THE CONFORMAL UNIVERSE I:
Theoretical Basis of Conformal General Relativity

(revised, amended and improved version)

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Abstract

This is the first of three papers on Conformal General Relativity (CGR), which aims to generalize Einstein’s General Relativity (GR) by requiring action–integral invariance under local scale transformations in addition to general coordinate transformations. The theory is constructed in the semiclassical approximation as a preliminary approach to a quantum theoretical implementation. The idea of a conformal extension of GR was advanced by Weyl in 1919 and fully developed by Cartan in the early 1920s. For several decades it had little impact on physics, as CGR implies that all fields have zero mass. Today this does not appear to be an unsurmountable difficulty since we know that mass parameters may result from the spontaneous breakdown of a symmetry. This paper is devoted to introducing the formalism necessary to implement this idea. The implementation of local conformal symmetry is carried out and a number of interesting consequences are reported, in particular: 1) CGR is equivalent to a conformal–invariant field theory equipped with a ghost scalar field of non–zero vacuum expectation value and invested with geometric meaning, here called the dilation field (ghosts are not so fatal to $S$–matrix unitarity if energy density is bounded from below); 2) the interaction of this field with a massless physical scalar field results in the production of a Higgs field of dynamically evolving mass capable of promoting a huge energy transfer from geometry to matter; 3) CGR is only possible in 4D–spacetime and satisfies the Mach–Einstein principle in the form clarified by Gürsey in 1963.
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References
1 Introduction: The problem of the Beginning

Difficulties with General Relativity (GR) come not only from the fact that its quantum theoretical implementation is still unsolved or controversial, but also because it does not explain the big bang. This and the following two papers – here referred to as Part II and Part III – describe an attempt to overcome the second difficulty by a conformal extension of the principle of GR, what provides an alternative to current models of inflation [1] and other approaches to conformal gravity [3] [4]. It does not conflict with Friedman–Robertson–Walker models [2], but rather it is a necessary premise for them.

The theory is constructed in the semiclassical approximation as the premise of a quantum theoretical implementation. To highlight the main aspects of the difficulty about explaining the big bang, let us start from a few cosmological considerations.

Astronomical observations corroborated by the heuristic Cosmological Principle of the non–existence of vantage points in the universe lead to the following statements. On the galaxy–cluster scale, and with respect to a privileged set of comoving observers, the universe appears to be homogeneous (same from point to point), isotropic (same view in all directions) and in uniform expansion. Observers are called comoving if each of them sees the others as moving with cosmic expansion (this definition is incomplete if the shapes of the space–like surfaces of comoving observers are not specified). Mutual distances among galaxy clusters appear to increase as if all pieces of matter accessible to our observation originated from an explosion diverging from a point in the finite or infinite past and were then subjected to forces depending on their relative distances. Of course, on intergalactic and galactic scales, expansion uniformity is hampered by the anisotropy of the gravitational field, caused by the inhomogeneity of matter distribution.

Gravitational equations alone, as described by standard GR, cannot explain the above–mentioned phenomena as they establish a relationship between ten local components of energy–momentum (EM) tensor and ten local components of gravitational field, whereas the number of Lagrangian degrees of freedom of the latter is twenty. To establish the properties of the universe on the large scale we should be able to assign appropriate values to the remaining ten degrees of freedom without having a priori any theoretical reason for preferring a particular assignment. Nor are we able to explain the smallness of the cosmological–constant, equivalent to a vacuum energy–density of about $10^{-47}$ GeV$^4$, which
is considered the cause of the slight acceleration of universe expansion.

But, even ignoring these aspects, the problem of the Beginning still appears of very difficult solution. If all the world lines of physical particles originated from a point–like event of spacetime, a huge amounts of matter and of negative gravitational energy should have been initially concentrated at the apex of a light–cone.

Alternatively, we may hypothesize that the universe originated from a spherical body of enormous mass – one might guess a huge black hole or a “cosmic bubble” – which would however raise other serious problems.

The source of all these difficulties is evident if we focus on the general structure of Einstein’s gravitational equation.

Let \( g_{\mu\nu}, R_{\mu\nu}, R, \Lambda, \Theta^M_{\mu\nu}, M_r, P = 2.43 \times 10^{18} \text{ GeV} \) and \( \kappa = 1/M_r^2 \simeq 1.685 \times 10^{-37} \text{ GeV}^{-2} \) be respectively: the metric tensor, Ricci tensor, Ricci scalar tensor, cosmological constant, EM tensor of matter fields, reduced Planck mass, and gravitational constant, then consider the structure of Einstein equation \( R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\Lambda = \kappa\Theta^M_{\mu\nu} \). To prevent misunderstandings, let us specify our mathematical conventions:

- The signature of \( g_{\mu\nu} \) is \((+,-,-,-)\) and the light speed is 1.
- \( \Theta^M_{\mu\nu} \) matches Hilbert’s definition
  \[
  \Theta^M_{\mu\nu}(x) = 2 \left[ \frac{\partial L^M(x)}{\delta g^{\mu\nu}(x)} - \frac{g_{\mu\nu}(x)}{2} L^M(x) \right],
  \]
  where \( L^M \) is the total Lagrangian density of the matter fields and \( x = \{x^0, x^1, x^2, x^3\} \) are spacetime–manifold parameters as contravariant coordinates.
- \( R_{\mu\nu} \) matches Landau–Lifchitz’ definition \( R_{\mu\nu} = R^\rho_{\mu\rho\nu} \[3, \] \), where
  \[
  R^\rho_{\mu\sigma\nu} = \partial_\sigma \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma} \Gamma^\rho_{\lambda\nu}
  \]
is the Riemann tensor and \( \Gamma^\rho_{\nu\sigma} = \frac{1}{2}g^{\rho\lambda}(\partial_\nu g_{\mu\lambda} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\mu\nu}) \) are the Christoffel symbols. Here and in the following, \( \partial_\mu f \equiv \partial f/\partial x^\mu \). Since \( R^\rho_{\mu\sigma\nu} \) is antisymmetric under the interchange \( \rho \leftrightarrow \mu \) or \( \sigma \leftrightarrow \rho \), the sign of our Ricci tensor is opposite to that of \( R_{\mu\nu} = R^\rho_{\nu\rho\mu} \), which is a definition preferred by others authors \[6, \[7].

- The sign of \( \Lambda \) is chosen so that \( \rho_{\text{vac}} = \Lambda/\kappa \) can be interpreted as the (positive) energy density of the vacuum. The expression \( \Theta^V_{\mu\nu}(x) = \rho_{\text{vac}} g_{\mu\nu}(x) \) will be called the EM tensor of the vacuum.
With these conventions, the gravitational equation of Einstein takes the form
\[ \Theta^M_{\mu\nu} - M_r^2 P \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \rho_{vac} g_{\mu\nu} = 0. \]

Note that, if we interpret
\[ \Theta^G_{\mu\nu} = -M_r^2 P \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \]
as the \textit{EM tensor of geometry}, the gravitational equation of GR takes the “democratic” form
\[ \Theta_{\mu\nu}(x) \equiv \Theta^M_{\mu\nu}(x) + \Theta^G_{\mu\nu}(x) + \Theta^V_{\mu\nu}(x) = 0, \quad (1) \]
where \( \Theta_{\mu\nu}(x) \) is the \textit{total EM tensor of matter, vacuum and geometry}.

Basing on these simple considerations, one may speculate that the universe originated from the vacuum state of an empty world, perhaps as a consequence of a nucleating event \cite{8} [9] capable of priming the spontaneous breakdown of some fundamental symmetry, and thence evolved under the effect of a powerful EM transfer from geometry to matter. But the idea of such a prodigious zero–sum game must be ruled out immediately from this standard GR theory as soon as we realize that the continuity equations
\[ D^\mu \Theta^G_{\mu\nu}(x) = 0, \quad D^\mu \Theta^V_{\mu\nu}(x) = 0, \quad D^\mu \Theta^M_{\mu\nu}(x) = 0, \]
where \( D^\mu \) are contravariant spacetime derivatives (see Appendix), hold separately; the first of them being imposed by the second Bianchi identities \cite{10}, the second because \( D^\lambda g_{\mu\nu}(x) \equiv 0 \) by definition of \( D^\lambda \) (see Appendix), and the third by consequence.

No matter how fanciful it may seem, the idea that the big bang might have been originated in this way has the merit of suggesting that a suitable modification of GR might account for an effective energy exchange between geometry and matter.

The purpose of this and next two papers is to prove that this is indeed the case for a theory based on the conformal extension of GR proposed by Weyl in 1919 \cite{11} and fully developed by Cartan on a purely geometric ground in the early 1920s \cite{12} \cite{13} \cite{14} \cite{15} – hereafter referred to as \textit{Conformal General Relativity} (CGR) – in which a new degree of freedom accounting for a possible scale expansion of spacetime geometry and matter generation is introduced. We will formulate the problem for spacetimes of dimension \( n > 2 \) (in short, \( nD \) spacetimes) precisely to prove that it can be implemented in 4D only.
The idea is not new. Several authors contributed to it since the early 1960s, in particular Gürsey (1963) \cite{16}, Schwinger (1969) \cite{17}, Fubini (1976) \cite{18}, Englert et al. (1976) \cite{19}, Brout et al. (1978, 1979) \cite{20} \cite{21} and many others, but our implementation is new.

2 Riemann manifolds and Cartan manifolds

Both GR and CGR describe the universe as grounded on a differentiable $n$D manifold parameterized by $n$ variables $x^\mu$ ($\mu = 0, 1, \ldots, n - 1$) and covered by a continuous set of local tangent spaces, which support isomorphic representations of a finite continuous group, called the fundamental group. The fundamental group of GR is the $n$D Poincaré group and that of CGR is its conformal extension. The structures of both groups can be easily determined basing on the local transformation properties of a fundamental tensor, respectively $g_{\mu\nu}(x)$ or $\tilde{g}_{\mu\nu}(x)$, of pseudo-Euclidean signature $(+,-\ldots,-)$.

The local representations of the fundamental group at any two points of the manifold are related to each other by a generally path–dependent law called the manifold connection, which describes the variations in fundamental–group representations as detectable from a local frame driven along the path. Connections which do not return the identity when local frames are driven along closed paths are said to possess a non–zero curvature.

In this general scheme, physical particles can be described as irreducible representations of the fundamental group which, in general, undergo path–dependent transformations when the local frame is driven along arbitrary manifold paths. Local connection curvatures are the group–transformation residues per unit area of connections carried along infinitesimal closed paths, and represent local forces acting on particles which may be grounded on the manifold.

Different groups and types of curvatures characterize different manifolds. Since in GR the fundamental group preserves the local metric character of spacetime geometry, the manifold is a Riemann manifold. Instead, the fundamental group of CGR is the group of conformal transformations in $n$D, which is the closest light–cone–preserving extension of the Poincaré group in $n$D containing the subgroup of local scale transformations, or dilations. We refer to this manifold as a Cartan manifold.
2.1 Riemann manifolds and GR gravitational equations

Riemann connections are characterized by the property that translations have zero curvature, while Lorentz rotations generally have not. Therefore, any infinitesimal round-trip of the local reference frame, from a point \( x \) to the same point \( x \), generally results in an infinitesimal Lorentz rotation of the frame at \( x \). This rotation, which accounts for the effects of gravitational forces and/or local-frame accelerations, is fully represented by the local components of a 4-index tensor \( R_{\mu\nu\rho\sigma} \), called the Riemann tensor.

Einstein’s General Principle of GR in \( n \) dimensions asserts the invariance under general coordinate transformations of the total action integral grounded on a \( nD \) Riemann manifold (with \( n > 2 \)) parameterized by \( n \) adimensional coordinates \( x^\mu, \mu = 0, 1, \ldots, n-1 \), which are presumed to map the entire manifold over a suitable topographic collection of connected charts. Actually, these transformations, which we denote by \( D: x^\mu \rightarrow \bar{x}^\mu(x) \), are assumed to be everywhere continuous and differentiable; in short, diffeomorphic. They are the analog of gauge transformations in electrodynamics. To fix the notation, the action of \( D \) on a scalar function \( f(x) \) will be denoted by \( f(x) \rightarrow \bar{f}(\bar{x}) = f[x(\bar{x})] \).

Let \( g_{\mu\nu}(x) \) be the fundamental tensor of the manifold and define \( g^{\mu\nu}(x) \) by equations \( g_{\mu\lambda}(x) g^{\lambda\nu}(x) = \delta^\nu_\mu \) (the Kronecker delta). Then, the square of the line element \( ds^2(x) = g_{\mu\nu}(x) dx^\mu dx^\nu \), as well as the volume element \( \sqrt{-g(x)} d^n x \), where \( g(x) \) is the determinant of matrix \( [g_{\mu\nu}(x)] \), are scalar functions of \( x \). Therefore \( D \) acts on these as follows

\[
g_{\mu\nu}(x) dx^\mu dx^\nu \rightarrow \bar{g}_{\mu\nu}(\bar{x}) d\bar{x}^\mu d\bar{x}^\nu; \quad \sqrt{-g(x)} d^n x \rightarrow \sqrt{-\bar{g}(\bar{x})} d^n \bar{x},
\]

where \( \bar{g}_{\mu\nu}(\bar{x}) = g_{\rho\sigma}[x(\bar{x})], \bar{g}(\bar{x}) = g[x(\bar{x})] \). From these we obtain

\[
\bar{g}_{\mu\nu}(\bar{x}) = g_{\rho\sigma}[x(\bar{x})] \frac{dx^\rho}{d\bar{x}^\mu} \frac{dx^\sigma}{d\bar{x}^\nu}; \quad \sqrt{-\bar{g}(\bar{x})} = \sqrt{-g[x(\bar{x})]} \frac{d^n \bar{x}}{d^n x}. \tag{2}
\]

The basic assumption of GR is that \( g_{\mu\nu}(x) \) itself is the gravitational field. Thus to obtain the dynamical equations of the gravitational field we must first represent the total action integral of the universe in the form \( A = A^M + A^G + A^V \), where

\[
A^M = \int \sqrt{-g} L^M (g^{\mu\nu}, \Psi, \partial_\lambda \Psi) \, d^n x; \tag{3}
\]

\[
A^G = -\frac{1}{2\kappa} \int \sqrt{-g} R(g^{\mu\nu}, \partial_\lambda g^{\mu\nu}) \, d^n x; \tag{4}
\]

\[
A^V = -\rho_{\text{vac}} \int \sqrt{-g} \, d^n x; \tag{5}
\]
are the action integrals respectively of matter, vacuum and geometry; $L^M$ is the Lagrangian density of matter–gravity interaction as a function of $g^{\mu\nu}$ and a set of matter fields $\Psi$; $\rho_{\text{vac}}$ is the cosmological constant as potential–energy density of the vacuum; $R$ is the Ricci scalar (with negative sign convention) as a function of $g^{\mu\nu}$ and $\partial_{\lambda}g^{\mu\nu}$.

Then we must derive the motion equation for $g_{\mu\nu}(x)$, i.e., the GR gravitational equation, by the variational equations

\[
\frac{\delta A}{\delta g^{\mu\nu}(x)} \equiv \frac{\delta (A^M + A^G + A^V)}{\delta g^{\mu\nu}(x)} = 0, \quad \frac{1}{\sqrt{-g(x)}} \frac{\delta \sqrt{-g(x)}}{\delta g^{\mu\nu}(x)} = -\frac{1}{2} g_{\mu\nu}(x).
\]

The first of these implies the invariance of $A$ under metric diffeomorphisms and the second follows from the well–known formula $\delta \det[M] = \det(M) \text{Tr}[M^{-1}\delta M]$, where $M$ is a squared matrix. Since, on the other hand, we have

\[
\begin{align*}
\frac{2}{\sqrt{-g}} \frac{\delta A^M}{\delta g^{\mu\nu}} &= 2 \left( \frac{\partial L^M}{\delta g^{\mu\nu}} - \frac{g^{\mu\nu}}{2} L^M \right) = \Theta^M_{\mu\nu}; \\
\frac{2}{\sqrt{-g}} \frac{\delta A^G}{\delta g^{\mu\nu}} &= -\frac{1}{\kappa} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = \Theta^G_{\mu\nu}; \\
\frac{2}{\sqrt{-g}} \frac{\delta A^V}{\delta g^{\mu\nu}} &= \rho_{\text{vac}} g_{\mu\nu};
\end{align*}
\]

the gravitational equation, can be put in the form $\Theta_{\mu\nu} = \Theta^M_{\mu\nu} + \Theta^G_{\mu\nu} + \Theta^V_{\mu\nu} = 0$, as anticipated in §1. The match between the signs of the three EM tensors and those of the action integrals is evident.

### 2.2 The General Principle of CGR

nD Cartan manifolds differ from nD Riemann manifolds in that the fundamental group of manifold connections is the conformal extension of the nD Poincaré group. For brevity, we refer to it as the conformal group. This group includes the subgroups of local dilations (9)–(11) and special conformal transformations – or elations (term coined by Cartan in 1922), whose actions on $x^\mu, ds(x)$, tensors and fields are widely described in §1 of Part II. The conformal extensions of Riemann connections will be called Cartan connections.

Cartan connections are characterized by the possibility of local–scale changes in the representations of geometric or physical objects driven along manifold’s paths. But, since we wish to exclude that round–trips carried out on the manifold may alter the size of a body, dilation connections must have zero curvature, which means that the size of
geometric or physical bodies is allowed to change only along open world lines. This property implies that the dilation connection is the gradient of a scalar field.

In § 1 of Part II, it will be proven that, if the dilation–connection curvature is zero, the elation–connection curvature too is zero. Hence, were it not for the existence of infinitely extending time–like paths, both dilation and elation connections would drive pure gauge transformations and therefore could be completely removed from the theory. In which case, CGR would only be a special case of GR. So, while metric connections are local properties of the manifold, dilation and elation connections are instead global.

The General Principle of CGR states the invariance of the action integral of matter and geometry under conformal diffeomorphisms of manifold parameters. These are obtained by combining the coordinate diffeomorphisms $x^\mu \rightarrow \bar{x}^\mu$ described in the previous subsection, with Weyl transformations, which are defined by the following action on the line element

\[ ds(x) \rightarrow d\bar{s}(x) = e^{\lambda(x)} ds(x), \]  

where $\lambda(x)$ is a real differentiable function of $x^\mu$. The exponential form of factor $e^{\lambda(x)}$ is intended to ensure that it is always positive. Correspondingly, the metric tensor and the volume element $\sqrt{-g(x)} d^n x$ undergo the following Weyl transformations

\[ g_{\mu\nu}(x) \rightarrow \tilde{g}_{\mu\nu}(x) = e^{2\alpha(x)} g_{\mu\nu}(x); \quad g^{\mu\nu}(x) \rightarrow \tilde{g}^{\mu\nu}(x) = e^{-2\lambda(x)} g^{\mu\nu}(x); \]

\[ \sqrt{-g(x)} d^n x \rightarrow \sqrt{-\tilde{g}(x)} d^n x = e^{\alpha(x)} \sqrt{-g(x)} d^n x. \]

The first basic assumption of CGR is that in the action integral of matter and geometry the fundamental tensor of the Cartan manifold has the general form

\[ \tilde{g}_{\mu\nu}(x) = e^{2\alpha(x)} g_{\mu\nu}(x), \]

where $g_{\mu\nu}(x)$ is a Riemannian metric tensor and $e^{\alpha(x)}$ is a dynamical dilation factor accounting for spacetime inflation, both of which are presumed to depend upon the details of matter distribution and dynamics. As in GR, $g_{\mu\nu}(x)$ contains the information about the gravitational forces locally acting on geometry and matter, whereas $e^{2\alpha(x)}$ contains the information about the global dilation properties of geometry and matter production during the inflationary epoch.

The second basic assumption is that CGR should rapidly converge to GR at the end of the inflationary epoch, because this is the geometry of our post–inflationary universe.
To be consistent with our idea of inflation, we postulate that $e^\alpha(x)$ is very small at the beginning of the inflation epoch, hence rapidly increasing and converging to one during the transition to the post-inflationary era. Correspondingly, $\alpha(x)$ is initially negative, then increasing and converging to zero at the inflationary epoch end. It is therefore clear that the preliminary question with CGR is about whether these properties can be really satisfied in the context of a suitable field-theoretic environment. The main result of our investigation reported in this first paper is that the answer to this question is not only positive but also quantitatively satisfactory, provided that the matter primarily generated by the inflation process is a cold gas of Higgs bosons.

The factorization of fundamental tensor (12) into global and local properties requires an important property of Cartan-manifold geometry. To ensure the global character of the dilation connection, factor $\alpha(x)$ must depend upon $x$ through a time-like function $\tau(x)$ defining a family of iso-dilation surfaces $\tau(x) = \text{const}$. Were it not so, the dilation connection could not be represented as the gradient of a scalar quantity. The only way to implement this property is that $\tau(x)$ be the length of the word line stemming from the origin $x = 0$ of a future light-cone and ending at a point $x$ of the future-cone interior. This implies that the Cartan manifold itself is entirely confined to the interior of the future-cone, thereby taking on the topological structure of a de Sitter spacetime. Further details regarding the determination of $\tau(x)$ are given in §7.3.

To clarify even better the significance of this factorization, let us perform a suitable diffeomorphic change of coordinates $x^\mu \rightarrow \tilde{x}^\mu$, along with a Weyl transformation of suitable scale factor $e^{\lambda(x)}$, so as to obtain $\sqrt{-g(x)} = 1$ or, equivalently, $\sqrt{-\tilde{g}(x)} = e^{\lambda(x)}$. Thus, provided that the coordinate diffeomorphisms remain restricted to the subgroup of unimodular coordinate diffeomorphisms $x^\mu \rightarrow \tilde{x}^\mu$, one degree of freedom can be regarded as permanently transferred from $g_{\mu\nu}(x)$ to the global scale factor $e^\alpha(x)$. The metric-tensor factor appearing in Eq.(12) can then be decomposed as

$$g_{\mu\nu}(x) = \gamma_{\mu\nu}(x) + h_{\mu\nu}(x), \quad \text{with} \quad \det[\gamma_{\mu\nu}(x)] = \det[\gamma_{\mu\nu}(x) + h_{\mu\nu}(x)] = -1.$$  \hspace{1cm} (13)

Here, $\gamma_{\mu\nu}(x)$ represents the metric tensor of the privileged reference frame determined by the global distribution of matter (see §7.1) and $h_{\mu\nu}(x)$ represents the gravitational field as a deviation from $\gamma_{\mu\nu}(x)$ determined by the local details of matter distribution. If $h_{\mu\nu}(x)$ are so small to be regarded as an infinitesimal perturbation of $\gamma_{\mu\nu}(x)$, the second part of
Eq. (13) implies \( g^{\mu\nu}(x) h_{\mu\nu}(x) \cong \gamma^{\mu\nu}(x) h_{\mu\nu}(x) = 0 \).

By combining unimodular coordinate transformations \( x^\mu \rightarrow \tilde{x}^\mu \) and Weyl transformations we obtain the gauge group of unimodular conformal diffeomorphisms of Cartan–manifold parameters. These acts on metric tensor \( g_{\mu\nu}(x) \) and its volume element \( \sqrt{-g(x)} \) as follows

\[
\begin{align*}
g_{\mu\nu}(x) & \rightarrow \tilde{g}_{\mu\nu}(\tilde{x}) = e^{2\tilde{\alpha}(\tilde{x})} \tilde{g}_{\rho\sigma}[\tilde{x}(\tilde{x})] \frac{dx^\rho}{d\tilde{x}^\mu} \frac{dx^\sigma}{d\tilde{x}^\nu}, \\
\sqrt{-g(x)} & \rightarrow \sqrt{-\tilde{g}(\tilde{x})} = e^{n\tilde{\alpha}(\tilde{x})} \sqrt{-\tilde{g}[\tilde{x}(\tilde{x})]} \frac{d^n \tilde{x}}{d^n x},
\end{align*}
\]

where \( \tilde{g}_{\mu\nu}[\tilde{x}(x)] \) and \( g[x(\tilde{x})] \) are defined as in Eq. (2) with \( \tilde{x} \) in place of \( \tilde{x} \). The reason why we have not replaced \( \sqrt{-g(x)} \) and \( \sqrt{-\tilde{g}[\tilde{x}(\tilde{x})]} \) with 1, is that these expressions depend on the free components of \( g_{\mu\nu}(x) \) and \( \tilde{g}_{\mu\nu}[\tilde{x}(\tilde{x})] \), respectively, regardless of the fact that these metric tensors are constrained by the unimodularity condition. In the next, if a function \( f(x) \) is adimensional, we shall write \( \tilde{f}(\tilde{x}) \) means \( f[x(\tilde{x})] \) (\( x^\mu \) are adimensional parameters).

Henceforth, to distinguish the mathematical formalisms of GR from that of CGR, we adopt the following convention: all quantities related with, or supported on, or belonging to the Cartan manifold will be superscripted by a tilde.

### 2.3 CGR on Cartan manifold

The convergence of CGR to GR at the post–inflationary limit suggests a method for deriving CGR action integrals from GR action integrals. What remain unspecified in this derivation are the properties that a GR action integral should have in order for the basic assumptions about CGR discussed in the previous subsection to be satisfied. In this subsection, we limit ourselves to outline the method for the sole purpose of showing in advance some conceptual difficulties that the reader may encounter in dealing with the mathematical formalism of CGR.

The prominent aspect of Cartan manifold representations is that scale factor \( e^{\alpha(x)} \) can be interpreted as the physical promoter of spacetime–scale expansion and matter generation. This is possible in CGR because each physical field \( \Psi \) of GR is replaced by its inflated counterpart \( \tilde{\Psi}(\tilde{x}) \), according to the rule

\[
\Psi(x) \rightarrow \tilde{\Psi}(\tilde{x}) = e^{w_{\Psi} \tilde{\alpha}(\tilde{x})} \Psi[x(\tilde{x})],
\]

where \( w_{\Psi} \) is a constant for the field \( \Psi \).
where \( w_\Psi \) is the dimension, or weight, of \( \Psi \); as if during inflation all fields were subjected to a sort of continuous renormalization. The transition from a Riemann to a Cartan manifold deeply alters the structure of action integrals also because all basic quantities and operators of standard tensor calculus: coordinate parameters \( x^\mu \), partial derivatives \( \partial_\mu \equiv \partial/\partial x^\mu \), metric tensor \( g_{\mu\nu} \), covariant derivatives \( D_\mu \), Christoffel symbols \( \Gamma^\lambda_{\mu\nu} \), Ricci tensor \( R_{\mu\nu} \), Ricci scalar \( R \) etc., are respectively replaced by their conformal (tilde) counterparts. These depend on both the standard tensors of GR and the scale factor \( e^{\alpha(x)} \) as described in Eqs. (A-12) to (A-22) of the Appendix and hereafter shortly listed:

\[
\begin{align*}
x^\mu &\to \tilde{x}^\mu = e^{-\tilde{\alpha}(\tilde{x})} x^\mu; \\
\partial_\mu &\to \tilde{\partial}_\mu = e^{-\tilde{\alpha}(\tilde{x})} \partial_\mu; \\
g_{\mu\nu} &\to \tilde{g}_{\mu\nu}; \\
D_\mu &\to \tilde{D}_\mu; \\
\Gamma^\lambda_{\mu\nu} &\to \tilde{\Gamma}^\lambda_{\mu\nu}; \\
R_{\mu\nu} &\to \tilde{R}_{\mu\nu}; \\
R &\to \tilde{R}.
\end{align*}
\]

By performing these replacements, the action–integrals of matter, vacuum and geometry \( A^M \), \( A^G \) and \( A^V \), described by Eqs. (3)–(4) of §2.1, are replaced by

\[
\begin{align*}
\tilde{A}^M &= \int \sqrt{-\tilde{g}} \tilde{L}^M (\tilde{g}^{\mu\nu}, \tilde{\dot{\Psi}}, \tilde{\partial}_\lambda \tilde{\Psi}) \, d^n\tilde{x}; \\
\tilde{A}^G &= -\frac{1}{2\kappa} \int \sqrt{-\tilde{g}} \tilde{R}(\tilde{g}^{\mu\nu}, \tilde{\partial}_\lambda \tilde{g}^{\mu\nu}) \, d^n\tilde{x}; \\
\tilde{A}^V &= -\rho_{\text{vac}} \int \sqrt{-\tilde{g}} \, d^n\tilde{x};
\end{align*}
\]

It is then evident that, when \( e^{\tilde{\alpha}(\tilde{x})} \) converges to 1 at the end of the inflation epoch, all tilde quantities of CGR, converge to the homologous standard quantities and action integrals of GR. Correspondingly, the EM tensors on the Cartan manifold

\[
\tilde{\Theta}^X_{\mu\nu}(\tilde{x}) = \frac{2}{\sqrt{-\tilde{g}(\tilde{x})}} \frac{\delta \tilde{A}^X}{\delta \tilde{g}^{\mu\nu}(\tilde{x})},
\]

where \( X \) stands for \( M, G \) or \( V \), also converge to their homologous \( \Theta^X_{\mu\nu}(x) \) of GR.

Clearly, this is possible because \( \tilde{A}^M, \tilde{A}^G \) and \( \tilde{A}^V \) preserve exactly the formal properties of \( A^M, A^G \) and \( A^V \). The physical properties are instead very different because the Lagrangian densities of the former contain the factor \( e^{n\tilde{\alpha}(\tilde{x})} \) coming from \( \sqrt{-\tilde{g}(\tilde{x})} \).

Since the way to pass from GR to CGR is so simple, one may believe that the structure of CGR is rather simple, after all. Actually it is not so because all complications related to scale–factor variability remain hidden. In particular, \( \rho_{\text{vac}} \) of GR cannot be regarded anymore as a cosmological constant. The main advantage of this approach to CGR is that the expression of the inflated action integral preserves the same form as that of GR.
2.4 CGR on Riemann manifold

Let us introduce in this subsection an alternative but equivalent approach to CGR, which is more satisfactory in regard to mathematical simplicity and symmetry, hence more suitable for practical computations, although at the price of a possible change of physical meaning.

It consists in transferring the role played by the scale factor $e^{a(x)}$ of the fundamental tensor \( (12) \), introduced in §2.2, to a scalar field $\sigma(x) = \sigma_0 e^{a(x)}$, where $\sigma_0$ is constant of dimension $-1$, to be interpreted as the physical promoter the inflation process. Let us call it the dilation field. Having thus freed the fundamental tensor from its scale factor, the theory of CGR remains grounded on a Riemann manifold of metric tensor $g_{\mu\nu}(x)$, what is a considerable simplification from a mathematical standpoint.

To realize how far-reaching this expediency is, let us consider what happens to the geometry action integral $\tilde{A}_G$ introduced in the previous subsection. Using Eq.(A-16) of Appendix, we see that Eq.(18) of the previous subsection undergoes the replacement

$$\sqrt{-\tilde{g}} \tilde{R} = \sqrt{-g} e^{(n-4)\alpha} \left\{ e^{2\alpha} R(n-1)(6-n) g^{\rho\sigma}(\partial_\rho e^\alpha)(\partial_\sigma e^\alpha) - 2(n-1) D_\mu(e^\alpha g^{\mu\nu}\partial_\nu) \right\},$$

which converges to the Lagrangian density $\sqrt{-g} R(x)$ of $A^G$ as $\sigma(x)$ converges to $\sigma_0$.

Using Eq.(A-18) of the Appendix with $n = 4$, we see that Eq.(18) takes the form

$$\tilde{A}_G = -\frac{1}{2\kappa} \int \sqrt{-\tilde{g}(\tilde{x})} \tilde{R}(\tilde{x}) d^4 \tilde{x} = -\frac{6}{\kappa \sigma_0^4} \int \sqrt{-g} \left\{ \frac{1}{2} g^{\rho\sigma}(\partial_\rho \sigma)(\partial_\sigma) + \frac{\sigma^2 R_{12}}{12} \right\} d^4 x,$$

where we got rid of surface term $\sqrt{-g} D_\mu(g^{\mu\nu} \sigma_\nu) \equiv \partial_\mu \left( \sqrt{-g} g^{\mu\nu} \sigma_\nu \right)$ by integration. Thus, provided that $\sigma_0 = \sqrt{6/\kappa} \equiv \sqrt{6} M_{rP}$, $\tilde{A}_G$ takes the simple form

$$A^G = -\int \sqrt{-g} \left[ \frac{1}{2} g^{\rho\sigma}(\partial_\rho \sigma)(\partial_\sigma) + \frac{\sigma^2 R_{12}}{12} \right] d^4 x,$$

which represents a scale–invariant Lagrangian density of a ghost scalar field $\sigma(x)$ interacting with the gravitational field through the term $\sigma^2 R/12$. Here $\tilde{A}^G$ is renamed as $A^G$ because the integration is carried out over a Riemann manifold. This result, which is only possible in the special case of four spacetime dimensions, is a mathematical miracle.

Performing the same substitutions in $\tilde{A}^M$ and $\tilde{A}^V$, we find that all constants with dimension $k$ are multiplied by $[\sigma(x)/\sigma_0]^k$. In particular: all mass terms $m_k$ possibly appearing in the Lagrangian density are replaced by $m_k \sigma(x)/\sigma_0$, fields $\tilde{\Psi}(\tilde{x})$ of dimension $n$ are replaced by $[\sigma(x)/\sigma_0]^n \Psi(x)$, $\tilde{\partial}_\mu$ is replaced by $[\sigma(x)/\sigma_0] \partial_\mu$ and $\rho_{vac}$ is replaced by

$$m_k \sigma(x)/\sigma_0 \Psi(x).$$
\( \rho_{\text{vac}}[\sigma(x)/\sigma_0]^n \). It is then clear that \( A^V \) cannot be interpreted as the energy density of the vacuum but rather as a contribution to the energy density of geometry. This is the price to pay for the removal of all dimensional constants.

Despite the presence of the gravitational coupling constant \( \kappa \equiv M_r^{-2} \) and other possible dimensional constants, the total action integral \( \tilde{A} = \tilde{A}^M + \tilde{A}^G + \tilde{A}^V \) grounded on a Cartan manifold is equivalent to a total action integral \( A \), grounded on a Riemann manifold, which is free from dimensional parameters but in which a new field \( \sigma(x) \) instead appears.

In §3 we shall prove that \( A \) is even manifestly invariant under the group of local conformal transformations. Because of this overall symmetry, the study of the behavior of matter and geometry during the inflationary epoch is greatly facilitated.

The possible interaction of \( \sigma(x) \) with all other fields \( \Psi(x) \) and \( g_{\mu\nu}(x) \) of the theory may result to a total Lagrangian density in which a clear-cut separation in three components \( A^M, A^G, A^V \) is no longer possible. In particular, as already shown, \( A^V \) cannot be regarded as the action integral of the vacuum. For this reason, we may express with all generality the conformal–invariant action integral derived from \( \tilde{A} \) in the form

\[
A = \int \sqrt{-g(x)} L(x) \, d^4x \equiv \int \sqrt{-\tilde{g}} L(\sigma, \partial_\lambda \sigma, \Psi, \partial_\lambda \Psi, g_{\mu\nu}, \partial_\lambda g_{\mu\nu}) \, d^4x. \tag{21}
\]

In summary, the total action integral of CGR can be expressed in two different but equivalent ways: either as a functional \( \tilde{A} \) of tilde quantities grounded on a Cartan manifold of fundamental tensor \( \tilde{g}_{\mu\nu}(x) \), and containing several dimensional constants, or as a functional \( A \) of non–tilde quantities and dilation field \( \sigma(x) \), grounded on a Riemann manifold of metric tensor \( g_{\mu\nu}(x) \), but entirely free from dimensional constants.

The relations between GR, CGR on Cartan manifold (CM) and CGR on Riemann manifold (RM) condense into the diagram

\[
\text{CGR on Cartan manifold} \xrightarrow{\text{equivalent}} \sigma(x) \rightarrow \sigma_0 \xrightarrow{\text{GR}} \text{CGR on Riemann manifold.}
\]

The possibility of two equivalent representations mark a substantial difference between CGR and GR. In §4 of Part II, we will discuss about how and why these representations are related to different interpretations of the physical events with respect to inertial reference frames themselves undergoing the effects of the inflationary scale expansion and with respect to those of the post–inflationary era, which behave as described in GR.
2.5 The survival of the cosmological constant

The idea of the perfect conformal invariance of CGR action integral on Riemann manifold is certainly very appealing, especially from a mathematical standpoint, but unfortunately it is marginally tarnished by a consequence of the unimodularity condition stated in §2.2.

The reason is that, both the Cartan and the Riemann manifold representations of CGR suffer the consequences of the unimodularity constraint $\sqrt{-g(x)} = 1$ caused by the factorization of the dilational degree of freedom. This fact alters the structure of the total action integral $A$ by the addition, to its Lagrangian density $L(x)$, of a mute term of the form $\lambda_c \left[ \sqrt{-g(x)} - 1 \right]$, where $\lambda_c$ is the Lagrangian multiplier of the constraint. So, the true action integral is not that given by the Eq.(21) of the previous section but is $A_{\lambda_c} = \int \left\{ \sqrt{-g(x)} L(x) + \lambda_c \left[ \sqrt{-g(x)} - 1 \right] \right\} d^n x$.

According to the Lagrange–multiplier theory [22], $\lambda_c$ is determined by the condition

$$\delta \int \sqrt{-g(x)} \left[ L(x) + \lambda_c \right] d^n x = 0; \quad \sqrt{-g(x)} = 1.$$ (22)

Since the variation with respect to $\sigma$ and $\Psi$ yields zero because of the motion equations and the gravitational field is still represented by $g_{\mu\nu}(x)$, Eq.(22) is equivalent to the gravitational equation

$$\frac{2}{\sqrt{-g(x)}} \delta A_{\lambda_c} = \Theta_{\mu\nu}(x) - \lambda_c g_{\mu\nu}(x) = 0 \quad \text{along with} \quad D^\mu \Theta_{\mu\nu}(x) = 0,$$ (23)

where $\Theta_{\mu\nu}(x)$ is the total EM tensor of CGR on the Riemann manifold and $D^\mu$ the contravariant derivatives in general coordinates defined in Eq.(A-21) of the Appendix. So, the cosmological–constant term, which seemed to have disappeared in the transition from GR to CGR, is reintroduced in another form by the unimodularity condition.

The relevant point regarding Eqs.(23) is that $\lambda_c g_{\mu\nu}(x)$ cancels exactly the cosmological–constant term possibly included in $\Theta_{\mu\nu}(x)$, which is indeed the case if, for instance, the initial value of $\sigma$, and consequently its VEV, does not vanish, while all other local quantities are negligible. In lack of this cancellation, the gravitational equation would be impossible.

Thus, in a certain sense, the presence of a cosmological constant in CGR is a direct consequence of the spontaneous breakdown of the conformal symmetry.

In §11 of Part III, we shall prove that Eqs.(22) and (23) are crucial for determining the cosmological constant of CGR in the semiclassical approximation.
3 Conformal–invariant action integrals

On an $n$D Riemann manifold, field dimensions are determined by the condition that the total dimension of conformal–invariant action integrand $\sqrt{-g(x)} L(x)$, where $L(x)$ is a Lagrangian density, be zero. Accordingly, scalar fields have dimension $1 - n/2$ and spinor fields dimension $(1 - n)/2$. Since spacetime parameters $x^\mu$ have dimension zero, partial derivatives $\partial_\mu$ too have dimension zero. Note that, since $ds^2(x) = g_{\mu\nu}(x) dx^\mu dx^\nu$ has dimension 2, $g_{\mu\nu}(x)$ has dimension 2, $\sqrt{-g(x)}$ dimension $n$ and Ricci scalar tensor $R(x)$ dimension $-2$. For consistency with expressions like $\partial_\mu - ig A_\mu(x)$, covariant gauge fields $A_\mu(x)$ have dimension zero. However, since $g_{\mu\nu}(x)$ has dimension $-2$, in virtue of equation $g_{\mu\lambda}(x) g^{\lambda\nu}(x) = \delta_\mu^\nu$, their contravariant counterparts $A^\mu(x)$ have dimension $-2$. Etc.

To state the main properties of conformal invariant action integrals, the following simple but important theorem is of help:

**A necessary (but not sufficient) condition for the action integral of a Lagrangian density $L(x)$, grounded on a $n$D Riemann manifold of metric tensor $g_{\mu\nu}(x)$, to be conformal invariant is not only that it is free from dimensional constants - which is quite obvious since conformal invariance implies global scale invariance - but also that the trace $\Theta(x) = g^{\mu\nu}(x) \Theta_{\mu\nu}(x)$ of its EM tensor $\Theta_{\mu\nu}(x)$ vanishes.**

**Proof:** Let $A = \int \sqrt{-g(x)} L(x) d^n x$ be the action integral. Conformal invariance implies the vanishing of the variation of $A$ under infinitesimal Weyl transformations of the form

$$g^{\mu\nu}(x) \rightarrow g^{\mu\nu}(x) + \delta \epsilon g^{\mu\nu}(x) \equiv g^{\mu\nu}(x) - 2 \epsilon(x) g^{\mu\nu}(x),$$

where $\epsilon(x)$ is an arbitrary infinitesimal function of $x$. We have, therefore,

$$\delta \epsilon A = -2 \int \sqrt{-g(x)} \epsilon(x) g^{\mu\nu}(x) \Theta_{\mu\nu}(x) d^n x = -2 \int \sqrt{-g(x)} \epsilon(x) \Theta(x) d^n x = 0,$$

from which equation $\Theta(x) = 0$ follows.

### 3.1 Action integrals of gauge fields are conformal invariant in 4D only

**Proof:** The Lagrangian density of a covariant Yang–Mills field $A^{YM}_\mu$ in $n$D is

$$L^{YM} = -\frac{1}{4} \text{Tr}[F_{\mu\nu} F^{\mu\nu}], \text{ with } F_{\mu\nu} = \tau_a F^a_{\mu\nu}, \quad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{ab}_c A^b_\mu A^c_\nu,$$
where $\tau_a$ are group-generator matrices, $f_{abc}^a$ group structure constants and $g$ the interaction constant, implying that the action integral has dimension $n - 4$. The symmetric EM tensor and its trace are respectively

$$\Theta_{\mu\nu}^{YM} = -\frac{1}{4} \text{Tr}[F_{\mu\lambda} F^\lambda_{\nu}] + \frac{1}{4} g_{\mu\nu} \text{Tr}[F_{\mu\nu} F^{\mu\nu}], \quad \Theta^{YM} = \left(\frac{n}{4} - 1\right) \text{Tr}[F_{\mu\nu} F^{\mu\nu}].$$

It is then evident that conformal invariance holds only for $n = 4$.

### 3.2 Action integrals of massless spinor fields are conformal invariant in any dimension

**Proof:** The Lagrangian density of a free fermion field $\psi$ of mass $M$ on the Riemann manifold has the form

$$L^F = \frac{i}{2} \left[ \bar{\psi} \left( \slashed{D} \psi \right) - \left( \slashed{D} \bar{\psi} \right) \psi \right] + M \bar{\psi} \psi,$$

where $\gamma^\mu(x) = \gamma^a e_a^\mu(x)$, $e_a^\mu(x)$ being the “vierbeins”, i.e., the $n$D analog of the “n–beins”, $\Gamma_\mu(x)$ are (anti–Hermitian) spin matrices, which are necessary to make partial derivatives $\partial_\mu$ covariant, and $\bar{\psi}$ and $\psi$ respectively the covariant Hermitian–conjugate of $\psi$ and $\psi$.

The motion equation is $i \slashed{D} \psi(x) = M \bar{\psi} \psi(x)$ and the EM tensor is

$$\Theta_{\mu\nu}^{F} = \frac{i}{4} \left[ \bar{\psi} \gamma_\mu \slashed{D}_\nu \psi + \bar{\psi} \gamma_\nu \slashed{D}_\mu \psi - (\slashed{D}_\mu \psi) \gamma_\nu \psi + (\slashed{D}_\nu \psi) \gamma_\mu \psi \right].$$

By index contraction of $\Theta_{\mu\nu}^{F}$, and exploitation of motion equations, we immediately obtain the trace $\Theta^{F} = M \bar{\psi} \psi$, clearly implying conformal invariance for $M = 0$ for any $n$.

Since for a gauge–field multiplet $A_\mu^a(x)$ and a spinor–field multiplet $\psi(x)$, the expression

$$A_\mu^a(x) \bar{\psi}(x) \tau_a \gamma^\mu(x) \psi(x)$$

is conformal invariant, it is easy to prove that conformal–invariant Lagrangian densities of spinor fields interacting with gauge fields exist only in 4D.

### 3.3 Action integrals of scalar fields are conformal invariant in 4D only

**Proof:** In $n$D, the more general Lagrangian density of a scalar field $\varphi$ with self–interaction constant $c$, interacting with the gravitational field through the metric tensor $g_{\mu\nu}(x)$ and its derivatives, and free from dimensional constants, has the general form

$$L_\varphi = \frac{1}{2} \left[ g^{\mu\nu} (\partial_\mu \varphi) (\partial_\nu \varphi) + a \varphi^2 R - \frac{c(n - 2)}{n} \varphi^{2n/(n-2)} \right].$$
where $R$ is the Ricci scalar and $a$ a suitable real constant. The motion equation is

$$D^2 \varphi - aR \varphi + c \varphi^{(n+2)(n-2)} = 0,$$

where $D^2 = D^\mu D_\mu$ and $D_\mu$ are respectively the covariant d’Alembert operator and the covariant derivatives constructed out of $g_{\mu\nu}$. The (improved) EM tensor is

$$\Theta^\varphi_{\mu\nu} = \left( \partial_\mu \varphi \right) \left( \partial_\nu \varphi \right) - \frac{g_{\mu\nu}}{2} \left[ g^\rho\sigma \left( \partial_\rho \varphi \right) \left( \partial_\sigma \varphi \right) - \frac{c(n-2)}{n} \varphi^{2n/(n-2)} \right] + a \left( g_{\mu\nu} D_2 - D_\mu \partial_\nu \right) \varphi^2 + a \varphi^2 \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right);$$

where Eq. (A-10) of Appendix is exploited, an integration by parts is performed, and a surface term suppressed.

By contraction with $g^{\mu\nu}$, and using the motion equation, we see that the trace of the EM vanishes only if $a = (n-2)/4(n-1)$, which implies that the action integral can be conformal invariant only if the Lagrangian density has the form

$$L_\varphi = \frac{1}{2} \left[ g_\mu^\nu \left( \partial_\mu \varphi \right) \left( \partial_\nu \varphi \right) + \frac{n-2}{4(n-1)} R \varphi^2 - \frac{c(n-2)}{n} \varphi^{2n/(n-2)} \right].$$

Now, Weyl transformation $\varphi(x) \rightarrow \tilde{\varphi}(x) = e^{-\alpha(x)} \varphi(x)$, along with Eqs. (10) and (11) of §2.2, and (A-14) of the Appendix, produce the transformation

$$\sqrt{-g} L_\varphi \rightarrow \sqrt{-\tilde{g}} \tilde{L}_\varphi = \sqrt{-g} e^{(n-4)\alpha} \left\{ L_\varphi - \frac{n-4}{8} \left[ n \varphi^2 \left( \partial^{\mu} \alpha \right) \left( \partial_\mu \alpha \right) - 2 \left( \partial^\mu \varphi^2 \right) \left( \partial_\mu \alpha \right) \right] - \frac{n-2}{4} D^\mu \left( \varphi^2 \partial_\mu \alpha \right) \right\},$$

showing that $\sqrt{-g(x)} e^{(n-4)\alpha(x)} D_\mu \left[ \varphi^2(x) \partial_\mu \alpha(x) \right]$ is a surface term if and only if $n = 4$. In summary, the only type of conformal–invariant action integral for a scalar field $\varphi$ interacting with the gravitational field is only possible in 4D and has the general form

$$L_\varphi = \frac{1}{2} g_\mu^\nu \left( \partial_\mu \varphi \right) \left( \partial_\nu \varphi \right) + \frac{R}{12} \varphi^2 - \frac{\lambda}{4} \varphi^4,$$

from which the motion equation $D^2 \varphi - R \varphi/6 + \lambda \varphi^3 = 0$ follows.

Since for a scalar field $\varphi(x)$ and a spinor field $\psi(x)$, the expression $\sqrt{-g(x)} \varphi(x) \tilde{\psi}(x) \psi(x)$ is conformal invariant, it is easy to prove that conformal–invariant Lagrangian densities of spinor fields interacting with scalar fields are only possible in 4D.
3.4 The action integral of a ghost scalar field has a geometric meaning

Basing on the results of the previous subsection, one might think that a gravitational equation similar to that found by Einstein comes spontaneously into play, provided that $\varphi$ has a non-zero vacuum expectation value (VEV). Unfortunately, however, in this way the gravitational coupling constant would have the wrong sign and consequently the gravitational field would be repulsive.

Instead, let us assume that the action integral is negative, i.e., has the form

$$A^\sigma = - \int \sqrt{-g} \left( g^{\mu \nu} (\partial_\mu \sigma) (\partial_\nu \sigma) + \frac{R \sigma^2}{6} + \frac{\bar{\lambda}}{2} \sigma^4 \right) d^4 x, \tag{25}$$

with $\bar{\lambda}$ a real constant and $\sigma(x)$ always positive (this is admissible because the motion equation is invariant under $\sigma \to -\sigma$). Therefore, without loss of generality, we can put $\sigma(x) = \sigma_0 e^{\alpha(x)}$ with $\sigma_0 > 0$. Note that, if $\bar{\lambda} > 0$, the potential–energy density $\bar{\lambda} \sigma^4(x)/4$ is always positive, which may have an important role in preventing the infinite growth of $\sigma(x)$, thus bounding the vacuum energy from below in suitable dynamical conditions.

Since the kinetic energy of $A^\sigma$ is negative, $\sigma(x)$ carries negative kinetic energy and therefore cannot be regarded as a physical field, but rather as a ghost scalar field potentially invested with geometric meaning. It is called the dilation field because, as we shall prove, it is the factor of spacetime–scale expansion during inflation.

Note: One should not worry too much about possible violations of $S$–matrix unitarity, because it has been recently proven that this does not happen if ghosts interact with physical fields so as to bound the Hamiltonian spectrum from below [26].

Now, the Ricci–scalar factor of Eq. (25) has the right sign for gravity to be attractive. Even better, let us put $\sigma_0 = \sqrt{6/\kappa} = \sqrt{6} M_P \simeq 6 \times 10^{18}$ GeV, where $\kappa$ is the gravitational coupling constant and perform the following Weyl transformations: $g^{\mu \nu}(x) \to \tilde{g}^{\mu \nu}(x) = e^{-2\alpha(x)} g^{\mu \nu}(x)$, $\sqrt{-g(x)} \to \sqrt{-\tilde{g}(x)} = e^{4\alpha(x)} \sqrt{-g(x)}$, for any local quantity $Q_n(x)$ of dimension $n$, $Q_n(x) \to \tilde{Q}_n(x) = e^{n\alpha(x)} Q_n(x)$ and, in particular, $\sigma(x) \to \tilde{\sigma}(x) = e^{-\alpha(x)} \sigma(x) \equiv \sigma_0$. Then we see that with these replacements $A^\sigma$ is transformed to

$$\tilde{A}^{(\sigma_0)} = - \int \frac{\sqrt{-\tilde{g}(x)}}{2\kappa} \tilde{R}(x) d^4 x - \frac{\bar{\lambda} \sigma_0^4}{4} \int \sqrt{-\tilde{g}(x)} d^4 x. \tag{26}$$

which is formally, but not substantially, equal to the corresponding expression of the Einstein–Hilbert action–integral of standard GR. Then, to the limit $e^{\alpha(x)} \to 1$, $\bar{\lambda} \sigma_0^4/4$, if not canceled by other terms, gives an improper contribution to the cosmological constant.
Thus, the conformal–invariant action integral $A^{(\sigma)}$ grounded on the Riemann manifold takes the form of a non–conformal–invariant action integral $\tilde{A}^{(\sigma_0)}$, for gravity and vacuum, grounded on the Cartan manifold. Note that choosing the positive sign for $\sigma_0$ is formally equivalent to assuming the spontaneous breakdown of conformal symmetry, in such a way that the degree of freedom of the ghost scalar field is incorporated into the determinant of the fundamental tensor on the Cartan manifold. We have therefore the following remarkable result (already known to Schwinger, 1969):

For $n = 4$, and for $n = 4$ only, the conformal–invariant action integral of a positive scalar–ghost field $\sigma(x) = \sigma_0 e^{\alpha(x)}$ on a Riemann manifold of metric $g_{\mu\nu}(x)$ is equivalent to a non–conformal–invariant action integral on a Cartan manifold with fundamental tensor $\tilde{g}_{\mu\nu}(x) = e^{2\alpha(x)} g_{\mu\nu}(x)$, in which the dimensional constant $\sigma_0$ plays the role of conformal–symmetry–breaking parameter.

Unfortunately, however, Eq.(25) makes sense only as a part of a more complex action integral. In fact, the solutions for $\sigma(x)$ to the motion equation derived from $A^{(\sigma)}$ are badly divergent. This raises the question of how the expression of $A^{(\sigma)}$ could be appropriately included in a more general action integral in order for the equation for $\sigma(x)$ to be convergent.

Indeed, more general sorts of conformal–invariant geometry action–integrals on 4D Riemann manifold may include a negative term proportional to the squared Weyl tensor $C^2(x)$, of the form described by Eq.(A-23) of Appendix, and a conformal–invariant potential–energy density term $V(\sigma^2, \varphi)$, where $\varphi$ is a suitable subset of all physical fields accounting for possible interactions between geometry and matter. For the sake of definiteness, we assume that $V(\sigma^2, \varphi)$ should vanish if $\sigma$ or $\varphi$ vanish. In consideration of this, it is suitable to represent the totality all matter fields by $(\varphi, \Psi)$, where $\Psi$ stands for the subset of matter fields different from $\varphi$.

$V(\sigma^2, \varphi)$ is of degree two in $\sigma$ since, otherwise, motion equations may permit $\sigma$ to change sign in the course of time, which would make it impossible to assume $\sigma(x) = \sigma_0 e^{\alpha(x)}$. Therefore, conformal–invariant interactions of fermions with $\sigma$ must be excluded because these are linear in $\sigma$.

Quadratic couplings with zero–mass vector fields, as for instance is $\sigma^2 g^{\mu\nu} A^a_\mu A^b_\nu$, must also be excluded, because a gauge–vector field cannot incorporate a scalar ghost as longitu-
dinal spin component via the Englert–Brout–Higgs mechanism [27] [28]. Hence, \( V(\sigma^2, \varphi) \) can only depend on a set of real or complex zero–mass scalar fields \( \varphi = \{ \varphi_1, \varphi_2, \ldots, \varphi_N \} \). Since all \( \varphi_i \) have dimension one, the only possibility is to put \( V(\sigma^2, \varphi) = \frac{1}{2} \sigma^2 c^{ij} \varphi_i^* \varphi_j \), where \([c^{ij}]\) is a suitable matrix (summation over repeated indices being understood).

In summary, the most general expression of a conformal–invariant geometry action–integral on the Riemann manifold must have the form

\[
A^{(\sigma, \varphi)} = - \int \sqrt{-g} \left[ \frac{1}{2\kappa} \tilde{R} - \frac{1}{2} \sigma^2 c^{ij} \tilde{\varphi}_i^* \tilde{\varphi}_j + \frac{\tilde{\lambda}}{2} \sigma^4 \right] d^4x, \tag{27}
\]

where \( \varphi_i \) are massless scalar fields. Here, the Lagrangian–density term proportional to the conformal–curvature tensor \( C^2(x) \), as described by Eq.(A-23) of the Appendix, is excluded for the reasons explained in the Appendix end.

By applying the Weyl transformations described in §2.2 of Appendix, and already used in Eq.(25) of §3.4, we obtain the the most general expression of the geometry action–integral on the Cartan manifold

\[
\tilde{A}^{(\sigma_0, \varphi)} = - \int \sqrt{-\tilde{g}} \left[ \frac{1}{2\kappa} \tilde{R} - \frac{1}{2} \sigma_0^2 c^{ij} \tilde{\varphi}_i^* \tilde{\varphi}_j + \frac{\tilde{\lambda}}{2} \sigma_0^4 \right] d^4x. \tag{28}
\]

In this representation, the conformal symmetry of \( A^{(\sigma, \varphi)} \) appears explicitly broken by the dimensional constant \( \sigma_0 \). Since the explicit dependence of the action integral on the dilation field is now disappeared, we may safely shift the Lagrangian–density terms \( \frac{1}{2} \sigma_0^2 c^{ij} \tilde{\varphi}_i^* \tilde{\varphi}_j \) and \( -\frac{\tilde{\lambda}}{2} \sigma_0^4 \) to the matter Lagrangian density.

As regards the action integral of matter on the Riemann manifold, let us denote by \( A^{(\varphi, \Psi)} \) the part of total action integral which depends on all physical fields \( (\varphi, \Psi) \), i.e., but not on \( \sigma(x) \) via \( V(\sigma^2, \varphi) \), and by \( L^{(\varphi, \Psi)}(x) \) its Lagrangian density. Then we have

\[
A^{(\varphi, \Psi)} = \int \sqrt{-g(x)} L^{(\varphi, \Psi)}(x) d^4x, \tag{29}
\]

as well as the corresponding action integral on the Cartan manifold

\[
\tilde{A}^{(\varphi, \tilde{\Psi})} = \int \sqrt{-\tilde{g}(x)} \tilde{L}^{(\varphi, \tilde{\Psi})}(x) d^4x. \tag{30}
\]

The most general form of total action integral of geometry and matter on the Riemann and the Cartan manifolds can then be respectively written as

\[
A = A^{(\sigma)} + A^{(\varphi, \Psi)} - \int \sqrt{-g(x)} V[\sigma^2(x), \varphi(x)] d^4x; \tag{31}
\]

\[
\tilde{A} = \tilde{A}^{(\sigma_0)} + \tilde{A}^{(\varphi, \tilde{\Psi})} - \int \sqrt{-\tilde{g}(x)} \tilde{V}[\sigma_0^2(x), \varphi(x)] d^4x; \tag{32}
\]
with $A^\sigma$ and $\tilde{A}^{\sigma_0}$ defined by Eqs. (25) and (26).

**In conclusion**

- Conformal invariance and 4-dimensionality of spacetime are closely related, since non-trivial semiclassical conformal–invariant actions exist only in 4D.
- Einstein’s GR can be incorporated into CGR, provided that conformal symmetry is spontaneously broken.
- Matter-field Lagrangian densities on the Riemann and Cartan manifolds maintain the same algebraic form, all quantities being replaced by tilde quantities, whereas the form of the geometric Lagrangian density changes considerably.
- In 4D, and only in 4D, the conformal symmetry of a conformal–invariant action integral on a Riemann manifold containing a scalar ghost field $\sigma(x) = \sigma_0 e^{\alpha(x)}$, $\sigma_0 > 0$, breaks down spontaneously to a Einstein–Hilbert action integral on the Cartan manifold, with $g(x) = -[\sigma(x)/\sigma_0]^8$ playing the role of the determinant of the fundamental tensor. Dimensional constant $\sigma_0$, which works as the conformal symmetry-breaking parameter, is related to gravitational constant $\kappa$ by equation $\sigma_0^2 = 6/\kappa = 6M_P^2$.

4 **Energy–momentum conservation**

Let $x^\mu$ be the spacetime coordinates of a 4D Riemann manifold, on which all local quantities of a field theory are grounded, and

$$\bar{x}^\mu(x) = x^\mu + \delta_\varepsilon x^\mu \equiv x^\mu + \varepsilon^\mu(x),$$

(33)

where $\varepsilon^\mu(x)$ are differentiable functions of $x^\nu$, an infinitesimal transformation of $x^\mu$. Let $g_{\rho\sigma}(x)$ and $\bar{g}_{\mu\nu}(\bar{x})$ be the metric tensor as a function of $x^\mu$ and of $\bar{x}^\mu$, respectively. From the first of Eqs. (2) of §2.1, we derive the corresponding variation of $g_{\mu\nu}(x)$

$$\delta_\varepsilon g_{\mu\nu}(x) \equiv \bar{g}_{\mu\nu}(\bar{x}) - g_{\mu\nu}(x) = -\varepsilon^\lambda(x) \partial_\lambda g_{\mu\nu}(x) - g_{\mu\lambda}(x) \partial_\nu \varepsilon^\lambda(x) - g_{\nu\lambda}(x) \partial_\mu \varepsilon^\lambda(x).$$

Using $g^{\mu\lambda}(x) g_{\lambda\nu}(x) = \delta_\nu^\mu$, where $\delta_\nu^\mu$ is Kronecker delta, we obtain

$$\delta_\varepsilon g^{\mu\nu}(x) = -g^{\mu\rho}(x) g^{\nu\sigma}(x) \delta_\varepsilon g_{\rho\sigma}(x) = g^{\mu\rho}(x) g^{\nu\sigma}(x) \varepsilon^\lambda(x) \partial_\lambda g_{\rho\sigma}(x) + g^{\mu\rho}(x) \partial_\rho \varepsilon^\nu(x) + g^{\mu\rho}(x) \partial_\rho \varepsilon^\nu(x).$$

(34)
Finally, using the well-known formula
\[ d \det(M) = \det(M) \text{Tr}[M^{-1} dM], \]
where \( M \) is any square matrix and \( \text{Tr}[\ldots] \) stands for matrix trace, we find
\[ \delta_x \sqrt{-g(x)} = - \frac{\sqrt{-g(x)}}{2} g_{\mu\nu}(x) \delta_x g^{\mu\nu}(x). \] (35)

To delimit the validity domain of our argument, we introduce the following definition:
An action integral is called complete if we can derive from it the complete motion equations of all fields contained in it.

Let \( \Psi(x) = \{\Psi_1(x), \Psi_2(x), \ldots, \Psi_M(x)\} \) and \( \Phi(x) = \{\Phi_1(x), \Phi_2(x), \ldots, \Phi_K(x)\} \) be respectively the internal and external fields, and let
\[ A = \int \sqrt{-g} L(g^{\mu\nu}, \partial_\lambda g^{\mu\nu}, \Psi, \partial_\lambda \Psi, \Phi) \, d^n x; \] (36)
be a complete action integral. Here, \( L \) is the Lagrangian density as a function of \( \Psi(x) \) and \( \Phi(x) \) of \( g_{\mu\nu}(x) \) and possibly of \( \partial_\lambda g^{\mu\nu}(x) \). Since, under the action of \( \delta_x x^\mu \), \( g_{\mu\nu}(x) \) undergoes the variation stated by Eq.(34) and \( \Psi_i(x) \) the variation
\[ \delta_x \Psi_i(x) = \left[ \partial_\mu \Psi_i(x) \right] \epsilon^\mu(x), \] (37)
we can express the invariance of the total action \( A \), generally depending through \( A \) on external fields \( \Psi_i(x) \), under arbitrary diffeorphisms of any bounded spacetime–region \( R \) of the Riemann manifold is expressed by the variational equation
\[ \delta_x A \equiv \int_R \frac{\delta A}{\delta g^{\mu\nu}(x)} \delta_x g^{\mu\nu}(x) \, d^n x + \int_R \frac{\delta A}{\delta \Psi_i(x)} \delta_x \Psi_i(x) \, d^n x = 0. \] (38)

Performing the variations and getting rid of surface terms, we obtain
\[ \frac{\delta_x A}{\delta_x \Psi_i(x)} = \sqrt{-g(x)} \left[ \frac{\partial L(x)}{\partial \Psi_i(x)} - \partial_\lambda \frac{\partial L^M(x)}{\partial_\lambda \Psi_i(x)} \right] = - \sqrt{-g(x)} F^i(x); \] (39)
\[ \frac{\delta_x A}{\delta_x g^{\mu\nu}(x)} = \frac{\partial \left[ \sqrt{-g(x)} L(x) \right]}{\partial g^{\mu\nu}(x)} - \partial_\lambda \frac{\partial \left[ \sqrt{-g(x)} L(x) \right]}{\partial_\lambda g^{\mu\nu}(x)} = \frac{\sqrt{-g(x)}}{2} \Theta_{\mu\nu}(x); \] (40)
where \( F^i[\Phi(x), x] \) are possible generalized external forces strictly depending on external fields \( \Phi_k(x), \Theta_{\mu\nu}(x) \) is the EM tensor according to Hilbert’s definition.

Then, we can rewrite Eq.(36) as
\[ \delta_x A = \int_R \sqrt{-g(x)} \left[ \frac{1}{2} \Theta_{\mu\nu}(x) \delta_x g^{\mu\nu}(x) - \Phi^i(x) \delta_x \Psi_i(x) \right] \, d^n x = 0. \] (41)
Let us prove that the EM tensor $\Theta_{\mu\nu}(x)$ of $A$ satisfies equation
\[
D^\mu \Theta_{\mu\nu}(x) = -\mathcal{F}^i[\Phi(x), x] \partial_\mu \Psi_i(x),
\] (42)
where $D^\mu$ are the contravariant derivatives. The right-hand side represents the work done by the generalized forces $\mathcal{F}^i[\Phi(x), x]$ to produce the diffeomorphic variations of $\Psi_i(x)$. So, if $A$ does not depend on external fields, $\Theta_{\mu\nu}(x)$ is covariantly conserved.

Now, using Eq.[A-1] of Appendix, we get $\partial_\lambda g_{\rho\sigma}(x) = \Gamma^\lambda_{\rho\sigma}(x) g_{\tau\sigma}(x) + \Gamma^\lambda_{\rho\sigma}(x) g_{\tau\rho}(x)$, where $\Gamma^\sigma_{\mu\nu}(x)$ are the Christoffel symbols. Consequently, Eq.(31) becomes
\[
\delta \varepsilon g^{\mu\nu}(x) = \left[ g^{\mu\rho}(x) \Gamma^\nu_{\rho\lambda}(x) + g^{\nu\rho}(x) \Gamma^\mu_{\rho\lambda}(x) \right] \varepsilon^\lambda(x) + g^{\mu\rho}(x) \partial_\rho \varepsilon^\nu(x) + g^{\nu\rho}(x) \partial_\rho \varepsilon^\mu(x).\] (43)

Exploiting the symmetry of $\Theta_{\mu\nu}(x)$, the following equations are then easily found
\[
\frac{1}{2} \sqrt{-g(x)} \Theta_{\mu\nu}(x) \delta \varepsilon g^{\mu\nu}(x) = \sqrt{-g(x)} \Theta^\rho_{\nu}(x) \left[ \partial_\rho \varepsilon^\nu(x) + \Gamma^\rho_{\mu\nu}(x) \varepsilon^\lambda(x) \right] = \partial_\rho \left[ \sqrt{-g(x)} \Theta^\rho_{\nu}(x) \varepsilon^\nu(x) \right] - \left\{ \partial_\rho \left[ \sqrt{-g(x)} \Theta^\rho_{\nu}(x) \right] - \sqrt{-g(x)} \Gamma^\rho_{\mu\nu}(x) \Theta^\nu_{\sigma}(x) \right\} \varepsilon^\nu(x).
\]
Replacing this expression in Eq.(11) and using the identity
\[
\partial_\rho \left[ \sqrt{-g(x)} \Theta^\rho_{\nu}(x) \right] - \sqrt{-g(x)} \Gamma^\rho_{\mu\nu}(x) \Theta^\nu_{\sigma}(x) \equiv D_\rho \left[ \sqrt{-g(x)} \Theta^\rho_{\nu}(x) \right] \equiv \sqrt{-g(x)} D_\rho \Theta^\rho_{\nu}(x)
\]
we arrive at rewriting Eq.(11) as
\[
\delta \varepsilon A = -\int_\mathcal{R} \sqrt{-g(x)} \left[ D^\mu \Theta_{\mu\nu}(x) + \Phi^i(x) \partial_\mu \Psi_i(x) \right] \varepsilon^\nu(x) d^3x + \frac{1}{2} \int_\mathcal{R} \partial_\rho \left[ \sqrt{-g(x)} \Theta^\rho_{\nu}(x) \varepsilon^\nu(x) \right] d^3x.
\]

Since $\varepsilon^\nu(x)$ is arbitrary inside $\mathcal{R}$ and vanishing at the boundary of $\mathcal{R}$ for any given $\mathcal{R}$, the last integration yields zero and so Eq.(12) immediately descends. Therefore, going back to Eqs. (41), (40) and (10) of §2.1, we also obtain the separate conservation laws
\[
D^\mu \Theta^M_{\mu\nu}(x) = 0; \quad D^\mu \Theta^G_{\mu\nu}(x) = 0; \quad D^\mu \Theta^V_{\mu\nu}(x) = 0.
\] (44)

These results can be extended to invariance under conformal diffeomorphisms, in which case $\tilde{\Theta}_{\mu\nu}(x)$ is replaced by the conformal EM tensor defined by Eq.(20) of §2.2, $D^\mu$ by $\tilde{D}^\mu$ defined by Eqs. (A-20) and (A-22) of the Appendix, Eq.(13) by
\[
\delta \tilde{\varepsilon} \tilde{g}^{\mu\nu}(x) = \left[ \tilde{g}^{\mu\rho}(x) \Gamma^\nu_{\rho\lambda}(x) + \tilde{g}^{\nu\rho}(x) \Gamma^\mu_{\rho\lambda}(x) \right] \varepsilon^\lambda(x) + \tilde{g}^{\mu\rho}(x) \partial_\rho \varepsilon^\nu(x) + \tilde{g}^{\nu\rho}(x) \partial_\rho \varepsilon^\mu(x).
\]
and Eq.(12) by $\tilde{D}^\mu \tilde{\Theta}_{\mu\nu}(x) = -\tilde{\mathcal{F}}^i[\tilde{\Phi}(x), x] \partial_\mu \tilde{\Psi}_i(x)$, where all quantities superscripted by a tilde are the conformal covariant or contravariant analogs of the non-superscripted ones.
5 Energy–momentum transportation

Let $A^A_0$ and $B^B_0$ be the complete action integrals on a Riemann manifold of two non-interacting fields $\varphi^A(x)$ and $\varphi^B(x)$, $L^A_0(x)$ and $L^B_0(x)$ their respective Lagrangian densities, $\Theta^A_{0\mu\nu}(x)$ and $\Theta^B_{0\mu\nu}(x)$ their respective EM tensors. As proven in the previous Section, invariance of $A^A_0$ and $B^B_0$ under metric diffeomorphisms entails the equations $D^\mu \Theta^A_{0\mu\nu}(x) = 0$ and $D^\mu \Theta^B_{0\mu\nu}(x) = 0$. We want to study how these equations change when the two fields interact through a potential–energy term $V(x) \equiv V[\varphi^A(x), \varphi^B(x)]$ obeying the conditions $V[\varphi^A(0), 0] = V[0, \varphi^B(x)] = 0$, so that the total action integral, the total Lagrangian density and the total EM tensors are $A = A^A_0 + A^B_0 - \int \sqrt{-g(x)} V(x) \, d^4x$, $L(x) = L^A_0(x) + L^B_0(x) - V(x)$, $\Theta_{\mu\nu}(x) = \Theta^A_{0\mu\nu}(x) + \Theta^B_{0\mu\nu}(x) + g_{\mu\nu}(x) V(x)$. Hence we have

$$D^\mu \Theta_{\mu\nu}(x) = D^\mu \Theta^A_{0\mu\nu}(x) + D^\mu \Theta^B_{0\mu\nu}(x) + \partial_\nu V(x) = 0,$$

but $D^\mu \Theta^A_{0\mu\nu}(x) \neq 0$ and $D^\mu \Theta^B_{0\mu\nu}(x) \neq 0$. We want to prove the equations

$$D^\mu \Theta^A_{0\mu\nu}(x) = \frac{\partial V(x)}{\partial \varphi^B(x)} \partial_\nu \varphi^B(x) = 0; \quad D^\mu \Theta^B_{0\mu\nu}(x) = \frac{\partial V(x)}{\partial \varphi^A(x)} \partial_\nu \varphi^A(x) = 0; \quad (45)$$

which means that $\varphi^B(x)$ and $\varphi^A(x)$ behave respectively as external fields for $A^A_0$ and $A^B_0$. Let us note that the action integrals

$$A^A = A^A_0 - \int \sqrt{-g(x)} V(x) \, d^4x; \quad A^B = A^B_0 - \int \sqrt{-g(x)} V(x) \, d^4x$$

can be regarded as complete actions integrals for $\varphi^A(x)$ and $\varphi^B(x)$, with $\varphi^B(x)$ and $\varphi^A(x)$ as external fields, respectively. Then, by applying the results of the previous Section, we have for the respective EM tensors the equations

$$\Theta^A_{\mu\nu}(x) = \Theta^A_{0\mu\nu}(x) + g_{\mu\nu}(x) V(x); \quad D^\mu \Theta^A_{\mu\nu}(x) = -\frac{\partial V(x)}{\partial \varphi^A(x)} \partial_\nu \varphi^A(x);$$

$$\Theta^B_{\mu\nu}(x) = \Theta^B_{0\mu\nu}(x) + g_{\mu\nu}(x) V(x); \quad D^\mu \Theta^B_{\mu\nu}(x) = -\frac{\partial V(x)}{\partial \varphi^B(x)} \partial_\nu \varphi^B(x).$$

Since

$$\partial_\nu V(x) = \frac{\partial V(x)}{\partial \varphi^A(x)} \partial_\nu \varphi^A(x) + \frac{\partial V(x)}{\partial \varphi^B(x)} \partial_\nu \varphi^B(x),$$

Eqs. (45) immediately follow.
6 Geometry–to–matter energy–transfer needs a Higgs field

Let $\Theta_{\mu \nu}^{(\sigma)}(x)$ and $\Theta_{\mu \nu}^{(\phi, \Psi)}(x)$ be the EM tensors of the action integrals $A^{(\sigma)}$ and $A^{(\phi, \Psi)}$, respectively described by Eq.(25) and Eq.(29) of the previous subsection. Then, in the absence of interaction between $\sigma(x)$ and $\phi(x)$, the following equations hold

$$D^\mu \Theta_{\mu \nu}^{(\sigma)}(x) = 0; \quad D^\mu \Theta_{\mu \nu}^{(\phi, \Psi)}(x) = 0.$$  \hspace{1cm} (46)

Now, let $\Theta_{\mu \nu}(x)$ be the EM tensor of a total action integral $A$ that includes the interaction term $V(x) = -\frac{1}{2} \sigma^2(x) c^{ij} \varphi_i^*(x) \varphi_j(x)$. As explained in the previous subsection, $A$ can be partitioned either as described by Eq.(31) or as $A = A^{(\sigma, \phi)} + A^{(\phi, \Psi)}$, where $A^{(\sigma, \phi)}$ is the complete action–integral of geometry described by Eq.(27). Correspondingly, we obtain the following conformal–invariant EM tensors

$$\Theta_{\mu \nu}(x) = \Theta_{\mu \nu}^{(\sigma)}(x) + \Theta_{\mu \nu}^{(\phi, \Psi)}(x) + \frac{1}{2} g_{\mu \nu}(x) \sigma^2(x) c^{ij} \varphi_i^*(x) \varphi_j(x);$$  \hspace{1cm} (47)

$$\Theta_{\mu \nu}(x) = \Theta_{\mu \nu}^{(\sigma, \phi)}(x) + \Theta_{\mu \nu}^{(\phi, \Psi)}(x);$$  \hspace{1cm} (48)

where $\Theta_{\mu \nu}^{(\sigma, \phi)}(x)$ is the EM tensor of $A^{(\sigma, \phi)}$. In consideration of the arguments discussed in [3] and exemplified by Eqs.(45), we find that Eqs.(46) are replaced by

$$D^\mu \Theta_{\mu \nu}^{(\phi, \Psi)}(x) = c^{ij} \varphi_i^*(x) \varphi_j(x) \sigma(x) \partial_\nu \sigma(x) = -D^\mu \Theta_{\mu \nu}^{(\sigma, \phi)}(x),$$  \hspace{1cm} (49)

$$D^\mu \Theta_{\mu \nu}^{(\sigma)}(x) = \frac{1}{2} \sigma^2(x) c^{ij} [\varphi_i^*(x) \partial_\nu \varphi_j(x) + \varphi_j^*(x) \partial_\nu \varphi_i(x)] = 0,$$  \hspace{1cm} (50)

clearly consistent with Eq.(47) and Eq.(48). In particular, Eqs.(49) describe the rate of EM transfer from geometry to matter and vice versa. They indicate that a positive energy transfer from geometry to matter occurs during the scale expansion of universe geometry, i.e., for $\partial_\nu \sigma(x) > 0$, provided that the eigenvalues of $[c^{ij}]$ are positive.

In order for CGR to approach GR in the post–inflation era, and therefore Eqs. [49] [50] approach Eqs. [44], both $\sigma(x)$ and $\varphi_i(x)$ must converge to constants at the end of inflation epoch.

To know if or to what extent this is possible, let us restrict our attention to the action integral in which only the interaction between $\sigma(x)$ and $\varphi_i$ and possible complex conjugates is represented, namely

$$A = \int \sqrt{-g} \left[ -g^{\mu \nu} (\partial_\mu \sigma)(\partial_\nu \sigma) - \frac{R}{6} \sigma^2 - \frac{\lambda}{2} \sigma^4 + \sigma^2 c^{ij} \varphi_i^* \varphi_j + g^{\mu \nu} \delta^{ij} (\partial_\mu \varphi_i^*) (\partial_\nu \varphi_j) + \frac{R}{6} \delta^{ij} \varphi_i^* \varphi_j - Q(\varphi, \varphi^*) \right] d^4 x,$$
where $\delta^{ij}$ is the Kronecker delta and $Q(\varphi, \varphi^*)$ is a real polynomial of fourth degree in $\varphi_i$ and $\varphi_i^*$, which represents self–interactions and possible mutual interactions of these fields.

Since many different choices are a priori possible for $c^{ij}$ and $Q(\varphi, \varphi^*)$, in the absence of any sufficient reason to favor a particular choice, we invoke the heuristic principle of maximum symmetry of fundamental laws by assuming that all of $\varphi_i$ are complex, $c^{ij} = c \delta^{ij}$ and $Q(\varphi, \varphi^*) = \frac{1}{2} \lambda |\varphi|^4$, where $|\varphi|^2 = \delta^{ij} \varphi_i^* \varphi_j = \sum_i |\varphi_i|^2$, with $c$ and $\lambda$ suitable real constants. This paves the way for a possible role of $\varphi_i$ in providing mass terms for gauge vector fields. Hence we have

$$A = \int \frac{\sqrt{g}}{2} \left[ \sum_i g^{\mu\nu} (\partial_\mu \varphi_i^*)(\partial_\nu \varphi_i) - \frac{\lambda}{2} |\varphi|^4 + c \sigma^2 |\varphi|^2 - \frac{\lambda}{2} \sigma^4 + \frac{R}{6} (|\varphi|^2 - \sigma^2) - g^{\mu\nu} (\partial_\mu \sigma)(\partial_\nu \sigma) \right] d^4x,$$

(51)

from which we derive the motion equations

$$D^2 \sigma - \frac{R}{6} \sigma - \frac{\lambda}{2} \sigma^3 + c \sigma |\varphi|^2 = 0; \quad D^2 \varphi_i - \frac{R}{6} \varphi_i + \lambda |\varphi|^2 \varphi_i - c \sigma^2 \varphi_i = 0 \text{ and c.c.}$$

Clearly, convergence to constant values of $\sigma$ and $\varphi_i$ for arbitrary initial conditions is possible provided that $R$ is constant and $c |\varphi|^2 - \frac{\lambda}{2} \sigma^2 = R/6$, $\lambda |\varphi|^2 - c \sigma^2 = R/6$.

Hence, either $R = 0$ and $\lambda = c^2$, or $R \neq 0$ and $c = \lambda = \frac{\lambda}{\sqrt{2}}$. Correspondingly, the potential–energy density of $\sigma–\varphi$ interaction takes the form

$$U(\sigma, \varphi) = \frac{\lambda}{4} (|\varphi|^2 - \frac{c}{\lambda} \sigma^2)^2 \quad \text{or} \quad U(\sigma, \varphi) = \frac{\lambda}{4} (|\varphi|^2 - \sigma^2)^2,$$

both of which, as functions of $|\varphi|$, exhibit a Mexican–hat profile of depth $\sqrt{c/\lambda} \sigma(x)$.

Since $A$ is invariant under the global–gauge group $U(N)$, by a suitable unitary transformation, we can bring any multiplet $\varphi$ to the standard form $\varphi_N(x) = \{0, \ldots, 0, \varphi(x)\}$, where $\varphi(x) = |\varphi(x)|$. Thus, Eq. (51) can be simply written as

$$A = \int \frac{\sqrt{g}}{2} \left[ g^{\mu\nu} (\partial_\mu \varphi_i)(\partial_\nu \varphi) - g^{\mu\nu} (\partial_\mu \sigma)(\partial_\nu \sigma) - \frac{\lambda}{2} \left( |\varphi|^2 - \frac{c}{\lambda} \sigma^2 \right)^2 - \frac{R}{6} (\sigma^2 - \frac{\lambda}{\sqrt{2}} \sigma^2) \right] d^4x.$$

We see that, with this global–gauge choice, $\varphi(x)$ behaves as a Higgs field of mass proportional to $\sigma(x)$, while the other $2N - 1$ corollary Nambu–Goldstone bosons of $\varphi(x)$ are transferred, once for ever, to the gauge vector sector via the couplings $A^k_\mu(x)$ $\tau_k \varphi_N(x)$, where $A^k_\mu(x)$ are $2N - 1$ gauge vector–fields and $\tau_k$ are the Lie algebra generators of $SU(N)$, where they become the promoters of other spontaneous symmetry–breakings [29].
Unfortunately, neither constant $R \neq 0$ nor $R = 0$ for ever are physically acceptable, since otherwise gravitational forces would be absent. Nor the first choice can be accepted for any $R$ depending on $x$ since otherwise $R$ would disappear from the motion equations as $\varphi - \sigma$ converges to zero, which is presumed to occur at the end of the inflation epoch. The second choice is conceivable for the time interval in which the matter field remains homogenous and isotropic, what is presumed to be the case during the inflation epoch. However, on the end of this epoch, these conditions cease because of the aggregating action of gravitational forces. If $R(x)$ is zero on average, small deviations from constant values of $\varphi$ and $\sigma$ are expected and are likely to be observable in the post-inflation era.

Passing from the Riemann to the Cartan manifold, the explicit interaction between geometry and matter disappears and we obtain

$$\tilde{A} = \int \frac{\sqrt{-\tilde{g}}}{2} \left[ \tilde{g}^{\mu\nu} (\partial_\mu \tilde{\varphi})(\partial_\nu \tilde{\varphi}) - \frac{\lambda}{2} \left( \tilde{\varphi}^2 - \frac{\mu_H^2}{2\lambda} \right)^2 - \frac{1}{6} \tilde{R} \left( \sigma_0^2 - \tilde{\varphi}^2 \right) \right] d^4x. \quad (52)$$

where we have put $c = \mu_H^2/2\sigma_0^2 \equiv \mu_H^2/12M^2_{rP}$, the terms depending on $\partial_\mu \sigma$ being absorbed into $\tilde{R}$. Here, $\mu_H = 126.5$ GeV is the mass of the physical Higgs field $\varphi_H(x) = \varphi(x) - \mu_H/\sqrt{2\lambda}$ and $M_{rP} = 2.4 \times 10^{18}$ GeV the reduced Planck mass. Since $\tilde{\varphi}$ ranges from zero to $\tilde{\varphi}_{\text{max}} \simeq \mu_H/\sqrt{2\lambda}$, we see that $\tilde{\varphi}^2(x)$ is always negligible compared to $M^2_{rP}$. Thus, the gravitational Lagrangian density is formally represented by $M^2_{rP} \tilde{R} \equiv \tilde{R}/\kappa$, as in GR.

An important point regarding the structure of Eq.(52) is that, to the limit $e^{\alpha(x)} \to 1$, $\tilde{\varphi}(x)$ evolves towards $\mu_H/\sqrt{2\lambda}$. Since in these conditions the interaction–potential term vanishes, the positive contribution to the cosmological constant coming from this term also vanish. However, in order that the cosmological constant be zero, the conformal Ricci tensor $\tilde{R}(x)$ itself should evolve towards zero, what is not true if the contribution to the cosmological constant due to the unimodularity condition, stated in §2.5, is not zero.

**In conclusion**

Energy transfer from geometry to matter can only be explained in the general framework of CGR provided under the following conditions: (1) the matter Lagrangian density on the Riemann manifold includes one or more (massless) scalar fields quadratically coupled with the dilation field; (2) the corresponding Lagrangian density on the Cartan manifold is formally equal to that of a Higgs boson field provided with a number of corollary Goldstone bosons. This makes CGR potentially capable of smoothly joining the Standard Model.
7 The Mach–Einstein–Gürsey principle

As already explained in §2.2, the fundamental tensor of a 4D Cartan manifold can be written as \( \tilde{g}_{\mu\nu}(x) = e^{2\alpha(x)} g_{\mu\nu}(x) \), with \( \alpha(x) \) chosen in such a way to fulfill the condition \( \sqrt{-g(x)} = 1 \) for the determinant \( g(x) \) of matrix \( [g_{\mu\nu}(x)] \). The physical meaning of this factorization was clarified by Gürsey in 1963.

According to the Mach–Einstein doctrine, here referred to as the Mach Principle, there exists in the universe a basic inertial frame which is globally determined by the distant bodies (it was traditionally called the reference frame of “fixed stars”, but today these should be more properly understood as galaxy clusters); the existence of such a frame being ensured by the observed simplicity of the universe on a sufficiently large scale. Unfortunately, this principle cannot be derived from Einstein’s equations, since in the theoretical framework of GR the inertial frame of a body only depends of the assumed metric tensor, independently of any consideration regarding the presence of other bodies.

To overcome this difficulty, Gürsey approached the problem as described in the next subsection. Unfortunately, this author could not realize that a generalization of GR may descend from the CGR principle in consequence of a spontaneous breakdown of conformal symmetry, since concepts of this sort were in gestation just in those years [30] [31].

7.1 The conformal background of the universe

To implement the Mach–Einstein principle within an extended GR framework, Gürsey proposed to proceed in three steps: (1) to find a way of separating local effects from the general cosmological structure due to the distribution of distant bodies, because all statements related to Mach’s principle involve such separation; (2) the boundary conditions being only meaningful in a definite coordinate system, we must be able to introduce privileged coordinate frames determined by the over–all cosmological structure that has been separated in the first step. These are the inertial frames that, according to Mach, are determined, to within a kinematical group, by the over–all distribution of matter; (3) to preserve the general covariance, we have to show that Machian boundary conditions can also be generalized to an arbitrary coordinate system, that is, to noninertial frames.

To satisfy these requirements, Gürsey hypothesized that the fundamental tensor of
spacetime geometry has a part $c_{\mu\nu}(x) = c(x) \eta_{\mu\nu}(x)$, where $\eta_{\mu\nu}(x)$ is a flat metric tensor, describing a conformally flat geometry, and another part describing the deviations from this uniform structure. The frame of distant bodies may then be defined as one in which $c_{\mu\nu}(x)$ takes a conformal form, so that light in this system travels on a straight line with constant velocity $c(=1)$. The boundary conditions for the metric then require that, with respect to the inertial frames of comoving observers, the fundamental tensor tends asymptotically to a conformal metric characteristic of a uniform cosmological structure. In this way, a transformation which takes the observer from an inertial to a non–inertial frame may be interpreted as a transformation which distorts the uniform and isotropic aspect of the cosmological background in such a way to resemble effects due to accelerations with respect to the “fixed–star” reference frame.

In Gürsey’s view, it is not this relative acceleration which produces the Machian forces, but rather it is the way in which $c(x)$ depends on $x^\mu$ that defines the inertial behavior of the observer with respect to the reference frame of heavy bodies on large scales, making these appear fixed when $c(x)$ appears maximally homogeneous and isotropic on large scales.

The only fundamental tensor consistent with this view is $\tilde{g}_{\mu\nu}(x) \equiv e^{2\alpha(x)} g_{\mu\nu}(x)$. The metric–tensor factor $g_{\mu\nu}(x)$ can then be written as

$$g_{\mu\nu}(x) = \gamma_{\mu\nu}(x) + h_{\mu\nu}(x),$$

with $\det[\gamma_{\mu\nu}(x)] = \det[\gamma_{\mu\nu}(x) + h_{\mu\nu}(x)] = -1$, where $\gamma_{\mu\nu}(x)$ describes the metric tensor of a flat spacetime and $h_{\mu\nu}(x)$ the gravitational field represented as a deviation from the flat geometry. Thus the information on the over–all cosmological structure is contained in $c_{\mu\nu}(x) = e^{2\alpha(x)} \gamma_{\mu\nu}(x)$ while that on the gravitational field is contained in $h_{\mu\nu}(x)$. For our needs we can assume $h_{\mu\nu}(x)$ to be very small, so that the gravitational field is represented as a small linear perturbation of the flat metric tensor, in which case the unimodularity condition stated in §2.2 implies $\gamma^{\mu\nu} h_{\mu\nu}(x) = 0$. This means, in practice, that non–linear gravitational effects are ignored and possible black holes are replaced by extended bodies of large mass.

To fulfill the Mach–Einstein principle, Gürsey assumed that $\tilde{g}_{\mu\nu}(x)$ obeys the following boundary condition at spatial infinity in the inertial frame

$$\tilde{g}_{\mu\nu}(x) \to e^{2\alpha(\tau)} \gamma_{\mu\nu}(x),$$

(53)

where $\tau = \sqrt{\eta_{\mu\nu} x^\mu x^\nu}$ is the kinematic–time coordinate of the world line of a de Sitter
spacetime and
\[ e^{\alpha(\tau)} = e^{\alpha(0)} \frac{1}{1 - K^2 \tau^2}, \quad (54) \]
where \( K \) is real for positive and imaginary for negative spatial curvature. This clearly implies that spacetime is confined to a future cone of origin \( \tau = 0 \) and conformally flat on the large scale, in contrast with the observed acceleration of universe expansion, and that its expansion rate is singular at some critical time \( \tau_c = 1/K \), which appears absurd.

To cure these inconveniences we propose a view that departs from Gürsey’s in four main aspects, hereafter briefly summarized but extensively discussed in the following:

1. At variance with Gürsey, we do not assume that the scale factor of the universe on the large scale has the form \((54)\). Instead, we propose that it is determined by the motion equation of the dilation field \( \sigma(x) = \sigma_0 e^{\alpha(x)} \) interacting with other fields, as described in §6, so that \( \alpha(x) \) be subjected to the boundary conditions
\[ \alpha(0) \ll 0, \quad \lim_{\tau \to \infty} \alpha(x) = 0, \quad (55) \]
(which are not granted a priori) and CGR converge to GR for \( \tau \to \infty \).

2. As proven in §2.2 of Parts II and in Part III, scale factor \( e^{\alpha(x)} \) is of de Sitter type only during a short kinematic–time interval \( \tau_c \), precisely from the instant of the spontaneous breakdown of conformal symmetry until the start of matter–generation process, after which it tends to increase with a decelerated time–course and become constant for large \( \tau \). Thus, in agreement with boundary conditions \((55)\), \( e^{\alpha(x)} \) is assumed to be initially very small, then increasing and converging to 1 as \( \tau \to \infty \).

3. In §7.4 it will be proven that the source of the dilation field during the acute stage of inflation is proportional to the trace of the matter EM tensor up to a constant factor. This relation, which only holds under the assumption of a spontaneous breakdown of conformal symmetry, was also inferred by Gürsey basing on heuristic arguments. As we shall see, it plays an important role in explaining the uniformization of matter density during inflation.

4. We do not exclude that the action integral may contain a small cosmological–constant term, which is equivalent to saying that the vacuum has a small energy density \( \rho_{\text{vac}} \).
7.2 General form of conformal geodesic equations

On the Cartan manifold, the motion of a point–like test particle under the action of the conformal gravitational field is governed by the conformal geodesic equation

\[ \frac{d^2x^\lambda}{ds^2} + \tilde{\Gamma}^\lambda_{\mu\nu} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0, \quad \text{or} \quad \frac{dx^\rho}{ds} \left( \tilde{D}_\rho \frac{dx^\lambda}{ds} \right) = 0 \quad \text{(self–parallelism condition)}, \]

where \( d\tilde{s} = \sqrt{\tilde{g}_{\mu\nu} dx^\mu dx^\nu} = (\sigma/\sigma_0) ds = e^\alpha ds \) is the proper–time element of the particle along its geodesic, \( \tilde{D}_\rho \) the covariant derivatives on the Cartan manifold and

\[ \tilde{\Gamma}^\lambda_{\mu\nu} = \delta^\lambda_{\mu} \partial_\nu \alpha + \delta^\lambda_{\nu} \partial_\mu \alpha - g_{\mu\nu} \partial^\lambda \alpha + \Gamma^\lambda_{\mu\nu}, \quad \text{with} \quad \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left( \partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right), \]

are the Christoffel symbols constructed out of \( \tilde{g}_{\mu\nu} \).

The explicit geodesic equation on the Riemann manifold then writes

\[ e^{-\alpha} \frac{d}{ds} \left( e^{-\alpha} \frac{dx^\lambda}{ds} \right) + 2 \partial_\mu \alpha \frac{dx^\mu}{ds} \frac{dx^\lambda}{ds} - \partial^\lambda \alpha + g^{\rho\lambda} \left( \partial_\mu g_{\rho\nu} - \frac{1}{2} \partial_\nu g_{\rho\mu} \right) \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0. \]

(57)

From this we obtain the contravariant 4D–acceleration of the test particle

\[ a^\lambda = \frac{d^2x^\lambda}{ds^2} = \partial^\lambda \alpha - \frac{d\alpha}{ds} \frac{d^2x^\lambda}{ds^2} - \frac{1}{2} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} \left( \partial_\mu g_{\rho\nu} - \frac{1}{2} \partial_\nu g_{\rho\mu} \right) u^\mu u^\nu. \]

(58)

where \( u^\mu = dx^\mu / ds \) is the contravariant 4D–velocity vector.

The first two terms on the right–hand side represent the contribution to acceleration due to the dilation field whereas the last term represents the contribution due to the gravitational field. It is then clear that the dilation field also exerts an inertial force, which disappears as \( \alpha \) approaches zero at the end of the inflationary epoch. Thus, in the post–inflation era the 4D acceleration of the test particle depends only on the gravitational field, as described by GR.

7.3 Synchronized comoving observers

These results give us the opportunity to introduce a precise definition of “comoving observers”. But in advance we need to introduce the \textit{hyperbolic coordinates} in flat spacetime \( \{ \tau, \hat{x} \} \), where \( \tau \) is the kinematic time and \( \hat{x} = \{ \rho, \vartheta, \phi \} \) the hyperbolic and Euler angles, which are related to Lorentzian coordinates \( x^\mu = \{ x^0, x^1, x^2, x^3 \} \) by \( x^0 = \tau \cosh \rho, \ x^1 = \tau \sinh \rho \sin \vartheta \cos \phi, \ x^2 = \tau \sinh \rho \sin \vartheta \sin \phi, \ x^3 = \tau \sinh \rho \cos \vartheta. \)
From these, the equations
\[
\tau = \sqrt{(x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2};
\]
\[
r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} = \tau \sinh \rho;
\]
are easily derived.

Hence, \(\tau = 0\) identifies the light cone of vertex \(V = \{0, 0, 0, 0\}\), \(\tau = \text{const.} > 0\) identifies a light–cone hyperboloid and \(r\) the radial distance of a point of coordinates \(\{\tau, \hat{x}\}\) from the light–cone axis.

The reader can verify that the hyperbolic–coordinate expression of the squared line element \(ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2\) takes the form
\[
ds^2 = d\tau^2 - \tau^2 [d\rho^2 + (\sinh \rho)^2 d\vartheta^2 + (\sinh \rho \sin \vartheta)^2 d\varphi^2].
\]

Now, let us define the hyperbolic coordinates for a curved spacetime. As well–known in tensor analysis, a complete set of geodesics stemming from a point \(V\) on a Riemannian manifold can be used to define a system of polar geodesic coordinates \([32]\), which are the analog of polar coordinates in Euclidean space (Fig.1).

These polar geodesics can be parameterized by hyperbolic coordinates as follows. Take as kinematic time of an event \(O\) the length \(\tau\) of the geodesic interval \(VO\), and as space–like parameters of \(O\) the hyperbolic and Euler “angles” \(\hat{x} = \{\rho, \vartheta, \varphi\}\) of the same geodesic measured in an infinitesimal neighborhood of \(V\).

Since for each polar geodesic we have \(d\tau/ds = 1\) and \(d\hat{x}^i/ds = 0\), we also have \(ds = d\tau\), \(d\hat{x}^i = 0\), hence \(\hat{x}^i = \text{constant}\), which allows us to label polar geodesics in a curved spacetime as \(\Gamma(\hat{x})\).
We can then cast the square of the line element in the form

\[ ds^2(x) = d\tau^2 - \tau^2 a_{ij}(\tau, \hat{x}) d\hat{x}^i d\hat{x}^j, \]

with initial conditions

\[ \lim_{\tau \to 0} a_{11} = 1; \quad \lim_{\tau \to 0} a_{22} = (\sinh \rho)^2; \quad \lim_{\tau \to 0} a_{33} = (\sinh \rho \sin \vartheta)^2; \quad \lim_{\tau \to 0} a_{ij} = 0 \ (i \neq j). \]

If the metric is not so curved as to require multi-chart representation, the information about the gravitational field is completely incorporated in functions \( a_{ij}(x) \).

The arc of geodesic \( OO' \) joining two infinitely close points \( O, O' \) of the same \( \tau \) is

\[ OO' = \tau \sqrt{a_{ij} d\hat{x}^i d\hat{x}^j} \]

and, consequently, we have

\[ \frac{ds}{d\tau} = \sqrt{1 - \tau^2 a_{ij}(\tau, \hat{x}) \left( \frac{d\hat{x}^i}{d\tau} \right) \left( \frac{d\hat{x}^j}{d\tau} \right)}. \]

The set of all points of the same time-like coordinate \( \tau \), for all polar geodesics in the spacetime region defined by a future cone, forms a 3D subspace \( \Sigma(\tau) \). A point \( O \), running along one of these geodesics, is intended to represent an observer on the Riemann manifold whose clock signs \( \tau \).

Since the observers of a universe expanding in a future cone of apical point \( V \) are called “comoving” provided that they move along their own polar geodesics stemming from \( V \), we can say that \( \Sigma(\tau) \) represents the set of synchronized comoving observers at kinematic time \( \tau \).

An important point regarding this spacetime partition is that the 3D volumes of \( \Sigma(\tau) \) surfaces are infinite, in contrast with those of the 3D spaces orthogonal to the inertial timeline in the standard representation of GR spacetime. This enables us to define the \textit{thermodynamic limit} of the universe in the 3D space of comoving observers, which is essential for describing the macroscopic evolution of the universe as a thermodynamic process.

If we assume that that the gravitational forces are negligible and matter distribution is homogeneous and isotropic all over each \( \Sigma(\tau) \), it is natural to assume that the expansion factor \( e^{\alpha(x)} \) also is the same all over \( \Sigma(\tau) \), i.e., \( \alpha(x) \equiv \alpha(\tau) \).
Since in this case the Riemann manifold is conformally flat, we can parameterize
the future cone by Lorentzian coordinates \( \{ x^0, \vec{x} \} \equiv \{ x^0, x^1, x^2, x^3 \} \), with metric \( \eta_{\mu\nu} = \text{diag}(1, -1, -1, -1) \), as shown in Fig. 2.

Figure 2: Lorentzian coordinates spanning a future cone in a conformally flat
spacetime. Synchronized comoving observers lie on the 3D hyperboloid \( \Sigma(\tau) \) at
kinematic–time distance \( x^0 = \tau \) from vertex \( V \). The position of an observer \( O \) on
\( \Sigma(\tau) \) is identified by the radial vector \( \vec{x} \).

We can thus express kinematic time as \( \tau = \sqrt{(x^0)^2 - |\vec{x}|^2} \), and therefore envisage
\( \vec{v}_O = -\vec{x}/x^0 \) as the contravariant velocity vector of comoving observer \( O \) at \( \{ x^0, \vec{x} \} \) and
\( v_O = |\vec{v}_O| \) its norm. The following relationships are then easily proven
\[
\frac{d\tau}{dx^0} = \frac{1}{\sqrt{1 - v_O^2}}; \quad \vec{\nabla}\tau = \frac{\vec{v}_O}{\sqrt{1 - v_O^2}},
\]
where \( \vec{\nabla} = \{ \partial^1, \partial^2, \partial^3 \} \) and \( \partial^i \equiv \eta^{ij} \partial_j = -\partial_i \), \( (i, j = 1, 2, 3) \).

Now, let us consider a test particle \( P \) moving along a geodesic \( \Gamma_P \), generally not
stemming from \( V \), let \( s_P \) be its proper time and \( x^a_P \) its spacetime coordinates. Hence,
\( ds_P \) is related to \( dx^a_P \) by \( ds_P^2 = g_{\mu\nu} dx^\mu_P dx^\nu_P \) and the contravariant components \( u^\mu_P \) of
4D–velocity are
\[
u_P \frac{dx^0_P}{ds_P} = \frac{1}{\sqrt{1 - v_P^2}}; \quad \vec{u}_P = \frac{dx^a_P}{ds_P} = \frac{dx^0_P}{dx^0_P} \frac{dx^0_P}{ds_P} = \frac{\vec{v}_P}{\sqrt{1 - v_P^2}},
\]
where \( \vec{v}_P \) is the 3D-velocity of the particle at \( x \). Since, by hypothesis, \( \alpha \) depends only on
\( \tau \) and \( g_{\mu\nu} = \eta_{\mu\nu} \), Eq. (58) of §7.2 becomes
\[
a^\lambda = \partial^\lambda \alpha - \frac{d\alpha}{ds} u^\lambda = \frac{d\alpha}{d\tau} \left( \partial^\lambda \tau - \frac{d\tau}{ds} u^\lambda \right).
\]

Using Eqs. (59) and
\[
\frac{d\tau}{ds} = \frac{d\tau}{dx^0} \frac{dx^0}{ds} = \frac{\sqrt{1 - v_O^2}}{\sqrt{1 - v_O^2}},
\]

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we obtain the 3D–acceleration
\[ \vec{a}_P = \frac{\vec{v}_O - \vec{v}_P \, d\alpha}{\sqrt{1 - v_O^2} \, d\tau}, \]
showing that, in general, the test particle is subjected to a viscous force proportional to the slope of dilation field profile which makes \( \vec{v}_P \) approach \( \vec{v}_O \). Hence, the dilation field in expansion behaves as a viscous medium which forces the matter to be at rest in the reference frame of comoving synchronized observers, which is precisely the privileged reference frame of fixed stars invoked by the Mach principle.

**In conclusion**

Whenever \( d\alpha/d\tau > 0 \) (in Part III it is proven that this condition is always satisfied) and provided that \( d\alpha/d\tau \) is sufficiently large, all particles are subjected to the viscous drag of the dilation field and consequently forced to be at rest in the reference frame of synchronous comoving observers. In these conditions, the universe is found in a frozen state and all currents vanish. At the end of this epoch, gravitation takes the upper hand on the viscous drag, charged currents start irradiating and the matter field starts collapsing and aggregate into hot macroscopic bodies.

### 7.4 Matter density homogenization during inflation

In the previous subsection, we assumed that during the inflation epoch the scale–expansion factor \( e^{\alpha(x)} \) is a pure function of \( \tau \), the kinematic time of comoving observers on the Riemann manifold, and we have proven that, during the inflation epoch, the viscous drag exerted by the dilation field in expansion forces all particles to stay at rest in the space \( \Sigma(\tau) \) of synchronized comoving reference frames. Unfortunately, this does not suffices to explain why matter should also remain homogeneous in spite of gravitational attraction.

In this section, the phenomenon of matter homogenization is explained as a consequence of the fact that, during the acute stage of inflation, all massive particles become sources of dilation–field potentials which are strongly repulsive. For our purposes, it is sufficient to improve the demonstration provided by Gürsey’s in 1963 that the dilation field \( \sigma_m(x) \) generated by a small body of mass \( m \) at rest acts repulsively on surrounding matter. To simplify the argument, the possible presence of a cosmological constant coming
from the unimodularity condition, as discussed in §2.5 will be ignored.

As seen in §3.4, the action integral of CGR may be written in two equivalent ways:

(1) on a 4D Riemann manifold of metric tensor \( g_{\mu\nu}(x) \), as a conformal invariant functional of the dilation field interacting with the gravitational field, through its couplings with the Ricci scalar tensor \( R(x) \), and a set of massless scalar field multiplet \( \varphi \), as described in §6;

(2) on a 4D Cartan manifold of fundamental tensor \( \tilde{g}_{\mu\nu}(x) \), as a non–conformal invariant functional formally – but not substantially – equal to an Einstein–Hilbert action integral.

In the second case, the action integral can be be written as \( \tilde{A} = \tilde{A}^G + \tilde{A}^M \), where

\[
\tilde{A}^G = \tilde{A}^{(\sigma_0)} + (\tilde{\lambda} \sigma_0^4/4) \int \sqrt{-\tilde{g}(x)} \, d^4x;
\]

\[
\tilde{A}^M = \tilde{A}^{(\varphi, \Psi)} - \int \sqrt{-\tilde{g}} \left[ V(\sigma_0^2, \tilde{\varphi}) - \tilde{\lambda} \sigma_0^4/4 \right] \, d^4x;
\]

\( \tilde{A}^{(\sigma_0)} \) and \( \tilde{A}^{(\varphi, \Psi)} \) being respectively given by Eq.(26) and (30) of §3.4.

This decomposition into two parts of \( \tilde{A} \) differs from the expression introduced by Eq.(32) of §3.4, by the transfer of the cosmological constant term \( - (\tilde{\lambda} \sigma_0^4/4) \int \sqrt{-\tilde{g}(x)} \, d^4x \) from \( \tilde{A}^{(\sigma_0)} \) to \( \tilde{A}^M \) and the inclusion of term \( - \int \sqrt{-\tilde{g}} V(\sigma_0^2, \tilde{\varphi}) \, d^4x \) into \( \tilde{A}^M \). In this new partition, the geometric action \( \tilde{A}^G \) takes the simple form \( -\kappa^{-1} \int \sqrt{-\tilde{g}(x)} \tilde{R}(x) \, d^4x \), while all physical fields, as well as the cosmological constant term interpreted as energy density of the vacuum, remain included in the matter action integral \( \tilde{A}^M \).

The variations of \( \tilde{A}^G \) and \( \tilde{A}^M \) with respect to \( \tilde{g}^{\mu\nu}(x) \) take then the form

\[
\frac{\delta \tilde{A}^G}{\delta \tilde{g}^{\mu\nu}(x)} = -\frac{\sqrt{-\tilde{g}}}{2\kappa} \left[ \tilde{R}_{\mu\nu}(x) - \frac{1}{2} \tilde{g}_{\mu\nu}(x) \tilde{R}(x) \right], \quad \frac{\delta \tilde{A}^M}{\delta \tilde{g}^{\mu\nu}(x)} = \frac{\sqrt{-\tilde{g}}}{2 \tilde{\sigma}_0} \tilde{\Theta}^M_{\mu\nu}(x),
\]

from which we derive the CGR gravitational equations on Cartan manifold

\[
\tilde{R}_{\mu\nu} - \frac{\tilde{g}_{\mu\nu}}{2} \tilde{R} = \kappa \tilde{\Theta}^M_{\mu\nu} \quad \text{and, consequently,} \quad \tilde{R} = -\kappa \tilde{\Theta}^M.
\]

These equations are formally identical to the corresponding Einstein–Hilbert GR equations (7) and (6); remember that our convention for the Ricci–tensor sign is opposite to that preferred by other authors, as explained in the Introduction. The difference, however, is made evident by using Eqs.(A-19) of the Appendix, which translates the same expression to the Riemann manifold in the form

\[
R_{\mu\nu} - \frac{g_{\mu\nu}}{2} R + \frac{1}{\sigma^2} \left[ 4 (\partial_\mu \sigma)(\partial_\nu \sigma) - g_{\mu\nu} g^{\rho\sigma} (\partial_\rho \sigma)(\partial_\sigma \sigma) \right] + \frac{2}{\sigma} (g_{\mu\nu} D^2 - D_\mu D_\nu) \sigma = \frac{6}{\sigma_0} \tilde{\Theta}^M_{\mu\nu} = \frac{6}{\sigma^2} \Theta^M_{\mu\nu}.
\]
where
\[ \Theta^M_{\mu\nu} = e^{-2\alpha} \tilde{\Theta}^M_{\mu\nu} = \frac{\sigma_0^2}{\sigma^2} \tilde{\Theta}^M_{\mu\nu}, \]
since \( \Theta^M_{\mu\nu} \) has dimension \(-2\).

By contraction with \( \tilde{g}^{\mu\nu} = e^{-2\alpha} g^{\mu\nu} \equiv (\sigma_0/\sigma)^2 g^{\mu\nu} \), we obtain
\[ -e^{2\alpha} \tilde{R} = -R + 6 e^{-\alpha} D^2 e^\alpha = \frac{6}{\sigma^2} \Theta^M, \quad \text{or} \quad D^2 \sigma = \frac{\Theta^M}{\sigma} + \frac{R}{6 \sigma}, \quad (62) \]
where \( \Theta^M = e^{-4\alpha} \tilde{\Theta}^M = (\sigma_0/\sigma)^4 \tilde{\Theta}^M \), which can be regarded as the motion equation for the dilation field, with ratio \( \Theta^M/\sigma \) playing the role of dilation–field source. Using this equation, we can rewrite Eq.(61) as
\[ R_{\mu\nu} = \frac{6}{\sigma^2} \left[ \Theta^M_{\mu\nu} - \frac{g_{\mu\nu}}{2} \Theta^M \right] - \frac{1}{\sigma^2} \left[ 4(\partial_\mu \sigma)(\partial_\nu \sigma) - g_{\mu\nu} g^{\rho\sigma}(\partial_\rho \sigma)(\partial_\sigma \sigma) \right] + \frac{1}{\sigma} \left( g_{\mu\nu} D^2 \sigma + 2 D_\mu \partial_\nu \sigma \right). \quad (63) \]

Therefore, Eqs. (61) and (62) range from
\[ \frac{1}{\sigma^2} \left[ 4(\partial_\mu \sigma)(\partial_\nu \sigma) - g_{\mu\nu} g^{\rho\sigma}(\partial_\rho \sigma)(\partial_\sigma \sigma) \right] - \frac{1}{\sigma} \left( g_{\mu\nu} D^2 \sigma + 2 D_\mu \partial_\nu \sigma \right) = \frac{6}{\sigma^2} \left[ \Theta^M_{\mu\nu} - \frac{g_{\mu\nu}}{2} \Theta^M \right], \quad D^2 \sigma = \frac{\Theta^M}{\sigma}, \quad (64) \]
which occurs if the conformal flatness condition \( R = 0 \) holds, to the familiar Einstein equations
\[ R_{\mu\nu} - \frac{g_{\mu\nu}}{2} R = \kappa \Theta^M_{\mu\nu}, \quad R = -\kappa \Theta^M, \]
which occurs when \( e^{\alpha(x)} \equiv \sigma(x)/\sigma_0 \) converges to 1 for \( x^0 \to \infty \).

As regards the first condition, if the curvature of the universe is negligible and the matter field is homogenous, isotropic, the gravitational forces vanish and \( R = 0 \). Normally, in GR, this equilibrium is instable and the matter field tends to collapse in clusters, making any possible initial state of homogeneity and isotropy rapidly disappear and both pressure and temperature increase. But, in CGR, where, as we are going to prove, possible matter field inhomogeneities are sources of dilation field components which act repulsively on the inhomogeneities themselves, thus preserving the condition \( R = 0 \) until the end of the inflation epoch, when the gravitational forces take the upper hand.

What we only need to prove is that this really happens at least during the main course of inflation. Indeed, let us consider a point–like particle of mass \( m \) propagating along a
geodesic \( x^\mu = z^\mu(\tilde{s}) \) on the Cartan manifold, where \( \tilde{s} \) is the proper time of the particle, i.e., a solution of Eq.(56) of §7.2.

Since the Cartan manifold representation preserves the formal properties of GR, let us transfer the expressions of the particle EM tensor and of its trace from standard relativistic mechanics to this representation, i.e., respectively,

\[
\Theta^m_{\mu\nu}(x) = m \delta^3(\vec{x} - \vec{z}(\tilde{s})) \tilde{u}_\mu(\tilde{s}) \tilde{u}_\nu(\tilde{s}) \frac{d\tilde{s}}{dx^0}, \quad \tilde{\Theta}^m(x) = m \delta^3(\vec{x} - \vec{z}(\tilde{s})) \frac{d\tilde{s}}{dx^0},
\]

where \( \delta^3(\vec{x}) = e^{-3\alpha(x)}\delta^3(\vec{x}) \) is the 3D Dirac delta on the space–like sections of the Cartan manifold and \( \tilde{u}_\mu(\tilde{s}) \) is the covariant 4–velocity of the particle at its proper time \( \tilde{s} \) (cf. Gürsey, 1963). In passing from the Cartan to the Riemann manifold, we must perform the following Weyl transformations

\[
\frac{d\tilde{s}}{dx^0} = e^{\alpha(x)} \frac{ds}{dx^0}, \quad \tilde{u}_\mu(\tilde{s}) = e^{-\alpha(x)} u_\mu(s), \quad \delta^3(\vec{x}) \frac{d\tilde{s}}{dx^0} = e^{-2\alpha(x)} \delta^3(\vec{x}),
\]

where \( s \) is the proper time of the particle on the Riemann manifold which corresponds to \( \tilde{s} \). These equations can easily be inferred since the Dirac delta has dimension \(-3\) and \( u^\mu = dx^\mu/ds, u_\mu = g_{\mu\nu}u^\nu \). Moreover, since in this transition any constant \( c_n \) of dimension \( n \) must be replaced by the corresponding conformal–covariant quantity \( c_n e^{m_\alpha} = c_n(\sigma/\sigma_0)^m \), we must replace \( m \) with \( \gamma_m \sigma(x) \), where \( \gamma_m = m/\sigma_0 \) is an adimensional constant. For instance, if \( m = 1 \text{ GeV} \), we have \( \gamma_m = 1.68 \times 10^{-19} \).

Consequently, in passing from the Cartan to the Riemann manifold the correct relations between the corresponding EM tensors are

\[
e^{2\alpha(x)} \Theta^m_{\mu\nu}(x) = \Theta^m_{\mu\nu}(x) = \gamma_m \sigma(x) \delta^3(\vec{x} - \vec{z}(s)) u_\mu(s) u_\nu(s) \frac{ds}{dx^0},
\]

\[
e^{4\alpha(x)} \tilde{\Theta}^m(x) = \Theta^m(x) = \gamma_m \sigma(x) \delta^3(\vec{x} - \vec{z}(s)) \frac{ds}{dx^0}.
\]

With these replacements the action integral of a point–like particle of mass \( m \) on the Riemann manifold along an arbitrary spacetime path \( \Gamma \) is \( A^m = \gamma_m \int_{\Gamma} \sigma(x) ds(x^\mu) \), which is manifestly conformal–invariant since \( \sigma(x)ds(x_\mu) \) has dimension zero.

If the particle is at rest at point \( x = 0 \) of polar geodesic \( \Gamma(0) \), as described in Fig.2 of §7.3 the tensors take the simple forms

\[
\tilde{\Theta}^m(x) = \tilde{\Theta}^m_{00}(x) = m \delta^3(\vec{x}), \quad \tilde{\Theta}^m_{i0}(x) = \tilde{\Theta}^m_{ij}(x) = 0 \ (i \neq j);
\]

\[
\Theta^m(x) = \Theta^m_{00}(x) = \gamma_m \sigma(x) \delta^3(\vec{x}), \quad \Theta^m_{0i}(x) = \Theta^m_{ij}(x) = 0 \ (i \neq j).
\]
Now assume that both the total EM–tensor trace and the Ricci scalar on the Riemann manifold split respectively into background components $\Theta^B(\tau)$ and $R^B(\tau)$, only depending on $\tau$, and local components $\Theta^m(x)$ and $R^m(\bar{x})$, only depending on the radial coordinate $\bar{x}$ of a point–like particle. However, since the background component is homogeneous and uniform in all 3D hyperboloids, in the absence of point–like particles, the fundamental tensor is conformally flat, which implies $R^B(\tau) = 0$ and $R^m(\bar{x})$ to only contribute to the Ricci scalar. In this case Eq.(62) takes the simple form
\[ D^2\sigma^B(\tau) = \frac{\Theta^B(\tau)}{\sigma^B(\tau)}, \]
where $\sigma^B(\tau)$ represents the dilation field of the background.

This equation shows that $\Theta^B(\tau)$ must vanish in the post–inflation era, since otherwise the boundary conditions (55) stated in §7.1 could not be fulfilled. Clearly, this is a very strong constraint of the theory, which indeed, as will be seen in Parts II and III, is fulfilled provided that $\sigma(x)$ interacts with a zero–mass scalar field $\varphi(x)$, as described in §6.

In the presence of the point–like particle, equation (62) takes instead the form
\[ D^2\sigma(x) = \frac{\Theta^B(\tau) + \Theta^m(x)}{\sigma(x)} + \frac{R^m(x)}{6}\sigma(x) = \frac{\Theta^B(\tau)}{\sigma(x)} + \frac{\gamma_m\delta^3(\bar{x})}{6} + \frac{R^m(x)}{6}\sigma(x), \]
where dilation field $\sigma(x) = \sigma^B(\tau) + \sigma^m(x)$ in turn splits.

Now, let us represent the gravitational field $h_{\mu\nu}$ as an infinitesimal perturbation $\delta g_{\mu\nu}$ of $g_{\mu\nu}$ and assume for $g_{\mu\nu}$ the Lorentz metric $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$, then, using Eq.(A-10) of the Appendix, we find $R^m = R^m_{\mu\nu} h^{\mu\nu} + \frac{1}{2}(g_{\mu\nu} D^2 - D_\mu D_\nu) h^{\mu\nu}$, which reduces to
\[ R^m = \frac{1}{2}\left[(\partial^2_0 - \Delta) h^m_{\mu\nu} - \partial_\mu \partial_\nu h^{\mu\nu}\right], \]
where $\Delta$ is the Poisson operator $\partial_1^2 + \partial_2^2 + \partial_3^2$, since $R^m_{\mu\nu}$ is infinitesimal. In standard GR with Lorentzian metric, $h^{\mu\nu}$ is related to the Newtonian potential $-\kappa m/4\pi r$, by equations $h^{00} = -2\kappa m/4\pi r$, $h^{ij} = -2\kappa m x^i x^j/4\pi r^3$, $h^{0i} = 0$, where $i, j$ are spatial indices and $r = |\bar{x}|$, so that the unimodularity condition $\eta_{\mu\nu} h^{\mu\nu} = 0$ stated in §2.2 be satisfied.

Actually, since we are working in the Riemann representation of CGR and $\sigma(\tau) = \sigma^B(\tau)$, $\kappa = 6/\sigma_0^2$ must be replaced by $6/|\sigma^B(\tau)|^2$ and $m = \gamma_m \sigma_0$ by $\gamma_m \sigma^B(\tau)$. So we have
\[ h^{00}(x) = -\frac{12}{\sigma^B(\tau)} \frac{\gamma_m}{4\pi r}, \quad h^{ij}(x) = -\frac{12}{\sigma^B(\tau)} \frac{\gamma_m}{4\pi r^3} x^i x^j, \quad h^{0i}(x) = 0, \quad \eta_{\mu\nu} h^{\mu\nu}(x) = 0. \]
Thus, on account of the unimodularity condition, Eq. (67) simplifies to
\[
R^m = -\frac{1}{2} \partial_\mu \partial_\nu h^{\mu \nu} = \frac{6 \gamma m}{4 \pi \tau} \delta^0_\mu \delta^0_\nu \frac{1}{\sigma_B} - \frac{6 \gamma m}{4 \pi \sigma_B} \partial_\mu \partial_\nu \frac{x^i x^j}{r^3} = \frac{6 \gamma m}{4 \pi \sigma_B} \frac{1}{\sigma_B} \delta^0_\mu \frac{1}{\sigma_B} - \frac{6 \gamma m}{\sigma_B} \delta^3(\vec{r}),
\] (68)
Inserting this into Eq. (66), we see that the \( \delta^3(\vec{r}) \)-terms disappear and we have
\[
D^2 \sigma(x) = \partial_\tau^2 \sigma^B(\tau) - \Delta \sigma^m(x) = \frac{\Theta^B(\tau)}{\sigma^B(\tau)} + \frac{\gamma m}{4 \pi \tau} \left[ \sigma^B(\tau) + \sigma^m(x) \right] \partial^2 \frac{1}{\sigma^B(\tau) + \sigma^m(x)}.
\] Here, we have replaced \( \partial_0 \) with \( \partial_\tau \) because \( x^0 \) coincides with \( \tau \) along the axis \( \vec{r} = 0 \) of the future cone. To the first order in \( \sigma^m(x) \), the part of this equation which depends on \( r \) is
\[
\Delta \sigma^m(\tau, \vec{r}) = -\frac{\gamma m}{4 \pi \tau} \sigma^B(\tau) \left[ 1 + \frac{\sigma^m(\tau, \vec{r})}{\sigma^B(\tau)} \right] \partial^2 \sigma^B(\tau) = -\frac{\gamma m \beta^B(\tau)}{4 \pi r} \left[ 1 + \frac{\sigma^m(\tau, \vec{r})}{\sigma^B(\tau)} \right] \frac{1}{\sigma^B(\tau)},
\] (69)
where \( \beta^B(\tau) = \sigma_B(\tau) \partial^2 \sigma_B(\tau)^{-1} \) and Eq. (65) has been used.

As shown in Part III, \( \beta^B(\tau) \) is always positive and rapidly converging to zero. Then, the solution to Eq. (69) is
\[
\sigma^m(r) = -\frac{\gamma m \beta^B(\tau)}{8 \pi} e^{-r/r_0(\tau)} r,
\] (70)
showing that the range \( r_0(\tau) = 8 \pi \sigma_B(\tau)/\gamma m \beta^B(\tau) \) becomes infinite in the post-inflation era. Any possible additional term only depending on \( \tau \) is absorbed into \( \sigma^B(\tau) \).

The radial force
\[
f^m(\tau, \vec{r}) \equiv -\nabla \sigma^m(r) = \frac{\gamma m \beta^B(\tau)}{8 \pi} e^{-r/r_0(\tau)} \left[ 1 - \frac{r}{r_0(\tau)} \right] \frac{1}{r},
\] (71)
is very strong during the acute stage of inflation since in this stage \( \beta^B(\tau) \) is very large.

### 7.5 Critical times of repulsion–to–attraction transitions during inflation

Consider a chargeless test–particle \( P_T \) traveling with velocity \( \vec{v} \) at variable distance \( \vec{r} \) from the axis \( \vec{r} = 0 \) of the future cone, which is the world line of a chargeless particle \( P_F \) of conformal mass \( m(\tau) = \gamma m \sigma^B(\tau) \). Assume that \( P_T \) undergoes the combined action of the viscous reaction of the background dilation field \( \sigma^B(\tau) \) and the repulsive and gravitational forces generated by \( P_F \). From Eq. (68) of §7.2, we can extract the 3D acceleration
\[
\vec{a} = -\frac{d \sigma^B}{d \tau} \vec{v} + \frac{\gamma m \beta^B(\tau)}{8 \pi} e^{-r/r_0(\tau)} \left[ 1 - \frac{r}{r_0(\tau)} \right] \frac{1}{r} - \frac{6 \gamma m}{4 \pi \sigma^B(\tau) r^3} \vec{r}.
\] (72)
The first term in the right–hand side represents the viscous force described by Eq. (60) of §7.3, the second represents the repulsive force described by Eq. (71) of the previous subsection and the third the conformal covariant expression of the Newtonian gravitational force $\vec{f}_G(\vec{r}) = -\kappa m \vec{r}/4\pi r^3$. Such expression being obtained by replacing $m = \gamma_m \sigma_B(\tau)$ with $\kappa = 6/[\sigma_B(\tau)]^2$ with $6/[\sigma_B(\tau)]^2$, as explained in the previous subsection.

Since the viscous force makes $\vec{v}$ quickly converge to zero, the acceleration reduces to

$$\vec{a} = \left(\frac{\gamma_m \beta B(\tau)}{8\pi} e^{-r/r_0(\tau)} \left[1 - \frac{r}{r_0(\tau)} - \frac{6 \gamma_m}{4\pi \sigma B(\tau) r^2}\right] \vec{r}\right),$$

which exhibits an alarming singularity at $r = 0$. Actually, $P_F$ cannot be a point–like particle, but rather a small body that can be represented as a hard sphere of radius $r_P$, so $P_T$ never reaches the singularity.

Whether the total force is repulsive or attractant, this only depends on the sign of the factor of $r/r_0(\tau)$ in the right–hand side of Eq. (73). Therefore, by solving equation

$$e^{-r_P/r_0(\tau)} \left[r_P^2 - \frac{r_P^3}{\sigma B(\tau) \beta B(\tau)}\right] = \frac{12}{\sigma B(\tau) \beta B(\tau)} \equiv \frac{12}{\partial^2[\sigma B(\tau)]^{-1}}$$

with respect to $\tau$, we obtain the kinematic time $\tau_{r_p}$ at which the sign inversion of acceleration occurs. As is evident, $\tau_{r_p}$ does not depend on the masses of $P_T$ and $P_F$.

Note that, for distances $r > r_P$ and $r/r_0(\tau)$ sufficiently small, repulsion has the upper hand on attraction, thus pushing $P_T$ towards the boundary of the repulsive region.

Now assume that the comoving space is populated by a gas of particles similar to $P_F$. Then, as long as the repulsive regions of the particles cover the entire space without leaving uncovered any interspaces, the gas remains homogeneously dispersed with approximately equal interparticle distances. But, as soon as the mean repulsion range goes down the mean interparticle distance, the gravitational forces enter in action causing the particles enclosed in their repulsive shells to collapse into irregular clusters.

Hence, there is a critical time $\tau_d < \tau_{r_p}$, depending on the density of this sort of coarse–grained matter field, at which the first inversion from repulsion to attraction occurs. As $\tau$ approaches $\tau_{r_p}$, the repulsion regions become smaller and smaller. At the end, when $\tau > \tau_{r_p}$, all particles confined in a such cluster tend to collapse into dense bodies.

So far, we have ignored the behavior of charged particles in the presence of repulsive and gravitational forces. In standard GR, the acceleration of a test–particle $P_T$ of charge $e$
and mass $\bar{m}$, subjected to the action of a charged fixed particle $P_F$ of charge $-e$ and mass $m \gg \bar{m}$ at $\vec{r} = 0$, is, in modulus, $a_e \simeq -e^2/4\pi\bar{m}r^2$. Its conformal covariant expression is therefore $a_e \simeq -e^2/4\pi\bar{m}\sigma^B(\tau)r^2$. Comparing this with the modulus of gravitational acceleration, as given by the last term in the right–hand side of Eq.(72), we see that the ratio is $a_e/a_g \simeq e^2/6\gamma_m\bar{\gamma}_m = e^2/\kappa m\bar{m} = e^2 M_{Pl}^2/m\bar{m}$, larger than $10^{40}$ for a proton–electron pair. Clearly, repulsion is insignificant in comparison to electrical attraction.

However, since electric forces decrease with distance and all particles are slowed down by viscosity, charged particles of both signs remain uniformly dispersed in the medium, thus preventing the mutual annihilation of particle–antiparticle pairs. This does not prevent quark aggregation because the strength of gluonic forces increases with distance.

In summary, we come to the following scenario. During the acute stage of inflation, a huge amount of Higgs bosons is created by energy transfer from geometry to matter. At the end of inflation epoch – when all particles produced by Higgs boson decay as prescribed by the Standard Model, the overwhelming majority of charged particle–anti–particle pairs annihilate each other with a huge production of photons and an unbalanced fraction of fermions only survives, provided that CP violation, under the conditions prescribed by Sakharov in 1967 [33], are fulfilled. After this stage, the high temperature of the plasma formed by charged and uncharged particles and photons is high enough to maintain, for an unspecified period, matter–field homogeneity and isotropy until the gravitational collapse of coarse–grained plasma assemblies starts.

**In conclusion**

*During the acute stage of inflation, all particles or particle aggregates of zero charge, are not only forced to be at rest all over the 3D space of synchronized comoving frames, but also repel each other uniformly and isotropically. This repulsion and the subsequent enhancement of temperature maintain the matter field perfectly homogeneous and isotropic all over the expanding 3D space of comoving frames. In the late inflationary epoch, when the increase in dilation–field amplitude becomes sufficiently small, and the repulsion fades away, all particles become more and more subjected to gravitational attraction and CGR tends to approach GR with increasing precision. This provide a simple explanation of why today the universe appears homogeneous and isotropic on the large scale.*
Appendix: Basic formulas of standard– and conformal–tensor calculus

For the sake of clarity, and for the purpose of indicating a few sign conventions, we list here the basic formulae of tensor calculus involved in this work. For simplicity, and because the subject is rarely used in this paper, we avoid dealing with spinors, vierbeins and their covariant derivatives. Let us start from the familiar standard–tensor calculus of GR:

- Christoffel symbols and their metric–tensor variations:

\[
\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left( \partial_{\mu} g_{\rho\nu} + \partial_{\nu} g_{\rho\mu} - \partial_{\rho} g_{\mu\nu} \right); \quad (A-1)
\]

\[
\delta \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left( D_{\mu} \delta g_{\rho\nu} + D_{\nu} \delta g_{\rho\mu} - D_{\rho} \delta g_{\mu\nu} \right);
\]

where \( D_{\mu} \) are the covariant derivatives constructed out from the \( \Gamma^\lambda_{\mu\nu} \), which are defined so as to satisfy Eqs.(A-5), and \( \delta g_{\mu\nu}(x) \) are small variations of \( g_{\mu\nu}(x) \).

For diagonal metrics \([g_{\mu\nu}] = \text{diag}[h_0, h_1, \ldots, h_{n-1}]\), Eqs.(A-1) simplify to

\[
\Gamma^{\rho}_{\mu\nu} = 0 \quad (\rho, \mu, \nu \neq), \quad \Gamma^{\rho}_{\mu\mu} = -\frac{\partial_{\rho} h_\mu}{2 h_\rho} \quad (\rho \neq \mu), \quad \Gamma^{\rho}_{\rho\nu} = \frac{\partial_{\nu} h_\rho}{2 h_\rho} \quad (\rho \neq \nu), \quad \Gamma^{\rho}_{\rho\rho} = \frac{\partial_{\rho} h_\rho}{2 h_\rho}, \quad (A-2)
\]

where repeated indices are not to be summed.

- Covariant and contravariant derivatives of mixed tensors \( T^{\sigma\ldots}_{\ldots\lambda\ldots} \):

\[
D_{\mu} T^{\sigma\ldots}_{\ldots\lambda\ldots} = \partial_{\mu} T^{\sigma\ldots}_{\ldots\lambda\ldots} + \Gamma^{\sigma}_{\mu\rho} T^{\rho\ldots}_{\ldots\lambda\ldots} + \cdots - \Gamma^{\sigma}_{\rho\lambda} T^{\rho\ldots}_{\ldots\mu\ldots} - \cdots \quad (A-3)
\]

\[
D^{\mu} T^{\sigma\ldots}_{\ldots\lambda\ldots} = \partial^{\mu} T^{\sigma\ldots}_{\ldots\lambda\ldots} + \Gamma^{\mu}_{\sigma\rho} T^{\rho\ldots}_{\ldots\lambda\ldots} + \cdots - \Gamma^{\mu}_{\rho\sigma} T^{\rho\ldots}_{\ldots\lambda\ldots} - \cdots \quad (A-4)
\]

where \( \Gamma^{\mu\sigma}_{\rho} = g^{\mu\nu} \Gamma^{\sigma}_{\nu\rho} \). Since the following identities hold:

\[
D_{\mu} g_{\nu\lambda} \equiv \partial_{\mu} g_{\nu\lambda} - \Gamma^{\rho}_{\mu\nu} g_{\rho\lambda} - \Gamma^{\rho}_{\mu\lambda} g_{\nu\rho} \equiv 0; \quad (A-5)
\]

\[
D_{\mu} g^{\sigma\lambda} \equiv \partial_{\mu} g^{\sigma\lambda} + \Gamma^{\sigma}_{\mu\rho} g^{\rho\lambda} + \Gamma^{\lambda}_{\mu\rho} g^{\rho\sigma} \equiv 0; \quad (A-6)
\]

we have \( D_{\mu} (g_{\nu\lambda} T^{\sigma\ldots}_{\ldots\lambda\ldots} = g_{\nu\lambda} D_{\mu} T^{\sigma\ldots}_{\ldots\lambda\ldots} \) and \( D_{\mu} (g^{\mu\lambda} T^{\sigma\ldots}_{\ldots\lambda\ldots} = g^{\mu\lambda} D_{\mu} T^{\sigma\ldots}_{\ldots\lambda\ldots} \) for any tensor \( T^{\sigma\ldots}_{\ldots\lambda\ldots} \). Since \( D^{\mu} \cdots = g^{\mu\nu} D_{\nu} \cdots = D_{\nu} g^{\mu\nu} \cdots \), the same property holds for contravariant derivatives. In short, both \( g_{\mu\nu} \) and \( g^{\mu\nu} \) and any function of these are “transparent” to covariant derivatives. In particular, \( D_{\mu} (\sqrt{-g} T^{\sigma\ldots}_{\ldots\lambda\ldots} = \sqrt{-g} D_{\mu} T^{\sigma\ldots}_{\ldots\lambda\ldots} \) where \( g \) is the determinant of matrix \([g_{\mu\nu}]\).
- Covariant divergence of a tensor $T^{\mu\nu\cdots\rho}$ and contravariant divergence of a tensor $T^\mu_{\nu\cdots\rho} \equiv g^{\mu\nu}T_{\nu\cdots\rho}$ write as follows

\[
D_\mu T^{\mu\nu\cdots\rho} = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} T^{\mu\nu\cdots\rho} \right) + \Gamma^\nu_{\mu\lambda} T^{\nu\lambda\cdots\rho} + \cdots + \Gamma^\rho_{\nu\lambda} T^{\mu\nu\cdots\lambda};
\]

\[
D^\mu T^{\mu\nu\cdots\rho} = \frac{1}{\sqrt{-g}} \partial^\mu \left( \sqrt{-g} T^{\mu\nu\cdots\rho} \right) - \Gamma^\lambda_{\mu\nu} T^{\mu\lambda\cdots\rho} - \cdots - \Gamma^\mu_{\nu\lambda} T^{\mu\nu\cdots\lambda}.
\]

- Riemann tensor and its metric–tensor variations:

\[
R^\rho_{\mu\nu\sigma} = \partial_\sigma \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\sigma} - \Gamma^\lambda_{\mu\sigma} \Gamma^\rho_{\lambda\nu};
\]

\[
\delta R^\rho_{\mu\nu\sigma} = \frac{1}{2} g^{\rho\lambda} \left( D_\sigma D_\mu \delta g_{\lambda\nu} + D_\sigma D_\nu \delta g_{\lambda\mu} - D_\sigma D_\lambda \delta g_{\mu\nu} + D_\nu D_\lambda \delta g_{\mu\sigma} - D_\mu D_\lambda \delta g_{\nu\sigma} - D_\nu D_\mu \delta g_{\lambda\sigma} \right);
\]

\[
\sum'_\lambda \text{ indicates the sum for } \lambda = 0, 1, \ldots, n-1 \text{ excluding } \lambda = \mu \text{ and } \lambda = \rho \text{ (don't worry about fractional exponents as these disappear after derivations).}
\]

- Covariant derivative commutators on covariant vectors $v_\rho$ and scalars $f$:

\[
(D_\mu D_\nu - D_\nu D_\mu) v_\rho = R^\rho_{\mu\nu\sigma} v_\sigma; \quad (D_\mu D_\nu - D_\nu D_\mu) f = 0. \tag{A-7}
\]

the second of which implies $D^2 D_\rho f = D_\rho D^2 f$.

- The Beltrami–d’Alembert operator on scalars and vectors:

\[
D^2 f \equiv D^\mu D_\mu f = \partial_\mu \partial^\mu f - \Gamma^\mu_{\rho\nu} \partial^\rho f = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \partial^\mu f \right); \tag{A-8}
\]

\[
D^2 v_\rho \equiv D_\mu D^\mu v_\rho = \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \partial^\mu v_\rho \right) - \Gamma^\lambda_{\rho\mu} \partial^\mu v_\lambda; \tag{A-9}
\]

because $\Gamma^\nu_{\nu\mu}(x) = \partial_\mu \ln \sqrt{-g(x)}$ as can be easily proved using Eq.(A-5) or (A-6) and the well–known formula $\partial_\mu \ln g(x) = g^{\alpha\sigma}(x) \partial_\mu g_{\alpha\sigma}(x) = -g_{\rho\sigma}(x) \partial_\mu g^{\rho\sigma}(x)$.
- Ricci tensors and their metric–tensor variations:

\[
R_{\mu\nu} \equiv R^\rho_{\mu\nu\rho} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\lambda_{\mu\rho} \Gamma^\rho_{\lambda\nu} - \Gamma^\lambda_{\mu\nu} \Gamma^\rho_{\lambda\rho} ;
\]

\[
\delta R_{\mu\nu} = \frac{1}{2} \left( D^\rho D_\mu \delta g_{\rho\nu} + D^\rho D_\nu \delta g_{\rho\mu} - D^2 \delta g_{\rho\rho} - D_\mu D_\nu g^{\rho\sigma} \delta g_{\rho\sigma} \right) ,
\]

\[
R \equiv R_{\mu\nu} g^{\mu\nu} , \quad \delta R = R_{\mu\nu} \delta g^{\mu\nu} + \frac{1}{2} (g_{\mu\nu} D^2 - D_\mu D_\nu) \delta g^{\mu\nu} ; \quad (A-10)
\]

The sign convention for the Riemann tensor is that of Eisenhart, but that of the Ricci tensors is opposite to Eisenhart’s, which is \( R_{\mu\nu} \equiv R^\rho_{\mu\nu\rho} = -R^\rho_{\mu\nu\rho} \), and matches Landau–Lifchitz (1970). The last of Eqs. (A-10) yields the useful formula:

\[
\frac{1}{\sqrt{-g}} \delta \frac{\delta}{\delta g^{\mu\nu}} \int \sqrt{-g} f R d^n x = f (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) + (g_{\mu\nu} D^2 - D_\mu D_\nu) f . \quad (A-11)
\]

For diagonal metrics \([g_{\mu\nu}] = \text{diag}[h_0, h_1, \ldots, h_{n-1}]\), we have \([34]\)

\[
R_{\mu\nu} = \frac{1}{8} \sum_{\sigma \neq \mu, \nu} \left[ (\partial_\mu \ln \frac{h_\sigma}{h_\nu} - \partial_\nu \ln \frac{h_\sigma}{h_\mu}) \partial_\rho \ln h_\sigma + (\mu \leftrightarrow \nu) \right] \quad (\mu \neq \nu) ;
\]

\[
R_{\mu\nu} = \frac{1}{4} (\partial_\mu \ln h_\mu - 2 \partial_\mu) \partial_\nu \ln \frac{g}{h_\mu} - \sum_{\sigma \neq \mu} \left[ (\partial_\sigma \ln h_\sigma)^2 + (\partial_\sigma \ln \frac{g}{h_\mu} + 2 \partial_\sigma) \partial_\rho h_\mu \right] ;
\]

where \( g = h_0 h_1 \cdots h_{n-1} \).

- Geometrical meaning of Ricci tensors: To clarify the geometrical meaning of \( R_{\mu\nu}(x) \) and \( R(x) \), let us solve equation

\[
[R_{\mu\nu}(x) - \rho(x) g_{\mu\nu}(x)] \lambda^\mu(x) \equiv R_{\mu\nu}(x) \lambda^\mu(x) - \rho(x) \lambda_\nu(x) = 0 ,
\]

for \( n \) solutions \( \lambda^\mu_k(x) \) respectively associated to eigenvalues \( \rho_k(x) (k = 1, 2, \ldots, n) \) and satisfying the orthonormalization conditions \( \lambda^\mu_k(x) \lambda_{\mu h}(x) = \delta_{k h} \). Then we can write \( R_{\mu\nu}(x) = \sum_k \rho_k(x) \lambda^\mu_k(x) \lambda_{\nu k}(x) \) and interpret \( \rho_k(x) \) as the spacetime curvature at \( x \) along the principal direction \( \lambda^\mu_k(x) \). The interesting formula \( R(x) = \sum_k \rho_k(x) \) then follows. Since it may happen that curvatures at \( x \) conjugate so as to make \( \sum_k \rho_k(x) = 0 \), we see that \( R(x) = 0 \) does not imply \( R_{\mu\nu}(x) = 0 \). But, since in general spacetime curvatures change unpredictably from point to point, it is very likely that this may happen only on zero–measure sets of spacetime points.

The conformal–tensor calculus of CGR is enriched by new properties, which are completely explained by the Weyl transformations of a few basic quantities. In particular, the
following ones are of decisive importance for our investigation. Taking as fundamental-
tensor variation the finite transformation $g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = e^{2\alpha(x)}g_{\mu\nu}(x)$, we obtain the transformations

\begin{equation}
\Gamma^\lambda_{\mu\nu} \to \tilde{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \delta^\lambda_\nu \partial_\mu \alpha + \delta^\lambda_\mu \partial_\nu \alpha - g_{\mu\nu} \partial^\lambda \alpha; \tag{A-12}
\end{equation}

\begin{equation}
R_{\mu\nu} \to \tilde{R}_{\mu\nu} = R_{\mu\nu} - (n - 2) g_{\mu\nu} g^{\sigma\tau} (\partial_\mu \alpha)(\partial_\tau \alpha) + (n - 2)(\partial_\mu \alpha)(\partial_\nu \alpha) - \\
(n - 2) D_\mu \partial_\nu \alpha - g_{\mu\nu} D^2 \alpha; \tag{A-13}
\end{equation}

\begin{equation}
R \to \tilde{R} = e^{-2\alpha} \left[ R - (n - 1)(n - 2) g^{\rho\sigma} (\partial_\rho \alpha)(\partial_\sigma \alpha) - 2(n - 1) D^2 \alpha \right]; \tag{A-14}
\end{equation}

where $\delta^\nu_\mu$ is the Kronecker delta (as in Eisenhart, 1949, p.90, but with opposite sign convention for $R_{\mu\nu}$). So, the conformal counterpart $\tilde{G}_{\mu\nu} \equiv \tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R}$ of Einstein’s tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ in nD is

\begin{equation}
\tilde{G}_{\mu\nu} = G_{\mu\nu} + \frac{(n - 2)(n - 3)}{2} g_{\mu\nu} g^{\rho\tau} (\partial_\rho \alpha)(\partial_\tau \alpha) + (n - 2)(\partial_\mu \alpha)(\partial_\nu \alpha) + \\
(n - 2) g_{\mu\nu} D^2 \alpha - (n - 2) D_\mu \partial_\nu \alpha. \tag{A-15}
\end{equation}

Using the identities

\begin{align*}
D_\mu \partial_\nu \alpha &= e^{-\alpha} D_\mu \partial_\nu e^\alpha - e^{-2\alpha} (\partial_\mu e^\alpha)(\partial_\nu e^\alpha) = e^{-2\alpha} D_\mu (e^\alpha \partial_\nu e^\alpha) - 2 e^{-2\alpha} (\partial_\mu e^\alpha)(\partial_\nu e^\alpha), \\
D^2 \alpha &= e^{-\alpha} D^2 e^\alpha - e^{-2\alpha} g^{\rho\sigma} (\partial_\rho e^\alpha)(\partial_\sigma e^\alpha) = e^{-2\alpha} D^\mu (e^\alpha \partial_\mu e^\alpha) - 2 e^{-2\alpha} g^{\rho\sigma} (\partial_\rho e^\alpha)(\partial_\sigma e^\alpha),
\end{align*}

Eqs. (A-14) and (A-15) can be cast into the forms

\begin{align*}
\tilde{R} &= e^{-2\alpha} R - e^{-4\alpha} \left[ (n - 1)(n - 4) g^{\rho\sigma} (\partial_\rho e^\alpha)(\partial_\sigma e^\alpha) + 2(n - 1) e^\alpha D^2 e^\alpha \right] \equiv \\
&= e^{-2\alpha} R + e^{-4\alpha} \left[ (n - 1)(6 - n) g^{\rho\sigma} (\partial_\rho e^\alpha)(\partial_\sigma e^\alpha) - 2(n - 1) D_\mu (e^\alpha g^{\mu\nu} \partial_\nu) \right]; \tag{A-16}
\end{align*}

\begin{align*}
\tilde{G}_{\mu\nu} &= G_{\mu\nu} + e^{-2\alpha} \left[ \frac{(n - 2)(n - 5)}{2} g_{\mu\nu} g^{\rho\tau} (\partial_\rho e^\alpha)(\partial_\tau e^\alpha) + 2(n - 2)(\partial_\mu e^\alpha)(\partial_\nu e^\alpha) \right] + \\
&= e^{-2\alpha} \left[ \frac{(n - 2)(n - 7)}{2} g_{\mu\nu} g^{\rho\tau} (\partial_\rho e^\alpha)(\partial_\tau e^\alpha) + 3(n - 2)(\partial_\mu e^\alpha)(\partial_\nu e^\alpha) \right] + \\
&= e^{-2\alpha} \left[ \frac{(n - 2)}{2} g_{\mu\nu} D_\rho (g^{\rho\sigma} \partial_\sigma e^\alpha) - D_\mu (e^\alpha \partial_\nu e^\alpha) \right]. \tag{A-17}
\end{align*}

In particular, putting $n = 4$ and $\sigma(x) = \sigma_0 e^{\alpha(x)}$ we obtain

\begin{equation}
\tilde{R} = e^{-2\alpha} R + \frac{6 e^{-4\alpha}}{\sigma^2_0} \left[ g^{\mu\nu} (\partial_\mu \sigma)(\partial_\nu \sigma) - D_\mu (g^{\mu\nu} \sigma \partial_\nu \sigma) \right] \equiv e^{-2\alpha} \left[ R - 6 \sigma^{-1} D^2 \sigma \right]; \tag{A-18}
\end{equation}
Conformal covariant and contravariant derivatives of conformal mixed tensors mimic the standard ones:

\[
\tilde{D}_\mu \tilde{T}^{\sigma...\lambda...} = \tilde{\partial}_\mu \tilde{T}^{\sigma...\lambda...} + \tilde{\Gamma}^{\sigma}_{\mu\rho} \tilde{T}^{\rho...\lambda...} + \cdots - \tilde{\Gamma}^{\rho}_{\mu\lambda} \tilde{T}^{\sigma...\rho...\cdots...} - \cdots
\]

(A-20)

\[
\tilde{D}^{\mu} \tilde{T}^{\sigma...\lambda...} = \tilde{g}^{\mu\nu} \tilde{T}^{\sigma...\nu...\lambda...} + \tilde{\Gamma}^{\sigma}_{\mu\rho} \tilde{T}^{\rho...\nu...\lambda...} + \cdots - \tilde{\Gamma}^{\rho}_{\mu\nu} \tilde{T}^{\sigma...\rho...\nu...\lambda...} - \cdots
\]

(A-21)

with \( \tilde{\partial}_\mu = e^{-\alpha} \partial_\mu \), \( \tilde{g}^{\mu\nu} = e^\alpha \partial^{\mu} \), \( \tilde{\Gamma}^{\sigma}_{\mu\rho} = \tilde{g}^{\mu\nu} \tilde{\Gamma}^{\sigma}_{\nu\rho} \). The vanishing of the fundamental–tensor covariant derivatives \( \tilde{D}_\mu \tilde{g}_{\lambda\lambda} = 0 \), and therefore the “transparency” properties \( \tilde{D}_\mu (\tilde{g}^{\lambda\nu} \tilde{T}^{\cdots\cdots}) = \tilde{g}^{\lambda\nu} \tilde{D}_\mu \tilde{T}^{\cdots\cdots} \), \( \tilde{D}_\mu (\sqrt{-\tilde{g}} \tilde{T}^{\cdots\cdots}) = \sqrt{-\tilde{g}} \tilde{D}_\mu \tilde{T}^{\cdots\cdots} \), still hold. In particular, the conformal covariant divergence of a conformal covariant tensor with two indices can be written as

\[
\tilde{D}^{\mu} \tilde{T}_{\mu\nu} = \frac{1}{\sqrt{-\tilde{g}}} \tilde{\partial}_\mu (\sqrt{-\tilde{g}} \tilde{g}^{\mu\nu} \tilde{T}_{\sigma\nu}) - \tilde{\Gamma}^{\sigma}_{\nu\lambda} \tilde{T}_{\sigma\lambda}.
\]

(A-22)

As already specified, symbols superscripted by a tilde in Eqs. (A-12)–(A-22) describe the structural changes of the basic tensors of absolute differential calculus when passing from GR to CGR.

Another important property of Weyl transformations regards the totally traceless part \( C_{\mu\nu\rho\sigma} \) of Riemann tensor \( R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \cdots \), known as the conformal–curvature tensor of Weyl, which satisfies the equations

\[
C_{\mu\nu\rho\sigma} g^{\mu\nu} = C_{\mu\nu\rho\sigma} g^{\nu\rho} = C_{\mu\nu\rho\sigma} g^{\mu\sigma} = C_{\mu\nu\rho\sigma} g^{\nu\sigma} = C_{\mu\nu\rho\sigma} g^{\rho\sigma} = C_{\mu\nu\rho\sigma} g^{\rho\sigma} = 0.
\]

The property consists precisely of the invariance of mixed tensor \( C_{\mu\nu\rho\sigma} = g^{\mu\lambda} C_{\lambda\nu\rho\sigma} \) under Weyl transformations, which may then be abbreviated to

\[
C_{\mu\nu\rho\sigma}(x) \rightarrow \tilde{C}_{\mu\nu\rho\sigma}(x) = C_{\mu\nu\rho\sigma}(x).
\]

This means that square of the Weyl–tensor \( C^2(x) = C_{\mu\nu\rho\sigma}(x) C^{\mu\nu\rho\sigma}(x) \) undergoes the Weyl transformation

\[
C^2(x) \rightarrow \tilde{C}^2(x) = e^{-4\alpha(x)} C^2(x).
\]
Consequently, the action integral
\[ A_n^C = -\frac{\beta^2}{2} \int \sqrt{-g(x)} C^2(x) d^n x, \] (A-23)
where \( \beta \) is a real constant, is conformal invariant for \( n = 4 \) only.

Note: About the possible presence of term \( A_4^C \) in the Einstein–Hilbert gravitational action–integral, let us point out that any small real value of \( \beta \) suffices to guarantee the renormalizability of quantum gravity, although at the price of introducing gravitational ghosts of mass \( M_G = M_{rP}/\beta \), where \( M_{rP} \) is the reduced Planck mass, which work in practice as Pauli–Villars regulators of graviton propagators.

After all, unitarity restricted to the Hilbert subspace of physical particles is patently violated in the description of the inflation process, if the evolution of the states of geometric degrees of freedom is totally ignored. In describing the inflation process as a transfer of energy from geometry to matter we find it necessary to introduce a ghost scalar field, which raises by itself the problem of unitarity violation.

In the current state of our investigation, we do not know whether, in a complete description of matter–geometry interaction, unitarity is preserved, thus we leave the problem unsolved. In any case, in regard to the presence of the Weyl–tensor term in the Lagrangian density, we will ignore the unitarity problem and, since we limit ourselves to the semiclassical approximation, in most cases, we will assume \( \beta = 0 \).

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