The extremal function associated to intrinsic norms

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Abstract

Through the study of the degenerate complex Monge-Ampère equation, we establish the optimal regularity of the extremal function associated to intrinsic norms of Chern-Levine-Nirenberg and Bedford-Taylor. We prove a conjecture of Chern-Levine-Nirenberg on the extended intrinsic norms on complex manifolds and verify Bedford-Taylor’s representation formula for these norms in general.

1. Introduction

For every compact complex manifold $M$ with boundary, Chern-Levine-Nirenberg defined in [9] an intrinsic norm on the homology groups $H_k(M, \mathbb{R})$ ($k = 1, \ldots, 2n - 1$), as the supremum of $C^2$ plurisubharmonic functions in a certain class. A similar norm $\tilde{N}$ was also introduced by Bedford-Taylor [7]. These norms are invariants of complex manifolds and decreasing under holomorphic mappings. In particular, the characterizations of these norms on $H_{2n-1}(M, \mathbb{R})$ are very important in the study of holomorphic mappings. Associated to these norms, there is an extremal function which satisfies the following homogeneous complex Monge-Ampère equation:

$$
\begin{cases}
(dd^c u)^n = 0 & \text{in } M^0 \\
u|_{\Gamma_1} = 1 \\
u|_{\Gamma_0} = 0,
\end{cases}
$$

(1.1)

where $d^c = i(\bar{\partial} - \partial)$, $M^0$ is the interior of $M$, and $\Gamma_1$ and $\Gamma_0$ are the corresponding outer and inner boundaries of $M$ respectively.

Based on the dual principle of the calculus of variations, Chern-Levine-Nirenberg conjectured that the intrinsic norm should be equal to a minimum of a certain other class of $C^2$ plurisubharmonic functions (see (1.9)). Under the

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assumption of $C^2$ regularity of the extremal function $u$ in (1.1), Bedford-Taylor verified the conjecture and observed the following important representation formula in [7],

\[
\tilde{N}(\{\Gamma_1\}) = \int_{\Gamma_1} \left( \frac{\partial u}{\partial r} \right)^n d^c r \land (dd^c r)^{n-1},
\]

where $\tilde{N}$ (see (1.6) and (1.7)) is the extended Chern-Levine-Nirenberg norm by Bedford-Taylor. Also, $\Gamma_1$ is the outer boundary of $M$ and $r$ is a defining function of $\Gamma_1$. The identity (1.2), together with Fefferman’s boundary regularity theorem ([10]), plays a crucial role in the study of holomorphic mappings on annular domains in $\mathbb{C}^n$ by Bedford-Burns in [2]. The idea of using the complex Monge-Ampère equation to study holomorphic mappings was discussed in [15] by Kerzman-Kohn-Nirenberg.

It is clear that the regularity of the extremal function is the key issue. As the equation (1.1) is a degenerate complex Monge-Ampère equation and a part of the boundary of the manifold $M$ is pseudoconcave, regularity becomes a difficult problem. The existence of Lipschitz solution was shown in [7], and the uniqueness of the solution of the equation is well known (see e.g., [4],[15] and [8]). In some special cases, for example on Reinhardt domains ([7]) or a perturbation of them ([1] and [17]), the extremal function is smooth. Unfortunately, the solution of (1.1) is not in $C^2$ in general. In [3], Bedford-Fornæss constructed counterexamples with solutions of exact $C^{1,1}$ regularity.

In this paper, we will prove the optimal $C^{1,1}$ regularity of the equation (1.1). Moreover, we will establish the formula (1.2) in general without the $C^2$ regularity assumption on the extremal function. And we will also verify the Chern-Levine-Nirenberg conjecture for $\tilde{N}$. We will adapt the subsolution method introduced by B. Guan and J. Spruck [12] and B. Guan [11] to deal with the degenerate complex Monge-Ampère equation. One of the crucial steps is the construction of a smooth subsolution in Proposition 1.1.

Before we state our main results, we first recall the intrinsic norms defined by Chern-Levine-Nirenberg, and the extensions of Bedford-Taylor.

Let $\gamma \in H_*(M, \mathbb{R})$ be a homology class in $M$; define

\[
N(\gamma) = \sup_{u \in \mathcal{F}} \inf_{T \in \gamma} |T(d^c u \land (dd^c u)^{k-1})|, \quad \text{if } \dim \gamma = 2k - 1;
\]

\[
N(\gamma) = \sup_{u \in \mathcal{F}} \inf_{T \in \gamma} |T(du \land d^c u \land (dd^c u)^{k-1})|, \quad \text{if } \dim \gamma = 2k,
\]

where $T$ runs over all currents which represent $\gamma$ and

\[
\mathcal{F} = \{u \in C^2(M) \mid u \text{ is pluri} \text{-subharmonic and } 0 < u < 1 \text{ on } M\}.
\]
The intrinsic norm $N$ may also be obtained as the supremum over the subclass of $C^2$ solutions of homogeneous complex Hessian equations,

$$F'_k = \{ u \in F \mid (dd^c u)^k = 0, \dim \gamma = 2k - 1, \text{ or } du \wedge (dd^c u)^k = 0, \dim \gamma = 2k \}.$$ 

When $k = 2n - 1$, elements of $F'_{2n-1}$ are plurisubharmonic functions satisfying the homogeneous complex Monge-Ampère equation

$$(dd^c u)^n = 0. \tag{1.5}$$

In a series of works [4]–[7], based on the fundamental Chern-Levine-Nirenberg inequality and Lelong’s work on positive currents, Bedford-Taylor developed weak solution theory for complex Monge-Ampère equations. They extended Monge-Ampère operators for locally bounded plurisubharmonic functions as positive currents, and obtained many important results. In [6] and [7], the variational properties of complex Monge-Ampère equations were studied. In particular in [7], Bedford-Taylor introduced the extended intrinsic norm $\tilde{N}$.

$$\tilde{N}\{\gamma\} = \sup_{u \in \tilde{F}} \inf_{T \in \gamma} |T(d^c u \wedge (dd^c u)^{k-1})|, \quad \text{if } \dim \gamma = 2k - 1, \tag{1.6}$$

$$\tilde{N}\{\gamma\} = \sup_{u \in \tilde{F}} \inf_{T \in \gamma} |T(du \wedge d^c u \wedge (dd^c u)^{k-1})|, \quad \text{if } \dim \gamma = 2k, \tag{1.7}$$

where the infimum this time is taken over smooth, compactly supported currents which represent $\gamma$ and

$$\tilde{F} = \{ u \in C(M) \mid u \text{ is plurisubharmonic }, 0 < u < 1 \text{ on } M \}.$$

It is shown that $\tilde{N} \geq N$ and $\tilde{N} < \infty$. $\tilde{N}$ also enjoys other similar properties of $N$. They are invariants of the complex structure, and decrease under holomorphic maps. These properties are useful in the study of holomorphic mappings. The intrinsic norms can be extended to intrinsic pseudo-metrics on the manifold which are closely related to Caratheodory and Kobayashi metrics.

It was observed in [9] that equation (1.5) also arises as the Euler equation for a stationary point of the convex functional

$$I(u) = \int_M du \wedge d^c u \wedge (dd^c u)^{n-1}. \tag{1.8}$$

Let $M$ be a closed complex manifold with smooth boundary $\partial M = \Gamma_1 \cup \Gamma_0$, and let

$$B = \{ u \in F \mid u = 1 \text{ on } \Gamma_1, u = 0 \text{ on } \Gamma_0 \}. \tag{1.9}$$

If $v \in B$, let $\gamma$ denote the $(2n - 1)$-dimensional homology class of the level hypersurface $v$ is constant. Then for all $T \in \gamma$, if $v$ satisfies $(dd^c v)^n = 0$,

$$\int_T dv \wedge (dd^c v)^{n-1} = \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1} = I(v).$$
The following conjecture was made in [9] by Chern-Levine-Nirenberg.

**Conjecture.** \( N \{ \Gamma_1 \} = \inf_{u \in \mathcal{B}} I(u). \)

Suppose \( M \) is of the following form,

\[
M = \Omega^* \setminus \left( \bigcup_{j=1}^{N} \Omega_j \right),
\]

where \( \Omega^*, \Omega_1, \ldots, \Omega_N \) are bounded strongly smooth pseudoconvex domains in \( \mathbb{C}^n \) where \( \Omega_1, \ldots, \Omega_N \) are pairwise disjoint, and \( \Omega_j \subset \Omega^* \), for all \( j = 1, \ldots, N \). Also, \( \bigcup_{j=1}^{N} \Omega_j \) is holomorphically convex in \( \Omega^* \) (this is a necessary condition in order for equation (1.1) to have a solution), and \( \Gamma_1 = \partial \Omega^* \) and \( \Gamma_0 = \bigcup_{j=1}^{N} \partial \Omega_j \).

For the solution \( u \) of (1.1), Bedford-Taylor proved in [7] that

\[
\tilde{N}(\{ \Gamma_1 \}) = \int_M d\omega \wedge (dd^c u)^{n-1},
\]

and

\[
\tilde{N}(\{ \Gamma_1 \}) = \inf_{v \in \tilde{\mathcal{B}}} \int_M d\omega \wedge (dd^c v)^{n-1},
\]

where

\[
\tilde{\mathcal{B}} = \{ u \in \tilde{\mathcal{F}} \| u \in \text{Lip}(M) \cup C^2(\partial M), \ u \geq 1 \text{ on } \Gamma_1, \ u \leq 0 \text{ on } \Gamma_0 \}.
\]

Furthermore, if the extremal function is in \( C^2(M) \), the identity (1.2) is valid. And if \( u \in C^1(M) \), then

\[
\tilde{N}(\{ \Gamma_1 \}) \leq \int_{\Gamma_1} \left( \frac{\partial u}{\partial r} \right)^n d^c r \wedge (dd^c r)^{n-1}.
\]

Concerning the manifold \( M \) in the conjecture, we remark that if \( \Gamma = \{ v = \text{constant} \} \) for some \( v \in \mathcal{B}, \ \Gamma \sim \{ v = 1 \} \sim \{ v = 0 \} \) in \( H_{2n-1}(M) \). The hypersurface \( \{ v = 1 \} \) is pseudoconvex, and \( \{ v = 0 \} \) is pseudoconcave. If \( M \) is embedded in \( \mathbb{C}^n \), \( v \) is strictly plurisubharmonic, and \( M \) must be of the form (1.10).

In fact, we will show the reverse is also true. The following proposition will be used in the estimation of the degenerate Monge-Ampère equations via the subsolution method.

**Proposition 1.1.** If \( M \) is of the form (1.10), there is \( v \in \text{PSH}(M^0) \cap C^\infty(M) \) such that \( v = 1 + cr \) near \( \Gamma_1 \) and \( v = c_j r_j \) near \( \partial \Omega_j, \ j = 1, 2, \ldots, N, \) for some positive constants \( c, c_1, \ldots, c_N \), where \( r \) and \( r_j \) are the defining functions of \( \Omega \) and \( \Omega_j, \ j = 1, \ldots, N, \) respectively. Furthermore,

\[
(dd^c v)^n > 0 \quad \text{in } M.
\]

Therefore, we will concentrate on the manifolds of the form (1.10). We now state our main results.
Theorem 1.1 (Regularity Theorem). If \( M \) is of the form (1.10), for the unique solution \( u \) of (1.1), there is a sequence \( \{u_k\} \subset B \) such that
\[
\|u_k\|_{C^2(M)} \leq C, \quad \text{for all } k, \quad \lim_{k \to \infty} \sup (dd^c u_k)^n = 0,
\]
and \( \lim_{k \to \infty} \|u_k - u\|_{C^{1,\alpha}(M)} = 0 \), for all \( 0 < \alpha < 1 \). In particular, \( u \in C^{1,1}(M) \).

As a consequence of the regularity theorem, we establish identity (1.2). This identity can be used to obtain the uniqueness of the extremal functions (e.g., Lemma 2.6 in [2]) and, in turn, to study holomorphic mappings. We will discuss these applications elsewhere.

Theorem 1.2. If \( M \) is of the form (1.10),
\[
\tilde{N}(\{\Gamma_1\}) = \int_{\Gamma_1} \left( \frac{\partial u}{\partial r} \right)^n d^c r \wedge (dd^c r)^{n-1}
\]
where \( r \) is the defining function of \( \Omega \).

The Chern-Levine-Nirenberg conjecture for \( \tilde{N} \) is also valid. That is, the class \( \mathcal{B} \) in (1.12) can be replaced by a smaller class \( \mathcal{B} \) defined in (1.9).

Theorem 1.3. If \( M \) is of the form (1.10), then
\[
\tilde{N}(\{\Gamma_1\}) = \inf_{v \in \mathcal{B}} \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1}.
\]

The rest of the paper is organized as follows. In Section 2, we establish \( C^2 \) a priori estimates for the solutions of the equation under the assumption of the existence of a strictly plurisubharmonic solution. Section 3 is devoted to the construction of the subsolution and the proofs of the main results.

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2. Estimates for the extremal function

Equation (1.1) is a degenerate complex Monge-Ampère equation. There have been many works in this direction. In particular, the work of Caffarelli-Kohn-Nirenberg-Spruck [8] establishes \( C^{1,1} \) regularity for solutions in strongly pseudoconvex domains with homogeneous boundary condition. On the other hand, the \( C^{1,1} \) regularity of the homogeneous complex Monge-Ampère equation on strongly pseudoconvex domains with arbitrary boundary data was obtained.
by Krylov [16]. There is also extensive literature on degenerate real Monge-Ampère equations (see [13] and [14] for the references). In our case, some pieces of the boundary are concave. This is the main technical difficulty of the problem. We will incorporate the subsolution method of [11] where interior regularity was treated for pluricomplex Green functions. The existence of a smooth subsolution $v$ of (2.1) in Proposition 1.1 is crucial in the proof of the theorem. This proposition will be proved in the next section.

To establish the existence of the solution $u \in C^{1,1}(M)$, we consider the following equation with parameter $0 \leq t < 1$,

$$
\begin{aligned}
(d\bar{d}u)^n &= (1-t)f_0 \\
u|_{\Gamma_1} &= 1 \\
u|_{\Gamma_0} &= 0,
\end{aligned}
$$

(2.1)

where $f_0 = \det(v_{ij})$ and $v$ is as in Proposition 1.1. When $0 \leq t < 1$, the equation is elliptic. We want to prove that equation (2.1) has a unique smooth solution with a uniform $C^{1,1}$ bound.

**Theorem 2.1.** With $M$ as in (1.10), there is a constant $C$ depending only on $M$ (independent of $t$) such that for each $0 \leq t < 1$, there is a unique smooth solution $u$ of (2.1) with

$$
\|u\|_{C^{1,1}(M)} \leq C.
$$

The a priori estimate (2.2) is the key to the proof of the main results. $C^{1,1}$ regularity of equation (1.1) can be deduced directly from the above theorem. In the following proof, we indicate $c$ to be the constant (which may vary line to line) depending only on $M$ (independent of $t$). We assume the existence of a subsolution $v$ in Proposition 1.1.

**Proof of Theorem 2.1.** Let $f_0 = \det(v_{ij})$. Then $f_0 > 0$, $f_0 \in C^\infty(M)$. We consider $0 \leq t \leq 1$.

$$
\begin{aligned}
\det(u_{ij}) &= (1-t)f_0 \\
u|_{\Gamma_1} &= 1 \\
u|_{\Gamma_0} &= 0.
\end{aligned}
$$

(2.3)

We show that for all $0 \leq t < 1$, there exists $u_t \in C^\infty$, $u_t$ plurisubharmonic, such that $u_t$ solves (2.3) and there exists $C > 0$, for all $0 \leq t < 1$

$$
\|u_t\|_{C^{1,1}(M)} \leq C.
$$

(2.4)

The uniqueness is a consequence of the comparison theorem for complex Monge-Ampère equations (see [4], [5] and [8]). We also note that if (2.4) holds, the equation is elliptic when $0 \leq t < 1$. By the Krylov-Evans theorem, $u_t \in C^\infty(\bar{M})$. Therefore, everything is reduced to the establishment of the a priori estimates (2.4). In the rest of the proof, we will drop the subindex $t$. 

**C⁰-estimates.** Since \( u \) is plurisubharmonic in \( M^0 \), and \( 0 \leq u \leq 1 \) on \( \partial M \), the maximum principle gives \( 0 \leq u(z) \leq 1 \) for all \( z \in M \).

**C¹-estimates.** For any first order differential operator \( D \) with constant real coefficients, we consider \( w = Du + Av \), where \( A \) is a constant to be picked up later. Apply \( D \) to the equation (2.1); we get

\[
\begin{align*}
\sum_{i,j} u^{\bar{i}j} (Du)^{\bar{i}j} &= Df_0 \frac{f_0}{f_0}.
\end{align*}
\]

We also have

\[
\begin{align*}
\sum_{i,j} u^{\bar{i}j} (Av)^{\bar{i}j} &\geq Ac \sum_{i} u^{\bar{i}i} \geq nAcf_0^{\frac{1}{n}}(t-1)^{-\frac{1}{n}} \geq nAcf_0^{\frac{1}{n}}.
\end{align*}
\]

If \( A \) is large,

\[
\sum_{i,j} u^{\bar{i}j} w^{\bar{i}j} > 0 \quad \text{in } M.
\]

By the Maximum Principle for elliptic operators,

\[
\max_{\bar{\Omega}} (Du + Av) = \max_{\partial M} (Du + Av).
\]

To estimate \( Du \) on \( \partial M \), let \( h \) be the solution of

\[
\begin{align*}
\Delta h &= 0 \quad \text{in } M^0, \\
\left. h \right|_{\Gamma_1} &= 1, \\
\left. h \right|_{\Gamma_0} &= 0.
\end{align*}
\]

Since \( \det(u_{ij}) \leq \det(v_{ij}) \),

\[
\begin{align*}
\Delta u &= \sum_{i=1}^{n} u_{ii} = nf_0^{\frac{1}{n}}(1-t)\frac{1}{n} > 0 = \Delta h
\end{align*}
\]

and

\[
\left. u \right|_{\partial M} = \left. v \right|_{\partial M} = h|_{\partial M},
\]

by the Comparison Principle, \( v(z) \leq u(z) \leq h(z) \), for all \( z \in M \). Therefore

\[
\begin{align*}
|Du(z)| &\leq \max (|Dv(z)|, |Dh(z)|) \leq c \quad \text{for all } z \in \partial M.
\end{align*}
\]

That is, \( \max_{\partial M} |Du| \leq c \). In turn,

\[
\begin{align*}
\begin{align*}
\max_{M} W &\leq c, \quad \text{and} \quad \max_{M} |Du| \leq c.
\end{align*}
\end{align*}
\]

**C²-estimates.** Since \( \det^{\frac{1}{n}} \) is concave,

\[
\begin{align*}
\sum_{i,j} u^{\bar{i}j} (D^2u)^{\bar{i}j} &\geq n \frac{D^2((f)^{\frac{1}{n}})}{(f)^{\frac{1}{n}}} = n \left. \frac{D^2f_0^{\frac{1}{n}}}{f_0^{\frac{1}{n}}} \right|_{\partial M}.
\end{align*}
\]
This yields

\[
    u^{ij}(D^2u + Av)_{ij} \geq n \frac{D^2f_0^\frac{1}{n}}{f_0^\frac{1}{n}} + Ac\Sigma u^{ii} \geq n \frac{D^2f_0^\frac{1}{n}}{f_0^\frac{1}{n}} + nAf_0^{-\frac{1}{n}} \geq n \frac{D^2f_0^\frac{1}{n}}{f_0^\frac{1}{n}} + nAf_0^{-\frac{1}{n}}.
\]

If we pick \( A \geq \frac{1}{c} \max_{z \in M} |D^2f_0^\frac{1}{n}| \), then

\[
    u^{ij}(D^2u + Av)_{ij} \geq 0 \quad \text{in } \Omega.
\]

By the Maximum Principle,

\[
    \max_M \{D^2u + Av\} = \max_{\partial M} \{D^2u + Av\}.
\]

(2.8)

In order to obtain \( C^2 \)-a priori estimates for \( u \), we need to get the estimates of the second derivatives of \( u \) on the boundary of \( M \). The boundary of \( M \) consists of pieces of compact strongly pseudoconvex and pseudoconcave hypersurfaces. The second derivative estimates on strongly pseudoconvex hypersurface were established in [8]. Therefore, we will concentrate on the pseudoconcave piece \( \Gamma_0 \) of the boundary. We follow the arguments in [11] with the aid of a strictly plurisubharmonic subsolution \( v \) of Proposition 1.1. Let \( h \) be the harmonic function of (2.5) in \( M \).

Suppose \( z_0 \in \Gamma_0 \), let \( z_1 = x_1 + \sqrt{-1}y_1, \ldots, z_{n-1} = x_{n-1} + \sqrt{-1}y_{n-1}, \) and \( z_n = x_n + \sqrt{-1}y_n \). We may assume \( z_0 = 0 \) and that \( \frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}, \frac{\partial}{\partial y_j} \) are tangential to \( \partial M \) at \( z_0 \). Set \( t_1 = x_1, \ t_2 = y_1, \ldots, t_{2n-3} = x_{n-1}, \ t_{2n-2} = y - n - 1, \ t_{2n-1} = y_n = t \). We also assume \( \frac{\partial}{\partial x_n} = -1 \) at \( z_0 = 0 \).

For all \( \varepsilon > 0 \), we let \( S_\varepsilon = B_\varepsilon(0) \cap M^0 \), and define,

\[
    w(z) = (v(z) - u(z)) + a(v(z) - h(z)) + bv^2(z).
\]

(2.9)

The constants \( a \) and \( b \) will be chosen later. The following lemma was proved by B. Guan (Lemma 2.1 in [11]). For completeness, we reproduce the proof (with some minor modification) here.

**Lemma 2.1.** Let \( \tilde{c} = \inf_{z \in M, \xi \in C^n \setminus \{0\}} \frac{\sum_{ij} v^i_j \xi_i \xi_j}{|\xi|^2} \). For suitable choices of positive constants \( a, b \) and \( \varepsilon \),

(2.10) \( \begin{array}{ccc} \text{i} & \text{L}w(z) \geq \frac{\tilde{c}}{4} \sum u^{ii}(z) & \text{in } S_\varepsilon, \\
\end{array} \)

(2.11) \( \begin{array}{ccc} \text{ii} & w(z) \leq 0 & \text{on } S_\varepsilon, \\
\end{array} \)
where
\[
L = \sum_{i,j} u^i \bar{u}^j \frac{\partial^2}{\partial z_i \partial \bar{z}_j}.
\]  \hfill (2.12)

**Proof of Lemma 2.1.** We first observe that,
\[
L(v - u) \geq Lv - n. \hfill (2.13)
\]
Also, as \(\text{tr}(v_{ij}) \geq n \tilde{c}\) (and by Hopf’s lemma),
\[
h(z) - v(z) \geq c_0 v(z), \quad \text{for all } z \in S_{\varepsilon}, \hfill (2.14)
\]
with some uniform constant \(c_0 > 0\). Furthermore,
\[
L(v - h) \geq -C_1 \sum u^i, \hfill (2.15)
\]
for some uniform positive constant \(C_1\) as \(h \in C^2\). We also have,
\[
L(v^2) = 2Lv + 2 \sum u^i \bar{v}^j \geq 0.
\]
Therefore, as \(a\) and \(b\) are positive,
\[
Lw \geq -(aC_1 \sum u^i + (1 + 2bV)Lv + 2b \sum u^i \bar{v}^j - n). \hfill (2.16)
\]
Now, as \(\frac{\partial v}{\partial x_n} = -1\) at \(z_0 = 0\) and \(Tv(0) = 0\) for any tangential vector field \(T\), if \(\varepsilon > 0\) is small,
\[
\sum u^i \bar{v}^j \geq u^n v_n \bar{v}^n - C_3 \varepsilon \sum u^i \hfill (2.17)
\]
for some positive constant \(C_3\) under control. By the geometric inequality,
\[
\tilde{c} \sum u^i \geq \frac{n(\frac{\tilde{c}}{q})^{\frac{n-1}{n}} (\frac{\tilde{c}}{q} + b)^{\frac{1}{n}}}{(\prod u^i)^{\frac{1}{n}}} \geq \frac{n(\frac{\tilde{c}}{q})^{\frac{n-1}{n}} b^{\frac{1}{n}}}{\det \frac{1}{n} u^i}. \hfill (2.18)
\]
Since \(\det(u_{ij}) = (1 - t)f_0 \leq f_0\), we pick \(b\) such that
\[
\inf_{z \in M} \left\{ \frac{n(\frac{\tilde{c}}{q})^{\frac{n-1}{n}} b^{\frac{1}{n}}}{\det \frac{1}{n} v^j(\bar{z})} \right\} \geq n. \hfill (2.19)
\]
Putting (2.17), (2.16) and (2.18) into (2.15), we obtain
\[
Lw \geq -(aC_1 + 2C_3 b \varepsilon + \tilde{c} \sum u^i + (1 + 2bv)Lv). \hfill (2.19)
\]
Since \(v > 0\) is dominated by \(\varepsilon\) in \(S_{\varepsilon}\), and \(Lv \geq \tilde{c} \sum u^i\), if we choose \(a = \frac{\tilde{c}}{1 + C_1}\) and \(\varepsilon\) small, (2.10) follows from (2.19).
To examine values of $w$ in $S_\varepsilon$, we note that $v = 0$ on $\partial \Gamma_0$. By Hopf’s lemma, if $\varepsilon$ is sufficiently small, for $c_0$ as in (2.14),

$$w \leq -ac_0v + bv^2 \leq 0.$$ 

The proof of the lemma is complete. \hfill \square

Now, back to the proof of Theorem 2.1: Let

$$T_{\alpha} = \frac{\partial}{\partial t_\alpha} - \frac{v_{\alpha}}{v_x} \frac{\partial}{\partial x_n}. \tag{2.20}$$

Let

$$\tilde{w}(z) = w(z) = \pm T_{\alpha}u(z) + u^2(z) + \tilde{C}w(z) - B|z|^2,$$

where $B$ and $\tilde{C}$ are certain constants to be picked later.

By [8],

$$L(\pm T_{\alpha}u(z) + u^2(z)) \geq -c(1 + \sum u_i^2),$$

in $S_\varepsilon$. From the above lemma, if we pick $\tilde{C}$ and $B$ large enough, with $\tilde{C}$ sufficiently larger than $B$ (as $\sum u_i^2 \geq \det^{-\frac{1}{n}} -1 u_{ij} \geq f_0$), we will have

(i) $L(\tilde{w})(z) \geq 0$ in $S_\varepsilon$, \tag{2.21}

(ii) $\tilde{w}(z) \leq 0$ on $\partial S_\varepsilon$, \tag{2.22}

Since $w(0) = 0$, we must have $w_{x_n}(0) \leq 0$. As a consequence, $|u_{x_n t_\alpha}(0)| \leq C$.

Near $\Gamma_0$, $u(z) = \sigma(z)v(z)$. But $u(z) \geq v(z) \geq 0$, for all $z \in M$, so that $\sigma(z) \geq 1$. On the other hand, if $\bar{n}$ is the normal to $\Gamma_0$,

$$u_n = \sigma v_n, \quad \text{for all } z \in \Gamma_0.$$

As $|Du|$ is bounded in $M$, $u_n \leq 0$ and $v_n = -1$ on $\Gamma_0$, so $1 \leq \sigma \leq c$. On $\Gamma_0$, we have

$$u_{ij}(z) = \sigma(z)v_{ij}(z), \quad \text{for all } i, j \leq n - 1.$$

It follows that

$$cI_{n-1} \geq u_{ij}(z) \geq v_{ij}(z) \geq I_{n-1}, \quad \text{for all } z \in \Gamma_0, \quad \text{for all } i, j \leq n - 1,$$

where $I_{n-1}$ is the $(n - 1)$-dim identity matrix.

This gives tangential second derivative bounds. To estimate $u_{x_n x_n}(0)$, we note that

$$u_{x_n x_n}(0) = 4u_{n \bar{n}}(0) - u_{n n}(0).$$

We only need to estimate $u_{n \bar{n}}(0)$. Since $u_{n \bar{n}}(0)$ and $u_{n n}(0)$ are bounded, $u_{n \bar{n}}(0)$ can be estimated by (2.23) and equation (2.3). In conclusion,

$$\|u\|_{C^2(M)} \leq c, \quad \text{for all } 0 \leq t < 1.$$ \tag{2.24}
Finally, the higher regularity for the solution of the equation (2.3) for $0 \leq t < 1$ follows from the Evans-Krylov theorem and the Schauder theorem. The existence follows from standard elliptic theory for the Dirichlet problem.

3. Existence of subsolutions

We construct in this section a strictly smooth plurisubharmonic function with the same boundary value as the solution $u$ of equation (1.1). We begin with some useful elementary lemmas.

**Lemma 3.1.** For all $\delta > 0$, there is an even function $h(t) \in C^\infty(\mathbb{R})$, such that

(i) $h(t) \geq |t|$, for all $t \in \mathbb{R}$, and $h(t) = |t|$, for all $|t| \geq \delta$;

(ii) $|h'(t)| \leq 1$ and $h''(t) \geq 0$, for all $t \in \mathbb{R}$ and $h'(t) \geq 0$, for all $t \geq 0$.

**Proof.** For all $\varepsilon > 0$, let $\rho_\varepsilon(s) = \frac{1}{\varepsilon^2} \rho\left(\frac{s}{\varepsilon}\right)$, where $\rho$ is a standard nonnegative smooth even function supported in $|s| \leq 1$ with $\int_{-\infty}^{\infty} \rho(s) ds = 1$. Set

$$\tilde{h}_\varepsilon(t) = \int_{-\infty}^{t} \rho_\varepsilon(s) ds - \int_{t}^{\infty} \rho_\varepsilon(s) ds.$$ 

Now,

$$\tilde{h}_\varepsilon'(t) = 2\rho_\varepsilon(t) \geq 0,$$

$$|\tilde{h}_\varepsilon(t)| \leq \int_{-\infty}^{t} \rho_\varepsilon(s) ds + \int_{t}^{\infty} \rho_\varepsilon(s) ds = \int_{-\infty}^{\infty} \rho_\varepsilon(s) ds = 1,$$

for all $t \in \mathbb{R}$.

Moreover,

$$\tilde{h}_\varepsilon(t) = 1, \text{ if } t > 2\varepsilon, \quad \text{and} \quad \tilde{h}_\varepsilon(t) = -1, \text{ if } t < -2\varepsilon.$$

We also have,

$$\tilde{h}_\varepsilon(t) \geq 0, \text{ if } t \geq 0, \quad \text{and} \quad \tilde{h}_\varepsilon(t) \leq 0, \text{ if } t \leq 0.$$

Since $\rho_\varepsilon(s)$ is nonnegative and even,

$$\tilde{h}_\varepsilon(t) = -\tilde{h}_\varepsilon(-t).$$

Now, for $\varepsilon = \frac{\delta}{2}$, set

$$h(t) = \int_{0}^{t} \tilde{h}_\varepsilon(s) ds + \left(1 - \int_{0}^{1} \tilde{h}_\varepsilon(s) ds\right).$$
We have
\[ h(t) = h(-t), \quad h'(t) = \tilde{h}_\varepsilon(t), \]
so that \( th'(t) \geq 0 \), for all \( t \). Also
\[ |h'(t)| = |\tilde{h}_\varepsilon(t)| \leq 1. \]
Since \( h(1) = 1 \) and \( h'(t) = 1 \), for all \( t \geq \delta \), we have \( h(t) = t \), for all \( t \geq \delta \).
Therefore, \( h(t) = |t| \), for all \( |t| \geq \delta \), as \( h \) is even. Finally
\[ h''(t) = \tilde{h}_\varepsilon'(t) = 2\rho_\varepsilon(t) \geq 0 \quad \text{for all} \quad t \in \mathbb{R}. \]

**Lemma 3.2.** Suppose \( \Omega \) is a domain of \( \mathbb{C}^n \). For \( f, g \in C^k(\Omega) \), \( k \geq 2 \), for all \( \delta > 0 \), there is an \( H \in C^k(\Omega) \) such that

(i) \( H \geq \max(f, g) \) and
\[ H(z) = \begin{cases} f(z), & \text{if } f(z) - g(z) > \delta, \\ g(z), & \text{if } g(z) - f(z) > \delta; \end{cases} \]

(ii) There exists \( |t(z)| \leq 1 \), such that
\[ (H_{ij}(z)) \geq \left( \frac{1 + t(z)}{2}f_{ij}(z) + \frac{1 - t(z)}{2}g_{ij}(z) \right), \quad \text{for all} \quad z \in \{|f - g| < \delta\}. \]

**Proof.** Note that \( \max(f, g) = \frac{f + g}{2} + \frac{|f - g|}{2} \). Let \( h \) be the function as in Lemma 3.1. We set
\[ H(z) = \frac{f(z) + g(z)}{2} + \frac{h(f(z) - g(z))}{2}. \]
It is obvious that \( H \) satisfies property (i). Now we calculate \( H_{ij}(z) \):
\[ H_{ij}(z) = \frac{f_{ij}(z) + g_{ij}(z)}{2} + \frac{1}{2} \left\{ h'((f(z) - g(z))(f_{ij}(z) - g_{ij}(z)) \\ + h''(f(z) - g(z))(f(z) - g(z))_i(f(z) - g(z))_j \right\} \geq \frac{1 + h'(f(z) - g(z))}{2}f_{ij}(z) + \frac{1 - h'(f(z) - g(z))}{2}g_{ij}(z). \]

We now construct a subsolution \( v \) in Proposition 2.1.

**Proof of Proposition 2.1.** A Lipschitz continuous subsolution can be constructed as in [7]. Let \( \psi_0, \ldots, \psi_N \) be the defining functions of \( \Omega^*, \Omega_1, \ldots, \Omega_N \) respectively, such that \( \psi_j \in C^\infty(\Omega_0) \) and \( dd^c \psi_j > 0 \) in a neighborhood \( U_j \) of \( \partial \Omega_j \), for all \( j = 1, \ldots, N \). (Note that \( \psi_j < 0 \) in \( \Omega_j \) and \( \psi_j \neq 0 \) on \( \partial \Omega_j \).) Since \( \bigcup_{j=1}^N \Omega_j \) are holomorphic convex in \( \Omega_0 \), there is a plurisubharmonic function
ψ in Ω₀, such that ψ(z) < 0 in a small neighborhood of \( \bigcup_{j=1}^{N} \partial \Omega_j \) and is positive outside of \( \bigcup_{j=1}^{N} \bar{U}_j \). We may assume that \( U_j \) are pair-wise disjoint for \( j = 1, \ldots, N \), and ψ \( \in C^\infty(\Omega_0) \) and \( dd^c \psi > 0 \) in \( \Omega_0 \). In each \( U_j \), we set

\[
V_j(z) = \max\{\varepsilon^2 \psi_j, \varepsilon \psi\}.
\]

If we pick \( \varepsilon \) small, we have \( V_j(z) = \varepsilon^2 \psi_j(z) \) in a small neighborhood of \( \partial \Omega_j \) in \( U_j \), and \( V_j(z) = \varepsilon \psi(z) \) away from that small neighborhood in \( U_j \). So, we may extend \( V_j \) to all of \( \Omega \) by defining \( V_j(z) = \varepsilon \psi(z) \) outside of \( U_j \) and \( V_j(z) \leq 1 \) on \( \Gamma_1 \). By applying Lemma 3.2 in \( U_j \), we obtain a smooth strictly plurisubharmonic function \( H_j \) such that \( H_j = 0 \) on \( \partial \Omega_j \) and \( H_j = \varepsilon \psi(z) \) outside of \( U_j \). Now, we pick \( \lambda \) large so that \( 1 + \lambda \psi_0 \) will be very negative in \( \bigcup_{j=1}^{N} \bar{U}_j \). Set

\[
H(z) = \max\{1 + \lambda \psi_0(z), H_1(z), \ldots, H_N(z)\}.
\]

The function \( H \) is taken as a maximum of smooth strongly plurisubharmonic functions, and at no point are there three equal functions involved. Therefore, we can apply Lemma 3.2 near the places where any two of these function meet to obtain a smooth strongly plurisubharmonic function \( v \), such that \( v(z) = 1 \) on \( \Gamma_1 \), \( v(z) = 0 \) on \( \Gamma_0 \) and \( (dd^c)^n v > 0 \) in \( M_0 \).

**Proof of the Regularity Theorem 1.1.** From the above proposition and Theorem 2.1, there is a sequence of strictly smooth plurisubharmonic functions \( \{u^1\} \) satisfying (2.1). By (2.2), there is a subsequence \( \{t_k\} \) that tends to 1, such that \( \{u_{t_k}\} \) converges to a plurisubharmonic function \( u \) in \( C^{1,\alpha}(M) \) for any \( 0 < \alpha < 1 \). By the Convergence Theorem for complex Monge-Ampère measures, \( u \) satisfies equation (1.1). Again by (2.2), \( u \in C^{1,1}(M) \).

**Proof of Theorem 1.2.** If we let \( \{u_k\} \) be as in the Regularity Theorem, we have

\[
\int_M du_k \wedge d^c u_k \wedge (dd^c u_k)^{n-1} = \int_{\Gamma_1} d^c u_k \wedge (dd^c u_k)^{n-1} - \int_M u_k (dd^c u_k)^n = \int_{\Gamma_1} \left( \frac{\partial u_k}{\partial r} \right)^n d^c r \wedge (dd^c r)^{n-1} - \int_M u_k (dd^c u_k)^n.
\]

Since \( u_k \to u \) in \( C^{1,\alpha}(M) \), \( \left( \frac{\partial u_k}{\partial r} \right)^n \to \left( \frac{\partial u}{\partial r} \right)^n \) uniformly on \( \Gamma_1 \). Therefore, by the Convergent Theorem for complex Monge-Ampère measures (see [5]), we get

\[
\tilde{N}(\{P_1\}) = \int_M du \wedge d^c u \wedge (dd^c u)^{n-1} = \int_{\Gamma_1} \left( \frac{\partial u}{\partial r} \right)^n d^c r \wedge (dd^c r)^{n-1}.
\]

**Proof of Theorem 1.3.** By [7],

\[
\tilde{N}(\{\Gamma_1\}) = \int_{\Gamma_1} d^c u \wedge (dd^c u)^{n-1} = \int_M du \wedge d^c u \wedge (dd^c u)^{n-1},
\]
and \( \tilde{N}(\{\Gamma_1\}) \geq N(\{\Gamma_1\}) \) by definition. Also by the Comparison Theorem ([4], [8]), for all \( v \in B \) if \( v \neq u \), one must have \( v < u \) in \( M^0 \). By Theorem 2.1 in [7],
\[
\int_M du \wedge d^c u \wedge (dd^c u)^{n-1} \leq \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1}.
\]
That is,
\[
\int_M du \wedge d^c u \wedge (dd^c u)^{n-1} \leq \inf_{v \in B} \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1}.
\]
On the other hand, by the Convergent Theorem for complex Monge-Ampère measures
\[
\liminf_{k \to \infty} \int_M du_k \wedge d^c u_k \wedge (dd^c u_k)^{n-1} = \int_M du \wedge d^c u \wedge (dd^c u)^{n-1}.
\]
That is,
\[
\tilde{N}(\Gamma_1) = \int_M du \wedge d^c u \wedge (dd^c u)^{n-1} = \inf_{v \in B} \int_M dv \wedge d^c v \wedge (dd^c v)^{n-1}. \quad \square
\]

Remark. The main results can be generalized to Stein manifolds without major changes. Estimates \( C^1 \) and \( C^2 \) can be obtained from the fact that there are finite global holomorphic and anti-holomorphic vector fields that generate \( T(M) \). Interior estimates \( C^2 \) also follow from Yau [18]. We can obtain \( C^2 \) boundary estimates using the same arguments as in this paper, as they are local estimates.

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