Some quasihomogeneous Legendrian varieties

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Abstract

We construct a family of examples of Legendrian subvarieties in some projective spaces. Although most of them are singular, a new example of smooth Legendrian variety in dimension 8 is in this family. The 8-fold has interesting properties: it is a compactification of the special linear group, a Fano manifold of index 5 and Picard number 1.

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1 Introduction

Real Legendrian subvarieties are classical objects of differential geometry and they have been investigated for ages. However, complex Legendrian subvarieties in a projective space (see §1.1.3 for the definition) are much more rigid and only few smooth and compact examples were known (see [Bry82], [LM04], [Buc07b]):

1. linear subspaces;
2. some homogeneous spaces called subadjoint varieties: the product of a line and a quadric \( \mathbb{P}^1 \times \mathbb{Q}^n \) and five exceptional cases:
   - twisted cubic curve \( \mathbb{P}^1 \subset \mathbb{P}^3 \),
   - Grassmannian \( Gr_L(3, 6) \subset \mathbb{P}^{13} \) of Lagrangian subspaces in \( \mathbb{C}^6 \),
   - full Grassmannian \( Gr(3, 6) \subset \mathbb{P}^{19} \),
   - spinor variety \( S_6 \subset \mathbb{P}^{31} \) (i.e. the homogeneous \( SO(12) \)-space parametrizing the vector subspaces of dimension 6 contained in a non-degenerate quadratic cone in \( \mathbb{C}^{12} \)) and
   - the 27-dimensional \( E_7 \)-variety in \( \mathbb{P}^{55} \) corresponding to the marked root:

\[ \cdots \]

3. every smooth projective curve admits a Legendrian embedding in \( \mathbb{P}^3 \) [Bry82];
4. a family of surfaces birational to the Kummer \( K3 \)-surfaces [LM04];
5. the blow up of \( \mathbb{P}^2 \) in three general points [Buc07b].

In this article we present a new example in dimension 8 (see theorem 1.4 (b)). Also we show how does the construction generalise to give new examples in dimensions 5 and 14 (see section 1.3) and finally we announce a result which will produce plenty of such examples (see section 1.4).

The original motivation for studying Legendrian subvarieties in a complex projective space comes from the studies of contact Fano manifolds\(^1\) (see [Wis00], [Keb01], [KPSW00]): The variety of tangent directions to the minimal rational curves through a fixed point on a contact Fano manifold make a Legendrian subvariety in the projectivisation of the fibre of contact distribution. The adjoint varieties (i.e. the closed orbit of the adjoint action of a simple Lie group \( G \) on \( \mathbb{P}(\mathfrak{g}) \)) are the only known examples of contact Fano manifolds and they give rise to the homogeneous Legendrian varieties\(^2\).

From our considerations here, some other potential applications come into the view — see sections 1.3 and 1.5.

Before we present our results precisely in section 1.2 we must introduce some notation. We need the notation of \{1.1.1-1.1.5\} to state the results and also \{1.1.6-1.1.10\} to prove them.

1.1 Notation and definitions

For this article we fix an integer \( m \geq 2 \).

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\(^1\) A complex projective manifold \( M \) of dimension \( 2n + 1 \) is called a contact manifold, if there exists a vector subbundle \( F \subset TM \) of rank \( 2n \), such that the map \( F \otimes F \rightarrow TM/F \) determined by the Lie bracket is nowhere degenerate. In such a case \( F \) is called a contact distribution. A projective manifold is Fano, if the anticanonical bundle is ample.

\(^2\) The groups of types \( B \) and \( D \) give rise to \( \mathbb{P}^1 \times \mathbb{Q}^n \). The five exceptional groups \( G_2, F_4, E_6, E_7, E_8 \) make the exceptional homogeneous Legendrian varieties. The groups of types \( A \) and \( C \) are somewhat special — see [LM04], [Buc06].
1.1.1 Vector space $V$

Let $V$ be a vector space over complex numbers $\mathbb{C}$ of dimension $2m^2$, which we interpret as a space of pairs of $m \times m$ matrices. The coordinates are: $a_{ij}$ and $b_{ij}$ for $i,j \in \{1, \ldots, m\}$. By $A$ we denote the matrix $(a_{ij})$ and similarly for $B$ and $(b_{ij})$.

By $\mathbb{P}(V)$ we mean the naive projectivisation of $V$, i.e. the quotient $(V \setminus \{0\})/\mathbb{C}^*$.

Given two $m \times m$ matrices $A$ and $B$, by $(A, B)$ we denote the point of the vector space $V$, while by $[A, B]$ we denote the point of the projective space $\mathbb{P}(V)$.

Sometimes, we will represent some linear maps $V \to V$ and some 2-linear forms $V \otimes V \to \mathbb{C}$ as $2m^2 \times 2m^2$ matrices. In such a case we will assume the coordinates on $V$ come in the lexicographical order: 

$a_{11}, \ldots, a_{1m}, a_{21}, \ldots, a_{mm}, b_{11}, \ldots, b_{1m}, b_{21}, \ldots, b_{mm}$.

1.1.2 Symplectic form $\omega$

On $V$ we consider the standard symplectic form $\omega((A, B), (A', B')) := \sum_{i,j} (a_{ij}b'_{ij} - a'_{ij}b_{ij}) = tr (A(B')^T - A'B^T)$. (1.1)

Further we set $J$ to be the matrix of $\omega$:

$$J := \begin{bmatrix} 0 & \text{Id}_{m^2} \\ -\text{Id}_{m^2} & 0 \end{bmatrix}.$$ 

1.1.3 Lagrangian and Legendrian subvarieties

A linear subspace $W \subset V$ is called Lagrangian if the $\omega$-perpendicular subspace $W^\perp$ is equal to $W$. Equivalently, $W$ is Lagrangian if and only if $\omega|_W \equiv 0$ and $\dim W$ is maximal possible, i.e. equal to $\frac{1}{2}\dim V$.

A subvariety $Z \subset V$ is called Lagrangian if for every smooth point $z \in Z$ the tangent space $T_zZ \subset V$ is Lagrangian. In particular, if $Z$ is Lagrangian, then $\dim Z = \frac{1}{2}\dim V = m^2$.

A subvariety $X \subset \mathbb{P}(V)$ is defined to be Legendrian if its affine cone $\hat{X} \subset V$ is Lagrangian. In particular, if $X$ is Legendrian, then $\dim X = \frac{1}{2}\dim V - 1 = m^2 - 1$.

1.1.4 Varieties $Y$, $X_{\text{inv}}(m)$ and $X_{\text{deg}}(m, k)$

We consider the following subvariety of $\mathbb{P}(V)$:

$$Y := \{ [A, B] \in \mathbb{P}(V) \mid AB^T = B^T A = \lambda^2 \text{Id}_m \text{ for some } \lambda \in \mathbb{C} \}.$$ (1.2)

The square at $\lambda$ seems to be irrelevant here, but it slightly simplifies the notation in the proofs of theorem 1.4(b) and proposition 2.4(ii).

Further we define two types of subvarieties of $Y$:

$$X_{\text{inv}}(m) := \left\{ [g, (g^{-1})^T] \in \mathbb{P}(V) \mid \det g = 1 \right\}$$

$$X_{\text{deg}}(m, k) := \left\{ [A, B] \in \mathbb{P}(V) \mid AB^T = B^T A = 0, \ \text{rk} A \leq k, \ \text{rk} B \leq m - k \right\}$$

where $k \in 0, 1, \ldots, m$. The varieties $X_{\text{deg}}(m, k)$ have been also studied by [Str82] and [MT99]. $X_{\text{inv}}(m)$ (especially $X_{\text{inv}}(3)$) is the main object of this article.
1.1.5 Automorphisms $\psi_\mu$

For any $\mu \in \mathbb{C}^*$ we let $\psi_\mu$ be the following linear automorphism of $V$:

$$\psi_\mu((A,B)) := (\mu A, \mu^{-1} B).$$

Also the induced automorphism of $\mathbb{P}(V)$ will be denoted in the same way:

$$\psi_\mu([A,B]) := [\mu A, \mu^{-1} B].$$

The notation introduced so far is sufficient to state the results of this paper (see section 1.2), but to prove them we need a few more notions.

1.1.6 Groups $G$ and $\tilde{G}$, Lie algebra $\mathfrak{g}$ and their representation

We set $\tilde{G} := \text{Gl}_m \times \text{Gl}_m$ and let it act on $V$ by:

$$(g,h) \in \tilde{G}, \quad (g,h) \cdot (A,B) := (g^T A h, g^{-1} B (h^{-1})^T).$$

This action preserves the symplectic form $\omega$.

We will mostly consider the restricted action of $G := \text{SL}_m \times \text{SL}_m < \tilde{G}$.

We also set $\mathfrak{g} := \mathfrak{sl}_m \times \mathfrak{sl}_m$ to be the Lie algebra of $G$ and we have the tangent action of $\mathfrak{g}$ on $V$:

$$(g,h) \cdot (A,B) = (g^T A + Ah, -gB - Bh^T).$$

Though we denote the action of the groups $G, \tilde{G}$ and the Lie algebra $\mathfrak{g}$ by the same · we hope it will not lead to any confusion. Also the induced action of $G$ and $\tilde{G}$ on $\mathbb{P}(V)$ will be denoted by ·.

1.1.7 Orbits $\mathcal{IN} \mathcal{V}^m$ and $\mathcal{DEG}_{k,l}^m$

We define the following sets:

$$\mathcal{IN} \mathcal{V}^m := \left\{ \left[ g, (g^{-1})^T \right] \in \mathbb{P}(V) \mid \det g = 1 \right\},$$

$$\mathcal{DEG}_{k,l}^m := \left\{ [A,B] \in \mathbb{P}(V) \mid AB^T = B^T A = 0, \ rk A = k, \ rk B = l \right\},$$

so that $X_{\text{inv}}(m) = \mathcal{IN} \mathcal{V}^m$ and $X_{\text{deg}}(m,k) = \mathcal{DEG}_{k,m}^m$.

Clearly, if $k + l > m$ then $\mathcal{DEG}_{k,l}^m$ is empty, so whenever speaking of $\mathcal{DEG}_{k,l}^m$ we will assume $k + l \leq m$.

1.1.8 Elementary matrices $E_{ij}$ and points $p_1$ and $p_2$

Let $E_{ij}$ be the elementary $m \times m$ matrix with unit in the $i^{th}$ row and the $j^{th}$ column and zeroes elsewhere.

We distinguish two points $p_1 \in \mathcal{DEG}_{1,0}^m$ and $p_2 \in \mathcal{DEG}_{0,1}^m$:

$$p_1 := [E_{mm}, 0] \quad \text{and} \quad p_2 := [0, E_{mm}]$$

These points will be usually chosen as nice representatives of the closed orbits $\mathcal{DEG}_{1,0}^m$ and $\mathcal{DEG}_{0,1}^m$. 
1.1.9 Tangent cone

We recall the notion of the tangent cone and a few among many of its properties. For more details and the proofs we refer to [Har95, lecture 20] and [Mum99, III.§3,§4].

For an irreducible Noetherian scheme $X$ over $\mathbb{C}$ and a closed point $x \in X$ we consider the local ring $\mathcal{O}_{X,x}$ and we let $m_x$ to be the maximal ideal in $\mathcal{O}_{X,x}$. Let

$$R := \bigoplus_{i=0}^{\infty} \left( m_x^i / m_x^{i+1} \right)$$

where $m_x^0$ is just the whole $\mathcal{O}_{X,x}$. Now we define the tangent cone $TC_x X$ at $x$ to $X$ to be $\text{Spec } R$.

If $X$ is a subscheme of an affine space $\mathbb{A}^n$ (which we will usually assume to be an affine piece of a projective space) the tangent cone at $x$ to $X$ can be understood as a subscheme of $\mathbb{A}^n$. Its equations can be derived from the ideal of $X$. For simplicity assume $x = 0 \in \mathbb{A}^n$ and then the polynomials defining $TC_0 X$ are the lowest degree homogeneous parts of the polynomials in the ideal of $X$.

Another interesting point-wise definition is that $v \in TC_0 X$ is a closed point if and only if there exists a holomorphic map $\varphi_v$ from the disc $D_t := \{ t \in \mathbb{C} : |t| < \delta \}$ to $X$, such that $\varphi_v(0) = 0$ and the first non-zero coefficient in the Taylor expansion in $t$ of $\varphi_v(t)$ is $v$, i.e.:

$$\varphi_v : D_t \to X \quad t \mapsto t^k v + t^{k+1} v_{k+1} + \ldots$$

We list some of the properties of the tangent cone, that will be used freely in the proofs:

1. The dimension of every component of $TC_x X$ is equal to the dimension of $X$.
2. $TC_x X$ is naturally embedded in the Zariski tangent space to $X$ at $x$ and $TC_x X$ spans the tangent space.
3. $X$ is regular at $x$ if and only if $TC_x X$ is equal (as scheme) to the tangent space.

1.1.10 Submatrices - extracting rows and columns

Assume $A$ is an $m \times m$ matrix and $I, J$ are two sets of indices of cardinality $k$ and $l$ respectively:

$$I := \{ i_1, i_2, \ldots, i_k | 1 \leq i_1 < i_2 < \ldots < i_k \leq m \},$$

$$J := \{ j_1, j_2, \ldots, j_l | 1 \leq j_1 < j_2 < \ldots < j_l \leq m \}.$$  

Then we denote by $A_{I,J}$ the $(m-k) \times (m-l)$ submatrix of $A$ obtained by removing rows of indices $I$ and columns of indices $J$. Also for a set of indices $I$ we denote by $I'$ the set of $m - k$ indices complementary to $I$.

We will also use a simplified version of the above notation, when we remove only a single column and single row: $A_{ij}$ denotes the $(m-1) \times (m-1)$ submatrix of $A$ obtained by removing $i$-th row and $j$-th column, i.e. $A_{ij} = A_{\{i\},\{j\}}$.

Also in the simplest situation where we remove only the last row and the last column, we simply write $A_m$, so that $A_m = A_{mm} = A_{\{m\},\{m\}}$. 
1.2 Main Results

In this note we give a classification of Legendrian subvarieties in $\mathbb{P}(V)$ that are contained in $Y$.

**Theorem 1.3.** Let projective space $\mathbb{P}(V)$, varieties $Y$, $X_{\text{inv}}(m)$, $X_{\text{deg}}(m,k)$ and automorphisms $\psi_\mu$ be defined as in §1.1.1-§1.1.5. Assume $X \subset \mathbb{P}(V)$ is an irreducible subvariety. Then $X$ is Legendrian and contained in $Y$ if and only if $X$ is one of the following varieties:

1. $X = \psi_\mu(X_{\text{inv}}(m))$ for some $\mu \in \mathbb{C}^*$ or
2. $X = X_{\text{deg}}(m,k)$ for some $k \in \{0,1,\ldots,m\}$.

The idea of the proof of theorem 1.3 is based on the observation that every Legendrian subvariety that is contained in $Y$ must be invariant under the action of group $G$. This is explained in section 2. A proof of the theorem is presented in section 3.1.

Also we analyse which of the above varieties appearing in 1. and 2. are smooth:

**Theorem 1.4.** With the definition of $X_{\text{inv}}(m)$ as in §1.1.4, the family $X_{\text{inv}}(m)$ contains the following varieties:

(a) $X_{\text{inv}}(2)$ is a linear subspace.
(b) $X_{\text{inv}}(3)$ is smooth, its Picard group is generated by a hyperplane section. Moreover $X_{\text{inv}}(3)$ is a compactification of $SL_3$ and it is isomorphic to a hyperplane section of Grassmannian $Gr(3,6)$. The connected component of $\text{Aut}(X_{\text{inv}}(3))$ is equal to $G = SL_3 \times SL_3$ and $X_{\text{inv}}(3)$ is not a homogeneous space.
(c) $X_{\text{inv}}(4)$ is the 15 dimensional spinor variety $S_6$.
(d) For $m \geq 5$, the variety $X_{\text{inv}}(m)$ is singular.

A proof of the theorem is explained in section 3.3. Variety $X_{\text{inv}}(3)$ is not yet described as a Legendrian subvariety variety, so it is our new smooth example of dimension 8.

**Theorem 1.5.** With the definition of $X_{\text{deg}}(m)$ as in §1.1.4, variety $X_{\text{deg}}(m,k)$ is smooth if and only if $k = 0$ , $k = m$ or $(m,k) = (2,1)$. In the first two cases, $X_{\text{deg}}(m,0)$ and $X_{\text{deg}}(m,m)$ are linear spaces, while $X_{\text{deg}}(2,1) \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$.

A proof of the theorem is presented in section 3.2.

The results of theorems 1.3, 1.4 can be generalised in (at least) three different directions:

1.3 Generalisation 1: Representation theory

The interpretation of theorem 1.3 (b) and (c) can be following: We take the exceptional Legendrian variety $Gr(3,6)$, slice it with a linear section and we get a description, that generalised to matrices of bigger size gives the bigger exceptional Legendrian variety $S_6$. Similar connection can be established between other exceptional Legendrian varieties.

For instance, assume that $V^{sym}$ is a vector space of dimension $2 \binom{m+1}{2}$, which we interpret as the space of pairs of $m \times m$ symmetric matrices $A, B$. Now in $\mathbb{P}(V^{sym})$ consider the subvariety $X_{\text{inv}}^{sym}(m)$, which is the closure of the following set:

$\{[A, A^{-1}] \in \mathbb{P}(V^{sym}) | A = A^T \text{ and } \det A = 1\}$.

This problem was suggested by Sung Ho Wang.
Theorem 1.6. All the varieties $X_{\text{inv}}^{\text{sym}}(m)$ are Legendrian and we have:

(a) $X_{\text{inv}}^{\text{sym}}(2)$ is a linear subspace.

(b) $X_{\text{inv}}^{\text{sym}}(3)$ is smooth and it is isomorphic to a hyperplane section of Lagrangian Grassmannian $\text{Gr}_L(3, 6)$.

(c) $X_{\text{inv}}^{\text{sym}}(4)$ is smooth and it is the 9 dimensional Grassmannian variety $\text{Gr}(3, 6)$.

(d) For $m \geq 5$, the variety $X_{\text{inv}}^{\text{sym}}(m)$ is singular.

The proof goes exactly as the proof of theorem 1.4.

Similarly, we can take $V_{\text{skew}}$ to be a vector space of dimension $2^{2m}$, which we interpret as the space of pairs of $2m \times 2m$ skew-symmetric matrices $A, B$. Now in $\mathbb{P}(V_{\text{skew}})$ consider subvariety $X_{\text{inv}}^{\text{skew}}(m)$, which is the closure of the following set:

$$\{(A, -A^{-1}) \in \mathbb{P}(V_{\text{skew}}) \mid A = -A^T \text{ and Pfaff } A = 1\}.$$

Theorem 1.7. All the varieties $X_{\text{inv}}^{\text{skew}}(m)$ are Legendrian and we have:

(a) $X_{\text{inv}}^{\text{skew}}(2)$ is a linear subspace.

(b) $X_{\text{inv}}^{\text{skew}}(3)$ is smooth and it is isomorphic to a hyperplane section of the spinor variety $S_6$.

(c) $X_{\text{inv}}^{\text{skew}}(4)$ is smooth and it is the 27 dimensional $E_7$ variety.

(d) For $m \geq 5$, the variety $X_{\text{inv}}^{\text{skew}}(m)$ is singular.

Here the only difference is that we replace the determinants by the Pfaffians of the appropriate submatrices and also for the previous cases we will be picking some diagonal matrices as nice representatives. Since there is no non-zero skew-symmetric diagonal matrix, we must modify a little bit our calculations, but there is no essential difference in the technique.

Neither $X_{\text{inv}}^{\text{sym}}(3)$ nor $X_{\text{inv}}^{\text{skew}}(3)$ have been described as smooth Legendrian sub-varieties.

Therefore we have established a new connection between the subadjoint varieties of the 4 exceptional groups $F_4$, $E_6$, $E_7$ and $E_8$. A similar connection was obtained by [LM02].

1.4 Generalisation 2: Hyperplane section

The variety $X_{\text{inv}}(3)$ is the first described example of smooth non-homogeneous Legendrian variety of dimension bigger than 2 (see [Bry82], [LM04], [Buc07b]). But this example is very close to a homogeneous one, namely isomorphic to a hyperplane section of $\text{Gr}(3, 6)$, which is a well known Legendrian variety. So a natural question arises, whether a general hyperplane section of other Legendrian varieties admits Legendrian embedding. The answer is yes and we explain it (as well as many conclusions from this surprisingly simple observation) in [Buc07a].

1.5 Generalisation 3: Group compactification

Theorem 1.4(b) says that $X_{\text{inv}}(3)$ is a smooth compactification of $\text{SL}_3$. In [Buc] we study a generalisation of this construction (which is not really related to Legendrian varieties) to find a family of compactifications of $\text{SL}_n$, which contains the smooth compactification of $\text{SL}_3$ and can be easily smoothened (by a single blow up of a closed orbit) for $n = 4$. 


\textbf{2 \ G-action and its orbits}

In [Buc06] we prove:

\textbf{Theorem 2.1.} Let $X \subset \mathbb{P}(V)$ be a Legendrian subvariety (see §1.1.3 for definition). Consider the following map:

$$H^0(O_{\mathbb{P}(V)}(2)) \simeq \text{Sym}^2 V^* \ni q = (x \mapsto x^T M(q)x) \mapsto 2J \cdot M(q) \in \mathfrak{sp}(V).$$

where $M(q)$ is the $(2m^2) \times (2m^2)$ matrix of $q$ and $J$ is the matrix of the symplectic form as in §1.1.1. Let $\mathcal{I}_2(X) \subset \text{Sym}^2 V^*$ be the vector space of quadrics containing $X$. Then:

- $\rho(\mathcal{I}_2(X))$ is a Lie subalgebra of $\mathfrak{sp}(V)$ tangent to a closed subgroup

$$\exp\left(\rho(\mathcal{I}_2(X))\right) < \text{Sp}(V).$$

- We have the natural action of $\text{Sp}(V)$ on $\mathbb{P}(V)$. The group $\exp\left(\rho(\mathcal{I}_2(X))\right)$ is the maximal connected subgroup in $\text{Sp}(V)$ which under this action preserves $X \subset \mathbb{P}(V)$.

\textbf{Proof.} See [Buc06] cor. 4.4, cor. 5.5, lem. 5.6.

Recall the definition of $Y$ in §1.1.4.

The following polynomials are in the homogeneous ideal of $Y$ (below $i, j$ are indices that run through $\{1, \ldots, m\}$, $k$ is a summation index):

$$\sum_{k=1}^{m} a_{ik} b_{jk} - \sum_{k=1}^{m} a_{1k} b_{1k} \quad (2.2a)$$

$$\sum_{k=1}^{m} a_{ik} b_{jk} \quad (2.2b)$$

$$\sum_{k=1}^{m} a_{k1} b_{k1} - \sum_{k=1}^{m} a_{k1} b_{k1} \quad (2.2c)$$

$$\sum_{k=1}^{m} a_{k1} b_{kj} \quad (2.2d)$$

These equations simply come from eliminating $\lambda$ from the defining equation of $Y$ — see equation (1.2).

For the statement and proof of the following proposition, recall our notation of §1.1.1, §1.1.2, §1.1.3, §1.1.5, §1.1.6 and §1.1.8

\textbf{Proposition 2.3.} Let $X \subset \mathbb{P}(V)$ be a Legendrian subvariety. If $X$ is contained in $Y$ then $X$ is preserved by the induced action of $G$ on $\mathbb{P}(V)$.

\textbf{Proof.} Let $\mathcal{I}_2(X)$ be as in the theorem 2.1 and define $\mathcal{I}_2(Y)$ analogously. Clearly $\mathcal{I}_2(Y) \subset \mathcal{I}_2(X)$. By theorem 2.1 it is enough to calculate that $\mathfrak{g} \subset \rho(\mathcal{I}_2(Y))$ or that the images of the quadrics $\mathbf{(2.2a)-2.2d)$ under $\rho$ generate $\mathfrak{g}$.

We deal in details of the proof only for $m = 2$. There is no difference between this case and the general one, except for the complexity of notation.

Let us take the quadric

$$q_{ij} := \sum_{k=1}^{m} a_{ik} b_{jk} = a_{11} b_{1j} + a_{12} b_{2j}$$
for any $i, j \in \{1, \ldots, m\} = \{1, 2\}$. Also let $Q_{ij}$ be the $2m^2 \times 2m^2$ symmetric matrix corresponding to $q_{ij}$. For instance:

$$Q_{12} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0
\end{bmatrix}.$$

So choose an arbitrary $(A, B) \in V$ and at the moment we want to think of it as of a single vertical $2m^2$-vector: $(A, B) = [a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}, b_{22}]^T$, so that the following multiplication makes sense:

$$\rho(q_{12}) = 2J \cdot Q_{12} \cdot (A, B) =$$

$$= \begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
a_{11} \\
a_{12} \\
a_{21} \\
a_{22} \\
b_{11} \\
b_{12} \\
b_{21} \\
b_{22}
\end{bmatrix}$$

Go back to the matrix notation:

$$\begin{bmatrix}
0 & 0 \\
0 & a_{11} \\
a_{12} \\
-b_{21} \\
-b_{22} \\
0 \\
0
\end{bmatrix}^T \cdot \begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}, \quad \begin{bmatrix}
0 & 0 \\
0 & a_{11} \\
a_{12} \\
-b_{21} \\
-b_{22} \\
0 \\
0
\end{bmatrix} = (E_{12}^T A, -E_{12} B)$$

Next in the ideal of $Y$ we have the following quadrics: $q_{ij}$ for $i \neq j$ (see (2.21)) and $q_{ii} - q_{11}$ (see (2.27)). By taking images under $\rho$ of the linear combinations of those quadrics we can get an arbitrary traceless matrix $g \in \mathfrak{sl}_m$ acting on $V$ in the following way:

$$g \cdot (A, B) = (g^T A, -g B).$$
Exponentiate this action of $\mathfrak{sl}_m$ to get the action of $\text{SL}_m$:

$$g \cdot (A, B) = (g^T A, g^{-1} B).$$

This proves that the action of subgroup $\text{SL}_m \times 0 < G = \text{SL}_m \times \text{SL}_m$ indeed preserves $X$ as claimed in the lemma. The action of the other component $0 \times \text{SL}_m$ is calculated in the same way, but using quadrics $(2.2c)$–$(2.2d)$. □

### 2.1 Invariant subsets

Recall our notation of §1.1.1, §1.1.4, §1.1.5, §1.1.6 and §1.1.7. Here we want to decompose $Y$ into a union of some $G$-invariant subsets, most of which are orbits.

**Proposition 2.4.**

(i) The sets $\mathcal{INV}^m$, $\psi_\mu(\mathcal{INV}^m)$ and $\mathcal{DEG}^{m}_{k,l}$ are $G$-invariant and they are all contained in $Y$.

(ii) $Y$ is equal to the union of all $\psi_\mu(\mathcal{INV}^m)$ (for $\mu \in \mathbb{C}^*$) and all $\mathcal{DEG}^{m}_{k,l}$ (for integers $k, l \geq 0$, $k + l \leq m$).

(iii) Every $\psi_\mu(\mathcal{INV}^m)$ is an orbit of the action of $G$. If $m$ is odd, then $\mathcal{INV}^m$ is isomorphic (as algebraic variety) to $\text{SL}_m$. Otherwise if $m$ is even, then $\mathcal{INV}^m$ is isomorphic to $(\text{SL}_m/\mathbb{Z}_2)$. In both cases, $\dim \psi_\mu(\mathcal{INV}^m) = \dim \mathcal{INV}^m = m^2 - 1$.

**Proof.** The proof of part (i) is an explicit verification from the definitions in §1.1.

To prove part (ii), assume $[A, B]$ is a point of $Y$, so $AB^T = B^T A = \lambda^2 \text{Id}_m$. First assume that the ranks of both matrices are maximal:

$$\text{rk } A = \text{rk } B = m.$$  

Then $\lambda$ must be non-zero and $B = \lambda^2 (A^{-1})^T$. Let $d := (\det A)^{-\frac{1}{2}}$ so that

$$\det(dA) = 1$$

and let $\mu := \frac{1}{dA}$. Then we have:

$$[A, B] = \left[ A, \lambda^2 (A^{-1})^T \right] = \left[ \frac{dA}{d\lambda}, d\lambda \left( (dA)^{-1} \right)^T \right] =$$

$$\left[ \mu(dA), \mu^{-1} \left( (dA)^{-1} \right)^T \right] = \psi_\mu \left( \left[ (dA), \left( (dA)^{-1} \right)^T \right] \right).$$

Therefore $[A, B] \in \psi_\mu(\mathcal{INV}^m)$.

Next, if either of the ranks is not maximal:

$$\text{rk } A < m \text{ or } \text{rk } B < m$$

then by $(1.2)$ we must have $AB^T = B^T A = 0$. So $[A, B] \in \mathcal{DEG}^{m}_{k,l}$ for $k = \text{rk } A$ and $l = \text{rk } B$.

Now we prove (iii). The action of $G$ commutes with $\psi_\mu$:

$$(g, h) \cdot \psi_\mu ([A, B]) = \psi_\mu ((g, h) \cdot [A, B]).$$

So to prove $\psi_\mu(\mathcal{INV}^m)$ is an orbit it is enough to prove that $\mathcal{INV}^m$ is an orbit, which follows from the definitions of the action and $\mathcal{INV}^m$. 


Proposition 2.5.

Also be read from the general description of group compactifications. We this is the compactification corresponding to the representation therein for the theory of equivariant compactifications. In the setup of [Tim03, §8], this is the compactification corresponding to the representation \( W \oplus W^* \), where \( W \) is the standard representation of \( \text{SL}_m \). Therefore some properties of \( X_{\text{inv}}(m) \) could also be read from the general description of group compactifications.

\[
\begin{align*}
\text{SL}_m & \rightarrow \mathcal{I}N\gamma^m \\
g & \mapsto [g,(g^{-1})^T]
\end{align*}
\]

If \( [g_1,(g_1^{-1})^T] = [g_2,(g_2^{-1})^T] \) then we must have \( g_1 = \alpha g_2 \) and \( g_1 = \alpha^{-1} g_2 \) for some \( \alpha \in \mathbb{C}^* \). Hence \( \alpha^2 = 1 \) and \( g_1 = \pm g_2 \). If \( m \) is odd and \( g_1 \in \text{SL}_m \) then \( -g_1 \notin \text{SL}_m \) so \( g_1 = g_2 \). So \( \mathcal{I}N\gamma^m \) is either isomorphic to \( \text{SL}_m \) or to \( \text{SL}_m/\mathbb{Z}_2 \) as stated.

\( \square \)

From proposition 2.4(ii) we conclude that \( X_{\text{inv}}(m) \) is an equivariant compactification of \( \text{SL}_m \) (if \( m \) is odd) or \( \text{SL}_m/\mathbb{Z}_2 \) (if \( m \) is even). See [Tim03] and references therein for the theory of equivariant compactifications. In the setup of [Tim03, §8], this is the compactification corresponding to the representation \( W \oplus W^* \), where \( W \) is the standard representation of \( \text{SL}_m \). Therefore some properties of \( X_{\text{inv}}(m) \) could also be read from the general description of group compactifications.

Proposition 2.5.

(i) The dimension of \( \mathcal{DEG}^m_{k,l} \) is \( (k+l)(2m-k-l) - 1 \). In particular, if \( k+l = m \) then the dimension is equal to \( m^2 - 1 \).

(ii) \( \mathcal{DEG}^m_{k,l} \) is an orbit of the action of \( G \), unless \( m \) is even and \( k = l = \frac{1}{2}m \).

(iii) If \( m \geq 3 \), then there are exactly two closed orbits of the action of \( G \): \( \mathcal{DEG}^m_{1,0} \) and \( \mathcal{DEG}^m_{0,1} \).

Proof. Part (i) follows from [Str82, prop 2.10].

For part (ii) let \( [A,B] \in \mathcal{DEG}^m_{k,l} \) be any point. By Gauss elimination and elementary linear algebra, we can prove that there exists \( (g,h) \in G \) such that \([A',B'] := (g,h) \cdot [A,B] \) is a pair of diagonal matrices. Moreover, if \( k+l < m \) then we can choose \( g \) and \( h \) such that:

\[
A' := \text{diag}(1,\ldots,1,0,\ldots,0,0,\ldots,0),
\]

\[
B' := \text{diag}(0,\ldots,0,1,\ldots,1,0,\ldots,0).
\]

Hence \( \mathcal{DEG}^m_{k,l} = G \cdot [A',B'] \) and this finishes the proof in the case \( k+l < m \).

So assume \( k+l = m \). Then we can choose \( (g,h) \) such that:

\[
A' := \text{diag}(1,\ldots,1,0,\ldots,0),
\]

\[
B' := \text{diag}(0,\ldots,0,d,\ldots,d),
\]

for some \( d \in \mathbb{C}^* \). If \( k \neq l \), then set \( e := d \frac{l}{k} \) and let

\[
g' := \text{diag}(e^k,\ldots,e^k,0,\ldots,0,0,\ldots,0),
\]

Clearly \( \det(g') = 1 \) and:

\[
(g', \text{Id}_m) \cdot [A',B'] = \begin{bmatrix}
\text{diag}(e^l,\ldots,e^l,0,\ldots,0), & \text{diag}(0,\ldots,0,e^k,\ldots,e^k) \\
\end{bmatrix}
\]

\[
\text{DEG}^m_{1,0} \quad \text{DEG}^m_{0,1}
\]

\[
\text{DEG}^m_{1,0} \quad \text{DEG}^m_{0,1}
\]
Some quasihomogeneous Legendrian varieties

where

\[ de^k = d^{1+\frac{k}{l}} = d^{\frac{1}{l}} e^l. \]

So rescaling we get:

\[
(g', \text{Id}_m) \cdot [A', B'] = \begin{bmatrix}
\text{diag}(1, \ldots, 1, 0, \ldots, 0), \\
\text{diag}(0, \ldots, 0, 1, \ldots, 1)
\end{bmatrix}
\]

and this finishes the proof of (ii).

For part (iii), denote by \( W_1 \) (respectively, \( W_2 \)) the standard representation of the first (respectively, the second) component of \( G = \text{SL}_m \times \text{SL}_m \). Then our representation \( V \) is isomorphic to \( (W_1 \otimes W_2) \oplus (W_1^* \otimes W_2^*) \). For \( m \geq 3 \) the representation \( W_i \) is not isomorphic to \( W_i^* \) and therefore \( V \) is a union of two irreducible non-isomorphic representations, so there are exactly two closed orbits of this action on \( \mathbb{P}(V) \). These orbits are simply \( \text{DEG}_{m,0}^* \) and \( \text{DEG}_{m,1}^* \).

\[ \square \]

2.2 Action of \( \tilde{G} \)

Recall the notation of \( \S 1.1.1, \S 1.1.6 \) and \( \S 1.1.7 \).

The action of \( \tilde{G} \) extends the action of \( G \), but it does not preserve \( X_{\text{inv}}(m) \). So we will only consider the action of \( \tilde{G} \) when speaking of \( X_{\text{deg}}(m, k) \).

We have properties analogous to proposition 2.5 (ii) and (iii) but with no exceptional cases:

Proposition 2.6.

(i) Every \( \text{DEG}_{m,k}^* \) is an orbit of the action of \( \tilde{G} \).

(ii) For every \( m \) there are exactly two closed orbits of the action of \( \tilde{G} \): \( \text{DEG}_{1,0}^* \) and \( \text{DEG}_{0,1}^* \).

Proof. It goes exactly as the proof of proposition 2.5 (ii) and (iii).

\[ \square \]

3 Legendrian varieties in \( Y \)

In this section we prove the main results of the article.

3.1 Classification

We start with proving the theorem 1.3. For this we use our notation of section 1.1.

Proof. First assume \( X \) is Legendrian and contained in \( Y \). If \( X \) contains a point \([A, B]\) where both \( A \) and \( B \) are invertible, then by proposition 2.3 it must contain the orbit of \([A, B]\), which by proposition 2.4 (ii) and (iii) is equal to \( \psi_\mu(\mathcal{L}N\mathcal{V}^m) \) for some \( \mu \in \mathbb{C}^* \). But dimension of \( X \) is \( m^2 - 1 \) which is exactly the dimension of \( \psi_\mu(\mathcal{L}N\mathcal{V}^m) \) (see proposition 2.4 (iii)), so

\[
X = \psi_\mu(\mathcal{L}N\mathcal{V}^m) = \psi_\mu(X_{\text{inv}}(m)).
\]

On the other hand if \( X \) does not contain any point \([A, B]\) where both \( A \) and \( B \) are invertible then in fact \( X \) is contained in the locus \( Y_0 := \{[A, B] : AB^T = \} \).
This locus is just the union of all $DEG_{k,l}^m$ and its irreducible components are the closures of $DEG_{k,m-k}^m$, which are exactly $X_{deg}(m,k)$. So in particular every irreducible component has dimension $m^2 - 1$ (see proposition 2.4(i)) and hence $X$ must be one of these components.

Therefore it remains to show that all these varieties are Legendrian.

The fact that $X_{deg}(m,k)$ is a Legendrian variety follows from Strickland’s proof that the affine cone over $X_{deg}(m,k)$ (or $W(k,m-k)$ in the notation of Strickland) is the closure of a conormal bundle. Conormal bundles are classical examples of Lagrangian varieties.

Since $\psi_\mu$ preserves the symplectic form $\omega$, it is enough to prove that $X_{inv}(m)$ is Legendrian.

The group $G$ acts symplectically on $V$ and the action has an open orbit on $X_{inv}(m)$ — see proposition 2.4(iii). Thus the tangent spaces to the affine cone over $X_{inv}(m)$ are Lagrangian if and only if just one tangent space at a point of the open orbit is Lagrangian.

So we take $[A,B] := [\text{Id}_m, \text{Id}_m]$. Now the affine tangent space to $X_{inv}(m)$ at $[\text{Id}_m, \text{Id}_m]$ is the linear subspace of $V$ spanned by $[\text{Id}_m, \text{Id}_m]$ and the image of the tangent action of the Lie algebra $\mathfrak{g}$. We must prove that for every four traceless matrices $g,h,g',h'$ we have:

$$\omega((g,h) \cdot [\text{Id}_m, \text{Id}_m], (g',h') \cdot [\text{Id}_m, \text{Id}_m]) = 0 \quad \text{and} \quad \omega((\text{Id}_m, \text{Id}_m), (g,h) \cdot [\text{Id}_m, \text{Id}_m]) = 0 \quad (3.1)$$

Equality (3.1) is true without the assumption that the matrices have trace 0:

$$\omega((g,h) \cdot [\text{Id}_m, \text{Id}_m], (g',h') \cdot [\text{Id}_m, \text{Id}_m]) =$$

$$= \omega\left((g^T + h, -(g + h^T)), ((g')^T + h', -(g' + (h')^T))\right) =$$

$$\text{by (1.1)} \quad \text{tr}\left(- (g^T + h)(g')^T + h' + (g + h^T)(g' + (h')^T)\right) =$$

$$= 0.$$

For equality (3.2) we calculate:

$$\omega((\text{Id}_m, \text{Id}_m), (g,h) \cdot [\text{Id}_m, \text{Id}_m]) =$$

$$= \omega\left((\text{Id}_m, \text{Id}_m), (g^T + h, -(g + h^T))\right) =$$

$$\text{by (1.1)} \quad -\text{tr}(g^T + h) - \text{tr}(g + h^T) = 0.$$

Hence we have proved that the closure of $\mathcal{INV}^m$ is Legendrian. □
3.2 Degenerate matrices

Recall our notation of $1.1.1$, $1.1.4$, $1.1.6$, $1.1.8$ and $1.1.9$.

By [Str22, prop. 1.3] the ideal of $X_{\text{deg}}(m, k)$ is generated by the coefficients of $AB^T$, the coefficients of $B^T A$, the $(k + 1) \times (k + 1)$-minors of $A$ and the $(m - k + 1) \times (m - k + 1)$-minors of $B$. In short we will say that the equations of $X_{\text{deg}}(m, k)$ are given by:

$$AB^T = 0, \quad B^T A = 0, \quad \text{rk}(A) \leq k, \quad \text{rk}(B) \leq m - k. \quad \text{(3.3)}$$

**Lemma 3.4.** Assume $m \geq 2$ and $1 \leq k \leq m - 1$. Then:

(i) The tangent cone to $X_{\text{deg}}(m, k)$ at $p_1$ is a product of a linear space of dimension $(2m - 2)$ and the affine cone over $X_{\text{deg}}(m - 1, k - 1)$.

(ii) $X_{\text{deg}}(m, k)$ is smooth at $p_1$ if and only if $k = 1$.

(iii) $X_{\text{deg}}(m, k)$ is smooth at $p_2$ if and only if $k = m - 1$.

**Proof.** We only prove (i) and (ii), while (i') and (ii') follow in the same way by exchanging $a_{ij}$ and $b_{ij}$. Consider equations (3.3) of $X_{\text{deg}}(m, k)$ restricted to the affine neighbourhood of $p_1$ obtained by substituting $a_{mm} = 1$. Taking the lowest degree part of these equations we get some of the equations of the tangent cone at $p_1$ (recall our convention on the notation of submatrices — see $1.1.10$):

$$b_{im} = b_{mi} = 0, \quad A_m B_{m}^T = 0, \quad B_{m}^T A_m = 0,$$

$$\text{rk} A_m \leq k - 1, \text{rk} B_m \leq m - k.$$  

These equations define the product of the linear subspace $A_m = B_m = 0, b_{im} = b_{mi} = 0$ and the affine cone over $X_{\text{deg}}(m - 1, k - 1)$ embedded in the set of those pairs of matrices, whose last row and column are zero: $a_{im} = a_{mi} = 0, b_{im} = b_{mi} = 0$. So the variety defined by those equations is irreducible and its dimension is equal to $(m - 1)^2 + 2m - 2 = m^2 - 1 = \dim X_{\text{deg}}(m, k)$. Since it contains the tangent cone we are interested in and by $1.1.9(1)$, they must coincide as claimed in (i).

Next (ii) follows immediately, since for $k = 1$ the equations above reduce to

$$b_{im} = b_{mi} = 0, \quad A_m = 0$$

and hence the tangent cone is just the tangent space, so $p_1$ is a smooth point of $X_{\text{deg}}(m, 1)$. Conversely, if $k > 1$ then $X_{\text{deg}}(m - 1, k - 1)$ is not a linear space, so by (i) the tangent cone is not a linear space either and $X$ is singular at $p_1$ — see $1.1.9(3)$.

$\square$

Now we can prove theorem $1.3$.

**Proof.** It is obvious from the definition of $X_{\text{deg}}(m, k)$, that $X_{\text{deg}}(m, 0) = \{ A = 0 \}$ and $X_{\text{deg}}(m, m) = \{ B = 0 \}$, so these are indeed linear spaces.

Therefore assume $1 \leq k \leq m - 1$. But $X_{\text{deg}}(m, k)$ is $G$ invariant (see proposition $2.6(i)$) and so is its singular locus $S$. Hence $X_{\text{deg}}(m, k)$ is singular if and only if $S$ contains a closed orbit of $G$.

So $X_{\text{deg}}(m, k)$ is smooth, if and only if it is smooth at both $p_1$ and $p_2$ (see proposition $2.6(ii)$), which (by lemma (ii) and (ii')) holds if and only if $k = 1$ and $m = 2$. 

To finish the proof, it remains to verify what kind of variety is $X_{\deg}(2, 1)$. Consider the following map:

$$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}(V) \simeq \mathbb{P}^7$$

$$[\mu_1, \mu_2, [\nu_1, \nu_2], [\xi_1, \xi_2] \mapsto [\xi_1 \left( \begin{array}{cc} \mu_1\nu_1 & \mu_1\nu_2 \\ \mu_2\nu_1 & \mu_2\nu_2 \end{array} \right), \xi_2 \left( \begin{array}{cc} \mu_2\nu_2 & -\mu_2\nu_1 \\ -\mu_1\nu_2 & \mu_1\nu_1 \end{array} \right)]$$

Clearly this is a Segre embedding in appropriate coordinates. The image of this embedding is contained in $X_{\deg}(2, 1)$ (see equation (3.5)) and since dimension of $X_{\deg}(2, 1)$ is equal to the dimension of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ we conclude the above map gives an isomorphism of $X_{\deg}(2, 1)$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. \hfill $\Box$

### 3.3 Invertible matrices

Recall the notation of $\{1,1,1\}, \{1,1,4\}, \{1,1,6\}, \{1,1,7\}$ and $\{1,1,9\}$. We wish to determine some of the equations of $X_{\text{inv}}(m)$. Clearly the equations of $Y$ (see (2.2)) are quadratic equations of $X_{\text{inv}}(m)$. To find other equations, we recall that

$$X_{\text{inv}}(m) := \left\{ [g, (g^{-1})^T] \in \mathbb{P}(V) \mid \det g = 1 \right\}$$

But for a matrix $g$ with determinant 1 we know that the entries of $(g^{-1})^T$ consist of the appropriate minors (up to sign) of $g$. Therefore we get many inhomogeneous equations satisfied by every pair $(g, (g^{-1})^T) \in V$ (recall our convention on the notation of submatrices — see (1.1.10):

$$\det(A_{ij}) = (-1)^{i+j}b_{ij} \quad \text{and} \quad a_{kl} = (-1)^{k+l} \det(B_{kl})$$

To make them homogeneous, multiply two such equations appropriately:

$$\det(A_{ij})a_{kl} = (-1)^{i+j+k+l}b_{ij}\det(B_{kl}). \quad (3.5)$$

These are degree $m$ equations, which are satisfied by the points of $X_{\text{inv}}(m)$ and we state the following theorem:

**Theorem 3.6.** Let $m = 3$. Then the quadratic equations (2.2a)–(2.2d) and the cubic equations (3.5) generate the ideal of $X_{\text{inv}}(3)$. Moreover $X_{\text{inv}}(3)$ is smooth.

**Proof.** It is enough to prove that the scheme $X$ defined by equations (2.2a)–(2.2d) and (3.5) is smooth, because the reduced subscheme of $X$ coincides with $X_{\text{inv}}(3)$.

The scheme $X$ is $G$ invariant, hence as in the proof of theorem 1.5 and by proposition 2.3(iii) it is enough to verify smoothness at $p_1$ and $p_2$. Since we have the additional symmetry here (exchanging $a_{ij}$’s with $b_{ij}$’s) it is enough to verify the smoothness at $p_1$.

Now we calculate the tangent space to $X$ at $p_1$ by taking linear parts of the equations evaluated at $a_{33} = 1$. From (2.2) we get that

$$b_{31} = b_{32} = b_{33} = b_{23} = b_{13} = 0.$$ 

Now from equations (3.5) for $k = l = 3$ and $i, j \neq 3$ we get the following evaluated equations:

$$a_{i'j'} - a_{i'3}a_{3j'} = \pm b_{ij}B_{33}$$

(where $i'$ is either 1 or 2, which ever is different than $i$ and analogously for $j'$) so the linear part is just $a_{i'j'} = 0$. Hence by varying $i$ and $j$ we can get

$$a_{11} = a_{12} = a_{21} = a_{22} = 0.$$
Therefore the tangent space has codimension at least 9, which is exactly the codimension of $X_{\text{inv}}(3)$ — see [2,iii]. Hence $X$ is smooth (in particular reduced) and $X = X_{\text{inv}}(3)$.

To describe $X_{\text{inv}}(m)$ for $m > 3$ we must find more equations.

There is a more general version of the above property of an inverse of a matrix with determinant 1, which is less popular.

**Proposition 3.7.**

(i) Assume $A$ is a $m \times m$ matrix of determinant 1 and $I, J$ are two sets of indices, both of cardinality $k$ (again recall our convention on indices and submatrices — see section 1.1.10). Denote by $B := (A^{-1})^T$. Then the appropriate minors are equal (up to sign):

$$\det A_{I,J} = (-1)^{\sum I + \sum J} \det B_{I',J'}.$$ 

(ii) The coordinate free way to express these equalities is following: if $W$ is a vector space of dimension $m$ and $f$ is a linear automorphism of $W$, let $\wedge^k f$ be the induced automorphism of $\wedge^k W$. If $\wedge^m f = \text{Id}_{\wedge^m W}$ then:

$$\wedge^{m-k} f = \wedge^k \left( \wedge^{m-1} f \right).$$

(iii) Consider the induced action of $G$ on the polynomials on $V$. Then the vector space spanned by the set of equations of (i) for a fixed $k$ is $G$ invariant.

**Proof.** Part (ii) follows explicitly from (i), since if $A$ is a matrix of $f$, then the terms of the matrices of the maps $\wedge^{m-k} f$ and $\wedge^k (\wedge^{m-1} f)$ are exactly the appropriate minors of $A$ and $B$.

Part (iii) follows easily from (ii).

As for (i), we only sketch the proof, leaving the details to the reader and his or her linear algebra students. Firstly, reduce to the case when $I$ and $J$ are just $\{1, \ldots, k\}$ and the determinant of $A$ is possibly $\pm 1$ (which is where the sign shows up in the equality). Secondly if both determinants $\det A_{I,J}$ and $\det B_{I',J'}$ are zero, then the equality is clearly satisfied. Otherwise assume for example $\det A_{I,J} \neq 0$. Then performing the appropriate row and column operations we can change $A_{I,J}$ into a diagonal matrix, $A_{I',J'}$ and $A_{I,J'}$ into the zero matrices and all these operations can be done without changing $B_{I',J'}$ nor $\det A_{I,J}$. Then the statement follows easily.

In particular we get:

**Corollary 3.8.** Assume $k, I$ and $J$ are as in proposition 3.7(i).

(a) If $m$ is even and $k = \frac{1}{2} m$, then the equation

$$\det A_{I,J} = (-1)^{\sum I + \sum J} \det B_{I',J'}$$

is homogeneous of degree $\frac{1}{2} m$ and it is satisfied by points of $X_{\text{inv}}(m)$.

(b) If $0 \leq k < \frac{1}{2} m$ and $l = \frac{1}{2} m - k$, then

$$(\det A_{I,J})^2 = (\det B_{I',J'})^2 \cdot (a_{11} b_{11} + \ldots + a_{1m} b_{1m})^l$$

is a homogeneous equation of degree $2(m - k)$ satisfied by points of $X_{\text{inv}}(m)$. 

Proof. Clearly both equations are homogeneous. If det $A = 1$ and $B = (A^{-1})^T$ then the following equations are satisfied:

$$\det A_{i,j} = (-1)^{\Sigma I + \Sigma J} \det B_{i',j'},$$  

(3.9)

$$1 = (a_{11}b_{11} + \ldots + a_{1m}b_{1m})^t,$$  

(3.10)

(equation (3.9) follows from proposition 3.7(i) and (3.10) follows from $AB^T = \text{Id}_m$).

Equation in (b) is just (3.9) squared multiplied side-wise by (3.10).

So both equations in (a) and (b) are satisfied by every pair $(A, (A - 1)^T)$ and by homogeneity also by $(\lambda A, \lambda (A^{-1})^T)$. Hence (a) and (b) hold on an open dense subset of $X_{\text{inv}}(m)$, so also on whole $X_{\text{inv}}(m)$.

\[\square\]

We know enough equations of $X_{\text{inv}}(m)$ to prove the theorem 1.4:

3.3.1 Case $m = 2$ — linear subspace

Proof. To prove (a) just take the linear equations from proposition 3.7(i) for $k = 1$:

$$a_{ij} = \pm b_{i'j'},$$

where $\{i, i'\} = \{j, j'\} = \{1, 2\}$.

\[\square\]

3.3.2 Case $m = 3$ — hyperplane section of $\text{Gr}(3, 6)$

Proof. For (b), $X_{\text{inv}}(3)$ is smooth by theorem 3.6 and it is a compactification of $\mathcal{I}N\mathcal{V}^3 \cong \text{SL}_3$ by proposition 2.4(i) and (iii).

Picard group of $X_{\text{inv}}(3)$. The complement of the open orbit

$$D := X_{\text{inv}}(3) \setminus \mathcal{I}N\mathcal{V}^3$$

must be a union of some orbits of $G$, each of them must have dimension smaller than $\dim \mathcal{I}N\mathcal{V}^3 = 8$. So by propositions 2.4(ii), (iii), 2.5(i) and (ii) the only candidates are $D\mathcal{E}G^3_{1,1}$, $D\mathcal{E}G^3_{0,1}$ and $D\mathcal{E}G^3_{1,0}$. We claim they are all contained in $X_{\text{inv}}(3)$. It is enough to prove that $D\mathcal{E}G^3_{1,1} \subset X_{\text{inv}}(3)$, since the other orbits are in the closure of $D\mathcal{E}G^3_{1,1}$. Take the curve in $X_{\text{inv}}(3)$ parametrised by:

$$\begin{pmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t^{-1} \end{pmatrix}, \begin{pmatrix} t^{-1} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & t \end{pmatrix}.$$

For $t = 0$ the curve meets $D\mathcal{E}G^3_{1,1}$, which finishes the proof of the claim.

Since $\dim D\mathcal{E}G^3_{1,1} = 7$ (see proposition 2.5(i)), $D$ is a prime divisor. We have $\text{Pic}(\text{SL}_3) = 0$ and by [Har77, prop. II.6.5(c)] the Picard group of $X_{\text{inv}}(3)$ is isomorphic to $\mathbb{Z}$ with the ample generator $[D]$.

Next we check that $D$ is linearly equivalent (as a divisor on $X_{\text{inv}}(3)$) to a hyperplane section $H$ of $X_{\text{inv}}(3)$. Since we already know that $\text{Pic}(X_{\text{inv}}(3)) = \mathbb{Z} \cdot [D]$, we must have $H \sim kD$ for some positive integer $k$. But there are lines contained in $X_{\text{inv}}(3)$ (for example those contained in $D\mathcal{E}G^3_{1,1} \cong \mathbb{P}^2 \times \mathbb{P}^2$). So let $L \subset X_{\text{inv}}(3)$ be any line and we intersect:

$$D \cdot L = \frac{1}{k} H \cdot L = \frac{1}{k}.$$  

But the result must be an integer, so $k = 1$ as claimed.

\[4\] Actually, the reader could also easily find explicitly some lines (or even planes) which intersect the open orbit and conclude that $X_{\text{inv}}(3)$ is covered by lines.
**Complete embedding.** Since $D$ itself is definitely not a hyperplane section of $X_{\text{inv}}(3)$, the conclusion is that the Legendrian embedding of $X_{\text{inv}}(3)$ is not given by a complete linear system. The natural guess for a better embedding is the following:

$$X' := \left\{ [1, g, \bigwedge^2 g] \in \mathbb{P}^{18} = \mathbb{P}(\mathbb{C} \oplus V) \mid \det g = 1 \right\}.$$

(we note that $\bigwedge^2 g = (g^{-1})^T$ for $g$ with $\det g = 1$) and one can verify that the projection from the point $[1, 0, 0] \in \mathbb{P}^{18}$ restricted to $X'$ gives an isomorphism with $X_{\text{inv}}(3)$.

The Grassmannian $Gr(3, 6)$ in its Plücker embedding can be described as the closure of:

$$\left\{ [1, g, \bigwedge^2 g, \bigwedge^3 g] \in \mathbb{P}^{19} = \mathbb{P}(\mathbb{C} \oplus V \oplus \mathbb{C}) \mid g \in M_{3 \times 3} \right\}$$

and we immediately identify $X'$ as the section $H := \left\{ \bigwedge^3 g = 1 \right\}$ of the Grassmannian.

Though it is not essential, we note that $H^1(O_{Gr(3, 6)}) = 0$ (see Kodaira vanishing theorem [Laz04, thm 4.2.1]) and hence the above embedding of $X_{\text{inv}}(3)$ is given by the complete linear system.

**Automorphism group.** It remains to calculate $Aut(X_{\text{inv}}(3))^0$ — the connected component of the automorphism group.

The tangent Lie algebra of the group of automorphisms of a complex projective manifold is equal to the global sections of the tangent bundle, see [AK95]. A vector field on $X_{\text{inv}}(3)$ is also a section of $TGr(3, 6)|_{X_{\text{inv}}(3)}$ and we have the following short exact sequence:

$$0 \rightarrow TGr(3, 6)(-1) \rightarrow TGr(3, 6) \rightarrow TGr(3, 6)|_{X_{\text{inv}}(3)} \rightarrow 0$$

The homogeneous vector bundle $TGr(3, 6)(-1)$ is isomorphic to $U^* \otimes Q \otimes \bigwedge^3 U$, where $U$ is the universal subbundle in $Gr(3, 6) \times \mathbb{C}^6$ and $Q$ is the universal quotient bundle. This bundle corresponds to an irreducible module of the parabolic subgroup in $SL_6$. Calculating explicitly its highest weight and applying Bott formula [DH95] we get that $H^1(TGr(3, 6)(-1)) = 0$. Hence every section of $TX_{\text{inv}}(3)$ extends to a section of $TGr(3, 6)$. In other words, if $P < Aut(Gr(3, 6)) \simeq \mathbb{P}GL_6$ is the subgroup preserving $X_{\text{inv}}(3) \subset Gr(3, 6)$, then the restriction map $P \rightarrow Aut(X_{\text{inv}}(3))^0$ is epimorphic.

The action of $SL_6$ on $\bigwedge^3 \mathbb{C}^6$ preserves the natural symplectic form $\omega'$:

$$\omega' : \bigwedge^2 \left( \bigwedge^3 \mathbb{C}^6 \right) \rightarrow \bigwedge^6 \mathbb{C}^6 \simeq \mathbb{C}.$$

Since the action of $P$ on $\mathbb{P}\left( \bigwedge^3 \mathbb{C}^6 \right)$ preserves the hyperplane $H$ containing $X_{\text{inv}}(3)$, it must also preserve $H^\perp$, i.e. $P$ preserves $[1, 0, 0, 1] \in \mathbb{P}^{19} = \mathbb{P}(\mathbb{C} \oplus V \oplus \mathbb{C})$. Therefore $P$ acts on the quotient $H/(H^\perp) = V$ and hence the restriction map factorises:

$$P \rightarrow Aut(\mathbb{P}(V), X_{\text{inv}}(3))^0 \rightarrow Aut(X_{\text{inv}}(3))^0.$$

By [Buc07b], group $Aut(\mathbb{P}(V), X_{\text{inv}}(3))^0$ is contained in the image of $Sp(V) \rightarrow \mathbb{P}GL(V)$, so by theorem 2.1, proposition 2.3 and theorem 3.6

$$Aut(\mathbb{P}(V), X_{\text{inv}}(3))^0 = G.$$
In particular $X_{\text{inv}}(3)$ cannot be homogeneous as it contains more than one orbit of the connected component of automorphism group.

We note that the fact that $X_{\text{inv}}(3)$ is not homogeneous can be also proved without calculating the automorphism group. Since $\text{Pic} X_{\text{inv}}(3) \simeq \mathbb{Z}$, it follows from [LM04 thm. 11], that $X_{\text{inv}}(3)$ could only be one of the subadjoint varieties. But none of them has $\text{Pic} \simeq \mathbb{Z}$ and dimension 8.

### 3.3.3 Case $m=4$ — spinor variety $S_6$

**Proof.** To prove (c) we only need to take 30 quadratic equations of $Y$ as in (2.2) and 36 quadratic equations from corollary 3.8 (a). By proposition 3.7(iii) the scheme $X$ defined by those quadratic equations is $G$-invariant. As in the proofs of theorems 1.5 and 3.6, we only check that $X$ is smooth at $p_1$ and $p_2$ and conclude it is smooth everywhere, hence those equations indeed define $X_{\text{inv}}(4)$.

Therefore $X_{\text{inv}}(4)$ is smooth, irreducible and its ideal is generated by quadrics, so it falls into the classification of [Buc06 thm. 5.11]. Hence we have two choices for $X_{\text{inv}}(4)$, whose dimension is 15: the product of a line and a quadric $\mathbb{P}^1 \times Q_{14}$ or the spinor variety $S_6$. The homogeneous ideal of polynomials vanishing on $\mathbb{P}^1 \times Q_{14} \subset \mathbb{P}^{31}$ is generated by $\dim(\text{SL}_2 \times \text{SO}_{16}) = 123$ linearly independent quadratic polynomials (see theorem 2.1, alternatively, one can calculate the equations explicitly — see [Buc05 §7.2]). So $X_{\text{inv}}(4)$, which by the above argument is generated by only 66 quadratic equations, must be isomorphic to $S_6$.

### 3.3.4 Case $m \geq 5$ — singular varieties

**Proof.** Finally we prove (d). We want to prove, that for $m \geq 5$ variety $X_{\text{inv}}(m)$ is singular at $p_1$. To do that, we calculate the reduced tangent cone

$$T := (TC_{p_1}X_{\text{inv}}(m))_{\text{red}}.$$

From equations (2.22) we easily get the following linear and quadratic equations of $T$ (again we suggest to have a look at [1.1.10]):

$$b_{im} = b_{mi} = 0, \quad A_mB^T_m = B^T_mA_m = \lambda^2 \text{Id}_{m-1}$$

for every $i \in \{1, \ldots, m\}$ and some $\lambda \in \mathbb{C}^*$.

Next assume $I$ and $J$ are two sets of indices both of cardinality $k = \left\lfloor \frac{1}{2}m \right\rfloor$ and such that neither $I$ nor $J$ contains $m$. Consider the equation of $X_{\text{inv}}(m)$ as in corollary 3.8(b):

$$(\det A_{I,J})^2 = (\det B_{I',J'})^2 \cdot (a_{11}b_{11} + \ldots a_{1m}b_{1m})^l.$$

To get an equation of $T$, we evaluate at $a_{mm} = 1$ and take the lowest degree part, which is simply $((A_m)_{I,J})^2 = 0$. Since $T$ is reduced, by varying $I$ and $J$ we get that:

$$\text{rk} A_m \leq m - 1 - k - 1 = \left\lfloor \frac{1}{2}m \right\rfloor - 2$$

and therefore also:

$$A_mB^T_m = B^T_mA_m = 0.$$

Hence $T$ is contained in the product of the linear space $W := \{A_m = 0, B = 0\}$ and the affine cone $\hat{U}$ over the union of $X_{\text{deg}}(m - 1, k)$ for $k \leq \left\lfloor \frac{1}{2}m \right\rfloor - 2$. We claim
that $T = W \times \hat{U}$. By proposition 2.5(i), every component of $W \times \hat{U}$ has dimension $2m - 2 + (m - 1)^2 = m^2 - 1 = \dim X_{\text{inv}}(m)$, so by §1.1.9(1) the tangent cone must be a union of some of the components. Therefore to prove the claim it is enough to find for every $k \leq \left\lceil \frac{1}{2}m \right\rceil - 2$ a single element of $DEG_{m-1}^{k,m-1-k-1}$ that is contained in the tangent cone.

So take $\alpha$ and $\beta$ to be two strictly positive integers such that

$$\alpha = \left(\frac{1}{2}m - k - 1\right) \beta$$

and consider the curve in $\mathbb{P}(V)$ with the following parametrisation:

$$\begin{bmatrix}
\text{diag}\{t_{\alpha}^{\alpha}, \ldots, t_{\alpha}^{\alpha+\beta}, \ldots, t_{\alpha}^{\alpha+\beta}, 1\}, \\
\text{diag}\{t_{\alpha}^{\alpha+\beta}, \ldots, t_{\alpha}^{\alpha+\beta}, t_{\alpha}^{\alpha}, \ldots, t_{\alpha}^{\alpha}, t_{2\alpha}^{2\alpha+\beta}\}
\end{bmatrix}.$$  

It is easy to verify that this family is contained in $\Lambda\mathcal{V}^m$ for $t \neq 0$ and as $t$ converges to 0, it gives rise to a tangent vector (i.e. an element of the reduced tangent cone - see point-wise definition in §1.1.9) that belongs to $DEG^{m-1}_{k,m-1-k-1}$.

So indeed $T = W \times \hat{U}$, which for $m \geq 5$ contains more than 1 component, hence cannot be a linear space. Therefore by §1.1.9(3) variety $X_{\text{inv}}(m)$ is singular at $p_1$.

**Remark 3.11.** Note that in both cases of $X_{\text{deg}}(m,k)$ and $X_{\text{inv}}(m)$, the reduced tangent cone is a Lagrangian subvariety in the fibre of the contact distribution. This is not accidental as explained in [Bucb].

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