Hypermächer, Bi-hypercomplex, 
Generalized Hyperkähler Structures and T-duality

Tetsuji Kimura\textsuperscript{a}, Shin Sasaki\textsuperscript{b} and Kenta Shiozawa\textsuperscript{c}

\textsuperscript{a} Center for Physics and Mathematics, Institute for Liberal Arts and Sciences, Osaka Electro-Communication University, Neyagawa, Osaka 572-8530, Japan

\textsuperscript{b} Department of Physics, Kitasato University
Sagamihara 252-0373, Japan

Abstract

We exploit the doubled formalism to study comprehensive relations among T-duality, complex and bi-hermitian structures \((J_+, J_-)\) in two-dimensional \(\mathcal{N} = (2,2)\) sigma models with/without twisted chiral multiplets. The bi-hermitian structures \((J_+, J_-)\) embedded in generalized Kähler structures \((J_+, J_-)\) are organized into the algebra of the tri-complex numbers. We write down an analogue of the Buscher rule by which the T-duality transformation of the bi-hermitian and Kähler structures are apparent. We also study the bi-hypercomplex and hyperkähler cases in \(\mathcal{N} = (4,4)\) theories. They are expressed, as a T-duality covariant fashion, in the generalized hyperkähler structures and form the split-bi-quaternion algebras. As a concrete example, we show the explicit T-duality relation between the hyperkähler structures of the KK-monopole (Taub-NUT space) and the bi-hypercomplex structures of the H-monopole (smeared NS5-brane). Utilizing this result, we comment on a T-duality relation for the worldsheet instantons in these geometries.
1 Introduction

It is well-known that the complex structures in spacetime geometries appear in association with supersymmetries. Target spaces of four-dimensional $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetric non-linear sigma models are described by Kähler and hyperkähler geometries, respectively \cite{1,2}. The same is true for theories with equivalent numbers of supersymmetries. For example, two-dimensional $\mathcal{N} = (2,2)$ and $\mathcal{N} = (4,4)$ theories require Kähler and hyperkähler geometries as target spaces of chiral multiplets. The geometries are characterized by complex structures $J$ or $J_a \ (a = 1, 2, 3)$ compatible with the target space metric $g_{\mu\nu}$. These facts are further generalized when the twisted chiral multiplets come in the theories together. For $\mathcal{N} = (2,2)$ theories with twisted chiral multiplets, the target spaces admit bi-hermitian structures $(J_+,J_-)$ \cite{3, 4}. They are commutative complex structures and are compatible with the metric. The geometries characterized by $(g_{\mu\nu},J_+,J_-)$ are called bi-hermitian manifolds. For $\mathcal{N} = (4,4)$ cases, the target spaces are bi-hypercomplex manifolds characterized by two commuting hypercomplex structures $(J_{a,+},J_{a,-}) \ (a = 1, 2, 3)$ compatible with the metric. These two-dimensional models may be identified with the worldsheet theories of fundamental strings on certain background spacetimes.

This is not the end of the story. Generalized geometry \cite{5}, developed for understanding the T-duality nature of target spaces $M$, plays an important role in relating these geometries. In \cite{6}, it is shown that the bi-hermitian structures $(J_+,J_-)$ and the bi-hypercomplex structures $(J_{a,+},J_{a,-})$ on the tangent bundle $TM$ are equivalent, via so-called the Gualtieri map, to the generalized Kähler structures $(\mathcal{J}_+,\mathcal{J}_-)$ and the generalised hyperkähler structures $(\mathcal{J}_{a,+},\mathcal{J}_{a,-})$ on the generalized tangent bundle $TM \oplus T^*M$, respectively. These relations are also studied...
at the level of supersymmetric sigma models [7–10]. A physical origin of this correspondence comes from the equivalence of the Lagrangian and the Hamiltonian formulations of sigma models [11,12]. See [13] and references therein for details.

One consequence of generalized geometry is that T-duality in string theory is realized in an apparent fashion. This becomes obvious when its connection to double field theory (DFT) [14] is revealed. DFT is developed from the doubled formalism [15, 16] where T-duality is realized manifestly. The dynamical fields in DFT are the generalized metric $\mathcal{H}_{MN}$ and the generalized dilaton $d$. They are subject to physical conditions known as the weak and the strong constraints. With these constraints, the spacetime metric $g_{\mu\nu}$, the NSNS $B$-field $B_{\mu\nu}$ and the dilaton $\phi$ are nicely packaged in $\mathcal{H}_{MN}$. They are defined on a 2D-dimensional para-Kähler or para-hermitian manifold $\mathcal{M}$ [17–19], that are sometimes called doubled space, where the Kaluza-Klein and the winding coordinates $x^M = (x^\mu, \tilde{x}_\mu)$ are naturally introduced. The famous Buscher rule of T-duality is realized as an $O(D,D)$ symmetry on the doubled tangent space $TM$. Due to the para-hermitian structure, $TM$ is decomposed into two transversal parts $L \oplus \tilde{L}$. The physical $D$-dimensional spacetime in $\mathcal{M}$ is defined by the foliated space $M$ for $L = TM$. The generalized and the doubled geometries are different since the former is build on the $D$-dimensional base space $M$ while the latter assumes the 2D-dimensional doubled base space $\mathcal{M}$. Although they are different, they are conceptually identified in the following sense. The weak and the strong constraints are trivially solved by $\mathcal{H}_{MN}$ and $d$ that depend on a half of the doubled coordinate $x^M$. This parameterizes the foliated space $M$. In other words, the constraints in DFT restrict the base space $\mathcal{M}$ to a $D$-dimensional subspace $M$ while keeping $TM$ intact. With this perception, $TM$ is identified with the generalized tangent bundle $TM \oplus T^*M$ through so-called the natural isomorphism [17, 20, 21]. In this sense, generalized geometry is implemented within the doubled geometry. Indeed, the $O(D,D)$ is the structure group of the generalized tangent bundle $TM \oplus T^*M$.

With these facts at hand, it is now legitimate to discuss the relation between the Kähler (hyperkähler) structures and the bi-hermitian (bi-hypercomplex) from the viewpoint of T-duality. Since the chiral and the twisted chiral multiplets are interchanged by T-duality [22–24], there should be an explicit transformation rule from the Kähler (hyperkähler) structures to the bi-hermitian (bi-hypercomplex), and vice versa, as target spaces of two-dimensional sigma models. In this paper, we establish these relations by working in the doubled formalism. We also note that the bi-hermitian (bi-hypercomplex) and the generalized (hyper)kähler structures involve interesting mathematical properties and they are relevant to T-duality of worldsheet instanton effects.

The organization of this paper is as follows. In the next section, we provide a brief overview on the two-dimensional $\mathcal{N} = (2, 2)$ and $\mathcal{N} = (4, 4)$ sigma models and introduce the bi-hermitian (bi-hypercomplex) structures in their target spaces. We then discuss the embedding of these
structures into the generalized complex and the hyperkähler structures on $TM \oplus T^*M$. We analyze hypercomplex algebras that the structures obey. In Section 3, we consider the T-duality transformations of the generalized complex structures. Using this, we extract the T-duality transformation of the bi-hermitian structures and write down the explicit “Buscher rule” of T-duality for them. In order to get a better understanding of the relation between the hyperkähler and bi-hypercomplex structures, Section 4 is devoted to an explicit example of T-duality for these structures. We focus on the T-duality between the KK- and the H-monopoles and show how the hyperkähler and the bi-hypercomplex structures are related. As a byproduct, we comment on the T-duality between the worldsheet instanton equations in these geometries in Section 5. Section 6 is devoted to conclusion and discussions.

2 $\mathcal{N} = (2, 2)$, $\mathcal{N} = (4, 4)$ sigma models and generalized (hyper)Kähler structures

In this section, we give a brief overview on two-dimensional $\mathcal{N} = (2, 2)$ and $\mathcal{N} = (4, 4)$ sigma models with chiral and twisted chiral multiplets and their target space geometries. The structures of the target spaces are well described by generalized geometry or doubled geometry. We introduce these notions in the following. For details, see [3] and references therein.

2.1 Supersymmetric sigma models and their target space geometries

The most general action containing two-dimensional $\mathcal{N} = (2, 2)$ chiral superfields $\hat{\Phi}^u (u = 1, \ldots, n)$ and twisted chiral superfields $\tilde{\chi}^p (p = 1, \ldots, m)$ is given by

$$S = \int d^2x d^2\theta d^2\bar{\theta} K(\hat{\Phi}^u, \tilde{\Phi}^\bar{u}, \tilde{\chi}^p, \tilde{\bar{\chi}}^{\bar{p}}).$$

(2.1)

Here $K$ is a real function. The action (2.1) is invariant under the following generalized Kähler transformation;

$$\delta K = \Lambda_1(\hat{\Phi}, \tilde{\chi}) + \Lambda_2(\tilde{\Phi}, \tilde{\bar{\chi}}) + \bar{\Lambda}_1(\bar{\tilde{\Phi}}, \tilde{\chi}) + \bar{\Lambda}_2(\bar{\tilde{\Phi}}, \tilde{\bar{\chi}}).$$

(2.2)

The bosonic part of the action (2.1) is found to be

$$S_{\text{bos}} = \frac{1}{2} \int d^2x \left[K_{u\bar{v}} \partial_a \varphi^u \bar{\partial} \varphi^v - K_{pq} \partial_a \chi^p \partial^b \chi^q + \epsilon^{ab} (K_{ap} \partial_a \varphi^u \partial_b \bar{\chi}^p + K_{pa} \partial_a \bar{\varphi}^u \partial_b \chi^p)\right],$$

(2.3)

where $\varphi^u$ and $\chi^p$ are the lowest components in the superfields $\hat{\Phi}^u$ and $\tilde{\chi}^p$, respectively. We have also defined $K_{u\bar{v}} = \frac{\partial^2 K}{\partial \varphi^u \partial \bar{\varphi}^v}$ and so on. In order that the theory is ghost free, we demand that $K_{pq}$ is negative-definite. Since the kinetic and the Wess-Zumino terms of the scalar fields define
the metric and the $B$-field in the target space, the presence of the twisted chiral multiplets is necessary to introduce the $B$-field and hence the torsion is given by $T = dB$.

We next examine supersymmetric transformations. Following [3], we first write down the action in terms of $\mathcal{N} = (1, 1)$ real superfields $\Phi^\mu (\mu = 1, \ldots, D)$:

$$S = -\frac{1}{4} \int \! d^2 x \, d^2 \theta \left[ g_{\mu \nu} (\Phi) D^\alpha \Phi^\mu D_\alpha \Phi^\nu + B_{\mu \nu} (\Phi) D^\alpha \Phi^\mu (\gamma_5 D)_\alpha \Phi^\nu \right].$$

(2.4)

Here $D_\alpha (\alpha = \pm)$ is $\mathcal{N} = (1, 1)$ supercovariant derivative, $g_{\mu \nu}$, $B_{\mu \nu} = -B_{\nu \mu}$ are the metric and the $B$-field in the target space and $\gamma_5 = \sigma^3$. The action is invariant under the gauge transformation $\delta B = d \xi$. In the following, the torsion in the target space is always given by $T = dB$. The action (2.4) is manifestly invariant under the $\mathcal{N} = (1, 1)$ supersymmetry transformation. We further require that the action is invariant under the following additional supersymmetry transformation;

$$\delta_\eta \Phi^\mu = -i (J_+)^\mu_{\nu} (\eta_+ D^- \Phi^\nu) + i (J_-)^\mu_{\nu} (\eta_- D^+ \Phi^\nu),$$

(2.5)

where $\eta_{\pm}$ are supersymmetry parameters and $(J_{\pm})^\mu_{\nu}(\Phi)$ are some matrices. By demanding that the commutator of the above transformations closes to translations on-shell, we find a condition on $\Phi^\mu$;

$$D_+ D_- \Phi^\mu + \Gamma^{(\pm)}_{\nu \rho} D_+ \Phi^\nu D_- \Phi^\rho = 0,$$

(2.6)

where $\Gamma^{(\pm)}_{\nu \rho} = \Gamma^{(0)}_{\nu \rho} \mp T^\nu_{\nu \rho}$ is the affine connection with the torsion $T$ and

$$\Gamma^{(0)}_{\nu \rho} = \frac{1}{2} g^{\mu \sigma} \left( \partial_\rho g_{\mu \sigma} + \partial_\nu g_{\rho \sigma} - \partial_\sigma g_{\nu \rho} \right).$$

(2.7)

is the Levi-Civita connection. By imposing the supersymmetry algebra, we find conditions on $J_{\pm}$;

$$(J_{\pm})^\nu_{\nu}, (J_{\pm})^\nu_{\rho} = -\delta^\nu_{\rho},$$

$$N_\pm = (J_{\pm}^\sigma_{\mu} \partial_{[\sigma]} (J_{\pm})^\rho_{[\nu]} - (J_{\pm})^\sigma_{\rho} \partial_{[\sigma]} (J_{\pm})^\nu_{[\mu]} = 0.$$  

(2.8)

Here $N_\pm$ are the Nijenhuis tensors associated with $J_{\pm}$ whose vanishing condition implies the integrability of $J_{\pm}$. Then, $J_{\pm}$ become the complex structures on the target space.

From the invariance of the action by the transformation (2.5), one finds the additional relations;

$$g_{\mu \rho} (J_{\pm})^\rho_{\nu} = -g_{\nu \rho} (J_{\pm})^\rho_{\nu},$$

$$\nabla_\mu^{(\pm)} (J_{\pm})^\nu_{\rho} = \partial_\mu (J_{\pm})^\nu_{\rho} + (J_{\pm})^\sigma_{\rho} \Gamma_{\mu \sigma}^{(\pm)} - (J_{\pm})^\nu_{\sigma} \Gamma_{\mu \sigma}^{(\pm)} = 0.$$  

(2.9)

They declare that the metric is hermitian with respect to $J_{\pm}$. The complex structures are covariantly constant for the affine connections $\Gamma^{(\pm)}$. Note that the metric is covariantly constant
with respect to both $\Gamma^{(\pm)}$. From the integrability of $J_{\pm}$ and the fact that they are covariantly constant, we have a non-trivial constraint on $J_{\pm}$:

$$(J_+ + J_-)^{\mu\nu\rho}(J_+ + J_-)^{\sigma\tau\kappa} = -(J_+ - J_-)^{\mu\nu\rho}(J_+ - J_-)^{\sigma\tau\kappa}.$$  \hspace{1cm} (2.10)

Here $(J_{\pm})_{\mu\nu\rho} = \frac{1}{2}\partial_{[\mu}(J_{\pm})_{\nu\rho]}$ and $(J_{\pm})_{\mu\nu} = -g_{\mu\rho}(J_{\pm})^{\rho\nu}$.

Now we demand that the two complex structures commute with each other;

$$[J_+, J_-] = 0.$$  \hspace{1cm} (2.11)

Then, we can define the following almost product structure;

$$\Pi^\mu_\nu = (J_+)^{\mu}_\rho(J_-)^{\rho}_\nu.$$  \hspace{1cm} (2.12)

By definition, $\Pi$ commutes with $J_{\pm}$ and satisfies the relation

$$\Pi^\mu_\rho\Pi^\rho_\nu = \delta^\mu_\nu.$$  \hspace{1cm} (2.13)

Therefore $\Pi$ is regarded as the almost real structure. One can show that when $J_{\pm}$ are integrable, then $\Pi$ is also. By the integrability of $\Pi$ and the hermiticity of $g_{\mu\nu}$ with respect to $J_{\pm}$, we find the relation

$$g_{\mu\rho}\Pi^\rho_\nu = g_{\nu\rho}\Pi^\rho_\mu.$$  \hspace{1cm} (2.14)

With these structures, we can choose $g_{\mu\nu}$ as a block diagonal form. By using (2.10), we obtain

$$g_{\mu\rho}\Pi^\rho_\nu = \frac{\partial^2 K}{\partial \Phi^\mu \partial \bar{\Phi}^{\bar{\mu}}}, \quad g_{\nu\bar{\mu}} = -\frac{\partial^2 K}{\partial \Phi^{\bar{\mu}} \partial \bar{\Phi}^\nu}. $$  \hspace{1cm} (2.15)

Here $K$ is the real function appeared in (2.1) and we have decomposed the indices $\mu, \nu = 1, \ldots, D = 2(m+n)$ into $p, q = 1, \ldots, m$ and $u, v = 1, \ldots, n$. The torsion is also determined by the function $K$ and the action (2.3) is recovered. This $K$ is determined uniquely up to the generalized Kähler transformation (2.2). From these structures we can read off the supersymmetry transformations;

$$\delta_\eta \Phi^u = J_+^u(v)(\eta_\alpha D_\alpha \Phi^v),$$

$$\delta_\eta \Phi^p = \tilde{J}_+^p(v)(\gamma_5 D_\alpha \Phi^q),$$

where $J_{\pm} = \tilde{J}_{\pm} J$. The superfields $\Phi^u$, $(u = 1, \ldots, n)$ and $\Phi^p$, $(p = 1, \ldots, m)$ are the $\mathcal{N} = (1, 1)$ components of the $\mathcal{N} = (2, 2)$ chiral superfields $\hat{\Phi}$ and the twisted chiral superfields $\hat{\chi}$. Note that $K$ is not a Kähler potential and the target space is not a Kähler manifold anymore. Since the target space has the two hermitian structures associated with the two complex structures $J_{\pm}$, this is called a bi-hermitian manifold.
In summary, the bi-hermitian structures are introduced as a way to realize the $\mathcal{N} = (2, 2)$ supersymmetry for theories with chiral and twisted chiral multiplets. In this case, the target manifold admits the metric $g_{\mu\nu}$, the $B$-field $B_{\mu\nu}$ and the torsion associated with the $B$-field $T = dB$. The bi-hermitian manifold is defined by two commuting complex structures $J_{\pm}$, and they are compatible with the metric $g(J_{\pm} \cdot, J_{\pm} \cdot) = g(\cdot, \cdot)$. Obviously, when the two complex structures coincide $J_+ = J_-$, the geometry becomes a Kähler manifold. This is realized only when the $B$-field is pure gauge and the torsion vanishes.

The same is true for $\mathcal{N} = (4, 4)$ chiral and twisted chiral multiplets. The $\mathcal{N} = (4, 4)$ supersymmetry requires the complex structures $J_a,_{\pm}$ ($a = 1, 2, 3$) satisfying the relations;

\[
(J_{a,\pm})^\mu_\nu (J_{b,\pm})^\nu_\rho + (J_{b,\pm})^\mu_\nu (J_{a,\pm})^\nu_\rho = -2\delta_{ab} \delta^\mu_\rho, \quad (a, b = 1, 2, 3),
\]
\[
[J_{a,\pm}, J_{b,\pm}] = 0.
\]

Each $J_{a,\pm}$ satisfies the quaternionic structures and the dimension of the target space is $4k$. The metric is hermitian with respect to $J_{a,\pm}$. Since the target space admits two commuting hyperkähler structures, the geometry is called a bi-hypercomplex manifold\(^1\). In this case, there are $3 \times 3 = 9$ almost product structures

\[ \Pi_{ab} = J_{a,\pm} J_{b,\pm}. \]

Similar to $\mathcal{N} = (2, 2)$ cases, an $\mathcal{N} = (4, 4)$ theory is governed by a real function $K$. When each complex structure coincides $J_{a,\pm} = J_{a,\mp}$ ($a = 1, 2, 3$) with each other, the geometry becomes a hyperkähler manifold.

Before we move to the next section, a comment is in order. We have discussed the bi-hermitian and the bi-hypercomplex structures where the complex structures $J_+, J_-$ commute with each other. However, this is not generically satisfied. It is known that when there are semi-chiral multiplets in sigma models, $J_+, J_-$ cease to be commuting and the bi-hermitian (bi-hypercomplex) geometry admits more general structures \([10, 25–29]\). In the following, we never assume the semi-chiral multiplets and consider commuting complex structures.

### 2.2 Generalized geometries

The bi-hermitian and the bi-hypercomplex structures of spacetime $M$ are well described in the context of generalized geometry or doubled geometry.

The generalized tangent bundle $\mathbb{T}M = TM \oplus T^*M$, or equivalently, the doubled tangent bundle $\mathbb{T}M$ is a natural arena to discuss T-duality relations of various quantities. For example, the metric $g_{\mu\nu}$, the $B$-field $B_{\mu\nu}$ and the dilation $\phi$ are organized into the generalized metric

\(^1\)This is also known as a hyperkähler with torsion geometry in vast literature.
$H_{MN}$ and the generalized dilation $d$;

$$H_{MN} = \left( g_{\mu\nu} - B_{\mu\rho}g^{\rho\sigma}B_{\sigma\nu} \right) \left( B_{\mu\rho}g^{\rho\nu}g_{\nu} \right), \quad e^{-2d} = \sqrt{-g}e^{-2\phi}, \quad (2.19)$$

where $M, N = 1, \ldots, 2D$. The T-duality transformations of $g_{\mu\nu}, B_{\mu\nu}$ and $\phi$ are read off from the $O(D, D)$ transformation of $H_{MN}$ and $d$;

$$H'_{MN} = O_M^{P}H_{PQ}O_N^{Q}, \quad e^{-2d'} = e^{-2d}. \quad (2.20)$$

Here the generalized dilation $d$ is invariant under the $O(D, D)$ rotation. Explicitly, the famous Buscher rule of the T-duality transformation [30]

$$g'_{ij} = g_{ij} - \frac{g_{iy}g_{jy} - B_{iy}B_{jy}}{g_{yy}}, \quad g'_{iy} = \frac{B_{iy}}{g_{yy}}, \quad g'_{yy} = \frac{1}{g_{yy}},$$

$$B'_{ij} = B_{ij} - \frac{B_{iy}g_{jy} - g_{iy}B_{jy}}{g_{yy}}, \quad B'_{iy} = \frac{g_{iy}}{g_{yy}}, \quad \phi' = \phi - \frac{1}{2}\log g_{yy}, \quad (i, j \neq y), \quad (2.21)$$

are obtained by the factorized $O(D, D)$ T-duality transformation;

$$\mathcal{O} = h_y = \begin{pmatrix} 1 - t_y & t_y \\ t_y & 1 - t_y \end{pmatrix}, \quad (t_y)^{\mu}_{\nu} = \delta^{\mu}_y\delta^\nu_y, \quad h^T_y = h_y. \quad (2.22)$$

Here $y$ is the isometry direction where the T-duality transformation is performed.

Just as $g_{\mu\nu}, B_{\mu\nu}, \phi$ in the spacetime $M$ are expressed in a T-duality covariant way, the complex structure $J$ and the Kähler form $\omega = -gJ$ are also given in the doubled formalism. A generalized almost complex structure $J$ is defined by an endomorphism $J: TM \to TM$ that preserves the inner product on $TM$ and squares to the minus identity $J^2 = -1_{2D}$. For simplicity, we consider $B_{\mu\nu} = 0$ for the time being. Given the structures $J$ and $\omega$ in $M$, one shows that

$$\mathcal{I}_J = \begin{pmatrix} J & 0 \\ 0 & -J^* \end{pmatrix}, \quad \mathcal{I}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \quad (2.23)$$

are generalized almost complex structures on $TM$. Here $J^* : T^*M \to T^*M$ is the adjoint of $J$. When $J$ is integrable and $\omega$ is closed, namely, the Nijenhuis tensor for $J$ vanishes and $d\omega = 0$, then $\mathcal{I}_J$ and $\mathcal{I}_\omega$ are Courant integrable, respectively [6]. Here the Courant integrability is defined by the involutivity of the $+i$-eigen bundle by $\mathcal{I}_J, \mathcal{I}_\omega$ for the Courant bracket. In this case, the generalized almost complex structures become generalized complex structures.

A generalized Kähler structure is defined by a pair of commutative generalized complex structures $(J_1, J_2)$ whose product $G = J_1J_2$ defines a positive-definite metric on $TM$. One easily finds that $\mathcal{I}_J$ commutes with $\mathcal{I}_\omega$ and $G = \mathcal{I}_J\mathcal{I}_\omega$ is positive-definite on $TM$. Then $(\mathcal{I}_J, \mathcal{I}_\omega)$ is the generalized Kähler structure. Since $\mathcal{I}_J$ commutes with $\mathcal{I}_\omega$, it is obvious that $G$ is the real
structure on $\mathbb{T}M$. The structures $(\mathcal{I}, \mathcal{J}, \mathcal{G})$ together with the identity $1_{2D}$ on $\mathbb{T}M$ form the algebra of the bi-complex numbers $\mathbb{C}_3$:

$$I^2 = I \omega = -1_{2D}, \quad G^2 = 1_{2D}, \quad I[I_\omega G = 1_{2D}. \quad (2.24)$$

We can also show that the bi-hermitian structures define generalized complex structures:

$$J = \frac{1}{2} \begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(J_+^* \pm J_-^*) \end{pmatrix}, \quad (2.25)$$

where $\omega_\pm = -gJ_\pm$ are the fundamental two-forms associated with the bi-hermitian structures $J_\pm$. Since they commute with each other $[J_+, J_-] = 0$, they define the generalized Kähler structure. In this sense, the set $(J_+, J_-)$ together with the metric $g_{\mu \nu}$ is equivalent to that in $(J_+, J_-)$. The map (2.25) is sometimes called the Gualtieri map. The physical origin of this correspondence is studied [7–12]. We find that each $(\mathcal{I}_+, \mathcal{I}_-) \otimes \mathcal{J}_\omega)$ and $(\mathcal{I}_-, \mathcal{I}_- \otimes \mathcal{J}_\omega)$ forms the algebra of the bi-complex numbers. Here $\mathcal{G} = \mathcal{I}_+ \mathcal{J}_\omega = \mathcal{I}_- \mathcal{J}_\omega$ is the common real structure. Furthermore, we find that there are additional real structures on $\mathbb{T}M$:

$$\mathcal{G}' = \mathcal{I}_+ \mathcal{I}_- = \mathcal{I}_\omega, \quad \mathcal{G}'' = \mathcal{I}_\omega \mathcal{J}_\omega I_+ = \mathcal{I}_\omega \mathcal{J}_\omega I_+ \cdot \quad (2.26)$$

Altogether, once again by incorporating the identity $1_{2D}$ on $\mathbb{T}M$, we obtain four real and four complex structures on $\mathbb{T}M$ from the given bi-hermitian structures on $M$. They obey an eight-dimensional algebra of so-called the tri-complex numbers $\mathbb{C}_3$.

The discussion is parallel in the bi-hypercomplex cases. One finds that the bi-hypercomplex structures $(J_{a,+}, J_{a,-})$ on $M$ defines generalized complex structures:

$$J_{a, \pm} = \frac{1}{2} \begin{pmatrix} J_{a, +} \pm J_{a, -} & -(\omega_{a, +}^{-1} \mp \omega_{a, -}^{-1}) \\ \omega_{a, +} \mp \omega_{a, -} & -(J_{a, -}^* \pm J_{a, +}^*) \end{pmatrix}, \quad (a = 1, 2, 3), \quad (2.27)$$

where $J_{a, \pm}$ are two commuting hyperkähler structures on $M$ and $\omega_{a, \pm} = -gJ_{a, \pm}$ are the associated fundamental two-forms. One finds that $J_{a, \pm}$ in (2.27) satisfy the following relations:

$$J_{a, +} J_{a, +} = -\delta_{ab} 1_{2D} + \epsilon_{abc} J_{c, +}, \quad J_{a, -} J_{b, -} = -\delta_{ab} 1_{2D} + \epsilon_{abc} J_{c, +}, \quad J_{a, +} J_{b, -} = \delta_{ab} \mathcal{G} + \epsilon_{abc} J_{c, -}, \quad J_{a, -} J_{b, -} = \delta_{ab} \mathcal{G} + \epsilon_{abc} J_{c, -}. \quad (2.28)$$

The generalized complex structures satisfying the relations (2.28) are known as the generalized hyperkähler structures [31]. We find that the algebra (2.28) is that of the split-bi-quaternions.

As we have discussed, the bi-hermitian and Kähler geometries are described by $\mathcal{N} = (2, 2)$ models with and without twisted chiral multiplets. Since the chiral and the twisted chiral multiplets are interchanged by T-duality, we next examine the explicit transformation rule for these geometries.
3 T-duality transformation between two different bi-hermitian structures

We start by the generalized complex structures associated with the bi-hermitian structures:

\[
\mathcal{J}_\pm = \frac{1}{2} e^B \begin{pmatrix} J_\pm + J_- & -(\omega_\pm^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(J_\pm^* \pm J_-^*) \end{pmatrix} e^{-B}, \quad e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \in O(D, D),
\]

(3.1)

where we have performed the \(B\)-transformation \(e^B\) to include the \(B\)-field. Note that in this case, the integrability of the generalized complex structures are defined by the \(H\)-twisted Courant bracket [6]. One finds that the expression (3.1) is decomposed as

\[
\mathcal{J}_+ = \frac{1}{2} e^B \left( \mathcal{I}_{J_+} + \mathcal{I}_{J_-} + \mathcal{I}_{\omega_+} - \mathcal{I}_{\omega_-} \right) e^{-B},
\]

(3.2)

\[
\mathcal{J}_- = \frac{1}{2} e^B \left( \mathcal{I}_{J_+} - \mathcal{I}_{J_-} + \mathcal{I}_{\omega_+} + \mathcal{I}_{\omega_-} \right) e^{-B}.
\]

(3.3)

Here, \(\mathcal{I}_{J_\pm}\) and \(\mathcal{I}_{\omega_\pm}\) are the generalized complex structures of the form;

\[
\mathcal{I}_{J_\pm} = \begin{pmatrix} J_\pm & 0 \\ 0 & J_-^* \end{pmatrix}, \quad \mathcal{I}_{\omega_\pm} = \begin{pmatrix} 0 & -\omega_\pm^{-1} \\ \omega_\pm & 0 \end{pmatrix}.
\]

(3.4)

We now perform the T-duality transformation along the \(y\)-direction. The generalized complex structures \(\mathcal{J}_+\) and \(\mathcal{J}_-\) are then transformed as

\[
\mathcal{J}_+^\prime = h_y \mathcal{J}_+ h_y = \frac{1}{2} h_y e^B \left( \mathcal{I}_{J_+} + \mathcal{I}_{J_-} + \mathcal{I}_{\omega_+} - \mathcal{I}_{\omega_-} \right) e^{-B} h_y, \quad (3.5)
\]

\[
\mathcal{J}_-^\prime = h_y \mathcal{J}_- h_y = \frac{1}{2} h_y e^B \left( \mathcal{I}_{J_+} - \mathcal{I}_{J_-} + \mathcal{I}_{\omega_+} + \mathcal{I}_{\omega_-} \right) e^{-B} h_y. \quad (3.6)
\]

Here, \(h_y \in O(D, D)\) is the factorized T-duality transformation given in (2.22). On the other hand, the generalized complex structures \(\mathcal{J}_+^\prime\) after the T-duality transformation are parameterized in such a way that

\[
\mathcal{J}_+^\prime = \frac{1}{2} e^{B'} \left( \mathcal{I}_{J_+^\prime} + \mathcal{I}_{J_-^\prime} + \mathcal{I}_{\omega_+^\prime} - \mathcal{I}_{\omega_-^\prime} \right) e^{-B'}, \quad (3.7)
\]

\[
\mathcal{J}_-^\prime = \frac{1}{2} e^{B'} \left( \mathcal{I}_{J_+^\prime} - \mathcal{I}_{J_-^\prime} + \mathcal{I}_{\omega_+^\prime} + \mathcal{I}_{\omega_-^\prime} \right) e^{-B'}, \quad (3.8)
\]

where \(J_\pm^\prime, \omega_\pm^\prime = -g_\mu J_\pm, \) and \(B'\) are components after the T-duality transformation. In particular, the metric \(g_\mu^\prime\) and the \(B\)-field \(B_\mu^\prime\) are given by the Buscher rule (2.21). We stress that \(\mathcal{I}_{J_\pm^\prime} \neq h_y \mathcal{I}_{J_\pm} h_y, \mathcal{I}_{\omega_\pm^\prime} \neq h_y \mathcal{I}_{\omega_\pm} h_y,\) and \(e^{B'} \neq h_y e^B h_y\) in general.

By using equations (3.5), (3.6), (3.7), and (3.8), we obtain

\[
\mathcal{J}_+^\prime + \mathcal{J}_-^\prime = e^{B'} \left( \mathcal{I}_{J_+^\prime} + \mathcal{I}_{\omega_+^\prime} \right) e^{-B'} = h_y e^B \left( \mathcal{I}_{J_+} + \mathcal{I}_{\omega_+} \right) e^{-B} h_y, \quad (3.9)
\]
Using the parameterizations (3.13) and (3.15), we find the structure and the fundamental two-form in the bi-hermitian geometry as

\[
J'_+ - J'_- = e^{B'} \left( I_{J'_+} - I_{J'_-} \right) e^{-B'} = h_y e^B \left( I_{J_-} - I_{J_-} \right) e^{-B} h_y.
\] (3.10)

Since \( I_{J'_\pm} \) and \( I_{\omega'\pm} \) are diagonal and off-diagonal block matrices respectively, we can read off the explicit forms of \( J'_\pm \) and \( \omega'_\pm \) from the most right-hand side parameterization in the above equations. Explicitly, by multiplying \( e^{-B'} \) from the left and \( e^{B'} \) from the right, in equations (3.9) and (3.10), we extract the transformation rules for \( J_\pm \) and \( \omega_\pm \).

We now explore the T-duality relation between two different bi-hermitian geometries. The T-duality transformation along the \( y \)-direction, from one bi-hermitian structure \((J_\pm, \omega_\pm)\) to the other \((J'_\pm, \omega'_\pm)\), is derived from the equations (3.9) and (3.10) as

\[
I_{J'_+} + I_{\omega'_+} = e^{-B'} h_y e^B \left( I_{J_+} + I_{\omega_+} \right) e^{-B} h_y e^{B'},
\] (3.11)

\[
I_{J'_-} - I_{\omega'_-} = e^{-B'} h_y e^B \left( I_{J_-} - I_{\omega_-} \right) e^{-B} h_y e^{B'}.
\] (3.12)

Here \( e^{B'} \) is a matrix given by

\[
e^{B'} = \begin{pmatrix}
\delta^i_j & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
B_{ij}' & B_{ij}' & \delta^i_j & 0 \\
B_{ij}' & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
\delta^i_j & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
B_{ij} - B_{ij} g_{yy}^{-1} g_{yy} - g_{ij} g_{yy}^{-1} B_{ij} + g_{ij} g_{yy}^{-1} \delta^i_j & 0 \\
-g_{yy}^{-1} g_{yy} & 0 & 0 & 1
\end{pmatrix}, \quad (i, j \neq y),
\] (3.13)

where we have used the Buscher rule (2.21). We decompose the components of the bi-complex structure and the fundamental two-form in the bi-hermitian geometry as

\[
(J_\pm)^{\mu\nu} = \begin{pmatrix}
(J_\pm)^{ij}_+ & (J_\pm)^{iy}_+ \\
(J_\pm)^{ij}_- & (J_\pm)^{iy}_-
\end{pmatrix}, \quad (\omega_\pm)^{\mu\nu} = \begin{pmatrix}
(\omega_\pm)^{ij}_+ & (\omega_\pm)^{iy}_+ \\
(\omega_\pm)^{ij}_- & (\omega_\pm)^{iy}_-
\end{pmatrix}.
\] (3.14)

Then \( I_{J_\pm} \) and \( I_{\omega_\pm} \) are expressed as

\[
I_{J_\pm} = \begin{pmatrix}
(J_\pm)^{ij}_+ & (J_\pm)^{iy}_+ & 0 & 0 \\
(J_\pm)^{ij}_- & (J_\pm)^{iy}_- & 0 & 0 \\
0 & 0 & -(J_\pm)^{ij}_+ & -(J_\pm)^{iy}_+ \\
0 & 0 & -(J_\pm)^{ij}_- & -(J_\pm)^{iy}_-
\end{pmatrix}, \quad I_{\omega_\pm} = \begin{pmatrix}
0 & 0 & -(\omega_\pm)^{ij}_+ & -(\omega_\pm)^{iy}_+ \\
0 & 0 & -(\omega_\pm)^{ij}_- & -(\omega_\pm)^{iy}_-
\end{pmatrix}.
\] (3.15)

Using the parameterizations (3.13) and (3.15), we find

\[
e^{-B'} h_y e^B I_{J_\pm} e^{-B} h_y e^{B'} = \begin{pmatrix}
(J_\pm)^{ij}_+ - (J_\pm)^{ij}_y g_{yy} g_{yy}^{-1} & 0 & 0 & (J_\pm)^{iy}_+ \\
B_{yik} (J_\pm)^{ij}_k g_{yy} g_{yy}^{-1} & 0 & -(J_\pm)^{ij}_y & B_{yik} + (J_\pm)^{iy}_y B_{yik} \\
((\omega_\pm)^{ij}_y B_{yj} - B_{yj} (\omega_\pm)^{ij}_y) g_{yy} & -g_{yy}^{-1} g_{yy} (J_\pm)^{ij}_y - (J_\pm)^{ij}_y & (J_\pm)^{iy}_y & (J_\pm)^{iy}_y - (J_\pm)^{iy}_y B_{yj}
\end{pmatrix}.
\] (3.16)
where we have used the compatibility condition \( J_±g = \omega_± \). Substituting the results of the calculations (3.16) and (3.17) into the equations (3.11) and (3.12), and comparing them with the parametrization in

\[
\mathcal{I}_{J±}' \pm \mathcal{I}ω±' = \begin{pmatrix}
(J±')ij & \mp \omega±'1\i
\pm \omega±' & -J±'
\end{pmatrix},
\]

we obtain the T-duality transformation rule of the bi-hermitian structures from \((J±, \omega±)\) to \((J±', \omega±')\):

\[
(J±')ij = (J±)ij - (J±)iy(gyj \mp Byj) gyy, \quad (J±')iy = \mp (J±)iy gyy,
\]

\[
(J±')yj = \pm (\omega±)yj + Byk (J±)k - (J±)k(gyj \mp Byj) gyy, \quad (J±')yj = \mp B_{yk}(J±k) gyy,
\]

\[
(\omega±')ij = (\omega±)ij - (\omega±)iy(gyj \mp Byj) + (gyj \pm Byj)(\omega±)yj gyy, \quad (\omega±')iy = \mp (\omega±)iy gyy.
\]

This exhibits the Buscher-like T-duality rule between two different bi-hermitian structures. The formula, derived from the doubled formalism, precisely agrees with the ones discussed in the context of worldsheet supersymmetry [32–35]. We find that an analogous formula holds for bi-hypercomplex cases. We note that an alternative Buscher-like rule for the components in the generalized almost complex structures is discussed in [36].

We focus on the relation between a Kähler geometry and a bi-hermitian geometry. In an (almost) Kähler geometry, we have \( B = 0 \) and the two (almost) complex structures \( J_+ \) and \( J_- \) coincide with each other; \( J_+ = J_- = J, \omega_+ = \omega_- = \omega \). Then, the Buscher-like T-duality rule from the Kähler structure \((J, \omega)\) to the bi-hermitian structure \((J±', \omega±')\) is given by

\[
(J±')ij = Jij - Jiygyj gyy, \quad (J±')iy = \mp Jiy gyy, \quad (J±')yj = \pm \omegayj, \quad (J±')yj = 0,
\]

\[
(\omega±')ij = \omegaij - \omegaiygyj + gyj\omegayj gyy, \quad (\omega±')iy = \mp \omegaiy gyy.
\]

The T-duality rule from the hyperkähler \((Ja, \omega_a), (a = 1, 2, 3)\) to the bi-hypercomplex \((Ja±, \omega_a±)\) is analogous.
We now calculate the square of the almost complex structures after the T-duality transformation. From the equation (3.20), the square of $J'_\pm$ can be calculated as

$$(J'_\pm)^2 = \left( J^i_k - \frac{J^i_y g_{yk}}{g_{yy}} \pm \frac{J^i_y}{g_{yy}} \right) \left( J^j_l - \frac{J^j_y g_{yl}}{g_{yy}} \pm \frac{J^j_y}{g_{yy}} \right) \pm \omega_{yj}$$

By the compatibility conditions $\omega = -gJ$ and $g = \omega J$, we have

$$(J'_\pm)^2 = \begin{pmatrix} J^i_k J^k_l - \frac{1}{g_{yy}} J^i_k J^k_y g_{yj} + J^i_y J^y_g J^j_l + J^j_y J^y_g J^i_l - \frac{J^i_y}{g_{yy}} \omega_{yj} & \mp \frac{J^i_y}{g_{yy}} \omega_{yj} \\ 0 & -1 \end{pmatrix}.$$  

Furthermore, by using the condition $J^2 = -1$, we find

$$(J'_\pm)^2 = \begin{pmatrix} -\delta^i_j & 0 \\ 0 & -1 \end{pmatrix} = -1.$$  

This shows that both of $J'_\pm$, after the T-duality transformation, satisfy the property of the almost complex structure.

We finally examine the commutativity of $J'_+$ and $J'_-$. The product of $J'_+$ and $J'_-$ is

$$(J'_+ J'_-) = \left( J^i_k - \frac{J^i_y g_{yk}}{g_{yy}} \pm \frac{J^i_y}{g_{yy}} \right) \left( J^j_l - \frac{J^j_y g_{yl}}{g_{yy}} \pm \frac{J^j_y}{g_{yy}} \right) \omega_{yj} = \begin{pmatrix} \frac{g_{yy}}{J^i_k J^k_l - \frac{1}{g_{yy}} J^i_k J^k_y g_{yj} + J^i_y J^y_g J^j_l + J^j_y J^y_g J^i_l - \frac{J^i_y}{g_{yy}} \omega_{yj} & \pm \frac{J^i_y}{g_{yy}} \omega_{yj} \\ 0 & -1 \end{pmatrix}.$$  

By using the compatibility condition $g = \omega J$, $\omega = -gJ$, and $J^2 = -1$, we obtain

$$(J'_+ J'_-) = \begin{pmatrix} J^i_k J^k_l + \frac{1}{g_{yy}} J^i_y J^y_g J^j_l - \frac{J^i_y}{g_{yy}} \omega_{yj} & 0 \\ 0 & 1 \end{pmatrix}.$$  

Then the relation $[J'_+, J'_-] = 0$ holds. Therefore we find that the commutativity of the bihermitian structures is preserved by T-duality transformations.

4 T-duality from KK- to H-monopole geometries

It is useful to introduce a concrete example in order to confirm the formulas in the previous section. In this section, we consider T-duality between the KK5- and the NS5-branes in
type II string theories. They are BPS solutions to type II supergravities preserving sixteen supersymmetries. A fundamental string propagating on these backgrounds preserves eight supersymmetries and its worldsheet theory has $\mathcal{N} = (4,4)$ supersymmetry. Then the geometries admit at least one hyperkähler structure. We note that the KK5- and the NS5-branes are incorporated into a single solution in the doubled space $\mathcal{M}$ [37]. This is known as the DFT monopole and the KK5- and the NS5-branes appear in a particular $O(D,D)$ frame. Then the DFT monopole geometry should have a generalized hypercomplex structure that encompasses bi-hypercomplex structures of spacetimes. The procedure discussed in the previous section is directly utilized to relate the bi-hypercomplex (hyperkähler) structures of the KK5- and the NS5-brane geometries. We elucidate this in the following.

The four-dimensional transverse geometry of the KK5-brane (also known as the KK-monopole) is given by

$$ds^2 = H \, dx_{123}^2 + H^{-1} (dx^4 + A_i \, dx^i)^2, \quad (i = 1, 2, 3),$$

where $H$ is a harmonic function in the flat three dimensions $(x^1, x^2, x^3)$ and $A_i (i = 1, 2, 3)$ is a vector potential satisfying the monopole equation $dA = \hat{*}_3 dH$. Here $\hat{*}_3$ is the Hodge star operator defined in the $(x^1, x^2, x^3)$ space. The isometry lies in the $x^4$-direction. This is the Euclidean Taub-NUT space admitting three complex structures $J_a (a = 1, 2, 3)$.

For later convenience we write the metric components;

$$g_{\mu \nu} = H^{-1} \begin{pmatrix} H^2 + A_1^2 & A_1 A_2 & A_1 A_3 & A_1 \\ A_1 A_2 & H^2 + A_2^2 & A_2 A_3 & A_2 \\ A_1 A_3 & A_2 A_3 & H^2 + A_3^2 & A_3 \\ A_1 & A_2 & A_3 & 1 \end{pmatrix},$$

and the three fundamental two-forms on the Taub-NUT space;

$$\omega_1 = dx^1 \wedge (dx^4 + A) + H \, dx^2 \wedge dx^3,$$

$$\omega_2 = dx^2 \wedge (dx^4 + A) + H \, dx^3 \wedge dx^1,$$

$$\omega_3 = dx^3 \wedge (dx^4 + A) + H \, dx^1 \wedge dx^2.$$  

They satisfy the definition of the hyperkähler structure $d\omega_i = 0$ when the monopole equation $dA = \hat{*}_3 dH$ is imposed. Then the explicit form of three complex structures is given by

$$(J_1)^{\mu \nu} = -H^{-1} \begin{pmatrix} A_1 & A_2 & A_3 & 1 \\ 0 & 0 & H & 0 \\ 0 & -H & 0 & 0 \\ -H^2 - A_1^2 & -A_1 A_2 + A_3 H & -A_1 A_3 - A_2 H & -A_1 \end{pmatrix},$$
\[(J_2)^{\mu \nu} = -H^{-1} \begin{pmatrix}
0 & 0 & -H & 0 \\
A_1 & A_2 & A_3 & 1 \\
H & 0 & 0 & 0 \\
-A_1A_2 - A_3H & -H^2 - A_2^2 & HA_1 - A_2A_3 & -A_2 \\
\end{pmatrix}, \]

\[(J_3)^{\mu \nu} = -H^{-1} \begin{pmatrix}
0 & H & 0 & 0 \\
-H & 0 & 0 & 0 \\
A_1 & A_2 & A_3 & 1 \\
A_2H - A_1A_3 & -HA_1 - A_2A_3 & -H^2 - A_3^2 & -A_3 \\
\end{pmatrix}. \tag{4.4}\]

One finds that this expression satisfies the quaternion algebra \(J_aJ_b = -\delta_{ab} + \epsilon_{abc}J_c.\)

Now we perform the T-duality transformation along the \(x^4\)-direction. By applying the Buscher rule (2.21) to the metric (4.1), we obtain

\[g_{\mu \nu}' = \begin{pmatrix}
H & 0 & 0 & 0 \\
0 & H & 0 & 0 \\
0 & 0 & H & 0 \\
0 & 0 & 0 & H \\
\end{pmatrix}, \quad B_{\mu \nu}' = \begin{pmatrix}
0 & 0 & 0 & A_1 \\
0 & 0 & 0 & A_2 \\
0 & 0 & 0 & A_3 \\
-A_1 & -A_2 & -A_3 & 0 \\
\end{pmatrix}, \quad e^{2\phi'} = H. \tag{4.5}\]

The geometry described by this metric, the \(B\)-field and the dilaton gives the four-dimensional transverse space of the smeared NS5-brane, which is also called the H-monopole.

We now examine the bi-hypercomplex structures of the H-monopole. By applying the T-duality rule (3.20) to the fundamental forms (4.3) and the three complex structures (4.4), we evaluate the bi-hypercomplex structures \((J_{a,1,\pm}', \omega_{a,1,\pm}')\) on the H-monopole geometry. The result is

\[J_{1,+}' = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix}, \quad J_{2,+}' = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix}, \quad J_{3,+}' = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 \\
\end{pmatrix}, \]

\[J_{1,-}' = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
\end{pmatrix}, \quad J_{2,-}' = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{pmatrix}, \quad J_{3,-}' = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}, \tag{4.6}\]

\[\omega_{1,+}' = \begin{pmatrix}
0 & 0 & 0 & -H \\
0 & 0 & H & 0 \\
0 & -H & 0 & 0 \\
H & 0 & 0 & 0 \\
\end{pmatrix}, \quad \omega_{2,+}' = \begin{pmatrix}
0 & 0 & -H & 0 \\
0 & 0 & 0 & -H \\
H & 0 & 0 & 0 \\
0 & H & 0 & 0 \\
\end{pmatrix}, \quad \omega_{3,+}' = \begin{pmatrix}
0 & H & 0 & 0 \\
-H & 0 & 0 & 0 \\
0 & 0 & 0 & -H \\
0 & 0 & H & 0 \\
\end{pmatrix}, \]

\[\omega_{1,-}' = \begin{pmatrix}
0 & 0 & 0 & H \\
0 & 0 & H & 0 \\
0 & -H & 0 & 0 \\
-H & 0 & 0 & 0 \\
\end{pmatrix}, \quad \omega_{2,-}' = \begin{pmatrix}
0 & 0 & -H & 0 \\
0 & 0 & 0 & H \\
H & 0 & 0 & 0 \\
0 & -H & 0 & 0 \\
\end{pmatrix}, \quad \omega_{3,-}' = \begin{pmatrix}
0 & H & 0 & 0 \\
-H & 0 & 0 & 0 \\
0 & 0 & 0 & H \\
0 & 0 & -H & 0 \\
\end{pmatrix}. \tag{4.7}\]
We find the complete agreement among the expressions (4.6), (4.7) and the ones in the literature [44]. We also confirm that these structures satisfy the compatibility condition
\[ \omega'_{a,\pm} = g', \quad (a: \text{no sum}). \] (4.8)
Furthermore, we find that \( J'_{a,\pm} \) are covariantly constant in the following sense;
\[ \nabla(+)_{\rho}(J'_{a,+})^\mu_\nu = \nabla_{\rho}(J'_{a,+})^\mu_\nu + \frac{1}{2} \left( (J'_{a,+})^\mu_\sigma g'^{\sigma\alpha} H'^{\rho\alpha} - (J'_{a,+})^\nu_\sigma g'^{\mu\alpha} H'^{\rho\alpha} \right) = 0, \]
\[ \nabla(-)_{\rho}(J'_{a,-})^\mu_\nu = \nabla_{\rho}(J'_{a,-})^\mu_\nu - \frac{1}{2} \left( (J'_{a,-})^\mu_\sigma g'^{\sigma\alpha} H'^{\rho\alpha} - (J'_{a,-})^\nu_\sigma g'^{\mu\alpha} H'^{\rho\alpha} \right) = 0, \] (4.9)
where \( \nabla \) involves the Levi-Civita connection compatible with the metric \( g'^{\mu\nu} \), and \( H'^{\mu\nu\rho} \) is the field strength of \( B'^{\mu\nu}_a \). They are nothing but the conditions (2.9) derived from the \( \mathcal{N} = (4,4) \) sigma models in Section 2.

5 Worldsheet instantons and T-duality

We have established the explicit T-duality relations between the Kähler (hyperkähler) and the bi-hermitian (bi-hypercomplex) geometries. In particular, the T-duality relation between the KK- and the H-monopole geometries are now apparent. An important notion that depends on these geometric structures is the worldsheet instantons in string theory [38]. In this section, we discuss its T-duality relation.

The worldsheet instanton equation is derived by the Bogomol’nyi completion of the Euclidean worldsheet action of string;
\[ S_E = \frac{T}{2} \int_{\Sigma} \sqrt{h} g_{\mu\nu} h^{ab} \partial_a X^\mu \partial_b X^\nu \]
\[ = \frac{T}{4} \int_{\Sigma} \sqrt{h} g_{\mu\nu} \left( \partial_a X^\mu \pm J^\mu_\rho \epsilon_{ac} \partial^c X^\rho \right) \left( \partial_b X^\nu \pm J^\nu_\sigma \epsilon_{bd} \partial^d X^\sigma \right) \]
\[ \pm \frac{T}{2} \int_{\Sigma} \sqrt{h} g_{\mu\nu} h^{ab} \epsilon_{ac} J^\mu_\rho \partial^c X^\rho \partial_b X^\nu, \] (5.1)
where \( T \) is the string tension, \( h_{ab} \) \((a,b = 1,2)\) is the metric of the worldsheet \( \Sigma \) and \( \epsilon_{ab} \) is the antisymmetric tensor on \( \Sigma \). \( g_{\mu\nu} \) and \( J^\mu_\nu \) are the metric and the complex structure of a Kähler geometry \( M \). The action is bounded from below when the following equation is satisfied;
\[ \partial_a X^\mu \pm J^\mu_\rho \epsilon_{ac} \partial^c X^\rho = 0. \] (5.2)
These are the worldsheet instanton equations. (Anti-)holomorphic maps \( X : \Sigma \rightarrow C^2 \) are solutions to these equations and they are called the worldsheet instantons. Here \( C^2 \) is a two-cycle in the spacetime \( M \). The action is given by
\[ S_E = \pm \int_{C^2} \omega_J. \] (5.3)
Here we employ a convention $T = 2$ and $(\omega_J)_{\mu\nu} = -g_{\mu\rho}J^{\rho\nu}$ is the Kähler form on $M$.

Now we consider the T-duality transformation of all the materials by introducing an isometry for the background along the $X^y$-direction, and making it be gauged. We introduce the Lagrange multiplier $\tilde{X}^y$ in the worldsheet action to ensure the vanishing gauge field strength and then integrate out $X^y$. The procedure results in the T-dualized background (2.21), which entails the bi-hermitian structures. This together with the transformation rules (3.20) for geometric structures leads to the instanton equations in the bi-hermitian geometry:

$$\partial_a X^\mu \pm J'^{\mu}_{\nu \epsilon a} \tilde{f} X^\rho = 0,$$

where $X^\mu = (X^i, \tilde{X}^y), (i \neq y)$. It is noteworthy that there is a one-to-two correspondence between the instantons in the Kähler and the bi-hermitian geometries. Among other things, we have two options for the topological terms associated with instantons;

$$S'_{E,+} = \pm \int_{C^2} \omega_{J^c} + i \int_{C^2} B', \quad S'_{E,-} = \pm \int_{C^2} \omega_{J^c} + i \int_{C^2} B'.$$  

The T-duality between the KK- and the H-monopole geometries goes in this class. Indeed, the worldsheet instanton effects of these geometries are intensively studied [39–43]. A well-known fact is that two-cycles that support instantons exist in the KK-monopole geometry but there are no such cycles in the T-dualized H-monopole side\(^2\). It has been discussed that instantons in the H-monopole geometry are point-like instantons with vanishing two-cycles [40, 41]. However, no explicit proof of this proposal has been known. The explicit T-duality relations of complex structures and this splitting effect of instantons would play an important role to understand this puzzle.

6 Conclusion and discussions

In this paper, we studied detailed relations among the Kähler (hyperkähler) structures, the bi-hermitian (bi-hypercomplex) structures, generalized or doubled geometries and T-duality. Although these notions were individually studied in various contexts, their comprehensive connections have been overlooked in the literature.

The bi-hermitian and the bi-hypercomplex geometries are the target spaces of two-dimensional $\mathcal{N} = (2, 2)$ and $\mathcal{N} = (4, 4)$ non-linear sigma models with twisted chiral (or hyper) multiplets. Since the chiral and the twisted chiral multiplets are interchanged by T-duality transformation in the two-dimensional theories, there are explicit T-duality relations between these geometries. Indeed, this has been intensively studied in the context of supersymmetric non-linear sigma models.

\(^2\)Strictly speaking, two-cycles exist only in the multi-centred Taub-NUT spaces. For the single-centred Taub-NUT space, we can define disk instantons instead, for which strings wrap on a cigar-type geometry. See [42, 43] for details.
On the other hand, the complex and the bi-hermitian structures are expressed in terms of generalized or doubled geometries where T-duality symmetry is more apparent. The bi-hermitian and the bi-hypercomplex structures are embedded into the generalized complex and the generalized hyperkähler structures, respectively. The T-duality transformation is implemented by the $O(D, D)$ rotations in the doubled formalism. We exhibited how to extract the transformation rule for the bi-hermitian (bi-hypercomplex) structures in the doubled formalism and wrote down the explicit Buscher-like formula in the component form. The result precisely reproduced the relations derived in the supersymmetric sigma models [32–35]. Although the formula itself has been known, the cumbersome procedures deriving the formula in sigma models or mathematically rigorous treatments [45] are drastically simplified in the doubled formalism. We also stress that our technique is available not only for geometries realized by supersymmetric sigma models but also for any geometries admitting appropriate bi-hermitian (bi-hypercomplex) structures. The bi-hermitian (bi-hypercomplex) structures of spacetimes are embedded in geometrical structures of the doubled space $\mathcal{M}$ and their transformations are just changes of basis for endomorphisms on $T\mathcal{M}$. Therefore our procedure is easily generalized to general $O(D, D)$ transformations rather than that based on the factorized T-duality developed in the context of sigma models.

The derived formula helps us to find how the geometric structures of spacetimes are interchanged via T-duality. Among other things, we showed the relations between the hyperkähler structure on the KK-monopole and the bi-hypercomplex structure on the H-monopole. By applying our formula to the KK-monopole geometry, we re-derived the bi-hypercomplex structure of the H-monopole first found in [44]. As a byproduct, we discussed the T-duality between the worldsheet instantons in the KK- and the H-monopole geometries. We found that there is a one-to-two correspondence between them. Since the worldsheet instantons cause drastic changes of spacetime geometries, we expect that our result would help to understand non-trivial relationships between T-dualized geometries.

We exhibited the bi-hypercomplex nature of the H-monopole (smeared NS5-brane) geometry with non-trivial torsion $T = H = dB$. By performing a T-duality transformation along a certain direction, we obtain the KK-monopole described by the hyperkähler Taub-NUT geometry without torsion. When we introduce an additional isometry and perform another T-duality transformation along this direction, we obtain the geometry of the exotic $5_2^2$-brane [46]. This geometry is again torsionful and preserve the same supersymmetry as in the H- and the KK-monopoles [47]. Therefore it is expected that the $5_2^2$-brane geometry admits bi-hypercomplex structures, at least locally\(^3\). This is also anticipated from the fact that the $5_2$-brane appears in a particular $O(D, D)$ frame of the DFT monopole [50]. We again stress that our Buscher-like

\(^3\)Notice that the single $5_2^2$-brane is not well-defined as a stand-alone object [48]. In order to discuss the globally well-defined structure of the system, we should incorporate two additional branes [49].
rule (3.20) derived from the doubled formalism is applicable, not limited to geometries realized by supersymmetric sigma models, to any geometries solving DFT (and hence supergravities). Apart from this fact, a sigma model description of the $5_2^2$-brane geometry [51], and the instantons in the $5_2^2$-brane geometry [52] are studied. It would be interesting to study their relations to the bi-hermitian geometry.

In order to get a better understanding of T-duality nature of geometric structures, it is useful to work in the doubled formalism. Indeed, the $O(D,D)$ and spacetime structures are incorporated into the Born geometry in a T-duality covariant way [20, 21, 53]. The T-dual properties of worldsheet instantons are best examined in the Born sigma model [54]. We found that the generalized Kähler and the generalised hyperkähler structures satisfy the algebras of the hypercomplex numbers. This implies an interesting mathematical property of the complex structures in the doubled space. We will report these relations in the near future.

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