Abstract
A physical system showing a classical (deterministic) behaviour to an observer can appear to be a quantum system to another observer unable to distinguish between some distinct states.

Summary 1 Quantum mechanics is a very precise and powerful physical theory but is accompanied with the negative hypothesis that the measuring process can have only an essentially statistical, nondeterministic character.

It is hard to believe that in the future this assumption will not be overcome or reduced in some way by new experiments or new theories: it does not seem there is any conclusive reason to exclude it.

But is a theory conceivable where the outcomes of the measurements are uniquely defined and the statistical previsions of quantum mechanics exactly respected?

From a mathematical viewpoint it is not difficult to produce an object with these properties. More difficult would be to justify in physical terms the artificial construction we propose; however we give a general argument showing how the interplay between the classical and quantum mechanics we offer is interpretable as the difference between an imaginary very expert observer and another non-expert observer compelled to confuse different states or different observables.

The main goal of this article is precisely to give a rigorous meaning to this interplay and a proof that the general quantum system, with all its states and observables, can be obtained from some classical system.

We cannot offer any physical representation for this classical system, we confine here proving that besides the well known theorems concerning the impossibility of hidden variables (cfr. [Neu], [J-P]) there is also room for a result in favor of the possibility.

All this is made inside the usual descriptions of the standard quantum physical systems via quantum logic (cfr. [Mac], [Jau], [Lud] etc.) and, except for the requirement of hidden variables, does not refer to any nonorthodoxical physical theory.
1 Reduction

Definition 2 A (model for a) classical physical system is a couple \((S, \mathcal{L})\) of a set \(S\) (the set of pure states) and a family \(\mathcal{L}\) of subsets of \(S\) (the family of propositions of \(S\)) distinguishing the elements of \(S\) (that is for every couple of different states there is a proposition in \(\mathcal{L}\) not containing both of them).

Every \(L\) in \(\mathcal{L}\) represents the subset where a proposition (an observable taking only the value 0 and 1) is true.

The hypothesis that \(\mathcal{L}\) distinguishes the elements of \(S\) is made to simplify the situation; we can always suppose this hypothesis verified because otherwise we pass to consider as states the classes of the following equivalence relation: two elements of \(S\) are equivalent if every time a proposition contains one of them it contains them both.

Example 3 Usually \(\mathcal{L}\), in the classical case, is the set of all parts of \(S\) or the set of all measurable Borel subset of a Borel family.

We are not going to make any restriction on the family \(\mathcal{L}\).

Definition 4 An observable for a classical system \((S, \mathcal{L})\) is a function \(f : S \rightarrow \mathbb{R}\) such that for every Borel subset \(B\) in the Borel family \(\mathcal{B}(\mathbb{R})\) of \(\mathbb{R}\) the inverse image \(f^{-1}(B)\) is in \(\mathcal{L}\).

Example 5 In particular the characteristic functions of the subsets \(L\) in \(\mathcal{L}\) are observable functions.

Let’s denote by \(\mathcal{F}\) the set of all the observable functions of the system \((S, \mathcal{L})\).

Remark 6 If \(f\) is an observable function of \((S, \mathcal{L})\) and \(g: \mathbb{R} \rightarrow \mathbb{R}\) is a Borel function the function \(g \circ f : S \rightarrow \mathbb{R}\) is an observable. In fact for every Borel subset \(B\) the set \((g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))\) is in \(\mathcal{L}\).

These definitions are given to idealize the situation of an observer (that we will call the precise observer) able to prepare with extreme precision a physical system in a variety of different (pure) states and to perform on the system.
several measures in such a way that when he prepares the state \( s \) and performs the observable \( f \) he can get the exact value \( f(s) \). Our observer knows all the time and exactly what state is preparing and what observable is performing.

To have a more precise idea of this kind of situation let us suppose that the precise observer has a very huge and efficient laboratory where he can prepare every sort of state of the physical system under consideration and use every sort of measuring apparatus: all he has to do is to give to the laboratory’s computer a “string” specifying completely and exactly the state to prepare and another “string” specifying the observable to measure and the fantastic laboratory does all the work.

The observer checks that given a ”state string” and an ”observable string” he always gets the same value and so he can state with certainty that the physical system considered is classical (deterministic).

Let us consider now another observer (that we will call the imprecise observer) studying the same physical system but with a poorer ability; this second observer can produce all the states and observables of the previous one but he does not know exactly what he makes: he gives a ”procedure” to produce a certain state and another ”procedure” to produce the measuring apparatus but if he repeats the given procedures he can get different values in a random and, for him, unavoidable way.

Let us suppose moreover that the precise observer can describe precisely what the problem is with the imprecise observer: when this second one chooses a ”procedure” he produces a state among several different ones in a given class of equivalence of \( S \) with a certain probability: there is an equivalence relation \((\text{confusion relation}) \, \mathcal{R}\) on the classical system \((S, \mathcal{L})\) and a probability measure \(\mu_p\) on every equivalence class \(p\) in the quotient set \(P = S/\mathcal{R}\). When the imprecise observer tries to prepare the system with a given procedure \(p\) he does not know which one of the states in the class \(p = \left\lfloor s \right\rfloor\) he is really preparing, therefore when he evaluates the observable \(f\) he can get any one of the values in the subset \(f(\left\lfloor s \right\rfloor)\). Making several trials he experiments all these values with different frequencies arriving in the end at the conclusion that the measure of the observable \(f\) on the ”preparation” \(p\) has a statistical character and that he cannot get anything more that the probability \(\pi(f, p, B)\) that the measure of \(f\) on \(p\) lies in the Borel subset \(B\) of \(\mathbb{R}\).

For the precise observer it is obvious that

\[
\pi(f, p, B) = \mu_p(f^{-1}(B) \cap p)
\]

If the imprecise observer is left unaware of his ”confusion” and convinced that he cannot get any more information on the system, he will decide, coherently,
not to distinguish between preparations or measuring apparatuses giving the same probabilities. Therefore he will define the following concept:

**Definition 7** A (model for a) **statistical physical system** is a triple \((P, O, \pi)\) of a set \(P\) (the set of statistical states), another set \(O\) (the set of statistical observables) and a function: \(\pi : O \times P \times B(\mathbb{R}) \rightarrow [0,1]\) (the probability that the measure of a observable on a state lies in a Borel subset of \(\mathbb{R}\)) such that

1. \(\pi(T', p, B) = \pi(T'', p, B)\) for every \(p\) in \(P\) and \(B\) in \(B(\mathbb{R})\) implies \(T' = T''\)
2. \(\pi(T, p', B) = \pi(T, p'', B)\) for every \(T\) in \(O\) and \(B\) in \(B(\mathbb{R})\) implies \(p' = p''\).

From the viewpoint of the precise observer this means that two states \(s'\) and \(s''\) are equivalent in the equivalence relation of confusion if and only if

\[
\mu_{[s']}(f^{-1}(B) \cap [s']) = \mu_{[s']}([f^{-1}(B) \cap [s''])
\]

for every \(f\) in \(F\) and every \(B\) in \(B(\mathbb{R})\).

Moreover the precise observer makes a discovery: the imprecise observer confuses not only the states but also the observables.

The set of statistical observables \(O\) is the quotient set of \(F\) modulo the equivalence relation stating that two functions \(f'\) and \(f''\) of \(F\) are equivalent if

\[
\mu_{[s]}(f'^{-1}(B) \cap [s]) = \mu_{[s]}(f''^{-1}(B) \cap [s])
\]

for every \(s\) in \(S\) and every \(B\) in \(B(\mathbb{R})\).

Therefore the precise observer can give the following:

**Definition 8** A **confusion** relation for a classical system \((S, L)\) is given assigning an equivalence relation \(R\) on \(S\) and a probability measure \(\mu_p\) on every equivalence class in such a way that for every couple of inequivalent elements \(s'\) and \(s''\) in \(S\) there exists a proposition \(L\) in \(L\) with \(\mu_{[s']}(L \cap [s']) \neq \mu_{[s'']}([L \cap [s'']])\).

It is clear that for every confusion relation \(R\) there is also defined an equivalence relation \(\mathcal{M}\) on the set \(F\) of observable functions by taking \(f'\mathcal{M} f''\) if:

\[
\mu_{[s]}(f'^{-1}(B) \cap [s]) = \mu_{[s]}(f''^{-1}(B) \cap [s])
\]

for every \(s\) in \(S\) and every \(B\) in \(B(\mathbb{R})\).

Therefore a statistical system is well defined by taking

1) \(\hat{S} = S/R\)
2) $\hat{F} = F/M$

3) $\hat{\mu} : \hat{F} \times \hat{S} \times B(\mathbb{R}) \to [0, 1]$ given by $\hat{\mu}([f],[s],B) = \mu_{[s]}(f^{-1}(B) \cap [s]).$

**Definition 9** Given a classical system $(S, \mathcal{L})$ and a confusion relation $(R, \{\mu_p\}_{p \in S/R})$, the statistical system $(F/M, S/R, \hat{\mu})$ is called the **system reduced** by the confusion relation.

**Remark 10** We call the procedure given above the reduction just because we pass from a state space to another making a quotient along (essentially) one-dimensional fibres as when we reduce a contact manifold producing a symplectic manifold.

When a statistical system can be obtained as a reduced system of a classical system there is at least a mathematical reason to talk of **hidden variables** (the "variables" describing the elements in each equivalence classes of the state set of the classical system): under every statistical state $p = [s]$ are "hidden" the elements of $[s]$, the "true states".

It is possible to make precise this assertion considering the following (cfr. [Jam] pag.262):

**Definition 11** Let $(O, P, \pi)$ be a statistical system, a **model for a system with hidden variables with respect to** $(O, P, \pi)$ is given assigning:

1. a set $S$ (the space of hidden states) a surjective map $\rho : S \to P$ (associating to a "hidden state" its "apparent state")

2. for each "apparent" state $p \in P$ a probability measure $\mu_p$ on $S$ (representing the probability to find in a measurable subset of $S$ a "hidden state" representing $p$); and

3. for each observable $T \in O$ a function $f_T : S \to \mathbb{R}$ (representing a classical observable giving the values that appears randomly for the statistical observable $T$) such that for every Borel subset $B$ of $\mathbb{R}$:

   $$\pi(T,p,B) = \mu_p(f_T^{-1}(B))$$

   (that is the probability that the value of $T$ on $p$ lies in $B$ is given by the probability to find a hidden state of $p$ between the states where the observable $f_T$ takes a value in $B$).
In fact, in the case of a reduced system, we can take as $\rho : S \to P$ the quotient map and for every $p$ in $P$ as probability measure $\mu_p$, the measure $\mu_p$ seen as a measure on all $S$ and not only on $\rho^{-1}(p)$.

We are going to prove that the general quantum system (given by a Hilbert space) is a reduced system of a classical system.

This kind of property is sometimes considered impossible to be proved or in contradiction with the principles of the standard quantum mechanics.

On the contrary the same property can be considered "well known" and quite obvious: if you want a "hidden variable " function giving the right statistical outcomes for a self-adjoint operator $T$ of the Hilbert space $H$ simply take the "quasi-inverse" function $f : P(H) \times [0, 1] \to \mathbb{R}$ (cfr. the proof of the following theorem.) defined by

$$f([h], t) = \sup \{ u : \langle E_{\leftarrow \infty, u}^T(h) \rangle \geq t \}.$$

It seems that all depends on what you mean. In this section we have just tried to suggest a plausible interpretation that can save determinism in observations.

## 2 The quantum system as a reduced system

**Definition 12** The (model) for the (irreducible) quantum system is given assigning:

1. the (complex) projective space $P(H)$ of a Hilbert space $H$ (of dimension at least two) as state space;
2. the set $SA(H)$ of self-adjoint operators on $H$ as observable space; and
3. the function $\pi : SA(H) \times P(H) \times \mathcal{B}(\mathbb{R}) \to [0, 1]$ defined by

$$\pi(T, [h], B) = \langle E_B^T(h) \rangle \frac{h, E_B^T(h)}{\langle h, h \rangle}$$

(where $E_B^T = \chi_B \circ T$ is the projector operator associated to the Borel subset $B$ of $\mathbb{R}$ in the spectral measure of $T$) as probability function.
Theorem 13  The quantum system is the reduced system of a classical system.

Proof. For every \([h]\) in \(\mathbf{P}(\mathbf{H})\) let us consider a complete separable metric space \(S_{[h]}\) with a Borel measure \(\mu_{[h]}\) such that \(\mu_{[h]}(S_{[h]}) = 1\) and \(\mu_{[h]}([s]) = 0\) for every \(s\) in \(S_{[h]}\). For every such space there is a measurable map \(\phi_{[h]} : S_{[h]} \rightarrow [0,1]\) such that \(\phi_{[h]}^{-1}([\lambda]) = \lambda\) where \(\lambda\) denotes the Lebesgue measure on the interval (cfr. [Roy] Thm. 9 pag. 327).

Let \(S\) be the disjoint union of the \([S_{[h]}]\) \([h] \in \mathbf{P}(\mathbf{H})\). We will call a subset \(L\) of \(S\) a proposition if and therefore for every Borel subset \(B\) such that \(\mu_{[h]}(L \cap S_{[h]}) = \langle E \rangle_{h}\) for every \([h]\) in \(\mathbf{P}(\mathbf{H})\).

Let \(\mathcal{L}\) be the set of all propositions in \(S\).

Every proposition \(L\) determines the set of all \([h]\) where \(\mu_{[h]}(L \cap S_{[h]}) = 1\) and therefore the projector \(E\). If we denote by \(\mathcal{E}\) the set of all projectors of \(\mathbf{H}\), a map \(\varepsilon : \mathcal{L} \rightarrow \mathcal{E}\) associating to a proposition its projector is well defined.

The map \(\varepsilon\) is surjective; fixed \(E\) is enough to take in \(S_{[h]}\) a measurable subset \(L_{[h]}\) such that \(\mu_{[h]}(L_{[h]} \cap S_{[h]}) = \langle E \rangle_{h}\) and then take \(L\) as the disjoint union of all the \(L_{[h]}\). It is not difficult to prove, in a similar way, that the propositions distinguish the elements in \(S\).

Let us denote by \(\mathcal{F}\), as usual, the set of all observable functions for \((S, \mathcal{L})\); we want to prove that to each of these functions \(f\) is associated a self-adjoint operator \(T\) such that

\[\mu_{[h]}(f^{-1}(B) \cap S_{[h]}) = \langle E^T_B \rangle_{h}\]

for every \([h]\) in \(\mathbf{P}(\mathbf{H})\) and \(B\) in \(\mathcal{B}(\mathbb{R})\).

For every real number \(t\) the proposition \(L_t = f^{-1}([-\infty,t])\) determines a projector \(E_t\). The family \(\{E_t\}_{t \in \mathbb{R}}\) is a spectral family of projectors of \(\mathbf{H}\) (cfr. [Wei] def. (7.11) pag. 180); in fact \(\langle E_t \rangle_{h} = (f|S_{[h]}; \mu_{[h]}([-\infty,t]))\) for every \(h\) and therefore the monotonicity, the left-continuity and the convergence to 0 and 1 properties for the projection operators follow from the analogous properties of cumulative distribution functions for Borel probability measures (cfr. [Roy] Lemma 10 pag. 262).

Hence the spectral family \(\{E_t\}_{t \in \mathbb{R}}\) defines a self-adjoint operator \(T\) such that for every \(t\) in \(\mathbb{R}\)

\[\mu_{[h]}(f^{-1}([-\infty,t]) \cap S_{[h]}) = \langle E^T_{-\infty,t} \rangle_{h}\]

and therefore for every Borel subset \(B\) of \(\mathbb{R}\)

\[\mu_{[h]}(f^{-1}(B) \cap S_{[h]}) = \langle E^T_B \rangle_{h}\].

The operator \(T\) is unambiguously defined by the function \(f\), let us denote by \(\tau : \mathcal{F} \rightarrow \mathcal{S}A(\mathbf{H})\) the map so defined. Let us prove this map is surjective.

For every \([h]\) let us denote by \(F_{[h]} : \mathbb{R} \rightarrow [0,1]\) the distribution function \(F_{[h]}(u) = \langle E^T_{-\infty,u} \rangle_{h}\); its induced Borel measure \(\nu_{F_{[h]}}\) has the property that \(\nu_{F_{[h]}}(B) = \langle E^T_B \rangle_{h}\) for every Borel subset \(B\).
Its quasi-inverse $\widetilde{F}_{[h]} : [0, 1] \rightarrow \mathbb{R}$ verifies $\widetilde{F}_{[h]}(\lambda) = \nu_{F_{[h]}}$ (cfr. [K-S] Thm. 4 pag. 94) and therefore $(\widetilde{F}_{[h]} \circ \phi_{[h]})_{\ast} (\mu_{[h]}) = \nu_{F_{[h]}}$, that is
\[
(\widetilde{F}_{[h]} \circ \phi_{[h]})_{\ast} \mu_{[h]}([a, b]) = F_{[h]}(b) - F_{[h]}(a) = \langle E^T_{[a, b]} \rangle
\]
for every $a < b$ in $\mathbb{R}$.

The function $f : S \rightarrow \mathbb{R}$ defined by $f(s) = \widetilde{F}_{[h]}(\phi_{[h]}(s))$ (where $[h]$ contains s) has the desired property: $\tau(f) = T$.

Let us prove that the reduced system of $(S, \mathcal{L})$ is the quantum system. Two elements $r$ in $[h]$ and $s$ in $[k]$ are equivalent if and only if $\mu_{[h]}(L \cap S_{[h]}) = \mu_{[k]}(L \cap S_{[k]})$ for every proposition $L$, therefore if and only if $\langle E \rangle_h = \langle E \rangle_k$ for every projector $E$ of $\mathbf{H}$, that is if and only if $[h] = [k]$.

Two functions $f$ and $g$ in $\mathcal{L}$ are equivalent if and only if $\langle E^{\tau(f)}_{B} \rangle_h = \langle E^{\tau(g)}_{B} \rangle_h$ for every $h$ and $B$. This means $E^{\tau(f)}_{B} = E^{\tau(g)}_{B}$ for every $B$, that is $\tau(f) = \tau(g)$.

In the end $\mu([f], [h], B) = \mu_{[h]}(f^{-1}(B) \cap S_{[h]}) = \langle E^{\tau(f)}_{B} \rangle_h = \pi(\tau(f), [h], B)$. □

From now on we will denote by $(S, \mathcal{L})$ a classical system giving the (irreducible) quantum system as reduction, by $\mathcal{F}$ its set of observable functions and by $\rho : S \rightarrow \mathbf{P}(\mathbf{H})$, $\tau : \mathcal{F} \rightarrow \mathcal{A}(\mathbf{H})$ and $\varepsilon : \mathcal{L} \rightarrow \mathcal{E}$ the quotient maps.

**Remark 14** If $L$ is in $\mathcal{L}$ then also $(S \setminus L)$ is in $\mathcal{L}$ and $\varepsilon(S \setminus L) = I - \varepsilon(L)$.

**Remark 15** If $T$ is a self-adjoint operator with spectral measure $\{B \rightarrow E_B\}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ is a Borel function then a self-adjoint operator $g(T)$ with spectral measure $\{B \mapsto E_{g^{-1}(B)}\}$ is well defined.

**Theorem 16** If $f$ is an observable function of a classical system $(S, \mathcal{L})$ reducing to the quantum system and $g : \mathbb{R} \rightarrow \mathbb{R}$ is any Borel function, it holds $\tau(g \circ f) = g(\tau(f))$.

**Proof.** The function $f$ gives the operator $\tau(f) = T$ with spectral measure $\{B \mapsto E_B = \varepsilon(f^{-1}(B))\}$. The observable function $g \circ f$ defines the spectral measure $\{B \mapsto \varepsilon(f^{-1}(g^{-1}(B)))\}$ and this is exactly the spectral measure of $g(T)$. □
Theorem 17 If $f$ is an observable function of a classical system $(S, \mathcal{L})$ reducing to the quantum system and $g : \mathbb{R} \rightarrow \mathbb{R}$ is any Borel function, it holds

1. $\langle g(\tau(f)) \rangle_h = \int_0^1 g(f([h], t)) \cdot d\lambda(t)$
2. $\langle \tau(f) \rangle_h = \int_0^1 f([h], t) \cdot d\lambda(t)$

Proof. The point (2) follows from (1) taking $g = id_{\mathbb{R}}$. Let us prove (1):

$$
\langle g(\tau(f)) \rangle_h = \int_{\mathbb{R}} g(u) \cdot d\nu_{(E^c(t))_{-\infty,1}} = \int_{\mathbb{R}} g(u) \cdot (d f_{[h]} \cdot \lambda_{[0,1]}) = \int_{[0,1]} g(u) \circ f_{[h]} \cdot d\lambda.
$$

Cfr. [Wei] Thm. 7.14(e) and [K-S] Cor. 3 pag. 93 for the passages.

Theorem 18 Given two projectors $E$ and $F$, if there exist two propositions $L$ and $M$ in $\mathcal{L}$ such that

1. the (finite) boolean algebra of subsets $\mathcal{A}$ generated by $L$ and $M$ is contained in $\mathcal{L}$ and
2. the map $\varepsilon : \mathcal{A} \rightarrow \mathcal{E}$ sends $L$ in $E$, $M$ in $F$ transforming the operation $\wedge$ in $\cap$, $\vee$ in $\cup$ and the complementation in the orthogonality then the projectors $E$ and $F$ are compatible (that is commute).

Proof. The union $L \cup M = L \cup (\overline{L} \cap M) = M \cup (\overline{M} \cap L)$ belongs to $\mathcal{L}$ and

$$
\varepsilon(L \cup M) = E \vee F = \varepsilon(L \cup (\overline{L} \cap M)) = E \vee (E' \wedge F) = \\
= \varepsilon(M \cup (\overline{M} \cap L)) = F \vee (F' \wedge E).
$$

This proves that $E$ and $F$ are compatible (cfr. [Jau] probl.2 of 5-8, pag. 87).

Remark 19 Therefore whenever you consider two noncommuting projectors $E$ and $F$ it is impossible to find two propositions $L$ and $M$ with the properties (1) and (2) of the previous theorem.
Definition 20  Let $E_1, E_2$ and $F_1, F_2$ be two couples of projectors in $\mathcal{E}$. We will say that the couples admit proposition intersections if there are two couples of propositions $A_1, A_2$ and $B_1, B_2$ with $\varepsilon(A_i) = E_i$ and $\varepsilon(B_i) = F_j$ for $i, j = 1, 2$ and such that the 16 intersections $A_i \cap B_j$, $\mathcal{C}A_i \cap B_j$, $\mathcal{C}A_i \cap \mathcal{C}B_j$, $\mathcal{C}B_j$ are all in $\mathcal{L}$ and $\varepsilon(A_i \cap B_j) = E_i \wedge F_j$, $\varepsilon(\mathcal{C}A_i \cap B_j) = (I - E_i) \wedge F_j$, $\varepsilon(A_i \cap \mathcal{C}B_j) = E_i \cap (I - F_j)$ and $\varepsilon(\mathcal{C}A_i \cap \mathcal{C}B_j) = (I - E_i) \cap (I - F_j)$.

Notation 21 In this situation we will consider the self-adjoint operators $T_{ij}(E, F) =$

$$
= E_i \wedge F_j + (I - E_i) \wedge (I - F_j) - (I - E_i) \wedge F_j - E_i \wedge (I - F_j) = \\
= \varepsilon(A_i \cap B_j) + \varepsilon(\mathcal{C}A_i \cap \mathcal{C}B_j) - \varepsilon(\mathcal{C}A_i \cap B_j) - \varepsilon(A_i \cap \mathcal{C}B_j).
$$

Theorem 22 If $E_1, E_2$ and $F_1, F_2$ are two couples of projectors in $\mathcal{E}$ admitting proposition intersections, then for every $h$ in $\mathbf{H} \setminus \{0\}$ it holds the inequality

$$
|\langle T_{11}(E, F) \rangle_h - \langle T_{12}(E, F) \rangle_h| + |\langle T_{21}(E, F) \rangle_h + \langle T_{22}(E, F) \rangle_h| \leq 2.
$$

Proof. The functions: $f_{ij} = \chi_{A_i \cap B_j} + \chi_{(S \setminus A_i) \cap (S \setminus B_j)} - \chi_{(S \setminus A_i) \cap B_j} - \chi_{A_i \cap (S \setminus B_j)}$ are functions on $S$ with $\int_0^1 \int_1 \lambda(t) \cdot d\lambda(t) = \langle T_{ij}(E, F) \rangle_h$. It is not difficult to check in $S$ the following equality: $|f_{11} - f_{12}| + |f_{21} + f_{22}| = 2$.

Therefore,

$$
|\langle T_{11}(E, F) \rangle_h - \langle T_{12}(E, F) \rangle_h| + |\langle T_{21}(E, F) \rangle_h + \langle T_{22}(E, F) \rangle_h| = \\
= \int_0^1 \int_1 \lambda(t) \cdot d\lambda(t) + \int_0^1 \int_1 \lambda(t) \cdot d\lambda(t) + \int_0^1 \int_1 \lambda(t) \cdot d\lambda(t) + \\
+ \int_0^1 \int_1 \lambda(t) \cdot d\lambda(t) \leq \int_0^1 |f_{11} - f_{12}| + |f_{21} + f_{22}||h(t)\cdot d\lambda(t) = 2. \quad \blacksquare
$$

Remark 23 The proof of the previous theorem mimics the usual one given to prove one of the Bell inequalities and in fact if you take in a quantum system two couples of projections not verifying the Bell inequality for some state, you have two couples of projections not admitting proposition intersections.

Problem 24 When we have two couples of projections not admitting proposition intersections we can consider the system as the reduction of a classical system and replace the projectors $E_1, E_2, F_1, F_2$ with some propositions $A_1, A_2, B_1, B_2$, but we do not dispose of, for example, the proposition $A_1 \cap B_1$ (or this intersection does not correspond to the projector $E_1 \wedge F_1$).

The absence of $A_1 \cap B_1$ is in contrast with the possibility, considered natural in a classical physical theory, to check "in the same time" two properties of a system.
This is undoubtedly strange and uncomfortable, however, if we take seriously the hypothesis of the precise observer, any objection to this eventuality cannot be considered definitive unless expressed in terms of his physics: in other words, we should be able first to know his description of the physical reality and how he can explain, for example, a possible absence of intersections.

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