A search for inverse magnetic catalysis in thermal quark-meson models

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We explore the parameter space of the two-flavor thermal quark-meson model and its Polyakov loop-extended version under the influence of a constant external magnetic field \( B \). We investigate the behavior of the pseudo critical temperature for chiral symmetry breaking taking into account the likely dependence of two parameters on the magnetic field: the Yukawa quark-meson coupling and the parameter \( T_0 \) of the Polyakov loop potential. Under the constraints that magnetic catalysis is realized at zero temperature and the chiral transition at \( B = 0 \) is a crossover, we find that the quark-meson model leads to thermal magnetic catalysis for the whole allowed parameter space, in contrast to the present picture stemming from lattice QCD.

I. INTRODUCTION

The phase diagram of magnetic quantum chromodynamics QCD, i.e. for strong interactions under the influence of an external classical Abelian magnetic field, is currently under construction. This phase diagram corresponds to a special case, as it does not suffer from the Sign Problem, and can be easily simulated on the lattice. So, from the theoretical point of view, it serves as a crucial check for effective models of QCD extended to regions not easily accessible by lattice simulations. From the experimental standpoint, this setup is also quite remarkable owing to the fact that strong magnetic fields are relevant in non-central heavy ion collisions, and play a major role in the possibility of observing the chiral magnetic effect (for a comprehensive review, see Ref. \cite{1}.)

The chiral and deconfining transitions under the effect of a magnetic background are, of course, amenable also to effective model descriptions \cite{2, 3}. Those models have predicted several outstanding new features to the thermodynamics of strong interactions, from shifting the chiral and the deconfinement crossover lines in the phase diagram \cite{4–32} to transforming the vacuum into a superconducting medium via \( \rho \)-meson condensation \cite{33, 34}, for high enough magnetic fields, i.e. a few times \( m_\pi^2 \). Nevertheless, the available lattice data \cite{35–39} contradicted essentially all predictions regarding the behavior of the pseudo critical lines for deconfinement and chiral symmetry restoration coming from chiral models (including their Polyakov loop extensions). The reason for this failure is unclear, but the fact that confinement is not properly captured in such chiral models might play a role \cite{17, 40}, as well as an apparently nontrivial chiral limit \cite{41}.

In this paper, we explore the parameter space of the two-flavor thermal quark-meson (QM) model and its Polyakov loop-extended version (PQM) under the influence of a constant external magnetic field \( B \). We investigate the behavior of the pseudo critical temperature for chiral symmetry breaking taking into account the likely dependence of two parameters on the magnetic field: the Yukawa quark-meson coupling and the parameter \( T_0 \) of the Polyakov loop potential. We scan an important part of the parameter spaces of these models, in order to check whether they can accommodate, at least qualitatively, the trend of inverse magnetic catalysis found in Ref. \cite{37}. In doing so, we keep two important constraints: (i) that magnetic catalysis is realized at zero temperature, and (ii) that the chiral transition at \( B = 0 \) is a crossover. In these two limits, i.e. zero temperature or zero magnetic field, chiral effective models usually produce results that are, at least qualitatively, in line with lattice QCD. We find, nevertheless, that the extensions considered by introducing a \( B \) dependence in the Yukawa coupling and in the parameter \( T_0 \) are not enough to account for the behavior of the critical temperature as found in lattice QCD simulations. We believe this should be also the case for other chiral models, which would signal the lack of some fundamental ingredient, possibly quark confinement.

The paper is organized as follows. In Section II we describe the thermal effective potential in the presence of a magnetic background. In Section III we consider the effects from the running of the Yukawa coupling, having the magnetic field as the momentum scale, in the QM model. In Section IV we investigate the consequences of making \( T_0 \) \( B \) dependent, in a fashion usually made with chemical potentials. Section V contains our summary.
TABLE I: Values of constants to which the parameters of the mesonic potential are adjusted, according to Ref. [33].

| Constant | $f_\sigma$ [MeV] | $m_\sigma$ [MeV] | $\lambda$ | $g_0$ | $m_f$ [MeV] | $q_u$ | $q_d$ |
|----------|-----------------|-----------------|---------|-------|-------------|-------|-------|
| Value    | 93              | 138             | 20      | 3.3   | 307         | +1/3  | −2/3  |

Let us first briefly review the Polyakov-Quark-Meson (PQM) model while we introduce our notation. The effective potential of the PQM model at finite temperature and magnetic field was calculated in [8]. At the mean-field level, it is a sum of four contributions,

$$V_{\text{eff}}(\sigma, \phi_1, \phi_2, B, T) = V_{\text{cl}}(\sigma) + V_P(\phi_1, \phi_2, T, T_0) + V_{\text{vac}}(\sigma, B) + V_{\text{para}}(\sigma, \phi_1, \phi_2, B, T),$$

(1)

where

$$V_{\text{eff}}(\sigma) = \frac{\lambda}{4}(\sigma^2 - v^2) - h\sigma$$

(2)

is the tree-level potential for the $\sigma$ field. The Polyakov loop potential (in the logarithmic parametrization) is given by [12]

$$\frac{V_P(\phi_1, \phi_2, T, T_0)}{T^4} = -\frac{a(T)}{2} L^* L + b(T) \log \left[1 - 6L^* L + 4(L^* L)^3 - 3(L^* L)^2\right].$$

(3)

The $T = 0$ contribution to the effective potential (with the normalization $V_{\text{vac}}(\sigma, 0) \equiv 0$) has the form

$$V_{\text{vac}}(\sigma, B) = -\frac{N_c}{2\pi^2} \sum_{f=u,d} |q_f B|^2 \left[\zeta(-1, x_f) - \frac{1}{2}(x_f^2 - x_f) \log x_f + \frac{x_f^2}{4}\right],$$

(4)

whereas $V_{\text{para}}(\sigma, \phi_1, \phi_2, B, T)$ is given by

$$V_{\text{para}} = -\frac{BT}{\pi^2} \sum_{f=u,d} \sum_{n=0}^{\infty} (2 - \delta_{n,0}) \sum_{i=1}^{3} \int_0^\infty dp \log \left(1 + 2e^{-\sqrt{p^2 + g^2\sigma^2 + 2n|q_f B|/T}} \cos \phi_i + e^{-2\sqrt{p^2 + g^2\sigma^2 + 2n|q_f B|/T}}\right),$$

(5)

and represents the paramagnetic (thermal) contribution to the effective potential.

In the previous formulas, $B$ is the magnetic field, $T$ is the temperature, $\sigma$ is the expectation value of the sigma meson field (the approximate order parameter for the chiral transition), $\phi_1, \phi_2,$ and $\phi_3$ are the (approximate) deconfinement order parameters related to the Polyakov loop $L$ (see, e.g., [8] for more details). The quantities $\lambda, v$ and $h$ are parameters of the mesonic self interaction [2], which are adjusted according to the constants on Table I and the functions $a(T)$ and $b(T)$ are defined as

$$a(T) = a_0 + a_1 \left(\frac{T_0}{T}\right) + a_2 \left(\frac{T_0}{T}\right)^2; \quad b(T) = b_3 \left(\frac{T_0}{T}\right)^3,$$

(6)

where the values of the parameters $a_0, a_1, a_2,$ and $b_3$ are listed in Table II. In eq. (4), we used $x_f := m_f^2/|q_f B| = g^2\sigma^2/|q_f B|,$ where $g$ is the meson-quark coupling and $q_f$ is the charge of the quark of flavor $f = u, d.$

Notice that the Quark-Meson (QM) model is recovered, as a particular case of the PQM model, when one sets $\phi_1 = \phi_2 = \phi_3 = 0$ (or, equivalently, $L = L^* = 1$) and neglects the Polyakov loop potential ($V_P \equiv 0$). In the PQM model, it can be shown that the reality of the minimum of the effective potential implies $\phi_3 = -(\phi_1 + \phi_2) = 0,$ so that there are actually only two order parameters in the model, $\sigma$ and $\phi \equiv \phi_1.$

TABLE II: Values of the parameters of the Polyakov loop potential [12].

| Constant | $a_0$ | $a_1$ | $a_2$ | $b_3$ | $T_0$ [MeV] |
|----------|-------|-------|-------|-------|-------------|
| Value    | $16\pi^2/45 \simeq 3.51$ | $-2.47$ | $15.2$ | $-1.75$ | $270$ |
The state of thermodynamic equilibrium is the minimum of the effective potential with respect to the order parameters $\sigma$ and $\phi$ at fixed external conditions ($T$ and $B$). One defines, then, the chiral condensate $\sigma(B,T)$, as well as the chiral susceptibility $\chi(B,T) = \partial \sigma(B,T)/\partial T$. The pseudo critical temperature for chiral symmetry breaking is identified as the peak of the chiral susceptibility $\chi(B,T)$. As discussed in Ref. [37], the pseudo critical temperature for the chiral and the deconfinement transitions split as the magnetic field increases sufficiently, a fact that is not in line with lattice results [37]. To keep our discussion simpler, we will focus our analysis on the chiral transition.

In the next two sections we promote two parameters to functions of the external magnetic field: the Yukawa coupling $g$ and the parameter $T_0$ of the Polyakov loop potential.

## III. B-DEPENDENT YUKAWA COUPLING IN THE QM MODEL

Let us first consider the modifications brought about by introducing a $B$ dependence in the Yukawa coupling, i.e. by considering $g = g(B)$. For that purpose, we adopt the QM model (which is equivalent to setting $\phi_1 = \phi_2 = \phi_3 = 0$ and $V_P \equiv 0$ in the PQM model). This will change our formulas (4) and (5) for the effective potential simply by the replacement $g \to g(B)$. In order to describe the parameter space of $g(B)$, we show the pseudo critical temperature as a function of the Yukawa coupling $g$ for different values of the applied field $B$, as seen in Fig. 1.

![Fig. 1: $T_c$ as a function of the coupling $g$ for various values of $B$. Black (continuous) line: $eB = 0$. Red (dashed) line: $eB = 5m_\pi^2$. Blue (dash-dotted) line: $eB = 10m_\pi^2$. Couplings higher than $g_c \approx 3.6$ lead to a first-order phase transition and are not shown in the plot. The vertical dotted line represents the value $g_0 = 3.3$, the physical value for $g$ in the vacuum. The vertical solid line corresponds to $g = 3.6$.](image)

Fig. 1 should be understood in the following way. Each continuous function $g(B)$ will correspond to some continuous path on the plot. All these paths must start on the continuous line ($eB = 0$), then proceed to some point on the dashed line ($eB = 5m_\pi^2$), then to the dotted line ($eB = 10m_\pi^2$), and so on, as $B$ increases. It is clear from Fig. 1 that any choice $g(B) = g_0$ (a constant) leads to an increasing $T_c(B)$, i.e. magnetic catalysis. The same behavior will set in if $g(B)$ is a decreasing function of $B$. The only possible way to find a decreasing $T_c(B)$ would be to let $g(B)$ increase with $B$.

However, as can be seen from Fig. 1, such a function would have to grow very strongly. Thus, the coupling $g$ would very soon reach the region $g \gtrsim 3.6$, where the transition becomes of first order. Lattice results [37] clearly show that the transition is a crossover for finite $B$ and, therefore, we do not consider first-order transitions as acceptable solutions in this paper. Besides, if $g(0) = 3.3$, as required by the parameter fit in the vacuum ($B = 0$ and $T = 0$), there is clearly no continuous $g(B)$ that would give rise to a decreasing $T_c(B)$ (at least not for a crossover, see Fig. 1). Therefore, we can conclude that, if one takes the usual parameter fixing in the vacuum, $g(0) = 3.3$, there is no continuous function $g(B)$ that could lead to inverse magnetic catalysis in the QM model at finite temperature and zero quark chemical potential, unless the chiral transition is of first order.

To illustrate the general trend discussed above, we consider two explicit examples. We show in Fig. 2 the evolution of the chiral order parameter $\sigma(B,T)$ as a function of the temperature for different values of $B$. We take the (increasing)
ansatz \( g(B) = g_0(1 + 0.01eB/v^2) \). For small values of the applied field, the transition is a crossover. However, as soon as \( B \) reaches a moderate value \( (eB \lesssim 5m_\pi^2) \), the chiral crossover turns into a first-order transition.

![Evolution of the normalized order parameter \( \sigma/v \) as a function of the temperature for a B-dependent Yukawa coupling \( g(B) = g_0(1 + 0.01eB/v^2) \). Notice the first-order phase transition at high \( B \) (corresponding to \( g > 3.6 \)).](image)

FIG. 2: Evolution of the normalized order parameter \( \sigma/v \) as a function of the temperature for a B-dependent Yukawa coupling \( g(B) = g_0(1 + 0.01eB/v^2) \). Notice the first-order phase transition at high \( B \) (corresponding to \( g > 3.6 \)).

IV. B-DEPENDENT \( T_0 \) IN THE PQM MODEL

Let us now consider the PQM model with the standard (fixed) Yukawa coupling \( g = g_0 = 3.3 \), but with a varying \( T_0 = T_0(B) \). As discussed in Ref. 42 44, the model parameter \( T_0 \) is equal to the deconfinement critical temperature in a pure gauge Polyakov loop model, \( T_0^{(pure)} = 270 \) MeV. However, due to the back reaction from the matter fields, \( T_0 \) should be lower than its pure gauge value when dynamical quarks are present. More specifically, one takes \( T_0 \) as the running scale in the perturbative renormalization group running of \( \alpha \), the strong coupling constant. Once \( \alpha \) depends also on the number of quark flavors, quark masses, and possibly on external constraints (e.g. on the chemical potential, critical temperature \( T \), field, how the pseudo critical temperature \( g \)) the same fashion as in the analysis of \( 39 \), the form of the Polyakov loop effective potential could be important for the mechanism of inverse magnetic catalysis around the transition temperature. A natural way to parametrize the \( B \) dependence of the Polyakov loop potential is through the \( T_0 \) parameter, described above. We investigate here whether or not this possibility can lead to inverse magnetic catalysis. We find that, within the PQM model (with all other parameters kept at their vacuum values), no function \( T_0(B) \) can lead to a decreasing \( T_c(B) \) for sufficiently high magnetic fields, such as those investigated in 42. Some choices for \( T_0(B) \) may, at most, make the pseudo critical temperature decrease for small values of \( B \). However, even for magnetic field intensities \( eB \sim 0.4 \) GeV/\( v \cdot 2 \sim 20 m_\pi^2 \), the catalysis-inducing vacuum term \( 4 \) dominates the whole picture and \( T_c(B) \) inevitably rises.

In order to see that the pseudo critical temperature tends to rise for every parametrization \( T_0(B) \), we proceed in the same fashion as in the analysis of \( g(B) \) in the previous section. Fig. 3 shows, for different intensities of magnetic field, how the pseudo critical temperature \( T_c \) depends on the value of \( T_0 \). For reference, we also show the value \( T_c(B = 0) = (173 \pm 8) \) MeV, obtained in the two-flavor lattice calculation of 45. In this diagram, each continuous function \( T_0(B) \) corresponds to some line connecting the curves shown. One can immediately realize that the pseudo critical temperature \( T_c(B) \) may go down only if \( T_0(B) \) decreases sufficiently fast. Such behavior, however, can not be sustained even at moderate intensities of the magnetic field, regardless of how fast \( T_0(B) \) may decrease as a function of \( B \).

Let us provide explicit examples. The choices \( b(N_f, 0, B) = b_0 - 60(eB)^2/m_\pi^2 \) and \( b(N_f, 0, B) = b_0 - 2\sqrt{eB}/m_\pi \), where \( b_0 := (11N_c - 2N_f)/6 \pi = 29/6 \pi \), lead to the functions \( T_0(B) \) shown in Fig. 4(a). The resulting normalized
FIG. 3: Pseudo critical temperature $T_c$ as a function of $T_0$ for several values of the applied magnetic field in the PQM model. The band $T_c = (173 \pm 8)\text{MeV}$ corresponds to the pseudo critical temperature found on the lattice for $N_f = 2$.

FIG. 4: (a) Value of the parameter $T_0$ for the two choices $b(N_f, 0, B) = b_0 - 60(eB)^2/m_\pi^2$ and $b(N_f, 0, B) = b_0 - 2\sqrt{eB}/m_\tau$. (b) The corresponding normalized phase diagram $T_c(B)/T_c(0)$, in comparison with lattice simulations [37] (gray band).

pseudo critical temperatures $T_c(B)/T_c(0)$ are shown\(^1\) in Fig. 4(b) and compared to the (also normalized) lattice phase diagram of [37]. As advertised, the pseudo critical temperature goes down for low values of magnetic field, following the same trend as the lattice result from [37], or maybe decreasing even faster. However, for moderate field intensities ($eB \lesssim 0.3\text{GeV}^2 \simeq 15m_\pi^2$ or lower in the examples shown), the pseudo critical temperature rises in spite of the very low values of $T_0(B)$ (see Fig. 4(a)). This behavior is due to the vacuum term Eq. (4), which induces the magnetic catalysis in the vacuum and dominates over the thermal contribution Eq. (5) for larger values of $B$.

\(^1\)Our calculation ceases at $eB \sim 0.3\text{GeV}^2$ due to numerical instabilities which arise when $T_0(B)$ is sufficiently small. Such instabilities are related to the Polyakov loop potential [37], which is ill-defined at $T_0 \to 0$. 

V. CONCLUSIONS

We examined the behavior of the pseudo critical temperature for the chiral transition in the QM and in the PQM model in the presence of a constant magnetic field background by allowing for two natural extensions of the parameter space: a $B$-dependent Yukawa coupling, $g(B)$, in the quark-meson model and a $B$-dependent Polyakov loop potential parameter, $T_0(B)$, in the PQM model.

We found that even strongly varying functions $g(B)$ and $T_0(B)$ are not able to reproduce (not even qualitatively) the phase diagram $T_c(B)$ from lattice simulations in the presence of a magnetic field up to fields $eB \sim 1\text{GeV}^2$. This robustness of the (P)QM model can be attributed to the vacuum term, Eq. (4), which is responsible for magnetic catalysis in the vacuum and still dominates the effective potential at finite temperature, even for moderate values of the magnetic field intensity.

Even though our analysis has been performed at mean-field level, we believe that our conclusions will not be changed if one considers fluctuations (as studied, e.g., in Refs. [10] and [47] for fixed values of the model parameters). Likewise, there is no reason to believe that other chiral models, such as the PNJL model, will behave differently.

The physical mechanism for inverse magnetic catalysis recently proposed in [39, 46] relies on a competition between valence and sea quarks. While the coupling of the magnetic field to valence quarks enhances magnetic catalysis, its interaction with sea quarks tends to decrease the chiral condensate, leading to inverse magnetic catalysis. In the context of chiral models, however, quarks are introduced as free quasiparticles, so that no degrees of freedom corresponding to sea quarks are present. We believe this can be an important, if not decisive, factor for the inability of chiral models to correctly describe the $T - B$ phase diagram.

It seems clear that further comparisons between predictions from these effective models and lattice data in systems containing other control parameters besides the temperature are crucial to test the reliability of those models. Besides the case with a magnetic field, these models seem to face difficulties describing lattice data also when one considers the dependence of the critical temperature on isospin and quark masses [48–50]. Apparently, the presently used chiral models miss some important physics to describe the properties of bulk QCD matter as extracted from lattice simulations when introducing new control parameters, here the magnetic field, even on a qualitative level. In view of this circumstance, it seems that extrapolations of these chiral models to regions of the QCD phase diagram not easily accessible by lattice simulations have to be taken with caution before these basic discrepancies are resolved.

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