ON CHOW WEIGHT STRUCTURES FOR cdh-MOTIVES WITH INTEGRAL COEFFICIENTS

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Abstract. The main goal of this paper is to define a certain Chow weight structure $w_{\text{Chow}}$ on the category $\mathcal{DM}_c(S)$ of (constructible) cdh-motives over an equicharacteristic scheme $S$. In contrast to the previous papers of D. Hébert and the first author on weights for relative motives (with rational coefficients), we can achieve our goal for motives with integral coefficients (if char $S = 0$; if char $S = p > 0$ then we consider motives with $\mathbb{Z}[\frac{1}{p}]$-coefficients). We prove that the properties of the Chow weight structures that were previously established for $\mathbb{Q}$-linear motives can be carried over to this "integral" context (and we generalize some of them using certain new methods). In this paper we mostly study the version of $w_{\text{Chow}}$ defined via "gluing from strata"; this enables us to define Chow weight structures for a wide class of base schemes.

As a consequence, we certainly obtain certain (Chow)-weight spectral sequences and filtrations on any (co)homology of motives.

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Introduction

In this paper we construct certain "weights" for $R$-linear motives over a scheme $S$. Here $S$ is an excellent finite dimensional Noetherian scheme of characteristic $p$ (that could be 0) and $R$ is a unital commutative associative coefficient ring; in the case $p > 0$ we require $p$ to be invertible in $R$. These weights are compatible with Deligne’s weights for constructible complexes of étale sheaves (see Remark 3.2.2(4) below).

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Now we explain this in more detail. Deligne’s weights for étale sheaves and mixed Hodge structures (and for the corresponding derived categories) are very important for modern algebraic geometry. So, lifting these weights to motives is an important part of the so-called Beilinson’s (motivic) dream. The 'classical' approach (due to Beilinson) to do this is to define a filtration on motives that would split Chow motives into their components corresponding to single (co)homology groups (i.e., it should yield the so-called Chow-Kunneth decompositions). Since the existence of Chow-Kunneth decompositions is very much conjectural, it is no wonder that this approach was not really successful (up to now); besides, it cannot work for $R$-linear motives if $R$ is not a $\mathbb{Q}$-algebra.

In [Bon10] an alternative method for defining weights for motives was proposed and successfully implemented: the so-called Chow weight structure on the triangulated category of Voevodsky motives $\mathcal{DM}_{gm}$ (with integral coefficients) over a characteristic 0 field was defined; the heart of this weight structure is the ”classical” category of Chow motives. Now, arbitrary weight structures yield functorial weight filtrations and weight spectral sequences for any (co)homological functor (from $\mathcal{DM}_{gm}$). These weight filtrations and spectral sequences generalize Deligne’s ones; note that they are also well-defined for any (co)homology with integral coefficients (this is a vast extension of the earlier results of [GiS96] on cohomology with compact support).

The next paper in this direction was [Bon11], where the Chow weight structure for $\mathbb{Z}[\frac{1}{p}]$-linear motives over a perfect field of characteristic $p$ was defined. At the same time, the theory of Voevodsky triangulated motivic categories over any ”more or less general” base scheme $S$ was (in [CiD09]) developed to the stage that the Chow weight structure for Beilinson motives over $S$ (i.e., for $S$-motives with rational coefficients) could be defined; the latter was done independently in [Heb11] and in [Bon14a] (see Remark 2.3.3(2) below).

The main goal of the current paper is to define the Chow weight structure on $\mathbb{Z}[\frac{1}{p}]$-linear motives (and more generally, on $R$-linear motives for any $\mathbb{Z}[\frac{1}{p}]$-algebra $R$) over any excellent finite dimensional Noetherian base scheme $S$ of characteristic $p$ (here we set $\mathbb{Z}[\frac{1}{p}] = \mathbb{Z}$ if $p = 0$, and consider $cdh$-motives with $R$-coefficients that were denoted by $\mathcal{DM}_{cdh}(S, R)$ in [CiD14]). To achieve this we use the ”gluing construction” of the Chow weight structure; this construction was described (for $\mathbb{Q}$-linear motives) in [Bon14a] §2.3 (whereas the method was first proposed in [Bon10 §8.2]). This required us to develop some new methods for studying morphisms between relative motives (in §1.3). We also note that all the properties and applications of the Chow weight structure described in [Bon14a] carry over to our ”integral” context. We apply some new arguments for studying the weight-exactness of motivic functors (in §2.2; following [Bon14b], we use Borel-Moore motives); they allow us to extend the corresponding results to not necessarily quasi-projective morphisms.

Thus this paper gives convenient tools for studying ”integral” (and torsion) weight phenomena for (equicharacteristic) schemes and motives. In particular, we obtain functorial ”Chow-weight” filtrations and spectral sequences (see §3.2). Note still that we are able to prove that ”explicit Chow motives” over $S$ yield a weight structure for $S$-motives (similarly to [Heb11] and [Bon14a §2.1]) only if $S$ is a ”smooth limit” of schemes of finite type over some field (see §2.3 and [Jin15]).

Lastly, we note that one can (certainly) consider motivic categories corresponding to Grothendieck topologies distinct from the $cdh$ one. In particular, a (not really ”successful”) attempt was made in [Bon13] to construct certain Chow weight structures on relative Nisnevich motivic categories (using their properties established in [CiD09]). Note yet that the Nisnevich motives are isomorphic to the $cdh$-ones over regular bases (see
[CiD14 Corollary 5.9)]; this is also expected to be true in general. On the other hand, though (\(R\)-linear) étale motivic categories (that were thoroughly studied in [CiD13]) enjoy several “nice” properties, there is no chance to define \(w_{\text{Chow}}\) for them unless \(\mathbb{Q} \subset R\), whereas in the latter case the relative motivic categories mentioned “do not depend on the choice of a topology” (if we compare cdh, Nisnevich, étale, and \(h\)-motives).

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1. Preliminaries

This section is mostly a recollection of basics on (relative \(cdh\))-motives and weight structures; yet the results of [13] and the methods of their proofs are (more or less) new.

1.1. Notation.

- For categories \(C, D\) we write \(D \subset C\) if \(D\) is a full subcategory of \(C\).
- For a category \(C\), and \(X, Y \in \text{Obj}\, C\), we denote by \(C(X, Y)\) the set of \(C\)-morphisms from \(X\) to \(Y\).
- An additive subcategory \(D\) of \(C\) is said to be Karoubi-closed in it if it contains all retracts of its objects in \(C\). The full subcategory of \(C\) whose objects are all retracts of objects of \(D\) (in \(C\)) will be called the Karoubi-closure of \(D\) in \(C\).
- \(\mathcal{C}\) will always denote some triangulated category; usually it will be endowed with a weight structure \(w\) (see Definition 1.4.1 below).
- For a set of objects \(C_i \in \text{Obj}\, \mathcal{C}\), \(i \in I\), we will denote by \(\langle C_i \rangle\) the smallest strictly full triangulated subcategory of \(\mathcal{C}\) containing all \(C_i\); for \(D \subset \mathcal{C}\) we will write \(\langle D \rangle\) instead of \(\text{Obj}\, D\). We will call the Karoubi-closure of \(\langle C_i \rangle\) in \(\mathcal{C}\) the triangulated category generated by \(C_i\) (recall that it is indeed triangulated).
- For \(X, Y \in \text{Obj}\, \mathcal{C}\) we will write \(X \perp Y\) if \(\mathcal{C}(X, Y) = \{0\}\). For \(D, E \subset \text{Obj}\, \mathcal{C}\) we will write \(D \perp E\) if \(X \perp Y\) for all \(X \in D, \, Y \in E\). For \(D \subset \mathcal{C}\) we will denote by \(D^\perp\) the class

\[
\{ Y \in \text{Obj}\, \mathcal{C} : X \perp Y \forall X \in D \}.
\]

Dually, \(D^\perp\) is the class \(\{ Y \in \text{Obj}\, \mathcal{C} : Y \perp X \forall X \in D \}\).
- We will say that some \(C_i \in \text{Obj}\, \mathcal{C}\), \(i \in I\), weakly generate \(\mathcal{C}\) if for \(X \in \text{Obj}\, \mathcal{C}\) we have: \(\mathcal{C}(C_i[j], X) = \{0\}\) \(\forall i \in I, \, j \in \mathbb{Z} \implies X = 0\) (i.e., if \(\{C_i[j] : j \in \mathbb{Z}\}^\perp\) contains only zero objects).
- \(M \in \text{Obj}\, \mathcal{C}\) will be called compact if the functor \(\mathcal{C}(M, -)\) commutes with all small coproducts that exist in \(\mathcal{C}\) (we will only consider compact objects in those categories that are closed with respect to arbitrary small coproducts).
- \(D \subset \text{Obj}\, \mathcal{C}\) will be called extension-stable if \(0 \in D\) and for any distinguished triangle \(A \to B \to C\) in \(\mathcal{C}\) we have: \(A, C \in D \implies B \in D\).
- We will call the smallest Karoubi-closed extension-stable subclass of \(\text{Obj}\, \mathcal{C}\) containing \(D\) the envelope of \(D\).
- We will sometimes need certain stratifications of a scheme \(S\). Recall that a stratification \(\alpha\) is a presentation of \(S\) as \(\bigcup S^\alpha_\ell\), where \(S^\alpha_\ell\), \(1 \leq \ell \leq n\), are pairwise disjoint locally closed subschemes of \(S\). Omitting \(\alpha\), we will denote by \(j_\ell : S^\alpha_\ell \to S\) the corresponding immersions. We do not demand the closure of each \(S^\alpha_\ell\) to be the union of strata (though we could do this); we will only assume that each \(S^\alpha_\ell\) is open in \(\bigcup_{\ell \geq \ell} S^\alpha_\ell\).
• Below we will identify a Zariski point (of a scheme $S$) with the spectrum of its residue field.
• $k$ is a prime field, $p = \text{char } k$ ($p$ may be 0).
• All the schemes we consider will be excellent, separated, Noetherian $k$-schemes (i.e., characteristic $p$ schemes) of finite Krull dimension (so, a "scheme" will always mean a scheme of this sort).
• A variety over a field $F/k$ is a (separated) reduced scheme of finite type over $\text{Spec } F$.
• $S_{\text{red}}$ will denote the reduced scheme associated with $S$.
• All morphisms of schemes considered below will be separated. They will also mostly be of finite type.
• Throughout the paper $R$ will be some fixed unital associative commutative algebra over $\mathbb{Z}[\frac{1}{p}]$ (we set $\mathbb{Z}[\frac{1}{p}] = \mathbb{Z}$ if $p = 0$).

1.2. On $cdh$-motives (after Cisinski and Deglise). We list some of the properties of the triangulated categories of $cdh$-motives (those are certain relative Voevodsky motives with $R$-coefficients described by Cisinski and Deglise). They are very much similar to the properties of Beilinson motives (i.e., of $\mathbb{Q}$-linear ones) that were established in [CiD09] and applied in [Heb11] and [Bon14a] for the construction of the corresponding Chow weight structures.

**Theorem 1.2.1.** Let $X$, $Y$ be (Noetherian finite dimensional excellent characteristic $p$) schemes; $f : X \to Y$ is a (separated) morphism of finite type.

1. For any $X$ there exists a tensor triangulated $R$-linear category $\mathcal{DM}(X)$ with the unit object $R_X$ (in [CD14] Definition 1.5) this category was denoted by $\mathcal{DM}_{cdh}(X, R)$; it is closed with respect to arbitrary small coproducts.
2. The (full) subcategory $\mathcal{DM}_c(X) \subset \mathcal{DM}(X)$ of compact objects is tensor triangulated, and $R_X \in \text{Obj } \mathcal{DM}_c(S)$. $\mathcal{DM}_c(X)$ weakly generates $\mathcal{DM}(X)$.
3. For any $f$ the following functors are defined: $f^* : \mathcal{DM}(Y) \to \mathcal{DM}(X) : f_*$ and $f'_* : \mathcal{DM}(X) \to \mathcal{DM}(Y) : f'_! ; f^*$ is left adjoint to $f_*$ and $f'_!$ is left adjoint to $f'_*$.
   We call these the motivic image functors. Any of them (when $f$ varies) yields a 2-functor from the category of (separated finite-dimensional excellent characteristic $p$) schemes with morphisms of finite type to the 2-category of triangulated categories. Besides, all motivic image functors preserve compact objects (i.e., they can be restricted to the subcategories $\mathcal{DM}_c(\cdot)$; they also commute with arbitrary (small) coproducts.
4. For a Cartesian square of morphisms of finite type
   \[
   \begin{array}{ccc}
   X' & \xrightarrow{f'} & Y' \\
   \downarrow{g'} & & \downarrow{g} \\
   X & \xrightarrow{f} & Y 
   \end{array}
   \]
   we have $g^* f_* \cong f'_! g'^*$ and $g'_* f'_! \cong f_! g_*$.
5. For any $X$ there exists a Tate object $R(1) \in \text{Obj } \mathcal{DM}_c(X)$; tensoring by it yields an exact Tate twist functor $-1$ on $\mathcal{DM}(X)$. This functor is an auto-equivalence of $\mathcal{DM}(X)$; we will denote the inverse functor by $-(-1)$. Tate twists commute with all motivic image functors mentioned (up to an isomorphism of functors). Besides, for $X = \mathbb{P}^1(Y)$ there is a functorial isomorphism $f_!(R_{\mathbb{P}^1(Y)}) \cong R_Y \bigoplus R_Y(-1)[-2]$.
6. $f^*$ is symmetric monoidal; $f^*(R_Y) = R_X$. 
Remark

These statements can (mostly) be found [CiD14]. Concretely:
(7) \( f_\ast \cong f_! \) if \( f \) is proper; \( f_! ^\dagger (\cdot) \cong f^*(\cdot)(s)[2s] \) if \( f \) is smooth (everywhere) of relative dimension \( s \). If \( f \) is an open immersion, we just have \( f_! ^\dagger = f^* \).
(8) If \( i : S' \to S \) is an immersion of regular schemes everywhere of codimension \( d \), then \( R_{S'}[-d][-2d] \cong i_!(R_S) \).
(9) If \( i : Z \to X \) is a closed immersion, \( U = X \setminus Z \), \( j : U \to X \) is the complementary open immersion, then the motivic image functors yield a gluing datum for \( DM_0(-) \) (in the sense of [BBDS15 §1.4.3]; see also [Bon10 Definition 8.2.1]). That means that (in addition to the adjunctions given by assertion 3) the following statements are valid.
(a) \( i_\ast \cong i_! \) is a full embeddings; \( j^* = j^! \) is isomorphic to the localization (functor) of \( DM(X) \) by \( i_\ast (DM(Z)) \).
(b) For any \( M \in \text{Obj} DM(X) \) the pairs of morphisms \( j_! j^! (M) \to M \to i_* i^*(M) \) and \( i^! i_* (M) \to M \to j_* j^*(M) \) can be uniquely completed to distinguished triangles (here the connecting morphisms come from the adjunctions of assertion 3).
(c) \( i^\dagger j_! = 0; i^! j_* = 0 \).
(d) All of the adjunction transformations \( i^* i_\ast \to 1_{DM(Z)} \to i^\dagger i_! \) and \( j^* j_\ast \to j^\dagger j_! \) are isomorphisms of functors.
(10) For the subcategories \( DM_c(-) \subset DM(-) \) the obvious analogue of the previous assertion is fulfilled.
(11) If \( f \) is a finite universal homeomorphism, then \( f^*, f_!, f^\dagger \), and \( f_! \) are equivalences of categories. Moreover, \( f^! R_Y \cong f^* R_Y = R_X \) and \( f_* R_X = f_! R_X \cong R_Y \).
(12) If \( S \) is of finite type over a field, \( DM_c(S) \) (as a triangulated category) is generated by \( \{ g_\ast (R_X)(r) \} \), where \( g : X \to S \) runs through all projective morphisms such that \( X \) is regular, \( r \in \mathbb{Z} \).
(13) Let a scheme \( S \) be the limit of an essentially affine (filtering) projective system of schemes \( S_\beta \) (for \( \beta \in B \)). Then \( DM_c(S) \) is isomorphic to the 2-colimit of the categories \( DM_c(S_\beta) \); in this isomorphism all the connecting functors are given by the corresponding motivic inverse image functors (cf. Remark 1.2.2(1) below).
(14) If \( S \) is smooth over \( k \) (or over any other perfect field), then for \( b, c, r \in \mathbb{Z} \) with \( r \geq 0 \) we have \( R_S(b)(2b) \perp R_S(c)[2c+r] \).

Proof. These statements can (mostly) be found [CiD14]. Concretely:
1. [CiD14 §1.6].
2. Immediate from the definition of \( DM(-) \) given in [CiD14 §1.5].
3. [CiD14 §1.6. Theorem 6.4].
4. [CiD14 Proposition 4.3, Theorem 5.1] yield that \( DM(-) \) is a motivic triangulated category; see [CiD09 2.4.45] for the definition and [CiD09 2.4.50] for a list of properties that includes the assertions desired.
5. [CiD14 Proposition 6.2].
6. This is an easy consequence of assertions 3 and 9; cf. (the proof of) Proposition 1.1.2(11) in [Bon14a].
7. By [CiD14 Proposition 7.1] the functors \( f^* \cong f^\dagger \) are equivalences of categories. Hence their (right and left) adjoints \( f_\ast \) and \( f_! \) are equivalences also.
8. [CiD14 Proposition 7.2].
9. [CiD14 Theorem 5.11].
10. [CiD14 Corollary 8.6, putting \( X = \text{Spec} k \)].

Remark 1.2.2. 1. In [CiD14] the functor \( g^* \) was constructed for any morphism \( g : Y' \to Y \) not necessarily of finite type; it preserves compact objects (see §6.1(ii) of ibid.) and unit
objects \( R_\ast \) (i.e., \( g^\ast(R_Y) = R_Y \)). Besides, for any such \( g \) and any morphism \( f : X \to Y \) of finite type we have an isomorphism \( g^\ast f_! \cong f_! g^\ast \) (for the corresponding \( f' \) and \( g' \); cf. part 3 of our theorem).

We also note: if \( f \) is a pro-open limit of immersions, then one can define \( f_! = f^\ast \) (in particular, one can define \( j_! \)) for the natural morphism \( j_K : K \to S \) if \( K \) is a Zariski point of a scheme \( S \); cf. [BBD82 §2.2.12]); \( f_! \) preserves compact objects. For the functors of this type we also have \( g^\ast f_! \cong f_! g^\ast \) (for \( g \) of finite type; see part 3 of our theorem once more); see [CI09 Proposition 4.3.14].

2. For any morphism \( f : X \to Y \) of finite type we set \( \mathcal{M}_f^BM(X) = f_! R_X \) (this is a certain Borel-Moore motif of \( X \); cf. [Bon14b], [Deg14], and [Lev98 §I.IV.2.4]). Note that Theorem 1.2.1 easily yields that \( \mathcal{M}_f^BM(X) \cong \mathcal{M}_\ell^BM(X_{red}) \) (recall that \( X_{red} \) is the reduced scheme associated with \( X \)). Besides, for any (separated) morphism \( g : Y' \to Y \) the previous part of this remark yields that \( g^\ast(\mathcal{M}_f^BM(X)) \cong \mathcal{M}_g^BM(X \times_Y Y') \).

**Lemma 1.2.3.** Let \( S = \cup S_\alpha^\circ \) be a stratification. Then the following statements are valid.

1. For any \( M, N \in \text{Obj} \mathcal{D}_M(S)(M, N) \) there exists a filtration of \( \mathcal{D}_M(S)(M, N) \) whose factors are certain subquotients of \( \mathcal{D}_M(S_\alpha^\circ)(j_\alpha^\ast(M), j_\alpha^\ast(N)) \). If \( M = R_S(a)[2a], N = R_S(b)[2b+r] \) for \( a, b, r \in \mathbb{Z}, S \) is regular, and all \( S_\alpha^\circ \) are regular and connected, then the factors of this filtration on \( \mathcal{D}_M(S)(M, N) \) are certain subquotients of \( \mathcal{D}_M(S_\alpha^\circ)(R_{S_\alpha^\circ}(b - a - c_\ell), R_{S_\alpha^\circ}(b - a - 2c_\ell)) \), where \( c_\ell \) is the codimension of \( S_\alpha^\circ \) in \( S \).

2. If \( g : S \to Y \) is a morphism of finite type then \( \mathcal{M}_f^BM(S) \) belongs to the envelope of \( \mathcal{M}_Y^BM(S_\alpha^\circ) \).

3. Let \( Z \subset X \) be a closed subscheme, where \( X \) is a scheme of finite type over \( S \); denote by \( U \subset X \) the complementary open subscheme. If \( Z \) and \( X \) are regular, \( Z \) is everywhere of codimension \( c \) in \( X \), then there is a distinguished triangle

\[ z_* R_Z(-c)[-2c] \to x_* R_X \to u_* R_U \]

\( \in \mathcal{D}_M(S) \), where \( z, x, u \) are the corresponding structure morphisms (to \( S \)).

**Proof.** 1. The second part of the assertion is an simple consequence of the first one (see Theorem 1.2.1(6,8)).

We prove the first statement by induction on the number of strata by definition (see §1.1) \( S_\alpha^\circ \) is open in \( S \), and the remaining \( S_\alpha^\circ \) yield a stratification of \( S \setminus S_\alpha^\circ \). We denote \( S \setminus S_\alpha^\circ \) by \( Z \), the (open) immersion \( S_\alpha^\circ \to S \) by \( j \) and the (closed) immersion \( Z \to S \) by \( i \).

Now, Theorem 1.2.1(9) yields a distinguished triangle

\[ j_\alpha^\ast j_\alpha^\ast(M) \to M \to i_\ast i^\ast(M). \]

Hence there exists a (long) exact sequence

\[ \cdots \to \mathcal{D}_M(S)(i_\ast i^\ast(M), N) \to \mathcal{D}_M(S)(M, N) \to \mathcal{D}_M(S)(j_\alpha^\ast j_\alpha^\ast(M), N) \to \cdots \]

The corresponding adjunctions of functors yield \( \mathcal{D}_M(S)(i_\ast i^\ast(M), N) \cong \mathcal{D}_M(Z)(i^\ast(M), i^! (N)) \) and \( \mathcal{D}_M(S)(j_\alpha^\ast j_\alpha^\ast(M), N) \cong \mathcal{D}_M(S_\alpha^\circ)(j_\alpha^\ast(M), j_\alpha^! (N)) \).

Now, by the inductive assumption the group \( \mathcal{D}_M(Z)(i^\ast(M), i^! (N)) \) has a filtration whose factors are certain subquotients of \( \mathcal{D}_M(S_\alpha^\circ)(j_\alpha^\ast(M), j_\alpha^! (N)) \) (for \( \ell \neq 1 \)). This concludes the proof.

2. We use the same induction and notation as in the previous proof. Considering the distinguished triangle (1.2) for \( M = R_S \) and applying \( g_! \) to it we obtain a distinguished triangle

\[ \mathcal{M}_f^BM(S_\alpha^\circ) \to \mathcal{M}_f^BM(S) \to \mathcal{M}_f^BM(Z). \]
In order to complete the inductive step it suffices to apply (the first statement in) Remark 1.2.2.

3. We can assume that $X$ is connected. Theorem 1.2.1(9) yields a distinguished triangle $i^!_VR_X \to R_X \to j_*j^!R_X (\cong j_*R_U)$ (see also part 6 of the theorem). If $Z$ and $X$ are regular, then $i^!_VR_X \cong i_*R_Z([-2c])$ (see part of the theorem). Hence the presence of $x_r$ to this distinguished triangle yields (1.1).

$\square$

1.3. Some orthogonality lemmas. The following motivic statements are very important for the current paper.

Lemma 1.3.1. If $S$ is a regular scheme, then for any $a, b, r \in \mathbb{Z}$ with $r > 0$ we have $\mathcal{D}_M(S)[a][2a] \perp \mathcal{D}_M(S)[b][2b + r]$.

Proof. We stratify $S$ as $\bigcup_{1 \leq \ell \leq n} S_\ell$ so that all $S_\ell$ are regular, affine and connected. For such a stratification (by Lemma 1.2.3(1)) we should prove

$\mathcal{D}_M(S_\ell)(a - \text{codim}_S S_\ell)[2a - 2 \text{codim}_S S_\ell], R_{S_\ell}(b - \text{codim}_S S_\ell)[2b - 2 \text{codim}_S S_\ell + r]) = \{0\}$.

Thus it is sufficient to prove the statement for strata. That is, we can assume that $S$ is regular and affine. Such a scheme $S$ can be presented as the inverse limit of regular schemes of finite type over $k$ (by the Popescu–Spivakovsky theorem; see [CD14, Theorem 4.1.5]). If $S = \varprojlim S_\beta$, then $\mathcal{D}_M(S)(R_S(b)[2b], R_S(c)[2c + r]) = \varinjlim \mathcal{D}_M(S_\beta)(R_S(b)[2b], R_S(c)[2c + r])$ by Theorem 1.2.1(13). In conclusion, we refer to part (14) of this theorem. $\square$

Remark 1.3.2. This continuity argument along with [CD14, Corollary 8.6] also easily yields that $\mathcal{D}_M(R_S, R_S(b)[2b + r])$ is isomorphic to the corresponding higher Chow group of $S$, i.e., it can be computed using the Bloch or Suslin complex (of codimension $b$ cycles in $S \times \Delta^-$) with $R$-coefficients, if $S$ is a regular affine scheme. This result cannot be automatically generalized to arbitrary (regular excellent finite-dimensional equicharacteristic) schemes since (to the knowledge of the authors) the Mayer–Vietoris property is not known for the higher Chow groups in this generality.

Lemma 1.3.3. Let $X$ and $Y$ be regular schemes, and $x : X \to S$ and $y : Y \to S$ be quasi-projective morphisms, $r, b, c \in \mathbb{Z}$.

Then $x^!(R_X)(b)[2b] \perp y_* (R_Y)(c)[2c + r]$ if $r > 0$.

Proof. By Theorem 1.2.1(5) we can assume that $b = 0$. Next, we have

$\mathcal{D}_M(x)(R_X, y_* (R_Y)(c)[2c + r]) \cong \mathcal{D}_M(x)(R_X, x'^!y_* (R_Y)(c)[2c + r])$

since $x'$ is left adjoint to $x_!$.

Thus we should prove that

$\mathcal{D}_M(x)(R_X, x'^!y_* (R_Y)(c)[2c + r]) = \{0\}$.

We argue somewhat similarly to [Bon14b, §2.1]. Let us make certain reduction steps.

Consider a factorization of $x$ as $X \xrightarrow{f} S' \xrightarrow{h} S$, where $h$ is smooth of dimension $q$, $f$ is an embedding, $S'$ is connected, and consider the corresponding diagram

$$
\begin{array}{ccc}
Z & \xrightarrow{f'} & Y' \\
\downarrow z & & \downarrow y' \\
X & \xrightarrow{f} & S'
\end{array}
$$

(upper row is the base change of the lower one to $Y$). Then we have $x'^!y_* (R_Y)(\ell)[2\ell] = f'^!h'^!y_* (R_Y)(\ell)[2\ell] \cong f'^!y_* (R_Y)(\ell)[2\ell]$, $\ell = \dim Y$ (by Theorem 1.2.1(14)). Parts 7 and 8 of
Theorem 1.2.4] allow us to transform this into $f^!y_*R_Y$. Hence below we may assume $x$ to be an embedding (since we can replace $S$ by $S'$ in the assertion). Besides, the isomorphism $x^!y_* \cong z_*x^*z^!_*$ for $z_Y = h_Y \circ f_Y$ yields that the group in question is zero if $Y$ lies over $S \setminus X$ (considered as a set); see Theorem 1.2.11.

Next we apply Lemma 1.2.3(1); it yields that it suffices to verify the statement for $Y$ replaced by the components of some regular connected stratification.

Now, we can choose a stratification of this sort such that each $Y$ lies either over $X$ or over $S \setminus X$. Therefore it suffices to verify our assertion in the case where $y$ factors through $x$. Moreover, since $x^!x_*$ is the identity functor on $\mathcal{D}\mathcal{M}_c(X)$ (in this case; see Theorem 1.2.1(9)), we may also assume that $X = S$. Next, applying the adjunction $y^* \dashv y_*$ we transform $\mathcal{D}\mathcal{M}(S)(R_S, y_*(R_Y)(c)[2c+r])$ into $\mathcal{D}\mathcal{M}(Y)(y^*(R_S), R_Y(c)[2c+r]) = \mathcal{D}\mathcal{M}(Y)(R_Y, R_Y(c)[2c+r])$. Thus it remains to apply the previous Lemma. \hfill \Box

1.4. Weight structures: short reminder. We recall some basics of the theory of weight structures.

Definition 1.4.1. (1) A pair of subclasses $C_{w \leq 0}, C_{w \geq 0} \subset \text{Obj } C$ will be said to define a weight structure $w$ for $C$ if they satisfy the following conditions:

(a) $C_{w \leq 0}, C_{w \geq 0}$ are Karoubi-closed in $C$ (i.e., contain all $C$-retracts of their objects).

(b) Semi-invariance with respect to translations: $C_{w \leq 0} \subset C_{w < 0}[1]$, $C_{w \geq 0} \supset C_{w > 0}[1]$.

(c) Orthogonality: $C_{w > 0} \perp C_{w \leq 0}[1]$.

(d) Weight decompositions: For any $M \in \text{Obj } C$ there exists a distinguished triangle

$$B \to M \to A \xrightarrow{f} B[1]$$

such that $A \in C_{w \geq 0}[1]$, $B \in C_{w \leq 0}$.

(2) The category $H_w \subset C$ whose objects are $C_{w = 0} = C_{w \geq 0} \cap C_{w \leq 0}$; $H_w(Z, T) = C(Z, T)$ for $Z, T \in C_{w = 0}$, will be called the heart of $w$.

(3) $C_{w \geq i}$ (resp. $C_{w \leq i}$, resp. $C_{w = i}$) will denote $C_{w \leq i}[i]$ (resp. $C_{w > i}[i]$, resp. $C_{w = i}$). We denote $C_{w \geq i} \cap C_{w \leq j}$ by $C_{[i, j]}$ (so it equals $\{0\}$ for $i > j$).

(4) We will say that $(C, w)$ is bounded if $\bigcup_{i \in \mathbb{Z}} C_{w = i} = \text{Obj } C = \cup_{i \in \mathbb{Z}} C_{w \geq i}$.

(5) Let $C$ and $C'$ be triangulated categories endowed with weight structures $w$ and $w'$, respectively; let $F : C \to C'$ be an exact functor. $F$ will be called left weight-exact (with respect to $w, w'$) if it maps $C_{w \leq 0}$ into $C'_{w' \leq 0}$; it will be called right weight-exact if it maps $C_{w \geq 0}$ into $C'_{w' \geq 0}$. $F$ is called weight-exact if it is both left and right weight-exact.

(6) Let $H$ be a full subcategory of a triangulated $C$. We will say that $H$ is negative if $\text{Obj } H \perp (\cup_{i > 0} \text{Obj } (H[i]))$.

(7) We call a category $\frac{A}{B}$ a factor of an additive category $A$ by its (full) additive subcategory $B$ if $\text{Obj } (\frac{A}{B}) = \text{Obj } A$ and $\frac{A}{B}(M, N) = A(M, N)/(\sum_{O \in \text{Obj } B} A(O, N) \circ A(M, O))$.

Now we recall those properties of weight structures that will be needed below (and that can be easily formulated). See [Bon14a] for the references to the proofs (whereas the fact that $H_w \cong \frac{H_w}{H_w^\perp}$ in the setting of assertion [8] was proved in [BoS15]).

Proposition 1.4.2. Let $C$ be a triangulated category.
(1) \((C_1, C_2)\) \((C_1, C_2 \subset \text{Obj } C)\) define a weight structure for \(C\) if and only if \((C_2^{\text{op}}, C_1^{\text{op}})\) define a weight structure for \(C^{\text{op}}\).

(2) Let \(w\) be a weight structure for \(C\). Then \(C_{w\geq 0} = (C_{w\leq -1})^{\perp}\) and \(C_{w\leq 0} = \perp C_{w\geq 1}\) (see [1.7]).

(3) Let \(w\) be a weight structure on \(C\). Then \(C_{w\leq 0}, C_{w\geq 0}\), and \(C_{w=0}\) are extension-stable.

(4) Suppose that \(v, w\) are weight structures for \(C\); let \(C_{v\leq 0} \subset C_{v\leq 0}\) and \(C_{v\geq 0} \subset C_{v\geq 0}\).

(5) Assume \(H \subset \text{Obj } C\) is negative and \(C\) is idempotent complete. Then there exists a unique weight structure \(w\) on the triangulated subcategory \(T\) of \(C\) generated by \(H\) such that \(H \subset T_{w=0}\). Its heart is the envelope \((\text{see [1.7]}\) \(C\); it is the idempotent completion of \(H\) if \(H\) is additive.

(6) For the weight structure mentioned in the previous assertion, \(T_{w\leq 0}\) is the envelope of \(\bigcup_{i \leq 0} H[i]\); \(T_{w\geq 0}\) is the envelope of \(\bigcup_{i \geq 0} H[i]\).

(7) Let \(C\) and \(D\) be triangulated categories endowed with weight structures \(w\) and \(v\), respectively. Let \(F : C \leftrightarrows D : G\) be adjoint functors. Then \(F\) is left weight-exact if and only if \(G\) is right weight-exact.

(8) Let \(w\) be a weight structure for \(C\); let \(D \subset C\) be a triangulated subcategory of \(C\). Suppose that \(w\) yields a weight structure for \(D\) \((\text{i.e., Obj } D \cap C_{w\leq 0}\) and \(\text{Obj } D \cap C_{w\geq 0}\) give a weight structure for \(D\)).

Then \(w\) also induces a weight structure on \(C/D\) (the localization, i.e., the Verdier quotient of \(C\) by \(D\)) in the following sense: the Karoubi-closures of \(C_{w\leq 0}\) and \(C_{w\geq 0}\) (considered as classes of objects of \(C/D\)) give a weight structure \(w'\) for \(C/D\) (note that \(\text{Obj } C = \text{Obj } C/D\)). Besides, \(\text{H}w'\) is naturally equivalent to \(\text{H}w/\text{H}w_D\) if \(w\) is bounded.

(9) Suppose that \(D \subset C\) is a full subcategory of compact objects endowed with a bounded weight structure \(w\). Suppose that \(D\) weakly generates \(C\); let \(C\) admit arbitrary (small) coproducts. Then \(w\) can be extended to a certain weight structure \(w\) for \(C\).

(10) Let \(D \xrightarrow{i} C \xleftarrow{j} \mathcal{E}\) be a part of a glueing datum \((\text{see Theorem [1.2.1]}\) \(D\)). Then for any pair of weight structures on \(D\) and \(\mathcal{E}\) (we will denote them by \(w_D\) and \(w_{\mathcal{E}}\), respectively) there exists a weight structure \(w\) on \(C\) such that both \(i^*\) and \(j^*\) are weight-exact (with respect to the corresponding weight structures). Besides, the functors \(i^*\) and \(j^*\) are right weight-exact (with respect to the corresponding weight structures); \(i^*\) and \(j^*\) are left weight-exact. Moreover,

\[ C_{w\geq 0} = C_1 = \{ M \in \text{Obj } C : i^!(M) \in D_{w_D\geq 0}, j^*(M) \in E_{w_{\mathcal{E}}\geq 0}\} \]

and

\[ C_{w\leq 0} = C_2 = \{ M \in \text{Obj } C : i^*(M) \in D_{w_D\leq 0}, j^*(M) \in E_{w_{\mathcal{E}}\leq 0}\}. \]

Lastly, \(C_1\) (resp. \(C_2\)) is the envelope of \(j_!(E_{w\leq 0}) \cup i_*(D_{w\leq 0})\) (resp. of \(j_!(E_{w\geq 0}) \cup i_*(D_{w\geq 0})\)).

(11) In the setting of the previous assertion, if \(w_D\) and \(w_{\mathcal{E}}\) are bounded, then \(w\) is bounded also. Besides, \(C_{w\leq 0}\) is the envelope of \(\{i_*(D_{w_D=-1}), j_!(E_{w_{\mathcal{E}}=-1}, l \leq 0\}\); \(C_{w\geq 0}\) is the envelope of \(\{i_*(D_{w_D=1}), j_!(E_{w_{\mathcal{E}}=1}, l \geq 0\}\).

(12) In the setting of assertion [11], the weight structure \(w\) described is the only weight structure for \(C\) such that both \(i^*\) and \(j^*\) are weight-exact.
Remark 1.4.3. Part § of the proposition can be re-formulated as follows. If $i_* : D \to C$ is an embedding of triangulated categories that is weight-exact (with respect to certain weight structures for $D$ and $C$), an exact functor $j^* : C \to E$ is equivalent to the localization of $C$ by $i_*(D)$, then there exists a unique weight structure $w'$ for $E$ such that the functor $j^*$ is weight-exact. If $w$ is bounded then $H_{w, E}$ is equivalent to $\frac{H_{w, E}}{i_*(H_{w, D})}$ (with respect to the natural functor).

2. On the Chow weight structures for relative motives

This is the main section of our paper. We define the Chow weight structures for relative motives using the "gluing construction" and study their properties. We also prove that the heart of $w_{\text{chow}}(S)$ consists of certain "Chow" motives if $S$ is a variety over a field (or if it is "pro-smooth affine" over a variety).

A substantial part of this section is just a "recombination" of (the corresponding parts of) [Bon14a, §2]; yet some of the arguments used in §2.2 are quite new (and rather interesting).

2.1. The construction of the Chow weight structure. First we describe certain candidates for $\mathcal{DM}_c(S)_{w_{\text{chow}} \geq 0}$ and $\mathcal{DM}_c(S)_{w_{\text{chow}} \leq 0}$; next we will prove that they yield a weight structure for $\mathcal{DM}_c(S)$ indeed. A reader interested in certain "motivation" for this construction is strongly recommended to look at (the remarks in) [Bon14a, §2.3].

For a scheme $X$ we will denote by $\mathcal{OP}(X)$ (resp. $\mathcal{ON}(X)$) the envelope (see [1,1] of $p_*(R_P)(s)[i+2s][\cong \mathcal{M}_X^{BM}(P)(s)[i+2s]$; see Remark [1.2.2]) in $\mathcal{DM}_c(X)$; here $p : P \to X$ runs through all morphisms to $X$ that can be factorized as $g \circ h$, where $h : P \to X'$ is a smooth projective morphism, $X'$ is a regular scheme, $g : X' \to X$ is a finite universal homeomorphism, $s \in \mathbb{Z}$, whereas $i \geq 0$ (resp. $i \leq 0$). We denote $\mathcal{OP}(X) \cap \mathcal{ON}(X)$ by $\mathcal{OZ}(X)$.

Remark 2.1.1. 1. Recall that for a morphism $f : Y \to X$ we have $f^*\mathcal{M}_X^{BM}(P) \cong \mathcal{M}_Y^{BM}(P_Y)$ (see Remark [1.2.2]). Next, suppose that for $X'/X$ as above the scheme $Y'_{\text{red}}$ associated with $Y' = X'_{\text{red}}$ is regular. Then the aforementioned remark immediately yields that $f^*\mathcal{M}_X^{BM}(P)(s)[2s] \in \mathcal{OZ}(Y)$.

Moreover, Theorem [1.2.1] easily yields that $f^!\mathcal{M}_X^{BM}(P) \in \mathcal{OZ}(Y)$ if $f$ induces an immersion $Y'_{\text{red}} \to X'$ of regular schemes. Indeed, consider the diagram

$$
\begin{align*}
\begin{array}{c}
P_{Y,\text{red}} \xrightarrow{p_Y} P_Y \xrightarrow{f_Y} P \\
\downarrow h_{Y,\text{red}} \downarrow h_Y \downarrow h
\end{array}
\end{align*}

(2.1)

Y'_{\text{red}} \xrightarrow{y'_Y} Y' \xrightarrow{f'} X'

\begin{align*}
\downarrow g_Y \downarrow g
\end{align*}

Y \xrightarrow{f} X

where $p_Y$ and $y'_Y$ are the corresponding nil-immersions. We can assume that $Y$ and $P$ are connected; hence $Y'_{\text{red}}$ and $P_{Y,\text{red}}$ are connected also. Denote the codimension of $Y'_{\text{red}}$ in $X'$ by $c$; then $p_Y \circ f_Y : P_{Y,\text{red}} \to P$ is an immersion of regular schemes of codimension $c$ also. Then we obtain: $f^*\mathcal{M}_X^{BM}(P) = f^*p_Y(R_P) \cong (g_Y \circ y'_Y)_*h_{Y,\text{red}},(p_Y \circ f_Y)^!R_P$. Using part § of the theorem, we transform this into $(g_Y \circ y'_Y)_*h_{Y,\text{red}},R_{P_{Y,\text{red}}}(-c)[-2c] \in \mathcal{OZ}(Y)$.

2. Certainly, for char $X = 0$ the universal homeomorphisms mentioned are just isomorphisms.
For a stratification $\alpha : S = \bigcup S^\alpha_\ell$ we denote by $\mathcal{OP}(\alpha)$ the class $\{M \in \text{Obj} \, \mathcal{DM}_c(S) : j^*_\ell(M) \in \mathcal{OP}(S^\alpha_\ell), 1 \leq \ell \leq n\}$; $\mathcal{ON}(\alpha) = \{M \in \text{Obj} \, \mathcal{DM}_c(S) : j^*_\ell(M) \in \mathcal{ON}(S^\alpha_\ell), 1 \leq \ell \leq n\}$.

We define the Chow weight structure for $\mathcal{DM}_c(S)$ as follows: $\mathcal{DM}_c(S)_{w_{\text{Chow}} \geq 0} = \cup_\alpha \mathcal{OP}(\alpha)$, $\mathcal{DM}_c(S)_{w_{\text{Chow}} \leq 0} = \cup_\alpha \mathcal{ON}(\alpha)$; here $\alpha$ runs through all stratifications of $S$.

**Lemma 2.1.2.** 1. Let $\delta$ be a (not necessarily regular) stratification of $S$; we denote the corresponding immersions $S^\delta_\ell \to S$ by $j^\delta_\ell$. Let $M$ be an object of $\mathcal{DM}_c(S)$. Suppose that $j^\delta_\ell(M) \in \mathcal{DM}_c(S^\delta_\ell)_{w_{\text{Chow}} \geq 0}$ (resp. $j^\delta_\ell(M) \in \mathcal{DM}_c(S^\delta_\ell)_{w_{\text{Chow}} \leq 0}$) for all $\ell$.

Then $M \in \mathcal{DM}_c(S)_{w_{\text{Chow}} \geq 0}$ (resp. $M \in \mathcal{DM}_c(S)_{w_{\text{Chow}} \leq 0}$).

2. For any immersion $j : V \to S$ we have $j_\ast(\mathcal{DM}_c(V)_{w_{\text{Chow}} \geq 0}) \subset \mathcal{DM}_c(S)_{w_{\text{Chow}} \geq 0}$ and $j_\ast(\mathcal{DM}_c(V)_{w_{\text{Chow}} \leq 0}) \subset \mathcal{DM}_c(S)_{w_{\text{Chow}} \leq 0}$.

3. For any $M \in \mathcal{DM}_c(S)_{w_{\text{Chow}} \leq 0}$ and $N \in \mathcal{DM}_c(S)_{w_{\text{Chow}} \geq 1}$ there exists a stratification $\alpha$ of $S$ such that $M \in \mathcal{ON}(\alpha)$, $N \in \mathcal{OP}(\alpha)$.

**Proof.** 1. We use induction on the number of strata in $\delta$. The 2-functoriality of motivic upper image functors yields: it suffices to prove the statement for $\delta$ consisting of two strata.

So, let $S = U \cup Z$, $U$ and $Z$ are disjoint, $U \neq \{0\}$ is open in $S$; we denote the immersions $U \to S$ and $Z \to S$ by $j$ and $i$, respectively. By the assumptions on $M$, there exist stratifications $\beta$ of $Z$ and $\gamma$ of $U$ such that $i^\ast(M) \in \mathcal{OP}(\beta)$ and $j^\ast(M) \in \mathcal{OP}(\gamma)$ (resp. $i^\ast(M) \in \mathcal{ON}(\beta)$ and $j^\ast(M) \in \mathcal{ON}(\gamma)$).

We take the union of $\beta$ with $\gamma$ and denote by $\alpha$ the stratification of $S$ obtained (for $\gamma = 1$ we put $S^\alpha_\ell = U^\alpha_\ell$ if $1 \leq \ell \leq \Gamma$ and $S^\alpha_\ell = Z^\beta_{\ell - 1}$ if $\ell > \Gamma$; note that we really obtain a stratification in our weak sense of this notion this way; see [1]. Then the 2-functoriality of $-^\ast$ (resp. of $-^\ast$) yields that $M \in \mathcal{OP}(\alpha)$ (resp. $M \in \mathcal{ON}(\alpha)$).

2. We choose a stratification $\delta$ containing $V$ (as one of the strata). So we assume that $V = S^\delta_v$ for some index $v$. Then it can be easily seen that $j^\delta_\ast j_v^\ast = 0 = j^\delta_\ast J_v^\ast$ for any $u \neq v$ and $j^\delta_\ast j_v^\ast \cong 1_{\mathcal{DM}_c(V)} \cong j^\delta_v j^\delta_v$ (see Theorem 2.1.9). Hence the result follows from assertion 1.

3. By Remark 2.1.1 it suffices to verify the following: if $\beta$, $\gamma$ are stratifications of $S$, and $S^\beta_\ell \to S^\gamma_\ell$, $S^\beta_\ell \to S^\gamma_\ell$ are (finite) sets of finite universal homomorphisms, then there exists a common subdivision $\alpha$ of $\beta$ and $\gamma$ such that all the (reduced) schemes $(S^\beta_\ell \times_S S^\gamma_\ell)_{\text{red}}$, $(S^\beta_\ell \times_S S^\gamma_\ell)_{\text{red}}$ are regular. To this end it obviously suffices to prove the following: if $f : Z \to S$ is an immersion, $g_i : T_i \to Z$ are some finite universal homeomorphisms, then there exists a stratification $\delta$ of $Z$ such that the schemes $T^\delta_\ell = (T_i \times_Z Z^\delta_\ell)_{\text{red}}$ are regular all $i$ and $\ell$.

We prove the latter statement by Noetherian induction. Suppose that it is fulfilled for any proper closed subscheme $Z'$ of $Z$. Since all $(T_i)_{\text{red}}$ are generically regular, we can choose a (sufficiently small) open non-empty subscheme $Z_1$ of $Z$ such that all of $(T_i \times_Z Z_1)_{\text{red}}$ are regular.

Next, apply the inductive assumption to the scheme $Z' = Z \setminus Z_1$ and the morphisms $g'_i = g_i \times_Z Z'$; we choose a stratification $\alpha'$ of $Z'$ such that all $T^\alpha'_{\ell'} = (T_i \times_Z Z^\alpha'_{\ell'})_{\text{red}}$ are regular. Then it remains to take the union of $Z_1$ with $\alpha'$, i.e., we consider the following stratification $\alpha$: $Z^\alpha_1 = Z_1$, and $Z^\alpha_\ell = Z^\alpha_{\ell - 1}$ for all $\ell > 1$.

**Theorem 2.1.3.** (1) The couple $(\mathcal{DM}_c(S)_{w_{\text{Chow}} \geq 0}, \mathcal{DM}_c(S)_{w_{\text{Chow}} \leq 0})$ yields a bounded weight structure $w_{\text{Chow}}$ for $\mathcal{DM}_c(S)$.

(2) $\mathcal{DM}_c(S)_{w_{\text{Chow}} \geq 0}$ (resp. $\mathcal{DM}_c(S)_{w_{\text{Chow}} \leq 0}$) is the envelope of $p_\ast(RP)(s)[2s + i]$ (resp. of $M^{BM}(P)(s)[2s - i]$) for $s \in \mathbb{Z}$, $i \geq 0$, and $p : P \to S$ being the composition of a
smooth projective morphism with a finite universal homeomorphism whose domain is regular and with an immersion.

(3) $w_{\text{Chow}}$ can be extended to a weight structure $w_{\text{Chow}}^{\text{big}}$ for the whole $\mathcal{DM}(S)$.

**Proof.** 1-2. We prove the statement by Noetherian induction. So, we suppose that assertions 1 and 2 are fulfilled for all proper closed subschemes of $S$. We prove them for $S$.

We denote the envelopes mentioned in assertion 2 by $(\mathcal{DM}_c(S)_{w_{\text{Chow}}^{\leq 0}}, \mathcal{DM}_c(S)_{w_{\text{Chow}}^{\geq 0}})$. We should prove that $w_{\text{Chow}}$ and $w_{\text{Chow}}^{\prime}$ yield coinciding weight structures for $\mathcal{DM}_c(S)$.

Obviously, $\mathcal{DM}_c(S)_{w_{\text{Chow}}^{\leq 0}}$, $\mathcal{DM}_c(S)_{w_{\text{Chow}}^{\geq 0}}$, $\mathcal{DM}_c(S)_{w_{\text{Chow}}^{\leq 0}}$ and $\mathcal{DM}_c(S)_{w_{\text{Chow}}^{\geq 0}}$ are Karoubi-closed in $\mathcal{DM}_c(S)$, and are semi-invariant with respect to translations (in the appropriate sense).

Now, Lemma 2.1.2 yields that $\mathcal{DM}_c(S)_{w_{\text{Chow}}^{\leq 0}} \subset \mathcal{DM}_c(S)_{w_{\text{Chow}}^{\leq 0}}$ and $\mathcal{DM}_c(S)_{w_{\text{Chow}}^{\geq 0}} \subset \mathcal{DM}_c(S)_{w_{\text{Chow}}^{\geq 0}}$. Hence in order to check that $w_{\text{Chow}}$ and $w_{\text{Chow}}^{\prime}$ are indeed weight structures, it suffices to verify the following:

(i) the orthogonality axiom for $w_{\text{Chow}}^{\prime}$;

(ii) any $M \in \text{Obj } \mathcal{DM}_c(S)$ possesses a weight decomposition with respect to $w_{\text{Chow}}^{\prime}$.

Thus these statements along with the boundedness of $w_{\text{Chow}}^{\prime}$ imply assertion 1. Besides, Proposition 1.4.2 yields that these two statements imply assertion 2 also, whereas in order to prove assertion 1 it suffices to verify the boundedness of $w_{\text{Chow}}^{\prime}$ (instead of that for $w_{\text{Chow}}^{\prime}$).

Now we verify (i). For some fixed $M \in \mathcal{DM}_c(S)_{w_{\text{Chow}}^{\leq 0}}$ and $N \in \mathcal{DM}_c(S)_{w_{\text{Chow}}^{\geq 1}}$ we check that $M \perp N$. By Lemma 2.1.3, we can assume that $M \in \mathcal{ON}(\alpha)$, $N \in \mathcal{OP}(\alpha)[1]$ for some stratification $\alpha$ of $S$. Hence it suffices to prove that $\mathcal{ON}(\alpha) \perp \mathcal{OP}(\alpha)[1]$ for any $\alpha$. The latter statement is an easy consequence of Lemmas 1.3.3 and 1.1.1.

Now we verify (ii) along with the boundedness of $w_{\text{Chow}}^{\prime}$. We choose some generic point $K$ of $S$, denote by $K^p$ its perfect closure, and by $j_{K^p}: K^p \to S$ the corresponding morphism. We fix some $M$. By Theorem 1.2.12, there exist some smooth projective varieties $P_i/K^p$, $1 \leq i \leq n$, (we denote the corresponding morphisms $P_i \to K^p$ by $p_i$) and some $s \in \mathcal{S}$ such that $j_{K^p}^*(M)$ belongs to the triangulated subcategory of $\mathcal{DM}_c(K^p)$ generated by $\{p_{is}(R_{P_i})(s)[2s]\}$. Now we choose some finite universal homeomorphism $K' \to K$ (i.e., a morphism of spectra of fields corresponding to a finite purely inseparable extension) such that $P_i$ are (and are smooth projective) over $K'$. By Theorem 1.2.13, for the corresponding morphisms $j_{K'}: K' \to S$ and $p'_i: P_{K'} \to K'$ we have the following: $j_{K'}^*(M)$ belongs to the triangulated subcategory of $\mathcal{DM}_c(K')$ generated by $\{p'_{is}(R_{P_{K'}})(s)[2s]\}$. Applying Zariski’s main theorem in the form of Grothendieck, we can choose a finite universal homeomorphism $g$ from a regular scheme $U'$ whose generic fibre is $K'$ to an open $U \subset S$ $(j: U \to S$ will denote the corresponding immersion) and smooth projective $h_i: P_{U'} \to U'$ such that the fibres of $P_{U'}$ over $K'$ are isomorphic to $P_{K'}$. Moreover, by Theorem 1.2.13 we can also assume that $(j \circ g)^*(M)$ belongs to the triangulated subcategory of $\mathcal{DM}_c(U')$ generated by $\{h_{is}(R_{P_{U'}})(s)[2s]\}$. Then Theorem 1.2.11 yields that $j^*(M)$ belongs to the triangulated subcategory of $\mathcal{DM}_c(U)$ generated by $\{(g \circ h_i)^*(R_{P_{U'}})(s)[2s]\}$.

Since $id_U$ yields a stratification of $U$, the set $\{(g \circ h_i)^*(R_{P_{U'}})(s)[2s]\}$ is negative in $\mathcal{DM}_c(U)$ (since $\mathcal{ON}(\alpha) \perp \mathcal{OP}(\alpha)[1]$ for any $\alpha$, as we have just proved). Therefore (by Proposition 1.4.25) there exists a weight structure $d$ for $D$ such that $D_{d\geq 0}$ (resp. $D_{d<0}$) is the envelope of $\cup_{n \geq 0} \{(g \circ h_i)^*(R_{P_{U'}})(s)[2s+n]\}$ (resp. of $\cup_{n \geq 0} \{(g \circ h_i)^*(R_{P_{U'}})(s)[2s-n]\}$). We also obtain that $D_{d\geq 0} \subset \mathcal{DM}_c(U)_{w_{\text{Chow}}^{\leq 0}}$ and $D_{d\leq 0} \subset \mathcal{DM}_c(U)_{w_{\text{Chow}}^{\geq 0}}$.\]
We denote $S \setminus U$ by $Z$ ($Z$ may be empty); $i : Z \to S$ is the corresponding closed immersion. By the inductive assumption, $w_{\text{Chow}}$ and $w_{\text{Chow}}'$ yield coinciding bounded weight structures for $\mathcal{D}M_c(Z)$.

We have a gluing datum $\mathcal{D}M_c(Z) \xrightarrow{i^*} \mathcal{D}M_c(S) \xrightarrow{j^*} \mathcal{D}M_c(U)$. We can ‘restrict it’ to a gluing datum

$$\mathcal{D}M_c(Z) \xrightarrow{i^*} j^{-1}(D) \xrightarrow{j_0^*} D$$

(see Proposition 1.4.2(10)), whereas $M \in \text{Obj}(j^{-1}(D))$; here $j_0^*$ is the corresponding restriction of $j^*$. Hence by Proposition 1.4.2(10) there exists a weight structure $w'$ such that the functors $i^*_w$ and $j_0^*$ are weight-exact (with respect to the weight structures mentioned). Hence there exists a weight decomposition $B \to M \to A$ of $M$ with respect to $w'$. Besides, there exist $m, n \in \mathbb{Z}$ such that $j_0^*(M) \in \mathcal{D}M_c(U)_{w' \geq m}$, $j_0^*(M) \in \mathcal{D}M_c(U)_{w' \leq n}$, $i^!(M) \in \mathcal{D}M_c(Z)_{w' \geq m}$, and $i^*(M) \in \mathcal{D}M_c(Z)_{w' \leq n}$. Hence $A[-1], M[-m] \in \mathcal{D}M_c(S)_{w' \geq 0}$, $B, M[-n] \in \mathcal{D}M_c(S)_{w' \leq 0}$; here we apply Proposition 1.4.2(11). So, we have verified (ii) and the boundedness of $w'_{\text{Chow}}$. As was shown above, this proves the finish of assertions 1-2.

3. Since $Hw_{\text{Chow}}$ generates $\mathcal{D}M_c(S)$, and $\mathcal{D}M_c(S)$ weakly generates $\mathcal{D}M(S)$ (by Theorem 1.4.2(2)), $Hw_{\text{Chow}}$ weakly generates $\mathcal{D}M(S)$. Hence the assertion follows immediately from assertion 1 and Proposition 1.4.2(9).

2.2. The main properties of $w_{\text{Chow}}(\cdot)$. Now we study (left and right) weight-exactness of the motivic image functors.

**Theorem 2.2.1.**

(1) The functors $-(b)[2b](= - \otimes R_S(b)[2b])$ are weight-exact with respect to $w_{\text{Chow}}$ for all $S$ and all $b \in \mathbb{Z}$.

(2) Let $f : X \to Y$ be a (separated) morphism of finite type.

(a) The functors $f^!$ and $f^*$ are right weight-exact; $f^*$ and $f^!$ are left weight-exact.

(b) Suppose moreover that $f$ is smooth. Then $f^*$ and $f^!$ are also weight-exact.

(c) If $S_{\text{red}}$ is regular then $R_S \in \mathcal{D}M_c(S)_{w_{\text{Chow}}=0}$.

(d) $\mathcal{D}M_c(S)_{w_{\text{Chow}} \leq 0}$ is the envelope of $\mathcal{M}_{S_{\text{BM}}}(T)(b)[2b - r]$ for $T$ running through all schemes of finite type over $S$, $b \in \mathbb{Z}$, $r \geq 0$; $\mathcal{D}M_c(S)_{w_{\text{Chow}} \geq 0}$ is the envelope of $t_*(R_T)(b)[2b + r]$ for $t : T \to S$ running through all morphisms of finite type with regular domains, $b \in \mathbb{Z}$, $r \geq 0$.

(e) Besides, the functor $g^*$ is right weight-exact for $g$ being an arbitrary (separated, not necessarily of finite type) morphism of schemes. It is weight-exact if $g$ is either (i) a (filtering) projective limit of smooth morphisms such that the corresponding connecting morphisms are smooth affine or (ii) a finite universal homeomorphism. In the latter case $g^!$ is weight-exact also.

(3) Let $i : Z \to X$ be a closed immersion; let $j : U \to X$ be the complementary open immersion.

(a) $Hw_{\text{Chow}}(U)$ is equivalent to the factor (in the sense of Definition 1.4.1(7)) of $Hw_{\text{Chow}}(X)$ by $i_*\left(Hw_{\text{Chow}}(Z)\right)$.

(b) For $M \in \text{Obj} \mathcal{D}M_c(X)$ we have: $M \in \mathcal{D}M_c(X)_{w_{\text{Chow}} \geq 0}$ (resp. $M \in \mathcal{D}M_c(X)_{w_{\text{Chow}} \leq 0}$) if and only if $j^!(M) \in \mathcal{D}M_c(U)_{w_{\text{Chow}} \geq 0}$ and $i^!(M) \in \mathcal{D}M_c(Z)_{w_{\text{Chow}} \geq 0}$ (resp. $j^*(M) \in \mathcal{D}M_c(U)_{w_{\text{Chow}} \leq 0}$ and $i^*(M) \in \mathcal{D}M_c(Z)_{w_{\text{Chow}} \leq 0}$).

(4) Let $S = \bigcup S_{\text{CHOW}}^\alpha$ be a stratification, $j^\ell : S_{\text{CHOW}}^\alpha \to S$ be the corresponding immersions. Then for $M \in \text{Obj} \mathcal{D}M_c(S)$ we have: $M \in \mathcal{D}M_c(S)_{w_{\text{Chow}} \geq 0}$ (resp. $M \in \mathcal{D}M_c(S)_{w_{\text{Chow}} \leq 0}$) if and only if $j^\ell_!(M) \in \mathcal{D}M_c(S_{\text{CHOW}}^\alpha)_{w_{\text{Chow}} \geq 0}$ (resp. $j^\ell_*(M) \in \mathcal{D}M_c(S_{\text{CHOW}}^\alpha)_{w_{\text{Chow}} \leq 0}$) for all $\ell$.

(5) For any $S$ we have $R_S \in \mathcal{D}M_c(S)_{w_{\text{Chow}} \leq 0}$. 

Proof. 1. Immediate from Theorem [2, 1.3(2)].

2. Let \( f \) be an immersion. Then the description of \( w_{\text{Chow}}(\cdot) \) given by Theorem [2, 1.3(2)] yields that \( f_* \) is left weight-exact and \( f^! \) is right weight-exact. Hence the corresponding adjunctions yield (by Proposition [1, 4.2(4)]) that \( f^! \) is left weight-exact and \( f^* \) is right weight-exact.

If \( f \) is smooth, then Theorem [2, 1.3(2)] (along with Theorem [1, 2.1(4)]) easily yields that \( f^* \) is left-weight exact and \( f^! \) is right weight-exact (since schemes that are smooth over regular bases are regular themselves). Hence part 7 of the theorem (along with assertion 1) implies that both of these functors are weight-exact (so, we obtain assertion 2b). Besides, adjunctions yield (by part (7) of Proposition [1, 4.2] that \( f^! \) is left weight-exact and \( f_* \) is right weight-exact.

Thus assertion 2a is valid for any quasi-projective \( f \) (since such an \( f \) can be presented as the composition of a closed immersion with a smooth morphism).

Now we verify assertion 2c. Let \( S_{\text{red}} \) be a regular scheme; denote by \( v \) the canonical immersion \( S_{\text{red}} \to S \). Then \( v_*(R_{S_{\text{red}}}) \in \mathcal{DM}_c(S)_{w_{\text{Chow}}=0} \) by Theorem [2, 1.3(2)]. Since \( v_*(R_{S_{\text{red}}}) \cong R_S \) by Theorem [1, 2.1(1)], we obtain the result.

Now we are able to prove assertion 2d. First we note (using Theorem [2, 1.3(2)]) that \( \mathcal{DM}_c(S)_{w_{\text{Chow}}<0} \) and \( \mathcal{DM}_c(S)_{w_{\text{Chow}}\geq 0} \) are subclasses of the corresponding envelopes. So, we should verify the converse inclusions. Note that any excellent Noetherian scheme admits a stratification the reductions of whose components are regular. Hence Lemma [1.2.3(2)] yields that it suffices to check the following: if \( T \) is a regular scheme of finite type over \( S \), \( b \in \mathbb{Z} \), and \( r \geq 0 \), then \( \mathcal{M}_c^{BM}(T)(b)[2b-r] \in \mathcal{DM}_c(S)_{w_{\text{Chow}}\leq 0} \) and \( t_*(R_T)(b)[2b+r] \in \mathcal{DM}_c(S)_{w_{\text{Chow}}\geq 0} \). Applying the lemma once more, we reduce the first of these inclusion statements to the case where \( T \) is quasi-projective over \( S \) (since any scheme of finite type over \( S \) possesses a stratification whose components are quasi-projective over \( S \)). Similarly, part 3 of the lemma allows us to assume that \( T \) is (regular and) quasi-projective over \( S \) in the second of these inclusion statements. Hence it suffices to note that \( R_T \in \mathcal{DM}_c(T)_{w_{\text{Chow}}=0} \) (by assertion 2c of our Theorem), and apply our assertion 1 along with assertion 2a (for the quasi-projective morphism \( t \)).

Now we return to the proof of assertion 2a for a general \( f \) (of finite type). Assertion 2d immediately yields the left weight-exactness of \( f_* \). Along with Theorem [1, 2.1(4)] it also easily yields the left weight-exactness of \( f^* \). Lastly, \( f_* \) and \( f^! \) are right weight-exact by Proposition [1, 4.2(7)].

The first statement in assertion 2e (also) easily follows from assertion 2d (along with Remark [1.2.2(2)]). Assertion 2d (along with Remark [1.2.2(1)]) also implies the weight-exactness of \( g^* \) in case (i) (since pro-smooth limits of regular schemes are regular). Lastly, \( g^* \cong g^* \) if \( g \) is a finite universal homeomorphism (see Theorem [1, 2.1(11)]); this finishes the proof of the assertion.

3. Since \( i_* \cong i_* \) in this case, the functor \( i_* \) is weight-exact by assertion 2a. The functor \( j^* \) is weight-exact by assertion 2b.

a). \( \mathcal{DM}_c(U) \) is the localization of \( \mathcal{DM}_c(X) \) by \( i_*(\mathcal{DM}_c(Z)) \) by Theorem [1, 2.1(10)]. Hence Proposition [1, 4.2(8)] yields the result (see Remark [1, 4.3] cf. also [Wil12, Theorem 1.7]).

b). Theorem [1, 2.1(10)] yields that \( w_{\text{Chow}}(X) \) is exactly the weight structure obtained by ‘gluing’ \( w_{\text{Chow}}(Z) \) with \( w_{\text{Chow}}(U) \) via Proposition [1, 4.2(10)] (here we use Theorem [1, 2.1(12)]). So we obtain the assertion desired (note that \( j^* = j^! \)).

4. The assertion can be easily proved by induction on the number of strata using assertion 3b.

5. Immediate from assertion 2c. \( \square \)
**Remark 2.2.2.** 1. Theorem [2.1.3](2) and assertion 2d of the previous theorem give two distinct descriptions of $(\mathcal{D}\mathcal{M}_c(S)_{w_{\text{Chow}} \geq 0}, \mathcal{D}\mathcal{M}_c(S)_{w_{\text{Chow}} \leq 0})$ as certain envelopes. It certainly follows that instead of all $T$ considered in the assertion mentioned it suffices to take only those $T$ that are quasi-projective over $S$.

2. One may apply the argument used in the proof of [Jin15] Lemma 2.23] to show that $\mathcal{M}^{BM}_S(X) \otimes \mathcal{M}^{BM}_S(Y) \cong \mathcal{M}^{BM}_S(X \times Y)$ for $X$ and $Y$ being any schemes of finite type over $S$ (note that loc. cit. itself gives this statement for $R = \mathbb{Q}$). It certainly follows that $\mathcal{D}\mathcal{M}_c(S)_{w_{\text{Chow}} \leq 0} \otimes \mathcal{D}\mathcal{M}_c(S)_{w_{\text{Chow}} \leq 0} \subset \mathcal{D}\mathcal{M}_c(S)_{w_{\text{Chow}} \leq 0}$.

Now we prove that positivity and negativity of objects of $\mathcal{D}\mathcal{M}_c(S)$ (with respect to $w_{\text{Chow}}$) can be ‘checked at points’; this is a motivic analogue of [BBD82] [§5.1.8].

**Proposition 2.2.3.** Let $S$ denote the set of (Zariski) points of $S$; for $K \in S$ we will denote the corresponding morphism $K \to S$ by $j_K$.

Then $M \in \mathcal{D}\mathcal{M}_c(S)_{w_{\text{Chow}} \leq 0}$ (resp. $M \in \mathcal{D}\mathcal{M}_c(S)_{w_{\text{Chow}} \geq 0}$) if and only if for any $K \in S$ we have $j_K^*(M) \in \mathcal{D}\mathcal{M}_c(K)_{w_{\text{Chow}} \leq 0}$ (resp. $j_K^*(M) \in \mathcal{D}\mathcal{M}_c(K)_{w_{\text{Chow}} \geq 0}$); see Remark 1.2.2(1).

**Proof.** If $M \in \mathcal{D}\mathcal{M}_c(S)_{w_{\text{Chow}} \leq 0}$ (resp. $M \in \mathcal{D}\mathcal{M}_c(S)_{w_{\text{Chow}} \geq 0}$) then Theorem 2.2.1(2e) (along with (2a)) yields that $j_K^*(M) \in \mathcal{D}\mathcal{M}_c(K)_{w_{\text{Chow}} \leq 0}$ (resp. $j_K^*(M) \in \mathcal{D}\mathcal{M}_c(K)_{w_{\text{Chow}} \geq 0}$) indeed.

We prove the converse implication by Noetherian induction. So, suppose that our assumption is true for motives over any closed subscheme of $S$, and that for some $M \in \text{Obj} \mathcal{D}\mathcal{M}_c(S)$ we have $j_K^*(M) \in \mathcal{D}\mathcal{M}_c(K)_{w_{\text{Chow}} \leq 0}$ (resp. $j_K^*(M) \in \mathcal{D}\mathcal{M}_c(K)_{w_{\text{Chow}} \geq 0}$) for any $K \in S$.

We should prove that $M \in \mathcal{D}\mathcal{M}_c(S)_{w_{\text{Chow}} \leq 0}$ (resp. $M \in \mathcal{D}\mathcal{M}_c(S)_{w_{\text{Chow}} \geq 0}$). By Proposition 1.4.2(2) it suffices to verify: for any $N \in \mathcal{D}\mathcal{M}_c(S)_{w_{\text{Chow}} \geq 1}$ (resp. for any $N \in \mathcal{D}\mathcal{M}_c(S)_{w_{\text{Chow}} \leq -1}$), and any $h \in \mathcal{D}\mathcal{M}_c(S)(M, N)$ (resp. any $h \in \mathcal{D}\mathcal{M}_c(S)(N, M)$) we have $h = 0$. We fix some $N$ and $h$.

By the ‘only if’ part of our assertion (that we have already proved) we have $j_K^*(N) \in \mathcal{D}\mathcal{M}_c(K)_{w_{\text{Chow}} \leq -1}$ (resp. $j_K^*(N) \in \mathcal{D}\mathcal{M}_c(K)_{w_{\text{Chow}} \geq 1}$); hence $j_K^*(h) = 0$. By Theorem 1.2.1[13] we obtain that $j^*(h) = 0$ for some open embedding $j : U \to S$, where $K$ is a generic point of $U$.

Now suppose that $h \neq 0$; let $i : Z \to S$ denote the closed embedding that is complementary to $j$. Then Lemma 1.2.3(2) yields that $\mathcal{D}\mathcal{M}_c(S)(i^*(M), i^!(N)) \neq \{0\}$ (resp. $\mathcal{D}\mathcal{M}_c(S)(i^*(N), i^!(M)) \neq \{0\}$). Yet $i^!(N) \in \mathcal{D}\mathcal{M}_c(Z)_{w_{\text{Chow}} \leq -1}$ (resp. $i^!(N) \in \mathcal{D}\mathcal{M}_c(Z)_{w_{\text{Chow}} \geq 1}$) by Theorem 2.2.1(b2), whereas $i^*(M) \in \mathcal{D}\mathcal{M}_c(Z)_{w_{\text{Chow}} \leq -1}$ (resp. $i^*(M) \in \mathcal{D}\mathcal{M}_c(Z)_{w_{\text{Chow}} \geq 1}$) by the inductive assumption. The contradiction obtained proves our assertion.

Lastly we prove that ‘weights are continuous’.

**Lemma 2.2.4.** Let $K$ be a generic point of $S$; denote the morphism $K \to S$ by $j_K$. Let $M$ be an object of $\mathcal{D}\mathcal{M}_c(S)$, and suppose that $j_K^*M \in \mathcal{D}\mathcal{M}_c(K)_{w_{\text{Chow}} \geq 0}$ (resp. $j_K^*M \in \mathcal{D}\mathcal{M}_c(K)_{w_{\text{Chow}} \leq 0}$). Then there exists an open immersion $j : U \to S$, $K \in U$, such that $j^*M \in \mathcal{D}\mathcal{M}_c(U)_{w_{\text{Chow}} \geq 0}$ (resp. $j^*M \in \mathcal{D}\mathcal{M}_c(U)_{w_{\text{Chow}} \leq 0}$).

**Proof.** First we treat the case $j_K^*(M) \in \mathcal{D}\mathcal{M}_c(K)_{w_{\text{Chow}} \geq 0}$. We consider a weight decomposition of $M[1]$: $B \xrightarrow{g} M[1] \to A \to B[1]$. We obtain that $j_K^*(g) = 0$ (since $\mathcal{D}\mathcal{M}_c(K)_{w_{\text{Chow}} \leq 0} \perp j_K^*(M)[1]$); hence (by Theorem 1.2.1[13]) there exists an open immersion $j : U \to S$ ($K \in U$) such that $j^*(g) = 0$. Thus $j^*M[1]$ is a retract of $j^*A$. Since $j^*A[-1] \in \mathcal{D}\mathcal{M}_c(U)_{w_{\text{Chow}} \geq 0}$ (see Theorem 2.2.1(2b)), and $\mathcal{D}\mathcal{M}_c(U)_{w_{\text{Chow}} \geq 0}$ is Karoubi-closed in $\mathcal{D}\mathcal{M}_c(U)$, we obtain the result.
2.3. Describing $Hw_{\text{Chow}}$ via Chow motives. Now we prove that for certain $S$ the heart of $w_{\text{Chow}}(S)$ has a quite "explicit" description in terms of certain Chow motives over $S$ (whence the name).

Remark 2.3.1. For a scheme $S$ define the category Chow($S$) of Chow motives over $S$ as the Karoubi-closure of $\{M^S_X(r)[2r]\}$ in $DM_c(S)$; here $f : X \to S$ runs through all proper morphisms such that $X$ is regular, $r \in \mathbb{Z}$.

Then Theorem 2.2.1(2c,2a) yields that Chow($S$) is an equivalence of categories. 

Proposition 2.3.2. Assume that $S$ can be presented as a filtered (projective) limit of varieties over some (not necessarily prime) field with smooth and affine transition morphisms. Then Chow($S$) = $Hw_{\text{Chow}}(S)$.

Proof. Since Chow($S$) is a filtered limit of Chow($S'$)s, it is negative (by the orthogonality axiom of weight structures). Hence Proposition 1.4.2 yields the existence of a weight structure $w$ on the triangulated subcategory $T$ of $DM_c(S)$ generated by Chow($S$) such that Chow($S$) $\cong$ $Hw$. Besides, parts 3 and 6 of the proposition yield that the embedding of $T$ into $DM_c(S)$ is weight-exact (with respect to $w$ and $w_{\text{Chow}}$). Hence part 4 of the proposition reduces the assertion to the fact that $T = DM_c(S)$, i.e., that Chow($S$) generates $DM_c(S)$.

By Theorem 1.2.1(12) the latter is true if $S$ is a variety itself. Hence it remains to pass to the ("pro-smooth affine") limit in this statement. By continuity (Theorem 1.2.1(13)) in order to achieve it suffices to note that pro-smooth affine morphisms respect Chow motives; the latter is immediate from Remark 1.2.2(2) (along with the fact that pro-smooth base change preserves regularity of schemes).

Remark 2.3.3. 1. This argument also shows that we could have considered only projective (regular) $X$ over $S$ in the definition of Chow($S$); we would have obtained the same category Chow($S$) (at least) when $S$ is as in Proposition 2.3.2.

2. Actually, the negativity of Chow($S$) in $DM_c(S)$ (for the "projective version" of the definition) follows immediately from Lemma 1.3.3. Thus we could have defined the corresponding "restriction" of $w_{\text{Chow}}$ (on a full subcategory of $DM_c(S)$) without any gluing arguments. Next, restricting ourselves to the case where $DM_c(S)$ is "Chow-generated", we could have easily applied the arguments used in the proof of Theorem 2.2.1 to this version of the Chow weight structure. This is (basically) the approach to the study of the Chow weight structures used in [Heb11] and [Bon14a] §2.1–2.2 (yet some of the methods used in the proof of Theorem 2.2.1 are "newer"; they were developed in [Bon14b] and in the current paper).

We chose not to apply this approach in the current paper since the class of base schemes for which we can use it is too small. The reason for this is that for $\mathbb{Q}$-linear motives one only needs certain alterations for the corresponding analogue of Theorem 1.2.1(12) (cf. [CiD09] §4.1]), whereas for $\mathbb{Z}^{[p]}$-linear motives one requires the so-called prime-to-$l$ alterations (for all primes $l \neq p$; cf. the proof of [CiD14] Proposition 7.2]) whose existence is only known in the context the aforementioned assertion.

3. Certainly, our proposition does not yield a "full description" of $Hw_{\text{Chow}}(S)$ since we have not "computed the morphisms" in Chow($S$). We note that the argument used in (the proof of) [Bon14a] Lemma 1.1.4(I.1]) allows one to express $DM_c(S)(M^S_{BM}(X)(r)[s], M^S_{BM}(X')(r')[s'])$ in terms of certain (Borel-Moore) motivic homology groups; here $r, s, r', s' \in \mathbb{Z}$, $X$ and $X'$
are regular schemes that are projective over $S$ (whereas $S$ is ”pro-smooth affine” over a
variety). Thus we can compute a ”substantial part” of morphism groups in Chow($S$). Yet
computing the composition operation for Chow($S$)-morphisms is a much more difficult
problem; for $R = \mathbb{Q}$ it was recently solved in [Jin15].

3. Applications to (co)homology of motives and other matters

In this section we list some immediate applications for our results (following [Bon14a]:
so a reader well acquainted with ibid. may skip this section completely or just have a
look at Proposition 3.2.3). Most of the statements below easily follow from the results
of [Bon10]; there is absolutely no problem to apply the corresponding arguments from
[Bon14a §3] (where the case $R = \mathbb{Q}$ was considered) for their proofs. The results seem
to be ”more interesting” in the case where $S$ is a pro-smooth affine limit of varieties (cf.
Proposition 2.3.2).

3.1. The weight complex functor for $\mathcal{DM}_{c}(S)$ and its Grothendieck group. We
note that the weight complex functor (whose ‘first ancestor’ was defined in [GiS96]) can
be defined for $\mathcal{DM}_{c}(S)$.

**Proposition 3.1.1.** (1) The embedding $H_{w,\text{Chow}}(S) \rightarrow K^{b}(H_{w,\text{Chow}}(S))$ factors through
a certain weight complex functor $t_{S} : \mathcal{DM}_{c}(S) \rightarrow K^{b}(H_{w,\text{Chow}}(S))$ which is exact and conservative.

(2) For $M \in \text{Obj}\mathcal{DM}_{c}(S)$, $i, j \in \mathbb{Z}$, we have $M \in \mathcal{DM}_{c}(S)[i,j]$ (see Definition
1.4.1(4)) if and only if $t_{S}(M) \in K(H_{w,\text{Chow}}(S))[i,j]$.

Now we calculate $K_{0}(\mathcal{DM}_{c}(S))$ and define a certain Euler characteristic for schemes
that are of finite type (and separated) over $S$.

**Proposition 3.1.2.** (1) We define $K_{0}(H_{w,\text{Chow}}(S))$ as the abelian group whose gen-
erators are $[M]$, $M \in \mathcal{DM}_{c}(S)_{w,\text{Chow}}=0$, and the relations are $[B] = [A] + [C]$ if $A, B, C \in \mathcal{DM}_{c}(S)_{w,\text{Chow}}=0$ and $B \cong A \oplus C$. For $K_{0}(\mathcal{DM}_{c}(S))$ we take similar

generators and set $[B] = [A] + [C]$ if $A \rightarrow B \rightarrow C \rightarrow A[1]$ is a distinguished

triangle.

Then the embedding $H_{w,\text{Chow}}(S) \rightarrow \mathcal{DM}_{c}(S)$ yields an isomorphism $K_{0}(H_{w,\text{Chow}}(S)) \cong
K_{0}(\mathcal{DM}_{c}(S))$.

(2) For the correspondence $\chi : X \mapsto [\mathcal{M}_{S}^{BM}(X)]$ from the class of schemes separated
of finite type over $S$ to $K_{0}(\mathcal{DM}_{c}(S)) \cong K_{0}(H_{w,\text{Chow}}(S))$ we have: $\chi(X \setminus Z) =
\chi(X) - \chi(Z)$ if $Z$ is a closed subscheme of $X$.

3.2. On Chow-weight spectral sequences and filtrations. Now we discuss (Chow)-
weight spectral sequences and filtrations for cohomology of motives. Certainly, here one
can pass to homology via obvious dualization (see Proposition 1.4.2(1)). We note that any
weight structure yields certain weight spectral sequences for any (co)homology theory;
the main distinction of the result below from the general case (i.e., from [Bon10 Theorem
2.4.2]) is that $T(H, M)$ always converges (since $w_{\text{Chow}}$ is bounded).

**Proposition 3.2.1.** Let $\mathcal{A}$ be an abelian category.

(1) Let $H : \mathcal{DM}_{c}(S) \rightarrow \mathcal{A}$ be a cohomological functor; for any $r \in \mathbb{Z}$ denote $H \circ [-r]$ by $H^{r}$. For an $M \in \text{Obj}\mathcal{DM}_{c}(S)$ we denote by $(M^{r})$ the terms of $t_{S}(M)$ (so,$\ M^{r} \in \mathcal{DM}_{c}(S)_{w,\text{Chow}}=0$; here we can take any possible choice of $t_{S}(M)$).

Then the following statements are valid.
(a) There exists a (Chow-weight) spectral sequence \( T = T(H, M) \) with \( E^r_1 = H^r(M^{-p}) \) \( \Rightarrow \) \( H^{p+q}(M) \); the differentials for \( E_1(T(H, M)) \) come from \( t_S(M) \).

(b) \( T(H, M) \) is \( \mathcal{D} \mathcal{M}_c(S) \)-functorial in \( M \) (and does not depend on any choices) starting from \( E_2 \).

(2) Let \( G : \mathcal{D} \mathcal{M}_c(S) \rightarrow \mathbb{A} \) be any contravariant functor. Then for any \( m \in \mathbb{Z} \) the object \( (W^m G)(M) = \text{Im}(G(w_{\text{Chow}, \geq m} M) \rightarrow G(M)) \) does not depend on the choice of \( w_{\text{Chow}, \geq m} M \); it is functorial in \( M \). We call the filtration of \( G(M) \) by \( (W^m G)(M) \) its Chow-weight filtration. If \( G \) is cohomological, it coincides with the filtration given by \( T(G, M) \).

Remark 3.2.2. 1. We obtain certain ("motivically functorial") Chow-weight spectral sequences and filtrations for any (co)homology of motives. In particular, we have them for étale and motivic cohomology of \( S \)-motives (with coefficients in an \( R \)-algebra). The corresponding functoriality results cannot be proved using 'classical' (i.e., Deligne's) methods, since the latter heavily rely on the degeneration of (an analogue of) \( T \) at \( E_2 \).

On the other hand, we probably do not have the "strict functoriality" property for this filtration unless \( T(H, M) \) degenerates for any \( M \in \text{Obj} \mathcal{D} \mathcal{M}_c(S) \) (see [Bon12, Proposition 3.1.2(II)]), whereas this degeneration is only known to hold for "more or less classical" \( \mathbb{Q} \)-linear cohomology theories only.

2. \( T(H, M) \) can be naturally described in terms of the virtual \( t \)-truncations of \( H \) (starting from \( E_2 \)); see [Bon10] \S 2.5] and [Bon14a].

3. For \( S \) being a pro-smooth affine limit of varieties (cf. \$2.3\)$ we obtain that the (co)homology of any \( M \in \text{Obj} \mathcal{D} \mathcal{M}_c(S) \) possesses a filtration by subfactors of (co)homology of regular projective \( S \)-schemes.

4. The arguments in [CGD13] \S 7.2] easily yield (for \( R \subset \mathbb{Q} \), \( p \neq l \in \mathbb{P} \)) the existence of a (covariant) étale realization functor from \( \mathcal{D} \mathcal{M}_c(S) \) into the corresponding category of Ekedahl's (étale, constructible) \( \mathbb{Q} \)-adic systems.

Note next that for \( S \) being a variety over \( k \) and \( p > 0 \) the target category is endowed with a certain weight filtration (that was defined in [BBD82], \$5\)); for \( p = 0 \) one can factor the realization functor mentioned through a certain category that is endowed with certain "weights" also (and was defined by A. Huber; see [Bon15, Proposition 2.5.1]). Moreover (see loc. cit.) in both cases the corresponding functors "respect weights". So, we obtain that \( w_{\text{Chow}} \) is closely related to Deligne's weights for constructible complexes of sheaves (that were crucial for defining "weights" in [Hub97] also). Yet (as explained in [Bon15, Remark 2.5.2]) the weight filtrations on the "étale categories" mentioned do not yield weight structures for them.

The functoriality of Chow-weight filtrations has quite interesting consequences.

Proposition 3.2.3. For a scheme \( X \) let \( H : \mathcal{D} \mathcal{M}_c(X) \rightarrow \mathbb{A} \) be a contravariant functor. For all \( N \in \mathcal{D} \mathcal{M}_c(U)_{w_{\text{Chow}} \leq 0} \) consider \( G(N) = (W^0 H)(j_!(N)) \subset H(j_!(N)) \).

(1) The following statements are fulfilled.

(a) \( G(N) \) is \( \mathcal{D} \mathcal{M}_c(U) \)-functorial in \( N \).

(b) \( G(N) \) is a quotient of \( H(M) \) for some \( M \in \mathcal{D} \mathcal{M}_c(X)_{w_{\text{Chow}} = 0} \).

(2) Let now \( N = j^*(M) (= j^!(M)) \) for \( M \in \mathcal{D} \mathcal{M}_c(X)_{w_{\text{Chow}} = 0} \). Then also the following is true.

(a) \( G(N) = \text{Im} H(M) \rightarrow H(j_! j^!(M)) \) (here we apply \( H \) to the morphism \( j_! j^!(M) \rightarrow M \) coming from the adjunction \( j_! \dashv j^! \)).
(b) Let \( H = \mathcal{DM}_c(X)(-; \mathbb{H}) \) for some \( \mathbb{H} \in \text{Obj} \mathcal{DM}(X) \). Then \( G(N) \cong \text{Im}(\mathcal{DM}_c(X)(M; \mathbb{H}) \to \mathcal{DM}(U)(N; j^*(\mathbb{H}))) \).

Remark 3.2.4. 1. Thus one may say that \((W^0H)(j_!(N))\) yields the 'integral part' of \( H(j_!(N)) \): we obtain the subobject of \( H^*(j_!(N)) \) that 'comes from any nice \( X \)-lift' of \( N \) if the latter exists, and a factor of \( H(M) \) for some \( M \in \mathcal{DM}_c(X)_{w\text{-Chow}=0} \) in general; cf. [Bei84] and [Sch00]. If \( N = \mathcal{M}^{BM}(T_U) \) for a regular \( T_U \) proper over \( U \), then any regular proper \(^\ast\) \( X \)-model \( T_X \) for \( T_U \) yields such a 'nice \( X \)-lift' for \( N \). Besides, in the setting of assertion 2b one can take \( \mathbb{H} \) representing \( \acute{e} \text{tale or motivic cohomology} \) (for an appropriate choice of coefficients and some fixed degree; so that \( \mathcal{DM}(S)(R_Y, f^*(\mathbb{H})) \) is the corresponding cohomology of \( Y \) for any \( f : Y \to X \)); then \( G(N) \) will be the image of this cohomology of \( T_X \) in the one of \( T_U \).

In this case one can also study the cohomology of an object \( N_K \) of \( \text{Chow}(K) \) (i.e., of an element of \( \mathcal{DM}_{c}(K)_{w\text{-Chow}=0} \)) for \( K \) being some generic point of \( X \). Indeed, any such \( N_K \) can be lifted to a Chow motif \( N \) over some \( U (K \in U, U \text{ is open in } X) \) by Remark 1.2.2(2) combined with Theorem 1.2.1[13] (since if \( N_K \) is a retract of \( \mathcal{M}^{BM}_K(T_K) \) for some regular variety \( T_K/K \), then \( T_K \) along with this retraction can be lifted to a regular \( T_U \) over some open \( U \subset X \) that contains \( K \)). Note that this construction enjoys 'the usual' \( \text{Chow}(K) \)-functoriality; this is an easy consequence of Theorem 1.2.1[13].

2. It could also be interesting to consider \( W^lH^*(j_i(N)) \) for \( l < 0 \).

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