Demonstration of quantum advantage in machine learning

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The main promise of quantum computing is to efficiently solve certain problems that are prohibitively expensive for a classical computer. Most problems with a proven quantum advantage involve the repeated use of a black box, or oracle, whose structure encodes the solution [1]. One measure of the algorithmic performance is the query complexity [2], i.e., the scaling of the number of oracle calls needed to find the solution with a given probability. Few-qubit demonstrations of quantum algorithms, such as Deutsch-Jozsa and Grover [1], have been implemented across diverse physical systems such as nuclear magnetic resonance [3–6], trapped ions [7], optical systems [8, 9], and superconducting circuits [10–12]. However, at the small scale, these problems can already be solved classically with a few oracle queries, and the attainable quantum advantage is modest [11, 12]. Here we solve an oracle-based problem, known as learning parity with noise [13, 14], using a five-qubit superconducting processor. Running classical and quantum [15] algorithms on the same oracle, we observe a large gap in query count in favor of quantum processing. We find that this gap grows by orders of magnitude as a function of the error rates and the problem size. This result demonstrates that, while complex fault-tolerant architectures will be required for universal quantum computing, a quantum advantage already emerges in existing noisy systems.

The limited size of engineered quantum systems and their extreme susceptibility to noise sources have made it hard so far to establish a clear advantage of quantum over classical computing. A promising avenue to highlight this separation is offered by a new family of algorithms designed for machine learning [16–19]. In this class of problems, artificial intelligence methods are employed to discern patterns in large amounts of data, with little or no knowledge of underlying models. A particular learning task, known as binary classification, is to identify an unknown mapping between a set of bits onto 0 or 1. An example of binary classification is identifying a hidden parity function [13, 14], defined by the unknown bit-string \( k \), which computes \( f(D, k) = D \cdot k \mod 2 \) on a register of \( n \) data bits \( D = \{ D_1, D_2, ..., D_n \} \) (Fig. 1a). The result, i.e., 0 (1) for even (odd) parity, is mapped onto the state of an additional bit \( A \). The learner has access to the output register of an example oracle circuit that implements \( f \) on random input states, on which he/she has no control. Repeated queries of the oracle allow the learner to reconstruct \( k \). However, any physical implementation suffers from errors, both in the oracle execution itself and in read-out of the register. In the presence of errors, the problem becomes hard. Assuming that every bit introduces an equal error probability, the best known algorithms have a number of queries growing as \( O(n) \) and runtime growing almost exponentially with \( n \) [13, 14, 20]. In view of the classical hardness of learning parity with noise (LPN), parity functions have been suggested as keys for secure and computationally easy authentication [21, 22].

The picture is different when the algorithm can process quantum superpositions of input states, i.e., when the oracle is implemented by a quantum circuit. In this case, applying a coherent operation on all qubits after an oracle query ideally creates the entangled state

\[
(\left| 0_A 0_D \right\rangle + \left| 1_A k_D \right\rangle)/\sqrt{2}.
\]

In particular, when \( A \) is measured to be in \( |1\rangle \), \( |D\rangle \) will be projected onto \( |k\rangle \). With constant error per qubit, learning from a quantum oracle requires a number of queries that scales as \( O(\log n) \), and has a total runtime that scales as \( O(n) \) [15]. This gives the quantum algorithm an exponential advantage in query complexity and a super-polynomial advantage in runtime.

In this work, we implement a LPN problem in a superconducting quantum circuit using up to five qubits, realizing the experiment proposed in Ref. 15. We construct a parity function with bit-string \( k \) using a series of CNOT gates between the ancilla and the data qubits (Fig. 1b). We then present two classes of learners for \( k \) and compare their performance. The first class simply structures a parity function with bit-string \( k \) by measuring the output qubits in the computational basis. Each measurement gives a number \( \hat{k} \) and the learner has access to the output qubit register. The second class performs some quantum computation (coherent operations), followed by classical analysis, to infer the solution. We show that the quantum approach outperforms the classical one in the number of queries required to reach a target error threshold, and that it is largely robust to noise added to the output qubit register.

The quantum device used in our experiment consists of five superconducting transmon qubits, \( A, D_1, ..., D_4, \)
and seven microwave resonators (Fig. 1c). Five of the resonators are used for individual control and readout of the qubits, to which they are dispersively coupled [24]. The center qubit $A$ plays the role of the result and is coupled to the data register $\{D_1, \ldots, D_n\}$ via the remaining two resonators. This coupling allows the implementation of cross-resonance (CR) gates [25] between $A$ (used as control qubit) and each $D_i$ (target), constituting the primitive two-qubit operation for the circuit in Fig. 1b (full gate decomposition in Extended Data Fig. 1). Each qubit state is read out by probing the dedicated resonator with a near-resonant microwave pulse. The output signals are then demodulated and integrated at room temperature to produce the homodyne voltages $\{V_{D_1}, \ldots, V_{D_n}, V_A\}$ (see Extended Data Fig. 2 for the detailed experimental setup).

To implement a uniform random example oracle for a particular $k$, we first prepare the data qubits in a uniform superposition (Fig. 1b). Preparing such a state ensures that all parity examples are produced with equal probability and is also key in generating a quantum advantage. We then implement the oracle as a series of CNOT gates, each having the same target qubit $A$ and a different control qubit $D_i$ for each $k_i = 1$. Finally, the state of all qubits is read out (with the optional insertion of Hadamard gates, see discussion below). The oracle mapping to the device is limited by imperfections in the two-qubit gates, with average fidelities $88 \sim 94\%$, characterized by randomized benchmarking [26] (see Extended Data Table 1). Readout errors in the register $\eta_{D_i}$, defined as the average probability of assigning a qubit to the wrong state, are limited to $20 \sim 40\%$ by the onset of inter-qubit cross-talk at higher measurement power (Extended Data Fig. 3). A Josephson parametric amplifier [27] in front of the amplification chain of $A$ suppresses its low-power readout error to $\eta_A = 5\%$.

Having implemented parity functions with quantum hardware, we now proceed to interrogate an oracle $N$ times and assess our capability to learn the corresponding $k$. We start with oracles with register size $n = 2$, involving $D_1, D_2,$ and $A$. We consider two classes of learning strategies, classical (C) and quantum (Q). In C, we perform a projective measurement of all qubits right after execution of the oracle. This operation destroys any coherence in the oracle output state, thus making any analysis of the result classical. The measured homodyne voltages $\{V_{D_1}, \ldots, V_{D_n}, V_A\}$ are converted into binary outcomes, using a calibrated set of thresholds (see Methods). Thus, for every query, we obtain a binary string $\{a, d_1, d_2\}$, where each bit is 0 (1) for the corresponding qubit detected in $|0\rangle$ ($|1\rangle$). Ideally, $a$ is the linear combination of $d_1, d_2$ expressed by the string $k$ (Fig. 1a). However, both the gates comprising the oracle and qubit readout are prone to errors (see Extended Data Table 1). To find the $k$ that is most likely to have produced our observations, at each query $m$ we compute the expected $\hat{a}_{k,m}$ for the measured $d_{1,m}, d_{2,m}$ and the 4 possible values of $k$. We then select the $k$ which minimizes the distance to the measured results $a_1, \ldots, a_N$ of $N$ queries, i.e., $\sum_N |\hat{a}_q - a_{i,k}|$ [13]. In the case of a tie, $k$ is randomly chosen among those producing the minimum distance. As expected, the error probability $p$ of obtaining the correct answer decreases with $N$ (Fig. 2a). Interestingly, the difficulty of the problem depends on $k$ and increases with the number of $k_i = 1$. This can be intuitively understood as needing to establish a higher correlation between data qubits when the weight of $k$ increases.

In our second approach (Q), while the oracle is left untouched, we apply local operations (Hadamard gates) to all qubits before measuring. Remarkably, this simple operation completely changes the statistics of the mea-
FIG. 2. Error probability $p$ to identify a 2-bit oracle $k$ as a function of the number of queries $N$. For both classical (a) and quantum (b) learners, one of the four oracles $k$ is applied, followed by the simultaneous measurement of all qubits. Hadamard gates are applied prior to measurement in the quantum case (Fig. 1b). See text for a description of the solvers in the two scenarios. Inset: number of queries $N_{1\%}(k)$ required to reach 1% error for the classical (empty bars) and quantum (solid) solver.

FIG. 3. Learning error probability $\bar{p}$ averaged over all the $n$-bit oracles $k$, for different $n$ and solvers. (a) $n = 2$, (b) $n = 3$. Making use of the analog measurements results $\{\langle V_{n1}\rangle,\ldots\langle V_{nN}\rangle, V_A\}$ (squares) improves over the digital solvers in Fig. 2 (circles) for both classical (empty symbols) and quantum (solid symbols) learning. The analog solver in Q proves to be the most efficient solution. Moreover, the gap between Q and C grows with $n$. The same dataset is used in Figs. 2 and 3, with $D_3$ ignored in the analysis for $n = 2$. See Extended Data Fig. 4 for the $p(N)$ corresponding to each 3-bit $k$. 

Postselect each oracle query on digital $a = 1$, but average all instances of $\{V_{D_1}\}$, and digitize the averages $\{\langle V_{D_i}\rangle\}$ instead of each observation (see Methods). For each $D_i$, the majority vote between $\approx N/2$ inaccurate observations is then replaced by a single vote with high accuracy. Using the analog results, not only does Q retain an advantage over C (smaller $p$ for given $N$), but it does so without introducing an overhead in classical processing.

The superiority of Q over C becomes even more evident when the oracle size $n$ grows from 2 to 3 data qubits (Fig. 3b). Whereas Q solutions are marginally affected, the best C solver demands almost an order of magnitude higher $N$ to achieve a target error. Maximizing the resources available in our quantum hardware, we observe an even larger gap for oracles with $n = 4$ (Extended Data Fig. 5), suggesting a continued increase of quantum advantage with the problem size.

As predicted, quantum parity learning surpasses classical learning in the presence of noise. To investigate the impact of noise on learning, we introduce additional readout error on either $A$ or on all $D_i$. This can be easily done by tuning the amplitude of the readout pulses, effectively decreasing the signal-to-noise ratio [28]. When the ancilla assignment error probability $\eta_A$ grows (Fig. 4a), the number of queries $N_{1\%}$ (the average of $N_{1\%}$ over all $k$) required by the C solver increases by up to 2 orders of magnitude in the measured range (see also Extended Data Fig. 6). Conversely, using Q, $N_{1\%}$ only changes by a factor of $\sim 3$. Key to this performance gap is the optimization of the digitization threshold for $\{\langle V_{D_i}\rangle\}$ at each value of $\eta_A$ (see Methods). When $\eta_A$ is increased, an interesting question is whether postselection on $V_A$...
with noise algorithm in a quantum setting. We have
widens with increasing \( \eta \) with average
outcomes. A quantum learner, with the ability of phys-
formance gap increases with added noise in the query
ing compared to its classical counterpart, where the per-
demonstrated a superior performance of quantum learn-
different solvers.)

Material for theoretical bounds on the scaling of \( \bar{N} \) queries, while \( Q' \) indiscriminately discards half of the
ancilla result (\( D \) or data (\( Q' \)) qubits, with \( n = 3 \).
eta is tuned by setting the readout power of the corresponding qubit(s). Empty (solid) circles correspond to the analog C (Q)
solver. (a), \( \bar{N} \) diverges for \( \eta_A \rightarrow 0.5 \) for C, while it stays limited for Q. When \( \eta_A \geq 0.25 \), it is preferable to ignore \( V_A \) altogether (\( Q' \), triangles). (b) Whereas both C and Q are severely affected by a noisy data register, Q remains superior and the performance gap increases with \( \eta_D \). See Methods for an explanation of the error bars.

remains always beneficial. In fact, for \( \eta_A > 0.25 \), it becomes more convenient to ignore \( V_A \) and use the totality of the queries (\( Q' \) in Fig. 4a).

Similarly, we step the readout error of the data qubits, with average \( \eta_D \), while setting \( \eta_A \) to the minimum. Not only does Q outperform C at every step, but the gap widens with increasing \( \eta_D \).

A numerical model including the measured \( \eta_A, \eta_D \), qubit decoherence, and gate errors modeled as depolarization noise (Extended Data Table 1) is in very good agreement with the measured \( N_{1\%} \) at all \( \eta_A, \eta_D \). This model allows us to extrapolate \( N_{1\%} \) to the extreme cases of zero and maximum noise. Obviously, when \( \eta_D = 0.5 \), readout of the data register contains no information, and \( N_{1\%} \) consequently diverges. On the other hand a random ancilla result (\( \eta_A = 0.5 \)) does not prevent a quantum learner from obtaining k. In this limit, the predicted factor of \( \sim 2 \) in \( N_{1\%} \) between Q and \( Q' \) can be intuitively understood as Q indiscriminately discards half of the queries, while \( Q' \) uses all of them. (See Supplementary Material for theoretical bounds on the scaling of \( \bar{N}_{1\%} \) for different solvers.)

In conclusion, we have implemented a learning parity with noise algorithm in a quantum setting. We have demonstrated a superior performance of quantum learning compared to its classical counterpart, where the performance gap increases with added noise in the query outcomes. A quantum learner, with the ability of physically manipulating the output of a quantum oracle, is expected to find the hidden key with a logarithmic number of queries and linear runtime as function of the problem size, whereas a passive classical observer would require a linear number of queries and nearly exponential runtime. We have shown that the difference in classical and quantum queries required for a target error rate grows with the oracle size in the experimentally accessible range, and that quantum learning is much more robust to noise. We expect that future experiments with increased oracle size will further demarcate a quantum advantage, in support of the predicted asymptotic behavior.

**METHODS**

**Pulse calibration.** Single- and two-qubit pulses are calibrated by an automated routine, executed periodically during the experiments. For each qubit, first the transition frequency is calibrated with Ramsey experiments. Second, \( \pi \) and \( \pi/2 \) pulse amplitudes are calibrated during a phase estimation protocol [29]. The pulse amplitudes, modulating a carrier through an I/Q mixer (Extended Data Fig. 2) are adjusted at every iteration of the protocol until the desired accuracy or signal-to-noise limit is reached. Pulses have a Gaussian envelope in the main quadrature and derivative-of-Gaussian in the other, with DRAG parameter [30] calibrated beforehand using a sequence amplifying phase errors [31]. CR gates are calibrated in a two-step procedure, determining first the optimum duration and then the optimum phase for a \( ZX_{90} \) unitary.

**Experimental setup.** A detailed schematic of the experimental setup is illustrated in Extended Data Fig. 2. For each qubit, signals for readout and control are delivered to the corresponding resonator through an individual line through the dilution refrigerator. For an efficient use of resources, we apply frequency division multiplexing [32] to generate the five measurement tones by sideband modulation of three microwave sources. Moreover, the same pair of BBN APS (custom arbitrary waveform generators) channels produce the readout pulses for \( \{D_1, D_2\} \), and another one for \( \{D_3, D_4\} \). Similarly, the output signals are pairwise combined at base temperature, limiting the number of HEMTs and digitizer channels to three. The attenuation on the input lines, distributed at different temperature stages, is a compromise between suppression of thermal noise impinging on the resonators (affecting qubit coherence) and the input power required for CR gates.

**Gate sequence.** CNOT gates can be decomposed in terms of CR gates using the relation CNOT\(_{12} = (Z_{90} \otimes X_{90}) \text{CR}_{12} \) [33]. Moreover, the role of control and target qubits are swapped, using CNOT\(_{12} = (H_1 \otimes H_2) \text{CNOT}_{21}(H_1 \otimes H_2) \). The first of these H gates is absorbed into state preparation for the LPN sequence (Figs. 1a and Extended Data Fig. 1). Similarly, when two CNOTs are executed back to back, two consecutive H gates on A are canceled out. In order to maintain the oracle identical in C and Q, we do not compile the H gates.
in the CNOTs with those applied before measurement in Q.

**Data analysis.** For each set of \{k, \eta_A, \eta_D\}, solver type, and register size n, we measure the result of 10,000 oracle queries. Each set is accompanied by n + 2 calibration points (averaged 10,000 times), providing the distributions of \(V_A, V_{D_1}, ..., V_{D_n}\) for the collective ground state and for single-qubit excitations (n data and 1 ancilla qubit). These distributions are then used to determine the optimum digitization threshold (for digital solvers) or as input to the Bayesian estimate in C. To obtain \(p(N)\), we resample the full data set with 1000 – 4000 random subsets of each size N.

Error bars are obtained by first computing the credible intervals for \(p\) at each set \{N, k, \eta_A, \eta_D\}. These intervals are computed with Jeffreys beta distribution prior Beta(\(\frac{1}{2}, \frac{1}{2}\)) for Bernoulli trials, with a credible level of 100% – (100% - 95%), \approx 99.36%. This ensures that, under a union bound, the average of estimates for 8 different keys is inside the credible interval with a probability of at least 95%. We then perform antitonic regression on the upper and lower bounds of the credible intervals to ensure monotonicity as function of \(N\), and then the intercept p = 0.01 for each k. The bounds on the value \(N_{1\%}\) averaged over the keys is computed by interval arithmetic on the credible intervals of \(N_{1\%}\) for each k.

**Classical solver with Bayesian estimate.** An improved classical solver for the LPN problem can be constructed when the oracle provides an analog output. Under the assumption of Gaussian distributions for each possible bit value, this improved solver corresponds to a Bayesian estimate of the key after a series of observations of the data and ancilla bits. More formally, taking a uniform prior distribution for all binary strings produced by the oracle, one computes the (unnormalized) posterior \(p(D_i)\) distribution for each data bit \(D_i\), the output of the oracle,

\[
p(D_i = b| V_{D_i}) = \frac{1}{2} \exp \left[ -\frac{(V_{D_i} - b)^2}{2 \sigma_i^2} \right]
\]

The (unnormalized) posterior distribution \(p_m(k| V_D, V_A)\) for the key k after the mth query, on the other hand, is given by

\[
p_m(k| V_D, V_A) = \exp \left[ -\frac{(V_A - D \cdot k)^2}{2 \sigma_A^2} \right] p(D| V_D) p_{m-1}(k),
\]

where \(p_0(k)\) is the prior distribution for each key. Here and above, \{V_{D_1}, ..., V_{D_n}, V_A\} are rescaled to have mean 0 and 1 for the corresponding qubit in |0\rangle and |1\rangle, respectively. Iterating this procedure (while updating \(p(k)\) at each iteration), and then choosing the most probable key \(k_{\text{Bayes}} = \arg \max_k p(k)\), one obtains an estimate for the key.

**Analog quantum solver with postselection on A.** While postselection on A is performed equally on both digital (Fig. 2) and analog (Figs. 3-4) Q solvers, in the analog case all postselected \{V_{D_i}\} are averaged together. Finally, the results \{\langle V_{D_i} \rangle\} are digitized to determine the most likely k. The choice of digitization threshold for each \(D_i\) depends on: a) the readout voltage distributions \(p_0\) and \(p_1\) for the two basis states, each characterized by a mean \(\mu\) and a variance \(\sigma^2\); b) \(\eta_A\). Ideally (\(\eta_A = 0\) and perfect oracle), the distribution of each query output \(V_{D_i}\) matches \(p_0\) \((p_1)\) for \(k_i = 0\) \((1)\). When \(\eta_A > 0\), the distribution for \(k_i = 1\) becomes the mixture \(\rho_{k_i=1} = \eta_A p_0 + (1 - \eta_A) p_1\). This mixture has mean \((1 - \eta_A) \mu_1 + \eta_A \mu_0\) and variance \((1 - \eta_A) \sigma_1^2 + \eta_A \sigma_0^2 - 2 \eta_A (1 - \eta_A) \mu_0 \mu_1\). Instead, \(\rho_{k_i=0} = \rho_0\) independently of \(\eta_A\). We approximate the expected distribution of the mean \(\langle V_{D_i} \rangle\) with a Gaussian having average and variance obtained from \(\rho_{k_i=0}(\rho_{k_i=1})\) for \(k_i = 0\) \((1)\). Finally, we choose the digitization threshold for \(V_{D_i}\) which maximally discriminates these two Gaussian distributions. We note that the number of queries scales the variance of both distributions equally and therefore does not affect the optimum threshold. Furthermore, this calibration protocol is independent of the oracle (see Extended Data Fig. 7).

**Analog quantum solver without postselection.** The analysis without ancilla (Q') closely follows the steps outlined in the last paragraph. For the purpose of extracting the optimum digitization thresholds, we consider \(\eta_A = 0.5\) in the expressions above. This corresponds to an equal mixture of \(p_0\) and \(p_1\) when \(k_i = 1\).

**Bounds on performance of the analog quantum solvers.** Here we demonstrate how the bounds from Ref. 15 can be easily adapted to the case where the solver uses analog voltage measurements. We consider both the case where experiments are postselected based on the digitized value of the ancilla (referred below as postselected soft averaging), and the case where the ancilla is ignored altogether. We consider different error rate for the ancilla and the data qubits.

**Postselected soft averaging.** In order to generalize the analysis in Ref. 15 to the postselected soft averaging case, we now need to take two types of data errors into account: depolarizing errors (our crude model for oracle errors), and measurement error (additive Gaussian noise).

First, postselection works identically to Ref. 15, since we treat the ancilla digitally. We note that, in this analysis, the ancilla error rate combines oracle errors and readout errors. Given \(n\) queries, \(n'\) are postselected according to the ancilla value \(V_A\), and \(s\) of this postselections are correct. Although \(s\) is unknown in an experiment, we condition our results on \(s\) being typical (i.e., we only consider the values of \(s\) that occur with probability higher than \(1 - \epsilon\) for some small \(\epsilon\)).

For the correct postselections, we have two possible voltage distributions for each \(D_i\), depending on whether the outcome is 0 or 1. The distribution of the outcomes will depend on whether we have one of the correct postselections, and on the value of ith key bit \(k_i\). If \(k_i = 0\), the
conditional voltage distributions, depending on whether we postselected correctly (stå) or not (ₓ), are

\[ \rho_{i|0}^{\text{stå}} \sim N(\eta_{i}s, s\sigma^2), \]
\[ \rho_{i|0}^{\text{x}} \sim N(\eta_{i}(N' - s), (N' - s)\sigma^2), \]

respectively, with \( N'(\mu, \sigma^2) \) the normal distribution with mean \( \mu \) and variance \( \sigma^2 \). Therefore, the overall distribution is

\[ \rho_{i|0} \sim N(\eta_{i}N', N'\sigma^2). \]

If the true bit value is 1, we have

\[ \rho_{i|1}^{\text{stå}} \sim N((1 - \eta_{i})s, s\sigma^2), \]
\[ \rho_{i|1}^{\text{x}} \sim N(\eta_{i}N' - s), (N' - s)\sigma^2), \]

and therefore

\[ \rho_{i|1} \sim N((1 - \eta_{i})s + \eta_{i}(N' - s), n'\sigma^2). \]

Now we must compute the optimal voltage threshold which determine the digital decision at each of the data qubits. If we define

\[ \mu_{i|j} = \mathbb{E}[\rho_{i|j}], \]

the threshold we must choose is

\[ T = \frac{1}{2}\mu_{i|0} + \frac{1}{2}\mu_{i|1} \]
\[ = s(1 - 2\eta_{i}) + 2\eta_{i}N'. \]

The complication is that this is conditioned on \( s \), but we will deal with that later, as the dependence on \( s \) also comes from the distribution of outcomes (not just the threshold). In the following we assume the value of \( s \) to be typical (i.e., \( s \) is contained in the region around the median excluding the distribution tails that add up to at most some small \( \epsilon \)). Under this assumption, we require that \( \mu_{i|0} \leq T \leq \mu_{i|1} \).

The probability of having the right answer at a particular bit is the probability that the averaged voltage is on the correct side of the threshold (above or below). If the true value of the bit is 0, i.e., if \( k_i = 0 \), given the threshold, we can compute

\[ \Pr(\rho_i \leq T|s, k_i = 0) = \Phi \left( \frac{T - \mu_{i|0}}{N'\sigma^2} \right) \]
\[ = 1 - Q \left( \frac{T - \mu_{i|0}}{N'\sigma^2} \right), \]

where \( \Phi \) is the cumulative distribution function for a normal distribution, and \( Q \) is the tail probability for the normal distribution. We can place a lower bound on \( \Pr(M_j \leq T|s, k_i = 0) \) with an upper bound on \( Q \). Note that, for the range of interest, the argument of \( Q \) is always positive, so we can use the bound

\[ Q(x) < \frac{1}{2} \exp \left( \frac{x^2}{2} \right), \quad x > 0 \]

and therefore

\[ \Pr(\rho_i \leq T|s) \geq 1 - \frac{1}{2} \exp \left[ - \frac{(T - \mu_{i|0})^2}{2N'\sigma^2} \right], \]

which is nearly what we want—we must now address the dependence on \( s \). One way to restrict the analysis to typical \( s \) is to require that, for \( \eta_A = \max\{\eta_A, 1 - \eta_A\} \), the probability

\[ \Pr(|s - \mu_s| < \delta'\mu_s) > 1 - 2 \exp \left( -\frac{\delta'^2 \eta_A N'}{3} \right) \]

is exponentially close to 1. This choice of \( \eta_A \) requires knowledge of the error rates in the ancilla so that, for example, one knows to postselect on 0 instead of 1 if \( \eta_A > 0.5 \).

In order to pick a lower bound valid for all typical thresholds and means, we choose the smallest \( |T - \mu_{i|0}| \) by choosing \( T \) and \( \mu_{j|0} \) independently from the typical sets. This leads to

\[ T - \mu_{i|0} > N'(1 - \delta') \left( \frac{1}{2} - \eta_{i} \right) \eta_{A} \]

and thus,

\[ \Pr(\rho_i \leq T|s) \geq 1 - \frac{1}{2} \exp \left[ -\frac{(N'(1 - \delta')^2 (\frac{1}{2} - \eta_{i})^2 \eta_{A}^2)}{2\sigma^2} \right] \]

so that, by the union bound,

\[ \Pr(\hat{a} \neq a|s) < \frac{n}{2} \exp \left[ -\frac{(N'(1 - \delta')^2 (\frac{1}{2} - \eta_{i})^2 \eta_{A}^2)}{2\sigma^2} \right] \]

and therefore the lower bound on the number of queries is

\[ N' > \frac{2\sigma^2}{(1 - \delta')^2 (\frac{1}{2} - \eta_{i})^2 \eta_{A}^2} \ln \frac{n}{2\delta}. \]

If \( k_i = 1 \), we take a similar approach, but the lower bound on the distance between the threshold and the mean is smaller, leading to

\[ N' > \frac{2\sigma^2}{(1 - 3\delta')^2 (\frac{1}{2} - \eta_{i})^2 \eta_{A}^2} \ln \frac{n}{2\delta}, \]

so clearly this is the worst case for \( k_i \).

If we want to bound \( N \) instead of \( N' \), we just remember that there is a 50% chance of collapsing into the informative branch of the state, and using the same typicality argument as before, we have

\[ N > \frac{4\sigma^2}{(1 - \delta')^2(1 - 3\delta')^2 (\frac{1}{2} - \eta_{i})^2 \eta_{A}^2} \ln \frac{n}{2\delta}. \]
where $\delta''$ measures how far from the mean $k$ is, with a corresponding Chernoff bound.

Analysis without postselection. The analysis is equivalent to the postselected case, but with $\eta_n = \frac{1}{2}$ and $N' = N$, since we keep all experiments and have a 50% chance of collapsing the state in the informative branch. All of this leads to

$$N > \frac{8\sigma^2}{(1-3\delta')^2(1/2 - \eta_D)^2} \ln \frac{n}{2\delta}.$$ 

We now see that depending of choices of $\delta'$ and $\delta''$, postselection may or may not lead to better bounds, but the asymptotic scaling is the same.

Complexity of digital classical solvers. Angluin and Laird [13] showed that learning with classification noise requires $O(n)$ queries as long as the classification error rate is below $\frac{1}{2}$, and propose an algorithm (disagreement minimization) that corresponds to solving an NP-complete problem. According to the exponential time hypothesis, it is widely believe that NP-complete problems can only be solved in exponential time. Note that, while the classification rate is nominally $\eta_A$ in our experiment, all errors (including $\eta_D$ and gate infidelities) can be combined onto an effective, $k$-dependent, single error rate.

Blum, Kalai, and Wasserman [14] devised a sub-exponential time algorithm for learning with classification errors, as long as the classification error rate is below $\frac{1}{2} - \frac{1}{\sqrt{n}}$ for $\delta < 1$, at the cost of increasing the query complexity to slightly sub-exponential scaling with $n$.

Later, Lyubashevsky [20] devised another slightly sub-exponential time algorithm for learning with classification errors, as long as the classification error rate is below $\frac{1}{2} - \frac{1}{\sqrt{n}}$ for $\delta < 1$, but bringing down the query complexity to $n^{1+\epsilon}$ for $\epsilon > 0$.

Note that the gains over exponential time scaling for these two algorithms are rather small – a reduction from $O(2^n)$ to $O(2^{n/2})$ and $O(2^{n^{1+\epsilon}})$, respectively.

For $n = 3$, the Blum-Kalai-Wasserman algorithm can only tolerate less than $\frac{2}{3} \approx 0.375$ classification error rate, while the Lyubashevskyy algorithm can only tolerate less than $\frac{1}{2} - \frac{1}{\sqrt{3}} \approx 0.033$ classification error rate. Lyubashevsky’s algorithm does not apply to any of the experiments discussed here because our classification error rates are too high. The Blum-Kalai-Wasserman algorithm only applies to some of the experiments discussed here, so for the sake of fair comparison across all error rates, we use Angluin and Laird’s disagreement minimization.

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CONTRIBUTIONS

C.A.R. and B.R.J. developed the BBN APS and the data acquisition software, D.R. carried out the experiment, D.R., M.P.S. and B.R.J. performed the data analysis, M.P.S. implemented the solvers and developed the theoretical models, D.R. and M.P.S. wrote the manuscript with comments from the other authors, A.W.C. and J.A.S. contributed to the initial design of the experiment, B.R.J., J.M.C. and J.M.G. supervised the project.

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1. Nielsen, M. A. & Chuang, I. L. Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
2. Cleve, R. An introduction to quantum complexity theory. In Collected Papers on Quantum Computation and Quantum Information Theory, 103–127 (World Scientific, 2001).
3. Jones, J. A., Mosca, M. & Hansen, R. H. Implementation of a quantum search algorithm on a quantum computer. Nature 393, 344–346 (1998).
4. Linden, N., Barjat, H. & Freeman, R. An implementation of the Deutsch-Jozsa algorithm on a three-qubit NMR quantum computer. J. Phys. Chem. 296, 61 – 67 (1998).
5. Chuang, I. L., Vandersypen, L. M. K., Zhou, X., Leung, D. W. & Lloyd, S. Experimental realization of a quantum algorithm. Nature 393, 143–146 (1998).
6. Chuang, I. L., Gershenfeld, N. & Kubinec, M. Experimental implementation of fast quantum searching. Phys. Rev. Lett. 80, 3408 (1998).
7. Guilde, S. et al. Implementation of the Deutsch-Jozsa algorithm on an ion-trap quantum computer. Nature 421, 48–50 (2003).
8. Takeuchi, S. Experimental demonstration of a three-qubit quantum computation algorithm using a single photon and linear optics. Phys. Rev. A 62, 032301 (2000).
9. Kwiat, P. G., Mitchell, J. R., Schwindt, P. D. D. & White, A. G. Grover’s search algorithm: an optical approach. J. Mod. Opt. 47, 257–266 (2000).
[10] DiCarlo, L. et al. Demonstration of two-qubit algorithms with a superconducting quantum processor. *Nature* **460**, 240 (2009).

[11] Yamamoto, T. et al. Quantum process tomography of two-qubit controlled-Z and controlled-NOT gates using superconducting phase qubits. *Phys. Rev. B* **82**, 184515 (2010).

[12] Dewes, A. et al. Quantum speeding-up of computation demonstrated in a superconducting two-qubit processor. *Phys. Rev. B* **85**, 140503 (2012).

[13] Angluin, D. & Laird, P. Learning from noisy examples. *Machine Learning* **2**, 343–370 (1988).

[14] Blum, A., Kalai, A. & Wasserman, H. Noise-tolerant learning, the parity problem, and the statistical query model. *J. ACM* **50**, 506–519 (2003).

[15] Cross, A. W., Smith, G. & Smolin, J. A. Quantum learning robust against noise. *Phys. Rev. A* **92**, 012327 (2015).

[16] Blais, A., Huang, R.-S., Wallraff, A., Girvin, S. M. & Schoelkopf, R. J. Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation. *Phys. Rev. A* **69**, 062320 (2004).

[17] Motzoi, F., Gambetta, J. M., Rebentrost, P. & Wilhelm, F. K. Simple pulses for elimination of leakage in weakly nonlinear qubits. *Phys. Rev. Lett.* **103**, 110501 (2009).

[18] Lucero, E. et al. Reduced phase error through optimized control of a superconducting qubit. *Phys. Rev. A* **82**, 042339 (2010).

[19] Jerger, M. et al. Frequency division multiplexing readout and simultaneous manipulation of an array of flux qubits. *Appl. Phys. Lett.* **101**, 042604 (2012).

[20] Chow, J. M. et al. Implementing a strand of a scalable fault-tolerant quantum computing fabric. *Nature Comm.* **5**, 4015 (2014).

[21] Kimmel, S., Low, G. H. & Yoder, T. J. Robust calibration of a universal single-qubit gate-set via robust phase estimation. arXiv:quant-ph/1502.02677 (2015).

[22] Petz, D. *SOFSEM 2012: Theory and Practice of Computer Science*, vol. 7147 of *Lecture Notes in Computer Science* (Springer Berlin Heidelberg, Berlin, Heidelberg, 2012).

[23] Córcoles, A. et al. Demonstration of a quantum error detection code using a square lattice of four superconducting qubits. *Nature Comm.* **6**, 6979 (2015).

[24] Blais, A., Huang, R.-S., Wallraff, A., Girvin, S. M. & Schoelkopf, R. J. Cavity quantum electrodynamics for superconducting electrical circuits: An architecture for quantum computation. *Phys. Rev. A* **69**, 062320 (2004).

[25] Rigetti, C. & Devoret, M. Fully microwave-tunable universal gates in superconducting qubits with linear couplings and fixed transition frequencies. *Phys. Rev. B* **81**, 134507 (2010).

[26] Magesan, E., Gambetta, J. M. & Emerson, J. Characterizing quantum gates via randomized benchmarking. *Phys. Rev. A* **85**, 042311 (2012).

[27] Hatridge, M., Vijay, R., Slichter, D. H., Clarke, J. & Siddiqi, I. Dispersive magnetometry with a quantum limited SQUID parametric amplifier. *Phys. Rev. B* **83**, 134501 (2011).

[28] Vijay, R., Slichter, D. H. & Siddiqi, I. Observation of quantum jumps in a superconducting artificial atom. *Phys. Rev. Lett.* **106**, 110502 (2011).
Extended Data Fig. 1. Circuit gate decomposition for 3-bit oracles. a $k = 111$, b, $k = 001$. CNOT gates [see Fig. 1(a)] are implemented by dressing the two-qubit primitives $CR_i = ZAX_{D_i} (\pi/2)$ with single-qubit gates (see Methods). Some of these gates cancel out with either state preparation (for $D_1$-$D_3$) or with those in a subsequent CNOT gate (for $A$) and are therefore not executed. Virtual $Z_{90}$ gates (not shown) are applied to $A$ after each CR gate. Dashed boxes indicate the Hadamard decomposition applied in Q. Pulse durations are not to scale. Note that in (b) the state preparation of $D_1$ and $D_2$ is moved after $CR_3$ to prevent dephasing induced by the off-resonant drive.

Extended Data Fig. 2. Experimental setup. Complete wiring of control and readout electronics inside and outside the Bluefors BF-LD400 dilution refrigerator (see Methods). Home-made Arbitrary Pulse Sequencers (BBN APS, each indicated by its 4 analog channels Ch1-Ch4) produce the waveforms for single-qubit measurement, control, and CR pulses. The readout signal for $A$ is boosted by a Josephson parametric amplifier (JPA) from UC Berkeley [27].
Extended Data Fig. 3. **Readout voltage distributions.** Normalized readout signals for $A, D_1, D_2, D_3$ for the 16 4-qubit computational states at optimum readout settings (comparable to Figs. 2-3) (a), and for the maximum $\eta_A$ (b) and $\eta_D$ (c) in Fig. 4a and b, respectively. Dots and error bars indicate averages and standard deviations, respectively. These data are taken in a subsequent cooldown of the same device under similar conditions, but with qubit transitions shifted up in frequency by $\sim 20$ MHz.

Extended Data Fig. 4. **Learning error $p$ for the individual 3-bit $k$.** The oracle queries are processed by the analog C (empty symbols) and Q (solid) solvers. The average errors are shown in Fig. 3b.
Extended Data Fig. 5. **Learning error $p$ for 4-bit oracles.** Only the oracles with $k_4 = 0$ could be implemented in this device.

Extended Data Fig. 6. **Average learning error $\bar{p}$ as a function of readout errors.** The outputs of 3-bit oracles are corrupted by increasing $\eta_A$ (a) or $\eta_D$ (b). The intercepts of these (and additional) curves with $\bar{p} = 0.01$ are shown in Fig. 4.
Extended Data Fig. 7. **Calibration of the digitization threshold $V_{Di}$ for the analog quantum solver Q.** For illustration purposes, we assume that the $\rho_0$ and $\rho_1$ (see Methods) have mean equal to 0 and 1, respectively, and variance equal to 0.25 in both cases. Ignoring oracle errors, $\rho_{k=0}$ (a) coincides with $\rho_0$, while $\rho_{k=1}$ (b) is a mixture of the two, with weights determined by the postselection error $\eta_A$, here 0 (red) or 0.2 (blue). (c) Distribution of the mean $\langle V_{Di} \rangle$. Increasing $\eta_A$ shifts the mean towards 0, thus decreasing the optimum discrimination threshold. Variances are arbitrarily scaled by a factor of 2, which does not affect the choice of threshold. The case without postselection on the ancilla (Q') corresponds to $\eta_A = 0.5$ (not shown) for the purpose of determining the threshold.

**Extended Data Table 1. Qubit and resonator parameters.** Single- and two-qubit gate fidelities are obtained by randomized benchmarking (RB) [26].

|                         | $A$   | $D_1$ | $D_2$ | $D_3$ | $D_4$ |
|-------------------------|-------|-------|-------|-------|-------|
| qubit frequency (GHz)   | 5.136 | 5.069 | 5.244 | 5.011 | 5.073 |
| readout resonator freq. (GHz) | 6.365 | 6.452 | 6.455 | 5.505 | 6.408 |
| relaxation time, $T_1$ (µs) | 24    | 38    | 37    | 40    | 41    |
| Ramsey decay time, $T_2$ (µs) | 10-25*| 36    | 41    | 38    | 50    |
| average assignment      | 0.05  | 0.24  | 0.16  | 0.21  | 0.43  |
| readout error, $\eta$   |       |       |       |       |       |
| gate duration (ns)       | 60    | 60    | 60    | 60    | 60    |
| single-qubit fidelity    | 0.993 | 0.998 | 0.998 | 0.998 | 0.998 |
| gate duration (ns)       | n.a.  | 300   | 340   | 1100  | -     |
| two-qubit fidelity       | n.a.  | 0.94  | 0.92  | 0.88  | -     |

*fluctuating