EXISTENCE AND COST OF BOUNDARY CONTROLS FOR A DEGENERATE/SINGULAR PARABOLIC EQUATION

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(Communicated by Piermarco Cannarsa)

Abstract. In this paper, we consider the following degenerate/singular parabolic equation
\[ u_t - (x^\alpha u_x)_x - \frac{\mu}{x^{2-\alpha}} u = 0, \quad x \in (0, 1), \quad t \in (0, T), \]
where \(0 \leq \alpha < 1\) and \(\mu \leq (1 - \alpha)^2/4\) are two real parameters. We prove
the boundary null controllability by means of a \(H^1(0, T)\) control acting either
at \(x = 1\) or at the point of degeneracy and singularity \(x = 0\). Besides we
give sharp estimates of the cost of controllability in both cases in terms of the
parameters \(\alpha\) and \(\mu\). The proofs are based on the classical moment method by
Fattorini and Russell and on recent results on biorthogonal sequences.

1. Introduction. The aim of this paper is to prove boundary null controllability for some degenerate/singular parabolic equation in \(1 - D\) and establish sharp estimates of the control cost. More precisely, we focus on the following degenerate/singular parabolic operator:
\[ P_{\alpha, \mu} u := u_t - (x^\alpha u_x)_x - \frac{\mu}{x^{2-\alpha}} u, \quad x \in (0, 1). \]
Observe that, when \( \mu = 0 \), this operator is purely degenerate:

\[
P_{\alpha,0}u = u_t - (x^\alpha u_x)_x, \quad x \in (0,1),
\]

whereas, when \( \alpha = 0 \), it becomes purely singular with a singularity that takes the form of an inverse square potential:

\[
P_{0,\mu}u = u_t - u_{xx} - \frac{\mu}{x^2} u, \quad x \in (0,1).
\]

Null controllability properties by means of a locally distributed control for such operators have been investigated in various papers. We refer the reader to the following pioneering contributions:

- In [10], the case of the purely degenerate operator \( P_{\alpha,0} \) has been analyzed. The author showed that null controllability under the action of an \( L^2 \) control supported in an open subset \( \omega \subset (0,1) \) holds true if and only if \( 0 \leq \alpha < 2 \). One distinguishes here the two cases \( 0 < \alpha < 1 \) and \( 1 \leq \alpha < 2 \) for well-posedness reasons: in the natural functional setting associated to the weakly degenerate operator, that is when \( 0 < \alpha < 1 \), the trace at \( x = 0 \) exists. So one can consider a Dirichlet condition at \( x = 0 \). On the contrary, the trace does not exist when \( \alpha \geq 1 \). Here the Dirichlet boundary condition needs to be changed by some Neumann-kind one. We also refer to [1, 9, 11, 36] for various related results.

- Concerning the inverse square singular operator \( P_{0,\mu} \), the first study was made in [44] and complemented in [19]. In these references, it was shown that null controllability with a control in \( L^2(\omega \times (0,T)) \) holds true if and only if \( \mu \leq \mu^* \) where \( \mu^* = 1/4 \) is the constant appearing in the well-known Hardy inequality

\[
\frac{1}{4} \int_0^1 z^2 x^{-2} dx \leq \int_0^1 z^2 dx.
\]

Also here one distinguishes the two cases \( \mu < \mu^* \) and \( \mu = \mu^* \) again for well-posedness reasons: the natural functional space in the critical case \( \mu = \mu^* \) differs slightly from the one in the general sub-critical case. Let us refer to [7, 15] for other similar situations.

- Finally, the controllability properties of the mixed degenerate/singular operator \( P_{\alpha,\mu} \) with locally distributed controls were studied in [43]. In this case, null controllability holds true if and only if

\[
0 \leq \alpha < 2 \quad \text{and} \quad \mu \leq \mu(\alpha)
\]

where

\[
\mu(\alpha) := \frac{(1-\alpha)^2}{4}
\]

is the constant appearing in the generalized Hardy inequality

\[
\frac{(1-\alpha)^2}{4} \int_0^1 z^2 x^{2-\alpha} dx \leq \int_0^1 x^\alpha z^2 dx. \quad (1.1)
\]

See also [22, 27] for other works on this theme.

Dealing with locally distributed controls, all these mentioned contributions are mainly based on a Carleman approach, suitably adapted for taking into account the degeneracy/singularity in the equation.
In the present paper, we turn to the case of a boundary control. This issue has been already analyzed in several contributions for the purely degenerate operator $P_{\alpha,0}$ and next for the purely singular one $P_{0,\mu}$. In more detail:

- For the purely degenerate operator $P_{\alpha,0}$, the first result of boundary controllability was obtained in [25] for a control acting at $x = 0$ and in the case of a weak degeneracy $0 \leq \alpha < 1$. It has been complemented in [12] where sharp estimates of cost of the control have been obtained. Next, in [14], the case of a strong degeneracy ($1 \leq \alpha < 2$) with a control acting at $x = 1$ has been studied, analyzing again both the existence and the cost of the control.
- For the purely singular operator $P_{0,\mu}$, the boundary controllability from $x = 1$ has been studied in [37] whereas the case of a control at $x = 0$ is treated in [8].

Nevertheless, to our knowledge, the question of the boundary controllability for the degenerate/singular operator $P_{\alpha,\mu}$ has never been addressed. The present paper fills this gap by providing these missing boundary controllability results, for controls acting either at $x = 1$ or at $x = 0$, where the degeneracy/singularity arises.

When addressing the boundary controllability problem, the approach by Carleman estimates presents some difficulties. Indeed, the specific weight functions introduced in [10, 43, 44] to deal with the degeneracy and/or the singularity do not provide suitable boundary terms. For this reason, the approach of the aforementioned papers is based on decomposition in series and the well-known moment method. This is the methodology that we will employ also in the present work.

The rest of the paper is organized as follows: in Section 2, we formulate precisely the problems we are going to study and we present our main theorems. Section 3 is devoted to the well-posedness of our problems. In Section 4, we introduce some preliminary results on the spectral properties of the operator $P_{\alpha,\mu}$ which will then be at the basis of our proofs. Moreover, we briefly describe the main procedure of the moment method. In Section 6 and 7, we present the proof of our main results, Theorem 2.1 and 2.2. Finally, in Section 8 we give some conclusive remarks and open problems.

2. Problem formulation and main results.

2.1. Description of the controllability problem. Let us describe more precisely the controllability problems we are interested in. First of all, we focus in this work on the case of a weak degeneracy, that is, $0 \leq \alpha < 1$. So throughout the paper, we assume that the parameters $\alpha$ and $\mu$ satisfy the following assumption:

$$0 \leq \alpha < 1 \quad \text{and} \quad \mu \leq \mu(\alpha) = \frac{(1 - \alpha)^2}{4}. \quad (2.1)$$

Moreover we will consider boundary controls acting either at $x = 1$ or at the degeneracy/singularity point $x = 0$.

2.1.1. Control acting away from the degenerate/singular point. We will first study the case of a boundary control acting at $x = 1$ (that is, away from the degenerate
and singular point): let \( u_0 \in L^2(0, 1) \), \( T > 0 \) and consider
\[
\begin{cases}
  u_t - (x^\alpha u_x)_x - \frac{\mu}{x^{2-\alpha}} u = 0, & (x, t) \in (0, 1) \times (0, T) := Q \\
  u(0, t) = 0, & t \in (0, T) \\
  u(1, t) = H(t), & t \in (0, T) \\
  u(x, 0) = u_0(x), & x \in (0, 1).
\end{cases}
\]
(2.2)

Here \( H \) represents some control term that aims to steer the solution to zero at time \( T \). Our first goal is to establish the existence of such control. Observe that the existence of a boundary control for (2.2) could be deduced from [44], using for example the standard argument that consists in extending the domain \((0, 1)\) into \((0, 1 + \eta)\), \( \eta > 0 \), applying the result of null controllability the equation in \((0, 1 + \eta)\) with a distributed control localized in \((1, 1 + \eta)\), and concluding by taking the trace at \( x = 1 \) of such a control function. However this method based on Carleman estimates would not provide optimal estimates of the cost of the control, in particular when \( \mu \to -\infty \) (see in [44, Remark 3.5]).

For this reason, we turn here to approaches based on decomposition in series and moment problems. These techniques have been developed by Fattorini-Russell [20, 21], and have been successfully applied/adapted to obtain sharp results in quite simple geometric situations. The interested reader may refer for example to [2, 5, 12, 16, 23, 25, 28, 30, 31, 32, 41, 42].

2.1.2. Control acting at the degenerate/singular point. Next we will turn to the case of a control acting at \( x = 0 \) (that is on the point of degeneracy and singularity). In this case, even the existence of a control is new since it cannot be deduced from the result of controllability by a locally distributed control. Moreover, as previously, we also aim at estimating precisely the cost of the control. The problem we consider here is:
\[
\begin{cases}
  u_t - (x^\alpha u_x)_x - \frac{\mu}{x^{2-\alpha}} u = 0, & (x, t) \in Q \\
  (x^{-\gamma} u)(0, t) = H(t), & t \in (0, T) \\
  u(1, t) = 0, & t \in (0, T) \\
  u(x, 0) = u_0(x), & x \in (0, 1).
\end{cases}
\]
(2.3)

Due to the presence of the singularity at \( x = 0 \), it is not possible to impose a standard non homogeneous Dirichlet boundary condition. For this reason, as in [8], we use the above weighted Dirichlet condition where the coefficient \( \gamma \) is defined by
\[
\gamma = \gamma(\alpha, \mu) := \frac{1 - \alpha}{2} - \frac{1}{2} \sqrt{(1 - \alpha)^2 - 4\mu} = \sqrt{\mu(\alpha)} - \sqrt{\mu(\alpha) - \mu}.
\]
(2.4)

Notice that we have
\[
\gamma(\alpha, 0) = 0 \quad \text{and} \quad \gamma(0, \mu) = \frac{1}{2} \left(1 - \sqrt{1 - 4\mu}\right),
\]
consistently with [8, 12].

2.2. Main results. We present here the main results of the paper. Let us recall that we are interested not only in proving the existence of a boundary control for \( P_{\alpha, \mu} \), but also in providing estimates for the controllability cost in terms of the parameters \( \alpha \) and \( \mu \). To this end, we first need to introduce the following notion of controllability cost.
For any $T > 0$, $0 \leq \alpha < 1$, $\mu \leq \mu(\alpha)$ and any initial datum $u_0 \in L^2(0,1)$, we introduce the set of admissible controls:

$$U_{ad}(\alpha, \mu, T, u_0) := \left\{ H \in H^1(0,T) \mid u^{(H)}(T) = 0 \right\},$$

where $u^{(H)}$ denotes the solution of (2.2) or of (2.3) corresponding to the control $H$. Then we consider the controllability cost for any $u_0 \in L^2(0,1)$

$$C^H_1(\alpha, \mu, T, u_0) := \inf_{H \in U_{ad}(\alpha, \mu, T, u_0)} \| H \|_{H^1(0,T)},$$

which is the minimal energy needed to drive the initial datum $u_0$ to zero. Let us mention that, in accordance with the results of [8, 12, 37], the controls that we will construct for the degenerate/singular operator $P_{\alpha, \mu}$ are $H^1(0,T)$-regular. As we shall see with more detail in Section 3, this regularity of our controls is motivated by the notion of weak solutions we will provide. Hence, the set of admissible controls and the controllability cost we just introduced are the suitable ones in our case. Finally, we define the global notion of controllability cost:

$$C^H_{bd-ctr}(\alpha, \mu, T) := \sup_{\| u_0 \|_{L^2(0,1)} = 1} C^H_1(\alpha, \mu, T, u_0).$$

2.2.1. Results for a control acting at $x = 1$. Our first main result, concerning the existence of a control for equation (2.2), will be the following.

**Theorem 2.1.** Let $0 \leq \alpha < 1$ and $\mu \leq \mu(\alpha)$. Given any $T > 0$ and $u_0 \in L^2(0,1)$, the following assertions hold:

(i) **Existence of a control.** There exists a control function $H \in H^1(0,T)$ such that the solution of (2.2) satisfies $u(x,T) = 0$.

(ii) **Upper bound of the cost.** There exists a constant $C_u > 0$, independent of $\alpha$, $\mu$ and $T$, such that the cost of null controllability for (2.2) satisfies

$$C^H_{bd-ctr}(\alpha, \mu, T) \leq C_u e^{\frac{\mu}{4}} \left[ 1 + \sqrt{\mu(\alpha) - \mu} \right] e^{-C_u \left[ 1 + \sqrt{\mu(\alpha) - \mu} \right]^2 T}.$$

(iii) **Lower bound of the cost.** Define

$$\nu(\alpha, \mu) := \frac{2}{2 - \alpha} \sqrt{\left( \frac{1 - \alpha}{2} \right)^2 - \mu} = \frac{2}{2 - \alpha} \sqrt{\mu(\alpha) - \mu}. \quad (2.5)$$

Then, there exists a constant $C_u > 0$, independent of $\alpha$, $\mu$ and $T$, such that the cost of null controllability for (2.2) satisfies:

- in the case
  $$\nu(\alpha, \mu) \in \left[ 0, \frac{1}{2} \right], \quad \text{that is, } \mu \in \left[ \frac{\alpha}{16} (3\alpha - 4), \mu(\alpha) \right],$$
  then
  $$C_{ctr-bd} \geq C_u e^{\frac{\mu}{4}} e^{-C_u \left[ 1 + \sqrt{\mu(\alpha) - \mu} \right]^2 T};$$

- in the case
  $$\nu(\alpha, \mu) \in \left[ \frac{1}{2}, +\infty \right), \quad \text{that is, } \mu \in \left( -\infty, \frac{\alpha}{16} (3\alpha - 4) \right],$$
  then
  $$C_{ctr-bd} \geq C_u e^{\frac{\mu}{4}} e^{-C_u \left[ 1 + \sqrt{\mu(\alpha) - \mu} \right]^2 T} e^{-C_u \left[ \sqrt{\mu(\alpha) - \mu} \right]^{1/3} \left( \ln \left[ \sqrt{\mu(\alpha) - \mu} \right] + \ln \frac{1}{\mu} \right)}.$$

The proof of Theorem 2.1 will be given in Section 6.
2.2. Results for a control acting at $x = 0$. The second main result of our work concerns the existence of a control for equation (2.3), and it reads as follows.

**Theorem 2.2.** Let $0 \leq \alpha < 1$ and $\mu < \mu(\alpha)$. Given any $T > 0$ and $u_0 \in L^2(0,1)$, the following assertions hold:

(i) **Existence of a control.** There exists a control function $H \in H^1(0,T)$ such that the solution of (2.3) satisfies $u(x,T) = 0$.

(ii) **Upper bound of the cost.** There exists a constant $C > 0$, independent of $\alpha$, $\mu$ and $T$, such that the cost of null controllability for (2.3) satisfies

$$C_{\text{ctr}} \leq C_u \frac{\Gamma(1 + \nu(\alpha, \mu))}{\sqrt{\mu(\alpha) + \mu(\alpha) - \mu}} e^{\frac{\nu}{\mu}} \left[ 1 + \sqrt{\mu(\alpha) - \mu} \right] e^{-C_u (1 + \sqrt{\mu(\alpha) - \mu})^2 T}.$$

(iii) **Lower bound of the cost.** There exists a constant $C_u > 0$, independent of $\alpha$, $\mu$ and $T$, such that the cost of null controllability for (2.3) satisfies:

- in the case $\nu(\alpha, \mu) \in \left[ 0, \frac{1}{2} \right]$, that is, $\mu \in \left[ \frac{\alpha}{16} (3\alpha - 4), \mu(\alpha) \right]$,

$$C_{\text{ctr}} \geq C_u \frac{\Gamma(1 + \nu(\alpha, \mu))}{\sqrt{\mu(\alpha) + \mu(\alpha) - \mu}} e^{\frac{\nu}{\mu}} T_{\text{ctr}} e^{\frac{\nu}{\mu}};$$

- in the case $\nu(\alpha, \mu) \in \left[ \frac{1}{2}, +\infty \right)$, that is, $\mu \in \left( -\infty, \frac{\alpha}{16} (3\alpha - 4) \right)$,

$$C_{\text{ctr}} \geq C_u e^{\frac{\nu}{\mu}} e^{-C_u [1 + \sqrt{\mu(\alpha) - \mu}]^2 T_{\text{ctr}} e^{\frac{\nu}{\mu}}} \frac{\sqrt{\mu(\alpha) + \mu(\alpha) - \mu} [\sqrt{\mu(\alpha) - \mu}]^{4/3} (\ln[\sqrt{\mu(\alpha) - \mu}] + \ln \frac{T_{\text{ctr}}}{T})}{\sqrt{\mu(\alpha) + \mu(\alpha) - \mu}}.$$

The proof of Theorem 2.2 will be given in Section 7.

2.3. Further comments on background and motivations. The origin of the problem of the cost of controllability (or of the minimal norm control) comes from control theory in finite dimension. This question has been solved in the case of ODE by Seidman in [39], where he considered the linear time-invariant finite-dimensional system:

$$x'(t) = Ax(t) + Bu(t), \quad t \in [0, T],$$

with $x \in \mathbb{R}^n$, $u \in L^2(0,T; \mathbb{R}^m)$, $A$ a $n \times n$ matrix and $B$ a $n \times m$ matrix with $m \leq n$.

In this framework, he obtained an estimate (which involves the Kalman’s rank of the system) of the blow-up rate of the minimal norm control when the control time $T$ goes to 0. An extended result allowing to consider all the $L^p(0,T)$-norms of the control instead of the only $L^2(0,T)$-norm is also given in [40].

Turning to the case of infinite-dimensional systems, we refer to [26] for an analysis of the lower bound of the controllability cost and to [3] for a work on a strongly damped wave equation.

Next, PDEs systems has also been studied. Let us mention the work by Lasiecka and Seidman [29] that provides optimal blow-up rates for a non-scalar thermoelastic system (heat conduction coupled with Kirchhoff or Euler-Bernoulli plate model). In this work, the authors consider the cases of interior and boundary controls and analyze the blow-up rate of the control cost both when the controllability time $T$
goes to 0 and when the coupling parameter $\alpha$ goes to 0 (a question that can only arise for systems). Still in the case of thermoelastic plates, Avalos and Lasiecka [4] also established optimal estimates of the blow-up of the cost function (both for a mechanic and a thermal control) as the controllability time $T$ goes to 0.

As for the motivation of studying the blow-up rates of controllability cost of PDEs (besides to be interesting in itself to evaluate the cost of controllability with respect to controllability time and to the parameters entering in the system), let us mention the link between this question and stochastic PDEs. As described in [17, 18], the cost of controllability is linked to the regularity of some Markov semi-group, including Ornstein-Uhlenbeck processes and related Kolmogorov equations. For some of these semi-groups (see e.g. [17, Theorem 8.3.3]), null controllability is equivalent to the differentiability and regularizing effect of the Ornstein-Uhlenbeck process. Moreover the regularity of solutions of the Kolmogorov equation depends on the asymptotic behavior of the minimal control norm as $T \to 0$.

Even in the deterministic case, it was shown in [24] that there is a connection between the asymptotic behavior of the cost and the regularity of the Bellman’s function (that describes the minimal time control).

In the present paper, we consider problems (2.2) and (2.3) with the parameters $\alpha$ and $\mu$. So it is natural here to investigate the null controllability cost as the control time $T$ goes to 0 (as usual) but also in terms of the two parameters $\alpha$ and $\mu$.

3. Well-posedness of the controllability problems. This section deals with the well-posedness of the models we are considering. To this end, let us first recall the functional framework in which we shall work.

3.1. Functional framework. We introduce here the suitable functional setting to deal with the degenerate/singular operator $P_{\alpha,\mu}$ (see [43]). For any $\mu \leq \mu(\alpha)$, we define

$$H^{1,\mu}_{\alpha,0}(0,1) := \left\{ u \in L^2(0,1) \cap H^{1}_{\text{loc}}((0,1]) \left| \int_{0}^{1} (x^{\alpha}u^2_x - \mu x^2 u^2) \, dx < +\infty \right. \right\}$$

and

$$H^{1,\mu}_{\alpha,0}(0,1) := \left\{ u \in \tilde{H}^{1,\mu}_{\alpha,0}(0,1) \left| u(0) = u(1) = 0 \right. \right\}.$$

In the case of a sub-critical parameter $\mu < \mu(\alpha)$, thanks to the generalized Hardy inequality (1.1), it is easy to see that $H^{1,\mu}_{\alpha,0}(0,1) = H^{1}_{\alpha,0}(0,1)$. On the contrary, for the critical value $\mu = \mu(\alpha)$, the space is enlarged (see [45] for this observation in the case $\alpha = 0$):

$$H^{1}_{\alpha,0}(0,1) \subsetneq H^{1,\mu(\alpha)}_{\alpha,0}(0,1).$$

Next we define

$$H^{2,\mu}_{\alpha,0}(0,1) := \left\{ u \in H^{1,\mu}_{\alpha,0}(0,1) \cap H^{2}_{\text{loc}}((0,1]) \left| (x^{\alpha}u_x)_x + \frac{\mu}{x^{2-\alpha}} u \in L^2(0,1) \right. \right\}.$$

Finally, we define the operator $A_{\alpha,\mu} : D(A_{\alpha,\mu}) \subset L^2(0,1) \to L^2(0,1)$ by

$$\left\{ D(A_{\alpha,\mu}) := H^{2,\mu}_{\alpha,0}(0,1) \cap H^{1,\mu}_{\alpha,0}(0,1) \right\}$$

$$\forall u \in D(A_{\alpha,\mu}), \quad A_{\alpha,\mu} u := (x^{\alpha}u_x)_x + \frac{\mu}{x^{2-\alpha}} u.$$
Notice that, if $u \in D(A_{\alpha,\mu})$, then $u$ satisfies the Dirichlet boundary conditions $u(0) = 0$ and $u(1) = 0$. Moreover the following result holds (see [43, Propositions 2,3 and 6]):

**Proposition 1.** $A_{\alpha,\mu} : D(A_{\alpha,\mu}) \subset L^2(0,1) \to L^2(0,1)$ is a self-adjoint negative operator with dense domain.

Hence, the operator $A_{\alpha,\mu}$ is the infinitesimal generator of an analytic semigroup of contractions $e^{tA_{\alpha,\mu}}$ on $L^2(0,1)$.

### 3.2. Homogeneous boundary conditions and a source term.

The above considerations allow us to state well-posedness results for the system with homogeneous boundary conditions and a source term. Under the assumption (2.1) and given an initial condition $w_0 \in L^2(0,1)$ and a source term $f \in L^2((0,1) \times (0,T))$, we consider the problem:

$$
\begin{align*}
&w_t - (x^\alpha w_x)_x - \frac{\mu}{x^{2-\alpha}} w = f(x,t), \quad (x,t) \in Q \\
&w(0,t) = 0, \quad t \in (0,T) \\
&w(1,t) = 0, \quad t \in (0,T) \\
&w(x,0) = w_0(x), \quad x \in (0,1).
\end{align*}
$$

(3.1)

The function $w \in C^0([0,T];L^2(0,1)) \cap L^2(0,T;H^1_{\alpha,\mu}(0,1))$ given by the variation formula

$$
w(x,t) = e^{tA_{\alpha,\mu}}w_0 + \int_0^t e^{(t-s)A_{\alpha,\mu}}f(x,s)ds.
$$

is called the mild solution of (3.1). We also say that a function $w \in C^0([0,T];H^1_{\alpha,\mu}(0,1)) \cap L^2(0,T;H^1_{\alpha,\mu}(0,1)) \cap L^2(0,T;D(A_{\alpha,\mu}))$ is a strict solution of (3.1) if it satisfies the equation almost everywhere in $(0,1) \times (0,T)$ and the boundary and initial conditions for all $t \in [0,T]$ and $x \in [0,1]$.

**Proposition 2.** If $w_0 \in H^1_{\alpha,\mu}(0,1)$, then the mild solution of (3.1) is also the unique strict solution.

The proof of Proposition 2 is given later in section 5.

### 3.3. Non homogeneous boundary condition at $x = 1$.

Next we turn to the boundary value problem (2.2). To define the solution of (2.2), we transform it into a problem with homogeneous boundary conditions and a source term. Let us introduce

$$
\forall x \in [0,1], \quad p(x) := x^q \quad \text{where} \quad q := \frac{1-\alpha}{2} + \sqrt{\mu(\alpha) - \mu}.
$$

Observe that $p(0) = 0$, $p(1) = 1$ and

$$
(x^\alpha p')'(x) + \frac{\mu}{x^{2-\alpha}} p(x) = 0.
$$

Formally, if $u$ is a solution of (2.2), then the function defined by

$$
v(x,t) = u(x,t) - \frac{p(x)}{p(1)} H(t) = u(x,t) - x^q H(t)
$$

(3.2)
is solution of
\[
\begin{aligned}
  v_t - (x^\alpha v_x)_x - \frac{\mu}{x^{2-\alpha}} v &= -\frac{p(x)}{p(1)} H'(t), \quad (x, t) \in Q \\
v(0, t) &= 0, \quad t \in (0, T) \\
v(1, t) &= 0, \quad t \in (0, T) \\
v(x, 0) &= u_0(x) - \frac{p(x)}{p(1)} H(0), \quad x \in (0, 1).
\end{aligned}
\]  

(3.3)

Reciprocally, given \( h \in L^2(0, T) \), consider the solution of
\[
\begin{aligned}
  v_t - (x^\alpha v_x)_x - \frac{\mu}{x^{2-\alpha}} v &= -\frac{p(x)}{p(1)} h(t), \quad (x, t) \in Q \\
v(0, t) &= 0, \quad t \in (0, T) \\
v(1, t) &= 0, \quad t \in (0, T) \\
v(x, 0) &= v_0(x), \quad x \in (0, 1).
\end{aligned}
\]

Then the function \( u \) defined by
\[ u(x, t) = v(x, t) + \frac{p(x)}{p(1)} \int_0^t h(\tau)d\tau \]
satisfies
\[
\begin{aligned}
  u_t - (x^\alpha u_x)_x - \frac{\mu}{x^{2-\alpha}} u &= 0, \quad (x, t) \in Q \\
u(0, t) &= 0, \quad t \in (0, T) \\
u(1, t) &= \int_0^t h(\tau)d\tau, \quad t \in (0, T) \\
u(x, 0) &= v_0(x), \quad x \in (0, 1).
\end{aligned}
\]

Let now \( H \in H^1(0, T) \) and \( u_0 \in L^2(0, 1) \). Define
\[ f(x, t) = -\frac{p(x)}{p(1)} H'(t) \quad \text{and} \quad v_0(x) = u_0(x) - \frac{p(x)}{p(1)} H(0). \quad (3.4) \]

Then, we have \( f \in L^2((0, 1) \times (0, T)) \) and \( v_0 \in L^2(0, 1) \). We can thus apply the notions of mild and weak solutions introduced in Section 3.2 to define in a suitable way the solution of (2.2).

**Definition 3.1.** We have the following notions of solution:

a) We say that \( u \in C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H^{1,\mu}_\alpha(0, 1)) \) is the mild solution of (2.2) if \( v \) defined by (3.2) and (3.4) is the mild solution of (3.3).

b) We say that \( u \in C^0([0, T]; H^{1,\mu}_\alpha(0, 1)) \cap H^1(0, T; L^2(0, 1)) \cap L^2(0, T; H^{2,\mu}_\alpha(0, 1)) \) is the strict solution of (2.2) if \( v \) defined by (3.2) and (3.4) is the strict solution of (3.3).

We deduce

**Proposition 3.** Assume that \( 0 \leq \alpha < 1 \) and \( \mu \leq \mu(\alpha) \).

a) Given \( u_0 \in L^2(0, 1) \) and \( H \in H^1(0, T) \), problem (2.2) admits a unique mild solution.

b) Given \( u_0 \in H^{1,\mu}_\alpha(0, 1) \) such that \( u_0(0) = 0 \) and \( H \in H^1(0, T) \) such that \( u_0(1) = H(0) \), problem (2.2) admits a unique strict solution. In particular, this holds true when \( u_0 \in H^{1,\mu}_\alpha(0, 1) \) and \( H \in H^1(0, T) \) is such that \( H(0) = 0 \).

The proof of Proposition 3 is given later in section 5.
3.4. **Non homogeneous boundary condition at** \( x = 0 \). Finally we study to the boundary value problem (2.3). To define its solution, as we did for (2.2) before, we transform it into a problem with homogeneous boundary conditions and a source term. Let us introduce

\[
\forall x \in [0, 1], \quad p(x) := 1 - x^q \quad \text{where} \quad q := 2\sqrt{\mu(\alpha) - \mu}. \tag{3.5}
\]

Observe that \( q = 0 \) in the critical case \( \mu = \mu(\alpha) \). (See also Remark 1 later). So we assume here that \( \mu < \mu(\alpha) \). Then notice that \( p(0) = 1, p(1) = 0 \). Moreover, one can readily check that

\[
\left[ x^\alpha (x^\gamma p)'' \right](x) + \frac{\mu}{x^{2-\alpha-\gamma}} p(x) = 0, \tag{3.6}
\]

where \( \gamma \) is the parameter introduced in (2.4). Formally, if \( u \) is a solution of (2.3), then the function defined by

\[
v(x, t) = u(x, t) - x^\gamma p(x) p(0) H(t) = u(x, t) - x^\gamma (1 - x^q) H(t) \tag{3.7}
\]

is solution of

\[
\begin{cases}
v_t - (x^\alpha v_x)_x - \frac{\mu}{x^{2-\alpha}} v = F(x, t), & (x, t) \in Q \\
v(0, t) = 0, & t \in (0, T) \\
v(1, t) = 0, & t \in (0, T) \\
v(x, 0) = v_0(x), & x \in (0, 1),
\end{cases} \tag{3.8}
\]

where we denoted

\[
F(x, t) := -x^\gamma \frac{p(x)}{p(0)} H'(t) \quad \text{and} \quad v_0(x) := u_0(x) - x^\gamma \frac{p(x)}{p(0)} H(0).
\]

Observe that (2.3) actually implies \((x^{-\gamma} v)(0, t) = 0\) which, in particular, gives \(v(0, t) = 0\) as written in (3.8). Indeed, notice that \(v\) in (3.7) is given explicitly by

\[
v(x, t) = \sum_{k \geq 1} v_k(t) \Phi_k(x),
\]

with

\[
v_k(t) := v_{k,0} e^{-\lambda_k t} + \int_0^t F_k(s) e^{-\lambda_k (t-s)} \, ds
\]

\[
v_{k,0} := \int_0^1 v_0(x) \Phi_k(x) \, dx
\]

\[
F_k(t) = \int_0^1 F(x, t) \Phi_k(x) \, dx
\]

where with \( \Phi_k \) and \( \lambda_k \) we denote the eigenfunctions and eigenvalues associated with \( P_{\alpha,\mu} \) (see Section 4 for more detail). Then, as \( x \to 0 \) we have

\[
x^{-\gamma} v(x, t) = \sum_{k \geq 1} v_k(t) x^{\frac{1-\alpha}{2} - \gamma} J_{\nu(\alpha,\mu)} \left( j_{\nu(\alpha,\mu),k} x^{\frac{\mu-\alpha}{2}} \right)
\]

\[
= \sum_{k \geq 1} v_k(t) x^{\sqrt{\mu(\alpha) - \mu} - \frac{\mu-\alpha}{2}} J_{\nu(\alpha,\mu)} \left( j_{\nu(\alpha,\mu),k} x^{\frac{\mu-\alpha}{2}} \right) \to 0.
\]

In view of that, we get

\[
x^{-\gamma} u(x, t) = x^{-\gamma} v(x, t) + \frac{p(x)}{\mu_0} H(t) \to H(t), \quad \text{as} \quad x \to 0.
\]
Reciprocally, given $h \in L^2(0, T)$, consider the solution of

$$
\begin{cases}
  v_t - (x^\alpha v_x)_x - \frac{\mu}{x^{2-\alpha}} v = -x^\gamma \frac{p(x)}{p(0)} h(t), & (x, t) \in Q \\
  v(0, t) = 0, & t \in (0, T) \\
  v(1, t) = 0, & t \in (0, T) \\
  v(x, 0) = v_0(x), & x \in (0, 1).
\end{cases}
$$

Then the function $u$ defined by

$$
 u(x, t) = v(x, t) + x^\gamma \frac{p(x)}{p(0)} \int_0^t h(\tau)d\tau
$$

satisfies

$$
\begin{cases}
  u_t - (x^\alpha u_x)_x - \frac{\mu}{x^{2-\alpha}} u = 0, & (x, t) \in Q \\
  (x^{-\gamma}u)(0, t) = \int_0^t h(\tau)d\tau, & t \in (0, T) \\
  u(1, t) = 0, & t \in (0, T) \\
  u(x, 0) = v_0(x), & x \in (0, 1).
\end{cases}
$$

Let now $H \in H^1(0, T)$ and $u_0 \in L^2(0, 1)$. Define

$$
 f(x, t) = -x^\gamma \frac{p(x)}{p(0)} H'(t) \quad \text{and} \quad v_0(x) = u_0(x) - x^\gamma \frac{p(x)}{p(0)} H(0).
$$

Then, we have $f \in L^2((0, 1) \times (0, T))$ and $v_0 \in L^2(0, 1)$. We can thus apply the notions of mild and weak solutions introduced in Section 3.2 to define in a suitable way the solution of (2.2).

**Definition 3.2.** We have the following notions of solution:

a) We say that $u \in C^0([0, T]; L^2(0, 1)) \cap L^2(0, T; H^{1, \mu}_\alpha(0, 1))$ is the mild solution of (2.3) if $v$ defined by (3.7) and (3.9) is the mild solution of (3.8).

b) We say that $u \in C^0([0, T]; H^{1, \mu}_\alpha(0, 1)) \cap H^1(0, T; L^2(0, 1)) \cap L^2(0, T; H^{2, \mu}_\alpha(0, 1))$ is the strict solution of (2.3) if $v$ defined by (3.7) and (3.9) is the strict solution of (3.8).

We deduce

**Proposition 4.** Assume that $0 \leq \alpha < 1$ and $\mu < \mu(\alpha)$.

a) Given $u_0 \in L^2(0, 1)$ and $H \in H^1(0, T)$, problem (2.3) admits a unique mild solution.

b) Given $u_0 \in H^{1, \mu}_\alpha(0, 1)$ and $H \in H^1(0, T)$ such that $(x^{-\gamma}u_0)(0) = H(0)$ and $u_0(1) = 0$, problem (2.3) admits a unique strict solution. In particular, this holds true when $u_0 \in H^{1, \mu}_\alpha(0, 1)$ such that $(x^{-\gamma}u_0)(0) = u_0(1) = 0$.

The proof of Proposition 4 is given later in section 5.

**Remark 1.** As a final remark we observe that, when $\mu = \mu(\alpha)$, the value of $q$ that we defined in (3.5) is zero. This means that, in the case of critical potentials, the change of variables introduced for defining the solution to our problem is the trivial one. Therefore, in what follows, when dealing with (2.3) we shall always assume $\mu < \mu(\alpha)$. Notice, however, that this assumption is not a limitation. Indeed, for critical potentials we do not expect our equation (2.3) to be well posed, at least not with the boundary conditions that we are imposing. This behavior had already
been observed in [8] for purely singular operators \((\alpha = 0)\), and a more detailed discussion on this point can be found in the Appendix A of that mentioned work.

4. Preliminary results and spectral properties of the corresponding operator. The strategy for proving Theorems 2.1 and 2.2 is based on the moment method (see [20, 21]) and requires the study of the associated Sturm-Liouville problem: one needs the expressions of the eigenvalues and eigenfunctions together with suitable estimates on the eigenvalues. We summarize here all these preliminary results that are useful to transform the controllability problems into moment problems and solve these questions.

In order to transform the question of null controllability into a moment problem, we first study the eigenvalue problem associated to the degenerate/singular operator \(P_{\alpha,\mu}\):

\[
\begin{cases}
P_{\alpha,\mu}\phi = -(x^\alpha \phi')' - \frac{\mu}{x^{2-\alpha}} \phi(x), & x \in (0, 1) \\
\phi(0) = 0 = \phi(1).
\end{cases}
\]  

(4.1)

We prove:

Proposition 5. Assume \(0 \leq \alpha < 1\) and \(\mu \leq \mu(\alpha)\) and define

\[\nu(\alpha, \mu) := \frac{2}{2-\alpha} \sqrt{\left(\frac{1-\alpha}{2}\right)^2 - \mu} = \frac{2}{2-\alpha} \sqrt{\mu(\alpha) - \mu}.
\]

For any \(\nu \geq 0\), we denote by \(J_\nu\) the Bessel function of first kind of order \(\nu\) and we denote

\[0 < j_{\nu,1} < j_{\nu,2} < \cdots < j_{\nu,k} < \cdots \to +\infty \text{ as } k \to +\infty
\]

the sequence of positive zeros of \(J_\nu\). Then the admissible eigenvalues \(\lambda\) for problem (4.1) are

\[\forall k \geq 1, \quad \lambda_{\alpha,\mu,k} = \left(\frac{2 - \alpha}{2}\right)^2 (j_{\nu(\alpha,\mu),k})^2
\]

(4.2)

and the corresponding (normalized) eigenfunctions are

\[\forall k \geq 1, \quad \Phi_{\alpha,\mu,k}(x) = \frac{\sqrt{2 - \alpha}}{|J'_{\nu(\alpha,\mu)}(j_{\nu(\alpha,\mu),k})|} x^{\frac{1-\alpha}{2}} J_{\nu(\alpha,\mu)} \left( j_{\nu(\alpha,\mu),k} x^{\frac{2-\alpha}{2}} \right).
\]

Moreover the family \((\Phi_{\alpha,\mu,k})_{k\geq1}\) forms an orthonormal basis of \(L^2(0, 1)\).

Proof. Let us first prove that that any admissible eigenvalue \(\lambda\) satisfies \(\lambda > 0\). Let \(\phi\) be an eigenfunction. Multiplying the equation by \(\phi\) and integrating by parts over \((0, 1)\), we get

\[
\int_0^1 \left(x^\alpha \phi_x^2 - \frac{\mu}{x^{2-\alpha}} \phi^2\right) dx = \lambda \int_0^1 \phi^2 dx.
\]

Using \(\mu \leq \mu(\alpha)\) and the generalized Hardy inequality (1.1), we have

\[\lambda \int_0^1 \phi^2 dx \geq \int_0^1 \left(x^\alpha \phi_x^2 - \frac{\mu(\alpha)}{x^{2-\alpha}} \phi^2\right) dx \geq 0.
\]

It follows that \(\lambda \geq 0\) since \(\phi \neq 0\). Assume now that \(\lambda = 0\). Then

\[
\int_0^1 \left(x^\alpha \phi_x^2 - \frac{\mu}{x^{2-\alpha}} \phi^2\right) dx = \lambda \int_0^1 \phi^2 dx = 0.
\]
This implies that $\phi \equiv 0$ since the left hand side of the above relation defines a norm on $H^{1,0}(0,1)$. (It is a consequence of (1.1) when $\mu < \mu(\alpha)$ and of [43, Theorem 2.2] when $\mu = \mu(\alpha)$). Since $\phi \not\equiv 0$, it follows that $\lambda > 0$.

In view of the above discussion, in what follows we will always assume $\lambda > 0$. Now, using the changes of variables

$$
\phi(x) = x^{\frac{1-\alpha}{2}} \psi \left( \frac{2}{2-\alpha} \sqrt{\lambda x^{\frac{2-\alpha}{\alpha}}} \right) \quad \text{and} \quad y = \frac{2}{2-\alpha} \sqrt{\lambda x^{\frac{2-\alpha}{\alpha}}},
$$

one can easily see that $\phi$ satisfies (4.1) if and only if $\psi$ is solution of

$$
\begin{align*}
\left\{ 
\begin{array}{l}
y^2 \psi''(y) + y \psi'(y) + (y^2 - \nu(\alpha, \mu)^2) \psi(y) = 0, \\ \psi(0) = 0 = \psi \left( \frac{2\sqrt{\lambda}}{2-\alpha} \right)
\end{array}
\right.,
\end{align*}
$$

Hence $\psi$ is a solution of the Bessel equation of order $\nu(\alpha, \mu)$. A fundamental system of solutions of the above Bessel equation is given by $\{J_{\nu(\alpha, \mu)}, Y_{\nu(\alpha, \mu)}\}$, where $J_{\nu(\alpha, \mu)}$ and $Y_{\nu(\alpha, \mu)}$ are the Bessel’s functions of order $\nu(\alpha, \mu)$, respectively of the first kind and of second kind. So $\Psi$ takes the form:

$$
\forall y \in \left( 0, \frac{2\sqrt{\lambda}}{2-\alpha} \right), \quad \Psi(y) = C J_{\nu(\alpha, \mu)}(y) + C' Y_{\nu(\alpha, \mu)}(y),
$$

for some $C, C' \in \mathbb{R}$. It is known that $J_{\nu(\alpha, \mu)}(0) = 0$ and $Y_{\nu(\alpha, \mu)}(0) = -\infty$ (see [34, sections 5.3 and 5.4]). In order to satisfy the boundary condition at $x = 0$, it follows that $C' = 0$. Thus

$$
\forall y \in \left( 0, \frac{2\sqrt{\lambda}}{2-\alpha} \right), \quad \Psi(y) = C J_{\nu(\alpha, \mu)}(y),
$$

with $C \neq 0$. Then the other boundary condition implies that

$$
J_{\nu(\alpha, \mu)} \left( \frac{2\sqrt{\lambda}}{2-\alpha} \right) = 0.
$$

So one has

$$
\frac{2\sqrt{\lambda}}{2-\alpha} = j_{\nu(\alpha, \mu), k},
$$

for some $k \in \mathbb{N}^*$. Therefore the set of admissible eigenvalues is given by

$$
\lambda_{\nu(\alpha, \mu), k} = \left( \frac{2-\alpha}{2} \right) ^2 j_{\nu(\alpha, \mu), k}^2, \quad k \in \mathbb{N}^*.
$$

As for the eigenfunctions, they take the form

$$
\forall k \geq 1, \quad \Phi_{\alpha, \mu, k}(x) = C_k x^{\frac{1-\alpha}{2}} J_{\nu(\alpha, \mu)} \left( j_{\nu(\alpha, \mu), k}^{2-\alpha} \right).
$$
It remains to show that \((\Phi_k)_{k \geq 1}\) forms an orthogonal family in \(L^2(0,1)\) and choose \(C_k\) so that it becomes normalized. For any \(n, m \geq 1\), let us compute
\[
\int_0^1 \Phi_{\nu(\alpha, \mu), n}(x) \Phi_{\nu(\alpha, \mu), m}(x) \, dx
\]
\[
= \frac{c_n c_m}{2 - \alpha} \int_0^1 x^{1-\alpha} J_{\nu(\alpha, \mu)} \left( j_{\nu(\alpha, \mu), n} x^{\frac{2-\alpha}{4}} \right) J_{\nu(\alpha, \mu)} \left( j_{\nu(\alpha, \mu), m} x^{\frac{2-\alpha}{4}} \right) \, dx
\]
\[
= \frac{2c_n c_m}{2 - \alpha} \int_0^1 y J_{\nu(\alpha, \mu)} \left( j_{\nu(\alpha, \mu), n} y \right) J_{\nu(\alpha, \mu)} \left( j_{\nu(\alpha, \mu), m} y \right) \, dy
\]
\[
= \frac{c_n c_m}{2 - \alpha} \delta_{nm} \left[ J_{\nu(\alpha, \mu) + 1} \left( j_{\nu(\alpha, \mu), n} \right) \right]^2,
\]
where we used the orthogonality property of Bessel’s functions (see [34, section 5.14]). Moreover, Bessel’s functions satisfy the identity (see [46, p. 45, equation (4.4))):
\[
x J'_\nu(x) - \nu J_\nu(x) = -x J_{\nu+1}(x),
\]
yielding \(J_{\nu+1}(j_{\nu,n}) = -J'_\nu(j_{\nu,n})\). It follows that
\[
\int_0^1 \Phi_{\nu(\alpha, \mu), n}(x) \Phi_{\nu(\alpha, \mu), m}(x) \, dx = \frac{c_n c_m}{2 - \alpha} \delta_{nm} \left[ J'_\nu(j_{\nu(\alpha, \mu), n}) \right]^2.
\]
Finally, choosing
\[
C_k = \frac{\sqrt{2 - \alpha}}{|J'_\nu(j_{\nu(\alpha, \mu), k})|},
\]
the family \((\Phi_k)_{k \geq 1}\) is orthonormal in \(L^2(0,1)\).

Next we give some estimates on the eigenvalues that will be useful in the analysis of the problem. Referring to [46, Section 15.53], we can give the following asymptotic expansion of the zeros of the Bessel function \(J_\nu\), for any fixed \(\nu \geq 0\):
\[
j_{\nu, k} = \left( k + \nu - \frac{1}{4} \right) \pi - \frac{4\nu^2 - 1}{8 \left( k + \nu - \frac{1}{4} \right)} \pi + O \left( \frac{1}{k^3} \right), \quad \text{as } k \to +\infty.
\]
Moreover, in what follows we will also need the following bounds on \(j_{\nu, k}\), which are provided in [35, Lemma 1]
\[
\begin{cases}
\forall \nu \in \left[ 0, \frac{1}{2} \right], & \forall k \geq 1, \quad \pi \left( k + \nu - \frac{1}{4} \right) \leq j_{\nu, k} \leq \pi \left( k + \nu - \frac{1}{8} \right), \\
\forall \nu \in \left[ \frac{1}{2}, +\infty \right), & \forall k \geq 1, \quad \pi \left( k + \nu - \frac{1}{8} \right) \leq j_{\nu, k} \leq \pi \left( k + \nu - \frac{1}{4} \right).
\end{cases}
\]

The inequalities above become exact when \(\nu = 1/2\) (which corresponds, according to (2.5), to \(\alpha = \mu = 0\)). We also recall the following result, whose proof is classical and can be found in [33, Proposition 7.8].

Lemma 4.1. Let \((j_{\nu, k})_{k \geq 1}\) be the sequence of positive zeros of the Bessel function \(J_\nu\). Then the following holds:

- The difference sequence \((j_{\nu, k+1} - j_{\nu, k})_k\) converges to \(\pi\) as \(k \to +\infty\).
- The sequence \((j_{\nu, k+1} - j_{\nu, k})_k\) is strictly decreasing if \(|\nu| > 1/2\), strictly increasing if \(|\nu| < 1/2\), and constant if \(|\nu| = 1/2\).
Lemma 4.2. We have the following bounds for the difference \( \sqrt{\lambda_{\alpha,\mu,k+1}} - \sqrt{\lambda_{\alpha,\mu,k}} \):

(i) When \( \nu(\alpha, \mu) \in [0, \frac{1}{2}) \) that is when \( \mu \in \left( \frac{\alpha}{16}(3\alpha - 4), \mu(\alpha) \right) \), then \( \forall k \geq 1, \)
\[
\frac{7\pi}{16} (2 - \alpha) \leq \sqrt{\lambda_{\alpha,\mu,k+1}} - \sqrt{\lambda_{\alpha,\mu,k}} \leq \frac{(2 - \alpha)}{2} \pi.
\]

(ii) When \( \nu(\alpha, \mu) \in \left[ \frac{1}{2}, +\infty \right) \) that is when \( \mu \in \left( -\infty, \frac{\alpha}{16}(3\alpha - 4) \right) \), then \( \forall k \geq 1, \)
\[
\frac{\pi}{2} (2 - \alpha) \leq \sqrt{\lambda_{\alpha,\mu,k+1}} - \sqrt{\lambda_{\alpha,\mu,k}} \leq \frac{(2 - \alpha)}{2} \left( j_{\nu(\alpha, \mu), 2} - j_{\nu(\alpha, \mu), 1} \right).
\]

Proof. Let us start with \( \nu(\alpha, \mu) \in [0, \frac{1}{2}) \). Concerning the lower bound, employing the estimates (4.3) we can easily obtain
\[
\sqrt{\lambda_{\alpha,\mu,k+1}} - \sqrt{\lambda_{\alpha,\mu,k}} = \frac{2 - \alpha}{2} \left( j_{\nu(\alpha, \mu), k+1} - j_{\nu(\alpha, \mu), k} \right)
\geq \frac{(2 - \alpha)\pi}{2} \left( \frac{\nu(\alpha, \mu)}{4} + \frac{7}{8} \right) \geq \frac{7\pi}{16} (2 - \alpha),
\]
since \( \nu(\alpha, \mu) \geq 0 \). Concerning the upper bound, thanks to Lemma 4.1 we immediately have that \( j_{\nu(\alpha, \mu), k+1} - j_{\nu(\alpha, \mu), k} < \pi \), which clearly implies
\[
\sqrt{\lambda_{\alpha,\mu,k+1}} - \sqrt{\lambda_{\alpha,\mu,k}} \leq \frac{\pi}{2} (2 - \alpha).
\]

For \( \nu(\alpha, \mu) \in \left[ \frac{1}{2}, +\infty \right) \), instead, thanks again to Lemma 4.1 we have that \( j_{\nu(\alpha, \mu), k+1} - j_{\nu(\alpha, \mu), k} > \pi \), which clearly implies
\[
\sqrt{\lambda_{\alpha,\mu,k+1}} - \sqrt{\lambda_{\alpha,\mu,k}} \geq \frac{\pi}{2} (2 - \alpha).
\]

Finally, the upper bound is again a consequence of Lemma 4.1:
\[
\sqrt{\lambda_{\alpha,\mu,k+1}} - \sqrt{\lambda_{\alpha,\mu,k}} = \frac{2 - \alpha}{2} \left( j_{\nu(\alpha, \mu), k+1} - j_{\nu(\alpha, \mu), k} \right)
\leq \frac{2 - \alpha}{2} \left( j_{\nu(\alpha, \mu), 2} - j_{\nu(\alpha, \mu), 1} \right),
\]
since the sequence \( \left( j_{\nu(\alpha, \mu), k+1} - j_{\nu(\alpha, \mu), k} \right) \) is nonincreasing in that case. Observe that it is the best upper bound (valid for any \( k \geq 1 \)) that one can obtain here. \( \square \)

Notice that, using the fact that \( 0 \leq \alpha < 1 \), one can deduce the following estimates that are also uniform with respect to \( \alpha \) and \( \mu \):

- when \( \nu(\alpha, \mu) \in \left[ 0, \frac{1}{2} \right) \),
  \[
  \forall k \geq 1, \quad \frac{7\pi}{16} \leq \sqrt{\lambda_{\alpha,\mu,k+1}} - \sqrt{\lambda_{\alpha,\mu,k}} \leq \pi; \quad (4.6)
  \]

- when \( \nu(\alpha, \mu) \in \left[ \frac{1}{2}, +\infty \right) \),
  \[
  \forall k \geq 1, \quad \frac{\pi}{2} \leq \sqrt{\lambda_{\alpha,\mu,k+1}} - \sqrt{\lambda_{\alpha,\mu,k}}. \quad (4.7)
  \]
On the other hand, let us observe that in the case \( \nu(\alpha, \mu) \in \left[ \frac{1}{2}, +\infty \right) \), the upper estimate given in Lemma 4.2 is not satisfactory. Indeed, one can quote the following inequality from [38]:

\[
\forall \nu > 0, \forall n \geq 1, \quad \nu - \frac{a_n}{2^{1/3}} \nu^{1/3} < j_{\nu, n} < \nu - \frac{a_n}{2^{1/3}} \nu^{1/3} + \frac{3}{20} a_n^2 \nu^{2/3},
\]

where \( a_n \) is the \( n \)-th negative zero of the Airy function. It follows that there exists \( \alpha > 0 \) such that

\[
j_{\nu, 2} - j_{\nu, 1} \sim a \nu^{1/3} \quad \text{as} \quad \nu \to +\infty.
\]

Consequently, for any \( \alpha \in [0, 1) \),

\[
j_{\nu(\alpha, \mu), 2} - j_{\nu(\alpha, \mu), 1} \sim a \nu(\alpha, \mu)^{1/3} \to +\infty \quad \text{as} \quad \mu \to -\infty. \tag{4.8}
\]

Moreover, this upper estimate being the best possible one valid for any \( k \geq 1 \) (see the proof of Lemma 4.2), it is of course not possible to improve it.

Therefore, in order to get sharp estimates of the cost of controllability, it will be important to provide some better upper estimates. To this end, we will use the following complementary asymptotic estimates that is only valid for \( k \) large enough but that has the advantage of being uniform with respect to \( \alpha \) and \( \mu \):

**Lemma 4.3.** When \( \nu(\alpha, \mu) \in \left[ \frac{1}{2}, +\infty \right) \), for any \( k > \nu(\alpha, \mu) \), we have

\[
\sqrt{\lambda_{\alpha, \mu, k+1}} - \sqrt{\lambda_{\alpha, \mu, k}} \leq 2\pi.
\]

**Proof.** It directly follows from the definition of \( \lambda_{\alpha, \mu, k} \) and Lemma 5.1 in [14] that says that the zeros the Bessel functions satisfy

\[
\forall \nu \geq \frac{1}{2}, \quad \forall k > \nu, \quad j_{\nu, k+1} - j_{\nu, k} \leq 2\pi.
\]

\( \square \)

5. **Proofs of Propositions 2, 3 and 4.**

5.1. **Proof of Proposition 2.** The operator \( A_{\alpha, \mu} \) generates an analytic semigroup of negative type on \( X = L^2(0, 1) \). Therefore, when \( w_0 \in D((-A_{\alpha, \mu})^{1/2}) \) and \( f \in L^2((0, T); X) = L^2(0, 1) \times (0, T) \), the solution \( w \) of (3.1), given by the variation of constants formula, satisfies (see for example [6]):

\[
w \in C^0([0, T]; D((-A_{\alpha, \mu})^{1/2}) \cap H^1(0, T; X) \cap L^2(0, T; D(A_{\alpha, \mu}))).
\]

Therefore in order to get Proposition 2, it suffices to prove that \( D((-A_{\alpha, \mu})^{1/2}) = H^1_{\alpha, \mu}(0, 1) \).

To this end, since the family \( \{ \Phi_{\alpha, \mu, k} \}_{k \geq 1} \) is an orthonormal basis of \( L^2(0, 1) \), for any \( u \in L^2(0, 1) \) we may write

\[
u = \sum_{k \geq 1} (u, \Phi_{\alpha, \mu, k}) \Phi_{\alpha, \mu, k}.
\]

Moreover, \( D((-A_{\alpha, \mu})^{1/2}) \) is defined by

\[
\left\{ u \in L^2(0, 1) \left| \sum_{k \geq 1} \lambda_{\alpha, \mu, k} (u, \Phi_{\alpha, \mu, k})^2 < +\infty \right. \right\}.
\]
We recall that, for any $u \in D(A_{\alpha,\mu})$, we have

$$
\int_{0}^{1} \left( x^\alpha u_x^2 - \frac{\mu}{x^{2-\alpha}} u^2 \right) dx = (-A_{\alpha,\mu}u, u)_{L^2(0,1)}.
$$

Hence

$$
\int_{0}^{1} \left( x^\alpha u_x^2 - \frac{\mu}{x^{2-\alpha}} u^2 \right) dx = \sum_{k \geq 1} \lambda_{\alpha,\mu,k} (u, \Phi_{\alpha,\mu,k})_{L^2(0,1)}^2.
$$

This relation still holds true by density argument for $u \in L^2(0,1) \cap H^1_{loc}((0,1])$. And it follows that $D((-A_{\alpha,\mu})^{1/2}) = H^1_{\alpha,0}(0,1)$.

5.2. **Proof of Proposition 3.** The proof of Proposition 3 follows noticing that $\tilde{H}(x,t) := \frac{p(x)}{p(1)} H(t)$ satisfies

$$
\tilde{H} \in C^0([0,T]; H^1_{\alpha,\mu}(0,1)) \cap H^1(0,T; L^2(0,1)) \cap L^2(0,T; H^2_{\alpha,\mu}(0,1)).
$$

Indeed, let us consider the two cases described in Proposition 3.

**Case a.** Assume that $u_0 \in L^2(0,1)$ and $H \in H^1(0,T)$. Then we observe that

$$
- \frac{p(x)}{p(1)} H'(t) \in L^2((0,1) \times (0,T)) \quad \text{and} \quad u_0(x) - \frac{p(x)}{p(1)} H(0) \in L^2(0,1).
$$

So problem (3.3) admits a unique mild solution

$$
u \in C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_{\alpha,\mu}(0,1))$$

that is given by the variation formula (see section 3.2). Next we set

$$
\begin{align*}
    u(x,t) &= \nu(x,t) + \frac{p(x)}{p(1)} H(t) = \nu(x,t) + \tilde{H}(x,t).
\end{align*}
$$

Thanks to the regularity of $\tilde{H}$, we have

$$
u \in C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H^1_{\alpha,\mu}(0,1))$$

which is the unique mild solution of (2.2).

**Case b.** Assume now that $u_0 \in H^1_{\alpha,\mu}(0,1)$ is such that $u_0(0) = 0$ and $H \in H^1(0,T)$ satisfies $H(0) = u_0(1)$. Then we still have

$$
- \frac{p(x)}{p(1)} H'(t) \in L^2((0,1) \times (0,T))
$$

and we can prove that

$$
\begin{align*}
    u_0(x) - \frac{p(x)}{p(1)} H(0) &\in H^1_{\alpha,\mu}(0,1). \quad \text{Indeed, this function belongs to } H^1_{\alpha,\mu}(0,1) \text{ since both functions } u_0 \text{ and } p \text{ belong to } H^1_{\alpha,\mu}(0,1). \text{ Moreover}
\end{align*}
$$

Moreover

$$
\begin{align*}
    u_0(0) - \frac{p(0)}{p(1)} H(0) &= 0 \quad \text{since } u_0(0) = 0 \text{ and } p(0) = 0 \quad \text{and}
\end{align*}
$$

$$
\begin{align*}
    u_0(1) - \frac{p(1)}{p(1)} H(0) &= u_0(1) - H(0) = 0 \quad \text{since } H(0) = u_0(1),
\end{align*}
$$
which shows that the function finally belongs to $H^{1,\mu}_{\alpha,0}(0,1)$. So, by Proposition 2, problem (3.3) admits a unique strict solution

$$v \in C^0([0,T]; H^{1,\mu}_{\alpha,0}(0,1)) \cap H^1(0,T; L^2(0,1)) \cap L^2(0,T; D(A_{\alpha,\mu})).$$

Next we set

$$u(x,t) = v(x,t) + \frac{p(x)}{p(1)} H(t) = v(x,t) + \tilde{H}(x,t).$$

Thanks to the regularity of $\tilde{H}$, we have

$$u \in C^0([0,T]; H^{1,\mu}_{\alpha,0}(0,1)) \cap H^1(0,T; L^2(0,1)) \cap L^2(0,T; H^2_{\alpha,\mu}(0,1))$$

which is the unique strict solution of (2.2).

5.3. Proof of Proposition 4. As previously, the proof of Proposition 4 follows immediately noticing that

$$\tilde{H}(x,t) := x^{\gamma} p(x) H(t)$$

satisfies

$$\tilde{H} \in C^0([0,T]; H^{1,\mu}_{\alpha,0}(0,1)) \cap H^1(0,T; L^2(0,1)) \cap L^2(0,T; H^2_{\alpha,\mu}(0,1)).$$

6. Proof of Theorem 2.1. This Section is devoted to the proof of our first result Theorem 2.1 on the boundary controllability for (2.2).

The proof will employ the classical moment method (see [20, 21]). This procedure is based on the explicit construction of the control $H$, given in terms of a family $(\sigma_{\alpha,\mu,m}(t))_{m \geq 1}$ biorthogonal in $L^2(0,T)$ to the family of real exponential $(e^{\lambda_{\alpha,\mu,n}})_{n \geq 1}$, that is

$$\forall m,n \geq 1, \quad \int_0^T \sigma_{\alpha,\mu,m}(t) e^{\lambda_{\alpha,\mu,n} t} dt = \delta_{mn} = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \quad (6.1)$$

In order to show the existence of such a sequence, we will use [12, Theorem 2.4], whose proof has been inspired by the works of Seidman-Avdonin-Ivanov [41] and Tucsnak-Tenenbaum [42]. In this part, it will be fundamental that the eigenvalues associated to our problem fulfill the gap conditions

$$\forall n \geq 1, \quad \sqrt{\lambda_{\alpha,\mu,n+1}} - \sqrt{\lambda_{\alpha,\mu,n}} \geq \gamma_{\min}.$$  

Furthermore, to obtain our estimates on the controllability cost, we also need to provide some sharp lower bound of the norm $\|\sigma_{\alpha,\mu,m}(t)\|_{L^2(0,T)}$, which will be obtained as a consequence of this second spectral estimate

$$\forall n \geq 1, \quad \sqrt{\lambda_{\alpha,\mu,n+1}} - \sqrt{\lambda_{\alpha,\mu,n}} \leq \gamma_{\max}. \quad (6.2)$$

When $\nu(\alpha,\mu) \in [0, \frac{1}{2}]$, (6.2) holds true for some $\gamma_{\max}$ that is independent of $\alpha$ and $\mu$. Here we will use [12, Theorem 2.5], inspired from Guichal [26].

When $\nu(\alpha,\mu) \in [\frac{1}{2}, +\infty)$, (6.2) still holds true but with $\gamma_{\max}$ that tends to $+\infty$ as $\mu \to -\infty$. So one could still use [12, Theorem 2.5] but this would not give a sharp estimate. For this reason, we complement (6.2) by the better asymptotic estimate given in Lemma 4.3. Then we will use [13, Theorem 2.2]. Therefore, according to the above discussion, our proof will be organized into the following steps:

- **Step 1.** Following the classical approach of [20, 21], we first reduce our control problem to a moment problem.
• **Step 2.** We give a formal solution, using the properties of the spectrum of the operator $P_{\alpha,\mu}$.

• **Step 3.** We prove the existence of the control, its regularity (in $H^1(0,T)$) and also give an upper bound of the cost of controllability.

• **Step 4.** We finally derive a lower bound of the cost of controllability.

Let $\alpha$ and $\mu$ be given such that $0 \leq \alpha < 1$ and $\mu \leq \mu(\alpha)$. For simplicity in the notations, we denote in the following by $\Phi_k$ (instead of $\Phi_{\alpha,\mu,k}$) and $\lambda_k$ (instead of $\lambda_{\alpha,\mu,k}$) the eigen-elements given by Proposition 5, and by $\sigma_m$ (instead of $\sigma_{\alpha,\mu,m}$) the biorthogonal family. Besides, also for simplicity in the notations, we will denote in a generic way by $C$ all the constants (independent of $k$, $\alpha$, $\mu$ and $T$) that appear in the calculus. We stress that the value of $C$ may change from line to line.

Finally, let us remark that all the proofs of the next sections deal with strict solution, but the same results can be obtained for mild solutions through a density argument.

6.1. **Reduction to a moment problem.** In this part, we treat the problem with formal computations. We will present a rigorous justification in a second moment.

Let us start expanding the initial condition $u_0 \in L^2(0,1)$ in the basis of the eigenfunctions $(\Phi_k)_{k \geq 1}$. Indeed, we know that there exists a sequence $(\rho^0_k)_{k \geq 1} \in \ell^2(\mathbb{N}^*)$ such that, for all $x \in (0,1)$,

$$u_0(x) = \sum_{k \geq 1} \rho^0_k \Phi_k(x), \quad \rho^0_k := \int_0^1 u_0(x) \Phi_k(x) \, dx, \quad k \geq 1.$$

Next, we expand also the solution $u$ to (2.2) as

$$u(x,t) = \sum_{k \geq 1} \beta_k(t) \Phi_k(x), \quad (x,t) \in Q,$$

with

$$\beta_k(t) := \int_0^1 u(x,t) \Phi_k(x) \, dx, \quad k \geq 1$$

and

$$\sum_{k \geq 1} \beta_k(t)^2 < +\infty.$$

Therefore, the controllability condition $u(x,T) = 0$ becomes

$$\forall k \geq 1, \quad \beta_k(T) = 0. \quad (6.3)$$

Moreover, we notice that the function $v_k(x,t) := \Phi_k(x) e^{\lambda_k(t-T)}$ solves the adjoint problem

$$\begin{cases}
  v_{k,t} + (x^\alpha v_{k,x})_x + \frac{\mu}{x^{2\alpha}} v_k = 0, \quad (x,t) \in Q \\
  v_k(0,t) = v_k(1,t) = 0, \quad t \in (0,T).
\end{cases} \quad (6.4)$$
Combining (2.2) and (6.4) we obtain

\[
0 = \int_Q \left[ v_k \left( u_t - (x^\alpha u_x)_x - \frac{\mu}{x^\beta} u \right) + u \left( v_{k,t} + (x^\alpha v_{k,x})_x + \frac{\mu}{x^\beta} v_k \right) \right] dx dt
\]

\[
= \int_0^1 v_k u \bigg|_0^T dx - \int_0^T x^\alpha u_x v_k \bigg|_0^1 dt + \int_0^T x^\alpha v_{k,x} u \bigg|_0^1 dt
\]

\[
= \int_0^1 v_k(x, T)u(x, T) dx - \int_0^1 v_k(x, 0)u_0(x) dx + \int_0^T H(t)v_{k,x}(1, t) dt
\]

\[
= \int_0^1 u(x, T)\Phi_k(x) dx - e^{-\lambda_k T} \int_0^1 u_0(x)\Phi_k(x) dx + e^{-\lambda_k T}\Phi_k(1) \int_0^T H(t)e^{\lambda_k t} dt
\]

\[
= \beta_k(T) - \rho_k e^{-\lambda_k T} + e^{-\lambda_k T}\Phi_k(1) \int_0^T H(t)e^{\lambda_k t} dt.
\]

Then, (6.3) yields

\[
\forall k \geq 1, \quad \Phi_k(1) \int_0^T H(t)e^{\lambda_k t} dt = \rho_k. \quad (6.5)
\]

On the other hand, since we are looking for a solution of the moment problem belonging to \(H^1(0, T)\), instead of (6.5) we would rather be interested in a condition involving the derivative of the function \(H\). This condition can be obtained integrating by parts in (6.5), as follows

\[
\int_0^T H(t)e^{\lambda_k t} dt = \frac{1}{\lambda_k} \left( H(t)e^{\lambda_k t} \bigg|_0^T - \frac{1}{\lambda_k} \int_0^T H'(t)e^{\lambda_k t} dt. \right)
\]

Therefore, \(H'(t)\) has to satisfy

\[
\forall k \geq 1, \quad -\frac{\Phi_k(1)}{\lambda_k} \int_0^T H'(t)e^{\lambda_k t} dt = \rho_k - \frac{\Phi_k'(1)}{\lambda_k} \left( H(T)e^{\lambda_k T} - H(0) \right). \quad (6.6)
\]

We will provide a solution to the above problem which satisfies \(H(0) = H(T) = 0\).

6.2. Formal solution of the moment problem. We exhibit here a formal solution of the moment problem (6.6).

6.2.1. Formal definition of the control \(H\). Set artificially \(\lambda_0 := 0\), so that we have now a sequence \((\lambda_k)_{k \geq 0}\). We assume for the moment that we are able to construct a family \((\sigma_m)_{m \geq 0}\) of functions \(\sigma_m \in L^2(0, T)\), which is biorthogonal to the family \((e^{\lambda_n t})_{n \geq 0}\). Observe that for \(n = 0\), using \(\lambda_0 = 0\), (6.1) implies

\[
\forall m \geq 1, \quad \int_0^T \sigma_m(t) dt = 0. \quad (6.7)
\]

Then let us define the function \(H\) as follows:

\[
H(t) := \int_0^t K(s) ds, \quad \text{with} \quad K(t) := -\sum_{k \geq 1} \frac{\lambda_k}{\Phi_k'(1)} \rho_k \sigma_k(t). \quad (6.8)
\]

It is straightforward that, if \(K \in L^2(0, T)\), then \(H \in H^1(0, T)\) with \(H(0) = 0\) and \(H'(t) = K(t)\). Moreover thanks to (6.7) we have, at least formally,

\[
H(T) = -\int_0^T \sum_{k \geq 1} \frac{\lambda_k}{\Phi_k'(1)} \rho_k \sigma_k(s) ds = -\int_0^T \sum_{k \geq 1} \frac{\lambda_k}{\Phi_k'(1)} \rho_k \int_0^T \sigma_k(s) ds = 0.
\]
Finally,

\[-\frac{\Phi_k'(1)}{\lambda_k} \int_0^T H'(t)e^{\lambda_k t} dt = -\frac{\Phi_k'(1)}{\lambda_k} \int_0^T K(t)e^{\lambda_k t} dt \]

\[= \frac{\Phi_k'(1)}{\lambda_k} \int_0^T \left( \sum_{\ell \geq 1} \frac{\lambda_\ell}{\Phi_k'(1)} \rho_\ell(t) \right) e^{\lambda_k t} dt = \frac{\Phi_k'(1)}{\lambda_k} \sum_{\ell \geq 1} \frac{\lambda_\ell}{\Phi_k'(1)} \rho_\ell \int_0^T \sigma_\ell(t)e^{\lambda_k t} dt \]

\[= \frac{\Phi_k'(1)}{\lambda_k} \sum_{\ell \geq 1} \frac{\lambda_\ell}{\Phi_k'(1)} \rho_\ell \delta_{k,\ell} = \rho_k^0, \]

and the moment problem (6.6) is formally satisfied.

6.2.2. If regular, the control \(H\) drives the solution from \(u_0\) to zero. Let us assume for now that \(K \in L^2(0, T)\) (and, consequently \(H\) introduced in (6.8) belongs to \(H^1(0, T)\)). We show here that \(H\) is able to drive the solution to (2.2) from the initial state \(u_0\) to zero in time \(T\). To this end, let us remind the change of variables

\[v(x, t) := u(x, t) - \frac{p(x)}{p(1)} H(t), \quad p(x) := x^q, \quad q = \frac{1 - \alpha}{2} + \sqrt{\mu(\alpha)} - \mu,\]

that transforms our original equation (2.2) in

\[
\begin{align*}
&v_t - (x^\alpha v_x)_x - \frac{\mu}{x^{2-\alpha}} v = -\frac{p(x)}{p(1)} K(t), \quad (x, t) \in Q \\
v(0, t) = v(1, t) = 0, & \quad t \in (0, T) \\
v(x, 0) = u_0(x), & \quad x \in (0, 1)
\end{align*}
\]

Now, for a fixed \(\varepsilon > 0\) we have

\[
\int_\varepsilon^1 \int_0^1 \frac{p(x)}{p(1)} K(t) \Phi_k(x) e^{\lambda_k t} dx dt = \int_\varepsilon^1 \int_0^1 \left( v_t - (x^\alpha v_x)_x - \frac{\mu}{x^{2-\alpha}} v \right) \Phi_k(x) e^{\lambda_k t} dx dt
\]

\[
= \int_0^1 v \Phi_k e^{\lambda_k t} \left|_\varepsilon^1 \right. dx + \int_\varepsilon^1 \int_0^1 v \left( -x^\alpha \Phi_k' - \frac{\mu}{x^{2-\alpha}} \Phi_k - \lambda_k \Phi_k \right) e^{\lambda_k t} dx dt
\]

\[
= e^{\lambda_k T} \int_0^1 v(x, T) \Phi_k(x) dx - e^{\lambda_k \varepsilon} \int_0^1 v(x, \varepsilon) \Phi_k(x) dx.
\]

Hence, taking the limit for \(\varepsilon \to 0^+\) we find

\[
\int_Q \frac{p(x)}{p(1)} K(t) \Phi_k(x) e^{\lambda_k t} dx dt = e^{\lambda_k T} \int_0^1 v(x, T) \Phi_k(x) dx - \rho_k^0.
\]

From this last identity and (6.8), it immediately follows

\[
e^{\lambda_k T} \int_0^1 v(x, T) \Phi_k(x) dx = \rho_k^0 + \left( \int_0^T K(t) e^{\lambda_k t} dt \right) \left( \int_0^1 \frac{p(x)}{p(1)} \Phi_k(x) dx \right)
\]

\[= \rho_k^0 - \frac{\lambda_k}{\Phi_k'(1)} \rho_k^0 \int_0^1 - \frac{p(x)}{p(1)} \Phi_k(x) dx.
\]
Moreover,
\[
\int_0^1 \frac{p(x)}{p(1)} \Phi_k(x) \, dx
\]
\[
= \frac{1}{\lambda_k} \int_0^1 \frac{p(x)}{p(1)} \lambda_k \Phi_k(x) \, dx = \frac{1}{\lambda_k} \int_0^1 \frac{p(x)}{p(1)} \left( (x^\alpha \Phi'_k(x))^\prime + \frac{\mu}{x^{2-\alpha}} \Phi_k(x) \right) \, dx
\]
\[
= \frac{1}{\lambda_k} \frac{1}{p(1)} \int_0^1 p(x) x^\alpha \Phi'_k(x) \, dx + \frac{1}{\lambda_k} \int_0^1 \frac{p(x)}{p(1)} \frac{\mu}{x^{2-\alpha}} \Phi_k(x) \, dx
\]
\[
= \frac{\Phi_k(1)}{\lambda_k} - \frac{1}{\lambda_k} \frac{1}{p(1)} \int_0^1 (x^\alpha p'(x))' + \mu x^{\alpha-2} p(x) \Phi_k(x) \, dx = \frac{\Phi_k(1)}{\lambda_k},
\]
since from the definition of \( p(x) \) it is straightforward to check that
\[
(x^\alpha p'(x))' + \mu x^{\alpha-2} p(x) = 0.
\]
Hence, we get
\[
e^{\lambda T} \int_0^1 v(x,T) \Phi_k(x) \, dx = 0,
\]
which of course implies \( v(x,T) = 0 \) and, since \( H(T) = 0 \), we can finally conclude that
\[
u(x,T) = v(x,T) + \frac{p(x)}{p(1)} H(T) = 0.
\]
At this stage, in order to prove point (i) of Theorem 2.1, it remains to prove the existence of a suitable biorthogonal family and to show that \( K \) belongs to \( L^2(0,T) \). This will be done in the next subsection together with the obtention of the upper bound of the cost of controllability.

6.3. Existence of the control, \( H^1 \) regularity and upper bound of the cost of controllability.

6.3.1. Existence of a suitable biorthogonal family. We will use the following result.

**Theorem 6.1.** (see [12, Theorem 2.4]) Assume that for all \( k \geq 0, \lambda_k \geq 0 \), and that there is some \( \gamma_{\min} > 0 \) such that
\[
\forall k \geq 0, \quad \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \gamma_{\min}.
\]
Then there exists a family \( (\sigma_m)_{m \geq 0} \) which is biorthogonal to the family \( (e^{\lambda T})_{k \geq 0} \) in \( L^2(0,T) \). Moreover, there exists some universal constant \( C_u \) independent of \( T \), \( \gamma_{\min} \) and \( m \) such that, for all \( m \geq 0 \), we have
\[
\| \sigma_m \|_{L^2(0,T)} \leq C_u e^{-2\lambda_{\min} T} e^{C_u \frac{\gamma_{\min}}{\lambda_{\min}} \sqrt{T}} B^*(T, \gamma_{\min}), \tag{6.9}
\]
with
\[
B^*(T, \gamma_{\min}) = \frac{C_u}{T} \max \left\{ T\gamma_{\min}^2, \frac{1}{T\gamma_{\min}^2} \right\}. \tag{6.10}
\]
Remark 2. [12, Theorem 2.4] is formulated in the following way:

\[ \| \sigma_m \|_{L^2(0,T)}^2 \leq C_u e^{-2\lambda_m T} e^{e^{\sqrt{\lambda_m}}} B(T, \gamma_{\min}), \]

with

\[ B(T, \gamma_{\min}) = \begin{cases} \left( \frac{1}{T} + \frac{1}{T^2} \right) e^{\frac{T}{\gamma_{\min}}} & \text{if } T \leq \gamma_{\min}^{-2}, \\
\frac{1}{\gamma_{\min}^2} & \text{if } T \geq \gamma_{\min}^{-2}, \end{cases} \]

and this is clearly equivalent to (6.9)-(6.10).

Using (4.6) and (4.7), the eigenvalues of the problem satisfy for all \( \mu \leq \mu(\alpha) \)

\[ \forall k \geq 1, \quad \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \geq \min \left\{ \frac{7\pi}{16} \cdot \frac{2}{\alpha}, \left( \frac{3\pi}{4} \right) \right\} = \frac{7\pi}{16}. \]

As before, define artificially \( \lambda_0 := 0 \). Then, for all \( \mu \leq \mu(\alpha) \),

\[ \sqrt{\lambda_1} - \sqrt{\lambda_0} = \frac{2 - \alpha}{2} j_{\nu(\alpha,\mu),1} \geq \frac{2 - \alpha}{2} \cdot \frac{3\pi}{4} = \frac{3\pi}{8} (2 - \alpha) \geq \frac{3\pi}{8}, \]

using the fact that, thanks to (4.3), one can easily prove that \( j_{\nu,1} \geq \frac{3\pi}{4} \) for all \( \nu \geq 0 \) and next using the fact that \( 2 - \alpha \geq 1 \). Therefore we can apply Theorem 6.1 to the family \((e^{\lambda_k t})_{k \geq 0}\) provided that we choose

\[ \gamma_{\min} = \min \left\{ \frac{7\pi}{16}, \frac{3\pi}{8} \right\} = \frac{3\pi}{8}. \]

We obtain that there exists a family \((\sigma_m)_{m \geq 0}\) biorthogonal to \((e^{\lambda_k t})_{k \geq 0}\) in \(L^2(0,T)\), and such that

\[ \| \sigma_m \|_{L^2(0,T)}^2 \leq C e^{-2\lambda_m T} e^{e^{\sqrt{\lambda_m}}} B(T) \leq C e^{-2\lambda_m T} e^{e^{\sqrt{\lambda_m}}} \tilde{B}(T), \quad (6.11) \]

with

\[ \tilde{B}(T) = \max \left\{ 1, \frac{1}{T^2} \right\} e^{\frac{T}{2}} \text{ for all } T > 0. \]

The form of \( \tilde{B}(T) \) easily follows from the definition of \( B(T, \gamma_{\min})^* \).

6.3.2. The control \( f \) belongs to \( H^1(0,T) \). We have to check that the control \( H \) defined as in (6.8) belongs to \( H^1(0,T) \). To this end, we are going to prove, instead, that the function \( K \) belongs to \( L^2(0,T) \). From (6.8) we have

\[ \| K \|_{L^2(0,T)} = \left\| \sum_{k \geq 1} \frac{\lambda_k}{\Phi_k(1)} \rho_k^2 \sigma_k \right\|_{L^2(0,T)} \leq \sum_{k \geq 1} \left| \rho_k^2 \right| \left| \frac{\lambda_k}{\Phi_k(1)} \right| \| \sigma_k \|_{L^2(0,T)}. \]

Let us compute the value of \( \Phi_k(1) \); we recall that

\[ \Phi_k(x) = C_k x^{1+\alpha} J_{\nu(\alpha,\mu)} \left( j_{\nu(\alpha,\mu),k} x^{\frac{2-\alpha}{4}} \right), \quad \text{with} \quad C_k = \frac{\sqrt{2-\alpha}}{|J_{\nu(\alpha,\mu)}(j_{\nu(\alpha,\mu),k})|.} \]

Thus, a direct computation gives

\[ \Phi_k'(x) = \frac{1-\alpha}{2} C_k x^{\frac{1-\alpha}{4}} J_{\nu(\alpha,\mu)} \left( j_{\nu(\alpha,\mu),k} x^{\frac{2-\alpha}{4}} \right) \]

\[ + \frac{2-\alpha}{2} C_k j_{\nu(\alpha,\mu),k} x^{\frac{1-\alpha}{4}} J_{\nu(\alpha,\mu)} \left( j_{\nu(\alpha,\mu),k} x^{\frac{2-\alpha}{4}} \right). \]
Hence
\[ \Phi_k'(1) = \left| \frac{1 - \alpha}{2} c_k j_{\nu(\alpha, \mu)}(\tilde{j}_{\nu(\alpha, \mu)}(k)) + \frac{2 - \alpha}{2} c_k j_{\nu(\alpha, \mu)}(k) j'_{\nu(\alpha, \mu)}(\tilde{j}_{\nu(\alpha, \mu)}(k)) \right| \]
\[ = \frac{(2 - \alpha)^3}{2} j_{\nu(\alpha, \mu)}(k). \]  
\hspace{1cm} (6.12)

Therefore, employing (6.12) and the explicit expression of the eigenvalues \( \lambda_k \) we obtain
\[ \left| \frac{\lambda_k}{\Phi_k'(1)} \right| = \left( \frac{2 - \alpha}{2} j_{\nu(\alpha, \mu)}(k) \right)^2 \frac{\sqrt{2 - \alpha}}{2} j_{\nu(\alpha, \mu)}(k) \leq \sqrt{2} j_{\nu(\alpha, \mu)}(k). \]

Consequently, employing (6.12) and the explicit expression of the eigenvalues \( \lambda_k \) we obtain
\[ \left| \frac{\lambda_k}{\Phi_k'(1)} \right| = \left( \frac{2 - \alpha}{2} j_{\nu(\alpha, \mu)}(k) \right)^2 \frac{\sqrt{2 - \alpha}}{2} j_{\nu(\alpha, \mu)}(k) \leq \sqrt{2} j_{\nu(\alpha, \mu)}(k). \]

Therefore, we get
\[ \|K\|_{L^2(0,T)} \leq \sqrt{2} \sum_{k \geq 1} |\rho_k| j_{\nu(\alpha, \mu)}(k) \|\sigma_k\|_{L^2(0,T)} \]
\[ \leq \sqrt{2} \left( \sum_{k \geq 1} |\rho_k|^2 \right)^{1/2} \left( \sum_{k \geq 1} j_{\nu(\alpha, \mu)}(k)^2 \|\sigma_k\|_{L^2(0,T)}^2 \right)^{1/2}. \]

Using the explicit expression of \( \lambda_k \), we get \( j_{\nu(\alpha, \mu)}(k) = 4\lambda_k/(2 - \alpha)^2 \leq 4\lambda_k \) since \( \alpha < 1 \). Hence, using also the estimate (6.11), we deduce that
\[ \|K\|_{L^2(0,T)} \leq C \|u_0\|_{L^2(0,1)} \left( \sum_{k \geq 1} \lambda_k e^{-2\lambda_k T} \rho_k e^{\sqrt{\lambda_k} B(T)} \right)^{1/2}, \]

which is finite. This implies that \( K \in L^2(0,T) \). Therefore we have \( H \in H^1(0,T) \) with of course \( H(0) = 0 \). And the fact that \( H(T) = 0 \) follows from (6.7) and (6.8).

6.3.3. Upper bound of the cost of controllability. As shown before, the function \( H \) defined in (6.8) is an admissible control. It follows that
\[ C_{ctr-bd} \leq \frac{\|H\|_{H^1(0,T)}}{\|u_0\|_{L^2(0,1)}} \leq \frac{\|K\|_{L^2(0,T)}}{\|u_0\|_{L^2(0,1)}}. \]

Hence
\[ C_{ctr-bd} \leq \frac{\sqrt{B(T)}}{e^\tau} \left( \sum_{k=1}^{\infty} \lambda_k e^{-2\lambda_k T} \rho_k e^{\sqrt{\lambda_k} B(T)} \right)^{1/2}. \]

Then let us write
\[ \rho_k \leq \lambda_k T + \frac{\rho'}{T}. \]

One deduces that
\[ C_{ctr-bd} \leq \frac{\sqrt{B(T)}}{e^\tau} \left( \sum_{k=1}^{\infty} \lambda_k e^{-\lambda_k T} \rho' \right)^{1/2} \]
\[ \leq \frac{\sqrt{B(T)}}{e^\tau} \left( \sum_{k=1}^{\infty} \frac{(2 - \alpha)^2}{4} j_{\nu(\alpha, \mu)}(k)^2 e^{-\frac{(2 - \alpha)^2}{4} j_{\nu(\alpha, \mu)}(k)^2 T} \right)^{1/2} \]
\[ \leq C e^\tau \left( \sum_{k=1}^{\infty} j_{\nu(\alpha, \mu)}(k)^2 e^{-\frac{(2 - \alpha)^2}{4} j_{\nu(\alpha, \mu)}(k)^2 T} \right)^{1/2}. \]
Next we use the following Lemma proved in [14]:

**Lemma 6.2.** There is some constant $C > 0$, independent of $\nu$ and of $Y$, such that:

$$\forall \nu \geq 0, \; \forall Y > 0, \; \sum_{k=1}^{\infty} \| u_k \|^2 e^{-\nu Y} \leq C \frac{1 + \nu^2}{Y^{3/2}} e^{-(1+\nu^2)Y}.$$  

Applying Lemma 6.2 with $Y = \frac{(2-\alpha)^2}{4}T$, it follows that

$$C_{etr-bd} \leq C e^{\frac{1 + \nu(\alpha, \mu)}{T}} e^{-\frac{1+\nu(\alpha, \mu)^2}{4}T} \leq C e^{\frac{1 + \nu(\alpha, \mu)}{T}} e^{-(1+\nu(\alpha, \mu)^2)T}.$$  

Next we use the following Lemma proved in [14]:

$$\forall k \geq 1, \; \Phi'_k(1) \int_0^T H_m(t) e^{\lambda_k t} dt = \rho_k^0 = \delta_{mk}.$$  

It follows that

$$\forall k \geq 1, \; \int_0^T \left( \Phi'_m(1) H_m(t) \right) e^{\lambda_k t} dt = \delta_{mk}. $$

In other words, the sequence $(\Phi'_m(1) H_m)_m$ is biorthogonal to the set $(e^{\lambda_k t})_{k \geq 1}.$

At this stage, we will distinguish the two following cases:

$$\nu(\alpha, \mu) \in \left[0, \frac{1}{2} \right] \quad \text{that is, } \mu \in \left[\frac{\alpha}{16} (3\alpha - 4), \mu(\alpha) \right]$$

and

$$\nu(\alpha, \mu) \in \left[\frac{1}{2}, +\infty \right) \quad \text{that is, } \mu \in \left(-\infty, \frac{\alpha}{16} (3\alpha - 4) \right].$$

**6.4. Lower bound of the cost of controllability.** Let us fix $m \geq 1$ and let us choose $u_0 = \Phi_m.$ Consider $H_m$ any control that drives the solution of (2.2) to zero in time $T$. Then (6.5) reads as

$$\forall k \geq 1, \; \Phi'_k(1) \int_0^T H_m(t) e^{\lambda_k t} dt = \rho_k^0 = \delta_{mk}. $$

It follows that

$$\forall k \geq 1, \; \int_0^T \left( \Phi'_m(1) H_m(t) \right) e^{\lambda_k t} dt = \delta_{mk}. $$

In this first case, we are going to use the following generalization of Guichal [26], proved in [12]:

**Theorem 6.3. ([12, Theorem 2.5]).** Assume that $\lambda_k \geq 0$ for all $k \geq 1$ and that there is some $0 < \gamma_{\min} \leq \gamma_{\max}$ such that

$$\forall k \geq 1, \; \gamma_{\min} \leq \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \leq \gamma_{\max}. \quad (6.13)$$

Then there exists $c_u > 0$ independent of $T$ and $m$ such that any family $(\sigma_m)_{m \geq 1}$ which is biorthogonal to the family $(e^{\lambda_k t})_{k \geq 1}$ in $L^2(0,T)$ satisfies

$$\| \sigma_m \|^2_{L^2(0,T)} \geq e^{-2\lambda_m T} e^{rac{1}{2\gamma_{\max} T}} b(T, \gamma_{\max}, m),$$

with

$$b(T, \gamma_{\max}, m) = \frac{c_u^2}{C(m, \gamma_{\max}, \lambda_1)^2 T} \left( \frac{1}{2\gamma_{\max} T} \right)^{2m} \frac{1}{(4\gamma_{\max}^2 T + 1)^2} \quad (6.14)$$

and

$$C(m, \gamma_{\max}, \lambda_1) = m!^{2m} \left[ \frac{2 \sqrt{\lambda_1}}{\gamma_{\max}} \right] + 1 \left( m + \left[ \frac{2 \sqrt{\lambda_1}}{\gamma_{\max}} \right] + 1 \right).$$
When $\nu(\alpha, \mu) \in [0, \frac{1}{2}]$, using (4.6), we see that assumption (6.13) is satisfied with 
\[ \gamma_{\min} := \frac{7\pi}{16}, \quad \text{and} \quad \gamma_{\max} := \pi. \]

So, using Theorem 6.3, we obtain that any family $(\sigma_m)_{m \geq 1}$ which is biorthogonal to the family $(e^{\lambda t})_{k \geq 1}$ in $L^2(0, T)$ satisfies:
\[ \|\sigma_m\|_{L^2(0, T)}^2 \geq e^{-2\lambda_m T} e^{\frac{1}{2\pi^2 T}} b(T, \gamma_{\max}, m), \]
where $b(T, \gamma_{\max}, m)$ is given in (6.14). Let us now apply this inequality for $m = 1$. It implies
\[ \|\sigma_1\|_{L^2(0, T)}^2 \geq e^{-2\lambda_1 T} e^{\frac{1}{2\pi^2 T}} b(T, \gamma_{\max}, 1). \quad (6.15) \]

Next, we observe that, for $\nu \in [0, \frac{1}{2}]$ and $n = 1$, (4.3) gives
\[ \frac{3\pi}{4} \leq \pi \left( \frac{3}{4} + \frac{\nu}{2} \right) \leq \pi \left( 1 + \frac{1}{4} \left( \nu - \frac{1}{2} \right) \right) \leq \pi. \]

It follows that
\[ \frac{9\pi^2}{64} \leq \left( \frac{2 - \alpha}{2} \right)^2 \left( \frac{3\pi}{4} \right)^2 \leq \lambda_1 \leq \left( \frac{2 - \alpha}{2} \right)^2 \pi^2 \leq \pi^2 \]

and
\[ \lambda_1 \leq \mathcal{C}(1 + \nu(\alpha, \mu))^2. \]

In particular, using $\lambda_1 \geq 9\pi^2 / 64$, we obtain that
\[ b(T, \gamma_{\max}, 1) \geq \frac{\mathcal{C}}{T^3(1 + T)^2}. \]

From (6.15), we deduce
\[ \|\sigma_1\|_{L^2(0, T)}^2 \geq \frac{\mathcal{C}}{T^3(1 + T)^2} e^{-2\lambda_1 T} e^{\frac{1}{2\pi^2 T}}. \]

Hence,
\[ \|\Phi'_1(1)H_1\|_{L^2(0, T)}^2 \geq \frac{\mathcal{C}}{T^3(1 + T)^2} e^{-2\lambda_1 T} e^{\frac{1}{2\pi^2 T}}. \]

From (6.12), one has
\[ |\Phi'_1(1)| = \left| \frac{(2 - \alpha)^{3/2} j_{\nu(\alpha, \mu), 1}}{2} \right| \leq \sqrt{2j_{\nu(\alpha, \mu), 1}} \leq \sqrt{2\pi}. \]

So we obtain that
\[ \mathcal{C}_{ctr-bd} \geq \frac{\mathcal{C}}{T^{3/2}(1 + T)} e^{-\lambda_1 T} e^{\frac{1}{T}}. \]

Finally, using the fact that $\lambda_1 \leq \mathcal{C}(1 + \nu(\alpha, \mu))^2$, we get
\[ \mathcal{C}_{ctr-bd} \geq \frac{\mathcal{C}}{T^{3/2}(1 + T)} e^{-\mathcal{C}(1 + \nu(\alpha, \mu))^2 T} e^{\frac{1}{T}} \geq C e^\frac{\mathcal{C}}{T^2} e^{-\mathcal{C}(1 + \nu(\alpha, \mu))^2 T}. \]

Since $2 - \alpha > 1$, we have $\nu(\alpha, \mu) \leq 2 \sqrt{\mu(\alpha) - \mu}$. Hence
\[ \mathcal{C}_{ctr-bd} \geq C e^\frac{\mathcal{C}}{T^2} \geq [1 + \sqrt{\mu(\alpha) - \mu}]^2 T, \]

which gives the result.
6.4.2. **Lower bound in the case** $\nu(\alpha, \mu) \in [1/2, +\infty)$. In this case, one still could apply Theorem 6.3. Indeed, assumption (6.13) is satisfied with

$$\gamma_{\text{min}} := \frac{\pi}{2}, \quad \gamma_{\text{max}} := \frac{2 - \alpha}{2} [j_{\nu(\alpha, \mu), 2} - j_{\nu(\alpha, \mu), 1}].$$

Nevertheless, since $\gamma_{\text{max}} \to +\infty$ as $\mu \to -\infty$ (as mentioned in (4.8)), this would not give the best possible result. On the other hand, from Lemma 4.3, one has

$$\forall k \geq N_*, \quad \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \leq \gamma_{\text{max}}^*$$

with

$$N_* := [\nu(\alpha, \mu)] + 1 \quad \text{and} \quad \gamma_{\text{max}}^* := 2\pi.$$

In that context, when there is a bad global upper gap $\gamma_{\text{max}}^*$, and a good (much smaller) asymptotic upper gap $\gamma_{\text{max}}^*$, it is interesting to use the following extension of Theorem 6.3:

**Theorem 6.4.** ([13, Theorem 2.2]) Assume that $\lambda_k \geq 0$ for all $k \geq 1$ and that there are $0 < \gamma_{\text{min}}^* \leq \gamma_{\text{max}}^* \leq \gamma_{\text{max}}$ such that

$$\forall k \geq 1, \quad \gamma_{\text{min}} \leq \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \leq \gamma_{\text{max}},$$

and

$$\forall k \geq N_*, \quad \sqrt{\lambda_{k+1}} - \sqrt{\lambda_k} \leq \gamma_{\text{max}}^*.$$  

Then any family $(\sigma_m)_{m \geq 1}$ which is biorthogonal to the family $(e^{\lambda_k t})_{k \geq 1}$ in $L^2(0, T)$ satisfies

$$\|\sigma_m\|^2_{L^2(0, T)} \geq e^{-2\lambda_m T} e^{T \gamma_{\text{max}}^*} b^*(T, \gamma_{\text{max}}^*, \gamma_{\text{max}}^*, N_*, \lambda_1, m)^2,$$

where $b^*$ is rational in $T$ (and explicitly given in [13, Lemma 4.4]).

We are here in position to apply Theorem 6.4. So we obtain that any family $(\sigma_m)_{m \geq 1}$ which is biorthogonal to the family $(e^{\lambda_k t})_{k \geq 1}$ in $L^2(0, T)$ satisfies

$$\|\sigma_m\|^2_{L^2(0, T)} \geq e^{-2\lambda_m T} e^{T \gamma_{\text{max}}^*} b^*(T, \gamma_{\text{max}}^*, \gamma_{\text{max}}^*, N_*, \lambda_1, m)^2,$$

where, when $m \leq N_*$, $b^*$ (see Lemma 4.4 in [13]) has the following explicit value of

$$b^*(T, \gamma_{\text{max}}^*, \gamma_{\text{max}}^*, N_*, \lambda_1, m) = c^* \frac{\sqrt{1 + T \lambda_1}}{\sqrt{T}} \frac{(T \gamma_{\text{max}}^*)^{K_s + K_s^* + 2} (T \gamma_{\text{max}}^*)^{K_s + K_s^* + 3}}{(1 + (T \gamma_{\text{max}}^*)^{N_* + K_s + K_s^* + 3})},$$

with

$$K_s = \left[\frac{2\sqrt{\lambda_1} + (N_* + m)\gamma_{\text{max}}}{\gamma_{\text{max}}^*}\gamma_{\text{max}}^*\right] - N_* + 2,$$

$$K_s^* = \left[\frac{\gamma_{\text{max}}^*(N_* - m)}{\gamma_{\text{max}}^*}\gamma_{\text{max}}^*\right] - N_* + 2,$$

$$c^* = \frac{1}{(N_* + K_s + K_s^* + 3)!} \frac{c_u(\gamma_{\text{max}}^*)^{2(N_* - 1)}}{c_{(+)}^* c_{(-)}^*},$$

and with

$$c_{(+)}^* = \left(\frac{\gamma_{\text{max}}^*}{\gamma_{\text{max}}^*}\right)^{N_* - 1} \left(\left[m + \frac{2\sqrt{\lambda_1}}{\gamma_{\text{max}}^*} + 1\right]! \left(\left[m + \frac{2\sqrt{\lambda_1} + (N_* + m)\gamma_{\text{max}}}{\gamma_{\text{max}}^*} + 1\right]! \left[2m + \frac{2\sqrt{\lambda_1}}{\gamma_{\text{max}}^*} + 1\right]!ight)ight),$$

and

$$c_{(-)}^* = \left(\frac{\gamma_{\text{max}}^*}{\gamma_{\text{max}}^*}\right)^{N_* - 1} \left(\left[m + \frac{2\sqrt{\lambda_1}}{\gamma_{\text{max}}^*} + 1\right]! \left(\left[m + \frac{2\sqrt{\lambda_1} + (N_* + m)\gamma_{\text{max}}}{\gamma_{\text{max}}^*} + 1\right]! \left[2m + \frac{2\sqrt{\lambda_1}}{\gamma_{\text{max}}^*} + 1\right]!\right)ight).$$
and
\[ \tau^{(-)} = \left( \frac{\gamma_{\text{max}}}{\gamma_{\text{max}}} \right)^{N_{\ast} - 1} \frac{(m - 1)! \left( N_{\ast} - m \right)!}{1 + \left[ \frac{\gamma_{\text{max}}}{\gamma_{\text{max}}} (N_{\ast} - m) \right]!} \cdot \]

In the above expressions, we take \( m = 1 \) and we only need to look at the behavior as \( \mu \to -\infty \) i.e. \( \nu(\alpha, \mu) \to +\infty \). This is possible to study (see [14]) and one obtains
\[ b^*(T, \gamma_{\text{max}}, \gamma^*, N_{\ast}, \lambda_1, 1) \geq e^{-\frac{C}{(1 + \nu(\alpha, \mu))^{4/3} (\ln \nu(\alpha, \mu) + \ln \frac{1}{T})}} \sqrt{1 + T} \sqrt{T}. \]

Consequently,
\[ \| \sigma_1 \|^2_{L^2(0, T)} \geq b(T, \alpha, \mu, 1)^2, \]
with
\[ b(T, \alpha, \mu, 1) := e^{-\lambda_1 T} e^{\frac{1}{4\pi T}} e^{-\frac{C}{(1 + \nu(\alpha, \mu))^{4/3} (\ln \nu(\alpha, \mu) + \ln \frac{1}{T})}} \sqrt{1 + T} \sqrt{T}. \]

Hence,
\[ \| \Phi_1(1) H_1 \|^2_{L^2(0, T)} \geq b(T, \alpha, \mu, 1). \]

This gives the following lower bound of the cost:
\[ C_{\text{ctr}} - bd \geq \frac{1}{|\Phi_1(1)|^2} b(T, \alpha, \mu, 1). \]

From (4.3), we have \( j\nu(\alpha, \mu), 1 \leq \mathcal{C}(1 + \nu(\alpha, \mu)) \). We deduce that \( \lambda_1 \leq \mathcal{C}(1 + \nu(\alpha, \mu))^2 \) and, using also (6.12), \( |\Phi_1(1)| \leq \mathcal{C}(1 + \nu(\alpha, \mu)) \). So, we get
\[ C_{\text{ctr}} - bd \geq \mathcal{C} e^{-\lambda_1 T} e^{\frac{1}{4\pi T}} e^{-\frac{C}{(1 + \nu(\alpha, \mu))^{4/3} (\ln \nu(\alpha, \mu) + \ln \frac{1}{T})}} \sqrt{1 + T} \sqrt{T} \]
\[ \geq \mathcal{C} e^{-\frac{C}{(1 + \nu(\alpha, \mu))^2 T}} e^{\frac{1}{4\pi T}} e^{-\frac{C}{(1 + \nu(\alpha, \mu))^{4/3} (\ln \nu(\alpha, \mu) + \ln \frac{1}{T})}} \sqrt{1 + T} \sqrt{T} \]
\[ \geq \mathcal{C} e^{-\mathcal{C} \left[ 1 + \sqrt{\nu(\alpha)} - \mu \right]^2 T} e^{-\mathcal{C} \left[ \nu(\alpha) - \mu \right]^{4/3} (\ln \sqrt{\nu(\alpha)} - \mu + \ln \frac{1}{T})}. \]

7. **Proof of Theorem 2.2.** This Section is devoted to the proof of Theorem 2.2 on the boundary controllability for (2.3). As for the case of a control acting at \( x = 1 \), the proof will be organized into the following steps:

- **Step 1.** The reduction to a moment problem.

- **Step 2.** Formal solution.

- **Step 3.** Existence and regularity of the control and upper bound of the cost.

- **Step 4.** Lower bound of the cost.

Moreover, in what follows we are not going to present all the details of our computations, since they are in many aspects similar to the ones in the previous sections.
7.1. Reduction to a moment problem. Once again, we expand the initial condition \( u_0 \in L^2(0,1) \) and the solution to (2.3) with respect to the basis of the eigenfunctions \( (\Phi_k)_{k \geq 1} \):

\[
u_0(x) = \sum_{k \geq 1} \rho_k \Phi_k(x), \quad u(x, t) = \sum_{k \geq 1} \beta_k(t) \Phi_k(x).
\]

Therefore, the controllability condition \( u(x, T) = 0 \) reads again as in (6.3). Besides, we notice again that the function \( v_k(x, t) := \Phi_k(x)e^{\lambda_k(t-T)} \) solves the adjoint problem (6.4). Hence, combining (2.3) and (6.4) we obtain

\[
0 = \int_Q \left[ v_k \left( u_t - (x^\alpha u_x)_x - \frac{\mu}{x^{2-\alpha}} u \right) + u \left( v_k, t + (x^\alpha v_{k,x})_x + \frac{\mu}{x^{2-\alpha}} v_k \right) \right] dx dt
\]

\[
= \int_0^1 v_k u \left[ T_0^T dx - \int_0^T x^\alpha u_{x} v_k \right] dt + \int_0^T x^\alpha v_{k,x} u \left. \right|_0^1 dt
\]

\[
= \int_0^1 v_k(x, T) u(x, T) dx - \int_0^1 v_k(x, 0) u_0(x) dx - \int_0^T H(t)(x^{\alpha+\gamma} v_{k,x})(0, t) dt
\]

\[
= \int_0^1 u(x, T) \Phi_k(x) dx - e^{-\lambda_k T} \int_0^1 u_0(x) \Phi_k(x) dx - e^{-\lambda_k T} r_k \int_0^T H(t)e^{\lambda_k t} dt
\]

\[
= \beta_k(T) - \rho_k e^{-\lambda_k T} - e^{-\lambda_k T} r_k \int_0^T H(t)e^{\lambda_k t} dt,
\]

where we have indicated

\[
r_k := \lim_{x \to 0^+} x^{\alpha+\gamma} \Phi_k'(x).
\]

Then, from the controllability condition (6.3) it follows that

\[
\forall k \geq 1, \quad -r_k \int_0^T H(t)e^{\lambda_k t} dt = \rho_k^0.
\]

On the other hand, since we are looking for a solution of the moment problem belonging to \( H^1(0, T) \), instead of (7.2) we would rather be interested in a condition involving the derivative of the function \( H \). This condition can be obtained once again integrating by parts as follows

\[
\int_0^T H(t)e^{\lambda_k t} dt = \frac{1}{\lambda_k} H(t)e^{\lambda_k t} \bigg|_0^T - \frac{1}{\lambda_k} \int_0^T H'(t)e^{\lambda_k t} dt.
\]

Therefore, the derivative \( H'(t) \) has to satisfy

\[
\forall k \geq 1, \quad \frac{r_k}{\lambda_k} \int_0^T H'(t)e^{\lambda_k t} dt = \rho_k^0 + \frac{r_k}{\lambda_k} \left( H(T)e^{\lambda_k T} - H(0) \right).
\]

Also in this case, we will provide a solution to the above problem which satisfies \( H(0) = H(T) = 0 \). To this end, we remark that the value of \( r_k \) can be computed explicitly, starting from (7.1) and using the definition of the Bessel function \( J_{\nu(\alpha, \mu)} \). In particular, one can readily verify that

\[
r_k = \frac{B_1(\alpha, \mu)}{\Gamma(1 + \nu(\alpha, \mu)) j_{\nu(\alpha, \mu)}},
\]

with

\[
B_1(\alpha, \mu) := \sqrt{\frac{2 - \alpha}{|j_{\nu(\alpha, \mu)}(j_{\nu(\alpha, \mu)}, k)|}} \left( \sqrt{\mu(\alpha)} + \sqrt{\mu(\alpha) - \mu} \right).
\]
7.2. Formal solution of the moment problem.

7.2.1. Formal definition of the control $H$. We present here the formal computations showing that the moment problem (7.3) has a solution $H$. To define this function $H$, we will employ again the biorthogonal a sequence $(\sigma_k)_{k \geq 1}$ that we introduced before, and whose existence is guaranteed by the gap conditions (4.4) and (4.5) and by Theorem 6.1. Now, let us define the function $H$ as follows:

$$H(t) := \int_0^t K(s) \, ds, \quad \text{with} \quad K(t) := \sum_{k \geq 1} \frac{\lambda_k}{r_k} \sigma_k(t). \quad (7.5)$$

It is straightforward that, if $K \in L^2(0,T)$, then we have $H \in H^1(0,T)$ with $H'(t) = K(t)$ and $H(0) = 0 = H(T)$. Moreover,

$$\frac{r_k}{\lambda_k} \int_0^T H'(t)e^{\lambda_k t} \, dt = \frac{r_k}{\lambda_k} \int_0^T K(t)e^{\lambda_k t} \, dt = \frac{r_k}{\lambda_k} \int_0^T \left( \sum_{\ell \geq 1} \frac{\lambda_\ell}{r_\ell} \sigma_\ell(t) \right) e^{\lambda_k t} \, dt$$

$$= \frac{r_k}{\lambda_k} \sum_{\ell \geq 1} \frac{\lambda_\ell}{r_\ell} \int_0^T \sigma_\ell(t)e^{\lambda_k t} \, dt = \frac{r_k}{\lambda_k} \sum_{\ell \geq 1} \frac{\lambda_\ell}{r_\ell} \rho_0 \delta_{k,\ell} = \rho_k,$$

and the moment problem (7.3) is formally satisfied.

7.2.2. If regular, the control $H$ drives the solution from $u_0$ to zero. We show here that the control $H$ that we introduced in (7.5) is able to drive the solution to (2.3) from the initial state $u_0$ to zero in time $T$. To this end, let us remind the change of variables

$$v(x,t) := u(x,t) - x^\gamma \frac{p(x)}{p(0)} H(t), \quad p(x) := 1 - x^q, \quad q = 2\sqrt{\mu(\alpha) - \mu},$$

that transforms our original equation (2.3) into

$$\begin{cases}
    v_t - (x^\alpha v_x)_x - \frac{\mu}{x^{2-\alpha}} v = -x^\gamma \frac{p(x)}{p(0)} K(t), & (x,t) \in Q \\
    v(0,t) = v(1,t) = 0, & t \in (0,T) \\
    v(x,0) = u_0(x), & x \in (0,1).
\end{cases}$$

Now, for a fixed $\varepsilon > 0$ we have

$$\int_\varepsilon^T \int_0^1 -x^\gamma \frac{p(x)}{p(0)} K(t) \Phi_k(x)e^{\lambda_k t} \, dx \, dt$$

$$= \int_\varepsilon^T \int_0^1 \left( v_t - (x^\alpha v_x)_x - \frac{\mu}{x^{2-\alpha}} v \right) \Phi_k(x)e^{\lambda_k t} \, dx \, dt$$

$$= \int_0^1 v \Phi_k e^{\lambda_k t} \int_\varepsilon^T \int_0^1 \left( -(x^\alpha v')' - \frac{\mu}{x^{2-\alpha}} \Phi_k - \lambda_k \Phi_k \right) e^{\lambda_k t} \, dx \, dt$$

$$= e^{\lambda_k T} \int_0^1 v(x,T) \Phi_k(x) \, dx - e^{\lambda_k \varepsilon} \int_0^1 v(x,\varepsilon) \Phi_k(x) \, dx.$$  

Hence, taking the limit for $\varepsilon \to 0^+$ we find

$$\int_Q -x^\gamma \frac{p(x)}{p(0)} K(t) \Phi_k(x)e^{\lambda_k t} \, dx \, dt = e^{\lambda_k T} \int_0^1 v(x,T) \Phi_k(x) \, dx - \rho_k^0.$$
Hence, we get since we already noticed that (see (3.6))

\[ e^{\lambda_k T} \int_0^1 v(x, T) \Phi_k(x) \, dx = \rho_k^0 + \left( \int_0^T K(t) e^{\lambda_k t} \, dt \right) \left( \int_0^1 -x^\gamma \frac{p(x)}{p(0)} \Phi_k(x) \, dx \right) \]

\[ = \rho_k^0 + \frac{\lambda_k}{r_k} \rho_k \int_0^1 -x^\gamma \frac{p(x)}{p(0)} \Phi_k(x) \, dx. \]

Moreover,

\[ \int_0^1 -x^\gamma \frac{p(x)}{p(0)} \Phi_k(x) \, dx \]

\[ = \frac{1}{\lambda_k} \int_0^1 -x^\gamma \frac{p(x)}{p(0)} \lambda_k \Phi_k(x) \, dx = \frac{1}{\lambda_k} \int_0^1 x^\gamma \frac{p(x)}{p(0)} \left( (x^\alpha \Phi_k'(x))' + \frac{\mu}{x^{2-\alpha}} \Phi_k(x) \right) \, dx \]

\[ = \frac{1}{\lambda_k} \frac{p(x)}{p(0)} x^{\alpha + \gamma} \Phi_k(x) \bigg|_0^1 - \frac{1}{\lambda_k} \int_0^1 \left( x^\gamma \frac{p(x)}{p(0)} \right)' x^\alpha \Phi_k(x) \, dx \]

\[ + \frac{1}{\lambda_k} \int_0^1 \left[ \left( x^\alpha (x^\gamma p(x))' \right)' + \mu x^{\alpha + \gamma - 2} \frac{p(x)}{p(0)} \right] \Phi_k(x) \, dx \]

\[ = - \frac{r_k}{\lambda_k} + \frac{1}{\lambda_k p(0)} \int_0^1 \left[ \left( x^\alpha (x^\gamma p(x))' \right)' + \mu x^{\alpha + \gamma - 2} \frac{p(x)}{p(0)} \right] \Phi_k(x) \, dx = - \frac{r_k}{\lambda_k}, \]

since we already noticed that (see (3.6))

\[ [x^\alpha (x^\gamma p')']'(x) + \frac{\mu}{x^{2-\alpha-\gamma}} p(x) = 0. \]

Hence, we get

\[ e^{\lambda_k T} \int_0^1 v(x, T) \Phi_k(x) \, dx = 0, \]

which of course implies \( v(x, T) = 0 \) and, since \( H(T) = 0 \), we can finally conclude that

\[ u(x, T) = v(x, T) + x^\gamma \frac{p(x)}{p(0)} H(T) = 0. \]

7.3. Existence of the control, \( H^1 \) regularity and upper bound of the cost of controllability. We have to check that the control \( H \) defined as in (6.8) belongs to \( H^1(0, T) \) and to obtain the upper bound for the controllability cost. To this end, as we did before, we are going to prove instead that the function \( K \) belongs to \( L^2(0, T) \).

In what follows, \( C_u \) denotes again a universal constant, independent of \( T, \gamma_{\text{max}}, \gamma_{\text{min}} \), and \( k \), which may change value even from line to line. From (7.5) we have

\[ \| K \|_{L^2(0, T)} = \left\| \sum_{k \geq 1} \frac{\lambda_k}{r_k} \rho_k^0 \sigma_k(t) \right\|_{L^2(0, T)} \leq \sum_{k \geq 1} |\rho_k^0| \frac{\lambda_k}{r_k} \| \sigma_k(t) \|_{L^2(0, T)}. \]
Moreover, employing the expression (4.2) of \( \lambda_k \) and the explicit expression of \( r_k \) given in (7.4) we obtain

\[
\left| \frac{\lambda_k}{r_k} \right| = \frac{(2 - \alpha)^{\frac{3}{2}} \Gamma(1 + \nu(\alpha, \mu)) |J'_\nu(\alpha, \mu)| |j_{\nu(\alpha, \mu)}|}{4 \left( \sqrt{\mu(\alpha)} + \sqrt{\mu(\alpha) - \mu} \right)} j_{\nu(\alpha, \mu)},
\]

\[
\leq c_u \frac{\Gamma(1 + \nu(\alpha, \mu))}{\sqrt{\mu(\alpha)} + \sqrt{\mu(\alpha) - \mu}} \mu(\alpha, \mu),
\]

since \( 0 \leq \alpha < 1 \) and \( |J'_\nu(\alpha, \mu)| |j_{\nu(\alpha, \mu)}| \leq 1 \) (see [12, Formula 79]). Therefore, we get

\[
\|K\|_{L^2(0,T)} \leq c_u \frac{\Gamma(1 + \nu(\alpha, \mu))}{\sqrt{\mu(\alpha)} + \sqrt{\mu(\alpha) - \mu}} \|u_0\|_{L^2(0,1)} \left( \sum_{k \geq 1} \mu(\alpha, \mu, k) \|\sigma_k(t)\|_{L^2(0,T)}^2 \right)^{\frac{1}{2}}.
\]

From here, proceeding as in Section 6.3.3, we can immediately conclude that \( K \in L^2(0,T) \) and we have the following estimate

\[
c_{ctr-bd} \leq c_u \frac{\Gamma(1 + \nu(\alpha, \mu))}{\sqrt{\mu(\alpha)} + \sqrt{\mu(\alpha) - \mu}} e^{\frac{c_u}{\nu(\alpha, \mu)}} \left( 1 + \sqrt{\mu(\alpha) - \mu} \right) e^{-c_u \left( 1 + \sqrt{\mu(\alpha) - \mu} \right)^2 T}.
\]

### 7.4. Lower bound of the cost of controllability

Fix \( m \geq 1 \) and consider the initial condition \( u_0 = \Phi_m \) in (2.2). Let \( H_m \) be any control that drives the solution of the equation to zero in time \( T \). Then, the moment condition (7.2) yields

\[
-r_k \int_0^T H_m(t)e^{\lambda_k t} dt = \rho_k = \int_0^1 u_0(x)\Phi_k(x) dx = \delta_{k,m}.
\]

Hence,

\[
\forall k \geq 1, \quad \int_0^T \left( r_m H_m(t) \right) e^{\lambda_k t} dt = \frac{\delta_{k,m}}{r_k} = \begin{cases} 1, & \text{if } k = m \\ 0, & \text{if } k \neq m. \end{cases}
\]

This means that the sequence \( (r_m H_m(t))_{t \geq 1} \) is biorthogonal to \( (e^{\lambda_k t})_{k \geq 1} \) in \( L^2(0,T) \). Now, as we did before, we choose \( m = 1 \) and we distinguish between the two cases

\[
\nu(\alpha, \mu) \in \left[ 0, \frac{1}{2} \right] \quad \text{and} \quad \nu(\alpha, \mu) \in \left[ \frac{1}{2}, +\infty \right).
\]

In the former one, employing (6.15) we have

\[
\|r_1 H_1(t)\|_{L^2(0,T)}^2 \geq e^{-2\lambda_1 T} e^{-\frac{1}{\gamma_{\max}}} b(T, \gamma_{\max}, 1),
\]

which yields

\[
\|H_1(t)\|_{L^2(0,T)}^2 \geq \frac{1}{|r_1|} e^{-2\lambda_1 T} e^{-\frac{1}{\gamma_{\max}}} b(T, \gamma_{\max}, 1).
\]

Now, thanks to (7.4) we obtain

\[
\frac{1}{|r_1|} = \frac{\Gamma(1 + \nu(\alpha, \mu)) |J'_\nu(\alpha, \mu)| |j_{\nu(\alpha, \mu)}|}{\sqrt{\mu(\alpha)} + \sqrt{\mu(\alpha) - \mu}} \frac{\sqrt{2 - \alpha}}{\sqrt{\mu(\alpha)}} j_{\nu(\alpha, \mu, 1)}.
\]
Moreover, since $0 \leq \alpha < 1$, employing (4.3) and the fact that $|J'_p(\nu(\alpha, 1))| \geq C$ with $C$ independent of $\mu$ (see [12, Corollary 2]), we also have

$$\frac{1}{|r_1|} \geq \frac{C_u}{\sqrt{\mu(\alpha)} + \sqrt{\mu(\alpha) - \mu}}$$

which yields

$$\|H_1(t)\|_{L^2(0, T)} \geq \frac{C_u}{\sqrt{\mu(\alpha)} + \sqrt{\mu(\alpha) - \mu}} e^{-2\lambda t} e^{\frac{1}{2}a_0 T} b(T, \gamma_{\max}, 1).$$

Proceeding now as in the proof of Theorem 2.1 it is easy to obtain our final estimate

$$C_{ctr-bd} \geq C_u \frac{1}{\sqrt{\mu(\alpha)} + \sqrt{\mu(\alpha) - \mu}} T e^{-C_u (1-\alpha)^2 T e^{-\frac{1}{4}T}}.$$

When $\nu(\alpha, \mu) \geq \frac{1}{2}$, instead, the lower bound reads as follows:

$$C_{ctr-bd} \geq C_u e^T e^{-C [1 + \sqrt{\mu(\alpha) - \mu}]^\frac{1}{2} T e^{-C (\sqrt{\mu(\alpha) - \mu})^{4/3} (\ln[\sqrt{\mu(\alpha) - \mu}] + \ln \frac{1}{T})}} \frac{1}{\sqrt{\mu(\alpha)} + \sqrt{\mu(\alpha) - \mu}}.$$

The proof of this fact is totally analogous to what we already did in the proof of Theorem 2.1 and we leave it to the reader.

8. Final comments and open questions. In this paper, we have analyzed the controllability properties of a degenerate/singular parabolic equation on the space interval $(0, 1)$. We have considered the two different situations of a boundary control acting at $x = 1$ or $x = 0$ (where the degeneracy/singularity occurs). In both cases, by means of the classical moment method, we have shown that the equation is null-controllable and we provided suitable estimates for the controllability cost.

We present hereafter a non-exhaustive list of comments and open problems related to our work.

1. As a first thing, we recall that in the present work we are not treating the strongly degenerate case $1 \leq \alpha < 2$.
   - When the control acts at $x = 1$, we expect that null controllability results can be obtained by combining our proofs with the ideas of [14]. Notwithstanding that, in order to keep the present paper of a reasonable length, we decided not to cover this case here.
   - When the control acts at $x = 0$, instead, this is an open question even in the purely degenerate case $\mu = 0$ ([14] deals only with controls in $x = 1$). Indeed, in this case one encounters difficulty already at the level of the well-posedness of the equation, due to the need to find a suitable boundary condition.

2. A second open problem is related to the obtaining of suitable Carleman estimates for boundary controllability. This is not an easy task. Indeed, the usual weights introduced in previous works ([7, 10, 11, 15, 19, 36, 43, 44]) for proving interior controllability are designed in such a way that all the boundary terms are greater or equal to zero, and can therefore be ignored. On the other hand, adapting these weights in order to keep the boundary terms and still be able to prove the Carleman is a quite cumbersome issue. Nevertheless, the interest in obtaining, if possible, a Carleman estimate for boundary controllability remains, and it is related to various further applications:
   - the treatment of equations with a nonlinear term;
• the possibility of considering general function \( a(x) \) (such as in [36]) instead of \( x^\alpha \) in the purely degenerate case (and even with a double degeneracy both at \( x = 0 \) and \( x = 1 \));
• the possibility of studying problems for a purely singular operator with two singular points at \( x = 0 \) and \( x = 1 \);
• the case of a degenerate/singular operator with \( \mu/x^{\beta} \) with \( \beta \leq 2 - \alpha \) (instead of \( \mu/x^{2-\alpha} \)). In this case (analyzed in [43] only limited to a locally distributed control), null controllability should be true for any \( \mu \) but it cannot be studied with the present method.

3. Finally, let us mention that, with similar techniques, one could study also the effect of a locally distributed control applied on the operator \( P_{\alpha,\mu} \). We did not consider this situation here in order to keep the present paper of a reasonable length. We chose to concentrate on the case of boundary controls which is more complex. We refer the interested reader to [14], where the study of the cost of boundary controls and also locally distributed controls have been made in the case of the operator \( P_{\alpha,0} \) (that is the purely degenerate case).

Acknowledgments. The authors wish to thank the referees for their valuable comments that helped to improve this manuscript.

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Received January 2020; revised December 2020.

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