B^\mu_\nu = 2\eta^\rho_{\mu} a_{\rho\nu} , \]

\[ G^\mu = a_{\mu\nu} : \eta^\rho_{\nu} + a_{\mu\nu} \eta^\rho_{\nu} \mu - \tilde{C}_\mu a_{\mu\nu} \eta^\rho_{\nu} + \eta^\rho_{\nu} a_{\rho\nu} C_\mu + E_\mu a_{\mu\nu} \eta^\rho_{\nu} + \eta^\rho_{\nu} a_{\mu\nu} E_\mu + 2\zeta_\mu a_{\mu\nu} \eta^\rho_{\nu} \, , \quad (6.18) \]

\[ \tilde{\mathcal{D}}_\mu = \nabla_\mu + \tilde{\mathcal{A}}_\mu = \nabla_\mu - \tilde{C}_\mu + \zeta_\mu + E_\mu \, , \]

\[ \mathcal{D}_\nu = \nabla_\nu + \mathcal{A}_\nu = \nabla_\nu + C_\nu + \zeta_\nu + E_\nu \, , \quad (6.19) \]

with \( C_\nu, \tilde{C}_\mu, \zeta \) defined in (6.10), (2.121) and \( E_\mu \)

\[ E_\mu = \mathcal{P}^{-1} \nabla_\mu \mathcal{P} \, . \quad (6.20) \]

More explicitly we can also write that

\[ \mathcal{L} = -a_{\mu\nu} \nabla_\mu \nabla_\nu + \mathcal{Y}_\mu \nabla_\mu + \mathcal{Z} \, , \quad (6.21) \]

where

\[ \mathcal{Y}_\mu = -a_{\mu\nu} \nabla_\mu \nabla_\nu + \tilde{C}_\nu a_{\mu\nu} - a_{\mu\nu} C_\nu - 2a_{\mu\nu} \zeta_\nu - a_{\mu\nu} E_\nu - E_\nu a_{\mu\nu} \, , \quad (6.22) \]

\[ \mathcal{Z} = -a_{\mu\nu} : C_\nu - a_{\mu\nu} C_\nu : \mu + \tilde{C}_\mu a_{\mu\nu} C_\nu - a_{\mu\nu} : \zeta_\nu - a_{\mu\nu} \zeta_\nu : \mu + \tilde{C}_\mu a_{\mu\nu} \zeta_\nu - \zeta_\nu a_{\mu\nu} C_\nu - \zeta_\nu a_{\mu\nu} \zeta_\nu - a_{\mu\nu} \zeta_\nu a_{\mu\nu} E_\nu - E_\nu a_{\mu\nu} C_\nu - \zeta_\nu a_{\mu\nu} E_\nu - \zeta_\nu a_{\mu\nu} \zeta_\nu E_\nu a_{\mu\nu} + \tilde{C}_\mu a_{\mu\nu} E_\nu - E_\mu a_{\mu\nu} E_\nu - \zeta_\nu a_{\mu\nu} E_\nu - \zeta_\nu a_{\mu\nu} \zeta_\nu E_\mu a_{\mu\nu} + Q \, . \quad (6.23) \]

Thus, by using the eq. (6.14) we obtain

\[ U(t|x, x) = \int_{\mathbb{R}^n} \frac{d\xi}{(2\pi)^n} e^{-i(H+K+L)} \cdot \left| \left. \frac{\partial}{\partial x} \right|_{x=x'} \right| , \quad (6.24) \]

which, by scaling the integration variable \( \xi \rightarrow t^{\frac{1}{2}} \xi \), takes the form

\[ U(t|x, x) = (4\pi t)^{-\frac{1}{2}} \int_{\mathbb{R}^n} \frac{d\xi}{\pi^2} \exp(-H - \sqrt{t}K - tL) \cdot \left| \left. \frac{\partial}{\partial x} \right|_{x=x'} \right| \, . \quad (6.25) \]
It is convenient, to rewrite this equation as

\[
U(t|x, x) = (4\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \frac{d\xi}{\pi^n} \ e^{-|\xi|^2} \exp(-\tilde{H} - \sqrt{t}K - tL) \cdot \bigg|_{x=x'},
\]

(6.26)

where $|\xi|^2 = g^{\mu\nu}\xi_\mu\xi_\nu$ and

\[
\tilde{H} = H - |\xi|^2.
\]

(6.27)

In order to evaluate the first three coefficients of the asymptotic expansion of (6.26) as $t \to 0$ we use the Volterra series for the exponent of a sum of two non-commuting operators in (2.164) to obtain

\[
\exp(-\tilde{H} - \sqrt{t}K - tL) = e^{-\tilde{H}} - \sqrt{t} \Omega + t\Psi + O(t^{\frac{3}{2}}),
\]

(6.28)

where

\[
\Omega = \int_0^1 dt_1 e^{-(1-t_1)\tilde{H}} K e^{-t_1\tilde{H}},
\]

(6.29)

and

\[
\Psi = \int_0^{t_2} dt_1 \int_0^{t_2} dt_2 e^{-(1-t_2)\tilde{H}} K e^{-(t_2-t_1)\tilde{H}} K e^{-t_1\tilde{H}} - \int_0^{t_2} dt_1 e^{-(1-t_1)\tilde{H}} L K e^{-t_1\tilde{H}}.
\]

(6.30)

We are only interested in the terms $a_0$ and $a_1$ of the heat kernel expansion, namely the terms of zero order and linear in the parameter $t$. These terms can be written, respectively, as

\[
a_0 = g^{\frac{1}{2}}\tilde{a}_0,
\]

(6.31)

\[
a_1 = g^{\frac{1}{2}}\tilde{a}_1,
\]

(6.32)

where

\[
\tilde{a}_0 = \int_{\mathbb{R}^n} \frac{d\xi}{\pi^n} g^{-\frac{1}{4}} e^{-|\xi|^2} \exp\{-\tilde{H}\} \cdot \bigg|_{x=x'},
\]

(6.33)

\[
\tilde{a}_1 = \int_{\mathbb{R}^n} \frac{d\xi}{\pi^n} g^{-\frac{1}{4}} e^{-|\xi|^2}\Psi \cdot \bigg|_{x=x'}.
\]

(6.34)
The term \( t^2 \) of the heat kernel expansion vanishes identically. This happens because the heat kernel coefficients are defined as \( \xi \)-integrals over the whole \( \mathbb{R}^n \) and the term \( \Omega \) in (6.29) is an odd function of \( \xi \).

6.2 Evaluation of the Heat Kernel Coefficients

6.2.1 Local Coefficient \( \tilde{a}_0 \)

We will evaluate the heat kernel coefficients \( A_0 \) and \( A_1 \) using the perturbation theory. The main idea is to introduce a small deformation parameter \( \lambda \) and evaluate the non-commutative corrections to the action of Spectral Matrix Gravity. For this purpose we write the matrix \( a_{\mu \nu} \) as

\[
a^{\mu \nu} = g^{\mu \nu} I + \lambda h^{\mu \nu},
\]

(6.35)

where \( h^{\mu \nu} \) is a traceless matrix-valued tensor field (a non-commutative perturbation of the Riemannian metric), satisfying

\[
\text{tr}_V h^{\mu \nu} = 0.
\]

(6.36)

Furthermore, we parameterize the matrix-valued density \( \rho \) introduced in (6.1) as

\[
\rho = g^{\frac{1}{2}} e^\phi e^{\lambda \sigma}.
\]

(6.37)

Here \( \sigma \) is a traceless matrix-valued scalar field and \( \phi \) is a scalar field. (Do not confuse it with the world function introduced in the previous sections!) Finally, we also decompose the endomorphism \( Q \),

\[
Q = q \cdot I + \lambda \Theta,
\]

(6.38)

where \( \Theta \) is a traceless matrix-valued scalar field.
Now we expand all the quantities in powers of $\lambda$. On doing so the matrix $\rho$ and its inverse read

$$
\rho = g\frac{1}{4} e^{\phi} \left(1 + \lambda \sigma + \frac{\lambda^2}{2} \sigma^2\right) + O(\lambda^3),
$$

$$
\rho^{-1} = g\frac{1}{4} e^{-\phi} \left(1 - \lambda \sigma + \frac{\lambda^2}{2} \sigma^2\right) + O(\lambda^3),
$$

and its derivative is

$$
g^{\frac{1}{2}} \rho_{\nu} = e^{\phi} \left[\lambda \sigma_{\nu} + \frac{\lambda^2}{2} (\sigma_{\nu} \sigma + \sigma \sigma_{\nu})\right] + e^{\phi} \varphi_{\nu} \left(1 + \lambda \sigma + \frac{\lambda^2}{2} \sigma^2\right) + O(\lambda^3). \tag{6.40}
$$

From the last two expressions one can easily evaluate the operators $C_{\nu}$ and $\bar{C}_{\nu}$ obtaining explicitly

$$
C_{\nu} = -\rho_{\nu}^{\nu} \rho^{-1} = -\phi_{\nu}^{\nu} - \lambda \sigma_{\nu}^{\nu} + \frac{\lambda^2}{2} [\sigma_{\nu}^{\nu}, \sigma] + O(\lambda^3), \tag{6.41}
$$

$$
\bar{C}_{\nu} = -\rho^{-1}_{\nu} \rho_{\nu}^{\nu} = -\phi_{\nu}^{\nu} - \lambda \sigma_{\nu}^{\nu} - \frac{\lambda^2}{2} [\sigma_{\nu}^{\nu}, \sigma] + O(\lambda^3). \tag{6.42}
$$

The operators $\tilde{H}$, $K$ and $L$ introduced above in (6.15), (6.16) and (6.17) depend on the deformation parameter $\lambda$ as well. By expanding them in terms of the deformation parameter we get

$$
\tilde{H} = H_0 + \lambda H_1,
$$

$$
K = K_0 + \lambda K_1 + \lambda^2 K_2 + O(\lambda^3),
$$

$$
L = L_0 + \lambda L_1 + \lambda^2 L_2 + O(\lambda^3), \tag{6.43}
$$
where

\[ H_0 = \xi_\alpha \xi_\beta (\eta^{\alpha}_\mu \eta^{\beta}_\nu g^{\mu \nu} - g^{\alpha \beta}) , \]  
(6.44)

\[ H_1 = \xi_\alpha \xi_\beta \eta^{\alpha}_\mu \eta^{\beta}_\nu h^{\mu \nu} , \]  
(6.45)

\[ K_0 = -i\xi_\alpha (2\eta^{\alpha}_\mu g^{\mu \nu} \nabla_\nu + 2\eta^{\alpha}_\mu g^{\alpha \nu} \zeta_\nu + \eta^{\alpha}_\mu \omega^{\mu}_\nu + E^{\mu} \eta^{\alpha}_\mu + \eta^{\alpha}_\mu E^{\mu}) , \]  
(6.46)

\[ K_1 = -i\xi_\alpha (2\eta^{\alpha}_\mu h^{\mu \nu} \nabla_\nu + \eta^{\alpha}_\mu h^{\mu \nu} \zeta_\nu + \eta^{\alpha}_\mu \omega^{\mu}_\nu + \eta^{\alpha}_\mu h^{\mu \nu} E_\nu 
+ \eta^{\alpha}_\mu \omega^{\alpha}_\nu E_\nu , \]  
(6.47)

\[ K_2 = -i\xi_\alpha [\eta^{\alpha}_\mu ([\sigma_\alpha \nu , h^{\mu \nu}] + [\sigma_\nu \mu , \sigma])] , \]  
(6.48)

\[ \mathcal{L}_0 = -\nabla^2 - (2\zeta_\mu + 2E_\mu) \nabla_\mu + \phi_\mu \omega^{\mu}_\nu - \zeta_\mu \omega^{\mu}_\nu \] 
\[ - \zeta_\mu \zeta^{\mu} - E_\mu \omega^{\mu}_\nu - E^{\mu} E_\mu - 2\zeta_\mu E^{\mu} + q , \]  
(6.49)

\[ \mathcal{L}_1 = -h^{\mu \nu} \nabla_\mu \nabla_\nu - [h^{\mu \nu} + 2\zeta_\mu , h^{\mu \nu} + (h^{\mu \nu} E_\nu + E_\nu h^{\mu \nu})] \nabla_\mu + \sigma_\mu \omega^{\mu}_\nu \] 
\[ + h^{\mu \nu} \phi_\mu \nu + h^{\mu \nu} \phi_\nu \mu + 2\phi_\mu \nu \sigma_\nu + \phi_\mu \nu h^{\mu \nu} \phi_\nu \nu - h^{\mu \nu} \zeta_\mu \nu \] 
\[ - h^{\mu \nu} \zeta_\mu \nu - \zeta_\mu h^{\mu \nu} \zeta^{\mu} - h^{\mu \nu} E_\nu - h^{\mu \nu} E_\nu \mu + [E_\mu , \sigma^{\mu}_\nu ] 
+ \phi_\nu \mu [E_\nu , h^{\mu \nu}] - E_\mu h^{\mu \nu} E_\nu - E_\mu h^{\mu \nu} \zeta_\mu \nu - \zeta_\mu h^{\mu \nu} E_\nu + \Theta , \]  
(6.50)

\[ \mathcal{L}_2 = ([h^{\mu \nu} , \sigma_\nu ] + [\sigma_\nu \mu , \sigma]) \nabla_\mu - \frac{1}{2} [\sigma_\mu \omega^{\mu}_\nu + h^{\mu \nu} \sigma_\nu + h^{\mu \nu} \sigma_\nu ] 
+ \sigma_\nu \nu \sigma_\nu + \sigma_\nu \nu h^{\mu \nu} \phi_\mu + \phi_\nu \mu h^{\mu \nu} \sigma_\nu - \frac{1}{2} [\sigma^{\mu}_\nu , \sigma] \zeta_\mu - \frac{1}{2} \zeta_\mu [\sigma^{\mu}_\nu , \sigma] 
+ \zeta_\mu h^{\mu \nu} \sigma_\nu - \sigma_\nu h^{\mu \nu} \zeta_\mu - \frac{1}{2} [\sigma_\mu \sigma_\nu , E_\mu + E_\nu (\sigma_\mu , \sigma)] 
+ E_\mu h^{\mu \nu} \sigma_\nu - \sigma_\nu h^{\mu \nu} E_\nu . \]  
(6.51)

In the framework of perturbation theory we write, then, the coefficients $\tilde{a}_0$ and $\tilde{a}_1$ of the heat kernel expansion in (6.33) and (6.34) in terms of the deformation.
parameter $\lambda$, namely

$$\tilde{a}_0(\lambda) = a_0^{(0)} + \lambda a_0^{(1)} + \lambda^2 a_0^{(2)} + O(\lambda^3)$$

$$\tilde{a}_1(\lambda) = a_1^{(0)} + \lambda a_1^{(1)} + \lambda^2 a_1^{(2)} + O(\lambda^3).$$

By using the explicit formulas obtained in (6.44) through (6.51), we will be able to evaluate all the coefficients of the Taylor expansions in (6.52).

Next, we introduce a notation that will be useful in the following calculations. Let $f$ be a function of $\xi$. We define the Gaussian average of the function $f$ as

$$\langle f \rangle = \int_{\mathbb{R}^n} \frac{d\xi}{\pi^n} g^{-\frac{1}{2}} e^{-\frac{1}{2} \xi^2} f(\xi).$$

(6.53)

The Gaussian averages of the polynomials are well known

$$\langle \xi_{\mu_1} \cdots \xi_{\mu_{2n+1}} \rangle = 0,$$

$$\langle \xi_{\mu_1} \cdots \xi_{\mu_{2n}} \rangle = \frac{(2n)!}{2^n n!} g(\mu_1 \mu_2 \cdots g_{\mu_{2n-\mu_{2n}}}),$$

(6.54)

where the parentheses ( ) denote the symmetrization over all the included indices.

For the coefficient of order zero of the heat kernel expansion we consider the first equation in (6.52). From the formula (6.33), it is easy to see that the only non-vanishing contribution to $\tilde{a}_0$ is

$$\tilde{a}_0 = \left( \left[ 1 - \lambda N + \frac{\lambda^2}{2} N^2 \right] \cdot I \right)_{x=x'} + O(\lambda^3).$$

(6.55)

By using the equations (6.44), (6.45) and after taking the coincidence limit we obtain the expression

$$\tilde{a}_0 = \left( \left[ I - \lambda h^{\mu \nu} \xi_\mu \xi_\nu + \frac{1}{2} \lambda^2 h^{\mu \nu} h^{\rho \sigma} \xi_\mu \xi_\rho \xi_\nu \xi_\sigma \right] \right) + O(\lambda^3),$$

(6.56)
and then, by performing the Gaussian averages, we get

\[ \tilde{a}_0 = 1 - \frac{\lambda}{2} h + \frac{\lambda^2}{8} (h^2 + 2 h^{\mu\nu} h_{\mu\nu}) + O(\lambda^3) , \]  
\[ (6.57) \]

where \( h = g_{\mu\nu} h^{\mu\nu} \). In order to evaluate the global coefficient \( A_0 \), we need the trace of \( (6.57) \). Since \( h^{\mu\nu} \) is traceless we immediately obtain

\[ \text{tr}_V \tilde{a}_0 = \text{tr}_V \left( 1 + \frac{\lambda^2}{8} h^2 + \frac{\lambda^2}{4} h^{\mu\nu} h_{\mu\nu} \right) + O(\lambda^3) . \]  
\[ (6.58) \]

### 6.2.2 Coincidence Limits

In this section we will list the various coincidence limits that we will use during the calculations performed in this chapter.

Through all this section the subscripts 0,1 and 2 will be used to denote terms of different order in the deformation parameter \( \lambda \). More precisely for any quantity \( X \) which contains different orders of \( \lambda \) we write

\[ X = X_0 + \lambda X_1 + \lambda^2 X_2 + O(\lambda^3) , \]

where \( X_0, X_1 \) and \( X_2 \) denote, respectively, the zeroth, first and second order in \( \lambda \).

We start with the coincidence limit of the operator \( \tilde{H} \) in \( (6.15) \) and its derivatives. More precisely we have

\[ [\tilde{H}] = \lambda \xi_\mu \xi_\nu h^{\mu\nu} . \]  
\[ (6.59) \]

For the first derivative we obtain

\[ [\tilde{H}; \mu] = \lambda \xi_\sigma h^{\alpha\beta} ; \mu \]  
\[ (6.60) \]

For the second derivative we get the following formula

\[ [\tilde{H}; \mu \nu] = -\frac{2}{3} \xi_\rho \xi^\rho R^\mu_{\rho \nu \sigma} - \frac{2}{3} \lambda \xi_\sigma \xi_\rho h^{\mu\nu} (R^\rho_{\mu \nu \sigma} + R^\rho_{\nu \sigma \rho}) + \lambda \xi_\rho \xi_\sigma h^{\alpha\beta} ; \mu \nu \]  
\[ (6.61) \]
Recall, now, the definition (6.16) for the operator $K$. The coincidence limits of the terms in $K$ are

\[
\begin{align*}
[B_{0}^{\rho^{'\nu}}] & = -2g^{\rho^{'\nu}}, \\
[B_{1}^{\rho^{'\nu}}] & = -2h^{\rho^{'\nu}}.
\end{align*}
\] (6.62)

For the derivatives of these quantities we have

\[
\begin{align*}
[(\nabla_{\mu} B_{0}^{\rho^{'\nu}})_{0}] & = 0, \\
[(\nabla_{\mu} B_{0}^{\rho^{'\nu}})_{1}] & = -2h^{\rho^{'\nu}};_{\mu}.
\end{align*}
\] (6.65)

For the other terms we have

\[
\begin{align*}
[G_{0}^{\rho^{'\nu}}] & = 0, \\
[G_{1}^{\rho^{'\nu}}] & = -h^{\rho^{'\nu}};_{\nu}.
\end{align*}
\] (6.67)

For the operator $L$ in (6.17) we need the following coincidence limits

\[
\begin{align*}
[(\nabla_{v} G_{0}^{\rho^{'\nu}})_{0}] & = \frac{1}{3}R_{\nu}^{\rho^{'\nu}} + R_{\nu}^{\rho^{'\nu}}, \\
[(\nabla_{v} G_{0}^{\rho^{'\nu}})_{1}] & = -h^{\rho^{'\nu}};_{\alpha\nu} - \frac{1}{3}h^{\rho^{'\alpha}} R_{\alpha\nu} + \frac{2}{3}h^{\rho^{'\alpha}} R_{\alpha v} + h^{\rho^{'\alpha}} R_{\alpha v}, \\
[(\nabla_{v} G_{0}^{\rho^{'\nu}})_{2}] & = -[\sigma;_{\alpha\nu}, h^{\rho^{'\alpha}}] - [\sigma;_{\alpha}, h^{\rho^{'\alpha}};_{\nu}] - [\sigma;_{\rho^{'\nu}}, \sigma] - [\sigma^{\rho^{'\nu}}, \sigma;_{\nu}] - \frac{1}{2} [\sigma;_{\nu}, \sigma].
\end{align*}
\] (6.69)
We also used, during the calculation, the coincidence limits for the derivatives of \( \mathcal{A}_\nu \), namely
\[
\left[ (\nabla_\mu \mathcal{A}_\nu)_0 \right] = \frac{1}{6} \phi_{\nu\mu} + \frac{1}{6} R_{\nu\mu} - \frac{1}{2} \mathcal{R}_{\nu\mu}, \tag{6.74}
\]
\[
\left[ (\nabla_\mu \mathcal{A}_\nu)_1 \right] = -\sigma_{\nu\mu}, \tag{6.75}
\]
\[
\left[ (\nabla_\mu \mathcal{A}_\nu)_2 \right] = \frac{1}{2} \left( [\sigma_{\nu\mu}, \sigma] + [\sigma_{\nu}, \sigma_{\mu}] \right). \tag{6.76}
\]

6.2.3 Local Coefficient \( \tilde{a}_1 \)

Now we evaluate the coefficient \( \tilde{a}_1 \). By using the expressions (6.34), (6.30) and (6.17) we have
\[
\text{tr}_V \tilde{a}_1 = \left\{ \left( \text{tr}_V [e^{-\hat{H} Q}] \right) + \left( \text{tr}_V \left[ \int_0^1 \int_0^{\tau_2} d\tau_1 e^{-(1-\tau_1)\hat{H}} \mathcal{K} e^{-\tau_1 \hat{H}} \right] \right) \right\} \bigg|_{x=x'}, \tag{6.77}
\]
where the first term of the expression (6.77) has been obtained by simply using the cyclic property of the trace.

In the following we will evaluate the terms in (6.77) separately. We start with the simplest of them, namely the one involving the endomorphism \( Q \). By using the Taylor expansion in \( \lambda \) of \( \hat{H} \) in (6.43) and the coincidence limits (2.119) we obtain
\[
\left\{ \left( \text{tr}_V [e^{-\hat{H} Q}] \right) \right\} \bigg|_{x=x'} = \left\{ \text{tr}_V \left( -Q + \lambda \xi_{\mu} \xi_{\rho} h^{\mu\nu} Q - \frac{\lambda^2}{2} \xi_{\mu} \xi_{\rho} \xi_{\sigma} \xi_{\tau} h^{\mu\nu} h^{\rho\sigma} Q \right) \right\} \bigg|_{x=x'} + O(\lambda^3). \tag{6.78}
\]
By expanding \( Q \) as in (6.38) and by performing the Gaussian averages we obtain
\[
\left\{ \text{tr}_V [e^{-\hat{H} Q}] \right\} \bigg|_{x=x'} = -N q + \lambda^2 \text{tr}_V \left( \frac{1}{2} h \Theta - \frac{1}{8} h^2 q - \frac{1}{4} h^{\mu\nu} h_{\mu\nu} q \right) + O(\lambda^3), \tag{6.79}
\]
where we used the property (6.36).

For the second term in equation (6.77) we get, by using the definition (6.16),

\[
\langle \text{tr} V \left[ \int_0^{\tau_2} d\tau_1 e^{-(1-\tau_2)\hat{H}} K e^{-(\tau_2-\tau_1)\hat{H}} K e^{-\tau_1\hat{H}} \right] \rangle \bigg|_{x=x'} = \\
-\langle \text{tr} V \left[ \xi_{\rho '} \xi_{\sigma '} \int_0^{\tau_2} d\tau_1 e^{-(1-\tau_2)\hat{H}} \left( B^{\rho ' \nu} \nabla_\nu e^{-(\tau_2-\tau_1)\hat{H}} B^{\sigma ' \mu} \nabla_\mu e^{-\tau_1\hat{H}} + G^{\rho '} e^{-(\tau_2-\tau_1)\hat{H}} B^{\sigma ' \mu} \nabla_\mu e^{-\tau_1\hat{H}} + G^{\rho '} e^{-(\tau_2-\tau_1)\hat{H}} G^{\sigma '} e^{-\tau_1\hat{H}} \right) \right] \rangle \bigg|_{x=x'}.
\]

It is straightforward to notice that in the last expression we need to compute first and second derivatives of the exponentials containing the operator \(\hat{H}\). These derivatives are computed by using integral representations, i.e. for the first derivative we have

\[
\nabla_\mu e^{-\tau \hat{H}} = -\beta_\mu(\tau) e^{-\tau \hat{H}},
\]

where

\[
\beta_\mu(\tau) = \int_0^\tau ds e^{-s \hat{H}} \hat{H}_{;\mu} e^{s \hat{H}}.
\]

This last integral can be evaluated by referring to the following formula and by integrating over \(s\)

\[
e^{-\hat{H}} \hat{H}_{;\mu} e^{\hat{H}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left[ \hat{H}_{;\cdots;\mu} \cdots \right].
\]

By expanding (6.82) in \(s\), up to the second order in \(\lambda\), we obtain

\[
\nabla_\mu e^{-\tau \hat{H}} = -\left( \tau \hat{H}_{;\mu} + \frac{1}{2} \tau^2 \left[ \hat{H}_{;\mu}, \hat{H} \right] \right) e^{-\tau \hat{H}} + O(\lambda^3).
\]
This last expression can be obtained by recalling that the coincidence limit for $\tilde{H}$ and its derivatives is of order $\lambda$ without the zeroth order term.

For the second derivative we write

$$\nabla_\mu \nabla_\nu e^{-\tau \tilde{H}} = -\int_0^\tau ds_1 e^{-(\tau-s_1)\tilde{H}} \tilde{H}_{;\mu\nu} e^{-s_1\tilde{H}} +$$

$$+ \int_0^\tau ds_2 \int_0^{s_2} ds_1 (e^{-(s_2-s_1)\tilde{H}} \tilde{H}_{;\nu} e^{-s_1\tilde{H}} \tilde{H}_{;\mu} e^{-(\tau-s_2)\tilde{H}} +$$

$$+ e^{-(\tau-s_2)\tilde{H}} \tilde{H}_{;\mu} e^{-(s_2-s_1)\tilde{H}} \tilde{H}_{;\nu} e^{-s_1\tilde{H}}). \quad (6.85)$$

We can express this formula in the same form as (6.84), i.e.

$$\nabla_\mu \nabla_\nu e^{-\tau \tilde{H}} = -\left(\tau \tilde{H}_{;\mu\nu} + \frac{1}{2} [\tilde{H}_{;\mu\nu}, \tilde{H}] - \frac{1}{2} \tau^2 \{\tilde{H}_{;\nu}, \tilde{H}_{;\mu}\} \right) e^{-\tau \tilde{H}} + O(\lambda^3). \quad (6.86)$$

Now that we have the expressions (6.81) through (6.86), we can substitute them in (6.80) and we can expand the remaining exponentials in $\tau$ up to orders $\lambda^2$. After the expansion of the exponentials and after taking the coincidence limit, we have to evaluate the double integrals of polynomials in $\tau_1$ and $\tau_2$ which will yield the numerical coefficients for the various terms in (6.80). The most general double integral that we need to evaluate is the following

$$I_1(\alpha, \beta, \gamma, \delta) = \int_0^1 d\tau_2 \int_0^{\tau_2} d\tau_1 \tau_1^\alpha (1-\tau_2)^\beta (\tau_2-\tau_1)^\gamma (1-\tau_2+\tau_1)^\delta, \quad (6.87)$$

with $(\alpha, \beta, \gamma, \delta)$ being positive integers so that the integral is well defined. The solution in closed form of (6.87) can be found as follows.

This integral is well defined for $\text{Re}(\alpha) > -1$, $\text{Re}(\gamma) > -1$ and $0 < \tau_2 < 1$. By using the integral representation of the hypergeometric function we can evaluate
the integral in $\tau_1$, i.e.

$$I_1 = \frac{\Gamma(1+\alpha)\Gamma(1+\gamma)}{\Gamma(2+\alpha+\gamma)} \int_0^1 d\tau_2 \tau_1^{1+\alpha+\gamma}(1-\tau_2)^{\beta+\delta} \, _2F_1\left(1+\alpha,-\delta,2+\alpha+\gamma;\frac{\tau_2}{1-\tau_2}\right).$$  

(6.88)

Now, by using the linear transformation formula for the hypergeometric function we get \[1\]

$$\, _2F_1\left(1+\alpha,-\delta,2+\alpha+\gamma;\frac{\tau_2}{1-\tau_2}\right) = (1-\tau_2)^{1+\alpha} \, _2F_1\left(1+\alpha,2+\alpha+\gamma+\delta,2+\alpha+\gamma;\tau_2\right).$$  

(6.89)

By substituting the last expression in the integral (6.88) we obtain

$$I_1 = \frac{\Gamma(1+\alpha)\Gamma(1+\gamma)}{\Gamma(2+\alpha+\gamma)} \times$$

$$\int_0^1 d\tau_2 \tau_1^{1+\alpha+\gamma}(1-\tau_2)^{1+\alpha+\beta+\delta} \, _2F_1\left(1+\alpha,2+\alpha+\gamma+\delta,2+\alpha+\gamma;\tau_2\right).$$  

(6.90)

This integral over $\tau_2$ is, now, of the following general form

$$I = \int_0^1 dx \, x^{c-1}(1-x)^{d-1} \, _2F_1(a,b,c;x),$$  

(6.91)

which is well define for $\text{Re}(c) > 0$ and $\text{Re}(d) > 0$ and has a solution in a closed form \[54\], namely

$$I = \frac{\Gamma(c)\Gamma(d)\Gamma(c+d-a-b)}{\Gamma(c+d-a)\Gamma(c+d-b)}. $$  

(6.92)

By using the results (6.92) in the integral (6.90), we get the solution (6.93), i.e.

$$I_1(\alpha,\beta,\gamma,\delta) = \frac{\Gamma(1+\alpha)\Gamma(1+\beta)\Gamma(1+\gamma)\Gamma(2+\alpha+\beta+\delta)}{\Gamma(3+\alpha+\beta+\gamma+\delta)\Gamma(2+\alpha+\beta)},$$  

(6.93)
where $\Gamma(x)$ is the Euler gamma function.

By using the technical details described above and the coincidence limits for the various terms in (6.80) (see Section 6.3.2), we obtain

$$
\left\langle \text{tr}_V \left[ \int_0^{\tau_2} d\tau_1 e^{-(1-\tau_2)\tilde{H}} K e^{-(\tau_2-\tau_1)\tilde{H}} K e^{-\tau_1\tilde{H}} \right] \right\rangle \bigg|_{x=x'} =
$$

$$
= \text{tr}_V (\Omega_0) + \lambda \text{tr}_V (\Omega_1) + \lambda^2 \text{tr}_V (\Omega_2) + O(\lambda^3),
$$

(6.94)

where

$$
\Omega_0 = \frac{1}{6} R ,
$$

(6.95)

$$
\begin{align*}
\Omega_1 &= \frac{1}{6} h^{\alpha \beta} - \frac{1}{6} h^{\mu \nu}_{: \mu \nu} - \frac{1}{12} h R + \frac{1}{6} h^{\mu \nu} R_{\mu \nu} , \\
\Omega_2 &= -\frac{1}{16} h_{: \mu \nu} h^{\mu \nu} + \frac{1}{6} h^{\mu \nu}_{: \mu \nu} h_{: \nu} - \frac{1}{8} h^{\mu \nu}_{: \rho} h_{\mu \nu}^{\rho} - \frac{1}{12} h h_{: \alpha}^{\alpha} + \frac{1}{12} h h_{: \mu \nu}^{\mu \nu} \\
&+ \frac{1}{6} h^{\mu \nu} h_{: \mu \nu} + \frac{1}{12} h h_{: \rho} h_{\mu \nu}^{\rho} - \frac{1}{12} h h_{: \mu \nu} R_{\mu \nu} + \frac{1}{48} h^2 R \\
&- \frac{1}{12} h h_{: \mu \nu} R_{\mu \nu} + \frac{1}{24} h^{\mu \nu} h_{: \mu \nu} R .
\end{align*}
$$

(6.96)

(6.97)

We can finally evaluate the last term in equation (6.77). By using the definitions (6.17) and (6.19) we can write that

$$
\left\langle \text{tr}_V \left[ \int_0^{\tau_2} d\tau_1 e^{-(1-\tau_2)\tilde{H}} \mathcal{D}_\mu a^{\mu \nu} \mathcal{D}_\nu e^{-\tau_1\tilde{H}} \right] \right\rangle \bigg|_{x=x'}
$$

$$
= \left\langle \text{tr}_V \left[ \int_0^{\tau_2} d\tau_1 e^{-(1-\tau_2)\tilde{H}} (a^{\mu \nu}_{: \mu} \nabla_\nu + a^{\mu \nu} \nabla_\mu \nabla_\nu + a^{\mu \nu}_{: \mu} \mathcal{A}_\nu \\
+ a^{\mu \nu} \mathcal{A}_\nu,_{\mu} + a^{\mu \nu} \mathcal{A}_\nu \nabla_\mu + \mathcal{A}_\nu a^{\mu \nu} \nabla_\nu + \mathcal{A}_\nu a^{\mu \nu} \mathcal{A}_\nu) \right] \right\rangle \bigg|_{x=x'} .
$$

(6.98)

In order to evaluate this term we use the derivatives in (6.84) and (6.86) and we expand the remaining exponentials of $\tilde{H}$ in $\tau$ up to terms in $\lambda^2$. During
the calculation the numerical coefficients of the various terms can be evaluated by referring to the following general integral

\[ I_2(\alpha, \beta) = \int_0^1 d\tau_{1} \tau_{1}^\alpha (1 - \tau_{1})^\beta, \quad (6.99) \]

where \((\alpha, \beta)\) are positive integers. The solution to this integral is easily found by recalling the integral representation of the hypergeometric function \([1]\)

\[ _2F_1(a, b, c ; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 dt \; t^{b-1}(1 - t)^{c-b-1}(1 - tz)^{-a}, \quad (6.100) \]

where \(\text{Re}(c) > 0\) and \(\text{Re}(b) > 0\). From this last general expression we obtain the integral \((6.99)\) by setting \(z = 0\), \(\alpha = b - 1\) and \(\beta = c - b - 1\). By recalling that \(_2F_1(a, b, c ; 0) = 1\), we finally get

\[ I_2(\alpha, \beta) = \frac{\Gamma(1 + \alpha)\Gamma(1 + \beta)}{\Gamma(2 + \alpha + \beta)} = B(1 + \alpha, 1 + \beta), \quad (6.101) \]

where \(B(a, b)\) denotes the Euler beta function.

The explicit form of \((6.98)\) can be obtained with the help of \((6.101)\) and the coincidence limits in Section 6.3.2. After a straightforward calculation one gets

\[ - \left\{ \text{tr}_V \left[ \int_0^1 d\tau_1 e^{-(1-\tau_1)\tilde{H}} \tilde{D}_\mu \tilde{a}^{\mu} \tilde{D}_\nu e^{-\tau_1\tilde{H}} \right] \right\}_{x=x'} = \]

\[ = \text{tr}_V (\Xi_0) + \lambda \text{tr}_V (\Xi_1) + \lambda^2 \text{tr}_V (\Xi_2) + O(\lambda^3), \quad (6.102) \]
where

\[
\Xi_0 = -\phi_{;\mu}^{\mu} - \phi_{;\mu}\phi^{\mu}, \quad (6.103)
\]

\[
\Xi_1 = -\sigma_{;\mu}^{\mu} - h^{\mu\nu}_{;\mu}\phi_{;\nu} - h^{\mu\nu}_{;\mu\nu} - 2\phi_{;\nu}^{\nu}\sigma_{;\nu} - \phi_{;\mu}h^{\mu\nu}\phi_{;\nu} \\
+ \frac{1}{2}h(\phi_{;\mu}^{\mu} + \phi_{;\mu}\phi^{\mu}) - \frac{1}{4}h^{\alpha\alpha}, \quad (6.104)
\]

\[
\Xi_2 = \frac{1}{8}hh^{\alpha\alpha} + \frac{1}{4}h^{\mu\nu}h^{\alpha\alpha}_{\mu\nu} + \frac{1}{12}h^{\mu\nu}_{;\mu} + \frac{1}{6}h^{\mu\nu}_{;\mu\nu}h^{\mu\nu,\rho} - \frac{1}{4}h^{\mu\nu}h_{;\mu}, \\
- \frac{1}{4}h_{;\muh;\nu + \frac{1}{2}h^{\mu\nu}_{;\mu}h_{;\nu} - h_{;\mu}h_{;\nu}^2\phi_{;\mu}^{\mu} - \frac{1}{4}h_{;\mu}\phi_{;\mu}^{\mu} \\
+ \frac{1}{2}h\sigma_{;\mu}^{\mu} + \frac{1}{2}h^{\mu\nu}\phi_{;\mu\nu} - \sigma_{;\mu}h^{\mu\nu}_{;\mu\nu} - \frac{1}{8}h^{2}\phi_{;\mu}^{\mu} - \frac{1}{4}h^{\mu\nu}_{;\mu\nu}h_{;\mu}\phi_{;\mu}^{\mu} \\
+ h\sigma_{;\mu}\phi_{;\mu}^{\mu} + \frac{1}{2}hh^{\mu\nu}\phi_{;\mu\nu} - \sigma_{;\mu}\phi_{;\mu}^{\mu} - 2h^{\mu\nu}\sigma_{;\nu}h_{;\mu\nu} \quad . \quad (6.105)
\]

In the notation of equation (6.52) we can write, now, the different contributions, in increasing order of \( \lambda \), to the coefficient \( \tilde{a}_1 \). In more details, by using the results (6.79), (6.95), (6.103) and recalling that

\[
\tilde{a}_1(\lambda) = a_1^{(0)}(\lambda) + \lambda a_1^{(1)}(\lambda) + \lambda^2 a_1^{(2)}(\lambda) + O(\lambda^3),
\]

we get

\[
a_1^{(0)} = \frac{1}{6}R - \phi_{;\mu}^{\mu} - \phi_{;\mu}\phi_{;\mu}^{\mu} - q . \quad (6.106)
\]

Moreover, by using (6.96) and (6.104) we obtain

\[
a_1^{(1)} = -\sigma_{;\mu}^{\mu} - h^{\mu\nu}_{;\mu}\phi_{;\nu} - h^{\mu\nu}_{;\mu\nu} - 2\phi_{;\nu}^{\nu}\sigma_{;\nu} - \phi_{;\mu}h^{\mu\nu}\phi_{;\nu} - \frac{1}{6}h^{\mu\nu}_{;\mu\nu} \\
+ \frac{1}{2}h(\phi_{;\mu}^{\mu} + \phi_{;\mu}\phi^{\mu}) - \frac{1}{12}h^{\alpha\alpha} + \frac{1}{12}hR. \quad (6.107)
\]

Finally, by combining the results in (6.79), (6.97) and (6.105), we have the following
expression for the term of order $\lambda^2$ in $\tilde{a}_1$, i.e.

$$ a_1^{(2)} = -h^{\mu\nu;\mu}\sigma;_{\nu} - h^{\mu\nu;\mu}_{;\nu} - \sigma;_{\nu} h^{\mu\nu,\sigma;_{\nu}} - \sigma;_{\nu} h^{\mu\nu,\phi;_{\nu}} - \phi;_{\nu} h^{\mu\nu,\sigma;_{\nu}} - \frac{1}{12} h^{\mu\nu;\mu} h_{\mu\nu} 
\quad + \frac{1}{2} \sigma;_{\nu} h^{\mu\nu;\phi;_{\nu}} + \frac{1}{2} h h^{\mu\nu;\phi;_{\nu}} h^{\mu\nu,\sigma;_{\nu}} + \frac{1}{12} h h^{\mu\nu,\phi;_{\nu}} h^{\mu\nu,\sigma;_{\nu}} - \frac{1}{12} h h^{\mu\nu,\phi;_{\nu}} h^{\mu\nu,\sigma;_{\nu}} 
\quad + \frac{1}{12} h h^{\mu\nu,\phi;_{\nu}} h^{\mu\nu,\sigma;_{\nu}} + \frac{1}{24} h h^{\mu\nu,\phi;_{\nu}} h^{\mu\nu,\sigma;_{\nu}} + \frac{1}{24} h h^{\mu\nu,\phi;_{\nu}} h^{\mu\nu,\sigma;_{\nu}} 
\quad - \frac{1}{8} h^2 q - \frac{1}{8} h^2 \phi;_{\nu} h^{\mu\nu,\phi;_{\nu}} - \frac{1}{8} h^2 \phi;_{\nu} h^{\mu\nu,\phi;_{\nu}} - \frac{1}{4} h h^{\mu\nu,\phi;_{\nu}} h^{\mu\nu,\phi;_{\nu}} - \frac{1}{4} h h^{\mu\nu,\phi;_{\nu}} h^{\mu\nu,\phi;_{\nu}} 
\quad - \frac{1}{4} h h^{\mu\nu,\phi;_{\nu}} h^{\mu\nu,\phi;_{\nu}} + \frac{1}{48} h h^{\mu\nu,\phi;_{\nu}} h^{\mu\nu,\phi;_{\nu}} + \frac{1}{48} h h^{\mu\nu,\phi;_{\nu}} h^{\mu\nu,\phi;_{\nu}} 
\quad + \frac{1}{24} h h^{\mu\nu,\phi;_{\nu}} h^{\mu\nu,\phi;_{\nu}} + \frac{1}{2} h^2 R . \quad (6.108) $$

### 6.3 Construction of the Action

In order to write the action of Spectral Matrix Gravity, we need to evaluate the global heat kernel coefficients $A_0$ and $A_1$. As we already mentioned above, the coefficients $A_k$ are expressed in terms of integrals of the local heat kernel coefficients $a_k$ (which are densities) or the coefficients $\tilde{a}_k$ (which are scalars). By using the equation (6.58), we get

$$ A_0 = \int_M dx g^{\frac{1}{2}} \text{tr} \left[ 1 + \frac{\lambda^2}{8} ( h^2 + 2 h_{\mu\nu} h^{\mu\nu} ) \right] + O(\lambda^3) . \quad (6.109) $$

Now we use the equations (6.52) and (6.106)-(6.108) to compute the coefficient $A_1$. By integrating by parts and by noticing that the trace of a commutator...
of any two matrices vanishes, up to terms of order $\lambda^2$, we obtain

$$A_1 = \int_M dx \, g^{\frac{1}{2}} \text{tr}_V \left( - q - \phi^{\mu \nu} \phi_{,\mu} + \frac{1}{6} R + \lambda^2 \left( - \sigma^{\mu \nu} \sigma_{,\mu} + \frac{1}{2} h \Theta ight) ight. $$

$$- \frac{1}{2} h^{\mu \nu} \sigma_{,\mu} + \frac{1}{2} h h^{\mu \nu} \phi_{,\mu} \phi_{,\nu} + h \sigma^{\mu \nu} \phi_{,\mu} - \frac{1}{2} h_{,\mu} h^{\mu \nu} \phi_{,\nu} - 2 \sigma_{,\mu} h^{\mu \nu} \phi_{,\nu}$$

$$- \frac{1}{12} h h^{\mu \nu} R_{\mu \nu} - \frac{1}{12} h^{\mu \nu} h^{\rho \sigma} R_{\mu \nu} - \frac{1}{2} h^{\mu \nu} h^{\rho \sigma} h_{\mu \nu} + \frac{1}{12} h^{\mu \nu} h_{\rho \sigma} h^{\rho \sigma} + \frac{1}{48} h^2 R $$

$$+ \frac{1}{24} h^{\mu \nu} h_{\mu \nu} R - \frac{1}{24} h_{,\mu} h^{\mu \nu} - \frac{1}{24} h^{\mu \nu} h_{\mu \nu} + \frac{1}{12} h^{\mu \nu} h^{\rho \sigma} h_{\mu \nu} - \frac{1}{8} h^2 q - \frac{1}{8} h^2 \phi_{,\mu} : \mu$$

$$- \frac{1}{8} h^2 \phi_{,\mu} \phi^{\mu} - \frac{1}{4} h^{\mu \nu} h_{\mu \nu} q - \frac{1}{4} h^{\mu \nu} h_{\mu \nu} \phi_{,\mu} : \rho - \frac{1}{4} h^{\mu \nu} h_{\mu \nu} \phi_{,\mu} \phi^{\mu} ) \right)$$

$$+ O(\lambda^3). \quad (6.110)$$

The invariant action functional is written as linear combination of the coefficients $A_0$ and $A_1$ as shown in (2.185)

$$S = \frac{1}{16\pi G} \int_M dx \, g^{\frac{1}{2}} \left( - 6 q - 6 \phi^{\mu \nu} \phi_{,\mu} + R - 2 \Lambda ight)$$

$$+ \frac{\lambda^2}{N} \text{tr}_V \left( - 6 \sigma^{\mu \nu} \sigma_{,\mu} - 3 h^{\mu \nu} \sigma_{,\mu} + 3 h \Theta + 3 h h^{\mu \nu} \phi_{,\mu} \phi_{,\nu} + 6 h \sigma^{\mu \nu} \phi_{,\mu} ight.$$

$$- 3 h^{\mu \nu} h_{,\mu} \phi_{,\nu} - 12 \sigma_{,\mu} h^{\mu \nu} \phi_{,\nu} - \frac{1}{2} h h^{\mu \nu} R_{\mu \nu} + \frac{1}{2} h^{\mu \nu} h^{\rho \sigma} R_{\rho \sigma} - \frac{1}{2} h^{\mu \nu} h^{\rho \sigma} R_{\rho \sigma}$$

$$- \frac{1}{2} h^{\mu \nu} h_{,\mu} \phi_{,\nu} + \frac{1}{2} h^{\mu \nu} h_{,\mu} \phi_{,\nu} - \frac{1}{8} h^2 R + \frac{1}{4} h^{\mu \nu} h_{\mu \nu} - \frac{1}{8} h_{,\mu} h^{\mu \nu} - \frac{1}{4} h^{\mu \nu} h_{\mu \nu} \phi_{,\rho} \phi^{\rho}$$

$$- \frac{3}{4} h^2 q - \frac{3}{4} h^2 \phi_{,\mu} : \mu - \frac{3}{4} h^2 \phi_{,\mu} : \mu - \frac{3}{2} h^{\mu \nu} h_{\mu \nu} q - \frac{3}{2} h^{\mu \nu} h_{\mu \nu} \phi_{,\rho} \phi^{\rho}$$

$$- \frac{3}{2} h^{\mu \nu} h_{\mu \nu} \phi_{,\mu} \phi^{\mu} - \frac{\Lambda}{4} h^2 - \frac{\Lambda}{2} h_{\mu \nu} h^{\mu \nu} \right) \right)$$

$$+ O(\lambda^3). \quad (6.111)$$

Obviously, the action functional that we obtained is invariant under the diffeomorphisms and the gauge transformation $h^{\mu \nu} \rightarrow U h^{\mu \nu} U^{-1}$. 
The next task is to find the equations of motion for the fields $\sigma$, $h_{\mu\nu}$ and $\phi$ by varying the action functional. In this way we will explicitly find the non-commutative corrections to Einstein’s equations.

6.4 The Equations of Motion

By performing the variation with respect to the field $\sigma$, we obtain the equation

$$4\Delta\sigma + \Delta h - 2h\Delta\phi - 2h^\nu\phi_{;\nu} + 4h_{\mu\nu}^\nu\phi_{;\mu} + 4h_{\mu\nu}^\mu\phi_{;\nu} + O(\lambda^3) = 0.$$  (6.112)

Here $\Delta$ is the Laplacian in the Euclidean case and the D’Alambertian in the pseudo-Euclidean case. For the matrix-valued field $h_{\mu\nu}$ we obtain the equation

$$g^{\mu\nu}\Delta\sigma + g^{\mu\nu}\Theta + h\phi_{;\nu}^\nu\phi_{;\mu} + g_{\mu\nu}^\nu h_{;\mu}^\nu\phi_{;\nu} + 2g_{\mu\nu}^\nu \sigma_{;\nu}^\nu \phi_{;\mu}$$

$$- h^{\mu\nu}(\phi_{;\nu}^\nu) - 4g^{\mu\nu}h_{;\mu}^\nu + g_{\mu\nu}^\nu h_{;\mu}^\nu + g^{\mu\nu}h_{;\mu}^\nu + \frac{1}{6}g_{\mu\nu}^\nu R_{\rho\sigma} - \frac{1}{6}g_{\mu\nu}^\nu R_{\rho\sigma} - \frac{1}{6}h_{\mu\nu}^\nu$$

$$+ \frac{1}{12}g_{\mu\nu}^\nu h R + \frac{1}{6}h_{\mu\nu}^\nu R + \frac{1}{12}g_{\mu\nu}^\nu \Delta h + \frac{1}{6}\Delta h_{\mu\nu} - \frac{1}{6}g_{\mu\nu}^\nu h q - \frac{1}{2}g_{\mu\nu}^\nu h \Delta\phi$$

$$- \frac{1}{2}g_{\mu\nu}^\nu h\phi_{;\nu}^\nu - h_{\mu\nu}^\nu q - h_{\mu\nu}^\nu \Delta\phi - h_{\mu\nu}^\nu \phi_{;\mu}^\nu - \frac{2}{3}g_{\mu\nu}^\nu h - \frac{2}{3}g_{\mu\nu}^\nu h + O(\lambda^3) = 0.$$  (6.113)

The variation of the action with respect to the scalar field $\phi$ yields

$$4\Delta\phi = -\frac{\lambda^2}{N} \text{tr}(2hh_{\mu\nu}^\nu\phi_{;\mu\nu} - 2h_{\mu\nu}^\nu h_{;\mu}^\nu\phi_{;\nu} - 2hh_{\mu\nu}^\nu\phi_{;\nu}$$

$$- 2h_{;\nu}^\nu\phi_{;\nu} + 2h_{;\nu}^\nu h_{;\mu}^\nu + h_{;\nu}^\nu h_{;\mu}^\nu + 4\sigma_{;\mu\nu}^\nu h_{;\mu}^\nu + 4\sigma_{;\mu\nu}^\nu h_{;\mu}^\nu$$

$$- \frac{1}{2}h\Delta h - \frac{1}{2}h_{;\mu}^\nu h_{;\nu}^\mu + hh_{;\mu}^\nu\phi_{;\mu}^\mu + \frac{1}{2}h\Delta\phi - h_{;\mu\nu}^\nu h_{;\mu\nu} - h_{;\mu\nu}^\nu h_{;\mu\nu}$$

$$+ 2h_{\mu\nu}^\nu h_{;\nu}^\mu\phi_{;\mu} + h_{;\mu\nu}^\nu h_{;\mu\nu} \Delta\phi) + O(\lambda^3).$$  (6.114)
The equation of motion for the field $g^{\mu\nu}$ can be written in the following form

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} = T^{\mu\nu} + \frac{\lambda^2}{N} \text{tr}_V \mathcal{A}^{\mu\nu}.$$  \hfill (6.115)

Here the tensor $T^{\mu\nu}$ is

$$T^{\mu\nu} = 6 \phi_{;\mu}^{;\nu} - 3 g^{\mu\nu} \phi_{;\rho}^{;\rho} - 3 q g^{\mu\nu},$$  \hfill (6.116)

which represents the stress-energy tensor for a massless scalar field. The tensor $\mathcal{A}^{\mu\nu}$ represents, instead, the stress-energy tensor for the fields $h^{\mu\nu}$ and $\sigma$.

The equation (6.115) is the main result of this chapter. As we can see, the new fields of our model, $h^{\mu\nu}$ and $\sigma$, contribute to modify the standard Einstein equations. More precisely they contribute to an additional term in the stress-energy tensor.

The tensor $\mathcal{A}^{\mu\nu}$ can be written as the sum of six terms:

$$\mathcal{A}^{\mu\nu} = A^{\mu\nu}_1 + A^{\mu\nu}_2 + A^{\mu\nu}_3 + A^{\mu\nu}_4 + A^{\mu\nu}_5 + A^{\mu\nu}_6.$$  \hfill (6.117)

In the first term we have only derivatives of the field $\sigma$

$$A^{\mu\nu}_1 = 6 \sigma_{;\mu}^{;\nu} - 3 g^{\mu\nu} \sigma_{;\rho}^{;\rho} - 6 h \sigma_{;\mu}^{;\nu} + 3 \sigma_{;\mu}^{;\mu} h_{;\nu} + 6 h^{\mu\nu} \sigma_{;\rho}^{;\rho} \phi_{;\mu}^{;\rho}$$

$$+ 3 \sigma_{;\rho}^{;\mu} h^{\rho\nu} + 3 g^{\mu\nu} \left( h \sigma_{;\rho}^{;\rho} \phi_{;\mu}^{;\rho} - \frac{1}{2} \sigma_{;\rho}^{;\mu} h_{;\nu} - 2 \sigma_{;\mu}^{;\rho} \phi_{;\nu} h^{\rho\sigma} \right).$$  \hfill (6.118)

The second term only contains derivatives of the scalar field $\phi$, namely

$$A^{\mu\nu}_2 = 3 h^{\rho\tau} \phi_{;\rho}^{;\tau} (h^{\mu\nu} + \frac{1}{2} g^{\mu\nu} h) + 3 \phi_{;\tau}^{;\mu} (h^{\rho\tau} \phi_{;\rho}^{;\nu} - \frac{1}{2} g^{\rho\tau} h^{\rho\nu} h_{;\rho})$$

$$- \frac{3}{2} \left( \phi_{;\rho}^{;\mu} + \phi_{;\mu}^{;\rho} \right) \left( 2 h^{\mu\nu} h_{;\rho}^{;\nu} + h h^{\mu\nu} + \frac{1}{4} h^2 g^{\mu\nu} + \frac{1}{2} g^{\mu\nu} h_{;\rho}^{;\nu} h_{;\rho} \right)$$

$$+ 3 h^{\rho\tau} h^{\nu\rho} \phi_{;\rho}^{;\rho} + \frac{3}{2} \phi_{;\mu}^{;\mu} + \phi_{;\nu}^{;\nu} \left( h_{;\rho}^{\rho} h_{;\sigma}^{\sigma} + \frac{1}{2} h^2 \right).$$  \hfill (6.119)
The third term only contains second derivatives of the matrix-valued tensor field \( h^{\mu\nu} \),

\[
\mathcal{G}_{(3)}^{\mu\nu} = \left. h^{\alpha\mu} \right|_{(\alpha \nu \rho \sigma)} + \frac{1}{2} h^{\alpha\mu} \left. \alpha \nu \rho \sigma \right|_{\nu \rho \sigma} - \frac{1}{2} h h^{\alpha\mu} \left. \rho \sigma \right|_{\alpha \nu} + \frac{1}{4} g^{\mu\nu} h^{\rho\sigma} \left. \rho \sigma \right|_{\nu \rho \sigma} \\
+ \frac{1}{2} h^{\alpha\mu} \left. \rho \sigma \right|_{\nu \rho \sigma} - \frac{1}{2} h h^{\alpha\mu} \left. \rho \sigma \right|_{\nu \rho \sigma} h_{\rho \sigma}^{\nu \rho \sigma} + h_{\rho \sigma}^{\nu \rho \sigma} h_{\rho \sigma}^{\nu \rho \sigma} + \frac{1}{2} h h^{\alpha\mu} \left. \rho \sigma \right|_{\nu \rho \sigma} + \frac{1}{2} h h^{\alpha\mu} \left. \rho \sigma \right|_{\nu \rho \sigma}.
\]  
(6.120)

The fourth term contains only first derivatives of \( h^{\mu\nu} \), namely

\[
\mathcal{G}_{(4)}^{\mu\nu} = -\frac{1}{2} h^{\alpha\mu} \left. \rho \sigma \right|_{\nu \rho \sigma} + \frac{1}{4} h h^{\alpha\mu} \left. \rho \sigma \right|_{\nu \rho \sigma} + \frac{1}{2} h h^{\alpha\mu} \left. \rho \sigma \right|_{\nu \rho \sigma} + \frac{1}{2} h h^{\alpha\mu} \left. \rho \sigma \right|_{\nu \rho \sigma}.
\]  
(6.121)

The fifth coefficient contains only first and second derivatives of \( h \)

\[
\mathcal{G}_{(5)}^{\mu\nu} = \frac{1}{4} h h^{\alpha\mu} \left. \rho \sigma \right|_{\nu \rho \sigma} + \frac{1}{4} h h^{\alpha\mu} \left. \rho \sigma \right|_{\nu \rho \sigma} + \frac{3}{8} h h^{\alpha\mu} \left. \rho \sigma \right|_{\nu \rho \sigma}.
\]  
(6.122)

The last term, \( \mathcal{R}_{(6)}^{\mu\nu} \), does not contain any derivative of \( h^{\mu\nu} \), namely

\[
\mathcal{R}_{(6)}^{\mu\nu} = 3 \Theta \left( h^{\mu\nu} + \frac{1}{2} g^{\mu\nu} h \right) - \frac{\Lambda}{2} \left[ \left( h^{\mu\nu} + \frac{1}{2} g^{\mu\nu} h \right) h + 2 h h^{\alpha\mu} h^{\rho\sigma} \right.
\]  
\[
- \frac{1}{2} R_{\alpha\beta} \left[ h^{\alpha\nu} h^{\beta\mu} + h^{\alpha\mu} h^{\beta\nu} + \frac{1}{2} h^{\alpha\nu} h^{\beta\mu} + \frac{1}{2} h^{\alpha\mu} h^{\beta\nu} + h^{\alpha\mu} h^{\beta\nu} \right] + h h^{\alpha\mu} h^{\beta\nu} + \frac{1}{4} g^{\mu\nu} h h^{\alpha\mu} h^{\beta\nu} R_{\alpha\beta\alpha\beta}
\]  
\[
+ \frac{1}{4} (R - 6) \left( g^{\mu\nu} h h^{\alpha\mu} h^{\beta\nu} + 4 h h^{\alpha\mu} h^{\beta\nu} \right) + \frac{1}{16} (R - 6) \left( g^{\mu\nu} h^2 + 4 h h^{\mu\nu} \right).
\]  
(6.123)
The dynamics described by the equations (6.112), (6.113), (6.114) and (6.115) can be studied by using an iterative method. Let us write the solution for the background fields $\phi$ and $g^{\mu\nu}$ as Taylor expansion in the deformation parameter $\lambda$ as follows

\[
\phi = \phi_0 + \lambda \phi_1 + \lambda^2 \phi_2 + O(\lambda^3),
\]
\[
g^{\mu\nu} = g^{\mu\nu}_0 + \lambda g^{\mu\nu}_1 + \lambda^2 g^{\mu\nu}_2 + O(\lambda^3).
\] (6.124)

By substituting these expressions in equations (6.114) and (6.115) we obtain, for the terms of order $\lambda^0$, the dynamical equations

\[
\Delta \phi = 0,
\]
\[
R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + \Lambda g^{\mu\nu} = 0.
\] (6.125)

As we can see from the last equations the term $g^{\mu\nu}_0$ is nothing but the solution of the ordinary Einstein equation in vacuum with cosmological constant. By substituting the solutions to (6.125) back into the equations of motion for the fields $\sigma$ and $h^{\mu\nu}$ we get equations of the form

\[
\Phi_1(g^{\mu\nu}_0, \phi_0)\sigma = O(\lambda^2),
\]
\[
\Phi_2(g^{\mu\nu}_0, \phi_0)h^{\mu\nu} = O(\lambda^2),
\] (6.126)

where $\Phi_1(g^{\mu\nu}_0, \phi_0)$ and $\Phi_2(g^{\mu\nu}_0, \phi_0)$ are linear second order partial differential operators. By iterating this process we can, in principle, find the solution to our dynamical equations in form of a Taylor series in $\lambda$.

### 6.5 Spectrum of Matrix Gravity on De Sitter Space

The action for Matrix Gravity obtained in the previous section is a functional of the fields $\phi$, $\sigma$, $h^{\mu\nu}$ and $g^{\mu\nu}$. The dynamics is described by a system of
non-linear partial differential equations coupled with each other. We analyze, now, the dynamics of the theory. For simplicity we will set, from now on, \( Q = 0 \). This particular value for the matrix-valued scalar \( Q \) will not affect our analysis.

As already mentioned above, from the equation of motion (6.114) for the field \( \phi \), we can see that to the zeroth order in the deformation parameter \( \lambda \) the field \( \phi \) satisfies the following equation

\[
\Delta \phi = 0 . \tag{6.127}
\]

As it is well known, the solution of the last equation represents a wave propagating in the whole space. Since we require that \( \phi \) vanishes at infinity, the only solution is \( \phi = O(\lambda^2) \) in the whole space.

With this solution for the field \( \phi \), the matrix-valued function \( \rho \) defined in (6.37) becomes

\[
\rho = e^{i\sigma} . \tag{6.128}
\]

A deeper analysis shows that the matrix-valued scalar field \( \sigma \) is not an independent field. Following [20, 21] the general form of \( \rho \) can be written as

\[
\rho = \omega^{-\frac{1}{4}} , \tag{6.129}
\]

where

\[
\omega = -\frac{1}{m!} \varepsilon_{\mu_1...\mu_m} \varepsilon_{\nu_1...\nu_m} a^{\mu_1\nu_1} \cdots a^{\mu_m\nu_m} . \tag{6.130}
\]

By using the decomposition (6.35) of \( h^{\mu\nu} \) in equation (6.130) we get the following formula, up to the term linear in the deformation parameter \( \lambda \),

\[
\sigma = -\frac{1}{4} h + \frac{\lambda}{8} h_{\mu\nu} h^{\mu\nu} + O(\lambda^2) . \tag{6.131}
\]
We can write down, now, the action by imposing the constraints $\phi = O(\lambda^2)$ and (6.131). The final result is the following

$$\begin{align*}
S &= \frac{1}{16\pi G} \int_M dx \frac{1}{g} \left\{ -6\phi^\mu_{\ ;\mu} + \frac{\lambda^2}{N} \text{tr} \left[ \frac{1}{4} g^{\mu\nu} h_{\ ;\mu} h_{\ ;\nu} \right] \right. \\
&\quad \left. - \frac{1}{2} h h^\mu_{\ ;\nu} R_{\mu\nu} + \frac{1}{2} h h^\mu_{\ ;\rho} R_{\mu\sigma\rho\nu} - \frac{1}{2} h h^\mu_{\ ;\mu} R_{\mu\nu} - \frac{1}{2} h h^\mu_{\ ;\mu} R_{\mu\nu} \right. \\
&\quad \left. + \frac{1}{2} h h^\mu_{\ ;\mu} R_{\mu\nu} + \frac{1}{8} h^2 R + \frac{1}{4} h h^\mu_{\ ;\mu} R - \frac{1}{4} h h^\mu_{\ ;\mu} R_{\mu\nu} \right. \\
&\quad \left. - \frac{\Lambda}{4} h^2 - \frac{\Lambda}{2} h h^\mu_{\ ;\mu} \right\} + O(\lambda^3). \quad (6.132)
\end{align*}$$

The action depends, now, only on the independent tensor fields $g^{\mu\nu}$, $h^{\mu\nu}$ and the scalar field $\phi$. Therefore, we will have only two equations that describe the dynamics of the theory. These dynamical equations can be easily derived from the ones given in the previous section by imposing the conditions (6.131) and $\phi = O(\lambda^2)$.

The action (6.132) and the equations of motions for the fields evaluated in the previous section, assume a simple form on maximally symmetric background geometries. As we mentioned in the previous section, the $\lambda^0$ term of the background field $g_{\mu\nu}^0$ is solution of the Einstein equations in vacuum with cosmological constant (6.125). In this section we consider the De Sitter solution to the equation (6.125). In this maximally symmetric case the Ricci and Riemann tensors take the following form

$$R^\mu_{\ ;\nu\alpha\beta} = \frac{1}{n(n-1)} (\delta^\mu_{\ ;\alpha} g_{\nu\beta} - \delta^\mu_{\ ;\beta} g_{\nu\alpha}) R \quad \text{and} \quad R_{\mu\nu} = \frac{1}{n} g_{\mu\nu} R. \quad (6.133)$$

The De Sitter metric gives a solution of the classical equations provided

$$R = \frac{2n}{n-2} \Lambda. \quad (6.134)$$
Here, and below, we restrict ourselves to the case $n > 2$. By substituting the expressions in equation (6.133) in the action (6.132), we find a form of the action functional valid in De Sitter geometry, namely

$$S = \frac{1}{16\pi G} \int_M dx \, g^{\frac{3}{2}} \left\{ -6\phi \phi_{,\mu} + R - 2\Lambda \right. \\
+ \left. \frac{\Lambda^2}{N} \text{tr} \left[ \frac{1}{4} h(-\Delta + \mu_1)h - \frac{1}{4} h_{\mu\nu}(-\Delta + \mu_2)h^{\mu\nu} + \frac{1}{2} h^{\mu\nu} ; h_{\mu\nu} ;^\rho - \frac{1}{2} h^{\mu\nu} ; h_{;\mu\nu} \right] \right\} + O(\Lambda^3),$$

(6.135)

where the terms $\mu_1$ and $\mu_2$ are defined as follows

$$\mu_1 = \frac{n^2 - 5n + 8}{2n(n-1)} R - \Lambda,$$

(6.136)

$$\mu_2 = -\frac{n - 3}{n - 1} R + 2\Lambda.$$

(6.137)

It is interesting, at this point of the discussion, to derive explicitly the spectrum of the theory. In order to achieve this result we need to decompose the field $h_{\mu\nu}$ in its irreducible modes: traceless transverse tensor mode, transverse vector mode, scalar mode and trace part. In other words we can write $h_{\mu\nu}$ as

$$h_{\mu\nu} = \tilde{h}_{\mu\nu}^\perp + \frac{1}{n} g_{\mu\nu} \varphi + 2\zeta_\perp^{\mu\nu} + \psi_{;\mu\nu},$$

(6.138)

where the scalar field $\varphi$ is defined as follows

$$\varphi = h - \Delta \psi,$$

and the fields $\tilde{h}_{\mu\nu}^\perp$ and $\zeta_\perp$ satisfy the conditions

$$\nabla^\mu \tilde{h}_{\mu\nu}^\perp = 0, \quad g^{\mu\nu} \tilde{h}_{\mu\nu}^\perp = 0, \quad \nabla^\mu \zeta_\perp = 0.$$
We can now substitute the expression (6.136) in the action (6.135), and evaluate the terms separately. Explicitly we obtain

\[
\int_M d^4x \, g^{\frac{1}{2}} \, h^{\mu\nu} \, \eta_{\mu\nu}^i \, ;^i = \int_M d^4x \, g^{\frac{1}{2}} \left[ -\frac{1}{n^2} \varphi \Delta \varphi - \frac{2}{n} \varphi \left( \Delta + \frac{R}{n} \right) \Delta \psi \\
+ \zeta_{\mu} \left( \Delta + \frac{R}{n} \right) \zeta^{+\mu} - \psi \left( \Delta + \frac{R}{n} \right) \Delta \psi \right], \quad (6.140)
\]

\[
\int_M d^4x \, g^{\frac{1}{2}} \, h^{\mu\nu} ;^i h_{\mu\nu}^i = \int_M d^4x \, g^{\frac{1}{2}} \left[ -\frac{1}{n} \varphi \Delta \varphi - \left( \frac{n+1}{n} \right) \varphi \left( \Delta + \frac{R}{n+1} \right) \Delta \psi \\
- \psi \left( \Delta + \frac{R}{n} \right) \Delta^2 \psi \right], \quad (6.141)
\]

\[
\int_M d^4x \, g^{\frac{1}{2}} \, h^{\mu\nu} \left( -\Delta + \mu_2 \right) h^{\mu\nu} \\
= \int_M d^4x \, g^{\frac{1}{2}} \left( h^{\mu\nu} \left( -\Delta + \mu_2 \right) h^{\mu\nu} + \frac{1}{n} \varphi \left( -\Delta + \mu_2 \right) \varphi \\
+ \frac{2}{n} \varphi \left( -\Delta + \mu_2 \right) \Delta \psi + 2 \zeta_{\mu} \left( \Delta + \frac{R}{n} \right) \left( \Delta - \mu_2 + \frac{n+1}{n(n-1)} R \right) \zeta^{+\mu} \\
- \psi \left[ \Delta^2 + \left( \frac{3R}{n} - \mu_2 \right) \Delta + \frac{R}{n} \left( \frac{2R}{n-1} - \mu_2 \right) \right] \Delta \psi \right], \quad (6.142)
\]

and finally

\[
\int_M d^4x \, g^{\frac{1}{2}} \, h \left( -\Delta + \mu_1 \right) h \\
= \int_M d^4x \, g^{\frac{1}{2}} \left[ \varphi \left( -\Delta + \mu_1 \right) \varphi + 2 \varphi \left( -\Delta + \mu_1 \right) \Delta \psi + \psi \left( -\Delta + \mu_1 \right) \Delta^2 \psi \right]. \quad (6.143)
\]

By using the decompositions (6.140) through (6.143) we rewrite the action
in terms of the irreducible modes of $h^{\mu\nu}$, namely

$$S = \frac{1}{16\pi G} \int_M dx g^{\frac{1}{2}} \left\{ -6\phi^{\mu\nu} + R - 2\Lambda + \frac{\lambda^2}{N} \text{tr} \left[ -\frac{1}{4} \hbar^{\perp}_{\mu\nu}(-\Delta + \mu_2)\hbar^{\perp}_{\mu\nu} \right] \ight. \left\} + O(\lambda^3) \right. \right].$$

It is straightforward to show now that on the mass shell, (6.134), the terms containing the fields $\zeta^{\perp}_{\mu}$ and $\psi$ vanish identically. More precisely we obtain the following form for the on-shell action functional

$$S \bigg|_{\text{on--shell}} = \frac{1}{16\pi G} \int_M dx g^{\frac{1}{2}} \left\{ \frac{4\Lambda}{n-2} + \frac{\lambda^2}{N} \text{tr} \left[ -\frac{1}{4} \hbar^{\perp}_{\mu\nu}(-\Delta + \frac{4\Lambda}{(n-1)(n-2)})\hbar^{\perp}_{\mu\nu} \right] \right. \left\} + O(\lambda^3) \right. \right].$$

Thus on the mass shell the only remaining fields are the (traceless) matrix-valued traceless transverse tensor $\hbar^{\perp}_{\mu\nu}$ and the (traceless) matrix-valued scalar field $\varphi$. This action looks exactly the same as in General Relativity, the only difference being that the fields are matrix-valued and traceless. Therefore, it describes $(N-1)$ spin-2 particles and $(N-1)$ spin-0 particles. Note also, that exactly as in General Relativity, the scalar conformal mode is unstable if the cosmological constant $\Lambda$ is assumed to be positive.
6.6 Concluding Remarks

In this chapter we studied a non-commutative deformation of General Relativity (called Spectral Matrix Gravity) proposed in [21] where the non-commutative limit has been explicitly evaluated. The approach of the paper [21] to construct the action for Matrix Gravity differs from the one proposed in [20, 19, 49]. In the latter the action of our model was a straightforward generalization of the Hilbert-Einstein action in which the measure and the scalar curvature were matrix-valued quantities. This last approach seems to have some intrinsic arbitrariness due to the freedom of choosing the particular form of the matrix-valued measure (for a discussion see [49]). In order to avoid these issues, in [21] the action of Matrix Gravity was defined as a linear combination of the first two global heat kernel coefficients (2.185) of a non-Laplace type partial differential operator.

By using the covariant Fourier transform method we were able to evaluate the coefficients $A_0$ and $A_1$, and as a result, the action functional within the perturbation theory in the deformation parameter $\lambda$. The main result of this chapter is the derivation of the modified Einstein equations in (6.115) in the weak deformation limit. In this case the pure non-commutative fields, namely $h^{\mu\nu}$ and $\sigma$, contribute to the right-hand side of the Einstein equation, that is, the stress-energy tensor. The explicit form of these non-commutative correction terms has been derived in (6.118) through (6.123) for the first time.

Some of the physical implications of Matrix Gravity have been extensively discussed in [19, 20, 21]. This theory exhibits non-geodesic motion which can be related to a violation of the equivalence principle, moreover, because of the new gauge symmetry, there are new physical conserved charges. At last, this theory
represents a consistent model of interacting spin-2 particles on curved space which usually was a problem. An interesting question is the limit as \( N \to \infty \) of our model, this might be related to matrix models and string theory.

As it is outlined in the introduction Matrix Gravity can be considered as a Gravitational Chromodynamics describing the gravitational interaction of a new degree of freedom that we call *gravitational color*. Whether or not it is related to the color of QCD is an open question. Let’s suppose for simplicity that it is the same, and that the gauge group of Matrix Gravity is nothing but \( SU(3) \). If one pushes this analogy with QCD to its logical limit then this would mean that the theory predicts that the *gravitational interaction of quarks depends on their colors*. Exactly as in QCD the strong interaction between quark of color \( i \) and a quark of color \( j \) is transmitted by gluon of type \((ij)\), the gravitational interaction between quark of color \( i \) and a quark of color \( j \) is transmitted by the graviton of type \((ij)\). In this case, all particles in the electro-weak sector, including photon, do not feel the gravitational color. In that sense it is ‘dark’. Notice that in the non-relativistic limit the Newtonian potential will also become ‘matrix-valued’. One can go even further. Since the usual (white) mass is determined by the sum of the color masses, one can assume that the color masses can be even negative. Then the gravitational interaction of such particles would include non only attractive forces but also repellent forces (antigravity?). This feature could then solve the mystery of singularities in General Relativity.

The consequences of our model in the ambit of cosmology are easily seen by inspecting equations (6.115) and (6.116). The deformation of the energy-momentum tensor in (6.116) is written only in terms of the non-commutative part...
$h^{\mu\nu}$. It would be interesting to study whether or not $h^{\mu\nu}$ could account for a field of negative pressure. If so, our model could describe the dynamics of dark energy. Furthermore, the distortion of the gravitation expansion of the universe due to the non-commutative degrees of freedom of the gravitational field will certainly have some effect on the anisotropy of the cosmic background radiation, nucleosynthesis and structure formation. Of course, a detailed analysis of these effects requires a careful study of the fluctuations in the early universe.

Of course the validity of these statements requires further investigations. The ultimate goal of this theory is to construct a consistent theory of the gravitational field which is compatible with the Standard Model and able to solve the current open issues which afflict General Relativity, i.e. the problems of the origin of dark matter and dark energy, the recent anomalies found in the solar system (Pioneer Anomaly, flyby anomaly, etc.) and last, but not least, the problem of quantization of the gravitational field.

In summary, we would like to stress that our model makes it possible to make a number of very specific predictions that can serve as experimental tests of the theory.
CHAPTER 7

KINEMATICS IN MATRIX GRAVITY

Abstract

We develop the kinematics in Matrix Gravity, which is a modified theory of gravity obtained by a non-commutative deformation of General Relativity. In this model the usual interpretation of gravity as Riemannian geometry is replaced by a new kind of geometry, which is equivalent to a collection of Finsler geometries with several Finsler metrics depending both on the position and on the velocity. As a result the Riemannian geodesic flow is replaced by a collection of Finsler flows. This naturally leads to a model in which a particle is described by several mass parameters. If these mass parameters are different then the equivalence principle is violated. In the non-relativistic limit this also leads to corrections to the Newton’s gravitational potential. We find the first and second order corrections to the usual Riemannian geodesic flow and evaluate the anomalous nongeodesic acceleration in a particular case of static spherically symmetric background.

5The material in this chapter has been published in General Relativity and Gravitation: I. G. Avramidi and G. Fucci, Kinematics in Matrix Gravity, Gen. Rel. Grav. (2008) DOI 10.1007/s10714-008-0713-6
7.1 Introduction

In this chapter we investigate the motion of test particles in an extended theory of gravity, called Matrix Gravity, proposed in a series of recent papers \[19, 20, 21\] and presented in Chapter 5. The main goal of the present chapter is to investigate the motion of test particles in a simple model of Matrix Gravity and study the non-geodesic corrections to General Relativity.

The outline is as follows. In Sect. 2. we develop the kinematics in Matrix Gravity. In Sect. 3. we compute the first and second order non-commutative corrections to the usual Riemannian geodesic flow. In Sect. 4 we find a static spherically symmetric solution of the dynamical equations of Matrix Gravity in a particular case of commutative $2 \times 2$ matrices. In Sect. 5 we evaluate the anomalous acceleration of test particles in this background. In Sect. 6 we discuss our results.

7.2 Kinematics in Matrix Gravity

7.2.1 Riemannian Geometry

Let us recall how the geodesic motion appears in General Relativity, that is, in Riemannian geometry (for more details, see \[20\]). First of all, let

\[
F(x, \xi) = \sqrt{-|\xi|^2},
\]

(7.1)

where $\xi_\mu$ is a non-vanishing cotangent vector at the point $x$, and $|\xi|^2 = g^{\mu\nu}(x)\xi_\mu\xi_\nu$ (recall that the signature of our metric is $(- + \cdots +)$). Obviously, this is a homogeneous function of $\xi$ of degree 1, that is,

\[
F(x, \lambda \xi) = \lambda F(x, \xi).
\]

(7.2)
Let
\[ H(x, \xi) = -\frac{1}{2}F^2(x, \xi) = \frac{1}{2}|\xi|^2. \] (7.3)
This is, of course, a homogeneous polynomial of \( \xi_\mu \) of order 2, and, therefore, the Riemannian metric can be recovered by
\[ g^{\mu\nu}(x) = \frac{\partial^2}{\partial \xi_\mu \partial \xi_\nu} H(x, \xi). \] (7.4)
Now, let us consider a Hamiltonian system with the Hamiltonian \( H(x, \xi) \)
\[ \frac{dx^\mu}{dt} = \frac{\partial H(x, \xi)}{\partial \xi_\mu} = g^{\mu\nu}(x)\xi_\nu, \] (7.5)
\[ \frac{d\xi_\mu}{dt} = -\frac{\partial H(x, \xi)}{\partial x^\mu} = -\frac{1}{2} \partial_\mu g^{\alpha\beta}(x)\xi_\alpha\xi_\beta. \] (7.6)
The trajectories of this Hamiltonian system are, then, nothing but the geodesics of the metric \( g_{\mu\nu} \). Of course, the Hamiltonian is conserved, that is,
\[ g^{\mu\nu}(x(t))\xi_\mu(t)\xi_\nu(t) = -E, \] (7.7)
where \( E \) is a constant parameter.

### 7.2.2 Finsler Geometry

As it is explained in [20, 21] Matrix Gravity is closely related to Finsler geometry [68] rather than Riemannian geometry. In this section we follow the description of Finsler geometry outlined in [68]. To avoid confusion we should note that we present it in a slightly modified equivalent form, namely, we start with the Finsler function in the cotangent bundle rather than in the tangent bundle.

Finsler geometry is defined by a Finsler function \( F(x, \xi) \) which is a homogeneous function of \( \xi_\mu \) of degree 1 and the Hamiltonian
\[ H(x, \xi) = -\frac{1}{2}F^2(x, \xi). \] (7.8)
Such Hamiltonian is still a homogeneous function of $\xi_\mu$ of degree 2, that is,

$$\xi_\mu \frac{\partial}{\partial \xi_\mu} H(x, \xi) = 2H(x, \xi), \quad (7.9)$$

but not necessarily a polynomial in $\xi_\mu$!

Now, we define a tangent vector $u$ by

$$u^\mu = \frac{\partial}{\partial \xi_\mu} H(x, \xi), \quad (7.10)$$

and the Finsler metric

$$G^{\mu\nu}(x, \xi) = \frac{\partial^2}{\partial \xi_\mu \partial \xi_\nu} H(x, \xi). \quad (7.11)$$

The difference with the Riemannian metric is, obviously, that the Finsler metric does depend on $\xi_\mu$, more precisely, it is a homogeneous function of $\xi_\mu$ of degree 0, i.e.

$$G^{\mu\nu}(x, \lambda \xi) = G^{\mu\nu}(x, \xi), \quad (7.12)$$

so that it depends only on the direction of the covector $\xi$ but not on its magnitude. This leads to a number of useful identities, in particular,

$$H(x, \xi) = \frac{1}{2} G^{\mu\nu}(x, \xi) \xi_\mu \xi_\nu, \quad (7.13)$$

and

$$u^\mu = G^{\mu\nu}(x, \xi) \xi_\nu. \quad (7.14)$$

Now, we can solve this equation for $\xi_\mu$ treating $u^\nu$ as independent variables to get

$$\xi_\mu = G_{\mu\nu}(x, u) u^\nu, \quad (7.15)$$

where $G_{\mu\nu}$ is the inverse Finsler metric defined by

$$G_{\mu\nu}(x, u) G^{\alpha\nu}(x, \xi) = \delta^\alpha_\mu. \quad (7.16)$$
By using the results obtained above we can express the Hamiltonian $H$ in terms of the vector $u^\mu$, more precisely we have

$$H(x, \xi(x, u)) = \frac{1}{2} G_{\mu\nu}(x, u) u^\mu u^\nu . \quad (7.17)$$

The derivatives of the Finsler metric obviously satisfy the identities

$$\frac{\partial}{\partial \xi_\alpha} G^{\beta\gamma}(x, \xi) = \frac{\partial}{\partial \xi_\beta} G^{\gamma\alpha}(x, \xi) = \frac{\partial}{\partial \xi_\gamma} G^{\alpha\beta}(x, \xi) , \quad (7.18)$$

$$\xi_\mu \frac{\partial}{\partial \xi_\mu} G^{\gamma\alpha}(x, \xi) = \xi_\mu \frac{\partial}{\partial \xi_\gamma} G^{\mu\alpha}(x, \xi) = 0 , \quad (7.19)$$

and, more generally,

$$\xi_\mu \frac{\partial^k}{\partial \xi_{\nu_1} \ldots \partial \xi_{\nu_k}} G^{\mu\alpha}(x, \xi) = 0 . \quad (7.20)$$

This means, in particular, that the following relations hold

$$\frac{\partial u^\mu}{\partial \xi_\alpha} = G^{\mu\alpha}(x, \xi) , \quad \frac{\partial \xi_\alpha}{\partial u^\mu} = G_{\mu\alpha}(x, u) . \quad (7.21)$$

It is easy to see that the metric $G_{\mu\nu}(x, u)$ is a homogeneous function of $u$ of degree 0, that is,

$$u^\mu \frac{\partial}{\partial u^\mu} G_{\mu\nu}(x, u) = 0 , \quad (7.22)$$

and, therefore, $H(x, \xi(x, u))$ is a homogeneous function of $u$ of degree 2. This leads to the identities

$$\xi_\mu = \frac{1}{2} \frac{\partial}{\partial u^\mu} H(x, \xi(x, u)) , \quad (7.23)$$

$$G_{\mu\nu}(x, u) = \frac{1}{2} \frac{\partial^2}{\partial u^\mu \partial u^\nu} H(x, \xi(x, u)) . \quad (7.24)$$

Finally, this enables one to define the Finsler interval

$$ds^2 = G_{\mu\nu}(x, \dot{x}) dx^\mu dx^\nu , \quad (7.25)$$
so that

\[ d\tau = \sqrt{-ds^2} = \sqrt{-G_{\mu\nu}(x, \dot{x})\dot{x}^\mu\dot{x}^\nu} \, dt = F(x, \xi(x, \dot{x})) \, dt, \quad (7.26) \]

where

\[ \dot{x}^\mu = \frac{dx^\mu}{dt}, \quad \xi_\mu = G_{\mu\nu}(x, \dot{x})\dot{x}^\nu. \quad (7.27) \]

By treating \( H(x, \xi) \) as a Hamiltonian we obtain a system of first order ordinary differential equations

\[ \frac{dx^\mu}{dt} = \frac{\partial H(x, \xi)}{\partial \xi_\mu}, \quad (7.28) \]
\[ \frac{d\xi_\mu}{dt} = -\frac{\partial H(x, \xi)}{\partial x^\mu}. \quad (7.29) \]

The trajectories of this Hamiltonian system naturally replace the geodesics in Riemannian geometry. Again, as in the Riemannian case, the Hamiltonian is conserved along the integral trajectories

\[ H(x(t), \xi(t)) = -E. \quad (7.30) \]

Of course, in the particular case, when the Hamiltonian is equal to \( H(x, \xi) = \frac{1}{2}|\xi|^2 \), all the constructions derived above reduce to the standard structure of Riemannian geometry.

### 7.2.3 Induced Finsler Geometry in Matrix Gravity

The kinematics in Matrix Gravity is defined as follows. In complete analogy with the above discussion we consider the matrix

\[ A(x, \xi) = a^{\mu\nu}(x)\xi_\mu\xi_\nu, \quad (7.31) \]
where \( a^{\mu \nu} \) is the matrix-valued metric \((6.35)\). As we mentioned in the introduction this expression has been already encountered in physics, in particular, in \([32]\) it is shown that it is the most general structure describing “analog models” for gravity.

This is a Hermitian matrix, so it has real eigenvalues \( h_i(x, \xi) \), \( i = 1, 2, \ldots, N \). We consider a generic case when the eigenvalues are simple. We note that the eigenvalues \( h_i(x, \xi) \) are homogeneous functions (but not polynomials!) of \( \xi \) of degree 2. Thus, each one of them, more precisely \( \sqrt{-h_i(x, \xi)} \), can serve as a Finsler function. In other words, we obtain \( N \) different Finsler functions, and, therefore, \( N \) different Finsler metrics. Thus, quite naturally, instead of a single Riemannian metric and a unique Riemannian geodesic flow there appears \( N \) Finsler metrics and \( N \) corresponding flows. In some sense, the noncommutativity leads to a “splitting” of a single geodesic to a system of close trajectories.

Now, to define a unique Finsler metric we need to define a unique Hamiltonian, which is a homogeneous function of the momenta of degree 2. It is defined in terms of the Finsler function as in \((7.8)\) which is a homogeneous function of the momenta of degree 1. To define a unique Finsler function we can proceed as follows. Let \( \mu_i, i = 1, \ldots, N \), be some dimensionless real parameters such that

\[
\sum_{i=1}^{N} \mu_i = 1, \tag{7.32}
\]

so that there are \( (N - 1) \) independent parameters. Then we can define the Finsler function by

\[
F(x, \xi) = \sum_{i=1}^{N} \mu_i \sqrt{-h_i(x, \xi)}. \tag{7.33}
\]

Notice that, in the commutative limit, as \( \kappa \to 0 \) and \( a^{\mu \nu} = g^{\mu \nu} \), all eigenvalues of the matrix \( A(x, \xi) \) degenerate to the same value, \( h_i(x, \xi) = |\xi|^2 \), and, hence, the Finsler
function becomes $F(x, \xi) = \sqrt{-|\xi|^2}$. In this case the Finsler flow degenerates to the usual Riemannian geodesic flow.

Next, we define the Hamiltonian according to eq. (7.8)

$$H(x, \xi) = -\frac{1}{2} \left( \mu \sqrt{-h_i(x, \xi)} \right)^2$$

$$= \frac{1}{2} \sum_{i=1}^{N} \mu_i^2 h_i(x, \xi) - \sum_{1 \leq i < j \leq N} \mu_i \mu_j \sqrt{h_i(x, \xi)h_j(x, \xi)}. \quad (7.34)$$

In a particular case, when all parameters $\mu_i$ are equal, i.e. $\mu_i = 1/N$, the Finsler function reduces to

$$F(x, \xi) = \frac{1}{N} \sum_{i=1}^{N} \sqrt{-h_i(x, \xi)} = \frac{1}{N} \text{tr} \sqrt{-A(x, \xi)}. \quad (7.35)$$

By using the decomposition of the matrix-valued metric $\alpha^{\mu\nu}$ (6.35) one can see that

$$\frac{1}{N} \text{tr} A(x, \xi) = |\xi|^2, \quad (7.36)$$

and, therefore,

$$\frac{1}{N} \sum_{i=1}^{N} h_i(x, \xi) = |\xi|^2. \quad (7.37)$$

Thus, we conclude that in this particular case

$$H(x, \xi) = \frac{1}{N} \left( \frac{1}{2} |\xi|^2 - \frac{1}{N} \sum_{1 \leq i < j \leq N} \sqrt{h_i(x, \xi)h_j(x, \xi)} \right). \quad (7.38)$$

It is difficult to give a general physical picture of these models since the Hamiltonian is non-polynomial in the momenta. Hamiltonian systems with homogeneous Hamiltonians have not been studied as thoroughly as the usual systems with quadratic Hamiltonians and a potential.
7.2.4 Kinematics

The problem is, now, how to use these mathematical tools to describe the motion of physical massive test particles in Matrix Gravity. The motion of a massive particle in the gravitational field is determined in General Relativity by the action which is proportional to the interval, that is,

\[
S_{\text{particle}} = - \int_{P_1}^{P_2} m \sqrt{-g_{\mu\nu}(x)d\hat{x}^\mu d\hat{x}^\nu} = - \int_{t_1}^{t_2} m \sqrt{-|\dot{x}|^2} dt , \tag{7.39}
\]

where \(m\) is the mass of the particle, \(P_1\) and \(P_2\) are the initial and the final position of the particle in the spacetime, \(t\) is a parameter, \(t_1\) and \(t_2\) are the initial and the final values, \(\dot{x}^\mu = \frac{dx^\mu}{dt}\) and \(|\dot{x}|^2 = g_{\mu\nu}(x)\dot{x}^\mu \dot{x}^\nu\). This action is, of course, reparametrization-invariant. So, as always, there is a freedom of choosing the parameter \(t\). We can always choose the parameter to be the affine parameter such that \(|\dot{x}|^2\) is constant, for example, if the parameter is the proper time \(t = \tau\), then \(|\dot{x}|^2 = -1\). The Euler-Lagrange equations for this functional are, of course,

\[
\frac{D\dot{x}^\nu}{dt} = \frac{d^2\dot{x}^\nu}{dt^2} + \Gamma^\nu_{\alpha\beta}(x) \dot{x}^\alpha \dot{x}^\beta = 0 , \tag{7.40}
\]

where \(\Gamma^\nu_{\alpha\beta}\) are the standard Christoffel symbols of the metric \(g_{\mu\nu}\). Of course, the equivalence principle holds since these equations do not depend on the mass.

In Matrix Gravity a particle is described instead of one mass parameter \(m\) by \(N\) different mass parameters

\[
m_i = m \mu_i , \tag{7.41}
\]

where

\[
m = \sum_{i=1}^{N} m_i . \tag{7.42}
\]
The parameters $m_i$ describe the “tendency” for a particle to move along the trajectory determined by the corresponding Hamiltonian $h_i(x, \xi)$. In the commutative limit we only observe the total mass $m$.

We define the Finsler function $F(x, \xi)$ and the Hamiltonian $H(x, \xi)$ as in eqs. (7.33) and (7.34). Then the action for a particle in the gravitational field has the form

$$S_{\text{particle}} = -\int_{t_1}^{t_2} mF(x, \xi(x, \dot{x})) \, dt. \quad (7.43)$$

Thus, the Finsler function $F(x, \xi(x, \dot{x}))$ (with the covector $\xi_\mu \text{ expressed in terms of the tangent vector } \dot{x}_\mu$) plays the role of the Lagrangian. To study the role of non-commutative corrections, it is convenient to rewrite this action in the form that resembles the action in General Relativity.

$$S_{\text{particle}} = -\int_{t_1}^{t_2} m_{\text{eff}}(x, \dot{x}) \sqrt{-|\dot{x}|^2} \, dt, \quad (7.44)$$

with some “effective mass” $m_{\text{eff}}(x, \dot{x})$ that depends on the location and on the velocity of the particle

$$m_{\text{eff}}(x, \dot{x}) = \sum_{i=1}^{N} m_i \sqrt{h_i(x, \xi(\dot{x}))} |\dot{x}|^2. \quad (7.45)$$

This action is again reparametrization-invariant. Therefore, we can choose the natural arc-length parameter so that $F(x, \xi(x, \dot{x})) = 1$. Then the equations of motion determined by the Euler-Lagrange equations have the same form

$$\frac{d^2 x^\mu}{dt^2} + \gamma_{\alpha\beta}(x, \dot{x}) \dot{x}^\alpha \dot{x}^\beta = 0, \quad (7.46)$$

where $\gamma_{\alpha\beta}(x, \dot{x})$ are the Finsler Christoffel coefficients defined by the equations that look identical to the usual equations but with the Finsler metric instead of the Rie-
mannian metric, that is,
\[ \gamma_{\alpha\beta}(x, \dot{x}) = \frac{1}{2} G^{\mu\nu}(x, \xi(x, \dot{x})) \left( \frac{\partial}{\partial x^\alpha} G_{\nu\beta}(x, \dot{x}) + \frac{\partial}{\partial x^\beta} G_{\nu\alpha}(x, \dot{x}) - \frac{\partial}{\partial x^\nu} G_{\alpha\beta}(x, \dot{x}) \right). \]
(7.47)

To study the role of non-commutative corrections it is convenient to rewrite these equations in a covariant form in the Riemannian language. In the commutative limit, as \( \kappa \to 0 \), we can expand all our constructions in power series in \( \kappa \) so that the non-perturbed quantities are the Riemannian ones. In particular, we have
\[ \gamma_{\alpha\beta}(x, \dot{x}) = \Gamma_{\alpha\beta}(x) + \theta_{\alpha\beta}(x, \dot{x}), \]
(7.48)
where \( \theta_{\alpha\beta} \) are some tensors of order \( \kappa \). Then the equations of motion can be written in the form
\[ \frac{Dv^\nu}{dt} = A_{\text{anom}}^\nu(x, \dot{x}), \]
(7.49)
where
\[ \frac{D\dot{x}^\nu}{dt} = \frac{d^2 x^\mu}{dt^2} + \Gamma_{\alpha\beta}(x) \dot{x}^\alpha \dot{x}^\beta \]
(7.50)
and
\[ A_{\text{anom}}^\nu(x, \dot{x}) = -\theta_{\alpha\beta}(x, \dot{x}) \dot{x}^\alpha \dot{x}^\beta, \]
(7.51)
is the anomalous nongeodesic acceleration.

### 7.3 Perturbation Theory

We see that the motion of test particles in matrix Gravity is quite different from that of General Relativity. The most important difference is that particles exhibit a non-geodesic motion. In other words, there is no Riemannian metric such that particles move along the geodesics of that metric. It is this anomalous acceleration that we are going to study in this chapter.
In the commutative limit the action of a particle in Matrix Gravity reduces to the action of a particle in General Relativity with the mass $m$ determined by the sum of all masses $m_i$. In this chapter we consider two different cases. In the first case, that we call the nonuniform model, we assume that all mass parameters are different, and in the second case, that we call the uniform model, we discuss what happens if they are equal to each other.

### 7.3.1 Nonuniform Model: First Order in $\kappa$

So, in this section we study the generic case when the parameters $\mu_i$ are different. As we already mentioned above, in this case the Finsler function $F(x, \xi)$ is given by (7.33). By using the decomposition $a^{\mu\nu} = g^{\mu\nu}I + \kappa h^{\mu\nu}$ of the matrix-valued metric $a^{\mu\nu}$ we have

$$A(x, \xi) = a^{\mu\nu}(x)\xi_\mu\xi_\nu = |\xi|^2 I + \kappa h^{\mu\nu}(x)\xi_\mu\xi_\nu.$$  

(7.52)

Therefore, the eigenvalues of the matrix $A(x, \xi)$ are

$$h_i(x, \xi) = |\xi|^2 + \kappa \lambda_i(x, \xi),$$  

(7.53)

where $\lambda_i(x, \xi)$ are the eigenvalues of the matrix $h^{\mu\nu}(x)\xi_\mu\xi_\nu$. In the first order in $\kappa$ we get the Finsler function

$$F(x, \xi) = \sqrt{-|\xi|^2} \left(1 + \kappa \frac{1}{2} \frac{P(x, \xi)}{|\xi|^2}\right) + O(\kappa^2),$$  

(7.54)

and the Hamiltonian

$$H(x, \xi) = \frac{1}{2}|\xi|^2 + \kappa \frac{1}{2} P(x, \xi) + O(\kappa^2),$$  

(7.55)
where
\[ P(x, \xi) = \sum_{i=1}^{N} \mu_i \lambda_i(x, \xi). \]  
(7.56)

By using the fact that \( P(x, \xi) \) is a homogeneous function of \( \xi \) of order 2, we find the Finsler metric
\[ G^{\mu \nu}(x, \xi) = g^{\mu \nu}(x) + \kappa q^{\mu \nu}(x, \xi) + O(\kappa^2), \]  
(7.57)
and its inverse
\[ G_{\mu \nu}(x, u) = g_{\mu \nu}(x) - \kappa q_{\mu \nu}(x, \xi(x, u)) + O(\kappa^2), \]  
(7.58)
where
\[ q^{\mu \nu}(x, \xi) = \frac{1}{2} \frac{\partial^2}{\partial \xi_\mu \partial \xi_\nu} P(x, \xi). \]  
(7.59)

Here the indices are raised and lowered with the Riemannian metric, and
\[ u^\mu(x, \xi) = G^{\mu \nu}(x, \xi) \xi_\nu, \quad \xi_\mu(x, u) = G_{\mu \nu}(x, u) u^\nu. \]  
(7.60)

Since \( P(x, \xi) \) is a homogeneous function of \( \xi \) of order 2 we have
\[ P(x, \xi) = q^{\mu \nu}(x, \xi) \xi_\mu \xi_\nu. \]  
(7.61)

Note that since \( \text{tr} h^{\mu \nu} = 0 \) the matrix \( h^{\mu \nu} \xi_\mu \xi_\nu \) is traceless, which implies that the sum of its eigenvalues is equal to zero. Thus, in the uniform case, when all mass parameters \( \mu_i \) are the same, the function \( P(x, \xi) \) vanishes. In this case the effects of non-commutativity are of the second order in \( \kappa \); we study this case in the next section.

We also note that
\[ |\xi|^2 = |u|^2 - 2\kappa P(x, \xi(x, u)) + O(\kappa^2). \]  
(7.62)
Thus, our Lagrangian is
\[
F(x, \xi(x, \dot{x})) = \sqrt{-|\dot{x}|^2} \left( 1 - \frac{\kappa}{2} \frac{1}{|\dot{x}|^2} P(x, \xi(x, \dot{x})) \right) + O(\kappa^2),
\]
(7.63)

Finally, we compute the Christoffel symbols to obtain
\[
\theta^\mu_{\alpha\beta}(x, \dot{x}) = -\frac{1}{2} g^\mu\nu \left( \nabla_\alpha q_{\beta\nu}(x, \dot{x}) + \nabla_\beta q_{\alpha\nu}(x, \dot{x}) - \nabla_\nu q_{\alpha\beta}(x, \dot{x}) \right) + O(\kappa^2),
\]
(7.64)

and the covariant derivatives are defined with the Riemannian metric.

Thus, the anomalous acceleration is
\[
A^\mu_{\text{anom}} = \frac{\kappa}{2} g^\mu\nu \left( 2 \nabla_\alpha q_{\beta\nu}(x, \dot{x}) - \nabla_\nu q_{\alpha\beta}(x, \dot{x}) \right) \dot{x}^\alpha \dot{x}^\beta + O(\kappa^2),
\]
(7.65)

### 7.3.2 Uniform Model: Second Order in $\kappa$

So, in this section we will simply assume that all mass parameters are equal, that is,
\[
m_i = \frac{m}{N}.
\]
(7.66)

In this case the Finsler function $F(x, \xi)$ is given by (7.35). By using the decomposition of the matrix-valued metric and the fact that $\text{tr} \, h^{\mu\nu} = 0$ we get the Finsler function
\[
F(x, \xi) = \sqrt{-|\dot{\xi}|^2} \left( 1 - \kappa^2 \frac{1}{8} S^{\mu\nu\alpha\beta}(x) \frac{\xi_\mu \xi_\nu \xi_\alpha \xi_\beta}{|\dot{\xi}|^4} \right) + O(\kappa^3),
\]
(7.67)

and the Hamiltonian
\[
H(x, \xi) = \frac{1}{2} |\dot{\xi}|^2 \left( 1 - \kappa^2 \frac{1}{8} S^{\mu\nu\alpha\beta}(x) \frac{\xi_\mu \xi_\nu \xi_\alpha \xi_\beta}{|\dot{\xi}|^4} \right) + O(\kappa^3),
\]
(7.68)

where
\[
S^{\mu\nu\alpha\beta} = \frac{1}{N} \text{tr} \left( h^{\mu\nu} h^{\alpha\beta} \right).
\]
(7.69)
By using the above, we compute the Finsler metric

\[ G^{\mu \nu}(x, \xi) = g^{\mu \nu}(x) - \kappa^2 \frac{1}{4} S^{\mu \nu \alpha \beta}(x) \frac{\xi_\alpha \xi_\beta}{|\xi|^2} + O(\kappa^3), \]  

(7.70)

and its inverse

\[ G_{\mu \nu}(x, u) = g_{\mu \nu}(x) + \kappa^2 \frac{1}{4} S_{\mu \nu \alpha \beta}(x) \frac{u^\alpha u^\beta}{|u|^2} + O(\kappa^3). \]  

(7.71)

We also note that

\[ |\xi|^2 = |u|^2 + \kappa^2 \frac{1}{2} S_{\mu \nu \alpha \beta}(x) \frac{u^\mu u^\nu u^\alpha u^\beta}{|u|^2} + O(\kappa^3). \]  

(7.72)

Thus, our Lagrangian is

\[ F(x, \xi(x, \dot{x})) = \sqrt{-|\dot{x}|^2 \left( 1 + \kappa^2 \frac{1}{8} S_{\mu \nu \alpha \beta}(x) \frac{\dot{x}^\mu \dot{x}^\nu \dot{x}^\alpha \dot{x}^\beta}{|\dot{x}|^4} \right) + O(\kappa^3), \]  

(7.73)

Finally, we compute the Christoffel symbols to obtain

\[ \theta^\mu_{\alpha \beta}(x, \dot{x}) = \kappa^2 \frac{1}{8} g^{\mu \nu} \left( \nabla^\alpha S_{\beta \nu \rho \sigma} + \nabla^\beta S_{\alpha \nu \rho \sigma} - \nabla^\nu S_{\alpha \beta \rho \sigma} \right) \frac{\dot{x}^\nu \dot{x}^\alpha \dot{x}^\sigma \dot{x}^\beta}{|\dot{x}|^2} + O(\kappa^3). \]  

(7.74)

Thus, the anomalous acceleration is

\[ A^\mu_{\text{anom}} = -\kappa^2 \frac{1}{8} g^{\mu \nu} \left( 2 \nabla^\alpha S_{\beta \nu \rho \sigma} - \nabla^\nu S_{\alpha \beta \rho \sigma} \right) \frac{\dot{x}^\nu \dot{x}^\alpha \dot{x}^\sigma \dot{x}^\beta}{|\dot{x}|^2} + O(\kappa^3), \]  

(7.75)

Notice that with our choice of the parameter \( t \) we have \( F(x, \xi(x, \dot{x})) = 1 \), and, therefore, in the equations of motion we can substitute with the same accuracy

\[ |\xi|^2 = -1 + O(\kappa^2), \quad |\dot{x}|^2 = -1 + O(\kappa^2). \]  

(7.76)

Therefore, we obtain finally

\[ A^\mu_{\text{anom}} = -\kappa^2 \frac{1}{8} g^{\mu \nu} \left( 2 \nabla^\alpha S_{\beta \nu \rho \sigma} - \nabla^\nu S_{\alpha \beta \rho \sigma} \right) \frac{\dot{x}^\nu \dot{x}^\alpha \dot{x}^\sigma \dot{x}^\beta}{|\dot{x}|^2} + O(\kappa^3). \]  

(7.77)
7.3.3 Non-commutative Corrections to Newton’s Law

Now, we will derive the non-commutative corrections to the Newton’s Law. We label the coordinates as

\[ x^0 = t, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \phi, \]

and consider the static spherically symmetric (Schwarzschild) metric

\[ ds^2 = -U(r)dt^2 + U^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2), \]

where

\[ U(r) = 1 - \frac{r_s}{r}, \quad r_s = 2GM, \]

and \( M \) is the mass of the central body. It is worth recalling that here \( t \) is the coordinate time. In the previous sections we used \( t \) to denote an affine parameter of the trajectory that we agreed to choose to be the proper time. In the present section we use \( \tau \) to denote the proper time and \( t \) to denote the coordinate time.

The motion of test particles in Schwarzschild geometry is very well studied in General Relativity, see, for example [82]. Assuming that the particle moves in the equatorial plane \( \theta = \pi/2 \) away from the center, that is, \( dr/d\tau > 0 \), the equations of motion have the following integrals [82]

\[ \dot{x}^0 = \frac{dt}{d\tau} = \frac{E}{mU(r)}, \quad \text{(7.81)} \]
\[ \dot{x}^1 = \frac{dr}{d\tau} = \sqrt{\frac{E^2}{m^2} - \left(1 + \frac{L^2}{m^2r^2}\right)U(r)}, \quad \text{(7.82)} \]
\[ \dot{x}^2 = \frac{d\theta}{d\tau} = 0, \quad \theta = \frac{\pi}{2}, \quad \text{(7.83)} \]
\[ \dot{x}^3 = \frac{d\phi}{d\tau} = \frac{L}{mr^2}, \quad \text{(7.84)} \]
where $m$, $L$, and $E$ are the mass of the particle, its orbital momentum and the energy.

In the non-relativistic limit for weak gravitational fields, assuming

$$E = m + E', \quad (7.86)$$

with $E' << m$, and $r >> r_g$ one can identify the coordinate time with the proper time, so that

$$\dot{x}^0 = \frac{dt}{d\tau} = 1. \quad (7.87)$$

Further, for the non-relativistic motion we have $\dot{r}$, $\dot{\theta}$, $\dot{\phi} << 1$, and the radial velocity reduces, of course, to the standard Newtonian expression

$$\dot{x}^1 = \frac{dr}{d\tau} = \sqrt{\frac{2E'}{m} - \frac{L^2}{m^2 r^2} + \frac{r_g}{r}}, \quad (7.88)$$

which for $L = 0$ becomes

$$\dot{x}^1 = \frac{dr}{d\tau} = \sqrt{\frac{2E'}{m} + \frac{r_g}{r}}, \quad (7.89)$$

It is worth stressing that the anomalous acceleration due to non-commutativity in the non-relativistic limit can be interpreted as a correction to the Newton’s Law. Assuming that a particle is moving in the equatorial plane, $\theta = \pi/2$, with zero orbital momentum, $\varphi = \text{const}$, the equation of motion is

$$\frac{d^2 r}{dt^2} = -\frac{\partial}{\partial r} V_{\text{eff}}(r) = -\frac{GM}{r^2} + A'_{\text{anom}}, \quad (7.90)$$

where in the uniform model

$$A'_{\text{anom}} = \frac{\kappa^2}{8} \partial_i S^{0000} + O(x^3), \quad (7.91)$$
with $S^{0000} = \frac{1}{8} \text{tr} h^{00} h^{00}$, and in the non-uniform model

$$A^r_{\text{anom}} = -\frac{\kappa}{2} \partial_r q^{00} + O(\kappa^2), \quad (7.92)$$

with $q^{00}$ being the component of the tensor $q^{\mu \nu}$ defined by (7.59). This gives the non-commutative corrections to Newton’s Law: in the uniform model,

$$V_{\text{eff}}(r) = -\frac{GM}{r} - \frac{\kappa^2}{8} S^{0000}(r) + O(\kappa^3), \quad (7.93)$$

and, in the nonuniform model,

$$V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{\kappa}{2} q^{00}(r) + O(\kappa^2). \quad (7.94)$$

Here, of course, the tensor components $S^{0000}$ and $q^{00}$ should be obtained by the solution of the non-commutative Einstein field equations (in the perturbation theory).

### 7.3.4 Static Spherically Symmetric Solutions

In the present chapter we study the effects of Matrix Gravity in the simplest possible case restricting ourselves to a commutative algebra. The commutativity assumption enormously simplifies the dynamical equations. By recalling equation (5.42) it is easy to show that in this case the dynamical equations look exactly as the Einstein equations in the vacuum

$$\mathcal{R}_{\mu \nu} = \Lambda g_{\mu \nu}, \quad (7.95)$$

where $\mathcal{R}_{\mu \nu}$ is the matrix-valued Ricci tensor defined by $\mathcal{R}_{\mu \nu} = \mathcal{R}^{\alpha}_{\mu \alpha \nu}$.

In this section we are going to study, in particular, a static spherically symmetric solution of the equation (7.95). We present the matrix-valued metric $a^{\mu \nu}$ by writing the “matrix-valued Hamiltonian”

$$a^{\mu \nu} \xi_{\mu} \xi_{\nu} = A(r)(\xi_0)^2 + B(r)(\xi_1)^2 + \frac{1}{r^2} \left[ (\xi_2)^2 + \frac{1}{\sin^2 \theta} (\xi_3)^2 \right], \quad (7.96)$$
or the “matrix-valued interval”

\[ b_{\mu\nu}dx^\mu dx^\nu = A^{-1}(r)dt^2 + B^{-1}(r)dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right), \quad (7.97) \]

where the coefficients \( A(r) \) and \( B(r) \) are commuting matrices that depend only on the radial coordinate \( r \). This simply means that we choose the following ansatz

\[
a^{00} = A, \quad a^{11} = B, \\
a^{22} = \frac{1}{r^2} \mathbb{1}, \quad a^{33} = \frac{1}{r^2 \sin^2 \theta} \mathbb{1}. \quad (7.98)
\]

Next, by computing the connection coefficients \( \mathcal{A}_\mu{}^\alpha{}^\nu \) and the matrix-valued Ricci tensor we obtain the equations of motion

\[
\mathcal{R}_{00} = A^{-1}B \left[ \frac{1}{2} A^{-1} A'' - \frac{3}{4} A^{-2} (A')^2 + \frac{1}{4} A^{-1} A' B^{-1} B' + \frac{1}{r^2} A^{-1} A' \right] = \Lambda A^{-1}, \quad (7.99)
\]
\[
\mathcal{R}_{11} = \frac{1}{2} A^{-1} A'' - \frac{3}{4} A^{-2} (A')^2 + \frac{1}{4} A^{-1} A' B^{-1} B' - \frac{1}{r^2} B^{-1} B' = \Lambda B^{-1}, \quad (7.100)
\]
\[
\mathcal{R}_{22} = -\frac{r}{2} B' - B + \frac{r}{2} BA^{-1} A' + \mathbb{1} = \Lambda r^2 \mathbb{1}, \quad (7.101)
\]
\[
\mathcal{R}_{33} = \sin^2 \theta \mathcal{R}_{22} = \Lambda r^2 \sin^2 \theta \mathbb{1}. \quad (7.102)
\]

where the prime denotes differentiation with respect to \( r \).

By using the equations (7.99) and (7.100) we find

\[ A^{-1} A' + B^{-1} B' = 0; \quad (7.103) \]

the general solution of this equation is

\[ A(r)B(r) = C_1, \quad (7.104) \]
where $C_1$ is an arbitrary constant matrix from our algebra. We require that at the spatial infinity as $r \to \infty$ the matrices $A$ and $B$ and, therefore, the matrix $C$ as well, are non-degenerate.

By using this relation we obtain further from eqs. (7.100) and (7.101) two compatible equations for the matrix $B$

$$B'' + \frac{2}{r}B' + 2\Lambda = 0, \quad (7.105)$$

and

$$rB' + B = (1 - \Lambda r^2)I. \quad (7.106)$$

The general solution of the eq. (7.106) is

$$B(r) = \left(1 - \frac{1}{3}\Lambda r^2\right)I + \frac{1}{r}C_2, \quad (7.107)$$

where $C_2$ is another arbitrary constant matrix from our algebra. It is not difficult to see that this form of the matrix $B$ also satisfies the eq. (7.105). The matrix $A$ is now obtained from the equation (7.104)

$$A(r) = C_1 \left[\left(1 - \frac{1}{3}\Lambda r^2\right)I + \frac{1}{r}C_2\right]^{-1}. \quad (7.108)$$

We will also require that in the limit $\kappa \to 0$ we should get the standard Schwarzschild solution with the cosmological constant

$$B(r) = -A^{-1}(r) = \left(1 - \frac{1}{3}\Lambda r^2 - \frac{r_g}{r}\right)I, \quad (7.109)$$

where $r_g$ is the gravitational radius of the central body of mass $M$,

$$r_g = 2GM, \quad (7.110)$$

that is, in that limit the matrices $E$ and $C$ should be

$$C_1 = -I, \quad C_2 = -r_g I. \quad (7.111)$$
7.3.5 2 × 2 Matrices

To be specific, we restrict ourselves further to real symmetric 2 × 2 matrices generated by
\[ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \] (7.112)

In this case the constant matrices \( C_1 \) and \( C_2 \) can be expressed in terms of four real parameters
\[ C_1 = \alpha I + \theta \tau, \quad C_2 = \mu I + L \tau, \] (7.113)
where \( \theta = \kappa \bar{\theta} \) and \( L = \kappa \bar{L} \) are the parameters of first order in the deformation parameter \( \kappa \). Here the parameters \( \alpha \) and \( \theta \) are dimensionless and the parameters \( \mu \) and \( L \) have the dimension of length.

Then the matrix \( B(r) \) has the form
\[ B(r) = \left( 1 - \frac{1}{3} \Lambda r^2 + \frac{\mu}{r} \right) I + \frac{L}{r} \tau. \] (7.114)

Next, noting that \( \tau^2 = \overline{I} \), and by using the relation
\[ (aI + b\tau)^{-1} = \frac{1}{a^2 - b^2} (aI - b\tau), \] (7.115)
we obtain the matrix \( A(r) \)
\[ A(r) = \phi(r) I + \psi(r) \tau, \] (7.116)
where
\[ \phi(r) = \frac{\alpha \left( 1 - \frac{1}{3} \Lambda r^2 \right) + \frac{\alpha \mu - \alpha L}{r}}{\left( 1 - \frac{1}{3} \Lambda r^2 + \frac{\mu}{r} \right)^2 - \frac{L^2}{r^2}}, \] (7.117)
\[ \psi(r) = \frac{\theta \left( 1 - \frac{1}{3} \Lambda r^2 \right) + \frac{\theta \mu - \alpha L}{r}}{\left( 1 - \frac{1}{3} \Lambda r^2 + \frac{\mu}{r} \right)^2 - \frac{L^2}{r^2}}. \] (7.118)
The parameters $\alpha, \theta, \mu$ and $L$ should be determined by the boundary conditions at spatial infinity. The question of boundary conditions is a subtle point since we do not know the physical nature of the additional degrees of freedom. We will simply require that the diagonal part of the metric is asymptotically De Sitter. This immediately gives

$$\alpha = -1.$$  \hspace{1cm} (7.119)

Now, we introduce a new parameter

$$r_0 = |\Lambda|^{-1/2},$$  \hspace{1cm} (7.120)

and require that for $r_g << r << r_0$, the diagonal part of the metric, more precisely, the function $\varphi(r)$ is asymptotically Schwarzschild, that is,

$$\varphi(r) = -1 \frac{r_g}{r} + O\left(\frac{r^2}{r_0^2}\right) + O\left(\frac{r^3}{r_0^2}\right).$$  \hspace{1cm} (7.121)

This fixes the parameter $\mu$

$$\mu = -r_g + \theta L.$$  \hspace{1cm} (7.122)

The parameters $\theta$ and $L$ remain undetermined.

Finally, by introducing new parameters

$$\rho = (1 + \theta^2)L - \theta r_g$$  \hspace{1cm} (7.123)

$$r_\pm = r_g - (\theta \pm 1)L$$  \hspace{1cm} (7.124)

we can rewrite our solution in the form

$$\varphi(r) = \frac{-r \left( r - \frac{1}{3} \Lambda r^3 - r_g + 2\theta L \right)}{\left[ r - \frac{1}{3} \Lambda r^3 - r_+ \right] \left[ r - \frac{1}{3} \Lambda r^3 - r_- \right]},$$  \hspace{1cm} (7.125)
\[
\psi(r) = \frac{r \left( \theta \left( r - \frac{1}{3} \Lambda r^3 \right) + \rho \right)}{\left[ r - \frac{1}{3} \Lambda r^3 - r_- \right] \left[ r - \frac{1}{3} \Lambda r^3 - r_+ \right]}. 
\] (7.126)

Of course, as \( \kappa \to 0 \) both parameters \( L = \kappa \bar{L} \) and \( \theta = \kappa \bar{\theta} \) vanish and we get the standard Schwarzschild solution with the cosmological constant.

Notice that the matrix-valued metric \( e^{\mu \nu} \) becomes singular when the matrices \( A \) and \( B \) are not invertible, that is, when

\[
\det A(r) = 0. 
\] (7.127)

The solutions of this equation are the roots of the cubic polynomials

\[
r - \frac{1}{3} \Lambda r^3 - r_- = 0 \quad \text{and} \quad r - \frac{1}{3} \Lambda r^3 - r_+ = 0 
\] (7.128)

Recall that the standard Schwarzschild coordinate singularity, which determines the position of the event horizon, is located at \( r = r_g \). The presence of singularities depends on the values of the parameters. We analyze, now, the first eq. in (7.128). In the case \( \Lambda \leq 0 \) the polynomial has one root if \( r_- > 0 \) and does not have any roots if \( r_- < 0 \). In the case \( \Lambda > 0 \) it is easy to see that: i) if \( r_- > (2/3)r_0 \), then there are no roots, ii) if \( 0 < r_- < (2/3)r_0 \), then the polynomial has two roots, and ii) if \( r_- < 0 \), then the polynomial has one root. The same applies to the second eq. in (7.128).

We emphasize that there are two cases without any singularities at any finite value of \( r \). This happens if either: a) \( \Lambda \leq 0 \) and \( r_+ < 0 \), or b) \( \Lambda > 0 \) and \( r_+ > (2/3)r_0 \). This can certainly happen for large values of \( |\theta| \) and \( |L| \). In particular, if \( \theta \) and \( L \) have the same signs and

\[
|\theta| > 1 + \frac{r_g}{|L|}, 
\] (7.129)
then both \( r_\pm \) are negative, \( r_\pm < 0 \), and if \( \theta \) and \( L \) have opposite signs and
\[
|\theta| > 1 + \frac{2r_0 - r_g}{|L|}, \quad (7.130)
\]
then \( r_\pm > (2/3)r_0 \). This is a very interesting phenomenon which is entirely new and due to the additional degrees of freedom.

We would like to clarify some points. The parameters \( \mu_i \) introduced in the previous sections describe the properties of the test particle, that is, the matter. The parameters \( \theta \) and \( \rho \) introduced in the static and spherically symmetric solution of non-commutative Einstein equations describe the properties of the gravitational field, that is, the properties of the source of the gravitational field, that is, the central body. The parameters \( \theta \) and \( \rho \) are not related to the parameters \( \mu_i \).

### 7.4 Anomalous Acceleration

In this section we are going to evaluate the anomalous acceleration of non-relativistic test particles in the static spherically symmetric gravitational field of a massive central body.

All we have to do is to evaluate the components of the anomalous acceleration \( (7.77) \). As we will see the only essential component of the anomalous acceleration is the radial one \( A'_\text{anom} \). All other components of the anomalous acceleration are negligible in this limit. As we will see below, the anomalous acceleration is caused by the radial gradient of the component \( h^{00} \) of the matrix-valued metric, which is
\[
\kappa h^{00} = \psi(r)\tau, \quad (7.131)
\]
where \( \psi(r) \) is given by \( (7.126) \). Our analysis is restricted to the perturbation theory in the deformation parameter \( \kappa \) (first order in \( \kappa \) in the non-uniform model and second
order in $\kappa$ in the uniform model). That is, we should expand our result in powers of $\rho$ and $\theta$ and keep only linear terms in the non-uniform model and quadratic terms in the uniform model.

For future use we write the function $\psi(r)$ in the first order in the parameter $\kappa$

$$\psi(r) = \frac{r \left[ \theta \left( r - \frac{1}{3} \Lambda r^3 \right) + \rho \right]}{\left( r - \frac{1}{3} \Lambda r^3 - \rho \right)^2} + O(\kappa^2),$$  
(7.132) and for $r << r_0$

$$\psi(r) = \frac{r (\theta r + \rho)}{(r - r_g)^2} + O(\kappa^2),$$  
(7.133)

and, finally, for $r_g << r << r_0$,

$$\psi(r) = \theta + \frac{\rho}{r} + O(\kappa^2).$$  
(7.134) We would like to emphasize at this point that the perturbation theory we are going to perform is only valid for small corrections. When the corrections become large we need to consider the exact equations of motion (7.51).

### 7.4.1 Uniform Model

In the non-relativistic limit the formula for the anomalous radial acceleration (7.91) gives

$$A^r_{\text{anom}} = \frac{1}{4} \psi(r) \psi'(r) + O(\kappa^3).$$  
(7.135)

The derivative of the function $\psi(r)$ is easily computed

$$\psi'(r) = \omega(r) \psi(r),$$  
(7.136)
where
\[ \omega(r) = \frac{1}{r} \left( \frac{\theta(1 - \Lambda r^2)}{r - \frac{1}{3} \Lambda r^3 - r_-} - \frac{1 - \Lambda r^2}{r - \frac{1}{3} \Lambda r^3 - r_+} \right). \] (7.137)

Thus, we obtain finally
\[ A'_{\text{anom}} = \frac{1}{4} \psi^2(r) \omega(r) + O(\kappa^3). \] (7.138)

Recall that the parameters \( \rho \) and \( \theta \) are of first order in \( \kappa \). Strictly speaking we should expand this formula in \( \rho \) and \( \theta \) keeping only quadratic terms; we get
\[ A'_{\text{anom}} = \frac{1}{4} \left[ \theta \left( r - \frac{1}{3} \Lambda r^3 \right) + \rho \right] r \left\{ \left( r - \frac{1}{3} \Lambda r^3 - r_g \right) \left[ \theta \left( 2r - \frac{4}{3} \Lambda r^3 \right) + \rho \right] \right. \\
-2r(1 - \Lambda r^2) \left[ \theta \left( r - \frac{1}{3} \Lambda r^3 \right) + \rho \right] \} + O(\kappa^3). \] (7.139)

For \( r \ll r_0 \) (that is, \( |\Lambda| r^2 \ll 1 \)) this becomes
\[ A'_{\text{anom}} = -\frac{1}{4} \left( r \left( \rho + \frac{\theta}{r} \right) \right) \left( \frac{\theta + 2 \rho r_g}{r^2} - \frac{2}{3} \Lambda r \right) + O(\kappa^3). \] (7.140)

We need to keep the term linear in \( \Lambda \) since we do not know the values of the parameters \( \theta \) and \( \rho \). Finally, for \( r_g \ll r \ll r_0 \) we obtain
\[ A'_{\text{anom}} = -\frac{1}{4} \left( \theta + \frac{\rho}{r} \right) \left( \frac{\rho + 2 \rho r_g}{r^2} - \frac{2}{3} \Lambda r \right) + O(\kappa^3). \] (7.141)

### 7.4.2 Non-uniform Model

Similarly, in the non-uniform model the anomalous acceleration is given by eq. (7.92). In the 2×2 matrix case considered above the eigenvalues of the matrix \( h^{\mu\nu} \epsilon_{\mu} \epsilon_{\nu} \) are
\[ \lambda_{1,2} = \pm \frac{1}{2} \text{tr} \left( h^{\mu\nu} \epsilon_{\mu} \epsilon_{\nu} \right). \] (7.142)
Therefore,
\[ P(\mathbf{x}, \xi) = \mu_1 \lambda_1 + \mu_2 \lambda_2 = \frac{1}{2} \text{tr} (h^{\mu \nu} \epsilon) \xi_\mu \xi_\nu, \]  
(7.143)

where
\[ \gamma = \mu_1 - \mu_2. \]  
(7.144)

Thus
\[ q^{\mu \nu} = \frac{\gamma}{2} \text{tr} (h^{\mu \nu} \epsilon). \]  
(7.145)

So, we obtain
\[ \kappa q^{00} = \gamma \psi (r). \]  
(7.146)

Thus
\[ A_{\text{anom}}^r = -\frac{1}{2} \gamma \psi' (r) + O(\kappa^2) \]
\[ = -\frac{1}{2} \gamma \psi (r) \omega (r) + O(\kappa^2). \]  
(7.147)

Now, we recall that \( \rho \) and \( \theta \) are of first order in \( \kappa \) and expand in powers of \( \rho \) and \( \theta \) keeping only linear terms
\[ A_{\text{anom}}^r = \frac{1}{2} \gamma \left[ (\rho + 2 \theta r_g) r + \rho r_g - \frac{2}{3} \theta \Lambda r^3 \right] \left( r - \frac{1}{3} \Lambda r^3 - r_g \right) \left( 2 r - \frac{4}{3} \Lambda r^3 + \rho \right) \]
\[ -2 r (1 - \Lambda r^2) \left[ \theta \left( r - \frac{1}{3} \Lambda r^3 + \rho \right) \right] + O(\kappa^2). \]  
(7.148)

In the case \( r << r_0 \) (when \( |\Lambda| r^2 << 1 \)) this takes the form
\[ A_{\text{anom}}^r = \frac{1}{2} \gamma \left[ (\rho + 2 \theta r_g) r + \rho r_g - \frac{2}{3} \theta \Lambda r^3 \right] \frac{(r_g - r)^3}{r^3} + O(\kappa^2). \]  
(7.149)

Finally, for \( r_g << r << r_0 \) we obtain
\[ A_{\text{anom}}^r = \frac{1}{2} \gamma \left[ \frac{(\rho + 2 \theta r_g)}{r^2} - \frac{2}{3} \theta \Lambda r \right] + O(\kappa^2). \]  
(7.150)
7.5 Conclusions

In this chapter we described the kinematics of test particles in the framework of a recently developed modified theory of gravitation, called Matrix Gravity [19, 20, 21]. We outlined the motivation for this theory, which is a non-commutative deformation of General Relativity. Matrix Gravity can be interpreted in terms of a collection of Finsler geometries on the spacetime manifold rather than in terms of Riemannian geometry. This leads, in particular, to a new phenomenon of splitting of Riemannian geodesics into a system of trajectories (Finsler geodesics) close to the Riemannian geodesic. More precisely, instead of one Riemannian metric we have several Finsler metrics and different mass parameters which describe the tendency to follow a particular Finsler geodesics determined by a particular Finsler metric. As a result the test particles exhibit a non-geodesic motion which can be interpreted in terms of an anomalous acceleration.

By using a commutative algebra we found a static spherically symmetric solution of the modified Einstein equations. In this case a completely new feature appears due to the presence of additional degrees of freedom. The coordinate singularities of our model depend on additional parameters (constants of integration). Interestingly, there is a range of values for these free parameters in which no singularity occurs. This is just one of the intriguing differences between Matrix Gravity and General Relativity.

The description of matter in Matrix Gravity needs additional study. In this chapter we studied just the behavior of classical test particles. We propose to describe a gravitating particle by several mass parameters rather than one parameter as in General Relativity. We considered two models of matter: a uniform one, in
which all mass parameters are equal, and a non-uniform one, in which the mass parameters are different. The choice of one model over the other should be dictated by physical reasons. It is worth emphasizing that in the generic non-uniform model the equivalence principle is violated.

The interesting question whether the matter is described by only one mass parameter or more than one mass parameters as well as the more general question of the physical origin of multiple mass parameters requires further study. Since we do not know much about the physical origin of the color masses, we do not assume that they are positive. We do not exclude the possibility that some of the mass parameters can be negative or zero. This would imply, of course, that in this theory there is also gravitational repulsion (antigravity). This could help solve the problem of the gravitational collapse in General Relativity, which is caused by the infinite gravitational attraction.
CHAPTER 8

A MODEL FOR THE PIONEER ANOMALY

Abstract

In a previous work we showed that massive test particles exhibit a non-
geodesic acceleration in a modified theory of gravity obtained by a non-commutative
deformation of General Relativity (so-called Matrix Gravity). We propose that this
non-geodesic acceleration might be the origin of the anomalous acceleration experi-
enced by the Pioneer 10 and Pioneer 11 spacecrafts.

8.1 Introduction

The Pioneer anomaly has been studied by many authors (see [2, 3, 65, 67, 57, 77] and the references in these papers) and it has a pretty strong experimental
status [60]. It exhibits itself in an anomalous acceleration of the Pioneer 10 and 11
spacecrafts in the range of distances between 20AU and 50AU (∼ 10^{14} cm) from the
Sun. The acceleration is directed toward the Sun and has a magnitude of [2, 3]

\[ A_{anom}' \approx (8.74 \pm 1.33) \times 10^{-8} \text{ cm/s}^2. \]  

(8.1)

In the last years there have been many attempts to explain the Pioneer anomaly by
modifying General Relativity (see, for example, [65] and the references therein).

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6The material in this chapter has been published as a preprint: I. G. Avramidi and G. Fucci, A
Model for the Pioneer Anomaly, arXiv: 0811.1573 [gr-qc]
However, there is also some evidence [77] that it could not be explained within standard General Relativity since it exhibits a non-geodesic motion. That is, it cannot be explained by just perturbing the Schwarzschild metric of the Solar system. It seems, from the analysis of the trajectories, that the spacecrafts do not move along the geodesics of any metric. Another puzzling fact is that there is no measurable anomaly in the motion of the planets themselves, which violates the equivalence principle. In other words, the heavy objects like the planets, with masses greater than $\sim 10^{27}$g, do not feel any anomaly while the smaller objects, like the Pioneer spacecrafts, with masses of order $\sim 10^5$g, do experience it.

There are also some interesting numerical coincidences regarding the Pioneer anomaly (noticed in [63] as well). Recall that the cosmological distance, which can be defined either by the Hubble constant $H$ or by the cosmological constant $\Lambda$, is of order

$$r_0 \sim \frac{c}{H} \sim \frac{1}{\sqrt{\Lambda}} \sim 10^{28} \text{cm} \quad (8.2)$$

and the Compton wavelength of the proton is of order

$$r_1 \sim \frac{\hbar}{m_p c} \sim 10^{-13} \text{cm} \quad . \quad (8.3)$$

Now, we easily see, first of all, that there is the following numerical relation

$$\left( \frac{r_1}{r_{\text{anom}}} \right) \sim \left( \frac{r_{\text{anom}}}{r_0} \right)^2 \quad , \quad (8.4)$$

where $r_{\text{anom}} \sim 10^{14}$cm is the distance at which the anomaly is observed. This means that

$$r_{\text{anom}} \sim \left( \frac{\hbar}{m_p c \Lambda} \right)^{1/3} \sim \left( \frac{\hbar c}{m_p H^2} \right)^{1/3} \quad . \quad (8.5)$$
Secondly, the characteristic distance determined by the value of the anomalous acceleration, \( A_{\text{anom}} \sim 10^{-8} \text{cm/sec}^2 \), is of the same order as the cosmological distance

\[
r_2 \sim \frac{c^2}{A_{\text{anom}}} \sim 10^{28} \text{cm},
\]

which simply means that

\[
A_{\text{anom}} \sim Hc \sim c^2 \sqrt{\Lambda}.
\]

It is very intriguing to speculate that the Pioneer effect is the result of some kind of interplay between the microscopic and cosmological effects at the macroscopic scales.

In this chapter we apply the investigation of motion of test particles in an extended theory of gravity, called Matrix Gravity, initiated in [26] to study the anomalous acceleration of Pioneer 10 and Pioneer 11 spacecrafts.

We would like to stress that this study is just a first attempt to analyze the phenomenological effects of Matrix Gravity. We do not claim that this simple model definitely solves the mystery of the anomaly. Our aim is just to propose another candidate for its origin. Only future tests and more detailed models can describe the Pioneer anomaly in full capacity. This work does not represent the final answer, but just a first attempt of studying this phenomenon within the framework of Matrix Gravity.

### 8.2 Anomalous Acceleration in Matrix Gravity

The anomalous non-geodesic acceleration was derived within perturbation theory in the deformation parameter in [26] and presented in the previous Chapter. We study the two cases mentioned above.
We consider a simple model of $2 \times 2$ real symmetric commutative matrices. The static spherically symmetric solution of the matrix Einstein equations for this model was obtained in [26]. By using the results on the motion of test particles obtained in Chapter 7, and by recalling that in the non-relativistic limit the only essential component of the anomalous acceleration is the radial one $A_{\text{anom}}^{r}$, we obtain for the uniform model

$$A_{\text{anom}}^{r} = \frac{1}{2} \frac{\partial}{\partial r} \left[ -\frac{1}{1 - \frac{r^2}{r_0^2}} + f_1(r) - 2\theta f_2(r) + \left( \frac{\theta}{2} f_1(r) + (1 + \theta^2) f_2(r) \right)^2 \right], \quad (8.8)$$

and for the non-uniform model,

$$A_{\text{anom}}^{r} = -\frac{1}{2} \frac{\partial}{\partial r} \left[ \frac{1}{1 - \frac{r^2}{r_0^2}} + (\gamma \theta - 1) f_1(r) + \gamma (1 + \theta^2) f_2(r) \right]. \quad (8.9)$$

In this last formulas we have introduced the functions

$$u(r) = 1 - \frac{1}{3} \Lambda r^2 - \frac{r_g}{r}, \quad (8.10)$$

and

$$f_1(r) = \frac{u(r)}{\left[ u(r) + (\theta + 1) \frac{L}{r} \right] \left[ u(r) + (\theta - 1) \frac{L}{r} \right]}, \quad (8.11)$$

$$f_2(r) = \frac{\frac{L}{r}}{\left[ u(r) + (\theta + 1) \frac{L}{r} \right] \left[ u(r) + (\theta - 1) \frac{L}{r} \right]}. \quad (8.12)$$

We would like to emphasize at this point that the perturbation theory is only valid for small corrections. Obviously, when the corrections become large one needs to consider the exact equations of motion.

### 8.3 Pioneer Anomaly

We have two free parameters in our model, $\theta$ and $L$ (and $\gamma$ in the non-uniform model). We estimate these parameters to match the value of the observed anomalous acceleration of the Pioneer spacecrafts.
First of all, we recall the observed value of the cosmological constant \( \Lambda \approx 2.5 \cdot 10^{-56} \text{cm}^{-2} \); therefore, \( r_0 \approx |\Lambda|^{-1/2} = 6.3 \cdot 10^{27} \text{cm} \), and the gravitational radius of the Sun \( r_g \approx 1.5 \cdot 10^5 \text{cm} \). The relevant scale of the Pioneer anomaly is \( r_{anom} \sim 10^{14} - 10^{15} \text{cm} \), therefore, we can restrict our analysis to the range \( r_g << r << r_0 \). The values of the dimensionless parameters are \( \frac{r_g}{r} \sim 10^{-8}, \frac{r}{r_0} \sim 10^{-15} \), and \( \frac{r}{r_0} \sim 10^{-23} \). We also recall that the value of the anomalous acceleration is \( A_{anom}' \approx 8.7 \cdot 10^{-8} \text{cm/s}^2 \).

We should stress that our analysis only applies to the range of distances relevant for the study of the Pioneer anomaly. Therefore, strictly speaking, from a formal point of view, one cannot extrapolate our equations beyond this interval. Since the parameters \( \frac{r_g}{r}, \frac{r}{r_0} \) and \( \frac{r}{r_0} \) are negligibly small (compared to 1) they can be omitted.

By using the eqs. (8.8) and (8.9), and by defining \( \rho = (1 + \theta^2)L - \theta r_g \), we obtain [26] (in the usual units, \( c \) being the speed of light) for \( r_g << r << r_0 \): in the uniform model,

\[
A_{anom}' = -\frac{c^2}{4} \left( \frac{\rho + 2\theta r_g}{r^2} - \frac{2}{3} \theta \Lambda r \right),
\]

and in the non-uniform model,

\[
A_{anom}' = \frac{c^2}{2} \gamma \left( \frac{\rho + 2\theta r_g}{r^2} - \frac{2}{3} \theta \Lambda r \right).
\]

**Uniform Model.** First, we restrict to the case of vanishing cosmological constant. Then the function (8.13) takes the form

\[
A_{anom}'(r) = -\frac{c^2}{4} \left( \frac{\rho + 2\theta r_g}{r} \right) \frac{\rho + 2\theta r_g}{r^2} \cdot
\]

It has an extremum if the signs of \( \theta \) and \( \rho \) are different, which occurs at \( r_* = -\frac{3 \rho}{2 \theta} \) and is equal to

\[
A_{anom}'(r_*) = -\frac{c^2}{27} \frac{\theta^3 (\rho + 2\theta r_g)}{\rho^2}.
\]
Now, we assume that $r_* \sim r_{anom} \sim 10^{14}$ cm and $A_{anom}^r(r_*) \sim -10^{-8}$ cm/sec$^2$ to estimate the parameters
\[ \rho \sim 10^7 \text{cm}, \quad \theta \sim -10^{-7}. \] (8.17)

If we leave the cosmological constant there is another range of parameters that should be investigated. Namely, when the term $\frac{2\theta}{r^2}$ becomes comparable with the term $\frac{\rho}{r^2}$. In this case the anomalous acceleration can be written, by dropping negligible terms, as
\[ A_{anom}^r(r) = -\frac{c^2}{4r_0} \left( \frac{\theta r_0}{r^2} + \frac{2\theta^2}{3r_0} r \right). \] (8.18)

We note that the term $\frac{c^2}{4r_0}$ gives the right magnitude of the anomalous acceleration. If we assume that the two terms in the parentheses are comparable at the characteristic length $r_{anom}$ and are of order 1, then we get an estimate
\[ \rho \sim \frac{r_{anom}^3}{r_0^2} \theta \quad \text{and} \quad \theta \sim \left( \frac{r_0}{r_{anom}} \right)^{1/3}, \] (8.19)
and, therefore,
\[ \rho \sim 10^{-7} \text{cm} \quad \text{and} \quad \theta \sim 10^7. \] (8.20)

**Nonuniform Model.** In the non-uniform model we have an additional parameter $\gamma$. The function has an extremum at
\[ r_* = \left( \frac{3\rho r_0^2}{\theta} \right)^{1/3}. \] (8.21)

Now, we assume that $r_* \sim r_{anom} \sim 10^{14}$ cm; then
\[ \frac{\rho}{\theta} = \frac{r_*^3}{3r_0^2} \sim 10^{-13} \text{cm}. \] (8.22)
Further, by assuming $A_{\text{anom}}(r_e) \sim -10^{-8}\text{cm/sec}^2$ and using the eq. we estimate the parameter $\gamma$

$$\gamma \sim 10^{13}.$$  \hspace{1cm} (8.23)

It is interesting to notice that, in this case, $\rho/\theta$ has the same order of magnitude of the Compton wavelength of the proton. Moreover, by using (8.22), we confirm the coincidence (8.5) mentioned in the introduction. This is very intriguing; it allows one to speculate that the anomalous acceleration could be a result of an interplay between the microscopic and macroscopic worlds; in other words, the Pioneer anomaly could be a quantum effect.

8.4 Conclusions

In this chapter we applied the kinematics of test particles in Matrix Gravity to the study of the Pioneer anomaly.

We considered two models: a uniform one, in which a particle is described by a single mass parameter, and a non-uniform one, in which a particle is described by multiple mass parameters. The choice of one model over the other should be dictated by physical reasons. The interesting question of whether the matter is described by only one mass parameter or more than one mass parameters requires further study. If the Pioneer anomaly is a new physical phenomenon we have to accept the fact that the equivalence principle does not hold. If this is the case, a model with different mass parameters (violating the equivalence principle) would be more appropriate to describe the motion of test particles in the Solar system.
CHAPTER 9

CONCLUSIONS

In this work we carried out a detailed study of two major subjects. The first part of this Dissertation is devoted to the study of non-perturbative aspects of quantum electrodynamics on Riemannian manifolds by mainly utilizing heat kernel asymptotic expansion techniques. The second part focuses on the analysis of low energy aspects of a newly developed theory of the gravitational field called Matrix Gravity.

In the following we will present a summary of the main results obtained in this work and we will also describe some ideas for future directions of research.

9.1 Summary of the main results

It is needless to say that non-perturbative results are of fundamental importance in Physics. On one hand, the techniques used to obtain such results are very interesting from the mathematical point of view and, on the other hand, completely new physical phenomena, which are not predicted in perturbation theory, can be discovered. Here, we will present a list of the main results obtained in this Dissertation.

1. We established the existence of a new type of non-perturbative asymptotic expansion for the heat kernel for the Laplacian, and its trace, on homogeneous Abelian bundles.
2. We developed a new perturbation theory for the Laplacian, the heat semigroup and the heat kernel for the case in which the $U(1)$ curvature (electromagnetic field) is much stronger than the Riemannian curvature.

3. We explicitly evaluated the universal tensor functions in the heat kernel coefficients. They are analytic functions of the dimensionless quantity $tF$. These universal functions were not known in the literature and they have been evaluated here for the first time.

4. We evaluated, for the first time, the first three non-diagonal heat kernel asymptotic coefficients, in (3.219), (3.220) and (3.221) in powers of the Riemannian curvature and in terms of the new universal tensor functions.

5. We found the first three diagonal heat kernel asymptotic coefficients in (3.224), (3.237) and (3.238) in powers of the Riemannian curvature and in terms of the new universal tensor functions. This result is completely new and it has been presented here for the first time in the literature.

6. We proved, here, that all the off-diagonal odd-order heat kernel asymptotic coefficients are odd order polynomials in the normal coordinate $u$, and, therefore, they vanish on the diagonal.

7. We developed an algebraic framework for the calculation of the heat kernel asymptotic coefficients which only relies on the algebra of the commutators of the relevant operators.

8. We proved, in this work, that the new non-perturbative heat kernel asymptotic coefficients can be written in terms of polynomials that we called generalized
Hermite polynomials. For these polynomials we found the generating function and evaluated explicitly the first six polynomials.

9. We proved a formula that gives the $n$-th power of the sum of two operators which satisfy the Heisenberg algebra, in terms of a finite sum involving powers of the operators and powers of their commutator.

10. We evaluated the imaginary part of the effective action for both scalar and spinor fields in a $n$-dimensional curved spacetime under the influence of a strong electromagnetic field.

11. We explicitly evaluated the coefficient linear in the Riemannian curvature, $b_2(t)$, of the heat kernel asymptotic expansion. By using this coefficient, we found the imaginary part of the effective action, of zeroth and first order in the Riemannian curvature, non-perturbative in the electromagnetic field, in $n$-dimensions.

12. We generalized the classical result obtained by Schwinger for the creation of pairs in the electromagnetic field: For the first time in literature, we found an expression for the creation of scalar and spinor particles in curved spacetime induced by the gravitational field. This essentially non-perturbative effect in curved spacetime was completely unknown and it has been found here for the first time.

13. We explicitly evaluated the imaginary part of the effective action for both scalar and spinor fields in a number of different interesting limiting cases:

   - The physical dimension: $n = 4$. 
- Massless limit (Supercritical Electric Field): \( m^2 \ll E, B \).
- Pure electric field: \( B \ll m^2, E \), in \( n \) dimensions and \( n = 4 \).

14. We have discovered new infrared divergences in the imaginary part of the effective action for massless spinor fields in four dimensions (or supercritical electric field), which is induced purely by the gravitational correction.

15. In this work, we obtained, for the first time in literature, the dynamical equations for Matrix General Relativity in absence of matter by varying a generalization of the Hilbert-Einstein action.

16. We proposed an action to describe non-commutative matter and we derived, for the first time, the non-commutative Einstein equations in presence of matter.

17. We evaluated the action of Spectral Matrix Gravity in the weak deformation limit up to the second order in the deformation parameter. Namely, up to quadratic terms in the Riemannian curvature.

18. We evaluated, for the first time, the dynamical equations for Spectral Matrix Gravity in the weak deformation limit, and we found the spectrum of the theory on a DeSitter background.

19. We proposed a model for kinematics of test particles in Matrix Gravity. In particular we proposed that test particles are described by several mass parameters and that they do not move along the geodesics predicted by General Relativity. That is, we predict the violation of the equivalence principle.
20. We evaluated, in perturbation theory, the non-geodesic part of the motion interpreted as an anomalous acceleration. We found the anomalous acceleration, up to the first order in the perturbation parameter, when all the mass parameters are the same (uniform model) and we also found the anomalous acceleration, up to the second order in the perturbation parameter, when they are all different (non-uniform model).

21. We found a static spherically symmetric solution of Matrix General Relativity in a simple model of Abelian $2 \times 2$ matrices.

22. In particular, we found that there exists a range for the free parameters in the theory for which there is no horizon in the static and spherically symmetric solution. This is a completely new feature, which is absent in General Relativity. In this particular Abelian case we computed explicitly the anomalous acceleration for both the uniform and non-uniform models.

23. We applied our results to the analysis of a recently found anomaly in the trajectories of the Pioneer 10 and 11 spacecrafts. We found that there is a range for the free parameters of our theory which can be adjusted to give the right order of magnitude for the Pioneer anomaly.

9.2 Future directions of research

The results in this Dissertation open a variety of different opportunities for future research. In the following we present a list of some of the ideas involving the results we obtained for the non-perturbative heat kernel asymptotic expansion on homogeneous Abelian bundles.
• The study of complex manifolds is very important in mathematics. An Hermitian manifold is a even-dimensional complex manifold on which a smooth Hermitian metric is defined. It can be shown that on this manifold there always exist an antisymmetric 2-form, called the Kähler form, which is obtained from the Hermitian metric. It is well known, then, that an Hermitian manifold is a Kähler manifold if the Kähler 2-form is closed. If we assume that the curvature on the Kähler manifold is small, we satisfy all the conditions for the analysis done in Chapter 3 by just replacing the electromagnetic 2-form with the Kähler 2-form. Such analysis would give the heat kernel asymptotic expansion coefficients for the Laplacian defined on Kähler manifolds. This would be an important study especially in connection with String Theory in which one often uses complex manifolds.

• Two dimensional quantum field theory has become very important over the past years. In fact, in two dimensions, one can obtain exact results for many cases of interest, which one can use to get a better understanding of how a particular quantum system would behave in four dimensions. Our results, when restricted to dimension two, would dramatically simplify. The heat kernel asymptotic coefficients that we found can be used to evaluate the effective action for quantum fields in two dimensional curved space under the influence of a strong electromagnetic field. In particular, the coefficient $b_2$ (linear in the Riemannian curvature) found here is responsible for the divergences in two dimensions and can be used to regularize the theory. Moreover, if one considers conformal fields, the same coefficients would give the conformal anomaly.
• During the evaluation of the non-perturbative heat kernel asymptotic coefficients, we introduced the generalized Hermite polynomials to represent the derivatives of the Schwinger heat kernel. These are polynomials in normal coordinates with coefficients given by analytic functions of the electromagnetic 2-form. It will be very interesting, from the mathematical point of view, to systematically study the most important properties of these polynomials. In particular, by finding the differential equations they satisfy one could attempt to prove some orthogonality relations between them. Moreover, it would be interesting to see if we can find some different representations, other than a Rodrigues-type formula find here, for the generalized Hermite polynomials. Lastly, it would be very interesting to find a recurrence relation especially for their derivatives.

• The Dirac operator has been intensively studied both in mathematics and theoretical physics. It is globally defined on a manifold $M$ provided that $M$ is orientable and has a spin structure. It can be proved that if $M$ is compact, the Dirac operator is elliptic, self-adjoint, and it has a discrete spectrum. Of particular relevance is the index of the Dirac operator because it gives a measure of the difference between left and right spinors due to the topology of the manifold $M$. This gives origin to the chiral anomaly [47, 66]. It can be proved that the index of an elliptic operator, and in particular of the Dirac operator, can be expressed in terms of the heat kernel. The results obtained in this work could be used in order to study the index of the Dirac operator in four dimensions under the condition of large parallel $U(1)$ curvature. It would be very interesting to analyze in what extent a strong electromagnetic field influences the number of left and right spinors and the chiral anomaly.
The non-perturbative results that we obtained earlier, are valid for an arbitrary Riemannian metric. Considering, in particular, Einstein manifolds, for which the metric satisfies Einstein’s equations, would be very interesting in astrophysical situations in which the electromagnetic field is stronger than the gravitational field. In fact, by specifying our result for the imaginary part of the effective action in the Schwarzschild metric, we could describe particle creation in the surrounding areas of objects like pulsars and magnetars. In particular, once the imaginary part of the effective action for scalar and spinor fields is known, one can evaluate the current of created particles in the neighborhood of such astrophysical objects.

Matrix Gravity is obviously of recent discovery, and much more research is needed to be done in order to unveil all of its features. In the following we list some ideas for future research.

- It is well known that General Relativity cannot be quantized in a consistent way, because Einstein’s theory is non-renormalizable. A satisfactory quantized theory of the gravitational field is the most important unsolved problem in present day theoretical physics. It would be interesting to study in detail whether or not Matrix Gravity can be successfully quantized. In fact, Matrix Gravity is nothing but a generalized \( \sigma \)-model, and therefore the problems of quantization are the same as a field theoretical \( \sigma \)-model.

- An interesting study would be the analysis of some simple non-commutative, static and symmetric solutions of the dynamical equations, like a non-commutative Schwarzschild metric and a non-commutative DeSitter metric. The first
would describe a non-commutative black hole while the second would represent a non-commutative cosmological model. The singularities in General Relativity appear when the geodesics in the spacetime cannot be prolonged to arbitrary values of their affine parameter, this means that all the geodesics converge to the singular point at a certain time. Since in our model the trajectory of particles is described by a bundle of trajectories, there is the possibility that the non-commutative black hole would be free of singularities. This would be an important feature of our model which is not present in General Relativity. By analyzing a non-commutative cosmological model, like DeSitter, one could try to understand if our model predicts the accelerated expansion of the Universe without relying on the concept of Dark Energy. Having a theory that does not rely on the mysterious Dark Energy to explain the accelerated expansion of the Universe would be of fundamental importance.

- Lastly, an important problem afflicting General Relativity is the flat rotational curves of galaxies. It is well known that the radial velocity of a non-relativistic fluid in rotation decreases, following a specific law, with the distance from the rotational axis. However, observations show that the radial velocity stays approximately constant regardless of the distance from the center. In order to explain this discrepancy with the theory, the idea of Dark Matter has been introduced. By looking at the rotational curves of galaxies one is able to derive the distribution of Dark Matter (which is unobservable) within the galaxy and in its neighborhoods. Obviously this idea is not satisfactory from a theoretical point of view. It would be interesting, then, to understand if a non-commutative model for galaxies could explain the flat rotational curves without relying on the concept of Dark Matter.
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