THE LAGRANGE BITOP ON $so(4) \times so(4)$
AND GEOMETRY OF THE PRYM VARIETIES

VLADIMIR DRAGOVIĆ AND BORISLAV GAJIĆ

Abstract. A four-dimensional integrable rigid-body system is considered and it is shown that it represents two twisted three-dimensional Lagrange tops. A polynomial Lax representation, which doesn’t fit neither in Dubrovin’s nor in Adler-van Moerbeke’s picture is presented. The algebro-geometric integration procedure is based on deep facts from the geometry of the Prym varieties of double coverings of hyperelliptic curves. The correspondence between all such coverings with Prym varieties split into two varieties of the same dimension and the integrable hierarchy associated to the initial system is established.

addresses:

V. D. International School for Advanced Studies, Via Beirut 2-4, Trieste, Italy

Mathematical Institute SANU, Kneza Mihaila 35, Belgrade, Yugoslavia

email vladad@mi.sanu.ac.yu and dragovic@sissa.it

B. G. Mathematical Institute SANU, Kneza Mihaila 35, Belgrade, Yugoslavia

email gajab@mi.sanu.ac.yu
1. Introduction

Starting from the end of 60’s, the Lax representation has been the one of the most powerful tools in the inverse scattering method for the integration of nonlinear differential equations, partial and ordinary as well. The Lax equation

$$\dot{L}(\lambda) = [L(\lambda), A(\lambda)]$$

with $L(\lambda), A(\lambda)$ being matrix polynomials in so called spectral parameter $\lambda$ were studied in the middle seventies by Dubrovin [12, 15]. That theory was based on the notion of the Baker-Akhiezer function, developed by Krichever and others from Novikov’s school (see [20, 14]). The $L - A$ pairs of such type and Dubrovin’s theory were used in the algebro-geometric integration of rigid-body motion in [21, 9].

Few years later, Adler and van Moerbeke presented a new version of such a theory in [1]. It was based on [22] and also applied in the integration of rigid-body motion in several cases [1, 25, 26].

Recently we have found a Lax representation of that, polynomial, form for the classical Hess-Apel’rot system, see [11]. Generalizing it, we constructed a new completely integrable system of the classical Euler-Poisson equations of motion of a heavy four-dimensional rigid body fixed at a point. Together with generalized Lagrange case and generalized symmetric case, which were introduced by Ratiu (see [25]), this system exosted the list of integrable Euler-Poisson equations with the $L$ operator, which is a quadratic polynomial in $\lambda$ of the form

$$L(\lambda) = \lambda^2 C + \lambda M + \Gamma.$$ 

The principal aim of this paper is to give algebro-geometric integration of this new system.

However, it appeared that this system did not fit exactly neither in Dubrovin’s nor in Adler-van Moerbeke’s picture. The matrix $L$ satisfies the condition

$$L_{12} = L_{21} = L_{34} = L_{43} = 0.$$ (1)

Such situation is explicitly excluded by Adler-van Moerbeke (see [1], Theorem 1) and not so explicitly by Dubrovin (see [12], Lemma 5 and Corollary). Up to our knowledge, examples which satisfy the condition of type (1) have not been studied before.

Analysis of the spectral curve and the Baker-Akhiezer function shows that dynamics of the system is related to certain Prym variety $\Pi$ (which splits according to the Mumford-Dalalian theory [24, 10, 28]) and evolution of divisors of some meromorphic differentials $\Omega^i_j$. Then the condition (1) requires that differentials

$$\Omega^1_2, \Omega^2_1, \Omega^3_4, \Omega^4_3$$

have to be holomorphic during the whole evolution. Compatibility of this requirement with dynamics possesses a strong constraint on the spectral curve: its theta divisor should contain some torus. In the case presented here such constraint appears to be satisfied according to Mumford’s relation (see [24])

$$\Pi^- \subset \Theta,$$ (2)
where $\Pi^-$ is a translation of the Prym variety $\Pi$.

The paper is organized as follows. In section 2 the definition of the system of the Euler-Poisson equations on $so(4) \times so(4)$ is given and few of its basic properties are listed such as the $L - A$ pair, a set of first integrals in involution. In the section 3 transformation of coordinates is performed in classical manner and the connection with the Lagrange top is presented. The spectral curve is described in section 4. In section 5 the Baker-Akhiezer function was studied. The next, section 6, contains analysis of the Prym variety $\Pi$ and via the Mumford-Dalalian theory, the connection of algebro-geometric and classical approach from section 3 is established. In the section 7 differentials $\Omega_{ij}$ are defined and the holomorphicity condition is derived from the condition (1). By using Mumford’s relation (2) formulae in the theta functions were derived. Necessary gluing of the infinite points of the spectral curve and passage to the generalized Jacobian is done in the section 8. The whole hierarchy of the Lagrange bitop is considered in the section 9. The higher operators $L_N$, are polynomials of order $N$ in $\lambda$. Their spectral curves are double covering of hyperelliptic curves of genus $2N - 1$; these coverings are defined by those divisors of order two which are of degree $2N$. We can conclude by saying that the Lagrange bitop hierarchy realizes all coverings of that kind.

2. The definition of the system and basic properties

The equations of motion of a heavy $n$-dimensional rigid body fixed at a point in the moving frame are:

\begin{align*}
\dot{M} &= [M, \Omega] + [\Gamma, \chi], \\
\dot{\Gamma} &= [\Gamma, \Omega],
\end{align*}

where the moving frame is such that the matrix $I$ is diagonal in it, $\text{diag}(I_1, \ldots, I_n)$. Here $M_{ij} = (I_i + I_j)\omega_{ij} \in so(n)$ is the kinetic momentum, $\Omega \in so(n)$ is the angular velocity, $\chi \in so(n)$ is a given constant matrix (describing a generalized center of the mass), $\Gamma \in so(n)$. Then $I_i + I_j$ are the principal inertia momenta. These equations are on the semidirect product $so(n) \times so(n)$ and they were introduced in [25].

We are going to consider a four-dimensional case of these equations defined by

\begin{align}
I_1 = I_2 &= a \\
I_3 = I_4 &= b
\end{align}

and

$$\chi = \begin{pmatrix}
0 & \chi_{12} & 0 & 0 \\
-\chi_{12} & 0 & 0 & 0 \\
0 & 0 & 0 & \chi_{34} \\
0 & 0 & -\chi_{34} & 0
\end{pmatrix}$$

with the conditions $a \neq b$, $\chi_{12}, \chi_{34} \neq 0$, $|\chi_{12}| \neq |\chi_{34}|$.

We will call this system the Lagrange bitop for the reasons we will explain at the end of section 3.

Proposition 1. [11]. The equations of motion (3) under the conditions (4) have an $L - A$ pair representation

\begin{align*}
\frac{d}{dt}L(\lambda) &= [L(\lambda), A(\lambda)] \\
L(\lambda) &= \lambda^2 C + \lambda M + \Gamma \\
A(\lambda) &= \lambda \chi + \Omega,
\end{align*}
where $C = (a + b)\chi$.

Before analysing the spectral properties of the matrices $L(\lambda)$, we will change the coordinates in order to diagonalize the matrix $C$. In this new basis the matrices $L(\lambda)$ have the form $\tilde{L}(\lambda) = U^{-1}L(\lambda)U$, where

$$U = \begin{pmatrix}
0 & 0 & i\sqrt{2} & \sqrt{2} \\
0 & 0 & \sqrt{2} & -i\sqrt{2} \\
i\sqrt{2} & \sqrt{2} & 0 & 0 \\
i\sqrt{2} & i\sqrt{2} & 0 & 0
\end{pmatrix}$$

After straightforward calculations, we have

$$\tilde{L}(\lambda) = \begin{pmatrix}
-i\Delta_{34} & 0 & -\beta_3^* + i\beta_4^* & i\beta_4 - \beta_4 \\
0 & i\Delta_{34} & -i\beta_3 + \beta_4 & -i\Delta_{34} - \beta_3 + i\beta_4 \\
\beta_3 - i\beta_4 & -i\beta_3 + \beta_4 & 0 & 0 \\
i\beta_3^* + \beta_4^* & \beta_3^* + i\beta_4^* & 0 & 0
\end{pmatrix}$$

where

$$\Delta_{12} = \lambda^2 C_{12} + \lambda M_{12} + \Gamma_{12},$$
$$\Delta_{34} = \lambda^2 C_{34} + \lambda M_{34} + \Gamma_{34},$$

$$\beta_3 = x_3 + \lambda y_3, \quad x_3 = \frac{1}{2} (\Gamma_{13} + i\Gamma_{23}),$$
$$\beta_4 = x_4 + \lambda y_4, \quad x_4 = \frac{1}{2} (\Gamma_{14} + i\Gamma_{24}),$$
$$\beta_3^* = \bar{x}_3 + \lambda \bar{y}_3, \quad y_3 = \frac{1}{2} (M_{13} + iM_{23}),$$
$$\beta_4^* = \bar{x}_4 + \lambda \bar{y}_4, \quad y_4 = \frac{1}{2} (M_{14} + iM_{24}).$$

The spectral polynomial

$$p(\lambda, \mu) = \det \left( \tilde{L}(\lambda) - \mu \cdot 1 \right)$$

has the form

$$p(\lambda, \mu) = \mu^4 + P(\lambda)\mu^2 + [Q(\lambda)]^2,$$

where

$$P(\lambda) = \Delta_{12}^2 + \Delta_{34}^2 + 4\beta_3\beta_4 + 4\beta_4\beta_3^*,$$
$$Q(\lambda) = \Delta_{12}\Delta_{34} + 2i(\beta_3^*\beta_4 - \beta_4\beta_3^*).$$

We can rewrite it in terms of $M_{ij}$ and $\Gamma_{ij}$:

$$P(\lambda) = A\lambda^4 + B\lambda^3 + D\lambda^2 + E\lambda + F,$$
$$Q(\lambda) = G\lambda^4 + H\lambda^3 + I\lambda^2 + J\lambda + K.$$
Their coefficients

\[
A = C_{12}^2 + C_{34}^2 = (C_+, C_+) + (C_-, C_-),
\]
\[
B = 2C_{34}M_{12} + 2C_{12}M_{12} = 2((C_+, M_+) + (C_-, M_-)),
\]
\[
D = M_{13}^2 + M_{14}^2 + M_{23}^2 + M_{12}^2 + M_{34}^2 + 2C_{12}G_{12} + 2C_{34}G_{34}
\]
\[= (M_+, M_+) + (M_-, M_-) + 2((C_+, C_+) + (C_-, C_-)),
\]
\[
E = 2G_{12}M_{12} + 2G_{13}M_{13} + 2G_{14}M_{14} + 2G_{23}M_{23} + 2G_{24}M_{24} + 2G_{34}M_{34}
\]
\[= 2((C_+, M_+) + (C_-, M_-)),
\]
\[
F = G_{12}^2 + G_{13}^2 + G_{14}^2 + G_{23}^2 + G_{24}^2 + G_{34}^2 = (\Gamma_+, \Gamma_+) + (\Gamma_-, \Gamma_-),
\]
\[
G = C_{12}C_{34} = (C_+, C_-),
\]
\[
H = C_{34}M_{12} + C_{12}M_{34} = (C_+, M_-) + (C_-, M_+),
\]
\[
I = C_{34}G_{12} + C_{12}G_{34} + M_{12}M_{34} + M_{23}M_{14} - M_{13}M_{24}
\]
\[= (C_+, \Gamma_-) + (C_-, \Gamma_+) + (M_+, M_-),
\]
\[
J = M_{34}G_{12} + M_{12}G_{34} + M_{14}G_{23} + M_{23}G_{14} - M_{13}M_{24} - M_{24}M_{13}
\]
\[= (M_+, \Gamma_-) + (M_-, \Gamma_+),
\]
\[
K = G_{34}G_{12} + G_{23}G_{14} - G_{13}G_{24} = (\Gamma_+, \Gamma_-).
\]

are integrals of motion of the system (3, 4). We used two vectors $M_+, M_− \in \mathbb{R}^3$ which correspond to $M_{ij} \in so(4)$ according to

\[
(M_+, M_-) \rightarrow \left( \begin{array}{cccc}
0 & -M_3^2 & M_2^3 & -M_1^3 \\
M_3^2 & 0 & -M_1^3 & -M_2^3 \\
-M_2^3 & M_1^3 & 0 & -M_3^3 \\
M_1^3 & M_2^3 & M_3^3 & 0
\end{array} \right)
\]

Here $M_j^i$ are the $j$-th coordinates of the vector $M_+$. The system (3, 4) is Hamiltonian with the Hamiltonian function

\[
\mathcal{H} = \frac{1}{2}(M_{13}\Omega_{13} + M_{14}\Omega_{14} + M_{23}\Omega_{23} + M_{12}\Omega_{12} + M_{34}\Omega_{34} + \chi_{12}\Gamma_{12} + \chi_{34}\Gamma_{34}
\]

The algebra $so(4) \times so(4)$ is 12 dimensional. The general orbits of the coadjoint action are 8 dimensional. According to [25], the Casimir functions are coefficients of $\lambda^0, \lambda^2, \lambda^4$ in the polynomials $|det\tilde{L}(\lambda)|^{1/2}$ and $-\frac{1}{2}Tr(\tilde{L}(\lambda))^2$.

Since

\[
|det\tilde{L}(\lambda)|^{1/2} = GL^4 + H\lambda^3 + I\lambda^2 + J\lambda + K,
\]
\[
-\frac{1}{2}Tr(\tilde{L}(\lambda))^2 = AX^4 + E\lambda + F,
\]
the Casimir functions are $J, K, E, F$. Nontrivial integrals of motion are $B, D, H, I$. They are in involution.

Nontrivial integrals of motion are $B, D, H, I$ are independent in the case $\chi_{12} \neq \pm \chi_{34}$. When $|\chi_{12}| = |\chi_{34}|$, then $2H = B$ or $2H = -B$ and there are only 3 independent integrals in involution.

So we have
Proposition 2 [11]. For $|\chi_{12}| \neq |\chi_{34}|$, the system (3, 4) is completely integrable in the Liouville sense.

There are two families of integrable Euler-Poisson equations introduced by Ratiu in [25]. The generalized symmetric case is defined by the conditions

$$I_1 = \cdots = I_n, \quad \chi \text{ arbitrary;}$$

and the generalized Lagrange case which is defined by

$$I_1 = I_2 = a, \quad I_3 = \cdots = I_n = b, \quad \chi_{ij} = 0 \text{ if } (i, j) \notin \{(1, 2), (2, 1)\}.$$

The system (3, 4) doesn’t fall in any of those families and together with them it makes the complete list of systems with the $L$ operator of the form

$$L(\lambda) = \lambda^2 C + \lambda M + \Gamma.$$

Proposition 3 [11]. If $\chi_{12} \neq 0$ then the Euler-Poisson equations (3) could be written in the form (5) (with arbitrary $C$) if and only if the equations (3) describe the generalized symmetric case, the generalized Lagrange case or the Lagrange bitop.

One can compare this with [24] (Theorem 15, ch. 53) and [16] (Example 2, p. 1451). The Lagrange bitop can be embedded, on the other hand, in the Bolsinov construction ([24], Theorem 17, ch. 53).

3. Classical integration and the Lagrange system

Starting from the well-known decomposition $so(4) = so(3) \oplus so(3)$, let us introduce

$$M_1 = \frac{1}{2}(M_+ + M_-), \quad M_2 = \frac{1}{2}(M_+ - M_-)$$

(and similar for $\Omega, \Gamma, \chi$), where $M_+, M_-$ are defined with (12). Equations (3) become

\begin{align*}
\dot{M}_1 &= 2(M_1 \times \Omega_1 + \Gamma_1 \times \chi_1) \\
\dot{M}_2 &= 2(M_2 \times \Omega_2 + \Gamma_2 \times \chi_2)
\end{align*}

and

\begin{align*}
\chi_1 &= (0, 0, -\frac{1}{2}(\chi_{12} + \chi_{34})) \\
\chi_2 &= (0, 0, -\frac{1}{2}(\chi_{12} - \chi_{34}))
\end{align*}

Since

$$M_+ = I_+ \Omega_+, \quad M_- = I_- \Omega_-,$$
where \( I_+ = \text{diag}(I_2 + I_3, I_1 + I_3, I_1 + I_2) \), \( I_- = \text{diag}(I_1 + I_4, I_2 + I_4, I_3 + I_4) \), the connection between \( M_1, M_2 \) and \( \Omega_1, \Omega_2 \) is

\[
M_1 = \frac{1}{2}[(I_+ + I_-)\Omega_1 + (I_+ - I_-)\Omega_2]
\]

\[
M_2 = \frac{1}{2}[(I_+ - I_-)\Omega_1 + (I_+ + I_-)\Omega_2].
\]

Using (4), we have

\[
M_1 = ((a + b)\Omega(1)_{11}, (a + b)\Omega(1)_{12}, (a + b)\Omega(1)_{13} + (a - b)\Omega(2)_{13})
\]

\[
M_2 = ((a + b)\Omega(2)_{21}, (a + b)\Omega(2)_{22}, (a - b)\Omega(1)_{13} + (a + b)\Omega(1)_{23}).
\]

We denoted with \( \Omega(ij) \) the \( j \) component of the vector \( \Omega \) and similar notation we use for \( \Gamma \) and \( \chi \).

If we denote \( \Omega_1 = (p_1, q_1, r_1) \), \( \Omega_2 = (p_2, q_2, r_2) \), then the first group of the equations (13) becomes

\[
\begin{align*}
\dot{p}_1 - mq_1r_2 &= -n_1\Gamma_{(1)2} \\
\dot{q}_1 + mp_1r_2 &= n_1\Gamma_{(1)1} \\
(a + b)\dot{r}_1 + (a - b)\dot{r}_2 &= 0
\end{align*}
\]

and

\[
\begin{align*}
\dot{p}_2 - mq_2r_1 &= -n_2\Gamma_{(2)2} \\
\dot{q}_2 + mp_2r_1 &= n_2\Gamma_{(2)1} \\
(a - b)\dot{r}_1 + (a + b)\dot{r}_2 &= 0
\end{align*}
\]

where

\[
m = -\frac{2(a - b)}{a + b}, \quad n_1 = -\frac{2\chi_{(1)3}}{a + b}, \quad n_2 = -\frac{2\chi_{(2)3}}{a + b}
\]

The integrals of motion are

\[
\begin{align*}
[(a + b)r_1 + (a - b)r_2)]\chi_{(1)3} &= f_{11} \\
(a + b)^2(p_1^2 + q_1^2) + [(a + b)r_1 + (a - b)r_2]^2 + 2(a + b)\chi_{(1)3}\Gamma_{(1)3} &= f_{12} \\
(a + b)p_1\Gamma_{(1)1} + (a + b)q_1\Gamma_{(1)2} + [(a + b)r_1 + (a - b)r_2]\Gamma_{(1)3} &= f_{13} \\
\Gamma_{(1)1}^2 + \Gamma_{(1)2}^2 + \Gamma_{(1)3}^2 &= 1 \\
[(a - b)r_1 + (a + b)r_2)]\chi_{(2)3} &= f_{21} \\
(a - b)^2(p_2^2 + q_2^2) + [(a - b)r_1 + (a + b)r_2]^2 + 2(a + b)\chi_{(2)3}\Gamma_{(2)3} &= f_{22} \\
(a + b)p_2\Gamma_{(2)1} + (a + b)q_2\Gamma_{(2)2} + [(a - b)r_1 + (a + b)r_2]\Gamma_{(2)3} &= f_{23} \\
\Gamma_{(2)1}^2 + \Gamma_{(2)2}^2 + \Gamma_{(2)3}^2 &= 1
\end{align*}
\]

Introducing \( \rho_i, \sigma_i \), defined with \( p_i = \rho_i \cos \sigma_i, q_i = \rho_i \sin \sigma_i \), using (15) and (16) after some calculations, we get

\[
\begin{align*}
\rho_1^2\dot{\sigma}_1 + mr_1\rho_1^2 &= n_1(\frac{f_{13}}{a + b} - \alpha_1\Gamma_{(1)3}) \\
[(\rho_1^2)^2] &= 4n_1^2\rho_1^2[1 - \frac{1}{n_1}(a_1 + \rho_1^2)^2] - 4n_1^2(\frac{f_{13}}{a + b} - \alpha_1a_1 - \frac{\alpha_1^2}{n_1}\rho_1^2)^2 \\
\rho_2^2\dot{\sigma}_2 + mr_2\rho_2^2 &= n_2(\frac{f_{23}}{a + b} - \alpha_2\Gamma_{(2)3}) \\
[(\rho_2^2)^2] &= 4n_2^2\rho_2^2[1 - \frac{1}{n_2}(a_2 + \rho_2^2)^2] - 4n_2^2(\frac{f_{23}}{a + b} - \alpha_2a_2 - \frac{\alpha_2^2}{n_2}\rho_2^2)^2
\end{align*}
\]

Lagrangian Bitop
where
\[ \alpha_1 = \frac{(a + b)r_1 + (a - b)r_2}{a + b} \quad \alpha_2 = \frac{(a + b)r_2 + (a - b)r_1}{a + b} \]
\[ a_i = \frac{\alpha_i^2(a + b)^2 - f_{i2}}{(a + b)^2} \quad i = 1, 2 \]

Let us denote \( u_1 = \rho_1^2, \ u_2 = \rho_2^2 \). From (17) we have
\[ a_i^2 = P_i(u_i), \quad i = 1, 2. \]

where
\[ P_i(u) = -4u^3 - 4u^2B_i + 4uC_i + D_i, \quad i = 1, 2. \]

and
\[ B_i = 2a_i + \alpha_i^2, \]
\[ C_i = n_i^2 - a_i^2 - 4\frac{\alpha_i \chi(i)3f_{i3}}{(a + b)^2} - 2\alpha_i^2a_i, \]
\[ D_i = -4\left(\frac{2\chi(i)3f_{i3}}{(a + b)^2} + \alpha_i a_i\right)^2, \quad i = 1, 2. \]

From the previous relations, we have
\[ \int \frac{du_1}{\sqrt{P_1(u_1)}} = t, \quad \int \frac{du_2}{\sqrt{P_2(u_2)}} = t. \]

So, the integration of the system (13) leads to the functions associated with the elliptic curves \( E_1, E_2 \), where \( E_i = E_i(\alpha_i, a_i, \chi(i)3, f_{i2}, f_{i3}) \) are given with:
\[ (19) \quad E_i : y^2 = P_i(u). \]

The equations (14) and (15) are very similar to those for the classical Lagrange system (see [17]). However, the system (14, 15) doesn’t split on two independent Lagrangian systems, since the third equations in (14) and (15) together give that \( r_1 \) and \( r_2 \) are constants. Also, in the definition of each of the curves \( E_1, E_2 \) both those constants are involved. That is the reason we refer to the system (3, 4) as nonsplitted Lagrange bitop. The formulae (17) are also very close to those for the Lagrange system. Although the Lagrange system has a long history (starting from 1788) and many important papers have been written about it, still there are some subtle questions and problems related to its integration (see [16, 5]). So, for the further analysis of the Lagrange bitop we pass to the algebro-geometric integration.

4. The first steps in an algebro-geometric integration procedure

The \( L(\lambda) \) matrix (5) for the Lagrange bitop (3, 4) is a quadratic polynomial in the spectral parameter \( \lambda \) with matrix coefficients. The general theories describing the isospectral deformations for polynomials with matrix coefficients were developed by Dubrovin [12, 15] in the middle of 70’s and by Adler, van Moerbeke [1] few years later. Dubrovin’s approach was based on the Baker-Akhiezer function and it was
applied in rigid body problems in [21, 9]. The other approach was based on [22]
and the connection with rigid body problems was given in [1, 25, 26].

As it will be shown below, none of these two theories can be directly applied in
our case. So, we are going to make certain modifications, and then we will integrate
the system (3, 4). As usual in the algebro-geometric integration, we consider the
spectral curve
\[ \Gamma : \det \left( \tilde{L}(\lambda) - \mu \cdot 1 \right) = 0. \]

By using (8, 9), we have
\[ (20) \Gamma : \mu^4 + \mu^2 \left( \Delta_{12}^2 + \Delta_{34}^2 + 4\beta_3^* \beta_4^* + 4\beta_4^* \beta_3^* \right) + \left[ \Delta_{12} \Delta_{34} + 2i(\beta_3^* \beta_4 - \beta_4^* \beta_3) \right]^2 = 0. \]

There is an involution
\[ \sigma : (\lambda, \mu) \rightarrow (\lambda, -\mu) \]
on the curve \( \Gamma \), which corresponds to the skew symmetry of the matrix \( L(\lambda) \). Denote
the factor-curve by \( \Gamma_1 = \Gamma / \sigma \).

**Lemma 1.** The curve \( \Gamma_1 \) is a smooth hyperelliptic curve of the genus \( g(\Gamma_1) = 3 \).
The arithmetic genus of the curve \( \Gamma \) is \( g_a(\Gamma) = 9 \).

**Proof.** The curve:
\[ \Gamma_1 : u^2 + P(\lambda)u + [Q(\lambda)]^2 = 0, \]
is hyperelliptic, and its equation in the canonical form is:
\[ u_1^2 = \frac{|P(\lambda)|^2}{4} - [Q(\lambda)]^2, \]
where \( u_1 = u + P(\lambda)/2 \). Since \( \frac{|P(\lambda)|^2}{4} - [Q(\lambda)]^2 \) is a polynomial of the degree 8, the
genus of the curve \( \Gamma_1 \) is \( g(\Gamma_1) = 3 \). The curve \( \Gamma \) is a double covering of \( \Gamma_1 \), and
the ramification divisor is of the degree 8. According to the Riemann-Hurwitz formula,
the arithmetic genus of \( \Gamma \) is \( g_a(\Gamma) = 9 \).

**Lemma 2.** The spectral curve \( \Gamma \) has four ordinary double points \( S_i, i = 1, \ldots, 4 \).
The genus of its normalization \( \tilde{\Gamma} \) is five.

**Proof.** From the equations
\[ \frac{\partial p(\lambda, \mu)}{\partial \lambda} = 0, \quad \frac{\partial p(\lambda, \mu)}{\partial \mu} = 0, \]
the four ordinary double points are \( S_k = (\lambda_k, 0), k = 1, \ldots, 4 \), where \( \lambda_k \) are zeroes of
the polynomial \( Q(\lambda) \). Thus, the genus of the curve \( \tilde{\Gamma} \) is \( g(\tilde{\Gamma}) = g_a(\Gamma) - 4 = 5 \).

**Lemma 3.** The singular points \( S_i \) of the curve \( \Gamma \) are fixed points of the involution \( \sigma \).
The involution \( \sigma \) exchanges the two branches of \( \Gamma \) at \( S_i \).

**Proof.** Fixed points of the \( \sigma \) are defined with \( \mu = 0 \), thus \( S_i \) are fixed points. Since
their projections on \( \Gamma_1 \) are smooth points, the involution \( \sigma \) exchanges the branches
of \( \Gamma \), which are given by the equation
\[ \mu^2 = \frac{-P(\lambda) + \sqrt{P^2(\lambda) - 4Q^2(\lambda)}}{2}. \]

In general, whenever the matrix \( L(\lambda) \) is antisymmetric, the spectral curve is
reducible in odd-dimensional case and singular in even-dimensional case.

Before starting the study of the analytic properties of the Baker-Akhiezer function,
let us give the formulae for (nonnormalized) eigen-vectors of the matrix \( L(\lambda) \).
Lemma 4. If the vector \( f = (f_1, f_2, f_3, f_4)^t \) is given by

\[
\begin{align*}
\begin{align*}
 f_1 &= (\Delta_{12}^2 + \mu^2)(i\Delta_{34} - \mu) - 2\mu(\beta_3\beta_4^* + \beta_4\beta_3^*) + 2\Delta_{12}(\beta_3\beta_4^* - \beta_4\beta_3^*) \\
f_2 &= 2\mu(\beta_3 - i\beta_4)(i\beta_3^* + \beta_4^*) \\
f_3 &= (-\beta_3 + i\beta_4)[(i\Delta_{12} - \mu)(i\Delta_{34} - \mu) + 2i(\beta_3\beta_4^* - \beta_4\beta_3^*)] \\
f_4 &= (i\beta_3^* + \beta_4^*)[(i\Delta_{12} + \mu)(i\Delta_{34} - \mu) + 2i(\beta_3\beta_4^* - \beta_4\beta_3^*)]
\end{align*}
\end{align*}
\]

then \( L(\lambda)f = \mu f \).

Corollary 1. The eigenvectors \( f' \) normalized by the condition

\[
\begin{align*}
 f'_1 + f'_2 + f'_3 + f'_4 &= 1,
\end{align*}
\]

have different values in the points \( S'_1, S'_2 \in \tilde{\Gamma} \), which cover the singular points \( S_i \in \Gamma \).

5. The Baker-Akhiezer function

The general integration technique based on the Baker-Akhiezer function was developed by Krichever (see [20], [14] and bibliography therein). The application on the matrix polynomials was done, as we said, by Dubrovin (see [12, 15]). Following those ideas, we consider the next eigen-problem

\[
\begin{align*}
\left( \frac{\partial}{\partial t} + \tilde{A}(\lambda) \right) \psi_k &= 0, \\
\tilde{L}(\lambda)\psi_k &= \mu_k\psi_k,
\end{align*}
\]

where \( \psi_k \) are the eigenvectors with the eigenvalue \( \mu_k \). Then \( \psi_k(t, \lambda) \) form \( 4 \times 4 \) matrix with components \( \psi^i_k(t, \lambda) \). Denote by \( \varphi^i_k \) corresponding inverse matrix. Let us introduce

\[
\Psi^i_j(t, \tau, (\lambda, \mu_k)) = \psi^i_k(t, \lambda) \cdot \varphi^j_k(\tau, \lambda)
\]

and

\[
g^i_j(t, (\lambda, \mu_k)) = \Psi^i_j(t, t, (\lambda, \mu_k)) = \psi^i_k(t, \lambda) \cdot \varphi^j_k(t, \lambda)
\]

(there is no summation on \( k \)) or, in other words

\[
g(t) = \psi_k(t) \otimes \varphi(t)^k.
\]

It is easy to check that the function \( \Psi^i_j(t, \tau, (\lambda, \mu_k)) \) satisfies

\[
\left( \frac{\partial}{\partial t} + \tilde{A}(\lambda) \right) \Psi(t, \tau, (\lambda, \mu_k)) = 0
\]

Then, if we denote with \( \Phi(t, \lambda) \) the fundamental solution of

\[
\left( \frac{\partial}{\partial t} + \tilde{A}(\lambda) \right) \Phi(t, \lambda) = 0,
\]
normalized with $\Phi(\tau) = 1$, there is a relation

$$\bar{\Psi}(t,\tau, (\lambda, \mu_k)) = \Phi(t,\lambda)g(\tau, (\lambda, \mu_k)).$$

Matrix $g$ is of rank 1, and we have

$$\frac{\partial \psi}{\partial t} = -\tilde{A}\psi, \quad \frac{\partial \varphi}{\partial t} = \varphi \tilde{A}, \quad \frac{\partial g}{\partial t} = [g, \tilde{A}].$$

We can consider vector-functions $\psi_k(t,\lambda) = (\psi_1^k(t,\lambda),\ldots,\psi_4^k(t,\lambda))^T$ as one vector-function $\psi(t, (\lambda, \mu)) = (\psi_1^1(t, (\lambda, \mu)),\ldots,\psi_4^4(t, (\lambda, \mu)))^T$ on the curve $\Gamma$ defined with $\psi_i(t, (\lambda, \mu_k)) = \psi_i^k(t,\lambda)$. The same we have for the matrix $\varphi_i^k$. The relations for the divisors of zeroes and poles of the functions $\psi_i$ and $\varphi_i$ in the affine part of the curve $\Gamma$ are:

\begin{equation}
(g_j^i)_{a} = d_j(t) + d^j(t) - D_r - D_s',
\end{equation}

where $D_r$ is the ramification divisor over $\lambda$ plane(see [12]) and $D_s$ is divisor of singular points, $D'_s \leq D_s$. One can easily calculate $\deg D_r = 16, \deg D_s = 8$.

The matrix elements $g_{ij}^k(t, (\lambda, \mu_k))$ are meromorphic functions on the curve $\Gamma$. We need their asymptotics in the neighborhoods of the points $P_k$, which cover the point $\lambda = \infty$. Let $\tilde{\psi}_k$ be the eigenvector of the matrix $\tilde{L}(\lambda)$ normalized in $P_k$ by the condition $\tilde{\psi}_k^k = 1$, and let $\tilde{\varphi}_k^k$ be the inverse matrix for $\tilde{\psi}_k^k$. We will use the following

**Lemma 5.** The matrix elements of $g$ have another decomposition

$$g_{ij}^k = \psi_{ki}^j \varphi_{kj}^k = \tilde{\psi}_{ki}^j \tilde{\varphi}_{kj}^k.$$

The proof of the Lemma is an immediate consequence of the proportionality of the vectors $\psi_k$ and $\tilde{\psi}_k$ (and $\varphi_k$ and $\tilde{\varphi}_k$).

**Lemma 6.** (a) The matrix $g$ has a representation

$$g = \frac{\mu^3 + a_1 \mu^2 + a_2 \mu + a_3}{\partial p(\lambda, \mu)/\partial \mu},$$

where

$$a_1 = L, \quad a_2 = PE + L^2, \quad a_3 = PL + L^3.$$

(b) For the Lax matrix $L$ from (3) it holds

$$a_3 = P(\lambda_i)L(\lambda_i) + L^3(\lambda_i) = 0,$$

for $\lambda_i : Q(\lambda_i) = 0$.

The proof of the Lemma follows from [12] and straightforward calculation. From the (a) part one can see that the matrix $g$ could have poles at the singular points of the spectral curve. But, from (b) we have
Corollary 2. The matrix $g$ doesn't have poles at the singular points of the curve $\Gamma$.

So, from now on, taking Corollaries 1 and 2 into account, we will consider all the functions in this section as functions on the normalization $\tilde{\Gamma}$ of the spectral curve $\Gamma$.

Since the functions $\tilde{\psi}_k^i$ and $\tilde{\varphi}_k^j$ are meromorphic functions in the neighborhood of the points $P_k$, their asymptotics can be calculated by expanding $\tilde{\psi}_k^i$ as a power series in $\frac{1}{\lambda}$ in the neighborhood of the point $\lambda = \infty$ around the vector $e_k$, where $e_k^i = \delta_k^i$. We get

$$
\left( \tilde{C} + \tilde{M} + \tilde{\Gamma} \right) \left( e_i + \frac{u_i}{\lambda} + \frac{v_i}{\lambda^2} + \frac{w_i}{\lambda^3} + \ldots \right)
= \left( \tilde{C}_{ii} + b_i \frac{1}{\lambda} + d_i \frac{1}{\lambda^2} + h_i \frac{1}{\lambda^3} + \ldots \right) \left( e_i + \frac{u_i}{\lambda} + \frac{v_i}{\lambda^2} + \frac{w_i}{\lambda^3} + \ldots \right),
$$

where the matrices $\tilde{C}$, $\tilde{M}$ and $\tilde{\Gamma}$ are defined by

$$
\tilde{L}(\lambda) = \lambda^2 \tilde{C} + \lambda \tilde{M} + \tilde{\Gamma}.
$$

Comparing the same powers of $\frac{1}{\lambda^\alpha}$, we get the system of the equations

$$
\begin{align*}
\tilde{C} e_i &= \tilde{C}_{ii} e_i \\
\tilde{C} u_i + \tilde{M} e_i &= \tilde{C}_{ii} u_i + b_i e_i \\
\tilde{C} v_i + \tilde{M} u_i + \tilde{\Gamma} e_i &= \tilde{C}_{ii} v_i + b_i u_i + d_i e_i \\
\tilde{C} w_i + \tilde{M} v_i + \tilde{\Gamma} u_i &= \tilde{C}_{ii} w_i + b_i v_i + d_i u_i + h_i e_i.
\end{align*}
$$

From the system (25), we have

$$
\begin{align*}
(u_i)_i &= 0, \quad (v_i)_i = 0, \quad (w_i)_i = 0 \\
(u_i)_j &= \frac{\tilde{M}_{ji}}{\tilde{C}_{ii} - \tilde{C}_{jj}} \quad j \neq i \\
(v_i)_j &= \frac{1}{\tilde{C}_{ii} - \tilde{C}_{jj}} \left( \sum_{k \neq i} \tilde{M}_{jk} \tilde{M}_{ki} - \tilde{M}_{ii} \tilde{M}_{ji} + \tilde{\Gamma}_{ji} \right) \\
(w_i)_j &= \frac{1}{\tilde{C}_{ii} - \tilde{C}_{jj}} \left[ \sum_{k \neq i} \tilde{M}_{jk} (v_i)_k + \sum_{k \neq i} \tilde{\Gamma}_{jk} (u_i)_k - b_i (v_i)_j - d_i (u_i)_j \right] \\
b_i &= \tilde{M}_{ii} \\
d_i &= \sum_{k \neq i} \frac{\tilde{M}_{ik} \tilde{M}_{ki}}{\tilde{C}_{ii} - \tilde{C}_{kk}} + \tilde{\Gamma}_{ii} \\
h_i &= \sum_{k \neq i} \tilde{M}_{ik} (v_i)_k + \sum_{k \neq i} \tilde{\Gamma}_{jk} (u_i)_k.
\end{align*}
$$
So the matrix $\tilde{\psi} = \{\tilde{\psi}_k^i\}$ in the neighborhood of $\lambda = \infty$ has the form:

$$\tilde{\psi} = 1 + \frac{u}{\lambda} + \frac{v}{\lambda^2} + \frac{w}{\lambda^3} + O\left(\frac{1}{\lambda^4}\right).$$  

(27)

Let the expansion of the matrix $\tilde{\phi} = \{\tilde{\phi}_k^i\}$ be

$$\tilde{\phi} = 1 + \frac{u_1}{\lambda} + \frac{v_1}{\lambda^2} + \frac{w_1}{\lambda^3} + O\left(\frac{1}{\lambda^4}\right).$$

(28)

From $\tilde{\psi} \cdot \tilde{\phi} = 1$ we obtain

$$u_1 = -u \quad \text{and} \quad v_1 = u^2 - v \quad \text{and} \quad w_1 = 2uv - u^3 - w.$$  

From $g = \tilde{\psi}_k^i \otimes \tilde{\phi}_k^i$ in the neighborhood of $\lambda = \infty$ it holds

$$g^j_i(t, (\lambda, \mu_k)) = \tilde{\psi}_k^i(t, \lambda)\tilde{\phi}_k^j(t, \lambda).$$

(29)

The last relation and (26) and (28) imply

$$g^j_i(t, (\lambda, \mu_k)) = 1 \quad \text{for } i = j = k.$$  

For $i = j \neq k$ and $i, k \in \{1, 2\}$ or $i, k \in \{3, 4\}$ we have

$$g^j_i(t, (\lambda, \mu_k)) = \frac{1}{\lambda^4}v_{ik}(v_1)_{ki} + O\left(\frac{1}{\lambda^5}\right).$$

So the functions $g^j_i$ have a zero of the fourth order at the point $P_k$. Here we used the fact that $\tilde{M}_{12} = 0, \tilde{M}_{34} = 0$.

On the other hand, for $i = j \neq k$ and $i \in \{1, 2\}, k \in \{3, 4\}$ or $i \in \{1, 2\}$ and $i \in \{3, 4\}$ we have

$$g^j_i(t, (\lambda, \mu_k)) = \frac{1}{\lambda^4}\frac{\tilde{M}_{ik}\tilde{M}_{ki}}{(C_{kk} - C_{ii})^2} + O\left(\frac{1}{\lambda^5}\right).$$

Thus, the functions $g^j_i$ in this case have a zero of the second order at $P_k$.

In the same manner, for $i \neq j, k \neq i, k \neq j, i, j \in \{1, 2\}$ or $i, j \in \{3, 4\}$, it holds

$$g^j_i(t, (\lambda, \mu_k)) = \frac{1}{\lambda^4}\frac{\tilde{M}_{ik}\tilde{M}_{kj}}{(C_{kk} - C_{ii})(C_{kk} - C_{jj})} + O\left(\frac{1}{\lambda^5}\right).$$

So $g^j_i$ have a zero of the second order at $P_k$.

For $i \neq j, k = i, i, j \in \{1, 2\}$ or $i, j \in \{3, 4\}$ we have

$$g^j_i(t, (\lambda, \mu_k)) = \frac{(v_1)_{ij}}{\lambda^2} + O\left(\frac{1}{\lambda^3}\right).$$
and \( g^j_i \) have a zero of the second degree at \( P_k \).

In the case \( i \neq j, k = j, i, j \in \{1, 2\} \) or \( i, j \in \{3, 4\} \) the functions \( g^j_i \) at \( P_k \) also have a zero of the second degree.

For \( i \neq j, k \neq i, k \neq j \) and \((i, j) \notin \{ (1, 2), (2, 1), (3, 4), (4, 3) \}\) we have

\[
g^j_i(t, (\lambda, \mu_k)) = \frac{\text{const}}{\lambda^3} + O \left( \frac{1}{\lambda^4} \right)
\]

So, the degree of a zero at \( P_k \) of the functions \( g^j_i \) in this case is \( 3 \).

In the last two cases the functions \( g^j_i \) have simple zeroes at \( P_k \):

for \( i \neq j, k = j, (i, j) \notin \{ (1, 2), (2, 1), (3, 4), (4, 3) \}\) the expansion is

\[
g^j_i(t, (\lambda, \mu_i)) = \frac{1}{\lambda} \frac{M_{ji}}{C_{ii} - C_{jj}} + O \left( \frac{1}{\lambda^2} \right)
\]

and similarly for \( i \neq j, k = i, (i, j) \notin \{ (1, 2), (2, 1), (3, 4), (4, 3) \}\) we have

\[
g^j_i(t, (\lambda, \mu_j)) = \frac{1}{\lambda} \frac{M_{ji}}{C_{jj} - C_{ii}} + O \left( \frac{1}{\lambda^2} \right)
\]

Summarizing, we get

\[\textbf{Lemma 7.} \] The functions \( g^j_i \) have the following divisors in infinity:

for \( i \neq j, (i, j) \notin \{ (1, 2), (2, 1), (3, 4), (4, 3) \}\)

\[
(g^j_i)_{\lambda=\infty} = 3 (P_1 + P_2 + P_3 + P_4) - 2P_i - 2P_j;
\]

for \( i, j \in \{1, 2\} \) or \( i, j \in \{3, 4\} \)

\[
(g^j_i)_{\lambda=\infty} = 2 (P_1 + P_2 + P_3 + P_4) \quad \text{for } i \neq j,
\]

\[
(g^j_i)_{\lambda=\infty} = 2 (P_1 + P_2 + P_3 + P_4) - 2P_i + 2P_j \quad \text{for } i = j,
\]

where \( l \neq i \) and \( l, i \in \{1, 2\} \) or \( l, i \in \{3, 4\} \).

Let us denote by \( \ddot{d}_j \) and by \( \ddot{d}^i \) the following divisors:

\[
\ddot{d}_1 = d_1 + P_2, \quad \ddot{d}_2 = d_2 + P_1, \quad \ddot{d}_3 = d_3 + P_4, \quad \ddot{d}_4 = d_4 + P_3,
\]

\[
\ddot{d}^1 = d^1 + P_2, \quad \ddot{d}^2 = d^2 + P_1, \quad \ddot{d}^3 = d^3 + P_4, \quad \ddot{d}^4 = d^4 + P_3.
\]

Finally, we have

\[\textbf{Proposition 4.}\]

a) The divisors of matrix elements of \( g \) are

\[
(g^j_i) = \ddot{d}^i + \ddot{d}_j - D_r + 2 (P_1 + P_2 + P_3 + P_4) - P_i - P_j
\]

b) The divisors \( \ddot{d}_i, \ddot{d}^i \) are of the same degree

\[
\text{deg } \ddot{d}_i = \text{deg } \ddot{d}^i = 5.
\]
Proof. a) is a consequence of the previous lemma. From \( \deg \tilde{d}_i = \deg \tilde{d}^i = \deg d_i + 1 = \deg d^i + 1 \) and \( \deg D_r = 16 \) using (16) we have b).

For the functions

\[
\tilde{\psi}^i(t, \tau, (\lambda, \mu_k)) = \frac{\tilde{\Psi}^i_j(t, \tau, (\lambda, \mu_k))}{\sum s g_j^s(\tau, (\lambda, \mu_k))}
\]

we have

\[
(32) \quad \tilde{\psi}^i(t, \tau, (\lambda, \mu_k)) = \sum_s \Phi^i_s(t, \lambda) h^s(\tau, (\lambda, \mu_k))
\]

where \( h^s \) are the eigenvector of \( L(\lambda) \) normalized by the condition

\[
\sum_s h^s(t, (\lambda, \mu_k)) = 1
\]

From (32) it follows that

\[
(33) \quad \tilde{\psi}^i(t, \tau, (\lambda, \mu_k)) = \sum_s \Phi^i_s(t, \lambda) \frac{\psi^s_k(\tau, \lambda)}{\sum_l \psi^l_k(\tau, \lambda)} = \frac{\psi^i_k(t, \lambda)}{\sum_l \psi^l_k(\tau, \lambda)}.
\]

**Proposition 5.** The functions \( \hat{\psi}^i \) satisfy the following properties

a) In the affine part of \( \tilde{\Gamma} \) the function \( \hat{\psi}^i \) has 4 time dependent zeroes which belong to the divisor \( d^i(t) \) defined by formula (24), and 8 time independent poles, e.g.

\[
\left( \tilde{\psi}^i(t, \tau, (\lambda, \mu_k)) \right)_a = d^i(t) - \bar{D}, \quad \deg \bar{D} = 8.
\]

b) At the points \( P_k \), the functions \( \hat{\psi}^i \) have essential singularities as follows:

\[
\hat{\psi}^i(t, \tau, (\lambda, \mu)) = \exp \left[ -(t - \tau) R_k \right] \hat{\alpha}^i(t, \tau, (\lambda, \mu))
\]

where \( R_k \) are given with

\[
R_1 = i \left( \frac{\chi_{34}}{z} + \omega_{34} \right), \quad R_2 = -R_1, \quad R_3 = i \left( \frac{\chi_{12}}{z} + \omega_{12} \right), \quad R_4 = -R_3
\]

and \( \hat{\alpha}^i \) are holomorphic in a neighborhood of \( P_k \),

\[
\hat{\alpha}^i(\tau, (\lambda, \mu)) = h^i(\tau, (\lambda, \mu))
\]

and

\[
\hat{\alpha}^i(t, \tau, P_k) = \delta^k_i + v^k(t) z + O(z^2),
\]

with

\[
v^k_i = \frac{M_{ki}}{C_{ii} - C_{kk}}.
\]
Proof. From (33) we see that $\hat{\psi}^j$ has $d^i(t)$ as a divisor of zeroes in the affine part. Also from (32) it follows that poles of $\hat{\psi}^j$ are poles of $h^s$, and do not depend on time. The functions $h^s$ are meromorphic on $\tilde{\Gamma}$, and they have the same number of poles and zeroes.

From (25) and (26), we have that
- $h^1$ has simple zeroes at $P_3$ and $P_4$, and the double zero at $P_2$;
- $h^2$ has simple zeroes at $P_3$ and $P_4$, and the double zero at $P_1$;
- $h^3$ has simple zeroes at $P_1$ and $P_2$, and the double zero at $P_4$;
- $h^4$ has simple zeroes at $P_1$ and $P_2$, and the double zero at $P_3$.

As in [20, 14], it could be proved that the functions $h^s$ have divisors of poles $\bar{D}, \deg \bar{D} = 8$. This proves the rest of a).

From
\[ \frac{\partial \ln \hat{\psi}^i(t, \tau, (\lambda, \mu_k))}{\partial t} = \hat{\psi}^i(t, (\lambda, \mu)) \frac{\psi^i(t, (\lambda, \mu))}{\psi^i(t, (\lambda, \mu))} = \sum A^i_k(t, \lambda)\psi^i_k(t, \lambda) \]

and asymptotics (26), (27) for $\hat{\psi}^i_k$ in a neighborhood of the points $P_k$, using the proportionality of $\psi^i_k$ and $\hat{\psi}^i_k$, we obtain
\[ \hat{\psi}^i(t, \tau) = \exp[(t - \tau)R_k]a^i(t, \tau). \]

Starting from the expansion in a neighborhood of $R_k$
\[ \psi^i_k(t, \lambda) = e^{R_k t}(a^i_k + v^i_k(t)z + O(z^2)), \quad \phi^i_k(t, \lambda) = e^{-R_k t}(b^i_k + w^i_k(t)z + O(z^2)), \]

using (30) we get $a^i_k = \delta^i_k$ and
\[ v^i_k = \frac{\tilde{M}_{ki}}{C_{ii} - C_{kk}}. \]

This finish the proof of the proposition.

Lemma 8. The following relation takes place on the Jacobian $\text{Jac}(\tilde{\Gamma})$:
\[ A(d^i(t) + \sigma d^i(t)) = A(d^i(t) + \sigma d^i(t)) \]

where $A$ is the Abel map from the curve $\tilde{\Gamma}$ to $\text{Jac}(\tilde{\Gamma})$.

Proof. Let us introduce functions $\phi^i(t, \tau, (\lambda, \mu)) = \frac{\psi^i(t, (\lambda, \mu))}{\psi^i(t, (\lambda, \mu))}$. Then $(\phi^i)_a = d^i(t) - d^i(\tau)$. Using the relation
\[ \sigma(P_1) = P_2, \quad \sigma(P_3) = P_4, \]

and statement b) in the Proposition 5 we see that $\sigma \phi^i \cdot \phi^i$ are meromorphic functions (they do not have the essential singularities in $P_k$). Consequently, for their divisors of zeroes and poles, it holds:
\[ (\sigma \phi^i \cdot \phi^i(t, \tau, (\lambda, \mu))) = d^i(t) + \sigma d^i(t) - d^i(t) - \sigma d^i(t). \]

Applying the Abel theorem, we finish the proof.
From the previous Lemma we see that the vectors \(\mathcal{A}(d^i(t))\) belong to some translation of the Prym variety \(\Pi = \text{Prym}(\widetilde{\Gamma} | \Gamma_1)\). More details concerning the Prym varieties one can find in [24, 23, 19, 7, 29, 28, 3, 6]. The natural question arises to compare two twodimensional tori \(\Pi\) and \(E_1 \times E_2\), where the elliptic curves \(E_i\) are defined in (19).

## 6. Geometry of the Prym variety \(\Pi\)

Together with the curve \(\Gamma_1\), one can consider curves \(C_1\) and \(C_2\) defined by the equations

\[
C_1 : v^2 = s\left(\frac{P(\lambda)}{2} + Q(\lambda)\right)
\]

\[
C_2 : v^2 = s\left(\frac{P(\lambda)}{2} - Q(\lambda)\right)
\]

where \(s\) is a constant to be fixed in the next Lemma.

**Lemma 9.** If \(s = 2/(a + b)^2\) in (34, 35) then the curves \(C_i\) are such that

\[E_i = \text{Jac}(C_i)\quad i = 1, 2.\]

**Proof.** The curves \(E_i\) from (19) can be represented in a canonical form

\[
y^2 = 4u^3 - g_2u - g_3, \quad i = 1, 2,
\]

where

\[g_{2i} = 4(\frac{B_i^2}{3} + C_i), \quad g_{3i} = 4(\frac{2B_i^3}{m27} + B_iC_i - \frac{D_i}{4}).\]

On the other hand, the equations for curves \(C_i\) are

\[y^2 = s(a_{0}^\pm \lambda^4 + a_{1}^\pm \lambda^3 + a_{2}^\pm \lambda^2 + a_{3}^\pm \lambda + a_{4}^\pm),\]

where

\[a_{0}^\pm = \frac{1}{2}(C_{12} \pm C_{34})^2,\]

\[a_{1}^\pm = (C_{12} \pm C_{34})(M_{12} \pm M_{34}),\]

\[a_{2}^\pm = ((M_{12} \pm M_{34})^2 + (M_{23} \pm M_{14})^2 + (M_{13} \mp M_{24})^2)/2 + (C_{12} \pm C_{34})(\Gamma_{12} \pm \Gamma_{34}),\]

\[a_{3}^\pm = (M_{12} \pm M_{34})(\Gamma_{12} \pm \Gamma_{34}) + (M_{23} \pm M_{14})(\Gamma_{23} \pm \Gamma_{14}) + (M_{13} \mp M_{24})(\Gamma_{13} \pm \Gamma_{24}),\]

\[a_{4}^\pm = 2.\]

The sign + corresponds to the curve \(C_1\) and − to \(C_2\).
Using the fact that the Jacobian of the curve given with the equation

$$y^2 = a\lambda^4 + 4b\lambda^3 + 6c\lambda^2 + 4d\lambda + e$$

is a canonical curve of the form (36) with

$$g_2 = ae - 4bd + 3c^2, \quad g_3 = ace + 2bcd - ad^2 - eb^2 - c^3,$$

we finish the proof of the Lemma by straightforward calculation.

Since the curve $\Gamma_1$ is hyperelliptic, in a study of the Prym variety $\Pi$ the Mumford-Dalalian theory can be applied (see [28, 24, 10]). Thus, using the previous Lemma, we come to

**Theorem 1.** The following relations take place:

a) The Prymian $\Pi$ is isomorphic to the product of the curves $E_i$:

$$\Pi = \text{Jac}(C_1) \times \text{Jac}(C_2).$$

b) The curve $\tilde{\Gamma}$ is the desingularization of $\Gamma_1 \times (P^1)C_2 = C_1 \times (P_1)\Gamma_1$.

c) The canonical polarization divisor $\Xi$ of $\Pi$ satisfies

$$\Xi = E_1 \times \Theta_2 + \Theta_1 \times E_2$$

where $\Theta_i$ is the theta-divisor of $E_i$.

**Proof.** The Prym variety $\Pi$ corresponds to the unramified double covering $\pi: \tilde{\Gamma} \to \Gamma_1$. This covering is determined by the divisor $D \in \text{Jac}_2(\Gamma_1)$, such that $2D = (\mu)$. So

$$D = R_1 + R_2 + R_3 + R_4 - 2(\bar{P}_1 + \bar{P}_3)$$

where $R_i = \pi(S_i)$ are the projections of the singular points on $\Gamma$ and $\bar{P}_i = \pi(P_i)$ are the projections of the infinite points on $\Gamma$.

On the other hand, the double covering over $\Gamma_1$ defined by the curves $C_1, C_2$ corresponds to the divisor $D_1 \in \text{Jac}_2(\Gamma_1)$

$$D_1 = X_1 + X_2 + X_3 + X_4 - 2(\bar{P}_1 + \bar{P}_3)$$

where $X_i$ are the branch points on $\Gamma_1$ defined by $P/2 - Q = 0$. Simple calculation shows that

$$\left(\frac{\mu + P/2}{\mu - Q}\right)^2 = \frac{P/2 - Q}{-2\mu}$$

holds on $\Gamma_1$. From the last relation it follows that the divisors $D$ and $D_1$ are equivalent.

The rest of the theorem now follows from the Mumford-Dalalian theory [10, 24, 28].

Theorem 1 explains the connection between the curves $E_1, E_2$ and the Prym variety $\Pi$. Further analysis of properties of the Prym varieties necessary for the understanding of dynamics of the Lagrange bitop will be done in the next section.
7. Isoholomorphisity condition, Mumford’s relation and integration using the Baker-Akhiezer function

According to the Proposition 5, the Baker-Akhiezer function $\Psi$ satisfies usual conditions of normalized $(n=)4$-point function on the curve of genus $g = 5$ with the divisor $\mathcal{D}$ of degree $\deg \mathcal{D} = g + n - 1 = 8$, see [14, 13]. And by the general theory, it should determine all dynamics uniquely. The basic question is why is such dynamics compatible with the condition (1)? In other words, why is the evolution of divisors $\tilde{d}_i(t)$ such that all the time $\tilde{d}_1$ contains $P_2$, $\tilde{d}_2$ contains $P_1$ and so on. To answer this question, let us consider the differentials $\Omega^i_j$

$$\Omega^i_j = g_{ij}d\lambda, \quad i, j = 1, \ldots, 4.$$  

It was proven by Dubrovin in the case of general position, that $\Omega^i_j$ is a meromorphic differential having poles at $P_i$ and $P_j$, with residuums $v^i_j$ and $-v^j_i$ respectively.

We have a simple

**Lemma 10.** The condition (1) is equivalent to

$$v^1_2 = v^2_1 = v^3_4 = v^4_3 = 0.$$  

From the Lemma 10 and Corollary 2, it follows

**Proposition 6.** The four differentials 

$$\Omega^1_2, \Omega^2_1, \Omega^3_4, \Omega^4_3$$  

are holomorphic during the whole evolution.

We can say that the condition (1) (together with the Corollary 2) implies iso-holomorphism. Let us recall the general formulae for $v$ from [13].

$$v^i_j = \frac{\lambda_i \theta(A(P_i) - A(P_j) + tU + z_0)}{\lambda_j \theta(tU + z_0) \epsilon(P_i, P_j)}, \quad i \neq j,$$

where $U = \sum x^{(k)}U^{(k)}$ is certain linear combination of $b$ periods $U^{(i)}$ of the differentials of the second kind $\Omega^{(1)}_{P_1}$, which have pole of order two at $P_i$; $\lambda_i$ are nonzero scalars, and

$$\epsilon(P_i, P_j) = \frac{\theta(\mathcal{F})(A(P_i) - P_j))}{(-\partial_{U^{(i)}} \theta(\mathcal{F})(0))^{1/2}(-\partial_{U^{(j)}} \theta(\mathcal{F})(0))^{1/2}}.$$  

(Here $\mathcal{F}$ is an arbitrary odd nondegenerate characteristics.)

**Proposition 7.** Holomorphicity of some of the differentials $\Omega^i_j$ implies that the theta divisor of the spectral curve contains some torus.

In a case of spectral curve which is a double unramified covering

$$\pi : \tilde{\Gamma} \to \Gamma_1;$$
with $g(\Gamma_1) = g, \ g(\tilde{\Gamma}) = 2g - 1$, as we have here, it is really satisfied that the theta divisor contains a torus, see [24]. More precisely, following [24], let us denote by $\Pi^-$ the set
\[ \Pi^- = \left\{ L \in P\iota e^{2g - 2}\tilde{\Gamma} \middle| \text{Nm}L = K_{\Gamma_1}, h^0(L) \text{ is odd} \right\}, \]
where $K_{\Gamma_1}$ is the canonical class of the curve $\Gamma_1$ and $\text{Nm} : \text{Pic} \tilde{\Gamma} \to \text{Pic} \Gamma_1$ is the norm map, see [24, 28] for details. For us, it is crucial that $\Pi^-$ is a translate of the Prym variety $\Pi$ and that Mumford's relation ([24]) holds
\[ \Pi^- \subset \Theta_{\tilde{\Gamma}}. \]

Let us denote
\[ U = i(\chi_{34}U^{(1)} - \chi_{34}U^{(2)} + \chi_{12}U^{(3)} - \chi_{12}U^{(4)}), \]
where $U^{(i)}$ is the vector of $\tilde{\ell}$ periods of the differential of the second kind $\Omega_{\omega_i}$, which is normalized by the condition that $\tilde{a}$ periods are zero. We suppose here that the cycles $\tilde{a}, \tilde{b}$ on the curve $\tilde{\Gamma}$ and $a, b$ on $\Gamma_1$ are chosen to correspond to the involution $\sigma$ and the projection $\pi$, see [2, 28]:
\begin{align*}
\sigma(\tilde{a}_k) &= \tilde{a}_{k+2}, \quad k = 1, 2; \\
\pi(\tilde{a}_0) &= a_0; \quad \pi(\tilde{b}_0) = 2b_0.
\end{align*}

The basis of normalized holomorphic differentials $[u_0, \ldots, u_5]$ on $\tilde{\Gamma}$ and $[v_0, v_1, v_2]$ on $\Gamma_1$ are chosen such that
\begin{align*}
\pi^*(v_0) &= u_0, \\
\pi^*(v_i) &= v_i + \sigma(v_i) = v_i + v_{i+2}, \quad i = 1, 2.
\end{align*}

Now we have

**Theorem 2.** If the vector $z_0$ in (38) corresponds to the translation of the Prym variety $\Pi$ to $\Pi^-$, and the vector $U$ is defined by (40) than the conditions (37) are satisfied.

**Proof.** Proof follows from the relations (38) and (40) and fact that $P_2 = \sigma(P_1)$ and $P_1 = \sigma(P_3)$.

**Proposition 8.** The explicit formula for $z_0$ is
\[ z_0 = \frac{1}{2}(\hat{\tau}_{00}, \hat{\tau}_{01}, \hat{\tau}_{02}, \hat{\tau}_{01}, \hat{\tau}_{02}), \]
where
\[ \hat{\tau}_{0i} = \int_{\tilde{b}_0} u_i, \quad i = 0, 1, 2. \]

The proof follows from [19], the Proposition 4.7.

The formulae for scalars $\lambda_i$ from the formula (38) will be given in the next section.
8. The evolution on the generalized Jacobian

The evolution on the Jacobian of the spectral curve, as we considered \( \text{Jac}(\tilde{\Gamma}) \) in the Section 7, gives the possibility to reconstruct the evolution of the Lax matrix \( L(\lambda) \) only up to the conjugation by diagonal matrices. That was one of the limitations of the Adler-van Moerbeke approach (see [1], Theorem 1). To overcome this problem, we are going to consider, following Dubrovin, the generalized Jacobian, obtained by gluing together the infinite points; in the present case \( P_1, P_2, P_3, P_4 : \)

\[
\text{Jac}(\tilde{\Gamma} \mid \{ P_1, P_2, P_3, P_4 \}).
\]

It can be understood as a set of classes of relative equivalence among the divisors on \( \tilde{\Gamma} \) of certain degree. Two divisors of the same degree \( D_1 \) and \( D_2 \) are called equivalent relative to the points \( P_1, P_2, P_3, P_4 \) if there exists a function \( f \) meromorphic on \( \tilde{\Gamma} \) such that \( (f) = D_1 - D_2 \) and \( f(P_1) = f(P_2) = f(P_3) = f(P_4) \).

The generalized Abel map is defined with

\[
\tilde{A}(P) = (A(P), \lambda_1(P), \ldots, \lambda_4(P)),
\]

where \( A(P) \) is the standard Abel map and

\[
\lambda_i(P) = \exp \int_{P_0}^{P} \Omega_{P_i Q_0}, \quad i = 1, \ldots, 4.
\]

Here \( \Omega_{P_i Q_0} \) denotes the normalized differential of the third kind, with poles at \( P_i \) and at arbitrary fixed point \( Q_0 \).

Then the generalized Abel theorem (see [19]) can be formulated as

**Lemma 11 (the generalized Abel theorem).** The divisors \( D_1 \) and \( D_2 \) are equivalent relative to the points \( P_1, P_2, P_3, P_4 \) if and only if there exist integervalued vectors \( N, M \) such that

\[
A(D_1) = A(D_2) + 2\pi N + BM,
\]

\[
\lambda_j(D_1) = c\lambda_j(D_2) \exp(M, A(D_2)), \quad j = 1, \ldots, 4
\]

where \( c \) is some constant and \( B \) is the period matrix of the curve \( \tilde{\Gamma} \).

The generalized Jacobi inverse problem can be formulated as the question of finding, for given \( z \), points \( Q_1, \ldots, Q_8 \) such that

\[
\sum_{i=1}^{8} A(Q_i) - \sum_{i=2}^{4} A(P_i) = z + K,
\]

\[
\lambda_j = c \exp \sum_{s=1}^{8} \int_{P_0}^{Q_s} \Omega_{P_j Q_0} + \kappa_j, \quad j = 1, \ldots, 4
\]

where the constants \( \kappa_j \) depend on the curve \( \tilde{\Gamma} \), the points \( P_1, P_2, P_3, P_4 \) and the choice of local parameters around them.

We will denote by \( Q_s \) the points which belong to the divisor \( \tilde{\mathcal{D}} \) from the Proposition 5, and by \( E \) the Pryme-form from [19]. Then we have
Proposition 9. The scalars $\lambda_j$ from the formula (38) are given with
\[ \lambda_j = \lambda_j^0 \exp \left( \sum_{k \neq j} i x^{(k)} \gamma_j^k \right), \]
where
\[ \lambda_j^0 = c \exp \left( \sum_{s=1}^8 \int_{P_0}^{Q_s} \Omega_{P_j} \Omega_0 + \kappa_j \right), \]
vector $\vec{x} = (x^{(1)}, \ldots, x^{(4)})$ is $t(\chi_{34}, -\chi_{34}, \chi_{12}, -\chi_{12})$ and
\[ \gamma_j^k = \frac{d}{dk_j} \ln E(P_i, P)|_{P=P_j}. \]

To give the formulae for the Baker-Akhiezer function, we need some notations. Let
\[ \alpha_j(\vec{x}) = \exp \left( i \sum \tilde{\gamma}_j^m x^{(m)} \right) \frac{\theta(z_0)}{\theta(i \sum x^{(k)} \Omega^{(k)} + z_0)}, \]
where
\[ \tilde{\gamma}_j^m = \int_{P_0}^{P_j} \Omega_{P_m}^{(1)}, \quad m \neq j, \]
and $\tilde{\gamma}_j^m$ is defined by the expansion
\[ \int_{P_0}^{P} \Omega_{P_m}^{(1)} = -k_m + \tilde{\gamma}_m^m + O(k_m^{-1}), \quad P \to P_m. \]

Denote
\[ \phi_j(\vec{x}, P) = \alpha_j(\vec{x}) \exp \left( -i \int_{P_0}^{P} x^{(m)} \Omega_{P_m}^{(1)} \frac{\theta(A(P) - A(P_j) - i \sum x^{(k)} \Omega^{(k)} - z_0)}{\theta(A(P) - A(P_j) - z_0)} \right). \]

Now we can state
Proposition 10. The Baker-Akhiezer function is given by
\[ \psi_j(\vec{x}, P) = \phi_j(\vec{x}, P) \frac{\chi_j^0 \theta(A(P-P_j) - z_0)}{\epsilon(P,P_j)} \frac{\chi_j^0 \theta(A(P-P_k) - z_0)}{\epsilon(P,P_k)}, \quad j = 1, \ldots, 4. \]

The proofs of the statements in this Section are standard from Dubrovin’s approach.

Having established how parameters of the formula (38) evolve, the reconstruction of the evolution of the phase space variables follows immediately from the Proposition 5. The generalized Liouville tori are four dimensional. Since two of the integrals of the motion of the Lagrange bitop are linear (see (11)), according to the well known fact of Classical Mechanics ([31, 3]) those generalized tori have twodimensional affine part. The two-dimensional compact part of such a torus
corresponds to the real part of the two-dimensional Prymian. The affine part corresponds to the odd part of the affine part of the generalized Jacobian

$$\text{Jac}(\tilde{\Gamma} | \{P_1, P_2, P_3, P_4\}) = \text{Jac}(\tilde{\Gamma}) \times \mathbb{C}^3.$$  

From the Theorem 1c, it follows that the reduction of the formulae can be done up to the elliptic theta functions on $E_i$ and exponential functions.

9. The Lagrange bitop hierarchy and equally splitted double hyperelliptic coverings

According to the Mumford - Dalalian theory (see [10, 24, 28]), double unramifide coverings over a hyperelliptic curve $y^2 = P_{2g+2}(x)$ of genus $g$ are in the correspondence with the divisions of the set of the zeroes of the polynomial $P_{2g+2}$ on two disjoint nonempty subsets with even number of elements. We will consider those coverings which correspond to the divisions on subsets with equal number of elements and we can call them equally splitted, since the Prym variety splits then as a sum of two varieties of equal dimension.

Now, let us consider with the fixed operator $A$ from (5) the whole hierarchy of systems defined by the Lax equations

$$\dot{L}^{(N)}_B = [L^{(N)}_B, A],$$  

where

$$L^{(N)}_B(\lambda) = \lambda^N B + \lambda^{N-1} M_1 + \cdots + M_N$$

is a polynomial in $\lambda$ of degree $N \geq 2$, and the matrix $B$ is proportional to the matrix $\chi$: $B = d\chi$. From the Lax equations we get the system

$$\frac{\partial M_N}{\partial t} = [M_N, \Omega],$$  

$$\frac{\partial M_k}{\partial t} + [\chi, M_{k+1}] = [M_k, \Omega],$$  

$$[\chi, M_1] = [B, \Omega].$$

Generalizing the situation from the Section 4, we see that the spectral curve $\Gamma_N$ is a singular curve of the form

$$p_N(\lambda, \mu) = \mu^4 + P_N(\lambda)\mu^2 + [Q_N(\lambda)]^2 = 0,$$

where the polynomials $P_N, Q_N$ have degree $\deg P_N = \deg Q_N = 2N$. So, its normalization is a double covering over the hyperelliptic curve

$$\mu_1^2 = \frac{P_N^2(\lambda)}{4} - Q_N^2(\lambda)$$

of genus $g_N = 2N - 1$. This covering corresponds to the division of the set of zeroes on subsets of zeroes of the polynomials $P_N/2 - Q_N$ and $P_N/2 + Q_N$. This is an equally splitted covering under the assumption $|\chi_{12}| \neq |\chi_{34}|$ we fixed at the beginning. It is easy to see that all equally splitted coverings can be realized in such a way. So we have
Theorem 3. The Lagrange bitop hierarchy realizes all equally splitted coverings over the hyperelliptic curves of genus greater than two.

Acknowledgement One of the authors (V. D.) has a great pleasure to thank Professor B. Dubrovin for stimulating discussions and Professor M. Narasimhan for helpful observations; his research was partially supported by SISSA and MURST Project Geometry of Integrable Systems. The research of both authors was partially supported by the Serbian Ministry of Science and Technology projects.

References

1. M. Adler and P. van Moerbeke. Linearization of Hamiltonian Systems, Jacobi Varieties and Representation Theory. *Advances in Math.* **38** (1980), 318-379.
2. E. Arbarello, M. Cornalba, P. A. Griffiths and J. Haris. *Geometry of algebraic curves* (Springer-Verlag, 1985).
3. V. I. Arnol’d. *Mathematical methods of classical mechanics* (Moscow: Nauka, 1989 [in Russian, 3rd edition]).
4. V. I. Arnol’d, V. V. Kozlov and A. I. Neishtadt. *Mathematical aspects of classical and celestial mechanics* / in *Dynamical systems III* (Berlin: Springer-Verlag, 1988).
5. M. Audin. *Spinning Tops* (Cambridge studies in Advanced Mathematics 51, 1996).
6. A. Beauville. Prym varieties and Schottky problem. *Inventiones Math.* **41** (1977), 149-196.
7. E. D. Belokolos, A. I. Bobenko, V. Z. Enol’skii, A. R. Its and V. B. Matveev. *Algebro-geometric approach to nonlinear integrable equations* (Springer series in Nonlinear dynamics, 1994).
8. A. I. Bobenko, A. G. Reyman, M. A. Semenov-Tien-Shansky. The Kovalewski top 99 years later *Commun. Math. Phys.* **122** (1989), 321-354.
9. O. I. Bogoyavlensky Integrable Euler equations on Li algebras arising in physical problems *Soviet Acad Izvestya* **48** (1984), 883-938 [in Russian]
10. S. G. Dalalian. Prym varieties of unramified double coverings of the hyperelliptic curves *Uspekhi Math. Naukh* **29** (1974) 165-166 [in Russian]
11. V. Dragović, B. Gajić: An L-A pair for the Hess-Apel’rot system and a new integrable case for the Euler-Poisson equations on so(4) × so(4). *Roy. Soc. of Edinburgh: Proc A* **131** (2001), 845-855.
12. B. A. Dubrovin. Vpolne integriruemye gamil’tonovy sistemy svyazanye s matirchnymi operatorami i Abelevy mnogoobrazyi. *Funk. Analiz i ego prilozhe-niya* **11**, (1977 [in Russian]), 28-41.
13. B. A. Dubrovin. Theta-functions and nonlinear equations. *Uspekhi Math. Nauk.* **36** (1981 [in Russian]), 11-80.
14. B. A. Dubrovin, I. M. Krichever and S. P. Novikov. Integrable systems I. in *Dynamical systems IV*, (Berlin: Springer-Verlag, ) 173-280.
15. B. A. Dubrovin, V. B. Matveev, S. P. Novikov. Nonlinear equations of Kortever-de Fries type, finite zone linear operators and Abelian varieties. *Uspekhi Math.*
LAGRANGE BITOP

Nauk. 31 (1976 [in Russian]), 55-136.
16 L. Gavrilov, A. Zhivkov. The complex geometry of Lagrange top. L’Enseignement Mathématique. 44 (1998), 133-170
17 V. V. Golubev. Lectures on integration of the equations of motion of a rigid body about a fixed point (Moskow: Gostenhizdat, 1953 [in Russian]; English translation: Philadelphia: Coronet Books, 1953).
18 P. A. Griffiths. Linearizing flows and a cohomological interpretation of Lax equations. American Journal of Math 107 (1983), 1445-1483.
19 J. D. Fay. Theta functions on Riemann surfaces, Lecture Notes in Mathematics, vol. 352, (Springer-Verlag), (1973)
20 I. M. Krichever. Algebro-geometric methods in the theory of nonlinear equations Uspekhi Math. Naukh 32 (1977) 183 - 208
21 S. V. Manakov. Remarks on the integrals of the Euler equations of the n-dimensional heavy top. Funkc. Anal. Appl. 10 (1976 [in Russian]), 93-94.
22 P. van Moerbeke and D. Mumford. The spectrum of difference operators and algebraic curves. Acta Math. 143 (1979), 93-154.
23 D. Mumford. Theta characteristics of an algebraic curve. Ann. scient. Ec. Norm. Sup. 4 serie 4(1971), 181-192.
24 D. Mumford. Prym varieties I. A collection of papers dedicated to Lipman Bers (Acad. Press.) New York, (1974), p. 325-350.
25 T. Ratiu. Euler-Poisson equation on Lie algebras and the N-dimensional heavy rigid body. American Journal of Math 104 (1982), 409-448.
26 T. Ratiu and P. van Moerbeke. The Lagrange rigid body motion. Ann. Ins. Fourier, Grenoble 32 (1982), 211-234.
27 A. G. Reyman and M. A. Semenov-Tian-Shansky: Lax representation with spectral parameter for Kovalevskaya top and its generalizations. Funkc. Anal. Appl. 22 (1988 [in Russian]), 87-88.
28 V. V. Shokurov. Algebraic curves and their Jacobians. in Algebraic Geometry III, (Berlin: Springer-Verlag, 1998 ) 219-261.
29 V. V. Shokurov. Distinguishing Prymians from Jacobians. Invent. Math. 65 (1981) 209-219
30 V. V. Trofimov and A. T. Fomenko. Algebra and geometry of integrable Hamiltonian differential equations (Moscow: Faktorial, 1995 [in Russian]).
31 Whittaker A treatise on the analytical dynamics of particles and rigid bodies, Cambridge at the University Press, (1952), p.456