The Hardy-Littlewood theorem for double Fourier-Haar series from Lebesgue spaces $L_{\bar{p}}[0,1]$ with mixed metric and from net spaces $N_{\bar{p},\bar{q}}(M)$

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Abstract. In terms of the Fourier-Haar coefficients, a criterion is obtained for the function $f(x_1,x_2)$ to belong to the net space $N_{\bar{p},\bar{q}}(M)$ and to the Lebesgue space $L_{\bar{p}}[0,1]^2$ with mixed metric, where $1 < \bar{p} < \infty$, $0 < \bar{q} \leq \infty$, $\bar{p} = (p_1,p_2)$, $\bar{q} = (q_1,q_2)$, $M$ is the set of all rectangles in $\mathbb{R}^2$. We proved the Hardy-Littlewood theorem for multiple Fourier-Haar series.

Key words: net space, Lebesgue space, anisotropic space, Fourier series, Haar system.

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1 Introduction

In studying the relationship between the integrability of a function and the summability of its Fourier coefficients, the most striking example is the Parseval equality

$$\int_0^1 |f(x)|^2 dx = \sum_{k=1}^{\infty} |a_k|^2,$$

where $a_k$ are Fourier coefficients by trigonometric system.

In the case, when $f \in L_p$, $p \neq 2$, here the Hardy-Littlewood inequalities hold: if $2 \leq p < \infty$, then

$$\|f\|_{L_p}^p \leq c_1 \sum_{k=1}^{\infty} k^{p-2}|a_k|^p,$$

if $1 \leq p \leq 2$, then

$$c_2 \sum_{k=1}^{\infty} k^{p-2}|a_k|^p \leq \|f\|_{L_p}^p.$$

For a function $f$ from $L_p$, the lower bounds for $p > 2$ and the upper bounds for $1 < p \leq 2$ are proved only under additional conditions.

Here we know the Hardy-Littlewood theorem \cite{1} for trigonometric series:

Let $1 < p < \infty$ and $f \sim \sum_{k=0}^{\infty} a_k \cos kx$. If $\{a_k\}_{k=0}^{\infty}$ is monotonically non-increasing sequence, or $f$ is monotone function, then for $f \in L_p[0,\pi]$ it is necessary and sufficient

$$\sum_{k=0}^{\infty} k^{p-2}|a_k|^p < \infty,$$

and the relation is fulfilled

$$\|f\|_{L_p}^p \asymp \sum_{k=0}^{\infty} k^{p-2}|a_k|^p.$$
As we can see, the conditions for monotone functions and functions with monotone coefficients to belong to the space $L_p$ are the same, namely, the convergence of the series:

$$\sum_{k=0}^{\infty} k^{p-2}|a_k|^p.$$ 

For series by Haar system the situation is different. P.L. Ul’yanov in [2] proved, that if the Fourier-Haar coefficients $\{a_k\}_{k=1}^{\infty}$ are monotonous, then in order to the function $f \in L_p(0,1]$ at $1 < p < \infty$ it is necessary and sufficient that $\{a_k\}_{k=1}^{\infty} \in l_2$, i.e. that the series $\sum_{k=1}^{\infty} |a_k|^2$ be converged.

Nursultanov E.D. and Aubakirov T.U. in paper [3] proved the following statement:

Let $1 < p < \infty$, $f$ is a monotone function. Then in order to $f \in L_p(0,1]$ it is necessary and sufficient that for the sequence of its Fourier-Haar coefficients $\{a_k\}_{k=0,j=1}^{\infty,2^k}$ the condition was met:

$$\left(\sum_{k=0}^{\infty} \left(2^k \left(\frac{1}{2} - \frac{1}{p}\right) \sup_{1 \leq j \leq 2^k} |a_j^k|\right)^p\right)^\frac{1}{p} < \infty.$$ 

For multiple trigonometric series, analogues of the Hardy-Littlewood theorem were obtained by F. Moritz [4], M.I. Dyachenko [5, 6].

If the coefficients $a = \{a_{k_1,k_2}\}_{k_1=0,k_2=0}^{\infty,\infty}$ are monotonic by each variable index $k_1, k_2$, then as showed M.I. Dyachenko [5], the convergence of the number series

$$\sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} k_1^{p-2} k_2^{p-2}|a_{k_1,k_2}|^p$$

is equivalent to $f \in L_p(\mathbb{T}^2)$ only when $\frac{4}{3} < p < \infty$ (in the case $2 \leq n$, for $\frac{2n}{n+1} < p < \infty$).

The goal of our work is to obtain the Hardy-Littlewood theorem for multiple Fourier-Haar series in Lebesgue spaces $L_p[0,1]^2$ with mixed metric and in anisotropic net spaces $N_{p,q}(M)$.

## 2 Main results

The Haar system is a system of functions $\chi = \{\chi_n(x)\}_{n=1}^{\infty}, x \in [0,1]$, in which $\chi_1(x) \equiv 1$, and the function $\chi_n(x)$ where $n = 2^k + j$, where $k = 0, 1, \ldots, j = 1, 2, \ldots, 2^k$ is defined as:

$$\chi_n(x) = \chi^j_k(x) = \begin{cases} 
2^{\frac{k}{2}}, & \frac{2j}{2^{k+1}} < x < \frac{2j-1}{2^{k+1}} \\
2^{\frac{k}{2}}, & \frac{2j-1}{2^{k+1}} < x < \frac{2j}{2^{k+1}} \\
0, & x \notin \left(\frac{j-1}{2^k}, \frac{j}{2^k}\right) 
\end{cases}$$
For the function \( f(x_1, x_2) \in L_1[0, 1]^2 \) consider its Fourier-Haar series

\[
f(x_1, x_2) \sim \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{j_1=1}^{2^{k_1}} \sum_{j_2=1}^{2^{k_2}} a_{k_1,k_2}^{j_1,j_2}(f) \chi_{k_1}^{j_1}(x_1) \chi_{k_2}^{j_2}(x_2),
\]

where \( a_{k_1,k_2}^{j_1,j_2}(f) \) are the Fourier-Haar coefficients of the function \( f(x_1, x_2) \) are defined as follows:

\[
a_{k_1,k_2}^{j_1,j_2} = \int_{0}^{1} \int_{0}^{1} f(x_1, x_2) \chi_{k_1}^{j_1}(x_1) \chi_{k_2}^{j_2}(x_2) dx_1 dx_2.
\]

For \( \bar{\sigma} = (\sigma_1, \sigma_2) \in \mathbb{R}^2 \), \( 1 \leq \bar{q} \leq \infty \) define space \( l_q^\bar{q}(l_\infty) \), as a set of all sequences \( a = \{a_{k_1,k_2}^{j_1,j_2} : k_i \in \mathbb{Z}_+, 1 \leq j \leq 2^{k_i}, i = 1, 2 \} \), for which is finite the norm

\[
\|a\|_{l_q^\bar{q}(l_\infty)} = \left( \sum_{k_1=0}^{\infty} \left( \sum_{k_2=0}^{\infty} \left( 2^{\sigma_1 k_1 + \sigma_2 k_2} \sup_{1 \leq j \leq \infty} |a_{k_1,k_2}^{j_1,j_2}| \right)^q \right)^{\frac{1}{q}} \right)^\frac{1}{\bar{q}},
\]

hereinafter expression \( \left( \sum_{k=0}^{\infty} b_k \right)^{\frac{1}{q}} \) in the case, when \( q = \infty \), is understood as \( \sup_{k \geq 0} b_k \).

Let \( M \) is set of all rectangles \( Q = Q_1 \times Q_2 \) from \( \mathbb{R}^2 \), for function \( f(x_1, x_2) \) integrable on each set \( Q \in M \) define

\[
\bar{f}(t_1, t_2; M) = \sup_{|Q_i| \geq t_i} \frac{1}{|Q_1||Q_2|} \left| \int_{Q_1} \int_{Q_2} f(x_1, x_2) dx_1 dx_2 \right|, \quad t_i > 0,
\]

where \( |Q_i| \) is the \( Q_i \) segment length.

Let \( 0 < \bar{p} = (p_1, p_2) < \infty \), \( 0 < \bar{q} = (q_1, q_2) \leq \infty \). By \( N_{\bar{p},\bar{q}}(M) \) denote the set of all functions \( f(x_1, x_2) \), for which

\[
\|f\|_{N_{\bar{p},\bar{q}}(M)} = \left( \int_{0}^{\infty} \left( \int_{0}^{\infty} \left( \frac{1}{t_1^{p_1}} \frac{1}{t_2^{p_2}} \bar{f}(t_1, t_2; M) \right)^{q_1} dt_1 \right)^{\frac{q_1}{m_1}} \frac{dt_2}{t_2} \right)^{\frac{1}{q_2}} < \infty,
\]

hereinafter, when \( q = \infty \), expression \( \left( \int_{0}^{\infty} (\varphi(t))^{\frac{q_1}{m_1}} \frac{dt}{t} \right)^{\frac{1}{q_2}} \) is understood as \( \sup_{t > 0} \varphi(t) \).

Spaces of this type were introduced and studied in [7] and were called net spaces. Net spaces are an important research tool in the theory of Fourier series, in operator theory and in other directions.

**Theorem 1.** Let \( 1 < \bar{p} < \infty \), \( 0 < \bar{q} \leq \infty \), \( \bar{\sigma} = \frac{1}{2} - \frac{1}{\bar{p}} \), \( M \) is the set of all rectangles in \( [0, 1]^2 \). Then, for \( f \in N_{\bar{p},\bar{q}}(M) \) it is necessary and sufficient that the sequence of its Fourier-Haar coefficients \( a = \{a_{k_1,k_2}^{j_1,j_2} : k_i \in \mathbb{Z}_+, 1 \leq j \leq 2^{k_i}, i = 1, 2 \} \) belonged to the space \( l_q^\bar{q}(l_\infty) \), and is fulfilled the relation

\[
\|f\|_{N_{\bar{p},\bar{q}}(M)} \asymp \|a\|_{l_q^\bar{q}(l_\infty)}.
\]
Note, that for spaces \( N_{p,q}(M) \), the relation (1) holds without any additional conditions on the function \( f \) and its Fourier coefficients. Thus, for the net spaces \( N_{p,q}(M) \), an analogue of Parseval’s equality holds for all \( 1 < \tilde{p} < \infty \).

Let \( 0 < \tilde{p} \leq \infty \). The space \( L_{\tilde{p}}[0,1]^2 \), called the Lebesgue space with a mixed metric, is defined as the set of functions \( f \) measurable on \([0, 1]^2\) for which is finite quantity

\[
\|f\|_{L_{\tilde{p}}[0,1]^2} := \left( \int_0^1 \left( \int_0^1 |f(x_1, x_2)|^{\tilde{p}_1} dx_1 \right)^{\frac{\tilde{p}_2}{\tilde{p}_1}} dx_2 \right)^{\frac{1}{\tilde{p}_2}}.
\]

The function \( f(x_1, x_2) \) is called monotonically non-increasing by each variable, if for \( 0 \leq y_1 \leq x_1 \) and \( 0 \leq y_2 \leq x_2 \) the inequality is holds

\[
f(x_1, x_2) \leq f(y_1, y_2).
\]

**Theorem 2.** Let \( 1 < \bar{p} < \infty, \tilde{\sigma} = \frac{1}{2} - \frac{1}{\bar{p}} \), \( f(x_1, x_2) \) is monotonically non-increasing function by each variable. Then, for \( f \in L_{\bar{p}}[0,1]^2 \) it is necessary and sufficient that the sequence of its Fourier-Haar coefficients

\[
a = \{a_{k_1k_2j}^{i_1i_2} : k_i \in \mathbb{Z}_+, 1 \leq j \leq 2^{k_i}, i = 1, 2\}
\]

belonged to the space \( l_\bar{p}^q(l_\infty) \), and is fulfilled the relation

\[
\|f\|_{L_{\bar{p}}[0,1]^2} \asymp \|a\|_{l_\bar{p}^q(l_\infty)}.
\]

**Remark 1.** The condition of a monotonically non-increasing function in Theorem 2 can be replaced by monotonicity in each variable.

## 3 Interpolation theorems for anisotropic spaces

In this section, we will consider an interpolation method for anisotropic spaces from the work [8]. This method is based on ideas from the works of Sparr G. [9], Fernandez D.L. [10] [11] [12] and others [13].

Let \( A_0 = (A_0^1, A_0^2) \), \( A_1 = (A_1^1, A_1^2) \) are two anisotropic spaces, \( E = \{\varepsilon = (\varepsilon_1, \varepsilon_2) : \varepsilon_1 = 0, \text{ or } \varepsilon_1 = 1, \varepsilon_2 = 0, \varepsilon_2 = 1, \varepsilon_i = 1, i = 1, 2\} \). For arbitrary \( \varepsilon \in E \) define the space \( A_\varepsilon = (A_\varepsilon^1, A_\varepsilon^2) \) with norm

\[
\|a\|_{A_\varepsilon} = \|\|a\|_{A_\varepsilon^1}\|_{A_\varepsilon^2}.
\]

A pair of anisotropic spaces \( A_0 = (A_0^1, A_0^2), A_1 = (A_1^1, A_1^2) \) will be called compatible, if there is a linear Hausdorff space, containing as subsets the spaces \( A_\varepsilon, \varepsilon \in E \).

Let \( 0 < \bar{\theta} = (\theta_1, \theta_2) < 1, 0 < \bar{q} = (q_1, q_2) \leq \infty \). By \( A_{\bar{\theta}, \bar{q}} = (A_0, A_1)_{\bar{\theta}, \bar{q}} \) we denote the linear subset \( \sum_{\varepsilon \in E} A_\varepsilon \), for whose elements it is true:

\[
\|a\|_{A_{\bar{\theta}, \bar{q}}} = \left( \int_0^\infty \left( \int_0^\infty \left( t_1^{-\theta_1} t_2^{-\theta_2} K(t_1, t_2) \right)^{q_1} dt_1 \right)^{\frac{1}{q_1}} dt_2 \right)^{\frac{1}{q_2}} < \infty,
\]

where

\[
K(t, a; A_0, A_1) = \inf \{ \sum_{\varepsilon \in E} t^\varepsilon \|a_\varepsilon\|_{A_\varepsilon} : a = \sum_{\varepsilon \in E} a_\varepsilon, a_\varepsilon \in A_\varepsilon \},
\]
where $t^c = t_1^{c_1} t_2^{c_2}$.

We need a theorem from [14], which we formulate for our case, when $A = l_\infty$.

**Theorem 3.** Let $\sigma_0 = (\sigma_0^1, \sigma_0^2) > \sigma_1 = (\sigma_1^1, \sigma_1^2)$, $0 \leq \bar{q}, \bar{q}_0, \bar{q}_1 \leq \infty$, $0 < \bar{\theta} = (\theta_1, \theta_2) < 1$ then will be true the equality

$$\left( l_{\bar{q}_0}^\sigma(l_\infty), l_{\bar{q}_1}^\sigma(l_\infty) \right)_{\bar{\theta}, \bar{q}} = l_{\bar{q}}^\bar{\sigma}(l_\infty),$$

where $\bar{\sigma} = (1 - \bar{\theta}) \bar{\sigma}_0 + \bar{\theta} \bar{\sigma}_1$.

The following interpolation theorem holds for anisotropic net spaces

**Theorem 4.** Let $M$ is the set of all rectangles in $\mathbb{R}^2$, $1 \leq \bar{p}_0 < \bar{p}_1 < \infty$, $1 \leq \bar{q}_0, \bar{q}_1 \leq \infty$, $0 < \bar{\theta} = (\theta_1, \theta_2) < 1$ then

$$(N_{\bar{p}_0, \bar{q}_0}(M), N_{\bar{p}_1, \bar{q}_1}(M))_{\bar{\theta}, \bar{q}} = N_{\bar{p}, \bar{q}}(M),$$

where $\frac{1}{\bar{p}} = \frac{1 - \bar{\theta}}{\bar{p}_0} + \frac{\bar{\theta}}{\bar{p}_1}$.

This theorem was proved in [15]. The method of proving theorem 4 differs from the proof of theorem 3. In the proof of Theorem 3, the property of ideality of the spaces $l_{\bar{q}}^\sigma(l_\infty)$ was essentially used, and the spaces $N_{\bar{p}, \bar{q}}(M)$ are not ideal. Therefore, the proof uses other ideas.

## 4 Proof of the theorem

Let function $f \in N_{\bar{p}, \infty}(M)$, $a(f) = \{a_{k_i}^{j_1,j_2} : k_i \in \mathbb{Z}_+, 1 \leq j \leq 2^{k_i}, i = 1, 2\}$ are its Fourier coefficients by the Haar system. Let us prove the inequality

$$\|a(f)\|_{l_{\bar{p}}^\sigma(l_\infty)} \leq C \|f\|_{N_{\bar{p}, \infty}(M)},$$

where $\bar{\sigma} = \frac{1}{2} - \frac{1}{\bar{p}}$, $1 < \bar{p} < \infty$.

By the definition of the space $l_{\bar{p}}^\sigma(l_\infty)$:

$$\|a(f)\|_{l_{\bar{p}}^\sigma(l_\infty)} = \sup_{k_i \geq 0} 2^{k_i} \left( \frac{1}{2} - \frac{1}{\bar{p}_1} \right)^{+} + k_2 \left( \frac{1}{2} - \frac{1}{\bar{p}_2} \right) \max_{1 \leq j \leq 2^{k_i}} |a_{k_i}^{j_1,j_2}|.$$

Note, that from the definition of the Fourier-Haar coefficients, we have

$$a_{k_1,k_2}^{j_1,j_2}(f) = 2^{k_1+\frac{k_2}{2}} \left[ \int_{(\Delta_{k_1}^{j_1})^+} \int_{(\Delta_{k_2}^{j_2})^+} f(x_1, x_2) dx_1 dx_2 - \int_{(\Delta_{k_1}^{j_1})^-} \int_{(\Delta_{k_2}^{j_2})^+} f(x_1, x_2) dx_1 dx_2 - \int_{(\Delta_{k_1}^{j_1})^+} \int_{(\Delta_{k_2}^{j_2})^-} f(x_1, x_2) dx_1 dx_2 + \int_{(\Delta_{k_1}^{j_1})^-} \int_{(\Delta_{k_2}^{j_2})^-} f(x_1, x_2) dx_1 dx_2 \right],$$
where
\[(\Delta_{j_1}^+) = \left( \frac{2j_1 - 2}{2^{k_1+1}}, \frac{2j_1 - 1}{2^{k_1+1}} \right), \quad (\Delta_{j_1}^-) = \left( \frac{2j_1 - 1}{2^{k_1+1}}, \frac{2j_1}{2^{k_1+1}} \right),\]
\[(\Delta_{j_2}^+) = \left( \frac{2j_2 - 2}{2^{k_2+1}}, \frac{2j_2 - 1}{2^{k_2+1}} \right), \quad (\Delta_{j_2}^-) = \left( \frac{2j_2 - 1}{2^{k_2+1}}, \frac{2j_2}{2^{k_2+1}} \right).\]

Then, given that the lengths of the segments \(|(\Delta)^+| = |(\Delta)^-| = \frac{1}{2^{k_1+1}}\), we get
\[
\|a(f)\|_{L^p_{\infty}}(l_\infty) \leq 4 \cdot 2^{-\frac{1}{p_1} - \frac{1}{p_2}} \sup_{Q_1 \times Q_2 \subseteq M} \frac{1}{|Q_1|^{\frac{1}{p_1}}} \frac{1}{|Q_2|^{\frac{1}{p_2}}} \left| \int_{Q_1} \int_{Q_2} f(x_1, x_2) dx_1 dx_2 \right| = \frac{2^{\frac{1}{p_1} + \frac{1}{p_2}}}{p} \|f\|_{N_{0, \infty}(M)}.
\]

We define the operator \(Tf = \{a_{j_1, j_2}^{j_1}\}_{j_1, j_2}(f)\). Let \(\bar{p}\) satisfies the condition of the theorem and \(\bar{p}_0, \bar{p}_1, \bar{p}, \bar{q}, \bar{\sigma}_1, \bar{\sigma}_2\) are such that \(1 < \bar{p}_0 < \bar{p} < \bar{p}_1 < \infty\), where \(\bar{p}_0 = (p_{0_1}, p_{0_2}), \bar{p}_1 = (p_{1_1}, p_{1_2})\) and \(\sigma_{0}^1 = \frac{1}{2} - \frac{1}{p_{0_1}}, \sigma_{0}^2 = \frac{1}{2} - \frac{1}{p_{0_2}}, \sigma_{1}^1 = \frac{1}{2} - \frac{1}{p_{1_1}}, \sigma_{1}^2 = \frac{1}{2} - \frac{1}{p_{1_2}}\).

Then it follows from the last inequality, that for a given operator:
\[
T : N_{(p_0^1, p_0^2), \infty}(M) \rightarrow l_{\infty}^{(\sigma_0^1, \sigma_0^2)}(l_{\infty}),
\]
\[
T : N_{(p_1^1, p_1^2), \infty}(M) \rightarrow l_{\infty}^{(\sigma_1^1, \sigma_1^2)}(l_{\infty}),
\]
\[
T : N_{(p_0^0, p_1^0), \infty}(M) \rightarrow l_{\infty}^{q}(l_{\infty}),
\]
\[
T : N_{(p_0^1, p_1^0), \infty}(M) \rightarrow l_{\infty}^{q}(l_{\infty}).
\]

Therefore,
\[
T : (N_{\bar{p}_0, \infty}(M), N_{\bar{p}_1, \infty}(M))_{\bar{\theta}, \bar{q}} \rightarrow (l_{\infty}^{\sigma_0}(l_{\infty}), l_{\infty}^{\sigma_1}(l_{\infty}))_{\bar{\theta}, \bar{q}}.
\]

According to the interpolation theorems [3] and [4] we have:
\[
(l_{\infty}^{\sigma_0}(l_{\infty}), l_{\infty}^{\sigma_1}(l_{\infty}))_{\bar{\theta}, \bar{q}} = l_{\bar{q}}^{\sigma}(l_{\infty}), \quad (N_{\bar{p}_0, \infty}(M), N_{\bar{p}_1, \infty}(M))_{\bar{\theta}, \bar{q}} = N_{\bar{p}, \bar{q}}(M).
\]

Therefore, \(T : N_{\bar{p}, \bar{q}}(M) \rightarrow l_{\infty}^{\sigma}(l_{\infty})\).

Thus, we get
\[
\|a(f)\|_{l_{\infty}^{\bar{q}}(l_{\infty})} \leq C \|f\|_{N_{\bar{p}, \bar{q}}(M)},
\]
where \(\frac{1}{\bar{p}} = \frac{1}{\bar{p}_0} + \frac{\bar{\theta}}{\bar{p}_1}, \bar{\sigma} = (1 - \bar{\theta})\bar{\sigma}_0 + \bar{\theta}\bar{\sigma}_1 = \frac{1}{2} - \frac{1}{\bar{p}}\).

Let us show the reverse inequality. Let \(N_1, N_2 \in \mathbb{N}, a = \{a_{k_1, k_2}^{j_1, j_2} : k_i \in \mathbb{Z}_+, 1 \leq j \leq 2^{k_i}, i = 1, 2\} \in l_{\bar{q}}^{\sigma}(l_{\infty}).\) Consider a polynomial
\[
S_{N_1, N_2}(a; x_1, x_2) = \sum_{k_1 = 0}^{N_1} \sum_{k_2 = 0}^{N_2} \sum_{j_1 = 1}^{2^{k_1}} \sum_{j_2 = 1}^{2^{k_2}} a_{k_1, k_2}^{j_1, j_2} \chi_{k_1}^{j_1}(x_1) \chi_{k_2}^{j_2}(x_2).
\]
\[ Q = Q_1 \times Q_2 \] an arbitrary rectangle from the net \( M \). Then

\[
\frac{1}{|Q_1|^{\frac{1}{p_1}}|Q_2|^{\frac{1}{p_2}}} \left| \int_{Q_2} \int_{Q_1} S_{N_1,N_2}(a; x_1, x_2) dx_1 dx_2 \right|
\leq \sum_{k_1=0}^{N_1} \sum_{k_2=0}^{N_2} \frac{1}{|Q_1|^{\frac{1}{p_1}}|Q_2|^{\frac{1}{p_2}}} \left| \int_{Q_2} \int_{Q_1} \sum_{j_1=1}^{2^{k_1}} \sum_{j_2=1}^{2^{k_2}} a_{k_1,k_2}^{j_1,j_2} \chi_{k_1}(x_1) \chi_{k_2}(x_2) dx_1 dx_2 \right|
= \sum_{k_1=0}^{N_1} \sum_{k_2=0}^{N_2} \frac{1}{|Q_1|^{\frac{1}{p_1}}|Q_2|^{\frac{1}{p_2}}} \left| \sum_{j_1=1}^{2^{k_1}} \sum_{j_2=1}^{2^{k_2}} a_{k_1,k_2}^{j_1,j_2} \int_{Q_2} \chi_{k_2}(x_2) dx_2 \int_{Q_1} \chi_{k_1}(x_1) dx_1 \right|.
\]

From the definitions of the functions \( \chi_{k_1}^{j_1}(x_1) \) and \( \chi_{k_2}^{j_2}(x_2) \) it follows that at most four terms in the sum

\[
\sum_{j_1=1}^{2^{k_1}} \sum_{j_2=1}^{2^{k_2}} a_{k_1,k_2}^{j_1,j_2} \int_{Q_2} \chi_{k_2}^{j_2}(x_2) dx_2 \int_{Q_1} \chi_{k_1}^{j_1}(x_1) dx_1
\]
are nonzero, namely, those terms where the supports of the functions \( \chi_{k_1}^{j_1}(x_1) \), \( \chi_{k_2}^{j_2}(x_2) \) contain, respectively, the ends of the segments \( Q_1, Q_2 \). Therefore,

\[
\frac{1}{|Q_1|^{\frac{1}{p_1}}|Q_2|^{\frac{1}{p_2}}} \left| \int_{Q_2} \int_{Q_1} S_{N_1,N_2}(a; x_1, x_2) dx_1 dx_2 \right|
\leq 4 \sum_{k_1=0}^{N_1} \sum_{k_2=0}^{N_2} \frac{1}{|Q_1|^{\frac{1}{p_1}}|Q_2|^{\frac{1}{p_2}}} \max_{1 \leq i \leq 2^{k_1}} a_{k_1,k_2}^{ji,j2} \min \left( |Q_1|, \frac{1}{2^{k_1}} \right) \min \left( |Q_2|, \frac{1}{2^{k_2}} \right).
\]

Note, that

\[
2^{k_1} \frac{1}{|Q_1|^{\frac{1}{p_1}}} \min \left( |Q_1|, \frac{1}{2^{k_1}} \right) \leq 2^{k_1} \left( \frac{1}{2} - \frac{1}{p_1} \right),
\]

therefore,

\[
\frac{1}{|Q_1|^{\frac{1}{p_1}}|Q_2|^{\frac{1}{p_2}}} \left| \int_{Q_2} \int_{Q_1} S_{N_1,N_2}(a; x_1, x_2) dx_1 dx_2 \right|
\leq 4 \sum_{k_1=0}^{N_1} \sum_{k_2=0}^{N_2} 2^{k_1} \left( \frac{1}{2} - \frac{1}{p_1} \right) + 2^{k_1} \left( \frac{1}{2} - \frac{1}{p_2} \right) \max_{1 \leq i \leq 2^{k_1}} a_{k_1,k_2}^{ji,j2}.
\]

Since the choice of the segments \( Q_1 \) and \( Q_2 \) is arbitrary, we obtain

\[
\|S_{N_1,N_2}(a; x_1, x_2)\|_{N_\infty} = \sup_{t_i > 0, i=1,2} \frac{1}{t_1^{\frac{1}{p_1}}t_2^{\frac{1}{p_2}}} \sup_{|Q_1| \geq t_1} \frac{1}{|Q_1||Q_2|} \int_{Q_2} \int_{Q_1} S_{N_1,N_2}(a; x_1, x_2) dx_1 dx_2
\]

\[
\leq \sup_{Q_1 \times Q_2 \in M} \frac{1}{|Q_1|^{\frac{1}{p_1}}|Q_2|^{\frac{1}{p_2}}} \left| \int_{Q_2} \int_{Q_1} S_{N_1,N_2}(a; x_1, x_2) dx_1 dx_2 \right|
\leq 4 \|a\|_{l_i}^{1}(t_\infty).
\]
Let $\bar{p}_0, \bar{p}_1, \sigma_0, \bar{\sigma}_1$ such that $1 < \bar{p}_0 < \bar{p} < \bar{p}_1 < \infty$, $\sigma_0 = \frac{1}{2} - \frac{1}{\bar{p}_0}$, $\bar{\sigma}_1 = \frac{1}{2} - \frac{1}{\bar{p}_1}$. Consider the operator $Ta = S_{N_1, N_2}(a; x_1, x_2)$. It follows from the last inequality that for this operator

$$T : l^{(\sigma_0, \sigma_2)}_1(l_\infty) \rightarrow N_{(\sigma_0, \sigma_2), \infty}(M),$$

$$T : l^{(\sigma_0, \sigma_1)}_1(l_\infty) \rightarrow N_{(\sigma_0, \sigma_1), \infty}(M),$$

$$T : l^{(\sigma_1, \sigma_2)}_1(l_\infty) \rightarrow N_{(\sigma_1, \sigma_2), \infty}(M),$$

$$T : l^{(\sigma_1, \sigma_1)}_1(l_\infty) \rightarrow N_{(\sigma_1, \sigma_1), \infty}(M).$$

Then

$$T : (l^{(\sigma_0, \infty)}_1, l^{(\sigma_1, \infty)}_1)_{\bar{p}, \bar{q}} \rightarrow (N_{\bar{p}, \infty}(M), N_{\bar{q}, \infty}(M))_{\bar{p}, \bar{q}},$$

Hence, we have

$$T : l^{(\sigma_0, \infty)}_1(l_\infty) \rightarrow N_{\bar{p}, \bar{q}}(M),$$

where $\frac{1}{\bar{p}} = \frac{1 - \bar{q}}{\bar{p}_0} + \frac{\bar{q}}{\bar{p}_1}$, $\bar{\sigma} = (1 - \bar{\theta})\sigma_0 + \bar{\theta}\bar{\sigma}_1 = \frac{1}{\bar{p}} - \frac{1}{\bar{q}}$ and therefore, the inequality holds

$$\|S_{N_1, N_2}(a; x_1, x_2)\|_{N_{\bar{p}, \bar{q}}(M)} \leq C\|a\|l^{(\sigma_0, \infty)}_1(l_\infty).$$

Next, we use the fact that the space $N_{\bar{p}, \bar{q}}(M)$ is a Banach space (see [16]) and therefore $S_{N_1, N_2}(a; x_1, x_2)$ converges to some function $f \in N_{\bar{p}, \bar{q}}(M)$ for $N_1, N_2 \rightarrow +\infty$.

## 5 Proof of the theorem

First, we give a lemma.

**Lemma 1.** Let $1 < p < \infty$, $\varphi \in L_p[0, 1]$, then

$$\|\varphi\|_{L_p[0, 1]} \geq \left( \sum_{k=0}^{\infty} \left( 2^{-\frac{k}{p}} \varphi^{**}(2^k) \right)^p \right)^{\frac{1}{p}},$$

where

$$\varphi^{**}(t) = \frac{1}{t} \int_0^t \varphi^*(s) ds = \sup_{|e|=t} \frac{1}{|e|} \int_e |\varphi(x)| dx.$$

**Proof.**

$$\|\varphi\|_{L_p[0, 1]} \geq \left( \int_0^1 (\varphi^{**}(t))^p dt \right)^{\frac{1}{p}} = \left( \sum_{k=0}^{\infty} \int_{2^{-(k+1)}}^{2^{-k}} (\varphi^{**}(t))^p dt \right)^{\frac{1}{p}} \geq \left( \sum_{k=0}^{\infty} 2^{-k} (\varphi^{**}(2^k))^p \right)^{\frac{1}{p}}.$$
Proof of the theorem

Note, that since \( f(x_1, x_2) \) is monotonic, we have

\[
|a_{j_{1j_2}}^{i_{1i_2}}| = \left| \int_{\Delta_{j_{1i_2}}^{k_1}} \int_{\Delta_{j_{1i_2}}^{k_2}} f(x_1, x_2) \chi_{j_{1i_2}}^{i_{1i_2}}(x_1, x_2) dx_1 dx_2 \right| \leq
\]

\[
\leq 2^{\frac{k_1}{2} + \frac{k_2}{2}} \int_{\Delta_{j_{1i_2}}^{k_1}} \int_{\Delta_{j_{1i_2}}^{k_2}} |f(x_1, x_2)| dx_1 dx_2 \leq
\]

\[
\leq 2^{\frac{k_1}{2} + \frac{k_2}{2}} \int_0^{2^{-k_2}} \int_0^{2^{-k_1}} f(x_1, x_2) dx_1 dx_2.
\]

Then from this estimate, the Minkowski inequalities and from the lemma imply

\[
\|a(f)\|_{L_p(\Omega)}) \leq \left( \sum_{k_2=0}^{\infty} \left( \sum_{k_1=0}^{\infty} \left( 2^{1 - \frac{1}{p_1}} k_1 + \left( 1 - \frac{1}{p_2} \right) k_2 \int_0^{2^{-k_2}} \int_0^{2^{-k_1}} f(x_1, x_2) dx_1 dx_2 \right)^{p_1} \right)^{\frac{1}{p_1}} \right)^{\frac{1}{p_2}} \leq
\]

\[
\leq \left( \sum_{k_2=0}^{\infty} \left( \sum_{k_1=0}^{\infty} \left( \left( 2^{1 - \frac{1}{p_1}} k_1 \int_0^{2^{-k_2}} \int_0^{2^{-k_1}} f(x_1, x_2) dx_1 dx_2 \right)^{p_1} \right)^{\frac{1}{p_1}} \right)^{\frac{1}{p_2}} \right)^{\frac{1}{p_2}} =
\]

\[
= \left( \sum_{k_2=0}^{\infty} \left( 2^{1 - \frac{k_2}{p_2}} \int_0^{2^{-k_2}} \varphi(x_2) dx_2 \right)^{\frac{1}{p_2}} \right) \leq \left( \sum_{k_2=0}^{\infty} \left( 2^{-\frac{k_2}{p_2}} \varphi^*(2^{-k}) \right)^{\frac{1}{p_2}} \right) \times
\]

\[
\times \left( \int_0^1 (\varphi^*(t))^{\frac{1}{p_2}} \right)^{\frac{1}{p_2}} = \|\varphi\|_{L_{p_2}},
\]

here

\[
\varphi(x_2) = \left( \sum_{k_1=0}^{\infty} \left( 2^{1 - \frac{k_1}{p_1}} \int_0^{2^{-k_1}} |f(x_1, x_2)| dx_1 \right)^{p_1} \right)^{\frac{1}{p_1}}.
\]

Similarly,

\[
\varphi(x_2) \leq \left( \int_0^1 |f(x_1, x_2)|^{p_1} dx_1 \right)^{\frac{1}{p_1}}.
\]

Thus, we get

\[
\|a(f)\|_{L_p(\Omega)} \leq c\|f\|_{L_{p_2}[0, 1]^2}.
\]

Let us show the reverse inequality. Since \( f(x_1, x_2) \) is a monotonically non-increasing by each variable function, then

\[
f(x_1, x_2) \leq \frac{1}{x_1 x_2} \int_0^{x_1} \int_0^{x_2} f(y_1, y_2) dy_1 dy_2 \leq \bar{f}(x_1, x_2; M).
\]
Therefore, from theorem 1 it follows

\[ \| f \|_{L_p[0,1]^2} \leq \| f \|_{N_{p,M}} \leq c \| a(f) \|_{l_q^p}(l_\infty). \]

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