UNIFORM STABILITY OF THE INVERSE SPECTRAL PROBLEM FOR A
CONVOLUTION INTEGRO-DIFFERENTIAL OPERATOR

Sergey Buterin

Abstract. The operator of double differentiation, perturbed by the composition of the
differentiation operator and a convolution one, on a finite interval with Dirichlet boundary
conditions is considered. We obtain uniform stability of recovering the convolution kernel from
the spectrum in a weighted $L_2$-norm and in a weighted uniform norm. For this purpose, we
successively prove uniform stability of each step of the algorithm for solving this inverse problem
in both the norms. Besides justifying the numerical computations, the obtained results reveal
some essential difference from the classical inverse Sturm–Liouville problem.

Key words: integro-differential operator, convolution, inverse spectral problem, uniform
stability, nonlinear integral equation, uniform norm

2010 Mathematics Subject Classification: 34A55 45J05 47G20

1. Introduction

In this paper, we establish stability of the inverse spectral problem for one important and
illustrative class of integro-differential operators. As is known, this property is vital for justi-
fying numerical algorithms and has a local nature, since it guaranties that ”small” deviations
of any fixed input data caused by measurement or truncation errors may lead only to ”small”
deviations of the solution. However, the target type of stability is stronger than the usual local
one and belongs to the so-called uniform stability, which involves uniform estimates.

Inverse spectral problems consist in recovering operators from their spectral characteristics.
The most complete results in the inverse spectral theory are known for differential operators (see
monographs [1–3] and references therein). The first substantial study in this direction (after the
pioneering work of Ambarzumian [5] recently having given a 90-year anniversary to this
subject) was carried out by Borg [6], who proved that the real-valued potential $q(x) \in L_2(0, \pi)$
in the Sturm–Liouville equation

$$-y'' + q(x)y = \lambda y, \quad 0 < x < \pi,$$

is uniquely determined by specifying the spectra $\{\lambda_{k,j}\}, \, j = 0, 1,$ of two boundary value
problems $\mathcal{L}_j(q), \, j = 0, 1,$ for equation (1) with one common boundary condition, say:

$$y(0) = y^{(j)}(\pi) = 0,$$

respectively. For complex-valued potentials, i.e. in the non-selfadjoint case, this uniqueness
result was generalized by Karaseva [7]. It is well-known that the following asymptotics holds:

$$\lambda_{k,j} = \rho_{k,j}^2, \quad \rho_{k,j} = k - \frac{j}{2} + \frac{\omega}{2k} + \frac{\varkappa_{k,j}}{k}, \quad \{\varkappa_{k,j}\} \in l_2, \quad k \geq 1, \quad j = 0, 1. \quad (2)$$

Here $\omega = \frac{1}{\pi} \int_0^\pi q(x) \, dx$. Borg [6] also established local solvability and local stability of the
Corresponding inverse problem. Specifically, the following theorem holds (see also [3]).

Theorem 1. For any model real-valued potential $q(x) \in L_2(0, \pi)$ there exists $\delta > 0$ such
that if arbitrary real sequences $\{\hat{\lambda}_{k,j}\}_{k \geq 1}, \, j = 0, 1,$ satisfy the condition

$$\Omega := \sqrt{\sum_{k=1}^{\infty} \left( |\lambda_{k,0} - \hat{\lambda}_{k,0}|^2 + |\lambda_{k,1} - \hat{\lambda}_{k,1}|^2 \right) \leq \delta,}$$

1Department of Mathematics, Saratov State University, email: buterinsa@info.sgu.ru
then there exists a unique function \( \tilde{q}(x) \in L_2(0, \pi) \) such that \( \{\tilde{\lambda}_{k,j}\}_{k \geq 1}, j = 0,1, \) are the spectra of the problems \( L_j(\tilde{q}) \), \( j = 0,1 \), respectively. Moreover,

\[
\|q - \tilde{q}\|_{L_2(0,\pi)} \leq C \Omega,
\]

where \( C \) depends only on \( q(x) \).

The original proof of Theorem 1 is applicable also for complex-valued potentials, but with the requirement of simplicity of the spectra. In [3], Theorem 1 was generalized for arbitrary multiple spectra, i.e. it remains completely true after replacing all entries of ”real” with ”complex”. However, in the self-adjoint case or, equivalently, when the function \( q(x) \) is real-valued, Marchenko and Ostrovskii [9] proved global solvability of this inverse problem stated in the following theorem.

**Theorem 2.** Two sequences \( \{\lambda_{k,0}\}_{k \geq 1} \) and \( \{\lambda_{k,1}\}_{k \geq 1} \) are the spectra of the boundary value problems \( \mathcal{L}_0(q) \) and \( \mathcal{L}_1(q) \), respectively, with a common real-valued potential \( q(x) \in L_2(0,\pi) \) if and only if they are real, have the asymptotics [4] with a common \( \omega \) and interlace in the following way: \( \lambda_{k,1} < \lambda_{k,0} < \lambda_{k+1,1} \), \( k \geq 1 \).

Global solvability of the inverse problem inspires one to request its uniform stability, i.e. when in estimate (3) the potential \( q(x) \) is not fixed. The question of uniform stability was raised by Savchuk and Shkalikov in [10], where they established, in particular, that for unfixed both \( q(x) \) and \( \tilde{q}(x) \) the constant \( C \) in (3) can increase only due to the following causes:

I. Deviations of the spectra from those corresponding to the zero potential become large;

II. Square roots of neighboring eigenvalues in any pair of problems become too close.

Thus, the following refinement of the stability part in Theorem 1 follows from results of [10].

**Theorem 3.** For any \( r > 0 \) and \( h \in (0, 1/2) \) there exists \( C = C(r, h) > 0 \) (i.e. depending only on \( r \) and \( h \)) such that estimate (3) holds as soon as:

(I) \( \|\{\lambda_{k,j}\}_k\|_2 \leq r \) and \( \|\{\tilde{\lambda}_{k,j}\}_k\|_2 \leq r \) for \( j = 0,1 \) and \( |\omega| \leq r \); as well as

(II) \( h \leq \rho_{k,1} + h \leq \rho_{k,0} \leq \rho_{k+1,1} - h \) and \( h \leq \tilde{\rho}_{k,1} + h \leq \tilde{\rho}_{k,0} \leq \tilde{\rho}_{k+1,1} - h \) for \( k \geq 1 \).

Here \( \tilde{\rho}_{k,j} \) and \( \tilde{\lambda}_{k,j} \) are analogous to \( \rho_{k,j} \) and \( \lambda_{k,j} \), respectively, but for the potential \( \tilde{q}(x) \).

Note that \( \tilde{\omega} \neq \omega \) implies \( \Omega = \infty \). In [10], a different metric is used, which, in particular, admits different mean values \( \omega \) and \( \tilde{\omega} \). Moreover, in [10], a uniform two-sided generalization of estimate (3) was obtained for potentials from a continuous scale of Sobolev spaces with different smoothness indices. Further aspects of uniform stability for the inverse Sturm–Liouville problem were studied in [11], while local stability in the uniform norm was obtained in [3].

Restriction (II) in Theorem 3 can be partially explained by the fact that unbounded rapprochement of eigenvalues related to different problems closes the situation when the inverse problem loses its solvability. We note that for uniform stability of the inverse problem for the integro-differential operator considered below no similar restriction is necessary.

For integro-differential operators and other classes of nonlocal ones, the classical methods that give global solution of inverse problems for differential operators (the transformation operator method [13] and the method of spectral mappings [34]) do not work. Various aspects of the inverse spectral theory for integro-differential operators were studied in [12, 39] and other works. In particular, in [15], developing the idea of Borg’s method, Yuuro proved local solvability and local stability for the inverse problem of recovering a convolutional perturbation of the Sturm–Liouville operator from the spectrum. Later on, in [19] the author proved global solvability of this inverse problem by another method developing the approach suggested in [17], where, in turn, a boundary value problem \( \mathcal{L} = \mathcal{L}(M) \) of the form

\[
\ell y := -y'' + \int_0^x M(x-t)y'(t)dt = \lambda y, \quad 0 < x < \pi, \quad y(0) = y(\pi) = 0
\]

\( 2 \)
was considered with a complex-valued function $M(x) \in L_{2,\pi} := \{f(x) : (\pi - x)f(x) \in L_2(0, \pi)\}$, and the following inverse problem was studied:

**Inverse Problem 1.** Given the spectrum $\{\lambda_k\}$ of $L$, find the function $M(x)$.

Here we obtain uniform stability of Inverse Problem 1. Note that even local stability of this inverse problem does not follow from results of [15]. The operator $\ell$ is quite illustrative, because any results obtained for $\ell$ can usually be generalized to more complicated classes of integro-differential operators. The following theorem is a fusion of Theorems 1.1 and 1.3 in [17].

**Theorem 4.** (i) For an arbitrary sequence of complex numbers $\{\lambda_k\}$ to be the spectrum of a boundary value problem $L$ (which, in turn, is determined by its spectrum uniquely) it is necessary and sufficient to have the asymptotics

$$\lambda_k = \rho_k^2, \quad \rho_k = k + \varsigma_k, \quad \{\varsigma_k\} \in l_2, \quad k \geq 1. \quad (5)$$

(ii) The function $M(x)$ satisfies the additional smoothness condition: $M(x) \in W^1_2[0, T]$ for each $T \in (0, \pi)$ and $M'(x) \in L_{2,\pi}$ if and only if

$$\lambda_k = \left(k + \frac{A}{k} + \frac{\varsigma_{k-1}}{k}\right)^2, \quad \{\varsigma_{k,1}\} \in l_2, \quad A - \text{const.}$$

Moreover, $M(0) = 2A$.

Furthermore, the function $M(x)$ can be found by the following algorithm.

**Algorithm 1.** Let the spectrum $\{\lambda_k\}$ of a certain problem $L(M)$ be given.

1) Construct the function $w(x)$ using the formulae

$$w(x) = \frac{2}{\pi} \sum_{k=1}^{\infty} k \Delta(k^2) \sin kx, \quad \Delta(\lambda) = \pi \prod_{k=1}^{\infty} \frac{\lambda_k - \lambda}{k^2}; \quad (6)$$

2) Find the function $N(x)$ by solving the nonlinear integral equation

$$w(\pi - x) = \sum_{\nu=1}^{\infty} \frac{(\pi - x)^\nu}{\nu!} N^{*\nu}(x), \quad (7)$$

where $f^{*1}(x) = f(x)$ and $f^{*(\nu+1)}(x) = f * f^{*\nu}(x)$, $\nu \geq 1$, while $f * g(x) = \int_0^x f(x - t)g(t) \, dt$;

3) Calculate the function $M(x)$ by the formula

$$M(x) = 2N(x) - \int_0^x N^{*2}(t) \, dt, \quad 0 < x < \pi. \quad (8)$$

The first part of Theorem 4 also was announced in [16]. Historically, it was the first result giving necessary and sufficient conditions for solvability of an inverse spectral problems for an integro-differential operator. Equation (7) is called main nonlinear integral equation of the inverse problem. In [17] (see also [40]), its global solvability was established, i.e. for any function $w(x) \in L_2(0, \pi)$ equation (7) has a unique solution $N(x) \in L_{2,\pi}$, which has played a crucial role in the proof of Theorem 4. Later on, developing the approach in [17] allowed researches to obtain global solution also for other classes of integro-differential operators [19, 23–25, 27, 30, 33–39]. For different classes of operators, the corresponding main equations take different forms, which makes it necessary to provide the proof of their solvability in each new case. In order to make it more convenient, in [41] a general approach has been developed for solving nonlinear equations of this type by introducing some abstract equation and proving its global solvability. Moreover, in [41] uniform stability of such nonlinear equations was established. In [42], solvability of
equation (7) was established in the class of entire functions of exponential type as soon as so is the free term \( w(x) \) and \( w(0) = 0 \). This fact along with the stability give an algorithm for solving equation (7) in \( L_{2,\pi} \), which can easily be implemented numerically.

The main result of the present paper is the following theorem, which gives uniform stability of Inverse Problem 1 in the weighted \( L_2 \)-norm as well as in the weighted uniform norm:

\[
\| f \|_{2,\pi} := \| (\pi - x) f(x) \|_2, \quad \| f \|_{\infty,\pi} := \| (\pi - x) f(x) \|_\infty,
\]

where we denoted \( \| \cdot \|_2 := \| \cdot \|_{L^2(0,\pi)} \) and \( \| \cdot \|_{\infty} := \| \cdot \|_{L^\infty(0,\pi)} \), while the corresponding distances between the spectra are determined as follows:

\[
\Lambda(\{\lambda_n\}, \{\tilde{\lambda}_n\}) := \sum_{k=1}^\infty \frac{|\lambda_k - \tilde{\lambda}_k|^2}{k^2}, \quad \Lambda_1(\{\lambda_n\}, \{\tilde{\lambda}_n\}) := \sum_{k=1}^\infty \frac{|\lambda_k - \tilde{\lambda}_k|}{k},
\]

where \( \{\lambda_k\} \) and \( \{\tilde{\lambda}_k\} \) are the spectra of the problems \( L(M) \) and \( L(\tilde{M}) \).

**Theorem 5.** For any \( r > 0 \) there exists \( C_r > 0 \) such that

\[
\| M - \tilde{M} \|_{2,\pi} \leq C_r \Lambda(\{\lambda_n\}, \{\tilde{\lambda}_n\}), \quad \| M - \tilde{M} \|_{\infty,\pi} \leq C_r \Lambda_1(\{\lambda_n\}, \{\tilde{\lambda}_n\})
\]

(9) as soon as \( \Lambda(\{\lambda_n\}, \{n^2\}) \leq r \) and \( \Lambda(\{\tilde{\lambda}_n\}, \{n^2\}) \leq r \).

The second inequality in (9) as well as other inequalities below with possibly infinite right-hand side mean that if it is finite, then so is the left-hand side and the two are related as stated. Note that under the additional smoothness conditions on \( M(x) \) and \( M(x) \) stated in the second part of Theorem 4, the value \( \Lambda_1(\{\lambda_n\}, \{\tilde{\lambda}_n\}) \) is finite always when \( M(0) = \tilde{M}(0) \). The proof of Theorem 5 is based on Algorithm 1 and is a direct corollary of Lemmas 1–4 below, which give stability of its steps 1–3 in appropriate metrics.

The paper is organized as follows. In the next section, we provide Lemma 1, which gives stability of the first step in Algorithm 1. In Section 3, we prove stability of steps 2) and 3) (Lemmas 2, 3 and 4, respectively). Throughout the paper, one and the same symbol \( C_r \) denotes different positive constants in estimates, which depend only on \( r \).

### 2. Stability of the characteristic function kernel

The function \( \Delta(\lambda) \) involved in the first step of Algorithm 1 is called characteristic function of the problem \( L \), whose eigenvalues coincide with zeros of \( \Delta(\lambda) \) with account of multiplicity. Originally, it is determined as \( \Delta(\lambda) = S(\pi, \lambda) \), where \( y = S(x, \lambda) \) is a solution of the equation in (4) under the initial conditions \( S(0, \lambda) = 0 \) and \( S'(0, \lambda) = 1 \). In [17], the representation

\[
\Delta(\lambda) = \frac{\sin \rho \pi}{\rho} + \int_0^\pi w(x) \frac{\sin \rho x}{\rho} \, dx, \quad w(x) \in L_2(0, \pi), \quad \rho^2 = \lambda,
\]

(10)

was established, which, in turn, gives the asymptotics (5) as well as the second formula in (6). Moreover, for any sequence of complex numbers \( \{\lambda_k\} \) of the form (5), the function \( \Delta(\lambda) \) constructed as the infinite product in (6) has the form (10) with a certain function \( w(x) \in L_2(0, \pi) \), which, in turn, is determined by the Fourier series in (6) (see Lemma 3.3 in [17]).

Consider another sequence \( \{\tilde{\lambda}_k\} \) of the form (5) along with the corresponding functions

\[
\tilde{\Delta}(\lambda) = \pi \prod_{k=1}^\infty \frac{\tilde{\lambda}_k - \lambda}{k^2} = \frac{\sin \rho \pi}{\rho} + \int_0^\pi \tilde{w}(x) \frac{\sin \rho x}{\rho} \, dx, \quad \tilde{w}(x) \in L_2(0, \pi).
\]

(11)

The following lemma gives uniform stabilities in \( L_2 \)-metric and \( L_\infty \)-metric of recovering the kernel \( w(x) \) from zeros \( \{\lambda_k\} \) of the function \( \Delta(\lambda) \).
Lemma 1. For any $r > 0$, the following estimates hold:

$$
\|w - \tilde{w}\|_2 \leq C_r \Lambda(\{\lambda_n\}, \{\tilde{\lambda}_n\}) \quad \text{and} \quad \|w - \tilde{w}\|_\infty \leq C_r \Lambda_1(\{\lambda_n\}, \{\tilde{\lambda}_n\})
$$

(12)

as soon as $\Lambda(\{\lambda_n\}, \{n^2\}) \leq r$ and $\Lambda(\{\tilde{\lambda}_n\}, \{n^2\}) \leq r$. Moreover, $\Lambda_1(\{\lambda_n\}, \{\tilde{\lambda}_n\}) < \infty$ implies $w(x) - \tilde{w}(x) \in C[0, \pi]$.

Before proceeding directly to the proof of Lemma 1, we provide several auxiliary assertions. First of all, we prove the first estimate in (12) in the particular case when $\tilde{\lambda}_k = k^2$, $k \geq 1$.

Proposition 1. For any $r > 0$, the estimate $\|w\|_2 \leq C_r \Lambda(\{\lambda_n\}, \{n^2\})$ is fulfilled as soon as $\Lambda(\{\lambda_n\}, \{n^2\}) \leq r$.

Proof. According to (9) and Parseval’s equality, we calculate

$$
\|w\|_2 = \sqrt{\frac{2}{\pi} \sum_{k=1}^{\infty} |k\Delta(k^2)|^2}, \quad k\Delta(k^2) = \pi k \prod_{j=1}^{\infty} \left(\lambda_j - k^2 \right) \left(\frac{j}{j^2 - k^2}\right) = a_k b_k \frac{\lambda_k - k^2}{k}, \quad a_k = \prod_{j \neq k}^{\infty} \frac{\lambda_j - k^2}{j^2 - k^2},
$$

$$
b_k = \pi \prod_{j \neq k}^{\infty} \frac{j^2 - k^2}{j^2} = k^2 \lim_{\rho \to k} \frac{\sin \rho \pi}{\rho(k + \rho)(k - \rho)} = -\frac{\pi}{2} \lim_{\rho \to k} \cos \rho \pi = (-1)^{k+1} \frac{\pi}{2}.
$$

Thus, it remains to prove that $|a_k| \leq C_r$ uniformly as $\Lambda(\{\lambda_n\}, \{n^2\}) \leq r$. For this purpose, we represent $a_k = a_k, a_k, a_k$, where

$$
a_k = \prod_{|j-k| \geq 2r} \left(1 + \frac{\lambda_j - j^2}{j^2 - k^2}\right), \quad a_k = \prod_{0<|j-k|<2r} \left(1 + \frac{\lambda_j - j^2}{j^2 - k^2}\right).
$$

Since

$$
\left|\frac{\lambda_j - j^2}{j^2 - k^2}\right| = \left|\frac{\lambda_j - j^2}{j} \frac{j}{j - k(j + k)} \Lambda(\{\lambda_n\}, \{n^2\}) \right| \leq \frac{1}{2} \quad \text{if} \quad |j-k| \geq 2r,
$$

we get the estimate

$$
|a_{k,1}| = \left|\exp \left( \sum_{|j-k| \geq 2r} \ln \left(1 + \frac{\lambda_j - j^2}{j^2 - k^2}\right) \right) \right| \leq \exp \left(2 \sum_{|j-k| \geq 2r} \left|\frac{\lambda_j - j^2}{j^2 - k^2}\right| \right),
$$

where the Cauchy–Bunyakovsky–Schwarz inequality implies

$$
\sum_{|j-k| \geq 2r} \left|\frac{\lambda_j - j^2}{j^2 - k^2}\right| \leq \sqrt{\sum_{|j-k| \geq 2r} \left|\frac{\lambda_j - j^2}{j^2}\right|^2 \sum_{|j-k| \geq 2r} \frac{j^2}{(j^2 - k^2)^2}} \leq \sqrt{\sum_{j \neq k}^{\infty} \frac{j^2}{(j^2 - k^2)^2}},
$$

while

$$
\sum_{j \neq k}^{\infty} \frac{j^2}{(j^2 - k^2)^2} = \sum_{j \neq k}^{\infty} \frac{j^2}{(j-k)^2(j+k)^2} < \sum_{j=1}^{k-1} \frac{1}{(k-j)^2} + \sum_{j=k+1}^{\infty} \frac{1}{(j-k)^2} < 2 \sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{3}.
$$

Finally, we get

$$
|a_{k,2}| \leq \prod_{0<|j-k|<2r} \left(1 + \frac{rj}{|j-k|(j+k)}\right) < (1 + r)^{4r-2},
$$

which finishes the proof.

In the general case, the proof begins with the following assertion.
Proposition 2. There exists a choice of \( \{ \var_x k \} \) in (3) such that for any \( r > 0 \) the estimate \(|\var_x k| \leq r\) holds for all \( k \in \mathbb{N} \) as soon as \( \Lambda(\{\lambda_n\}, \{n^2\}) \leq r \).

Proof. Putting \( \varepsilon_k := (\lambda_k - k^2)/k \), we arrive at \(|\varepsilon_k| \leq \Lambda(\{\lambda_n\}, \{n^2\}) \leq r \). Further, we have \( \lambda_k = k^2 + k \varepsilon_k = (k + \var_x k)^2 \), where \( \var_x k = k(\sqrt{1 + \varepsilon_k/k} - 1) \) and \( \Re \sqrt{\gamma} \geq 0 \). Hence, \(|\var_x k| = |\varepsilon_k|/|\sqrt{1 + \varepsilon_k/k} + 1| \leq |\varepsilon_k| \leq r \).

In what follows, without loss of generality we assume that \( r \in \mathbb{N} \).

Proposition 3. For any \( r \in \mathbb{N} \), the estimate \(|\lambda_j - k^2| \geq 4jr\) holds as soon as \( |j - k| \geq 6r \) and \( \Lambda(\{\lambda_n\}, \{n^2\}) \leq r \).

Proof. According to (5) and Proposition 2 we have \(|\lambda_j - k^2| = |j - k + \var_x j||j + k - \var_x j| \geq (|j - k| - r)(j + k - r)\). Thus, it is sufficient to prove that

\[
(|j - k| - r)(j + k - r) \geq 4jr. \tag{13}
\]

Let \( j \geq k \). Then (13) is equivalent to the inequality \( j^2 - 6jr - k^2 + r^2 \geq 0 \), which is, obviously, fulfilled for \( j - k \geq 6r \), by virtue of the inequalities \( j \geq k + 6r > 3r + \sqrt{8r^2 + k^2} \).

Further, let \( k > j \). Then (13) is equivalent to the inequality \( j^2 + 4jr - (k - r)^2 \leq 0 \), which, in turn, holds for \( k - j \geq 5r \), because

\[
j \leq k - 5r = -2r + \sqrt{4r^2 - 4r(k - r) + (k - r)^2} < -2r + \sqrt{4r^2 + (k - r)^2}.
\]

Thus, for \( |j - k| \geq 6r \) inequality (13) is proven. \( \Box \)

For \( r, k \in \mathbb{N} \) we introduce the sets

\( \Omega_r(k) := \{ j : |j - k| < 6r, j \in \mathbb{N} \} \), \( \Omega'_r(k) := \Omega_r(k) \setminus \{k\} \), \( \Theta_r(k) := \{ j : |j - k| \geq 6r, j \in \mathbb{N} \} \).

Obviously, \( \Omega_r(k) \cup \Theta_r(k) = \mathbb{N} \). We also put \( \alpha_{r,k} := \# \Omega_r(k) = \min\{k, 6r\} + 6r - 1 \leq 12r - 1 \).

Denote

\[
\sigma_{r,k}(\lambda) := \prod_{j \in \Omega_r(k)} \frac{\lambda_j - \lambda}{j^2}, \quad \tilde{\sigma}_{r,k}(\lambda) := \prod_{j \in \Omega'_r(k)} \frac{\tilde{\lambda}_j - \lambda}{j^2}.
\]

Proposition 4. For any \( r \in \mathbb{N} \), the estimates

\[
|\sigma_{r,k}(k^2)| \leq \frac{C_r}{k^{\alpha_{r,k}}}|\lambda_k - k^2|, \quad |\sigma_{r,k}(k^2) - \tilde{\sigma}_{r,k}(k^2)| \leq \frac{C_r}{k^{\alpha_{r,k}}} \sum_{j \in \Omega_r(k)} \frac{|\lambda_j - \tilde{\lambda}_j|}{j}, \quad k \in \mathbb{N}, \tag{14}
\]

are fulfilled as soon as \( \Lambda(\{\lambda_n\}, \{n^2\}) \leq r \) and \( \Lambda(\{\tilde{\lambda}_n\}, \{n^2\}) \leq r \).

Proof. We have

\[
\sigma_{r,k}(k^2) = \frac{\lambda_k - k^2}{k^2} \cdot \prod_{j \in \Omega'_r(k)} \frac{\lambda_j - k^2}{j^2}. \tag{15}
\]

Since \( j \in \Omega_r(k) \) is equivalent to the inequalities \( \max\{0, k - 6r\} < j < k + 6r \), we have

\[
|\lambda_j - k^2| = |j^2 - k^2 + 2j \var_x j + \var_x j^2| < 6r(2k + 6r) + 2k + 6r)r + r^2 \leq C_r, \quad j \in \Omega_r(k), \quad k \in \mathbb{N}, \tag{16}
\]

and

\[
\frac{1}{j} \leq \frac{1}{\max\{0, k - 6r\} + 1} \leq \frac{6r}{k}, \quad j \in \Omega_r(k), \quad k \in \mathbb{N}. \tag{17}
\]

Substituting estimates (16) and (17) into (15), we get

\[
|\sigma_{r,k}(k^2)| \leq \frac{|\lambda_k - k^2|}{k^2} \cdot \left(\frac{36r^2C_r}{k}\right)^{\alpha_{r,k} - 1},
\]

6
which coincides with the first estimate in (14). Further, it is easy to check that

\[ \sigma_{r,k}(k^2) - \tilde{\sigma}_{r,k}(k^2) = \sum_{j \in \Theta_r(k)} \tilde{\sigma}_{r,k,j}(k^2) \frac{\lambda_j - \lambda_j}{j^2} \sigma_{r,k,j}(k^2), \]  

where

\[ \sigma_{r,k,j}(\lambda) = \prod_{\nu=j+1}^{k+6r-1} \frac{\lambda_{\nu} - \lambda}{\nu^2}, \quad \tilde{\sigma}_{r,k,j}(\lambda) = \prod_{\nu=\max\{0,k-6r\}+1}^{j-1} \frac{\lambda_{\nu} - \lambda}{\nu^2}. \]

According to (16) and (17), we have

\[ |\sigma_{r,k,j}(k^2)\tilde{\sigma}_{r,k,j}(k^2)| \leq \left( \frac{36r^2C}{k} \right)^{\alpha_{r,k}-1}, \]

which along with (17) and (18) give the second estimate in (14).

\[ \square \]

Denote

\[ \Delta_k(\lambda) := \frac{\Delta(\lambda)}{\sigma_{r,k}(\lambda)} = \pi \prod_{j \in \Theta_r(k)} \frac{\lambda_j - \lambda}{j^2}, \quad \tilde{\Delta}_k(\lambda) := \frac{\tilde{\Delta}(\lambda)}{\sigma_{r,k}(\lambda)} = \pi \prod_{j \in \Theta_r(k)} \frac{\tilde{\lambda}_j - \lambda}{j^2}. \]  

**Proposition 5.** For any \( r \in \mathbb{N} \), the estimate

\[ |\Delta_k(k^2)| \leq C_r k^{\alpha_{r,k}-1}, \quad k \in \mathbb{N}, \]  

holds as soon as \( \Lambda(\{\lambda_n\}, \{n^2\}) \leq r \).

**Proof.** Put \( \varepsilon_r := 8r + 1 \). Then, in particular, according to (5) and Proposition 2, we have \( |\rho_k - k| < \varepsilon_r \) for all \( k \in \mathbb{N} \). Hence, the maximum modulus principle gives

\[ |\Delta_k(k^2)| < \max_{|\rho - \rho_k| = \varepsilon_r} \left| \frac{\Delta(\rho^2)}{\sigma_{r,k}(\rho^2)} \right| = \max_{|\rho - \rho_k| = \varepsilon_r} \left| \Delta(\rho^2) \prod_{j \in \Theta_r(k)} \frac{j^2}{\lambda_j - \rho^2} \right|. \]  

(21)

By virtue of representation (10) and Proposition 1, we have the estimate

\[ |\Delta(\rho^2)| \leq \frac{A_{r,C}}{|\rho| + r + \varepsilon_r}, \quad |\text{Im}\rho| \leq C, \]

where \( A_{r,C} \) depends only on \( r \) and \( C \). The latter estimate holds also if \( |\rho - \rho_k| = \varepsilon_r \) for any \( k \in \mathbb{N} \), because in this case we have \( |\text{Im}\rho| \leq |\text{Im}\rho_k| + |\text{Im}(\rho - \rho_k)| \leq C := r + \varepsilon_r \). Furthermore, according to Proposition 2, for \( |\rho - \rho_k| = \varepsilon_r \) we have the estimate

\[ \frac{1}{|\rho| + r + \varepsilon_r} \leq \frac{1}{|\rho_k| - |\rho - \rho_k| + r + \varepsilon_r} = \frac{1}{|\rho_k| + r} \leq \frac{1}{k}, \quad k \in \mathbb{N}. \]

Thus, we get the estimate

\[ |\Delta(\rho^2)| \leq \frac{C_r}{k}, \quad |\rho - \rho_k| = \varepsilon_r, \quad k \in \mathbb{N}. \]  

(22)

Further, we have

\[ \prod_{j \in \Theta_r(k)} j^2 < (k + 6r)^{2\alpha_{r,k}} \leq C_r k^{2\alpha_{r,k}}, \quad k \in \mathbb{N}. \]  

(23)

Moreover, if \( |\rho - \rho_k| = \varepsilon_r \) and \( j \in \Theta_r(k) \), then we also have

\[ |\rho - \rho_j| \geq \varepsilon_r - |\rho_j - \rho_k| \geq \varepsilon_r - |j - k| - |x_j - x_k| > \varepsilon_r - 8r = 1, \quad k \geq 1, \]
\[ |\rho + \rho_j| \geq |\rho_j + \rho_k| - \varepsilon_r \geq j + k - 2r - \varepsilon_r > 2k - 8r - \varepsilon_r \]
\[ = 2k - 2\varepsilon_r + 1 = \left(2 - \frac{2\varepsilon_r - 1}{k}\right)k \geq \left(2 - \frac{2\varepsilon_r - 1}{\varepsilon_r}\right)k = k^2, \quad k \geq \varepsilon_r. \]
Hence, we have \(|\lambda_j - \rho^2| \geq k/\varepsilon_r\) as soon as \(|\rho - \rho_k| = \varepsilon_r, j \in \Omega_r(k)\) and \(k \geq \varepsilon_r\), which along with (21)–(23) give (20) for \(k \geq \varepsilon_r\). Further, for \(k = 1, \varepsilon_r - 1\) and \(j \in \Omega_r(k)\) we have
\[ j < k + 6r \leq 14r, \quad |\lambda_j - \rho^2| = |\rho - \rho_j||\rho + \rho_j| \geq (16r - j - r)^2 = r^2 \quad \text{for} \quad |\rho| = 16r. \]
Hence, for \(k < \varepsilon_r\), estimate (20) follows from the following rough estimate:
\[ |\Delta_k(k^2)| < \max_{|\rho|=16r} \left| \Delta(\rho^2) \prod_{j \in \Omega_r(k)} \frac{j^2}{\lambda_j - \rho^2} \right| \leq 14^{2r-2} \max_{|\rho|=16r} |\Delta(\rho^2)| \leq C_r, \quad k = 1, \varepsilon_r - 1. \]
Thus, we arrive at (20) for all \(k \in \mathbb{N}\).

Denote
\[ \theta_k := \sum_{j \in \Theta_r(k)} \frac{|\lambda_j - \lambda_j|}{\lambda_j - k^2}, \quad k \in \mathbb{N}. \]
The following proposition gives an estimate for the sequence \(\{\theta_k\}\) in the \(l_{\infty}\)-norm.

**Proposition 6.** For any \(r \in \mathbb{N}\), the estimate
\[ \sup_{k \in \mathbb{N}} \theta_k \leq C_r \Lambda(\{\lambda_n\}, \{\tilde{\lambda}_n\}) \]
is fulfilled as soon as \(\Lambda(\{\lambda_n\}, \{n^2\}) \leq r\).

**Proof.** The Cauchy–Bunyakovsky–Schwarz inequality gives
\[ \theta_k \leq \sqrt{\sum_{j \in \Theta_r(k)} \frac{j^2}{|\lambda_j - k^2|^2} \sum_{j \in \Theta_r(k)} \frac{|\lambda_j - \tilde{\lambda}_j|^2}{j^2}} \leq \alpha_k \Lambda(\{\lambda_n\}, \{\tilde{\lambda}_n\}), \quad \alpha_k = \sqrt{\sum_{j \in \Theta_r(k)} \frac{j^2}{|\lambda_j - k^2|^2}}. \]
Thus, it remains to prove that \(\alpha_k \leq C_r\) for \(k \in \mathbb{N}\). We have \(\alpha_k^2 = \eta_k + \zeta_k\), where \(\eta_k = 0\) for \(k \leq 6r\) and
\[ \eta_k = \sum_{j=1}^{k-6r} \frac{j^2}{|\lambda_j - k^2|^2} \leq \sum_{j=1}^{k-6r} \frac{j^2}{(\rho_j - k)^2(\rho_j + k)^2} \leq \sum_{j=1}^{k-6r} \frac{(k-6r)^2}{(k-j-r)^2(k+j-r)^2} \]
\[ < \sum_{j=1}^{k-6r} \frac{1}{(k-j-r)^2} = \sum_{j=5r}^{k-r} \frac{1}{j^2} < \frac{\pi^2}{6}, \quad k > 6r, \]
while
\[ \zeta_k = \sum_{j=k+6r}^{\infty} \frac{j^2}{|\lambda_j - k^2|^2} \leq \sum_{j=k+6r}^{\infty} \frac{j^2}{(j-k-r)^2(j+k-r)^2}, \quad k \in \mathbb{N}. \]
Since
\[ \frac{j}{j+k-r} \leq \frac{1}{1-\frac{r-1}{j+5r}} \leq \frac{1}{1-\frac{r-1}{1+6r}} = \frac{1+6r}{2+5r} < \frac{6}{5}, \quad j \geq 1+6r, \]
we arrive at
\[ \zeta_k \leq \frac{36}{25} \sum_{j=k+6r}^{\infty} \frac{1}{(j-k-r)^2} = \frac{36}{25} \sum_{j=1}^{\infty} \frac{1}{(j+5r-1)^2} < \frac{6\pi^2}{25}, \]
which finishes the proof. \(\square\)
Finally, we estimate the sequence \( \{\theta_k\} \) also in the \( l_2 \)-norm.

**Proposition 7.** For any \( r > 0 \), the estimate

\[
\sqrt{\sum_{k=1}^{\infty} \theta_k^2} \leq C_r \Lambda_1(\{\lambda_n\}, \{\tilde{\lambda}_n\})
\]

is fulfilled as soon as \( \Lambda(\{\lambda_n\}, \{n^2\}) \leq r \).

**Proof.** For \( k \in \mathbb{N} \), putting

\[
\beta_{k,j} := \left| \frac{\lambda_j - \tilde{\lambda}_j}{\lambda_j - k^2} \right|, \quad j \in \Theta_r(k), \quad \beta_{k,j} := 0, \quad j \in \Omega_r(k),
\]

and using the generalized Minkowski inequality, we get

\[
\sqrt{\sum_{k=1}^{\infty} \theta_k^2} = \sqrt{\sum_{k=1}^{\infty} \left( \sum_{j \in \Theta_r(k)} \left| \frac{\lambda_j - \tilde{\lambda}_j}{\lambda_j - k^2} \right| \right)^2} \leq \sum_{k=1}^{\infty} \left( \sum_{j=1}^{\infty} \beta_{k,j} \right)^2 \leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \beta_{k,j}^2.
\]

Thus, it is sufficient to prove that \( \gamma_j \leq C_r \) for \( j \in \mathbb{N} \). We have

\[
\gamma_j = \left| \frac{\lambda_j - \tilde{\lambda}_j}{\lambda_j - k^2} \right|, \quad \gamma_j = \left| \frac{\lambda_j - \tilde{\lambda}_j}{\lambda_j - k^2} \right|.
\]

and

\[
\mu_j = \sum_{k=j+6r}^{\infty} \frac{j^2}{|\lambda_j - k^2|^2} \leq \sum_{k=j+6r}^{\infty} \frac{j^2}{(|\rho_j| + k)^2(|\rho_j| - k)^2} \leq \frac{j^2}{(|\rho_j| + j + 6r)^2} \sum_{k=j+6r}^{\infty} \frac{1}{(|\rho_j| - k)^2}
\]

\[
< \sum_{k=j+6r}^{\infty} \frac{1}{(k-j-r)^2} = \sum_{k=1}^{\infty} \frac{1}{(k+5r-1)^2} < \frac{\pi^2}{6},
\]

while \( \xi_j = 0 \) for \( j \geq 6r \) and

\[
\xi_j = \sum_{k=1}^{j-6r} \frac{j^2}{|\lambda_j - k^2|^2} \leq \sum_{k=1}^{j-6r} \frac{j^2}{(|\lambda_j| - k^2)^2} = \sum_{k=1}^{j-6r} \frac{j^2}{(|\rho_j| + k)^2(|\rho_j| - k)^2}
\]

\[
\leq \frac{j^2}{|\lambda_j| + 1} \sum_{k=1}^{j-6r} \frac{1}{(j-k-r)^2} = \frac{j^2}{|\lambda_j| + 1} \sum_{k=1}^{j-6r} \frac{1}{k^2} < \frac{\pi^2}{6} \frac{j^2}{|\lambda_j| + 1}, \quad j > 6r.
\]

It remains to note that \( (|\lambda_j| + 1)^{-1}j^2 \leq r^2 \) for \( j \leq r \) and

\[
\frac{j^2}{|\lambda_j| + 1} < \frac{j^2}{(j-r)^2} = \frac{1}{(1 - \frac{r}{j})^2} \leq \frac{1}{(1 - \frac{r}{r+1})^2} = (r + 1)^2
\]

for \( j \geq r + 1 \). \( \square \)

Now we are in position to give the proof of Lemma 1.

**Proof of Lemma 1.** By virtue of (23) and (21) along with Parseval’s equality, we have

\[
\hat{w}(x) = \frac{2}{\pi} \sum_{k=1}^{\infty} k \hat{\Delta}(k^2) \sin kx, \quad \|\hat{w}\|_2 = \sqrt{\frac{2}{\pi} \sum_{k=1}^{\infty} |k \hat{\Delta}(k^2)|^2}, \quad (24)
\]
where \( \hat{w}(x) = w(x) - \tilde{w}(x) \) and \( \hat{\Delta}(\lambda) = \Delta(\lambda) - \bar{\Delta}(\lambda) \). According to (19), we arrive at
\[
\hat{\Delta}(k^2) = \Delta_k(k^2) - \sigma_{r,k}(k^2) \left( 1 - \frac{\bar{\Delta}_k(k^2)}{\Delta_k(k^2)} \right),
\]  
where
\[
\left| 1 - \frac{\bar{\Delta}_k(k^2)}{\Delta_k(k^2)} \right| = \left| 1 - \prod_{j \in \Theta_r(k)} \frac{\hat{\lambda}_j - k^2}{\lambda_j - k^2} \right| = \left| 1 - \exp \left( \sum_{j \in \Theta_r(k)} \ln \left( 1 - \frac{\lambda_j - \hat{\lambda}_j}{\lambda_j - k^2} \right) \right) \right|,
\]
where, in turn, by virtue of Proposition 3, we have
\[
\left| \frac{\lambda_j - \hat{\lambda}_j}{\lambda_j - k^2} \right| = \left( \frac{\lambda_j - \hat{\lambda}_j}{\lambda_j - k^2} \right) \leq \frac{\Lambda(\{\lambda_n\}, \{n^2\}) + \Lambda(\{\hat{\lambda}_n\}, \{n^2\})}{4r} \leq \frac{1}{2}, \quad j \in \Theta_r(k).
\]
Thus, we get
\[
\left| 1 - \frac{\bar{\Delta}_k(k^2)}{\Delta_k(k^2)} \right| \leq \sum_{\nu=1}^{\infty} \frac{2^\nu}{\nu!} \left( \sum_{j \in \Theta_r(k)} \left| \frac{\lambda_j - \hat{\lambda}_j}{\lambda_j - k^2} \right| \right)^\nu = \sum_{\nu=1}^{\infty} \frac{(2\theta_k)^\nu}{\nu!} \leq 2\theta_k \exp(2\theta_k),
\]
which along with (25) and Propositions 4 and 5 give
\[
\sum_{k=1}^{\infty} a_k \leq C_r \sum_{j \in \Omega_r(k)} \left( \frac{2}{j} \right) \frac{\left| \lambda_j - \hat{\lambda}_j \right|}{\lambda_j - k^2} + C_r \theta_k \exp(2\theta_k) \frac{\left| \lambda_k - k^2 \right|}{k}.
\]  
Since
\[
\left( \sum_{k=1}^{n} a_k \right)^2 \leq n \sum_{k=1}^{n} a_k^2, \quad \sum_{k=1}^{\infty} \sum_{j \in \Omega_r(k)} a_{k,j} = \sum_{k,j \in \mathbb{N}, \left| k-j \right| < 6r} a_{k,j}, \quad \sum_{k=1}^{\infty} \sum_{j \in \Omega_r(k)} a_{k,j} = \sum_{j=1}^{\infty} \sum_{k \in \Omega_r(j)} a_{k,j}
\]
for, in particular, any non-negative summands, we have
\[
\sqrt{\sum_{k=1}^{\infty} \left( \sum_{j \in \Omega_r(k)} \left| \frac{\lambda_j - \hat{\lambda}_j}{j} \right| \right)^2} \leq \sqrt{\sum_{k=1}^{\infty} \sum_{j \in \Omega_r(k)} \frac{\left| \lambda_j - \hat{\lambda}_j \right|^2}{j^2}} = \sqrt{\sum_{j=1}^{\infty} \frac{\left| \lambda_j - \hat{\lambda}_j \right|^2}{j^2} \sum_{k \in \Omega_r(j)} \alpha_{r,k}} \leq (12r - 1) \Lambda(\{\lambda_n\}, \{\lambda_n\}),
\]
which along with (26) and Proposition 6 give
\[
\sqrt{\sum_{k=1}^{\infty} \left| k \hat{\Delta}(k^2) \right|^2} \leq C_r \left( 12r - 1 + C_r \Lambda(\{\lambda_n\}, \{n^2\}) \right) \Lambda(\{\lambda_n\}, \{\hat{\lambda}_n\}).
\]
Taking the second equality in (24) into account, we arrive at the first estimate in (12).

Further, by virtue of the first equality in (24) along with (26) and the Cauchy–Bunyakovsky–Schwarz inequality, we get
\[
\| \hat{w} \| \leq \frac{2}{\pi} \sum_{k=1}^{\infty} k \left| k \hat{\Delta}(k^2) \right| \leq C_r \sum_{k=1}^{\infty} \sum_{j \in \Omega_r(k)} \left| \frac{\lambda_j - \hat{\lambda}_j}{j} \right| + C_r \Lambda(\{\lambda_n\}, \{n^2\}) \sum_{k=1}^{\infty} \frac{\theta_k^2}{k}.
\]
According to (27) and Proposition 7, we obtain
\[\sum_{k=1}^{\infty} k|\hat{\Delta}(k^2)| \leq C_r \sum_{j=1}^{\infty} \sum_{k \in \Omega_r(j)} |\lambda_j - \tilde{\lambda}_j| + C_r A_1(\{\lambda_n\}, \{\tilde{\lambda}_n\}) \leq 12r C_r A_1(\{\lambda_n\}, \{\tilde{\lambda}_n\}),\]
which along with (28) give the second estimate in (12). Finally, note that \(A_1(\{\lambda_n\}, \{\tilde{\lambda}_n\}) < \infty\) implies the uniform convergence of the series in (24), which gives \(\hat{w}(x) \in C[0, \pi]\).

3. Stability of steps 2) and 3) in Algorithm 1

Denote \(h(x) := (\pi - x)N(x)\) and consider the nonlinear operator
\[\mathcal{D}h(x) := \sum_{\nu=2}^{\infty} \frac{(\pi - x)^{\nu}}{\nu!} N^{*\nu}(x).\]
Then the main equation (7) takes the form \(w(\pi - x) = h(x) + \mathcal{D}h(x)\), while the operator \(\mathcal{D}\) belongs to the class \(\mathcal{E}_{r,1}\) in [11] and satisfies Condition \(A\) therein. Thus, the following lemma, giving uniform stability of equation (7) in the metric of \(L_{2,\pi}\), follows from Corollary 1 in [11].

**Lemma 2.** For any \(r > 0\) the estimate \(\|N - \tilde{N}\|_{2,\pi} \leq C_r \|w - \tilde{w}\|_2\) holds as soon as \(\|w\|_2 \leq r\) and \(\|\tilde{w}\|_2 \leq r\), where \(N(x)\) is the solution of equation (7), while \(\tilde{N}(x)\) is the one of the equation
\[\tilde{w}(\pi - x) = \sum_{\nu=1}^{\infty} \frac{(\pi - x)^{\nu}}{\nu!} \tilde{N}^{*\nu}(x).\]

The following lemma gives uniform stability of equation (7) in the weighted uniform norm.

**Lemma 3.** For any \(r > 0\) the estimate \(\|N - \tilde{N}\|_{\infty,\pi} \leq C_r \|w - \tilde{w}\|_\infty\) holds as soon as \(\|w\|_2 \leq r\) and \(\|\tilde{w}\|_2 \leq r\), where \(N(x)\) and \(\tilde{N}(x)\) are solutions of equations (7) and (29).

**Proof.** Subtracting (29) from (7), we get
\[\hat{w}(\pi - x) = \hat{h}(x) + \int_0^x K(x, t) \hat{h}(t) \, dt,\]
where \(\hat{w}(x) = w(x) - \tilde{w}(x)\), \(\hat{h}(x) = h(x) - \tilde{h}(x)\), \(\tilde{h}(x) = (\pi - x)\tilde{N}(x)\) and
\[K(x, t) = \frac{1}{\pi - t} \sum_{\nu=2}^{\infty} \frac{(\pi - x)^{\nu}}{\nu!} f_{\nu}(x - t), \quad f_{\nu} = N^{*\nu-1} + \sum_{j=1}^{\nu-2} N^{*j} \tilde{N}^{*(\nu-1-j)} + \tilde{N}^{*(\nu-1)}.\]
By virtue of Lemma 2, there exists \(r_0 > 0\) depending only on \(r\) such that \(\|w\|_2 \leq r\) implies \(\|h(x)\|_2 = \|N\|_{2,\pi} \leq r_0\). Let \(R(x, t)\) be the resolvent kernel for \(K(x, t)\), i.e.
\[\hat{h}(x) = \hat{w}(\pi - x) + \int_0^x R(x, t) \hat{w}(\pi - t) \, dt, \quad R(x, t) = -K(x, t) - \int_t^x K(x, \tau) R(\tau, t) \, d\tau.\]
Thus, since \(\|N - \tilde{N}\|_{\infty,\pi} = \|\hat{h}\|_\infty\), it is sufficient to prove the estimate
\[A_R \leq C_r, \quad \text{where} \quad A_G := \text{ess sup}_{0 < x < \pi} \sqrt{\int_0^x |G(x, t)|^2 \, dt}.\]

Indeed, the second equality in (30) implies the estimate
\[|R(x, t)| \leq |K(x, t)| + A_K \sqrt{\int_t^x |R(\tau, t)|^2 \, d\tau},\]
which implies $A_R \leq (1 + R)A_K$, where $R = \|R(x, t)\|_{L_2(0, \pi)^2}$. Moreover, by virtue of Lemma 1 in [31], we have $R \leq F(K)$, where $F(x) = x + \sum_{n=0}^{\infty} \frac{x^{n+2}}{\sqrt{n!}}$ and $K = \|K(x, t)\|_{L_2(0, \pi)^2}$. Thus, it remains to note that $K \leq \sqrt{\pi}A_K$ and

$$(\pi - x)^{\nu-1} |f_\nu(x - t)| \leq \nu(\pi - x)^{\nu-1} N_1^{\nu-1}(x - t) \leq \nu h_1^{\nu-1}(x - t),$$

where $N_1(x) = |N(x)| + |\tilde{N}(x)|$ and $h_1(x) = (\pi - x)N_1(x) = |h(x)| + |\tilde{h}(x)|$. Hence, we have $\|h_1\|_2 \leq 2r_0$ and arrive at the estimates

$$|K(x, t)| \leq \sum_{\nu=2}^{\infty} \frac{h_1^{\nu-1}(x - t)}{\nu!}, \quad A_K \leq \sum_{\nu=1}^{\infty} \frac{|h_1^{\nu}|}{\nu!} \leq \sum_{\nu=1}^{\infty} \frac{\pi^{\nu-1}\|h_1\|_2^{\nu}}{\nu!} \leq \frac{\exp(2r_0\sqrt{\pi}) - 1}{\sqrt{\pi}},$$

which finish the proof of (31).

The following lemma gives the uniform stabilities of the third step in Algorithm 1.

**Lemma 4.** For any $r > 0$ the following estimates hold:

$$\|M - \tilde{M}\|_{2, \pi} \leq C_r \|N - \tilde{N}\|_{2, \pi}, \quad \|M - \tilde{M}\|_{\infty, \pi} \leq C_r \|N - \tilde{N}\|_{\infty, \pi}$$

(32)

as soon as $\|N\|_{2, \pi} \leq \rho$ and $\|\tilde{N}\|_{2, \pi} \leq \rho$, where the function $M(x)$ is determined by $N(x)$ via formula (3), while $\tilde{M}(x)$ is determined by $\tilde{N}(x)$ via the analogous formula

$$\tilde{M}(x) = 2\tilde{N}(x) - \int_0^x \tilde{N}^2(t) \, dt.$$  

(33)

**Proof.** For briefness, we denote $\tilde{N}(x) := N(x) - \tilde{N}(x)$ and $\tilde{M}(x) := M(x) - \tilde{M}(x)$. Then, subtracting (33) from (3), we get

$$\tilde{M}(x) = 2\tilde{N}(x) - 1 * (N^2 - \tilde{N}^2)(x) = 2\tilde{N}(x) - 1 * f_2 * \tilde{N}(x),$$

(34)

where $f_2(x) = N(x) + \tilde{N}(x)$ and, hence, $\|f_2\|_{2, \pi} \leq 2r$. Thus, we arrive at the estimate

$$(\pi - x)|1 * f_2 * \tilde{N}(x)| \leq \int_0^x |f_2(t)| \, dt \int_0^{x-t} (\pi - \tau)|\tilde{N}(\tau)| \, d\tau$$

$$\leq \int_0^x |f_2(t)| \sqrt{\int_0^{x-t} (\pi - \tau)^2 |\tilde{N}(\tau)|^2 \, d\tau} \, \sqrt{x - \tau} \, dt \leq \|\tilde{N}\|_{2, \pi} \int_0^x |f_2(t)| \sqrt{x - \tau} \, dt.$$  

(35)

We also get

$$\int_0^\pi \left( \int_0^x |f_2(t)| \sqrt{x - \tau} \, dt \right)^2 \, dx \leq \int_0^\pi x \, dx \int_0^x |f_2(t)|^2 (x - t) \, dt$$

$$= \int_0^\pi |f_2(x)|^2 \, dx \int_x^\pi \tau(t - \tau) \, d\tau \leq \pi \int_0^\pi (\pi - x)|f_2(x)|^2 \, dx \int_x^\pi \tau \, dt = \pi \|f_2\|_{2, \pi}^2 \leq 4\pi r^2,$$

which along with (34) and (35) give the first estimate in (32) with $C_r = 2 + 2r\sqrt{\pi}$.

Further, by virtue of (34), we get the estimate

$$(\pi - x)|\tilde{M}(x)| \leq 2(\pi - x)|\tilde{N}(x)| + \|\tilde{N}\|_{\infty, \pi} \int_0^x (x - t)|f_2(t)| \, dt,$$

which gives $\|\tilde{M}\|_{\infty, \pi} \leq 2(1 + r\sqrt{\pi})\|\tilde{N}\|_{\infty, \pi}$ and finishes the proof.  

**Funding.** This work was supported by Grant 20-31-70005 of the Russian Foundation for Basic Research.
References

[1] Marchenko V.A. *Sturm–Liouville Operators and Their Applications*, Naukova Dumka, Kiev (1977) (Russian); English transl., Birkhauser, 1986.

[2] Levitan B.M. *Inverse Sturm–Liouville Problems*, Nauka, Moscow (1984) (Russian); English transl., VNU Sci. Press, Utrecht, 1987.

[3] Freiling G. and Yurko V. *Inverse Sturm–Liouville Problems and Their Applications*, Huntington, NY, Nova Science Publishers, 2001.

[4] Yurko V.A. *Method of Spectral Mappings in the Inverse Problem Theory*, Inverse and Ill-posed Problems Series, Utrecht, VSP, 2002.

[5] Ambarzumian V. Über eine Frage der Eigenwerttheorie, Z. Phys. 53 (1929) 690–695.

[6] Borg G. Eine Umkehrung der Sturm–Liouvilleleschen Eigenwertaufgabe, Acta Math. 78 (1946) 1–96.

[7] Karaseva T.M. On the inverse Sturm–Liouville problem for a non-Hermitian operator, Mat. Sbornik 32 (1953) no.74, 477–484. (Russian)

[8] Buterin S. and Kuznetsova M. On Borg’s method for non-selfadjoint Sturm–Liouville operators // Anal. Math. Phys. 9 (2019) no.4, 2133–2150.

[9] Marchenko V.A. and Ostrovskii I.V. A characterization of the spectrum of the Hill operator, Mat. Sb. 97 (1975), 540–606; English transl. in Math. USSR-Sb. 26 (1975), no.4, 493–554.

[10] Savchuk A.M. and Shkalikov A.A. Inverse problems for Sturm–Liouville operators with potentials in Sobolev spaces: Uniform stability, Funk. Anal. i ego Pril. 44 (2010) no.4, 34–53; English transl. in Funk. Anal. Appl. 44 (2010) no.4, 270–285.

[11] Savchuk A.M. and Shkalikov A.A. Recovering a potential of the Sturm–Liouville problem from finite sets of spectral data, Amer. Math. Soc. Transl. 233 (2014) 14pp.

[12] Malamud M.M. On some inverse problems, Boundary Value Problems of Mathematical Physics, Kiev, 1979, 116–124.

[13] Yurko V.A. Inverse problem for integro-differential operators of the first order, Functional Analysis, Ul’janovsk, 1984, 144–151.

[14] Eremin M.S. An inverse problem for a second-order integro-differential equation with a singularity, Diff. Uravn. 24 (1988) no.2, 350–351.

[15] Yurko V.A. An inverse problem for integro-differential operators, Mat. Zametki, 50 (1991), no.5, 134–146 (Russian); English transl. in Math. Notes 50 (1991), no. 5–6, 1188–1197.

[16] Buterin S.A. Recovering a convolution integro-differential operator from the spectrum, Matematika. Mekhanika, vol. 6, Saratov Univ., Saratov, 2004, pp. 15–18.

[17] Buterin S.A. On an inverse spectral problem for a convolution integro-differential operator, Results Math. 50 (2007) no.3-4, 173–181.

[18] Kuryshova Ju.V. Inverse spectral problem for integro-differential operators, Mat. Zametki 81 (2007) no.6, 855–866; English transl. in Math. Notes 81 (2007) no.6, 767–777.
Buterin S.A. On the reconstruction of a convolution perturbation of the Sturm–Liouville operator from the spectrum, Diff. Uravn. 46 (2010) no.1, 146–149 (Russian); English transl. in Diff. Eqns. 46 (2010) no.1, 150–154.

Kuryshova Yu.V. and Shieh C.-T. An inverse nodal problem for integro-differential operators, J. Inverse and Ill-Posed Problems 18 (2010) no.4, 357–369.

Wang Y. and Wei G. The uniqueness for Sturm–Liouville problems with aftereffect, Acta Math Sci. 32A (2012) no.6, 1171–1178.

Yurko V.A. An inverse spectral problems for integro-differential operators, Far East J. Math Sci. 92 (2014) no.2, 247–261.

Buterin S.A. and Choque Rivero A.E. On inverse problem for a convolution integro-differential operator with Robin boundary conditions, Appl. Math. Lett. 48 (2015) 150–155.

Buterin S.A. and Sat M. On the half inverse spectral problem for an integro-differential operator, Inverse Problems in Science and Engineering 25 (2017) no.10, 1508–1518.

Bondarenko N. and Buterin S. On recovering the Dirac operator with an integral delay from the spectrum, Results Math. 71 (2017) no.3-4, 1521–1529.

Yurko V.A. Inverse spectral problems for first order integro-differential operators, Boundary Value Probl. (2017) 2017:98, 1–7, https://doi.org/10.1186/s13661-017-0831-8

Bondarenko N. and Buterin S. An inverse spectral problem for integro-differential Dirac operators with general convolution kernels, Applicable Analysis (2018), 17pp. https://doi.org/10.1080/00036811.2018.1508653

Buterin S.A. On inverse spectral problems for first-order integro-differential operators with discontinuities, Appl. Math. Lett. 78 (2018), 65–71.

Buterin S.A. Inverse spectral problem for Sturm–Liouville integro-differential operators with discontinuity conditions, Sovr. Mat. Fundam. Napravl. 64 (2018) no.3, 427–458; Engl. transl. in J. Math. Sci. (to appear)

Bondarenko N.P. An inverse problem for an integro-differential operator on a star-shaped graph, Math. Meth. Appl. Sci. 41 (2018) no.4, 1697–1702.

Buterin S.A. and Vasiliev S.V. On uniqueness of recovering the convolution integro-differential operator from the spectrum of its non-smooth one-dimensional perturbation, Boundary Value Probl. (2018) 2018:55, 1–12, https://doi.org/10.1186/s13661-018-0974-2.

Zolotarev, V.A. Inverse spectral problem for the operators with non-local potential, Mathematische Nachrichten (2018), 1–21, DOI: https://doi.org/10.1002/mana.201700029.

Ignatiev M. On an inverse spectral problem for the convolution integro-differential operator of fractional order, Results Math. (2018) 73:34, 8pp.

Ignatiev M. On an inverse spectral problem for one integro-differential operator of fractional order, J. Inverse and Ill-posed Probl. 27 (2019) no.1, 17–23.

Bondarenko N.P. An inverse problem for the integro-differential Dirac system with partial information given on the convolution kernel, J. Inverse Ill-Posed Probl. 27 (2019) no.2, 151–157.
[36] Bondarenko N.P. An inverse problem for an integro-differential pencil with polynomial eigenparameter-dependence in the boundary condition, Anal. Math. Phys. 9 (2019) no.4, 2227–2236.

[37] Bondarenko N.P. An inverse problem for an integro-differential equation with a convolution kernel dependent on the spectral parameter, Results Math. (2019) 74:148, 1–7.

[38] Bondarenko N.P. An inverse problem for the second-order integro-differential pencil, Tamkang J. Math. 50 (2019) no.3, 223–231.

[39] Buterin S.A. An inverse spectral problem for Sturm–Liouville-type integro-differential operators with Robin boundary conditions, Tamkang J. Math. 50 (2019) no.3, 207–221.

[40] Buterin S.A. Inverse spectral reconstruction problem for the convolution operator perturbed by a one-dimensional operator, Matem. Zametki 80 (2006) no.5, 668–682 (Russian); English transl. in Math. Notes 80 (2006) no.5, 631–644.

[41] Buterin S. and Malyugina M. On global solvability and uniform stability of one nonlinear integral equation, Results Math. (2018) 73:117, 1–19.

[42] Buterin S.A. and Terekhin P.A. On solvability of one nonlinear integral equation in the class of analytic functions, Applied Mathematics Letters 96 (2019) 27–32.