Orthogonal Polynomials with Periodically Modulated Recurrence Coefficients in the Jordan Block Case II

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Abstract
We study Jacobi matrices with $N$-periodically modulated recurrence coefficients when the sequence of $N$-step transfer matrices is convergent to a non-trivial Jordan block. In particular, we describe asymptotic behavior of their generalized eigenvectors, we prove convergence of $N$-shifted Turán determinants as well as of the Christoffel–Darboux kernel on the diagonal. Finally, by means of subordinacy theory, we identify their absolutely continuous spectrum as well as their essential spectrum. By quantifying the speed of convergence of transfer matrices we were able to cover a large class of Jacobi matrices. In particular, those related to generators of birth–death processes.

Keywords Orthogonal polynomials · Asymptotics · Turán determinants · Christoffel–Darboux kernel

Mathematics Subject Classification Primary 47B36; Secondary 42C05

1 Introduction

Consider two sequences $a = (a_n : n \in \mathbb{N}_0)$ and $b = (b_n : n \in \mathbb{N}_0)$ such that $a_n > 0$ and $b_n \in \mathbb{R}$ for all $n \geq 0$. Let $A$ be the closure in $\ell^2(\mathbb{N}_0)$ of the operator acting by the
matrix

\[ A = \begin{pmatrix}
    b_0 & a_0 & 0 & 0 & \cdots \\
    a_0 & b_1 & a_1 & 0 & \cdots \\
    0 & a_1 & b_2 & a_2 & \cdots \\
    0 & 0 & a_2 & b_3 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix} \]

on finitely supported sequences. The operator \( A \) is called Jacobi matrix and its Jacobi parameters are the sequences \( a \) and \( b \). Recall that \( \ell^2(\mathbb{N}_0) \) is the Hilbert space of square summable complex-valued sequences with the scalar product

\[ \langle x, y \rangle_{\ell^2(\mathbb{N}_0)} = \sum_{n=0}^{\infty} x_n y_n. \]

Its standard orthonormal basis will be denoted by \((\delta_n : n \in \mathbb{N}_0)\). Namely, \( \delta_n \) is the sequence having 1 on the \( n \)-th position and 0 elsewhere.

Let us observe that the operator \( A \) is always symmetric. However, if \( A \) is unbounded, that is at least one of the sequences \( a \) and \( b \) is unbounded, it does not have to be self-adjoint. If it is self-adjoint, then one can define a Borel probability measure \( \mu \) as

\[ \mu(\cdot) = \langle EA(\cdot)\delta_0, \delta_0 \rangle_{\ell^2} \]

where \( EA \) is the spectral resolution of the identity of \( A \). Then the sequence of polynomials \((p_n : n \in \mathbb{N}_0)\) satisfying

\[
  p_0(x) = 1, \quad p_1(x) = \frac{x - b_0}{a_0},
  a_{n-1} p_{n-1}(x) + b_n p_n(x) + a_n p_{n+1}(x) = x p_n(x), \quad n \geq 1.
\]

is an orthonormal basis in \( L^2(\mathbb{R}, \mu) \), that is the Hilbert space of square integrable complex-valued functions with the scalar product

\[ \langle f, g \rangle_{L^2(\mathbb{R}, \mu)} = \int_{\mathbb{R}} f(x) \overline{g(x)} \mu(dx). \]

Moreover, the operator \( U : \ell^2(\mathbb{N}_0) \to L^2(\mathbb{R}, \mu) \) defined on the basis vectors by

\[ U\delta_n = p_n \]

is unitary and satisfies

\[ (UAU^{-1}f)(x) = xf(x) \]
for every \( f \in L^2(\mathbb{R}, \mu) \) such that \( xf \in L^2(\mathbb{R}, \mu) \), see [45, Section 6] for more details. It follows that the spectral properties of \( A \) are intimately related to the properties of \( \mu \). For example, \( \sigma_{\text{ess}}(A) \) is the set of accumulation points of \( \text{supp}(\mu) \). Furthermore, if

\[
\mu = \mu_{\text{ac}} + \mu_{\text{sing}}
\]

is the Lebesgue decomposition of \( \mu \) into the absolutely continuous and the singular parts with respect to the Lebesgue measure, then \( \sigma_{\text{ac}}(A) = \text{supp}(\mu_{\text{ac}}) \) and \( \sigma_{\text{sing}}(A) = \text{supp}(\mu_{\text{sing}}) \).

Jacobi matrices are thoroughly studied. In the bounded case, let us only refer to the recent monograph [50] and to the references therein. For unbounded case, see e.g. [10, 16, 21, 39, 42, 57–60] and the references therein. In this article we consider unbounded Jacobi matrices only.

An interesting class of unbounded Jacobi matrices is related to the so-called birth–death processes (see, e.g. [46]), that is stationary Markov processes with the discrete state space \( \mathbb{N}_0 \). According to [24] generators of birth–death processes correspond to the Jacobi parameters

\[
a_n = \sqrt{\lambda_n \mu_{n+1}}, \quad b_n = -\lambda_n - \mu_n \]

where positive sequences \( (\lambda_n : n \in \mathbb{N}_0) \) and \( (\mu_n : n \in \mathbb{N}_0) \) are called the birth and death rates, respectively. The simplest case is when \( \lambda_n = \mu_{n+1} \), which we call symmetric. In particular, we can consider the following example.

**Example 1.1** Let \( \kappa \in (1, 2) \) and set

\[
a_n = (n + 1)^\kappa, \quad b_n = -(n + 1)^\kappa - n^\kappa.
\]

Then \( \lambda_n = (n + 1)^\kappa, \mu_n = n^\kappa \).

Another interesting class of unbounded Jacobi matrices has been recently studied in [66].

**Example 1.2** For \( \kappa \in (1, \infty) \) and \( f, g > -1 \) we set

\[
a_n = (n + 1)^\kappa \left( 1 + \frac{f}{n + 1} \right), \quad b_n = -2(n + 1)^\kappa \left( 1 + \frac{g}{n + 1} \right).
\]

In particular, in [66], spectral properties of \( A \) has been described if \( \kappa \in \left( \frac{3}{2}, \infty \right) \) and \( \kappa + 2g - 2f \neq 0 \).

Let us observe that in both examples the Jacobi parameters satisfy

\[
\lim_{n \to \infty} a_n = \infty, \quad \lim_{n \to \infty} \frac{a_{n-1}}{a_n} = 1, \quad \lim_{n \to \infty} \frac{b_n}{a_n} = -2. \quad (1.1)
\]

The aim of this article is to study spectral properties of \( A \) as well as the asymptotic behavior of the associated orthogonal polynomials \( (p_n : n \geq 0) \) for a large subclass of
Jacobi parameters satisfying (1.1) containing sequences from Examples 1.1 and 1.2 as special cases. In fact, in this article we will go beyond (1.1) by allowing the sequences \((a_n : n \in \mathbb{N}_0)\) and \((b_n : n \in \mathbb{N}_0)\) to be asymptotically periodic. To be more precise, given \(N\) a positive integer, we say that Jacobi parameters \((a_n : n \in \mathbb{N}_0)\) and \((b_n : n \in \mathbb{N}_0)\) are \(N\)-periodically modulated if there are two \(N\)-periodic sequences \((\alpha_n : n \in \mathbb{Z})\) and \((\beta_n : n \in \mathbb{Z})\) of positive and real numbers, respectively, such that

\[
\begin{align*}
\text{(a)} & \quad \lim_{n \to \infty} a_n = \infty, \\
\text{(b)} & \quad \lim_{n \to \infty} \left| \frac{\alpha_{n-1}}{\alpha_n} - \frac{\alpha_{n-1}}{\alpha_{n+1}} \right| = 0, \\
\text{(c)} & \quad \lim_{n \to \infty} \left| \frac{\beta_n}{\alpha_n} - \frac{b_n}{a_n} \right| = 0.
\end{align*}
\]

It turns out that spectral properties of \(N\)-periodically modulated Jacobi matrices depend on the matrix \(X_0(0)\) where for any \(n \geq 0\) we have set

\[
\mathcal{X}_n(x) = \mathcal{B}_{N+n-1}(x) \mathcal{B}_{N+n-2}(x) \cdots \mathcal{B}_n(x)
\]

where

\[
\mathcal{B}_j(x) = \begin{pmatrix} 0 & 1 \\ \frac{\alpha_{j-1}}{\alpha_j} & x - \beta_j \frac{1}{\alpha_j} \end{pmatrix}.
\]

More specifically, we can distinguish four cases:

I. if \(|\text{tr} \mathcal{X}_0(0)| < 2\), then under some regularity assumptions on Jacobi parameters one has that \(\sigma(A) = \mathbb{R}\), and it is purely absolutely continuous, see e.g. \([19, 21, 54, 56, 57]\);

II. if \(|\text{tr} \mathcal{X}_0(0)| = 2\), then we have two subcases:

(a) if \(\mathcal{X}_0(0)\) is diagonalizable then under some regularity assumptions on Jacobi parameters there is a compact interval \(I \subset \mathbb{R}\) such that \(A\) is purely absolutely continuous on \(\mathbb{R} \setminus I\), and it is purely discrete in the interior of \(I\), see e.g. \([5–7, 10, 11, 17, 18, 23, 43, 54, 55, 60]\);

(b) if \(\mathcal{X}_0(0)\) is not diagonalizable then the only situation which was known is the case when either the essential spectrum of \(A\) is empty or it is a half-line, see e.g. \([4, 8, 9, 20, 22, 33, 34, 36, 37, 39, 44, 51, 61, 66]\);

III. if \(|\text{tr} \mathcal{X}_0(0)| > 2\), then under some regularity assumptions on Jacobi parameters the essential spectrum of \(A\) is empty, see e.g. \([16, 21, 38, 60, 64]\);

Observe that in case I the absolutely continuous spectrum fills the whole real line, whereas in the case III it is empty. This phenomenon was originally observed in \([21]\) and it was called spectral phase transition of the first type. Notice that the case II corresponds to the point where the actual phase transition occurs. In fact, in \([21, \text{Section 5}]\) the task of analyzing the case II was formulated as a very interesting open problem, whose analysis required finding new tools. Nowadays, the case II.a is quite well-understood, see \([58, 60]\). Therefore, in this article we are exclusively interested in the case II.b, which for \(N = 1\) and \(\alpha_n \equiv 1, \beta_n \equiv -2\) covers (1.1).
Let us introduce an auxiliary positive sequence $\gamma = (\gamma_n : n \in \mathbb{N}_0)$ tending to infinity. In Examples 1.1 and 1.2 we take $\gamma_n = a_n$ and $\gamma_n = n + 1$, respectively. We say that $N$-periodically modulated Jacobi parameters $(a_n), (b_n)$ are $\gamma$-tempered if the sequences

$$
\left( \sqrt{\gamma_n} \left( \frac{\alpha_{n-1}}{\alpha_n} - \frac{a_{n-1}}{a_n} \right) : n \in \mathbb{N} \right), \left( \sqrt{\gamma_n} \left( \frac{\beta_n}{\alpha_n} - \frac{b_n}{a_n} \right) : n \in \mathbb{N} \right), \left( \frac{\gamma_n}{a_n} : n \in \mathbb{N} \right)
$$

belongs to $D^1_N$. Let us recall that a sequence $(x_n : n \in \mathbb{N})$ belongs to $D^1_N$ if

$$
\sum_{n=1}^{\infty} |x_{n+N} - x_n| < \infty.
$$

About the sequence $\gamma$ we assume that

$$
\left( \sqrt{\gamma_n} \left( \sqrt{\frac{\alpha_{n-1}}{\alpha_n}} - \sqrt{\frac{\gamma_{n-1}}{\gamma_n}} \right) : n \in \mathbb{N} \right), \left( \frac{1}{\sqrt{\gamma_n}} : n \in \mathbb{N} \right) \in D^1_N,
$$

and

$$
\lim_{n \to \infty} \left( \sqrt{\gamma_{n+N}} - \sqrt{\gamma_n} \right) = 0.
$$

Moreover, we impose that

$$
\gamma_n (1 - \varepsilon [\mathcal{X}_n(0)]_{11}) \left( \frac{\alpha_{n-1}}{\alpha_n} - \frac{a_{n-1}}{a_n} \right)

- \gamma_n \varepsilon [\mathcal{X}_n(0)]_{21} \left( \frac{\beta_n}{\alpha_n} - \frac{b_n}{a_n} \right) : n \in \mathbb{N} \right) \in D^1_N
$$

where $\varepsilon = \text{sign}(\text{tr} \mathcal{X}_0(0))$. To formulate the main results of this paper, we need further definitions. For $x \in \mathbb{C}$ and $n \in \mathbb{N}_0$ we define the transfer matrix by

$$
B_n(x) = \begin{pmatrix} 0 & 1 \\ -\frac{a_{n-1}}{a_n} & \frac{x - b_n}{a_n} \end{pmatrix}.
$$

We use the convention that $a_{-1} := 1$. Moreover, for a matrix

$$
Y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}
$$

we set $[Y]_{ij} = y_{ij}$. The discriminant of $Y$ is defined as $\text{discr} Y = (\text{tr} Y)^2 - 4 \det Y$.

The first main result of this article identifies the absolutely continuous and the essential spectrum of the studied class of Jacobi matrices.
Theorem A Let $N$ be a positive integer. Let $(\gamma_n)$ be a sequence of positive numbers tending to infinity and satisfying (1.2) and (1.3). Let $(a_n)$ and $(b_n)$ be $\gamma$-tempered $N$-periodically modulated Jacobi parameters such that $\mathcal{X}_0(0)$ is a non-trivial parabolic element. Suppose that (1.4) holds true with $\varepsilon = \text{sign}(\text{tr} \mathcal{X}_0(0))$. Set

$$X_n(x) = B_{n+N-1}(x)B_{n+N-2}(x) \ldots B_{n+1}(x)B_n(x).$$

(1.5)

Then the limit

$$\tau(x) = \frac{1}{4} \lim_{n \to \infty} \frac{\gamma_n + N - 1}{a_n + N - 1} \text{discr} X_n(x), \quad x \in \mathbb{R},$$

(1.6)

exists and defines a polynomial of degree at most one. Let

$$\Lambda_- = \tau^{-1}((0, \infty)) \quad \text{and} \quad \Lambda_+ = \tau^{-1}((-\infty, 0)).$$

If $\Lambda_- \cup \Lambda_+ \neq \emptyset$ and $A$ is self-adjoint then\(^1\)

$$\sigma_{\text{sing}}(A) \cap \Lambda_- = \emptyset \quad \text{and} \quad \sigma_{\text{ac}}(A) = \sigma_{\text{ess}}(A) = \text{cl}(\Lambda_-).$$

Let us emphasize that Theorem A excludes the case $\Lambda_- = \Lambda_+ = \emptyset$, that is $\tau \equiv 0$. Moreover, Theorem A implies that the operator $A$ is not semi-bounded if $\Lambda_+ = \emptyset$ and $\Lambda_- \neq \emptyset$, because $\Lambda_- = \mathbb{R} = \sigma(A)$. However, it is unclear under what hypotheses the operator $A$ is semi-bounded when $\Lambda_+ \neq \emptyset$. Recall that in the case III a characterization of semi-boundedness of the operator $A$ was given in [38].

The condition (1.4) might look rather restrictive. However, it is always satisfied by Jacobi parameters studied in [61] as well as for generators of symmetric birth–death processes (cf. Remark 11.5). Hence, Theorem A can be applied to Jacobi parameters described in Example 1.1 where for $\gamma_n = a_n$ we get $\tau(x) = x$. Moreover, if $N = 1$ the condition (1.4) reduces to

$$\left( \gamma_n \left( 1 + \frac{a_n - 1}{a_n} + \varepsilon \frac{b_n}{a_n} \right) \right) \in \mathcal{D}_1.$$

Therefore, Theorem A can be applied to Jacobi parameters given in Example 1.2 where for $\gamma_n = n + 1$ we obtain $\tau(x) \equiv -\kappa - 2g + 2f$.

The proof of Theorem A uses the theory of subordinacy. It was first developed in [14] for one-dimensional Schrödinger operators on the real half-line, and later adapted to other classes of operators, see e.g. the survey [13] for more details. In particular, the extension to Jacobi matrices has been accomplished in [26]. The theory of subordinacy links asymptotic behavior of generalized eigenvectors to spectral properties of Jacobi matrices. Let us recall that a sequence $(u_n : n \in \mathbb{N}_0)$ is a generalized eigenvector associated to $x \in \mathbb{C}$, and corresponding to $\eta \in \mathbb{R}^2 \setminus \{0\}$, if the sequence of vectors

$$\tilde{u}_0 = \eta,$$

\(^1\) By cl$(X)$ we denote the closure of the set $X$.\(^\square\) Springer
\[ \tilde{u}_n = \left( \frac{u_{n-1}}{u_n} \right), \quad n \geq 1, \]
satisfies
\[ \tilde{u}_{n+1} = B_n(x)\tilde{u}_n, \quad n \geq 0. \] (1.7)

We often write \((u_n(\eta, x) : n \in \mathbb{N}_0)\) to indicate the dependence on the parameters. In particular, the sequence of orthogonal polynomials \((p_n(x) : n \in \mathbb{N}_0)\) is the generalized eigenvector associated to \(\eta = (0, 1)^t\) and \(x \in \mathbb{C}\). Motivated by [49, Section 8] it will be convenient to define (generalized) Christoffel–Darboux kernel by
\[ K_n(x, y; \eta) = \sum_{j=0}^{n} u_j(\eta, x)u_j(\eta, y), \quad x, y \in \mathbb{R}, \quad \eta \in \mathbb{R}^2 \setminus \{0\}. \]

Suppose that \(A\) is self-adjoint. According to [26, Theorem 3], if for some compact interval with non-empty interior \(K \subset \mathbb{R}\),
\[ \liminf_{n \to \infty} \frac{K_n(x, x; \eta)}{K_n(x, x; \tilde{\eta})} < \infty \quad \text{for any } x \in K \text{ and } \eta, \tilde{\eta} \in S^1, \] (1.8)
where by \(S^1\) we denote the unit sphere in \(\mathbb{R}^2\), then the measure \(\mu\) is absolutely continuous on \(K\), and \(K \subset \text{supp}(\mu)\). Consequently, \(A\) is absolutely continuous on \(K\), and \(K \subset \sigma_{\text{ac}}(A)\). This theory became a standard approach to spectral analysis of Jacobi matrices. It has also been observed that by imposing some uniformity conditions to (1.8) more detailed information on the density of \(\mu\) can be obtained, see the references in [13, Section 4]. In the present article we shall show that for any compact interval \(K \subset \Lambda_\tau\) the following stronger version of (1.8) holds true
\[ \sup_{n \in \mathbb{N}_0} \sup_{x \in K} \sup_{\eta, \tilde{\eta} \in S^1} \frac{K_n(x, x; \eta)}{K_n(x, x; \tilde{\eta})} < \infty. \] (1.9)

In view of [2] (see also [32] for a different proof in a more general setup) the condition (1.9) implies existence of positive constants \(c_1, c_2\) such that the density of \(\mu, \mu'\), satisfies
\[ c_1 < \mu'(x) < c_2 \quad \text{for almost all } x \in K, \text{ with respect to the Lebesgue measure.} \]

Finally, in [47], the following consequence of subordinacy theory has been established: if \(A\) is self-adjoint and for some \(K \subset \mathbb{R}\) there is a function \(\eta : K \to \mathbb{R}^2 \setminus \{0\}\) such that
\[ \sum_{n=0}^{\infty} \sup_{x \in K} |u_n(\eta(x), x)|^2 < \infty, \] (1.11)
then \(K \cap \sigma_{\text{ess}}(A) = \emptyset\). In Theorem 4.1, with a help of a recently obtained variant of discrete Levinson’s type theorem (see [60]), we show that (1.11) holds for every
compact interval $K \subset \Lambda_+$. The fact that (1.9) holds for every compact interval $K \subset \Lambda_-$ is a consequence of the following theorem.

Theorem B Let $N$ be a positive integer. Let $(\gamma_n)$ be a sequence of positive numbers tending to infinity and satisfying (1.2) and (1.3). Let $(a_n)$ and $(b_n)$ be $\gamma$-tempered $N$-periodically modulated Jacobi parameters such that $\mathcal{X}_0(0)$ is a non-trivial parabolic element. Suppose that (1.4) holds true with $\varepsilon = \text{sign}(\text{tr } \mathcal{X}_0(0))$. Set

$$\rho_n = \sum_{j=0}^{n} \frac{\sqrt{\alpha_j \gamma_j}}{a_j}.$$ 

If $\Lambda_- \neq \emptyset$, then $A$ is self-adjoint if and only if $\rho_n \to \infty$. If it is the case, then the limit

$$\lim_{n \to \infty} \frac{1}{\rho_n} K_n(x, x; \eta)$$

exists locally uniformly with respect to $(x, \eta) \in \Lambda_- \times S^1$, and defines a continuous positive function.

Example 1.3 Let $\kappa \in (1, \frac{3}{2}]$ and $f, g > -1$ be such that

$$\kappa + 2g - 2f < 0.$$ 

We set

$$a_n = (n + 1)^\kappa \left( 1 + \frac{f}{n + 1} \right), \quad b_n = 2(n + 1)^\kappa \left( 1 + \frac{g}{n + 1} \right).$$

Since $\kappa > 1$, the Carleman condition is not satisfied, that is

$$\sum_{n=0}^{\infty} \frac{1}{a_n} < \infty.$$ 

As it is easy to check, we can apply Theorems A and B to the above Jacobi parameters, which leads to the conclusion that the corresponding Jacobi operator $A$ is self-adjoint and $\sigma_{ac}(A) = \mathbb{R}$.

Example 1.3 is inspired by examples given by Kostyuchenko–Mirzoev in [28] who provided Jacobi parameters giving rise to self-adjoint Jacobi operators violating the Carleman’s condition. Later the original Kostyuchenko–Mirzoev class was somewhat extended and it was proven that one usually has $\sigma_{ess}(A) = \emptyset$, see e.g. [17, Section 2.2] and [60, Section 6.2]. To the best of our knowledge Jacobi parameters described in Example 1.3 provide the first instances of Jacobi operators violating the Carleman’s condition such that $\sigma_{ac}(A) = \mathbb{R}$. In contrast, a construction of self-adjoint Jacobi matrices with $\sigma_{ac}(A) = [0, \infty)$ violating Carleman’s condition is well-known, see e.g. [9].
To prove Theorem B, we first determine asymptotic behavior of generalized eigenvectors. Then we apply a non-trivial averaging procedure to it. The asymptotic formula is given in the following theorem.

**Theorem C** Let $N$ be a positive integer. Let $(\gamma_n)$ be a sequence of positive numbers tending to infinity and satisfying (1.2) and (1.3). Let $(a_n)$ and $(b_n)$ be $\gamma$-tempered $N$-periodically modulated Jacobi parameters such that $X_0(0)$ is a non-trivial parabolic element. Suppose that (1.4) holds true with $\varepsilon = \text{sign}(\text{tr} X_0(0))$. If $\Lambda_+ \neq \emptyset$, then for each $i \in \{0, 1, \ldots, N-1\}$ and every compact interval $K \subset \Lambda_-$, there are a continuous function $\varphi_i : S^1 \times K \to \mathbb{C}$ and $j_0 \geq 1$ such that

$$\lim_{j \to \infty} \sup_{(\eta, x) \in S^1 \times K} \left| \frac{d_j N+i-1}{\sqrt{j N+i-1}} u_{j N+i}(\eta, x) \right| - |\varphi_i(\eta, x)| \sin \left( \sum_{k=j_0}^{j-1} \theta_{k;i}(x) + \arg \varphi_i(\eta, x) \right) = 0$$

where $\theta_{k;i} : K \to \mathbb{R}$ are some explicit continuous functions. Moreover, $\varphi_i(\eta, x) = 0$ for some (and then for all) $(\eta, x) \in S^1 \times K$ if and only if $[X_i(0)]_{21} = 0$.

The proof of Theorem C is based on uniform diagonalization of transfer matrices which has been already used in [61]. However, in the current setup we were not able to relate $|\varphi_i(\eta, x)|$ to the density of $\mu$. Hence, in order to prove that $\varphi_i(\eta, x) \neq 0$ provided $[X_i(0)]_{21} \neq 0$, we needed an additional argument based on a consequence of the following theorem (see Corollary 6.3 for details) which studies convergence of generalized $N$-shifted Turán determinants. The latter are defined as

$$\mathcal{D}_n(\eta, x) = \det \begin{pmatrix} u_{n+N-1}(\eta, x) & u_{n-1}(\eta, x) \\ u_{n+N}(\eta, x) & u_n(\eta, x) \end{pmatrix} = u_n(\eta, x)u_{n+N-1}(\eta, x) - u_{n-1}(\eta, x)u_{n+N}(\eta, x)$$

where $(u_n(\eta, x) : n \in \mathbb{N}_0)$ is the generalized eigenvector associated to $x \in \mathbb{R}$, and corresponding to $\eta \in \mathbb{R}^2 \setminus \{0\}$. The (classical) shifted Turán determinants correspond to $\eta = (0, 1)^t$. They were defined for the first time in [65] for $N = 1$, and then generalized in [12] to $N \geq 1$. In [65] they were instrumental in studying the zeros of the Legendre polynomials where it was observed that they are non-negative on the support of their orthogonality measure, see also [25] for later developments. As it was shown in [40, Theorem 7.34] and [12, Theorem 6], if $\text{supp}(\mu)$ is compact, the asymptotic behavior of shifted Turán determinants is usually closely related to the density of $\mu$, see [30, 31] and the survey [41]. The extension of the above phenomena to measures with unbounded support has been accomplished in [54, 55, 57, 61]. For these reasons the following theorem is an important result on its own.

**Theorem D** Let $N$ be a positive integer. Let $(\gamma_n)$ be a sequence of positive numbers tending to infinity and satisfying (1.2) and (1.3). Let $(a_n)$ and $(b_n)$ be $\gamma$-tempered $N$-periodically modulated Jacobi parameters such that $X_0(0)$ is a non-trivial parabolic
element. Suppose that (1.4) holds true with \(\varepsilon = \text{sign}(\text{tr}\ X_0(0))\). If \(\Lambda_\pm \neq \emptyset\), then for each \(i \in \{0, 1, \ldots, N - 1\}\) the limit
\[
\lim_{\eta \to \infty} \left| a_{n+N+1} \sqrt{\gamma_{n+N+1} - 1} D_n(\eta, x) \right|
\] (1.13)
exists locally uniformly with respect to \((x, \eta) \in \Lambda_\pm \times S^1\) and defines a continuous positive function.

Let us remark that the first order asymptotics of generalized eigenvectors provided by Theorem C is insufficient to prove (1.13). It is an open problem whether, similarly to [57–59, 61], one can relate the value of (1.13) to the density of the measure \(\mu\). We hope to return to this problem in the future.

In this article, we also consider \(\ell^1\)-type perturbations of Jacobi parameters \(a, b\) satisfying hypotheses of Theorem A. Namely, in Sect. 10, we study Jacobi parameters \(\tilde{a}, \tilde{b}\) of the form
\[
\tilde{a}_n = a_n(1 + \xi_n), \quad \tilde{b}_n = b_n(1 + \zeta_n),
\]
where \((\sqrt{n}\xi_n), (\sqrt{n}\zeta_n) \in \ell^1\). We show that for sequences \(\tilde{a}\) and \(\tilde{b}\) the analogues of Theorems A–C hold true. In particular, we can treat the following Jacobi parameters

**Example 1.4** For \(\kappa \in (1, \infty)\) and \(f, g \in \mathbb{R}\) we set
\[
\tilde{a}_n = (n + 1)^\kappa \left(1 + \frac{f}{n + 1} + \xi_n\right), \quad \tilde{b}_n = 2(n + 1)^\kappa \left(1 + \frac{g}{n + 1} + \zeta_n\right),
\]
where \((\sqrt{n}\xi_n), (\sqrt{n}\zeta_n) \in \ell^1\) and \(\kappa + 2g - 2f \neq 0\).

Jacobi parameters considered in Example 1.4 under the additional restrictions \(\kappa \in \left(\frac{3}{2}, \infty\right)\) and \(\xi_n, \zeta_n = O(n^{-2})\), have recently been studied in [66].

Before we close the introduction, let us mention some of the approaches used in the literature for analysis of the case II.b. In [9] it was observed that a certain class of Jacobi matrices related to birth–death processes can be studied by considering the restriction to a subspace of \(\ell^2\) of the square of Jacobi matrices belonging to the case I with \(b_n \equiv 0\). This method is particularly effective in describing \(\sigma_{ac}(A)\). Next, in [36], asymptotics of generalized eigenvectors was studied by the reduction to the analysis of a discrete variant of Ricatti equation, whereas in [44, 51] the analysis was possible by applying Birkhoff–Adams theorem. Further, in [39] by adaptation of Kooman method (see [27]) and the approach of [1] it was possible to obtain asymptotic behavior of generalized eigenvectors for \(x \in \mathbb{C}\setminus\{0\}\) as well as continuity of the density of the measure \(\mu\). A very important class of methods is motivated by the technique introduced by Harris and Lutz in [15]. In these methods for a given \(i \in \{0, 1, \ldots, N - 1\}\) one consider the "change of variables"

\[
\vec{u}_{nN+i} = Z_n \vec{v}_n, \quad n \geq 0
\] (1.14)
for some invertible matrices \(Z_n\). Then by (1.7) and (1.5) the sequence \((\vec{v}_n)\) satisfies the equation

\[
\vec{v}_{n+1} = Z_{n+1}^{-1} X_{nN+i} Z_n \vec{v}_n, \quad n \geq 0.
\] (1.15)
The matrices $Z_n$ are chosen in a way that one can apply to the system (1.15) Levinson’s theorem. Then thanks to the relation (1.14), the asymptotics of $(\tilde{u}_nN+i : n \geq n_0)$. The success of this approach depends on the properties of the matrices $Z_n$. In [20] the construction of these matrices were motivated by a formal WKB method in which, by means of an ansatz, one guesses the form of the solution. This approach was later extended in [4, 22, 33, 37]. It should be emphasized that the resulting matrices $Z_n$ were complex-valued, oscillating and unbounded.

In this work, we start by extending techniques which were successful in the prequel [61]. Namely, we construct matrices $Z_n$ such that the system (1.15) satisfies hypotheses of a uniform discrete Levinson’s theorem so it belongs to Harris–Lutz paradigm. However, our matrices are very simple and explicit (see (3.1)), real and convergent (obviously to a singular matrix). These features lead to greater applicability of our approach than in the previous works. Since Jacobi parameters considered in this paper are more “singular” than in [61], we were forced to use a more general and delicate change of variables, so that we can exploit the condition (1.4) to “smooth them out”. Using our change of variables, the spectral properties of $A$ on $\Lambda_+$ can be derived analogously to [61]. On $\Lambda_-$ the situation is much more involved. Namely, in [61], in order to prove that $\mu$ is absolutely continuous on every compact $K \subset \Lambda_-$, we used an explicit sequence of probability measures $(\mu_n : n \in \mathbb{N})$ which converges weakly to $\mu$, and such that the sequence of their densities converges uniformly on $K$ to a continuous positive function. In the present paper this approach does not work anymore. To get around of this issue we apply the subordinacy theory. This requires to analyze the asymptotic behavior of Christoffel–Darboux kernel which was possible thanks to the asymptotics obtained in Theorem C. All of this reduces the problem to study averages of highly oscillatory sums. For this reason we develop Lemma 8.1, which might be of independent interest.

The method of asymptotic analysis of generalized eigenvectors is similar to [61]. However, in the present situation we had to find another argument showing positivity of the function $|\varphi_i|$. Previously, by using the convergence of densities of the sequence $(\mu_n : n \in \mathbb{N})$, we were able to explicitly compute the value of $|\varphi_i|$ in terms of $\mu'$. In the present work we use certain algebraic properties of $\varphi_i$ together with Theorem D, see Claim 7.2 for details. Let us emphasize that the method of subordinacy gives the bound (1.10) only, which is weaker than the continuity of $\mu'$. The drawback of the current approach compared to [61] is that we do not get a constructive method to approximate the density of $\mu$. In the forthcoming article [62], by linking the asymptotic behavior of zeros of the polynomials $(p_n : n \in \mathbb{N}_0)$ with the value of (1.12), we managed to prove that, under certain additional hypotheses, the density of the measure $\mu$ for Jacobi matrices satisfying Theorem B is a continuous positive function on $\Lambda_-$. The article is organized as follows: In Sect. 2 we fix notation and we formulate basic facts. Section 3 is devoted to our change of variables. In Sect. 4 we study spectral properties of $A$ on $\Lambda_+$. Next, in Sect. 5 we describe uniform diagonalization of transfer matrices on $\Lambda_-$, which is used in the rest of the article. The proof of Theorem D is presented in Sect. 6. Next, in Sect. 7, we prove Theorem C. Section 8 is devoted to the proof of Theorem B. In Sect. 9 we study the self-adjointness of $A$. The extensions of Theorems A–C to $\ell^1$-type perturbations is achieved in Sect. 10. Finally, in Sect. 11, we present more concrete classes of sequences to illustrate results of this article.
Notation

By \( \mathbb{N} \) we denote the set of positive integers and \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). Throughout the whole article, we write \( A \lesssim B \) if there is an absolute constant \( c > 0 \) such that \( A \leq cB \). We write \( A \asymp B \) if \( A \lesssim B \) and \( B \lesssim A \). Moreover, \( c \) stands for a positive constant whose value may vary from occurrence to occurrence. For any compact set \( K \), by \( o_K(1) \) we denote the class of functions \( f_n : K \rightarrow \mathbb{R} \) such that \( \lim_{n \rightarrow \infty} f_n = 0 \) uniformly on \( K \).

2 Preliminaries

In this section we fix the notation which is used in the rest of the article.

2.1 Stolz Class

In this section we define a proper class of slowly oscillating sequences which is motivated by [52], see also [57, Section 2]. Let \( V \) be a normed space. We say that a sequence \( (x_n : n \in \mathbb{N}) \) of vectors from \( V \) belongs to \( \mathcal{D}_r(V) \) for certain \( r \in \mathbb{N} \), if it is bounded and for each \( j \in \{1, \ldots, r\} \),

\[
\sum_{n=1}^{\infty} \| \Delta^j x_n \|^r < \infty
\]

where

\[
\Delta^0 x_n = x_n, \\
\Delta^j x_n = \Delta^{j-1} x_{n+1} - \Delta^{j-1} x_n, \quad j \geq 1.
\]

If \( V \) is the real line with Euclidean norm we abbreviate \( \mathcal{D}_r = \mathcal{D}_r(V) \). Given a compact set \( K \subset \mathbb{C} \) and a normed vector space \( R \), we denote by \( \mathcal{D}_r(K, R) \) the case when \( V \) is the space of all continuous mappings from \( K \) to \( R \) equipped with the supremum norm. Let us recall that \( \mathcal{D}_r(V) \) is an algebra provided \( V \) is a normed algebra. Let \( N \) be a positive integer. We say that a sequence \( (x_n : n \in \mathbb{N}) \) belongs to \( \mathcal{D}_r^N(V) \), if for any \( i \in \{0, 1, \ldots, N-1\} \),

\[
(x_{nN+i} : n \in \mathbb{N}) \in \mathcal{D}_r(V).
\]

Again, \( \mathcal{D}_r^N(V) \) is an algebra provided \( V \) is a normed algebra. In what follows we shall use \( \mathcal{D}_r^1(V) \) only.

2.2 Finite Matrices

By \( \text{Mat}(2, \mathbb{C}) \) and \( \text{Mat}(2, \mathbb{R}) \) we denote the space of \( 2 \times 2 \) matrices with complex and real entries, respectively, equipped with the spectral norm. Next, \( \text{GL}(2, \mathbb{R}) \) and
SL(2, \mathbb{R}) consist of all matrices from Mat(2, \mathbb{R}) which are invertible and of determinant equal 1, respectively. A matrix \( X \in SL(2, \mathbb{R}) \) is a non-trivial parabolic if it is not a multiple of the identity and \( |\text{tr } X| = 2 \).

Let \( X \in \text{Mat}(2, \mathbb{C}) \). By \( X^t \) we denote the transpose of the matrix \( X \). Let us recall that symmetrization and the discriminant are defined as

\[
sym(X) = \frac{1}{2} X + \frac{1}{2} X^*, \quad \text{and} \quad \text{discr}(X) = (\text{tr } X)^2 - 4 \det X,
\]

respectively. Here \( X^* \) denotes the Hermitian transpose of the matrix \( X \).

By \( \{e_1, e_2\} \) we denote the standard orthonormal basis of \( \mathbb{C}^2 \), i.e.

\[
e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]

Lastly, for a sequence of square matrices \((C_n : n_0 \leq n \leq n_1)\) we set

\[
\prod_{k=n_0}^{n_1} C_k = C_{n_1} C_{n_1-1} \cdots C_{n_0}.
\]

### 2.3 Generalized Eigenvectors

A sequence \((u_n : n \in \mathbb{N}_0)\) is a generalized eigenvector associated to \( x \in \mathbb{C} \) and corresponding to \( \eta \in \mathbb{R}^2 \setminus \{0\} \), if the sequence of vectors

\[
\bar{u}_0 = \eta,
\]

\[
\bar{u}_n = \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}, \quad n \geq 1,
\]

satisfies

\[
\bar{u}_{n+1} = B_n(x) \bar{u}_n, \quad n \geq 0,
\]

where \( B_n \) is the transfer matrix defined as

\[
B_0(x) = \begin{pmatrix} 0 & 1 \\ -\frac{1}{a_0} & \frac{x-b_0}{a_0} \end{pmatrix}, \quad B_n(x) = \begin{pmatrix} 0 & 1 \\ -\frac{1}{a_{n-1}} & \frac{x-b_n}{a_n} \end{pmatrix}, \quad n \geq 1.
\]

To indicate the dependence on the parameters, we write \((u_n(\eta, x) : n \in \mathbb{N}_0)\). In particular, the sequence of orthogonal polynomials \((p_n(x) : n \in \mathbb{N}_0)\) is the generalized eigenvector associated to \( \eta = e_2 \) and \( x \in \mathbb{C} \).
2.4 Periodic Jacobi Parameters

By \((\alpha_n : n \in \mathbb{Z})\) and \((\beta_n : n \in \mathbb{Z})\) we denote \(N\)-periodic sequences of real and positive numbers, respectively. For each \(k \geq 0\), let us define polynomials \((p_n^k : n \in \mathbb{N}_0)\) by relations

\[
p_0^k(x) = 1, \quad p_1^k(x) = \frac{x - \beta_k}{\alpha_k},
\]

\[
\alpha_{n+k-1} p_{n-1}^k(x) + \beta_{n+k} p_n^k(x) + \alpha_{n+k} p_{n+1}^k(x) = x p_n^k(x), \quad n \geq 1.
\]

Let

\[
\mathfrak{B}_n(x) = \left( \begin{array}{cc} 0 & 1 \\ \frac{\alpha_{n-1}}{\alpha_n} & \frac{x - \beta_n}{\alpha_n} \end{array} \right), \quad \text{and} \quad \mathfrak{X}_n(x) = \prod_{j=n}^{N+n-1} \mathfrak{B}_j(x), \quad n \in \mathbb{Z}.
\]

By \(\mathfrak{A}\) we denote the Jacobi matrix corresponding to

\[
\begin{pmatrix}
\beta_0 & \alpha_0 & 0 & 0 & \ldots \\
\alpha_0 & \beta_1 & \alpha_1 & 0 & \ldots \\
0 & \alpha_1 & \beta_2 & \alpha_2 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\]

2.5 Tempered Periodic Modulations

Let \(N\) be a positive integer. We say that Jacobi parameters \((\alpha_n : n \in \mathbb{N}_0)\) and \((\beta_n : n \in \mathbb{N}_0)\) are \(N\)-periodically modulated if there are two \(N\)-periodic sequences \((\alpha_n : n \in \mathbb{Z})\) and \((\beta_n : n \in \mathbb{Z})\) of positive and real numbers, respectively, such that

\[
\begin{align*}
(a) \quad & \lim_{n \to \infty} \alpha_n = \infty, \\
(b) \quad & \lim_{n \to \infty} \left| \frac{\alpha_{n-1}}{\alpha_n} - \frac{a_{n-1}}{a_n} \right| = 0, \\
(c) \quad & \lim_{n \to \infty} \left| \frac{\beta_n}{\alpha_n} - \frac{b_n}{a_n} \right| = 0.
\end{align*}
\]

In this article we are mostly interested in tempered \(N\)-periodically modulated Jacobi parameters, i.e. we assume that there is a sequence of positive numbers \((\gamma_n : n \in \mathbb{N}_0)\) tending to infinity and satisfying

\[
\left( \sqrt{\gamma_n} \left( \sqrt[4]{\alpha_{n-1}} - \sqrt[4]{\alpha_n} \right) : n \in \mathbb{N} \right), \left( \frac{1}{\sqrt{\gamma_n}} : n \in \mathbb{N} \right) \in \mathcal{D}_1^N, \quad (2.3)
\]

and

\[
\lim_{n \to \infty} \left( \sqrt{\gamma_{n+N}} - \sqrt{\gamma_n} \right) = 0, \quad (2.4)
\]

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such that

\[
\left(\sqrt{\gamma_n} \frac{\alpha_{n-1} - a_{n-1}}{\alpha_n} : n \in \mathbb{N}\right), \left(\sqrt{\gamma_n} \frac{b_n - b_n}{\alpha_n} : n \in \mathbb{N}\right),
\]

\[
\left(\frac{\gamma_n}{\alpha_n} : n \in \mathbb{N}\right) \in D_1^N.
\]  

(2.5)

In view of (2.5), there are two \(N\)-periodic sequence \((s_n : n \in \mathbb{Z})\) and \((r_n : n \in \mathbb{Z})\) such that

\[
\lim_{n \to \infty} \left| \sqrt{\alpha_n} \gamma_n \left(\frac{\alpha_{n-1} - a_{n-1}}{\alpha_n} \right) - s_n \right| = \lim_{n \to \infty} \left| \sqrt{\alpha_n} \gamma_n \left(\frac{b_n - b_n}{\alpha_n} \right) - r_n \right| = 0.  
\]  

(2.6)

From (2.3) it stems that

\[
\lim_{n \to \infty} \left| \frac{\alpha_{n-1}}{\alpha_n} - \frac{\gamma_{n-1}}{\gamma_n} \right| = 0.  
\]  

(2.7)

Hence, there is \(t \geq 0\), such that

\[
\lim_{n \to \infty} \frac{\gamma_n}{\alpha_n} = t.  
\]  

(2.8)

Let us observe that, if \(t > 0\) then with no lose of generality we can assume that \(t = 1\) and \(\gamma_n \equiv a_n\). Therefore, in what follows we shall assume that \(t \in \{0, 1\}\).

Let us define the \(N\)-step transfer matrix by

\[
X_n = B_{n+N-1} B_{n+N-2} \cdots B_{n+1} B_n.
\]

Observe that for each \(i \in \{0, 1, \ldots, N - 1\},\)

\[
\lim_{j \to \infty} B_{jN+i}(x) = \mathcal{B}_i(0)
\]

and

\[
\lim_{j \to \infty} X_{jN+i}(x) = \mathcal{X}_i(0)
\]

locally uniformly with respect to \(x \in \mathbb{C}\). In the whole article we assume that the matrix \(\mathcal{X}_0(0)\) is a non-trivial parabolic element of \(\text{SL}(2, \mathbb{R})\). Let \(T_0\) be a matrix so that

\[
\mathcal{X}_0(0) = \varepsilon T_0 \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} T_0^{-1}
\]  

(2.9)

where

\[
\varepsilon = \text{sign}(\text{tr} \mathcal{X}_0(0)).
\]  

(2.10)

Since

\[
\mathcal{X}_i(0) = \mathcal{B}_{i-1}(0) \cdots \mathcal{B}_0(0) \mathcal{X}_0(0) \mathcal{B}_0^{-1}(0) \cdots \mathcal{B}_{i-1}^{-1}(0),
\]  

(2.11)
by taking
\[ T_i = \mathcal{B}_{i-1}(0) \cdots \mathcal{B}_0(0) T_0, \quad (2.12) \]
we obtain
\[ \mathfrak{x}_i(0) = \varepsilon T_i \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix} T_i^{-1}. \]

Hence,
\[ \begin{align*}
[\mathfrak{x}_i(0)]_{11} &= \frac{\varepsilon}{\det T_i} (\det T_i - ([T_i]_{11} + [T_i]_{12})([T_i]_{21} + [T_i]_{22})) \\
[\mathfrak{x}_i(0)]_{12} &= \frac{\varepsilon}{\det T_i} ([T_i]_{11} + [T_i]_{12})^2 \\
[\mathfrak{x}_i(0)]_{21} &= \frac{\varepsilon}{\det T_i} (-([T_i]_{21} + [T_i]_{22})^2) \\
[\mathfrak{x}_i(0)]_{22} &= \frac{\varepsilon}{\det T_i} (\det T_i + ([T_i]_{11} + [T_i]_{12})([T_i]_{21} + [T_i]_{22})).
\end{align*} \]

In particular,
\[ \frac{([T_i]_{11} + [T_i]_{12})([T_i]_{21} + [T_i]_{22})}{\det T_i} = 1 - \varepsilon [\mathfrak{x}_i(0)]_{11} \quad (2.13) \]
and
\[ \frac{([T_i]_{21} + [T_i]_{22})^2}{\det T_i} = -\varepsilon [\mathfrak{x}_i(0)]_{21}. \quad (2.14) \]

We often assume that
\[ \left( \gamma_n \left( 1 - \varepsilon [\mathfrak{x}_n(0)]_{11} \right) \left( \frac{\alpha_{n-1}}{\alpha_n} - \frac{a_{n-1}}{a_n} \right) \right. \]
\[ \left. - \gamma_n \varepsilon [\mathfrak{x}_n(0)]_{21} \left( \frac{\beta_n}{\alpha_n} - \frac{b_n}{a_n} \right) : n \in \mathbb{N} \right) \in \mathcal{D}_1^N. \quad (2.15) \]

Therefore, there is \( N \)-periodic sequence \((u_n : n \in \mathbb{N}_0)\) such that
\[ \lim_{n \to \infty} \left| \gamma_n \left( 1 - \varepsilon [\mathfrak{x}_n(0)]_{11} \right) \left( \frac{\alpha_{n-1}}{\alpha_n} - \frac{a_{n-1}}{a_n} \right) \right. \]
\[ \left. - \gamma_n \varepsilon [\mathfrak{x}_n(0)]_{21} \left( \frac{\beta_n}{\alpha_n} - \frac{b_n}{a_n} \right) - u_n \right| = 0. \quad (2.16) \]

Let us define
\[ \tau(x) = \frac{1}{4} \Theta^2 - \nu(x) \quad (2.17) \]
where
\[ \nu(x) = \text{te} \sum_{n=0}^{N-1} \frac{[\mathfrak{x}_{n'}(0)]_{21}}{\alpha_{n'-1}} - \Phi, \quad (2.18) \]
and
\[
\mathfrak{U} = \sum_{i' = 0}^{N-1} \frac{u_{i'}}{\alpha_{i'-1}}, \quad \text{and} \quad \mathfrak{S} = \sum_{i' = 0}^{N-1} \frac{s_{i'}}{\alpha_{i'-1}}.
\tag{2.19}
\]

In view of [61, Proposition 2.1],
\[
\sum_{i' = 0}^{N-1} \frac{[\mathcal{X}_{i'}(0)]_{21}}{\alpha_{i'-1}} = - \operatorname{tr} \mathcal{X}'_{0}(0),
\tag{2.20}
\]

thus
\[
\tau(x) = \frac{1}{4} \mathfrak{S}^2 + \mathfrak{U} + \frac{t \varepsilon (\operatorname{tr} \mathcal{X}'_{0}(0))}{x}.
\tag{2.21}
\]

The following proposition answers the question when \( \mathcal{X}_{0}(0) \) is a non-trivial parabolic element of \( \text{SL}(2, \mathbb{R}) \).

**Proposition 2.1** Suppose that \( |\operatorname{tr} \mathcal{X}_{0}(0)| = 2 \). Then \( \mathcal{X}_{0}(0) \) is a non-trivial parabolic element of \( \text{SL}(2, \mathbb{R}) \) if and only if \( \operatorname{tr} \mathcal{X}'_{0}(0) \neq 0 \).

**Proof** The matrix \( \mathcal{X}_{0}(0) \) is a trivial parabolic element if and only if \( \mathcal{X}_{0}(0) = \varepsilon \text{ Id} \) where \( \varepsilon \) is defined in (2.10). Then by (2.11) we get \( \mathcal{X}_{i}(0) = \varepsilon \text{ Id} \) for all \( i \in \{0, 1, \ldots, N-1\} \). Consequently, by (2.20), \( \operatorname{tr} \mathcal{X}_{0}(0) = 0 \). On the other hand, if \( \mathcal{X}_{0}(0) \neq \varepsilon \text{ Id} \), then thanks to [55, Proposition 3] at least one of the numbers \( [\mathcal{X}_{0}(0)]_{21} \) and \( [\mathcal{X}_{1}(0)]_{21} \) is non-zero.

In view of [61, Proposition 2.2] we have
\[
\sum_{i' = 0}^{N-1} \frac{[\mathcal{X}_{i'}(0)]_{21}}{\alpha_{i'-1}} = |\operatorname{tr} \mathcal{X}'_{0}(0)|,
\]

thus \( \operatorname{tr} \mathcal{X}'_{0}(0) \neq 0 \). \( \square \)

If \( t \neq 0 \), then thanks to Proposition 2.1 we have \( \operatorname{tr} \mathcal{X}'(0) \neq 0 \), so we can set
\[
x_{0} = -\frac{\mathfrak{U} + \frac{1}{4} \mathfrak{S}^2}{\varepsilon \operatorname{tr} \mathcal{X}'_{0}(0)},
\tag{2.22}
\]

and \( \Lambda = \mathbb{R} \backslash \{x_{0}\} \). Otherwise, we shall assume that \( \tau \neq 0 \) and we set \( \Lambda = \mathbb{R} \).

**Proposition 2.2** If (2.7) is satisfied, then
\[
\mathfrak{S} = \lim_{n \to \infty} \sqrt{\gamma_{n}} \left( 1 - \frac{a_{n}}{a_{n+N}} \right).
\tag{2.23}
\]

In particular, \( \mathfrak{S} \geq 0 \).

**Proof** Let us first observe that
\[
1 - \frac{a_{n}}{a_{n+N}} = \frac{a_{n}}{a_{n+N}} - \frac{a_{n}}{a_{n+N}}
\]

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Thus
\[
\sqrt{\frac{\gamma_n}{\alpha_n}} \left( 1 - \frac{a_n}{a_n+N} \right) = \sum_{j=0}^{N-1} \frac{\alpha_{n+j+1}}{\alpha_{n+N}} \frac{a_n}{a_n+j} \frac{1}{\alpha_n} \sqrt{\frac{\gamma_n}{\alpha_{n+j+1}\gamma_{n+j+1}}} \times \left( \frac{\alpha_{n+j}}{\alpha_{n+j+1}} - \frac{a_{n+j}}{a_{n+j+1}} \right).
\]

Hence, by the hypothesis (b), (2.7) and (2.6),
\[
\lim_{n \to \infty} \sqrt{\frac{\gamma_n}{\alpha_n}} \left( 1 - \frac{a_n}{a_n+N} \right) = \lim_{n \to \infty} \sum_{j=0}^{N-1} \frac{\alpha_{n+j+1}}{\alpha_{n+N}} \frac{a_n}{a_n+j} \frac{1}{\alpha_n} \sqrt{\frac{\gamma_n}{\alpha_{n+j+1}\gamma_{n+j+1}}} \times \left( \frac{\alpha_{n+j}}{\alpha_{n+j+1}} - \frac{a_{n+j}}{a_{n+j+1}} \right) = \mathcal{S}
\]
and (2.23) follows. To see the last statement, we assume, contrary to our claim, that
\[\mathcal{S} < 0.\]
Then there is \(n_0 > 0\) such that for \(n \geq n_0\),
\[a_n - a_{n+N} > 0.\]
Therefore, for each \(n \geq n_0\),
\[a_n \leq \max_{0 \leq i \leq N-1} a_{n_0+i} < \infty,\]
which contradicts the hypothesis (a). \(\square\)

3 The Shifted Conjugation

In this section we freely use the notation introduced in Sect. 2. Fix \(i \in \{0, 1, \ldots, N-1\}\) and set
\[
Z_j = T_i \left( \frac{1}{e^\vartheta_j} \frac{1}{e^{-\vartheta_j}} \right)
\]
where \(T_i\) has been defined in (2.9) and (2.12), and
\[
\vartheta_j(x) = \frac{\alpha_{i-1} \tau(x)}{\sqrt{\gamma_{i-1}N+i-1}}.
\]
The proof of the following theorem is a generalization of [61, Theorem 3.2].

**Theorem 3.1** Let $N$ be a positive integer and $i \in \{0, 1, \ldots, N - 1\}$. Let $(\gamma_n : n \in \mathbb{N})$ be a sequence of positive numbers tending to infinity and satisfying (1.2) and (1.3). Let $(a_n : n \in \mathbb{N}_0)$ and $(b_n : n \in \mathbb{N}_0)$ be $\gamma$-tempered $N$-periodically modulated Jacobi parameters such that $\mathcal{X}_0(0)$ is a non-trivial parabolic element. Suppose that (1.4) holds true with $\varepsilon = \text{sign}(\text{tr} \mathcal{X}_0(0))$. Then for any compact interval $K \subset \Lambda$,

$$Z_j^{-1} Z_{j+1} = \text{Id} + \frac{\alpha_{i-1}}{\gamma(j+1)^{N+i-1}} Q_j$$

(3.3)

where $(Q_j)$ is a sequence from $D_1(K, \text{Mat}(2, \mathbb{R}))$ convergent uniformly on $K$ to the zero matrix.

**Proof** In the proof we denote by $(\delta_j)$ a generic sequence from $D_1$ tending to zero which may change from line to line. By a straightforward computation we obtain

$$Z_j^{-1} Z_{j+1} = \frac{1}{\det Z_j} \begin{pmatrix} e^{-\theta_j} & -1 \\ -e^{\theta_j} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ e^{\theta_{j+1}} & e^{-\theta_{j+1}} \end{pmatrix}$$

$$= \frac{1}{e^{-\theta_j} - e^{\theta_j}} \begin{pmatrix} f_j & g_j \\ \tilde{g}_j & \tilde{f}_j \end{pmatrix}$$

where

$$f_j = e^{-\theta_j} - e^{\theta_{j+1}}, \quad g_j = e^{-\theta_j} - e^{-\theta_{j+1}}$$

$$\tilde{g}_j = -e^{\theta_j} + e^{\theta_{j+1}}, \quad \tilde{f}_j = -e^{\theta_j} + e^{-\theta_{j+1}}.$$

Observe that

$$\gamma_n \left( \frac{1}{\sqrt{\gamma_n}} - \frac{1}{\sqrt{\gamma_{n+N}}} \right) = \left( \sqrt{\gamma_{n+N}} - \sqrt{\gamma_n} \right) \frac{\sqrt{\gamma_n}}{\gamma_{n+N}}.$$
Similarly, $N$-periodicity of $(\alpha_n)$ and (2.3) leads to
\[
\left(\sqrt{\frac{\gamma_{n-1}}{\gamma_n}} : n \in \mathbb{N}\right) \in D_1^N.
\]
For fixed $j \in \mathbb{N}$, we have
\[
\sqrt{\frac{\gamma_n}{\gamma_{n+j}}} = \sqrt{\frac{\gamma_n}{\gamma_{n+1}}} \sqrt{\frac{\gamma_{n+1}}{\gamma_{n+2}}} \cdots \sqrt{\frac{\gamma_{n+j-1}}{\gamma_{n+j}}}.
\]
thus
\[
\left(\sqrt{\frac{\gamma_n}{\gamma_{n+j}}} : n \in \mathbb{N}\right) \in D_1^N. \quad (3.6)
\]
Hence, by (3.4)–(3.6)
\[
\frac{1}{\sqrt{\gamma_{(j+2)N+i-1}}} = \frac{1}{\sqrt{\gamma_{(j+1)N+i-1}}} + \frac{1}{\gamma_j N} \delta_j,
\]
and so
\[
\vartheta_{j+1} = \vartheta_j + \frac{1}{\gamma_j N} \delta_j. \quad (3.7)
\]
Next, we compute
\[
e^{\vartheta_{j+1}} = 1 + \vartheta_{j+1} + \frac{1}{2} \vartheta_{j+1}^2 + \frac{1}{\gamma_j N} \delta_j
\]
and
\[
e^{-\vartheta_j} = 1 - \vartheta_j + \frac{1}{2} \vartheta_j^2 + \frac{1}{\gamma_j N} \delta_j.
\]
Hence,
\[
f_j = 1 - \vartheta_j + \frac{1}{2} \vartheta_j^2 - \left(1 + \vartheta_{j+1} + \frac{1}{2} \vartheta_{j+1}^2\right) + \frac{1}{\gamma_j N} \delta_j
\]
\[
= -2 \vartheta_j + \frac{1}{\gamma_j N} \delta_j.
\]
Since $\frac{x}{\sinh(x)}$ is an even $C^2(\mathbb{R})$ function, we have
\[
\frac{\vartheta_j}{\sinh(\vartheta_j)} = 1 + \frac{1}{\sqrt{\gamma_j N}} \delta_j.
\]
Therefore,
\[
\frac{1}{e^{-\vartheta_j} - e^{\vartheta_j}} f_j = \frac{f_j}{-2 \vartheta_j \sinh(\vartheta_j)} \vartheta_j
\]
\[
\left( 1 + \frac{1}{\sqrt{\gamma} j N} \delta_j \right) \left( 1 + \frac{1}{\sqrt{\gamma} j N} \delta_j \right) = 1 + \frac{1}{\sqrt{\gamma} j N} \delta_j.
\]

Analogously, we treat \( g_j \).Namely, we write

\[
g_j = 1 - \vartheta_j + \frac{1}{2} \vartheta_j^2 - \left( 1 - \vartheta_{j+1} + \frac{1}{2} \vartheta_{j+1}^2 \right) + \frac{1}{\gamma j N} \delta_j
\]

\[
= \frac{1}{\gamma j N} \delta_j.
\]

Hence,\[
\frac{1}{e^{-\vartheta_j} - e^{\vartheta_j}} g_j = \frac{1}{\sqrt{\gamma j N}} \delta_j.
\]

Similarly, we can find that\[
\frac{1}{e^{-\vartheta_j} - e^{\vartheta_j}} \tilde{f}_j = 1 + \frac{1}{\sqrt{\gamma j N}} \delta_j,
\]

\[
\frac{1}{e^{-\vartheta_j} - e^{\vartheta_j}} \tilde{g}_j = \frac{1}{\sqrt{\gamma j N}} \delta_j.
\]

Consequently,\[
Z_j^{-1} Z_{j+1} = \text{Id} + \sqrt{\frac{\alpha_i - 1}{\gamma_{(j+1)N+i-1}}} Q_j
\]

where \((Q_j)\) is a sequence from \(\mathcal{D}_1(K, \text{Mat}(2, \mathbb{R}))\) for any compact interval \(K \subset \Lambda\) convergent to the zero matrix proving the formula (3.3). \(\Box\)

**Theorem 3.2** Let \(N\) be a positive integer and \(i \in \{0, 1, \ldots, N - 1\}\). Let \((\gamma_n : n \in \mathbb{N})\) be a sequence of positive numbers tending to infinity and satisfying (1.2) and (1.3). Let \((a_n : n \in \mathbb{N}_0)\) and \((b_n : n \in \mathbb{N}_0)\) be \(\gamma\)-tempered \(N\)-periodically modulated Jacobi parameters such that \(X_0(0)\) is a non-trivial parabolic element. Suppose that (1.4) holds true with \(\varepsilon = \text{sign}(\text{tr} X_0(0))\). Then for any compact interval \(K \subset \Lambda\),

\[
Z_{j+1}^{-1} X_{jN+i} Z_j = \varepsilon \left( \text{Id} + \sqrt{\frac{\alpha_i - 1}{\gamma_{(j+1)N+i-1}}} R_j \right)
\]

where \((R_j)\) is a sequence from \(\mathcal{D}_1(K, \text{Mat}(2, \mathbb{R}))\) convergent uniformly on \(K\) to

\[
\mathcal{R}_i(x) = \frac{\sqrt{\tau(x)}}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} - \frac{\nu(x)}{2\sqrt{\tau(x)}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} - \frac{\xi}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.
\]
In particular, $\text{discr } R_i = 4\tau(x)$.

**Proof** In the following argument, we denote by $(\delta_j)$ and $(\xi_j)$ generic sequences tending to zero from $D_1$ and $D_1(K, \text{Mat}(2, \mathbb{R}))$, respectively, which may change from line to line.

Observe that by (1.2)

$$
\lim_{j \to \infty} \frac{Y_{jN+i'}-1}{Y_{jN+i'}} = \frac{\alpha_{i'-1}}{\alpha_i'},
$$

and

$$
\left(\frac{Y_{jN+i'}-1}{Y_{jN+i'}} : j \in \mathbb{N}\right) \in D_1.
$$

Hence,

$$
\frac{1}{Y_{jN+i'}} = \frac{1}{Y_{jN+i'}-1} \frac{Y_{jN+i'}-1}{Y_{jN+i'}-1} \frac{1}{Y_{jN+i'}-1} \left(\frac{Y_{jN+i'}-1}{Y_{jN+i'}-1} - \frac{\alpha_{i'-1}}{\alpha_i'}\right) \frac{1}{\delta_j},
$$

Consequently,

$$
\frac{\alpha_{i'}}{Y_{jN+i'}} = \frac{\alpha_{i-1}}{Y(j+1)N+i-1} + \frac{1}{\delta_j}. \tag{3.9}
$$

Next, for $x \in K$, we have

$$
\frac{x}{a_{jN+i'}} = \frac{x}{a_{jN+i'}} \frac{Y_{jN+i'}}{Y_{jN+i'}} \frac{Y_{jN+i'}-1}{Y_{jN+i'}-1} \frac{1}{Y_{jN+i'}-1} \left(\frac{Y_{jN+i'}-1}{Y_{jN+i'}-1} - \frac{t}{a_{jN+i'}}\right) \frac{1}{\delta_j}, \tag{3.10}
$$

Let

$$
\xi_n = \frac{\alpha_{n-1}}{\alpha_n} - \frac{a_{n-1}}{a_n}, \quad \text{and} \quad \xi_n = \frac{\beta_n}{\alpha_n} - \frac{b_n}{a_n}.
$$

Using (3.10), the matrix $B_{jN+i'}$ can be written as

$$
\begin{pmatrix}
-\frac{\alpha_{i'-1}}{\alpha_i'} + \frac{\xi_{jN+i'}}{Y_{jN+i'}} + \frac{1}{Y_{jN+i'}} \delta_j - \frac{\beta_{i'}}{\alpha_i'} + \frac{x t}{Y_{jN+i'}} & \frac{1}{Y_{jN+i'}} \delta_j
\end{pmatrix}
$$
\[
\begin{aligned}
&= \left( -\frac{\alpha'_{i-1}}{\alpha_i'} - \frac{\beta_{i-1}}{\alpha_i'} \right) + \left( \frac{0}{\alpha_i'} \right) \frac{x_{\xi_j N+i'} \gamma_j N+i'}{\zeta_{j N+i'} + \zeta_{j N+i'}} + \frac{1}{\gamma_j N+i'} E_j
&= \left( -\frac{\alpha'_{i-1}}{\alpha_i'} - \frac{\beta_{i-1}}{\alpha_i'} \right) \left\{ \begin{array}{c} \text{Id} + \frac{\alpha_{i'}}{\alpha_{i'-1}} \left( \begin{array}{cc} -\frac{\beta_{i'}}{\alpha_{i'}} & -1 \\ \alpha_{i'-1} & 0 \end{array} \right) \left( \begin{array}{cc} \xi_{j N+i'} & \zeta_{j N+i'} \\ 0 & \gamma_j N+i' \end{array} \right) \\
\end{array} \right\} \left\{ \begin{array}{c} \text{Id} - \frac{\alpha_{i'}}{\alpha_{i'-1}} \left( \xi_{j N+i'} \zeta_{j N+i'} + \frac{x_t}{\gamma_j N+i'} \right) + \frac{1}{\gamma_j N} E_j \end{array} \right\}.
\end{aligned}
\]

Hence,

\[
X_{j N+i} = B_{j N+i + N-1} \cdots B_{j N+i + 1} B_{j N+i}
= \mathcal{X}_i(0) \left\{ \begin{array}{c} \text{Id} - \sum_{i'=i}^{N+i-1} \frac{\alpha_{i'}}{\alpha_{i'-1}} \left( \mathcal{B}_{i'-1}(0) \cdots \mathcal{B}_i(0) \right)^{-1} \\
\times \left( \begin{array}{cc} \xi_{j N+i'} & \zeta_{j N+i'} + \frac{x_t}{\gamma_j N+i'} \\ 0 & 0 \end{array} \right) \left( \mathcal{B}_{i'-1}(0) \cdots \mathcal{B}_i(0) \right) + \frac{1}{\gamma_j N} E_j \end{array} \right\},
\]

and so

\[
Z_{j+1}^{-1} X_{j N+i} Z_j = Z_{j+1}^{-1} \mathcal{X}_i(0) Z_j \left\{ \begin{array}{c} \text{Id} - \sum_{i'=i}^{N+i-1} \frac{\alpha_{i'}}{\alpha_{i'-1}} \left( e^{\theta_j} - e^{-\theta_j} \right)^{-1} T_{i'-1} \\
\times \left( \begin{array}{cc} \xi_{j N+i'} & \zeta_{j N+i'} + \frac{x_t}{\gamma_j N+i'} \\ 0 & 0 \end{array} \right) T_{i'} \left( e^{\theta_j} - e^{-\theta_j} \right) + \frac{1}{\sqrt{\gamma_j N}} E_j \end{array} \right\}.
\]

To find the asymptotic of the first factor, we write

\[
Z_{j+1}^{-1} \mathcal{X}_i(0) Z_j = \frac{e}{e^{-\theta_j} - e^{\theta_j}} \left( \begin{array}{cc} f_j & g_j \\ \tilde{g}_j & \tilde{f}_j \end{array} \right)
\]

where

\[
\begin{aligned}
f_j &= e^{\theta_j} - \theta_j + 1 - 2e^{\theta_j},
&\quad g_j = e^{-\theta_j} - \theta_j + 1 - 2e^{-\theta_j},
\tilde{g}_j &= -e^{\theta_j} + \theta_j - 1 + 2e^{\theta_j},
\tilde{f}_j &= -e^{-\theta_j} + \theta_j - 1 + 2e^{-\theta_j}.
\end{aligned}
\]

Since by (3.7)

\[
e^{\theta_j} - \theta_j + 1 = 1 + \frac{1}{\gamma_j N} \delta_j, \quad \text{and} \quad e^{\theta_j} = 1 + \theta_j + \frac{1}{2} \theta_j^2 + \frac{1}{\gamma_j N} \delta_j,
\]

 Springer
we get

\[ f_j = 1 + 1 - 2 \left(1 + \vartheta_j + \frac{1}{2} \vartheta_j^2 \right) + \frac{1}{\gamma_{jN}} \delta_j \]

\[ = -2\vartheta_j - \vartheta_j^2 + \frac{1}{\gamma_{jN}} \delta_j. \]

Moreover,

\[ \frac{\vartheta_j}{\vartheta_{j+1}} = 1 + \frac{1}{\sqrt{\gamma_{jN}}} \delta_j. \]

Thus

\[
\frac{1}{e^{-\vartheta_{j+1}} - e^{\vartheta_{j+1}}} f_j = \frac{f_j}{-2\vartheta_j \vartheta_{j+1}} \frac{\vartheta_j}{\vartheta_{j+1}} \frac{\vartheta_{j+1}}{\sinh \vartheta_{j+1}}
= \left(1 + \frac{1}{2} \vartheta_j + \frac{1}{\sqrt{\gamma_{jN}}} \delta_j \right) \left(1 + \frac{1}{\sqrt{\gamma_{jN}}} \delta_j \right)
= 1 + \vartheta_j + \frac{1}{\sqrt{\gamma_{jN}}} \delta_j.
\] (3.11)

Analogously, we can find that

\[ \tilde{f}_j = -2\vartheta_j + \vartheta_j^2 + \frac{1}{\gamma_{jN}} \delta_j, \]

and

\[
\frac{1}{e^{-\vartheta_{j+1}} - e^{\vartheta_{j+1}}} \tilde{f}_j = 1 - \frac{1}{2} \vartheta_j + \frac{1}{\sqrt{\gamma_{jN}}} \delta_j.
\] (3.12)

Next, we write

\[ g_j = 1 - \vartheta_j - \vartheta_{j+1} + \frac{1}{2} (\vartheta_j + \vartheta_{j+1})^2 + 1 - 2 \left(1 - \vartheta_j + \frac{1}{2} \vartheta_j^2 \right) + \frac{1}{\gamma_{jN}} \delta_j \]

\[ = \vartheta_j^2 + \frac{1}{\gamma_{jN}} \delta_j, \]

thus

\[
\frac{1}{e^{-\vartheta_{j+1}} - e^{\vartheta_{j+1}}} g_j = -\frac{1}{2} \vartheta_j + \frac{1}{\sqrt{\gamma_{jN}}} \delta_j.
\] (3.13)

Similarly, we get

\[ \tilde{g}_j = -\vartheta_j^2 + \frac{1}{\gamma_{jN}} \delta_j, \]
and so
\[
\frac{1}{e^{-\theta_{j+1}} - e^{\theta_{j+1}}} \tilde{g}_j = \frac{1}{2} \vartheta_j + \frac{1}{\sqrt{\nu_j N}} \delta_j.
\] (3.14)

Consequently, by (3.11)–(3.14) we obtain
\[
Z_{j+1}^{-1} X_i(0) Z_j = \varepsilon \left\{ \text{Id} + \frac{1}{2} \vartheta_j \left( \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right) + \frac{1}{\sqrt{\nu_j N}} \mathcal{E}_j \right\}.
\] (3.15)

Next, we observe that
\[
e^{\vartheta_j} = 1 + \delta_j, \quad \frac{\vartheta_j}{\sinh \vartheta_j} = 1 + \delta_j,
\]
thus in view of (2.6), for each \(i' \in \{0, 1, \ldots, N-1\}\) we have
\[
\begin{align*}
\left( \frac{1}{e^{\vartheta_j}} & - \frac{1}{e^{-\vartheta_j}} \right)^{-1} T_{i'}^{-1} \left( \begin{array}{c} \xi_{j_N+i'} \\ 0 \end{array} \right) T_{i'} \left( \begin{array}{c} \xi_{j_N+i'} \\ 0 \end{array} \right)
&= - \frac{1}{2\vartheta_j} \gamma_{j_N+i'} \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) T_{i'}^{-1} \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right) T_{i'} \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) + \frac{1}{\sqrt{\nu_j N}} \mathcal{E}_j
\\
&= - \frac{1}{2\vartheta_j} \gamma_{j_N+i'} \left( \left[ T_{i'} \right]_{21} + \left[ T_{i'} \right]_{22} \right)^2 \left( \begin{array}{cc} 1 & 1 \\ -1 & -1 \end{array} \right) + \frac{1}{\sqrt{\nu_j N}} \mathcal{E}_j
\\
&= \frac{1}{2\vartheta_j} \gamma_{j_N+i'} \left( \varepsilon [X_{i'}(0)]_{21} \right) \left( \begin{array}{cc} 1 & 1 \\ -1 & -1 \end{array} \right) + \frac{1}{\sqrt{\nu_j N}} \mathcal{E}_j.
\end{align*}
\]

We write
\[
\begin{align*}
\left( \frac{1}{e^{\vartheta_j}} & - \frac{1}{e^{-\vartheta_j}} \right)^{-1} T_{i'}^{-1} \left( \begin{array}{cc} \xi_{j_N+i'} & \xi_{j_N+i'} \\ 0 & 0 \end{array} \right) T_{i'} \left( \begin{array}{cc} \xi_{j_N+i'} & \xi_{j_N+i'} \\ 0 & 0 \end{array} \right)
&= \frac{1}{e^{-\vartheta_j} - e^{\vartheta_j}} \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) T_{i'}^{-1} \left( \begin{array}{cc} \xi_{j_N+i'} & \xi_{j_N+i'} \\ 0 & 0 \end{array} \right) T_{i'} \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right)
\\
&+ \frac{1}{e^{-\vartheta_j} - e^{\vartheta_j}} \left( \begin{array}{cc} e^{-\vartheta_j} & 0 \\ 1 - e^{\vartheta_j} & 0 \end{array} \right) T_{i'}^{-1} \left( \begin{array}{cc} \xi_{j_N+i'} & \xi_{j_N+i'} \\ 0 & 0 \end{array} \right) T_{i'} \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right)
\\
&+ \frac{1}{e^{-\vartheta_j} - e^{\vartheta_j}} \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) T_{i'}^{-1} \left( \begin{array}{cc} \xi_{j_N+i'} & \xi_{j_N+i'} \\ 0 & 0 \end{array} \right) T_{i'} \left( \begin{array}{cc} e^{\vartheta_j} & 0 \\ 0 & e^{-\vartheta_j} - 1 \end{array} \right)
\\
&+ \frac{1}{\sqrt{\nu_j N}} \mathcal{E}_j.
\end{align*}
\]

We observe that
\[
\frac{1}{e^{-\vartheta_j} - e^{\vartheta_j}} \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) T_{i'}^{-1} \left( \begin{array}{cc} \xi_{j_N+i'} & 0 \\ 0 & 0 \end{array} \right) T_{i'} \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right)
\]
\[= \frac{\xi_{jN+i'}}{e^{-\theta_j} - e^{\theta_j}} \frac{([T_{i'}]_{11} + [T_{i'}]_{12})([T_{i'}]_{21} + [T_{i'}]_{22})}{\det T_{i'}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \]

and

\[= \frac{\xi_{jN+i'}}{e^{-\theta_j} - e^{\theta_j}} (1 - \varepsilon[\mathcal{X}_{i'}(0)]_{11}) \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} , \]

Hence, by (1.4) and (2.16),

\[ \frac{1}{e^{-\theta_j} - e^{\theta_j}} \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) T_{i'}^{-1} \left( \begin{array}{cc} \xi_{jN+i'} & \xi_{jN+i'} \\ 0 & 0 \end{array} \right) T_{i'} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \]

and

\[= \frac{1}{2} \frac{\bar{u}_{i'}}{\gamma_{jN+i'}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \frac{1}{\sqrt{\gamma_{jN}}} E_j. \]

Next, let us notice that

\[\left( \begin{array}{c} e^{\theta_j} - 1 \\ e^{\theta_j} - e^{-\theta_j} \end{array} : j \in \mathbb{N} \right) \in \mathcal{D}_1, \]

and

\[\lim_{j \to \infty} \frac{e^{\theta_j} - 1}{e^{\theta_j} - e^{-\theta_j}} = \frac{1}{2}. \]

Therefore, by (2.6), we get

\[= \frac{1}{2} \left( \begin{array}{c} e^{\theta_j} - 1 \\ e^{\theta_j} - 0 \\ 0 \\ 0 \end{array} \right) T_{i'}^{-1} \left( \begin{array}{c} s_{i'} \\ \sqrt{\gamma_{jN+i'}} \gamma_{jN+i'} \\ 0 \\ 0 \end{array} \right) T_{i'} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1}{\sqrt{\gamma_{jN}}} E_j \]

and

\[\frac{1}{e^{-\theta_j} - e^{\theta_j}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} T_{i'}^{-1} \left( \begin{array}{cc} \xi_{jN+i'} & 0 \\ 0 & 0 \end{array} \right) T_{i'} \begin{pmatrix} 0 & 0 \\ e^{\theta_j} - 1 & e^{-\theta_j} - 1 \end{pmatrix} \]
In view of (2.13), (2.14) and (2.16), we have

\[
e^{-\theta_j} - e^{\theta_j} = \frac{\mathbf{s}_{i'}}{2\sqrt{\alpha_i'' Y_{jN+i'}}} \frac{[T_{i'}]_{21} + [T_{i'}]_{22}}{\det T_{i'}} (1 - 1) + \frac{1}{\sqrt{Y_jN}} \mathcal{E}_j,
\]

and

\[
e^{-\theta_j} - e^{\theta_j} = \frac{\mathbf{v}_{i'}}{2\sqrt{\alpha_i'' Y_{jN+i'}}} \frac{[T_{i'}]_{21} + [T_{i'}]_{22}}{\det T_{i'}} (1 - 1) + \frac{1}{\sqrt{Y_jN}} \mathcal{E}_j.
\]

In view of (2.13), (2.14) and (2.16), we have

\[
\mathbf{s}_{i'} ([T_{i'}]_{11} + [T_{i'}]_{12}) = -\mathbf{v}_{i'} ([T_{i'}]_{11} + [T_{i'}]_{22}),
\]

thus by (3.16) and (3.17) we get

\[
\frac{1}{e^{-\theta_j} - e^{\theta_j}} \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) T_{i'}^{-1} \left( \begin{array}{cc} \mathbf{s}_{i'} & 0 \\ 0 & \mathbf{v}_{i'} \end{array} \right) T_{i'} \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) = \frac{\mathbf{s}_{i'}}{2\sqrt{\alpha_i'' Y_{jN+i'}}} \frac{[T_{i'}]_{21} + [T_{i'}]_{22}}{\det T_{i'}} (1 - 1) + \frac{1}{\sqrt{Y_jN}} \mathcal{E}_j,
\]

and

\[
\frac{1}{e^{-\theta_j} - e^{\theta_j}} \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) T_{i'}^{-1} \left( \begin{array}{cc} \mathbf{s}_{i'} & 0 \\ 0 & \mathbf{v}_{i'} \end{array} \right) T_{i'} \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) = \frac{\mathbf{s}_{i'}}{2\sqrt{\alpha_i'' Y_{jN+i'}}} \frac{[T_{i'}]_{21} + [T_{i'}]_{22}}{\det T_{i'}} (1 - 1) + \frac{1}{\sqrt{Y_jN}} \mathcal{E}_j.
\]
Hence,
\[
\frac{1}{e^{-\vartheta_j} - e^{\vartheta_j}} \left( \begin{array}{cc} e^{-\vartheta_j} - 1 & 0 \\ 1 & e^{-\vartheta_j} \end{array} \right) T_{i'}^{-1} \left( \begin{array}{cc} \xi_{jN+i'} & \zeta_{jN+i'} \\ 0 & 0 \end{array} \right) T_{i'} \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \\
+ \frac{1}{e^{-\vartheta_j} - e^{\vartheta_j}} \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) T_{i'}^{-1} \left( \begin{array}{cc} \xi_{jN+i'} & \zeta_{jN+i'} \\ 0 & 0 \end{array} \right) T_{i'} \left( \begin{array}{cc} 0 & 0 \\ e^{\vartheta_j} - 1 & e^{-\vartheta_j} - 1 \end{array} \right)
\]
\[= \frac{s_{i'}}{2\sqrt{\alpha_{i'}\gamma_{jN+i'}}} \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) + \frac{1}{\sqrt{\gamma_jN}} E_j.
\]

Summarizing, we obtain
\[
\left( \begin{array}{cc} 1 & e^{\vartheta_j} \\ e^{-\vartheta_j} & 1 \end{array} \right)^{-1} T_{i'}^{-1} \left( \begin{array}{cc} \xi_{jN+i'} & \zeta_{jN+i'} + \frac{x_t}{\gamma_{jN+i'}} \\ 0 & 0 \end{array} \right) T_{i'} \left( \begin{array}{cc} 1 & e^{-\vartheta_j} \\ e^{\vartheta_j} & 1 \end{array} \right)
\]
\[= \frac{1}{2\vartheta_j} \frac{t_x(e[X_i(0)]_{21}) - u_{i'}}{\gamma_{jN+i'}} \left( \begin{array}{cc} 1 & 1 \\ -1 & -1 \end{array} \right) \\
+ \frac{s_{i'}}{2\sqrt{\alpha_{i'}\gamma_{jN+i'}}} \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) + \frac{1}{\sqrt{\gamma_jN}} E_j.
\]

By (3.9) and (2.18), we have
\[
\sqrt{\gamma_{(j+1)N+i-1}} \sum_{i'=i}^{N+i-1} \frac{\alpha_{i'}}{\alpha_{i'-1}} \frac{\alpha_{i'}}{\gamma_{jN+i'}} x_t(e[X_i'(0)]_{21}) - u_{i'}
\]
\[= \sqrt{\frac{\alpha_{i'-1}}{\gamma_{(j+1)N+i-1}}} \vartheta_j + \frac{1}{\sqrt{\gamma_jN}} \delta_j,
\]
and
\[
\sum_{i'=i}^{N+i-1} \frac{\alpha_{i'}}{\alpha_{i'-1}} \frac{s_{i'}}{\sqrt{\alpha_{i'}\gamma_{jN+i'}}} = \sqrt{\frac{\alpha_{i-1}}{\gamma_{(j+1)N+i-1}}} S + \frac{1}{\sqrt{\gamma_jN}} \delta_j.
\]

Therefore,
\[
\sum_{i'=i}^{N+i-1} \frac{\alpha_{i'}}{\alpha_{i'-1}} \left( \begin{array}{cc} 1 & 1 \\ e^{\vartheta_j} & e^{-\vartheta_j} \end{array} \right)^{-1} T_{i'}^{-1} \left( \begin{array}{cc} \xi_{jN+i'} & \zeta_{jN+i'} - \frac{x_t}{\gamma_{jN+i'}} \\ 0 & 0 \end{array} \right) T_{i'} \left( \begin{array}{cc} 1 & 1 \\ e^{\vartheta_j} & e^{-\vartheta_j} \end{array} \right)
\]
\[= \vartheta_j \frac{\nu}{2 |x|} \left( \begin{array}{cc} 1 & 1 \\ -1 & -1 \end{array} \right) + \sqrt{\frac{\alpha_{i-1}}{\gamma_{(j+1)N+i-1}}} S \left( \begin{array}{cc} 1 & -1 \\ -1 & 1 \end{array} \right) + \frac{1}{\sqrt{\gamma_jN}} E_j.
\]

Finally, by (3.15) we get
\[
Z_{j+1}^{-1} X_{jN+i} Z_j = \varepsilon \left\{ \Id + \sqrt{\frac{\alpha_{i-1}}{\gamma_{(j+1)N+i-1}}} \mathcal{R}_i + \frac{1}{\sqrt{\gamma_jN}} E_j \right\}
\]
where
\[
R_i(x) = \frac{\sqrt{|\tau(x)|}}{2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} - \frac{\upsilon(x)}{2\sqrt{|\tau(x)|}} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} - \frac{S}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}
\]
which finishes the proof.

**Corollary 3.3** Suppose that the hypotheses of Theorem 3.2 are satisfied. Then
\[
\lim_{j \to \infty} \gamma(j+1)N+i \text{ discr } (X_{jN+i}) = 4\tau \alpha_{i-1}
\]
locally uniformly on \(\Lambda\) where \(\tau\) is defined in (2.17).

**Proof** We write
\[
Z_j^{-1}X_{jN+i}Z_j = (Z_j^{-1}Z_j) (Z_{j+1}^{-1}X_{jN+i}Z_j),
\]
thus by Theorems 3.1 and 3.2, we obtain
\[
\varepsilon Z_j^{-1}X_{jN+i}Z_j = \left( \text{Id} + \sqrt{\frac{\alpha_{i-1}}{\gamma(j+1)N+i}} Q_j \right) \left( \text{Id} + \sqrt{\frac{\alpha_{i-1}}{\gamma(j+1)N+i}} R_j \right)
\]
\[
= \text{Id} + \sqrt{\frac{\alpha_{i-1}}{\gamma(j+1)N+i}} (Q_j + R_j) + \frac{\alpha_{i-1}}{\gamma(j+1)N+i} Q_j R_j.
\]
Hence,
\[
\frac{\gamma(j+1)N+i-1}{\alpha_{i-1}} \text{ discr}(X_{jN+i}) = \text{discr} \left( R_j + Q_j + \sqrt{\frac{\alpha_{i-1}}{\gamma(j+1)N+i}} Q_j R_j \right),
\]
and consequently,
\[
\lim_{j \to \infty} \frac{\gamma(j+1)N+i-1}{\alpha_{i-1}} \text{ discr}(X_{jN+i}) = \text{discr}(R_i).
\]
Since \(\text{discr} R_i = 4\tau\), the conclusion follows.

4 Essential Spectrum

In this section we want to understand spectral properties of Jacobi operators corresponding to tempered \(N\)-periodically modulated sequences. We set
\[
\Lambda_- = \tau^{-1}((-\infty, 0)), \quad \text{and} \quad \Lambda_+ = \tau^{-1}((0, +\infty)).
\]
Observe that, if \(t = 0\) then at least one of the sets \(\Lambda_-\) or \(\Lambda_+\) is empty. Similar to the proof of [61, Theorem 4.1] we use the shifted conjugation (Theorems 3.1, 3.2) together with a variant of a Levison’s theorem for discrete systems developed in [60, Theorem 4.4].
Theorem 4.1  Let $N$ be a positive integer. Let $(\gamma_n : n \in \mathbb{N})$ be a sequence of positive numbers tending to infinity and satisfying (1.2) and (1.3). Let $(a_n : n \in \mathbb{N}_0)$ and $(b_n : n \in \mathbb{N}_0)$ be $\gamma$-tempered $N$-periodically modulated Jacobi parameters such that $X_0(0)$ is a non-trivial parabolic element. Suppose that (1.4) holds true with $\varepsilon = \text{sign}(\text{tr} X_0(0))$. If the operator $A$ is self-adjoint then

$$\sigma_{\text{ess}}(A) \cap \Lambda_+ = \emptyset.$$ 

Proof  Let us fix a compact interval $K \subset \Lambda_+$ and $i \in \{0, 1, \ldots, N - 1\}$. We set

$$Y_j = Z_{j+1}^{-1} X_{jN+i} Z_j$$

where $Z_j$ is the matrix defined in (3.1). By Theorem 3.2, we have

$$Y_j = \varepsilon \left( \text{Id} + \sqrt{\frac{\alpha_{i-1}}{\gamma_{(j+1)N+i-1}}} R_j \right)$$  \hspace{1cm} (4.1)$$

where $(R_j : j \in \mathbb{N})$ is a sequence from $\mathcal{D}_1(K, \text{Mat}(2, \mathbb{R}))$ convergent uniformly on $K$ to the matrix $\mathcal{R}_i$ given by the formula (3.8). By Corollary 3.3, there are $j_0 \geq 0$ and $\delta > 0$, so that for all $j \geq j_0$ and $x \in K$,

$$\text{discr} R_j(x) \geq \delta.$$  \hspace{1cm} (4.2)$$

In particular, the matrix $R_j(x)$ has two eigenvalues

$$\xi_j^+(x) = \frac{\text{tr} R_j(x) + \sqrt{\text{discr} R_j(x)}}{2} \quad \text{and} \quad \xi_j^-(x) = \frac{\text{tr} R_j(x) - \sqrt{\text{discr} R_j(x)}}{2}.$$$$

In view of (4.1), the matrix $Y_j(x)$ has eigenvalues

$$\lambda_j^+ = \varepsilon \left( 1 + \sqrt{\frac{\alpha_{i-1}}{\gamma_{(j+1)N+i-1}}} \xi_j^+ \right) \quad \text{and} \quad \lambda_j^- = \varepsilon \left( 1 + \sqrt{\frac{\alpha_{i-1}}{\gamma_{(j+1)N+i-1}}} \xi_j^- \right).$$

By possible increasing $j_0$ we can guarantee that

$$|\lambda_j^-(x)| \leq |\lambda_j^+(x)|$$  \hspace{1cm} (4.3)$$

for all $j \geq j_0$ and $x \in K$.

Now, by the Stolz–Cesàro theorem (see e.g. [35, Section 3.1.7]), (1.3) implies that

$$\lim_{j \to \infty} \frac{\sqrt{\gamma_{jN+i}}}{j} = \lim_{j \to \infty} \left( \sqrt{\gamma_{(j+1)N+i}} - \sqrt{\gamma_{jN+i}} \right) = 0,$$  \hspace{1cm} (4.4)$$
and so

\[ \sum_{j=0}^{\infty} \frac{1}{\sqrt{\gamma_{jN+i}}} = \infty. \]

Therefore, by (4.2), we can apply [60, Theorem 4.4] to the system

\[ \Psi_{j+1} = Y_j \Psi_j. \]  \hspace{1cm} (4.5)

Consequently, there are \((\Psi_j^- : j \geq j_0)\) and \((\Psi_j^+ : j \geq j_0)\) such that

\[
\begin{align*}
\lim_{j \to \infty} \sup_{x \in K} \left\| \frac{\Psi_j^+(x)}{\prod_{k=j_0}^{j-1} \lambda_k^+(x)} - v^+(x) \right\| &= 0, \quad \text{and} \\
\lim_{j \to \infty} \sup_{x \in K} \left\| \frac{\Psi_j^-(x)}{\prod_{k=j_0}^{j-1} \lambda_k^-(x)} - v^-(x) \right\| &= 0
\end{align*}
\]  \hspace{1cm} (4.6)

where \(v^-(x)\) and \(v^+(x)\) are continuous eigenvectors of \(R_i(x)\) corresponding to

\[
\xi^+(x) = \frac{\text{tr} R_i(x) + \sqrt{\text{discr} R_i(x)}}{2} \quad \text{and} \quad \xi^-(x) = \frac{\text{tr} R_i(x) - \sqrt{\text{discr} R_i(x)}}{2}.
\]

Since \(\tau(x) > 0\), by means of (3.8) one can verify that

\[
v_1^+(x) + v_2^+(x) \neq 0, \quad \text{and} \quad v_1^-(x) + v_2^-(x) \neq 0. \]  \hspace{1cm} (4.7)

Indeed, otherwise \(e_1 - e_2\) would be an eigenvector of \(R_i(x)\), but

\[
R_i(x)(e_1 - e_2) = (\sqrt{\tau(x)} - \Theta)e_1 + (\sqrt{\tau(x)} + \Theta)e_2 = (\sqrt{\tau(x)} - \Theta)(e_1 - e_2) + 2\sqrt{\tau(x)}e_2,
\]

thus \(\tau(x) = 0\), which is impossible.

Now, by (4.5) the sequences \(\Phi_j^\pm = Z_j \Psi_j^\pm\) satisfy

\[ \Phi_{j+1} = X_{jN+i} \Phi_j, \quad j \geq j_0. \]

We set

\[
\phi_1^\pm = B_1^{-1} \cdots B_{j_0}^{-1} \Phi_j^\pm
\]

and

\[
\phi_{n+1}^\pm = B_n \phi_n^\pm, \quad n > 1.
\]
Then for \( jN + i' > j_0N + i \) with \( i' \in \{0, 1, \ldots, N - 1\} \), we get

\[
\phi_{jN+i'}^\pm = \begin{cases}
B_{jN+i}^{-1} B_{jN+i+1}^{-1} \cdots B_{jN+i-1}^{-1} \Phi_j^\pm & \text{if } i' \in \{0, 1, \ldots, i-1\}, \\
\Phi_j^\pm & \text{if } i' = i, \\
B_{jN+i'-1} B_{jN+i'-2} \cdots B_{jN+i} \Phi_j^\pm & \text{if } i' \in \{i+1, \ldots, N-1\}.
\end{cases}
\]

Since for \( i' \in \{0, 1, \ldots, i-1\} \),

\[
\lim_{j \to \infty} B_{jN+i}^{-1} B_{jN+i+1}^{-1} \cdots B_{jN+i-1}^{-1} = \mathcal{B}_{i'}^{-1}(0) \mathcal{B}_{i'+1}^{-1}(0) \cdots \mathcal{B}_{i}^{-1}(0),
\]

and

\[
\lim_{j \to \infty} Z_j v^\pm = (v_1^\pm + v_2^\pm) T_i(e_1 + e_2),
\]

we obtain

\[
\lim_{j \to \infty} \sup_k \left\| \frac{\phi_{jN+i'}^\pm}{\prod_{k=0}^{j-1} \lambda_k^\pm} - (v_1^\pm + v_2^\pm) T_i(e_1 + e_2) \right\| = 0. \tag{4.8}
\]

Analogously, we can show that (4.8) holds true also for \( i' \in \{i+1, \ldots, N-1\} \).

Since \((\phi_n^\pm : j \in \mathbb{N})\) satisfies (1.2), the sequence \((u_n^\pm(x) : n \in \mathbb{N}_0)\) defined as

\[
u_n^\pm(x) = \begin{cases}
(\phi_n^\pm(x), e_1) & \text{if } n = 0, \\
(\phi_n^\pm(x), e_2) & \text{if } n \geq 1,
\end{cases}
\]

is a generalized eigenvector associated to \( x \in K \), provided that \((u_0^\pm, u_1^\pm) \neq 0 \) on \( K \).

Suppose on the contrary that there is \( x \in K \) such that \( \phi_n^\pm(x) = 0 \). Hence, \( \phi_n^\pm(x) = 0 \) for all \( n \in \mathbb{N} \), thus by (4.7) and (4.8) we must have \( T_0(e_1 + e_2) = 0 \) which is impossible since \( T_0 \) is invertible.

Consequently, \((u_n^+(x) : n \in \mathbb{N}_0)\) and \((u_n^-(x) : n \in \mathbb{N}_0)\) are two generalized eigenvectors associated with \( x \in K \) with different asymptotic behavior, thus they are linearly independent.

Now, let us suppose that \( A \) is self-adjoint. By the proof of [47, Theorem 5.3], if

\[
\sum_{n=0}^{\infty} \sup_{x \in K} |u_n^-(x)|^2 < \infty \tag{4.9}
\]

then \( K \cap \sigma_{\text{ess}}(A) = \emptyset \), and since \( K \) is any compact subset of \( \Lambda_+ \) this implies that \( \sigma_{\text{ess}}(A) \cap \Lambda_+ = \emptyset \). Hence, it is enough to show (4.9). Let us observe that by (4.8), for each \( i' \in \{0, 1, \ldots, N-1\}, j > j_0, \) and \( x \in K \),

\[
|u_{jN+i'}^-(x)| \leq c \prod_{k=j_0}^{j-1} |\lambda_k^-(x)|. \tag{4.10}
\]
Since \((R_j : j \in \mathbb{N})\) converges to \(R_i\) uniformly on \(K\), and

\[
\lim_{n \to \infty} \gamma_n = \infty,
\]

there is \(j_1 \geq j_0\), such that for \(j > j_1\),

\[
\sqrt{\frac{\alpha_{i-1}}{\gamma(j+1)N+i-1}} \left( |\text{tr} R_j(x)| + \sqrt{\text{discr} R_j(x)} \right) \leq 1.
\]

Therefore, for \(j \geq j_1\),

\[
|\lambda_j^-(x)| = 1 + \sqrt{\frac{\alpha_{i-1}}{\gamma(j+1)N+i-1}} \frac{\text{tr} R_j(x) - \sqrt{\text{discr} R_j(x)}}{2}.
\]

By (3.8) and Proposition 2.2, \(\text{tr} R_i = -\mathcal{G} \leq 0\), thus (4.4) implies that

\[
\lim_{j \to \infty} j \sqrt{\frac{\alpha_{i-1}}{\gamma(j+1)N+i-1}} \frac{\text{tr} R_j(x) - \sqrt{\text{discr} R_j(x)}}{2} = -\infty.
\]

In particular, there is \(j_2 \geq j_1\) such that for all \(j > j_2\),

\[
\sup_{x \in K} |\lambda_j^-(x)| \leq 1 - \frac{1}{j}.
\]

Consequently, by (4.10), there is \(c' > 0\) such that for all \(i' \in \{0, 1, \ldots, N-1\}\) and \(j > j_2\),

\[
\sup_{x \in K} |u_{jN+i'}^-(x)| \leq c \prod_{k=j_2}^{j-1} \left( 1 - \frac{1}{k} \right) \leq \frac{c'}{j},
\]

which leads to (4.9) and the theorem follows. \(\square\)

**Remark 4.2** In Sect. 9 we characterize when \(A\) is self-adjoint. In particular, Theorem 9.1 settles the problem when \(\Lambda_- \neq \emptyset\). If \(\Lambda_- = \emptyset\) but \(\Lambda_+ \neq \emptyset\), the formula (9.5) is a necessary and sufficient condition for self-adjointness of \(A\).

### 5 Uniform Diagonalization

Fix a positive integer \(N\) and \(i \in \{0, 1, \ldots, N-1\}\). Let \((\gamma_n : n \in \mathbb{N})\) be a sequence of positive numbers tending to infinity and satisfying (1.2) and (1.3). Let \((a_n : n \in \mathbb{N}_0)\) and \((b_n : n \in \mathbb{N}_0)\) be \(\gamma\)-tempered \(N\)-periodically modulated Jacobi parameters such that \(\mathcal{X}_0(0)\) is non-diagonalizable and let \(\varepsilon = \text{sign}(\text{tr} \mathcal{X}_0(0))\). Suppose that (1.4) holds
true. Assume that $\Lambda_\neq \emptyset$. Let us consider a compact interval in $\Lambda_\neq$ and a generalized eigenvector $(u_n : n \in \mathbb{N}_0)$ associated to $x \in K$ and corresponding to $\eta \in S^1$. We set

$$Y_j = Z_{j+1}^{-1} X_{jN+i} Z_j$$ (5.1)

and

$$\tilde{v}_j(\eta, x) = Z_j^{-1}(x) \tilde{u}_{jN+i}(\eta, x)$$ (5.2)

where $Z_j$ is defined in (3.1). In view of Theorem 3.2, we have

$$Y_j = \epsilon \left( \text{Id} + \sqrt{\frac{\alpha_i-1}{Y(j+1)N+i-1}} R_j \right)$$

where $(R_j)$ is a sequence from $\mathcal{D}_1(K, \text{Mat}(2, \mathbb{R}))$ convergent to $R_i$ given by (3.8). Since $\text{discr} \ R_i < 0$ on $K$

$$|[R_i(x)]_{12}| > 0$$

and there are $\delta > 0$ and $j_0 \geq 1$ such that for all $j \geq j_0$ and $x \in K$,

$$\text{discr} \ R_j(x) < -\delta, \quad \text{and} \quad |[R_j(x)]_{12}| > \delta.$$ 

Therefore, $R_j(x)$ has two eigenvalues $\xi_j(x)$ and $\overline{\xi_j(x)}$ where

$$\xi_j(x) = \frac{\text{tr} \ R_j(x) + i\epsilon \sqrt{-\text{discr} \ R_j(x)}}{2}. \quad (5.3)$$

Moreover,

$$R_j = C_j \begin{pmatrix} \xi_j & 0 \\ 0 & \overline{\xi_j} \end{pmatrix} C_j^{-1}$$

where

$$C_j = \begin{pmatrix} 1 & 1 \\ \frac{\xi_j - [R_j]_{11}}{[R_j]_{12}} & \frac{1}{\overline{\xi_j} - [R_j]_{11}} \end{pmatrix}.$$ 

Using (5.1), $Y_j(x)$ has two eigenvalues $\lambda_j(x)$ and $\overline{\lambda_j(x)}$ where

$$\lambda_j(x) = \epsilon \left( 1 + \sqrt{\frac{\alpha_i-1}{Y(j+1)N+i-1}} \xi_j(x) \right).$$

Moreover,

$$Y_j = C_j D_j C_j^{-1}$$ (5.4)
where
\[ D_j = \begin{pmatrix} \lambda_j & 0 \\ 0 & \lambda_j \end{pmatrix}. \] (5.5)

Theorem 3.2 implies that \((C_j : j \geq j_0)\) and \((D_j : j \geq j_0)\) belong to \(D_1(K, \text{Mat}(2, \mathbb{C}))\). By (3.8), there is a mapping \(C_\infty : K \to \text{GL}(2, \mathbb{C})\) such that
\[ \lim_{j \to \infty} C_j = C_\infty \]
uniformly on \(K\).

**Claim 5.1** There is \(c > 0\) such that for all \(j \geq L > j_0\),
\[ \|\vec{v}_j\| \leq c \left( \prod_{k=L}^{j-1} \|D_k\| \right) \|\vec{v}_L\| \]
uniformly on \(\mathbb{S}^1 \times K\).

For the proof, we write
\[ \vec{v}_j = Y_{j-1} \cdots Y_{j_0} \vec{v}_L. \]
thus
\[ \|\vec{v}_j\| \leq \|Y_{j-1} \cdots Y_L\| \|\vec{v}_L\|. \]

Next,
\[ Y_{j-1} \cdots Y_L = C_{j-1} \left( \prod_{k=L}^{j-1} (D_k C_k^{-1} C_{k-1}) \right) C_{L-1}^{-1} \]
and so
\[ \|Y_{j-1} \cdots Y_L\| \leq \|C_{j-1}\| \left( \prod_{k=L}^{j-1} \|D_k C_k^{-1} C_{k-1}\| \right) \|C_{L-1}^{-1}\| \]
\[ \leq c \prod_{k=L}^{j-1} \|D_k\| \]
where the last estimate follows by [57, Proposition 1], proving Claim 5.1.

Next, we show the following statement.

**Claim 5.2** We have
\[ \lim_{j \to \infty} \frac{a_{jN+i-1}}{\sqrt{\gamma_{jN+i-1}}} \prod_{k=j_0}^{j-1} |\lambda_k|^2 = \frac{a_{j_0N+i-1} \sinh \theta_{j_0}}{\sqrt{\alpha_{i-1}} |\tau|} > 0 \]
uniformly on $K \subset \Lambda_-$.

By (5.4) and (5.5),

$$|\lambda_k(x)|^2 = \det D_k(x) = \det Y_j(x),$$

which together with (5.1) gives

$$|\lambda_k(x)|^2 = \frac{\sinh \vartheta_k(x)}{\sinh \vartheta_{k+1}(x)} \cdot \frac{a_{jN+i-1}}{a_{(j+1)N+i-1}}.$$

Hence,

$$\prod_{k=j_0}^{j-1} |\lambda_k(x)|^2 = \frac{\sinh \vartheta_{j_0}(x)}{\sinh \vartheta_j(x)} \cdot \frac{a_{j_0N+i-1}}{a_{jN+i-1}},$$

and since

$$\lim_{j \to \infty} \frac{\sinh \vartheta_j(x)}{\vartheta_j(x)} = 1$$

the claim follows.

### 6 Generalized Shifted Turán Determinants

In this section we study generalized $N$-shifted Turán determinants. Namely, for $\eta \in \mathbb{R}^2 \setminus \{0\}$ and $x \in \mathbb{R}$ we consider

$$S_n(\eta, x) = a_{n+N-1} \sqrt{\gamma_{n+N-1}} \langle E\tilde{u}_{n+N}, \tilde{u}_n \rangle$$

where $(\tilde{u}_n : n \in \mathbb{N}_0)$ corresponds to a generalized eigenvector associated to $x$ and corresponding to $\eta$, and

$$E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

**Theorem 6.1** Let $N$ be a positive integer and $i \in \{0, 1, \ldots, N-1\}$. Let $(\gamma_n : n \in \mathbb{N})$ be a sequence of positive numbers tending to infinity and satisfying (1.2) and (1.3). Let $(a_n : n \in \mathbb{N}_0)$ and $(b_n : n \in \mathbb{N}_0)$ be $\gamma$-tempered $N$-periodically modulated Jacobi parameters such that $X_0(0)$ is a non-trivial parabolic element. Suppose that (1.4) holds true with $\varepsilon = \text{sign}(\text{tr} X_0(0))$. Then the sequence $(|S_{jN+i}| : j \in \mathbb{N})$ converges locally uniformly on $S^1 \times \Lambda_-$ to a positive continuous function.
Proof We use the uniform diagonalization described in Sect. 5. Let us define
\[
\tilde{S}_j = a_{(j+1)N+i-1} \sqrt{\gamma_j} \langle E \tilde{v}_{j+1}, \tilde{v}_j \rangle.
\]
(6.1)

The first step is to show that \((\tilde{S}_j : j \geq j_0)\) is asymptotically close to \((S_{jN+i} : j \geq j_0)\).

Claim 6.2 We have
\[
\lim_{j \to \infty} |S_{jN+i} - \tilde{S}_j| = 0
\]
uniformly on \(S^1 \times K\).

For the proof we write
\[
S_{jN+i} = a_{(j+1)N+i-1} \sqrt{\gamma_j} \langle E \tilde{v}_{j+1}, \tilde{v}_j \rangle
\]
= \(a_{(j+1)N+i-1} \sqrt{\gamma_j} \langle Z_j^* E Z_j, \tilde{v}_j \rangle\)
= \(a_{(j+1)N+i-1} \sqrt{\gamma_j} \langle \det Z_j \rangle \langle E Z_j^{-1} Z_{j+1} \tilde{v}_j, \tilde{v}_j \rangle\)
where we have used that for any \(Y \in \text{GL}(2, \mathbb{R})\),
\[
(Y^{-1})^* E = \frac{1}{\det Y} E Y.
\]
(6.2)

Now, by Theorem 3.1
\[
S_{jN+i} - \tilde{S}_j = a_{(j+1)N+i-1} \sqrt{\gamma_j} \langle E (Z_j^{-1} Z_{j+1} - \text{Id}) \tilde{v}_{j+1}, \tilde{v}_j \rangle
\]
= \(a_{(j+1)N+i-1} \sqrt{\gamma_j} \langle \det Z_j \rangle \langle E Z_j^{-1} Z_{j+1} \tilde{v}_j, \tilde{v}_j \rangle\).

Observe that by (5.5) and (5.4)
\[
\|D_k\|^2 = |\lambda_k|^2 = \lambda_k \lambda_k = \det Y_k.
\]

Therefore, by (5.1),
\[
\prod_{k=j_0}^{j-1} \|D_k\|^2 = \det Z_{j_0} \frac{a_{j_0N+i-1}}{\det Z_j a_{jN+i-1}}.
\]

Next, in view of Claim 5.1, for \(j \geq j_0\),
\[
\|\tilde{v}_j\|^2 \lesssim \prod_{k=j_0}^{j-1} \|D_k\|^2 \lesssim \frac{1}{a_{jN+i-1} |\det Z_j|}.
\]
Hence,
\[ |S_{j+i} - \tilde{S}_j| \lesssim a_{j+1}N_{i-1}^N \sqrt{\alpha_{i-1}} \det Z_j \|Q_j\| \|v_j\|^2 \]
\[ \lesssim \|Q_j\| \]
and the claim follows by Theorem 3.1.

We show next that the sequence \((\tilde{S}_j : j \geq j_0)\) converges uniformly on \(S^1 \times K\) to a positive continuous function. By (6.1) and (6.2), we have
\[ \tilde{S}_j = a_{j+1}N_{i-1}^N \sqrt{\gamma_{j+1}N_{i-1}^N \det Z_{j+1}} (E \bar{v}_{j+1}, Y_j^{-1} \bar{v}_{j+1}) \]
\[ = a_{j+1}N_{i-1}^N \sqrt{\gamma_{j+1}N_{i-1}^N \det Z_{j+1}} (Y_j^{-1} \ast E \bar{v}_{j+1}, \bar{v}_{j+1}) \]
\[ = a_{j+1}N_{i-1}^N \sqrt{\gamma_{j+1}N_{i-1}^N \det Z_j \cdot Y_j^{-1}} (EY_j \bar{v}_{j+1}, \bar{v}_{j+1}), \]
and since
\[ \det Y_j = \det(Z_{j+1}^{-1}X_{j+i}Z_j), \]
we obtain
\[ \tilde{S}_{j+1} = a_{j+1}N_{i-1}^N \sqrt{\gamma_{j+2}N_{i-1}^N \det Z_{j+1}} (EY_{j+1} \bar{v}_{j+1}, \bar{v}_{j+1}). \]

By (6.1) we have
\[ \tilde{S}_{j+1} - \tilde{S}_j = \varepsilon a_{j+1}N_{i-1}^N \sqrt{\gamma_{j+1}N_{i-1}^N \det Z_{j+1}} \left\{ EW_j \bar{v}_{j+1}, \bar{v}_{j+1} \right\} \]
where
\[ W_j = \sqrt{\frac{\alpha_{i-1}}{\gamma_{j+1}N_{i-1}^N}} \left( \frac{a_{j+2}N_{i-1}^N}{a_{j+1}N_{i-1}^N} R_{j+1} - \frac{a_{j+1}N_{i-1}^N}{a_{jN_{i-1}^N}} R_j \right), \]
Hence,
\[ W_j = \sqrt{\frac{\alpha_{i-1}}{\gamma_{j+1}N_{i-1}^N}} \left( \alpha_{i-1}\left( \frac{a_{j+2}N_{i-1}^N}{a_{j+1}N_{i-1}^N} R_{j+1} - \frac{a_{j+1}N_{i-1}^N}{a_{jN_{i-1}^N}} R_j \right) \right), \]
and so
\[ \|W_j\| \lesssim \sqrt{\frac{\alpha_{i-1}}{\gamma_{j+1}N_{i-1}^N}} \left( \|\Delta \left( \frac{a_{j+1}N_{i-1}^N}{a_{jN_{i-1}^N}} \right) \| + \|\Delta R_j\| \right). \]
Therefore,

\[ \left| \tilde{S}_{j+1} - \tilde{S}_j \right| \lesssim a_{(j+1)N+i-1} \left( \left| \Delta \left( \frac{a_{(j+1)N+i-1}}{a_{jN+i-1}} \right) \right| + \| \Delta R_j \| \right) \| \tilde{v}_{j+1} \|^2. \]

On the other hand, by (6.3),

\[ \tilde{S}_j = \varepsilon a_{(j+1)N+i-1} \sqrt{\gamma(j+1)N+i-1} (\det Z_{j+1}) \times \frac{a_{(j+1)N+i-1}}{a_{jN+i-1}} \sqrt{\gamma(j+1)N+i-1} \{ E R_j \tilde{v}_{j+1}, \tilde{v}_{j+1} \}. \]

Since

\[ \lim_{j \to \infty} \text{sym}(E R_j) = \text{sym}(E R_i) \]

\[ = \frac{\sqrt{|\tau|}}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} - \frac{\nu}{2\sqrt{|\tau|}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} - \frac{\xi}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (6.4) \]

and the matrix on the right-hand side of (6.4) has determinant equal to \(-\tau > 0\), we obtain

\[ |\tilde{S}_j| \gtrsim a_{(j+1)N+i-1} |(\det Z_{j+1})| \cdot |\tilde{v}_{j+1}|^2. \]

Consequently, we arrive at

\[ \left| \tilde{S}_{j+1} - \tilde{S}_j \right| \lesssim \left( \left| \Delta \left( \frac{a_{(j+1)N+i-1}}{a_{jN+i-1}} \right) \right| + \| \Delta R_j \| \right) |\tilde{S}_j|. \]

Since \( \tilde{S}_j \neq 0 \) on \( K \), we get

\[ \sum_{j=j_0}^{\infty} \sup_{\eta \in S^1} \sup_{x \in K} \frac{|\tilde{S}_{j+1}(\eta, x)|}{|\tilde{S}_j(\eta, x)|} - 1 \lesssim \sum_{j=j_0}^{\infty} \left| \Delta \left( \frac{a_{(j+1)N+i-1}}{a_{jN+i-1}} \right) \right| + \sup_{x \in K} \| \Delta R_j(x) \|, \]

which implies that the product

\[ \prod_{k=j_0}^{\infty} \left( 1 + \frac{|\tilde{S}_{k+1}| - |\tilde{S}_k|}{|\tilde{S}_k|} \right) \]

converges uniformly on \( S^1 \times K \) to a positive continuous function. Because

\[ \left| \frac{\tilde{S}_j}{\tilde{S}_{j_0}} \right| = \prod_{k=j_0}^{j-1} \left( 1 + \frac{|\tilde{S}_{k+1}| - |\tilde{S}_k|}{|\tilde{S}_k|} \right), \]

the same holds true for the sequence \( (\tilde{S}_j : j \geq j_0) \). In view of Claim 6.2, the proof is completed. \( \square \)
Corollary 6.3 Suppose that the hypotheses of Theorem 6.1 are satisfied. Then for any compact \( K \subset \Lambda_\infty \) there is a constant \( c > 1 \) such that for any generalized eigenvector \( \tilde{u} \) associated with \( x \in K \) and corresponding to \( \eta \in S^1 \), we have
\[
c^{-1} \leq \frac{a_{j+1}N+i-1}{\sqrt{\gamma(j+1)N+i-1}} \| \tilde{v}_j \|^2 \leq c
\]
where \( \tilde{v}_j = Z_j^{-1}u_{jN+i} \).

**Proof** By (6.1) and Theorem 3.2 we have
\[
\tilde{S}_j = a_{j+1}N+i-1 \sqrt{\gamma(j+1)N+i-1} (\det Z_j) \langle EY_j \tilde{v}_j, \tilde{v}_j \rangle
\]
\[
= \varepsilon a_{j+1}N+i-1 \sqrt{\alpha_{i-1}} (\det Z_j) \langle ER_j \tilde{v}_j, \tilde{v}_j \rangle.
\]
Hence, by (6.4), we have
\[
|\tilde{S}_j| \asymp a_{j+1}N+i-1 |\det Z_j| |\tilde{v}_j|^2.
\]
Observe that
\[
\lim_{j \to \infty} \sqrt{\gamma(j+1)N+i-1} \det Z_j = \lim_{j \to \infty} \frac{-2 \sinh \vartheta_j \vartheta_j \sqrt{\gamma(j+1)N+i-1}}{\vartheta_j \sqrt{\alpha_{i-1}} |\tau(x)|}
\]
\[
= -2 \sqrt{\alpha_{i-1}} |\tau(x)|
\]
uniformly on \( K \). By the fact that \( \tilde{S}_j \) is uniformly convergent on \( S^1 \times K \) to a positive function, the conclusion follows. \( \square \)

7 Asymptotics of the Generalized Eigenvectors

In this section we study the asymptotic behavior of generalized eigenvectors. We keep the notation introduced in Sect. 5.

**Theorem 7.1** Let \( N \) be a positive integer. Let \( (\gamma_n : n \in \mathbb{N}) \) be a sequence of positive numbers tending to infinity and satisfying (1.2) and (1.3). Let \( (a_n : n \in \mathbb{N}_0) \) and \( (b_n : n \in \mathbb{N}_0) \) be \( \gamma \)-tempered \( N \)-periodically modulated Jacobi parameters such that \( \tilde{z}_0(0) \) is a non-trivial parabolic element. Suppose that (1.4) holds true with \( \varepsilon = \text{sign}(\text{tr} \tilde{z}_0(0)) \). Then for each \( i \in \{0, 1, \ldots, N-1\} \) and every compact interval \( K \subset \Lambda_\infty \), there are \( j_0 \in \mathbb{N} \) and a continuous function \( \varphi : S^1 \times K \to \mathbb{C} \) such that for every generalized eigenvector \( u_n : n \in \mathbb{N}_0 \),
\[
\lim_{j \to \infty} \sup_{\eta \in S^1} \sup_{x \in K} \left| \frac{\sqrt{\gamma(j+1)N+i-1}}{\prod_{k=j_0}^{j-1} \lambda_k(x)} (u_{j+1}N+i(\eta, x) - \lambda_j(x)u_{jN+i}(\eta, x)) - \varphi(\eta, x) \right| = 0.
\]
Moreover, \( \varphi(\eta, x) = 0 \) if and only if \( [X_i(0)]_{21} = 0 \). Furthermore,

\[
\frac{u_{jN+i}(\eta, x)}{\prod_{k=j_0}^{j-1} |\lambda_k(x)|} = \frac{|\varphi(\eta, x)|}{\sqrt{\alpha_{j-1} |\tau(x)|}} \sin \left( \sum_{k=j_0}^{j-1} \theta_k(x) + \arg \varphi(\eta, x) \right) + E_j(\eta, x)
\]

where

\[
\theta_k(x) = \arccos \left( \frac{\text{tr } Y_k(x)}{2\sqrt{\det Y_k(x)}} \right)
\]

and

\[
\lim_{j \to \infty} \sup_{\eta \in S^1} \sup_{x \in K} |E_j(\eta, x)| = 0.
\]

**Proof** In the proof we use the uniform diagonalization constructed in Sect. 5. For \( j > j_0 \), we set

\[
\phi_j = \frac{u_{(j+1)N+i} - \overline{\lambda_j} u_{jN+i}}{\prod_{k=j_0}^{j-1} \lambda_k}.
\]

Let us observe that there is \( c > 0 \) such that for all \( j \in \mathbb{N} \), and \( x \in K \),

\[
\left\| Z_j^t e_2 - \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ \end{pmatrix} T_j^t e_2 \right\| \leq c \vartheta_j.
\]

We show that the sequence \( (\sqrt{Y_{(j+1)N+i-1}} \phi_j : j > j_0) \) converges uniformly on \( K \).

By (5.5), \( \|D_j\| = |\lambda_j| \), thus by Claim 5.1

\[
\left| u_{(j+1)N+i} - \langle \tilde{v}_j, Z_j^t e_2 \rangle \right| = \left| \langle \tilde{v}_{j+1}, (Z_{j+1}^t - Z_j^t) e_2 \rangle \right|
\leq \| \tilde{v}_{j+1} \| \cdot |\vartheta_{j+1} - \vartheta_j|
\lesssim \left( \prod_{k=j_0}^{j-1} |\lambda_k| \right) |\vartheta_{j+1} - \vartheta_j|.
\]

Hence,

\[
\lim_{j \to \infty} \sqrt{Y_{(j+1)N+i-1}} \left| u_{(j+1)N+i} - \langle \tilde{v}_j, Z_j^t e_2 \rangle \right| = 0
\]

uniformly on \( K \). Next, by (5.4) we write

\[
(Y_j - \overline{\lambda_j} \text{Id})\tilde{v}_j = C_j \begin{pmatrix} \lambda_j - \overline{\lambda_j} & 0 \\ 0 & 0 \end{pmatrix} C_j^{-1} \tilde{v}_j,
\]
thus by (7.2) we obtain
\[
\left\| (Y_j - \lambda_j \text{Id})\bar{v}_j, Z_j e_2 \right\| - \left\| (Y_j - \lambda_j \text{Id})\bar{v}_j, \left( \begin{array}{c} 1 \\ 1 \end{array} \right) T' e_2 \right\|
\leq \theta_j |\lambda_j - \bar{\lambda}_j| \cdot \|\bar{v}_j\| \lesssim \theta_j \left( \prod_{k=j_0}^{j-1} |\lambda_k| \right)
\]
where in the last estimate we have used
\[
|\lambda_j - \bar{\lambda}_j| = |2i \Im(\lambda_j)| = \sqrt{\frac{\alpha_{i-1}}{\gamma(j+1)N+i-1}} \sqrt{-\text{disc} R_j}.
\]
Hence, it is enough to show that the sequence \((\bar{\phi}_j : j > j_0)\) where
\[
\bar{\phi}_j = \frac{\sqrt{\gamma(j+1)N+i-1}}{\prod_{k=j_0}^{j-1} \lambda_k} \left( (Y_j - \lambda_j \text{Id})\bar{v}_j, \left( \begin{array}{c} 1 \\ 1 \end{array} \right) T' e_2 \right)
\]
converges uniformly on \(S^1 \times K\). To do so, for a given \(\epsilon > 0\) there is \(L_0 > j_0\) such that for all \(L \geq L_0\) we have
\[
\sum_{k=L-1}^{\infty} \sup_K \|\Delta C_k\| < \epsilon.
\]
For \(j \geq L\), we set
\[
\psi_{j:L} = \frac{\sqrt{\gamma(j+1)N+i-1}}{\prod_{k=j_0}^{j-1} \lambda_k} \left( C_j (D_j - \lambda_j \text{Id}) \left( \prod_{k=L}^{j-1} D_k \right) C^{-1}_{L-1} \bar{v}_L, \left( \begin{array}{c} 1 \\ 1 \end{array} \right) T' e_2 \right).
\]
Observe that
\[
(Y_j - \lambda_j \text{Id})\bar{v}_j = (Y_j - \lambda_j \text{Id}) \left( \prod_{k=L}^{j-1} Y_k \right) \bar{v}_L = C_j (D_j - \lambda_j \text{Id}) C^{-1}_j \left( \prod_{k=L}^{j-1} C_k D_k C^{-1}_k \right) \bar{v}_L = C_j (D_j - \lambda_j \text{Id}) D_j^{-1} \left( \prod_{k=L}^{j} D_k C^{-1}_k C^{-1}_{k-1} \right) C^{-1}_{L-1} \bar{v}_L.
\]
Hence, by [57, Proposition 1], there is \(c > 0\) such that for all \(j \geq L \geq j_0\),
\[
\left\| \prod_{k=L}^{j} D_k C_k C^{-1}_{k-1} \right\| - \left\| \prod_{k=L}^{j} D_k \right\| \leq c \prod_{k=L}^{j} |\lambda_k| \cdot \sum_{k=L-1}^{\infty} \sup_K \|\Delta C_k\|.
\]
Thus, by Claim 5.1, (7.4) and the fact that \( \| D_k \| = |\lambda_k| \), we obtain

\[
|\tilde{\phi}_j - \psi_{j;L}| \leq c \sum_{k=L-1}^{\infty} \sup_k \| \Delta C_k \| \leq c \epsilon,
\]

(7.5)

for all \( j \geq L \). Hence, for all \( n > m \geq L \),

\[
|\tilde{\phi}_n - \tilde{\phi}_m| \leq c \epsilon + |\psi_n;L - \psi_m;L|.
\]

Therefore, our task is reduced to showing that the sequence \((\psi_j;L : j \geq L)\) converges uniformly on \( K \). Since, by (7.4)

\[
\sqrt{Y(N+i-1)} \prod_{k=L}^{j-1} \frac{D_k}{\lambda_k} = i \sqrt{\alpha_{i-1}} \sqrt{-\extrm{discr} R_j} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \prod_{k=L}^{j-1} \frac{D_k}{\lambda_k}
\]

we get uniformly on \( K \)

\[
\lim_{j \to \infty} \psi_{j;L} = \frac{i \sqrt{\alpha_{i-1}} \sqrt{-\extrm{discr} R_i}}{\prod_{k=j_0}^{L-1} \lambda_k} \begin{pmatrix} C_{\infty}^{-1} \tilde{\psi}_L, (1 & 1) T_i^t e_2 \end{pmatrix}
\]

(7.6)

where

\[
C_{\infty} = \lim_{j \to \infty} C_j = \left( \frac{1}{[R_i]_{11}}, \frac{1}{[R_i]_{12}} \right).
\]

(7.7)

Thus, we have proved that both sequences \((\tilde{\phi}_j : j > j_0)\) and \((\psi_{j;L} : j \geq L)\) converge uniformly on \( K \). Let us denote its limits by \( \tilde{\phi}_\infty \) and \( \psi_{\infty;L} \), respectively. By (7.5), for all \( L \geq L_0 \) we have

\[
|\tilde{\phi}_\infty - \psi_{\infty;L}| \leq c \epsilon.
\]

(7.8)

Let us observe that

\[
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} C_{\infty} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = ([C_{\infty}]_{11} + [C_{\infty}]_{21}) \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.
\]

By (7.7) the expression on the right-hand side has non-zero imaginary part. Thus from (7.6) we can write

\[
\psi_{\infty;L} = \frac{h}{\prod_{k=j_0}^{L-1} \lambda_k} \begin{pmatrix} C_{L-1}^{-1} \tilde{\psi}_L, (1 & 1) T_i^t e_2 \end{pmatrix}
\]

\[
= \frac{h}{\prod_{k=j_0}^{L-1} \lambda_k} ([T_i]_{21} + [T_i]_{22}) (C_{L-1}^{-1} \tilde{\psi}_L, e_1)
\]

(7.9)
for some function $h$ without zeros on $\mathbb{S}^1 \times K$. Thus, by (2.14), if $[\mathcal{X}_i(0)_{21} = 0$, then $\psi_{\infty; L} \equiv 0$ for all $L$. Consequently, by (7.8), $\tilde{\phi}_{\infty} \equiv 0$ on $\mathbb{S}^1 \times K$. On the other hand, if $[\mathcal{X}_i(0)_{21} \neq 0$, then the following claim holds true.

**Claim 7.2** For each $(\eta, x) \in \mathbb{S}^1 \times K$,

$$\lim \inf_{L \to \infty} |\psi_{\infty; L}(\eta, x)| > 0.$$

On the contrary, let us suppose that there are $\eta \in \mathbb{S}^1$, $x \in K$ and a sequence $(L_j : j \in \mathbb{N})$ such that

$$\lim_{j \to \infty} L_j = \infty,$$

and

$$\lim_{j \to \infty} \psi_{\infty; L_j}(\eta, x) = 0.$$

Setting $\tilde{v}_L = v_L^1 e_1 + v_L^2 e_2$, we have

$$\langle C_{L-1}^{-1} \tilde{v}_L, e_1 \rangle = \left( \begin{array}{c} \frac{\mathcal{X}_{L-1} - [R_{L-1}]_{11}}{[R_{L-1}]_{12}} \left( [R_{L-1}]_{11} - [R_{L-1}]_{12} \right) v_L^1 - v_L^2 \\
1 \end{array} \right) \tilde{v}_L, e_1 \rangle$$

$$= \frac{\mathcal{X}_{L-1} - [R_{L-1}]_{11}}{[R_{L-1}]_{12}} v_1^L - v_2^L.$$

Hence, by (7.9), we obtain

$$\lim_{j \to \infty} \frac{1}{\prod_{k=j_0}^{L_j-1} |\lambda_k(x)|} \left( \frac{\mathcal{X}_{L_j-1}(x) - [R_{L_j-1}(x)]_{11}}{[R_{L_j-1}(x)]_{12}} v_1^{L_j}(\eta, x) - v_2^{L_j}(\eta, x) \right) = 0.$$

In view of (5.2), $\tilde{v}_{L_j}(\eta, x)$ is a real vector, thus by taking imaginary parts of the bracket, we conclude that

$$\lim_{j \to \infty} \frac{v_1^{L_j}(\eta, x)}{\prod_{k=j_0}^{L_j-1} |\lambda_k(x)|} = 0.$$

Hence,

$$\lim_{j \to \infty} \frac{v_2^{L_j}(\eta, x)}{\prod_{k=j_0}^{L_j-1} |\lambda_k(x)|} = 0,$$

which in view of Claim 5.2 contradicts to Corollary 6.3 proving the claim.
Next, let us consider \( \eta \in S^1 \) and \( x \in K \). By Claim 7.2,
\[
A = \liminf_{L \to \infty} |\psi_\infty;L(\eta, x)| > 0. \tag{7.10}
\]
Taking \( \epsilon = \frac{A}{2c} \), by (7.8), for all \( L \geq L_0 \),
\[
|\tilde{\phi}_\infty(\eta, x)| \geq |\psi_\infty;L(\eta, x)| - c\epsilon = |\psi_\infty;L(\eta, x)| - \frac{A}{2}.
\]
Thus, in view of (7.10),
\[
|\tilde{\phi}_\infty(\eta, x)| \geq \frac{A}{2}.
\]
Consequently, \( \tilde{\phi}_\infty \) cannot be zero on \( S^1 \times K \) provided that \( [X_i(0)]_{21} \neq 0 \).

In view of (7.3) there is a function \( \varphi : S^1 \times K \to \mathbb{R} \), such that
\[
\varphi = \lim_{j \to \infty} \sqrt{(j+1)N+i-1}\phi_j
\]
uniformly on \( S^1 \times K \). In fact, one has \( \varphi = \tilde{\phi}_\infty \). In particular, we obtain
\[
\lim_{j \to \infty} \sup_{\eta \in S^1} \sup_{x \in K} \left| \sqrt{(j+1)N+i-1}\frac{u(j+1)N+i-1(\eta, x) - \lambda_j(\eta)}{\prod_{k=0}^{j-1} |\lambda_k(x)|} - \varphi(\eta, x) \prod_{k=0}^{j-1} \frac{\lambda_k(x)}{|\lambda_k(x)|} \right| = 0.
\]

Since \( u_n(\eta, x) \in \mathbb{R} \), by taking imaginary part we conclude that
\[
\lim_{j \to \infty} \sup_{\eta \in S^1} \sup_{x \in K} \left| \frac{\sqrt{\alpha_i-1}}{2} \sqrt{-\text{discr} R_j(x)} \frac{u(j+1)N+i-1(\eta, x)}{\prod_{k=0}^{j-1} |\lambda_k(x)|} - |\varphi(\eta, x)| \sin \left( \sum_{k=0}^{j-1} \theta_k(x) + \arg \varphi(\eta, x) \right) \right| = 0
\]
where we have also used that
\[
-\sqrt{(j+1)N+i-1}\Im(\lambda_j(x)) = \frac{\sqrt{\alpha_i-1}}{2} \sqrt{-\text{discr} R_j(x)}.
\]
Lastly, observe that
\[
\left| \frac{1}{\sqrt{-\text{discr} R_j(x)}} - \frac{1}{2|\tau(x)|} \right| \leq \sum_{k=j}^{\infty} \| \Delta R_k(x) \|.
\]
which completes the proof. □

Remark 7.3 There is \( i \in \{0, 1, \ldots, N - 1 \} \) such that \( |\varphi_i(\eta, x)| > 0 \) for all \( x \in K \) and \( \eta \in S^1 \). Indeed, by [55, Proposition 3], if \( [X_{i-1}(0)]_{21} = 0 \) and \( [X_i(0)]_{21} = 0 \), then \( X_i(0) \) is a multiple of identity which is a trivial parabolic element. Contradiction.

8 Christoffel–Darboux Kernel on the Diagonal

In this section we study the asymptotic behavior of generalized Christoffel–Darboux kernel on the diagonal. Given Jacobi parameters \((a_n : n \in \mathbb{N}_0)\) and \((b_n : n \in \mathbb{N}_0)\), and \( \eta \in S^1 \), we set

\[
K_n(x, y; \eta) = \sum_{m=0}^{n} u_m(x, \eta)u_m(y, \eta), \quad x, y \in \mathbb{R},
\]

where \((u_n(x, \eta) : n \in \mathbb{N}_0)\) is generalized eigenvector associated to \( x \) and corresponding to \( \eta \). Let

\[
\rho_n = \sum_{m=0}^{n} \frac{\sqrt{\alpha_m \gamma_m}}{a_m}.
\]

For \( N \)-periodically modulated Jacobi parameters we also study

\[
K_{i;j}(x, y; \eta) = \sum_{k=0}^{j} u_{kN+i}(x, \eta)u_{kN+i}(y, \eta), \quad x, y \in \mathbb{R},
\]

where \( i \in \{0, 1, \ldots, N - 1 \} \). Let

\[
\rho_{i;j} = \sum_{k=1}^{j} \frac{\sqrt{\gamma_{kN+i}}}{a_{kN+i}}.
\]

Lemma 8.1 Let \((\gamma_n : n \in \mathbb{N})\) and \((a_n : n \in \mathbb{N})\) be sequences of positive numbers such that

\[
\lim_{n \to \infty} \gamma_n = \infty, \quad \sum_{n=0}^{\infty} \frac{\sqrt{\gamma_n}}{a_n} = \infty.
\]

Suppose that \((\xi_n : n \in \mathbb{N})\) is a sequence of real functions on some compact set \( K \subset \mathbb{R}^d \) such that

\[
\lim_{n \to \infty} \sup_{x \in K} \left| \sqrt{\gamma_n} \xi_n(x) - \psi(x) \right| = 0
\]
for certain function $\psi : K \to (0, \infty)$ satisfying
\[ c^{-1} \leq \psi(x) \leq c, \quad \text{for all } x \in K. \]

We set
\[
\Xi_n(x) = \sum_{j=0}^{n} \xi_j(x), \quad \text{and} \quad \Delta_n = \sum_{j=0}^{n} \frac{\sqrt{\gamma_j}}{a_j}.
\]

If
\[
\left( \frac{\gamma_n}{a_n} : n \in \mathbb{N} \right) \in \mathcal{D}_1, \quad \text{(8.1)}
\]
then
\[
\lim_{n \to \infty} \frac{1}{\Delta_n} \sum_{k=0}^{n} \frac{\sqrt{\gamma_k}}{a_k} \cos \left( \Xi_k(x) \right) = 0 \quad \text{(8.2)}
\]
uniformly with respect to $x \in K$.

**Proof** First, we write
\[
\left| \sum_{k=0}^{n} \frac{\sqrt{\gamma_k}}{a_k} \cos \left( \Xi_k(x) \right) - \sum_{k=0}^{n} \frac{\gamma_k \xi_k(x)}{a_k \psi(x)} \cos \left( \Xi_k(x) \right) \right|
\]
\[
\leq \sum_{k=0}^{n} \frac{\sqrt{\gamma_k}}{a_k} \left| 1 - \frac{\sqrt{\gamma_k}}{\psi(x)} \xi_k(x) \right|.
\]

Since by the Stolz–Cesàro theorem
\[
\lim_{n \to \infty} \frac{1}{\Delta_n} \sum_{k=0}^{n} \frac{\sqrt{\gamma_k}}{a_k} \left| 1 - \frac{\sqrt{\gamma_k}}{\psi(x)} \xi_k(x) \right| = \lim_{n \to \infty} \left| 1 - \frac{\sqrt{\gamma_k}}{\psi(x)} \xi_k(x) \right| = 0,
\]
uniformly with respect to $x \in K$, we obtain
\[
\lim_{n \to \infty} \frac{1}{\Delta_n} \sum_{k=0}^{n} \frac{\sqrt{\gamma_k}}{a_k} \cos \left( \Xi_k(x) \right) = \lim_{n \to \infty} \frac{1}{\Delta_n} \sum_{k=0}^{n} \frac{\gamma_k \xi_k(x)}{a_k \psi(x)} \cos \left( \Xi_k(x) \right).
\]

Next, we observe that
\[
\left| \sum_{k=1}^{n} \frac{\gamma_k}{a_k} \left( \xi_k(x) \cos \left( \Xi_k(x) \right) - \int_{\Xi_{k-1}(x)}^{\Xi_k(x)} \cos(t) \, dt \right) \right|
\]
\[
\leq \sum_{k=1}^{n} \frac{\gamma_k}{a_k} \int_{\Xi_{k-1}(x)}^{\Xi_k(x)} \left| \cos(t) - \cos \left( \Xi_k(x) \right) \right| \, dt \leq \frac{1}{2} \sum_{k=1}^{n} \frac{\gamma_k}{a_k} |\xi_k(x)|^2.
\]
In view of the Stolz–Cesàro theorem
\[
\lim_{n \to \infty} \frac{1}{\Delta_n} \sum_{k=1}^{n} \frac{\gamma_k}{a_k} |\xi_k(x)|^2 = \lim_{n \to \infty} \sqrt{\gamma_n} |\xi_n(x)|^2 = 0,
\]
thus
\[
\lim_{n \to \infty} \frac{1}{\Delta_n} \sum_{k=0}^{n} \frac{\sqrt{\gamma_k}}{a_k} \cos(\Xi_k(x)) = \lim_{n \to \infty} \frac{1}{\Delta_n} \sum_{k=1}^{n} \frac{\gamma_k}{a_k} (\sin \Xi_k(x) - \sin \Xi_{k-1}(x)).
\]

Now, by the summation by parts we get
\[
\sum_{k=1}^{n} \frac{\gamma_k}{a_k} (\sin \Xi_k(x) - \sin \Xi_{k-1}(x)) = \frac{\gamma_n}{a_n} \sin \Xi_n(x) - \frac{\gamma_1}{a_1} \sin \Xi_0(x)
+ \sum_{k=1}^{n-1} \left( \frac{\gamma_k}{a_k} - \frac{\gamma_{k+1}}{a_{k+1}} \right) \sin \Xi_k(x).
\]

Thus, by (8.1),
\[
\sup_{x \in K} \left| \sum_{k=1}^{n} \frac{\gamma_k}{a_k} (\sin \Xi_k(x) - \sin \Xi_{k-1}(x)) \right| \leq 2 \frac{\gamma_1}{a_1} + 2 \sum_{k=1}^{\infty} \left| \frac{\gamma_{k+1}}{a_{k+1}} - \frac{\gamma_k}{a_k} \right|.
\]
Consequently,
\[
\lim_{n \to \infty} \frac{1}{\Delta_n} \sum_{k=1}^{n} \frac{\gamma_k}{a_k} (\sin \Xi_k(x) - \sin \Xi_{k-1}(x)) = 0
\]
and the lemma follows.

**Theorem 8.2** Let $N$ be a positive integer. Let $(\gamma_n : n \in \mathbb{N})$ be a sequence of positive numbers tending to infinity and satisfying (1.2) and (1.3). Let $(a_n : n \in \mathbb{N}_0)$ and $(b_n : n \in \mathbb{N}_0)$ be $\gamma$-tempered $N$-periodically modulated Jacobi parameters such that $\mathcal{X}_0(0)$ is a non-trivial parabolic element. Suppose that (1.4) holds true with $\varepsilon = \text{sign}(\text{tr} \mathcal{X}_0(0))$. If
\[
\lim_{n \to \infty} \rho_n = \infty,
\]
then there is $n_0 \geq 1$ such that for all $n \geq n_0$,
\[
\lim_{n \to \infty} \frac{1}{\rho_n} K_n(x, x; \eta) = \frac{1}{2N} \sum_{i=0}^{N-1} |\varphi_i(\eta, x)|^2 a_{j_0N+i-1} \sinh \varphi_{j_0N+i-1}(x) \left( \sqrt{\alpha_{i-1}|\tau(x)|} \right)^3
\]
locally uniformly with respect to $(x, \eta) \in \Lambda_- \times S^1$. 
Proof Let $K$ be a compact interval with non-empty interior contained in $\Lambda_{-}$. By Theorem 7.1 and Claim 5.2, there is $j_0 \geq 1$ such that for $x \in K$, $\eta \in S^1$, and $k > j_0$,

$$
\frac{a_{kN+i-1}}{\sqrt{\alpha_{i-1}}Y(k+1)N+i-1}u_{kN+i}^2(\eta, x)
= \frac{|\varphi_i(\eta, x)|^2a_{j_0N+i-1} \sinh \vartheta_{j_0N+i}(x)}{2(\sqrt{\alpha_{i-1}}|\tau(x)|)^3} \sin^2 \left( \sum_{\ell=j_0}^{k-1} \vartheta_{\ell N+i}(x) + \arg \varphi_i(\eta, x) \right)
+ E_{kN+i}(\eta, x)
$$

where

$$
\lim_{k \to \infty} \sup_{\eta \in S^1} \sup_{x \in K} |E_{kN+i}(\eta, x)| = 0.
$$

Therefore,

$$
\sum_{k=j_0+1}^{j} u_{kN+i}^2(\eta, x)
= \frac{|\varphi_i(\eta, x)|^2a_{j_0N+i-1} \sinh \vartheta_{j_0N+i}(x)}{2(\sqrt{\alpha_{i-1}}|\tau(x)|)^3} \sum_{k=j_0+1}^{j} \frac{\sqrt{\alpha_{i-1}}Y(k+1)N+i-1}{a_{(k+1)N+i-1}}
\times \left( 1 - \cos \left( 2 \sum_{\ell=j_0}^{k-1} \vartheta_{\ell N+i}(x) + 2 \arg \varphi_i(\eta, x) \right) \right)
+ \sum_{k=j_0+1}^{j} \frac{\sqrt{\alpha_{i-1}}Y(k+1)N+i-1}{a_{(k+1)N+i-1}} E_{kN+i}(\eta, x).
$$

We claim that

$$
\lim_{j \to \infty} \frac{1}{\sqrt{\alpha_{i-1}}\rho_{i-1;j}} K_{i;j}(x, x; \eta) = \frac{|\varphi_i(\eta, x)|^2a_{j_0N+i-1} \sinh \vartheta_{j_0N+i-1}(x)}{2(\sqrt{\alpha_{i-1}}|\tau(x)|)}
$$

uniformly with respect to $(x, \eta) \in K \times S^1$. To see this, we observe that by the Stolz–Cesàro theorem,

$$
\lim_{j \to \infty} \frac{1}{\rho_{i-1;j}} \sum_{k=j_0+1}^{j} \frac{\sqrt{Y(k+1)N+i-1}}{a_{(k+1)N+i-1}} E_{kN+i}(\eta, x)
= \lim_{j \to \infty} \sqrt{\frac{Y(j+1)N+i-1}{Y_{jN+i-1}}} \cdot \frac{a_{jN+i-1}}{a_{(j+1)N+i-1}} E_{jN+i}(\eta, x) = 0.
$$
Since there is $c > 0$ such that
\[
\sup_{\eta \in S^1} \sup_{x \in K} \sum_{k=0}^{j_0} u_k^2 N^{i+1}(\eta, x) \leq c
\]
to prove (8.4) it is enough to show that
\[
\lim_{j \to \infty} \frac{1}{\rho_{i-1}; j} \sum_{k=j_0+1}^{j} \sqrt{\alpha_{i-1} \gamma_{(k+1)N+i-1}} \frac{a_{(k+1)N+i-1}}{\alpha_{i-1}} \times \cos \left( 2 \sum_{\ell=j_0}^{k-1} \theta_{\ell N+i} (x) + 2 \arg \varphi_i (\eta, x) \right) = 0
\]
uniformly with respect to $(x, \eta) \in K \times S^1$. Observe that (8.5) is an easy consequence of Lemma 8.1, provided we show the following statement.

Claim 8.3 For all $i \in \{0, 1, \ldots, N-1\}$,
\[
\lim_{j \to \infty} \sqrt{\frac{\gamma_{(j+1)N+i-1}}{\alpha_{i-1}}} \theta_{j N+i} (x) = \sqrt{|\tau(x)|}
\]
uniformly with respect to $x \in K$.

Using the notation introduced in Sect. 5, Theorem 3.2 gives
\[
\lim_{j \to \infty} Y_j = \varepsilon \text{ Id}
\]
locally uniformly on $\Lambda_\varepsilon$. In particular,
\[
\lim_{j \to \infty} \frac{\text{tr } Y_j (x)}{2 \sqrt{\det Y_j (x)}} = \varepsilon.
\]
Since
\[
\lim_{t \to 1} \frac{\arccos t}{\sqrt{1 - t^2}} = 1,
\]
we obtain
\[
\lim_{j \to \infty} \left( 1 - \left( \frac{\text{tr } Y_j (x)}{2 \sqrt{\det Y_j (x)}} \right)^2 \right)^{-1/2} \theta_j (x) = 1.
\]
Let us observe that, by Theorem 3.2,
\[
\sqrt{1 - \left( \frac{\text{tr } Y_j (x)}{2 \sqrt{\det Y_j (x)}} \right)^2} = \frac{\sqrt{\text{discr } Y_j (x)}}{2 \sqrt{\det Y_j (x)}}
\]
\[ = \sqrt{\frac{\alpha_{i-1}}{Y_{(j+1)N+i-1}}} \sqrt{-\text{discr} R_j(x)} \]  \\

Hence,
\[
\lim_{j \to \infty} \sqrt{\frac{Y_{(j+1)N+i-1}}{\alpha_{i-1}}} \sqrt{1 - \left( \frac{\text{tr} Y_j(x)}{2 \sqrt{\det Y_j(x)}} \right)^2} = \sqrt{|\tau(x)|}, \tag{8.6}
\]
proving the claim.

To complete the proof of the theorem we write
\[
K_{jN+i}(x, x; \eta) = \sum_{i' = 0}^{N-1} K_{i'; j}(x, x; \eta) + \sum_{i' = i+1}^{N-1} \left( K_{i'; j-1}(x, x; \eta) - K_{i'; j}(x, x; \eta) \right).
\]

Observe that by Theorem 7.1,
\[
\sup_{\eta \in S} \sup_{x \in K} |K_{i'; j-1}(x, x; \eta) - K_{i'; j}(x, x; \eta)| = \sup_{\eta \in S} \sup_{x \in K} |u_{jN+i'}(\eta, x)|^2 \leq c.
\]

Moreover, by [59, Proposition 3.7] and (2.7), for \(m, m' \in \mathbb{N}_0\),
\[
\lim_{j \to \infty} \frac{a_{jN+m'}}{a_{jN+m}} = \lim_{j \to \infty} \frac{Y_{jN+m'}}{Y_{jN+m}} = \frac{\alpha_{m'}}{\alpha_m},
\]
thus, by the Stolz–Cesàro theorem,
\[
\lim_{j \to \infty} \frac{\rho_{i-1; j}}{\rho_{jN+i'}} = \lim_{j \to \infty} \frac{\sqrt{Y_{jN+i-1}}}{\alpha_{jN+i-1}} \sum_{k=1}^{N} \frac{\sqrt{\alpha_{jN+i'-k}Y_{jN+i'-k}}}{a_{jN+i'-k}} = \frac{1}{N\sqrt{\alpha_{i-1}}}.
\]

Hence, by (8.4)
\[
\lim_{j \to \infty} \frac{1}{\rho_{jN+i}} K_{jN+i}(x, x; \eta) = \lim_{j \to \infty} \sum_{i' = 0}^{N-1} \frac{1}{\rho_{i'-1; j}} K_{jN+i'}(x, x; \eta) \cdot \frac{\rho_{i'-1; j}}{\rho_{jN+i}}
\]
\[
= \sum_{i' = 0}^{N-1} \frac{|\varphi_{i'}(\eta, x)|^2 a_{jN+i'-1} \sinh \vartheta_{j0N+i'-1}(x)}{2N(\sqrt{\alpha_{i'-1}|\tau(x)|})^3}
\]
and the theorem follows. \qed
Remark 8.4 In view of Remark 7.3 by Theorem 8.2, all generalized eigenvectors are not square-summable, hence by \[45, Theorem 6.16\] the operator $A$ is self-adjoint. Next, by [3, Theorem 2.1], we conclude that $\mu$ is absolutely continuous on $\Lambda_-$ and its density $\mu'$ has the property that for every compact interval $K \subset \Lambda_-$ with non-empty interior there is $c > 0$ such that

$$c^{-1} \leq \mu'(x) \leq c$$

for almost all $x \in K$ (with respect to the Lebesgue measure). Consequently, we have $\sigma_{ac}(A) \supset \text{cl}(\Lambda_-)$. In view of Theorem 4.1 we actually have $\sigma_{ac}(A) = \sigma_{ess}(A) = \text{cl}(\Lambda_-)$.

9 The Self-Adjointness of $A$

In this section we study the conditions that guarantee that the operator $A$ is self-adjoint. The first theorem covers the case when $\Lambda_- \neq \emptyset$.

Theorem 9.1 Let $N$ be a positive integer. Let $(\gamma_n : n \in \mathbb{N})$ be a sequence of positive numbers tending to infinity and satisfying (1.2) and (1.3). Let $(a_n : n \in \mathbb{N}_0)$ and $(b_n : n \in \mathbb{N}_0)$ be $\gamma$-tempered $N$-periodically modulated Jacobi parameters such that $\mathcal{X}_0(0)$ is a non-trivial parabolic element. Suppose that (1.4) holds true with $\varepsilon = \text{sign}(\text{tr} \mathcal{X}_0(0))$. If $\Lambda_- \neq \emptyset$, then the Jacobi operator $A$ associated with $(a_n : n \in \mathbb{N}_0)$ and $(b_n : n \in \mathbb{N}_0)$ is self-adjoint if and only if

$$\sum_{n=0}^{\infty} \frac{\sqrt{\gamma_n}}{a_n} = \infty. \quad (9.1)$$

Proof The case when (9.1) is satisfied is covered by Remark 8.4. Assume now that (9.1) is not satisfied. Let $x \in \Lambda_-$. By Theorem 7.1 and Claim 5.1, there are $j_0 \geq 1$ and $c > 0$ such that for all $j \geq j_0, i \in \{0, 1, \ldots, N-1\}$, and $\eta \in S^1$,

$$|u_{jN+i}(\eta, x)|^2 \leq c \frac{\sqrt{\gamma_{jN+i-1}}}{a_{jN+i-1}}.$$

Hence, every generalized eigenvector associated to $x$ is square-summable. In view of [45, Theorem 6.16] the operator $A$ is not self-adjoint. This completes the proof.

The next theorem covers the case when $\Lambda_- = \emptyset$ but $\Lambda_+ \neq \emptyset$.

Theorem 9.2 Let $N$ be a positive integer. Let $(\gamma_n : n \in \mathbb{N})$ be a sequence of positive numbers tending to infinity and satisfying (1.2) and (1.3). Let $(a_n : n \in \mathbb{N}_0)$ and $(b_n : n \in \mathbb{N}_0)$ be $\gamma$-tempered $N$-periodically modulated Jacobi parameters such that $\mathcal{X}_0(0)$ is a non-trivial parabolic element. Suppose that (1.4) holds true with $\varepsilon = \text{sign}(\text{tr} \mathcal{X}_0(0))$. If $\Lambda_- = \emptyset$ but $\Lambda_+ \neq \emptyset$, then $\Lambda_+ = \mathbb{R}$, and

(i) if $-\mathcal{S} + \sqrt{\mathcal{S}^2 + 4\mathcal{U}} < 0$ then the operator $A$ is not self-adjoint;
(ii) if \(-S + \sqrt{S^2 + 4U} > 0\) then the operator \(A\) is self-adjoint.

Moreover, if the operator \(A\) is self-adjoint then \(\sigma_{\text{ess}}(A) = \emptyset\).

**Proof** If \(\Lambda_- = \emptyset\) then \(t = 0\) and so \(\Lambda_+ = \mathbb{R}\). Let \(i = 0\) and \(K = \{0\}\). We can repeat the first part of the proof of Theorem 4.1. Now, by (4.6) and (4.8), there are \(j_1 \geq j_0\) and \(c > 0\) such that for all \(j \geq j_1\),

\[
|u_{jN}(0)|^2 + |u_{jN-1}(0)|^2 = \|\phi_{jN}(0)\|^2 \geq c \prod_{k=j_0}^{j} |\lambda_k^+(0)|^2.
\]

(9.2)

Moreover, for all \(j \geq j_1\) and \(i' \in \{0, 1, \ldots, N-1\}\),

\[
\|\phi_{jN+i'}(0)\|^2 \leq c \prod_{k=j_0}^{j} |\lambda_k^-(0)|^2, \quad \text{and}
\]

\[
\|\phi_{jN+i'}(0)\|^2 \leq c \prod_{k=j_0}^{j} |\lambda_k^+(0)|^2.
\]

(9.3)

By (4.3) we obtain

\[
\sum_{j=j_1}^{\infty} \sum_{i'=0}^{N-1} |u_{jN+i'}(0)|^2 \leq c \sum_{j=j_1}^{\infty} \sum_{i'=0}^{N-1} |u_{jN+i'}(0)|^2.
\]

(9.4)

Hence, the operator \(A\) is self-adjoint if and only if there is \(j_0 > 0\) such that

\[
\sum_{j=j_0}^{\infty} \prod_{k=j_0}^{j} |\lambda_k^+(0)|^2 = \infty.
\]

(9.5)

Indeed, if (9.5) is satisfied then by (9.2) the generalized eigenvector \((u_n^+(0) : n \in \mathbb{N}_0)\) is not square-summable, thus by [45, Theorem 6.16], the operator \(A\) is self-adjoint. On the other hand, if (9.5) is not satisfied, then by (9.3) and (9.4), all generalized eigenvectors associated to 0 are square-summable, thus by [45, Theorem 6.16], the operator \(A\) is not self-adjoint. The second part of the theorem follows by Theorem 4.1.

Since \((\gamma_{jN} : j \in \mathbb{N})\) approaches infinity, there is \(j_0 \geq 1\) such that

\[
|\lambda_j^+(0)| = 1 + \sqrt{\frac{\alpha_{N-1}}{\gamma_{(j+1)N-1}}} \text{tr} R_j(0) + \sqrt{\text{discr} R_j(0)}
\]

Next, we observe that

\[
\lim_{j \to \infty} \frac{\sqrt{\gamma_{jN}}}{j} = 0.
\]
Let us consider the case (i). Because \((R_N(j) : j \in \mathbb{N})\) converges to \(R_0(0)\), there is \(j_0 \geq j_0\) such that for all \(j \geq j_1\),

\[
j \sqrt{\frac{\mathcal{K}_{N-1}}{\gamma(j+1)N-1}} \left( \frac{\text{tr} R_j(0) + \sqrt{\text{discr} R_j(0)}}{2} \right) < -1,
\]

hence

\[
|\lambda_j^+(0)| \leq 1 - \frac{1}{j}.
\]

Consequently,

\[
\prod_{k=j_1}^j |\lambda_k^+(0)| \leq \prod_{k=j_1}^j \left( 1 - \frac{1}{k} \right) = \frac{j_1 - 1}{j},
\]

that is (9.5) is not satisfied and so the operator \(A\) is not self-adjoint.

The reasoning in the case (ii) is analogous. Namely, there is \(j_1 \geq j_0\) such that for all \(j \geq j_1\),

\[
j \sqrt{\frac{\mathcal{K}_{N-1}}{\gamma(j+1)N-1}} \left( \frac{\text{tr} R_j(0) + \sqrt{\text{discr} R_j(0)}}{2} \right) > 1,
\]

hence

\[
|\lambda_j^+(0)| \geq 1 + \frac{1}{j},
\]

and so

\[
\prod_{k=j_1}^j |\lambda_k^+(0)| \geq \prod_{k=j_1}^j \left( 1 + \frac{1}{k} \right) = \frac{j + 1}{j_1}.
\]

Therefore, (9.5) is satisfied and the operator \(A\) is self-adjoint. \(\Box\)

**Remark 9.3** If in Theorem 9.2 one has \(\Lambda_- = \emptyset\) and \(\Lambda_+ \neq \emptyset\), then \(A\) is self-adjoint if and only if (9.5) holds true. Let us emphasize that we cannot treat the case \(\Lambda_- = \Lambda_+ = \emptyset\), that is \(\tau \equiv 0\).

### 10 The \(\ell^1\)-Type Perturbations

In this section we show how to get the main results of the paper in the presence of certain size \(\ell^1\) perturbations. Let \((\tilde{a}_n : n \in \mathbb{N}_0)\) and \((\tilde{b}_n : n \in \mathbb{N}_0)\) be Jacobi parameters.
satisfying
\[ \tilde{a}_n = a_n(1 + \xi_n), \quad \tilde{b}_n = b_n(1 + \zeta_n) \]

where \((a_n : n \in \mathbb{N}_0)\) and \((b_n : n \in \mathbb{N}_0)\) are \(\gamma\)-tempered \(N\)-periodically modulated Jacobi parameters such that \(\mathfrak{X}_0(0)\) is a non-trivial parabolic element, and \((\xi_n : n \in \mathbb{N}_0)\) and \((\zeta_n : n \in \mathbb{N}_0)\) are certain real sequences satisfying
\[ \sum_{k=1}^{\infty} \sqrt{\gamma_n} (|\xi_n| + |\zeta_n|) < \infty. \]

We follow the reasoning explained in [61, Section 9].

Fix a compact set \(K \subset \mathbb{R}\). Let us denote by \((\Delta_n)\) any sequence of \(2 \times 2\) matrices such that
\[ \sum_{n=0}^{\infty} \sup_{K} \| \Delta_n \| < \infty. \]

We notice that
\[ \tilde{B}_n(x) = B_n(x) + \frac{\sqrt{\gamma_n}}{a_n} \Delta_n(x) \quad (10.1) \]

where
\[ \tilde{B}_0(x) = \begin{pmatrix} 0 & 1 \\ - \frac{1}{\tilde{a}_0} & \frac{x - \tilde{a}_0}{\tilde{a}_0} \end{pmatrix}, \]
\[ \tilde{B}_n(x) = \begin{pmatrix} 0 & 1 \\ - \frac{\tilde{a}_{n-1}}{\tilde{a}_n} & \frac{x - \tilde{a}_n}{\tilde{a}_n} \end{pmatrix}, \quad n \geq 1. \]

Moreover, for
\[ \tilde{X}_n = \tilde{B}_{n+N-1} \tilde{B}_{n+n-2} \ldots \tilde{B}_n \]

we have
\[ \tilde{X}_n - X_n = \sum_{k=n}^{n+N-1} \frac{\sqrt{\gamma_n}}{a_n} \tilde{B}_{n+N-1} \ldots \tilde{B}_{k+1} \Delta_k B_{k-1} \ldots B_n, \]

which together with
\[ \sup_{n \in \mathbb{N}_0} \sup_{x \in K} (\| B_n(x) \| + \| \tilde{B}_n(x) \|) < \infty, \]

implies that
\[ \tilde{X}_n = X_n + \frac{\sqrt{\gamma_n}}{a_n} \Delta_n. \quad (10.2) \]
Suppose that \( K \subset \Lambda_+ \). Then, by Theorem 3.2,
\[
Z_{j+1}^{-1} \tilde{X}_{jN+i} Z_j = \varepsilon \left( \text{Id} + \frac{\alpha_{i-1}}{\sqrt{Y_{(j+1)N+i-1}}} R_j \right) + \frac{\sqrt{Y_{jN+i}}}{a_{jN+i}} Z_{j+1}^{-1} \Delta_{jN+i} Z_j.
\]

Since there is \( c > 0 \) such that for all \( j \in \mathbb{N} \),
\[
\sup_K \| Z_{j+1}^{-1} \| \leq c \sqrt{Y_{jN+i}}, \quad \text{and} \quad \sup_K \| Z_j \| \leq c,
\]
by setting
\
V_j = \varepsilon \frac{\sqrt{Y_{jN+i}}}{a_{jN+i}} Z_{j+1}^{-1} \Delta_{jN+i} Z_j
\
we get
\[
Z_{j+1}^{-1} \tilde{X}_{jN+i} Z_j = \varepsilon \left( \text{Id} + \frac{\alpha_{i-1}}{\sqrt{Y_{(j+1)N+i-1}}} R_j + V_j \right)
\]
where \((R_j)\) is a sequence from \( \mathcal{D}_1(K, \text{Mat}(2, \mathbb{R})) \) convergent uniformly on \( K \) to \( \mathcal{R}_i \), and
\[
\sum_{j=1}^{\infty} \sup_K \| V_j \| < \infty. \tag{10.3}
\]

If \( (\sqrt{Y_n}) \) is sublinear and \( (\sup_K \| \Delta_n \|) \) belongs to \( \ell^1 \), for each subsequence there is a further subsequence \( (L_j : j \in \mathbb{N}_0) \) such that
\[
\sup_K \| \Delta_{L_j} \| \leq c \frac{1}{\sqrt{Y_{L_j+N-1}}}. \]
Consequently, we can find subsequence \( L_j \equiv i \mod N \) such that
\[
\sup_K \| V_{L_j} \| \leq c \frac{\sqrt{Y_{L_j+N-1}}}{a_{L_j+N-1}}. \]
Since [60, Theorem 4.4] allows perturbation satisfying (10.3) we can repeat the proof of Theorem 4.1 to get the following statement.

**Theorem 10.1** Let \( N \) be a positive integer. Let \((\gamma_n : n \in \mathbb{N})\) be a sequence of positive numbers tending to infinity and satisfying (1.2) and (1.3). Let \( \tilde{A} \) be the Jacobi operator associated with Jacobi parameters \((\tilde{a}_n : n \in \mathbb{N}_0)\) and \((\tilde{b}_n : n \in \mathbb{N}_0)\) such that
\[
\tilde{a}_n = a_n (1 + \xi_n), \quad \tilde{b}_n = b_n (1 + \zeta_n),
\]
where \((a_n : n \in \mathbb{N}_0)\) and \((b_n : n \in \mathbb{N}_0)\) are \(\gamma\)-tempered \(N\)-periodically modulated Jacobi parameters such that \(\tilde{X}_0(0)\) is a non-trivial parabolic element. Suppose that (1.4) holds true with \(\varepsilon = \text{sign}(\text{tr} \tilde{X}_0(0))\). If

\[
\sum_{n=0}^{\infty} \sqrt{\gamma_n} (|\xi_n| + |\zeta_n|) < \infty
\]

for certain real sequences \((\xi_n : n \in \mathbb{N}_0)\) and \((\zeta_n : n \in \mathbb{N}_0)\), then

\[
\sigma_{\text{ess}}(\tilde{A}) \cap \Lambda_+ = \emptyset.
\]

Next, let us consider a compact set \(K \subset \Lambda_+ - \Lambda_-\). By Theorem 7.1 and Claim 5.1, there is \(c > 0\) such that for all \(n \in \mathbb{N}_0\),

\[
\sup_K \|B_n B_{n-1} \ldots B_0\| \leq c \gamma_n^{1/4} a_n^{-1/2}, \tag{10.4}
\]

and since \(\det B_n = \frac{a_{n-1}}{a_n}\), we get

\[
\sup_K \|B_n B_{n-1} \ldots B_0\|^{-1} \leq c \gamma_n^{1/4} a_n^{1/2}. \tag{10.5}
\]

Moreover, by (10.1)

\[
\tilde{B}_n \ldots \tilde{B}_1 \tilde{B}_0 = \tilde{B}_n \ldots \tilde{B}_1 B_0 \left( \text{Id} + \gamma_0^{1/2} a_0^{-1} B_0^{-1} \Delta_0 \right)
\]

\[
= \tilde{B}_n \ldots \tilde{B}_2 B_1 B_0
\]

\[
\times \left( \text{Id} + \gamma_1^{1/2} a_1^{-1} (B_1 B_0)^{-1} \Delta_1 B_0 \right) \left( \text{Id} + \gamma_0^{1/2} a_0^{-1} B_0^{-1} \Delta_0 \right)
\]

\[
= B_n \ldots B_1 B_0
\]

\[
\times \prod_{j=0}^{n} \left( \text{Id} + \gamma_j^{1/2} a_j^{-1} (B_j \ldots B_1 B_0)^{-1} \Delta_j (B_{j-1} \ldots B_1 B_0) \right)
\]

thus by (10.4) and (10.5)

\[
\|\tilde{B}_n \ldots \tilde{B}_1 \tilde{B}_0\| \leq \|B_n \ldots B_1 B_0\| \prod_{j=0}^{n} \left( 1 + \gamma_j^{1/2} a_j^{-1} \|(B_j \ldots B_1 B_0)^{-1}\| \right)
\]

\[
\times \|B_{j-1} \ldots B_1 B_0\| \cdot \|\Delta_j\|
\]

\[
\leq \|B_n \ldots B_1 B_0\| \prod_{j=0}^{n} \left( 1 + c \gamma_j^{3/4} \gamma_{j-1}^{1/4} a_j^{-3/2} a_{j-1}^{1/2} \|\Delta_j\| \right)
\]

\[
\leq \|B_n \ldots B_1 B_0\| \exp \left( c \sum_{j=0}^{n} \|\Delta_j\| \right).
\]
Hence,
\[
\sup_K \| \tilde{B}_n \ldots \tilde{B}_1 \tilde{B}_0 \| \leq c \gamma_n^{1/4} a_n^{-1/2}.
\tag{10.6}
\]
Next, let us introduce the following sequence of matrices
\[
M_j = (B_j B_{j-1} \ldots B_0)^{-1} (\tilde{B}_j \tilde{B}_{j-1} \ldots \tilde{B}_0).
\tag{10.7}
\]
Since
\[
M_{j+1} - M_j = (B_{j+1} B_j \ldots B_0)^{-1} (\tilde{B}_{j+1} - B_{j+1}) (\tilde{B}_j \tilde{B}_{j-1} \ldots \tilde{B}_0),
\]
by (10.1), (10.5) and (10.6), we obtain
\[
\sup_K \| M_{j+1} - M_j \| \leq c \sup_K \| \Delta_{j+1} \|.
\]
Therefore, the sequence of matrices \((M_j)\) converges uniformly on \(K\) to certain continuous mapping \(M\), and
\[
\sup_K \| M - M_j \| \leq c \sum_{k=j+1}^{\infty} \sup_K \| \Delta_k \|. 
\tag{10.8}
\]
Observe that for each \(x \in K\) the matrix \(M(x)\) is non-degenerate. Indeed, we have
\[
\det M(x) = \lim_{j \to \infty} \det M_j(x) = \lim_{j \to \infty} \frac{a_j}{\tilde{a}_j} = 1.
\]
Given \(\eta \in \mathbb{S}^1\), we set
\[
\eta_n = \frac{M_{n-1} \eta}{\| M_{n-1} \eta \|}.
\]
Let us denote by \((\tilde{a}_n(\eta, x) : n \in \mathbb{N}_0)\) generalized eigenvector associated to \(x \in \mathbb{R}\) and \(\eta \in \mathbb{S}^1\) and generated by \((\tilde{a}_n : n \in \mathbb{N}_0)\) and \((\tilde{b}_n : n \in \mathbb{N}_0)\). Notice that for all \(n \in \mathbb{N}\) and \(x \in K\), by (2.1) and (10.7), we have
\[
\tilde{u}_n(\eta_n(x), x) = \frac{1}{\| M_{n-1}(x) \eta \|} \tilde{u}_n(\eta, x).
\tag{10.9}
\]
By Theorem 7.1 and Claim 5.1,
\[
\sup_{n \in \mathbb{N}} \sup_{x \in K} \frac{a_{n+N-1}}{\sqrt{\gamma_n+N-1}} \| \tilde{u}_n(\eta_n(x), x) \| < \infty.
\]
\[\textcircled{\text{Springer}}\]
which together with (10.9) implies that
\[
\sup_{n \in \mathbb{N}} \sup_{x \in K} \frac{\tilde{a}_{n+N-1}}{\sqrt{\gamma_{n+N-1}}} \left\| \tilde{u}_n(\eta, x) \right\|^2 < \infty .
\] (10.10)

**Theorem 10.2** Let \( N \) be a positive integer. Let \((\gamma_n : n \in \mathbb{N})\) be a sequence of positive numbers tending to infinity and satisfying (1.2) and (1.3). Let \( \tilde{A} \) be the Jacobi operator associated with Jacobi parameters \((\tilde{a}_n : n \in \mathbb{N}_0)\) and \((\tilde{b}_n : n \in \mathbb{N}_0)\) such that
\[
\tilde{a}_n = a_n (1 + \xi_n), \quad \tilde{b}_n = b_n (1 + \zeta_n),
\]
where \((a_n : n \in \mathbb{N}_0)\) and \((b_n : n \in \mathbb{N}_0)\) are \(\gamma\)-tempered \(N\)-periodically modulated Jacobi parameters such that \(X_0(0)\) is a non-trivial parabolic element. Suppose that (1.4) holds true with \(\varepsilon = \text{sign} (\text{tr} X_0(0))\). If
\[
\sum_{n=0}^{\infty} \sqrt{\gamma_n} (|\xi_n| + |\zeta_n|) < \infty
\]
for certain real sequences \((\xi_n : n \in \mathbb{N}_0)\) and \((\zeta_n : n \in \mathbb{N}_0)\), then for each compact interval \(K \subset \Lambda_-\), there are \(j_0 \in \mathbb{N}\) and a continuous function \(\tilde{\phi} : \mathbb{S}^1 \times K \to \mathbb{R}\) such that
\[
\sqrt{\frac{\tilde{a}_{jN+i-1}}{\sqrt{\gamma_{jN+i-1}}} \tilde{u}_{jN+i}(\eta, x)} = |\tilde{\phi}(\eta, x)| \sin \left( \sum_{k=j_0}^{j} \theta_k(x) + \arg \tilde{\phi}(\eta, x) \right) + E_j(\eta, x)
\] (10.11)
where \(\theta_k\) are given in (7.1) and
\[
\lim_{j \to \infty} \sup_{\eta \in \mathbb{S}^1} \sup_{x \in K} |E_j(\eta, x)| = 0.
\]
Moreover, \(\tilde{\phi}(\eta, x) = 0\) if and only if \(\|X_i(0)\|_2 = 0\).

**Proof** Fix a compact set \(K \subset \Lambda_-\). Since
\[
\left\| M_{jN+i-1}(x) \eta \right\| = \left\| M(x) \eta \right\| + o_K(1)
\]
and
\[
\varphi(\eta_{jN+i}(x), x) = \varphi(\eta(x), x) + o_K(1)
\]
where
\[
\eta(x) = \frac{M(x) \eta}{\left\| M(x) \eta \right\|},
\]
by (10.9) and Theorem 7.1, we obtain
\[
\frac{\tilde{u}_{jN+i}(\eta, x)}{\prod_{k=j_0}^{j-1} |\lambda_k(x)|} = \|M_{jN+i-1}(x)\eta\| \frac{|\varphi(\eta_{jN+i}(x), x)|}{\sqrt{\alpha_{i-1}}|\tau(x)|} \\
\times \sin \left( \sum_{k=j_0}^{j-1} \theta_k(x) + \arg \varphi(\eta_{jN+i}(x), x) \right) + o_K(1)
\]
\[
= \|M(x)\eta\| \\
\times \frac{|\varphi(\eta(x), x)|}{\sqrt{\alpha_{i-1}}|\tau(x)|} \sin \left( \sum_{k=j_0}^{j-1} \theta_k(x) + \arg \varphi(\eta(x), x) \right) + o_K(1).
\]
In view of Claim 5.2 we conclude the proof. \( \Box \)

Now by repeating the proofs of Theorems 8.2 and 9.1, the asymptotic formula (10.11) leads to the following statement.

**Theorem 10.3** Let \( N \) be a positive integer. Let \( (\gamma_n : n \in \mathbb{N}) \) be a sequence of positive numbers tending to infinity and satisfying (1.2) and (1.3). Let \( \tilde{A} \) be the Jacobi operator associated with Jacobi parameters \( (\tilde{a}_n : n \in \mathbb{N}_0) \) and \( (\tilde{b}_n : n \in \mathbb{N}_0) \) such that

\[
\tilde{a}_n = a_n(1 + \xi_n), \quad \tilde{b}_n = b_n(1 + \zeta_n),
\]

where \( (a_n : n \in \mathbb{N}_0) \) and \( (b_n : n \in \mathbb{N}_0) \) are \( \gamma \)-tempered \( \mathbb{N} \)-periodically modulated Jacobi parameters such that \( \mathcal{X}_0(0) \) is a non-trivial parabolic element. Suppose that (1.4) holds true with \( \varepsilon = \text{sign}(\text{tr} \mathcal{X}_0(0)) \). Assume that

\[
\sum_{n=0}^{\infty} \sqrt{\gamma_n} (|\xi_n| + |\zeta_n|) < \infty
\]

for certain real sequences \( (\xi_n : n \in \mathbb{N}_0) \) and \( (\zeta_n : n \in \mathbb{N}_0) \). Set

\[
\hat{\rho}_n = \sum_{j=0}^{n} \frac{\sqrt{\alpha_j \gamma_j}}{\tilde{a}_j}.
\]

If \( \Lambda_- \neq \emptyset \) then the Jacobi operator \( \tilde{A} \) associated to parameters \( \tilde{a} \) and \( \tilde{b} \) is self-adjoint if and only if \( \hat{\rho}_n \to \infty \). If it is the case then the limit

\[
\lim_{n \to \infty} \frac{1}{\hat{\rho}_n} K_n(x, x; \eta)
\]

exists locally uniformly with respect to \( (x, \eta) \in \Lambda_- \times S^1 \) and defines a continuous positive function.
11 Examples

11.1 Classes of Sequences

11.1.1 Kostyuchenko–Mirzoev

Let \( N \) be a positive integer and suppose that \((\alpha_n)\) and \((\beta_n)\) are \(N\)-periodic Jacobi parameters. We define
\[
\begin{align*}
a_n &= \alpha_n \hat{a}_n \left( 1 + \frac{f_n}{\delta_n} \right), \\
b_n &= \beta_n \hat{a}_n \left( 1 + \frac{g_n}{\delta_n} \right),
\end{align*}
\]
where \((f_n), (g_n)\) satisfy
\[
\lim_{n \to \infty} |f_n - \bar{f}_n| = 0, \quad \lim_{n \to \infty} |g_n - \bar{g}_n| = 0,
\]
for some \(N\)-periodic sequences \((\bar{f}_n), (\bar{g}_n)\), and \((\hat{a}_n), (\delta_n)\) are positive sequences such that
\[
\sum_{n=0}^{\infty} \frac{1}{\hat{a}_n} < \infty, \quad \lim_{n \to \infty} \delta_n = \infty, \quad \text{and}
\]
\[
\lim_{n \to \infty} \delta_n \left( 1 - \frac{\hat{a}_{n+1}}{\hat{a}_n} \right) = \kappa > 0.
\]
The sequences \((a_n)\) and \((b_n)\) satisfying (11.1)–(11.3) are called \(N\)-periodically modulated Kostyuchenko–Mirzoev Jacobi parameters. This class has been studied before, see e.g. [17, 48, 60, 66].

11.1.2 Symmetric Birth–Death Processes

In [29, Section 2] it is shown that generators of a birth–death process are unitarily equivalent to Jacobi matrices of the form
\[
\begin{align*}
a_n &= \sqrt{\lambda_n \mu_{n+1}}, \\
b_n &= -\lambda_n - \mu_n,
\end{align*}
\]
where \((\lambda_n : n \in \mathbb{N}_0)\) and \((\mu_{n+1} : n \in \mathbb{N}_0)\) are some positive sequences. When \(\lambda_n = \mu_{n+1}\), then we obtain a particularly simple class of Jacobi parameters
\[
b_n = -a_{n-1} - a_n.
\]
If (11.5) is satisfied, we shall refer to Jacobi parameters \((a_n)\) and \((b_n)\) as corresponding to a symmetric birth–death process. This class has been studied before, see e.g. [8–10, 42, 53]. In fact, in view of Proposition 11.1 below, instead of (11.4) it is sufficient to consider Jacobi parameters
\[
\begin{align*}
a_n &= \sqrt{\lambda_n \mu_{n+1}}, \\
b_n &= \lambda_n + \mu_n.
\end{align*}
\]
Proposition 11.1 Let \((a_n : n \in \mathbb{N}_0)\) and \((b_n : n \in \mathbb{N}_0)\) be sequences of positive and real numbers respectively. Let \(A\) and \(\hat{A}\) be Jacobi matrices with Jacobi parameters \((a_n : n \in \mathbb{N}_0)\), \((b_n : n \in \mathbb{N}_0)\) and \((a_n : n \in \mathbb{N}_0)\), \((-b_n : n \in \mathbb{N}_0)\), respectively. Then
\[
\hat{A} = U(-A)U^{-1},
\]
where \(U : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)\) is a unitary operator defined by \((Ux)_n = (-1)^nx_n\).

The proof of Proposition 11.1 is just a simple computation, see e.g. [9, Lemma 3.5] or [10, Proposition 3.5] for more details.

11.2 The General N

11.2.1 Kostyuchenko–Mirzoev’s Class

Remark 11.2 Let \(N\) be a positive integer and let \((\alpha_n)\) and \((\beta_n)\) be \(N\)-periodic Jacobi parameters such that \(X_0(0)\) is a non-trivial parabolic element. Consider the sequences \((a_n), (b_n)\) satisfying (11.1)–(11.3), where
\[
(\delta_n \frac{\hat{a}_n}{a_n}, (\frac{\delta_n}{1 + \frac{\delta_n}{a_n}}), (f_n), (g_n) \in \mathcal{D}_1^N,
\]
and
\[
(\delta_n - \delta_{n-1}, (\frac{1}{\sqrt{\delta_n}}) \in \mathcal{D}_1^N.
\]

Then for \(\gamma_n = \alpha_n\delta_n\), the hypotheses of Theorem 3.2 are satisfied. Moreover,
\[
\tau(x) \equiv N\kappa - \varepsilon \sum_{i=0}^{N-1} \frac{\beta_i}{\alpha_{i-1}} [X_i(0)]_{21}(\delta_i - \frac{\delta_i}{\alpha_i}).
\]

To see this, let us first observe that
\[
\frac{\gamma_n}{a_n} = \frac{\delta_n}{a_n} \frac{1}{1 + \frac{f_n}{\delta_n}};
\]
which belongs to \(\mathcal{D}_N^1\). Next, we write
\[
\frac{\alpha_{n-1}}{\alpha_n} - \frac{a_{n-1}}{a_n} = \frac{\alpha_{n-1}}{\alpha_n} \left(1 - \frac{\hat{a}_{n-1}}{a_n} \frac{f_{n-1}}{\delta_{n-1}} \frac{1}{1 + \frac{f_n}{\delta_n}}\right)
\]
\[
= \frac{\alpha_{n-1}}{\alpha_n \delta_n} \left(\delta_n \left(1 - \frac{\hat{a}_{n-1}}{a_n} \frac{f_{n-1}}{\delta_{n-1}} \frac{1}{1 + \frac{f_n}{\delta_n}}\right) + \hat{a}_{n-1} \frac{f_n}{\delta_n} \frac{1}{1 + \frac{f_n}{\delta_n}}\right).
\]
Hence,

\[ \left( \gamma_n \left( \frac{\alpha_{n-1}}{\alpha_n} - \frac{a_{n-1}}{a_n} \right) \right) \in D_1^N. \]

Moreover,

\[ \lim_{n \to \infty} \gamma_n \left( \frac{\alpha_{n-1}}{\alpha_n} - \frac{a_{n-1}}{a_n} \right) = \alpha_{i-1}(\kappa + \hat{f}_i - \hat{g}_{i-1}). \quad (11.9) \]

Analogously, we write

\[ \frac{\beta_n}{\alpha_n} \frac{b_n}{a_n} = \frac{\beta_n}{\alpha_n} \frac{1}{\delta_n} \left( 1 - \frac{\delta_n}{\delta_n} \right) = \frac{\beta_n}{\alpha_n} \frac{1}{\delta_n} \frac{1}{1 + \frac{\delta_n}{\delta_n}}, \]

thus

\[ \left( \gamma_n \left( \frac{\beta_n}{\alpha_n} - \frac{b_n}{a_n} \right) \right) \in D_1^N \]

and

\[ \lim_{n \to \infty} \gamma_n \left( \frac{\beta_n}{\alpha_n} - \frac{b_n}{a_n} \right) = \beta_i (\hat{f}_i - \hat{g}_i). \quad (11.10) \]

Next, we easily compute that

\[ \sqrt{\frac{\alpha_{n-1}}{\alpha_n}} \sqrt{\gamma_n} - \sqrt{\gamma_{n-1}} = \sqrt{\frac{\alpha_{n-1}}{\alpha_n} \frac{\delta_n - \delta_{n-1}}{\delta_n} \frac{1}{1 + \frac{\delta_n}{\delta_n}}}. \]

Consequently, all the hypotheses of Theorem 3.2 are satisfied. Moreover, by (11.9) and (11.10), we obtain

\[ s_n \equiv 0, \quad r_n \equiv 0 \]

and

\[ u_n = \alpha_{n-1}(\kappa + \hat{f}_n - \hat{f}_{n-1})(1 - \varepsilon[\mathcal{X}_n(0)_{11}]) - \varepsilon \beta_n (\hat{f}_n - \hat{g}_n)[\mathcal{X}_n(0)]_{21}. \]

To compute the value of \( t \), observe that by (11.7) the sequence \( (\delta_n - \delta_{n-1} : n \in \mathbb{N}) \) is bounded and by (11.3) \( \delta_n \to \infty \). Thus,

\[ \lim_{n \to \infty} \frac{\delta_{n-1}}{\delta_n} = \lim_{n \to \infty} \left( 1 - \frac{\delta_n - \delta_{n-1}}{\delta_n} \right) = 1. \]

Next, by (11.3)

\[ \lim_{n \to \infty} \frac{\hat{a}_{n-1}}{\hat{a}_n} = 1 - \lim_{n \to \infty} \frac{\delta_n (1 - \frac{\hat{a}_{n-1}}{\hat{a}_n})}{\delta_n} = 1 - \kappa \lim_{n \to \infty} \frac{1}{\delta_n} = 1. \]
This together with (11.8) implies that
\[ t = \lim_{n \to \infty} \frac{\delta_n}{\hat{a}_n}. \]
exists. If we had \( t > 0 \), then there would exist \( n_0 \in \mathbb{N} \) and a constant \( c > 0 \) such that for all \( n \geq n_0 \)
\[ \frac{1}{\hat{a}_n} \geq c \frac{1}{\delta_n}. \]
Consequently, by (11.3)
\[ \sum_{n=0}^{\infty} \frac{1}{\delta_n} < \infty. \quad (11.11) \]
On the other hand, we have
\[ \delta_n \leq \delta_0 + \sum_{k=0}^{n-1} |\delta_{k+1} - \delta_k|. \]
Thus by the boundedness of \((\delta_n - \delta_{n-1} : n \in \mathbb{N})\) we get that for some \( c' > 0 \) one has \( \delta_n \leq c'(n + 1) \). It leads to a contradiction with (11.11). Hence, \( t = 0 \), which easily gives the formula for \( \tau \).

11.2.2 Symmetric Birth–Death Class

**Lemma 11.3** Let \( N \) be a positive integer. Suppose that \((\tilde{\alpha}_n)\) is a \( 2N \) periodic sequence of positive numbers such that
\[ \tilde{\alpha}_0\tilde{\alpha}_2 \ldots \tilde{\alpha}_2N-2 = \tilde{\alpha}_1\tilde{\alpha}_3 \ldots \tilde{\alpha}_2N-1. \quad (11.12) \]
Set
\[ \alpha_n = \tilde{\alpha}_{2n+1}\tilde{\alpha}_{2n+2}, \quad \beta_n = \tilde{\alpha}_2^2 + \tilde{\alpha}_{2n+1}^2. \quad (11.13) \]
Then \( \text{tr} \mathcal{X}_0(0) = 2\varepsilon \) where \( \varepsilon = (-1)^N \). Moreover,
\[ \text{tr} \mathcal{X}_0'(0) = -\varepsilon \sum_{i=0}^{N-1} \frac{1}{\alpha_i} \sum_{k=0}^{N-1} \prod_{j=i}^{i+k-1} \left( \frac{\tilde{\alpha}_{2j}}{\tilde{\alpha}_{2j+1}} \right)^2 \quad (11.14) \]
and
\[ (1 - \varepsilon[\mathcal{X}_n(0)]_{11}) \frac{\alpha_n^{-1}}{\alpha_n} - \frac{\tilde{\alpha}_2^n}{\tilde{\alpha}_{2n+1}\tilde{\alpha}_{2n+2}} \varepsilon[\mathcal{X}_n(0)]_{21} \equiv 0. \quad (11.15) \]

**Proof** We start with the following Claim, which is inspired by [63, Lemma 2].

\[ \Box \text{ Springer} \]
Claim 11.4. Let $\ell \geq 0$ and let $\left(p_{n}^{[\ell]} : n \geq 0\right)$ be the sequence of orthogonal polynomials associated with recurrence coefficients $(\alpha_{n+\ell} : n \geq 0)$ and $(\beta_{n+\ell} : n \geq 0)$, where $(\alpha_{n} : n \geq 0)$ and $(\beta_{n} : n \geq 0)$ satisfy (11.13). Then

$$p_{n}^{[\ell]}(0) = \frac{\tilde{\alpha}_{2\ell}}{\tilde{\alpha}_{2\ell+2}} \sum_{k=0}^{n} \left(w_{k}^{[\ell]}\right)^{2}, \quad (11.16)$$

where

$$w_{k}^{[\ell]} = (-1)^{k} \prod_{j=\ell}^{\ell+k-1} \frac{\tilde{\alpha}_{2j}}{\tilde{\alpha}_{2j+1}}.$$

To see this we reason by induction over $n \in \mathbb{N}_{0}$. For $n = 0$ and $n = 1$ the formula (11.16) can be checked by direct computations. Next, let us observe that

$$- \tilde{\alpha}_{2\ell+2k-1} w_{k}^{[\ell]} = \tilde{\alpha}_{2\ell+2k-2} w_{k-1}^{[\ell]}, \quad k \geq 1. \quad (11.17)$$

By the recurrence relation we have

$$\alpha_{n+\ell} p_{n+1}^{[\ell]}(0) = -\beta_{n+\ell} p_{n}^{[\ell]}(0) - \alpha_{n+\ell-1} p_{n-1}^{[\ell]}(0), \quad n \geq 1.$$

Hence, by the induction hypothesis, (11.17) and (11.13) we obtain

$$\alpha_{n+\ell} p_{n+1}^{[\ell]}(0)$$

$$= -\beta_{n+\ell} \frac{\tilde{\alpha}_{2\ell}}{\tilde{\alpha}_{2\ell+2}} \sum_{k=0}^{n} \left(w_{k}^{[\ell]}\right)^{2} - \frac{\tilde{\alpha}_{2\ell}}{\tilde{\alpha}_{2\ell+2}} \sum_{k=0}^{n-1} \left(w_{k}^{[\ell]}\right)^{2}$$

$$= \frac{\tilde{\alpha}_{2\ell}}{w_{n+1}^{[\ell]}} \left(\frac{\tilde{\alpha}_{2\ell+2n}^{2} + \tilde{\alpha}_{2\ell+2n+1}^{2}}{\tilde{\alpha}_{2\ell+2n+1}} \sum_{k=0}^{n} \left(w_{k}^{[\ell]}\right)^{2} - \frac{\tilde{\alpha}_{2\ell+2n}^{2}}{\tilde{\alpha}_{2\ell+2n+1}} \sum_{k=0}^{n-1} \left(w_{k}^{[\ell]}\right)^{2}\right)$$

$$= \frac{\tilde{\alpha}_{2\ell}}{w_{n+1}^{[\ell]}} \left(\tilde{\alpha}_{2\ell+2n}^{2} w_{n+1}^{[\ell]} + \tilde{\alpha}_{2\ell+2n+1}^{2} \sum_{k=0}^{n} \left(w_{k}^{[\ell]}\right)^{2}\right)$$

and the conclusion follows by once again using (11.13).

Next, in view of [55, Proposition 3] we have

$$X_{n}(0) = \left(\frac{\alpha_{n-1}}{\alpha_{n}} p_{N-1}^{[n+1]}(0) \quad p_{N}^{[n]}(0) \quad p_{N-1}^{[n]}(0) \quad p_{N}^{[n+2]}(0)\right). \quad (11.18)$$
Thus, by (11.16),
\begin{align*}
\text{tr } \mathcal{X}_0(0) &= p_N^{[0]}(0) - \frac{\alpha_{N-1}}{\alpha_N} p_{N-2}^{[1]}(0) \\
&= \frac{1}{w_N^{[0]}} \sum_{k=0}^{N} \left( w_k^{[0]} \right)^2 - \frac{\alpha_{N-1}}{\alpha_N} \tilde{\alpha}_2 \sum_{k=0}^{N-2} \left( w_k^{[1]} \right)^2.
\end{align*}

Observe that
\begin{equation}
\label{eq:w_k1}
w_k^{[1]} = -\frac{\tilde{\alpha}_1}{\tilde{\alpha}_0} w_{k+1}^{[0]}, \quad k \geq 0.
\end{equation}

Therefore, by combining (11.17), (11.19), (11.13) and using 2\(N\)-periodicity of \((\tilde{\alpha}_n)\) we arrive at
\begin{align*}
\text{tr } \mathcal{X}_0(0) &= \frac{1}{w_N^{[0]}} \left( \sum_{k=0}^{N} \left( w_k^{[0]} \right)^2 - \sum_{k=1}^{N-1} \left( w_k^{[0]} \right)^2 \right) \\
&= \frac{1}{w_N^{[0]}} \left( 1 + \left( w_N^{[0]} \right)^2 \right) = 2(-1)^N
\end{align*}

where the last equality follows from (11.16) and (11.12).

In view of [61, Proposition 2.1], (11.18) gives
\begin{align*}
\text{tr } \mathcal{X}'_0(0) &= \sum_{i=0}^{N-1} \frac{p_{N-1}^{[i+1]}(0)}{\alpha_i} \\
&= -\varepsilon \sum_{i=0}^{N-1} \frac{\tilde{\alpha}_2}{\alpha_i} \tilde{\alpha}_{2i-1} \sum_{k=0}^{N-1} \left( w_k^{[1]} \right)^2
\end{align*}

where in the last equality we have used (11.16) and (11.17). Now, (11.14) is an easy consequence of (11.16). Since \(|\text{tr } \mathcal{X}_0(0)| = 2\) and \(\text{tr } \mathcal{X}'_0(0) \neq 0\), Proposition 2.1 implies that \(\mathcal{X}_0(0)\) is a non-trivial parabolic element.

It remains to prove (11.15). Observe that by (11.16), (11.17), (11.13) and 2\(N\)-periodicity of \((\tilde{\alpha}_n)\) we get
\begin{align*}
\frac{\tilde{\alpha}_2^{2n}}{\tilde{\alpha}_{2n+1}} p_{n+1}^{[n+1]}(0) + \frac{\alpha_{n-1}}{\alpha_n} p_{n+1}^{[n]}(0) &= \frac{\tilde{\alpha}_{2n}}{\tilde{\alpha}_{2n+1} w_{N-1}^{[n+1]}} \sum_{k=0}^{N-1} \left( w_k^{[n+1]} \right)^2 - \sum_{k=0}^{N-2} \left( w_k^{[n+1]} \right)^2 \\
&= \frac{\tilde{\alpha}_{2n}}{\tilde{\alpha}_{2n+1} w_{N-1}^{[n+1]}} = -w_{N}^{[n+1]} = -\varepsilon.
\end{align*}
Hence, by (11.18),

\[
(1 - \varepsilon[\mathcal{X}_n(0)]_{11}) \frac{\alpha_{n-1}}{\alpha_n} - \frac{\tilde{\alpha}_{2n}^2}{\alpha_{2n+1} \alpha_{2n+2}} \varepsilon[\mathcal{X}_n(0)]_{21} = \frac{\alpha_{n-1}}{\alpha_n} (1 - \varepsilon^2) = 0
\]

which completes the proof. \(\square\)

**Remark 11.5** Let \(N\) be a positive integer. Let \((\alpha_n)\) be a positive \(N\)-periodic sequence. Suppose that \((\gamma_n)\) is a positive sequence satisfying

\[
\left(\frac{\sqrt{\alpha_{n-1}}}{\alpha_n} \sqrt{\gamma_n} - \sqrt{\gamma_{n-1}}\right), \left(\frac{1}{\alpha_n} \sqrt{\gamma_n} - \frac{1}{\gamma_n} - 1\right) \in \mathcal{D}^N_1,
\]

and

\[
\lim_{n \to \infty} \left(\sqrt{\gamma_{n+N}} - \sqrt{\gamma_n}\right) = 0, \quad \lim_{n \to \infty} \gamma_n = \infty.
\]

Let us set

\[
a_n = \gamma_n, \quad b_n = \gamma_{n-1} + \gamma_n.
\]

Then \(\beta_n = \alpha_{n-1} + \alpha_n\) and the hypotheses of Theorem 3.2 are satisfied with

\[
\tau_i = s_i = 2\sqrt{\alpha_{i-1}} \lim_{n \to \infty} \left(\frac{\sqrt{\alpha_{n-1}}}{\alpha_n} \sqrt{\gamma_n} - \sqrt{\gamma_{n-1}}\right), \quad (11.20)
\]

and

\[
t = 1, \quad u_i \equiv 0. \quad (11.21)
\]

In particular,

\[
\tau(x) = -\left(\sum_{i=0}^{N-1} \frac{\alpha_{i-1}}{\alpha_i}\right) \left(\sum_{i=0}^{N-1} \frac{1}{\alpha_i}\right) \cdot x. \quad (11.22)
\]

To see this, let us define

\[
\tilde{\alpha}_{2n+1} = \tilde{\alpha}_{2n+2} = \sqrt{\alpha_n}, \quad n \in \mathbb{Z}. \quad (11.23)
\]

By Lemma 11.3, \(\mathcal{X}_0(0)\) is a non-trivial parabolic element with \(\text{tr} \mathcal{X}_0(0) = 2\varepsilon\) for \(\varepsilon = (-1)^N\). Next, we have

\[
\frac{\beta_n}{\alpha_n} = \frac{b_n}{a_n} = \frac{\alpha_{n-1} + \alpha_n}{\alpha_n} - \frac{a_{n-1} + a_n}{\alpha_n}
\]

\[
= \left(\frac{\alpha_{n-1}}{\alpha_n} - \frac{a_{n-1}}{a_n}\right). \quad (11.24)
\]
Hence, by (11.15) and (11.23),

\[
\left( \frac{\alpha_{n-1}}{\alpha_n} - \frac{a_{n-1}}{a_n} \right) \left( 1 - \varepsilon [X_n(0)]_{11} \right) - \varepsilon \left( \frac{\beta_n}{\alpha_n} - \frac{b_n}{a_n} \right) [X_n(0)]_{21}
\]

\[
= \left( \frac{\alpha_{n-1}}{\alpha_n} - \frac{a_{n-1}}{a_n} \right) \left( 1 - \varepsilon [X_n(0)]_{11} - \varepsilon [X_n(0)]_{21} \right) \equiv 0.
\]

In particular, the left-hand side belongs to \( D_N^1 \) and \( u \equiv 0 \).

Let us observe that

\[
\frac{\alpha_{n-1}}{\alpha_n} - \frac{a_{n-1}}{a_n} = \left( \frac{\alpha_{n-1}}{\alpha_n} - \frac{a_{n-1}}{a_n} \right) \left( \frac{\alpha_{n-1}}{\alpha_n} + \frac{a_{n-1}}{a_n} \right) = \frac{1}{\sqrt{a_n}} \left( \sqrt{\frac{\alpha_{n-1}}{\alpha_n} - \sqrt{a_{n-1}}} \right) \left( \sqrt{\frac{\alpha_{n-1}}{\alpha_n} + \sqrt{a_{n-1}}} \right).
\]

Hence,

\[
\sqrt{\alpha_n a_n} \left( \frac{\alpha_{n-1}}{\alpha_n} - \frac{a_{n-1}}{a_n} \right) = \sqrt{\alpha_n} \left( \sqrt{\frac{\alpha_{n-1}}{\alpha_n} - \sqrt{a_{n-1}}} \right) \left( \sqrt{\frac{\alpha_{n-1}}{\alpha_n} + \sqrt{a_{n-1}}} \right). \tag{11.25}
\]

In particular, the left-hand side of (11.25) belongs to \( D_N^1 \). Moreover, we get

\[
\delta_i = 2\sqrt{\alpha_{i-1}} \lim_{n \to \infty} \left( \frac{\alpha_{n-1}}{\alpha_n} \sqrt{\gamma_n} - \sqrt{\gamma_{n-1}} \right),
\]

which together with (11.24) gives (11.20). Finally, by (11.23) and (11.14) we get

\[
\text{tr} \ X'_0(0) = -\varepsilon \sum_{i=0}^{N-1} \frac{1}{\alpha_i} \sum_{k=0}^{N-1} \frac{\alpha_{i-1}}{\alpha_{i+k-1}} = -\varepsilon \sum_{i=0}^{N-1} \frac{\alpha_{i-1}}{\alpha_i} \sum_{k=0}^{N-1} \frac{1}{\alpha_k}.
\]

By Proposition 2.2 we obtain \( S = 0 \). Hence, in view of (2.21) the formula (11.22) follows.

11.3 \( N = 1 \)

In this section we specify our results to \( N = 1 \).

**Remark 11.6** Suppose that for some \( \varepsilon \in \{-1, 1\} \)

\[
\left( \sqrt{\gamma_n} \left( \frac{a_{n-1}}{a_n} - 1 \right) \right), \left( \sqrt{\gamma_n} \left( \frac{b_n}{a_n} + 2\varepsilon \right) \right), \left( \gamma_n \left( 1 + \frac{a_{n-1}}{a_n} + \frac{b_n}{a_n} \varepsilon \right) \right), \left( \frac{\gamma_n}{\alpha_n} \right) \in D_1,
\]

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where \((\gamma_n)\) is a positive sequence satisfying
\[
\left( \sqrt{\gamma_n} - \sqrt{\gamma_{n-1}} \right), \left( \frac{1}{\sqrt{\gamma_n}} \right) \in D_1,
\]
and
\[
\lim_{n \to \infty} \left( \sqrt{\gamma_n} - \sqrt{\gamma_{n-1}} \right) = 0, \quad \lim_{n \to \infty} \gamma_n = \infty.
\]
Let

\[
\begin{align*}
 s &= \lim_{n \to \infty} \sqrt{\gamma_n} \left( \frac{a_{n-1}}{a_n} - 1 \right), \quad \tau = \lim_{n \to \infty} \sqrt{\gamma_n} \left( \frac{b_n}{a_n} + 2\varepsilon \right), \quad t = \lim_{n \to \infty} \frac{\gamma_n}{a_n} \\
 u &= \lim_{n \to \infty} \gamma_n \left( 1 + \frac{a_{n-1}}{a_n} + \varepsilon \frac{b_n}{a_n} \right).
\end{align*}
\]

Then
\[
\tau(x) = xt\varepsilon - u + \frac{1}{4}s^2.
\]

In particular, if \(\tau(x)\) is not identically zero, then the hypotheses of Theorems 3.1 and 3.2 are satisfied.

**Remark 11.7** Suppose that sequences \((\tilde{\xi}_n)\) and \((\tilde{\zeta}_n)\) satisfy
\[
(\sqrt{\delta_n \tilde{\xi}_n}), (\sqrt{\delta_n \tilde{\zeta}_n}) \in \ell^1.
\]

Then
\[
\begin{align*}
(1 + \frac{f_n}{\delta_n})(1 + \tilde{\xi}_n) &= 1 + \frac{f_n}{\delta_n} + \xi_n, \quad \text{and} \\
(1 + \frac{g_n}{\delta_n})(1 + \tilde{\zeta}_n) &= 1 + \frac{g_n}{\delta_n} + \zeta_n,
\end{align*}
\]

where \((\sqrt{\delta_n \xi_n}), (\sqrt{\delta_n \zeta_n}) \in \ell^1\). Thus \(\ell^1\)-type perturbations of (11.1) cover the Jacobi parameters of the form
\[
\tilde{\alpha}_n = \alpha_n \left( 1 + \frac{f_n}{\delta_n} + \xi_n \right), \quad \tilde{b}_n = -2\varepsilon\alpha_n \left( 1 + \frac{g_n}{\delta_n} + \zeta_n \right),
\]

where \((\sqrt{\delta_n \xi_n}), (\sqrt{\delta_n \zeta_n}) \in \ell^1\).

**Example 11.8** The case when \(\hat{\alpha}_n = (n + 1)^\kappa\) for some \(\kappa > \frac{3}{2}\) and \(\delta_n = n + 1\) was considered in [66] when \(\tau(x) \neq 0\). More specifically, it was assumed that
$$a_n = (n + 1)^\kappa \left(1 + \frac{f}{n + 1} + O(n^{-2})\right), \quad b_n = -2\varepsilon (n + 1)^\kappa \left(1 + \frac{g}{n + 1} + O(n^{-2})\right).$$

In view of Remarks 11.7 and 11.2 the above Jacobi parameters are covered by the present article. Let us emphasize that we can take any $\kappa > 1$ and more general perturbations $(\xi_n)$ and $(\zeta_n)$.

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