Exponential convergence of a dissipative quantum system towards finite-energy grid states of an oscillator

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Abstract

Based on the stabilizer formalism underlying Quantum Error Correction (QEC), the design of an original Lindblad master equation for the density operator of a quantum harmonic oscillator is proposed. This Lindblad dynamics stabilizes exactly the finite-energy grid states introduced in 2001 by Gottesman, Kitaev and Preskill for quantum computation. Stabilization results from an exponential Lyapunov function with an explicit lower-bound on the convergence rate. Numerical simulations indicate the potential interest of such autonomous QEC in presence of non-negligible photon-losses.

1. Introduction

Quantum Error Correction (QEC) represents a much sought-after target in the road towards large-scale quantum computations. Indeed, decoherence affecting early quantum computing platforms limits their ability to carry out interesting computations. However, the threshold theorem [1] states that the use of quantum error correcting codes could allow for arbitrarily long reliable quantum computations, provided the noise levels affecting the hardware could be kept below a threshold depending on the considered code. A major issue for QEC is the huge resource overhead associated with the use of error correcting codes [2] and recent years have seen a growing number of encoding proposals aim at reducing this overhead, such as the so-called cat code [3, 4], binomial code [5] or GKP code [6]. In particular, recent experiments in superconducting circuits [7] and trapped ions [8] demonstrated the generation and stabilization of the finite-energy grid states underlying the GKP encoding, sparking a renewed interest for its use for quantum computation (see e.g. recent reviews [9, 10]). From a control theoretical perspective, QEC is a feedback-loop. Usual QEC is a discrete-time process based on a static output-feedback where the measured error syndrome (a classical output signal) indicates which correcting unitary transformation has to be applied via a specific short time-pulse on the classical control-input signal; in that case the controller is a classical system. On the other hand, autonomous QEC or reservoir engineering QEC is a continuous-time process where the controller is a dissipative quantum system coupled to the system storing quantum information. The idea of exploiting quantum dissipation goes back to optical pumping [11]. In [12] the potential interest of such dissipation engineering is highlighted for quantum state preparation and computation.

Continuous-time stabilization through dissipation engineering has been experimentally demonstrated for cat codes (see e.g. [13, 14, 15]) and theoretically contemplated for GKP states in [16] where numerical simulations based on a Lindblad master equation with two dissipation operators indicate the potential interest of this approach; however, the authors did not investigate convergence rates of the proposed dynamics or energy boundedness along trajectories. Here we go further and propose a set of four dissipation Lindblad operators exponentially stabilizing finite-energy GKP states introduced in [6]. In section 2, we develop, for a square lattice, a heuristic method to design Lindblad dynamics (3) with an explicit lower bound on the convergence rate. Section 4, devoted to numerical simulations with...
2. Lindblad dissipators derived from infinite-energy stabilizer generators

Set \( \eta = 2\sqrt{\pi} \) and consider the Hermitian phase-space operators of a quantum harmonic oscillator \( Q \) and \( P \) satisfying \( [Q, P] = i \). By Glauber identity \( e^{\pm iQ} \) and \( e^{\pm iP} \) commute. The four commuting operators \( e^{iQ}, e^{-iQ}, e^{iP}, \) and \( e^{-iP} \) are called the infinite-energy GKP stabilizers and their common eigenspace associated to the eigenvalue \(+1\) is called the infinite-energy GKP codespace. These four stabilizer operators are the independent generators of the stabilizer group \( \{e^{iQ}e^{iP} | (n, m) \in \mathbb{Z}^2 \} \).

In the \( q \)-representation, \( P \equiv -i \frac{d}{dq} \) hence \( e^{\pm iP} \equiv e^{\pm \frac{\pi}{2} \eta} \) corresponds to a constant shift of \( \pm \eta \) on \( q \). Thus, \( e^{iQ}e^{iP} \) applied on the wave function \( |\psi\rangle \equiv (\psi(q))_{q \in \mathbb{R}} \) reads
\[
e^{iQ}e^{iP} |\psi\rangle \equiv \left( e^{i\eta |\eta\rangle |q + m\eta\rangle} \right)_{q \in \mathbb{R}}. \tag{1}
\]

Solving for the \( +1 \)-eigenstates of (1), we find that the infinite-energy GKP codespace is of dimension 2 and spanned by two Dirac combs, the even comb \( \sum_{k \in \mathbb{Z}} \delta(q - 2k) \) and the odd comb \( \sum_{k \in \mathbb{Z}} \delta(q - (2k + 1)) \) located at odd multiples of \( \frac{2\pi}{\eta} \) \((\delta \) stands for the Dirac distribution).

In 6, section VI [see also 10 for a recent exposure], consider \( E_{\text{even}} = e^{-\frac{1}{2}(Q^2 + P^2)} \), a regularizing Hermitian operator with \( 0 < \varepsilon \ll 1 \). In the \( q \)-representation \( E_{\text{even}} \) corresponds to the convolution with the Mehler kernel
\[
K(q, q', \varepsilon) = \frac{\exp(-\eta q' q/\varepsilon)}{\sqrt{2\pi \sinh 2\varepsilon}} \exp \left( -\frac{(q - q')^2}{2 \tanh 2\varepsilon} \right).
\]

Thus, \( E_{\text{even}} |\psi\rangle \) reads \( \int_{\mathbb{R}} K(q, q', \varepsilon) |\psi(q')\rangle dq' \) applied to the even and odd combs yields the following coherent superpositions of Gaussian squeezed states of finite-energy (average photon-number around \( 1/\langle 2\varepsilon \rangle \)):

\[
|\text{even}_\varepsilon\rangle = \left( \sum_{k \in \mathbb{Z}} \frac{\sinh(2k)}{\sqrt{2\pi \sinh 2\varepsilon}} e^{-i\eta \varepsilon k \sinh(2k)/\sqrt{2\pi \sinh 2\varepsilon}} \right)_{q \in \mathbb{R}} e^{\varepsilon \eta^2/2}.
\]

\[
|\text{odd}_\varepsilon\rangle = \left( \sum_{k \in \mathbb{Z}} \frac{\sinh(2k+1)}{\sqrt{2\pi \sinh 2\varepsilon}} e^{-i\eta \varepsilon (2k+1) \sinh(2k+1)/\sqrt{2\pi \sinh 2\varepsilon}} \right)_{q \in \mathbb{R}} e^{\varepsilon \eta^2/2}.
\]

With \( 0 < \varepsilon \ll 1 \) these two finite-energy and smooth quantum states approximate generators of the infinite-energy GKP codespace. We introduce an orthonormal basis of their span, defined by \( |0_\varepsilon\rangle \propto |\text{even}_\varepsilon\rangle \) and \( |1_\varepsilon\rangle \propto |\text{odd}_\varepsilon\rangle - \tfrac{|\text{even}_\varepsilon\rangle}{|\text{even}_\varepsilon\rangle} \langle \text{even}_\varepsilon | \langle \text{even}_\varepsilon | \). By construction, \( |0_\varepsilon\rangle \) and \( |1_\varepsilon\rangle \) belong to the kernel of the following four non-Hermitian operators derived from the infinite-energy stabilizer operators:

\[
V_1 = E_{\text{even}} e^{iQ} E^{-1}_{\varepsilon} - I, \quad V_2 = E_{\text{even}} e^{-iQ} E^{-1}_{\varepsilon} - I, \\
V_3 = E_{\text{even}} e^{iP} E^{-1}_{\varepsilon} - I, \quad V_4 = E_{\text{even}} e^{-iP} E^{-1}_{\varepsilon} - I.
\]

Using

\[
E_{\text{even}} Q E^{-1}_{\varepsilon} = \cosh(\varepsilon) Q + i \sinh(\varepsilon) P \equiv R \\
E_{\text{even}} P E^{-1}_{\varepsilon} = -i \sinh(\varepsilon) Q + \cosh(\varepsilon) P \equiv S
\]

these four operators read

\[
V_1 = e^{i\eta R} - I, \quad V_2 = e^{-i\eta S} - I, \\
V_3 = e^{-i\eta R} - I, \quad V_4 = e^{-i\eta S} - I. \tag{2}
\]

Since \( |R, S\rangle = i \) and \( \eta^2 = 4\pi \), for any \( k, \ell \), \( V_k \) and \( V_\ell \) commute. Then any density operator \( \rho \) having his support in span\( \{|0_\varepsilon\rangle, |1_\varepsilon\rangle\} \) is a steady state of the following Lindblad master equation:

\[
\frac{d}{dt} \rho = \sum_{k=1}^{4} \mathcal{D}_{V_k} (\rho) \equiv \mathcal{L}_\varepsilon (\rho) \tag{3}
\]

where \( \mathcal{D}_V (\rho) \equiv V \rho V^\dagger - \frac{1}{2} (V^\dagger V \rho + \rho V^\dagger V) / 2 \). Next section provides a first formal analysis ensuring the exponential convergence of the above dynamical system towards the finite-energy GKP codespace, i.e., towards the set of density operators \( \rho \) with range in span\( \{|0_\varepsilon\rangle, |1_\varepsilon\rangle\} \).

3. Exponential convergence

The rigorous functional analysis framework is not addressed here: calculations are led as if the dimension of the underlying Hilbert space were finite. The \textit{a priori} estimate we obtain constitutes a first step towards a fully rigorous mathematical analysis that we plan to develop in future publications.

Theorem 1. Consider \( W = \sum_{k=1}^{4} V_k \) where the \( V_k \) are given by (2). Then for any time-varying density operator \( \rho(t) \) satisfying (3), we have, for \( \eta = 2\sqrt{\pi} \) and \( \varepsilon \in (0, \frac{1}{2\eta}] \),

\[
\frac{d}{dt} \text{Tr}(W \rho(t)) \leq -\kappa(\varepsilon, \eta) \text{Tr}(W \rho(t))
\]

with \( \kappa(\varepsilon, \eta) > 0 \) given by

\[
\kappa(\varepsilon, \eta) = (\sinh(\eta^2 s) - \sin(\eta^2 c))(1 - e^{-3\eta^2 s/2}) - (\cosh(\eta^2 s) - \cos(\eta^2 c))(1 + e^{-3\eta^2 s/2}) \tag{4}
\]

where \( s = \sinh(2\varepsilon) \) and \( c = \cosh(2\varepsilon) \). For \( 0 < \varepsilon \ll 1 \) and \( \eta = 2\sqrt{\pi} \), we have \( \kappa(\varepsilon, \eta) = 2\eta^2 \varepsilon^2 + O(\varepsilon^3) \).
The detailed proof is quite technical. It is given in appendix A and relies on Glauber identity. This theorem implies that, for any initial density operator \( \rho(0) \), \( 0 \leq \text{Tr}(W\rho(t)) \leq \text{Tr}(W\rho(0))e^{-\kappa_{1}\varepsilon t} \). Thus, \( \lim_{t \to +\infty} \text{Tr}(W\rho(t)) = 0 \). Since, for all \( t \geq 0 \), \( \rho(t) \geq 0 \) and \( W \geq 0 \), the support of \( \rho(t) \) converges to \( \ker W = \text{span}([0_e], [1_e]) \), the finite-energy GKP codespace. Since any operator with support in \( \ker W \) belongs to \( \ker \mathcal{L}_\varepsilon \), \( \rho(t) \) exponentially converges to a steady state of (3).

Moreover, \( \ker \mathcal{L}_\varepsilon \) coincides with operators having their support in \( \ker W \). Thus, \( \ker \mathcal{L}_\varepsilon \) is of real dimension 4, spanned by density operators with support on \( \text{span}([0_e], [1_e]) \).

**Remark 1.** The a priori estimate of theorem 1 is also valid when \( \eta = \sqrt{2\pi} \). Then \( \ker W \) is spanned by a single wave function corresponding to the regularization of the Dirac comb \( \sum_{k \in \mathbb{Z}} \delta(q-k\sqrt{2\pi}) \) and colinear to \( \sum_{k \in \mathbb{Z}} e^{2\pi i k y} \sqrt{2\pi \sinh 2\pi} e^{-\frac{q-k\sqrt{2\pi}}{2\pi \sinh 2\pi}^2} \).

Such grid states are certainly to be considered as interesting resources in metrology to measure simultaneously the commuting modular observables derived from \( e^{\pm m Q} \) and \( e^{\pm m P} \) and thus to avoid the Heisenberg uncertainty principle attached to measurements of \( Q \) and \( P \) (see [17, chapter V, section 4]).

Consider

\[
S_0 = |0_e\rangle \langle 0_e| + |1_e\rangle \langle 1_e|, \quad S_x = |1_e\rangle \langle 0_e| + |0_e\rangle \langle 1_e|, \quad S_y = i|1_e\rangle \langle 0_e| - i|0_e\rangle \langle 1_e|, \quad S_z = |0_e\rangle \langle 0_e| - |1_e\rangle \langle 1_e|.
\]

Since \( \ker \mathcal{L}_\varepsilon \) is of real dimension 4, the kernel \( \ker \mathcal{L}_\varepsilon^m \) of its adjoint \( \mathcal{L}_\varepsilon^m \) for the Frobenius product is also of dimension 4. It is spanned by four independent Hermitian invariant operators, \( I \) (conservation of the trace) and

\[
J_\xi = \lim_{t \to +\infty} e^{J_\xi t}(S_\xi), \quad \xi = x, y, z.
\]

Since the spectra of \( S_x \), \( S_y \), and \( S_z \) are \( \{-1, 0, 1\} \), the spectra of \( J_x \), \( J_y \), and \( J_z \) are inside \([-1, 1]\) [18]. Then for any operator \( \rho \), we have (see e.g. [19]):

\[
\lim_{t \to +\infty} e^{J_\xi t}(\rho) = \frac{S_0 + \text{Tr}(J_x \rho) S_x + \text{Tr}(J_y \rho) S_y + \text{Tr}(J_z \rho) S_z}{2}.
\]

The quantities \( \text{Tr}(J_x \rho) \), \( \text{Tr}(J_y \rho) \), \( \text{Tr}(J_z \rho) \) can be seen as the Bloch coordinates of a logical qubit encoded in the density operator \( \rho \), as they always satisfy \( (\text{Tr}(J_x \rho))^2 + (\text{Tr}(J_y \rho))^2 + (\text{Tr}(J_z \rho))^2 \leq 1 \).

**4. Simulations with photon-loss errors**

The above formulae are used in our simulations to compute numerically \( S_\xi \) and \( J_\xi \) just by numerical time.
integration of (3) and of its adjoint. A Galerkin approximation is used with Fock subspace $|n⟩$ $0≤n≤n^*$ where $|n⟩$ is the state with $n$ photons [20]. Since the average number of photons on the finite-energy GKP codospace is around $1/(2ε)$, $n^*$ has to be much larger than $1/ε$. We have observed numerically that taking $n^*$ around $20/ε$ is enough since higher values do not change the results. On figure 1, we have performed simulations for $ε = 1/10$, $1/20$ and $1/30$. All simulations start with $ρ_0 = |0⟩⟨0|$ on the finite-energy GKP codospace, i.e. with logical coordinates $Tr(J_xρ_0) = Tr(J_yρ_0) = 0$ and $Tr(J_zρ_0) = 1$. All simulations include photon-loss errors at a rate $κ_1 = ε/5$ scaled as $10%$ of the inverse of the average number of photon in $|0⟩$ and $|1⟩$. The Lindblad master equations numerically solved are of two kinds:

$$\frac{d}{dt}ρ = \begin{cases} \mathcal{L}_ε(ρ) + \frac{ε}{5} \mathcal{D}_a(ρ), & \text{curve label } "on"; \\ \frac{ε}{5} \mathcal{D}_a(ρ), & \text{curve label } "off", \end{cases}$$

where $a = (Q+ip)/\sqrt{2}$ is the annihilation operator. We observe a strong suppression of errors in presence of the engineered dissipation $\mathcal{L}_ε$. Other simulations not presented here with local phase-space operators (polynomial of low-order in $P$ and $Q$) such as $\mathcal{D}_a′$, $\mathcal{D}_Q$ and $\mathcal{D}_P$ instead of $\mathcal{D}_a$, exhibit a similar strong decrease of the decoherence rate when $ε$ is decreased.

5. Concluding remarks

The guarantee of exponential stability provided by theorem 1, combined with the numerically observed efficient protection against local errors in phase space, motivates the following issue: how to physically implement the autonomous stabilization scheme attached to the Lindblad master equation (3)? Quantum superconducting circuits [7] and trapped ions [8] appear as promising platforms for this task.

The strong impact of $ε$ close to $0^+$ on the decoherence rates is an indication of some exponential behaviour in the protection against local errors. This point will be investigated in future works.

Notice the analogy between the Lyapunov function $W$ in theorem 1 and the Lyapunov function $(a′−αT)^T(a′−αT)$ introduced in [21] for the Lindblad master equation corresponding to multi-photon pumping and cat-qubits [3]: $\frac{d}{dt}ρ = \mathcal{D}_{a′−αT}(ρ)$ with $r ∈ N^*$ and $α ∈ C$. As already done in [21] for cat-qubits, we expect to provide in forthcoming publications a fully rigorous and functional analysis proof of well-posedness and exponential convergence of the infinite-dimensional initial-value problem (3).

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A. Proof of theorem 1

Elementary numerical computations combined with the asymptotics $2n^4\epsilon^2$ around $0^+$ ensure that $κ > 0$ when $η = 2\sqrt{κ}$ and $ε \in (0, 1/2]$. 

Formally $\frac{d}{dt} \text{Tr}(Wp(t)) = \text{Tr} (p(t) \sum_{k=1}^4 R^*_k(W))$ where the adjoints $R^*_k$ of the super operators $R_k$ are given by

$$R^*_k(W) \triangleq V^*_kWW_k - (V^*_k V_k W + W V^*_k V_k)/2$$

$$\equiv (V^*_k[W, V_k] + [V^*_k, W] V_k)/2.$$

Since for any $k$ and $\ell$, $[V_\ell, V_k] = 0$, we have $[V^*_k V_\ell, V_k] = [V^*_k, V_k] V_\ell$. Thus

$$\frac{d}{dt} W \triangleq \sum_{k=1}^4 R^*_k(W) = \sum_{k,\ell} V^*_k [V^*_k, V_\ell] V_\ell.$$

Let us introduce the notation

$$R_1 = R, R_2 = S, R_3 = -R, R_4 = -S$$

such that

$$V_k = e^{i\eta R_k} - I.$$

Exploiting Glauber identity, we get

$$e^{i\eta A}e^{i\eta B} = e^{-\eta^2/2}[A, B]e^{i\eta(A+B)}$$

for operators $A, B$ such that $[A, [A, B]] = [B, [A, B]] = 0$, from which

$$[e^{i\eta A}, e^{i\eta B}] = e^{i\eta B} (e^{-\eta^2/2}[A, B] - I) e^{i\eta A}.$$

With $A, B \in \{R, S, R^\dagger, S^\dagger\}$ and commutations

$$[R, R^\dagger] = [S, S^\dagger] = \sinh(2\epsilon)I \text{ and } [R, S^\dagger] = i\cosh(2\epsilon)I,$$

we get

$$\frac{d}{dt} W = \sum_{k,\ell} V^*_k [V^*_k, V_\ell] V_\ell$$

$$= \sum_{k,\ell} V^*_k [e^{-i\eta R_1}, e^{i\eta R_3}] V_\ell$$

$$= \sum_{k,\ell} V^*_k e^{i\eta R_3} \left( e^{\eta^2/2}[R_1, R_3] - I \right) e^{-i\eta R_1} V_\ell$$

$$= \sum_{k,\ell} W^*_k T_{k,\ell} W_\ell$$

where

$$W_k \triangleq e^{-i\eta R_k} V_k$$

and

$$T_{k,\ell} \triangleq e^{\eta^2/2}[R_1, R_3] - I$$

are scalar coefficients forming the entries of the Hermitian circulant matrix:

$$T = \begin{pmatrix}
-1 + e^{-\eta^2}s & -1 + e^{\eta^2}s & -1 + e^{-\eta^2}c & -1 + e^{\eta^2}c \\
-1 + e^{\eta^2}s & -1 + e^{-\eta^2}s & -1 + e^{-\eta^2}c & -1 + e^{\eta^2}c \\
-1 + e^{-\eta^2}s & -1 + e^{\eta^2}s & -1 + e^{-\eta^2}c & -1 + e^{\eta^2}c \\
-1 + e^{\eta^2}s & -1 + e^{-\eta^2}s & -1 + e^{-\eta^2}c & -1 + e^{\eta^2}c
\end{pmatrix}$$

with $s = \sin(2\epsilon)$ and $c = \cosh(2\epsilon)$. This matrix admits the spectral decomposition $T = \sum_{k} \lambda_k w_k^* w_k$ where

$$w_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \text{ with } \lambda_1 = 2(\cosh(\eta^2 s) - \cos(\eta^2 c))$$

$$w_2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \text{ with } \lambda_2 = 2(\cosh(\eta^2 s) + \cos(\eta^2 c) - 2)$$

$$w_3 = (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}) \text{ with } \lambda_3 = -2(\sinh(\eta^2 s) - \sin(\eta^2 c))$$

$$w_4 = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}) \text{ with } \lambda_4 = -2(\sinh(\eta^2 s) + \sin(\eta^2 c)).$$

Simple numerical computations show that for $\eta = 2\sqrt{κ}$ and $\eta \epsilon \in (0, 1/2]$ one has

$$\lambda_4 \leq \lambda_3 \leq 0 \leq \lambda_2 \leq \lambda_1.$$

With

$$F_1 = \frac{1}{2} (W_1 - W_2 + W_3 - W_4)$$

$$F_2 = \frac{1}{2} (W_1 + W_2 + W_3 + W_4)$$

$$F_3 = \frac{1}{2} (W_1 - iW_2 - W_3 + iW_4)$$

$$F_4 = \frac{1}{2} (W_1 + iW_2 - W_3 - iW_4)$$

we have

$$\frac{d}{dt} W = \sum_{k} \lambda_k F_k^* F_k \leq \lambda_4 (F_1^* F_1 + F_2^* F_2) + \lambda_3 (F_3^* F_3 + F_4^* F_4).$$

With

$$F_1^* F_1 + F_2^* F_2 = \frac{1}{2} (W_1 + W_3)^* (W_1 + W_3)$$

$$+ (W_2 + W_4)^* (W_2 + W_4)$$

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and
\[ F_3^† F_3 + F_4^† F_4 = \frac{1}{2} \left( (W_1 - W_3)^\dagger (W_1 - W_3) + (W_2 - W_4)^\dagger (W_2 - W_4) \right) \]

one gets
\[ \frac{d}{dt} W \leq \frac{\lambda_1}{2} (W_1 + W_3)^\dagger (W_1 + W_3) + \ldots \text{non-negative}. \]

This means that
\[ e^{\frac{3\eta^2 s}{4} \cosh(3\eta \sinh(\epsilon))} \geq \pm e^{\eta \cosh(\epsilon) Q} (\cosh(3\eta \sinh(\epsilon)) Q) \]

We conclude by changing \( \epsilon \) to \(-\epsilon\).

Similarly, we have
\[ \frac{\lambda_1}{2} (W_2 + W_4)^\dagger (W_2 + W_4) + \frac{\lambda_2}{2} (W_2 - W_4)^\dagger (W_2 - W_4) \leq -2e^{-\eta^2 s/8} \kappa(\eta, \epsilon) (\epsilon^{\eta R}/2 - e^{-\eta R}/2) \]
\[ \ldots \cosh(3\eta \sinh(\epsilon) Q) (e^{\eta s/2} - e^{-\eta s/2}). \]

We have also
\[ W = \]
\[ 2e^{-\eta^2 s/8} (\epsilon^{\eta R}/2 - e^{-\eta R}/2) \cosh(\eta \sinh(\epsilon) P) (\epsilon^{\eta R}/2 - e^{-\eta R}/2) \]
\[ + 2e^{-\eta^2 s/8} (\epsilon^{\eta S}/2 - e^{-\eta S}/2) \cosh(\eta \sinh(\epsilon) Q) (\epsilon^{\eta S}/2 - e^{-\eta S}/2). \]

Since \( \kappa(\eta, \epsilon) > 0 \) we have \( \frac{d}{dt} W \leq -\kappa(\eta, \epsilon) W \)

B. An operator inequality

**Lemma 1.** Take two operators Hermitian \( Q \) and \( P \) such that \([Q, P] = iI\). Then \( \forall \eta, \epsilon \in \mathbb{R} \)
\[ e^{-3\eta^2 s/4 \cosh(3\eta \sinh(\epsilon) Q)} \geq \pm \cos(\eta \cosh(\epsilon) Q). \]

**Proof.** Set \( R = \cosh(\epsilon) Q + i \sinh(\epsilon) P \) and \( \Lambda_\pm = e^{\eta R} \pm e^{\eta R} e^{-\eta R} \) then usual computations based on Glauber identity yield
\[ \Lambda_\pm \Lambda_\pm = \]
\[ 2e^{-\eta^2 s/8} \left( \cosh(3\eta \sinh(\epsilon) P) \pm e^{-3\eta^2 s/4 \cosh(3\eta \sinh(\epsilon) Q)} \right). \]

With \( \lambda_1 - \lambda_1 \leq 0, \epsilon > 0 \) and lemma 1, we have
\[ (-\lambda_1 + \lambda_3)e^{-3\eta^2 s/4 \cosh(3\eta \sinh(\epsilon) Q)} \leq (\lambda_1 - \lambda_3)e^{-3\eta^2 s/4 \cosh(3\eta \sinh(\epsilon) P)}. \]

Consequently
\[ \frac{\lambda_1}{2} (W_1 + W_3)^\dagger (W_1 + W_3) + \frac{\lambda_2}{2} (W_2 - W_4)^\dagger (W_2 - W_4) \leq -2e^{-\eta^2 s/8} \kappa(\eta, \epsilon) (\epsilon^{\eta R}/2 - e^{-\eta R}/2) \]
\[ \ldots \cosh(3\eta \sinh(\epsilon) Q) (\epsilon^{\eta R}/2 - e^{-\eta R}/2). \]

with
\[ \kappa(\eta, \epsilon) = -\frac{1}{2} \lambda_3 (1 - e^{-\frac{3\eta^2 s}{2}}) - \frac{1}{2} \lambda_1 (1 + e^{-\frac{3\eta^2 s}{2}}) \]
\[ = (\sinh(\eta^2 s) - \sin(\eta^2 c))(1 + e^{-3\eta^2 s/2}) \]