Quantum Stochastic Generators

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Abstract

We discuss stochastic derivations, stochastic Hamiltonians and the flows that they generate, algebraic fluctuation-dissipation theorems, etc., in a language common to both classical and quantum algebras. It is convenient to define distinct notions of time-ordered exponentials to take account of the breakdown of the Leibniz rule in the Itô calculus. We introduce a notion of quantum Stratonovich calculus and show how it relates to Stratonovich-Dyson time ordered exponentials. We then use it to demonstrate a natural way to add stochastic derivations.

1 Introduction

Symmetries play a considerable role in Mathematical Physics, particularly when determining dynamical flows with required invariant properties. Much of our insight comes from being able to go from the infinitesimal generators (having desirable symmetry features) to the flows themselves. This intuition requires, to a large extent, the generators to be derivations. When we consider stochastic flows, the Itô calculus loses the Leibniz rule and with it much of our physical intuition. This is why the Stratonovich calculus is usually preferred by physicists. In addition, the Wong-Zakai approximation theory tells us that we can approximate a stochastic flow with a differentiable random flow: if the approximations have some symmetry property then the limit Stratonovich equations will retain this property before it is typically lost in the conversion to Itô form.

The aim of this article is to describe algebraic notions such as derivations, their stochastic analogues, dissipation, etc., in a common language that applies to classical and quantum systems. Whereas there have been many attempts to construct general classical analogues to quantum stochastic flows, the spirit of the quantum flows is arguably best captured by stochastic flows on symplectic manifolds preserving the Poisson brackets. This was investigated by Sinha and we develop somewhat the algebraic similarities. Given the suitability of $C^*$-algebras for modelling quantum mechanical variables it is natural to consider algebras of functions on Poisson manifolds as an intermediary between the commutative and noncommutative cases. This view is strengthened considerably by
the deep analogies known to exist between Poisson and operator algebras \[3\], \[4\]. It is natural to return to these analogies when considering stochastic flows describing irreversible dynamical evolutions.

One of the most notable omissions from quantum probability has been a general theory of a quantum Stratonovich calculus. This is largely due to the bias, already prevalent in classical probability, towards the Itô calculus on the grounds that of martingale stability. Nevertheless, in problems, such as stochastic process on manifolds \[5\] \[6\], the Stratonovich calculus is used in on entirely equivalent footing as the Itô calculus and, because it does maintain the Leibniz rule, allows one to see the underlying differential geometric structure. Motivated to develop these ideas for algebras of quantum observables, the author considered Stratonovich integral for quantum diffusions \[7\], however, in the later development of this ideas, much of the physical insight came from limit theorems - weak coupling and low density - and this lead to a formulation in terms of quantum white noises: see \[8\], \[9\]. Formally, the familiar features, like a Hamiltonian nature for the flow, are evident in the Weyl ordered form (the analogue of Stratonovich version) while the Wick ordering distorts these features and produced the analogue of the Itô version. While quantum white noises offer the most intuitive approach, and should amenable to a rigorous treatment in some extension of the Hida theory, they have not found favour with either the quantum probability or the mathematical physics community. However, it is possible to give a sufficient account of events from within the quantum Itô calculus, and this is what we address in this paper.

With regards to applications, it is convenient to distinguish three different notions of time-ordered exponentials when exponentiating quantum stochastic integrals. The usual Dyson form, with Itô differentials, does not exponentiate a derivation-valued process to a homomorphic map. Instead we must either use Stratonovich differentials or exponentiate over time steps. The former strategy was applied using quantum white noises in a series of papers by the author \[8\], \[10\], while the latter was investigated in \[11\] and \[12\]. These alternatives to the Itô-Dyson form coincide in the special case of quantum diffusions.

Let \(\mathfrak{A}\) be the *-algebra of operators modelling a system and its environment. The space of all linear maps \(L : \mathfrak{A} \mapsto \mathfrak{A}\), having the reality property \(L (X^\dagger) = L (X)\dagger\), will be denoted by \(\mathcal{L}(\mathfrak{A})\). The dissipation of such a map is defined to be the bilinear mapping \(\mathcal{D}_L : \mathfrak{A} \times \mathfrak{A} \mapsto \mathfrak{A}\) given by

\[
\mathcal{D}_L (X, Y) = L(XY) - (LX)Y - X(LY).
\] (1)

If \(\mathcal{D}_L\) is zero then \(L\) is called a derivation. If \(v \in \mathcal{L}(\mathfrak{A})\) is a derivation on the algebra if we have the Leibniz property

\[
v(XY) = v(X)Y + Xv(Y),
\]

for all \(X, Y \in \mathfrak{A}\). Let us introduce the notations

\[
v^{\circ n} = v \circ \cdots \circ v, \quad e^{tv} (\cdot) = \sum_{n=0}^{\infty} \frac{t^m}{n!} v^{\circ n} (\cdot),
\]
then \( v^m (XY) = \sum_m (\binom{n}{m}) v^m (X) v^{n-m} (Y) \) and
\[
e^{tu} (XY) = e^{tu} (X) Y + X e^{tu} (Y).
\]

Therefore, derivations act as the generators of flow maps that preserve the algebraic structure (homomorphisms).

We introduce the following causal structure: for each \( t \geq 0 \), let \( \mathfrak{A}_t \) be the *-subalgebra of \( \mathfrak{A} \) modelling the system and its environment up to time \( t \), then we assume that we have the isotony condition \( \mathfrak{A}_s \subset \mathfrak{A}_t \) whenever \( s < t \). The family \( \{ \mathfrak{A}_t : t \geq 0 \} \) is called a filtration. Let \( \mathbb{E}_t : \mathfrak{A} \mapsto \mathfrak{A}_t \) be a projective conditional expectation.

A flow is a family \( \{ \Phi_{t,s} : t \geq s \geq 0 \} \) of maps in \( \mathcal{L} ( \mathfrak{A} ) \) with the properties

1. \( \Phi_{r,t} \circ \Phi_{r,s} = \Phi_{t,s} \) whenever \( t \geq r \geq s \);
2. \( \lim_{t \rightarrow s} \Phi_{t,s} = id \), the identity map on \( \mathfrak{A} \).

The flow is said to be adapted to the filtration if \( \Phi_{t,s} (\mathfrak{A}_s) \subset \mathfrak{A}_t \) whenever \( s < t \) and is said to be homomorphic if \( \Phi_{t,s} (XY) = \Phi_{t,s} (X) \Phi_{t,s} (Y) \) for all \( X, Y \in \mathfrak{A}_s \).

Given a flow, we define the maps \( \{ L_{t,s} : t \geq s \geq 0 \} \) of maps in \( \mathcal{L} ( \mathfrak{A} ) \) by
\[ L_{t,s} = \Phi_{t,s} - id. \]

Let us write \( dL_t \) for \( L_{t+dt,t} \) and we may think of this as an \( \mathcal{L} ( \mathfrak{A} ) \)-valued measure which we refer to as a stochastic generator. Given two such measures \( dL_t \) and \( dL'_t \), we define their mutual quadratic variation over time interval \([S,T]\) as
\[
\int_S^T dL_t \circ dL'_t = \lim_{|P(S,T)| \to 0^+} \sum_{(t_j,t_{j+1}) \in P(S,T)} L (t_{j+1} + t_j, t_j) \circ L' (t_{j+1} + t_j, t_j)
\]

where we have the limit over all partitions \( P (S,T) \) of \([S,T]\) into sub-intervals of maximum length \(|P(S,T)|\). In general, for a family \( \{ K_{t,s} : t \geq s \geq 0 \} \) of maps in \( \mathcal{L} ( \mathfrak{A} ) \), if we have that, for every finite interval \([S,T]\),
\[
\lim_{|P(S,T)| \to 0^+} \sum_{(t_j,t_{j+1}) \in P(S,T)} K (t_{j+1} + t_j, t_j) = 0
\]

then we write \( dK_t = o (dt) \). We say that a flow is regular, or deterministic, if \( dL_t \circ dL_t = o (dt) \). In general, however, this does not hold and we typically encounter the Itô rule for differentials of compositions, viz.
\[
d(L \circ L') \equiv (dL) \circ L' + L \circ (dL') + (dL) \circ (dL')
\]

with the last term, the Itô correction, being non-zero. (Here “\( \equiv \)” means equal up to terms of order \( o (dt) \).)

A flow \( \Phi \) is said to admit a forward derivative \( u_t (\cdot) \in \mathcal{L} ( \mathfrak{A} ) \) if the following limits exist
\[
u_t (X) := \lim_{\tau \to 0^+} \frac{1}{\tau} \mathbb{E}_t [ \Phi_{t+t, t} X - X ]
\]
for each $t \geq 0, X \in \mathfrak{A}_d$. The forward derivative will generally not be a derivation! If the flow admits a forward derivative $u_t$ then a difference martingale $\mathcal{M}_{t,t_0}(\cdot) \in \mathcal{L}(\mathfrak{A}_{t_0})$ is defined by

$$\Phi_{t,t_0}(X) = X + \int_{t_0}^t u_s(X) \, ds + \mathcal{M}_{t,t_0}(X), \quad (t_0 < t).$$

For $s < r < t, \mathcal{M}_{t,s} = \mathcal{M}_{t,r} + \mathcal{M}_{r,s}$.

### 1.1 Itô-Dyson Exponentials

The flow can be reconstructed from $dL_t$. We have the (Itô) differential equations

$$d\Phi_{t,t_0}(\cdot) = dL_t \circ \Phi_{t,t_0}(\cdot)$$

with initial condition $\lim_{t \downarrow t_0} \Phi_{t,t_0} = id$. We write

$$\Phi_{t,t_0} = \tilde{T}_{ID} \exp \left\{ \int_{t_0}^t dL \right\}$$

where $\tilde{T}_{ID}$ is the Dyson time-ordering symbol and we refer to right hand sides as Itô-Dyson time-ordered exponentials. They can be developed as an, at least formal, Picard series

$$\Phi_{t,t_0}(X) = X + \int_{t_0}^t dL_{t_1}(X) + \int_{t_0}^t dL_{t_2} \left( \int_{t_0}^{t_2} dL_{t_1} \right) (X) + \cdots$$

$$= \sum_{n=0}^{\infty} \int_{\Delta_n(t_0,t)} dL_{t_n} \circ \cdots \circ dL_{t_1} (X). \quad (6)$$

Here $\Delta_n(a,b)$ denotes the simplex consisting of all $n$–tuples $(t_n, \cdots, t_1)$ ordered so that $b \geq t_n \geq \cdots \geq t_1 \geq a$. So we have, again formally,

$$\tilde{T}_{ID} \exp \left\{ \int_{t_0}^t dL \right\} = \sum_{n=0}^{\infty} \int_{\Delta_n(t_0,t)} dL_{t_n} \circ \cdots \circ dL_{t_1}.$$  

Because the Itô rule means a breakdown of the Leibniz identity, it typically means that if $dL_t(\cdot)$ is equivalent to a derivation, but is not regular, then the flow $\Phi$ we construct will fail to be homomorphic. Indeed, the condition on $dL_t$ in order for the flow to be a family homomorphisms is that its dissipation balances its fluctuations

$$\mathfrak{D}_{dL_t} = (dL_t) \otimes (dL_t).$$

More explicitly, $dL_t(XY) = dL_t(X)Y + XdL_t(Y) + dL_t(X)dL_t(Y)$. 

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1.2 Exponentiated Dyson Exponentials

We may refer to \( \Phi_{t,t_0} = e^{(t-t_0)v} \) as the autonomous flow generated by velocity field \( v \). More generally, we could take a family \( \{v_t : t \geq 0\} \) of derivations and consider the flow generated by \( dL_t(\cdot) = v_t(\cdot)\ dt \), that is, \( v_t(\cdot) \) is the instantaneous velocity field. This time, we can write the non-autonomous flow as \( \Phi_{t,t_0} = \tilde{T}_{ID} \exp \left\{ \int_{t_0}^t v_s\ ds \right\} \). We observe that

\[
\Phi_{t+dt,t} (\cdot) = id (\cdot) + v_t (\cdot)\ dt = e^{v(\cdot)dt} + o(dt).
\]

Unfortunately this cannot be the case when we consider differential generators \( dL_t \) that are not regular. The reason is that, because of the Itô calculus, compositions \( (dL_t)^n \) need not be \( o(dt) \) for \( n \geq 2 \). Let us write in general

\[
dH_t = e^{dL_t} - id = \sum_{n \geq 1} (dL_t)^n
\]

and define the exponentiated Dyson, or ED-type, time-ordered exponential to be \( \tilde{T}_{ED} \exp \left\{ \int_{t_0}^t dL \right\} = \tilde{T}_{ID} \exp \left\{ \int_{t_0}^t dH \right\} \). This time, if the \( dL_t \) are derivations then the \( e^{dL_t}(\cdot) \) will behave as homomorphisms, and so too will

\[
\tilde{T}_{ED} \exp \left\{ \int_{t_0}^t dL \right\} = \tilde{T}_{ID} \exp \left\{ \int_{t_0}^t (e^{dL} - id) \right\}.
\]

1.3 Stratonovich-Dyson Exponentials

As we have seen above, the Itô calculus implies that the Dyson time-ordered exponential \( \tilde{T}_{ID} \exp \left\{ \int_{t_0}^t dL \right\} \) will not generally yield a homomorphism if the \( dL_t(\cdot) \) are derivations. There is another strategy for producing homomorphisms and that is to use the Stratonovich calculus instead of the Itô one. The Itô rule (for ordinary products) is that

\[
d (X_t Y_t) = X_t (dY_t) + (dX_t) Y_t + (dX_t) (dY_t).
\]

The Leibniz rule may be formally restored as \( d (X_t Y_t) = X_t \ast (dY_t) + (dX_t) \ast Y_t \) where we introduce Stratonovich differentials \( X_t \ast (dY_t) = X_t (dY_t) + \frac{1}{2} (dX_t) (dY_t) \).

For compositions of infinitesimal generators we have the similar relation

\[
d (X_t \circ Y_t) = X_t \circ (dY_t) + (dX_t) \circ Y_t + (dX_t) \circ (dY_t),
\]

with the Leibniz rule salvaged as

\[
d (X_t \circ Y_t) = X_t \ast (dY_t) + (dX_t) \ast Y_t
\]

where now \( X_t \ast (dY_t) = X_t \circ (dY_t) + \frac{1}{2} (dX_t) \circ (dY_t) \) and \( (dX_t) \ast Y_t = (dX_t) \circ Y_t + \frac{1}{2} (dX_t) \circ (dY_t) \).

We define the Stratonovich-Dyson, or SD-type, time-ordered exponential \( \Phi_{t,t_0} = \tilde{T}_{SD} \exp \left\{ \int_{t_0}^t dL_s \right\} \) to be the solution to the integro-differential equation

\[
\Phi_{t,t_0} (X) = X + \int_{t_0}^t dG_s (\Phi_{s,t_0} (X))
\]
where $dG_t$ is the Stratonovich differential

$$dG_t \circ \Phi = dL_t \odot \Phi = dL_t \circ \Phi + \frac{1}{2} dL_t \circ dL_t \circ \Phi.$$ 

that is

$$dG_t = dL_t + \frac{1}{2} (dL_t)^{o2}.$$ 

(10)

We have now defined $\Phi_{t,t_0} = \tilde{T}_{SD} \exp \left\{ \int_{t_0}^t dL_s \right\}$ as the solution to

$$\Phi_{t+dt,t_0} (\cdot) = dL_t \odot \Phi_{t,t_0} (\cdot) = dG_t \circ \Phi_{t,t_0} (\cdot)$$

with $\lim_{t \downarrow t_0} \Phi_{t,t_0} (\cdot) = id$. As the Leibniz rule is observed in the Stratonovich calculus, $\tilde{T}_{SD} \exp \left\{ \int_{t_0}^t dL_s \right\}$ will be a homomorphism when the $dL_t$ are derivations.

1.4 Properties

If the flow is regular, say $dL_t (\cdot) \equiv v_t (\cdot) dt$ for some $v_t \in \mathcal{L} (\mathfrak{A})$ and $(dL_t)^{o2} \equiv 0$ for $n \geq 2$, then the various notion of time-ordered exponentials coincide. We say that the flow is a diffusion if $(dL_t)^{o2}$ is equivalent to $A_t (\cdot) dt$ for some $A_t \in \mathcal{L} (\mathfrak{A})$, however, $(dL_t)^{o2} \equiv 0$ for $n \geq 3$. In this case, we have the truncation

$$dH_t = e^{dL_t} - id = dL_t + \frac{1}{2} (dL_t)^{o2}$$

and so $dH_t = dG_t$ and therefore

$$\tilde{T}_{ED} \exp \left\{ \int_{t_0}^t dL_s \right\} = \tilde{T}_{SD} \exp \left\{ \int_{t_0}^t dL_s \right\}.$$

This is however a fluke which cannot be expected to hold for stochastic processes other than diffusions.

The $ED$-type exponential occurs when we consider discrete time-approximations. We have for instance

$$\tilde{T}_{ED} \exp \left\{ \int_{t_0}^t dL \right\} = \lim_{\max \{t_{j+1} - t_j \} \to 0} \exp \left\{ \Phi_{t_N,t_{N-1}} \circ \cdots \circ \Phi_{t_1,t_0} \right\}$$

$$= \lim_{\max \{t_{j+1} - t_j \} \to 0} \exp \left\{ \int_{t_{N-1}}^{t_N} dL_s \right\} \circ \cdots \circ \exp \left\{ \int_{t_0}^{t_1} dL_s \right\}$$

where the limit is over all partitions $t = t_N > t_{N-1} > \cdots > t_1 > t_0$. Let us suppose that the stochastic generator has essentially commutative increments, that is,

$$[\Phi_{t,s}, \Phi_{t',s'}] = 0$$

whenever the intervals $(s,t)$ and $(s',t')$ do not overlap, then

$$\tilde{T}_{ED} \exp \left\{ \int_{t_0}^t dL \right\} = \exp \Phi_{t,t_0}.$$
The SD-type exponential occurs when we wish to approximate stochastic flows by regular ones. Let $v_{s}^{(\lambda)}(\cdot)$ be velocity fields parametrized by $t \geq 0$ and $\lambda > 0$. Suppose that we have the limit

$$
\lim_{\lambda \to 0} \int_{t_0}^{t} v_{s}^{(\lambda)}(\cdot) \, ds = \int_{t_0}^{t} dL_s(\cdot)
$$

where $dL_t$ is possibly stochastic. We may then expect the limit

$$
\lim_{\lambda \to 0} \tilde{T}_{ID} \exp \left\{ \int_{t_0}^{t} v_{s}^{(\lambda)} \, ds \right\} = \tilde{T}_{SD} \exp \left\{ \int_{t_0}^{t} dL_s \right\}.
$$

Evidently we should not have expected the Dyson exponential on the right hand side as limit should be a homomorphism. In the theory of approximating stochastic differential equations by ordinary differential equations, it is well known that the Stratonovich calculus is the one that best anticipates the limit form.

1.5 Examples from Mathematical Physics.

**Classical mechanics** Let $\mathfrak{A}$ be the $C^\infty$ functions on a manifold $M$. Here $\mathfrak{A}$ is a commutative algebra (though not generally a $C^*$-algebra) with respect to pointwise multiplication, and derivations correspond to the tangent vector fields. In this case, the dissipation is known by several different names: the Gamma operator, l’opérateur carré du champ, the cometric operator, etc. (see Meyer’s appendix to [6]).

If, however, $M$ is also endowed with a Poisson brackets which is, of course, a bilinear mapping. We may take our product to be the anti-symmetric, non-associative one given by the choice $f \star g \equiv \{ f, g \}$. In this case the $\star$-automorphisms are those maps that preserve the Poisson brackets. Likewise $\mathcal{D}_L$ will now determine the extent to which a semi-group $e^{tL}$ destroys the Poisson structure: explicitly, we have $\mathcal{D}_L(f,g) = L(\{ f, g \}) - \{ Lf, g \} - \{ f, Lg \}$. Given a real function $h \in \mathfrak{A}$, the Hamiltonian vector field $X_h$ generated by $h$ is defined by $X_h(f) := \{ f, h \}$. By the Jacobi property of Poisson brackets, $X_h$ will be a Poisson-derivation

$$
X_h \{ f, g \} = \{ X_h f, g \} + \{ f, X_h g \};
$$

and by the Leibniz property of Poisson brackets it will also be a tangent vector field. The Poisson manifold is said to be Poisson-simple if the only maps having Poisson-dissipation zero are the Hamiltonian vector fields [3]. For symplectic manifolds, all Poisson derivations are locally Hamiltonian generated.

**Quantum Mechanics** Next let $\mathfrak{A}$ be an algebra of operators acting on a Hilbert space $\mathcal{H}$, with the taking of adjoints $\dagger$ as the usual involution. The operator product is non-commutative and to a certain extent carries out the role played by both pointwise multiplication and Poisson brackets in classical
mechanics. The linear maps on $\mathfrak{A} \mapsto \mathfrak{A}$ are sometimes referred to as super-operators. For a self-adjoint operator $h \in \mathfrak{A}$, we define the super-operator $X_h$ by $X_h(f) := \frac{1}{i}[f, h]$. If $\mathfrak{A}$ is a $W^*$-algebra, or a unital simple $C^*$-algebra, then it is well-known that all real derivations take this Hamiltonian form \[14\]. The dissipation of a super-operator, with respect to the operator product, was introduced by Lindblad \[15\] as a key ingredient in analyzing generators of completely positive semi-groups.

2 Quantum Stochastic Flows

Let $\mathfrak{H} = \Gamma \left( L^2(\mathbb{R}^+, dt) \right)$ be the Fock space over square-integrable functions of positive time and let $\mathfrak{H}_t = \Gamma \left( L^2([0, t], dt) \right)$. The family $\{\mathfrak{H}_t : t \geq 0\}$ then gives a filtration of Hilbert subspaces of $\mathfrak{H}$. Fixing an initial Hilbert space $\mathfrak{h}$, we consider the filtration of $\mathfrak{A} = \mathfrak{B}(\mathfrak{h} \otimes \mathfrak{H})$ specified by $\mathfrak{A}_t := \mathfrak{B}(\mathfrak{h} \otimes \mathfrak{H}_t)$.

The quartet of fundamental quantum stochastic processes \[16\] may be denoted by $\{A^{\alpha \beta}_t : t \geq 0\}$: these are $A^{00}_t = t$ (time), $A^{10}_t = A_t$ (creation), $A^{11}_t = A_t^\dagger$ (annihilation) and $A^{11}_t = \Lambda_t$ (conservation). Their Itô table is then
\[
dA^{\alpha 1}_t dA^{1\beta}_t = dA^{\alpha \beta}_t,
\]
with all other second order differentials vanishing. This may be written explicitly as \[16\]

| $\times$ | $dA^\dagger$ | $d\Lambda$ | $dA$ | $dt$ |
|---------|----------------|----------|------|------|
| $dA^\dagger$ | 0 | 0 | 0 | 0 |
| $d\Lambda$ | $dA^\dagger$ | $d\Lambda$ | 0 | 0 |
| $dA$ | $dt$ | $dA$ | 0 | 0 |
| $dt$ | 0 | 0 | 0 | 0 |

We have the adjoint relations $\left(A^{\alpha \beta}_t\right)^\dagger = A^{\beta \alpha}_t$: that is, $\Lambda_t$ is self-adjoint and $A^{\dagger}_t$ is indeed the adjoint of $A_t$. We remark that the combinations $Q_t = A_t + A^\dagger_t$ and $N_t = \Lambda_t + A_t + A^\dagger_t + t$ give representations for the Wiener and Poisson processes respectively when we specify the Fock vacuum as state.

A closed evolution restricted to $\mathfrak{A}_{[0]} = \mathfrak{B}(\mathfrak{h})$ may be described by the family of unitaries
\[
U_{t, t_0} = \mathbf{T}_{ID} \exp \left\{ -i \int_{t_0}^t H_s ds \right\}
\]
where $\{H_s : s \geq 0\}$ is a family of self-adjoint operators forming what we usually call a time-dependent Hamiltonian.
Our aim is to construct unitary quantum stochastic processes on $\mathfrak{A}$ of the form

$$U_{t,t_0} = \tilde{T}_{ID} \exp \left\{ -i \int_{t_0}^t dG_s \right\} = \tilde{T}_{SD} \exp \left\{ -i \int_{t_0}^t dE_s \right\} \quad (11)$$

where

$$dG_t = G_{\alpha\beta} \otimes dA_{\alpha\beta}^t, \quad dE_t = E_{\alpha\beta} \otimes dA_{\alpha\beta}^t.$$

(We use a summation convention that repeated Greek indices are summed over values 0 and 1.) We take the coefficients $G_{\alpha\beta}$ and $E_{\alpha\beta}$ to be bounded operators on $\mathfrak{h}$. Special choices of the coefficients should lead to stochastic evolutions driving by either Wiener or Poisson Noise, however, it is known that classical stochastic processes do not account for all the quantum stochastic evolutions we would wish to consider, and so we work with all four fundamental processes.

### 2.1 Quantum Stratonovich Calculus

We do not aim a full generalization of the Stratonovich prescription to quantum stochastic calculus, however, we shall give the algebraic rules which tell us how to transform certain Itô integrals into Stratonovich ones in a manner that parallels the classical theory. There are two main surprises: the first being that it can be done; the second being that there is an ambiguity in the definition.

Let $\{ X_t : t \geq 0 \}$ be a family of operators (a quantum stochastic process) on $\mathfrak{h} \otimes \mathfrak{h}$. If we have that $X_t$ acts nontrivially only on the subspace $\mathfrak{h} \otimes \mathfrak{h}_t$ then we say that the process is adapted. Our definition of Stratonovich differentials will amount to

$$\begin{align*}
(dX_t) * X_t &= (dX_t) Y_t + \kappa (dX_t) (dY_t), \\
X_t * (dY_t) &= X_t (dY_t) + \kappa^* (dX_t) (dY_t),
\end{align*} \quad (12)$$

where $\kappa$ is a complex number with $\text{Re} \, \kappa = \frac{1}{2}$. Evidently we have $(dX_t) * X_t + X_t * (dY_t) = d (X_t Y_t)$ from the quantum Itô formula [16]. The fact that we may choose the imaginary part of $\kappa$ means that we have a “gauge freedom” which in many respects is similar to that in the Tomita-Takesaki theory. The symmetric choice would be $\kappa = \frac{1}{2}$, however, its explanation is as a damping constant, and physically this may be complex.

The stochastic Schrödinger equation from above is then (ignoring the $t_0$ dependence)

$$dU_t = -i (dG_t) U_t = -i (dE_t) * U_t \quad (13)$$

and we would like to determine how to transform between Itô and Stratonovich forms. Let us begin by computing $(dA_{\alpha\beta}^t) * U_t = (dA_{\alpha\beta}^t) U_t + \kappa (dA_{\alpha\beta}^t) (dU_t)$.\)
We have

\[(dA^\alpha_\beta)_t (dU_t) = -i (dA^\alpha_\beta_\mu_\nu) (G_{\mu\nu} \otimes dA^\mu_\nu_\nu)_t U_t\]

\[= -i\delta_{1\beta} (G_{1\nu} \otimes dA^\alpha_\nu_\nu)_t U_t\]

from which we see that

\[
\begin{align*}
(dA_\alpha)_t U_t &= (1 - i\kappa G_{11}) (dA_\alpha)_t U_t - i\kappa G_{10} \left( dA^\alpha_\beta_\mu \right) U_t \\
(dA^\alpha_\beta)_t U_t &= (dA^\alpha_\beta)_t (dA^\alpha_\beta)_t U_t \\
(dA_\alpha)_t U_t &= (1 - i\kappa G_{11}) (dA_\alpha)_t U_t - i\kappa G_{10} (dt)_t U_t \\
(dt)_t U_t &= (dt)_t U_t
\end{align*}
\]

and inversely

\[
\begin{align*}
(dA_\alpha)_t U_t &= (1 - i\kappa G_{11})^{-1} \left[ (dA_\alpha)_t U_t + i\kappa G_{10} (dt)_t U_t \right] \\
(dA_\alpha)_t U_t &= (1 - i\kappa G_{11})^{-1} \left[ (dA_\alpha)_t U_t + i\kappa G_{10} \left( dA^\alpha_\beta_\mu \right) U_t \right]
\end{align*}
\]

We can next of all read off the relationship between the \( G_{\alpha\beta} \) and the \( E_{\alpha\beta} \) by comparing coefficients:

\[
\begin{align*}
E_{11} &= \frac{G_{11}}{1 - i\kappa G_{11}} \\
E_{10} &= \frac{1}{1 - i\kappa G_{11}} G_{10} \\
E_{01} &= G_{01} \frac{1}{1 - i\kappa G_{11}} \\
E_{00} &= G_{00} + i\kappa G_{01} \frac{1}{1 - i\kappa G_{11}} G_{10}
\end{align*}
\]

or inversely

\[
\begin{align*}
G_{11} &= \frac{E_{11}}{1 + i\kappa E_{11}} \\
G_{10} &= \frac{1}{1 + i\kappa E_{11}} E_{10} \\
G_{01} &= E_{01} \frac{1}{1 + i\kappa E_{11}} \\
G_{00} &= E_{00} + i\kappa E_{01} \frac{1}{1 + i\kappa E_{11}} E_{10}
\end{align*}
\]

The relationship between the Itô and Stratonovich coefficients may be written more compactly using the following remarkable formulas

\[
\begin{align*}
E_{\alpha\beta} &= G_{\alpha\beta} + i\kappa G_{\alpha\beta_1} \frac{1}{1 - i\kappa G_{11}} G_{1\beta}, \\
G_{\alpha\beta} &= E_{\alpha\beta} - i\kappa E_{\alpha\beta_1} \frac{1}{1 + i\kappa E_{11}} E_{1\beta}, \quad (14)
\end{align*}
\]
showing a duality between them. In particular the operators \(1 - i\kappa G_{11}\) and \(1 + i\kappa E_{11}\) are required to be invertible and inverse to each other.

The conditions for unitarity are the isometric and co-isometric properties \(U_t^\dagger U_t = 1 = U_t U_t^\dagger\). In differential terms this is

\[
0 = d\left(U_t^\dagger U_t\right) = \left(dU_t^\dagger\right)U_t + U_t^\dagger \left(dU_t\right) + \left(dU_t^\dagger\right)\left(dU_t\right)
\]

with \(0 = d\left(U_t U_t^\dagger\right)\) similarly. From the Stratonovich form \(dU_t = -i (dE_t) U_t\) we see that it is enough to ask that \((dE_t)^\dagger = dE_t\). The unitarity of the process should come down to the conditions

\[
(E_{\alpha\beta})^\dagger = E_{\beta\alpha}. \tag{16}
\]

The process \(U_{t,t_0} = \tilde{T}_{SD} \exp\left\{ -i \int_{t_0}^t dE_s \right\}\) is evidently unitary because we are using the Stratonovich calculus to time-order explicitly unitary components. If we use the quantum Itô calculus with \(dU_t = -i (dG_t) U_t\) we see that

\[
0 = d\left(U_t^\dagger U_t\right) = U_t^\dagger \left[i \left(dG_t^\dagger\right) - i \left(dG_t\right) - \frac{1}{2} \left(dG_t\right)^\dagger \left(dG_t\right)\right] U_t
\]

which implies

\[
0 = i \left(dG_t^\dagger\right) - i \left(dG_t\right) - \frac{1}{2} \left(dG_t\right)^\dagger \left(dG_t\right)
\]

\[
= \left(i G_{\alpha\beta}^\dagger - i G_{\alpha\beta} - \frac{1}{2} G_{1\alpha}^\dagger G_{1\beta}\right) \otimes dA_t^{\alpha\beta}.
\]

It turns out that the four equations \(0 = i G_{\beta\alpha}^\dagger - i G_{\alpha\beta} - \frac{1}{2} G_{1\alpha}^\dagger G_{1\beta}\) guarantee both the isometric and co-isometric properties, and therefore unitarity. It is well-known that the general solution to this equation is

\[
G_{00} = H - i \frac{1}{2} K^\dagger K
\]

\[
G_{10} = K
\]

\[
G_{01} = K^\dagger W
\]

\[
G_{11} = i (W - 1)
\]

with \(W\) unitary, \(H\) self-adjoint, and \(K\) bounded but otherwise arbitrary. One can readily check that the coefficients \(G_{\alpha\beta}\) will satisfy these conditions once \((E_{\alpha\beta})^\dagger = E_{\beta\alpha}\) with the explicit choices

\[
W = \frac{1 - i\kappa^* E_{11}}{1 + i\kappa E_{11}},
\]

\[
K = \frac{1}{1 + i\kappa E_{11}} E_{10},
\]

\[
H = E_{00} + \Im \kappa E_{01} \frac{1}{1 + i\kappa E_{11}} E_{10}.
\]
So, although the stochastic generator \( dG_t \) does not look self-adjoint, the process \( U_{t,t_0} = \mathbf{T}_{t,t_0} \exp \{-i \int_{t_0}^t dG_s \} \) is nevertheless unitary.

The flow is then given by the family of maps

\[
\Phi_{t,t_0} (X) = U_{t,t_0}^\dagger (X) U_{t,t_0}
\]

which defines a system of homomorphic flows on \( \mathfrak{A} \). The quantum stochastic differential equation for \( \Phi_{t,t_0} (X) \) is

\[
d\Phi_{t,t_0} = dL_t \circ \Phi_{t,t_0}
\]

with

\[
dL_t (\cdot) = L_{t,\alpha\beta} (\cdot) \otimes dA_{t,\alpha\beta}
\]

where \( L_{t,\alpha\beta} (X) = i (G_{t,\alpha\beta}^\dagger X - iX (G_{t,1\alpha}^\dagger - \frac{1}{2} (G_{t,1\alpha}^\dagger X (G_{t,1\beta}^\dagger) and G_{t,\alpha\beta}^\dagger = \Phi_{t,t_0} (G_{t,\alpha\beta}) \).

The stochastic derivation property is then stated in the form

\[
\mathcal{D} L_{\alpha\beta} = L_{\alpha 1} \otimes L_{1\beta};
\]

that is, \( L_{\alpha\beta} (XY) - L_{\alpha\beta} (X) Y - X L_{\alpha\beta} (Y) = L_{\alpha 1} (X) L_{1\beta} (Y) \). These are the well-known structure equations for non-commutative flows \[17\].

2.2 Remarks

In principle, the algebraic manipulations can be extended to time-dependent \( E_{\alpha\beta} \in \mathfrak{B} (\mathfrak{h}) \) and more generally to adapted coefficients. We may also generalize to \( N \) Bose noises \( A_{t,\alpha\beta} \) where now the Greek indices run over \( 0, 1, \cdots N \) and this leads to a tensorial version of the equations (14). In this case we are free to introduce additional gauge degrees of freedom.

We mention that we have the following approximations theorem which is the quantum analogue of the Wong-Zakai result and which justifies our construction of quantum Stratonovich calculus. Let \( a_t (\lambda) \) be Bose fields on a Fock space \( \delta_{\lambda}^{(R)} \) correspond to some physical system which we shall refer to as the reservoir. we have chosen to parameterize them by time \( t \) and also a scale parameter \( \lambda > 0 \). For \( \lambda \) fixed, we consider canonical commutation relations of the type

\[
[a_t (\lambda), a_s^\dagger (\lambda)] = G_{\lambda} (t - s)
\]

where \( G_{\lambda} \) is the two point function and is assumed to be a regular function of the time difference. We shall assumed that \( G_{\lambda} (\cdot) \) is integrable with \( \int_{-\infty}^{\infty} G_{\lambda} = 1 \) and we naturally require that

\[
G_{\lambda} (-\tau) = G_{\lambda} (\tau)^*.
\]

Let \( \kappa = \int_{0}^{\infty} G_{\lambda} \) then \( \text{Re} \kappa = \frac{1}{2} \) and \( \int_{-\infty}^{0} G_{\lambda} = \kappa^* \).
We now consider what happens if, in the limit $\lambda \to 0$, we have
\[ G_{\lambda} (\tau) \to \delta (\tau) \] (18)
in the sense of Schwartz distribution. In particular, we wish to study the asymptotic behavior of the unitary $U_t (\lambda)$ coupling a given system with state space $h$ to the reservoir with an interaction Hamiltonian $\Upsilon_t (\lambda)$. Here $U_t (\lambda)$ is given as the solution to
\[ U_t (\lambda) = 1 - i \int_0^t \Upsilon_s (\lambda) U_s (\lambda) \, ds. \]

**Theorem 1** [18] For the interactions on $h \otimes S_R^{(\lambda)}$ of the type
\[ \Upsilon_t (\lambda) = E_{11} \otimes a_1^t (\lambda) a_t (\lambda) + E_{10} \otimes a_1^t (\lambda) + E_{01} \otimes a_t (\lambda) + E_{00} \otimes 1 \] (19)
with $E_{\alpha\beta} \in \mathcal{B}(h)$, $E_{11}$ and $E_{00}$ self-adjoint, $E_{10} = E_{01}^\dagger$, and $\|\kappa E_{11}\| < 1$, the weak matrix limit of $U_t (\lambda)$ is described by the unitary quantum stochastic process $U_t$ given by the Stratonovich-Dyson time-ordered exponential
\[ U_t = \hat{T}_{SD} \exp \{-i \int_{t_0}^t dE_s\} \]
with $dE_t = E_{\alpha\beta} \otimes dA_t^{\alpha\beta}$.

The condition $\|\kappa E_{11}\| < 1$ is required to ensure that multiple scatterings diminish rather than augment amplitudes - it also allows (14) to be expanded in a geometric series. A simpler version of this result, applicable for commuting coefficients only, was given in [19]. The convergence also applies to the Heisenberg dynamics and so we get convergence of the regular pre-limit flow to the stochastic quantum flow. A similar set of formula arise when the $a_1^t (\lambda)$ are replaced by Fermion fields [20]: the fundamental processes $A_t^{\alpha\beta}$ now being the Fermi analogues.

### 2.3 Addition Rules for Stochastic Derivations

Let $\{dL_t^{(n)}\}_n$ be a finite collection of stochastic derivations: their sum is not typically another stochastic derivation. In general, $\sum_n dL_t^{(n)} + dF$ defines a stochastic derivation only for some suitable choice of “Itô” correction $F$. For the quantum problem, we realize each stochastic derivation $dL_t$ as a function of the operators $E_{\alpha\beta}$, i.e. $dL_t = dG_t (E_{\alpha\beta})$.

The natural procedure is then to consider the total stochastic Hamiltonian
\[ dE_t = \sum_n dE_t^{(n)} = \sum_n E_{\alpha\beta}^{(n)} \otimes dA_t^{\alpha\beta}. \] The corresponding stochastic derivation is then
\[ dG_t = dG \left( \sum_n E_{\alpha\beta}^{(n)} \right). \] (20)
2.3.1 Examples

i) When the $E_{11}^{(n)} = 0$, the relation is simply

$$K = \sum_n K^{(n)}; \quad H = \sum_n H^{(n)} - \text{Re}\kappa \sum_\alpha K^{(\alpha)} K^{(\alpha)^\dagger}.$$

ii) (For simplicity, take $\kappa = \frac{1}{2}$.) Let $W^{(a)}, W^{(b)}$ be commutative unitaries related to $E^{(a)}_{11}$ and $E^{(b)}_{11}$ by the preceding relations. That is, $E^{(\alpha)}_{11} = 2i\frac{W^{(\alpha)} - 1}{W^{(\alpha)} + 1}$. The composite unitary is $W = \frac{1-i(E^{(a)}_1 + E^{(b)}_1)/2}{1+i(E^{(a)}_1 + E^{(b)}_1)/2}$ which, after some algebra, becomes

$$W = W^{(a)} \frac{(3 + W^{(a)} + W^{(b)} - W^{(a)}W^{(b)})^I}{(3 + W^{(a)} + W^{(b)} - W^{(a)}W^{(b)})} W^{(b)}.$$

In [13], a formula $K := L + M + [[L, M]]$ for the sum of two stochastic derivations, $L$ and $M$, is given. There the bracket is $[[L, M]]$ is their mutual quadratic variation defined by

$$[[L, M]](t, dt) = dL_t \circ dM_t$$

where $\circ$ denotes composition in $\mathcal{L}(\mathfrak{A})$. We have developed a generalization to arbitrary many summands.

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