Mixed-integer bilevel representability

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Received: 11 August 2018 / Accepted: 12 August 2019 / Published online: 27 August 2019
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Abstract
We study the representability of sets by extended formulations using mixed-integer bilevel programs. We show that feasible regions modeled by continuous bilevel constraints (with no integer variables), complementarity constraints, and polyhedral reverse convex constraints are all finite unions of polyhedra. Conversely, any finite union of polyhedra can be represented using any one of these three paradigms. We then prove that the feasible region of bilevel problems with integer variables exclusively in the upper level is a finite union of sets representable by mixed-integer programs and vice versa. Further, we prove that, up to topological closures, we do not get additional modeling power by allowing integer variables in the lower level as well. To establish the last statement, we prove that the family of sets that are finite unions of mixed-integer representable sets (up to topological closures) forms an algebra of sets; i.e., this family is closed under finite unions, intersections and complementation.

Keywords Bilevel programming · Representability · Stackelberg games

Mathematics Subject Classification 90C11 · 91A05 · 90C99

Electronic supplementary material The online version of this article (https://doi.org/10.1007/s10107-019-01424-w) contains supplementary material, which is available to authorized users.

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1 Introduction

This paper studies mixed-integer bilevel linear (MIBL) programs of the form

$$\begin{align*}
\max_{x,y} & \quad c^\top x + d^\top y \\
\text{s.t.} & \quad Ax + By \leq b \\
& \quad y \in \arg\max_y \left\{ f^\top y : Cx + Dy \leq g, y_i \in \mathbb{Z} \text{ for } i \in I_F \right\} \\
& \quad x_i \in \mathbb{Z} \text{ for } i \in I_L
\end{align*}$$

(1)

where $x$ and $y$ are finite-dimensional real decision vectors, $b$, $c$, $d$, $f$, and $g$ are finite-dimensional vectors of parameters, and the constraint matrices $A$, $B$, $C$, and $D$ of parameters have conforming dimensions. The decision-maker who determines $x$ is called the leader, while the decision-maker who determines $y$ is called the follower. The sets $I_L$ and $I_F$ are subsets of the index sets of $x$ and $y$ (respectively) that determine which leader and follower decision variables are integers. The follower solves the lower level problem

$$\begin{align*}
\max_y & \quad f^\top y \\
\text{s.t.} & \quad Dy \leq g - Cx \\
& \quad y_i \in \mathbb{Z} \text{ for } i \in I_F
\end{align*}$$

(2)

for a given choice of $x$ by the leader. The lower level problem is captured by the arg max in (1).

Bilevel programming has a long history, with traditions in theoretical economics (see, for instance, [29], which originally appeared in 1975) and operations research (see, for instance, [10,22]). While much of the research community’s attention has focused on the continuous case, there is a growing literature on bilevel programs with integer variables, starting with early work in the 1990s by Bard and Moore [3,30] through a more recent surge of interest [12,15,18,24,25,32,34,35,38,40]. Research has primarily focused on algorithmic concerns, with a recent emphasis on leveraging advancements in cutting plane techniques (a tool that is also used for solving the continuous case, see, for instance, [1]). Typically, these algorithms restrict how variables appear in the problem. For instance, [12,38] consider the setting where all variables are integer-valued. Fischetti et al. [18] allow for continuous variables but restrict the leader’s continuous variables from entering the follower’s problem. Another stream of papers studies questions of computational complexity in the (mixed-)integer setting, also often restricting the appearance of integer variables (see, for instance, [11,12,23,24]). This line of work builds on classical results on the computational complexity of bilevel linear programming (see, for instance, [2,16,22]). Finally, Vicente et al. [36] consider the existence of optimal solutions under restricted types of integrality constraints. For instance, they show that in the setting where the leader’s variables are all continuous and the follower’s variables are all bounded and integer, there may not exist any bilevel optimal solutions.
To our knowledge, the literature has not undertaken a thorough study of general MIBL programs with no restrictions on the variables and constraints. The contribution of this paper is to ask and answer a simple question: what types of sets can be modeled as feasible regions (or possibly projections of feasible regions) of such general MIBL programs? Put in the standard terminology of the optimization literature: what sets are \textit{MIBL-representable}? We thus shift the focus from algorithmic and complexity concerns to questions of representation.

The classical paper of Jeroslow and Lowe \cite{20} provides a characterization of sets that can be represented by mixed-integer linear feasible regions. They show that a set is the projection of the feasible region of a mixed-integer linear problem (termed \textit{MILP-representable}) if and only if it is the Minkowski sum of a finite union of polytopes and a finitely-generated integer monoid (concepts defined carefully below). This result is the gold standard in the theory of representability, as it precisely defines the limits of mixed-integer programming as a modeling framework. Jeroslow and Lowe’s result also serves as inspiration for recent interest in the representability of a variety of optimization paradigms. See the paper by Vielma \cite{37} for a review of the literature until 2015 and \cite{5,26–28} for examples of more recent work.

To our knowledge, questions of representability have not even been explicitly asked of \textit{continuous bilevel linear} (CBL) programs where $I_L = I_F = \emptyset$ in (1). Accordingly, our initial focus concerns characterizations of CBL-representable sets. In the first key result of our paper (Theorem 18), we show that every CBL-representable set can also be modeled as the feasible region of a linear complementarity (LC) problem (in the sense of \cite{14}). Indeed, we show that both CBL-representable sets and LC-representable sets are precisely finite unions of polyhedra. Our proof method works through a connection to superlevel sets of piecewise linear convex functions (what we term \textit{polyhedral reverse-convex sets}) that alternately characterize finite unions of polyhedra. In other words, an arbitrary finite union of polyhedra can be modeled as a continuous bilevel program, a linear complementarity problem, or an optimization problem over a polyhedral reverse-convex set.

A natural question arises: can one relate CBL-representability and MILP-representability? Despite some connections between CBL programs and MILPs (see, for instance, \cite{2}), the collection of sets they represent are incomparable (see Corollary 23 below). The Jeroslow–Lowe characterization of MILP-representability as the finite union of polytopes summed with a finitely-generated monoid has a fundamentally different geometry than CBL-representability as a finite union of polyhedra. It is thus natural to conjecture that MIBL-representability should involve some combination of the two geometries. We show that this intuition is \textit{roughly} correct, with an important caveat.

A complicating aspect of MIBL programs, noticed early on in \cite{30}, is that the feasible region of a MIBL program may not be \textit{topologically closed} (perhaps the simplest example illustrating this fact is Example 1.1 of \cite{23}). This possibility throws a wrench in the classical narrative of representability, which has largely focused on closed sets. The recent work of Lubin et al. in \cite{27} is careful to study representability by \textit{closed} convex sets. This focus is entirely justified. Closed sets are indeed of most interest to the working optimizer and modeler, since sets that are not closed may fail to have desirable optimality properties (such as existence of optimal solutions). Accordingly,
we focus our investigation on closures of MIBL-representable sets. We provide a complete characterization of these sets as unions of finitely many MILP-representable sets (Theorem 20). This is the second key result of the paper. This characterization conforms to the rough intuition of the last paragraph: MIBL-representable sets are finite unions of other objects, but instead of these objects being polyhedra as in the case of CBL-programs, they are MILP-representable sets, reflecting the inherent integrality of MIBL programs.

To prove this characterization, we develop a generalization of Jeroslow and Lowe’s theory to mixed integer sets to generalized polyhedra, which are finite intersections of closed and open halfspaces. It is the non-closed nature of generalized polyhedra that allows us to study the non-closed feasible regions of MIBL-programs. Specifically, these tools arise when we take the value function approach to bilevel programming, as previously studied in [15,25,35,39,41]. Here, we leverage the characterization of Blair [8] of the value function of the mixed-integer program in the lower level problem (2). Blair’s characterization leads us to analyze superlevel and sublevel sets of Chvátal functions. A Chvátal function is (roughly speaking) a linear function with integer rounding (a more formal definition appears later). Basu et al. [5] show that superlevel sets of Chvátal functions are MILP-representable. Sublevel sets are trickier than superlevel sets, but for a familiar reason—they are not always closed. This is not an accident. The non-closed nature of mixed-integer bilevel sets, generalized polyhedra, and sublevel sets of Chvátal functions are all tied together in a key technical result that shows that sublevel sets of Chvátal functions are precisely finite unions of generalized mixed-integer linear representable (GMILP-representable) sets (Theorem 22). This result is the key to establishing our characterization of MIBL-representability.

In fact, showing that the sublevel set of a Chvátal function is the finite union of GMILP-representable sets is a corollary of a more general result. We show that the collection of sets that are finite unions of GMILP-representable sets is an algebra (closed under unions, intersections, and complements). We believe this result is of independent interest.

Our representability results in the mixed-integer case require rationality assumptions on the data. This is an inevitable consequence when dealing with mixed-integer sets. Even the classical result of Jeroslow and Lowe [20] requires rationality assumptions on the data. Without this assumption the result does not hold. The key issue is that the structure of mixed-integer sets (and their projections) can be hard to comprehend unless additional assumptions are made. The most common assumption is the rationality of all input data (see [17] for a discussion of some of these issues).

Summary of contributions We provide geometric characterizations of CBL-representability and MIBL-representability (where the latter is up to closures) in terms of finite unions of polyhedra and finite unions of MILP-representable sets, respectively. In the process of establishing these results, we develop a theory of representability of mixed-integer sets in generalized polyhedra and show that finite unions of GMILP-representable sets form an algebra. This last result implies that finite unions of MILP-representable sets also form an algebra (up to closures).

We organize the rest of the paper as follows. The main definitions needed to state our main results are found in Sect. 2, followed in Sect. 3 by self-contained statements
of these main results. Section 4 contains our analysis of continuous bilevel sets and their representability. Section 5 explores representability in the mixed-integer setting.

Some notation The following basic concepts are entirely standard, but their notation less so. We state our notation for clarity and completeness. Let \( \mathbb{R} \), \( \mathbb{Q} \), and \( \mathbb{Z} \) denote the set of real numbers, rational numbers, and integers, respectively. Given a subset \( S \) of \( \mathbb{R}^n \) for some integer \( n \), the interior and closure of \( S \) are denoted \( \text{int}(S) \) and \( \text{cl}(S) \), respectively. For any set \( S \subseteq \mathbb{R}^n \), the complement is denoted \( S^c := \{ x \in \mathbb{R}^n : x \notin S \} \). The set of all conic combinations of the elements of \( S \) is called the cone of \( S \) and denoted \( \text{cone}(S) \). A cone \( C \) is pointed if for \( x \neq 0, x \in C \) implies \( -x \notin C \). A cone \( C \) is simplicial if the extreme rays of \( C \) are linearly independent. The set of all conic combinations with integer multipliers is called the monoid generated by \( S \) (or sometimes the integer cone of \( S \)) and denoted \( \text{monoid}(S) \). For any convex set \( K \subseteq \mathbb{R}^n \), the recession cone of \( K \) is denoted \( \text{rec}(K) \). The affine hull \( \text{aff}(S) \) of \( S \) is the intersection of all affine sets containing \( S \). The relative interior \( \text{relint}(S) \) of \( S \) is the interior of \( S \) in the relative topology of \( \text{aff}(S) \). The ball \( B(c, r) = \{ x \in \mathbb{R}^n : ||x - c|| \leq r \} \) in the closed ball of radius \( r \) centered at \( c \). We use \( \text{proj}_x \{ (x, y) : (x, y) \in S \} \) to denote the projection of the set \( S \) on to the space of \( x \) variables. The standard inner product on \( \mathbb{R}^n \) is denoted \( \langle u, v \rangle \) for \( u, v \in \mathbb{R}^n \). We use \( [k] \) to denote the set \( \{1, 2, 3, \ldots, k\} \) for any natural number \( k \).

2 Key definitions

This section provides the definitions needed to understand the statements of our main results, collected in Sect. 3. Concepts that appear only in the proofs of these results are defined later as needed. We begin with formal definitions of the types of sets we study.

Definition 1 (Mixed-integer bilevel linear set) A set \( S \subseteq \mathbb{R}^{n_L + n_f} \) is called a mixed-integer bilevel linear (MIBL) set if there exist \( A \in \mathbb{R}^{m_L \times n_L}, B \in \mathbb{R}^{m_L \times n_f}, b \in \mathbb{R}^{m_L}, f \in \mathbb{R}^{n_f}, D \in \mathbb{R}^{m_f \times n_f}, C \in \mathbb{R}^{m_f \times n_L} \) and \( g \in \mathbb{R}^{m_f} \) such that

\[
S = S^1 \cap S^2 \cap S^3, \tag{3a}
\]

\[
S^1 = \left\{ (x, y) \in \mathbb{R}^{n_L + n_f} : Ax + By \leq b \right\}, \tag{3b}
\]

\[
S^2 = \left\{ (x, y) \in \mathbb{R}^{n_L + n_f} : y = \arg \max_i \left\{ f^\top y : Dy \leq g - Cx, \; y_i \in \mathbb{Z} \; \text{for} \; i \in \mathcal{I}_F \right\} \right\}, \tag{3c}
\]

\[
S^3 = \left\{ (x, y) \in \mathbb{R}^{n_L + n_f} : x_i \in \mathbb{Z} \; \text{for} \; i \in \mathcal{I}_L \right\}, \tag{3d}
\]

where \( \mathcal{I}_L \subseteq [n_L], \mathcal{I}_F \subseteq [n_f] \). Further, we call \( S \) a

- continuous bilevel linear (CBL) set if it has a representation with \( |\mathcal{I}_L| = |\mathcal{I}_F| = 0 \);
- bilevel linear with integer upper level (BLUI) set if it has a representation with \( |\mathcal{I}_F| = 0 \).

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We use the notation $\mathcal{P}_{\mathbb{R}}^{\mathbb{R}}$ to denote the family of CBL sets, $\mathcal{P}_{\mathbb{R}}^{\mathbb{Z} \times \mathbb{R}}$ to represent the family of BLUI sets and $\mathcal{P}_{\mathbb{Z} \times \mathbb{R}}^{\mathbb{Z} \times \mathbb{R}}$ to represent the family of mixed-integer bilevel sets. The superscripts in this notation indicate the types of variables of the leader, the subscripts indicate the types of variables of the follower. Such sets are labeled rational if all the entries in $A$, $B$, $C$, $D$, $b$, $f$, $g$ are rational. In the same vein, the family of polyhedra will be denoted by $\mathcal{P}_{\mathbb{R}}^{\mathbb{R}}$ and the family of mixed-integer points in polyhedra will be denoted by $\mathcal{P}_{\mathbb{Z} \times \mathbb{R}}^{\mathbb{Z} \times \mathbb{R}}$. There are no subscripts in this notation, indicating that these problems are single level optimization problems.

**Definition 2** (Linear complementarity sets) A set $S \subseteq \mathbb{R}^n$ is a linear complementarity (LC) set, if there exist $M \in \mathbb{R}^{n \times n}$, $q \in \mathbb{R}^n$ and $A$, $b$ of appropriate dimensions such that

$$S = \left\{ x \in \mathbb{R}^n : x \geq 0, \ Mx + q \geq 0, \ x^\top (Mx + q) = 0, \ Ax \leq b \right\}.$$

Sometimes, we represent this using the alternative notation

$$0 \leq x \perp Mx + q \geq 0 \quad Ax \leq b.$$

The family of LC sets is denoted $\mathcal{LC}$. An LC set is rational if all the entries in $A$, $M$, $b$, $q$ are rational.

As an example, the set of all $n$-dimensional binary vectors is a linear complementarity set. Indeed, this set can be modeled as $0 \leq x \perp (1 - x) \geq 0$, where $1$ denotes the vector of all ones.

**Definition 3** (Polyhedral convex function) A function $f : \mathbb{R}^n \to \mathbb{R}$ is a polyhedral convex function with $k$ pieces if there exist $\alpha^1, \ldots, \alpha^k \in \mathbb{R}^n$ and $\beta_1, \ldots, \beta_k \in \mathbb{R}$ such that

$$f(x) = \max_{j=1}^k \left\{ \langle \alpha^j, x \rangle - \beta_j \right\}.$$

A polyhedral convex function is rational if all the entries in the affine functions are rational.

Note that a polyhedral convex $f$ is a maximum of finitely many affine functions. Hence $f$ is always a convex function.

**Definition 4** (Polyhedral reverse-convex set) A set $S \in \mathbb{R}^n$ is a polyhedral reverse-convex (PRC) set if there exist $n' \in \mathbb{Z}_+$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$ and polyhedral convex functions $f_i$ for $i \in [n']$ such that

$$S = \left\{ x \in \mathbb{R}^n : Ax \leq b, \ f_i(x) \geq 0 \text{ for } i \in [n'] \right\}.$$

The family of PRC sets is denoted $\mathcal{PRC}$. A PRC set is rational if all the entries in $A$, $b$ are rational, and the polyhedral convex functions $f_i$ are all rational.
One of the distinguishing features of MIBL sets is their potential for not being closed. This was discussed in the introduction and a concrete example is provided in Lemma 49. To explore the possibility of non-closedness, we introduce the notion of generalized polyhedra.

**Definition 5** (Generalized, closed, and relatively open polyhedra) A generalized polyhedron is the intersection of finitely many open and closed halfspaces. A bounded generalized polyhedron is called a generalized polytope. The intersection of finitely many closed halfspaces is called a closed polyhedron, or simply a polyhedron. A bounded closed polyhedron is a closed polytope, or simply a polytope. A relatively open polyhedron $P$ is a generalized polyhedron where $P = \text{relint}(P)$. If a relatively open polyhedron $P$ is bounded, we call it a relatively open polytope. Such sets are called rational if all the defining halfspaces (open or closed) can be expressed using affine functions with rational data.

Note that the closure of a generalized or relatively open polyhedron is a closed polyhedron. Singletons are, by definition, relatively open polyhedra.

**Definition 6** (Generalized, closed and relatively open mixed-integer sets) A generalized (respectively, closed and relatively open) mixed-integer set is the set of mixed-integer points in a generalized (respectively, closed and relatively open) polyhedron. We often refer to closed mixed-integer sets simply as mixed-integer sets. A generalized (respectively, closed and relatively open) mixed integer set is rational if the corresponding generalized (respectively, closed and relatively open) polyhedron is rational.

Our primary focus is to explore how collections of the above objects can be characterized and are related to one another. To facilitate this investigation, we employ the following notation and vocabulary. Let $\mathcal{T}$ be a family of sets. These families include objects of different dimensions. For instance, the family of polyhedra includes polyhedra in $\mathbb{R}^2$ as well as those in $\mathbb{R}^3$. We will not make explicit reference to the ambient dimension of a member of the family $\mathcal{T}$ unless it is unclear from the context. Let $\text{cl}(\mathcal{T})$ denote the family of the closures of all sets in $\mathcal{T}$. When referring to the rational members of a family (as per definitions above), we use the notation $\mathbb{Q}\mathcal{T}$.

We are not only interested in the above sets, but also linear transformations of these sets. The concept of representability captures this notion.

**Definition 7** (Representability) Given a family of sets $\mathcal{T}$, $S$ is called a $\mathcal{T}$-representable set or representable by $\mathcal{T}$ if there exists a $T \in \mathcal{T}$ and a linear transform $L$ such that $S = L(T)$. The collection of all such $\mathcal{T}$-representable sets is denoted $\text{repr}(\mathcal{T})$. We use the notation $\mathbb{Q}\text{repr}(\mathbb{Q}\mathcal{T})$ to denote the images of the rational sets in $\mathcal{T}$ under rational linear transforms, i.e., those linear transforms that can be represented using rational matrices.\(^1\)

**Remark 8** The standard definition of representability in the optimization literature uses coordinate projections as opposed to linear transforms in general. However, under a

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\(^1\) We never refer to non-rational linear transforms of rational sets, nor to rational linear transforms of sets in a family $\mathcal{T}$ that are not rational. Accordingly, we do not introduce any notation for these contingencies.
mild assumption on the family \( T \), it can be shown that \( \text{repr} (T) \) is simply the collection of sets that are coordinate projections of sets in \( T \). Indeed, since coordinate projections are linear transforms, we certainly get all coordinate projections in \( \text{repr} (T) \). Now consider a set \( S \in \text{repr} (T) \), i.e., there exists a set \( T \in \tilde{T} \) and a linear transform \( L \) such that \( S = L(T) \). Observe that \( S = \text{proj}_x \{ (x, y) : x = L(y), \ y \in T \} \). Thus, if \( T \) is a family that is closed under addition of free variables (like the set \{\( (x, y) : y \in T \))\), and under intersecting with affine subspaces (like \( x = L(y) \) above), then \( \text{repr} (T) \) does not contain anything beyond coordinate projections of sets in \( T \). All families considered in this paper are easily verified to satisfy these conditions.

One can immediately observe that \( T \subseteq \text{repr} (T) \) since the linear transform can be chosen as the identity transform. However, inclusion may or may not be strict. For example, it is well known that if \( T \) is the set of all polyhedra, then \( \text{repr} (T) = T \). However, if \( T \) is the family of all mixed-integer sets then \( T \subsetneq \text{repr} (T) \).

Continuing to introduce our notation, the family of all finite unions of sets in \( T \) is denoted \( \text{disj} (T) \) (where \( \text{disj} \) connotes “disjunction”). Specifically, we use \( \text{disj} (P^{\mathbb{R}}) \) to denote the family of finite unions of polyhedra and \( \text{disj} (P^{\mathbb{Z} \times \mathbb{R}}) \) to denote the family of finite unions of mixed-integer sets \( P^{\mathbb{Z} \times \mathbb{R}} \). We use \( \tilde{P}^{\mathbb{Z} \times \mathbb{R}} \) to denote the family of generalized mixed-integer sets and \( \tilde{\text{disj}} \left( \tilde{P}^{\mathbb{Z} \times \mathbb{R}} \right) \) to denote the family of sets that can be written as finite unions of sets in \( \tilde{P}^{\mathbb{Z} \times \mathbb{R}} \). Finally, \( B \) denotes the family of all bounded sets. All of this notation is summarized in Table 1.

**Lemma 9** The family \( \text{repr} (\text{disj} (P^{\mathbb{Z} \times \mathbb{R}})) \) is exactly the family of finite unions of sets in \( \text{repr} (\tilde{P}^{\mathbb{Z} \times \mathbb{R}}) \); that is, \( \text{repr} (\text{disj} (P^{\mathbb{Z} \times \mathbb{R}})) = \text{disj} (\text{repr} (\tilde{P}^{\mathbb{Z} \times \mathbb{R}})) \). Similarly, \( \text{repr} (\text{disj} (\tilde{P}^{\mathbb{Z} \times \mathbb{R}})) = \text{disj} (\text{repr} (\tilde{P}^{\mathbb{Z} \times \mathbb{R}})) \). The statements also holds for the families of sets with rational description, i.e., \( \tilde{Q} \)-\text{repr} (\text{disj} (\tilde{Q}P^{\mathbb{Z} \times \mathbb{R}})) = \text{disj} (\tilde{Q} \)-\text{repr} (\text{disj} (\tilde{Q}P^{\mathbb{Z} \times \mathbb{R}})) \) and \( \tilde{Q} \)-\text{repr} (\text{disj} (\tilde{Q}P^{\mathbb{Z} \times \mathbb{R}})) = \text{disj} (\tilde{Q} \)-\text{repr} (\text{disj} (\tilde{Q}P^{\mathbb{Z} \times \mathbb{R}})) \).

**Remark 10** Due to Lemma 9, we will interchangeably use the notation \( \text{repr} (\text{disj} (P^{\mathbb{Z} \times \mathbb{R}})) \) and the phrase “finite unions of MILP-representable sets” (similarly, \( \text{repr} (\text{disj} (\tilde{P}^{\mathbb{Z} \times \mathbb{R}})) \) and “finite unions of sets in \( \text{repr} (\tilde{P}^{\mathbb{Z} \times \mathbb{R}}) \)” without further comment in the remainder of this paper.

Finally, we introduce concepts that are used in describing characterizations of these families of sets. One concept used to articulate the “integrality” inherent in many of these families is the following.

**Definition 11** (Monoid) A set \( C \subseteq \mathbb{R}^n \) is a monoid if \( 0 \in C \) and for all \( x, y \in C \), \( x + y \in C \). A monoid is finitely generated if there exist \( r^1, r^2, \ldots, r^k \in \mathbb{R}^n \) such that

\[
C = \left\{ x : x = \sum_{i=1}^{k} \lambda_i r_i \text{ where } \lambda_i \in \mathbb{Z}_+ ; \forall i \in [k] \right\}.
\]
We denote the right-hand side of the above as monoid \( \{ r_1, \ldots, r^k \} \). A monoid \( C \) is pointed if \( \text{cone}(C) \) is a pointed cone. A finitely generated monoid is rational if the generators \( r_1, \ldots, r^k \) are all rational vectors. The family of all finitely generated monoids is denoted \( \text{MON} \) and finitely generated monoid with rational vectors is denoted \( \text{Q-MON} \).

The following seminal result from Jeroslow and Lowe [20] shows that a rational MILP-representable set is the Minkowski sum of a finite union of rational polytopes and a rational finitely generated monoid.

**Theorem 12** [20] Let \( T \in \text{Q-repr} \left( \mathcal{P}^{\mathbb{Z} \times \mathbb{R}} \right) \). Then,

\[
T = P + C
\]

for some \( P \in \text{disj} \left( \mathcal{P}^{\mathbb{R}} \right) \) and some rational \( C \in \text{Q-MON} \).

We also define three families of functions that provide an alternative vocabulary for describing “integrality”; namely, Chvátal functions, Gomory functions and Jeroslow functions.
functions. These families derive significance here from their ability to articulate value functions of integer and mixed-integer programs (as seen in [6–9]). See also [41] for additional discussion of value functions for bilevel programs with pure integer variables.

**Chvátal functions** are defined recursively by using linear combinations and floor ([·]) operators on other Chvátal functions, assuming that the set of affine linear functions are Chvátal functions. We formalize this using a binary tree construction as below. We adapt the definition from Basu et al. [5].

**Definition 13** (Chvátal functions [5]) A Chvátal function \( \psi : \mathbb{R}^n \to \mathbb{R} \) is constructed as follows. We are given a finite binary tree where each node is either: (i) a leaf node which corresponds to an affine linear function on \( \mathbb{R}^n \) with rational coefficients; (ii) a node with one child and a corresponding edge labeled by either \([·]\) or a non-negative rational number; or (iii) a node with two children with two corresponding edges labeled by a non-negative rational number. Start at the root node and recursively form functions corresponding to subtrees rooted at its children using the following rules.

1. If the root has no children then it is a leaf node corresponding to an affine linear function with rational coefficients. Then \( \psi \) is the affine linear function.
2. If the root has a single child, recursively evaluating a function \( g \), and the edge to the child is labeled as \([·]\), then \( \psi(x) = [g(x)] \). If the edge is labeled by a non-negative number \( \alpha \), define \( \psi(x) = \alpha g(x) \).
3. Finally, if the root has two children, containing functions \( g_1, g_2 \) and edges connecting them labeled with non-negative rationals, \( a_1, a_2 \), then \( \psi(x) = a_1 g_1(x) + a_2 g_2(x) \).

We call the number of \([·]\) operations in a binary tree used to represent a Chvátal function the **order** of this binary tree representation of the Chvátal function. Note that a given Chvátal function may have alternative binary tree representations with different orders.

**Definition 14** (Gomory functions) A Gomory function \( G \) is the pointwise minimum of finitely many Chvátal functions. That is,

\[
G(x) := \min_{i=1}^{k} \psi_i(x)
\]

where \( \psi_i \) is a Chvátal function for all \( i \in [k] \).

Gomory functions are then used to build Jeroslow functions, as originally defined in [8].

**Definition 15** (Jeroslow function) Let \( G \) be a Gomory function. For any invertible matrix \( E \), and any vector \( x \), define \( [x]_E := E [E^{-1} x] \). Let \( I \) be a finite index set and

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2 The definition in [5] used \( \lceil·\rceil \) as opposed to \([·]\). We make this change in this paper to be consistent with Jeroslow and Blair’s notation.

3 The “order” of a Chvátal function’s representation is called its “ceiling count” in [5].
let \( \{ E_i \}_{i \in I} \) be a set of \( n \times n \) invertible rational matrices indexed by \( I \), and \( \{ w_i \}_{i \in I} \) be a set of rational vectors in \( \mathbb{R}^n \) indexed by \( I \). Then \( J : \mathbb{R}^n \to \mathbb{R} \) is a Jeroslow function if

\[
J(x) := \max_{i \in I} \left\{ G(\lfloor x \rfloor E_i) + w_i^\top (x - \lfloor x \rfloor E_i) \right\}.
\]

**Remark 16** We have explicitly allowed only rational entries in the data defining Chvátal, Gomory, and Jeroslow functions. This is standard in the literature since the natural setting for these functions and their connection to mixed-integer optimization uses rational data.

**Remark 17** It follows from Definitions 13–15 that the family of Chvátal functions, Gomory functions and Jeroslow function are all closed under composition with affine functions and addition of affine functions.

An important result by Blair in [8] is that the value function of a mixed integer program with rational data is a Jeroslow function. This result allows us to express the lower-level optimality condition captured in the bilevel constraint (3c). This is a critical observation for our study of MIBL-representability.

### 3 Main results

Our main results concern the relationship between the sets defined in Table 1 and the novel machinery we develop to establish these relationships.

**Representability results** First, we explore bilevel sets with only continuous variables. We show that the sets represented by continuous bilevel constraints, linear complementarity constraints and polyhedral reverse convex constraints are all equivalent and equal to the family of finite unions of polyhedra.

**Theorem 18** The following holds:

\[
\text{repr} \left( \mathcal{PR} \right) = \text{repr} \left( \mathcal{LC} \right) = \text{repr} \left( \mathcal{PRC} \right) = \mathcal{PRC} = \text{disj} \left( \mathcal{PR} \right)
\]

**Remark 19** The rational version of Theorem 18 holds:

\[
\text{Q-repr} \left( \mathcal{Q-PR} \right) = \text{Q-repr} \left( \mathcal{Q-LC} \right) = \text{Q-repr} \left( \mathcal{Q-PRC} \right) = \mathcal{Q-PRC} = \text{disj} \left( \mathcal{Q-PR} \right)
\]

We will not explicitly prove the rational version; the proof below can be adapted to the rational case without any difficulty.

Our next set of results concern representability by bilevel problems with integer variables. In contrast to Theorem 18, rationality of the data is an important assumption in this setting. This is to be expected: this assumption is crucial whenever one deals with mixed-integer points, as mentioned in the introduction. We show that, with integrality constraints in the upper level only, bilevel representable sets correspond to finite unions
of MILP-representable sets. Further allowing integer variables in the lower level may yield sets that are not necessarily closed. However, we show that the closure of sets are again finite unions of MILP-representable sets. These containments are captured in the following result.

**Theorem 20** The following holds:

\[
\mathbb{Q}^{-}\text{repr}\left(\mathbb{Q}^{-}\mathcal{P}_{\mathbb{Z} \times \mathbb{R}}\right) \supseteq \text{cl}\left(\mathbb{Q}^{-}\text{repr}\left(\mathbb{Q}^{-}\mathcal{P}_{\mathbb{Z} \times \mathbb{R}}\right)\right) = \mathbb{Q}^{-}\text{repr}\left(\text{disj}\left(\mathbb{Q}^{-}\mathcal{P}_{\mathbb{Z} \times \mathbb{R}}\right)\right)
\]

Behind the proof of this result are two novel technical results that we believe have interest in their own right. The first concerns an algebra of sets that captures, to some extent, the inherent structure that arises when bilevel constraints and integrality interact. Recall that an algebra of sets is a collection of sets that is closed under taking complements and finite unions. It is trivial to observe that finite unions of generalized polyhedra form an algebra, i.e., a family that is closed under finite unions, finite intersections and complements. We show that a similar result holds even for finite unions of generalized mixed-integer representable sets.

**Theorem 21** For any natural number \(n\), the family of sets

\[
\left\{ S \subseteq \mathbb{R}^n : S \in \mathbb{Q}^{-}\text{repr}\left(\text{disj}\left(\mathbb{Q}^{-}\mathcal{P}_{\mathbb{Z} \times \mathbb{R}}\right)\right) \right\}
\]

is an algebra of subsets of \(\mathbb{R}^n\).

The connection of the above algebra to optimization is made explicit in the following theorem, our second novel technical result used in the proof of Theorem 20.

**Theorem 22** Let \(\psi : \mathbb{R}^n \rightarrow \mathbb{R}\) be a Chvátal, Gomory, or Jeroslow function. Then (i) \(\{ x : \psi(x) \leq 0 \}\) (ii) \(\{ x : \psi(x) \geq 0 \}\) (iii) \(\{ x : \psi(x) = 0 \}\) (iv) \(\{ x : \psi(x) < 0 \}\) and (v) \(\{ x : \psi(x) > 0 \}\) are elements of \(\mathbb{Q}^{-}\text{repr}\left(\text{disj}\left(\mathbb{Q}^{-}\mathcal{P}_{\mathbb{Z} \times \mathbb{R}}\right)\right)\).

As observed below Definition 15, the family of Jeroslow functions capture the properties of the bilevel constraint (3c) and thus Theorem 22 is critical in establishing Theorem 20.

**Relationship between the families**

In view of our representability results above, one may wonder what is the relationship between the family \(\text{repr}\left(\mathcal{P}_{\mathbb{Z} \times \mathbb{R}}\right)\) of mixed-integer representable sets, the family \(\text{repr}\left(\mathcal{P}_{\mathbb{R} \times \mathbb{R}}\right)\) of continuous bilevel representable sets, and the family \(\text{repr}\left(\mathcal{P}_{\mathbb{Z} \times \mathbb{R}}\right)\) of mixed-integer bilevel representable sets. The following corollary and Fig. 1 make this clear.

**Corollary 23** The following relations hold:

\[
\text{repr}\left(\mathcal{P}_{\mathbb{R} \times \mathbb{R}}\right) \setminus \text{repr}\left(\mathcal{P}_{\mathbb{Z} \times \mathbb{R}}\right) \neq \emptyset,
\]

\(\Rightarrow\) Springer
Fig. 1 Venn diagram illustrating the relationship between families of sets. The set $A$ corresponds to $\mathbb{Q}$-repr $\left(\mathbb{Q} - \mathbb{P} \mathbb{Z} \times \mathbb{R}\right)$, $B$ corresponds to $\mathbb{Q}$-repr $\left(\mathbb{Q} - \mathbb{P} \mathbb{R}\right)$, $C$ corresponds to compact sets in $\mathbb{Q}$-repr $\left(\mathbb{Q} - \mathbb{P} \mathbb{Z} \times \mathbb{R}\right)$, $D$ corresponds to $\mathbb{Q}$-repr $\left(\mathbb{Q} - \mathbb{P} \mathbb{Z} \times \mathbb{R}\right)$, $E$ corresponds to $\mathbb{Q}$-repr $\left(\mathbb{Q} - \mathbb{P} \mathbb{Z} \times \mathbb{R}\right)$. Each region shown in the set is non-empty. Examples showing nonemptiness of the regions $A \backslash B$, $B \backslash A$ and $D \backslash (A \cup B)$ are shown in the proof of Corollary 23. The relationship $C \subset A \cap B$ follows from Corollary 24. Any cone is an example of a set in $(A \cap B) \backslash C$. An example of a set in $E \backslash D$ is shown in Lemma 49 (see also Theorem 20). Finally, we also claim that every set in $E \backslash D$ is not closed; in fact, $D$ is precisely the collection of the closure of the sets in $E$ (see Theorem 20).

$$\text{repr} \left(\mathbb{P} \mathbb{Z} \times \mathbb{R}\right) \cap \text{repr} \left(\mathbb{P} \mathbb{R}\right) \neq \emptyset,$$

$$\text{repr} \left(\mathbb{P} \mathbb{Z} \times \mathbb{R}\right) \cap \left(\text{repr} \left(\mathbb{P} \mathbb{Z} \times \mathbb{R}\right) \cup \text{repr} \left(\mathbb{P} \mathbb{R}\right)\right) \neq \emptyset.$$

The examples that establish Corollary 23 have rational data. It turns out that these examples arise because of unboundedness. If we impose boundedness of the sets along with rationality of the description, then these examples disappear.

**Corollary 24** The following holds:

$$\mathbb{Q} \text{- repr} \left(\mathbb{Q} - \mathbb{P} \mathbb{R}\right) \cap B = \mathbb{Q} \text{- repr} \left(\mathbb{Q} - \mathbb{P} \mathbb{R}\right) \cap B = \mathbb{Q} \text{- repr} \left(\mathbb{Q} - \mathbb{P} \mathbb{R}\right) \cap B = \mathbb{Q} \text{- repr} \left(\mathbb{Q} - \mathbb{P} \mathbb{R}\right) \cap B = \mathbb{Q} \text{- repr} \left(\mathbb{Q} - \mathbb{P} \mathbb{R}\right) \cap B.$$

**4 Representability of continuous bilevel sets**

The goal of this section is to prove Theorem 18. We establish the result across several lemmata. First, we show that sets representable by polyhedral reverse convex constraints are unions of polyhedra and vice versa. Then we show three inclusions, namely

\[ \text{repr} \left(\mathbb{P} \mathbb{Z} \times \mathbb{R}\right) \cap \text{repr} \left(\mathbb{P} \mathbb{R}\right) \neq \emptyset, \]

\[ \text{repr} \left(\mathbb{P} \mathbb{Z} \times \mathbb{R}\right) \cap \left(\text{repr} \left(\mathbb{P} \mathbb{Z} \times \mathbb{R}\right) \cup \text{repr} \left(\mathbb{P} \mathbb{R}\right)\right) \neq \emptyset. \]
Lemma 25 If \( T^1 \) and \( T^2 \) are two families of sets such that \( T^1 \subseteq \text{repr}(T^2) \) then \( \text{repr}(T^1) \subseteq \text{repr}(T^2) \). Moreover, the rational version holds, i.e., \( \mathbb{Q} \cdot T^1 \subseteq \mathbb{Q} \cdot \text{repr}(T^2) \) implies \( \mathbb{Q} \cdot \text{repr}(\mathbb{Q} \cdot T^1) \subseteq \mathbb{Q} \cdot \text{repr}(\mathbb{Q} \cdot T^2) \).

**Proof** Let \( T \in \text{repr}(T^1) \). This means there is a linear transform \( L^1 \) and \( T^1 \in T^1 \) such that \( T = L^1(T^1) \). Since \( T^1 \in \text{repr}(T^2) \) by assumption, there exists a linear transform \( L^2 \) and \( T^2 \in T^2 \) such that \( T^1 = L^2(T^2) \). So \( T = L^1(T^1) = L^1(L^2(T^2)) = (L^1 \circ L^2)(T^2) \), implying \( T \in \text{repr}(T^1) \) and proving the result. The rational version follows by restricting all linear transforms and sets to be rational. \(\square\)

We now establish the first building block of Theorem 18.

**Lemma 26** The following holds:

\[
\text{disj} \left( \mathcal{P}^\mathbb{R} \right) = \mathcal{P}^\mathcal{R}C = \text{repr}(\mathcal{P}^\mathcal{R}C).
\]

**Proof** We start by proving the first equivalence. Consider the \( \supseteq \) direction first. Let \( S \in \mathcal{P}^\mathcal{R}C \). Then \( S = \{ x \in \mathbb{R}^n : Ax < b, f_i(x) \geq 0 \text{ for } i \in [n'] \} \) for some polyhedral convex functions \( f_i \) with \( k_i \) pieces each. First, we show that \( S \) is a finite union of polyhedra. Choose one halfspace from the definition of each of the functions \( f_i(x) = \max_{j=1}^{k_i} \{ \langle \alpha^{ij}, x \rangle \} \) (i.e., \( \{ x : \langle \alpha^{ij}, x \rangle \geq 0 \} \) for some \( j \) and each \( i \)) and consider their intersection. This gives a polyhedron. There are exactly \( K = \prod_{i=1}^{n'} k_i \) such polyhedra. We claim that \( S \) is precisely the union of these \( K \) polyhedra, intersected with \( \{ x : Ax \leq b \} \) (clearly, the latter set is in \( \text{disj}(\mathcal{P}^\mathbb{R}) \)). Suppose \( x \in S \). Then \( Ax \leq b \). Also, since \( f_i(x) \geq 0 \), we have \( \max_{j=1}^{k_i} \{ \langle \alpha^{ij}, x \rangle - \beta_{ij} \} \geq 0 \). This means for each \( i \), there exists a \( j \) such that \( \langle \alpha^{ij}, x \rangle - \beta_{ij} \geq 0 \). The intersection of all such halfspaces is one of the \( K \) polyhedra defined earlier. Conversely, suppose \( x \) is in one of these \( K \) polyhedra (the one defined by \( \langle \alpha^{ij}, x \rangle - \beta_{ij} \geq 0 \) for \( i \in [n'] \)) intersected with \( \{ x : Ax \leq b \} \). Then \( f_i(x) = \max_{j=1}^{k_i} \{ \langle \alpha^{ij}, x \rangle - \beta_{ij} \} \geq \langle \alpha^{ij}, x \rangle - \beta_{ij} \geq 0 \) and thus \( x \in S \). This shows that \( \mathcal{P}^\mathcal{R}C \subseteq \text{disj}(\mathcal{P}^\mathbb{R}) \).

Conversely, suppose \( P \in \text{disj}(\mathcal{P}^\mathbb{R}) \) and is given by \( P = \bigcup_{i=1}^{k} P_i \) and \( P_i = \{ x : A^i x \leq b^i \} \) where \( b^i \in \mathbb{R}^{m_i} \). Let \( a_{ij} \) refer to the \( j \)-th row of \( A^i \) and \( b_j^i \) the \( j \)-th coordinate of \( b^i \). Let \( \Omega \) be the Cartesian product of the index sets of constraints, i.e., \( \Omega = \{ 1, \ldots, m_1 \} \times \{ 1, \ldots, m_2 \} \times \cdots \times \{ 1, \ldots, m_k \} \). For any \( \omega \in \Omega \), define the following function:

\[
f_{\omega}(x) = \max_{i=1}^{k} \left\{ -\langle a_{i\omega_i}, x \rangle + b_{i\omega_i}^i \right\},
\]

where \( \omega_i \) denotes the index chosen in \( \omega \) from the set of constraints in \( A^i x \leq b^i \), \( i = 1, \ldots, k \). This construction is explicitly illustrated in Example 30 and captured in Fig. 2. Let \( T = \{ x : f_{\omega}(x) \geq 0, \forall \omega \in \Omega \} \in \mathcal{P}^\mathcal{R}C \). We now claim that \( P = T \). If \( \bar{x} \in P \) then there exists an \( i \) such that \( \bar{x} \in P_i \), which in turn implies that \( b^i - A^i \bar{x} \geq 0 \).
However each of the $f_\omega$ contains at least one of the rows from $b^i - A^i \bar{x}$, and that is non-negative. This means each $f_\omega$, which are at least as large as any of these rows, are non-negative. This implies $P \subseteq T$. Now suppose $\bar{x} \notin P$. This means in each of the $k$ polyhedra $P_i$, at least one constraint is violated. Now consider the $f_\omega$ created by using each of these violated constraints. Clearly for this choice, $f_\omega(x) < 0$. Thus $\bar{x} \notin T$. This shows $T \subseteq P$ and hence $P = T$. This finally shows $\text{disj}(P^R) \subseteq \text{PRC}$. Combined with the argument in the first part of the proof, we have $\text{PRC} = \text{disj}(P^R)$.

Now consider the set $\text{repr}(\text{disj}(P^R))$. A linear transform of a union of finitely many polyhedra is a union of finitely many polyhedra. Thus $\text{repr}(\text{disj}(P^R)) = \text{disj}(P^R)$. But $\text{disj}(P^R) = \text{PRC}$ and so $\text{repr}(\text{PRC}) = \text{repr}(\text{disj}(P^R)) = \text{disj}(P^R) = \text{PRC}$, proving the remaining equivalence in the statement of the lemma.

Next, we show that any set representable using polyhedral reverse convex constraints is representable using continuous bilevel constraints. To achieve this, we give an explicit construction of the bilevel set.

**Lemma 27** The following holds:

$$\text{repr}(P^R^C) \subseteq \text{repr}(P^R).$$

**Proof** Suppose $S \in P^R^C$. Then,

$$S = \{x \in \mathbb{R}^n : Ax \leq b, \ f_i(x) \geq 0 \text{ for } i \in [n']\}$$

for some $n'$, $A$, $b$ and polyhedral convex functions $f_i : \mathbb{R}^n \to \mathbb{R}$ for $i \in [n']$. Further, let us explicitly write $f_i(x) = \max_{j=1}^{k_i} \{\langle \alpha^i j, x \rangle - \beta_{ij} \}$ for $j \in [k_i]$ for $i \in [n']$. Now, consider the following CBL set $S' \subseteq \mathbb{R}^n \times \mathbb{R}^{n'}$ where $(x, y) \in S'$ if
\[ Ax \leq b \]
\[ y \geq 0 \]
\[ y \in \arg \max_y \left\{ -\sum_{i=1}^{n'} y_i : y_i \geq \langle \alpha^{ij}, x \rangle - \beta_{ij} \text{ for } j \in [k_i], i \in [n'] \right\}. \tag{4} \]

A key observation here is that in (4), the condition \( y_i \geq \langle \alpha^{ij}, x \rangle - \beta_{ij} \) for all \( j \in [k_i] \) is equivalent to saying \( y_i \geq f_i(x) \). Thus (4) can be written as \( y \in \arg \min_y \left\{ \sum_{i=1}^{n'} y_i : y_i \geq f_i(x), i \in [n'] \right\} \). Since we are minimizing the sum of coordinates of \( y \), this is equivalent to saying \( y_i = f_i(x) \) for \( i \in [n'] \). However, we have constrained \( y \) to be non-negative. So, if \( S'' = \{ x \in \mathbb{R}^n : \exists y \in \mathbb{R}^{n'} \text{ such that } (x, y) \in S' \} \), it naturally follows that \( S = S'' \). Thus \( S \in \text{repr}(P_{\mathbb{R}}) \), proving the inclusion \( \mathcal{P} \subseteq \text{repr}(P_{\mathbb{R}}) \). The result then follows from Lemma 25. \( \square \)

The next result shows that a given CBL-representable set can be represented as a LC-representable set.

**Lemma 28** The following holds:
\[ \text{repr}(P_{\mathbb{R}}) \subseteq \text{repr}(LC). \]

**Proof** See supplementary material. \( \square \)

Finally, to complete the cycle of containment, we show that a set representable using linear complementarity constraint can be represented as a polyhedral reverse convex set.

**Lemma 29** The following holds:
\[ \text{repr}(LC) \subseteq \text{repr}(\mathcal{P}). \]

**Proof** Let \( p, q \in \mathbb{R} \). Notice that \( 0 \leq p \perp q \geq 0 \iff p \geq 0, q \geq 0, \max\{-p, -q\} \geq 0 \). Consider any \( S \in LC \). Then
\[ S = \{ x : 0 \leq x \perp Mx + q \geq 0, Ax \leq b \} = \{ x : Ax \leq b, x \geq 0, Mx + q \geq 0, f_i(x) \geq 0 \}, \]
where \( f_i(x) = \max\{-x_i, -[Mx + q]_i\} \). Clearly, each \( f_i \) is a polyhedral convex function and hence by definition \( S \) is a polyhedral reverse-convex set. This implies \( S \in \mathcal{P} \). So, \( LC \subseteq \mathcal{P} \implies \text{repr}(LC) \subseteq \text{repr}(\mathcal{P}). \) \( \square \)

With the previous lemmata in hand we can now establish Theorem 18.

**Proof of Theorem 18** Follows from Lemmata 29–26.

For added intuition, we provide an example of the transformations discussed in Lemmata 26–29.
Example 30 Consider the set shown in Fig. 2.

\[ S = [1, 2] \cup \{3\} \cup [4, \infty) \]  

This can be written as a PRC set by using the construction in Lemma 26 as follows.

\[ S_{PRC} = \{ x : f_1(x) \geq 0; f_2(x) \geq 0; f_3(x) \geq 0 \} \]

where

\[ f_1(x) = x - 1 \]
\[ f_2(x) = \max \{-x + 2, x - 3\} \]
\[ f_3(x) = \max \{-x + 3, x - 4\} \]

One can check the equality, \( S = S_{PRC} \). Now, we show that this region is indeed bilevel representable, motivated by the construction in Lemma 27.

\[ S_{PR} = \left\{ x : \exists y \in \mathbb{R}^2 \text{ such that } x - 1 \geq 0; y_1 \geq 0; y_2 \geq 0; y \in \arg\min_y \{ y_1 + y_2 : y_1 \geq -x + 2; y_1 \geq x - 3; y_2 \geq -x + 3, y_2 \geq x - 4 \} \right\} \]

Again, one can observe that the value of \( y_1, y_2 \) that will satisfy the bilevel constraint is \( y_1 = f_2(x) \) and \( y_2 = f_3(x) \). We have constraints that \( f_1(x) = x - 1 \geq 0 \) and both \( y_1, y_2 \geq 0 \), showing that \( S_{PR} = S_{PRC} \). Now, we show that \( S_{PR} \) is in \( \mathcal{L}C \) by the construction in Lemma 28. We define \( S_{LC} \) to be the projection of the following set over the space of \( x \) variables.

\[ x - 1 \geq 0 \]
\[ 0 \leq y_1 \perp 1 - \delta_1 - \delta_2 \geq 0 \]
\[ 0 \leq y_2 \perp 1 - \delta_3 - \delta_4 \geq 0 \]
\[ 0 \leq \delta_1 \perp x + y_1 - 2 \geq 0 \]
\[ 0 \leq \delta_2 \perp -x + y_1 + 3 \geq 0 \]
\[ 0 \leq \delta_3 \perp x + y_2 - 3 \geq 0 \]
\[ 0 \leq \delta_4 \perp -x + y_2 + 4 \geq 0 \]

Finally, this can be shown to be a set in \( \mathcal{P}RC \), by setting \( f_i = \max\{-a_i, -b_i\} \) where \( a_i, b_i \) are the linear forms on either side of the \( \perp \) side in (6b) to (6g). We have the linear constraint in (6a), and the entire set can be projected over the space of \( x \) variables, completing the chain of constructive transformations.

5 Representability of mixed-integer bilevel sets

The goal of this section is to prove Theorem 20. Again, we establish the result over a series of lemmata. In Sect. 5.1, we show that the family of sets that are repre-
sentable by rational MIBL sets with integer variables only in the upper-level (BLUI sets) is equal to the family of finite unions of rational MILP-representable sets (that is, \( Q\text{-repr} \left( Q-\mathcal{P}_{\mathbb{Z} \times \mathbb{R}} \right) = Q\text{-repr} \left( \text{disj} \left( Q-\mathcal{P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \)). In Sect. 5.2 we prove the important intermediate result Theorem 21 that the family \( Q\text{-repr} \left( \text{disj} \left( Q-\mathcal{P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \) is an algebra of sets. This pays off in Sect. 5.4, where we establish the remaining containments of Theorem 20 using the properties of this algebra and the characterization of value functions of mixed-integer programs in terms of Jeroslow functions, which we study in Sect. 5.3.

5.1 Mixed-integer bilevel sets with continuous lower level

We prove the equivalence \( Q\text{-repr} \left( Q-\mathcal{P}_{\mathbb{Z} \times \mathbb{R}} \right) = Q\text{-repr} \left( \text{disj} \left( Q-\mathcal{P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \) from Theorem 20 by establishing two containments in the following two lemmas.

**Lemma 31** The following holds:

\[
Q\text{-repr} \left( Q-\mathcal{P}_{\mathbb{Z} \times \mathbb{R}} \right) \subseteq Q\text{-repr} \left( \text{disj} \left( Q-\mathcal{P}_{\mathbb{Z} \times \mathbb{R}} \right) \right)
\]

**Proof** Suppose \( T \in Q-\mathcal{P}_{\mathbb{Z} \times \mathbb{R}} \). Then \( T = \{ x : x \in T' \} \cap \{ x : x_j \in \mathbb{Z}, \forall j \in \mathcal{I}_L \} \), for some \( T' \in Q-\mathcal{P}_{\mathbb{R}} \) and for some \( \mathcal{I}_L \) (we are abusing notation here slightly because we use the symbol \( x \) now to denote the entire set of leader and follower variables in the bilevel problem). By Theorem 18, \( T' \in \text{disj} \left( \mathcal{P}_{\mathbb{R}} \right) \). Thus we can write \( T' = \bigcup_{i=1}^{k} T'_i \) where each \( T'_i \) is a polyhedron. Now \( T = \left( \bigcup_{i=1}^{k} T'_i \right) \cap \{ x : x_j \in \mathbb{Z}, \forall j \in \mathcal{I}_L \} = \bigcup_{i=1}^{k} \left( \{ x : x_j \in \mathbb{Z}, \forall j \in \mathcal{I}_L \} \cap T'_i \right) \) which is in \( \text{disj} \left( \mathcal{P}_{\mathbb{Z} \times \mathbb{R}} \right) \) by definition. \( \square \)

**Lemma 32** The following holds:

\[
Q\text{-repr} \left( \text{disj} \left( Q-\mathcal{P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \subseteq Q\text{-repr} \left( Q-\mathcal{P}_{\mathbb{Z} \times \mathbb{R}} \right).
\]

**Proof** First we prove the following claim, which we use to prove the lemma.

**Claim 32.1:** Let \( \tilde{T} \subseteq \mathbb{R}^{n' + n''} \) and \( \tilde{T} \in Q\text{-repr} \left( Q-\mathcal{P}_{\mathbb{R}} \right) \). Then \( \tilde{T} \cap (\mathbb{R}^{n'} \times \mathbb{Z}^{n''}) \in Q\text{-repr} \left( Q-\mathcal{P}_{\mathbb{Z} \times \mathbb{R}} \right) \).

**Proof of Claim 32.1:** Since \( \tilde{T} \in Q\text{-repr} \left( Q-\mathcal{P}_{\mathbb{R}} \right), \) from definition, we can write

\[
\tilde{T} = \left\{ \left( x, z \right) = L(u, v) : Au + Bv \leq b; \right\}
\]

\[
v \in \arg \max_{v'} \left\{ f^T v' : Dv' \leq g - Cu \right\}
\]

where \( A, B, C, D, b, f, g, L \) are all rational matrices of appropriate dimensions. Let \( T := \left\{ \left( x, z \right) : (x, z) \in \tilde{T}; \ z \in \mathbb{Z}^{n''} \right\} = \tilde{T} \cap (\mathbb{R}^{n'} \times \mathbb{Z}^{n''}). \) Observe that \( T = \text{proj}_{x,z}(X), \) where
\[
X := \left\{(x, z, u, v) : v \in \operatorname{arg \max}_{v'} \left\{ f^T v' : Dv' \leq g - Cu - 0x - 0z \right\} \right\}
\]

(7)

So, by definition, \( T \in \mathbb{Q} \)-repr \((\mathbb{Q} \text{- } P_{\mathbb{Z} \times \mathbb{R}})\), which proves the claim.

Let \( S \in \mathbb{Q} \)-repr \((\text{disj}(\mathbb{Q} \text{- } P_{\mathbb{Z} \times \mathbb{R}}))\); then, by definition, there exist polyhedra \( \tilde{S}_i \) for \( i = 1 \) to \( k \) such that \( S \) is the linear image under the linear transform \( L \) of the mixed integer points in the union of the \( \tilde{S}_i \). Let \( \tilde{S} = \bigcup_{i=1}^{k} \tilde{S}_i \) and so \( S = \left\{ L(y, z) : (y, z) \in \tilde{S} \cap (\mathbb{R}^n \times \mathbb{Z}^d) \right\} \). By Theorem 18, \( \tilde{S} \in \mathbb{Q} \)-repr \((\mathbb{Q} \text{- } P_{\mathbb{Z} \times \mathbb{R}})\).

And by Claim 32.1 and Lemma 25, \( S \in \mathbb{Q} \)-repr \((\mathbb{Q} \text{- } P_{\mathbb{Z} \times \mathbb{R}})\).

\[\square\]

5.2 The algebra \(\mathbb{Q} \)-repr \((\text{disj}(\mathbb{Q} \text{- } P_{\mathbb{Z} \times \mathbb{R}}))\)

The main result in this section is that the family of sets that are finite unions of sets representable by mixed-integer points in generalized polyhedra forms an algebra (in the sense of set theory). To do so, we develop some additional theory for generalized polyhedra, as defined in Sect. 2, by adapting some standard results from the usual theory for closed polyhedra.

From [31, Theorem 6.6] we obtain the following:

**Lemma 33** [31] Let \( Q \subseteq \mathbb{R}^n \) be a relatively open polyhedron and \( L \) is any linear transformation. Then \( L(Q) \) is a relatively open polyhedron. If \( Q \) and \( L \) are both rational, then \( L(Q) \) is also rational.

We now prove a preliminary result on the path to generalizing Theorem 12 to generalized polyhedra.

**Lemma 34** Let \( Q \subseteq \mathbb{R}^n \times \mathbb{R}^d \) be a rational generalized polyhedron. Then \( Q \cap (\mathbb{Z}^n \times \mathbb{R}^d) \) is a union of finitely many sets, each of which is the Minkowski sum of a relatively open rational polytope and a rational finitely generated monoid.

**Proof** See supplementary material.

The following is an analog of Jeroslow and Lowe’s fundamental result (Theorem 12) applied to the generalized polyhedral setting.

**Theorem 35** The following are equivalent:

1. \( S \in \mathbb{Q} \)-repr \((\text{disj}(\mathbb{Q} \text{- } P_{\mathbb{Z} \times \mathbb{R}}))\).
2. \( S \) is a finite union of sets, each of which is the Minkowski sum of a rational relatively open polytope and a rational finitely generated monoid, and
3. \( S \) is a finite union of sets, each of which is the Minkowski sum of a rational generalized polytope and a rational finitely generated monoid.
Proof (1) \(\implies\) (2): Observe from Lemma 33 that a (rational) linear transform of a (rational) relatively open polyhedron is a (rational) relative open polyhedron, and by definition of a (rational) monoid, a (rational) linear transform of a (rational) monoid is a (rational) monoid. Now from Lemma 34, the result follows.

(2) \(\implies\) (3): This is trivial since every (rational) relatively open polyhedron is a (rational) generalized polyhedron.

(3) \(\implies\) (1): This follows from the observation that the Minkowski sum of a rational generalized polytope and a rational monoid is a rational generalized mixed-integer representable set. A formal proof could be constructed following the proof of Theorem 12 given in the original [20] or [13, Theorem 4.47]. These results are stated for the case of closed polyhedra but it is straightforward to observe that their proofs equally apply to the generalized polyhedra setting with only superficial adjustments. We omit those minor details for brevity.

Remark 36 The rationality assumption cannot be removed from Lemma 34, and hence cannot be removed from Theorem 35. This is one of the places where the rationality assumption plays a crucial role; see also Remark 43.

Now, we prove that if we intersect sets within the family of generalized MILP-representable sets, then we remain in that family.

Lemma 37 Let \(S\) and \(T\) be sets in \(\mathbb{R}^n\) where \(S, T \in \text{repr} \left( \tilde{P}_{\mathbb{Z} \times \mathbb{R}} \right) \). Then \(S \cap T \in \text{repr} \left( \tilde{P}_{\mathbb{Z} \times \mathbb{R}} \right) \). The rational version holds, i.e., one can replace \(\text{repr} \left( \tilde{P}_{\mathbb{Z} \times \mathbb{R}} \right)\) by \(\mathbb{Q}\)-\text{repr} \left( \mathbb{Q}-\tilde{P}_{\mathbb{Z} \times \mathbb{R}} \right) \) in the statement.

Proof See supplementary material.

An immediate corollary of the above lemma is that the class \(\text{repr} \left( \text{disj} \left( \tilde{P}_{\mathbb{Z} \times \mathbb{R}} \right) \right)\) is closed under finite intersections.

Lemma 38 Let \(S\) and \(T\) be sets in \(\mathbb{R}^n\) where \(S, T \in \text{repr} \left( \text{disj} \left( \tilde{P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \). Then \(S \cap T \in \text{repr} \left( \text{disj} \left( \tilde{P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \). The rational version holds, i.e., one can replace \(\text{repr} \left( \text{disj} \left( \tilde{P}_{\mathbb{Z} \times \mathbb{R}} \right) \right)\) by \(\mathbb{Q}\)-\text{repr} \left( \text{disj} \left( \mathbb{Q}-\tilde{P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \) in the statement.

Proof See supplementary material.

To understand the interaction between generalized polyhedra and monoids, we review a few standard terms and results from lattice theory. We refer the reader to [4,13,33] for more comprehensive treatments of this subject.
**Definition 39** *(Lattice)* Given a set of linearly independent vectors $d^1, \ldots, d^r \in \mathbb{R}^n$, the lattice generated by the vectors is the set

$$
\Lambda = \left\{ x : x = \sum_{i=1}^{r} \lambda_i d^i, \lambda_i \in \mathbb{Z} \right\}
$$

We call the vectors $d^1, \ldots, d^r$ as the generators of the lattice $\Lambda$ and denote it by $\Lambda = \mathbb{Z}(d^1, \ldots, d^r)$.

**Definition 40** *(Fundamental parallelepiped)* Given $\Lambda = \mathbb{Z}(d^1, \ldots, d^r) \subseteq \mathbb{R}^n$, we define the fundamental parallelepiped of $\Lambda$ (with respect to the generators $d^1, \ldots, d^r$) as the set

$$
\Pi(d^1, \ldots, d^r) := \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^{r} \lambda_i d^i, 0 \leq \lambda_i < 1 \right\}.
$$

We prove the following two technical lemmata, which are crucial in proving that $\mathbb{Q}$-repr $\left( \text{disj} \left( \mathbb{Q}-\mathbb{P}_{\mathbb{Z} \times \mathbb{R}} \right) \right)$ is an algebra. The lemmata prove that $(P + M)^c$, where $P$ is a rational polytope and $M$ is a finitely generated rational monoid, is in $\mathbb{Q}$-repr $\left( \text{disj} \left( \mathbb{Q}-\mathbb{P}_{\mathbb{Z} \times \mathbb{R}} \right) \right)$. The first lemma (Lemma 41) proves this under the assumption that $M$ is generated by linearly independent vectors. The second lemma (Lemma 42) uses this preliminary result to prove it for a general monoid. The proof of the first lemma is based on the following observations.

(i) If the polytope $P$ is contained in the fundamental parallelepiped $\Pi$ of the lattice generated by the monoid $M$, then the complement of $P + M$ is just $(\Pi \setminus P) + M$ along with everything outside $\text{cone}(M)$.

(ii) The entire lattice can be written as a disjoint union of finitely many cosets with respect to an appropriately chosen sublattice. The sublattice is chosen such that its fundamental parallelepiped contains $P$ (after a translation). Then combining the finitely many cosets with the observation in (i), we obtain the result.

Observation (i) is needed to avoid overlap between distinct translates of $P$ in $P + M$. The linear independence of the generators of the monoid is important to be able to use the fundamental parallelepiped in this way. The proof also has to deal with the technicality that the monoid (and the lattice generated by it) need not be full-dimensional.

**Lemma 41** Let $M \subseteq \mathbb{R}^n$ be a monoid generated by a linearly independent set of vectors $M_g = \{m^1, \ldots, m^k\}$ with rational entries and let $P \in \mathcal{B} \cap \mathbb{Q}$-$\widehat{\mathbb{P}}_{\mathbb{R}}$. Then $(P + M)^c \in \mathbb{Q}$-repr $\left( \text{disj} \left( \mathbb{Q}-\mathbb{P}_{\mathbb{Z} \times \mathbb{R}} \right) \right)$.

**Proof** Suppose $k \leq n$. We now choose rational vectors $\tilde{m}^{k+1}, \ldots, \tilde{m}^{n}$, a scaling factor $\alpha \in \mathbb{Z}_+$ and a translation vector $f \in \mathbb{R}^n$ such that the following all hold:

(i) $\tilde{M}_g := \{m^1, \ldots, m^k, \tilde{m}^{k+1}, \ldots, \tilde{m}^{n}\}$ forms an orthogonal basis of $\mathbb{R}^n$. 

$$
\text{Springer}
$$
(ii) \( \{\hat{m}^i\}_{i=k+1}^n \) are orthogonal to each other and each is orthogonal to the space spanned by \( M_g \).

(iii) \( f + P \) is contained in the fundamental parallelepiped defined by the vectors
\[
\hat{M}_g := \alpha M_g \cup \{\hat{m}^{k+1}, \ldots, \hat{m}^n\}.
\]

Such a choice is always possible because of the boundedness of \( P \) and by utilizing the Gram–Schmidt orthogonalization process (one checks that the Gram–Schmidt procedure produces rational vectors if one starts with any rational basis; thus \( \{\hat{m}^{k+1}, \ldots, \hat{m}^n\} \) can be taken to be rational). Since we are interested in proving inclusion in \( \mathbb{Q} \)-repr \( \left( \text{disj} \left( \mathbb{Q} \cdot P \times \mathbb{R}^1 \right) \right) \), which is closed under translations, we can assume \( f = 0 \) without loss of generality.

Define \( \tilde{A} := Z(\hat{M}_g) \) and \( \hat{M} := \text{monoid}(\hat{M}_g) \). Define \( \tilde{\Lambda} := Z(\hat{M}_g) \subseteq \tilde{A} \) and \( \bar{M} := \text{monoid}(\hat{M}_g) \subseteq \hat{M} \). Moreover, linear independence of \( \tilde{M}_g \) and \( \bar{M}_g \) implies that \( \hat{M} = \tilde{\Lambda} \cap \text{cone}(\hat{M}_g) \) and \( \bar{M} = \tilde{\Lambda} \cap \text{cone}(\bar{M}_g) \). All of these together imply

Claim 41.1: \( \bar{M} = \tilde{\Lambda} \cap \hat{M} \).

Proof of Claim 41.1: \( \bar{M} = \tilde{\Lambda} \cap \text{cone}(\hat{M}_g) = \tilde{\Lambda} \cap \text{cone}(\hat{M}_g) = (\tilde{\Lambda} \cap \tilde{A}) \cap \text{cone}(\hat{M}_g) = \tilde{\Lambda} \cap (\tilde{\Lambda} \cap \text{cone}(\hat{M}_g)) = \tilde{\Lambda} \cap \hat{M} \), where the second equality follows from the fact that cone(\( \hat{M}_g \)) = cone(\( M_g \)).

Claim 41.2: \( \Pi_{\hat{M}_g} + \bar{M} = \text{cone}(\hat{M}_g) \). Moreover, given any element in \( x \in \text{cone}(\hat{M}_g) \), there exist unique \( u \in \Pi_{\hat{M}_g} \) and \( v \in \bar{M} \) such that \( x = u + v \).

Proof of Claim 41.2: Suppose \( u \in \Pi_{\hat{M}_g} \) and \( v \in \bar{M} \), then both \( u \) and \( v \) are non-negative combinations of elements in \( \hat{M}_g \). So clearly \( u + v \) is also a non-negative combination of those elements. This proves the forward inclusion. To prove the reverse inclusion, let \( x \in \text{cone}(\hat{M}_g) \). Then \( x = \sum_{i=1}^k \lambda_i m^i + \sum_{i=k+1}^n \lambda_i \hat{m}^i \) where \( \lambda_i \in \mathbb{R}_+ \). But now, we can write
\[
x = \left( \sum_{i=1}^k \left\lfloor \frac{\lambda_i}{\alpha} \right\rfloor \alpha m^i + \sum_{i=k+1}^n \lfloor \lambda_i \rfloor \hat{m}^i \right) + \left( \sum_{i=1}^k \left( \frac{\lambda_i}{\alpha} - \left\lfloor \frac{\lambda_i}{\alpha} \right\rfloor \right) \alpha m^i + \sum_{i=k+1}^n (\lambda_i - \lfloor \lambda_i \rfloor) \hat{m}^i \right)
\]
where the term in the first parentheses is in \( \bar{M} \) and the term in the second parentheses is in \( \Pi_{\hat{M}_g} \). Uniqueness follows from linear independence arguments, thus proving the claim.

Claim 41.3: \( \left( \Pi_{\hat{M}_g} + \bar{M} \right) \setminus (P + \bar{M}) = (\Pi_{\hat{M}_g} \setminus P) + \bar{M} \).

Proof of Claim 41.3:

First, consider \( x = u + v \in \left( \Pi_{\hat{M}_g} \setminus P \right) + \bar{M} \) such that \( u \in \Pi_{\hat{M}_g} \setminus P \), \( v \in \bar{M} \). Then \( x \in \left( \Pi_{\hat{M}_g} + \bar{M} \right) \) trivially. To show \( x \notin P + \bar{M} \), suppose \( x = u' + v' \) for some \( u' \in P \subseteq \Pi_{\hat{M}_g} \) and \( v' \in \bar{M} \). Now, observe from the uniqueness in Claim 41.2,
$u' = u$, and hence $u' \notin P$, which contradicts our assumption that $u' \in P$. This shows
$x \notin P + \widetilde{M}$ and hence $x \in (\Pi_{\mathbb{M}_g} + \widetilde{M}) \setminus (P + \widetilde{M})$.

Now consider $x \in (\Pi_{\mathbb{M}_g} + \widetilde{M}) \setminus (P + \widetilde{M})$. Thus, $x = u + v$ for some $u \in \Pi_{\mathbb{M}_g}$
and $v \in \widetilde{M}$. If $u \in P$ then $x = u + v \in P + \widetilde{M}$ contradicting the assumption that
$x \notin P + \widetilde{M}$. Thus, $u \notin P$, i.e., $u \in \Pi_{\mathbb{M}_g} \setminus P$. Thus, $x = u + v \in (\Pi_{\mathbb{M}_g} \setminus P) + \widetilde{M}$.

Define $S = \Pi_{\mathbb{M}_g} \cap \tilde{\Lambda}$. Then we have $\tilde{\Lambda} = S + \Lambda$ [4, Theorem VII.2.5]. Moreover,
for every $s \in S$, $(s + \Lambda) \cap \widetilde{M} = s + (\Lambda \cap \widetilde{M})$.

\[
P + \widetilde{M} = P + (\tilde{\Lambda} \cap \widetilde{M}) \\
= \bigcup_{s \in S} (P + ((s + \Lambda) \cap \widetilde{M})) \\
= \bigcup_{s \in S} (P + (\Lambda \cap \widetilde{M}) + s) \\
= \bigcup_{s \in S} (P + \widetilde{M} + s),
\]

where the last equality follows from Claim 41.1. The intuition behind the above equation is illustrated in Fig. 3.

We will first establish that $(P + \widetilde{M})^c \in Q\text{-}\text{repr}\left(\text{disj}\left(Q\text{-}\mathbb{P}_{\mathbb{Z} \times \mathbb{R}}\right)\right)$. By taking complements in (9), we obtain that $(P + \widetilde{M})^c = \bigcap_{s \in S} (P + \widetilde{M} + s)^c$. But from Lemma 37, and from the finiteness of $S$, if we can show that $(P + \widetilde{M} + s)^c$ is in

\[\text{(a)} P \text{ is shown in gray and the translates of } P \text{ along } \widetilde{M} \text{ are shown. The fundamental parallelepiped } \Pi_{\tilde{\Lambda}} \text{ is also shown to contain } P.\]

\[\text{(b) } P + ((s + \Lambda) \cap \widetilde{M}) \text{ is shown for each } s \in S. \text{ The crosses correspond to the translation of } \widetilde{M} \text{ along each } s \in S. \text{ The union of everything in Figure 3b is Figure 3a.}\]

\[\text{Fig. 3 Intuition for the set } S \text{ such that } \Lambda + S = \tilde{\Lambda} \text{ to assist in explaining the proof of Claim 41.3} \]
Since each $s \in S$ induces only translations, without loss of generality, we may only consider the case where $s = 0$. Since we have $P + M \subseteq \text{cone}(\tilde{M}_g)$, we have

$$(P + M)^c = \text{cone}(\tilde{M}_g)^c \cup (\text{cone}(\tilde{M}_g) \setminus (P + M)) = \text{cone}(\tilde{M}_g)^c \cup \left( \left( \prod_{\mathbb{M}_g} + \tilde{M} \right) \setminus (P + \tilde{M}) \right),$$

which follows from Claim 41.2. Continuing from (10):

$$(P + \tilde{M})^c = \text{cone}(\tilde{M}_g)^c \cup \left( \left( \prod_{\mathbb{M}_g} \setminus P \right) + \tilde{M} \right),$$

which follows from Claim 41.3.

The first set $\text{cone}(\tilde{M}_g)^c$ in (5.2) belongs to $\mathbb{Q}$-\text{repr}$(\text{disj} (\mathbb{Q} \cdot P^{\mathbb{Z} \times \mathbb{R}}))$ since the complement of a cone is a finite union of generalized polyhedra. In the second set $(\prod_{\mathbb{M}_g} \setminus P) + \tilde{M}$, $\prod_{\mathbb{M}_g}$ and $P$ are rational generalized polytopes, and hence $\prod_{\mathbb{M}_g} \setminus P$ is a finite union of rational generalized polytopes, $(\prod_{\mathbb{M}_g} \setminus P) + \tilde{M}$ is a set in $\mathbb{Q}$-\text{repr}$(\text{disj} (\mathbb{Q} \cdot P^{\mathbb{Z} \times \mathbb{R}}))$ by Theorem 35. Thus, $(P + \tilde{M})^c \in \mathbb{Q}$-\text{repr}$(\text{disj} (\mathbb{Q} \cdot P^{\mathbb{Z} \times \mathbb{R}}))$ (note that Lemma 9 shows that $\mathbb{Q}$-\text{repr}$(\text{disj} (\mathbb{Q} \cdot P^{\mathbb{Z} \times \mathbb{R}}))$ is closed under unions).

We now finally argue that $(P + M)^c$ belongs to $\mathbb{Q}$-\text{repr}$(\text{disj} (\mathbb{Q} \cdot P^{\mathbb{Z} \times \mathbb{R}}))$. Let $A^1 := (P + \tilde{M})^c$. For each vector $\tilde{m}^i$ for $i = k + 1, \ldots, n$ added to form $\tilde{M}_g$ from $M_g$, define $H^i$ as follows:

$$H^i = \left\{ x : \langle \tilde{m}^i, x \rangle \geq \| \tilde{m}^i \|^2_2 \right\}.$$

Now let $A^2 := \bigcup_{i = k+1}^n H^i$. Note that $A^2$ is a finite union of halfspaces and hence $A^2 \in \mathbb{Q}$-\text{repr}$(\text{disj} (\mathbb{Q} \cdot P^{\mathbb{Z} \times \mathbb{R}}))$ (note that we use rationality of $(\tilde{m}^{k+1}, \ldots, \tilde{m}^n)$).

We claim $(P + M)^c = A^1 \cup A^2$. This suffices to complete the argument since we have shown $A^1$ and $A^2$ are in $\mathbb{Q}$-\text{repr}$(\text{disj} (\mathbb{Q} \cdot P^{\mathbb{Z} \times \mathbb{R}}))$ and thus so is their union.

First, we show that $A^1 \cup A^2 \subseteq (P + M)^c$, i.e., $P + M \subseteq A^1 \cap A^2$. Let $x \in P + M$. Since $M \subseteq \tilde{M}$ we have $x \in P + \tilde{M}$. Thus, $x \notin A^1$. Further, since $x \in P + M$ we may write $x = u + v$ with $u \in P$ and $v \in M$ where $u = \sum_{i=1}^k \mu_i \alpha m^i + \sum_{i=k}^n \mu_i \tilde{m}^i$, $v = \sum_{j=1}^k \lambda_j m^j$ with $0 \leq \mu < 1$ and $\lambda_j \in \mathbb{Z}_+$, since $P \subseteq \prod_{\mathbb{M}_g}$. So for all $i$, $\langle \tilde{m}^i, u \rangle = \mu_i \| \tilde{m}^i \|^2_2 < \| \tilde{m}^i \|^2_2$. This is because we have $\tilde{m}^i$ is orthogonal to every
vector \( m^i \) and \( \tilde{m}^j \) for \( i \neq j \). Hence, \( \langle \tilde{m}^i, u + v \rangle < \| \tilde{m}^i \|^2_2 \). This follows because \( \tilde{m}^i \) is orthogonal to the space spanned by the monoid \( M \ni v \). Thus \( x \notin A^2 \). So we now have \( P + M \subseteq A^1_i \cap A^2_j \) and so \( A^1 \cup A^2 \subseteq (P + M)^c \).

Conversely, suppose \( x \notin P + M \). If, in addition, \( x \notin P + \tilde{M} \) then \( x \in A^1 \) and we are done. However, if \( x = u + v \in P + (\tilde{M} \setminus M) \) with \( u \in P \) and \( v \in \tilde{M} \setminus M \). This means \( v = \sum_{j=1}^{k} \lambda_j m^j + \sum_{j=k+1}^{n} \lambda_j \tilde{m}^j \) with \( \lambda_j \in \mathbb{Z}_+ \) for all \( j = 1, \ldots, n \) and \( \lambda_j > 1 \) for some \( j \in \{ k + 1, \ldots, n \} \) and we can write \( u = \sum_{j=1}^{k} \mu_j \alpha m^j + \sum_{j=k+1}^{n} \mu_j \tilde{m}^j \) with \( 0 \leq \mu \leq 1 \). So \( \langle \tilde{m}^i, u \rangle = \langle \tilde{m}^i, \mu_i \tilde{m}^i \rangle \geq 0 \) and \( \langle \tilde{m}^j, v \rangle = \langle \tilde{m}^j, \lambda_j \tilde{m}^j \rangle > \| \tilde{m}^j \|^2_2 \). So \( u + v \in H^j \subseteq A^2 \). Thus we have the reverse containment and hence the result. \( \Box \)

**Lemma 42** Let \( P \subseteq \mathbb{R}^n \) be a rational generalized polytope and \( M \subseteq \mathbb{R}^n \) be a rational, finitely generated monoid. Then \( S = (P + M)^c \subseteq \mathbb{Q} \text{-repr} \left( \text{disj} \left( \mathbb{Q} \text{-repr} (\text{disj} \left( \mathbb{Q} \text{-repr} (P \cap \mathbb{Z} \times \mathbb{R})) \right) \right) \right) \).

**Proof** Define \( C := \text{cone}(M) \). Consider a triangulation \( C = \bigcup_i C_i \), where each \( C_i \) is simplicial. Now, \( M_i := M \cap C_i \) is a monoid for each \( i \) (one simply checks the definition of a monoid) and moreover, it is a pointed monoid because \( C_i \) is pointed and \( \text{cone}(M_i) = C_i \). Since every extreme ray of \( C_i \) has an element of \( M_i \) on it. Observe that \( M = \bigcup_i M_i \). By Theorem 4, part 1, in [21], each of the \( M_i \) are finitely generated. By part 3) of the same theorem, each \( M_i \) can be written as \( M_i = \bigcup_{j=1}^{w_i} (p^i,j + \tilde{M}_i) \) for some finite vectors \( p^i,1, \ldots, p^i,w_i \subseteq M_i \), where \( \tilde{M}_i \) is the monoid generated by the elements of \( M_i \) lying on the extreme rays of \( C_i \). Now,

\[
P + M = \bigcup_i (P + M_i) = \bigcup_i \bigcup_{j=1}^{w_i} (P + (p^i,j + \tilde{M}_i)).
\]

Thus by Lemma 38, it suffices to show that \( (P + (p^i,j + \tilde{M}_i))^c \) is in \( \mathbb{Q} \)-repr \( \left( \text{disj} \left( \mathbb{Q} \text{-repr} (P \cap \mathbb{Z} \times \mathbb{R}) \right) \right) \). Since \( \tilde{M}_i \) is generated by rational linearly independent vectors and \( P \) is a rational generalized polytope, we have our result from Lemma 41. \( \Box \)

**Remark 43** We do not see a way to remove the rationality assumption in Lemma 42, because it uses Theorem 4 in [21] that assumes that the monoid is rational and finitely generated. This is the other place where rationality becomes a crucial assumption in the analysis (see also Remark 36). On the other hand, the rationality assumption in Lemma 41 can be relaxed by a more careful argument, invoking the direction of Theorem 35 which does not need the rationality assumption.

**Lemma 44** If \( S \in \mathbb{Q} \)-repr \( \left( \text{disj} \left( \mathbb{Q} \text{-repr} (P \cap \mathbb{Z} \times \mathbb{R}) \right) \right) \) then \( S^c \in \mathbb{Q} \)-repr \( \left( \text{disj} \left( \mathbb{Q} \text{-repr} (P \cap \mathbb{Z} \times \mathbb{R}) \right) \right) \).

**Proof** By Theorem 35, \( S \) can be written as a finite union \( S = \bigcup_{j=1}^{t} S_j \), with \( S_j = P_j + M_j \), where \( P_j \) is a rational generalized polytope and \( M_j \) is a rational,
Lemma 46

Proof of Theorem 21

⊓⊔

result.

Sc-finitely generated monoid. Observe \( S_j^c = (P_j + M_j)^c \), which by Lemma 42, is in \( \mathbb{Q} \)-repr \( \left( \text{disj} \left( \mathbb{Q} - P \mathbb{Z} \times \mathbb{R} \right) \right) \). Now by De Morgan’s law, \( S^c = \left( \bigcup_j S_j \right)^c = \bigcap_j S_j^c \). By Lemma 38, \( \mathbb{Q} \)-repr \( \left( \text{disj} \left( \mathbb{Q} - P \mathbb{Z} \times \mathbb{R} \right) \right) \) is closed under intersections, and we have the result.

Proof of Theorem 21

We recall that a family of sets \( \mathcal{F} \) is an algebra if the following two conditions hold. (i) \( S \in \mathcal{F} \implies S^c \in \mathcal{F} \) and (ii) \( S, T \in \mathcal{F} \implies S \cup T \in \mathcal{F} \). For the class of interest, the first condition is satisfied from Lemma 44. The second condition is satisfied by Lemma 9 which shows that \( \mathbb{Q} \)-repr \( \left( \text{disj} \left( \mathbb{Q} - P \mathbb{Z} \times \mathbb{R} \right) \right) \) is the same as \textit{finite unions} of sets in \( \mathbb{Q} \)-repr \( \left( \mathbb{Q} - P \mathbb{Z} \times \mathbb{R} \right) \).

5.3 Value function analysis

We now discuss the three classes of functions defined earlier, namely, Chvátal functions, Gomory functions and Jeroslow functions, and show that their sublevel, superlevel and level sets are all elements of \( \mathbb{Q} \)-repr \( \left( \text{disj} \left( \mathbb{Q} - P \mathbb{Z} \times \mathbb{R} \right) \right) \). This is crucial for studying bilevel integer programs via a value function approach to handling the lower-level problem using the following result.

Theorem 45 (Theorem 10 in [8]) For every rational matrices \( A, B \) of appropriate dimensions, there exists a Jeroslow function \( J \), such that for any \( b \) for which the mixed integer program \( \max_{x, y} \{ c^T x + d^T y : Ax + By = b; \ (x, y) \geq 0, \ x \in \mathbb{Z}^m \} \) is feasible, its optimal value is \( J(b) \).

Lemma 46

Let \( \psi : \mathbb{R}^n \to \mathbb{R} \) be a Chvátal function. Then (i) \( \{ x : \psi(x) \geq 0 \} \), (ii) \( \{ x : \psi(x) \leq 0 \} \), (iii) \( \{ x : \psi(x) = 0 \} \), (iv) \( \{ x : \psi(x) < 0 \} \), and (v) \( \{ x : \psi(x) > 0 \} \) are all in \( \mathbb{Q} \)-repr \( \left( \text{disj} \left( \mathbb{Q} - P \mathbb{Z} \times \mathbb{R} \right) \right) \).

Proof (i) We prove by induction on the order \( \ell \in \mathbb{Z}^+ \) used in a binary tree representation of \( \psi \) (see Definition 13). For \( \ell = 0 \), a Chvátal function is a rational linear inequality and hence \( \{ x : \psi(x) \geq 0 \} \) is a rational halfspace, which is clearly in \( \mathbb{Q} \)-repr \( \left( \text{disj} \left( \mathbb{Q} - P \mathbb{Z} \times \mathbb{R} \right) \right) \). Assuming that the assertion is true for all orders \( \ell \leq k \), we prove that it also holds for order \( \ell = k + 1 \). By Basu et al. [5, Theorem 4.1], we can write \( \psi(x) = \psi_1(x) + [\psi_2(x)] \) where \( \psi_1 \) and \( \psi_2 \) are Chvátal functions with representations of order no greater than \( k \). Hence,

\[
\{ x : \psi(x) \geq 0 \} = \{ x : \psi_1(x) + [\psi_2(x)] \geq 0 \} = \{ x : \exists y \in \mathbb{Z}, \ \psi_1(x) + y \geq 0, \ \psi_2(x) \geq y \}.
\]

We claim equivalence because, suppose \( \overline{x} \) is an element of the set in RHS with some \( \overline{y} \in \mathbb{Z} \), \( \overline{y} \) is at most \( [\psi_2(x)] \). So if \( \psi_1(\overline{x}) + \overline{y} \geq 0 \), we immediately have
\( \psi_1(\overline{x}) + \lfloor \psi_2(\overline{x}) \rfloor \geq 0\) and hence \( \overline{x} \) is in the set on LHS. Conversely, if \( \overline{x} \) is in the set on LHS, then choosing \( \overline{y} = [\overline{x}] \) satisfies all the conditions for the sets in RHS, giving the equivalence. Finally, observing that the RHS is an intersection of sets which are already in \( \mathbb{Q} \)-\text{repr} \( \left( \text{disj} \left( \mathbb{Q} \overline{-P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \) by the induction hypothesis and the fact that \( \mathbb{Q} \)-\text{repr} \( \left( \text{disj} \left( \mathbb{Q} \overline{-P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \) is an algebra by Theorem 21, we have the result.

(ii) By similar arguments as (i), the statement is true for \( \ell = 0 \). For positive \( \ell \), we proceed by induction using the same construction. Now,

\[
\{ x : \psi(x) \leq 0 \} = \{ x : \psi_1(x) + \lfloor \psi_2(x) \rfloor \leq 0 \}
\]

\[
= \{ x : \exists y \in \mathbb{Z}, \, \psi_1(x) + y \leq 0, \, \psi_2(x) \geq y, \, \psi_2(x) < y + 1 \}. 
\]

The last two conditions along with integrality on \( y \) ensures \( y = \lfloor \psi_2(x) \rfloor \). Note that \( \{ x : \psi_2(x) - y \geq 0 \} \) is in \( \mathbb{Q} \)-\text{repr} \( \left( \text{disj} \left( \mathbb{Q} \overline{-P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \) by (i). Similarly \( \{ x : \psi_2(x) - y - 1 \geq 0 \} \) is in \( \mathbb{Q} \)-\text{repr} \( \left( \text{disj} \left( \mathbb{Q} \overline{-P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \). Since \( \mathbb{Q} \)-\text{repr} \( \left( \text{disj} \left( \mathbb{Q} \overline{-P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \) is an algebra (cf. Theorem 21), its complement is in \( \mathbb{Q} \)-\text{repr} \( \left( \text{disj} \left( \mathbb{Q} \overline{-P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \) and hence we have \( \{ x : \psi_2(x) < y + 1 \} \) is closed under finite intersections, the result follows.

(iii) Set defined by (iii) is an intersection of sets defined in (i) and (ii).

(iv)–(v) Sets defined here are complements of sets defined in (i)–(ii).

\[\square\]

**Lemma 47** Let \( G : \mathbb{R}^n \to \mathbb{R} \) be a Gomory function. Then (i) \( \{ x : G(x) \geq 0 \} \), (ii) \( \{ x : G(x) \leq 0 \} \), (iii) \( \{ x : G(x) = 0 \} \), (iv) \( \{ x : G(x) < 0 \} \), (v) \( \{ x : G(x) > 0 \} \) are all in \( \mathbb{Q} \)-\text{repr} \( \left( \text{disj} \left( \mathbb{Q} \overline{-P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \).

**Proof** Let \( G(x) = \min_{i=1}^{k} \psi_i(x) \), where each \( \psi_i \) is a Chvátal function.

(i) Note that \( \{ x : G(x) \geq 0 \} = \bigcap_{i=1}^{k} \{ x : \psi_i(x) \geq 0 \} \in \mathbb{Q} \)-\text{repr} \( \left( \text{disj} \left( \mathbb{Q} \overline{-P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \) since each individual set in the finite intersection is in \( \mathbb{Q} \)-\text{repr} \( \left( \text{disj} \left( \mathbb{Q} \overline{-P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \) by Lemma 46 and \( \mathbb{Q} \)-\text{repr} \( \left( \text{disj} \left( \mathbb{Q} \overline{-P}_{\mathbb{Z} \times \mathbb{R}} \right) \right) \) is closed under intersections by Lemma 38.
(ii) Note that $G(x) \leq 0$ if and only if there exists an $i$ such that $\psi_i(x) \leq 0$. So 
\[ \{ x : G(x) \leq 0 \} = \bigcup_{i=1}^k \{ x : \psi_i(x) \leq 0 \} \in \mathbb{Q} - \text{repr} \left( \text{disj} \left( \mathbb{Q} - \mathbb{P}^{\mathbb{Z} \times \mathbb{R}} \right) \right) \] 
since each individual set in the finite union is in $\mathbb{Q} - \text{repr} \left( \text{disj} \left( \mathbb{Q} - \mathbb{P}^{\mathbb{Z} \times \mathbb{R}} \right) \right)$ by Lemma 46, and $\mathbb{Q} - \text{repr} \left( \text{disj} \left( \mathbb{Q} - \mathbb{P}^{\mathbb{Z} \times \mathbb{R}} \right) \right)$ is an algebra by Theorem 21.

(iii) This is the intersection of sets described in (i) and (ii).

(iv)–(v) Sets defined here are complements of sets defined in (i)–(ii).

\[ \square \]

**Lemma 48** Let $J : \mathbb{R}^n \to \mathbb{R}$ be a Jeroslow function. Then (i) $\{ x : J(x) \geq 0 \}$, (ii) $\{ x : J(x) \leq 0 \}$, (iii) $\{ x : J(x) = 0 \}$, (iv) $\{ x : J(x) < 0 \}$, and (v) $\{ x : J(x) > 0 \}$ are all in $\mathbb{Q} - \text{repr} \left( \text{disj} \left( \mathbb{Q} - \mathbb{P}^{\mathbb{Z} \times \mathbb{R}} \right) \right)$.

**Proof** (i) Let $J(x) = \max_{i \in \mathcal{I}} G \left( [x]_{E_i} \right) + w_i^\top (x - [x]_{E_i})$ be a Jeroslow function, where $G$ is a Gomory function, $\mathcal{I}$ is a finite set, $\{ E_i \}_{i \in \mathcal{I}}$ is set of rational invertible matrices indexed by $\mathcal{I}$, and $\{ w_i \}_{i \in \mathcal{I}}$ is a set of rational vectors indexed by $\mathcal{I}$. Since we have a maximum over finitely many sets, from the fact that $\mathbb{Q} - \text{repr} \left( \text{disj} \left( \mathbb{Q} - \mathbb{P}^{\mathbb{Z} \times \mathbb{R}} \right) \right)$ is an algebra, it suffices to show 
\[ \{ x : G([x]_E) + w^\top (x - [x]_E) \geq 0 \} \in \mathbb{Q} - \text{repr} \left( \text{disj} \left( \mathbb{Q} - \mathbb{P}^{\mathbb{Z} \times \mathbb{R}} \right) \right) \] 
for arbitrary $E$, $w$ and Gomory function $G$. Observe that 
\[ \left\{ x : G([x]_E) + w^\top (x - [x]_E) \geq 0 \right\} = \text{proj}_x \left\{ (x, y^1, y^2, y^3) : \begin{array}{l} G(y^1) + y^2 \geq \varepsilon \\
 y^1 = [x]_E \\
 y^2 = \langle w, y^3 \rangle \\
 y^3 = x - y^1 \end{array} \right\} \]
and the set being projected in the right hand side above is equal to the following intersection 
\[ \left\{ (x, y^1, y^2, y^3) : G(y^1) + y^2 \geq 0 \right\} \]
\[ \cap \left\{ (x, y^1, y^2, y^3) : E^{-1} y^1 = \lfloor E^{-1} x \rfloor \right\} \]
\[ \cap \left\{ (x, y^1, y^2, y^3) : y^2 = \langle w, y^3 \rangle \right\} \]
\[ \cap \left\{ (x, y^1, y^2, y^3) : y^3 = x - y^1 \right\} . \]
Since each of the sets in the above intersection belong to $\mathbb{Q} - \text{repr} \left( \text{disj} \left( \mathbb{Q} - \mathbb{P}^{\mathbb{Z} \times \mathbb{R}} \right) \right)$ by Lemmata 46 and 47 (also noting Remark 17), and $\mathbb{Q} - \text{repr} \left( \text{disj} \left( \mathbb{Q} - \mathbb{P}^{\mathbb{Z} \times \mathbb{R}} \right) \right)$ is an algebra by Theorem 21, we obtain the result.

\[ \square \]
(ii) As in (i), since we have maximum over finitely many sets, from the fact that
\( Q - \text{repr} \left( \text{disj} \left( Q - \widetilde{P} Z \times R \right) \right) \) is an algebra (Theorem 21), it suffices to show
\[
\{ x : G(\lfloor x \rfloor_E) + w^\top (x - \lfloor x \rfloor_E) \leq 0 \} \in Q - \text{repr} \left( \text{disj} \left( Q - \widetilde{P} Z \times R \right) \right)
\]
for arbitrary \( E, w \) and Gomory function \( G \). The same arguments as before pass through, except for replacing the \( \geq \) in the first constraint with \( \leq \).

(iii) This is the intersection of sets described in (i) and (ii).

(iv)–(v) Sets defined here are complements of sets defined in (i)–(ii).

\[ \square \]

**Proof of Theorem 22** Follows from Lemmata 46–48.

\[ \square \]

### 5.4 General mixed-integer bilevel sets

We start by quoting an example from [23] showing that the MIBL set need not even be a closed set. This is the first relation in Theorem 20, showing a strict containment.

**Lemma 49** [23, Example 1.1] *The following holds:*

\[
\text{repr} \left( P Z \times R \right) \setminus \text{cl} \left( \text{repr} \left( P Z \times R \right) \right) \neq \emptyset.
\]

**Proof** The following set \( T \) (refer Fig. 4) is in \( \text{repr} \left( P Z \times R \right) \setminus \text{cl} \left( \text{repr} \left( P Z \times R \right) \right) \):

\[
T = \left\{ (x, y) \in \mathbb{R}^2 : x \geq 0, \ y \in \arg \min_y \{ y : y \geq x, \ 0 \leq y \leq 1, \ y \in \mathbb{Z} \} \right\}.
\]

**Fig. 4** The set \( T \) used in the proof of Lemma 49. The dotted lines and the dashed lines depict the upper and lower level constraints respectively. The continuous line show the set. A shaded circle at the end of the line segment implies that the endpoint is part of the set. An unshaded circle at the end of the line segment shows that the point is not a part of the set.
By definition, $T \in \text{repr} \left( P_{Z\times R}^{Z \times R} \right)$. Observe that the bilevel constraint is satisfied only if $y = \lceil x \rceil$. So $T = \{(0, 0)\} \cup \{(0, 1) \times \{1\}\}$. So $T$ is not a closed set. Observing that every set in $\text{cl} \left( \text{repr} \left( P_{Z\times R}^{Z \times R} \right) \right)$ is closed, $T \notin \text{cl} \left( \text{repr} \left( P_{Z\times R}^{Z \times R} \right) \right)$. The example can be extended to arbitrary dimensions by using Cartesian products of the set $T$ with itself or with any other polyhedron. \hfill \Box

We now develop the tools to establish Theorem 20.

**Lemma 50** The following holds: $\mathbb{Q}\text{-}\text{repr} \left( \mathbb{Q}\text{-}\text{P}_{Z\times R}^{Z \times R} \right) \subseteq \mathbb{Q}\text{-}\text{repr} \left( \text{disj} \left( \mathbb{Q}\text{-}\text{P}_{Z\times R}^{Z \times R} \right) \right)$.

**Proof** Recall from Definition 1 an element $S$ of $\mathbb{Q}\text{-}\text{P}_{Z\times R}^{Z \times R}$ consists of the intersection $S^1 \cap S^2 \cap S^3$ (with rational data). From Theorem 45, $S^2$ can be rewritten as $\{(x, y) : f^T y \geq J(g - Cx)\}$ for some Jeroslow function $J$. Thus, from Lemma 48, $S^2 \in \mathbb{Q}\text{-}\text{repr} \left( \text{disj} \left( \mathbb{Q}\text{-}\text{P}_{Z\times R}^{Z \times R} \right) \right)$. Moreover, $S^1, S^3 \in \mathbb{Q}\text{-}\text{repr} \left( \text{disj} \left( \mathbb{Q}\text{-}\text{P}_{Z\times R}^{Z \times R} \right) \right)$ since they are either rational polyhedra or mixed-integer points in rational polyhedra.

Thus, $S = S^1 \cap S^2 \cap S^3 \in \mathbb{Q}\text{-}\text{repr} \left( \text{disj} \left( \mathbb{Q}\text{-}\text{P}_{Z\times R}^{Z \times R} \right) \right)$ by Theorem 21, proving the inclusion. This shows that $\mathbb{Q}\text{-}\text{P}_{Z\times R}^{Z \times R} \subseteq \mathbb{Q}\text{-}\text{repr} \left( \text{disj} \left( \mathbb{Q}\text{-}\text{P}_{Z\times R}^{Z \times R} \right) \right)$, and by Lemma 25 the result follows. \hfill \Box

Lemma 50 gets us close to showing $\text{cl} \left( \mathbb{Q}\text{-}\text{repr} \left( \mathbb{Q}\text{-}\text{P}_{Z\times R}^{Z \times R} \right) \right) \subseteq \mathbb{Q}\text{-}\text{repr} \left( \text{disj} \left( \mathbb{Q}\text{-}\text{P}_{Z\times R}^{Z \times R} \right) \right)$, as required in Theorem 20. Indeed, we can immediately conclude from Lemma 50 that $\text{cl} \left( \mathbb{Q}\text{-}\text{repr} \left( \mathbb{Q}\text{-}\text{P}_{Z\times R}^{Z \times R} \right) \right) \subseteq \text{cl} \left( \mathbb{Q}\text{-}\text{repr} \left( \text{disj} \left( \mathbb{Q}\text{-}\text{P}_{Z\times R}^{Z \times R} \right) \right) \right)$.

The next few results build towards showing that $\text{cl} \left( \mathbb{Q}\text{-}\text{repr} \left( \text{disj} \left( \mathbb{Q}\text{-}\text{P}_{Z\times R}^{Z \times R} \right) \right) \right) \left( \mathbb{Q}\text{-}\text{P}_{Z\times R}^{Z \times R} \right)$. The latter is intuitive since closures of generalized polyhedra are closed polyhedra. As we shall see, the argument is a bit more delicate than this simple intuition. We first recall a couple of standard results on the closure operation $\text{cl}$(·).

**Lemma 51** If $S_1, \ldots, S_k \subseteq \mathbb{R}^n$ then $\text{cl} \left( \bigcup_{i=1}^n S_i \right) = \bigcup_{i=1}^n \text{cl} \left( S_i \right)$.

**Proof** See supplementary material. \hfill \Box

**Lemma 52** Let $A, B$ be sets such that $\text{cl}(A)$ is compact and $B$ is closed. Then $\text{cl}(A + B) = \text{cl}(A) + B$.

**Proof** See supplementary material. \hfill \Box

**Lemma 53** The following holds: $\text{cl} \left( \mathbb{Q}\text{-}\text{repr} \left( \text{disj} \left( \mathbb{Q}\text{-}\text{P}_{Z\times R}^{Z \times R} \right) \right) \right) = \mathbb{Q}\text{-}\text{repr} \left( \text{disj} \left( \mathbb{Q}\text{-}\text{P}_{Z\times R}^{Z \times R} \right) \right)$. 

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Proof The $\supseteq$ direction is trivial because sets in $Q$-repr $(\operatorname{disj} (Q^- \mathcal{P}^Z \times \mathbb{R}))$ are closed and a closed polyhedron is a type of generalized polyhedron. For the $\subseteq$ direction, let $S \in \operatorname{cl} \left( Q^- \operatorname{repr} \left( \operatorname{disj} (Q^- \mathcal{P}^Z \times \mathbb{R}) \right) \right)$; that is, $S = \operatorname{cl}( \bigcup_{i=1}^k (P_i + M_i) )$ for some $P_i$ that are finite unions of rational generalized polytopes and $M_i$ that are finitely generated rational monoids (we have used Lemma 9 here). By Lemma 51, this equals $\bigcup_{i=1}^k \operatorname{cl} (P_i + M_i)$. Observe that $P_i$ is a finite union of generalized polytopes and are hence bounded. Thus their closures are compact. Also $M_i$ are finitely generated monoids and are hence closed. Thus, by Lemma 52, we can write this is equal to $\bigcup_{i=1}^k \operatorname{cl} (P_i) + M_i$. But by Theorem 12, each of these sets $\operatorname{cl} (P_i) + M_i$ is in $Q$-repr $(Q^- \mathcal{P}^Z \times \mathbb{R})$. Thus, their finite union is in $Q$-repr $(\operatorname{disj} (Q^- \mathcal{P}^Z \times \mathbb{R}))$ by Lemma 9.
\[\square\]

Corollary 54 The following holds: $\operatorname{cl} \left( Q^- \operatorname{repr} \left( Q^- \mathcal{P}^Z \times \mathbb{R} \right) \right) \subseteq Q^- \operatorname{repr} \left( \operatorname{disj} (Q^- \mathcal{P}^Z \times \mathbb{R}) \right)$.

Proof Follows from Lemmata 50 and 53. $\square$

We are now ready to prove the main result of the section.

Proof of Theorem 20 The strict inclusion follows from Lemma 49. The equality $Q^- \operatorname{repr} \left( Q^- \mathcal{P}^Z \times \mathbb{R} \right) = Q^- \operatorname{repr} \left( \operatorname{disj} (Q^- \mathcal{P}^Z \times \mathbb{R}) \right)$ is obtained from Lemmata 31 and 32. For the equality $\operatorname{cl} \left( Q^- \operatorname{repr} \left( Q^- \mathcal{P}^Z \times \mathbb{R} \right) \right) = Q^- \operatorname{repr} \left( \operatorname{disj} (Q^- \mathcal{P}^Z \times \mathbb{R}) \right)$, the inclusion $\supseteq$ follows from the equality $Q^- \operatorname{repr} \left( Q^- \mathcal{P}^Z \times \mathbb{R} \right) = Q^- \operatorname{repr} \left( \operatorname{disj} (Q^- \mathcal{P}^Z \times \mathbb{R}) \right)$ just established above and the facts that BLUI sets are MIBL sets and sets in $Q^- \operatorname{repr} \left( \operatorname{disj} (Q^- \mathcal{P}^Z \times \mathbb{R}) \right)$ are closed. The reverse inclusion is immediate from Corollary 54. $\square$

We now give three concrete examples to establish Corollary 23. First, we illustrate a set that is CBL-representable but not MILP-representable. Next, we illustrate an example of a set which is MILP-representable but not CBL-representable. Finally, we provide an example of a set that is BLUI-representable but neither MILP-representable nor CBL-representable.

Proof of Theorem 23 We construct a set $T \in \operatorname{repr} (\mathcal{P}^Z_\mathbb{R})$ as follows. Consider the following set $T' \in \mathcal{P}^Z_\mathbb{R}$ given by $(x, y, z_1, z_2) \in \mathbb{R}^4$ satisfying:

$$(y, z_1, z_2) \in \operatorname{arg min} \left\{ \begin{array}{l} z_1 \geq x \\ z_1 \geq -x \\ z_2 \leq x \\ z_2 \leq -x \\ y \leq z_1 \\ y \geq z_2 \end{array} \right\}$$

with no upper-level constraints. Consider $T = \{ (x, y) \in \mathbb{R}^2 : (x, y, z_1, z_2) \in T' \}$ illustrated in Fig. 5. Note that $T \notin \operatorname{repr} (\mathcal{P}^Z_\mathbb{R})$. We claim $T \notin \operatorname{repr} (\mathcal{P}^Z_\mathbb{R})$. Suppose it is. Then by Theorem 12, $T$ is the Minkowski sum of a finite union of polytopes.
and a monoid. Note that \( \{(x, x) : x \in \mathbb{R}\} \subset T \) which implies \((1, 1)\) is an extreme ray and \(\lambda(1, 1)\) should be in the integer cone of \( T \) for some \(\lambda > 0\). Similarly \(\{(-x, x) : x \in \mathbb{R}\} \subset T \) which implies \((-1, 1)\) is an extreme ray and \(\lambda'(-1, 1)\) should be in the integer cone of \( T \) for some \(\lambda' > 0\). Both the facts imply, for some \(\lambda'' > 0\), the point \((0, \lambda'') \in T\). But no such point is in \( T \) showing that \( T \notin \text{repr}(\mathcal{P}_{\mathbb{Z} \times \mathbb{R}}) \).

Conversely, consider the set of integers \( \mathbb{Z} \subset \mathbb{R} \). Clearly, \( \mathbb{Z} \) is not a finite union of polyhedra, but \( \mathbb{Z} \in \text{repr}(\mathcal{P}_{\mathbb{Z} \times \mathbb{R}}) \) trivially.
Finally, we provide an example of a set \( T \in \mathbb{Q} \text{- repr} \left( \mathbb{Q} \text{-} \mathcal{P}_{\mathbb{Z} \times \mathbb{R}} \right) \setminus (\mathbb{Q} \text{- repr} (\mathbb{Q} \text{-} \mathcal{P}_{\mathbb{Z} \times \mathbb{R}}) \cup \mathbb{Q} \text{- repr} (\mathbb{Q} \text{-} \mathcal{P}_{\mathbb{R}})) \). Consider the set (shown in Fig. 6)

\[
T = (\mathbb{Z} \times \{0\}) \cup (\{0\} \times \mathbb{Z})
\]

Clearly this is a union of two sets from \( \mathbb{Q} \text{- repr} (\mathbb{Q} \text{-} \mathcal{P}_{\mathbb{Z} \times \mathbb{R}}) \) but \( T \notin \mathbb{Q} \text{- repr} (\mathbb{Q} \text{-} \mathcal{P}_{\mathbb{Z} \times \mathbb{R}}) \).

It is also possible to show \( T \notin \mathbb{Q} \text{- repr} (\mathbb{Q} \text{-} \mathcal{P}_{\mathbb{R}}) \) since it has infinitely many connected components. But we show \( T \in \mathbb{Q} \text{- repr} \left( \mathbb{Q} \text{-} \mathcal{P}_{\mathbb{Z} \times \mathbb{R}} \right) \) by the following construction.

\[
\tilde{T} = \{(x, y) \in \mathbb{Z}^2 \times \mathbb{R}^4 : y \geq 0 \}
\]

\[
\begin{align*}
&y \in \arg \min_y \quad \left\{ y_1 + y_2 + y_3 + y_4 : \\
&\quad y_1 \geq x_1 \quad y_1 \geq x_2 \\
&\quad y_2 \geq x_1 \quad y_2 \geq -x_2 \\
&\quad y_3 \geq -x_1 \quad y_3 \geq x_2 \\
&\quad y_4 \geq -x_1 \quad y_4 \geq -x_2
\right\}
\end{align*}
\]

Observe that we can obtain the solution to the lower-level problem from the constraints. The solution to the lower-level problem is given by \( y_1 = \max(x_1, x_2), \quad y_2 = \max(x_1, -x_2), \quad y_3 = \max(-x_1, x_2) \) and \( y_4 = \max(-x_1, -x_2) \). Note that each component of \( y \) is required to be non-negative. Thus each of these non-negativity constraints (along with the bilevel constraint) refers to a unique set of three quadrants in \( \mathbb{R}^2 \). Their intersection is nothing but the coordinate axes. The integrality constraints on \( x \) give the required set \( T \), since we can observe, \( T = \{x \in \mathbb{R}^2 : (x, y) \in \tilde{T}\} \).

These examples can be generalized to arbitrary dimension. For example, one can consider the set obtained by a Cartesian product of the given example set with a polyhedra of arbitrary dimension. The resulting set will have the same representability since the Cartesian product operation only corresponds to adding new variables and linear constraints that do not impact representability in the senses we consider here. \( \Box \)

These examples show that the issue of comparing MILP-representable sets and CBL-representable sets arises in how these two types of sets can become unbounded. MILP-representable sets are unbounded in “integer” directions from a single integer cone, while CBL-representable sets are unbounded in “continuous” directions from potentially more than one distinct recession cones of polyhedra. However, restricting to bounded sets is not enough to show the equivalences in Corollary 24. Consider the following set: \( S = \{x \in \mathbb{R} : x = \sqrt{2}z_1 - z_2, \quad 0 \leq x \leq 1, \quad z_1, z_2 \in \mathbb{Z}_+\} \). Clearly, this is a bounded MILP representable set. However, \( S \) is a dense set of points in the \([0, 1]\) interval and so the closure of \( S \) is \([0, 1]\), but the set itself cannot be represented as a finite union of polytopes and so is not CBL-representable. Restricting to both bounded and rational sets we can show the equivalence of MILP-representable sets and CBL-representable sets.

**Proof of Theorem 24** The first three equalities follow trivially from Theorem 18. To prove that \( \mathbb{Q} \text{- repr} \left( \text{disj} \left( \mathbb{Q} \text{-} \mathcal{P}_{\mathbb{R}} \right) \right) \cap \mathcal{B} = \mathbb{Q} \text{- repr} \left( \mathbb{Q} \text{-} \mathcal{P}_{\mathbb{Z} \times \mathbb{R}} \right) \cap \mathcal{B}, \)
observe from Theorem 12 that any set in \( \mathbb{Q} \)-repr \((\mathbb{Q} \setminus \mathbb{P} \times \mathbb{R})\) is the Minkowski sum of a finite union of polytopes and a monoid. Observing that \( T \in \mathbb{Q} \)-repr \((\mathbb{Q} \setminus \mathbb{P} \times \mathbb{R})\) is bounded if and only if the monoid is a singleton set containing only the zero vector, the equality follows.

Finally, from Theorem 20, \( \mathbb{Q} \)-repr \((\mathbb{Q} \setminus \mathbb{P} \times \mathbb{R})\) contains unions of sets in \( \mathbb{Q} \)-repr \((\mathbb{Q} \setminus \mathbb{P} \times \mathbb{R})\). But the only bounded sets in \( \mathbb{Q} \)-repr \((\mathbb{Q} \setminus \mathbb{P} \times \mathbb{R})\) are finite union of polytopes. And a finite union of such sets is again just a finite union of polytopes, giving nothing additional. \( \square \)

Acknowledgements The authors are immensely grateful to the Associate Editor and 3 reviewers for their insightful and meticulous comments. The notational suggestions from the AE and several other valuable suggestions from the AE and the reviewers went a long way in improving the readability of the paper. The first and third authors were supported by NSF Grant CMMI1452820 and ONR Grant N000141812096. The second author thanks the University of Chicago Booth School of Business for its generous financial support.

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