Density fluctuations in the presence of spinodal instabilities

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Density fluctuations resulting from spinodal decomposition in a non-equilibrium first-order chiral phase transition are explored. We show that such instabilities generate divergent fluctuations of conserved charges along the isothermal spinodal lines appearing in the coexistence region. Thus, divergent density fluctuations could be a signal not only for the critical end point but also for the first order phase transition expected in strongly interacting matter. We also compute the mean-field critical exponent at the spinodal lines. Our analysis is performed in the mean-field approximation to the NJL model formulated at finite temperature and density. However, our main conclusions are expected to be generic and model independent.

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One of the central questions addressed in the context of QCD is the phase structure and the phase diagram of strongly interacting matter at finite temperature and baryon number density [1]. Based on calculations in effective models and on universality arguments one finds that the order of the transition from the hadronic to the quark gluon plasma phase depends on the number of quark flavors and on the value of the quark masses [2, 3, 4, 5, 6, 7, 8, 9, 10]. For physical values of the parameters one expects that the transition at high temperature and low net baryon number density is continuous. In the opposite limit of low temperature and large density the QCD phase transition is expected to be first order. This suggests that the phase diagram exhibits a critical end point (CEP), where the first order chiral transition of QCD terminates [3, 11]. Recent, however still preliminary, results obtained in first principle calculations in QCD, in lattice gauge theory (LGT), confirm the existence of such a point in the temperature $T$ and chemical potential $\mu_q$ plane [12, 13, 14].

The search for the CEP has recently attracted considerable attention. It is of particular interest to identify the position of the critical end point in the phase diagram and to study generic properties of thermodynamic quantities in its vicinity. The qualitative behavior of physical observables and their dependence on thermodynamic variables in the critical region can be studied in effective chiral models. However, a quantitative description of the thermodynamics near the phase transition can only be obtained theoretically by solving QCD ab initio in LGT or phenomenologically in the context of experimental studies of heavy ion collisions.

To locate the phase transition line in the QCD phase diagram one needs observables that are sensitive probes of the critical structure. Modifications in the magnitude of fluctuations or the corresponding susceptibilities have been suggested as a possible signal for deconfinement and chiral symmetry restoration [3, 11]. In this context, fluctuations related to conserved charges are of particular interest [15]. The fluctuations of baryon number and electric charge diverge at the critical end point while they are finite along the cross over and first order phase boundaries [3, 9, 11]. Consequently, singular fluctuations of baryon number and electric charge as well as a non monotonic behavior of these fluctuations as functions of the collision energy in heavy ion collisions have been proposed as possible signals for the QCD critical end point [3, 9, 11]. However, the finiteness of the fluctuations along the first order transition depend on the assumption that this transition appears in equilibrium.

A first order phase transition is intimately linked with the existence of a convex anomaly in the thermodynamic pressure [17], which can be uncovered only in non-equilibrium systems. There is an interval of energy density or baryon number density where the derivative of the pressure is positive, $\partial P/\partial V > 0$. This anomalous behavior characterizes a region of instability in the $(T, n_q)$-plane, where $n_q$ is the net quark number density. This region is bounded by the spinodal lines, where the pressure derivative with respect to volume vanishes. The derivative taken at constant temperature and that taken at constant entropy define the isothermal and isentropic spinodal lines, respectively.

The consequences of spinodal decomposition have been discussed in connection with the chiral/deconfinement phase transition in heavy ion collisions [17, 18, 19, 20, 21, 22]. Furthermore, spinodal decomposition plays a crucial role in the description of the 1st order nuclear liquid-gas transition in low energy nuclear collisions [17, 21]. It has also been argued that in the region of phase coexistence, a phase separation can lead to an enhancement of baryon [18] and strangeness fluctuations [19].

In this letter we consider the fluctuations of conserved charges along the spinodal lines, expected at finite net baryon density in the QCD phase diagram. We show that if the chiral phase transition is first order, then the fluctuations of the net densities of baryon number and electric charge diverge along the isothermal spinodal lines. Consequently, large fluctuations of these quantities may be a signal for a first order phase transition in the QCD.
medium. We also compute the mean-field critical exponent of the quark susceptibility at the spinodal lines and compare with that at the CEP.

In our study of fluctuations across the first order chiral phase transition we adopt the Nambu–Jona-Lasinio (NJL) model. For two quark flavors and three colors the NJL Lagrangian reads \[23, 24]:
\[
\mathcal{L} = \bar{\psi}(i\slashed{D} + m + \mu \gamma_0)\psi + G_S \left[ (\bar{\psi}\psi)^2 + \left(\bar{\psi}i\gamma_5\psi\right)^2 \right],
\]
where \(m = \text{diag}(m_u, m_d)\) is the current quark mass, \(\mu = \text{diag}(\mu_u, \mu_d)\) are the quark chemical potentials and \(\vec{\tau}\) are Pauli matrices. The strength of the interactions between the constituent quarks is controlled by the coupling \(G_S\Lambda^2 = 2.44\) with the three momentum cut-off \(\Lambda = 587.9\) MeV, introduced to regularize the ultraviolet divergences. The parameters are fixed to reproduce the mass, \(\Lambda = 587\) MeV.

In the mean field approximation the thermodynamics of the NJL model is, for an isospin symmetric system, given by the thermodynamic potential \[21, 22]:
\[
\Omega(T, \mu; M)/V = \frac{(M - m)^2}{4G_S} - 12 \int \frac{d^3\vec{p}}{(2\pi)^3} \left[ E(\vec{p}) - T \ln(1 - n^{(+)}(\vec{p}, T, \mu)) - T \ln(1 - n^{(-)}(\vec{p}, T, \mu)) \right]
\]
where \(M = m - 2G_S(\bar{\psi}\psi)\) is the dynamical quark mass, \(E(\vec{p}) = \sqrt{\vec{p}^2 + M^2}\) is the quasiparticle energy and \(n^{(\pm)}(\vec{p}, T, \mu) = \left(1 + \exp\left[(E(\vec{p}) \mp \mu)/T\right]\right)^{-1}\) are the quark/antiquark distribution functions.

The dynamical mass \(M\) is obtained self consistently from the stationarity condition \(\partial \Omega/\partial M = 0\), which implies
\[
M = m + 24G_S \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{M}{E} \left[1 - n^{(+)} - n^{(-)}\right].
\]
The relevant thermodynamic observables, the quark number density and the corresponding susceptibility, are given by
\[
n_q = -\frac{\partial \Omega}{\partial \mu}, \quad \chi_{\mu\mu} = \frac{\partial n_q}{\partial \mu}. \tag{4}
\]

The NJL model has a generic, QCD like, phase diagram. It exhibits a critical end point that separates the cross over from the first order chiral phase transition. The relevant part of the phase diagram in the \((T, n_q)\)-plane is shown in Fig. 1. If the first order phase transition takes place in equilibrium, there is a coexistence region, which ends at the critical end point. However, in a non-equilibrium first order phase transition, the system supercools/superheats and, if driven sufficiently far from equilibrium, it becomes unstable due to the convex anomaly in the thermodynamic pressure. In other words, in the coexistence region there is a range of densities and temperatures, bounded by the spinodal lines, where the spatially uniform system is mechanically unstable. The location of the spinodal lines is determined by the conditions
\[
\left(\frac{\partial P}{\partial V}\right)_T = 0 \quad \text{and} \quad \left(\frac{\partial P}{\partial V}\right)_S = 0, \tag{5}
\]
for the isothermal and isentropic spinodal lines, respectively. Both these lines are shown in Fig. 1. From the thermodynamic relation
\[
\left(\frac{\partial P}{\partial V}\right)_T = \frac{\left(\frac{\partial P}{\partial V}\right)_S}{C_V} + \frac{T}{C_V} \left[\left(\frac{\partial P}{\partial T}\right)_V\right]^2, \tag{6}
\]
it is clear that the isentropic spinodal lines are located inside the instability region bounded by the isothermal spinodals and that the two sets of lines coincide at \(T = 0\). In the mean field approximation the isothermal instabilities disappear at the CEP, where the first order transition ends, while the isentropic spinodal curves join below the CEP. We note that the isentropic spinodal lines will probably be modified considerably when fluctuations are properly included. This is because the specific heat \(C_V\) diverges at the CEP \[1\], while in the mean-field approximation it remains finite. It then follows from \(\partial P/\partial V\) that both the isothermal and the isentropic pressure derivatives in \(\partial P/\partial T\) vanish at the CEP. Consequently, in a more complete description, the isentropic spinodal lines are expected to move up in temperature and to also join at the CEP.

Since the isothermal spinodal lines join at the CEP, as shown in Fig. 1, it is natural to explore how the charge fluctuations develop when going beyond the critical end point.
fluctuations, assuming a typical wave vector $q = 100 - 300$ MeV and using the results of [25] for the collisionless regime, yields a characteristic growth time $\tau \sim 1 - 10$ fm in agreement with [26]. We note that the detailed calculations of ref. [25] show that the growth rates of the most unstable mode deep inside the spinodal region depends only weakly on the collision rate. Thus, our estimate is expected to be valid also in the intermediate and hydrodynamic regimes. Within the growth time an initial fluctuation grows by a factor $e$. Since this time is smaller than or on the order of the typical expansion time scale in relativistic heavy-ion collisions, we conclude that the spinodal instability may lead to observable fluctuations. A more quantitative estimate of the expected fluctuations in heavy-ion collisions would require a kinetic calculation, e.g. along the lines of ref. [29].

The behavior of $\chi_{\mu\mu}$ seen in Fig. 2 is a direct consequence of the thermodynamic relations

$$\left( \frac{\partial P}{\partial V} \right)_T = -\frac{n_q^2}{V} \chi_{\mu\mu},$$

$$\left( \frac{\partial P}{\partial V} \right)_S = -\frac{n_q^2}{V} \chi_{TT} - \frac{2\chi_{\mu T}}{n_q} + \left( \frac{\chi_{\mu T}}{n_q} \right)^2 \chi_{\mu\mu},$$

which connect the pressure derivatives with the susceptibilities $\chi_{xy} = -\partial^2 Q/\partial x \partial y$. Along the isothermal spinodal lines the pressure derivative in 7 vanishes. Thus, for non-vanishing density $n_q$, $\chi_{\mu\mu}$ must diverge to satisfy 7. Furthermore, since the pressure derivative $\partial P/\partial V|_T$ changes sign when crossing the spinodal line, there must be a corresponding sign change in $\chi_{\mu\mu}$, as seen in Fig. 2. Due to the linear relation between $\chi_{\mu\mu}$, the isovector susceptibility $\chi_I$ and the charge susceptibility $\chi_Q$ [27], the charge fluctuations are also divergent at the isothermal spinodal line. Thus, in heavy-ion collisions, fluctuations of the baryon number and electric charge could show enhanced fluctuations, as a signal of the spinodal decomposition. The spinodal phase separation can also lead to fluctuations in strangeness [14] and isospin densities [30].

At the isentropic spinodal line the baryon number susceptibility is in general finite. This is also true for the other susceptibilities appearing in Eq. 8. The isentropic spinodal line [4] corresponds to a zero of the numerator in 8 and of the velocity of hydrodynamic sound waves. In the hydrodynamic limit the instability sets in at the isothermal spinodal, but the growth rate becomes large only at the isentropic spinodal line [25].

In the case of an equilibrium first order phase transition, the density fluctuations do not diverge; in the coexistence region the susceptibility is a linear combination of the positive susceptibilities above and below the phase boundary. Thus, the fluctuations increase as one approaches the CEP along the first order transition and decrease again in the cross over region. This led to the prediction of a non-monotonous behavior of the fluctuations with increasing beam energy as a signal for the existence of a CEP [11, 28]. We stress that strictly...
speaking this is relevant only for the idealized situation where the first order phase transition takes place in equilibrium. In the more realistic non-equilibrium system one expects fluctuations in a larger region of the phase diagram, i.e., over a broader range of beam energies, due to the spinodal instabilities.

The critical exponent at the isothermal spinodal line is found to be $\gamma = 1/2$, with $\chi_{\mu\mu} \sim (\mu - \mu_c)^{-\gamma}$, while $\gamma = 2/3$ at the CEP, in agreement with the mean-field results [9, 16]. Thus, the singularities at the two spinodal lines conspire to yield a somewhat stronger divergence as they join at the CEP. The exponents are renormalized by fluctuations, but the smooth evolution of the singularity from the spinodal lines to the CEP, illustrated in Fig. 3, is expected to be generic.

We have shown that the net quark number fluctuations diverge at the isothermal spinodal lines of the first order chiral phase transition [31]. As the system crosses this line, it becomes unstable with respect to spinodal decomposition. The unstable region is in principle reachable in non-equilibrium systems, created e.g., in heavy ion collisions. This means that large fluctuations of the density are expected not only at the second order critical end point but also at a non-equilibrium first order transition. In fact, the signal from the first-order transition may be much stronger than that from the CEP. Model calculations suggest that the critical region of enhanced susceptibility around the CEP is fairly small [9, 16, 28], while it is large in the spinodal region, where the fluctuations appear due to the divergence of $\chi_{\mu\mu}$ and due to the mechanical instability of the system. We stress that there is a close relation between the singularities at the CEP and the spinodal lines.

The properties of different susceptibilities were obtained within the NJL model in the mean field approximation. However, the singular properties of charge susceptibilities in the presence of spinodal instabilities are quite general. They appear due to the straightforward thermodynamic relation between the pressure derivatives and charge susceptibilities. Thus, although the NJL model is non-confining, the results presented in this paper are expected to be robust on a qualitative level.

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significantly, when these effects are properly accounted for. See refs. \[1, 29\]\[31\] The divergence of $\chi_{\mu\mu}$ along the spinodal lines implies a corresponding divergence of the electric charge susceptibility $\chi_Q$ due to the relation \[27\].