Tunneling electro-conductance of atomic Bose condensates

V. M. Akulin, Yu. E. Lozzovik, I. E. Mazets, A. G. Rudavets, and A. Sarfati

1 Laboratoire Aimé Cotton, CNRS, Campus d’Orsay, 91405, Orsay, France
2 Institute of Spectroscopy, 142190, Moscow Reg. Troitsk, Russia
3 A.F. Ioffe Physics-Technical Institute, St. Petersburg 194021, Russia
4 Atom-Institut der Österreichischen Universität, TU Wien, A-1020 Vienna, Austria and
5 Moscow Institute of Physics and Technology, 141700 Dolgoprudny, Russia

(Date text June 30, 2008)

We consider interaction of an electron with a Bose condensate of atoms having electron affinity. Though states of the electron attached to atoms form a continuous band, tunneling through this band is strongly suppressed by quantum fluctuations of the condensate density. We adapt standard field theory methods originally developed for description of a particle propagating through a disordered potential and present an exactly soluble analytical model of the process. In contrast with the standard description, we take into account inelastic processes associated with quantum transitions in the condensate. Possibilities of the experimental observation of the phenomenon are discussed.

I. WHY ELECTRO-CONDUCTANCE OF ATOMIC CONDENSATES?

Of course a simple curiosity first of all. But in order to provide a deeper motivation, we review the milestones in the history of Bose condensation from the first days till present time, trying to convince readers that this question appears as a natural step in the development of this fundamental field of research.

In 1924 S. N. Bose [1] has calculated partition of the radiation field (Termodynamische Wahrscheinlichkeit für Strahlungsfeld) employing combinatorial formula for indistinguishable elements. Two weeks later Einstein [2] has generalized this approach to the case of material particles forming an ideal one-atom gases and has noticed that it yields a decrease of the number of particles at a given velocity as compared to the prediction of the Maxwell law. Six months later, Einstein had stated directly that Bose distribution below a certain temperature is incompatible with the conservation of the number of particle, unless one assumes that some fraction of particles is condensed in the state with zero momentum. These classical results were purely thermodynamical being based only on the notion of the size of phase volume per one quantum state. The De Broglie wave aspects of the condensation have not been discussed.

The fundamental aspects of the field theory required for description of atomic Bose-Einstein condensates, such as collective quantum states [3], weakly interacting quasiparticles [4], and the mean-field description [5, 6, 7], have been developed in the middle of XX-th century in the context of superfluidity. But only at the end of the century it become possible to address the condensation of one-atom gases experimentally [8, 9, 10].

For the radiation field, the situation turned out to be the opposite, – discovery of masers and lasers took place in years 50-th that is much earlier than understanding of a deep analogy between lasing from one hand side and phase transitions [11] from the other hand side, including the Bose condensation of photons [12] as a particular case of phase transitions. By now it became clear that the coherent laser radiation results from condensation of photons in the same mode of an optical resonator with an active media inside. But as in the case of Bose condensation of gases, the first explanations of lasing were also almost thermodynamical being based on the rate equation and relations between the Einstein’s kinetic coefficients for spontaneous and induced radiations. The quantum state aspects of the radiation field have not been addressed.

However, the rate equations alone cannot account for the coherence properties of the radiation, since they completely ignore the important phase relations among the quantum states of the field. The consistent quantum-mechanical consideration of lasing shows that the resulting quantum state of the electromagnetic field mode approaches a coherent Glauber state [13] with minimum energy-phase uncertainty, which goes to the classical limit [14] when the number of quanta in the mode increases. Therefore within the kinetic approach, the statement that the induced photons are all coherent, whatever it means, has been taken as a sort of axiom, whereas the classical description [13] of the electromagnetic field above the lasing threshold was considered as satisfactory for all practical purposes. Quantum nature of the radiation [15] was only addressed in the context of the radiation noises and the photon counting statistics [16].

Though the theory required for the description of atomic Bose condensates has been developed earlier, from the standpoint of quantum-classical correspondence it is constructed in the same way as the laser theory. Based on the Ginzburg-Landau approach to phase transitions [5, 18], it relies on the collective atomic amplitude satisfying Gross-Pitaevskii equation. This amplitude being a quantum object by itself, still should be considered as a classical field in the context of quantum variables of individual atoms, - following Dirac [19], the corresponding quantum field operators in the second quantization representation are replaced by a large classical field plus relatively small quantum fluctuations. By making use of the Bogolyubov transformation, these fluctuations are partially taken into account and result in a
Vlasov-like dispersion relation between the quasiparticles energies and the momenta, while the rest of the contribution accounting for quantum noises is usually ignored.

The collective amplitude approach is absolutely adequate for the present experimental situation, and traditionally it is employed for predictions and explanations of many fascinating results observed experimentally, such as condensate interference, sound, excited states, hydrodynamic condensate motion, vortices, vortices, vortices, and Bose condensate theory, one inevitably arrives at a point where consideration of the essentially quantum fluctuations becomes crucial. One of the examples is the condensation dynamics. According to the Liouville theorem, the condensation associated with a shrinking of the occupied phase volume cannot occur in a closed system – one needs to invoke a dissipation and hence fluctuations associated with the dissipation in virtue of the fluctuation-dissipation theorem. In the context of lasing this problem has been intensively studied in years 60-th and by now is well presented in textbooks. Recently a consistent description of the Bose condensation dynamics in the spirit of the laser theory has been developed.

In this work we consider another example, where quantum field fluctuations play the dominating role. It concerns an electron tunneling through the Bose condensate at the energy close to the electron affinity $E_a$ of the condensed atoms. We show that the electro-conductance measured in the presence of long-range interatomic interactions. Moreover, this amplitude satisfying the nonlinear Schrödinger equation, enforces the variety of possible effects that can be observed. Still, most of these phenomena remain classical from the viewpoint of the second quantization, since they do not address the essentially quantum-field-theoretical aspects of the atomic Bose condensates, such as quantum fluctuations.

Thermodynamical fluctuations have been considered at the very early stage of the development of Bose condensates theory, including their De Broglie wave aspects. Also the kinetic aspects of Bose condensates have already been considered. Moreover, following natural logics of the development of Bose condensate theory, one inevitably arrives at a point where consideration of the essentially quantum fluctuations becomes crucial. One of the examples is the condensation dynamics. According to the Liouville theorem, the condensation associated with a shrinking of the occupied phase volume cannot occur in a closed system – one needs to invoke a dissipation and hence fluctuations associated with the dissipation in virtue of the fluctuation-dissipation theorem. In the context of lasing this problem has been intensively studied in years 60-th and by now is well presented in textbooks. Recently a consistent description of the Bose condensation dynamics in the spirit of the laser theory has been developed.

We note that several questions about charged condensates have already been addressed. Transformations induced by the presence of atomic ions has been considered in Ref. The question of a highly efficient charge exchange in cold gases closely relevant to the tunneling conductance of condensates was the subject of Ref. Eigenstates and dynamics of an electrons trapped by an atomic Bose condensate has been considered in Ref. in the framework of the mean field approximation and the scattering length model of the atom-electron interaction. All these papers were focus on the case where the charge is completely localized inside the condensate. However since the polarization energy of atoms usually exceed considerably the condensate chemical potential, by putting a charge inside a cold gas one either destroys completely the condensed phase or, in the most optimistic scenario, triggers a long-lasting process of relaxation to a new metastable condensed configuration. In contrast, the tunneling conductance does not imply localization of the charge inside the condensate, and therefore it appears as the most gentle way of addressing the essentially quantum properties of condensates by means of tools being relevant to the case of an electron moving in a classical disordered potential have to be modified in order to include the case of random potentials created by the essentially quantum fluctuations. Indeed, according to the Born-Oppenheimer principle, the electron sees not the quantum averages of the atomic positions given by the mean field but their instantaneous values. The latter, when being in generic positions, form a disordered media for the tunneling electron. The averaging over the atomic quantum state should therefore be performed only for the final result of consideration of the electronic part of problem – for the probability of tunneling through the disordered media.

Drawing parallels with the laser physics, one can say in a very broad sense, that the discreetness of atoms plays a role similar to the shot noise of a laser beam photons registered by a radiation detector. However it turns out that the naive analog of antibunching in photon counting, namely the binary correlation of the atomic positions in condensate, is not the parameter that governs the tunneling conductance. The latter depends on the size of collective quantum density fluctuations. Focussing at the simplest case, which can be exactly solved analytically, we consider only the situation where the ensemble of condensed atoms is in a coherent state. From the second quantization point of view, this state is the direct analog of the Glauber state for the laser field with the average number $N$ of photons. An analog of spontaneous photons, the non-condensed fraction of the gas, is ignored. For solving the electronic part of the problem, we also employ the simplest version of $\sigma$-model, which corresponds to Gaussian unitary ensembles. We believe that in spite of such restrictive assumptions, this consideration still remains general enough to reveal the main qualitative effects associated with electro-conductance of Bose condensates.

We consider two tools that have emerged from this theory: the concept of tunneling transparency and the $\sigma$-model technique, although both these
an electrical interaction.

The paper is organized as follows. In Sect II we present a heuristic picture of the tunneling through a condensate, which not pretending to be rigorous gives an intuitive idea about the process. In Sect III we introduce the main formal tools required for consideration, the Hamiltonian, the initial conditions, and describe a representation most adequate to the case of the electron energy close to $E_a$, based on the quantum fields of atoms and a negative ion. In Sect IV we consider the process with the help of the mean-field approximation and determine the typical time and energy domains where the tunneling can occur. In Sect V we turn to the role of the quantum density fluctuations and show that they crucially modify the mean-field picture. Trying to make this section as simple for reading as possible, we have moved to Appendix XI the most important but rather technical part of the calculations based on the $\sigma$-model. In Sect VI we continue by considering the regime required for the observation of the tunneling and estimate the atomic density, the typical tunneling time, the typical current, and typical losses of the condensed atoms during the tunneling along with there energy dependence. It turns out that in order to clearly observe the phenomenon one needs to increase the densities of the atomic condensate by a few orders of magnitude with respect to ones experimentally available at the moment. We also discuss some requirements that the leads materials should satisfy in order to ensure efficient transport of electrons to the condensate. In Sect VII we conclude by discussing the main results obtained and the feasibility of experimental observation of the phenomenon.

II. A HEURISTIC DESCRIPTION

The tunneling regime implies that the wave function of the tunneling electron forms a wave packet of a size much larger then the size of the condensate, such that no considerable charge present at the atoms during the elementary act of tunneling. This is illustrated in Fig 1a. The electron wave packet comes from the right to the point $r_a$, where the right lead approaches the condensate, and mainly scatters back. Still a small part of the wave packet amplitude tunnels to the left point $r_b$, where the second lead approaches the condensate. The tunneling occur through a number of the negative ion states corresponding to the electron location near the positions $r_n$ of the condensate atoms. Each of the ionic states is formed as a result of the attraction of the electron by a short-range affinity potentials, which we chose in the form $E_a \delta(r-r_n)$. Penetration of the electron through the gas is a coherent process resulting from the interference of all possible tunneling trajectories.

We incorporate the well-known approach to the quantum tunneling electro-conductence through a disordered media resulting from the interference of many possible tunneling trajectories into the quantum consideration of the condensate. Briefly, the idea of the description is following. As the first step, for a fixed realization of random potential, one writes a formal expression

$$G_A(\{U(r)\}, \eta, r_a, r_b)G_R(\{U(r)\}, \eta, r_b, r_a)$$

in terms of advanced and retarded Green’s functions for the probability of tunneling of an electron at energy $\eta$ from an initial point $r_a$ to a final point $r_b$ through the domain of random potential $U(r)$, which includes interference of all possible tunneling trajectories. As the second step, one takes an average over all possible realization of the random potential given by the functional integral

$$\int G_A(U, \eta, r_a, r_b)G_R(U, \eta, r_b, r_a)W(U)DU(r)$$

with a functional weight $W(U)$ that gives the probability of realization of the potential $U(r)$. This allows one to get rid of unimportant details related to particular realizations and retain only the universal contribution of the trajectory interference. After such averaging, the problem can usually be traced analytically. In one- and two-dimensional setting, the tunneling is considerably affected by the phenomenon of strong and weak quantum localization, respectively, that result from the interference of self intersecting trajectories.

Tunneling of an electron through the Bose condensate must be very similar to the tunneling through a disordered potential

$$U(r) = \sum_{n=1}^{N} E_n \delta(r - r_n)$$

FIG. 1: Tunneling conductance of Bose condensate. A wave packet of the tunneling electron is much larger compared to the size of the condensate, such that the electron mainly remains within the leads and no charge present on atoms during the process. The Fermi energies of the leads $E_f$ are close to the electron affinity $E_a$ of the atoms. (a) Energy diagram of the leads and states corresponding to various locations of the negative ion. (b) Wave packet of the tunneling electron. (c) General view of the setting.
with a random distribution of all $N$ atomic positions $r_n$, although this process should have an important difference, – the average has to be performed not over an ensemble of the all possible realization of the random potential, but over the quantum distribution of the condensate atoms. If after the tunneling, the condensate remains in the initial quantum state, the analogy is complete, whereas the quantum atomic density distribution $\rho(\{r_n\})$ plays the role of weight function:

$$\int G_A(\{r_n\}, \eta, r_a, r_b)G_R(\{r_n\}, \eta, r_b, r_a)\rho(\{r_n\})d^N r.$$  

The situation gets much richer when one considers the tunneling accompanied by quantum transitions in the condensate. In this case, one has to develop an approach, which takes into account all possible final states of the condensate, including the outcomes that correspond to a partial or complete condensate destruction. In the framework of the Born-Oppenheimer separation of the electronic and atomic motions, the tunneling probability associated with the condensate transition from the initial state with the condensate wave function $\psi_{in}(\{r_n\})$ to the final state $\psi_{fn}(\{r_n\})$ reads

$$\int \psi_{fn}(\{r_n\})G_A(\{r_n\}, \eta, r_a, r_b)\psi_{in}^*(\{r_n\})d^N r \tag{3}$$

$$\times \int \psi_{fn}^*(\{r_n\})G_R(\{r_n\}, \eta, r_b, r_a)\psi_{in}(\{r_n\})d^N r,$$

and the summation over all possible final states yields the overall tunneling probability.

Let us consider a heuristic picture of such processes comparing several situations shown in Fig. 2. Let us assume that the condensate is initially in a pure quantum state given by the wave function $\psi_{in}(\{r_n\})$. Distribution of the atomic density in this state results from the superposition of the probability amplitudes of various atomic configurations, including the rare ones that correspond to high probability of the electron tunneling. In other words, for the condensate in a pure quantum state, the highly conducting configurations may appear as a result of unlikely quantum fluctuations of the atomic density. In Fig. 2a we show one of such unlikely configurations, – some of the condensate atoms form a one dimensional equidistant chain, which serves as a conducting trail connecting the points $r_a$ and $r_b$. Tunneling through such a chain results from the constructive interference of all electron trajectories going along the trail (one of them is shown in Fig. 2d)), and the overall tunneling time approximately equals to the time of tunneling between neighboring atoms multiplied by the number of atoms in the chain.

Evidently, the probability of an exactly equidistant chain is vanishing small, but it increases exponentially when we consider configurations where the atomic positions may deviate from the equidistant ones, as shown in Fig. 2b). However, such deviations destroy the constructive interference of the tunneling trajectories, and the chain conductance gets exponentially suppressed by the effect of Anderson localization. When the allowed non-equidistance in the chain increases, the effect of exponentially growing probability is compensated by the exponential decrease of its tunneling transparency, such that the net contributions of different configurations becomes comparable. To answer the question “which sort of the trail configurations gives the main contribution?”, one has to take into account more subtle dependences, and in particular the dependence on the spatial dimensionality of the problem. Moreover, one cannot exclude a priori the dominating contribution of numerous interfering “diffusive” trails, shown in Fig. 2c).

**FIG. 2:** Tunneling trails for various atomic configurations resulting from quantum density fluctuations in the initial quantum state of BEC. Interference of all the trajectories that go (forward and back) along the same trail yields the net trail conductance. (a) A rare highly conducting fluctuation: atoms are aligned in an equidistant one dimensional chain. (b) A more probable fluctuation: the tunneling chain is not equidistant. The conduction is thus suppressed by Anderson localization. (c) The most likely situation: many trails, each of which is weakly conducting, contribute to the overall conductance. (d) A trajectory can go forward and back along the trail.
given by the inverse Fourier transform of the Green’s function \[ \int dq \exp[iq \cdot \epsilon(q)] G_R(q, \epsilon(q, \eta, r, r_a)) \]. As the result of measurement, the atoms localized at the tunneling trail got extracted from the condensate, and the elementary act of tunneling thus implies evaporation of all the chain atoms. In contrast, in the opposite limit when the dominating contribution comes from interference of numerous long trajectories (Fig. 2c), the “which way” question remains without answer, and the correspondent condensate state reduction is much less pronounced. It can result in evaporation of just a few atoms per the tunneling event. In order to understand which scenario takes place in reality, one has to solve the problem analytically.

III. HAMILTONIAN AND THE INITIAL STATES

Let us start the analytic consideration of the condensate conductance by reviewing the relevant fundamentals of the quantum many-body theory of Bose liquids. The description rely on the quantum field theory, where the atomic variables are given in terms of the bosonic second quantization operators acting on the vacuum state \(|0\rangle\): the creation field operator \(\hat{\Phi}^+(r)\) and the annihilation field operator \(\hat{\Phi}(r)\). The atomic density at the point \(r\) corresponds to the mean value of the operator \(\hat{n}(r) = \hat{\Phi}^+(r) \hat{\Phi}(r)\). With the help of Fourier transformation one introduces the corresponding field operators \(\hat{\Phi}^+(p), \hat{\Phi}(p)\) and \(\hat{n}(p) = \hat{\Phi}^+(p) \hat{\Phi}(p)\) in the momentum representation. The Hamiltonian of the ensemble of cold atoms

\[ H_0 = \int \hat{\Phi}^+(p) \epsilon_p \hat{\Phi}(p) d^4p + \gamma \int \hat{\Phi}^+(r) \hat{\Phi}(r) \hat{\Phi}^+(r) \hat{\Phi}(r) d^4r \]

involves the single particle kinetic energy given by the dispersion law \(\epsilon_p\) and the local binary interaction characterized by a coupling constant \(\gamma = \frac{4\pi \hbar^2 a}{m}\) given in terms of the scattering length \(a\). Here \(M\) is the atomic mass, and \(\hbar\) is the Planck constant, which we set to unity hereafter unless the contrary is said explicitly.

In the limit of large number of particles \(N \to \infty\) one can employ the semiclassical approximation for the field variables assuming the condensate in a coherent field state 

\[ |\psi(\epsilon)\rangle = \exp \left\{ \int \left[ A(r) \hat{\Phi}^+(r) + A^*(r) \hat{\Phi}(r) \right] d^4r \right\} |0\rangle, \]

which results from the application of the field displacement operator

\[ \hat{U}_d(A(r)) = e^{i \int [A(r) \hat{\Phi}^+(r) + A^*(r) \hat{\Phi}(r)] d^4r}. \]

The variational principle \(\delta A(r) \langle A(r) | H_0 - E | A(r) \rangle = 0\) then yields the Gross-Pitaevskii equation

\[ EA(r) = \epsilon(\hat{p}) A(r) + \gamma |A(r)|^2 A(r) \]

for the displacement field eigen functions \(A(r)\). Here \(\epsilon(\hat{p})\) is the dispersion law \(\epsilon_p\), where the momentum is replaced by the momentum operator. In the presence of an external field the potential energy term has also to be included.

The lowest energy solution of the Gross-Pitaevskii equation gives the fundamental state of the condensate. However, in zero order Born-Oppenheimer approximation the excitations do not directly contribute to the condensate conductance as long as the main parts of atoms remain in the fundamental state. They may manifest themselves indirectly, via the mean density \(\langle n(r) \rangle\) and the mean fluctuation of the particle number \(\delta n = \left( \langle n^2(r) \rangle - \langle n(r) \rangle^2 \right)^{1/2}\). Therefore here, we ignore the excitations and consider the technically simplest case of a coherent initial state of the condensate

\[ |\text{in}\rangle = \exp \left\{ \int \left[ A \hat{\Phi}^+(r) + A^* \hat{\Phi}(r) \right] d^4r \right\} |0\rangle, \]

with a uniform mean density \(n = AA^*\) and Gaussian local fluctuations \(\delta n \propto n^{1/2}\). Moreover, remaining within the Born-Oppenheimer approximation, we will ignore dynamics of the Bose condensate during the electron tunneling process, and hence the initial state of the condensate Eq. (5) is the only ingredient required for the further consideration.

Note that the initial state Eq. (5) results from the displacement of the condensate vacuum state \(|0\rangle\) in the phase space of the second quantization, which is given by the displacement operator \(\hat{U}_d \equiv \hat{U}_d(A(r) = \text{const})\) of Eq. (4) with a spatially uniform displacement field \(A(r)\), whereas the vacuum state corresponds to zero average number of atoms and only allows for the quantum fluctuations of the atomic density.

Let us now turn to the description of the tunneling electron. The fermionic field creation \(\hat{\psi}^+(r)\) and annihilation \(\hat{\psi}(r)\) operators and their counterparts \(\hat{\psi}^+(p)\) and \(\hat{\psi}(p)\) in the momentum representation together with the potential energy Eq. (2) yield the electronic part of the Hamiltonian

\[ \hat{H}_e = \int \hat{\psi}^+(p) \frac{\hbar^2}{2m} \hat{\psi}(p) d^4p + \int \hat{\Phi}^+(r') \hat{\Phi}(r') \delta (r - r') \hat{\psi}^+(r) \hat{\psi}(r) d^4r d^4r' \]

which also includes interaction with the atoms. Here \(m\) is the electron mass. However, this form of the Hamiltonian is not very convenient for the description of the slow electron tunneling at the energies close to \(E_a\). In fact, the electron states relevant to the tunneling conductance correspond to the wave functions that are strongly localized near the positions of the condensate atoms, and therefore the mean values of both kinetic and the potential parts of Eq. (4) are large, and just their difference is of the order of the inverse tunneling time. In high energy physics this situation corresponds to the well-known positronium
problem. Acting by the analogy, we introduce the second
quantization operators of the negative ions

\[ \hat{\Psi}(r) = \hat{\Phi}(r) \int \varphi(r-r') \hat{\psi}(r') d^4r' \]

\[ \hat{\Psi}^+(r) = \hat{\Phi}^+(r) \int \varphi(r-r') \hat{\psi}^+(r') d^4r' \]

(7)

with the help of the wave function \( \varphi(r-r') \) of the electron
taken at an isolated atom. In this representation the Hamiltonian Eq.(6) shifted
by the electron affinity \( E_a \) takes the form

\[ \hat{H}_e = \int_V d^4r \int_V d^4r_1 \hat{\Psi}^+(r) \hat{\Phi}(r)V(r-r_1) \hat{\Psi}(r_1) \hat{\Phi}^+(r_1) \]

(8)

where \( V \) is the system volume. The interaction operator \( V(r) \) has the explicit
physical meaning of the electron tunneling probability amplitude between two
condensate atoms at a distance \( r \). This amplitude is given by the electron affinity
multiplied by the overlap of electron wave functions \( \varphi \) centered at different atoms.

It is convenient to work the momentum representation, where the field operators
are related to that in the coordinate representation by the Fourier
transformations

\[ \hat{\Psi}(p) = \frac{1}{(2\pi)^\frac{d}{2}} \int e^{i pr} \hat{\Psi}(r) d^d r \]

\[ \hat{\Psi}(p) = \frac{1}{(2\pi)^\frac{d}{2}} \int e^{-i pr} \hat{\Psi}(r) d^d r. \]

In this representation the interaction operator

\[ V(p) = \int d^d r e^{-i pr} V(r) \]

it has an explicit form

\[ V(p) = \frac{\kappa^2 \Lambda}{p^2 + \kappa^2}, \]

(9)

identical for all one, two, and three dimensional cases. Here \( \kappa = \sqrt{2mE_a} \) is a parameter in the asymptotic expression \( \exp(-\kappa r) \) of the atomic electron wave function
corresponding to the electron affinity \( E_a \), and \( \Lambda \sim 2E_a/d \) is a parameter related to the shape of the bound
electron wave function \( \varphi(r-r_n) \) near the neutral atom. Note that in the coordinate representation the kernel of
the interaction operator reads

\[ V(r) = \frac{1}{(2\pi)^d} \int e^{i pr} V(p) d^d p = \begin{cases} \Lambda \kappa^2 e^{-\kappa |r|} & \text{for 1D} \\ \frac{\Lambda \kappa^2}{\pi |r|} K_0(\kappa |r|) & \text{for 2D} \\ \frac{\Lambda \kappa^2}{\pi |r|^2} e^{-\kappa |r|} & \text{for 3D} \end{cases} \]

(10)

where \( K_0 \) is the Bessel function.

Numerical value of the parameters \( \Lambda \) and \( \kappa \) entering the interaction
Eq.(10) can be found either by direct calculation in the framework of the model of \( \delta \)-potential, or
by comparison of the calculated charge exchange cross
section with the experimentally observed values. The last option better takes into account the realistic
potential that binds the electron to the neutral atom. For Cs atoms with the electron affinity \( E_a = 0.39 eV \) in the
dimensional units one finds

\[ \kappa = \frac{\sqrt{2mdE_a}}{\hbar} = 2.26 \times 10^9 \text{[m}^{-1}] \].

(11)

The coupling constant \( \Lambda \) can be found from the value of
the velocity dependent resonant charge transfer cross section

\[ \sigma(v) = \int_0^\infty 2\pi r^2 \sin^2 \left\{ \int_0^\infty \frac{\Lambda \kappa^2}{\pi \hbar r^2 + v^2 t^2} e^{-\kappa \sqrt{r^2 + v^2 t^2}} dt \right\} dr, \]

(12)

which in the limit of high velocity \( v \) yields

\[ \sigma(v) = \frac{4\Lambda^2 \kappa^2}{\pi v^2 \hbar^2}. \]

(13)

By comparing this expression with the experimental data of Ref.\[68] \( \sigma = 500 \pi A^2 \) at the kinetic energy \( 1KeV \ (v = 3.8 \times 10^{5} m/s) \) one finds

\[ \Lambda = 3.9 \times 10^{-47} \left( \frac{m^5}{s^2} \right) kg, \]

and hence for the 3D case

\[ \frac{\Lambda \kappa^2}{\pi |r|} e^{-\kappa |r|} \approx 0.9 [eV] \exp \left( \frac{-2.26 \times 10^9 r \ [m]}{2.26 \times 10^9 r \ [m]} \right). \]

Note that the \( \delta \)-potential model yields smaller tunneling amplitudes, which we will employ later for the pessimistic estimations.

Hamiltonian Eq.(8) with the interaction Eq.(10) and the initial state of the condensate Eq.(5) have to be
complemented by the initial \( \hat{\Psi}^+(r_n) |0\>_i \) and the final \( \langle 0| \hat{\Psi}(r_k) \) states of the ion that correspond to the elementary act of the conduction, where \( |0\>_i \) is the ionic vacuum state. Here we consider only the case of a single
electron tunneling, leaving the problem of the Cooper pairs tunneling for future studies. Therefore the ion
density operator - \( \hat{\Psi}(r) \hat{\Psi}^+(r) \) and the atom density operator \( \hat{\Phi}(r) \hat{\Phi}^+(r) \) given in terms of the field operators satisfy the normalization conditions

\[ \langle 0| \int_V \hat{\Psi}(r) \hat{\Psi}^+(r) d^d r |0\>_i = 1 \]

(14)

\[ \langle 0| \int_V \hat{\Psi}(p) \hat{\Psi}^+(p) \frac{d^d p}{(2\pi)^d} |0\>_i = 1 \]

(15)

\[ \int_A \langle A(r) | \hat{\Phi}(r) \hat{\Phi}^+(r) | A(r) \rangle \ d^d r = N \]

(16)

\[ \int_A \langle A(p) | \hat{\Phi}(p) \hat{\Phi}^+(p) | A(p) \rangle \ \frac{d^d p}{(2\pi)^d} = N \]

(17)
IV. MEAN FIELD APPROXIMATION FOR THE TUNNELING DYNAMICS

We now consider the probability of the electron tunneling through the Bose condensate in the framework of the simplest mean field model. The results of these calculations are strictly speaking incorrect, since they get strongly modified when, in the next Section, we take into account quantum fluctuations of the condensate. Still the model is useful for identification of typical tunneling times and typical structures of the tunneling probability dependence on the electron energy. Though in the chosen regime of large electron wave packets (Fig. 1), the electron is never completely localized in the condensate, we will employ the dynamic formulation of the problem where the electron is initially localized at an atom near the point $r_a$ and, as a result of sequential tunneling from one atom to another, reaches an atom near the point $r_b$. This representation technically is much more convenient for certain calculations, while the results for the regime of large wave packets can be obtained by proper time convolutions. In other words, when considering the tunneling electron we neglect its Coulombic effect on the translational degrees of freedom of the condensate atoms.

The initial state of the system

$$|in\rangle = \hat{\Phi}(r_a)\hat{\Phi}^+(r_a)e^{i[\mathcal{A}\phi^+(r)+\mathcal{A}^*\phi(r)]d^4r}|0\rangle_{i}$$

(18)

reveals the fact that the electron by entering the condensate in the state Eq. (5) annihilates an atom in the point $r_a$ and creates an ion at its’ place. By the analogy, the final state of the system just before the electron leaves the condensate reads

$$\langle fn| = \langle 0|_i|0\rangle e^{-i\int [\mathcal{A}\phi^+(r)+\mathcal{A}^*\phi(r)]d^4r}\hat{\Phi}^+(r_b)\hat{\Phi}(r_b).$$

(19)

The time dependent tunneling probability amplitude given by the retardation Green’s function $G_R(t, r_a, r_b) = \langle fn| \exp\left\{-it\hat{H}\right\}|in\rangle$ has the explicit form

$$G_R(t, r_a, r_b) = \langle 0|_i|0\rangle e^{-i\int [\mathcal{A}\phi^+(r)+\mathcal{A}^*\phi(r)]d^4r}\hat{\Phi}^+(r_b)\hat{\Phi}(r_b)$$

$$\exp\left\{-it\int d^4r d^4r_1\hat{\Psi}^+(r_1)\phi(r)V(r-r_1)\hat{\psi}(r_1)\hat{\Phi}^+(r_1)\right\}$$

$$\hat{\Phi}(r_a)\hat{\Phi}^+(r_a)e^{i\int [\mathcal{A}\phi^+(r)+\mathcal{A}^*\phi(r)]d^4r}|0\rangle_{i}.$$\hspace{1cm} (20)

Now we employ the representation suggested by the unitary displacement operator $\hat{U}_d$ of Eq. (14), with which the allowance for the bosonic commutation relations $[\hat{\Phi}(r), \hat{\Phi}^+(r')] = \delta(r-r')$, results in the transformations

$$\hat{\Phi}^+(r') \rightarrow \hat{\Phi}^+(r') - iA^*$$

$$\hat{\Phi}(r') \rightarrow \hat{\Phi}(r') + iA$$

(21)

of condensate field operators. This yields

$$G_R(t) = \langle 0|_i|0\rangle \left(\hat{\Phi}^+(r_b) - iA^*\right)\hat{\Phi}(r_b)$$

$$\exp\left\{-it\int d^3r\phi^+(r)\phi(r)V(r-r_1)\hat{\psi}(r_1)\hat{\Phi}^+(r_1)\right\}$$

$$\left(\hat{\Phi}(r_a) + iA\right)\hat{\psi}(r_a)|0\rangle|0\rangle_i.$$\hspace{1cm} (22)

where the Green’s function arguments $r_a$ and $r_b$ are omitted.

The mean field approximation implies that one simply ignores the fluctuations and hence neglects the quantum field operators $\hat{\Phi}$ and $\hat{\Phi}^+$ acting on the vacuum state as compared to the average macroscopic amplitudes $A^* = A = \sqrt{n}$ given by the condensate average density $n$. The electron tunneling amplitude Eq. (22) in the momentum representation thus takes the form

$$G_R(t) = \langle 0|_i|0\rangle \hat{\Phi}(p)exp\left[ip(r_a - r_b)\right]$$

$$\exp\left\{-int\right\} \hat{\Psi}^+(p)V(p)\hat{\psi}(p)\right\}$$

$$\left(\hat{\Phi}(r_a) + iA\right)\hat{\psi}(r_a)|0\rangle.$$\hspace{1cm} (23)

Substituting the explicit expression Eq. (9) for $V(p)$ to Eq. (23), after straightforward calculations of the matrix element one obtains

$$G_R(t, R) = \frac{n}{(2\pi)^d} \int \exp\left[-\frac{it}{\hbar} + ipR\right] d^4p,$$\hspace{1cm} (24)

where $R = r_a - r_b$ and we have employed the natural units of the problem: the length unit $1/\kappa \sim 0.45 \times 10^{-9}m$ and the time unit $(2\pi)^d/h/n \sim 5 \times 10^{-8}s$. The numerical values are given for the case of 3D setting and for the condensate density $n \sim 10^{13}cm^{-3}$.

In the natural units, the density $n$ becomes a small dimensionless parameter $\sim 10^{-9}$ that equals to the expectation number of atoms in the volume $\kappa^3$. The Green’s function is thus proportional to $n$. This factor comes from the initial states and reflects a small probability amplitude to have the condensate atoms in the vicinities of the point electrodes. Evidently, this small pre-exponential factor increases considerably when one integrates over the leads surfaces: for the surface of the order of the condensate cross-section $S \sim 10^{-14}m^2$ the enhancement factor can reach $\sim 10^5$. One may interpret this factor by saying that the number of tunneling channels increases proportionally to the lead surface areas. Note that the transition probability scales as the condensate density square, which is a natural consequence of the condensate coherence resulting in the constructive interference of different channels.

The large parameter $R$ allows one to evaluate the integral Eq. (24) with the help of the stationary phase method. The saddle point is given by the conditions

$$R = -\frac{2tp_x}{(p_z^2 + 1)^2} \text{; } p_y = p_z = 0 \hspace{1cm} (25)$$

where $p_x, p_y, p_z$ are the components of the electron momentum. The pre-exponential factor becomes

$$\exp\left[-\frac{it}{\hbar} + ipR\right] \sim \exp\left[-i\frac{t}{\hbar} \sqrt{4tp_x^2 + 1}\right] \sim \exp\left[i\frac{t}{\hbar} \sqrt{4tp_x^2 + 1}\right].$$

This yields

$$G_R(t) \sim \frac{n}{(2\pi)^d} \int \exp\left[i\frac{t}{\hbar} \sqrt{4tp_x^2 + 1}\right] d^4p.$$
Alternatively, this integral can be evaluated numerically, with the allowance for the fact that the contribution from the asymptotic $p \to \infty$ corresponding to the Fourier transform of $\delta$ function has to be subtracted for numerical convergency. The dependence of $|G_R(t, R)|^2$ on scaled time and distance is shown in Fig. 3 for 1D. The left plot is done with the help of the saddle point approximation, whereas the right one is an exact calculations based on the representation of the integral in Eq. (24)

$$G_R(t, R) = n \sqrt{\frac{1}{2\pi R}} \sum_{k=0}^{\infty} \frac{(-itR/2)^k}{k!(k-1)!} K_{\frac{3}{2}-k}(R)$$

as a converging series of the Bessel functions $K_{\nu}(R)$, which results from casting of $\exp \left[\frac{-it}{(p^2 + 1)^{-1}}\right]$ in Taylor series followed by exact integration over $p$ of each term of the series. For other dimensions just a power of time enters the denominator of the pre-exponential factor. The saddle point conditions Eq. (25) help to understand why the tunneling starts at a time, which is linearly proportional to $R$: at the real axis $p$, the combination $-2p_x (p_x^2 + 1)^{-2}$ has the maximum $3\sqrt{3}/8$, and hence the saddle point locates at the real momentum axis only when the ratio $R/t$ is less than this number. Note that the contribution of only small $p$ yields the free particle propagator

$$G_R(t, R) = \frac{n}{(-4\pi it)^{d/2}} \exp \left[-it - \frac{R^2}{4t}\right].$$

As it has already been mentioned in the beginning of this Section, the time dependent approach is not directly relevant to case of large size of the tunneling wave packets, whereas the energy representation only has a physical meaning. By applying the Fourier transformation in time \( \int_0^{\infty} dt \exp[-i\epsilon t] \) to Eq. (24) and neglecting the singular terms at $R = 0$ we find the energy dependent transition probability amplitudes

$$G_R(\epsilon, R)_{1D} = \frac{i}{2\epsilon^2} \sqrt{\frac{\epsilon}{\epsilon + 1}} \exp \left(-R\sqrt{\frac{\epsilon + 1}{\epsilon}}\right)$$

$$G_R(\epsilon, R)_{2D} = \frac{i}{2\pi \epsilon^2} K_0 \left(R\sqrt{\frac{\epsilon + 1}{\epsilon}}\right)$$

$$G_R(\epsilon, R)_{3D} = \frac{i}{4\pi \epsilon^2} \exp \left(-R\sqrt{\frac{\epsilon + 1}{\epsilon}}\right),$$

which yield probabilities depicted in Fig. 4. Note that in the dimensional units, the point $\epsilon = 0$ corresponds to the electron affinity energy, and the point $\epsilon = -1$ is shifted with respect to this value by $-n\Lambda/(2\pi)^d$. The physically meaningful domain corresponds to $\epsilon$ close to $-1$ where the typical momentum is small as compared to $\kappa$, which is consistent with the free particle expression Eq. (27). Moreover, one finds a smaller cutoff momentum $p_c \sim n^{-d}$ for the electron by imposing the requirement that the phase space integral \( \int_0^{\infty} \int_{-\infty}^{\infty} \frac{1}{(2\pi)^d} \) equals the total number of the electron quantum states, that is the number of atoms $N$. Going beyond this momentum is inconsistent with the formfactor approximation Eq. (7).

Note that the correlation effects in the interatomic distance discussed in Ref. [63].

V. THE ROLE OF QUANTUM DENSITY FLUCTUATIONS

The mean field picture gets drastically changed when the quantum fluctuations of the condensate density are taken into account. The fluctuations cause scattering of the electron in the course of tunneling, while the energy and the momentum transfer associated with this scattering induce quantum transitions in the condensate. Analytical consideration of this regime is technically rather involved and rely on the functional integration over the regular and Grassmann fields. This method, known as $\sigma$-model, is widely employed for description of disordered
systems, and we just adapt it to for case of averaging over the condensate quantum state instead of the ensemble averaging, for which it has been originally developed.

We start with the technically more simple time-dependent version of Eq. (30) for the tunneling probability

\[ P(t, \tau, z) = \sum_{fn} \langle in | G_A(\tau) | fn \rangle e^{i\delta N} \langle fn | G_R(t) | in \rangle, \]

where \( \delta N \) denotes the difference in the occupation numbers between the initial \( | in \) and the final \( | fn \) states of the condensate. This function allows one to find several important physical quantities. In particular, the Fourier transform of the generating function

\[ P(\eta, z) = \int_{-\infty}^{\infty} P(t, \tau, z) \exp[-i\eta(t - \tau)] dt d\tau \]

at \( z = 0 \) equals to the net electron tunneling probability at the energy \( \eta \) with the allowance of all possible changes of the condensate quantum state induced by the tunneling. The derivative \( \partial_{\tau} P(\eta, z) \big|_{z=0} \) gives the number of atoms that have left the condensate as a result of the tunneling. The probability of tunneling with no condensate transition corresponds to \( P(\eta, -\infty) \).

The explicit form of the generating function Eq. (29) corresponding to the initial state of condensate Eq. (30) reads

\[ P(\theta, z) = \langle 0 | 00 | \hat{U}_d^+ \hat{\Psi}(r_a) \hat{\Phi}^+(r_a) e^{i\theta} \hat{R}_A \hat{\Phi}^+(r_b) \hat{\Psi}(r_b) e^{-i\theta} \hat{R}_R \hat{\Phi}^+(r_a) \hat{\Psi}(r_a) \hat{U}_d | 00 \rangle, \]

where \( \hat{U}_d \) is the displacement operator Eq. (4), \( \hat{N} = \int dr \hat{\Phi}^+(r) \hat{\Phi}(r) \) is the number of particles operator,

\[ \hat{R}_A = \int dr dr_1 \hat{\Phi}^+(r_1) \hat{\Phi}(r_1) V(r - r_1) \hat{\Phi}^+(r) \hat{\Psi}(r), \]

\[ \hat{R}_R = \int dr dr_1 \hat{\Phi}^+(r_1) \hat{\Phi}(r_1) V(r - r_1) \hat{\Phi}^+(r) \hat{\Psi}(r) \]

are Hamiltonians for the advanced and retarded Green’s functions respectively, field operators with bars correspond to the advanced Green’s function, and the vacuum state \( | 00 \rangle \) is the direct product of the vacuum states for \( \hat{\Psi} \) and \( \hat{\Phi} \).

In the first line of Eq. (30) one recognizes the combination \( \langle in | G_A(\tau) \) in the last line of this equation stands the combination \( G_R(t) | in \), while the middle line gives the expression for \( \sum_{fn} \langle fn | e^{i\delta N} \langle fn | \). Let us explain the physical meaning of the last combination in more detail. The initial coherent state of condensate is constructed by applying the displacement operator Eq. (4) to the condensate vacuum (no atoms only vacuum fluctuations). If during the electron tunneling no change of this state occur, the inverse displacement operator restores the vacuum and the operator \( \exp z \int \hat{\Phi}^+(r) \hat{\Phi}(r) dr \) makes no action. If a change occur, the inverse displacement does not restore the vacuum, but yields a state \( | x \) with the expectation value \( \langle x | \int \hat{\Phi}^+(r) \hat{\Phi}(r) dr | x \rangle \) equal to average number \( \delta N \) of the atoms extracted from the condensate. Hence, application of the operator \( \exp z \int \hat{\Phi}^+(r) \hat{\Phi}(r) dr \) yields the required factor \( e^{iz\delta N} \).

For description of the effect of atomic density fluctuations on the electron tunneling probability, one needs to separate the condensate and the electron motions. There is a standard way to do it with the help of functional integration. One represents the evolution operator for a bipartite system as a product of two commuting operators each of which acts on only one of the parts. Each of these operators depends however on a common function, a sort of external field, which serves as a variable for the subsequent functional integration. In this way one can first calculate the required matrix elements for the initial coherent condensate state of the atomic part and find as the result a weight functional for the electron motion. Thus obtained weight functional is an analog of the distribution functional \( W(U) \) in Eq. (11) of random potentials employed in the theory of disordered conductors, and therefore further calculations can be done by the analogy, – with the help of a technique known as the non-linear \( \sigma \)-model.

Details of these rather long analytic calculations are given in the Appendix Sect (X) The final expression for the generating function has an explicit form

\[ P(\zeta, \eta, z) = -2G_A \left( R \sqrt{\frac{\eta + i\eta_s}{2}} \right) G_R \left( R \sqrt{\frac{\eta - i\eta_s}{2}} \right) \]

\[ \text{Re} \int_{-1}^{1} dZ \int_{-\infty}^{\infty} dw e^{-N^2(1-e^{-z})^2 \epsilon^2 (Z^2 - w^2)^2 - i2N\zeta_0(Z - w)} \]

in terms of the electron energy \( \eta = \frac{\epsilon + \xi}{2} \) and the frequency \( \zeta = \frac{\epsilon - \xi}{2} \). Here \( \eta_s = \text{Im} \sqrt{\eta - \frac{1}{2}} \) is a parameter related to the electron states density profile, and Green’s functions Eq. (22) given explicitly in Eq. (31) in fact coincide with that given by Eq. (27) where the arguments \( \sqrt{\frac{\epsilon + \xi}{\epsilon}} \) are simply replaced by the combination \( i(\eta + i\eta_s)^{1/2}/\sqrt{2} \). Results for the net tunneling probability \( P(t \rightarrow \infty, \eta) = P(0, \eta, 0) \) and the number \( \delta N(t \rightarrow \infty, \eta) = P^*_0(0, \eta, 0) \) of atoms extracted from the condensate are depicted in Fig. (X) for 2D case along with the probability suggested by the mean field model. As it will now be explained, the dimensionality of the problem plays here just a marginal
role, and does not affect the conductance property in the domain, which gives the main contribution to the tunneling probability.

Let us give a qualitative picture of the tunneling process emerging from the analytical consideration. The mean field model considers atoms as uniformly distributed in space and relies on the atomic density smeared over the space. The tunneling electron mobility then results from the dispersion law Eq. (4), which close to the lower spectrum edge can be considered as the kinetic energy of a free particle with a large effective mass, whereas the potential energy is ignored. However within the Born-Oppenheimer approximation, the electron potential energy created by frozen atoms, on the contrary, is a strongly irregular function composed by δ-like spikes placed at generic atomic positions in space. Were these atoms be indeed fixed in space, the tunneling electron would experience the Anderson localization effect resulting from the quantum interference of many different multiple scattering trajectories. At the low-energy edge, the electron state density spectrum would correspond to short Anderson localization lengths, and the tunneling would be strongly suppressed. The difference between this case and the case under consideration is in the fact that the frozen atoms are not fixed, and even without changing their average space positions they still can change quantum states by changing their momenta as the result of the multiple electron scattering. This destroys the electron trajectory interference and the related influence of the dimensionality of the problem. It is therefore not surprising that the result can be found in the framework of zero-dimensional σ-model, while the role of the dimensionality is reduced to purely geometric factors. We note that in such a situation the Green’s functions keep almost the same form of the coordinate dependence just experiencing a shift of the energy scale and acquiring imaginary parts of the energy arguments corresponding to the rate of quantum transitions among the states of condensate.

Suppression of the Anderson localization for an electron tunneling through the condensate can also be viewed by analogy to the quantum molecular transitions. In a molecule with fixed atomic positions, the electronic energies take discrete values thus forming electronic terms, which can be seen as a consequence of the interference of electron trajectories. A weak external fields is unable to induce purely electronic transitions without involving the nuclear motion, unless the resonance conditions are satisfied, - the interference prevents electrons from changing of the quantum states. The resonances implies exact tuning of the field frequency, which for given positions of atoms can be achieved only for a specific choice a pair of the electronic states. However for the polyatomic molecules, when one takes into account the nuclear motion with a very dense vibrational spectrum, the resonance conditions can be matched for many electronic terms at once. In other words, the quantum interference of the purely electronic motion gets destroyed by the vibrations. The transition probability is then governed by the Frank-Condon factors resulting from the overlaps of the vibrational eigen functions corresponding to different electronic terms. For the condensate, the role of the Frank-Condon factors play the matrix elements Eq. (3), whereas the role of the vibrational energy numbers play the second quantization occupation numbers of the condensate modes.

Let us now compare and contrast the results of the mean field model and the 0D σ-model, illustrated in Fig.6. Near the bottom spectrum edge the density of states in the mean field approximation correspond to that of free particle, while the allowance of the quantum fluctuations results in a Wigner semicircular density distribution. In both cases the tunneling probability is proportional to the product of the coordinate dependent advance $G_A$ and and retarded $G_R$ Green’s functions. However, for the mean field model, they both are harmonic functions of the distance $r$ and correspond to the free particle motion, while the allowance for density fluctuations makes $G_A$ and $G_R$ exponentially decaying with $r$. In the mean field model, $G_A (\xi, r)$ and $G_R (\varepsilon, r)$ depending on different energy variables $\xi$ and $\varepsilon$, respectively, after the inverse Fourier transformation yield two wave packet propagators related by the complex conjugation condition $G_A (t, r) = G_R^*(t, r)$. The result of σ-model is different: $G_A (\eta, r)$ and $G_R (\eta, r)$ being exponentially decreasing complex conjugate functions of $r$ both depend on the same energy variable $\eta$, while the frequency dependence of the type $-2n^2 (g(\eta)\zeta^2)$ factors out and yields a linear time dependence as a common cofactor of the product $G_A G_R$. Here $g(\eta)$ is the density of states at the energy $\eta$ (see Eq. (15)). The situation thus resembles behavior of the tunneling evanescent waves under a potential barrier in the regular formulation of this quantum-mechanical
and always gives unity. However, the time at which the tunneling reaches the stationary value equals the Heisenberg time given by the state density \( g(\eta) \), and thus the smaller is the state density the shorter is the saturation time. This results is the consequence of the random matrix theory where all spectral properties and their time dependent counterparts are governed exclusively by the local density of states.

One has to emphasize an important qualitative difference between the electron moving in a disordered potential and the electron tunneling through the condensate. For the condensate, one of the channels contributing the electron transport should be avoided, if possible. This is the channel when the electron makes not a virtual but a real transition to some point in between the leads, which implies the electron charge localization in this point and hence provokes complete destruction of the condensate. It does not contribute to the process of tunneling. In contrast, this processes does give the dominating contribution to electronic diffusion in disordered potentials, and, evidently, is influenced by the space dimensionality. In other words, the tunneling regime requires the condensate size being of the order of the electron mean free path in the corresponding diffusion setting. Due to this reason, utilization of \( O \) dimensional \( \sigma \)-model for the description of the tunneling looks as a reasonable approach.

VI. THE ROLE OF ATOMIC AND THE LEAD MATERIAL PROPERTIES, AND THE TUNNELING EFFICIENCY IN NUMBERS

In order to discuss the possibility and requirements for an experimental observation of the electron tunneling through a condensate, we now present the typical probability and time scales of the tunneling process in the
dimensional units along with the related spectral characteristics. Since the objective is to describe the electron transition from one lead to the other, the corresponding probability amplitudes should be given in terms of electronic states, and hence the ion point-to-point Green's functions Eqs. (24,34) have to be projected to the lead electron wave functions $\varphi_1^*(r_b)$ and $\varphi_1(r_a)$ with the help of Eq. (4), which relates position $r$ of the negative ion with the wave function $\varphi(r-r')$ of the attached electron. The main contribution to the projection comes from the evanescent tails of $\varphi_1$ integrated over two volumes $dz_a dS_a$ and $dz_b dS_b$ near surface of the lead $a$ of area $S_a$ and near surface of the lead $b$ of area $S_b$, respectively.

Among all possible $\varphi_1$, the maximum overlap in the directions $z_{a,b} \perp S_{a,b}$ eventually comes from the electronic surface states $\varphi_{sl}(r_{a,b})$, while the overlap integrals

$$\phi(r_a) = \int d\mathbf{r}_a \varphi^*(r_a - r_a') \varphi_{sl}(r_a')$$

$$\phi^*(r_b) = \int d\mathbf{r}_b \varphi(r_b - r_b') \varphi_{sl}^*(r_b')$$

playing role of the initial and the final ionic wave functions are well-localized in the vicinity of the surfaces. If we assume a typical size of the evanescent tails of the order of $\kappa^{-1}$ and take the lead areas big enough $S_a \sim S_b \gg \kappa^{-1}$, which implies many condensate atoms on average located at a distance $\leq \kappa^{-1}$ from the surface, the pre-exponential factor $n$ in Eqs. (24,34) can be replaced by a large parameter $N_e = S\kappa^{-1}$ that has a physical meaning of the number of tunneling channels. Note that "on average" is understood here not in the statistical but in the quantum sense. Different tunneling channels therefore interfere, whereas the results of such an interference for 3D are given by the integrals

$$G_R = \int dS_a dS_b \frac{iN_e \phi^*(r_b) \phi(r_a)}{4\pi\kappa |r_a - r_b|} e^{-i(r_a - r_b)\kappa \sqrt{\varepsilon}}$$

$$G_R = \int dS_a dS_b \frac{N_e \phi^*(r_b) \phi(r_a)}{4\pi\kappa |r_a - r_b|} e^{-i(r_a - r_b)\kappa \left(\frac{1}{2} - i\frac{\varepsilon}{\kappa^2}\right)}$$

corresponding to the mean field and the $\sigma$ models, respectively. The amplitudes $\phi$ are now defined at the surfaces and are normalized by the condition $\int \phi^* \phi dS = 1$, while the energies $\varepsilon$ and $\eta$ are chosen close to the corresponding condensate conducting band edges. One can move the big distance $R = |r_a - r_b|$ in front of the integrals and obtain

$$G_R(\varepsilon, R) = iN_e \frac{\phi^* (\varepsilon) \phi (\varepsilon)}{4\pi\kappa R}$$

$$G_R(\eta, R) = N_e \frac{\phi^* (\eta) \phi (\eta)}{4\pi\kappa R} e^{-\frac{\eta}{\kappa^2} R},$$

where the surface functions are now in the momentum representation.

One sees that the tunneling probability depends on the surface electron wave functions that, in turn, are governed by the band structure of the bulk material. For a broad-band semiconductor with a low electric polarizability, the surface electron may form a Rydberg-like band spectra bounded to the surface with an energy $E_s$ due to the presence of the weak image force. For a small $E_s$, the wave functions corresponding to these surface bands are rather extended in the direction normal to the surface. In this case, the number of the channels involved in tunneling yet increases. However, it may require compensation for eventual energy mismatches between the band energies and the atomic affinity, which in principle can be done by a proper periodic external perturbation. One can also argue that the materials with low electric susceptibility have less chances to destroy the condensate by the Van der Waals interaction.

Detailed consideration of the optimum choice of the material is a separate task deserving a special consideration. Here we will not dwell on this interesting issue and simply take the factors $\phi$ in Eq. (33) of the order of unity. Still it worth mentioning that different models address different types of the surface wave functions. The mean field model by considering the tunneling electron as a free propagating particle of a large mass imposes the phase matching conditions between the plane waves of the tunneling and the surface electrons. Due to the difference in effective masses of these electrons, the conditions are fulfilled near the bottoms of the relevant bands, – the tunneling band and the surface band corresponding to small momenta. The bottoms of the bands thus have to coincide in this case. Since now the energy only enters the arguments of $\phi$, the tunneling dynamics is governed by the shape of the surface state density profile near zero momenta. In contrast, the $\sigma$-model for the fluctuations results in irregular localized eigenfunctions of the tunneling band that are correlated only at the atomic scale of sizes, and therefore the main contribution comes from the surface states with the momenta of the order of $\kappa$. The time dependence is formed by the tunneling band level statistics. Note that in order to be consistent within the tunneling regime approximation, in both cases we have to consider the tunneling times as short compared to the duration of the surface electron wave packet.

Let us now estimate the numerical values for the tunneling probability and the typical tunneling time at the example of Cs. For typical dimensionless energies $\eta \sim 1/R$, both models yield

$$P \simeq \left( \frac{N_e}{4\pi\kappa R} \right)^2 \simeq \left( \frac{S\kappa}{4\pi\kappa^2 R} \right)^2 \left( \frac{S\kappa}{4\pi\kappa^2 R} \right)^2$$

that for $\kappa = 2.26 \times 10^9 \text{m}^{-1}$, $n \sim 10^{13} \text{cm}^{-3}$, $S \sim 10^{-14} \text{m}^2$, $R \sim 10^{-7} \text{m}$ amounts to

$$P \simeq 2.4 \times 10^{-16}.$$
lead. The tunneling time corresponding to this density is definitely superior to the time of tunneling between two atoms at a distance $n^{-1/3}$, which for $n \sim 10^{15} \text{cm}^{-3}$ amounts to a completely unrealistic number

$$T_{iat} = \frac{E_a}{\hbar} e^{-\kappa n^{-1/3}} \simeq 10^{41} \text{sec}.$$

(35)

Therefore at the condensate densities $n \sim 10^{15} \text{cm}^{-3}$ experimentally accessible at the moment, the tunneling electro-conductance is absolutely impossible.

However, $T_{iat}$ exponentially decreases with the interatomic distance and gets in the reasonable domain only for the densities $n \sim 10^{19} \text{cm}^{-3}$ corresponding to the interatomic distance of 100 Å, where one finds the time

$$T_{iat} \simeq 7 \times 10^{-5} \text{sec}$$

and the correspondent tunneling probability

$$P \simeq 2.4 \times 10^{-6}.$$ 

Therefore in the regime of dense condensate with interatomic distance of the order of the scattering length $a \sim 50 \text{Å}$, the tunneling approaches a realistic limit. In the mean-field approximation, the typical time for tunneling through all the condensate volume is of the order of $\kappa R T_{iat} \simeq 16 \text{msec}$. In order to get an idea about the corresponding tunneling current, one has to multiply the probability by the electron charge, and divide by the typical tunneling time, which again yields a small number $J = 2.3 \times 10^{-23} \text{A}$ that can be increased up to $4 \times 10^{-18} \text{A}$ by increasing the atomic density up to the limit $n \sim 10^{19} \text{cm}^{-3}$ suggested by the scattering length. Evidently, this regime remains outside the limits of experimental feasibility and still implies an individual electron counting. Unfortunately such densities are unrealistic due to three-body recombination and dimers formation the condensate gets destroyed at a much shorter time scale and at much smaller densities.

Still the situation is not completely hopeless, since the electron affinity enters the exponent in Eq.35. Therefore, the tunneling probability and the corresponding current increase drastically for the atoms with low electronic affinity, such as Ca, where $E_a \simeq 20 \text{meV}$ [69,70].

At the same atomic density, the current is of the order of $J \approx 6 \times 10^{-2} \text{pA}$, the tunneling time $\kappa R T_{iat} \simeq 6 \times 10^{-12} \text{sec}$, and the tunneling probability $P \sim 10^{-1}$, which can be considered as the most optimistic estimation of the tunneling process efficiency. Even at the condensate density $n \sim 10^{15} \text{cm}^{-3}$, one finds $P \sim 10^{-9}$, while the tunneling time remains at the observation limit $\kappa R T_{iat} \simeq 10^{-2} \text{sec}$. Moreover, estimations show that at this densities the three-body recombination rate $K_3 n^2$ with $K_3 \sim 0.4 \times 10^{-3} \text{cm}^3/\text{sec}$ is of the order of 0.04 sec$^{-1}$, which is still small enough to be ignored. However, condensation of rare-earth gases is a much more challenging task, although the possibility to control the scattering length in a wide region $a \sim 10^3 - 10^4 \text{Å}$ by making use of

Feichbach resonances has been studied theoretically [71] and addressed experimentally [72].

Let us now turn to the spectral properties of the tunneling process, where the difference between the mean field and the $\sigma$-model is the most pronounced. From the typical size of the tunneling time one sees that the spectral characteristics manifest themselves in the radio frequency domain, in other words they correspond to the energy scale of $10^{-9} - 10^{-12} \text{eV}$, which practically excludes the experimental studies based on the naive electric circuit setting shown in Fig.11. However, it becomes possible to employ high-Q sources of electromagnetic radiation at the frequency domain required for the compensation of the difference between the surface band energy and the electron affinity of atoms and emulate in this way small voltage differences applied to the leads, as it is illustrated in Fig.11. This setting allows one to trace the spectral dependence of the tunneling probability and thus to distinguish between the mean field and the $\sigma$-model predictions.

According to Eqs.(33), the tunneling probability for the mean field model is energy independent as long as we remain near the lowest edge of the tunneling band and discard the effect of the surface band structure. Allowance for the density fluctuations results in an exponential root energy dependence

$$P(\eta, R) = \frac{1}{N} \sqrt{2\eta} \left( \frac{N_e}{4\pi \kappa R} \right)^2 e^{-2\sqrt{2\eta R \kappa}}$$

which does not depend on the frequency $\zeta$, thus it is not associated with the inverse of a tunneling time and can therefore be interpreted as an inhomogeneous spectral line. The line width $\delta \nu \simeq \delta \eta/T_{iat}$ strongly depends on the electron affinity energy and on the condensate density.
via the dependence of $T_{\text{tot}}$ on these quantities. Numerical values of $\delta \nu$ estimated for Ca and Cs are depicted in Fig.10 which shows that the width $\delta \nu$ is in the domain of a typical diode laser linewidth for calcium already at $n \sim 10^{16}$ cm$^{-3}$. Note that the tunneling time $\sim N\sqrt{\eta}$

![FIG. 10: The inhomogeneous linewidth $\delta \nu$ as a function of the condensate density.](image)

suggested by $\sigma$-model (see Eq. (28) with $q_s \sim \sqrt{\eta}$ where $\eta \rightarrow \eta - \sqrt{2}$) is not related with the inhomogeneous line width by the uncertainty relations, and in the dimensional units it is of the order of $T \sim N(R\kappa)^{-1} T_{\text{tot}}$.

We now consider the frequency dependence of the number of atoms kicked out from the condensate as the result of electron tunnel. Equation (??) with the allowance of Eq.(54) for 3D Green’s function can be written in the form

$$\delta N = \frac{q_s N_c^2}{2 (N\gamma q_s)^3} \int G_A G_R d\nu \sim \frac{q_s N_c^2}{2 (N\gamma q_s)^3} q_s^{-d/2},$$

which takes into account the number $N_c$ of the tunneling channels. One sees that this dependence is a rather fast inverse power function of the This form also shows that the condensate losses $\delta N$ per electron tunneling event decreases with increasing number of the condensate atoms. By taking $q_s \sim 1/R$, $N_c \sim N/\kappa\kappa$ one finds

$$\delta N \sim \frac{1}{N\gamma^3} (R\kappa)^{d/2},$$

and assuming $\gamma \sim 1$ arrives at $\delta N \sim \frac{1}{N\gamma^3} (R\kappa)^{d/2}$. This number is of the order of $10^2$ for $R\kappa \sim 10^4$ and $N \sim 10^6$.

VII. RESULTS AND DISCUSSION

Though our consideration seems to pose more questions than answer, we start this section by emphasizing the novelty of the problem: in the most general formulation, it addresses properties of a particle that experience an essentially quantum multiple scattering off a material field in a predetermined, in our case coherent, quantum state. The novelty has two facets, one of which relates to the properties of the scattered particle that in the presence of inelastic channels manifest some deviations from a similar problem of quantum transport in disordered media, while the other facet relates with possibility to reveal certain properties of an atomic condensate, such as the size of fluctuations, by incorporating its quantum field to electric circuits.

The main result of the paper is following. In spite of the fact that in Bose condensate all atoms are in the same quantum state with a wave function extending at microscopic distances, the tunneling electron feels the discrete nature of individual atoms associated with the quantum density fluctuations, and as a consequence, the tunneling gets strongly inhibited by the quantum localization effect. In other words, a simple mean-field picture that considers atomic condensate as a uniform support for the moving electron cannot correspond to reality. Though the allowance for irreversible transitions in the condensate partially destroys the electron interference and thereby facilitate the tunneling, it does not remove the localization effect completely. In the accordance with this picture is the fact that the state density profile of the tunneling electron band has the Wigner semicircular form in contrast to one corresponding to a free particle propagation. Superposition of this profile with the profile of the localization length determines the spectral line of the conductance efficiency.

The main message emerging from the consideration of relevant numbers is that the electron tunneling through a condensate is not a completely impossible process, and it can, at least in principle, be realized in a regime that approaches the domain of parameters currently available experimentally. The main problem is to identify atomic species that have weakly bounded negative ions. The typically large $(\sim 10^3 A^2)$ charge-exchange cross-section of negative ions in normal conditions that can yet increase for cold collisions are still not sufficient to ensure the efficient electron tunneling transport at typical interatomic distances of currently existing condensates. Nevertheless, since the charge exchange rate exponentially increases with decreasing electron binding energy in negative ions, in some cases of closed shells atoms with low polarizability, like Ca for instance, the tunneling regime can be achieved in principle, and the question is thus get moved to the possibility to condense these atoms.

Note that under condition of the exponential dependence of the tunneling probability on the interatomic distance, the role of careful calculation of specific atomic parameters, such as the actual shape of wave function of the attached electron, based on standard methods of atomic physics may becomes important. It may change essentially the estimated required atomic density, as it was the case for the presented calculations performed with
and without the parameter $\Lambda$. One should also have in mind the case of $He$ atoms with closed $s$-shell that do not have electronic affinity at all since their polarization potential is not strong enough to accommodate a bound state of electron. Still they are able to create a disordered potential for the tunneling electron at the energies close to zero. This potential will also modify the state density profile at this region. All the calculations based on $\sigma$-model thus remain valid, and the only difference will be in the sizes of the time and energy units, that presumably should correspond to a more efficient tunneling. Of course, one needs to know precise shape of the electron binding potential of atoms and the corresponding scattering length in order to make more accurate predictions.

Not a less important question is the efficiency of interface between the electric leads and the condensate. In order to ensure an efficient tunneling from the leads to the condensate without serious destruction of the latter, one needs to have electronic surface states rather extended in the direction perpendicular to the surface. Reliable determination of such states is a subject of serious numerical work, but at the first glance one can hope that the broad-band semiconductors may offer such a possibility. One also can consider the possibility to employ carbon nanotubes for the purpose, having in mind that the questions about the dielectric properties Ref.\[73\] and Casimir potentials for the atom-tube interaction Ref.\[74\] have already been considered in details for such systems. However, for the case of nanotubes one has to expect considerable decrease of the number of the tunneling channels, which is a direct consequence of the small size of the surface area.

It also worth mentioning that application of periodic external fields may not only compensate for existing mismatches among the surface state band and the condensate tunneling electron band, but also offer additional enhancement of the transition between these bands when goes through highly excited and therefore highly extended intermediate surface states. Moreover for a multifrequency resonance setting, it can also serve as an efficient tool, which reduces the influence of the thermal effects in the leads by satisfying strict resonance conditions at the final or intermediate states of the excitation cascade.

Looking at prospective, one can believe that the present consideration may also become relevant in the context of condensed Rydberg atoms, where the conductance might exist due to the partially occupied bands of delocalized Rydberg states. However, in order to consider this case, one needs to allow for the electronic many-body phenomena, such like Cooper pairs, that have been omitted in the present work.

One should also have in mind that there exist other physical systems, such as Bose condensates of excitons at the interface between two semiconductors, where similar phenomena can take place.

VIII. ACKNOWLEDGEMENTS

V.M.A deeply appreciate numerous explanations of the details of nonlinear $\sigma$-model technique given by Yan Fyodorov and Vladimir Kravtsov. Finansial support of IN-TAS grant is greatly acknowledged by all the authors.

IX. APPENDIX

Here we show how to separate the atomic field operators from the ionic ones. Prior to the separation, we make some transformations that simplify the calculations. Consider Taylor expansions of the time dependent exponents in Eq.\[36\]. Each term of the expansion is a sequence of the operator products

$$\frac{(-it)^k}{k!} \int \prod_i^{k} dr_1 \ldots \hat{\Psi}^+(r_3)\hat{\Phi}(r_3)V(r_3 - r_2)\hat{\Phi}^+(r_2)\hat{\Psi}(r_2)\hat{\Psi}^+(r_2)\hat{\Phi}(r_2) V(r_2 - r_1)\hat{\Phi}^+(r_1)\hat{\Psi}(r_1)$$

where the left argument of factor must coincide with the right argument of the next operator, as it is emphasized in Eq.\[36\] for the second argument by the underlining brace. This condition is the direct consequence of the assumption that we have only one tunneling electron. The condensate operators also correspond to the same coordinates, and hence they can be grouped together when the Taylor series are summed back to the exponent. In other words, operators $\hat{\Phi}^+(r)$ can be moved to $\hat{\Phi}(r)$ comprising the number of particles operator in each point where the ion field operator $\hat{\Psi}$ is present. The same is valid for $\hat{\Psi}$ operators. Equation \[36\] thus takes the form
\[ P(t, \tau, z) = \langle 0 | \langle 0 | e^{-i \int (A \dot{\Phi}^+ + A^* \Phi) dr} \hat{\Psi}(r_a) \Phi(r_a) e^{i \int \dot{\Psi}^+ (r_1) V(r-r_1) \psi(r) \dot{\phi}(r) dr} \hat{\Phi}(r_b) e^{i \int \dot{\Phi}^+ (r_2) \Phi(r_2) dr} \hat{\psi}(r_b) e^{-it \int \dot{\psi}^+ (r_1) V(r-r_1) \psi(r) \dot{\phi}(r) dr} \hat{\phi}(r_b) \hat{\psi}(r_b) e^{-it \int \dot{\phi}^+ (r_2) \Phi(r_2) dr} \hat{\phi}(r_b) \hat{\Phi}(r_b) \rangle_0 | 0 \rangle_i | 0 \rangle_i . \]

The general idea underlying the separation of the ionic and condensate fields rely on functional integrals, and in particular, on the functional Fourier transformation. The main steps of this procedure are as follows. A functional \( \Omega(\tilde{X}(x) \tilde{Y}(x)) \) depending on a product of two commuting operator fields \( \tilde{X}(x) \) and \( \tilde{Y}(x) \) can be written as a functional integral

\[
\int \mathcal{D}u(x) \Omega(\tilde{X}(x)u(x)) \delta_F \left( \tilde{Y}(x) - u(x) \right)
\]

where the functional Dirac \( \delta \)-function has the Fourier representation

\[
\delta_F (f(x) - g(x)) = \int e^{i2\pi \int dz (f(x) - g(x))u^*(x)} \mathcal{D}u^*(x).
\]

Here \( u(x) \) and \( u^*(x) \) are two independent real-valued functional variables. By combining these two expressions one represents \( \Omega(\tilde{X}(x) \tilde{Y}(x)) \) as a functional integral of a product, where one factor depends only on the operator field \( \tilde{X}(x) \) and the other only on the operator field \( \tilde{Y}(x) \):

\[
\int \Omega(\tilde{X}(x)u(x)) e^{i2\pi \int dz [\tilde{Y}(x) - u(x)]u^*(x)} \mathcal{D}u(x) \mathcal{D}u^*(x).
\]

In particular, two factors in Eq. (37),

\[
e^{i\pi \int \dot{\Psi}^+ (r_1) V(r-r_1) \Psi(r) \dot{\phi}(r) dr}
\]

and

\[
e^{-it \int \dot{\psi}^+ (r_1) V(r-r_1) \psi(r) \dot{\phi}(r) dr}
\]

can be represented in the form

\[
\int \left[ e^{i\pi \int \dot{\Psi}^+ (r_1) V(r-r_1) \Psi(r) u(r) dr} + e^{i2\pi \int dz [\tilde{Y}(x) - u(x)]u^*(x)} \mathcal{D}u(x) \mathcal{D}u^*(x) \right]
\]

and

\[
\int e^{-it \int \dot{\psi}^+ (r_1) V(r-r_1) \psi(r) v(r) dr} + e^{i2\pi \int dz [\tilde{Y}(x) - u(x)]u^*(x)} \mathcal{D}u(x) \mathcal{D}u^*(x) \]

respectively.

By regrouping the operators, one obtains Eq. (37) as a functional integral of a product of three matrix elements each of which is taken by the corresponding vacuum state

\[
P(t, \tau, z) = \int \mathcal{D}v(x) \mathcal{D}v^*(x) \mathcal{D}u(x) \mathcal{D}u^*(x) e^{-i2\pi v^*(r) v e^{-i2\pi u^*(r) u} \langle Cd \rangle G_A(\tau) G_R(t)} ,
\]

where

\[
\mathcal{G}_A = \langle 0 | \bar{\Psi}(r_a) e^{i\pi \int \dot{\Psi}^+ (r_1) V(r-r_1) \Psi(r) u(r) \dot{\phi}(r) dr} \mathcal{D}u(x) \mathcal{D}u^*(x) \hat{\psi}(r_b) e^{-it \int \dot{\psi}^+ (r_1) V(r-r_1) \psi(r) \dot{\phi}(r) dr} \hat{\phi}(r_b) \rangle_0,
\]

\[
\mathcal{G}_R = \langle 0 | \bar{\Phi}(r_b) e^{-it \int \dot{\Phi}^+ (r_2) \Phi(r_2) \dot{\phi}(r) dr} \hat{\phi}(r_b) \rangle_0
\]

are the advanced and retarded Green’s functions calculated over the vacuum states of the ion, and the matrix element over the condensate vacuum

\[
\langle Cd \rangle = \langle 0 | e^{i\pi \int \dot{\Psi}^+ (r_1) V(r-r_1) \Psi(r) \dot{\phi}(r) dr} \mathcal{D}u(x) \mathcal{D}u^*(x) \hat{\psi}(r_b) e^{-it \int \dot{\psi}^+ (r_1) V(r-r_1) \psi(r) \dot{\phi}(r) dr} \hat{\phi}(r_b) e^{i2\pi \int dz [\tilde{Y}(x) - u(x)]u^*(x)} \mathcal{D}u(x) \mathcal{D}u^*(x) \rangle_0 .
\]

The bar over the Green’s functions \( \mathcal{G}_A \) and \( \mathcal{G}_R \) shows that they are calculated for the operators that include fields \( u(r) \) and \( v(r) \), respectively.

We calculate \( \langle Cd \rangle \) with the help of two types of operator relations: the Baker-Campbell-Hausdorff rules

\[
e^{A+B} = e^B e^A e^{[A,B]/2}
\]

for the operators with c-number commutator, and the relations

\[
e^{z \int d\tau \dot{\Phi}^+ (r) \delta (r) \dot{\phi}(r) \Phi(r) e^{-z \int d\tau \dot{\phi}^+ (r) \delta (r) \phi(r)} = e^{z \dot{\Phi}^+ (r)}
\]

\[
e^{z \int d\tau \dot{\Phi}^+ (r) \delta (r) \dot{\phi}(r) \Phi(r) e^{-z \int d\tau \dot{\phi}^+ (r) \delta (r) \phi(r)} = e^{-z \dot{\Phi}^+ (r)}
\]

for the bosonic operators. With the help of Eq. (40) one can move the factors depending on the number operators towards the right vacuum state, where they just result in factors 1, and obtain

\[
\langle 0 | e^{-i \int (A e^{-i2\pi v^*(r) \dot{\phi}(r) + A^* e^{i2\pi u^*(r) \dot{\phi}(r)} dr} \dot{\Phi}^+ (r_a) \Phi(r_a) dr} e^{i \int (A \dot{\Phi}^+ (r) + A^* \dot{\phi}(r) dr} \Phi(r_a) e^{-i \int (A e^{-i2\pi v^*(r) \dot{\phi}(r) + A^* e^{i2\pi u^*(r) \dot{\phi}(r)} dr} \dot{\Phi}^+ (r_a) \Phi(r_a) dr} \Phi(r_a) e^{i \int (A \dot{\Phi}^+ (r) + A^* \dot{\phi}(r) dr} \Phi(r_a) e^{-i \int (A e^{-i2\pi v^*(r) \dot{\phi}(r) + A^* e^{i2\pi u^*(r) \dot{\phi}(r)} dr} \dot{\Phi}^+ (r_a) \Phi(r_a) dr} \Phi(r_a) \rangle_0 .
\]
Ignoring the quantum fluctuations in the pre-exponential factors yields
\[
\langle Cd \rangle = n^2 \langle 0 | e^{-i\int (Ae^{-i2\pi u^*(r)\hat{\Phi}^+(r) + A^*e^{i2\pi u^*(r)}\hat{\Phi}(r))dr} \\
e^{i\int (A\hat{\Phi}^+(r) + A^*\hat{\Phi}(r))dr} e^{-i\int (Ae^{i\hat{\Phi}^+(r) + A^*e^{-i\hat{\Phi}(r))dr}} e^{i\int (Ae^{-i2\pi u^*(r)\hat{\Phi}^+(r) + A^*e^{-i2\pi u^*(r)}\hat{\Phi}(r))dr} | 0 \rangle.
\]

With the help of Eq. (40) the exponential terms can be normally ordered, that is rearranged in such a way that all \( \hat{\Phi} \) are to the right with respect to \( \hat{\Phi}^+ \). All the terms containing normally ordered field operators vanish. The c-number commutators appearing in the course of the normal ordering yield a factor \( e^{w(u^*(r),v^*(r))} \), which reads
\[
w(u^*(r),v^*(r)) = n \int dr (e^{2\pi u^*(r)} + e^{2\pi v^*(r)} - 2 + e^{2(1 - e^{2\pi u^*(r)})}(1 - 1)}
\]
The functional Fourier transform
\[
W_{u,v} = \int \mathcal{D}v^*(x)\mathcal{D}u^*(x)e^{w-2\xi \int dr [u(r)u^*(r) + v^*(r)]}.
\]
of \( e^{w(u^*(r),v^*(r))} \) plays role of the weight factor for functional integration over the fields \( u(r) \) and \( v(r) \). However, it has a structure inconvenient for calculations. Technically is much simpler to perform the functional integration over the fields \( u \) and \( v \) first, and then integrate over \( u^*(r) \) and \( v^*(r) \). Equation (40) thus takes the form
\[
P(t,\tau,\varepsilon) = n^2 \int \mathcal{D}v^*(x)\mathcal{D}u^*(x)\mathcal{D}v(x)\mathcal{D}u(x) \\
\mathcal{G}_R(\varepsilon) \mathcal{G}_A(\tau) e^{w-2\xi \int dr [u(r)u^*(r) + v^*(r)]}.
\]
One can perform the Fourier transformation over times \( t \) and \( \tau \) and write
\[
P(\varepsilon,\xi,\omega) = n^2 \int W_{u,v} \mathcal{D}v(x)\mathcal{D}u(x) \\
\mathcal{G}_A(\xi) \mathcal{G}_R(\omega) e^{w-2\xi \int dr [u(r)u^*(r) + v^*(r)]}.
\]
where the Green’s functions can be given explicitly
\[
\mathcal{G}_A(\xi) = \langle 0 | \hat{\Psi}(r_a) \xi - \hat{H}_u^{-1} \hat{\Psi}^+ (r_b) | 0 \rangle
\]
\[
\mathcal{G}_R(\omega) = \langle 0 | \hat{\Psi}(r_b) \omega - \hat{H}_v^{-1} \hat{\Psi}^+ (r_a) | 0 \rangle
\]
in terms of the field operators
\[
\hat{H}_u = \int d\tau d\tau_1 \hat{\Psi}^+ (r_1) V(r-r_1) \hat{\Psi}(r) u(r) \\
\hat{H}_v = \int d\tau d\tau_1 \hat{\Psi}^+ (r_1) V(r-r_1) \hat{\Psi}(r) v(r),
\]
that differ from the Hamiltonians Eq. (41) having classical fields \( u(r) \) and \( v(r) \) instead of the quantum condensate fields \( \hat{\Phi}^+(r) \) and \( \hat{\Phi}(r) \).

In order to find the average of \( \mathcal{G}_A(\xi) \mathcal{G}_R(\varepsilon) \), we follow the prescription well elaborated in the theory of disordered systems and express the Green’s functions with the help of functional superintegrals over supervector fields \( \lambda(r) \) and \( \mu(r) \). For the advanced and retarded Green’s functions these fields are grouped to the line vectors
\[
\lambda(r) = (\lambda_u(r) \lambda_v(r)) ; \mu(r) = (\mu_u(r) \mu_v(r))
\]
respectively, and the conjugated columns vectors
\[
\overline{\lambda}(r) = \left( \overline{\lambda}_u(r) \overline{\lambda}_v(r) \right) ; \overline{\mu}(r) = \left( \overline{\mu}_u(r) \overline{\mu}_v(r) \right).
\]
This integral over regular(bosonic) functional variables \( \lambda_u(r) \) and \( \mu_u(r) \) and anticommuting Grassmannian (fermionic) variables \( \lambda_v(r) \) and \( \mu_v(r) \) read
\[
\mathcal{G}_R(\varepsilon) = \int \mathcal{D}\lambda(r)\mathcal{D}\overline{\lambda}(r) e^{i\xi \mathcal{A}_f(r_a)\lambda_f(r_b)} \\
\mathcal{G}_A(\xi) = \int \mathcal{D}\mu(r)\mathcal{D}\overline{\mu}(r) e^{-i\xi \mathcal{A}_f(r_a)\mu_f(r_b)}.
\]
where the conjugated variables denoted by bars and the actions have the form
\[
S_x = \int d\tau d\lambda(r) \overline{\lambda}(r) - \int d\tau d\tau_1 \lambda(r_1) \overline{V}(r-r_1) \overline{\lambda}(r)v(r) \\
S_\xi = \int d\tau d\mu(r) \overline{\mu}(r) - \int d\tau d\tau_1 \mu(r_1) \overline{V}(r-r_1) \overline{\mu}(r)u(r).
\]
Here the products \( \lambda(r) \overline{\lambda}(r) \) and \( \mu(r) \overline{\mu}(r) \) stand for the scalar products \( \lambda_u(r) \overline{\lambda}_u(r) + \lambda_v(r) \overline{\lambda}_v(r) \) and \( \mu_u(r) \overline{\mu}_u(r) + \mu_v(r) \overline{\mu}_v(r) \), respectively. The functional differentials \( \mathcal{D}\lambda(r)\overline{\lambda}(r) \) stand for \( \mathcal{D}\lambda_u(r)\mathcal{D}\overline{\lambda}_u(r)\mathcal{D}\lambda_f(r)\mathcal{D}\overline{\lambda}_f(r) \), and \( \mathcal{D}\mu(r)\overline{\mu}(r) \) for \( \mathcal{D}\mu_u(r)\mathcal{D}\overline{\mu}_u(r)\mathcal{D}\mu_f(r)\mathcal{D}\overline{\mu}_f(r) \), and the operator \( \overline{V} \) reads
\[
\overline{V}(r-r_1) = \left( \begin{array}{cc} V(r-r_1) & 0 \\ 0 & V(r-r_1) \end{array} \right).
\]
We substitute Eq. (43) with the actions Eq. (45) to Eq. (42) and perform the integration over the fields \( u \) and \( v \) first. This integration yields \( \delta \xi \) functions with the arguments \( 2\pi v^*(x) - \int d\tau \lambda(r_1) \overline{V}(r-r_1) \overline{\lambda}(r) \) and \( 2\pi u^*(x) + \int d\tau \mu(r_1) \overline{V}(r-r_1) \overline{\mu}(r) \). Performing the integration over the variables \( u^* \) and \( v^* \) one finds
\[ P(\xi, \varepsilon, z) = n^2 \int \mathcal{D}\lambda(r) \mathcal{D}(\overline{\lambda})(r) e^{i \int \! dr \lambda(r) \overline{\lambda}(r) \overline{\lambda}(r_0)\lambda(r_b)} \int \mathcal{D}\mu(r) \mathcal{D}(\overline{\mu})(r) e^{-i \int \! dr \mu(r) \overline{\mu}(r) \mu_f(r_b)\overline{\mu}_f(r_a)} \]
\[
\exp \left\{ n \int \! dr \left( e^{i \int \! dr_1 \lambda(r_1) V(r-r_1)\overline{\lambda}(r)} + e^{-i \int \! dr_1 \mu(r_1) V(r-r_1)\overline{\mu}(r)} - 2 \right) \right\}
\]
\[
\exp \left\{ n \int \! dr \left( 1 - e^{i \int \! dr_1 \lambda(r_1) V(r-r_1)\overline{\lambda}(r)} \right) \left( 1 - e^{-i \int \! dr_1 \mu(r_1) V(r-r_1)\overline{\mu}(r)} \right) e^{i \varepsilon} \right\} \]  

(46)

Thus far the calculations were exact. At this point one has to make an approximation, since exact calculation of the functional integrals are further impossible. We notice that one fold integrals with the exponentially decreasing kernel \( V(r-r_1) \) are of the order on the atomic density \( n \), which is a small dimensionless parameter. By the second order Taylor expansion of the terms entering the exponents in the second and third lines of Eq. (46), we obtain

\[ P(\xi, \varepsilon, z) = n^2 \int \mathcal{D}\lambda(r) \mathcal{D}(\overline{\lambda})(r) \mathcal{D}\mu(r) \mathcal{D}(\overline{\mu})(r) \overline{\lambda}(r_0)\lambda(r_b)\mu_f(r_b)\overline{\mu}_f(r_a) e^{i \int \! dr \lambda(r)\overline{\lambda}(r) - i \int \! dr \mu(r)\overline{\mu}(r)} 
\]
\[
e^{-\frac{i}{2} \int \! dr_1 dr_2 \left[ \lambda(r_1) V(r-r_1)\overline{\lambda}(r) - \mu(r_1) V(r-r_1)\overline{\mu}(r) \right] \left[ \lambda(r_2) V(r-r_2)\overline{\lambda}(r) + \mu(r_2) V(r-r_2)\overline{\mu}(r) \right]} \]  

(47)

Let us rewrite this expression in a more compact way by introducing two supervectors in the functional coordinate space - the column vector \( \overline{\phi}(r) \) defined by their coordinate representations as

\[ \overline{\phi}(r) = (r| \overline{\phi}) = \begin{pmatrix} \overline{\lambda}_b(r) \\ \overline{\mu}_b(r) \\ \overline{\lambda}_f(r) \\ \overline{\mu}_f(r) \end{pmatrix} \]

and the corresponding line vector \( \langle \phi \rangle \)

\[ \phi(r') = \langle \phi|r'\rangle = \begin{pmatrix} \lambda_b(r') \\ \mu_b(r') \\ \lambda_f(r') \\ \mu_f(r') \end{pmatrix} \]

In this notations Eq. (47) takes the form

\[ P(\xi, \varepsilon, z) = n^2 \int \mathcal{D}\langle \phi \rangle \mathcal{D}(\overline{\phi}) \lambda_f(r_0)\lambda(r_b)\mu_f(r_b)\overline{\mu}_f(r_a) \exp \left[ \langle \phi | (i\zeta + i\eta \Sigma_z + i\eta \Sigma_x V) | \overline{\phi} \rangle \right] \]
\[
\exp \left[ -\frac{n}{2} \left( 1 - e^z \right) \int \! dr_1 dr_2 dr \left( \phi(r_2) V(r_2-r) \Sigma_z \overline{\phi}(r) \right) \left( \phi(r_1) V(r_1-r) \Sigma_z \overline{\phi}(r) \right) \right] 
\]
\[
\exp \left[ -\frac{n}{2} \left( 1 + e^{-z} \right) \int \! dr_1 dr_2 dr \left( \phi(r_2) V(r_2-r) \overline{\phi}(r) \right) \left( \phi(r_1) V(r_1-r) \overline{\phi}(r) \right) \right] , \]

where two matrices

\[ \langle r' | V | r \rangle = V(r'-r) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]  

and \( \langle r' | \Sigma_z | r \rangle = \delta(r'-r) \Sigma_z = \delta(r'-r) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \)

and two variables \( \eta = \frac{\varepsilon + \zeta}{2} \) and \( \zeta = \frac{\varepsilon - \zeta}{2} \) have been introduced. Here by \( \mathcal{D}|\phi\rangle \mathcal{D}(\overline{\phi}) \) we denote the functional integration by all eight components of the vectors \( |\phi\rangle \) and \( \langle \overline{\phi} | \).
Let us rewrite Eq. (18) in an even shorter way by introducing an object similar to the density matrix – the direct product
\[ \rho = | \phi \rangle \otimes \langle \phi |. \]
This allows one to put Eq. (47) in the form
\[ P(\zeta, \eta, z) = n^2 \int D [\bar{\phi}] D [\phi] DQ(r) e^{-\frac{\nu}{4} \text{Str} Q^2} e^{\text{Str}(i\zeta \mu \Sigma_{\mu} + i\eta \Sigma_{\eta} V)\rho} e^{-\frac{\nu}{2} \text{Str} r V \Sigma_{\eta} \rho V \Sigma_{\eta} - \frac{\nu}{2} (1 + e^z) \text{Str} r V \rho V}. \]

The symbol Str stands for the regular trace operator with respect to the coordinate dependence and with respect to regular vector components and implies change of the sign for the Grassmann components. By Str we denote the supertrace operation, which involves only the matrix variables and does not imply integration over the coordinates, it amounts to the difference of the traces of the boson-boson and fermion-fermion blocks.

The next step is application of the Hubbard-Stratanovich separation of the terms quadratic in \( \rho \). This can be done by introducing an superintegral over a supermatrix \( Q \) which, generally speaking depends only the coordinate \( r \), that is \( \langle r' | Q | r \rangle = \delta(r' - r) \hat{Q} \) with
\[ \hat{Q} = \left( \begin{array}{cccc}
Q_{\lambda \lambda_{\lambda_{\lambda}}} & Q_{\lambda \lambda_{\mu_{\mu}}} & Q_{\lambda \lambda_{\lambda_{\mu}}} & Q_{\lambda \lambda_{\mu_{\mu}}} \\
Q_{\mu_{\mu} \lambda_{\lambda}} & Q_{\mu_{\mu} \mu_{\mu}} & Q_{\mu_{\mu} \lambda_{\mu}} & Q_{\mu_{\mu} \mu_{\mu}} \\
Q_{\lambda_{\mu} \lambda_{\lambda}} & Q_{\lambda_{\mu} \mu_{\mu}} & Q_{\lambda_{\mu} \lambda_{\mu}} & Q_{\lambda_{\mu} \mu_{\mu}} \\
Q_{\mu_{\mu} \mu_{\mu}} & Q_{\mu_{\mu} \mu_{\mu}} & Q_{\mu_{\mu} \mu_{\mu}} & Q_{\mu_{\mu} \mu_{\mu}}
\end{array} \right), \]

where the matrix elements \( Q_{i,j} = Q_{i,j}(r) \) are the regular functions of \( r \) in the diagonal \( 2 \times 2 \) blocks (boson-boson and fermion-fermion) and the Grassmann functions of \( r \) in the off-diagonal (boson-fermion) blocks. The third line of Eq. (49) then takes the form
\[ \exp \left\{ -\frac{\nu}{4} \text{Str} [1 + e^z] \rho V \Sigma_{\eta} \rho V \Sigma_{\eta} + (1 + e^z) \rho V \rho V \right\} = \int DQ(r) \exp \left\{ -\frac{\nu}{2} \text{Str} Q^2 - i \frac{\nu}{2} \text{Str} VQ' \rho \right\} \]

where \( \langle r' | Q' | r \rangle = \delta(r' - r) \hat{Q}' \) with
\[ \hat{Q}' = \left( \begin{array}{cccc}
-\frac{Q_{\lambda \lambda_{\lambda_{\lambda}}}}{i e^{z/2} Q_{\lambda \lambda_{\mu_{\mu}}}} & \frac{Q_{\lambda \lambda_{\mu_{\mu}}}}{i e^{z/2} Q_{\lambda \lambda_{\mu_{\mu}}}} & \frac{Q_{\lambda \lambda_{\lambda_{\mu}}}}{i e^{z/2} Q_{\lambda \lambda_{\mu_{\mu}}}} & \frac{Q_{\lambda \lambda_{\mu_{\mu}}}}{i e^{z/2} Q_{\lambda \lambda_{\mu_{\mu}}}} \\
\frac{Q_{\mu_{\mu} \lambda_{\lambda}}}{i e^{z/2} Q_{\mu_{\mu} \mu_{\mu}}} & \frac{Q_{\mu_{\mu} \mu_{\mu}}}{i e^{z/2} Q_{\mu_{\mu} \mu_{\mu}}} & \frac{Q_{\mu_{\mu} \lambda_{\mu}}}{i e^{z/2} Q_{\mu_{\mu} \mu_{\mu}}} & \frac{Q_{\mu_{\mu} \mu_{\mu}}}{i e^{z/2} Q_{\mu_{\mu} \mu_{\mu}}} \\
\frac{Q_{\lambda_{\mu} \lambda_{\lambda}}}{i e^{z/2} Q_{\lambda_{\mu} \mu_{\mu}}} & \frac{Q_{\lambda_{\mu} \mu_{\mu}}}{i e^{z/2} Q_{\lambda_{\mu} \mu_{\mu}}} & \frac{Q_{\lambda_{\mu} \lambda_{\mu}}}{i e^{z/2} Q_{\lambda_{\mu} \mu_{\mu}}} & \frac{Q_{\lambda_{\mu} \mu_{\mu}}}{i e^{z/2} Q_{\lambda_{\mu} \mu_{\mu}}} \\
\frac{Q_{\mu_{\mu} \mu_{\mu}}}{i e^{z/2} Q_{\mu_{\mu} \mu_{\mu}}} & \frac{Q_{\mu_{\mu} \mu_{\mu}}}{i e^{z/2} Q_{\mu_{\mu} \mu_{\mu}}} & \frac{Q_{\mu_{\mu} \mu_{\mu}}}{i e^{z/2} Q_{\mu_{\mu} \mu_{\mu}}} & \frac{Q_{\mu_{\mu} \mu_{\mu}}}{i e^{z/2} Q_{\mu_{\mu} \mu_{\mu}}}
\end{array} \right). \]

In these notations Eq. (49) reads
\[ P(\zeta, \eta, z) = n^2 \int D [\bar{\phi}] D [\phi] DQ(r) e^{-\frac{\nu}{4} \text{Str} Q^2} \text{Str}(i\zeta \mu \Sigma_{\mu} + i\eta \Sigma_{\eta} V)\rho \]
\[ \text{exp} \left\{ -\frac{\nu}{4} \text{Str} \left[ (1 + e^z) \rho V \Sigma_{\eta} \rho V \Sigma_{\eta} + (1 + e^z) \rho V \rho V \right] \right\} = \int DQ(r) \exp \left\{ -\frac{\nu}{2} \text{Str} Q^2 - i \frac{\nu}{2} \text{Str} VQ' \rho \right\} \]

where \( \langle r' | Q' | r \rangle = \delta(r' - r) \hat{Q}' \) with
\[ \hat{Q}' = \left( \begin{array}{cccc}
-\frac{Q_{\lambda \lambda_{\lambda_{\lambda}}}}{i e^{z/2} Q_{\lambda \lambda_{\mu_{\mu}}}} & \frac{Q_{\lambda \lambda_{\mu_{\mu}}}}{i e^{z/2} Q_{\lambda \lambda_{\mu_{\mu}}}} & \frac{Q_{\lambda \lambda_{\lambda_{\mu}}}}{i e^{z/2} Q_{\lambda \lambda_{\mu_{\mu}}}} & \frac{Q_{\lambda \lambda_{\mu_{\mu}}}}{i e^{z/2} Q_{\lambda \lambda_{\mu_{\mu}}}} \\
\frac{Q_{\mu_{\mu} \lambda_{\lambda}}}{i e^{z/2} Q_{\mu_{\mu} \mu_{\mu}}} & \frac{Q_{\mu_{\mu} \mu_{\mu}}}{i e^{z/2} Q_{\mu_{\mu} \mu_{\mu}}} & \frac{Q_{\mu_{\mu} \lambda_{\mu}}}{i e^{z/2} Q_{\mu_{\mu} \mu_{\mu}}} & \frac{Q_{\mu_{\mu} \mu_{\mu}}}{i e^{z/2} Q_{\mu_{\mu} \mu_{\mu}}} \\
\frac{Q_{\lambda_{\mu} \lambda_{\lambda}}}{i e^{z/2} Q_{\lambda_{\mu} \mu_{\mu}}} & \frac{Q_{\lambda_{\mu} \mu_{\mu}}}{i e^{z/2} Q_{\lambda_{\mu} \mu_{\mu}}} & \frac{Q_{\lambda_{\mu} \lambda_{\mu}}}{i e^{z/2} Q_{\lambda_{\mu} \mu_{\mu}}} & \frac{Q_{\lambda_{\mu} \mu_{\mu}}}{i e^{z/2} Q_{\lambda_{\mu} \mu_{\mu}}} \\
\frac{Q_{\mu_{\mu} \mu_{\mu}}}{i e^{z/2} Q_{\mu_{\mu} \mu_{\mu}}} & \frac{Q_{\mu_{\mu} \mu_{\mu}}}{i e^{z/2} Q_{\mu_{\mu} \mu_{\mu}}} & \frac{Q_{\mu_{\mu} \mu_{\mu}}}{i e^{z/2} Q_{\mu_{\mu} \mu_{\mu}}} & \frac{Q_{\mu_{\mu} \mu_{\mu}}}{i e^{z/2} Q_{\mu_{\mu} \mu_{\mu}}}
\end{array} \right). \]

The main features of the tunneling conductance can be found in the framework of the simplest version of this technique, which ignores the coordinate dependence \( Q_{i,j}(r) = Q_{i,j} \) of the matrix elements Eq. (50), (51) It is applicable when the perturbation of the mean field regime by the condensate density fluctuations is strong enough. In complete analogy with the situation known in Quantum Mechanics, where a strong time-dependent perturbation induce quantum transitions between slightly detuned levels as if it were resonant. In such a situation, only the integral of the time dependence enters the result, while the particular form of the time dependence plays no role and can be taken uniform. Here, by the analogy, we have to compute the interactions resulting in two last terms in the exponent in the second line of Eq. (52). Since we consider tunneling at a long distance \( R \), the typical difference of the momenta of the advanced and retarded Green’s functions is small, and according to Eq. (9) the typical variation of the interaction associated with the term \( \text{Str} \Sigma_{\eta} V \) can be estimated as \( \mathcal{N} \partial_p V(p) R^{-1} \sim \mathcal{N} AR^{-1} \). The fluctuation-induced coupling associated with the term \( \text{Str} VQ' \) can be estimated from the condition \( \frac{n}{4} \text{Str} Q^2 \sim \frac{n}{4} \text{Str} \hat{Q}' \sim 1 \), that
is $\hat{Q} \sim 1/\sqrt{N}$, and hence this coupling is of the order of $\Lambda\sqrt{N}$. Therefore, the fluctuations can be considered as uniform for $NR^{-2} \sim nR \ll 1$, which we assume to be the case for the system under consideration.

The assumption of spacial uniform matrix elements allows one to employ the momentum representation where two momenta $p$ and $q$ corresponding to the advanced and retarded Green’s functions, respectively. This simplifies which allows one to write Eq.(54) in the form

$$\Lambda \eta \sim \sqrt{N}$$

where $\zeta$ and $\eta$ are neglected when compared to unity. After having shifted the matrix $\hat{Q}$ by $\zeta \hat{S}_a$, one obtains a form of this expression

$$P(\zeta, \eta, z) = e \oint \frac{dpdq}{(2\pi)^2} e^{iR(p-q)} \int D\phi D\bar{\phi}$$

where $\zeta$ and $\eta$ were neglected when compared to unity. After having shifted the matrix $\hat{Q}$ by $\zeta \hat{S}_a$, one obtains a form of this expression

$$P(\zeta, \eta, z) = e \oint \frac{dpdq}{(2\pi)^2} e^{iR(p-q)} \int D\phi D\bar{\phi}$$

convenient for further calculations.

We did not yet specified the precise form of the matrix $\hat{Q}$ that allows one to change order of the integration in Eq.(60) and evaluate the integral over $D\phi D\bar{\phi}$ first. The standard approach suggests to express this matrix in terms of the generators of the superalgebra $osp(1,1|2)$ and make use of the parametrization

$$\hat{Q} = \tilde{P} \hat{P} \tilde{P}^{-1},$$

where $\tilde{P}$ is a supermatrix

$$\tilde{P} = \begin{pmatrix} q_1 & 0 & \pi_1 & 0 \\ 0 & q_2 & 0 & \pi_2 \\ \kappa_1 & 0 & \imath q_1 & 0 \\ 0 & \kappa_2 & 0 & \imath q_2 \end{pmatrix}$$

and $\tilde{P}$ is a pseudounitary rotation

$$\tilde{T} = \hat{U} \hat{M}$$

with

$$\hat{M} = \begin{pmatrix} \cosh \Theta & e^{-i\Phi} \sinh \Theta & 0 & 0 \\ e^{-i\Phi} \sinh \Theta & \cosh \Theta & 0 & 0 \\ 0 & 0 & \cos \theta & \imath e^{-i\phi} \sin \theta \\ 0 & 0 & \imath e^{i\phi} \sin \theta & \cos \theta \end{pmatrix}$$

$$\hat{U} = \begin{pmatrix} 1 - \frac{\pi_2}{2} & 0 & -\pi_2 & 0 \\ 0 & 1 + \frac{\pi_2}{2} & 0 & -i\beta \\ \alpha & 0 & 1 + \frac{\pi_2}{2} & 0 \\ 0 & i\beta & 0 & 1 - \frac{\pi_2}{2} \end{pmatrix}.$$
We apply this parametrization to the generating function in the form Eq.\textcolor{red}{(63)} and perform the transformations Eqs.\textcolor{red}{(54)\textcolor{red}{55)}. Evaluation of the integral over $\mathcal{D}\mathcal{D}\varphi$ yields

\[
P(\zeta, \eta, z) = \partial_{\mathcal{J}_a, \mathcal{J}_b} \int \frac{dp dq}{(2\pi)^{2d}} e^{i R(p-q)} \int d\hat{Q} e^{-N\left(\frac{1}{2} - \zeta \mathcal{S}\hat{Q}^2 + \frac{1}{2} - \zeta \mathcal{S}(\hat{Q}\Sigma_z)^2\right)} - \zeta \mathcal{S}\Sigma_z \hat{Q}}
\]

By substituting $\hat{Q}' = -\sqrt{\Sigma_z} \hat{Q} \sqrt{\Sigma_z}$ and by moving $\sqrt{\Sigma_z}$ under the supertrace operators one finds

\[
P(\zeta, \eta, z) = \partial_{\mathcal{J}_a, \mathcal{J}_b} \int \frac{dp dq}{(2\pi)^{2d}} e^{i R(p-q)} \int d\hat{Q} e^{-N\left(\frac{1}{2} - \zeta \mathcal{S}\hat{Q}^2 + \frac{1}{2} - \zeta \mathcal{S}(\hat{Q}\Sigma_z)^2\right)} - \zeta \mathcal{S}\Sigma_z \hat{Q}}
\]

which after taking the derivatives

\[
P(\zeta, \eta, z) = \frac{n^2}{4} \int d\hat{Q} e^{-f \frac{\mathcal{V}^2}{2\Sigma_z} \text{Str ln}[\eta - p^2 - \hat{Q}]} e^{N\zeta \mathcal{S}\Sigma_z \hat{Q} - N\frac{1}{2} - \zeta \mathcal{S}\hat{Q}^2 - N\frac{1}{2} - \zeta \mathcal{S}(\hat{Q}\Sigma_z)^2}
\]

allows one to explicitly perform the integration over $dp dq$ and results in

\[
P(\zeta, \eta, z) = \int d\hat{Q} e^{-f \frac{\mathcal{V}^2}{2\Sigma_z} \text{Str ln}[\eta - p^2 - \hat{Q}]}
\]

where by the analogy to Eq.\textcolor{red}{(28)} the Green’s functions (depending on the matrix $\hat{Q}$) read

\[
G_{A,R}(\hat{Q}) = \frac{ne^{\frac{\mathcal{V}^2}{2\Sigma_z}}}{\sqrt{\eta - p^2}} \quad \text{for } 1D
\]

\[
G_{A,R}(\hat{Q}) = \frac{nK_c(\pm 1)\sqrt{\eta - p^2}}{4\pi R} \quad \text{for } 2D
\]

\[
G_{A,R}(\hat{Q}) = \frac{ne^{\frac{\mathcal{V}^2}{2\Sigma_z}}}{\sqrt{\eta - p^2}} \quad \text{for } 3D,
\]

and the choice of signs has to be done such that to assure the convergency.

By moving $\hat{T}$ under the supertraces and superdeterminants, one can obtain Eq.\textcolor{red}{(63)} in the form

\[
P(\zeta, \eta, z) = \int d\hat{P} d\hat{T} e^{-f \frac{\mathcal{V}^2}{2\Sigma_z} \text{Str ln}[\eta - p^2 - \hat{P}]}
\]

\[
e^{-N\zeta \mathcal{S}\Sigma_z \hat{P} - N\frac{1}{2} - \zeta \mathcal{S}\hat{P}^2 - N\frac{1}{2} - \zeta \mathcal{S}2\mathcal{S}(\hat{P}\Sigma_z)^2}
\]

\[
\Sigma_z \hat{P} \quad \text{Str}_{\mathcal{J}_a} G\left(\hat{P}\right) \quad \text{Str}_{\mathcal{J}_b} G\left(\hat{P}\right)
\]

\[
\text{(66)}
\]

\[
\text{convenient for integration over the matrix } \hat{P}. \text{ Here the integration over } d\hat{P} d\hat{T} \text{ has to be done according to the measure Eq.\textcolor{red}{(63)}, the matrices with primes read}
\]

\[
\hat{\Sigma}_z' = \begin{pmatrix}
cosh 2\Theta & -e^{2i\Phi} \sinh 2\Theta & 0 & 0 \\
e^{-2i\Phi} \sinh 2\Theta & -cosh 2\Theta & 0 & 0 \\
0 & 0 & \cos 2\Theta & -e^{-2i\Phi} \sin 2\Theta \\
0 & 0 & i e^{2i\Phi} \sin 2\Theta & -\cos 2\Theta
\end{pmatrix}
\]

\[
\hat{\Sigma}_a' = \begin{pmatrix}
\pi \alpha \cosh^2\Theta & -\frac{\pi \alpha}{2} e^{2i\Phi} \sinh 2\Theta & -\pi \cos \Theta & 0 \\
\frac{\pi}{2} e^{-2i\Phi} \sinh 2\Theta & -\pi \cosh^2\Theta & -\pi \cos \Theta & 0 \\
-\alpha \cos \Theta \cos \Theta & \alpha \sin \Theta e^{2i\Phi} \cos \Theta & \cos 2\Theta & -e^{-2i\Phi} \sin 2\Theta \\
-\alpha e^{2i\Phi} \cosh \Theta \sin \Theta & \alpha e^{2i\Phi} \cosh \Theta \sin \Theta & -e^{-2i\Phi} \sin 2\Theta & -\cos 2\Theta
\end{pmatrix}
\]

\[
\hat{\Sigma}_b' = \begin{pmatrix}
\frac{\pi}{2} \beta \cosh^2\Theta & -\frac{\pi}{2} e^{2i\Phi} \sinh 2\Theta & -\beta \cosh 2\Theta & 0 \\
\frac{\pi}{2} e^{-2i\Phi} \sinh 2\Theta & -\frac{\pi}{2} \cosh^2\Theta & -\beta \cosh 2\Theta & 0 \\
-\beta e^{-2i\Phi} \cosh \Theta \sin \Theta & \beta e^{-2i\Phi} \cosh \Theta \sin \Theta & \cos 2\Theta & -e^{-2i\Phi} \sin 2\Theta \\
\beta e^{2i\Phi} \cosh \Theta \cos \Theta & -\beta \cosh 2\Theta & -e^{2i\Phi} \sin 2\Theta & -\cos 2\Theta
\end{pmatrix}
\]

\[
\text{where by the analogy to Eq.\textcolor{red}{(28)}, the Green’s functions (depending on the matrix $\hat{Q}$) read}
\]

\[
G_{A,R}(\hat{Q}) = \frac{ne^{\frac{\mathcal{V}^2}{2\Sigma_z}}}{\sqrt{\eta - p^2}} \quad \text{for } 1D
\]

\[
G_{A,R}(\hat{Q}) = \frac{nK_c(\pm 1)\sqrt{\eta - p^2}}{4\pi R} \quad \text{for } 2D
\]

\[
G_{A,R}(\hat{Q}) = \frac{ne^{\frac{\mathcal{V}^2}{2\Sigma_z}}}{\sqrt{\eta - p^2}} \quad \text{for } 3D,
\]
and the terms in the exponent have the explicit form
\[
\text{Str} \frac{\delta^2}{\delta z^2} \tilde{F} = (q_1 - q_2) \cos 2\theta - i (\tilde{q}_1 - \tilde{q}_2) \cos 2\theta
\]
\[
\text{Str} \tilde{F}^2 = q_1^2 + q_2^2 + \tilde{q}_1^2 + \tilde{q}_2^2 + 2\kappa_1 \kappa_1 + 2\kappa_2 \kappa_2
\]
\[
\text{Str} \left( \frac{\delta^2}{\delta z^2} \tilde{F} \right)^2 = (q_1^2 + q_2^2) \cosh 2\theta - 2q_1 q_2 \sinh^2 2\theta + (\tilde{q}_1^2 + \tilde{q}_2^2) \cos 2\theta + 2\tilde{q}_1 \tilde{q}_2 \sin^2 2\theta +
2(\kappa_1 \kappa_1 + \kappa_2 \kappa_2) \cosh 2\theta \cos 2\theta - 2i \left( e^{2i(\Phi + \phi)} \kappa_2 \kappa_2 + e^{-2i(\Phi + \phi)} \kappa_1 \kappa_1 \right) \sin 2\theta \sin 2\theta.
\]

The integral over \(d\tilde{F}\) can be evaluated by the saddle point method. The main contribution, generally speaking, comes from a saddle point where the matrix elements \(q_1, q_2, \tilde{q}_1, \tilde{q}_2\) of \(\tilde{F}\) are shifted to the complex plane, whence the matrix \(\tilde{F}\) has a diagonal form \(\tilde{P}_s\) no longer corresponding to Eq. (73) and independent on Grassmann variables. The latter, as well as the deviations of the regular variables are considered as perturbations. Note that the pre-exponential factor does not change as the result of integration over these perturbations, since the Grassmann integration around \(\tilde{P}_s\) compensate for the pre-factor resulting from the integration over the regular deviations. Also note that the among different possibilities, the saddle point has to be chosen in such a way that the integral over \(d\tilde{F}\) converges. Moreover, the choice \(q_1 = -q_2 = i\tilde{q}_1 = -i\tilde{q}_2 = i\kappa\) at the saddle point allows one to get rid of the large terms \(N\text{Str} \tilde{P}_s^2\) and \(\text{Str} \ln(\eta - q^2 - \tilde{P}_s)\), and Eq. (74) takes the form
\[
P(\zeta, \eta, z) = n^2 \int d\tilde{F} e^{-2N\zeta \eta^2 (\cosh 2\theta - \cos 2\theta)}
\]
\[
e^{-2N\left(1-e^{-z}\right)\eta^2 \left(\cosh^2 2\theta - \cos^2 2\theta\right)}
\]
\[
\text{Str} \frac{\delta^2}{\delta z^2} G_{A,R} \left( \tilde{P}_s \right) \text{ Str} \frac{\delta^2}{\delta z^2} G_{A,R} \left( \tilde{P}_s \right)
\]
\[
= \left( \frac{1}{\pi^2} \cosh 2\theta - \cos 2\theta \right)^2 \left( G_{A,R} \right)^2
\]
where the choice between the advanced and retarded Green’s functions depends on the position of the matrix element \(q_1 = \pm i\kappa\) on the complex plane.

Since \(\text{Str} \frac{\delta^2}{\delta z^2} G = \alpha(\cosh 2\theta - \cos 2\theta) G_{A,R}^2 - G_{A,R} - \sin^2 \theta (G_{R,G} - G_{A})\) and \(\text{Str} \frac{\delta^2}{\delta z^2} \tilde{G} = \beta(\cosh 2\theta - \cos 2\theta) G_{A,R}^2 + \sin^2 \theta (G_{R,G} - G_{A})\) by integration over \(d\Phi d\phi d\tilde{A} d\tilde{\phi} d\tilde{R} d\tilde{\phi} d\tilde{R}\), one can explicitly find the pre-exponential factors in Eq. (74)
\[
\text{Str} \frac{\delta^2}{\delta z^2} G \text{ Str} \frac{\delta^2}{\delta z^2} \tilde{G} = -\left( \frac{1}{\pi^2} \cosh 2\theta - \cos 2\theta \right)^2 \left( G_{A,R} \right)^2
\]
with the Green’s functions
\[
G_{A} \rightarrow \frac{n e^{-i\pi/4} \sqrt{\gamma}}{4\pi R} \quad G_{R} \rightarrow \frac{n e^{-i\pi/4} \sqrt{\gamma}}{2\sqrt{\gamma}} \quad \text{for } 1D
\]
\[
G_{A} \rightarrow \frac{n e^{-i\pi/4} \sqrt{\gamma}}{4\pi R} \quad G_{R} \rightarrow \frac{n e^{-i\pi/4} \sqrt{\gamma}}{2\sqrt{\gamma}} \quad \text{for } 2D
\]
\[
G_{A} \rightarrow \frac{n e^{-i\pi/4} \sqrt{\gamma}}{2\pi R} \quad G_{R} \rightarrow \frac{n e^{-i\pi/4} \sqrt{\gamma}}{4\pi R} \quad \text{for } 3D.
\]

The replacement
\[
\cosh 2\theta \rightarrow Z, \cos 2\theta \rightarrow w
\]
followed by simple algebra makes the explicit expressions Eq. (71) for the generating function more compact
\[
P(\zeta, \eta, z) = \int d\tilde{F} e^{-2N\zeta \eta^2 (\cosh 2\theta - \cos 2\theta)}
\]
\[
e^{-2N\zeta \eta^2 (Z - w)} (G_{A} - G_{R})^2.
\]

At the limit \(z \rightarrow 0\) the position of the saddle point is independent from \(Z\) and \(w\), which immediately yields the tunneling probability
\[
P(\zeta, 0) = \frac{(G_{A} - G_{R})^2}{2} \int d\tilde{F} e^{-2N\zeta \eta^2 (Z - w)} (G_{A} - G_{R})^2
\]
\[
\int d\tilde{F} e^{-2N\zeta \eta^2 (Z - w)} (Z^2 - w^2)
\]
that have been kicked out from the condensate as the result of the tunneling.

Calculation of the integrals Eq. (72) for the probability results in
\[
P(\zeta, 0) = \frac{(G_{A} - G_{R})^2}{2} \frac{8 \sin^2 \left( N \zeta q_s \right)}{2N^2 \zeta^2 q_s^2},
\]

It is expedient to show in Fig. 11 the Fourier transforms over the frequency \(\zeta\) of the coefficients in front of the \((G_{A} - G_{R})^2\) in Eq. (76). The electron tunnelling probability through the condensate linearly grows as function of time and saturates at the Heisenberg time, which is a natural consequence of the employed 0D model of fluctuations (coordinate independent matrix \(Q\)). However, the typical saturation time \(t \sim N q_s\) depends on the energy \(\eta\) via the energy dependence of the position \(q_s(\eta)\) of the saddle point position in the complex plane, which we will find later on. Note that, a rapidly oscillating part of the coordinate dependence, which is present due to the interference beats between \(G_{A}\) and \(G_{R}\) for the leads size small compared to \(1/\sqrt{\eta}\), must disappear for large leads, and the combination \((G_{A} - G_{R})^2\) has to be replaced by \(2G_{A}G_{R}\).
Let us now calculate the number of atoms kicked out of the condensate. At this stage we have to take into account the fact that the frequency \( \zeta = \omega - i\gamma \) has an imaginary part \(-i\gamma\) originating from the fact that the electron leaves the condensate once it has reached the outgoing lead. This implies that the electron spends just a finite period of time in the condensate and therefore kicks just a finite number of atoms out of the latter. Technically, this manifests itself in the divergency of the integral for \( P'_{\pm}(\zeta, \eta, 0) \) emerging from Eq. (73) for the case \( \gamma = 0 \). By direct integration over \( dZdw \) in Eq. (75) for the case of large leads one finds

\[
P'_{\pm}(\zeta, \eta, 0) = -\frac{G_A G_R}{\pi^2 (2Nq_s\zeta)^3}.
\]

Fourier transformation of Eqs. (76) over \( \zeta \) has the form

\[
P(t, \eta, 0) = G_A G_R \frac{(t \Theta (t) - (t - 2Nq_s) \Theta (t - 2Nq_s))}{2N^2 q_s^3},
\]

\[
P'_{\pm}(t, \eta, 0) = -\frac{q_s G_A G_R \eta^2 \Theta (t)}{(Nq_s)^3},
\]

where \( \Theta (x) \) is the step function. Note that the time dependence of the tunneling probability comes from the spectrum statistics; it is typical of the chosen Gaussian unitary ensemble (GUE). In Fig. 12 we present these two dependences for \( \gamma \neq 0 \). The finite \( \gamma \) results in the net tunneling probability

\[
\int \gamma P(t, \eta, 0) dt = G_A G_R \frac{1 - e^{-2N\gamma q_s}}{2Nq_s}
\]

and the total number of depleted atoms

\[
\int P'_{\pm}(t, \eta, 0) dt = \frac{q_s}{4(N\gamma q_s)^3} \int G_A G_R dV
\]

It corresponds to the situation where the destructive interference increases in the course of time and stops the tunneling process after a time of the order at the Heisenberg time. At longer times, the electron does not present in the condensate and hence does not result in atomic losses.

We now concentrate on the position of the saddle point. To this end one has to consider the saddle-point conditions suggested by the requirement

\[
\frac{\partial}{\partial \eta} \left( N q_s^2 + \int \frac{V dp'}{(2\pi)^d} \ln |p - p'^2 - q_s| \right) = 0,
\]

where the cut-off momentum is given by the condition \( \frac{V}{(2\pi)^d} \int dp' = N \). One can consider the dimensionless density \( n = N/V \) as a small parameter. Taking the derivative one finds

\[
2nq_s = \int \frac{dp'}{(2\pi)^d (\eta - p'^2 - q_s)}.
\]

The left hand side of the equation contains the small parameter, and therefore the right hand side also should be small. The contribution to the integral on the right hand side comes from the domain of small momenta \( p'^2 - n^2/4 \) while \( \eta \sim q_s \sim 1 \). We therefore can neglect \( p'^2 \) in the denominator and arrive at

\[
2nq_s = \frac{1}{(\eta - q_s)} \int \frac{dp'}{(2\pi)^d} = \frac{n}{(\eta - q_s)},
\]

which immediately results in

\[
q_s = \frac{\eta \pm \sqrt{\eta^2 - 2}}{2}
\]

that is in the dependence typical of the random matrices and resulting in the Wigner semicircle law for the state density. Imaginary part of this quantity is the parameter \( q_s \) entering Eqs. (72), while the real part (which have been earlier ignored in the expressions for \( P \) and \( \delta \mathcal{N} \)) just gives a shift of the energy scale. Substitution of the
result Eq. (83) to Eqs. (72) yields

\[ G_{A;R} = \frac{n^{2-1/2}}{\sqrt{\pi} \sqrt{\eta^2 - 2}} \exp \left[ \pm i R \left( -\sqrt{\eta^2 - 2} \right)^{3/2} \right] \] for 1D

\[ G_{A;R} = \frac{n}{2\pi R} K_0 \left( \pm \sqrt{\eta^2 - 2} \right)^{1/2} \] for 2D

\[ G_{A;R} \to \frac{n}{4\pi R} \exp \left[ \pm i R \left( -\sqrt{\eta^2 - 2} \right)^{1/2} \right] \] for 3D.

(84)

Now one has to substitute the obtained Green’s functions to the expression Eqs. (84-81) for the net tunneling probability and the total number of atoms kicked out from the condensate.

\[ \int \gamma P(t, \eta, 0) dt = G_A G_R \frac{1 - e^{-2N\gamma q_s}}{2Nq_s} \] (85)

and the total number of depleted atoms

\[ \int P'_z(t, \eta, 0) dt d\mathcal{V} = \frac{q_s}{4 (N\gamma q_s)} \int G_A G_R d\mathcal{V} \] (86)

These results are shown in Fig. (13) One sees that the dependences \( P(\eta) \) are not too much sensitive to the dimensionality of the problem, and the main contribution to the conductance comes from the edge of the electron levels band, which is formed as the result of the perturbation of the mean-field band by the strong quantum fluctuations of the atomic density in the condensate. The dependences \( \delta N(\eta) \) have more pronounced dependence on energy via dependence of the volume occupied by the tunneling electron on the spatial dimensionality.

[1] S. N. Bose Wärme- gleichgewicht in Strahlungsfeld bei Anwesenheit von Materie, Zeitschrift für Physik, 24, 384-393, (1924)
[2] A. Einstein, Quantentheorie des einatomigen idealen Gases, Sitzungsberichte der Preussischen Akademie der Wissenschaften (Berlin), Physikalisch-mathematische Klasse, vom 10 Juli 1924, 261-267,(1924), and vom 8 Januar 1925, 33-14, (1925).
[3] L. Landau, Theory of the Superfluidity of Helium II, Phys. Rev. 60, 356-358 (1941)
[4] N.N. Bogolyubov, Izv. Akad. Nauk SSSR, Ser. Fiz. 11, 77 (1947)
[5] V.L. Ginzburg and L.D. Landau, Zh. Eksp. Teor. Fiz. 20, 1064 (1950);
[6] L.P. Pitaevskii, Vortex lines in an imperfect Bose gas, Zh. Eksp. Teor. Fiz., 1961, V.40, 646-651.(Sov. Phys. JETP, 13, 451-454,(1961));
[7] E.P. Gross, Structure of a quantized vortex in boson systems, Nuovo Cimento, 20 (3), 454-477, (1961)
[8] K.B. Davis, M.O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, and W. Ketterle Bose-Einstein Condensation in a Gas of Sodium Atoms. Physical Review Letters 75, 3969-3973 (1995)
[9] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E. A. Cornell, Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor Science 269, 198 - 201 (1995)
[10] C. C. Bradley, C. A. Sackett, J. J. Tollett, and R. G. Hulet, Evidence of Bose-Einstein Condensation in an Atomic Gas with Attractive Interactions, Phys. Rev. Lett. 75, 1687 - 1690, (1995);
[11] V. De Georgio and M. O. Scully, Analogy between the Laser Threshold Region and a Second-Order Phase Transition, Phys. Rev. A2, 1170-1177, (1970)
[12] A. N. Oraevsky, Bose condensate from the standpoint of laser physics, QUANTUM ELECTRONICS, 31 (12), 1038-1057, (2001)
[13] R. J. Glauber, Coherent and Incoherent States of the Radiation Field, Phys. Rev. 131, 2766 - 2788 (1963)
[14] E. C. G. Sudarshan, Equivalence of Semiclassical and Quantum Mechanical Descriptions of Statistical Light Beams, Phys. Rev. Lett. 10, 277 - 279 (1963)
[15] Willis E. Lamb, Jr. Theory of an Optical Maser, Phys. Rev. 134, A1429 - A1450 (1964)
[16] L. Mandel and E. Wolf, Coherence Properties of Optical Fields, Rev. Mod. Phys. 37, 231 - 287 (1965)
[17] M. Lax, Quantum Noise. IV. Quantum Theory of Noise Sources, Phys. Rev. 145, 110 - 129 (1966)

[18] V. L. Ginzburg and L. P. Pitaevskii, On the theory of superfluidity, Zh. Eksp. Teoret. Fiz., 34, 1240-1245, (1958) (Soviet Physics. JETP 7, 858-861, (1958))

[19] Discussing the replacement of the field operators by their classical mean values, N.N Bogolyubov in Ref. 12 gives a reference to §63 Waves and Bose-Einstein particles of the second edition of Dirac's textbook Principles of Quantum Mechanics. However, this section has been removed and replaced by the author in the later editions of the book.

[20] Review of the results concerning Bose Condensation is not the subject of our paper. Therefore, not pretending to give a comprehensive list of publications on this issue, we just mention the book edited by A. Griffin, D. W. Snoke, and S. Stringari Bose-Einstein Condensation Cambridge University Press, New York, 1995, which contains a collection of papers addressing the main fundamental aspects of the problem, and the review paper by P. Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, Theory of Bose-Einstein condensation in trapped gases Rev. Mod. Phys., 71, (1999)

[21] M. R. Andrews, C. G. Townsend, H.-J. Miesner, D. S. Durfee, D. M. Kurn, and W. Ketterle, Observation of interference between two Bose condensates Science 275: 697-641 (1997)

[22] Y. Castin and J. Dalibard, The relative phase of two Bose-Einstein condensates, Phys. Rev. A 55, 4330 (1997)

[23] A. Sinatra, P. Fedichev, Y. Castin, J. Dalibard, and G. Shlyapnikov, Dynamics of two interacting Bose-Einstein condensates, Phys. Rev. Lett. 82, 251 (1999)

[24] M.R. Matthews, B.P. Anderson, P.C. Haljan, D.S. Hall, C.E. Wieman, E.A. Cornell, Vortices in a Bose-Einstein Condensate, Phys. Rev. Lett. 83, 2498-2501 (1999)

[25] K.W. Madison, F. Chevy, W. Hohlbein, and J. Dalibard, Vortices in a stirred Bose-Einstein condensate, Phys. Phys. Rev. Lett. 84, 806 - 809 (2000)

[26] M. R. Andrews, D. M. Kurn, H.-J. Miesner, D. S. Durfee, C. G. Townsend, S. Inouye, and W. Ketterle, Propagation of Sound in a Bose-Einstein Condensate, Phys. Rev. Lett. 79, 553 - 556 (1997)

[27] V. M. Perez-Garcia, H. Michinel, J. I. Cirac, M. Lewenstein, and P. Zoller, Low Energy Excitations of a Bose-Einstein Condensate: A Time-Dependent Variational Analysis, Phys. Rev. Lett. 77, 5320 - 5323 (1996)

[28] D. H. J. O’Dell, S. Giovanazzi, and C. Eberlein, Exact Hydrodynamics of a Trapped Dipolar Bose-Einstein Condensate, Phys. Rev. Lett. 92, 250401 (2004)

[29] D. S. Lee, C. Y. Lin, and R. J. Rivers, Derivation of hydrodynamics for the gapless mode in the BEC-BCS crossover from the exact one-loop effective action, Phys Rev Lett. 98, 020603, (2007)

[30] Z. Hadzibabic, P. Krüger, M. Cheneau, S. P. Rath, and J. Dalibard, The trapped two-dimensional Bose gas : from Bose-Einstein condensation to Berezinskii-Kosterlitz-Thouless physics.

[31] U. Gavish, and Yvan Castin, Matter-Wave Localization in Disordered Cold Atom Lattices, Phys. Rev. Lett. 95, 020401 (2005)

[32] R. C. Kuhn, C. Miniatura, D. Delande, O. Sigwarth, and C. A. Müller Localization of Matter Waves in Two-Dimensional Disordered Optical Potentials, Phys. Rev. Lett. 95, 250403 (2005)

[33] L. Sanchez-Palencia, D. Clement, P. Lugan, P. Bouyer, G. V. Shlyapnikov, and A. Aspect Anderson Localization of Expanding Bose-Einstein Condensates in Random Potentials, Phys. Rev. Lett. 98, 210401 (2007)

[34] S. E. Skipetrov, A. Minguzzi, B. A. van Tiggeleen, and B. Shapiro, Anderson Localization of a Bose-Einstein Condensate in a 3D Random Potential, Phys. Rev. Lett. 100, 165301 (2008)

[35] L. Fallani, J. E. Lye, V. Guarrera, C. Fort, and M. Inguscio, Ultracold Atoms in a Disordered Crystal of Light: Towards a Bose Glass, Phys. Rev. Lett. 98, 130404 (2007)

[36] T. Schulte, S. Drenkelforth, J. Kruse, W. Ertmer, J. Arlt, K. Sacha, J. Zakrzewski, and M. Lewenstein, Routes Towards Anderson-Like Localization of Bose-Einstein Condensates in Disordered Optical Lattices, Phys. Rev. Lett. 95, 170411 (2005)

[37] M. Polini, R. Fazio, A. H. MacDonald, and M. P. Tosi, Realization of Fully Frustrated Josephson-Junction Arrays with Cold Atoms, Phys. Rev. Lett. 95, 010401 (2005)

[38] B. T. Seaman, M. Kraner, D. Z. Anderson, and M. J. Holland, Atomtronics: ultracold atom analogs of electronic devices. Phys. Rev. A 75, 023615, (2007)

[39] A. Ruschhaupt and J.G. Muga. Three-dimensional effects in atom diodes: Atom-optical devices for one-way motion. Phys. Rev. A 76, 013619, (2007)

[40] J.A. Stickney and A.A. Zozulya. Transistorlike behavior of a Bose–Einstein condensate in a triple-well potential. Phys. Rev. A 75 013608, (2007).

[41] J. Chabe, G. Lemarie, B. Gremaud, D. Delande, P. Szriftgiser, J.-C. Garreau, Experimental observation of the Anderson transition with atomic matter waves, arXiv:0709.4320v1;

[42] C. Fort, L. Fallani, V. Guarrera, J. E. Lye, M. Modugno, D. S. Wiersma, and M. Inguscio, Effect of Optical Disorder and Single Defects on the Expansion of a Bose-Einstein Condensate in a One-Dimensional Waveguide, Phys. Rev. Lett. 95, 170410 (2005)

[43] M. Rizzi, D. Rossini, G. De Chiara, S. Montangero, and R. Fazio, , Phase Diagram of Spin-1 Bosons on One-Dimensional Lattices, Phys. Rev. Lett. 95, 240404 (2005)

[44] L. Sanchez-Palencia1, and L. Santos, Bose-Einstein condensates in optical quasicrystal lattices, Phys. Rev. A 72, 053607 (2005)

[45] E. Altman, A. Polkovnikov, E. Demler, B. I. Halperin, and M. D. Lukin, Superfluid-insulator transition in a moving system of interacting bosons, Phys.Rev.Lett. 95, 020402 (2005)

[46] M. Lewenstein, A. Sanpera, V. Ahufinger, B. Damski, A. Sen, U. Sen, Ultracold atomic gases in optical lattices: mimicking condensed matter physics and beyond. Advances in Physics 56, 243-379 (2007)

[47] T. Schulte, S. Drenkelforth, G. Kleine Böning, W. Ertmer, J. Arlt, M. Lewenstein, and L. Santos, Dynamics of Bloch oscillations in disordered lattice potentials, Phys. Rev. A 77, 023610 (2008)

[48] R. Furth, , Zs. Phys. 48, 323,-(1928); Zs. Phys. 50, 310-(1928)

[49] F. London, On the Bose-Einstein Condensation, Phys. Rev. 54, 947 - 954 (1938)

[50] A. D. Galanin, Density fluctuations in an ideal Bose-Einstein gas, J. Exptl. Theoret. Phys. 10, 1267-1282 (1940)

[51] Yu. Kagan and B. V. Svistunov, Evolution of large-scale correlations in a strongly nonequilibrum Bose condensation process, Pis’sma Zh. Eksp. Teor. Fiz. 67, 495-501
(1998)

[52] Yu. Kagan and B. V. Svistunov, Evolution of Correlation Properties and Appearance of Broken Symmetry in the Process of Bose-Einstein Condensation Phys. Rev. Lett.79, 3331-3334, (1997)

[53] Yu. Kagan and A. E. Muryshev, The spectral properties of non-condensate particles in Bose-condensed atomic hydrogen Physics Letters A 278, 159-164, (2000).

[54] K. Groral, M. Gajda, and K. Rzazewski, Multi-mode description of an interacting Bose-Einstein condensate, OPTICS EXPRESS 8, 92-98 (2001).

[55] M. Scully and W. E. Lamb, Jr., Quantum Theory of an Optical Maser, Phys. Rev. Lett. 16, 853 - 855 (1966).

[56] M. O. Scully and M. S. Zubairy, Quantum Optics, Cambridge University Press, 1997.

[57] Vitaly V. Kocharovsky, Vladimir V. Kocharovsky, M. Holthaus, C. H. Raymond Ooi, A. Svidzinsky, W. Ketterle, and M. O. Scully, Fluctuations in Ideal and Interacting Bose-Einstein Condensates: From the laser phase transition analogy to squeezed states and Bogoliubov quasiparticles, Advances in Atomic, Molecular and Optical Physics. 53, 291 (2006).

[58] V.V. Kocharovsky, V.V. Kocharovsky, and M.O. Scully, Condensate statistics in interacting and ideal dilute Bose gases, Phys. Rev. Lett. 84, 2306-2309 (2000).

[59] P. W. Anderson, Absence of diffusion in certain random lattices, Phys. Rev. 109, 1492-1505 (1958).

[60] I. M. Lifshits, S. A. Gredeskul, L. A. Pastur, Introduction to the theory of disordered systems.- New-York: Ets. Wilet, 1988. 462p, §30

[61] K. Efetov, Supersymmetry in Disorder and Chaos, Cambridge University Press 1997.

[62] Note that the situation differs from the inelastic tunneling of electrons through mesoscopic quantum dots with the allowance of the electron-phonon interaction, where the tunneling rate is of the order of the phonon frequencies. This case is discussed for example in the paper by K. Haule and J. Bonca, Inelastic tunneling through mesoscopic structures, Phys. Rev. B 59, 13087-13093 (1999).

[63] S. Cowell, H. Heiselberg, I. E. Mazets, J. Morales, V. R. Pandharipande, and C. J. Pethick, Cold Bose Gases with Large Scattering Lengths, Phys. Rev. Lett. 88, 210403 (2002).

[64] R. Cote, V. Kharchenko, and M. D. Lukin, Mesoscopic molecular ions in Bose-Einstein condensates, Phys. Rev. Lett. 89, 093001 (2002).

[65] R. Côté, From Classical Mobility to Hopping Conductivity: Charge Hopping in an Ultracold Gas, Phys. Rev. Lett. 85, 5316-5319 (2000).

[66] A. M. Dykhne and A. G. Rudavets, Broken Symmetry and Coherence of Molecular Vibrations in Tunnel Transistors, in "Decoherence, Entanglement and Information Protection in Complex Quantum Systems, (V. M. Akulin, A. Sarfati, G. Kurizki, and S. Pellegrin, Ed.) NATO Science Series, Mathematics, Physics and Chemistry, 189, p. 635-676, Springer 2005.

[67] In the textbook by E. M. Lifshits and L. P. Pitaevskii, Statistical Physics Part 2 §26 the c-number A is denoted by Ξ.

[68] Yu. F. Bydin, Zh. Eksperim. i Teor. Fiz. 46, 1612 (1964) [JETP 19, 1091 (1964)].

[69] V. V. Petrunin, H. H. Andersen, P. Balling, and T. Andersen, Structural Properties of the Negative Calcium Ion: Binding Energies and Fine-Structure Splitting, Phys. Rev. Lett. 76, 744-747 (1996).

[70] B. Bussery-Honvault, J.-M. Launay and R. Moszynski, Phys. Rev. A 68, 032718 (2003).

[71] R. Ciurylo, E. Tiesinga, and P. S. Julienne, Optical tuning of the scattering length of cold alkaline-earth-metal atoms, Phys. Rev. A 71, 030701 (2005).

[72] D. P. Hansen, J. R. Mohr, and A. Hemmerich, Magnetic trapping of metastable calcium atoms, PRA 67, 021401(R), (2003).

[73] B. Kozinsky and N. Marzari, Static dielectric properties of carbon nanotubes from first principles cond-mat/0602599v1

[74] R. Ferranti, S. Scheel, and P. L. Knight, Trapping cold atoms near carbon nanotubes: Thermal spin flips and Casimir-Polder potential, PRA 75 062905 (2007).

[75] That is, involving only a single Q configuration stem from the semiclassical trace or Itzykson–Zuber integral formuluæ.