Non-abelian monopole solutions in arbitrary gauge

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The Georgi-Glashow model equations of motion are examined by general static spherically symmetric real and complex parametrizations of gauge fields in arbitrary gauge. Their connection with the known ’t Hooft-Polyakov and Julia-Zee equations is shown without any gauge-fixing, elucidating so the gauge invariant meaning of corresponding functions. The obtained systems of equations have a new exact analytical solutions for complex non-abelian monopole and dyon fields with real finite or zero energy densities.

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1. Introduction

The seminal conception of abelian Dirac monopole today is 90 years old. It plays an important role in a number of field theoretic considerations, where it can account for the quantization of the electric charge. It has got its second life in non-abelian gauge theories with (or without) spontaneous symmetry breaking according to Higgs mechanism. It was shown that these theories naturally incorporate the Dirac monopole as abelian projection of their non-abelian monopole solutions, which, in turn, can catalyze the decay of proton or provide quarks confinement.

For the stationary case with $\partial^0 \to 0$, where $x^\mu = (x^0, x)$, $x = r \hat{n}$ for $r = |x|$, the time independence presupposes a choice of Lorentz frame in which the fields are at rest. The classical field the absence of “chromo-electric” field strength $E^0_a(x) = 0$ implies the temporal gauge condition to non-abelian gauge potentials $A^0_a(x) = (A^0_a(x), A_a(x))$ as $A^0_a(x) = 0$. The well known classical static ’t Hooft-Polyakov solution in fact satisfies an additional transverse gauge condition $(n \cdot A_a(x)) = 0$ and Coulomb gauge condition $(\nabla \cdot A_a(x)) = 0$. The last condition excludes from the potential $A^0_a(x)$ not only the longitudinal structure $n^a n^0$ but also the transverse one $(\delta^{ja} - n^j n^a)$, inevitably accompanied it (cf. [11], [12] below).

In the following sections it is shown how the consistent removing of those gauge constraints extends and generalizes the interpretation of the functions of ’t Hooft-Polyakov and Julia-Zee solutions and leads to new exact classical solutions for com-
plex non-abelian monopole and dyon fields with real finite or zero energy densities.

2. Georgi-Glashow model and ‘t Hooft-Polyakov ansatz

We shall start with reminding of group representation, covariant derivative $\hat{D}^\mu$, field strength $F^\mu_{\nu}$, Lagrangian, equations of motion, energy density and vacuum expectation for $SO(3)$ Georgi-Glashow model[13] This model contains the gauge potentials $A^\mu_a(x)$ and scalar Higgs fields $h_a(x)$ in adjoint representation for $A^\mu_a = A^\mu_a T^a$ and $\tilde{h} = h_a T^a$, that for $\hat{\partial}^\mu \equiv (\partial^\mu, -\mathbf{\nabla})$ and generators $T^a$ obey the relations[14,15]

$$[T^a, T^b] = i \varepsilon^{abc} T^c, \quad (T^a)_{bc} = -i \varepsilon^{abc}, \quad \text{in adjoint representation,}$$

$$\hat{D}^\mu = \hat{\partial}^\mu - i e A^\mu, \quad \left(\hat{D}^\mu \tilde{h}\right)_a \equiv \left[\hat{D}^\mu, \tilde{h}\right]_a = \hat{\partial}^\mu h_a + e \varepsilon^{abc} A^\mu_b h_c,$$

$$\varepsilon_{\mu\nu\sigma\kappa} \left(\hat{D}^\nu \tilde{G}^{\sigma\kappa}\right)_a = 0, \quad \Theta^{\mu\nu}(x) = -G_{a\kappa a\sigma} G^{\mu\nu a\kappa} + \left(\hat{D}^\mu \tilde{h}\right)_a \left(\hat{D}^\nu \tilde{h}\right)_a - \eta^{\mu\nu} L(x),$$

$$\Theta^{00}(x) = \frac{1}{2} \left(\langle E_a^2 \rangle - \langle B_a^2 \rangle \right)^2 + \left(\langle \hat{D}^\mu \tilde{h}\rangle_a \right)^2 + \left(\langle \hat{D}^\nu \tilde{h}\rangle_a \right)^2 = \frac{1}{2} \left(\langle (h_b) - \mu^2 \rangle^2 \right)^2,$$

with $E_a^2 = -G_{a\mu}^{} G^{\mu}_{a}$, $B_a^l = -\frac{1}{2} \varepsilon^{jkl} G_{a}^{jk}$, $\langle \hbar_a(x) | 0 \rangle = \delta^{a3}$, $\varepsilon^{jkl} F^{2} = M^2$.

Here $M$ is the mass acquired by the gauge fields $A^\mu_a$, $A^\mu_b$ after absorption of Higgs fields $h_1, h_2$. The remaining massless component $A^\mu_a$ corresponds to residual $U(1)$ gauge symmetry responding for existence of abelian monopole in this theory and is connected with residual freedom of rotation around third axis in isotopic (“color”) space, defined[12,13] by vacuum expectation $\langle \hbar_a(x) \rangle$ in the unitary gauge [3].

The static ‘t Hooft-Polyakov solution for $A^\mu_a(x) = 0$ with $(n \cdot A_a(x)) = 0$, $(\mathbf{\nabla} \cdot A_a(x)) = 0$ is defined[17] by two real functions $K(r)$ and $H(r)$ as

$$A^\mu_a(x) = -\varepsilon^{jkl} \frac{n^k}{er} [1 - K(r)], \quad h_a(x) = n^a \frac{H(r)}{er},$$

$$G^{jk}_a (x) = \varepsilon^{jkl} B^l_a (x) = \frac{n^l n^a}{er^2} \left\{ n^l n^a \left[ K^2(r) - 1 \right] + (\delta^{la} - n^l n^a) r \frac{dK(r)}{dr} \right\}.$$
3. Spherically symmetric gauge fields

We consider the most general static spherically symmetric (ss) ansatz for the gauge field, as a superposition of three independent ss-structures, mutually orthogonal in configuration and isotopic spaces simultaneously, for \( x = r \mathbf{n} \), with \( \nabla r = \mathbf{n} \):

\[
A_i^a(x) = x^{jk} n^k \frac{\gamma(r)}{er} - n^j n^a \frac{\alpha(r)}{er} - (\delta^{ja} - n^j n^a) \frac{\beta(r)}{er},
\]

so that now:

\[
(n \cdot A_a(x)) = -n^a \frac{\alpha(r)}{er}, \quad (\nabla \cdot A_a(x)) = \frac{n^a}{er} \left[ 2\beta(r) - \partial_r (r \alpha(r)) \right],
\]

where the second longitudinal structure \( n^j n^a \) also is orthogonal to the both transverse ones \( \varepsilon^{jk} n^k \) and \( (\delta^{ja} - n^j n^a) \) separately in configuration and isotopic spaces.

Without the loss of generality, the gauge invariance of the theory allows to impose the conditions onto the function \( \alpha(r) \) to be real and regular everywhere including infinity. The conditions for the functions \( \beta(r), \gamma(r) \) will be given below.

The conditions of spherical symmetry for ss- vector field (11) as well as for scalar field [9] mean their invariance under rotation \( R(g) \) in configuration space and global gauge transformation \( V(g) \) simultaneously, with the same arbitrary rotation parameters from diagonal subgroup \( g \in SO(3) \) of such simultaneous rotations in both configuration and isotopic spaces:

\[
V(g) [R(g) \hat{A}(R^{-1}(g)x)] V^{-1}(g) = \hat{A}(x), \quad V(g) \hat{h}(R^{-1}(g)x) V^{-1}(g) = \hat{h}(x),
\]

which holds for each of these three ss-structures separately. These ss-structures define also the “chromo-magnetic” field [3, 8], where, unlike the potentials (11), the longitudinal structure \( n^j n^a \) appears as gauge (i.e. \( \alpha \))-independent and the prime means the derivative with respect to \( r \) also for the function \( Y(r) = 1 + \gamma(r) \):

\[
B_i^a(x) = \frac{n^j n^a}{er^2} \left[ 1 - Y^2 - \beta^2 \right] - \frac{(\delta^{ja} - n^j n^a)}{er^2} (rY' + \alpha \beta) - \frac{\varepsilon^{jk} n^k}{er^2} (r\beta' - \alpha Y).
\]

With the same ansatz [9] for the Higgs field, by means of independence and orthogonality of the above three ss-structures, the equations of motion [3] lead to the following system of four equations, it would seem, for four unknown functions:

\[
r^2 \frac{d}{dr} \left( \frac{rY' + \alpha \beta}{r} \right) = Y (Y^2 + \beta^2 + H^2 - 1) - \alpha (r\beta' - \alpha Y),
\]

\[
r^2 \frac{d}{dr} \left( \frac{r\beta' - \alpha Y}{r} \right) = \beta (Y^2 + \beta^2 + H^2 - 1) + \alpha (rY' + \alpha \beta),
\]

\[
\beta (rY' + \alpha \beta) = Y (r\beta' - \alpha Y),
\]

\[
r^2 \frac{d^2 H}{dr^2} = H \left\{ \frac{\lambda}{e^2} [H^2 - (Mr)^2] + 2 (Y^2 + \beta^2) \right\}.
\]

It may be obtained also by variation with respect to the corresponding functions \( Y, \beta, \alpha, H \) of the energy functional [12] for the energy density [7] with the gauge field [11] and strength [14] (for corresponding stress tensor \( \Theta^{jk}_m(x) \) see the Appendix,
Eqs. (A.1), (A.2) with $E_a = 0$ and $J = 0$):

$$E_m[\Theta] = \int d^3x \Theta_{m0}(x) \rightarrow 4\pi \int_0^\infty r^2 dr \Theta_{m0}(r) =$$

$$= \frac{4\pi}{e^2} \int_0^\infty \frac{dr}{r^2} \left\{ \frac{1}{2} \left[ 1 - Y^2 - \beta^2 \right]^2 + (rY' + \alpha \beta)^2 + (r\beta' - \alpha Y)^2 + \frac{1}{2} \left[ H^2 - (Mr)^2 \right]^2 \right\}. \quad (19)$$

Note that this quantity and its density $\Theta_{m0}(r)$ being observables should be real and gauge (i.e. $\alpha$)-independent also for the complex unobservable gauge-dependent non-abelian gauge potentials (11) and field strengths (14).

For $Y \neq 0$ and $Y^2 + \beta^2 \neq 0$ the third Eq. (17) is immediately integrated for

$$\frac{\beta(r)}{Y(r)} = \tan \omega(r), \quad r\omega'(r) = \alpha(r), \quad \omega(r) = \int_\xi^r \frac{\alpha(\xi)}{\xi} \pmod{\pi}. \quad (20)$$

Then the symmetry of the system (15)–(18) under the simultaneous changing of $Y \mapsto \pm \beta$ and $\beta \mapsto \mp \tilde{Y}$ can be realised in the form:

$$Y(r) = K(r) \cos \omega(r), \quad \beta(r) = K(r) \sin \omega(r), \quad \text{where} \quad Y^2 + \beta^2 = K^2 > 0, \quad (22)$$

with the real functions $K(r)$ and $\omega(r)$, providing this symmetry for real $Y, \beta$ by the simple shifting $\omega \mapsto \tilde{\omega} \mp \pi/2$. Then each of the first two Eqs. (15), (16) and the fourth Eq. (18) are reduced exactly to the first and second ‘t Hooft-Polyakov equations for the functions $K(r)$ and $H(r)$ respectively, being the same as those for the ‘t Hooft-Polyakov ansatz (9):

$$r^2 \frac{d^2 K}{dr^2} = K \left[ K^2 + H^2 - 1 \right], \quad r^2 \frac{d^2 H}{dr^2} = H \left\{ \frac{\lambda}{e^2} \left[ H^2 - (Mr)^2 \right] + 2K^2 \right\}. \quad (23)$$

They are of course reproduced by the system of Eqs. (15)–(18) for $\alpha = \beta = 0$ and $Y \mapsto K$. The functions $\alpha, \omega$ describing the gauge arbitrariness (12) could not be defined by any dynamical equations. Nevertheless, their inclusion demonstrates the gauge invariant meaning of ‘t Hooft-Polyakov functions $K(r)$ and $H(r)$. The gauge-dependent non-abelian “chromo-magnetic” field strength (14) now reads as

$$B^a_{x}(x) = \frac{n^j n^a}{er^2} \left[ 1 - K^2 \right] - \Xi^{ja}(n, \omega) \frac{rK'}{er^2}, \quad (24)$$

with

$$\Xi^{ja}(n, \omega) = (\delta^{ja} - n^j n^a) \cos \omega + \varepsilon^{jka} n^k \sin \omega, \quad (25)$$

as transverse gauge-dependent tensor. For $\omega(r) = 0$ this strength obviously returns to the ‘t Hooft-Polyakov strength (10) as its natural gauge-dependent generalization.
The full energy (20) for the solution (22), (24) also takes its known form in independent of \( \omega, \alpha \) (stress tensor \( \Theta_{jk}^m(x) \) see in (A.4), (A.6)–(A.10) with \( J = 0 \)):

\[
E_m[\Theta] = \frac{4\pi e^2}{c^2} \int_0^\infty \frac{dr}{r^2} \left\{ \frac{1}{2} [1 - K^2]^2 + (rK')^2 + \frac{1}{2} \left( rH' - H \right)^2 + H^2 K^2 + \frac{\lambda}{4e^2} \left( H^2 - (Mr)^2 \right)^2 \right\}.
\]

(26)

The boundary conditions to nonlinear Eqs. (23) is more subtle question. The asymptotics at \( r \to \infty \) is strictly determined by topological features of monopole solutions as vanishing of \( K(r) \) and \( S(r) \). With the help of this conditions one can linearize the Eqs. (23) at \( r \to \infty \) to obtain that for \( m = f \sqrt{2 \lambda} \) and \( \lambda \geq 0 \):

\[
K(r) \sim e^{-Mr} \to 0, \quad H(r) \mp Mr \sim e^{-mr} \to 0, \quad \text{i.e.} \quad S(r) \sim \frac{e^{-mr}}{Mr} \to 0,
\]

(27)
as \( r \to \infty \), where \( m \geq 0 \) is associated with the mass of third Higgs boson after spontaneous symmetry breaking and quantisation over Higgs vacuum.

The boundary conditions at \( r \to 0 \), preserving the finite monopole mass as its finite full energy at rest can be written as:

\[
K(r) - 1 \sim O(r), \quad H(r) \sim O(r), \quad \text{i.e.} \quad S(r) \sim O(1).
\]

(28)

In order to preserve the regularity of the fields at the origin \( r \to 0 \) they should be imposed in a stronger form:

\[
K(r) - 1 \sim O(r^2), \quad H(r) \sim O(r^2), \quad \text{i.e.} \quad S(r) - 1 \sim O(r).
\]

(29)

The conditions (27) and (29) are fulfilled e.g. for known exact BPS-solution in the limit \( \lambda \to 0 \). However, for the solutions with \( \lambda > 0 \) only the existense theorem but not the uniqueness theorem is known for both the conditions (27) and (28) or (29) simultaneously. On the other hand, the conditions (27) are enough to provide a finite abelian magnetic charge. So, the constraints (28) or (29) at \( r = 0 \) do not have the same importance as the first one (27) at \( r \to \infty \), at least as long as we remain in the rest frame of the monopole field. The well known example of such kind is the non-abelian Wu-Yang monopole solution in SU(2) Yang-Mills gauge theory.

4. Other monopole solutions

Another possibility to realise the observed symmetry of the system is to impose \( \beta = \eta Y \) with \( \eta = \text{const.} \) Then the Eq. (17) gives \( \eta^2 = -1 \). So we have two sets of complex classical solutions for every \( \eta = \mp i \). Their complexification thus naturally arises from Euler equations without any changing of the Lagrangian (4) and energy density (7). So, the third equation of the system holds identically, whereas each of the first two Eqs. (15), (16) now coincides with the following:

\[
r^2 \frac{d}{dr} \left( \frac{rY' + \eta \alpha Y}{r} \right) = Y \left( H^2 - 1 \right) - \eta \alpha \left( rY' + \eta \alpha Y \right).
\]

(30)
For the both cases, with real Higgs field i.e. real $H(r)$, it is convenient to parameterize complex solutions of (30) by two real functions $N(r)$ and $\omega(r)$ (with the omitted indexes $\pm$ for $\eta = \mp i$) as

$$Y_{\pm}(r) = N(r)e^{\pm i\omega(r)}, \quad \beta_{\pm}(r) = \mp iY_{\pm}(r), \quad \text{with} \quad Y_{\pm}^2 + \beta_{\pm}^2 = 0. \quad (31)$$

The Eq. (30) for both solutions then reads as

$$r^2 N'' - \left[(H^2 - 1) + (r\omega' - \alpha)^2\right] N = \mp i \left[2rN' - Nr \frac{d}{dr}(r\omega' - \alpha)\right]. \quad (32)$$

Obviously its right hand side now is real, while its left one is pure imaginary. So both of them turn to zero separately. To fulfill this it is enough for the second relation (21) hold again as $r\omega' = \alpha$, thus eliminating the dependence on indexes $\pm$ of functions $N, \omega$ as well as any gauge dependence of the obtained system of equations (the function $N$ was introduced only as real but not as positive):

$$r^2 N'' - (H^2 - 1) N = 0, \quad r^2 H'' = \frac{\lambda}{e^2} [H^2 - (Mr)^2] H. \quad (33)$$

Its explicit exact solution for both monopole (+) and antimonopole (−) cases, satisfies to boundary conditions (21), (23) for $H(r)$ and to asymptotic condition for $N(r)$ similar to that for $K(r)$ (27), and has the following form:

$$H(r) = \pm Mr \equiv \pm \zeta, \quad \text{whence} \quad N(r) = N_0 \left(\frac{2\zeta}{\pi}\right)^{1/2} K_{i\varrho}(\zeta), \quad \text{for} \quad \varrho = \frac{\sqrt{3}}{2}, \quad (34)$$

where for $\zeta > 0$ $K_{i\varrho}(\zeta)$ is the Macdonald function\(^{29}\) which remains the same and real for both imaginary indices $\pm i\varrho$. So for arbitrary dimensionless constant $N_0 > 0$ the function $N(r)$ behaves as $(\Gamma(z)$- is Euler gamma-function)

$$N(r) \rightarrow \begin{cases} N_0 e^{\zeta}, & \text{with } \zeta = Mr, \quad \text{as} \quad r \rightarrow \infty; \\ N_0 \left(\frac{2\zeta}{\pi \sinh \pi \varrho}\right)^{1/2} \cos \left[\Im \ln \Gamma(i\varrho)\right] - \varrho \ln \left(\frac{\zeta}{2}\right), & \text{as} \quad r \rightarrow 0. \quad (35)\end{cases}$$

Similarly to the above real case (22), the complex non abelian gauge field (11) takes now the form with all three gauge-dependent ss- components:

$$A^j_{a\pm}(x) = e^{jka}n^k \left(\frac{N \epsilon^{i\omega}}{er} - n^i n^a \alpha(r) - \left(\delta^{ja} - n^j n^a\right) \frac{(\mp i)N \epsilon^{i\omega}}{er}\right). \quad (36)$$

Corresponding complex “chromo-magnetic” field strength (14) becomes

$$B^j_a(x) = n^i n^a \left(\frac{(\delta^{ja} - n^j n^a)}{er^2} - \frac{\epsilon^{jka}n^k}{er^2} \left(\pm i\right) r N' \epsilon^{i\omega}\right), \quad (37)$$

where the first gauge-independent longitudinal part exactly reproduces the non-abelian Wu-Yang monopole solution\(^\dagger\) while the two gauge-dependent transverse contributions exponentially disappear at $r \rightarrow \infty$ according to (34). The complex structure of these contributions in (37) in some sense is similar to real and imaginary axises of the complex plane. These “axises” are mutually exchanged by the shifting $\omega \mapsto \omega \mp \pi/2$ in accordance with above symmetry of equations of motion.
In view of (28), (A.3), (A.6), the energy-momentum tensor densities are also reduced exactly to that of non-abelian Wu-Yang monopole, since in (29), (A.2) all the terms excluding the first one vanish all together on the solution (31), (34), (37):

$$\Theta^{\mu 0}(r) = \frac{1}{2} e^2 r^4, \quad \Theta^{\mu k}(x) = \left( \frac{\delta^{jk}}{3} - n^j n^k \right) \frac{1}{e^2 r^4} \neq \frac{1}{6} \frac{1}{e^2 r^4}. \quad (38)$$

It is worthwhile to remind that the 't Hooft abelian projection of non-abelian monopoles solutions gives the same Dirac abelian monopole field for any one of them. It is a simple matter to show that, in accordance with topological nature of magnetic charge $g = \pm 4 \pi / e$, the 't Hooft abelian fields strength tensor:\(46, 47\)

$$\mathcal{F}^{\mu \nu} = \frac{h_a G^{\mu \nu}}{(h_b h_b)^{1/2}} - \frac{\varepsilon^{abc} h_a}{e (h_d h_d)^{3/2}} \left( \hat{D}^{\mu} \hat{h} \right)_b \left( \hat{D}^{\nu} \hat{h} \right)_c,$$
for $j, k, l = 1, 2, 3$, \quad (39)

$$\mathcal{F}^{j0} = \mathcal{E}^j, \quad \mathcal{F}^{jk} = -\varepsilon^{jkl} \mathcal{B}^l,$$ \quad gives $\mathcal{B}(x) = \pm \frac{n}{e r^2} = \frac{g x}{4 \pi r^2}$,

(40) for arbitrary functions $H \geq 0$ and $\alpha, \beta, Y$ in (9), (11), (14), giving also the same energy-momentum tensor (39) for the abelian Dirac monopole field (40). Inspite of that the solutions (37), as non-abelian Wu-Yang solution, for both abelian and non-abelian monopole lead to the identically divergent full energy (19), (38), such kind of fields are used in a number of modern field theoretical considerations: (22-26)

5. Other dyon solutions

According to Julia and Zee (10) the temporal stationary $ss$- component of gauge field, with the same 3D-space components (11), appears in the chosen rest frame as

$$A^0_a(x) = n^a \frac{J(r)}{r}, \quad \text{leaving} \quad \left( \hat{D}^0 \hat{h} \right)_a = 0, \quad \text{whence} \quad \left( \hat{D}^j \hat{E}^j \right)_a = 0, \quad (41)$$

for $\left( \hat{D}^j \hat{h} \right)_a = -n^j n^a \frac{d}{dr} \left( \frac{H}{e r^2} \right) - \left( \delta^{ja} - n^j n^a \right) \frac{H Y}{e r^2} - \varepsilon^{jka} n^k \frac{H \beta}{e r^2}$,

and $E^j_a = G^{j0}_a = -n^a \frac{d}{dr} \left( \frac{J(r)}{e r^2} \right) - \left( \delta^{ja} - n^j n^a \right) \frac{J Y}{e r^2} - \varepsilon^{jka} n^k \frac{J \beta}{e r^2}$,

(42) as “chromo-electric” field. With the same ansatz (9) for the Higgs fields this explicitly demonstrates Julia-Zee correspondence (10). The equations of motion (5) give now the system of five equations:

$$r^2 \frac{d}{dr} \left( \frac{r Y' + \alpha \beta}{r} \right) = Y \left( r^2 + \beta^2 + H^2 - J^2 - 1 \right) - \alpha \left( r Y' - \alpha Y \right), \quad (44)$$

$$r^2 \frac{d}{dr} \left( \frac{r \beta - \alpha Y}{r} \right) = \beta \left( r^2 + \beta^2 + H^2 - J^2 - 1 \right) + \alpha \left( r Y' + \alpha \beta \right), \quad (45)$$

$$\beta \left( r Y' + \alpha \beta \right) = Y \left( r Y' - \alpha Y \right), \quad (46)$$

$$r^2 \frac{d^2 H}{dr^2} = H \left\{ \frac{\lambda}{e^2} \left[ H^2 - (M r)^2 \right] + 2 \left( Y^2 + \beta^2 \right) \right\}, \quad (47)$$

$$r^2 \frac{d^2 J}{dr^2} = 2 J \left( Y^2 + \beta^2 \right). \quad (48)$$
The third Eq. (46) remains unchanged as Eq. (17) of previous system, as a direct consequence of chosen reference frame. Indeed, with the use of Eqs. (41), (43), (14), for the momentum density of (6) and for the corresponding full momentum one has:

$$\Theta_{jk}^{o} (x) \rightarrow (E_a \times B_a)^j = \varepsilon^{jkl} E_{a}^{k} B_{a}^{l} = \frac{2n_{j}}{c^{2}r^{4}} J \{ Y (r \beta' - \alpha Y) - \beta (rY' + \alpha \beta) \},$$

(J = 0 for $$\Theta_{m}^{o} (x)$$), whence $$P_{m,a}^{j} (x) = \int d^{3}x \Theta_{m,a}^{o} (x) \rightarrow 0,$$

in accordance with the chosen rest frame for both monopole and dyon cases. In the same terms the energy functional corresponding to density (50) now takes the form:

$$E_{d} [\Theta] = \frac{4\pi}{e^{2}} \int_{0}^{\infty} \frac{dr}{r^{2}} \left\{ \frac{1}{2} \left[ 1 - Y^{2} - \beta^{2} \right]^{2} + (rY' + \alpha \beta)^{2} + (r\beta' - \alpha Y)^{2} + \frac{1}{2} (rH' - H)^{2} \right\}.$$  

Corresponding stress tensor $$\Theta_{jk}^{d} (x)$$ is given by (A.1), (A.2) in Appendix. Similarly to monopole case the real substitution (22) reduces the system (44)–(48) and the energy (51) to those obtained by Julia and Zee (11–14) correspondingly as:

$$r^{2}K'' = K \left[ K^{2} + H^{2} - J^{2} - 1 \right], \quad r^{2}J'' = 2JK^{2},$$  

$$r^{2}H'' = H \left\{ \frac{\lambda}{e^{2}} [H^{2} - (Mr)^{2} + 2K^{2}] \right\},$$

and as (corresponding tensor $$\Theta_{jk}^{d} (x)$$ is given by (A.4), (A.6)–(A.10) in Appendix)

$$E_{d} [\Theta] = \frac{4\pi}{e^{2}} \int_{0}^{\infty} \frac{dr}{r^{2}} \left\{ \frac{1}{2} \left[ 1 - K^{2} \right]^{2} + (rK')^{2} + \frac{1}{2} (rH' - H)^{2} + \frac{1}{2} (rJ' - J)^{2} + (H^{2} + J^{2})K^{2} + \frac{\lambda}{4e^{2}} [H^{2} - (Mr)^{2}]^{2} \right\}.$$  

The reduction to these equations also takes place again for $$\alpha = \beta = 0$$ and $$Y \rightarrow K$$. In order to preserve the energy finite, the boundary conditions (27), (28) or (29) should be supplemented now with the conditions (11–14) for the function $$J(r)$$:

$$J (r) \rightarrow Cr + Q, \quad as \quad r \rightarrow \infty, \quad and \quad J (r) \sim O(r), \quad or \quad J (r) \sim O(r^{2}), \quad as \quad r \rightarrow 0.$$  

The first of them, thanks to Eqs. (52), changes also the asymptotic conditions (27) for $$K(r)$$ as $$r \rightarrow \infty$$ to (11)

$$K (r) \sim e^{-r \sqrt{M^{2} - \omega^{2}}}, \quad and \quad hence \quad J (r) - Cr + Q \sim e^{-2r \sqrt{M^{2} - \omega^{2}}},$$

Similarly to “chromo-magnetic” strength (24), (25), the “chromo-electric” strength (43) with real substitution (22), becomes a gauge-dependent generalization (with $$J \rightarrow H$$ for ($$\tilde{D}^{0}$$) of Julia-Zee expression (11) corresponding here to $$\omega = 0$$:

$$E_{a}^{J} (x) = -n^{j}n^{0} \frac{d}{dr} \left( \frac{J}{er} \right) - \Xi^{in} (n, \omega) \frac{JK}{er^{2}}.$$  

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Similarly to previous section, the third Eq. (45) here induces again the complex substitution $\beta = \eta Y$ with $\eta = \mp i$, which again reduces the first two Eqs. (44), (46) to one and the same equation for the complex function $Y(r)$:

$$r^2 \frac{d}{dr} \left( \frac{rY' + \alpha Y}{r} \right) = Y \left( H^2 - J^2 - 1 \right) - \eta \alpha \left( rY' + \alpha Y \right).$$

For the same complex substitution $N \pm = Ne^{-\eta \omega}$. this equation similarly transforms to

$$r^2 N'' - \left( (H^2 - J^2 - 1) + \left( r\omega' - \alpha \right)^2 \right) N = \mp i \left( 2rN' - N + r \frac{d}{dr} \right) \left( r\omega' - \alpha \right).$$

Here, for the same reasons as above, the second relation (21) for real $J^2$ leads again to the gauge-independent system of three equations, instead of (54), (55):

$$r^2 N'' - (H^2 - J^2 - 1) N = 0, \quad r^2 H'' = \frac{\lambda}{\varepsilon^2} \left[ H^2 - (Mr)^2 \right] H, \quad r^2 J'' = 0.$$  

Its explicit exact solution similar to (54) now reads (for real $C^2, Q^2, CQ$) as:

$$H(r) = \pm Mr, \quad J(r) = Cr + Q, \quad \text{whence, for } \zeta = r\sqrt{M^2 - C^2} > 0,$$

where $W_{\kappa,\pi}(z)$ is Whittaker function (29). So for arbitrary dimensionless constant $N_0 > 0$, the function $N_d(r)$ behaves similar to function $K(r)$ in (53) as

$$N_d(r) \rightarrow \begin{cases} N_0 (2\zeta)^{-\kappa} e^{-\zeta}, & \text{as } r \rightarrow \infty; \\ N_0 \left| \frac{\Gamma(2\zeta)}{\Gamma\left(\frac{1}{2} - \kappa + i\zeta\right)} \right| \left(2\zeta\right)^{1/2} \cos \left[ \Phi_{\kappa}(\overline{\zeta}) - \overline{\zeta} \ln(2\zeta) \right], & \text{as } r \rightarrow 0, \end{cases}$$

for $\Phi_{\kappa}(\overline{\zeta}) = \text{Im} \left\{ \ln \left[ \frac{\Gamma(2\zeta)}{\Gamma\left(\frac{1}{2} - \kappa + i\zeta\right)} \right] \right\}$, $M^2 - C^2 > 0$, with $C^2, Q^2 \geq 0$.  

For $C = 0$, i.e. $\kappa = 0$, $\overline{\zeta} \rightarrow \zeta$, with (29) $W_{\kappa,\pi}(2\zeta) = \sqrt{2\zeta/\pi} K_\zeta(\zeta)$, this solution for the dyon function $N_d(r)$ reduces to the previous monopole one (53), (54) with $g \rightarrow \overline{\zeta}$.  

In terms of functions $N \rightarrow N_d$ and $\omega$, the complex non-abelian gauge field (59) and magnetic strength (64) remain the same as for monopole. Their complex structure, as well as for the “chromo-electric” field strength (43), now, instead of (24), can be adsorbed into the complex transverse gauge-dependent tensor:

$$B^a_{\pm} = (\delta^a_{jk} n^k) e^r - n^j n^a \omega(r) e^r \pm i Y^a_{\pm} \frac{N_d(r)}{er}, \quad \frac{\hat{D}_a}{er} = - Y^a_{\pm} \frac{HN_d}{er^2},$$

$$B^a_{\pm} = \frac{n^j n^a}{er^2} - Y^a_{\pm} \frac{N_d(r)^2}{er^2}, \quad F^a_{\pm} = - n^j n^a \frac{d}{dr} \left( \frac{j}{er} \right) - Y^a_{\pm} \frac{J N_d}{er^2}. $$
According to (3), (6), (11) they obey one and the same “self-dual-like” relations:
\[
\left(\hat{D}_\pm \hat{E}_\pm\right)_a = 0, \quad \left(\hat{D}_\pm \hat{B}_\pm\right)_a = 0, \quad \left(\hat{D}_\pm \hat{A}_\pm\right)_a = 0. \tag{68}
\]

The obtained pairs of complex conjugate solutions (31), (61), (62), (65)–(67) belong to the complexified algebra \(so(3, C)\) which has the same generators as its original real form \(so(3, R) \equiv so(3)\). They show that some of the symmetry restrictions that are imposed on the fields from the very beginning by the reasons of formal mathematic convenience in any case should be relaxed for the specific classical solutions, as a rule, due to their dynamical properties (e.g. the translation invariance already for the real original ‘t Hooft-Polyakov monopole solution (9), (10)). Nevertheless, this relaxation may not destroy at all the symmetry structure of the model, but only change the way of its realisation, e.g., as when instead of the one real solution (11), (22), (24), (25), (27) the above two complex conjugate ones appear. The field of specific classical solution itself is not the group parameter of gauge transformations, but it defines a specific equivalence class with respect to them. As shown in Appendix B, both equivalence classes for above real and complex original solutions (\(\{0\}\) for \(\omega = 0\)) are given by the one and the same automorphism of those different elements of the same algebra \(so(3, C) \approx so(3)\). Both these different classes of spherically symmetric solutions are defined by the same spherically symmetric gauge transformation \(U(x) \in SO(3)\), where \(U(x) = \exp \{i\theta^a(x)T^a\} \) with \(\theta^a(x) = n^a\omega(r)\), for arbitrary real regular gauge function \(\omega(r)\), as:
\[
\hat{A}_\mu^{[0]}(x) \mapsto \hat{A}_\mu^{(\omega)}(x) = U(x)\left(\hat{A}_\mu^{[0]}(x) + \frac{i}{e} \partial_\mu\right) U^{-1}(x), \quad \hat{A}_0^{(\omega)}(x) = \hat{A}_0^{[0]}(x). \tag{69}
\]
\[
\hat{G}_{\mu\nu}^{[0]}(x) \mapsto \hat{G}_{\mu\nu}^{(\omega)}(x) = U(x)\hat{G}_{\mu\nu}^{[0]}(x) U^{-1}(x), \quad \text{i.e.} \quad \hat{B}^{[0]}(x) \mapsto \hat{B}^{(\omega)}(x), \tag{70}
\]
\[
\hat{E}_{\mu}^{[0]}(x) \mapsto \hat{E}_{\mu}^{(\omega)}(x), \quad \hat{h}_{[0]}(x) \mapsto \hat{h}_{(\omega)}(x) = U(x)\hat{h}_{[0]}(x) U^{-1}(x) = \hat{h}_{[0]}(x). \tag{71}
\]

Thus, the relaxation of the symmetry restrictions concerns only the dynamical structure of the solutions, and not their gauge-dependent structure parameterising this gauge equivalence. So, the complex classical solutions to dynamical equations for \(A_\mu^a\) or \(G_{\mu\nu}^a\) do not mean here the complexification of gauge group \(SO(3) \equiv SO(3, R)\) to \(SO(3, C)\) and do not destroy the gauge symmetry of the model.

The abelian electric strength arises as a longitudinal field from (61), (65)–(67). For (61), (67) it is generated by point electric charge \(q = Qg\) with magnetic \(g = \pm 4\pi/e\):
\[
\mathcal{E}^j = \frac{h_a E^j}{(h_a h_b)^{1/2}} = \pm \frac{n^j}{e} \frac{d}{dr} \left(\frac{J}{r}\right), \quad \mathcal{E}(x) = \pm \frac{Q}{e} \frac{n^j}{r^2} = q \mathcal{B}(x), \quad q = \int d^3 x (\nabla \cdot \mathcal{E}). \tag{72}
\]

Since the solution (61) for \(J(r)\) does not satisfy to boundary condition (55) at the origin, the electric charge here can not be expressed as the known integral (11, 10) over variable \(r\) only, giving here formally only zero results: \(q \to 0\), if \((K^2 \equiv Y^2 + \beta^2)\)
\[
q = \pm \frac{4\pi}{e} \int_0^\infty r dr \frac{d^2 J}{dr^2} = \pm \frac{8\pi}{e} \int_0^\infty r dr JK^2 = - \frac{4\pi}{Me} \int_0^\infty dr \frac{d}{dr} \left( r H \frac{d}{dr} \left(\frac{J}{r}\right) \right). \tag{73}
\]
The reason is that for \( x = r \) the operator expression \( \nabla_x \equiv n \partial_r + (1/r) \partial_n \) takes place only on the fields that are regular enough at \( r = 0 \), or excluding this point.

The obtained complex dyon solutions \((31), (61), (62), (65)–(67)\), as well as the monopol ones \((34)–(37)\), are independent of \( \lambda \) at all. Unlike the (also generalized here) ‘t Hooft-Polyakov and Julia-Zee solutions \((24), (57)\), they both additively contain the Wu-Yang-like monopole solution as their exact longitudinal parts, that simultaneoulsy are their exact asymptotic items at \( r \to \infty \) for both “chromo-magnetic” and “chromo-electric” fields:

\[
B_j^\pm(x) \to \frac{n^j n^a}{er^2}, \quad E_j^\pm(x) \to \frac{Q n^j n^a}{er^2} \approx Q B_j^\pm(x).
\]  

This approximate asymptotic relation transforms to the identity \((72)\) for abelian strengths corresponding to complex solutions \((66), (67)\). For \( Q = \mp i \) it also mean\((72)\) the approximate asymptotically self- or anti-self-dual non-abelian fields respectively. According to \((63)\), this approximate (anti) self-duality of the fields \((67)\) takes place for \( r > R_0 \) with the “self-duality radius”, defined by \( R_0^{-1} = \sqrt{M^2 - C^2} > 0 \).

According to fixed boundary conditions at \( r \to \infty \) \((55), (56)\), the generalized Julia-Zee dyon solutions \((24), (57)\) have the same asymptotic behavior \((74)\) with the same asymptotical self-duality relation, when \( Q = \mp i \). The corrections to this relation, coming as \((66)\) from functions \( K(r) \) and \( J(r) \), are of the order of \( e^{-r/R_0} \) for transversal parts of the field strengths and of the order of \( e^{-2r/R_0} \) for their longitudinal parts. However, for the obtained here complex solution \((67)\) the corrections to longitudinal part are absent at all.

The abelian strengths for generalized Julia-Zee dyon solutions \((24), (57)\) are, of course, the same as given by Julia and Zee\((1)\) there are the magnetic field \((40)\) of point-like Dirac monopole and electric field given by the first relation \((72)\) only. However, this electric field\((1)\) here is not the field of point-like electric charge from the second Eq. \((72)\). This equation becomes now only the approximate asymptotic relation between the abelian electric and magnetic strengths, and the full electric charge is given here by Eqs. \((73)\).

On the solutions \((61)–(67)\) the energy-momentum tensor \((51), (A.2), (A.6)\) coincides again with the abelian one for the fields \((72)\) and can be written as

\[
\Theta_{\mu\nu}^0(r) = \frac{(1 + Q^2)}{2 e^2 r^4}, \quad \Theta_{\mu\nu}^j(x) = \left( \frac{\delta_{jk}}{3} - n^j n^k \right) \frac{(1 + Q^2)}{e^2 r^4} + \frac{\delta_{jk}}{6} \frac{(1 + Q^2)}{e^2 r^4}.
\]

Thus, the obtained complex solutions have real energy-momentum density \( \Theta_{\mu\nu}^\mu(x) \), but their full energies divergent at the origin similar to Wu-Yang solution. At the same time, the expressions \((75)\) represent the asymptotic form of energy-momentum tensor of Julia-Zee solution at \( r \to \infty \). Indeed, according to \((27), (54)\), at least for \( \lambda > 0 \), the corrections to this asymptotic form of \( \Theta_{\mu\nu}^0(r) \) in \((74)\) and \( \Theta_{\mu\nu}^j(x) \) \((A.4)\) are, evidently, of exponential type only, of the orders of \( e^{-2r/R_0} \) and \( e^{-2mr} \).

With the help of corresponding embedding\((20)\) of algebra \( so(3) \), the similar solutions may be obtained for the more wide gauge groups.\((10)\)
6. Conclusions and outlook

The parity symmetry arguments\textsuperscript{11} as (4.13) of Goddard and Olive\textsuperscript{12}

\[
A_j^a(-x) \mapsto -A_j^a(x), \quad A_0^a(-x) \mapsto A_0^a(x), \quad h_a(-x) \mapsto h_a(x),
\]

which reduce the potential (11) to the ‘t Hooft ansatz (9), do not seem to be well
justified for the configuration and isotopic spaces simultaneously (especially if one
compare their ansatz (4.15) to (4.13)), because the meaning of “parity symmetry”
in isotopic (color) space is not well defined.\textsuperscript{17,22}

In fact, here we deal simply with the
additional Coulomb and transverse gauge conditions removing the functions \(\alpha(r)\) and \(\beta(r)\) according to Eqs. (12). We have shown here that the removing of that
constraints can be described in analytical way. It leads to explicit generalization of
the known ‘t Hooft-Polyakov and Julia-Zee solutions for arbitrary gauge and gives
the new complex \(\lambda\)- independent solutions (61), (62), (66), (67) for non-abelian
monopole and dyon fields, that have simple real energy-momentum tensor (75) and
are tightly connected with the known non-abelian Wu-Yang solution.\textsuperscript{15,16}

Nevertheless, one can also obtain zero energy-momentum tensor (75) i.e. solution
with zero mass and stress, choosing the parametres of complex non-abelian solution
(61), (62), (75) for real \(J^2\) as \(Q = \mp i\), with \(\Re C = 0\). This, at first sight exotic,
formal choice corresponds to exact (anti) self-duality (74) of longitudinal parts of
the strengths and delivers the absolute minimum to the energy (51), (75) of the so-
lution as \(E_d[\Theta] = 0\). For this choice, \(r_0 = \pm 1/2\), and this solution describes massless
complex self-dual dyon saturating the Bogomol’nyi bound\textsuperscript{16,19} \(E_d[\Theta] \geq f\sqrt{g^2 + q^2}\)
by real magnetic charge \(g\), equipped with imaginary “electric” charge \(q = Qg\) with
\(Q^2 = -1\). For abelian strengths this corresponds formally to pure imaginary “elec-
tric field” (72) of such complex dyon. The remaining arbitrary imaginary parameter
\(C\) of this self-dual dyon solution governs the value of its self-duality radius \(R_0\). E.g.
for \(C = 0\), the asymptotics (63) at \(r \to \infty\) becomes a simple exact solution for
\(N_\theta(r) = N_0 W_{0,1/2}(\zeta) = N_0 e^{-\kappa}\) with \(R_0 = 1/M\), similar to (55). For \(C \to \pm i\infty\)
independently of \(Q\), with \(\kappa \to \pm iQ\), there follows from the asymptotics (63), that
\(N_\theta(r) = N_0 W_{\kappa,i}(\zeta) \to 0\) as \(\zeta \to +\infty\), giving \(R_0 = 0\). So the non-abelian Wu-Yang
monopole solution is reproduced exactly by magnetic strength (67) for this choice
of parameter \(C\), and not only asymptotically as \(r \to \infty\) in (74). In other words, for
\(Q = \mp i\) with that choice of \(C\), the (anti) self-duality becomes an exact feature for
both non-abelian and abelian strengths (67), (72) simultaneously.

On the other hand, the case \(Y = \beta = 0\) leads to the same non-zero exact dyon
solution of Georgi-Glashow model, given by Eqs. (61), (62), (66), (67) for \(N_0 = 0\).
So, it is also given by the relations (74), becoming the exact equalities. For real \(Q\), it
may also be considered formaly as a real particular solution of Julia-Zee equations
(52), (53) with \(K(r) = 0\). However, it becomes again (anti) self-dual and massless
for \(Q = \mp i\). So, for classical solutions to non-abelian equations of motions for \(A^\mu\)
there are at least two independent sources of complexity: from \(A_0^0(x)\) and/or \(A_0^a(x)\).

The discussed massless complex (anti) self-dual dyon solutions can have a particu-
lar interest e.g. as a possible carriers of dark energy.\textsuperscript{34–36} Indeed, due to their
masslessness it is impossible to connect any Lorentz reference frame to them. Thanks to the pure imaginary value of corresponding interaction forces, their imaginary “electric” charges by definition should be fully invisible for usual electric and magnetic charges and currents, and thus for usual Maxwell electromagnetic fields. Furthermore, since the usual charged pairs creation is a pure electric effect with probability proportional to $E^2$, it is then impossible for pure imaginary “electric” charges and fields with $Q = \mp i$. That means also the absence for them of the usual loop corrections as well as of the known Gribov confinement of massless electric charges. Moreover, for all known classical, quantum mechanical and quantum field cases the relative abelian dynamic of two such, even complex, dyons in their center of mass frame is described by two combinations of their charges:

\[ C = g_1 g_2 + q_1 q_2 = g_1 g_2 (1 + Q_1 Q_2), \quad \mathcal{M} = g_1 g_2 - q_1 q_2 = g_1 g_2 (Q_1 - Q_2). \]

Thus, when $Q_1 = Q_2$ for $Q_{1,2}^2 = -1$, both the “Coulomb” hermitian $C$ and the “magnetic” non hermitian $\mathcal{M}$ contributions exactly disappear. That means that for two such complex dyons, for the same sign of their real magnetic charges $g_{1,2}$, their mutual magnetic repulsion may be either twice in $C/r_{12}$ by respective Coulomb “electric” interaction of the imaginary “electric” charges with opposite signs or exactly compensated by this interaction for the dyons with the same sign of “electric” charges. For the same reasons, the mutual attraction of two real magnetic charges $g_{1,2}$ with opposite signs in (40) also may be either twice or exactly compensated by Coulomb interaction of imaginary “electric” charges of those complex self-dual dyons. For the both last cases of compensation with $C = 0$, the value $\mathcal{M} = 0$ automatically, and the motion of self-dual dyons becomes completely free.

Therefore, the filling of the vacuum with those complex self-dual dyons with the one and the same sign of their real magnetic charges and with the one and the same sign of their imaginary “electric” charges, with constant particle $\bar{n}_\nu$ and energy $\bar{\rho}_\nu$ densities, naturally gives the necessary linear with distance macroscopic effect of antigravity for usual matter. So, being completely unobservable, such complex dyons can serve as effective carriers of dark energy as a kinetic energy of their free motion with speed of light, manifesting in the constant energy density $\bar{\rho}_\nu$ of that “dyonic vacuum medium”. Considering this density as “equilibrium” energy density of the spinless complex dyon “gas”, with the conventional value $34 \bar{\rho}_\nu \sim 10^{-46} \text{ (Gev)}^4$, one finds $\bar{n}_\nu \approx 1, 2 \pi^{-2} (30 \pi^{-2} \bar{\rho}_\nu)^{3/4} \sim 10^6 \text{ cm}^{-3}$, as estimation for corresponding “equilibrium” particle density. Of course, this is very rough estimate, because the pressure of that “vacuum medium” should be $p = -\bar{\rho}_\nu$, and not as for the free equilibrium gas with $p = \bar{\rho}/3$.

Besides, any simply connected compact macroscopic domain of such a “dyonic vacuum medium” with only the opposite sign of imaginary “electric” charges will “float up” as a whole in the first one also with a force, obviously linear with distance. Perhaps this can be used to explain different inflation rates at different scales. Here it is also natural to suppose, that contributions of non hermitian “magnetic” forces proportional to $\mathcal{M}$ vanish on macroscopic averaging over their directions.
real dyons they disappear as pure gauge since the value of $M$ is quantised [2, 4, 10].

On the other hand, the gauge-independent potential and strength decompositions, with exactly isolated non-abelian Wu-Yang-monopole-like contributions obtained here for the solutions (66), (67), are similar to gauge-independent decompositions of these quantities obtained in QCD [24, 25]. They are used there for explanation of monopoles condensation, leading to color confinement due to dual Meissner effect [24]. Since the non-abelian monopole and antimonopole are physically indistinguishable in QCD, they should have the same abelian magnetic charges $g$. Thus, if the last QCD decompositions [24, 25] can be reduced to the first ones $\lambda$-independent (61), (66), (67) together with the self-duality relations like (72), (74), also with pure imaginary values of abelian “electric” charge for $Q = \mp i$, then the both types of “vacuum media” with the massless complex self-dual dyons, those for dark energy and for color confinement, will coincide.

Appendix A.

For stationary spherically symmetric case (9), (41) the stress tensor is defined by (6), (8) as follows:

$$\Theta_{jk}(x) = \delta_{jk}\left\{ \frac{1}{2} \left[ (E_a)^2 + (B_a)^2 \right] - \frac{1}{2} \left[ (\hat{D}^a\hat{h})_a \right]^2 - \frac{\lambda}{4} \left( h_a h_a - f^2 \right)^2 \right\} - E_j a E_k a - B_j a B_k a + (\hat{D}^j h)_a (\hat{D}^k h)_a. \right.$$  

(A.1)

Substitution of Higgs fields (9) and strengths (14), (42), (43) transforms it to:

$$\Theta_{djk}(x) = \left( \frac{\delta_{jk}}{2} - n^j n^k \right) \left\{ \left[ 1 - Y^2 - \beta^2 \right]^2 + \left[ \left( \frac{J}{er} \right)^2 \right] - \left[ \left( \frac{H}{er} \right)^2 \right] \right\} +$$

$$+ \frac{n^j n^k}{e^2 r^4} \left\{ (r Y') + (r') \alpha \beta + (r' - r) \alpha Y + (J^2 - H^2) \left( Y^2 + \beta^2 \right) \right\} -$$

$$- \frac{n^j n^k}{e^2 r^4} \left[ H^2 - (M r)^2 \right]^2. \right.$$  

(A.2)

For the complex solution the substitution (31), (61), (62) immediately reduces this to expression (75) in the following simple form, explicitly showing its conservation:

$$\Theta_{djk}(x) = \left( \frac{\delta_{jk}}{2} - n^j n^k \right) \frac{(1 + Q^2)}{e^2 r^4}, \quad \partial_j \Theta_{djk}(x) = 0, \quad \text{for} \quad r \neq 0. \right.$$  

(A.3)

For $Q^2 = 0$ it is reduced to the corresponding monopole stress tensor (38).

The real substitution (22) simplifies tensor (A.2) to the following one

$$\Theta_{djk}(x) = \left( \frac{\delta_{jk}}{2} - n^j n^k \right) \left\{ \left[ 1 - K^2 \right]^2 + \left[ \left( \frac{J}{er} \right)^2 \right] - \left[ \left( \frac{H}{er} \right)^2 \right] \right\} +$$

$$+ \frac{n^j n^k}{e^2 r^4} \left\{ (r K')^2 + (J^2 - H^2) K^2 \right\} - \frac{n^j n^k}{4 e^4 r^4} \left[ H^2 - (M r)^2 \right]^2, \right.$$  

(A.4)
corresponding to Julia-Zee solution\(^{11}\) For \(J = 0\) it is reduced to monopole stress tensor \(\Theta^j_m(x)\) corresponding to \('t\) Hooft-Polyakov solution\(^{44}\).

The given form of Eqs. (38), (75) reflects the fact that any symmetric tensor \(\Theta^j_k\), e.g. \(\Theta^j_k(x) = \delta^j_k u(r) + n_j n_k \sigma(r)\), may be identically decomposed into the mutually orthogonal traceless and diagonal parts as

\[
\Theta^j_k = \left(\Theta^j_k - \frac{\delta^j_k}{3} Tr(\Theta)\right) + \frac{\delta^j_k}{3} Tr(\Theta), \quad \text{what gives (A.5)}
\]

\[
\Theta^j_k(x) = \left(n_j n_k - \frac{\delta^j_k}{3}\right) \sigma(r) + \frac{\delta^j_k}{3} \Pi(r), \quad \text{for } \Pi(r) = u(r) + \frac{1}{3} \sigma(r). \quad (A.6)
\]

According to M. Polyakov\(^{85}\) \(\Pi(r)\) represents the distribution of pressure, whereas \(\sigma(r)\) gives the distribution of shear forces.

For the complex dyon solution from (A.3), (A.6), in view of (75), it follows that

\[
e^2 r^4 \sigma_d(r) = -(1 + Q^2), \quad e^2 r^4 \Pi_d(r) = \frac{1}{6} (1 + Q^2). \quad (A.7)
\]

From Eq. (A.4) for Julia-Zee dyon it follows, that:

\[
e^2 r^4 u_d(r) = \frac{1}{2} \left(1 - K^2\right)^2 + \left(r J - J^2 - (r H' - H)\right)^2 - \frac{\lambda}{4 e^2} \left[H^2 - (M r)^2\right]^2, \quad (A.8)
\]

\[
e^2 r^4 \sigma_d(r) = \left(r K'\right)^2 + \left(J^2 - H^2\right) K^2 - \left[1 - K^2\right] + \left(r J - J^2 - (r H' - H)^2\right) \left[1 - K^2\right] - \frac{\lambda}{4 e^2} \left[H^2 - (M r)^2\right]^2 + \left(1 - K^2\right) + \left(J^2 - H^2\right) K^2. \quad (A.9)
\]

The analysis of stability conditions corresponding to these values for both Julia-Zee and \('t\) Hooft-Polyakov solutions was done recently in the work of Iu. Panteleeva\(^{46}\).

**Appendix B.**

The infinitesimal versions of the first of Eqs. (69)

\[
A^{a(1)}_\mu(x) \rightarrow A^{a(\omega)}_\mu(x) = A^{a(1)}_\mu(x) + e^{a b c A^{b(1)}_\mu(x)} \theta^c(x) + \frac{1}{e} \partial_\mu \theta^a(x), \quad (B.1)
\]

immediately coincide with that for \(^{11}, \quad 21\), \(^{22}\) and \(^{60}\) respectively, with the one and the same infinitesimal \(\theta^a(x) = n^a \omega(r)\). According to \(^{9}, \quad 11\) for the vector-matrix \(\mathcal{T} = e_j \delta^{ja} T^a\) defined in configuration space with Cartesian basis \(e_j\), by writing \(\theta^a(x) T^a = \omega(r) (n \cdot \mathcal{T})\) and \(U(x) = \exp \{i \omega(r) (n \cdot \mathcal{T})\}\), one can get the other relations \(^{69}, \quad 70, \quad 71\) from the known rotation formulae \(^{39, \quad 44}\) that for arbitrary fixed vectors \(n, s,\) with \([n, T^a] = [s, T^a] = 0\) and algebra \(^{11}\) look as:

\[
e^{i \omega(n \cdot \mathcal{T})} (n \cdot \mathcal{T}) e^{-i \omega(n \cdot \mathcal{T})} = (n \cdot \mathcal{T}), \quad \text{but } e^{i \omega(n \cdot \mathcal{T})} (s \cdot \mathcal{T}) e^{-i \omega(n \cdot \mathcal{T})} = (s \cdot \mathcal{T}) \cos \omega + (s \times n) \cdot \mathcal{T} \sin \omega,
\]

whence: \(e_j \gamma^{ja}_\pm(n, \omega) T^a \equiv \gamma^{ja}_\pm = e^{i \omega(n \cdot \mathcal{T})} \gamma^{ja}_\pm e^{-i \omega(n \cdot \mathcal{T})}, \quad (B.3)\)

and obviously the same for \(\bar{\zeta}^{ja}(n, \omega)\) \(^{25}, \quad 65\).
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