Non-diagonal open spin-1/2 $XXZ$ quantum chains by separation of variables: complete spectrum and matrix elements of some quasi-local operators

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Abstract. The integrable quantum models, associated with the transfer matrices of the 6-vertex reflection algebra for spin-1/2 representations, are studied in this paper. In the framework of Sklyanin’s quantum separation of variables (SOV), we provide the complete characterization of the eigenvalues and eigenstates of the transfer matrix and the proof of the simplicity of the transfer matrix spectrum. Moreover, we use these integrable quantum models as further key examples for which to develop a method in the SOV framework to compute matrix elements of local operators. This method is based on the resolution of the quantum inverse problem (i.e. the reconstruction of local operators in terms of the quantum separate variables) plus the computation of the action of separate covectors on separate vectors. In particular, for these integrable quantum models, which in the homogeneous limit reproduce the open spin-1/2 $XXZ$ quantum chains with non-diagonal boundary conditions, we have obtained the SOV-reconstruction for a class of quasi-local operators and determinant formulae for the covector–vector actions. As a consequence of these findings we provide one determinant formula for the matrix elements of this class of reconstructed quasi-local operator on transfer matrix eigenstates.

Keywords: integrable spin chains (vertex models), quantum integrability (Bethe ansatz)
1. Introduction

In this paper we analyse the lattice quantum integrable models characterized in the quantum inverse scattering method (QISM) [3]–[8] by (boundary) monodromy matrices which satisfy the reflection algebra [9]–[12] w.r.t. the 6-vertex $R$-matrix solution of the Yang–Baxter equation. The prototypical elements in this class of quantum integrable models are the open $XXZ$ spin-$1/2$ quantum chains, reproduced under the homogeneous

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limit of the representations of this reflection algebra on the spin-1/2 quantum chains. In the special representations corresponding to the open \(XXZ\) chain with diagonal boundary matrices\(^1\) the system has been largely analysed in the framework of the algebraic Bethe ansatz (ABA)\(^2\) \[3, 4\] with results going from the spectrum \[9\]–\[12\] up to the correlation functions\(^3\) \[15\], where the Lyon group method\(^4\) \[18, 19\] has been generalized to the reflection algebra case in the ABA framework.

The situation is more complicated in the case of general non-diagonal boundary matrices; technical difficulties arise and the ABA method can be applied only limitedly to non-diagonal boundary matrices which satisfy one constraint\(^5\) \(\geq 1\) which allows the definition of reference states by gauge transformations \[24, 25\]. Under the same constraint on the boundary parameters the method based on the combined use of the fusion of transfer matrices \[26, 27\] and the truncation identity for the root of the unit case \[28\] has also been developed for open \(XXZ\) spin-1/2 quantum chains. In \[29\] these truncation identities were first observed for general values of the boundary parameters and then in \[30\] they were used to construct \(TQ\)-functional relations to study the transfer matrix eigenvalues in the constrained case; see also \[31\] for the same type of eigenvalue analysis under different constraints on the boundary parameters. However, in this class of non-diagonal boundaries the use of the algebraic Bethe ansatz is complicated due to the fact that two sets of Bethe ansatz equations are generally required in order to have some numerical evidence of the completeness of the spectrum description \[32\]. The other problem in the ABA framework is that a determinant formula for the scalar products is missing as soon as non-diagonal boundary matrices are considered. Some analysis for solving this last problem has been addressed in the paper \[33\] where the partition function for the dynamical diagonal open case was considered. According to the standard techniques in the ABA framework, this is the first step towards the determination of the scalar product formula in a determinant form.

The state of the art for the general unconstrained non-diagonal boundary matrices is even more fragmented in the framework of Bethe ansatz analysis\(^6\), \(^7\); only the eigenvalue analysis is implemented by the fusion procedure for special values (roots of the unit case) of the anisotropy parameter \[37\] while for the other cases (non-roots of the unit case) the construction of the \(Q\)-operator is mainly at a conjecture level \[38\]. Once again the

\(^1\) Numerical 2 \(\times\) 2 matrix solutions of the reflection equations, located at the ending points of the quantum chain.

\(^2\) Let us recall that in the half-infinite volume case (with one boundary) similar multiple integral formulae have been previously derived in \[13, 14\] by using the \(q\)-vertex operator method.

\(^3\) Always in the ABA framework; see also \[16, 17\] for the extension of this method to the higher spin quantum chains.

\(^4\) It is also worth citing that the spectrum of the Temperley–Lieb loop model with open boundaries has been investigated by Bethe ansatz techniques in \[20, 21\]. The relevance of this analysis for open \(XXZ\) spin-1/2 quantum chains follows as under the same constraint on the boundary parameters its spectrum is contained in that of the loop model.

\(^5\) It is worth mentioning that by using a different approach, based on the representation theory of the so-called \(q\)-Onsager algebra, the spectrum of the open \(XXZ\) spin-1/2 quantum chains with general non-diagonal boundary conditions has been characterized in \[34, 35\] in terms of the roots of certain characteristic polynomials.

\(^6\) It is also worth citing that in \[36\] a functional method based only on the use of the Yang–Baxter algebra has been introduced and applied to the eigenvalue analysis of the open \(XXZ\) spin-1/2 chains for general anisotropy and boundary parameters. In fact in \[36\] the same method has been used to analyse also the closed \(XXZ\) spin-1/2 chains with general antiperiodic boundary conditions.
completeness of the spectrum description is verified only by some numerical analysis and the absence of eigenstate construction is the first fundamental missing step towards the matrix elements of local operators.

The circumstances that for general non-diagonal boundary matrices the ABA method does not work while for the constrained ones, for which it works, it is instead missing a scalar product formula have so far prevented further generalization of the method described in [15] to the reflection algebra representations for non-diagonal boundary matrices. However, we are in the position to develop a different approach based on Sklyanin’s quantum separation of variables (SOV) method [39, 40], which can be used to get the exact characterization of their spectrum (eigenvalues and eigenstates) and the computation of their form factors. Let us comment that in the special case of the spin-1/2 representations of the rational 6-vertex reflection algebra (XXX spin-1/2 quantum chain) the construction of the functional version of the separation of variables of Sklyanin has been implemented in [41, 42]; i.e. in these papers a representation of this algebra on a space of symmetric functions has been derived\(^8\). However, the explicit construction of the SOV representation and of the transfer matrix eigenstates in the original Hilbert space of the quantum chain are not provided there.

Sklyanin’s SOV seems to be a more efficient method to analyse the spectral problem w.r.t. other methods like the algebraic Bethe ansatz. It works for a large class of integrable quantum models; it leads to both the eigenvalues and the eigenstates of the transfer matrix providing a complete characterization of the spectrum under simple requirements\(^9\). Moreover, in all the integrable quantum models analysed by SOV in the series of papers [46]–[49] it was possible to show that the transfer matrix forms a complete\(^{10}\) set of commuting conserved charges of the quantum model. Finally, the approach that we are going to present also allows the characterization of the matrix elements of local operators of the quantum model. Let us say that this approach can be considered as the generalization to the SOV framework of the Lyon group method. It has already been implemented in [2] for the XXZ spin-1/2 quantum chain [50] with antiperiodic boundary conditions. Note that this last model was previously analysed by the Baxter \(Q\)-operator [51] technique and Sklyanin’s functional separation of variables for the XXX chain [40] extended in [52] to the XXZ case. Instead, the results obtained in [2] go from the complete characterization of the spectrum up to the calculation of the form factors of the local spin \(\sigma^a_n\) in a determinant form. This approach was originally introduced in [1] for the lattice quantum sine-Gordon model [4, 8].

Finally, let us comment that the analysis of these systems with non-diagonal boundary conditions is in particular interesting due to the relevant physical applications to systems in non-equilibrium like the asymmetric simple exclusion processes (ASEP) as it allows the description of systems for which the particle number is not conserved\(^{11}\). Then, the lack of knowledge on the spectrum in this general framework, the complications emerging in the use of the algebraic Bethe ansatz (constraints and the absence of a scalar product formula)

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\(^8\) See also [43] for the analysis by the functional SOV of the related but more general spin-boson model.

\(^9\) Note that on the contrary in the ABA framework a proof of completeness has been given only for some models; for example see [44] for the XXX Heisenberg model and [45] for the nonlinear quantum Schrödinger model, and the references contained in these papers.

\(^{10}\) Note that this is the natural quantum analogue of the definition of classical complete integrability.

\(^{11}\) For relevant references on this subject from the point of view of the connection to quantum integrable models see [53]–[55] and references therein.
plus the physical relevance of these open chains with non-diagonal boundary conditions make it clear how great the impact brought to this research area by the solution of these systems by our approach of quantum separation of variables can be.

1.1. Outline of the paper

The paper is organized as follows. In section 2, we introduce the trigonometric 6-vertex spin-1/2 representations of the reflection algebra for the most general boundary conditions. There, the first fundamental properties of the generators of this algebra are reproduced by simple generalization of Sklyanin’s original results [12] to the general non-diagonal boundary conditions. In section 3, we generalize to the reflection algebra the definition of quantum separate variables and explicitly construct their eigenbasis for the two classes of transfer matrices analysed in both the left and right spaces of the representations. This allows us to define in this new basis the SOV-representations of the generators of the reflection algebra. In section 4, the coupling of left and right eigenstates of the quantum separate variables is defined leading to the decomposition of the identity in the SOV-basis. In section 5, we use the SOV representation of the reflection algebra generators to characterize completely the spectra of the two classes of transfer matrices and to prove their simplicity. In particular, we show that all their eigenvalues and eigenstates are defined by the solutions of a discrete system of equations (not directly of Bethe ansatz type) in a well defined class of functions. In section 6, we introduce the definition of left and right separate states which in particular describe transfer matrix eigenstates. Then we compute the scalar product of these states, i.e. the action of the left separate states on the right ones. In section 7, we present our construction of a class of quasi-local operators in terms of the quantum separate variables. In section 8, the matrix elements of these classes of local operators are computed. Finally, section 9 contains our conclusion and outlook.

2. The reflection algebra and the open spin-1/2 XXZ quantum chain

In this section we describe a class of quantum integrable models characterized in the framework of the quantum inverse scattering method by monodromy matrices $\mathcal{U}(\lambda)$ which are solutions of the following reflection equation:

$$R_{12}(\lambda - \mu)\mathcal{U}_1(\lambda) R_{12}(\lambda + \mu)\mathcal{U}_2(\mu) = \mathcal{U}_2(\mu) R_{12}(\lambda + \mu)\mathcal{U}_1(\lambda) R_{12}(\lambda - \mu) \quad (2.1)$$

w.r.t. the 6-vertex trigonometric solution of the Yang–Baxter equation, the $R$-matrix

$$R_{12}(\lambda) \equiv \begin{pmatrix} \sinh(\lambda + \eta) & 0 & 0 & 0 \\ 0 & \sinh \lambda & \sinh \eta & 0 \\ 0 & \sinh \eta & \sinh \lambda & 0 \\ 0 & 0 & 0 & \sinh(\lambda + \eta) \end{pmatrix} \in \text{End}(R_1 \otimes R_2), \quad (2.2)$$

where $R_x \cong \mathbb{C}^2$ is a two-dimensional linear space.
2.1. Highest weight representations of the 6-vertex reflection algebra on spin-1/2 chains

Let $K(\lambda; \zeta, \kappa, \tau)$ be the following (general non-diagonal boundary) matrix:

$$K(\lambda; \zeta, \kappa, \tau) = \frac{1}{\sinh \zeta} \begin{pmatrix} \sinh(\lambda + \zeta) & \kappa e^\tau \sinh 2\lambda \\ \kappa e^{-\tau} \sinh 2\lambda & \sinh(\zeta - \lambda) \end{pmatrix},$$

where $\zeta, \kappa$ and $\tau$ are arbitrary complex parameters. It is the most general scalar solution of the 6-vertex trigonometric reflection equation

$$R_{12}(\lambda - \mu)K_1(\lambda)R_{12}(\lambda + \mu)K_2(\mu) = K_2(\mu)R_{12}(\lambda + \mu)K_1(\lambda)R_{12}(\lambda - \mu).$$

Then, following Sklyanin [12], it is possible to construct in the $2^N$-dimensional representation space

$$\mathcal{R}_N \equiv \otimes_{n=1}^N R_n$$

two classes of solutions to the same reflection equation (2.4). In order to do so, let us define

$$K_-(\lambda) = K(\lambda - \eta/2; \zeta_-, \kappa_-, \tau_-), \quad K_+(\lambda) = K(\lambda + \eta/2; \zeta_+, \kappa_+, \tau_+),$$

where $\zeta_\pm, \kappa_\pm, \tau_\pm$ are arbitrary complex parameters, and the (bulk) monodromy matrix

$$M_0(\lambda) = R_{0N}(\lambda - \xi_N - \eta/2) \cdots R_{02}(\lambda - \xi_2 - \eta/2) R_{01}(\lambda - \xi_1 - \eta/2),$$
$$M(\lambda) = (-1)^N \sigma_0^y M_0^0(-\lambda) \sigma_0^y,$$

$M_0(\lambda) \in \text{End}(R_0 \otimes \mathcal{R}_N)$, solution of the 6-vertex Yang–Baxter equation

$$R_{12}(\lambda - \mu)M_1(\lambda)M_2(\mu) = M_2(\mu)M_1(\lambda)R_{12}(\lambda - \mu).$$

Now we can define the following (boundary) monodromy matrices $U_\pm(\lambda) \in \text{End}(R_0 \otimes \mathcal{R}_N)$ as follows:

$$U_-(\lambda) = M_0(\lambda)K_-(\lambda)\hat{M}_0(\lambda) = \begin{pmatrix} A_-(\lambda) & B_-(\lambda) \\ C_-(\lambda) & D_-(\lambda) \end{pmatrix},$$
$$U_0(\lambda) = M_0^0(\lambda)K_0^0(\lambda)\hat{M}_0^0(\lambda) = \begin{pmatrix} A_0(\lambda) & B_0(\lambda) \\ C_0(\lambda) & D_0(\lambda) \end{pmatrix},$$

then $U_-(\lambda)$ and $U_+(\lambda) \equiv U_0^0(-\lambda)$ define two classes of solutions of the reflection equation (2.4).

As is standard in the quantum inverse scattering method and as was proven in [12], from these monodromy matrices it is possible to define a commuting family of transfer matrices $T(\lambda) \in \text{End}(\mathcal{R}_N)$ as follows:

$$T(\lambda) \equiv \text{tr}_0\{K_+(\lambda) M(\lambda) K_-(\lambda)\hat{M}(\lambda)\} = \text{tr}_0\{K_+(\lambda)U_-(\lambda)\} = \text{tr}_0\{K_-(\lambda)U_+(\lambda)\}.$$

The problems that we address in this paper are the complete characterization of the spectrum (eigenvalue and eigenstates) of this transfer matrix and the computation of

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12 The representation space of a spin-1/2 quantum chain of $N$ local sites each one associated to a two-dimensional local space $R_n$.

13 Note that here we have chosen a shifted definition of the inhomogeneity w.r.t. the one used in [15]; in this case the homogeneous limit corresponds to $\xi_m = 0$ for $m = 1, \ldots, N$. 

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some matrix elements of quasi-local operators for two quite general classes of non-diagonal boundary matrices \( K_{\pm}(\lambda) \). It is then worth recalling that the open spin-1/2 XXZ quantum chain, with the most general non-diagonal integrable boundary conditions, is characterized by the following Hamiltonian:

\[
H_{ND} = \sum_{i=1}^{N-1} (\sigma^x_i \sigma^x_{i+1} + \sigma^y_i \sigma^y_{i+1} + \cosh \eta \sigma^z_i \sigma^z_{i+1})
+ \frac{\sinh \eta}{\sinh \zeta_-} [\sigma^z_i \cosh \zeta_- + 2 \kappa \cosh \tau_- + i \sigma^y_i \sinh \tau_-]
+ \frac{\sinh \eta}{\sinh \zeta_+} [\sigma^z_N \cosh \zeta_+ + 2 \kappa (\sigma^y_N \cosh \tau_+ + i \sigma^y_N \sinh \tau_+)].
\] (2.12)

The Hamiltonian is reproduced in the homogeneous limit by the following derivative of the transfer matrix (2.11):

\[
H_{ND} = \frac{2(\sinh \eta)^{1-2N}}{\text{tr}\{K_{ND}(\eta/2)\} \text{tr}\{K_{ND}(\eta/2)\}} \frac{d}{d\lambda} T(\lambda) \bigg|_{\lambda=\eta/2} + \text{constant.}
\] (2.13)

2.2. First fundamental properties

Here, some important properties on the generators of the reflection algebra \( A_{\pm}(\lambda), B_{\pm}(\lambda), C_{\pm}(\lambda) \) and \( D_{\pm}(\lambda) \) are given as they will play a fundamental role in the solution of the transfer matrix \( T(\lambda) \) spectral problem.

**Proposition 2.1** (\( U_- \)-reflection algebra). In the reflection algebra generated by the elements of \( U_- (\lambda) \) the quantum determinant

\[
\det_q U_- (\lambda) \equiv \sinh(2\lambda - 2\eta)[A_- (\lambda + \eta/2)A_+ (-\lambda + \eta/2) - B_- (\lambda + \eta/2)C_- (-\lambda + \eta/2)]
= \sinh(2\lambda - 2\eta)[D_- (\lambda + \eta/2)D_+ (-\lambda + \eta/2) - C_- (\lambda + \eta/2)B_- (-\lambda + \eta/2)]
\] (2.14)

is central,

\[
[\det_q U_- (\lambda), U_- (\mu)] = 0.
\] (2.15)

Moreover, it admits the following explicit expression:

\[
\det_q U_- (\lambda) = \sinh(2\lambda - 2\eta)A_- (\lambda + \eta/2)A_- (-\lambda + \eta/2),
\] (2.17)

where

\[
A_- (\lambda) \equiv g_- (\lambda) a(\lambda) d(-\lambda), \quad d(\lambda) \equiv a(\lambda - \eta),
\]

\[
a(\lambda) \equiv \prod_{n=1}^{N} \sinh(\lambda - \xi_n + \eta/2),
\] (2.18)

and

\[
g_\pm (\lambda) \equiv \frac{\sinh(\lambda \mp \alpha_\pm \eta/2) \cosh(\lambda \mp \beta_\pm \eta/2)}{\sinh \alpha_\pm \cosh \beta_\pm},
\] (2.19)

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where $\alpha_\pm$ and $\beta_\pm$ are defined in terms of the boundary parameters by
\[
\sinh \alpha_\pm \cosh \beta_\pm \equiv \frac{\sinh \zeta_\pm}{2K_\pm}, \quad \cosh \alpha_\pm \sinh \beta_\pm \equiv \frac{\cosh \zeta_\pm}{2K_\pm}.
\] (2.20)

Moreover, the generator families $A_-(\lambda)$ and $D_-(\lambda)$ are related by the following parity relation:
\[
D_-(\lambda) = \frac{\sinh(2\lambda - \eta)}{\sinh 2\lambda} A_-(\lambda) + \frac{\sinh \eta}{\sinh 2\lambda} A_-(\lambda),
\] (2.21)
while for the other two families the following parity relations hold:
\[
B_-(\lambda) = -\frac{\sinh(2\lambda + \eta)}{\sinh(2\lambda - \eta)} B_-(-\lambda), \quad C_-(\lambda) = -\frac{\sinh(2\lambda + \eta)}{\sinh(2\lambda - \eta)} C_-(-\lambda).
\] (2.22)

**Proof.** This proposition is just a rephrasing and a simple extension to the case of a general non-diagonal $K_-(\lambda)$ boundary matrix of the results stated in propositions 5–7 of Sklyanin’s article [12]. The quantum determinant as defined in formulae (38) \[12\]–(42) \[12\],
\[
\det_q U_- (\lambda) \equiv \tilde{U}_-(\lambda - \eta/2) U_- (\lambda + \eta/2) = U_- (\lambda + \eta/2) \tilde{U}_-(\lambda - \eta/2)
\] (2.23)
\[
= A_-(\lambda + \eta/2) \tilde{D}_-(\lambda - \eta/2) - B_-(\lambda + \eta/2) \tilde{D}_-(\lambda - \eta/2),
\] (2.24)
is central independently of $K_-(\lambda)$ being diagonal or non-diagonal. Here, we have used the definition (43) \[12\],
\[
\tilde{U}_-(\lambda) \equiv \frac{\text{tr}_{12} R_{12}(-\eta) U_- (\lambda) R_{21}(2\lambda)}{\sinh \eta} = \begin{pmatrix} \tilde{D}_-(\lambda) & -\tilde{B}_-(\lambda) \\ -\tilde{C}_-(\lambda) & \tilde{A}_-(\lambda) \end{pmatrix}
\] (2.25)
\[
\equiv \begin{pmatrix} D_- (\lambda) \sinh 2\lambda - A_- (\lambda) \sinh \eta & -\sinh(2\lambda + \eta) B_- (\lambda) \\ -\sinh(2\lambda + \eta) B_- (\lambda) & A_- (\lambda) \sinh 2\lambda - D_- (\lambda) \sinh \eta \end{pmatrix}.
\] (2.26)

So, we can also write it as it follows:
\[
\det_q U_- (\lambda) \equiv \det_q K_- (\lambda) \det_q M_0 (\lambda) \det_q M_0 (-\lambda),
\] (2.27)
where $\det_q M(\lambda)$ is the (bulk) quantum determinant of the Yang–Baxter algebra,
\[
\det_q M_0 (\lambda) = A(\lambda + \eta/2) D(\lambda - \eta/2) - B(\lambda + \eta/2) C(\lambda - \eta/2)
\] (2.28)
\[
= a(\lambda + \eta/2) d(\lambda - \eta/2).
\] (2.28)

Here, we have denoted
\[
M_0 (\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix},
\] (2.29)
and
\[
\det_q K_- (\lambda) = (K_-)_{1,1} (\lambda + \eta/2) (K_-)_{2,2} (\lambda - \eta/2) - (K_-)_{1,2} (\lambda + \eta/2) (K_-)_{2,1} (\lambda - \eta/2)
\] (2.30)
\[
= -\frac{\sinh(2\lambda - 2\eta)}{\sinh^2 \zeta_-} (\sinh(\lambda + \zeta_-) \sinh(\lambda - \zeta_-) + \kappa^2 \sinh^2 2\lambda),
\] (2.30)
where
\[
\widetilde{K}_-(\lambda) \equiv \sinh(2\lambda - \eta)\sigma_y^0 K_{(s)}^0(-\lambda)\sigma_y^0.
\] (2.31)

Then the explicit expression (2.17) follows observing that it holds that
\[
det_q K_-(\lambda) = \sinh(2\lambda - 2\eta)g_-(\lambda + \eta/2)g_-(\lambda + \eta/2),
\] (2.32)
when we use the parameters \(\alpha_-\) and \(\beta_-\).

Finally, it is simple to remark that formula (46)\cite{12} is equivalent to the symmetry properties (2.21) and (2.22), which in turn imply the expressions for the quantum determinant (2.14) and (2.15) when used to rewrite formula (2.24). \(\square\)

It is worth remarking that similar statements hold for the reflection algebra generated by \(\mathcal{U}_+(\lambda)\). In fact, they are simply consequences of the previous proposition when it is taken into account that \(\mathcal{U}_+^0(-\lambda)\) satisfies the same reflection equation of \(\mathcal{U}_-(\lambda)\).

**Proposition 2.2 (\(\mathcal{U}_+\)-reflection algebra).** In the reflection algebra generated by the elements of \(\mathcal{U}_+(\lambda)\) the quantum determinant
\[
det_q \mathcal{U}_+(\lambda) = \sinh(2\lambda + 2\eta)[\mathcal{A}_+(\lambda - \eta/2)\mathcal{A}_+(\lambda-\eta/2) - \mathcal{B}_+(\lambda-\eta/2)\mathcal{C}_+(\lambda-\eta/2)]
\] (2.33)
\[
= \sinh(2\lambda + 2\eta)[\mathcal{D}_+(\lambda-\eta/2)\mathcal{D}_+(\lambda-\eta/2) - \mathcal{C}_+(\lambda-\eta/2)\mathcal{B}_+(\lambda-\eta/2)]
\] (2.34)
is central,
\[
[\det_q \mathcal{U}_+(\lambda), \mathcal{U}_+(\mu)] = 0.
\] (2.35)

Moreover, it admits the following explicit expression:
\[
det_q \mathcal{U}_+(\lambda) = \sinh(2\lambda + 2\eta)\mathcal{D}_+(\lambda + \eta/2)\mathcal{D}_+(\lambda - \eta/2),
\] (2.36)
where the function \(\mathcal{D}_+(\lambda)\) is defined by
\[
\mathcal{D}_+(\lambda) = g_+(\lambda)a(-\lambda)d(\lambda),
\] (2.37)
where \(g_+(\lambda)\) is defined in (2.19). Moreover, the generator families \(\mathcal{A}_+(\lambda)\) and \(\mathcal{D}_+(\lambda)\) are related by the following parity relation:
\[
\mathcal{D}_+(\lambda) = \frac{\sinh(2\lambda + \eta)}{\sinh 2\lambda} \mathcal{A}(-\lambda) - \frac{\sinh \eta}{\sinh 2\lambda} \mathcal{A}_+(\lambda),
\] (2.38)
while for the other two families the following parity relations hold:
\[
\mathcal{B}_+(-\lambda) = -\frac{\sinh(2\lambda - \eta)}{\sinh(2\lambda + \eta)} \mathcal{B}_+(\lambda), \quad \mathcal{C}_+(-\lambda) = -\frac{\sinh(2\lambda - \eta)}{\sinh(2\lambda + \eta)} \mathcal{C}_+(\lambda).
\] (2.39)

**Proof.** Here, we only need to notice that
\[
det_q K_+(\lambda) = (K_+)^{1,1}_1(\lambda - \eta/2)(\widetilde{K}_+)^{1,2}_1(\lambda + \eta/2) - (K_+)^{1,2}_1(\lambda - \eta/2)(\widetilde{K}_+)^{2,1}_1(\lambda + \eta/2)
\] (2.40)
\[
= -\frac{\sinh(2\lambda + 2\eta)}{\sinh^2 \zeta_+} (\sinh(\lambda + \zeta_+) \sinh(\lambda - \zeta_+) + \kappa^2 \sinh^2 2\lambda),
\] (2.40)
where
\[ \widetilde{K}_+^{(\pm)}(\lambda) \equiv \sinh(2\lambda + \eta)\sigma_{0}^{y}K_+^{(\pm)}(-\lambda)\sigma_{0}^{y} \] (2.41)
can be written in the form
\[ \det_{q} K_+^{(\pm)}(\lambda) = \sinh(2\lambda + 2\eta)g_+(\lambda - \eta/2)g_+(\lambda - \eta/2), \] (2.42)
where the parameters \( \alpha_+ \) and \( \beta_+ \), entering in the function \( g_+(\lambda) \), are defined in terms of the parameters of the \( K_+^{(\pm)}(\lambda) \) boundary matrix in (2.20).

Let us introduce the following notations:
\[ K_\pm^{(\pm)}(\lambda) \equiv \frac{1}{\sinh \zeta_\pm} \left( \begin{array}{cc} \sinh(\lambda + \zeta_\pm \pm \eta/2) & \kappa_\pm \epsilon^{\mp \pm} \sinh(2\lambda + \pm \eta) \\ \kappa_\pm \epsilon^{-\mp \pm} \sinh(2\lambda + \pm \eta) & \sinh(\zeta_\pm \mp \eta/2 - \lambda) \end{array} \right) = \begin{pmatrix} a_\pm(\lambda) & b_\pm(\lambda) \\ c_\pm(\lambda) & d_\pm(\lambda) \end{pmatrix}, \] (2.43)
and by using these notations let us rewrite the transfer matrix (2.11) in the following two equivalent forms:
\[ T^{(\pm)}(\lambda) = T^{(\pm)}_\lambda(\lambda) + b_\mp(\lambda) C_\pm(\lambda) + c_\mp(\lambda) B_\pm(\lambda), \] (2.44)
where
\[ T^{(\pm)}_\lambda(\lambda) \equiv a_\mp(\lambda) A_\pm(\lambda) + d_\mp(\lambda) D_\pm(\lambda), \] (2.45)
is the transfer matrix\(^{14}\) of the system with diagonal matrix \( K_\pm^{(\pm)}(\lambda) \), respectively. Then we have the following corollary.

**Corollary 2.1.** \( T^{(\pm)}_\lambda(\lambda) \) admits the following explicitly even forms w.r.t. the spectral parameter \( \lambda \):
\[ T^{(\pm)}_\lambda(\lambda) \equiv a_\pm(\lambda) A_\pm(\lambda) + a_\mp(\lambda) A_\pm(-\lambda) \]
\[ = d_\pm(\lambda) D_\pm(\lambda) + d_\mp(\lambda) D_\pm(-\lambda), \] (2.47)
where
\[ a_\pm(\lambda) \equiv \frac{\sinh(2\lambda \pm \eta) \sinh(\lambda + \zeta_\pm \mp \eta/2)}{\sinh 2\lambda \sinh \zeta_\pm}, \] (2.48)
\[ d_\pm(\lambda) \equiv \frac{\sinh(2\lambda \pm \eta) \sinh(\zeta_\pm - \lambda \pm \eta/2)}{\sinh 2\lambda \sinh \zeta_\pm}. \] (2.49)
Moreover, the most general transfer matrix is also even in the spectral parameter \( \lambda \),
\[ T(-\lambda) = T(\lambda). \] (2.50)

**Proof.** By using the formulae (2.21) and (2.38) to rewrite \( T^{(\pm)}_\lambda(\lambda) \) only in terms of \( A_\pm(\lambda) \) or only in terms of \( D_\pm(\lambda) \) after some simple algebra we get our formulae (2.46) and (2.47), respectively. Then the parity (2.50) of the transfer matrix \( T(\lambda) \) follows by

\(^{14}\)Note that \( T^{(+)}_\lambda(\lambda) \) corresponds to the \( K_-(\lambda) \) diagonal while \( K_+(\lambda) \) is left general as well as \( T^{(-)}_\lambda(\lambda) \) corresponding to the \( K_+(\lambda) \) diagonal while \( K_-(\lambda) \) is left general.

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remarking that the parity properties

\[ b_\pm (-\lambda) C_\pm (-\lambda) = b_\pm (\lambda) C_\pm (\lambda), \quad c_\pm (-\lambda) B_\pm (-\lambda) = c_\pm (\lambda) B_\pm (\lambda) \]

are just a rewriting of the known properties (2.22) and (2.39).

**Proposition 2.3.** The monodromy matrix \( \mathcal{U}_\pm (\lambda) \) satisfies the following transformation properties under Hermitian conjugation.

(I) Under the condition \( \eta \in \mathbb{R} \) (massless regime), it holds that

\[ \mathcal{U}_\pm (\lambda)^\dagger = [\mathcal{U}_\pm (-\lambda^*)]^\dagger_0, \quad \text{for} \ \{i\tau_\pm, i\kappa_\pm, i\xi_\pm, \ldots, i\xi_N\} \in \mathbb{R}^{N+3}. \]

(II) Under the condition \( \eta \in \mathbb{R} \) (massive regime), it holds that

\[ \mathcal{U}_\pm (\lambda)^\dagger = [\mathcal{U}_\pm (\lambda^*)]^\dagger_0, \quad \text{for} \ \{\tau_\pm, \kappa_\pm, \xi_\pm, \ldots, \xi_N\} \in \mathbb{R}^{N+3}. \]

Under the same conditions on the parameters of the representation it holds that

\[ \mathcal{T}(\lambda)^\dagger = \mathcal{T}(\lambda^*), \]

i.e. \( \mathcal{T}(\lambda) \) defines a one-parameter family of normal operators which are self-adjoint for \( \lambda \) both real and imaginary.

**Proof.** This proposition gives the generalization to the case of general non-diagonal boundary conditions of the transformation properties under Hermitian conjugation proven in [15] for the diagonal case. The proof is given by using the following transformation properties under Hermitian conjugation.

(I) For \( \{i\eta, i\tau_\pm, i\kappa_\pm, i\xi_\pm, \ldots, i\xi_N\} \in \mathbb{R}^{N+3} \) then it holds that

\[ R_{0n}(\lambda - \xi_1 - \eta/2)^\dagger = \sigma_0^y R_{0n}(\lambda^* - \xi_1 - \eta/2)\sigma_0^y = -[R_{0n}(-\lambda^* + \xi_1 - \eta/2)]^\dagger_0, \]
\[ K_{\pm}(\lambda)^\dagger = [K_{\pm}(-\lambda^*)]^\dagger_0. \]

(II) For \( \{\eta, \tau_\pm, \kappa_\pm, \xi_\pm, \ldots, \xi_N\} \in \mathbb{R}^{N+3} \) then it holds that

\[ R_{0n}(\lambda - \xi_1 - \eta/2)^\dagger = -\sigma_0^y R_{0n}(\lambda^* - \xi_1 - \eta/2)\sigma_0^y = [R_{0n}(\lambda^* + \xi_1 - \eta/2)]^\dagger_0 \]
\[ K_{\pm}(\lambda)^\dagger = [K_{\pm}(\lambda^*)]^\dagger_0. \]

This can be verified by direct calculations. From these it follows that

\[ M(\lambda)^\dagger = \left( -\frac{\eta}{\eta^*} \right)^N \sigma_0^y M \left( -\left( \frac{\eta}{\eta^*} \right) \lambda^* \right) \sigma_0^y = \left( \frac{\eta}{\eta^*} \right)^N \left[ \hat{M} \left( \left( \frac{\eta}{\eta^*} \right) \lambda^* \right) \right]^\dagger_0 \]

for the (bulk) monodromy matrix and so

\[ \mathcal{U}_\pm (\lambda)^\dagger = \left\{ \left[ \hat{M} \left( \left( \frac{\eta}{\eta^*} \right) \lambda^* \right) \right]^\dagger_0 \left[ K_{\pm} \left( \left( \frac{\eta}{\eta^*} \right) \lambda^* \right) \right]^\dagger_0 \left[ M \left( \left( \frac{\eta}{\eta^*} \right) \lambda^* \right) \right]^\dagger_0 \right\}_q-\text{reverse\,-\,order} \]

where the notation \( q\,-\text{reverse\,-\,order} \) is referred to the reverse order in the generators of the Yang–Baxter algebras. In more detail, the matrix elements of the \( 2 \times 2 \) matrix \( \mathcal{U}_\pm (\lambda)^\dagger \) are computed by using the normal matrix products in the auxiliary space.
0, as indicated inside the brackets on the rhs of (2.60), then in each element of the matrix $U_\pm(\lambda)\dagger$ the matrix elements of $[M((\eta/\eta^*)^s)\tau_0]$ are put to the right of those of $[M((\eta/\eta^*)^s)\tau_0]$. It is then simple to verify that the r.h.s. of (2.60) with this prescribed order coincides with

$$
\left[ U_\pm \left( \left( \frac{\eta}{\eta^*} \right)^s \lambda^s \right) \right]^{\tau_0},
$$

which proves both (2.52) and (2.53). Then by using these last two formulae and the transformation properties (2.56) and (2.58), we get

$$
\mathcal{T}(\lambda)^\dagger = \text{tr}_0 \left\{ \left[ K_\mp \left( \left( \frac{\eta}{\eta^*} \right)^s \lambda^s \right) \right]^{\tau_0} \left[ U_\pm \left( \left( \frac{\eta}{\eta^*} \right)^s \lambda^s \right) \right]^{\tau_0} \right\} = \mathcal{T} \left( \left( \frac{\eta}{\eta^*} \right)^s \lambda^s \right)_{2.50} = \mathcal{T}^*(\lambda).
$$

(2.62)

3. SOV-representations for the $\mathcal{T}(\lambda)$-spectral problem

The method to construct quantum separation of variable (SOV) representations for the spectral problem of the transfer matrices associated with the representations of the Yang–Baxter algebra has been defined by Sklyanin in [39, 40]. Here, we show that there are quite general representations of the reflection algebra with non-diagonal boundary matrices for which the quantum SOV-representations can be constructed by adapting Sklyanin’s method. In more detail, the following theorem holds.

Theorem 3.1. Let the inhomogeneities $\{\xi_1, \ldots, \xi_N\} \in \mathbb{C}^N$ satisfy the following conditions:

$$
\xi_a \neq \xi_b + r\eta \quad \forall \ a \neq b \in \{1, \ldots, N\} \quad \text{and} \quad r \in \{-1, 0, 1\},
$$

(3.1)

then the commuting families of generators of the reflection algebra $B_\pm(\lambda)$ and $C_\pm(\lambda)$ are diagonalizable and have a simple spectrum.

Moreover, denoted with $\epsilon \in \{+,-\}$, the following statements hold.

(I) The representations for which the commuting family $B_\epsilon(\lambda)$ is diagonal define the quantum SOV-representations for the spectral problem of the transfer matrix $\mathcal{T}(\lambda)$ associated with the boundary matrix $K_{-\epsilon}(\lambda)$ diagonal or lower triangular while $K_{\epsilon}(\lambda)$ is a general non-lower triangular; i.e. the SOV-representations for

$$
\mathcal{T}(\lambda) \equiv \mathcal{T}_{(\epsilon)}(\lambda) + c_{-\epsilon}(\lambda) B_{\epsilon}(\lambda), \quad \text{for} \quad b_{-\epsilon}(\lambda) = 0 \quad \text{and} \quad b_{\epsilon}(\lambda) \neq 0.
$$

(3.2)

(II) The representations for which the commuting family $C_\epsilon(\lambda)$ is diagonal define the quantum SOV-representations for the spectral problem of the transfer matrix $\mathcal{T}(\lambda)$ associated with the boundary matrix $K_{-\epsilon}(\lambda)$ diagonal or upper triangular while $K_{\epsilon}(\lambda)$ is a general non-upper triangular one; i.e. the SOV-representations for

$$
\mathcal{T}(\lambda) \equiv \mathcal{T}_{(\epsilon)}(\lambda) + b_{-\epsilon}(\lambda) C_{\epsilon}(\lambda), \quad \text{for} \quad c_{-\epsilon}(\lambda) = 0 \quad \text{and} \quad c_{\epsilon}(\lambda) \neq 0.
$$

(3.3)

The proof of the theorem will be given in the following sections by explicit construction of the $B_\pm(\lambda)$-eigenbasis and the solution by quantum separation of variables of the spectral problem for the transfer matrix $\mathcal{T}(\lambda)$ in the class of representations of the reflection algebra listed in point (I) of the theorem. The Hermitian conjugation properties lead to
the construction of the $C_{\pm}(\lambda)$-eigenbasis and then imply that the theorem holds also for the class of representations listed in point (II) of the theorem.

3.1. Left and right representations of the reflection algebras

Let us give some more detail on the space of the representation of the spin-$1/2$ quantum chain. As this is the same representation space used in [2] for the 6-vertex Yang–Baxter algebra similar notation will be given here.

Let us introduce the standard spin basis for the two-dimensional linear space $R_n$, the quantum space at the site $n$ of the chain, whose elements are the $\sigma^z_n$-eigenvectors $|k,n\rangle$, characterized by

$$\sigma^z_n|k,n\rangle = k|k,n\rangle, \quad k \in \{-1,1\}. \quad (3.4)$$

Similarly, the $\sigma^z_n$-eigencovectors $\langle k,n|$, characterized by

$$\langle k,n|\sigma^z_n = k\langle k,n|, \quad k \in \{-1,1\} \quad (3.5)$$

define a basis in $L_n$, the dual space of $R_n$. Then, $2^N$-dimensional representations with $N+6$ parameters (the inhomogeneities and the boundary parameters) of the reflection algebra are defined in the left (covectors) and right (vectors) linear spaces

$$\mathcal{L}_N \equiv \otimes_{n=1}^N L_n, \quad \mathcal{R}_N \equiv \otimes_{n=1}^N R_n. \quad (3.6)$$

Moreover, $\mathcal{R}_N$ is naturally provided with the structure of Hilbert space by introducing the scalar product characterized by the following action on the spin basis:

$$\left( \otimes_{n=1}^N |k_n,n\rangle, \otimes_{n=1}^N |k'_n,n\rangle \right) \equiv \prod_{n=1}^N \delta_{k_n,k'_n} \quad \forall k_n,k'_n \in \{-1,1\}. \quad (3.7)$$

3.2. $B_-$-SOV-representations of the reflection algebra

In this subsection we construct the left and right SOV-representations of the reflection algebra generated by $U_-(\lambda)$ by constructing the left and right $B_-(\lambda)$-eigenbasis.

**Theorem 3.2.** (I) Left $B_-(\lambda)$ SOV-representations. If (3.1) is satisfied and $b_-(\lambda) \neq 0$, then the states

$$\langle -,h_1,\ldots,h_N| \equiv \frac{1}{N_-} \langle 0| \prod_{n=1}^N \left( \frac{A_-(\eta/2 - \xi_n)}{A_-(\eta/2 - \xi_n)} \right)^{h_n}, \quad (3.8)$$

where

$$\langle 0| \equiv \otimes_{n=1}^N \langle 1,n|, \quad N_- = \left[ \prod_{1 \leq b < a \leq N}(\eta_{ab}^{(1)} - \eta_{ba}^{(1)}) \right]^{1/2}, \quad (3.9)$$

and

$$\eta_{ab}^{(h_n)} \equiv \cosh 2[\xi_n + (h_n - \frac{1}{2})\eta], \quad (3.10)$$

$h_n \in \{0,1\}, n \in \{1,\ldots,N\}$, define a $B_-(\lambda)$-eigenbasis of $\mathcal{L}_N$,

$$\langle -,h|B_-(\lambda) = B_-(\lambda)\langle -,h|, \quad (3.11)$$

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where \( \langle -, \hbar \rangle \equiv \langle -, h_1, \ldots, h_N \rangle \) for \( \hbar \equiv (h_1, \ldots, h_N) \) and

\[
B_-(\hbar)(\lambda) \equiv \kappa_\hbar^\alpha \frac{\sinh(2\hbar - \eta)}{\sinh \zeta_\hbar} a_h(\lambda) a_h(-\lambda),
\]

with

\[
a_h(\lambda) \equiv \prod_{n=1}^N \sinh(\lambda - \xi_n - (h_n - \frac{1}{2})\eta).
\]

On the generic state \( \langle -, \hbar \rangle \), the action of the remaining reflection algebra generators is defined by

\[
\langle -, \hbar \rangle |A_- (\lambda) = \sum_{a=1}^{2N} \frac{\sinh(2\lambda - \eta) \sinh(\lambda + \xi_a(h_a))}{\sinh(2\xi_a(h_a)) - \eta \sinh 2\xi_a(h_a)} \times \prod_{b=1}^N \frac{\cosh 2\lambda - \cosh 2\xi_b(h_b)}{\cosh 2\xi_b(h_b) - \cosh 2\xi_a(h_a)} A_- (\xi_a(h_a))
\]

\[
\times \langle -, \hbar | T_a^\varphi \rangle + \det M(0) \cosh(\lambda - \eta/2) \prod_{a=1}^N \frac{\cosh 2\lambda - \cosh 2\xi_b(h_b)}{\cosh \eta - \cosh 2\xi_b(h_b)} \langle -, \hbar \rangle
\]

\[
+ (-1)^N \coth \zeta_- \det M(i\pi/2) \sinh(\lambda - \eta/2) \times \prod_{b=1}^N \frac{\cosh 2\lambda - \cosh 2\xi_b(h_b)}{\cosh \eta + \cosh 2\xi_b(h_b)} \langle -, \hbar \rangle,
\]

where

\[
\xi_n^{(h_n)} = \varphi_n \left[ \xi_n + (h_n - \frac{1}{2})\eta \right] \quad \text{for} \quad h_n \in \{0, 1\} \quad \text{and} \quad \forall n \in \{1, \ldots, 2N\},
\]

\[
\varphi_a = 1 - 2\theta(a-N) \quad \text{with} \quad \theta(x) = \{0 \text{ for } x \leq 0, 1 \text{ for } x > 0\},
\]

and

\[
\langle -, h_1, \ldots, h_a, \ldots, h_N | T_a^\pm \rangle = \langle -, h_1, \ldots, h_a \pm 1, \ldots, h_N \rangle.
\]

Indeed, the representation of \( \mathcal{D}_-(\lambda) \) follows from the identity (2.21) while \( \mathcal{C}_-(\lambda) \) is uniquely defined by the quantum determinant relation.

(II) Right \( \mathcal{B}_-(\lambda) \) SOV-representations. If (3.1) is satisfied and \( b_-(\lambda) \neq 0 \), the states

\[
|-, h_1, \ldots, h_N \rangle \equiv \frac{1}{N-} \prod_{n=1}^N \left( \frac{D_-(\xi_n + \eta/2)}{k_n^{-}(\xi_n - \xi_n)} \right)^{(1-h_n)} |0\rangle,
\]

where

\[
|0\rangle \equiv \otimes_{n=1}^N | -1, n \rangle, \quad k_n^{-} = \frac{\sinh(2\xi_n + \eta)}{\sinh(2\xi_n - \eta)}.
\]

\[
h_n \in \{0, 1\}, n \in \{1, \ldots, N\}, \text{ define a } \mathcal{B}_-(\lambda)-\text{eigenbasis of } \mathcal{R}_N,
\]

\[
\mathcal{B}_-(\lambda)|-, \hbar \rangle = |-, \hbar \rangle \mathcal{B}_-(\lambda).
\]
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On the generic state \(|-\hbar\rangle\), the action of the remaining reflection algebra generators is defined by

\[
D_-(\lambda) |-\hbar\rangle = \sum_{a=1}^{2N} T_a^- \varphi_a |-\hbar\rangle \frac{\sinh(2\lambda - \eta) \sinh(\lambda + \zeta^{(h_a)}_a)}{\sinh(2\zeta^{(h_a)}_a) - \eta} \sinh(2\zeta^{(h_a)}_a),
\]

\[
\times \prod_{b \neq a \mod N} \frac{\cosh 2\lambda - \cosh 2\zeta^{(h_b)}_b}{\cosh 2\zeta^{(h_b)}_a - \cosh 2\zeta^{(h_b)}_b} \det q^{M_0(0)} \frac{\cosh(\lambda - \zeta^{(h_b)}_b)}{\cosh \eta - \cosh 2\zeta^{(h_b)}_b} \sinh(2\zeta^{(h_b)}_a - \eta) \sinh 2\zeta^{(h_b)}_a \times N \prod_{b=1}^N \frac{\cosh 2\lambda - \cosh 2\zeta^{(h_b)}_b}{\cosh \eta + \cosh 2\zeta^{(h_b)}_b},
\]

where

\[
D_-(\zeta^{(h_a)}_a) = (k_a^-)^{2\varphi_a} A_-(\zeta^{(h_a)}_a) - 2\varphi_a \xi_a,
\]

\[
T_a^\pm |-, h_1, \ldots, h_a, \ldots, h_N\rangle = |-, h_1, \ldots, h_a \pm 1, \ldots, h_N\rangle.
\]

Indeed, the representation of \(A_-(\lambda)\) follows from the identity (2.21) while \(C_-(\lambda)\) is uniquely defined by the quantum determinant relation.

**Proof of (I).** It is worth writing explicitly the (boundary–bulk) decomposition of the reflection algebra generator,

\[
B_-(\lambda) = -a_-(\lambda)A(\lambda)B(-\lambda) + b_-(\lambda)A(\lambda)A(-\lambda) - c_-(\lambda)B(\lambda)B(-\lambda) + d_-(\lambda)B(\lambda)A(-\lambda),
\]

in terms of the generators of the Yang–Baxter algebra. Then, the following well known properties:

\[
\langle 0 | A(\lambda) = a(\lambda) \langle 0 |, \quad \langle 0 | D(\lambda) = d(\lambda) \langle 0 |, \quad \langle 0 | B(\lambda) = 0, \quad \langle 0 | C(\lambda) \neq 0.
\]

with

\[
a(\lambda) \equiv \prod_{n=1}^N \sinh(\lambda - \xi_n + \eta/2), \quad d(\lambda) \equiv \prod_{n=1}^N \sinh(\lambda - \xi_n - \eta/2),
\]

imply that \(\langle 0 |\) is a \(B_- (\lambda)\)-eigenstate with non-zero eigenvalue,

\[
\langle 0 | B_- (\lambda) \equiv B_{-0}(\lambda) \langle 0 |.
\]

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Now by using the reflection algebra commutation relations
\[
\mathcal{A}_-(\lambda_2)\mathcal{B}_-(\lambda_1) = \frac{\sinh(\lambda_1 - \lambda_2 + \eta)\sinh(\lambda_2 - \lambda_1 - \eta)}{\sinh(\lambda_1 - \lambda_2)\sinh(\lambda_1 + \lambda_2)}\mathcal{B}_-(\lambda_1)\mathcal{A}_-(\lambda_2) + \frac{\sinh(\lambda_1 - \lambda_2 - \eta)\sinh\eta}{\sinh(\lambda_2 - \lambda_1)\sinh 2\lambda_1}\mathcal{B}_-(\lambda_2)\mathcal{A}_-(\lambda_1) - \frac{\sinh\eta}{\sinh(\lambda_1 + \lambda_2)\sinh 2\lambda_1}\mathcal{B}_-(\lambda_2)\mathcal{D}_-(\lambda_1)
\]
(3.27)
we can follow step by step the proof given in [2] to prove the validity of (3.11) and (3.20). Under the condition (3.1), these relations also imply that each set of states \(\langle -, h |\) and \(|-, h\rangle\) forms a set of \(2^N\) independent states, i.e. a \(\mathcal{B}_-(\lambda)\)-eigenbasis of \(\mathcal{L}_N\) and \(\mathcal{R}_N\) respectively.

The action of \(\mathcal{A}_-(\zeta_b^{(h)})\) for \(b \in \{1, \ldots, 2N\}\) follows by the definition of the states \(\langle -, h |\) and \(|-, h\rangle\), the reflection algebra commutation relations (3.27) and the quantum determinant relations. Moreover, by using the identities
\[
\mathcal{U}_-(\eta/2) = \det_q M(0) I_0, \quad \mathcal{U}_-(\eta/2 + i\pi/2) = i \coth\zeta_- \det_q M(i\pi/2)\sigma_0^z,
\]
(3.28)
and remarking that \(\mathcal{A}_-(\lambda)\) has the following functional dependence w.r.t. \(\lambda\):
\[
\mathcal{A}_-(\lambda) = \sum_{\alpha=0}^{2N+1} e^{(2\alpha-2N+1)\lambda}\mathcal{A}_-\alpha,
\]
(3.29)
we get the following interpolation formula for the action on \(\langle -, h |\):
\[
\langle -, h |\mathcal{A}_-(\lambda) = \sum_{\alpha=0}^{2N+1} \prod_{b=1}^{2N+2} \frac{\sinh(\lambda - \zeta_b^{(h)})}{\sinh(\zeta_a^{(h_a)} - \zeta_b^{(h_b)})}\mathcal{A}_-(\zeta_a^{(h_a)})\langle -, h |T_a^{-\varphi_a}
+ \det_q M(0) \prod_{b=1}^{2N+2} \frac{\sinh(\lambda - \zeta_b^{(h)})}{\sinh(\zeta_b^{(1)})}\langle -, h |\n+ i \coth\zeta_- \det_q M(i\pi/2) \prod_{b=1}^{2N+2} \frac{\sinh(\lambda - \zeta_b^{(h)})}{\sinh(\zeta_b^{(1)} - \zeta_b^{(h_b)})}\langle -, h |,
\]
(3.30)
where
\[
\zeta_{2N+1}^{(1)} = \eta/2, \quad \zeta_{2N+2}^{(1)} = \eta/2 + i\pi/2, \quad \det_q M(\lambda) = a(\lambda + \eta/2)d(\lambda - \eta/2),
\]
(3.31)
and we have denoted \(\zeta_{2N+1}^{(1)} = \zeta_{2N+1}^{(h_a)}\) for \(b = 1, 2\) for any \(\mathcal{B}_-\)-eigenstate. Then, it is a simple exercise to rewrite this in the form (3.14).

**Proof of (II).** The proof is given along the same lines as those for point (I) of the theorem; we just need to make the following remarks. First of all, as
\[
A(\lambda)|0\rangle = d(\lambda)|0\rangle, \quad D(\lambda)|0\rangle = a(\lambda)|0\rangle, \quad B(\lambda)|0\rangle = 0, \quad C(\lambda)|0\rangle \neq 0,
\]
(3.32)
then \(|0\rangle\) is a \(\mathcal{B}_-(\lambda)\)-eigenstate with non-zero eigenvalue,
\[
\mathcal{B}_-(\lambda)|0\rangle = B_{-1}(\lambda)|0\rangle.
\]
(3.33)
Now all we need are the following reflection algebra commutation relations:

\[ B_-(\lambda_1) D_-(\lambda_2) = \frac{\sinh(\lambda_1 - \lambda_2 + \eta) \sinh(\lambda_2 + \lambda_1 - \eta)}{\sinh(\lambda_1 - \lambda_2) \sinh(\lambda_1 + \lambda_2)} D_-(\lambda_2) B_-(\lambda_1) \]

\[ - \frac{\sinh \eta \sinh(\lambda_2 + \lambda_1 - \eta)}{\sinh(\lambda_1 - \lambda_2) \sinh(\lambda_2 + \lambda_1)} D_-(\lambda_1) B_-(\lambda_2) \]

\[ - \frac{\sinh \eta}{\sinh(\lambda_1 + \lambda_2)} A_-(\lambda_1) B_-(\lambda_2). \]  

(3.34)

By using them we get the following interpolation formula for the action on \(|-, h\rangle\):

\[ D_-(\lambda)|-, h\rangle = \sum_{a=1}^{2N} T_a^{-\varphi} |-, h\rangle \prod_{b=1}^{2N+2} \frac{\sinh(\lambda - \zeta_b^{(h_a)})}{\sinh(\zeta_a^{(h_a)} - \zeta_b^{(h_a)})} D_-(\zeta_a^{(h_a)}) \]

\[ + |-, h\rangle \det M(0) \prod_{b=1}^{2N+2} \frac{\sinh(\lambda - \zeta_b^{(h_a)})}{\sinh(\zeta_b^{(h_1)} - \zeta_b^{(h_2)})} \]

\[ - i|-, h\rangle \coth \zeta_- \det M(i\pi/2) \prod_{b=1}^{2N+2} \frac{\sinh(\lambda - \zeta_b^{(h_b)})}{\sinh(\zeta_b^{(h_1)} - \zeta_b^{(h_2)})}, \]  

(3.35)

which can be rewritten in the form (3.21).

\[ \square \]

### 3.3. \( B_+ \)-SOV representations of the reflection algebra

In this subsection we construct the left and right SOV-representations of the reflection algebra generated by \( \mathcal{U}_+(\lambda) \) by constructing the left and right \( B_+(\lambda) \)-eigenbases.

**Theorem 3.3.** (1) **Left \( B_+ \)-SOV-representations.** If (3.1) is satisfied and \( b_+(\lambda) \neq 0 \), then the states

\[ \langle +, h_1, \ldots, h_N | \equiv \frac{1}{N_+} \langle 0 | \prod_{n=1}^{N} \left( \frac{D_+(-\zeta_n^{(1)})}{D_+(-\zeta_n^{(1)})} \right)^{(1-h_n)} \left( 1-h_n \right), \]  

(3.36)

where

\[ N_+ = \left[ \langle 0 | \prod_{n=1}^{N} \frac{D_+(-\zeta_n^{(1)})}{D_+(-\zeta_n^{(1)})} \right]^{1/2} \]  

(3.37)

\( h_n \in \{0, 1\}, n \in \{1, \ldots, N\}, \) define a \( B_+ \)-eigenbasis of \( \mathcal{L}_N \),

\[ \langle +, h | \equiv \langle +, h_1, \ldots, h_N | \text{ for } h \equiv (h_1, \ldots, h_N) \]  

where

\[ B_+(\lambda) \equiv \kappa_+ e^{\varphi} \frac{\sinh(2\lambda + \eta)}{\sinh \zeta_+} a_h(\lambda) a_h(-\lambda). \]  

(3.39)
On the generic state \( \langle +, h \rangle \), the action of the remaining reflection algebra generators is defined by

\[
\langle +, h \rangle |D_+(\lambda) = \sum_{a=1}^{2N} \frac{\sinh(2\lambda + \eta) \sinh(\lambda + \zeta_a^{(h_a)})}{\sinh(2\zeta_a^{(h_a)} + \eta) \sinh 2\zeta_a^{(h_a)}} \times \prod_{b=1 \atop b \neq a \mod N}^{N} \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(h_b)}}{\cosh 2\zeta_a^{(h_a)} - \cosh 2\zeta_b^{(h_b)}} D_+(\zeta_a^{(h_a)})
\]

\[
\times \langle +, h | T_a^{\pm} + \det M(0) \cosh(\lambda + \eta/2) \prod_q^N \cosh 2\lambda - \cosh 2\zeta_b^{(h_b)} \rangle q \cosh \eta - \cosh 2\zeta_b^{(h_b)} \langle +, h | + (-1)^{N+1} \coth \zeta_+ \det M(i\pi/2) \sinh(\lambda + \eta/2) \rangle
\]

\[
\times \prod_{b=1}^N \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(h_b)}}{\cosh \eta + \cosh 2\zeta_b^{(h_b)}}, \tag{3.40}
\]

where

\[
\langle +, h_1, \ldots, h_a, \ldots, h_N | T_a^{\pm} = \langle +, h_1, \ldots, h_a + 1, \ldots, h_N |. \tag{3.41}
\]

Indeed, the representation of \( \mathcal{A}_+(\lambda) \) follows from the identity (2.38) while \( \mathcal{C}_+(\lambda) \) is uniquely defined by the quantum determinant relation.

(II) Right \( \mathcal{B}_+(\lambda) \) SOV-representations. If (3.1) is satisfied and \( b_+ (\lambda) \neq 0 \), then the states

\[
| +, h_1, \ldots, h_N \rangle \equiv \frac{1}{N_+} \prod_{n=1}^N \left( \frac{\mathcal{A}_+(\zeta_n^{(0)})}{k_n^{(+)} D_+(-\zeta_n^{(1)})} \right)^{h_n} |0\rangle, \tag{3.42}
\]

where

\[
k_n^{(+)} = \frac{\sinh(2\zeta_n - \eta)}{\sinh(2\zeta_n + \eta)}. \tag{3.43}
\]

\( h_n \in \{0, 1\}, n \in \{1, \ldots, N\} \), define a \( \mathcal{B}_+(\lambda) \)-eigenbasis of \( \mathcal{R}_N \),

\[
\mathcal{B}_+(\lambda)| +, h \rangle = | +, h \rangle B_{+, h} (\lambda), \tag{3.44}
\]

On the generic state \( | +, h \rangle \), the action of the remaining reflection algebra generators is defined by

\[
\mathcal{A}_+(\lambda)| +, h \rangle = \sum_{a=1}^{2N} T_a^{\pm} | +, h \rangle \frac{\sinh(2\lambda + \eta) \sinh(\lambda + \zeta_a^{(h_a)})}{\sinh(2\zeta_a^{(h_a)} + \eta) \sinh 2\zeta_a^{(h_a)}} \times \prod_{b=1 \atop b \neq a \mod N}^N \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(h_b)}}{\cosh 2\zeta_a^{(h_a)} - \cosh 2\zeta_b^{(h_b)}} A_+(\zeta_a^{(h_a)})
\]

\[
+ | +, h \rangle \det M(0) \cosh(\lambda + \eta/2) \prod_q^N \cosh 2\lambda - \cosh 2\zeta_b^{(h_b)} \rangle q \cosh \eta - \cosh 2\zeta_b^{(h_b)} \langle +, h |
\]

\[
+ (-1)^{N+1} | +, h \rangle \coth \zeta_+ \det M(i\pi/2) \sinh(\lambda + \eta/2) \times \prod_{b=1}^N \frac{\cosh 2\lambda - \cosh 2\zeta_b^{(h_b)}}{\cosh \eta + \cosh 2\zeta_b^{(h_b)}}, \tag{3.45}
\]
where
\[
A_+(\zeta^{(h_a)}_a) = \left( k^{(+)\lambda}_a \right)^{\varphi_a} D_+^{(\zeta^{(h_a)}_a - 2\varphi_a\zeta_a)},
\]
\[
T^+_a \left[ +, h_1, \ldots, h_a, \ldots, h_N \right] = \left[ +, h_1, \ldots, h_a \pm 1, \ldots, h_N \right].
\] (3.46)

Indeed, the representation of \( D_+^{+}(\lambda) \) follows from the identity (2.38) while \( C_+^{+}(\lambda) \) is uniquely defined by the quantum determinant relation.

**Proof of (I).** The proof is given along the same lines as the previous theorem; we just need to make the following remarks. First of all let us give the (boundary–bulk) decomposition of the reflection algebra generator,
\[
\mathcal{B}_+^{+}(\lambda) = B^{+}(\lambda) D^{-(\lambda)} d_+^{+}(\lambda) + D^{+}(\lambda) D^{-(\lambda)} b_+^{+}(\lambda)
\] in terms of the generators of the Yang–Baxter algebra. Then, the properties (3.24) imply that \( \langle 0 \rangle \) is a \( \mathcal{B}_+^{+}(\lambda) \)-eigenstate with non-zero eigenvalue \( B_+^{+}(\lambda) \). The proof of point (I) is based on the following reflection algebra commutation relations:
\[
D_+^{+}(\lambda_1) \mathcal{B}_+^{+}(\lambda_2) = \frac{\sinh(\lambda_1 - \lambda_2 + \eta) \sinh(\lambda_2 + \lambda_1 + \eta)}{\sinh(\lambda_1 - \lambda_2) \sinh(\lambda_1 + \lambda_2)} \mathcal{B}_+^{+}(\lambda_2) D_+^{+}(\lambda_1)
\]
\[
- \frac{\sinh(2\lambda_2 + \eta) \sinh \eta}{\sinh(\lambda_1 - \lambda_2) \sinh 2\lambda_2} \mathcal{B}_+^{+}(\lambda_1) D_+^{+}(\lambda_2)
\]
\[
- \frac{\sinh \eta \sinh(2\lambda_2 + \eta)}{\sinh(\lambda_1 + \lambda_2) \sinh 2\lambda_2} \mathcal{B}_+^{+}(\lambda_1) D_+^{+}(-\lambda_2),
\] (3.48)
and on the identities
\[
\mathcal{U}_+(-\eta/2) = \det_q M(0) I_0, \quad \mathcal{U}_+(-\eta/2 + i\pi/2) = i \coth_{+} \det_q M(i\pi/2) \sigma_0.
\] (3.49)
By using them and the fact that \( D_+^{+}(\lambda) \) has the following functional dependence w.r.t. \( \lambda \):
\[
D_+^{+}(\lambda) = \sum_{a=0}^{2N+1} q^{(2a-2N+1)\lambda} D_+, \quad \] (3.50)
we get the following interpolation formula for the action on \( \langle +, h \rangle \):
\[
\langle +, h | D_+^{+}(\lambda) \rangle = \sum_{a=1}^{2N} \prod_{b=1}^{2N+2} \frac{\sinh(\lambda - \zeta^{(h_a)}_b)}{\sinh(\zeta^{(h_a)}_a - \zeta^{(h_b)}_b)} D_+^{+}(\zeta^{(h_a)}_a) \langle +, h | T^{\varphi_a}_a \rangle
\]
\[
+ \det_q M(0) \prod_{b=1}^{2N+2} \frac{\sinh(\lambda - \zeta^{(h_b)}_a)}{\sinh(\zeta^{(1)}_{2N+1} - \zeta^{(h_a)}_b)} \langle +, h | \rangle
\]
\[
- i \coth_{+} \det_q M(i\pi/2) \prod_{b=1}^{2N+1} \frac{\sinh(\lambda - \zeta^{(1)}_{2N+1})}{\sinh(\zeta^{(1)}_{2N+2} - \zeta^{(h_a)}_b)} \langle +, h | ,
\] (3.51)
where we have denoted \( \zeta^{1}_{2N+1}(h_{2N+1}) = \zeta^{1}_{2N+1}(h_{2N+1}) \) for \( b = 1, 2 \) and
\[
\zeta^{1}_{2N+1} = -\eta/2, \quad \zeta^{1}_{2N+2} = -\eta/2 + i\pi/2.
\] (3.52)
Then, it is a simple exercise to rewrite this in the form (3.40). \qed
Let us present the main properties of the $B_+$.  

4.1. Change of basis properties when the gauge in the SOV-representations is chosen. 

completely fixed by the left and right SOV-representations of the Yang–Baxter algebras $B_+$ of proposition 2.3 these results correspond to the computations of scalar products between corresponding basis. It is worth remarking that for the Hermitian conjugation properties computed in this way allowing us to write the decomposition of the identity in the

The action of a generic left $B_+$-eigenstate starting from the original spin basis, which can be rewritten in the form (3.45).

$$B_+(\lambda_2)A_+(\lambda_1) = \frac{\sinh(\lambda_1 - \lambda_2 + \eta) \sinh(\lambda_2 + \lambda_1 + \eta)}{\sinh(\lambda_1 - \lambda_2) \sinh(\lambda_1 + \lambda_2)} A_+(\lambda_1)B_+(\lambda_2)$$

$$+ \frac{\sinh \eta \sinh(2\lambda_2 + \eta)}{\sinh(\lambda_2 - \lambda_1) \sinh 2\lambda_2} A_+(\lambda_2)B_+(\lambda_1)$$

$$+ \frac{\sinh \eta \sinh(2\lambda_2 + \eta)}{\sinh(\lambda_1 + \lambda_2) \sinh 2\lambda_2} A_+(-\lambda_2)B_+(\lambda_1).$$

(3.53)

By using them we get the following interpolation formula for the action on $|+, h\rangle$:

$$A_+(\lambda)|+, h\rangle = \sum_{a=1}^{2N} T^{(\lambda)}_a |+, h\rangle \prod_{b=1}^{2N+2} \frac{\sinh(\lambda - \zeta^{(b)}_a)}{\sinh(\zeta^{(b)}_a - \zeta^{(b)}_b)} A_+^{(b)}$$

$$+ |+, h\rangle \det M(0) \prod_{b=1}^{2N+2} \frac{\sinh(\lambda - \zeta^{(b)}_a)}{\sinh(\zeta^{(b)}_a - \zeta^{(b)}_b)}$$

$$+ i|+, h\rangle \coth \zeta_+ \det M(i\pi/2) \prod_{b=1}^{2N+1} \frac{\sinh(\lambda - \zeta^{(b)}_b)}{\sinh(\zeta^{(b)}_a - \zeta^{(b)}_b)},$$

(3.54)

which can be rewritten in the form (3.45).

4. SOV-decomposition of the identity

The action of a generic left $B_+$-eigenstate on a generic right $B_+$-eigenstate is here computed in this way allowing us to write the decomposition of the identity in the corresponding basis. It is worth remarking that for the Hermitian conjugation properties of proposition 2.3 these results correspond to the computations of scalar products between $B_+$-eigenvectors and $C_-$-eigenvectors. We show that up to an overall constant these are completely fixed by the left and right SOV-representations of the Yang–Baxter algebras when the gauge in the SOV-representations is chosen.

4.1. Change of basis properties

Let us present the main properties of the $2^N \times 2^N$ matrices $U^{(L,e)}$ and $U^{(R,e)}$,

$$\langle \epsilon, h \rangle = \langle h | U^{(L,e)} = \sum_{i=1}^{2^N} U^{(L,e)}_{x(h),i} | x^{-1}(i) \rangle$$

$$| \epsilon, h \rangle = U^{(R,e)} | h \rangle = \sum_{i=1}^{2^N} U^{(R,e)}_{x(h)} | x^{-1}(i) \rangle,$$

which define the change of basis to the SOV-basis starting from the original spin basis,

$$\langle h \rangle \equiv \bigotimes_{n=1}^N (2h_n - 1, n) \quad \text{and} \quad | h \rangle \equiv \bigotimes_{n=1}^N | 2h_n - 1, n \rangle,$$

(4.1)
where \( \varpi \) is the following natural isomorphism between the sets \( \{0, 1\}^N \) and \( \{1, \ldots, 2^N\} \):

\[
\varpi : \mathbf{h} \in \{0, 1\}^N \to \varpi(\mathbf{h}) \equiv 1 + \sum_{a=1}^{N} 2^{(a-1)}h_a \in \{1, \ldots, 2^N\}. \tag{4.3}
\]

Note that the matrices \( U^{(L,e)} \) and \( U^{(R,e)} \) are invertible matrices for the diagonalizability of \( \mathcal{B}_e(\lambda) \),

\[
U^{(L,e)} \mathcal{B}_e(\lambda) U^{(R,e)} = \Delta_{\mathcal{B}_e}(\lambda) U^{(R,e)} \mathcal{B}_e(\lambda) U^{(L,e)}, \quad \mathcal{B}_e(\lambda) U^{(R,e)} = U^{(R,e)} \Delta_{\mathcal{B}_e}(\lambda). \tag{4.4}
\]

Here \( \Delta_{\mathcal{B}_e}(\lambda) \) is the \( 2^N \times 2^N \) diagonal matrix whose elements, for the simplicity of the \( \mathcal{B}_e \)-spectrum, read

\[
(\Delta_{\mathcal{B}_e}(\lambda))_{i,j} \equiv \delta_{i,j} B_{e,\varpi^{-1}(i)}(\lambda) \quad \forall i, j \in \{1, \ldots, 2^N\}. \tag{4.5}
\]

Moreover, the following proposition holds.

**Proposition 4.1.** The \( 2^N \times 2^N \) matrix

\[
M^{(e)} \equiv U^{(L,e)} U^{(R,e)} \tag{4.6}
\]

is diagonal and it is characterized by

\[
M^{(e)}_{\varpi(h),\varpi(h')} = \langle \epsilon, \mathbf{h}| \epsilon, \mathbf{h}' \rangle = \prod_{1 \leq b < a \leq N} \frac{1}{\eta_{(h_a)} - \eta_{(h_b)}}, \tag{4.7}
\]

**Proof.** Note that as this is the action of a left \( \mathcal{B}_e \)-eigenstate on a right \( \mathcal{B}_e \)-eigenstate, corresponding to different \( \mathcal{B}_e \)-eigenvalues, this implies that the matrix \( M^{(e)} \) is diagonal; then to compute its diagonal elements we compute the matrix elements \( \theta^{(-)}_a \equiv \langle -, h_1, \ldots, h_a = 0, \ldots, h_N| A_- (\xi_a + \eta/2)|-, h_1, \ldots, h_a = 1, \ldots, h_N \rangle \), where \( a \in \{1, \ldots, N\} \). Using the left action of the operator \( A_- (\xi_a + \eta/2) \) we get

\[
\theta^{(-)}_a = A_- (\eta/2 - \xi_a) \frac{\sinh \eta}{\sinh(2\xi_a - \eta)} \prod_{b=1}^{N} \frac{\cosh 2\zeta_a^{(1)} - \cosh 2\zeta_b^{(0)}}{\cosh 2\zeta_a^{(0)} - \cosh 2\zeta_b^{(1)}} \times \langle -, h_1, \ldots, h_a = 1, \ldots, h_N|-, h_1, \ldots, h_a = 1, \ldots, h_N \rangle, \tag{4.8}
\]

while using the decomposition (2.21) and the fact that

\[
\mathcal{D}_- (-\xi_a - \eta/2)|-, h_1, \ldots, h_a = 1, \ldots, h_N \rangle = 0 \tag{4.9}
\]

it holds that

\[
A_- (\xi_a + \eta/2)|-, h_1, \ldots, h_a = 1, \ldots, h_N \rangle = k_a \frac{\sinh \eta}{\sinh(2\xi_a + \eta)} A_- (\eta/2 - \xi_a)|-, h_1, \ldots, h_a = 0, \ldots, h_N \rangle, \tag{4.10}
\]

and then we get

\[
\theta^{(-)}_a = \frac{\sinh \eta}{\sinh(2\xi_a - \eta)} A_- (\eta/2 - \xi_a) (-, h_1, \ldots, h_a = 0, \ldots, h_N|-, h_1, \ldots, h_a = 0, \ldots, h_N \rangle, \tag{4.11}
\]

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so that it holds that
\[
\langle - , h_1 , \ldots , h_a = 1 , \ldots , h_N | - , h_1 , \ldots , h_a = 1 , \ldots , h_N \rangle
\frac{\langle - , h_1 , \ldots , h_a = 0 , \ldots , h_N | - , h_1 , \ldots , h_a = 0 , \ldots , h_N \rangle}{\prod_{b=1 \atop b \neq a}^{N}} \cosh \frac{2 \zeta_a^{(0)}}{\cosh \frac{2 \zeta_b^{(h_b)}}{\cosh \frac{2 \zeta_a^{(1)}}{\cosh \frac{2 \zeta_b^{(h_b)}}}}},
\]
from which one can prove
\[
\langle - , h_1 , \ldots , h_a = 0 , \ldots , h_N | - , h_1 , \ldots , h_a = 0 , \ldots , h_N \rangle = \prod_{1 \leq b < a \leq N} \frac{\eta^{(1)}_a - \eta^{(1)}_b}{\eta^{(h_b)}_a - \eta^{(h_a)}_b}.
\]
This proves the proposition for \( \epsilon = - \), as
\[
\langle - , 1 , \ldots , 1 | - , 1 , \ldots , 1 \rangle = \prod_{1 \leq b < a \leq N} \frac{1}{\eta^{(1)}_a - \eta^{(1)}_b},
\]
by our definition of the normalization \( N_- \). Similarly for \( \epsilon = + \), we compute the matrix elements \( \theta^{(+)}_a \equiv \langle + , h_1 , \ldots , h_a = 1 , \ldots , h_N | D_+ (\xi_a - \eta/2) | + , h_1 , \ldots , h_a = 0 , \ldots , h_N \rangle \), where \( a \in \{ 1 , \ldots , N \} \). Then using the left action of the operator \( D_+ (\xi_a - \eta/2) \) we get
\[
\theta^{(+)}_a = - D_+ (\xi_a - \eta/2) \frac{\sinh \eta}{\sinh(2 \xi_a + \eta)} \prod_{b=1 \atop b \neq a}^{N} \cosh \frac{2 \zeta_a^{(0)}}{\cosh \frac{2 \zeta_b^{(h_b)}}{\cosh \frac{2 \zeta_a^{(1)}}{\cosh \frac{2 \zeta_b^{(h_b)}}}}}
\times \langle + , h_1 , \ldots , h_a = 0 , \ldots , h_N | + , h_1 , \ldots , h_a = 0 , \ldots , h_N \rangle,
\]
while using the decomposition (2.38) and the fact that
\[
A_+ (\xi_a + \eta/2) | + , h_1 , \ldots , h_a = 0 , \ldots , h_N \rangle = 0
\]
it holds that
\[
D_+ (\xi_a - \eta/2) | + , h_1 , \ldots , h_a = 0 , \ldots , h_N \rangle = - \frac{\theta_a^{(+)} \sinh \eta}{\sinh(2 \xi_a - \eta)} D_+ (\xi_a - \eta/2)
\times | + , h_1 , \ldots , h_a = 1 , \ldots , h_N \rangle,
\]
and then we get
\[
\theta_a = - \frac{\sinh \eta}{\sinh(2 \xi_a + \eta)} D_+ (\xi_a - \eta/2) \langle + , h_1 , \ldots , h_a = 1 , \ldots , h_N | + , h_1 , \ldots , h_a = 1 , \ldots , h_N \rangle,
\]
and so
\[
\langle + , h_1 , \ldots , h_a = 1 , \ldots , h_N | + , h_1 , \ldots , h_a = 1 , \ldots , h_N \rangle \frac{\langle + , h_1 , \ldots , h_a = 0 , \ldots , h_N | + , h_1 , \ldots , h_a = 0 , \ldots , h_N \rangle}{\prod_{b=1 \atop b \neq a}^{N}} \cosh \frac{2 \zeta_a^{(0)}}{\cosh \frac{2 \zeta_b^{(h_b)}}{\cosh \frac{2 \zeta_a^{(1)}}{\cosh \frac{2 \zeta_b^{(h_b)}}}}},
\]
from which we have
\[
\frac{\langle + , h_1 , \ldots , h_a = 1 , \ldots , h_N | + , h_1 , \ldots , h_N \rangle}{\langle + , 0 , \ldots , 0 | + , 0 , \ldots , 0 \rangle} = \prod_{1 \leq b < a \leq N} \frac{\eta^{(0)}_a - \eta^{(0)}_b}{\eta^{(h_b)}_a - \eta^{(h_a)}_b},
\]
where \( \eta^{(h)}_a \equiv \sinh(2 \eta_a \xi_a + \eta) \) for \( h = 0 , 1 \).
which proves the proposition, as
\[
\langle +, 0, \ldots, 0 | +, 0, \ldots, 0 \rangle = \prod_{1 \leq b < a \leq N} \frac{1}{\eta_a^{(0)} - \eta_b^{(0)}},
\]
by our definition of the normalization \( N_+ \).

\[ \square \]

4.2. SOV-decomposition of the identity

The following spectral decomposition of the identity \( I \):
\[
I \equiv \sum_{i=1}^{2^N} \mu_i | \epsilon, \varepsilon^{-1}(i) \rangle \langle \epsilon, \varepsilon^{-1}(i) |,
\]
can be given in terms of the left and right SOV-bases, where \( \mu_i \equiv \langle \epsilon, \varepsilon^{-1}(i) | \epsilon, \varepsilon^{-1}(i) \rangle^{-1} \) is the analogue of the so-called Sklyanin’s measure\(^{15} \) in our 6-vertex reflection algebra representations. Now using the result of the previous section we can explicitly write
\[
I \equiv \sum_{h_1, \ldots, h_N=0}^{1} \prod_{1 \leq b < a \leq N} (\eta_{a}^{(h_a)} - \eta_{b}^{(h_a)}) | \epsilon, h_1, \ldots, h_N \rangle \langle \epsilon, h_1, \ldots, h_N |.
\]

5. SOV-characterization of the \( T(\lambda) \)-spectrum

It is worth mentioning that for the class of boundary conditions here analysed previous results in the literature can be extracted particularizing the analysis implemented in [36]–[38]. There the eigenvalue analysis is presented while the analysis of eigenstates and in particular of the matrix elements of local operators is beyond the means of the methods used. Here, we characterize in the \( B_{\pm} \)-SOV-representations the spectrum (eigenvalues and eigenstates) of the transfer matrix \( T(\lambda) \) for the class of boundary conditions listed in theorem 3.1. The results of this section and those of the following ones will show that the SOV approach here introduced has the advantage of providing matrix elements of local operators as soon as the eigenvalues of the transfer matrix are known.

Let us start by giving the following characterization.

**Lemma 5.1.** Let us denote by \( \Sigma_T \) the set of eigenvalue functions of the transfer matrix \( T(\lambda) \); then any \( \tau(\lambda) \in \Sigma_T \) is an even function of \( \lambda \) of the form
\[
\tau(\lambda) = 2 \sinh(\lambda - \eta/2) \sinh(\lambda + \eta/2) \coth \zeta_\lambda \coth \zeta_{\pm} \det M(i \pi/2)
+ 2 \cosh(\lambda - \eta/2) \cosh(\lambda + \eta/2) \det M(0)
+ \sinh(2\lambda - \eta) \sinh(2\lambda + \eta) \sum_{b=1}^{N} c_b^2 \cosh 2\lambda)^{b-1}.
\]

**Proof.** The transfer matrix \( T(\lambda) \) is an even function of \( \lambda \) so the same is true for the \( \tau(\lambda) \in \Sigma_T \). Moreover, from the identities (3.28) and (3.49), after some simple computation

\[ ^{15} \text{Sklyanin’s measure was first introduced by Sklyanin in the quantum Toda chain [39]; see also [56, 57] for further discussions.} \]
the following identities are derived:
\[ T(\pm \eta/2) = a_x(\epsilon \eta/2) \det_q M(0) = 2 \cosh \eta \det_q M(0), \]
\[ T(\pm (\eta/2 - i\pi/2)) = i a_x(\epsilon \eta/2 - \epsilon i \pi/2) \coth \zeta_+ \det_q M(i\pi/2) \]
\[ = -2 \cosh \eta \coth \zeta_- \coth \zeta_+ \det_q M(i\pi/2), \]
both for \( \epsilon = +, - \). These identities together with the known functional form of \( T(\lambda) \) w.r.t. \( \lambda \) imply the statement in the lemma.

5.1. The transfer matrix spectrum in the \( B_-\)-SOV-representations

In this section we characterize the spectrum of the transfer matrix
\[ T_-(\lambda) \equiv \hat{T}_-(\lambda) + c_+ \hat{\lambda} \mathcal{B}_-(\lambda), \]
associated with the representations of the reflection algebra under the following class of boundary parameters:
\[ b_+(\lambda) = 0 \quad \text{and} \quad b_-(\lambda) \neq 0. \]

**Theorem 5.1.** If the condition (3.1) is satisfied, then \( T_- (\lambda) \) has a simple spectrum and \( \Sigma_{T_-} \) coincides with the solutions of the discrete system of equations,
\[ \tau_- (\pm a^{(0)}_a) \tau_- (\pm \zeta^{(1)}_a) = A_- (\zeta^{(1)}_a) A_- (-\zeta^{(0)}_a), \quad \forall a \in \{1, \ldots, N\}, \]
in the class of functions of the form (5.1), where the coefficient \( A_- (\lambda) \) is defined by
\[ A_- (\lambda) \equiv a_+ (\lambda) A_- (\lambda), \]
and satisfies the quantum determinant condition
\[ a_+ (\lambda) a_+ (-\lambda + \eta) \det_q U_- (\lambda - \eta/2) = \sinh (2\lambda - \eta) A_- (\lambda) A_- (-\lambda + \eta). \]

(I) The vector
\[ |\tau_- \rangle = \sum_{h_1, \ldots, h_N = 0}^1 \prod_{a=1}^N \hat{Q}_{\tau_-}(\zeta^{(1)}_a) \prod_{1 \leq b < a \leq N} (\hat{a}_b^{(h_a)} - \hat{\eta}_b^{(h_b)}) |-, h_1, \ldots, h_N \rangle \]
defines, uniquely up to an overall normalization, the right \( T_- \)-eigenstate corresponding to \( \tau_- (\lambda) \in \Sigma_{T_-} \). The coefficients in (5.9) are characterized by
\[ \hat{Q}_{\tau_-}(\zeta^{(1)}_a)/\hat{Q}_{\tau_-}(\zeta^{(0)}_a) = \tau_- (\zeta^{(1)}_a)/A_- (-\zeta^{(0)}_a). \]

(II) The covector
\[ \langle \tau_- | = \sum_{h_1, \ldots, h_N = 0}^1 \prod_{a=1}^N \hat{Q}_{\tau_-}(\zeta^{(1)}_a) \prod_{1 \leq b < a \leq N} (\hat{a}_b^{(h_a)} - \hat{\eta}_b^{(h_b)}) \langle -, h_1, \ldots, h_N | \]
defines, uniquely up to an overall normalization, the left \( T_- \)-eigenstate corresponding to \( \tau_- (\lambda) \in \Sigma_{T_-} \). The coefficients in (5.11) are characterized by
\[ \hat{Q}_{\tau_-}(\zeta^{(1)}_a)/\hat{Q}_{\tau_-}(\zeta^{(0)}_a) = a_1^{(-1)} k_1^{(-1)} \tau_- (\zeta^{(0)}_a)/A_- (\zeta^{(1)}_a), \]
where
\[ \alpha_n^{-} = \frac{a_+(\zeta_n^{(1)})}{d_+(-\zeta_n^{(0)})} = \frac{d_+(\zeta_n^{(1)})}{a_+(-\zeta_n^{(0)})} = \frac{\sinh(2\zeta_n + 2\eta)}{k_n^- \sinh(2\zeta_n - 2\eta)}. \] (5.13)

**Proof.** In the \( \mathcal{B}_- \)-SOV representations the spectral problem for \( \mathcal{T}_-(\lambda) \) is reduced to a discrete system of \( 2^N \) Baxter-like equations
\[ \tau_-(\zeta_n^{(h_n)}))\Psi_{\tau_-}(h) = A_-(-\zeta_n^{(h_n)}))\Psi_{\tau_-}(T_n^-(h)) + A_-(-\zeta_n^{(h_n)})\Psi_{\tau_-}(T_n^+(h)) \] (5.14)
for any \( n \in \{1, \ldots, N\} \) and \( h \in \{0, 1\}^N \) in the coefficients (wavefunctions)
\[ \Psi_{\tau_-}(h) \equiv \langle -, h_1, \ldots, h_N | \tau_- \rangle \] (5.15)
of the \( \mathcal{T}_- \)-eigenstate \( |\tau_- \rangle \) associated with \( \tau_- (\lambda) \in \Sigma_{\mathcal{T}_-} \); here, we have used the notations
\[ T_n^\pm(h) \equiv (h_1, \ldots, h_n \pm 1, \ldots, h_N). \] (5.16)

As
\[ A_-(-\zeta_n^{(0)}) = A_-(-\zeta_n^{(1)}) = 0, \] (5.17)
the previous system of equations (5.14) is equivalent to the following system of homogeneous equations:
\[ \left( \begin{array}{cc} \tau_-(\zeta_n^{(0)}) & -A_-(-\zeta_n^{(0)}) \\ -A_-(-\zeta_n^{(1)}) & \tau_-(\zeta_n^{(1)}) \end{array} \right) \left( \begin{array}{c} \Psi_{\tau_-}(h_1, \ldots, h_n = 0, \ldots, h_1) \\ \Psi_{\tau_-}(h_1, \ldots, h_n = 1, \ldots, h_1) \end{array} \right) = \left( \begin{array}{c} 0 \\ 0 \end{array} \right), \] (5.18)
for any \( n \in \{1, \ldots, N\} \) with \( h_{m \neq n} \in \{0, 1\} \). The condition \( \tau_- (\lambda) \in \Sigma_{\mathcal{T}_-} \) implies that the determinants of the \( 2 \times 2 \) matrices in (5.18) must be zero for any \( n \in \{1, \ldots, N\} \), which is equivalent to (5.6). Moreover, the rank of the matrices in (5.18) is 1 as
\[ A_-(-\zeta_n^{(0)}) \neq 0 \quad \text{and} \quad A_-(-\zeta_n^{(1)}) \neq 0, \] (5.19)
and then (up to an overall normalization) the solution is unique,
\[ \frac{\Psi_{\tau_-}(h_1, \ldots, h_n = 1, \ldots, h_1)}{\Psi_{\tau_-}(h_1, \ldots, h_n = 0, \ldots, h_1)} = \frac{\tau_-(\zeta_n^{(0)})}{A_-(-\zeta_n^{(0)})}, \] (5.20)
for any \( n \in \{1, \ldots, N\} \) with \( h_{m \neq n} \in \{0, 1\} \). So for fixed \( \tau_- (\lambda) \in \Sigma_{\mathcal{T}_-} \) there exists (up to normalization) one and only one corresponding \( \mathcal{T}_- \)-eigenstate \( |\tau_- \rangle \) with coefficients of the factorized form given in (5.9)–(5.10); i.e. the \( \mathcal{T}_- \)-spectrum is simple.

Vice versa, if \( \tau_- (\lambda) \) is in the set of functions (5.1) and satisfies (5.6), then the state \( |\tau_- \rangle \) defined by (5.9)–(5.10) satisfies
\[ \langle -, h_1, \ldots, h_N | \mathcal{T}_-(\zeta_n^{(h_n)})|\tau_- \rangle = \tau_-(\zeta_n^{(h_n)})\langle -, h_1, \ldots, h_N | \tau_- \rangle \quad \forall \ n \in \{1, \ldots, N\} \] (5.21)
for any \( \mathcal{B}_- \)-eigenstate \( |-, h_1, \ldots, h_N \rangle \) and this implies
\[ \langle -, h_1, \ldots, h_N | \mathcal{T}_-(\lambda)|\tau_- \rangle = \tau_-(\lambda)\langle -, h_1, \ldots, h_N | \tau_- \rangle \quad \forall \ \lambda \in \mathbb{C}, \] (5.22)
i.e. \( \tau_- (\lambda) \in \Sigma_{\mathcal{T}_-} \) and \( |\tau_- \rangle \) is the corresponding \( \mathcal{T}_- \)-eigenstate.

Concerning the left \( \mathcal{T}_- \)-eigenstates the proof is complete; as above we have just to remark that in this case the matrix elements
\[ \langle \tau_- | \mathcal{T}_-(\zeta_n^{(h_n)})|-, h_1, \ldots, h_N \rangle \] (5.23)
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can be computed by using the right $\mathcal{B}_-$-representation
\[
\tau_- (\zeta_n^{(h_n)}) \bar{\Psi}_{\tau_-} (\mathbf{h}) = \mathcal{D}_- (\zeta_n^{(h_n)}) \bar{\Psi}_{\tau_-} (\mathbf{T}_n^- (\mathbf{h})) + \mathcal{D}_- (-\zeta_n^{(h_n)}) \bar{\Psi}_{\tau_-} (\mathbf{T}_n^+ (\mathbf{h}))
\]
for all $n \in \{1, \ldots, N\}$,
(5.24)
where
\[
\bar{\Psi}_{\tau_-} (\mathbf{h}) \equiv \langle \tau_- | -, h_1, \ldots, h_N \rangle,
\]
and the coefficient $\mathcal{D}_-$ reads
\[
\mathcal{D}_- (\lambda) \equiv d_+ (\lambda) \mathcal{D}_- (\lambda).
\]
(5.26)

Some explanations and comments on the statements and results of the previous theorem are probably in order. First of all, by our statement ‘complete characterization of the spectrum’ we intend to say that we have proven the complete equivalence of the transfer matrix spectral problem (eigenvalues and eigenstates) with the well defined mathematical problem of classifying all the solutions to the discrete system of equations (5.6) in the class of functions (5.1). Let us mention that this is not the case for the other methods that can be applied for these boundary conditions for which the proof that they indeed describe the entire spectrum of the transfer matrix has to be separately derived. Moreover, it is worth saying that a direct consequence of the previous theorem is that whenever we can prove independently the diagonalizability of the transfer matrix, as for example happens for self-adjoint boundary conditions (2.52)–(2.54), then the system (5.6) is forced to have $2^N$ distinct solutions in the class of functions (5.1) and the sets of eigenstates (5.9) and (5.11) define respectively right and left bases of the space of the representation. However, it must be remarked that it becomes a more complicated task to find all solutions of this system of equations in that given class of functions as the system size increases and this system of equations is not directly equivalent to a system of Bethe equations. Then, in particular, in view of the analysis of the continuum and/or infinite volume limit it is central to get a reformulation of this SOV spectrum characterization by functional equations. The construction of a Baxter $Q$-operator can play an important role in achieving this aim. Let us recall that a $Q$-operator is a one-parameter operator family which satisfies properties of the type
\[
[\mathcal{T}_-(\lambda), Q(\lambda)] = [Q(\lambda), Q(\mu)] = 0,
\]
\[
\mathcal{T}_-(\lambda) Q(\lambda) = \alpha(\lambda) Q(\lambda + \eta) + \beta(\lambda) Q(\lambda - \eta),
\]
where $\alpha(\lambda)$ and $\beta(\lambda)$ are some characteristic functions of the constructed $Q$-operator. Indeed, if this functional equation coincides with the discrete system (5.14) in the spectrum of the $\mathcal{B}_-$-zeros, we can use the $Q$-operator to reformulate by functional equations the SOV spectrum characterization; this approach was for example used in [46]. A Baxter $Q$-operator can be a priori introduced by adapting to the boundary conditions considered in this paper the construction presented in [38] for general values of the anisotropy $\eta$ and for general boundary conditions; however, it is worth recalling that the existence of such a $Q$-operator is mainly at a conjecture level.

Let us mention that it can indeed be interesting to compute the eigenvalues numerically for a small system size; in particular, to explicitly verify the diagonalizability of these transfer matrices under the most general boundary conditions analysed in this paper.

\[\text{doi:10.1088/1742-5468/2012/10/P10025} \]
In our next paper, we will follow a different approach to introduce such a functional reformulation of the SOV spectrum. In particular, in the roots of the unit case for the anisotropy parameter \( \eta = 2i\pi p'/p \), with \( p \) and \( p' \) coprime integers, we will adapt the approach first introduced in [47]. The functional equation there derived for the transfer matrix eigenvalues will be shown to have the same form as that deduced in [37] for the most general boundary conditions. Moreover, we will show the polynomiality, in the exponential of the spectral parameter \( \lambda \), of the constructed Baxter \( Q \)-operator, which will also allow us to prove the completeness of the spectrum description by the solutions to the associated system of Bethe ansatz type equations.

5.2. The transfer matrix spectrum in the \( B_+ \)-SOV-representations

In this section we characterize the spectrum of the transfer matrix

\[
T_+ (\lambda) \equiv T_\downarrow^{(\lambda)} (\lambda) + c_- (\lambda) B_+ (\lambda)
\]

associated with the representations of the reflection algebra under the following class of boundary parameters:

\[
b_- (\lambda) = 0 \quad \text{and} \quad b_+ (\lambda) \neq 0.
\]

Let us now write the left and right eigenstates of the transfer matrices (5.28) in the \( B_+ \)-SOV-representation.

**Theorem 5.2.** If (3.1) is satisfied, then \( T_+ (\lambda) \) has a simple spectrum and \( \Sigma_{T_+} \) coincides with the solutions of the discrete system of equations

\[
\tau_+ (\pm \zeta_a^{(0)}) \tau_+ (\pm \zeta_a^{(1)}) = D_+ (\zeta_a^{(0)}) d_+(\zeta_a^{(1)}), \quad \forall a \in \{1, \ldots, N\},
\]

in the class of functions of the form (5.1), where the coefficient \( D_+ (\lambda) \) is defined by

\[
D_+ (\lambda) \equiv d_- (\lambda) D_+ (\lambda),
\]

and satisfies the quantum determinant condition

\[
d_- (\lambda - \eta/2) d_- (-\lambda - \eta/2) \det \mathcal{U}_+ (\lambda) = \sinh(2\lambda + 2\eta) D_+ (\lambda - \eta/2) D_+ (-\lambda - \eta/2).
\]

(I) The vector

\[
| \tau_+ \rangle = \sum_{h_1, \ldots, h_N = 0}^N \prod_{a=1}^N Q_{\tau_+} (\zeta_a^{(h_a)}) \prod_{1 \leq b < a \leq N} (\zeta_a^{(h_a)} - \zeta_b^{(h_b)}) |+, h_1, \ldots, h_N \rangle
\]

defines, uniquely up to an overall normalization, the right \( T_+ \)-eigenstate corresponding to \( \tau_+ (\lambda) \in \Sigma_{T_+} \). The coefficients in (5.33) are characterized by

\[
Q_{\tau_+} (\zeta_a^{(1)}) / Q_{\tau_+} (\zeta_a^{(0)}) = \tau_+ (\zeta_a^{(0)}) / D_+ (\zeta_a^{(0)}).
\]

(II) The covector

\[
\langle \tau_+ | = \sum_{h_1, \ldots, h_N = 0}^N \prod_{a=1}^N \bar{Q}_{\tau_+} (\zeta_a^{(h_a)}) \prod_{1 \leq b < a \leq N} (\zeta_a^{(h_a)} - \zeta_b^{(h_b)}) \langle +, h_1, \ldots, h_N |\]

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defines, uniquely up to an overall normalization, the left \( T_+ \)-eigenstate corresponding to \( \tau_+(\lambda) \in \Sigma_{T_+} \). The coefficients in (5.35) are characterized by

\[
\bar{Q}_{T_+}(\zeta_a^{(1)})/\bar{Q}_{T_+}(\zeta_a^{(0)}) = \tau_+(\zeta_a^{(0)})/\left((\alpha^{(+)}_a)k^{(+)}_a\right)\bar{D}_+(-\zeta_a^{(1)}),
\]

where

\[
\alpha^{(+)}_a = \frac{a_-(-\zeta_a^{(1)})}{d_-(\zeta_a^{(1)})} = \frac{d_-(\zeta_a^{(0)})}{a_-(\zeta_a^{(1)})} = \frac{\sinh(2\xi_n - 2\eta)}{k^{(+)}_n \sinh(2\xi_n + 2\eta)}.
\]

**Proof.** If we take a \( T_+ \)-eigenstate \(|\tau_+\rangle\) corresponding to the eigenvalue \( \tau_+(\lambda) \in \Sigma_{T_+} \), it has the coefficients

\[
\Psi_{\tau_+}(h) \equiv \langle +, h_1, \ldots, h_N | \tau_+ \rangle
\]

in the SOV-basis, which satisfies the following discrete system of Baxter-like equations:

\[
\tau_+(\zeta_n^{(0)})\Psi_{\tau_+}(h) = D_+(\zeta_n^{(0)})\Psi_{\tau_+}(T_n^+(h)) + D_+(\zeta_n^{(1)})\Psi_{\tau_+}(T_n^-(h)),
\]

for any \( n \in \{1, \ldots, N\} \) and \( h \in \{0, 1\}^N \). We can rewrite this as the following system of homogeneous equations:

\[
\begin{pmatrix}
\tau_+(\zeta_n^{(0)}) & -D_+(\zeta_n^{(0)}) \\
-D_+(-\zeta_n^{(1)}) & \tau_+(\zeta_n^{(1)})
\end{pmatrix}
\begin{pmatrix}
\Psi_{\tau_+}(h_1, \ldots, h_n = 0, \ldots, h_1) \\
\Psi_{\tau_+}(h_1, \ldots, h_n = 1, \ldots, h_1)
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix},
\]

where

\[
D_+(\zeta_n^{(0)}) = D_+(\zeta_n^{(1)}) = 0, \quad D_+(\zeta_n^{(0)}) \neq 0 \quad \text{and} \quad D_+(\zeta_n^{(1)}) \neq 0,
\]

from which the condition \( \tau_+(\lambda) \in \Sigma_{T_+} \) directly implies (5.6), and moreover it has to hold that

\[
\frac{\Psi_{\tau_+}(h_1, \ldots, h_n = 1, \ldots, h_1)}{\Psi_{\tau_+}(h_1, \ldots, h_n = 0, \ldots, h_1)} = \frac{\tau_+(\zeta_n^{(0)})}{D_+(\zeta_n^{(1)})}
\]

for any \( n \in \{1, \ldots, N\} \) with \( h_n \neq \in \{0, 1\} \). This fixes the factorized form given in (5.33)–(5.34) for the \( T_+ \)-eigenstate \(|\tau_+\rangle\) and implies the simplicity of the \( T_+ \)-spectrum. Taking a \( \tau_+(\lambda) \) solution of (5.6) in the class of functions (5.1) and constructing \(|\tau_+\rangle\) by (5.33)–(5.34), then the proof that \( \tau_+(\lambda) \in \Sigma_{T_+} \) and \(|\tau_+\rangle\) is the corresponding \( T_+ \)-eigenstate can be given following the same steps presented in theorem 5.1. Concerning the left \( T_- \)-eigenstates the construction is carried out as above; we have just to remark that in this case the matrix elements

\[
\langle \tau_+ | T_+(\zeta_n^{(h)}) | +, h_1, \ldots, h_N \rangle
\]

can be computed by using the right \( \mathcal{B}_- \)-representation

\[
\tau_+(\zeta_n^{(h)}) \bar{\Psi}_{\tau_+}(h) = A_+(\zeta_n^{(h)}) \bar{\Psi}_{\tau_+}(T_n^+(h)) + A_+(\zeta_n^{(h)}) \bar{\Psi}_{\tau_+}(T_n^-(h)),
\]

\[\forall \; n \in \{1, \ldots, N\},\]

where

\[
\bar{\Psi}_{\tau_+}(h) \equiv \langle \tau_+ |-, h_1, \ldots, h_N \rangle,
\]

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and the coefficient $A_+(\lambda)$ reads
\[ A_+(\lambda) \equiv a_-(\lambda)A_+(\lambda). \] (5.46)

\section{Scalar products}

The presentation will be made simultaneously for $B_\epsilon$-SOV-representations with $\epsilon = +, -$.

\begin{proposition}
Let $\langle \alpha_\epsilon \rangle$ be an arbitrary covector and $|\beta_\epsilon \rangle$ be an arbitrary vector of separate forms.
\[ \langle \alpha_\epsilon \rangle = \sum_{h_1, \ldots, h_N = 0}^{1} \prod_{a=1}^{N} \alpha_{\epsilon,a}(s_a^{(h_a)}) \prod_{1 \leq b < a \leq N} \left( \eta_a^{(h_b)} - \eta_b^{(h_a)} \right) \langle \epsilon, h_1, \ldots, h_N \rangle, \] (6.1)
\[ |\beta_\epsilon \rangle = \sum_{h_1, \ldots, h_N = 0}^{1} \prod_{a=1}^{N} \beta_{\epsilon,a}(c_a^{(h_a)}) \prod_{1 \leq b < a \leq N} \left( \eta_a^{(h_b)} - \eta_b^{(h_a)} \right) |\epsilon, h_1, \ldots, h_N \rangle \] (6.2)
in the $B_\epsilon$-eigenbasis, then the action of $\langle \alpha_\epsilon \rangle$ on $|\beta_\epsilon \rangle$ reads
\[ \langle \alpha_\epsilon | \beta_\epsilon \rangle = \det_\mathbb{N} \| M_{a,b}^{(\alpha_\epsilon, \beta_\epsilon)} \| \text{ with } M_{a,b}^{(\alpha_\epsilon, \beta_\epsilon)} = \sum_{h=0}^{1} \alpha_{\epsilon,a}(s_a^{(h)}) \beta_{\epsilon,a}(c_a^{(h)}) (\eta_a^{(h)})^{b-1}. \] (6.3)

Moreover, if $\tau_\epsilon(\lambda) \neq \tau'_\epsilon(\lambda) \in \Sigma_{T_\epsilon}$, the action of the $T_\epsilon$-eigencovector $\langle \tau_\epsilon \rangle$ on the $T_\epsilon$-eigenvector $|\tau'_\epsilon \rangle$ is zero, and in particular it holds that
\[ \sum_{b=1}^{N} M_{a,b}^{(\tau_\epsilon, \tau'_\epsilon)} c_b^{(\tau_\epsilon, \tau'_\epsilon)} = 0 \quad \forall a \in \{1, \ldots, N\}, \] (6.4)
where the $c_b^{(\tau_\epsilon, \tau'_\epsilon)}$ are defined by
\[ \tau_\epsilon(\lambda) - \tau'_\epsilon(\lambda) \equiv \sinh(2\lambda - \eta) \sinh(2\lambda + \eta) \sum_{b=1}^{N} c_b^{(\tau_\epsilon, \tau'_\epsilon)} (\cosh 2\lambda)^{b-1}. \] (6.5)

\begin{proof}
The formula (4.7) and the SOV-decomposition of the states $\langle \alpha_\epsilon \rangle$ and $|\beta_\epsilon \rangle$ imply
\[ \langle \alpha_\epsilon | \beta_\epsilon \rangle = \sum_{h_1, \ldots, h_N = 0}^{1} V(\eta_1^{(h_1)}, \ldots, \eta_N^{(h_N)}) \prod_{a=1}^{N} \alpha_{\epsilon,a}(c_a^{(h_a)}) \beta_{\epsilon,a}(\eta_a^{(h_a)}), \] (6.6)
where $V(x_1, \ldots, x_N) \equiv \prod_{1 \leq b < a \leq N} (x_a - x_b)$ is the Vandermonde determinant which for the multilinearity of the determinant implies (6.3).

The $T_\epsilon$-eigenstates $\langle \tau_\epsilon \rangle$ and $|\tau'_\epsilon \rangle$ are left and right separate states, so the action $\langle \tau_\epsilon | \tau'_\epsilon \rangle$ is also given by (6.3) and to prove $\langle \tau_\epsilon | \tau'_\epsilon \rangle = 0$ for $\tau_\epsilon(\lambda) \neq \tau'_\epsilon(\lambda) \in \Sigma_{T_\epsilon}$ we have just to prove (6.4). It is simple to remark that
\[ \sum_{b=1}^{N} M_{a,b}^{(\tau_\epsilon, \tau'_\epsilon)} c_b^{(\tau_\epsilon, \tau'_\epsilon)} = \sum_{h=0}^{1} \frac{Q_{\tau'_\epsilon}(\zeta_a^{(h)}) Q_{\tau_\epsilon}(\zeta_a^{(h)}) (\tau_\epsilon(\zeta_a^{(h)}) - \tau'_\epsilon(\zeta_a^{(h)}))}{\sinh(2\zeta_a^{(h)} - \eta) \sinh(2\zeta_a^{(h)} + \eta)}, \] (6.7)
\end{proof}

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so in the case $\epsilon = -$, we can use equations (5.10) and (5.12) to rewrite
\[
Q_{\tau_-}(\zeta_a^{(1)}_a)Q_{\tau_-}(\zeta_a^{(0)}_a)(\tau_-(\zeta_a^{(1)}_a) - \tau_-'(\zeta_a^{(1)}_a)) = k_a\alpha_a A_-(\zeta_a^{(0)}_a) Q_{\tau_-}(\zeta_a^{(1)}_a)Q_{\tau_-}(\zeta_a^{(0)}_a)
- A_-(\zeta_a^{(1)}_a) Q_{\tau_-}(\zeta_a^{(0)}_a)Q_{\tau_-}(\zeta_a^{(1)}_a),
\]
and
\[
Q_{\tau_-}(\zeta_a^{(0)}_a)Q_{\tau_-}(\zeta_a^{(0)}_a)(\tau_-(\zeta_a^{(0)}_a) - \tau_-'(\zeta_a^{(0)}_a)) = (k_a\alpha_a)^{-1} [A_-(\zeta_a^{(1)}_a) Q_{\tau_-}(\zeta_a^{(0)}_a)Q_{\tau_-}(\zeta_a^{(1)}_a)
- k_a\alpha_a A_-(\zeta_a^{(0)}_a) Q_{\tau_-}(\zeta_a^{(1)}_a)Q_{\tau_-}(\zeta_a^{(0)}_a)].
\]
Then by substituting them in (6.7) we get (6.4). Similarly, in the case $\epsilon = +$, we can use equations (5.34) and (5.36) to rewrite
\[
Q_{\tau_+}(\zeta_a^{(1)}_a)Q_{\tau_+}(\zeta_a^{(1)}_a)(\tau_+(\zeta_a^{(1)}_a) - \tau_0'(\zeta_a^{(1)}_a)) = (k_a\alpha_a)^{-1} [D_+(\zeta_a^{(0)}_a) Q_{\tau_+}(\zeta_a^{(0)}_a)Q_{\tau_+}(\zeta_a^{(1)}_a)
- k_a\alpha_a D_+(\zeta_a^{(0)}_a) Q_{\tau_+}(\zeta_a^{(0)}_a)Q_{\tau_+}(\zeta_a^{(1)}_a)],
\]
and
\[
Q_{\tau_+}(\zeta_a^{(0)}_a)Q_{\tau_+}(\zeta_a^{(0)}_a)(\tau_+(\zeta_a^{(0)}_a) - \tau_0'(\zeta_a^{(0)}_a)) = k_a\alpha_a (D_+(\zeta_a^{(1)}_a) Q_{\tau_+}(\zeta_a^{(0)}_a)Q_{\tau_+}(\zeta_a^{(1)}_a)
- D_+(\zeta_a^{(0)}_a) Q_{\tau_+}(\zeta_a^{(1)}_a)Q_{\tau_+}(\zeta_a^{(0)}_a)).
\]
Then by substituting them in (6.7) we get (6.4). \qed

It is worth remarking that the vector $(|\epsilon, \alpha|)\in \mathcal{R}_N$ is of separate form in the $\zeta$-eigenbasis thanks to the Hermitian conjugation properties of the Yang–Baxter generators. Then the previous result describes also the scalar products for these states. The determinant formulae obtained here can then be considered as the SOV analogues of Slavnov’s scalar product formula [58, 18] which holds in the framework of the algebraic Bethe ansatz.

7. Boundary reconstructions of strings of local operators

Here we present identities between couples of (boundary) generators of the reflection algebra and (bulk) generators of the Yang–Baxter algebra which can be used to reconstruct any local operator in terms of the boundary operators. This programme is explained and developed for simple strings of local operators like
\[
\sigma_1^+ \cdots \sigma_n^+, \quad \sigma_n^- \cdots \sigma_N^+.
\]

7.1. Mixed bulk and $\mathcal{U}_\pm$-boundary reconstructions

Let us start by recalling the bulk reconstruction formulae.

**Proposition 7.1 ( [18]).** Let $x_n \in \text{End}(R_n)$ be the generic local operator in the local quantum space $R_n$, then it admits the following reconstruction in terms of the generators of the Yang–Baxter algebra:
\[
x_n \equiv \prod_{a=1}^{n-1} T(\zeta_a^{(1)}_a) \det_q M(\zeta_n) - \text{tr}_0(M_0(\zeta_0^{(0)}_0)\sigma_0^+ x_0^0 \sigma_0^-) \prod_{a=1}^{n-1} T^{-1}(\zeta_a^{(1)}_a) \quad (7.2)
\]

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\[ T^{-1}(\zeta_n^{(1)}) \text{tr}_0(M_0(\zeta_n^{(0)})\sigma^ y_0 x_0^a \sigma^ y_0) \frac{T(\zeta_n^{(1)})}{\det_q M(\xi_n)} \prod_{a=n+1}^N T(\zeta_a^{(1)}) \]  (7.3)

\[ = \prod_{a=1}^{n-1} T(\zeta_a^{(1)}) \text{tr}_0(M_0(\zeta_n^{(1)})x_0) \frac{T(\zeta_0^{(0)})}{\det_q M(\xi_n)} \prod_{a=1}^{n-1} T^{-1}(\zeta_0^{(1)}) \]  (7.4)

\[ = \prod_{a=n+1}^N T^{-1}(\zeta_a^{(1)}) \frac{T(\zeta_0^{(0)})}{\det_q M(\xi_n)} \text{tr}_0(M_0(\zeta_n^{(1)})x_0) \prod_{a=n+1}^N T(\zeta_a^{(1)}), \]  (7.5)

where we have used the identities

\[ T(\zeta_a^{(1)}) T(\zeta_0^{(0)}) = \det_q M(\xi_n). \]  (7.6)

In [59] a reconstruction of local operators which uses the reconstruction of the propagator by the (bulk) transfer matrix \( T(\zeta^{(b)}) \) of the Yang–Baxter algebra and the elements of the (boundary) reflection algebra \( \mathcal{U}_\pm(\lambda) \) has been derived. In this subsection we reproduce this result and provide another three equivalent reconstructions.

**Proposition 7.2.** Let \( x_n \in \text{End}(R_n) \) be the generic local operator in the local quantum space \( R_n \), then it admits the following reconstructions in terms of the generators of the (boundary) reflection algebra \( \mathcal{U}_\pm(\lambda) \):

\[ x_n = \prod_{a=1}^{n-1} T(\zeta_a^{(1)}) \text{tr}_0(\mathcal{U}_+(\zeta_n^{(1)})x_0) \frac{T_+(\zeta_0^{(0)})}{\det_q \mathcal{U}_+(\xi_n)} \prod_{a=1}^{n-1} T^{-1}(\zeta_0^{(1)}) \]  (7.7)

\[ = \prod_{a=1}^{n-1} T(\zeta_a^{(1)}) \frac{T_+(\zeta_n^{(1)})}{\det_q \mathcal{U}_+(\xi_n)} \text{tr}_0(\mathcal{U}_+(\zeta_0^{(0)})x_0) \prod_{a=1}^{n-1} T^{-1}(\zeta_0^{(1)}), \]  (7.8)

and the following ones in terms of the generators of the (boundary) reflection algebra \( \mathcal{U}_-(\lambda) \):

\[ x_n = \prod_{a=n+1}^N T^{-1}(\zeta_a^{(1)}) \text{tr}_0(\mathcal{U}_-(\zeta_n^{(0)})\sigma^ y_0 x_0^a \sigma^ y_0) \frac{T_-(\zeta_0^{(1)})}{\det_q \mathcal{U}_-(\xi_n)} \prod_{a=n+1}^N T(\zeta_a^{(1)}) \]  (7.9)

\[ = \prod_{a=n+1}^N T^{-1}(\zeta_a^{(1)}) \frac{T_-(\zeta_0^{(0)})}{\det_q \mathcal{U}_-(\xi_n)} \text{tr}_0(\mathcal{U}_-(\zeta_n^{(1)})x_0) \prod_{a=n+1}^N T(\zeta_a^{(1)}). \]  (7.10)

Here, we have denoted\(^{17}\)

\[ \tilde{T}_\pm(\lambda) = \tilde{a}_\pm(\lambda) \mathcal{A}_\pm(\lambda) + \tilde{a}_\mp(\lambda) \mathcal{A}_\mp(\lambda), \]  (7.11)

\[ \tilde{a}_\pm(\lambda) \equiv a_\pm(\lambda)|_{\xi_+ = i\pi/2} = d_\pm(\lambda)|_{\xi_+ = i\pi/2} = \cos(\lambda \mp \eta/2) \frac{\sinh(2\lambda \pm \eta)}{\sinh 2\lambda}, \]  (7.12)

\(^{17}\)Note that

\[ \tilde{T}_-(\lambda) \equiv \text{tr}_0(\mathcal{U}_-(\lambda)K_+(\lambda)|_{\zeta_+ = i\pi/2}) = \cosh(\lambda + \eta/2)(\mathcal{A}_-(\lambda) + \mathcal{D}_-(\lambda)), \]

\[ \tilde{T}_+(\lambda) \equiv \text{tr}_0(\mathcal{U}_+(\lambda)K_-(\lambda)|_{\zeta_- = i\pi/2}) = \cosh(\lambda - \eta/2)(\mathcal{A}_+(\lambda) + \mathcal{D}_+(\lambda)). \]
The first reconstruction in terms of \( U_+ \) is the result proven in proposition 1 of [59]; similarly it is possible to prove our second reconstruction in terms of \( U_- \). Let us prove here the reconstructions in terms of \( U_- \); by definition of the propagator operator it holds that

\[
\prod_{a=1}^{n} T(\zeta_{n}^{(1)}) \text{tr}_0(U_-(\lambda)\sigma_0^y x_0^a \sigma_0^y) = (-1)^n \text{tr}_0(L_{0,n}(\lambda) \cdots L_{01}(\lambda) L_{0N}(\lambda) \cdots L_{0n+1}(\lambda) K_-(\lambda)\sigma_0^y \\
\times L_{0n+1}^0(-\lambda) \cdots L_{0n}^0(-\lambda) L_{01}^0(-\lambda) \cdots L_{0n}^0(-\lambda) x_0^a \sigma_0^y) \prod_{a=1}^{n} T(\zeta_{a}^{(1)}),
\]

(7.15)

where \( L_{0a}(\lambda) \equiv R_{0a}(\lambda - \xi_a - \eta/2) \) for any \( a \in \{1, \ldots, N\} \).

Then, the rhs of (7.15) computed in \( \lambda = -\zeta_n^{(1)} \) reads

\[
\text{tr}_0(U_-(\lambda)\sigma_0^y x_0 \prod_{a=1}^{n} T(\zeta_{a}^{(1)})),
\]

(7.17)

where

\[
L_{0n}^0(\zeta_{n}^{(1)}) x_0^a \sigma_0^y = \sinh \eta [x_0^a P_{0n}^0 \sigma_0^y = \sinh \eta P_{0n}^0 \sigma_0^y x_0^a.
\]

(7.18)

Similarly, the rhs of (7.15) computed in \( \lambda = -\zeta_n^{(1)} \) reads

\[
x_n \text{tr}_0(U_-(\lambda)\sigma_0^y \prod_{a=1}^{n} T(\zeta_{a}^{(1)})),
\]

(7.19)

where

\[
x_0^a \sigma_0^y L_{0n}(\zeta_{n}^{(0)}) = -\sinh \eta [P_{0n} \sigma_0^y x_0^a = x_n^a \sigma_0^y L_{0n}(\zeta_{n}^{(0)}).
\]

(7.20)

By using these formulae the reconstructions (7.9) and (7.10) simply follow. □

The previous proposition naturally implies the following.

**Corollary 7.1.** The following annihilation identities hold.

1. For the generators of the reflection algebra \( U_-(\lambda) \)

\[
\mathcal{A}_-(\zeta_{n}^{(0)}) \mathcal{C}_-(\pm \zeta_{n}^{(1)}) = \mathcal{A}_-(\zeta_{n}^{(0)}) \mathcal{D}_-(\pm \zeta_{n}^{(1)}) = 0,
\]

(7.21)

\[
\mathcal{A}_-(\zeta_{n}^{(1)}) \mathcal{C}_-(\pm \zeta_{n}^{(0)}) = \mathcal{A}_-(\zeta_{n}^{(1)}) \mathcal{D}_-(\zeta_{n}^{(0)}) = 0,
\]

(7.22)

\[
\mathcal{D}_-(\zeta_{n}^{(0)}) \mathcal{B}_-(\pm \zeta_{n}^{(1)}) = \mathcal{D}_-(\zeta_{n}^{(0)}) \mathcal{A}_-(\pm \zeta_{n}^{(1)}) = 0,
\]

(7.23)

\[
\mathcal{D}_-(\zeta_{n}^{(1)}) \mathcal{B}_-(\zeta_{n}^{(0)}) = \mathcal{D}_-(\zeta_{n}^{(1)}) \mathcal{A}_-(\zeta_{n}^{(0)}) = 0,
\]

(7.24)

\[
\mathcal{B}_-(\zeta_{n}^{(0)}) \mathcal{B}_-(\pm \zeta_{n}^{(1)}) = \mathcal{B}_-(\zeta_{n}^{(0)}) \mathcal{A}_-(\pm \zeta_{n}^{(1)}) = \mathcal{B}_-(\pm \zeta_{n}^{(1)}) \mathcal{A}_-(\zeta_{n}^{(0)}) = 0,
\]

(7.25)
and
\[ C_-(\pm \zeta_n^{(0)})C_-(\pm \zeta_n^{(1)}) = C_-(\pm \zeta_n^{(0)})D_-(\zeta_n^{(1)}) = C_-(\pm \zeta_n^{(1)})D_-(\zeta_n^{(0)}) = 0. \] (7.26)

(II) For the generators of the reflection algebra \( \mathcal{U}_+(\lambda) \)
\[
\begin{align*}
\mathcal{A}_+(-\zeta_n^{(0)})\mathcal{B}_+(-\zeta_n^{(0)}) &= \mathcal{A}_+(-\zeta_n^{(0)})\mathcal{D}_+(-\zeta_n^{(0)}) = 0, \\
\mathcal{A}_+(-\zeta_n^{(1)})\mathcal{B}_+(-\zeta_n^{(0)}) &= \mathcal{A}_+(-\zeta_n^{(0)})\mathcal{D}_+(-\zeta_n^{(0)}) = 0, \\
\mathcal{D}_+(-\zeta_n^{(0)})\mathcal{C}_+(\zeta_n^{(1)}) &= \mathcal{D}_+(-\zeta_n^{(0)})\mathcal{A}_+(\zeta_n^{(1)}) = 0, \\
\mathcal{D}_+(-\zeta_n^{(1)})\mathcal{C}_+(\zeta_n^{(0)}) &= \mathcal{D}_+(\zeta_n^{(1)})\mathcal{A}_+(-\zeta_n^{(0)}) = 0, \\
\mathcal{B}_+(\zeta_n^{(0)})\mathcal{B}_+(\zeta_n^{(1)}) &= \mathcal{B}_+(\zeta_n^{(1)})\mathcal{D}_+(\zeta_n^{(1)}) = \mathcal{B}_+(\zeta_n^{(1)})\mathcal{D}_+(-\zeta_n^{(0)}) = 0,
\end{align*}
\]
(7.27)–(7.31)
and
\[ C_+(\pm \zeta_n^{(0)})C_+(\pm \zeta_n^{(1)}) = C_+(\pm \zeta_n^{(0)})A_+(\zeta_n^{(1)}) = C_+(\pm \zeta_n^{(1)})A_+(-\zeta_n^{(0)}) = 0. \] (7.32)

**Proof.** The proof of these annihilation identities can be made along the same lines as used for the bulk case. Let us sketch it here. By using the reconstruction formulae of the previous proposition and the identities
\[
\mathcal{T}_\pm(\zeta_n^{(1)}) = \det_q \mathcal{U}_\pm(\zeta_n),
\] (7.33)

it also holds that
\[
\begin{align*}
\prod_{a=1}^{n-1} T^{-1}(\zeta_n^{(1)}) x_n^{(3)} \prod_{a=1}^{n-1} T(\zeta_n^{(1)}) &= \frac{\text{tr}_0(\mathcal{U}_+(\zeta_n^{(1)}) x_n^{(1)}) \text{tr}_0(\mathcal{U}_+(\zeta_n^{(0)}) x_n^{(2)})}{\det_q \mathcal{U}_+(\zeta_n)} \\
&= \frac{\mathcal{T}_+(\zeta_n^{(1)}) \text{tr}_0(\mathcal{U}_+(\zeta_n^{(0)}) x_n^{(1)}) \text{tr}_0(\mathcal{U}_+(\zeta_n^{(1)}) x_n^{(2)}) \mathcal{T}_+(\zeta_n^{(0)})}{(\det_q \mathcal{U}_+(\zeta_n))^2},
\end{align*}
\] (7.34)
and
\[
\begin{align*}
\prod_{a=n+1}^{N} T(\zeta_n^{(1)}) y_n^{(3)} \prod_{a=n+1}^{N} T^{-1}(\zeta_n^{(1)}) &= \frac{\text{tr}_0(\mathcal{U}_-(\zeta_n^{(0)}) y_n^{(1)}) \text{tr}_0(\mathcal{U}_-(\zeta_n^{(1)}) y_n^{(2)})}{\det_q \mathcal{U}_-(\zeta_n)} \\
&= \frac{\mathcal{T}_-(\zeta_n^{(0)}) \text{tr}_0(\mathcal{U}_-(\zeta_n^{(0)}) y_n^{(1)}) \text{tr}_0(\mathcal{U}_-(\zeta_n^{(1)}) y_n^{(2)}) \mathcal{T}_-(\zeta_n^{(0)})}{(\det_q \mathcal{U}_-(\zeta_n))^2},
\end{align*}
\] (7.36)

where we have defined
\[
x_n^{(3)} \equiv x_n^{(1)} x_n^{(2)}, \quad y_n^{(3)} \equiv \sigma_0^y (y_n^{(2)}) y_n^{(1)} \sigma_0^y.
\] (7.38)

We can now use these formulae to derive all the annihilation formulae.

### 7.2. \( \mathcal{U}_{+-} \)-boundary reconstruction

Let us remark that the bulk reconstruction formulae of proposition 7.1, the corresponding annihilation identities\(^{18}\) and the Yang–Baxter commutation relation imply that the generic

\(^{18}\) See lemma 5.1 of [15].
string of local operators of the form
\[
\prod_{a=1}^{n} x_a
\]  
(7.39)
can indeed be represented as linear combinations of the following strings of bulk operators:
\[
\left( \frac{\text{tr}_0(M_0(\zeta_1^{(1)})y_0^{(1)})}{\det_q M(\xi_1)} \right) \left( \frac{\text{tr}_0(M_0(\zeta_2^{(1)})y_0^{(2)})}{\det_q M(\xi_2)} \right) \cdots \\
\times \left( \frac{\text{tr}_0(M_0(\zeta_n^{(1)})y_0^{(n)})}{\det_q M(\xi_n)} \right).
\]  
(7.40)

Then the following identities between bulk and \(U_+\)-boundary generators:
\[
\text{tr}_0(M_0(\zeta_n^{(1)})y_0) \frac{T(\zeta_n^{(0)})}{\det_q M(\xi_n)} = \frac{\text{tr}_0(\mathcal{U}_+(\zeta_n^{(1)})y_0)}{\det_q \mathcal{U}_+(\xi_n)} \frac{T_+(\zeta_n^{(0)})}{\det_q \mathcal{U}_+(\xi_n)}
\]  
(7.41)

imply that we can also write (7.39) by a linear combination of the following strings of \(U_+\)-boundary generators:
\[
\left( \frac{\text{tr}_0(\mathcal{U}_+(\zeta_1^{(1)})y_0^{(1)})}{\det_q \mathcal{U}_+(\xi_1)} \right) \left( \frac{\text{tr}_0(\mathcal{U}_+(\zeta_2^{(1)})y_0^{(2)})}{\det_q \mathcal{U}_+(\xi_2)} \right) \cdots \\
\times \left( \frac{\text{tr}_0(\mathcal{U}_+(\zeta_n^{(1)})y_0^{(n)})}{\det_q \mathcal{U}_+(\xi_n)} \right).
\]  
(7.42)

Similarly, the identities between bulk and \(U_-\)-boundary generators
\[
\text{tr}_0(M_0(\zeta_n^{(0)})y_0) \frac{T(\zeta_n^{(1)})}{\det_q M(\xi_n)} = \frac{\text{tr}_0(\mathcal{U}_-(\zeta_n^{(0)})y_0)}{\det_q \mathcal{U}_-(\xi_n)} \frac{T_-(\zeta_n^{(1)})}{\det_q \mathcal{U}_-(\xi_n)}
\]  
(7.43)

imply that the generic string of local operators of the form
\[
\prod_{a=n}^{N} x_a
\]  
(7.44)
can indeed be represented as linear combinations of the following strings of \(U_-\)-boundary generators:
\[
\left( \frac{\text{tr}_0(\mathcal{U}_-(\zeta_N^{(0)})y_0^{(N)})}{\det_q \mathcal{U}_-(\xi_N)} \right) \left( \frac{\text{tr}_0(\mathcal{U}_-(\zeta_{N-1}^{(0)})y_0^{(N-1)})}{\det_q \mathcal{U}_-(\xi_{N-1})} \right) \cdots \\
\times \left( \frac{\text{tr}_0(\mathcal{U}_-(\zeta_n^{(0)})y_0^{(n)})}{\det_q \mathcal{U}_-(\xi_n)} \right).
\]  
(7.45)

Here we show explicitly that these reconstructions work for some special strings of local operators.
Proposition 7.3. Let us consider the open XXZ spin chain with general $K_-$ and diagonal or triangular $K_+$; then the following boundary reconstruction holds:

$$\sigma_n^- \cdots \sigma_1^- = (-1)^{N+1-n} \prod_{a=n}^N \frac{\bar{a}_+(\zeta_n^{(1)})}{a_-(\zeta_n^{(1)})} \prod_{n \leq a < b \leq N} \frac{\sinh(\xi_a + \xi_b - \eta)}{\sinh(\xi_a + \xi_b)} \times B_-(\zeta_N^{(0)}) \cdots B_-(\zeta_n^{(0)}) \frac{T_-(\zeta_N^{(1)})}{\det_q \mathcal{U}_-(\zeta_N)} \cdots \frac{T_-(\zeta_n^{(1)})}{\det_q \mathcal{U}_-(\zeta_n)}. \quad (7.46)$$

Let us consider the open XXZ spin chain with general $K_+$ and diagonal or triangular $K_-';$ then the following boundary reconstruction holds:

$$\sigma_1^- \cdots \sigma_n^- = \prod_{a=1}^n \frac{d_-(\zeta_0^{(0)})}{d_-(\zeta_a^{(0)})} \prod_{1 \leq a < b \leq n} \frac{\sinh(\xi_a + \xi_b + \eta)}{\sinh(\xi_a + \xi_b)} \times B_+(\zeta_1^{(1)}) \cdots B_+(\zeta_n^{(1)}) \frac{T_+(\zeta_1^{(0)})}{\det_q \mathcal{U}_+(\zeta_1)} \cdots \frac{T_+(\zeta_n^{(0)})}{\det_q \mathcal{U}_+(\zeta_n)}. \quad (7.47)$$

**Proof.** Let us prove explicitly the first reconstruction as for the second one we can proceed similarly. First of all by using the bulk reconstruction of proposition 7.1 one gets

$$\sigma_n^- \cdots \sigma_1^- = (-1)^{N+1-n} B(\zeta_N^{(0)}) \cdots B(\zeta_n^{(0)}) \frac{T(\zeta_N^{(1)})}{\det_q M_0(\zeta_N)} \cdots \frac{T(\zeta_n^{(1)})}{\det_q M_0(\zeta_n)}. \quad (7.48)$$

Now by using the bulk annihilation identities $B(\zeta_N^{(0)})D(\zeta_N^{(1)}) = 0$, we get the set of identities

$$B(\zeta_N^{(0)}) \cdots B(\zeta_n^{(0)})T(\zeta_N^{(1)}) = B(\zeta_N^{(0)}) \cdots B(\zeta_n^{(0)})A(\zeta_N^{(1)})$$

$$= \frac{\sinh(\xi_n - \xi_N - \eta)}{\sinh(\xi_n - \xi_N)} B(\zeta_N^{(0)}) \cdots B(\zeta_n^{(0)})A(\zeta_N^{(1)})B(\zeta_n^{(0)})$$

$$= \prod_{a=n}^{N-1} \frac{\sinh(\xi_a - \xi_N - \eta)}{\sinh(\xi_a - \xi_N)} B(\zeta_N^{(0)})A(\zeta_N^{(1)})B(\zeta_n^{(0)}) \cdots B(\zeta_n^{(0)})$$

$$= \prod_{a=n}^{N-1} B(\zeta_N^{(0)})T(\zeta_N^{(1)})B(\zeta_n^{(0)}) \cdots B(\zeta_n^{(0)}), \quad (7.49)$$

where to commute $B(\zeta_n^{(0)})$ and $A(\zeta_N^{(1)})$ we have used the Yang–Baxter commutation relation

$$B(\zeta_N^{(0)})A(\zeta_N^{(1)}) = \frac{\sinh(\xi_n - \xi_N - \eta)}{\sinh(\xi_n - \xi_N)} A(\zeta_N^{(1)})B(\zeta_n^{(0)}) + \frac{\sinh \eta}{\sinh(\xi_n - \xi_N)} B(\zeta_N^{(1)})A(\zeta_n^{(0)}), \quad (7.50)$$

which together with the bulk annihilation identity $B(\zeta_n^{(0)})B(\zeta_n^{(1)}) = 0$ implies the above second identity; the third one is obtained by reiterating the commutation and finally the fourth one by using once again the annihilation identity $B(\zeta_N^{(0)})D(\zeta_N^{(1)}) = 0$. Repeating the same procedure to commute the other transfer matrices through the product of the $B$-operators we get

\[\text{doi:10.1088/1742-5468/2012/10/P10025} \]

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\[ \sigma_n^{\cdots} \sigma_n^{-} = (-1)^{N+1-n} \prod_{n \leq a < b \leq N} \frac{\sinh(\xi_a - \xi_b - \eta)}{\sinh(\xi_a - \xi_b)} \frac{B_-(\xi_n^T(\xi_n^1))}{\det_q M_0(\xi_n)} \cdots \frac{B_-(\xi_n^T(\xi_n^1))}{\det_q M_0(\xi_n)} , \quad (7.51) \]

and then by using the boundary–bulk identities (7.43) we can rewrite it in the form

\[ \sigma_n^{\cdots} \sigma_n^{-} = (-1)^{N+1-n} \prod_{n \leq a < b \leq N} \frac{\sinh(\xi_a - \xi_b - \eta)}{\sinh(\xi_a - \xi_b)} \frac{B_-(\xi_n^T(\xi_n^1))}{\det_q \mathcal{U}_- (\xi_n)} \cdots \frac{B_-(\xi_n^T(\xi_n^1))}{\det_q \mathcal{U}_- (\xi_n)} . \quad (7.52) \]

We can now use the boundary annihilation identities (7.25) and the reflection algebra commutation relations to move all the transfer matrices \( \mathcal{T}_- (\xi_n^1) \) to the right and transform them into \( \mathcal{T}_- (\xi_n^1) \). In more detail, from the annihilation identities and the definition of the transfer matrices it holds that

\[ B_-(\xi_n^0) \mathcal{T}_- (\xi_n^1) = \frac{\tilde{a}_+ (\xi_n^1)}{a_+ (\xi_n^1)} B_-(\xi_n^0) \mathcal{T}_- (\xi_n^1) , \quad (7.53) \]

and so the first transfer matrix on the right in (7.52) can be rewritten in the desired form. Now let us apply the procedure to the product \( B_-(\xi_n^0) \mathcal{T}_- (\xi_n^1) B_-(\xi_n^0) \) to rewrite it in the desired form \( B_-(\xi_n^0) B_-(\xi_n^0) \mathcal{T}_- (\xi_n^1) \). The annihilation identity (7.25) implies

\[ B_-(\xi_n^0) \mathcal{T}_- (\xi_n^1) B_-(\xi_n^0) = \tilde{a}_+ (\xi_n^1) B_-(\xi_n^0) A_-(\xi_n^0) B_-(\xi_n^0) \quad (7.54) \]

\[ = \frac{\sinh(\xi_n - \xi_n+1)}{\sinh(\xi_n - \xi_n+1 - \eta)} \frac{\sinh(\xi_n + \xi_n+1)}{\sinh(\xi_n + \xi_n+1)} \]

\[ \times B_-(\xi_n^0) B_-(\xi_n^0) A_-(\xi_n^1) . \quad (7.55) \]

In the second identity we have used the reflection algebra commutation relation

\[ A_-(\xi_n^0) = \frac{\sinh(\xi_n - \xi_n+1)}{\sinh(\xi_n - \xi_n+1 - \eta)} \frac{\sinh(\xi_n + \xi_n+1)}{\sinh(\xi_n + \xi_n+1)} B_-(\xi_n^0) A_-(\xi_n^0) \]

\[ - \frac{\sinh(2\xi_n - 2\eta)}{\sinh(\xi_n - \xi_n+1 - \eta) \sinh(2\xi_n - \eta)} B_-(\xi_n^1) A_-(\xi_n^0) \]

\[ - \frac{\sinh(\xi_n + \xi_n+1)}{\sinh(\xi_n + \xi_n+1) \sinh(2\lambda_1)} B_-(\xi_n^0) \mathcal{D}_- (\xi_n^0) \quad (7.56) \]

and the fact that due to the presence of \( B_-(\xi_n^0) \) the second and third terms on the right of this formula give zero when inserted in (7.52). Now by using the commutativity of the \( B_- \) -generators and once again the annihilation identities it holds that

\[ B_-(\xi_n^0) B_-(\xi_n^0) A_-(\xi_n^0) = \frac{B_-(\xi_n^0) B_-(\xi_n^0) \mathcal{T}_- (\xi_n^1)}{\tilde{a}_+ (\xi_n^1)} , \quad (7.57) \]

so that we have accomplished our task for \( N = n+1 \) having the product \( \mathcal{T}_- (\xi_n^1) \mathcal{T}_- (\xi_n^1) \) to the right of (7.52). Reiterating this procedure for the remaining transfer matrices \( \mathcal{T}_- (\xi_n^1) \) for \( a \in \{ n + 2, \ldots, N \} \) we obtain our result.

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8. Matrix elements

8.1. From the $B_-$-SOV representation

Here we consider the transfer matrices (5.4), then the following proposition holds.

Proposition 8.1. Let $\langle \tau_- \rangle$ and $|\tau'_-\rangle$ be a generic couple of left and right $T_-$-eigenstates, then we have

$$
\langle \tau_- | \sigma_n^- \cdots \sigma_N^- | \tau'_- \rangle = \frac{-(\kappa_- e^{\tau_-} \sinh \eta/2N \sinh \zeta_-)^{N-n}}{V(\eta_1^{(0)}, \ldots, \eta_N^{(0)})} \prod_{n\leq a < b \leq N} \frac{\sinh(\xi_a + \xi_b - \eta)}{\sinh(\xi_a + \xi_b)} \det \| \Sigma^{(-,n,-\tau'_-)}_{a,b} \| \nonumber
\times \prod_{a=n}^{N} \frac{\bar{a}_+(\xi^{(1)}_a)\tau'_-(\xi^{(1)}_a)Q_{\tau'_-}(\xi^{(1)}_a)Q_{\tau_-}(\xi^{(1)}_a)}{a_+(\xi^{(1)}_a) \det q \mathcal{U}_-(\xi_a)}.
$$

where $\| \Sigma^{(-,n,-\tau'_-)}_{a,b} \|$ is the $(2N-n) \times (2N-n)$ matrix of elements,

$$
\Sigma^{(-,n,-\tau'_-)}_{a,b} \equiv M^{(\tau_-,-\tau'_-)}_{a,b} \quad \text{for} \quad a \in \{1, \ldots, n-1\}, \quad b \in \{1, \ldots, 2N-n\},
$$

$$
\Sigma^{(-,n,-\tau'_-)}_{a,b} \equiv \left( \eta_a^{(0)} \right)^{(b-1)} \quad \text{for} \quad a \in \{n, \ldots, N\}, \quad b \in \{1, \ldots, 2N-n\},
$$

$$
\Sigma^{(-,n,-\tau'_-)}_{a,b} \equiv \left( \eta_a^{(1)} \right)^{(b-1)} \quad \text{for} \quad a \in \{N+1, \ldots, 2N-n\}, \quad b \in \{1, \ldots, 2N-n\}.
$$

Proof. Here we use the reconstruction (7.46) for $\sigma_n^- \cdots \sigma_N^-$, then we can act with the product of transfer matrices on the right $T_-$-eigenstate $|\tau'_-\rangle$ and we are left with the following computation:

$$
\mathcal{B}_-(\xi^{(0)}_N) \cdots \mathcal{B}_-(\xi^{(0)}_n)|\tau'_-\rangle.
$$

From the decomposition of $|\tau'_-\rangle$ in the $B_-$-eigenstates, we get

$$
\mathcal{B}_-(\xi^{(0)}_N) \cdots \mathcal{B}_-(\xi^{(0)}_n)|\tau'_-\rangle = \prod_{a=n}^{N} Q_{\tau'_-}(\xi^{(1)}_a)
\times \sum_{h_1, \ldots, h_{n-1}=0}^{1} \prod_{a=1}^{n-1} Q_{\tau'_-}(\xi^{(h_a)}_a) \prod_{a=n}^{N} \mathcal{B}_{-\{h_1, \ldots, h_{n-1}, 1, \ldots, 1\}}(\xi^{(0)}_a)
\times V(\eta_1^{(h_1)}, \ldots, \eta_{n-1}^{(h_{n-1})}, \eta_n^{(1)}, \ldots, \eta_N^{(1)})|h_1, \ldots, h_{n-1}, 1, \ldots, 1\rangle.
$$

Let us rewrite the $B_-$-eigenvalues in terms of the $\eta_a^{(h_a)}$,

$$
B_{-\lambda}(\lambda) = \frac{(-1)^N \kappa_- e^{\tau_-} \sinh(2\lambda - \eta)}{2^N \sinh \zeta_-} \prod_{a=1}^{N} (\cosh 2\lambda - \eta_a^{(h_a)}),
$$

and then we have

doi:10.1088/1742-5468/2012/10/P10025
\[
\prod_{a=n}^{N} B_{-\{h_1,\ldots,h_{n-1},1,\ldots,1\}}(\zeta_a^{(0)}) V(\eta_1^{(h_1)}, \ldots, \eta_{n-1}^{(h_{n-1})}, \eta_n^{(1)}, \ldots, \eta_N^{(1)})
\]

\[
= \frac{((-1)^{n-1} \kappa e^{-\sinh(\eta/2N\sinh(\zeta))} \binom{N}{n-1}}{V(\eta_1^{(0)}, \ldots, \eta_N^{(0)})} \prod_{a=n}^{N} \sinh(2(\xi_a - \eta))
\times V(\eta_1^{(h_1)}, \ldots, \eta_{n-1}^{(h_{n-1})}, \eta_n^{(0)}, \ldots, \eta_N^{(0)}, \eta_n^{(1)}, \ldots, \eta_N^{(1)}).
\] (8.8)

Using this last formula and taking the scalar product we obtain our result. \[\square\]

8.2. From the $B_+$-SOV representation

Here we consider the transfer matrices (5.28), then the following proposition holds.

**Proposition 8.2.** Let us consider the open XXZ chain with transfer matrix $\mathcal{T}_+(\lambda)$ (5.28) and let $\langle \tau_+ |$ and $| \tau'_+ \rangle$ be a generic couple of left and right $\mathcal{T}_+$-eigenstates, then we have

\[
\langle \tau_+ | \sigma^1 \cdots \sigma^N | \tau'_+ \rangle = \frac{((-1)^{N_n+1} e^{\tau_+} \sinh(\eta/2N) \sinh(\zeta_+))^n}{V(\eta_1^{(1)}, \ldots, \eta_N^{(1)})} \times \prod_{1 \leq a < b \leq n} \sinh(\xi_a + \xi_b + \eta) \sinh(\xi_a + \xi_b) \det_{N+n} \| \Sigma_{a,b}^{(+,n,\tau_+,\tau'_+)} \|
\times \prod_{a=1}^{n} \frac{\hat{d}_-(-\zeta_0^{(0)} \tau'_+ (\zeta_0^{(0)})) Q_{\tau_+} (\zeta_0^{(0)}) Q_{\tau'_+} (\zeta_0^{(0)}) \sinh(2\xi_a)}{\hat{d}_-(\zeta_0^{(0)}) \det_\eta \bar{U}_+ (\xi_a)},
\] (8.9)

where $\| \Sigma_{a,b}^{(+,n,\tau_+,\tau'_+)} \|$ is the $(N+n) \times (N+n)$ matrix of elements,

\[
\Sigma_{a,b}^{(+,n,\tau_+,\tau'_+)} \equiv \left( \eta_a^{(1)} \right)^{(b-1)} \text{ for } a \in \{1, \ldots, n\}, \quad b \in \{1, \ldots, N+n\},
\] (8.10)

\[
\Sigma_{a,b}^{(+,n,\tau_+,\tau'_+)} \equiv \left( \eta_a^{(0)} \right)^{(b-1)} \text{ for } a \in \{n+1, \ldots, 2n\}, \quad b \in \{1, \ldots, N+n\},
\] (8.11)

\[
\Sigma_{a,b}^{(+,n,\tau_+,\tau'_+)} \equiv \mathcal{M}_{a-n,b}^{(\tau_+,\tau'_+)} \text{ for } a \in \{2n+1, \ldots, N+n\}, \quad b \in \{1, \ldots, N+n\}.
\] (8.12)

**Proof.** Here we use the reconstruction (7.47) for $\sigma^1 \cdots \sigma^N$, then we can act with the product of transfer matrices on the right $\mathcal{T}_+$-eigenstate $| \tau'_+ \rangle$ and we are left with the following computation:

\[
B_+ (\zeta_0^{(0)}) \cdots B_+ (\zeta_n^{(0)}) | \tau'_+ \rangle.
\] (8.13)

From the decomposition of $| \tau'_+ \rangle$ in the $B_+$-eigenstates, we get

\[
B_+ (\zeta_0^{(1)}) \cdots B_+ (\zeta_n^{(1)}) | \tau'_+ \rangle = \prod_{a=1}^{n} Q_{\tau_+} (\zeta_a^{(1)})
\times \sum_{h_n+1,\ldots,h_n=0}^{N} \prod_{a=n+1}^{N} Q_{\tau'_+} (\zeta_a^{(h_a)}) \prod_{a=1}^{n} B_+ (0, \ldots, 0, h_n+1, \ldots, h_n) (\zeta_a^{(1)})
\times V(\eta_1^{(0)}, \ldots, \eta_n^{(0)}, \eta_n^{(h_{n+1})}, \ldots, \eta_N^{(h_n)}) |0, \ldots, 0, h_n+1, \ldots, h_N\rangle.
\] (8.14)
Let us rewrite the $B_+\text{-eigenvalues}$ in terms of the $\eta^{(h_a)}_a$, 

$$B_+\text{-eigenvalues} \equiv \frac{\kappa_+ e^{\tau_+} \sinh(2\lambda + \eta)}{2^N \sinh \zeta} \prod_{a=1}^{N} (\cosh 2\lambda - \eta^{(h_a)}_a),$$  \hspace{1cm} (8.15)  

and then we have 

$$\prod_{a=n}^{N} B_+\{0, \ldots, 0, h_{n+1}, \ldots, h_N\}(\zeta^{(1)}_a) V(\eta^{(0)}_1, \ldots, \eta^{(0)}_n, \eta^{(h_{n+1})}_{n+1}, \ldots, \eta^{(h_N)}_N)$$ 

$$= \frac{((-1)^N \kappa_+ e^{\tau_+} \sinh \eta/2^N \sinh \zeta)}{V(\eta^{(1)}_1, \ldots, \eta^{(1)}_n)} \prod_{a=n}^{N} \sinh 2\zeta_a \sinh 2(\zeta_a + \eta)$$ 

$$\times V(\eta^{(1)}_1, \ldots, \eta^{(1)}_n, \eta^{(0)}_1, \ldots, \eta^{(0)}_n, \eta^{(h_{n+1})}_{n+1}, \ldots, \eta^{(h_N)}_N).$$ \hspace{1cm} (8.16)  

Using this last formula and taking the scalar product we obtain our result. \hfill \Box  

9. Conclusion and outlook  

We have analysed the integrable quantum models associated with the transfer matrices corresponding to one general non-diagonal and one diagonal or triangular boundary matrices. For these integrable quantum models, defining in the homogeneous limit the open spin-1/2 XXZ quantum chains in the same class of non-diagonal boundary matrices, we have obtained, in theorems 5.1 and 5.2, the complete SOV-characterization of the transfer matrix eigenvalues and eigenstates, the proof of the simplicity of the spectrum and, in proposition 6.1, determinant formulae of $N \times N$ matrices for the scalar products of separate states. Finally, in propositions 8.1 and 8.2, the matrix elements of a class of quasi-local operators have been characterized on the transfer matrix eigenstates in determinant form by using the reconstruction of these operators by Sklyanin’s quantum separate variables provided in proposition 7.3. The relevance of these findings in the framework of non-equilibrium systems like the partial asymmetric simple exclusion processes (PASEP) will be described in our next two papers where further matrix elements of quasi-local operators will be characterized.  

Let us recall that in the literature of quantum integrable models there exist different applications of separation of variable methods for computing the matrix elements of local operators. An important example is presented in Smirnov’s paper [56], where determinant formulae for the matrix elements of a conjectured basis of local operators have been derived in Sklyanin’s SOV framework for the quantum integrable Toda chain [39]. There is a strong analogy among Smirnov’s formulae, those that we have here derived and more generally those which appear in the papers [1, 2, 49]. The main differences in all these formulae are due to model dependent features, like the nature of the spectrum of the quantum separate variables. In fact, it is worth citing also the results of [60] on the form factors of the restricted sine-Gordon at the reflectionless points in the S-matrix formulation [61]–[64]. The form factors there derived\(^\text{19}\) can be represented once again as

\(^{19}\text{Note that recently in [65] these results have been connected to those obtained by the introduction of a fermionic basis of quasi-local operators in the infinite volume limit of the XXZ spin-1/2 chain.}\)
determinants. The connection with SOV emerges on the basis of a semi-classical analysis, used as a tool to overcome the problem\(^{20}\) of the local field identification.

Let us comment that in this paper we have followed a different approach for the reconstruction of local operators w.r.t. that used in [15]. Apart from the different framework, ABA in [15] and SOV in this paper, we have decided to reconstruct local operators directly by the quantum separate variables of the 6-vertex reflection algebra and not in terms of those of the 6-vertex Yang–Baxter algebra. The main motivation to do this is related to the increased complexity of the functional relations among the generators of these two algebras in the general non-diagonal cases which makes computation of the action of the quantum separate variables of the 6-vertex Yang–Baxter algebra more complicated than the eigenstates of the 6-vertex reflection algebra transfer matrices. Our current approach is of course more natural as the action of the quantum separate variables of the reflection algebra on the corresponding transfer matrix eigenstates has a simpler form. However, it is worth commenting that this reconstruction programme is not yet completed as we have so far constructed explicitly only some classes of quasi-local operators by our approach, but we believe it to be possible to use this type of reconstruction to characterize all matrix elements of local operators. Moreover, it is important to say that in this paper we have not addressed one central issue for the computation of the transfer matrix eigenvalues, i.e. the derivation of an equivalent functional equation reformulation of the SOV spectrum characterization. Indeed, it has to be noticed that the spectrum characterization here derived is not a standard one in the Bethe ansatz framework as the discrete systems of equations (5.6) and (5.30) are not of Bethe ansatz type. We will address this central issue in a future publication adapting for the present model at the roots of the unit case the approach introduced in [47]. This will lead us also to the construction of the Baxter \(Q\)-operator and to a proof of the completeness of the spectrum description by solutions to a system of Bethe ansatz type equations.

Let us point out that one important motivation of our paper is to derive form factors of local operators expressed by determinant formulae as were obtained in [2] for the 6-vertex transfer matrix with antiperiodic boundary conditions. Indeed, the knowledge of the form factors of local operators is an important step towards the complete solution of the quantum model as the form factors represent efficient numerical tools for the computation of two point correlation functions. In fact, we can rewrite correlation functions in spectral series of form factors and then we can try to use the same approach as developed in [75] in the ABA framework. This is a concrete project as also in our SOV framework it will be possible eventually to have representations for the scalar products and complete characterization of the transfer matrix spectrum in terms of solutions of a system of Bethe equations.

Finally, let us comment that the analysis developed in this paper defines the required setup to extend the results on the spectrum characterization and the scalar product formulae in the SOV framework to the most general non-diagonal spin-1/2 open \(XXZ\) and \(XYZ\) quantum chains. Indeed, the so-called Baxter’s gauge transformations can

\(^{20}\)Let us recall that this is a longstanding problem in the S-matrix formulation. The description of massive IQFTs as (superrenormalizable) perturbations of conformal field theories by relevant local fields [66, 67] has been at the origin of the attempt to classify the local field content of massive theories (the set of the solutions to the form factor equations [68, 69]) by that of the corresponding ultraviolet conformal field theories. Several results are known which confirm this characterization, see for example [70]–[74].
be used also in the reflection algebra framework to reduce the spectral problem to one analysable by SOV. In more detail, the transfer matrices of both 8-vertex and 6-vertex reflection algebras associated with the most general integrable boundary matrices can be reduced by gauge transformations to those of a dynamical 6-vertex reflection algebra of elliptic and trigonometric type, respectively, with one triangular boundary matrix. The implementation of the SOV analysis for the spectral problem of these dynamical 6-vertex systems is currently under study in collaboration with N Kitanine and it consists of the generalization to the dynamical case of the SOV results derived in this paper for the standard reflection algebra.

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