The Wiener Test for Higher Order Elliptic Equations

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1. Introduction. Wiener’s criterion for the regularity of a boundary point with respect to the Dirichlet problem for the Laplace equation [W] has been extended to various classes of elliptic and parabolic partial differential equations. They include linear divergence and nondivergence equations with discontinuous coefficients, equations with degenerate quadratic form, quasilinear and fully nonlinear equations, as well as equations on Riemannian manifolds, graphs, groups, and metric spaces (see [LSW], [FJK], [DMM], [LM], [KM], [MZ], [AH], [Lab], [TW] to mention only a few). A common feature of these equations is that all of them are of second order, and Wiener type characterizations for higher order equations have been unknown so far. Indeed, the increase of the order results in the loss of the maximum principle, Harnack’s inequality, barrier techniques, and level truncation arguments, which are ingredients in different proofs related to the Wiener test for the second order equations.

In the present work we extend Wiener’s result to elliptic differential operators $L(\partial)$ of order $2m$ in the Euclidean space $\mathbb{R}^n$ with constant real coefficients

$$L(\partial) = (-1)^m \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta} \partial^{\alpha+\beta}.$$ 

We assume without loss of generality that $a_{\alpha\beta} = a_{\beta\alpha}$ and $(-1)^m L(\xi) > 0$ for all nonzero $\xi \in \mathbb{R}^n$. In fact, the results can be extended to equations with variable (for example, Hölder continuous) coefficients in divergence form but we leave aside this generalization to make exposition more lucid.

We use the notation $\partial$ for the gradient $(\partial_{x_1}, \ldots, \partial_{x_n})$, where $\partial_{x_k}$ is the partial derivative with respect to $x_k$. By $\Omega$ we denote an open set in $\mathbb{R}^n$ and by $B_\rho(y)$ the ball $\{x \in \mathbb{R}^n : |x - y| < \rho\}$, where $y \in \mathbb{R}^n$. We write $B_\rho$ instead of $B_\rho(O)$.

Consider the Dirichlet problem

$$L(\partial)u = f, \quad f \in C^\infty_0(\Omega), \quad u \in H^m(\Omega),$$

where we use the standard notation $C^\infty_0(\Omega)$ for the space of infinitely differentiable functions in $\mathbb{R}^n$ with compact support in $\Omega$ as well as $H^m(\Omega)$ for the completion of $C^\infty_0(\Omega)$ in the energy norm.

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We call the point $O \in \partial \Omega$ regular with respect to $L(\partial)$ if for any $f \in C_0^\infty(\Omega)$ the solution of (1) satisfies
\[ \lim_{\Omega \ni x \to O} u(x) = 0. \] (2)

For $n = 2, 3, \ldots, 2m - 1$ the regularity is a consequence of the Sobolev imbedding theorem. Therefore, we suppose that $n \geq 2m$. In the case $m = 1$ the above definition of regularity is equivalent to that given by Wiener.

The following result coincides with Wiener’s criterion in the case $n = 2$ and $m = 1$.

**Theorem 1** Let $2m = n$. Then $O$ is regular with respect to $L(\partial)$ if and only if
\[ \int_0^1 C_{2m}(B_\rho \setminus \Omega) \rho^{-1} d\rho = \infty. \] (3)

Here and elsewhere $C_{2m}$ is the potential-theoretic Bessel capacity of order $2m$ (see [AHed]). If $n = 2m$ and $O$ belongs to a continuum contained in the complement of $\Omega$, condition (3) holds.

The case $n > 2m$ is more delicate because no result of Wiener’s type is valid for all operators $L(\partial)$ (see [MN]). To be more precise, even the vertex of a cone can be irregular with respect to $L(\partial)$ if the fundamental solution of $L(\partial)$:
\[ F(x) = F(x/|x|)|x|^{2m-n}, \quad x \in \mathbb{R}^n \setminus O, \] (4)
changes sign. Examples of operators $L(\partial)$ with this property were given in [MN] and [D]. For instance, according to [MN] the vertex of a sufficiently thin 8-dimensional cone $K$ is irregular with respect to the operator
\[ L(\partial)u := 10\partial_{x_8}^4 u + \Delta^2 u, \quad u \in \dot{H}^2(\mathbb{R}^8 \setminus K). \]

In the sequel, Wiener’s type characterization of regularity for $n > 2m$ is given for a subclass of the operators $L(\partial)$ called *positive with the weight $F$*. This means that for all real-valued $u \in C_0^\infty(\mathbb{R}^n \setminus O)$,
\[ \int_{\mathbb{R}^n} L(\partial)u(x) \cdot u(x) F(x) \, dx \geq c \sum_{k=1}^m \int_{\mathbb{R}^n} |
abla_k u(x)|^2 |x|^{2k-n} \, dx, \] (5)
where $\nabla_k$ is the gradient of order $k$, i.e. $\nabla_k = \{ \partial^\alpha \}$ with $|\alpha| = k$.

The positivity of the left-hand side in (5) is equivalent to the inequality
\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{L(\xi) + L(\eta)}{L(\xi - \eta)} f(\xi) f(\eta) d\xi d\eta > 0 \]
for all non-zero $f \in C_0^\infty(\mathbb{R}^n)$.

**Theorem 2** Let $n > 2m$ and let $L(\partial)$ be positive with weight $F$. Then $O$ is regular with respect to $L(\partial)$ if and only if
\[ \int_0^1 C_{2m}(B_\rho \setminus \Omega) \rho^{2m-n-1} d\rho = \infty. \] (6)
Note that in direct analogy with the case of the Laplacian we could say, in Theorems 1 and 2, that $O$ is irregular with respect to $L(\partial)$ if and only if the set $\mathbb{R}^n \setminus \Omega$ is $2m$-thin in the sense of linear potential theory [L], [AHed].

Let, for example, the exterior of $\Omega$ contain the region

$$\{ x : 0 < x_n < 1, \quad (x_1^2 + \ldots + x_{n-1}^2)^{1/2} < f(x_n) \},$$

where $f$ is an increasing function such that $f(0) = f'(0) = 0$. Then the point $O$ satisfies (6) if and only if

$$\int_0^1 \log f(\tau)^{-1} \tau^{-1} d\tau = \infty \quad \text{for } n = 2m + 1$$

and

$$\int_0^1 f(\tau) \tau^{2m-n} d\tau = \infty \quad \text{for } n \geq 2m + 2.$$

Since, obviously, the operator $L(\partial)$ of the second order is positive with the weight $F$, Wiener’s result for $n > 2$ is contained in Theorem 2.

We note that the pointwise positivity of $F$ follows from (5), but the converse is not true. In particular, the $m$-harmonic operator with $2m < n$ satisfies (5) if and only if $n = 5, 6, 7$ for $m = 2$ and $n = 2m + 1, 2m + 2$ for $m > 2$ (see [M2], where the proof of sufficiency of (5) is given for $(-\Delta)^m$ with $m$ and $n$ as above, and also [E] dealing with the sufficiency for noninteger powers of the Laplacian in the intervals $(0, 1)$ and $[n/2 - 1, n/2]$).

We state some auxiliary assertions of independent interest which concern the so called $L$-capacitary potential $U_K$ of the compact set $K \subset \mathbb{R}^n$, $n > 2m$, i.e. the solution of the variational problem

$$\inf \{ \int_{\mathbb{R}^n} L(\partial) u \cdot u \, dx : \ u \in C^\infty_0(\mathbb{R}^n), u = 1 \text{ in vicinity of } K \}.$$

These assertions are used in the proof of necessity in Theorem 2.

By the $m$-harmonic capacity $\text{cap}_m(K)$ of a compact set $K$ we mean

$$\inf \left\{ \sum_{|\alpha| = m} \frac{m!}{\alpha!} ||\partial^\alpha u||^2_{L^2(\mathbb{R}^n)} : \ u \in C^\infty_0(\Omega), u = 1 \text{ in vicinity of } K \right\}. \quad (7)$$

**Lemma 1** Let $\Omega = \mathbb{R}^n$, $2m < n$. For all $y \in \mathbb{R}^n \setminus K$

$$U_K(y) = 2^{-1}U_K(y)^2 + \int_{\mathbb{R}^n} \sum_{m \geq j \geq 1} \sum_{|\mu| = |\nu| = j} \partial^\mu U_K(x) \cdot \partial^\nu U_K(x) \cdot \mathcal{P}_{\mu\nu}(\partial) F(x - y) \, dx,$$

where $\mathcal{P}_{\mu\nu}(\zeta)$ are homogeneous polynomials of degree $2(m - j)$, $\mathcal{P}_{\mu\nu} = \mathcal{P}_{\nu\mu}$ and $\mathcal{P}_{\alpha\beta}(\zeta) = a_{\alpha\beta}$ for $|\alpha| = |\beta| = m$. 

Corollary 1 Let $\Omega = \mathbb{R}^n$ and $2m < n$. For all $y \in \mathbb{R}^n \setminus K$ there holds the estimate
\[
|\nabla^j U_K(y)| \leq c_j \text{dist}(y, K)^{2m-n-j} \text{cap}_m K, \tag{9}
\]
where $j = 0, 1, 2, \ldots$ and $c_j$ does not depend on $K$ and $y$.

By $M$ we denote the Hardy-Littlewood maximal operator.

Corollary 2 Let $2m < n$ and let $0 < \theta < 1$. Also let $K$ be a compact subset of $\overline{B}_\rho \setminus B_{\theta \rho}$. Then the $L$-capacitary potential $U_K$ satisfies
\[
M |\nabla^l U_K(0)| \leq c_\theta \rho^{2m-l-n} \text{cap}_m K, \tag{10}
\]
where $l = 0, 1, \ldots, m$ and $c_\theta$ does not depend on $K$ and $\rho$.

Let $L(\partial)$ be positive with the weight $F$. Then identity (8) implies that the $L$-capacitary potential of a compact set $K$ with positive $m$-harmonic capacity satisfies
\[
0 < U_K(x) < 2 \quad \text{on} \quad \mathbb{R}^n \setminus K. \tag{11}
\]
In general, the bound 2 in (11) cannot be replaced by 1.

Proposition 1 If $L = \Delta^{2m}$, then there exists a compact set $K$ such that
\[
(U_K - 1)_{\mathbb{R}^n \setminus K}
\]
changes sign in any neighbourhood of a point of $K$.

We give a lower pointwise estimate for $U_K$ stated in terms of capacity (compare with the upper estimate (9)).

Proposition 2 Let $n > 2m$ and let $L(\partial)$ be positive with the weight $F$. If $K$ is a compact subset of $B_d$ and $y \in \mathbb{R}^n \setminus K$, then
\[
U_K(y) \geq c (|y| + d)^{2m-n} \text{cap}_m K.
\]

Sufficiency in Theorem 2 follows from the next assertion which is of interest in itself.

Lemma 2 Let $2m < n$ and let $L(\partial)$ be positive with the weight $F$. Also let $u \in H^m(\Omega)$ satisfy $L(\partial)u = 0$ on $\Omega \cap B_{2R}$. Then, for all $\rho \in (0, R),$
\[
\sup \{|u(p)|^2 : p \in \Omega \cap B_{\rho}\} + \int_{\Omega \cap B_{\rho}} \sum_{k=1}^{m} \frac{|\nabla^k u(x)|^2}{|x|^{n-2k}} \, dx
\]
\[
\leq c_1 M_R(u) \exp \left(-c_2 \int_{\rho}^{R} \frac{\text{cap}_m(\overline{B}_\tau \setminus \Omega)}{\tau^{n-2m+1}} \, d\tau \right), \tag{12}
\]
where $c_1$ and $c_2$ are positive constants, and
\[
M_R(u) = R^{-n} \int_{\Omega \cap (B_{2R} \setminus B_R)} |u(x)|^2 \, dx.
\]
The present work gives answers to some questions posed in [M2]. I present several simply formulated unsolved problems.

1. Is it possible to replace the positivity of $L(\partial)$ with the weight $F(x)$ by the positivity of $F(x)$ in Theorem 2?

A particular case of this problem is the following one.

2. Does Theorem 2 hold for the operator $(-\Delta)^m$, where $n \geq 8$, $m = 2$ and $n \geq 2m + 3$, $m > 2$? (13)

The next problem concerns Green's function $G_m$ of the Dirichlet problem for $(-\Delta)^m$ in an arbitrary domain $\Omega$.

3. Prove or disprove the estimate

$$|G_m(x, y)| \leq c(m, n) \frac{|x - y|^{n-2m}}{|x - y|^{n-2m}},$$

where $c(m, n)$ is independent of $\Omega$ and $m$ and $n$ are the same as in [M3].

For $n = 5, 6, 7, m = 2$ and $n = 2m + 1$, $2m + 2$, $m > 2$ estimate (13) was proved in [M3]. In the sequel, by $u$ we denote a solution in $\hat{H}^m(\Omega)$ of the equation

$$(-\Delta)^m u = f \quad \text{in} \quad \Omega.$$ (15)

Clearly, (14) leads to the following estimate of the maximum modulus of $u$

$$\|u\|_{L_\infty} \leq c(m, n, \text{mes}_n(\Omega))\|f\|_{L_p(\Omega)},$$

where $p > n/2m$. However, the validity of this estimate for the same $n$ and $m$ as in (13) is an open problem. Moreover, the following questions arise.

4. Let $m = 2$, $n \geq 8$, and let $\Omega$ be an arbitrary bounded domain. Is $u$ uniformly bounded in $\Omega$ for any $f \in C^\infty_0(\Omega)$?

5. Let $m > 2$ and $n \geq 2m + 3$. Also, let $\partial \Omega$ have a conic singularity. Is $u$ uniformly bounded in $\Omega$ for any $f \in C^\infty_0(\Omega)$?

For $m = 2$, the affirmative answer to the last question is given in [MP].

I formulate two related open problems.

6. Let $m = 2$ and $n = 2$. Is $u$ Lipschitz up to the boundary of an arbitrary bounded domain, for any $f \in C^\infty_0(\Omega)$?

7. Let $m = 2$ and $n \geq 3$. Does $u$ belong to the class $C^{1,1}(\Omega)$ for any $f \in C^\infty_0(\Omega)$ if $\Omega$ is convex?

According to [KoM], the last is true in the two-dimensional case.

I conclude with the following variant of the Phragmén-Lindelöf principle (see [M3]).

**Proposition 3** Let either $n = 5, 6, 7$, $m = 2$ or $n = 2m + 1$, $2m + 2$, $m > 2$. Further, let $\eta u \in \hat{H}^m(\Omega)$ for all $\eta \in C^\infty(\mathbb{R}^n)$, $\eta = 0$ near $O$. If

$$\Delta^m u = 0 \quad \text{on} \quad \Omega \cap B_1,$$

then $u \equiv 0$ in $\Omega$.
then either \( u \in \dot{H}^m(\Omega) \) and 

\[
\limsup_{\rho \to 0} \sup_{B_\rho \cap \Omega} |u(x)| \exp \left( c \int_\rho^1 \operatorname{cap}_m(B_\rho \setminus \Omega) \frac{d\rho}{\rho} \right) < \infty
\]

or 

\[
\liminf_{\rho \to 0} \rho^{n-2m} M_\rho(u) \exp \left( -c \int_\rho^1 \operatorname{cap}_m(B_\rho \setminus \Omega) \frac{d\rho}{\rho} \right) > 0.
\]

It would be interesting to extend this assertion to other values of \( n \) and \( m \).

References

[AH] Adams, D.R. & Herd, A., The necessity of the Wiener test for some semi-linear elliptic PDE. Ind. Univ. Math. J., 41 (1992), 109–124.

[AHed] Adams, D.R. & Hedberg, L.I., Function Spaces and Potential Theory. Springer-Verlag, Berlin, 1996.

[DMM] Dal Maso, G. & Mosco, U., Wiener criteria and energy decay for relaxed Dirichlet problems. Arch. Rational Mech. Anal., 95 (1986), 345–387.

[D] Davies, E.B., Limits in \( L^p \) regularity of self-adjoint elliptic operators. J. Diff. Equations, 135:1 (1997), 83–102.

[E] Eilertsen, S., On weighted positivity and the Wiener regularity of a boundary point for the fractional Laplacian. Ark. för Mat., 38:1 (2000), 53–75.

[FKJ] Fabes, E.G., Jerison, D. & Kenig, C., The Wiener test for degenerate elliptic equations. Ann. Inst. Fourier (Grenoble), 32 (1982), 151–182.

[KM] Kilpeläinen, T. & Malý, J., The Wiener test and potential estimates for quasi-linear elliptic equations. Acta Math., 172 (1994), 137–161.

[KoM] Kozlov, V.A. & Maz’ya, V., Boundary behavior of solutions to linear and nonlinear elliptic equations in plane convex domains. Math. Research Letters, 8 (2001), 1–5.

[L] Landkof, N.S., Foundations of Modern Potential Theory. Nauka, Moscow, 1966 (Russian). English translation: Springer-Verlag, Berlin, 1972.

[Lab] Labutin, D.A. Potential estimates for a class of fully nonlinear elliptic equations. Duke Math. J., 111 (2002), 1–49.

[LM] Lindqvist, P. & Martio, O., Two theorems of N. Wiener for solutions of quasilinear elliptic equations, Acta Math., 155 (1985), 153–171.

[LSW] Littman, W., Stampacchia, G. & Weinberger, H.F., Regular points for elliptic equations with discontinuous coefficients. Ann. Scuola Norm. Sup. Pisa, Ser. III, 17 (1963), 43–77.

[MZ] Malý, J. & Ziemer, W.P. Regularity of Solutions of Elliptic Partial Differential Equations. Mathematical Surveys and Monographs, 51, AMS, Providence, RI, 1997.

[M1] Maz’ya, V., The Dirichlet problem for elliptic equations of arbitrary order in unbounded regions. Dokl. Akad. Nauk SSSR, 150 (1963), 1221–1224 (Russian). English translation in Soviet Math., 4 (1963), 860–863.
[M2] Maz’ya, V., Unsolved problems connected with the Wiener criterion. In The Legacy of Norbert Wiener: A Centennial Symposium, Proc., Cambridge, Massachusetts, Amer. Math. Soc., 199–208, 1994.

[M3] Maz’ya, V., On Wiener’s type regularity of a boundary point for higher order elliptic equations. Nonlinear Analysis, Function Spaces and Applications VI, 119–155. Proceedings of the Spring School held in Prague, May 31–June 6, 1998. Prague, 1999.

[MN] Maz’ya, V. & Nazarov, S., The apex of a cone can be irregular in Wiener’s sense for a fourth-order elliptic equation. Mat. Zametki, 39:1 (1986), 24–28 (Russian). English translation in Math. Notes, 39 (1986), 14–16.

[MP] Maz’ya, V. & Plamenevskii, B. A., On the maximum principle for the biharmonic equation in a domain with conical points. Izv. Vyssh. Uchebn. Zaved. Mat. (1981) no. 2, 52–59 (Russian). English translation in Soviet Math. (Iz. VUZ), 25 (1981), 61–70.

[TW] Trudinger, N.S. & Wang, X.-J., On the weak continuity of elliptic operators and applications to potential theory. Amer. J. Math., 124 (2002), 369–410.

[W] Wiener, N., The Dirichlet problem. J. Math. and Phys., 3 (1924), 127–146. Reprinted in Norbert Wiener: Collected Works with Commentaries, vol. 1, 394–413, MIT Press, Cambridge, Massachusetts, 1976.