A Note on Combinatorial Derivation

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Abstract

Given an infinite group $G$ and a subset $A$ of $G$ we let $\Delta(A) = \{ g \in G : |gA \cap A| = \infty \}$ (this is sometimes called the combinatorial derivation of $A$). A subset $A$ of $G$ is called large if there exists a finite subset $F$ of $G$ such that $FA = G$. We show that given a large set $X$, and a decomposition $X = A_1 \cup \ldots \cup A_n$, there must exist an $i$ such that $\Delta(A_i)$ is large. This answers a question of Protasov. We also answer a number of related questions of Protasov.

1 Introduction

For a subset $A$ of an infinite group $G$ we denote

$$\Delta(A) = \{ g \in G : |gA \cap A| = \infty \}.$$ 

This is sometimes called the combinatorial derivation of $A$. We note that $\Delta(A)$ is a subset of $AA^{-1}$, the difference set of $A$. It can sometimes be useful to consider $\Delta(A)$ as the elements that appear in $AA^{-1}$ ‘with infinite multiplicity’. In [3] Protasov analysed a series of results on the subset combinatorics of groups (see the survey [4]) with reference to the function $\Delta$, and asked a number of questions. In this note we present answers to some of those questions.

A subset $A$ of $G$ is said to be [2]:

- large if there exists a finite subset $F$ of $G$ such that $FA = G$;
- $\Delta$-large if there exists a finite subset $F$ of $G$ such that $F\Delta(A) = G$.

Protasov asked [3]:

**Question A.** Is every large subset of an arbitrary infinite group $G$ $\Delta$-large?

**Question B.** Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that, for any group $G$ and any partition $G = A_1 \cup \ldots \cup A_n$, there exists an $i$ and a subset $F$ of $G$ such that $G = F\Delta(A_i)$ and $|F| \leq f(n)$?

**Question C.** Let $G$ be an infinite group. Given $G = A_1 \cup \ldots \cup A_n$, such that $A_i = A_i^{-1}$ and $e \in A_i$ for $i \in \{1, \ldots , n\}$, does there exist an $i$ and an infinite subset $X$ of $G$ such that $X \subseteq A_i$ and $\Delta(X) \subseteq A_i$?

A subset $A$ of $G$ is said to be sparse if for every infinite subset $X$ of $G$, there exists a non-empty finite subset $F$ of $X$ such that $\bigcap_{g \in F} gA$ is finite. A subset $A$ of $G$ is said to be $\nabla$-thin if either $A$ is finite, or there exists an $n \in \mathbb{N}$ such that $\Delta^n(A) = \{ e \}$, where $\Delta^n$ denotes the iterated application of $\Delta$.

Protasov also asked:

**Question D.** Is every $\nabla$-thin subset of a group $G$ sparse?

In Section 2 we present answers to the four questions posed by Protasov. We say a subset $A$ of $G$ is cofinite if there exists a finite subset $H$ of $G$ such that $A = G \setminus H$. Our main result is:
Theorem 1. Let $G$ be an infinite group. Given a subset $X$ of $G$ such that there exists a finite subset $F$ of $G$ such that $FX$ is cofinite, and a decomposition $X = A_1 \cup \ldots \cup A_n$, then there exists an $i$ and a subset $F'$ of $G$ such that $|F'| \leq |F|(|F| + 1)^{2n-3}$ and $F' \Delta(A_i) = G$.

This provides a positive answer to Questions A and B. We also show:

Theorem 2. Let $G$ be an infinite group. Given a subset $A$ of $G$ and a countable subset $X$ of $\Delta(A)$ such that $X = X^{-1}$ and $e \in X$, there exists a subset $Y$ of $A$ such that $\Delta(Y) = X$.

Since for all infinite sets $A$ we have that $e \in \Delta(A)$, this provides a positive answer to Question C. Finally we also show:

Theorem 3. Let $G$ be an infinite group, $A$ a subset of $G$. Then if $A$ is $\nabla$-thin, then $A$ is sparse.

2 Results

We will start by considering Question A. For a subset $A$ of $G$ and a finite subset $F$, it is easy to see that $\Delta(FA) = F \Delta(A) F^{-1}$. Therefore for abelian groups it is apparent that if $A$ is large, with say $FA = G$, we have that $F^{-1}F \Delta(A) = \Delta(G) = G$, and so $A$ is $\Delta$-large.

Lemma 4. Let $G$ be an infinite group, $A$ a subset of $G$. If there exists a finite subset $F$ of $G$ such that $FA$ is cofinite, then $F \Delta(A) = G$.

Proof. We would like to find some finite set $X = \{x_1, \ldots, x_k\}$, such that the set of translates $\{x_1 A, x_2 A, \ldots, x_k A\}$ has the property that, for any $g \in G$, we must have $|gA \cap x_i A| = \infty$ for some $i$. Then, for all $g \in G$ we would have that $|x_i^{-1} gA \cap A| = \infty$ for some $i$ and so $x_i^{-1} g A \in \Delta(A)$ and so $g \in X \Delta(A)$. Therefore we could conclude that $X \Delta(A) = G$. Let $F = \{f_1, \ldots, f_k\}$.

Since $FA$ is cofinite, there exists some finite subset $H$ of $G$ such that $f_1 A \cup \ldots \cup f_k A = FA = G \setminus H$. Therefore we see that for any $g \in G$ there must exist an $i$ such that $|gA \cap f_i A| = \infty$. Therefore $F$ satisfies the property above, and so $F \Delta(A) = G$.

We note that Question A follows from Lemma 4 as a simple corollary.

Corollary 5. Let $G$ be an infinite group, $A$ a subset of $G$. Then if $A$ is large, then $A$ is $\Delta$-large.

There are various concepts of “small” for subsets of groups. A subset $A$ of $G$ is said to be $[2]$:

- small if $(G \setminus A) \cap L$ is large for every large subset $L$ of $G$;
- $P$-small if there exists an injective sequence $(g_n)_{n=1}^\infty$ in $G$ such that $g_i A \cap g_j A = \phi$ for all $i, j$;
- almost $P$-small if there exists an injective sequence $(g_n)_{n=1}^\infty$ in $G$ such that $|g_i A \cap g_j A| < \infty$ for all $i, j$;
- weakly $P$-small if for every $n \in \mathbb{N}$ there exists distinct elements $g_1, g_2, \ldots, g_n$ in $G$ such that $g_i A \cap g_j A = \phi$ for all $i, j$.

Protasov also asked:

Question E. Is every nonsmall subset of an arbitrary infinite group $G$ $\Delta$-large?

We can also use the method in Lemma 4 to show that sets which are not almost $P$-small are $\Delta$-large.

Theorem 6. Let $G$ be an infinite group, $A$ a subset of $G$. Then if $A$ is not almost $P$-small, then $A$ is $\Delta$-large.

Proof. Take a maximal set $F = \{f_1, \ldots, f_k\}$ such that $|f_i A \cap f_j A| < \infty$ for all $i, j$. Such a set exists and is finite since $A$ is not almost $P$-small. Then, for all $g \in G$ we must have that $|gA \cap f_i A| = \infty$ for some $i$, since $F$ is maximal. Hence $f_i^{-1} g A \in \Delta(A)$ and so $G = F \Delta(A)$.

We note however that there do exist sets $A$ which are not weakly $P$-small (and so also not $P$-small), but which are still not $\Delta$-large.
Example 7. Consider the group \((\mathbb{Z},+\)) \(|\mathcal{A}\{10^n, 10^n + n : n \in \mathbb{N}\}\). Clearly any translate of \(A\) has non-empty intersection with \(A\), and so \(A\) cannot be weakly P-small. However \(\Delta(A) = \{0\}\) since each difference only appears a finite number of times in \(A\).

It remains to show that sets which are not small are \(\Delta\)-large. We will be able to show this using ideas that are used in answering Question B.

To motivate the proof we first consider the case \(n = 2\). Given a decomposition of \(G\) into two sets \(A \cup B\) what does it mean if \(\Delta(A) \neq \emptyset\)? Well in that case we have some \(g \in G\), \(g \notin \Delta(A)\). Therefore there are only a finite number of \(h \in G\) such that the group elements \(g h^{-1}\) are both members of \(A\), since each such \(h\) is in \(g A \cap A\). Therefore there is some finite subset \(H\) of \(G\) such that for all \(h \in G \setminus H\), either \(h \in B\) or \(g h^{-1} \in B\). But then we have that \(\{e, g\} B = B \cup gB = G \setminus H\) and so, by Lemma 4 \(\{e, g\} \Delta(B) = G\). This idea motivates the following lemma which will be key to answering Question B.

**Lemma 8.** Let \(G\) be an infinite group. Let \(X\) be a subset of \(G\) such that there exists finite subsets \(F, H_1\) of \(G\) such that \(X = G \setminus H_1\). Then given a decomposition of \(X\) into two sets \(X = A \cup B\), either \(F \Delta(A) = G\) or there exists \(g \in G\) and a finite subset \(H_2\) of \(X\) such that \((gF \cup \{e\}) B = X \setminus H_2\).

**Proof.** Let \(F = \{f_1, \ldots, f_k\}\). If \(F \Delta(A) \neq G\), then there exists \(g \in G\), \(g \notin F \Delta(A)\), that is, \(\bigcup_{i=1}^{k} f_i^{-1} g \notin \Delta(A)\) for \(i = 1, \ldots, k\). Now, as before, for each \(i\) there are only finitely many \(h \in X\) such that both \(h\) and \(f_i^{-1} g h \in A\). Also we claim that there are only finitely many \(h\) such that none of the group elements \(f_1^{-1} g h, \ldots, f_k^{-1} g h\) are in \(X\). Indeed, since if \(F^{-1} g h \cap X = \emptyset\) then we have that \(g h \cap F X = g h \cap G \setminus H_1 = \emptyset\) and so \(g h \in g^{-1} H_1\).

Therefore we have that there exists some finite subset \(H_2\) of \(X\) such that for all \(h \in X \setminus H_2\) and for all \(i\), no pair \(f_i^{-1} g h\) are both in \(A\), and at least one of the group elements \(f_i^{-1} g h\) is in \(X\). Therefore we have that \(B \cup g^{-1} F B = X \setminus H_2\). \(\Box\)

We note at this point that this lemma allows us to settle the final part of Question E, whether or not a subset \(A\) of \(G\) which is not small, must be \(\Delta\)-large.

**Corollary 9.** Let \(G\) be an infinite group, \(A\) a subset of \(G\). Then if \(A\) is not small, then \(A\) is \(\Delta\)-large.

**Proof.** If \(A\) is not small then there exists a large set \(L\) such that \((G \setminus A) \cap L\) is not large. Without loss of generality we let assume that \(A \subset L\). Then let \(L = (L \setminus A) \cup A\), and there exists a finite subset \(F\) of \(G\) such that \(F L = G\). Therefore, by Lemma 8 if \(F \Delta(A) \neq G\), then there exists \(g \in G\) and a finite subset \(H_2\) of \(L\) such that \((gF \cup \{e\})(L \setminus A) = L \setminus H_2\). Therefore there is some finite subset \(H_3\) of \(G\) such that \(F(gF \cup \{e\})(L \setminus A) = G \setminus H_3\). However it is then clear that there exists some finite subset \(F'\) of \(G\) such that \(F'(L \setminus A) = G\), however by assumption \(L \setminus A\) was not large, and so \(F \Delta(A) = G\). Therefore \(A\) is \(\Delta\)-large. \(\Box\)

We can apply Lemma 8 inductively to prove the main result in the note.

**Proof of Theorem 7.** We induct on \(n\). The case \(n = 1\) follows from Lemma 4. Given that the result holds for all \(k < n\), and given a subset \(X\) of \(G\) and finite subsets \(F, H_1\) of \(G\) such that \(FX = G \setminus H_1\) and a decomposition \(X = A_1 \cup \ldots \cup A_n\). We have, by Lemma 8 that either \(F \Delta(A_1) = G\), or there exists \(g \in G\) and a finite subset \(H_2\) of \(G\) such that \((gF \cup \{e\})(A_2 \cup \ldots \cup A_n) = X \setminus H_2\). We apply the induction hypothesis to the set \(Y = (A_2 \cup \ldots \cup A_n)\). Note that there exists a finite subset \(H_3\) of \(G\) such that \(F(gF \cup \{e\})Y = G \setminus H_3\). Thus we have that there exists an \(i\) and a subset \(F'\) of \(G\) such that

\[
|F'| \leq |F|(|F| + 1)(|F|(|F| + 1) + 1)^2n-2-1 \leq |F|(|F| + 1)((|F| + 1)^2)^{2n-2-1}
\]

and \(F' \Delta(A_1) = G\). \(\Box\)

It is known [1] that given a decomposition \(G = A_1 \cup \ldots \cup A_n\), there exists an \(i\) and a subset \(F\) of \(G\) such that \(|F| \leq 2^{n-1}-1\) and \(F(A_iA_i^{-1}) = G\). As \(\Delta(A)\) is a subset of \(AA^{-1}\) the following Corollary strengthens this result.
**Corollary 10.** Let $G$ be an infinite group. Given a decomposition $G = A_1 \cup \ldots \cup A_n$ then there exists an $i$ and a subset $F$ of $G$ such that $|F| \leq 2^{|\omega^{n-1}|}$ and $F \Delta(A_i) = G$.

**Proof.** We apply Theorem 1 with $X = G$ and $F = \{e\}$ to get the result stated. $\square$

Theorem 1 says that if we decompose any large set into a finite number of pieces, at least one of the parts must be $\Delta$-large. However there do exist $\Delta$-large sets which decompose into two sets which are not $\Delta$-large.

**Example 11.** Consider the group $(\mathbb{Z}, +)$. Let $A = \{10^n : n \in \mathbb{N}\}$ and let $B = \{10^1 + 1\} \cup \{10^2 + 1\} \cup \{10^3 + 2\} \cup \{10^4 + 1\} \cup \{10^5 + 2\} \cup \{10^6 + 3\} \cup \{10^7 + 1\} \ldots$. Then if $X = A \cup B$ we see immediately that $\Delta(X) = \mathbb{Z}$, but $\Delta(A) = \Delta(B) = \phi$.

There also exist decompositions of large sets into a finite number of sets, none of which are large.

**Example 12.** Consider the free group on 2 elements, $F(a, b)$. If we denote by $aSb$ the set of reduced words in $F(a, b)$ that start with $a$ and end with $b$, then it is clear that $aSb$ is not large. Indeed no finite set of translates can contain the words $a^n$ for all $n$. However we can decompose $F(a, b)$ as

$$F(a, b) \setminus \{e\} = \bigcup_{x, y \in \{a, a^{-1}, b, b^{-1}\}} xSy,$$

none of which are large.

We now consider Question C. In [3] it is shown that for all infinite groups $G$, and all subsets $A$ of $G$ such that $A = A^{-1}$ and $e \in A$, there exists some subset $X$ of $G$ such that $\Delta(X) = A$. Using a similar construction we are able to prove Theorem 2.

**Proof of Theorem 2.** Let $X = \{x_1, x_2, \ldots\}$ and let $Z_0 = Y_0 = \phi$. We define a new sequence. For all $n \in \mathbb{N}$ let

$$w_i = x_{i, \frac{n(n - 1)}{2} + i} \text{ for } n(n - 1) + 1 \leq i \leq n(n + 1).$$

That is, $w_1 = x_1$, $w_2 = x_1, w_3 = x_2, w_4 = x_1, w_5 = x_2, w_6 = x_3, w_7 = x_1, \ldots$. Our plan is to add pairs of elements to $Y$ such that each pair has difference $w_i$, but introduces no new differences with the elements already in $Y$. Given $Y_{i-1} = \bigcup_{z = 0}^{n+1} Z_i$ we want to inductively find $Z_i = \{z_i, w_i, z_i\} \subset A$ such that $(Z_iY_{i-1}^{-1} \cup Y_{i-1}Z_i^{-1}) \cap Y_{i-1}Y_{i-1}^{-1} = \phi$. Equivalently $Z_i$ needs to avoid the finite set $Y_{i-1}Y_{i-1}^{-1}$. This is always possible since $w_i \in \Delta(A)$ and so the number of such pairs is infinite. We let $Y = \bigcup_{i=1}^{\infty} Y_i$ and see that $\Delta(Y) = X$ as claimed. $\square$

We note at this point that, perhaps surprisingly, it is necessary for the subset $X$ in Theorem 2 to be countable.

**Proposition 13.** There exists a group $G$, a subset $A$ of $G$, and a subset $Y$ of $\Delta(A)$ such that there does not exist any subset $X$ of $A$ with $\Delta(X) = Y$.

**Proof.** Let $\kappa = (2^{\mathfrak{c}})^+$ (so assuming CH we would have that $\kappa = \mathfrak{c}$), and let $\alpha$ be the initial ordinal of cardinality $\kappa$. Consider the group $G = (\mathbb{Z}_2)^{\kappa}$. That is, $G$ is the direct product of $\kappa$ copies of $\mathbb{Z}_2$. Let $x_i$, for $i \leq \alpha$, be the element which is 1 in the $i$th copy of $\mathbb{Z}_2$ and 0 elsewhere. Consider the set

$$A = \{x_n + x_i : 1 \leq n < \omega, \omega \leq i \leq \alpha\} \cup \{x_n : 1 \leq n < \omega\}.$$

We see that the set $X = \{x_i : \omega \leq i \leq \alpha\}$ is a subset of $\Delta(A)$, however we claim that there does not exist any subset $Y$ of $A$ such that $\Delta(Y) = X$. Indeed, suppose such a subset exists. Let $i$ be such that $\omega \leq i \leq \alpha$. Since $x_i \in \Delta(Y)$ we have that the set of $n < \omega$ such that both $x_n$ and $x_n + x_i$ are in $Y$ is infinite. Let us call this set $L_i$. Since $2^{\mathfrak{c}} < \kappa$ we must have that there exist $i, j \leq \alpha$ such that $L_i = L_j$. But then we also have that $x_i + x_j \in \Delta(Y)$, contradicting our initial assumption. $\square$

This phenomenon arises since in the definition of the function $\Delta$ we only require that the intersection $|gA \cap A|$ is infinite. If instead we were to consider a more general function

$$\Delta[G](A) = \{g \in G : |gA \cap A| = |G|\},$$
an analogous version of Theorem 3 could be proved, by the same argument, for sets of larger cardinality. Indeed much of the work in this note, including more general versions of Lemma 3 and Theorem 1 can be easily adapted to state results in this framework. Given Theorem 3 we can answer Question C, albeit in a slightly trivial way.

Corollary 14. Let $G$ be an infinite group, $G = A_1 \cup \ldots \cup A_n$, $A_i = A_i^{-1}$, $e \in A_i$ for $i \in \{1, \ldots, n\}$. Then there exists an $i$ and an infinite subset $X$ of $G$ such that $X \subseteq A_i$ and $\Delta(X) \subseteq A_i$.

Proof. By Theorem 3 we have that, within any infinite set $A$ there exists a subset $X$ of $A$ with $\Delta(X) = \{e\}$. At least one of the $A_i$ must be infinite, and therefore such an $X$ satisfies the statement.

Finally we turn to Question D.

Proof of Theorem 3. We will prove that every set which is not sparse is not $\nabla$-thin. Given $A \subseteq G$ which is not sparse then there exists some infinite subset $X$ of $G$ such that for every finite subset $F$ of $X \cap_{g \in F} gA$ is finite. In particular for every pair $g_i, g_j \in X$ we have that $|g_i A \cap g_j A| = \infty$ and so $X^{-1} X \subseteq \Delta(A)$.

However for an infinite set $X$ we claim that $X^{-1} X \subseteq \Delta(X^{-1} X)$. Indeed given $g_i^{-1} g_j \in X^{-1} X$ we have that

$$|g_i^{-1} g_j X^{-1} X \cap X^{-1} X| = |g_i X^{-1} X \cap g_j X^{-1} X| \geq |X \cap X| = |X| = \infty,$$

and so $g_i^{-1} g_j \in \Delta(X^{-1} X)$. Therefore if $A$ is not sparse we have that $X^{-1} X \subseteq \Delta^n(A)$ for all $n \in \mathbb{N}$, and so $\Delta^n(A) \neq \emptyset$ for any $n$.

However, there exist 2-sparse sets which are not $\nabla$-thin.

Example 15. Consider the group $(\mathbb{Z},+)$. Let

$$A = \{0\} \cup \{1\} \cup \{10\} \cup \{11\} \cup \{10^2\} \cup \{10^2 + 3\} \cup \{10^3\} \cup \{10^3 + 1\} \cup \{10^4\} \cup \{10^4 + 3\} \cup \{10^5\} \cup \{10^5 + 5\} \cup \{10^6\} \cup \{10^6 + 1\} \cup \ldots$$

Then we see that $\Delta(A) = \{0\} \cup \{2n + 1 : n \in \mathbb{Z}\}$, and so $\Delta^n(A) = \{2n : n \in \mathbb{Z}\}$ for all $n \geq 2$. However, given any infinite subset $X$ of $G$ we must have two numbers $a$ and $b$ in $X$ whose difference is even. Then $aA \cap bA$ is finite, since there are only a finite number of even differences in $A$ less than any given number.

Finally we mention an open problem. Corollary 10 says that if an infinite group $G$ is split into a finite number of sets, then one of those sets must be $\Delta$-large. For groups of larger cardinality can we prove similar results? For example:

**Question F.** Let $G$ be an infinite group, with $|G| = \kappa$. Given $\mu < \kappa$, $|I| = \mu$, and a decomposition $G = \bigcup_{i \in I} A_i$, does there exist an $i \in I$ and a finite subset $F$ of $G$ such that $F \Delta(A_i) = G$?

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