Matrices and α-Stable Bipartite Graphs

Vadim E. Levit and Eugen Mandrescu
Department of Computer Science
Holon Academic Institute of Technology
52 Golomb Str., P.O. Box 305
Holon 58102, ISRAEL
{levitv, eugen}@barley.cteh.ac.il

Abstract

A square \((0,1)\)-matrix \(X\) of order \(n \geq 1\) is called fully indecomposable if there exists no integer \(k\) with \(1 \leq k \leq n - 1\), such that \(X\) has a \(k\) by \(n-k\) zero submatrix. The reduced adjacency matrix of a bipartite graph \(G = (A, B, E)\) (having \(A \cup B = \{a_1, ..., a_m\} \cup \{b_1, ..., b_n\}\) as vertex set, and \(E\) as edge set), is \(X = [x_{ij}], 1 \leq i \leq m, 1 \leq j \leq n\), where \(x_{ij} = 1\) if \(a_ib_j \in E\) and \(x_{ij} = 0\) otherwise. A stable set of a graph \(G\) is a subset of pairwise nonadjacent vertices. The stability number of \(G\), denoted by \(\alpha(G)\), is the cardinality of a maximum stable set in \(G\). A graph is called α-stable if its stability number remains the same upon both the deletion and the addition of any edge. We show that a connected bipartite graph has exactly two maximum stable sets that partition its vertex set if and only if its reduced adjacency matrix is fully indecomposable. We also describe a decomposition structure of α-stable bipartite graphs in terms of their reduced adjacency matrices. On the base of these findings we obtain both new proofs for a number of well-known theorems on the structure of matrices due to Brualdi (1966), Marcus and Minc (1963), Dulmage and Mendelsohn (1958), and some generalizations of these statements. Several new results on α-stable bipartite graphs and their corresponding reduced adjacency matrices are presented, as well. Two kinds of matrix product are also considered (namely, Boolean product and Kronecker product), and their corresponding graph operations. As a consequence, we obtain a strengthening of one Lewin’s theorem claiming that the product of two fully indecomposable matrices is a fully indecomposable matrix.

1 Introduction

Throughout this paper \(G = (V, E)\) is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set \(V = V(G)\) and edge set \(E = E(G)\). If \(A\) is a subset of vertices, \(G[A]\) is the subgraph of \(G\) spanned by \(A\), i.e., the graph having \(A\) as its vertex set, and containing all the edges of \(G\) connecting vertices of \(A\). By \(G - W\) we mean either the subgraph \(G[V - W]\), if \(W \subset V(G)\), or the partial subgraph of \(G\) obtained by deleting the edges from \(W\), whenever \(W \subset E(G)\) (we use \(G - a\), if \(W = \{a\}\)). If \(A, B\) are disjoint subsets of \(V\), then \((A, B)\) stands for
the set \( \{ e = ab : a \in A, b \in B, e \in E \} \). The neighborhood of a vertex \( v \in V \), denoted by \( N(v) \), is the set of vertices adjacent to \( v \). For any \( A \subset V(G) \), we denote \( N_G(A) = \bigcup \{ N(x) : x \in A \} \), or, if no ambiguity, \( N(A) \). A subset \( D \subset V(G) \) is said to be 2-dominating in \( G \) if \( |N(v) \cap D| \geq 2 \), for any vertex \( v \in V - D \). A stable set (i.e., a set containing pairwise nonadjacent vertices) of maximum size will be referred to as a maximum stable set of \( G \). The stability number of \( G \), denoted by \( \alpha(G) \), is the cardinality of a maximum stable set of \( G \). A perfect matching is a set of non-incident edges of \( G \) covering all its vertices.

A bipartite graph is a triple \( G = (A, B, E) \), where \( E \) is its edge set and \( \{A, B\} \) is its bipartition; if \( |A| = |B| \), then \( G \) is called balanced bipartite. If \( A, B \) are as the only two maximum stable sets of \( G \), then it is a bistable bipartite graph.

A graph \( G = (V, E) \) is called:

(i) \( \alpha^- \)-stable if \( \alpha(G - e) = \alpha(G) \) is valid for any \( e \in E \), [3];

(ii) \( \alpha^+ \)-stable if \( \alpha(G + e) = \alpha(G) \) holds for any \( e \notin E \), \( e = xy \) and \( x, y \in V \), [3];

(iii) \( \alpha \)-stable if it is both \( \alpha^- \)-stable and \( \alpha^+ \)-stable, [1].

Let \( G = (A, B, E) \) be a bipartite graph, where \( A = \{a_1, a_2, ..., a_m\} \) and also \( B = \{b_1, b_2, ..., b_n\} \). Then \( G \) can be characterized by its adjacency matrix, which is a square \((0,1)\)-matrix of order \( m + n \)

\[
\begin{bmatrix}
O & X \\
X^t & O
\end{bmatrix},
\]

where \( X = [x_{ij}], 1 \leq i \leq m, 1 \leq j \leq n \), with \( x_{ij} = 1 \) if \( a_i b_j \in E \) and \( x_{ij} = 0 \) otherwise. \( X \) is called the reduced adjacency matrix of the bipartite graph \( G \). Any \((0,1)\)-matrix of size \( m \) by \( n \) is the reduced adjacency matrix of a bipartite graph. If \( G \) is balanced bipartite, then its reduced adjacency matrix is a square \((0,1)\)-matrix of order \( n = |A| = |B| \). The term rank \( \rho = \rho(X) \) of a \((0,1)\)-matrix \( X \) of size \( m \) by \( n \) is the maximal number of 1’s of \( X \) with no two of 1’s on a line (i.e., on a row or on a column). A collection of \( n \) elements of a square \((0,1)\)-matrix \( X \) of order \( n \) is called a diagonal of \( X \) provided no two elements belong to the same row or column of \( X \). A nonzero diagonal of \( X \) is a diagonal not containing any 0’s.

A square \((0,1)\)-matrix \( X \) of order \( n \) is called partly decomposable if \( n = 1 \) and its unique entry is zero, or \( n > 1 \) and there exists an integer \( k \) with \( 1 \leq k \leq n - 1 \), such that \( X \) has a \( k \) by \( n - k \) zero submatrix. A square matrix is fully indecomposable provided it is not partly decomposable, [13]. By permuting the lines of \( X \), the partly decomposable matrix \( X \) can be written in the form

\[
\begin{bmatrix}
X_1 & O \\
X_2 & X_3
\end{bmatrix},
\]

where \( O \) is a zero matrix of size \( k \) by \( n - k \), \( X_1 \) and \( X_3 \) are square matrices of orders \( k \) and \( n - k \), respectively.

Decomposition structures of \( \alpha^+ \)-stable and \( \alpha \)-stable bipartite graphs were first established in Levit and Mandrescu [12]. On the base of these findings we obtain both new proofs for several well-known theorems on the structure of matrices due to Brualdi [1], [2], [3], [4], Marcus and Mine [14], Dulmage and Mendelsohn [15], and also some generalizations of these statements. Some new results on reduced adjacency
matrices of $\alpha$-stable bipartite graphs are presented, as well. For example, we show that a connected bipartite graph has exactly two maximum stable sets that partition its vertex set if and only if its reduced adjacency matrix is fully indecomposable.

The paper is organized as follows: for the sake of self-consistency, section 2 contains a series of results referring to the structure of bistable, $\alpha^+\,$-stable, and $\alpha^-\,$-stable bipartite graphs. We use these findings further, in section 3, proving some corresponding assertions for reduced adjacency matrices associated with bipartite graphs. Sections 4 and 5 are dealing with two different kinds of matrix product, (namely, Boolean and Kronecker), and the corresponding graph operations.

## 2 $\alpha$-Stable bipartite graphs

Haynes et al. proved the following theorem, describing stability properties of general graphs.

**Theorem 2.1** \[10\] A graph $G$ is:

(i) $\alpha^-$-stable if and only if each of its maximum stable sets is a 2-dominating set in $G$;

(ii) $\alpha^+$-stable if and only if no pair of vertices is contained in all its maximum stable sets.

Using Theorem 2.1, we proved the following result from \[11\], which in particular, is valid for trees, as Gunther et al. show in \[9\].

**Theorem 2.2** \[11\] If $G$ is a connected bipartite graph, then the following assertions are equivalent:

(i) $G$ is $\alpha^+$-stable;

(ii) $G$ has a perfect matching;

(iii) $G$ possesses two maximum stable sets that partition its vertex set.

Figure 1 illustrates some basic differences between $\alpha^+$-stable and $\alpha^-$-stable graphs. Namely, both are bipartite, but $G_1$ is $\alpha^+$-stable and non-$\alpha^-$-stable (it has a perfect matching and a non-2-dominating maximum stable set), while $G_2$ is $\alpha^-$-stable (its unique maximum stable set is 2-dominating), and non-$\alpha^+$-stable (it has no perfect matching).

![Figure 1: $\alpha^+$-stable and $\alpha^-$-stable bipartite graphs $G_1$ and $G_2$.](image)

**Lemma 2.3** If $G = (A, B, E)$ is an $\alpha$-stable graph, and $S$ is a maximum stable set of $G$ meeting both $A$ and $B$, then the subgraph $H = G[(S \cap A) \cup (B - S)]$ is $\alpha$-stable.
Theorem 2.2. For any ∈ G is also 2-dominating, since for any ∈ G are vertex-disjoint, bistable bipartite and α are stable, because for any ∈ G, subgraphs: H, such that both X and X ∩ (B - S) are non-empty. S' = X ∪ S is clearly a maximum stable set of G, and therefore, we have: |N(a) ∩ X| = |N(a) ∩ X| = |N(a) ∩ S| ≥ 2, for any a ∈ S and |N(b) ∩ X| = |N(b) ∩ X| = |N(a) ∩ S| ≥ 2, for any b ∈ B - S - X, i.e., X is 2-dominating in H. Consequently, H is α-stable, by Theorem 2.1.

Proposition 2.4 A connected bipartite graph G is α-stable if and only if G can be decomposed as G = G1 ∪ G2 ∪ ... ∪ Gk, k ≥ 1, such that all Gi = (Ai, Bi, Ei), 1 ≤ i ≤ k, are vertex-disjoint, bistable bipartite and α-stable.

Proof. If G = (A, B, E) has A and B as its only two maximum stable sets, then G itself is bistable bipartite and α-stable. Otherwise, let S be a maximum stable set of G, such that both S = S ∩ A and S = S ∩ B are non-empty. By Lemma 2.3, the subgraphs: H1 = G[(S ∩ A) ∪ (B - S)] and H2 = G[(A - S) ∪ (S ∩ B)] are α-stable. If they both have only two maximum stable sets, then they build the decomposition needed. Otherwise, we continue with this decomposition procedure, until all the subgraphs we obtain are α-stable and have exactly two maximum stable sets. After a finite number of subpartitions, we get a decomposition G = G1 ∪ G2 ∪ ... ∪ Gk, k ≥ 1, such that every Gi = (Ai, Bi, Ei), 1 ≤ i ≤ k, has only Ai and Bi as its maximum stable sets. Then G is α-stable, since it has at least one perfect matching, namely, M = \{Mi, 1 ≤ i ≤ k\}, such that Mi is a maximum stable set of G i.

According to Theorem 2.4, it suffices to show that any maximum stable set S of G is also 2-dominating in G. For S = A (and analogously for S = B), suppose S is not 2-dominating. Hence, there is a vertex b ∈ Bi such that |S ∩ N(b)| = |{a}| = 1. Clearly a ∈ Ai, and this implies that Ai ∪ \{b\} is a third maximum stable set in Gi, which contradicts the fact that Gi is bistable. Thus, Ai (also Bi) and S are 2-dominating in Gi, G respectively.

Suppose S meets both A and B. We claim that if i ∈ \{1, ..., k\} and S ∩ Ai ≠ \emptyset, then S ∩ Ai ⊆ S (similarly, if S ∩ Bi ≠ \emptyset, i ∈ \{1, ..., k\} then Bi ⊆ S). Otherwise, if there is some j ∈ \{1, ..., k\}, such that both S ∩ Aj and S ∩ Bj are non-empty, we have:

|S ∩ Aj| + |S ∩ Bj| < |Aj| = α(Gj) and |S ∩ Aj| + |S ∩ Bj| ≤ |Ai| = α(Gi), for i ≠ j.

Hence, we arrive at the following contradiction:

α(G) = |S| = |S ∩ A1| + |S ∩ B1| + ... + |S ∩ Ak| + |S ∩ Bk| < |A1| + ... + |Ak| = α(G).
Let $v \in A \cup B - S$. The vertex $v \in B_i$ for some $i \in \{1, \ldots, k\}$. Hence, $S \cap B_i = \emptyset$. Consequently, $A_i \subseteq S$, and $|S \cap N(v)| \geq |A_i \cap N(v)| \geq 2$, since $A_i$ is 2-dominating in $G_i$. Finally, $S$ is also 2-dominating in $G$, and this completes the proof. $$\blacksquare$$

An example of this decomposition is presented in Figure 2. $G = G_1 \cup G_2$ is $\alpha$-stable bipartite and both $G_1, G_2$ are bistable bipartite.

![Figure 2: An example of decomposition: $G = G_1 \cup G_2$ and $G_1, G_2$ are bistable.](image)

**Theorem 2.5** If $G = (A, B, E)$ is a bipartite graph with at least 4 vertices, then the following conditions are equivalent (see examples of a bistable bipartite graph and a non-bistable bipartite graph in Figure 3):

(i) $G$ is bistable bipartite;

(ii) $G$ is $\alpha^+$-stable and $G - a - b$ is $\alpha^+$-stable, for any $a \in A$ and $b \in B$;

(iii) for any $a \in A$ and $b \in B$, $G - a - b$ has a perfect matching;

(iv) $G$ is connected and any of its edges is contained in a perfect matching of $G$;

(v) $|N(X)| > |X|$, for any proper subset $X$ of $A$ and of $B$.

**Proof.** (i) $\Rightarrow$ (ii) According to Theorem 2.2, $G$ is $\alpha^+$-stable. Let $a \in A, b \in B$ and $H = G - \{a, b\}$. It suffices to show that $\alpha(H) = |A - \{a\}| = |B - \{b\}|$. Suppose, on the contrary, that $\alpha(H) = \alpha(G)$; then there is a stable set $S$ in $H$, such that $\alpha(H) = |S|$. Consequently, $S$ is a third maximum stable set in $G$, in contradiction with the premises on $G$.

(ii) $\Rightarrow$ (i) Clearly, $G$ is connected and $\alpha^+$-stable. By Theorem 2.2, we obtain that $\alpha(G) = |A| = |B|$. Let $S$ be a third maximum stable set in $G, a \in A - S$ and $b \in B - S$. $H = G - \{a, b\}$ is $\alpha^+$-stable and $\alpha(H) = |A - \{a\}| = |B - \{b\}| = \alpha(G) - 1$, by the hypothesis. Since $S$ is stable in $H$, we obtain the following contradiction $\alpha(G) = |S| \leq \alpha(H) = \alpha(G) - 1$. Consequently, $G$ has only $A$ and $B$ as maximum stable sets.

(ii) $\Leftrightarrow$ (iii) It is true, according to Theorem 2.2.

(iii) $\Rightarrow$ (iv) $G$ is connected, since otherwise for $a, b$ in different color classes and different connected components, $G - a - b$ has no perfect matching, contradicting the assumption on $G - a - b$. Let $ab$ be an arbitrary edge of $G$ and $M$ be a perfect matching in $G - a - b$, which exists according to hypothesis. Hence, $M \cup \{ab\}$ is a perfect matching in $G$ containing $ab$.

(iv) $\Rightarrow$ (i) Suppose, on the contrary, that $G$ has a maximum stable set $S$ meeting both $A$ and $B$. If denote $S_A = S \cap A$ and $S_B = S \cap B$, then in any perfect matching of $G$, the sets $S_A$ and $S_B$ are matched respectively with $B - S_B, A - S_A$. Consequently, we obtain that no edge $ab$ joining a vertex $a \in A - S_A$ with some vertex $b \in B - S_B$ (such an edge must exist, because $G$ is connected) belongs to some perfect matching of $G$, contradicting the assumption on $G$. Therefore, $G$ is bistable bipartite.
(i) ⇒ (v) Clearly, \( \alpha(G) = |A| = |B| \). Suppose that there is some proper subset \( X \) of \( A \) such that \( |N(X)| \leq |X| \). Consequently, \((X, B - N(X)) = \emptyset \), and hence, \( S = X \cup (B - N(X)) \) is stable in \( G \) with \( |S| = |X| + |B - N(X)| \geq |X| + |A - X| = \alpha(G) \). Thus, since \( S \) meets both \( A \) and \( B \), we infer that \( S \) is a third maximum stable set of \( G \), and this is a contradiction, because \( G \) is bistable. An analogous proof can be obtained if \( X \subset B \).

(v) ⇒ (i) If \(|N(X)| > |X| \) holds for any proper subset \( X \) of \( A \) and of \( B \), it follows that \( |A| = |B| \leq \alpha(G) \). Assume that some maximum stable set \( S \) of \( G \) meets both \( A \) and \( B \). Then we obtain the following contradiction:

\[
\alpha(G) = |S| = |S \cap A| + |S \cap B| < |N(S \cap A)| + |S \cap B| \leq |B| \leq \alpha(G).
\]

Consequently, \( G \) must be bistable bipartite. ■

The graph \( G_1 \) in Figure 3 is non-bistable, since it has 3 maximum stable sets, but \( G_2 \) is bistable.

![Figure 3: \( G_1 \) is non-bistable, \( G_2 \) is bistable.](image)

**Corollary 2.6** If \( G \) is a bistable bipartite graph with at least 4 vertices, then

\[ \cap\{M : M \text{ is a perfect matching of } G\} = \emptyset. \]

**Proof.** By Theorem 2.3, \( G \) is \( \alpha^+ \)-stable, and therefore, it has perfect matchings. Suppose, on the contrary, that there exists \( ab \in \cap\{M : M \text{ is a perfect matching of } G\} \). If \( x \in N(a) - \{b\} \), then, according to Theorem 2.5, \( H = G - a - x \) is \( \alpha^+ \)-stable and thus, it has a perfect matching \( M_0 \), which matches \( b \) with some \( y \in N(b) - \{a\} \). Hence, \( M_0 \cup \{ax\} \) is a perfect matching of \( G \) and \( ab \notin M_0 \cup \{ax\} \), contradicting the assumption on \( ab \). Therefore, we have \( \cap\{M : M \text{ is a perfect matching of } G\} = \emptyset. \) ■

**Proposition 2.7** A connected bipartite graph \( G \) is \( \alpha \)-stable if and only if it has perfect matchings and \( \cap\{M : M \text{ is a perfect matching of } G\} = \emptyset. \)

**Proof.** By Proposition 2.4, \( G \) may be decomposed as \( G = G_1 \cup G_2 \cup \ldots \cup G_k, k \geq 1 \), such that each \( G_i = (A_i, B_i, E_i), 1 \leq i \leq k \), is bistable bipartite. Taking into account Corollary 2.6 and the fact that

\[ \cup\{M_i : M_i \text{ is a perfect matching of } G_i, 1 \leq i \leq k\} \]

is a perfect matching in \( G \), we get that \( \cap\{M : M \text{ is a perfect matching of } G\} = \emptyset. \)

Conversely, we claim first that from any vertex are issuing at least two edges contained in some perfect matchings of \( G \). Otherwise, there is a vertex \( v \) in \( G \), so that only one edge, say \( vw \), is contained in a perfect matching of \( G \); such an edge
must exist, because $G$ has perfect matchings. Moreover, since $v$ is matched with a vertex by each such matching, we infer that $vw$ belongs to all perfect matchings of $G$, in contradiction with $\cap \{M : M is a perfect matching of G\} = \emptyset$. Assume, on the contrary, that $G$ is not $\alpha$-stable, i.e., $G$ is not $\alpha^+$-stable, since by Theorem 2.2, $G$ is $\alpha^+$-stable. Therefore, there is a maximum stable set $S$, meeting both $A$ and $B$, and a vertex, say $a \in A$, such that $|N(a) \cap S| = |\{b\}| = 1$. Since from $a$ are issuing at least two edges contained in different perfect matchings of $G$, we infer that there is at least a vertex $c \in N(a) \cap (B - S)$, such that $ac$ is in a perfect matching $M$ of $G$. Hence, since $|A - S \cap A - \{a\}| < |S \cap B|$, some vertex in $S \cap B$ must be matched by $M$ with some vertex in $S \cap A$, thus contradicting the stability of $S$. Therefore, $G$ is $\alpha$-stable.

Proposition 2.8 A connected balanced bipartite graph $G$ is $\alpha^+$-stable if and only if it admits a decomposition as $G = G_1 \cup \ldots \cup G_k$, all $G_i$ being vertex-disjoint and bistable bipartite.

Proof. Let $H_0 = G[M_0]$ and $H_1 = G - H_0$, where

$$M_0 = \cap \{M : M is a perfect matching of G\} = \emptyset.$$ 

Clearly, $H_1$ has $\cap \{M : M is a perfect matching of H_1\} = \emptyset$, while $H_0$ is either empty or a disjoint union of $K_2$. According to Propositions 2.7 and 2.4, any connected component of $H_1$ has a decomposition in bistable bipartite subgraphs. Therefore, $G$ admits a decomposition as $G = G_1 \cup \ldots \cup G_k$, all $G_i$ being vertex-disjoint and bistable bipartite.

Conversely, if $G = G_1 \cup \ldots \cup G_k$, and all $G_i$ are bistable bipartite, then each $G_i$ has at least a perfect matching $M_i$, and

$$\cup \{M_i : M_i is a perfect matching of G_i, 1 \leq i \leq k\}$$

is a perfect matching in $G$. Consequently, by Theorem 2.2, $G$ is $\alpha^+$-stable.

In Figure 4 is presented an example of decomposition of an $\alpha^+$-stable bipartite graph into vertex-disjoint and bistable bipartite components: $G = G_1 \cup G_2 \cup G_3$.

![Figure 4: An example of decomposition into bistable components: $G = G_1 \cup G_2 \cup G_3$.](image)

3 Matrices and bipartite graphs

It is not difficult to see that the unity matrix $I_n, n \geq 1$, is the reduced adjacency matrix of $nK_2$, i.e., of the graph consisting of $n$ disjoint copies of $K_2$. Moreover, we have:
Lemma 3.1 A bipartite graph \( G \) is disconnected if and only if its adjacency matrix \( X \) can be written as

\[
\begin{bmatrix}
X_1 & O & O & \ldots & O \\
O & X_2 & O & \ldots & O \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
O & O & \ldots & O & X_k
\end{bmatrix},
\]

(1)

where the blocks \( X_1, X_2, \ldots, X_k \) are the adjacency matrices corresponding respectively to the \( k \geq 2 \) connected components of \( G \).

Lemma 3.2 Let \( S \) be a proper subset of the vertex set of graph \( G = (A, B, E) \), with \( p + q \) vertices, where \( p = |S \cap A| \geq 1 \) and \( q = |S \cap B| \geq 1 \). Then \( S \) is stable in \( G \) if and only if its reduced adjacency matrix \( X \) can be written as

\[
\begin{bmatrix}
X_1 & O \\
X_2 & X_3
\end{bmatrix},
\]

where \( O \) is a \( p \) by \( q \) zero matrix.

Proof. By using an appropriate indexing for \( A \) and for \( B \), we may suppose that \( S \cap A = \{a_1, \ldots, a_p\} \) and \( S \cap B = \{b_{n-q+1}, \ldots, b_n\} \). Therefore, \( S \) is stable in \( G \) if and only if \( x_{ij} = 0 \) for any \( i \in \{1, \ldots, p\} \) and \( j \in \{n - q + 1, \ldots, n\} \), i.e., \( X \) has exactly the form announced above.

Proposition 3.3 Let \( G = (A, B, E) \) be a connected balanced bipartite graph with \( 2n \) vertices and \( X \) be its reduced adjacency matrix. Then \( G \) has a stable set of \( n \) vertices that meets both \( A \) and \( B \) if and only if \( X \) is partly decomposable.

Proof. If \( p = |S \cap A| \), then \( q = |S \cap B| = n - p \), and by Lemma 3.2, we obtain \( X \) in the form

\[
\begin{bmatrix}
X_1 & O \\
X_2 & X_3
\end{bmatrix},
\]

where \( O \) is a \( p \) by \( n - p \) zero matrix, \( 1 \leq p \leq n - 1 \), i.e., \( X \) is partly decomposable.

Proposition 3.4 A balanced bipartite graph is bistable if and only if its reduced adjacency matrix is fully indecomposable.

Proof. Since a bistable bipartite graph \( G = (A, B, E) \) is connected and has only \( A \) and \( B \) as maximum stable sets, Proposition 3.3 ensures that its reduced adjacency matrix can not be partly decomposable. The converse is clear.

Following the terminology from \[8\], let us recall that for a balanced bipartite graph \( G = (A, B, E) \), a cover is a pair of subsets \( A_0, B_0 \) of \( A, B \) respectively, such that for every edge \( ab \in E \), either \( a \in A_0 \) or \( b \in B_0 \). \( G \) is cover irreducible if its only minimum

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8
covers are \{A, ∅\} and \{∅, B\}. The reduced adjacency matrix of a cover irreducible bipartite graph is a \textit{cover irreducible matrix}. On the other hand, a bipartite graph \(G\) is called \textit{elementary}, \cite{14}, if the set, containing any of its edges that appears in at least one perfect matching, forms a connected subgraph of \(G\). It is shown, \cite{14}, that elementary bipartite graphs and the cover irreducible bipartite graphs are the same. It turns out that bistable bipartite graphs are exactly cover irreducible bipartite graphs, and fully indecomposable matrices coincide with cover irreducible matrices. Our approach is based, in principal, on the \textit{bistable property}. Combining Theorem \ref{thm:2.2} and Proposition \ref{prop:3.4}, we get the following result from \cite{6}:

\textbf{Corollary 3.5} Let \(G = (A, B, E)\) be a balanced bipartite graph with \(2n\) vertices and \(X\) be its reduced adjacency matrix. Then \(X\) is fully indecomposable if and only if \(G\) is connected and any of its edges belongs to a perfect matching of \(G\).

We obtain a simple proof for the following characterization of fully indecomposable matrices, due to Marcus and Minc, \cite{15}, and Brualdi, \cite{1}.

\textbf{Theorem 3.6} A \((0, 1)\)-matrix \(X\) of order \(n \geq 2\) is fully indecomposable if and only if every 1 of \(X\) belongs to a nonzero diagonal and every 0 of \(X\) belongs to a diagonal whose other elements equal 1.

\textbf{Proof.} Let \(G = (A, B, E)\) be a balanced bipartite graph with \(|A| = |B| = n\), having \(X\) as its reduced adjacency matrix. Then, according to Proposition \ref{prop:3.4} and Theorem \ref{thm:2.2}, \(X\) is fully indecomposable if and only if \(G - a - b\) is \(\alpha^+\)-stable for any \(a \in A\) and \(b \in B\), i.e., for any \(i, j \in \{1, \ldots, n\}\), the submatrix \(Y\), obtained by deleting the row \(i\) and the column \(j\) of \(X\), has a nonzero diagonal, and this completes the proof.

Another consequence is the following result of Marcus and Minc from \cite{14}.

\textbf{Corollary 3.7} A fully indecomposable \((0, 1)\)-matrix \(X\) of order \(n\) contains at most \(n(n - 2)\) zero entries.

\textbf{Proof.} Let \(G = (A, B, E)\) be a balanced bipartite graph with \(X\) as its reduced adjacency matrix. By Proposition \ref{prop:3.4}, \(G\) is bistable and according to Theorem \ref{thm:2.2}(v), \(|N(v)| \geq 2\) holds for any vertex \(v\) of \(G\). Consequently, any row of \(X\) cannot have more than \(n - 2\) zeros, and hence \(X\) cannot contain more than \(n(n - 2)\) zero entries. On the other hand, \(C_{2n}, n \geq 2\), is bistable and its reduced adjacency matrix has exactly \(n(n - 2)\) zero entries.

A \((0, 1)\)-matrix of order \(n \geq 2\) has \textit{total support} provided each of its 1’s belongs to a nonzero diagonal. As a consequence, we get the following result from \cite{6}.

\textbf{Proposition 3.8} \cite{6} Let \(X\) be a \((0, 1)\)-matrix of order \(n \geq 2\) with total support, and let \(G\) be the bipartite graph whose reduced adjacency matrix is \(X\). Then \(G\) is connected if and only if \(X\) is fully indecomposable.
Proof. Clearly, $X$ is with total support if and only if any edge of $G$ is contained in a perfect matching of $G$. Therefore, taking into account Theorem 2.5 and Proposition 3.4, we get that: $G$ is connected $\iff$ $G$ is bistable $\iff$ $X$ is fully indecomposable.

We can now characterize the bipartite graphs whose reduced adjacency matrix is with total support.

**Proposition 3.9** The reduced adjacency matrix $X$ of a bipartite graph $G$ has total support if and only if all connected components of $G$ are bistable bipartite.

**Proof.** If $G$ is connected, then according to Proposition 3.4, $X$ has total support if and only if $G$ is bistable. If $G$ is disconnected, Lemma 3.1 implies that $X$ can be written in the form (1), and then $X$ has total support if and only if all the blocks $X_1, ..., X_k$ have total support, i.e., according to Propositions 3.4 and 3.8, all connected components of $G$ are bistable bipartite.

**Proposition 3.10** Let $G$ be a balanced bipartite graph with $2n$ vertices and $X$ be its reduced adjacency matrix. Then the following assertions are equivalent:

(i) $G$ is $\alpha^+$-stable;
(ii) $X$ has a nonzero diagonal;
(iii) $\rho(X) = n$;
(iv) $\text{per}(X) > 0$.

**Proof.** By Theorem 2.2, $G$ is $\alpha^+$-stable if and only if it has a perfect matching, i.e., its reduced adjacency matrix $X$ has a nonzero diagonal, and this is equivalent to both (iii) and (iv).

The following result due to Minc is an immediate consequence of the above proposition.

**Corollary 3.11** A $(0, 1)$-matrix $X$ of order $n \geq 2$ is fully indecomposable if and only if every $(n-1)$-square submatrix $Y$ of $X$ has $\text{per}(Y) > 0$.

**Proof.** Suppose $X$ is the reduced adjacency matrix of the balanced bipartite graph $G = (A, B, E)$. According to Proposition 3.4, $X$ is fully indecomposable if and only if $G$ is bistable bipartite, and by Theorem 2.5, this happens if and only if $G - a - b$ is $\alpha^+$-stable, for any $a \in A$ and $b \in B$, i.e., by virtue of the Proposition 3.10, $\text{per}(Y) > 0$ holds for any $(n - 1)$-square submatrix $Y$ of $X$.

**Theorem 3.12** Let $G$ be a balanced bipartite graph with $2n$ vertices and $X$ be its reduced adjacency matrix. Then $G$ is $\alpha$-stable if and only if $X$ can be written as

\[
\begin{bmatrix}
X_1 & X_{12} & X_{13} & \ldots & X_{1k} \\
O & X_2 & X_{23} & \ldots & X_{2k} \\
& \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots \\
& & & O & O & \ldots & X_k
\end{bmatrix},
\]

where $X_1, ..., X_k$ are fully indecomposable matrices of order at least 2.
Proof. By Proposition 2.4, $G$ is $\alpha$-stable if and only if it admits a decomposition as $G = G_1 \cup \ldots \cup G_k$, where all $G_i, 1 \leq i \leq k$, are simultaneously $\alpha$-stable and bistable balanced bipartite. Hence, using an appropriate indexing for the vertices of $G$, $X$ can be written in the form (2), with $X_1, \ldots, X_k$ as reduced adjacency matrices corresponding to $G_1, \ldots, G_k$, and therefore being fully indecomposable, by Proposition 3.4. Each $X_i$ is of order at least two, since it corresponds to $G_i$, which is a bistable bipartite and $\alpha$-stable graph, i.e., it has at least 4 vertices.

Theorem 3.13 Let $G$ be a balanced bipartite graph with $2n$ vertices and $X$ be its reduced adjacency matrix. Then $G$ is $\alpha^\pm$-stable if and only if $X$ can be written in the form (2), where $X_1, \ldots, X_k$ are fully indecomposable matrices.

Proof. By Proposition 2.4, $G$ is $\alpha^\pm$-stable if and only if it admits a decomposition as $G = G_1 \cup \ldots \cup G_k$, where all $G_i, 1 \leq i \leq k$, are bistable balanced bipartite. Hence, using an appropriate indexing for the vertices of $G$, $X$ can be written according to the form (2), with $X_1, \ldots, X_k$ as reduced adjacency matrices corresponding to $G_1, \ldots, G_k$, and therefore being fully indecomposable, by Proposition 3.4.

As a consequence, we obtain:

Theorem 3.14 (Dulmage and Mendelsohn, [7], Brualdi, [1]). Let $X$ be a $(0,1)$-matrix of order $n$ with term rank $\rho(X)$ equal to $n$. Then there exist permutation matrices $P$ and $Q$ of order $n$ and an integer $k \geq 1$ such that $PAQ$ has the form (2), where all $X_1, \ldots, X_k$ are square fully indecomposable matrices.

Proof. Let $G$ be a bipartite graph, whose reduced adjacency matrix is $X$. By Proposition 3.9, $G$ is $\alpha^\pm$-stable, and according to Proposition 2.4, it admits a decomposition as $G = G_1 \cup \ldots \cup G_k$, all $G_i$ being bistable bipartite. Hence, using an appropriate indexing for the vertices of $G$, $X$ can be written according to the form (2), with $X_1, \ldots, X_k$ as reduced adjacency matrices corresponding to $G_1, \ldots, G_k$. Proposition 3.4 ensures that $X_1, \ldots, X_k$ are fully indecomposable.

Corollary 3.15 Let $X$ be a $(0,1)$-matrix of order $n$ with $\rho(X) = n$. Then the following assertions are true:

(i) the intersection of all nonzero diagonals of $X$ is empty if and only if all $X_i$ in the matrix (3) are of order at least 2;

(ii) the number of 1 by 1 blocks $X_i$ in the matrix (3) is equal to the number of common elements of all nonzero diagonals of $X$.

Corollary 3.16 (Brualdi, [3]) Let $X$ be a square $(0,1)$-matrix of order $n$ and let $X_{ij}$ denote the matrix obtained from $X$ by striking the $i$-th row and the $j$-th column. Then $X$ is fully indecomposable if and only if per($X_{ij}$) > 0.

Proof. Let $G = (A, B, E)$ be a bipartite graph whose reduced adjacency matrix is $X$. By Proposition 3.4, $X$ is fully indecomposable if and only if $G$ is bistable, i.e., for any $a \in A$ and $b \in B, G - a - b$ has a perfect matching (according to Theorem 2.5).
that is, by Theorem 2.2, the matrix $X_{ab}$ has positive permanent.

We end this section with the following characterization of the reduced adjacency matrix corresponding to an $\alpha$-stable bipartite graph.

**Proposition 3.17** Let $G$ be a balanced bipartite graph and $X$ be its reduced adjacency matrix. Then $G$ is $\alpha$-stable if and only if for any non-zero entry $x_{ij}$ of $X$ there exists a non-zero diagonal of $X$ that does not contain it.

**Proof.** According to Proposition 2.7, $G$ is $\alpha$-stable if and only if it has perfect matchings and $\cap\{M : M is a perfect matching of G\} = \emptyset$, that is $G$ has perfect matchings and for any of its edges $e$ there is a perfect matching $M$ such that $e \notin M$. In other words, if and only if for any non-zero entry $x_{ij}$ of $X$ there exists a non-zero diagonal of $X$ that does not contain it.

## 4 Boolean product of matrices

Let $G = (A, B, E)$ and $H = (B, C, F)$ be two balanced bipartite graphs on $2n$ vertices. We define the **join** of $G$ with $H$ as the graph $P = G \ast H = (A, C, W)$, where $ac \in W$ if and only if there is $b \in B$, such that $ab \in E$ and $bc \in F$. The **Boolean matrix product** of two $(0, 1)$-matrices $X, Y$ is a $(0, 1)$-matrix denoted by $X \bullet Y$ and having the same zero and non-zero entries as the usual matrix product $XY$; the term **Boolean** refers actually to the property of the Boolean addition operation: $1 + 1 = 1$; for an example, see Figure 5. Using this notation we have the following:

**Lemma 4.1** If $X, Y$ are respectively, the reduced adjacency matrices of the balanced bipartite graphs $G$ and $H$, then the Boolean matrix product $Z = X \bullet Y$ is the reduced adjacency matrix of the graph $P = G \ast H$.

**Proof.** If $X = (x_{ij}), Y = (y_{ij})$ and $Z = (z_{ij})$, then clearly we have:

$z_{ij} = \sum_{k=1}^{n} x_{ik} y_{kj} \neq 0 \iff$ there exists $k \in \{1, ..., n\}$ such that $x_{ik} = y_{kj} = 1 \iff$ there is some $b_k \in B$, so that $a_i b_k \in E$ and $b_k c_j \in F \iff a_i c_j \in W$.

$$X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad Y = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad Z = X \bullet Y = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Figure 5: The graphs join operation and its corresponding Boolean matrix product.
Remark 4.1 \( X \cdot Y \) is fully indecomposable if and only if \( XY \) is fully indecomposable.

Corollary 4.2 Any balanced bipartite graph \( G \) on \( 2n, n \geq 1 \), vertices is isomorphic to \( G * nK_2 \).

Proposition 4.3 Let \( G = (A, B, E) \) and \( H = (B, C, F) \) be balanced bipartite graphs.

(i) If \( G \) and \( H \) are \( \alpha^+ \)-stable, then \( G * H \) is \( \alpha^+ \)-stable.

(ii) If one of \( G, H \) is \( \alpha^+ \)-stable and the other is bistable bipartite, then \( G * H \) is bistable bipartite.

(iii) If \( G \) and \( H \) are bistable bipartite, then \( G * H \) is also bistable bipartite.

Proof. (i) Taking into account the definition of \( * \)-operation, it is clear that \( G * H \) has a perfect matching, whenever both \( G \) and \( H \) have a perfect matching. Hence, Theorem 2.2 implies that \( G * H \) is \( \alpha^+ \)-stable whenever \( G \) and \( H \) are both \( \alpha^+ \)-stable.

(ii) Suppose that \( G \) is \( \alpha^+ \)-stable and \( H \) is bistable bipartite. If \( D \) is an arbitrary proper subset of \( A \) or of \( C \), then according to Theorem 2.5 and Hall’s marriage theorem we get: \( |D| < |N_G(D)| \leq |N_H(N_G(D))| = |N_{G*H}(D)| \), i.e., \( G * H \) is bistable, by virtue of the same Theorem 2.5.

The assertion (iii) is a consequence of (ii).

Corollary 4.4 Let \( X, Y \) be \((0,1)\)-matrices of order \( n \). If \( \text{per}(X) > 0 \) and \( Y \) is fully indecomposable, then \( XY \) is fully indecomposable.

Corollary 4.5 (Lewin, [13]) The product of any finite number of fully indecomposable matrices is a fully indecomposable matrix.

Proof. Clearly, it is sufficient to prove the statement for two matrices, say \( X \) and \( Y \). Let \( G = (A, B, E) \) and \( H = (B, C, F) \) be balanced bipartite graphs, having \( X, Y \) respectively, as reduced adjacency matrices. Lemma 1.1 implies that \( X \cdot Y \) is the reduced adjacency matrix of the graph \( G * H \). By Proposition 1.4, \( G \) and \( H \) are bistable bipartite, and according to Proposition 1.3, \( G * H \) is also bistable bipartite. Hence, Proposition 1.4 ensures that \( X \cdot Y \) is fully indecomposable. Therefore, \( XY \) is fully indecomposable, as well.

Corollary 4.6 (Marcus and Minc, [13]) If \( X \) is a fully indecomposable \((0,1)\)-matrix, then \( XX^\dagger \) is fully indecomposable.

5 Kronecker product of matrices

Let \( G = (A, B, E) \) and \( H = (C, D, F) \) be two balanced bipartite graphs on \( 2n \) vertices. The Kronecker product of graphs \( G \) and \( H \) is the graph \( K = G \otimes H = (A \times C, B \times D, U) \), where \( (a, c)(b, d) \in U \) if and only if \( ab \in E \) and \( cd \in F \). In these notations we have the following:
Lemma 5.1 If $X, Y$ are respectively, the reduced adjacency matrices of the balanced bipartite graphs $G$ and $H$, then the Kronecker matrix product $Z = X \otimes Y$ is the reduced adjacency matrix of the graph $K = G \otimes H$.

Proof. If $X = (x_{ij}), Y = (y_{ij})$ and $Z = (z_{ij})$, then we have:

$$z_{ij} = z_{(k-1)m+p,(r-1)m+q} = x_{kr}y_{pq} = 1 \iff x_{kr} = 1 \text{ and } y_{pq} = 1$$

$\iff a_kb_r \in E$ and $c_pd_q \in F \iff (a_k, c_p)(b_r, d_q) \in U$, i.e., $Z$ is the reduced adjacency matrix of $K$. $\blacksquare$

Proposition 5.2 If $G = (A, B, E)$ and $H = (C, D, F)$ are $\alpha^+$-stable, then their Kronecker product $K = G \otimes H$ is also $\alpha^+$-stable.

Proof. Let $\{(a_i, b_i) : 1 \leq i \leq n\}$ and $\{(c_j, b_j) : 1 \leq j \leq m\}$ be perfect matchings in $G, H$ respectively, which exist by virtue of Theorem 2.2. Hence, according to the same theorem, $K$ is also $\alpha^+$-stable, since $\{(a_i, c_j)(b_i, d_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$ is a perfect matching of $K$. $\blacksquare$

Corollary 5.3 Let $X, Y$ be two $(0,1)$-matrices of order $n, m$, respectively. Then

$$\rho(X \otimes Y) \geq \rho(X)\rho(Y), \text{ and if } \rho(X) = n, \rho(Y) = m, \text{ then } \rho(X \otimes Y) = \rho(X)\rho(Y).$$

Proof. Let $G = (A, B, E)$ and $H = (C, D, F)$ be bipartite graphs having $X, Y$ as reduced adjacency matrices, respectively. If the edge sets

$$\{a_ib_i : 1 \leq i \leq \rho(X)\} \text{ and } \{c_jb_j : 1 \leq j \leq \rho(Y)\}$$

are maximum matchings in $G, H$ respectively, then

$$M = \{(a_i, c_j)(b_i, d_j) : 1 \leq i \leq \rho(X), 1 \leq j \leq \rho(Y)\}$$

is a matching in $G \otimes H$, and consequently $\rho(X \otimes Y) \geq |M| \geq \rho(X)\rho(Y)$. If $\rho(X) = n$ and $\rho(Y) = m$, i.e., both $G$ and $H$ have perfect matchings, then $M$ is a perfect matching in $G \otimes H$, and this ensures that $\rho(X \otimes Y) = \rho(X)\rho(Y)$. $\blacksquare$

Proposition 5.4 If $G = (A, B, E)$ is $\alpha$-stable and $H = (C, D, F)$ is $\alpha^+$-stable, then their Kronecker product $K = G \otimes H$ is $\alpha$-stable.

Proof. Let $X, Y, Z$ be the corresponding reduced adjacency matrices of $G, H$ and $K$. By Proposition 5.17, for any non-zero entry $z_{ij} = z_{(k-1)m+p,(r-1)m+q} = x_{kr}y_{pq}$ of $Z$, there is a non-zero diagonal $\{x_{1i}, x_{2i}, ..., x_{ni}\}$ of $X$ that does not contain $x_{kr}$, and clearly the blocks $\{x_{1i}Y, x_{2i}Y, ..., x_{ni}Y\}$ contain one non-zero diagonal of $Z$, since $Y$ has at least a non-zero diagonal. According to Proposition 3.17, $K$ is $\alpha$-stable. $\blacksquare$

Corollary 5.5 The Kronecker product of two $\alpha$-stable bipartite graphs is $\alpha$-stable.

In [1], Bruhat proved that:

Theorem 5.6 The Kronecker product of two fully indecomposable matrices is a fully indecomposable matrix.

As a consequence, we get:

Corollary 5.7 The Kronecker product of two bistable bipartite graphs is a bistable bipartite graph.
6 Conclusions

In this paper we investigated the intimate relationship existing between the structure of both $\alpha^+$-stable and $\alpha$-stable bipartite graphs, and the structure of their corresponding reduced matrices. The mutual transfer of the results was done via the following bridge: bistable bipartite graphs vis-a-vis fully indecomposable matrices.

On the base of this duality, we have obtained new proofs and extensions of several well-known theorems on matrices, and on the other hand, new characterizations of $\alpha^+$-stable or $\alpha$-stable bipartite graphs.

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