Power-law liquid in cuprate superconductors from fermionic unparticles

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Recent photoemission spectroscopy measurements [arXiv:1509.01611] on cuprate superconductors have inferred that over a wide range of doping, the imaginary part of the electron self-energy scales as \( \Sigma'' \sim (\omega^2 + \pi^2 T^2)^\alpha \) with \( \alpha = 1 \) in the overdoped Fermi-liquid state and \( \alpha < 0.5 \) in the optimal to underdoped regime. We show that this non-Fermi-liquid scaling behavior can naturally be explained by the presence of a scale-invariant state of matter known as unparticles. We evaluate analytically the electron self-energy due to interactions with fermionic unparticles. We find that, in agreement with experiments, the imaginary part of the self-energy scales with respect to temperature and energy as \( \Sigma'' \sim T^{2+2\alpha} \) and \( \omega^{2+2\alpha} \), where \( \alpha \) is the anomalous dimension of the unparticle propagator. In addition, the calculated occupancy and susceptibility of fermionic unparticles, unlike those of normal fermions, have significant spectral weights even at high energies. This unconventional behavior is attributed to the branch cut in the unparticle propagator which broadens the unparticle spectral function over a wide energy range and non-trivially alters the scattering phase space by enhancing (suppressing) the intrinsic susceptibility at low energies for negative (positive) \( \alpha \). Our work presents new evidence suggesting that unparticles might be important low-energy degrees of freedom in strongly coupled systems such as the cuprate superconductors.

I. INTRODUCTION

Understanding the physics of cuprate superconductors involves identifying the low-energy degrees of freedom that can reproduce the bizarre features of the normal state which traditionally include \( T \)-linear resistivity, pseudogap, Fermi arcs, etc. Adding to the complexity are the recent angle-resolved photoemission spectroscopy (ARPES) measurements\(^1\) of the cuprates that revealed that its well-known \( T \)-linear resistivity can be construed as a slice of a unified power-law scaling behavior. Over a wide range of doping levels, the measured scattering rates in the non-superconducting state scale with respect to temperature and frequency as \( \Sigma'' \sim (\omega^2 + \pi^2 T^2)^\alpha \), with only the scaling exponent \( a \) varying with doping. The power-law smoothly varies from Fermi-liquid-like at overdoping, to one with \( a \sim 0.5 \) representing \( T \)-linear scattering rate at optimal doping, and to \( a \lesssim 0.5 \) at underdoping. Such a non-Fermi-liquid state of matter is dubbed a power-law liquid.

Theoretically, mechanisms yielding similar non-Fermi liquid scalings have been extensively studied\(^2-11\). In a marginal Fermi liquid\(^2\), a polarizability proportional to \( \omega/T \) leads to \( T \)-linear resistivity, while a \( \omega \)-wave Pomeronchuk instability in two dimensions\(^3\) yields self-energies with \( \omega^{2/3} \) and \( T^{2/3} \) dependence. In addition, similar behaviors can also be obtained by coupling quasi-particles with gauge bosons\(^4\), Goldstone bosons\(^5\), and critical bosons\(^6\) near a quantum critical point\(^7\). Furthermore, strong coupling theories using the anti-de Sitter spacetime (AdS)/conformal field theory (CFT) correspondence\(^8\) and Gutzwiller projection in hidden Fermi liquid theory\(^9\) also exhibit \( T \)-linear resistivity. In particular, the spectral functions calculated within the AdS/CFT formalism can also exhibit a range of power-law scaling when the scaling dimension of the boundary fermionic operator is tuned continuously\(^10,11\).

Because of the recent unified scaling observations, it is natural to invoke a scale-invariant sector such as unparticles as the effective low-energy degrees of freedom in the cuprates. Proposed a decade ago as a scale-invariant sector within the standard model\(^12\), unparticles can emerge in strong coupling theories as low-energy degrees of freedom. Exhibiting features similar to those of a fractional number of invisible massless particles\(^12\), unparticles are an incoherent state of matter that lack any particle-like behavior. They can be construed as a product of states with a continuous distribution of masses\(^13-15\) and can be constructed from theories in AdS\(^16\).

While extensively studied in high-energy physics, unparticles remain relatively new in condensed matter physics. In the context of the cuprates, unparticles have been proposed to explain the absence of Luttinger’s theorem in the pseudogap phase\(^17\) using zeros in the Green function\(^18\) and have also been found to yield unusual superconducting properties\(^17,19,20\) and optical conductivity\(^21\).

In this paper, we show analytically that interactions between electrons and fermionic unparticles can produce the power-law liquid revealed in the cuprates by recent ARPES experiments\(^1\). This paper is a follow-up to our recent paper that focused on bosonic unparticles\(^22\). Here we find that, in agreement with the experiments, the electron self-energy due to interactions with fermionic unparticles exhibits power-law scaling with respect to both energy and temperature: \( \Sigma'' \sim \omega^{2+2\alpha} \) and \( T^{2+2\alpha} \), where \( \alpha \) is the anomalous scaling of the unparticle propagator. In addition, we find that the occupancy number and susceptibility of fermionic unparticles, unlike those of normal fermions, have significant spectral weights even at high energies. These unconventional behaviors can be attributed to the branch cut...
in the unparticle propagator which broadens the unparticle spectral function over a wide energy range, and non-trivially alters the scattering phase space by enhancing (suppressing) the intrinsic susceptibility at low energies for negative (positive) $\alpha$.

II. ELECTRON-FERMIONIC UNPARTICLE SCATTERING

A. Model

We consider a system of electrons in the presence of a background of fermionic unparticles. The action of the system in Matsubara-Fourier space is given by

\[
S = T \sum_n \sum_p \psi_n^\dagger(p) G_0^{-1}(p, i\omega_n) \psi_n(p) + T \sum_n \sum_p \phi_n^\dagger(p) G_\alpha^{-1}(p, i\omega_n) \phi_n(p) + U T^3 \sum_{m,n,l,k,p,q} \psi_{m-l}^\dagger(k-q) \phi_{n+l}^\dagger(p+q) \phi_n(p) \psi_m(k),
\]

where $\psi$ is the non-relativistic electron field, $\phi$ is the fermionic unparticle field, $G_0$ is the bare electron Green function

\[
G_0(p, i\omega_n) = \frac{1}{i\omega_n - E_p},
\]

and $G_\alpha$ is the fermionic unparticle Green function

\[
G_\alpha(k, i\omega_n) = \frac{1}{(i\omega_n - \epsilon_k + \mu)^{1-\alpha}}.
\]

Here, $\epsilon_k$ is the unparticle energy spectrum, $1 - \alpha$ is the scaling exponent, and $\mu$ is the chemical potential. When $\alpha = 0$, the Green function reduces to that of a normal particle. In addition, $U$ is the interaction between electrons and unparticles, and $T$ is the temperature. The subscripts of the fields denote the dependence on the Matsubara frequency. In this model, the fermionic unparticles are assumed to exist up to a UV momentum cutoff, $\Lambda$ because they represent a low-energy description of some microscopic theory. For the unparticle Green function to be scale-invariant, we set $\mu = 0$ when $\alpha \neq 0$. While the literature in high-energy physics considers fermionic unparticles as relativistic four-spinors within the standard model\cite{23,24}, here in the context of the cuprates, we consider them as non-relativistic fermions. For simplicity, we also omit the normalization factor and the effects of spins.

In this paper, we focus on unparticles with $-1 < \alpha < 1$. In this case, instead of a simple pole, the unparticle Green function has a branch cut, which we choose to be along the negative energy axis. That is, the branch cut of $z^{1-\alpha}$ is chosen to be along $-\infty < z < 0$ with the phase angle defined in the range $-\pi < \theta < \pi$. Fig. 1 shows that, compared to particles, the spectral function of unparticles

\[
A_\alpha(k, \omega) \equiv -\frac{1}{\pi} \text{Im} G_\alpha(k, \omega + i\eta) = \frac{1}{\pi} |\sin(\pi\alpha)| \frac{\theta(\epsilon_k - \omega)}{|\epsilon_k - \omega|^{1-\alpha}}
\]

remains divergent at $\omega = \epsilon_k$, but has a broadened peak due to the presence of the branch cut, representing the incoherence of unparticles. Here $\theta(x)$ is the Heaviside step function. It is precisely the modeling of the broad incoherent background in the electron spectral function that unparticles are tailored to handle.

Figure 1: (a) The spectral function $A_\alpha(\omega)$ of unparticles compared to that of particles. As $\alpha$ deviates from zero, the delta peak in the spectral function broadens due to the branch cut in the Green function. (b) The energy and momentum dependence of the unparticle spectral function for a quadratic energy spectrum $\epsilon_k \sim k^2$ with $\alpha = 0.5$. The broadening of the spectral function reflects the incoherent nature of unparticles.

B. Electron self-energy

For a constant interaction $U$ between electrons and fermionic unparticles, Fig. 2 illustrates the lowest-order contribution to the electron self-energy $\Sigma(k, i\omega_n)$. This
can be written as

\[
\Sigma (k, i\omega_n) = -U^2 \sum_q T \sum_{i\omega_m} G_0 (k - q, i\omega_n - i\omega_m) \\
\chi_\alpha (q, i\omega_m),
\]  

where

\[
\chi_\alpha (q, i\omega_m) = \sum_p T \sum_{i\omega_n} G_\alpha (p, i\omega_n) G_\alpha (p - q, i\omega_n - i\omega_m)
\]

is the unparticle susceptibility, and \(G_0 (p, i\omega_m)\) is the electron Green function. While unparticle-particle interactions in the standard model are constrained by experiments to be weak\(^{12}\), the coupling strength \(U\) here in the cuprates can be significant.

\[\Sigma'' (k, \omega) = -U^2 \sum_{Pq} \tilde{S}''_{\alpha} (\omega, \epsilon_{p-q}, E_{k-q}, \omega),\]

where

\[
\tilde{S}''_{\alpha} (\epsilon_1, \epsilon_2, \epsilon_3, \omega) = \left[ n_B (\omega - \epsilon_3) + n_F (\omega - \epsilon_3) \right] \left[ \tilde{\kappa}_\alpha (\epsilon_1 - \omega + \epsilon_3) - \kappa_\alpha (\epsilon_1 - \omega + \epsilon_3) \right], \tag{8}
\]

\[
\tilde{\kappa}_\alpha (\epsilon, \epsilon') = \pi \int_{-\infty}^{\infty} dz \ n_F (z) \ A_\alpha (z - \epsilon) \ A_\alpha (z - \epsilon'). \tag{9}
\]

Here, \(n_{F/B} (z) = (e^z/T \pm 1)^{-1}\) is the Fermi (Bose) distribution, and \(E_k\) is the electron energy spectrum. To elucidate the analytic structure of \(\tilde{S}''_{\alpha}\), we note that, for \(\alpha > 0\) in the \(T \to 0\) limit, the integral \(\tilde{\kappa}\) evaluates to the closed-form expression

\[
\tilde{\kappa} (\epsilon, \epsilon') = \frac{1}{1 - 2\alpha} \left[ \frac{1}{\pi} \sin^2 \left( \pi \alpha \right) \right]^{1 - 2\alpha} 2F_1 \left[ 1 - \alpha, \frac{1 - \alpha}{2}; \frac{3 - \alpha}{2}; \frac{\epsilon - \epsilon'}{\xi (\epsilon, \epsilon')} \right], \tag{10}
\]

where \(2F_1 (a, b; c; z)\) is the hypergeometric function, and \(\xi (\epsilon, \epsilon') = \max (|\epsilon - \epsilon'|, \epsilon + \epsilon').\) As in the Fermi liquid case, \(\alpha = 0,\)

\[
\tilde{S}''_{\text{FL}} (\epsilon_1, \epsilon_2, \epsilon_3, \omega) = \pi \delta (\epsilon_1 - \epsilon_2 - \omega + \epsilon_3) \\
\[ \theta (-\epsilon_1) \theta (-\epsilon_3) \theta (\epsilon_2) \theta (\epsilon_1) \theta (\epsilon_3) \theta (-\epsilon_2) \right], \tag{11}
\]

we find that the analogous expression for unparticles \(\tilde{S}''_{\alpha} (\epsilon_1, \epsilon_2, \epsilon_3, \omega)\) can be nonzero for other values of energies \(\epsilon_1, \epsilon_2, \epsilon_3, \omega\). These features are illustrated in Fig. 3. These nonzero values provide additional contributions to the electron self-energy, and can be attributed to the broadening of the unparticle spectral function illustrated in Fig. 1.
parameter 

general form of the electron self-energy 

Next, to determine the scaling form of the electron self-energy in the \( T \to 0 \) limit, we note that the unparticle spectral function scales as

\[
A_\alpha(\lambda \omega) = \frac{1}{\pi} \lim_{\eta \to 0} \text{Im} G_\alpha(\lambda \omega + i\eta) = \frac{1}{\pi} \lambda^{-1+\alpha} \lim_{\eta \to 0} \text{Im} G_\alpha(\omega + i\eta) = \lambda^{-1+\alpha} A_\alpha(\omega) = \lambda^{-1+\alpha} A_\alpha(\omega).
\] (12)

Consequently, we have

\[
\bar{k}_{\alpha}(\lambda \epsilon, \lambda \epsilon') = \lambda^{-1+2\alpha} \bar{k}_{\alpha}(\epsilon, \epsilon'),
\] (13)

\[
\bar{S}''_{\alpha}(\lambda \epsilon_1 \lambda \epsilon_2, \lambda \epsilon_3, \lambda \omega) = \lambda^{-1+2\alpha} \bar{S}''_{\alpha}(\epsilon_1, \epsilon_2, \epsilon_3, \omega). (14)
\]

Then, approximating the density of states to be constant near the Fermi level, we find that the imaginary part of the electron self-energy in the \( T \to 0 \) limit becomes

\[
\Sigma''(k, \omega) = -U^2 \sum_{p_1 p_2 p_3} \delta_{p_1+p_3,p_2} \bar{S}''_{\alpha}(\epsilon_1, \epsilon_2, E_{p_3}, \omega)
\]

\[
\sim -U^2 \int d\epsilon_1 d\epsilon_2 dE \bar{S}''_{\alpha}(\epsilon_1, \epsilon_2, E, \omega),
\] (15)

which scales with respect to energy \( \omega \) as

\[
\Sigma''(k, \lambda \omega) = \lambda^{2+2\alpha} \Sigma''(k, \omega).
\] (16)

Therefore, the electron self-energy due to electron-unparticle interactions behaves as \( \Sigma'' \sim \omega^{2+2\alpha} \) at low temperatures, deviating from the Fermi liquid behavior of \( \Sigma''_{FL} \sim \omega^2 \). In the \( \omega \to 0 \) limit, a similar argument shows that \( \Sigma'' \sim T^{2+2\alpha} \) at low energies. Summarized in Fig. 4, these scaling behaviors of the electron self-energy are our main result; they hold for \(-1 < \alpha < 1\), and do not depend on the specific form of the electron energy spectrum, \( E_k \). For \( \alpha \lesssim 0 \), this non-Fermi-liquid state of matter quantitatively corresponds to the power-law liquid revealed in the cuprates by the recent ARPES measurements\(^1\).

**C. Susceptibility**

The scaling behavior of the electron self-energy can be traced back to the unparticle susceptibility \( \chi_{\alpha} \) defined by Eq. 6. From the calculations in Appendix A, its imaginary part after analytic continuation \( i\omega_n \to \omega + i\eta \) can be written as

\[
\chi''_{\alpha}(q, \omega) = \sum_{p} S''_{\alpha}(\epsilon_p, \epsilon_{p-q}, \omega),
\] (17)

where

\[
S''_{\alpha}(\epsilon_1, \epsilon_2, \omega) = \bar{k}_{\alpha}(\epsilon_1, \epsilon_2 + \omega) - \bar{k}_{\alpha}(\epsilon_2, \epsilon_1 - \omega). (18)
\]

The function \( \bar{k}_{\alpha} \) is defined above by Eq. 9. Fig. 5 illustrates the unparticle susceptibility in the \( q \to 0 \) and \( T \to 0 \) limit for a quadratic energy spectrum \( \epsilon_k \sim k^2 \) in...
two dimensions. We note three features distinctive from the analogous free electron susceptibility. First, the unparticle susceptibility is nonzero despite the chemical potential being restricted to be zero on account of scale invariance. This is unlike normal particles for which a zero chemical potential necessarily implies that there is zero filling and hence zero susceptibility. Second, the unparticle susceptibility does not have a cutoff at high energies. Third, from

\[ \chi''(q=0,\lambda \omega) = \hat{d}_p S''_\alpha(p^2, p^2, \lambda \omega) \]

\[ = \pi \int_0^\infty d\epsilon \chi''_\alpha(\epsilon, \epsilon, \lambda \omega). \]

\[ = \lambda^{2\alpha} \chi''_\alpha(0, \lambda \omega), \quad (19) \]

we see that the susceptibility scales as \( \chi''(0, \omega) \sim \omega^{2\alpha} \). Such a scaling form ensures that when \( \alpha < 0 \) (\( \alpha > 0 \)), the susceptibility is enhanced (suppressed) at low energies, as shown in Fig. 5. Such an enhancement (suppression) is crucial for the increased (decreased) scattering rate, as quantified by the electron self-energy in the previous subsection. These features completely violate the usual susceptibility sum rule and can be attributed to the broadening of the unparticle spectral function. As \( |\alpha| \) decreases, the features become less pronounced, as expected.

Similar non-Fermi liquid behavior induced by the enhancement of low energy susceptibility also occurs, for example, in systems where large portions of the Fermi surface are nested with a single nesting wave vector \(^{25,26}\), and in multiband models with orbital fluctuations \(^{27}\). Additionally, the self-energy of a Fermi liquid in the presence of weak impurities has an imaginary part of the form \( \Sigma'' \sim (E - E_F)^{d/2} \), where \( d \) is the spatial dimension \(^{28}\). Non-Fermi liquid behavior in this case can also be understood as an enhancement in the low energy spectrum of the susceptibility \(^{28}\).

**D. Occupancy**

In Fermi liquid theory, the quadratic scaling of the electron self-energy follows from a phase-space argument involving the occupancy of electrons. Therefore, it can be illuminating to explore how this argument is modified in the case of unparticles by computing the occupancy for unparticles,

\[ n_\alpha(\epsilon_p) = \int_{-\infty}^\infty dz n_F(z) A_\alpha(z - \epsilon_p) e^{i\phi}. \quad (20) \]

Fig. 6 shows that in the \( T \to 0 \) limit, unlike the Fermi distribution for particles, the occupancy of unparticles is significant even when \( \epsilon_p \) is large. This counterintuitive result can be understood by noting that the occupancy number measures the filling of states at momentum \( p \), instead of at energy \( \epsilon_p \). This distinction is important because, unlike the particle case, the unparticle spectral function is broadened over a wide energy range. Consequently, even unparticles with a large \( \epsilon_p \) possess a significant amount of low energy states that are filled at low temperatures. For \( \alpha < 0 \), these states enlarge the scattering phase space in the electron self-energy by enhancing the low energy susceptibility bubble, resulting in the non-Fermi liquid behavior described in the preceding section. In addition, the occupancy is notably non-symmetric, reflecting the particle-hole asymmetry of the unparticle Green function. This enhancement of
phase space undoubtedly reflects the enhanced scattering rate that ultimately grows linearly with temperature as opposed to the standard $T^2$ in the Fermi liquid case.

Figure 6: The energy $\epsilon_p$ dependence of the unparticle occupancy at $T = 0$. The significant occupancy at large $\epsilon_p$ differs from the Fermi distribution.

III. DISCUSSION AND CONCLUSIONS

As discussed in Ref. 1, a sublinear scaling of the electron self-energy can be interpreted as having a vanishing quasiparticle residue $Z$ in Fermi liquid theory. This signifies that interactions with fermionic unparticles with $\alpha < -0.5$ cause electrons to behave completely incoherently, which is unsurprising given the nature of unparticles. Nevertheless, since $\Sigma''(\omega = 0, T = 0) = 0$, the Fermi surface remains sharp.

We can similarly calculate the self-energy of unparticles due to self interactions, that is, when the electron line in Fig. 2 is replaced by another unparticle line. Naively, we expect the self-energy to scale as $\Sigma'' \sim \omega^{2+3\alpha}$ and $T^{2+3\alpha}$. This result, as well as the susceptibility and occupancy calculated above, can in principle be observed experimentally. However, any meaningful comparison with experimental observations would require further knowledge about the form of couplings between unparticles and external fields.

Our recent paper22 studied the effects of bosonic unparticles on the electron self-energy. While similar scaling behaviors were obtained, there are a few subtle differences. First, while a unitarity bound constrains the scaling dimension of bosonic unparticles, we do not know of any such constraint for fermionic unparticles. This freedom allows for a more qualitative agreement with experiments. Second, unlike the results in the bosonic case, there is no dependence on the dimensionality in the scaling of $\Sigma''$. This state of affairs obtains because we approximate the density of states of both electrons and fermionic unparticles to be constant near the Fermi level. Third, our susceptibility plots in Fig. 5 differ from that in Ref. 22, because a nonzero chemical potential was previously adopted.

In conclusion, we showed analytically that interactions between electrons and fermionic unparticles—a scale-invariant state of matter—can produce the power-law scaling revealed in the cuprates by recent ARPES experiments. In particular, we found that, at low temperatures and energies, the electron self-energy due to interactions with fermionic unparticles exhibits power-law scaling with respect to energy and temperature: $\Sigma'' \sim \omega^{2+2\alpha}$ and $T^{2+2\alpha}$, where $\alpha$ is the anomalous scaling of the unparticle propagator. This non-Fermi-liquid behavior can be attributed to the broadening of the unparticle spectral function over a wide energy range, which drastically alters the scattering phase space by enhancing (suppressing) the intrinsic susceptibility at low energies for negative (positive) $\alpha$. Although unparticles have zero chemical potential as required by scale invariance, they nonetheless can contribute to the electron self-energy due to the same broadening. Our results present new evidence suggesting that unparticles might be important low-energy degrees of freedom in the cuprates, and should inspire the interpretation of other experimental data using unparticles.

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Appendix A: Analytic evaluation of Matsubara sums

1. Susceptibility

The unparticle susceptibility defined by Eq. 6 involves the fermionic Matsubara sum

$$S_\alpha(\epsilon_1, \epsilon_2, i \omega_n) = T \sum_{i \omega_m} G_\alpha(i \omega_m - \epsilon_1) G_\alpha(i \omega_m - i \omega_n - \epsilon_2),$$
where $i\omega_n$ is a bosonic Matsubara frequency. Using Cauchy’s residue theorem, we rewrite the Matsubara sum as

$$S_{\alpha} (\epsilon_1, \epsilon_2, i\omega_n) = -\frac{1}{2\pi i} \oint_C dz \, n_F (z) G_\alpha (z - \epsilon_1) G_\alpha (z - i\omega_n - \epsilon_2),$$

where $n_F (z) = (e^{z/T} + 1)^{-1}$ is the Fermi distribution. Since the integrand is analytic except along $\text{Im}z = 0$ and $\text{Im}z = i\omega_n$, we use the contour $C$ illustrated in Fig. 7a.

Figure 7: The contours used to evaluate the Matsubara sums in (a) the unparticle susceptibility, and (b) the electron self-energy.

The integrals along the paths at large radius vanish when $\alpha < 1/2$. For $\alpha \geq 1/2$, a convergence factor $e^{\pm z_0}$ can be included so that the same integrals vanish. Consequently, the nonvanishing contributions to the contour integral are those along the branch cuts:

$$I_{b1} (\epsilon_1, \epsilon_2, i\omega_n) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dz \, n_F (z) \left[ G_\alpha (z^+ - \epsilon_1) - G_\alpha (z^- - \epsilon_1) \right] G_\alpha (z - i\omega_n - \epsilon_2)$$

$$= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dz \, n_F (z) 2i\text{Im} \left[ G_\alpha (z^+ - \epsilon_1) \right] G_\alpha (z - i\omega_n - \epsilon_2)$$

$$= \int_{-\infty}^{\infty} dz n_F (z) A_\alpha (z - \epsilon_1) G_\alpha (z - i\omega_n - \epsilon_2),$$

$$I_{b2} (\epsilon_1, \epsilon_2, i\omega_n) = -\frac{1}{2\pi i} \int_{-\infty + i\omega_n}^{\infty + i\omega_n} dz \, n_F (z) G_\alpha (z + i\omega_n - \epsilon_1) \left[ G_\alpha (z^+ - \omega - \epsilon_2) - G_\alpha (z^- - i\omega_n - \epsilon_2) \right]$$

$$= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dz \, n_F (z + i\omega_n) G_\alpha (z + i\omega_n - \epsilon_1) \left[ G_\alpha (z^+ - \epsilon_2) - G_\alpha (z^- - \epsilon_2) \right]$$

$$= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dz \, n_F (z) G_\alpha (z + i\omega_n - \epsilon_1) 2i\text{Im}G_\alpha (z^+ - \epsilon_2)$$

$$= \int_{-\infty}^{\infty} dz \, n_F (z) G_\alpha (z + i\omega_n - \epsilon_1) A_\alpha (z - \epsilon_2).$$

Here $z^\pm = z \pm i\eta$, with $\eta > 0$ being a positive real infinitesimal. After analytic continuation $i\omega_n \rightarrow \omega + i\eta$, the imaginary part of the Matsubara sum becomes

$$\text{Im}S_{\alpha} (\epsilon_1, \epsilon_2, \omega + i\eta) = \int_{-\infty}^{\infty} dz \, n_F (z) \left[ A_\alpha (z - \epsilon_1) \text{Im}G_\alpha (z - \omega - i\eta - \epsilon_2) + \text{Im}G_\alpha (z + \omega + i\eta - \epsilon_1) A_\alpha (z - \epsilon_2) \right]$$

$$= \int_{-\infty}^{\infty} dz \, n_F (z) \left[ A_\alpha (z - \epsilon_1) \pi A_\alpha (z - \omega - \epsilon_2) - \pi A_\alpha (z + \omega - \epsilon_1) A_\alpha (z - \epsilon_2) \right]$$

$$= \pi \int_{-\infty}^{\infty} dz \, n_F (z) A_\alpha (z - \epsilon_1) A_\alpha (z - \omega - \epsilon_2) - (\epsilon_1 \leftrightarrow \epsilon_2, \omega \rightarrow -\omega)$$

$$\equiv \tilde{\kappa}_\alpha (\epsilon_1, \epsilon_2 + \omega) - \tilde{\kappa}_\alpha (\epsilon_2, \epsilon_1 - \omega),$$
where we have defined

$$\tilde{\kappa}_\alpha (\epsilon, \epsilon') = \pi \int_{-\infty}^{\infty} dz \ n_F (z) A_\alpha (z - \epsilon) A_\alpha (z - \epsilon').$$

For $\alpha > 0$, we can evaluate this exactly in the $T \to 0$ limit using the unparticle spectral function in Eq. 4:

$$\tilde{\kappa}_\alpha (\epsilon, \epsilon') = \frac{1}{\pi} \sin^2 (\pi \alpha) \int_{-\infty}^{\min(\epsilon, 0, \epsilon')} dz \frac{1}{(\epsilon - z)^{1-\alpha} (\epsilon' - z)^{1-\alpha}}$$

$$= \frac{1}{\pi} \sin^2 (\pi \alpha) \left[ \frac{2}{\xi (\epsilon, \epsilon')} \right]^{1-2\alpha} \int_0^1 dt \left[ t^{-\frac{1}{2} - \alpha} \right]$$

$$= \frac{1}{\pi} \sin^2 (\pi \alpha) \left[ \frac{2}{\xi (\epsilon, \epsilon')} \right]^{1-2\alpha} _2F_1 \left[ 1 - \alpha, \frac{1}{2} - \alpha; \frac{3}{2} - \alpha; \frac{\epsilon - \epsilon'}{\xi (\epsilon, \epsilon')} \right],$$

where $\xi (\epsilon, \epsilon') = \max (|\epsilon - \epsilon'|, \epsilon + \epsilon')$, and $_2F_1 (a, b; c, z)$ is the hypergeometric function.

### 2. Self-energy

The electron self-energy defined by Eq. 5 involves the bosonic Matsubara sum

$$\tilde{S}_\alpha (\epsilon_1, \epsilon_2, \epsilon_3, i\omega_n) = T \sum_{\omega_{m'}} G_0 (i\omega_n - i\omega_{m'} - \epsilon_3) S_\alpha (\epsilon_1, \epsilon_2, i\omega_{m'})$$

$$= \frac{1}{2\pi i} \oint_{C'} dz' n_B (z') G_0 (i\omega_n - z' - \epsilon_3) S_\alpha (\epsilon_1, \epsilon_2, z') + TG_0 (i\omega_n - \epsilon_3) S_\alpha (\epsilon_1, \epsilon_2, 0),$$

where $i\omega_n$ is a fermionic Matsubara frequency, and $n_B (z) = (e^{z/T} - 1)^{-1}$ is the Bose distribution. Since the integrand has a branch cut along Im$z' = 0$ and a pole on a line Im$z' = i\omega_n$, we adopt the contour $C'$ shown in Fig. 7b. If $\alpha < 1$, the integrals along the paths at large radius vanish.

The integrals along the small circle of radius $r$ around the origin require special consideration. Since $S_\alpha (\epsilon_1, \epsilon_2, z')$ is analytic in the upper (lower) half plane, we see that $S_\alpha (\epsilon_1, \epsilon_2, z') \to S_\alpha (\epsilon_1, \epsilon_2, 0)$ in the limit $z' \to 0$ for $z'$ in the same domain. Hence, as the radius $r \to 0$, the integral along the small circle reduces to

$$\frac{1}{2\pi i} \oint_{|z'|=r} dz' n_B (z') G_0 (i\omega_n - z' - \epsilon_3) S_\alpha (\epsilon_1, \epsilon_2, z') \sim \left[ \frac{1}{2\pi i} \oint_{|z'|=r} dz' n_B (z') \right] G_0 (i\omega_n - \epsilon_3) S_\alpha (\epsilon_1, \epsilon_2, 0)$$

$$= -TG_0 (i\omega_n - \epsilon_3) S_\alpha (\epsilon_1, \epsilon_2, 0),$$

which exactly cancels the $i\omega_{m'} = 0$ term in $\tilde{S}_\alpha (\epsilon_1, \epsilon_2, \epsilon_3, i\omega_n)$. This cancellation can be physically motivated. First, notice that the imaginary part of the term contains the factor $\delta (\omega - \epsilon_3)$ after analytic continuation. Then, since $\omega = \epsilon_3$ corresponds to no energy transfer between unparticles and electrons, such a term understandably should not contribute to the electron self-energy.

Then, the nonvanishing contributions to $\tilde{S}_\alpha$ are simply the integrals along the lines Im$z' = 0$ and Im$z' = i\omega_n$.
\[\tilde{I}_{b1}(\epsilon_1, \epsilon_2, \epsilon_3, i\omega_n) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dz' n_B(z') G_0(i\omega_n - z', \epsilon_3) \left[ \tilde{S}_\alpha(\epsilon_1, \epsilon_2, z'') - \tilde{S}_\alpha(\epsilon_1, \epsilon_2, z'') \right] \]

\[\tilde{I}_{b2}(\epsilon_1, \epsilon_2, \epsilon_3, i\omega_n) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} dz' n_B(z') G_0(i\omega_n - z', \epsilon_3) 2i\text{Im}\tilde{S}_\alpha(\epsilon_1, \epsilon_2, z'') \]

Here, \( A_0 = -\frac{1}{2} \text{Im} G_0 \) is the spectral function of \( G_0 \), and we take principal values of the integrals in \( \tilde{I}_{b1} \) due to the cancellation mentioned above. Then, analytic continuation \( i\omega_n \rightarrow \omega + i\eta \) gives

\[\text{Im}\tilde{I}_{b1}(\epsilon_1, \epsilon_2, \epsilon_3, \omega + i\eta) = -\int_{-\infty}^{\infty} dz' n_B(z') A_0(\omega - z', \epsilon_3) \left[ \tilde{\kappa}_\alpha(\epsilon_1, \epsilon_2 + z') - \tilde{\kappa}_\alpha(\epsilon_2, \epsilon_1 - z') \right],\]

\[\text{Im}\tilde{I}_{b2}(\epsilon_1, \epsilon_2, \epsilon_3, \omega + i\eta) = -\int_{-\infty}^{\infty} dz' n_F(z') A_0(-z', \epsilon_3) \left[ \tilde{\kappa}_\alpha(\epsilon_1, \epsilon_2 + z' + \omega) - \tilde{\kappa}_\alpha(\epsilon_2, \epsilon_1 - z' - \omega) \right],\]

\[\text{Im}\tilde{S}(\epsilon_1, \epsilon_2, \epsilon_3, \omega + i\eta) = -\int_{-\infty}^{\infty} dz' [n_B(z') + n_F(z' - \omega)] A_0(\omega - z', \epsilon_3) \left[ \tilde{\kappa}_\alpha(\epsilon_1, \epsilon_2 + z') - \tilde{\kappa}_\alpha(\epsilon_2, \epsilon_1 - z') \right] \]

\[\text{Im}\tilde{S}_\alpha(\epsilon_1, \epsilon_2, \epsilon_3, \omega + i\eta) = [n_B(\omega - \epsilon_3) + n_F(-\epsilon_3)] \left[ \tilde{\kappa}_\alpha(\epsilon_2, \epsilon_1 - \omega + \epsilon_3) - \tilde{\kappa}_\alpha(\epsilon_1, \epsilon_2 + \omega - \epsilon_3) \right].\]
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