Uniform Regularity Estimates in Homogenization Theory of Elliptic System with Lower Order Terms

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Abstract

In this paper, we extend the uniform regularity estimates obtained by M. Avellaneda and F. Lin in [3,6] to the more general second order elliptic systems in divergence form \{L_ε, ε > 0\}, with rapidly oscillating periodic coefficients. We establish not only sharp \(W^{1,p}\) estimates, Hölder estimates, Lipschitz estimates and non-tangential maximal function estimates for the Dirichlet problem on a bounded \(C^1,η\) domain, but also a sharp \(O(ε)\) convergence rate in \(H^1_0(Ω)\) by virtue of the Dirichlet correctors. Moreover, we define the Green’s matrix associated with \(L_ε\) and obtain its decay estimates. We remark that the well known compactness methods are not employed here, instead we construct the transformations (1.11) to make full use of the results in [3,6].

1 Introduction and main results

The main purpose of this paper is to study the uniform regularity estimates for second order elliptic systems with lower order terms, arising in homogenization theory. More precisely, we consider

\[ L_ε = -\text{div} [A(x/ε) \nabla + V(x/ε)] + B(x/ε) \nabla + c(x/ε) + \lambda I, \]

where \(λ\) is a constant, and \(I = (δ^{αβ})\) denotes the identity matrix. In a special case, let \(A = I = 1\), \(V = B\), \(c = 0\), and \(W = \text{div}(V)\), the operator \(L_ε\) becomes

\[ L_ε = -\Delta + \frac{1}{ε} W(x/ε) + \lambda, \]

where \(W\) is the rapidly oscillating potential term (see [7, pp.91]). It is not hard to see that the uniform regularity estimates obtained in this paper are not trivial generalizations of [3,6], and they are new even for \(L_ε\).

Let 1 \(≤ i, j \leq d\), 1 \(≤ α, β \leq m\), where \(d \geq 3\) denotes the dimension, and \(m \geq 1\) is the number of equations in the system. Suppose that the measurable functions \(A = (a^{αβ}_{ij}) : \mathbb{R}^d \rightarrow \mathbb{R}^{m^2 \times d^2}\), \(V = (V^{αβ}_i) : \mathbb{R}^d \rightarrow \mathbb{R}^{m^2 \times d}\), \(B = (B^{αβ}_i) : \mathbb{R}^d \rightarrow \mathbb{R}^{m^2 \times d}\), \(c = (c^{αβ}) : \mathbb{R}^d \rightarrow \mathbb{R}^{m^2}\) satisfy the following conditions:

- the uniform ellipticity condition

\[ μ|ξ|^2 ≤ a^{αβ}_{ij}(y)ξ^α_i ξ^β_j ≤ μ^{-1}|ξ|^2 \quad \text{for } y \in \mathbb{R}^d \text{ and } ξ = (ξ^α) \in \mathbb{R}^{md}, \text{ where } μ > 0; \quad (1.1) \]

(The summation convention for repeated indices is used throughout.)
Throughout this paper, we always assume that $\Omega$ is a bounded domain with $\eta \in [\tau, 1]$, and $L_\epsilon = -\nabla[A(x/\epsilon)\nabla]$ is the elliptic operator from [3,6], unless otherwise stated.

The main idea of this paper is to find the transformations (1.11) between two solutions corresponding to $L_\epsilon$ and $L_\sigma$ such that the regularity results of $L_\epsilon$ can be applied to $L_\sigma$ directly. Particularly, in order to handle the boundary Lipschitz estimates, we define the Dirichlet correctors $\Phi_{\epsilon,k} = (\Phi_{\alpha\beta})$, $0 \leq k \leq d$, with $\beta$ the boundedness condition

$$\max\{\|V\|_{L^\infty}(\mathbb{R}^d), \|B\|_{L^\infty(\mathbb{R}^d)}, \|c\|_{L^\infty(\mathbb{R}^d)}\} \leq \kappa_1, \quad \text{where } \kappa_1 > 0;$$

the regularity condition

$$\max\{\|A\|_{C^{0,\gamma}(\mathbb{R}^d)}, \|V\|_{C^{0,\gamma}(\mathbb{R}^d)}, \|B\|_{C^{0,\gamma}(\mathbb{R}^d)}\} \leq \kappa_2, \quad \text{where } \tau \in (0, 1) \text{ and } \kappa_2 > 0.$$

Set $\kappa = \max\{\kappa_1, \kappa_2\}$, and we say $A \in \Lambda(\mu, \tau, \kappa)$ if $A = A(y)$ satisfies conditions (1.1), (1.2) and (1.3). Throughout this paper, we always assume that $\Omega$ is a bounded $C^1,\eta$ domain with $\eta \in [\tau, 1]$, and $L_\epsilon = -\nabla[A(x/\epsilon)\nabla]$ is the elliptic operator from [3,6], unless otherwise stated.

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and \(B^{\alpha,p}(\partial\Omega; \mathbb{R}^m)\) denotes the \(L^p\) Besov space of order \(\alpha\) (see [1]). We mention that for ease of notations we say the constant \(C\) depends on \(\omega\) instead of \(\omega(t)\) in the rest of the paper. We will prove Theorem 1.1 in Section 3 by using bootstrap and duality arguments. We mention that there are no periodicity or regularity assumptions on the coefficients of the lower order terms in Theorems 1.1 and 1.2. The estimate (1.8) still holds when \(\Omega\) is a bounded \(C^1\) domain (see [39]).

Results of the \(W^{1,p}\) estimates for elliptic or parabolic equations with VMO coefficients can be found in [10,12,16,34,35]. In the periodic setting, similar estimates for parabolic systems, elasticity systems, and Stokes systems were obtained by [17,18,21], respectively. Also, the uniform \(W^{1,p}\) estimates for \(L_\varepsilon\) with almost periodic coefficients were shown in [2] recently.

\textbf{Theorem 1.2} (Hölder estimates). Suppose that the coefficients of \(L_\varepsilon\) satisfy the same conditions as in Theorem 1.1. Let \(p > d\), \(f = (f^\alpha) \in L^p(\Omega; \mathbb{R}^{md})\), \(F \in L^q(\Omega; \mathbb{R}^m)\), and \(g \in C^{0,\sigma}(\partial\Omega; \mathbb{R}^m)\), where \(q = \frac{pd}{p+d}\), and \(\sigma = 1 - d/p\). Then the weak solution \(u_\varepsilon\) to (1.7) satisfies the uniform estimate

\[\|u_\varepsilon\|_{C^{0,\sigma}(\Omega)} \leq C\left\{\|f\|_{L^p(\Omega)} + \|F\|_{L^q(\Omega)} + \|g\|_{C^{0,\sigma}(\partial\Omega)}\right\},\]

where \(C\) depends on \(\mu, \omega, \kappa, \lambda, p, \sigma, d, m\) and \(\Omega\).

The estimate (1.9) is sharp in terms of the Hölder exponent of \(g\). If \(g \in C^{1,1}(\partial\Omega; \mathbb{R}^m)\), (1.9) is just Corollary 3.8. The uniform Hölder estimates for \(L_\varepsilon\) were given in [3] by the compactness method which also works for non-divergence form elliptic equations (see [4]). However, we cannot derive the sharp estimate by simply applying this method. So we turn to study the Green’s matrix \(G_\varepsilon(x,y)\) associated with \(L_\varepsilon\) and obtain the decay estimates

\[|G_\varepsilon(x,y)| \leq \frac{C}{|x-y|^{d-2}} \min\left\{1, \frac{d_x^\sigma}{|x-y|^\sigma}, \frac{d_y^\sigma}{|x-y|^\sigma}, \frac{d_x^\sigma d_y^\sigma}{|x-y|^{\sigma + \sigma}}\right\}, \quad \forall x, y \in \Omega, \quad x \neq y,
\]

where \(\sigma, \sigma' \in (0,1), d_x = \text{dist}(x, \partial\Omega)\) denotes the distance between \(x\) and \(\partial\Omega\), and \(C\) is independent of \(\varepsilon\) (see Theorem 3.11). Then we prove Theorem 1.2 through a subtle argument developed by Z. Shen [40], where he proved a similar result for \(L_\varepsilon\) in the almost periodic setting.

The existence and some related properties of the Green’s matrix with respect to \(L_1\) were studied by S. Hofmann and S. Kim [26]. We also refer the reader to [27] for parabolic systems, and [23, 36] for the scalar case.

\textbf{Theorem 1.3} (Lipschitz estimates). Suppose that \(A \in \Lambda(\mu, \tau, \kappa), V\) satisfies (1.2), (1.4), \(B\) and \(c\) satisfy (1.3), and \(\lambda \geq \lambda_0\). Let \(p > d\) and \(0 < \sigma \leq \eta\). Then for any \(f \in C^{0,\sigma}(\Omega; \mathbb{R}^{md}), F \in L^p(\Omega; \mathbb{R}^m)\), and \(g \in C^{1,\sigma}(\partial\Omega; \mathbb{R}^m)\), the weak solution to (1.7) satisfies the uniform estimate

\[\|
abla u_\varepsilon\|_{L^\infty(\Omega)} \leq C \left\{\|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} + \|g\|_{C^{1,\sigma}(\partial\Omega)}\right\},\]

where \(C\) depends on \(\mu, \tau, \kappa, \lambda, p, d, m, \sigma, \eta\) and \(\Omega\).

The estimate (1.10) can not be improved even if the coefficients of \(L_\varepsilon\) and \(\Omega\) are smooth, since the corrector \(\chi_0\) defined in (2.1) is a counter example. Here we use two important transformations

\[u_\varepsilon = [I + \varepsilon \chi_0(x/\varepsilon)] v_\varepsilon, \quad \text{and} \quad u_\varepsilon = \Phi_{\varepsilon,0} v_\varepsilon\] \hspace{1cm} (1.11)

to deal with the interior and global Lipschitz estimates, respectively. We explain the main idea as follows:

\[(D_1) \begin{cases} L_\varepsilon(u_\varepsilon) = \text{div}(f) + F \quad \text{in } \Omega, \\ u_\varepsilon = g \quad \text{on } \partial\Omega, \end{cases} \quad (D_2) \begin{cases} L_\varepsilon(v_\varepsilon) = \text{div}(\tilde{f}) + \tilde{F} \quad \text{in } \Omega, \\ v_\varepsilon = \Phi_{\varepsilon,0} v_\varepsilon \quad \text{on } \partial\Omega. \end{cases}\]

Note that \(\Phi_{\varepsilon,0}\) is not periodic, which is the main difficulty to overcome. So we rewrite (D_1) as (D_2) to keep \(L_\varepsilon\) periodic, while the price to pay is that the new source term \(\tilde{f}\) involves \(\nabla u_\varepsilon\). As we mentioned before, there is no uniformly bounded Hölder estimate for \(\nabla u_\varepsilon\). Fortunately, it follows from Theorem 1.2 that

\[\|
abla \Phi_{\varepsilon,0}\|_{C^{0,\sigma_1}(\Omega)} = O(\varepsilon^{-\sigma_1}) \quad \text{and} \quad \|
abla u_\varepsilon\|_{C^{0,\sigma_1}(\Omega)} = O(\varepsilon^{-\sigma_2}) \quad \text{as } \varepsilon \to 0,\]

where \(\sigma_1, \sigma_2\) are the Hölder exponents in (1.10) and (1.12), respectively.
where $0 < \sigma_1 < \sigma_2 < 1$ are independent of $\varepsilon$ (see Lemma 4.9 and 4.10). Together with an important consequence of Lemma 4.8

$$\|\Phi_{\varepsilon,0} - I\|_{L^\infty(\Omega)} = O(\varepsilon) \quad \text{as } \varepsilon \to 0,$$

(1.13)

we obtain that $\tilde{f}$ is uniformly Hölder continuous through the observation that the convergence rate in (1.13) is faster than the divergence rate in (1.12) as $\varepsilon \to 0$. Also, Theorem 1.1 implies $\tilde{F} \in L^p(\Omega)$ with $p > d$. Thus we can employ the results in [3] immediately, and the proof of Theorem 1.3 is finalized by a suitable extension technique.

We remark that the compactness argument for our elliptic systems is also valid, however it would be much more complicated. For more references, C. E. Kenig and C. Prange [32] established uniform Lipschitz estimates with more general source terms in the oscillating boundaries setting, and the same type of results for parabolic systems and Stokes systems were shown in [17][24], respectively.

**Theorem 1.4** (Nontangential maximal function estimates). Suppose that $A \in \Lambda(\mu, \tau, \kappa)$, $V, B$ satisfy (1.2) and (1.4), $c$ satisfies (1.3), and $\lambda \geq \lambda_0$. Let $1 < p < \infty$, and $u_\varepsilon$ be the solution of the $L^p$ Dirichlet problem $L_\varepsilon(u_\varepsilon) = 0$ in $\Omega$ and $u_\varepsilon = g$ on $\partial\Omega$ with $(u_\varepsilon)^* \in L^p(\partial\Omega)$, where $g \in L^p(\partial\Omega; \mathbb{R}^m)$ and $(u_\varepsilon)^*$ is the nontangential maximal function. Then

$$\| (u_\varepsilon)^* \|_{L^p(\partial\Omega)} \leq C_p \| g \|_{L^p(\partial\Omega)},$$

(1.14)

where $C_p$ depends on $\mu, \tau, \kappa, \lambda, d, m, p, \eta$ and $\Omega$. Furthermore, if $g \in L^\infty(\partial\Omega; \mathbb{R}^m)$, we have

$$\| u_\varepsilon \|_{L^\infty(\Omega)} \leq C \| g \|_{L^\infty(\partial\Omega)},$$

(1.15)

where $C$ depends on $\mu, \tau, \kappa, \lambda, d, m, \eta$ and $\Omega$.

The estimate (1.15) is known as the Agmon-Miranda maximum principle, and $(u_\varepsilon)^*$ is defined in (4.40). We remark that the proof of Theorem 1.4 is motivated by [4][31][38]. Define the Poisson kernel associated with $L_\varepsilon$ as

$$P_\varepsilon^{\alpha\beta}(x, y) = -n_j(x) a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial}{\partial x_i} \{ G_\varepsilon^{\alpha\gamma}(x, y) \} - n_j(x) B_{ij}^{\alpha\beta}(x/\varepsilon) G_\varepsilon^{\alpha\gamma}(x, y),$$

where $n_j$ denotes the $j$th component of the outward unit normal vector of $\partial\Omega$. Due to Theorem 1.3 we obtain $|\nabla x \nabla y G_\varepsilon(x, y)| \leq C |x - y|^{-d}$ for $x, y \in \Omega$, and $x \neq y$ (see Lemma 4.11), which implies the decay estimate of $P_\varepsilon$ (see (4.37)). Thus the solution $u_\varepsilon$ can be formulated by (4.38). Note that $P_\varepsilon$ is actually closely related to the adjoint operator $L_\varepsilon^*$ (see Remark 2.3). That is the reason why we additionally assume (1.2) and (1.4) for $B$ in this theorem. We refer the reader to Remark 4.12 for more references on Theorem 1.4.

**Theorem 1.5** (Convergence rates). Let $\Omega$ be a bounded $C^{1,1}$ domain. Suppose that the coefficients of $L_\varepsilon$ satisfy the same conditions as in Theorem 1.3 and $B, c$ additionally satisfy the periodicity condition (1.2). Let $u_\varepsilon$ be the weak solution to $L_\varepsilon(u_\varepsilon) = F$ in $\Omega$ and $u_\varepsilon = 0$ on $\partial\Omega$, where $F \in L^2(\Omega; \mathbb{R}^m)$. Then we have

$$\| u_\varepsilon - \Phi_{\varepsilon,0} u - (\Phi_{\varepsilon,k} - P_{\varepsilon,k}) \frac{\partial u^\beta}{\partial x_k} \|_{H^1_0(\Omega)} \leq C \varepsilon \| F \|_{L^2(\Omega)},$$

(1.16)

where $u$ satisfies $L_0(u) = F$ in $\Omega$ and $u = 0$ on $\partial\Omega$. Moreover, if the coefficients of $L_\varepsilon$ satisfy (1.1) - (1.4), then

$$\| u_\varepsilon - u \|_{L^p(\Omega)} \leq C \varepsilon \| F \|_{L^p(\Omega)}$$

(1.17)

holds for any $F \in L^p(\Omega; \mathbb{R}^m)$, where $q = \frac{pd}{d-p}$ if $1 < p < d$, $q = \infty$ if $p > d$, and $C$ depends on $\mu, \tau, \kappa, \lambda, m, d, p$ and $\Omega$. 


We mention that the estimates \([1.16]\) and \([1.17]\) are sharp in terms of the order of \(\varepsilon\). The ideas in the proof are mainly inspired by \([29,30]\). It is easy to see \(\|u_\varepsilon-u\|_{L^2(\Omega)} = O(\varepsilon)\) is a direct corollary of \([1.16]\) or \([1.17]\). In the case of \(p = d\), \([1.17]\) is shown in Remark 5.3. Moreover, in the sense of “operator error estimates” the convergence rate like \([1.17]\) can also be expressed by \(\|\mathcal{L}_\varepsilon^{-1} - \mathcal{L}_0^{-1}\|_{L^p \rightarrow L^q} \leq C\varepsilon\), where \(\| \cdot \|_{L^p \rightarrow L^q}\) is referred to as the \((L^p \rightarrow L^q)\)-operator norm.

The convergence rates are active topics in homogenization theory. Decades ago, the \(L^2\) convergence rates were obtained in \([7,25]\) for scalar cases due to the maximum principle. At the beginning of 2000’s, the operator-theoretic (spectral) approach was successfully introduced by M. Sh. Birman and T. A. Suslina \([8,9]\) to investigate the convergence rates (operator error estimates) for the problems in the whole space \(L^2(\mathbb{R}^d)\). They obtained the sharp convergence rates \(O(\varepsilon)\) in the \((L^2 \rightarrow L^2)\)-operator norm and \((L^2 \rightarrow H^1)\)-operator norm for a wide class of matrix strongly elliptic second order self-adjoint operators, respectively. These results were extended to second order strongly elliptic systems including lower order terms in \([44]\). Recently, C. E. Kenig, F. Lin and Z. Shen \([29]\) developed the \(L^2\) convergence rates for elliptic systems on Lipschitz domains with either Dirichlet or Neumann boundary data by additionally assuming regularity and symmetry conditions, while T.A. Suslina \([42,43]\) also obtained similar results on a bounded \(C^{1,1}\) domain without any regularity assumption on the coefficients. We refer the reader to \([13,19,20,37,40]\) and references therein for more results.

In the end, we comment that the above five theorems are still true for \(d = 1, 2\). Since we usually have a different method to treat the cases \(d \geq 3\) and \(d = 1, 2\) (for example, see \([26, pp.2]\)), we omit the discussion about the cases of \(d = 1, 2\) here.

## 2 Preliminaries

Define the correctors \(\chi_k = (\chi_k^{\alpha\beta}), \ 0 \leq k \leq d\), associated with \(\mathcal{L}_\varepsilon\) as follows:

\[
\begin{aligned}
L_1(\chi_k) &= \text{div}(V) \quad \text{in } \mathbb{R}^d, \\
\chi_k &\in H^1_{\text{per}}(Y; \mathbb{R}^{m^2}) \quad \text{and } \int_Y \chi_k dy = 0
\end{aligned}
\]  

for \(k = 0\), and

\[
\begin{aligned}
L_1(\chi_k + P_k^\beta) &= 0 \quad \text{in } \mathbb{R}^d, \\
\chi_k^\beta &\in H^1_{\text{per}}(Y; \mathbb{R}^m) \quad \text{and } \int_Y \chi_k^\beta dy = 0
\end{aligned}
\]  

for \(1 \leq k \leq d\), where \(Y = [0,1]^d \cong \mathbb{R}^d/\mathbb{Z}^d\), and \(H^1_{\text{per}}(Y; \mathbb{R}^m)\) denotes the closure of \(C^\infty_{\text{per}}(Y; \mathbb{R}^m)\) in \(H^1(Y; \mathbb{R}^m)\). Note that \(C^\infty_{\text{per}}(Y; \mathbb{R}^m)\) is the subset of \(C^\infty(Y; \mathbb{R}^m)\), which collects all \(Y\)-periodic vector-valued functions (see \([14, pp.56]\)). By asymptotic expansion arguments, we obtain the homogenized operator

\[
\mathcal{L}_0 = -\text{div}(\hat{A}\nabla + \hat{V}) + \hat{B}\nabla + \hat{c} + \lambda I,
\]

where \(\hat{A} = (\hat{a}_{ij}^{\alpha\beta}), \hat{V} = (\hat{v}_i^{\alpha\beta}), \hat{B} = (\hat{B}_i^{\alpha\beta})\) and \(\hat{c} = (\hat{c}^{\alpha\beta})\) are given by

\[
\begin{aligned}
\hat{a}_{ij}^{\alpha\beta} &= \int_Y [a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma} \frac{\partial \chi_k^{\gamma\beta}}{\partial y_k}] dy, \\
\hat{v}_i^{\alpha\beta} &= \int_Y [v_i^{\alpha\beta} + a_{ij}^{\alpha\gamma} \frac{\partial \chi_k^{\gamma\beta}}{\partial y_j}] dy, \\
\hat{B}_i^{\alpha\beta} &= \int_Y [B_i^{\alpha\beta} + B_j^{\alpha\gamma} \frac{\partial \chi_k^{\gamma\beta}}{\partial y_j}] dy, \\
\hat{c}^{\alpha\beta} &= \int_Y [c^{\alpha\beta} + B_i^{\alpha\gamma} \frac{\partial \chi_k^{\gamma\beta}}{\partial y_i}] dy.
\end{aligned}
\]  

The proof is left to readers (see \([7, pp.103]\) or \([28, pp.31]\)).

**Remark 2.1.** It follows from the conditions \([1.1]\) and \([1.2]\) that \(\mu |\xi|^2 \leq \hat{a}_{ij}^{\alpha\beta} \xi_i \xi_j^{\alpha\beta} \leq \mu_1 |\xi|^2\) holds for any \(\xi = (\xi^{\alpha}) \in \mathbb{R}^{md}\), where \(\mu_1\) depends only on \(\mu\). Moreover, if \(a_{ij}^{\alpha\beta} = a_{ji}^{\alpha\beta}\), then \(\mu |\xi|^2 \leq \hat{a}_{ij}^{\alpha\beta} \xi_i \xi_j^{\alpha\beta} \leq \mu^{-1} |\xi|^2\) (see \([7, pp.23]\)). This illustrates that the operator \(\mathcal{L}_0\) is still elliptic.
Remark 2.2. We introduce the following notations for simplicity. We write \( \chi_{k, \varepsilon}(x) = \chi_k(x/\varepsilon) \), \( A_{\varepsilon}(x) = A(x/\varepsilon) \), \( V_{\varepsilon}(x) = V(x/\varepsilon) \), \( B_{\varepsilon}(x) = B(x/\varepsilon) \), \( c_{\varepsilon}(x) = c(x/\varepsilon) \), and their components follow the same abbreviated way. Note that the abbreviations are not applied to \( \Phi_{k, \varepsilon}(x) \) or \( \Psi_{\varepsilon, k}(x) \).

Let \( B = B(x, r) = B_r(x) \), and \( KB = B(x, Kr) \) denote the concentric balls as \( K > 0 \) varies, where \( r < 1 \) in general. We say that \( \Omega \) is a bounded \( C^1 \), \( r \) domain, if there exist \( r_0 > 0 \), \( M_0 > 0 \) and \( \{ P_i : i = 1, 2, \ldots, n \} \subset \partial \Omega \) such that \( \partial \Omega \subset \bigcup_{i=1}^{n_1} B(P_i, r_0) \) and for each \( i \), there exists a function \( \psi_i \in C^{3, \eta} (\mathbb{R}^{d-1}) \) and a coordinate system, such that \( B(P_i, C_0 r_0) \cap \Omega = B(0, C_0 r_0) \cap \{ (x', x_d) \in \mathbb{R}^d : x' \in \mathbb{R}^{d-1} \text{ and } x_d > \psi_i(x') \} \), where \( C_0 = 10(M_0 + 1) \) and \( \psi_i \) satisfies

\[
\psi_i(0) = 0, \quad \text{and} \quad \| \psi_i \|_{C^{3, \eta}(\mathbb{R}^{d-1})} \leq M_0. \tag{2.4}
\]

We set \( D(r) = D(r, \psi) = \{ (x', x_d) \in \mathbb{R}^d : |x'| < r \text{ and } \psi(x') < x_d < \psi(x') + C_0 r \} \) and \( \Delta(r) = \Delta(r, \psi) = \{ (x', \psi(x')) \in \mathbb{R}^d : |x'| < r \} \). In the paper, we say the constant \( C \) depends on \( \Omega \), which means \( C \) involves both \( M_0 \) and \( |\Omega| \). This is especially important when we do near boundary regularity estimates. Here \( |\Omega| \) denotes the volume of \( \Omega \). We also mention that for any \( E \subset \Omega \), we write \( f_E = \int_E f(x) dx = \frac{1}{|E|} \int_E f(x) dx \), and the subscript of \( f_E \) is usually omitted.

Remark 2.3. Let \( L_{\varepsilon}^* \) be the adjoint of \( L_{\varepsilon} \), given by

\[
[L_{\varepsilon}^*(u_\varepsilon)]^\beta = -\frac{\partial}{\partial x_j} \left\{ a_{ij}^{\varepsilon}(x/\varepsilon) \frac{\partial u_\varepsilon^\alpha}{\partial x_i} + B_j^{\alpha\beta}(x/\varepsilon) u_\varepsilon^\alpha \right\} + V_i^{\alpha\beta}(x/\varepsilon) \frac{\partial v_\varepsilon^\alpha}{\partial x_i} + c_{\varepsilon}^{\alpha\beta}(x/\varepsilon) v_\varepsilon^\alpha + \lambda v_\varepsilon^\beta.
\]

Then we define the bilinear form associated with \( L_{\varepsilon} \) as

\[
B_{\varepsilon}[u_\varepsilon, \phi] = \int_\Omega \left\{ a_{ij}^{\varepsilon}(x/\varepsilon) \frac{\partial u_\varepsilon^\alpha}{\partial x_i} + B_j^{\alpha\beta}(x/\varepsilon) u_\varepsilon^\alpha \right\} \frac{\partial \phi^\alpha}{\partial x_j} dx + \int_\Omega \left\{ B_i^{\varepsilon \alpha\beta} \frac{\partial u_\varepsilon^\alpha}{\partial x_i} + c_{\varepsilon}^{\alpha\beta}(x/\varepsilon) u_\varepsilon^\alpha + \lambda u_\varepsilon^\beta \right\} \phi^\alpha dx
\]

and the conjugate bilinear form with respect to \( L_{\varepsilon}^* \) as

\[
B_{\varepsilon}^*[v_\varepsilon, \phi] = \int_\Omega \left\{ a_{ij}^{\varepsilon}(x/\varepsilon) \frac{\partial v_\varepsilon^\alpha}{\partial x_i} + B_j^{\varepsilon \alpha\beta}(x/\varepsilon) v_\varepsilon^\alpha \right\} \frac{\partial \phi^\alpha}{\partial x_j} dx + \int_\Omega \left\{ B_i^{\varepsilon \alpha\beta} \frac{\partial v_\varepsilon^\alpha}{\partial x_i} + c_{\varepsilon}^{\alpha\beta}(x/\varepsilon) v_\varepsilon^\alpha + \lambda v_\varepsilon^\beta \right\} \phi^\alpha dx \tag{2.5}
\]

for any \( u_\varepsilon, v_\varepsilon, \phi \in H_0^1(\Omega; \mathbb{R}^m) \). It follows that \( B_{\varepsilon}[u_\varepsilon, v_\varepsilon] = B_{\varepsilon}^*[v_\varepsilon, u_\varepsilon] \) and

\[
< L_{\varepsilon}(u_\varepsilon), v_\varepsilon > = \int_\Omega (A_{\varepsilon} \nabla u_\varepsilon + V_{\varepsilon} u_\varepsilon) \nabla v_\varepsilon dx - \int_\Omega u_\varepsilon \text{div}(B_{\varepsilon} v_\varepsilon) dx + \int_\Omega (c_{\varepsilon} + \lambda) u_\varepsilon v_\varepsilon dx = < u_\varepsilon, L_{\varepsilon}^*(v_\varepsilon) > . \tag{2.6}
\]

Lemma 2.4. Let \( \Omega \) be a Lipschitz domain. Suppose that \( A \) satisfies the ellipticity condition \( [1.1] \), and other coefficients of \( L_{\varepsilon} \) satisfy \( [1.3] \). Then we have the following properties: for any \( u, v \in H_0^1(\Omega; \mathbb{R}^m) \),

\[
|B_{\varepsilon}[u, v]| \leq C \| u \|_{H_0^1(\Omega)} \| v \|_{H_0^1(\Omega)}, \quad \text{and} \quad c_0 \| u \|_{H_0^1(\Omega)}^2 \leq B_{\varepsilon}[u, u], \quad \text{whenever} \quad \lambda \geq \lambda_0, \tag{2.7}
\]

where \( \lambda_0 = \lambda_0(\mu, \kappa, m, d) \) is sufficiently large. Note that \( C \) depends on \( \mu, \kappa, m, d, \Omega \), while \( c_0 \) depends on \( \mu, \kappa, m, d, \Omega \).

Theorem 2.5. The coefficients of \( L_{\varepsilon} \) and \( \lambda_0 \) are as in Lemma 2.4. Suppose \( F \in H^{-1}(\Omega; \mathbb{R}^m) \) and \( g \in H^{\frac{1}{2}}(\partial \Omega; \mathbb{R}^m) \). Then the Dirichlet boundary value problem \( L_{\varepsilon}(u_\varepsilon) = F \) in \( \Omega \) and \( u_\varepsilon = g \) on \( \partial \Omega \) has a unique weak solution \( u_\varepsilon \in H^1(\Omega) \), whenever \( \lambda \geq \lambda_0 \), and the solution satisfies the uniform estimate

\[
\| u_\varepsilon \|_{H^1(\Omega)} \leq C \left\{ \| F \|_{H^{-1}(\Omega)} + \| g \|_{H^{\frac{1}{2}}(\partial \Omega)} \right\}, \tag{2.8}
\]

where \( C \) depends only on \( \mu, \kappa, m, d \) and \( \Omega \). Moreover, with one more the periodicity condition \( [1.2] \) on the coefficients of \( L_{\varepsilon} \), we then have \( u_\varepsilon \rightharpoonup u \) weakly in \( H^1(\Omega; \mathbb{R}^m) \) and strongly in \( L^2(\Omega; \mathbb{R}^m) \) as \( \varepsilon \to 0 \), where \( u \) is the weak solution to the homogenized problem \( L_0(u) = F \) in \( \Omega \) and \( u = g \) on \( \partial \Omega \).
Remark 2.6. The proof of Lemma 2.4 follows from the same argument in the scalar case (see [13] [22]). Theorem 2.5 involves the uniqueness and existence of the weak solution to (1.7), and the so-called homogenization theorem associated with $L_\varepsilon$. The proof of Theorem 2.5 follows from Lemma 2.4 Lax-Milgram theorem, Tartar’s method of oscillating test functions (see [7] pp.103 or [28] pp.31]). We refer the reader to [14] for more details on the Tartar’s method. We also mention that all the results in Lemma 2.4 and Theorem 2.5 still hold for $L_\varepsilon^*$ and $B_\varepsilon^*$

Lemma 2.7 (Caccioppoli’s inequality). Suppose that $A$ satisfies (1.1), and other coefficients satisfy (1.3). Assume that $u_\varepsilon \in H^1_0(\Omega; \mathbb{R}^m)$ is a weak solution to $L_\varepsilon(u_\varepsilon) = \text{div}(f) + F$ in $\Omega$, where $f = (f_\alpha^\varepsilon) \in L^2(\Omega; \mathbb{R}^{md})$ and $F \in L^3(\Omega; \mathbb{R}^m)$ with $q = \frac{2d}{d+2}$. Then for any $B \subset 2B \subset \Omega$, we have the uniform estimate

$$
\left( \int_B |\nabla u_\varepsilon|^2 dx \right)^{\frac{1}{2}} \leq \frac{C}{r} \left( \int_{2B} |u_\varepsilon|^2 dx \right)^{\frac{1}{2}} + C \left( \int_{2B} |f|^2 dx \right)^{\frac{1}{2}} + rC \left( \int_{2B} |F|^q dx \right)^{\frac{1}{q}},
$$

(2.9)

where $C$ depends only on $\mu, \kappa, \lambda, m, d$.

Proof. The proof is standard, and we provide a proof for the sake of completeness. Let $\phi \in C_0^1(\Omega)$ be a cut-off function satisfying $\phi = 1$ in $B$, $\phi = 0$ outside $2B$, and $|\nabla \phi| \leq 2/r$. Then let $\varphi = \phi^2 u_\varepsilon$ be a test function, it follows that

$$
\int_\Omega \left[ A^{\alpha\beta}_\varepsilon \nabla u_\varepsilon^{\beta} + V^{\alpha\beta}_\varepsilon u_\varepsilon^{\beta} \right] \nabla u_\varepsilon^{\alpha} \phi^2 dx + 2 \int_\Omega \left[ A^{\alpha\beta}_\varepsilon \nabla u_\varepsilon^{\beta} + V^{\alpha\beta}_\varepsilon u_\varepsilon^{\beta} \right] \nabla \phi u_\varepsilon^{\alpha} \phi dx + \int_\Omega B^{\alpha\beta}_\varepsilon \nabla u_\varepsilon^{\beta} u_\varepsilon^{\alpha} \phi^2 dx + \int_\Omega c^{\alpha\beta}_\varepsilon u_\varepsilon^{\beta} \phi^2 + \lambda |u_\varepsilon|^2 \phi^2 dx = \int_\Omega F^{\alpha} u_\varepsilon^{\alpha} \phi^2 dx - \int_\Omega f^{\alpha} \nabla u_\varepsilon^{\alpha} \phi^2 dx - 2 \int_\Omega f^{\alpha} \nabla \phi u_\varepsilon^{\alpha} \phi dx, \quad \text{in } \Omega.
$$

By using the ellipticity condition and Young’s inequality with $\delta$, we have

$$
\frac{\mu}{4} \int_\Omega \phi^2 |\nabla u_\varepsilon|^2 dx + (\lambda - C') \int_\Omega \phi^2 |u_\varepsilon|^2 dx \leq C \int_\Omega |\nabla \phi|^2 |u_\varepsilon|^2 dx + C \int_\Omega \phi^2 |f|^2 dx + \int_\Omega \phi^2 |F||u_\varepsilon| dx,
$$

(2.10)

where $C' = C'(\mu, \kappa, m, d)$. This together with

$$
\int_\Omega \phi^2 |F||u_\varepsilon| dx \leq \left( \int_\Omega (\phi|u_\varepsilon|)^2 dx \right)^{\frac{1}{2}} \left( \int_\Omega (\phi|F|)^q dx \right)^{\frac{1}{q}} \leq C \left( \int_\Omega |\nabla (\phi u_\varepsilon)|^2 dx \right)^{\frac{1}{2}} \left( \int_\Omega (\phi|F|)^q dx \right)^{\frac{1}{q}}.
$$

(2.11)

gives (2.9), where $2^* = \frac{2d}{(d-2)}$, and we use Hölder’s inequality, Sobolev’s inequality, and Young’s inequality in order. \hfill \Box

Remark 2.8. In fact, (2.9) is the interior $W^{1,2}$ estimate. By the same argument, we can also derive the near boundary Caccioppoli’s inequality for the weak solution to $L_\varepsilon(u_\varepsilon) = \text{div}(f) + F$ in $D(4r)$ and $u_\varepsilon = 0$ on $\Delta(4r)$,

$$
\left( \int_{D(r)} |\nabla u_\varepsilon|^2 dx \right)^{\frac{1}{2}} \leq \frac{C}{r} \left( \int_{D(2r)} |u_\varepsilon|^2 dx \right)^{\frac{1}{2}} + C \left( \int_{D(2r)} |f - \bar{f}|^2 dx \right)^{\frac{1}{2}} + rC \left( \int_{D(2r)} |F|^q dx \right)^{\frac{1}{q}}.
$$

(2.11)

We point out that the constant $C$ in (2.9) or (2.11) depends only on $\mu, \kappa, m, d$, whenever $\lambda \geq \lambda_0 \geq C'$.

Remark 2.9. Suppose that $A \in \Lambda(\mu, \tau, \kappa)$, and $V$ satisfies (1.2) and (1.4). In view of the interior Schauder estimate (see [27]), we obtain

$$
\max_{0 \leq k \leq d} \{ \|\chi_k\|_{L^\infty(\gamma_\varepsilon)}, \|\nabla \chi_k\|_{L^\infty(\gamma_\varepsilon)}, \|\nabla \chi_k||C^\alpha(\gamma_\varepsilon) \} \leq C(\mu, \tau, \kappa, m, d),
$$

(2.12)
We give a proof of (2.15) for the sake of completeness. In view of (2.1) and (2.3), we have
\[ p \geq d \]
Sobolev embedding theorem, provided \( U \) satisfies (2.13) gives
\[ \frac{\partial}{\partial y_i} \{ \chi_{ij} \} = 0. \]
It follows from (2.3) that
\[ \text{where } C.E. \text{ Kenig, F. Lin, and Z. Shen } [29, 30] \text{ showed that there exists } E_{1ij} \in H^1(Y), \text{ such that} \]
\[ \frac{\partial}{\partial y_i} \{ \chi_{ij} \} = \beta_i \frac{\partial}{\partial y_j} \{ \chi_{ij} \} \]
\[ \text{Note that } \frac{\partial}{\partial y_i} \{ \chi_{ij} \} = \beta_i \frac{\partial}{\partial y_j} \{ \chi_{ij} \} \]
\[ \text{there exists } F_{ki} \in H^1(Y) \text{ such that} \]
\[ U_i^{\alpha \gamma} = \frac{\partial}{\partial y_k} \{ F_{ki} \}, \quad F_{ki}^{\alpha \gamma} = -F_{ik}^{\alpha \gamma} \quad \text{and} \quad \| F_{ki}^{\alpha \gamma} \|_{L^\infty(Y)} \leq C(\mu, \omega, \kappa, m, d). \]
We give a proof of (2.15) for the sake of completeness. In view of (2.1) and (2.3), we have \( \int_Y U_i^{\alpha \gamma}(y) \, dy = 0 \) and \( \frac{\partial}{\partial y_i} \{ U_i^{\alpha \gamma} \} = 0 \). Then there exists a unique solution \( \theta_i^{\alpha \gamma} \in H^1(Y) \) satisfying
\[ \Delta \theta_i^{\alpha \gamma} = U_i^{\alpha \gamma} \quad \text{in } \mathbb{R}^d, \quad \int_Y \theta_i^{\alpha \gamma} = 0 \]
(see [11, 14]). Let \( F_{ki}^{\alpha \gamma} = \frac{\partial}{\partial y_k} \{ \theta_i^{\alpha \gamma} \} - \frac{\partial}{\partial y_i} \{ \theta_i^{\alpha \gamma} \} \), and obviously \( F_{ki}^{\alpha \gamma} = -F_{ik}^{\alpha \gamma} \). We mention that \( \theta_i^{\alpha \gamma} \in H^2(Y) \), which implies \( F_{ki}^{\alpha \gamma} \in H^1(Y) \). Next, we verify \( F_{ki}^{\alpha \gamma} \) is bounded, which is equivalent to
\[ \max_{1 \leq i \leq d} \left\{ \| \nabla \theta_i^{\alpha \gamma} \|_{L^\infty(Y)} \right\} \leq C(\mu, \omega, \kappa, m, d). \]
Observe that (2.13) gives \( U_i^{\alpha \gamma} \in H^1(Y) \), then the estimate (2.16) follows from the \( L^p \) estimates and the Sobolev embedding theorem, provided \( p > d \). Moreover, we have
\[ \frac{\partial}{\partial y_k} \{ F_{ki}^{\alpha \gamma} \} = \Delta \{ \theta_i^{\alpha \gamma} \} - \frac{\partial}{\partial y_k} \frac{\partial}{\partial y_i} \{ \theta_i^{\alpha \gamma} \} = U_i^{\alpha \gamma}. \]
Note that \( \frac{\partial}{\partial y_k} \{ U_i^{\alpha \gamma} \} = 0 \) in \( \mathbb{R}^d \), which implies \( \Delta \theta_i^{\alpha \gamma} = 0 \). In view of Liouville’s theorem (see [15]), we have \( \frac{\partial}{\partial y_k} \{ \theta_i^{\alpha \gamma} \} = C \), therefore \( \frac{\partial}{\partial y_k} \frac{\partial}{\partial y_i} \{ \theta_i^{\alpha \gamma} \} = 0 \), and we complete the proof of (2.15).
In addition, we define the auxiliary functions \( \theta_i^{\alpha \gamma} \) and \( \zeta^{\alpha \gamma} \) as follows:
\[ \Delta \theta_i^{\alpha \gamma} = W_i^{\alpha \gamma} := \tilde{W}_i^{\alpha \gamma} - B_i^{\alpha \gamma}(y) \frac{\partial}{\partial y_j} \{ \chi_{ij} \} \quad \text{in } \mathbb{R}^d, \quad \int_Y \theta_i^{\alpha \gamma} \, dy = 0, \]
\[ \Delta \zeta^{\alpha \gamma} = Z^{\alpha \gamma} := \tilde{Z}^{\alpha \gamma} - \sigma^{\alpha \gamma}(y) - B_i^{\alpha \gamma}(y) \frac{\partial}{\partial y_i} \{ \chi_0 \} \quad \text{in } \mathbb{R}^d, \quad \int_Y \zeta^{\alpha \gamma} \, dy = 0. \]
It follows from (2.13) that \( \int_Y W_i^{\alpha \gamma}(y) \, dy = 0 \) and \( \int_Y Z^{\alpha \gamma}(y) \, dy = 0 \), which implies the existence of \( \theta_i^{\alpha \gamma} \) and \( \zeta^{\alpha \gamma} \). By the same argument, it follows from (2.13) that \( \max \{ \| \nabla \theta_i^{\alpha \gamma} \|_{L^\infty(Y)}, \| \nabla \zeta^{\alpha \gamma} \|_{L^\infty(Y)} \} \leq C(\mu, \omega, \kappa, d, m) \).
We end this remark by a summary. Suppose that \(A \in \text{VMO}(\mathbb{R}^d)\), and the coefficients of \(L\) satisfy (1.1) \(- (1.3)\), then we have

\[
\max_{1 \leq i,j,k \leq d} \left\{ \|E^{\alpha \gamma}_{iij}\|_{L^\infty(Y)}, \|E^{\alpha \gamma}_{kij}\|_{L^\infty(Y)}, \|\nabla \partial^\alpha\gamma_i\|_{L^\infty(Y)}, \|\nabla \lambda_i\|_{L^\infty(Y)} \right\} \leq C(\mu, \omega, \kappa, m, d). \tag{2.18}
\]

In the special case of \(m = 1\) or \(d = 2\), the estimate (2.18) still holds without any regularity assumption on \(A\).

We now introduce the Lipschitz estimate and Schauder estimate that will be frequently employed later. Let \(L(u) = -\text{div}(A \nabla u) + L(u) = -\text{div}(A \nabla u + Vu) + B \nabla u + (c + \lambda I)u\). Then we have the following results:

**Lemma 2.10.** Let \(\Omega\) be a bounded \(C^{1,\tau}\) domain. Suppose \(A\) satisfies (1.1) and (1.4). Let \(u\) be the weak solution to \(L(u) = \text{div}(f) + F\) in \(\Omega\) and \(u = 0\) on \(\partial\Omega\), where \(f \in C^{0,\sigma}(\Omega; \mathbb{R}^d)\) with \(\sigma \in (0, \tau]\), and \(F \in L^p(\Omega; \mathbb{R}^m)\) with \(p > d\). Then we have:

(i) the Schauder estimate

\[
[\nabla u]_{C^{0,\sigma}(\Omega)} \leq C \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \right\}; \tag{2.19}
\]

(ii) the Lipschitz estimate

\[
\|\nabla u\|_{L^\infty(\Omega)} \leq C \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \right\}, \tag{2.20}
\]

where \(C\) depends on \(\mu, \tau, \kappa, \sigma, m, d, p\) and \(\Omega\). Moreover, if \(u = g\) on \(\partial\Omega\) with \(g \in C^{1,\sigma}(\partial\Omega; \mathbb{R}^m)\), then we have

\[
[\nabla u]_{C^{0,\sigma}(\Omega)} \leq C \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} + \|g\|_{C^{1,\sigma}(\partial\Omega)} \right\}. \tag{2.21}
\]

**Proof.** The results are standard, and we provide a proof for the sake of completeness. For (i), we refer the reader to [21] pp.75-95 for the details. For (ii), due to the properties of Green function (denoted by \(G(x,y)\)) associated with \(L\) and \(|G(x,y)| \leq C|x - y|^{2d - 2}\), \(|\nabla_x G(x,y)\| \leq C|x - y|^{-1-d}\) and \(|\nabla_x \nabla_y G(x,y)\| \leq C|x - y|^{-d}\) (the existence of \(G(x,y)\) is included in [26] Theorem 4.1), we have

\[
|\nabla u(x) - \nabla u(y)| = \left| \int_\Omega \nabla_x \nabla_y G(x,y) \left[ f(y) - f(x) \right] dy - f(x) \right| \int_\Omega \nabla_x \nabla_y G(x,y) dy + \int_\Omega \nabla_x G(x,y) F(y) dy \right|\]

\[
= \left| - \int_\Omega \nabla_x \nabla_y G(x,y) \left[ f(y) - f(x) \right] dy - f(x) \right| \int_\Omega \nabla_x \nabla_y G(x,y) dy + \int_\Omega \nabla_x G(x,y) F(y) dy \right|\]

\[
= \left| - \int_\Omega \nabla_x \nabla_y G(x,y) \left[ f(y) - f(x) \right] dy + \int_\Omega \nabla_x G(x,y) F(y) dy \right|
\]

for any \(x \in \Omega\). Note that we use the integration by parts in the second equality, and the fact of \(|\nabla_x G(x, \cdot) = 0\) on \(\partial\Omega\) in the last equality. This implies (2.20). Finally we can use the extension technique to obtain the estimate (2.21). (Its proof is similar to that in the proof of Theorem 1.1 and we refer the reader to [22] pp.136-138 for the extension lemmas.) \(\square\)

**Remark 2.11.** Set \(U(x, r) = \Omega \cap B(x, r)\) for any \(x \in \Omega\). Let \(\varphi \in C_0^\infty(2B)\) be a cut-off function satisfying \(\varphi = 1\) in \(B\), \(\varphi = 0\) outside \(3/2B\), and \(|\nabla \varphi| \leq C/r\). Let \(w = u \varphi\), where \(u\) is given in Lemma 2.10. Then we have

\[
L(w) = \text{div}(f \varphi) - f \cdot \nabla \varphi + F \varphi - \text{div}(A \nabla \varphi u) - A \nabla u \nabla \varphi \quad \text{in} \quad \Omega, \quad w = 0 \quad \text{on} \quad \partial\Omega.
\]

We apply the estimate (2.20) to the above equation with \(r = 1\), and obtain

\[
\|\nabla u\|_{L^\infty(U(x, 1))} \leq C \left\{ \|u\|_{C^{0,\sigma}(U(x, 2))} + \|u\|_{L^p(U(x, 2))} + \|f\|_{L^\infty(U(x, 2))} + \|F\|_{L^p(U(x, 2))} \right\} \leq C \left\{ \|u\|_{W^{1,\sigma}(U(x, 2))} + \|f\|_{L^\infty(U(x, 2))} + \|F\|_{L^p(U(x, 2))} \right\} \tag{2.22}
\]
where \( s = \max\{p, \lfloor \frac{d}{1-\sigma} \rfloor + 1 \} \) (\( \lfloor \cdot \rfloor \) is the integer part of \( \frac{d}{1-\sigma} \)), and we employ the fact of \( \|u\|_{C^0,\sigma(U(x,2))} \leq C\|u\|_{W^{1,s}(U(x,2))} \). On account of

\[
\|\nabla u\|_{L^q(U(x,2))} \leq C\{\|\nabla u\|_{L^2(U(x,4))} + \|f\|_{L^p(U(x,4))} + \|F\|_{L^q(U(x,4))}\},
\]

where \( q = \frac{pd}{d+p} \) (see [21, Theorem 7.2]), we have

\[
\|u\|_{W^{1,q}(U(x,2))} \leq C\{\|u\|_{W^{1,2}(U(x,4))} + \|f\|_{L^p(U(x,4))} + \|F\|_{L^q(U(x,4))}\}
\] (2.23)

where we use \( \|u\|_{L^q(U(x,2))} \leq C\{\|\nabla u\|_{L^2(U(x,2))} + \|u\|_{L^2(U(x,2))}\} \) in the above inequality (see (3.5)). Combining (2.22) and (2.23), we have

\[
\|\nabla u\|_{L^\infty(U(x,1))} \leq C\{\|\nabla u\|_{L^2(U(x,4))} + \|u - \bar{u}\|_{L^2(U(x,4))} + \|f\|_{L^\infty(U(x,4))} + \|\tilde{f}\|_{C^0,\sigma(U(x,4))} + \|\tilde{F}\|_{L^p(U(x,4))}\}
\]

Note that if \( U(x,4) \subset \Omega \), then \( v = u - \bar{u} \) is still a solution to \( L(u) = \text{div}(f) + F \) in \( \Omega \), where \( \bar{u} = \int_{U(x,4)} u \, dy \). In this case, the above estimate becomes

\[
\|\nabla u\|_{L^\infty(U(x,1))} \leq C\{\|\nabla u\|_{L^2(U(x,4))} + \|u - \bar{u}\|_{L^2(U(x,4))} + \|f\|_{L^\infty(U(x,4))} + \|\tilde{f}\|_{C^0,\sigma(U(x,4))} + \|\tilde{F}\|_{L^p(U(x,4))}\}.
\] (2.24)

Next, we let \( v(y) = u(ry) \), where \( y \in U(x,4) \). Then we have \( \tilde{L}(v) = \text{div}(\tilde{f}) + \tilde{F} \) in \( U(x,4) \) and \( v = 0 \) on \( \partial \Omega \cap U(x,4) \), where \( \tilde{L} = \frac{\partial}{\partial r} - \alpha^2 \frac{\partial}{\partial y} \), \( \tilde{f}(y) = r f(ry) \) and \( \tilde{F}(y) = r^2 F(ry) \). It follows from (2.24) that

\[
\|\nabla v\|_{L^\infty(U(x,1))} \leq C\{\|\nabla v\|_{L^2(U(x,4))} + \|\tilde{f}\|_{L^\infty(U(x,4))} + \|\tilde{F}\|_{L^p(U(x,4))}\},
\]

and by change of variable, we have

\[
\|\nabla u\|_{L^\infty(U(x,r))} \leq C\left\{\left(\int_{U(x,4r)} |\nabla u|^2 dy\right)^{1/2} + \|f\|_{L^\infty(U(x,4r))} + r^\sigma [f]_{C^0,\sigma(U(x,4r))} + r^\sigma \int_{U(x,4r)} |F|^p dy\right\}^{1/p}.
\]

By a covering technique (shown in the proof of Theorem 4.3), we have

\[
\|\nabla u\|_{L^\infty(U)} \leq C\left\{\left(\int_{2U} |\nabla u|^2 dy\right)^{1/2} + \|f\|_{L^\infty(2U)} + r^\sigma [f]_{C^0,\sigma(2U)} + r^\sigma \int_{2U} |F|^p dy\right\}^{1/p}.
\] (2.25)

Here \( U \) is the abbreviation of \( U(x, r) \) and \( 2U = U(x, 2r) \). We mention that all the above proof is so-called localization argument, which gives a way to obtain “local estimates” (such as interior estimates and boundary estimates) from corresponding “global estimates”. The main point is based on cut-off function coupled with rescaling technique. So, on account of the estimate (2.19), following the same procedure as before, it is not hard to derive

\[
[\nabla u]_{C^0,\sigma(U)} \leq C r^{-\sigma} \left\{\left(\int_{2U} |\nabla u|^2 dy\right)^{1/2} + \|f\|_{L^\infty(2U)} + r^\sigma [f]_{C^0,\sigma(2U)} + r^\sigma \int_{2U} |F|^p dy\right\}^{1/p},
\]

and by (2.21),

\[
[\nabla u]_{C^0,\sigma(D(r))} \leq C r^{-\sigma} \left\{\left(\int_{D(2r)} |\nabla u|^2 dy\right)^{1/2} + r^{-1}\|g\|_{L^\infty(\Delta(2r))} + \|\nabla g\|_{L^\infty(\Delta(2r))} + r^\sigma [\nabla g]_{C^0,\sigma(\Delta(2r))}\right\}
\] (2.27)

holds for \( u \) satisfying \( L(u) = 0 \) in \( D(2r) \) and \( u = g \) on \( \Delta(2r) \). We note that the estimate (2.25) is of help to arrive at (2.29), and the extension technique (see [22] pp.136) is used in (2.27). The details of the proof are omitted. Finally we remark that (2.27) is exactly the Schauder estimate at boundary, which can be directly proved (see [21, Theorem 5.21]).
Lemma 2.12. Let $\Omega$ be a bounded $C^{1,\tau}$ domain, and $\sigma \in (0, \tau]$. Suppose that $A,V$ satisfy (1.1) and (1.3), and $B,c$ satisfy (1.3). Let $u$ be the weak solution to $\mathcal{L}(u) = \text{div}(f) + F$ in $\Omega$ and $u = 0$ on $\partial \Omega$, where $f,F$ satisfy the same conditions as in Lemma 2.10. Then we have

(i) the Lipschitz estimate

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \right\};$$

(ii) the Schauder estimate

$$[\nabla u]_{C^{0,\sigma}(\Omega)} \leq C \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \right\},$$

where $C$ depends on $\mu, \tau, \kappa, m, d, \sigma, p$ and $\Omega$.

Proof. The results are classical, and we offer a sketch of the proof. First we rewrite $\mathcal{L}(u) = \text{div}(f) + F$ as $L(u) = \text{div}(f + V u) - B \nabla u - (c + \lambda I) u + F$ in $\Omega$. It follows from (2.20) that

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C \left\{ [f]_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} + \|u\|_{C^{0,\sigma}(\Omega)} + \|u\|_{W^{1,p}(\Omega)} \right\}$$

$$\leq C \left\{ [f]_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} + 2\|\nabla u\|_{L^\infty(\Omega)} \right\}\|u\|_{L^\infty(\Omega)}^{1-\sigma} + \|u\|_{W^{1,p}(\Omega)}$$

$$\leq C \left\{ [f]_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} + \|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{L^\infty(\Omega)} + \|u\|_{W^{1,p}(\Omega)} \right\} + \frac{1}{2}\|\nabla u\|_{L^\infty(\Omega)},$$

where we use Young’s inequality in the last inequality. By the Sobolev embedding theorem we have $\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)}$ for $p > d$. This together with (2.20) leads to

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C \left\{ [f]_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} + \|u\|_{W^{1,p}(\Omega)} \right\}$$

$$\leq C \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \right\},$$

where we use the $W^{1,p}$ estimate with $1 < p < \infty$ in the last inequality, which can be derived by a similar argument as in the proof of Theorem 1.1 or see [11][21][22].

It remains to show (ii). In view of (2.19) and (2.28), we obtain

$$[\nabla u]_{C^{0,\sigma}(\Omega)} \leq C \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} + \|u\|_{L^\infty(\Omega)} + \|\nabla u\|_{L^\infty(\Omega)} + \|u\|_{W^{1,p}(\Omega)} \right\}$$

$$\leq C \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \right\},$$

where we also use $W^{1,p}$ estimate in the last inequality. The proof is completed. \(\square\)

Remark 2.13. Let $u$ be given in Lemma 2.12. Applying the localization argument (see Remark 2.11) to the estimates (2.28) and (2.29), we can similarly give the corresponding local estimates:

$$\|\nabla u\|_{L^\infty(U)} \leq C \left\{ r^{-1} \left( \int_{2U} |u|^2\,dy \right)^{1/2} + \|f\|_{L^\infty(2U)} + r^\sigma [f]_{C^{0,\sigma}(2U)} + r \left( \int_{2U} |F|^p\,dy \right)^{1/p} \right\},$$

and

$$[\nabla u]_{C^{0,\sigma}(U)} \leq C r^{-\sigma} \left\{ r^{-1} \left( \int_{2U} |u|^2\,dy \right)^{1/2} + \|f\|_{L^\infty(2U)} + r^\sigma [f]_{C^{0,\sigma}(2U)} + r \left( \int_{2U} |F|^p\,dy \right)^{1/p} \right\},$$

where $C$ depends on $\mu, \tau, \kappa, m, d, p, \sigma$ and $M_0$. We mention that in the proof of (2.31), we also need $W^{1,p}$ estimate like [21] Theorem 7.2 for $\mathcal{L}$. It can be established by using the bootstrap method which is exactly shown in the proof of Theorem 3.3 so we do not repeat them. The remainder of the argument is analogous to that in Remark 2.11.
3 \(W^{1,p}\) estimates & Hölder estimates

**Lemma 3.1.** Let \(2 \leq p < \infty\). Suppose that \(A \in \text{VMO}(\mathbb{R}^d)\) satisfies (1.1) and (1.2). Assume \(f = (f^\alpha) \in L^p(\Omega; \mathbb{R}^d)\), and \(F \in L^q(\Omega; \mathbb{R}^m)\) with \(q = \frac{pd}{p+d}\). Then the weak solution to \(L\varepsilon(u\varepsilon) = \text{div}(f) + F\) in \(\Omega\) and \(u\varepsilon = 0\) on \(\partial\Omega\) satisfies the uniform estimate

\[
\|\nabla u\varepsilon\|_{L^p(\Omega)} \leq C\{\|f\|_{L^p(\Omega)} + \|F\|_{L^q(\Omega)}\},
\]

where \(C\) depends only on \(\mu, \omega, \kappa, p, q, d, m\) and \(\Omega\).

**Remark 3.2.** The estimate (3.1) actually holds for \(1 < p < \infty\), where \(q = \frac{pd}{p+d}\) if \(p > \frac{d}{d-1}\), and \(q > 1\) if \(1 < p \leq \frac{d}{d-1}\). In the case of \(F = 0\), (3.1) is shown in [6]. If \(F \neq 0\), we can derive the above result by the duality argument applied in Lemma 3.7. The same method may be found in [16][31]. Besides, we refer the reader to [39] for the sharp range of \(p\)'s on Lipschitz domains.

**Theorem 3.3.** (Interior \(W^{1,p}\) estimates). Let \(2 \leq p < \infty\). Suppose that \(A \in \text{VMO}(\mathbb{R}^d)\) satisfies (1.1), (1.2), and other coefficients satisfy (1.3). Assume that \(u\varepsilon \in H^1_{\text{loc}}(\Omega; \mathbb{R}^m)\) is a weak solution to \(L\varepsilon(u\varepsilon) = \text{div}(f) + F\) in \(\Omega\), where \(f \in L^p(\Omega; \mathbb{R}^d)\) and \(F \in L^q(\Omega; \mathbb{R}^m)\) with \(q = \frac{pd}{p+d}\). Then, we have \(\nabla u\varepsilon \in L^p(\Omega)\) and the uniform estimate

\[
\left(\int_B \left|\nabla u\varepsilon\right|^p dx\right)^{\frac{1}{p}} \leq \frac{C}{r}\left(\int_{2B} \left|u\varepsilon\right|^2 dx\right)^{\frac{1}{2}} + C\left\{\left(\int_{2B} |f|^p dx\right)^{\frac{1}{p}} + r\left(\int_{2B} |F|^q dx\right)^{\frac{1}{q}}\right\}
\]

for any \(B \subset 2B \subset \Omega\) with \(0 < r \leq 1\), where \(C\) depends only on \(\mu, \omega, \kappa, \lambda, p, m, d\).

**Proof.** By rescaling we may assume \(r = 1\). In the case of \(p = 2\), (3.2) follows from Lemma 2.7. Next, we will prove (3.2) for \(p \in [2, p_\star]\), where \(p_\star = \frac{2d}{d-2k_0}\) and \(0 < d/2 \leq k_0 + 1\). To do so, let \(u\varepsilon = \varphi\varepsilon\), where \(\varphi \in C_0^\infty(2B)\) is a cut-off function satisfying \(\varphi = 1\) in \(B\), \(\varphi = 0\) outside \(3/2B\), and \(\|\nabla \varphi\| \leq C\). We rewrite the original systems as

\[
-\text{div}(A\varepsilon \nabla u\varepsilon) = \text{div}(f \varphi) - f \cdot \nabla \varphi + F\varphi + \tilde{F} \quad \text{in } \Omega,
\]

where

\[
\tilde{F}^\alpha = \text{div}(V^\varepsilon \alpha \beta \varepsilon u^\beta \varepsilon - A^\alpha \beta \varepsilon \nabla \varphi u^\beta \varepsilon - V^\varepsilon \alpha \beta \varepsilon \nabla \varphi \varepsilon u^\beta \varepsilon - B^\varepsilon \alpha \beta \varepsilon \nabla \varphi \varepsilon u^\beta \varepsilon - c^\varepsilon \alpha \beta \varepsilon \nabla \varphi \varepsilon u^\beta \varepsilon - \lambda w^\varepsilon \alpha \varepsilon.
\]

Hence it follows from (2.9), (3.1), (3.6), and Hölder’s inequality that

\[
\|\nabla u\varepsilon\|_{L^p(B)} \leq C\{\|u\varepsilon\|_{L^p(2B)} + \|\nabla u\varepsilon\|_{L^q(2B)} + \|u\varepsilon\|_{L^q(2B)} + \|f\|_{L^p(2B)} + \|F\|_{L^q(2B)}\}
\]

We first check the special case of \(p = p_\star\) to obtain the final step \(k_0\) of iteration, and then verify the above inequality for any \(p \in [2, p_\star]\). Second, it is not hard to extend the range of \(p\)'s to \([2, p_\star]\) by at most \(k_0\) times of iteration. The rest of the proof is to extend the \(p\)'s range to \(p < \infty\). Indeed it is true, since

\[
\|\nabla u\varepsilon\|_{L^p(B)} \leq C\{\|u\varepsilon\|_{L^q(2B)} + \|u\varepsilon\|_{L^q(2B)} + \|f\|_{L^p(2B)} + \|F\|_{L^q(2B)}\}
\]

and \(q < d \in [2, p_\star]\), which is exactly the start point for iterations due to the previous case.

Hence, let \(N = 4^{k_0}\), we have proved

\[
\|\nabla u\varepsilon\|_{L^p(B)} \leq C\{\|u\varepsilon\|_{L^q(NB)} + \|f\|_{L^p(NB)} + \|F\|_{L^q(NB)}\}
\]

for any \(2 \leq p < \infty\) in the case of \(r = 1\). We remark that (i) the estimate (3.3) uniformly holds for \(\varepsilon > 0\); (ii) the constant in (3.3) can be given by \(C \leq C(\mu, \omega, m, d, p)\{\|A\|_{L^\infty(\mathbb{R}^d)} + \|V\|_{L^\infty(\mathbb{R}^d)} + \|B\|_{L^\infty(\mathbb{R}^d)} + \|c\|_{L^\infty(\mathbb{R}^d)} + \lambda\}^{k_0+2}\). The two points make the rescaling argument valid when we study the estimate (3.2) for \(0 < r < 1\).
We now let \(v_\varepsilon(x) = u_\varepsilon(rx),\) where \(x \in B_1\) and \(r \in (0, 1)\). Hence we have
\[
\tilde{\mathcal{L}}(v_\varepsilon) = -\text{div}[A(rx/\varepsilon)\nabla v_\varepsilon + \tilde{V}(rx/\varepsilon)v_\varepsilon] + \tilde{B}(rx/\varepsilon)\nabla v_\varepsilon + \tilde{c}(rx/\varepsilon)v_\varepsilon + \tilde{\lambda}v_\varepsilon = \text{div}(\tilde{f}) + \tilde{F} \quad \text{in NB}_1, \tag{3.4}
\]
where
\[
\tilde{V}(x) = rV(x), \quad \tilde{B} = rB(x), \quad \tilde{c} = r^2c(x), \quad \tilde{\lambda} = r^2\lambda, \quad \tilde{f} = rf(rx), \quad \tilde{F} = r^2F(rx).
\]
It is clear to see that the coefficients of \(\tilde{\mathcal{L}}\) satisfy the same assumptions as \(\mathcal{L}\) in this theorem. Set \(\varepsilon' = \varepsilon/r,\) and applying (3.3) directly, we obtain
\[
\|\nabla v_{\varepsilon'}\|_{L^p(B_1)} \leq C\left\{\|v_{\varepsilon'}\|_{L^2(\text{NB}_1)} + \|\tilde{f}\|_{L^p(\text{NB}_1)} + \|	ilde{F}\|_{L^q(\text{NB}_1)}\right\},
\]
where \(C\) is the same constant as in (3.3). This implies
\[
\|\nabla u_\varepsilon\|_{L^p(B,r)} \leq C\left\{r^{-1+\frac{d}{p}-\frac{d}{q}}\|u_\varepsilon\|_{L^2(\text{NB},r)} + \|f\|_{L^p(\text{NB},r)} + r^{1+\frac{d}{p}-\frac{d}{q}}\|F\|_{L^q(\text{NB},r)}\right\}.
\]
Finally, for any \(B\) with \(0 < r \leq 1\), we choose the small ball with \(r/N\) radius to cover \(B_r\). Hence we have
\[
\|\nabla u_\varepsilon\|_{L^p(B,r)} \leq C\left\{r^{-1+\frac{d}{p}-\frac{d}{q}}\|u_\varepsilon\|_{L^2(B,2r)} + \|f\|_{L^p(B,2r)} + r^{1+\frac{d}{p}-\frac{d}{q}}\|F\|_{L^q(B,2r)}\right\},
\]
and this gives (3.2). We complete the proof. \(\square\)

**Remark 3.4.** Here we introduce two elementary interpolation inequalities used in the above proof. Let \(u \in W^{1,p}(\Omega; \mathbb{R}^m)\) with \(2 \leq p < \infty\), then for any \(\delta > 0\), there exists a constant \(C_\delta\) depending on \(\delta, p, d, m\) and \(\Omega\), such that
\[
\|u\|_{L^p(\Omega)} \leq \delta\|\nabla u\|_{L^p(\Omega)} + C_\delta\|u\|_{L^2(\Omega)}. \tag{3.5}
\]
The estimate (3.5) can be easily derived by contradiction argument (or see [1]). As a result, we have
\[
\|u\|_{L^p(\Omega)} \leq C\|u\|_{W^{1,p}(\Omega)} \leq C\|\nabla u\|_{L^q(\Omega)} + C\|u\|_{L^2(\Omega)} \tag{3.6}
\]
for \(1 \leq p < \infty\) and \(q = \frac{pd}{p+d}\), where \(C\) depends on \(p, d, m\) and \(\Omega\).

**Corollary 3.5.** Suppose that the coefficients of \(\mathcal{L}_\varepsilon\) satisfy the same conditions as in Theorem 3.3. Let \(p > d\) and \(\sigma = 1 - d/p\). Assume that \(u_\varepsilon \in H^1_0(\Omega; \mathbb{R}^m)\) is a weak solution of \(\mathcal{L}_\varepsilon(u_\varepsilon) = \text{div}(f) + F\) in \(\Omega\), where \(f \in L^p(\Omega; \mathbb{R}^{md})\) and \(F \in L^q(\Omega; \mathbb{R}^m)\) with \(q = \frac{pd}{p+d} > \frac{d}{2}\). Then we have
\[
\|u_\varepsilon\|_{C^{0,\sigma}(B)} \leq C r^{-\sigma}\left\{\left(\int_{2B} |u_\varepsilon|^2 dx\right)^{\frac{1}{2}} + r\left(\int_{2B} |f|^p dx\right)^{\frac{1}{p}} + r^2\left(\int_{2B} |F|^q dx\right)^{\frac{1}{q}}\right\}, \tag{3.7}
\]
for any \(B \subset 2B \subset \Omega\) with \(0 < r \leq 1\). In particular, for any \(s > 0\),
\[
\|u_\varepsilon\|_{L^\infty(B)} \leq C\left\{\left(\int_{2B} |u_\varepsilon|^s dx\right)^{\frac{1}{s}} + r\left(\int_{2B} |f|^p dx\right)^{\frac{1}{p}} + r^2\left(\int_{2B} |F|^q dx\right)^{\frac{1}{q}}\right\}, \tag{3.8}
\]
where \(C\) depends only on \(\mu, \omega, \kappa, \lambda, p, m, d\).

**Proof.** Assume \(r = 1\). It follows from the Sobolev embedding theorem and Remark 3.4 that
\[
\|u_\varepsilon\|_{C^{0,\sigma}(B)} \leq C\|u_\varepsilon\|_{W^{1,p}(B)} \leq C\|\nabla u_\varepsilon\|_{L^p(B)} + C\|u_\varepsilon\|_{L^2(B)}.
\]
Then it follows from (3.2) and rescaling arguments that
\[
\|u_\varepsilon\|_{C^{0,\sigma}(B)} \leq C r^{-\sigma}\left(\int_{2B} |u_\varepsilon|^2 dx\right)^{\frac{1}{2}} + C r^{1-\sigma}\left\{\left(\int_{2B} |f|^p dx\right)^{\frac{1}{p}} + r\left(\int_{2B} |F|^q dx\right)^{\frac{1}{q}}\right\}.
\]
where \( \sigma = 1 - d/p \). Moreover, for any \( x \in B \) we have
\[
|u_\varepsilon(x)| \leq |u_\varepsilon(x) - \bar{u}_\varepsilon| + |\bar{u}_\varepsilon| \leq \|u_\varepsilon\|_{C^0,\sigma(B)} + \left( \int_B |u_\varepsilon(y)|^2 dy \right)^{1/2}
\]
from Hölder’s inequality. This gives
\[
\|u_\varepsilon\|_{L^\infty(B)} \leq C \left( \int_{2B} |u_\varepsilon|^2 dx \right)^{1/2} + Cr \left\{ \left( \int_{2B} |f|^p dx \right)^{1/p} + r \left( \int_{2B} |F|^q dx \right)^{1/q} \right\}.
\]
Moreover, by the iteration method (see [21] pp.184), we have (3.8).

**Lemma 3.6.** Suppose that the coefficients of \( \mathcal{L}_\varepsilon \) satisfy the same assumptions as in Theorem 3.3. Let \( 1 < p < \infty \), \( f = (f^\alpha) \in L^p(\Omega; \mathbb{R}^m) \). Then there exists a unique \( u_\varepsilon \in W^{1,p}_0(\Omega; \mathbb{R}^m) \) such that \( \mathcal{L}_\varepsilon(u_\varepsilon) = \text{div}(f) \) in \( \Omega \) and \( u_\varepsilon = 0 \) on \( \partial\Omega \), whenever \( \lambda \geq \lambda_0 \). Moreover, the solution satisfies the uniform estimate
\[
\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)},
\]
where \( \lambda_0 \) is given in Lemma 2.4 and \( C \) depends on \( \mu, \omega, \kappa, \lambda, p, d, m \) and \( \Omega \).

**Proof.** In the case of \( p = 2 \), it follows from Theorem 2.5 that there exists a unique solution \( u_\varepsilon \in H^1_0(\Omega; \mathbb{R}^m) \) satisfying the uniform estimate \( \|\nabla u_\varepsilon\|_{L^2(\Omega)} \leq C\|f\|_{L^2(\Omega)} \). For \( p > 2 \), the uniqueness and existence of the weak solution is reduced to the case of \( p = 2 \). We rewrite the original systems as
\[
\mathcal{L}_\varepsilon(u_\varepsilon) = \text{div}(f + V_\varepsilon u_\varepsilon) - B_\varepsilon \nabla u_\varepsilon - (c_\varepsilon + \lambda)u_\varepsilon.
\]
Applying (2.5), (3.1) and Sobolev’s inequality, we obtain
\[
\|\nabla u_\varepsilon\|_{L^{p_k_0}(\Omega)} \leq C\{\|f\|_{L^{p_k_0}(\Omega)} + \|u_\varepsilon\|_{L^{p_k_0}(\Omega)} + \|\nabla u_\varepsilon\|_{L^{p_k_0-1}(\Omega)}\} \leq C\{\|f\|_{L^{p_k_0}(\Omega)} + \|\nabla u_\varepsilon\|_{L^{p_k_0-1}(\Omega)}\}
\]
where \( p_k_0 = \frac{2d}{2d-k_0} \), and \( k_0 \) is a positive integer such that \( k_0 < d/2 \leq k_0 + 1 \). We claim that (3.10) holds for any \( p \in [2, p_{k_0}] \). Indeed, let \( T(f) = \nabla u_\varepsilon \), then together with \( \|T\|_{L^2 \to L^2} \leq C \) and \( \|T\|_{L^{p_{k_0}} \to L^{p_{k_0}}} \leq C \), the Marcinkiewicz interpolation theorem gives \( \|T\|_{L^q \to L^p} \leq C \) (see [11]). Moreover, for any \( p > p_{k_0} \), we still have \( \|\nabla u_\varepsilon\|_{L^q(\Omega)} \leq C\{\|f\|_{L^p(\Omega)} + \|\nabla u_\varepsilon\|_{L^q(\Omega)}\} \leq C\|f\|_{L^p(\Omega)} \), since \( q < d \).

By the duality argument, we can derive (3.9) for \( p \in (1, 2) \). Let \( h = (h^\beta_\varepsilon) \in C^1(\Omega; \mathbb{R}^{md}) \), and \( v_\varepsilon \) be the weak solution to \( \mathcal{L}_\varepsilon^*(v_\varepsilon) = \text{div}(h) \) in \( \Omega \) and \( v_\varepsilon = 0 \) on \( \partial\Omega \). Hence, in view of the previous result, we have \( \|\nabla v_\varepsilon\|_{L^{p'}(\Omega)} \leq C\|h\|_{L^{p'}(\Omega)} \) for any \( p' > 2 \). Moreover, if \( f \in C^0(\Omega; \mathbb{R}^{md}) \), there exists the weak solution \( u_\varepsilon \in H^1_0(\Omega; \mathbb{R}^m) \) to the original systems. Then it follows from Remark 2.3 that
\[
\int_{\Omega} \nabla u_\varepsilon h dx = -\int_{\Omega} u_\varepsilon \mathcal{L}_\varepsilon^*(v_\varepsilon) dx = -\int_{\Omega} \mathcal{L}_\varepsilon(u_\varepsilon)v_\varepsilon dx = \int_{\Omega} f \nabla v_\varepsilon dx.
\]
This gives \( \|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C\|f\|_{L^p(\Omega)} \), where \( p'/(p' - 1) \). By the density argument, we can verify the existence of solutions in \( W^{1,p}_0(\Omega) \) for general \( f \in L^p(\Omega; \mathbb{R}^{md}) \), as well as the uniqueness for \( 1 < p < 2 \). The proof is complete.

**Lemma 3.7.** Suppose that the coefficients of \( \mathcal{L}_\varepsilon \) satisfy the same conditions as in Lemma 3.6. Let \( 1 < p < \infty \). Then for any \( F \in L^q(\Omega; \mathbb{R}^m) \), where \( q = pd/(p+d) \) if \( p > d/(d+1) \), and \( q > 1 \) if \( 1 < p \leq d/(d-1) \), there exists a unique solution \( u_\varepsilon \in W^{1,p}_0(\Omega; \mathbb{R}^m) \) to \( \mathcal{L}_\varepsilon(u_\varepsilon) = F \) in \( \Omega \) and \( u_\varepsilon = 0 \) on \( \partial\Omega \), satisfying the uniform estimate
\[
\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C\|F\|_{L^q(\Omega)},
\]
where \( C \) depends only on \( \mu, \omega, \kappa, \lambda, p, q, d, m \) and \( \Omega \).
Proof. We prove this lemma by the duality argument. The uniqueness is clearly contained in Lemma 3.6 and the existence of the solution \( u_\varepsilon \) follows from the density and Theorem 2.5. The rest of the proof is to establish (3.11).

Consider the dual problem for any \( f \in C_0^d(\Omega; \mathbb{R}^{md}) \), there exists the unique \( v_\varepsilon \in H^1_0(\Omega; \mathbb{R}^m) \) to \( \mathcal{L}_\varepsilon(v_\varepsilon) = \text{div}(f) \) in \( \Omega \) and \( v_\varepsilon = 0 \) on \( \partial\Omega \). Note that it follows from Lemma 3.6 that \( \|\nabla v_\varepsilon\|_{L^p' (\Omega)} \leq C \|f\|_{L^p (\Omega)} \). Then we have

\[
\int_\Omega \nabla u_\varepsilon f dx = - \int_\Omega u_\varepsilon \mathcal{L}_\varepsilon^*(v_\varepsilon) dx = - \int_\Omega \mathcal{L}_\varepsilon(u_\varepsilon) v_\varepsilon dx = - \int_\Omega F v_\varepsilon dx,
\]

and

\[
\left\| \int_\Omega \nabla u_\varepsilon f dx \right\| \leq \|f\|_{L^p(\Omega)} \|v_\varepsilon\|_{L^p'(\Omega)} \leq C \|f\|_{L^p(\Omega)} \|\nabla v_\varepsilon\|_{L^p'(\Omega)} \leq C \|f\|_{L^q(\Omega)} \|f\|_{L^{p'}(\Omega)}.
\]

Note that \( \frac{1}{q} = \frac{1}{p'} - \frac{1}{n} \) if \( p < d \), \( 1 < q' < \infty \) if \( p' = d \), and \( q' = \infty \) if \( p' > d \). In other words, \( q = \frac{pd}{p+d} \) if \( p > \frac{d-1}{d} \), and \( q > 1 \) if \( 1 < p \leq \frac{d-1}{d} \). Finally we obtain \( \|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C \|f\|_{L^q(\Omega)} \).

Proof of Theorem 1.1. In the case of \( g = 0 \), we write \( v_\varepsilon = u_{\varepsilon,1} + u_{\varepsilon,2} \), where \( u_{\varepsilon,1} \) and \( u_{\varepsilon,2} \) are the solutions in Lemma 3.9 and 3.7 respectively. Then we have

\[
\|\nabla v_\varepsilon\|_{L^p(\Omega)} \leq \|\nabla u_{\varepsilon,1}\|_{L^p(\Omega)} + \|\nabla u_{\varepsilon,2}\|_{L^p(\Omega)} \leq C\{\|f\|_{L^p(\Omega)} + \|F\|_{L^q(\Omega)}\}.
\]

For \( g \neq 0 \), consider the homogeneous Dirichlet problem \( \mathcal{L}_\varepsilon(w_\varepsilon) = 0 \) in \( \Omega \) and \( w_\varepsilon = g \) on \( \partial\Omega \), where \( g \in B^{1-1/p,p}(\partial\Omega; \mathbb{R}^m) \). By the properties of boundary Besov space, there exists \( G \in W^{1,p}(\Omega; \mathbb{R}^m) \) such that \( G = g \) on \( \partial\Omega \) and \( \|G\|_{W^{1,p}(\Omega)} \leq C\|g\|_{B^{1-1/p,p}(\partial\Omega)} \). Let \( h_\varepsilon = w_\varepsilon - G \), we have

\[
\mathcal{L}_\varepsilon(h_\varepsilon) = \text{div}(A_\varepsilon \nabla G + V_\varepsilon G) - B_\varepsilon \nabla G - (c_\varepsilon + \lambda) G \quad \text{in} \quad \Omega, \quad h_\varepsilon = 0 \quad \text{on} \quad \partial\Omega.
\]

Recall the case of \( g = 0 \), in which there exists the unique weak solution \( h_\varepsilon \in W_0^{1,p}(\Omega; \mathbb{R}^m) \), satisfying the uniform estimate \( \|\nabla h_\varepsilon\|_{L^p(\Omega)} \leq C\|G\|_{W^{1,p}(\Omega)} \leq C\|g\|_{B^{1-1/p,p}(\partial\Omega)} \) for \( 1 < p < \infty \). This implies

\[
\|\nabla w_\varepsilon\|_{L^p(\Omega)} \leq \|\nabla h_\varepsilon\|_{L^p(\Omega)} + \|\nabla G\|_{L^p(\Omega)} \leq C\|g\|_{B^{1-1/p,p}(\partial\Omega)}.
\]

Finally, let \( u_\varepsilon = v_\varepsilon + w_\varepsilon \). Combining (3.12) and (3.13), we have

\[
\|\nabla u_\varepsilon\|_{L^p(\Omega)} \leq C\{\|f\|_{L^p(\Omega)} + \|F\|_{L^q(\Omega)} + \|g\|_{B^{1-1/p,p}(\partial\Omega)}\},
\]

where \( C \) depends only on \( \mu, \omega, \kappa, \lambda, p, q, d, m \) and \( \Omega \). We complete the proof.

\[
\textbf{Corollary 3.8.} \quad \text{Suppose that the coefficients of } \mathcal{L}_\varepsilon \text{ satisfy the same conditions as in Theorem 1.1. Set } d < p < \infty \text{ and } \sigma = 1 - d/p. \text{ Let } f = (f^\alpha) \in L^p(\Omega; \mathbb{R}^{md}), F \in L^q(\Omega; \mathbb{R}^m) \text{ with } q = \frac{pd}{p+d}, \text{ and } g \in C^{0,1}(\partial\Omega; \mathbb{R}^m). \text{ Then the unique solution } u_\varepsilon \text{ to (1.7) satisfies the uniform estimate}
\]

\[
\|u_\varepsilon\|_{C^{0,\sigma}(\Omega)} \leq C\{\|f\|_{L^p(\Omega)} + \|F\|_{L^q(\Omega)} + \|g\|_{C^{0,1}(\partial\Omega)}\},
\]

where \( C \) depends only on \( \mu, \omega, \kappa, \lambda, p, q, d, m, \) and \( \Omega \).

\[
\textbf{Proof.} \quad \text{Due to the extension theorem (see [22, pp.136])}, \text{ there exists an extension function } G \in C^{0,1}(\Omega; \mathbb{R}^m) \text{ such that } G = g \text{ on } \partial\Omega \text{ and } \|G\|_{C^{0,1}(\Omega)} \leq C\|g\|_{C^{0,1}(\partial\Omega)}. \text{ This also implies } \|G\|_{W^{1,p}(\Omega)} \leq C\|g\|_{C^{0,1}(\partial\Omega)} \text{ for any } p \geq 1. \text{ Let } v_\varepsilon, w_\varepsilon \text{ be the weak solutions to the following Dirichlet problems:}
\]

\[
(i) \quad \begin{cases} \mathcal{L}_\varepsilon(v_\varepsilon) = \text{div}(f) + F \quad \text{in} \quad \Omega, \\ v_\varepsilon = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \quad (ii) \quad \begin{cases} \mathcal{L}_\varepsilon(w_\varepsilon) = 0 \quad \text{in} \quad \Omega, \\ w_\varepsilon = g \quad \text{on} \quad \partial\Omega, \end{cases}
\]

respectively. For (i), it follows from the Sobolev embedding theorem and Theorem 1.1 that \( \|v_\varepsilon\|_{C^{0,\sigma}(\Omega)} \leq C\|\nabla v_\varepsilon\|_{L^p(\Omega)} \leq C\{\|f\|_{L^p(\Omega)} + \|F\|_{L^q(\Omega)}\}. \) For (ii), by setting \( h_\varepsilon = w_\varepsilon - G \), we have \( \mathcal{L}_\varepsilon(h_\varepsilon) = -\mathcal{L}_\varepsilon(G) \) in \( \Omega \) and \( h_\varepsilon = 0 \) on \( \partial\Omega \). Therefore \( \|h_\varepsilon\|_{C^{0,\sigma}(\Omega)} \leq C\|G\|_{W^{1,p}(\Omega)} \leq C\|g\|_{C^{0,1}(\partial\Omega)} \), which implies \( \|w_\varepsilon\|_{C^{0,\sigma}(\Omega)} \leq C\|g\|_{C^{0,1}(\partial\Omega)} \). Let \( u_\varepsilon = v_\varepsilon + w_\varepsilon \). Combining the estimates related to \( v_\varepsilon \) and \( w_\varepsilon \), we derive the estimate (3.14).
Remark 3.9. Assume the same conditions as in Corollary 3.8 let \( u_\varepsilon \) be a weak solution to (1.7). Then by the same argument as in the proof of Theorem 3.3 we obtain the near boundary Hölder estimate

\[
\|u_\varepsilon\|_{C^{0,\sigma}(\partial r)} \leq C r^{-\sigma} \left\{ \left( \int_{D(2r)} |u_\varepsilon|^2 \, dx \right)^{\frac{1}{2}} + r \|g\|_{C^{0,1}(\Delta(2r))} + r \left( \int_{D(2r)} |f|^p \, dx \right)^{\frac{1}{p}} + r^2 \left( \int_{D(2r)} |F|^q \, dx \right)^{\frac{1}{q}} \right\},
\]

where \( \sigma = 1 - d/p \), and \( C \) depends on \( \mu, \omega, \kappa, \lambda, p, d, m \) and \( \Omega \).

Remark 3.10. In the following, we frequently use the abbreviated writing like \( f_{\Omega} F(x, y) = f_{\Omega} F(\cdot, y) \).

Theorem 3.11. Suppose the same conditions on \( L_\varepsilon \) as in Theorem 1.1 and \( \lambda \geq \lambda_0 \). Then there exists a unique Green’s matrix \( G_\varepsilon : \Omega \times \Omega \rightarrow \mathbb{R}^{m^2} \) such that \( G_\varepsilon(\cdot, y) \in H^1(\Omega \setminus B_r(y); \mathbb{R}^{m^2}) \cap W^{1, s}(\Omega; \mathbb{R}^{m^2}) \) with \( s \in [1, \frac{d}{d-1}] \) for each \( y \in \Omega \) and \( r > 0 \), and \( B^*_\varepsilon[G^\varepsilon(\cdot, y), \phi] = \phi \cdot (y) \), \( \forall \phi \in W^{1,p}_0(\Omega; \mathbb{R}^m) \) with \( p > d \).

Particularly, for any \( F \in L^q(\Omega; \mathbb{R}^m) \) with \( q > d/2 \),

\[
u_\varepsilon^r(y) = \int_{\Omega} G^\varepsilon(x, y) F^\alpha(x, y) \, dx
\]

satisfies \( \mathcal{L}_\varepsilon(\nu_\varepsilon^r) = F \) in \( \Omega \) and \( u_\varepsilon = 0 \) on \( \partial \Omega \). Moreover, let \( *G_\varepsilon(\cdot, x) \) be the adjoint Green’s matrix of \( G_\varepsilon(\cdot, y) \), then \( G_\varepsilon(x, y) = [ *G_\varepsilon(y, x) ]^* \) and for any \( \sigma, \sigma' \in (0, 1) \), the following estimates

\[
|G_\varepsilon(x, y)| \leq \frac{C}{|x - y|^{d-2}} \min \left\{ 1, \frac{d_x^\sigma}{|x - y|^\sigma}, \frac{d_y^\sigma}{|x - y|^\sigma}, \frac{d_x^\sigma d_y^\sigma}{|x - y|^{\sigma + \sigma'}} \right\}
\]

hold for any \( x, y \in \Omega \) and \( x \neq y \), where \( d_x = \text{dist}(x, \partial \Omega) \), and \( C \) depends only on \( \mu, \omega, \kappa, \lambda, d, m \) and \( \Omega \).

Lemma 3.12 (Approximating Green’s matrix). Assume the same conditions as in Theorem 3.11. Define the approximating Green’s matrix \( G_{\rho, \varepsilon} \) as

\[
B^*_\varepsilon[G^\varepsilon(\cdot, y), u] = \int_{\Omega \setminus \rho(y)} u_\varepsilon \, dx, \quad \forall u \in H^1_0(\Omega; \mathbb{R}^m),
\]

where \( 1 \leq \gamma \leq m, \) and \( \Omega_\rho(y) = \Omega \cap B_\rho(y) \). Then if \( |x - y| < d_y/2 \), we have the uniform estimate

\[
|G^\varepsilon_{\rho, \varepsilon}(x, y)| \leq \frac{C}{|x - y|^{d-2}}, \quad \forall \rho < |x - y|/4,
\]

where \( C \) depends only on \( \mu, \omega, \kappa, \lambda, d, m \) and \( \Omega \). Moreover, for any \( s \in [1, \frac{d}{d-1}] \), we have

\[
\sup_{\rho > 0} \|G^\varepsilon_{\rho, \varepsilon}(\cdot, y)\|_{W^{1,s}_0(\Omega)} \leq C(\mu, \omega, \kappa, \lambda, d, m, s, \Omega, d_y).
\]

Proof. First of all, we show \( G_{\rho, \varepsilon}(x, y) = [ G^\varepsilon_{\rho, \varepsilon}(x, y) ] \) is well defined. Let \( I(u) = \int_{\Omega \setminus \rho(y)} u^\gamma \, dx \), then \( I \in H^{-1}(\Omega; \mathbb{R}^m) \) and \( |I(u)| \leq C|\Omega_\rho(y)|^{-1/2} \|u\|_{H^1_0(\Omega)} \) with \( 2^* = \frac{2d}{d-2} \). It follows from Theorem 2.5 that there exists a unique \( G^\varepsilon_{\rho, \varepsilon}(\cdot, y) \in H^1_0(\Omega; \mathbb{R}^m) \) satisfying (3.20) and

\[
\|\nabla G^\varepsilon_{\rho, \varepsilon}(\cdot, y)\|_{L^2(\Omega)} \leq C|\Omega_\rho(y)|^{-\frac{1}{2^*}}.
\]

For any \( F \in C_0^{\infty}(\Omega; \mathbb{R}^m) \), consider \( \mathcal{L}_\varepsilon(u_\varepsilon) = F \) in \( \Omega \) and \( u_\varepsilon = 0 \) on \( \partial \Omega \). There exists the unique solution \( u_\varepsilon \in H^1_0(\Omega; \mathbb{R}^m) \) such that

\[
\int_{\Omega} FG_{\rho, \varepsilon}(\cdot, y) = B_{\varepsilon}[u_\varepsilon, G_{\rho, \varepsilon}(\cdot, y)] = B^*_\varepsilon[G^\varepsilon_{\rho, \varepsilon}(\cdot, y), u_\varepsilon] = \int_{\Omega_\varepsilon(y)} u_\varepsilon^\gamma.
\]
Suppose $\text{supp}(F) \subseteq B \subseteq \Omega$, where $B = B_R(y)$. Then it follows from \[2.8\], \[3.3\] and \[3.21\] that
\[
\left| \int \Omega F G_{\rho,\varepsilon}^\gamma (\cdot, y) \right| \leq \| u_\varepsilon \|_{L^\infty(1/4B)} \leq C \left[ \left( \int_{1/2B} \| u_\varepsilon \|_{L^1}^2 \right)^{\frac{1}{2}} + R^2 \left( \int_{1/2B} |F|^p \right)^{\frac{1}{p}} \right] \leq CR^2 \left( \int_B |F|^p \right)^{\frac{1}{p}}
\]
for any $\rho < R/4$ and $p > d/2$. This implies
\[
\left( \int_B |G_{\rho,\varepsilon}^\gamma (\cdot, y)|^q \right)^{\frac{1}{q}} \leq CR^{2-d}, \quad \forall \, R \leq d_y, \quad \forall \, q \in \left[ 1, \frac{d}{d-2} \right].
\]

Now we turn to \[3.21\]. Set $r = |x - y|$, and $r \leq d_y/2$. In view of \[3.20\], $G_{\rho,\varepsilon}(x, y)$ actually satisfies $L_\varepsilon [G_{\rho,\varepsilon}^\gamma (\cdot, y)] = 0$ in $B_{\frac{r}{2}}(y) \setminus B_{\frac{r}{2}}(y)$. By using \[3.3\] again, we obtain
\[
|G_{\rho,\varepsilon}^\gamma (x, y)| \leq C \int_{B_{\frac{r}{2}}(x)} |G_{\rho,\varepsilon}^\gamma (\cdot, y)| \leq C \int_{B_{2R}(y)} |G_{\rho,\varepsilon}^\gamma (\cdot, y)| \leq C |x - y|^{2-d}
\]
for any $\rho < |x - y|/4$, where $C$ depends only on $\mu, \omega, \kappa, \lambda, d, m$ and $\Omega$. Therefore, we will prove \[3.22\]. Step one, we verify the following estimates,
\[
\int_{\Omega \setminus B(y,R)} |\nabla G_{\rho,\varepsilon}^\gamma (\cdot, y)|^2 \leq CR^{2-d}, \quad \int_{\Omega \setminus B(y,R)} |G_{\rho,\varepsilon}^\gamma (\cdot, y)|^2 \leq CR^{-d}, \quad \forall \, \rho > 0, \quad \forall \, R < d_y/4. \tag{3.25}
\]

On the one hand, let $\varphi \in C_0^1(\Omega)$ be a cut-off function satisfying $\varphi \equiv 0$ on $B(y, R)$, $\varphi \equiv 1$ outside $B(y, 2R)$, and $|\nabla \varphi| \leq C/R$. Choose $u = \varphi^2 G_{\rho,\varepsilon}^\gamma (\cdot, y)$ in \[3.20\] and $\lambda \geq \lambda_0$. It follows from \[2.11\] and \[3.21\] that
\[
\int_{\Omega} \varphi^2 |\nabla G_{\rho,\varepsilon}^\gamma (\cdot, y)|^2 \leq C \int_{\Omega} |\nabla \varphi|^2 |G_{\rho,\varepsilon}^\gamma (\cdot, y)|^2 \leq \frac{C}{R^2} \int_{2B(\lambda/2)} |x - y|^{2(2-d)} \leq CR^{2-d}, \quad \forall \, \rho < R/4. \tag{3.26}
\]

On the other hand, it follows from \[3.24\] that
\[
\int_{\Omega \setminus B(y,R)} |\nabla G_{\rho,\varepsilon}^\gamma (\cdot, y)|^2 \leq \int_{\Omega} |\nabla G_{\rho,\varepsilon}^\gamma (\cdot, y)|^2 \leq CR^{-d}, \quad \forall \, \rho \geq R/4.
\]

Thus we have the first inequality of \[3.25\] for all $\rho > 0$.

For the second estimate in \[3.25\], we observe
\[
\int_{\Omega} |\sigma G_{\rho,\varepsilon}^\gamma (\cdot, y)|^2 \leq C \left( \int_{\Omega} |\nabla \varphi G_{\rho,\varepsilon}^\gamma (\cdot, y)|^2 \right)^{-\frac{1}{4}} \leq C \left( \int_{\Omega} |\nabla \varphi G_{\rho,\varepsilon}^\gamma (\cdot, y)|^2 + |\sigma \nabla G_{\rho,\varepsilon}^\gamma (\cdot, y)|^2 \right)^{\frac{1}{4}} \leq CR^{-d} \tag{3.27}
\]
for any $\rho < R/4$, where we use Sobolev’s inequality in the first inequality and \[3.20\] in the last inequality. We remark that the constant $C$ does not involve $R$. In the case of $\rho \geq R/4$, since $G_{\rho,\varepsilon}^\gamma (\cdot, y) = 0$ on $\partial \Omega$, we have
\[
\int_{\Omega \setminus B(y,R)} |G_{\rho,\varepsilon}^\gamma (\cdot, y)|^2 \leq \int_{\Omega} |G_{\rho,\varepsilon}^\gamma (\cdot, y)|^2 \leq C \left( \int_{\Omega} |\nabla G_{\rho,\varepsilon}^\gamma (\cdot, y)|^2 \right)^{\frac{1}{4}} \leq CR^{-d},
\]
where we use Sobolev’s inequality in the second inequality and \[3.24\] in the last inequality. This together with \[3.27\] leads to
\[
\int_{\Omega \setminus B(y,R)} |G_{\rho,\varepsilon}^\gamma (\cdot, y)|^2 \leq CR^{-d}, \quad \forall \, \rho > 0, \quad \forall \, R < d_y/4.
\]

We now address ourselves to the uniform estimates of $G_{\rho,\varepsilon}^\gamma (\cdot, y)$ and $\nabla G_{\rho,\varepsilon}^\gamma (\cdot, y)$ with respect to parameter $\rho$. In the case of $t > (d_y/4)^{1-d}$, we obtain
\[
\left| \{ x \in \Omega : |\nabla G_{\rho,\varepsilon}^\gamma (\cdot, y)| > t \} \right| \leq CR^d + t^2 \int_{\Omega \setminus B(y,R)} |\nabla G_{\rho,\varepsilon}^\gamma (\cdot, y)|^2 \leq Ct^{\frac{d}{d-1}}, \quad \forall \, \rho > 0. \tag{3.28}
\]
For $t > (d_y/4)^{2-d}$, it follows that
\[ \left| \left\{ x \in \Omega : |G_{\rho,\varepsilon}(\cdot, y)| > t \right\} \right| \leq Ct^{-\frac{3}{d-2}}, \quad \forall \, \rho > 0. \] (3.29)

Then in view of \((3.28)\) and \((3.29)\), we have
\[ \int_{\Omega} |G_{\rho,\varepsilon}(\cdot, y)|^s \leq C d_y^{s(2-d)} + C \int_{(d_y/4)^{2-d}}^{\infty} t^{s-1} \cdot t^{-\frac{d}{d-2}} dt \leq C \left[ d_y^{s(2-d)} + d_y^{s(2-d)+d} \right] \]
for $s \in [1, \frac{d}{d-2})$, and
\[ \int_{\Omega} |\nabla G_{\rho,\varepsilon}(\cdot, y)|^s \leq C \left[ d_y^{s(1-d)} + d_y^{s(1-d)+d} \right] \]
for $s \in [1, \frac{d}{d-1})$, where $C$ depends only on $\mu, \omega, \kappa, \lambda, d, m, s$ and $\Omega$. Combining the two inequalities above, we have \((3.22)\), and the proof is complete. □

**Proof of Theorem 3.11** From the uniform estimate \((3.22)\), it follows that there exist a subsequence of \(\{G_{\rho_n,\varepsilon}(\cdot, y)\}_{n=1}^{\infty}\) and \(G_{\varepsilon}(\cdot, y)\) such that for any $s \in (1, \frac{d}{d-1})$,
\[ G_{\rho_n,\varepsilon}(\cdot, y) \rightharpoonup G_{\varepsilon}(\cdot, y) \quad \text{weakly in } W^{1,s}_0(\Omega; \mathbb{R}^m) \quad \text{as } n \to \infty. \] (3.30)

Hence, we have
\[ B_{\varepsilon}^*[G_{\varepsilon}(\cdot, y), \phi] = \lim_{n \to \infty} B_{\varepsilon}^*[G_{\rho_n,\varepsilon}(\cdot, y), \phi] = \lim_{n \to \infty} \int_{\Omega_{\rho_n}(y)} \phi^\gamma = \phi^\gamma \]
for any $\phi \in W^{1,p}_0(\Omega; \mathbb{R}^m)$ with $p > d$, where we use the definition of the approximating Green’s matrix. Due to Theorem 1.1, there exists the weak solution \(u_{\varepsilon} \in W^{1,p}_0(\Omega; \mathbb{R}^m)\) satisfying \(\mathcal{L}_{\varepsilon}(u_{\varepsilon}) = F\) in $\Omega$ and $u_{\varepsilon} = 0$ on $\partial \Omega$ for any $F \in L^2(\Omega; \mathbb{R}^m)$ with $p > d$. Thus we obtain
\[ u_{\varepsilon}(y) = B_{\varepsilon}^*[G_{\varepsilon}(\cdot, y), u_{\varepsilon}] = B_{\varepsilon}[u_{\varepsilon}, G_{\varepsilon}(\cdot, y)] = \int_{\Omega} G_{\varepsilon}(\cdot, y) F. \]

We now verify the uniqueness. If \(\widehat{G}_{\varepsilon}(\cdot, y)\) is another Green’s matrix of \(\mathcal{L}_{\varepsilon}\), we then have \(\widehat{u}_{\varepsilon} = \int_{\Omega} \widehat{G}_{\varepsilon}(\cdot, y) F\). It follows from the uniqueness of the weak solution that \(\int_{\Omega} [\widehat{G}_{\varepsilon}(\cdot, y) - G_{\varepsilon}(\cdot, y)] F = 0\) for any $F \in L^2(\Omega; \mathbb{R}^m)$, hence \(\widehat{G}_{\varepsilon}(\cdot, y) = G_{\varepsilon}(\cdot, y)\) a.e. in $\Omega$.

Next, let \(G_{\theta,\varepsilon}(\cdot,x)\) denote the approximating adjoint of \(G_{\rho,\varepsilon}(\cdot, y)\), which satisfies
\[ B_{\varepsilon}[G_{\theta,\varepsilon}(\cdot, x), u] = \int_{\Omega_{\rho}(x)} u^\theta, \quad \forall \, u \in H^1_0(\Omega; \mathbb{R}^m). \] (3.31)

By the same argument, we can derive the existence and uniqueness of \(G_{\theta,\varepsilon}(\cdot,x)\), as well as the estimates similar to \((3.21)\) and \((3.22)\). Thus for any $\rho, \varrho > 0$, we obtain
\[ \int_{\Omega_{\rho}(y)} *G_{\theta,\varepsilon}(z,x)dz = B_{\varepsilon}^*[G_{\rho,\varepsilon}(\cdot, y), G_{\theta,\varepsilon}(\cdot, x)] = B_{\varepsilon}[G_{\theta,\varepsilon}(\cdot, x), G_{\rho,\varepsilon}(\cdot, y)] = \int_{\Omega_{\rho}(x)} G_{\rho,\varepsilon}(z,y)dz. \]

Note that \(\mathcal{L}_{\varepsilon}[G_{\theta,\varepsilon}(\cdot, x)] = 0\) in $\Omega \setminus B_{\rho}(x)$ and \(\mathcal{L}_{\varepsilon}[G_{\rho,\varepsilon}(\cdot, y)] = 0\) in $\Omega \setminus B_{\varrho}(y)$. In view of Corollary 3.5, \(G_{\theta,\varepsilon}(\cdot, x)\) and \(G_{\rho,\varepsilon}(\cdot, y)\) are locally H"older continuous. Therefore, we have \(*G_{\theta,\varepsilon}(y,x) = G_{\theta,\varepsilon}(y,x)\) as $\rho \to 0$ and $\varrho \to 0$, which implies \(G_{\varepsilon}(x,y) = [G_{\varepsilon}(x,y)]^*\) for any $x, y \in \Omega$ and $x \neq y$.

Let $r = |x - y|$ and $F \in C^0(\Omega_{r/2}(x))$. Assume \(u_{\varepsilon}\) is the solution of \(\mathcal{L}_{\varepsilon}(u_{\varepsilon}) = F\) in $\Omega$ and $u_{\varepsilon} = 0$ on $\partial \Omega$. Then we have \(u_{\varepsilon}(y) = \int_{\Omega} G_{\varepsilon}(z,y)F(z)dz\). Since \(\mathcal{L}_{\varepsilon}(u_{\varepsilon}) = 0\) in $\Omega \setminus \Omega_{r/2}(x)$, it follows from Corollary 3.5 that
\[ |u_{\varepsilon}(y)| \leq C \left( \int_{\Omega_{r/2}(y)} |u_{\varepsilon}(z)|^2 dz \right)^{\frac{1}{2}} \leq C r^{1 - \frac{d}{4}} \left( \int_{\Omega} |u_{\varepsilon}(z)|^2 dz \right)^{\frac{1}{2}}. \]
\[
C r^{-\frac{d}{2}} \left( \int_{\Omega} |\nabla u_\varepsilon(z)|^2 \, dz \right)^{\frac{1}{2}} \leq C r^{-\frac{d}{2}} \|F\|_{L^{\frac{2d}{d+2}}(\Omega)} \leq C r^{-\frac{d}{2}} \|F\|_{L^2(\Omega_\varepsilon(x))},
\]
where we use Hölder’s inequality in the second inequality and Sobolev’s inequality in the third inequality, as well as the estimate (3.11) with \( p = 2 \) in the fourth inequality. This implies
\[
\left( \int_{\Omega_\varepsilon(x)} |G_\varepsilon(z,y)|^2 \, dz \right)^{\frac{1}{2}} \leq C r^{2-d}.
\]
(3.32)

Note that \( L_\varepsilon[\mathcal{G}_\varepsilon(\cdot, y)] = 0 \) in \( \Omega \setminus B(y, r) \) for any \( r > 0 \). So in the case of \( r \leq 3d_x \), it follows from (3.18) and (3.32) that
\[
|G_\varepsilon(x, y)| \leq C \left( \int_{\Omega_\varepsilon(x)} |G_\varepsilon(z, y)|^2 \, dz \right)^{\frac{1}{2}} \leq C \frac{C d}{|x-y|^{d-2}}.
\]

For \( r > 3d_x \), in view of (3.16) and (3.32), for any \( \sigma \in (0, 1) \), we have
\[
|G_\varepsilon(x, y)| = |G_\varepsilon(x, y) - G_\varepsilon(x, \bar{y})| \leq C \left( \frac{|x-\bar{y}|}{r} \right)^{\sigma} \left( \int_{\Omega_\varepsilon(x)} |G_\varepsilon(z, y)|^2 \, dz \right)^{\frac{1}{2}} \leq C \frac{C d}{|x-y|^{d-2+\sigma}},
\]
where \( \bar{y} \in \partial \Omega \) such that \( d_x = |x-\bar{y}| \). By the same argument, we can obtain similar results for \( \star \mathcal{G}_\varepsilon(\cdot, x) \).

Remark 3.13. We will see in Section 4 that the estimates (3.19) actually hold for \( \sigma = \sigma' = 1 \), which are
\[
|G_\varepsilon(x, y)| \leq \frac{C d}{|x-y|^{d-2}} \min \left\{ 1, \frac{d_x}{|x-y|} \right\}. \]

Let \( r = |x-y| \), and \( \bar{y} \in \partial \Omega \) such that \( d_y = |y-\bar{y}| \). In the case of \( d_y < r/6 \), due to \( G_\varepsilon(x, \cdot) = 0 \) on \( \partial \Omega \), we have
\[
|G_\varepsilon(x, y)| = |G_\varepsilon(x, y) - G_\varepsilon(x, \bar{y})| \leq \|\nabla G_\varepsilon(x, \cdot)\|_{L^\infty(\Omega_\varepsilon(y))} |y-\bar{y}| \leq \frac{C d_y}{r} \left( \int_{\Omega_\varepsilon(y)} |G_\varepsilon(x, z)|^2 \, dz \right)^{\frac{1}{2}} \leq \frac{C d_y}{|x-y|^{d-1}},
\]
where we employ the estimate (4.36) in the second inequality, and (3.32) in the last one. In the case of \( d_y \geq r/6 \), we can straightforward derive the above estimate from the estimate \( |G_\varepsilon(x, y)| \leq C |x-y|^{2-d} \).

Similarly, we can derive
\[
|G_\varepsilon(x, y)| = |\mathcal{G}_\varepsilon(y, x)| \leq C d_x d_y |x-y|^{1-d}.
\]

Then we plug the above estimate back into the last inequality of (3.33), and obtain \( |G_\varepsilon(x, y)| \leq C d_x d_y |x-y|^{1-d} \).

Remark 3.14. The main idea in the proofs of Theorem 3.11 and Lemma 3.12 can be found in [3, 26]. We comment that the indices \( \sigma \) and \( \sigma' \in (0, 1) \) can be equal, which actually come from the Hölder estimate with zero boundary data. Equipped with the estimate (3.19), it is possible to arrive at the sharp Hölder estimate with nonzero boundary data.
Proof of Theorem 1.2. We first assume that \( u_{x,1} \) satisfies \( L(x,u_{x,1}) = 0 \) in \( \Omega \) and \( u_{x,1} = g \) on \( \partial \Omega \). Let \( v \) be the extension function of \( g \), satisfying \( \Delta v = 0 \) in \( \Omega \) and \( v = g^\alpha \) on \( \partial \Omega \). For any \( x \in \Omega \), set \( B = B(x,d_x) \). We have the estimate

\[
|\nabla v(x)| \leq C \left( \int_B |\nabla v|^2 dy \right)^{\frac{1}{2}} \leq \frac{C}{d_x} \left( \int_B |v(y) - v(x)|^2 dy \right)^{\frac{1}{2}} \leq C d_x^{\sigma-1} [v]_{C^{0,\sigma}(\Omega)} \leq C d_x^{\sigma-1} \|g\|_{C^{0,\sigma}(\partial \Omega)} \quad (3.34)
\]

for any \( \sigma \in (0,1) \), where we use the (interior) Lipschitz estimate (2.25) in the first inequality, Cacciopoli’s inequality in the second inequality, and the Hölder estimate: \( [v]_{C^{0,\sigma}(\Omega)} \leq C \|g\|_{C^{0,\sigma}(\partial \Omega)} \) in the last inequality. By normalization we may assume \( \|g\|_{C^{0,\sigma}(\partial \Omega)} = 1 \). Let \( w_x = u_{x,1} - v \), then \( L_x(w_x) = -L_x(v) \) in \( \Omega \) and \( w_x = 0 \) on \( \partial \Omega \). It follows from (3.18) that

\[
w_x(y) = - \int_\Omega \nabla G(x,y) [A_x \nabla v + V_x v] dx - \int_\Omega G(x,y) [B_x \nabla v + (c_x + \lambda)v] dx,
\]

which implies

\[
|w_x(y)| \leq C \int_\Omega |\nabla G(x,y)||[dx]^{\sigma-1} dx + C \int_\Omega |G(x,y)||[dx]^{\sigma-1} dx + C \int_\Omega \left( |\nabla G(x,y)| + |G(x,y)| \right) dx
\]

where we use the estimate (3.34).

To estimate \( I_1 \), set \( r = d_y/2 \). It follows from (2.9) and (3.19) that

\[
\int_{B(y,r)} |\nabla G(x,y)|[dx]^{\sigma-1} dx \leq C \sum_{j=0}^\infty (2^{-j}r)^d \int_{B(y,2^{-j}r) \setminus B(y,2^{-j-1}r)} |\nabla G(x,y)|^2 dx \frac{1}{r^{\sigma-1}}
\]

\[
\leq C \sum_{j=0}^\infty (2^{-j}r)^{d-1} \int_{B(y,2^{-j+1}r) \setminus B(y,2^{-j-2}r)} |G(x,y)|^2 dx \frac{1}{r^{\sigma-1}} \leq C r^\sigma,
\]

Next, we address ourselves to the integral on \( \Omega \setminus B(y,r) \). Let \( Q \) be a cube in \( \mathbb{R}^d \) with the property that \( 3Q \subset \Omega \setminus \{y\} \), and \( l(Q), \text{dist}(Q, \partial \Omega) \) are comparable, where \( l(Q) \) denotes the side length of \( Q \) (see [11, pp.167]). Thus, for fixed \( z \in Q \) there exist \( c_1, c_2 > 0 \) such that \( c_1 |z - y| \leq |x - y| \leq c_2 |z - y| \) for any \( x \in Q \), and we have

\[
\int_Q |\nabla G(x,y)|[dx]^{\sigma-1} dx \leq C[l(Q)]^{\sigma-2}[Q] \left( \int_{2Q} |G(x,y)|^2 dx \right)^{\frac{1}{2}}
\]

\[
\leq C[l(Q)]^{\sigma+\sigma_1-2}[Q] \frac{r^{\sigma_2}}{|z - y|^{d-2+\sigma_1+\sigma_2}} \leq C r^{\sigma_2} \int_Q \frac{[dx]^{\sigma+\sigma_1-2}}{|z - y|^{d-2+\sigma_1+\sigma_2}} dx,
\]

where we use the estimate (2.9) in the first inequality, the estimate (3.19) in the second one, and the Chebyshev’s inequality in the last one. (Note that \( \sigma_1 \) and \( \sigma_2 \) will be given later.) By decomposing \( \Omega \setminus B(y,r) \) as a non-overlapping union of cubes \( Q \) (see [11, pp.167-170]), we then obtain

\[
\int_{\Omega \setminus B(y,r)} |\nabla G(x,y)|[dx]^{\sigma-1} dx \leq C r^{\sigma_2} \int_{\Omega \setminus B(y,r)} \frac{[dx]^{\sigma+\sigma_1-2}}{|x - y|^{d-2+\sigma_1+\sigma_2}} dx =: I_{11} + I_{12} + I_{13}.
\]

Note that we add the additional distance \( r \) in the denominator in the second inequality and therefore the corresponding domain of integral becomes the union of \( \Sigma_0 = B(y,r) \), \( \Sigma_1 = \cup_{j=0}^\infty \Omega_j \) and \( \Sigma_2 = \cup_{j=0}^\infty \cup_m^\infty \Omega_{j,m} \), where

\[
\Omega_j = \Omega \cap \{2^j r \leq |x - y| \leq 2^{j+1} r \} \cap \{2^j r \leq d_x \leq 2^{j+1} r + 2r \},
\]

\[
\Omega_{j,m} = \Omega \cap \{2^j r \leq |x - y| \leq 2^{j+1} r \} \cap \{2^{-m-1}(2^j r) \leq d_x \leq 2^{-m}(2^j r) \}.
\]
Then a routine computation gives rise to $I_{11} \leq C r^\sigma$,

$$I_{12} \leq C r^\sigma \sum_{j=0}^{\infty} \frac{(2^j r)^{\sigma+\sigma_1-2}}{(2^j r)^{d-2+\sigma_1+\sigma_2}} \cdot (2^j r)^d \leq C r^\sigma \sum_{j=0}^{\infty} (2^j)^{\sigma-\sigma_2} \leq C r^\sigma,$$

$$I_{13} \leq C r^\sigma \sum_{j=0}^{\infty} \sum_{m=0}^{\infty} \frac{(2^{-m-1}(2^j r))^{\sigma+\sigma_1-1}}{(2^j r)^{d-2+\sigma_1+\sigma_2}} \cdot (2^j r)^{d-1} \leq C r^\sigma \sum_{m=0}^{\infty} (2^{-m})^{\sigma+\sigma_1-1} \sum_{j=0}^{\infty} (2^j)^{\sigma-\sigma_2} \leq C r^\sigma,$$

provided we choose $\sigma_1, \sigma_2 \in (0, 1)$ such that $\sigma_1 + \sigma > 1$ and $\sigma_2 < \sigma$. Combining the above estimates, we obtain $I_1 \leq C [d_y]^\sigma$. In view of (3.19), we obtain

$$I_2 \leq C [d_y]^\sigma \int_\Omega \frac{1}{|x-y|^{d-1}} dx \leq C [d_y]^\sigma,$$

and $I_3 \leq C (I_1 + I_2)$. Hence, for any $y \in \Omega$ we have

$$|w_\epsilon(y)| \leq I_1 + I_2 + I_3 \leq C [d_y]^\sigma. \quad (3.35)$$

Consider three cases: (1) $|x-y| \leq d_x/4$; (2) $|x-y| \leq d_y/4$; (3) $|x-y| > \max\{d_x/4, d_y/4\}$. In the first case, let $r = d_x$. In view of (3.31) and (3.35) we first have

$$\sup_{z \in B(x, r/2)} |\nabla v(z)| \leq C r^{\sigma-1} \quad \text{and} \quad \sup_{z \in B(x, r/2)} |w_\epsilon(z)| \leq C r^\sigma. \quad (3.36)$$

It follows from (3.31), (3.35) and (3.36) that

$$|w_\epsilon(x) - w_\epsilon(y)| \leq [w_\epsilon]_{C^{0,\sigma}(B(x, d_x/4))} |x-y|^\sigma \leq C |x-y|^\sigma \left\{ \frac{1}{2} r^{-\sigma} \left( \int_{2B} |w_\epsilon|^2 dz \right)^{1/2} + r^{-\sigma} \left( \int_{2B} (\nabla v|^p + |v|^p) dz \right)^{1/2} + r^{-2\sigma} \left( \int_{2B} (\nabla v|^q + |v|^q) dz \right)^{1/2} \right\} \leq C |x-y|^\sigma,$$

where we also use the fact $\|v\|_{L^\infty(\Omega)} \leq C$ (the maximum principle) in the last inequality. It is clear to see that we can handle the second case in the same manner.

In the case of (3), we derive

$$|w_\epsilon(x) - w_\epsilon(y)| \leq |w_\epsilon(x)| + |w_\epsilon(y)| \leq C |x-y|^\sigma$$

from (3.35). Thus we have proved $\|w_\epsilon\|_{C^{0,\sigma}(\Omega)} \leq C \|g\|_{C^{0,\sigma}(\partial\Omega)}$. This together with $\|v\|_{C^{0,\sigma}(\Omega)} \leq C \|g\|_{C^{0,\sigma}(\partial\Omega)}$ gives

$$\|u_{\epsilon,1}\|_{C^{0,\sigma}(\Omega)} \leq \|w_\epsilon\|_{C^{0,\sigma}(\Omega)} + \|v\|_{C^{0,\sigma}(\Omega)} \leq C \|g\|_{C^{0,\sigma}(\partial\Omega)}.$$

In addition, assume that $u_{\epsilon,2}$ satisfies $L_\epsilon(u_{\epsilon,2}) = \text{div} f + F$ in $\Omega$ and $u_{\epsilon,2} = 0$ on $\partial\Omega$. It follows from Corollary 3.8 that $\|u_{\epsilon,2}\|_{C^{0,\sigma}(\Omega)} \leq C \{ \|f\|_{L^p(\Omega)} + \|F\|_{L^q(\Omega)} \}$. Let $u_\epsilon = u_{\epsilon,1} + u_{\epsilon,2}$, we finally obtain (1.9) and complete the proof.

## 4 Lipschitz estimates & Nontangential maximal function estimates

**Lemma 4.1.** Suppose $A \in \Lambda(\mu, \tau, \kappa)$. Let $p > d$ and $\nu \in (0, \eta]$. Assume that $f = (f_i^\nu) \in C^{0,\nu}(\Omega; \mathbb{R}^{md})$, $F \in L^p(\Omega; \mathbb{R}^m)$ and $g \in C^{1,\nu}(\partial\Omega; \mathbb{R}^m)$. Then the unique solution $u_\epsilon$ to $L_\epsilon(u_\epsilon) = \text{div}(f) + F$ in $\Omega$ and $u_\epsilon = g$ on $\partial\Omega$ satisfies the uniform estimate

$$\|\nabla u_\epsilon\|_{L^\infty(\Omega)} \leq C \left\{ \|f\|_{C^{0,\nu}(\Omega)} + \|F\|_{L^p(\Omega)} + \|g\|_{C^{1,\nu}(\partial\Omega)} \right\}, \quad (4.1)$$

where $C$ depends only on $\mu, \tau, \kappa, m, d, \eta, p, \nu$ and $\Omega$. 
Proof. See \cite{[32]} Remark 16]. In fact, \eqref{1.1} is a updated version of \cite{[3]} Theorem 2].

Lemma 4.2. Suppose \( A \in \Lambda(\mu, \tau, \kappa) \). Let \( \Gamma_\varepsilon(x, y) \) denote the fundamental solution of \( L_\varepsilon \), then we have

\[
\max\{\|\nabla_\varepsilon \Gamma_\varepsilon(x, z)\|, \|\nabla_\varepsilon \nabla_\varepsilon \Gamma_\varepsilon(x, z)\|\} \leq C|x - z|^{1-d}, \quad \|\nabla_\varepsilon \nabla_\varepsilon \Gamma_\varepsilon(x, z)\| \leq C|x - z|^{-d}, \tag{4.2}
\]

where \( C \) depends only on \( \mu, \tau, \kappa, m, d \).

Proof. See \cite{[33]} pp.6].

Lemma 4.3. Suppose \( A \in \Lambda(\mu, \tau, \kappa) \). Let \( F \in L^p(\Omega; \mathbb{R}^m) \) with \( p > d, \ f \in C^0(\Omega; \mathbb{R}^{nd}) \) with \( \sigma \in (0, 1) \), and \( u_\varepsilon \in H^1_\text{loc}(\Omega; \mathbb{R}^m) \) be the weak solution of \( L_\varepsilon(u_\varepsilon) = \text{div}(F) + F \) in \( \Omega \). Then for any \( B \subset 2B \subset \Omega \), we have \( \|\nabla u_\varepsilon\| \in L^\infty(B) \) and the uniform estimate

\[
\|\nabla u_\varepsilon\|_{L^\infty(B)} \leq \frac{C}{r} \left( \int_{2B} |u_\varepsilon|^2 \right)^{\frac{1}{2}} + C \left( \|f\|_{L^\infty(2B)} + r^\sigma \|f\|_{C^0(\Omega; \mathbb{R}^{nd})} + r \left( \int_{2B} |F|^p \right)^{\frac{1}{p}} \right), \tag{4.3}
\]

where \( C \) depends only on \( \mu, \tau, \kappa, p, m, d \) and \( \sigma \).

Proof. By rescaling we may assume \( r = 1 \), and \( \varphi \in C_0^\infty(2B) \) is a cut-off function such that \( \varphi = 1 \) on \( 5/4B \), \( \varphi = 0 \) outside \( 3/2B \), and \( |\nabla \varphi| \leq C \). Then we have

\[
-\text{div}[A_\varepsilon^{\alpha \beta} \nabla (\varphi u_\varepsilon^\beta)] = \text{div}(f^\alpha \varphi) + F^\alpha \varphi - f^\alpha \nabla \varphi - A_\varepsilon^{\alpha \beta} \nabla \varphi u_\varepsilon^\alpha \nabla \varphi - \text{div}(A_\varepsilon^{\alpha \beta} \nabla \varphi u_\varepsilon^\alpha). \]

It follows from the fundamental solution that for any \( x \in B \),

\[
u_\varepsilon(x) = -\int_{2B} \nabla_y \Gamma_\varepsilon(x, y) f(y) \varphi(y) dy + \int_{2B} \Gamma_\varepsilon(x, y) [F \varphi - f \nabla \varphi - A_\varepsilon \nabla u_\varepsilon \nabla \varphi] dy + \int_{2B} \nabla_y \Gamma_\varepsilon(x, y) A_\varepsilon \nabla \varphi u_\varepsilon dy, \tag{4.4}
\]

where we use integration by part in first and third term in right hand side of the above equality. Then differentiating both sides of \( [4.4] \) with respect to \( x \) gives

\[
\nabla u_\varepsilon(x) = -\int_{2B} \nabla_x \nabla_y \Gamma_\varepsilon(x, y) \left[ f(y) \varphi(y) - f(x) \varphi(x) \right] dy - f(x) \varphi(x) \int_{\partial(2B)} \nabla_x \Gamma_\varepsilon(x, y)n(y) dS(y)
\]

\[
+ \int_{2B} \nabla_x \Gamma_\varepsilon(x, y) [F \varphi - f \nabla \varphi - A_\varepsilon \nabla u_\varepsilon \nabla \varphi] dy + \int_{2B} \nabla_x \varphi u_\varepsilon dy,
\]

where \( dS \) denotes the surface measure of \( \partial \Omega \), and \( n \) is the outward unit normal to \( \partial(2B) \). We refer the reader to \cite{[22]} pp.55] for the skill used above.

Hence, in view of \eqref{4.2}, we obtain

\[
|\nabla u_\varepsilon(x)| \leq C \left\{ \int_{2B} \frac{|f(y)| |\varphi(y)| - f(x) \varphi(x)|}{|x - y|^d} dy + \sup_{x \in B} |f(x) \varphi(x)| \int_{|x - y| = 1} \frac{dS(y)}{|x - y|^d - 1} \right\}
\]

\[
+ \int_{2B} \frac{|F(y)|}{|x - y|^d - 1} dy + \int_{\left( 3/2B \right) \backslash \left( 5/4B \right)} \frac{|f(y)| + |\nabla u_\varepsilon(y)|}{|x - y|^d - 1} dy + \int_{\left( 3/2B \right) \backslash \left( 5/4B \right)} \frac{u_\varepsilon(y)}{|x - y|^d} dy, \]

where we use the observation of \( \nabla \varphi = 0 \) on \( 5/4B \) and \( \varphi = 0 \) outside \( 3/2B \) in last two terms. This leads to

\[
|\nabla u_\varepsilon(x)| \leq C \left\{ \|f\|_{C^0, \sigma(2B)} + \|f\|_{L^\infty(2B)} + \|F\|_{L^p(2B)} + \left( \int_{3/2B} |\nabla u_\varepsilon|^2 dy \right)^{\frac{1}{2}} + \left( \int_{2B} |u_\varepsilon|^2 dy \right)^{\frac{1}{2}} \right\}
\]

for any \( x \in B \). Then it follows from the Caccioppoli’s inequality that

\[
\|\nabla u_\varepsilon\|_{L^\infty(B)} \leq C \{ \|u_\varepsilon\|_{L^2(2B)} + \|F\|_{C^0, \sigma(2B)} + \|F\|_{L^p(2B)} \},
\]

where \( p > d \), and \( C \) depends on \( \mu, \tau, \kappa, p, d, m \) and \( \sigma \). By using the rescaling technique, \cite{[4.3]} can be easily derived.
Theorem 4.4. (Interior Lipschitz estimates). Suppose that $A \in \Lambda(\mu, \tau, \kappa)$, $V$ satisfies (1.2), (1.4), and $B, c$ satisfy (1.3). Let $p > d$ and $\sigma \in (0, 1)$. Assume that $u_\varepsilon \in H^1_{loc}(\Omega; \mathbb{R}^m)$ is a weak solution to $\mathcal{L}_\varepsilon(u_\varepsilon) = \text{div}(f) + F$ in $\Omega$, where $f \in C^{0, \sigma}(\Omega; \mathbb{R}^{md})$ and $F \in L^p(\Omega; \mathbb{R}^m)$. Then for any $B \subset 2B \subset \Omega$ with $0 < r \leq 1$, we have the uniform estimate
\[
\|\nabla u_\varepsilon\|_{L^\infty(B)} \leq C \left( \frac{1}{r} \left( \int_{\partial B} |u_\varepsilon|^2 \right)^{\frac{1}{2}} + C \left\{ \|f\|_{L^\infty(2B)} + r^\sigma |f|_{C^{0, \sigma}(2B)} + r \left( \int_{2B} |F|^p \right)^{\frac{1}{p}} \right\} \right),
\]
where $C$ depends only on $\mu, \tau, \kappa, \lambda, p, d, m$ and $\sigma$.

Proof. We only need to prove (4.5) in the case of $\varepsilon < \varepsilon_0$, where $\varepsilon_0$ will be given later. Since the estimate (4.5) immediately follows from the classical results when $\varepsilon \geq \varepsilon_0$. Consider the transformation $T(x, \varepsilon) = [T^{\alpha\beta}(x, \varepsilon)]$ as follows
\[
u_\varepsilon^\beta = T^{\alpha\beta}(x, \varepsilon) v_\delta^\gamma = \left[ \delta^{\alpha\gamma} + \varepsilon \chi_0^{\alpha\gamma}(x/\varepsilon) \right] v_\delta^\gamma.
\]
In view of (2.13), it is not hard to see $T(x, \varepsilon)$ is a diagonally dominant matrix whenever $\varepsilon < \varepsilon_1 = \varepsilon_1(\mu, \tau, \kappa, m, d)$. Hence we have the existence of $T^{-1}(x, \varepsilon)$,
\[
\frac{1}{2} \leq \|T(\cdot, \varepsilon)\|_{L^\infty(\Omega)} \leq \frac{3}{2} \quad \text{and} \quad 2/3 \leq \|T^{-1}(\cdot, \varepsilon)\|_{L^\infty(\Omega)} \leq 2 \quad \text{for} \quad \varepsilon \in (0, \varepsilon_1),
\]
where
\[
\|T(\cdot, \varepsilon)\|_{L^\infty(\Omega)} = \sup_{x \in \Omega} \|T(x, \varepsilon)\|_{\infty} \quad \text{and} \quad \|T(x, \varepsilon)\|_{\infty} = \max_{1 \leq \alpha, \beta \leq m} |T^{\alpha\beta}(x, \varepsilon)|.
\]
Moreover, in view of
\[
T^{-1}(x, \varepsilon) - T^{-1}(y, \varepsilon) = T^{-1}(x, \varepsilon) \left[ T(y, \varepsilon) - T(x, \varepsilon) \right] T^{-1}(y, \varepsilon),
\]
we have
\[
\|T^{-1}(x, \varepsilon) - T^{-1}(y, \varepsilon)\|_{\infty} \leq \|T^{-1}(x, \varepsilon)\|_{\infty} \|T^{-1}(y, \varepsilon)\|_{\infty} \|T(y, \varepsilon) - T(x, \varepsilon)\|_{\infty} \leq C|y - x|^{\sigma}
\]
for any $\sigma \in (0, 1]$ and $x, y \in B \subset \Omega$, where we use $|T(\cdot, \varepsilon)|_{C^{0, \sigma}(\Omega)} \leq C \varepsilon^{1-\sigma} |\chi_0|_{C^{0, \sigma}(Y)} \leq C(\mu, \tau, \kappa, m, d, \sigma)$ which follows from (2.12) and (2.13). Thus we obtain
\[
\|T^{-1}(\cdot, \varepsilon)\|_{C^{0, \sigma}(\Omega)} = \|T^{-1}(\cdot, \varepsilon)\|_{L^\infty(\Omega)} + \|T^{-1}(\cdot, \varepsilon)\|_{C^{0, \sigma}(\Omega)} \leq \max \{2, C(\mu, \tau, \kappa, \sigma, m, d)\}
\]
for $\varepsilon \in (0, \varepsilon_1)$.

Consider the new system
\[
-\text{div}(\tilde{A}_\varepsilon \nabla v_\varepsilon) = \text{div}(\tilde{f}) + \tilde{F} \quad \text{in} \quad \Omega,
\]
where $\tilde{A}_\varepsilon^{\alpha\gamma} = A_\varepsilon^{\alpha\beta} \left[ \delta^{\beta\gamma} + \varepsilon \chi_0^{\beta\gamma} \right]$, 
\[
\tilde{f}^{\alpha} = f^{\alpha} + \varepsilon v_\varepsilon^{\alpha\beta} \chi_0^{\beta\gamma} v_\varepsilon^{\gamma} \quad \text{and} \quad \tilde{F}^{\alpha} = F^{\alpha} + A_\varepsilon^{\alpha\beta} \nabla \chi_0^{\beta\gamma} v_\varepsilon^{\gamma} + V_\varepsilon^{\alpha\beta} \nabla v_\varepsilon^{\gamma} - B_\varepsilon^{\alpha\beta} \nabla u_\varepsilon^{\gamma} - (c_\varepsilon^{\alpha\beta} + \lambda \delta^{\alpha\beta}) u_\varepsilon^{\beta}.
\]
Obviously, there exists $\varepsilon_2 = \varepsilon_2(\mu, \tau, \kappa, d, m)$ such that $\tilde{A} \in \Lambda(\frac{\mu}{\varepsilon_2}, \tau, \kappa + 1)$ whenever $\varepsilon \leq \varepsilon_2$.

Let $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$ and $\varepsilon \in (0, \varepsilon_0]$. For any $r \in (0, 1]$, it follows from (4.3) that
\[
\|\nabla v_\varepsilon\|_{L^\infty(B)} \leq C \left\{ r^{-1} \|v_\varepsilon\|_{L^\infty(2B)} + \|\tilde{f}\|_{L^\infty(2B)} + r^\sigma |\tilde{f}|_{C^{0, \sigma}(2B)} + r \left( \int_{2B} |\tilde{F}|^p \right)^{\frac{1}{p}} \right\},
\]
where $\sigma' = \min\{\sigma, \tau\}$. For convenience, we denote
\[
\mathcal{R}(nB) = \frac{1}{r} \left( \int_{nB} |u_\varepsilon|^2 \right)^{\frac{1}{2}} + \|f\|_{L^\infty(nB)} + r \left( \int_{nB} |F|^p \right)^{\frac{1}{p}}.
\]
Hence, in view of (3.7), (3.8), (4.6) and (4.8), we obtain
\[
\|v_\varepsilon\|_{L^\infty(2B)} \leq \|T^{-1}(\cdot, \varepsilon)\|_{L^\infty(2B)} \|u_\varepsilon\|_{L^\infty(2B)} \leq C r R(4B),
\]
\[
[v_\varepsilon]_{C^{0,\sigma'}(2B)} \leq [T^{-1}(\cdot, \varepsilon)]_{C^{0,\sigma'}(2B)} \|u_\varepsilon\|_{L^\infty(2B)} + \|T^{-1}(\cdot, \varepsilon)\|_{L^\infty(2B)} [v_\varepsilon]_{C^{0,\sigma'}(2B)} \leq C \{r + r^{1-\sigma'}\} R(4B),
\]
\[
\left(\int_{2B} |\nabla v_\varepsilon|^p \, dx\right)^{\frac{1}{p}} \leq \|T^{-1}(\cdot, \varepsilon)\|_{L^\infty(2B)} \|u_\varepsilon\|_{L^\infty(2B)} + \|T^{-1}(\cdot, \varepsilon)\|_{L^\infty(2B)} \left(\int_{2B} |\nabla u_\varepsilon|^p \, dx\right)^{\frac{1}{p}} \leq C \{r + 1\} R(4B),
\]
(4.10)
where we use Theorem 3 to estimate the term of \(\left(\int_{2B} |\nabla u_\varepsilon|^p \, dx\right)^{\frac{1}{p}}\).

Note that since \(\sigma'\) we have
\[
\left|\int_{2B} F \, dx\right| \leq \left(\int_{2B} |F|^p \, dx\right)^{\frac{1}{p}} + C \left(\int_{2B} |\nabla v_\varepsilon|^p \, dx\right)^{\frac{1}{p}} + C \|u_\varepsilon\|_{L^\infty(2B)}
\]
and
\[
\left(\int_{2B} |F|^p \, dx\right)^{\frac{1}{p}} \leq \left(\int_{2B} |F|^p \, dx\right)^{\frac{1}{p}} + C \left(\int_{2B} |\nabla v_\varepsilon|^p \, dx\right)^{\frac{1}{p}} + C \|u_\varepsilon\|_{L^\infty(2B)}
\]
Combining (4.9), (4.10), (4.12) and (4.13), we have
\[
\|\nabla v_\varepsilon\|_{L^\infty(2B)} \leq C \left\{\|f\|_{L^\infty(2B)} + r^{\sigma'} [f]_{C^{0,\sigma'}(2B)} + r \left(\int_{2B} |F|^p \, dx\right)^{\frac{1}{p}} \right\} + C \{1 + r + r^{1+\sigma'} + r^2\} R(4B),
\]
where \(C\) depends on \(\mu, \tau, \kappa, \lambda, \sigma, p, m, d\). This, together with (2.12), (4.7) and (4.11), gives
\[
\|\nabla v_\varepsilon\|_{L^\infty(B_r(x_0))} \leq \|\nabla T(\cdot, \varepsilon)\|_{L^\infty(B_r(x_0))} \|v_\varepsilon\|_{L^\infty(B_r(x_0))} + \|T(\cdot, \varepsilon)\|_{L^\infty(B_r(x_0))} \|\nabla v_\varepsilon\|_{L^\infty(B_r(x_0))} \leq C \{R(4B) + r^{\sigma'} [f]_{C^{0,\sigma}(4B)}\}.
\]
Note that since \(\sigma' \leq \sigma\) we have
\[
r^{\sigma'} [f]_{C^{0,\sigma'}(4B)} \leq C r^{\sigma'} [f]_{C^{0,\sigma}(4B)}.
\]
For any \(B_r(x_0) \subset B_{2r}(x_0) \subset \Omega\), there exist \(\{B^*_r(x_i)\}_{i=1}^N\) and \(x_i \in B_r(x_0)\) such that \(B_r(x_0) \subset \bigcup_{i=1}^N B^*_r(x_i)\). It is clear to see \(B_r(x_i) \subset B_{2r}(x_0)\) for any \(x_i \in B_r(x_0)\). Hence we have
\[
\|\nabla u_\varepsilon\|_{L^\infty(B_r(x_0))} \leq \max_{1 \leq i \leq N} \left\{\|\nabla u_\varepsilon\|_{L^\infty(B^*_r(x_i))}\right\} \leq C \{R(B_r(x_i)) + r^{\sigma'} [f]_{C^{0,\sigma}(B_r(x_i))}\}
\]
\[
\leq C \left\{r^{-1} \left(\int_{B_{2r}(x_0)} |u_\varepsilon|^2 \, dx\right)^{\frac{1}{2}} + \|f\|_{L^\infty(B_{2r}(x_0))} + r^{\sigma'} [f]_{C^{0,\sigma}(B_{2r}(x_0))} + r \left(\int_{B_{2r}(x_0)} |F|^p \, dx\right)^{\frac{1}{p}} \right\},
\]
and we complete the proof. \(\Box\)

To prove the global Lipschitz estimates, we study some properties of the Dirichlet correctors \(\Phi_{\varepsilon,k}\), \(0 \leq k \leq d\), which actually play a similar role as \(\chi_k\) in the interior Lipschitz estimates.
Lemma 4.5. Suppose $A \in \Lambda(\mu, \tau, \kappa)$. Let $g \in C^{0,1}(\partial \Omega; \mathbb{R}^m)$, and $u_\varepsilon \in H^1(\Omega; \mathbb{R}^m)$ be the solution of $L_\varepsilon(u_\varepsilon) = 0$ in $\Omega$ and $u_\varepsilon = g$ on $\partial \Omega$. Then for any $Q \subset \partial \Omega$ and $0 \leq r < \text{diam}(\Omega)$,

$$\left( \int_{B(Q,r) \cap \Omega} |\nabla u_\varepsilon|^2 \, dx \right)^{1/2} \leq C \left\{ \|\nabla g\|_{L^\infty(\partial \Omega)} + \varepsilon^{-1}\|g\|_{L^\infty(\partial \Omega)} \right\},$$

(4.14)

where $C$ depends only on $\mu, \tau, \kappa, d, m$ and $\Omega$.

Lemma 4.6. Let $A \in \Lambda(\mu, \tau, \kappa)$. Then we have

$$\|\Phi_{\varepsilon,k}^0 - P_k^0\|_{L^\infty(\Omega)} \leq C\varepsilon \quad \text{and} \quad \|\nabla \Phi_{\varepsilon,k}\|_{L^\infty(\Omega)} \leq C$$

(4.15)

for $1 \leq k \leq d$, where $C$ depends only on $\mu, \tau, \kappa, d, m, \eta$ and $\Omega$.

Remark 4.7. Lemmas 4.5 and 4.6 were proved in [38], as well as in [3]. Here we omit the proof.

Lemma 4.8. Assume that $A \in \Lambda(\mu, \tau, \kappa)$, and $V$ satisfies (1.2), (1.4). Then for any $\sigma \in (0, 1]$, we have

$$\|\Phi_{\varepsilon,0} - I\|_{L^\infty(\Omega)} \leq C\varepsilon, \quad \|\Phi_{\varepsilon,0} - I\|_{C^{0,\sigma}(\Omega)} \leq C\varepsilon^{1-\sigma} \quad \text{and} \quad \|\nabla \Phi_{\varepsilon,0}\|_{L^\infty(\Omega)} \leq C,$$

(4.16)

where $C$ depends only on $\mu, \tau, \kappa, d, m, \eta$ and $\Omega$.

Proof. Let $r_0 = \text{diam}(\Omega)$. If $\varepsilon \geq cr_0$, (4.16) follows from the classical boundary Lipschitz estimates for elliptic system in divergence form with the Hölder continuous coefficients. If $0 < \varepsilon < cr_0$, consider

$$u_\varepsilon(x) = \Phi_{\varepsilon,0}(x) - I - \varepsilon \chi_0(x/\varepsilon).$$

Then $L_\varepsilon(u_\varepsilon) = L_\varepsilon(\Phi_{\varepsilon,0}) - L_\varepsilon[\varepsilon \chi_0(x/\varepsilon)] = 0$ in $\Omega$, and $u_\varepsilon = -\varepsilon \chi_0(x/\varepsilon)$ on $\partial \Omega$. Hence, it follows from the Agmon-Miranda maximum principle (see [3] Theorem 3 or [38] Remark 3.4.4]) that

$$\sup_{x \in \Omega} |u_\varepsilon(x)| \leq C \sup_{x \in \partial \Omega} |u_\varepsilon(x)| \leq C\varepsilon \sup_{x \in \partial \Omega} \|\chi_0\|_{L^\infty(\mathbb{R}^d)} \leq C\varepsilon,$$

which implies $\|\Phi_{\varepsilon,0} - I\|_{L^\infty(\Omega)} \leq C\varepsilon$. Additionally, for any $\sigma \in (0, 1)$, in view of Theorem 1.2, we have $\|\Phi_{\varepsilon,0} - I\|_{C^{0,\sigma}(\Omega)} \leq C\varepsilon^{1-\sigma}$. Note that $L_\varepsilon$ is the special case of $\mathcal{L}_\varepsilon$, and $C$ depends only on $\mu, \tau, \kappa, d, m, \sigma$ and $\Omega$.

Moreover, it follows from Lemma 4.5 that

$$\left( \int_{D(Q,r)} |\nabla u_\varepsilon|^2 \, dx \right)^{1/2} \leq C,$$

which implies

$$\left( \int_{D(Q,r)} |\nabla \Phi_{\varepsilon,0}|^2 \, dx \right)^{1/2} \leq C$$

(4.17)

for any $Q \subset \partial \Omega$, and $\varepsilon \leq r < r_0$. By the interior Lipschitz estimate, we have

$$\sup_{\{d_x \geq \varepsilon\} \cap \Omega} |\nabla u_\varepsilon(x)| \leq \frac{C}{\varepsilon} \left( \int_{\Omega} |u_\varepsilon|^2 \, dy \right)^{1/2} \leq C,$$

which gives

$$\sup_{\{d_x \geq \varepsilon\} \cap \Omega} |\nabla \Phi_{\varepsilon,0}(x)| \leq C.$$  

(4.18)

In the case of $\{d_x < \varepsilon\} \cap \Omega$, we apply the blow-up argument. Let

$$v(x) = \frac{1}{\varepsilon} \Phi_{\varepsilon,0}(\varepsilon x) - \frac{1}{\varepsilon} I,$$
then we have
\[
\begin{cases}
L_1(v) = \text{div}(V) & \text{in } \Omega_\varepsilon, \\
v = 0 & \text{on } \partial\Omega_\varepsilon,
\end{cases}
\]
where \(\Omega_\varepsilon = \{x \in \mathbb{R}^d : \varepsilon x \in \Omega\}\). Note that although the character of the boundary varies, \(\Omega_\varepsilon\) is still a bounded \(C^{1,\eta}\) domain. The boundary functions of \(\Omega_\varepsilon\) are denoted by \(\psi_{i,\varepsilon}(x) = \psi_i(\varepsilon x), i = 1, \cdots, n_0\) (recall Remark 2.22), and we fortunately have \(\|\psi_{i,\varepsilon}\|_{C^{1,\eta}(\mathbb{R}^{d-1})} \leq \varepsilon^{1+\eta}\|\psi_i\|_{C^{1,\eta}(\mathbb{R}^{d-1})} \leq \varepsilon^{1+\eta}M_0\). Hence, it follows from the (boundary) Lipschitz estimate (2.25) that
\[
\sup_{B(0,1) \cap \Omega_\varepsilon} |\nabla v| \leq C \left\{ \left( \int_{B(0,2) \cap \Omega_\varepsilon} |\nabla v|^2 \, dy \right)^{\frac{1}{2}} + \|V\|_{C_0^0(\mathbb{R}^d)} \right\}.
\]
This implies
\[
\sup_{x \in D(0,\varepsilon)} |\nabla \Phi_{\varepsilon,0}(x)| \leq C \left\{ \left( \int_{D(0,2\varepsilon)} |\nabla \Phi_{\varepsilon,0}|^2 \, dy \right)^{\frac{1}{2}} + \|V\|_{C_0^0(\mathbb{R}^d)} \right\} \leq C. \tag{4.19}
\]
Note that we choose \(\varepsilon = 2\varepsilon\) in (4.17) to give the last inequality. Combining (4.18) and (4.19), we have \(\|\nabla \Phi_{\varepsilon,0}\|_{L^\infty(\Omega)} \leq C\), and this also implies the second estimate of (4.16) for \(\sigma = 1\). We thus complete the proof. \(\square\)

**Lemma 4.9.** Assume the same conditions as in Lemma 4.8. Then we have
\[
\|\nabla \Phi_{\varepsilon,0}\|_{C_0^0(\mathbb{R}^d)} \leq C \max\{\varepsilon^{-\tau}, 1\}. \tag{4.20}
\]
Furthermore, \(\Phi_{\varepsilon,0}^{-1}\) exists and satisfies the following estimates:
\[
2/3 \leq \|\Phi_{\varepsilon,0}^{-1}\|_{L^\infty(\Omega)} \leq 2, \quad \|\nabla (\Phi_{\varepsilon,0}^{-1})\|_{L^\infty(\Omega)} \leq C, \tag{4.21}
\]
whenever \(\varepsilon \leq \varepsilon_0\), where \(\varepsilon_0 = \varepsilon_0(\mu, \tau, \kappa, d, m, \eta)\) is sufficiently small, and \(C\) depends only on \(\mu, \tau, \kappa, d, m, \eta\) and \(\Omega\).

**Proof.** Let \(\tilde{\Phi}_{\varepsilon,0} = \Phi_{\varepsilon,0} - I\), then \(L_{\varepsilon}(\tilde{\Phi}_{\varepsilon,0}) = \text{div}(V_\varepsilon)\) in \(\Omega\), and \(\tilde{\Phi}_{\varepsilon,0} = 0\) on \(\partial\Omega\). We first prove (4.20) in the case of \(\varepsilon < 1\). Set \(U(\varepsilon) = \Omega \cap B(P, \varepsilon)\) for any \(P \in \Omega\). By translation we may assume \(P = 0\). In view of the Schauder estimate (2.26) and Lemma 4.8 we have
\[
\left[ \nabla \Phi_{\varepsilon,0} \right]_{C_0^0(U(\varepsilon))} \leq C \varepsilon^{-\tau} \left\{ \left( \int_{U(2\varepsilon)} |\nabla \Phi_{\varepsilon,0}|^2 \right)^{\frac{1}{2}} + \|V_\varepsilon\|_{L^\infty(U(2\varepsilon))} + \varepsilon^\tau [V_\varepsilon]_{C_0^0(U(2\varepsilon))} \right\}
\leq C \varepsilon^{-\tau} \left\{ \|\nabla \Phi_{\varepsilon,0}\|_{L^\infty(\Omega)} + \|V\|_{C_0^0(\mathbb{R}^d)} \right\} \leq C \varepsilon^{-\tau},
\]
where \(C\) depends only on \(\mu, \tau, \kappa, d, \eta\) and \(M_0\). Note that \(\left[ \nabla \Phi_{\varepsilon,0} \right]_{C_0^0(U(\varepsilon))} = \left[ \nabla \Phi_{\varepsilon,0} \right]_{C_0^0(U(\varepsilon))}\). Thus by a covering argument (see [22] pp.98), we obtain \(\|\nabla \Phi_{\varepsilon,0}\|_{C_0^0(\mathbb{R}^d)} \leq C \varepsilon^{-\tau}\). The case of \(\varepsilon > 1\) is trivial, since we can derive (4.20) by using the Schauder estimates (2.19) directly.

Next we prove (4.21). It follows from (4.16) that \(\|\Phi_{\varepsilon,0}\|_{L^\infty(\Omega)} \leq C \varepsilon\). Since \(\Phi_{\varepsilon,0} = I + \tilde{\Phi}_{\varepsilon,0}\), we know that there exists \(\Phi_{\varepsilon,0}^{-1} \in L^\infty(\Omega)\) such that
\[
1/2 \leq \|\Phi_{\varepsilon,0}\|_{L^\infty(\Omega)} \leq 3/2 \quad \text{and} \quad 2/3 \leq \|\Phi_{\varepsilon,0}^{-1}\|_{L^\infty(\Omega)} \leq 2
\]
whenever \(\varepsilon \leq \varepsilon_0(\mu, \tau, \kappa, d, m, \Omega)\), and \(\varepsilon_0\) is sufficiently small.

Due to
\[
\Phi_{\varepsilon,0}^{-1}(x) - \Phi_{\varepsilon,0}^{-1}(y) = \Phi_{\varepsilon,0}^{-1}(x) \left[ \Phi_{\varepsilon,0}(y) - \Phi_{\varepsilon,0}(x) \right] \Phi_{\varepsilon,0}^{-1}(y),
\]
we have
\[
|\Phi_{\varepsilon,0}^{-1}(x) - \Phi_{\varepsilon,0}^{-1}(y)| \leq \|\Phi_{\varepsilon,0}^{-1}\|_{L^\infty(\Omega)} \|\Phi_{\varepsilon,0}^{-1}\|_{L^\infty(\Omega)} \|\nabla \Phi_{\varepsilon,0}\|_{L^\infty(\Omega)} |x - y| \leq C|x - y|
\]
for \(x, y \in \Omega\), and this implies \(\|\nabla \Phi_{\varepsilon,0}^{-1}\|_{L^\infty(\Omega)} \leq C\). The proof is complete. \(\square\)
Lemma 4.10. (A nonuniform estimate). Suppose that $A \in \Lambda(\mu, \tau, \kappa)$, $V$ satisfies (1.2), (1.4), and $B, c$ satisfy (1.3). Let $p > d$ and $\sigma \in (0, \tau]$. Assume $f \in C^{0,\sigma}(\Omega; \mathbb{R}^m)$ and $F \in L^p(\Omega; \mathbb{R}^m)$, then the weak solution to $L_\varepsilon(u_\varepsilon) = \text{div}(f) + F$ in $\Omega$ and $u_\varepsilon = 0$ on $\partial\Omega$ satisfies the estimate

$$
\|\nabla u_\varepsilon\|_{C^{0,\frac{\sigma}{2}}(\Omega)} \leq C \max \{ \varepsilon^{\frac{\sigma}{2}}, 1 \} \{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \},
$$

where $C$ depends only on $\mu, \tau, \kappa, \lambda, p, d, m, \eta$ and $\Omega$.

Proof. If $\varepsilon \geq 1$, (4.22) follows directly from the Schauder estimate (2.29) and the Lipschitz estimate (2.28).

In the case of $0 < \varepsilon < 1$, the main idea is based upon the following interpolation inequality

$$
\|\nabla u_\varepsilon\|_{C^{0,\frac{\sigma}{2}}(\Omega)} \leq 2 \|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \|\nabla u_\varepsilon\|_{C^{0,\sigma}(\Omega)}.
$$

Set $U(\varepsilon) = \Omega \cap B(P, \varepsilon)$ for any $P \in \overline{\Omega}$, and by translation we may assume $P = 0$. We first study $\|\nabla u_\varepsilon\|_{L^\infty(\Omega)}$ through the uniform Hölder estimates. To do so, let $v_\varepsilon = u_\varepsilon - [I + \varepsilon \chi_0(x/\varepsilon)]u_\varepsilon(0)$. Hence we have

$$
L_\varepsilon(v_\varepsilon) = \text{div}(f) + F + [\text{div}(v_\varepsilon \chi_0) - B_\varepsilon \nabla y \chi_0 - (c_\varepsilon + \lambda I)(I + \varepsilon \chi_0)]u_\varepsilon(0) \quad \text{in} \quad \Omega,
$$

where $y = x/\varepsilon$. If $0 \in \partial\Omega$, we have $v_\varepsilon = 0$ on $\partial\Omega$. From the Lipschitz estimate (2.31) on $\varepsilon$ scale, we obtain

$$
\|\nabla v_\varepsilon\|_{L^\infty(U(\varepsilon))} \leq C \left\{ \varepsilon \left( \int_{U(2\varepsilon)} |v_\varepsilon|^2 \right)^{\frac{1}{2}} + \varepsilon \left[ \|f\|_{C^{0,\sigma}(U(2\varepsilon))} + \varepsilon \left( \int_{U(2\varepsilon)} |F|^p \right) \right] + \varepsilon \right\}
$$

$$
\leq \frac{C}{\varepsilon} \left( \int_{U(2\varepsilon)} |v_\varepsilon - u_\varepsilon(0)|^2 \right)^{\frac{1}{2}} + C \|u_\varepsilon\|_{L^\infty(\Omega)} + C \|f\|_{C^{0,\sigma}(\Omega)} + C \varepsilon \|F\|_{L^p(U(\varepsilon))}
$$

$$
\leq \frac{C}{\varepsilon^{1-\sigma}} \|u_\varepsilon\|_{C^{0,\sigma}(\Omega)} + C \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \right\},
$$

where we use (3.14) in the third inequality. This implies

$$
\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \leq \|\nabla v_\varepsilon\|_{L^\infty(\Omega)} + \|\nabla \chi_0\|_{L^\infty(y)} \|u_\varepsilon\|_{L^\infty(\Omega)} \leq C \varepsilon^{-1} \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \right\}.
$$

Next, it directly follows from the Schauder estimate (2.32) that

$$
\|\nabla u_\varepsilon\|_{C^{0,\frac{\sigma}{2}}(U(\varepsilon))} \leq C \left\{ \frac{1}{\varepsilon^{1+\sigma}} \left( \int_{U(2\varepsilon)} |u_\varepsilon|^2 \right)^{\frac{1}{2}} + \varepsilon \left[ \|f\|_{C^{0,\sigma}(U(2\varepsilon))} + \varepsilon^{-\sigma} \|F\|_{L^\infty(U(2\varepsilon))} \right] \right\}
$$

$$
\leq \frac{C}{\varepsilon^{1+\sigma}} \left\{ \|u_\varepsilon\|_{L^\infty(\Omega)} + \|f\|_{C^{0,\sigma}(\Omega)} \right\} + C \left( \int_{\Omega} |F|^p \right)^{\frac{1}{p}}
$$

$$
\leq \frac{C}{\varepsilon^{1+\sigma}} \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \right\},
$$

where $C$ depends on $\mu, \tau, \kappa, \lambda, p, \sigma, m, d, \eta$ and $\Omega$. By a covering argument (see [22] pp.98), we have

$$
\|\nabla u_\varepsilon\|_{C^{0,\sigma}(\Omega)} \leq C \varepsilon^{-1-\sigma} \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \right\}.
$$

Finally, the estimate (4.22) follows from (4.23), (4.24) and (4.25). We complete the proof. □

Proof of Theorem 1.3. In the case of $g = 0$, we only need to consider the following transformation

$$
u_\varepsilon^g(x) = \Phi_\varepsilon^g y_\varepsilon^g(x)$$

(4.26)
for $\varepsilon < \varepsilon_*$, where $\varepsilon_* = \min\{\varepsilon_0, \varepsilon_1, \varepsilon_2\}$, and $\varepsilon_0$ is given in Lemma 4.9 and $\varepsilon_1, \varepsilon_2$ can be chosen later. Since it is clear to see that the estimate (1.10) immediately follows from the Lipschitz estimate (2.28) for $\varepsilon \geq \varepsilon_*$. Then the Dirichlet problem (1.7) can be transformed into

$$
\begin{cases}
L_\varepsilon(v_\varepsilon) = \text{div}(\tilde{f}) + \tilde{F} & \text{in } \Omega, \\
v_\varepsilon = 0 & \text{on } \partial \Omega,
\end{cases}
$$

where

$$
\tilde{f}^\alpha = f^\alpha + A_{\varepsilon}^{\alpha\beta}(\Phi_{\varepsilon,0}^{\beta\gamma} - \delta^{\beta\gamma})\nabla v_\varepsilon^\gamma + V_{\varepsilon}^{\alpha\beta}(\Phi_{\varepsilon,0}^{\beta\gamma} - \delta^{\beta\gamma})v_\varepsilon^\gamma,
$$

$$
\tilde{F}^\alpha = F^\alpha + A_{\varepsilon}^{\alpha\beta}\Phi_{\varepsilon,0}^{\beta\gamma}\nabla v_\varepsilon^\gamma + V_{\varepsilon}^{\alpha\beta}\nabla v_\varepsilon^\gamma - B_{\varepsilon}^{\alpha\beta}\nabla u_\varepsilon^\beta - (c_\varepsilon^\alpha + \lambda\delta^{\alpha\beta})u_\varepsilon^\beta.
$$

It follows from Theorem 1.1 and Corollary 3.8 that

$$
\max \left\{ \|\nabla u_\varepsilon\|_{L^p(\Omega)}, \|u_\varepsilon\|_{C^{0,\sigma}(\Omega)} \right\} \leq C \left\{ \|f\|_{L^\infty(\Omega)} + \|F\|_{L^p(\Omega)} \right\},
$$

This, together with Lemma 4.9 gives

$$
\max \left\{ \|\nabla v_\varepsilon\|_{L^p(\Omega)}, \|v_\varepsilon\|_{C^{0,\sigma}(\Omega)} \right\} \leq C \left\{ \|f\|_{L^\infty(\Omega)} + \|F\|_{L^p(\Omega)} \right\},
$$

where $\sigma' = 1 - d/p$, and $C$ depends on $\mu, \tau, \kappa, \lambda, \sigma, p, d, m$ and $\Omega$. Here we use $v_\varepsilon = \Phi_{\varepsilon,0}^{-1}u_\varepsilon$, and $\nabla v_\varepsilon = \nabla(\Phi_{\varepsilon,0}^{-1})u_\varepsilon + \Phi_{\varepsilon,0}^{-1}\nabla u_\varepsilon$. However we need to rewrite $\nabla v_\varepsilon = -\nabla\Phi_{\varepsilon,0}v_\varepsilon + (I - \Phi_{\varepsilon,0})\nabla v_\varepsilon + \nabla u_\varepsilon$ to handle the Hölder norm of $\nabla v_\varepsilon$. Set $\nu = \min\{\tau, \sigma, \sigma'\}/4$ and $\nu' = \max\{\tau, 1 - 2\nu\}$. Note that $0 < \nu' < 1$, and we obtain

$$
\begin{align*}
[\nabla v_\varepsilon]_{C^{0,\sigma}(\Omega)} & \leq \left[ \nabla\Phi_{\varepsilon,0} \right]_{C^{0,\nu}(\Omega)} \|v_\varepsilon\|_{L^\infty(\Omega)} + \|\nabla\Phi_{\varepsilon,0}\|_{L^\infty(\Omega)} \|v_\varepsilon\|_{C^{0,\nu}(\Omega)} + \left[ I - \Phi_{\varepsilon,0} \right]_{C^{0,\nu}(\Omega)} \|v_\varepsilon\|_{L^\infty(\Omega)} \\
& \quad + \|I - \Phi_{\varepsilon,0}\|_{L^\infty(\Omega)} \|\nabla v_\varepsilon\|_{C^{0,\nu}(\Omega)} \\
& \leq C(\varepsilon^{-\tau} + \varepsilon^{2\nu-1}) \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \right\} + C\varepsilon^{1-\nu} \|\nabla v_\varepsilon\|_{L^\infty(\Omega)} + C\varepsilon \|\nabla v_\varepsilon\|_{C^{0,\nu}(\Omega)}
\end{align*}
$$

where we apply (4.16), (4.20), (4.22) and (4.28) to the second inequality. This implies

$$
[\nabla v_\varepsilon]_{C^{0,\sigma}(\Omega)} \leq C\varepsilon^{-\nu'} \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \right\} + C\|\nabla v_\varepsilon\|_{L^\infty(\Omega)}
$$

whenever $\varepsilon < \varepsilon_1$, where $\varepsilon_1 = \min\{1/(2C), 1\}$. Hence, we have

$$
\begin{align*}
\|\tilde{f}\|_{C^{0,\nu}(\Omega)} & \leq \|f\|_{C^{0,\sigma}(\Omega)} + C\varepsilon \|\nabla v_\varepsilon\|_{L^\infty(\Omega)} + \|A_\varepsilon(\Phi_{\varepsilon,0} - I)\nabla v_\varepsilon\|_{C^{0,\nu}(\Omega)} + \|V_\varepsilon(\Phi_{\varepsilon,0} - I)v_\varepsilon\|_{C^{0,\nu}(\Omega)} \\
& \leq C\varepsilon^{1-\nu'} \|\nabla v_\varepsilon\|_{L^\infty(\Omega)} + C \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \right\},
\end{align*}
$$

where we use (1.4), (4.10), (4.20), (4.28) and (4.29) in the second inequality. In view of (4.28), we also have

$$
\|\tilde{F}\|_{L^p(\Omega)} \leq \|F\|_{L^p(\Omega)} + C \left\{ \|\nabla v_\varepsilon\|_{L^p(\Omega)} + \|\nabla u_\varepsilon\|_{L^p(\Omega)} + \|u_\varepsilon\|_{L^p(\Omega)} \right\} \leq C \left\{ \|f\|_{L^\infty(\Omega)} + \|F\|_{L^p(\Omega)} \right\}.
$$

We now apply Lemma 4.1 to (4.27) and obtain

$$
\|\nabla v_\varepsilon\|_{L^\infty(\Omega)} \leq C \left\{ \|\tilde{f}\|_{C^{0,\nu}(\Omega)} + \|\tilde{F}\|_{L^p(\Omega)} \right\} \\
\leq C\varepsilon^{1-\nu'} \|\nabla v_\varepsilon\|_{L^\infty(\Omega)} + C \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \right\},
$$

which gives $\|\nabla v_\varepsilon\|_{L^\infty(\Omega)} \leq C \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \right\}$, whenever $\varepsilon < \varepsilon_2 = \min\{1/(2C)^{1-\nu'}, 1\}$. So we have

$$
\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \leq \|\nabla(\Phi_{\varepsilon,0}v_\varepsilon)\|_{L^\infty(\Omega)} \leq C \left\{ \|f\|_{C^{0,\sigma}(\Omega)} + \|F\|_{L^p(\Omega)} \right\},
$$

where $C$ depends only on $\mu, \tau, \kappa, \lambda, p, \sigma, d, m, M_0, \eta$ and $|\Omega|$. 

In the case of \( g \neq 0 \), consider the homogeneous system \( \mathcal{L}_\varepsilon(u_\varepsilon) = 0 \) in \( \Omega \) and \( u_\varepsilon = g \) on \( \Omega \), where \( g \in C^{1,\sigma}(\partial \Omega; \mathbb{R}^m) \) with \( \sigma \in (0, \eta] \). Let \( h_\varepsilon \) be the extension function of \( g \), satisfying

\[
- \text{div}(A_\varepsilon \nabla h_\varepsilon) = 0 \quad \text{in} \ \Omega \quad \text{and} \quad h_\varepsilon = g \quad \text{on} \ \partial \Omega.
\]

It follows from Lemma 4.11 that \( \|\nabla h_\varepsilon\|_{L^\infty(\Omega)} \leq C\|g\|_{C^{1,\sigma}(\partial \Omega)} \), where \( C \) depends only on \( \mu, \tau, \kappa, \sigma, d, m, \eta \) and \( \Omega \). Let \( \varrho = \min\{\tau, \sigma\} \). By the argument applied to Lemma 4.10, we obtain \( \|\nabla h_\varepsilon\|_{C^{0,\varrho}(\Omega)} \leq C\epsilon^{-1-\varrho}\|g\|_{C^{1,\varrho}(\partial \Omega)} \).

Indeed, due to (2.26) we have

\[
[\nabla h_\varepsilon]_{C^{0,\varrho}(B(P,\varepsilon))} \leq C\epsilon^{-\varrho}(\int_{B(P,2\varepsilon)}|\nabla h_\varepsilon|^2)^{\frac{1}{2}} \leq C\epsilon^{-\varrho}\|\nabla h_\varepsilon\|_{L^\infty(\Omega)} \leq C\epsilon^{-\varrho}\|g\|_{C^{1,\varrho}(\partial \Omega)}
\]

for any \( B(P,2\varepsilon) \subset \Omega \), while for the boundary estimates, it follows from the (boundary) Schauder estimate (2.27) that

\[
[\nabla h_\varepsilon]_{C^{0,\varrho}(D(\varepsilon))} \leq \frac{C}{\varepsilon^{\varrho}} \left( C\epsilon^{-\varrho}\|\nabla h_\varepsilon\|_{L^\infty(D(2\varepsilon))} + \|g\|_{C^{1,\varrho}(\partial \Omega)} \right) \leq C\epsilon^{-1-\varrho}\|g\|_{C^{1,\varrho}(\partial \Omega)}
\]

for any \( P \in \partial \Omega \), where \( C \) depends on \( \mu, \tau, \kappa, \varrho, d, m, M_0, \eta \), and \( |\Omega| \). Thus we have

\[
[\nabla h_\varepsilon]_{C^{0,\frac{\varrho}{2}}(\Omega)} \leq 2\|\nabla h_\varepsilon\|_{L^\infty(\Omega)} \leq C\epsilon^{-\frac{1+\varrho}{2}}\|g\|_{C^{1,\varrho}(\partial \Omega)}.
\] (4.31)

Set \( w_\varepsilon^\beta(x) = u_\varepsilon^\beta(x) - \Phi^\beta_\varepsilon(x)h_\varepsilon(x) \), we obtain

\[
\begin{aligned}
\mathcal{L}_\varepsilon(w_\varepsilon) &= \text{div}(\tilde{f}) + \tilde{F}, & \text{in} & \ \Omega, \\
\quad w_\varepsilon &= 0, & \text{on} & \ \partial \Omega,
\end{aligned}
\]

where

\[
\tilde{f}^\alpha = A^\alpha_\varepsilon(\Phi^\beta_\varepsilon - \delta^\beta)\nabla h_\varepsilon + V^\alpha_\varepsilon(\Phi^\gamma_\varepsilon - \delta^\gamma)h_\varepsilon, \\
\tilde{F}^\alpha = A^\alpha_\varepsilon(\delta^\beta)\nabla h_\varepsilon + V^\alpha_\varepsilon \nabla h_\varepsilon - B^\alpha_\varepsilon \nabla(\Phi^\gamma_\varepsilon) - (c^\alpha_\varepsilon + \lambda \delta^\beta)\Phi^\beta_\varepsilon h_\varepsilon.
\]

Now, let \( \nu = \varrho/2 \). In view of (4.16) and (4.31), we have

\[
\|\tilde{f}\|_{C^{0,\nu}(\Omega)} \leq C\|g\|_{C^{1,\varrho}(\partial \Omega)} \quad \text{and} \quad \|\tilde{F}\|_{L^p(\Omega)} \leq C\|g\|_{C^{1,\varrho}(\partial \Omega)}.
\]

Note that \( \|h_\varepsilon\|_{L^\infty(\Omega)} \leq C\|g\|_{L^\infty(\partial \Omega)} \), where \( C \) depends only on \( \mu, \tau, \kappa, \sigma, d, m \) and \( \Omega \). Hence, recalling (4.30), we have

\[
\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \leq \|\nabla w_\varepsilon\|_{L^\infty(\Omega)} + \|\nabla(\Phi^\gamma_\varepsilon)h_\varepsilon\|_{L^\infty(\Omega)} \leq C\{\|\tilde{f}\|_{C^{0,\nu}(\Omega)} + \|\tilde{F}\|_{L^p(\Omega)} + \|g\|_{C^{1,\varrho}(\partial \Omega)}\} \leq C\|g\|_{C^{1,\varrho}(\partial \Omega)}.
\] (4.32)

Finally, (1.10) follows from (4.30) and (4.32) by writing \( u_\varepsilon = u_{\varepsilon,1} + u_{\varepsilon,2} \), where \( u_{\varepsilon,1}, u_{\varepsilon,2} \) respectively satisfy the homogeneous and non-homogeneous systems (see 3.15). The proof is complete. \( \square \)

**Lemma 4.11.** Suppose that the coefficients of \( \mathcal{L}_\varepsilon \) satisfy the same conditions as in Theorem 1.4. Then \( \mathcal{G}_\varepsilon(x,y) \) has the following estimates:

\[
|\nabla_x \nabla_y \mathcal{G}_\varepsilon(x,y)| \leq \frac{C}{|x-y|^{d}},
\] (4.33)

\[
|\nabla_x \mathcal{G}_\varepsilon(x,y)| \leq \frac{C}{|x-y|^{d-1}} \min\left\{ 1, \frac{d_y}{|x-y|} \right\} \quad \text{and} \quad |\nabla_y \mathcal{G}_\varepsilon(x,y)| \leq \frac{C}{|x-y|^{d-1}} \min\left\{ 1, \frac{d_x}{|x-y|} \right\}
\] (4.34)

for any \( x, y \in \Omega \) and \( x \neq y \), where \( C \) depends only on \( \mu, \tau, \kappa, \lambda, d, m, \eta \) and \( \Omega \).
Proof. For any \(x, y \in \Omega\), let \(r = |x - y|\). Due to

\[
\mathcal{L}_\varepsilon^*[G_\varepsilon(\cdot, y)] = 0 \quad \text{in } \Omega \setminus B(y, \rho)
\]

for any \(\rho > 0\), it follows from (1.10) and (3.32) that

\[
|\nabla_x G_\varepsilon(x, y)| \leq \|\nabla G_\varepsilon(\cdot, y)\|_{L^\infty(\Omega_\varepsilon^*)} \leq \frac{C}{r} \left(\int_{\Omega_\varepsilon^*} |G_\varepsilon(z, y)|^2 dz\right)^{1/2} \leq Cr^{1-d},
\]

where \(x\) can be on \(\partial \Omega\). By applying the localization technique (as shown in Remark 2.13 to (1.10), we have

\[
\|\nabla u_\varepsilon\|_{L^\infty(\Omega_\varepsilon^*)} \leq \frac{C}{r} \left(\int_{\Omega_\varepsilon^*} |u_\varepsilon|^2 dy\right)^{1/2}
\]

for \(u_\varepsilon\) satisfying \(\mathcal{L}_\varepsilon(u_\varepsilon) = 0\) in \(\Omega_\varepsilon^*(x)\) and \(u_\varepsilon = 0\) on \(\partial(\Omega_\varepsilon^*(x)) \cap \partial \Omega\). (We remark that we just consider the estimate at boundary, and the interior one directly follows from (1.13).) So, we can derive the second inequality of (4.36).

For the adjoint Green’s matrix \(\mathcal{G}_\varepsilon(\cdot, x)\), we have \(|\nabla_y \mathcal{G}_\varepsilon(x, y)| = |\nabla_y^* \mathcal{G}_\varepsilon(y, x)| \leq Cr^{1-d}\) by the same argument. Moreover, since \(\nabla_y \mathcal{G}_\varepsilon(\cdot, y)\) still satisfies (1.35) for any \(\rho > 0\), and \(\nabla_y \mathcal{G}_\varepsilon(\cdot, y) = 0\) on \(\partial \Omega\), we obtain

\[
|\nabla_x \nabla_y \mathcal{G}_\varepsilon(x, y)| \leq \frac{C}{r} \left(\int_{\Omega_\varepsilon^*} |z - y|^{2(1-d)} dz\right)^{1/2} \leq Cr^{-d},
\]

where \(r/2 < |z - y| < 2r\). Observe that \(\nabla_y \mathcal{G}_\varepsilon(\cdot, y) = 0\) and \(\nabla_x (\mathcal{G}_\varepsilon(\cdot, x)) = 0\) on \(\partial \Omega\), we have

\[
|\nabla_y \mathcal{G}_\varepsilon(x, y)| = |\nabla_y \mathcal{G}_\varepsilon(x, y) - \nabla_y \mathcal{G}_\varepsilon(\bar{x}, y)| \leq |\nabla_x \nabla_y \mathcal{G}_\varepsilon(x, y)||x - \bar{x}| \leq \frac{Cd_x}{|x - y|^d},
\]

where \(\bar{x} \in \partial \Omega\) such that \(d_x = |x - \bar{x}|\). Similarly, we have \(|\nabla_x \mathcal{G}_\varepsilon(x, y)| \leq Cd_y|x - y|^{-d}\), and the proof is complete. \(\square\)

Proof of Theorem 1.4. Define the conormal derivative \(\frac{\partial}{\partial \nu}, (\frac{\partial}{\partial \nu})^*\) corresponding to \(\mathcal{L}_\varepsilon\) and \(\mathcal{L}_\varepsilon^*\) as follows:

\[
\frac{\partial}{\partial \nu} = -n_i A_{ij}(x/\varepsilon) \frac{\partial}{\partial x_j} - n_i V_i(x/\varepsilon), \quad \left(\frac{\partial}{\partial \nu}\right)^* = -n_j A_{ij}(x/\varepsilon) \frac{\partial}{\partial x_i} - n_j B_j(x/\varepsilon),
\]

where \(n = (n_1, \cdots, n_d)\) denotes the outward unit normal vector to \(\partial \Omega\). Thus, define the Poisson kernel \(\mathcal{P}_\varepsilon(\cdot, y) = [\mathcal{P}_{\gamma\beta}(\cdot, y)]\) associated with \(\mathcal{L}_\varepsilon\) as

\[
\mathcal{P}_{\gamma\beta}(x, y) = \left(\frac{\partial}{\partial \nu}\right)^* [G_{\alpha\gamma}(x, y)] = -n_j(x) a_{ij}^{\alpha\beta}(x/\varepsilon) \frac{\partial}{\partial x_i}\{G_{\alpha\gamma}(x, y)\} - n_j(x) B_j^{\alpha\beta}(x/\varepsilon) G_{\alpha\gamma}(x, y)
\]

for \(y \in \Omega\) and \(x \in \partial \Omega\). It follows from (4.34) that

\[
|\mathcal{P}_{\varepsilon}(x, y)| \leq Cd_y|x - y|^{-d},
\]

where \(C\) depends only on \(\mu, \tau, \kappa, \lambda, d, m, \eta\) and \(\Omega\). Thus for any \(g \in L^p(\partial \Omega; \mathbb{R}^m)\) with \(p \in (1, \infty]\), the solution to \(\mathcal{L}_\varepsilon(u_\varepsilon) = 0\) in \(\Omega\) and \(u_\varepsilon = g\) on \(\partial \Omega\) can be written by

\[
u_\varepsilon(y) = \int_{\partial \Omega} \mathcal{P}_\varepsilon(x, y) g(x) dS(x)
\]

for any \(y \in \Omega\), and it follows from (4.37) that

\[
|\nu_\varepsilon(y)| \leq Cd_y \int_{\partial \Omega} \frac{|g(x)|}{|x - y|^d} dS(x)
\]
Recall that the nontangential maximal function of $u_\varepsilon$ is defined by
\[
(u_\varepsilon)^*(Q) = \sup \{ |u_\varepsilon(x)| : x \in \Omega \text{ and } |x - Q| \leq N_0 \text{dist}(x, \partial\Omega) \} \quad \text{for } Q \in \partial\Omega,
\]
where $N_0 = N_0(\Omega) > 1$ is sufficiently large. Hence, if $|y - x_0| \leq N_0 d_y$ for some $x_0 \in \partial\Omega$, then we have
\[
|u_\varepsilon(y)| \leq C \int_{\partial\Omega \cap B(x_0, r)} |g(x)| dS(x) + C d_y \sum_{j=0}^{\infty} \int_{\Sigma_j} \frac{|g(x)| dS(x)}{|x - y|^2}
\leq C \left\{ \int_{\partial\Omega \cap B(x_0, r)} |g(x)| dS(x) \right\}
\leq C \sup_{0 < r < \text{diam}(\Omega)} \left\{ \int_{\partial\Omega \cap B(x_0, r)} |g(x)| dS(x) \right\}
\]
where $r = d_y/2$ and $\Sigma_j = \partial\Omega \cap \{ B(x_0, 2^{j+1}r) \setminus B(x_0, 2^j r) \}$. Note that
\[
\mathcal{M}_{\partial\Omega}(|g|)(x_0) = \sup_{0 < r < \text{diam}(\Omega)} \left\{ \int_{\partial\Omega \cap B(x_0, r)} |g(x)| dS(x) \right\}
\]
is the Hardy-Littlewood maximal function of $g$ on $\partial\Omega$. Thus it is not hard to see that
\[
(u_\varepsilon)^*(x_0) \leq C \mathcal{M}_{\partial\Omega}(|g|)(x_0).
\]

Due to the $L^p$ bounded properties of the Hardy-Littlewood maximal operator: $\|\mathcal{M}_{\partial\Omega}(|g|)\|_{L^p(\partial\Omega)} \leq C\|g\|_{L^p(\partial\Omega)}$ for $1 < p \leq \infty$ (see [11] pp.5), the estimates (1.14) and (1.15) can be derived immediately.

Now, we turn back to verify (3.18). Let $R = d_y/2$, and $\varphi \in C_0^\infty(B(y, R))$ be a cut-off function such that $\varphi = 1$ in $B(y, R/4)$ and $\varphi = 0$ outside $B(y, R/2)$. Then, since $L_\varepsilon(u_\varepsilon) = 0$, we have $L_\varepsilon(\varphi u_\varepsilon) = -L_\varepsilon((1 - \varphi)u_\varepsilon)$ in $\Omega$ and $\varphi u_\varepsilon = 0$ on $\partial\Omega$. Hence, in view of (3.18), we obtain
\[
(\varphi u_\varepsilon)(y) = -\int_\Omega G_\varepsilon(\cdot, y) L_\varepsilon((1 - \varphi)u_\varepsilon)
\leq -\lim_{r \to 0} \left\{ \int_{\Omega \setminus B(y, r)} G_\varepsilon(\cdot, y) L_\varepsilon((1 - \varphi)u_\varepsilon) - \int_{\Omega \setminus B(y, r)} L_\varepsilon^* G_\varepsilon(\cdot, y) (1 - \varphi)u_\varepsilon \right\}
\leq \int_{\partial\Omega} \mathcal{P}_\varepsilon(\cdot, y) [(1 - \varphi)u_\varepsilon] + \lim_{r \to 0} \int_{\partial B(y, r)} G_\varepsilon(\cdot, y) \frac{\partial}{\partial \nu} [(1 - \varphi)u_\varepsilon] - \lim_{r \to 0} \int_{\partial B(y, r)} \mathcal{P}_\varepsilon(\cdot, y) [(1 - \varphi)u_\varepsilon]
\leq \int_{\partial\Omega} \mathcal{P}_\varepsilon(\cdot, y) [(1 - \varphi)u_\varepsilon].
\]
Note that $L_\varepsilon^* G_\varepsilon(\cdot, y) = 0$ in $\Omega \setminus B(y, r)$ for any $r > 0$, and $(1 - \varphi)u_\varepsilon \equiv 0$ in $B(y, R/4)$. The proof is complete. \hfill \Box

**Remark 4.12.** Note that the same type of results for $L_\varepsilon$ with Dirichlet boundary conditions and Neumann boundary conditions were shown in [5] Theorem 3 and [31] Theorem 1.3, respectively. Also, we refer the reader to [22] Theorem 1.3 for the same type of result in the almost periodic setting. In the case of $m = 1$, when we derive the estimate (1.15) with $C = 1$, there is no regularity condition on the coefficients of $L_\varepsilon$, but some additional conditions on $V$ are inevitably required even when $\lambda \geq \lambda_0$, and $\varepsilon = 1$ (see [22] pp.179).

## 5 Convergence rates

**Lemma 5.1.** Suppose that $A \in \Lambda(\mu, \tau, \kappa)$, and $V$ satisfies (1.2) and (1.4). Let
\[
\Psi_{\varepsilon, 0}^\alpha(x) = \Phi_{\varepsilon, 0}^\alpha(x) - \delta^\alpha - \varepsilon \chi_0^\alpha(x/\varepsilon), \quad \Psi_{\varepsilon, k}^\beta(x) = \Phi_{\varepsilon, k}^\beta(x) - P_k^\beta - \varepsilon \chi_k^\beta(x/\varepsilon).
\]
Then we have
\[
\max_{0 \leq k \leq d} \|\Psi_{\varepsilon,k}\|_{L^\infty(\Omega)} \leq C\varepsilon, \quad \max_{0 \leq k \leq d} \|\nabla \Psi_{\varepsilon,k}(x)\| \leq C \min \left\{ 1, \varepsilon d_x^{-1} \right\},
\]
(5.1)
where \(C\) depends only on \(\mu, \tau, \kappa, m, d\) and \(\Omega\).

**Proof.** By the definition of Dirichlet correctors \(\Phi_{\varepsilon,k}\), we have \(L_\varepsilon(\Psi_{\varepsilon,k}) = 0\) in \(\Omega\) and \(\Psi_{\varepsilon,k} = -\varepsilon \chi_{k,\varepsilon}\) on \(\partial \Omega\). Thus it follows from the interior Lipschitz estimate \((4.3)\) and Agmon-Miranda maximum principle (see \([3]\)) that,
\[
|\nabla \Psi_{\varepsilon,k}(x)| \leq \frac{C}{d_x} \left( \int_{B(x,d_x)} |\Psi_{\varepsilon,k}|^2 \, dy \right)^{1/2} \leq C\varepsilon d_x^{-1}, \quad \forall x \in \Omega.
\]
The rest parts of the lemma follow from Lemmas \((4.6)\) and \((4.8)\) and Remark \((2.9)\) \(\Box\)

**Lemma 5.2.** Suppose that \(u_\varepsilon \in H^1(\Omega)\), \(u \in H^2(\Omega)\) and \(L_\varepsilon(u_\varepsilon) = L_0(u)\) in \(\Omega\). Let
\[
w_\varepsilon^\beta = u_\varepsilon^\beta - \Phi_{\varepsilon,0}^\beta u^\gamma - \left[ \Phi_{\varepsilon,k}^\beta - x_k \delta^\beta \right] \frac{\partial u^\gamma}{\partial x_k},
\]
where \(1 \leq k \leq d\). Then
\[
[L_\varepsilon(w_\varepsilon)]^\alpha = \frac{\partial}{\partial x_i} \{ \tilde{E}_{\varepsilon,ki,\varepsilon}^{\alpha\gamma} \frac{\partial^2 u^\gamma}{\partial x_i \partial x_j} + \tilde{F}_{\varepsilon,ki,\varepsilon}^{\alpha\gamma} \frac{\partial u^\gamma}{\partial x_i} + \tilde{H}_{\varepsilon,ki,\varepsilon}^{\alpha\gamma} u^\gamma \} + a^{\alpha\beta}_{ij,\varepsilon} \left\{ \frac{\partial \Phi_{\varepsilon,k}^{\beta\gamma}}{\partial x_j} \frac{\partial^2 u^\gamma}{\partial x_i \partial x_j} + \frac{\partial \Phi_{\varepsilon,k}^{\beta\gamma}}{\partial x_i} \frac{\partial u^\gamma}{\partial x_j} + \frac{\partial \Phi_{\varepsilon,k}^{\beta\gamma}}{\partial x_j} \frac{\partial u^\gamma}{\partial x_i} \right\} - B^{\alpha\beta}_{i,\varepsilon} \left\{ \frac{\partial \Phi_{\varepsilon,k}^{\beta\gamma}}{\partial x_i} \frac{\partial u^\gamma}{\partial x_i} + \frac{\partial \Phi_{\varepsilon,k}^{\beta\gamma}}{\partial x_i} \frac{\partial u^\gamma}{\partial x_i} \right\} - [c^{\alpha\beta}_{\varepsilon} + \lambda \delta^{\alpha\beta}] \left\{ \frac{\Phi_{\varepsilon,k}^{\beta\gamma} - x_k \delta^\beta}{\partial x_i} \frac{\partial u^\gamma}{\partial x_i} + \frac{\Phi_{\varepsilon,k}^{\beta\gamma} - x_k \delta^\beta}{\partial x_i} u^\gamma \right\} - \varepsilon \frac{\partial \Phi_{\varepsilon,k}^{\beta\gamma}}{\partial y_i} \frac{\partial u^\gamma}{\partial x_i} - \varepsilon \frac{\partial \Phi_{\varepsilon,k}^{\beta\gamma}}{\partial y_i} \frac{\partial u^\gamma}{\partial x_i} \right\},
\]
(5.2)
where \(y = x/\varepsilon\), and
\[
\tilde{E}_{\varepsilon,ki,\varepsilon}^{\alpha\gamma} = \varepsilon \tilde{E}_{\varepsilon,ki,\varepsilon}^{\alpha\gamma} + a^{\alpha\beta}_{ij,\varepsilon} \left[ \Phi_{\varepsilon,k}^{\beta\gamma} - x_k \delta^\beta \right],
\]
\[
\tilde{F}_{\varepsilon,ki,\varepsilon}^{\alpha\gamma} = \varepsilon \tilde{F}_{\varepsilon,ki,\varepsilon}^{\alpha\gamma} + a^{\alpha\beta}_{ij,\varepsilon} \left[ \Phi_{\varepsilon,k}^{\beta\gamma} - x_k \delta^\beta \right] + \varepsilon \delta^{\alpha\beta} \frac{\partial u^\beta}{\partial x_i},
\]
\[
\tilde{H}_{\varepsilon,ki,\varepsilon}^{\alpha\gamma} = V^{\alpha\beta}_{ij,\varepsilon} \left[ \Phi_{\varepsilon,k}^{\beta\gamma} - x_k \delta^\beta \right] + \varepsilon \delta^{\alpha\beta} \frac{\partial u^\beta}{\partial y_i},
\]
Note that \(\tilde{E}_{\varepsilon,ki,\varepsilon}^{\alpha\gamma}, \tilde{F}_{\varepsilon,ki,\varepsilon}^{\alpha\gamma}, \tilde{H}_{\varepsilon,ki,\varepsilon}^{\alpha\gamma}, \tilde{\zeta}_{\varepsilon}^{\alpha\gamma}\) are defined in Remark \((2.9)\).

**Proof.** From \(L_\varepsilon(u_\varepsilon) = L_0(u)\), it follows that
\[
[L_\varepsilon(w_\varepsilon)]^\alpha = [L_0(u)]^\alpha - [L_\varepsilon(\Phi_{\varepsilon,0} u)]^\alpha - [L_\varepsilon((\Phi_{\varepsilon,k} - x_k I) \frac{\partial u}{\partial x_k})]^\alpha
\]
\[
= - \frac{\partial}{\partial x_i} \left\{ a^{\alpha\beta}_{ij,\varepsilon} \frac{\partial u^\beta}{\partial x_j} \right\} + \left( \tilde{B}^{\alpha\beta}_{i,\varepsilon} - \tilde{V}^{\alpha\beta}_{i,\varepsilon} \right) \frac{\partial u^\beta}{\partial x_i} + \left[ c^{\alpha\beta}_{\varepsilon} + \lambda \delta^{\alpha\beta} \right] u^\beta - \frac{\partial \Phi_{\varepsilon,k}^{\beta\gamma}}{\partial y_i} \frac{\partial u^\gamma}{\partial x_i} - \frac{\partial \Phi_{\varepsilon,k}^{\beta\gamma}}{\partial y_i} \frac{\partial u^\gamma}{\partial x_i} \right\},
\]
(5.3)
where \(I_1 = L_\varepsilon(\Phi_{\varepsilon,0} u)\) and \(I_2 = L_\varepsilon((\Phi_{\varepsilon,k} - x_k I) \frac{\partial u}{\partial x_k})\). By the definition of \(\Phi_{\varepsilon,k}\), and \(\chi_k\), \(0 \leq k \leq d\), we obtain
\[
[I_1]^\alpha = - \frac{\partial}{\partial x_i} \left\{ a^{\alpha\beta}_{ij,\varepsilon} \frac{\partial u^\beta}{\partial x_j} \right\} - \frac{\partial}{\partial x_i} \left\{ a^{\alpha\beta}_{ij,\varepsilon} \left[ \Phi_{\varepsilon,k}^{\beta\gamma} - x_k \delta^\beta \right] \frac{\partial u^\gamma}{\partial x_j} \right\} - a^{\alpha\beta}_{ij,\varepsilon} \frac{\partial \Phi_{\varepsilon,k}^{\beta\gamma}}{\partial x_j} \frac{\partial u^\gamma}{\partial x_i}
\]
\[
- \frac{\partial}{\partial x_i} \left\{ c^{\alpha\beta}_{\varepsilon} \left[ \Phi_{\varepsilon,k}^{\beta\gamma} - x_k \delta^\beta \right] u^\gamma \right\} + B^{\alpha\beta}_{i,\varepsilon} \frac{\partial \Phi_{\varepsilon,k}^{\beta\gamma}}{\partial x_i} \frac{\partial u^\gamma}{\partial x_i} + \left[ c^{\alpha\beta}_{\varepsilon} + \lambda \delta^{\alpha\beta} \right] \Phi_{\varepsilon,k}^{\beta\gamma} u^\gamma \]
where $y = x/\varepsilon$. Put $I_1$ and $I_2$ into (5.3), and then we have

$$[L_\varepsilon(w_\varepsilon)]^a = -\frac{\partial}{\partial x_i}\left\{a_{ij,e}^a \frac{\partial u^\gamma}{\partial x_j}\right\} + \frac{\partial}{\partial x_i}\left\{a_{ij,e}^a [\Phi^\beta_{\varepsilon,0} - \delta^\beta_{\varepsilon,0}] \frac{\partial u^\gamma}{\partial x_j}\right\} + a_{ij,e}^a [\Phi^\beta_{\varepsilon,0} - \delta^\beta_{\varepsilon,0}] \frac{\partial u^\gamma}{\partial x_j},$$

$$+ \left[c_{e}^{\alpha \beta} + \lambda \delta^\beta_{\varepsilon,0}\right] \left[\Phi^\beta_{\varepsilon,0} - \delta^\beta_{\varepsilon,0}\right] u^\gamma - U^\alpha_{i,e} (y) \frac{\partial u^\gamma}{\partial x_i} + W^\alpha_{i,e} \frac{\partial u^\gamma}{\partial x_i} + Z^\alpha_{e} (5.4)$$

where $b_{ij,e}^a$, $U^\alpha_{i,e}$, $W^\alpha_{i,e}$, $Z^\alpha_{e}$ are defined in Remark 2.9. Besides, the following identities hold.

$$-\frac{\partial}{\partial x_i}\left\{b_{ij,e}^a (y) \frac{\partial u^\gamma}{\partial x_j}\right\} = -\varepsilon \frac{\partial}{\partial x_i}\left\{E_{kij,e}^\alpha \frac{\partial u^\gamma}{\partial x_j}\right\} = -\varepsilon \frac{\partial}{\partial x_i}\left\{E_{kij,e}^\alpha \frac{\partial u^\gamma}{\partial x_j} + F_{kij,e}^\gamma \frac{\partial u^\gamma}{\partial x_j}\right\} = \varepsilon \frac{\partial}{\partial x_i}\left\{E_{kij,e}^\alpha \frac{\partial u^\gamma}{\partial x_j}\right\};$$

$$U^\alpha_{i,e} (y) \frac{\partial u^\gamma}{\partial x_i} = \varepsilon \frac{\partial}{\partial x_k}\left\{F_{kij,e}^\alpha \frac{\partial u^\gamma}{\partial x_i}\right\} = \varepsilon \frac{\partial}{\partial x_k}\left\{F_{kij,e}^\gamma \frac{\partial u^\gamma}{\partial x_i}\right\} = \varepsilon \frac{\partial}{\partial x_k}\left\{F_{kij,e}^\gamma \frac{\partial u^\gamma}{\partial x_i}\right\};$$

$$W^\alpha_{i,e} \frac{\partial u^\gamma}{\partial x_k} = \varepsilon \frac{\partial}{\partial x_i}\left\{\frac{\partial u^\gamma}{\partial y_i}\right\} = \varepsilon \frac{\partial}{\partial x_i}\left\{\frac{\partial u^\gamma}{\partial x_i}\right\} = \varepsilon \frac{\partial}{\partial x_i}\left\{\frac{\partial u^\gamma}{\partial x_i}\right\};$$

$$Z^\alpha_{e} \frac{\partial u^\gamma}{u^\gamma} = \varepsilon \frac{\partial}{\partial x_i}\left\{\frac{\partial u^\gamma}{\partial y_i}\right\} = \varepsilon \frac{\partial}{\partial x_i}\left\{\frac{\partial u^\gamma}{\partial x_i}\right\} = \varepsilon \frac{\partial}{\partial x_i}\left\{\frac{\partial u^\gamma}{\partial x_i}\right\}.$$

These together with (5.4) give the formula (5.2), and we complete the proof.

**Proof of Theorem 1.5.** Let $w_{\varepsilon,1}, w_{\varepsilon,2} \in H^1_0(\Omega; \mathbb{R}^m)$ satisfy $w^\beta_{\varepsilon} = w^\beta_{\varepsilon,1} + w^\beta_{\varepsilon,2}$, such that

$$L_\varepsilon(w_{\varepsilon,1}) = L_\varepsilon(w_{\varepsilon}) - \Theta \quad \text{in} \ \Omega, \quad L_\varepsilon(w_{\varepsilon,2}) = \Theta \quad \text{in} \ \Omega, \quad (5.5)$$

where $w_{\varepsilon}$ is given in Lemma 5.2, and $\Theta = (\Theta^\alpha)$ satisfies
\[ \Theta^\alpha = a_{ij, \varepsilon}^{\alpha \beta} \left\{ \frac{\partial \Psi_{x_j}^{\beta}}{\partial x_j} \frac{\partial^2 u^\gamma}{\partial x_j \partial x_i} + \frac{\partial \Psi_{x_i}^{\alpha}}{\partial x_j} \frac{\partial u^\gamma}{\partial x_j} \right\} - B_{ij, \varepsilon}^{\alpha \beta} \left\{ \frac{\partial \Psi_{x_k}^{\beta}}{\partial x_k} \frac{\partial u^\gamma}{\partial x_i} + \frac{\partial \Psi_{x_i}^{\alpha}}{\partial x_k} \frac{\partial u^\gamma}{\partial x_k} \right\}. \]

For the first equation of (5.5), it immediately follows from Theorem 1.1, Lemmas 4.6, 4.8 and Remark 2.9 that
\[ \|\nabla w_{\varepsilon, 1}\|_{L^p(\Omega)} \leq C \varepsilon \|u\|_{W^{2,p}(\Omega)}. \quad (5.6) \]
For the second equation, in view of (2.7), we have
\[ c_0\|\nabla w_{\varepsilon, 2}\|_{L^2(\Omega)} \leq B_{\varepsilon}[w_{\varepsilon, 2}, w_{\varepsilon, 2}] = \int_{\Omega} \Theta^\alpha w_{\varepsilon, 2}^\alpha. \]
For the right hand side, it follows from (5.1) and Cauchy’s inequality that
\[ \int_{\Omega} \Theta^\alpha w_{\varepsilon, 2}^\alpha \leq C \varepsilon \int_{\Omega} (|\nabla^2 u| + |\nabla u| + |u|) |w_{\varepsilon, 2}| \frac{dx}{dx} \leq C \varepsilon \|u\|_{H^2(\Omega)} \|\nabla w_{\varepsilon, 2}\|_{L^2(\Omega)}, \]
where we use Hardy’s inequality in the last inequality. Hence we have
\[ \|w_{\varepsilon}\|_{H^1(\Omega)} \leq C \{\|\nabla w_{\varepsilon, 1}\|_{L^2(\Omega)} + \|\nabla w_{\varepsilon, 2}\|_{L^2(\Omega)}\} \leq C \varepsilon \|u\|_{H^2(\Omega)}, \]
and this gives the estimate (1.10).

We now turn to show the estimate (1.15). First, by recalling the estimate (3.33), we have
\[ |G_\varepsilon(x, y)| \leq \frac{C_d}{|x - y|^{d-1}}. \quad (5.7) \]
Additionally, in view of (5.1), we have
\[ |\Theta(y)| \leq \frac{C \varepsilon}{d^y} \{|\nabla^2 u(y)| + |\nabla u(y)| + |u(y)|\} \quad (5.8) \]
for any \( y \in \Omega. \)

Since
\[ w_{\varepsilon, 2}(x) = \int_{\Omega} G_\varepsilon(x, y)^{\alpha \beta} \Theta^\alpha(y) dy \quad \text{for} \ x \in \Omega, \]
it follows from (5.7) and (5.8) that
\[ |w_{\varepsilon, 2}(x)| \leq C \varepsilon \int_{\Omega} \frac{1}{|x - y|^{d-1}} \{|\nabla^2 u(y)| + |\nabla u(y)| + |u(y)|\} dy. \quad (5.9) \]
Thus by the Hardy-Littlewood-Sobolev theorem of fractional integration (see [11, pp.119]), we obtain
\[ \|w_{\varepsilon, 2}\|_{L^q(\Omega)} \leq C \varepsilon \|u\|_{W^{2,p}(\Omega)} \]
with \( 1/q = 1/p - 1/d \) when \( 1 < p < d \). For \( p > d \), we can straightforward use Hölder’s inequality to arrive at
\[ \|w_{\varepsilon, 2}\|_{L^\infty(\Omega)} \leq C \varepsilon \|u\|_{W^{2,p}(\Omega)}. \]

Besides, it follows from (5.6) and the Sobolev inequality that \( \|w_{\varepsilon, 1}\|_{L^q(\Omega)} \leq C \|\nabla w_{\varepsilon, 1}\|_{L^p(\Omega)} \leq C \varepsilon \|u\|_{W^{2,p}(\Omega)}. \)
Thus we obtain
\[ \|w_{\varepsilon}\|_{L^q(\Omega)} \leq \left\{ \|w_{\varepsilon, 1}\|_{L^q(\Omega)} + \|w_{\varepsilon, 2}\|_{L^q(\Omega)} \right\} \leq C \varepsilon \|u\|_{W^{2,p}(\Omega)}, \]
which implies the estimate (1.17), and the proof is complete. \( \Box \)
Remark 5.3. In view of Lemma 4.11 and Lemma 5.2, we can actually derive the following estimates by the arguments developed in [30],

\[ |G_\varepsilon(x, y) - G_0(x, y)| \leq \frac{C\varepsilon}{|x - y|^{d-1}}, \quad \forall \ x, y \in \Omega \text{ and } x \neq y. \]

Then we have

\[ \|u_\varepsilon - u\|_{L^\infty(\Omega)} \leq C\varepsilon \left[ \ln(R_0\varepsilon^{-1} + 2) \right]^{\frac{1}{p} - \frac{1}{2}} \|F\|_{L^p(\Omega)}, \]

where \( R_0 \) denotes the diameter of \( \Omega \). Moreover, let \( \Omega \) be a bounded \( C^2,\eta \) domain, we have

\[ \left| \frac{\partial}{\partial x_i} \left\{ \mathcal{G}^{\alpha\beta}_\varepsilon(x, y) \right\} - \frac{\partial}{\partial x_i} \left\{ \Phi^{\alpha\gamma}_\varepsilon(x, y) \right\} \mathcal{G}^{\gamma\beta}_0(x, y) - \frac{\partial}{\partial x_i} \left\{ \Phi^{\alpha\gamma}_\varepsilon(x, y) \right\} \frac{\partial}{\partial x_k} \left\{ \mathcal{G}^{\alpha\gamma}_0(x, y) \right\} \right| \leq C\varepsilon \ln(\varepsilon^{-1}|x - y| + 2) \]

for any \( x, y \in \Omega \) and \( x \neq y \). Then it follows that for any \( 1 < p < \infty \),

\[ \|u_\varepsilon - \Phi_\varepsilon,0u - (\Phi^{\beta}_\varepsilon,k - P^\beta_k) \frac{\partial u_\varepsilon}{\partial x_k}\|_{W_0^{1,p}(\Omega)} \leq C\varepsilon \left[ \ln(R_0\varepsilon^{-1} + 2) \right]^{\frac{1}{p} - \frac{1}{2}} \|F\|_{L^p(\Omega)}, \]

where \( C \) depends only on \( \mu, \tau, \kappa, \lambda, m, d, p \) and \( \Omega \). The details are left to readers (or see [30]).

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