Abstract—Polar codes are introduced for discrete memoryless broadcast channels. For $m$-user deterministic broadcast channels, polarization is applied to map uniformly random message bits from $m$ independent messages to one codeword while satisfying broadcast constraints. The polarization-based codes achieve rates on the boundary of the private-message capacity region for two-user noisy broadcast channels, polar implementations are presented for two information-theoretic schemes: i) Cover’s superposition codes; ii) Marton’s codes. Due to the structure of polarization, constraints on the auxiliary and channel-input distributions are identified to ensure proper alignment of polarization indices in the multi-user setting. The codes achieve rates on the capacity boundary of a few classes of broadcast channels (e.g., binary-input stochastically degraded). The complexity of encoding and decoding is $O(n \log n)$ where $n$ is the block length. In addition, polar code sequences obtain a stretched-exponential decay of $O(2^{-n^\beta})$ of the average block error probability where $0 < \beta < \frac{1}{2}$.

Index Terms—Polar Codes, Deterministic Broadcast Channel, Cover’s Superposition Codes, Marton’s Codes.

I. INTRODUCTION

INtroduced by T. M. Cover in 1972, the broadcast problem consists of a single source transmitting $m$ independent private messages to $m$ receivers through a single discrete, memoryless, broadcast channel (DM-BC) [1]. The private-message capacity region is known if the channel structure is deterministic, degraded, less-noisy, or more-capable [2]. For general classes of DM-BCs, there exist inner bounds such as Marton’s inner bound [3] and outer bounds such as the Nair-El-Gamal outer bound [4]. One difficult aspect of the broadcast problem is to design an encoder which maps $m$ independent messages to a single codeword of symbols which are transmitted simultaneously to all receivers. Several codes relying on random binning, superposition, and Marton’s strategy have been analyzed in the literature (see e.g., the overview in [5]).

A. Overview of Contributions

The present paper focuses on low-complexity codes for broadcast channels based on polarization methods. Polar codes were invented originally by Arıkan and were shown to achieve the capacity of binary-input, symmetric, point-to-point channels with $O(n \log n)$ encoding and decoding complexity where $n$ is the code length [6]. In this paper, we obtain the following results.

- Polar codes for deterministic, linear and non-linear, binary-output, $m$-user DM-BCs (cf. [7]). The capacity-achieving broadcast codes implement low-complexity random binning, and are related to polar codes for other multi-user scenarios such as Slepian-Wolf distributed source coding [8], [9], and multiple-access channel (MAC) coding [10]. For deterministic DM-BCs, the polar transform is applied to channel output variables. Polarization is useful for shaping uniformly random message bits from $m$ independent messages into non-equiprobable codeword symbols in the presence of hard broadcast constraints. As discussed in Section I-B1 and referenced in [11–13], it is difficult to design low-complexity parity-check (LDPC) codes or belief propagation algorithms for the deterministic DM-BC due to multi-user broadcast constraints.
- Polar codes for general two-user DM-BCs based on Cover’s superposition coding strategy. In the multi-user setting, constraints on the auxiliary and channel-input distributions are placed to ensure alignment of polarization indices. The achievable rates lie on the boundary of the capacity region for certain classes of DM-BCs such as binary-input stochastically degraded channels.
- Polar codes for general two-user DM-BCs based on Marton’s coding strategy. In the multi-user setting, due to the structure of polarization, constraints on the auxiliary and channel-input distributions are identified to ensure alignment of polarization indices. The achievable rates lie on the boundary of the capacity region for certain classes of DM-BCs such as binary-input semi-deterministic channels.
- For the above broadcast polar codes, the asymptotic decay of the average error probability under successive cancelation decoding at the broadcast receivers is established to be $O(2^{-n^\beta})$ where $0 < \beta < \frac{1}{2}$. The error probability is analyzed by averaging over polar code ensembles. In addition, properties such as the chain rule of the Kullback-Leibler divergence between discrete probability.
measures are exploited.

Throughout the paper, for different broadcast coding strategies, a systems-level block diagram of the communication channel and polar transforms is provided.

B. Relation to Prior Work

1) Deterministic Broadcast Channels: The deterministic broadcast channel has received considerable attention in the literature (e.g. due to related extensions such as secure broadcast, broadcasting with side information, and index coding [14], [15]). Several practical codes have been designed. For example, the authors of [11] propose sparse linear coset codes to emulate random binning and survey propagation to enforce broadcast channel constraints. In [12], the authors propose enumerative source coding and Luby-Transform codes for deterministic DM-BCs specialized to interference-management scenarios. Additional research includes reinforced belief propagation with non-linear coding [13]. To our knowledge, polarization-based codes provide provable guarantees for achieving rates on the capacity-boundary in the general case.

2) Polar Codes for Multi-User Settings: Subsequent to the derivation of channel polarization in [6] and the refined rate of polarization in [16], polarization methods have been extended to analyze multi-user information theory problems. In [10], a joint polarization method is proposed for m-user MACs with connections to matroid theory. Polar codes were extended for several other multi-user settings: arbitrarily-permutated parallel channels [17], degraded relay channels [18], cooperative relaying [19], and wiretap channels [20]–[22]. In addition, several binary multi-user communication scenarios including the Gelfand-Pinsker problem, and Wyner-Ziv problem were analyzed in [23] Chapter 4. Polar codes for lossless and lossy source compression were investigated respectively in [8] and [24]. In [8], source polarization was extended to the Slepian-Wolf problem involving distributed sources. The approach is based on an “onion-peeling” encoding of sources, whereas a joint encoding is proposed in [25]. In [9], a unified approach is provided for the Slepian-Wolf problem based on generalized monotone chain rules of entropy. To our knowledge, the design of polarization-based broadcast codes is relatively new.

3) Binary vs. q-ary Polarization: The broadcast codes constructed in the present paper for DM-BCs are based on polarization for binary random variables. However, in extending to arbitrary alphabet sizes, a large body of prior work exists and has focused on generalized constructions and kernels [26], and generalized polarization for q-ary random variables and q-ary channels [27]–[30]. The reader is also referred to the monograph in [31] containing a clear overview of polarization methods.

C. Notation

An index set \( \{1, 2, \ldots, m\} \) is abbreviated as \([m]\). An \( m \times n \) matrix array of random variables is comprised of variables \( Y_i(j) \) where \( i \in [m] \) represents the row and \( j \in [n] \) the column. The notation \( Y_{1:k}^{k+\ell} \doteq \{Y_i(k), Y_i(k+1), \ldots, Y_i(\ell)\} \) for \( k \leq \ell \). When clear by context, the term \( Y_i^n \) represents \( Y_1^1:n \). In addition, the notation for the random variable \( Y_i^n \) is used interchangeably with \( Y_i^n \). The notation \( f(n) = \mathcal{O}(g(n)) \) means that there exists a constant \( k \) such that \( f(n) \leq k g(n) \) for sufficiently large \( n \). For a set \( S \), \( \text{clo}(S) \) represents set closure, and \( \text{co}(S) \) the convex hull operation over set \( S \). Let \( h_b(x) = -x \log_2(x) - (1-x) \log_2(1-x) \) denote the binary entropy function. Let \( a \ast b \doteq (1-a) b + a(1-b) \).

II. MODEL

Definition 1 (Discrete, Memoryless Broadcast Channel): The discrete memoryless broadcast channel (DM-BC) with \( m \) broadcast receivers consists of a discrete input alphabet \( \mathcal{X} \), discrete output alphabets \( \mathcal{Y}_i \) for \( i \in [m] \), and a conditional distribution \( P_{Y_1,Y_2,\ldots,Y_m|X}(y_1,y_2,\ldots,y_m|x) \) where \( x \in \mathcal{X} \) and \( y_i \in \mathcal{Y}_i \).

Definition 2 (Private Messages): For a DM-BC with \( m \) broadcast receivers, there exist \( m \) private messages \( \{W_i\}_{i \in [m]} \) such that each message \( W_i \) is composed of \( n R_i \) bits and \( \{W_1, W_2, \ldots, W_m\} \) is uniformly distributed over \([2^{n R_1}] \times [2^{n R_2}] \times \cdots \times [2^{n R_m}] \).

Definition 3 (Channel Encoding and Decoding): For the DM-BC with independent messages, let the vector of rates \( \bar{R} \doteq [ R_1 \ R_2 \ \ldots \ R_m ]^T \). An \((\bar{R},n)\) code for the DM-BC consists of one encoder \( x^n : [2^{n R_1}] \times [2^{n R_2}] \times \cdots \times [2^{n R_m}] \to \mathcal{X}^n \), and \( m \) decoders specified by \( \hat{W}_i : \mathcal{Y}_i^n \to [2^{n R_i}] \) for \( i \in [m] \). Based on received observations \( \{Y_i(j)\}_{j \in [n]} \), each decoder outputs a decoded message \( \hat{W}_i \).

Definition 4 (Average Probability of Error): The average probability of error \( P_e^{(n)} \) for a DM-BC code is defined to be the probability that the decoded message at all receivers is not equal to the transmitted message,

\[
P_e^{(n)} = \mathbb{P} \left\{ \bigvee_{i=1}^m \hat{W}_i (\{Y_i(j)\}_{j \in [n]}) \neq W_i \right\}.
\]

Definition 5 (Private-Message Capacity Region): If there exists a sequence of \((\bar{R},n)\) codes with \( P_e^{(n)} \to 0 \), then the rates \( \bar{R} \in \mathbb{R}_+^m \) are achievable. The private-message capacity region is the closure of the set of achievable rates.

III. DETERMINISTIC BROADCAST CHANNELS

Definition 6 (Deterministic DM-BC): Define \( m \) deterministic functions \( f_i(x) : \mathcal{X} \to \mathcal{Y}_i \) for \( i \in [m] \). The deterministic
DM-BC with \( m \) receivers is defined by the following conditional distribution

\[
P_{Y_1, Y_2, \ldots, Y_m|X}(y_1, y_2, \ldots, y_m|x) = \prod_{i=1}^m \delta[y_i = f_i(x)].
\]

**A. Capacity Region**

**Proposition 1 (Marton [32], Pinsker [33]):** The capacity region of the deterministic DM-BC includes those rate-tuples \( \bar{R} \in \mathbb{R}_+^m \) in the region

\[
\mathcal{E}_{DET-BC} \triangleq \text{co} \left( \bigcup_{X, (Y_i)_{i \in [m]}} \mathfrak{R}(X, (Y_i)_{i \in [m]}) \right),
\]

where the polyhedral region \( \mathfrak{R}(X, (Y_i)_{i \in [m]}) \) is given by

\[
\mathfrak{R} \triangleq \left\{ \bar{R} \left| \sum_{i \in S} R_i < H(Y_{\pi(i)}|Y_{\pi(k)})_{k=1:i-1} \right. \forall S \subseteq [m] \right\}.
\]

The union in Eqn. (2) is over all random variables \( X, Y_1, Y_2, \ldots, Y_m \) with joint distribution induced by \( P_X(x) \) and \( Y_i = f_i(X) \).

**Example 1 (Blackwell Channel):** In Figure 1 the Blackwell channel is depicted with \( \mathcal{X} = \{0, 1, 2\} \) and \( \mathcal{Y}_i = \{0, 1\} \). For any fixed distribution \( P_X(x) \), it is seen that \( P_{Y_1, Y_2}(y_1, y_2) \) has zero mass for the pair \((1, 0)\). Let \( \alpha \in [\frac{1}{2}, \frac{2}{3}] \). Due to the symmetry of this channel, the capacity region is the union of two regions,

\[
\{(R_1, R_2) : R_1 \leq h_b(\alpha), R_2 \leq h_b(\frac{\alpha}{2}), R_1 + R_2 \leq h_b(\alpha + \alpha), \}
\]

\[
\{(R_1, R_2) : R_1 \leq h_b(\frac{\alpha}{2}), R_2 \leq h_b(\alpha), R_1 + R_2 \leq h_b(\alpha + \alpha), \}
\]

where the first region is achieved with input distribution \( P_X(0) = P_X(1) = \frac{1}{2} \), and the second region is achieved with \( P_X(1) = P_X(2) = \frac{1}{2} \) [2]. The sum rate is maximized for a uniform input distribution which yields a pentagonal achievable rate region: \( R_1 \leq h_b(\frac{1}{3}), R_2 \leq h_b(\frac{1}{3}), R_1 + R_2 \leq \log_2 3 \). Figure 1 illustrates the capacity region.

**B. Main Result**

**Theorem 1 (Polar Code for Deterministic DM-BC):** Consider an \( m \)-user deterministic DM-BC with arbitrary discrete input alphabet \( \mathcal{X} \), and binary output alphabets \( Y_i \in \{0, 1\} \). Fix input distribution \( P_X(x) \) where \( x \in \mathcal{X} \) and constant \( 0 < \beta < \frac{1}{2} \). Let \( \pi : [m] \to [m] \) be a permutation on the index set of receivers. Let the vector

\[
\bar{R} \triangleq \left[ R_{\pi(1)} \ R_{\pi(2)} \ldots R_{\pi(m)} \right]^T.
\]

There exists a sequence of polar broadcast codes over \( n \) channel uses which achieves rates \( \bar{R} \) where the rate for receiver \( \pi(i) \in [m] \) is bounded as

\[
0 \leq R_{\pi(i)} < H(Y_{\pi(i)}|Y_{\pi(k)})_{k=1:i-1}.
\]

The average error probability of this code sequence decays as \( P_e(n) = O(2^{-n^\beta}) \). The complexity of encoding and decoding is \( O(n \log n) \).

**Remark 1:** To prove the existence of low-complexity broadcast codes, a successive randomized protocol is introduced in Section V-A which utilizes \( o(n) \) bits of randomness at the encoder. A deterministic encoding protocol is also presented.

**Remark 2:** The achievable rates for a fixed input distribution \( P_X(x) \) are the vertex points of the polyhedral rate region defined in (3). To achieve non-vertex points, the following coding strategies could be applied: time-sharing; rate-splitting for the deterministic DM-BC [34]; polarization by Arıkan utilizing generalized chain rules of entropy [9]. For certain input distributions \( P_X(x) \), as illustrated in Figure 1 for the Blackwell channel, a subset of the achievable vertex points lie on the capacity boundary.

**Remark 3:** Polarization of channels and sources extends to \( q \)-ary alphabets (see e.g. [27]). Similarly, it is entirely possible to extend Theorem 1 to include DM-BCs with \( q \)-ary output alphabets.
IV. OVERVIEW OF POLARIZATION METHOD
FOR DETERMINISTIC DM-BCS

For the proof of Theorem 1, we utilize binary polarization theorems. By contrast to polarization for point-to-point channels, in the case of deterministic DM-BCs, the polar transform is applied to the output random variables of the channel.

A. Polar Transform

Consider an input distribution \( P_X(x) \) to the deterministic DM-BC. Over \( n \) channel uses, the input random variables to the channel are given by

\[
X^{1:n} = \{X^1, X^2, \ldots, X^n\},
\]

where \( X^j \sim P_X \) are independent and identically distributed (i.i.d.) random variables. The channel output variables are given by \( Y_i(j) = f_i(X(j)) \) where \( f_i(\cdot) \) are the deterministic functions to each broadcast receiver. Denote the random matrix of channel output variables by

\[
Y \triangleq \begin{bmatrix}
Y_1^1 & Y_2^1 & Y_3^1 & \cdots & Y_n^1 \\
Y_1^2 & Y_2^2 & Y_3^2 & \cdots & Y_n^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
Y_1^n & Y_2^n & Y_3^n & \cdots & Y_m^n
\end{bmatrix},
\]

where \( Y \in \mathbb{F}_2^{n \times n} \). For \( n = 2^j \) and \( \ell \geq 1 \), the polar transform is defined as the following invertible linear transformation,

\[
U = YG_n
\]

where \( G_n \triangleq \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \otimes \log_2 n \). The matrix \( G_n \in \mathbb{F}_2^{n \times n} \) is formed by multiplying a matrix of successive Kronecker matrix-products (denoted by \( \otimes \)) with a bit-reversal matrix \( B_n \) introduced by Arikan [8]. The polarized random matrix \( U \in \mathbb{F}_2^{n \times n} \) is indexed as

\[
U \triangleq \begin{bmatrix}
U_1^1 & U_2^1 & U_3^1 & \cdots & U_n^1 \\
U_1^2 & U_2^2 & U_3^2 & \cdots & U_n^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
U_1^n & U_2^n & U_3^n & \cdots & U_m^n
\end{bmatrix}.
\]

B. Joint Distribution of Polarized Variables

Consider the channel output distribution \( P_{Y_1Y_2\ldots Y_m} \) of the deterministic DM-BC induced by input distribution \( P_X(x) \). The \( j \)-th column of the random matrix \( Y \) is distributed as \( (Y_1^j, Y_2^j, \ldots, Y_m^j) \sim P_{Y_1Y_2\ldots Y_m} \). Due to the memoryless property of the channel, the joint distribution of all output variables is

\[
P_{Y_1^nY_2^n\ldots Y_m^n}(y_1^n, y_2^n, \ldots, y_m^n) = \prod_{j=1}^{n} P_{Y_1Y_2\ldots Y_m}(y_1^j, y_2^j, \ldots, y_m^j).
\]

The joint distribution of the matrix variables in \( Y \) is characterized easily due to the i.i.d. structure. The polarized random matrix \( U \) does not have an i.i.d. structure. However, one way to define the joint distribution of the variables in \( U \) is via the polar transform equation (5). An alternate representation is via a decomposition into conditional distributions as follows.

\[
P_{U_1^nU_2^n\ldots U_m^n}(u_1^n, u_2^n, \ldots, u_m^n) = \prod_{i=1}^{m} \prod_{j=1}^{n} P(u_i(j)\mid u_1^{1:j-1}, \{u_k^{1:n}\}_{k\in[1:i-1]}).
\]

As derived by Arikan in [8] and summarized in Section IV-E, the conditional probabilities in (8) and associated likelihoods may be computed using a dynamic programming method which “divides-and-conquers” the computations efficiently.

C. Polarization of Conditional Entropies

Proposition 2 (Polarization [8]): Consider the pair of random matrices \( (Y, U) \) related through the polar transformation in (5). For \( i \in [m] \) and any \( \epsilon \in (0, 1) \), define the set of indices

\[
A_i^{(\epsilon)} \triangleq \left\{ j \in [n] : H(U_i(j)\mid U_{1:j-1}^{1:1}, \{Y_k^{1:n}\}_{k\in[1:i-1]} \geq 1 - \epsilon \right\}.
\]

Then in the limit as \( n \to \infty \),

\[
\frac{1}{n} \left| A_i^{(\epsilon)} \right| \to H(Y_i\mid Y_1Y_2\ldots Y_{i-1}).
\]

1The abbreviated notation of the form \( P(a|b) \) which appears in [8] indicates \( P_{A|B}(a|b) \), i.e. the conditional probability \( P\{A = a|B = b\} \) where \( A \) and \( B \) are random variables.
For sufficiently large $n$, Theorem 2 establishes that there exist approximately $nH(Y_1 Y_2 \cdots Y_{n-1})$ indices per row $i \in [m]$ of random matrix $U$ for which the conditional entropy is close to 1. The total number of indices in $U$ for which the conditional entropy terms polarize to 1 is approximately $nH(Y_1 Y_2 \cdots Y_m)$. The polarization phenomenon is illustrated in Figure 2.

Remark 4: Since the polar transform $G_n$ is invertible, $\{U^{1:n}_k\}_{k \in [1:n-1]}$ are in one-to-one correspondence with $\{Y^{1:n}_k\}_{k \in [1:n-1]}$. Therefore the conditional entropies $H(U_i(j)|U^{1:j-1}_i, \{Y^{1:n}_k\}_{k \in [1:n-1]})$ also polarize to 0 or 1.

D. Rate of Polarization

The Bhattacharyya parameter of random variables is closely related to the conditional entropy. The parameter is useful for characterizing the rate of polarization.

Definition 7 (Bhattacharyya Parameter): Let $(T, V) \sim P_{T,V}$ where $T \in \{0, 1\}$ and $V \in \mathcal{V}$ where $\mathcal{V}$ is an arbitrary discrete alphabet. The Bhattacharyya parameter $Z(T|V) \in [0, 1]$ is defined

$$Z(T|V) = 2 \sum_{v \in \mathcal{V}} P_V(v) \sqrt{P_{T|V}(0|v) P_{T|V}(1|v)}. \quad (11)$$

As shown in Lemma 11 of Appendix A, $Z(T|V) \to 1$ implies $H(T|V) \to 1$, and similarly $Z(T|V) \to 0$ implies $H(T|V) \to 0$ for a binary random variable. Based on the Bhattacharyya parameter, the following theorem specifies sets $\mathcal{M}^{(n)}_t \subseteq [n]$ that will be called message sets.

Proposition 3 (Rate of Polarization): Consider the pair of random matrices $(Y, U)$ related through the polar transformation in 5. Fix constants $0 < \beta < \frac{1}{2}$, $\tau > 0$, $i \in [m]$. Let $\delta_n = 2^{-\beta}$ be the rate of polarization. Define the index sets

$$\mathcal{M}^{(n)}_i \triangleq \left\{ j \in [n] : Z(U_i(j)|U^{1:j-1}_i, \{Y^{1:n}_k\}_{k \in [1:i-1]}) \geq 1 - \delta_n \right\}. \quad (12)$$

Then there exists an $N_o = N_o(\beta, \tau)$ such that

$$\frac{1}{n} |\mathcal{M}^{(n)}_i| \geq H(Y_1 Y_2 \cdots Y_{i-1}) - \tau, \quad (13)$$

for all $n > N_o$.

The proposition is established via the Martingale Convergence Theorem by defining a super-martingale with respect to the Bhattacharyya parameters [6] [8]. The rate of polarization is characterized by Arikan and Telatar in 16.

Remark 5: The message sets $\mathcal{M}^{(n)}_i$ are computed “offline” only once during a code construction phase. The sets do not depend on the realization of random variables. In the following Section V-E a Monte Carlo sampling approach for estimating Bhattacharyya parameters is reviewed. Other highly efficient algorithms are known in the literature for finding the message indices (see e.g. Tal and Vardy [35]).

E. Estimating Bhattacharyya Parameters

As shown in Lemma 11 in Appendix A one way to estimate the Bhattacharyya parameter $Z(T|V)$ is to sample from the distribution $P_{T,V}(t, v)$ and evaluate $E_{T,V} \sqrt{\varphi(T, V)}$. The function $\varphi(t, v)$ is defined based on likelihood ratios

$$L(v) \triangleq \frac{P_{T|V}(0|v)}{P_{T|V}(1|v)},$$

$$L^{-1}(v) \triangleq \frac{P_{T|V}(1|v)}{P_{T|V}(0|v)}.$$  

Similarly, to determine the indices in the message sets $\mathcal{M}^{(n)}_i$ defined in Proposition 3 the Bhattacharyya parameters $Z(U_i(j)|U^{1:j-1}_i, \{Y^{1:n}_k\}_{k \in [1:i-1]})$ must be estimated efficiently. For $n \geq 2$, define the likelihood ratio

$$L^{(i,j)}(U_i(j)|U^{1:j-1}_i, \{Y^{1:n}_k\}_{k \in [1:i-1]}) \triangleq \frac{P(U_i(j) = 0 U^{1:j-1}_i = u^{1:j-1}_i, \{Y^{1:n}_k = y^{1:n}_k\}_{k \in [1:i-1]})}{P(U_i(j) = 1 U^{1:j-1}_i = u^{1:j-1}_i, \{Y^{1:n}_k = y^{1:n}_k\}_{k \in [1:i-1]})}.$$  

The dynamic programming method given in 8 allows for a recursive computation of the likelihood ratio. Define the following sub-problems

$$\Xi_1 = L^{(i,j)}(u^{1:2j-2}_i \oplus u^{1:2j-2}_i, \{y^{1:n}_k\}_{k \in [1:i-1]}),$$

$$\Xi_2 = L^{(i,j)}(u^{1:2j-2}_i, \{y^{1:n}_k\}_{k \in [1:i-1]}),$$

where the notation $u^{1:2j-2}_i \oplus u^{1:2j-2}_i$ represents the odd and even indices respectively of the sequence $u^{1:2j-2}_i$. The recursive computation of the likelihoods is characterized by

$$L^{(i,2j-1)}(u^{1:2j-2}_i, \{y^{1:n}_k\}_{k \in [1:i-1]}) = \Xi_1 \Xi_2 + 1 \Xi_1 + \Xi_2,$$

$$L^{(i,2j)}(u^{1:2j-1}_i, \{y^{1:n}_k\}_{k \in [1:i-1]}) = (\Xi_1)^\gamma \Xi_2,$$

where $\gamma = 1$ if $u_i(2j - 1) = 0$ and $\gamma = -1$ if $u_i(2j - 1) = 1$. In the above recursive computations, the base case is for sequences of length $n = 2$.

V. PROOF OF THEOREM 11

The proof of Theorem 11 is based on binary polarization theorems as discussed in Section V. The random coding arguments of C. E. Shannon prove the existence of capacity-achieving codes for point-to-point channels. Furthermore, random binning and joint-typicality arguments suffice to prove the existence of capacity-achieving codes for the deterministic DM-BC. However, it is shown in this section that there exist capacity-achieving polar codes for the binary-output deterministic DM-BC.
A. Broadcast Code Based on Polarization

The ordering of the receivers’ rates in $\hat{R}$ is arbitrary due to symmetry. Therefore, let $\pi(i) = i$ be the identity permutation which denotes the successive order in which the message bits are allocated for each encoder. The encoder must map $m$ independent messages $(W_1, W_2, \ldots, W_m)$ uniformly distributed over $[2^nR_1] \times [2^nR_2] \times \cdots \times [2^nR_m]$ to a codeword $x^n \in \mathcal{X}^n$. To construct a codeword for broadcasting $m$ independent messages, the following binary sequences are formed at the encoder: $u_{i,1}^n, u_{i,2}^n, \ldots, u_{i,m}^n$. To determine a particular bit $u_{i,j}$ in the binary sequence $u_{i}^n$, if $j \in \mathcal{M}_i(n)$, the bit is selected as a uniformly distributed message bit intended for receiver $i \in [m]$. As defined in Proposition 3 the message set $\mathcal{M}_i(n)$ represents those indices for bits transmitted to receiver $i$. The remaining non-message indices in the binary sequence $u_{i}^n$ for each user $i \in [m]$ are computed either according to a deterministic or random mapping.

1) Deterministic Mapping: Consider a class of deterministic boolean functions indexed by $i \in [m]$ and $j \in [n]$:

$$
\psi^{(i,j)} : \{0,1\}^{n(\max(0, i-1)) + j - 1} \rightarrow \{0,1\}.
$$

As an example, consider the deterministic boolean function based on the maximum a posteriori polar coding rule.

$$
\psi^{(i,j)}_{MAP} (u_{i,1}^{1:j-1}, \{y_{k}^{1:n}\}_{k \in [i:i-1]}) \triangleq \arg\max_{u \in \{0,1\}} \{ \mathbb{P}\left( U_i(j) = u \left| U_i^{1:j-1} = u_{i,1}^{1:j-1}, \{Y_k^{1:n} = y_{k}^{1:n}\}_{k \in [i:i-1]} \right. \right) \}.
$$

(16)

2) Random Mapping: Consider a class of random boolean functions indexed by $i \in [m]$ and $j \in [n]$:

$$
\psi^{(i,j)} : \{0,1\}^{n(\max(0, i-1)) + j - 1} \rightarrow \{0,1\}.
$$

As an example, consider the random boolean function

$$
\psi^{(i,j)}_{RAND} (u_{i,1}^{1:j-1}, \{y_{k}^{1:n}\}_{k \in [i:i-1]}) \triangleq \begin{cases} 
0, & \text{w.p. } \lambda_0(u_{i,1}^{1:j-1}, \{y_{k}^{1:n}\}_{k \in [i:i-1]}), \\
1, & \text{w.p. } 1 - \lambda_0(u_{i,1}^{1:j-1}, \{y_{k}^{1:n}\}_{k \in [i:i-1]}),
\end{cases}
$$

(18)

where

$$
\lambda_0(u_{i,1}^{1:j-1}, \{y_{k}^{1:n}\}_{k \in [i:i-1]}) \triangleq \mathbb{P}\left( U_i(j) = 0 \left| U_i^{1:j-1} = u_{i,1}^{1:j-1}, \{Y_k^{1:n} = y_{k}^{1:n}\}_{k \in [i:i-1]} \right. \right).
$$

The random boolean function $\psi^{(i,j)}_{RAND}$ may be thought of as a vector of Bernoulli random variables indexed by the input to the function. Each Bernoulli random variable of the vector has a fixed probability of being one or zero that is well-defined.

3) Mapping From Messages To Codeword: The binary sequences $u_{i}^{1:n}$ for $i \in [m]$ are formed successively bit by bit. If $j \in \mathcal{M}_i(n)$, then the bit $u_{i,j}$ is one message bit from the uniformly distributed message $W_i$ intended for user $i$. If $j \notin \mathcal{M}_i(n)$, $u_{i,j} = \psi^{(i,j)}_{MAP}(u_{i,1}^{1:j-1}, \{y_{k}^{1:n}\}_{k \in [i:i-1]})$ in the case of a deterministic mapping, or $u_{i,j} = \psi^{(i,j)}_{RAND}(u_{i,1}^{1:j-1}, \{y_{k}^{1:n}\}_{k \in [i:i-1]})$ in the case of a random mapping. The encoder then applies the inverse polar transform for each sequence: $y_{i}^{1:n} = u_{i}^{1:n} G_{i}^{-1}$. The codeword $x^n$ is formed symbol-by-symbol as follows:

$$
x(j) \in \bigcap_{i=1}^{m} \left[ f_i^{-1}(y_i(j)) \right].
$$

If the intersection set is empty, the encoder declares a block error. A block error only occurs at the encoder.

4) Decoding at Receivers: If the encoder succeeds in transmitting a codeword $x^n$, each receiver obtains the sequence $y_{i}^{1:n}$ noiselessly and applies the polar transform $G_i$ to recover $u_{i}^{1:n}$ exactly. Since the message indices $\mathcal{M}_i(n)$ are known to each receiver, the message bits in $u_{i}^{1:n}$ are decoded correctly by receiver $i$.

B. Total Variation Bound

While the deterministic mapping $\psi^{(i,j)}_{MAP}$ performs well in practice, the average probability of error $P_e(n)$ of the coding scheme is more difficult to analyze in theory. The random mapping $\psi^{(i,j)}_{RAND}$ at the encoder is more amenable to analysis via the probabilistic method. Towards that goal, consider the following probability measure defined on the space of tuples of binary sequence:

$$
Q(u_1^n, u_2^n, \ldots, u_m^n) \triangleq \prod_{i=1}^{m} \prod_{j=1}^{n} Q(u_{i,j}\mid u_{i,1}^{1:j-1}, \{u_{k}^{1:n}\}_{k \in [i:i-1]}).
$$

(19)

where the conditional probability measure

$$
Q(u_{i,j} \mid u_{i,1}^{1:j-1}, \{u_{k}^{1:n}\}_{k \in [i:i-1]}) \triangleq \begin{cases} 
\frac{1}{2}, & \text{if } j \in \mathcal{M}_i(n), \\
P\left( U_i(j) = 0 \mid U_i^{1:j-1} = u_{i,1}^{1:j-1}, \{Y_k^{1:n} = y_{k}^{1:n}\}_{k \in [i:i-1]} \right), & \text{otherwise}. 
\end{cases}
$$

The probability measure $Q$ defined in (19) is a perturbation of the joint probability measure $P$ defined in (5) for the random variables $U_i(j)$. The only difference in definition between $P$ and $Q$ is due to those indices in message set $\mathcal{M}_i(n)$. The following lemma provides a bound on the total variation distance between $P$ and $Q$.

\[ A \] A related proof technique was provided for lossy source coding based on polarization in a different context [24]. In the present paper, a different proof is supplied that utilizes the chain rule for KL-divergence.
Lemma 1: (Total Variation Bound) Let probability measures $P$ and $Q$ be defined as in (8) and (19) respectively. Let $0 < \beta < 1$. For sufficiently large $n$, the total variation distance between $P$ and $Q$ is bounded as

$$\sum_{\{u_k^{1:n}\}_{k \in [m]}} \left| P(\{u_k^{1:n}\}_{k \in [m]}) - Q(\{u_k^{1:n}\}_{k \in [m]}) \right| \leq 2^{-n^\beta}.$$ 

Proof: See Section 4 of the Appendices.

C. Analysis of the Average Probability of Error

For the $m$-user deterministic DM-BC, an error event occurs at the encoder if a codeword $x^n$ is unable to be constructed symbol by symbol according to the broadcast protocol described in Section V-A. Define the following set consisting of $m$-tuples of binary sequences,

$$\mathcal{T} \triangleq \left\{ (y_1^n, y_2^n, \ldots, y_m^n) : \exists j \in [n], \sum_{i=1}^m f_i^{-1}(y_i(j)) = 0 \right\}.$$ 

(20)

The set $\mathcal{T}$ consists of those $m$-tuples of binary output sequences which are inconsistent due to the properties of the deterministic channel. In addition, due to the one-to-one correspondence between sequences $u_i^{1:n}$ and $y_i^{1:n}$, denote by $\mathcal{T}$ the set of $m$-tuples $(u_1^{1:n}, u_2^{1:n}, \ldots, u_m^{1:n})$ that are inconsistent.

For the broadcast protocol, the rate $R_i = \frac{1}{n} |\mathcal{M}_i^{(n)}|$ for each receiver. Let the total sum rate for all broadcast receivers be $R_\Sigma = \sum_{i \in [m]} R_i$. If the encoder uses a fixed deterministic map $\psi(i,j)$ in the broadcast protocol, the average probability of error is

$$P_e^{(n)}\left[ \{\psi(i,j)\} \right] = \frac{1}{2^{nR_\Sigma}} \sum_{\{u_k^{1:n}\}_{k \in [m]}} \left[ \prod_{i \in [m]} \mathbb{I}[\psi(i,j)(u_i^{1:n})_{k \in [1:i-1]} = u_i(j)] \right].$$

(21)

In addition, if the random maps $\Psi(i,j)$ are used at the encoder, the average probability of error is a random quantity given by

$$P_e^{(n)}\left[ \{\Psi(i,j)\} \right] = \frac{1}{2^{nR_\Sigma}} \sum_{\{u_k^{1:n}\}_{k \in [m]}} \left[ \prod_{i \in [m]} \mathbb{I}[\psi(i,j)(u_i^{1:n})_{k \in [1:i-1]} = u_i(j)] \right].$$

(22)

Instead of characterizing $P_e^{(n)}$ directly for deterministic maps, the analysis of $P_e^{(n)}\left[ \{\Psi(i,j)\} \right]$ leads to the following lemma.

Lemma 2: Consider the broadcast protocol of Section V-A. Let $R_i = \frac{1}{n} |\mathcal{M}_i^{(n)}|$ for $i \in [m]$ be the broadcast rates selected according to the criterion given in (12) in Proposition 3. Then for $0 < \beta < 1$ and sufficiently large $n$,

$$\mathbb{E}_{\{\Psi(i,j)\}}\left[ P_e^{(n)}\left[ \{\Psi(i,j)\} \right] \right] < 2^{-n^\beta}.$$ 

Proof: Let probability measure $Q$ matches the desired calculation exactly. Step (23) is due to the fact that the probability measure $P$ has zero mass over $m$-tuples of binary sequences that are inconsistent. Step (24) follows directly from Lemma 1. Lastly, since the expectation over random maps $\{\Psi(i,j)\}$ of the average probability of error decays stretched-exponentially, there must exist a set of deterministic maps which exhibit the same behavior.

VI. NOISY BROADCAST CHANNELS

SUPERPOSITION CODING

Coding for noisy broadcast channels is now considered using polarization methods. By contrast to the deterministic case, a decoding error event occurs at the receivers on account of the randomness due to noise. For the remaining sections, it is assumed that there exist $m = 2$ users in the DM-BC. The private-message capacity region for the DM-BC is unknown even for binary input, binary output two-user channels such as the skew-symmetric DM-BC. However, the private-message capacity region is known for specific classes.

A. Special Classes of Noisy DM-BCs

Definition 8: The two-user physically degraded DM-BC is a channel $P_{Y_1Y_2|X}(y_1, y_2|x)$ for which $X = Y_1 - Y_2$ form a Markov chain, i.e., one of the receivers is statistically stronger than the other:

$$P_{Y_1Y_2|X}(y_1, y_2|x) = P_{Y_1|X}(y_1|x)P_{Y_2|Y_1}(y_2|y_1).$$

(26)

Definition 9: A two-user DM-BC $P_{Y_1Y_2|X}(y_1, y_2|x)$ is stochastically degraded if its conditional marginal distributions.
The converse statements for (30), (32), and (33) do not hold in general. Figure 3 illustrates the different types of broadcast channels as a hierarchy.

**Proof:** See Section III of the Appendices.

**B. Cover’s Inner Bound**

Superposition coding involves one auxiliary random variable $V$ which conveys a “cloud center” or a coarse message decoded by both receivers. One of the receivers then decodes an additional “satellite codeword” conveyed through $X$ containing a fine-grain message that is superimposed upon the coarse information.

**Proposition 4 (Cover’s Inner Bound):** For any two-user DM-BC, the rates $(R_1, R_2) \in \mathbb{R}_+^2$ in the region $\mathcal{R}(X, V, Y_1, Y_2)$ are achievable where

$$\mathcal{R}(X, V, Y_1, Y_2) \triangleq \left\{ R_1, R_2 \mid R_1 \leq I(X; Y_1|V), \right.$$

$$R_2 \leq I(V; Y_2),$$

$$R_1 + R_2 \leq I(X; Y_1) \bigg\}.$$  

and where random variables $X, V, Y_1, Y_2$ obey the Markov chain $V \rightarrow X \rightarrow (Y_1, Y_2)$.

**Remark 6:** Cover’s inner bound is applicable for any broadcast channel. By symmetry, the following rate region is also achievable: $R_1, R_2 \mid R_1, R_2 \leq I(X; Y_2|V), R_1 \leq I(V; Y_1), R_1 + R_2 \leq I(X; Y_2)$ for random variables obeying the Markov chain $V \rightarrow X \rightarrow (Y_1, Y_2)$.

**Remark 7:** The inner bound is the capacity region for degraded, less-noisy, and more-capable DM-BCs (i.e. Class I through Class IV as shown in Figure 3). For the degraded and less-noisy special classes, the capacity region is simplified to $\{ R_1, R_2 \mid R_1 \leq I(X; Y_1|V), R_2 \leq I(V; Y_2) \}$. To see this, note that $I(V; Y_2) \leq I(V; Y_1)$ which implies $I(V; Y_2) + I(X; Y_1|V) \leq I(V; Y_1) + I(X; Y_1|V) = I(X; Y_1)$. Therefore the sum-rate constraint $R_1 + R_2 \leq I(X; Y_1)$ of the rate-region in (34) is automatically satisfied.

**Example 2 (Binary Symmetric DM-BC):** The two-user binary symmetric DM-BC consists of a binary symmetric channel with flip probability $p_1$ denoted as $\text{BSC}(p_1)$ and a second channel $\text{BSC}(p_2)$. Assume that $p_1 < p_2 < \frac{1}{2}$ which implies stochastic degradation as defined in (27). For $\alpha \in [0, \frac{1}{2}]$, Cover’s superposition inner bound is the region,

$$\left\{ R_1, R_2 \mid R_1 \leq h_b(\alpha \ast p_1) - h_b(p_1), \right.$$

$$R_2 \leq 1 - h_b(\alpha \ast p_2) \bigg\}.$$  

The above inner bound is determined by evaluating (34) where $V$ is a Bernoulli random variable with $P_V(v) = \frac{1}{2}$, $X = V \oplus S$, and $S$ is a Bernoulli random variable with $P_{S_1}(1) = \alpha$. Figure 4 plots this rectangular inner bound for two different values $\alpha = \frac{1}{5}$ and $\alpha = \frac{1}{4}$. The corner points of this rectangle given in (35) lie on the capacity boundary.

**Example 3 (DM-BC with BEC($\epsilon$) and BSC($p$) [36]):** Consider a two-user DM-BC comprised of a BSC($p$) from $X$ to $Y_1$ and a BEC($\epsilon$) from $X$ to $Y_2$. Then it can be shown that the following cases hold:
However, the listed property exists a sequence of polar broadcast codes over For extension to have a larger alphabet in the case of broadcast channels. An inner bound is to superimpose two messages of information to guarantee that polarization indices are less noisy than \( Y_1 \) but not less noisy. The channel does not belong to the special classes.

The capacity region for all channel parameters for this example is achieved using superposition coding.

C. Main Result

Theorem 2 (Polarization-Based Superposition Code):
Consider any two-user DM-BC with binary input alphabet \( X = \{0, 1\} \) and arbitrary output alphabets \( Y_1, Y_2 \). There exists a sequence of polar broadcast codes over \( n \) channel uses which achieves the following rate region

\[
\mathcal{R}(V, X, Y_1, Y_2) \triangleq \left\{ (R_1, R_2) \left| R_1 \leq I(X; Y_1|V), \quad R_2 \leq I(V; Y_2) \right. \right\},
\]

(36)

where random variables \( V, X, Y_1, Y_2 \) have the following listed properties:

- \( V \) is a binary random variable.
- \( P_{Y_1|V}(y_1|v) \succ P_{Y_2|V}(y_2|v) \).
- \( V \rightarrow X \rightarrow (Y_1, Y_2) \) form a Markov chain.

For \( 0 < \beta < \frac{1}{2} \), the average error probability of this code sequence decays as \( P_e(n) = \mathcal{O}(2^{-n^\beta}) \). The complexity of encoding and decoding is \( \mathcal{O}(n \log n) \).

Remark 8: The requirement that auxiliary \( V \) is a binary random variable is due to the use of binary polarization theorems in the proof. Indeed, the auxiliary \( V \) may need to have a larger alphabet in the case of broadcast channels. An extension to \( q \)-ary random variables is entirely possible if \( q \)-ary polarization theorems are utilized.

Remark 9: The requirement that \( V \rightarrow X \rightarrow (Y_1, Y_2) \) holds is standard for superposition coding over noisy channels. However, the listed property \( P_{Y_1|V}(y_1|v) \succ P_{Y_2|V}(y_2|v) \) is due to the structure of polarization and is used in the proof to guarantee that polarization indices are aligned. If both receivers are able to decode the coarse message carried by the auxiliary random variable \( V \), the polarization indices for the coarse message must be nested for the two receivers’ channels.

VII. PROOF OF THEOREM 2

The block diagram for polarization-based superposition coding is given in Figure 5. Similar to random codes in Shannon theory, polarization-based codes rely on \( n \)-length i.i.d. statistics of random variables; however, a specific polarization structure based on the chain rule of entropy allows for efficient encoding and decoding. The key idea of Cover’s inner bound is to superimpose two messages of information onto one codeword.

A. Polar Transform

Consider the i.i.d. sequence of random variables \( (V^j, X^j, Y_1^j, Y_2^j) \sim P_V(v)P_{X|V}(x|v)P_{Y_1,Y_2|X}(y_1,y_2|x) \) where the index \( j \in [n] \). Let the \( n \)-length sequence of auxiliary and input variables \( (V^j, X^j) \) be organized into the random matrix

\[
\Omega \triangleq \begin{bmatrix}
X^1 & X^2 & X^3 & \cdots & X^n \\
V^1 & V^2 & V^3 & \cdots & V^n
\end{bmatrix}.
\]

(37)

Applying the polar transform to \( \Omega \) results in the random matrix \( U \equiv \Omega G_n \). Let the random variables in the random matrix \( U \) be indexed as follows:

\[
U = \begin{bmatrix}
U_1^1 & U_1^2 & U_1^n \\
U_2^1 & U_2^2 & U_2^n
\end{bmatrix}.
\]

(38)

The above definitions are consistent with the block diagram given in Figure 5 (and noting that \( G_n = G_n^{-1} \)). The polar transform extracts the randomness of \( \Omega \). In the transformed domain, the joint distribution of the random variables in \( U \) is given by

\[
P_{U_1^n,U_2^n}(u_1^n, u_2^n) \triangleq P_{X^n,V^n}(u_1^nG_n, u_2^nG_n).
\]

(39)
For polar coding purposes, the joint distribution is decomposed as follows,

\[
P_{U_1U_2}(u_1^n, u_2^n) = P_{U_2}(u_2^n)P_{U_1|U_2}(u_1^n | u_2^n) = \prod_{j=1}^{n} P(u_2(j) | u_2^{1:j-1}) P(u_1(j) | u_1^{1:j-1}, u_2^n). \tag{40}
\]

The conditional distributions may be computed efficiently using recursive protocols as already mentioned. The polarized variables in \(U\) are not i.i.d. random variables.

### B. Polarization Theorems Revisited

**Definition 12 (Polarization Sets for Superposition Coding):** Let \(V^n, X^n, Y_1^n, Y_2^n\) be the sequence of random variables as introduced in Section VII-A. In addition, let \(U^n_1 = X^nG_n\) and \(U^n_2 = Y^nG_n\). Let \(\delta_n = 2^{-n^\beta}\) for \(0 < \beta < \frac{1}{2}\). The following polarization sets are defined:

\[
\mathcal{H}_{X|V}^{(n)}(j) = \{ j \in [n] : Z(U_1(j))U_1^{1:j-1}, V^n \geq 1 - \delta_n \},
\]

\[
\mathcal{L}_{X|V,Y_1}^{(n)}(j) = \{ j \in [n] : Z(U_1(j))U_1^{1:j-1}, V^n, Y_1^n \leq \delta_n \},
\]

\[
\mathcal{L}_{V|Y_1}^{(n)}(j) = \{ j \in [n] : Z(U_2(j))U_2^{1:j-1}, Y_1^n \leq \delta_n \},
\]

\[
\mathcal{H}_{V}^{(n)}(j) = \{ j \in [n] : Z(U_2(j))U_2^{1:j-1} \geq 1 - \delta_n \},
\]

\[
\mathcal{L}_{V|Y_2}^{(n)}(j) = \{ j \in [n] : Z(U_2(j))U_2^{1:j-1}, Y_2^n \leq \delta_n \}.
\]

**Definition 13 (Message Sets for Superposition Coding):** In terms of the polarization sets given in Definition 12, the following message sets are defined:

\[
\mathcal{M}_{1}^{(n)} = \mathcal{H}^{(n)} \cap \mathcal{L}_{V|Y_1}^{(n)}, \tag{41}
\]

\[
\mathcal{M}_{1}^{(n)} = \mathcal{H}^{(n)} \cap \mathcal{L}_{X|V,Y_1}^{(n)}, \tag{42}
\]

\[
\mathcal{M}_{2}^{(n)} = \mathcal{H}^{(n)} \cap \mathcal{L}_{V|Y_2}^{(n)}. \tag{43}
\]

**Proposition 5 (Polarization):** Consider the polarization sets given in Definition 12 and the message sets given in Definition 13 with parameter \(\delta_n = 2^{-n^\beta}\) for \(0 < \beta < \frac{1}{2}\). Fix a constant \(\tau > 0\). Then there exists an \(N_0 = N_0(\beta, \tau)\) such that

\[
\frac{1}{n} |\mathcal{M}_{1}^{(n)}| \geq (H(X|V) - H(X|V, Y_1)) - \tau,
\]

\[
\frac{1}{n} |\mathcal{M}_{2}^{(n)}| \geq (H(V) - H(V|Y_2)) - \tau,
\]

for all \(n > N_0\).

**Lemma 4:** Consider the message sets defined in Definition 13. If the property \(P_{Y_1|V}(y_1|v) > P_{Y_2|V}(y_2|v)\) holds for conditional distributions \(P_{Y_1|V}(y_1|v)\) and \(P_{Y_2|V}(y_2|v)\), then the Bhattacharyya parameters

\[
Z(U_2(j)U_2^{1:j-1}, Y_1^n) \leq Z(U_2(j)U_2^{1:j-1}, Y_2^n)
\]

for all \(j \in [n]\). As a result,

\[
\mathcal{M}_{1}^{(n)} \subseteq \mathcal{M}_{2}^{(n)}.
\]

**Proof:** The proof follows from Lemma 12 and repeated application of Lemma 13 in Appendix A. □

### C. Broadcast Encoding Based on Polarization

The polarization theorems of the previous section are useful for defining a multi-user communication system as diagrammed in Figure 5. The broadcast encoder must map two independent messages \((W_1, W_2)\) uniformly distributed over \([2^nR_1] \times [2^nR_2]\) to a codeword \(x^n \in X^n\) in such a way that the decoding at each separate receiver is successful. The achievable rates for a particular block length \(n\) are

\[
R_1 = \frac{1}{n} |\mathcal{M}_{1}^{(n)}|,
\]

\[
R_2 = \frac{1}{n} |\mathcal{M}_{2}^{(n)}|.
\]

To construct a codeword, the encoder first produces two binary sequences \(u_1^n \in \{0,1\}^n\) and \(u_2^n \in \{0,1\}^n\). To determine \(u_1(j)\) for \(j \in \mathcal{M}_{1}^{(n)}\), the bit is selected as a uniformly distributed message bit intended for the first receiver. To determine \(u_2(j)\) for \(j \in \mathcal{M}_{2}^{(n)}\), the bit is selected as a uniformly distributed message bit intended for the second receiver. The remaining non-message indices of \(u_1^n\) and \(u_2^n\) are computed according to deterministic or random functions which are shared between the encoder and decoder.

1) **Deterministic Mapping:** Consider the following deterministic boolean functions indexed by \(j \in [n]\):

\[
\psi_1^{(j)} : \{0,1\}^{n+j-1} \rightarrow \{0,1\},
\]

\[
\psi_2^{(j)} : \{0,1\}^{j-1} \rightarrow \{0,1\}.
\]
As an example, consider the deterministic boolean functions based on the maximum a posteriori polar coding.

\[
\psi_1^{(j)}(u_1^{1:j-1}, v^n) \triangleq \arg\max_{u \in \{0,1\}} \{\Pr(U_1(j) = u | U_1^{1:j-1} = u_1^{1:j-1}, V^n = v^n)\},
\]

(48)

\[
\psi_2^{(j)}(u_2^{1:j-1}) \triangleq \arg\max_{u \in \{0,1\}} \{\Pr(U_2(j) = u | U_2^{1:j-1} = u_2^{1:j-1})\}.
\]

(49)

2) Random Mapping: Consider the following class of random boolean functions indexed by \( j \in [n] \):

\[
\Psi_1^{(j)}: \{0,1\}^{n+j-1} \to \{0,1\},
\]

(50)

\[
\Psi_2^{(j)}: \{0,1\}^{j-1} \to \{0,1\}.
\]

(51)

As an example, consider the random boolean functions

\[
\Psi_1^{(j)}(u_1^{1:j-1}, v^n) \triangleq \begin{cases} 
0, & \text{w.p. } \lambda_0(u_1^{1:j-1}, v^n), \\
1, & \text{w.p. } 1 - \lambda_0(u_1^{1:j-1}, v^n), 
\end{cases}
\]

(52)

\[
\Psi_2^{(j)}(u_2^{1:j-1}) \triangleq \begin{cases} 
0, & \text{w.p. } \lambda_0(u_2^{1:j-1}), \\
1, & \text{w.p. } 1 - \lambda_0(u_2^{1:j-1}),
\end{cases}
\]

(53)

where

\[
\lambda_0(u_1^{1:j-1}) = \Pr(U_2(j) = 0 | U_1^{1:j-1} = u_1^{1:j-1}).
\]

\[
\lambda_0(u_2^{1:j-1}) = \Pr(U_1(j) = 0 | U_1^{1:j-1} = u_1^{1:j-1}, V^n = v^n).
\]

The random boolean functions \( \Psi_1^{(j)} \) and \( \Psi_2^{(j)} \) may be thought of as a vector of independent Bernoulli random variables indexed by the input to the function. Each Bernoulli random variable of the vector is zero or one with a fixed probability.

3) Protocol: The encoder constructs the sequence \( u_2^n \) first using the message bits \( W_2 \) and either (49) or (53). Next, the sequence \( v^n = u_2^n G_n \) is created. Finally, the sequence \( u_1^n \) is constructed using the message bits \( W_1 \), the sequence \( v^n \), and either the deterministic maps defined in (48) or the randomized maps in (53). The transmitted codeword is \( x^n = u_1^n G_n \).

D. Broadcast Decoding Based on Polarization

1) Decoding At First Receiver: Decoder \( D_1 \) decodes the binary sequence \( \hat{u}_2^n \) first using its observations \( y_1^n \). It then reconstructs \( \hat{v}_1^n = \hat{u}_2^n G_n \). Using the sequence \( \hat{v}_1^n \) and observations \( y_1^n \), the decoder reconstructs \( \hat{u}_1^n \). The message \( W_1 \) is located at the indices \( j \in \mathcal{M}_1^{(n)} \) in the sequence \( \hat{u}_1^n \). More precisely, define the following deterministic polar decoding functions:

\[
\xi_1^{(j)}(u_2^{1:j-1}, y_1^n) \triangleq \arg\max_{u \in \{0,1\}} \{\Pr(U_2(j) = u | U_2^{1:j-1} = u_2^{1:j-1}, V^n = y_1^n)\}.
\]

(54)

\[
\xi_2^{(j)}(u_1^{1:j-1}, y_1^n) \triangleq \arg\max_{u \in \{0,1\}} \{\Pr(U_1(j) = u | U_1^{1:j-1} = u_1^{1:j-1}, V^n = y_1^n)\}.
\]

(55)

The decoder \( D_1 \) reconstructs \( \hat{u}_2^n \) bit-by-bit successively as follows using the identical shared random mapping \( \psi_2^{(j)} \) (or possibly the identical shared mapping \( \psi_2^{(j)} \)) used at the encoder:

\[
\hat{u}_2(j) = \begin{cases} 
\xi_2^{(j)}(\hat{u}_2^{1:j-1}, y_1^n), & \text{if } j \in \mathcal{M}_2^{(n)}, \\
\psi_2^{(j)}(\hat{u}_2^{1:j-1}), & \text{otherwise}.
\end{cases}
\]

(56)

If Lemma \( \text{H} \) holds, note that \( \mathcal{M}_1^{(n)} \subseteq \mathcal{M}_2^{(n)} \). With \( \hat{u}_2^n \), decoder \( D_1 \) reconstructs \( \hat{v}_1^n = \hat{u}_2^n G_n \). Then the sequence \( \hat{u}_1^n \) is constructed bit-by-bit successively as follows using the identical shared random mapping \( \psi_1^{(j)} \) (or possibly the identical shared mapping \( \psi_1^{(j)} \)) used at the encoder:

\[
\hat{u}_1(j) = \begin{cases} 
\xi_1^{(j)}(\hat{u}_1^{1:j-1}, \hat{v}_1^n, y_1^n), & \text{if } j \in \mathcal{M}_1^{(n)}, \\
\psi_1^{(j)}(\hat{u}_1^{1:j-1}), & \text{otherwise}.
\end{cases}
\]

(57)

2) Decoding At Second Receiver: The decoder \( D_2 \) decodes the binary sequence \( \hat{u}_2^n \) using observations \( y_2^n \). The message \( W_2 \) is located at the indices \( j \in \mathcal{M}_2^{(n)} \) of the sequence \( \hat{u}_2^n \). More precisely, define the following polar decoding functions

\[
\xi_2^{(j)}(u_2^{1:j-1}, y_2^n) \triangleq \arg\max_{u \in \{0,1\}} \{\Pr(U_2(j) = u | U_2^{1:j-1} = u_2^{1:j-1}, Y^n = y_2^n)\}.
\]

(58)

The decoder \( D_2 \) reconstructs \( \hat{u}_2^n \) bit-by-bit successively as follows using the identical shared random mapping \( \psi_2^{(j)} \) (or possibly the identical shared mapping \( \psi_2^{(j)} \)) used at the encoder:

\[
\hat{u}_2(j) = \begin{cases} 
\xi_2^{(j)}(\hat{u}_2^{1:j-1}, y_2^n), & \text{if } j \in \mathcal{M}_2^{(n)}, \\
\psi_2^{(j)}(\hat{u}_2^{1:j-1}), & \text{otherwise}.
\end{cases}
\]

(59)

Remark 10: The encoder and decoders execute the same protocol for reconstructing bits at the non-message indices. This is achieved by applying the same deterministic maps \( \psi_1^{(j)} \) and \( \psi_2^{(j)} \) or randomized maps \( \psi_1^{(j)} \) and \( \psi_2^{(j)} \).
E. Total Variation Bound

To analyze the average probability of error \( P_e(n) \) via the probabilistic method, it is assumed that both the encoder and decoder share the randomized mappings \( \Psi_1^{(j)} \) and \( \Psi_2^{(j)} \). Define the following probability measure on the space of tuples of binary sequences.

\[
Q(u^n_1, u^n_2) \triangleq \prod_{j=1}^{n} Q(u^{1:j-1}_j | u^{1:j}_j) Q(u^1_j | u^{1:j-1}_j, \overline{u}^{1:j}_j)
\]

(60)

In (60), the conditional probability measures are defined as

\[
Q(u^{1:j-1}_j | u^{1:j}_j) \triangleq \begin{cases} 
\frac{1}{2}, & \text{if } j \in \mathcal{M}^{(n)}_1, \\
P(u^{1:j-1}_j | u^{1:j}_j), & \text{otherwise.}
\end{cases}
\]

\[
Q(u^1_j | u^{1:j-1}_j, u^n_2) \triangleq \begin{cases} 
\frac{1}{2}, & \text{if } j \in \mathcal{M}^{(n)}_1, \\
P(u^1_j | u^{1:j-1}_j, u^n_2), & \text{otherwise.}
\end{cases}
\]

The probability measure \( Q \) defined in (60) is a perturbation of the joint probability measure \( P_{U^n|V^n}(u^n_1, u^n_2) \) in (49). The only difference in definition between \( P \) and \( Q \) is due to those indices in message sets \( \mathcal{M}^{(n)}_1 \) and \( \mathcal{M}^{(n)}_2 \). The following lemma provides a bound on the total variation distance between \( P \) and \( Q \). The lemma establishes the fact that inserting uniformly distributed message bits in the proper indices \( \mathcal{M}^{(n)}_1 \) and \( \mathcal{M}^{(n)}_2 \) at the encoder does not perturb the statistics of the \( n \)-length random variables too much.

**Lemma 5: (Total Variation Bound)** Let probability measures \( P \) and \( Q \) be defined as in (49) and (60) respectively. Let \( 0 < \beta < 1 \). For sufficiently large \( n \), the total variation distance between \( P \) and \( Q \) is bounded as

\[
\sum_{u^n_1 \in \{0,1\}^n} \left| P_{U^n|V^n}(u^n_1, u^n_2) - Q(u^n_1, u^n_2) \right| \leq 2^{-n^\beta}.
\]

**Proof:** See Section C of the Appendices.

F. Error Sequences

The decoding protocols for \( D_1 \) and \( D_2 \) were established in Section VII. To analyze the probability of error of successive cancelation (SC) decoding, consider the sequences \( u^n_1 \) and \( u^n_2 \) formed at the encoder, and the resulting observations \( y^n_1 \) and \( y^n_2 \) received by the decoders. It is convenient to group the sequences together and consider all tuples \((u^n_1, u^n_2, y^n_1, y^n_2)\).

Decoder \( D_1 \) makes an SC decoding error on the \( j \)-th bit for the following tuples:

\[
\mathcal{T}_{1v}^i \triangleq \left\{ (u^n_1, u^n_2, y^n_1, y^n_2) : P_{U_2^n|V_1^n}(u_2^{1:j-1}, y^n_1) \leq P_{U_2^n|V_1^n}(u_2^{1:j-1} + 1, y^n_1) \right\},
\]

(61)

The set \( \mathcal{T}_{1v}^i \) represents those tuples causing an error at \( D_1 \) in the case \( u_2^{1:j} \) is inconsistent with respect to observations \( y^n_1 \) and the decoding rule. The set \( \mathcal{T}_{1}^i \) represents those tuples causing an error at \( D_1 \) in the case \( u_1^{1:j} \) is inconsistent with respect to \( u^n = u^n_2 G, y^n_1 \), and the decoding rule. Similarly, decoder \( D_2 \) makes an SC decoding error on the \( j \)-th bit for the following tuples:

\[
\mathcal{T}_{2}^i \triangleq \left\{ (u^n_1, u^n_2, y^n_1, y^n_2) : P_{U_2^n|V_1^n}(u_2^{1:j-1}, y^n_2) \leq P_{U_2^n|V_1^n}(u_2^{1:j-1} + 1, y^n_2) \right\}.
\]

(61)

The set \( \mathcal{T}_{2}^i \) represents those tuples causing an error at \( D_2 \) in the case \( u_2^{1:j} \) is inconsistent with respect to observations \( y^n_2 \) and the decoding rule. Since both decoders \( D_1 \) and \( D_2 \) only declare errors for those indices in the message sets, the set of tuples causing an error is

\[
\mathcal{T}_v \triangleq \bigcup_{j \in \mathcal{M}^{(n)}_1} \mathcal{T}_{1v}^i \bigcup_{j \in \mathcal{M}^{(n)}_2} \mathcal{T}_{2}^i \bigcup_{j \in \mathcal{M}^{(n)}_1} \mathcal{T}_1^i.
\]

(62)

The complete set of tuples causing a broadcast error is

\[
\mathcal{T} \triangleq \mathcal{T}_v \cup \mathcal{T}_1 \cup \mathcal{T}_2.
\]

(65)

The goal is to show that the probability of choosing tuples of error sequences in the set \( \mathcal{T} \) is small under the distribution induced by the broadcast code.

G. Average Error Probability

Denote the total sum rate of the broadcast protocol as \( R_S = R_1 + R_2 \). Consider first the use of fixed deterministic maps \( \psi_1^{(j)} \) and \( \psi_2^{(j)} \) shared between the encoder and decoders. Then the probability of error of broadcasting the two messages at
rates $R_1$ and $R_2$ is given by
\[
P_c^{(n)} \left[ \{ \psi_1^{(j)}, \psi_2^{(j)} \} \right] = \frac{1}{2^n R_2} \sum \left[ \prod_{j \in [n]} \mathbb{1}_{\psi_2^{(j)}(u_2^{(j-1)}) = u_2(j)} \prod_{j \in [n]} \mathbb{1}_{\psi_1^{(j)}(u_1^{(j-1)}, u_2) = u_1(j)} \right].
\]

If the encoder and decoders share randomized maps $\psi_1^{(j)}$ and $\psi_2^{(j)}$, then the average probability of error is a random quantity determined as follows
\[
P_c^{(n)} \left[ \{ \psi_1^{(j)}, \psi_2^{(j)} \} \right] = \frac{1}{2^n R_2} \sum \left[ \prod_{j \in [n]} \mathbb{1}_{\psi_2^{(j)}(u_2^{(j-1)}) = u_2(j)} \prod_{j \in [n]} \mathbb{1}_{\psi_1^{(j)}(u_1^{(j-1)}, u_2) = u_1(j)} \right].
\]

By averaging over the randomness in the encoders and decoders, the expected block error probability $P_c^{(n)} \left[ \{ \psi_1^{(j)}, \psi_2^{(j)} \} \right]$ is upper bounded in the following lemma.

**Lemma 6:** Consider the polarization-based superposition code described in Section VIII-C and Section VIII-D. Let $R_1$ and $R_2$ be the broadcast rates selected according to the Bhattacharyya criterion given in Proposition 5. Then for $0 < \beta < 1$ and sufficiently large $n$, $\mathbb{E}_{\{\psi_1^{(j)}, \psi_2^{(j)}\}} \left[ P_c^{(n)} \left[ \{ \psi_1^{(j)}, \psi_2^{(j)} \} \right] \right] < 2^{-n^\beta}$.

**Proof:** See Section C of the Appendices.

If the average probability of error decays to zero in expectation over the random maps $\{\psi_1^{(j)}\}$ and $\{\psi_2^{(j)}\}$, then there must exist at least one fixed set of maps for which $P_c^{(n)} \rightarrow 0$.

### VIII. Noisy Broadcast Channels

#### Marton’s Coding Scheme

##### A. Marton’s Inner Bound

For general noisy broadcast channels, Marton’s inner bound involves two correlated auxiliary random variables $V_1$ and $V_2$ [3]. The intuition behind the coding strategy is to identify two “virtual” channels, one from $V_1$ to $Y_1$, and the other from $V_2$ to $Y_2$. Somewhat surprisingly, although the broadcast messages are independent, the auxiliary random variables $V_1$ and $V_2$ may be correlated to increase rates to both receivers.

While there exist generalizations of Marton’s strategy, the basic version of the inner bound is presented in this section.

**Proposition 6 (Marton’s Inner Bound):** For any two-user DM-BC, the rates $(R_1, R_2) \in \mathbb{R}_+^2$ in the pentagonal region $\mathcal{R}(X, V_1, V_2, Y_1, Y_2)$ are achievable where
\[
\mathcal{R}(X, V_1, V_2, Y_1, Y_2) \triangleq \left\{ R_1, R_2 \mid R_1 \leq I(V_1; Y_1), \\
R_2 \leq I(V_2; Y_2), \\
R_1 + R_2 \leq I(V_1; Y_1) + I(V_2; Y_2) - I(V_1; V_2) \right\}.
\]

and where $X, V_1, V_2, Y_1, Y_2$ have a joint distribution given by $P_{V_1V_2}(v_1, v_2)P_{X|V_1V_2}(x|v_1, v_2)P_{Y_1Y_2}(y_1, y_2|x)$.

**Remark 11:** It can be shown that for Marton’s inner bound there is no loss of generality if $P_{X|V_1V_2}(x|v_1, v_2) = \mathbb{1}_{x = \phi(v_1, v_2)}$ where $\phi(v_1, v_2)$ is a deterministic function [2 Section 8.3]. Thus, by allowing a larger alphabet size for the auxiliaries, $X$ may be a deterministic function of auxiliaries $(V_1, V_2)$. Marton’s inner bound is tight for the class of semi-deterministic DM-BCs for which one of the outputs is a deterministic function of the input.

#### B. Main Result

**Theorem 3 (Polarization-Based Marton Code):** Consider any two-user DM-BC with arbitrary input and output alphabets. There exist sequences of polar broadcast codes over $n$ channel uses which achieve the following rate region
\[
\mathcal{R}(V_1, V_2, X, Y_1, Y_2) \triangleq \left\{ R_1, R_2 \mid R_1 \leq I(V_1; Y_1), \\
R_2 \leq I(V_2; Y_2) - I(V_1; V_2) \right\},
\]

where random variables $V_1, V_2, X, Y_1, Y_2$ have the following listed properties:
- $V_1$ and $V_2$ are binary random variables.
- $P_{V_2|V_1}(v_2|v_1) = P_{V_1V_2}(v_1, v_2)$.
- For a deterministic function $\phi : \{0, 1\}^2 \rightarrow \mathcal{X}$, the joint distribution of all random variables is given by
\[
P_{V_1V_2X|Y_1Y_2}(v_1, v_2, x, y_1, y_2) = P_{V_1V_2}(v_1, v_2) \mathbb{1}_{x = \phi(v_1, v_2)} P_{Y_i|X}(y_1, y_2|x).
\]

For $0 < \beta < 1$, the average error probability of this code sequence decays as $P_c^{(n)} = \mathcal{O}(2^{-n^\beta})$. The complexity of encoding and decoding is $\mathcal{O}(n \log n)$.

**Remark 12:** The listed property $P_{V_2|V_1}(v_2|v_1) \geq P_{V_1V_2}(v_1, v_2)$ is necessary in the proof due to polarization-based codes requiring an alignment of polarization indices.

In addition, it is difficult even to evaluate Marton’s inner bound for general channels due to the need for proper cardinality bounds on the auxiliaries [37]. These issues lie outside the scope of the present paper.
The property is a natural restriction since it also implies that $I(Y_2; V_2) > I(V_1; V_2)$ so that $R_2 > 0$. However, certain joint distributions on random variables are not permitted using the analysis of polarization presented here. It is not clear whether a different approach obviates the need for an alignment of indices.

Remark 13: By symmetry, the rate tuple $(R_1, R_2) = (I(V_1; Y_1) - I(V_1; V_2), I(V_2; Y_2))$ is achievable with low-complexity codes under similar constraints on the joint distribution of $V_1, V_2, X, Y_1, Y_2$. The rate tuple is a corner point of the pentagonal rate region of Marton’s inner bound given in \(66\).

IX. PROOF OF THEOREM 5

The block diagram for polarization-based Marton coding is given in Figure 6. Marton’s strategy differs from Cover’s superposition coding with the presence of two auxiliaries and the function $\phi(v_1, v_2)$ which forms the codeword symbol-by-symbol. The polar transform is applied to each $n$-length i.i.d. sequence of auxiliary random variables.

A. Polar Transform

Consider the i.i.d. sequence of random variables $(V_1^n, V_2^n, X^n, Y_1^n, Y_2^n)$ where $P_{V_1^n, V_2^n} (v_1^n, v_2^n)$ is the joint distribution of the variables in $V$ is given by

$$P_{U_1^n, U_2^n} (u_1^n, u_2^n) \triangleq P_{V_1^n, V_2^n} (u_1^n G_n, u_2^n G_n).$$

(70)

However, for polar coding purposes, the joint distribution is decomposed as follows,

$$P_{U_1^n, U_2^n} (u_1^n, u_2^n) = P_{U_1^n} (u_1^n) P_{U_2^n | U_1^n} (u_2^n | u_1^n) = \prod_{j=1}^{n} P(u_1(j) | u_1^{j-1}) P(u_2(j) | u_2^{j-1}, u_1^n).$$

(71)

The above conditional distributions may be computed efficiently using recursive protocols. The polarized random variables of $U$ do not have an i.i.d. distribution.

B. Effective Channel

Marton’s achievable strategy establishes virtual channels for the two receivers via the function $\phi(v_1, v_2)$. The virtual channel is given by

$$P_{Y_1^n, Y_2^n | V_1^n, V_2^n} (y_1^n, y_2^n | v_1^n, v_2^n) \triangleq P_{Y_1^n, Y_2^n | X} (y_1^n, y_2^n | \phi(v_1^n, v_2^n)).$$

Due to the memoryless property of the DM-BC, the effective channel between auxiliaries and channel outputs is given by

$$P_{Y_1^n, Y_2^n | V_1^n, V_2^n} (y_1^n, y_2^n | v_1^n, v_2^n) \triangleq \prod_{i=1}^{n} P_{Y_1^n | X} (y_1(i) | \phi(v_1(i), v_2(i))).$$

The polarization-based Marton code establishes a different effective channel between polar-transformed variables and the channel outputs. The effective polarized channel is

$$P_{Y_1^n, Y_2^n | V_1^n, V_2^n} (y_1^n, y_2^n | u_1^n, u_2^n) \triangleq P_{Y_1^n, Y_2^n | U_1^n, U_2^n} (y_1^n, y_2^n | u_1^n G_n, u_2^n G_n).$$

(72)

C. Polarization Theorems Revisited

**Definition 14 (Polarization Sets for Marton Coding):** Let $V_1^n, V_2^n, X^n, Y_1^n, Y_2^n$ be the sequence of random variables as introduced in Section IX-A. In addition, let $U_1^n = V_1^n G_n$ and
Consider the polarization sets defined in Proposition 14. For the message sets given in Definition 14, the following is highlighted by the vertical red rectangles. At finite codelength $= n$, exact alignment is not possible due to partially-polarized indices pictured in gray.

$$U_2^n = V_2^nG_n.$$ Let $\delta_n = 2^{-n^\beta} < 0 < \beta < \frac{1}{2}$. The following polarization sets are defined:

$$\mathcal{H}^{(n)}_{V_1} \triangleq \left\{ j \in [n] : Z\left(U_1(j) | U_1^{j-1} \right) \geq 1 - \delta_n \right\},$$

$$\mathcal{L}^{(n)}_{V_1 | Y_1} \triangleq \left\{ j \in [n] : Z\left(U_1(j) | U_1^{j-1}, Y_1^n \right) \leq \delta_n \right\},$$

$$\mathcal{H}^{(n)}_{V_2 | V_1} \triangleq \left\{ j \in [n] : Z\left(U_2(j) | U_2^{j-1}, V_1^n \right) \geq 1 - \delta_n \right\},$$

$$\mathcal{L}^{(n)}_{V_2 | V_1} \triangleq \left\{ j \in [n] : Z\left(U_2(j) | U_2^{j-1}, V_1^n \right) \leq \delta_n \right\},$$

$$\mathcal{H}^{(n)}_{V_2 | Y_2} \triangleq \left\{ j \in [n] : Z\left(U_2(j) | U_2^{j-1}, Y_2^n \right) \geq 1 - \delta_n \right\},$$

$$\mathcal{L}^{(n)}_{V_2 | Y_2} \triangleq \left\{ j \in [n] : Z\left(U_2(j) | U_2^{j-1}, Y_2^n \right) \leq \delta_n \right\}.$$

**Definition 15 (Message Sets for Marton Coding):** In terms of the polarization sets given in Definition 14, the following message sets are defined:

$$\mathcal{M}_{1}^{(n)} \triangleq \mathcal{H}^{(n)}_{V_1} \cap \mathcal{L}^{(n)}_{V_1 | Y_1},$$

$$\mathcal{M}_{2}^{(n)} \triangleq \mathcal{H}^{(n)}_{V_2 | V_1} \cap \mathcal{L}^{(n)}_{V_2 | Y_2}.\quad (73)$$

**Proposition 7 (Polarization):** Consider the polarization sets given in Definition 14 and the message sets given in Definition 15 with parameter $\delta_n = 2^{-n^\beta}$ for $0 < \beta < \frac{1}{2}$. Fix a constant $\tau > 0$. Then there exists an $N_0 = N_0(\beta, \tau)$ such that

$$\frac{1}{n} \left| \mathcal{M}_{1}^{(n)} \right| \geq \left( H(V_1) - H(V_1 | Y_1) \right) - \tau,\quad (75)$$

$$\frac{1}{n} \left| \mathcal{M}_{2}^{(n)} \right| \geq \left( H(V_2) - H(V_2 | V_1) \right) - \tau,\quad (76)$$

for all $n > N_0$.

**Lemma 7:** Consider the polarization sets defined in Proposition 7. If the property $P_{Y_2 | V_2}(y_2 | v_2) > P_{V_1 | V_2}(v_1 | v_2)$ holds for conditional distributions $P_{Y_2 | V_2}(y_2 | v_2)$ and $P_{V_1 | V_2}(v_1 | v_2)$, then $I(V_2; Y_2) > I(V_1; V_2)$ and the Bhattacharyya parameters

$$Z\left(U_2(j) | U_2^{1:j-1}, V_2^n \right) \leq Z\left(U_2(j) | U_2^{1:j-1}, Y_1^n \right)$$

for all $j \in [n]$. As a result,

$$\mathcal{L}^{(n)}_{V_2 | V_1} \subseteq \mathcal{L}^{(n)}_{V_2 | Y_2},$$

$$\mathcal{H}^{(n)}_{V_2 | Y_2} \subseteq \mathcal{H}^{(n)}_{V_2 | V_1}.$$

**Proof:** The proof follows from Lemma 12 and repeated application of Lemma 13 in Appendix A.

**Remark 14:** The alignment of polarization indices characterized by Lemma 7 is diagrammed in Figure 7. The alignment ensures the existence of polarization indices in the set $\mathcal{M}_{2}^{(n)}$ for the message $W_2$ to have a positive rate $R_2 > 0$. The indices in $\mathcal{M}_{2}^{(n)}$ represent those bits freely set at the broadcast encoder and simultaneously those bits that may be decoded by $D_2$ given its observations.

**D. Partially-Polarized Indices**

As shown in Figure 7 for the Marton coding scheme, exact alignment of polarization indices is not possible. However, the alignment holds for all but $\Theta(n)$ indices. The sets of partially-polarized indices shown in Figure 7 are defined as follows.

**Definition 16 (Sets of Partially-Polarized Indices):**

$$\Delta_1 \triangleq \left[ n \right] \setminus \left( \mathcal{H}^{(n)}_{V_2 | V_1} \cap \mathcal{L}^{(n)}_{V_2 | V_1} \right),$$

$$\Delta_2 \triangleq \left[ n \right] \setminus \left( \mathcal{H}^{(n)}_{V_2 | Y_2} \cap \mathcal{L}^{(n)}_{V_2 | Y_2} \right).\quad (77)$$

As implied by Ankan’s polarization theorems, the number of partially-polarized indices is negligible asymptotically as $n \to \infty$. For an arbitrarily small $\eta > 0$,

$$\frac{\left| \Delta_1 \cup \Delta_2 \right|}{n} \leq \eta,\quad (79)$$
for all \( n \) sufficiently large enough. As will be discussed, providing these \( o(n) \) bits as “genie-given” bits to the decoders results in a rate penalty; however, the rate penalty is negligible for sufficiently large code lengths.

### E. Broadcast Encoding Based on Polarization

As diagrammed in Figure 6, the broadcast encoder must map two independent messages \((W_1, W_2)\) uniformly distributed over \([2^{nR_1}] \times [2^{nR_2}]\) to a codeword \(x^n \in \mathcal{X}^n\) in such a way that the decoding at each separate receiver is successful. The achievable rates for a particular block length \( n \) are

\[
R_1 = \frac{1}{n} |\mathcal{M}_1^{(n)}|, \\
R_2 = \frac{1}{n} |\mathcal{M}_2^{(n)}|.
\]

To construct a codeword, the encoder first produces two binary sequences \( u_1^n \in \{0, 1\}^n \) and \( u_2^n \in \{0, 1\}^n \). To determine \( u_1(j) \) for \( j \in \mathcal{M}_1^{(n)} \), the bit is selected as a uniformly distributed message bit intended for the first receiver. To determine \( u_2(j) \) for \( j \in \mathcal{M}_2^{(n)} \), the bit is selected as a uniformly distributed message bit intended for the second receiver. The remaining non-message indices of \( u_1^n \) and \( u_2^n \) are decided randomly according to the proper statistics as will be described in this section. The transmitted codeword is formed symbol-by-symbol via the \( \phi \) function,

\[
\forall j \in [n]: x(j) = \phi(v_1(j), v_2(j))
\]

where \( v_1^n = u_1^n G_n \) and \( v_2^n = u_2^n G_n \). A valid codeword sequence is always guaranteed to be formed unlike in the case of coding for deterministic broadcast channels.

1) Random Mapping: To fill in the non-message indices, we define the following random mappings. Consider the following class of random boolean functions where \( j \in [n] \):

\[
\Psi_1^{(j)}: \{0, 1\}^{j-1} \rightarrow \{0, 1\}, \\
\Psi_2^{(j)}: \{0, 1\}^{n+j-1} \rightarrow \{0, 1\}, \\
\Gamma: [n] \rightarrow \{0, 1\}.
\]

More concretely, we consider the following specific random boolean functions based on the statistics derived from polarization methods:

\[
\Psi_1^{(j)}(u_1^{1:j-1}) \triangleq \begin{cases} 
0, & \text{w.p. } \lambda_0(u_1^{1:j-1}), \\
1, & \text{w.p. } 1 - \lambda_0(u_1^{1:j-1}), 
\end{cases} \tag{83}
\]

\[
\Psi_2^{(j)}(u_2^{1:j-1}, v_1^n) \triangleq \begin{cases} 
0, & \text{w.p. } \lambda_0(u_2^{1:j-1}, v_1^n), \\
1, & \text{w.p. } 1 - \lambda_0(u_2^{1:j-1}, v_1^n)
\end{cases} \tag{84}
\]

\[
\Gamma(j) \triangleq \begin{cases} 
0, & \text{w.p. } \frac{1}{2}, \\
1, & \text{w.p. } \frac{1}{2}.
\end{cases} \tag{85}
\]

where

\[
\lambda_0(u_1^{1:j-1}) \triangleq \mathbb{P}(U_1(j) = 0 | U_1^{1:j-1} = u_1^{1:j-1}).
\]

\[
\lambda_0(u_2^{1:j-1}, v_1^n) \triangleq \mathbb{P}(U_2(j) = 0 | U_2^{1:j-1} = u_2^{1:j-1}, V_1^n = v_1^n).
\]

For a fixed \( j \in [n] \), the random boolean functions \( \Psi_1^{(j)}, \Psi_2^{(j)} \) may be thought of as a vector of independent Bernoulli random variables indexed by the input to the function. Each Bernoulli random variable of the vector is zero or one with a fixed well-defined probability that is efficiently computable. The random boolean function \( \Gamma \) may be thought of as an \( n \)-length vector of Bernoulli(\( \frac{1}{2} \)) random variables.

2) Encoding Protocol: The broadcast encoder constructs the sequence \( u_1^n \) bit-by-bit successively,

\[
u_1(j) = \begin{cases} 
W_1 \text{ message bit, } & \text{if } j \in \mathcal{M}_1^{(n)}, \\
\Psi_1^{(j)}(u_1^{1:j-1}), & \text{otherwise.}
\end{cases} \tag{86}
\]

The encoder then computes the sequence \( v_1^n = u_1^n G_n \). To generate \( v_2^n \), the encoder constructs the sequence \( u_2^n \) (given \( v_1^n \)) as follows,

\[
u_2(j) = \begin{cases} 
W_2 \text{ message bit, } & \text{if } j \in \mathcal{M}_2^{(n)}, \\
\Gamma(j), & \text{if } j \in \mathcal{H}_{2|V_1} \setminus \mathcal{M}_2^{(n)}, \\
\Psi_2^{(j)}(u_2^{1:j-1}, v_1^n), & \text{otherwise.}
\end{cases} \tag{87}
\]

Then the sequence \( v_2^n = u_2^n G_n \). The randomness in the above encoding protocol over non-message indices ensures that the pair of sequences \((u_1^n, u_2^n)\) has the correct statistics as if drawn from the joint distribution of \((U_1^n, U_2^n)\). In the last step, the encoder transmits a codeword \( x^n \) formed symbol-by-symbol:

\[
x(j) = \phi(v_1(j), v_2(j)) \text{ for all } j \in [n].
\]

For \( j \in \Delta_2 \), where \( \Delta_2 \) is the set of partially-polarized indices defined in (78), the encoder records the realization of \( u_2(j) \). These indices will be provided to the second receiver’s decoder \( D_2 \) as “genie-given” bits.

### F. Broadcast Decoding Based on Polarization

1) Decoding At First Receiver: Decoder \( D_1 \) decodes the binary sequence \( \tilde{u}_1^n \) using its observations \( y_1^n \). The message \( W_1 \) is located at the indices \( j \in \mathcal{M}_1^{(n)} \) in the sequence \( \tilde{u}_1^n \).

More precisely, we define the following deterministic polar decoding function for the \( j \)-th bit:

\[
\xi_{u_1}^{(j)}(u_1^{1:j-1}, y_1^n) \triangleq \arg \max_{u \in \{0, 1\}} \mathbb{P}(U_1(j) = u | U_1^{1:j-1} = u_1^{1:j-1}, Y_1^n = y_1^n).
\] \tag{88}

Decoder \( D_1 \) reconstructs \( \tilde{u}_1^n \) bit-by-bit successively as follows using the identical random mapping \( \Psi_1^{(j)} \) at the encoder:

\[
\tilde{u}_1(j) = \begin{cases} 
\xi_{u_1}^{(j)}(u_1^{1:j-1}, y_1^n), & \text{if } j \in \mathcal{M}_1^{(n)}, \\
\Psi_1^{(j)}(u_1^{1:j-1}), & \text{otherwise.}
\end{cases} \tag{89}
\]
Given that all previous bits \( \hat{u}_{i}^{1:j-1} \) have been decoded correctly, decoder \( D_1 \) makes a mistake on the \( j \)-th bit \( \hat{u}_{i}(j) \) only if \( j \in \mathcal{M}_{1}^{(n)} \). For the remaining indices, the decoder produces the same bit produced at the encoder due to the shared random maps.

2) Decoding At Second Receiver: The decoder \( D_2 \) decodes the binary sequence \( \hat{u}^{n}_{2} \) using observations \( y^{n}_{2} \). The message \( W_{2} \) is located at the indices \( j \in \mathcal{M}_{2}^{(n)} \) of the sequence \( \hat{u}^{n}_{2} \). Define the following deterministic polar decoding functions

\[
\xi^{(j)}_{\hat{u}^{n}_{2}}(u^{1:j-1}_{2}, y^{n}_{2}) = \arg\max_{u \in \{0,1\}} \left\{ P \left( U_{2}(j) = u \middle| U_{2}^{1:j-1} = u^{1:j-1}_{2}, Y^{n}_{2} = y^{n}_{2} \right) \right\}.
\]

Decoder \( D_2 \) reconstructs \( \hat{u}^{n}_{2} \) bit-by-bit successively as follows using the identical shared random mapping \( \Gamma \) used at the encoder. Including all but \( o(n) \) of the indices,

\[
\hat{u}_{2}(j) = \begin{cases} 
\xi^{(j)}_{\hat{u}^{n}_{2}}(u^{1:j-1}_{2}, y^{n}_{2}), & \text{if } j \in \mathcal{L}_{V_{2}}^{(n)}, \\
\Gamma(j), & \text{if } j \in \mathcal{H}_{V_{2}}^{(n)}. 
\end{cases}
\]

(90)

For those indices \( j \in \Delta_{2} \) where \( \Delta_{2} \) is the set of partially-polarized indices defined in (78), the decoder \( D_{2} \) is provided with “genie-given” bits from the encoder. Thus, all bits are decoded, and \( D_{2} \) only makes a successive cancelation error for those indices \( j \in \mathcal{L}_{V_{2}}^{(n)} \). Communicating the genie-given bits from the encoder to decoder results in a rate penalty. However, since the number of genie-given bits scales asymptotically as \( o(n) \), the rate penalty can be made arbitrarily small.

Remark 15: It is notable that decoder \( D_{2} \) reconstructs \( \hat{u}^{n}_{2} \) using only the observations \( y^{n}_{2} \). At the encoder, the sequence \( u^{n}_{2} \) was generated with the realization of a sequence \( v^{n}_{2} \) as given in (87). However, decoder \( D_{2} \) does not reconstruct the sequence \( \hat{v}^{n}_{2} \). From this operational perspective, Marton’s scheme differs crucially from Cover’s superposition scheme because there does not exist the notion of a “stronger” receiver which reconstructs all the sequences decoded at the “weaker” receiver.

G. Total Variation Bound

To analyze the average probability of error \( P_{e}^{(n)} \), it is assumed that both the encoder and decoder share the randomized mappings \( \Psi_{1}^{(j)} \), \( \Psi_{2}^{(j)} \), and \( \Gamma \) (where \( \Psi_{2}^{(j)} \) is not utilized at decoder \( D_{2} \)). Define the following probability measure on the space of tuples of binary sequences,

\[
Q_{1}(u^{1:n}_{1}, u^{n}_{2}) \triangleq Q(u^{1:n}_{1})Q(u^{n}_{2})
\]

\[
\prod_{j=1}^{n} Q(u_{1}(j)|u^{1:j-1}_{1})Q(u_{2}(j)|u^{1:j-1}_{2}, u^{n}_{1}),
\]

(92)

where the conditional probability measures are defined as

\[
Q(u_{1}(j)|u^{1:j-1}_{1}) \triangleq \begin{cases} 
\frac{1}{2}, & \text{if } j \in \mathcal{M}_{1}^{(n)}, \\
\frac{1}{2}P(u_{1}(j)|u^{1:j-1}_{1}), & \text{otherwise.}
\end{cases}
\]

\[
Q(u_{2}(j)|u^{1:j-1}_{2}, u^{n}_{1}) \triangleq \begin{cases} 
\frac{1}{2}, & \text{if } j \in \mathcal{M}_{2}^{(n)}, \\
\frac{1}{2}P(u_{2}(j)|u^{1:j-1}_{2}, u^{n}_{1}), & \text{otherwise.}
\end{cases}
\]

The probability measure \( Q \) defined in (92) is a perturbation of the joint probability measure \( Q_{1}^{U_{1}^{n}U_{2}^{n}}(u^{1:n}_{1}, u^{n}_{2}) \) in (77). The only difference in definition between \( P \) and \( Q \) is due to those indices in message sets \( \mathcal{M}_{1}^{(n)} \) and \( \mathcal{H}_{V_{2}|V_{1}}^{(n)} \) (note: \( \mathcal{M}_{1}^{(n)} \subseteq \mathcal{H}_{V_{2}|V_{1}}^{(n)} \)). The following lemma provides a bound on the total variation distance between \( P \) and \( Q \). The lemma establishes the fact that inserting uniformly distributed message bits in the proper indices \( \mathcal{M}_{1}^{(n)} \) and \( \mathcal{M}_{2}^{(n)} \) (or the entire set \( \mathcal{H}_{V_{2}|V_{1}}^{(n)} \)) at the encoder does not perturb the statistics of the \( n \)-length random variables too much.

Lemma 8: (Total Variation Bound) Let probability measures \( P \) and \( Q \) be defined as in (77) and (92) respectively. Let \( 0 < \beta < 1 \). For sufficiently large \( n \), the total variation distance between \( P \) and \( Q \) is bounded as

\[
\sum_{u^{1:n}_{1} \in \{0,1\}^{n}} \left| Q_{1}^{U_{1}^{n}U_{2}^{n}}(u^{1:n}_{1}) - Q(u^{1:n}_{1}, u^{n}_{2}) \right| \leq 2^{-n^{\beta}}.
\]

Proof: Omitted. The proof follows via the chain rule for KL-divergence and is identical to the previous proofs of Lemma 11 and Lemma 5.

H. Error Sequences

The decoding protocols for \( D_{1} \) and \( D_{2} \) were established in Section IX-F. To analyze the probability of error of successive cancelation (SC) decoding, consider the sequences \( u^{n}_{1} \) and \( u^{n}_{2} \) formed at the encoder, and the resulting observations \( y^{n}_{1} \) and \( y^{n}_{2} \) received by the decoders. The effective polarized channel \( P_{U_{1}|U_{2}^{n}Y_{2}^{n}}(y^{1:n}_{1}, y^{n}_{2}|u^{1:n}_{1}, u^{n}_{2}) \) was defined in (72) for a fixed \( \phi \) function. It is convenient to group the sequences together and consider all tuples \( (u^{1:n}_{1}, u^{n}_{2}, y^{1:n}_{1}, y^{n}_{2}) \).

Decoder \( D_{1} \) makes an SC decoding error on the \( j \)-th bit for the following tuples:

\[
\mathcal{T}_{1}^{j} \triangleq \begin{cases} 
(u^{1:n}_{1}, u^{n}_{2}, y^{1:n}_{1}, y^{n}_{2}); \\
P_{U_{1}^{1:j-1}|1:y^{n}_{1}}(u_{1}(j)|u^{1:j-1}_{1}, y^{n}_{1}) \leq P_{U_{1}^{1:j-1}|1:y^{n}_{1}}(u_{1}(j) + 1|u^{1:j-1}_{1}, y^{n}_{1})
\end{cases}
\]

(93)

The set \( \mathcal{T}_{1}^{j} \) represents those tuples causing an error at \( D_{1} \) in the case \( u_{1}(j) \) is inconsistent with respect to observations \( y^{n}_{1} \).
and the decoding rule. Similarly, decoder $D_2$ makes an SC decoding error on the $j$-th bit for the following tuples:

$$\mathcal{T}_2^j \triangleq \left\{ (u_1^n, u_2^n, y_1^n, y_2^n) : P_{U_2|U_2^{-j-1}Y_2^n}(u_2^j y_1^n, y_2^n) \right\},$$

$$P_{U_2|U_2^{-j-1}Y_2^n}(u_2^j y_1^n, y_2^n) \leq P_{U_2|U_2^{-j-1}Y_2^n}(u_2^{j-1} y_1^n, y_2^n) + P_{U_2|U_2^{-j-1}Y_2^n}(u_2^j y_1^n, y_2^n) \right\}.$$

The set $\mathcal{T}_2^j$ represents those tuples causing an error at $D_2$ in the case $u_2(j)$ is inconsistent with respect to observations $y_2^n$ and the decoding rule. The set of tuples causing an error is

$$\mathcal{T}_1 \triangleq \bigcup_{j \in M_1^{(n)}} \mathcal{T}_1^j,$$

$$\mathcal{T}_2 \triangleq \bigcup_{j \in M_2^{(n)}} \mathcal{T}_2^j,$$

$$\mathcal{T} \triangleq \mathcal{T}_1 \cup \mathcal{T}_2.$$ (96)

The goal is to show that the probability of choosing tuples of error sequences in the set $\mathcal{T}$ is small under the distribution induced by the broadcast code.

I. Average Error Probability

If the encoder and decoders share randomized maps $\Psi_1^{(j)}$, $\Psi_2^{(j)}$, and $\Gamma$, then the average probability of error is a random quantity determined as follows

$$P_e^{(n)}\left[\{\Psi_1^{(j)}, \Psi_2^{(j)}, \Gamma\}\right] = \sum_{(u_1^n, u_2^n, y_1^n, y_2^n) \in \mathcal{T}} P_{Y_1^n Y_2^n U_1^n U_2^n}(y_1^n, y_2^n | u_1^n, u_2^n) \cdot \frac{1}{2^n R_1} \prod_{j \in [n]: j \notin M_1^{(n)}} \gamma_j(\Psi_1^{(j)}(u_1^{j-1}) = u_1(j)) \cdot \frac{1}{2^n R_2} \prod_{j \in [n]: j \notin M_2^{(n)}} \gamma_j(\Psi_2^{(j)}(u_2^{j-1}, u_1^n \Gamma) = u_2(j)) \cdot \prod_{j \in [n]: j \notin M_1^{(n)} \cup M_2^{(n)}} \gamma_j(\Gamma(j) = u_2(j)).$$

By averaging over the randomness in the encoders and decoders, the expected block error probability $P_e^{(n)}\left[\{\Psi_1^{(j)}, \Psi_2^{(j)}, \Gamma\}\right]$ is upper bounded in the following lemma.

Lemma 9: Consider the polarization-based Marton code described in Section IX-E and Section IX-F. Let $R_1$ and $R_2$ be the broadcast rates selected according to the Bhattacharyya criterion given in Proposition 7. Then for $0 < \beta < 1$ and sufficiently large $n$,

$$\mathbb{E}_{\{\Psi_1^{(j)}, \Psi_2^{(j)}, \Gamma\}} \left[P_e^{(n)}\left[\{\Psi_1^{(j)}, \Psi_2^{(j)}, \Gamma\}\right]\right] < 2^{-n^\beta}.$$  

Proof: See Section 12 of the Appendices.

If the average probability of block error decays to zero in expectation over the random maps $\{\Psi_1^{(j)}\}$, $\{\Psi_2^{(j)}\}$, and $\Gamma$, then there must exist at least one fixed set of maps for which $P_e^{(n)} \to 0$. Hence, polar codes for Marton’s inner bound exist under suitable restrictions on distributions and they achieve reliable transmission according to the advertised rates (except for a small set of $o(n)$ polarization indices as is discussed next).

J. Rate Penalty Due to Partial Polarization

Lemma 9 is true assuming that decoder $D_2$ obtains “ genie-given” bits for the set of indices $\Delta_2$ defined in (78). The set $\Delta_2$ represents those indices that are partially-polarized and which cause a slight misalignment of polarization indices in the Marton scheme. Fortunately, the set $\Delta_2$ contains a vanishing fraction of indices: $\frac{1}{n} |\Delta_2| \leq \eta$ for $\eta > 0$ arbitrarily small and $n$ sufficiently large. Therefore, a two-phase strategy suffices for sending the “ genie-given” bits. In the first phase of communication, the encoder sends several $n$-length blocks while decoder $D_2$ waits to decode. After accumulating several blocks of output sequences, the encoder transmits all the known bits in the set $\Delta_2$ for all the first-phase transmissions. The encoder and decoder can use any reliable point-to-point polar code with non-vanishing rate for communication. Having received the “ genie-aided” bits in the second-phase, the second receiver then decodes all the first-phase blocks. The number of blocks sent in the first-phase is $\mathcal{O}(\frac{1}{\eta})$. The rate penalty is $\mathcal{O}(\eta)$ where $\eta$ can be made arbitrarily small. A similar argument was provided in [24] for designing polar codes for the Gelfand-Pinsker problem.

X. Conclusion

Coding for broadcast channels is fundamental to our understanding of communication systems. Broadcast codes based on polarization methods achieve rates on the capacity boundary for several classes of DM-BCs. In the case of $m$-user deterministic DM-BCs, polarization of random variables from the channel output provides the ability to extract uniformly random message bits while maintaining broadcast constraints at the encoder. As referenced in the literature, maintaining multi-user constraints for the DM-BC is a difficult task for traditional belief propagation algorithms and LDPC codes.

For two-user noisy DM-BCs, polar codes were designed based on Marton’s coding strategy and Cover’s superposition strategy. Constraints on auxiliary and input distributions were placed in both cases to ensure alignment of polarization indices in the multi-user setting. The asymptotic behavior of the average error probability was shown to be $P_e^{(n)} = \mathcal{O}(2^{-n^\beta})$ with an encoding and decoding complexity of $\mathcal{O}(n \log n)$. The next step is to supplement the theory with experimental evidence of the error-correcting capability of polar codes over simulated channels for finite code lengths. The results demonstrate that polar codes have a potential for use in several network communication scenarios.
APPENDIX A
POLAR CODING LEMMAS

The following lemmas provide a basis for proving polar coding theorems. A subset of the lemmas were proven in different contexts, e.g., channel vs. source coding, and contain citations to references.

Lemma 10: Consider two random variables \( X \in \{0, 1\} \) and \( Y \in \mathcal{Y} \) with joint distribution \( P_{X,Y}(x,y) \). Let \( Q(x|y) = \frac{1}{2} \) denote a uniform conditional distribution for \( x \in \{0, 1\} \) and \( y \in \mathcal{Y} \). Then the following identity holds.

\[
D \left( P_{X|Y}(x|y) \biggm| Q(x|y) \right) = 1 - H(X|Y). \tag{97}
\]

**Proof:** The identity follows from standard definitions of entropy and Kullback-Leibler distance.

\[
H(X|Y) = \sum_{y \in \mathcal{Y}} P_Y(y) \sum_{x \in \{0,1\}} P_{X|Y}(x|y) \log_2 \frac{1}{P_X(x|y)}
\]

\[
= \sum_{y \in \mathcal{Y}} P_Y(y) \sum_{x \in \{0,1\}} P_{X|Y}(x|y) \log_2 \frac{1}{Q(x|y)}
\]

\[
- \sum_{y \in \mathcal{Y}} P_Y(y) \sum_{x \in \{0,1\}} P_{X|Y}(x|y) \log_2 \frac{P_X(x|y)}{Q(x|y)}
\]

\[
= \sum_{y \in \mathcal{Y}} P_Y(y) \left[ 1 - \sum_{x \in \{0,1\}} P_{X|Y}(x|y) \log_2 \frac{P_X(x|y)}{Q(x|y)} \right]
\]

\[
= 1 - D \left( P_{X|Y}(x|y) \biggm| Q(x|y) \right).
\]

Lemma 11 (Estimating The Bhattacharyya Parameter):
Let \( (T,V) \sim P_{T,V}(t,v) \) where \( T \in \{0,1\} \) and \( V \in \mathcal{V} \) where \( \mathcal{V} \) is an arbitrary discrete alphabet. Define a likelihood function \( L(v) \) and inverse likelihood function \( L^{-1}(v) \) as follows.

\[
L(v) \triangleq \frac{P_{T|V}(0|v)}{P_{T|V}(1|v)}, \quad L^{-1}(v) \triangleq \frac{P_{T|V}(1|v)}{P_{T|V}(0|v)}
\]

To account for degenerate cases in which \( P_{T|V}(t|v) = 0 \), define the following function,

\[
\varphi(t,v) \triangleq \begin{cases} 
0 & \text{if } \mathbb{I}[P_{T|V}(t|v)=0] \\
L(v) & \text{if } \mathbb{I}[P_{T|V}(t|v)>0] \text{ and } \mathbb{I}_{\{t=1\}} \\
L^{-1}(v) & \text{if } \mathbb{I}[P_{T|V}(t|v)>0] \text{ and } \mathbb{I}_{\{t=0\}}
\end{cases}
\]

In order to estimate \( Z(T|V) \in [0,1] \), it is convenient to sample from \( P_{T|V}(t,v) \) and express \( Z(T|V) \) as an expectation over random variables \( T \) and \( V \).

\[
Z(T|V) = \mathbb{E}_{T,V} \sqrt{\varphi(T,V)}.
\tag{99}
\]

**Proof:** The following forms of the Bhattacharyya parameter are equivalent.

\[
Z(T|V) \triangleq \sum_{v \in \mathcal{V}} P_{V}(v) \sqrt{P_{T|V}(0|v)P_{T|V}(1|v)}
\]

\[
= \sum_{v \in \mathcal{V}} P_{V}(v) \sum_{t \in \{0,1\}} P_{T|V}(t|v) \sqrt{1 - P_{T|V}(t|v)}
\]

\[
= \sum_{t \in \{0,1\}} \sum_{v \in \mathcal{V}} P_{T|V}(t|v) \sqrt{1 - P_{T|V}(t|v)}
\]

\[
= \mathbb{E}_{T,V} \sqrt{\varphi(T,V)}.
\]

Lemma 12 (Stochastic Degradation (cf. [23])): Consider discrete random variables \( V, Y_1, \) and \( Y_2 \). Assume that \(|\mathcal{V}| = 2\) and that discrete alphabets \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \) have an arbitrary size. Then

\[
P_{Y_1|V}(y_1|v) > P_{Y_2|V}(y_2|v) \Rightarrow Z(V|Y_2) \geq Z(V|Y_1). \tag{100}
\]

**Proof:** Beginning with the definition of the Bhattacharyya parameter leads to the following derivation:

\[
Z(V|Y_2) \triangleq \sum_{y_2} \sqrt{P_{V|Y_2}(0,y_2)P_{V|Y_2}(1,y_2)}
\]

\[
= \sum_{y_2} \sqrt{P_{V}(0)P_{V}(1)\sum_{y_1} P_{Y_1|V}(y_1|0)\tilde{P}_{Y_2|Y_1}(y_2|y_1)} \cdot \sum_{y_1} P_{Y_1|V}(y_1|0)\tilde{P}_{Y_2|Y_1}(y_2|y_1)
\]

Then applying the Cauchy–Schwarz inequality yields

\[
Z(V|Y_2) \geq \sum_{y_2} \sqrt{P_{V}(0)P_{V}(1)\sum_{y_1} P_{Y_1|V}(y_1|0)\tilde{P}_{Y_2|Y_1}(y_2|y_1)} \cdot \sum_{y_1} P_{Y_1|V}(y_1|0)\tilde{P}_{Y_2|Y_1}(y_2|y_1)
\]

\[
= \sqrt{P_{V}(0)P_{V}(1)\sum_{y_1} P_{Y_1|V}(y_1|0)\tilde{P}_{Y_2|Y_1}(y_2|y_1)} \cdot \sqrt{P_{Y_1|V}(y_1|0)P_{Y_1|V}(y_1|1)}.
\]
Intercalating the order of summations yields
\[
Z(V|Y_2) 
\geq 2 \sqrt{P_V(0)P_V(1)} \left[ \sum_{y_1} \sqrt{P_{Y_1|V}(y_1|0)P_{Y_1|V}(y_1|1)} \right] 
\cdot \sum_{y_2} \tilde{P}_{Y_2|Y_1}(y_2|y_1) 
= Z(V|Y_1).
\]

**Lemma 13 (Successive Stochastic Degradation (cf. [23]))**: Consider a binary random variable \( V \), and discrete random variables \( Y_1 \) with alphabet \( \mathcal{Y}_1 \), and \( Y_2 \) with alphabet \( \mathcal{Y}_2 \). Assume that the joint distribution \( P_{V,Y_1,Y_2} \) obeys the constraint \( P_{Y_1|V}(y_1|v) \geq P_{Y_2|V}(y_2|v) \). Consider two \( i.i.d \) random copies \( (V^1,Y_1^1,Y_2^1) \) and \( (V^2,Y_1^2,Y_2^2) \) distributed according to \( P_{V,Y_1,Y_2} \). Define two binary random variables \( U^1 \triangleq V^1 \oplus V^2 \) and \( U^2 \triangleq V^2 \). Then the following holds
\[
Z(U^1|Y_2^{2|1}) \geq Z(U^1|Y_1^{2|1}), \quad Z(U^2|U^1,Y_1^{2|1}) \geq Z(U^2|U^1,Y_1^{1|2}). \quad (101)
\]

**Proof**: Given the assumptions, the following stochastic degradation conditions hold:
\[
P_{Y_1^2|V^1}(y_1^2|u^1) \geq P_{Y_2^1|V^1}(y_2^1|u^1), \quad (103)
P_{Y_2^1|V^2}(y_2^1|u^2) \geq P_{Y_2^2|V^2}(y_2^2|u^2), \quad (104)
\]
The goal is to derive new stochastic degradation conditions for the polarized conditional distributions. The binary random variables \( U^1 \) and \( U^2 \) are not necessarily independent Bernoulli\((\frac{1}{2})\) variables. Taking this into account,
\[
P_{Y_2^1,Y_2^2|U^1}(y_2^1,y_2^2|u^1) 
= \frac{1}{P_{U^1}(u^1)} \sum_{u^2 \in \{0,1\}} P_{V,Y_2}(u^1 \oplus u^2, y_2^1) P_{V,Y_2}(u^2, y_2^2) 
= \frac{1}{P_{U^1}(u^1)} \sum_{u^2 \in \{0,1\}} \left[ P_{Y_2^1|V^1}(y_2^1|u^1 \oplus u^2) P_{V^1}(u^1 \oplus u^2) 
\cdot P_{Y_2^2|V^2}(y_2^2|u^2) P_{V^2}(u^2) \right].
\]
Applying the property due to the assumption in (103),
\[
P_{Y_2^1,Y_2^2|U^1}(y_2^1,y_2^2|u^1) 
= \frac{1}{P_{U^1}(u^1)} \sum_{u^2 \in \{0,1\}} \left[ P_{V^1}(u^1 \oplus u^2) P_{V^2}(u^2) 
\cdot \sum_{a \in \mathcal{Y}_1} P_{Y_2^1|V^1}(a|u^1 \oplus u^2) \tilde{P}_{Y_2^1|Y_1^1}(y_2^1|a) 
\cdot \sum_{b \in \mathcal{Y}_1} P_{Y_2^2|V^2}(b|u^2) \tilde{P}_{Y_2^2|Y_1^2}(y_2^2|b) \right].
\]
Intercalating the order of summations and grouping the terms representing \( P_{Y_1^1,Y_2^1|U^1}(y_1^1,y_2^1|u^1) \) yields the following
\[
P_{Y_2^1,Y_2^2|U^1}(y_2^1,y_2^2|u^1) 
= \sum_{a \in \mathcal{Y}_1,b \in \mathcal{Y}_1} P_{Y_2^1|Y_1^1}(a,b|u^1) \tilde{P}_{Y_2^1|Y_1^1}(y_2^1|a) \tilde{P}_{Y_2^2|Y_1^2}(y_2^2|b). \]
The above derivation proves that
\[
P_{Y_2^1,Y_2^2|U^1}(y_2^1,y_2^2|u^1) \succ P_{Y_2^1,Y_2^2|U^1}(y_2^1,y_2^2|u^1). \]

**Lemma 12** this concludes the proof for the ordering of the Bhattacharyya parameters given in (101).

In a similar way, it is possible to show that
\[
P_{Y_2^1,Y_2^2|U^1,Y_2^2}(y_2^1,y_2^2,u^1|u^2) 
= \frac{1}{P_{U^2}(u^2)} \left[ P_{V^2}(u^1 \oplus u^2) P_{V^2}(u^2) \right] 
\cdot \sum_{a \in \mathcal{Y}_1} P_{Y_2^1|V^1}(a|u^1 \oplus u^2) \tilde{P}_{Y_2^1|Y_1^1}(y_2^1|a) 
\cdot \sum_{b \in \mathcal{Y}_1} P_{Y_2^2|V^2}(b|u^2) \tilde{P}_{Y_2^2|Y_1^2}(y_2^2|b). \]

Applying the property due to the assumption in (102),
\[
P_{Y_2^1,Y_2^2|U^1,Y_2^2}(y_2^1,y_2^2,u^1|u^2) 
= \frac{1}{P_{U^2}(u^2)} \left[ P_{V^2}(u^1 \oplus u^2) P_{V^2}(u^2) \right] 
\cdot \sum_{a \in \mathcal{Y}_1} P_{Y_2^1|V^1}(a|u^1 \oplus u^2) \tilde{P}_{Y_2^1|Y_1^1}(y_2^1|a) 
\cdot \sum_{b \in \mathcal{Y}_1} P_{Y_2^2|V^2}(b|u^2) \tilde{P}_{Y_2^2|Y_1^2}(y_2^2|b) \right]. \]

Intercalating the order of the terms and grouping the terms representing \( P_{Y_1^1,Y_2^1|U^1,Y_2^1}(y_1^1,y_2^1,u^1|u^2) \) yields the following
\[
P_{Y_2^1,Y_2^2|U^1,Y_2^2}(y_2^1,y_2^2,u^1|u^2) 
= \sum_{a \in \mathcal{Y}_1,b \in \mathcal{Y}_1} P_{Y_2^1|Y_1^1}(a,b|u^1,u^2) \tilde{P}_{Y_2^1|Y_1^1}(y_2^1|a) \tilde{P}_{Y_2^2|Y_1^2}(y_2^2|b) \right]. \]

The above derivation proves that
\[
P_{Y_2^1,Y_2^2|U^1,Y_2^2}(y_2^1,y_2^2,u^1|u^2) \succ P_{Y_2^1,Y_2^2|U^1,Y_2^2}(y_2^1,y_2^2,u^1|u^2). \]

Combined with Lemma 12 this concludes the proof for the ordering of the Bhattacharyya parameters given in (102).
Lemma 14 (Pinck's Inequality): Consider two discrete probability measures $P(y)$ and $Q(y)$ for $y \in \mathcal{Y}$. The following inequality holds for a constant $\kappa \triangleq 2 \ln 2$.

$$\sum_{y \in \mathcal{Y}} |P(y) - Q(y)| \leq \sqrt{\kappa D(P(y)||Q(y))}.$$  

Lemma 15 (Arıkan [8]): Consider two discrete random variables $X \in \{0, 1\}$ and $Y \in \mathcal{Y}$. The Bhattacharyya parameter and conditional entropy are related as follows.

$$Z(X|Y)^2 \leq H(X|Y)$$

$$H(X|Y) \leq \log_2(1 + Z(X|Y))$$

Lemma 16 (Bhattacharyya vs. Entropy Parameters): Consider two discrete random variables $X \in \{0, 1\}$ and $Y \in \mathcal{Y}$. For any $0 < \delta < \frac{1}{2}$,

$$Z(X|Y) \geq 1 - \delta \Rightarrow H(X|Y) \geq 1 - 2\delta.$$ 

$$Z(X|Y) \leq \delta \Rightarrow H(X|Y) \leq \log_2(1 + \delta).$$

Proof: Due to Lemma 15, $H(X|Y) \geq Z(X|Y)^2 \geq (1 - \delta)^2 \geq 1 - 2\delta + \delta^2 \geq 1 - 2\delta$. It follows that if $Z(X|Y) \geq 1 - \delta$ and $\delta \to 0$, then $H(X|Y) \to 1$ as well. Similarly, due to Lemma 15 taking constant $\kappa = \frac{1}{1 + \delta}$ and using the series expansion of $\log_2(1 + \delta)$, if $Z(X|Y) \leq \delta$ then $H(X|Y) \leq \log_2(1 + \delta) = \kappa \left( \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\delta^k}{k} \right) \leq \kappa \delta$. It follows that if $Z(X|Y) \leq \delta$ and $\delta \to 0$, then $H(X|Y) \to 0$ as well. \qed

Appendix B

Proof of Lemma 11

The total variation bound of Lemma 11 is decomposed in a simple way due to the chain rule for Kullback-Leibler distance between discrete probability measures. The joint probability measures $P$ and $Q$ were defined in [8] and [19] respectively. According to definition, if $P(\{u_i^{1:n}\}_{i \in [m]}) > 0$ then $Q(\{u_i^{1:n}\}_{i \in [m]}) > 0$. Therefore the Kullback-Leibler distance $D(P||Q)$ is well-defined and upper bounded as follows.

$$D(P(\{u_i^{1:n}\}_{i \in [m]})||Q(\{u_i^{1:n}\}_{i \in [m]}))$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} D\left( P \left( u_i(j) \big| \{u_i^{1:j-1}\}_{i \in [k]} \right) \right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \left[ D\left( P \left( u_i(j) \big| \{u_i^{1:j-1}\}_{i \in [k]} \right) \right) \right]_{(105)}$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} \left[ 0 \right] \left( \mathcal{M}_i^{(n)} \right)$$

$$\leq \sum_{i=1}^{m} \sum_{j=1}^{n} \left( \mathcal{M}_i^{(n)} \right)$$

The equality in (108) is due to the chain rule for Kullback-Leibler distance. The equality in (106) is valid because for indices $j \notin \mathcal{M}_i^{(n)}$, $P(\{u_i(j)\}_{i \in [k]} \{u_i^{1:n}\}_{k \in [i-1]}) = Q(\{u_i(j)\}_{i \in [k]} \{u_i^{1:n}\}_{k \in [i-1]})$. The equality in (107) is valid due to Lemma 10 and the fact that $Q(\{u_i(j)\}_{i \in [k]} \{u_i^{1:n}\}_{k \in [i-1]}) = \frac{1}{2}$ for indices $j \notin \mathcal{M}_i^{(n)}$. The equality in (108) follows due to the one-to-one correspondence between variables $\mathcal{M}_i^{(n)}$ and $\{Y_i^{1:n}\}_{k \in [i-1]}$. The last inequality (109) follows from Lemma 16 due to the fact that $Z\left( U_i(j) \big| U_i^{1:j-1}, \{Y_i^{1:n}\}_{k \in [i-1]} \right) \geq 1 - \delta_n$ for indices $j \in \mathcal{M}_i^{(n)}$.

To finish the proof of Lemma 11,

$$\sum_{\{u_i^{1:n}\}_{k \in [m]}} \left[ P(\{u_i^{1:n}\}_{k \in [m]}) - Q(\{u_i^{1:n}\}_{k \in [m]}) \right]$$

$$\leq \sqrt{\kappa D\left( P \left( \{u_i^{1:n}\}_{k \in [m]} \right) \bigg| Q \left( \{u_i^{1:n}\}_{k \in [m]} \right) \right)}$$

$$\leq \sqrt{\kappa \sum_{i=1}^{m} 2\delta_n \left| \mathcal{M}_i^{(n)} \right|}$$

$$\leq \sqrt{(2\kappa)(m \cdot \delta_n)(2^{-n^\alpha})}.$$  

The inequality in (110) is due to Pinsker’s inequality given
in Lemma 14. The inequality in (111) was proven in (109). Finally for \(\beta' \in (\beta, \frac{1}{2})\), \((2\kappa)(m \cdot n)(2^{-n^2}) < 2^{-n^2}\) for sufficiently large \(n\). Hence the total variation distance is bounded by \(O(2^{-n^2})\) for any \(0 < \beta < \frac{1}{2}\).

**APPENDIX C**

**SUPERPOSITION CODING**

The total variation bound of Lemma 5 is decomposed in a simple way due to the chain rule for Kullback-Leibler distance between discrete probability measures. The joint probability measures \(P\) and \(Q\) were defined in (30) and (60) respectively. According to definition, if \(P_{U_1^T U_2^n} (u_1^n, u_2^n) > 0\) then \(Q(u_1^n, u_2^n) > 0\). Therefore the Kullback-Leibler distance \(D(P||Q)\) is well-defined. Applying the chain rule,

\[
D(P_{U_1^T U_2^n} (u_1^n, u_2^n) \parallel Q(u_1^n, u_2^n)) = \sum_{j=1}^{n} D(P(u_1(j)|u_1^{1:j-1}) \parallel Q(u_1(j)|u_1^{1:j-1}) + \sum_{j=1}^{n} D(P(u_2(j)|u_2^{1:j-1}, u_1^n) \parallel Q(u_2(j)|u_2^{1:j-1}, u_1^n)) \]

\[
\quad = \sum_{j \in M_1^{(n)}} D(P(u_1(j)|u_1^{1:j-1}) \parallel Q(u_1(j)|u_1^{1:j-1})) + \sum_{j \in M_2^{(n)}} D(P(u_2(j)|u_2^{1:j-1}, u_1^n) \parallel Q(u_2(j)|u_2^{1:j-1}, u_1^n))
\]

Applying Lemma 16, the one-to-one relation between \(U_1^T\) and \(V^n\), and Lemma 16 leads to the following result.

\[
D(P_{U_1^T U_2^n} (u_1^n, u_2^n) \parallel Q(u_1^n, u_2^n)) = \sum_{j \in M_1^{(n)}} [1 - H(U_1(j)|U_1^{1:j-1})] + \sum_{j \in M_2^{(n)}} [1 - H(U_2(j)|U_2^{1:j-1}V^n)] \leq 2\delta_n \left| M_1^{(n)} + M_2^{(n)} \right|
\]

Using identical arguments as applied in the proof of Lemma 11, the total variation distance between \(P\) and \(Q\) is bounded as \(O(2^{-n^2})\).

To prove Lemma 6, the expectation of the average probability of error of the polarization-based superposition code is written as

\[
E_{\{\Psi_j^{(1)}, \Psi_j^{(2)}\}} \left[ P_e(n) \left( \{ \Psi_j^{(1)}, \Psi_j^{(2)} \} \right) \right] = \sum_{\{u_1^n, u_2^n, y_1^n, y_2^n\} \in T} \left[ \prod_{j \in [n]} \mathbb{P} \left( \Psi_j^{(1)} = 1 \right) \right] \left[ \mathbb{P} \left( \Psi_j^{(2)} = u_2^{1:j-1} = u_2^n \right) \right]
\]

From the definitions of the random boolean functions \(\Psi_j^{(1)}\) in (52) and \(\Psi_j^{(2)}\) in (53), it follows that

\[
\mathbb{P} \left( \Psi_1^{(1)} = u_1^{1:j-1} \right) = u_1(j)
\]

\[
\mathbb{P} \left( \Psi_2^{(1)} = u_2^{1:j-1} \right) = u_2(j)
\]

The expression for the expected average probability of error is then simplified by substituting the definition for \(Q(u_1^n, u_2^n)\) provided in (60) as follows,

\[
E_{\{\Psi_j^{(1)}, \Psi_j^{(2)}\}} \left[ P_e(n) \left( \{ \Psi_j^{(1)}, \Psi_j^{(2)} \} \right) \right] = \sum_{\{u_1^n, u_2^n, y_1^n, y_2^n\} \in T} \left[ P_{Y_1^n Y_2^n U_1^T U_2^n} (y_1^n, y_2^n | u_1^n, u_2^n) Q(u_1^n, u_2^n) \right]
\]

The next step in the proof is to split the error term \(E_{\{\Psi_j^{(1)}, \Psi_j^{(2)}\}} \left[ P_e(n) \left( \{ \Psi_j^{(1)}, \Psi_j^{(2)} \} \right) \right]\) into two main parts, one part due to the error caused by polar decoding functions, and the other part due to the total variation distance between probability measures.

\[
E_{\{\Psi_j^{(1)}, \Psi_j^{(2)}\}} \left[ P_e(n) \left( \{ \Psi_j^{(1)}, \Psi_j^{(2)} \} \right) \right] = \sum_{\{u_1^n, u_2^n, y_1^n, y_2^n\} \in T} \left[ P_{Y_1^n Y_2^n U_1^T U_2^n} (y_1^n, y_2^n | u_1^n, u_2^n) \right] \left[ Q(u_1^n, u_2^n) - P_{U_1^T U_2^n} (u_1^n, u_2^n) + P_{U_1^T U_2^n} (u_1^n, u_2^n) \right]
\]

\[
\leq \sum_{\{u_1^n, u_2^n, y_1^n, y_2^n\} \in T} \left[ \sum_{u_1^n \in (0,1)^n} \left[ P_{U_1^T U_2^n} (u_1^n, u_2^n) - Q(u_1^n, u_2^n) \right] \right] + \sum_{u_2^n \in (0,1)^n} \left[ P_{U_1^T U_2^n} (u_1^n, u_2^n) - Q(u_1^n, u_2^n) \right].
\]
it remains to upper bound the error term due to the polar decoding functions. Towards this end, note first that \( \mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2, \mathcal{T}_1 = \cup_j \mathcal{T}_i^j \), for \( j \in \mathcal{M}_2^{(n)} \subseteq \mathcal{M}_1^{(n)} \), \( \mathcal{T}_i = \cup_j \mathcal{T}_i^j \) for \( j \in \mathcal{M}_1^{(n)} \), and \( \mathcal{T}_2 \subseteq \cup_j \mathcal{T}_2^j \) for \( j \in \mathcal{M}_2^{(n)} \). It is convenient to bound each type of error bit by bit successively at both decoder \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) as follows.

\[
\mathcal{E}_{1v}^j \triangleq \sum_{(u_1^n, u_2^n, y_1^n, y_2^n) \in \mathcal{T}_i^j} \mathcal{P}(u_1^n, u_2^n, y_1^n, y_2^n) = \sum_{(u_1^n, u_2^n, y_1^n, y_2^n) \in \mathcal{T}_i^j} \mathcal{P}(u_1^n, u_2^n, y_1^n, y_2^n)
\]

In this form, it is possible to upper bound the error term \( \mathcal{E}_{1v}^j \) with the corresponding Bhattacharyya parameter as follows,

\[
\mathcal{E}_{1v}^j = \sum_{u_1^n, u_2^n, y_1^n, y_2^n} \mathcal{P}(u_1^n, u_2^n, y_1^n, y_2^n)
\]

\[
\mathcal{E}_{1v}^j = \sum_{u_1^n, u_2^n, y_1^n, y_2^n} \mathcal{P}(u_1^n, u_2^n, y_1^n, y_2^n)
\]

This concludes the proof demonstrating that the expected average probability of error is upper bounded as \( \mathcal{O}(2^{-n^*}) \).

**APPENDIX D**

**MARTON CODING**

To prove Lemma 9 the expectation of the average probability of error of the polarization-based Marton code is written as

\[
\mathbb{E}_{\{\psi_{1/2}^{(j)}, \Gamma\}}[\mathcal{P}_e^{(n)}(\{\psi_{1/2}^{(j)}, \Gamma\})] =
\]

\[
\sum_{\{u_1^n, u_2^n, y_1^n, y_2^n\} \in \mathcal{T}} \mathcal{P}(u_1^n, u_2^n, y_1^n, y_2^n)
\]

\[
\frac{1}{2^{nR_2}} \prod_{j \in [n]: j \notin \mathcal{M}_1^{(n)}} \mathbb{P}(\Gamma(j) = u_2(j))
\]

\[
\prod_{j \in [n]: j \notin \mathcal{M}_1^{(n)}} \mathbb{P}(\psi_{1/2}^{(j)}(u_1^{j-1}, u_1^n G_n) = u_2(j))
\]

The expression is then simplified by substituting the definition of \( Q(u_1^n, u_2^n) \) provided in (92), and then splitting the error term into two parts:

\[
\mathbb{E}_{\{\psi_{1/2}^{(j)}, \Gamma\}}[\mathcal{P}_e^{(n)}(\{\psi_{1/2}^{(j)}, \Gamma\})] =
\]

\[
\sum_{\{u_1^n, u_2^n, y_1^n, y_2^n\} \in \mathcal{T}} \mathcal{P}(u_1^n, u_2^n, y_1^n, y_2^n) Q(u_1^n, u_2^n)
\]

\[
+ \sum_{\{u_1^n, u_2^n, y_1^n, y_2^n\} \in \mathcal{T}} \mathcal{P}(u_1^n, u_2^n, y_1^n, y_2^n) - Q(u_1^n, u_2^n)
\]

Therefore, the total error due to decoding at the receivers is upper bounded as

\[
\mathcal{E} \triangleq \sum_{\{u_1^n, u_2^n, y_1^n, y_2^n\} \in \mathcal{T}} \mathcal{P}(u_1^n, u_2^n, y_1^n, y_2^n)
\]

\[
\leq \sum_{j \in [n]: j \notin \mathcal{M}_1^{(n)}} Z(u_1^n, u_2^n, y_1^n, y_2^n)
\]

\[
+ \sum_{j \in [n]: j \notin \mathcal{M}_1^{(n)}} Z(u_1^n, u_2^n, y_1^n, y_2^n)
\]

\[
\leq \delta_n \left[ |\mathcal{M}_1^{(n)}| + |\mathcal{M}_2^{(n)}| \right]
\]

\[
\leq 3n\delta_n.
\]
The error term pertaining to the total variation distance was already upper bounded as in Lemma 8. The error due to successive cancelation decoding at the receivers is upper bounded as follows.

\[
\mathcal{E} \triangleq \sum_{(u_1^n, v_1^n, y_1^n) \in \mathcal{T}} P_{U_1^n, U_2^n, Y_1^n} P_{Y_2}^n (u_1^n, u_2^n, y_1^n, y_2^n) 
\leq \sum_{j \in \mathcal{V}_1 \setminus \mathcal{V}_2} Z(U_1^n U_2^{j-1}) + \sum_{j \in \mathcal{V}_2 \setminus \mathcal{V}_1} Z(U_2^n U_1^{j-1}),
\]

\[
\leq \delta_n \left( |\mathcal{M}_1(n)| + |\mathcal{M}_2(n)| \right) 
\leq 2n \delta_n.
\]

This concludes the proof demonstrating that the expectation of the average probability of block error is upper bounded as \(\mathcal{O}(2^{-n^3})\).

**APPENDIX E
PROOF OF LEMMA 8**

The implication in (30) follows since \(X - Y_1 - Y_2\) means that \(P_{Y_1|X}(y_1|x) = \sum_{y_2} P_{Y_2|X}(y_2|x) P_{Y_1|Y_2}(y_1|y_2).\) The implication in (31) follows by observing that

\[
P_{Y_2|Y_1}(y_1|y_2) = \sum_{y_1} P_{Y_2|X}(y_2|x) P_{Y_1|Y_2}(y_1|y_2) \quad \text{for all } x \in X.
\]

Similarly, \(P_{Y_1|Y_2}(y_2|y_2) = P_{Y_1|X}(y_2|y_2)\) for any \(v \in X.\) Due to the assumed stochastic degradedness condition \(P_{Y_1|X}(y_1|v) = \sum_{y_2} P_{Y_2|X}(y_2|v) P_{Y_1|Y_2}(y_1|y_2),\) for any \(v \in X,
\]

\[
P_{Y_2|X}(y_2|v) = P_{Y_2|Y_1}(y_2|v) = \sum_{y_1} P_{Y_1|X}(y_1|v) P_{Y_2|Y_1}(y_2|y_1) \quad \text{for all } y_1, y_2 \in Y.
\]

Therefore the stochastic degradedness property \(P_{Y_1|X}(y_1|x) \succ P_{Y_1|Y_2}(y_1|y_2)\) must hold as well. The statement of (31) means that Class I and Class II are equivalent as shown in Figure 3. The implication in (32) follows because assuming the stochastic degradedness property \(P_{Y_1|X}(y_1|v) \succ P_{Y_2|Y_1}(y_2|v)\) holds for all \(P_{X|V}(x|v),\) there exists a \(Y_1\) such that \(V - Y_1 - Y_2\) form a Markov chain and \(P_{Y_1|V}(y_1|v) = P_{Y_1|X}(y_1|v)\) for all \(P_{X|V}(x|v)\). By the data processing inequality, \(I(V; Y_1) \geq I(V; Y_2).\) If \(P_{Y_1|V}(y_1|v) = P_{Y_1|Y_2}(y_1|v),\) then \(P_{Y_1|Y_1}(v, y_1) = P_{Y_1|Y_2}(v, y_1)\) for all \(P_{X}(v).\) It follows that for all \(P_{X}(v),\) the mutual information \(I(V; Y_1) = I(V; Y_2).\) The implication in (33) follows by setting \(P_{X}(v, x) = \mathbb{1}_{[v=x]} P_{X}(x)\) and letting \(V = \mathcal{X}.\) Then for any \(v \in \mathcal{X},
\]

\[
P_{X}(v, y_1) = \sum_{x \in \mathcal{X}} P_{X}(v, x) P_{Y_1|X}(y_1|x) \quad \text{for all } x \in \mathcal{X}
\]

Similarly for any \(v \in \mathcal{X}, P_{X}(v, y_2) = P_{X}(v, y_2).\) Therefore for the particular choice of \(P_{X}(v, x) = \mathbb{1}_{[v=x]} P_{X}(x),\)

\[I(V; Y_1) = I(X; Y_1)\] and \(I(V; Y_2) = I(X; Y_2).\) The converse statements for (30), (32), and (33) do not hold due to a counterexample involving a DM-BC comprised of a binary erasure channel BEC(\(\epsilon\)) and a binary symmetric channel BSC(\(p\)) as described in Example 3.

**REFERENCES**

[1] T. M. Cover, “Broadcast channels,” IEEE Transactions on Information Theory, vol. 18, pp. 2–14, January 1972.

[2] A. E. Gamal and Y.-H. Kim, Network Information Theory. New York: Cambridge University Press, 2011.

[3] K. Marton, “A coding theorem for the discrete memoryless broadcast channel,” IEEE Transactions on Information Theory, vol. 25, pp. 306–311, May 1979.

[4] C. Nair and A. E. Gamal, “An outer bound to the capacity region of the broadcast channel,” IEEE Transactions on Information Theory, vol. 53, pp. 350–355, January 2007.

[5] T. M. Cover, “Comments on broadcast channels,” IEEE Transactions on Information Theory, vol. 44, pp. 2524–2530, October 1998.

[6] E. Arikan, “Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels,” IEEE Transactions on Information Theory, vol. 55, pp. 3051–3073, July 2009.

[7] N. Goela, E. Abbe, and M. Gastpar, “Polar codes for the deterministic broadcast channel,” in Proc. International Zurich Seminar on Communications, (Zurich, Switzerland), pp. 51–54, February 2012.
[8] E. Arikan, “Source polarization,” in Proc. of the IEEE International Symposium on Information Theory, June 2010.

[9] E. Arikan, “Polar coding for the slepian-wolf problem based on monotone chain rules,” in Proc. of the International Symposium on Information Theory, July 2012.

[10] E. Abbe and E. Telatar, “Polar codes for the m-user multiple access channel,” IEEE Transactions on Information Theory, vol. 58, pp. 5437–5448, August 2012.

[11] W. Yu and M. Aksenovic, “Coding for the blackwell channel: A survey propagation approach,” in Proc. of the IEEE International Symposium on Information Theory, (Adelaide, Australia), September 2005.

[12] T. P. Coleman, E. Martinian, M. Effros, and M. Medard, “Interference management via capacity-achieving codes for the deterministic broadcast channel,” in Proc. IEEE Information Theory Workshop, pp. 23–27, September 2005.

[13] A. Braunstein, F. Kayhan, G. Montorsi, and R. Zecchina, “Encoding for the blackwell channel with reinforced belief propagation,” in Proc. of the IEEE International Symposium on Information Theory, (Nice, France), 2007.

[14] Z. Bar-Yossef, Y. Birk, T. S. Jayram, and T. Kol, “Index coding with side information,” IEEE Transactions on Information Theory, vol. 57, no. 3, pp. 1479–1494, 2011.

[15] S. E. Rouayheb, A. Sprintson, and C. Georghiades, “On the index coding problem and its relation to network coding and matroid theory,” IEEE Transactions on Information Theory, vol. 56, no. 7, pp. 3187–3195, 2010.

[16] E. Arikan and E. Telatar, “On the rate of channel polarization,” in Proc. of the IEEE International Symposium on Information Theory, (Seoul, South Korea), July 2009.

[17] E. Hof, I. Sason, and S. Shamai, “Polar coding for reliable communications over parallel channels,” CoRR, vol. abs/1005.2770, 2010.

[18] M. Karzand, “Polar codes for degraded relay channels,” in Proc. International Zurich Seminar on Communications, (Zurich, Switzerland), pp. 59–62, February 2012.

[19] R. Blasco-Serrano, R. Thobaben, M. Andersson, V. Rathi, and M. Skoglund, “Polar codes for cooperative relaying,” IEEE Transactions on Communications, vol. 60, pp. 3263–3273, November 2012.

[20] M. Andersson, V. Rathi, R. Thobaben, J. Kilewer, and M. Skoglund, “Nested polar codes for wiretap and relay channels,” IEEE Communications Letters, vol. 14, pp. 752–754, August 2010.

[21] H. Mahdavifar and A. Vardy, “Achieving the secrecy capacity of wiretap channels using polar codes,” IEEE Transactions on Information Theory, vol. 57, pp. 6428–6443, October 2011.

[22] O. Koyluoglu and H. El Gamal, “How to construct polar codes,” IEEE Transactions on Information Forensics and Security, vol. 7, pp. 1472–1483, October 2012.

[23] S. B. Korada, Polar Codes for Channel and Source Coding. PhD thesis, EPFL, 2009.

[24] S. B. Korada and R. L. Urbanke, “Polar codes are optimal for lossy source coding,” IEEE Transactions on Information Theory, vol. 56, no. 4, pp. 1751–1768, 2010.

[25] E. Abbe, “Randomness and dependencies extraction via polarization,” in Proc. of the Information Theory and Applications (ITA) Workshop, (San Diego, California), February 2011.

[26] A. Korada, E. Sasoglu, and R. Urbanke, “Polar codes: Characterization of exponent, bounds, and constructions,” IEEE Transactions on Information Theory, vol. 56, pp. 6253–6264, December 2010.

[27] E. Sasoglu, E. Telatar, and E. Arikan, “Polarization for arbitrary discrete memoryless channels,” CoRR, vol. abs/0908.0302, 2009.

[28] R. Mori and T. Tanaka, “Channel polarization on q-ary discrete memoryless channels by arbitrary kernels,” in Proc. IEEE International Symposium on Information Theory, pp. 894–898, June 2010.

[29] A. Sahebi and S. Pradhan, “Multilevel polarization of polar codes over arbitrary discrete memoryless channels,” in Proc. of the IEEE International Symposium on Information Theory, 2011.

[30] W. Park and A. Barg, “Polar codes for q-ary channels, q = 2^n,” IEEE Transactions on Information Theory, vol. 59, pp. 955–969, February 2013.

[31] E. Sasoglu, “Polarization and polar codes,” Foundations and Trends in Comm. and Information Theory, vol. 8, no. 4, pp. 259–381, 2012.