Near-Horizon Conformal Symmetry and Black Hole Entropy

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Abstract

Near an event horizon, the action of general relativity acquires a new asymptotic conformal symmetry. Using two-dimensional dilaton gravity as a test case, I show that this symmetry results in a chiral Virasoro algebra with a calculable classical central charge, and that Cardy’s formula for the density of states reproduces the Bekenstein-Hawking entropy. This result lends support to the notion that the universal nature of black hole entropy is controlled by conformal symmetry near the horizon.

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1. Introduction

Since the seminal work of Bekenstein \[1\] and Hawking \[2\] in the early 1970s, we have understood that black holes are thermodynamic objects, with characteristic temperatures and entropies. The Bekenstein-Hawking entropy depends on both Planck’s constant $\hbar$ and Newton’s gravitational constant $G$, and offers one of the few known “windows” into quantum gravity. In particular, an understanding of the microscopic statistical mechanics of black hole thermodynamics may give us valuable information about the fundamental degrees of freedom of quantized general relativity. Until quite recently, though, standard derivations of the Bekenstein-Hawking entropy involved only macroscopic thermodynamics, and a statistical mechanical description was more a hope than a reality.

In the past few years, this situation has changed dramatically. Today, indeed, we face the opposite problem: we have many candidate descriptions of black hole statistical mechanics, all of which yield the same entropy despite counting very different states. In particular, there are two string theoretical descriptions, one based on counting D-brane states \[3\] and another involving a dual conformal field theory \[4\]; an approach in loop quantum gravity that counts spin network states \[5\]; and a slightly more obscure method \[6\] based on Sakharov’s old idea of induced gravity \[7\]. The problem of “universality” is to explain why these approaches agree, and why they agree with the original semiclassical computations \[2, 8\] that know nothing of the details of quantum gravity.

One possible answer is that black hole thermodynamics may be controlled by a symmetry inherited from the classical theory. This idea has its roots in an observation by Strominger \[9\] and Birmingham et al. \[10\] that black hole entropy in three spacetime dimensions can be obtained from Cardy’s formula \[11\] for the density of states of a two-dimensional conformal field theory at the “boundary” of spacetime. A number of authors have tried to extend such arguments to black holes in arbitrary dimensions \[12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23\], but while these calculations seem to have the right “flavor,” none is yet fully satisfactory \[24, 25, 26, 27, 28\]. In particular, all such proposals so far require awkward boundary conditions at black hole horizons, and most have serious difficulties in two spacetime dimensions, where there does not seem to be enough room at the one-dimensional horizon for the required degrees of freedom.

In this paper, I point out three new ingredients that lead to an improved description of the near-horizon symmetries of a black hole, and show how they may overcome these difficulties. The new ingredients are the following:

1. Conformal symmetry: In the presence of a stationary black hole—or, more generally, a black hole with a momentarily stationary region near its horizon—the Einstein-Hilbert action of general relativity acquires a new conformal symmetry. Indeed, let $\Delta$ be a segment of such a horizon (see figure 1), and let $\mathcal{N}$ be a “momentarily stationary” neighborhood, that is, a neighborhood admitting a Killing vector $\chi^a$ for which $\Delta$ is a Killing horizon. If $f$ is an arbitrary smooth function that vanishes outside $\mathcal{N}$, then under the transformation

$$g_{ab} \rightarrow \nabla_c (f \chi^c) g_{ab} \quad (1.1)$$
the action in $n$ dimensions transforms, up to possible boundary terms, as

$$\delta I = \frac{1}{16\pi G} \int_C \nabla_c (f \chi^c) g^{ab} G_{ab} \epsilon = \frac{1}{16\pi G} \frac{n-2}{2} \int_N f \chi^c \nabla_c R \epsilon = 0, \quad (1.2)$$

where $\epsilon$ is the volume element and the last equality follows from the fact that $\chi^a$ is a Killing vector. The addition of matter to the Lagrangian will not change this result, as long as the matter fields have the same symmetries as the metric in $N$.

For (1.1) to be a genuine symmetry, of course, it must preserve the relevant space of fields; that is, the new metric must also admit a Killing vector in $N$. It is straightforward to check that this will be the case if

$$\left( \chi^a \nabla_a \right)^2 f = 0. \quad (1.3)$$

Below, we shall generalize this argument to the case of an asymptotic symmetry, for which $(\chi^a \nabla_a)^2 f(x)$ approaches zero as $x$ approaches $\Delta$.

The transformation (1.1) is not a symmetry of the full Einstein-Hilbert action, of course, since a generic metric admits no local Killing vector. If one is interested in quantum gravitational questions about black holes, though, one should restrict the action to field configurations in which a black hole is present [29]. For such configurations, a symmetry of the form (1.1) is present at least as an asymptotic symmetry.

2. **Horizon symplectic form:** In the presence of a horizon, the canonical symplectic form of general relativity—that is, roughly, the Poisson brackets—picks up a new contribution from the horizon. This is most easily seen in the covariant canonical formalism [31], in which the symplectic form for a collection of fields $\phi$ is given by an integral

$$\Omega[\phi; \delta_1 \phi, \delta_2 \phi] = \int_C \omega[\phi; \delta_1 \phi, \delta_2 \phi] \quad (1.4)$$

of a closed form $\omega$ over a (partial) Cauchy surface $C$. Consider the two surfaces $C_1$ and $C_2$ of figure 1. The fact that $\omega$ is a closed form ensures that

$$\Omega_{C_1}[\phi; \delta_1 \phi, \delta_2 \phi] = \Omega_{C_2}[\phi; \delta_1 \phi, \delta_2 \phi] + \int_{\Delta \cap C_2} \Delta \cap C_1 \omega[\phi; \delta_1 \phi, \delta_2 \phi], \quad (1.5)$$
where the integral on the right-hand side is over the portion of the horizon joining $C_1$ and $C_2$. For the isolated horizon boundary conditions of Ref. [30], the restriction of $\omega$ to $\Delta$ is exact, and the horizon integral can be absorbed in $\Omega$. In general, though, there is no reason to expect such a simple outcome. Instead, to define a symplectic structure that is independent of the choice of Cauchy surface $C$, one must choose a “reference” cross-section $S$ of the horizon and define

$$
\hat{\Omega}_C[\phi; \delta_1 \phi, \delta_2 \phi] = \int_C \omega[\phi; \delta_1 \phi, \delta_2 \phi] + \int_{\Delta \cap C} \omega[\phi; \delta_1 \phi, \delta_2 \phi] \quad (1.6)
$$

where the second integral is over the portion of the horizon connecting $S$ and $C_1$. This term is already implicit in [30], where the boundary contribution to $\Omega$ is fixed in terms of a reference cross-section that is used to determine the relevant “integration constant.”

The Poisson brackets thus include a contribution from the horizon itself. As we shall see below, for an asymptotic symmetry of the sort we are interested in here, this horizon contribution will dominate.

3. **Asymptotic symmetry:** The horizon of a generic black hole need not have a stationary neighborhood $\mathcal{N}$. The nonexpanding horizon boundary conditions of Ashtekar et al. [30], for example, require a Killing vector only on the horizon itself. What we really need is the notion of an asymptotic symmetry, in which the spacetime is “almost” stationary as one approaches the horizon.

Traditionally, an “asymptotic symmetry” in general relativity has meant an exact symmetry, i.e., a diffeomorphism, that preserves some extra asymptotic structure. Here we have a slightly different situation: a symmetry of the action that may be exact only at the horizon, but that can be made arbitrarily good by shrinking the neighborhood $\mathcal{N}$ in which the parameter $f$ has its support. This circumstance is probably best viewed as an instance of a weakly broken symmetry. In particular, we can find an approximate Killing vector $\chi^a$ near the horizon (e.g., in the manner of [32]) and a metric $\bar{g}$ for which $\chi^a$ is an exact Killing vector, and write $g = \bar{g} + h$, where $h = 0$ at the horizon. The Lagrangian $L[\bar{g} + h]$ is then invariant up to terms of order $h$, and it may be shown that the would-be Noether current for the transformation (1.1) is conserved up to terms of order $h$. While more work is required to fully understand this sort of symmetry, it is reasonably clear that if $h$ is sufficiently smooth, an asymptotic symmetry near the horizon should become an exact symmetry for fields located on the horizon itself.

2. **The two-dimensional black hole**

We can now ask whether the new symmetry (1.1) places any restrictions on black hole thermodynamics. In general, one ought not expect a symmetry to determine anything as “microscopic” as a density of states. There is one important exception, though: for a one- or two-dimensional conformal symmetry described by a Virasoro algebra

$$
[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0} \quad (2.1)
$$
with central charge $c$, the Cardy formula \[11, 33\] tells us that the number of states having eigenvalue $\Delta$ of $L_0$ goes asymptotically as

$$\rho(\Delta) \sim \exp \left\{ 2\pi \sqrt{\frac{c_{\text{eff}} \Delta}{6}} \right\} \quad \text{with } c_{\text{eff}} = c - 24\Delta_0,$$

where $\Delta_0$ is the lowest eigenvalue of $L_0$. The question is thus whether the symmetry \([1.1]\) can be described by such an algebra.

To answer this question, it is useful to focus on a particular example, two-dimensional dilaton gravity. This is not as restrictive as it may seem, since general relativity in any dimension can be dimensionally reduced via a Kaluza-Klein mechanism to two-dimensional gravity coupled to “matter” fields, and I shall argue below that the extra fields do not affect the conclusions. Still, this work should be considered a first step, which can presumably be considerably generalized.

The action for dilaton gravity can be written in the form \[34\]

$$I = \int L = \frac{1}{2G} \int \left( \phi R + \frac{1}{\ell^2} V[\phi] \right) \epsilon,$$  

where $\epsilon$ is the two-dimensional volume form and $V$ is an arbitrary function of the dilaton field $\phi$. (The kinetic term for $\phi$ has been absorbed into $\phi R$ by field redefinition.) Strictly speaking, one cannot define the expansion of a null congruence in two dimensions, but the analog in dilaton gravity is

$$\vartheta = \frac{1}{\phi} \ell^a \nabla_a \phi$$

where $\ell^a$ is the null normal. All known exact black hole solutions, including dimensionally reduced descriptions of higher-dimensional black holes, have null horizons with vanishing $\vartheta$.

As in previous work \[12, 13\], we will start with a “stretched horizon,” in this case a null surface $\tilde{\Delta}$ with null normal $\ell^a$ for which $\vartheta$ is small but nonzero. Near a genuine horizon, we can take $\vartheta$ to be a measure of how far we have “stretched” away; in the end, we will take the limit $\vartheta \to 0$.

In two dimensions, the vector $\ell^a$ determines a unique “orthogonal” null vector $n^a$, such that $\ell^a n_a = -1$. We extend $n^a$ from $\tilde{\Delta}$ by requiring that

$$n^a \nabla_a n_b = 0,$$  

from which it follows that

$$\nabla_a \ell_b = -\kappa n_a \ell_b, \quad \nabla_a n_b = \kappa n_a n_b$$

where $\kappa$ is the “surface gravity.” Note, though, that unlike a timelike or spacelike unit vector, a null normal does not have a fixed normalization: by rescaling $\ell^a \to f \ell^a$, one can change $\kappa$ almost arbitrarily on a fixed null surface $\tilde{\Delta}$ \[30\],

$$\kappa \to \ell^a \nabla_a f + \kappa f, \quad n^a \nabla_a f = 0,$$  

\[2.7\]
where the last condition ensures that (2.3) is preserved. We shall use this freedom below to choose a convenient form for $\kappa$.

Observe from (2.6) that

$$\nabla_a \ell_b + \nabla_b \ell_a = \kappa g_{ab}, \quad (2.8)$$

so $\ell_a$ is a conformal Killing vector. We will see later that the natural scaling of $\ell_a$ leads to a surface gravity $\kappa$ proportional to $\vartheta$, so $\ell_a$ is actually an approximate Killing vector near the horizon.

The application of the transformation (1.1) to two dimensions is a bit tricky, both because a new field $\phi$ is present and because the field equations of dilaton gravity differ from those of ordinary general relativity. In general, we should expect $\phi$ as well as $g_{ab}$ to transform, and it is easy to check that under a transformation

$$\delta g_{ab} = \ell^c \nabla_c (\ell^d f) g_{ab} = (\ell^c \nabla_c f + \kappa f) g_{ab}$$

$$\delta \phi = (\ell^c \nabla_c h + \kappa h), \quad (2.9)$$

the Lagrangian (2.3) satisfies $\delta L \sim \vartheta$. We thus have an asymptotic symmetry in the sense described earlier. In particular, by restricting $f$ and $h$ to have their support in a small region near a horizon, we can make the variation $\delta I$ arbitrarily small. For now, the relationship of $f$ and $h$ will remain unspecified; we shall see later that the choice that makes the transformation (2.9) canonical actually implies that $\delta L \sim \vartheta^2$.

Equation (2.9) is not enough to determine the separate variations of $\ell^a$ and $n^a$. This is to be expected, since the normalization of $\ell^a$ is not fixed; the only restriction, from (2.6), is that $n^a \nabla_a (n_b \delta \ell^b) = 0$. We are thus free to choose $\delta \ell^a = 0$, which then implies that

$$\ell_a \delta n^a = \ell^c \nabla_c f + \kappa f$$

$$\delta \kappa = \ell^b \nabla_b (\ell^c \nabla_c f + \kappa f)$$

$$\delta s = \ell^a \nabla_a (\ell^b \nabla_b h + \kappa h) \quad (2.10)$$

where $s = \ell^a \nabla_a \phi = \vartheta \phi$. It follows that

$$[\delta_1, \delta_2] g_{ab} = (\ell^c \nabla_c \{f_1, f_2\} + \kappa \{f_1, f_2\} ) g_{ab} \quad \text{with} \quad \{f_1, f_2\} = (\ell^a \nabla_a f_1) f_2 - (\ell^a \nabla_a f_2) f_1, \quad (2.11)$$

giving the standard conformal algebra.

To express the transformations (2.9) in Hamiltonian form, we need the symplectic form $\Omega$ of (1.6). This can be computed by Wald’s methods [31, 34]. For variations that have their support only in a small neighborhood $N$ of $\tilde{\Delta}$, the main contribution will come from the integral along $\tilde{\Delta}$. Restricting the symplectic form of Ref. [34] to $\tilde{\Delta}$, one finds that

$$\hat{\Omega} = \frac{1}{2G} \int_{\tilde{\Delta}} (\ell^a \nabla_a (\delta_1 \phi) \ell_b \delta_2 n^b - \ell^a \nabla_a (\delta_2 \phi) \ell_b \delta_1 n^b) \hat{\epsilon}, \quad (2.12)$$

where $\hat{\epsilon} = n$ is the induced volume element on $\tilde{\Delta}$. Since $\tilde{\Delta}$ is null, one can integrate by parts, and use (2.10) to obtain

$$\hat{\Omega}[\delta_1, \delta_2] = -\frac{1}{2G} \int_{\Delta} (\delta_1 \phi \delta_2 \kappa - \delta_2 \phi \delta_1 \kappa) \hat{\epsilon}. \quad (2.13)$$
3. Hamiltonian and Virasoro algebra

The next question is whether the transformation (2.9) is canonical, that is, whether it is generated by a “Hamiltonian” \( L \). Such a Hamiltonian must satisfy

\[
\delta L[f, h] = \hat{\Omega} [\delta, \delta f, h] = -\frac{1}{2G} \int_{\tilde{\Delta}} \left[ \delta \phi \ell^b \nabla_b (\ell^c \nabla_c f + \kappa f) - \delta \kappa (\ell^c \nabla_c h + \kappa h) \right] \hat{\epsilon} \tag{3.1}
\]

where again \( s = \ell^a \nabla_a \phi \) and I have integrated by parts to obtain the last equality. The variation \( \delta \) can be thought of as an exterior derivative on the space of fields, and the integrability condition for (3.1) is that \( \delta^2 L[f, h] = 0 \). If we assume that the parameters \( f \) and \( h \) are field-independent, so \( \delta f = \delta h = 0 \), it is easy to see that this condition requires that \( \delta s \) be proportional to \( \delta \kappa \). In particular, this proportionality must hold for variations of the form (2.9), and this, together with the requirement that \( f \) and \( h \) be field-independent, implies that

\[
\frac{\kappa}{s} = \text{constant on } \tilde{\Delta}, \tag{3.2}
\]

\[
s f = \kappa h. \tag{3.3}
\]

Despite appearances, (3.2) is not a real restriction on the geometry, since it can always be satisfied by rescaling \( \ell^a \) as in (2.7). As noted earlier, this relation makes the transformation (2.9) an even better approximate symmetry. Indeed, it may be checked that now

\[
\delta \int \phi R \hat{\epsilon} = 2 \int \left[ n^a \nabla_a f (\ell^a \nabla_a s - \kappa s) + n^a \nabla_a \left( \frac{s}{\kappa} f \right) \ell^a \nabla_a \kappa \right] \hat{\epsilon}, \tag{3.4}
\]

and the integrand goes as \( \vartheta^2 \) near \( \Delta \). While no corresponding suppression appears automatically in the potential term in (2.3), the variation of that term can easily be arranged to be of order \( \vartheta^2 \) by an appropriate choice of \( \kappa/s \) on \( \tilde{\Delta} \).

With the relation (3.3) between \( h \) and \( f \), (3.1) can be easily integrated, yielding

\[
L[f] = \frac{1}{2G} \int_{\tilde{\Delta}} s (2 \ell^a \nabla_a f + \kappa f) \hat{\epsilon} = -\frac{1}{2G} \int_{\tilde{\Delta}} (2 \ell^a \nabla_a s - \kappa s) f \hat{\epsilon}. \tag{3.5}
\]

We must next choose a basis for the functions \( f \) on \( \tilde{\Delta} \). Since the normalization of \( \ell^a \) is not fixed—even (3.2) determines it only up to a constant—the corresponding light cone coordinate has no intrinsic physical meaning. There is, however, a natural coordinate on \( \tilde{\Delta} \), the dilaton \( \phi \) itself, which by the two-dimensional version of the Raychaudhuri equation should be monotonic on \( \tilde{\Delta} \). Let

\[
z = e^{2 \pi i \phi/\phi_+}, \tag{3.6}
\]

where \( \phi_+ \) is the value of \( \phi \) on the horizon, so \( z \to 1 \) at \( \Delta \). We can then choose a basis of functions to be proportional to \( z^n \), with the proportionality constants determined from (2.11) and the requirement that the \( f_n \) satisfy the standard \( \text{Diff}(S^1) \) commutation relations:

\[
f_n = \frac{\phi_+}{2\pi s} z^n, \quad \{f_m, f_n\} = i(m - n) f_{m+n}. \tag{3.7}
\]

*This choice is almost unique, in that \( \phi_+ \) is the only natural quantity in the theory having the right dimension. In principle, though, we could have chosen \( w = z^n \) to define our modes. This would leave the central charge (3.12) unchanged, but would shift the Hamiltonian (3.10).*
Note that with this choice, the consistency condition (1.3) is satisfied asymptotically:

$$(\ell^a \nabla_a)^2 f_n = -\frac{2\pi n^2}{\varphi_+} sz^n \to 0 \text{ as } \tilde{\Delta} \to \Delta.$$  \hfill (3.8)

In terms of these modes, the Hamiltonian (3.5) becomes

$$L[f_n] = -\frac{1}{2G} \kappa \int_\Delta \left( 1 + 2 \frac{\ell^a \nabla_a \nabla_b \phi}{\kappa} \right) s f_n \phi_+ \frac{dz}{2\pi i z^3}.$$  \hfill (3.9)

On shell, though, \(\nabla_a \nabla_b \phi \propto g_{ab}\) \cite{34}, so the last term in (3.9) vanishes, giving

$$L[f_n] = -\frac{1}{2G} \kappa \phi_+^2 \delta_{n0}.$$  \hfill (3.10)

It remains for us to compute the Poisson brackets \(\{L[f_m], L[f_n]\}\). This is most easily done directly from equation (3.1), and a straightforward computation yields

$$\{L[f_m], L[f_n]\} = \delta_{f_m} L[f_n] = -\frac{2\pi i s}{G \kappa} n^3 \delta_{m+n,0}.$$  \hfill (3.11)

This may be recognized as the expression for a central term in the Virasoro algebra, with central charge

$$c = -\frac{24\pi s}{G \kappa}.$$  \hfill (3.12)

Inserting (3.10) and (3.11) into the Cardy formula (2.2), we obtain a density of states

$$\log \rho(L_0) = \frac{2\pi \phi_+}{G},$$  \hfill (3.13)

giving exactly the standard Bekenstein-Hawking entropy for the two-dimensional dilaton black hole \cite{34}.

In contrast to previous work on Virasoro algebras at the horizon, this derivation has the nice feature that the central charge (3.12) does not depend on the particular black hole being considered. The algebra may therefore be viewed as a universal one, with different black holes represented by different values (3.10) of \(L_0\).

For simplicity, I have dealt only with two-dimensional black holes. An extension to higher dimensions would clearly be of interest. As noted above, though, higher-dimensional general relativity may be dimensionally reduced in the manner of Kaluza and Klein to two-dimensional dilaton gravity coupled with extra “matter” fields (see, for example, \cite{35}). It is fairly easy to see that these added terms cannot contribute to the classical central charge (3.12), although they might give quantum corrections. The algebra derived here is thus more universal than it might seem.

As also noted above, we should probably worry further about the making the notion of an “asymptotic symmetry” used here more rigorous. It may be useful to exploit a generalization of the symmetry (1.1) that exists in the presence of a conformal Killing vector \(\eta^a\),

$$\nabla_a \eta_b + \nabla_b \eta_a = \kappa g_{ab}.$$  \hfill (3.14)
It is not hard to check that the transformation

$$g_{ab} \rightarrow \left( \eta^c \nabla_c f + \frac{n-2}{2} \kappa f \right) g_{ab}$$

leaves the Einstein-Hilbert action invariant provided that $f$ is chosen to satisfy

$$\int g^{ab} \nabla_a \kappa \nabla_b f \epsilon = 0.$$  \hspace{1cm} (3.16)

Moreover, if the original metric $g_{ab}$ admits a conformal Killing vector, it is easily checked that the transformed metric does as well. Maintaining the condition (3.16) is more complicated, but at least one solutions exists: if both $\kappa$ and $f$ are functions of a single null coordinate $v$, (3.16) holds automatically, and is preserved by (3.15). Work on understanding the implications of this extended symmetry is in progress.

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