A question on generalization of partition functions of CY 3-folds in String Theory

Mohammad Reza Rahmati

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Abstract

This is an expository article on the topological string partition function promoting an extension of the partition function of open Gromov-Witten theory of CY 3-folds defined by the trace of vertex operators. We also give a brief survey of their connection to the theory of Hilbert scheme of points on surface. Specifically; we apply infinitely many Cassimir operators twisted to the vertex operator computing the amplitude. The case of finite number of twists has been well discussed in the mathematics and Physics literature.

1 Introduction

The vertex operators provide a representation theoretic and the combinatorial framework for the generating series in string theory. To a CY two generating series can be associated; one the topological vertex partition function and the other, the Gauge index generating series that can be obtained as the elliptic genus of certain Hilbert scheme of points on surface. The topological vertex partition function is obtained from a framed gluing of topological vertex associated to different $\mathbb{C}^3$ patches of the CY 3-fold. The partition function has various interpretations which are somehow related to each other. One can mention; BPS content of the Gauge actions, Fock space formalism of Hilbert scheme of points, Knot invariants, GW correlation functions, Generating series of quiver Gauge theory, etc.

The vertex operators act on the Fock space of infinite dimension and their trace produces generating series which are crucial in quantum Physics. The vertex operator provide a powerful tool to compute string theory amplitudes. We follow a computation of the character of the infinite wedge representation (Fock space) \[8\] where a product formula is established for the character. Bloch-Okounkov prove that the associated character is quasimodular.

In various cases the partition function can be formulated through the Fock space structure on $\bigoplus_n H(n)$ where $H(n)$ stands for the cohomology of the Hilbert scheme of $n$-points on $\mathbb{C}^2$, \[7\]. The generating series appears as a correlation functions involving chern characters of the Hilbert scheme associated to cohomology classes on the CY 3-fold $X$. The GW-theory of CY 3-folds can be explained by the $\chi_y$-genus of the Hilbert scheme of points on surface.

In general the successive application of commutation rule of the boson fields interacting to the vertex operator produces product factors [see Appendix in \[2\]]. The product formulas for the string partition functions are specially of interests. The GopaKumar-Vafa invariants can be obtained from these product formulas.

We consider the partition functions as characters of representations and specifically as trace of vertex operators, twisted by one or more Cassimir operators. We give a natural extension of the trace to the case when infinitely many Cassimirs. We ask of a representation and Physics theoretic interpretations for the trace of the extended vertex operator.
2 Hilbert Scheme of points on Surface

We briefly follow [7] on basics of Hilbert scheme of points on surface. Our purpose is to present the Fock space structure on the cohomology of Hilbert scheme of points on surface, as an alternative way to produce partition functions of CY 3-folds. Fix a quiver \((I, H)\), with vertex set \(I\) and edges \(H\). We shall consider the quiver variety

\[ M_\zeta(v, w) = \mu^{-1}(\zeta)_{ss}^{ss}/GL_v C \]

where \(v \in \mathbb{Z}_+^I, w \in \mathbb{Z}_+^I \) are dimension and framing vectors and

\[ \zeta = (\zeta_i) \in \mathbb{C}^I \mapsto \bigoplus_{i \in I} \zeta_i \text{Id}_{V_i} \in Z(gl_v) \]

\((GL_v = \prod_i GL(V_i))\), and \(Z(.)\) is the center. The map \(\mu\) is defined as

\[ \mu : M(v, w) \rightarrow gl_v \]

\[ \mu(B, a, b) = \left( \sum_{\text{in}(h) = i} \epsilon(h) B_h B_{\bar{h}} + a_i b_i \right) \in \bigoplus_i gl(V_i) = gl_v \]

where

\[ M(v, w) = \bigoplus_{h \in H} \text{Hom}(V_{\text{out}(h)}, V_{\text{in}(h)}) \bigoplus (\text{Hom}_{i \in H}(W_i, V_i) \oplus \text{Hom}(V_i, W_i)) \]

The variety \(M_\zeta(v, w)\) can be interpreted as the moduli of representations of the quiver \((I, H)\). If \(A = (a_{ij})\) be the adjacency matrix of the quiver \((I, H)\), then \(C = 2Id - A\) is a Cartan matrix, and

\[ R_+ = \{ \beta = (\beta_i) \in \mathbb{Z}_+^I | \beta^T C \beta \leq 2 \} \]

is a set of positive roots of a Lie algebra. Lets for simplicity \((I, H)\) be a quiver of \(\tilde{A}_{l-1}\)-type with cyclic orientation, That is the associated lie algebra is

\[ \tilde{sl}_l = sl_l[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d \]

Set

\[ w_0 = (1, 0, ..., 0) \in \mathbb{Z}^I, \]

\[ \zeta_0 = (0, \zeta_\mathbb{R}), \]

\[ \zeta_\mathbb{R} \in \{ \eta = (\eta_i) \in \mathbb{R}^I | \eta_i < 0, \forall i \} \]

Let \(e_i\) be the \(i\)-th coordinate vector of \(Z^I\). Define

\[ L_i(v) = \{(\varrho_1, \varrho_2) | \varrho_1 \subset \varrho_2, \varrho_1 \in M_{\zeta_0}(v, w_0), \varrho_2 \in M_{\zeta_0}(v + e_i, w_0) \} \]

\[ \subset M_{\zeta_0}(v, w_0) \times M_{\zeta_0}(v + e_i, w_0) \]

Consider the two projections \(p_1, p_2\) from \(M_{\zeta_0}(v, w_0) \times M_{\zeta_0}(v + e_i, w_0)\) onto the first and second factors. For \(\alpha \in H^*(M_{\zeta_0}(v, w_0))\) and \(\beta \in H^*(M_{\zeta_0}(v + e_i, w_0))\) we can define a representation

\[ e_i \mapsto E_i : (\beta \mapsto (-1)^{u_{i+1} + w_i} p_{1*}(p_2^* \beta L_i(v))) \]

\[ f_i \mapsto F_i : (\alpha \mapsto (-1)^{u_{i+1} + w_i} p_{2*}(p_1^* \alpha L_i(v))) \]
It is possible to choose $\zeta_0$ such that

$$M_{\zeta_0}(v, w_0) \cong X_0^{[n]}$$  \hspace{1cm} (2.10)

Then one gets naturally the action on $X_0^{[n]}$. Set

$$H_{X}^{i} = H^{i}(X^{[n]}, C), \quad H_{X}^{\lambda} = \bigoplus_{i=0}^{4n} H_{X}^{i}, \quad H_{X} = \bigoplus_{n} H_{X}^{n}$$  \hspace{1cm} (2.11)

and

$$|0\rangle = 1 \in H^0(X^{[0]}) = \mathbb{C}$$  \hspace{1cm} (2.12)

Define the cycles

$$Q^{[m+n,m]} = \{(\xi, \epsilon, \eta) \in X^{[m+n]} \times X \times X^{[m]} | \xi \supset \eta, \text{Supp}(I_\eta / I_\xi) = \{x\} \}$$

where $m \geq 0$, $Q^{[m,m]} = 0$. One has dim $Q^{[m+n,m]} = 2m + n + 1$. If $\alpha \in H^*(X)$, for each $n \geq 0$ define the two creation and annihilation correspondences

$$a_{-n}(\alpha) : a \mapsto p_1(Q^{[m+n,m]}:p_0^\alpha.p_2^\alpha)$$
$$a_n(\alpha) : b \mapsto p_2(Q^{[m+n,m]}:p_0^\alpha.p_3^\alpha)$$

For $n \neq 0$, the correspondences $a_{-n}(\alpha)$ have bidegree $(n, 2n - 2 + |\alpha|)$. We have $a_{-n}(\alpha)^\dagger = (-1)^n a_n(\alpha)$ with respect to the cup product. They satisfy the commutation relation

$$[a_m(\alpha), a_n(\alpha)] = -m\delta_{m,-n}(\alpha, \beta)$$

of Heisenberg type. $H_X$ is an irreducible module over Heisenberg algebra generated by $a_n(\alpha)$ with highest weight $|0\rangle = 1 \in H^0(X^{[0]})$. It is linearly generated by

$$a_{-n_1}(\alpha_1) \cdots a_{-n_k}(\alpha_k)|0\rangle, \quad n_1, \ldots, n_k > 0, \alpha_1, \ldots, \alpha_n \in H^*(X)$$

One can consider the above elements factored by the image of the map $\tau_k : H^*(X) \to H^*(X^k)$ induced from the diagonal embedding $\tau_k : X \to X^k$. We set

$$a_{n_1} \cdots a_{n_k}(\tau_k^\alpha) = \sum a_{n_1}(\alpha_{j,1}) \cdots a_{n_k}(\alpha_{j,k}), \quad \tau_k^\alpha = \sum_j \alpha_{j,1} \otimes \cdots \otimes \alpha_{j,n}$$

which is well defined cf. loc. cit. We define the normal ordering product

$$: a_{m_1} a_{m_2} : = \begin{cases} a_{m_1} a_{m_2}, & m_1 \leq m_2 \\ a_{m_1} a_{m_2}, & m_1 \geq m_2 \end{cases}$$

Set

$$\mathcal{L}_n = -1/2 \sum_{m \in \mathbb{Z}} : a_m a_{-m} :$$

These operators satisfy the Virasoro relations, and also act on the former operators as shifts

$$[\mathcal{L}_m(\alpha), \mathcal{L}_n(\beta)] = (m-n)\mathcal{L}_{m+n}(\alpha \beta) + \frac{m^3 - m}{12}\delta_{m,-n}(\int_X e_X \alpha \beta)Id_{H_X}$$

$$[\mathcal{L}_m(\alpha), a_n(\beta)] = -na_{m+n}(\alpha \beta)$$

where $e_X$ is the Euler class.
Let $L$ be a line bundle on $X$. Define a virtual vector bundle $E_L$ on $X^{[k]} \times X^{[l]}$ of rank $k + l$ by

$$E_L|_{(\xi, \eta)} = \chi(O, L) - \chi(I_{\eta}, I_{\xi} \otimes L)$$

(2.21)

There is an Ext operator of Carlson

$$W(L, z) : H_X[[z, z^{-1}]] \to H_Z[[z, z^{-1}]]$$

$$\langle W(L, z)\eta, \xi \rangle = \int_{X^{[k]} \times X^{[l]}} (\eta \otimes \xi) c_{k+l}(E_L)$$

(2.22)

Define the vertex operators $\Gamma_\pm(L, z) = \exp\left(\sum_n \frac{z^n}{n} a_{\pm n}(L)\right)$. They satisfy the commutation relation

$$\Gamma_+(L) \Gamma_-(L') = (1 - \frac{y}{x}) (L, L') \Gamma_-(L') \Gamma_+(L)$$

(2.23)

We can write the Carlson Ext operator in terms of vertex operators

$$W(L, z) = \Gamma_-(L - K_X, z) \Gamma_+(\pm L)$$

(2.24)

where $K_X$ is the canonical class.

Consider

$$Z_n = \{ (\xi, x) \subset X^{[n]} \times X| x \in \text{Supp}(\xi) \}$$

(2.25)

with two projections $p_1, p_2$. For $\gamma \in H^*(X)$ set

$$G(\gamma)_n = p_{1*} (ch(O_{Z_n}) p_2^* \text{td}(X) p_2^* \gamma) \in H^*(X^{[n]}), \quad n > 0$$

(2.26)

and write

$$G(\gamma)_n = \bigoplus_i G_i(\gamma)_n, \quad G_i(\gamma)_n \in H^{s+2i}(X^{[n]})$$

(2.27)

For a sheaf $F$ on $X$ define

$$F^{[n]} = p_{1*} p_2^* F$$

(2.28)

This operation reduces to $K$-groups. Then one can show that

$$\sum_n c(L^{[n]}) = \exp\left(\sum_{r \geq 1} \frac{(-1)^{r-1}}{r} a_{-r}(c(L))\right) |0\rangle$$

(2.29)

By setting $c_h(L^{[n]} = \sum_i c_i(L^{[n]}) h^i$, We can write this identity as a generating series

$$\sum_n c_h(L^{[n]} w^n) = \exp\left(\sum_{r \geq 1} \frac{(-1)^{r-1}}{r} a_{-r}(c_h(L)) w^r\right) |0\rangle = c_h(L) \exp(a_{-1}(1_X) w) |0\rangle$$

(2.30)

It satisfies the differential equation

$$\frac{d}{dw} Z = C_h(L) Z, \quad C_h(L) = c(L) a_{-1}(1_X) c(L)^{-1}$$

(2.31)
Let \( L \) be a line bundle on \( X \) and \( ch_k(L[n]) \) be the \( k \)-th chern character of \( L[n] \). Define the generating series

\[
\langle ch_{k_1}^{L_1} \ldots ch_{k_n}^{L_n} \rangle := \sum_{n \geq 0} q^n \left( \int_{X[n]} ch_{k_1}(L_1[n_1]) \ldots ch_{k_n}(L_n[n_k]) c(T_{X[n]}) \right)
\]

\[
\langle ch_{k_1}^{L_1} \ldots ch_{k_n}^{L_n} \rangle = \frac{\langle ch_{k_1}^{L_1} \ldots ch_{k_n}^{L_n} \rangle}{\langle 1 \rangle} (2.32)
\]

\[
\langle 1 \rangle = \langle q : q \rangle_{\infty}^{\chi(X)}, \quad (a : q)_n = \prod_{i=0}^{n} (1 - aq^i)
\]

These correlation functions are conjecturally multiple \( q \)-zeta values, i.e. a \( q \)-deformation of usual multiple zeta values. They are equal to the following generating series up to a factor. Let \( \alpha_1, \ldots, \alpha_n \in H^*(X) \),

\[
F_{\alpha_1 \ldots \alpha_n}^{k_1 \ldots k_n} = \sum_n q^n \left( \int_{X[n]} \prod_{i=1}^{n} G_{k_i}(\alpha_i, n) c(T_{X[n]}) \right) = \text{Tr} \left( q^n W(\Sigma, s) \prod_i L_{k_i}(\alpha_i) \right) (2.33)
\]

They appear as the constant coefficient in a more general generating series, cf. [7].

### 3 Topological String partition functions of CY 3-folds

The materials of this section are well known. The references are [1, 35, 34, 33, 31, 30, 26, 27, 28, 25, 24, 23, 22, 6, 5, 4, 3, 2]. We explain the topological string partition function of CY 3-folds and the method to write it in terms of topological vertex. We employ the concept of toric varieties and their fan diagrams. Specifically we shall consider the web diagrams of CY 3-folds on the plane.

Let \( T = (\mathbb{C}^\times)^p \) be the \( p \)-dimensional torus. It is the complexification of \( U(1)^p \). Assume \( \Sigma \) be a collection of cones called a fan in \( N_\mathbb{R} = \mathbb{R}^m \) and \( M = N^\vee \) the dual. Associated to a fan \( \Sigma \) we can define a variety

\[
M_\Sigma = \frac{\mathbb{C}^m \setminus Z(\Sigma)}{T_0}
\]

where \( T_0 \) is a product of torus with a finite abelian group. The equivalence classes turns out to be

\[
(z_1, \ldots, z_m) \sim (\lambda Q_a^1 z_1, \ldots, \lambda Q_a^m z_m), \lambda \in \mathbb{C}^\times
\]

and the relation

\[
\sum Q_a^i v_i = 0, \quad \text{fan condition} (3.3)
\]

where \( v_i \) are generators of \( \Sigma \) (called fan condition). If \( D_i \) be the divisor defined by \( z_i = 0 \), the canonical bundle of \( M_\Sigma \) can be written as \( K_M = \mathcal{O}(-\sum D_i) \). It is trivial if \( \Sigma D_i \sim 0 \). This condition can be interpreted as the existence of a lattice point \( m \in (\mathbb{R}^m)^\vee \) such that \( \langle v_i, m \rangle = 1 \) for all \( i \). Thus \( M_\Sigma \) is CY iff the vectors \( v_i \) all lie in a hyperplane. It follows that

\[
\sum_i Q_a^i v_i = 0, \quad \text{CY condition} (3.4)
\]
Thus a toric CY is never compact.

We will deal with CY 3-folds. In this case the 3-dimensional fan is projected on a 2-dimensional graph, called the web diagram, and \(T_0 = (\mathbb{C}^*)^{m-3} \times \text{finite abelian gp.}\) There is a moment map

\[
\mu : M_\Sigma \to \Delta_M
\]

where \(\Delta_M\) is the polytope of \(M\), i.e. dual to the fan \(\sigma\). The moment map is given by \(m-3\) maps \(\mu_a : \mathbb{C}^m \to \mathbb{C}, \ a = 1, 2, ..., m-3\), and is described by the equations

\[
\sum_{i=1}^{m} Q_a^i |z_i|^2 = \text{Re}(t_a), \quad t_a \in \mathbb{C}
\]

called Witten D-term equations. We have an action of \(U(1)^{m-3}\) on coordinates \(z_j\)

\[
z_j \mapsto \exp(iQ_a^i \alpha_a) z_j, \quad a = 1, 2, ..., m-3
\]

It turns out that

\[
M_\Sigma = \frac{\bigcap_{i=1}^{m-3} \mu_a^{-1}(\text{Re}(t_a))}{T_0}
\]

is a CY 3-fold and \(t_a\) are complexified Kahler parameters. We look at CY 3-folds as \(T^3 = (S^1)^3\)-fibrations over 3-dimensional base with corners. Introduce new variables

\[
(p_1, \theta_1), ..., (p_k, \theta_k), \quad p_i = |z_i|^2, \quad z_i = |z_i|e^{i\theta_i}
\]

The 3-fold is then parametrized by the coordinates \(p_i\) and \(\theta_i\), where the latter describes \(T^3\). The CY 3-fold is constructed by gluing many open \(\mathbb{C}^3\)-patches to each other along edges. The web diagram of CY 3-folds records the information of a framing when gluing \(\mathbb{C}^3\)-patches. The frame is settled by choosing a vector \(f_i\) perpendicular to the vectors \(v_i\) of the fan. The choice of framing affects the partition function of the CY, however here we assume the frame is chosen to be standard, i.e. \(f_i \perp v_i = 1\) for all \(i\).

**Fermionic model:** \([26]\) To each \(\mathbb{C}^3\)-patch of the CY variety we associate a partition function called topological vertex \(C_{R_1,R_2,R_3}\) which can be written in the following two equivalent forms

\[
\sum_{R_1,R_2,R_3} C_{R_1,R_2,R_3}^{f_1,f_2,f_3} \prod_{i=1}^{3} Tr_{R_i} V_i = \sum_{\overline{k}(1), \overline{k}(2), \overline{k}(3)} C_{\overline{k}(1), \overline{k}(2), \overline{k}(3)}^{f_1,f_2,f_3} \frac{1}{n_{\overline{k}(i)}} \prod_{i=1}^{3} Tr_{\overline{k}(i)} V_i
\]

where \(Tr_{\overline{R}} V\) denotes the trace of \(V\) as appears in the representation \(R\), and \(\chi_R(C(\overline{k}))\) is the character of the symmetric group calculated at the conjugacy class \(C(\overline{k})\). The sum in the left hand side runs over all the representations \(R_1, R_2, R_3\) of the symmetric group \(S_N\) for all \(N\). The trace of representations of the symmetric groups can also be stated via the conjugacy classes and we simply have

\[
Tr_{\overline{R}} V = \sum_R \chi_R(C(\overline{k})) Tr_{\overline{R}} V
\]

The two sums are related by the following identity

\[
C_{R_1,R_2,R_3}^{f_1,f_2,f_3} = \sum_{\overline{k}(1), \overline{k}(2), \overline{k}(3)} C_{\overline{k}(1), \overline{k}(2), \overline{k}(3)}^{f_1,f_2,f_3} \prod_{i=1}^{3} \frac{\chi_{R_i}(C(\overline{k}(i)))}{n_{\overline{k}(i)}}
\]
The $V_i$ are holonomy variables, given by the determinant of the Wilson line $V_i = P \int_{\Gamma_i} A_i$. In fact one notes that in counting the holomorphic maps from curves of different genus to a CY 3-fold, these curves must go to the vertices or edges of the web diagram, and to illustrate them, they wrap around the edges with different holonomies. The change of the framing affect as

$$C^{f_1-n_1v_1,f_2-n_2v_2,f_3-n_3v_3}_{R_1,R_2,R_3} = (-1)^{\sum n_i l(R_i) q_i} \sum n_i k(R_i)/2 C_{R_1,R_2,R_3}^{f_1,f_2,f_3}$$

(3.13)

[See [26] for the definition of indices $l(R_i), k(R_i)$ and $n_i$.] The topological vertex $C_{R_1,R_2,R_3}$ is invariant under circle change of the representations $R_i$, i.e they have cyclic symmetry. We have the rule

$$v_1 \rightarrow -v_1 \rightsquigarrow C_{R_1,R_2,R_3} \rightarrow (-1)^{l(R_i)} C_{R_1,R_2,R_3}$$

(3.14)

In the gluing of two graphs $\gamma_1$ and $\gamma_2$ with partition functions $Z(\gamma_1)$ and $Z(\gamma_2)$ we produce terms as

$$\sum_Q Z(\Gamma_1)(-1)^{l(Q)} e^{l(Q)t} Z(\Gamma_2) q^t \quad \rightsquigarrow \quad \sum_k Z(\Gamma_1) \exp\left(-l(k) t\right) \prod_j k_j / k_j - Z(\Gamma_2)$$

(3.15)

The rules to write the partition function in terms of topological vertices are as follows,

1. The edges of the graph are labeled by integral vectors $v_i$. To each edge asociate a representation $R_i$.

2. For smooth CY the graph can be divided into trivalent vertices corresponding to $\mathbb{C}^3$-patches.

3. To each vertex there associates an ordered triple $(v_i, v_j, v_k)$ by reading counterclockwise.

4. If all the edges are incoming associate $C_{R_i,R_j,R_k}$ to $(v_i, v_j, v_k)$, otherwise replace the corresponding representation by its transpose times $(-1)^{l(R_i)}$.

5. If the vertex $(v_i, v_j, v_k)$ shares the $i$-th edge with the vertex $(v_i, v_j', v_k')$, we glue the amplitudes by summing over the representations on the $i$-th edge

$$\sum_{R_i} C_{R_i,R_k,R_i} (e^{-l(R_i) t} (-1)^{(n_i+1) l(R_i)} q^{-n_i k(R_i)/2}) C_{R_i,R_j,R_k}$$

(3.16)

where $n_i = |v_k \wedge v_k'|$.

6. The length of edge can be read from the Witten D-term equation on the Kahler moduli of $X$. The edges of the graph are straight lines on the plane with rational slope. To the $i$-th edge in the $(p_i, q_i)$-direction of length $x_i$ we associate a Kahler parameter $t_i = x_i / \sqrt{p_i^2 + q_i^2}$.

7. For non-compact edges the corresponding representation is trivial, denoted $R = 0, \emptyset, \bullet$.

8. If there are more than one $D$-branes say $n$ on the edge, we have contributions of $n$ open strings stretching between $D$-branes. The effect of integrating out these strings is

$$\exp\left(-\sum_{m=1}^{n} \frac{1}{m} tR U_1^m U_2^m\right) = \sum_{R} (-1)^{l(R)} tr R U_1 tR U_2$$. It produces contributions of the form

$$\sum_{R_i, Q_{a,i}^L, Q_{a,i}^R} C_{R_i,R_k,R_i} (e^{-L(i)} q^{f(i)}) C_{R_i,R_j,R_k} \prod_{a=1}^{n} tr Q_{a,i}^L V_a tr Q_{a,i}^R V_a^{-1}$$

(3.17)
where the exponents in the middle term is read from the Young diagram of the representations $R_i, R_i', Q_{a,i}^L, Q_{a,i}^R$, cf. [26].

(9) The topological vertex is given by

$$C_{\lambda \mu \nu} = q^{[k(\mu)/2]} S_{\nu'}(q^{-\rho}) \sum_{\eta} S_{\lambda'/\eta}(q^{-\nu'-\rho}) S_{\mu/\eta}(q^{-\nu'-\rho})$$  \hspace{1cm} (3.18)

where $q^{-\nu'-\rho} = \{q^{-\nu_i+(2i-1)/2}\}_{i=1,2,...}$, $q = e^{2\pi i \epsilon}$, and $\| \lambda \|= \sum_2^\lambda 2$, $k(\mu) = |\mu| - |\mu|$. [26]

We explain another model to express the partition function of CY 3-folds involving Boson operators. We shall consider the Hilbert space $\mathcal{H}$ of states to be generated by

$$| \bar{k} \rangle = \prod_j \alpha_{k,j} |0\rangle, \quad \bar{k} = (k_1, k_2, ..., k_m)$$  \hspace{1cm} (3.19)

is a vector involving winding numbers. On $\mathcal{H}^\otimes \mathcal{H}^3$ we define an element

$$|P\rangle = \exp \left(-t \sum_j \frac{1}{n_j} \alpha_{-j} \alpha_j \right) |0\rangle \otimes |0\rangle = \sum_{\bar{k}} e^{-l(\bar{k})t} \frac{(-1)^h}{n_{\bar{k}}} |\bar{k}\rangle \otimes \langle \bar{k}|, \quad n_{\bar{k}} = \prod_j k_j! j^{k_j}$$  \hspace{1cm} (3.20)

called propagator. The topological vertex is defined as a state in $\mathcal{H}^\otimes \mathcal{H}^3$. If we write the partition function $Z$ formally as

$$Z = \sum_{\bar{k}} C_{\bar{k}_1, \bar{k}_2, \bar{k}_3} \frac{1}{n_{\bar{k}_1}} Tr_{\bar{k}_1} V_1 Tr_{\bar{k}_2} V_2 Tr_{\bar{k}_3} V_3$$  \hspace{1cm} (3.21)

Thinking of topological vertex $C$ as a state in $\mathcal{H}^\otimes \mathcal{H}^3$ we write $Z$ by

$$Z = \sum_{\bar{k}_1} Tr_{\bar{k}_1} V_1 Tr_{\bar{k}_2} V_2 Tr_{\bar{k}_3} V_3 \frac{1}{n_{\bar{k}_1}} \langle \bar{k}_1 | \otimes \langle \bar{k}_2 | \otimes \langle \bar{k}_3 | C \rangle$$  \hspace{1cm} (3.22)

It follows from the last formula that $C$ can be written in the form

$$|C\rangle = \exp \left( \sum_{\bar{k}} \frac{1}{n_{\bar{k}}} Tr_{\bar{k}} V^\otimes n \alpha_{\bar{k}} \right) |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3$$  \hspace{1cm} (3.23)

We may formally write $Z = \langle V_1 | \otimes \langle V_2 | \otimes \langle V_3 | C \rangle$ where

$$V = \langle 0 | \exp \left( \sum \frac{1}{n} Tr V^\otimes n \alpha_n \right)$$  \hspace{1cm} (3.24)

Example. The partition function of the web diagram obtained by gluing three $\mathbb{C}^3$ patches along a triangle is

$$Z = \sum_{R_1, R_2, R_3} (-1)^{l(R_i)} q^{k(R_i) C_{R_2 R_3} C_{R_3 R_1}}$$  \hspace{1cm} (3.25)
• The partition function of an \((M,N)\)-web diagram with \(M\) vertical and \(N\) horizontal hexagons is written as

\[
Z(M,N) = \sum_{\alpha} \prod_{a=1}^{N} Q_{\alpha_a}^{\alpha^a} \prod_{\mu,\nu} (1 - Q_{\nu_{\mu_a}}) \prod_{a} C_{\mu_{\nu_{\alpha_a}}(t,q)} \prod_{b} C_{\nu_{\mu_b}}(t,q) \tag{3.26}
\]

where \(\alpha^a\) is a set of \(M\) partitions \(\alpha_1^a, \ldots, \alpha_M^a\), and \(\beta_b = \alpha_b^{a+1}\). [2].

4 Vertex Operator formalism of Partition Functions

The material of this section are well known. The references are [14, 29, 30] as well as many other available texts. We include this section as part of the task to make our terminology more concrete and understandable. We begin with the definition of the Fock space. The Fock space \(\mathfrak{F} = \bigwedge V\) is the vector space spanned by semi-infinite wedge product of a fixed basis of the infinite dimensional vector space \(V = \sum_{i \in 1/2+\mathbb{Z}} \mathbb{C}v_i\), i.e. monomials \(v_{i_1} \wedge v_{i_2} \wedge \ldots\) such that

- \(i_1 > i_2 > \ldots\)
- \(i_j = i_{j-1} - 1/2\) for \(j \gg 0\)

We have the creation operators and the annihilation operators

\[
\begin{align*}
\psi_k : v_{i_1} \wedge v_{i_2} \wedge \ldots & \mapsto v_{i_1} \wedge v_{i_2} \wedge \ldots \\
\psi^*_k : v_{i_1} \wedge v_{i_2} \wedge \ldots & \mapsto (-1)^l v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge \hat{v}_{i_l=k} \wedge \ldots
\end{align*}
\]  

The monomials can be parametrized by partitions

\[
|\lambda\rangle = v_\lambda = \lambda_1 - 1/2 \wedge \lambda_2 - 3/2 \wedge \ldots \tag{4.2}
\]

We can also write this using Frobenius coordinates of partitions

\[
|\lambda\rangle = \prod_{i=1}^{l} \psi^*_{a_i} \psi^*_{b_i} |0\rangle, \quad a_i = \lambda_i - i + 1/2, \quad b_i = \lambda'_i - i + 1/2 \tag{4.3}
\]

We can define operators

\[
\alpha_n = \sum_{k \in 1/2+\mathbb{Z}} \psi_{k+n}^* \psi_k^* \tag{4.4}
\]

They satisfy \([\alpha_n, \psi_k] = \psi_{k+n}, [\alpha_n, \psi_k^*] = -\psi_{k-n}\). The operators of the form

\[
\Gamma_+(x) = \exp \left( \sum_{n \geq 1} \frac{x^n}{n} \alpha_n \right), \quad \Gamma_-(x) = \exp \left( \sum_{n > 0} \frac{x^n}{n} \alpha_{-n} \right) \tag{4.5}
\]

are called vertex operators. They are adjoint with respect to the natural inner product. We have a commutation relation

\[
\Gamma_+(x)\Gamma_-(y) = (1 - xy)\Gamma_-(y)\Gamma_+(x) \tag{4.6}
\]
We have

\[ \Gamma_+(x) v_\mu = \sum_{\lambda \geq \mu} s_{\lambda/\mu}(x) v_\lambda \] (4.7)

Vertex operators provide powerful tools to express partitions.

\[ \Gamma_+(1) |\mu\rangle = \sum_{\lambda \geq \mu} |\lambda\rangle \]
\[ \Gamma_-(1) |\mu\rangle = \sum_{\lambda \subset \mu} |\lambda\rangle \] (4.8)

For example we may write the McMahon function as

\[ Z = \sum_{\text{3-dim partitions}} q^\text{boxes} = \langle \prod_{t=0}^{\infty} q^{L_0} \Gamma_+(1) q^{L_0} (\prod_{t=-\infty}^{-1} \Gamma_-(1) q^{L_0}) \rangle \\
= \langle \prod_{n>0} \Gamma_+(q^{-n-1/2}) \prod_{n>0} \Gamma_-(q^{-n-1/2}) \rangle \] (4.9)

We may divide a 3-dimensional partitions into slices of two dimensional partitions, for instance along the diagonals or any other way this could be done. In this way the vertex operator divides into multiplication of many vertex operators of the slices,

\[ Z(\{x^\pm_m\}) = \langle \prod_{u_1<m<u_{i+1}} \Gamma_+(x^-_m) \prod_{v_i<m<v_i} \Gamma_-(x^+_m) \rangle = \langle \prod_{u_0<m<u_n} \Gamma_-(x^e(m)) \rangle \] (4.10)

In this way one can obtain product formulas such as

\[ Z(\{x^\pm_m\}) \prod_{m_1<m_2} (1 - x^-_{m_1} x^+_{m_2}) \] (4.11)

For example we may choose

\[ \{x^+_m\} = \{t^i q^v | i = 1, 2, \ldots \} \]
\[ \{x^-_m\} = \{t^{-1} q^{-v} | j = 1, 2, \ldots \} \] (4.12)

Then we get

\[ Z_{\lambda\mu\nu}(t, q) = \langle \prod_{u_0<m<u_n} \Gamma_-(x^e(m)) \rangle \] (4.13)

The partition function can also be read by putting a wall on the distance \( M \) along one of the axis. Then using commutation (7.6), we have expressions of the form

\[ Z = \langle \prod_{0<m<\infty} \Gamma_-(x^+_m) \prod_{-\infty<m<0} \Gamma_+(x^-_m) \rangle \]
\[ = \prod_{l_1=1}^{\infty} \prod_{l_2=1}^{M} (1 - x^+_{m_l_1 - 1/2} x^-_{m_l_2 + 1/2})^{-1} (\prod_{0<m<\infty} \Gamma_+(x^-_m) \prod_{-\infty<m<0} \Gamma_-(x^+_m)) \] (4.14)

The last factor in paranthesis is equal to 1, and we obtain a product formula.
Example. The refined partition function of the CY 3-fold double \( \mathbb{P}^1 \) (3 \( \mathbb{C}^3 \) patches where each of the outer two glued to the middle one along a a compact leg of length \( Q_i, \ i = 1, 2 \)) can be written as

\[
Z' = \left\{ \prod_{0 < m < L} \Gamma_+(x_m^+) \prod_{-M < m < 0} \Gamma_-(x_m^-) \right\}
\]

\[
= \sum_{\lambda} \langle 0 | \prod_{0 < m < L} \Gamma_-(x_m^+)|\lambda\rangle \prod_{-M < m < 0} \Gamma_+(x_m^-)|0\rangle
\]

\[
= \prod_{i} \prod_{j} \frac{(1 - Q_1 t_i^{1/2} q^j^{1/2})(1 - Q_2 t_i^{-1/2} q^{-j/2})}{(1 - Q_1 Q_2 t_i^{-1} q^j)}
\]

(4.15)

where \( \lambda \) runs over partitions in the range, and

\[
\{x_m^+\} = \{t, t^2, \ldots, t^L\}
\]

\[
\{x_m^-\} = \{1, q, q^2, \ldots, q^{M-1}\}
\]

(4.16)

\[
Z' = Z_{\text{McMahon}} Z_{\text{refined}}. \ [12].
\]

Remark 4.1. The quotient \( Z_{\mu \nu} \frac{\omega}{\omega_{\mu \nu}} = C_{\mu \nu} \) is called refined topological vertex. The aforementioned technics also applies to the refined version of topological vertex.

To give some sense of computations we calculate the trace of a general vertex operator action on the Fock space \( \mathfrak{F} \), cf. [14]. We have the following formula for the trace of a vertex operator acting on \( \mathfrak{F} = \bigwedge^\infty \bigoplus_{n=1}^\infty \mathfrak{F}_{\chi_n} |0\rangle \)

\[
\text{Tr} \left( q^{L_0} \exp \left( \sum_n A_n \alpha_n \right) \exp \left( \sum_n B_n \alpha_n \right) \right) = \prod_{n} \sum_{k} \sum_{l=0}^{\infty} \frac{n! A_n^l B_n^l}{l!} q^{nk} \left( \frac{k}{l} \right)
\]

(4.17)

where \( L_0 \) is the charge operator, \( q^{L_0}|\lambda\rangle = |\lambda||\lambda\rangle \).

To prove the formula, denote the operator in the trace by \( T \). We have the isomorphism

\[
\bigwedge V = \bigotimes_{n=1}^\infty \bigoplus_{k=0}^\infty \alpha_{-n}^k |0\rangle
\]

(4.18)

which implies

\[
\text{Tr}(T) = \prod_{n=1}^\infty \text{Tr} \left( T\big| \bigoplus_{k=0}^\infty \alpha_{-n}^k |0\rangle \right)
\]

\[
= \prod_{n} \sum \langle \alpha_{-n}^k |0\rangle q^{L_0} e^{A_n \alpha_n} e^{B_n \alpha_n} |\alpha_{-n}^k |0\rangle
\]

\[
= \prod_{n} \sum \frac{A_n^l B_n^m}{l! m!} q^{n(l-m+k)} \langle \alpha_{-n}^k |0\rangle |\alpha_{-n}^l \alpha_{n}^m |\alpha_{n}^k |0\rangle
\]

\[
= \prod_{n} \sum \frac{A_n^l B_n^l}{l!} q^{nk} \langle \alpha_{-n}^k |0\rangle |\alpha_{-n}^l \alpha_{n}^l |\alpha_{n}^k |0\rangle
\]

\[
= \prod_{n} \sum \sum_{k} \sum_{l=0}^{\infty} \frac{n! A_n^l B_n^l}{l!} q^{nk} \left( \frac{k}{l} \right)
\]

(4.19)
5 Generalization

In this section we extend the partition function of a CY 3-fold as the trace vertex operators twisted by certain Cassimir operators, and present our main result. Our motivation is a computation on the character of the infinite wedge Fock space in \[8\]. Let \( \mathcal{F} = \bigwedge V \) be the Fock space on a fixed basis of \( V = \bigoplus_{j \in \mathbb{Z}} \mathbb{C} v_j \). Then the character of \( \mathcal{F} \) as a representation of \( \mathfrak{gl}_\infty \) is given by

\[
\text{ch}(\mathfrak{gl}_\infty, \mathcal{F}) = \prod_{n \geq 0} (1 + q_0 q_1^{n+1/2} q_2^{n+3/2} \ldots)(1 + q_0^{-1} q_1^{n-1/2} q_2^{-1/2} \ldots) \tag{5.1}
\]

where \( q_j = e^{2\pi i \tau_j} \). The partition function of \( U(1) \) theory can be written in the form

\[
Z(\tau, m, \epsilon) = Tr \left( Q_{\tau}^{L_0} \exp \left( \sum_{n \geq 1} \frac{Q_m^n - 1}{n(q^{n/2} - q^{-n/2})} \alpha_n \right) \exp \left( \sum_{n \geq 1} \frac{Q_{-m}^{-1}}{n(q^{n/2} - q^{-n/2})} \alpha_{-n} \right) \right) \tag{5.2}
\]

Using the commutation relation of \( \alpha_{\pm} \) it can be written as

\[
Z(\tau, m, \epsilon) = \prod_k (1 - Q_k^{L_0})^{-1} \prod_{i,j} \frac{(1 - Q_k Q_m^{-1} q^{i+j-1})(1 - Q_k^{-1} Q_m q^{i+j-1})}{(1 - Q_k^{i+j-1})} \tag{5.3}
\]

cf. \[12\]. The partition function in (8.4) can be generalized to

\[
Z(\tau, m, \epsilon, t) = Tr \left( Q_{\tau}^{L_0} e^{\sum_n t_n L_n} \exp \left( \sum_{n \geq 1} \frac{Q_m^n - 1}{n(q^{n/2} - q^{-n/2})} \alpha_n \right) \exp \left( \sum_{n \geq 1} \frac{Q_{-m}^{-1}}{n(q^{n/2} - q^{-n/2})} \alpha_{-n} \right) \right) \tag{5.4}
\]

In the limit \( m \mapsto 0 \) we obtain

\[
Z(\tau, m = 0, \epsilon, t) = Tr \left( Q_{\tau}^{L_0} e^{\sum_n t_n L_n} \right) \tag{5.5}
\]

We can write the partition function in Remark 8.2 in terms of the Gromov-Witten potentials

\[
Z(\tau, m, \epsilon) = \exp \left( \sum_{g \geq 0} e^{2g-2} F_g \right) \tag{5.6}
\]

where

\[
e^{F_1} = \prod_k (1 - Q_k^{L_0})^{-1} \left( \frac{(1 - Q_k^2) Q_m^{-1} (1 - Q_k Q_m)^2}{(1 - Q_k^4)^{1/24}} \right) \tag{5.7}
\]

One may ask if the above equation is equal to 6.8 with \( q_2 = q_3 = \ldots = 1 \).

**Problem:** Consider the following trace as a twist of the two former ones,

\[
Tr \left( \exp \left( \sum_{j \geq 0} 2\pi i L_j \right) \exp \left( \sum_{n \geq 0} A_n \alpha_{-n} \right) \exp \left( \sum_{n \geq 0} B_n \alpha_n \right) \right) \tag{5.8}
\]

We pose the following questions;

- How to compute the trace in terms of the former traces.
• How the last trace is related to the former two characters in Theorems 8.1 and 8.2. What is the representation theory interpretation of that.

• In case that the coefficients $A_n, B_n$ are suitably chosen what is the Physical interpretation of the trace in terms of string theory partition functions.

• Is there any product formula for the trace.

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