A tutorial on the range variant of asymmetric numeral systems

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Abstract
This paper is intended to be an accessible introduction to the range variant of Asymmetric Numeral Systems (rANS). This version of ANS can be used as a drop in replacement for traditional arithmetic coding (AC). Implementing rANS is more straightforward than AC, and this paper includes pseudo-code which could be converted without too much effort into a working implementation. An example implementation, based on this tutorial, is available at https://raw.githubusercontent.com/j-towns/ans-notes/master/rans.py. After reading (and understanding) this tutorial, the reader should understand how rANS works, and be able to implement it and prove that it attains a near optimal compression rate.

1 Introduction
We are interested in algorithms for lossless compression of sequences of data. The range variant of asymmetric numeral systems (ANS) is such an algorithm, and, like arithmetic coding (AC), it is close to optimal in terms of compression rate (Duda, 2009). The key difference between ANS and AC is that ANS is last-in-first-out (LIFO), or ‘stack-like’, while AC is first-in-first-out (FIFO), or ‘queue-like’.

ANS comprises two functions, which we denote \texttt{push} and \texttt{pop}, for encoding and decoding, respectively (the names refer to the analogous stack operations). The \texttt{push} function accepts some pre-compressed information \(m\) (short for ‘message’), and a symbol \(x\) to be compressed, and returns a piece of compressed information, \(m'\). Thus it has the signature

\[
\texttt{push} : (m, x) \mapsto m'.
\] (1)

The new compressed state, \(m'\), contains precisely the same information as the pair \((m, x)\), and therefore \texttt{push} can be inverted, giving a decoder mapping. The decoder, \texttt{pop}, maps from \(m'\) back to \(m, x\):

\[
\texttt{pop} : m' \mapsto (m, x).
\] (2)
The functions `push` and `pop` are inverse to one another, so $\text{push}(\text{pop}(m)) = m$ and $\text{pop}(\text{push}(m, x)) = (m, x)$.

Encoding and decoding both require knowledge of some model over symbols. We use $\mathcal{A} = \{a_1, \ldots, a_I\}$ to denote the alphabet from which symbols $x$ are drawn. We denote the probability mass function of the model $P$. Later we will need to assume that all probability masses are quantized to some precision $r_p$, i.e. that there exist integers $p_1, \ldots, p_I$ such that $P(a_i) = \frac{p_i}{2^{r_p}}$ for each $i = 1, \ldots, I$.

## Building an encoder/decoder pair

The problem which the ANS encoder solves is

**Problem 1** Given a sequence of random variables $X_1, \ldots, X_N$, find an invertible algorithm which will map any sample $x_1, \ldots, x_N$ to a binary message $b$, such that the length of $b$ is close to the information content $h(x_1, \ldots, x_N)$.

Given that the algorithm is invertible, we can reformulate Problem 1 in terms of the inverse. This leads to a different, but equivalent, problem statement:

**Problem 2** Given a sequence of random variables $X_1, \ldots, X_N$, find an invertible algorithm which maps a source of bits to a sequence $x_1, \ldots, x_N$, such that the number of bits observed is close to the information content $h(x_1, \ldots, x_N)$.

Our description of the details of ANS focuses on the decoding algorithm, because this leads to a more straightforward presentation. We describe the decoder and show that it solves Problem 2, then we show how to invert it to form an encoder.

The decoding algorithm we describe will be formed from a series of ANS `pop` operations, its inverse will be formed from ANS `push` operations.

### 2.1 The structure of the message

We use a pair $m = (s, t)$ as the data structure for $m$. The element $t$ is a stack of unsigned integers of some fixed precision $r_t$. This stack has its own push and pop operations, which we denote `stack_push` and `stack_pop` respectively. The element $s$ is an unsigned integer with precision $r_s$ where $r_s > r_t$. We need $s$ to be large to ensure our decoding is accurate, and so we also impose the constraint $s \geq 2^{r_s - r_t}$, more detail on how and why we do this is given below. In our implementation we use $r_t = 32$ and $r_s = 64$. The stack $t$, along with its pop operation, can model the ‘source of bits’ from our problem statement above.

### 2.2 Constructing the pop operation

Our strategy for performing a decode with `pop` will be to firstly to extract a symbol from $s$. We do this using a bijective function $d : \mathbb{N} \to \mathbb{N} \times \mathcal{A}$, which takes an integer $s$ as input and returns a pair $(s', x)$, where $s'$ is an integer and $x$ is a symbol. Thus `pop` begins
def pop(s, t):
    s', x = d(s)

We design the function d so that if \( s \geq 2^r \epsilon - r_t \), then
\[
\log s' \geq \log s - h(x) + \log(1 - \epsilon)
\] (3)

where \( \epsilon := 2^{r_t + r_t - r_s} \). We give details of d and prove eq. (3) below. Note that for small \( \epsilon \) we have \( \log(1 - \epsilon) \approx -\frac{\epsilon}{\ln 2} \), and thus this term is small.

After extracting a symbol using d, we check whether \( s' \) is below \( 2^r \epsilon - r_t \), and if it is we stack_pop an integer from t and move its contents into the lower order bits of \( s' \). We refer to this as ‘renormalization’. Having done this, we return the new message and the symbol x. The full definition of pop is thus

\[
\text{def pop}(s, t):
\text{    } s', x = d(s)
\text{    } s, t = \text{renorm}(s', t)
\text{    } \text{return } (s, t), x
\]

Renormalization is necessary to ensure that the \( s \) returned by pop satisfies \( s \geq 2^r \epsilon - r_t \) and is therefore is large enough that eq. (3) holds at the start of any future pop operation. The renorm function has a while loop, which pushes elements from t into the lower order bits of s until s is full to capacity. To be precise:

\[
\text{def renorm}(s, t):
\text{    } # While s has space for another element from t
\text{    } \text{while } s < 2^r \epsilon - r_t:
\text{    } \text{    } # Pop an element t_{top} from t
\text{    } \text{    } t, t_{top} = \text{stack_pop}(t)
\text{    } \text{    } # and push t_{top} into the lower bits of s
\text{    } \text{    } s = 2^r t \cdot s + t_{top}
\text{    } \text{return } s, t
\]

The condition \( s < 2^r \epsilon - r_t \) guarantees that \( 2^r t \cdot s + t_{top} < 2^r s \), and thus there can be no loss of information resulting from overflow. We also have
\[
\log(2^r t \cdot s + t_{top}) \geq r_t + \log s.
\] (4)

Applying this inequality repeatedly, once for each iteration of the while loop in renorm, we have
\[
\log s \geq \log s' + r_t \cdot [\# \text{ elements popped from } t]
\] (5)

where \( s, t = \text{renorm}(s', t) \) as in the definition of pop.
2.3 Popping in sequence

We now directly tackle the setup described in Problem 2, performing a sequence of pop operations to decode a sequence of data. We suppose that we are given some initial ‘message’ \(m_0 = (s_0, t_0)\), where \(2^{r_t - r_s} \leq s_0 < 2^{r_s}\) and \(t_0\) is a stack of infinite depth, modelling the ‘source of bits’ from the problem statement.

For \(n = 1 \ldots N\), we let \((s_n, t_n), x_n = \text{pop}(s_{n-1}, t_{n-1})\), where each pop operation uses the corresponding distribution \(P(x_n | x_1, \ldots, x_{n-1})\).

Now, applying eq. (3) and eq. (5) to the \(n\)th pop gives

\[
\log s_{n+1} \geq \log s_n - h(x_n | x_1, \ldots, x_{n-1}) + b_n + \log(1 - \epsilon) \quad (6)
\]

where \(b_n\) is the number of bits popped/observed from the stack \(t\) in the \(n\)th pop step. Applying eq. (6) recursively, for \(n = 1, \ldots, N\), yields

\[
\log s_N \geq \log s_0 - h(x_1, \ldots, x_N) + \sum_{n=1}^{N} b_n + N \log(1 - \epsilon) \quad (7)
\]

which can be rearranged to give

\[
\sum_{n=1}^{N} b_n \leq h(x_1, \ldots, x_N) - N \log(1 - \epsilon) + r_t \approx h(x_1, \ldots, x_N) + N \epsilon + r_t \quad (8)
\]

since \(\log(1 - \epsilon) \approx \epsilon\) for small \(\epsilon\) and \(\log s_N - \log s_0 < r_t\). Thus rANS solves problem 2, the number of bits observed from \(t\) is ‘close’ to \(h(x_1, \ldots, x_N)\) in the sense that the difference is no more than an additive constant, \(r_t\), plus a term which grows linearly with \(N\), but with a very small coefficient \(\epsilon\). In our implementation we use \(r_p = 16\) or less, and thus \(\epsilon \leq \frac{2^{16+32-64}}{\ln 2} \approx 2.2 \times 10^{-5}\).

It now remains for us to describe the function \(d\) and show that it satisfies eq. (3), as well as showing how to invert pop to form an encoder.

2.4 The function \(d\)

The function \(d : \mathbb{N} \to \mathbb{N} \times \mathcal{A}\) must be a bijection, and we aim for \(d\) to satisfy eq. (3), and thus \(P(x) \approx \frac{d}{s}\). Achieving this is actually fairly straightforward. One way to define a bijection \(d : s \mapsto (s', x)\) is to start with a mapping \(\tilde{d} : s \mapsto x\), with the property that none of the preimages \(\tilde{d}^{-1}(x) := \{n \in \mathbb{N} : \tilde{d}(n) = x\}\) are finite for \(x \in \mathcal{A}\). Then let \(s'\) be the index of \(s\) within the (ordered) set \(\tilde{d}^{-1}(x)\), with indices starting at 0. Equivalently, \(s'\) is the number of integers \(n\) with \(0 \leq n < s\) and \(d(n) = x\).

With this setup, the ratio

\[
\frac{s'}{s} = \frac{|\{n \in \mathbb{N} : n < s, d(n) = x\}|}{s} \quad (9)
\]

is the density of numbers which decode to \(x\), within all the natural numbers less \(s\). For large \(s\) we can ensure that this ratio is close to \(P(x)\) by setting \(\tilde{d}\) such
that numbers which decode to a symbol $x$ are distributed within the natural numbers with density close to $P(x)$.

To do this, we partition $\mathbb{N}$ into finite ranges of equal length, and treat each range as a model for the interval $[0,1]$, with sub-intervals within $[0,1]$ corresponding to each symbol, and the width of each sub-interval being equal to the corresponding symbol’s probability (see fig. 1). To be precise, the mapping $\tilde{d}$ can then be expressed as a composition $\tilde{d} = \tilde{d}_2 \circ \tilde{d}_1$, where $\tilde{d}_1$ does the partitioning described above, and $\tilde{d}_2$ assigns numbers within each partition to symbols (sub-intervals). So

$$\tilde{d}_1(s) := s \mod 2^r. \quad (10)$$

Using the shorthand $\bar{s} := \tilde{d}_1(a)$, and defining

$$c_j := \begin{cases} 0 & \text{if } j = 1 \\ \sum_{k=1}^{j-1} p_k & \text{if } j = 2, \ldots, I \end{cases} \quad (11)$$

as the (quantized) cumulative probability of symbol $a_{j-1}$,

$$\tilde{d}_2(\bar{s}) := a_i \text{ where } i := \max\{j : c_j \leq \bar{s}\}. \quad (12)$$

Figure 1 illustrates this mapping, with a particular probability distribution, for the range $s = 64, \ldots, 71$.

| $s$  | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 |
|------|----|----|----|----|----|----|----|----|
| $s \mod 2^r$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| $x$  | $a$ | $b$ | $c$ | $d$ |
|      | 0  | 1  |

Figure 1: Showing the correspondence between $s$, $s \mod 2^r$ and the symbol $x$.

The interval $[0,1] \subset \mathbb{R}$ is modelled by the set of integers $\{0,1,\ldots,2^r - 1\}$. In this case $r_p = 3$ and the probabilities of each symbol are $P(a) = \frac{1}{8}$, $P(b) = \frac{2}{8}$, $P(c) = \frac{3}{8}$ and $P(d) = \frac{1}{8}$.

### 2.5 Computing $s'$

The number $s'$ was defined above as ‘the index of $s$ within the (ordered) set $\tilde{d}^{-1}(x)$, with indices starting at 0’. We now derive an expression for $s'$ in terms of $s$, $p_i$ and $c_i$, where $i = \max\{j : c_j \leq \bar{s}\}$ (as above), and we prove eq. (3).

Our expression for $s'$ is a sum of two terms. The first term counts the entire intervals, corresponding to $a_i$, which are below $s$. This is equal to $p_i \cdot (s \div 2^r)$, where $\div$ denotes integer division, discarding any remainder. The second term counts our position within the current interval, which is $s - c_i \equiv s \mod 2^r - c_i$. Thus

$$s' = p_i \cdot (s \div 2^r) + s \mod 2^r - c_i. \quad (13)$$
This expression is straightforward to compute. Moreover from this expression it is straightforward to prove eq. (3). Firstly, taking the log of both sides of eq. (13) and using the fact that \( s \mod 2^{r_p} - c_i \geq 0 \) gives
\[
\log s' \geq \log(p_i \cdot (s \div 2^{r_p})).
\] (14)
then by the definition of \( \div \), we have \( s \div 2^{r_p} > \frac{s}{2^{r_p}} - 1 \), and thus
\[
\log s' \geq \log \left( p_i \left( \frac{s}{2^{r_p}} - 1 \right) \right)
\] (15)
\[
\geq \log s - h(x) + \log \left( 1 - \frac{2^{r_p}}{s} \right)
\] (16)
using the fact that \( P(x) = \frac{p_i}{2^{r_p}} \). Finally, using \( s \geq 2^{r_p-\epsilon} \),
\[
\log s' \geq \log s - h(x) + \log(1 - \epsilon)
\] (17)
as required.

By choosing \( r_s - r_t \) to be reasonably large (it is equal to 32 in our implementation), we ensure that \( \frac{s}{2^{r_p}} \) is very close to \( P(x) \). This behaviour can be seen visually in fig. 2 which shows the improvement in the approximation for larger \( s \).

### 2.6 Pseudocode for \( d \)

We now have everything we need to write down a procedure to compute \( d \). We assume access to a function \( f_P : \bar{s} \mapsto a_i, c_i, p_i \), where \( i \) is defined above.
This function clearly depends on the distribution $P$, and its computational complexity is equivalent to that of computing the CDF and inverse CDF of $P$. For many common distributions, the CDF and inverse CDF have straightforward closed form expressions, which don’t require an explicit sum over $i$.

We compute $d$ as follows:

```python
def d(s):
    \bar{s} = s \mod 2^r
    x, c, p = f_p(\bar{s})
    s' = p \cdot (s \div 2^r) + \bar{s} - c
    return s', x
```

### 2.7 Inverting the decoder

Having described a decoding process which appears not to throw away any information, we now derive the inverse process, `push`, and show that it is computationally straightforward.

The `push` function has access to the symbol $x$ as one of its inputs, and must do two things. Firstly it must `stack_push` the correct number of elements to $t$ from the lower bits of $s$. Then it must reverse the effect of $d$ on $s$, returning a value of $s$ identical to that before `pop` was applied.

Thus, on a high level, the inverse of the function `pop` can be expressed as

```python
def push((s, t), x):
    p, c = g_P(x)
    s', t = renorm_inverse(s, t; p)
    s = d^{-1}(s'; p, c)
    return s, t
```

where $g_P : x \mapsto p_i, c_i$ with $i$ as above, similarly to $f_P$ in that it is analogous to computing the quantized CDF and mass function $P$. The function $d^{-1}$ is really a pseudo-inverse of $d$; it is the inverse of $s \mapsto d(s, x)$, holding $x$ fixed.

As mentioned above, `renorm_inverse` must `stack_push` the correct amount of data from the lower order bits of $s$ into $t$. A necessary condition which the output of `renorm_inverse` must satisfy is

$$2^{r_i - r_s} \leq d^{-1}(s'; p, c) < 2^{r_s} \quad (18)$$

This is because the output of `push` must be a valid message, as described in Section 2.1, just as the output of `pop` must be.

The expression for $s'$ in eq. (13) is straightforward to invert, yielding a formula for $d^{-1}$:

$$d^{-1}(s'; p, c) = 2^{r_p} \cdot (s' \div p) + s' \mod p + c \quad (19)$$
We can substitute this into eq. (18) and simplify:

\[
2^{r_s - r_t} \leq 2^{r_p} \cdot (s' \div p) + s' \mod p + c < 2^{r_s}
\]  
\[
\iff 2^{r_s - r_t} \leq 2^{r_p} \cdot (s' \div p) < 2^{r_s}
\]  
\[
\iff p \cdot 2^{r_s - r_t - r_p} \leq s' < p \cdot 2^{r_s - r_p}
\]

Roughly speaking, `renorm_inverse` should move data from the lower order bits of \(s'\) into \(t\) (decreasing \(s'\)) until eq. (22) is satisfied. To be specific:

```python
def renorm_inverse(s', t):
    while s' ≥ p · 2^{r_s - r_p}:
        t = stack_push(t, s' mod 2^{r_t})
        s' = s' ÷ 2^{r_t}
    return s', t
```

However, as mentioned above, although eq. (22) is a necessary condition which \(s'\) must satisfy, it isn’t immediately clear that it’s sufficient. Is it possible that we need to continue the while loop in `renorm_inverse` past the first time that \(s' < p \cdot 2^{r_s - r_p}\)? In fact this can’t be the case, because \(s' ÷ 2^{r_t}\) decreases \(s'\) by a factor of at least \(2^{r_t}\), and thus as we iterate the loop above we will land in the interval specified by eq. (22) at most once. This guarantees that the \(s\) that we recover from `renorm_inverse` is the correct one.

### 3 Further reading

Since its invention by Duda, 2009, ANS appears not to have gained widespread attention in academic literature, despite being used in various state of the art compression systems. At the time of writing, a search on Google Scholar for the string “asymmetric numeral systems” yields 148 results. For comparison, a search for “arithmetic coding”, yields ‘about 44,000’ results. As far as I’m aware, ANS has not yet made it into any textbooks.

However, for those wanting to learn more there is a huge amount of material on different variants of ANS in Duda, 2009 and Duda et al., 2015. A parallelized implementation based on SIMD instructions can be found in Giesen, 2014 and a version which encrypts the message whilst compressing in Duda and Niemiec, 2016. An extension of ANS to models with latent variables was developed by Townsend et al., 2019.

Duda maintains a list of ANS implementations at [https://encode.su/threads/2078-List-of-Asymmetric-Numeral-Systems-implementations](https://encode.su/threads/2078-List-of-Asymmetric-Numeral-Systems-implementations).

### References

Duda, Jarek (2009). *Asymmetric Numeral Systems*. arXiv: [0902.0271 [cs, math]](http://arxiv.org/abs/0902.0271) (visited on 10/26/2019).
Duda, Jarek and Niemiec, Marcin (2016). Lightweight Compression with Encryption Based on Asymmetric Numeral Systems. In arXiv: 1612.04662 [cs, math] URL: http://arxiv.org/abs/1612.04662 (visited on 01/24/2020).

Duda, Jarek, Tahboub, Khalid, Gadgil, Neeraj J., and Delp, Edward J. (2015). The Use of Asymmetric Numeral Systems as an Accurate Replacement for Huffman Coding. In 2015 Picture Coding Symposium (PCS). 2015 Picture Coding Symposium (PCS), pp. 65–69. DOI: 10.1109/PCS.2015.7170048.

Giesen, Fabian (2014). Interleaved Entropy Coders. In arXiv: 1402.3392 [cs, math] URL: http://arxiv.org/abs/1402.3392 (visited on 01/24/2020).

Townsend, James, Bird, Thomas, and Barber, David (2019). Practical Lossless Compression with Latent Variables Using Bits Back Coding. In International Conference on Learning Representations. URL: https://openreview.net/forum?id=ryE981R5tm.