

Djamila Benterki · Nasser-Eddine Tatar

Stabilization of a nonlinear Euler–Bernoulli beam

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Abstract In this work, we study the vibration control of a flexible mechanical system. The dynamic of the problem is modeled as a viscoelastic nonlinear Euler–Bernoulli beam. To suppress the undesirable transversal vibrations of the beam, we adopt a control at the right boundary of the beam. This control law is simple to implement. We prove uniform stability of the system using a viscoelastic material, the multiplier method and some ideas introduced in [20]. It is shown that a large range of rates of decay of the energy can be achieved through a determined class of kernels. Unlike most of the existing classes in the market, ours are not necessarily strictly decreasing.

Mathematics Subject Classification 34K30 · 35R09 · 35R10

1 Introduction

Flexible systems exert an increasing influence in different industries and fields. For instance, we may cite flexible manipulators, flexible robot arm and marine risers for oil and gas transportation. The vibration problem of flexible systems has become a crucial topic of research. It is a widespread phenomena in engineering. The origin of these vibrations and their nature might be different. They can cause numerous harmful effects on the production process, including the damage of the equipment with significant financial consequences. There are many approaches to deal with vibration and stabilize flexible systems. Boundary control is the most practical and efficient one. In reference [5], active boundary controls to reduce vibration of an Euler–Bernoulli beam systems in one dimension are considered. In [12], nonlinear vibrations and stability issues are studied. In [4], boundary controllers are used to reduce the vibration of a coupled nonlinear flexible marine riser. Reference [11] considered an adaptive boundary control for an axially moving belt system to eliminate the vibration. In [8] using the direct method of Lyapunov, the exponential stability of a closed-loop system is proven with the help of boundary controls. Kelleche and Tatar [9] designed a nonlinear boundary control for a viscoelastic flexible system. Park et al. [14] studied the Euler–Bernoulli beam equation with memory, they proved the existence and the exponential stability of solutions for the problem

\[ v_{tt}(x, t) + v_{xxxx}(x, t) - \int_0^t k(t-s)v_{xxxx}(s)ds + g(v_t(x, t)) = 0, \text{ in } (0, L) \times [0, \infty), \]

D. Benterki (✉)
Faculty of Mathematics and Informatics, University Mohamed El Bachir El Ibrahimi, Bordj Bou Arreridj, Algeria
E-mail: benterkidj@yahoo.fr

N.-E. Tatar
Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia
E-mail: tatarn@Kfupm.edu.sa
under the boundary and initial conditions

\[
\begin{cases}
\v_{xx}(L, t) = v_t(0, t) = v(0, t) = 0, \forall t \geq 0, \\
\v_{xxx}(L, t) - \int_0^t k(t-s)\v_{xxx}(L, s)ds = u(t), \forall t \geq 0, \\
v(x, 0) = v_0(x), \v_t(x, 0) = v_1(x), x \in (0, L),
\end{cases}
\]

where

\[
\begin{align*}
&u(t) = \tilde{h}(t)v_t(L, t), \\
&h_t(t) = rv_t^2(L, t), h(0) = h_0 > 0, t \geq 0, r > 0.
\end{align*}
\]

They supposed that the kernel \( k \) verifies

\[-c_1k(t) \leq k_t(t) \leq -c_2k(t), \quad 0 < k_{tt}(t) \leq c_3k(t)\]

for some \( c_i > 0, i = 1, ..., 3 \). Furthermore, a similar result in [15] was established under a boundary control

\[
\begin{cases}
u_t(t) = h(t)v_t(L, t) + \mu(t)\sin t, \quad t \geq 0, \\
h_t(t) = rv_t^2(L, t), h(0) = h_0 > 0, \quad t \geq 0, r > 0, \\
\mu_t(t) = v_t(L, t)\sin t, \quad \mu(0) = \mu_0.
\end{cases}
\]

Seghour et al. [17] investigated the following system

\[
\begin{align*}
&\rho\v_{tt}(x, t) + EI\v_{xxx}(x, t) - EI\int_0^t k(t-s)\v_{xxx}(s)ds - T\v_{xx}(x, t) = 0, \\
&\text{in } (0, L) \times [0, \infty), \\
&\v_{xx}(L, t) = v_x(0, t) = v(0, t) = 0, \forall t \geq 0, \\
&-EI\v_{xxx}(L, t) + EI\int_0^t k(t-s)\v_{xxx}(L, s)ds + Tv_x(x, t) = u(t) - d_v\v_t(L, t) \\
&-M\v_{ttt}(L, t), \quad t \geq 0, \\
v(x, 0) = v_0(x), \v_t(x, 0) = v_1(x), x \in (0, L),
\end{align*}
\]

where the positives coefficients \( d_v, M \) represents the vessel damping and the mass of the surface vessel. They showed an exponential decay result for solutions with the following conditions on the kernel:

\[0 < k'(t) + mk(t) \leq \zeta(t), \quad m \geq 0 \quad t \geq 0\]

for some positive function \( \zeta(t) \) and

\[
\begin{align*}
u(t) = \frac{-K}{v_t(L, t)} \left\{ v_{xx}(L, t) + v(L, t) + \int_0^t k(t-s)\v_{xxx}(L, s)ds \right\}^2, K > 0, \quad t \geq 0,
\end{align*}
\]

Moreover, in [18], the authors considered a similar problem under \( u(t) = d_v\v_t(L, t) \), for kernels \( k \) verifying

\[k'(t) \leq \zeta(t), \quad k(t-s) \geq \eta(t) \int_t^\infty k(t-\tau)d\tau, \quad t \geq 0\]

for some function \( \eta(t) \). Later, the authors in [1] established uniform stability of the same problem for kernels satisfying

\[k'(t) \leq 0, \quad \gamma(t)\zeta(t) \in L^1(0, \infty)\]

In [3], the authors considered the vibrating flexible beam system
The arbitrary decay of the energy result is shown in Section 3.

\[ \psi \] function of the beam. We show an arbitrary decay result for problem (1)–(3) with weaker hypotheses on the relaxation

The variance length envisaged with the tension force will be assumed to be weak compared to the overall length

exponentially decaying functions only. Relaxation functions that can have zero derivatives on certain subsets of

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a wide range of materials with various viscoelastic properties can be used in modern engineering.

They imposed a linear control force at the boundary to achieve the exponential stability of the system.

Motivated by this work [3], the objective of the present paper is to consider the nonlinear viscoelastic Euler–Bernoulli beam equation

\[
\rho A v_{tt}(x, t) + EI v_{xxxx}(x, t) - P_0 v_{xx}(x, t) - \frac{3}{2} EA v_{x}(x, t) v_x^2(x, t) = 0, \\
in (0, L) \times [0, \infty),
\]

\[ v_{xx}(0, t) = v_{xx}(L, t) = v(0, t) = 0, \forall t \geq 0 \]

\[ -EI v_{xxxx}(L, t) + P_0 v_x(L, t) + \frac{1}{2} EA v_x^3(L, t) = -K_1 v_t(L, t), \forall t \geq 0, \ K_1 > 0. \]

The initial conditions are

\[ v(x, 0) = v_0(x), \ v_t(x, 0) = v_1(x), x \in (0, L), \]

where

\( EI \): the flexural rigidity

\( \rho A \): the mass per unit length

\( EA \): the axial stiffness

\( v(x, t) \): the transverse displacement and

\( P_0 \): the tension force

The variance length envisaged with the tension force will be assumed to be weak compared to the overall length of the beam. We show an arbitrary decay result for problem (1)–(3) with weaker hypotheses on the relaxation function \( \psi \) than the existing ones for similar problems. Namely, we do not limit ourselves to polynomially or exponentially decaying functions only. Relaxation functions that can have zero derivatives on certain subsets of \((0, \infty)\) are considered, see [20–22]. We assume that the zone where the kernel is flat and is small. Consequently, a wide range of materials with various viscoelastic properties can be used in modern engineering.

The rest of our paper is arranged as follows: In Section 2, we give some useful lemmas needed for our result. The arbitrary decay of the energy result is shown in Section 3.

2 Notation and main results

We introduce the following notation

\[
(\psi \square f)(t) = \int_0^L \int_0^t \psi(t - \tau) \left[ f(t, x) - f(\tau, x) \right]^2 d\tau dx
\]

\[
(\psi \ast f)(t) = \int_0^L \int_0^t \psi(t - \tau) f(\tau, x) d\tau dx, \ t \geq 0.
\]

For the kernel \( \psi \) we assume:
(H1) \( \psi : \mathbb{R}_+ \to \mathbb{R}_+ \) is a differentiable function satisfying
\[
0 < k = \int_0^{+\infty} \psi(s)\,ds < 1.
\]

(H2) \( \psi'(t) \leq 0 \) for almost all \( t \geq 0 \).

(H3) There exists a positive increasing function \( \theta(t) \) such that \( \frac{\theta'(t)}{\theta(t)} = u(t) \) is a decreasing function and
\[
\int_0^{+\infty} \psi(s)\theta(s)\,ds < +\infty.
\]

We denote
\[
\mathcal{V} = \{ v \in H^2(0, L) \mid v(0) = 0 \},
\]
\[
\mathcal{H} = \{ v \in \mathcal{V} \cap H^4(0, L) \mid v_{xx}(0) = v_{xx}(L) = 0 \}
\]
and \((., .), \| \|\) the inner product and the norm of the space \( L^2(0, L) \), respectively. The existence result for our problem (1)-(3) can be proved by Faedo–Galerkin method, the reader may consult [14].

**Theorem 1** Suppose that (H1)-(H3) are satisfied. If \( (v_0, v_1) \in \mathcal{H} \times L^2(0, L) \), then there exists a unique solution \( v \) of problem (1)-(3), in the sense that for \( T > 0 \), \( v \in L^\infty([0, T], \mathcal{H}), v_t \in L^\infty([0, T], \mathcal{V}), v_{tt} \in L^2([0, T], L^2(0, L)). \)

Moreover, we have \( v \in C([0, T), \mathcal{V}), v_t \in C([0, T), L^2(0, L)). \)

We define the (classical) energy of problem (1)-(3) by
\[
E(t) = \frac{1}{2} \left[ \rho A \| v_t(t) \|^2 + EI \| v_{xx}(t) \|^2 + P_0 \| v_x(t) \|^2 + \frac{EA}{4} \| v^2_x(t) \|^2 \right].
\]

Then, the time derivative of energy is equal to
\[
E'(t) = P_0 \int_0^t \psi(t-s) (v_x(s), v_{xt}(s))\,ds - K_1 v_t^2(L, t), \quad t \geq 0. \tag{5}
\]

It is easy to see that
\[
\int_0^t \psi(t-s) (v_x(s), v_{xt}(s))\,ds = -\frac{1}{2} (\psi \Box v_x)'(t) + \frac{1}{2} (\psi' \Box v_x)(t)
\]
\[
+ \frac{1}{2} \frac{d}{dt} \left( \| v_x(t) \|^2 \int_0^t \psi(s)\,ds \right) - \frac{1}{2} \psi(t) \| v_x(t) \|^2, \quad t \geq 0.
\]

Then, we consider the modified energy
\[
e(t) = \frac{\rho A}{2} \| v_t(t) \|^2 + \frac{EI}{2} \| v_{xx}(t) \|^2 + \frac{P_0}{2} \left( 1 - \int_0^t \psi(s)\,ds \right) \| v_x(t) \|^2
\]
\[
+ \frac{EA}{8} \| v^2_x(t) \|^2 + \frac{P_0}{2} (\psi \Box v_x)(t), \quad t \geq 0. \tag{6}
\]

By differentiation, we obtain
\[
e'(t) = P_0 \left( \psi \Box v_x \right)(t) - \frac{P_0}{2} \psi(t) \| v_x(t) \|^2 - K_1 v_t^2(L, t), \quad t \geq 0. \tag{7}
\]
If our non-negative relaxation function satisfies $\psi' \leq 0$, it follows that $e(t)$ is nonincreasing and uniformly bounded above by $e(0) = E(0)$. Next, we introduce the functionals

$$
\varphi_1(t) = \rho A \zeta \int_0^t \int_0^L x v_t(x, t) v_x(x, t) dx - \rho A \int_0^t \int_0^t \psi(t - s)(v(t) - v(s)) ds dx,
$$

$$
\varphi_2(t) = P_0 \int_0^t \int_0^L K_0(t - s) v_x(s)^3 ds dx,
$$

$$
\varphi_3(t) = EI \int_0^t \int_0^L K_0(t - s) v_{xx}(s)^2 ds dx + \frac{EA}{2} \int_0^t \int_0^L K_0(t - s) v_x(s)^4 ds dx,
$$

where $\zeta$ is a positive constant to be determined later,

$$
K_0(t) = \theta^{-1}(t) \int_t^\infty \psi(s) \theta(s) \, ds
$$

and $\theta(t)$ is specified below. We define the second modified functional by

$$
L(t) = e(t) + \sum_{i=1}^3 \lambda_i \varphi_i(t), \quad t \geq 0
$$

for $\lambda_i > 0$, $i = 1, 2, 3$ to be specified later. Our first result shows that this functional is an appropriate one to consider.

**Proposition 2** There exist $q_i > 0$, $i = 1, 2$ such that

$$
q_1 \left( e(t) + \varphi_2(t) + \varphi_3(t) \right) \leq L(t) \leq q_2 \left( e(t) + \varphi_2(t) + \varphi_3(t) \right), \quad t \geq 0.
$$

**Proof** It is easy to see, from the above definitions, that

$$
\varphi_1(t) \leq \frac{\rho A}{2} \zeta L \| v_t(t) \|^2 + \frac{\rho A}{2} \zeta L \| v_x(t) \|^2 + \frac{\rho A}{2} \| v_t(t) \|^2 + \frac{\rho A c_p}{2} \frac{P_0}{P_0} k (\psi v_x)(t) \\
\leq \frac{\rho A}{2} (1 + \zeta L) \| v_t(t) \|^2 + \frac{\rho A}{2} \frac{P_0}{P_0} \zeta L \| v_x(t) \|^2 + \frac{\rho A c_p}{2} \frac{P_0}{P_0} k (\psi v_x)(t) \\
\leq c_1 \left( \frac{\rho A}{2} \| v_t(t) \|^2 + \frac{P_0}{2} \| v_x(t) \|^2 + \frac{P_0}{2} k (\psi v_x)(t) \right),
$$

where $c_1 = \max(1 + \zeta L, \frac{\rho A \zeta L}{P_0}, \frac{\rho A c_p}{P_0})$. With these in mind, we have

$$
L(t) \leq (1 + \lambda_1 c_1) \frac{\rho A}{2} \| v_t(t) \|^2 + \frac{EI}{2} \| v_{xx}(t) \|^2 + \frac{EA}{8} \| v_x(t) \|^2 \\
+ \left( 1 - \int_0^t \psi(s) ds + \lambda_1 c_1 \right) \frac{P_0}{2} \| v_x(t) \|^2 + (1 + \lambda_1 c_1 k) \frac{P_0}{2} (\psi v_x)(t) \\
+ \lambda_2 \varphi_2(t) + \lambda_3 \varphi_3(t)
$$

and

$$
2L(t) \geq (1 - c_1 \lambda_1) \rho A \| v_t(t) \|^2 + (1 - k - \lambda_1 c_1) \frac{P_0}{2} \| v_x(t) \|^2 \\
+ (1 - \lambda_1 c_1 k) \frac{P_0}{2} (\psi v_x)(t) + EI \| v_{xx}(t) \|^2 \\
+ \frac{EA}{4} \| v_x(t) \|^2 + 2\lambda_2 \varphi_2(t) + 2\lambda_3 \varphi_3(t), \quad t \geq 0.
$$
Therefore, $q_1 \left(e(t) + \varphi_2(t) + \varphi_3(t)\right) \leq L(t) \leq q_2 \left(e(t) + \varphi_2(t) + \varphi_3(t)\right)$ for some constant $q_i > 0$ and $\lambda_1$ such that
\[
\lambda_1 < \frac{1-k}{c_1}.
\]

The next result [21] gives a better estimate for
\[
\int_0^L v_x \int_0^t \psi(t-s)v_x(s)dsdx.
\]

**Lemma 3** We have for a continuous function $\psi$ on $[0, \infty)$ and $v \in H^1(0, L)$
\[
\int_0^L v_x \int_0^t \psi(t-s)v_x(s)dsdx
\]
\[
= \frac{1}{2} \left( \int_0^t \psi(s)ds \right) \|v_x\|^2 + \frac{1}{2} \int_0^t \psi(t-s) \|v_x(s)\|^2 ds - \frac{1}{2} (\psi \Box v_x)(t), t \geq 0.
\]

**3 Asymptotic behavior**

In this section we state and show our result. To this end we require some notation. For every measurable set $A \subset \mathbb{R}^+$, we define the probability measure
\[
\hat{\Psi}(A) = \frac{1}{1-k} \int_A \psi(s)ds
\]
where $k = \int_0^\infty \psi(s)ds$. The flatness set and the flatness rate of $\psi$ are (respectively) defined by
\[
\mathcal{F}_\psi = \{ s \in \mathbb{R}^+, \psi(s) > 0 \text{ and } \psi'(s) = 0 \}
\]
and
\[
\mathcal{R}_\psi = \hat{\Psi}(\mathcal{F}_\psi).
\]

Let $t^* > 0$ and $\int_0^{t^*} \psi(s)ds = \psi_* > 0$.

**Theorem 4** Let us suppose that $\psi$ and $\theta$ satisfy the hypotheses (H1)–(H3) and $\mathcal{R}_\psi < \frac{1}{4}$. Then, there exist positive constants $C$ and $\nu$ such that
\[
e(t) \leq C \theta(t)^{-\nu}, t \geq 0.
\]

**Proof** A differentiation of $\varphi_1(t)$, with respect to $t$ along the solution of (1)-(4), gives
\[
\varphi_1'(t) = \zeta \rho A \int_0^L [xv_{tt}v_x + xv_tv_{xt}] dx
\]
\[
- \rho A \int_t^0 v_t \left[ \int_0^t \psi'(t-s)(v(t) - v(s))dsdx + v_t \int_0^t \psi(s)ds \right] dx
\]
\[
- \rho A \int_0^{t_t} v_{tt} \int_0^t \psi(t-s)(v(t) - v(s))dsdx, t \geq 0.
\]
We decompose the first integral into

\[
\rho A \int_0^L [x v_{tt} v_x + x v_t v_{xt}] \, dx = \rho A \int_0^L x v_t v_{xt} \, dx + \rho A \int_0^L x v_{tt} v_x \, dx .
\] (13)

Clearly

\[
I_1 = \rho A \int_0^L x \frac{d(v_x^2)}{2 \, dx} \, dx = \rho A \int_0^L 2 L v_x^2 (L, t) - \frac{\rho A}{2} v_t^2, \quad t \geq 0
\] (14)

and

\[
I_2 = \int_0^L \left( -EI v_{xxxx}(x, t) + P_0 v_{xx}(x, t) - P_0 \int_0^t \psi(t-s) v_{xx}(s) \, ds + \frac{1}{2} EA(v_x^3)_x \right) x v_x \, dx
\]

\[
= -EI \int_0^L x v_x v_{xxxx}(x, t) \, dx + P_0 \int_0^L x v_x v_{xx}(x, t) \, dx
\]

\[
- P_0 \int_0^L x v_x \left[ \psi(t-s) v_{xx}(s) \, ds + \frac{EA}{2} \int_0^L x v_x (v_x^2)_x \, dx, \quad t \geq 0.
\] (15)

Moreover,

\[
I_{21} = -EI L v_x (L, t) v_{xxx} (L, t) + EI \int_0^L (x v_{xx} + v_x) v_{xxx} (x, t) \, dx
\]

\[
= -EI L v_x (L, t) v_{xxx} (L, t) + EI \int_0^L \frac{x d}{2 \, dx} (v_x^2)
\]

\[
+ EI v_x (x, t) v_{xx} (x, t) \Big|_0^L - EI \int_0^L v_{xx}^2 \, dx
\]

\[
= -EI L v_x (L, t) v_{xxx} (L, t) + EI \frac{x}{2} v_{xx}^2 (x, t) \Big|_0^L - \frac{3}{2} EI \| v_{xx} \|_2
\]

that is

\[
I_{21} = -EI L v_x (L, t) v_{xxx} (L, t) - \frac{3}{2} EI \| v_{xx} \|_2 , \quad (15)
\]

\[
I_{22} = \frac{P_0}{2} \int_0^L x \frac{d}{dx} v_x^2 (x, t) \, dx = \frac{P_0}{2} L v_x^2 (L, t) - \frac{P_0}{2} \| v_x \|_2^2 . \quad (16)
\]
and

\[
I_{23} = -P_0 L v_\chi(L, t) \int_0^t \psi(t-s) v_\chi(L, s) \, ds + P_0 \int_0^L v_\chi \int_0^t \psi(t-s) v_\chi(s) \, ds \, dx
\]

\[ + P_0 \int_0^L x v_{xx} \int_0^t \psi(t-s) v_\chi(s) \, ds \, dx. \]

Using Young and Cauchy–Schwartz inequality we estimate the integral

\[
P_0 \int_0^L x v_{xx} \int_0^t \psi(t-s) v_\chi(s) \, ds \, dx \leq \frac{E I}{2} \|v_{xx}(t)\|^2 + \frac{P_0^2 L^2 k}{2E I} \int_0^t \psi(t-s) \|v_\chi(s)\|^2 \, ds.
\]

Next, Lemma 2 yields

\[
P_0 \int_0^L v_\chi \int_0^t \psi(t-s) v_\chi(s) \, ds \, dx = \frac{P_0}{2} \left( \int_0^t \psi(s) \, ds \right) \|v_\chi\|^2
\]

\[ + \frac{P_0}{2} \int_0^t \psi(t-s) \|v_\chi(s)\|^2 \, ds - \frac{P_0}{2} (\psi \Box v_\chi)(t), \quad t \geq 0.
\]

Then

\[
I_{23} \leq -P_0 L v_\chi(L, t) \int_0^t \psi(t-s) v_\chi(L, s) \, ds + \frac{E I}{2} \|v_{xx}\|^2 + \frac{P_0}{2} k \|v_\chi\|^2
\]

\[ + \frac{P_0}{2} \left( \frac{P_0}{E I} k L^2 + 1 \right) \int_0^t \psi(t-s) \|v_\chi(s)\|^2 \, ds - \frac{P_0}{2} (\psi \Box v_\chi)(t) \quad (17)
\]

and

\[
I_{24} = \frac{E A}{2} L v_\chi^4(L, t) - \frac{E A}{2} \int_0^L (v_\chi + x v_{xx}) v_\chi^3 \, dx
\]

\[ = \frac{E A}{2} L v_\chi^4(L, t) - \frac{E A}{2} \int_0^L v_\chi^4(x, t) \, dx - \frac{E A}{8} \int_0^L x \frac{d}{dx} (v_\chi^4)
\]

\[ = \frac{E A}{2} L v_\chi^4(L, t) - \frac{E A}{2} \int_0^L v_\chi^4(x, t) \, dx - \frac{E A}{8} L v_\chi^4(L, t) + \frac{E A}{8} \int_0^L v_\chi^4(x, t) \, dx
\]

or

\[
I_{24} = \frac{3E A}{8} L v_\chi^4(L, t) - \frac{3E A}{8} \|v_\chi\|^2. \quad (18)
\]

Taking into account (14)–(18), we have

\[
I_\xi \leq -\xi \frac{P A}{2} \|v_\eta\|^2 - \xi E I \|v_{xx}(t)\|^2 - \xi \frac{P_0}{2} (1 - k) \|v_\chi(t)\|^2 - \frac{3}{8} E A \|v_\tau^2(t)\|^2
\]
Moreover, by Young’s inequality

\[+\frac{\zeta P_0}{2} \left( \frac{P_0}{EI} k L^2 + 1 \right) \int_0^t \psi(t-s) \|v_x(s)\|^2 ds - \frac{\zeta P_0}{2} (\psi \Box v_x)(t) \, dx\]

\[+ \left( -EI \, v_{xxx}(L, t) + \frac{P_0}{2} v_x(L, t) + \frac{3EA}{8} EAv_x^3(L, t) - P_0 \int_0^t \psi(t-s)v_x(L, s) \, ds \right) \times L \zeta v_x(L, t) + \frac{\rho A}{2} L v_x^2(L, t).\]

The boundary control gives us

\[
\begin{align*}
L \zeta v_x(L, t) = & -EI \, v_{xxx}(L, t) + \frac{P_0}{2} v_x(L, t) + \frac{3EA}{8} EAv_x^3(L, t) - P_0 \int_0^t \psi(t-s)v_x(L, s) \, ds \\
& \times \left( -EI \, v_{xxx}(L, t) + \frac{P_0}{2} v_x(L, t) + \frac{3EA}{8} EAv_x^3(L, t) - P_0 \int_0^t \psi(t-s)v_x(L, s) \, ds \right) \\
& \times L \zeta v_x(L, t) - \frac{EA}{8} L \zeta v_x^3(L, t) - \frac{P_0}{2} \zeta L v_x^2(L, t) \\
& = L \zeta K_1 v_1(L, t)v_x(L, t) - \frac{EA}{8} \zeta L v_x^3(L, t) - \frac{P_0}{2} \zeta L v_x^2(L, t), \quad t \geq 0.
\end{align*}
\]

Moreover, by Young’s inequality

\[\zeta L K_1 v_1(L, t)v_x(L, t) \leq \frac{\zeta L K_1}{2} v_1^2(L, t) + \frac{\zeta L K_1}{2} v_x^2(L, t)\]

and, therefore,

\[I_\zeta \leq - \frac{\zeta P_0}{2} \|v_1\|^2 - \zeta EI \|v_{xx}(t)\|^2 - \frac{P_0}{2} (1 - k) \|v_x(t)\|^2 - \frac{3}{8} EA \|v_x^2(t)\|^2\]

\[+ \frac{P_0}{2} \left( \frac{P_0}{EI} k L^2 + 1 \right) \int_0^t \psi(t-s) \|v_x(s)\|^2 ds - \frac{P_0}{2} (\psi \Box v_x)(t)\]

\[+ \frac{\zeta L}{2} (\rho A + K_1) v_1^2(L, t) + \frac{\zeta L}{2} (K_1 - P_0) v_x^2(L, t) - \zeta \frac{EA}{8} L v_x^3(L, t). \tag{19}\]

Notice that

\[-\rho A \int_0^L v_t \left[ \int_0^t \psi(t-s)(v(t) - v(s)) ds dx + v_t \int_0^t \psi(s) ds \right] dx\]

\[\leq \rho A \left( \delta_1 - \int_0^t \psi(s) ds \right) \|v_t\|^2 - \frac{\rho \psi(0)}{4 \delta_1} (\psi \Box v_x). \tag{20}\]
After substitution of $-\rho A v_{tt}$ from (1) and integrating by part, we obtain

\[-\rho A \int_0^L v_{tt} \int_0^t \psi(t-s)(v(t) - v(s)) ds \, dx\]

\[= (EI v_{xxx}(L, t) - P_0 v_x(L, t) - \frac{EA}{2} v_x^3(L, t) + P_0 \int_0^t \psi(t-s)v_x(L, s) ds)\]

\[\times \int_0^t \psi(t-s)(v(L, t) - v(L, s)) ds\]

\[+ \int_0^t \left(-EI v_{xxx}(t) + \frac{EA}{2} v_x^3(t) + P_0 v_x(t) - P_0 \int_0^t \psi(t-s)v_x(s) ds\right)\]

\[\times \left(\int_0^t \psi(t-s)(v_x(t) - v_x(s)) ds\right) dx\]

\[= K_1 v_1(L, t) \int_0^t \psi(t-s)(v(L, t) - v(L, s)) ds\]

\[-\int_0^L EI v_{xxx}(t) \left(\int_0^t \psi(t-s)(v_x(t) - v_x(s)) ds\right) dx\]

\[+ \frac{EA}{2} \int_0^L v_x^3(t) \left(\int_0^t \psi(t-s)(v_x(t) - v_x(s)) ds\right) dx\]

\[+ P_0 \left(1 - \int_0^t \psi(s) ds\right) \int_0^L v_x(t) \left(\int_0^t \psi(t-s)(v_x(t) - v_x(s)) ds\right) dx\]

\[+ P_0 \left(\int_0^t \psi(t-s)(v_x(t) - v_x(s)) ds\right)^2 dx\]

\[= -K_1 v_1(L, t) \int_0^t \psi(t-s)(v(L, t) - v(L, s)) ds + J_1 + J_2 + J_3 + J_4, \ t \geq 0.\]

Again utilizing Young’s inequality, we get

\[-K_1 v_1(L, t) \int_0^t \psi(t-s)(v(L, t) - v(L, s)) ds\]

\[= -K_1 v_1(L, t)v(L, t) \int_0^t \psi(s) ds + K_1 v_1(L, t) \int_0^t \psi(t-s)v(L, s) ds\]

\[\leq \frac{K_1}{2} (k+1) v_t^2(L, t) + \frac{kK_1}{2} v^2(L, t) + \frac{K_1}{2} \left(\int_0^t \psi(t-s)v(L, s) ds\right)^2, \ t \geq 0. \quad (21)\]
For the second and the third term in (21), we have
\[
\frac{K_1 k}{2} v^2(L, t) \leq \frac{K_1 k}{2} L \| v_x \|^2,
\]
\[
\frac{K_1}{2} \left( \int_0^t \psi(t-s)v(L, s)ds \right)^2 = \frac{K_1}{2} \left( \int_0^t \psi(t-s) \int_0^L v_x(x, s)dxds \right)^2,
\]
and
\[
\frac{K_1}{2} \left( \int_0^t \psi(t-s)v(L, s)ds \right)^2 \leq \frac{K_1}{2} L k \int_0^t \psi(t-s) \| v_x(s) \|^2 ds, \ t \geq 0.
\]
Hence,
\[
-K_1 v_t(L, t) \int_0^t \psi(t-s) (v(L, t) - v(L, s))ds
\]
\[
\leq \frac{K_1}{2} (k + 1) v_t^2(L, t) + \frac{K_1 k}{2} L \| v_x \|^2 + \frac{K_1}{2} L k \int_0^t \psi(t-s) \| v_x(s) \|^2 ds.
\]
(22)

For $\delta_2 > 0$, we can write
\[
J_1 = \int_0^L EI v_{xx}(t) \left( \int_0^t \psi(t-s)(v_{xx}(t) - v_{xx}(s))ds \right) dx
\]
\[
\leq EI \left( \int_0^t \psi(s)ds + \delta_2 \right) \| v_{xx}(t) \|^2 + \frac{EI}{4\delta_2} \left( \int_0^t \psi(s)ds \right) \int_0^t \psi(t-s) \| v_{xx}(s) \|^2 ds
\]
and
\[
J_2 = \frac{EA}{2} \int_0^t \psi(s)ds \| v_x^2(t) \|^2 - \frac{EA}{2} \int_0^L v_x^2(t) \int_0^t \psi(t-s) v_x(s) v_x(t) ds dx
\]
\[
= \frac{EA}{2} \int_0^t \psi(s)ds \| v_x^2(t) \|^2 - \frac{EA}{2} \int_0^L v_x^2(t) \int_0^t \psi^{1/2}(t-s) v_x(s) \psi^{1/2}(t-s) v_x(t) ds dx
\]
\[
\leq \frac{EA}{2} k \| v_x^2(t) \|^2 + \frac{EA}{4} \left( \int_0^L (v_x^2(t))^2 dx \right)^{1/2} \left( \int_0^t \psi^{1/2}(t-s) v_x(s) \psi^{1/2}(t-s) v_x(t) ds dx \right)^{1/2}
\]
\[
\leq \frac{EA}{2} k \| v_x^2(t) \|^2 + \frac{EA}{4} \| v_x^2(t) \|^2 + \frac{EA}{4} \int_0^L \int_0^t \psi^{1/2}(t-s) v_x(s) \psi^{1/2}(t-s) v_x(t) ds dx
\]
\[
\leq \frac{EA}{2} k \| v_x^2(t) \|^2 + \frac{EA}{4} \| v_x^2(t) \|^2 + \frac{EA}{4} \int_0^L \int_0^t \psi(t-s) v_x^2(s) ds dx + \frac{k}{\delta_3} \int_0^t \psi(t-s) \| v_x^2(s) \|^2 ds
\]
\[
\leq \frac{EA}{2} \left( k + \frac{k}{2\delta_3} + \frac{1}{2} \right) \| v_x^2(t) \|^2 + \frac{EA}{16} \delta_3 k^2 \int_0^t \psi(t-s) \| v_x^2(s) \|^2 ds.
\]
Now we proceed to estimate $J_3$. For all measurable sets $A$ and $F$ such that $A = \mathbb{R}^+ \setminus F$, we see that

$$J_3 = P_0 \left( 1 - \int_0^t \psi(s)ds \right) \int_0^L v_x(t) \left( \int_{A \cap [0,t]} \psi(t-s)v_x(t) - v_x(s)ds \right) dx$$

$$+ P_0 \left( 1 - \int_0^t \psi(s)ds \right) \int_0^L v_x(t) \left( \int_{F \cap [0,t]} \psi(t-s)v_x(t) - v_x(s)ds \right) dx, \quad t \geq 0.$$

We denote $Q_t = Q \cap [0, t]$. Using Lemma 2, we obtain for $\delta_4 > 0$

$$P_0 \left( 1 - \int_0^t \psi(s)ds \right) \int_0^L v_x(t) \left( \int_{A_t} \psi(t-s)v_x(t) - v_x(s)ds \right) dx$$

$$\leq P_0 \left( 1 - \int_0^t \psi(s)ds \right) \left( \delta_4 \| v_x(t) \|^2 + \frac{k}{4\delta_4} \int_0^L \int_{A_t} \psi(t-s)(v_x(t) - v_x(s))^2 ds dx \right),$$

clearly

$$\int_0^L v_x(t) \left( \int_{F_t} \psi(t-s)v_x(t) - v_x(s)ds \right) dx$$

$$= \left( \int_{F_t} \psi(s)ds \| v_x(t) \|^2 \right) \left( \int_0^L v_x(t) \left( \int_{F_t} \psi(t-s)v_x(s)ds \right) dx \right)$$

and

$$- \int_0^L v_x(t) \left( \int_{F_t} \psi(t-s)v_x(s)ds \right) dx$$

$$\leq \frac{1}{2} \int_{F_t} \psi(s)ds \| v_x(t) \|^2 + \frac{1}{2} \int_{F_t} \psi(t-s) \| v_x(s) \|^2 ds, \quad t \geq 0.$$

Thus,

$$\int_0^L v_x(t) \left( \int_{F_t} \psi(t-s)v_x(t) - v_x(s)ds \right) dx$$

$$\leq \frac{3}{2} k \int_{F_t} \left( \mathcal{F}_\psi \| v_x(t) \|^2 \right) + \frac{1}{2} \int_{F_t} \psi(t-s) \| v_x(s) \|^2 ds,$$
Taking into account (19)-(22) and the above estimations of $\hat{\psi}$ is defined in (11). We end up with

\[
J_3 \leq P_0 (1 - \psi_\star) \left( \delta_4 + \frac{3}{2} k \hat{\psi} (\mathcal{F}_\phi) \right) \| v_\star (t) \|^2 \\
+ P_0 (1 - \psi_\star) \frac{k}{4 \delta_4} \int_0^L \int_{\mathcal{A}_t} \psi (t - s) (v_\star (t) - v_\star (s))^2 \, ds \, dx \\
+ \frac{P_0}{2} (1 - \psi_\star) \int_{\mathcal{F}_t} \psi (t - s) \| v_\star (s) \|^2 \, ds, \ t \geq t_\star.
\]

For $\delta_5 \geq 0$, we have

\[
J_4 \leq P_0 \left( 1 + \frac{1}{\delta_3} \right) k \int_0^L \int_{\mathcal{A}_t} \psi (t - s) (v_\star (t) - v_\star (s))^2 \, ds \, dx \\
+ P_0 \left( 1 + \delta_5 \right) k \hat{\psi} (\mathcal{F}_\phi) \int_0^L \int_{\mathcal{F}_t} \psi (t - s) (v_\star (t) - v_\star (s))^2 \, ds \, dx, \ t \geq t_\star.
\]

Taking into account (19)-(22) and the above estimations of $J_1$, $J_2$, $J_3$, $J_4$, we obtain

\[
\varphi_1 (t) \leq \rho A \left( \delta_1 - \psi_\star - \frac{\varsigma}{2} \right) \| \nu_t \|^2 + EI (k + \delta_2 - \varsigma) \| v_{xx} \|^2 \\
+ P_0 \left( - \frac{\varsigma}{2} (1 - k) + \frac{K_1 k L}{2P_0} + (1 - \psi_\star) \left( \delta_4 + \frac{3}{2} k \hat{\psi} (\mathcal{F}_\phi) \right) \right) \| \nu_t \|^2 \\
+ \left( \frac{\varsigma}{2} \frac{P_0}{EI} k L^2 + 1 \right) \int_0^t \psi (t - s) \| v_\star (s) \|^2 \, ds \\
+ \frac{EA}{2} \left( k + \frac{k}{2 \delta_3} - \frac{3 \varsigma}{4} \right) \| v_\star^2 \|^2 + \frac{EA \delta_3}{16} k^2 \int_0^t \psi (t - s) \| v_\star^2 (s) \|^2 \, ds \\
+ \frac{1}{2} (\rho A \varsigma L + K_1 (k + 1 + \varsigma L)) v_\star^2 (L, t) + \frac{\varsigma L}{2} (K_1 - P_0) v_\star^2 (L, t) \\
- \frac{\varsigma E A}{8} v_\star^4 (L, t) - \frac{c_p \psi (0)}{4 \delta_1} (\psi' \vartriangledown v_\star) (t) + \frac{k EI}{4 \delta_2} \int_0^t \psi (t - s) \| v_{xx} (s) \|^2 \, ds \\
- \frac{\varsigma P_0}{2} (\psi' \vartriangledown v_\star) (t) + P_0 k \left( \frac{1 - \psi_\star}{4 \delta_4} + 1 + \frac{1}{\delta_3} \right) \int_0^L \psi (t - s) (v_\star (t) - v_\star (s))^2 \, ds \, dx \\
+ P_0 (1 + \delta_5) k \hat{\psi} (\mathcal{F}_\phi) \int_0^L \int_{\mathcal{F}_t} \psi (t - s) (v_\star (t) - v_\star (s))^2 \, ds \, dx \\
+ \frac{P_0}{2} (1 - \psi_\star) \int_{\mathcal{F}_t} \psi (t - s) \| v_\star (s) \|^2 \, ds, \ t \geq t_\star.
\]

(23)
Further, a differentiation of $\varphi_2(t)$ yields

$$\varphi'_2(t) = P_0K_\theta(0)\|v_x\|^2 + P_0\int_0^t K'_\theta(t - s)\|v_x(s)\|^2\,ds$$

$$= P_0K_\theta(0)\|v_x\|^2 - P_0\int_0^t \frac{\theta'(t - s)}{\theta(t - s)}K_\theta(t - s)\|v_x(s)\|^2\,ds - P_0\int_0^t \psi(t - s)\|v_x(s)\|^2\,ds$$

$$\leq P_0K_\theta(0)\|v_x\|^2 - P_0u(t)\int_0^t K_\theta(t - s)\|v_x(s)\|^2\,ds - P_0\int_0^t \psi(t - s)\|v_x(s)\|^2\,ds, \quad t \geq 0.$$  

(24)

Regarding $\varphi'_3(t)$ it appears that

$$\varphi'_3(t) = EI K_\theta(0)\|v_{xx}\|^2 + EI\int_0^t K'_\theta(t - s)\|v_{xx}(s)\|^2\,ds + \frac{EA}{2}K_\theta(0)\|v_x^2\|^2$$

$$+ \frac{EA}{2}\int_0^t K'_\theta(t - s)\|v_x^2(s)\|^2\,ds,$$

that is

$$\varphi'_3(t) \leq EI K_\theta(0)\|v_{xx}\|^2 + \frac{EA}{2}K_\theta(0)\|v_x^2\|^2 - EI\int_0^t \psi(t - s)\|v_{xx}(s)\|^2\,ds$$

$$- \frac{EA}{2}u(t)\int_0^t K_\theta(t - s)\|v_x^2(s)\|^2\,ds - \frac{EA}{2}\int_0^t \psi(t - s)\|v_x^2(s)\|^2\,ds$$

$$- EI u(t)\int_0^t K_\theta(t - s)\|v_{xx}(s)\|^2\,ds, \quad t \geq 0.$$  

(25)

Collecting the estimations (7), (23)–(25), we find for $t \geq t_*$

$$L'(t) \leq P_0\left(\frac{1}{2} - \lambda_1 \frac{c_p \psi(0)}{4\delta_1}\right)(\psi'\Box v_x)(t) + \rho A\lambda_1(\delta_1 - \psi - \frac{\xi}{2})\|v_x\|^2$$

$$+ P_0\left(-\frac{\xi}{2}(1 - k)\lambda_1 + \frac{K_1k}{2}L\lambda_1 + \lambda_1(1 - \psi_*) (\delta_4 + \frac{3}{2}k \psi (F_\psi) + \lambda_2 K_\theta(0))\right)\|v_x\|^2$$

$$+ EI ((k + \delta_2 - \xi)\lambda_1 + \lambda_3 K_\theta(0))\|v_{xx}\|^2 + EI\left(\frac{k}{4\delta^2}\lambda_1 - \lambda_3\right)\int_0^t \psi(t - s)\|v_{xx}(s)\|^2\,ds$$

$$+ \frac{EA}{2}\left(k^2\delta^3 \lambda_1 - \lambda_3\right)\int_0^t \psi(t - s)\|v_{xx}(s)\|^2\,ds$$

$$- \left(K_1 - \left(\frac{K_1}{2} (k + 1 + \xi L) + \frac{\rho A\xi L}{2}\right)\lambda_1\right)v_x^2(L, t)$$

$$+ \frac{EA}{2}\left(\lambda_1(k + \frac{k}{2\delta_3} + \frac{1}{2} - \frac{3\xi}{4}) + \lambda_3 K_\theta(0)\right)\|v_x\|^2$$

$$- \frac{\xi}{8}L\lambda_1 v_x^2(L, t) - \frac{\xi L}{2}\lambda_1(P_0 - K_1)v_x^2(L, t) - \frac{P_0}{2}\lambda_1\xi(\psi\Box v_x)(t)$$

$$- \frac{\xi}{8}L\lambda_1 v_x^2(L, t) - \frac{\xi L}{2}\lambda_1(P_0 - K_1)v_x^2(L, t) - \frac{P_0}{2}\lambda_1\xi(\psi\Box v_x)(t)$$
we take
\[ A \]

Choosing
\[
\lim \frac{\hat{\psi}}{N} = \frac{\hat{\psi}}{N}\]

For \( n \in \mathbb{N} \), we introduce the sets [16]
\[ A_n = \{ s \in \mathbb{R}^+ : n\psi'(s) + \psi(s) \leq 0 \} . \]

Notice that
\[ \cup_n A_n = \mathbb{R}^+ \setminus \{ \mathcal{F}_n \cup \mathcal{N}_\psi \} , \]
where \( \mathcal{N}_\psi \) is the null set in which \( \psi' \) is not defined. The complement of \( A_n \) in \( \mathbb{R}^+ \) is denoted by \( \mathcal{F}_n = \mathbb{R}^+ \setminus \mathcal{A}_n \). It appears that \( \lim_{n \to \infty} \hat{\psi}(\mathcal{F}_n) = \hat{\psi}(\mathcal{F}_\psi) \) since \( \mathcal{F}_{n+1} \subset \mathcal{F}_n \) for all \( n \) and \( \cap_n \mathcal{F}_n = \mathcal{F}_\psi \cup \mathcal{N}_\psi \). Then, we take \( \mathcal{A}_n = A \), \( \mathcal{F}_n = F \), and we select \( \lambda_1 \leq \frac{\delta_1}{c_p \psi(0)} \), so that
\[ \frac{1}{2} - \lambda_1 \frac{c_p \psi(0)}{4 \delta_1} \geq \frac{1}{4}. \]
Choosing
\[ \lambda_3 = \frac{k^2 \delta_3}{8}, \lambda_1, \lambda_3 = \frac{2}{k}, \delta_2 = 1, \]
we may write
\[
L'(t) \leq \rho A \left( \delta_1 - \psi_\ast - \frac{\zeta}{2} \lambda_1 \| v_\| \right) - \frac{\zeta L}{2} \lambda_1 (P_0 - K_1) v_\| (L, t)
\]
\[ - \left( K_1 - \left( \frac{k}{2} (k + 1 + K_1 \psi) + \frac{P_0 \psi}{2} \right) \lambda_1 \right) \psi_\| (L, t) - \frac{EA}{8} L \lambda_1 \psi_\| (L, t) \]
\[ + P_0 \left( - \frac{\zeta}{2} (1 - k) \lambda_1 + \frac{K_1}{2} P_0 \right) \psi_\| (L, t) + \lambda_1 (1 - \psi_\ast) \psi_\| (0, t) \]
\[ + \int_0^t \psi_\| (t - s) v_\| (t) ds \]
\[ + \int_0^t \psi_\| (t - s) v_\| (t) ds \]
\[ + \lambda_1 k P_0 \left( \frac{1}{\delta_1} + 1 + \frac{1}{\delta_5} \right) \int_0^t \psi_\| (t - s) (v_\| (t) - v_\| (s))^2 ds \]
\[
- \lambda_2 \psi_\| (t) - \lambda_3 \psi_\| (t), \ t \geq t_\ast
\]
(26)
we choose $\zeta = 1 + 2k$ so that
\[
\begin{align*}
\frac{k^2}{4} + k + \frac{1}{2} - 3\zeta + k\frac{1}{4}K_\theta(0) &< 0, \\
1 - k + \frac{1}{k} &< 4, \\
1 - \zeta + \frac{k}{4}K_\theta(0) &< 0.
\end{align*}
\]

Also, we need $K_\theta(0) \leq \min \left\{ 1 - k + \frac{1}{k} \right\}$ and $\delta_1 = \frac{\psi_* + 1 + 2k}{2}$.

For small $\delta_5$ and $t^*$, $n$ large enough, we see that if $\tilde{\psi}(F_n) < \frac{1}{4}$ then
\[ (1 + \delta_5)k \tilde{\psi}(F_n) - \frac{1}{2} (1 + 2k) < 0 \]
and
\[ \frac{3}{2}k (1 - \psi_*) \tilde{\psi}(F_n) < \sigma (1 + 2k) \left( \frac{1 - k}{2} \right) \]
with $\sigma = \frac{3k (1 - \psi_*)}{4 (1 - k) (1 + 2k)}$. For the relation
\[ \lambda_2 K_\theta(0) \leq (1 - \sigma) (1 + 2k) \left( \frac{1 - k}{2} \right) \lambda_1 - \left( \frac{K_1k}{2P_0} L + (1 - \psi_*) \delta_4 \right) \lambda_1 \]
to hold, it suffices that
\[ \lambda_2 K_\theta(0) \leq (1 - \sigma) (1 + 2k) \left( \frac{1 - k}{3} \right) \lambda_1 - \frac{K_1k}{2P_0} L \lambda_1 \]
\[ \leq \frac{4 + k (1 + 3\psi_* - 8k - 6K_1L/P_0) \lambda_1}{12} \]
with $\delta_4$ small enough. Taking $K_1 \leq \min \left\{ P_0, \frac{P_0}{30L} \right\}$, we get
\[ \lambda_2 K_\theta(0) \leq \frac{4 + 4k/5 + 3k\psi_* - 8k^2}{12}. \]

This is possible if $\psi_* > \frac{8k^2 - 4 - 4k/5}{3k}$.

We also need $\lambda_1$ so small that
\[ \lambda_1 kP_0 \left( \frac{1 - \psi_*}{4\delta_4} + (1 + \frac{1}{\delta_5}) \right) - \frac{1}{4n} < 0, \]
\[ \lambda_1 \left\{ [(2k + 1) L + k + 1] + 30L^2 \frac{\rho A}{P_0} (1 + 2k) \right\} < 2. \]

As a consequence of the above consideration,
\[ L'(t) \leq -c_1 \frac{\psi_*}{2} \rho A\lambda_1 \| v_t \|^2 - c_2 P_0 \| v_x \|^2 - c_3 EI \| v_{xx} \|^2 - c_4 EA \| v_x \|^2 \]
\[ -c_5 P_0 \int_0^t (\psi \delta v_x) \ dx - \lambda_2 u(t) \phi_2(t) - \lambda_3 u(t) \phi_3(t), \ t \geq t_* \]
for some positive constants $c_i$, $i = 1, \ldots, 5$. For $\lambda_1$ even smaller if necessary, we get
\[ L'(t) \leq -C_1 e(t) - \lambda_2 u(t) \phi_2(t) - \lambda_3 u(t) \phi_3(t), \ t \geq t_. \]

(27)
where $C_1$ is some positive constant. As $u(t)$ is nonincreasing, we have $u(t) \leq u(0)$ for all $t \geq t_*$. Then (27) becomes

$$L'(t) \leq -\frac{C_1}{u(0)} u(t)e(t) - \lambda_2 u(t)\varphi_2(t) - \lambda_3 u(t)\varphi_3(t), \quad t \geq t_*.$$ 

By Proposition 1, we obtain

$$L'(t) \leq -C_2 u(t)L(t)$$

for some positive constant $C_2$. Integrating (28) over $[t_*, t]$ yields

$$L(t) \leq e^{-\int_{t_*}^t u(s) ds} L(t_*), \quad t \geq t_*.$$ 

Then using inequality (9) of Proposition 1, we find

$$q_1(e(t) + \varphi_2(t) + \varphi_3(t)) \leq e^{-\int_{t_*}^t u(s) ds} L(t_*) , \quad t \geq t_*.$$ 

The continuity of $E(t)$ over the interval $[0, t_*]$ makes it possible to deduce

$$e(t) \leq \frac{C}{\theta(t)\nu}, \quad t \geq 0$$

for some positive constants $C$ and $\nu$. \[\square\]

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