CORRELATIONS BETWEEN PRIMES IN SHORT INTERVALS
ON CURVES OVER FINITE FIELDS

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ABSTRACT. We prove an analogue of the Hardy-Littlewood conjecture on the asymptotic distribution of prime constellations in the setting of short intervals in function fields of smooth projective curves over finite fields.

1. INTRODUCTION

In recent work [2], Bary-Soroker and the first author use their work with Rosenzweig [3] on the asymptotic distribution of primes inside short intervals in $\mathbb{F}_q[t]$ to establish a natural counterpart, over the field $\mathbb{F}_q(t)$, to the still unsolved Hardy-Littlewood conjecture. In [4], the two present authors show how to extend the results of [3] to give asymptotic distributions of primes in short intervals on the complement of a very ample divisor in a smooth, geometrically irreducible projective curve $C$ over a finite field $\mathbb{F}_q$. In the present paper, we apply ideas from [2] and [4] to prove a natural counterpart to the Hardy-Littlewood conjecture on the complement of a very ample divisor in a smooth, geometrically irreducible projective curve $C$ over a finite field $\mathbb{F}_q$.

1.1. The Hardy-Littlewood conjectures for short intervals. Let $K$ be a number field, $\mathcal{O}_K$ its ring of integers, and $N(h) \triangleq \#\mathcal{O}_K/(h)$ the norm on elements $h \in \mathcal{O}_K$. Given an $n$-tuple $\sigma = (\sigma_1, \ldots, \sigma_n)$ of elements in $\mathcal{O}_K$, denote by $\pi_{K,\sigma}(x)$ the $n$-tuple principal prime counting function

$$\pi_{K,\sigma}(x) \triangleq \# \{ h \in \mathcal{O}_K : 2 < N(h) \leq x, \text{ and each } (h+\sigma_i) \subset \mathcal{O}_K \text{ is prime} \}.$$

The Hardy-Littlewood $n$-tuple conjecture [11] (henceforth the HL conjecture), asserts that:

**Conjecture 1.1.1 (Hardy-Littlewood).** The function $\pi_{Q,\sigma}$ satisfies the asymptotic formula

$$\pi_{Q,\sigma}(x) \sim \mathcal{G}(\sigma) \frac{x}{(\log x)^n}, \quad x \to \infty$$

for a positive constant $\mathcal{G}(\sigma)$.

Despite much numerical verification and many proof attempts, the HL conjecture remains open. Results toward the conjecture include the work of Goldston, Pintz and Yildirim [8], Zhang [18] and Maynard [14]. See Graville [9] for a partial history of the problem.

One can also formulate short interval variants of the HL conjecture. As in [4, Discussion before Conjecture 1.3.1], the ambiguity in defining subsets of $\mathcal{O}_K$ in terms of the norm on $K$ leads to at least two possible formulations. Fix a $n$-tuple $\sigma = (\sigma_1, ..., \sigma_n)$ of elements in $\mathcal{O}_K$. Then each real number $1 > \varepsilon > 0$ determines a family of intervals $I(x, \varepsilon) \triangleq [x-x^\varepsilon, x+x^\varepsilon] \subset \mathbb{R}$, with corresponding prime counting function

$$\pi_{K,\sigma}(I(x, \varepsilon)) \triangleq \# \{ N_K(h) \in I(x, \varepsilon) : \text{each } (h+\sigma_i) \subset \mathcal{O}_K \text{ is prime} \}.$$

On the other hand, if $S = \{ \text{infinite places of } K \}$ and $\varepsilon_S = (\varepsilon_p)_{p \in S}$ is an $\#S$-tuple of real numbers $0 < \varepsilon_p > 0$, then for each $b \in \mathcal{O}_K$, we can define

$$I(b, \varepsilon_S) \triangleq \{ a \in \mathcal{O}_K : |a-b|_p \leq |b|_p^{\varepsilon_p} \text{ for each } p \in S \},$$

**Conjecture 1.1.1**
with corresponding prime counting function
\[ \pi_{\mathcal{K}, \sigma}(I(b, \varepsilon_S)) = \# \{ a \in I(b, \varepsilon_S) : \text{ each } (a + \sigma_i) \subset \mathcal{O}_K \text{ is prime} \}. \]

**Conjecture 1.1.2. (i).** The function \( \pi_{\mathcal{K}, \sigma}(I(x, \varepsilon)) \) satisfies
\[ \pi_{\mathcal{K}, \sigma}(I(x, \varepsilon)) \sim \mathcal{G}(\sigma) \frac{\# I(x, \varepsilon)}{(\log x)^n}, \quad x \to \infty, \]
where \( \mathcal{G}(\sigma) \) is some positive constant depending on \( \sigma \) and on the class number of \( K \).

**Conjecture 1.1.2. (ii).** The function \( \pi_{\mathcal{K}, \sigma}(I(b, \varepsilon_S)) \) satisfies
\[ p^n_{\mathcal{K}, \sigma}(I(b, \varepsilon_S)) \sim \mathcal{G}(\sigma) \frac{\# I(b, \varepsilon_S)}{(\log N_K(b))^n}, \quad N_K(b) \to \infty, \]
where \( \mathcal{G}(\sigma) \) is some positive constant depending on \( \sigma \) and on the class number of \( K \).

Gross and Smith [10] present numeric evidence for Conjecture 1.1.2.(i) for certain number fields and reasonable sets, which are regions that may be interpreted as short intervals. Based on this, they present a conjecture similar to Conjecture 1.1.2.(i) above.

**Theorem 1.1.3.** Fix an integer \( B > n \). Then:

(i) (Pollack [16, 17], Bary-Soroker [6], Carmon [7]). The function \( \pi_{q,t}(k) \) satisfies
\[ \pi_{q,t}(k) = \frac{q^k}{k^n} \left( 1 + O_B(q^{-1/2}) \right) \]
uniformly in \( f_1, \ldots, f_n \) with degrees bounded by \( B \) as \( q \to \infty \).

(ii) (Bank & Bary-Soroker [2, Theorem 1.1]). The function \( \pi_{q,t}(I(f_0, \varepsilon)) \) satisfies
\[ \pi_{q,t}(I(f_0, \varepsilon)) = \frac{\# I(f_0, \varepsilon)}{\prod_{i=1}^n \deg(f_0 + f_i)} \left( 1 + O_B(q^{-1/2}) \right) \]
uniformly for all \( f_1, \ldots, f_n \) of degree at most \( B \), and for all monic polynomials \( f_0 \) satisfying \( \frac{2}{\varepsilon} < \deg f_0 < B \), as \( q \to \infty \) an odd prime power.

1.2. **Main results: Correlation of primes in short intervals on curves.** Let \( C \) be a smooth projective geometrically irreducible curve over a finite field \( \mathbb{F}_q \). Fix an effective divisor \( D = m_1p_1 + \cdots + m_sp_s \) on \( C \). Let \( \sigma = (\sigma_1, \ldots, \sigma_n) \) be an \( n \)-tuple of rational functions \( \sigma_i \) on \( C \), regular on \( C \setminus E \equiv C \setminus \text{supp}(E) \). Define an \( n \)-tuple principal prime counting function
\[ \pi_{C, \sigma}(E) \equiv \# \left\{ h \in \mathbb{P}^1(C) \text{ such that } \div(h)_E = E, \text{ and each } \right. \]
\[ \left. \text{function } h + \sigma_1, \ldots, h + \sigma_n \text{ generates a prime ideal in the ring of regular functions on } C \setminus E \right\}. \tag{2} \]

**Remark 1.2.1.** The condition that \( \div(h)_E = E \) should be thought of as analogous to the condition \( \deg h = k \) in (1). In the special case where \( C = \mathbb{P}^1 \), the ring of rational functions on \( C \) regular away from the point \( \infty \in \mathbb{P}^1 \) is the polynomial ring \( \mathbb{F}_q[t] \). Thus when \( E = \infty \), the function \( \pi_{C, \sigma} \) defined in (2) reduces to the function \( \pi_{q,t} \) defined in (1).
Conjecture 1.2.2. The function $\pi_{C,\sigma}(E)$ satisfies

$$\pi_{C,\sigma}(E) \sim \mathcal{S}(\sigma) \left( \frac{q^{\deg E}}{(\deg E)^n} \right)^n, \quad q \to \infty$$

for a positive constant $\mathcal{S}(\sigma)$ depending on $\sigma$, uniformly in $\sigma_1, \ldots, \sigma_n$ with bounded degree.

Definition 1.2.3. Given a regular function $f$ on $C \setminus E$, the interval (of size $E$ around $f$) is the set

$$I(f, E) \triangleq \left\{ \text{regular functions on } C \setminus E \text{ such that } \nu_p(h - f) \geq -m_i \text{ for all } 1 \leq i \leq s \right\}$$

for each $p$. Define the interval $I(f, E)$ as a short interval if the order of the pole of $f$ at each $p_i$ is at least $m_i$, and strictly greater than $m_i$ for at least one $p_i$. Define

$$\pi_{C,\sigma}(I(f, E)) \triangleq \# \left\{ h \in I(f, E) \text{ such that } h + \sigma_1, \ldots, h + \sigma_n \text{ generate prime ideals in the ring} \right\}.$$

Our main result is an analogue of Conjecture 1.1.2 that extends Theorem 1.1.3.(ii) to curves of arbitrary genus over $\mathbb{F}_q$. Fix a smooth projective geometrically irreducible curve $C$ over $\mathbb{F}_q$. Let $E = m_1p_1 + \cdots + m_sp_s$ be an effective divisor on $C$, and let $f_0, \sigma_1, \ldots, \sigma_n$ be distinct regular functions on $C \setminus E$ satisfying $-\nu_p(f_0) > m_q$ and $\nu_p(f_0) \neq \nu_p(\sigma_i)$ for each $1 \leq i \leq n$.

Theorem A. Fix an integer $B > n$. If $\text{char } \mathbb{F}_q \neq 2$ and $E \geq 3E_0$ for some effective divisor $E_0$ on $C$ with $\deg E_0 \geq 2g + 1$, then the asymptotic formula

$$\pi_{C,\sigma}(I(f_0, E)) = \frac{\#I(f_0, E)}{\prod_{i=1}^n \deg(f_0 + \sigma_i)} \left( 1 + O_B(q^{-1/2}) \right)$$

holds uniformly for all $E$ and $f_0, \sigma_1, \ldots, \sigma_n$ as above satisfying $\deg(\text{div}(f_0 + \sigma_i)|_E) < B$, and as $q \to \infty$ an odd prime power.

1.3. Outline of paper. Our strategy in proving Theorem A is similar in spirit to [2]. In §3 we briefly review the necessary divisor theory and show that the splitting fields of distinct linear functions, evaluated at the generic element $f_A$ of a short interval, are linearly disjoint. We use this to show that the Galois group of the product of several linear functions evaluated at $f_A$ is isomorphic to a direct product of symmetric groups. In §3 we use a Chebotarev-type density theorem to estimate $\pi_{C,\sigma}(I(f_0, E))$. In §3.2, we prove the main Theorem A as a special case of the more general Theorem B, which deals with general factorization types.

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2. Galois group calculation

2.1. Relevant background. We make use of the theory of divisors on algebraic varieties. Necessary background appears in [4, §2], with further details in [12, II.6]. For each point $p \in C$, let $\nu_p$ denote valuation at $p$, applied to both functions and divisors on $C$. 

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For the remainder of the present §2, we fix an effective very ample divisor $E$ on $C$ and a function $f_0$ regular on $C \setminus E$ satisfying

$$-\nu_p(f_0) > \nu_p(E) \quad \text{for all } p \in \text{supp}(E).$$

We require a somewhat stronger positivity condition on $E$ than “effective and very ample.” Namely, assume that there exists a divisor $E_0$ on $C$ such that

$$E \geq 3E_0. \quad (3)$$

Observe however that if $E$ is effective and very ample but fails to satisfy (3), then we can replace $E$ by $3E$ to achieve (3). Let $\sigma = (\sigma_1, \ldots, \sigma_n)$ be an $n$-tuple of distinct rational functions on $C$, each regular on $C \setminus E$, and each satisfying the inequalities

$$\nu_p(\sigma_i) \neq \nu_p(f_0) \quad \text{and} \quad -\nu_p(\sigma_i) > \nu_p(E), \quad \text{for all } p \in \text{supp}(E). \quad (4)$$

For each $\sigma_i$, define the monic linear polynomial

$$L_i(X) \overset{\text{def}}{=} \sigma_i + X \quad \text{in} \quad \mathbb{F}_q(C)[X].$$

Let $R$ denote the ring of regular functions on $C \setminus E$. Fix a basis $\{1, g_1, \ldots, g_m\}$ of $H^0(C, \mathcal{O}(E))$ once and for all, let $\mathbb{F}_q(A) = \mathbb{F}_q(A_0, \ldots, A_m)$ denote the field of rational functions in $m + 1$ variables, and define $\mathbb{A}^{m+1} \overset{\text{def}}{=} \text{Spec} \mathbb{F}_q[A] = \text{Spec} \mathbb{F}_q[A_0, \ldots, A_m]$. On the trivial family of curves $(C \setminus E) \times_{\mathbb{F}_q} \mathbb{A}^{m+1} = \text{Spec} R[A] = \text{Spec} R \otimes_{\mathbb{F}_q} \mathbb{F}_q[A]$, we have the regular function

$$\mathcal{J}_A \overset{\text{def}}{=} f_0 + A_0 + \sum_{j=1}^m A_j g_j.$$

For each $\mathbb{F}_q$-rational point $a \in \mathbb{A}^{m+1}$, the restriction of $\mathcal{J}_A$ to $(C \setminus E) \times_{\mathbb{F}_q} \text{Spec} \kappa(a)$ defines a regular function $\mathcal{J}_a$ on $C \setminus E$.

Let $K/\mathbb{F}_q(A)$ be an algebraic extension. For an ideal $\mathcal{J}$ in the Dedekind domain $K \otimes_{\mathbb{F}_q(A)} R(\mathcal{A})$, let

$$\mathcal{J} = \mathfrak{P}_1^{e_1} \cdots \mathfrak{P}_\ell^{e_\ell} \quad (5)$$

denote the prime decomposition of $\mathcal{J}$.

**Definition 2.1.1.** (i) If $e_1 = \cdots = e_\ell = 1$ in (5), then we define the splitting field of $\mathcal{J}$ over $K$, denoted split($\mathcal{J}$) or split($\mathcal{J}/K$), to be the composite

$$\text{split}(\mathcal{J}) \overset{\text{def}}{=} \text{split}(\mathfrak{P}_1) \cdots \text{split}(\mathfrak{P}_\ell),$$

where for each $1 \leq i \leq n$, split($\mathfrak{P}_i$) denotes the normal closure of $\kappa(\mathfrak{P}_i)$ in $\mathbb{F}_q(A)$.

(ii) If each extension $\kappa(\mathfrak{P}_i)/K$ is separable, then the composite extension split($\mathcal{J}$)/$K$ is normal, and we define the Galois group of $\mathcal{J}$ to be

$$\text{Gal}(\mathcal{J}/K) \overset{\text{def}}{=} \text{Gal}($$split($\mathcal{J}$)/$K$).$$

2.2. **Linearly disjointness of splitting fields.** For each $1 \leq i \leq n$, the rational function $L_i(\mathcal{J}_A)$ on $C_{\mathbb{F}_q(A)}$ determines a morphism

$$L_i(\mathcal{J}_A) : C_{\mathbb{F}_q(A)} \longrightarrow \mathbb{F}_q(A)$$

and a field extension

$$\mathbb{F}_q(A) \hookrightarrow \mathbb{F}_q(C)(A_0, \ldots, A_m). \quad (6)$$

Likewise, if we define $\mathbb{F}_q(A') \overset{\text{def}}{=} \mathbb{F}_q(A_1, \ldots, A_m)$, then the rational function

$$\Psi_i \overset{\text{def}}{=} \sigma_i + f_0 + \sum_{j=1}^m A_j g_j = L_i(\mathcal{J}_A) - A_0.$$
on $C_{q}(A')$ provides a morphism

$$\Psi_i : C_{q}(A') \rightarrow \mathbb{P}^1_{q}(A')$$

and field extension

$$\mathbb{F}_q(A')(t) \hookrightarrow \mathbb{F}_q(C)(A_1, \ldots, A_m). \quad (7)$$

**Lemma 2.2.1.** The field extension (6) is separable.

*Proof.* The assumption (4) implies that for each $p \in \text{supp}(E)$, we have

$$\nu_p(\sigma_i + f_0) = \min\{\nu_p(\sigma_i), \nu_p(f_0)\} < -\nu_p(E). \quad (8)$$

This makes $I(\sigma_i + f_0, E)$ a short interval in the sense of [4, corrected Definition 1.4.1]. Thus by [4, Lemmas 3.3.1 & 3.3.3], the variety $V(L_i(\mathcal{F}_A)) \subset (C\setminus E)_{\mathbb{F}_q(A)}$ consists of a single point $\Psi_i$ such that the field extension

$$\mathbb{F}_q(A) \hookrightarrow \kappa(\Psi_i) \quad (9)$$

is separable. Under the identification

$$\mathbb{F}_q(A) \sim \mathbb{F}_q(A')(t)$$

$$A_0 \mapsto -t, \quad (10)$$

the extensions (7) and (9) become isomorphic. Thus (7) is separable. \hfill \Box

**Remark 2.2.2.** Let $\Psi_i$ be the underlying point of $V(\mathcal{F}_A)$ as in the proof of Lemma 2.2.1. By Lemma 2.2.1, we can consider the splitting field

$$\text{split}(L_i(\mathcal{F}_A)) \cong \text{split}(\kappa(\Psi_i)/\mathbb{F}_q(A))$$

obtained as the normal closure of the extension (6), as defined in [4, Definition 3.2.1]. Let $d_i$ denote the discriminant of the extension $\text{split}(L_i(\mathcal{F}_A))/\mathbb{F}_q(A)$, as defined in [15, §III.3, Definition (2.8)] for instance. The discriminant is a fractional ideal in $\mathbb{F}_q(A)$ that restricts to an actual ideal in $\mathbb{F}_q(A')[A_0]$. Because $\mathbb{F}_q(A')[A_0]$ is a principal ideal domain, there exists a function $D_i \in \mathbb{F}_q(A')[A_0]$, well defined up to multiplication by elements in $\mathbb{F}_q(A')^\times$, such that

$$d_i = (D_i) \in \mathbb{F}_q(A')[A_0].$$

Via the identification (10), [15, §III.3, Corollary (2.12)] implies that the map $\Psi_i$ is ramified at a point $x$ in $C_{q}(A')\setminus E_{q}(A')$ if and only if $D_i$ vanishes at the point $\Psi_i(x)$ inside $\text{Spec} \ \mathbb{F}_q(A')[A_0] \subset \mathbb{P}^1_{q}(A').$

**Notation 2.2.3.** Let $\Omega^1_C$ denote the sheaf of Kähler differentials on $C$. For each $1 \leq i \leq n$, the functions $\Psi_i$ and $L_i(\mathcal{F}_A)$ are regular on $(C\setminus E)_{\mathbb{F}_q(A')}$ and $(C\setminus E)_{\mathbb{F}_q(A')}$, respectively. Let $d\Psi_i$ and $dL_i(\mathcal{F}_A)$ denote their differentials, and note that

$$d\Psi_i = dL_i(\mathcal{F}_A) \quad \text{on} \quad C_{q}(A')\setminus E_{q}(A).$$

Given a field extension $K/\mathbb{F}_q$, a section $\omega \in \Gamma(C_K \setminus E_K, \Omega^1_{C_K})$, and any point $x \in C_K \setminus E_K$, let $\omega|_x$ denote the restriction of $\omega$ to the fiber $(\Omega^1_{C_K})_x$.

**Lemma 2.2.4.** In the notation of Remark 2.2.2, $D_1$ and $D_2$ are relatively prime in the polynomial ring $\mathbb{F}_q(A')[A_0]$ if and only if the system of equations

$$d\Psi_1|_x = 0;$$
$$d\Psi_2|_y = 0; \quad (11)$$
$$\Psi_1(x) = \Psi_2(y),$$

has no solution in pairs of (not necessarily distinct) points $x, y \in (C\setminus E)_{\mathbb{F}_q(A')}$.
Proof. Because $\Psi^{-1}_i(\infty) = \text{supp}(E_{q}(A'))$, Remark 2.2.2 implies that $D_1$ and $D_2$ are relatively prime if and only if the branch divisors of $\Psi_1$ and $\Psi_2$ are disjoint in $\text{Spec} \mathbb{F}_q(A')$. Recall that the support of the branch divisor of $\Psi_1$ is the set of all $p \in \mathbb{P}^1_{E_{q}(A')}$ such that
\[ p = \Psi_1(x) \quad \text{and} \quad d\Psi_1|_x = 0 \]
for some $x \in C_{E_{q}(A')}$. Thus a point $p$ of $\text{Spec} \mathbb{F}_q(A')$ in the support of both branch divisors $\Psi_1$ and $\Psi_2$ is any point satisfying $\Psi_1(x) = p = \Psi_2(y)$ and $d\Psi_1|_x = 0 = d\Psi_2|_y$ for some (not necessarily distinct) pair of points $x, y \in C_{E_{q}(A')} \setminus \mathbb{F}_{q}(A')$. □

**Proposition 2.2.5.** If char $\mathbb{F}_q \neq 2$, then the system of equations (11) has no solution in points $x, y \in C_{E_{q}(A')} \setminus \mathbb{F}_{q}(A')$.

Proof. Since $\Psi_i = L_i(\mathcal{F}_A) - A_0$, it suffices to prove that the system of equations
\[
\begin{align*}
dL_1(\mathcal{F}_A)|_x &= 0 \\
dL_2(\mathcal{F}_A)|_y &= 0 \\
L_1(\mathcal{F}_A)(x) &= L_2(\mathcal{F}_A)(y)
\end{align*}
\]
has no solutions over $\mathbb{F}_{q}(A)$.

As in the proof of [4, Proposition 4.3.3], choose an effective very ample divisor $E_0$ satisfying (3), define $m_0 \overset{\text{def}}{=} \dim H^0(C, \mathcal{O}(E_0)) - 1$, and let $C \subset \mathbb{P}^{m_0}$ be the closed embedding determined by $E_0$.

Assume first that $x \neq y$. For each pair of distinct points $\xi, \eta \in C_{\mathbb{F}_q \setminus \mathbb{F}_q}$, let $t$ be a linear form on $\mathbb{P}^{m_0}$ such that $t(\xi) \neq t(\eta)$. The assumption on $E_0$ gives us a new $\mathbb{F}_q$-linear basis
\[
\{1, t, t^2, \ldots, t^\ell, g_{\ell+1}, \ldots, g_m\}
\]
of $H^0(C_{\mathbb{F}_q \setminus \mathbb{F}_q}, \mathcal{O}(E_{q}))$,
and a new coordinate system $A_0, B_1, \ldots, B_m$ on $A_{m+1}$ such that
\[
L_i(\mathcal{F}_A) = \sigma_i + f_0 + A_0 + B_1t + \cdots + B_{\ell}t^{\ell} + B_{\ell+1}g_{\ell+1} + \cdots + B_m g_m
\]
for each $i = 1, 2$. For $i = 1, 2$, define $\Phi_i \overset{\text{def}}{=} L_i(\mathcal{F}_A) - B_1t - B_2t^2$. As in the proof of [4, Proposition 4.3.3], we can choose a Zariski opens $U_{\xi, \eta}$ that provide a covering of
\[
((C_{\mathbb{F}_q \setminus \mathbb{F}_q} \times (C_{\mathbb{F}_q \setminus \mathbb{F}_q})) \setminus \{\text{diagonal}\}),
\]
such that an $\mathbb{F}_q(A)$-valued solution $(u, v) \in U_{\xi, \eta}$ to (12) is the same thing as a solution to the single equation
\[
\det(u, v) \overset{\text{def}}{=} \det \begin{pmatrix}1 & 2t(u) & \varphi_1(u) \\ 1 & 2t(v) & \varphi_2(v) \\ t(v) - t(u) & t(v)^2 - t(u)^2 & c(u, v) \end{pmatrix} = 0,
\]
where $\varphi_i = \frac{\partial \Phi_i}{\partial u}$ for $i = 1, 2$, and where $c(u, v) \overset{\text{def}}{=} \Phi_1(u) - \Phi_2(v)$. A direct calculation gives
\[
\det(u, v) = \left((t(v) - t(u)) \left(2c(u, v) + (t(u) - t(v)) (\varphi_1(u) + \varphi_2(v))\right)\right).
\]
By the same reasoning as in [4, Proposition 4.3.3], it suffices to prove that $\det(u, v)$ cannot be 0 identically on $U_{\xi, \eta}$.

If $n \geq 3$, then the coefficient of $B_3$ in
\[
2c(u, v) + (t(u) - t(v)) (\varphi_1(u) + \varphi_2(v))
\]
is $2(t(u)^3 - t(v)^3) + (t(u) - t(v))3(t(u)^2 + t(v)^2)$, which is not identically 0. Since $t(u) \neq t(v)$, this implies that $\det(u, v)$ is not identically zero on $U_{\xi'}$.

Assume next that $x = y$. If the pair $x = y \in C_{E_q}(A) \cup E_q(A)$ is a solution to (12), then $L_1(\mathcal{F}_A)(x) = L_2(\mathcal{F}_A)(x)$, implying that $\sigma_1 - \sigma_2(x) = 0$. Because $\sigma_1$ and $\sigma_2$ are distinct regular functions on $C_{E_q}(A) \cup E_q(A)$, this implies that $x \in C_{E_q}(A) \cup E_q(A)$. This allows us to choose a linear form $t$ on $\mathbb{F}_q^m$ such that $t(x) \neq 0$. Using this linear form $t$, choose an $\mathbb{F}_q$-linear basis (13). This choice lets us write $L_i(\mathcal{F}_A)$ as in (14). The solution $x$ satisfies $dL_1(\mathcal{F}_A)|_x = 0$. But our choice of $t$ lets us write

$$0 = dL_1(\mathcal{F}_A)|_x = \frac{d\sigma_1}{dt}(x) + \frac{df_0}{dt}(x) + B_1 + \sum_{j=2}^\ell jB_j t^{j-1}(x) + \sum_{j=\ell+1}^m B_j \frac{dg_j}{dt}(x)$$

and so

$$-B_1 = \frac{d\sigma_1}{dt}(x) + \frac{df_0}{dt}(x) + \sum_{j=2}^\ell jB_j t^{j-1}(x) + \sum_{j=\ell+1}^m B_j \frac{dg_j}{dt}(x).$$

However, the right hand side of this last equation does not involve $B_1$, contradicting the fact that $x \in C_{E_q}(A) \cup E_q(A)$. □

**Corollary 2.2.6.** If $\text{char } \mathbb{F}_q \neq 2$, then the splitting fields split($L_i(\mathcal{F}_A)$), for $1 \leq i \leq n$, are linearly disjoint over $\mathbb{F}_q(A)$.

**Proof.** As noted in the proof of Lemma 2.2.1, for each $p \in \text{supp}(E)$ and for each $1 \leq i \leq n$, the conditions (4) imply that the inequality (8) holds. Thus the hypotheses of [4, Proposition 4.3.3 and Theorem 4.1.1] are satisfied and we have $\text{Gal} \left( L_i(\mathcal{F}_A), \mathbb{F}_q(A) \right) \cong S_{k_i}$ with

$$k_i \overset{\text{def}}{=} \text{deg } \text{div}(f_0 + \sigma_i)|_{C \setminus E} = \text{deg } \text{div}(L_i(f_0))|_{C \setminus E}.$$ 

Because we have inclusions $\text{Gal} \left( L_i(\mathcal{F}_A), \mathbb{F}_q(A) \right) \subseteq \text{Gal} \left( L_i(\mathcal{F}_A), \mathbb{F}_q(A) \right) \hookrightarrow S_{k_i}$, this implies

$$\text{Gal} \left( L_i(\mathcal{F}_A), \mathbb{F}_q(A) \right) \cong S_{k_i}.$$ 

Let $A_{k_i} \subset S_{k_i}$ denote the alternating group on $k_i$ letters. Its fixed field is the quadratic extension

$$\text{split}(L_i(\mathcal{F}_A))^{A_{k_i}} \cong \mathbb{F}_q(A)(\sqrt{D_i}),$$

where $D_i \in \mathbb{F}_q(A')[A_0]$ is an element generating the discriminant ideal $\mathfrak{d}_i$ as in Remark 2.2.2. By [5, Lemma 3.3], it suffices to prove that the fields $\mathbb{F}_q(A)(\sqrt{D_i})$, for $1 \leq i \leq n$, are linearly disjoint.

Without loss of generality, consider $i = 1, 2$. By Lemma 2.2.4 and Proposition 2.2.5, the elements $D_1, D_2 \in \mathbb{F}_q(A')[A_0]$ have no common prime factors. Likewise, [4, Proposition 4.3.3] implies that both $D_1$ and $D_2$ are square free. Because the fields $\mathbb{F}_q(A)(\sqrt{D_1})$ and $\mathbb{F}_q(A)(\sqrt{D_2})$ are degree-2 extensions of $\mathbb{F}_q(A)$, their intersection $\mathbb{F}_q(A)(\sqrt{D_1}) \cap \mathbb{F}_q(A)(\sqrt{D_2})$ is either $\mathbb{F}_q(A)$ itself, or else the two field extensions coincide: $\mathbb{F}_q(A)(\sqrt{D_1}) = \mathbb{F}_q(A)(\sqrt{D_2})$. The latter is the case if and only if the product $D_1D_2 \in \mathbb{F}_q(A')[A_0]$ contains the square of a prime factor, contradicting the fact that $D_1$ and $D_2$ are square free and relatively prime in $\mathbb{F}_q(A')[A_0]$. □

**Corollary 2.2.7.** For each algebraic extension $K/\mathbb{F}_q$, we have natural group isomorphisms

$$\text{Gal} \left( \prod_{i=1}^n L_i(\mathcal{F}_A), K(A) \right) \cong \prod_{i=1}^n \text{Gal} \left( L_i(\mathcal{F}_A), K(A) \right) \cong S_{k_1} \times \cdots \times S_{k_n}. \quad \square$$
3. Counting Argument and Proof of Theorem A

In this section we prove Theorem A the counting Proposition 3.1.4. The latter provides an asymptotic formula for the number of \( \mathbb{F}_q \)-valued points \( a \in \mathbb{A}^{m+1} \) for which each of the elements \( L_1(\mathcal{F}_A), \ldots, L_n(\mathcal{F}_A) \) generates a prime ideal in \( R \). The formulation of Proposition 3.1.4 should be viewed as an explicit Chebotarev theorem. Our proof makes use of [1, Appendix A and Theorem 3.1] and [4, Proposition 5.1.4].

3.1. Factorization types and general counting argument. Suppose given an \( \mathbb{F}_q \)-rational point \( a \in \mathbb{A}^{m+1} \). If \( R/(L_i(\mathcal{F}_A)) \) is a separable \( \mathbb{F}_q \)-algebra, then because \( R \) is a Dedekind domain, the ideal \( (L_i(\mathcal{F}_A)) \subset R \) admits a prime factorization

\[
(L_i(\mathcal{F}_A)) = f_{i1} \cdots f_{i\ell_i},
\]

such that each residue field \( \kappa(f_{ij}) = R/(f_{ij}) \) is a separable extension of \( \mathbb{F}_q \). The conditions on \( f_0 \) and \( \sigma_i \) guarantee that in this case

\[
k_i = \deg \text{div}(L_i(\mathcal{F}_A))|_{C \setminus E} = \deg \text{div}(f_{i1})|_{C \setminus E} + \cdots + \deg \text{div}(f_{i\ell_i})|_{C \setminus E}.
\]

(15)

**Definition 3.1.1.** Given an \( \mathbb{F}_q \)-rational point \( a \in \mathbb{A}^{m+1} \), if \( R/(L_i(\mathcal{F}_A)) \) is a separable \( \mathbb{F}_q \)-algebra, the factorization type \( \lambda_{i,a} \) of \( L_i(\mathcal{F}_A) \) is the partition of \( k_i \) given in (15).

The \( n \)-tuple factorization type counting function of \( \mathcal{L} = (L_1, \ldots, L_n) \) for a fixed \( n \)-tuple \( \lambda = (\lambda_1, \ldots, \lambda_n) \), where each \( \lambda_i \) is a partition of \( k_i = \deg L_i(f_0) \), is the assignment \( \pi_{C,\mathcal{L}}(\cdot; \lambda) \) taking the short interval \( I(f_0, E) \) to the value

\[
\pi_{C,\mathcal{L}}(I(f_0, E); \lambda) \overset{def}{=} \# \{ a \in \mathbb{A}^{m+1} : R/(L_i(\mathcal{F}_A)) \text{ is separable and } \lambda_{i,a} = \lambda_i \text{ for } 1 \leq i \leq n \}.
\]

**Definition 3.1.2.** Given a positive integer \( N \) and a permutation \( \tau \in S_N \), the partition type of \( \tau \), denoted \( \lambda_\tau \), is the partition of \( N \) determined by the cycle decomposition of \( \tau \). Having fixed a subgroup \( G \subseteq S_N \), for each partition \( \lambda \) of \( N \) we define

\[
P(\lambda) \overset{def}{=} \frac{# \{ \tau \in G | \lambda_\tau = \lambda \}}{|G|}.
\]

In other words, \( P(\lambda) \) is the probability that a given permutation in \( G \) has partition type \( \lambda \).

**Remark 3.1.3.** Given two positive integers \( N_1 \) and \( N_2 \), subgroups \( G_1 \subseteq S_{N_1} \) and \( G_2 \subseteq S_{N_2} \), and partitions \( \lambda_1 \) of \( N_1 \) and \( \lambda_2 \) of \( N_2 \), we write \( P(\lambda_1) \) and \( P(\lambda_2) \) for the respective probabilities, without explicit reference to the groups \( G_1 \) and \( G_2 \).

**Proposition 3.1.4.** Define

\[
\mathcal{L}(\mathcal{F}_A) \overset{def}{=} L_1(\mathcal{F}_A) \cdots L_n(\mathcal{F}_A) \in R[\mathcal{A}].
\]

Let \( G = \text{Gal}(\mathcal{L}(\mathcal{F}_A), \mathbb{F}_q(\mathcal{A})) \), and fix a partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \) of

\[
N \overset{def}{=} \deg \text{div}(L_1(f_0) \cdots L_n(f_0))|_{C \setminus E}.
\]

Then under the assumptions in §2.1, there exists a constant \( c = c(B) \) such that

\[
|\pi_{C,\mathcal{L}(\mathcal{F}_A)}(I(f_0, E); \lambda) - P(\lambda)q^{m+1}| \leq cq^{m+1/2}.
\]

**Remark 3.1.5.** Consider the \( \mathbb{F}_q \)-scheme \( V(\mathcal{L}(\mathcal{F}_A)) \). By [4, Remark 5.1.5], for each \( 1 \leq i \leq n \) there exist an affine open \( U_i = \text{Spec} \mathcal{A}_i \subset \mathbb{A}^{m+1} \) and a monic separable polynomial \( S_i(t) \in \mathcal{A}_i[t] \) such that

\[
V(L_i(\mathcal{F}_A))_{U_i} \cong \text{Spec} \mathcal{A}_i[t]/(S_i(t)).
\]

Hence there exists an affine subscheme \( U = \text{Spec} \mathcal{A} \subset \bigcap_{i=1}^n U_i \) such that

\[
V(S(t))_{U} \cong V(\mathcal{L}(\mathcal{F}_A))_{U}, \quad \text{for} \quad S(t) \overset{def}{=} S_1(t) \cdots S_n(t).
\]
Proof of Proposition 3.1.4. Let $U = \text{Spec } \mathcal{A}$ and $\mathcal{S}(t) \in \mathcal{A}[t]$ as in Remark 3.1.5. Then there exists some element $a \in \mathbb{F}_q(A)^\times$ such that $a\mathcal{S}(t) \in \mathbb{F}_q[A][t]$. Since $V(\mathcal{S}(t)) \cong V(\mathcal{L}(\mathcal{F}_A))$ over $U$, we have that $\text{Gal}(a\mathcal{S}(t), \mathbb{E}_q(\mathcal{A})) \cong G$. Thus by [1, Theorem 3.1], Proposition 3.1.4 holds with $a\mathcal{S}(t)$ in place of $\mathcal{L}(\mathcal{F}_A)$ for some constant $c_1(B)$. Interpreting the closed complement $Z \cong \mathbb{A}^{m+1} \setminus U$ as an $m$-cycle in $\mathbb{A}^{m+1}$, [13, Lemma 1] implies that there exists some constant $c_2(B)$ such that $\#Z(\mathbb{F}_q) \leq c_2(B)q^m$. Finally,

$$\left| \pi_{C,\mathcal{L}(\mathcal{F}_A)}(I(f_0, E); \lambda) - P(\lambda)q^{m+1} \right| \leq c_1(B) q^{m} + c_2(B) q^m$$

where $c(B) = c_1(B) + c_2(B)$. \hfill \Box

3.2. Proof of Theorem A. We obtain Theorem A as a specific case of the following more general Theorem B, which deals with arbitrary factorization types.

**Theorem B.** In the conditions and notation of Theorem A, let $\lambda$ be a partition as in Proposition 3.1.4. Then

$$\pi_{C,\mathcal{L}(\mathcal{F}_A)}(I(f_0, E); \lambda) = P(\lambda_1) \cdots P(\lambda_n) \#I(f_0, E)\left(1 + O_B(q^{-1/2})\right).$$

**Proof of Theorem B.** By Proposition 2.2.7, $\text{Gal}(\mathcal{L}(\mathcal{F}_A), \mathbb{F}_q(\mathcal{A})) \cong S_{k_1} \times \cdots \times S_{k_n}$. Since $P(\lambda) = P(\lambda_1) \cdots P(\lambda_n)$ and $\#I(f_0, E) = q^{m+1}$, Proposition 3.1.4 gives

$$\pi_{C,\mathcal{L}(\mathcal{F}_A)}(I(f_0, E); \lambda) = P(\lambda_1) \cdots P(\lambda_n) q^{m+1} + O_B(q^{m+1/2})$$

$$= P(\lambda_1) \cdots P(\lambda_n) \#I(f_0, E)\left(1 + O_B(q^{-1/2})\right),$$

as desired. \hfill \Box

**Proof of Theorem A.** In Theorem B, take each $\lambda_i$ to be the partition of $k_i$ into a single cell. Then, $P(\lambda_i) = \frac{1}{k_i} = \frac{1}{\deg(f_0 + \sigma_i)}$ and so

$$\pi_{C,\mathcal{L}(\mathcal{F}_A)}(I(f_0, E); \lambda) = P(\lambda_1) \cdots P(\lambda_n) q^{m+1} + O_B(q^{m+1/2})$$

$$= \frac{\#I(f_0, E)}{\prod_{i=1}^n \deg(f_0 + \sigma_i)_{C \cap E}} \left(1 + O_B(q^{-1/2})\right),$$

as desired. \hfill \Box

**Remark 3.2.1.** In the formulation of the Hardy-Littlewood Conjecture 1.1.1, one can consider polynomial functions more general than the monic linear functions $L_i(X) = \sigma_i + X \in \mathcal{O}_K[X]$. Some of the resulting variant forms of Conjecture 1.1.1 are well-known conjectures or results in their own right. For example, Conjecture 1.1.1 becomes the quantitative Goldbach conjecture if we set $n = 2$ with $L_1(X) = X$ and $L_2(X) = \sigma - X$, for $\sigma \in \mathcal{O}_K$. If one takes $L_1(X) = X$ and $L_2(X) = \sigma_0 + \sigma_1 X$ for $\sigma_0, \sigma_1 \in \mathcal{O}_K$, then Conjecture 1.1.1 provides an asymptotic count of primes in an arithmetic progression.

Over $\mathbb{F}_q(t)$, Bary-Soroker and the first author establish a version of Theorem A for non-monic linear functions $\sigma_1 X + \sigma_10 \in \mathbb{F}_q[t][X]$. We expect that this and other interesting and important variants of Theorem A hold on curves of higher genus over $\mathbb{F}_q$. We plan to investigate these questions in future work.
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