AdS$_3$/CFT$_2$ correspondence at finite temperature

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Abstract

The AdS/CFT correspondence is established for the AdS$_3$ space compactified on a solid torus with the CFT field on the boundary. Correlation functions that correspond to the bulk theory at finite temperature are obtained in the regularization a’la Gubser, Klebanov, and Polyakov. The BTZ black hole solutions in AdS$_3$ are T-dual to the solution in the AdS$_3$ space without singularity.

1 Introduction

The AdS/CFT correspondence [1, 2, 3] has been verified for interacting field cases [4, 5] (three- and four-graviton scattering, etc.) and it is interesting to check it also in cases where the space–time geometry is more involved than the spherical one. Various approaches to this problem were proposed in [7] and [6].

In [7], it was proposed, starting from a two-dimensional (compact) manifold $M$, to consider a theory on the space $M \times \mathbb{R}_+$ endowed with the AdS metric. From the topological standpoint, this, however, results in a singularity as $s \in \mathbb{R} \rightarrow +\infty$ (because $M$ is not necessarily simply connected), and one must impose an additional condition on the fields of the theory (the fast decreasing at infinity) in order to make the field configuration smooth.

In [6], we considered the massless scalar field theory on AdS$_3$ space. (The method can be generalized to the massive modes of AdS$_n \times S^{d+1}$ space.) In this paper, we also consider the case of a general (not necessarily rectangular) torus and discuss the physical content of the results obtained. We also consider the case of a homogeneously compactified AdS$_3$ manifold without (topological) singularity in the interior (as there is no boundary as $s \rightarrow \infty$; our calculations technically resemble the bulk calculations of [6]). We show that the classical scalar field theory on the AdS$_3$ manifold in the bulk gives us the appropriate quantum correlators on the boundary.

The compactification corresponds to considering the finite temperature case, so we calculate correlators of boundary fields for the AdS$_3$ space at finite temperature [8]. Following [8], one must take into account all possible solutions of the Einstein gravity that have the anti-de Sitter metric at infinity. In the AdS$_3$ case, however, the black-hole solutions of Hawking and Page [4, 8] turn out [10, 11] to be T-dual to the case of a pure AdS$_3$ space without internal singularities; this holds for the corresponding correlation functions as well.

In Sec. 2, we recall the general structure of AdS$_3$ manifolds and introduce the $\varepsilon$-cone regularization in order to make the volume and the boundary area finite. In Sec. 3, we review the relationship between the black hole and empty space solutions of the AdS$_3$ gravity. In Secs. 4 and 5, we formulate the scalar field theory in AdS$_3$, while the corresponding correlation functions are calculated by using an improved technique in Sec. 6. Separate investigation is performed for the case where a conical singularity (the angle deficit) is developed in the vicinity of a (unique) closed geodesic inside the solid torus; the correlation functions are found for the case where the angle deficit is a rational number (Sec. 7). A brief discussion is in Sec. 8.

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2 Geometry of AdS$_3$ manifolds

The group $SL(2, \mathbb{C})$ of conformal transformations of the complex plane admits the continuation to the upper half-space $\mathbb{H}^+_3$ endowed with the constant negative curvature (AdS$_3$ space). In the Schottky uniformization picture, Riemann surfaces of higher genera can be obtained from $\mathbb{C}$ by factorizing it over a finitely generated free-acting discrete subgroup $\Gamma \subset SL(2, \mathbb{C})$. Therefore, we can continue the action of this subgroup to the whole AdS$_3$ and, after factorization, obtain a three-dimensional manifold of constant negative curvature (an AdS$_3$ manifold) whose boundary is (topologically) a two-dimensional Riemann surface $\mathbb{C}$.

We consider the simplest case of a genus one AdS$_3$ manifold obtained after the identification

$$(w, \overline{w}, s) \sim (qw, \overline{qw}, s/q),$$

(2.1)

where $q = e^{a+ib}$ is the modular parameter, $a, b \in \mathbb{R}$, $a > 0$, $w, \overline{w} = x + iy, x - iy$ are the coordinates on $\mathbb{C}$, and $s$ is the third coordinate on AdS$_3$.

Adopting the AdS/CFT correspondence approach, we should first regularize expressions in order to make them finite. In the AdS$_n$ case with the brane singularity at infinity, this was done by setting the boundary data on the $\varepsilon$-plane $\mathbb{C}$ rather than at infinity (zero plane). However, in our case, we cannot take an $\varepsilon$-plane because it is not invariant w.r.t. transformations (2.1). Instead, we can set the boundary data on the $\varepsilon$-cone—the set of points

$$z/r = \varepsilon, \quad r^2 = w\overline{w} + s^2.$$}

Given the boundary data on this cone, we fix the problem setting—the Laplace equation then has a unique solution (the Dirichlet problem on a compact manifold).

Geometrically, performing the $\varepsilon$-cone regularization and factorizing over the group $\Gamma$ of transformations (2.1), we obtain the solid torus on whose boundary (the two-dimensional torus) the CFT fields dwell. The “center” of the torus is a unique closed geodesic, which has the length $\log |q|$ (the image of the vertical half-line $z = \varepsilon = 0$), while the AdS-invariant distance $r$ from this geodesic to the image of the $\varepsilon$-cone is constant, $\cosh r = 1/\varepsilon$.

3 Black hole solution in AdS$_3$ in spherical coordinates

In order to operate with the cone geometry, it is convenient to reformulate the free-field problem on AdS$_3$ in the spherical coordinates $r \equiv e^\tau, \theta, \varphi$ in which relations (2.1) read

$$(\tau, \theta, \varphi) \sim (\tau + ma, \theta, \varphi + mb + 2\pi n), \quad m, n \in \mathbb{Z},$$

(3.1)

$$(d\tau^2 + d\theta^2 + \sin^2 \theta d\varphi^2) / \cos^2 \theta,$$

where $d\tau^2$ is the invariant interval squared. On the cone $\sin \theta = \varepsilon$, the usual toric periodic conditions are imposed on fields depending on the two-dimensional variables $\tau$ and $\varphi$.

We prefer to work in the standard spherical coordinates $s = r \cos \theta, \quad y = r \sin \theta \sin \varphi$, and $x = r \sin \theta \cos \varphi$. Upon the identification

$$\rho = b \tan \theta, \quad t = b \log r, \quad \varphi = \varphi$$

(3.2)

we obtain the AdS$_3$ metric in the standard form $\mathbb{C}$.

$$ds^2 = \left(1 + \frac{\rho^2}{b^2}\right) d\tau^2 + \frac{1}{1 + \rho^2/b^2} d\rho^2 + \rho^2 d\varphi^2,$$

(3.3)

with fields to be regular at $\rho = 0$.

The remarkable fact is that the spinning black hole solution metric (with the nonzero angular momentum $J$ and mass $M$) given by the interval

$$ds^2 = (N^\perp)^2 d\rho^2 + (N^\perp)^{-2} d\tau^2 + r^2 (d\phi + N^\phi d\rho)^2,$$

$$N^\perp = \left(-M + \frac{r^2}{4\tau} - \frac{J^2}{4\tau^2}\right)^{1/2},$$

$$N^\phi = -\frac{J}{2\tau^2},$$

(3.4)
with the following indentifications imposed:
\[
\phi \sim \phi + 2\pi, \quad \left(\frac{\rho}{\phi}\right) \sim \left(\frac{\rho}{\phi}\right) + \frac{2\pi}{M} \left(\frac{r_+}{|r_-|/l}\right),
\]
\[
r^2_{\pm} = \frac{Ml^2}{2} \left[1 \pm \left(1 + \frac{J^2}{M^2l^2}\right)^{1/2}\right],
\]
(3.5)
can be reduced by the coordinate transformations
\[
\varphi = \frac{r_+}{l^2} \phi + \frac{|r_-|}{l} \phi, \quad \tau = \frac{r_+}{l} \phi - \frac{|r_-|}{l^2} \rho, \quad \theta = \arcsin \sqrt{\frac{r^2 - r_+^2}{r^2 - r_-^2}}
\]
(3.6)
to coordinates (3.1) with
\[a = 2\pi r_+/l \quad \text{and} \quad b = 2\pi |r_-|/l\]
[10].

Therefore, metric (3.4) with the respective (two-dimensional) radial and angular coordinates \(\rho\) and \(\phi\) becomes metric (3.1) with the respective coordinates \(\tau\) and \(\varphi\). Note that (for \(J = 0\)) radial and angular coordinates readily interchange and we obtain the torus that is \(T\)-dual to the initial one.

4 Scalar field on AdS$_3$ in spherical coordinates

The action of the massless scalar field \(\Phi\) on AdS$_3$ in coordinates (3.1) is
\[
\int \frac{dx\,dy\,ds}{s^3} \{s^2\partial_s \Phi \partial_s \Phi + s^2(\partial_x \Phi \partial_x \Phi + \partial_y \Phi \partial_y \Phi)\} = \int \tan \theta d\tau d\theta d\varphi \left\{\partial_{\tau} \Phi \partial_{\tau} \Phi + \partial_{\theta} \Phi \partial_{\theta} \Phi + \frac{1}{\sin^2 \theta} \partial_{\varphi} \Phi \partial_{\varphi} \Phi\right\}
\]
(4.1)
It admits the variable separation,
\[
\Phi(\tau, \theta, \varphi) = \sum_{k,m \in \mathbb{Z}} \Phi_{k,m}(\tau) Y_{k,m}(\sin \theta) X_m(\varphi).
\]
(4.2)
Taking into account the periodicity conditions
\[
\Phi(\tau + \log |q|, \theta, \varphi + \arg q) = \Phi(\tau, \theta, \varphi), \quad \Phi(\tau, \theta, \varphi + 2\pi) = \Phi(\tau, \theta, \varphi),
\]
(4.3)
where \(q\) is the modular parameter of the torus,
\[
q = e^{a+ib}, \quad a, b \in \mathbb{R},
\]
(4.4)
we obtain
\[
X_m(\varphi) = e^{im\varphi}, \quad \partial^2 \Phi X_m = -m^2 X_m;
\]
(4.5)
\[
\Phi_{k,m}(\tau) = e^{i\left[-\frac{mb}{2a} + \frac{\pi k}{a}\right] \tau}, \quad \partial^2 \Phi_{k,m}(\tau) = -\frac{w^2}{4} \Phi_{k,m}(\tau),
\]
(4.6)
\[
w = -\frac{mb}{2a} + \frac{\pi k}{a}.
\]
(4.7)
For the function \(Y_{k,m}(\sin \theta)\), the eigenvalue problem is
\[
\frac{\cos \theta}{\sin \theta} \partial_{\theta} \left(\frac{\sin \theta}{\cos \theta} \partial_{\theta} Y_{k,m}(\sin \theta)\right) - \frac{m^2}{\sin^2 \theta} Y_{k,m}(\sin \theta) = \frac{w^2}{4} Y_{k,m}(\sin \theta).
\]
(4.8)
We substitute \(\rho\) for \(\sin \theta\) and consider the problem on the interval \(1 - \varepsilon \geq \rho \geq 0\) with the regularity condition at \(\rho = 0\),
\[
Y''_{k,m}(\rho) + \frac{1}{\rho} Y'_{k,m}(\rho) - \frac{m^2}{\rho^2(1 - \rho^2)} Y_{k,m}(\rho) = \frac{w^2/4}{1 - \rho^2} Y_{k,m}(\rho).
\]
(4.9)
Equation (4.9) can be reduced to the standard hypergeometric equation whose general solution that is regular at \( \theta = 0 \) yields

\[
\Phi(\tau, \theta, \varphi) = \sum_{m, k \in \mathbb{Z}} e^{im\varphi} e^{i\left(\frac{mb}{2} + \frac{2\pi k}{2}\right)} C_{k,m} \times Y_{k,m}(\sin \theta),
\]

where \( C_{k,m} \) are the mode amplitudes.

Expression (4.10) is singular at \( z = 1 \) and we must find its asymptotic behavior for \( z = 1 - \varepsilon \). For \( c - a - b \in \mathbb{Z} \), the exact relation is (see, e.g., formula 7.3.1.31 from [14])

\[
2F_1(a, b; c; z) = \sum_{k=0}^{+\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k, \quad (a)_k \equiv \prod_{i=0}^{k-1} (a + i),
\]

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\[
2F_1(a, b; c; z) \bigg|_{c=a+b+m} = \Gamma \left[ m, a + b + m \right] \sum_{k=0}^{m-1} \frac{(a)_k(b)_k}{k!(1-m)_k} (1-z)^{-k} - \Gamma \left[ a + b + m, a, b \right] (z-1)^m \sum_{k=0}^{m} \frac{(a+m)_k(b+m)_k}{k!(m+k)!} (1-z)^k \times \left[ \log(1-z) - \Psi(k+1) - \Psi(k+m+1) + \Psi(a+k+m) + \Psi(b+k+m) \right]. \tag{4.11}
\]

Here \( \Psi(x) \) is the logarithmic derivative of the \( \Gamma \)-function. In the massless case, \( m = 1 \) in (4.11).

Coefficients \( C_{k,m} \) determine the boundary values of the field \( \Phi \). Then, for action (4.1), we obtain

\[
\int_0^\alpha d\tau \int_0^{2\pi} d\varphi \int_{\sin \theta = 0}^{\sin \theta = 1 - \varepsilon} d\theta \frac{\sin \theta}{\cos \theta} \left\{ \partial_\tau \Phi \partial_\tau \Phi + \partial_\theta \Phi \partial_\theta \Phi + \frac{1}{\sin^2 \theta} \partial_\varphi \Phi \partial_\varphi \Phi \right\} = \int_0^\alpha d\tau \int_0^{2\pi} d\varphi \Phi(\tau, 1 - \varepsilon, \varphi) \sin \theta \frac{\partial}{\partial \sin \theta} \Phi(\tau, \sin \theta, \varphi) \bigg|_{\sin \theta = 1 - \varepsilon}. \tag{4.12}
\]

Keeping only the logarithmically divergent and finite parts as \( \varepsilon \to 0 \) and using the standard formulas for \( \Psi \)-functions, we obtain action (4.1) in the form of the mode expansion \( (C_{k,m} \equiv C_{-k,-m}) \)

\[
- \sum_{k,m \in \mathbb{Z}} |C_{k,m}|^2 \left( \frac{m^2}{4} + w^2 \right) \left[ \log \varepsilon + \Psi \left(1 + m/2 + iw\right) + \Psi \left(1 + m/2 - iw\right) + \Psi \left(1 - m/2 + iw\right) + \Psi \left(1 - m/2 - iw\right) - 4\Psi(1) \right], \tag{4.13}
\]

where \( w \) and \( q \) are in (4.7), (4.4).

5 Massive modes

Including into consideration the “internal” (compact) degrees of freedom (e.g., assuming the compact manifold to be a sphere \( S^{d+1} \)), we obtain an additional term in initial action (4.1),

\[
\int \frac{dx \, dy \, ds}{s^3} \left\{ s^2 \partial_\theta \Phi \partial_\theta \Phi + s^2 (\partial_\varphi \Phi \partial_\varphi \Phi + \partial_\theta \Phi \partial_\theta \Phi) + l(l+d)\Phi^2 \right\} = \int \tan \theta \, d\tau \, d\theta \, d\varphi \left\{ \partial_\theta \Phi \partial_\theta \Phi + \partial_\varphi \Phi \partial_\varphi \Phi + \frac{1}{\sin^2 \theta} \partial_\varphi \Phi \partial_\varphi \Phi + \frac{l(l+d)}{\cos^2 \theta} \Phi^2 \right\}. \tag{5.1}
\]
Separating the variables as above, we obtain that the restrictions on the “torus” coordinates $\varphi$ and $\tau$ are as in (4.5) and only the equation for the $\theta$-component is changed,

$$\frac{\cos \theta}{\sin \theta} \partial_\varphi \left( \frac{\sin \theta}{\cos \theta} \partial_\theta Y_{k,m} \right) - \frac{m^2 Y_{k,m}}{\sin^2 \theta} - \frac{\l(l+d)Y_{k,m}}{\cos^2 \theta} = 0. \quad (5.2)$$

Introducing a new quantity $\rho$,

$$4\rho(\rho - 1) = l(l + d), \quad \rho > 0. \quad (5.3)$$

the solution to (5.2) that is regular at $\theta = 0$ becomes

$$Y_{k,m}(\theta) = (\cos \theta)^{\frac{1}{2} - \frac{2\rho}{\theta}} \left( \frac{|m|}{2} + 1 - \rho + iw, \frac{|m|}{2} + 1 - \rho - iw; |m| + 1; \sin^2 \theta \right), \quad (5.4)$$

where again $w = -\frac{mb}{2a} + \frac{\pi k}{a}$.

Important particular case is $d = 2$ (AdS$_3 \times SU^3$) where $\rho = l/2 + 1$ and we must use formula (4.11) in order to find the asymptotic behavior. In this case, the nonlocality is again encoded in the $\Psi$-function terms.

### 6 Correlation function for massless modes

Turn now to expression (4.13). The mode multiplier is the Fourier transform of the correlation function of the boundary CFT field on the torus. We are interested in the nonlocal contribution to Green’s functions on torus coming from this formula and disregard all local terms, which can be removed by a proper renormalizing procedure. (Therefore, we consider two-point correlation functions rather than Green’s functions for a two-dimensional problem; the latter are distributions and necessarily demand local contributions to be taken into account.)

The $m$-dependence in (4.13) is analytic and we may represent the $\Psi$-function using the formula

$$\Psi(1 + a) - \Psi(1 + b) = \sum_{j=0}^{\infty} \left( \frac{1}{1 + j + b} - \frac{1}{1 + j + a} \right).$$

Let us find the CFT correlation functions that follow from (4.13). The term $\left( \frac{m^2}{4} + w^2 \right)$ in front of the summand is nothing but the Laplacian action on the Riemann surface, i.e., we obtain that the correlation function is

$$G(\tau, \varphi) = \frac{1}{4} (\partial^2_\tau + \partial^2_\varphi) \sum_{m, k \in \mathbb{Z}} e^{im\varphi + i\left[\frac{-mb}{2a} + \frac{\pi k}{2a}\right] \tau} \left[ 2 \log \varepsilon + \sum_{(\pm), \pm} \Psi(1(\pm)m/2 \pm iw) - 4\Psi(1) \right] =$$

$$= \frac{1}{4} (\partial^2_\tau + \partial^2_\varphi) \sum_{m, k \in \mathbb{Z}} e^{im\varphi + i\left[\frac{-mb}{2a} + \frac{\pi k}{2a}\right] \tau} \times$$

$$\times \left[ 2 \log \varepsilon + \sum_{(\pm), \pm} \sum_{i=1}^{\infty} \left( \frac{4}{7} - \frac{l(\pm) m}{7} \pm \frac{1}{l(\pm) m/2 \pm \frac{i\pi k}{2a}} \right) \right]. \quad (6.1)$$

The sign $\sum_{(\pm), \pm}$ in (6.1) denotes the sum over four terms with all possible appearances of “+” and “−” signs. We distinguish between two appearances of $\pm$ signs by taking one of them in parentheses. Constant and polynomial terms are irrelevant to our discussion as they produce only local contributions.

First, we take the sum over $k$,

$$\sum_{k=-\infty}^{\infty} e^{2\pi ik\tau/\alpha} \frac{e^{2\pi ik\tau/\alpha}}{l(\pm) m/2 + \left( \frac{ik}{2a} - \frac{\pi k}{2a} \right)} = f(\tau/\alpha). \quad (6.2)$$
The function \( f(\tau/a) \) is periodic under the shift \( \tau \rightarrow \tau + 1 \) and satisfy the functional equation
\[
\pm \frac{1}{2a} \partial_{\tau/a} f(\tau/a) + \left( l(\pm) \frac{m}{2} \mp \frac{imb}{2a} \right) f(\tau/a) = \delta^1_{\Pi}(\tau/a), \tag{6.3}
\]
where \( \delta^1_{\Pi}(x) \) is the periodic \( \delta \)-function with the unit period. A (unique) solution to (6.3) that is periodic in \( \tau \) is a saw-tooth-like exponential curve
\[
f(\tau/a) = A e^{\pi \tau/a} \quad \text{for} \quad \tau/a \in (0,1),
\]
which is to be continued periodically to the whole \( \mathbb{R} \). Equation (6.3) gives
\[
\chi = \mp 2a(l(\pm)m/2) + imb \quad \text{and} \quad A = \frac{\pm 2a}{1 - e^{\mp 2a(l(\pm)m/2) + imb}}.
\]
Therefore, the remaining nonlocal terms are combined into the sum
\[
\sum_{m \in \mathbb{Z}} \sum_{l \pm (\pm)} e^{im\varphi - imb/2} \sum_{\pm, (\pm)} \frac{\pm 2a e^{\mp 2(l(\pm)m/2)\tau + imb/2}}{1 - e^{\mp 2a(l(\pm)m/2) + imb}}, \quad 0 \leq \tau < a. \tag{6.4}
\]
Expression (6.4) becomes
\[
\sum_{m \in \mathbb{Z}} \sum_{l \pm (\pm)} \Re \left[ -2a e^{mz + 2il\varphi} e^{imb/2} + 2a e^{mz - 2il\varphi} e^{-imb/2} \right], \quad z \equiv \tau + i\varphi. \tag{6.5}
\]
It is convenient to represent the denominators in (6.5) as
\[
\frac{1}{1 - q^m e^{i\chi}} = \left\{ \begin{array}{ll} 
\sum_{p=0}^{\infty} q^{mp} e^{-ip\chi} & \text{for} \quad |q| > 1, \quad m < 0, \\
-\sum_{p=0}^{\infty} q^{-mp} e^{-ip\chi} & \text{for} \quad |q| > 1, \quad m > 0.
\end{array} \right. \tag{6.6}
\]
Now, it is easy to sum over \( l \) and \( m \) in (6.5) (the term with \( m = 0 \) vanishes). We obtain
\[
G(z, \overline{z}) = \frac{1}{4} \partial_{\tau} \partial_{\overline{\rho}} \sum_{p=0}^{\infty} \frac{1}{q^p e^{-z} - 1} \cot(\varphi - bp) - 2a \sum_{p=0}^{\infty} \frac{1}{q^p e^{-z} - 1} \cot(\varphi - bp) \tag{6.7}
\]
and it is straightforward to take the derivatives. The answer is
\[
G(z, \overline{z}) = \frac{a}{4} \sum_{p=\infty}^{\infty} \frac{1}{\sinh^2 \frac{z + ap + ib}{2} \sinh^2 \frac{\overline{z} + ap - ib}{2}}. \tag{6.8}
\]
This is the correlation function of two Yang–Mills tensor field insertions on torus obtained in \( \Re \) for \( b = 0 \). At singularity points, it has the proper behavior \( G(r) \sim 1/r^4 \), which is character for the weight two CFT fields.

## 7 Exact correlation function for the conical singularity case

Using our technique, we are able to find the correlation functions also in the case where a conical singularity appears at the axis \( \theta = 0 \) of metric (6.1). This corresponds to the following double periodic conditions for the variable \( z = \tau + i\varphi \):
\[
z \rightarrow z + \frac{2\pi i}{a}, \quad z \rightarrow z + \frac{2\pi}{ad} \xi, \quad \text{where} \quad \xi = a + ib, \quad a, b, d \in \mathbb{R}. \tag{7.1}
\]
Actually, we can calculate the correlation function in the most general case with torsion where \( \tau \) variable is also shifted by the first transformation. Then, the two-dimensional eigenfunctions are
\[
\Phi_{k,l}(\tau, \varphi) = e^{i(a\xi + c\varphi + d(\xi - b\varphi))} \tag{7.2}
\]
(in (7.3), we set \( c = 0 \)) and the formula for the function \( Y_{k,m}(\sin \theta) \) is exactly (1.14) where we must set
\[
m = a\ell + kc, \quad w = -ib + kd.
\]
Then, acting similarly to (6.1)–(6.6), we obtain the expression with the single remaining summation over \( n, \)
\[
G(z, \bar{z}) = \frac{2\pi}{d} \partial_x \partial_{\bar{z}} \sum_{n=-\infty}^{\infty} \left( \frac{1}{e^{\frac{a(x+\bar{z})+4\pi in}{dz}} - 1} \cdot e^{\frac{a(x+\bar{z})+4\pi in}{dz}} - 1 \right),
\]
where \( \xi \equiv a + ib, \rho \equiv c + id, \) and \( \Delta \equiv ad + bc. \)

From now on, we restrict the consideration only to the case without torsion, \( c = 0. \) Also, we assume \( a \in \mathbb{Z}_+ \) to be a positive integer. Then, (7.4) becomes
\[
G(z, \bar{z}) = \frac{\pi}{2da} \sum_{n=-\infty}^{\infty} \sum_{j=0}^{a-1} \frac{1}{\sinh^2\left( \frac{\pi}{a} + \frac{\pi n}{a} + \frac{\pi j}{a} \right)} \sinh^2\left( \frac{\pi}{d} + \frac{\pi n}{ad} - \frac{\pi j}{a} \right)
\]
\[
(7.5)
\]
The formula (7.5) resembles (6.8) with one, but important, difference. We can write (6.8) as
\[
\sum_{p,m_1,m_2 \in \mathbb{Z}} \frac{4a}{(z + (a + ib)p + 2\pi im_1)^2(\bar{z} + (a - ib)p + 2\pi im_2)^2}.
\]
Then, turning to (7.3), we obtain after rescaling \( z \rightarrow z/a, \)
\[
\sum_{p,m_1,m_2 \in \mathbb{Z}} \sum_{j=0}^{a-1} \frac{8\pi a^3}{d} \frac{1}{(z + 2\pi \xi p + 2\pi i(j + m_1 a))^2(\bar{z} + 2\pi \xi \bar{p} - 2\pi i(j + m_2 a))^2}.
\]
\[
(7.7)
\]
In expression (7.6), the singularities in \( z \) and \( \bar{z} \) come always at the same \( p, \) but at all possible (independent) shifts in the angular variable \( \varphi. \) Expression (7.7) enjoys the first of these properties but angular shifts are no more independent—the mismatch is always an multiple of \( a \) and, for instance, in the limit \( a \rightarrow \infty, \) the zero mismatch term prevails, which automatically produces a \( T \)-dual answer.

8 Discussion

Having an AdS asymptotic metric in the boundary domain, we can continue it in a bulk in (at least two) different ways: as an empty space or as a space with the BTZ black hole singularity. From the CFT standpoint, these two continuations are connected by the \( T \)-duality transformation; therefore, adopting a viewpoint that we must take a sum over all possible continuations of the AdS metric (related, probably, to instantonic modes) for obtaining a proper correlation function on the boundary, we obtain, instead of (6.8), the sum of (6.8) and its \( T \)-dual.
Obtained expression (6.8) describes the two-point correlator \( \langle G^2_{\mu\nu}(0,0)G^2_{\mu\nu}(z,z) \rangle \) of the Yang–Mills tensor insertions at nonzero temperature, which, from the conformal field theory standpoint, corresponds to the case of a rectangular torus. This demonstrates again that the AdS/CFT correspondence holds in our case where no singularity at the AdS time infinity is assumed. A more interesting (but far more involved technically) is the problem of verifying this correspondence in actual gravitational calculations of a multi-point correlation functions.

The last problem to discuss is the \( T \)-duality under the modular transformation \( a \to \frac{1}{a} \) in different prescriptions. In the prescription of \( |7| \), it is possible to obtain the solid torus by factorizing the AdS\(_3\) space with respect to the action of the Abelian group generated by two shifts (parabolic elements of \( SL(2,\mathbb{C}) \)): \( z \to z + 1 \) and \( z \to z + i\xi \) (the third coordinate, \( s \), is invariant). Then, the fundamental domain is the half-infinite cylinder \( 0 < \Re z \leq 1, 0 < \Im z \leq \xi, \varepsilon < s < M \) and, geometrically, this corresponds (in the three-dimensional picture) to the closed torus inside which there is a singular (as \( M \to \infty \)) torus of AdS radius \( \sim \frac{1}{M} \) and length \( \xi/M \). Then, in the limit \( \varepsilon \to 0, M \to \infty \), the corresponding temperature correlation function becomes

\[
G_{Bon}(z,\tau) \sim \sum_{n,m=-\infty}^{\infty} \frac{1}{(z + n + i\xi m)^2(\tau + n - i\xi m)^2},
\]

which apparently differs from (6.8) but coincides with (7.3) in the limit \( a \to \infty \) (with the proper rescaling of \( \varphi \) and the Green’s function). This, however, is not very surprising as the case \( a \to \infty \) corresponds to the case where the factorization group is generated by two parabolic (oricyclic) elements of the \( SL(2,\mathbb{C}) \) group.

Already in the free field theory, there remain questions on the mass spectrum, on the generalization to higher dimensions, etc. Of special interest is the question how to consider solid Riemann surfaces of higher genera. The construction works well in this case, but the \( \varepsilon \)-regularized surface, or the boundary of the integration domain, cannot be described in the invariant distance terms (the structure of the set of closed geodesics becomes very involved already starting from genus two); however, we hope that one can obtain a proper answer using an approximation technique. In the present calculations, we disregard all local contributions. However, these contributions becomes important when considering additional boundary terms in the initial action. It would be interesting to check whether the Hamiltonian prescription of \( |10| \) holds in this case.

It is also interesting to relate the obtained correlation functions (6.8) and (7.5) with the \( S \)-matrix scattering problem in AdS\(_3\) \( |17| \).

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