SIGN CHANGES OF FOURIER COEFFICIENTS OF SIEGEL CUSP FORMS OF DEGREE TWO ON HECKE CONGRUENCE SUBGROUPS

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ABSTRACT. In this article, we give a lower bound on the number of sign changes of Fourier coefficients of a non-zero degree two Siegel cusp form of even integral weight on a Hecke congruence subgroup. We also provide an explicit upper bound for the first sign change of Fourier coefficients of such Siegel cusp forms. Explicit upper bound on the first sign change of Fourier coefficients of a non-zero Siegel cusp form of even integral weight on the Siegel modular group for arbitrary genus were dealt in an earlier work of Choie, the first author and Kohnen.

1. INTRODUCTION AND STATEMENTS OF THE THEOREMS

The arithmetic of Fourier coefficients of cusp forms has been the focus of study for some time now. When these coefficients are real, studying distribution of their sign has become an active area of research in recent times. For example, in the case of elliptic cusp forms with real coefficients, this has been studied in \cite{1, 4, 5, 8, 15, 17}.

The question of infinitely many sign changes of Fourier coefficients of Siegel cusp forms with real Fourier coefficients has been studied in \cite{7} and sign changes of Hecke eigenvalues of Siegel cusp forms of degree two was studied in \cite{9}. The first sign change question for Hecke eigenvalues for Siegel cusp forms of genus two was addressed by Kohnen and the second author \cite{11}.

In this article, we give a lower bound on the number of sign changes in short intervals of Fourier coefficients of non-zero Siegel cusp forms of even integral weight and degree 2 on the Hecke congruence subgroup $\Gamma_0^{(2)}(N)$ with real Fourier coefficients. In order to deduce this result, we need to prove a corresponding result for elliptic cusp forms of square free level. This result seems to be new even in the case of elliptic cusp forms (see Theorem 4) and requires us to redo an earlier work of Rankin \cite{18} with explicit dependence on the weight and level of the elliptic cusp form (see Proposition 21). If the elliptic cusp form is a normalised new form of square free level $N$, then Kohnen, Lau and Shparlinski \cite{12} give a better lower bound.

On another direction, recently Choie, the first author and Kohnen \cite{2} gave an explicit upper bound for the first sign change of Fourier coefficients of non-zero Siegel cusp forms of even...
integral weight on the symplectic group $\Gamma_g := \text{Sp}_g(\mathbb{Z}) \subset \text{GL}_{2g}(\mathbb{Z})$ of arbitrary genus $g \geq 2$ with real Fourier coefficients. In this article, we provide an explicit upper bound for the first sign change of Fourier coefficients of degree two non-zero Siegel cusp form of even integral weight on a Hecke congruence subgroup of $\text{Sp}_2(\mathbb{Z})$ with real Fourier coefficients.

In order to state our theorems, we now fix some notations. For a natural number $N$, let

$$\Gamma_0^{(2)}(N) := \{ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}_2(\mathbb{Z}) \mid C \equiv 0 \pmod{N} \}$$

be the Hecke congruence subgroup of $\Gamma_2$ of level $N$ and $\mathcal{H}_2$ be the Siegel upper-half space of degree 2 consisting of all symmetric $2 \times 2$ complex matrices whose imaginary parts are positive definite. For $k \in \mathbb{N}$, let $S_k(\Gamma_0^{(2)}(N))$ be the space of Siegel cusp forms of weight $k$ on $\Gamma_0^{(2)}(N)$. It is well known that

$$\Psi_1(N) := [\Gamma_2 : \Gamma_0^{(2)}(N)] = N^3 \prod_{p \mid N, \ p \text{ prime}} (1 + 1/p)(1 + 1/p^2).$$

Any $F \in S_k(\Gamma_0^{(2)}(N))$ has a Fourier expansion

$$F(Z) := \sum_{T > 0} a(T) e^{2\pi i \text{tr} (TZ)},$$

where $Z \in \mathcal{H}_2$ and $T$ runs over all positive definite half-integral $2 \times 2$ matrices. In this set-up, we have the following theorems.

**Theorem 1.** Let $k, N$ be natural numbers with $k$ even and $N$ square-free. Also let $F$ be a non-zero Siegel cusp form of weight $k$ and degree 2 on the Hecke congruence subgroup $\Gamma_0^{(2)}(N)$ with real Fourier coefficients $a(T)$ for $T > 0$. For any $\epsilon > 0$, let $h := x^{13/14} + \epsilon$. Then there exists $T_1 > 0, T_2 > 0$ with $\text{tr} T_i \in (x, x + h]$, $i = 1, 2$ such that $a(T_1) > 0$ and $a(T_2) < 0$ for any

$$x \gg \epsilon^k 4.2 \cdot N^{8.4} \cdot \log^{80} k \cdot e^{(c_7 \log(N+1))},$$

where $c_7 > 0$ is an absolute constant and the constant in $\gg$ depends only on $\epsilon$.

As an immediate corollary, we have

**Corollary 2.** Let $k, N$ be natural numbers with $k$ even and $N$ square-free. Also let $F$ be a non-zero Siegel cusp form of weight $k$ and degree 2 on the Hecke congruence subgroup $\Gamma_0^{(2)}(N)$ with real Fourier coefficients $a(T)$ for $T > 0$. Then for any sufficiently small $\epsilon > 0$ and

$$x \gg \epsilon^k 4.2 \cdot N^{8.4} \cdot \log^{80} k \cdot e^{(c_7 \log(N+1))}$$

with $c_7 > 0$ an absolute constant, there exists at least $\gg x^{1/14 - \epsilon}$ many $T > 0$ with $\text{tr} T \in (x, 2x]$ and $a(T) < 0$. The same lower bound holds for the number of $T > 0$ with $\text{tr} T \in (x, 2x]$ and $a(T) > 0$. 


Theorem 3. Let $k, N$ be natural numbers with $k$ even and $N$ square-free. Also let $F$ be a non-zero Siegel cusp form of weight $k$ and degree 2 on the Hecke congruence subgroup $\Gamma_0^{(2)}(N)$ with real Fourier coefficients $a(T)$ for $T > 0$ at infinity. Then there exists $T_1 > 0, T_2 > 0$ with

$$\text{tr } T_1, \text{ tr } T_2 \ll k^5 (\log k)^{26} N^{39/2} e^{c_2 \log(N+1) \log\log(N+2)}$$

and $a(T_1) > 0, a(T_2) < 0$. Here $c_2$ as well as the constant in $\ll$ are absolute.

The paper is organized as follows. In the next section, we give a proof of Theorem 3 since it is relatively easier. In the penultimate section, using strong convexity principle, we give a lower bound on the number of sign changes in short intervals of Fourier coefficients of elliptic cusp forms of square-free level with real Fourier coefficients. In the final section, we give a proof of Theorem 1.

2. PROOF OF THEOREM 3

By the given hypothesis, we have

$$F(Z) := \sum_{T > 0} a(T) e^{2\pi i \text{tr } (TZ)} ,$$

where $Z \in H_2$ and $T$ runs over positive definite half-integral $2 \times 2$ matrices. Write

$$T = \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix}$$

with $m, n \in \mathbb{N}$ and $r \in \mathbb{Z}$ with $r^2 < 4nm$ and

$$Z := \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix} ,$$

where $\tau, \tau' \in \mathbb{H}, z \in \mathbb{C}$ with $\Im \tau \Im \tau' - \Im^2 z > 0$. Then

$$F(Z) := \sum_{m \geq 1} \phi_m(\tau, z) e^{2\pi i m \tau'} ,$$

where

$$\phi_m(\tau, z) = \sum_{\substack{n \geq 1, r \in \mathbb{Z}, \\ r^2 < 4nm}} c(n, r) e^{2\pi i (n\tau + rz)} \quad \text{and} \quad c(n, r) := a \begin{pmatrix} n & r/2 \\ r/2 & m \end{pmatrix} .$$

We claim that $\phi_m$ is a Jacobi cusp form of weight $k$ and index $m$ on $\Gamma_0(N) \ltimes \mathbb{Z}^2$. 
For $M := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ and $(\lambda, \mu) \in \mathbb{Z}^2$, the matrices

$$\gamma(M) := \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in Sp_2(\mathbb{Z})$$

and

$$M_{\lambda,\mu} := \begin{pmatrix} 1 & 0 & 0 & \mu \\ \lambda & 1 & \mu & 0 \\ 0 & 1 & \mu & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \in \Gamma_0^{(2)}(N).$$

Note that if $M \in \Gamma_0(N)$, then $\gamma(M) \in \Gamma_0^{(2)}(N)$. These matrices $\gamma(M)$ and $M_{\lambda,\mu}$ act on $H_2$ as follows

$$(\tau, z, \tau') \mapsto \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d}, \tau' - \frac{cz^2}{c\tau + d} \right),$$

$$(\tau, z, \tau') \mapsto (\tau, z + \lambda\tau + \mu, \tau' + 2\lambda z + \lambda^2 \tau).$$

Then

$$F|\gamma(M)(Z) = \sum_{m \geq 1} (\phi_m | M)(\tau, z) e^{2\pi im\tau'},$$

and

$$F|M_{\lambda,\mu}(Z) = \sum_{m \geq 1} (\phi_m | [\lambda, \mu])(\tau, z) e^{2\pi im\tau'} = \sum_{m \geq 1} \phi_m(\tau, z) e^{2\pi im\tau'},$$

as $F|M_{\lambda,\mu}(Z) = F(Z)$. Also

$$F|\gamma(M)(Z) = \sum_{T > 0} b(T, \gamma(M)) e^{\frac{2\pi i}{N} \text{tr}(TZ)},$$

where $T$ runs over positive definite half-integral $2 \times 2$ matrices. Write $T := \begin{pmatrix} n & r/2 \\ r/2 & m_1 \end{pmatrix}$, where $n, m_1 > 0$ and $r^2 < 4nm_1$ with $r \in \mathbb{Z}$ and as before $Z := \begin{pmatrix} \tau & z \\ z & \tau' \end{pmatrix}$ with $\tau, \tau' \in \mathcal{H}, z \in \mathbb{C}$ and $\Im \tau \Im \tau' - \Im^2 z > 0$. Then

$$F|\gamma(M)(Z) = \sum_{T > 0} b(T, \gamma(M)) e^{\frac{2\pi i}{N} \text{tr}(TZ)}$$

$$= \sum_{n, m_1 > 0} b \left( \begin{pmatrix} n & r/2 \\ r/2 & m_1 \end{pmatrix}, \gamma(M) \right) e^{\frac{2\pi i}{N}(n\tau + rz + m_1\tau')}.$$

Comparing this with equation (2), we see that $m_1 \equiv 0 \mod N$ and hence

$$n \geq 1, r \in \mathbb{Z}, r^2 < 4nm_1 \sum_{n \geq 1, r \in \mathbb{Z}, r^2 < 4nmN} b \left( \begin{pmatrix} n & r/2 \\ r/2 & Nm \end{pmatrix}, \gamma(M) \right) e^{\frac{2\pi i}{N}(n\tau + rz)}.$$
Hence

\[(\phi_m | M)(\tau, z) = \sum_{n \geq 1, r \in \mathbb{Z}, \ r^2 < 4mnN} b(n, r) e^{2\pi i (n\tau + rz)},\]

where

\[b(n, r) := b \left( \left( \frac{n}{r/2}, \frac{r/2}{Nm} \right), \gamma(M) \right).\]

Further for \(M \in \Gamma_0(N),\) one has

\[\sum_{m \geq 1} (\phi_m | M)(\tau, z) e^{2\pi im\tau'} = F | \gamma(M)(Z) = F(Z) = \sum_{m \geq 1} \phi_m(\tau, z) e^{2\pi im\tau'}.\]

Thus \(\phi_m\) is Jacobi cusp form of weight \(k\) and index \(m\) on \(\Gamma_0(N) \ltimes \mathbb{Z}^2.\)

Since we can replace \(F\) by \(-F,\) it is sufficient to show that there exists a \(T > 0\) with \(\text{tr} \ T\) in the given range such that \(a(T) < 0.\) We know from [19, 20] that there exists a \(T_0 > 0\)

\[\text{tr} \ T_0 \leq \frac{4}{3\sqrt{3\pi}} k\Psi_1(N)\]

such that \(a(T_0) \neq 0.\) Write \(T_0 := \left( \frac{n_0}{r_0/2}, \frac{r_0/2}{m_0} \right) > 0.\) Since \(a(T_0) \neq 0,\) we have \(\phi_{m_0}(\tau, z)\) is not the zero function. Further, \(k\) is even and \(-I \in \Gamma_0(N)\) implies that \(\phi_{m_0}(\tau, z)\) is an even function of \(z\) for fixed \(\tau.\)

Suppose that

\[\phi_{m_0}(\tau, z) = \sum_{n \geq 1, r \in \mathbb{Z}, \ r^2 < 4nm_0} c(n, r) q^n \zeta^r, \quad \text{where} \quad q := e^{2\pi i \tau}, \ \zeta := e^{2\pi iz}.\]

be the expansion of \(\phi_{m_0}\) at infinity and

\[\phi_{m_0}(\tau, z) := \sum_{\nu \geq 0} \chi_\nu(\tau) z^\nu\]

be the Taylor series expansion of \(\phi_{m_0}\) around \(z = 0.\) Since \(\phi_{m_0}\) is not the zero function, all \(\chi_\nu\)’s can not be zero. If \(\alpha\) is the smallest non-negative integer such that \(\chi_\alpha(\tau)\) is not the zero function, then it follows from [3], page 31 (see also [2]) that \(\chi_\alpha(\tau)\) is a non-zero cusp form of weight \(k + \alpha\) on \(\Gamma_0(N).\)

Moreover, by Theorem 1.2, page 10 in [3], it follows that

\[\alpha \leq 2m_0 < 2 \text{tr} \ T_0 \leq \frac{8}{3\sqrt{3\pi}} k\Psi_1(N).\]

Note that \(\alpha\) is even as \(\phi_{m_0}\) is an even function of \(z\) for fixed \(\tau.\) Now by differentiating both sides of equation (5) with respect to \(z\) and then evaluating at \(z = 0\) and using equation (4), we see that for \(\tau \in \mathcal{H},\)

\[\chi_\alpha(\tau) = \sum_{n \geq 1} B(n) q^n,\]
where
\[ B(n) := \begin{cases} \sum_{r \in \mathbb{Z}} c(n, r) & \text{if } \alpha = 0, \\ \frac{1}{\sqrt{(2\pi)^d}} \sum_{r \in \mathbb{Z}} c(n, r) r^{\alpha} & \text{if } \alpha > 0. \end{cases} \]

In the above situation, either \( i^\alpha = 1 \) or \( i^\alpha = -1 \). If \( i^\alpha = 1 \), then by the work of Choie and Kohnen, there exists \( n_1 \ll \Psi_2(k + \alpha, N) \) such that \( B(n_1) < 0 \), where
\[ \Psi_2(k, N) := k^3 N^4 \log^{10}(kN) e^{c_1 \log(N+1)} \max \left( \prod_{p|N} \frac{\log(kN)}{\log p}, k^2 N^{1/2} \log^{16}(kN) \right) \]
and \( c_1 \) is an absolute constant. In the case \( i^\alpha = -1 \), again by the same result of Choie and Kohnen, we can find \( n_2 \ll \Psi_2(k + \alpha, N) \) such that \( B(n_2) > 0 \). In both the cases using (6), there exists an \( r_1 \) (resp. \( r_2 \)) such that \( c(n_1, r_1) < 0 \) (resp. \( c(n_2, r_2) < 0 \)). Thus
\[ c(n_i, r_i) = a \left( \begin{pmatrix} n_i & r_i/2 \\ r_i/2 & m_0 \end{pmatrix} \right) < 0, \quad \text{where } i = 1, 2. \]

Now for \( i = 1, 2 \), we have
\[ \text{tr} \left( \begin{pmatrix} n_i & r_i/2 \\ r_i/2 & m_0 \end{pmatrix} \right) = n_i + m_0 \ll \Psi_2(k + \alpha, N) + \text{tr } T_0 \leq \Psi_2(k + \alpha, N) + \frac{4}{3\sqrt{3\pi}} k \Psi_1(N) \ll \Psi_2(k + \alpha, N), \]
where
\[ k + \alpha \leq k + 2m_0 < k + \frac{8k}{3\sqrt{3\pi}} \Psi_1(N) < 2k \Psi_1(N). \]
Hence
\[ \text{tr} \left( \begin{pmatrix} n_i & r_i/2 \\ r_i/2 & m_0 \end{pmatrix} \right) \ll \Psi_2(2k \Psi_1(N), N) \ll \Psi_2(k \Psi_1(N), N). \]

Further, for \( \epsilon > 0 \)
\[ \nu(N) \leq (1 + \epsilon) \frac{\log N}{\log \log N} \]
for all \( N \geq N(\epsilon) \) (see [23], page 83 for a proof). Now \( \ell := k \Psi_1(N) > N^3 \) and \( \frac{\log x}{\log \log x} \) is an increasing function for \( x \geq 16 \). Thus for all \( N \geq N_0(\epsilon) \geq 16 \),
\[ \prod_{p|N \text{ prime}} \frac{\log(\ell N)}{\log p} \leq \frac{1}{\log 2} (\log(\ell N))^{\nu(N)} \leq \frac{1}{2} e^{(1+\epsilon) \frac{\log \ell}{\log \log \ell} (\log \log(\ell N))} \leq \frac{1}{2} e^{(1+\epsilon)^2 \log \ell} \]
and hence
\[ \Phi_\ell(N) := \prod_{p|N \text{ prime}} \frac{\log(\ell N)}{\log p} \ll \ell^2 N^{1/2} \log^{16}(\ell N), \]
where the constant in \( \ll \) is absolute. This implies that

\[
\Psi_2(k\Psi_1(N), N) \ll e^{c_1 \frac{\log(N+1)}{\log(\log(N+2))}} k^5 N^{39/2} \log^{26}(kN^4) \log \log^{10}(N + 2) \\
\ll k^5 (\log k)^{26} N^{39/2} e^{c_2 \frac{\log(N+1)}{\log(\log(N+2))}},
\]

where \( c_2 > 0 \) is an absolute constant. The last inequality is true as for \( k \geq 2 \) and \( N \geq 1 \), we see that

\[
\log \log(N + 2) \ll \frac{\log(N+1)}{\log(\log(N+2))} \quad \text{and} \quad \log(kN^4) \ll (\log k) \frac{\log(N+1)}{\log(\log(N+2))}.
\]

This completes the proof of Theorem 3.

3. Number of Sign Changes of Fourier Coefficients in Short Intervals for Elliptic Cusp Forms

Throughout this section, we denote a prime number by \( p \). In order to prove Theorem 1, we will need to prove a corresponding theorem for arbitrary non-zero elliptic cusp form of weight \( k \) on \( \Gamma_0(N) \) having real Fourier coefficients. In particular, we prove the following theorem.

**Theorem 4.** Let \( f \) be a non-zero cusp form of even weight \( k \) on \( \Gamma_0(N) \), where \( N \) is square free. Assume that the Fourier coefficients \( \beta(n) \) of \( f \) at infinity are real. Then for any \( \epsilon > 0 \) and for

\[
x \gg_{\epsilon} k^{42} N^{84} \log^{80} k \cdot e^{c_7 \frac{\log(N+1)}{\log(\log(N+2))}} \text{ with } c_7 > 0 \text{ an absolute constant},
\]

there exists \( n_1, n_2 \in (x, x + h] \) with \( h := x^{13/14 + \epsilon} \) such that \( \beta(n_1) > 0, \beta(n_2) < 0 \). The constant in \( \gg \) depends only on \( \epsilon \).

In order to prove this theorem, we use strong convexity principle. Further, we need the following additional notations and lemmas from the works of [1, 10].

Let \( \mathcal{H} \) be the Poincaré upper half plane, \( S_k(N) \) be the space of cusp forms of weight \( k \) for \( \Gamma_0(N) \) and \( S_k^{\text{new}}(N) \) be the subspace of new forms of \( S_k(N) \). If \( f, g \in S_k(N) \), then we normalize the Petersson inner product on \( S_k(N) \) by

\[
< f, g > := \frac{1}{[\Gamma_1 : \Gamma_0(N)]} \int_{\Gamma_0(N) \backslash \mathcal{H}} f(z) \overline{g(z)} y^k \frac{dx dy}{y^2},
\]

where \( z := x + iy \). Further, if

\[
f(z) := \sum_{n \geq 0} a_f(n) q^n \in S_k(N), \quad q := e^{2\pi i z}, \quad z \in \mathcal{H},
\]

then we denote by

\[
\lambda_f(n) := \frac{a_f(n)}{n^{(k-1)/2}},
\]
the normalized Fourier coefficient of \( f \). By a normalized Hecke eigen form \( f \in S_k^\text{new}(N) \), we mean \( \lambda_f(1) = 1 \) and \( \lambda_f(n) \) are normalized Hecke eigenvalues. Hence by Deligne’s bound, one has

\[ |\lambda_f(n)| \leq \tau(n), \quad \text{for all } n \in \mathbb{N} \]

when \( f \) is a normalized Hecke eigen form in \( S_k^\text{new}(N) \) and \( \tau(n) \) is the number of divisors of \( n \). For \( f, g \in S_k(N) \) and \( s \in \mathbb{C} \) with \( \sigma := \Re(s) > 1 \), one defines the Rankin-Selberg zeta function attached to \( f, g \) as

\[ R_{f,g}(s) := \sum_{n \geq 1} \frac{\lambda_f(n) \lambda_g(n)}{n^s}. \]

It is known that the completed Rankin-Selberg zeta function

\[ R_{f,g}^*(s) := N^s(2\pi)^{-2s} \Gamma(s) \Gamma(s + k - 1) \zeta_N(2s) R_{f,g}(s), \]

where

\[ \zeta_N(s) := \prod_{\text{prime } p \mid N} (1 - p^{-s})^{-1} \]

has meromorphic continuation to \( \mathbb{C} \) with possible simple poles at \( s = 0, 1 \) with

\[ \text{Res}_{s=1} R_{f,g}(s) = \frac{12}{(k-1)!} (\frac{4\pi}{2})^{k-1} < f, g >. \]

If \( f, g \in S_k(N) \) with \( < f, g > = 0 \), then \( R_{f,g}^*(s) \) is holomorphic everywhere and is of finite order. When \( f = g \in S_k^\text{new}(N) \) is a normalized Hecke eigenform, then

\[ R_{f,f}(s) := \prod_{p \mid N} (1 + p^{-s})^{-1} \frac{\zeta(s)}{\zeta(2s)} L(\text{sym}^2 f, s), \]

where \( L(\text{sym}^2 f, s) \) is the symmetric square \( L \)-function attached to \( f \) (see [6,10] for details). We now list the lemmas which will be important for our theorem.

**Lemma 5.** [Choie-Kohnen (see page 523 of [1])] Let \( f \in S_{k_1}^\text{new}(N_1) \), \( g \in S_{k_2}^\text{new}(N_2) \) be normalized Hecke eigenforms with normalized eigenvalues \( \lambda_f(n) \) and \( \lambda_g(n) \) respectively. Also let \( N \) be a square-free integer such that \( N_1, N_2 | N \). If \( \delta \mid \frac{N_1}{N} \), \( \delta' \mid \frac{N}{N_2} \) and \( T := \gcd(\delta, \delta') \), then

\[ R_{f|V_{\delta},g|V_{\delta'}}(s) = \left( \frac{\delta' T}{T} \right)^{-s-k+1} \prod_{p \mid T} \frac{\lambda_f(p) - \lambda_g(p) p^{-s}}{1 - p^{-2s}} \cdot \prod_{p \mid \delta'} \frac{\lambda_g(p) - \lambda_f(p) p^{-s}}{1 - p^{-2s}} R_{f,g}(s), \]

where

\[ (f|V_\delta)(z) := f(\delta z) \quad \text{for } z \in \mathcal{H}. \]
Lemma 6. [Choie-Kohnen (see page 524 of [1])] For square-free $N$, the space $S_k(N)$ is the orthogonal direct sum of the one-dimensional subspaces spanned by

$$F := f \prod_{p|N} (1 + \epsilon_f,p p^{k/2} V_p),$$

where $f$ runs over all normalized Hecke eigenforms in $S_k^{\text{new}}(d)$ for all $d|N$ and $\epsilon_f,p$ runs over the signs $\pm 1$.

Lemma 7. [Kohnen-Sengupta [10]] For a square-free integer $N$, let $f \in S_k^{\text{new}}(N)$ be a normalized Hecke eigen form. Then for all $t \in \mathbb{R}$ and $1 < \delta < \frac{1}{\log 2}$, we have

$$|L(\text{sym}^2 f, \delta + it)| \ll \frac{\delta}{(\delta - 1)^3},$$

and

$$|L(\text{sym}^2 f, 1 - \delta + it)| \ll \frac{\delta}{(\delta - 1)^3} \cdot (kN)^{2\delta - 1} \cdot |1 + it|^{3(\delta - \frac{1}{2})}.$$  

As an immediate Corollary, we have

Corollary 8. For a square-free integer $N$, let $f \in S_k^{\text{new}}(N)$ be a normalized Hecke eigen form. Then for all $t \in \mathbb{R}$ and $\Delta = \frac{1}{100}$, we have

$$(11) \quad |L(\text{sym}^2 f, 1 + \Delta + it)| \ll 1$$

and

$$|L(\text{sym}^2 f, -\Delta + it)| \ll (kN)^{1+2\Delta} \cdot |1 + it|^{3(1+2\Delta)}.$$  

We now state a result of Rademacher which plays a pivotal role in our work.

Proposition 9. [Rademacher [21]] Suppose that $g(s)$ is continuous on the closed strip $a \leq \sigma \leq b$ and holomorphic and of finite order on $a < \sigma < b$. Further, suppose that

$$|g(a + it)| \leq E|P + a + it|^\alpha, \quad |g(b + it)| \leq F|P + b + it|^\beta,$$

where $E, F$ are positive constants and $P, \alpha$ and $\beta$ are real constants that satisfy

$$P + a > 0, \quad \alpha \geq \beta.$$

Then for all $a < \sigma < b$ and $t \in \mathbb{R}$, we have

$$|g(\sigma + it)| \leq (E|P + \sigma + it|^\alpha)^{\frac{b-a}{b-a}} \cdot (F|P + \sigma + it|^\beta)^{\frac{b-a}{b-a}}.$$  

Proposition 10. For square-free integer $N$, let $f \in S_k^{\text{new}}(N)$ be a normalized Hecke eigen form. Then for all $-\Delta < \sigma < 1 + \Delta$ with $\Delta = \frac{1}{100}$ and for any $t \in \mathbb{R}$, we have

$$|L(\text{sym}^2 f, \sigma + it)| \ll (kN)^{(1+\Delta - \sigma)} \cdot (3 + |t|)^{\frac{3}{2}(1+\Delta - \sigma)}.$$  

In particular, for all $t \in \mathbb{R}$, we have

$$(12) \quad |L(\text{sym}^2 f, \frac{3}{4} + it)| \ll (kN)^{\frac{1}{4} + \Delta} \cdot (3 + |t|)^{\frac{3}{2} + \frac{3\Delta}{2}}.$$
Proof. Applying Proposition 9 with
\[ a = -\Delta, \quad b = P = 1 + \Delta, \]
\[ E = E_1(kN)^{1+2\Delta}, \quad \alpha = \frac{3}{2}(1 + 2\Delta), \quad \beta = 0 \]
and \( F, E_1 \) are absolute constants and finally using Corollary 8, we get our result. \( \square \)

We can now deduce the following corollary.

**Corollary 11.** For square-free integer \( N \), let \( f \in S_k^{\text{new}}(N) \) be a normalized Hecke eigen form. Then for all \( t \in \mathbb{R} \) and any \( \epsilon > 0 \), we have
\[
| R_{f,f} \left( \frac{3}{4} + it \right) | \ll_{\epsilon} (kN)^{\frac{5}{4} + \Delta} \cdot e^{\left( c \sqrt{\frac{\log(N+1)}{\log \log(N+2)}} \right) \cdot (3 + |t|)^{\frac{11}{12} + \frac{2\Delta}{3} + \epsilon},
\]
where \( c > 0 \) is an absolute constant and \( \Delta = \frac{1}{100} \).

**Proof.** Recall that
\[
R_{f,f}(s) = \prod_{\substack{p \text{ prime} \atop p \mid N}} (1 + p^{-s})^{-1} \cdot \frac{\zeta(s)}{\zeta(2s)} \cdot L(\text{sym}^2 f, s).
\]
To complete the proof, we use Proposition 10 in addition to the results that
\[
\left| \zeta \left( \frac{3}{4} + it \right) \right| \ll_{\epsilon} |1 + it|^{\frac{5}{12} + \frac{\Delta}{2}}, \quad \left| \frac{1}{\zeta \left( \frac{3}{2} + \frac{3}{4} + \epsilon \right) (3 + |t|)^\epsilon} \right| \ll_{\epsilon} 1 + it \ll_{\epsilon}
\]
and for \( \sigma = 3/4 \)
\[
\prod_{\substack{p \text{ prime} \atop p \mid N}} (1 + p^{-s})^{-1} \ll \prod_{\substack{p \text{ prime} \atop p \mid N}} (1 + p^{-3/2})^{-1} \ll e^{c \sqrt{\frac{\log(N+1)}{\log \log(N+2)}}},
\]
where \( c > 0 \) is an absolute constant. For a proof of the last inequality, see page 180 of [10]. \( \square \)

In fact, we can prove the following general statement.

**Proposition 12.** For a square-free integer \( N \), let \( f \in S_k^{\text{new}}(N) \) be a normalized Hecke eigen form. Also let \( \epsilon > 0 \). Then for \( \frac{1}{2} \leq \sigma < 1 + \Delta \) with \( \Delta = \frac{1}{100} \) and \( t \in \mathbb{R} \) with \( |t| \gg 1 \), we have
\[
| R_{f,f} (\sigma + it) | \ll_{\epsilon} (kN)^{(1+\Delta-\sigma)} \cdot e^{c \sqrt{\frac{\log(N+1)}{\log \log(N+2)}} \cdot (3 + |t|)^{\frac{3}{2}(1+\Delta-\sigma) + \max \{ \frac{1-\sigma}{3}, 0 \} + \epsilon},
\]
where \( c > 0 \) is an absolute constant. Further for any \( t \in \mathbb{R}, \epsilon > 0 \), we have
\[
\left| R_{f,f} \left( \frac{1}{2} + \frac{\Delta}{2} + it \right) \right| \ll_{\epsilon} (kN)^{\frac{1+\Delta}{2}} \cdot e^{c \sqrt{\frac{\log(N+1)}{\log \log(N+2)}} \cdot (3 + |t|)^{\frac{3}{2}(1+\Delta) + \frac{2\Delta}{3} + \epsilon},
\]
where \( c > 0 \) is an absolute constant.
Proof. Note that for \( \frac{1}{2} \leq \sigma < 1 + \Delta \), one has

\[
\prod_{p \text{ prime} \mid N} (1 + p^{-s})^{-1} \ll e^{c \sqrt{\frac{\log(N+1)}{\log\log(N+2)}}},
\]

where \( c > 0 \) is an absolute constant. We now use the following estimates (see pages 145-146 of [23])

\[
|\zeta(\sigma + it)| \ll |t|^\frac{3\sigma}{2} + \frac{s}{2} \quad \text{and} \quad \left| \frac{1}{\zeta(2\sigma + 2it)} \right| \ll |t|^\frac{3s}{2}
\]

for \( t \in \mathbb{R} \) with \( |t| \gg 1 \) and \( 1/2 \leq \sigma \leq 1 \) to complete the proof of the first part. For the second part of the Proposition, we proceed exactly as in Corollary [11] and use the estimates

\[
\left| \zeta\left(\frac{1}{2}(1 + \Delta) + it\right) \right| \ll |t|^\frac{3s}{200} + \epsilon \quad \text{and} \quad \left| \frac{1}{\zeta(1 + \Delta + 2it)} \right| \ll 1
\]

for any \( t \in \mathbb{R} \). \( \square \)

Now we would like to estimate the Rankin-Selberg \( L \)-function for two distinct newforms \( f \) and \( g \). Here we have the following Proposition.

**Proposition 13.** For square-free integers \( N_1, N_2 \), let \( f \in S_k^{\text{new}}(N_1) \), \( g \in S_k^{\text{new}}(N_2) \) be normalized Hecke eigen forms with \( f \neq g \). Then for any \( t \in \mathbb{R} \), \( \Delta = \frac{1}{100} \) and \( N := \text{lcm}(N_1, N_2) \), we have

\[
|\zeta_N(2 + 2\Delta + 2it) \cdot R_{f,g}(1 + \Delta + it)| \ll 1
\]

and

\[
|\zeta_N(-2\Delta + 2it) \cdot R_{f,g}(-\Delta + it)| \ll \tau(N) \cdot (kN)^{1/2 + 2\Delta} \cdot N^\Delta \cdot |1 + it|^{2 + 4\Delta},
\]

where \( \tau(N) \) is the number of divisors of \( N \) and for \( s \in \mathbb{C} \) with \( \Re(s) > 1 \), one defines

\[
\zeta_N(s) = \prod_{p \text{ prime} \mid D} (1 - p^{-s})^{-1}.
\]

**Proof.** The first inequality follows as both \( \zeta_N(2 + 2\Delta + 2it) \) and \( R_{f,g}(1 + \Delta + it) \) are absolutely convergent. Using the functional equation (see [14] for details)

\[
\prod_{p \text{ prime} \mid D} (1 - w_p v_p p^{-s})^{-1} R^*_{f,g}(s) = \prod_{p \text{ prime} \mid D} (1 - w_p v_p p^{-(1-s)})^{-1} R^*_{f,g}(1-s),
\]

we have

\[
\zeta_N(2 - 2s) \cdot R_{f,g}(1 - s) = (2\pi)^{2 - 4s} \cdot N^{2s - 1} \cdot \frac{\Gamma(s)}{\Gamma(1-s)} \cdot \frac{\Gamma(k - 1 + s)}{\Gamma(k - s)}
\]

\[
\cdot \prod_{p \text{ prime} \mid D} \left(1 - \frac{1}{w_p v_p p^{s-1}} \right) \cdot \zeta_N(2s) \cdot R_{f,g}(s),
\]
where $D := \gcd(N_1, N_2)$ and $w_p, v_p \in \{\pm 1\}$ are the eigenvalues of $f$ and $g$ respectively for the corresponding Atkin-Lehner involutions. Now using Stirling’s formula (see page 178 of [10]), one has
\[
\left| \frac{\Gamma(k + \Delta + it)}{\Gamma(k - 1 - \Delta + it)} \right| \ll k^{1+2\Delta} |1 + it|^{1+2\Delta},
\]
and
\[
\left| \frac{\Gamma(1 + \Delta + it)}{\Gamma(-\Delta + it)} \right| \ll |1 + it|^{1+2\Delta} \quad \text{for all } t \in \mathbb{R}.
\]
Also for all $t \in \mathbb{R}$, one has
\[
| \prod_{p \text{ prime}} (1 - w_p v_p p^{-1-\Delta+it}^{-1}) | \leq \prod_{p \text{ prime}} \left( 1 - \frac{1}{p^{1+\Delta}} \right)^{-1} \ll 1,
\]
and
\[
| \prod_{p \text{ prime}} (1 - w_p v_p p^{\Delta+it}) | \ll \prod_{p \text{ prime}} (1 + p^\Delta) \ll \prod_{p \text{ prime}} 2p^\Delta = \tau(D). D^\Delta.
\]
Hence we have
\[
|\zeta_N(-2\Delta + 2it) \cdot R_{f,g}(-\Delta + it)| \ll \tau(N). (kN)^{1+2\Delta}. N^\Delta \cdot |1 + it|^{2+4\Delta}.
\]
This completes the proof of the Proposition. □

Now using Proposition 9 and Proposition 13 we get

**Proposition 14.** For square-free integers $N_1, N_2$, let $f \in S_k^{\text{new}}(N_1)$, $g \in S_k^{\text{new}}(N_2)$ be normalized Hecke eigen forms with $f \neq g$ and $N := \text{lcm} (N_1, N_2)$. Then for any $t \in \mathbb{R}$ and $-\Delta < \sigma < 1 + \Delta$ with $\Delta = \frac{1}{100}$, we have
\[
|\zeta_N(2\sigma + 2it) \cdot R_{f,g}(\sigma + it)| \ll (kN)^{1+\Delta-\sigma}. (\tau(N). N^\Delta)^{\frac{1+\Delta-\sigma}{1+2\Delta}}. (3 + |t|)^{2(1+\Delta-\sigma)}.
\]
In particular for any $t \in \mathbb{R}$, $\epsilon > 0$ and $\frac{1}{2} \leq \sigma < 1 + \Delta$, we have
\[
|R_{f,g}(\sigma + it)| \ll \epsilon \cdot (kN)^{1+\Delta-\sigma}. (\tau(N). N^\Delta)^{\frac{1+\Delta-\sigma}{1+2\Delta}}.
\]
\[
\log \log(N + 2) \cdot (3 + |t|)^{2(1+\Delta-\sigma)+\epsilon}.
\]

**Proof.** From the theory of newforms, we know that $<f, g> = 0$. In this case, we know that $\zeta_N(2s)R_{f,g}(s)$ is entire and is of finite order (see [13][14] for details). Set
\[
a = -\Delta, \quad b = P = 1 + \Delta,
\]
\[
E = E_1(kN)^{1+2\Delta}. \tau(N). N^\Delta, \quad \alpha = 2 + 4\Delta, \quad \beta = 0
\]
and $F, E_1$ are absolute constants. We now use Proposition 13 and apply Proposition 9 in the strip $a \leq \sigma \leq b$ to conclude the first part of our result. Further note that for $\frac{1}{2} \leq \sigma < 1 + \Delta$, one has
\[
|\zeta_N(2\sigma + 2it)^{-1}| \ll \prod_{p \text{ prime}} \left( \frac{1 + p^{-1}}{|\zeta(2\sigma + 2it)|} \right) \ll \epsilon \log \log(N + 2) \cdot |1 + it|^\epsilon.
\]
This completes the proof of (14).

As an immediate Corollary, we have

**Corollary 15.** For square-free integers $N_1, N_2$, let $f \in S_{k}^{\text{new}}(N_1)$, $g \in S_{k}^{\text{new}}(N_2)$ be normalized Hecke eigen forms with $f \neq g$ and $N := \text{lcm}(N_1, N_2)$. Then for any $t \in \mathbb{R}$ and $\Delta = \frac{1}{100}$, we have

$$R_{f,g} \left( \frac{1}{2} + \frac{\Delta}{2} + it \right) \ll_{\epsilon} (kN)^\frac{1}{2}(1+\Delta) \cdot (\tau(N) \cdot N^\Delta)^\frac{(1+\Delta)}{2(1+2\Delta)} \cdot \log \log(N+2) \cdot (3 + |t|)^{1+\Delta+\epsilon}$$

$$R_{f,g} \left( \frac{3}{4} + it \right) \ll_{\epsilon} (kN)^{\frac{3}{4}+\Delta} \cdot (\tau(N) \cdot N^\Delta)^\frac{(1+4\Delta)}{2(1+2\Delta)} \cdot \log \log(N+2) \cdot (3 + |t|)^{\frac{1}{2}+2\Delta+\epsilon}$$

Combining Proposition 14 and Proposition 12, we now have the following statement.

**Proposition 16.** For a square-free integer $N$ and integers $N_1, N_2 | N$, let $f \in S_{k}^{\text{new}}(N_1)$, $g \in S_{k}^{\text{new}}(N_2)$ be normalized Hecke eigen forms, not necessarily distinct. Also let $\epsilon > 0$. Then for $\frac{1}{2} \leq \sigma < 1 + \Delta$ with $\Delta = \frac{1}{100}$ and $t \in \mathbb{R}$ with $|t| \gg 1$, we have

$$|R_{f,g}(\sigma + it)| \ll_{\epsilon} \left( kN \right)^{1+\Delta-\sigma} \cdot (\tau(N) \cdot N^\Delta)^\frac{1+\Delta-\sigma}{2(1+2\Delta)} \cdot e^{(c_1 \sqrt{\frac{\log(N+1)}{\log \log(N+2)}})} \cdot (3 + |t|)^{2(1+\Delta-\sigma) + \epsilon},$$

where $c_1 > 0$ is an absolute constant.

Combining Proposition 12, Corollary 11 and Corollary 15, we now get

**Proposition 17.** For a square-free integer $N$ and integers $N_1, N_2 | N$, let $f \in S_{k}^{\text{new}}(N_1)$, $g \in S_{k}^{\text{new}}(N_2)$ be normalized Hecke eigen forms, not necessarily distinct. Then for any $\epsilon > 0$, $t \in \mathbb{R}$, $\Delta = \frac{1}{100}$, we have

$$R_{f,g} \left( \frac{1}{2} + \frac{\Delta}{2} + it \right) \ll_{\epsilon} \left( kN \right)^{\frac{1}{2}(1+\Delta)} \cdot (\tau(N) \cdot N^\Delta)^\frac{1+\Delta}{2(1+2\Delta)} \cdot e^{(c_1 \sqrt{\frac{\log(N+1)}{\log \log(N+2)}})} \cdot (3 + |t|)^{1+\Delta+\epsilon}$$

and

$$R_{f,g} \left( \frac{3}{4} + it \right) \ll_{\epsilon} \left( kN \right)^{\frac{3}{4}+\Delta} \cdot (\tau(N) \cdot N^\Delta)^\frac{1+4\Delta}{2(1+2\Delta)} \cdot e^{(c_1 \sqrt{\frac{\log(N+1)}{\log \log(N+2)}})} \cdot (3 + |t|)^{\frac{1}{2}+2\Delta+\epsilon},$$

where $c_1 > 0$ is an absolute constant.

Further, we have the following Proposition.

**Proposition 18.** For square-free integer $N$ and $d, d' | N$, let $f \in S_{k}^{\text{new}}(d)$, $g \in S_{k}^{\text{new}}(d')$ be normalized Hecke eigen forms. For $\epsilon_{f,p}, \epsilon_{g,p} \in \{\pm 1\}$, define

$$F := f \mid \prod_{p|\Delta} (1 + \epsilon_{f,p}p^{k/2}V_p) \quad \text{and} \quad G := g \mid \prod_{p|\Delta} (1 + \epsilon_{g,p}p^{k/2}V_p)$$

as in Lemma 3. Also let $\epsilon > 0$. Then for $\frac{1}{2} \leq \sigma < 1 + \Delta$ with $\Delta = \frac{1}{100}$ and $t \in \mathbb{R}$ with $|t| \gg 1$, we have

$$|R_{F,G}(\sigma + it)| \ll_{\epsilon} \left( kN \right)^{1+\Delta-\sigma} \cdot N^{\frac{(1+\Delta-\sigma)}{2(1+2\Delta)}} \cdot \frac{1}{2} \cdot e^{(c_2 \sqrt{\frac{\log(N+1)}{\log \log(N+2)}})} \cdot (3 + |t|)^{2(1+\Delta-\sigma)+\epsilon},$$
where \( c_2 > 0 \) is an absolute constant. Further, we have for all \( t \in \mathbb{R} \)

\[
\left| R_{F,G} \left( \frac{1}{2} (1 + \Delta) + it \right) \right| \ll \epsilon \left( kN \right)^{\frac{1}{2}(1+\Delta)} \cdot N^{\frac{\Delta(1+\Delta)}{2(1+2\Delta)} + \frac{1}{2}} \cdot \epsilon^{-\left( c_2 \log(N+1) \right)} \cdot (3 + |t|)^{1+\Delta + \epsilon},
\]

and

\[
\left| R_{F,G} \left( \frac{3}{4} + it \right) \right| \ll \epsilon \left( kN \right)^{1+\Delta} \cdot N^{\frac{\Delta(1+4\Delta)}{4(1+2\Delta)} + \frac{1}{2}} \cdot \epsilon^{-\left( c_2 \log(N+1) \right)} \cdot (3 + |t|)^{\frac{1}{2} + 2\Delta + \epsilon},
\]

where the constant \( c_2 > 0 \) is absolute.

**Proof.** Note that

\[
R_{F,G}(\sigma + it) = \sum_{\substack{\delta \Delta \Delta' \sigma \sigma' \delta \delta' \sigma \sigma' \delta \delta' \sigma \sigma' \delta \delta'}} \epsilon_{f,\delta} \epsilon_{g,\delta'} \left( \frac{\delta\delta'}{T} \right)^{k/2} R_{f|V_\sigma|V_{\sigma'}}(\sigma + it),
\]

where

\[
\epsilon_{f,\delta} = \prod_{p|\delta} \epsilon_{f,p} \quad \text{and} \quad \epsilon_{g,\delta'} = \prod_{p|\delta'} \epsilon_{g,p}.
\]

We would now like to find bounds for \( R_{f|V_\sigma|V_{\sigma'}}(\sigma + it) \) when \( 1/2 \leq \sigma < 1 + \Delta \) and any \( t \in \mathbb{R} \) with \( |t| \gg 1 \). For \( T := \gcd(\delta, \delta') \), we have

\[
\prod_{p \mid \delta} \left| \frac{\lambda_f(p) - \lambda_g(p)}{p^{-\sigma}} \right| \ll 4^{\nu(\frac{T}{\delta})},
\]

and

\[
\prod_{p \mid \delta'} \left| \frac{\lambda_g(p) - \lambda_f(p)}{p^{-\sigma}} \right| \ll 4^{\nu(\frac{T}{\delta'})},
\]

where \( \nu(N) \) is the number of distinct prime factors of \( N \). Also

\[
\left| \prod_{p \mid N} \left( 1 - p^{-2\sigma - 2it} \right)^{-1} \right| \leq \prod_{p \mid N} \left( 1 - p^{-2\sigma} \right)^{-1} \leq \prod_{p \mid N} \left( 1 - p^{-1} \right)^{-1} \ll \prod_{p \mid N} \left( 1 + p^{-1} \right) \ll \log \log(N + 2).
\]

Hence by applying Lemma 5, we get

\[
\left| R_{f|V_\sigma|V_{\sigma'}}(\sigma + it) \right| \ll \left( \frac{\delta\delta'}{T} \right)^{-k/2} \cdot 4^{\nu(N)} \cdot \log \log(N + 2) \cdot |R_{f,g}(\sigma + it)|
\]
for all $\frac{1}{2} \leq \sigma < 1 + \Delta$ with $\Delta = \frac{1}{100}$ and any $t \in \mathbb{R}$ with $|t| \gg 1$. Thus for $\sigma$ and $t$ in the above mentioned range and for any $\epsilon > 0$, we get

$$|R_{F,G}(\sigma + it)| \ll_{\epsilon} 4^{\nu(N)} . (kN)^{1+\Delta-\sigma} . \tau(N). N^\Delta \frac{1+\Delta-\sigma}{1+2\Delta} \cdot \log \log(N+2) \cdot e^{(c_1 \sqrt{\frac{\log(N+1)}{\log\log(N+2)}})} \cdot \left( \sum_{\delta | N} \delta \delta' \right)^{-\frac{k}{2} + \frac{\Delta}{2}} . \sqrt{T} . (3 + |t|)^{2(1+\Delta-\sigma)+\epsilon}$$

by noting that

$$\left( \frac{\delta \delta'}{T^2} \right)^{-\frac{k}{2} + \frac{\Delta}{2}} \leq 1.$$ 

Finally using the inequality (see page 533 of [1])

$$\left( \sum_{\delta | N} \delta \delta' \right)^{-\frac{k}{2} + \frac{\Delta}{2}} \leq \sqrt{N} \cdot \tau^2(N),$$

where $\tau(N)$ is the number of divisors of $N$ and

$$4^{\nu(N)} \cdot \log \log(N+2) \cdot e^{(c_1 \sqrt{\frac{\log(N+1)}{\log\log(N+2)}})} \cdot \tau(N)^{1+\Delta-\sigma} \cdot \left( \sum_{\delta | N} \sqrt{\gcd(\delta, \delta')} \right) \ll \sqrt{N} \cdot e^{(c_2 \frac{\log(N+1)}{\log\log(N+2)})} \cdot (3 + |t|)^{2(1+\Delta-\sigma)+\epsilon},$$

where $c_2 > 0$ is an absolute constant, we get for any $\epsilon > 0$, $1/2 \leq \sigma < 1 + \Delta$ and $t \in \mathbb{R}$ with $|t| \gg 1$ that

$$|R_{F,G}(\sigma + it)| \ll_{\epsilon} (kN)^{1+\Delta-\sigma} \cdot N^\Delta \frac{1+\Delta-\sigma}{1+2\Delta} \cdot \sqrt{N} \cdot e^{(c_2 \frac{\log(N+1)}{\log\log(N+2)})} \cdot (3 + |t|)^{2(1+\Delta-\sigma)+\epsilon}.$$

This completes the proof of the first part of the theorem. Proceeding as in the first part and applying Proposition[17] we get for any $t \in \mathbb{R}$ and $\epsilon > 0$ that

$$\left| R_{F,G} \left( \frac{1}{2}(1 + \Delta) + it \right) \right| \ll_{\epsilon} (kN)^{\frac{1}{2}(1 + \Delta)} \cdot N^\Delta \frac{1+\Delta}{1+2\Delta} \cdot \sqrt{N} \cdot e^{(c_2 \frac{\log(N+1)}{\log\log(N+2)})} \cdot (3 + |t|)^{1+\Delta+\epsilon},$$

and

$$\left| R_{F,G} \left( \frac{3}{4} + it \right) \right| \ll_{\epsilon} (kN)^{\frac{3}{4} + \Delta} \cdot N^\Delta \frac{1+\Delta}{1+2\Delta} \cdot \sqrt{N} \cdot e^{(c_2 \frac{\log(N+1)}{\log\log(N+2)})} \cdot (3 + |t|)^{\frac{3}{4} + 2\Delta+\epsilon},$$

where the constant $c_2 > 0$ is absolute. This completes the proof of the Proposition.

We also need the following lemma of Choie and Kohnen.

**Lemma 19.** [Choie-Kohnen (see page 534 of [1])] Let $N$ be square-free and $d | N$. Also let $f \in S^\text{new}_k(d)$ be a normalized Hecke eigen form and $F$ be as in Lemma[6] Then

$$1 \ll \frac{(4\pi)^{k-1}}{(k-1)!} . \log(kN) . \prod_{p | N} \left( 1 + \frac{1}{p} \right) . e^{(c \sqrt{\frac{\log(N+1)}{\log\log(N+2)}})} \cdot < F, F >.$$
where \( \tilde{c} > 0 \) is an absolute constant.

Finally, we have the following Propositions which are important to complete the proof of Theorem 4.

**Proposition 20.** For a square-free \( N \), let \( f \) be a non-zero cusp form of weight \( k \) for \( \Gamma_0(N) \) with real Fourier coefficients. Let

\[
D(k, N) := \frac{2\pi^2 (4\pi)^{k-1}}{(k-1)!} \prod_{p \mid N} \left(1 + \frac{1}{p}\right)
\]

and

\[
(15) \quad \tilde{f} := \frac{f}{\sqrt{D(k, N)||f||}},
\]

where \( ||.|| \) is the Petersson norm. If for \( n \in \mathbb{N} \), \( \lambda_{\tilde{f}}(n) \) are normalized Fourier coefficients of \( \tilde{f} \), then for any \( 0 < \epsilon < \frac{12}{25} \) and \( \Delta = \frac{1}{100} \), we have

\[
\sum_{n \leq x} \lambda^2_{\tilde{f}}(n) \log \left(\frac{x}{n}\right) = \frac{6}{\pi^2} \prod_{p \mid N} \left(1 + \frac{1}{p}\right)^{-1} x + O_{\epsilon} \left((kN)^{\frac{k}{4} + \Delta} \cdot N^{\frac{1+4\Delta}{4(1+2\Delta)}} \cdot \sqrt{N} \cdot e^{c_3 \log(N+1)} \cdot \log k \cdot x^{3/4}\right),
\]

where \( c_3 > 0 \) is an absolute constant.

**Proof.** Note that if \( a_f(n) \) are the Fourier coefficients of \( f \), then \( \lambda_f(n) := a_f(n)/n^{\frac{k-1}{2}} \). We use Perron’s formula to write

\[
\sum_{n \leq x} \lambda^2_f(n) \log \left(\frac{x}{n}\right) = \frac{1}{2\pi i} \int_{1+\frac{3}{4}+i\infty}^{1+\frac{3}{4}+i\infty} R_{f,\tilde{f}}(s) \frac{x^s}{s^2} ds,
\]

where \( R_{f,\tilde{f}}(s) \) is the Rankin-Selberg \( L \)-function attached to \( \tilde{f} \). Then shifting the line of integration and using equations (8) and (15), we get

\[
\sum_{n \leq x} \lambda^2_f(n) \log \left(\frac{x}{n}\right) = \frac{6}{\pi^2} \prod_{p \mid N} \left(1 + \frac{1}{p}\right)^{-1} x + \frac{1}{2\pi i} \int_{\frac{3}{4}-i\infty}^{\frac{3}{4}+i\infty} R_{f,\tilde{f}}(s) \frac{x^s}{s^2} ds.
\]

We write

\[
\tilde{f} := \sum_{\tau=1}^{d_{k,N}} \alpha_{F_\tau} F_\tau,
\]

where \( d_{k,N} := \dim S_k(N) \) and \( \{F_\tau\}_{1 \leq \tau \leq d_{k,N}} \) is the special orthogonal basis of \( S_k(N) \) (in some fixed order) mentioned in Lemma 6. Set

\[
A_{\tilde{f}} := \sum_{\tau=1}^{d_{k,N}} |\alpha_{F_\tau}|.
\]
Since
\[ R_{\tilde{f}, \tilde{f}}(s) = \sum_{\tau, \tau'} \alpha_{\tau} \overline{\alpha}_{\tau'} R_{F_{\tau}, F_{\tau}'}(s), \]
using Proposition 18 we get
\[ \left| R_{F,G} \left( \frac{3}{4} + it \right) \right| \ll \epsilon \cdot d_{k,N} \cdot (kN)^{\frac{1}{4} + \Delta} \cdot N^{\frac{\Delta(1+4\Delta)}{4(1+2\Delta)}} \cdot \sqrt{N} \cdot e^{\left( \frac{\log(N+1)}{\log\log(N+2)} \right)} \cdot (3 + |t|)^{\frac{1}{2} + 2\Delta + \epsilon} \]
for all \( t \in \mathbb{R} \). We know by Chebyshef’s inequality that
\[ A_f^2 \leq d_{k,N} \sum_{\tau=1}^{d_{k,N}} |\alpha_{\tau}|^2 \]
and hence for any \( t \in \mathbb{R} \)
\[ \left| R_{\tilde{f}, \tilde{f}} \left( \frac{3}{4} + it \right) \right| \ll \epsilon \cdot d_{k,N} \cdot (kN)^{\frac{1}{4} + \Delta} \cdot N^{\frac{\Delta(1+4\Delta)}{4(1+2\Delta)}} \cdot \sqrt{N} \cdot e^{\left( \frac{\log(N+1)}{\log\log(N+2)} \right)} \cdot (3 + |t|)^{\frac{1}{2} + 2\Delta + \epsilon} \cdot \sum_{\tau=1}^{d_{k,N}} |\alpha_{\tau}|^2. \]
Note that
\[ \langle \tilde{f}, \tilde{f} \rangle = \sum_{\tau=1}^{d_{k,N}} |\alpha_{\tau}|^2 < F_{\tau}, F_{\tau} >. \]
since the basis \( \{ F_{\tau} \}_{1 \leq \tau \leq d_{k,N}} \) is orthogonal. Using this along with equation (15) and Lemma 19, we can write for any \( t \in \mathbb{R} \)
\[ (17) \quad \left| R_{\tilde{f}, \tilde{f}} \left( \frac{3}{4} + it \right) \right| \ll \epsilon \cdot d_{k,N} \cdot (kN)^{\frac{1}{4} + \Delta} \cdot N^{\frac{\Delta(1+4\Delta)}{4(1+2\Delta)}} \cdot \sqrt{N} \cdot e^{\left( \frac{\log(N+1)}{\log\log(N+2)} \right)} \cdot (3 + |t|)^{\frac{1}{2} + 2\Delta + \epsilon} \cdot \sum_{\tau=1}^{d_{k,N}} |\alpha_{\tau}|^2 \]
where the constants \( \tilde{c}_2 > 0 \) is absolute. Since
\[ (18) \quad d_{k,N} \ll k \cdot N \cdot \log\log(N + 2) \]
and choosing $\epsilon$ with $0 < \epsilon < \frac{12}{25}$, we now have

$$
\sum_{n \leq x} \lambda_f^2(n) \log \left( \frac{x}{n^2} \right) = \frac{6}{\pi^2} \prod_{p \mid \text{prime} \mid N} \left( 1 + \frac{1}{p} \right)^{-1} x + O_\epsilon \left( (kN)^{\frac{5}{2} + \Delta} \cdot N^{\Delta(1+2\Delta)} \cdot \sqrt{N} \cdot \epsilon \left( \frac{c_3 \log(N+1)}{\log \log(N+2)} \right) \cdot \log k \cdot x^{3/4} \right),
$$

where $c_3 > 0$ is an absolute constant. This completes the proof of the Proposition. \hfill \Box

**Proposition 21.** For a square-free $N$, let $f$ be a non-zero cusp form of weight $k$ for $\Gamma_0(N)$ with real Fourier coefficients. Also let $\tilde{f}$ be as in Proposition 20 and $\lambda_f(n)$ be the normalized $n$-th Fourier coefficient of $\tilde{f}$. We have

$$
\sum_{n \leq x} \lambda_f^2(n) = \frac{6}{\pi^2} \prod_{p \mid \text{prime} \mid N} \left( 1 + \frac{1}{p} \right)^{-1} x + O_\epsilon \left( (kN)^{\frac{5}{2} + \Delta} \cdot N^{\Delta(1+2\Delta)} \cdot \sqrt{N} \cdot \epsilon \left( \frac{c_4 \log(N+1)}{\log \log(N+2)} \right) \cdot \log k \cdot x^{3/4} \right),
$$

where $0 < \epsilon < \frac{97}{400}$, $\Delta = \frac{1}{100}$ and $c_4 > 0$ is an absolute constant.

**Proof.** If $a_f(n)$ are the Fourier coefficients of $\tilde{f}$, then $\lambda_f(n) := a_f(n)/n^{k/2}$. As in Proposition 20 we write

$$
\tilde{f} := \sum_{\tau=1}^{d_{k,N}} \alpha_\tau F_\tau,
$$

where $d_{k,N} := \dim S_k(N)$ and $\{F_\tau\}_{1 \leq \tau \leq d_{k,N}}$ is the special orthogonal basis of $S_k(N)$ (in some fixed order) mentioned in Lemma 6 and set

$$
A_\tilde{f} := \sum_{\tau=1}^{d_{k,N}} |\alpha_\tau|.
$$

If $F_\tau = f/\prod_{p \mid d} (1 + \epsilon_{f,p} k/2 V_p)$, where $d \mid N$ and $\epsilon_{f,\delta} = \prod_{p \mid \delta} \epsilon_{f,p}$, then

$$
|\lambda_{F_\tau}(n)| = \left| \sum_{\delta \mid N} \epsilon_{f,\delta} \delta^{1/2} \lambda_f \left( \frac{n}{\delta} \right) \right| \ll (\tau(N) \cdot \sqrt{N}) \cdot \tau(n),
$$

where $\tau(N)$ is the number of divisors of $N$ and $\lambda_f(n/\delta) = 0$ if $\delta \nmid n$. Hence

$$
|\lambda_f(n)|^2 = \left| \sum_{\tau=1}^{d_{k,N}} \alpha_\tau \lambda_{F_\tau}(n) \right|^2 \ll \sum_{1 \leq \tau, \tau' \leq d_{k,N}} |\alpha_\tau \alpha_{\tau'}| \cdot |\lambda_{F_\tau}(n) \lambda_{F_{\tau'}}(n)| \ll \tau^2(N) \cdot A_\tilde{f}^2 \cdot \tau^2(n) \ll k \cdot N^2 \cdot \log k \cdot e \left( \epsilon' \frac{\log(N+1)}{\log \log(N+2)} \right) \cdot n^{\epsilon_1},
$$
for any $\epsilon_1 > 0$ and absolute constant $c' > 0$. Now by Perron’s formula (see page 67 of [17]) and for $x \not\in \mathbb{N}$, we have

$$
\sum_{n \leq x} \lambda_f^2(n) = \frac{1}{2\pi i} \int_{1+\Delta/2-iT}^{1+\Delta/2+iT} R_{f,f}(s) \frac{x^s}{s} ds + O \left( k \cdot N^2 \cdot \log k \cdot e^{c' \log(N+1) \log(N+2)} \cdot \frac{x^{1+\Delta/2+\epsilon}}{T} \right),
$$

where $R_{f,f}(s)$ is the Rankin-Selberg $L$-function attached to $f, \Delta = \frac{1}{100}$ and $1 \leq T \leq x$ to be chosen later. Then shifting the line of integration and using equations (8) and (15), we get

$$
\sum_{n \leq x} \lambda_f^2(n) = \frac{6}{\pi^2} \prod_{p \mid N} (1 + 1/p)^{-1} x + \frac{1}{2\pi i} \int_{1+\Delta/2-iT}^{1+\Delta/2+iT} R_{f,f}(s) \frac{x^s}{s} ds
$$

where the constants $c'$ are absolute. Now by Perron’s formula (see page 67 of [17]) and choosing $T = x^{1/4}$, we get

$$
\sum_{n \leq x} \lambda_f^2(n) = \frac{6}{\pi^2} \prod_{p \mid N} (1 + 1/p)^{-1} x + \frac{1}{2\pi i} \int_{1+\Delta/2-iT}^{1+\Delta/2+iT} R_{f,f}(s) \frac{x^s}{s} ds
$$

where $0 < \epsilon < 97/400$ and $c_4 > 0$ is an absolute constant. This completes the proof. □
Proposition 22. For a square-free $N$, let $f$ be a non-zero cusp form of weight $k$ for $\Gamma_0(N)$ with real Fourier coefficients. Also let $\tilde{f}$ be as in Proposition 20 and $\lambda_{\tilde{f}}(n)$ be the normalized $n$-th Fourier coefficient of $\tilde{f}$. We then have

$$
\sum_{n \leq x} \lambda_f(n) \log \left( \frac{x}{n} \right) \ll \sqrt{k \log k} \cdot N^{\frac{5}{2} + \frac{\Delta}{2}} \cdot x^\frac{1}{2},
$$

$$
\sum_{n \leq x} \lambda_{\tilde{f}}(n) \ll \sqrt{k \log k} \cdot N^{\frac{5}{2} + \frac{\Delta}{2}} \cdot e^{(a_1 \frac{\log(N+1)}{\log(N+2)}) \cdot x^\frac{1}{2}},
$$

where $\Delta = \frac{1}{100}$ and $a_1, a_2 > 0$ are absolute constants.

We need the following Lemmas to prove Proposition 22.

Lemma 23. For a square-free $N$, let $f$ be a normalized newform of weight $k$ for $\Gamma_0(N)$. Then for any $-\Delta < \sigma < 1 + \Delta$ with $\Delta = \frac{1}{100}$ and any $t \in \mathbb{R}$, we have

$$
|L(f, \sigma + it)| \ll N^{\frac{1+\Delta-\sigma}{2}} \cdot (3 + |t|)^{1+\Delta-\sigma},
$$

where the constant in $\ll$ is absolute.

Proof of Lemma 23. It is easy to see that

$$
|L(f, 1 + \Delta + it)| \ll 1, \quad \forall \in \mathbb{R},
$$

where $L(f, s)$ is the Hecke $L$-function attached to $f$. Now using the functional equation

$$
L^*(f, s) = L^*(f, 1 - s),
$$

where $L^*(f, s) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f, s)$, we get

$$
|L(f, -\Delta + it)| \ll N^{1/2+\Delta} \cdot |1 + it|^{1+2\Delta}
$$

for any $t \in \mathbb{R}$. Finally using Proposition 9 we get for any $-\Delta < \sigma < 1 + \Delta$ and any $t \in \mathbb{R}$ that

$$
|L(f, \sigma + it)| \ll N^{\frac{(1+\Delta-\sigma)}{2}} \cdot (3 + |t|)^{1+\Delta-\sigma},
$$

where the constant in $\ll$ is absolute. This completes the proof of Lemma 23.

Lemma 24. For square-free integer $N$ and $d | N$, let $f \in S^\text{new}_k(d)$ be normalized Hecke eigen forms. For $\epsilon_{f,p} \in \{\pm 1\}$, define

$$
F := f \mid \prod_{p | \delta} (1 + \epsilon_{f,p} V_p^{k/2}) = \sum_{\delta | \Delta} \epsilon_{f,\delta} V_\delta^{k/2} f | V_\delta, \quad \epsilon_{f,\delta} := \prod_{p | \delta} \epsilon_{f,p}
$$

as in Lemma 6. Also let $\epsilon > 0$. Then for $-\Delta < \sigma < 1 + \Delta$ with $\Delta = \frac{1}{100}$ and any $t \in \mathbb{R}$, we have

$$
|L(F, \sigma + it)| \ll \tau_{1/2-\sigma}(N) \cdot N^{\frac{(1+\Delta-\sigma)}{2}} \cdot (3 + |t|)^{1+\Delta-\sigma},
$$

where the constant in $\ll$ is absolute and $\tau_\ell(N) := \sum_{d | N} d^\ell$. 

Proof of Lemma 24. By definition, for any $z \in \mathcal{H}$, we have
\[
F(z) = \sum_{\delta|\frac{N}{2}} \epsilon_f,\delta \delta^{k/2}(f|V\delta)(z) = \sum_{\delta|\frac{N}{2}} \epsilon_f,\delta \delta^{k/2} f(\delta z).
\]
This implies that
\[
\lambda_F(n) = \sum_{\delta|\frac{N}{2}} \epsilon_f,\delta \sqrt{\delta} \lambda_f\left(\frac{n}{\delta}\right).
\]
Hence for any $s \in \mathbb{C}$ with $\Re(s) > 1$, we have
\[
L(F,s) = (\sum_{\delta|\frac{N}{2}} \epsilon_f,\delta \delta^{-s+1/2}) L(f,s).
\]
Thus for any $0 < \sigma < 1 + \Delta$ and any $t \in \mathbb{R}$, we have
\[
|L(F,\sigma + it)| \ll \tau_{1/2-\sigma}(N) \cdot |L(f,\sigma + it)| \ll \tau_{1/2-\sigma}(N) \cdot N^{\frac{1+\Delta-\sigma}{2}} \cdot (3 + |t|)^{1+\Delta-\sigma}.
\]
This completes the proof of Lemma 24.

We now complete the proof of Proposition 22.

Proof. As before, we write
\[
\tilde{f} := \sum_{\tau=1}^{d_{k,N}} \alpha_{\tau} F_{\tau},
\]
where $d_{k,N} := \dim S_k(N)$ and $\{F_{\tau}\}_{1 \leq \tau \leq d_{k,N}}$ is the special orthogonal basis of $S_k(N)$ (in some fixed order) mentioned in Lemma 6 and set
\[
A_{\tilde{f}} := \sum_{\tau=1}^{d_{k,N}} |\alpha_{\tau}|.
\]
Since
\[
L(\tilde{f},s) = \sum_{\tau} \alpha_{\tau} L(F_{\tau},s),
\]
using Lemma 24 we get for any $-\Delta < \sigma < 1 + \Delta$ and any $t \in \mathbb{R}$
\[
|L(\tilde{f},\sigma + it)| \ll A_{\tilde{f}} \cdot \tau_{1/2-\sigma}(N) \cdot N^{\frac{1+\Delta-\sigma}{2}} \cdot (3 + |t|)^{1+\Delta-\sigma}
\]
(22)
\[
\ll \sqrt{k} \cdot \sqrt{\log k} \cdot e^{\left(b_1 \sqrt{\frac{\log(N+1)}{\log \log(N+2)}}\right)} \cdot \tau_{1/2-\sigma}(N) \cdot N^{\frac{1+\Delta-\sigma}{2}} \cdot (3 + |t|)^{1+\Delta-\sigma},
\]
where $b_1$ is an absolute constant and also we have used the estimate
\[
A_{\tilde{f}} \ll \sqrt{k} N \log k \cdot e^{\left(b_1 \sqrt{\frac{\log(N+1)}{\log \log(N+2)}}\right)}.
\]
In particular, for $\sigma = \frac{1}{5}$, we have
\[
|L(\tilde{f}, \frac{1}{5} + it)| \ll \sqrt{k} \log k \cdot N^{\frac{k}{5} + \frac{5}{2}} \cdot e^{\left(b_1 \sqrt{\frac{\log(N+1)}{\log \log(N+2)}}\right)} \cdot (3 + |t|)^{\frac{k}{5}+\Delta} \quad \forall \ t \in \mathbb{R},
\]
(23)
where $a_1$ is an absolute constant. Then
\[
\sum_{n \leq x} \lambda_f(n) \log \left( \frac{x}{n} \right) = \frac{1}{2\pi i} \int_{1+\frac{\Delta}{2}-iT}^{1+\frac{\Delta}{2}+iT} L(\tilde{f}, s) \frac{x^s}{s} \, ds.
\]

Now by shifting the line of integration to the line $\sigma = \frac{1}{5}$, we get
\[
\sum_{n \leq x} \lambda_f(n) \log \left( \frac{x}{n} \right) \ll \sqrt{k \log k} \cdot N^{\frac{\alpha}{5} + \frac{\Delta}{5}} \cdot \exp\left(a_1 \frac{\log(N+1)}{\log\log(N+2)}\right) \cdot x^{\frac{1}{5}}.
\]

For the second part of the Proposition, we proceed as in Proposition 21. First note that
\[
|\lambda_f(n)| = \left| \sum \alpha_\tau \lambda_{F_\tau}(n) \right| \ll A_f \cdot \tau(N) \cdot \sqrt{N} \cdot \tau(n)
\]
where $a'_1$ is an absolute constant and hence by Perron’s formula for $x \notin \mathbb{Z}$, we have
\[
\sum_{n \leq x} \lambda_f(n) = \frac{1}{2\pi i} \int_{1+\frac{\Delta}{2}-iT}^{1+\frac{\Delta}{2}+iT} L(\tilde{f}, s) \frac{x^s}{s} \, ds + O \left( \sqrt{k \log k} \cdot N \cdot \exp\left(a'_1 \frac{\log(N+1)}{\log\log(N+2)}\right) \cdot x^{1+\Delta} \right),
\]
where $1 \leq T \leq x$ to be chosen later. Then shifting the line of integration, we get
\[
\frac{1}{2\pi i} \int_{1+\frac{\Delta}{2}-iT}^{1+\frac{\Delta}{2}+iT} L(\tilde{f}, s) \frac{x^s}{s} \, ds = \frac{1}{2\pi i} \left( \int_{1+\frac{\Delta}{2}-iT}^{\frac{1}{5}-iT} + \int_{\frac{1}{5}-iT}^{\frac{1}{5}+iT} + \int_{\frac{1}{5}+iT}^{1+\frac{\Delta}{2}+iT} \right) L(\tilde{f}, s) \frac{x^s}{s} \, ds
\]
\[
= Y_1 + Y_2 + Y_3, \quad \text{say}.
\]
Using equation (23), we get
\[
|Y_2| \ll \sqrt{k \log k} \cdot N^{\frac{\alpha}{5} + \frac{\Delta}{5}} \cdot \exp\left(a'_1 \frac{\log(N+1)}{\log\log(N+2)}\right) \cdot x^{\frac{1}{5} - \Delta} \cdot T^{1/5}.
\]

Now using (22), we get
\[
|Y_1 + Y_3| \ll \sqrt{k \log k} \cdot \exp\left(a_2 \frac{\log(N+1)}{\log\log(N+2)}\right) \cdot N^{\frac{\alpha}{5} + \frac{\Delta}{5}} \cdot \frac{1}{T} \left\{ \max_{\frac{1}{5} \leq \sigma \leq 1+\frac{\Delta}{2}} \left( \frac{1}{T} \right) \right\} \cdot T^{1+\Delta},
\]
where the constant in $\ll$ and the constant $a_2 > 0$ are absolute. Now by choosing $T = x^{1/2}$, we get
\[
\sum_{n \leq x} \lambda_f(n) \ll \sqrt{k \log k} \cdot \exp\left(a_2 \frac{\log(N+1)}{\log\log(N+2)}\right) \cdot N^{\frac{\alpha}{5} + \frac{\Delta}{5}} \cdot x^{\frac{1}{2} + \Delta}.
\]

This completes the proof. \hfill \Box
We are now in a position to complete the proof of Theorem 4.

**Proof of Theorem 4.** Let $D(k, N)$ be as in Proposition 20 and as before we normalise $f$ by the following condition

\[(26) \quad D(k, N). \langle f, f \rangle = 1,\]

where $\langle, \rangle$ is the Petersson inner product. This means we replace $f$ by

\[\tilde{f} := \frac{f}{\sqrt{D(k, N)||f||}}.\]

Let $\tilde{\beta}(n)$ be the Fourier coefficients of $\tilde{f}$ and $\tilde{\beta}(n) := \lambda_f(n)n^{k-1}$. Note that $\tilde{\beta}(n)$ and $\lambda_f(n)$ have the same sign. Hence it is sufficient to show the sign change of $\lambda_f(n)$ in the desired range.

Assume the contrary, i.e. $\lambda_f(n)$ has constant sign for $n \in (x, x + h]$, where $h = x^{13/14+\epsilon}$ and $\epsilon > 0$. Without loss of generality, we can assume that $0 < \epsilon < 1/200$ as constant sign in a bigger interval implies constant sign in a smaller interval. Since we can always replace $\tilde{f}$ by $-\tilde{f}$, we can assume that $\lambda_f(n) \geq 0$ for all $n \in (x, x + h]$.

By Cauchy-Schwartz inequality, we have

\[(27) \quad \sum_{x < n \leq x + h} \lambda_f^2(n) \log^2 \left( \frac{x + h}{n} \right) \leq \left( \sum_{x < n \leq x + h} |\lambda_f(n)| \log^2 \left( \frac{x + h}{n} \right) \right)^{1/2} \left( \sum_{x < n \leq x + h} |\lambda_f(n)|^3 \log^2 \left( \frac{x + h}{n} \right) \right)^{1/2} \]

\[= \left( \sum_{x < n \leq x + h} \lambda_f(n) \log^2 \left( \frac{x + h}{n} \right) \right)^{1/2} \left( \sum_{x < n \leq x + h} \lambda_f^3(n) \log^2 \left( \frac{x + h}{n} \right) \right)^{1/2},\]

by our assumption on $\lambda_f(n)$. We will estimate the left hand side of (27) from below and the two terms on the right hand side from above and derive a contradiction for sufficiently large $x$. Note that

\[\sum_{x < n \leq x + h} \lambda_f^2(n) \log^2 \left( \frac{x + h}{n} \right) = \sum_{n \leq x + h} \lambda_f^2(n) \log^2 \left( \frac{x + h}{n} \right) - \sum_{n \leq x} \lambda_f^2(n) \log^2 \left( \frac{x + h}{n} \right)\]

\[= \sum_{n \leq x + h} \lambda_f^2(n) \log^2 \left( \frac{x + h}{n} \right) - \sum_{n \leq x} \lambda_f^2(n) \log^2 \left( \frac{x}{n} \right) - \log^2 \left( 1 + \frac{h}{x} \right) \sum_{n \leq x} \lambda_f^2(n)\]

\[\quad - 2 \log \left( 1 + \frac{h}{x} \right) \sum_{n \leq x} \lambda_f^2(n) \log \left( \frac{x}{n} \right).\]
We know from page 538 of [1] that

\[ \sum_{n \leq y} \lambda_j^2(n) \log^2 \left( \frac{y}{n} \right) = \frac{12}{\pi^2} \prod_{p \mid N \text{ prime}} (1 + 1/p)^{-1} y + O \left( k^{3/2} \cdot N^2 \cdot \log^5 k \cdot \sqrt{\Phi_k(N)} \cdot e^{\tilde{c}_1 \log(N+1) \log \log(N+2) \cdot y^{1/2}} \right), \]

where \( \Phi_k(N) \) is as in (2) and \( \tilde{c}_1 > 0 \) is an absolute constant. Hence we have

\[ \sum_{n \leq x+h} \lambda_j^2(n) \log^2 \left( \frac{x+h}{n} \right) - \sum_{n \leq x} \lambda_j^2(n) \log^2 \left( \frac{x}{n} \right) = \frac{12}{\pi^2} \prod_{p \mid N \text{ prime}} (1 + 1/p)^{-1} h + O \left( k^{3/2} \cdot N^2 \cdot \log^5 k \cdot \sqrt{\Phi_k(N)} \cdot e^{\tilde{c}_1 \log(N+1) \log \log(N+2) \cdot x^{1/2}} \right). \]

By Proposition 20 and Proposition 21, we know that

\[ \sum_{n \leq x} \lambda_j^2(n) \log \left( \frac{x}{n} \right) = \frac{6}{\pi^2} \prod_{p \mid N \text{ prime}} (1 + 1/p)^{-1} x + O \left( (kN)^{\frac{1}{2} + \Delta} \cdot N^{\frac{\Delta(1+\Delta)}{4(1+2\Delta)}} \cdot \sqrt{N} \cdot c_3 \log(N+1) \cdot \log k \cdot x^{3/4} \right) \]

\[ \sum_{n \leq x} \lambda_j^2(n) = \frac{6}{\pi^2} \prod_{p \mid N \text{ prime}} (1 + 1/p)^{-1} x + O \left( (kN)^{\frac{3}{2} + \Delta} \cdot N^{\frac{\Delta(1+\Delta)}{2(1+2\Delta)}} \cdot \sqrt{N} \cdot \log k \cdot c_4 \log(N+1) \cdot \log \log(N+2) \cdot x^{\frac{3}{2}(1+\Delta)+\epsilon} \right), \]

where \( c_3 > 0, c_4 > 0 \) are absolute constants and \( \Delta = \frac{1}{100} \). Thus by using the above identities, we get

\[ \sum_{x < n \leq x+h} \lambda_j^2(n) \log^2 \left( \frac{x+h}{n} \right) = \frac{12}{\pi^2} \prod_{p \mid N \text{ prime}} (1 + 1/p)^{-1} h - \log^2 \left( 1 + \frac{h}{x} \right) \sum_{n \leq x} \lambda_j^2(n) - 2 \log \left( 1 + \frac{h}{x} \right) \sum_{n \leq x} \lambda_j^2(n) \log \left( \frac{x}{n} \right) + O \left( k^{3/2} \cdot N^2 \cdot \log^5 k \cdot \sqrt{\Phi_k(N)} \cdot e^{\tilde{c}_1 \log(N+1) \log \log(N+2) \cdot x^{1/2}} \right) \]

\[ = \frac{2}{\pi^2} \prod_{p \mid N \text{ prime}} (1 + 1/p)^{-1} \frac{h^3}{x^2} + O \left( (kN)^{\frac{3}{2} + \Delta} \cdot N^{\frac{\Delta(1+\Delta)}{2(1+2\Delta)}} \cdot \sqrt{N} \Phi_k(N) \cdot \log^5 k \cdot e^{\tilde{c}_4 \log(N+1) \log \log(N+2) \cdot x^{5/7+4\epsilon}} \right), \]

where the last equality follows from the identity

\[ \log(1 + \frac{h}{x}) = \frac{h}{x} - \frac{h^2}{2x^2} + \frac{h^3}{3x^3} + O \left( \frac{h^4}{x^4} \right) \]
for $x \gg 1$ and $\tilde{c}_5 > 0$ is an absolute constant. Thus we have

$$\sum_{x < n \leq x + h} \lambda_f^2(n) \log^2 \left( \frac{x + h}{n} \right) \gg \frac{h^3}{x^2 \log \log (N + 2)};$$

where the constant in $\gg$ is absolute and

$$x \gg \epsilon \left( (kN)^{\frac{3}{2} + \frac{\Delta}{2}} \cdot N^{\frac{\Delta(1 + \Delta)}{2(1 + 2\Delta)}} \cdot \sqrt{N\Phi_k(N)} \cdot \log^5 k \cdot e^{(c_5 \log(N + 1)) / \log(N + 2)} \right)^{\frac{1}{1 - 14\epsilon}},$$

where $c_5 > 0$ is an absolute constant. From (24), we get

$$\sum_{x < n \leq x + h} \lambda_f(n)^3 \log^2 \left( \frac{x + h}{n} \right) \ll A(k, N) \cdot x \log^9 x,$$

where

$$A(k, N) := k^{3/2} \cdot N^3 \cdot \log^{5/2} k \cdot e^{c_6' \log(N + 1) / \log(N + 2)}$$

and the constant $c_6' > 0$ and the constant in $\ll$ are absolute. Now from page 540, equation (8.7) of [1] and using Proposition 22, we get

$$\left| \sum_{x < n \leq x + h} \lambda_j(n) \log^2 \left( \frac{x + h}{n} \right) \right| = \left| \sum_{n \leq x + h} \lambda_j(n) \log^2 \left( \frac{x + h}{n} \right) - \sum_{n \leq x} \lambda_j(n) \log^2 \left( \frac{x + h}{n} \right) - \sum_{n \leq x} \lambda_j(n) \log \left( \frac{x}{n} \right) \right|$$

$$\ll A(k, N) \cdot x^{1/2} + \left| 2 \log \left( 1 + \frac{h}{x} \right) \sum_{n \leq x} \lambda_j(n) \log \left( \frac{x}{n} \right) \right|$$

$$+ \left| \log^2 \left( 1 + \frac{h}{x} \right) \sum_{n \leq x} \lambda_j(n) \right|$$

$$\ll k^{3/2} \cdot N^3 \cdot \log^{5/2} k \cdot e^{c_6' \log(N + 1) / \log(N + 2)} \cdot x^{1/2},$$

where the constant $c_6' > 0$ and the constant in $\ll$ are absolute. Therefore an upper bound for the right hand side of the inequality (27) is $\ll \epsilon x^{3/4 + \epsilon}$ which contradicts the lower bound in (29) when

$$x \gg \epsilon \max \left\{ \left[ k^{3/2} \cdot N^3 \cdot \log^{5/2} k \cdot e^{c_6' \log(N + 1) / \log(N + 2)} \right]^{\frac{28}{1 + 14\epsilon}}, \right.$$  

$$\left[ (kN)^{\frac{3}{2} + \frac{\Delta}{2}} \cdot N^{\frac{\Delta(1 + \Delta)}{2(1 + 2\Delta)}} \cdot \sqrt{N\Phi_k(N)} \cdot \log^5 k \cdot e^{(c_5 \log(N + 1)) / \log(N + 2)} \right]^{\frac{14}{1 - 14\epsilon}} \right\},$$

where $c_5, c_6 > 0$ are absolute constants. Note that for any sufficiently small $\eta$, one knows that $\log \Phi_k(N) \leq (1.7 + \eta) \log N$ if $N \geq k \geq 16$ and $\log \Phi_k(N) \leq (1.7 + \eta) \log k$ if $k \geq N \geq 16$. Hence
for any $0 < \epsilon < 1/200$, one has

$$\max \left\{ \left[ k^{3/2} \cdot N^{3} \cdot \log^{5/2} k \cdot e^{c_{6} \log(N+1) \log log(N+2)} \right]^{28 \epsilon}, \right.$$  

$$\left[ (kN)^{3/2 + \Delta} \cdot N^{\Delta(1+\Delta)} \cdot \sqrt{N\Phi_k(N)} \cdot \log^{5} k \cdot e^{(c_{5} \log(N+1) \log log(N+2))} \right]^{14 \epsilon + 1 - 14 \epsilon} \right\} \ll k^{42} \cdot N^{84} \cdot \log^{80} k \cdot e^{(c_{7} \log(N+1) \log log(N+2))},$$

where $c_{7} > 0$ is an absolute constant. Thus there exists $n_{1} \in (x, x + h]$ such that $\lambda_{f}(n_{1}) < 0$ and therefore $\beta(n_{1}) < 0$. This concludes the proof of the theorem.

4. PROOF OF THEOREM 1

Let the notations be as in Theorem 3, $\epsilon > 0$ and

$$x \gg \epsilon \cdot k^{42} \cdot N^{84} \cdot \log^{80} k \cdot e^{(c_{7} \log(N+1) \log log(N+2))},$$

where $c_{7} > 0$ is an absolute constant. Also let $\epsilon_{1} := \epsilon/4$ and $h_{1} := x^{13/14+\epsilon_{1}}$. Using Theorem 4 on $\chi_{\alpha}$, we get $n_{1}, n_{2} \in (x, x + h_{1}]$ such that $B(n_{1}) > 0$ and $B(n_{2}) < 0$. Then arguing as in the proof of Theorem 3, there exist a pair $(n, r)$ such that $c(n, r) < 0$ where $n = n_{1}$ or $n = n_{2}$ and $r = r_{1}$ or $r = r_{2}$. Since

$$c(n, r) = a\left(\begin{pmatrix} n & r/2 \\ r/2 & m_{0} \end{pmatrix}\right),$$

we have

$$a\left(\begin{pmatrix} n & r/2 \\ r/2 & m_{0} \end{pmatrix}\right) < 0.$$  

Now

$$\text{tr} \left(\begin{pmatrix} n & r/2 \\ r/2 & m_{0} \end{pmatrix}\right) = n + m_{0}$$

with $x < n + m_{0} \leq x + h_{1} + c(k, N)$, where

$$c(k, N) := \frac{4}{3\sqrt{3\pi}} kN^{3} \prod_{p|N} (1 + 1/p)(1 + 1/p^{2}).$$

Hence

$$n + m_{0} \leq x + 2h_{1} \leq x + h.$$

This completes the proof of Theorem 1.

Acknowledgements. We would like to thank S. Böcherer and W. Kohnen for providing us certain references. We also thank C. Poor for providing us the bound in equation (14). The authors would like to thank the referee for his/her suggestions which improved the exposition of the
paper and also for pointing out an inaccuracy in the previous version of the paper. We would also like to thank R. Balasubramanian, Biplab Paul and Purusottam Rath for going through an earlier version of the paper. Part of the work was done when the first author was visiting ICTP as a regular associate and would like to thank ICTP for the excellent working facilities.

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