Research article

Estimation of finite population mean under PPS in presence of maximum and minimum values

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Abstract: This article deals with the estimation of the finite population mean under probability proportional to size (PPS) sampling using information on the auxiliary variable along with the rank of the auxiliary variable. We propose a ratio, product and regression type estimators by incorporating the maximum and minimum values of the study variable and the auxiliary variable. The mathematical expressions of the proposed estimators are derived up to first order of approximation. Efficiency comparisons are made on the basis of real data sets.

Keywords: PPS; mean squared error; auxiliary variable; maximum and minimum values
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1. Introduction

In the survey sampling literature, researchers have attempted to obtain estimates for population quantities such as mean, total, median, etc., that possess maximum statistical properties. For this
purpose, a representative part of the population is needed. When the population of interest is homogeneous, then one can use a simple random sampling scheme for selecting units. On the other hand, when sampling units vary considerably in size, then units may be selected with probability proportional to size (PPS). Probability proportional to size sampling, usually called as PPS sampling, is an unequal probability sampling scheme, in which the probability of selection for each sampling unit in the population is proportional to an auxiliary variable. Let $Y$ be a variable under consideration and $X$ be a supplementary information. For instance, let us consider that we want to estimate the population in the villages in a particular district. Then we would select a variable on which we have information as the auxiliary variable, e.g.,

(a) Size of each village in the district (correlated with one study variable $= 0.75$, say).

(b) Number of households in each town in the district (correlation with a study variable $= 0.95$, say).

Based on the above information, we would select the ancillary variable that has the maximum correlation with the study variable. Thus the variable at (b) may be more useful as auxiliary variable when selecting a sample using probability proportional to size with replacement sampling.

Similarly, surveys in relation to income of households may differ in sizes; for a medical survey related to the number of patients, health units may vary in sizes. Similarly, in an agriculture context, fields may vary in sizes. Villages with larger geographical areas are likely to have large populations and covered large areas under food crops (see [21]). The number of persons in the previous period may be taken as a measure of size related to surveys of socio-economic characters, which are likely to be related to population (see [14]).

The use of auxiliary information can be used either at selection stage or at estimation stage, or at both stages. [15] proposed alternative estimators in PPS sampling for multiple characteristics. [22] proposed the regression type estimator with PPS sampling. The readers are also referred to the papers by [2, 16, 18, 20], and the references cited therein. The use of auxiliary information may increase the precision of estimators of the unknown population parameters such as population mean, variance, correlation coefficient, etc. Some common estimators that utilize the information about the auxiliary variable are highly correlated with the study variable. When the correlation between the study variable and the auxiliary variable is high, in such situation, the rank of the auxiliary variable is also correlated with the study variable.

Recently, many authors have proposed ratio type estimators by means of transforming the auxiliary variables. The readers can explore these research findings by looking [5, 6, 10, 11, 19, 23]. For estimating the finite population mean using maximum and minimum values, see [1, 3, 4, 7, 8, 9, 12, 13], among other.

In this article, we propose ratio, product and regression type estimators for estimating the finite population mean under PPS sampling scheme, using maximum and minimum values. Consider a finite population $U = \{1, 2, ..., N\}$. Let $y_i$ and $(x_i, z_i)$ be the values of the study variable ($y$) and the auxiliary variables ($x, z$), respectively. Let $r_{xi}$ be the rank of the auxiliary variable corresponding to rank of $x_i$, i.e., $(R_x)$.

Let a sample of size $n$ is selected with probability proportional to size $z_i$ with replacement (PPSWR), i.e., $P_i = \frac{z_i}{\sum_{i=1}^{N} z_i}$.

Suppose that $u_i$ and $v_i$ are the study and auxiliary for the PPS sampling. Let $v_i^*$ denote the rank of $v_i$,
that is

\[ u_i = \frac{y_i}{NP_i}, \quad v_i = \frac{x_i}{NP_i}, \quad v_i^* = \frac{r_i(x_i)}{NP_i}, \]

\[ \bar{u} = \lambda \sum_{i=1}^{n} u_i, \quad \bar{v} = \lambda \sum_{i=1}^{n} v_i, \quad \bar{v}^* = \lambda \sum_{i=1}^{n} v_i^*. \]

Let

\[ e_0 = \frac{\bar{u} - \bar{Y}}{\bar{Y}}, \quad e_1 = \frac{\bar{v} - \bar{X}}{\bar{X}}, \quad e_2 = \frac{\bar{v}^* - \bar{R}_x}{\bar{R}_x}. \]

such that

\[ E(e_o) = E(e_1) = E(e_2) = 0. \]

Also

\[ E(e_0^2) = \lambda C^2_u, \quad E(e_1^2) = \lambda C^2_v, \quad E(e_2^2) = \lambda C^2_v^*, \quad E(e_0 e_1) = \lambda \rho_{uv} C_u C_v, \]

\[ E(e_0 e_2) = \lambda \rho_{uv} C_u C_v^*, \quad E(e_1 e_2) = \lambda \rho_{vv'} C_v C_v', \quad C^2_u = \frac{S^2_u}{\bar{Y}^2}, \quad C^2_v = \frac{S^2_v}{\bar{X}^2}, \quad C^2_v' = \frac{S^2_v'}{\bar{R}_x^2}, \]

where

\[ \lambda = \frac{1}{n}, S^2_u = \sum_{i=1}^{N} P_i (u_i - \bar{Y})^2, \quad S^2_v = \sum_{i=1}^{N} P_i (v_i - \bar{X})^2, \quad S^2_v^* = \sum_{i=1}^{N} P_i (v_i^* - \bar{R}_x)^2 \]

\[ S_{uv} = \sum_{i=1}^{N} P_i (u_i - \bar{Y})(v_i - \bar{X}), \quad S_{uv^*} = \sum_{i=1}^{N} P_i (u_i - \bar{Y})(v_i^* - \bar{R}_x), \]

\[ S_{vv'} = \sum_{i=1}^{N} P_i (v_i - \bar{X})(v_i^* - \bar{R}_x), \]

\[ \rho_{uv} = \frac{\sum_{i=1}^{N} P_i (u_i - \bar{Y})(v_i - \bar{X})}{S_u S_v}, \quad \rho_{uv^*} = \frac{\sum_{i=1}^{N} P_i (u_i - \bar{Y})(v_i^* - \bar{R}_x)}{S_u S_{v^*}}, \]

\[ \rho_{vv'} = \frac{\sum_{i=1}^{N} P_i (v_i - \bar{X})(v_i^* - \bar{R}_x)}{S_v S_{v'}}. \]

Many real data sets carry unexpected large \((y_{max})\) or small \((y_{min})\) values. In estimation of finite population mean, the results will be sensitive when such types of values occur. Under these circumstances, when there exist \(y_{max}\) and \(y_{min}\), then the results will be either overestimated or underestimated. To handle such type of situation, [17] suggested the following unbiased estimator for the estimation of finite population mean using maximum and minimum values:

\[ \bar{y}_s = \begin{cases} 
\bar{y} + c, & \text{if samples contain } y_{min} \text{ but not } y_{max} \\
\bar{y} - c, & \text{if samples contain } y_{max} \text{ but not } y_{min} \\
\bar{y}, & \text{for all other samples.}
\end{cases} \]
The variance of \( \bar{y}_s \) is given by

\[
V(\bar{y}_s) = \lambda S^2_y - \frac{2\lambda nc}{N-1}(y_{\text{max}} - y_{\text{min}} - nc),
\]

(1.1)

where \( S^2_y \) is the population variance, and \( c \) is a constant. The optimum value of \( c \) is

\[
c_{\text{opt}} = \frac{(y_{\text{max}} - y_{\text{min}})^2}{2n}.
\]

The minimum variance of \( \bar{y}_s \) is specified by

\[
V(\bar{y}_s)_{\text{min}} = V(\bar{y}) - \lambda \frac{(y_{\text{max}} - y_{\text{min}})^2}{2(N-1)},
\]

which is always smaller than the variance of \( \bar{y} \). The usual ratio estimator under probability proportional to size (PPS) is

\[
\bar{y}_{R(pps)} = \frac{\bar{y}(\bar{X})}{\bar{V}}.
\]

(1.2)

The bias and MSE of \( \bar{y}_{R(pps)} \) up to the first order of approximation are given by

\[
\text{Bias}(\bar{y}_{R(pps)}) \approx \lambda \left( S^2_v + S^2_{\nu r} + \rho_{\nu r} S_v S_{\nu r} - \rho_{\nu r} S_u S_{\nu r} \right)
\]

(1.3)

and

\[
\text{MSE}(\bar{y}_{R(pps)}) \approx \lambda \left[ S^2_u + R^2_1 S^2_v + R^2_2 S^2_{\nu r} + 2R_1 R_2 S_{\nu r} - 2R_1 S_{\nu r} - 2R_2 S_{\nu r} \right],
\]

(1.4)

respectively, where \( R_1 = \frac{\bar{y}}{X} \) and \( R_2 = \frac{\bar{y}}{R}_s \). The usual product estimator under PPS is given by

\[
\bar{y}_{P(pps)} = \frac{\bar{y}(\bar{X})}{\bar{V}} = \frac{\bar{y}}{R}_s.
\]

(1.5)

The bias and MSE of \( \bar{y}_{P(pps)} \) up to first order of approximation are given by

\[
\text{Bias}(\bar{y}_{P(pps)}) \approx \lambda \left( S^2_v + S^2_{\nu r} + \rho_{\nu r} S_v S_{\nu r} + \rho_{\nu r} S_u S_{\nu r} \right)
\]

(1.6)

and

\[
\text{MSE}(\bar{y}_{P(pps)}) \approx \lambda \left[ S^2_u + R^2_1 S^2_v + R^2_2 S^2_{\nu r} + 2R_1 R_2 S_{\nu r} + 2R_2 S_{\nu r} + 2R_1 S_{\nu r} \right],
\]

(1.7)

respectively. The usual regression estimator for estimating the unknown population mean under PPS sampling scheme is

\[
\bar{y}_{lr(pps)} = \bar{u} + b_1(\bar{X} - \bar{v}) + b_2(\bar{R}_s - \bar{v}'),
\]

(1.8)

where \( b_1 = \frac{\rho_{\nu r}^2}{\rho_{\nu r}^2} \), and \( b_2 = \frac{\rho_{\nu r}^2}{\rho_{\nu r}^2} \), are the sample regression coefficients.

The MSE of \( \bar{y}_{lr(pps)} \) up to first order of approximation is obtained as

\[
\text{MSE}(\bar{y}_{lr(pps)}) = \lambda S^2_u \left( 1 - \rho_{\nu r}^2 - \rho_{\nu r}^2 + 2\rho_{\nu r} \rho_{\nu r} \rho_{\nu r} \right).
\]

(1.9)
2. Suggested estimators

Following the lines of [17], we propose a ratio, product and regression type estimators under PPS sampling utilizing the auxiliary variable along with rank of the auxiliary variable having stronger correlation with the study variable. We also incorporate the minimum and maximum values of the study and the auxiliary variables.

2.1. First situation

When the correlation between the study variable and the auxiliary variable is positive, then, for the selection of larger value of the auxiliary variable, a larger value of the study variable is to be selected. And, for the selection of the smaller value of the auxiliary variable, a smaller value of the study variable is to be selected. To utilize such type of information, we suggest the ratio type estimator using the auxiliary variable and rank of the auxiliary variable as

$$\hat{Y}_{R(pps)} = \bar{u} \left( \frac{\bar{X}}{\bar{v}_{c1}} \right) \left( \frac{\bar{R}_x}{\bar{R}_v} \right)$$

(2.1)

or

$$\hat{Y}_{R(pps)} = \begin{cases} (\bar{u} + c_1) \left( \frac{\bar{X}}{\bar{v}_{c1}} \right) \left( \frac{\bar{R}_x}{\bar{R}_v} \right), & \text{if samples contain } u_{min} \text{ and } v_{min}, v_{min}^* \\ (\bar{u} - c_1) \left( \frac{\bar{X}}{\bar{v}_{c1}} \right) \left( \frac{\bar{R}_x}{\bar{R}_v} \right), & \text{if samples contain } u_{max} \text{ and } v_{max}, v_{max}^* \\ \bar{u} \left( \frac{\bar{X}}{\bar{v}_{c1}} \right) \left( \frac{\bar{R}_x}{\bar{R}_v} \right), & \text{for all other samples} \end{cases}$$

(2.2)

The regression type estimator is

$$\hat{Y}_{R1(pps)} = \bar{u}_{c11} + b_1 (\bar{X} - \bar{v}_{c21}) + b_2 (\bar{R}_x - \bar{R}_v)$$

(2.3)

where

$$b_1 = \frac{c_{v2}}{c_{x}}, \quad b_2 = \frac{c_{v3}}{c_{x}}, \quad \bar{u}_{c11} = \bar{u} + c_1, \quad \bar{v}_{c21} = \bar{v} + c_2, \quad \bar{v}_{c31} = \bar{v} + c_3.$$

If the sample contains $u_{min}$ and $(v_{min}, v_{min}^*)$, then $\bar{u}_{c21} = \bar{u} - c_1$, $\bar{v}_{c31} = \bar{v} - c_2$, $\bar{v}_{c3} = \bar{v} - c_3$. If the samples contain $u_{max}$ and $(v_{max}, v_{max}^*)$, then $\bar{u}_{c11} = \bar{u}$, $\bar{v}_{c21} = \bar{v}$, $\bar{v}_{c31} = \bar{v}$, for all other samples (here, we mean that, if we can take any value of sample, the ratio estimator gives us good result in term of MSEs as compared to the usual ratio and product estimators using two auxiliary variables.)

2.2. Second situation

While in this situation, when the correlation between the study variable and the auxiliary variable is negative, then, for the selection of larger value of the auxiliary variable, the smaller value of the study variable is to be selected. And for the selection of the smaller value of the auxiliary variable, the larger value of the study variable is to be selected. In such situation the proposed product type estimator using the auxiliary variable and rank of the auxiliary variable $(x)$ is given by

$$\hat{Y}_{P(pps)} = \bar{u}_{c12} \left( \frac{\bar{v}_{c22}}{\bar{X}} \right) \left( \frac{\bar{v}_{c2}^*}{\bar{R}_x} \right)$$

(2.4)

or

$$\hat{Y}_{P(pps)} = \begin{cases} (\bar{u} + c_1) \left( \frac{\bar{X}}{\bar{v}_{c22}} \right) \left( \frac{\bar{R}_x}{\bar{R}_v} \right), & \text{if samples contain } u_{min} \text{ and } v_{max}, v_{max}^* \\ (\bar{u} - c_1) \left( \frac{\bar{X}}{\bar{v}_{c22}} \right) \left( \frac{\bar{R}_x}{\bar{R}_v} \right), & \text{if samples contain } u_{max} \text{ and } v_{min}, v_{min}^* \\ \bar{u} \left( \frac{\bar{X}}{\bar{v}_{c22}} \right) \left( \frac{\bar{R}_x}{\bar{R}_v} \right), & \text{for all other samples} \end{cases}$$

(2.5)
The regression type estimator is

\[ \hat{Y}_{\text{r2}(	ext{pps})} = \hat{u}_{c1} + b_1(\bar{X} - \hat{v}_{c2}) + b_2(\bar{R}_x - \bar{v}_{c2}), \] (2.6)

where \((\hat{u}_{c1} = \bar{u} + c_1, \hat{v}_{c2} = \bar{v} - c_2, \bar{v}_{c2} = \bar{v} + c_3)\).

If the samples contain \(u_{\text{min}}\) and \(v_{\text{max}}\) and \(v_{\text{min}}\), then \((\hat{u}_{c1} = \bar{u} - c_1, \hat{v}_{c2} = \bar{v} + c_2, \bar{v}_{c2} = \bar{v} + c_3)\) for all types of samples. Also \(c_1, c_2, c_3\) are unknown constants. To obtain the biases and mean squared errors, we use the following relative errors terms and their expectations:

\[ e_0 = \frac{\bar{u}_{c1} - \hat{Y}}{\bar{Y}}, \quad e_1 = \frac{\bar{u}_{c2} - \hat{X}}{\bar{X}}, \quad e_2 = \frac{\bar{v}_{c3} - \bar{R}_x}{\bar{R}_x}, \]

such that \(E(e_0) = E(e_1) = E(e_2) = 0\),

\[ E(e_0^2) = \frac{\lambda}{\bar{Y}^2} \left[ S_u^2 - \frac{2nc_1}{N-1} (u_{\text{max}} - u_{\text{min}} - nc_1) \right], \]

\[ E(e_1^2) = \frac{\lambda}{\bar{X}^2} \left[ S_v^2 - \frac{2nc_2}{N-1} (v_{\text{max}} - v_{\text{min}} - nc_2) \right], \]

\[ E(e_2^2) = \frac{\lambda}{\bar{R}_x^2} \left[ S_{v'}^2 - \frac{2nc_3}{N-1} (v_{\text{max}}' - v_{\text{min}}' - nc_3) \right]. \]

Expressing (2.1) in terms of \(e_0, e_1\) and \(e_2\), we have

\[ \hat{Y}_{\text{r2}(	ext{pps})} = \hat{Y}(1 + e_0)(1 + e_1)^{-1}(1 + e_2)^{-1}. \] (2.7)

Expressing (2.7) up to first order of approximation, we have

\[ \hat{Y}_{\text{r2}(	ext{pps})} - \hat{Y} \approx \hat{Y}\left(e_0 - e_1 - e_2 + e_0^2 + e_1^2 + e_1 e_2 - e_0 e_2 - e_0 e_1\right). \] (2.8)

Taking expectation on both sides of (2.8), we have

\[ \text{Bias}(\hat{Y}_{\text{r2}(	ext{pps})}) \approx \frac{\lambda}{\bar{Y} R_x^2} \left[ S_{v'}^2 - \frac{2nc_2}{N-1} (v_{\text{max}}' - v_{\text{min}}' - nc_2) \right] + R_2 \left[ S_{v'}^2 - \frac{2nc_3}{N-1} (v_{\text{max}}' - v_{\text{min}}' - nc_3) \right] - R_1 R_2 \left[ S_{uv'} - \frac{n}{N-1} c_3(v_{\text{max}}' - v_{\text{min}}' - nc_3) \right] \]

\[ - \left( R_1 R_2 \left[ S_{uv} - \frac{n}{N-1} (v_{\text{max}} - v_{\text{min}}) + c_2(v_{\text{max}}' - v_{\text{min}}') - 2nc_2 c_3 \right] \right) \]

\[ - R_1 \left[ S_{uv} - \frac{n}{N-1} c_2(v_{\text{max}} - v_{\text{min}}) + c_1(v_{\text{max}}' - v_{\text{min}}') - 2nc_1 c_3 \right] \]

\[ - R_2 \left[ S_{uv'} - \frac{n}{N-1} c_3(v_{\text{max}} - v_{\text{min}}) - 2nc_1 c_2 \right] \]
Squaring (2.8), and then taking expectations, we have

\[
MSE\left(\hat{Y}_{Rpp}\right) \approx \left[ \lambda \left( S_u^2 + R_1^2 S_{\nu}^2 + R_2^2 S_{\nu'}^2 + 2 R_1 R_2 S_{\nu \nu'} - 2 R_2 S_{\nu \nu'} \right) \\
- \frac{2n\lambda (c_1 R_2 c_3 - R_1 c_2)}{N - 1} \left\{ (u_{\text{max}} - u_{\text{min}}) - R_1 (v_{\text{max}} - v_{\text{min}}) \\
- R_2 (v_{\text{max}}^* - v_{\text{min}}^*) - n (c_1 - R_2 c_3 - R_1 c_2) \right\} \right].
\]

(2.9)

Differentiating (2.9) with respect to \( c_1, c_2, \) and \( c_3 \), we have

\[
c_{1\text{opt}} = \frac{u_{\text{max}} - u_{\text{min}}}{2n}, \\
c_{2\text{opt}} = \frac{v_{\text{max}} - v_{\text{min}}}{2n}, \\
c_{3\text{opt}} = \frac{v_{\text{max}}^* - v_{\text{min}}^*}{2n}.
\]

Substituting the optimum value of \( c_1, c_2 \) and \( c_3 \) in (2.9), we get the minimum \( MSE \) of \( \hat{Y}_{Rpp} \) given by

\[
MSE\left(\hat{Y}_{Rpp}\right)_{\text{min}} \approx MSE\left(\bar{Y}_{Rpp}\right) - \frac{\lambda}{2(N - 1)} \left\{ (u_{\text{max}} - u_{\text{min}}) - R_1 (v_{\text{max}} - v_{\text{min}}) \\
- R_2 (v_{\text{max}}^* - v_{\text{min}}^*) \right\}^2,
\]

(2.10)

where

\[
MSE\left(\bar{Y}_{Rpp}\right) = \lambda \left[ S_u^2 + R_1^2 S_{\nu}^2 + R_2^2 S_{\nu'}^2 + 2 R_1 R_2 S_{\nu \nu'} - 2 R_2 S_{\nu \nu'} - 2 R_1 S_{\nu \nu'} \right].
\]

Similarly, the bias and minimum \( MSE \) of product estimator in PPS sampling scheme is given by

\[
\text{Bias}(\hat{Y}_{ppp}) \approx \lambda \left[ \frac{R_1}{\bar{X}\bar{R}} \left\{ S_v^2 - \frac{2 n c_2}{N - 1} (v_{\text{max}} - v_{\text{min}} - n c_2) \right\} + R_2 \left\{ S_{\nu'}^2 - \frac{2 n c_3}{N - 1} (v_{\text{max}}^* - v_{\text{min}}^* - n c_3) \right\} + R_1 R_2 \left\{ S_{\nu \nu'} - \frac{n}{N - 1} c_3 (v_{\text{max}} - v_{\text{min}}) + c_2 (v_{\text{max}}^* - v_{\text{min}}^*) - 2 n c_2 c_3 \right\} + R_1 \left\{ S_{av} - \frac{n}{N - 1} c_2 (u_{\text{max}} - u_{\text{min}}) + c_1 (v_{\text{max}} - v_{\text{min}}) - 2 n c_1 c_2 \right\} + R_2 \left\{ S_{av'} - \frac{n}{N - 1} c_3 (u_{\text{max}} - u_{\text{min}}) + c_2 (v_{\text{max}}^* - v_{\text{min}}^*) - 2 n c_2 c_1 \right\} \right].
\]

and

\[
MSE\left(\hat{Y}_{ppp}\right)_{\text{min}} \approx MSE\left(\bar{Y}_{ppp}\right) - \frac{\lambda}{2(N - 1)} \left\{ (u_{\text{max}} - u_{\text{min}}) + R_1 (v_{\text{max}} - v_{\text{min}}) \\
+ R_2 (v_{\text{max}}^* - v_{\text{min}}^*) \right\}^2,
\]

(2.11)
where 
\[ \text{MSE} \left( \bar{Y}_{R(pps)} \right) = \lambda \left[ S_u^2 + R_1^2 S_v^2 + R_2^2 S_{uv}^2 + 2 R_1 R_2 S_{uvv} + 2 R_2 S_{uvv} \right]. \]

The minimum \( \text{MSE} \) of regression estimator in case of positive correlation is given by
\[
\text{MSE} \left( \hat{Y}_{lr(pps)} \right)_{\min} \equiv \text{MSE} \left( \bar{Y}_{lr(pps)} \right) - \frac{\lambda}{2(N-1)} \left[ (u_{\max} - u_{\min}) - \beta_1 (v_{\max} - v_{\min}) \right. \\
\left. - \beta_2 (v^*_{\max} - v^*_{\min}) \right]^2, \tag{2.12}
\]
where
\[ \text{MSE} \left( \bar{Y}_{lr(pps)} \right) = \lambda S_u^2 \left( 1 - \rho_{uv}^2 - \rho_{uv}^2 + 2 \rho_{uv} \rho_{uv}^* \right) \]
and \( \beta_1 \) and \( \beta_2 \) are the population regression coefficients.

Similarly, the minimum \( \text{MSE} \) of \( \hat{Y}_{lr(pps)} \) in case of negative correlation is
\[
\text{MSE} \left( \hat{Y}_{lr(pps)} \right)_{\min} \equiv \text{MSE} \left( \bar{Y}_{lr(pps)} \right) - \frac{\lambda}{2(N-1)} \left[ (u_{\max} - u_{\min}) + \beta_1 (v_{\max} - v_{\min}) \right. \\
\left. + \beta_2 (v^*_{\max} - v^*_{\min}) \right]^2. \tag{2.13}
\]

A general form for the \( \text{MSE} \) for the situation of both positive and negative correlation between the study and the auxiliary variable is given by
\[
\text{MSE} \left( \hat{Y}_{lr(pps)} \right)_{\min} \equiv \text{MSE} \left( \bar{Y}_{lr(pps)} \right) - \frac{\lambda}{2(N-1)} \left[ (u_{\max} - u_{\min}) - |\beta_1| (v_{\max} - v_{\min}) \right. \\
\left. - |\beta_2| (v^*_{\max} - v^*_{\min}) \right]^2. \tag{2.14}
\]

3. Comparison of estimators

In this section, we compare the proposed estimators with usual ratio, product and regression estimators under PPS sampling scheme.

3.1. Condition (i)

By (1.4) and (2.10), we get
\[
\left[ \text{MSE} \left( \bar{Y}_{R(pps)} \right) - \text{MSE} \left( \hat{Y}_{R(pps)} \right)_{\min} \right] \geq 0,
\]
if
\[
\left[ (u_{\max} - u_{\min}) - R_1 (v_{\max} - v_{\min}) - R_2 (v^*_{\max} - v^*_{\min}) \right]^2 \geq 0.
\]

3.2. Condition (ii)

By (1.7) and (2.11), we obtain
\[
\left[ \text{MSE} \left( \bar{Y}_{P(pps)} \right) - \text{MSE} \left( \hat{Y}_{P(pps)} \right)_{\min} \right] \geq 0,
\]
if
\[
\left[ (u_{\max} - u_{\min}) + R_1 (v_{\max} - v_{\min}) + R_2 (v^*_{\max} - v^*_{\min}) \right]^2 \geq 0.
\]
3.3. **Condition (iii)**

By (1.9) and (2.14), it comes

\[
MS E(\hat{y}_{lr(pps)}) - MSE(\hat{y}_{lr(g)(pp)})_{min} \geq 0,
\]

if

\[
\left[ (u_{max} - u_{min}) - |\beta_1| (v_{max} - v_{min}) - |\beta_2| (v^*_{max} - v^*_{min}) \right]^2 \geq 0.
\]

We observe that the proposed estimators perform better than the existing estimators if above conditions (i)–(iii) are satisfied.

4. **Empirical study**

We consider four real data sets for numerical comparisons which are described below.

**Population 1:** [Source: [14]]

\[y=\text{Cultivation wheat in the region during 1964.}\]
\[x=\text{Cultivation wheat in the region during 1963.}\]
\[z=\text{Cultivated area in 1961.}\]
\[R_x=\text{Rank of the auxiliary variable (x).}\]
\[N = 34, \ n = 6\]
\[\bar{Y} = 199.4412, \ \bar{X} = 208.8824, \ \bar{Z} = 747.5882, \ \bar{R}_x = 17.47059,\]
\[C_u = 0.3630288, \ C_v = 0.3401158, \ C^*_v = 0.3598615,\]
\[C^2_u = 0.1317899, \ C^2_v = 0.1156788, \ C^2_v = 0.1295003,\]
\[\rho_{uv} = 0.7014932, \ \rho_{uv^*} = 0.4524924, \ \rho_{vv^*} = 0.5274431,\]
\[S_{uv} = 3387.898, \ S_{uv^*} = 233.6354, \ S_{vv^*} = 202.4327,\]
\[S^2_u = 3825.127, \ S^2_v = 3689.709, \ S^2_v = 27.96725,\]
\[\beta_1 = 0.918202, \ \beta_2 = 7.238203,\]
\[u_{max} = 315.2996, \ u_{min} = 46.72426,\]
\[v_{max} = 352.3937, \ v_{min} = 38.93689,\]
\[v^*_{max} = 32.81399, \ v^*_{min} = 7.457239.\]

**Population 2:** [Source: Abreu-Lima and Joao(2009) Hospital, Faculty of Medicine, Porto University, Portugal].

VCG(vectocardiograms) data set:

\[http://archive.ics.uci.edu/ml/datasets/vcg.\]

\[y=\text{The electrical axis corresponds to the half-area vector in the horizontal QRS Loop.}\]
\[x=\text{The electrical axis is determined by the amplitudes of the wave peaks (x,y) in the horizontal QRS Loop.}\]
\[z=\text{The electrical axis is determined by the time integrals ratio along x and y in the horizontal QRS Loop.}\]
\[R_x=\text{Rank of the auxiliary variable (x).}\]
\[N = 120, \ n = 15\]
\[\bar{Y} = 149.2458, \ \bar{X} = 151.0833, \ \bar{Z} = 154.3958, \ \bar{R}_x = 60.44167,\]
\[
C_u = 0.7646985, \quad C_v = 0.3704849, \quad C^*_v = 0.5443165, \\
C^2_u = 0.5847638, \quad C^2_v = 0.1372591, \quad C^2_v = 0.2962804, \\
\rho_{uv} = 0.2013749, \quad \rho_{uv^2} = 0.06761159, \quad \rho_{vv^2} = 0.1100916, \\
S^2 = 1327.191, \quad S^2_{uv^2} = 325.0099, \quad S^2_{vv^2} = 240.8604, \\
S^2_u = 1918.083, \quad S^2_v = 1593.75, \quad S^2_v = 284.4695, \\
\beta_1 = 0.8327473, \quad \beta_2 = 1.142512, \\
u_{max} = 1389.562, \quad u_{min} = 0.441131, \\
v_{max} = 0.441131, \quad v_{min} = 1.317745, \\
v^*_{max} = 463.1875, \quad v^*_{min} = 0.4392485.
\]

**Population 3:** [Source: Herbert(2009), DEQ, Faculty of Engineering, Porto University, Portugal].

Wines dataset

http://archive.ics.uci.edu/ml/datasets/wine.

y=Aspartame.

x=Leucine.

z=Isoleucine.

R_s=Rank of the Leucine.

N = 67, \ n = 8

\[Y = 23.63433, \quad \bar{X} = 20.59851 \quad Z = 9.792537, \quad \bar{R}_x = 34, \]

\[C_u = 0.53356861, \quad C_v = 0.6869816, \quad C^*_v = 0.7242662, \]

\[C^2_u = 0.2869596, \quad C^2_v = 0.4719437, \quad C^2_v = 0.5245616, \]

\[\rho_{uv} = 0.3171296, \quad \rho_{uv^2} = 0.4236072, \quad \rho_{vv^2} = 0.4236072, \]

\[S^2_{av} = 71.13702, \quad S^2_{uv^2} = 128.748, \quad S^2_{vv^2} = 180.1726, \]

\[S^2_u = 93.83272, \quad S^2_v = 107.6524, \quad S^2_v = 336.2949, \]

\[\beta_1 = 0.6608032, \quad \beta_2 = 0.3828427, \]

\[u_{max} = 122.2254, \quad u_{min} = 5.092119, \]

\[v_{max} = 131.6552, \quad v_{min} = 0, \]

\[v^*_{max} = 213.9851, \quad v^*_{min} = 1.052961. \]

**Population 4:** [Source: [18]].

y=Number of tube wells.

x=Irrigated area (in hectares) for 69 villages of Doraha development block of Punjab, India.

z=Number of tractors.

R_s=Rank of the auxiliary variable (x).

N = 69, \ n = 8

\[Y = 135.2609, \quad \bar{X} = 345.7536, \quad \bar{Z} = 21.23188, \quad \bar{R}_s = 34.95652, \]

\[C_u = 0.6701015, \quad C_v = 0.481404, \quad C^*_v = 0.6560908, \]

\[C^2_u = 0.049036, \quad C^2_v = 0.2317498, \quad C^2_v = 0.3076422, \]

\[\rho_{uv} = 0.1814586, \quad \rho_{uv^2} = 0.1015504, \quad \rho_{vv^2} = 0.4086618, \]

\[S^2_{av} = 3442.406, \quad S^2_{uv^2} = 1740.037, \quad S^2_{vv^2} = 237.9886, \]

\[S^2_u = 2529.325, \quad S^2_v = 17050.9, \quad S^2_v = 329.5376, \]

\[\beta_1 = 0.20189, \quad \beta_2 = 0.7221896, \]

\[u_{max} = 859.8913, \quad u_{min} = 57.45098, \]

\[v_{max} = 1167.754, \quad v_{min} = 121.0217, \]

\[v^*_{max} = 115.5958, \quad v^*_{min} = 4.246377. \]
The results are given in Table 1.

### Table 1. MSE values of all the considered estimators.

| Estimator   | Population 1 | Population 2 | Population 3 | Population 4 |
|-------------|--------------|--------------|--------------|--------------|
| $\hat{y}_{R}(pps)$ | 805.8873     | 143.7090     | 42.90262     | 1080.554     |
| $\hat{y}_{P}(pps)$ | 550.8713     | 107.0965     | 11.79146     | 1079.505     |
| $\hat{y}_{l}(pps)$ | 4503.025     | 707.3308     | 128.4614     | 1987.579     |
| $\hat{\hat{y}}_{P}(pps)$ | 4248.009     | 670.7184     | 97.3502      | 1986.530     |
| $\hat{y}_{l}(pps)$ | 406.7386     | 122.4856     | 10.3137      | 245.8052     |
| $\hat{\hat{y}}_{l}(g)(pps)$ | 304.4522     | 88.85371     | 7.832164     | 55.42954     |

In Table 1, we observed that (MSEs) of the proposed estimators are smaller than the corresponding existing estimators, for all four populations. The performance of the regression estimator is the best among all other estimators.

### 5. Conclusions

In this paper, we have proposed ratio, product, and regression type estimators in the presence of maximum and minimum values using the auxiliary variable and rank of the auxiliary variable $X$ under PPS sampling scheme. The bias and mean squared error of the proposed estimators were derived under the first degree of approximation. Based on the theoretical and numerical investigations, it is observed that the proposed estimators are more efficient than the corresponding existing estimators for all populations which are used here. The performance of the suggested regression estimator is the best than existing estimators in terms of MSEs. Categorically, we recommend the use of our proposed estimators over the existing estimators considered in this paper for the new survey for estimating the finite population mean under probability proportional to size.

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### Conflict of interest

The authors declare no conflict of interest.

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