NONCOMMUTATIVE MIRROR SYMMETRY FOR PUNCTURED SURFACES

RAF BOCKLANDT

ABSTRACT. In [7] Abouzaid, Auroux, Efimov, Katzarkov and Orlov showed that the
wrapped Fukaya Categories of punctured spheres are derived equivalent to the categories
of singularities of a superpotential on certain crepant resolutions of toric 3 dimensional sin-
gularities. We generalize this result to other punctured Riemann surfaces and reformulate
it in terms of certain noncommutative algebras coming from dimer models. In particular,
given any consistent dimer model we can look at a subcategory of noncommutative ma-
trix factorizations and show that this category is $A_\infty$-isomorphic to a subcategory of the
wrapped Fukaya category of a punctured Riemann surface. The connection between the
dimer model and the punctured Riemann surface then has a nice interpretation in terms of
a duality on dimer models. Finally, we tie this result to the classical commutative mirror
symmetry.

1. INTRODUCTION

Originally homological mirror symmetry was developed by Kontsevich [30] as a frame-
work to explain the similarities between the symplectic geometry and algebraic geometry of
certain Calabi-Yau manifolds. More precisely its main conjecture states that for any com-
 pact Calabi-Yau manifold with a complex structure $X$, one can find a mirror Calabi-Yau
manifold $X'$ equipped with a symplectic structure such that the derived category of coherent
sheaves over $X$ is equivalent to the zeroth homology of the Fukaya category of $X'$. The
latter is a category that represents the intersection theory of Lagrangian submanifolds of
$X'$.

$$D^b\text{Coh } X \sim H^0\text{Fuk } X'$$

Over the years it turned out that this conjecture is part of a set of equivalences which are
much broader than the compact Calabi-Yau setting [26, 20, 1, 5, 6]. Removing the com-
pactness or Calabi-Yau condition often makes the mirror a singular object, which physicists
call a Landau-Ginzburg model [36, 37].

A Landau-Ginzburg model $(X, W)$ is a pair of a smooth space $X$ and a complex valued
function $W : X \to \mathbb{C}$, which is called the potential. On the algebraic side we associate to it
the dg-category of matrix factorizations $MF(X, W)$. Its objects are diagrams $P_0 \xrightarrow{W} P_1$, where $P_i$ are vector bundles and the composition of the maps results in multiplication with $W$. The morphisms are morphisms between these vector bundles equiped with a natural differential.

On the symplectic side we demand that the Lagrangians have their boundary fixed on
$W^{-1}(0)$. On the other hand if $X'$ is noncompact we need to tweak the notion of the Fukaya
category, by imposing certain conditions on the behaviour of the Langrangians near infinity
and using a Hamiltonian flow to adjust the intersection theory. This gives us the notion of
a wrapped Fukaya category [2].

In [7] Abouzaid, Auroux, Efimov, Katzarkov and Orlov proved an instance of mirror
symmetry between such objects. On the symplectic side they considered a sphere with
$k$ punctures and on the algebraic side they considered a special Landau Ginzburg model
on a certain toric quasiprojective noncompact Calabi Yau threefold and they proved an
equivalence between the derived wrapped Fukaya category of the former and the derived
category of matrix factorizations of the latter.

In this paper we aim to generalize their result to all Riemann surfaces with $k \geq 3$
punctures. On the algebraic side though we will not construct classical Landau-Ginzburg
models but instead we will look at noncommutative Landau-Ginzburg models \cite{38}. This means that we replace the commutative space $X$ with a noncommutative Calabi-Yau algebra $A$ and the potential will be a central element. The category of matrix factorizations needs also an adjustment: instead of vector bundles we must take projective modules.

The Calabi-Yau algebras under consideration will come from dimer models, which are certain quivers embedded in a Riemann surface. Such algebras, known as Jacobi algebras, also have a canonical choice of potential $\ell$ coming from the faces in which the quiver splits the Riemann surface.

The result we obtain is that for any consistent dimer model $Q$, we can find a full subcategory $\text{mf}(Q)$ of the category of all matrix factorizations of its Jacobi algebra $\text{MF}(\text{Jac} Q, \ell)$ which is $A_\infty$-isomorphic to a full subcategory $\text{fuk}(Q)$ of the wrapped Fukaya category $\text{Fuk}(S)$ of a punctured Riemann surface $S$. The category $\text{fuk}(Q)$ is constructed using a new dimer model $\tilde{Q}$, embedded in the closure of $S$ such that its vertices are the punctures. The new dimer can be obtained explicitly from the original dimer by a process called dimer duality.

We illustrate how our viewpoint and the approach in \cite{7} fit together. The simplest example gives us an equivalence between the sphere with 3 punctures $SS^3$ and the Landau-Ginzburg model $(\mathbb{C}^3, xyz)$. Commutatively this corresponds to the category of singularities of the three standard planes in affine 3-space. Noncommutatively\footnote{Note that in this example the Jacobi algebra $\text{Jac} Q \cong \mathbb{C}[X,Y,Z]$ is not noncommutative, but for all other dimer models it is.} we can see this as a dimer model on the torus with a superpotential $\ell$ coming from the faces.

\[
\begin{array}{c|c|c}
\text{Commutative version} & \text{Noncommutative version} \\
H^0\text{MF}(\mathbb{C}^3, xyz) & H^0\text{Fuk}(SS^3) & \text{mf}(Q) \cong A_\infty \text{fuk}(Q) \\
\end{array}
\]

On the right hand side $Q$ is embedded in a torus while its dual, $\tilde{Q}$, sits in a sphere and its 3 vertices are the 3 punctures in the commutative picture. The dual can be obtained by flipping over the clockwise faces, reversing their arrows and gluing everything back again.

By construction $\text{fuk}(Q)$ will generate the full wrapped Fukaya category. On the algebra side it is not completely clear whether $\text{mf}(Q)$ generates the full category of matrix factorizations $\text{MF}(\text{Jac} Q, \ell)$ because unless $Q$ sits inside a torus, the Jacobi algebra is not Noetherian. However if $Q$ does sit inside a torus, we will show that $\text{mf}(Q)$ indeed generates $\text{MF}(\text{Jac} Q, \ell)$. A result of Ishii and Ueda \cite{23} also implies that $\text{MF}(\text{Jac} Q, \ell) \cong \text{MF}(\tilde{X}, \ell)$ where $\tilde{X}$ is a crepant resolution of the center $\text{Spec} \mathbb{C}[\text{Jac} Q]$, so we recover the commutative result.

The outline of the paper is as follows: first we review some of the basics of $A_\infty$-structures, quivers, dimer models and some algebras associated to them. We combine these subjects to look at $A_\infty$-structures on certain dimer models, called rectified dimers. We classify some of these $A_\infty$-structures and then we turn to both sides of mirror symmetry. First we show that the Fukaya category associated to any polyhedral subdivision of a Riemann-surface gives rise to one of the $A_\infty$-structures we considered. Secondly we show that matrix factorizations of a consistent dimer model also give rise to such an $A_\infty$-structure and finally we tie the two sides together by constructing an explicit duality on dimer models. We end with a discussion about the connection between the commutative results in \cite{7}.
and the noncommutative geometry we employed. In view of readability of the paper, we deferred the proof of the classification of $A_\infty$-structures to an appendix.

2. $A_\infty$-CATEGORIES

In this section we will introduce the basics of $A_\infty$-categories. For more information we refer to [27, 32].

An $A_\infty$-category $C$ with degrees in $\mathbb{Z}_2$ consists of the following data:

- a set of objects $0b\ C$,
- for each pair of object $X, Y \in 0b\ C$ a complex $\mathbb{Z}_2$-graded vector space $\text{Hom}_C(X, Y)$,
- for each sequence of $n+1$ objects $X_0, \ldots, X_k \in 0b\ C$ a multilinear map $\mu_k$ of degree $2-k$
  $$\mu_k : \text{Hom}_C(X_1, X_0) \otimes \cdots \otimes \text{Hom}_C(X_k, X_{k-1}) \to \text{Hom}_C(X_k, X_0)$$

such that the identities
  $$[M_k] \sum_{s+i+t=k} (-1)^s \mu(f_1, \ldots, f_s, \mu(s+1, \ldots, f_s+i), f_{s+i+1}, \ldots, f_k) = 0$$

hold for any sequence of homogeneous homomorphisms $X_0 \xrightarrow{f_1} X_1 \cdots X_{k-1} \xrightarrow{f_k} X_k$. In this expression the sign takes the form $\sum s + lt + (2-l)(\deg f_1 + \cdots + \deg f_k)$. For $k = 1$, the identity becomes $\mu \mu(f) = 0$, so each hom-space can be considered as a complex with $d : f \mapsto \mu(f)$.

An $A_\infty$-category is called strictly unital if for every object $X$ there is an element $1_X \in \text{Hom}_C(X, X)$ of degree 0 such that

- $\mu(a, 1_X) = a$ if $a$ is a homomorphism with source $X$ and $\mu_2(1_X, a) = a$ is a homomorphism with target $X$.
- $\mu(a_1, \ldots, a_n) = 0$ if $n \neq 2$ and one of the $a_i = 1_X$.

If $\mu_i = 0$ for all $i \neq 2$ a strictly unital $A_\infty$-category is the same as a $\mathbb{Z}_2$-graded ordinary category. If $\mu_2 = 0$ for all $i \geq 3$ we get a dg-category.

Aside 2.1. Just like for ordinary categories we define an $A_\infty$-algebra as an $A_\infty$-category with one object and identifying the algebra with the hom-space from this object to itself. We can turn any $A_\infty$-category into an $A_\infty$-algebra by taking the direct sum of all hom spaces in the original category and extending the products multilinearly and setting products of maps that do not concatenate zero.

An $A_\infty$-functor $F : A \to B$ between two $A_\infty$-categories consists of

- a map $F_0 : 0b\ A \to 0b\ B$,
- for each sequence of $k+1$ objects $X_0, \ldots, X_k \in 0b\ A$ a linear map of degree $1-k$
  $$F_k : \text{Hom}_C(X_1, X_0) \otimes \cdots \otimes \text{Hom}_C(X_k, X_{k-1}) \to \text{Hom}_C(F_0 X_k, F_0 X_0)$$

(subject to the following identities)
  $$[M_k] \sum_{0 \leq s \leq t \leq k} \pm F(f_1, \ldots, f_s, \mu(f_{s+1}, \ldots, f_t), f_{t+1}, \ldots, f_k)$$
  $$\sum_r \sum_{1 \leq i_1 \leq \cdots \leq i_r \leq k} \pm \mu(F(f_1, \ldots, f_{i_1}), \ldots, F(f_{i_r+1}, \ldots, f_k))$$

for any sequence of homogeneous homomorphisms $X_0 \xrightarrow{f_1} X_1 \cdots X_{k-1} \xrightarrow{f_k} X_k$. An $A_\infty$-functor is called strict if $F_i = 0$ for $i > 1$, it is called an isomorphism if $F_0$ is a bijection and all $F_i$ are isomorphisms and it is called a quasi-isomorphism if $F_0$ is a bijection and the $F_1$ induce isomorphisms on the level of the homology of $d = \mu_1$.

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2Everything in this section can be generalized to $A_\infty$-categories with degrees in $\mathbb{Z}$ or any other cyclic group.
2.1. Minimal models. An \( A_\infty \)-category is called minimal if \( \mu_1 = 0 \). Note that unlike general \( A_\infty \)-categories, minimal \( A_\infty \)-categories are also genuine categories if we put \( f_1 f_2 := \mu(f_1, f_2) \) and forget the higher \( \mu \)'s. This is because in this case \([M_3]\) becomes the standard associativity identity.

An \( A_\infty \)-structure on an ordinary \( \mathbb{Z}_2 \)-graded \( \mathcal{C} \)-linear category \( \mathcal{C} \) is a set of multiplications \( \mu \) that turn \( \mathcal{C} \) into a minimal \( A_\infty \)-category such that as an ordinary category it is identical to the category structure of \( \mathcal{C} \).

**Theorem 2.2** (Kadeishvili). Let \( \mathcal{C} \) be an \( A_\infty \)-category and denote by \( H \mathcal{C} \) the ‘category’ with the same objects but as hom-spaces the homology of the hom-spaces in \( \mathcal{C} \). There is an \( A_\infty \)-structure on \( H \mathcal{C} \) such that there is an object-preserving quasi-isomorphism between \( H \mathcal{C} \) and \( \mathcal{C} \). This \( A_\infty \)-structure on \( H \mathcal{C} \) is unique up to object-preserving \( A_\infty \)-isomorphism.

How do we construct this new \( A_\infty \)-structure? In order to do this we use a graphical method from \([31]\). Set \( d = \mu_1 \) and for each \( \text{Hom}_\mathcal{C}(X, Y) \) choose a map \( h \) of degree \(-1\) on \( A \) such that
\[
h^2 = 0 \quad \text{and} \quad dhd = d.
\]
We will call this map a codifferential. The map \( \pi := 1 - dh - hd \) is a projection and we can identify \( \text{Im} \pi \) with \( \text{Hom}_\mathcal{M}(X, Y) \) because \( d\pi = \pi d = 0 \) and if \( dx = 0 \) then \( \pi x = x - d(hdx) \). Let \( i \) be the embedding \( \text{Hom}_\mathcal{C}(X, Y) = \text{Im} \pi \subset \text{Hom}_\mathcal{C}(X, Y) \).

Given a rooted tree \( \mathcal{T} \) with \( k + 1 \) leaves we can define a multilinear map
\[
m^\mathcal{T}: \text{Hom}_\mathcal{C}(X_1, X_0) \otimes \cdots \otimes \text{Hom}_\mathcal{C}(X_k, X_{k-1}) \to \text{Hom}_\mathcal{C}(F_0 X_k, F_0 X_0)
\]
by interpreting every leaf as the map \( i \), every internal node as a map \( h \mu \) and the root as \( \pi \mu \).

The new multiplication \( \mu^R_n \) is then defined as the sum \( \sum_T n m^\mathcal{T} \) over all rooted with \( n \) leaves for \( n > 1 \) and \( \mu^R_1 = 0 \).

2.2. Twisted completion. Given an \( A_\infty \)-category \( \mathcal{C} \) we define a twisted object \([27]\) as a pair \((M, \delta)\), where \( M = \mathbb{N}[0b \mathcal{C} \times \mathbb{Z}_2] \) is a formal sum of objects shifted by elements in \( \mathbb{Z}_2 \).

We will write such a sum as \( v_1[i_1] \oplus \cdots \oplus v_k[i_k] \) where the \( v_j \) are objects and the \( i_j \) shifts.

The map \( \delta \) is an upper triangular \( k \times k \)-matrix with entries \( \delta_{st} \in \text{Hom}_\mathcal{C}(v_s, v_t) \) of degree 1 and subject to the identity
\[
\sum_{n=1}^{\infty} (-1)^{n(n-1)/2} \mu_n(\delta, \ldots, \delta) = 0
\]
where we extended \( \mu_n \) to matrices in the standard way.

The homomorphism space between two such objects \((M, \delta)\) and \((M', \delta')\) is given by
\[
\bigoplus_{r, s} \text{Hom}(v_r, v'_s)[i_r - i_s]
\]
which we equip with an \( A_\infty \)-structure as follows:
\[
\mu(f_1, \ldots, f_n) := \sum_{t=0}^{\infty} \sum_{i_0 + \cdots + i_n = t} \pm \mu(\delta_i, \ldots, \delta_i f_1, \delta_i, \ldots, \delta_i f_n, \delta_i, \ldots, \delta).
\]
The \( \pm \)-sign is calculated by multiplying with a factor \((-1)^{n+t-k}\) for each \( \delta \) in the expression on position \( k \).

The \( A_\infty \)-category of twisted objects and their homomorphism spaces is denoted by \( H \mathcal{C} \). It also has a minimal model, which we denote by \( H \mathcal{C} \). Note that because it is a minimal model, \( H \mathcal{C} \) is a genuine category.
Aside 2.3. If $A$ is a genuine $\mathbb{C}$-linear category with a finite number of objects, such that the $\text{Hom}_A(X, Y)$ are finite dimensional and contain no nontrivial idempotents, then we can consider $A$ as a path algebra of a quiver with relations. We can construct the derived category $\text{DModA}$ and look at the smallest triangulated subcategory generated by $A$ as a module over itself. We can construct a $\mathbb{Z}$-graded category from this by putting $\text{Hom}(X, Y) := \oplus_{i \in \mathbb{Z}} \text{Hom}_{\text{DModA}}(X, Y[i])$.

On the other hand we can view $A$ as an $\mathbb{A}_\infty$-category with degrees in $\mathbb{Z}$, by putting all of $\text{Hom}_A(X, Y)$ in degree 0. It makes sense to look at $\text{HTwA}$ and it turns out that this category is equivalent to the category we defined above (see [27]). In this light $\text{HTwC}$ can be seen as a useful generalization of the derived category of an algebra.

3. Embedded quivers and dimers

As usual a quiver $Q$ is a finite (or locally finite) oriented graph. We denote the set of vertices by $Q_0$, the set of arrows by $Q_1$ and the maps $h, t$ assign to each arrow its head and tail. A nontrivial path $p$ is a sequence of arrows $a_k \cdots a_0$ such that $t(a_i) = h(a_{i+1})$. We write $p[i]$ to denote the arrow $a_i$ and we set $h(p) = h(p[k])$ and $t(p) = t(p[0])$.

A trivial path is just a vertex. A path $p$ is called cyclic if $h(p) = t(p)$ and the equivalence class of a cyclic path under cyclic permutation is called a cycle.

The path category $\mathbb{C}Q$ is the category with as objects the vertices, as homomorphisms linear combinations of paths and as composition concatenation.

Given a quiver $Q$, one can construct its double, $\bar{Q}$, which has the same number of vertices but for every arrow $a \in Q_1$ we add an extra arrow $a^{-1}$ with $h(a) = t(a^{-1})$ and $t(a) = h(a^{-1})$. The weak path category of $Q$ is the following quotient

$$\mathbb{C}\bar{Q} := \frac{\mathbb{C}Q}{\langle aa^{-1} = h(a), a^{-1}a = t(a) | a \in Q_0 \rangle}$$

A weak path in $Q$ is a path in $\bar{Q}$, viewed as a homomorphism in the weak path category. We also speak of weak arrows and cycles and if we want to stress that a path or cycle is not weak we will call it real.

A quiver is called embedded in a compact orientable surface without boundary $S$, if $Q_0$ is a discrete subset of $S$ and every arrow is a smooth embedding $a : [0, 1] \to S$ such that $h(a) = a(1)$ and $t(a) = a(0)$ and different arrows only intersect in end points. We identify $a^{-1}$ with the map $a$ in reverse direction. We also denote the surface in which the quiver embeds by $|Q|$. If $Q$ is a quiver embedded in $S$ and $\pi : \hat{S} \to S$ is the universal cover of $S$, we can construct a new quiver $\hat{Q}$ which is called the universal cover of $Q$. Its vertices are all points in $\hat{S}$ that map onto vertices in $Q$, and its arrows are maps that push forward to arrows in $Q$.

We say that the quiver $Q$ splits the surface $S$ if the complement of the quiver consists of a disjoint union of open disks. Each of these disks is bounded by a weak path $c$ that goes around it in counterclockwise direction. Such a cycle is called a boundary cycle and we collect these cycles in a set $Q^+_2$.

A dimer model [17, 14, 16] is a quiver that splits a surface for which every boundary cycle is either real or the inverse of a real cycle. The real cycles that are boundary cycles are called the positive cycles and are grouped in the set $Q^+_2 := Q^+_2 \cap CQ$. Negative cycles are those for which the inverse is a boundary cycle and we set $Q^-_2 := (Q^+_2)^{-1} \cap CQ$. Note that the orientability of the surface implies that every arrow will be contained in one positive cycle and one negative.

Example 3.1. We give 4 examples of embedded quivers that split a surface. The first 3 are embedded in a torus, the last in a double torus. Arrows and vertices with the same label are
identified.

The last 3 are dimer models.

The Jacobi category of a dimer model is the quotient of the path category by the ideal generated by relations of the form $r_a := r_+ - r_-$ where $r_+a \in Q_1^+$ and $r_-a \in Q_1^-$ for some arrow $a \in Q_1$:

$$\text{Jac}(Q) := \frac{\mathbb{C}Q}{\langle r_a | a \in Q_1 \rangle}.$$  

For the last dimer model in example 3.1 we have $Q_2^+ = \{abxcd\}$ and $Q_2^- = \{baxdc\}$ and

$$\text{Jac}(Q) := \frac{\mathbb{C}(a, b, c, d, x)}{\langle bxcdb, xcdab, cdab - dcba, dabx - baxd, abxc - cbax \rangle}.$$  

Aside 3.2. The reason these Jacobi algebras are important is because they appear as non-commutative analogues of Calabi-Yau manifolds. A compact 3-Calabi-Yau manifold can be defined as a smooth variety $X$ which has the following duality

$$\text{Ext}_X^i(\mathcal{F}, \mathcal{G}) = \text{Ext}_X^{3-i}(\mathcal{G}, \mathcal{F})^*$$

for all coherent sheaves $\mathcal{F}, \mathcal{G}$ on $X$.

Similarly a 3-Calabi-Yau algebra [15] can be defined such that it has a similar duality

$$\text{Ext}_X^i(M, N) = \text{Ext}_X^{3-i}(N, M)^*$$

for all finite dimensional left $A$-modules.

Not every Jacobi algebra will be Calabi-Yau, this will depend on the structure of the dimer model. To characterize such dimer models we introduce a notion of consistency. Several different notions are available in the literature [21, 11, 12, 35, 24, 10] but we will restrict to one: zigzag consistency.

Fix a dimer model $Q$ and its universal cover $\tilde{Q}$. For any arrow $\tilde{a} \in \tilde{Q}_1$ we can construct its zig ray $Z_{\tilde{a}}^+$. This is an infinite path

$$\ldots \tilde{a}_2 \tilde{a}_1 \tilde{a}_0$$

such that $\tilde{a}_0 = \tilde{a}$ and $\tilde{a}_{i+1} \tilde{a}_i$ sits in a positive cycle if $i$ is even and in a negative cycle if $i$ is odd. Similarly the zag ray $Z_{\tilde{a}}^-$ is the path where $\tilde{a}_{i+1} \tilde{a}_i$ sits in a positive cycle if $i$ is odd and in a negative cycle if $i$ is even. The projection of a zig or a zag ray down to $Q$ will give us a cyclic path because $Q$ is finite. Such a cyclic path will be called a zigzag cycle. A dimer model is called zigzag consistent if for every arrow $\tilde{a}$ the zig and the zag ray only meet in $\tilde{a}$:

$$Z_{\tilde{a}}^- [i] = Z_{\tilde{a}}^+ [j] \implies i = j = 0.$$  

In example 3.1 the second and fourth quiver are consistent, while the third quiver is not because $Z_{\tilde{a}}^- [3] = Z_{\tilde{a}}^+ [3] = \tilde{y}$. Note that a dimer model on a sphere is never zigzag consistent as its universal cover is finite.

**Theorem 3.3.** [10] If a dimer model is zigzag-consistent then its Jacobi Algebra is 3-Calabi-Yau.
4. Rectified Dimers and Gentle Categories

Given a quiver $Q$ that splits the surface we construct a dimer model, which we call the rectified dimer $RQ$.

- The vertices of $RQ$ are the centres of the arrows in $Q$
  \[ RQ_0 := \{ v_a := a(1/2) | a \in Q_1 \}. \]
- The arrows of $RQ$ are segments $\alpha = \tilde{v}_a \tilde{v}_b$ connecting the centres of two (weak) arrows that follow each other clockwise in one of the boundary cycles.

It is easy to check that this is indeed a dimer model. Each of the boundary cycles of $Q$ will give a positive cycle in $RQ$ and each of the vertices in $RQ$ will give us a negative cycle. In Euclidean geometry the process of cutting of the vertices of a polyhedron at the midpoints of the edges is called rectification, hence the name.

To each of the arrows we assign a $\mathbb{Z}_2$-degree. If $a b r$ is a boundary cycle of $Q$ and $a$ and $b$ are both real or both weak we give the arrow $\alpha = \tilde{v}_a \tilde{v}_b$ degree 1. Otherwise we give it degree 0. Note that if $Q$ is already a dimer then all arrows in $RQ$ have degree 1.

**Example 4.1.** Below are two examples of rectification. In the first we rectify a quiver on a torus with one vertex and two loops to obtain a dimer model with two vertices and four arrows. In the second example we rectify a tetrahedron to obtain an octahedron.

To indicate the grading we marked the degree zero arrows. The unmarked arrows all have degree 1.

Instead of associating to this dimer its Jacobi category, we will look at a different category. The gentle category of a rectified dimer is the quotient of the path category by the ideal generated by relations of the form $\alpha \beta$ where $\alpha$ and $\beta$ are consecutive arrows of a positive cycle.

\[ \text{Gtl}(RQ) := \frac{\mathbb{C}RQ}{(\alpha \beta | \exists \rho : \alpha \beta \rho \in RQ^2_2)} \]

**Remark 4.2.** The gentle category of a rectified dimer is a generalization of the gentle algebra of a triangulation which was introduced by Assem, Brüstle, Plamondon and Charbonneau-Jodoin in [3]. If the original quiver that splits the surface comes from a triangulation, these two algebras coincide. One can easily check that $\text{Gtl}(RQ)$ is a gentle algebra in the sense of [3] and [4].

We end this section with a lemma on the structure of $\text{Gtl}(RQ)$ that will be useful later on.

**Lemma 4.3.** If $v, w$ are two vertices in the rectified dimer $RQ$ then

\[ \text{vGtl}(RQ)w = \begin{cases} \mathbb{C}[c] & \text{if } c \in \mathbb{C}[c_1, c_2]/(c_1c_2) \\ 0 & \text{if } v = w \text{ and } c_1, c_2 \text{ are the two negative cycles through } v \\ \text{otherwise} & \end{cases} \]

Moreover all negative cycles have even degree.

**Proof.** Any nonzero nontrivial path can only be extended in one way to a nonzero path that is one arrow longer: one must add the arrow that sits in the same negative cycle as the last arrow. So, if $p$ is a nonzero path then all its arrows must be contained in one negative cycle. This implies the statement above.
The degree of a negative cycle is even because if on changes the direction in one of the arrows in the original dimer, the degree of the cycle doesn’t change (2 rectified arrows change degree). If all original arrows arrive in the original vertex representing the negative cycle then all rectified arrows in this cycle have zero degree. \( \square \)

We will now describe a specific \( A_\infty \)-structure on \( \text{Gtl}(RQ) \), which can be constructed inductively. For any sequence of paths \( p_1, \ldots, p_k \) and any cycle \( b_1 \ldots b_l \in RQ_2^+ \) with \( h(b_1) = t(p_i) \) we set

\[
\mu(p_1, \ldots, p_i b_1, b_2, \ldots, b_{l-1}, b_l p_{i+1}, \ldots, p_k) := (-1)^s \mu(p_1, \ldots, p_k).
\]

with sign convention \( s = l(p_1 + \cdots + p_i + k - i) \).

\[
\mu \left( \begin{array}{ccc}
\ldots & \ldots & \ldots \\
& & \\
& & \\
& & \\
& & \\
\ldots & \ldots & \ldots
\end{array} \right) = \pm \mu \left( \begin{array}{ccc}
\ldots & \ldots & \ldots \\
& & \\
& & \\
& & \\
& & \\
\ldots & \ldots & \ldots
\end{array} \right)
\]

For \( k > 2 \) we set \( \mu(q_1, \ldots, q_k) = 0 \) if we cannot perform any reduction of the form above and for \( k = 2 \) we use the ordinary product on \( \text{Gtl}(RQ) \). This \( A_\infty \)-structure is an example of a broader class of \( A_\infty \)-structures \( \mu^\kappa \), parametrized by maps \( Q_2^+ \times \mathbb{N} \rightarrow \mathbb{C} \), introduced in the appendix. In particular \( \mu = \mu^\kappa \) with \( \kappa(c, i) = 1 \) if \( i = 1 \) and \( \kappa(c, i) = 0 \) if \( i > 1 \).

**Theorem 4.4.**

1. \( \mu \) defines an \( A_\infty \)-structure on \( \text{Gtl}(RQ) \).
2. If \( \mu \) is an \( A_\infty \)-structure on \( \text{Gtl}(RQ) \) such that for every cycle \( c = a_1 \ldots a_l \in RQ_2^+ \) there is a \( \lambda_c \in \mathbb{C} \) such that

\[
\mu(a_1, \ldots, a_l) = \begin{cases} 
\lambda_c h(a_i) & j - i + 1 = 0 \text{ mod } l \\
0 & j - i + 1 \neq 0 \text{ mod } l
\end{cases}
\]

then \( \mu \) is \( A_\infty \)-isomorphic to \( \mu \).

**Proof.** This is a combination of theorem 9.9, lemma 9.10 and theorem 9.12. Proofs of these results can be found in the appendix [2]. \( \square \)

Different quivers in the same surface will give different gentle \( A_\infty \)-categories, but these categories are closely related. In fact if we go to the twisted completion the difference disappears.

**Lemma 4.5.** Suppose \( Q \) is an embedded quiver that splits the surface \( [Q] \) and \( \alpha \) is one of the arrows of \( Q \). If \( Q' \) is the new quiver obtained by changing the direction of \( \alpha \) (i.e. swapping its head and tail) then \( \text{Tw Gtl}(RQ), \mu \) and \( \text{Tw Gtl}(RQ'), \mu \) are isomorphic \( A_\infty \)-categories.

**Proof.** Let \( v_0 \) be the object in \( \text{Gtl}(RQ) \) corresponding to the arrow \( \alpha \) we want to reverse. We denote the corresponding object in \( \text{Gtl}(RQ') \) as \( v_0' \). All other objects we denote by \( v_i \) in both categories. We now define a strict functor \( \mathcal{F} : \text{Tw Gtl}(RQ) \rightarrow \text{Tw Gtl}(RQ') \):

- \( \mathcal{F}_0(v_0[0] \oplus v_0[1] \oplus \text{rest, } \delta) = (v_0'[0] \oplus v_0'[1] \oplus \text{rest, } \delta) \)
- \( \mathcal{F}_1(f) = f \)

It is easy to check that the degrees match up and that this is an isomorphism. \( \square \)

**Lemma 4.6.** Suppose \( Q \) is an embedded quiver that splits the surface \( [Q] \). Suppose that \( a_1 \ldots a_k \) is a boundary cycle and let \( b \) be a new arrow in this face connecting \( h(a_1) \) and \( h(a_k) \) with \( 2 < i < k - 1 \). Denote the quiver obtained by adding \( b \) to \( Q \) as \( Q' \), then \( \text{H Tw Gtl}(RQ) \) and \( \text{H Tw Gtl}(RQ') \) are equivalent as \( A_\infty \)-categories.

**Proof.** Let \( v_0 \) be the object in \( \text{Gtl}(RQ') \) corresponding to the arrow \( b \) we want to add. Denote the object corresponding to the arrow \( a_j \) by \( v_j \) in both categories. We use the Greek letters \( \alpha_j \) to denote he arrow between \( v_{j+1} \) and \( v_j \) and the arrow \( \beta_0 \) connects \( v_1 \) with \( v_0 \) and \( \beta_i \) connects \( v_i \) to \( v_i \).
We can identify $v_0$ with the object

$$w = \left( v_1[1] + \cdots + v_s[1], \delta := \begin{pmatrix} 0 & \alpha_1 \\ \vdots & \vdots \\ 0 & \alpha_{s-1} \end{pmatrix} \right).$$

We have a map $f_1 : v_0 \to w$ given by $(\beta_0 0 \cdots 0)^T$ and a map $f_2 : w \to v_0$ given by $(0 \cdots 0 \beta_i)$.

These maps are each other’s inverses in $\mathcal{H} \mathcal{T} \mathcal{W} \mathcal{G} \mathcal{T} \mathcal{L} (\mathcal{R} \mathcal{Q}^\prime)$. Clearly $\mu_2(f_1, f_2) = \mu(f_1, \delta, \ldots, \delta, f_2) = \mu(\beta_0, \alpha_1, \ldots, \alpha_s, \beta_i) = 1_{w}$. On the other hand we have

$$\mu(f_2, f_1) = \mu(f_2, f_1, \delta, \ldots, \delta) + \mu(\delta, f_2, f_1, \delta, \ldots, \delta) + \cdots + \mu(\delta, \ldots, \delta, f_2, f_1)$$

$$= \left( \begin{array}{ccc} \mu(\alpha_1, \ldots, \alpha_{s-1}, \beta_i, \beta_0) \\ \mu(\alpha_1, \ldots, \alpha_{s-1}, \beta_i, \beta_0, \alpha_1) \\ \vdots \\ \mu(\beta_i, \beta_0, \alpha_1, \ldots, \alpha_{s-1}) \end{array} \right)$$

$$= \left( \begin{array}{c} v_1 \\ v_2 \\ \vdots \\ v_s \end{array} \right) = 1_{w}.$$

Because the twisted completion is itself closed under twisted completion, $\mathcal{H} \mathcal{T} \mathcal{W} \mathcal{G} \mathcal{T} \mathcal{L} (\mathcal{R} \mathcal{Q}^\prime)$ is a full subcategory of $\mathcal{H} \mathcal{T} \mathcal{W} \mathcal{G} \mathcal{T} \mathcal{L} (\mathcal{R} \mathcal{Q})$. By construction the latter is also a full subcategory of the former, so they are equivalent.

**Corollary 4.7.** As an $A_{\infty}$-category $\mathcal{H} \mathcal{T} \mathcal{W} \mathcal{G} \mathcal{T} \mathcal{L} (\mathcal{R} \mathcal{Q})$, $\mu$ only depends on the genus of the surface and the number of the vertices of $Q$.

**Proof.** By adding enough arrows we can turn $Q$ into a triangulation, so the statement only needs to be proven for triangulations. By [19] we know that two triangulations of punctured surfaces can be turned into each other by a process of mutation. This process removes an arrow to create a quadrangle and then puts in a new arrow. By lemma 4.6 this does not change $\mathcal{T} \mathcal{W} \mathcal{G} \mathcal{T} \mathcal{L} (\mathcal{R} \mathcal{Q})$. \qed

## 5. The relation with the wrapped Fukaya category

A Liouville manifold is a manifold $S$ with punctures equipped with a 1-form $\theta$ such that $\omega = d\theta$ is a symplectic form. The symplectic forms allows us to transform the 1-form into a vector field which generates a flow called the Liouville flow and we demand that the Liouville flow points towards the punctures near them. To any Liouville manifold one can associate an $A_{\infty}$-category called the wrapped Fukaya category $\mathcal{F} \mathcal{u} \mathcal{k} \mathcal{a} \mathcal{y}$. We start with objects that are graded exact Lagrangian submanifolds which are invariant under the Liouville flow outside a compact subset of $S$.

The space of morphisms between two Lagrangians $\mathcal{L}_0$ and $\mathcal{L}_1$ is defined as the vector space spanned by the intersection points between $\mathcal{L}_1$ and the time 1 flow of $\mathcal{L}_0$ for a Hamiltonian $H : S \to \mathbb{R}$ which is quadratic at infinity.

$$\text{Hom}(\mathcal{L}_0, \mathcal{L}_1) = C^0(\mathcal{L}_0 \cap \mathcal{L}_1) \text{ where } dH = \omega(\frac{d}{dt} \phi^t, -)$$

Note that such intersection points are also in one to one correspondences with time one chords between $\mathcal{L}_0$ and $\mathcal{L}_1$. These are curves $\gamma : [0, 1] \to S : t \mapsto \phi^t(\gamma(0))$ such that $\gamma(0) \in \mathcal{L}_0$ and $\gamma(1) \in \mathcal{L}_1$.

Consider a sequence of intersection points $\mathcal{L}_0 \xrightarrow{p_k} \mathcal{L}_{k-1} \cdots \mathcal{L}_2 \xrightarrow{p_1} \mathcal{L}_1$. The product is defined as

$$\mu_n(p_k, \ldots, p_1) = \sum_{\mathcal{L}_0 \xrightarrow{\phi^t} \mathcal{L}_k} m_q.$$
if we look at the intersection points between $\phi^t L_0$ and $L_k$ and vary $t$ from $k$ to 1. Finally, to get the full wrapped Fukaya category we take the twisted completion\(^3\). The definition of the wrapped Fukaya category has a lot of technical issues for which we refer to [39] and [2].

In the case of a punctured surface, such Lagrangians can be seen as curves connecting the punctures. By tweaking the Liouville structure on the punctured surface we can ensure that each arrow $a_i$ of an embedded quiver is an exact Lagrangian $L_i$ and the punctures of the surface match the vertices of the quiver.

The Hamiltonian will make the flow spiral around the punctures, as illustrated below.

This implies that for every natural number $n$ there is a time one chords between the two Lagrangians that circle around the puncture counterclockwise $n$ times. Likewise in the algebra $\mathfrak{gt}1(RQ)$, there is also exactly one path between the two vertices that circle around the puncture $n$ times. This means that there is a bijection between $\text{Hom}_{\mathfrak{fruk}}(L_i, L_j)$ and $\nu_n \mathfrak{gt}1(RQ) \nu_{a_i}$. So, considered as a vector space the full subcategory of $F_Q \subset \text{Fuk}_S \setminus Q_0$ containing the objects $L_i$ and the category $\mathfrak{gt}1(RQ)$ are isomorphic.

**Theorem 5.1.** Consider a quiver $Q$ that splits a surface $S$ and put a Liouville structure on $S \setminus Q_0$.

1. The full subcategory $\mathfrak{fruk}(Q)$ of the wrapped Fukaya category $\text{Fuk}_S \setminus Q_0$ with objects the arrows of $Q$ is $\mathbb{A}_\infty$-isomorphic to $\mathfrak{gt}1(RQ)$ with its generic $\mathbb{A}_\infty$-structure.
2. $\text{Fuk}_S \setminus Q_0$ is equivalent to $\text{Tw} \mathfrak{gt}1(RQ), \mu$.

**Proof.** (Sketch) We already know that there is a linear isomorphism, so we only need to check the ordinary multiplications. The pictures of the compositions are easier to parse if we go to the universal cover of a closed neighborhood of a puncture. As such a neighborhood is a cylinder, the universal cover will be the upper half plane. The original Lagrangians will be vertical half lines and the flowed lines can be seen as sloped curves.

We can now draw a picture for the ordinary multiplication and the product of all arrows around a positive cycle in $RQ$. The immersed polygon is shaded in both pictures.

---

\(^3\)In principle we need to take the Karoubi completion, which ensures that direct summands of objects will also be objects.
Finally all products of shorter or longer sequences of arrows around a positive cycle are zero because there is no appropriate intersection point or there is a convexity problem. This establishes the first part of the theorem.

The second part is basically a consequence of the fact that $\mathcal{H} \mathcal{T}H \mathcal{G}t \mathcal{I}l (RQ)$ does not depend on the quiver. Given any Lagrangian in $\text{Fuk}(S \setminus Q_0)$ we can add Lagrangians until we get a quiver $Q$ that splits the surface. This implies that this particular Lagrangian sits in $\mathcal{H} \mathcal{T}H \mathcal{G}t \mathcal{I}l (RQ)$. So every Lagrangian sits in a $\mathcal{H} \mathcal{T}H \mathcal{G}t \mathcal{I}l (RQ)$ and $\mathcal{H} \mathcal{T}H \mathcal{G}t \mathcal{I}l (RQ) \subset \text{Fuk}(S \setminus Q_0) \subset \mathcal{H} \mathcal{T}H \mathcal{G}t \mathcal{I}l (RQ)$. □

6. CONSISTENT DIMER MODELS AND MATRIX FACTORIZATIONS

6.1. Matrix factorizations in general. Consider either an algebra $A$ or a smooth algebraic variety $X$. A potential is defined as a central element $W \in Z(A)$ in the former case or a polynomial function $X \to \mathbb{C}$. The pair $(X, W)$ is called a commutative Landau-Ginzburg model, the pair $(\mathfrak{a}, W)$ is a noncommutative Landau-Ginzburg model.

A matrix factorization of a Landau-Ginzburg model is a diagram $\vec{P}$

$$
\begin{array}{ccc}
P_0 & \xrightarrow{p_0} & P_1 \\
p_1 & \xleftarrow{p_0} & P_1
\end{array}
$$

where $P_i$ are projective $A$-modules or vector bundles over $X$ such that $p_0p_1 = W$ and $p_1p_0 = W$.

Given 2 matrix factorizations $\vec{P}$ and $\vec{Q}$ we define $\text{Hom}(\vec{P}, \vec{Q})$ as the following $\mathbb{Z}_2$-graded space of module morphisms/sheaf morphisms

$$
\text{Hom}(\vec{P}, \vec{Q}) = \text{Hom}_e(P_0, Q_0) \oplus \text{Hom}_e(P_1, Q_1) \oplus \text{Hom}_o(P_0, Q_1) \oplus \text{Hom}_o(P_1, Q_0).
$$

On this space we have a differential of odd degree:

$$
d \begin{pmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{pmatrix} := \begin{pmatrix} 0 & p_0 \\ p_1 & 0 \end{pmatrix} \begin{pmatrix} f_{00} & f_{01} \\ f_{10} & f_{11} \end{pmatrix} - \begin{pmatrix} f_{00} & -f_{01} \\ -f_{10} & f_{11} \end{pmatrix} \begin{pmatrix} 0 & q_0 \\ q_1 & 0 \end{pmatrix}.
$$

It is easy to check that $d^2 = 0$.

The category of all matrix factorizations is denoted by $\mathcal{MF}(A, W)$ or $\mathcal{MF}(X, W)$ and it has the structure of a $\mathbb{Z}_2$-graded dg-category or $A_{\infty}$-category. In many cases it is interesting to look at small subcategories of this $A_{\infty}$-category.

6.2. The $A_{\infty}$-algebra for a consistent dimer model. Let $Q$ be a zigzag consistent dimer model on a surface (with nonzero genus) and $A$ its Jacobi algebra. The Jacobi algebra has a central element coming from the cycles

$$
\ell := \sum_{v \in Q_0} c_v
$$

where $c_v \in Q_2$ is a cycle starting in $v$. Note that the relations in the Jacobi algebra ensure that this expression does not depend on the choices of $v$ and is indeed central.

For each arrow $a$ we can define a matrix factorization of $\ell$

$$
\vec{P}_a := Ah(a) \xrightarrow{a} At(a)
$$

where $\bar{a}$ is defined such that $\bar{a}\bar{a} \in Q_2$. 
We define $\mathfrak{mf}(Q)$ as the dg-category with as objects the $\tilde{P}_a$ and morphisms as in the previous section.

If $\tilde{Q}$ is the universal cover of $Q$ and $\pi$ the cover map, then we can also define matrix factorizations in the universal cover $\tilde{P}_a$ for every $\tilde{a} \in \tilde{Q}_1$. By lifting paths to the universal cover we can easily see that

$$\text{Hom}(\tilde{P}_a, \tilde{P}_b) = \bigoplus_{b, \pi(b) = b} \text{Hom}(\tilde{P}_{\tilde{a}}, \tilde{P}_b).$$

where we have chosen a fixed lift of $a$ and vary over the lifts of $b$. This identification is also compatible with the differentials on both sides.

As we already know $\mathfrak{mf}(Q)$ is a dg-algebra and hence its homology $\text{H} \mathfrak{mf}(Q)$ allows an $A_\infty$-structure. Because the original dg-structure is compatible with the lift to the universal cover, this $A_\infty$-structure will also be compatible.

6.3. A nice basis for $\text{H} \mathfrak{mf}(\tilde{Q})$. Recall that the zig ray $Z_{a_0}^+$ (zag ray $Z_{a_0}^-$) of an arrow $a_0$ is the infinite sequence $\ldots a_2 a_1 a_0$ of arrows in the universal cover, $\tilde{Q}$, such that $a_{i+1} a_i$ sits in a positive cycle if $i$ is even (odd) and in a negative cycle if $i$ is odd (even). Given a piece of a zig ray $Z = a_u \ldots a_1 a_0$ with even length, we define the left (right) opposite path as the path formed by all arrows that are not in $Z$ but are in positive (negative) cycles which meet $Z$. If $Z$ has odd length we take the left opposite path $a_{u-1} \ldots a_0$ and the right opposite path of $a_u \ldots a_1$.

We can put these two opposite paths in the correct entries of a $2 \times 2$-matrix and add an appropriate minus sign on the upper right entry.

$$\text{gives } \zeta(Z) = \begin{pmatrix} 0 & -\text{opp}_1 \\ \text{opp}_2 & 0 \end{pmatrix}$$

$$\text{gives } \zeta(Z) = \begin{pmatrix} \text{opp}_2 & 0 \\ 0 & \text{opp}_1 \end{pmatrix}$$

In this way we obtain an element in $\zeta(Z) \in \text{Hom}(P_{a_u}, P_{a_0})$ which is an off-diagonal or diagonal matrix, depending on whether the length of $Z$ is even or odd.

**Lemma 6.1.** If $Q$ is a consistent dimer model on a surface with nonpositive Euler characteristic and $Z$ is a zigzag path then both opposite paths are minimal (i.e. not a multiple of $\ell$).

**Proof.** From [12] we know there is a unique minimal path between every two vertices in the universal cover and such that every other path between these vertices is a power of $\ell$ times this path.

Let $p$ be the minimal path in the universal cover that connects $h(Z)$ to $t(Z)$ and let $a$ be the last arrow of $Z$. First suppose that $p$ does not cross $Z$ and assume we cannot apply any reductions to $p$ that make the piece that $p$ and $Z$ cut out smaller. We will prove the lemma by induction on the number of faces inside this piece.

If this number is 1 we are done because then the path is precisely the opposite path. If the number is bigger than 1, then the other zigzag path $Z'$ that goes through the penultimate arrow of $Z$ must cut the piece in two. Let $b$ be the arrow where $Z'$ intersects $p$ and denote by $p'$ the path which consists of the piece of $p$ before $b$. If $p$ is minimal then so must $p' a$ be and by induction it must be the path that goes opposite to $Z'$.

The arrow $b$ makes that $bp'$ and a fortiori $p$ can be reduced.
If \( p \) does cross \( Z \) in an arrow or a vertex, we can apply the above argument for each of the pieces separately and get the same result. \( \square \)

**Lemma 6.2.** The \( \zeta(Z) \) for which \( Z \) starts in \( a \) and ends in \( b \) form a basis for \( \mathrm{Hom}(\bar{P}_b, \bar{P}_a) \).

**Proof.** Because of \( Z_2 \)-degree and path-degree reasons we can look for basis elements of the forms
\[
\begin{pmatrix}
  f_{00} & 0 \\
  0 & f_{11}
\end{pmatrix}, \quad \begin{pmatrix}
  0 & f_{01} \\
  f_{10} & 0
\end{pmatrix}
\]
where the \( f_{ij} \) are paths up to a sign. We will only treat the first case. Being in the kernel of \( d \) implies that \( f_{11} = a^{-1}(f_{00})b = (\bar{a}f_{00})\bar{b}^{-1} \in \text{Jac} (\bar{Q}) \). These two conditions are the same as asking that \( \bar{a}f_{00}b \) is a multiple of \( \ell \). Being in the image of \( d \) implies that \( f_{00} \) is a left multiple of \( a \) or a right multiple of \( \bar{b} \) (which is equivalent to \( f_{11} \) being a right multiple of \( b \) or a left multiple of \( \bar{a} \)).

So we are looking for paths \( f_{00} \) such that \( \bar{a}f_{00}b \) is a multiple of \( \ell \) but neither \( \bar{a}f_{00} \) or \( f_{00}b \) is a multiple of \( \ell \). In particular this means that \( f_{00} \) and \( f_{11} \) are both minimal paths.

Now look at the paths \( f_{00} \) and \( f_{11} \) in the universal cover. If these 2 paths intersect at a vertex \( v \) we can split \( f_{ii} \) in two \( f_{iv} v f_{vi} \). Now \( \bar{a}f_{0v} \) and \( f_{1v} \) are both minimal and run between the same vertices so they must be the same. But then \( f_{11} = a\bar{a}f_{0v} f_{v1} \) is a left multiple of \( \bar{a} \) which is impossible. So \( a, b, f_{00} \) and \( f_{11} \) bound a simply connected piece \( S \) in the universal cover. After applying the relations we can assume that this piece is as small as possible.

Look at the zigzag path \( Z \) that starts from \( a \) and enters this simply connected piece. This zigzag path must leave the piece at an arrow \( c \).

If \( c \) is in \( f_{00} \) then we can split \( f_{00} = f_{0c} c f_{c0} \). Because of minimality \( f_{0c} \) and the opposite path running along the zigzag path from \( h(a) \) to \( h(c) \) must be equal. But then \( f_{0c} c \) ends in \( \bar{d} \) where \( d \) precedes \( c \) in the zigzag path. This contradicts the fact that we chose \( S \) as small as possible. Similarly \( Z \) cannot leave \( S \) through an arrow of \( f_{11} \), so \( b \) must lie on the zigzag path through \( a \). In order for \( Z \) to leave \( S \) at \( b \), we must have that \( Z[i] = b \) with \( b \) even. \( \square \)

**Corollary 6.3.** In the universal cover \( \bar{Q} \) the space \( \mathrm{Hom}(\bar{P}_b, \bar{P}_a) \) is either zero dimensional or one dimensional, depending on whether \( \bar{b} \) sits in a zigzag ray from \( \bar{a} \) or not.

Now we are going to express the \( A_\infty \)-products in terms of these bases. To do this we will expand this basis to a full basis of \( \mathrm{Hom}(\bar{P}_b, \bar{P}_a) \).

Consider the set of matrices
\[\begin{pmatrix}
  f_{00} & 0 \\
  0 & f_{01}
\end{pmatrix}, \quad \begin{pmatrix}
  0 & f_{01} \\
  f_{10} & 0
\end{pmatrix}\] where \( f_{00} \) and \( f_{01} \) are paths.
Theorem 6.6.

\[
\begin{pmatrix} 0 & 0 \\ 0 & f_{11} \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ f_{10} & 0 \end{pmatrix}
\]
where \( f_{11} \) and \( f_{10} \) are paths and \( af_{11}b^{-1}, af_{10}b^{-1} \in \text{Jac}(Q) \)
and let \( U \) be the span of these matrices.

By construction it is clear that the matrices above are linearly independent and

\[
\text{Hom}(P_a, P_b) = U \oplus \mathbb{H} \text{Hom}(P_a, P_b) \oplus d(U).
\]

This split allows us to define a codifferential

\[
h : \text{Hom}(P_a, P_b) \to \text{Hom}(P_a, P_b) : u_1 \oplus \zeta(Z_a \pm \jmath) \oplus du_2 \mapsto u_2 \oplus 0 \oplus 0
\]

Lemma 6.4.

- \( dhd = d \),
- \( h^2 = 0 \),
- \( h(\zeta(Z)) = 0 \).
- \( h \) respects the split \( \text{Hom}(P_a, \overline{P}_b) = \bigoplus_{b, a(b) = b} \text{Hom}(P_a, \overline{P}_b) \).

Proof. These facts follow easily from the construction. \( \square \)

We are now going to determine the \( A_\infty \)-structure on \( \text{Hom}(Q) \). First we start with the ordinary multiplication.

Theorem 6.5. Suppose \( Z_1 \) is a zigzag path from \( b \) to \( a \) and \( Z_2 \) is a zigzag path from \( a \) to \( c \)

\[
\mu(\zeta(Z_1), \zeta(Z_2)) = \begin{cases} 
\zeta(Z_2a^{-1}Z_1) & \text{if } Z_2a^{-1}Z_1 \text{ is a zigzag path} \\
0 & \text{otherwise}
\end{cases}
\]

Proof. From the formula for the \( A_\infty \)-structure we get

\[
\mu(\zeta(Z_1), \zeta(Z_2)) = (1 - hd + dh)(\zeta(Z_1)\zeta(Z_2))
\]

Now \( \zeta(Z_1)\zeta(Z_2) \) is a matrix that either consists of 2 paths (possibly with signs) on the
off-diagonals or on the diagonals. The expression \((1 - hd + dh)(\zeta(Z_1)\zeta(Z_2))\) will be
nonzero only if these paths are opposite paths of some other zigzag path.

If \( Z_2a^{-1}Z_1 \) is a zigzag path, one can easily check that \( \zeta(Z_1)\zeta(Z_2) = \zeta(Z_2a^{-1}Z_1) \) and
hence \( \mu(\zeta(Z_1), \zeta(Z_2)) = \zeta(Z_2a^{-1}Z_1) \).

If \( Z_1 \) and \( Z_2 \) are zigzag paths in different directions, we will show that \( \mathbb{H} \text{Hom}(\overline{P}_b, \overline{P}_c) = 0 \) in the universal cover. If this were not the case then there must be a zigzag path \( Z \) from \( c \) to \( b \).

If \( Z_2 \) lies to the left of \( Z_1 \) then in order for \( Z \) to intersect with \( Z_2 \), \( Z \) must also lie to
the left of \( Z_1 \) and therefore the length of \( Z_1 \) must be odd.

![Diagram](image)

For similar reasons the length of \( Z_2 \) must be odd while the length of \( Z \) must be even.
This is impossible because of the degree of \( \mu(\zeta(Z_1), \zeta(Z_2)) \) would then be even while the
degree of \( \zeta(Z) \) is odd. \( \square \)

Theorem 6.6. Let \( a_1 \ldots a_k \) be a positive or negative cycle in \( Q \) then

\[
\mu(\zeta(a_1a_2), \zeta(a_2a_3), \ldots, \zeta(a_{l-1}a_l)) = \begin{cases} 
0 & l \neq k \\
\zeta(a_l) & l = k
\end{cases}
\]
\textbf{Proof.} First note that
\[ \zeta(a_i a_{i+1}) = \begin{pmatrix} 0 & -t(a_i) \\ a_{i+2} \ldots a_{i+k-1} & 0 \end{pmatrix}. \]
Therefore
\[ h(\zeta(a_i a_{i+1}) \zeta(a_{i+1} a_{i+2})) = \begin{pmatrix} 0 & 0 \\ -a_{i+3} \ldots a_{i+k-1} & 0 \end{pmatrix} \]
and by induction
\[ h(h(\ldots h(\zeta(a_i a_{i+1}) \zeta(a_{i+1} a_{i+2}) \zeta(a_{i+2} a_{i+3}) \ldots) \zeta(a_{i+r} a_{i+r+1})) = \begin{pmatrix} 0 & 0 \\ -a_{i+r+2} \ldots a_{i+k-1} & 0 \end{pmatrix}. \]
if \( r + 2 \leq k - 1 \) and zero otherwise.

If we multiply two such expressions together we get zero. If we multiply one such expression on the left with \( \zeta(a_{i-1} a_i) \) we get a matrix with only the upper left diagonal nonzero. If we apply \( h \) to this we also get zero.

The only trees that might give a nonzero contribution to \( \mu \) are
\[ \pi(h(\ldots h(h(\zeta(a_1 a_2) \zeta(a_2 a_3) \zeta(a_1 a_2 a_3) \ldots) \zeta(a_r a_{r+1}))) \]
\[ \pi(\zeta(a_1 a_2) h(\ldots (h(\zeta(a_2 a_3) \zeta(a_1 a_2 a_3) \ldots) \zeta(a_r a_{r+1}))) \]
Both expressions can only give a nonzero result if \( r = k \). Indeed by degree reasons these expressions are even: \( r \deg \zeta(a_1 a_2) + (r - 2) \deg h + \deg \pi = 0 \), but the only path that goes opposite a zigzag path of odd length which is also a subpath of \( a_1 \ldots a_{i+r} \) is the trivial path and this happens when \( r = 0 \mod k \). If \( r > k + 1 \) the expression \( h(h(\ldots h(\ldots)))) \) will be zero.

If \( r = k \) then we see that
\[ h(\ldots h(h(\zeta(a_1 a_2) \zeta(a_2 a_3) \zeta(a_1 a_2 a_3) \ldots) \zeta(a_r a_{r+1})) + \zeta(a_1 a_2) h(\ldots (h(\zeta(a_2 a_3) \zeta(a_1 a_2 a_3) \ldots) \zeta(a_r a_{r+1}) \]
\[ = \begin{pmatrix} h(a_1) & 0 \\ 0 & t(a_1) \end{pmatrix} = \zeta(a_1). \]
Because the latter is a basis element of the homology, \( \pi(\zeta(a_1)) = \zeta(a_1) \). \qed

7. Dimer duality as mirror symmetry

Let \( Q \) be any, not necessarily consistent, dimer. We define its mirror dimer \( \hat{Q} \) as follows

1. The vertices of \( \hat{Q} \) are the zigzag cycles of \( Q \).
2. The arrows of \( \hat{Q} \) are the arrows of \( Q \), \( h(a) \) is the zigzag cycle coming from the zig ray, and \( t(a) \) is the cycle coming from the zag ray.
3. The positive faces of \( \hat{Q} \) are the positive faces of \( Q \).
4. The negative faces of \( \hat{Q} \) are the negative faces of \( Q \) in reverse order.

We illustrate this with a couple of examples:
Remark 7.1. The dual can also be obtained by cutting the dimer along the arrows, flipping over the clockwise faces, reversing their arrows and gluing everything back again. This construction is basically the same construction that was introduced by Feng, He, Kennaway and Vafa in [13] applied to all possible dimers.

Lemma 7.2. $Q \mapsto \overline{Q}$ is an involution on the set of all dimer models.

Proof. Let $Z = a_1 \ldots a_k$ be a zigzag cycle in $Q$. If $i$ is odd then $a_i a_{i+1}$ sits in a positive cycle in $Q$ while $a_i a_{i+1}$ sits in a negative cycle of $\overline{Q}$ when $i$ is even.

This implies that $a_i a_{i+1}$ sits in a positive cycle in $Q$ for $i$ odd while $a_{i+1} a_i$ sits in a negative cycle of $Q$ when $i$ is even. Hence in $Q$ the odd arrows of $Z$ must have the same head in $Q$ which is equal to the tail of the even arrows. This means there is a well defined map from the zigzag paths in $Q$ and the vertices of $Q$. Because $a$ is an even arrow in its zig cycle and an odd in its zag cycle, under this map the zig cycle and the zag cycle of $a$ in $\overline{Q}$ correspond to the original head and tail. Finally this map is a bijection because given any vertex $v \in Q$ we can make a zigzag cycle in $\overline{Q}$ by listing all arrows incident with $v$ in a clockwise direction.

Theorem 7.3. If $Q$ is a consistent dimer then $\mathcal{Hmf}(Q)$ is $\mathbb{A}_\infty$-isomorphic to $\mathcal{Gt}_1(RQ), \mu$.

Proof. The underlying category of $\mathcal{Hmf}(Q)$ is indeed isomorphic to $\mathcal{Gt}_1(RQ)$. The paths in $\mathcal{Gt}_1(RQ)$ are those that cycle around vertices of $Q$, which are precisely the zigzag paths of $Q$. Two paths multiply to zero in $\mathcal{Gt}_1(RQ)$ if they go around different vertices in $Q$ just as the product of $\zeta(p_1) \cdot \zeta(p_2)$ is zero if they belong to different zigzag cycles. Therefore we clearly have an isomorphism of categories.

Finally the $\mathbb{A}_\infty$-structure is $\mathbb{A}_\infty$-isomorphic to $\mu$ because of lemma 6.6 and theorem 4.4.

Corollary 7.4. For any consistent dimer $Q$ we have that $\mathcal{Hmf}(Q)$ and $\mathcal{Fuk}(\mathcal{Gt}_1) \setminus \mathcal{Gt}_1(Q_0)$ are $\mathbb{A}_\infty$-isomorphic $\mathbb{A}_\infty$-categories.

Note that the above result applies to all consistent dimers, not only those on the torus. This means that the Jacobi algebra of these dimers is not necessarily Noetherian. So in some cases this result does not derive from a commutative instance of mirror symmetry in the background. However if one restricts to dimer models on a torus it is possible to recover the commutative picture from the dimer itself.

8. RECOVERING THE COMMUTATIVE PICTURE

Let $Q$ be a consistent dimer model on a torus and let $A = \text{Jac}(Q)$ be its Jacobi algebra. Fix a vertex $o$ which we will call the trivial vertex. Now choose 2 cyclic paths $x, y$ through $o$ that are not multiples of $\ell$ and form a basis for the homology of the torus. Finally let $z$ be any positive cycle through $o$.

A perfect matching $\mathcal{P}$ is a subset of the arrows such that every positive or negative cycle contains exactly one arrow of $\mathcal{P}$. Every perfect matching defines a degree function on $A$...
by setting the degree of an arrow equal to 1 if it is in the matching and zero otherwise. A collection of perfect matchings \( \{ \mathcal{P}_1, \ldots, \mathcal{P}_k \} \) is called \( \theta \)-stable\(^4\) if there is a path from \( o \) to every vertex that has degree zero for all the matchings in the collections.

For every stable collection \( S = \{ \mathcal{P}_1, \ldots, \mathcal{P}_k \} \) we define a cone in \( N = \mathbb{R}^3 \)

\[
\sigma_S = \mathbb{R}^+(\mathcal{P}_1(x), \mathcal{P}_1(y), \mathcal{P}_1(z)) + \cdots + \mathbb{R}^+(\mathcal{P}_k(x), \mathcal{P}_k(y), \mathcal{P}_k(z))
\]

**Theorem 8.1** (Ishii-Ueda-Mozgovoy). The collection of cones \( \sigma_S \) where \( S \) is a stable set of matchings forms a fan and the toric variety of this fan, \( \tilde{X} \) is a crepant resolution of \( X = \text{Spec} \mathbb{Z}(A) \).

If we intersect the fan with the plane at height \( z = 1 \), we get a convex polygon, which is subdivided in triangles and on each integral point of the polygon sits a unique perfect matching. These lattice points will also give us a basis for the toric divisors of \( \tilde{X} \). So any line bundle over \( \tilde{X} \) can be identified with a \( \mathbb{Z} \)-linear combination of stable perfect matchings. In this language we have the following:

**Theorem 8.2** (Ishii-Ueda-Mozgovoy-Bender). There is an equivalence of categories \( \mathcal{F} : \text{D}^b \text{Mod} A \to \text{D}^b \text{Coh} \tilde{X} \), which maps the projective \( vA \) to the line bundle \( \sum \mathcal{P}_i(p_v)_! \mathcal{P}_i \) where \( p_v \) is any path from \( o \) to \( v \) and the sum runs over all stable matchings.

In \([22]\) Ishii and Ueda use this result to get an equivalence of categories between the corresponding categories of matrix factorizations.

**Theorem 8.3** (Ishii-Ueda-[23]). There is an equivalence of categories \( \mathcal{F} : \text{H}^0 \text{MF}(A, \ell) \to \text{H}^0 \text{MF}(\tilde{X}, \ell) \).

The last theorem implies that any of the matrix factorizations \( \tilde{P}_\sigma \) is transformed into a matrix factorization of line bundles \( \mathcal{L}_\sigma \) over \( \tilde{X} \).

We illustrate the theorems with one particular dimer model. We take vertex 1 to be the trivial one.

| Dimer | Stable Matchings | Fan | Linebundles |
|-------|-----------------|-----|------------|
| ![Dimer Diagram](image) | \( \mathcal{P}_1 = \{ b, w \} \) | ![Fan Diagram](image) | \( \mathcal{L}_1 = \mathcal{O} \) |
|       | \( \mathcal{P}_2 = \{ d, z \} \) |   | \( \mathcal{L}_2 = \mathcal{O}_{\mathcal{V}_2} \) |
|       | \( \mathcal{P}_3 = \{ a, y \} \) |   | \( \mathcal{L}_3 = \mathcal{O}_{\mathcal{V}_1 + \mathcal{V}_2} \) |
|       | \( \mathcal{P}_4 = \{ c, x \} \) |   | \( \mathcal{L}_4 = \mathcal{O}_{\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_4} \) |
|       | \( \mathcal{P}_5 = \{ a, b \} \) |   |           |

The only thing that is missing to get an equivalence

\[
\text{H}^0 \text{MF}(\tilde{X}, \ell) \cong \text{H}^0 \text{Fuk}(\tilde{O}) \setminus \mathcal{O}_0
\]

is a proof that the objects \( \mathcal{L}_\sigma \) generate \( \text{H}^0 \text{MF}(\tilde{X}, \ell) \). We will briefly sketch a proof of this. In \([36]\) Orlov proved that we have an equivalence of categories

\[
\text{H}^0 \text{MF}(\tilde{X}, \ell) \cong \frac{\text{D}^b \text{Coh}(\tilde{X}_0)}{\text{Perf}(\tilde{X}_0)} \text{ with } \tilde{X}_0 := \ell^{-1}(0).
\]

The second category is the quotient of the derived category of the zero fiber of \( \ell \) by its perfect complexes. This category is also called the category of singularities of \( \tilde{X}_0 \). The functor that realizes this equivalence maps a matrix factorization \( P_0 \xrightarrow{p_0} P_1 \) to \( \text{Cok} p_0 \).

If we can prove that \( \text{D}^b \text{Coh}(\ell^{-1}(0)) \) is generated by the \( \text{Cok} \mathcal{F} a \) for \( a \in \mathcal{Q}_1 \) we are done.

For each cone \( \sigma \) of co-dimension \( i \) in the fan we get a dimension \( i \) closed subvariety \( \mathcal{Y}_\sigma \) of \( \tilde{X}_0 \) which is the intersection of the toric invariant divisors that generate the cone. If we can find for each \( \sigma \) an object \( \mathcal{M}_\sigma \) that is supported on an open subvariety of \( \mathcal{Y}_\sigma \), then

\(\text{(continued)}\)

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the collection \( M \) will generate \( D^b \text{Coh}(\tilde{X}_0) \). This is true because \( D^b \text{Coh}(\tilde{X}_0) \) is generated by the subcategories \( D^b \text{Coh} \{\tilde{X}_0\} \), containing the objects with homology supported on the \( Y_\sigma \). In fact one only needs the \( Y_\sigma \) with codimension zero because these are the connected components of \( \tilde{X}_0 \). Each of the \( Y_\sigma \) \( \cup \tau \neq \sigma \)\( Y_\tau \) is affine and hence \( M \) together with the subcategories \( D^b \text{Coh} Y_\sigma \cap Y_\tau \)\( \tilde{X}_0 \) generate the whole \( D^b \text{Coh} Y_\sigma \)\( \tilde{X}_0 \).

Let \( \sigma \) be a cone generated by \( \{P_{i_1}, \ldots, P_{i_k}\} \). Consider \( M_\sigma = \bigoplus_{a \in \cup P_{i_k}} \text{Coka} \). This is a sheaf over \( \tilde{X}_0 \) which has maximal rank at the locus where all \( a \in \cup P_{i_k} \) are zero. This is precisely \( Y_\sigma \) and hence \( U_\sigma \) generates a sheaf \( M_\sigma \) which has rank one on an open subspace of \( Y_i \) and is zero elsewhere.

All this allows us to conclude:

**Theorem 8.4.** Given a consistent dimer model on a torus and \( \tilde{X} \) a crepant resolution of \( \text{Spec} Z(\text{Jac} Q) \) we have the following equivalences.

\[
H^0 \text{MF}(\tilde{X}, \ell) \cong H^0 \text{MF}(\text{Jac} Q, \ell) \cong H^0 \text{Fuk}(\mathcal{O}) \setminus \mathcal{O}_0
\]

We end with a graphical connection between the surface of the mirror dimer and the polygon constructed from the original dimer.

**Theorem 8.5.** Let \( Q \) be a consistent dimer on a torus. The mirror dimer \( \mathcal{O} \) gives us a surface \( |\mathcal{O}| \setminus \mathcal{O}_0 \) of genus \( g \) with \( k \) punctures, where \( g \) is the number of internal lattice points of the polygon associated to \( Q \) and \( k \) is the number of boundary lattice points.

**Proof.** For every lattice point in the polygon there is precisely one stable perfect matching. If we take two consecutive stable perfect matchings \( P_1, P_2 \) on the boundary, then the arrows in \( P_1 \cup P_2 \setminus P_1 \cap P_2 \) form a zigzag path and any zigzag path can be constructed in this way (for a proof of this see [13]). This implies there is a bijection that maps the line segments on the boundary between the lattice points to zigzag paths and hence the number of zigzag paths equals the number of boundary lattice points.

The equivalence between the \( K \)-groups of \( D^b \text{Coh} \tilde{X} \) and \( D^b \text{ModJac} (\mathcal{O}) \) shows that the number of maximal cones in the fan of \( \tilde{X} \) equals the number of vertices in \( Q \). The former equals twice the area of the polygon because the fan cuts the polygon in elementary triangles. By Pick’s theorem this area is \( B/2 + I - 1 \) where \( B \) is the number of boundary lattice points and \( I \) the number of internal lattice points. From this discussion we get that

\[
\# \mathcal{O}_0 = \# \{\text{zigzag paths in } Q\} = B \quad \text{and} \quad \# \mathcal{O}_0 = B + 2I - 2.
\]

The genus of \( |\mathcal{O}| \) is

\[
g = 1 - \# \mathcal{O}_0 - \# \mathcal{O}_1 + \# \mathcal{O}_2
\]

\[
= 1 - (\# Q_0 - (2I + 2) - \# Q_1 + \# Q_2)
\]

\[
= 1 - \frac{(2I + 2) + 0}{2} = I
\]

The first step follows because \( \# \mathcal{O}_i = \# Q_i \) for \( i = 1, 2 \), while the second step is a consequence of the fact that \( Q \) sits in a torus. \( \square \)

**Remark 8.6.** This well known by physicists and can be found in [13]. Graphically we can say that the mirror surface \( |\mathcal{O}| \setminus \mathcal{O}_0 \) is a tubular neighborhood of the dual diagram of the fan associated to \( \tilde{X} \).

![Diagram](image-url)
9. Appendix: Hochschild Cohomology and $A_{\infty}$-structures

In this appendix we look at the connection between Hochschild cohomology and $A_{\infty}$-structures and calculate the Hochschild cohomology of the gentle categories coming from rectified dimers.

Then we will define certain $A_{\infty}$-structures on these gentle categories and use our calculation to find criteria for when a given $A_{\infty}$-structure will be $A_{\infty}$-isomorphic to one of our defined ones.

In this appendix we will assume that $Q$ itself is a rectified dimer, so in particular every vertex has 2 arrows arriving and two arrows leaving.

9.1. Hochschild cohomology and $A_{\infty}$-structures. The discussion in this section closely matches [7]. If we are interested in $A_{\infty}$-structures on an ordinary category $B = \mathcal{Gt}1\ Q$ up to $A_{\infty}$-isomorphism, we need to study the Hochschild cohomology of $B$.

A length $n$ multi-functor $F$ consists of linear maps for each sequence of $n + 1$ objects $X_0, \ldots, X_n \in \mathcal{B}$. 

$$F : \text{Hom}_\mathcal{B}(X_1, X_0) \otimes \cdots \otimes \text{Hom}_\mathcal{B}(X_n, X_{n-1}) \to \text{Hom}_\mathcal{C}(X_n, X_0).$$

The set of all length $n$-multifunctors forms a $\mathbb{Z}_2$-graded vector space which we denote by $\mathcal{M}^n$.

We can construct a differential $d : \mathcal{M}^n \to \mathcal{M}^{n+1}$

$$dF(f_1, \ldots, f_{n+1}) = f_1 F(f_2, \ldots, f_{n+1})$$

$$- F(f_1 f_2, \ldots, f_{n+1}) + F(f_1, f_2 f_3, \ldots, f_{n+1}) \cdots \pm F(f_1, \ldots, f_{n-1} f_n)$$

$$\mp F(f_1, \ldots, f_{n-1}) f_n$$

The Hochschild cohomology of $B$ is defined as the cohomology of the complex $\mathcal{M}^*$:

$$\text{HochC}(B) := H(\mathcal{M}^*).$$

The importance of the Hochschild cohomology of $C$ can be seen by the following 2 lemmas.

**Lemma 9.1.** [7] Let $\mu_i, i \leq k$ be a sequence of multifunctors with $\mu_1 = 0$ and $\mu_2 = \cdot$ then we can rewrite $[M_k]$ as

$$d\mu_k = \Phi$$

where $\Phi$ is an expression calculated from the $\mu_i, i < k$ and $d\Phi = 0$ if all $[M_i]$ for $i < k$ hold.

**Lemma 9.2.** [7] Let $\mu$ and $\nu$ be two $A_{\infty}$-structures on $B$ and let $F_i, i \leq k$ be a sequence of multifunctors with $F_1 = 1$ then we can rewrite $[F_{k+1}]$ as

$$dF_k = \Psi$$

where $\Psi$ is an expression calculated from the $\mu_i$, the $\nu_i$ and the $F_i, i < k$. Moreover $d\Psi = 0$ if all $[F_i]$ for $i < k$ hold.

These lemmas imply that we can find a solution for $\mu_k$ (or $F_k$) if and only if the homology class of $\Phi$ (or $\Psi$) is trivial. So calculating the Hochschild cohomology of $B$ is crucial if one wants to classify $A_{\infty}$-structures on $B$.

9.2. Hochschild cohomology of the gentle categories. Now we will calculate the Hochschild cohomology of the gentle category $B = \mathcal{Gt}1\ (Q)$, which we will view as an algebra in this section. For this we will use a minimal bimodule resolution for $B$, which can be obtained by a result by Bardzel [8].

**Theorem 9.3** (Bardzel). Suppose $B$ is the path algebra of a quiver $Q$ modulo relations of length two. Let $Z_k \subset \mathbb{C} Q$ be the vector space spanned by paths of length $k$ of which all subpaths of length 2 are zero in $B$.

The terms in minimal bimodule resolution $P^*$ are given by $P_k = B \otimes Z_k \otimes B$ where the tensor product is taken over $\mathbb{C} Q_0$. The maps between the terms have the following form

$$1 \otimes b_1 \ldots b_k \otimes 1 \mapsto b_1 \otimes b_2 \ldots b_k \otimes 1 - (-1)^k \otimes b_1 \ldots b_{k-1} \otimes b_k$$
As is well known the Hochschild cohomology of $B$ is the cohomology of the complex

$$\text{Hom}_{B \otimes B^{opp}}(P^*, B).$$

Note that both this complex and $P^*$ are $\mathbb{Z}Q_1$-graded with degree zero differentials. Moreover in the case of these gentle algebras, the bimodule resolution also splits as a direct sum where each summand $P^*_c$ only contains the paths that go around one cycle $c \in Q_2^1$. So let us focus on one such cycle $c = b_1 \ldots b_l$ and set $b_k$ to be $b_k \mod l$ for all $k \in \mathbb{Z}$.

Clearly

$$\text{Hom}_{B \otimes B^{opp}}(B \otimes b_1 \ldots b_j \otimes B, B) \cong h(b_j)Bt(b_j)$$

and we will use the notation $[b_i \ldots b_j \rightarrow x]$ for the morphism that maps $b_i \ldots b_j$ to $x \in h(b_i)Bt(b_j)$. By lemma 4.3 the space $h(b_i)Bt(b_j)$ is only nonzero in 3 cases.

1. If $|b_i \ldots b_j| = -1 \mod l$ then $h(b_j)Bt(b_j) = b_{i-1}^{-1}zC[z]$ where $z$ is the unique negative cycle through $b_{i-1}$.

2. If $|b_i \ldots b_j| = 0 \mod l$ then $h(b_j)Bt(b_j) = h(b_j)C[z_1, z_2]/(z_1z_2)$ where $z_1, z_2$ are the two negative cycles that start in $h(b_i)$.

3. If $|b_i \ldots b_j| = 1 \mod l$ then $h(b_j)Bt(b_j) = b_jC[z]$ where $z$ is the unique negative cycle through $b_j$.

With the notation above, the differential on $\text{Hom}_{B \otimes B^{opp}}(P^*, B)$ becomes

$$d[b_i \ldots b_j \rightarrow x] = [b_{i+1} \ldots b_j \rightarrow b_{i+1}x] - (-1)^{j-i}b_{i+1}^{-1}[b_{i+1} \ldots b_j + 1 \rightarrow x_{b_j+1}]$$

(1) If $|b_i \ldots b_j| = j-i + 1 = -1 \mod l$ then

$$d[b_i \ldots b_j \rightarrow z^{u}] = [b_{i-1} \ldots b_j \rightarrow b_{i-1}^{-1}z^{u}] = b_{i-1}^{-1}[b_{i+1} \ldots b_j + 1 \rightarrow z^{u}] - (-1)^{j-i}[b_{i+1} \ldots b_j + 1 \rightarrow b_{i-1}^{-1}z^{u}b_{j+1}].$$

Both terms only appear for the differential of $[b_i \ldots b_j \rightarrow b_{i-1}^{-1}z^{u}]$, so $d$ acts injectively on $\text{Hom}(P^{n+1}_c, B)$.

(2) If $|b_i \ldots b_j| = 0 \mod l$ and $x = z_1^u$ or $x = z_2^v$, $u > 0$ then one term of $d[b_i \ldots b_j \rightarrow x]$ is zero and the other nonzero. Every such term appears exactly twice:

$$\begin{cases} 
    d[b_i \ldots b_j \rightarrow x] = \pm d[b_{i+1} \ldots b_{j-1} \rightarrow b_{i+1}^{-1}x_{b_j-1}] & \text{if } b_{i+1} \text{ sits in } x \\
    d[b_i \ldots b_j \rightarrow x] = \pm d[b_{i+1} \ldots b_{j+1} \rightarrow b_{i+1}^{-1}x_{b_j}] & \text{if } b_{i+1} \text{ sits in } x
\end{cases}$$

The sum/difference gives us an element in the kernel which is in the image of $d$.

For $x = h(b_j)$ we get that both terms are nonzero and it is easy to check that each term appears twice: once for $d[b_{i+1} \ldots b_j \rightarrow h(b_j)]$ and once for $d[b_{i+1} \ldots b_j \rightarrow h(b_{i+1})]$ and therefore the homology in degree $i-j+1 = 0 \mod l$ is generated by elements of the form

$$\sum_{0 \leq u \leq j-i} (-1)^{u(j-i)}[b_{i+u} \ldots b_{j+u} \rightarrow h(b_{i+u})]$$

(3) If $|b_i \ldots b_j| = 1 \mod l$ then $d$ acts as zero because for any $x = b_i z^u$ the terms of the differential each contain two consecutive arrows of $c$. As we have seen above, each term $[b_{i+1} \ldots b_j \rightarrow b_i z^u]$ is in the image of $d$ if $u > 0$. If $u = 0$ then $[b_{i+1} \ldots b_j \rightarrow b_i] - (-1)^{j-i+1}[b_{i+1} \ldots b_j \rightarrow b_j]$ is in the image of $d$, clearly this implies that

$$\sum_{0 \leq u \leq j-i} (-1)^{u(j-i+1)}[b_{i+u} \ldots b_{j+u} \rightarrow b_{i+u}]$$

is the only element we need to add to the image to span the whole space.

From this discussion, we can conclude that

**Proposition 9.4.** The homology of $\text{Hom}(P^*_c, B)$ in degrees $nl - 1$ is zero, in degrees $nl$ it is one and spanned by

$$\Omega^{n,l}_{B^c} = \sum_{i=1}^{nl} (-1)^{i(nl-1)}[b_i \ldots b_{i+nl-1} \rightarrow h(b_i)].$$
and in degrees \( nl + 1 \) it is one and spanned by
\[
\Omega^c_{nl}(b_1, \ldots, b_{nl-1}) = \sum_{i=1}^{nl} (-1)^{(nl+1)i} \cdot [b_i, \ldots, b_{i+nl} \rightarrow b_i].
\]

Now we go back to the original viewpoint of the Hochschild cohomology. Given a multifunctor \( \Psi \) of length \( k \) with \( d\Psi = 0 \), how can we detect whether its cohomology class is zero? The idea is to look at the image of \( k \)-tuples of the form \( (b_1, \ldots, b_k) \) where the \( b_i \) are consecutive arrows in a positive cycle:

**Lemma 9.5.** Let \( \Psi \) be a multifunctor of length \( u \) with \( d\Psi = 0 \) then \( \Psi \in \text{Ker}d \) if \( \Psi(b_1, \ldots, b_n) \) has no length 0 terms or length 1 terms for every \( (b_1, \ldots, b_k) \) where the \( b_i \) are consecutive arrows in a positive cycle.

**Proof.** For \( u = 0 \mod l \) the homology class \( \Omega^c_{nl}(b_1, \ldots, b_{nl-1}) = h(b_i) \) contains a length 0 term. Now if \( \Pi \) is any multifunctor of length \( nl - 1 \) then one can check that
\[
d\Pi(b_1, \ldots, b_{i+nl-1}) = b_i \Pi(b_{i+1}, \ldots, b_{i+nl-1}) - 0 + \cdots + 0 + 2 \Pi(b_1, \ldots, b_{i+nl-2}) b_{i+nl-1}
\]
which has no terms of length 0 or 1 because neither \( \Pi(b_{i+1}, \ldots, b_{i+nl-1}) \) nor \( \Pi(b_1, \ldots, b_{i+nl-2}) \) can be a vertex.

Similarly for \( u = 1 \mod l \), if \( \Pi \) is any multifunctor of length \( nl \) then we can write
\[
d\Pi(b_1, \ldots, b_{i+nl}) = \alpha_i b_i + \cdots \end{equs}
where \( \alpha_i \in \mathbb{C} \). Now one easily checks that \( \sum_{i=1}^{nl} (-1)^{(nl+1)i} \alpha_i = 0 \). But if we replace \( d\Pi \) by \( \Omega^c_{nl} \) we get that \( \sum_{i=1}^{nl} (-1)^{(nl+1)i} \alpha_i = nl \neq 0 \).

Another ingredient we will need is an interesting collection of elements in \( \text{Ker}d \). Choose for each arrow \( a \) and each \( n \geq 1 \) a number \( \zeta_{n,a} \in \mathbb{C} \). Define the multifunctor \( g_k \) on \( Gt(\mathbb{Q}) \) by \( g_k(x_1, \ldots, x_k) = \zeta_{n,a} \text{sat} \) if \( x_1 = sa, x_k = at \) and for \( i \in [2, k-1] \) all \( x_i \) are arrows and \( ax_2 \ldots x_{k-1} \) is the \( n^\text{th} \) power of a positive cycle in \( \mathbb{Q} \). For other cases where the \( x_i \) are paths set \( g_k \) zero and extend it multilinearly.

**Lemma 9.6.** \( dg_k = 0 \).

**Proof.** Note that \( g_k \) is well defined because through every arrow there is only one positive cycle, so \( \text{sat} \) is uniquely determined by \( x_1, \ldots, x_k \).

Let \( u_1, \ldots, u_{k+1} \) be paths and suppose \( dg_k(u_1, \ldots, u_{k+1}) \neq 0 \), then there must at least be one term in the expression that is nonzero.

- If \( u_1 g_k(u_2, \ldots, u_{k+1}) \neq 0 \) then \( u_2 = sa, u_{k+1} = at \) for some \( a \) and \( u_1 \text{sat} \neq 0 \).
- The second term is \( -g_k(u_1 u_2, \ldots, u_{k+1}) = -\zeta_{n,a} u_1 \text{sat} \).
- The last term is nonzero because \( u_k \) is an arrow \( b \) such that \( a \) and \( b \) sit in a positive cycle and hence \( ba = 0 \) but if \( g_k(u_1, u_2, \ldots, u_{k+1}) \neq 0 \) then \( u_1 = s'b \) which contradicts \( u_1 \text{sat} \neq 0 \). Finally all intermediate terms are also zero because they contain the product of two consecutive arrows in a positive cycle.
- If \( g_k(u_1 u_2, \ldots, u_{k+1}) \neq 0 \) then \( u_1 u_2 = sa \) (and hence \( u_2 = s'a \)) for some \( a \) and \( sat = u_1 s'at \neq 0 \). The first term is \( u_1 g_k(s_2, \ldots, u_{k+1}) = \zeta_{n,a} u_1 s'at \).
- Like in the previous paragraph all other terms are zero.
- Similarly the last and penultimate term also cancel out.
- If an intermediate term \( g_k(u_1, \ldots, u_i u_{i+1}, \ldots, u_{k+1}) \neq 0 \) then either \( u_i \) or \( u_{i+1} \) is a vertex. In the former case the previous intermediate term cancels it and in the latter case the next term does the job.

\[ \square \]

9.3. Some \( A_\infty \)-structures on \( Gt(\mathbb{Q}) \). We will describe some specific \( A_\infty \)-structures on \( Gt(\mathbb{Q}) \), which can be constructed inductively. Choose a map \( \kappa : Q_2^+ \times \mathbb{N} \rightarrow \mathbb{C} \).

For any cycle \( b_1 \ldots b_i \) in \( Q_2^+ \), power \( n \), paths \( p_1, \ldots, p_k \) and location \( i \in [1, k] \) such that

A \( p_i b_1 \neq 0 \) and \( b_i p_{i+1} \neq 0 \),
B \( p_i \neq h(b_i) \) if \( i = 1 \) and \( p_{i+1} \neq t(b_i) \) if \( i + 1 = k \).
\[ \mu^\kappa(p_1, \ldots, p_k) = (-1)^n \kappa(b_1 \ldots b_n) \mu(p_1, \ldots, p_k) \]

with sign convention \( s = n ! (p_1 + \cdots + p_i + k - i) \).

\[ \mu^\kappa \left( \begin{array}{c} \vdots \\ n \times \end{array} \right) = \pm \kappa \left( \begin{array}{c} \vdots \\ n \end{array} \right) \mu^\kappa \left( \begin{array}{c} \vdots \\ \end{array} \right) \]

For \( u > 2 \) we set \( \mu^\kappa(q_1, \ldots, q_u) = 0 \) if we cannot perform any reduction of the form above.

**Lemma 9.7.** The rule above makes \( \mu^\kappa \) well defined for all \( u \).

**Proof.** We have to check that if there are two positive cycles \( a_r, \ldots a_1 \) and \( b_s \ldots b_1 \) that can be used to reduce the product, the end result will not depend on the order of the reduction. If we look at an entry that is equal to an arrow, we can uniquely identify the cycle and the power we need to reduce this arrow away. This is because each arrow sits in just one or with sign convention \( \mu \)

In both cases it is clear that, up to a sign, first reducing \( a_r \ldots a_1 \) and then \( b_s \ldots b_1 \) gives the same result as reducing in the opposite order. To show that the signs are the same, let us look at the first \( \mu_{r+s-1-4} \). If we first reduce \( a_1 \ldots a_r \) and then \( b_1 \ldots b_s \), we get a total sign

\[ s_{\text{tot}} = r(p_1 + \cdots + p_i + k - i + 1 + s - 2) + s(p_1 + \cdots + p_j + k - j - 1). \]

If we do it the other way round we get

\[ s_{\text{tot}} = s(p_1 + \cdots + p_i + a_1 + \cdots + a_r + k - j - 1) + r(p_1 + \cdots + p_i + k - i - 1). \]

Because of the construction of the gradation in section \([\text{section}] \) \( a_1 \ldots a_r \) has odd degree if and only if \( r \) is odd. Therefore \( s(a_1 + \cdots + a_r) + r(s - 2) = 0 \mod 2 \). The second sign calculation is similar. \( \square \)

**Lemma 9.8.** The \( \mu_u := \mu_u^\kappa \), \( u > 2 \) have the following properties

1. \( \mu(p_1, \ldots, p_u) \) is homotopic to \( p_1 \cdots p_u \) viewed as a path in \( |Q| \setminus Q_0 \).
2. If \( \mu(p_1, \ldots, p_u) = \pm q \) is not a trivial path then either \( p_1 := q \) or \( q = p_u^\kappa q \). In the first case for any \( q' \) such that \( q' p_1 \neq 0 \) we have \( \mu(q' p_1^\kappa, \ldots, p_u) = q' \mu(p_1^\kappa, \ldots, p_u) \) in the second we have \( \mu(p_1, \ldots, p_u, q') = \mu(p_1, \ldots, p_u) q' \) if \( p_u^\kappa q' \neq 0 \).
3. \( \mu(p_1, \ldots, p_u) = 0 \) if \( p_i p_{i+1} \neq 0 \) for some \( i < u \) or in particular when \( p_i \) is trivial.

**Proof.** We prove this by induction on the number of cycles we need to reduce \( \mu_u \) to an ordinary product.

1. The first statement clearly holds for \( \mu_2 \) and by induction for higher multiplications because any positive cycle \( a_1 \cdots a_t \) is contractible in \( |Q| \setminus Q_0 \).
2. If there is just one cycle then \( \mu(p_1 a_1, \ldots, a_t p_2) = p_1 p_2 \) but \( a_1 \) and \( a_t \) sit in a different negative cycle so \( p_1 p_2 \) is zero unless one of the 2 paths has length zero. It is also clear that if \( p_1 \) has nonzero length then \( \mu(q p_1 a_1, \ldots, a_t p_2) = q p_1 \) and a similar statement holds for nontrivial \( p_2 \).

If the statement holds up to \( k \) reductions then it also holds for \( k + 1 \) reductions because the first and last path in \( \mu \) after a reduction are subpaths starting from the outer ends of the first and last paths of the unreduced \( \mu \).
3 If there is just one cycle and \( \mu(p_1a_1, \ldots, a_lp_2) \neq 0 \) then \( a_ia_{i+1} = 0 \). If the statement holds up to \( k \) reductions then it also holds for \( k + 1 \) reductions because \( a_ia_{i+1} = 0 \) for any positive cycle.

\[ \]

**Proposition 9.9.** The products \( \mu^n \) turn \( \mathfrak{gt}_1 \) into an \( \mathcal{A}_\infty \)-category.

**Proof.** We will show that the identity \([M_k]\)

\[
\sum_{s+r+t=k} (-1)^{s+r+(2-r)(p_1+\cdots+p_s)} \mu(p_1, \ldots, p_s, \mu_r(p_{s+1}, \ldots, p_{s+r}), p_{s+r+1}, \ldots, p_k) = 0
\]

holds for all possible collections of paths \( p_1, \ldots, p_k \) with \( t(p_i) = h(p_{i+1}) \). We will do this using induction on \( k \).

For \( k \leq 3 \) the identities hold because \( \mathfrak{gt}_1 \) is an associative algebra with zero differential. Suppose now that the identity \([M_j]\) holds for all \( j < k \).

Because \( k > 3 \), every term in \( M_k \) will have at least one higher order multiplication in it. So if there is a nonzero term, there is at least one cycle we can reduce, so we can assume that the sequence of paths looks like

\[
p_1, \ldots, p_ia_1, a_2, \ldots, a_{nl}p_{l+1}, \ldots, p_k.
\]

Suppose now that we have a term that is nonzero. If the inner \( \mu \) is completely contained in the cycle then it can only be nonzero if \( n > 1 \) and \( \mu \) covers a subcycle. In that case this inner \( \mu \) returns a trivial path for one of the entries of the outer \( \mu \), which will then be zero.

If the inner \( \mu \) overlaps partially with the cycle then we can consider two situations:

A If the outer \( \mu \) is a higher multiplication we get expressions like

\[
\mu(\ldots, \mu(\ldots, a_j, a_{j+1}), \ldots) \text{ or } \mu(\ldots, a_j, \mu(a_{j+1}), \ldots).
\]

In the first case \( \mu(\ldots, a_j) \) must evaluate to something ending in \( a_j \) otherwise we cannot reduce \( a_{j+1} \).

If \( j \geq 2 \), \( \mu(\ldots, a_j) \) we first reduce \( \mu(\ldots, a_j) \) until we get to the reduction that will remove \( a_{j-1} \). If this reduction does not remove \( a_j \) then we end up with a higher order \( \mu \) that contains a trivial path which is impossible because we assumed the term was nonzero. If this reduction does remove \( a_j \) the only way we could get something nonzero is if the situation looked like

\[
\mu(qa_{j+1}, a_{j+2}, \ldots, a_j) \to \mu(q, t(a_j)) = q.
\]

But as \( q \) must end in \( a_j \) we have that \( qa_{j+1} = 0 \). Hence if the term is nonzero we must assume \( j = 1 \) and then for the same reason no reduction for the inner \( \mu \) can reduce \( a_1 \).

Similarly for \( \mu(\ldots, a_j, \mu(a_{j+1}), \ldots) \) we have that \( j > nl - 2 \) and no inner reduction can reduce \( a_{nl} \).

B If the outer \( \mu \) is the ordinary multiplication then it looks like

\[
\mu(\mu(\ldots, a_{nl-2}, a_{nl-1}), a_{nl}p_k)
\]

or a similar expression with the inner \( \mu \) on the right. Now reduce the inner \( \mu \) until we get to the reduction that gets rid of \( a_{nl-2} \). It must also remove \( a_{nl-1} \) because otherwise we get a higher multiplication with a trivial path. This implies that the nonzero term looks like

\[
\mu(\mu(q_1a_{nl-1}q_2, \ldots, a_{nl-2}, a_{nl-1}), a_{nl}p_k)
\]

(The inner reduction first gets rid of \( q_2, \ldots, q_n \) and after that there can only be \( nl \) terms in the inner \( \mu \) because otherwise reducing \( a_{nl-1}a_{nl-1} \) will give a higher multiplication with a trivial path).
If \(q_2, \ldots, q_n\) is nontrivial, we pick instead of \(a_1 \ldots a_{nl}\) a cycle in there. This cycle will then fall under case \([A]\). If \(q_2, \ldots, q_n\) is trivial our sequence of paths looks like
\[
p_1a_{nl}, a_1, a_2, \ldots, a_{nl-1}, a_np_2^*.
\]
It is easy to check that for such cases \([M_k]\) holds.

If our sequence is not of the form \((*)\) all nonzero terms fall in 4 categories.
- The inner \(\mu\) contains no part of the cycle \(a_1, \ldots, a_{nl}\).
- The inner \(\mu\) contains the whole cycle \(a_1, \ldots, a_{nl}\).
- The first entry of the inner \(\mu\) is \(a_{nl}p_{i+1}\) and the inner \(\mu\) evaluates to a right multiple of \(a_{nl}\).
- The last entry of the inner \(\mu\) is \(p_{i}a_1\) and the inner \(\mu\) evaluates to a left multiple of \(a_1\).

In each of these cases this term equals
\[
\pm \kappa(a_1 \ldots a_t, n)\mu(p_1, \ldots, p_s, \mu(p_{s+1}, \ldots, p_{s+r}), p_{s+r+1}, \ldots, p_k)
\]
for the appropriate \(s, r, t\). Different terms will give different simplified terms. Putting a factor \(\pm \kappa(a_1 \ldots a_t, n)\) in front we get that the expression \([M_{k+nl-2}]\) for
\[
p_1, \ldots, p_{i}a_1, a_2, \ldots, a_{nl}p_{i+1}, \ldots, p_k
\]
is \(\pm \kappa(a_1 \ldots a_t, n)\) times \([M_k]\) for \(p_1, \ldots, p_k\).

The fact that the signs match up requires some computation. In particular we need to show that the product of the sign in \([M_{k+nl-2}]\), the sign of \(\kappa\) and the sign in \([M_k]\) does not depend on \(s, r, t\) and the 4 different cases (note that the first case splits in 2 depending on whether the cycle comes before or after the inner \(\mu\)).

We will only do this calculation for the first case with \(i < s\) (i.e. the cycle comes before \(\mu\))
\[
s + (nl - 2) + rt + (2 - r)(p_1 + \cdots + p_s + a_1 + \cdots + a_{nl})
\]
\[
+ s + rt + (2 - r)(p_1 + \cdots + p_s) + nl(p_1 + \cdots + p_t) + s + t + 1 - i
\]
\[
= (nl - 2) + (2 - r)(a_1 + \cdots + a_{nl}) + nl(s + t + 1 - i + p_1 + \cdots + p_t)
\]
\[
= nl(r + s + t + 1 - i + p_1 + \cdots + p_t) \mod 2
\]
and the second case (when the cycle is inside the inner \(\mu\): \(s + t > i > s\))
\[
s + (r + nl - 2)t + (2 - r - nl + 2)(p_1 + \cdots + p_s)
\]
\[
+ s + rt + (2 - r)(p_1 + \cdots + p_s) + nl(p_s+1 + \cdots + p_t) + r + i - s
\]
\[
= nl - nl(p_1 + \cdots + p_s) + nl(r - i + s + p_{s+1} + \cdots + p_t)
\]
\[
= nl(r + s + t + 1 - i + p_1 + \cdots + p_t) \mod 2
\]

Given a map \(\rho : \text{RQ}_1 \to \mathbb{C}^*\) we get an autoequivalence of the category \(\text{FukQ}\)
\[
f_\rho : \text{Gt1}(Q) \to \text{Gt1}(Q) : a \mapsto \rho(a)a.
\]
This autoequivalence turns an \(A_\infty\)-structure defined by \(\kappa\) into a new one, which is defined by \(\kappa' : Q^+_2 \to \mathbb{C} : a_1 \ldots a_k \mapsto \kappa(a_1 \ldots a_k)\rho(a_1) \cdots \rho(a_k)\).
Lemma 9.10. Let $\kappa_1, \kappa_2 : \mathbb{Q}_2^+ \times \mathbb{N} \to \mathbb{C}$ be two maps for which $\kappa_i(c, n) = 0$ as soon as $n > 1$. The two $\mathbb{A}_\infty$-structures defined by $\kappa_1$ and $\kappa_2$ are isomorphic if $\forall c \in R\mathbb{Q}_2^+ : \kappa_1(c) = 0 \iff \kappa_2(c) = 0$.

Proof. Choose an arrow $a_c$ in each positive cycle $c$ with $\kappa_1(c, 1) \neq 0$ and define $\rho(a_c) = \kappa_2(c, 1)\kappa_1(c, 1)^{-1}$ and set all other $\rho(a) = 1$. This gives an isomorphism between the two $\mathbb{A}_\infty$-structures. $\square$

9.4. Identifying $\mathbb{A}_\infty$-structures on $\mathcal{G}_t\mathcal{L}(Q)$.

Lemma 9.11. If $\mu$ is an $\mathbb{A}_\infty$-structure on $\mathcal{G}_t\mathcal{L}(Q)$ and $a_1, \ldots, a_l$ is a positive cycle and we can find a $\kappa \in \mathbb{C}$ such that

$$\mu(a_1, \ldots, a_{nl}) = \kappa h(a_1)$$

then

$$\mu(a_1, \ldots, a_{nl+1}) = \kappa h(a_1).$$

Proof. This follows straight from the identity $[M_{nl+1}]$ applied to the paths $a_1, \ldots, a_{nl}, a_1$:

$$\mu(a_1, \mu(a_2, \ldots, a_{nl})), a_1) - \mu(\mu(a_1, \ldots, a_{nl}), a_1) = 0.$$

$\square$

Theorem 9.12. Let $\mathcal{G}_t\mathcal{L}(Q)$ be the gentle category coming from a rectified dimer $Q$. Let $\mu$ be an $\mathbb{A}_\infty$-structure on $\mathcal{G}_t\mathcal{L}(Q)$ such that for every positive cycle $a_1 \ldots a_l$ we have

$$\mu(a_1, \ldots, a_j) = \begin{cases} \kappa(a_1 \ldots a_i, n)h(a_i) & j - i + 1 = nl \\ 0 & \text{otherwise}, \end{cases}$$

where

$$\kappa : \mathbb{Q}_2^+ \times \mathbb{N} \to \mathbb{C} : (a_1 \ldots a_i, n) \mapsto \mu(a_1, \ldots, a_{nl}).$$

is a map. Then $\mu$ is $\mathbb{A}_\infty$-isomorphic to $\mu^a$.

Proof. In order to prove that $\mu^a$ and $\mu$ are isomorphic we construct an $\mathbb{A}_\infty$-functor $\mathcal{F}$ with $\mathcal{F}_1 = 1$. We show that we can do this by constructing the $\mathcal{F}_i$ one at a time. Suppose we have constructed $\mathcal{F}_i$ for $i < r$.

Now look at the identity $[F_n]$. We already know it is of the form

$$d\mathcal{F}_n = \Psi$$

with $d\Psi = 0$. In order to show that we can find an $\mathcal{F}_n$ we have to show that $\Psi \in \text{Im}d$ or equivalently that the homology class of $\Psi$ is zero. If we prove that $\Psi(a_1, \ldots, a_{i+nl-1})$ and $\Psi(a_1, \ldots, a_{i+nl})$ contain no length 0 or length 1 terms, then by lemma 9.5 we are done.

The terms in the expression $\Psi(a_1, \ldots, a_r)$ are of the forms

- $\mu(a_1, \ldots, a_{r-i}) - \mu^a(a_1, \ldots, a_{r-i})$. This expression contains no length 0 or length 1 terms by lemma 9.11.
- $\mathcal{F}(a_1, \ldots, a_j, \mu(a_{j+1}, \ldots, a_u), a_{u+1}, \ldots, a_r)$. Such a term is zero because by the condition in the theorem $\mu(a_{j+1}, \ldots, a_u)$ can only be zero or a multiple of $h(a_{j+1})$. But then $\mathcal{F}$ contains an idempotent so it is zero by lemma 9.13.
- $\mu^a(\mathcal{F}(a_1, \ldots, a_{i_1}), \ldots, \mathcal{F}(a_1, \ldots, a_{i_r}))$. If $\mathcal{F}(a_1, \ldots, a_{i+s})$ is nonzero then by lemma 4.3 $s$ must be $-2, -1, 0 \mod l$.
  - If it is $-2 \mod l$ then $\mathcal{F}(a_1, \ldots, a_{i+s})$ sits in $\text{Hom}(h(a_i), t(a_1))$, so all its terms have higher length.
  - If it is $-1 \mod l$ then the degree of $\mathcal{F}$ ($s$) and the degree of $a_i, \ldots, a_{i+s}$ ($s + 1$) imply that $\mathcal{F}(a_1, \ldots, a_{i+s})$ has odd degree, which contradicts lemma 4.3.
  - If it is 0 $\mod l$ then $\mathcal{F}(a_1, \ldots, a_{i+s})$ sits in $\text{Hom}(t(a_1), h(a_i))$ and can have a term which is a multiple of $a_i$ but we can always add an element in $\ker d$ to $f$ so by lemma 9.6 we can assume this term is zero.

In other words all terms are zero or have higher length. By construction $\mu^a$ of such an expression is zero, because there is no position where we can start to reduce it.
Lemma 9.13. Let \((F_1)\) be a partial \(A\) functor between 2 \(A\)-structures \(\mu^1\) and \(\mu^2\), for which \(F_1\) is the identity functor. For each object \(X\) we have that
\[
F_k(u_1, \ldots, u_{l-1}, 1_X, u_{l+1}, \ldots, u_k) = 0
\]
where the \(u_i\) are any maps.

Proof. We first show that \(F_k(1_X, \ldots, 1_X) = 0\). If \(k\) is even this is true because \(\deg F = 1 - k\) is odd and there are no odd elements in \(\text{Hom}(X, X)\). If \(k\) is odd then we use induction: suppose it holds for \(i < k\). Now
\[
dF_k(1_X, \ldots, 1_X) = 1_X F_k(1_X, \ldots, 1_X) - F_k(1_X \cdot 1_X, \ldots, 1_X) \cdots \pm F_k(1_X, \ldots, 1_X) 1_X
\]
by induction.

because the number of terms is odd. But on the other hand we know that because \(F\) is a partial functor the following identity holds:
\[
dF_k(1_X, \ldots, 1_X) = \sum_{r+s+t=k} \pm F(1_X, \ldots, 1_X, \mu^1(1_X, \ldots, 1_X), 1_X, \ldots, 1_X)
\]
\[
- \sum_{i_1+\cdots+i_u=k} \pm \mu^2(F(1_X, \ldots, 1_X), \cdots, F(1_X, \ldots, 1_X)).
\]
each of these terms is zero because both the \(\mu^1(1_X, \ldots, 1_X)\) are zero and \(F(1_X, \ldots, 1_X) = 0\) by induction.

To prove that \(F_k(u_1, \ldots, u_{l-1}, 1_X, u_{l+1}, \ldots, u_k) = 0\) we use double induction: on \(k\) and on the number of entries \(u_i\) that are not the identity. Now
\[
dF_k(u_1, \ldots, u_{l-1}, 1_X, u_{l+1}, \ldots, u_k) = + u_1 F_k(u_2, \ldots, 1_X, 1_X, u_k) - \ldots
\]
\[
\pm F_k(u_1, \ldots, u_{l-1}, 1_X, u_{l+1}, \ldots, u_k)
\]
\[
\pm F_k(u_1, \ldots, u_{l-1}, 1_X, u_{l+1}, \ldots, u_k)
\]
\[
+ \ldots F_k(u_2, \ldots, 1_X, 1_X, \ldots, u_{k-1}) u_k
\]
The first and last line are zero by induction because they have even more entries that are the identity, the intermediate lines contain 3 terms with alternating sign that are equal to
\[
F_k(u_1, \ldots, u_{l-1}, 1_X, u_{l+1}, \ldots, u_k).
\]
Again we can look at the identity for the partial \(A\)-functor
\[
dF_k(\ldots) = \sum_{r+s+t=k} \pm F(\ldots, \mu^1(\ldots), \ldots) - \sum_{i_1+\cdots+i_u=k} \pm \mu^2(F(\ldots), \ldots, F(\ldots)).
\]
Each term of it is zero: either because it contains an \(f_i\) with \(i < k\) with a \(1_X\) entry or a \(\mu^1\) or \(\mu^2\) with a \(1_X\) entry. \(\square\)

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RAF BOCKLANDT, SCHOOL OF MATHEMATICS AND STATISTICS, HERSCHEL BUILDING, NEWCASTLE UNIVERSITY, NEWCASTLE UPON TYNE, NE1 7RU, UK
E-mail address: raf.bocklandt@gmail.com