Reconciliation of Approaches to the Semantics of Logics without Distribution

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In memoriam Jon Michael Dunn

Abstract
This article clarifies and indeed completes an approach (initiated by Dunn and this author several years ago and again pursued by the present author over the last three years or so) to the relational semantics of logics that may lack distribution (Dunn’s non-distributive gaggl es). The approach uses sorted frames with an incidence relation on sorts (polarities), equipped with additional sorted relations, but, in the spirit of Occam’s razor principle, it drops the extra assumptions made in the generalized Kripke frames approach, initiated by Gehrke, that the frames be separated and reduced (RS-frames). We show in this article that, despite rejecting the additional frame restrictions, all the main ideas and results of the RS-frames approach relating to the semantics of non-distributive logics are captured in this simpler framework. This contributes in unifying the research field, and, in an important sense, it complements and completes Dunn’s gaggle theory project for the particular case of logics that may drop distribution.

1 Preliminaries

1.1 A Note on Motivation
The motivation for this article is twofold. First, it aims at complementing Dunn’s gaggle-theory project [2, 10] by addressing the case of logics that may lack distribution. This has been the topic of recent research both by the present author and by researchers in the RS-frames approach, initiated by Gehrke [14]. The second motivation relates to clarifying points of convergence and divergence between our approach and that of RS-frames. Dunn himself seems to have stood at the junction of both, as he has contributed, with this author, the lattice representation result [28] on which our approach is based, while having also contributed in applying the RS-frames approach to his gaggle-theory project, with Gehrke and Palmigiano [11]. Indeed we conclude that apart from dropping
the “RS” from the “RS-frames” approach, which is to say the assumptions that frames (polarities with relations) are *Separated and Reduced*, the two approaches have nearly identical objectives and nearly identical techniques, though they have developed separately. In our concluding remarks we point at some issues that seem to indicate that the approach we have taken may be better suited for some purposes (though no doubt the same can be said for the RS-approach, for some other purposes).

### 1.2 Normal Lattice Expansions

In [10], Dunn introduced the notion of a distributoid, as a distributive lattice with various operations that in each argument place either distribute or co-distribute over either meets or joins, always returning the same type of lattice operation (always a meet, or always a join). To define technically the extended notion, to which we refer as a normal lattice expansion, let \( \{1, \partial\} \) be a 2-element set, \( \mathcal{L}^1 = \mathcal{L} \) and \( \mathcal{L}^\partial = \mathcal{L}^{op} \) (the opposite lattice, order reversed). Extending the Jónsson-Tarski terminology [31], a function \( f : \mathcal{L}_1 \times \cdots \times \mathcal{L}_n \rightarrow \mathcal{L}_{n+1} \) is *additive*, or a normal operator, if it distributes over finite joins of the lattice \( \mathcal{L}_i \), for each \( i = 1, \ldots, n \), delivering a join in \( \mathcal{L}_{n+1} \).

**Definition 1.1.** An \( n \)-ary operation \( f \) on a bounded lattice \( \mathcal{L} \) is a normal lattice operator of distribution type \( \delta(f) = (i_1, \ldots, i_n; i_{n+1}) \in \{1, \partial\}^{n+1} \) if it is a normal additive function \( f : \mathcal{L}^{i_1} \times \cdots \times \mathcal{L}^{i_n} \rightarrow \mathcal{L}^{i_{n+1}} \) (distributing over finite joins in each argument place), where each \( i_j \), for \( j = 1, \ldots, n + 1 \), is in the set \( \{1, \partial\} \), i.e. \( \mathcal{L}^{i_j} \) is either \( \mathcal{L} \), or \( \mathcal{L}^\partial \).

**Example 1.1.** A modal normal diamond operator \( \diamond \) is a normal lattice operator of distribution type \( \delta(\diamond) = (1; 1) \), i.e. \( \diamond : \mathcal{L} \rightarrow \mathcal{L} \), distributing over finite joins of \( \mathcal{L} \). A normal box operator \( \Box \) is also a normal lattice operator in the sense of Definition 1.1 of distribution type \( \delta(\Box) = (\partial; \partial) \), i.e. \( \Box : \mathcal{L}^\partial \rightarrow \mathcal{L}^\partial \) distributes over finite joins of \( \mathcal{L}^\partial \), which are then just meets of \( \mathcal{L} \).

An FLew-algebra (also referred to as a full \( \text{BCK} \)-algebra, or a commutative integral residuated lattice) \( \mathcal{A} = (L, \land, \lor, 0, 1, \circ, \rightarrow) \) is a normal lattice expansion, where \( \delta(\circ) = (1, 1; 1) \), \( \delta(\rightarrow) = (1, \partial; \partial) \), i.e. \( \circ : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L} \) and \( \rightarrow : \mathcal{L} \times \mathcal{L}^\partial \rightarrow \mathcal{L}^\partial \) are both normal lattice operators with the familiar distribution properties. Drophing exchange, \( \circ \) has two residuals \( \leftarrow, \rightarrow \), one in each argument place, where \( \delta(\leftarrow) = (\partial, 1; \partial) \), i.e. \( \leftarrow : \mathcal{L}^\partial \times \mathcal{L} \rightarrow \mathcal{L}^\partial \).

De Morgan Negation \( \sim \) is a normal lattice operator and it has both the distribution type \( \delta_1(\sim) = (1; \partial) \) and \( \delta_2(\sim) = (\partial; 1) \), as it switches both joins to meets and meets to joins.

The Grishin operators [19] \( \leftarrow, \star, \rightarrow \), satisfying the familiar co-residuation conditions \( a \geq c \leftarrow b \iff a \ast b \geq c \) iff \( b \geq a \rightarrow c \) have the respective distribution properties, which are exactly captured by assigning to them the distribution types \( \delta(\ast) = (1, \partial; \partial) \) (\( \ast \) behaves like a binary box operator), \( \delta(\leftarrow) = (1, \partial; 1) \) and \( \delta(\rightarrow) = (\partial, 1; 1) \).
Dunn’s distributoid’s \cite{10} are the special case of a normal lattice expansion where the underlying lattice is distributive. BAO’s (Boolean Algebras with Operators) \cite{31,32} are the special case where the underlying lattice is a Boolean algebra and all normal operators distribute over finite joins of the Boolean algebra, i.e. they are all of distribution types of the form \((1, \ldots, 1; 1)\). For BAO’s there is no need to consider operators of other distribution types, as they can be obtained by composition of operators with Boolean complementation. For example, in studying residuated Boolean algebras \cite{33}, Jónsson and Tsinakis introduce a notion of conjugate operators and they show that intensional implications (division operations) \(\setminus/\) (the residuals of the product operator \(\circ\)) are interdefinable with the conjugates (at each argument place) \(\triangleleft,\triangleright\) of \(\circ\), i.e. \(a\setminus b = (a \triangleright b)^{-}\) and \(a \triangleright b = (a\setminus b)^{-}\) (and similarly for \(\triangleright,\setminus\), see \cite{33} for details). Note that \(\triangleleft,\triangleright\) are not operators, whereas \(\triangleleft,\triangleright\) are.

The relational representation of BAO’s in \cite{31}, extending Stone’s representation \cite{37} of Boolean algebras using the space of ultrafilters of the algebra, forms the technical basis of the subsequently introduced by Kripke possible worlds semantics, with its well-known impact on the development of normal modal logics. Dunn’s approach and objective in \cite{10} has been to achieve the same unified semantic treatment for the logics of distributive lattices with various quasioperators, now based on the Priestley representation \cite{35} of distributive lattices in ordered Stone spaces (simplifying Stone’s original representation \cite{36} of distributive lattices), using the space of prime filters, and abstracting over various specific results in the semantics of distributive, non-classical logics, notably Relevance Logics.

For non-distributive lattices, Urquhart pioneered a representation theorem \cite{39}, using the space of maximally disjoint filter-ideal pairs. Over the years, Urquhart’s representation has proven notoriously difficult to work with, though some authors, including Dunn himself (with Allwein) \cite{11}, as well as Dintsch, Orlowska, Radzikowska and Vakarelov \cite{12} have based a semantic treatment of specific systems on it. Hartonas and Dunn \cite{27}, published in (1997) \cite{28}, provide a lattice representation and duality result based on the representation of semilattices and of Galois connections and abstracting over Goldblatt’s \cite{17} representation of ortholattices, replacing orthocomplementation with the trivial Galois connection (the identity map \(\iota : L \rightarrow (L^\circ)^\partial\)). Hartonas \cite{20} presents another lattice representation, extended to include various lattice expansions. Both \cite{20,28} form the background of a representation and duality result for normal lattice expansions \cite{21}, extending the representation of \cite{28}.

The bulk of Dunn’s work on gaggles predates the extension of the theory of canonical extensions to bounded lattices advanced by Gehrke and Harding in \cite{15}, extending the Jónsson-Tarski results for perfect extensions of Boolean algebras \cite{31} and the Gehrke-Jónsson following extension to distributive lattice expansions \cite{16}. Subsequently, Gehrke \cite{14} proposed generalized Kripke frames (RS-frames), based on Hartung’s lattice representation \cite{30}, as a suitable framework for the relational semantics of logics lacking distribution \cite{4}, including full Linear Logic \cite{8}. The RS-frames approach to the semantics of logics without
distribution was further developed by Palmigiano and co-workers, notably Conradie [5,7,8].

2 Polarities with Relations

2.1 Definitions, Notational Conventions and Basic Facts

Let \{1, \partial\} be a set of sorts and \((Z_1, Z_0)\) a sorted set. Base sorted frames \(\mathfrak{F} = (X, \perp, Y) = (Z_1, \perp, Z_0)\) are triples consisting of nonempty sets \(Z_1 = X, Z_0 = Y\) and a binary relation \(\sqsubseteq \subseteq X \times Y\), which will be referred to as the Galois relation of the frame. It generates a Galois connection \((\_\perp^\mid : \mathcal{P}(X) \cong \mathcal{P}(Y)^\partial : \_\perp^\dagger(\_))\) \((V \subseteq U^\perp \iff U \subseteq V^\dagger)\), defined by

\[
U^\perp = \{ y \in Y \mid \forall x \in U \ x \perp y \} = \{ y \in Y \mid U \subseteq y \}
\]

\[
\dagger V = \{ x \in X \mid \forall y \in V \ x \perp y \} = \{ x \in X \mid x \perp V \}
\]

Triples \((X, R, Y), R \subseteq X \times Y\), where \(R\) is treated as the Galois relation of the frame, are variously referred to in the literature as polarities, after Birkhoff [3], as formal contexts, in the Formal Concept Analysis (FCA) tradition [13], or as object-attribute (categorization, classification, or information) systems [34,44,10], or as generalized Kripke frames [14], or as polarity frames in the bi-approximation semantics of [25].

A subset \(A \subseteq X\) will be called stable if \(A = \_\perp^\dagger(A^\perp)\). Similarly, a subset \(B \subseteq Y\) will be called co-stable if \(B = (\_\perp^\dagger B)^\perp\). Stable and co-stable sets will be referred to as Galois sets, disambiguating to Galois stable or Galois co-stable when needed and as appropriate.

\(G(X), G(Y)\) designate the families (complete lattices) of stable and co-stable sets, respectively. Note that the Galois connection restricts to a duality of the complete lattices of Galois stable and co-stable sets \((\_\perp^\mid : G(X) \cong G(Y)^\partial : \_\perp^\dagger(\_))\).

Preorder relations are induced on each of the sorts, by setting for \(x, z \in X\), \(x \preceq z\) \(\iff \{ x \}^\perp \subseteq \{ z \}^\perp\) and, similarly, for \(y, v \in Y\), \(y \preceq v\) \(\iff \_\perp^\dagger(y) \subseteq \_\perp^\dagger(v)\). A (sorted) frame is called separated if the preorders \(\preceq\) (on \(X\) and on \(Y\)) are in fact partial orders \(\leq\). Note that if the frame is separated, then \(\Gamma x = \Gamma z\) \(\iff x = z\) and \(\_\perp^\dagger(y) = \_\perp^\dagger(v)\) \(\iff y = v\). Thus we can identify \(X\) and \(Y\) with the corresponding subsets of \(G(X), G(Y)\). Moreover, we can identify \(Z = X \cup Y\) with the family of sets \(\{ \Gamma x \mid x \in X\} \cup \{ \_\perp^\dagger(y) \mid y \in Y\} \subseteq G(X)\). The following result is due to Gehrke ([13], Proposition 2.7, Corollary 2.11).

**Proposition 2.1** (Gehrke [14]). In a separated frame \((X, \perp, Y)\) the set \(Z = X \cup Y\) is partially ordered by the relation \(\preceq\) defined for \(x, z \in X\) and \(y, v \in Y\) by

\[
\begin{align*}
    x \preceq y & \iff \Gamma x \subseteq \_\perp^\dagger(y) \iff x \perp y \\
    x \preceq z & \iff \Gamma x \subseteq \Gamma z \iff z \preceq x \\
    y \preceq v & \iff \_\perp^\dagger(y) \subseteq \_\perp^\dagger(v) \iff y \preceq v \\
    y \preceq x & \iff \_\perp^\dagger(y) \subseteq \Gamma x \iff \forall u \in X \forall w \in Y \ (u \perp y \land x \perp w \rightarrow u \perp w)
\end{align*}
\]

Moreover, \(G(X)\) is the Dedekind-MacNeille completion \(\overline{Z}\) of \((Z, \preceq)\). \(\square\)
Generalized Kripke frames, introduced by Gehrke \[14\], are separated and reduced polarities (RS-frames), where the latter is defined by the conditions

1. \( \forall x \in X \exists y \in Y \ (x \not\leq y \land \forall z \in X (z \leq x \land z \neq x \rightarrow z \leq y)) \)
2. \( \forall y \in Y \exists x \in X \ (x \not\leq y \land \forall v \in Y (y \leq v \land y \neq v \rightarrow x \leq v)) \)

The concept goes back to Wille’s Formal Concept Analysis (FCA) framework \[13\] and Hartung’s lattice representation theorem \[30\]. Gehrke observes that being reduced means that all the elements of \( X \) are completely join irreducible in \( X \) (equivalently, in \( Z = \mathcal{G}(X) \)) and, dually, that all the elements of \( Y \) are completely meet irreducible in \( Y \) (equivalently in \( Z = \mathcal{G}(X) \)). This turns the poset \( (Z, \leq) \) to what is called a perfect poset \[11\] (join-generated by the set \( J^\infty(Z) \) of its join irreducibles and meet-generated by the set \( M^\infty(Z) \) of its meet irreducibles), which can be represented as the two-sorted frame \( (J^\infty(Z), M^\infty(Z), \leq) \).

To model additional logical operators, RS-frames are equipped with relations subject to the requirement that all their sections be stable sets. For example, to model the Lambek calculus product operator \( \circ \), RS-frames are equipped with a relation \( S \subseteq Y \times (X \times X) \), all sections of which are required to be stable, and an operation \( \otimes \) is generated on stable sets by defining (see \[14\])

\[
A \otimes C = \{ y \in Y \mid \forall x \in A \forall z \in C \ ySxz \} = \{ u \in X \mid \forall y \in Y (\forall x, z \in X (x \in A \land z \in C \rightarrow ySxz) \rightarrow u \leq y) \}
\]

In the canonical frame (constructed as the Hartung representation \[30\] of the Lindenbaum-Tarski algebra \( \mathcal{L} \) of the logic), the relation \( S \) is defined by the condition \( ySxz \) iff \( \forall a, c \in \mathcal{L} (a \in x \land c \in z \rightarrow a \circ c \in y) \).

In our own approach, initiated with Dunn and the lattice representation result of \[27, 28\] and gradually developed in the last three years or so by this author \[21–24, 26, 29\], we have preferred to apply Occam’s razor principle and reject any property of frames that can be rejected while still allowing for the derivation of the needed results. Therefore we work with polarities that need not be separated, or reduced. Section stability for the additional relations, however, is retained as a requirement.\[4\]

In the rest of this section we specify the dual objects (polarities with relations) of normal lattice expansions, thereby a class of frames for logics that may lack distribution is described. Representation issues (for completeness arguments) is discussed in the next section and it amounts to an extension of the lattice representation published by this author with Dunn \[28\]. We do not specify any particular logical signature (except for assuming that conjunction, disjunction and logical constants for truth and falsity are present). The logical setting is very much the same as that detailed by Conradie and Palmigiano in \[6\] and the reader is referred to this article for a syntactic description of the logics. We save some space by working here with the algebraic structures corresponding

\[4\]The usefulness of this property has not been fully acknowledged in some of our previous writings, as we did not make section stability explicit. We fill in for this gap in this article.
to logics that may lack distribution (LE-logics, in the terminology of [3]), i.e. with normal lattice expansions.

**Remark 2.2 (Notational Conventions).** For a sorted relation \( R \subseteq \Pi_{j=1}^{n+1} Z_{ij} \), where \( i_j \in \{1, \partial\} \) for each \( j \) (and thus \( Z_{ij} = X \) if \( i_j = 1 \) and \( Z_{ij} = Y \) when \( i_j = \partial \)), we make the convention to regard it as a relation \( R \subseteq Z_{i_{n+1}} \times \Pi_{j=1}^{n} Z_{ij} \), we agree to write its sort type as \( \sigma(R) = (i_{n+1}; i_1 \ldots i_n) \) and for a tuple of points of suitable sort we write \( uR^i_{u_1} \ldots u_n \) for \( (u, u_1, \ldots, u_n) \in R \). We often display the sort type as a superscript, as in \( R^\alpha \). Thus, for example, \( R^\partial \partial \) is a subset of \( Y \times (X \times Y) \). In writing then \( yR^\partial \partial x \) it is understood that \( x \in X = Z_1 \) and \( y, v \in Y = Z_\partial \). The sort superscript is understood as part of the name designation of the relation, so that, for example, \( R^{111}, R^\partial \partial \) name two different relations.

We use \( \Gamma \) to designate upper closure \( \Gamma U = \{ z \in X \mid \exists x \in U \; x \preceq z \} \), for \( U \subseteq X \), and similarly for \( U \subseteq Y \). \( U \) is *increasing* (an upset) iff \( U = \Gamma U \). For a singleton set \( \{x\} \subseteq X \) we write \( \Gamma x \), rather than \( \Gamma (\{x\}) \) and similarly for \( \{y\} \subseteq Y \).

We typically use the standard FCA [13] priming notation for each of the two Galois maps \( ^\alpha (\cdot) \), \( (\cdot)^\alpha \). This allows for stating and proving results for each of \( \mathcal{G}(X), \mathcal{G}(Y) \) without either repeating definitions and proofs, or making constant appeals to duality. Thus for a Galois set \( G, G' = G^\alpha \), if \( G \in \mathcal{G}(X) \) (\( G \) is a Galois stable set), and otherwise \( G' = ^\alpha G \), if \( G \in \mathcal{G}(Y) \) (\( G \) is a Galois co-stable set).

For an element \( u \) in either \( X \) or \( Y \) and a subset \( W \), respectively of \( Y \) or \( X \), we write \( uW \), under a well-sorting assumption, to stand for \( u \sqsubseteq W \) (which stands for \( u \subseteq w \), for all \( w \in W \)), or \( W \sqsubseteq u \) (which stands for \( w \subseteq u \), for all \( w \in W \)), where well-sorting means that either \( u \in X \sqcap W \), or \( W \subseteq X \) and \( u \in Y \), respectively. Similarly for the notation \( u\sqcup v \), where \( u, v \) are elements of different sort.

We designate \( n \)-tuples (of sets, or elements) using a vectorial notation, setting \( (G_1, \ldots, G_n) = \hat{G} \in \prod_{j=1}^{n} \mathcal{G}(Z_{ij}) \), \( \hat{U} \in \prod_{j=1}^{n} \mathcal{G}(Z_{ij}) \), \( \hat{u} \in \prod_{j=1}^{n} Z_{ij} \) (where \( i_j \in \{1, \partial\} \)). Most of the time we are interested in some particular argument place \( 1 \leq k \leq n \) and we write \( \hat{G}[F]_k \) for the tuple \( \hat{G} \) where \( G_k = F \) (or \( G_k \) is replaced by \( F \)). Similarly \( \hat{u}[x]_k \) is \( (u_1, \ldots, u_{k-1}, x, u_{k+1}, \ldots, u_n) \). For brevity, we write \( \hat{u} \approx v \) for the pointwise ordering statements \( u_1 \preceq v_1, \ldots, u_n \preceq v_n \). We also let \( \hat{u} \in \hat{W} \) stand for the conjunction of componentwise membership \( u_j \in W_j \), for all \( j = 1, \ldots, n \).

To refer to sections of relations (the sets obtained by leaving one argument place unfilled) we make use of the notation \( \hat{u}_{[\cdot]}^k \) which stands for the \((n-1)\)-tuple \( (u_1, \ldots, u_{k-1}, [\cdot], u_{k+1}, \ldots, u_n) \) and similarly for tuples of sets, extending the membership convention for tuples to cases such as \( \hat{u}_{[\cdot]}^k \in \hat{F}_{[\cdot]}^k \) and similarly for ordering relations \( \hat{u}_{[\cdot]}^k \preceq \hat{v}_{[\cdot]}^k \). We also quantify over tuples (with, or without a hole in them), instead of resorting to an iterated quantification over the elements of the tuple, as for example in \( \exists \hat{u}_{[\cdot]}^k \in \hat{F}_{[\cdot]}^k \exists v, w \in G \; wR\hat{u}_{[\cdot]}^k \).

**Lemma 2.3.** Let \( \mathfrak{F} = (X, \pm, Y) \) be a polarity and \( u \in Z = X \cup Y \).

1. \( \pm \) is increasing in each argument place
2. \( (\Gamma u)' = \{u\}' \) and \( \Gamma u = \{u\}'' \) is a Galois set. 
3. Galois sets are increasing, i.e. \( u \in G \) implies \( \Gamma u \subseteq G \)

4. For a Galois set \( G \), \( G = \bigcup_{u \in G} \Gamma u \)

5. For a Galois set \( G \), \( G = \bigvee_{u \in G} \Gamma u = \bigcap_{y \in G} \{v\}' \).

6. For a Galois set \( G \) and any set \( W \), \( W'' \subseteq G \) iff \( W \subseteq G \).

Proof. By simple calculation. Proof details are included in [21], Lemma 2.2. For claim 4, \( \bigcup_{u \in G} \Gamma u \subseteq G \) by claim 3 (Galois sets are upsets). In claim 5, given our notational conventions, the claim is that if \( G \in \mathcal{G}(X) \), then \( G = \bigcap_{G \in \mathcal{G}(Y)} \{y\}' \) and if \( G \in \mathcal{G}(Y) \), then \( G = \bigcap_{x \in \mathcal{X}} \{x\}' \). \( \square \)

For the purposes of this article, the following definition of closed and open elements suffices.

**Definition 2.4** (Closed and Open Elements). The principal upper sets of the form \( \Gamma x \), with \( x \in X \), will be called closed, or filter elements of \( \mathcal{G}(X) \), while sets of the form \( ^{\downarrow}y \), with \( y \in Y \), will be referred to as open, or ideal elements of \( \mathcal{G}(X) \). Similarly for \( \mathcal{G}(Y) \). A closed element \( \Gamma u \) is clopen iff there exists an element \( v \), with \( u|v \), such that \( \Gamma u \asymp \{v\}' \).

By Lemma 23, the closed elements of \( \mathcal{G}(X) \) join-generate \( \mathcal{G}(X) \), while the open elements meet-generate \( \mathcal{G}(X) \) (similarly for \( \mathcal{G}(Y) \)).

**Definition 2.5** (Galois Dual Relation). For a relation \( R \), of sort type \( \sigma \), its Galois dual relation \( R' \) is the relation defined by \( uR'\bar{v} \iff \forall w (wR\bar{v} \rightarrow u|w) \). In other words, \( R'\bar{v} = (R\bar{v})' \).

For example, given a relation \( R^{111} \) its Galois dual is the relation \( R^{011} \) where for any \( x, z \in X \), \( R^{011}xz = (R^{111}xz)^{\downarrow} = \{y \in Y \mid \forall u \in X (uR^{111}xz \rightarrow u \downarrow y)\} \) and, similarly, for a relation \( S^{01\partial} \) its Galois dual is the relation \( S^{11\partial} \) where for any \( x \in X, v \in Y \) we have \( S^{11\partial}xv = (S^{01\partial}xv)^{\downarrow} \), i.e. \( xS^{11\partial}xv \) holds iff for all \( y \in Y \), if \( yS^{01\partial}xv \) obtains, then \( x \downarrow y \).

**Definition 2.6** (Sections of Relations). For an \((n + 1)\)-ary relation \( R^n \) and an \( n \)-tuple \( \bar{u} \), \( R^n\bar{u} = \{w \mid wR^n\bar{u}\} \) is the section of \( R^n \) determined by \( \bar{u} \). To designate a section of the relation at the \( k \)-th argument place we let \( \bar{u}[\_k \rangle \bar{u} \) be the tuple with a hole at the \( k \)-th argument place. Then \( wR^n\bar{u}[\_k \rangle \bar{u} = \{v \mid wR^n\bar{u}[v\_k] \subseteq Z_{ik} \) is the \( k \)-th section of \( R^n \).

### 2.2 Image Operators, Conjugates and Residuals

If \( R^n \) is a relation on a sorted frame \( \mathcal{F} \), of some sort type \( \sigma = (i_{n+1}; i_1 \ldots i_n) \), then as in the unsorted case, \( R^n \) (but we shall drop the displayed sort type when clear from context) generates a (sorted) image operator \( \alpha_R \), defined by (1), of sort \( \sigma(\alpha_R) = (i_1, \ldots, i_n; i_{n+1}) \), defined by the obvious generalization of the Jonsson-Tarski image operators [31].

\[
\alpha_R(\bar{W}) = \{w \in Z_{i_{n+1}} \mid \exists \bar{w} \ (wR\bar{w} \wedge \bigwedge_{j=1}^{i_n} (w_j \in W_j))\} = \bigcup_{\bar{w} \in \bar{W}} R\bar{w} \quad (1)
\]
where for each \( j \), \( W_j \subseteq Z_{i_j} \).

Thus \( \alpha_R \) is a normal and completely additive function in each argument place, therefore it is residuated, i.e. for each \( k \) there is a set-operator \( \beta_R^k \) satisfying the condition:

\[
\alpha_R(\bar{W}[V]_k) \subseteq U \quad \text{iff} \quad V \in \beta_R^k(\bar{W}[U]_k)
\]

(2)

Hence \( \beta_R^k(\bar{W}[U]_k) \) is the largest set \( V \) s.t. \( \alpha_R(\bar{W}[V]_k) \subseteq U \) and it is thereby definable by

\[
\beta_R^k(\bar{W}[U]_k) = \bigcup \{ V \mid \alpha_R(\bar{W}[V]_k) \subseteq U \}
\]

(3)

Let \( \overline{\alpha}_R \) be the closure of the restriction of \( \alpha_R \) to Galois sets \( \tilde{F} \),

\[
\overline{\alpha}_R(\tilde{F}) = (\alpha_R(\tilde{F}))'' = \left( \bigcup_{j=1,\ldots,n} R\bar{w} \right)'' = \bigvee_{\bar{w} \in \tilde{F}} (R\bar{w})''
\]

(4)

where \( F_j \in \mathcal{G}(Z_{i_j}) \), for each \( j \in \{1,\ldots,n\} \). The operator \( \overline{\alpha}_R \) is sorted and its sorting is inherited from the sort type of \( R \). For example, if \( \sigma(R) = (\partial;11) \), \( \alpha_R : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathcal{P}(Y) \), hence \( \overline{\alpha}_R : \mathcal{G}(X) \times \mathcal{G}(X) \rightarrow \mathcal{G}(Y) \). Single sorted operations \( \overline{\alpha}_R : \mathcal{G}(X) \times \mathcal{G}(X) \rightarrow \mathcal{G}(X) \) and \( \overline{\alpha}_R^d : \mathcal{G}(Y) \times \mathcal{G}(Y) \rightarrow \mathcal{G}(Y) \) can be then extracted by composing appropriately with the Galois connection: \( \overline{\alpha}_R^d(A,C) = (\overline{\alpha}_R(A,C))'' \) (where \( A,C \in \mathcal{G}(X) \)) and, similarly, \( \overline{\alpha}_R^d(B,D) = \overline{\alpha}_R(B',D') \) (where \( B,D \in \mathcal{G}(Y) \)). Similarly for the \( n \)-ary case.

**Definition 2.7 (Complex Algebra).** Let \( \mathfrak{F} = (X,\pm,Y,R) \) be a polarity with a relation \( R \) of some sort \( \sigma(R) = (i_{n+1};i_1\cdots i_n) \). The **complex algebra** of \( \mathfrak{F} \) is the structure \( \mathfrak{F}^+ = (\mathcal{G}(X),\overline{\alpha}_R^d) \) and its **dual complex algebra** is the structure \( \mathfrak{F}^0 = (\mathcal{G}(Y),\overline{\alpha}_R^d) \).

Most of the time we work with the dual sorted algebra \( \mathfrak{F}^0 : \mathcal{G}(X) = \mathcal{G}(Y)^0 : \mathfrak{F}^0 \), as it allows for considering sorted operations that distribute over joins in each argument place (which are either joins of \( \mathcal{G}(X) \), or of \( \mathcal{G}(Y) \), depending on the sort type of the operation). Single-sorted normal operators are then extracted in the complex algebra by composition with the Galois maps, as indicated above.

**Remark 2.8 (Objective).** The primary objective of the current section is to specify conditions under which the residuation structure \( \alpha_R \dashv \beta_R^k \) is preserved under the restriction and closure operation described above so that the sorted operator \( \overline{\alpha}_R \) on Galois sets is residuated, hence it distributes over arbitrary joins of Galois sets. The notion of conjugate operators we next introduce is useful in this context. Conjugates were introduced in [33] for residuated Boolean algebras. We generalize here to the sorted case, using the duality provided by the Galois connection rather than by classical complementation.

**Definition 2.9 (Conjugates).** Let \( \alpha \) be an image operator (generated by some relation \( R \)) of sort type \( \sigma(\alpha) = (i_1,\ldots,i_k,\ldots,i_{n};i_{n+1}) \) and \( \overline{\alpha} \) the closure of its restriction to Galois sets in each argument place, as defined above. A function \( \overline{\sigma} \) on Galois sets, of sort type

\[
\sigma(\overline{\sigma}) = (i_1,\ldots,i_{k-1},\overline{i_{n+1}},i_{k+1},\ldots,i_n;\overline{i_k})
\]
(where \(i_j = \partial\) if \(i_j = 1\) and \(i_j = 1\) when \(i_j = \partial\)) is a **conjugate** of \(\overline{\alpha}\) at the \(k\)-th argument place (or a \(k\)-conjugate) iff the following condition holds
\[
\overline{\alpha}(\bar{F}) \subseteq G \text{ iff } \overline{\alpha}^k(\bar{F}[G']_k) \subseteq F'_k
\]  
for all Galois sets \(F_j \in \mathcal{G}(Z_{i_j})\) and \(G \in \mathcal{G}(Z_{i_{n+1}})\).

It follows from the definition of a conjugate function that \(\overline{\gamma}\) is a \(k\)-conjugate of \(\overline{\alpha}\) iff \(\overline{\alpha}\) is one of \(\overline{\gamma}\) and we thus call \(\overline{\alpha}, \overline{\gamma}\) \(k\)-conjugates. Note that the priming notation for both maps of the duality \((\cdot) = \mathcal{G}(X) \in \mathcal{G}(Y)^\partial : \gamma(\cdot)\) packs together, in one form, four distinct (due to sorting) cases of conjugacy.

**Example 2.1.** In the case of a ternary relation \(R^{111}\) of the indicated sort type, an image operator \(\alpha_R = \Theta : \mathcal{P}(X) \times \mathcal{P}(X) \rightarrow \mathcal{P}(X)\) is generated. Designate the closure of its restriction to Galois stable sets by \(\Theta : \mathcal{G}(X) \times \mathcal{G}(X) \rightarrow \mathcal{G}(X)\).

Then \(\overline{\Theta} = \Theta\) is of sort type \(\sigma(\Theta) = (1,1;1)\). If \(\overline{\Theta}_R = \Theta : \mathcal{G}(X) \times \mathcal{G}(Y) \rightarrow \mathcal{G}(Y)\), with \(\sigma(\Theta) = (1,\partial;\partial)\), then \(\Theta, \Theta\) are **conjugates** iff for any Galois stable sets \(A, F, C \in \mathcal{G}(X)\) it holds that \(A\Theta F \subseteq C\) iff \(A \Theta C \subseteq F'\).

Note that, given an operator \(\Theta : \mathcal{G}(X) \times \mathcal{G}(Y) \rightarrow \mathcal{G}(Y)\), if we now define \(\Rightarrow : \mathcal{G}(X) \times \mathcal{G}(X) \rightarrow \mathcal{G}(X)\) by \(A \Rightarrow C = (A \Theta C)\) it is immediate that \(\Theta, \Rightarrow\) are conjugates iff \(\Theta, \Rightarrow\) are residuated. In other words
\[
A\Theta F \subseteq C \text{ iff } A \Theta C' \subseteq F' \text{ iff } F \in A \Rightarrow C
\]

**Lemma 2.10.** The following are equivalent.

1) \(\overline{\alpha}_R\) distributes over any joins of Galois sets at the \(k\)-th argument place
2) \(\overline{\alpha}_R\) has a \(k\)-conjugate \(\overline{\gamma}_R^k\) defined on Galois sets by
\[
\overline{\gamma}_R^k(\bar{F}) = \bigcap \{G \mid \overline{\alpha}_R(\bar{F}[G']_k) \subseteq F'_k\}
\]
3) \(\overline{\alpha}_R\) has a \(k\)-residual \(\overline{\beta}_R^k\) defined on Galois sets by
\[
\overline{\beta}_R^k(\bar{F}[G]_k) = (\overline{\alpha}_R^k(\bar{F}[G']_k))^\prime = \sqrt{\{G' \mid \overline{\alpha}_R(\bar{F}[G']_k) \subseteq F'_k\}}
\]

**Proof.** Existence of a \(k\)-residual is equivalent to distribution over arbitrary joins and the residual is defined by
\[
\overline{\beta}_R^k(\ldots, F_{k-1}, H, F_{k+1}, \ldots) = \sqrt{\{G \mid \overline{\alpha}_R(\ldots, F_{k-1}, G, F_{k+1}, \ldots) \subseteq H\}}
\]
We show that the distributivity assumption 1) implies that 2) and 3) are equivalent, i.e. that
\[
\overline{\alpha}_R(\bar{F}[G]_k) \subseteq H \text{ iff } \overline{\gamma}_R^k(\bar{F}[H']_k) \subseteq G' \text{ iff } G \subseteq \overline{\beta}_R^k(\bar{F}[H]_k)
\]
We illustrate the proof for the unary case only, as the other parameters remain idle in the argument.
Assume \( \overline{\alpha}_R(G) \subseteq H \) and let \( \overline{\gamma}_R(H') = \cap \{ E \mid \overline{\alpha}_R(E') \subseteq H \} \), a Galois set by definition, given that \( G, H, E \) are assumed to be Galois sets. Then \( G' \) is in the set whose intersection is taken. Hence \( \overline{\gamma}_R(H') \subseteq G' \) follows from the definition of \( \overline{\gamma}_R \). It also follows by definition that \( G \subseteq \overline{\gamma}_R(H) = (\overline{\gamma}_R(H'))' \).

Assuming \( G \in \overline{\gamma}_R(H) \) we obtain by definition that \( G \in (\overline{\gamma}_R(H'))' \), hence \( G \subseteq \overline{\gamma}_R(E') \subseteq H \), using the definition of \( \overline{\gamma}_R \) and duality. Hence by the distributivity assumption \( \overline{\alpha}_R(G) \subseteq \overline{\gamma}_R(E') \mid \overline{\alpha}_R(E') \subseteq H \). This establishes that \( \overline{\gamma}_R(G) \subseteq H \) iff \( \overline{\gamma}_R(H') \subseteq G' \) iff \( G \in \overline{\gamma}_R(H) \), qed.

**Definition 2.11.** We let \( \beta^k_{R} \) be the restriction of \( \beta^k_R \) of equation (3) to Galois sets, according to its sort type, explicitly defined by (6):

\[
\beta^k_{R} \left( \overline{E}[G] \right) = \bigcup \{ F \in G(Z_{i_k}) \mid \alpha_R(\overline{E}[F]_{i_k}) \subseteq G \} \quad (6)
\]

**Theorem 2.12.** If \( \overline{\alpha}_R \) is residuated in the \( k \)-th argument place, then \( \beta^k_{R} \) is its residual and \( \beta^k_{R} \left( \overline{E}[G] \right) \) is a Galois set, i.e. the union in equation (6) is actually a join in \( \mathcal{G}(\overline{Z}_{i_k}) \).

**Proof.** We illustrate the proof for the unary case only, since the other parameters that may exist remain idle in the argument. In the unary case, \( \beta^k_{R}\left( [E] \right) = \cup \{ F \mid \alpha_R(F) \subseteq G \} \), for Galois sets \( F, G \).

Note first that \( \overline{\alpha}_R(F) \subseteq G \) iff \( F \in \beta^k_{R}(G) \). Left-to-right is obvious by definition and by the fact that for a Galois set \( G \) and any set \( U, U'' \subseteq G \) iff \( U \subseteq G \). If \( F \subseteq \beta^k_{R}(G) \subseteq \beta^k_R(G) \), then by residuation \( \alpha_R(F) \subseteq G \). Given that \( G \) is a Galois set, it follows \( \overline{\alpha}_R(F) \subseteq G \).

If indeed \( \overline{\alpha}_R \) is residuated on Galois sets with a map \( \beta^k_{R} \), then the residual is defined by \( \beta^k_{R}(G) = \cup \{ F \mid \alpha_R(F) \subseteq G \} \), and this is precisely the closure of \( \beta^k_{R}(G) = \cup \{ F \mid \alpha_R(F) \subseteq G \} \) and this is actually a join in \( \mathcal{G}(\overline{Z}_{i_k}) \).

**Lemma 2.13.** \( \beta^k_{R} \) is equivalently defined by (7) and by (8)

\[
\beta^k_{R} \left( \overline{E}[G]_{i_k} \right) = \bigcup \{ F \mid \alpha_R(\overline{E}[F]_{i_k}) \subseteq G \} \quad (7)
\]

\[
\beta^k_{R} \left( \overline{E}[G]_{i_k} \right) = \{ u \in Z_{i_k} \mid \alpha_R(\overline{E}[u]_{i_k}) \subseteq G \} \quad (8)
\]

**Proof.** \( \beta^k_{R} \) is defined by equation (6), so if \( u \in \beta^k_{R} \left( \overline{E}[G] \right) \), let \( F \in \overline{G}(\overline{Z}_{i_k}) \) be such that \( u \in F \) and \( \alpha_R(\overline{E}[F]_{i_k}) \subseteq G \). Then \( \Gamma u \subseteq F \) and by monotonicity of \( \alpha_R \) we have \( \alpha_R(\overline{E}[\Gamma u]_{i_k}) \subseteq \alpha_R(\overline{E}[F]_{i_k}) \subseteq G \) and this establishes the left-to-right inclusion for the first identity of the lemma. The converse inclusion is obvious since \( \Gamma u \) is a Galois set.

For the second identity, the inclusion right-to-left is obvious. Now if \( u \) is such that \( \alpha_R(\overline{E}[\Gamma u]_{i_k}) \subseteq G \) and \( u \leq w \), then \( \Gamma w \subseteq \Gamma u \) and then by monotonicity of \( \alpha_R \) it follows that \( \alpha_R(\overline{E}[\Gamma w]_{i_k}) \subseteq \alpha_R(\overline{E}[\Gamma u]_{i_k}) \subseteq G \).
This shows that \( \bigcup \{ \Gamma u \in \mathcal{G}(Z_{i_k}) \mid \alpha_R(\hat{E}[\Gamma u]_k) \subseteq G \} \) is contained in the set \( \{ u \in Z_{i_k} \mid \alpha_R(\hat{E}[\Gamma u]_k) \subseteq G \} \), and given the first part of the lemma, the second identity obtains as well. \( \square \)

**Definition 2.14** (Conjugate Relations). Let \( \mathfrak{g} = (X, \pm, Y, R, S) \), where \( \sigma(R) = (i_{n+1}; i_1 \cdots i_k \cdots i_n) \), \( \sigma(S) = (t_{n+1}; t_1 \cdots t_k \cdots t_n) \), where \( t_{n+1} = i_k, t_j = i_{n+1} \) and for \( j \notin \{k, n+1\}, t_j = i_j \). Let \( \alpha_R \) and \( \eta_S \) be the generated image operators and \( \overline{\alpha_R}, \overline{\eta_S} \) be the closures of their restriction to Galois sets.

The relations \( R, S \) will be called \( k \)-conjugate relations iff the Galois set operators \( \overline{\alpha_R}, \overline{\eta_S} \) are \( k \)-conjugates (Definition 2.9), i.e. just in case (given that \( G, F'_k \) are Galois sets) \( \alpha_R(\hat{F}) \subseteq G \) iff \( \eta_S(\hat{F}[G'_k]) \subseteq F'_k \).

**Lemma 2.15.** Let \( \mathfrak{g} = (X, \pm, Y, R, S) \), where \( \sigma(R) = (i_{n+1}; i_1 \cdots i_k \cdots i_n) \), \( \sigma(S) = (t_{n+1}; t_1 \cdots t_k \cdots t_n) \), where \( t_{n+1} = i_k, t_j = i_{n+1} \) and for \( j \notin \{k, n+1\}, t_j = i_j \). Assume that the \( k \)-th sections of the Galois dual relation \( R' \) of \( R \) are Galois sets. Let \( T \) be the relation defined, for \( w \in Z_{\overline{\alpha_R}} \), by \( vT\hat{p}[w]_k \) iff \( w \in (vR'\hat{p}[\_]_k)' \) iff \( \forall u \in F_k (vR'\hat{p}[u]_k \rightarrow u[w]) \).

If the constraint \( \Box \) holds in the frame, then \( R \) and \( S \) are \( k \)-conjugates.

\[
\forall v \in Z_{\overline{\alpha_R}} \forall \hat{p}[\_]_k \in Z_{\overline{\alpha_R}}(\hat{p}[\_]_k \in Z_{\overline{\alpha_R}}(vT\hat{p}[w]_k \leftrightarrow w\hat{s}[v]_k)) \tag{9}
\]

**Proof.** We have

\[
\begin{align*}
\alpha_R(\hat{F}) \subseteq G \quad & \text{iff} \quad \bigcup \{ p \in \hat{F} \mid (G' \subseteq R'p) \} \\
& \text{iff} \quad \forall v \in Z_{\overline{\alpha_R}}(G[v] \rightarrow vR'p) \\
& \text{iff} \quad \forall \hat{p} \in \hat{F} \rightarrow (G' \subseteq R'\hat{p}) \\
& \text{iff} \quad \forall \hat{p} \in \hat{F} \quad \forall v \in Z_{\overline{\alpha_R}}(G[v] \rightarrow vR'\hat{p}) \\
& \text{iff} \quad \forall \hat{p} \in \hat{F} \quad \forall v \in Z_{\overline{\alpha_R}}(\hat{p}[\_]_k \in Z_{\overline{\alpha_R}}(G[v] \rightarrow vR'\hat{p}[p]_k)) \tag{2.9}
\end{align*}
\]

(Using the hypothesis that the \( k \)-th sections of \( R' \) are Galois sets)

\[
\begin{align*}
& \text{iff} \quad \forall \hat{p}[\_]_k \forall v \in Z_{\overline{\alpha_R}}(\hat{p}[\_]_k \in Z_{\overline{\alpha_R}}(G[v] \rightarrow (vR'\hat{p}[\_]_k) \subseteq F'_k)) \\
& \text{iff} \quad \forall \hat{p}[\_]_k \forall v \in Z_{\overline{\alpha_R}}(\hat{p}[\_]_k \in Z_{\overline{\alpha_R}}(G[v] \rightarrow (vR'\hat{p}[\_]_k) \subseteq F'_k)) \\
& \text{iff} \quad \forall \hat{p}[\_]_k \forall v \in Z_{\overline{\alpha_R}}(vT\hat{p}[w]_k \rightarrow F_k[w]) \tag{2.9}
\end{align*}
\]

On the other hand, we have

\[
\begin{align*}
& \eta_S(\hat{F}[G'_k]) \subseteq F'_k \quad \text{iff} \quad \bigcup \{ v \in \hat{F}[G'_k] \mid S\hat{p}[v]_k \subseteq F'_k \} \\
& \text{iff} \quad \forall \hat{p}[\_] \forall v \in Z_{\overline{\alpha_R}}(wS\hat{p}[v]_k \rightarrow F_k[w]) \\
& \text{and thus the claim of the lemma is proved.} \quad \square
\end{align*}
\]

**Theorem 2.16.** Let \( \mathfrak{g} = (X, \pm, Y, R) \) be a frame with an \((n+1)\)-ary sorted relation, of some sort \( \sigma(R) = (i_{n+1}; i_1 \cdots i_n) \). If for any \( w \in Z_{\overline{\alpha_R}} \) and any \((n-1)\) tuple \( \hat{p}[\_]_k \) with \( p_j \in Z_{i_j} \), for each \( j \in \{1, \ldots, n\} \setminus \{k\} \), the sections \( wR'\hat{p}[\_]_k \) of the Galois dual relation \( R' \) of \( R \) are Galois sets, then \( \overline{\alpha_R} \) distributes at the \( k \)-th argument place over arbitrary joins in \( \mathcal{G}(Z_{i_k}) \).
Proof. Define the relation $T$ from $R$ as in the statement of Lemma 2.15

$$vT \overline{p}[w]_k \iff w \in (v R' \overline{p}[_]_k)'$$

Then use equation (9), repeated below, as a definition for a relation $S$

$$\forall v \in Z_{\overline{w}_n+1} \forall \overline{p}[_]_k \in Z_{\overline{s}_i}[_]_k \forall w \in Z_{\overline{w}_n}(vT \overline{p}[w]_k \leftrightarrow wS \overline{p}[v]_k)$$

Note that the sort type of $S$, as defined, is $\sigma(S) = (t_{n+1}; t_1 \cdots t_k \cdots t_n)$, where $t_{n+1} = \overline{t}_k, t_k = \overline{t}_{n+1}$ and for $j \notin \{k, n + 1\}, t_j = i_j$. By the proof of Lemma 2.15 the relations $R$ and $S$ are $k$-conjugates. Consequently, by Lemma 2.10 $\overline{a}_R$ distributes at the $k$-th argument place over arbitrary joins in $G(Z_k)$ and it has a $k$-residual which, by Theorem 2.12 is precisely the restriction to Galois sets $\beta_R^k$ (defined by equation (10), equivalently by Lemma 2.13) of the $k$-residual $\beta_R^k$ of the image operator $\alpha_R$.

By composition with the Galois connection, single-sorted operators $\overline{\pi}_R, \overline{\sigma}_R$ can be obtained on $G(X)$ and $G(Y)$, respectively. Given that the Galois connection is a duality between Galois stable and co-stable sets, completely normal lattice operators (dual to each other) are obtained on $G(X)$ and $G(Y)$, respectively. Therefore we have proven the following result.

Corollary 2.17. Let $\mathfrak{F} = (X, \perp, Y, (R_p)_{p \in P})$ be a polarity with relations indexed in some set $P$, of varying sort types $\sigma_p (p \in P)$ and such that every section of the Galois dual relations $R_p' (p \in P)$ is a Galois set. Then the dual complex algebra $\mathfrak{F}^*$ of $\mathfrak{F}$ is a normal lattice expansion where a relation of sort type $(i_{n+1}; i_1 \cdots i_n)$ determines a completely normal lattice operator of distribution type $(i_1, \ldots, i_n; i_{n+1})$.

3 Representation of Normal Lattice Expansions

A bounded lattice expansion is a structure $\mathcal{L} = (L, \leq, \wedge, \vee, 0, 1, \mathcal{F}_1, \mathcal{F}_\partial)$, where $\mathcal{F}_1$ consists of normal lattice operators $f$ of distribution type $\delta(f) = (i_1, \ldots, i_n; 1)$ (i.e. of output type 1), while $\mathcal{F}_\partial$ consists of normal lattice operators $h$ of distribution type $\delta(h) = (t_1, \ldots, t_n; \partial)$ (i.e. of output type $\partial$). For representation purposes, nothing depends on the size of the operator families $\mathcal{F}_1$ and $\mathcal{F}_\partial$ and we may as well assume that they contain a single member, say $\mathcal{F}_1 = \{f\}$ and $\mathcal{F}_\partial = \{h\}$. In addition, nothing depends on the arity of the operators, so we may assume they are both $n$-ary.

3.1 Canonical Frame Construction

The canonical frame is constructed as follows, based on [20][21][27][28].

First, the base polarity $\mathfrak{F} = (\text{Filt}(\mathcal{L}), \perp, \text{Idl}(\mathcal{L}))$ consists of the sets $X = \text{Filt}(\mathcal{L})$ of filters and $Y = \text{Idl}(\mathcal{L})$ of ideals of the lattice and the relation $\perp \subseteq \text{Filt}(\mathcal{L}) \times \text{Idl}(\mathcal{L})$ is defined by $x \perp y$ if $x \cap y \neq \emptyset$, while the representation map
ζ_1 sends a lattice element \( a \in L \) to the set of filters that contain it, \( \zeta_1(a) = \{ x \in X \mid a \in x \} = \Gamma x_a \). Similarly, a co-representation map \( \zeta_0 \) is defined by \( \zeta_0(a) = \{ y \in Y \mid a \in y \} = \Gamma y_a \). It is easily seen that \( (\zeta_1(a))^\prime = \zeta_0(a) \) and, similarly, \( (\zeta_0(a))^\prime = \zeta_1(a) \). The images of \( \zeta_1, \zeta_0 \) are precisely the families (sublattices of \( G(X), G(Y) \), respectively) of clopen elements of \( G(X), G(Y) \), since clearly \( \Gamma x_a = ^\# \{ y_a \} \) and \( \Gamma y_a = \{ x_a \} ^\# \). For further details the reader is referred to [27][28].

Second, for each normal lattice operator a relation is defined, such that if \( \delta = (i_1, \ldots, i_n; i_{n+1}) \) is the sort type of the operator, then \( \sigma = (i_{n+1}; i_1 \ldots, i_n) \) is the sort type of the relation. Without loss of generality, we have restricted to the families of operators \( F_1 = \{ f \} \) and \( F_0 = \{ h \} \), so that we shall define two corresponding relations \( R, S \) of respective sort types \( \sigma(R) = (1; i_1 \ldots, i_n) \) and \( \sigma(S) = (\partial; t_1 \ldots, t_n) \), where for each \( j, i_j \) and \( t_j \) are in \( \{ 1, \partial \} \). In other words:

\[
R \subseteq X \times \prod_{j=1}^{j=n} Z_{i_j} \quad S \subseteq Y \times \prod_{j=1}^{j=n} Z_{t_j}
\]

To define the relations, we use the point operators introduced in [20] (see also [21]). In the generic case we examine, we need to define two sorted operators

\[
\hat{f} : \prod_{j=1}^{j=n} Z_{i_j} \rightarrow Z_1 \quad \hat{h} : \prod_{j=1}^{j=n} Z_{t_j} \rightarrow Z_0
\]

(recall that \( Z_1 = X, Z_0 = Y \))

Assuming for the moment that the point operators have been defined, the canonical relations \( R, S \) are defined by

\[
xR\hat{u} \quad \text{iff} \quad \hat{f}(\bar{u}) \subseteq x \quad (\text{for } x \in X \text{ and } \bar{u} \in \prod_{j=1}^{j=n} Z_{i_j})
\]

\[
yS\hat{v} \quad \text{iff} \quad \hat{h}(\bar{v}) \subseteq y \quad (\text{for } y \in Y \text{ and } \bar{v} \in \prod_{j=1}^{j=n} Z_{t_j})
\]

(10)

Returning to the point operators and letting \( x_e, y_e \) be the principal filter and principal ideal, respectively, generated by a lattice element \( e \), these are uniformly defined as follows, for \( \bar{u} \in \prod_{j=1}^{j=n} Z_{i_j} \), and \( \bar{v} \in \prod_{j=1}^{j=n} Z_{t_j} \)

\[
\hat{f}(u_1, \ldots, u_n) = \bigvee \{ x_{f(a_1, \ldots, a_n)} \mid \bigwedge_j (a_j \in u_j) \} = \bigvee \{ x_{f(\bar{a})} \mid \bar{a} \in \bar{u} \}
\]

\[
\hat{h}(v_1, \ldots, v_n) = \bigvee \{ y_{h(a_1, \ldots, a_n)} \mid \bigwedge_j (a_j \in v_j) \} = \bigvee \{ y_{h(\bar{a})} \mid \bar{a} \in \bar{v} \}
\]

(11)

In other words, \( \hat{f}(\bar{u}) = \{ (f(\bar{a}) \mid \bar{a} \in \bar{u}) \} \) is the filter generated by the set \( \{ f(\bar{a}) \mid \bar{a} \in \bar{u} \} \) and similarly \( \hat{h}(\bar{v}) \) is the ideal generated by the set \( \{ h(\bar{a}) \mid \bar{a} \in \bar{v} \} \).

**Example 3.1** (FL_{ew}). We consider as an example the case of associative, commutative, integral residuated lattices \( L = (L, \sqsubseteq, \land, \lor, 0, 1, \circ, \to) \), the algebraic models of \( \text{FL}_{ew} \) (the associative full Lambek calculus with exchange...
and weakening), also referred to in the literature as full BCK. By residuation of $\circ \rightarrow$, the distribution types of the operators are $\delta(\circ) = (1,1;1)$ and $\delta(\rightarrow) = (1,\partial;\partial)$. Let $(\text{Filt}(\mathcal{L}), \perp, \text{Idl}(\mathcal{L}))$ be the canonical frame of the bounded lattice $(L, \leq, \land, \lor, 0, 1)$. Designate the corresponding canonical point operators by $\circ$ and $\rightarrow$, respectively. They are defined by

\[
\begin{align*}
\text{x } \circ \text{z} & = \bigvee \{x_{a,c} \mid a \in x \land c \in z\} \in \text{Filt}(\mathcal{L}) \quad (x, z \in \text{Filt}(\mathcal{L})) \\
\text{x } \rightarrow \text{v} & = \bigvee \{y_{a,c} \mid a \in x \land c \in v\} \in \text{Idl}(\mathcal{L}) \quad (x \in \text{Filt}(\mathcal{L}), v \in \text{Idl}(\mathcal{L}))
\end{align*}
\]

where recall that we write $x, y, z$ for the principal filter and ideal, respectively, generated by the lattice element $e$, so that $x \circ z \in \text{Filt}(\mathcal{L})$, while $(x \rightarrow v) \in \text{Idl}(\mathcal{L})$.

The relations $R^{111}, S^{111}$ are then defined by

\[
\begin{align*}
u R^{111} x \text{z} \iff x \circ z \subseteq u \\
y S^{111} x \text{v} \iff (x \rightarrow v) \subseteq y
\end{align*}
\]

of sort types $\sigma(R) = (1;11)$ and $\sigma(S) = (\partial;1\partial)$. The canonical FL_{ew}-frame is therefore the structure $\mathfrak{F} = (\text{Filt}(\mathcal{L}), \perp, \text{Idl}(\mathcal{L}), R^{111}, S^{111})$.

### 3.2 Properties of the Canonical Frame

**Lemma 3.1.** The following hold for the canonical frame.

1. The frame is separated
2. For $\bar{u} \in \prod_{i=1}^{m} Z_{i} \text{ and } \bar{v} \in \prod_{j=1}^{n} Z_{j}$ the sections $R\bar{u}$ and $S\bar{v}$ are closed elements of $G(X)$ and $G(Y)$, respectively
3. For $x \in X, y \in Y$, the $n$-ary relations $xR, yS$ are decreasing in every argument

**Proof.** For 1), just note that the ordering $\subseteq$ is set-theoretic inclusion (of filters, and of ideals, respectively), hence separation of the frame is immediate.

For 2), by the definition of the relations, $R\bar{u} = \{x \mid \bar{f}(\bar{u}) \subseteq x\} \in G(X)$ and similarly for $S\bar{v}$.

For 3), if $w \subseteq u_{k}$, then $\{x_{f(a_{1}, \ldots, a_{n})} \mid a_{k} \in w \land \land_{j \neq k}(a_{j} \in u_{j})\}$ is a subset of the set $\{x_{f(a_{1}, \ldots, a_{n})} \mid \land_{j}(a_{j} \in u_{j})\}$, hence taking joins it follows that $\bar{f}(\bar{u}[w]_{k}) \subseteq \bar{f}(\bar{u})$. By definition, if $xR\bar{u}$ holds, then we obtain $\bar{f}(\bar{u}[w]_{k}) \subseteq \bar{f}(\bar{u}) \subseteq x$, hence $xR\bar{u}[w]_{k}$ holds as well. Similarly for the relation $S$.

**Lemma 3.2.** In the canonical frame, $xR\bar{u}$ holds if and only if $\forall \bar{a} \in L^{n} (\bar{a} \in \bar{u} \rightarrow f(\bar{a}) \in x)$. Similarly, $yS\bar{v}$ holds if and only if $\forall \bar{a} \in L^{n} (\bar{a} \in \bar{u} \rightarrow h(\bar{a}) \in y)$.

**Proof.** By definition $xR\bar{u}$ holds if $\bar{f}(\bar{u}) \subseteq x$, where $\bar{f}(\bar{u})$, by its definition is the filter generated by the elements $f(\bar{a})$, for $\bar{a} \in \bar{u}$, hence clearly $\bar{a} \in \bar{u}$ implies $f(\bar{a}) \in x$. Similarly for the relation $S$.

**Lemma 3.3.** Where $R', S'$ are the Galois dual relations of the canonical relations $R, S$, $yR'\bar{u}$ holds if $\bar{f}(\bar{u}) \perp y$ iff $\exists \bar{b}(\bar{b} \in \bar{u} \land f(\bar{b}) \in y)$. Similarly, $xS'\bar{v}$ holds iff $x \perp h(\bar{v})$ iff $\exists \bar{c}(\bar{c} \in \bar{v} \land h(\bar{c}) \in x)$. 
Proof. By definition of the Galois dual relation, \( yR\bar{u} \) holds iff for all \( x \in X \), if \( xR\bar{u} \) obtains, then \( x \parallel y \). By definition of the canonical relation \( R \), for any \( x \in X \), \( xR\bar{u} \) holds iff \( \bar{f}(\bar{u}) \subseteq x \) and thereby \( \bar{f}(\bar{u})R\bar{u} \) always obtains. Hence, \( yR\bar{u} \) is equivalent to \( \forall x \in X \ (\bar{f}(\bar{u}) \subseteq x \rightarrow x \cap y \neq \emptyset) \), from which it follows that \( \bar{f}(\bar{u}) \perp y \) iff \( yR\bar{u} \) obtains.

To show that \( yR\bar{u} \) holds iff \( \exists \bar{a} \in \bar{u} \land f(\bar{a}) \in y \), since the direction from right to left is trivially true, assume \( yR\bar{u} \), or, equivalently by the argument given above, assume that \( \bar{f}(\bar{u}) \perp y \), i.e. \( \bar{f}(\bar{u}) \cap y \neq \emptyset \) and let \( e \in \bar{f}(\bar{u}) \cap y \). By \( e \in \bar{f}(\bar{u}) \) and definition of \( \bar{f}(\bar{u}) \) as the filter generated by the set \( \{ f(\bar{a}) \mid \bar{a} \in \bar{u} \} \), let \( \bar{a}^1, \ldots, \bar{a}^s \), for some positive integer \( s \), be \( n \)-tuples of lattice elements (where \( \bar{a}^r = (a^r_1, \ldots, a^r_n) \), for \( 1 \leq r \leq s \) such that \( f(\bar{a}^1) \land \cdots \land f(\bar{a}^s) \leq e \) and \( a^r_j \in u_j \) for each \( 1 \leq r \leq s \) and \( 1 \leq j \leq n \). Recall that the distribution type of \( f \) is \( \delta(f) = (i_1, \ldots, i_n; 1) \), where for \( j = 1, \ldots, n \) we have \( i_j \in \{ 1, \partial \} \) and define elements \( b_1, \ldots, b_n \) as follows.

\[
 b_j = \begin{cases} 
 a^1_j \land \cdots \land a^n_j & \text{if } i_j = 1 \\
 a^1_j \lor \cdots \lor a^n_j & \text{if } i_j = \partial 
\end{cases}
\]

When \( i_j = 1 \), \( f \) is monotone at the \( j \)-th argument place, \( u_j \) is a filter and \( b_j \leq a^r_j \in u_j \), for all \( r = 1, \ldots, s \), so that \( b_j = a^1_j \land \cdots \land a^n_j \in u_j \). Similarly, when \( i_j = \partial \), \( f \) is antitone at the \( j \)-th argument place, while \( u_j \) is an ideal, so that \( b_j = a^1_j \lor \cdots \lor a^n_j \in u_j \). This shows that \( \bar{b} \in \bar{u} \) and it remains to show that \( f(\bar{b}) \in y \). We argue that \( f(\bar{b}) \leq f(\bar{a}^1) \land \cdots \land f(\bar{a}^s) \leq e \) and the desired conclusion follows by the fact that \( e \in y \), an ideal.

For any \( 1 \leq r \leq s \), let \( \bar{a}^r[b_j]^{i_j=1}_{j} \) be the result of replacing \( a^r_j \) by \( b_j \) in the tuple \( \bar{a}^r \) and in every position \( j \) from 1 to \( n \) such that \( i_j = 1 \) in the distribution type of \( \bar{f} \). Since \( b_j \leq a^r_j \) and \( f \) is monotone at any such \( j \)-th argument place, it follows that \( f(\bar{a}^r[b_j]^{i_j=1}_{j}) \leq f(\bar{a}^r) \), for all \( 1 \leq r \leq s \).

In addition, for any \( 1 \leq r \leq s \), let \( \bar{a}^r[b_j]^{i_j=1}_{j}[b_j]^{i_j'=\partial}_{j'} \) be the result of replacing \( a^r_j \) by \( b_j \) in the tuple \( \bar{a}^r[b_j]^{i_j=1}_{j} \) and in every position \( j' \) from 1 to \( n \) such that \( i_j' = \partial \) in the distribution type of \( f \). Since \( b_j \leq a^r_j \) and \( f \) is antitone at any such \( j' \)-th argument place, it follows that \( f(\bar{a}^r[b_j]^{i_j=1}_{j}[b_j]^{i_j'=\partial}_{j'}) \leq f(\bar{a}^r[b_j]^{i_j=1}_{j}) \leq f(\bar{a}^r) \), for all \( 1 \leq r \leq s \). Since \( \bar{a}^r[b_j]^{i_j=1}_{j}[b_j]^{i_j'=\partial}_{j'} = \bar{b} \) we obtain that

\[
 f(\bar{b}) = f(\bar{a}^r[b_j]^{i_j=1}_{j}[b_j]^{i_j'=\partial}_{j'}) \leq f(\bar{a}^r[b_j]^{i_j=1}_{j}) \leq f(\bar{a}^r) \land \cdots \land f(\bar{a}^s) \leq e
\]

hence \( f(\bar{b}) \in y \) and this completes the proof, as far as the relation \( R \) is concerned.

The argument for the relation \( S \) is similar and can be safely left to the reader.

\[\square\]

**Lemma 3.4.** In the canonical frame, all sections of the Galois dual relations \( R', S' \) of the canonical relations \( R, S \) are Galois sets.

**Proof.** There are two cases to handle, one for each of the relations \( R', S' \), with two subcases for each one, depending on whether \( i_k \) is 1, or \( \partial \).
Case of the relation $R'$: We have $R'\overline{u} = (R\overline{u})'$, by definition, so the section $R'\overline{u}$ is a Galois (co-stable) set. It remains to be shown that for any $y \in Y$ and $\overline{u}[z]_k$, the $k$-th section $y R' \overline{u}[z]_k$ is a Galois set, for any $1 \leq k \leq n$. There are two subcases to consider, accordingly as $i_k = 1$, or $i_k = \partial$ and recall that $\delta(f) = (i_1, \ldots, i_n; 1)$. Hence if $i_k = 1$, then $f$ is monotone and it distributes over finite joins at the $k$-th argument place and if $i_k = \partial$, then $f$ is antitone and it co-distributes at the $k$-th argument place over finite meets (turning them to joins).

Subcase $i_k = 1$:
Then $y R' \overline{u}[z]_k \subseteq X = \text{Filt}(L)$.

Let $v$ be the ideal generated by the set $V = \{b \in L \mid \exists a \overline{u}[z]_k \ f(a[b]_k) \in y\}$. For any $x \in X$ such that $y R' \overline{u}[x]_k$ holds, by Lemma 3.3 we have $\overline{f}(\overline{u}[x]_k) + y$, equivalently, $\exists a \overline{u}[z]_k \exists b \ (a[b]_k \in \overline{u}[x]_k \land f(a[b]_k) \in y)$. Thus $b \in x \cap v$ and so $y R' \overline{u}[z]_k \subseteq v$.

We assume $z \in (y R' \overline{u}[z]_k)''$ and we need to show that $y R' \overline{u}[z]_k$. The assumption implies that $z \in v$, i.e. for some lattice element $e$ we have $e \in z \cap v$. By $e \in v$, let $b_1, \ldots, b_s \in V \subseteq v$, for some positive integer $s$, be elements such that $e \leq b_1 \lor \cdots \lor b_s \in v$, since $v$ is an ideal.

Since $b_1, \ldots, b_s \in V$, there are tuples of lattice elements $\overline{e}[z]_k$ such that $f(\overline{e}[b_r]_k) \in y$, for each $1 \leq r \leq s$. Considering the distribution type of $f$ and as in the proof of Lemma 3.3 define the tuple of elements $\overline{a} = (a_1, \ldots, a_n)$ by

$$a_j = \begin{cases} c_j^1 \land \cdots \land c_j^s & \text{if } i_j = 1 \\ c_j^1 \lor \cdots \lor c_j^s & \text{if } i_j = \partial \end{cases}$$

For each $1 \leq r \leq s$ let $\overline{e}'[b_r]_k[a_j]_j^{i_j=1}$ be the result of replacing in $\overline{e}'[b_r]_k$ all $c_j^r$ by $a_j$ whenever $i_j = 1$ in the distribution type of $f$. Then for each $r$ as above and by monotonicity of $f$ at the $j$-th argument place whenever $i_j = 1$ we have $f(\overline{e}'[b_r]_k[a_j]_j^{i_j=1}) \leq f(\overline{e}'[b_r]_k) \in y$, an ideal, hence $f(\overline{e}'[b_r]_k[a_j]_j^{i_j=1}) \in y$. Let also $\overline{e}'[b_r]_k[a_j]_j^{i_j=1}[a_{j'}]_{j'}^{i_{j'}=\partial}$ be the result of further replacing in $\overline{e}'[b_r]_k[a_j]_j^{i_j=1}$ all $c_j^r$ by $a_{j'}$ whenever $i_{j'} = \partial$ in the distribution type of $f$. Since $f$ is antitone at the $j'$-th position when $i_{j'} = \partial$ (given that the output type of $f$ is assumed to be $i_{\eta+1} = 1$), we obtain $f(\overline{e}'[b_r]_k[a_j]_j^{i_j=1}[a_{j'}]_{j'}^{i_{j'}=\partial}) \leq f(\overline{e}'[b_r]_k[a_j]_j^{i_j=1}) \leq f(\overline{e}'[b_r]_k) \in y$ and so $f(\overline{e}'[b_r]_k[a_j]_j^{i_j=1}[a_{j'}]_{j'}^{i_{j'}=\partial}) \in y$. But, having performed substitutions in all places (except for the $k$-th) $\overline{e}'[b_r]_k[a_j]_j^{i_j=1}[a_{j'}]_{j'}^{i_{j'}=\partial} = \overline{a}[b_r]_k$, for each $1 \leq r \leq s$.

It follows from the above that, for each $1 \leq r \leq s$ we have $f(\overline{a}[b_r]_k) \in y$, an ideal, hence also $\forall_{r=1}^s f(\overline{a}[b_r]_k) \in y$. By case assumption, $i_k = 1$, hence $f$ distributes over finite joins in the $k$-th argument place and then we obtain that $f(\overline{a}[b_1 \lor \cdots \lor b_s]_k) = \forall_{r=1}^s f(\overline{a}[b_r]_k) \in y$. By $e \leq b_1 \lor \cdots \lor b_s$ and monotonicity of $f$ at the $k$-th argument place we obtain $f(\overline{a}[e]_k) \leq f(\overline{a}[b_1 \lor \cdots \lor b_s]_k) \in y$, hence also $f(\overline{a}[e]_k) \in y$.

Therefore, there exists $\overline{a}[e]_k \in \overline{a}[z]_k$ such that $f(\overline{a}[e]_k) \in y$ which, by Lemma 3.3 is equivalent to $y R' \overline{u}[z]_k$. This shows that $(y R' \overline{u}[z]_k)'' \subseteq y R' \overline{u}[z]_k$, so the
section has been shown to be a Galois (stable) set.

The subcase \( i_k = \partial \) and the case of the relation \( S \) are treated similarly and we leave details to the interested reader.

The canonical frame for a lattice expansion \( L = (L, \leq, \wedge, \vee, 0, 1, f, h) \), where \( \delta(f) = (i_1, \ldots, i_n; 1) \) and \( \delta(h) = (t_1, \ldots, t_n; \partial) \) \((i_j, t_j \in \{1, \partial\})\) is the structure \( L_\partial = \mathcal{F} = (Filt(L), \perp, Idl(L), R, S) \). By Proposition 5.4, the canonical relations \( R, S \) are compatible with the Galois connection generated by \( \perp \subseteq X \times Y \), in the sense that all sections of their Galois dual relations are Galois sets. Set operators \( \alpha_R, \eta_S \) are defined as in Section 2.2 and we let \( \overline{\alpha}_R, \overline{\eta}_S \) be the closures of their restrictions to Galois sets (according to their distribution types). Note that \( \overline{\alpha}_R(F) \in \mathcal{G}(X) \), while \( \overline{\eta}_S(G) \in \mathcal{G}(Y) \), given the output types of \( f, h \) (alternatively, given the sort types of \( R, S \)).

It follows from Theorem 2.16 and Lemma 3.4 that the sorted operators \( \overline{\alpha}_R, \overline{\eta}_S \) on Galois sets distribute over arbitrary joins of Galois sets (stable or co-stable, according to the sort types of \( R, S \)) in each argument place.

Note that \( \overline{\alpha}_R, \overline{\eta}_S \) are sorted maps, taking their values in \( \mathcal{G}(X) \) and \( \mathcal{G}(Y) \), respectively. We define single-sorted maps on \( \mathcal{G}(X) \) (analogously for \( \mathcal{G}(Y) \)) by composition with the Galois connection

\[
\overline{\alpha}_f(A_1, \ldots, A_n) = \overline{\alpha}_R(\ldots, A_j, \ldots, A_j', \ldots) \underset{i_j=1}{\underset{\text{i.e.}}{\text{\&}}} \underset{i_r=\partial}{\text{\&}} (A_1, \ldots, A_n \in \mathcal{G}(X)) \tag{12}
\]

\[
\overline{\eta}_h(B_1, \ldots, B_n) = \overline{\eta}_R(\ldots, B_r, \ldots, B_j', \ldots) \underset{i_r=\partial}{\underset{\text{i.e.}}{\text{\&}}} \underset{i_j=1}{\text{\&}} (B_1, \ldots, B_n \in \mathcal{G}(Y)) \tag{13}
\]

Given that the Galois connection is a duality of Galois stable and Galois co-stable sets, it follows that the distribution type of \( \overline{\alpha}_f \) is that of \( f \) and that \( \overline{\alpha}_f \) distributes, or co-distributes, over arbitrary joins and meets in each argument place, according to its distribution type, returning joins in \( \mathcal{G}(X) \). Similarly, for \( \overline{\eta}_h \). Thus, the lattice representation maps \( \zeta_1 : (L, \leq, \wedge, \vee, 0, 1) \rightarrow \mathcal{G}(X) \) and \( \zeta_\partial : (L, \leq, \wedge, \vee, 0, 1) \rightarrow \mathcal{G}(Y) \) are extended to maps \( \zeta_1 : L \rightarrow \mathcal{G}(X) \) and \( \zeta_\partial : L \rightarrow \mathcal{G}(Y) \) by setting

\[
\zeta_1(f(a_1, \ldots, a_n)) = \overline{\alpha}_f(\zeta_1(a_1), \ldots, \zeta_1(a_n)) = \overline{\alpha}_R(\ldots, \zeta_1(a_j), \ldots, \zeta_\partial(a_r), \ldots) \underset{i_j=1}{\underset{\text{i.e.}}{\text{\&}}} \underset{i_r=\partial}{\text{\&}} \tag{14}
\]

\[
\zeta_\partial(f(a_1, \ldots, a_n)) = \overline{\alpha}_f(\zeta_1(a_1), \ldots, \zeta_1(a_n)) \tag{15}
\]

It has been therefore established that there exists a map from normal lattice expansions to polarities with relations, as specified in the following concluding result.

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Corollary 3.5. Given a normal lattice expansion \( \mathcal{L} = (L, \leq, \wedge, \vee, 0, 1, \mathcal{F}_1, \mathcal{F}_0) \), where \( \mathcal{F}_1, \mathcal{F}_0 \) are families of normal lattice operators of output types 1 and \( \emptyset \), respectively, the dual frame \( \mathcal{L}' \) of the lattice expansion \( \mathcal{L} \) is a polarity with additional relations, where for a normal lattice operator \( f \) of distribution type \( (i_1, \ldots, i_n : i_{n+1}) \) the corresponding frame relation \( R_f \) is of sort type \( (i_{n+1} : i_1 \cdots i_n) \) and where all sections of its Galois dual relation \( R'_f \) are Galois sets. \( \square \)

3.3 Representation, Canonical Extensions and RS-Frames

A canonical lattice extension is defined in [15] as a pair \( (\alpha, C) \) where \( C \) is a complete lattice and \( \alpha \) is an embedding of a lattice \( \mathcal{L} \) into \( C \) such that the following density and compactness requirements are satisfied

- (density) \( \alpha[\mathcal{L}] \) is dense in \( C \), where the latter means that every element of \( C \) can be expressed both as a meet of joins and as a join of meets of elements in \( \alpha[\mathcal{L}] \).
- (compactness) for any set \( A \) of closed elements and any set \( B \) of open elements of \( C \), \( \wedge A \leq \vee B \) iff there exist finite subcollections \( A' \subseteq A, B' \subseteq B \) such that \( \wedge A' \leq \vee B' \)

where the closed elements of \( C \) are defined in [15] as the elements in the meet-closure of the representation map \( \alpha \) and the open elements of \( C \) are defined dually as the join-closure of the image of \( \alpha \).

Proposition 3.6. \( \mathcal{G}(X) \) (the lattice of Galois stable subsets of the set of filters) is a canonical extension of the (bounded) lattice \( (L, \leq, \wedge, \vee, 0, 1) \).

Proof. This was shown by Gehrke and Harding in [15]. More precisely, existence of canonical extensions is proven in [15] by demonstrating that the compactness and density requirements are satisfied in the representation due to Hartonas and Dunn [28], which is precisely the representation presented in Section 3.1. \( \square \)

Proposition 3.7. The canonical representations of the normal lattice operators \( f, h \), of respective output types 1, \( \emptyset \), as defined by the equations (14) and (15), are the \( \sigma \) and \( \pi \)-extension, respectively, of \( f, h \).

Proof. In the representation of Section 3.1 the closed and open elements of \( \mathcal{G}(X) \) are the sets of the form \( \Gamma x (x \in X) \) and \( \overline{\{y\}} (y \in Y) \), respectively. For a unary lattice operator \( f : \mathcal{L} \rightarrow \mathcal{L} \), its \( \sigma \)-extension in a canonical extension \( C \) of the lattice \( \mathcal{L} \) is defined in [15] by equation (16), where \( K \) is the set of closed elements of \( C \) and \( \emptyset \) is its set of open elements

\[
\begin{align*}
   f_\sigma(k) &= \bigwedge \{ f(a) \mid k \leq a \in L \} & f_\sigma(u) &= \bigvee \{ f_\sigma(k) \mid K \ni k \leq u \} \\
   f_\pi(o) &= \bigvee \{ f(a) \mid L \ni a \leq o \} & f_\pi(u) &= \bigwedge \{ f_\pi(o) \mid u \leq o \in \emptyset \}
\end{align*}
\]

(16) (17)

where in these definitions \( \mathcal{L} \) is identified with its isomorphic image in \( C \) and \( a \in \mathcal{L} \) is then identified with its representation image.

Working concretely with the canonical extension of [28], the \( \sigma \) extension \( f_\sigma : \mathcal{L}_\sigma \rightarrow \mathcal{L}_\sigma \) of a monotone map \( f \) as in equation (14) and the dual \( \sigma \)-extension
Theorem 3.4 (Canonical Relations in the RS-Frames Approach). In modeling the Lambek calculus product operator \( \circ \), of distribution type \((1, 1; 1)\) (see Example 3.1), the canonical relation \( R^{111} \) was defined by (using the point operators) \( uRxz \) iff \( x \otimes z \subseteq u \), where \( x \otimes z \) is the filter generated by the elements \( a \circ c \), with \( a \in x \) and \( c \in z \). By Lemma 3.2 specialized to this case, this amounts to the classical definition of a canonical relation, familiar from the Boolean and distributive case, by the clause

\[
uRxz \iff \forall a, c \in L \ (a \in x \land c \in z \rightarrow a \circ c \in u)
\]
By Lemma 3.3 specialized to the particular case, \( yR'xz \) holds iff \( x \oplus z \perp y \) iff \( \exists a, c \in L \) \( (a \in x \land c \in z \land a \circ c \in y) \), which is precisely Gehrke’s [14] canonical relation definition for the Lambek product operator.

This is no isolated matter, as in the RS-frames approach the choice is made to work directly with a relation that can be nevertheless defined as the Galois dual of a classically defined accessibility relation. This is also witnessed by the way Goldblatt [18] proceeds, generalizing on Gehrke’s [14], to define relations and set-operators in a frame. Indeed, examining (for his case of interest) additive operators \( F \) on stable sets he defines a relation \( S_F \) by setting \( S_F \bar{z} \) iff \( F(\Gamma z_1, \ldots, \Gamma z_n) \perp y \) (which in the case of a binary, completely additive operator \( F \), is equivalent in the canonical frame to \( z_1 \oplus z_2 \perp y \)). It is merely a matter of choice and convenience, given the purpose at hand, which relation to decide to work with. Gehrke [14] does indeed point out that instead of using the relation \( S \subseteq Y \times (X \times X) \) defined as above (for the Lambek product operator), one could use a relation \( R \subseteq X \times (X \times X) \), which is actually the Galois dual of \( S \), but she does not dwell much on the matter.

Though, to the best of this author’s knowledge, it has not been made explicit in the RS-frames approach how relations are to be defined corresponding to arbitrary normal lattice operators in general (but only in cases of particular examples), the relations on an RS-frame corresponding to normal lattice operators of some distribution type \( (i_1, \ldots, i_n; i_{n+1}) \) are the Galois duals of our canonical accessibility relations, hence they are systematically of sort type \( (i_{n+1}; i_1 \cdots i_n) \), and operators are defined from them. For example, Goldblatt [18] (generalizing Gehrke’s [14] definition for the Lambek product operator) defines from a relation \( S \subseteq Y \times X \) an operator \( F_S \) by setting

\[
F_S^*(\bar{F}) = \bigcap \{ S\bar z \mid \bar z \in \bar F \} \\
F_S(\bar{F}) = \frac{1}{\bigwedge} F_S^*(\bar{F}) = \bigvee \{ \frac{1}{\bigwedge} (S\bar z) \mid \bar z \in \bar F \} = \frac{1}{\bigwedge} (S\bar z) \quad (18)
\]

The relation \( R \) defined as the Galois dual of \( S \), i.e. \( R\bar z = \frac{1}{\bigwedge} (S\bar z) \) is precisely a relation of sort type \( (i_{n+1}; i_1 \cdots i_n) \) and, assuming section stability, \( R \) and \( S \) are each other’s Galois dual. Therefore we obtain

\[
F_S(\bar{F}) = \bigvee \frac{1}{\bigwedge} (S\bar z) = \bigvee \frac{1}{\bigwedge} R\bar z = \left( \bigcup \frac{1}{\bigwedge} R\bar z \right)'' \quad (19)
\]

A comparison of equations (14) and (20) reveals then that the two definitions are variants of each other.

4 Conclusions

We have argued that the two approaches, the one developed in this article (concluding and completing our previous recent work on the subject) and the RS-frames approach really only differ in whether the polarity is assumed to be separated and reduced or not. The results of this article have shown that
nothing is lost by dropping these additional assumptions, as far as the semantics of logics without distribution is concerned.

There are three points of interest, however, that are worth making.

First, a Stone type duality for RS-frames (essentially for Hartung’s lattice representation) has encountered difficulties, similar to these encountered in extending Urquhart’s representation to a full Stone duality. In [21], we have developed a duality result for normal lattice expansions, extending the representation of [28]. In view of Goldblatt’s recent proposal [18] of a notion of bounded morphisms for polarities, this result, combined with the results of this article, can be improved to a Stone duality for normal lattice expansions with bounded morphisms as the morphisms in the dual category of polarities. We shall not dwell on this more here, for lack of space.

Second, the approach we have presented in this article allows for relating the logic of non-distributive lattices to the sorted, residuated (poly)modal logic of polarities with relations, where the residuated pair of modal operators is generated by the complement of the Galois relation of the frame. Preliminary results in this direction have been reported in [25, 26] by this author, but the area is far from fully explored. Regarding non-distributive logics as fragments of sorted, residuated (poly)modal logics allows for importing techniques and results from modal logic in the field of logics lacking distribution.

Finally, we believe that the semantic framework presented in this article fully complements Dunn’s gaggle theory project and in an important sense it completes the project for the case of non-distributive logical calculi.

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