THE SYSTEMS WITH ALMOST BANACH-MEAN EQUICONTINUITY FOR ABELIAN GROUP ACTIONS

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Abstract In this paper, we present the concept of Banach-mean equicontinuity and prove that the Banach-, Weyl- and Besicovitch-mean equicontinuities of a dynamic system of Abelian group action are equivalent. Furthermore, we obtain that the topological entropy of a transitive, almost Banach-mean equicontinuous dynamical system of Abelian group action is zero. As an application of our main result, we show that the topological entropy of the Banach-mean equicontinuous system under the action of an Abelian group is zero.

Key words Abelian group action; Banach mean equicontinuous; Banach mean density; independence set

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1 Introduction

Ergodic theory and topological dynamics are two branches of the modern theory of dynamical systems. The first, though not in its broadest definition, deals with group actions on a probability measure space in a measure-preserving way; the second deals with the action of groups on a compact metric space as groups of homeomorphisms. In this paper, we discuss problems that exist under the framework of countable group action on the compact metric spaces which constitute the fundamental objects of study in the field of dynamical systems.

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It is well known that equicontinuous systems have simple dynamical behaviors. A dynamical system is called equicontinuous if the collection of maps defined by the action of the group is a uniformly equicontinuous family. Equicontinuous systems are dynamically the ‘simplest’ ones; in fact, there is a complete classification of equicontinuous minimal systems.

Mean equicontinuity has attracted interest in recent years due to its connections with the ergodic properties of measurable dynamical systems, i.e. dynamical systems equipped with an invariant probability measure. In particular, it has been shown that using a measure theoretic version of mean equicontinuity, one can characterize when a measure-preserving system has a discrete spectrum [12] and when the maximal equicontinuous factor is actually an isomorphism [5, 17].

The concept of mean equicontinuity comes in two variants: one is called Weyl-mean equicontinuity and the other Besicovitch-mean equicontinuity. The concepts of Weyl- and Besicovitch-mean equicontinuity were introduced in [17] for $\mathbb{Z}$-actions. In fact, in this case the notion of Besicovitch-mean equicontinuity is immediately seen to be equivalent to the concept of mean Lyapunov-stability, which was already introduced in 1951 by Fomin [9] in the context of $\mathbb{Z}$-actions with a discrete spectrum. Later, a first systematic treatment was carried out by Auslander [1].

Answering an open question in [24], it was proved by Li, Tu and Ye in [17] that every invariant measure of a mean equicontinuous system of integer group action has a discrete spectrum. Localizing the notion of mean equicontinuity, they introduced notions of almost mean equicontinuity and almost Weyl-mean equicontinuity. In [17] they proved that a system with the former property may have positive entropy while a system with the latter property must have zero entropy.

Concerning abelian group action, mean equicontinuity and its relation to the spectral theory of dynamical systems (in particular, to discrete spectrum) has been studied by various groups [11–13]. In the minimal case and as regards the action of the Abelian group, Fuhrmann, Gröger and Lenz [11] concluded that mean equicontinuity is equivalent to a discrete spectrum with continuous eigenfunctions.

Inspired by these previous papers, we will discuss the dynamical properties of countable Abelian group action systems. In this paper, we introduce the concept of Banach-mean equicontinuity regarding group action dynamical systems; this is broader than Weyl- and Besicovitch-mean equicontinuity, and not limited to the dynamical systems of amenable group actions. Moreover, we prove that the above three concepts are equivalent when the dynamic system is an Abelian group action. Furthermore, we introduce the concept of almost Banach-mean equicontinuity for a countable Abelian group action system and obtain the following main result:

**Theorem 1.1** Let $G$ be a countably infinite Abelian group, let $X$ be a compact metric space without isolated points, and let the action $G \actson X$ be transitive. If the action $G \actson X$ is almost Banach-mean equicontinuous, then

$$h_{\text{top}}(X,G) = 0.$$  

As an application of our main result, we prove that the topological entropy of the Banach-mean equicontinuous system under the action of an Abelian groups is zero.

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Theorem 1.2 Let $G$ be a countably infinite Abelian group, let $X$ be a compact metric space, and let $G \curvearrowright X$ be a continuous action. If $G \curvearrowright X$ is Banach-mean equicontinuous, then

$$h_{\text{top}}(X, G) = 0.$$ 

The paper is organized as follows: we begin in Section 2 by recalling some basic notations, definitions and results regarding group action systems. In Section 3 we relate the concept and basic propositions of the amenable group. Section 4 is devoted to the concepts of Banach-, Besicovitch- and Weyl-mean equicontinuity for amenable group actions. In this section we prove that the three concepts are equivalent when the dynamic system is an Abelian group action system. In Section 5 we introduce the concept of the Weyl-mean sensitivity of an amenable group action system. In this section, we obtain a dichotomy result related to Weyl-mean equicontinuity and Weyl-mean sensitivity for when a dynamical system is transitive. In Section 6, we give the proof of our main results. Finally, in Section 7, we apply our main result to prove the topological entropy of the Banach-mean equicontinuous system under the action of an Abelian group is zero.

2 Preliminaries

In this section, we recall some basic notations, definitions, and results. We refer the reader to the textbook [21] for information on group action.

By referring to an action of the group $G$ with identity $e$ on a set $X$, we mean a map $\alpha : G \times X \to X$ such that, writing the first argument as a subscript, $\alpha_s(\alpha_t(x)) = \alpha_{st}(x)$ and $\alpha_e(x) = x$ for all $x \in X$ and $s, t \in G$. Most of the time we will write the action as $G \curvearrowright X$, with the image of a pair $(s, x)$ written as $sx$. For a set $A \subseteq X$ and $s \in G$ and $x \in X$ we write

$$sA = \{sx : x \in A\}, \quad Gx = \{gx : g \in G\}.$$ 

The $G$-orbit of a point $x \in X$ is the set $Gx$.

In this paper, we call a topological space $X$ equipped with a continuous action $G \curvearrowright X$ the group action system, and denote it by $(X, G)$.

Definition 2.1 The action $G \curvearrowright X$ is (topologically) transitive if, for all nonempty open sets $U, V \subset X$, there exists an $s \in G$ such that $sU \cap V \neq \emptyset$. The point $x \in X$ is transitive if $Gx = X$. Denote by $\text{Tran}(X, G)$ the set of all transitive points.

The following proposition in [21] suggested that, when $X$ is metrizable, transitivity can be thought of as a generic version of minimality in the sense of a Baire category:

Proposition 2.2 ([21, Proposition 7.9]) Suppose that $X$ is metrizable. Then the following are equivalent:

1. the action $G \curvearrowright X$ is transitive;
2. there is a dense orbit;
3. the set of points in $X$ with dense orbit is a dense $G_\delta$.

Definition 2.3 A point $x \in X$ is recurrent if for every neighbourhood $U$ of $x$, the set $\{s \in G : sx \in U\}$ is infinite. Denote by $\text{Re}(X, G)$ the set of all recurrent points.

Proposition 2.4 ([21, Proposition 7.11]) Suppose that the action $G \curvearrowright X$ is transitive and that $X$ is metrizable and has no isolated points. Then the set of recurrent points in $X$ is
a dense \( G_d \).

**Definition 2.5** Let \( X \) be a set. A collection \( \{(A_{i,1}, \ldots, A_{i,k}) : i \in I\} \) of \( k \)-tuples of subsets of \( X \) is said to be independent if \( \bigcap_{i \in F} A_{i,\omega(i)} \neq \emptyset \) for every nonempty finite set \( F \subseteq I \) and \( \omega \in \{1, \ldots, k\}^F \).

**Definition 2.6** Let \( G \triangleleft X \) be an action and \( A = (A_1, \ldots, A_k) \) a tuple of subsets of \( X \). We say that a set \( J \subseteq G \) is an independence set for \( A \) if the collection \( \{(s^{-1}A_1, \ldots, s^{-1}A_k) : s \in J\} \) is independent.

**Definition 2.7** Let \( G \) be a group. Denote by \( \text{Fin}(G) \) the family of all non-empty finite subsets of \( G \). Let \( E \subseteq G \) be a subset of \( G \). The upper Banach density of \( E \) is defined as

\[
\text{BD}^*(E) = \inf \left\{ \sup_{g \in G} \frac{|E \cap Fg|}{|Fg|} : F \in \text{Fin}(G) \right\}.
\]

The lower Banach density of \( E \) is given by \( \text{BD}_*(E) = 1 - \text{BD}^*(G \setminus E) \).

Clearly one has \( \text{BD}_*(E) \leq \text{BD}^*(E) \). If \( \text{BD}_*(E) = \text{BD}^*(E) \), then we say that there exists the Banach density of \( E \) and denote it by \( \text{BD}(E) \).

From the above definitions it is easy to see that the Banach upper density has a right shift invariant property. For the sake of completeness, we give a proof here.

**Proposition 2.8** \( \text{BD}^*(Es) = \text{BD}^*(E) \) for any \( s \in G \) and \( E \) subset of \( G \).

**Proof** By the symmetry of the pair of sets \( Es \) and \( E \), it is sufficient to prove that \( \text{BD}^*(Es) \leq \text{BD}^*(E) \). Let \( F \) be any nonempty finite subset of \( G \). From the definition of the Banach upper density of \( Es \), one has

\[
\text{BD}^*(Es) \leq \sup_{g \in G} \frac{|Es \cap Fg|}{|Fg|} = \sup_{g \in G} \frac{|E \cap Fgs^{-1}|}{|Fgs^{-1}|} = \sup_{t \in G} \frac{|E \cap Ft|}{|Ft|}.
\]

The arbitrariness of \( F \) implies that \( \text{BD}^*(Es) \leq \text{BD}^*(E) \). Hence the proposition is obtained. \( \square \)

It is not difficult to observe the following result:

**Lemma 2.9** Let \( F, F_1, F_2 \) be subsets of \( G \) and \( s \in G \). Then,

1. if \( F_1 \) has a Banach density of one and \( F_1 \subseteq F_2 \), then so does \( F_2 \);
2. if \( F \) has a Banach density of one, then \( G \setminus F \) is a set of Banach density zero;
3. if \( F_1 \) and \( F_2 \) have a Banach density of one, then so does \( F_1 \cap F_2 \);
4. if \( F \) has a Banach density of one, then so does \( Fs \).

## 3 Amenable Group

This section is devoted to the class of amenable groups. This is a class of groups that plays an important role in many areas of mathematics, such as ergodic theory, harmonic analysis, dynamical systems, geometric group theory, probability theory and statistics.

Let \( G \) be a group. A mean for \( G \) on \( \ell^\infty(G) \) is a unital positive linear function \( \sigma : \ell^\infty(G) \to \mathbb{C} \) (unital means that \( \sigma(1) = 1 \)). The mean \( \sigma \) is left invariant if \( \sigma(sf) = \sigma(f) \) for all \( s \in G \) and \( f \in \ell^\infty(G) \), where \( (sf)(t) = f(s^{-1}t) \) for all \( t \in G \).

**Definition 3.1** The group \( G \) is said to be amenable if there is a left invariant mean on \( \ell^\infty(G) \).
The above definition of a countable amenable group \( G \) is equivalent to the existence of a sequence of finite subsets \( \{F_n\} \) of \( G \) which is asymptotically invariant, i.e.,

\[
\lim_{n \to \infty} \frac{|gF_n \Delta F_n|}{|F_n|} = 0 \quad \text{for all } g \in G,
\]

where \( gF_n = \{gf : f \in F_n\} \), \( \cdot \) denotes the cardinality of a set, and \( \Delta \) is the symmetric difference. Such a sequence is called a (left) Følner sequence.

The class of amenable groups contains, in particular, all finite groups, all Abelian groups and, more generally, all solvable groups. In this paper, we need the following theorem:

**Theorem 3.2** ([6, Theorem 4.6.1]) Every Abelian group is amenable.

**Definition 3.3** Let \( F \) and \( A \) be nonempty finite subsets of \( G \). We say that \( A \) is \((F, \varepsilon)\)-invariant if

\[
|s \in A : Fs \subseteq A| \geq (1 - \varepsilon)|A|.
\]

**Definition 3.4** Let \( f \) be a real-valued function on the set of all finite subsets of \( G \). We say that \( f(A) \) converges to a limit \( L \) as \( A \) becomes more and more invariant if, for every \( \varepsilon > 0 \), there are a finite set \( F \subseteq G \) and a \( \delta > 0 \) such that

\[
|f(A) - L| < \varepsilon \quad \text{for every nonempty } (F, \delta)-\text{invariant finite set } A \subseteq G.
\]

**Theorem 3.5** ([21, Theorem 4.38]) Suppose that \( G \) is amenable. Let \( \phi \) be a \([0, \infty)\)-valued function on the set of all finite subsets of \( G \) such that

1. \( \phi(As) = \phi(A) \) for all finite \( A \subseteq G \) and \( s \in G \);
2. \( \phi(A \cup B) \leq \phi(A) + \phi(B) \) for all finite \( A, B \subseteq G \) (subadditivity).

Then \( \phi(A)/|A| \) converges to a limit as \( A \) becomes more and more invariant.

Let \( G \rhd X \) be an action and \( A = (A_1, \ldots, A_k) \) a tuple of subsets of \( X \). It is readily seen that the function

\[
\varphi_A(F) = \max\{|F \cap J| : J \text{ is an independence set for } A\}
\]

on the collection of nonempty finite subsets of \( G \) satisfies the two conditions in Theorem 3.5, so that the quantity \( \varphi_A(F)/|F| \) converges as \( F \) becomes more and more invariant (Definition 3.4), and the limit is equal to \( \inf_F \varphi_A(F)/|F| \) where \( F \) ranges over all nonempty finite subsets of \( G \).

**Definition 3.6** For a finite tuple \( A = (A_1, \ldots, A_k) \) of subsets of \( X \), we define the independence density \( I(A) \) of \( A \) to be the above limit.

**Proposition 3.7** ([21, Proposition 12.7]) Let \( A = (A_1, \ldots, A_k) \) be a tuple of subsets of \( X \) and let \( d > 0 \). Then the following are equivalent:

1. \( I(A) \geq d \);
2. there are a Følner sequence \( \{F_n\} \) and an independence set \( J \) for \( A \) such that

\[
\lim_{n \to \infty} |F_n \cap J|/|F_n| \geq d.
\]

In what follows, we recall some notions which were introduced in [20].

Let \( E \subseteq G \). The upper asymptotic density of \( E \) with respect to a Følner sequence \( F = \{F_n\}_{n \in \mathbb{N}} \), denoted by \( \overline{d}_F(E) \), is defined by

\[
\overline{d}_F(E) = \limsup_{n \to \infty} \frac{|E \cap F_n|}{|F_n|}.
\]
Similarly, the lower asymptotic density of $E$ with respect to a Følner sequence $\mathcal{F} = \{F_n\}_{n \in \mathbb{N}}$, denoted by $d_\mathcal{F}(E)$, is defined by

$$d_\mathcal{F}(E) = \liminf_{n \to \infty} \frac{|E \cap F_n|}{|F_n|}.$$ 

One may say that $E$ has an asymptotic density $d_\mathcal{F}(E)$ of $E$ with respect to a Følner sequence $\mathcal{F} = \{F_n\}_{n \in \mathbb{N}}$ if $d_\mathcal{F}(E) = d_\mathcal{F}(E)$, where $d_\mathcal{F}(E)$ is equal to this common value.

Let $\{F_n\}_{n \in \mathbb{N}}$ be a Følner sequence of the amenable group $G$ and $E \subseteq G$. For the upper Banach density of $E$ please refer to Definition 2.7. Meanwhile, we have the following formula for the properties of upper Banach density (see [8, Lemma 2.9]):

$$\text{BD}^*(E) = \lim_{n \to \infty} \sup_{g \in G} \frac{|E \cap F_n g|}{|F_n g|}.$$ 

As for the relationship between upper Banach density and upper asymptotic density, we have the following formula (see [4, Lemma 3.3]):

$$\text{BD}^*(E) = \sup_{\mathcal{F}} \limsup_{n \to \infty} \frac{|E \cap F_n|}{|F_n|}. \quad (3.1)$$

Here the supremum is taken over all Følner sequences $\mathcal{F} = \{F_n\}_{n \in \mathbb{N}}$ of $G$.

Throughout this paper, $G$ is a countable amenable group and $G \curvearrowright X$ is a continuous action on a compact metric space. We write $\Delta_k(X)$ for the diagonal $\{(x,\ldots,x) : x \in X\}$ in $X^k$.

**Definition 3.8** We call a tuple $x = (x_1,\ldots,x_k) \in X^k$ an IE-tuple (or IE-pair if $k = 2$) if, for every product neighbourhood $U_1 \times \ldots \times U_k$ of $x$, the tuple $(U_1\ldots U_k)$ has positive independence density. We denote the set of IE-tuples of length $k$ by $\text{IE}_k(X,G)$.

In this paper, we need the following theorem:

**Theorem 3.9** ([21, Theorem 12.19]) $\text{IE}_2(X,G) \setminus \Delta_2(X)$ is nonempty if and only if $h_{\text{top}}(X,G) > 0$.

4 Besicovitch-, Weyl- and Banach-Mean Equicontinuity

In a 2005 study of a dynamical system with bounded complexity (defined by using the mean metrics), Huang, Li, Thouvenot, Xu and Ye [18] introduced a notion called “equicontinuity in the mean”. In 2015, Li, Tu and Ye [17] showed that for a minimal system, the notions of mean equicontinuity and equicontinuity in the mean are equivalent for $\mathbb{Z}$-actions. The concepts of Besicovitch- and Weyl-mean equicontinuity were introduce, in [17] for $\mathbb{Z}$-actions, and in [11] for amenable actions.

In this paper, we give a notion of Banach-mean equicontinuity on a dynamical system for a group action. For countable amenable group action systems, we show that two concepts, Weyl- and Banach-mean equicontinuity are equivalent. By the results of [11], we also know that the concepts of Besicovitch-, Weyl- and Banach-mean equicontinuity are the same for Abelian group action systems.

**Definition 4.1** Let $G$ be a discrete group and let $\text{Fin}(G)$ be the family of all non-empty finite subsets of $G$. Let $X$ be a compact metric space with metric $d$. For $x, y \in X$, we denote

$$\overline{d}(x, y) = \inf_{F \in \text{Fin}(G)} \sup_{g \in G} \frac{1}{|F|} \sum_{t \in F_g} d(tx, ty).$$

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We say that the action $G \actson X$ is Banach-mean equicontinuous or simply $B$-mean equicontinuous if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\overline{D}(x,y) < \varepsilon$ whenever $d(x,y) < \delta$ for $x,y \in X$.

A point $x \in X$ is called a Banach-mean equicontinuous point if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $y \in B(x,\delta)$,

$$\overline{D}(x,y) < \varepsilon.$$ 

We say that the action $G \actson X$ is almost Banach-mean equicontinuous if the group action system $(X,G)$ has at least one Banach-mean equicontinuous point.

By the compactness of $X$, it is easy to see that the action $G \actson X$ is Banach-mean equicontinuous if and only if every point in $X$ is a Banach-mean equicontinuous point.

In [11], Fuhrmann, Gröger and Lenz introduced the concepts Besicovitch- and Weyl-mean equicontinuity for amenable group action systems.

**Definition 4.2** Let $G$ be an amenable group and $\mathcal{F} = \{F_n\}_{n \in \mathbb{N}}$ be a Følner sequence of $G$. We say that the action $G \actson X$ is Besicovitch-$\mathcal{F}$-mean equicontinuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$D_{\mathcal{F}}(x,y) := \limsup_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} d(gx,gy) < \varepsilon$$

for all $x,y \in X$ with $d(x,y) < \delta$. The dependence on the Følner sequence immediately motivates the next definition. We say that the action $G \actson X$ is Weyl-mean equicontinuous if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x,y \in X$ with $d(x,y) < \delta$, we have

$$D(x,y) := \sup \{ D_{\mathcal{F}}(x,y) : \mathcal{F} \text{ is a Følner sequence} \} < \varepsilon. \quad (4.1)$$

A point $x \in X$ is called a Weyl-mean equicontinuous point if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $y \in B(x,\delta)$,

$$D(x,y) < \varepsilon.$$ 

We say that the action $G \actson X$ is almost Weyl-mean equicontinuous if the group action system $(X,G)$ has at least one Weyl-mean equicontinuous point.

Before we can proceed, a few comments are in order. First, note that $\overline{D}$, $D_{\mathcal{F}}$ and $D$ are pseudometric. Moreover, as is not hard to see, $D$ is $G$-invariant; that is, $D(gx,gy) = D(x,y)$ for all $x,y \in X$ and $g \in G$.

In what follows, for the amenable group action system, we will see that the Banach pseudometric $\overline{D}(\cdot,\cdot)$ is equal to the Weyl pseudometric $D(\cdot,\cdot)$.

**Theorem 4.3** Let $G$ be a countable amenable group, let $X$ be a compact metric space and let $G \actson X$ be a group action. Then

$$\overline{D}(x,y) = D(x,y) \quad \text{for any pair } x,y \in X.$$

**Proof** Let $x,y \in X$. First, we show that $D(x,y) \leq \overline{D}(x,y)$.

Let $\varepsilon > 0$. From the definition of $\overline{D}(x,y)$, there is a nonempty finite subset $F \in \text{Fin}(G)$ such that

$$\sup_{g \in G} \frac{1}{|F|} \sum_{t \in Fg} d(tx,ty) < \overline{D}(x,y) + \varepsilon.$$
Let \( \{F_n\}_{n \in \mathbb{N}} \) be a Følner sequence of \( G \). In what follows we will show that

\[
\limsup_{n \to \infty} \frac{1}{|F_n|} \sum_{t \in F_n} d(tx, ty) \leq \overline{D}(x, y) + \varepsilon.
\]

Take \( g \in G \). For every \( h \in F_n \), one has

\[
\frac{1}{|F|} \sum_{s \in F_{hg}} d(sx, sy) \leq \sup_{g' \in G} \frac{1}{|F|} \sum_{s \in F_{g'}} d(sx, sy) < \overline{D}(x, y) + \varepsilon.
\]

Thus it follows that

\[
\sum_{h \in F_n} \sum_{s \in F_{hg}} d(sx, sy) < |F_n||F|\overline{D}(x, y) + \varepsilon.
\]

We denote that \( \alpha(h, t) = d((th)g, (th)g) \) for \( h \in F_n \) and \( t \in F \). Then the above inequality can be re-written as

\[
\sum_{h \in F_n} \sum_{t \in F} \alpha(h, t) < |F_n||F|\overline{D}(x, y) + \varepsilon.
\]

It is clear that there is \( t' \in F \) such that

\[
\sum_{h \in F_n} \alpha(h, t') = \min \left\{ \sum_{h \in F_n} \alpha(h, t) : t \in F \right\}.
\]

Therefore, we get

\[
(\overline{D}(x, y) + \varepsilon)|F_n||F| > \sum_{h \in F_n} \sum_{t \in F} \alpha(h, t) = \sum_{t \in F} \sum_{h \in F_n} \alpha(h, t) \geq |F| \sum_{h \in F_n} \alpha(h, t'),
\]

which implies that

\[
\sum_{s \in F_{tg}} d(sx, sy) = \sum_{h \in F_n} \alpha(h, t') < |F_n|\overline{D}(x, y) + \varepsilon.
\]

Note that

\[
\sum_{s \in F_{tg}} d(sx, sy) \leq \sum_{s \in F_{t'g}} d(sx, sy) + \sum_{s \in F_{t'g} \triangle F_{tg}} d(sx, sy)
\]

\[
< |F_n|\overline{D}(x, y) + \varepsilon + |t'|F_n \triangle F_{tg} \cdot \text{diam}(X)
\]

\[
= |F_n|\overline{D}(x, y) + \varepsilon + |t'|F_n \triangle F_{tg} \cdot \text{diam}(X),
\]

where \( \text{diam}(X) \) is the diameter of the compact metric space \((X, d)\). Since \( F_n \) is a Følner sequence, we have that

\[
\limsup_{n \to \infty} \frac{1}{|F_n|} \sum_{s \in F_{tg}} d(sx, sy) \leq \overline{D}(x, y) + \varepsilon + \limsup_{n \to \infty} \frac{|t'|F_n \triangle F_{tg}|}{|F_n|} \cdot \text{diam}(X) = \overline{D}(x, y) + \varepsilon.
\]

From the arbitrariness of the Følner sequence \( \{F_n\} \), we get

\[
\sup_{\{F_n\}} \limsup_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} d(gx, gy) \leq \overline{D}(x, y) + \varepsilon,
\]

where the supremum is taken over all Følner sequences of \( G \); that is

\[
D(x, y) \leq \overline{D}(x, y) + \varepsilon.
\]

The arbitrariness of \( \varepsilon \) implies that

\[
D(x, y) \leq \overline{D}(x, y).
\]
Suppose that \( D(x_0, y_0) < \overline{D}(x_0, y_0) \) for some \( x_0, y_0 \in X \). In what follows, we will obtain a contradiction.

We choose two real numbers \( \eta_1, \eta_2 \in \mathbb{R} \), such that
\[
D(x_0, y_0) < \eta_1 < \eta_2 < \overline{D}(x_0, y_0).
\] (4.2)

Let \( \{F_n\}_{n \in \mathbb{N}} \) be a Følner sequence of the amenable group \( G \). Note that \( F_n \) is a nonempty finite subset of \( G \) for each \( n \in \mathbb{N} \). From the definition of \( \overline{D}(x_0, y_0) \), we have
\[
\sup_{g \in G} \frac{1}{|F_n|} \sum_{t \in F_n g} d(tx_0, ty_0) \geq \overline{D}(x_0, y_0) > \eta_2.
\]
Thus, for each \( n \in \mathbb{N} \), there exists \( g_n \in G \) such that
\[
\frac{1}{|F_n g_n|} \sum_{t \in F_n g_n} d(tx_0, ty_0) > \eta_2.
\]
Set \( H_n = F_n g_n \). Since \( F' := \{H_n\}_{n \in \mathbb{N}} \) is also a (left) Følner sequence of \( G \), we get that
\[
\eta_1 > D(x_0, y_0) \geq D_{F'}(x_0, y_0) = \limsup_{n \to \infty} \frac{1}{|H_n|} \sum_{g \in H_n} d(gx_0, gy_0) \geq \eta_2.
\]
This is a contradiction. Hence the theorem is proved. \( \square \)

From the above Theorems, it follows that the concepts of Banach- and Weyl-mean equicontinuity are equivalent for the amenable group action system.

**Corollary 4.4** Let \( G \) be a countable amenable group, let \( X \) be a compact metric space and let \( G \curvearrowright X \) be a group action. Then \( G \curvearrowright X \) is Banach-mean equicontinuous if and only if \( G \curvearrowright X \) is Weyl-mean equicontinuous.

According to the theorem on the independence of Følner sequences for an amenable group in [11, Theorem 1.3, p. 6], we can get the following result:

**Theorem 4.5** Let \( G \) be a countable Abelian group and \( G \curvearrowright X \) be a dynamical system. Then the following three statements are equivalent:

1. \( G \curvearrowright X \) is Banach-mean equicontinuous;
2. \( G \curvearrowright X \) is Weyl-mean equicontinuous;
3. \( G \curvearrowright X \) is Besicovitch-\( F \)-mean equicontinuous for some left Følner sequence \( F \).

Let \( G \) be a countable amenable group and let \( G \curvearrowright X \) be a group action. Let \( E \) denote the set of all Weyl-mean equicontinuous points of the group action system \( (X, G) \). For every \( \varepsilon > 0 \), let
\[
E_\varepsilon = \{ x \in X : \exists \delta > 0, \forall y, z \in B(x, \delta), D(y, z) < \varepsilon \}.
\] (4.3)

For the Weyl-mean equicontinuous points we have the following proposition:

**Proposition 4.6** Let \( G \) be a countable amenable group, let \( G \curvearrowright X \) be a group action and let \( \varepsilon > 0 \). Then \( E_\varepsilon \) is open and \( s E_{\varepsilon/2} \subseteq E_\varepsilon \) for all \( s \in G \). Moreover, \( E = \bigcap_{m=1}^{\infty} E_{\varepsilon m} \) is a \( G_\delta \) subset of \( X \).

**Proof** Let \( x \in E_\varepsilon \). Choose \( \delta > 0 \) satisfying the condition from the definition of \( E_\varepsilon \) for \( x \). Fix \( y \in B(x, \delta/2) \). If \( z, w \in B(y, \delta/2) \), then \( z, w \in B(x, \delta) \), so \( D(z, w) < \varepsilon \). This shows that \( B(x, \delta/2) \subseteq E_\varepsilon \), and hence, \( E_\varepsilon \) is open.

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Let $s \in G$. Suppose that $x \in s \mathcal{E}_{\varepsilon/2}$, so $s^{-1}x \in \mathcal{E}_{\varepsilon/2}$. Choose $\delta > 0$ satisfying the condition from the definition of $\mathcal{E}_{\varepsilon/2}$ for $s^{-1}x$; that is, for all $y, z \in B(s^{-1}x, \delta)$, one has $D(y, z) < \varepsilon/2$. By the continuity of the map $s^{-1}$, there exists $\eta > 0$ satisfying the condition from the definition of $\mathcal{E}_{\varepsilon/2}$ for $s^{-1}x$; that is, for all $y, z \in B(s^{-1}x, \delta)$, one has $D(y, z) < \varepsilon/2$.

Let $u, v \in B(x, \eta)$. Then $s^{-1}u, s^{-1}v \in B(s^{-1}x, \delta)$.

Let $F = \{F_n\}_{n \in \mathbb{N}}$ be any Følner sequence of $G$. Note that $F_s = \{F_n s\}_{n \in \mathbb{N}}$ is also a (left) Følner sequence of $G$. Thus, we have

$$D_F(u, v) = \limsup_{n \to \infty} \frac{1}{|F_n|} \sum_{g \in F_n} d(gss^{-1}u, gss^{-1}v)$$

$$= \limsup_{n \to \infty} \frac{1}{|F_n s|} \sum_{t \in F_n s} d(t(s^{-1}u), t(s^{-1}v))$$

$$= D_{Fs}(s^{-1}u, s^{-1}v) \leq D(s^{-1}u, s^{-1}v) < \varepsilon/2.$$ 

The arbitrariness of the Følner sequence $F$ indicates that $D(u, v) \leq \varepsilon/2 < \varepsilon$, which implies that $x \in \mathcal{E}_\varepsilon$. Hence we get $s \mathcal{E}_{\varepsilon/2} \subseteq \mathcal{E}_\varepsilon$.

If $x \in X$ belongs to all $\mathcal{E}_{m \varepsilon}$, then clearly, $x \in \mathcal{E}$.

Conversely, if $x \in \mathcal{E}$ and $m \geq 1$, there exists $\delta > 0$ such that $D(x, y) < 1/2m$ for all $y \in B(x, \delta)$. If $y, z \in B(x, \delta)$, then

$$D(y, z) \leq D(y, x) + D(x, z) < \frac{1}{m}.$$ 

Thus $x \in \mathcal{E}_{\frac{1}{m}}$. Therefore we get $\mathcal{E} = \bigcap_{m=1}^{\infty} \mathcal{E}_{\frac{1}{m}}$. Hence, the proof is completed. 

5 Weyl-Mean Sensitivity

Let $X$ be a compact metric space. Recall that a subset of $X$ is called residual if it contains the intersection of a countable collection of dense open sets. By the Baire category theorem, a residual set is also dense in $X$.

Definition 5.1 Let $G \acts X$ be a continuous action and let $x \in X$ be a point. We say that the point $x$ is a Weyl-mean sensitive point if there exists $\delta > 0$ such that, for every $\varepsilon > 0$, there is $y \in B(x, \varepsilon)$ satisfying

$$D(x, y) > \delta.$$ 

For the definition of the function $D(\cdot, \cdot)$, please refer to (4.1).

We say that the action $G \acts X$ is Weyl-mean sensitive if every point $x \in X$ is a Weyl-mean sensitive point.

Proposition 5.2 Let $G$ be a countable amenable group and let $X$ be a compact metric space. Let $G \acts X$ be a transitive action. Then,

1. The set of Weyl-mean equicontinuous points is either empty or residual. If, in addition, the action $G \acts X$ is almost Weyl-mean equicontinuous, then every transitive point is Weyl-mean equicontinuous.

2. If the action $G \acts X$ is minimal and almost Weyl-mean equicontinuous, then it is Weyl-mean equicontinuous.
Proof If $\mathcal{E}_\varepsilon$ is empty for some $\varepsilon > 0$, then the set of Weyl-mean equicontinuous points $\mathcal{E}$ is empty.

Now, we assume that every $\mathcal{E}_\varepsilon$ is nonempty. Then, for each $\varepsilon > 0$, $\mathcal{E}_\varepsilon$ is a nonempty open subset of $X$. In what follows we show that every $\mathcal{E}_\varepsilon$ is dense. Let $U$ be any nonempty open subset of $X$. By the transitivity of the action $G \curvearrowright X$, noting that $\mathcal{E}_{\varepsilon/2}$ is a nonempty open subset of $X$ and considering Proposition 4.6, there exists $s \in G$ such that $\emptyset \neq U \cap s \mathcal{E}_{\varepsilon/2} \subseteq U \cap \mathcal{E}_\varepsilon$.

Hence $\mathcal{E}$ is either empty or residual, by the Baire Category Theorem. If $\mathcal{E}$ is residual, then every $\mathcal{E}_\varepsilon$ is open and dense. Let $x \in X$ be a transitive point and $\varepsilon > 0$. Then there exists some element $s \in G$ such that $s x \in \mathcal{E}_{\varepsilon/2}$, and, by Proposition 4.6, $x \in s^{-1} \mathcal{E}_{\varepsilon/2} \subseteq \mathcal{E}_\varepsilon$. Thus $x \in \mathcal{E}$. □

Proposition 5.3 Let $G$ be a countable amenable group and let $X$ be a compact metric space. Let $G \curvearrowright X$ be a continuous action. If there exists $\delta > 0$ such that for every non-empty open subset $U$ of $X$ there are $x,y \in U$ satisfying $D(x,y) > \delta$, then the group action $G \curvearrowright X$ is Weyl-mean sensitive.

Proof Suppose that there exists $\delta > 0$ such that for any non-empty open subset $U$ of $X$, there are $u,v \in U$ satisfying $D(u,v) > \delta$.

Let $x \in X$ and $\varepsilon > 0$. Then $B(x,\varepsilon) \neq \emptyset$ and $B(x,\varepsilon)$ is open subset of $X$. Then there exist $y,z \in B(x,\varepsilon) \subseteq X$ satisfying $D(y,z) > \delta$.

Thus we have either that $D(x,y) > \frac{\delta}{2}$ or $D(x,z) > \frac{\delta}{2}$. This implies that the group action $G \curvearrowright X$ is Weyl-mean sensitive. □

Proposition 5.4 Let $G$ be a countable amenable group and let $X$ be a compact metric space. Let the action $G \curvearrowright X$ be transitive. If there exists a transitive point which is a Weyl-mean sensitive point, then the action $G \curvearrowright X$ is Weyl-mean sensitive.

Proof Let $x \in X$ be a Weyl-mean sensitivity point. Thus there exists $\delta > 0$ such that, for every $\varepsilon > 0$, there is $y \in B(x,\varepsilon)$ satisfying $D(x,y) > \delta$.

Take a non-empty open subset $U$ of $X$. Since $x$ is a transitive point, there exists $s \in G$ such that $sx \in U$; that is, $x \in s^{-1}U$. Furthermore, as $s^{-1}U$ is open, there is $\epsilon > 0$ such that $B(x,\epsilon) \subseteq s^{-1}U$; that is, $sB(x,\epsilon) \subseteq U$. By the assumption that $x$ is a Weyl-mean sensitivity point, there exists $y \in B(x,\epsilon)$ satisfying that $D(x,y) > \delta$. By the definition of $D(x,y)$, there is a (left) Følner sequence $\mathcal{F} = \{F_n\}_{n \in \mathbb{N}}$ of $G$ such that $D_{\mathcal{F}}(x,y) > \delta$.

Let $u = sx$, $v = sy$. Noting that $\mathcal{F}s^{-1} = \{F_n s^{-1}\}_{n \in \mathbb{N}}$ is also a (left) Følner sequence and $u,v \in U$, then

$$D(u,v) \geq D_{\mathcal{F}s^{-1}}(u,v) = D_{\mathcal{F}s^{-1}}(sx,sv) = D_{\mathcal{F}}(x,y) > \delta.$$ Thus the group action $G \curvearrowright X$ is Weyl-mean sensitive, by Proposition 5.3. □

Theorem 5.5 Let $G$ be a countable amenable group and let $X$ be a compact metric space. If the action $G \curvearrowright X$ is transitive, then the action $G \curvearrowright X$ is either almost Weyl-mean equicontinuous or Weyl-mean sensitive.

Proof Let $x \in X$ be a transitive point. If $x$ is a Weyl-mean sensitivity point, then the action $G \curvearrowright X$ is Weyl-mean sensitive, by Proposition 5.4. If $x$ is not a Weyl-mean sensitive point, then it is a Weyl-mean equicontinuous point. Thus the action $G \curvearrowright X$ is almost Weyl-mean equicontinuous. □
Corollary 5.6 Let $G$ be a countable amenable group and let $X$ be a compact metric space. Let $G \acts X$ be a minimal system. Then the action $G \acts X$ is either Weyl-mean sensitive or Weyl-mean equicontinuous.

6 The Proof of Main Theorem

To prove our main theorem we need some preparation. For the following result, please see [17, Proposition 5.8]:

Proposition 6.1 Let $(X, \beta, \mu)$ be a probability space, and let $\{E_i\}_{i=1}^{\infty}$ be a sequence of measurable sets with $\mu(E_i) \geq a > 0$ for some constant $a$ and any $i \in \mathbb{N}$. Then, for any $k \geq 1$ and $\epsilon > 0$, there is $N = N(a, k, \epsilon)$ such that, for any tuple $\{s_1 < s_2 < \ldots < s_n\}$ with $n \geq N$, there exist $1 \leq t_1 < \ldots < t_k \leq n$ with

$$\mu(E_{s_{t_1}} \cap E_{s_{t_2}} \cap \ldots \cap E_{s_{t_k}}) \geq a^k - \epsilon.$$ 

Let $(X, d)$ be a compact metric space, with Borel $\sigma$-algebra $\mathcal{B}$. Denote by $\mathcal{M}(X)$ the space of Borel probability measures on $X$. Our main interest is the weak-* topology of space $\mathcal{M}(X)$. This is standard (see Pathasarathy [22]).

Theorem 6.2 Let $X$ be compact metric space and let $\{\mu_n\}$ be a sequence of probability measures in $\mathcal{M}(X)$. Let $\mu \in \mathcal{M}(X)$. Then the following statements are equivalent:

1. $\{\mu_n\}$ converges to $\mu$ with weak-* topology in $\mathcal{M}(X)$;
2. $\lim_{n \to \infty} \int f d\mu_n = \int f d\mu$ for all $f \in C(X)$;
3. $\limsup_{n \to \infty} \mu_n(C) \leq \mu(C)$ for every closed set $C$;
4. $\liminf_{n \to \infty} \mu_n(G) \geq \mu(G)$ for every open set $G$;
5. $\lim_{n \to \infty} \mu_n(A) = \mu(A)$ for every Borel set $A$ whose boundary has $\mu$-measure 0.

In order to obtain our results, we need the following fundamental fact:

Fact 6.3 Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers. Suppose that $\lim_{n \to \infty} b_n$ exists and that $\limsup_{n \to \infty} a_n$ is finite. Then

$$\limsup_{n \to \infty} (a_n + b_n) = \limsup_{n \to \infty} a_n + \lim_{n \to \infty} b_n.$$ 

The next result is the well-known Furstenberg corresponding principle [10] of the amenable group version.

Proposition 6.4 Let $G$ be a countable amenable group and let $F$ be a subset of $G$ with $\text{BD}^*(F) > 0$. Then for any $k \geq 1$ and $\epsilon > 0$, there is $N = N(\text{BD}^*(F), k, \epsilon)$ such that, for any $n \geq N$ and any tuple $\{s_1, s_2, \ldots, s_n\} \subseteq G$, there exist $1 \leq t_1 < t_2 < \ldots < t_k \leq n$ such that

$$\text{BD}^*(s_{t_1}^{-1} F \cap s_{t_2}^{-1} F \cap \ldots \cap s_{t_k}^{-1} F) \geq (\text{BD}^*(F))^k - \epsilon.$$ 

Proof Let $K = \{0, 1\}$ be a finite alphabet. We define the map $\Sigma : G \times K^G \to K^G$ by $\Sigma(g, x) := x \circ R_g$ for all $g \in G$ and $x \in K^G$. Here $R_g : G \to G$ is defined by $R_g(h) := hg$ for all $h \in G$. Let $s \in G$ and $x \in K^G$. Thus $sx(g) = x(gs)$ for all $g \in G$.

Take $\xi \in K^G$ satisfying $\xi(s) = 0$ for all $s \in F$ and $\xi(s) = 1$ for all $s \in G \setminus F$. Denote by

$$X = \{s\xi : s \in G\}$$ 

the orbit closure with respect to $K^G$. 

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It is clear that $X$ is a compact metric space. Meanwhile, it follows that the set \( \{ s \in G : \xi(s) = 0 \} = F \) has positive upper Banach density.

Let \( \{ H_n \} \) be a (left) Følner sequence for $G$. By a formula of upper Banach density (see [8, Lemma 2.9]), we have that

$$\lim_{n \to \infty} \sup_{g \in G} \frac{|F \cap H_n g|}{|H_n g|} = BD^*(F) > 0.$$

Let $\kappa > 0$. For the above limit equation, there is $N_1 \in \mathbb{N}$ such that, for every $n \geq N_1$, one has that

$$BD^*(F) - \frac{\kappa}{2} < \sup_{g \in G} \frac{|F \cap H_n g|}{|H_n g|} < BD^*(F) + \frac{\kappa}{2}.$$

Thus, for each $n \geq N_1$, there is $g_n \in G$ such that

$$BD^*(F) - \kappa < \sup_{g \in G} \frac{|F \cap H_n g|}{|H_n g|} - \frac{\kappa}{2} < \frac{|F \cap H_n g_n|}{|H_n g_n|}.$$

This means that

$$\lim_{n \to \infty} \frac{|F \cap H_n g_n|}{|H_n g_n|} = BD^*(F) > 0.$$

Set $F_n = H_n g_n$ for all $n \in \mathbb{N}$. Then \( \{ F_n \}_{n \in \mathbb{N}} \) is a (left) Følner sequence with

$$\lim_{n \to \infty} \frac{|F_n \cap F|}{|F_n|} = BD^*(F). \quad (6.1)$$

Define a sequence of probability measures in $\mathcal{M}(X)$ as

$$\mu_n := \frac{1}{|F_n|} \sum_{s \in F_n} \delta_{s \xi},$$

where $\delta_{s \xi}$ is Dirac measure at the point $s \xi$ in $X$.

Since $\mathcal{M}(X)$ is a compact metrizable space (see Theorem 6.3 in [22], p43), there exists a subsequence \( \{ \mu_{n_l} \}_{l \in \mathbb{N}} \) that converges to a probability measure $\nu$ with weak-* topology in $\mathcal{M}(X)$; that is,

$$\mu_{n_l} \overset{\text{weak-*}}{\longrightarrow} \nu.$$

In what follows we will show that $\nu$ is a $G$-invariant probability measure; that is, $\nu = g \nu$ for each $g \in G$.

Let $g \in G$. From (3) of Theorem 6.2, it is easy to check that

$$g \mu_{n_l} \overset{\text{weak-*}}{\longrightarrow} g \nu. \quad (6.2)$$

**Claim**  For any Borel set $B$ of $X$, one has that

$$\lim_{l \to \infty} (g \mu_{n_l}(B) - \mu_{n_l}(B)) = 0. \quad (6.3)$$

In fact, since \( \{ F_{n_l} \}_{l \in \mathbb{N}} \) is a Følner sequence of $G$, we have that

$$|g \mu_{n_l}(B) - \mu_{n_l}(B)| = |\mu_{n_l}(g^{-1}B) - \mu_{n_l}(B)|$$

$$= \frac{1}{|F_{n_l}|} \left( \sum_{s \in F_{n_l}} \delta_{s \xi}(g^{-1}B) - \sum_{s \in F_{n_l}} \delta_{s \xi}(B) \right).$$
\[
\frac{1}{|F_n|} \left( \sum_{s \in F_n} \delta_{(s \xi)}(B) - \sum_{s \in F_n} \delta_s(\xi)(B) \right) \\
= \frac{1}{|F_n|} \left( \sum_{s \in F_n} \delta_{s \xi}(B) - \sum_{s \in F_n} \delta_s(\xi)(B) \right) \\
\leq \frac{|gF_n \Delta F_n|}{|F_n|} \to 0 \quad \text{as} \quad l \to \infty.
\]

Hence the claim is obtained.

Now we will prove that \( g \nu = \nu \). Let \( C \) be any closed subset of \( X \). From (6.3), Fact 6.3 and (3) of Theorem 6.2, we have that

\[
\lim_{l \to \infty} \sup g \mu_{n_l}(C) = \lim_{l \to \infty} \sup g \mu_{n_l}(C) + \lim_{l \to \infty} \left( \mu_{n_l}(C) - g \mu_{n_l}(C) \right) = \lim_{l \to \infty} \sup \mu_{n_l}(C) \leq \nu(C).
\]

Applying (3) of Theorem 6.2 again, we get \( g \mu_{n_l} \xrightarrow{\text{weak-*}} \nu \). Combining this with (6.2), we have that \( g \nu = \nu \). Thus \( \nu \) is a G-invariant measure.

Denote by \( e \) the unit element of the group \( G \). We define \( A(0) = \{ \eta \in K^G : \eta(e) = 0 \} \cap X \).

Since \( \{ \eta \in K^G : \eta(e) = 0 \} \) is a clopen subset of \( K^G \), it follows that \( A(0) \) is a clopen subset of \( X \). Therefore the boundary of \( A(0) \) is an empty set; that is, \( \partial(A(0)) = \emptyset \). From (5) of Theorem 6.2, we have that

\[
\nu(A(0)) = \lim_{l \to \infty} \mu_{n_l}(A(0)) = \lim_{l \to \infty} \frac{1}{|F_n|} \sum_{s \in F_n} \delta_s(\xi)(A(0)).
\]

Note that \( s \xi \in A(0) \iff \xi(s) = 0 \iff s \in F, \) so, by (6.1), we get that

\[
\nu(A(0)) = \lim_{l \to \infty} \frac{1}{|F_n|} \sum_{s \in F_n} \delta_s(\xi)(A(0)) = \lim_{l \to \infty} \frac{|F \cap F_n|}{|F_n|} = \text{BD}^*(F).
\]  

(6.4)

Since \( G \) is countable, we list \( G \) as \( G = \{ g_i \}_{i=1}^{\infty} \). Denote that \( E_i = g_i^{-1}A(0) \) for each \( i \in \mathbb{N} \). Owing to \( \nu \) being \( G \)-invariant, we deduce that \( \nu(E_i) = \nu(g_i^{-1}A(0)) = \nu(A(0)) = \text{BD}^*(F) \) for each \( i \in \mathbb{N} \).

Let \( k \geq 1 \) and \( \epsilon > 0 \). From Proposition 6.1, there is \( N = N(\text{BD}^*(F), k, \epsilon) \) such that, for any \( n \geq N \) and any tuple \( \{ s_1, s_2, \ldots, s_n \} \subseteq G \), there exist \( \{ 1 \leq t_1 < t_2 < \cdots < t_k \leq n \} \) satisfying

\[
\nu(s_{t_1}^{-1}A(0) \cap s_{t_2}^{-1}A(0) \cap \ldots \cap s_{t_k}^{-1}A(0)) \geq \nu(A(0))^k - \epsilon.
\]  

(6.5)

Set \( B = s_{t_1}^{-1}A(0) \cap s_{t_2}^{-1}A(0) \cap \ldots \cap s_{t_k}^{-1}A(0) \). It is easy to see that

\[
s \xi \in s_{t_i}^{-1}A(0) \iff (s_i, s) \in A(0) \iff (s_i, s) \xi(e) = 0 \iff \xi(s_i, s) = 0 \iff s \in s_{t_i}^{-1}F.
\]

Since \( A(0) \) is a clopen subset of \( X \), the set \( B \) is also clopen in \( X \). Therefore, the boundary of \( B \) is an empty set; that is, \( \partial(B) = \emptyset \). Thus, by equation (3.1) and (5) of Theorem 6.2, we have

\[
\text{BD}^* \left( s_{t_1}^{-1}F \cap s_{t_2}^{-1}F \cap \ldots \cap s_{t_k}^{-1}F \right) \geq \lim_{l \to \infty} \frac{|F_{n_l} \cap (s_{t_1}^{-1}F \cap s_{t_2}^{-1}F \cap \ldots \cap s_{t_k}^{-1}F)|}{|F_{n_l}|} \\
= \lim_{l \to \infty} \frac{1}{|F_{n_l}|} \sum_{s \in F_{n_l}} \delta_s(\xi)(B) = \lim_{l \to \infty} \mu_{n_l}(B) \\
= \nu(B).
\]  

(6.6)
Applying (6.4), (6.5) and (6.6), we get
\[ \text{BD}^* \left( s_{t_1}^{-1}F \cap s_{t_2}^{-1}F \cap \ldots \cap s_{t_k}^{-1}F \right) \geq \text{BD}^*(F)^k - \epsilon. \]
Hence the proposition is obtained. \( \square \)

**Lemma 6.5** Let \( G \) be a countably infinite amenable group, let \( S \) be a subset of \( G \) with \( \text{BD}^*(S) > 0 \), and let \( W \subseteq G \) be an infinite set (i.e. \( |W| = \infty \)). Then there exist two distinct elements \( l_1, l_2 \in W \) such that
\[ \text{BD}^*(l_1^{-1}S \cap l_2^{-1}S) \geq \frac{\text{BD}^*(S)^2}{2}. \]

**Proof** Set \( W = \{s_1, s_2, \ldots, s_n, \ldots\} \) (\( s_i \neq s_j \) for \( i \neq j \)). By Proposition 6.4, taking
\[ k = 2, \epsilon = \text{BD}^*(S)^2/2, \quad \text{and} \quad N = N(\text{BD}^*(S), 2, \text{BD}^*(S)^2/2) \]
as in Proposition 6.4, for the tuple \( \{s_1, s_2, \ldots, s_n\} \subseteq W \) with \( n \geq N \), there exist \( 1 \leq t_1 < t_2 \leq n \) such that
\[ \text{BD}^*(s_{t_1}^{-1}S \cap s_{t_2}^{-1}S) \geq \frac{1}{2} \text{BD}^*(S)^2 > 0. \]
Let \( l_1 = s_{t_1} \) and \( l_2 = s_{t_2} \). Then the proof is completed. \( \square \)

The main result in this section is

**Theorem 6.6** Let \( G \) be a countably infinite Abelian group and let \( X \) be a compact metric space without isolated points. Suppose that the action \( G \curvearrowright X \) is transitive. If \( h_{\text{top}}(X, G) > 0 \), then the action \( G \curvearrowright X \) is Weyl-mean sensitive.

**Proof** It suffices to prove that there exists a transitive point \( x_0 \) which is not a Weyl-mean equicontinuous point, by Proposition 5.4.

As \( G \) is Abelian, the group \( G \) is an amenable group. Since \( h_{\text{top}}(X, G) > 0 \), and by Theorem 3.9 and Definition 3.8, there exists an IE pair \((x_1, x_2) \in IE_2(X, G) \setminus \triangle_2(X)\) satisfying, for any nonempty open neighborhood \( V_1 \times V_2 \ni (x_1, x_2) \), that \( A' = (V_1, V_2) \) has positive independent density, i.e.,
\[ I(A') = \inf_F \frac{\varphi_{A'}(F)}{|F|} > 0, \]
where \( \varphi_{A'}(F) = \max\{|F \cap J| : J \text{ is an independent set for } A'\} \), and \( F \) ranges over all nonempty finite subsets of \( G \). Since \( x_1 \neq x_2 \), we choose two open sets \( U_i (i = 1, 2) \) which are in the neighborhood of \( x_1 \) with
\[ d(U_1, U_2) > 2\delta_0 > 0 \quad \text{and} \quad I(A) = \inf_F \frac{\varphi_A(F)}{|F|} > 0. \quad (6.7) \]
Here \( A = (U_1, U_2) \). Thus, by Proposition 3.7 and (3.1), there exists an independent set \( J \) for \( A \) such that
\[ \text{BD}^*(J) \geq I(A) > 0. \]

Since \( G \curvearrowright X \) is transitive and \( X \) has no isolated points, by Propositions 2.2 and 2.4, we know that the set \( \text{Tran}(X, G) \) of points in \( X \) with dense orbit and the set \( \text{Re}(X, G) \) of recurrent points are both dense \( G_\delta \) sets of \( X \). Applying the Baire category theorem, we have that \( \text{Tran}(X, G) \cap \text{Re}(X, G) \) is also a dense \( G_\delta \) set of \( X \), which means that we have \( \text{Tran}(X, G) \cap \text{Re}(X, G) \neq \emptyset \).
Let \( x_0 \in \text{Tran}(X, G) \cap \text{Re}(X, G) \). In what follows, we will show that \( x_0 \) is not a Weyl-mean equicontinuous point.

For each \( \delta > 0 \), denote that
\[
G(x, B(x, \delta)) = \{ s \in G : sx \in B(x, \delta) \}.
\]
The cardinality of the set \( G(x_0, B(x_0, \delta)) \) is infinite because \( x_0 \) is a recurrent point.

Take \( m_0 \in \mathbb{N} \) satisfying
\[
\frac{1}{m_0} \leq \frac{\delta_0}{\delta} \text{BD}^*(J)^2.
\]  
(6.8)
Here \( \delta_0 \) is defined as in (6.7). Recall that, from (4.3),
\[
\mathcal{E}_{1/m_0} = \left\{ x \in X : \exists \delta > 0, \forall y, z \in B(x, \delta), \text{D}(y, z) < 1/m_0 \right\}.
\]
The rest of the proof we will establish the following assertion:

**Claim** \( x_0 \notin \mathcal{E}_{1/m_0} \).

Suppose that
\[
x_0 \in \mathcal{E}_{1/m_0}.
\]  
(6.9)
Then there exists \( \delta^* > 0 \) depending on \( x_0 \) and \( m_0 \) such that
\[
\text{D}(y, z) < \frac{1}{m_0} \quad \text{for all} \quad y, z \in B(x, \delta^*).
\]  
(6.10)
Let
\[
0 < \delta' < \min\{\delta_0, \delta^*\}.
\]  
(6.11)
Recall that \( \text{BD}^*(J) > 0 \) for the independent set \( J \) for \( A \). It follows from Lemma 6.5 that there are two distinct elements,
\[
l_1, l_2 \in G(x_0, B(x_0, \delta')), \tag{6.12}
\]
such that
\[
\text{BD}^*(l_1^{-1}J \cap l_2^{-1}J) \geq \frac{1}{2} \text{BD}^*(J)^2.
\]
For each \( g \in l_1^{-1}J \cap l_2^{-1}J \), there exist \( g_1, g_2 \in J \) such that \( g = l_1^{-1}g_1 = l_2^{-1}g_2 \). Thus we define a map
\[
\varphi : l_1^{-1}J \cap l_2^{-1}J \to 2^J \quad \text{by} \quad \varphi(g) = \{ g_1, g_2 \} = \{ l_1g, l_2g \}.
\]
Given \( s \in l_1^{-1}J \cap l_2^{-1}J \), it is clear that the number of elements \( g \in l_1^{-1}J \cap l_2^{-1}J \) such that \( \varphi(g) \cap \varphi(s) \neq \emptyset \) is at most three. Indeed, \( g \) can only take \( s, l_1^{-1}l_2s, \text{or} \ l_2^{-1}l_1s \).

Let \( H \) be a maximal subset of \( l_1^{-1}J \cap l_2^{-1}J \) with the property that, for every pair \( g, s \in H \) and \( g \neq s \), \( \varphi(g) \cap \varphi(s) = \emptyset \) (Zorn’s Lemma guarantees the existence of the set \( H \)). Now we claim that
\[
l_1^{-1}J \cap l_2^{-1}J \subseteq H \cup l_1^{-1}l_2H \cup l_2^{-1}l_1H.
\]
Otherwise, there exists an element \( g^* \in l_1^{-1}J \cap l_2^{-1}J \), but \( g^* \notin H \cup l_1^{-1}l_2H \cup l_2^{-1}l_1H \). Thus
\[
\varphi(g^*) \cap \varphi(h) = \emptyset \quad \text{for all} \quad h \in H. \tag{6.13}
\]
Indeed, if \( \varphi(g^*) \cap \varphi(h_0) \neq \emptyset \) for some \( h_0 \in H \), then, by the above argument, we know that \( g^* = h_0 \), \( g^* = l_1^{-1}l_2h_0 \), or \( g^* = l_2^{-1}l_1h_0 \), which contradicts the fact that \( g^* \notin H \cup l_1^{-1}l_2H \cup l_2^{-1}l_1H \).

Hence, by (6.13), we deduce that the set \( H \cup \{ g^* \} \) satisfies the property that, for every pair
For $g, s \in H \cup \{g^*\}$ and $g \neq s$, $\varphi(g) \cap \varphi(s) = \emptyset$. Noting that $g^* \notin H$ and $g^* \in l_1^{-1}J \cap l_2^{-1}J$, we can see that this contradicts the fact that the set $H$ is a maximal subset $l_1^{-1}J \cap l_2^{-1}J$ with such a property. 

Hence, we get 

$$l_1^{-1}J \cap l_2^{-1}J \subseteq H \cup l_1^{-1}l_2H \cup l_2^{-1}l_1H.$$ 

(6.14) 

According to Proposition 2.8 and the fact that $G$ is abelian, one has that 

$$\text{BD}^*(H) = \text{BD}^*(l_1^{-1}l_2H) = \text{BD}^*(l_2^{-1}l_1H).$$ 

Combining this with (6.14), it follows that 

$$\text{BD}^*(l_1^{-1}J \cap l_2^{-1}J) \leq \text{BD}^*(H) + \text{BD}^*(l_1^{-1}l_2H) + \text{BD}^*(l_2^{-1}l_1H) = 3\text{BD}^*(H).$$ 

Therefore, we have 

$$\text{BD}^*(H) \geq \frac{1}{3}\text{BD}^*(l_1^{-1}J \cap l_2^{-1}J) \geq \frac{1}{6}\text{BD}^*(J)^2.$$ 

(6.15) 

Recall that $G$ is an amenable group as $G$ is abelian and Theorem 3.2. By (3.1), we know that 

$$\text{BD}^*(H) = \sup_{F} \limsup_{n \to \infty} \frac{|H \cap F_n|}{|F_n|},$$ 

(6.16) 

where the supremum is taken over all Følner sequences $F = \{F_n\}_{n \in \mathbb{N}}$ of $G$. Thus, by (6.15) and (6.16), there is a Følner sequence $\{F_n\}$ of $G$ satisfying 

$$\limsup_{n \to \infty} \frac{|H \cap F_n|}{|F_n|} > \frac{1}{7}\text{BD}^*(J)^2.$$ 

Therefore, there exists a subsequence $\{m_n\}_{n=1}^{\infty}$ of $\mathbb{N}$ such that $m_n < m_{n+1}$, $m_n \geq n$ and 

$$\frac{|F_{m_n} \cap H|}{|F_{m_n}|} \geq \frac{1}{7}\text{BD}^*(J)^2.$$ 

(6.17) 

We denote that $J_1 = l_1H$ and that $J_2 = l_2H$. Since $H \subseteq l_1^{-1}J \cap l_2^{-1}J$, we immediately have $J_1 \cup J_2 \subseteq J$. Furthermore, we have that $J_1 \cap J_2 = \emptyset$. Indeed, if $J_1 \cap J_2 \neq \emptyset$, then there are $h_1, h_2 \in H$ such that $l_1h_1 = l_2h_2$. As $l_1 \neq l_2$, it follows that $h_1 \neq h_2$. Note that 

$$\varphi(h_1) = \{l_1h_1, l_2h_1\} = \{l_2h_2, l_1h_1\} \quad \text{and} \quad \varphi(h_2) = \{l_1h_2, l_2h_2\}.$$ 

Thus $\varphi(h_1) \cap \varphi(h_2) \neq \emptyset$ and $h_1 \neq h_2 \in H$, which contradicts the definition of $H$. Hence, $J_1 \cap J_2 = \emptyset$.

Let $n \in \mathbb{N}$. Inequality (6.17) implies that $F_{m_n} \cap H \neq \emptyset$. Denote that 

$$T_n = F_{m_n} \cap H.$$ 

Then we define the maps $\psi_i : T_n \to J_i (i = 1, 2)$ as follows: 

$$\psi_i(s) := l_is \quad \text{for} \quad s \in T_n.$$ 

It is easy to see that 

$$\psi_i(T_n) \subseteq J_i \subseteq J \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad \psi_1(T_n) \cap \psi_2(T_n) = \emptyset \quad \text{as} \quad J_1 \cap J_2 = \emptyset.$$ 

From the definition of the independent set of $J$ for $A = (U_1, U_2)$ (see Definition 2.5 and Definition 2.6), we get that 

$$\left( \bigcap_{s \in T_n} (\psi_1(s))^{-1}U_1 \right) \cap \left( \bigcap_{s \in T_n} (\psi_2(s))^{-1}U_2 \right) \neq \emptyset;$$
Therefore, we have
\[
\left( \bigcap_{s \in T_n} (\psi_1(s))^{-1}U_1 \right) \cap \left( \bigcap_{s \in T_n} (\psi_2(s))^{-1}U_2 \right) \text{ is an nonempty open set of } X.
\]

Recall that \( X = \overline{Gx_0} \). Hence, we get that
\[
Gx_0 \cap \left( \bigcap_{s \in T_n} (\psi_1(s))^{-1}U_1 \right) \cap \left( \bigcap_{s \in T_n} (\psi_2(s))^{-1}U_2 \right) \neq \emptyset.
\]

Choose a point \( y_n \in X \) with
\[
y_n \in Gx_0 \cap \left( \bigcap_{s \in T_n} (\psi_1(s))^{-1}U_1 \right) \cap \left( \bigcap_{s \in T_n} (\psi_2(s))^{-1}U_2 \right).
\]
Thus
\[
y_n = g_n x_0 \quad \text{for some } g_n \in G. \quad (6.18)
\]

Moreover, for each \( g \in T_n \), since \( G \) is abelian, one has that
\[
g(l_1 y_n) = (l_1 g) y_n = \psi_1(g) y_n \in U_1 \quad \text{and} \quad g(l_2 y_n) = l_2 g y_n = \psi_2(g) y_n \in U_2. \quad (6.19)
\]
Combining this with 6.18 and with \( G \) being an Abelian group, we get that
\[
(g_n g)(l_1 x_0) = (g l_1) y_n \in U_1 \quad \text{and} \quad (g_n g)(l_2 x_0) = (g l_2) y_n \in U_2.
\]
Therefore, we obtain that, for each \( g \in T_n \),
\[
g_n g \in G(l_1 x_0, U_1) \cap G(l_2 x_0, U_2);
\]
that is,
\[
g_n T_n = g_n (F_{m_n} \cap H) \subseteq G(l_1 x_0, U_1) \cap G(l_2 x_0, U_2). \quad (6.20)
\]
Recall that \( d(U_1, U_2) > 2\delta_0 \). Hence, one has
\[
d(s l_1 x_0, s l_2 x_0) > 2\delta_0 \quad \text{for every } s \in g_n T_n = g_n (F_{m_n} \cap H). \quad (6.21)
\]
Therefore, we have
\[
\limsup_{n \to \infty} \frac{1}{|g_n F_{m_n}|} \sum_{s \in g_n F_{m_n}} d(s(l_1 x_0), s(l_2 x_0)) 
\geq \limsup_{n \to \infty} \frac{1}{|g_n F_{m_n}|} \sum_{s \in g_n (F_{m_n} \cap H)} d(s(l_1 x_0), s(l_2 x_0)) 
\geq \limsup_{n \to \infty} \frac{1}{|g_n F_{m_n}|} \cdot 2\delta_0 \cdot |g_n (F_{m_n} \cap H)| 
= \limsup_{n \to \infty} 2\delta_0 \frac{|F_{m_n} \cap H|}{|F_{m_n}|} \quad \text{(by 6.17)} 
> \frac{\delta_0}{l} BD^*(J)^2. \quad (6.22)
\]
Denote that
\[
\mathcal{F}' = \{ g_n F_{m_n} \}_{n \in \mathbb{N}}.
\]
Since \( \{ F_n \}_{n \in \mathbb{N}} \) is a Følner sequence of the Abelian group \( G \), \( \mathcal{F}' \) is also a Følner sequence of \( G \).
Inequality (6.22) shows that
\[
D(l_1 x_0, l_2 x_0) \geq D_{\mathcal{F}'}(l_1 x_0, l_2 x_0) > \frac{\delta_0}{l} BD^*(J)^2. \quad \text{(see Definition 4.2)} \quad (6.23)
\]
Meanwhile, the above inequality implies that \( l_1x_0 \neq l_2x_0 \).

Recall that, from (6.12) and (6.11),

\[
  l_1, l_2 \in G(x_0, B(x_0, \delta')) \quad \text{and} \quad \delta' < \delta^*;
\]

that is,

\[
  l_1x_0, l_2x_0 \in B(x_0, \delta^*). \tag{6.24}
\]

Combining this with (6.23), (6.24) and (6.8), one has that

\[
  l_1x_0, l_2x_0 \in B(x_0, \delta^*) \quad \text{and} \quad D(l_1x_0, l_2x_0) > \frac{1}{m_0}.
\]

This contradicts inequality (6.10). Hence we obtain that

\[
  x_0 \notin \mathcal{E}_{1/m_0}. \tag{6.25}
\]

Recall that \( \mathcal{E} \) denotes the set of all Weyl-mean equicontinuous points of the group action system \((X, G)\). From Proposition 4.6, we know that

\[
  \mathcal{E} = \bigcap_{m=1}^{\infty} \mathcal{E}_{1/m}. \tag{6.26}
\]

By (6.25) and (6.26), we get

\[
  x_0 \notin \mathcal{E}.
\]

Therefore, \( x_0 \) is not a Weyl-mean equicontinuous point of \( G \acts X \). Therefore, \( x_0 \) is a Weyl-mean sensitive point of \( G \acts X \). By the assumption that \( G \acts X \) is transitive and Proposition 5.4, we deduce that \( G \acts X \) is Weyl-mean sensitive.

Hence, the theorem is proved. \( \square \)

**Proof of Theorem 1.1** The proof follows from Theorem 6.6, Theorem 5.5, Theorem 3.2 and Theorem 4.3. \( \square \)

### 7 An Application

In order to get our result, we need to establish the following concepts and theorems:

**Definition 7.1** ([21]) By a p.m.p. (probability-measure-preserving) action of \( G \), we mean an action of \( G \) on a standard probability space \((X, \mu)\) by measure-preserving transformations. In this case, we will combine the notation and simply write \( G \acts (X, \mu) \).

Given an action \( G \acts X \) on a compact metric space \( X \), we say that a set \( A \subseteq X \) is \( G \)-invariant if \( GA = A \), which is equivalent to \( GA \subseteq A \). When the action is probability-measure preserving and \( A \) is a measurable set, we interpret \( G \)-invariance to mean that \( GA = A \) modulo a null set, i.e., \( \mu(sA \triangle A) = 0 \) for all \( s \in G \).

**Definition 7.2** ([21]) The action \( G \acts (X, \mu) \) is said to be ergodic if \( \mu(A) = 0 \) or 1 for every \( G \)-invariant measurable set \( A \subseteq X \).

Any dynamical system with an amenable group action admits invariant probability measures and the ergodic measures can be shown to be the extremal points of the set of invariant probability measures (see, for example, the monographs [7, 25]). Let \( \mathcal{M}(X) \), \( \mathcal{M}_G(X) \) and \( \mathcal{M}_G^e(X) \) denote the sets of all Borel probability measures on \( X \), the \( G \)-invariant regular Borel probability measures on \( X \), and the ergodic measures in \( \mathcal{M}_G(X) \), respectively.
Proposition 7.3 ([21, Proposition 2.5]) For a p.m.p. action $G \curvearrowright (X, \mu)$, the following are equivalent:

1. the action is ergodic;
2. $\mu(A) = 0$ or $1$ for every measurable set $A \subseteq X$ satisfying $sA = A$ for all $s \in G$ (i.e., $G$-invariance in the strict sense);
3. for all sets $A, B \subseteq X$ of positive measure, there is an $s \in G$ such that $\mu(sA \cap B) > 0$.

Now, we recall the concept of amenable measure entropy (see [16] and [21]).

Let $G$ be an amenable group and let $G \curvearrowright (X, \mu)$ be a p.m.p. action. Let

$\mathcal{P} = \{A_1, A_2, \ldots, A_n\}$

be a finite partition of $X$ and let $F$ be a nonempty finite subset of $G$. Setting $\mathcal{P}^F$ for the join

$\bigvee_{s \in F} s^{-1} \mathcal{P}$,

$h(\mathcal{P}) = \inf_{F} \frac{1}{|F|} H(\mathcal{P}^F)$,

where $F$ ranges over nonempty finite subsets of $G$ and

$H(\mathcal{P}) = \sum_{i=1}^{n} -\mu(A_i) \log \mu(A_i)$.

The entropy of the action $G \curvearrowright (X, \mu)$ is

$h_{\mu}(X, G) = \sup_{\mathcal{P}} h(\mathcal{P})$,

where $\mathcal{P}$ ranges over all finite partitions of $X$.

The support of a measure $\mu \in \mathcal{M}(X)$, denoted by $\text{supp}(\mu)$, is the smallest closed subset $C$ of $X$ such that $\mu(C) = 1$ (see [23]); that is,

$\text{supp}(\mu) = \bigcap_{K \text{ is closed and } \mu(K) = 1} K$.

The following fact is well known:

Fact 7.4 We have that

$\text{supp}(\mu) = \{x \in X : \text{ for every open neighborhood } U \text{ of } x, \mu(U) > 0\}$

$= X \setminus \bigcup \{U \subset X : U \text{ is open and } \mu(U) = 0\}$.

Topological entropy is related to measure entropy by the variational principle which asserts that for a continuous map on a compact metric space, the topological entropy equals the supremum of the measure entropy taken over all the invariant probability measures. The following is a statement of the variational principle of the version of the amenable group action that we need in this paper:

Theorem 7.5 ([19, Theorem 5.2]) (Variational principle of topological entropy) Let $G$ be an amenable group and let $X$ be a compact metric space. Then

$h_{\text{top}}(X, G) = \sup_{\mu \in \mathcal{M}_G(X)} h_{\mu}(X, G) = \sup_{\mu \in \mathcal{M}_G^e(X)} h_{\mu}(X, G)$.

As an application of our main result, we have
Theorem 7.6 Let $G$ be a countable Abelian group, let $X$ be a compact metric space, and let $G \curvearrowright X$ be a continuous action. If $G \curvearrowright X$ is Banach-mean equicontinuous, then

$$h_{\text{top}}(X, G) = 0.$$ 

Proof Let $\mu$ be an ergodic invariant measure on the action $G \curvearrowright X$. Denote by $X_0 = \text{supp}(\mu)$ the support of the ergodic invariant measure $\mu$. It is clear that $X_0$ is a $G$-invariant closed subset of $X$ and that $G \curvearrowright (X_0, \mu)$ is also ergodic. Moreover, we have that $h_\mu(X, G) = h_\mu(X_0, G)$.

In what follows we show that $h_\mu(X_0, G) = 0$.

Let $U, V$ be any pair nonempty open sets of $X_0$. Then $\mu(U) \mu(V) > 0$, on account of $\text{supp}(\mu) = X_0$ and Fact 7.4. Thus there is an element $s \in G$ such that $\mu(U \cap sV) > 0$ on account of $G \curvearrowright (X_0, \mu)$ being ergodic and Proposition 7.3; that is,

$$U \cap sV \neq \emptyset.$$ 

Note that $X_0$ is a compact metric space. Hence the action $G \curvearrowright X_0$ is topological transitive.

Now we divide things into two cases to complete our proof.

Case 1 $X_0$ has no isolated points.

Since $G \curvearrowright X$ is Banach-mean equicontinuous, it is clear that $G \curvearrowright X_0$ is also Banach-mean equicontinuous. By Theorem 1.1 and because $X_0$ has no isolated points, we get that

$$h_{\text{top}}(X_0, G) = 0.$$ 

Note that $\mu|_{X_0}$ is an ergodic measure of $G \curvearrowright X_0$. Then, by Theorem 7.5, we obtain that

$$h_\mu(X_0, G) = 0.$$ 

Therefore, we have that

$$h_\mu(X, G) = h_\mu(X_0, G) = 0.$$ 

Recall that $\mu$ be any ergodic invariant measure on the action $G \curvearrowright X$. Again applying Theorem 7.5, we deduce that

$$h_{\text{top}}(X, G) = 0.$$ 

Case 2 $X_0$ has isolated points.

Suppose that $x_0 \in X_0$ is an isolated point of $X_0$, so the single point set $\{x_0\}$ is an open set of $X_0$. Let $V \subseteq X_0$ be any open set. Since the action $G \curvearrowright X_0$ is topological transitive, there is $s \in G$ such that $sx_0 \in V$. This fact implies that the orbit of $x_0$ is dense in $X_0$, that is, $Gx_0 = X_0$.

Note that $x_0 \in \text{supp}(\mu)$ and that the single point set $\{x_0\}$ is an open set. Thus one has $\mu(\{x_0\}) > 0$. Since $\mu(X_0) = 1$, we deduce that the cardinality of the set $Gx_0$ is finite (i.e., $|Gx_0| < \infty$). Combining this with $Gx_0 = X_0$, we get that the cardinality of the space $X_0$ is finite (i.e., $|X_0| < \infty$). By the definition of topological entropy, it is easy to see that

$$h_{\text{top}}(X_0, G) = 0.$$ 

In what follows, with an argument similar to that in Case 1, we can obtain that

$$h_{\text{top}}(X, G) = 0.$$ 

Hence the theorem is proved.

\[ \square \]
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