THE COUNTING VERSION OF A PROBLEM OF ERDŐS

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Abstract. A set $A$ of natural numbers possesses property $P_h$, if there are no distinct elements $a_0, a_1, \ldots, a_h \in A$ with $a_0$ dividing the product $a_1 a_2 \ldots a_h$. Erdős determined the maximum size of a subset of $\{1, \ldots, n\}$ possessing property $P_2$. More recently, Chan, Győri and Sárközy [9] solved the case $h = 3$, finally the general case also got resolved by Chan [8], the maximum size is $\pi(n) + \Theta((n^{2/(h+1)}) (\log n))$.

In this note we consider the counting version of this problem and show that the number of subsets of $\{1, \ldots, n\}$ possessing property $P_2$ is $T(n) \cdot e^{\Theta((n^{2/3}/(\log n)^2))}$ for a certain function $T(n) \approx (3.517 \ldots) \pi(n)$. For $h > 2$ we prove that the number of subsets possessing property $P_h$ is $T(n) \cdot e^{\sqrt{n}(1+o(1))}$.

This is a rare example in which the order of magnitude of the lower order term in the exponent is also determined.

1. Introduction

We say that a set $A \subseteq \mathbb{N}$ possesses property $P_h$, if there are no distinct elements $a_0, a_1, \ldots, a_h \in A$ with $a_0$ dividing the product $a_1 a_2 \ldots a_h$. Let us denote the set of those subsets of a set $S$ that possess property $P_h$ by $P_h(S)$.

Property $P_h$ was introduced by Erdős [12] back in 1938, who studied the maximum size of a subset of $[n] := \{1, \ldots, n\}$ possessing property $P_2$. He proved that the extremal size is $\pi(n) + \Theta((n^{2/3}/(\log n)^2))$, that is, besides showing that the main term is $\pi(n)$, he could determine the lower order term up to a constant factor.

More recently, Chan, Győri and Sárközy [9] determined the lower order term for $h = 3$ too, then finally Chan [8] resolved the general case $h > 1$ by showing that the extremal size is $\pi(n) + \Theta((n^{2/(h+1)}) (\log n)^2)$. Chan even studied the dependence of the constant on $h$ hidden in the $\Theta_h$ notion (assuming $n$ is sufficiently larger than $h$). This dependence was determined (up to a constant factor) by Sándor and the first author [19] establishing that for $n$ sufficiently larger than $h$ the extremal size is $\pi(n) + \Theta((n^{2/(h+1)}) (\log n)^2)$.

Given now the satisfying answer on how large a subset of $[n]$ possessing property $P_h$ could be, a natural next step is to estimate how many subsets of $[n]$ possess property $P_h$, that is, how large $P_h([n])$ is. This is the question we are concerned about in this paper.

Indeed, enumerating subsets of $[n]$ satisfying various properties was initiated by Cameron and Erdős [7] in the 80s. In particular they considered the enumeration problem of primitive sets (note that a set is primitive if it possesses property $P_1$). The nature of this problem, and also the applied techniques, obtained results are different from the case of $P_h$ with $h \geq 2$. For more on the case of primitive sets we refer to the papers [1, 17, 18, 23].
Another related question is enumerating multiplicative Sidon subsets of \([n]\), also initiated by Cameron and Erdős. Recently, Liu and the first author [16] proved that the number of multiplicative Sidon subsets of \([n]\) is \(R(n) \cdot 2^{\Theta(n^{3/4}/(\log n)^{3/2})}\) for a certain function \(R(n) \approx 2^{1.815\pi(n)}\) which they specified. That is, the order of magnitude of the lower order term in the exponent is also determined.

1.1. Main result. Let \(H_h(n) := |\mathcal{P}_h([n])|\) denote the number of those subsets of \([n]\) that possess property \(P_h\).

**Theorem 1.1.** There exist positive constants \(c_1\) and \(c_2\) such that, for the number of those subsets of \([n]\) that possess property \(P_2\), we have

\[
T(n) \cdot e^{c_1 n^{2/3}/\log n} \leq H_2(n) \leq T(n) \cdot e^{c_2 n^{2/3}/\log n},
\]

if \(n\) is sufficiently large.

Let \(h \geq 3\) be an integer. For large enough \(n\), the number of subsets of \([n]\) possessing property \(P_h\) satisfies

\[
T(n) \cdot e^{\sqrt{n} e^{-11 \sqrt{n} \log n/\log n}} \leq H_h(n) \leq T(n) \cdot e^{\sqrt{n} e^{4 \sqrt{n} \log n/\log n}},
\]

where

\[
T(n) := \prod_{\sqrt{n} < p \leq n, \ p \ prime} ([n/p] + 1).
\]

A more explicit formula for the function \(T(n)\) is

\[
T(n) = e^{O(n^{1/2})} \cdot \prod_{i=1}^{n^{1/2}} (1 + 1/i)^{\pi(n/i)}.
\]

A more crude estimate is \(T(n) = (\alpha + o(1))^{\pi(n)}\), where

\[
\alpha := \prod_{i=1}^{\infty} (1 + 1/i)^{1/i} = 3.517\ldots
\]

Theorem [11] is another rare example of an enumeration result in which the correct order of magnitude of the lower order term is given. Moreover, in the case \(h > 2\) even the lower order term (in the exponent) is determined up to a \(1 + o(\log \log n/\log n)\) factor.

1.2. Related results. The past decade has witnessed rapid development in enumeration problems in combinatorics. In particular, a related problem of enumerating additive Sidon sets, i.e. sets with distinct sums of pairs, and its generalisation to the so-called \(B_h\)-sets was studied by Dellamonica, Kohayakawa, Lee, Rödl and Samotij [10, 11, 15]. For more recent results on enumerating sets with additive constraints, see e.g. [3, 4, 5, 13, 14, 20, 22]. Many of these counting results use the theory of hypergraph containers introduced by Balogh, Morris and Samotij [6], and independently by Saxton and Thomason [21]. We refer the readers to [6, 21] for more literature on enumeration problems on graphs and other settings.

**Organisation of the paper.** Section 2 sets up notations and tools needed for the proof. In Section 3 we prove Theorem 1.1. Some concluding remarks are given in Section 4.

**Asymptotic notations.** Throughout the paper we will use the standard notation \(\ll, \gg\) and respectively \(O\) and \(\Omega\) is applied to positive quantities in the usual way. That is, \(X \gg Y,\)
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\[ Y \ll X, \quad X = \Omega(Y) \text{ and } Y = O(X) \] all mean that \( X \geq cY \), for some absolute constant \( c > 0 \). If both \( X \ll Y \) and \( Y \ll X \) hold we write \( X = \Theta(Y) \). If the constant \( c \) depends on a quantity \( h \), we write \( X \gg h Y \), \( Y = \Omega h(Y) \), and so on.

## 2. Preliminary lemmas

Throughout the paper we will need some bounds on the prime-counting function \( \pi(x) \). The following standard bound will be enough for our purposes:

**Lemma 2.1.** If \( x \) is sufficiently large, then

\[
\frac{x}{\log x} + \frac{x}{(\log x)^2} \leq \pi(x) \leq \frac{x}{\log x} + \frac{2x}{(\log x)^2}.
\]

For proving Theorem 1.1 we will also use multiplicative bases and two lemmas from [19].

We say that the set \( B \subseteq \mathbb{Z}^+ \) forms a multiplicative basis of order \( h \) of a set \( S \), if every element \( s \in S \) can be written as the product of \( h \) members of \( B \). In particular, \( B \) is a multiplicative basis of order \( h \) for \( [n] \) if \( [n] \subseteq B^h \), that is, if each positive integer up to \( n \) can be expressed as a product of \( h \) (not necessarily distinct) elements of \( B \).

We will consider multiplicative bases of minimum size, the existence of such “small” bases is guaranteed by the following lemma.

**Lemma 2.2.** Let \( h \geq 2 \) be an integer. There exists a multiplicative basis \( B \) of order \( h \) for \( [n] \) of size

\[
|B| = \pi(n) + O_h \left( \frac{n^{2/(h+1)}}{(\log n)^2} \right).
\]

**Proof.** This follows from Theorem 1 of [19].

Finally, the lemma below describes a connection between a set possessing property \( \mathcal{P}_h \) and a multiplicative basis of order \( h \).

**Lemma 2.3.** Let \( A \subseteq [n] \) be a set possessing property \( \mathcal{P}_h \) and \( B \subseteq [n] \) be a multiplicative basis of order \( h \) for \( [n] \). Then there exists an injective mapping \( \varphi : A \rightarrow B \) such that for \( \varphi(a) = b \) there exist integers \( b_2, \ldots, b_h \in B \) such that \( a = bb_2 \ldots b_h \).

**Proof.** This is a special case of Lemma 12 in [19].

## 3. Proof of Theorem 1.1

### 3.1. First we consider the case \( h = 2 \).

#### 3.1.1. Lower bound

For obtaining the lower bound we will use linear hypergraphs. Let us recall that a hypergraph is linear if each pair of hyperedges intersects in at most one vertex.

Let us consider sets of the form \( A = A_1 \cup A_2 \subseteq [n] \), where:

- In \( A_1 \) each element has a prime factor from \((\sqrt{n}, n]\) and for every prime \( p \in (\sqrt{n}, n]\) the number of multiples of \( p \) contained in \( A_1 \) is at most one.
- The set \( A_2 \) can be obtained in the following way. We take a 3-uniform linear hypergraph \( G \) with vertex set \( V = \{ p : p \text{ is a prime, } p \in (n^{1/3}/2, n^{1/3}) \} \) and edge set \( E \). Let \( A_2 = \{ pqr : \{ p, q, r \} \in E \} \). That is, \( A_2 \) contains integers that cannot be written as a product of three distinct primes such that these three primes form a hyperedge of \( G \).
We claim that \( A = A_1 \cup A_2 \) possesses property \( \mathcal{P}_2 \). If \( a_0 \in A_1 \), then there is a prime \( p \in (\sqrt{n}, n] \) such that \( p \mid a_0 \) and \( p \nmid a \), if \( a \in A \setminus \{a_0\} \), so \( a_0 \nmid a_1 a_2 \). If \( a_0 \in A_2 \), then \( a_0 = pqr \) for three primes \( p, q, r \in (n^{1/3}/2, n^{1/3}) \). Assume that \( a_0 | a_1 a_2 \) for some \( a_1, a_2 \in A \setminus \{a_0\} \). We can assume that \( a_1 \) is divisible by at least two of the primes \( p, q, r \), for instance, \( pq^2 | a_1 \). If \( a_1 \in A_2 \), then this contradicts the linearity of \( G \). If \( a_1 \in A_1 \), then \( a_1 \) has one prime factor larger than \( \sqrt{n} \) and two prime factors \( (p \text{ and } q) \) from \( (n^{1/3}/2, n^{1/3}) \), which is a contradiction again.

Hence, each set that can be obtained as \( A = A_1 \cup A_2 \) possesses property \( \mathcal{P}_2 \). Note that for different pairs \( (A_1, A_2) \) we get different sets \( A = A_1 \cup A_2 \).

The number of choices for \( A_1 \) is
\[
T(n) = \prod_{\sqrt{n} < p \leq n, \text{prime}} (\lceil n/p \rceil + 1),
\]
since for each prime \( p \in (\sqrt{n}, n] \) we can include in \( A_1 \) either one of the \( \lceil n/p \rceil \) multiples of \( p \) (up to \( n \)) or none of them, the choices for different primes are independent from each other.

The number of choices for \( A_2 \) is the number of linear 3-uniform hypergraphs on vertex set \( V \). By dropping out at most three elements from \( V \) we get a set of cardinality \( |V'| \equiv 1, 3 \pmod{6} \). The number of linear 3-uniform hypergraphs on \( V \) is at least the number of Steiner Triple Systems on \( V' \) which is known \cite{24} to be \( 2^{\Theta(|V'|^2 \log |V'|)} \).

Hence, the following lower bound is obtained:
\[
\prod_{\sqrt{n} < p \leq n, \text{prime}} (\lceil n/p \rceil + 1) \cdot e^{\Theta(n^{2/3}/\log n)} \leq H_2(n).
\]

3.1.2. Upper bound. Now, we continue with the upper bound. According to Lemma \cite{22} there exists a multiplicative basis \( B = P \cup X \) of order 2, where
\[
P = \{p : p \text{ is a prime, } p \in (\sqrt{n}, n]\}
\]
and
\[
|X| \ll n^{2/3}/(\log n)^2.
\]
(Note that a multiplicative basis for \( [n] \) must contain all the primes up to \( n \), therefore the above \( P \) is a subset of any multiplicative basis \( B \), and we can set \( X := B \setminus P \) for a multiplicative basis \( B \) of minimum size.)

Also, by Lemma \cite{23} if \( A \) possesses \( \mathcal{P}_2 \), then there is an injective mapping \( \varphi : A \to B \), such that for any \( \varphi(a) = b \) we have \( b \mid a \). Let \( A_P \), resp. \( A_X \), be the set of elements mapped (by \( \varphi \)) to \( P \), resp. \( X \):
\[
A_P = \varphi^{-1}(P), \quad A_X = \varphi^{-1}(X).
\]

The number of choices for \( A_P \) is at most
\[
\prod_{\sqrt{n} < p \leq n, \text{prime}} (\lceil n/p \rceil + 1) = T(n).
\]

As \( |A_X| = |X| \ll n^{2/3}/(\log n)^2 \), the number of choices for \( A_X \) is at most \( 2^{O(n^{2/3}/\log n)} \), therefore, the number of choices for \( A = A_P \cup A_X \) is at most
\[
T(n) \cdot e^{O(n^{2/3}/\log n)},
\]
as needed.
3.2. Now, let \( h \geq 3 \) be any integer.

3.2.1. Lower bound. Let us consider sets \( A \) where each element has exactly one prime factor from \( (\sqrt{n} \log n, n]\) and for every prime \( p \in (\sqrt{n} \log n, n]\) the number of multiples of \( p \) contained in \( A \) is at most one.

Note that these sets satisfy the required property for every \( h \). Indeed, let \( a_0, a_1, \ldots, a_h \in A \) be distinct, then there exists a prime \( p \in (\sqrt{n} \log n, n]\) which divides \( a_0 \) and does not divide any of \( a_1, a_2, \ldots, a_h \), thus \( a_0 \nmid a_1 a_2 \ldots a_h \).

Let us give a lower bound on the number of these sets.

If \( p \in [\sqrt{n} \log n, n] \), then the number of choices (for the multiple of \( p \)) is \( \left\lfloor \frac{n}{p} \right\rfloor + 1 \) (it is also possible that none of the multiples of \( p \) is chosen to be in \( A \)). These can be chosen independently, since none of them is divisible by any other prime from \( (\sqrt{n} \log n, \sqrt{n} \log n, n] \).

Now, if \( p \in (\sqrt{n} \log n, \sqrt{n} \log n) \), then we have to exclude those multiples of \( p \) that have another prime factor \( q \in (\sqrt{n} \log n, \sqrt{n} \log n) \).

Note that \( q \leq n/p \). That is, the number of choices (for sufficiently large \( n \)) is at least

\[
\left[ \frac{n}{p} \right] + 1 - \sum_{q \leq n/p, \text{prime}} \left\lfloor \frac{n}{p} \right\rfloor = \left( [n/p] + 1 \right) \left( 1 - \frac{5 \log \log n}{\log n} \right),
\]

since

\[
\sum_{q \leq n/p, \text{prime}} \frac{1}{q} \leq \frac{1}{q \leq \sqrt{n} \log n, \text{prime}} \leq \frac{5 \log \log n}{\log n},
\]

according to Mertens’ theorem which states that for some constant \( M > 0 \) we have

\[
\sum_{q \leq x, \text{prime}} \frac{1}{q} = \log \log x + M + O \left( \frac{1}{\log x} \right).
\]

The choices are independent from each other.

Hence, the number of choices for the set \( A \) (for sufficiently large \( n \)) is at least

\[
(3.1) \quad \left( 1 - \frac{5 \log \log n}{\log n} \right)^{\pi(\sqrt{n} \log n) - \pi(\sqrt{n} / \log n)} \prod_{\sqrt{n} / \log n < p \leq n, p \text{ prime}} ([n/p] + 1).
\]

Observe that for large enough \( n \)

\[
(3.2) \quad 1 - \frac{5 \log \log n}{\log n} \geq e^{-11 \log \log n / 2 \log n},
\]

since for sufficiently small positive \( x \) we have \( 1 - x \geq e^{-1.1x} \). Also, Lemma 2.1 yields (for large enough \( n \)) that

\[
(3.3) \quad \pi(\sqrt{n} \log n) - \pi(\sqrt{n} / \log n) \leq 2 \sqrt{n}.
\]

According to (3.2) and (3.3) we obtain that

\[
(1 - \frac{5 \log \log n}{\log n})^{\pi(\sqrt{n} \log n) - \pi(\sqrt{n} / \log n)} \geq e^{-11 \sqrt{n} \log \log n / \log n}.
\]
Thus by using Lemma 2.1 again we obtain that for large enough \( n \)
\[
\prod_{\sqrt{n}/\log n < p \leq n, \ p \ \text{prime}} ([n/p] + 1) \geq T(n) \cdot \sqrt{n}^{\pi(\sqrt{n}) - \pi(\sqrt{n}/\log n)} \geq T(n) \cdot e^{\sqrt{n}}.
\]

Therefore, (3.1) yields that
\[
H_h(n) \geq T(n)e^{\sqrt{n}e^{-\frac{11\sqrt{n}\log n}{\log n}}}
\]

3.2.2. Upper bound. According to Lemma 2.2, there exists a multiplicative basis \( B = P \cup X \) of order \( h \), where
\[
P = \{ p : p \text{ is a prime, } p \in (n^{2/(h+1)}/\log n, n]\}
\]
and
\[
|X| \ll n^{2/(h+1)/(\log n)^2}.
\]

Also, by Lemma 2.3, if \( A \in \mathcal{P}_h([n]) \), then there is an injective mapping \( \varphi : A \to B \) satisfying that if \( \varphi(a) = b \), then \( b \mid a \). Let \( A_P \), resp. \( A_X \), be the set of elements mapped to \( P \), resp. \( X \):
\[
A_P = \varphi^{-1}(P), \quad A_X = \varphi^{-1}(X).
\]

The number of choices for \( A_P \) is at most
\[
(3.4) \prod_{n^{2/(h+1)}/\log n < p \leq n, \ p \ \text{prime}} ([n/p] + 1) = T(n) \cdot \prod_{n^{2/(h+1)}/\log n < p \leq \sqrt{n}, \ p \ \text{prime}} ([n/p] + 1).
\]

Observe that by Lemma 2.1 for large enough \( n \)
\[
(3.5) \prod_{n^{2/(h+1)}/\log n < p \leq \sqrt{n}/\log n, \ p \ \text{prime}} ([n/p] + 1) \leq n^{3\sqrt{n}/(\log n)^2} = e^{3\sqrt{n}/\log n}
\]
and
\[
(3.6) \prod_{\sqrt{n}/\log n < p \leq \sqrt{n}, \ p \ \text{prime}} ([n/p] + 1) \leq (\sqrt{n} \log n + 1)^{\pi(\sqrt{n})} \leq e^{((\log n)/2 + \log \log n + 1)(2\sqrt{n}/\log n + 8\sqrt{n}/(\log n)^2)}
\]
\[
\leq e^{\sqrt{n}e^{3\sqrt{n}\log n/\log n}}.
\]

As \( |A_X| \leq |X| \ll n^{2/(h+1)/\log n)^2} \), the number of choices for \( A_X \) is at most \( 2O(n^{2/(h+1)/\log n}) \leq e^{O(\sqrt{n}/\log n)} \), therefore, by (3.4), (3.5) and (3.6) the number of choices for \( A = A_P \cup A_X \) is – assuming that \( n \) is sufficiently large – at most
\[
T(n) \cdot e^{\sqrt{n}e^{4\sqrt{n}\log n/\log n}}.
\]

Remark 3.1. The obtained estimation is more precise in the case \( h \geq 3 \). Let us explain the reason for this, and look at the main difference between the cases \( h = 2 \) and \( h \geq 3 \).

The contribution from the product \( \prod_{p \mid n}([n/p] + 1) \), where \( p \) is taken from the interval, say, \((t, 2t)\) with \( t \sim n^\alpha \) is \( e^{\Theta(n^n)} \). Because of this, in case of \( h = 2 \) we could cut at any \( n^\alpha \) with \( \alpha \in [1/3, 2/3] \) in the definition of \( A_1 \), since the contribution from the product taken for \( p \in (n^{1/3}, n^{2/3-\varepsilon}) \) is negligible compared to the contribution of \( A_2 \). To achieve better bounds, one would have to study and understand the count corresponding to the “\( A_2 \)-part” better.
The analogue of $A_2$ could also be considered in the case $h \geq 3$ (by taking $(h + 1)$-uniform linear hypergraphs on $V = \{p : p$ is a prime, $p \in (n^{1/h}/2, n^{1/h})\})$, but this would only give an $e^{\Theta(n^{2/(h+1)/\log n})}$ factor. While the term $\Theta(n^{2/(h+1)/\log n})$ in the exponent turns out to be the second order term in the case $h = 2$, for $h \geq 3$ it is negligible compared to the additional contribution obtained by cutting lower than $\sqrt{n}$ in the definition of $A_1$. As it turns out from the calculation, we already get the precise lower order term if we cut at $\sqrt{n}/\log n$, though some additional care was needed, as an element of $[n]$ might have more than one prime factors larger than $\sqrt{n}/\log n$.

4. Concluding remarks

In this paper, we determine the number of those subsets of $[n]$ that possess property $P_h$, giving bounds that are optimal up to a constant factor in the exponent of the lower order term $e^{\Theta(n^{2/(h+1)/\log n})}$ for $h = 2$ and $e^{\sqrt{n}(1+o(1))}$ for $h > 2$.

A natural extension of these results would be to count those subsets of $[n]$ that satisfy the following property. Let us say that $A$ satisfies property $P_{r,s}$, if there are no distinct elements $a_1, a_2, \ldots, a_{r+s} \in A$ with $a_1 \ldots a_r \mid a_{r+1} \ldots a_{r+s}$. However, even determining the extremal size for subsets of $[n]$ seem to be difficult. The smallest interesting case is $r = 2, s = 3$; in this case we know that the largest possible size of a subset of $[n]$ possessing property $P_{2,3}$ is between $\pi(n) + n^{1/2+o(1)}$ and $\pi(n) + n^{2/3+o(1)}$.

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