Logarithmic Conformal Null Vectors and SLE

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February 7, 2022

Abstract

Formal Loewner evolution is connected to conformal field theory. In this letter we introduce an extension of Loewner evolution, which consists of two coupled equations and connect the martingales of these equations to the null vectors of logarithmic conformal field theory.

Keywords: conformal field theory, SLE equation

1 Introduction

In the last two decades, conformal field theory in two dimensions has found many applications in different areas of physics, such as string theory, critical phenomena and condensed matter physics. In particular, the minimal models introduced in [1] reveal many exact solutions to various two dimensional phase transitions like Ising model at critical point or three critical Ising model and so on. In such models, the crucial role is played by the so called null vectors. In addition these standard transitions, conformal field theory has shown its strength in another type of critical phenomena, known as geometric critical phenomena. The most famous models of this type are percolation and self avoiding random walks. These are random spatial processes, where either the probability distribution is determined by equilibrium statistical mechanics, but main question is about the geometrical properties of the model (as in perculation), or the probability distribution is itself geometrical in nature (as in self avoiding walks).

On the other hand, in recent years another way to attack such problems has been proposed: Stochastic Lowener Evolution (SLE) [2]. SLE is a probabilistic approach to study scaling behavior of geometrical models. For a review see [3, 4, 5, 6]. SLE’s can be simply stated as conformally covariant processes, defined on the upper half plane, which describe the evolution of random domains, called SLE hulls. These random domains represent critical clusters.

This idea was first developed by Schramm [2]. He showed that under assumption of conformal invariance, the scaling limit of self avoiding random walks is SLE2. (SLE’s are parameterized by a real number k, and are abbreviated as SLEk) Also he claimed without proof that SLE6 is the scaling limit of critical percolation. The claim was proved later on by Smirnov [7]. He showed that in the scaling limit of percolation, conformal invariance exists and also using the new technic proved cardy’s formula [8, 9].

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An explicit relationship between SLE and CFT was discovered by Bauer and Bernard [10, 11, 12, 13]. They coupled SLE\(_k\) to boundary conformal field theory with specific central charge depending on the parameter \(k\). These CFT’s live on the complement of SLE hulls in the upper half plane. In such situations, boundary states emerge on the boundary conformal field theory. The good point is that these states are zero modes of the SLE\(_k\) evolution, that is they are conserved in mean. This means that all components of these states are local martingales of SLE\(_k\) and hence one is able to compute crossing probabilities in purely algebraic terms. The key point of these results is existence of null vectors, just as in the case of minimal models.

In this letter we attempt to connect SLE to a certain type of CFT’s, known as logarithmic conformal field theories. These theories were first introduced by Gurarie [14] and have found many applications in many various areas of physics. For example see [15, 16] and references therein. The difference between an LCFT and a CFT, lies in the appearance of logarithmic as well powers in the singular behavior of the correlation functions. In an LCFT, degenerate groups of operators may exist which all have the same conformal weight. They form a Jordan cell under the action of \(L_0\). This leads to very strange properties of these theories.

In this letter we first define SLE\(_k\) and compactly, then we review shortly Bauer and Bernard method in connecting SLE\(_k\) to CFT. Then we propose a modified version of SLE equation, which in fact consists of two coupled equations, and connect the two coupled Loewner’s equations to logarithmic conformal field theory.

### 2 Chordal Stochastic Loewner Evolutions and CFT

SLE is a growth process defined via conformal maps which are solutions of Loewner equation

\[
\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}
\]

in which \(\xi_t\) is a Gaussian random variable obeying the familiar Langevin’s equations of Brownian motion started from \(B_{t=0} = 0\). The autocorrelation of such random variables is defined to be

\[
E[\xi_t \xi_s] = k \min(t, s),
\]

which means that \(\xi_t\) is proportional to a normalized Brownian motion \(B_t\), that is : \(\xi(t) = \sqrt{k}B_t\).

The solution of (1) exists as long as \(g_t(z) - \xi_t\) is non zero. Let \(\tau(z)\) be the first time such that zero is a limit point of \(g_t(z) - \xi_t\) as \(t \to \tau\). So, for each point \(z\), the map exists if \(t < \tau(z)\). This means that at every time, only for a subset of the upper half complex plane (\(H\)) the map could be defined. We denote these subsets by \(K_t\)

\[
K_t = \{ z \in H : \tau(z) \leq t \}.
\]

These subsets are called hulls of SLE evolution. Using these sets one can define new sets \(H_t \equiv H/K_t\) (the complement of \(K_t\) in upper half plane \(H\)) which are open are and conformally equivalent to \(H\) via the mapping \(g_t\). In other words at each subsequent time \(t\), \(g_t\) defines a new mapping region \(H_t\), which is mapped into \(H\). The important property of \(g_t\) is that the mapping sets \(H_t\) continually get smaller and smaller as \(t\) gets larger. If \(\xi(t)\) is continuous, singularities in \(g_t(z)\) trace out a continues curve in \(H\) which are called trace and denoted by \(\gamma(t)\). It can be proved that the trace obeys the equation

\[
\gamma(t) = \lim_{\epsilon \to 0} g_t^{-1}(i\epsilon).
\]

The trace is a path which shows where the singularities arise and what points have been removed from the mapping region.
To connect SLE\(_k\) to CFT, it’s useful to define new map \(f_t(z) \equiv g_t(z) - \xi_t\) which satisfies the new stochastic differential equation

\[
\text{df}_t = \frac{2dt}{f_t} - d\xi_t
\]

The mapping \(f_t\) belongs to the group \(N_-\) of germs of holomorphic functions at \(\infty\) and have the form \(z + \sum_{m \leq -1} f_m z^{m+1}\). The members of the group of germs act on themselves: for \(f, g \in N_-\) one can define \(\gamma_f, g \equiv g \circ f\) with the property \(\gamma_{gof} = \gamma_f \cdot \gamma_g\). Now we will take an arbitrary function \(F\) in \(N_-\) and use Ito’s formula for equation (5) to obtain

\[
d\gamma_f, F = (\gamma_f, F)(\frac{2dt}{f_t} - d\xi_t) + \frac{f}{2}(\gamma_f, F'').
\]

By applying \(\gamma_f^{-1}\) to both sides of this formula we have

\[
\gamma_f^{-1}d\gamma_f = 2dt\left(\frac{2}{z}\partial_z + \frac{k}{2} \partial_z^2\right) - d\xi_t \partial_z
\]

The key idea is to associate to any \(\gamma_f \in N_-\) an operator \(G_f\) acting on appropriate representation of Virasoro group. So, the operator \(G_f\) should satisfy the following equation

\[
G_f^{-1}dG_f = dt(-2L_{-2} + \frac{k}{2}L_{-1}^2) + d\xi_t L_{-1}.
\]

Using this formula Bauer and Bernard have shown how SLE\(_k\) couples to CFT. They showed if \(|\omega\rangle\) be the highest weight vector of an irreducible Virasoro representation with central charge \(c_k = \frac{(3k-8)(6-k)}{2k}\) and conformal weight \(h_k = \frac{4k}{2k}\), which means it has a singular vector of level 2 in its descendants, then the time expectation value \(E[G_f | \omega]\) is a martingale of the SLE\(_k\) evolution, that is

\[
E[G_f | \omega] = G_s | \omega\rangle,
\]

where the time averaging is for all times less than \(s\). This means that correlation functions of the conformal field theory in \(H_t\) are time independent and equal to their value at \(t = 0\). Let’s what does the state \(G_t | \omega\rangle\) mean. Suppose \(|\omega\rangle\) be a boundary changing operator in \(H\), then one can show that the equivalent operator in \(H_t\) is just \(G_t | \omega\rangle\). In fact \(G_t | \omega\rangle\) is a generating function for all conserved quantities in chordal SLE.

### 3 SLE\(_k\) and LCFT

Before establishing a connection between LCFT and SLE\(_k\), we briefly recall a powerful method introduced in [18] for investigating LCFT’s. The method is based on a nilpotent weight, \(\theta\), added to the ordinary weight of a primary field. Also a compound field is defined via \(\Phi(z, \theta) = \phi(z) + \theta \psi(z)\), where \(\phi\) and \(\psi\) are the usual logarithmic pairs. This new compound field is easily seen to transform under scaling as

\[
\Phi(\lambda z, \theta) = \lambda^{-\left(h + \frac{\theta}{2}\right)} \Phi(z, \theta).
\]

With such definitions, one is able to extend all the calculations done in ordinary CFT to LCFT, just by adding a nilpotent variable where ever the weight of the fields are found.

Following the same scheme, one can generalize Loewner equation in the following way

\[
df_t(\theta) = \frac{2dt}{f_t(\theta)} - b(\theta) dB_t
\]
where all the quantities depending on $\theta$ can be expanded in powers of $\theta$, as an example we have $f_t(\theta) = f_t^{(1)} + \theta f_t^{(2)}$. The equation itself also can be expanded in terms of the nilpotent variable so that two distinct equation is obtained, on for $f^{(1)}$ and the other for $f^{(2)}$

$$df_t^{(1)} = \frac{2dt}{f_t^{(1)}} - b^{(1)} dB_t$$

$$df_t^{(2)} = -\frac{2f_t^{(2)}dt}{(f_t^{(1)})^2} - b^{(2)} dB_t \tag{12}$$

These two equations are the simplest generalization of the standard Loewner equation using $\theta$ formalism. Note that the prefactor of noise should also depend on $\theta$, as it is related to the weight of the primary field. Also note that the second equation is coupled to the first one and can not be considered independently.

Now we should see what happens to the function $F$, which belong to germs of holomorphic functions. Because we have two distinct equations, we will have two distinct functions, say $F^{(1)}$ and $F^{(2)}$, also we have to define two distinct variables $z$ and $w$ instead of one which happened to be $z$. The variable $z$ evolves with the first equation of (12) and $w$ evolves with the second equation of (12). In face one can construct a compound variable $Z(\theta) = z + \theta w$ just as the same we did for $f^{(1)}$ and $f^{(2)}$. It is simply observed from equations (12) that $F^{(1)}$ just depends on $z$ and $F^{(2)}$ depends both on $z$ and $w$. With a straightforward calculation, the generalized Ito formula turn out to be

$$dF(\theta) = \left(-2l_{-2} + \frac{1}{2}b(\theta)^2 l_{-1}^2\right) F(\theta) + b(\theta) l_{-1} F(\theta) dB_t \tag{13}$$

where $l_n = -z^{n+1} \frac{\partial}{\partial z} - (n + 1)z^n w \frac{\partial}{\partial w}$ satisfy the Witt algebra $[l_n, l_m] = (m - n) l_{n+m}$. These operators have good representation in $\theta$ language, just note that

$$\frac{\partial}{\partial z} = \frac{\partial Z(\theta)}{\partial z} \frac{\partial}{\partial Z(\theta)}, \quad \frac{\partial}{\partial w} = \frac{\partial Z(\theta)}{\partial w} \frac{\partial}{\partial Z(\theta)} = \theta \frac{\partial}{\partial Z(\theta)}. \tag{14}$$

Now by the definition of $l_n$ we have $l_n = -[Z(\theta)]^{n+1} \partial / \partial Z(\theta)$. The operators $l_n$ have the counterparts in the Hilbert space of a conformal field theory, $L_n$, which satisfy the well known Virasoro algebra. So, just as in the case of ordinary CFT, we can define proper Virasoro transformations associated with these new functions. Theses Virasoro operators should satisfy the following equation

$$dG(\theta) = G(\theta) \left(-2L_{-2} + \frac{1}{2}b(\theta)^2 L_{-1}^2\right) + G(\theta) b(\theta) L_{-1} dB_t, \tag{15}$$

which is again just like the one derived in CFT case, of course with $\theta$ modifications.

Before stating the connection between LCFT and the generalized SLE equation, we briefly recall how singular vectors are investigated in the context of LCFT’s. The problem of singular vectors in LCFT’s have been considered in several papers [17, 18], we will follow mainly [18] as the language is close to the one we have used here. In an LCFT, the representation of Virasoro algebra is constructed from a compound highest weight vector $|\omega + \theta\rangle = |\omega^{(1)}\rangle + \theta |\omega^{(2)}\rangle$, with the properties

$$L_0 |\omega + \theta\rangle = (h + \theta) |\omega + \theta\rangle,$$

$$L_n |\omega + \theta\rangle = 0 \quad (n \geq 1). \tag{16}$$

This means that $|\omega^{(1)}\rangle$ and $|\omega^{(2)}\rangle$ have the same weight and form a $2 \times 2$ Jordan cell. All the descendants are produced by applying $L_{-n}$’s to these states or equivalently to $|\omega + \theta\rangle$ with $n > 0$. 

4
There may be some representations, in which some of the descendants are perpendicular to all other vectors including themselves. These are the singular vectors. As an example with proper adjustment of $h$, the weight of the primary field and the central charge, the vector
\[
|\chi(\theta)\rangle = \left(-2 L_{-2} + \frac{3}{2(h + \theta) + 1} L_{-1}^2\right) |\omega + \theta\rangle
\] (17)
turns out to be a singular one. If you compare the term appearing in parenthesis with the one obtained in equation (15), you’ll find a fascinating similarity, just take $b(\theta)^2 = \frac{3}{2(h + \theta) + 1}$ then, you will have completely the same combination (We have dropped the last term in (15) for the moment). Let’s operate on both sides of (15) with $|\omega + \theta\rangle$. gathering all the terms we arrive at
\[
dG(\theta)|\omega + \theta\rangle = G(\theta)b(\theta)L_{-1}dB_t|\omega + \theta\rangle.
\] (18)
From the definition of Ito integrals, $G(\theta)$ is independent of $dB_t$, so the time average of the right hand side is vanishes. This means that the time average of changes in $G(\theta)|\omega + \theta\rangle$ is zero and it is a martingale of the model.

**Acknowledgement:** We would like to thank M. Saadat and M. R. Rahimi-Tabar for their helpful discussions.

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