A-priori pole selection for reduced models in vibro-acoustics

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Large models of complex dynamic systems can be evaluated efficiently using model order reduction methods. Many techniques, for example the iterative rational Krylov algorithm (IRKA), rely on a set of expansion points chosen before the reduction procedure. The number and location of the expansion points has a major impact on the quality of the resulting reduced model and the convergence of the algorithm. Based on the system’s geometry and material, the number of modes in a certain frequency range can be computed using wave equations. This mode count allows the choice of both a reasonable size for the reduced model as well as a reasonable distribution of initial expansion points, which improves the convergence of IRKA. Using the mode count in a specific frequency range, a reduced model approximating the full model only in this frequency range can be generated.

1 Introduction

To model the response of vibro-acoustic systems under high frequent excitation using the finite element method (FEM), a very fine spatial discretization has to be used; this results in large and computationally expensive models. Model order reduction (MOR) methods can be used to reduce the computational effort while preserving the full system’s frequency response. MOR methods for the efficient computation of systems modeling various physical phenomena exist and are based on different mathematical tools. In the following, we will consider a moment-matching method based on Krylov subspaces for second order dynamic systems [1].

This method is not depending on classical damping mechanisms (e.g. proportional damping) and can therefore be used for systems with spatially varying damping characteristics. This is often the case for vibro-acoustic problems. In the vibro-acoustic setting, it is often sufficient to compute the system response in a defined frequency range, where e.g. the damping material is active. We present a strategy to efficiently generate reduced models of vibro-acoustic systems, which are valid in a specific frequency range. For this, we use characteristics of the system, which are known a priori.

2 Reduced order modeling for vibro-acoustic systems

The vibration of a structure with \( n \) degrees of freedom under a harmonic excitation with frequency \( \omega \) and a single input and a single output is modeled using second order dynamic systems of the form

\[
\begin{align*}
\left(-\omega^2 M + i\omega C + K\right) u &= f \\
y &= b^T u.
\end{align*}
\]

\( M, C, K \in \mathbb{R}^{n \times n} \) are mass, damping, and stiffness matrix respectively, \( f, u, b \in \mathbb{R}^n \) are force, displacement, and output vector, and \( y \) is the system output. The response function is given by \( h_1(\omega) = b^T \left(-\omega^2 M + i\omega C + K\right)^{-1} f \). MOR methods project the large system matrices onto a lower dimensional subspace using projection matrices \( V, W \in \mathbb{R}^{n \times r} \), where \( r \ll n \):

\[
M_r = V^T M V, \quad C_r = V^T C V, \quad K_r = V^T K V, \quad f_r = W^T f, \quad b_r = b V.
\]

The reduced system’s transfer function is given by \( h_r(\omega) = b_r^T \left(-\omega^2 M_r + i\omega C_r + K_r\right)^{-1} f_r \) and the reduction bases \( V, W \) have to be chosen such that \( h_r(\omega) \approx h(\omega) \). In the vibro-acoustic setting, the system matrices are symmetric positive definite, so we set \( W = V \) to preserve this property [2].

A method to find a reduction basis for second order systems is the iterative rational Krylov algorithm (SO-IRKA) [2], which iteratively finds optimal expansion points for a Hermitian interpolation of the full system. The algorithm needs an initial set of \( r \) expansion points \( s = i\omega \) at which the system is evaluated; these evaluations form the reduction basis:

\[
V = \left[\left(s_r^2 M + s_r C + K\right)^{-1} f, \ldots, \left(s_1^2 M + s_1 C + K\right)^{-1} f\right].
\]

In each IRKA iteration, the expansion points for the next step are chosen as the mirror images of the reduced system’s poles. Solving the eigenproblem for second order systems results in \( 2r \) eigenvalues, so to keep the reduced model from doubling each iteration, only half of them are selected as expansion points for the next iteration [2]. If we choose only the mirror images
of eigenvalues lying in a specific frequency region as new expansion points, the reduced model is valid in this frequency region. This choice is only possible due to the second order structure, so the strategy differs from the use of frequency-limited Gramians for first order systems proposed in [3,4].

3 A strategy to choose initial IRKA expansion points

The dynamic behavior of vibro-acoustic systems is characterized by the number of modes for specific frequencies. This mode count $N$ can be derived from system geometry, material parameters, and the boundary conditions using wave equations. For a rectangular plate in bending excited by a harmonic load with frequency $\omega$, $N$ can be computed with the equations found in Chapter 8 of [5] using the plate’s area, thickness, material density, Poisson’s ratio, Young’s modulus, and boundary conditions. Relations for other geometries can also be found in [5]. The modes of a system and the optimal expansion points coincide [6], so using the mode count as a tool to distribute initial IRKA expansion points reduces the necessary iterations until convergence and therefore the computational cost, as $r$ solutions of the full system have to be computed during each iteration.

We propose the following strategy to choose initial expansion points for SO-IRKA: We first choose the frequency range $[\omega_{\text{min}}, \omega_{\text{max}}]$ in which the reduced model should be valid. Now the mode count in this frequency region is evaluated. As many expansion points as computed modes are distributed linearly within $[\omega_{\text{min}}, \omega_{\text{max}}]$ and SO-IRKA is started. After each iteration, the expansion points are updated using only eigenvalues lying within $[\omega_{\text{min}}, \omega_{\text{max}}]$ to make the reduced model valid in this range. Figure 1 illustrates the choice of expansion points and compares the reduced model to the full solution.

![Figure 1](image_url)

**Fig. 1:** Left: Mode count function (—) and chosen expansion points (•). Middle: Frequency response of the reduced (—) and full (—) model. Right: Relative error between reduced and full model. The gray area in all three plots (■) shows the frequency range to which the reduced model is tuned.

In this example, an aluminum plate discretized by FEM with order $n = 1596$ is clamped along all edges and subjected to a randomly distributed loading orthogonal to the plate. The normalized energy is computed by normalizing the averaged kinetic and potential energies with respect to the input power. The reduced model of dimension $r = 24$ approximates the full solution within $[\omega_{\text{min}}, \omega_{\text{max}}]$ with a relative error of $\frac{||h(\omega) - h_r(\omega)||}{||h(\omega)||} = 1.00e^{-7}$ in this range. The model converged after the first iteration for a tolerance of $1e - 5$, so the mode count function estimated the location of the transfer function poles well.

4 Conclusion

We presented a method to generate reduced order models for vibro-acoustic systems valid in a specific frequency range. In order to accelerate IRKA convergence, we used the mode count, which is available for structures with various geometries. A rapid convergence and a low error in the frequency range of interest were achieved. If some poles of the system are heavily damped, the mode count overestimates the number of necessary expansion points, but IRKA still provides a reduced model with good approximation at convergence. Insights about the damping mechanism can help adjusting the mode count.

References

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