Classification of Conic Bundles on a Rational Elliptic Surface in any Characteristic

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Abstract

Let $X$ be a rational elliptic surface with elliptic fibration $\pi : X \to \mathbb{P}^1$ over an algebraically closed field $k$ of any characteristic. Given a conic bundle $\varphi : X \to \mathbb{P}^1$ we use numerical arguments to classify all possible fibers of $\varphi$ and study the interplay between singular fibers of $\pi$ and $\varphi$.

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1 Introduction

Let $S$ be a smooth, projective surface over a field $k$. We define a conic bundle on $S$ in the most natural way, namely as a morphism onto a curve $S \to C$ whose general fiber is a smooth, rational curve. A prominent case where conic bundles arise is in the classification of minimal models of $k$-rational surfaces. Iskovskikh showed [Isk79, Theorem 1] that if $S$ is a $k$-minimal rational surface, then $S$ is either (a) $\mathbb{P}^2_k$, (b) a quadric on $\mathbb{P}^3_k$, (c) a del Pezzo surface, or (d) $S$ admits a conic bundle such that its singular fibers are isomorphic to a pair of lines meeting at a point. The latter was named a standard conic bundle by Manin and Tsfasman [MT86, Subsection 2.2]. We remark that the notion of standard conic bundle is often extended to higher dimension and plays a role in the classification of threefolds over $\mathbb{C}$ in the Minimal Model Program (see, for example, [Sar80], [Isk87] and the survey [Pro18]).

We are concerned more specifically with conic bundles on a rational surface $S$ over a field $k$ of any characteristic with an elliptic fibration $\pi : X \to \mathbb{P}^1$, i.e. a genus 1 fibration with a section. This is motivated by results and techniques from [Sal12], [LS20], [GS17], [GS20] and [AGL16], which we now briefly explain.

In [Sal12], $k$ is a number field and the goal is to study sets of fibers $\pi^{-1}(t)$ with $t \in \mathbb{P}^1(k)$ such that the Mordell-Weil rank $r_t$ of the fiber is greater than the generic rank $r$ of the fibration. The presence of a bisection, i.e. a rational curve $C \subset X$ such that $C \cdot (-K_X) = 2$ induces a family of bisections, which turns out to be a conic bundle on $X$. The author explores the existence of such conic bundles and proves that the sets $\{r_t \geq r+1\}$ and $\{r_t \geq r+2\}$ are infinite under certain hypothesis. This strategy was later refined in [LS20] — using, for example, one conic bundle instead of two — in order to study the structure of these sets in view of the Hilbert property [Ser08, Chapter 3]. Further developments are found in [CS22], where special conic bundles are considered in order to study the set $\{r_t \geq r+3\}$, and also in [HS19], where many ideas are extended from elliptic fibrations to families of Abelian varieties over a rational curve.

When $k$ is algebraically closed with characteristic zero, the existence of conic bundles on $X$ is also used in [GS17], [GS20] in order to classify elliptic fibrations on K3 surfaces which are quadratic covers of $X$. More precisely, given a degree two morphism $f : \mathbb{P}^1 \to \mathbb{P}^1$ ramified away from nonreduced fibers of $\pi$, the induced K3 surface is $X' := X \times_f \mathbb{P}^1$. The base change also gives rise to an elliptic fibration $\pi' : X' \to \mathbb{P}^1$ and a degree two map $f' : X' \to X$. By composition with $f'$, every conic bundle on $X$ induces a genus 1 fibration on $X'$, oftentimes having a section, in which case we get elliptic fibrations distinct from $\pi$.

In a different context, conic bundles also appear in [AGL16]. Here $k = \mathbb{C}$ and the goal is to find generators for the Cox ring $\mathcal{R}(X) := \bigoplus_{[D]} H^0(X, \mathcal{O}_X(D))$, where $[D]$ runs through $\text{Pic}(X)$. Given a rational elliptic fibration $\pi : X \to \mathbb{P}^1$, the ring $\mathcal{R}(X)$ is finitely generated if and only if $X$ is a Mori Dream Space [HK00, Proposition 2.9], which in turn is equivalent to $\pi$ having generic rank zero [AL13, Corollary 5.4]. Assuming this is the case, the authors show that in many configurations of $\pi$ each minimal set of generators of $\mathcal{R}(X)$ must contain an element $g \in H^0(X, \mathcal{O}_X(D))$, where $D$ is a fiber of a conic bundle on $X$, whose possibilities are explicitly described.

This argues for a detailed analysis of conic bundles on rational elliptic surfaces, which is what we aim for. We explore ideas that are either implicit or mentioned in passing throughout [Sal12], [LS20], [GS17], [GS20], [AGL16] and make generalizations whenever they apply.

The paper is organized as follows. We set the preliminaries in Section 2, where we define an elliptic surface, state some of its properties, define conic bundles and list some general facts about
divisors, fibers and linear systems. Section 3 is dedicated to showing that conic bundles can be characterized numerically through a bijective correspondence with certain classes of the Néron-Severi group, in a sense made precise in Theorem 3.8. In Section 4 we prove our first result, which is the complete classification of conic bundle fibers in arbitrary characteristic.

**Theorem 4.2.** Let $X$ be a rational elliptic surface with elliptic fibration $\pi : X \to \mathbb{P}^1$ and let $\varphi : X \to \mathbb{P}^1$ be a conic bundle. If $D$ is a fiber of $\varphi$, then the intersection graph of $D$ fits one of the following types. Conversely, if $D$ fits any of these types, then $|D|$ induces a conic bundle $X \to \mathbb{P}^1$.

| Type       | Intersection Graph |
|------------|--------------------|
| 0          | $\begin{array}{c} \star \end{array}$ |
| $A_2$      | $\begin{array}{c} 1 \hspace{1cm} 1 \end{array}$ |
| $A_n \ (n \geq 3)$ | $\begin{array}{c} 1 \hspace{0.5cm} 1 \hspace{0.5cm} 1 \hspace{0.5cm} 1 \hspace{0.5cm} 1 \hspace{0.5cm} 1 \end{array}$ |
| $D_3$      | $\begin{array}{c} 1 \hspace{0.5cm} 2 \hspace{0.5cm} 1 \end{array}$ |
| $D_m \ (m \geq 4)$ | $\begin{array}{c} 2 \hspace{0.5cm} 2 \hspace{0.5cm} 2 \hspace{0.5cm} 2 \hspace{0.5cm} 2 \hspace{0.5cm} 1 \hspace{0.5cm} 1 \end{array}$ |

$\star$ smooth, irreducible curve of genus zero  
$\circ (-1)$-curve (section of $\pi$)  
$\bullet (-2)$-curve (component of a reducible fiber of $\pi$)

In Section 5 we investigate how the fibers of $\pi$ affect the possibilities for conic bundles on $X$ and obtain our second result.

**Theorem 5.2.** Let $X$ be a rational elliptic surface with elliptic fibration $\pi : X \to \mathbb{P}^1$. Then the following statements hold:

a) $X$ admits a conic bundle with an $A_2$ fiber $\Rightarrow \pi$ is not extremal (i.e. has positive generic rank).
b) $X$ admits a conic bundle with an $A_n \ (n \geq 3)$ fiber $\Leftrightarrow \pi$ has a reducible fiber other than $\Pi^*$.  
c) $X$ admits a conic bundle with a $D_3$ fiber $\Leftrightarrow \pi$ has at least two reducible fibers.  
d) $X$ admits a conic bundle with a $D_m \ (m \geq 4)$ fiber $\Leftrightarrow \pi$ has a nonreduced fiber or a fiber $I_n \ (n \geq 4)$.  

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At last, in Section 6, we describe a method for constructing conic bundles from pencils of plane curves with genus 0, which we apply in Section 7 to produce examples.

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2 Set up

In what follows, all surfaces are projective and smooth over an algebraically closed field $k$ of any characteristic, unless otherwise specified. We always use $X$ to denote an elliptic surface. In Subsection 2.1 we define elliptic surfaces and refer to classical results. In Subsection 2.2 we cover some well-known facts about rational elliptic surfaces and present two additional lemmas. Conic bundles are defined in Subsection 2.3.

2.1 Elliptic surfaces

Definition 2.1. A surface $X$ is called an elliptic surface if there is smooth projective curve $C$ and a surjective morphism $\pi : X \to C$, called an elliptic fibration, such that

i) The fiber $\pi^{-1}(t)$ is a smooth genus 1 curve for all but finitely many $t \in C$.

ii) (existence of a section) There is a morphism $\sigma : C \to X$ such that $\pi \circ \sigma = \text{id}_C$, called a section.

iii) (relative minimality) No fiber of $\pi$ contains an exceptional curve in its support (i.e., a smooth rational curve with self-intersection $-1$).

Remark 2.2. Condition iii) can be understood as an extra hypothesis for $\pi$. For our purposes it is a natural one, since it assures that the fibers are as in Kodaira’s classification (Theorem 2.4) and that some convenient properties hold for rational elliptic surfaces (see Theorem 2.8 and Theorem 2.9).

Remark 2.3. Given $\sigma : C \to X$ as in Definition 2.1, the curve $P := \sigma(C)$ on $X$ is also called a section. The image $\sigma(C)$ is isomorphic to $C$ and meets a general fiber of $\pi$ at one point. Conversely, for every smooth curve $P \subset X$ meeting a general fiber of $\pi$ at one point, there is a morphism $\sigma : C \to X$ such that $\pi \circ \sigma = \text{id}_C$ and $\sigma(C) = P$ [SS10, Subsection 3.4].
We refer to some classical results on elliptic surfaces.

**Theorem 2.4.** [Tat75, Section 6] Let $\pi : X \to C$ be an elliptic fibration. If $F$ a singular fiber of $\pi$, then all possibilities for $F$ are listed below. The symbols $\tilde{A}_n, \tilde{D}_n, \tilde{E}_n$ indicate the type of extended Dynking diagram formed by the intersection graph of $F$.

Moreover, when $F$ is irreducible (namely of types $I_1$ and II), it is a singular, integral curve of arithmetic genus 1. When $F$ is reducible, all members in its support are smooth, rational curves with self-intersection $-2$.

**Remark 2.5.** Tate’s algorithm for identifying singular fibers involves local Weierstrass forms and is valid over perfect fields (e.g. algebraically closed). This local analysis can be pathological in characteristics 2, 3 but even then all possible singular fibers are still covered by the list in Theorem 2.4 [Tat75, Section 6]. We note that for non-perfect fields, new fiber types may occur (see [Szy04]). In this paper we do not deal with local Weierstrass forms but only with the numerical behavior of fibers as divisors, so we are free to use Theorem 2.4 over algebraically closed fields of any characteristic.

**Theorem 2.6.** [Shi90, Thm. 3.1] On an elliptic surface $X$, algebraic and numerical equivalences coincide, i.e. $D_1, D_2 \in \text{Div}(X)$ are equivalent in $\text{NS}(X) \iff D_1 \cdot D = D_2 \cdot D$ for every $D \in \text{Div}(X)$. 

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![Dynkin diagrams for singular fibers](image-url)
2.2 Rational elliptic surfaces

**Definition 2.7.** We say that $X$ is a rational elliptic surface if there is an elliptic fibration $\pi : X \to C$ and $X$ is a rational surface.

We exhibit a standard method for constructing rational elliptic surfaces. Since our base field $k$ is algebraically closed, every elliptic surface over $k$ can be obtained using this method (Theorem 2.8), which is not the case over arbitrary fields.

The construction works as follows. Let $F, G$ be cubics on $\mathbb{P}^2_k$, at least one of them smooth. The intersection $F \cap G$ has nine points counted with multiplicity and the pencil of cubics $P := \{sF + tG = 0 \mid (s : t) \in \mathbb{P}^1\}$ has $F \cap G$ as base locus. Let $\phi : \mathbb{P}^2 \dasharrow \mathbb{P}^1$ be the rational map associated to $P$. The blowup $\pi : X \to \mathbb{P}^2$ at the base locus resolves the indeterminacies of $\phi$ and yields a surface with an elliptic fibration $\pi : X \to \mathbb{P}^1$.

\[ X \xrightarrow{p} \mathbb{P}^2 \dasharrow \mathbb{P}^1 \]

By construction, $X$ is rational. Conversely, every rational elliptic surface over $k$ (which is algebraically closed) can be obtained by this method.

**Theorem 2.8.** [CD89, Theorem 5.6.1] Every rational elliptic surface over an algebraically closed field is isomorphic to the blowup of $\mathbb{P}^2$ at the base locus of a pencil of cubics.

We mention some other distinguished features of rational elliptic surfaces.

**Theorem 2.9.** [SS10, Section 8.2] Let $X$ be a rational elliptic surface and $\pi : X \to \mathbb{P}^1$ its elliptic fibration. Then

\[ \text{i) } \chi(X, \mathcal{O}_X) = 1, \text{ where } \chi(X, \mathcal{O}_X) = h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X). \]

\[ \text{ii) } -K_X \text{ is linearly equivalent to any fiber of } \pi. \text{ In particular, } -K_X \text{ is nef.} \]

\[ \text{iii) Every section of } \pi \text{ is an exceptional curve (smooth, rational curve with self-intersection } -1). \]

We include two elementary results. Lemma 2.10 provides a simple test to detect fibers of the elliptic fibration and Lemma 2.11 describes the negative curves on a rational elliptic surface.

**Lemma 2.10.** Let $\pi : X \to \mathbb{P}^1$ be a rational elliptic fibration and $E$ an integral curve in $X$. If $E \cdot K_X = 0$, then $E$ is a component of a fiber of $\pi$. If moreover $E^2 = 0$, then $E$ is a fiber.

**Proof.** If $P \in E$, then the fiber $F := \pi^{-1}(\pi(P))$ intersects $E$ at $P$, i.e. $E \cap F \neq \emptyset$. On the other hand, $-K_X$ is linearly equivalent to $F$ by Theorem 2.9, so $E \cdot F = -E \cdot K_X = 0$. Hence $E$ must be a component of $F$. Assuming moreover that $E^2 = 0$, we prove that $E = F$. In case $F$ is smooth, this is clear. So we assume $F$ is singular and analyze its Kodaira type according to Theorem 2.4. Notice that $F$ is not reducible, otherwise $E$ would be a $(-2)$-curve, which contradicts $E^2 = 0$. Hence $F$ is either of type $I_1$ or $II$. In both cases $F$ is an integral curve, therefore $E = F$. ■
Lemma 2.11. Let \( \pi : X \to \mathbb{P}^1 \) be a rational elliptic fibration. Every negative curve on \( X \) is either a \((-1)\)-curve (section of \( \pi \)) or a \((-2)\)-curve (component from a reducible fiber of \( \pi \)).

Proof. Let \( E \) be any integral curve in \( X \) with \( E^2 < 0 \). By Theorem 2.9, \( -K_X \) is nef and linearly equivalent to any fiber of \( \pi \). So \( E \cdot (-K_X) \geq 0 \) and by adjunction \( 2p_a(E) - 2 = E^2 + E \cdot K_X < 0 \), which only happens if \( p_a(E) = 0 \). Consequently \( E^2 = -1 \) or \(-2\). In case \( E^2 = -2 \) we have \( E \cdot K_X = 0 \), so \( E \) is fiber a component by Lemma 2.10 and this fiber is reducible by Theorem 2.4. If \( E^2 = -1 \), by adjunction \( E \cdot (-K_X) = 1 \). But \(-K_X \) is lineary equivalent to any fiber, so \( E \) meets a general fiber at one point, therefore \( E \) is a section by Remark 2.3. ■

2.3 Conic bundles

The following definition of conic bundle is essentially identical to the ones in [LS20, Definition 2.5], [GS17, Definition 3.1]. The difference here is that \( k \) is algebraically closed with arbitrary characteristic.

Definition 2.12. A conic bundle on a surface \( S \) is a surjective morphism onto a smooth curve \( \varphi : S \to C \) whose general fiber is a smooth, irreducible curve of genus 0.

Remark 2.13. When \( S \) is rational, \( C \) is isomorphic to \( \mathbb{P}^1 \) by Lüroth’s theorem.

Remark 2.14. Definition 2.12 is more general than the one of standard conic bundle in [MT86, Subsection 2.2], in which every singular fiber is isomorphic to a pair of lines meeting at a point. Indeed we show in Theorem 4.2 that such a singular fiber is one among four possible types using our definition.

Remark 2.15. Definition 2.12 is geometric in nature. In Section 3 we show that a conic bundle may be equivalently defined numerically by a certain Néron-Severi class \( [D] \in \text{NS}(X) \).

2.4 General facts about divisors, fibers and linear systems

In this subsection we let \( S \) be a smooth projective surface (not necessarily elliptic) over an algebraically closed field. We list some elementary results involving divisors, fibers of morphisms \( S \to \mathbb{P}^1 \) and linear systems, none of which depends on the theory of elliptic surfaces.

Lemma 2.16. Let \( D \) be an effective divisor such that \( D \cdot C = 0 \) for all \( C \in \text{Supp} \, D \). Then \( D \) is nef and \( D^2 = 0 \).

Proof. To see that \( D \) is nef, just notice that \( D \cdot C' \geq 0 \) for every \( C' \notin \text{Supp} \, D \). To prove that \( D^2 = 0 \) let \( D = \sum n_i C_i \), where each \( C_i \) is in \( \text{Supp} \, D \). Then \( D^2 = D \cdot (\sum n_i C_i) = \sum n_i D \cdot C_i = 0 \). ■

The next lemma explores some properties of fibers of a surjective morphism \( f : S \to \mathbb{P}^1 \).

Lemma 2.17. Let \( F \) be a fiber of a surjective morphism \( f : S \to \mathbb{P}^1 \). Then the following hold:

a) \( F \cdot C = 0 \) for all \( C \in \text{Supp} \, F \).

b) If \( F \) is connected and \( E \) is a divisor such that \( \text{Supp} \, E \subset \text{Supp} \, F \), then \( E^2 \leq 0 \) and \( E^2 = 0 \) if and only if \( E = rF \) for some \( r \in \mathbb{Q} \).

c) If \( F_1, \ldots, F_n \) are the connected components of \( F \), then each \( F_i \) is nef with \( F_i^2 = 0 \) and \( F_i \cdot K_S \in 2\mathbb{Z} \).
Proof. a) Taking an arbitrary $C \in \text{Supp } F$ and another fiber $F' \neq F$, we have $F \cdot C = F' \cdot C = 0$.

b) This is Zariski’s lemma [Pet95, Ch. 6, Lemma 6].

c) Fix $i$. To prove that $F_i$ is nef and $F_i^2 = 0$ we show that $F_i \cdot C_i = 0$ for any $C_i \in \text{Supp } F_i$ then apply Lemma 2.16. Indeed, if $j \neq i$, the components $F_i, F_j$ are disjoint, so $F_i \cdot C_i = 0$. Since $F$ is a fiber and $C_i \in \text{Supp } F_i \subset \text{Supp } F$, then $F \cdot C_i = 0$ by a). Hence $F_i \cdot C_i = (F_1 + \ldots + F_n) \cdot C_i = F \cdot C_i = 0$, as desired. For the last part, by Riemann-Roch $F_i \cdot K_S = 2(\chi(S, O_S) - \chi(S, O_S(F_i))) \in 2\mathbb{Z}$. ■

This last lemma is a property of linear systems with no fixed components.

**Lemma 2.18.** Let $E, E'$ be effective divisors such that $E' \leq E$ and that the linear systems $|E|, |E'|$ have the same dimension. If $|E|$ has no fixed components, then $E' = E$.

**Proof.** The fact that $E' \leq E$ implies that $H^0(S, O_S(E'))$ is a subspace of $H^0(S, O_S(E))$. By hypothesis these spaces have the same dimension, so $H^0(S, O_S(E')) = H^0(S, O_S(E))$. Hence

$$|E| = \{ E + \text{div}(f) \mid f \in H^0(S, O_S(E)) \}$$

$$= \{ E' + \text{div}(f) + (E - E') \mid f \in H^0(S, O_S(E')) \}$$

$$= |E'| + (E - E').$$

Assuming $|E|$ has no fixed components, we must have $E - E' = 0$. ■

### 3 Numerical characterization of conic bundles

Let $\pi : X \to \mathbb{P}^1$ be a rational elliptic fibration. We give a characterization of conic bundles on $X$ which shows the numerical nature of conic bundles on rational elliptic surfaces. The motivation for this comes from the following. Let $\varphi : X \to \mathbb{P}^1$ be a conic bundle and $C$ a general fiber of $\varphi$, which is a smooth, irreducible curve of genus zero. Clearly $C$ is a nef divisor with $C^2 = 0$ and by adjunction $C \cdot (-K_X) = 2$. These three numerical properties are enough to prove that $|C|$ is a base point free pencil and consequently the induced morphism $\varphi_{|C|} : X \to \mathbb{P}^1$ is precisely $\varphi$.

Conversely, let $D$ be a nef divisor with $D^2 = 0$ and $D \cdot (-K_X) = 2$. Since numerical and algebraic equivalence coincide by Theorem 2.6, it makes sense to consider the class $[D] \in \text{NS}(X)$. The natural question is whether $[D]$ induces a conic bundle on $X$. The answer is yes, moreover there is a natural correspondence between such classes and conic bundles (Theorem 3.8), which is the central result of this section.

In order to prove this correspondence we need a numerical analysis of a given class $[D] \in \text{NS}(X)$ so that we can deduce geometric properties of the induced morphism $\varphi_{|D|} : X \to \mathbb{P}^1$, such as connectivity of fibers (Proposition 3.5) and composition of their support (Proposition 3.4). These properties are also crucial to the classification of fibers in Section 4.

**Definition 3.1.** A class $[D] \in \text{NS}(X)$ is called a conic class when

i) $D$ is nef.

ii) $D^2 = 0$.

iii) $D \cdot (-K_X) = 2$. 

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Lemma 3.2. Let \([D] \in \text{NS}(X)\) be a conic class. Then \(|D|\) is a base point free pencil and therefore induces a surjective morphism \(\varphi_{|D|} : X \to \mathbb{P}^1\).

Proof. By [Har95, Theorem III.1(a)], \(|D|\) is base point free and \(h^1(X, D) = 0\). We have \(\chi(X, \mathcal{O}_X) = 1\) by Theorem 2.9, and Riemann-Roch gives \(h^0(X, D) + h^2(X, D) = 2\), so we only need to prove \(h^2(X, D) = 0\). Assume by contradiction that \(h^2(X, D) \geq 1\). By Serre duality \(h^0(K_X - D) \geq 1\), so \(K_X - D\) is linearly equivalent to an effective divisor. Since \(D\) is nef, \((K_X - D) \cdot D \geq 0\), which contradicts \(D^2 = 0\) and \(D \cdot (−K_X) = 2\). ■

Remark 3.3. It follows from Lemma 3.2 that a conic class has an effective representative.

Notice that we do not know a priori that the morphism \(\varphi_{|D|} : X \to \mathbb{P}^1\) in Lemma 3.2 is a conic bundle. At this point we can only say that if \(C\) is a smooth, irreducible fiber of \(\varphi_{|D|}\), then \(g(C) = 0\) by adjunction. However it is still not clear whether a general fiber of \(\varphi_{|D|}\) is irreducible and smooth. We prove that this is the case in Proposition 3.6. In order to do that we need information about the components of \(D\) from Proposition 3.4 and the fact that \(D\) is connected from Proposition 3.5.

Proposition 3.4. Let \([D] \in \text{NS}(X)\) be a conic class. If \(D\) is an effective representative, then every curve \(E \in \text{Supp } D\) is a smooth rational curve with \(E^2 \leq 0\).

Proof. Take an arbitrary \(E \in \text{Supp } D\). By Lemma 3.2, \(D\) is a fiber of the morphism \(\varphi_{|D|} : X \to \mathbb{P}^1\) induced by \(|D|\), hence \(D \cdot E = 0\) by Lemma 2.16. Assuming \(E^2 > 0\) by contradiction, the fact that \(D^2 = 0\) implies that \(D\) is numerically equivalent to zero by the Hodge index theorem [Har77, Thm. V.1.9, Rmk. 1.9.1]. This is absurd because \(D \cdot (−K_X) = 2 \neq 0\), so indeed \(E^2 \leq 0\).

To show that \(E\) is a smooth rational curve, it suffices to prove that \(p_a(E) = 0\). By Theorem 2.9, \(−K_X\) is linearly equivalent to any fiber of \(\pi\), in particular \(−K_X\) is nef and \(E \cdot K_X \leq 0\). By adjunction \(2p_a(E) - 2 = E^2 + E \cdot K_X \leq 0\), so \(p_a(E) \leq 1\). Assume by contradiction that \(p_a(E) = 1\). This can only happen if \(E^2 = E \cdot K_X = 0\), so \(E\) is a fiber of \(\pi\) by Lemma 2.10. In this case \(E\) is linearly equivalent to \(−K_X\), so \(D \cdot E = D \cdot (−K_X) = 2\), which contradicts \(D \cdot E = 0\). ■

While Proposition 3.4 provides information about the support of \(D\), the next proposition states that \(D\) is connected, which is an important fact about the composition of \(D\) as a whole.

Proposition 3.5. Let \([D] \in \text{NS}(X)\) be a conic class. If \(D\) is an effective representative, then \(D\) is connected.

Proof. Let \(D = D_1 + ... + D_n\), where \(D_1, ..., D_n\) are connected components. By Lemma 2.17 c) each \(D_i\) is nef with \(D_i^2 = 0\) and \(D_i \cdot K_X \in 2\mathbb{Z}\). Since \(−K_X\) is nef by Theorem 2.9 and \(D \cdot (−K_X) = 2\), then \(D_{i_0} \cdot (−K_X) = 2\) for some \(i_0\) and \(D_i \cdot (−K_X) = 0\) for \(i \neq i_0\). In particular \([D_{i_0}] \in \text{NS}(X)\) is a conic class. By Lemma 2.18, both \(|D_i|, |D_{i_0}|\) are pencils, so \(D = D_{i_0}\) by Lemma 2.18. ■

We use Propositions 3.4 and 3.5 to conclude that \(\varphi_{|D|} : X \to \mathbb{P}^1\) is indeed a conic bundle.
**Proposition 3.6.** Let $[D] \in \text{NS}(X)$ be a conic class. Then all fibers of $\varphi_{[D]} : X \to \mathbb{P}^1$ are smooth, irreducible curves of genus 0 except for finitely many which are reducible and supported on negative curves. In particular, $\varphi_{[D]}$ is a conic bundle.

**Proof.** Let $F$ be a fiber of $\varphi_{[D]}$. Since $F$ is linearly equivalent to $D$, then $[F] = [D] \in \text{NS}(X)$, so $F$ is connected by Proposition 3.5. By Proposition 3.4 every $E \in \text{Supp} F$ has $g(E) = 0$ and $E^2 \leq 0$.

First assume $E^2 = 0$ for some $E \in \text{Supp} F$. Since $F$ is a connected fiber of $\varphi_{[D]}$, then $E = rF$ for some $r \in \mathbb{Q}$ by Lemma 2.17. Because $F \cdot (-K_X) = 2$, we have $E \cdot K_X = -2r$. By adjunction, $r = 1$, so $F = E$ is a smooth, irreducible curve of genus 0.

Now assume $E^2 < 0$ for every $E \in \text{Supp} F$. Then $F$ must be reducible, otherwise $E^2 = 0$. This gives a contradiction. Conversely, if $F$ is reducible, then $E^2 < 0$ for all $E \in \text{Supp} F$, otherwise $E^2 = 0$ for some $E$ and by the last paragraph $F$ is irreducible, which is a contradiction.

This shows that either $F$ is smooth, irreducible of genus 0 or $F$ is reducible, in which case $F$ is supported on negative curves. We are left to show that $\varphi_{[D]}$ has finitely many reducible fibers.

Assume by contradiction that there is a sequence $\{F_n\}_{n \in \mathbb{N}}$ of distinct reducible fibers of $\varphi_{[D]}$. In particular each $F_n$ is supported on negative curves, which are either $(-1)$-curves (sections of $\pi$) or $(-2)$-curves (components of reducible fibers) by Theorem 2.11.

Since $\pi$ has finitely many singular fibers, the number of $(-2)$-curves in $X$ is finite, so there are finitely many $F_n$ with $(-2)$-curves in its support. Excluding such $F_n$, we may assume all members in $\{F_n\}_{n \in \mathbb{N}}$ are supported on $(-1)$-curves. For each $n$, take $P_n \in \text{Supp} F_n$. The fibers $F_n, F_m$ are disjoint, so $P_n, P_m$ are disjoint when $n \neq m$. If we successively contract the exceptional curves $P_1, \ldots, P_t$, we are still left with an infinite set $\{P_n\}_{n > t}$ of exceptional curves, so we cannot reach a minimal model for $X$, which is absurd.

**Remark 3.7.** In characteristic zero the proof of Proposition 3.6 can be made simpler by applying Bertini’s theorem, which guarantees that the general fiber is smooth from the fact that $X$ is smooth.

We now prove the numerical characterization of conic bundles.

**Theorem 3.8.** Let $\pi : X \to \mathbb{P}^1$ be a rational elliptic fibration. If $[D] \in \text{NS}(X)$ is a conic class, then $[D]$ is a base point free pencil whose induced morphism $\varphi_{[D]} : X \to \mathbb{P}^1$ is a conic bundle. Moreover, the map $[D] \mapsto \varphi_{[D]}$ has an inverse $\varphi \mapsto [F]$, where $F$ is any fiber of $\varphi$. This gives a natural correspondence between conic classes and conic bundles.

**Proof.** Given a conic class $[D] \in \text{NS}(X)$, by Proposition 3.6 the general fiber of $\varphi_{[D]} : X \to \mathbb{P}^1$ is a smooth, irreducible curve of genus 0, so $\varphi_{[D]}$ is a conic bundle.

Conversely, if $\varphi : X \to \mathbb{P}^1$ is a conic bundle and $C$ is a smooth, irreducible fiber of $\varphi$, in particular $C^2 = 0$ and $g(C) = 0$. Clearly $C$ is nef, so $[C] \in \text{NS}(X)$ is a conic class. Conversely, any other fiber $F$ of $\varphi$ is linearly equivalent to $C$, therefore $[F] = [C] \in \text{NS}(X)$ and the map $\varphi \mapsto [F]$ is well defined.

We verify that the maps are mutually inverse. Given a class $[D]$ we may assume $D$ is effective since $[D]$ is a pencil, so $D$ is a fiber of $\varphi_{[D]}$ and $\varphi_{[D]}$ is sent back to $[D]$. Conversely, given a conic bundle $\varphi$ with a fiber $F$, then $\varphi_{[F]}$ coincides with $\varphi$ tautologically, so $[F]$ is sent back to $\varphi$. ■
Corollary 3.9. Every fiber of a conic bundle $\varphi : X \to \mathbb{P}^1$ is connected. Moreover, all fibers $\varphi$ are smooth, irreducible curves of genus 0 except for finitely many which are reducible and supported on negative curves.

Proof. Let $F$ be any fiber of $\varphi$. By Theorem 3.8, $[F]$ is a conic class and $\varphi|_F : X \to \mathbb{P}^1$ is precisely $\varphi$. Then $F$ is connected by Proposition 3.5 and the rest follows from Proposition 3.6. $\blacksquare$

Corollary 3.9 is the starting point for the classification of conic bundle fibers in Section 4.

4 Classification of conic bundle fibers

Let $\pi : X \to \mathbb{P}^1$ be a rational elliptic fibration. In this section we give a complete description of fibers of a conic bundle based on Corollary 3.9 in Section 3.

Lemma 4.1. Let $\varphi : X \to \mathbb{P}^1$ be a conic bundle and $D$ any fiber of $\varphi$. Then $D$ is connected and

(i) $D$ is either a smooth, irreducible curve of genus 0, or

(ii) $D = P_1 + P_2 + D'$, where $P_1, P_2$ are $(-1)$-curves (sections of $\pi$), not necessarily distinct, and $D'$ is either zero or supported on $(-2)$-curves (components of reducible fibers of $\pi$).

Proof. By Corollary 3.9, all fibers of $\varphi$ fall into category (i) except for finitely many that are reducible and supported on negative curves. Let $D$ be one of such finitely many.

From Lemma 2.11, Supp $D$ has only $(-1)$-curves (sections of $\pi$) or $(-2)$-curves (components of reducible fibers of $\pi$). By adjunction, if $C \in \text{Supp } D$ is a $(-2)$-curve, then $C \cdot (-K_X) = 0$, and if $P \in \text{Supp } D$ is a $(-1)$-curve, then $P \cdot (-K_X) = 1$. But $D \cdot (-K_X) = 2$, so we must have $D = P_1 + P_2 + D'$ with the desired composition. $\blacksquare$

At this point we have enough information about the curves that support a conic bundle fiber. It remains to investigate their multiplicities and how they intersect one another.
Theorem 4.2. Let $X$ be a rational elliptic surface with elliptic fibration $\pi : X \rightarrow \mathbb{P}^1$ and let $\varphi : X \rightarrow \mathbb{P}^1$ be a conic bundle. If $D$ is a fiber of $\varphi$, then the intersection graph of $D$ fits one of the types below. Conversely, if the intersection graph of a divisor $D$ fits any of these types, then $|D|$ induces a conic bundle $\varphi|_D : X \rightarrow \mathbb{P}^1$.

| Type   | Intersection Graph |
|--------|--------------------|
| 0      | $\ \ast$           |
| $A_2$  | $\circ \ 1 \ \circ$ |
| $A_n$  | $\circ \ 1 \circ \ 1 \circ \ 1 \circ$ for $n \geq 3$ |
| $D_3$  | $\circ \ 2 \ \circ$ |
| $D_m$  | $\circ \ 2 \circ \ 2 \circ \ 1 \circ$ for $m \geq 4$ |

* smooth, irreducible curve of genus 0
○ $(-1)$-curve (section of $\pi$)
∙ $(-2)$-curve (component of a reducible fiber of $\pi$)

Terminology. Before we prove Theorem 4.2, we introduce a natural terminology for dealing with the intersection graph of $D$. When $C, C' \in \text{Supp} \ D$ are distinct, we say that $C'$ is a neighbour of $C$ when $C \cdot C' > 0$. If $C$ has exactly one neighbour, we call $C$ an extremity. We denote the number of neighbours of $C$ by $n(C)$. A simple consequence of these definitions is the following lemma.

Lemma 4.3. If $D = \sum_i n_i E_i$ is a fiber of a morphism $X \rightarrow \mathbb{P}^1$, then $n(E_i) \leq -n_i^2 E_i^2$ for every $i$.

Proof. By definition of $n(E_i)$, clearly $n(E_i) \leq \sum_{j \neq i} E_i \cdot E_j$. Since $D \cdot E_i = 0$ by Lemma 2.17, then

\[0 = D \cdot E_i = \sum_j n_j E_j \cdot E_i = n_i E_i^2 + \sum_{j \neq i} n_j E_j \cdot E_i \geq n_i E_i^2 + \sum_{j \neq i} E_j \cdot E_i \geq n_i E_i^2 + n(E_i).\]
Proof of Theorem 4.2. We begin by the converse. If \( D \) fits one of the types, we must prove \([D] \in \text{NS}(X)\) is a conic class, so that \( \varphi[D] : X \to \mathbb{P}^1 \) is a conic bundle by Theorem 3.8. A case-by-case verification gives \( D \cdot C = 0 \) for all \( C \in \text{Supp} \ D \). Since \( D \) is effective, it is nef with \( D^2 = 0 \) by Lemma 2.16. The condition \( D \cdot (-K_X) = 2 \) is satisfied in type 0 by adjunction. We have \( D \cdot (-K_X) = 2 \) in types \( A_2, A_4 \) for they contain two distinct sections of \( \pi \) and also in types \( D_3, D_m \) for they contain a section with multiplicity 2. Hence \([D] \in \text{NS}(X)\) is a conic class, as desired.

Now let \( D \) be a fiber of \( \varphi \). By Lemma 4.1, \( D \) is connected and has two possible forms. If \( D \) is irreducible, we get type 0. Otherwise \( D = P_1 + P_2 + D' \), where \( P_1, P_2 \) are \((-1\text{-})\)-curves and \( D' \) is either zero or supported on \((-2\text{-})\)-curves. If \( D' = \sum_i n_i C_i \) then \( n(C_i) \leq 2n_i \) by Lemma 4.3. The bounds for \( n(P_1), n(P_2) \) depend on whether i) \( P_1 \neq P_2 \) or ii) \( P_1 = P_2 \). In what follows Lemma 2.17 a) is used implicitly several times.

i) \( P_1 \neq P_2 \). In this case \( n(P_1) \leq 1 \) and \( n(P_2) \leq 1 \). Since \( D \) is connected, both \( P_1, P_2 \) must have some neighbour, so \( n(P_1) = n(P_2) = 1 \), therefore \( P_1, P_2 \) are extremities. If the extremities \( P_1, P_2 \) meet, they form the whole graph, so \( D = P_1 + P_2 \). This is type \( A_2 \).

If \( P_1, P_2 \) do not meet, by connectedness there must be a path on the intersection graph joining them, say \( P_1, C_1, ..., C_k, P_2 \). Since \( P_1 \) is an extremity, it has only \( C_1 \) as a neighbour, so \( 0 = D \cdot P_1 = -1 + n_1 \) gives \( n_1 = 1 \). Moreover \( n(C_1) \leq 2 \) and by the position of \( C_1 \) in the path we have \( n(C_1) = 2 \). We prove by induction that \( n_i = 1 \) and \( n(C_i) = 2 \) for all \( i = 1, ..., k \). Assume this is true for \( i = 1, ..., \ell < k \). Then \( 0 = D \cdot C_\ell = 1 - 2 + n_{\ell+1} \), so \( n_{\ell+1} = 1 \). Moreover, \( n(C_{\ell+1}) \leq 2 \) and by the position of \( C_{\ell+1} \) in the path we have \( n(C_{\ell+1}) = 2 \). So the graph is precisely the chain \( P_1, C_1, ..., C_k, P_2 \). This is type \( A_n \) (\( n \geq 3 \)).

ii) \( P_1 = P_2 \). In this case \( n(P_1) \leq 2 \). We cannot have \( n(P_1) = 0 \), otherwise \( D^2 = (2P_1)^2 = -4 \), so \( n(P_1) = 1 \) or 2. If \( P_1 \) has two neighbours, say \( C_1, C_2 \), then \( 0 = D \cdot P_1 = -2 + n_1 + n_2 \), which only happens if \( n_1 = n_2 = 1 \). Moreover, \( n(C_1) \leq 2 \), so \( C_1 \) can possibly have another neighbour \( C_3 \) in addition to \( P_1 \). But then \( D \cdot P_1 = 0 \) gives \( n_3 = 0 \), which is absurd, so \( C_1 \) has only \( P_1 \) as a neighbour. By symmetry \( C_2 \) also has only \( P_1 \) as a neighbour. This is type \( D_3 \).

Finally let \( n(P_1) = 1 \) and \( C_1 \) be the only neighbour of \( P_1 \). Then \( C_1 \cdot P_1 = 0 \) when \( i > 1 \). Notice that \( C_1, C_i \) come from the same fiber of \( \pi \), say \( F \), otherwise \( C_i \) would be in a different connected component as \( C_1, P_1 \), which contradicts \( D \) being connected. The possible Dynkin diagrams for \( F \) are listed in Theorem 2.4. Since \( P_1 \) intersects \( F \) in a simple component, the possibilities are

In all diagrams above, \( D \cdot P_1 = 0 \) gives \( n_1 = 2 \). In the third diagram, \( D \cdot C_1 = 0 \) gives \( n_2 + n_m = 2 \), which only happens if \( n_2 = n_m = 1 \). But \( D \cdot C_2 = 0 \) gives \( n_3 = 0 \), which is absurd, so we rule out the third diagram. For the first two diagrams we proceed by induction: if \( k < m \) and \( n_1 = ... = n_k = 2 \), then \( D \cdot C_k = 0 \) gives \( n_{k+1} = 2 \), therefore \( n_1 = ... = n_m = 2 \). But in the first diagram \( D \cdot C_m = 0 \) gives \( n_m = 1 \), which is absurd, so the first diagram is also ruled out.

Now let \( C_{m+1}, C_{m+2} \) be the first elements in branches 1 and 2 respectively. Then \( D \cdot C_m = 0 \) gives \( n_{m+1} = n_{m+2} = 1 \). Consequently \( n(C_{m+1}) \leq 2 \) and \( n(C_{m+2}) \leq 2 \). If \( C_{m+1} \) has another neighbour \( C_{m+3} \) in addition to \( C_m \), then \( D \cdot C_{m+1} = 0 \) gives \( n_{m+3} = 0 \), which is absurd, so \( C_{m+1} \) is an extremity. By symmetry, \( C_{m+2} \) is also an extremity. This is type \( D_m \) (\( m \geq 4 \)).
5 Fibers of conic bundles vs. fibers of the elliptic fibration

Let $X$ be a rational elliptic surface with elliptic fibration $\pi : X \to \mathbb{P}^1$. The existence of a conic bundle $\varphi : X \to \mathbb{P}^1$ with a given fiber type is strongly dependent on the fiber configuration of $\pi$. This relationship is explored in Theorem 5.2, which provides simple criteria to identify when a certain fiber type is possible. Before we prove it, we need the following result about the existence of disjoint sections.

**Lemma 5.1.** If $X$ is a rational elliptic surface with at least two sections, then there exists a pair of disjoint sections.

**Proof.** Let $\text{MW}(\pi)$ be the Mordell-Weil group of $\pi$, whose neutral section we denote by $O$. By [OS91, Thm. 2.5], $\text{MW}(\pi)$ is generated by sections which are disjoint from $O$. Then there must be a generator $P \neq O$ disjoint from $O$, otherwise $\text{MW}(\pi) = \{O\}$, which contradicts the hypothesis. ■

We now state and prove the main result of this section.

**Theorem 5.2.** Let $X$ be a rational elliptic fibration with elliptic fibration $\pi : X \to \mathbb{P}^1$. Then the following statements hold:

a) $X$ admits a conic bundle with an $A_2$ fiber $\Rightarrow$ $\pi$ is not extremal (i.e. has positive generic rank).

b) $X$ admits a conic bundle with an $A_n$ ($n \geq 3$) fiber $\iff$ $\pi$ has a reducible fiber distinct from $\Pi^*$. 

c) $X$ admits a conic bundle with a $D_3$ fiber $\iff$ $\pi$ has at least two reducible fibers.

d) $X$ admits a conic bundle with a $D_m$ ($m \geq 4$) fiber $\iff$ $\pi$ has a nonreduced fiber or a fiber $I_n$ ($n \geq 4$).

**Proof.** a) If $\pi$ is extremal, then all sections of $\pi$ are torsion sections, therefore no two of them ever meet [SS19, Corollary 8.30], which makes type $A_2$ impossible.

b) Assume $X$ admits a conic bundle with an $A_n$ ($n \geq 3$) fiber. Since type $A_n$ contains a $(-2)$-curve, by Lemma 2.11, $\pi$ has a reducible fiber $F$. We claim that $F$ is not of type $\Pi^*$. Indeed, if this were the case, the Mordell-Weil group $\text{MW}(\pi)$ would be trivial [Per90], which is impossible, since the $A_n$ fiber of the conic bundle contains two distinct sections. Conversely, assume $\pi$ has a reducible fiber $F$ other than $\Pi^*$. Then $\text{MW}(\pi)$ is not trivial [Per90] and by Lemma 5.1 we may find two disjoint sections $P, P'$. Let $C, C' \in \text{Supp} \ F$ be the components hit by $P, P'$. Since $F$ is connected, there is a path $C, C_1, ..., C_\ell, C'$ in the intersection graph of $F$. Let $D := P + C + C_1 + ... + C_\ell + C' + P'$. By Theorem 4.2, $\varphi|_D : X \to \mathbb{P}^1$ is a conic bundle and $D$ is an $A_n$ fiber of it.
c) Assume $X$ admits a conic bundle with a $D_3$ fiber $D = C_1 + 2P + C_2$, where $C_1, C_2$ are $(-2)$-curves and $P$ is a section with $C_1 \cdot P = C_2 \cdot P = 1$. Since $P$ hits each fiber of $\pi$ at exactly one point, then $C_1, C_2$ must come from two distinct reducible fibers of $\pi$. Conversely, let $F_1, F_2$ be two reducible fibers of $\pi$. If $P$ is a section, then $P$ hits $F_i$ at some $(-2)$-curve $C_i \in \text{Supp } F_i$. Let $D := C_1 + 2P + C_2$. By Theorem 4.2, $\varphi|_D : X \to \mathbb{P}^1$ is a conic bundle and $D$ is a $D_3$ fiber of it.

![Diagram of fibers $F_1$ and $F_2$]

**Remark 5.3.** The converse of Theorem 5.2 a) is not true. Indeed, let $\pi$ with fiber configuration $(\text{III}^*, 3\text{I}_1)$, in which case the generic rank is 1. If $X$ admitted a conic bundle with an $A_2$ fiber, there would be sections $P_1, P_2$ with $P_1 \cdot P_2 = 1$. However, by analyzing the Mordell-Weil lattice of $X$ and using the height formula [SS19, Thm. 6.24], one finds that $P_1 \cdot P_2 = \frac{n^2 - 4}{4}$ or $\frac{n^2 - 1}{4}$ for some $n \in \mathbb{Z}$. In both cases $P_1 \cdot P_2 = 1$ is impossible. A further study on the possible intersection numbers of sections is found in [Cos22].
6 Construction of conic bundles from a pencil of genus zero curves

Let $X$ be a rational elliptic surface with elliptic fibration $\pi : X \to \mathbb{P}^1$. As described in Subsection 2.2, $\pi$ is induced by a pencil of cubics $\mathcal{P}$ from the blowup $p : X \to \mathbb{P}^2$ of the base locus of $\mathcal{P}$. We describe a method for constructing a conic bundle $\varphi : X \to \mathbb{P}^1$ from a pencil of curves with genus zero on $\mathbb{P}^2$.

**Construction.** Let $\mathcal{Q}$ be a pencil of conics (or a pencil of lines) given by a dominant rational map $\psi : \mathbb{P}^2 \to \mathbb{P}^1$ with the following properties:

(a) $\psi^{-1}(t)$ is smooth for all but finitely many $t \in \mathbb{P}^1$, i.e. the general member of $\mathcal{Q}$ is smooth.

(b) The indeterminacy locus of $\psi$ (equivalently, the base locus of $\mathcal{Q}$) is contained in the base locus of $\mathcal{P}$, which includes infinitely near points.

Now define a surjective morphism $\varphi : X \to \mathbb{P}^1$ by the composition

$$
\begin{array}{ccc}
X & \xrightarrow{p} & \mathbb{P}^2 \\
\downarrow & & \downarrow \psi \\
\mathbb{P}^1 & \xrightarrow{\varphi} & \mathbb{P}^1
\end{array}
$$

Notice that $\varphi$ is a well defined conic bundle. Indeed, by property (b) the points of indeterminacy of $\psi$ are blown up under $p$, so $\varphi$ is a morphism. By property (a) the general fiber of $\varphi$ is a smooth, irreducible curve. Since $\mathcal{Q}$ is composed of conics (or lines), the general fiber of $\varphi$ has genus zero.

**Illustration.** Let $C$ be a smooth cubic and let $L_1, L_2, L_3$ be concurrent lines. Define $\mathcal{P}$ as the pencil generated by $C$ and $L_1 + L_2 + L_3$. Take points $P_1, P_2 \in L_1 \cap C$ and $P_3, P_4 \in L_3 \cap C$ and let $\mathcal{Q}$ be pencil of conics through $P_1, P_2, P_3, P_4$. In the following picture, $Q \in \mathcal{Q}$ is a general conic, so the strict transform of $Q$ under $p$ is a general fiber of the conic bundle $\varphi : X \to \mathbb{P}^1$.

**Remark 6.1.** Since the base points $P_1, P_2, P_3, P_4$ of $\mathcal{Q}$ are blown up under $p$, then the pullback pencil $p^* \mathcal{Q}$ has four fixed components, namely the exceptional divisors $E_1, E_2, E_3, E_4$. By eliminating these we obtain a base point free pencil $p^* \mathcal{Q} - E_1 - E_2 - E_3 - E_4$, which is precisely the one given by $\varphi : X \to \mathbb{P}^1$. 
7 Examples

Let \( \pi : X \to \mathbb{P}^1 \) be a rational elliptic fibration. For simplicity we assume \( \text{char}(k) = 0 \) throughout Section 7, although similar constructions are possible in positive characteristic. We construct examples of conic bundles using the method described in Section 6.

Notation.

\begin{align*}
\mathcal{P} & \quad \text{pencil of cubics on } \mathbb{P}^2 \text{ inducing } \pi. \\
p & \quad \text{blowup } p : X \to \mathbb{P}^2 \text{ at the base locus of } \mathcal{P}. \\
Q & \quad \text{pencil of conic (or lines) on } \mathbb{P}^2. \\
Q & \quad \text{conic in } Q, \text{ not necessarily smooth.} \\
L & \quad \text{line in } Q. \\
\varphi & \quad \text{conic bundle } \varphi : X \to \mathbb{P}^1 \text{ induced by } Q \text{ (see Section 6).} \\
D & \quad \text{singular fiber of } \varphi \text{ such that } \varphi(D) = Q \text{ (or } \varphi(D) = L). \\
D' & \quad \text{another singular fiber of } \varphi.
\end{align*}

Remark 7.1. Since \( D \) is a fiber of \( \varphi \), by the correspondence in Proposition 3.8 the morphism \( \varphi|_D : X \to \mathbb{P}^1 \text{ induced by } |D| \) coincides with \( \varphi \) itself.

Remark 7.2. To simplify our notation, the strict transform of a curve \( E \subset \mathbb{P}^2 \) under \( p \) is also denoted by \( E \) instead of the usual \( \tilde{E} \).

7.1 Extreme cases

By Theorem 5.2, there are two extreme cases in which \( X \) can only admit conic bundles with exactly one type of singular fiber.

1. When \( \pi \) has a II* fiber: \( X \) only admits conic bundles with singular fibers of type \( D_m \) (\( m \geq 4 \)).
2. When \( \pi \) has no reducible fibers: \( X \) only admits conic bundles with singular fibers of type \( A_2 \).

In Persson’s classification [Per90], case (1) corresponds to the first two entries in the list (the ones with trivial Mordell-Weil group) and case (2) corresponds to the last six (the ones with maximal Mordell-Weil rank). Examples 7.1.1 and 7.1.2 illustrate cases (1) and (2) respectively.
**Example 7.1.1.** We consider $\pi$ with configuration $(\Pi^*, \Pi)$. By Theorem 5.2, $X$ can only admit conic bundles with singular fibers of type $D_m$ ($m \geq 4$). We construct the elliptic surface $X$ by blowing up the base locus $\{9P_1\}$ of the pencil of cubics $\mathcal{P}$ induced by a smooth cubic $C$ and a triple line $3L$. Let $\mathcal{Q}$ be the pencil of lines through $P_1$. Then $D := p^*L - E_1$ is a curve in the base point free pencil $p^* \mathcal{Q} - E_1$, which induces the conic bundle $\varphi_{|D|} : X \to \mathbb{P}^1$. The curve $D$ is a $D_9$ fiber of $\varphi_{|D|}$.

Remark 7.3. The conic bundle in Example 7.1.1 is in fact the only conic bundle on $X$. This follows from the fact that $\Pi^*$ is the only reducible fiber of $\pi$ and that $E_9$ is the only section of $X$ since the Mordell-Weil group $\text{MW}(\pi)$ is trivial [Per90]. By examining the intersection graph of $\Pi^*$, $D$ is the only divisor on $X$ forming a fiber of type $D_m$ ($m \geq 4$).

Remark 7.4. The case when $\pi$ has configuration $(\Pi^*, 2I_1)$ admits a similar construction and Remark 7.3 also applies.
**Example 7.1.2.** We take \( \pi \) with configuration \((I, 10I_1)\). By Theorem 5.2, \( X \) can only admit conic bundles with singular fibers of type \( A_2 \). We construct the elliptic surface \( X \) by blowing up the base locus \( \{P_1, ..., P_9\} \) of the pencil of cubics \( \mathcal{P} \) induced by \( C \) and \( C' \). Let \( \mathcal{Q} \) be the pencil of lines through \( P_1 \). Then \( D := p^* L - E_1 \) is a curve in the base point free pencil \( p^* \mathcal{Q} - E_1 \), which induces the conic bundle \( \varphi_{[D]} : X \rightarrow \mathbb{P}^1 \). The curve \( D \) is an \( A_2 \) fiber of \( \varphi_{[D]} \).

\[ C : y^2 z - 4x^3 + 4xz^2 = 0. \]
\[ C' : y^2 z - 4x^3 + 4xz^2 + (127/100)(xz^2 - 4y^3 + 4yz^2) = 0. \]
\[ L : \text{line through } P_1, P_2. \]
\[ \mathcal{P} : \text{pencil of cubics induced by } C \text{ and } C'. \]
\[ \mathcal{Q} : \text{pencil of lines through } P_1. \]
\[ D = L + E_2, \text{ type } A_2. \]
\[ \text{Sequence of contractions: } E_9, E_8, E_7, E_6, E_5, E_4, E_3, E_2, E_1. \]

**Remark 7.5.** \( D \) is not the only singular fiber of the conic bundle in Example 7.1.2. In fact, each line \( L_{1i} \) joining \( P_1, P_i \) for any \( i = 2, ..., 9 \) induces an \( A_2 \) fiber of \( \varphi_{[D]} \), namely \( L_{1i} + E_i \).

**Remark 7.6.** Since the conic bundle in Example 7.1.2 only admits singular fibers of type \( A_2 \) and such fibers are isomorphic to a pair of lines meeting at a point, this is a *standard conic bundle* in the sense of Manin and Tsfasman [MT86, Subsection 2.2].
7.2 Mixed fiber types

Example 7.2.1. We take \( \pi \) with configuration \((IV, II, 6)\). By Theorem 5.2, \( X \) admits only conic bundles with singular fibers of types \( A_2 \) or \( A_n \) \((n \geq 3)\). We construct a conic bundle with singular fibers of types \( A_2 \) and \( A_3 \). We construct the elliptic surface \( X \) by blowing up the base locus \( \{P_1, \ldots, P_9\} \) of the pencil of cubics induced by \( C \) and \( L_1 + L_2 + L_3 \). Let \( Q \) be the pencil of lines through \( P_1 \). Then \( D := p^*L - E_1 \) and \( D' := p^*L' - E_1 \) are curves in the base point free pencil \( p^*Q - E_1 \), which induces the conic bundle \( \varphi|_D : X \to \mathbb{P}^1 \). The curves \( D, D' \) are fibers of \( \varphi|_D \) of type \( A_2, A_3 \) respectively.

\[
\begin{align*}
C & : x^3 + y^3 - y^2z = 0. \\
L_1 & : y + 2x - z = 0. \\
L_2 & : y + 4x - 2z = 0. \\
L_3 & : y + 8x - 4z = 0. \\
L & : \text{line through } P_1, P_4. \\
L' & = L_1. \\
P & : \text{pencil of cubics induced by } C \text{ and } L_1 + L_2 + L_3. \\
Q & : \text{pencil of lines through } P_1. \\
D & = L + E_4, \text{ type } A_2. \\
D' & = E_6 + L' + E_9, \text{ type } A_3. \\
\text{Sequence of contractions: } & E_9, E_8, E_7, E_6, E_5, E_4, E_3, E_2, E_1.
\end{align*}
\]

Remark 7.7. In addition to \( D \) and \( D' \), the conic bundle in Example 7.2.1 has five other singular fibers, each of type \( A_2 \). Namely, each line \( L_{1i} \) joining \( P_1, P_i \) with \( i \in \{2, 3, 5, 7, 8\} \) induces the \( A_2 \) fiber \( L_{1i} + E_i \).
Example 7.2.2. We take $\pi$ with configuration $(I_7, \Pi, 3I_1)$. By Theorem 5.2, $X$ admits conic bundles with singular fibers of types $A_n$ ($n \geq 3$), $D_m$ ($m \geq 4$) and possibly $A_2$. We construct a conic bundle with types $A_2, A_4, D_4$. The elliptic surface $X$ is constructed by blowing up the base locus $\{2P_1, 3P_2, 2P_3, P_4, P_5\}$ of the pencil of cubics $\mathcal{P}$ induced by $C$ and $L_1 + L_2 + L_3$. Let $Q$ be the pencil of lines through $P_1$. Then $D := p^*L - E_1$, $D' := p^*L' - E_1$ and $D'' := p^*L'' - E_1$ are curves in the base point free pencil $p^*Q - E_1$, which induces the conic bundle $\varphi_{|D|} : X \to \mathbb{P}^1$. The curves $D, D', D''$ are fibers of $\varphi_{|D|}$ of types $D_4, A_4, A_2$ respectively.

\[C : yz^2 - 2x^2(x - z) = 0.\]
\[L_1 : y = 0.\]
\[L_2 : 2x - 9y - 2z = 0.\]
\[L_3 : 4x + 9y = 0.\]
\[L = L_1.\]
\[L' = L_2.\]
\[L'' : \text{line joining } P_1, P_5.\]
\[\mathcal{P} : \text{pencil of cubics induced by } C \text{ and } L_1 + L_2 + L_3.\]
\[Q : \text{pencil of lines through } P_1.\]
\[D = 2G_2 + 2F_3 + (E_2 + L), \text{ type } D_4.\]
\[D' = E_4 + L' + E_3 + F_3, \text{ type } A_4.\]
\[D'' = L'' + E_5, \text{ type } A_2.\]
Sequence of contractions: $E_5, E_4, F_3, E_3, G_2, F_2, E_2, F_1, E_1$. 

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Example 7.2.3. We take $\pi$ with configuration $(I_2^*, III, I_1^1)$. By Theorem 5.2, $X$ admits conic bundles with singular fibers of types $A_n \ (n \geq 3)$, $D_3$, $D_m \ (m \geq 4)$ and possibly $A_2$. We construct a conic bundle with singular fiber of types $A_3$, $D_3$, $D_5$. The elliptic surface $X$ is constructed by blowing up the base locus $\{P_1, ..., P_9\}$ of the pencil of cubics $P$ induced by $C$ and $L_1 + L_2 + L_3$. Let $Q$ be the pencil of lines through $P_1$. Then $D := p^* L - E_1$, $D' := p^* L' - E_1$ and $D'' := p^* L'' - E_1$ are curves in the base point free pencil $p^* Q - E_1$, which induces the conic bundle $\varphi_{|D|} : X \to \mathbb{P}^1$. The curves $D$, $D'$, $D''$ are fibers of $\varphi_{|D|}$ of types $A_3$, $D_3$, $D_5$ respectively.

$C : x^3 + y^3 - y^2 z = 0.$
$L_1 : y - z = 0.$
$L_2 : x = 0.$
$L_3 : y + 2x - z = 0.$
$L := L_2.$
$L' := L_3.$
$L'' := L_1.$

$P$ : pencil of cubics induced by $C$ and $L_1 + L_2 + L_3$.

$Q$ : pencil of lines through $P_1$.

$D = E_2 + L + E_3$, type $A_3$.

$D' = E_4 + 2F_4 + L'$, type $D_3$.

$D'' = 2I_1 + 2H_1 + 2G_1 + (L'' + F_1)$, type $D_5$.

Sequence of contractions: $F_4, E_4, E_3, E_2, I_1, H_1, G_1, F_1, E_1$. 

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