Matrix spherical analysis on nilmanifolds

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Abstract. Given a nilpotent Lie group $N$, a compact subgroup $K$ of automorphisms of $N$ and an irreducible unitary representation $(\tau, W_\tau)$ of $K$, we study conditions on $\tau$ for the commutativity of the algebra of $\text{End}(W_\tau)$-valued integrable functions on $N$, with an additional property that generalizes the notion of $K$-invariance. A necessary condition, proved by F. Ricci and A. Samanta, is that $(K \ltimes N, K)$ must be a Gelfand pair. In this article we determine all the commutative algebras from a particular class of Gelfand pairs constructed by J. Lauret.

1 Introduction

Let $N$ be a connected and simply connected nilpotent Lie group endowed with a left-invariant Riemannian metric determined by an inner product $\langle \cdot, \cdot \rangle$ on its Lie algebra $n$. The Riemannian manifold $(N, \langle \cdot, \cdot \rangle)$ is said to be a homogeneous nilmanifold (cf. [L1]). Its isometry group is given by the semidirect product $K \ltimes N$, where $K$ is the group of orthogonal automorphisms of $N$ (cf. [Wi]). The action of $K$ on $N$ will be denoted by $k \cdot x$. We say that $(K, N)$ (or $(K \ltimes N, K)$) is a Gelfand pair when the convolution algebra of $K$-invariant integrable functions on $N$ is commutative. Let $(\tau, W_\tau)$ be an irreducible unitary representation of $K$. We say that $(K, N, \tau)$ is a commutative triple when the convolution algebra of $\text{End}(W_\tau)$-valued integrable functions on $N$ such that $F(k \cdot x) = \tau(k)F(x)\tau(k)^{-1}$ (for all $k \in K$ and $x \in N$) is commutative. Finally, we say that $(K, N, \tau)$ is a strong Gelfand pair if $(K, N, \tau)$ is commutative for every $\tau \in \hat{K}$ (where $\hat{K}$ denotes the set of equivalence classes of irreducible unitary representations of $K$). It is shown in [RS] that if there exists $\tau \in \hat{K}$ such that $(K, N, \tau)$ is a commutative triple, then $(K, N)$ is a Gelfand pair.

Starting from a faithful real representation $V$ of a compact Lie algebra $g$, J. Lauret constructs in [L] a two-step nilpotent Lie group $N(g, V)$ and he gives a full classification of the Gelfand pairs $(K, N(g, V))$. The aim of this article is to determine the nontrivial commutative triples occurring for the indecomposable Gelfand pairs given there. We will restrict our attention to the groups $N(g, V)$ having square integrable representations. In these cases we will see that it is possible to reduce the problem to studying the commutativity of triples $(K', H, \tau|_{K'})$, where $H$ is a Heisenberg group and $K'$ is a subgroup of the orthogonal automorphisms of $H$ contained in $K$ and $\tau|_{K'}$ denotes the restriction of $\tau$ to $K'$.

The main goal of this paper will be to prove the following theorem.

**Theorem 1.** Let $(K, N(g, V))$ be an indecomposable Gelfand pair such that $N(g, V)$ has a square integrable representation. The complete list of commutative triples $(K, N(g, V), \tau)$ with $\tau \in \hat{K}$ is the following:
2 SOME PRELIMINARY RESULTS

• \((\text{SU}(2) \times \text{Sp}(n), N(\text{su}(2), (\mathbb{C}^2)^n), \tau), \ n \geq 1\) (Heisenberg-type), where \(\mathbb{C}^2\) denotes the standard representation of \(\text{su}(2)\) regarded as a real representation (and \(\text{su}(2)\) acts on \((\mathbb{C}^2)^n\) component-wise) and \(\tau \in \text{SU}(2)\) or \(\tau \in \text{Sp}(n)\) with highest weight associated to a constant partition of length at most \(n\).

• \((\text{SU}(n) \times S^1, N(\text{su}(n), \mathbb{C}^n), \tau), \ n \geq 3\), where \(\mathbb{C}^n\) denotes the standard representation of \(\text{su}(n)\) regarded as a real representation and \(\tau\) is a character of \(S^1\).

• \((\text{SU}(n) \times S^1, N(u(n), \mathbb{C}^n), \tau), \ n \geq 3\), where \(\mathbb{C}^n\) denotes the standard representation of \(u(n)\) regarded as a real representation and \(\tau\) is a character of \(S^1\).

• \((\text{SU}(2) \times \text{U}(k) \times \text{Sp}(n), N(u(2), (\mathbb{C}^2)^k \oplus (\mathbb{C}^2)^n), \tau), \ k \geq 1, n \geq 0\), where the center of \(u(2)\) acts non-trivially only on \((\mathbb{C}^2)^k\), \((\mathbb{C}^2)^n\) denotes the representation of \(\text{su}(2)\) stated in the first item and \(u(2)\) acts component-wise on \((\mathbb{C}^2)^k\) in the standard way regarded as a real representation and \(\tau \in \hat{U}(k)\).

• \((K, N(\mathfrak{g}, V), \tau)\) where:
  
  - \(\mathfrak{g} := \text{su}(m_1) \oplus \cdots \oplus \text{su}(m_\beta) \oplus \text{su}(2) \oplus \cdots \oplus \text{su}(2) \oplus \mathfrak{c}\), where there are \(\alpha\) copies of \(\text{su}(2)\), \(m_i \geq 3\) for all \(1 \leq i \leq \beta\) and \(\mathfrak{c}\) is an abelian component.

  - \(V := \mathbb{C}^{m_1} \oplus \cdots \oplus \mathbb{C}^{m_\beta} \oplus \mathbb{C}^{2k_1 + 2n_1} \oplus \cdots \oplus \mathbb{C}^{2k_\alpha + 2n_\alpha}\), where \(k_j \geq 1\) and \(n_j \geq 0\) for all \(1 \leq j \leq \alpha\).

  - \(\mathfrak{g}\) is acting on \(V\) as follows: For each \(1 \leq i \leq \beta + \alpha\), \(\mathfrak{c}\) has a maximal subspace, denoted by \(\mathfrak{c}_i\), and with \(\dim(\mathfrak{c}_i) = 1\), acting non-trivially only on one component of \(V\). For \(1 \leq i \leq \beta\), \(\text{su}(m_i) \oplus \mathfrak{c}_i\) acts non-trivially only on \(\mathbb{C}^{m_i}\) and for \(\beta + 1 < i \leq \beta + \alpha\), \(\text{su}(2) \oplus \mathfrak{c}_i\) acts non-trivially only on \(\mathbb{C}^{2k_i + 2n_i}\).

  - \(K = G \times U\) where
    
    - \(G := \text{SU}(m_1) \times \cdots \times \text{SU}(m_\beta) \times \text{SU}(2) \times \cdots \times \text{SU}(2)\), with \(\alpha\) copies of \(\text{SU}(2)\) and
    
    - \(U := S^1 \times \cdots \times S^1 \times U(k_1) \times \text{Sp}(m_1) \times \cdots \times U(k_\alpha) \times \text{Sp}(n_\alpha)\), with \(\beta\) copies of \(S^1\).

  - \(\tau \in \hat{S}^1 \times \cdots \times \hat{S}^1 \times \hat{U}(k_1) \times \cdots \times \hat{U}(k_\alpha)\).

• \((U(n), N(\mathbb{R}, \mathbb{C}^n))\), \(n \geq 1\) (Heisenberg group) is a strong Gelfand pair (proved in \([Y]\)).

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2 Some preliminary results

Let \(N\) be a connected and simply connected nilpotent Lie endowed with a left-invariant Riemannian metric determined by an inner product \((\cdot, \cdot)\) on its Lie algebra. We denote by \(\hat{N}\) the set of equivalence classes of irreducible unitary representations of \(N\). The Plancherel theorem states that there exists a measure \(\mu\) on \(\hat{N}\) such that

\[
\int_{\hat{N}} |f(x)|^2 dx = \int_{\hat{N}} \|\rho(f)\|_{HS}^2 d\mu(\rho),
\]

for all \(f \in L^1(N) \cap L^2(N)\), where \(dx\) is the Haar measure on \(N\), \(HS\) stands for the Hilbert-Schmidt norm, and for \(\rho \in \hat{N}\), \(\rho(f)\) is the operator defined by \(\rho(f) := \int_N f(x)\rho(x)dx\). It has a natural extension
to matrix-valued functions as follows. Let $M_m(\mathbb{C})$ denote the set of all square $m \times m$ matrices over $\mathbb{C}$.

**Theorem 2.** Let $F \in L^1(N, M_m(\mathbb{C})) \cap L^2(N, M_m(\mathbb{C}))$. Then

$$\|F\|_2^2 = \int_{\hat{N}} \|\rho(F)\|_{HS}^2 \, d\mu(\rho),$$

where $\rho(F)$ is the operator defined by $\rho(F) := \int_N \rho(x) \otimes F(x) \, dx$.

**Proof.** Let $\{e_i\}_{i=1}^m$ be a basis of $\mathbb{C}^m$. The classical Plancherel theorem holds for all the matrix entries $F_{i,j}(x) := \langle F(x)e_j, e_i \rangle$ of $F$. Therefore

$$\|F\|_2^2 := \sum_{i,j=1}^m \|F_{i,j}\|_2^2 = \sum_{i,j=1}^m \int_{\hat{N}} \|\rho(F_{i,j})\|_{HS}^2 \, d\mu(\rho).$$

For each $(\rho, H_\rho) \in \hat{N}$, let $\{h_\alpha\}$ be an orthonormal basis of the Hilbert space $H_\rho$, whence

$$\sum_{i,j=1}^m \int_{\hat{N}} \|\rho(F_{i,j})\|_{HS}^2 \, d\mu(\rho) = \sum_{i,j=1}^m \int_{\hat{N}} \sum_{\alpha,\beta} |\langle \rho(F_{i,j})h_\alpha, h_\beta \rangle|^2 \, d\mu(\rho)$$

$$= \sum_{i,j=1}^m \int_{\hat{N}} \sum_{\alpha,\beta} \int_N |\langle \rho(x)h_\alpha, h_\beta \rangle\langle F(x)e_j, e_i \rangle dx|^2 \, d\mu(\rho)$$

$$= \int_{\hat{N}} \sum_{i,j=1}^m \sum_{\alpha,\beta} \int_N (\langle \rho(x) \otimes F(x)(h_\alpha \otimes e_j), h_\beta \otimes e_i \rangle dx|^2 \, d\mu(\rho)$$

$$= \int_{\hat{N}} \sum_{i,j=1}^m \sum_{\alpha,\beta} |\langle \rho(F)(h_\alpha \otimes e_j), h_\beta \otimes e_i \rangle|^2 \, d\mu(\rho)$$

$$= \int_{\hat{N}} \|\rho(F)\|_{HS}^2 \, d\mu(\rho).$$

\[ \square \]

Let $Z$ be the center of $N$. A representation $(\rho, H_\rho) \in \hat{N}$ is said to be **square integrable** if its matrix coefficients $(u, \rho(x)v)$, for $u, v \in H_\rho$, are square integrable functions on $\hat{N}$ module $Z$. We denote by $\hat{N}_{sq}$ the subset of $\hat{N}$ of square integrable classes. Theorem 14.2.14 in [Wo] states that if $N$ has a square integrable representation, its Plancherel measure is concentrated on $\hat{N}_{sq}$.

Let $K$ be a compact subgroup of automorphisms of $N$. On the one hand it is shown in [BJR] that if $(K, N)$ is a Gelfand pair, then $N$ must be abelian or two-step nilpotent. On the other hand, in [RS] it is proved that if there exists $\tau \in \hat{K}$ such that $(K, N, \tau)$ is a commutative triple, then $(K, N)$ must be a Gelfand pair. For these reasons we deal only with two-step nilpotent groups $N$. Moreover, we assume that $N$ has a square integrable representation.
Let \( \mathfrak{n} \) be the Lie algebra of \( N \) with Lie bracket \([.,.]\). Consequently, \( \mathfrak{n} = \mathfrak{z} \oplus V \) where \( \mathfrak{z} \) is its center and \( V \) is the orthogonal complement of \( \mathfrak{z} \). The group \( N \) acts naturally on its Lie algebra \( \mathfrak{n} \) by the adjoint action \( \text{Ad} \). Also, \( N \) acts on \( \mathfrak{n}^* \), the real dual space of \( \mathfrak{n} \), by the dual or contragredient representation of the adjoint representation \( \text{Ad}^*(n)\lambda := \lambda \circ \text{Ad}(n^{-1}) \) (for all \( n \in N \) and \( \lambda \in \mathfrak{n}^* \)). Fixed \( \lambda \in \mathfrak{n}^* \) nontrivial, let \( O(\lambda) := \{ \text{Ad}^*(n)\lambda | n \in N \} \) be its coadjoint orbit.

From Kirillov's theory there is a correspondence between \( \hat{N} \) and the set of coadjoint orbits. Let \( B_\lambda \) be the skew symmetric bilinear form on \( \mathfrak{n} \) given by

\[
B_\lambda(X, Y) := \lambda([X, Y]) \quad (X, Y \in \mathfrak{n}).
\]

Let \( \mathfrak{m} \subset \mathfrak{n} \) be a maximal isotropic subalgebra in the sense that \( B_\lambda(X, Y) = 0 \) for all \( X, Y \in \mathfrak{m} \) and let \( M := \exp(\mathfrak{m}) \). Defining on \( M \) the character \( \chi_\lambda(\exp(Y)) := e^{i\lambda(Y)} \) (for all \( Y \in \mathfrak{m} \)), the irreducible representation \( \rho_\lambda \in \hat{N} \) associated to \( O(\lambda) \) is the induced representation \( \rho_\lambda := \text{Ind}_M^N(\chi_\lambda) \).

Let \( X_\lambda \in \mathfrak{z} \) be the representative of \( \lambda_{ij} \) (the restriction of \( \lambda \) to \( \mathfrak{z} \)), that is, \( \lambda(Y) = \langle Y, X_\lambda \rangle \) for all \( Y \in \mathfrak{z} \). We can split \( \mathfrak{z} = \mathbb{R}X_\lambda \oplus \mathfrak{z}_\lambda \), where \( \mathfrak{z}_\lambda := \text{Ker}(\lambda_{ij}) \) is the orthogonal complement of \( \mathbb{R}X_\lambda \) on \( \mathfrak{z} \).

One immediately sees that \( \rho_\lambda \in \overline{N}_{sq} \) implies that \( B_\lambda \) is non-degenerate on \( V \) and that the orbits are maximal, i.e., they are of the form \( O(\lambda) = \lambda \oplus V^* \). Indeed, let \( a_\lambda \) be the subspace of \( V \) where \( B_\lambda \) is degenerate, i.e.,

\[
a_\lambda := \{ u \in V | B_\lambda(u, v) = 0 \ \forall v \in V \}
\]

and let \( b_\lambda \) be the subspace of \( V \) where \( B_\lambda \) is non-degenerate. Consider \( \mathfrak{n}_\lambda := a_\lambda \oplus b_\lambda \oplus \mathbb{R}X_\lambda \) and \( N_\lambda := \exp(\mathfrak{n}_\lambda) \). We equipped \( a_\lambda \) with the trivial Lie bracket and \( \mathfrak{h}_\lambda := b_\lambda \oplus \mathbb{R}X_\lambda \) with the bracket given by

\[
[X, Y]_{b_\lambda} := B_\lambda(X, Y)X_\lambda.
\]

We observe that \( \mathfrak{h}_\lambda \) is a Heisenberg algebra and we denote by \( H_\lambda \) the corresponding Heisenberg group. Let \( A_\lambda := \exp(a_\lambda) \). The representation \( \rho_\lambda \) is trivial on \( Z_\lambda := \exp(\mathfrak{z}_\lambda) \). Thus it factors through \( N_\lambda \) and identifying \( N_\lambda \) with the product group \( A_\lambda \times H_\lambda \) we can write

\[
\rho_\lambda(a, n) = \chi(a)\rho_\lambda(n),
\]

where \( \chi \) is a unitary character of \( A_\lambda \) and \( \rho_\lambda \) is an irreducible unitary representation of \( H_\lambda \). Thus, \( \rho_\lambda \) cannot be square integrable unless \( a_\lambda \equiv \{0\} \).

The converse assertion is also true. That is, if the orbits are maximal (or equivalently, if \( B_\lambda \) is non-degenerate on \( V \)), then \( \rho_\lambda \) is square integrable. In fact, \( \mathfrak{n}_\lambda = \mathbb{R}X_\lambda \oplus V \). As before, since the character \( \chi_\lambda \) is trivial on \( \mathfrak{z}_\lambda \), the representation \( \rho_\lambda \) acts trivially on \( \mathfrak{z}_\lambda \) and it is not important. In this case \( N_\lambda \) is isomorphic to a Heisenberg group, so \( \rho_\lambda \) is a square integrable representation.

This is a particular case of the following general result (cf. [Wo, Theorem 14.2.6]). If \( N \) is a connected and simply connected nilpotent Lie group, the following conditions are equivalent:

(i) \( [\rho_\lambda] \in \overline{N}_{sq} \).

(ii) The orbit \( O(\lambda) \) is completely determined by \( \lambda_{ij} \).
(iii) $B_\lambda$ is non-degenerate on $\mathfrak{n}/\mathfrak{j}$.

Finally, we remark that in this context the bilinear form $B_\lambda$ restricted to $V \times V$ defines the symplectic form associated to the Heisenberg group $N_\lambda$, so $\dim(V) = 2m$ for some positive integer $m$ and the vector space $\mathfrak{m}$ is isomorphic to $\mathfrak{j} \oplus \mathbb{R}^m$.

Now we concentrate on the action of the compact subgroup $K$ of orthogonal automorphisms of $N$ (we make no distinction between automorphisms of $N$ and $\mathfrak{n}$). The group $K$ acts on $\tilde{N}$ in the following way: given $k \in K$ and $(\rho_\lambda, H_{\rho_\lambda}) \in \tilde{N}$, $\rho_\lambda^k(x) := \rho_\lambda(k \cdot x)$ defines an irreducible unitary representation of $N$, which may or may not be equivalent to $\rho_\lambda$. On the one hand, let $K_{\rho_\lambda} := \{ k \in K | \rho_\lambda^k \sim \rho_\lambda \}$ be the stabilizer of $\rho_\lambda$. For $k \in K_{\rho_\lambda}$, there exists a (unique up to a unitary factor) unitary operator $\omega_\lambda(k)$ on $H_{\rho_\lambda}$ which intertwines $\rho_\lambda$ with $\rho_\lambda^k$, i.e., $\rho_\lambda^k(x) = \omega_\lambda(k)\rho_\lambda(x)\omega_\lambda(k)^{-1}$ for all $x \in N$. It defines a genuine unitary representation $K_{\rho_\lambda}$ on $H_{\rho_\lambda}$ (cf. [BJR1, Lemma 2.3])

On the other hand, let $K_{X_\lambda}$ be the stabilizer of $X_\lambda$ (with respect to the action of $K$ on $\mathfrak{n}$). For the case where $\rho_\lambda \in \tilde{N}_{sq}$, it is easy to see that $K_{\rho_\lambda}$ coincides with $K_{X_\lambda}$ and we denote them by $K_\lambda$. Observe that, for all $u, v \in V$ and for all $k \in K_\lambda$,

$$B_\lambda(k \cdot u, k \cdot v) = \langle X_\lambda, [k \cdot u, k \cdot v] \rangle = \langle X_\lambda, k \cdot [u, v] \rangle = \langle k^{-1} \cdot X_\lambda, [u, v] \rangle$$

where in the second equality we used that $K$ acts on $\mathfrak{n}$ by automorphisms. As a result, $K_\lambda$ is a subgroup of the symplectic group $\text{Sp}(V, (B_\lambda)_{V \times V})$. Moreover, since $K_\lambda$ is compact, we can assume that it is a subgroup of the unitary group $U(m) \subset \text{Sp}(V, (B_\lambda)_{V \times V})$. At this point we emphasize that the representation $\omega_\lambda$ of $K_\lambda$ coincides with the *metaplectic representation* associated to the Heisenberg group $N_\lambda$.

### 2.1 Localization

Let $(\tau, W_\tau) \in \tilde{\mathcal{K}}$. We denote by $L^1_\tau(N, \text{End}(W_\tau))$ the space of End$(W_\tau)$-valued integrable functions $F$ on $N$ such that

$$F(k \cdot n) = \tau(k)F(n)\tau(k)^{-1} \quad (k \in K, \ n \in N).$$

It is an algebra with the convolution product given by

$$(F \ast G)(x) := \int_N F(xy^{-1})G(y)dy \quad (x \in N; \ F, G \in L^1_\tau(N, \text{End}(W_\tau))).$$

**Definition 1.** $(K, N, \tau)$ (or $(K \ltimes N, K, \tau)$) is a **commutative triple** if the algebra $L^1_\tau(N, \text{End}(W_\tau))$ is commutative.

**Theorem 3.** *(Reduction to Heisenberg groups)* Let $N$ be a connected and simply connected real two-step nilpotent Lie group which has a square integrable representation. Let $K$ be a compact subgroup of orthogonal automorphisms of $N$ and let $(\tau, W_\tau) \in \tilde{\mathcal{K}}$. Then

(i) $(K, N)$ is a Gelfand pair if and only if $(K_\lambda, N_\lambda)$ is a Gelfand pair for every $\rho_\lambda \in \tilde{N}_{sq}$ (scalar case).
(ii) \((K, N, \tau)\) is a commutative triple if and only if \((K_\lambda, N_\lambda, \tau_{|K_\lambda})\) is a commutative triple for every \(\rho_\lambda \in \hat{N}_{sq}\) (matrix case).

**Proof.**

(i) We recall that, by hypothesis, \(N\) has a Plancherel measure concentrated on \(\hat{N}_{sq}\) and thus we apply the argument given in [BJR1] pages 571-574.

(ii) A generalization of the ideas given in [BJR1] proves the second statement. In order to do that generalization we use [RS, Theorem 6.1] which asserts that \((K, N, \tau)\) is a commutative triple if and only if \(\omega_\lambda \otimes (\tau_{|K_\lambda})\) is multiplicity free for each \(\rho_\lambda \in \hat{N}\).

We assume first that \((K, N, \tau)\) is a commutative triple. In particular, \(\omega_\lambda \otimes (\tau_{|K_\lambda})\) is multiplicity free for all \(\rho_\lambda \in \hat{N}_{sq}\). Therefore [RS, Theorem 6.1] states that \((K_\lambda, N_\lambda, \tau_{|K_\lambda})\) is a commutative triple for all \(\rho_\lambda \in \hat{N}_{sq}\).

Conversely, let \(F, G \in L^1_r(N, \text{End}(W_\nu))\). It is clear that if \(\rho \in \hat{N}\), \(\rho(F * G) = \rho(F)\rho(G)\). If we see that \(\rho_\lambda(F)\) commutes with \(\rho_\lambda(G)\) for all \(\rho_\lambda \in \hat{N}_{sq}\), applying the Plancherel theorem (Theorem [2] and [WG, Theorem 14.2.14]) we will have shown \(F * G = G * F\) a.e. Let \(F \in L^1_r(N, \text{End}(W_\nu))\). We know that the operator \(\rho_\lambda(F)\) intertwines \(\omega_\lambda \otimes (\tau_{|K_\lambda})\) with itself (cf. [RS, Lemma 6.2]). Since, for each \(\rho_\lambda \in \hat{N}_{sq}\), \((K_\lambda, N_\lambda, \tau_{|K_\lambda})\) is a commutative triple, \(\omega_\lambda \otimes (\tau_{|K_\lambda})\) is multiplicity free. Therefore by Schur lemma, \(\rho_\lambda(F)\) is a multiple of the identity operator on each irreducible component of \(H_{\rho_\lambda}\) and the conclusion follows immediately.

With this result we reduce the study of \((K, N, \tau)\) “localizing” the problem to each triple \((K_\lambda, N_\lambda, \tau_{|K_\lambda})\), for \(\lambda\) in correspondence with \(\rho_\lambda \in \hat{N}_{sq}\).

We will assume that \(\rho_\lambda\) is acting on the Fock space \(\mathcal{F}_\lambda\) which consist, for \(\lambda > 0\) (respectively \(\lambda < 0\)), of holomorphic functions (respectively antiholomorphic) on \(\mathbb{C}^m\) square integrable with respect to the measure \(e^{-\frac{1}{2}|z|^2}\). The space \(\mathcal{P}(\mathbb{C}^m)\) of polynomials on \(\mathbb{C}^m\) is dense on \(\mathcal{F}_\lambda\). Thereupon we consider the metaplectic action of \(U(m)\) on \(\mathcal{P}(\mathbb{C}^m)\) given by \((\omega(k)(p))(z) := p(k^{-1}z)\) and we will work with \(\omega|_{K_\lambda}\) instead of \(\omega_\lambda\).

### 2.2 Criteria for a family of two-step nilpotent Lie groups

In this subsection we introduce a subclass of two-step homogeneous nilmanifolds with an analog construction to that of H-type groups. This construction is very well explained in the papers [L] and [LT] by J. Lauret. Start from a real faithful representation \((\pi, V)\) of a compact Lie algebra \(\mathfrak{g}\) (i.e., \(\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{c}\) where \(\mathfrak{c}\) is the center of \(\mathfrak{g}\) and \(\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]\) is a compact semisimple Lie algebra). Let \(\mathfrak{n} := \mathfrak{g} \oplus V\) be the two-step nilpotent Lie algebra with center \(\mathfrak{g}\) and Lie bracket defined on \(V\) by \(\langle [u, v], X \rangle_{\mathfrak{g}} := \langle \pi(X)u, v \rangle_V\) for all \(u, v \in V, X \in \mathfrak{g}\), where the inner products \(\langle \cdot, \cdot \rangle_\mathfrak{g}\) and \(\langle \cdot, \cdot \rangle_V\) are \(\text{ad}(\mathfrak{g})\)-invariant and \(\pi(\mathfrak{g})\)-invariant respectively. These inner products define an inner product \(\langle \cdot, \cdot \rangle_{\mathfrak{g}}\) on...
integrable representation of $N$ of $K, N, \tau$. Now we assume that $(\pi, V)$ is multiplicity free. If $(\omega, \tau)$ is a commutative triple if and only if $(\omega \otimes \tau)_{|_{X \times U}}$ is multiplicity free. In the next theorem we state a similar criterion in the case when $g$ is a semisimple Lie algebra.

**Theorem 4.** Let $g$ be a semisimple Lie algebra and let $T$ be a maximal torus of $G$. Assume that $N = N(g, V)$ has a square integrable representation. The triple $(K, N, \tau)$ is commutative if and only if $(\omega \otimes \tau)_{|_{T \times U}}$ is multiplicity free.

**Proof.** If $(\omega \otimes \tau)_{|_{T \times U}}$ is multiplicity free, then for each $X_\lambda \in g$, $(\omega \otimes \tau)_{|_{K_}\lambda}$ is multiplicity free since $C_G(X_\lambda)$ contains a maximal torus of $G$.

Now we assume that $(K, N, \tau)$ is a commutative triple. We recall that $X \in g$ is said to be a regular element of $g$ when $C_G(X)$ is a maximal torus and it is very well known that the set of regular elements of $g$ is open and dense in $g$. We only need to find a regular element in $g$ corresponding to a square integrable representation of $N$.

For each $\lambda \in n^*$ there is defined a canonical function $P(\lambda)$ which is a homogeneous polynomial on $n^*$ called the Pfaffian, which only depends on $\lambda|_g$ (for a reference see [Wo]). It is proved in [Wo, Theorem 14.2.10] that there exists a bijection from $\{\lambda \in g^* | P(\lambda) \neq 0\}$ onto $N_{sq}$ given by

$$\phi(\lambda|_g) = \rho_\lambda$$
Moreover, Theorem 14.2.14 in [Wo] asserts that the Plancherel measure of $N$ is concentrated on $\widehat{N}_{\text{sq}}$ and that its image under the map $\phi^{-1}$ is a positive multiple of $|P(X)|dX$, where $dX$ is the Lebesgue measure in $g$.

In particular, the set $\{\lambda \in g^* \mid P(\lambda) \neq 0\}$ is in correspondence with the set $\{X_\lambda \in g \mid \pi(X_\lambda) \text{ is invertible}\}$. Since it is an open set, there is a regular element on it. \hfill $\square$

**Corollary 1.** Let $g$ be a non-abelian compact Lie algebra (i.e. $g' \neq \{0\}$). Assume that there exists $X \in g'$ such that $\pi(X)$ is invertible. The triple $(K, N, \tau)$ is commutative if and only if $(\omega \otimes \tau)|_{T \times U}$ is multiplicity free, where $T \subset G$ is a maximal torus of $G$.

**Proof.** As in the previous proof, if $(\omega \otimes \tau)|_{T \times U}$ is multiplicity free, $(K, N, \tau)$ is commutative. Conversely, by the hypothesis $\{X \in g' \mid \pi(X) \text{ is invertible}\}$ is a non-empty open set, so it has a regular element and the conclusion follows as in Theorem 1. \hfill $\square$

The group $N(g, V)$ is said *decomposable* if $N(g, V)$ is a direct product of Lie groups of the form

$$N(g, V) = N(h_1, V_1) \times N(h_2, V_2).$$

Otherwise, we will say that $N(g, V)$ is *indecomposable*.

We now recall the classification, due to J. Lauret, of all the Gelfand pairs of the form $(G \times U, N(g, V))$ where $N(g, V)$ is indecomposable and has a square integrable representation (cf. [L, Remark 3]).

(I) $(SU(2) \times Sp(n), N(su(2), (\mathbb{C}^2)^n)), \ n \geq 1$ (Heisenberg-type), where $su(2)$ acts on $(\mathbb{C}^2)^n$ as $Im(\mathbb{H})$ acts component-wise on $\mathbb{H}^n$ by the quaternion product on the left side, where $\mathbb{H}$ denotes the quaternions and $Im(\mathbb{H})$ the imaginary quaternions.

(II) $(\text{Spin}(4) \times Sp(k_1) \times Sp(k_2), N(su(2) \oplus su(2), (\mathbb{C}^2)^{k_1} \oplus \mathbb{R}^4 \oplus (\mathbb{C}^2)^{k_2})), \ k_1 + k_2 \geq 1$, where the real vector space $\mathbb{R}^4 = (\mathbb{C}^2 \oplus \mathbb{C}^2)_{\mathbb{R}}$ denotes the standard representation of $so(4) = su(2) \oplus su(2)$ and the first copy of $su(2)$ acts only on $(\mathbb{C}^2)^{k_1}$ and the second one only on $(\mathbb{C}^2)^{k_2}$.

(III) $(Sp(2) \times Sp(n), N(sp(2), (\mathbb{C}^4)^n)), \ n \geq 1$, where $sp(2)$ acts component-wise on $(\mathbb{H}^2)^n$ in the standard way (identifying $\mathbb{H}^2$ with $\mathbb{C}^4$).

(IV) $(SO(2n), N(so(2n), \mathbb{R}^{2n})), \ n \geq 2$ (free two-step nilpotent Lie groups), where $\mathbb{R}^{2n}$ denotes the standard representation of $so(2n)$.

(V) $(SU(n) \times S^1, N(su(n), \mathbb{C}^n)), \ n \geq 3$, where $\mathbb{C}^n$ denotes the standard representation of $su(n)$ regarded as a real representation.

(VI) $(SU(n) \times S^1, N(u(n), \mathbb{C}^n)), \ n \geq 3$, where $\mathbb{C}^n$ denotes the standard representation of $u(n)$ regarded as a real representation.

(VII) $(SU(2) \times U(k) \times Sp(n), N(u(2), (\mathbb{C}^2)^k \oplus (\mathbb{C}^2)^n)), \ k \geq 1, n \geq 0$, where the center of $u(2)$ acts non-trivially only on $(\mathbb{C}^2)^k$, in fact, $(\mathbb{C}^2)^n$ denotes the representation of $su(2)$ described in item (I) and $u(2)$ acts component-wise on $(\mathbb{C}^2)^k$ in the standard way regarded as a real representation.

(VIII) $(G \times U, N(g, V))$ where:
2 SOME PRELIMINARY RESULTS

- \( g := \mathfrak{su}(m_1) \oplus \ldots \oplus \mathfrak{su}(m_\beta) \oplus \mathfrak{su}(2) \oplus \ldots \oplus \mathfrak{su}(2) \oplus \mathfrak{c} \), with \( \alpha \) copies of \( \mathfrak{su}(2) \), \( m_i \geq 3 \) for all \( 1 \leq i \leq \beta \) and \( \mathfrak{c} \) is an abelian component.
- \( V := \mathbb{C}^{m_1} \oplus \ldots \oplus \mathbb{C}^{m_\beta} \oplus \mathbb{C}^{2k_1+2n_1} \oplus \ldots \oplus \mathbb{C}^{2k_\alpha+2n_\alpha} \), where \( k_j \geq 1 \) and \( n_j \geq 0 \) for all \( 1 \leq j \leq \alpha \).
- \( \mathfrak{g} \) acts on \( V \) as follows: for each \( 1 \leq i \leq \beta + \alpha \), \( \mathfrak{c} \) has a maximal subspace, denoted by \( \mathfrak{c}_i \) and with \( \dim(\mathfrak{c}_i) = 1 \), acting non-trivially only on one component of \( V \). For \( 1 \leq i \leq \beta \), \( \mathfrak{su}(m_i) \oplus \mathfrak{c}_i \) acts non-trivially only on \( \mathbb{C}^{m_i} \) (as the representation stated in item (VI)) and for \( \beta + 1 < i \leq \beta + \alpha \), \( \mathfrak{su}(2) \oplus \mathfrak{c}_i \) acts non-trivially only on \( \mathbb{C}^{2k_i+2n_i} \) (as the representation stated in item (VII)).
- \( U := S^1 \times \ldots \times S^1 \times U(k_1) \times \text{Sp}(n_1) \times \ldots \times U(k_\alpha) \times \text{Sp}(n_\alpha) \), with \( \beta \) copies of \( S^1 \).

(IX) \( (U(n), N(\mathbb{R}, \mathbb{C}^n)) \), \( n \geq 1 \) (Heisenberg group).

**Remark 1.** We observe that for all the cases stated in the above list there exists \( X \in \mathfrak{g}' \) satisfying \( \text{Ker}(\pi(X)) = \{0\} \).

Our goal is to consider, for each case, an arbitrary irreducible unitary representation \( \tau \) of \( K = G \times U \) and determine if the resulting triple is commutative. The last item of the list was studied in [Y] where it was proved that the triples \( (U(n), N(\mathbb{R}, \mathbb{C}^n), \tau) \) are commutative for all \( \tau \in \hat{U}(n) \), i.e., when \( N \) is the Heisenberg group and \( K \) is the unitary group we have a strong Gelfand pair.

We remark that we are excluding two of the cases given in [L, Remark 3] since the corresponding nilpotent group does not have square integrable representations. They still may give rise to commutative triples.

### 2.3 Background about tensor product of irreducible representations relative to the symplectic group

A partition is a monotone decreasing finite sequence \( \sigma = (\sigma_1, \sigma_2, \ldots) \) where its entries \( \sigma_i \) are non-negative integers. The set of all partitions is denoted by \( \mathbb{P} \). The length of \( \sigma \), denoted by \( l(\sigma) \), is the number of nonzero entries of \( \sigma \), and the size of \( \sigma \), denoted by \( |\sigma| \), is the sum of its entries. A partition \( \sigma \) is often identified with its Young diagram, which is a left-justified array of \( |\sigma| \) cells with \( \sigma_i \) cells in the \( i \)-th row. The conjugate partition \( \sigma' \) of \( \sigma \) is the partition whose diagram is obtained by reflecting the diagram of \( \sigma \) along the main diagonal. For two partitions \( \sigma \) and \( \varsigma \), we write \( \varsigma \subset \sigma \) if \( \varsigma_i \leq \sigma_i \) for all \( i \) and the skew diagram \( \sigma/\varsigma \) is defined to be the set that remains when we make the difference of the two diagrams. The size of this skew diagram is defined by \( |\sigma/\varsigma| = |\sigma| - |\varsigma| \). The skew diagram \( \sigma/\varsigma \) is said to be a horizontal \( k \)-strip if it contains at most one cell in each column and \( |\sigma/\varsigma| = k \). Note that \( \sigma/\varsigma \) is a horizontal strip if and only if \( \sigma_1 \geq \varsigma_1 \geq \sigma_2 \geq \varsigma_2 \geq \ldots \).

Let \( \mathfrak{h} \) be the Cartan subalgebra of \( \mathfrak{sp}(n) \) consisting of the diagonal matrices (over \( \mathbb{C} \)) of the form

\[
\mathfrak{h} = \left\{ H = \begin{pmatrix} h_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & h_n \end{pmatrix} \mid h_1, \ldots, h_n \in \mathbb{C} \right\}.
\]
Let \( \delta_{i,j} \) denote the Kronecker delta. Writing \( H_i := \delta_{i,i} - \delta_{n+i,n+i} \), we have that \( \mathfrak{h} \) is a complex vector space generated by \( \{H_1, \ldots, H_n\} \). Let \( \{L_1, \ldots, L_n\} \) be its dual basis in the dual space \( \mathfrak{h}^* \), so \( \langle L_i, H \rangle = h_i \) for all \( H \in \mathfrak{h} \). The fundamental weights of \( \mathfrak{sp}(n) \) are given by \( L_1, L_1 + L_2, \ldots, L_1 + \ldots + L_n \) and the simple roots are \( L_1 - L_2, \ldots, L_{n-1} - L_n, 2L_n \). From the theorem of the highest weight, every irreducible representation of \( \mathfrak{sp}(n) \) is in correspondence with a nonnegative integer linear combination of the fundamental weights. Hence \( \eta \in \widehat{\mathfrak{sp}(n)} \) can be parametrized in terms of the weights \( \{L_i\} \) as \( (\eta_1, \ldots, \eta_n) \) where \( \eta_i \in \mathbb{Z}_{\geq 0} \forall i \) and \( \eta_1 \geq \eta_2 \geq \ldots \geq \eta_n \). (For a reference see, for example, [K] or [FH].)

By abuse of notation, for each \( \eta \in \widehat{\mathfrak{sp}(n)} \) we will denote its partition with the same letter \( \eta \). Apart from that, we will denote the representation \( \eta \in \widehat{\mathfrak{sp}(n)} \) by \( \eta(\eta_1,\ldots,\eta_n) \) to emphasize that the representation \( \eta \) is in correspondence with the partition \( (\eta_1, \ldots, \eta_n) \).

The decomposition of the tensor product of irreducible representations of \( \mathfrak{sp}(n) \) is developed in [KT] and in the recent works [HLLS] and [O].

The irreducible unitary representations of \( \text{Sp}(n) \) on the space of the homogeneous polynomials over \( \mathbb{C}^{2n} \) of degree \( r \) and \( s \) respectively are associated to the partitions \( (r) \) and \( (s) \) of length one and they are denoted by \( \eta(r) \) and \( \eta(s) \). The formula of the decomposition of their tensor product is given in [KT], page 510] for \( r \geq s \):

\[
\eta(r) \otimes \eta(s) = \bigoplus_{i=0}^{s} \bigoplus_{j=0}^{j} \eta(r+s-j,i,j-i).
\]  

(1)

The tensor product of a representation \( \eta(1,\ldots,1) \) of length \( r > 1 \) (denoted also by \( \eta(1^r) \)) by a representation \( \eta(s) \) (with \( s > 1 \)) is given in [KT], page 510] as well:

\[
\eta(1^r) \otimes \eta(s) = \eta(s+1,1^{r-1}) \oplus \eta(s,1^r) \oplus \eta(s-1,1^{r-1}) \oplus \eta(s,1^{r-2}).
\]  

(2)

Finally, we enunciate a “universal” Pieri rule due to S. Okada.

**Theorem 5.** [O, Proposition 3.1] For an arbitrary irreducible representation (or partition) \( \eta \) and a nonnegative integer \( s \),

\[
\eta \otimes \eta(s) = \sum_{\sigma \in \mathcal{P} \text{ of length at most } n} M_{\eta,s}^{\sigma} \eta(\sigma),
\]  

(3)

where \( M_{\eta,s}^{\sigma} \) denotes the number of partitions \( \zeta \) such that \( \eta/\zeta \) and \( \sigma/\zeta \) are horizontal strips and \( |\eta/\zeta| + |\sigma/\zeta| = s \).

**Corollary 2.** Let \( \eta \) be an arbitrary irreducible representation of \( \text{Sp}(n) \), then \( \eta \) appears in the decomposition into irreducible factors of \( \eta \otimes \eta(2) \).

**Proof.** We use the “universal” Pieri rule with \( s = 2 \). The cardinal number \( M_{\eta,2}^{\eta} \) is not zero since considering \( \zeta = (\eta_1, \ldots, \eta_m - 1) \) we have that \( \eta/\zeta \) is a horizontal strip (there is only one box in the Young diagram) and \( 2|\eta/\zeta| = 2 \).

**Corollary 3.** Let \( \eta \in \widehat{\mathfrak{sp}(n)} \) corresponding to the partition \( (\eta_1, \ldots, \eta_m) \), with \( \eta_m \neq 0 \), \( 0 \leq m \leq n \). The representation \( \eta \otimes \eta(s) \) is multiplicity free for all \( s \in \mathbb{Z}_{\geq 0} \) if and only if \( \eta_i = \eta_j \) for all \( 1 \leq i, j \leq m \).

Note that, from [1] and [2], the corollary holds in the particular cases \( \eta = \eta(r) \) (for some \( r \in \mathbb{Z}_{\geq 0} \)) and \( \eta = \eta(1^{r'}) \) (for some \( r' \in \mathbb{Z}_{\geq 0} \)).
Proof. We assume first that $s = 2$ and that $\eta_i > \eta_{i+1}$, i.e. $\eta_i - 1 \geq \eta_{i+1}$, for some $1 \leq i \leq m$. From \cite{22}, $\eta$ appears in the decomposition of $\eta \otimes \eta(2)$ at least twice: considering $\varsigma_1 = (\eta_1, ..., \eta_m - 1)$ and $\varsigma_2 = (\eta_1, ..., \eta_{i-1}, \eta_i - 1, \eta_{i+1}, ..., \eta_k)$ we have that $\eta/\varsigma_1$ and $\eta/\varsigma_2$ are 1-horizontal strips.

Therefore we have proved that $\eta$ must satisfy necessarily the condition $\eta_i = \eta_j$ for all $1 \leq i, j \leq m$. Now we will prove that this condition is sufficient.

Let $(a, a, ..., a)$ be the partition of length $0 \leq m \leq n$ associated to $\eta$ for some $a \in \mathbb{N}$. Let $\varsigma$ be a partition such that $\eta/\varsigma$ is a horizontal strip. Since $\varsigma$ must satisfy $\eta_1 \geq \varsigma_1 \geq ... \geq \varsigma_{m-1} \geq \eta_m \geq \varsigma_m$, we have that $\varsigma$ is equal to $\varsigma_j := (a, ..., a, a - j)$ for some $0 \leq j \leq a$. Now we fix an arbitrary non-negative integer $s$ and let $\sigma$ be an irreducible representation which appears in the decomposition of $\eta \otimes \eta(s)$. Accordingly, there is $0 \leq j \leq a$ such that $\sigma/\varsigma_j$ is an $(s - j)$-horizontal strip. Note that the partition $(\sigma_1, ..., \sigma_l)$ associated to $\sigma$ must satisfy: $l \leq m + 1$, $\sigma_2 = ... = \sigma_{m-1} = a$, $\sigma_1 \geq a$, $(a - j) \leq \sigma_m \leq a$ and $0 \leq \sigma_{m+1} \leq (a - j)$. Therefore $|\sigma/\varsigma_j| = \sigma_1 - a + \sigma_m - (a - j) + \sigma_{m+1}$ and thus $\sigma_1 + \sigma_m + \sigma_{m+1} = s + 2a - 2j$. From the last equality, $\sigma$ appears only once in the decomposition of $\eta \otimes \eta(s)$: indeed, if we assume that it appears two times (or more), then there are $\varsigma_j$ and $\varsigma_k$, as above, with $j \neq k$, such that $\sigma/\varsigma_j$ and $\sigma/\varsigma_k$ are $(s - j)$ and $(s - k)$ horizontal strips respectively, hence $s + 2a - 2j = s + 2a - 2k$ (so $j = k$) and we have a contradiction. \hfill $\blacksquare$

3 Case by case analysis

In this paragraph we explore each of the cases given in Section 2.2 in order to derive commutative triples.

Case (I): Heisenberg type

Let $\mathbb{H}$ denote the quaternions and $\text{Im}(\mathbb{H})$ the imaginary quaternions. For $n \geq 1$, the group $N(\mathfrak{su}(2), (\mathbb{C}^2)^n)$ is the H-type group with Lie algebra $\text{Im}(\mathbb{H}) \oplus \mathbb{H}^n$ where the real representation $\pi$ of $\text{Im}(\mathbb{H})$ on $\mathbb{H}^n$ is given by the quaternion product on the left side on each component,

$$\pi(z)(v) := (zv_1, ..., zv_n) \quad (v = (v_1, ..., v_n) \in \mathbb{H}, \ z \in \text{Im}(\mathbb{H})) \quad (4)$$

i.e. $\mathfrak{su}(2)$ acts on $(\mathbb{C}^2)^n$ as $\text{Im}(\mathbb{H})$ acts naturally on each coordinate of $\mathbb{H}^n$ (identifying $\text{Im}(\mathbb{H})$ with $\mathfrak{su}(2)$ and $\mathbb{H}^n$ with $(\mathbb{C}^2)^n$).

The unitary group of intertwining operators of $\pi$ is $\text{Sp}(n)$ that acts on $\mathbb{H}^n$ on the right side and $K = \text{SU}(2) \times \text{Sp}(n)$. A maximal torus on $\text{SU}(2)$ is given by $T^1 := \{ (e^{i\theta}, 0) | \theta \in \mathbb{R} \}$. Note that the action of $T^1 \times \text{Sp}(n)$ on $\mathcal{P}((\mathbb{C}^2)^n)$ (the space of polynomials on $(\mathbb{C}^2)^n$) is $\mathbb{C}$-linear whereas the action of $K$ is not.

The metaplectic representation $\omega$ of $T^1 \times \text{Sp}(n)$ on $\mathcal{P}((\mathbb{C}^2)^n)$ decomposes into

$$\omega = \bigoplus_{s \in \mathbb{Z}_{\geq 0}} \chi_s \otimes \eta(s), \quad (5)$$

where $\chi_s(\theta) := e^{-is\theta}$ and $\eta(s)$ denotes the irreducible representation of $\text{Sp}(n)$ on the space $\mathcal{P}_s((\mathbb{C}^2)^n)$ of homogeneous polynomials of degree $s$. 
A representation $\tau \in \widehat{\text{SU}(2)} \times \widehat{\text{Sp}(n)}$ is given by the tensor product $\tau = \nu_k \otimes \eta$, where $(\nu_k, P_k(\mathbb{C}^2)) \in \widehat{\text{SU}(2)}$ is the well known representation of $\text{SU}(2)$ on the homogeneous polynomials on $\mathbb{C}^2$ of degree $k$ and $\eta \in \widehat{\text{Sp}(n)}$. When we restrict $\tau$ to $T^1 \times \text{Sp}(n)$ we have

$$(\nu_k \otimes \eta)|_{T^1 \times \text{Sp}(n)} = \bigoplus_{i=0}^{k} \chi_{k-2i} \otimes \eta. \quad (6)$$

In consequence

$$(\omega \otimes \nu_k \otimes \eta)|_{T^1 \times \text{Sp}(n)} = \bigoplus_{s=0}^{\infty} \left( \bigoplus_{i=0}^{k} \chi_{s+k-2i} \otimes \eta(s) \right) \otimes \eta. \quad (7)$$

**Proposition 1.** The triple $(\text{SU}(2) \times \text{Sp}(n), N(\text{su}(2), (\mathbb{C}^2)^n), \tau)$ is commutative if and only if:

(i) $\tau \in \widehat{\text{SU}(2)}$ or

(ii) $\tau \in \widehat{\text{Sp}(n)}$ and corresponds to a partition of the form $(a, a, ..., a)$, of length at most $n$.

**Proof.** Let $\tau = \nu_k \otimes \eta \in \widehat{\text{SU}(2)} \times \widehat{\text{Sp}(n)}$.

From (7), if $\tau = \nu_k$, the assertion is easily proved since

$$(\omega \otimes \nu_k)|_{T^1 \times \text{Sp}(n)} = \bigoplus_{s=0}^{\infty} \left( \bigoplus_{i=0}^{k} \chi_{s+k-2i} \otimes \eta(s) \right) \otimes \eta.$$ 

is multiplicity free. For the case $\tau = \eta$, from (7) we arrive at

$$(\omega \otimes \eta)|_{T^1 \times \text{Sp}(n)} = \bigoplus_{s=0}^{\infty} \chi_s \otimes \eta(s) \otimes \eta.$$ 

It is multiplicity free if and only if $\eta(s) \otimes \eta$ is multiplicity free for all $s \in \mathbb{Z}_{\geq 0}$. From Corollary 3 this holds if and only if $\eta$ corresponds to a partition of the form $(a, a, ..., a)$ of length at most $n$.

Now we assume $\eta$ and $\nu_k$ are both nontrivial and from Theorem 4 we will prove that $(\omega \otimes \nu_k \otimes \eta)|_{T^1 \times \text{Sp}(n)}$ is not multiplicity free.

If we consider $s = 0$ in (7), we pick $\chi_k \otimes \eta$. On the other hand, if we consider $s = 2$, we pick $\chi_k \otimes \eta(2) \otimes \eta$. From Corollary 2, $\eta$ appears in the decomposition into irreducible factors of the tensor product $\eta(2) \otimes \eta$. Therefore $\eta$ appears at least twice in the decomposition of $(\omega \otimes \nu_k \otimes \eta)|_{T^1 \times \text{Sp}(n)}$. \(\square\)

**Case (II)**

In this case $g = \text{su}(2) \oplus \text{su}(2)$ and it acts on $V = (\mathbb{C}^2)^{k_1} \oplus \mathbb{R}^4 \oplus (\mathbb{C}^2)^{k_2}$ (with $k_1 + k_2 \geq 1$) by

(i) the first copy $\text{su}(2)$ of $g$ acts only on $(\mathbb{C}^2)^{k_1}$ as $\text{sp}(1)$ acts on $\mathbb{H}^{k_1}$ in the natural way, component-wise, whereas the second copy $\text{su}(2)$ of $g$ acts analogously only on $(\mathbb{C}^2)^{k_2}$ and

(ii) $g = \text{so}(4)$ acts naturally on $\mathbb{R}^4$ (since $\text{su}(2) \oplus \text{su}(2) = \text{so}(4)$).
Thus, the connected unitary group of intertwining operators of this action is $U = \text{Sp}(k_1) \times \text{Sp}(k_2)$.

Let $T^1$ be a torus of $\text{SU}(2)$. The metaplectic representation of $T^1 \times T^1 \times U$ on $\mathcal{P}((\mathbb{C}^2)^{k_1} \oplus \mathbb{R}^4 \oplus (\mathbb{C}^2)^{k_2})$ decomposes in the following way:

(i) The action of $T^1 \times \text{Sp}(k_1)$ on $\mathcal{P}((\mathbb{C}^2)^{k_1})$ decomposes without multiplicity into irreducible representations as

$$\bigoplus_{r \in \mathbb{Z}_{\geq 0}} \chi_r \otimes \eta(r),$$  

(8)

where $\chi_r(\theta) = e^{-r \theta}$ (for all $\theta \in T^1$) and $\eta(r)$ is the classical representation of $\text{Sp}(k_1)$ on $\mathcal{P}_r((\mathbb{C}^2)^{k_1})$.

The action of $T^1 \times \text{Sp}(k_2)$ on $\mathcal{P}((\mathbb{C}^2)^{k_2})$ decomposes analogously

$$\bigoplus_{s \in \mathbb{Z}_{\geq 0}} \chi_s \otimes \eta(s),$$  

(9)

where here $\eta(s)$ is the classical representation of $\text{Sp}(k_2)$ on $\mathcal{P}_s((\mathbb{C}^2)^{k_2})$.

(ii) The action of $T^1 \times T^1 = \{(e^{i\theta_1} 0 \ 0 \ e^{i\theta_2}) \mid \theta_1, \theta_2 \in \mathbb{R}\}$ on $\mathcal{P}(\mathbb{C}^2)$ decomposes without multiplicity into the following sum of characters,

$$\bigoplus_{l_1,l_2 \in \mathbb{Z}_{\geq 0}} \chi(l_1,l_2),$$  

(10)

where $\chi(l_1,l_2)(\theta_1, \theta_2) = e^{-l_1 i \theta_1} e^{-l_2 i \theta_2}$ (for all $(\theta_1, \theta_2) \in T^1 \times T^1$).

Therefore from (8), (9) and (10),

$$\omega|_{T^1 \times T^1 \times U} = \bigoplus_{r,s,l_1,l_2 \in \mathbb{Z}_{\geq 0}} \chi(r+l_1,s+l_2) \otimes \eta(r) \otimes \eta(s).$$  

(11)

Let $\tau = \nu_m \otimes \nu_n \otimes \eta_1 \otimes \eta_2$ be an irreducible unitary representation of $K = \text{SU}(2) \times \text{SU}(2) \times U$, where $\nu_m$ and $\nu_n$ are the classical irreducible unitary representations of $\text{SU}(2)$ on $\mathcal{P}_m(\mathbb{C}^2)$ and $\mathcal{P}_n(\mathbb{C}^2)$ respectively, and $\eta_1$ and $\eta_2$ are arbitrary irreducible unitary representations of $\text{Sp}(k_1)$ and $\text{Sp}(k_2)$ respectively. When we restrict $\tau$ to $T^1 \times T^1 \times U$ we obtain the following decomposition,

$$\left(\bigoplus_{i=0}^{m} \chi_{m-2i}\right) \otimes \left(\bigoplus_{j=0}^{n} \chi_{n-2j}\right) \otimes \eta_1 \otimes \eta_2.$$  

(12)

Therefore from (11) and (12),

$$(\omega \otimes \tau)|_{T^1 \times T^1 \times U} = \bigoplus_{r,s,l_1,l_2 \in \mathbb{Z}_{\geq 0}} \chi(r+l_1,m-2i,s+l_2,n-2j) \otimes \eta(r) \otimes \eta_1 \otimes \eta(s) \otimes \eta_2.$$  

(13)

**Proposition 2.** The triple $(\text{SU}(2) \times \text{SU}(2) \times \text{Sp}(k_1) \times \text{Sp}(k_2), N(\text{su}(2) \oplus \text{su}(2), (\mathbb{C}^2)^{k_1} \oplus \mathbb{R}^4 \oplus (\mathbb{C}^2)^{k_2}), \tau)$ is commutative if and only if $\tau$ is the trivial representation.

**Proof.** Let $\tau = \nu_m \otimes \nu_n \otimes \eta_1 \otimes \eta_2$ as above. If $\nu_m$ is nontrivial, i.e. $m \geq 1$, considering in (13) $l_1 = 0, i = 0$ and then $l_1 = 2, i = 1$, we obtain multiplicity in the decomposition into irreducible representations of $(\omega \otimes \tau)|_{T^1 \times T^1 \times U}$. Analogously, if $\nu_n$ is nontrivial, i.e. $n \geq 1$, we have multiplicity in (13).

If $\eta_1 \in \text{Sp}(k_1)$ is nontrivial, by the same argument given in Proposition 1 (that uses Corollary 2), considering on the one hand $r = 0, l_1 = 2$ and on the other hand $r = 2, l_1 = 0$ on (13), we have multiplicity in $(\omega \otimes \tau)|_{T^1 \times T^1 \times U}$. The same holds for a nontrivial $\eta_2 \in \text{Sp}(k_2)$. \qed
3 CASE BY CASE ANALYSIS

Case (III)

This case consists of $\mathfrak{g} = \mathfrak{sp}(2)$ and $\mathfrak{n} = \mathfrak{sp}(2) \oplus (\mathbb{H}^2)^n$ (with $n \geq 1$), where we are identifying $\mathbb{C}^4$ with $\mathbb{H}^2$ (as real vector spaces). The real representation $\pi$ of $\mathfrak{sp}(2)$ on $(\mathbb{H}^2)^n$ is similar to the one described in the case (I):

$$\pi(z)(v) := (zv_1, ..., zv_n) \quad (v = (v_1, ..., v_n) \in \mathbb{H}^2, \ z \in \mathfrak{sp}(2)).$$

(14)

As the group of unitary intertwining operators of $(\pi, (\mathbb{H}^2)^n)$ is $\text{Sp}(n)$ (acting on the right side as a $\mathbb{C}$-linear action), the group $K$ is $\text{Sp}(2) \times \text{Sp}(n)$.

Let $T^2 := \{(e^{i\theta_1} 0 0 \ e^{i\theta_2}) \mid \theta_1, \theta_2 \in \mathbb{R}\}$ be a maximal torus on $\text{Sp}(2)$.

Since the natural action of $\mathfrak{sp}(2)$ on $\mathbb{H}^2$ is irreducible, Schur’s lemma implies that each intertwining operator $A \in \text{Sp}(n)$ of the action of $\mathfrak{sp}(2)$ on $\mathcal{P}((\mathbb{H}^2)^n)$, $A : (\mathbb{H}^2)^n \rightarrow (\mathbb{H}^2)^n$, has the following matrix representation

$$[A] = \begin{pmatrix}
    a_{11}I & a_{12}I & \cdots & a_{1n}I \\
    a_{21}I & a_{22}I & \cdots & a_{2n}I \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1}I & a_{n2}I & \cdots & a_{nn}I
\end{pmatrix}, \quad \text{where } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and } a_{i,j} \in \mathbb{H}.
$$

We can deduce that the action of $\text{Sp}(n)$ on $\mathcal{P}((\mathbb{C}^4)^n)$ splits as $\mathcal{P}((\mathbb{C}^2)^n) \otimes \mathcal{P}((\mathbb{C}^2)^n)$ and also it can be written as $\bigoplus_{r,s \in \mathbb{Z}_{\geq 0}} \mathcal{P}_r((\mathbb{C}^2)^n) \otimes \mathcal{P}_s((\mathbb{C}^2)^n)$.

Therefore the metaplectic representation $\omega$ of $T^2 \times \text{Sp}(n)$ on $\mathcal{P}((\mathbb{C}^4)^n)$ decomposes into

$$\omega|_{T^2 \times \text{Sp}(n)} = \bigoplus_{r,s \in \mathbb{Z}_{\geq 0}} \chi_{(r,s)} \otimes \eta_{(r)} \otimes \eta_{(s)},$$

where $\chi_{(r,s)}(\theta_1, \theta_2) = e^{-ir\theta_1}e^{-is\theta_2}$ and where $\eta_{(r)}$ and $\eta_{(s)}$ denote the classical representations of $\text{Sp}(n)$ on the homogeneous polynomials on $(\mathbb{C}^2)^n$ of degree $r$ and $s$ respectively. Moreover, from (1), we obtain the multiplicity free decomposition

$$\omega = \bigoplus_{r,s \in \mathbb{Z}_{\geq 0}} \chi_{(r,s)} \otimes \left( \bigotimes_{j=0}^{s} \bigotimes_{i=0}^{j} \eta_{(r+s-j-i,j-i)} \right).$$

(15)

Proposition 3. The triple $(\text{Sp}(2) \times \text{Sp}(n), N(\mathfrak{sp}(2), (\mathbb{H}^2)^n), \tau)$ is commutative if and only if $\tau$ is the trivial representation.

Proof. Let $\tau \in \text{Sp}(2) \times \text{Sp}(n)$. Then $\tau = \kappa \otimes \eta$, where $\kappa \in \text{Sp}(2)$ and $\eta \in \text{Sp}(n)$.

First, let $\eta \in \text{Sp}(n)$ be nontrivial. Fix $r = 1$ and $s = 1$ in (15). On the one hand take in (15) $i = j = 0$ and on the other hand, $i = j = 1$. As a result, $\chi_{(1,1)} \otimes (\kappa_{12}) \otimes \eta$ and $\chi_{(1,1)} \otimes (\kappa_{12}) \otimes (\eta_{(2)} \otimes \eta)$ appear in $(\omega \otimes \tau)|_{T^2 \times \text{Sp}(n)}$. From Corollary 2 we know that $\eta$ appears in the decomposition of $\eta_{(2)} \otimes \eta$. As we expected, $(\omega \otimes \tau)|_{T^2 \times \text{Sp}(n)}$ is not multiplicity free. This leads us to assume the component $\eta$ to be trivial.
Now, let $\kappa \in \widehat{\text{Sp}(2)}$ be nontrivial. We can associate to $\kappa$ the partition $(\kappa_1, \kappa_2)$. From the theorem of the highest weight, every weight of $\kappa$ is given by the highest weight minus nonnegative sums of the simple roots $L_1 - L_2$ and $2L_2$. Let $\chi_{(\kappa_1, \kappa_2)}$ and $\chi_{(\kappa_1-l, \kappa_2+l-2m)}$ be two different characters in the decomposition of $\kappa|_{\mathbb{T}_2}$ for some nonnegative integers $l$ and $m$. In the decomposition of $\omega|_{\mathbb{T}_2 \times \text{Sp}(n)} \otimes \kappa|_{\mathbb{T}_2}$ appear (in particular) the following sums of irreducible representations

$$
\left( \bigoplus_{r,s \in \mathbb{Z}_{\geq 0}} \chi_{(\kappa_1+r, \kappa_2+s)} \otimes \eta(r) \otimes \eta(s) \right) \oplus \left( \bigoplus_{r',s' \in \mathbb{Z}_{\geq 0}} \chi_{(\kappa_1-l+r', \kappa_2+l-2m+s')} \otimes \eta(r') \otimes \eta(s') \right).
$$

We choose $r' = l$, $s' = 2m$, $r = 0$ and $s = l$. They satisfy

$$
\begin{cases}
  r = -l + r' \\
  s = l - 2m + s'
\end{cases}
$$

(16)

The representation $\eta(l)$ appears in $\eta(r) \otimes \eta(s)$ and in $\eta(r') \otimes \eta(s')$. This holds since from (1),

$$
\eta(r) \otimes \eta(s) = \eta(0) \otimes \eta(l) = \eta(l),
$$

$$
\eta(r') \otimes \eta(s') = \eta(l) \otimes \eta(2m) = \bigoplus_{j=0}^{\max(l,2m)} \bigoplus_{i=0}^{j} \eta(l+2m-j, i, j, i),
$$

and taking $j = i = m$ we see that $\eta(l)$ appears in $\eta(r') \otimes \eta(s')$.

In conclusion, when $\tau$ is not the trivial representation, $(\omega \otimes \tau)|_{\mathbb{T}_2 \times \text{Sp}(n)}$ is not multiplicity free and by Theorem 4 the triple is not commutative.

**Case (IV): free two-step nilpotent Lie groups**

We will study the pairs $(\text{SO}(2n), N(\mathfrak{so}(2n), \mathbb{R}^{2n}))$, for $n \geq 2$. Here the representation $\pi$ is the natural action of $\mathfrak{so}(2n)$ on $\mathbb{R}^{2n}$ and therefore $K = \text{SO}(2n)$.

We fix a maximal torus on $\text{SO}(2n)$,

$$
T^n := \left\{ \begin{pmatrix}
\cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\
\sin(\theta_1) & \cos(\theta_1) & 0 & 0 \\
& & \ddots & \\
0 & 0 & \cos(\theta_n) & -\sin(\theta_n) \\
0 & 0 & \sin(\theta_n) & \cos(\theta_n)
\end{pmatrix} \mid \theta_1, \ldots, \theta_n \in \mathbb{R} \right\}.
$$

The metaplectic representation of $T^n$ on $\mathcal{P}(\mathbb{C}^n)$ is the usual one and it decomposes without multiplicity into a direct sum of characters

$$
\omega|_{T^n} = \bigoplus_{(k_1, \ldots, k_n) \in \mathbb{Z}_{\geq 0}} \chi_{(k_1, \ldots, k_n)},
$$

(17)

where $(\chi_{(k_1, \ldots, k_n)}(\theta_1, \ldots, \theta_n))(z_1^{k_1} \ldots z_n^{k_n}) = e^{-ik_1\theta_1}z_1^{k_1} \ldots e^{-ik_n\theta_n}z_n^{k_n}$.

Let $(\tau, W_\tau) \in \widehat{\text{SO}(2n)}$ be a nontrivial representation. We must analyze the decomposition into irreducible factors of $\tau$ restricted to the torus $T^n$ and then verify whether $\omega|_{T^n} \otimes \tau|_{T^n}$ is multiplicity free or not:
(i) If $\tau_{\varphi}$ is not multiplicity free, then $\omega_{\varphi} \otimes \tau_{\varphi}$ is not multiplicity free either.

(ii) Assume that $\tau_{\varphi}$ is multiplicity free. Since $\tau$ is not the trivial representation, $\tau_{\varphi}$ decomposes into a sum of characters (as many as its dimension). Let $\chi(u_1,\ldots,u_n)$ and $\chi(v_1,\ldots,v_n)$ be two different characters on $\tau_{\varphi}$ determined by the $n$-tuples of integers $(u_1,\ldots,u_n)$ and $(v_1,\ldots,v_n)$. Let $a := \min\{s_i, r_i: 1 \leq i \leq n\}$ and for each $1 \leq i \leq n$, consider nonnegative integers $s_i := a - r_i$ and $l_i := a - s_i$. Since $\chi(k_1,\ldots,k_n)$ and $\chi(l_1,\ldots,l_n)$ appear in $\omega_{\varphi}$, then $\chi(a,\ldots,a)$ appears in the decomposition of $\omega_{\varphi} \otimes \chi(u_1,\ldots,u_n)$ and in the decomposition of $\omega_{\varphi} \otimes \chi(v_1,\ldots,v_n)$, so $\chi(a,\ldots,a)$ appears at least twice in $\omega_{\varphi} \otimes \tau_{\varphi}$.

In conclusion, by Theorem 4, we have proved the following result.

**Proposition 4.** The triple $(SO(2n), N(so(2n), R^{2n}), \tau)$ is commutative if and only if $\tau$ is the trivial representation.

**Cases (V) and (VI)**

Let $n \geq 3$. In case (V), $n = su(n) \oplus C^n$, the representation $\tau$ of $su(n)$ on $C^n$ is the canonical and then $K = SU(n) \times S^1$. In case (VI), $\mathfrak{g} = u(n)$ is not semisimple and it is acting on $C^n$ in the natural way. The subgroup of automorphisms $K$ is the same as in case (V), i.e., $K = SU(n) \times S^1$.

Every $\tau \in \hat{K}$ is given by $\tau = \nu \otimes \chi_r$, for some $\nu \in SU(n)$ and some character $\chi_r$ of $S^1$ with $r \in Z$.

**Proposition 5.** The triple $(SU(n) \times S^1, N(su(n), C^n), \tau)$ is commutative if and only if $\tau \in \hat{S^1}$.

**Proof.** Let $T^{n-1}$ be a maximal tours in $SU(n)$. In order to use Theorem 4, we analyze how to decompose the metaplectic representation of $T^{n-1} \times S^1$ acting on $P(C^n)$. We obtain

$$\omega_{\varphi} \otimes \chi_r = \bigoplus_{m_1,\ldots,m_n \in Z_{\geq 0}} \chi(m_1,\ldots,m_n),$$

where $(\chi(m_1,\ldots,m_n)(\theta_1,\ldots,\theta_n))(z_1^{m_1},\ldots,z_n^{m_n}) = e^{-i m_1 \theta_1} z_1^{m_1} \cdots e^{-i m_n \theta_n} z_n^{m_n}$.

Let $\tau = \nu \otimes \chi_r$ as above. $\tau$ restricted to $T^{n-1} \times S^1$ decomposes into a sum of characters of the form $\chi_{k_1,\ldots,k_{n-1}}$ for certain $k_1,\ldots,k_{n-1} \in Z$.

It is clear that if $\nu$ is the trivial representation of $SU(n)$, $(\omega \otimes \chi_r)_{\varphi} \otimes \chi_r$ decomposes into a sum of characters without multiplicity since

$$(\omega \otimes \chi_r)_{\varphi} \otimes \chi_r = \bigoplus_{m_1,\ldots,m_n \in Z_{\geq 0}} \chi(m_1,\ldots,m_n,\nu+r),$$

We suppose $(\nu, W_{\nu})$ is nontrivial, so $\dim(W_{\nu}) > 1$. Let $\chi_{k_1,\ldots,k_{n-1}}$ and $\chi_{l_1,\ldots,l_{n-1}}$ be two different characters that appear in $\nu_{\varphi}$. As a result, the characters

$$\bigoplus_{m_1,\ldots,m_n \in Z_{\geq 0}} \chi(m_1+k_1,\ldots,m_n-k_{n-1},m_n+r)$$

and

$$\bigoplus_{m_1,\ldots,m_n \in Z_{\geq 0}} \chi(m_1+l_1,\ldots,m_n-l_{n-1},m_n+r).$$
appear in the decomposition of \((\omega \otimes \tau \otimes \chi_r)|_{\Gamma_{n-1} \times \mathbb{Z}}\). We obtain multiplicity in the decomposition of \((\omega \otimes \nu \otimes \chi_r)|_{\Gamma_{n-1} \times \mathbb{Z}}\) when we choose, for example, \(m_n = \tilde{m}_n = 0\) and for \(1 \leq i \leq n - 1\):

\[
\begin{cases}
  m_i = k_i - l_i ; \tilde{m}_i = 0 & \text{if } k_i - l_i \geq 0 \\
  m_i = 0 ; \tilde{m}_i = l_i - k_i & \text{if } k_i - l_i < 0.
\end{cases}
\]

\[
(19)
\]

\[\square\]

**Proposition 6.** The triple \((SU(n) \times S^1, N(u(n), \mathbb{C}^n), \tau)\) is commutative if and only if \(\tau \in \mathcal{S}^1\).

**Proof.** This result holds from the previous theorem and Corollary [1].

\[\square\]

**Case (VII)**

In the item (VII) of the list \(\mathfrak{g} = u(2)\) acts on \(V = (\mathbb{C}^2)^k \oplus (\mathbb{C}^2)^n\) by \(\pi\) defined in the following way:

(i) \(u(2)\) acts on each of the \(k\) components of \((\mathbb{C}^2)^k\) in the natural way,

(ii) in \((\mathbb{C}^2)^n\) the center of \(u(2)\) acts trivially and the semisimple part acts as \(sp(1)\) (or \(Im(\mathbb{H})\)) on the left side in each of the \(n\) components of \((\mathbb{C}^2)^n\).

Therefore the unitary group of intertwining operators is \(U = U(k) \times Sp(n)\). The nilpotent group \(N(u(2), (\mathbb{C}^2)^k \oplus (\mathbb{C}^2)^n))\) has Lie algebra \(n = u(2) \oplus (\mathbb{C}^2)^k \oplus (\mathbb{C}^2)^n\) and its group of orthogonal automorphisms is \(K = SU(2) \times U(k) \times Sp(n)\). A maximal torus on \(SU(2)\) is given by \(T^1 = \{(e^{i\theta} 0_0 | \theta \in \mathbb{R})\}\) and it is in correspondence with \(\{e^{i\theta} | \theta \in \mathbb{R}\}\) in \(Sp(1)\).

The metaplectic representation \(\omega\) of \(T^1 \times U\) acts on \(\mathcal{P}((\mathbb{C}^2)^k \oplus (\mathbb{C}^2)^n)\) in the following way:

(i) In order to understand the action of \(U(k)\) on \(\mathcal{P}((\mathbb{C}^2)^k)\) we must think each element \(A \in U(k)\) as an (unitary) intertwining operator of the action of \(u(2)\) on \(\mathcal{P}((\mathbb{C}^2)^k)\): since the natural action of \(u(2)\) on \(\mathbb{C}^2\) is irreducible, Schur’s lemma implies that \(A\) is a linear operator \(A : (\mathbb{C}^2)^k \to (\mathbb{C}^2)^k\) whose matrix representation is

\[
[A] = \begin{pmatrix}
  a_{11}I & a_{12}I & \ldots & a_{1k}I \\
  a_{21}I & a_{22}I & \ldots & a_{2k}I \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{k1}I & a_{k2}I & \ldots & a_{kk}I
\end{pmatrix}, \text{ where } I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } a_{i,j} \in \mathbb{C}.
\]

From here the action of \(U(k)\) on \(\mathcal{P}((\mathbb{C}^2)^k)\) splits as \(\mathcal{P}((\mathbb{C}^2)^k) \otimes \mathcal{P}((\mathbb{C}^2)^k)\) and also it can be written as \(\bigoplus_{r,s \in \mathbb{Z}_{\geq 0}} \mathcal{P}_r((\mathbb{C}^2)^k) \otimes \mathcal{P}_s((\mathbb{C}^2)^k)\). Although \(\mathcal{P}_r((\mathbb{C}^2)^k) \otimes \mathcal{P}_s((\mathbb{C}^2)^k)\) is not irreducible as \(U(k)\)-module, it is multiplicity free. Moreover, from [1], \((U(n), N(\mathbb{R}, \mathbb{C}^n))\) is a strong Gelfand pair. In particular, for each \(s \in \mathbb{Z}_{\geq 0}\), the sum \(\bigoplus_{r \in \mathbb{Z}_{\geq 0}} \mathcal{P}_r((\mathbb{C}^2)^k) \otimes \mathcal{P}_s((\mathbb{C}^2)^k)\) is multiplicity free as \(U(k)\)-module.

Apart from that, \(T^1 \subset SU(2)\) acts on each polynomial \(p \in \mathcal{P}((\mathbb{C}^2)^k)\) by

\[
p(u_1, v_1, \ldots, u_k, v_k) \mapsto p(e^{i\theta}u_1, e^{-i\theta}v_1, \ldots, e^{i\theta}u_k, e^{-i\theta}v_k), \quad (u_i, v_i \in \mathbb{C}).
\]

Therefore the action of \(T^1 \times U(k)\) on \(\mathcal{P}((\mathbb{C}^2)^k)\) decomposes without multiplicity into

\[
\bigoplus_{r,s \in \mathbb{Z}_{\geq 0}} \chi_{r-s} \otimes U(r) \otimes U(s),
\]

\[
(20)
\]
denoting by $u_r$ and $u_s$ the classical irreducible representations of $U(k)$ on $P_r(C^k)$ and $P_s(C^k)$ respectively and where $\chi_{r-s}(e^{i\theta}) = e^{-(r-s)i\theta}$.

According to [Y], for each $s \in Z_{\geq 0}$,

$$\bigoplus_{r \in Z_{\geq 0}} \chi_r \otimes u_r \otimes u_s$$

is multiplicity free. Hence

$$\bigoplus_{r,s \in Z_{\geq 0}} \chi_{r-s} \otimes u_r \otimes u_s = \bigoplus_{s \in Z_{\geq 0}} X_{-s} \otimes \left( \bigoplus_{r \in Z_{\geq 0}} \chi_r \otimes u_r \otimes u_s \right)$$

is also multiplicity free.

(ii) For each $j \in Z_{\geq 0}$ the action of $Sp(n)$ on $P_j(C^{2n})$, denoted by $\eta_j$, is irreducible. $T^1 \subset Sp(1)$ acts on each homogeneous polynomial by its degree, in the sense that if $p \in P_j(C^{2n})$ the action is $(e^{i\theta}, p) \mapsto e^{-j\theta}p$. Therefore the action of $T^1 \times Sp(n)$ on $P(C^{2n})$ decomposes into irreducible representations as

$$\bigoplus_{j \in Z_{\geq 0}} \chi_j \otimes \eta_j.$$  \hspace{1cm} (21)

In conclusion, the metaplectic representation decomposes multiplicity free into

$$\omega_{|_{T^1 \times U}} = \left( \bigoplus_{r,s \in Z_{\geq 0}} \chi_{r-s} \otimes u_r \otimes u_s \right) \otimes \left( \bigoplus_{j \in Z_{\geq 0}} \chi_j \otimes \eta_j \right)$$  \hspace{1cm} (22)

(but we must keep in mind that each term $u_r \otimes u_s$ is not irreducible).

An irreducible unitary representation of $K$ is given by $\tau = \nu_d \otimes v \otimes \eta$, where $(\nu_d, P_d(C^2))$ is the well known irreducible unitary representation of $SU(2)$ on the homogeneous polynomials on $C^2$ of degree $d$, $v \in \widehat{U(k)}$ and $\eta \in Sp(n)$. This representation restricted to $T^1 \times U$ decomposes into irreducible representations as

$$\tau_{|_{T^1 \times U}} = (\bigoplus_{i=0}^d \chi_{d-2i}) \otimes v \otimes \eta.$$

Therefore $\omega \otimes \tau$ restricted to $T^1 \times U$ decomposes into

$$\left( \omega \otimes \tau \right)_{|_{T^1 \times U}} = \bigoplus_{r,s,j \in Z_{\geq 0}} \chi_{r-s+j+d-2i} \otimes u_r \otimes u_s \otimes v \otimes \eta_j \otimes \eta.$$  \hspace{1cm} (23)

**Proposition 7.** The triple $(SU(2) \times U(k) \times Sp(n), N(u(n), (C^2)^k \oplus (C^2)^n), \tau)$ is commutative if and only if $\tau \in \widehat{U(k)}$.

**Proof.** Let $\tau = \nu_d \otimes v \otimes \eta \in SU(2) \times \widehat{U(k)} \times Sp(n)$.

Let $\eta$ be nontrivial. From Corollary [24] $\eta$ appears in the decomposition into irreducible representations of $\eta_{(2)} \otimes \eta$. Thus, considering on the one hand $j = 0, r = 1, s = 0$ and on the other hand $j = 2, r = 0, s = 1$, we obtain repetition in [23].
Let $\nu d$ be nontrivial, i.e., $d \geq 1$. We obtain repetition of representations in (23) considering, on the one hand $i = 0$ and $j = 0$ and on the other hand $i = 1$ and $j = 2$.

In conclusion, if $\nu d$ or $\eta$ are nontrivial, from Corollary 1, we obtain non-commutative triples.

Assume $\nu d$ and $\eta$ are trivial. We will show that the decomposition

$$(\omega \otimes \tau)|_{T^1 \times U(k) \times Sp(n)} = \bigoplus_{r,s,j \in \mathbb{Z} \geq 0} \chi_{r+s+j} \otimes \nu(r) \otimes \nu(s) \otimes \nu \otimes \eta(j).$$

is multiplicity free by noting that its restriction to $T^1 \times Z(U(k)) \times Sp(n)$ is multiplicity free, where $Z(U(k))$ is the center of $U(k)$. Let $\chi_t$ be the character associated to $\nu|_{Z(U(k))}$. To each representation $\nu(r) \in \widehat{U(k)}$ acting on the space of homogeneous polynomials of degree $r$, the character associated corresponds to the degree $r$, i.e., $(\nu(r))|_{Z(U(k))} = \chi_r$. Thus,

$$(\omega \otimes \nu)|_{T^1 \times Z(U(k)) \times Sp(n)} = \bigoplus_{r,s,j \in \mathbb{Z} \geq 0} \chi_{r+s+j} \otimes \chi_{r+s+t} \otimes \eta(j). \quad (24)$$

The parameter $l$ is fixed and we could assume that the parameter $j$ is also fixed because when $j$ runs on $\mathbb{Z} \geq 0$ the factor $\eta(j)$ changes. Therefore we obtain multiplicity in (24) if there exist parameters $r, s, r', s' \in \mathbb{Z} \geq 0$, $r \neq r'$, $s \neq s'$, such that

$$\begin{cases} r - s = r' - s' \\ r + s = r' + s' \end{cases}$$

and this is not possible.

In conclusion, if $\nu m$ or $\eta$ are trivial, from Corollary 1, we obtain commutative triples. \qed

Case (VIII)

In this case the Lie algebra $g$ will not be semisimple. Let $\alpha$ and $\beta$ be two nonnegative integers and let $\{m_i\}_{i=1}^\beta$ be integers greater or equal than 3. We take

$$g := \mathfrak{su}(m_1) \oplus \ldots \oplus \mathfrak{su}(m_{\beta}) \oplus \mathfrak{su}(2) \oplus \ldots \oplus \mathfrak{su}(2) \oplus \mathfrak{c},$$

where $\mathfrak{c}$ is its center and there are $\alpha$ copies of $\mathfrak{su}(2)$. The abelian component satisfies that $1 \leq \dim(\mathfrak{c}) < \alpha + \beta$. Let us consider the vector spaces

$$V_1 := C^{m_1}, \ldots, V_\beta := C^{m_\beta}, V_{\beta+1} := C^{2k_1+2n_1}, \ldots, V_{\beta+\alpha} := C^{2k_\alpha+2n_\alpha},$$

where $k_j \geq 1$ and $n_j \geq 0$ for all $1 \leq j \leq \alpha$ and let $V := \bigoplus_{i=1}^{\beta+\alpha} V_i$. The nilpotent group $N(g, V)$ is given in the following way:

- For each $1 \leq i \leq \beta + \alpha$, $\mathfrak{c}$ has a maximal subspace, denoted by $\mathfrak{c}_i$, of dimension one acting non-trivially on $V_i$. 
For $1 \leq i \leq \beta$, $\mathfrak{su}(m_i) \oplus \mathfrak{c}_i$ acts on $V_i$ as in the case (VI). That is, since $\mathfrak{su}(m_i) \oplus \mathfrak{c}_i$ is isomorphic to $\mathfrak{u}(m_i)$, it acts in the natural way on $\mathbb{C}^{m_i}$. Apart from that, $\mathfrak{su}(m_i) \oplus \mathfrak{c}_i$ acts trivially on $V_j$ for all $j \neq i$.

For $\beta + 1 < i \leq \beta + \alpha$, $\mathfrak{su}(2) \oplus \mathfrak{c}_i$ acts on $V_i$ as in the case (VII). That is, it acts naturally as $\mathfrak{u}(2)$ on $(\mathbb{C}^2)^{k_i}$ and $\mathfrak{su}(2)$ acts naturally as $\mathfrak{sp}(1)$ on $(\mathbb{H})^{n_i}$ (which is isomorphic to $(\mathbb{C}^2)^{n_i}$ a real vector spaces). Also, $\mathfrak{su}(2) \oplus \mathfrak{c}_i$ acts trivially on $V_j$ for all $j \neq i$.

The connected group of unitary intertwining operators is

$$U := S^1 \times \ldots \times S^1 \times U(k_1) \times \text{Sp}(n_1) \times \ldots \times U(k_\alpha) \times \text{Sp}(n_\alpha),$$

where there are $\beta$ copies of $S^1$.

Let us consider first the case where $g = \mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{c}$ and $V = \mathbb{C}^m \oplus (\mathbb{C}^2)^k \oplus (\mathbb{C}^2)^n$, for some integers $m \geq 3$, $k \geq 1$ and $n \geq 0$. Here we have $K = SU(m) \times SU(2) \times S^1 \times U(k) \times \text{Sp}(n)$. Let $T^{m-1}$ be a maximal torus of $SU(m)$ and $T^1$ a maximal torus of $SU(2)$.

The metaplectic representation $\omega$ of $T^{m-1} \times T^1 \times S^1 \times U(k) \times \text{Sp}(n)$ acts on $\mathcal{P}(\mathbb{C}^m \oplus \mathbb{C}^{2k} \oplus \mathbb{C}^{2n})$ in the following way,

(i) $T^1 \times U(k) \times \text{Sp}(n)$ acts as in the case (VII) on $\mathcal{P}(\mathbb{C}^{2k} \oplus \mathbb{C}^{2n})$ and

(ii) $T^{m-1} \times S^1$ is an $m$-dimensional torus in $U(m)$ that acts on $\mathcal{P}(\mathbb{C}^m)$ in the natural way.

Thus, the representation of $T^{m-1} \times T^1 \times S^1 \times U(k) \times \text{Sp}(n)$ decomposes into

$$\left( \bigoplus_{j_1, \ldots, j_m \in \mathbb{Z}_{\geq 0}} \chi_{(j_1, \ldots, j_m)} \right) \otimes \left( \bigoplus_{r, s \in \mathbb{Z}_{\geq 0}} \chi_{r-s} \otimes v(r) \otimes v(s) \right) \otimes \left( \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \chi_j \otimes \eta(j) \right),$$

where in the first factor of the tensor product we have a sum of characters of the form $\chi_{(j_1, \ldots, j_m)}(\theta_1, \ldots, \theta_m) := e^{-j_1 \theta_1} \cdots e^{-j_m \theta_m}$, whereas the second factor, $\left( \bigoplus_{r, s \in \mathbb{Z}_{\geq 0}} \chi_{r-s} \otimes v(r) \otimes v(s) \right) \otimes \left( \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \chi_j \otimes \eta(j) \right)$, is the same as in the case (VII). Remember that each term $v(r) \otimes v(s)$ is not irreducible, but anyway $\omega|_{T^{m-1} \times T^1 \times S^1 \times U(k) \times \text{Sp}(n)}$ is multiplicity free [Y].

**Proposition 8.** The triple $(SU(m) \times SU(2) \times S^1 \times U(k) \times \text{Sp}(n), N(\mathfrak{su}(m) \oplus \mathfrak{su}(2) \oplus \mathfrak{c}, \mathbb{C}^m \oplus (\mathbb{C}^2)^k \oplus (\mathbb{C}^2)^n, \tau)$ is commutative if and only if $\tau \in S^1 \times U(k)$.

**Proof.** An irreducible representation of $K$ is given by the tensor product $\tau = \nu \otimes \nu_d \otimes \chi_t \otimes v \otimes \eta$, where $\nu \in \hat{SU}(m)$, $(\nu_d, \mathcal{P}_d(\mathbb{C}^2))$ is the well known irreducible representation of $SU(2)$ on the homogeneous polynomials on $\mathbb{C}^2$ of degree $d$, $\chi_t$ is a character of $S^1$ ($t \in \mathbb{Z}$), $\nu \in \hat{U}(k)$ and $\eta \in \hat{\text{Sp}(n)}$.

First of all note that if $\nu_d$ or $\eta$ are nontrivial, from Proposition [7] we have multiplicity in the decomposition of $\omega \otimes \tau$ when we restrict it to $T^{m-1} \times T^1 \times S^1 \times U(k) \times \text{Sp}(n)$. Hence we suppose $\nu_d$ and $\eta$ trivial.
Now, let $\nu \in \widehat{SU}(m)$ be nontrivial. From the theorem of the highest weight, $\nu$ can be parametrized in terms of a partition $(\nu_1, \ldots, \nu_{m-1})$ and every weight of $\nu$ is of the form

$$\nu - a := (\nu_1 - a_1, \nu_2 + a_1 - a_2, \ldots, \nu_{n-2} + a_{n-3} - a_{n-2}, \nu_{m-1} + a_{m-1}),$$

where $a_i \in \mathbb{Z}_{\geq 0}$ for all $1 \leq i \leq m - 1$. Let $(\nu_1, \ldots, \nu_{m-1})$ and $\nu - a$ be two different weights of the representation $\nu$ with $a_i \in \mathbb{Z}_{\geq 0}$ $\forall i$. In the decomposition of $\omega \otimes (\nu \otimes \chi_t \otimes \nu)$ restricted to $T^{m-1} \times T^1 \times \mathbb{S}^1 \times U(k) \times \text{Sp}(n)$ we have, in particular, the following terms

$$\left( \bigoplus_{j_1, \ldots, j_m \in \mathbb{Z}_{\geq 0}} \chi_{\nu + (j_1, \ldots, j_m)} \otimes \chi_{j_m + t} \right) \otimes \left( \bigoplus_{r, s \in \mathbb{Z}_{\geq 0}} \chi_{r-s} \otimes \nu_{(r)} \otimes \nu_{(s)} \otimes \nu \right) \otimes \left( \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \chi_j \otimes \eta_{(j)} \right)$$

and

$$\left( \bigoplus_{j'_1, \ldots, j'_{m-1} \in \mathbb{Z}_{\geq 0}} \chi_{\nu - a + (j'_1, \ldots, j'_{m-1})} \otimes \chi_{j'_{m-1} + t} \right) \otimes \left( \bigoplus_{r, s \in \mathbb{Z}_{\geq 0}} \chi_{r-s} \otimes \nu_{(r)} \otimes \nu_{(s)} \otimes \nu \right) \otimes \left( \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \chi_j \otimes \eta_{(j)} \right).$$

We obtain multiplicity considering for example,

$$\begin{cases} j_1 = 0, & j'_1 = a_1, \\ j_i = a_i - a_{i+1}, & j'_i = 0 \quad \text{if} \quad a_i - a_{i+1} \geq 0 \quad \text{for} \quad 1 < i < m - 1, \\ j_i = 0, & j'_i = -(a_i - a_{i+1}) \quad \text{if} \quad a_i - a_{i+1} < 0 \quad \text{for} \quad 1 < i < m - 1, \\ j_{m-1} = a_{m-1}, & j'_{m-1} = 0. \end{cases}$$

In conclusion, if $\nu_d$, $\eta$ or $\nu$ are nontrivial, from Corollary 1 we obtain non-commutative triples. This leads us to suppose that $\nu_d$, $\eta$ and also $\nu$ are trivial representations.

Finally, for any $\chi_t \in \widehat{S^1}$ and any $\nu \in \widehat{U(k)}$, we must analyze

$$\left( \bigoplus_{j_1, \ldots, j_m \in \mathbb{Z}_{\geq 0}} \chi_{(j_1, \ldots, j_m + t)} \right) \otimes \left( \bigoplus_{r, s \in \mathbb{Z}_{\geq 0}} \chi_{r-s} \otimes \nu_{(r)} \otimes \nu_{(s)} \otimes \nu \right) \otimes \left( \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \chi_j \otimes \eta_{(j)} \right). \quad (26)$$

The sum $\bigoplus_{j_1, \ldots, j_m \in \mathbb{Z}_{\geq 0}} \chi_{(j_1, \ldots, j_m + t)}$ is multiplicity free for all $t \in \mathbb{Z}$ and from Proposition 1 the decomposition $\left( \bigoplus_{r, s \in \mathbb{Z}_{\geq 0}} \chi_{r-s} \otimes \nu_{(r)} \otimes \nu_{(s)} \otimes \nu \right) \otimes \left( \bigoplus_{j \in \mathbb{Z}_{\geq 0}} \chi_j \otimes \eta_{(j)} \right)$ is multiplicity free for all $\nu \in \widehat{U(k)}$.

Thus, by Corollary 1 we obtain commutative triples when $\tau = \chi_t \otimes \nu$. \hfill \Box

With slight changes in the proof of the above proposition we can easily finish the case (VIII).

**Proposition 9.** The triple $(K, N(g, V), \tau)$ is commutative if and only if $\tau = \chi_{t_1} \otimes \ldots \otimes \chi_{t_\beta} \otimes v_1 \otimes \ldots \otimes v_\alpha$, where $\chi_{t_i} \in \widehat{S^1}$ $(t_i \in \mathbb{Z})$ for all $1 \leq i \leq \beta$ and $v_j \in \widehat{U(k_j)}$ for all $1 \leq j \leq \alpha$.

**Summary**

Putting together Proposition 1 to 9 we obtain the complete list of nontrivial commutative triples $(G \times U, N(g, V), \tau)$ as in Theorem 1.
(SU(2) \times \text{Sp}(n), N(\mathfrak{su}(2), (\mathbb{C}^2)^n), \tau), \text{ for all } \tau \in \widehat{\text{SU}(2)} \text{ and for all } \tau \in \widehat{\text{Sp}(n)} \text{ associated to a constant partition of length at most } n, \text{ where } n \geq 1,

(SU(n) \times S^1, N(\mathfrak{su}(n), \mathbb{C}^n), \tau), \text{ for all } \tau \in \widehat{S^1}, \text{ where } n \geq 3,

(SU(n) \times S^1, N(\mathfrak{u}(n), \mathbb{C}^n), \tau), \text{ for all } \tau \in \widehat{S^1}, \text{ where } n \geq 3,

(SU(2) \times U(k) \times \text{Sp}(n), N(\mathfrak{u}(n), (\mathbb{C}^2)^k \oplus (\mathbb{C}^2)^n), \tau), \text{ for all } \tau \in \widehat{U(k)}, \text{ where } k \geq 1, n \geq 0,

(G \times U, N(\mathfrak{g}, V), \tau) \text{ where }
\begin{align*}
\mathfrak{g} &= \mathfrak{su}(m_1) \oplus \ldots \oplus \mathfrak{su}(m_\beta) \oplus \mathfrak{su}(2) \oplus \ldots \oplus \mathfrak{su}(2) \oplus \mathfrak{c} \text{ with } \mathfrak{c} \text{ an abelian component}, \\
G &= \text{SU}(m_1) \times \ldots \times \text{SU}(m_\beta) \times \text{SU}(2) \times \ldots \times \text{SU}(2), \\
V &= \mathbb{C}^{m_1} \oplus \ldots \oplus \mathbb{C}^{m_\beta} \oplus \mathbb{C}^{2k_1+2n_1} \oplus \ldots \oplus \mathbb{C}^{2k_\alpha+2n_\alpha} \text{ and} \\
U &= S^1 \times \ldots \times \widehat{S^1} \times \text{Sp}(n_1) \times \ldots \times \text{Sp}(n_\alpha)
\end{align*}

for all \( \tau \in \widehat{S^1} \otimes \ldots \otimes \widehat{S^1} \otimes \widehat{\text{U}(k_1)} \otimes \ldots \otimes \widehat{\text{U}(k_\alpha)} \), where \( m_j \geq 3 \) for all \( 1 \leq j \leq \beta, k_i \geq 1, n_i \geq 0 \) for all \( 1 \leq i \leq \alpha \) and

(U(n), N(\mathbb{R}, \mathbb{C}^n), \tau), \text{ for all } \tau \in \widehat{U(n)}, \text{ where } n \geq 1 \text{ (proved by Yakimova in [Y]).}

Remark 2. The Gelfand pair \((U(n), N(\mathbb{R}, \mathbb{C}^n))\) (where \(N(\mathbb{R}, \mathbb{C}^n)\) is the Heisenberg group and \(U(n)\) is its maximal group of orthogonal automorphisms) is the only strong Gelfand pair in this family.

Remark 3. Let \( U \) be the connected component of the unitary intertwining operators of \((\pi, V)\). Assume that \( U \) is a direct product of groups and \( U(n) \) appears in \( U \) for some \( n \geq 1 \). For \( \tau \in \widehat{U(n)} \) the triple \((G \times U, N(\mathfrak{g}, V), \tau)\) is a commutative triple. (See cases (V) to (IX).)

Theorem 6. Let \( N(\mathfrak{g}, V) \) be indecomposable which has a square integrable representation. The nontrivial triple \((G \times U, N(\mathfrak{g}, V), \tau)\) is commutative if and only if one of the following holds:

(i) \( U \) can be written as a direct product of groups \( U = L \times M \) where \( L := U(n_1) \times \ldots \times U(n_m) \) (for some \( m \geq 1 \) and \( n_i \geq 1 \)) and \( \tau \in \widehat{L} \).

(ii) \( N(\mathfrak{g}, V) \) is the H-type group and \( \tau \) is as in Proposition [I].

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