Reproducing formulas for generalized translation invariant systems on locally compact abelian groups

Jakobsen, Mads Sielemann; Lemvig, Jakob

Published in:
American Mathematical Society. Transactions

Link to article, DOI:
10.1090/tran/6594

Publication date:
2016

Document Version
Peer reviewed version

Citation (APA):
Jakobsen, M. S., & Lemvig, J. (2016). Reproducing formulas for generalized translation invariant systems on locally compact abelian groups. American Mathematical Society. Transactions, 368(12), 8447-8480. https://doi.org/10.1090/tran/6594

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.
Reproducing formulas for generalized translation invariant systems on locally compact abelian groups

Mads Sielemann Jakobsen*, Jakob Lemvig†

October 23, 2014

Abstract: In this paper we connect the well established discrete frame theory of generalized shift invariant systems to a continuous frame theory. To do so, we let $\Gamma_j$, $j \in J$, be a countable family of closed, co-compact subgroups of a second countable locally compact abelian group $G$ and study systems of the form $\bigcup_{j \in J} \{g_{j,p}(\cdot - \gamma)\}_{\gamma \in \Gamma_j, p \in P_j}$ with generators $g_{j,p}$ in $L^2(G)$ and with each $P_j$ being a countable or an uncountable index set. We refer to systems of this form as generalized translation invariant (GTI) systems. Many of the familiar transforms, e.g., the wavelet, shearlet and Gabor transform, both their discrete and continuous variants, are GTI systems. Under a technical $\alpha$-local integrability condition ($\alpha$-LIC) we characterize when GTI systems constitute tight and dual frames that yield reproducing formulas for $L^2(G)$. This generalizes results on generalized shift invariant systems, where each $P_j$ is assumed to be countable and each $\Gamma_j$ is a uniform lattice in $G$, to the case of uncountably many generators and (not necessarily discrete) closed, co-compact subgroups. Furthermore, even in the case of uniform lattices $\Gamma_j$, our characterizations improve known results since the class of GTI systems satisfying the $\alpha$-LIC is strictly larger than the class of GTI systems satisfying the previously used local integrability condition. As an application of our characterization results, we obtain new characterizations of translation invariant continuous frames and Gabor frames for $L^2(G)$. In addition, we will see that the admissibility conditions for the continuous and discrete wavelet and Gabor transform in $L^2(\mathbb{R}^n)$ are special cases of the same general characterizing equations.

1 Introduction

In harmonic analysis one is often interested in determining conditions on generators of function systems, e.g., Gabor and wavelet systems, that allow for reconstruction of any function in a given class of functions from its associated transform via a reproducing formula. The work of Hernández, Labate, and Weiss [30] and of Ron and Shen [46] on generalized shift invariant systems in $L^2(\mathbb{R}^n)$ presented a unified theory for many of the familiar discrete transforms, most notably the Gabor and the wavelet transform. The generalized shift invariant systems are collections of functions of the form $\bigcup_{j \in J} \{T_\gamma g_j\}_{\gamma \in \Gamma_j}$, where $J$ is a countable index set, $T_\gamma$ denotes translation by $\gamma$, $\Gamma_j$ a full-rank lattice in $\mathbb{R}^n$, and $\{g_j\}_{j \in J}$ a subset of $L^2(\mathbb{R}^n)$. Here, the word “shift” is

2010 Mathematics Subject Classification. Primary: 42C15, 43A32, 43A70, Secondary: 43A60, 46C05.

Key words and phrases. continuous frame, dual frames, dual generators, g-frame, Gabor frame, generalized shift invariant system, generalized translation invariant system, LCA group, Parseval frame, wavelet frame

*Technical University of Denmark, Department of Applied Mathematics and Computer Science, Matematiktorvet 303B, 2800 Kgs. Lyngby, Denmark, E-mail: mjsaj@dtu.dk

†Technical University of Denmark, Department of Applied Mathematics and Computer Science, Matematiktorvet 303B, 2800 Kgs. Lyngby, Denmark, E-mail: jakle@dtu.dk
used since the translations are discrete and the word “generalized” since the shift lattices $\Gamma_j$ are allowed to change with the parameter $j \in J$. The main result of Hernández, Labate, and Weiss 30 is a characterization, by so-called $t_\alpha$-equations, of all functions $g_j$ that give rise to isometric, isomorphic transforms, called Parseval frames in frame theory.

The goal of this work is to connect the discrete transform theory of generalized shift invariant systems to a continuous/integral transform theory. In doing so, the scope of the “unified approach” started in 30,16 will be vastly extended. What more is, this new theory will cover “intermediate” steps, the semi-continuous transforms, and we will do so in a very general setting of square integrable functions on locally compact abelian groups. In particular, we recover the usual characterization results for discrete and continuous Gabor and wavelet systems as special cases. For discrete wavelets in $L^2(\mathbb{R})$ with dyadic dilation, this result was obtained in 1995, independently by Gripenberg 23 and Wang 48, and it can be stated as follows. Define the translation operator $T_b f(x) = f(x - b)$ and dilation operator $D_\alpha f(x) = |a|^{-1/2} f(x/a)$ for $b \in \mathbb{R}, a \neq 0$. The discrete wavelet system $\{T_{2^j k} D_{2^j} \psi\}_{j,k \in \mathbb{Z}}$ generated by $\psi \in L^2(\mathbb{R})$ is indeed a generalized shift invariant system with $J = \mathbb{Z}$, $\Gamma_j = 2^j \mathbb{Z}$, and $g_j = D_{2^j} \psi$. Now, the linear operator $W_d$ defined by

$$W_d : L^2(\mathbb{R}) \rightarrow \ell^2(\mathbb{Z}^2), \quad W_d f(j, k) = \langle f, T_{2^j k} D_{2^j} \psi \rangle$$

is isometric, isomorphic if, and only if, for all $\alpha \in \bigcup_{j \in \mathbb{Z}} 2^{-j} \mathbb{Z}$, the following $t_\alpha$-equations hold:

$$t_\alpha := \sum_{j \in \mathbb{Z} : a \in 2^{-j} \mathbb{Z}} \hat{\psi}(2^j \xi) \bar{\psi}(2^j (\xi + \alpha)) = \delta_{\alpha, 0} \quad \text{for a.e. } \xi \in \mathbb{R}, \quad (1.1)$$

where $\hat{\mathbb{R}}$ denotes the Fourier domain. In the language of frame theory, we say that generators $\psi \in L^2(\mathbb{R})$ of discrete Parseval wavelet frames have been characterized by $t_\alpha$-equations.

Calderón 6 discovered in 1964 that any function $\psi \in L^2(\mathbb{R})$ satisfying the Calderón admissibility condition

$$\int_{\mathbb{R} \setminus \{0\}} \frac{|\hat{\psi}(a \xi)|^2}{|a|} \, da = 1 \quad \text{for a.e. } \xi \in \mathbb{R} \quad (1.2)$$

leads to reproducing formulas for the continuous wavelet transform. To be precise, the linear operator $W_c$ defined by

$$W_c : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R} \setminus \{0\} \times \mathbb{R}, \frac{da db}{a^2}), \quad W_c f(a, b) = \langle f, T_b D_a \psi \rangle$$

is isometric, isomorphic if, and only if, the Calderón admissibility condition holds. We will see that the Calderón admissibility condition is nothing but the $t_\alpha$-equation (there is only one!) for the continuous wavelet system. Similar results hold for the Gabor case; here the continuous transform is usually called the short-time Fourier transform. Actually, the theory is not only applicable to the Gabor and wavelet setting, but to a very large class of systems of functions including shearlet and wave packet systems, which we shall call generalized translation invariant systems. We refer the reader to the classical texts 12,14,27 and the recent book 38 for introductions to the specific cases of Gabor, wavelet, shearlet and wave packet analysis.

In 36, Kutyniok and Labate generalized the results of Hernández, Labate, and Weiss to generalized shift invariant systems $\bigcup_{j \in J} \{T_{\gamma_j} g_j\}_{\gamma_j \in \Gamma_j} \subset L^2(G)$, where $G$ is a second countable locally compact abelian group and $\Gamma_j$ is a family of uniform lattices (i.e., $\Gamma_j$ is a discrete subgroup and the quotient group $G/\Gamma_j$ is compact) indexed by a countable set $J$. The main goal of the present paper is to develop the corresponding theory for semi-continuous and continuous frames in $L^2(G)$. In order to achieve this, we will allow non-discrete translation groups $\Gamma_j$, and we
will allow for each translation group to have uncountable many generators, indexed by some index set $P_j$, $j \in J$. We say that the corresponding family $\bigcup_{j \in J} \{T_j g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ in $L^2(G)$ is a \textit{generalized translation invariant system}. To be precise, we will, for each $j \in J$, take $P_j$ to be a $\sigma$-finite measure space with measure $\mu_{P_j}$ and $\Gamma_j$ to be closed, co-compact (i.e., the quotient group $G/\Gamma_j$ is compact) subgroups. We mention that \textit{any} locally compact abelian group has a co-compact subgroup, namely the group itself. On the other hand, there exist groups that do not contain uniform lattices, e.g., the $p$-adic numbers. Thus, the theory of generalized translation invariant systems is applicable to a larger class of locally compact abelian groups than the theory of generalized shift invariant systems.

The two wavelet cases described above fit our framework. The discrete wavelet system can be written as $\bigcup_{j \in \mathbb{Z}} \{T_j(D_j \psi)\}_{\gamma \in \mathbb{Z}/\mathbb{Z}}$, so we see that $P_j$ is a singleton and $\mu_{P_j}$ a weighted counting measure for each $j \in J = \mathbb{Z}$, and that there are countably many different (discrete) $\Gamma_j$. For the continuous wavelet system on the form $\{T_j(D_j \psi)\}_{\gamma \in \mathbb{R}, p \in \mathbb{R}\setminus\{0\}}$, we have that $J$ is a singleton, e.g., $\{j_0\}$ since there is only one translation subgroup $\Gamma_{j_0} = \mathbb{R}$. On the other hand, here $P_{j_0}$ is uncountable and $\mu_{P_{j_0}}$ a weighted Lebesgue measure. We stress that our setup can handle countable many (distinct) $\Gamma_j$ and countable many $P_j$, each being uncountable.

The characterization results in \cite{30,36} rely on a technical condition on the generators and the translation lattices, the so-called \textit{local integrability condition}. This condition is straightforward to formulate for generalized translation invariant systems, however, we will replace it by a strictly weaker condition, termed $\alpha$ \textit{local integrability condition}. Therefore, even for generalized shift invariant systems in the euclidean setting, our work extends the characterization results by Hernández, Labate, and Weiss \cite{30}. Under the $\alpha$ integrability condition, we show in Theorem 3.5 that $\bigcup_{j \in J} \{T_j g_{j,p}\}_{\gamma \in \Gamma_j, p \in P_j}$ is a Parseval frame for $L^2(G)$, that is, the associated transform is isometric, isomorphic if, and only if,

$$t_\alpha := \sum_{j \in J: \alpha \in \Gamma_j^\perp} \int_{P_j} \overline{g_{j,p}(\omega)} g_{\gamma,j,p}(\omega + \alpha) \, d\mu_{P_j}(p) = \delta_{\alpha,0} \quad \text{a.e. } \omega \in \hat{G}$$

for every $\alpha \in \bigcup_{j \in J} \Gamma_j^\perp$, where $\Gamma_j^\perp = \{\omega \in \hat{G} : \omega(x) = 0 \text{ for all } x \in \Gamma_j\}$ denotes the annihilator of $\Gamma_j$. Now, returning to the two main examples of this introduction, the discrete and continuous wavelet transform, we see why the number of the $t_\alpha$-equations in (1.1) and (1.2) are so different. In the discrete case the corresponding union of the annihilators of the translation groups is $\bigcup_{j \in \mathbb{Z}} \mathbb{Z}/\mathbb{Z}$, while in the continuous case the annihilator of $\mathbb{R}$ is simply $\{0\}$, which corresponds to only one $t_\alpha$-equation ($\alpha = 0$).

Finally, as Kutyniok and Labate \cite{36} restrict their attention to Parseval frames, there are currently no characterization results available for \textit{dual} (discrete) frames in the setting of locally compact abelian groups. Hence, one additional objective of this paper is to prove characterizing equations for \textit{dual} generalized translation invariant frames to remedy this situation.

For a related study of reproducing formulas from a purely group representation theoretical point of view, we refer to the work of Führ \cite{20}, and De Mari, De Vito \cite{15}, and the references therein.

The paper is organized as follows. We recall some basic theory about locally compact abelian groups and introduce the generalized translation invariant systems in Section 2.1 and 2.2 respectively. Additionally, in Section 2.3 we give a short introduction to the theory of continuous frames and g-frames. In Section 3 we present our main characterization result for dual generalized translation invariant frames (Theorem 3.4) and, as corollary, then for Parseval frames (Theorem 3.5). In Section 3.2 and 3.3 we relate several conditions used in our main results.
Finally, we consider the special case of translation invariant systems and apply our characterization results on concrete groups and to concrete examples in Sections 3.4 and 4. Specifically, we consider discrete and continuous wavelet systems in $L^2(\mathbb{R}^n)$, shearlets in $L^2(\mathbb{R}^2)$, discrete, semi-continuous and continuous Gabor frames on LCA groups and GTI systems over the $p$-adic integers and numbers.

During the final stages of this project, we realized that Bownik and Ross [4] have completed a related investigation. As they consider and characterize the structure of translation invariant subspaces on locally compact abelian groups, their results do not overlap with our results in any way. However, they do consider translations along a closed, co-compact subgroup. We adopt their terminology of translation invariance, in place of shift invariance, to emphasize the fact that $\Gamma_j$ need not be discrete.

2 Preliminaries

In the following sections we set up notation and recall some useful results from Fourier analysis on locally compact abelian (LCA) groups and continuous frame theory. Furthermore, we will prove two important lemmas, Lemma 2.2 and 2.4.

2.1 Fourier analysis on locally compact abelian groups

Throughout this paper $G$ will denote a second countable locally compact abelian group. We note that the following statements are equivalent: (i) $G$ is second countable, (ii) $L^2(G)$ is separable, (iii) $G$ is metrizable and $\sigma$-compact. Note that the metric on $G$ can be chosen to be translation invariant.

To $G$ we associate its dual group $\hat{G}$ consisting of all characters, i.e., all continuous homomorphisms from $G$ into the torus $\mathbb{T} \cong \{z \in \mathbb{C} : |z| = 1\}$. Under pointwise multiplication $\hat{G}$ is also a locally compact abelian group. We will use addition and multiplication as group operation in $G$ and $\hat{G}$, respectively. Note that in the introduction we used addition as group operation in $\hat{G}$. By the Pontryagin duality theorem, the dual group of $\hat{G}$ is isomorphic to $G$ as a topological group, i.e., $\hat{\hat{G}} \cong G$. We recall the well-known facts that if $G$ is discrete, then $\hat{G}$ is compact, and vice versa.

We denote the Haar measure on $G$ by $\mu_G$. The (left) Haar measure on any locally compact group is unique up to a positive constant. From $\mu_G$ we define $L^1(G)$ and the Hilbert space $L^2(G)$ over the complex field in the usual way.

For functions $f \in L^1(G)$ we define the Fourier transform

$$\mathcal{F} f(\omega) = \hat{f}(\omega) = \int_G f(x) \overline{\omega(x)} \, d\mu_G(x), \quad \omega \in \hat{G}.$$ 

If $f \in L^1(G)$, $\hat{f} \in L^1(\hat{G})$, and the measure on $G$ and $\hat{G}$ are normalized so that the Plancherel theorem holds (see [32 (31.1)]), the function $f$ can be recovered from $\hat{f}$ by the inverse Fourier transform

$$f(x) = \mathcal{F}^{-1} \hat{f}(x) = \int_{\hat{G}} \hat{f}(\omega) \omega(x) \, d\mu_{\hat{G}}(\omega), \quad x \in G.$$ 

From now on we always assume that the measure on a group $\mu_G$ and its dual group $\mu_{\hat{G}}$ are normalized this way, and we refer to them as dual measures. As in the classical Fourier analysis $\mathcal{F}$ can be extended from $L^1(G) \cap L^2(G)$ to an isometric isomorphism between $L^2(G)$ and $L^2(\hat{G})$.

On any locally compact abelian group $G$, we define the following two linear operators. For $a \in G$, the operator $T_a$, called translation by $a$, is defined by

$$T_a : L^2(G) \to L^2(G), \quad (T_a f)(x) = f(x - a), \quad x \in G.$$
For $\chi \in \hat{G}$, the operator $E_\chi$, called modulation by $\chi$, is defined by

\[ E_\chi : L^2(G) \to L^2(G), \quad (E_\chi f)(x) = \chi(x)f(x), \quad x \in G. \]

Together with the Fourier transform $\mathcal{F}$, the two operators $E_\chi$ and $T_a$ share the following commutator relations: $T_a E_\chi = \chi(a) E_\chi T_a$, $\mathcal{F} T_a = E_{a^{-1}} \mathcal{F}$, and $\mathcal{F} E_\chi = T_\chi \mathcal{F}$.

For a subgroup $H$ of an LCA group $G$, we define its annihilator as

\[ H^\perp = \{ \omega \in \hat{G} : \omega(x) = 1 \text{ for all } x \in H \}. \]

The annihilator $H^\perp$ is a closed subgroup in $\hat{G}$, and if $H$ is closed, then $\hat{H} \cong \hat{G}/H^\perp$ and $\hat{G}/H \cong \hat{H}^\perp$.

We will repeatedly use Weil’s formula; it relates integrable functions over $\hat{G}$ with functions on the quotient space $G/H$.

We mention the following results concerning Weil’s formula [44].

**Theorem 2.1.** Let $H$ be a closed subgroup of $G$. Let $\pi_H : G \to G/H$, $\pi_H(x) = x + H$ be the canonical map from $G$ onto $G/H$. If $f \in L^1(G)$, then the following holds:

(i) The function $\hat{x} \mapsto \int_H f(x + h) \, d\mu_H(h)$, $\hat{x} = \pi_H(x)$ defined almost everywhere on $G/H$, is integrable.

(ii) (Weil’s formula) Let two of the Haar measures on $G$, $H$ and $G/H$ be given, then the third can be normalized such that

\begin{equation}
\int_G f(x) \, d\mu_G(x) = \int_{G/H} \int_H f(x + h) \, d\mu_H(h) \, d\mu_{G/H}(\hat{x}). \tag{2.1}
\end{equation}

(iii) If (2.1) holds, then the respective dual measures on $\hat{G}, H^\perp \cong \hat{G}/H^\perp, \hat{G}/H^\perp \cong \hat{H}$ satisfy

\begin{equation}
\int_{\hat{G}} \hat{f}(\omega) \, d\mu_{\hat{G}}(\omega) = \int_{\hat{G}/H^\perp} \int_{H^\perp} \hat{f}(\omega \gamma) \, d\mu_{H^\perp}(\gamma) \, d\mu_{\hat{G}/H^\perp}(\hat{\omega}). \tag{2.2}
\end{equation}

**Remark 1.** Since a Haar measure and its dual are chosen so that the Plancherel theorem holds we have the following uniqueness result: If two of the measures on $G$, $H$, $G/H$, $\hat{G}$, $H^\perp$ and $\hat{G}/H^\perp$ are given, and these two are not dual measures, by requiring Weil’s formulas (2.1) and (2.2), all other measures are uniquely determined.

For more information on harmonic analysis on locally compact abelian groups, we refer the reader to the classical books [17,31,32,44].

For a Borel set $E \subset \hat{G}$ with $\mu_{\hat{G}}(E) = 0$, we define:

\[ \mathcal{D} = \{ f \in L^2(G) : \hat{f} \in L^\infty(\hat{G}) \text{ and } \text{supp} \hat{f} \text{ is compact in } \hat{G} \setminus E \}. \tag{2.3} \]

It is not difficult to show that $\mathcal{D}$ is dense in $L^2(G)$ exactly when $\mu_{\hat{G}}(E) = 0$. We will frequently prove our results on $\mathcal{D}$ and extend by a density argument. The role of the set $E$ is to allow for “blind spots” of transforms – a term coined by Führ [21]. We will let $E$ be an unspecified set satisfying $\mu_{\hat{G}}(E) = 0$; the specific choice of $E$ depends on the application, e.g., in the Gabor and wavelet case [30] one would usually take $E = \emptyset$ and $E = \{0\}$, respectively.

The following result relies on Weil’s formula and will play an important part of the proofs in Section 3.
Lemma 2.2. Let $H$ be a closed subgroup of an LCA group $G$ with Haar measure $\mu_H$. Suppose that $f_1, f_2 \in \mathcal{D}$ and $\varphi, \psi \in L^2(G)$. Then

$$\int_H \langle f_1, T_h \varphi \rangle \langle T_h \psi, f_2 \rangle \, d\mu_H(h) = \int_G \int_{H^\perp} \hat{f}_1(\omega) \overline{\hat{f}_2(\omega \alpha)} \hat{\varphi}(\omega) \hat{\psi}(\omega \alpha) \, d\mu_{H^\perp}(\alpha) \, d\mu_{\hat{G}}(\omega).$$

Proof. Let $h \in H$. An application of the Plancherel theorem together with Weil’s formula yields

$$\langle f_1, T_h \varphi \rangle = \langle \hat{f}_1, \hat{T}_h \hat{\varphi} \rangle = \langle \hat{f}_1, E_{-h} \hat{\varphi} \rangle = \int_{\hat{G}/H^\perp} \hat{f}_1(\omega \gamma) \overline{\hat{\varphi}(\omega \gamma)} \hat{\omega}(h) (\gamma \omega) \, d\mu_{\hat{G}/H^\perp}(\omega) = \int_{\hat{H}} \hat{\omega}(h) \int_{H^\perp} \hat{f}_1(\omega \gamma) \overline{\hat{\varphi}(\omega \gamma)} \, d\mu_{H^\perp}(\gamma) \, d\mu_{\hat{H}}(\omega),$$

where we tacitly used that $\hat{G}/H^\perp \cong \hat{H}$. A similar calculation can be done for $\langle T_h \psi, f_2 \rangle$. To ease notation, we define $\langle \hat{f}, \hat{\varphi} \rangle(\omega, H^\perp) = \int_{H^\perp} \hat{f}(\omega \gamma) \overline{\hat{\varphi}(\omega \gamma)} \, d\mu_{H^\perp}(\gamma)$ for $f \in \mathcal{D}$. Again, by the Plancherel theorem and Weil’s formula we have

$$\int_H \langle f_1, T_h \varphi \rangle \langle T_h \psi, f_2 \rangle \, d\mu_H(h) = \int_{\hat{G}/H^\perp} \left( \int_{H^\perp} \hat{f}_1(\omega \gamma) \overline{\hat{\varphi}(\omega \gamma)} \, d\mu_{H^\perp}(\gamma) \right) \left( \int_{H^\perp} \hat{f}_2(\omega \beta) \overline{\hat{\psi}(\omega \beta)} \, d\mu_{H^\perp}(\beta) \right) \, d\mu_{\hat{G}/H^\perp}(\omega)$$

$$= \left( \left[ \mathcal{F}^{-1} \hat{f}_1, \hat{\varphi}(\cdot, H^\perp), \mathcal{F}^{-1} \hat{f}_2, \hat{\psi}(\cdot, H^\perp) \right]_{L^2(\hat{G})} \right) = \left( \left[ \hat{f}_1, \varphi(\cdot, H^\perp), \hat{f}_2, \psi(\cdot, H^\perp) \right]_{L^2(\hat{H})} \right)$$

$$= \int_{\hat{G}/H^\perp} \left( \int_{H^\perp} \hat{f}_1(\omega \gamma) \overline{\hat{\varphi}(\omega \gamma)} \, d\mu_{H^\perp}(\gamma) \right) \left( \int_{H^\perp} \hat{f}_2(\omega \beta) \overline{\hat{\psi}(\omega \beta)} \, d\mu_{H^\perp}(\beta) \right) \, d\mu_{\hat{G}/H^\perp}(\omega).$$

Here $\mathcal{F}$ denotes the Fourier transform on $H$.



2.2 Definition of generalized translation invariant systems

Let $J \subset \mathbb{Z}$ be a countable index set. For each $j \in J$, let $P_j$ be a countable or an uncountable index set, let $g_{j,p} \in L^2(G)$ for $p \in P_j$, and let $\Gamma_j$ be a closed, co-compact subgroup in $G$. Recall that co-compact subgroups are subgroups of $G$ for which $G/\Gamma_j$ is compact. For a compact abelian group, the group is metrizable if, and only if, the character group is countable \cite{31} (24.15]). Hence, since $G/\Gamma_j$ is compact and metrizable, the group $G/\Gamma_j \cong \Gamma_j^\perp$ is discrete and countable. Unless stated otherwise we equip $\Gamma_j^\perp$ with the counting measure and assume a fixed Haar measure $\mu_G$ on $G$. By Remark 1 this uniquely determines the measures on $\Gamma_j, G/\Gamma_j, \hat{G}$, and $\hat{G}/\Gamma_j^\perp$. 

\begin{rem}
\end{rem}
The generalized translation invariant (GTI) system generated by \( \{ g_{j,p} \}_{p \in P, j \in J} \) with translation along closed, co-compact subgroups \( \{ \Gamma_j \}_{j \in J} \) is the family of functions \( \bigcup_{j \in J} \{ T_{\gamma} g_{j,p} \}_{\gamma \in \Gamma_j, p \in P_j} \). To ease notation, we will suppress the dependence of \( j \) in \( g_{j,p} \) and write the GTI system as \( \bigcup_{j \in J} \{ T_{\gamma} g_{j,p} \}_{\gamma \in \Gamma_j, p \in P_j} \).

If we take \( \Gamma = \Gamma_j \) for each \( j \in J \), we obtain a translation invariant (TI) system in the sense that \( f \in \bigcup_{j \in J} \{ T_{\gamma} g_{j,p} \}_{\gamma \in \Gamma_j, p \in P_j} \) implies \( T_{\gamma} f \in \bigcup_{j \in J} \{ T_{\gamma} g_{j,p} \}_{\gamma \in \Gamma_j, p \in P_j} \) for all \( \gamma \in \Gamma \). However, generalized translation invariant systems are more general than translation invariant systems since we allow for a different subgroup for each set of generators \( \{ g_{j,p} \}_{p \in P_j} \).

When each \( P_j \) is countable and each \( \Gamma_j \) is a uniform lattice, i.e., a discrete, co-compact subgroup, we recover the generalized shift invariant (GSI) systems considered in \([36]\). However, we note that there exist locally compact abelian groups that do not contain any uniform lattices.

As an example we mention the \( p \)-adic integers, the \( LCA \) groups will have only trivial examples of uniform lattices, e.g., the neutral element, but have plenty non-trivial co-compact subgroups, see Example \([10]\) in Section \([4]\).

Finally, as an alternative generalization of uniform lattices, we mention the idea of so-called quasi-lattices, see \([28,29]\). In contrast to closed, co-compact subgroups, quasi-lattices are discrete subsets in \( G \) that are not necessarily groups.

### 2.3 Frame theory

The central concept of this section is that of a continuous frame. The definition is as follows.

**Definition 2.3.** Let \( \mathcal{H} \) be a complex Hilbert space, and let \( (M, \Sigma_M, \mu_M) \) be a measure space, where \( \Sigma_M \) denotes the \( \sigma \)-algebra and \( \mu_M \) the non-negative measure. A family of vectors \( \{ f_k \}_{k \in M} \) is called a continuous frame for \( \mathcal{H} \) with respect to \((M, \Sigma_M, \mu_M)\) if

1. \( k \mapsto f_k \) is weakly measurable, i.e., for all \( f \in \mathcal{H} \), the mapping \( M \to \mathbb{C}, k \mapsto \langle f, f_k \rangle \) is measurable, and
2. there exist constants \( A, B > 0 \) such that

\[
A \| f \|^2 \leq \int_M |\langle f, f_k \rangle|^2 \, d\mu_M(k) \leq B \| f \|^2 \quad \text{for all } f \in \mathcal{H}. \tag{2.4}
\]

The constants \( A \) and \( B \) are called frame bounds.

**Remark 2.** As we will only consider separable Hilbert spaces in this paper, we can replace weak measurability of \( k \mapsto f_k \) with (strong) measurability with respect to the Borel algebra in \( \mathcal{H} \) by Pettis’ theorem.

In cases where it will cause no confusion, we will simply say that \( \{ f_k \}_{k \in M} \) is a frame for \( \mathcal{H} \).

If \( \{ f_k \}_{k \in M} \) is weakly measurable and the upper bound in the above inequality \( \tag{2.4} \) holds, then \( \{ f_k \}_{k \in M} \) is said to be a Bessel family with constant \( B \). A frame \( \{ f_k \}_{k \in M} \) is said to be tight if we can choose \( A = B \); if, furthermore, \( A = B = 1 \), then \( \{ f_k \}_{k \in M} \) is said to be a Parseval frame.

Two Bessel families \( \{ f_k \}_{k \in M} \) and \( \{ g_k \}_{k \in M} \) are said to be dual frames if

\[
\langle f, g \rangle = \int_M \langle f, g_k \rangle \, d\mu_M(k) \quad \text{for all } f, g \in \mathcal{H}. \tag{2.5}
\]

In this case we say that the following assignment

\[
f = \int_M \langle f, g_k \rangle g_k \, d\mu_M(k) \quad \text{for } f \in \mathcal{H}, \tag{2.6}
\]

...
hods in the weak sense. Equation (2.6) is often called a reproducing formula for $f \in \mathcal{H}$. The following argument shows that two such dual frames indeed are frames, and we shall say that the frame $\{f_k\}_{k \in M}$ is dual to $\{g_k\}_{k \in M}$, and vice versa. We need to show that both Bessel families $\{f_k\}_{k \in M}$ and $\{g_k\}_{k \in M}$ satisfy the lower frame bound. By taking $f = g$ in (2.5) and using the Cauchy-Schwarz inequality, we have

$$
\|f\|^2 = \int_M \langle f, f_k \rangle \langle g_k, f \rangle \, d\mu_M(k) \leq \left( \int_M |\langle f, f_k \rangle|^2 \, d\mu_M(k) \right)^{1/2} \left( \int_M |\langle f, g_k \rangle|^2 \, d\mu_M(k) \right)^{1/2} 
$$

$$
\leq \left( \int_M |\langle f, f_k \rangle|^2 \, d\mu_M(k) \right)^{1/2} \sqrt{B_g} \|f\|.
$$

In the last step we used that $\{g_k\}_{k \in M}$ has an upper frame bound $B_g$. Rearranging the terms in the above inequality gives

$$
\frac{1}{B_g} \|f\|^2 \leq \int_M |\langle f, f_k \rangle|^2 \, d\mu_M(k).
$$

Hence, the Bessel family $\{f_k\}_{k \in M}$ satisfies the lower frame condition and is a frame. A similar argument shows that $\{g_k\}_{k \in M}$ satisfies the lower frame condition. This completes the argument. Moreover, by a polarization argument, it follows that two Bessel families $\{f_k\}_{k \in M}$ and $\{g_k\}_{k \in M}$ are dual frames if, and only if,

$$
\langle f, f \rangle = \int_M \langle f, f_k \rangle \langle f_k, f \rangle \, d\mu_M(k) \quad \text{for all } f \in \mathcal{H}.
$$

We mention that to a given frame for $\mathcal{H}$ one can always find at least one dual frame. For more information on (continuous) frames, we refer to [1, 2, 8, 18, 22, 34].

To a frame $\{f_k\}_{k \in M}$ for $\mathcal{H}$, we associate the frame transform given by

$$
\mathcal{H} \rightarrow L^2(M, \mu_M), \quad f \mapsto (k \mapsto \langle f, f_k \rangle).
$$

As mentioned in the introduction, this transform is isometric, isomorphic if, and only if, the family $\{f_k\}_{k \in M}$ is a Parseval frame. A similar conclusion holds for a pair of dual frames.

Let $(M_1, \Sigma_1, \mu_1)$ and $(M_2, \Sigma_2, \mu_2)$ be measure spaces. We say that a family $\{f_k\}_{k \in M_1}$ in the Hilbert space $\mathcal{H}$ is unitarily equivalent to a family $\{g_k\}_{k \in M_2}$ in the Hilbert space $\mathcal{K}$ if there is a point isomorphism $\iota: M_1 \rightarrow M_2$, i.e., $\iota$ is a (measurable) bijection such that $\iota(\Sigma_1) = \Sigma_2$ and $\mu_1 \circ \iota^{-1} = \mu_2$, a unitary mapping $U: \mathcal{K} \rightarrow \mathcal{H}$, and measurable mapping $M_1 \rightarrow \mathbb{C}, k \mapsto c_k$ with $|c_k| = 1$ such that $f_k = c_k U g_{\iota(k)}$ for all $k \in M_1$. This notion of unitarily equivalence generalizes a similar concept from [4]. Unitarily equivalence is important to us since it preserves many of the properties we are interested in, e.g., the frame property, including the frame bounds. The following lemma tells us that “pairwise” unitarily equivalence preserves the property of being dual frames.

**Lemma 2.4.** Let $\{f_k\}_{k \in M_1}$ and $\{\tilde{f}_k\}_{k \in M_1}$ be families in $\mathcal{H}$, and let $\{g_k\}_{k \in M_2}$ and $\{\tilde{g}_k\}_{k \in M_2}$ be families in $\mathcal{K}$. Suppose that

$$
f_k = c_k U g_{\iota(k)} \quad \text{and} \quad \tilde{f}_k = c_k U \tilde{g}_{\iota(k)}
$$

for some point isomorphism $\iota: M_1 \rightarrow M_2$, a unitary mapping $U: \mathcal{K} \rightarrow \mathcal{H}$, and a measurable mapping $M_1 \rightarrow \mathbb{C}, k \mapsto c_k$ with $|c_k| = 1$ for $k \in M_1$. Then $\{f_k\}_{k \in M_1}$ and $\{\tilde{f}_k\}_{k \in M_1}$ are dual frames with respect to $(M_1, \Sigma_1, \mu_1)$ if, and only if, $\{g_k\}_{k \in M_2}$ and $\{\tilde{g}_k\}_{k \in M_2}$ are dual frames with respect to $(M_2, \Sigma_2, \mu_2)$. 


Proof. Assume that \( \{f_k\}_{k \in M_1} \) and \( \{\tilde{f}_k\}_{k \in M_1} \) are a pair of dual frames. Since the composition of measurable functions is again measurable, then by our assumptions it follows that \( \{g_k\}_{k \in M_2} \) and \( \{\tilde{g}_k\}_{k \in M_2} \) are weakly measurable. They are obviously Bessel families. For \( f \in \mathcal{K} \) and \( g \in H \) we compute:

\[
(f, U^* g) = (U f, g) = \int_{M_1} \langle U f, \tilde{f}_k \rangle \langle f_k, g \rangle \, d\mu_1(k) = \int_{M_1} \langle U f, c_k U \tilde{g}_i(k) \rangle \langle c_k U g_i(k), g \rangle \, d\mu_1(k)
\]

\[
= \int_{M_1} \langle f, \tilde{g}_i(k) \rangle \langle g_i(k), U^* g \rangle \, d\mu_1(k) = \int_{M_2} \langle f, \tilde{g}_k \rangle \langle g_k, U^* g \rangle \, d\mu_2(k),
\]

where the last equality follows from the properties of the point isomorphism. Since \( U^* \) is invertible on all of \( \mathcal{K} \), this implies that \( \{g_k\}_{k \in M_2} \) and \( \{\tilde{g}_k\}_{k \in M_2} \) are dual frames. The opposite implication follows by symmetry. \( \square \)

If \( \mu_M \) is the counting measure and \( \Sigma_M = 2^M \) is a discrete \( \sigma \)-algebra, we say that \( \{f_k\}_{k \in M} \) is a \textit{discrete frame} whenever \( \lbrack 2.4 \rbrack \) is satisfied; for this measure space, any family of vectors is obviously weakly measurable. For discrete frames, equation \( \lbrack 2.6 \rbrack \) holds in the usual strong sense, i.e., with (unconditional) convergence in the \( H \) norm.

Lastly, we combine the notion of continuous frames with that of generalized frames, also known as g-frames. Let \( (M_j, \Sigma_j, \mu_j) \) be a measure space for each \( j \in J \), where \( J \subset \mathbb{Z} \) is a countable index set. We will say that a union \( \bigcup_{j \in J} \{f_{j,k}\}_{k \in M_j} \) is a g-frame for \( H \), or simply a frame, with respect to \( \{L^2(M_j, \mu_j) : j \in J \} \) if

(a) \( k \mapsto f_{j,k}, M_j \to H \) is measurable for each \( j \in J \), and

(b) there exist constants \( A, B > 0 \) such that

\[
A \|f\|^2 \leq \sum_{j \in J} \int_{M_j} |\langle f, f_{j,k} \rangle|^2 \, d\mu_{M_j}(k) \leq B \|f\|^2 \text{ for all } f \in H. \tag{2.7}
\]

The above definition and statements about continuous frames carry over to continuous g-frames; we refer to the original paper by Sun \[47\] for a detailed account of g-frames. Lemma \[2.4\] is also easily transferred to this new setup. We will repeatedly use that it is sufficient to verify the various frame properties on a dense subset of \( H \). The precise statement is as follows.

**Lemma 2.5.** Let \( D \) be a dense subset of \( H \), and let \( (M_j, \mu_j) \) be a measure space for each \( j \in J \).

(i) Suppose that \( \bigcup_{j \in J} \{f_{j,k}\}_{k \in M_j} \) and \( \bigcup_{j \in J} \{g_{j,k}\}_{k \in M_j} \) are Bessel families in \( H \). If, for \( f \in D \),

\[
\langle f, f \rangle = \sum_{j \in J} \int_{M_j} |\langle f, f_{j,k} \rangle|^2 \, d\mu_{M_j}(k), \tag{2.8}
\]

then equation \( \lbrack 2.8 \rbrack \) holds for all \( f \in H \), i.e., \( \bigcup_{j \in J} \{f_{j,k}\}_{k \in M_j} \) and \( \bigcup_{j \in J} \{g_{j,k}\}_{k \in M_j} \) are dual frames.

(ii) Suppose that \( (M_j, \mu_{M_j}) \) are \( \sigma \)-finite and \( \bigcup_{j \in J} \{f_{j,k}\}_{k \in M_j} \) weakly measurable. If, for \( f \in D \),

\[
\langle f, f \rangle = \sum_{j \in J} \int_{M_j} |\langle f, f_{j,k} \rangle|^2 \, d\mu_{M_j}(k), \tag{2.9}
\]

then equation \( \lbrack 2.9 \rbrack \) holds for all \( f \in H \), i.e., \( \bigcup_{j \in J} \{f_{j,k}\}_{k \in M_j} \) is a Parseval frame.
Proof. (i): The first statement follows by a straightforward generalization of the proof of the same result for discrete frames [19, Lemma 7]. The duality of $\bigcup_{j \in J} \{f_{j, k}\}_{k \in M_j}$ and $\bigcup_{j \in J} \{g_{j, k}\}_{k \in M_j}$ follows then by polarization.

(ii): Without loss of generality we can assume that the measure space $(M_j, \mu_{M_j})$ is bounded for each $j \in J$. By use of Lebesgue’s bounded convergence theorem, equation (2.9) for $f \in \mathcal{D}$ implies that $\bigcup_{j \in J} \{f_{j, k}\}_{k \in M_j}$ is a Bessel family on all of $\mathcal{H}$; a similar argument can be found in the proof of [43, Proposition 2.5]. The result now follows from (i). \[ \square \]

3 Generalized translation invariant systems

In this section we will work with generalized translation invariant systems $\bigcup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$ introduced in Section 2.2, in the setting of continuous $g$-frames. In order to do this, we let

In this section we will work with generalized translation invariant systems

\begin{enumerate}
  
  \item[(I)] $\{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$ is a frame for $L^2(G)$. Such functions are sometimes called Carathéodory functions, and

\item[(II)] $\{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$ is jointly measurable on $(M_j, \Sigma_{M_j}) = (P_j \times \Gamma_j, \Sigma_{P_j} \otimes B_{\Gamma_j})$. Thus, the family of functions $\bigcup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$ is automatically weakly measurable. A generalized translation invariant system is therefore a frame for $L^2(G)$ if (2.7) is satisfied with respect to the measure spaces $(M_j, \Sigma_{M_j}, \mu_{M_j})$. Similar conclusions are valid with respect to generalized translation invariant systems being Bessel families, Parseval frames, etc. Let us here just observe that for dual frames $\bigcup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in P_j}$ and $\bigcup_{j \in J} \{T_\gamma h_p\}_{\gamma \in \Gamma_j, p \in P_j}$, we have the reproducing formula

\[ f = \sum_{j \in J} \int_{P_j} \int_{\Gamma_j} \langle f, T_\gamma g_p \rangle T_\gamma h_p \, d\mu_{\Gamma_j}(\gamma) \, d\mu_{P_j}(p) \text{ for } f \in L^2(G), \]

where the measure on $\Gamma_j$ is chosen so that the measure on $\Gamma_j^\perp$ is the counting measure.

Remark 3. In Section 3 we always assume the three standing hypotheses. However, in many special cases these assumptions are automatically satisfied:

\begin{enumerate}
  \item[(a)] When $P_j$ is countable for all $j \in J$, we will equip it with a scaled counting measure $k \mu_{\mu}$, $k > 0$, and the discrete $\sigma$-algebra $2^P_j$. If all $P_j$, $j \in J$, are countable, all three standing hypotheses therefore trivially hold.

  \item[(b)] If $P_j$ is a second countable metric space for all $j \in J$ and if $p \mapsto g_p$ is continuous, then the standing hypotheses (II) and (III) are satisfied. Hence, if $P_j$ is also a subset of $G$ or $\hat{G}$ equipped with their respective Haar measure, then all three standing hypotheses hold.

\end{enumerate}
The main characterization results are stated in Theorem 3.4 and 3.5. These results rely on the following technical assumption.

**Definition 3.1.** We say that two generalized translation invariant systems \( \cup_{j \in J} \{ T_{\gamma} g_p \}_{\gamma \in \Gamma_j, p \in P_j} \) and \( \cup_{j \in J} \{ T_{\gamma} h_p \}_{\gamma \in \Gamma_j, p \in P_j} \) satisfy the **dual \( \alpha \) local integrability condition** (dual \( \alpha \)-LIC) if, for all \( f \in D \),

\[
\sum_{j \in J} \int_{P_j} \sum_{a \in \Gamma_j^1} \int_{G} \left| \hat{f}(\omega) \hat{f}(\omega \alpha) \hat{g}_p(\omega) \hat{h}_p(\omega \alpha) \right| d\mu_{\hat{G}}(\omega) d\mu_{P_j}(p) < \infty. \tag{3.1}
\]

In case \( g_p = h_p \) we refer to (3.1) as the **\( \alpha \) local integrability condition** (\( \alpha \)-LIC) for the generalized translation invariant system \( \cup_{j \in J} \{ T_{\gamma} g_p \}_{\gamma \in \Gamma_j, p \in P_j} \).

The \( \alpha \)-LIC should be compared to the local integrability condition for generalized shift invariant systems introduced in [30] for \( L^2(\mathbb{R}^n) \) and in [36] for \( L^2(G) \). For generalized translation invariant systems \( \cup_{j \in J} \{ T_{\gamma} g_p \}_{\gamma \in \Gamma_j, p \in P_j} \) the **local integrability conditions** (LIC) becomes

\[
\sum_{j \in J} \int_{P_j} \sum_{a \in \Gamma_j^1} \int_{\text{supp} f} \left| \hat{f}(\omega) \hat{g}_p(\omega) \right|^2 d\mu_{\hat{G}}(\omega) d\mu_{P_j}(p) < \infty \quad \text{for all} \quad f \in D. \tag{3.2}
\]

Since the integrands in (3.1) and (3.2) are measurable on \( P_j \times \hat{G} \), we are allowed to reorder sums and integrals in the local integrability conditions.

We will see (Lemma 3.9 and Example 3.1) that the LIC implies the \( \alpha \)-LIC, but not vice versa. Moreover, we mention that dual local integrability conditions have not been considered in the literature before. The following simple observation will often be used.

**Lemma 3.2.** The following assertions are equivalent:

(i) The systems \( \cup_{j \in J} \{ T_{\gamma} g_p \}_{\gamma \in \Gamma_j, p \in P_j} \) and \( \cup_{j \in J} \{ T_{\gamma} h_p \}_{\gamma \in \Gamma_j, p \in P_j} \) satisfy the dual \( \alpha \)-LIC,

(ii) for each compact subset \( K \subseteq \hat{G} \setminus E \)

\[
\sum_{j \in J} \int_{P_j} \sum_{a \in \Gamma_j^1} \int_{K \cap \alpha^{-1} K} \left| \hat{g}_p(\omega) \hat{h}_p(\omega \alpha) \right| d\mu_{\hat{G}}(\omega) d\mu_{P_j}(p) < \infty.
\]

**Proof.** To show that (i) implies (ii), let \( K \) be any compact subset in \( \hat{G} \) and define \( \hat{f} = 1_K \). Then, by assumption,

\[
\sum_{j \in J} \int_{P_j} \sum_{a \in \Gamma_j^1} \int_{G} \left| \hat{f}(\omega) \hat{f}(\omega \alpha) \hat{g}_p(\omega) \hat{h}_p(\omega \alpha) \right| d\mu_{\hat{G}}(\omega) d\mu_{P_j}(p)
\]

\[
= \sum_{j \in J} \int_{P_j} \sum_{a \in \Gamma_j^1} \int_{K \cap \alpha^{-1} K} \left| \hat{g}_p(\omega) \hat{h}_p(\omega \alpha) \right| d\mu_{\hat{G}}(\omega) d\mu_{P_j}(p) < \infty.
\]

To show that (ii) implies (i), take \( f \in D \) and denote supp \( \hat{f} \) by \( K \). Note that \( \hat{f} \in L^\infty(\hat{G}) \). Hence, we find that

\[
\sum_{j \in J} \int_{P_j} \sum_{a \in \Gamma_j^1} \int_{G} \left| \hat{f}(\omega) \hat{f}(\omega \alpha) \hat{g}_p(\omega) \hat{h}_p(\omega \alpha) \right| d\mu_{\hat{G}}(\omega) d\mu_{P_j}(p)
\]
In a similar way, we see that \( \bigcup_{j \in J} \{ T_{\gamma} g_p \}_{\gamma \in \Gamma_j, p \in P_j} \) satisfies the local integrability condition if, and only if, for each compact subset \( K \subseteq \hat{G} \setminus E \)

\[
\sum_{j \in J} \int_{P_j} \int_{\Gamma_j} \left| \hat{g}_p(\omega) \right|^2 \, d\mu_{\hat{G}}(\omega) \, d\mu_{P_j}(p) < \infty.
\]  

(3.3)

Inspired by the definition of the Calderón sum in wavelet theory, we will say that the term \( \sum_{j \in J} \int_{P_j} |\hat{g}_p(\omega)|^2 \, d\mu_{P_j}(p) \) is the Calderón integral. The next result shows that the Calderón integral is bounded if the generalized translation invariant system is a Bessel family. From this it follows that the \( t_\alpha \)-equations (3.6) are well-defined. We remark that Proposition 3.3 generalizes [36, Proposition 3.6] and [30, Proposition 4.1] from the uniform lattice setting where each \( P_j \) is countable to the setting of generalized translation invariant systems.

**Proposition 3.3.** If the generalized translation invariant system \( \bigcup_{j \in J} \{ T_{\gamma} g_p \}_{\gamma \in \Gamma_j, p \in P_j} \) is a Bessel family with bound \( B \), then

\[
\sum_{j \in J} \int_{P_j} \left| \hat{g}_p(\omega) \right|^2 \, d\mu_{P_j}(p) \leq B \quad \text{for a.e. } \omega \in \hat{G}.
\]  

(3.4)

**Proof.** We begin by noting that the Calderón integral in (3.4) is well-defined by our standing hypothesis (\( \text{III} \)). We assume without loss of generality that \( J = \mathbb{Z} \). From the Bessel assumption on \( \bigcup_{j \in J} \{ T_{\gamma} g_p \}_{\gamma \in \Gamma_j, p \in P_j} \), we have

\[
\sum_{|j| \leq M} \int_{P_j} \int_{\Gamma_j} |\langle f, T_{\gamma} g_p \rangle|^2 \, d\mu_{\Gamma_j}(\gamma) \, d\mu_{P_j}(p) \leq B \| f \|^2
\]

for every \( M \in \mathbb{N} \) and all \( f \in L^2(\hat{G}) \). By Lemma 2.2 we then get

\[
\sum_{|j| \leq M} \int_{P_j} \sum_{\alpha \in \Gamma_j} \int_{\hat{G}} \hat{f}(\omega) \hat{f}(\omega \alpha) \hat{g}_p(\omega) \hat{g}_p(\omega \alpha) \, d\mu_{\hat{G}}(\omega) \, d\mu_{P_j}(p) \leq B \| f \|^2
\]  

(3.5)

for every \( M \in \mathbb{N} \) and all \( f \in D \). Assume towards a contradiction that there exists a Borel subset \( N \subseteq \hat{G} \) of positive measure \( \mu_{\hat{G}}(N) > 0 \) for which

\[
\sum_{j \in J} \int_{P_j} |\hat{g}_p(\omega)|^2 \, d\mu_{P_j}(p) > B \quad \text{for a.e. } \omega \in N.
\]  

In [36] it is assumed that \( N \) contains an open ball, but this needs not be the case. However, since \( \hat{G} \) is \( \sigma \)-compact, there exists a compact set \( K \) so that \( \mu_{\hat{G}}(K \cap N) > 0 \). Set \( \delta_M := \inf \{ d(\alpha, 1) : \alpha \in \Gamma_j \setminus \{ 1 \}, |j| \leq M \} \). For any discrete subgroup \( \Gamma \) there exists a \( \delta > 0 \) such that \( B(x, \delta) \cap \Gamma = \{ x \} \) for \( x \in \Gamma \), where \( B(x, \delta) \) denotes the open ball of radius \( \delta \) and center \( x \). It follows that \( \delta_M > 0 \) since \( \delta_M \) is the smallest of such radii about \( x = 1 \) from a finite union of discrete subgroups \( \Gamma_j \). Let \( \mathcal{O} \) be an open covering of \( K \) of sets with diameter strictly less than \( \delta_M/2 \).
Since a finite subset of $\mathcal{O}$ covers $K$, there is an open set $B \in \mathcal{O}$ so that $\mu_{\hat{G}}(B \cap K \cap N) > 0$. Define $f \in L^2(G)$ by

$$\hat{f} = 1_{B \cap K \cap N}.$$  

By Remark 4 below, we can assume that $E$ does not intersect the closure of $B \cap K \cap N$. Therefore, $f \in D$ and by our assumption we have

$$\sum_{|j| \leq M} \int_{P_j} \sum_{\alpha \in \Gamma_j} \int_{\hat{G}} \hat{f}(\omega) \overline{\hat{g}_p(\omega) \hat{g}_p(\omega \alpha)} \, d\mu_{\hat{G}}(\omega) \, d\mu_{P_j}(p)$$

$$= \int_{\hat{G}} |\hat{f}(\omega)|^2 \sum_{|j| \leq M} \int_{P_j} |\hat{g}_p(\omega)|^2 \, d\mu_{P_j}(p) \, d\mu_{\hat{G}}(\omega),$$

where the change of the order of integration above is justified by an application of the Fubini-Tonelli theorem together with the Bessel assumption (3.4) and our standing hypotheses (I) and (III). By letting $M$ tend to infinity, we see that

$$\sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j} \int_{\hat{G}} \hat{f}(\omega) \overline{\hat{g}_p(\omega) \hat{g}_p(\omega \alpha)} \, d\mu_{\hat{G}}(\omega) \, d\mu_{P_j}(p) > B \|f\|^2,$$

which contradicts (3.5).

Remark 4. In case $E$ intersects the closure of $A := B \cap K \cap N$ in the proof of Proposition 3.3 one needs to approximate the function $f$ with functions from $D$ as defined in (2.3). As we will use such arguments several times in the remainder of this paper, let us consider how to do such a modification in this specific case. Define $E_A = E \cap A$ and

$$F_n = \{ \omega \in A : \inf \{ d(\omega, a) : a \in E_A \} < \frac{1}{n} \},$$

for each $n \in \mathbb{N}$. Define $\hat{f}_n = 1_{A \setminus F_n} \in D$. Since $F_{n+1} \subset F_n$ and $\mu_{\hat{G}}(F_1) < \infty$, we have

$$\|\hat{f} - \hat{f}_n\| = \mu_{\hat{G}}(F_n) \rightarrow \mu_{\hat{G}}(\cap_{n \in \mathbb{N}} F_n) = \mu_{\hat{G}}(E_A) = 0 \quad \text{as } n \rightarrow \infty,$$

where $\hat{f} = 1_{B \cap K \cap N}$. Finally, we use $\hat{f}_n$ in place of $\hat{f}$ in the final argument of the proof above, and let $n \rightarrow \infty$.

3.1 Characterization results for dual and Parseval frames

We are ready to prove the first of our main results, Theorem 3.4. Under the technical dual $\alpha$-LIC assumption we characterize dual generalized translation invariant frames in terms of $t_\alpha$-equations. We stress that these GTI systems are dual frames with respect to $\{L^2(M_j, \mu_j) : j \in J\}$ defined in the previous section. Recall that we assume a Haar measure on $G$ to be given, and that we equip every $\Gamma_j^\perp \subset \hat{G}$ with the counting measure.

Theorem 3.4. Suppose that $\cup_{j \in J} \{ T_j g_p \}_{\gamma \in \Gamma_j, p \in P_j}$ and $\cup_{j \in J} \{ T_j h_p \}_{\gamma \in \Gamma_j, p \in P_j}$ are Bessel families satisfying the dual $\alpha$-LIC. Then the following statements are equivalent:

(i) $\cup_{j \in J} \{ T_j g_p \}_{\gamma \in \Gamma_j, p \in P_j}$ and $\cup_{j \in J} \{ T_j h_p \}_{\gamma \in \Gamma_j, p \in P_j}$ are dual frames for $L^2(G)$,

(ii) for each $\alpha \in \bigcup_{j \in J} \Gamma_j^\perp$ we have

$$t_\alpha(\omega) := \sum_{j \in J : \alpha \in \Gamma_j^\perp} \int_{P_j} \overline{\hat{g}_p(\omega)} \hat{h}_p(\omega \alpha) \, d\mu_{P_j}(p) = \delta_{\alpha, 1} \quad a.e. \, \omega \in \hat{G}.$$  

(3.6)
Proof. Let us first show that the $t_\alpha$-equations are well-defined. Take $B$ to be a common Bessel bound for the two GTI families. By two applications of the Cauchy-Schwarz inequality and Proposition 3.3 we find that

$$
\sum_{j \in J : \alpha \in \Gamma_j^+} \int_{P_j} |\hat{g}_p(\omega)||\hat{h}_p(\omega\alpha)| \, d\mu_{P_j}(p) 
\leq \sum_{j \in J} \int_{P_j} |\hat{g}_p(\omega)|^2 \, d\mu_{P_j}(p) \leq B,
$$

for a.e. $\omega \in \hat{G}$. This shows that the $t_\alpha$-equations are well-defined and converge absolutely.

For $f \in D$, define the function

$$
w_f : G \to \mathbb{C}, \quad w_f(x) := \sum_{j \in J} \int_{P_j} \langle T_x f, T_x g_p \rangle \, d\mu_{P_j}(\gamma) \, d\mu_{P_j}(p). \tag{3.7}
$$

By Lemma 2.2 and the calculation $\overline{T_x f(\omega)} T_x f(\omega) = \overline{\alpha(x) f(\omega)} f(\omega)$, we have

$$
w_f(x) = \sum_{j \in J} \int_{P_j} \int_{\hat{G}} \sum_{\alpha \in \Gamma_j^+} \alpha(x) \overline{\hat{f}(\omega)} \overline{\hat{g}_p(\omega)} \hat{h}_p(\omega, \alpha) \, d\mu_{G}(\omega) \, d\mu_{P_j}(p).
$$

Let $\varphi_{\alpha,j}(p, \omega)$ denote the innermost summand in the right hand side expression above. By our standing hypothesis $[\text{III}]$, the function $\varphi_{\alpha,j}$ is $(\Sigma P_j \otimes B_G)$-measurable for each $\alpha$. Applying Beppo Levi’s theorem to the dual $\alpha$ local integrability condition yields that the function $\sum_\alpha \varphi_{\alpha,j}$ belongs to $L^1(P_j \times \hat{G})$ for each $j \in J$. An application of Fubini’s theorem now gives:

$$
w_f(x) = \sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^+} \int_{\hat{G}} \mathbf{1}_{\Gamma_j^+}(\alpha) \varphi_{\alpha,j}(p, \omega) \, d\mu_{P_j}(p) \, d\mu_{\hat{G}}(\omega).
$$

Lebesgue’s dominated convergence theorem then yields:

$$
w_f(x) = \sum_{j \in J} \sum_{\alpha \in \cup_{j \in J} \Gamma_j^+} \alpha(x) \int_{\hat{G}} \mathbf{1}_{\Gamma_j^+}(\alpha) \overline{\hat{f}(\omega)} \overline{\hat{g}_p(\omega)} \hat{h}_p(\omega, \alpha) \, d\mu_{P_j}(p) \, d\mu_{\hat{G}}(\omega).
$$

By the dual $\alpha$ local integrability condition the summand belongs to $\ell^1(J \times \cup_{j \in J} \Gamma_j^+)$ and we can therefore interchange the order of summations. Further, by Lebesgue’s bounded convergence theorem, we can interchange the sum over $j \in J$ and the integral over supp $f \subset \hat{G}$. Hence,

$$
w_f(x) = \sum_{\alpha \in \cup_{j \in J} \Gamma_j^+} \alpha(x) \int_{\hat{G}} \overline{\hat{f}(\omega)} \overline{\hat{g}_p(\omega)} \hat{h}_p(\omega, \alpha) \, d\mu_{\hat{G}}(\omega)
$$

Finally, we arrive at:

$$
w_f(x) = \sum_{\alpha \in \cup_{j \in J} \Gamma_j^+} \alpha(x) \hat{\omega}(\alpha), \quad \text{where} \quad \hat{\omega}(\alpha) := \int_{\hat{G}} \overline{\hat{f}(\omega)} \overline{\hat{g}_p(\omega)} t_\alpha(\omega) \, d\mu_{\hat{G}}(\omega). \tag{3.8}
$$
From the previous calculations and the dual $\alpha$-LIC, it follows that the convergence in (3.8) is absolute. By the Weierstrass M-test, we see that $w_f$ is the uniform limit of a generalized Fourier series and thus an almost periodic, continuous function.

We start by showing the implication (ii)$\Rightarrow$(i). Inserting (3.6) into (3.8) for $x = 0$ yields

$$w_f(0) = \sum_{j \in J} \int_{\Gamma_j} \langle f, T_\gamma g_p \rangle \langle T_\gamma h_p, f \rangle \, d\mu_{\gamma_j}(\gamma) \, d\mu_{\gamma_j}(p) = \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^+} \alpha(0) \int_{\hat{G}} \hat{f}(\omega) \hat{f}(\omega \alpha) \delta_{\alpha,1} \, d\mu_{\hat{G}}(\omega) = \langle f, f \rangle,$$

and (i) follows by Lemma 2.5(i).

For the converse implication (i)$\Rightarrow$(ii), we have

$$w_f(x) = \sum_{j \in J} \int_{\Gamma_j} \langle T_\gamma f, T_\gamma g_p \rangle \langle T_\gamma h_p, T_\gamma f \rangle \, d\mu_{\gamma_j}(\gamma) \, d\mu_{\gamma_j}(p) = \|f\|^2$$

for each $f \in D$. Consider now the function $z(x) := w_f(x) - \|f\|^2$. We have shown that $w_f$ is continuous and by construction $z$ is identical to the zero function. Additionally, since $w_f$ equals an absolute convergent, generalized Fourier series, also $z$ can be expressed as an absolute convergent generalized Fourier series $z(x) = \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^+} \alpha(x) \hat{z}(\alpha)$, with

$$\hat{z}(\alpha) = \begin{cases} \int_{\hat{G}} \hat{f}(\omega)^2 t_1(\omega) \, d\mu_{\hat{G}}(\omega) - \|f\|^2 & \text{for } \alpha = 1, \\ \int_{\hat{G}} \hat{f}(\omega) \hat{f}(\omega \alpha) t_\alpha(\omega) \, d\mu_{\hat{G}}(\omega) & \text{for } \alpha \in \bigcup_{j \in J} \Gamma_j^+ \setminus \{1\}. \end{cases}$$

By the uniqueness theorem for generalized Fourier series [13, Theorem 7.12], the function $z(x)$ is identical to zero if, and only if, $\hat{z}(\alpha) = 0$ for all $\alpha \in \bigcup_{j \in J} \Gamma_j^+$. In case $\alpha = 1$ we have $\int_{\hat{G}} \hat{f}(\omega)^2 t_1(\omega) \, d\mu_{\hat{G}}(\omega) = 0$ for $f \in D$. Hence, since $D$ is dense in $L^2(G)$, we conclude that $t_1(\omega) = 1$ for a.e. $\omega \in \hat{G}$. For $\alpha \in \bigcup_{j \in J} \Gamma_j^+ \setminus \{1\}$, we have

$$\int_{\hat{G}} \hat{f}(\omega) \hat{f}(\omega \alpha) t_\alpha(\omega) \, d\mu_{\hat{G}}(\omega) = 0. \quad (3.9)$$

Define the multiplication operator $M_{T_\alpha} : L^2(G) \to L^2(G)$ by $M_{T_\alpha} \hat{f}(\omega) = \hat{t}_\alpha(\omega) \hat{f}(\omega)$. This linear operator is bounded since by Proposition 3.3 $t_\alpha(\omega) \in L^\infty(\hat{G})$. We can now rewrite the left hand side of (3.9) as an inner-product:

$$\langle \hat{f}, M_{T_\alpha^{-1}} \hat{f} \rangle_{L^2(\hat{G})} = 0,$$

where $f \in D$. Since $D$ is dense in the complex Hilbert space $L^2(G)$, this implies that $M_{T_\alpha}^{-1} T_\alpha = 0$. After multiplication with $T_\alpha$ from the right, we have $M_{T_\alpha}^{-1} T_\alpha = 0$ and therefore $t_\alpha = 0$. \hfill $\square$

From Theorem 3.4 we easily obtain the corresponding characterization for tight frames. We state it for Parseval frames only as it is just a matter of scaling.

**Theorem 3.5.** Suppose that the generalized translation invariant system $\cup_{j \in J} \{T_\gamma g_p\}_{\gamma \in \Gamma_j, p \in \pi_j}$ satisfies the $\alpha$ local integrability condition. Then the following assertions are equivalent:
Consider the generalized translation invariant system

\[ \bigcup_{j \in J} \{ T_\gamma g_p \}_{\gamma \in \Gamma_j, p \in P_j} \]

is a Parseval frame for \( L^2(G) \),

(ii) for each \( \alpha \in \bigcup_{j \in J} \Gamma_j^\perp \) we have

\[
t_\alpha := \sum_{j \in J, \alpha \in \Gamma_j^\perp} \int_{P_j} |\hat{g}_p(\omega)| |\hat{g}_p(\omega \alpha)| \, d\mu_{P_j}(p) = \delta_{\alpha,1} \quad \text{a.e. } \omega \in \hat{G}.
\]

**Proof.** We first remark that the integrals in (ii) indeed converge absolutely. This follows from two applications of the Cauchy-Schwarz’ inequality (as in the proof of Theorem 3.4), which gives:

\[
\sum_{j \in J, \alpha \in \Gamma_j^\perp} \int_{P_j} |\hat{g}_p(\omega)| |\hat{g}_p(\omega \alpha)| \, d\mu_{P_j}(p) \leq 1.
\]

In view of Theorem 3.4, we only have to argue that the assumption on the Bessel family can be omitted. If we assume (i), then clearly \( \bigcup_{j \in J} \{ T_\gamma g_p \}_{\gamma \in \Gamma_j, p \in P_j} \) is a Bessel family and (ii) follows from Theorem 3.4.

Suppose that (ii) holds. Formula (3.8) is still valid, where \( w_f \) is defined as in (3.7) with \( h_p = g_p \). Setting \( x = 0 \) in (3.8) yields

\[
\|f\|^2 = \sum_{j \in J} \int_{P_j} \|f_j(p)\|^2 d\mu_{P_j}(p) \quad \text{for all } f \in D.
\]

Finally, we conclude by Lemma 2.5(ii) that \( \bigcup_{j \in J} \{ T_\gamma g_p \}_{\gamma \in \Gamma_j, p \in P_j} \) is a Parseval frame for \( L^2(G) \). \( \square \)

By virtue of Lemma 2.4, we have the following extension of Theorem 3.4 and 3.5.

**Corollary 3.6.** The characterization results in Theorem 3.4 and 3.5 extend to systems that are unitarily equivalent to generalized translation invariant systems.

### 3.2 On sufficient conditions and the local integrability conditions

Let us now turn to sufficient conditions for a generalized translation invariant system to be a Bessel family or a frame. Proposition 3.7 is a generalization of the results in, e.g., [10] and [9], which state the corresponding result for GSI systems in the euclidean space and locally compact abelian groups, respectively. The result is as follows.

**Proposition 3.7.** Consider the generalized translation invariant system \( \bigcup_{j \in J} \{ T_\gamma g_p \}_{\gamma \in \Gamma_j, p \in P_j} \).

(i) If

\[ B := \text{ess sup}_{\omega \in \hat{G}} \sum_{j \in J} \sum_{\alpha \in \Gamma_j^\perp} |\hat{g}_p(\omega)| |\hat{g}_p(\omega \alpha)| \, d\mu_{P_j}(p) < \infty, \quad (3.10) \]

then \( \bigcup_{j \in J} \{ T_\gamma g_p \}_{\gamma \in \Gamma_j, p \in P_j} \) is a Bessel family with bound \( B \).

(ii) Furthermore, if also

\[ A := \text{ess inf}_{\omega \in \hat{G}} \left( \sum_{j \in J} \int_{P_j} |\hat{g}_p(\omega)|^2 \, d\mu_{P_j}(p) - \sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp \setminus \{0\}} |\hat{g}_p(\omega)| |\hat{g}_p(\omega \alpha)| \, d\mu_{P_j}(p) \right) \]

then \( \bigcup_{j \in J} \{ T_\gamma g_p \}_{\gamma \in \Gamma_j, p \in P_j} \) is a frame for \( L^2(G) \) with bounds \( A \) and \( B \).
Proof. With a few adaptations the result follows from the corresponding proofs in \cite{9} and \cite{10}.

We refer to (3.10) as the absolute CC-condition, or for short, CC-condition \cite{7}. Proposition \cite{3.7} is useful in applications as a mean to verify that a given family indeed is Bessel, or even a frame. Moreover, in relation to the characterizing results in Theorem 3.4 and 3.5, the CC-condition (3.10) is sufficient for the \(\alpha\)-LIC to hold. In contrast, we remark that, by Example \cite{1} in Section 3.3, the CC-condition does not imply the LIC.

**Lemma 3.8.** If \(\bigcup_{j \in J} \{T_j \gamma_p\}_{\gamma \in \Gamma_j, p \in P_j}\) and \(\bigcup_{j \in J} \{T_j h_p\}_{\gamma \in \Gamma_j, p \in P_j}\) satisfy

\[
\text{ess sup}_{\omega \in \hat{G}} \sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^+} \left| \hat{g}_p(\omega) \hat{h}_p(\omega) \right| \ d\mu_{\hat{G}}(\omega) \ d\mu_{P_j}(p) < \infty
\]

and

\[
\text{ess sup}_{\omega \in \hat{G}} \sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^+} \left| \hat{g}_p(\omega) \hat{h}_p(\omega) \right| \ d\mu_{\hat{G}}(\omega) \ d\mu_{P_j}(p) < \infty,
\]

then the dual \(\alpha\) local integrability condition is satisfied. Furthermore, if \(\bigcup_{j \in J} \{T_j g_p\}_{\gamma \in \Gamma_j, p \in P_j}\) satisfies the CC-condition (3.10), then the \(\alpha\) local integrability condition is satisfied.

**Proof.** By applications of Cauchy-Schwarz’ inequality, we find

\[
\sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^+} \int_{\hat{G}} |\hat{f}(\omega) \hat{g}_p(\omega) \hat{h}_p(\omega)| \ d\mu_{\hat{G}}(\omega) \ d\mu_{P_j}(p)
\]

\[
\leq \left[ \sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^+} \int_{\hat{G}} \left| \hat{f}(\omega) \right|^2 |\hat{g}_p(\omega) \hat{h}_p(\omega)| \ d\mu_{\hat{G}}(\omega) \ d\mu_{P_j}(p) \right]^{1/2}
\]

\[
\times \left[ \sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^+} \int_{\hat{G}} \left| \hat{f}(\omega) \right|^2 |\hat{g}_p(\omega) \hat{h}_p(\omega)| \ d\mu_{\hat{G}}(\omega) \ d\mu_{P_j}(p) \right]^{1/2}
\]

\[
= \left[ \int_{\hat{G}} \left| \hat{f}(\omega) \right|^2 \sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^+} |\hat{g}_p(\omega) \hat{h}_p(\omega)| \ d\mu_{\hat{G}}(\omega) \ d\mu_{P_j}(p) \right]^{1/2}
\]

\[
\times \left[ \int_{\hat{G}} \left| \hat{f}(\omega) \right|^2 \sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^+} |\hat{g}_p(\omega) \hat{h}_p(\omega)| \ d\mu_{\hat{G}}(\omega) \ d\mu_{P_j}(p) \right]^{1/2} < \infty.
\]

Finally, we show that the LIC implies the (dual) \(\alpha\)-LIC. The precise statement is as follows.

**Lemma 3.9.** If both \(\bigcup_{j \in J} \{T_j \gamma_p\}_{\gamma \in \Gamma_j, p \in P_j}\) and \(\bigcup_{j \in J} \{T_j h_p\}_{\gamma \in \Gamma_j, p \in P_j}\) satisfy the local integrability condition (3.2), then \(\bigcup_{j \in J} \{T_j g_p\}_{\gamma \in \Gamma_j, p \in P_j}\) and \(\bigcup_{j \in J} \{T_j h_p\}_{\gamma \in \Gamma_j, p \in P_j}\) satisfy the dual \(\alpha\) local integrability condition. In particular, if \(\bigcup_{j \in J} \{T_j g_p\}_{\gamma \in \Gamma_j, p \in P_j}\) satisfies the local integrability condition, then it also satisfies the \(\alpha\) local integrability condition.

**Proof.** By use of Cauchy-Schwarz’ inequality and \(2|c| \leq |c|^2 + |d|^2\), we have

\[
\sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^+} \int_{\hat{G}} |\hat{f}(\omega) \hat{g}_p(\omega) \hat{h}_p(\omega)| \ d\mu_{\hat{G}}(\omega) \ d\mu_{P_j}(p)
\]
\[ \leq \sum_{j \in J} \int_{\Gamma_j} \sum_{\alpha \in \Gamma_j^\perp} \left( \int_{\text{supp} f} |\hat{f}(\omega)\hat{h}_p(\omega\alpha)|^2 d\mu_{\hat{G}}(\omega) \right)^{1/2} \left( \int_{\text{supp} f} |\hat{f}(\omega\alpha)\hat{g}_p(\omega)|^2 d\mu_{\hat{G}}(\omega) \right)^{1/2} d\mu_p(p) \]
\[ = \sum_{j \in J} \int_{\Gamma_j} \sum_{\alpha \in \Gamma_j^\perp} \left( \int_{\text{supp} f} |\hat{f}(\omega^{-1})\hat{h}_p(\omega)|^2 d\mu_{\hat{G}}(\omega) \right)^{1/2} \left( \int_{\text{supp} f} |\hat{f}(\omega\alpha)\hat{g}_p(\omega)|^2 d\mu_{\hat{G}}(\omega) \right)^{1/2} d\mu_p(p) \]
\[ \leq \frac{1}{2} \sum_{j \in J} \int_{\Gamma_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{\text{supp} f} |\hat{f}(\omega^{-1})\hat{h}_p(\omega)|^2 d\mu_{\hat{G}}(\omega) d\mu_p(p) \]
\[ + \frac{1}{2} \sum_{j \in J} \int_{\Gamma_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{\text{supp} f} |\hat{f}(\omega\alpha)\hat{g}_p(\omega)|^2 d\mu_{\hat{G}}(\omega) d\mu_p(p) < \infty, \]
and the statements follow. \( \square \)

The relationships between the various conditions considered above are summarized in the diagram below. To simplify the presentation we do not consider dual frames. An arrow means that the assumption at the tail of the arrow implies the assumption at the head. A crossed out arrow means that one can find a counter example for that implication; clearly, implications to the left in the top line are also not true in general.

\[
\begin{array}{c}
\text{CC} \xrightarrow{\text{Bessel}} \text{Calderón integral} < B \\
\text{LIC} \xrightarrow{\alpha\text{-LIC}} (t_\alpha\text{-eqns.} \iff \text{Parseval})
\end{array}
\]

The crossed out arrows are shown by Example 1 and Example 2 in the next section.

### 3.3 Two examples on the role of the local integrability conditions

In this section we consider two key examples. Both examples take place in \( \ell^2(\mathbb{Z}) \); however, they can be extended to \( L^2(\mathbb{R}) \), see [3]. The first example, Example 1, shows that for a GTI system the \( \alpha \) local integrability condition is strictly weaker than the local integrability condition.

**Example 1.** Let \( G = \mathbb{Z}, N \in \mathbb{N}, N \geq 2 \) and consider the co-compact subgroups \( \Gamma_j = N^j\mathbb{Z}, j \in \mathbb{N} \). Note that \( \hat{G} \) can be identified with the half-open unit interval \( [0,1) \) under addition modulo one. To each \( \Gamma_j \) we associate \( N^j \) functions \( g_{j,p} \), for \( p = 0, 1, \ldots, N^j - 1 \). Each function \( g_{j,p} \) is defined by its Fourier transform

\[ \hat{g}_{j,p} = (N - 1)^{1/2} N^{-j/2} 1_{[p/N^j,(p+1)/N^j)}. \]

The factor \((N - 1)^{1/2}\) is for normalization purposes and does not play a role in the calculations. The annihilator of each \( \Gamma_j \) is given by \( \Gamma_j^\perp = N^{-j}\mathbb{Z} \cap [0,1) \). Note that the number of elements in \( \Gamma_j^\perp \) is \( N^j \). We equip both \( G \) and \( \Gamma_j^\perp \) with the counting measure, this implies that the measure on \( \Gamma_j \) is the counting measure multiplied by \( N^j \). For the generalized translation invariant system

\[ \cup_{j \in \mathbb{N}} \{ T_{\gamma^j} g_{j,p} \}_{\gamma \in \Gamma_j, p=0,1,\ldots,N^j-1} \]

we show the following: (i) the LIC is violated, (ii) the \( \alpha \)-LIC holds, (iii) the system is a Parseval frame for \( \ell^2(\mathbb{Z}) \). It then follows from Theorem 3.5 that the \( t_\alpha \)-equations are satisfied.

Ad (i). In order for the LIC to hold we need

\[ \sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} \sum_{\alpha \in \Gamma_j^\perp} \int_{K \cap (K-\alpha)} |\hat{g}_{j,p}(\omega)|^2 d\omega < \infty \]
for all compact $K \subseteq [0, 1)$, see Lemma 3.2. In particular for $K = \hat{G}$, we find
\[
\sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} \sum_{a \in \Gamma_j^+} \int_0^1 |\hat{g}_{j,p}(\omega)\hat{g}_{j,p}(\omega + \alpha)| \, d\omega < \infty.
\]
Therefore, the local integrability condition is not satisfied.

Ad (ii). By Lemma 3.2, it suffices to show that
\[
\sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} \sum_{a \in \Gamma_j^+} \int_{\hat{G} \setminus (\hat{G} - \alpha)} |\hat{g}_{j,p}(\omega)\hat{g}_{j,p}(\omega + \alpha)| \, d\omega \leq \infty.
\]
Due to the support of $\hat{g}_{j,p}$, we have $|\hat{g}_{j,p}(\omega)\hat{g}_{j,p}(\omega + \alpha)| = 0$ for $\alpha \in \Gamma_j^+ \setminus \{0\}$. We thus find that
\[
\sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} \sum_{a \in \Gamma_j^+} \int_0^1 |\hat{g}_{j,p}(\omega)\hat{g}_{j,p}(\omega + \alpha)| \, d\omega = \sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} \int_0^1 |\hat{g}_{j,p}(\omega)|^2 \, d\omega
\]
\[
= (N-1) \sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} N^{-2j} = (N-1) \sum_{j=1}^{\infty} N^{-j} = 1.
\]

Ad (iii). Note that $\sum_{p=0}^{N^j-1} |\hat{g}_{j,p}(\omega)|^2 = (N-1) N^{-j} \mathbb{1}_{[0,1)}(\omega)$ for $\omega \in [0,1)$ and for all $j \in \mathbb{N}$. Using the frame bound estimates from Proposition 3.7, we have
\[
B = \operatorname{ess sup}_{\omega \in [0,1)} \sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} \sum_{a \in \Gamma_j^+} |\hat{g}_{j,p}(\omega)\hat{g}_{j,p}(\omega + \alpha)|
\]
\[
= \operatorname{ess sup}_{\omega \in [0,1)} \sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} |\hat{g}_{j,p}(\omega)|^2 = \operatorname{ess sup}_{\omega \in [0,1)} (N-1) \sum_{j=1}^{\infty} N^{-j} \mathbb{1}_{[0,1)}(\omega) = 1.
\]
In the same way, for the lower frame bound, we find
\[
A = \operatorname{ess inf}_{\omega \in [0,1)} \left( \sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} |\hat{g}_{j,p}(\omega)|^2 - \sum_{j=1}^{\infty} \sum_{p=0}^{N^j-1} \sum_{a \in \Gamma_j^+ \setminus \{0\}} |\hat{g}_{j,p}(\omega)\hat{g}_{j,p}(\omega + \alpha)| \right) = 1.
\]
These calculations also show that $\cup_{j \in \mathbb{N}} \{ \sum_{j \in \mathbb{N} \setminus \{0,1\}} T_jg_{j,p} \}_{j \in \mathbb{N}, p=0,1,\ldots,N^j-1}$ is actually a union over $j \in \mathbb{N}$ of tight frames $\{ T_jg_{j,p} \}_{j \in \mathbb{N}, p=0,1,\ldots,N^j-1}$ each with frame bound $N^{-j}$. Furthermore, we see that the CC-condition is satisfied, even though the LIC fails. Hence, the CC-condition does not imply LIC (however, by Lemma 3.3, it does imply the $\alpha$-LIC).

The following example is inspired by similar constructions in [5] and [36]. It shows two points. Firstly, the $\alpha$ local integrability condition cannot be removed in Theorem 3.5. Secondly, it is possible for a GTI Parseval frame to satisfy the $t_n$-equations even though the $\alpha$ local integrability condition fails. We show these observations in the reversed order.
Example 2. Let \( G = \mathbb{Z} \) and for each \( m \in \mathbb{Z} \) and \( k \in \mathbb{N} \), let \( [m]_k \) denote the residue class of \( m \) modulo \( k \). Then, for \( \tau_j = 2^{j-1} - 1 \), \( j \in \mathbb{N} \),
\[
\mathbb{Z} = \bigcup_{j \in \mathbb{N}} [\tau_j]_{2^j} = [0]_2 \cup [1]_4 \cup [3]_8 \cup [7]_{16} \cup [15]_{32} \ldots ,
\]
where the union is disjoint. Now set \( g_j = N^{-j/2} \mathbb{1}_{\tau_j} \) and \( \Gamma_j = N^j \mathbb{Z} \) for \( N = 2 \). The GTI system \( \bigcup_{j \in \mathbb{N}} \{ T_\gamma g_j \} \) is essentially a reordering of the standard orthonormal basis \( \{ e_k \} \) for \( \ell^2(\mathbb{Z}) \).

The factor \( N^{-j/2} \) in the definition of \( g_j \) is due to the fact that we equip \( \Gamma_j \) with the counting measure. This implies that the measure on \( \Gamma_j \) becomes \( N^j \) times the counting measure. One can now show that this GTI system does not satisfy the \( \alpha \)-LIC. However, the system does indeed satisfy the \( t_\alpha \)-equations. For \( \alpha = 0 \):
\[
\sum_{j=1}^\infty |\hat{g}_j(\omega)|^2 = \sum_{j=1}^\infty 2^{-j} |e^{2\pi i \tau_j \omega}|^2 = \frac{1}{2 - 1} = 1,
\]
and for \( \alpha = k/2^j \) with \( k \) odd,
\[
\sum_{j \in J: \alpha \in \Gamma_j^*} \hat{g}_j(\omega) \bar{\hat{g}}_j(\omega + \alpha) = \sum_{j \in J} 2^{-j} e^{-2\pi i k \frac{1}{2^j} (2^j - 1)} = e^{2\pi i k 2^{-j}} \sum_{j=1}^\infty 2^{-j} e^{-2\pi i 2^{j-1}} = e^{2\pi i k 2^{-j}} \left( -2^{-j} + \sum_{j=1}^\infty 2^{-j} \right) = 0.
\]

If one uses \( N \geq 3, N \in \mathbb{N} \) in place of \( N = 2 \), then the \( \alpha \)-LIC is still not satisfied. However, even though for suitably chosen \( \tau_j \) (the formula is more complicated than for \( N = 2 \), see [3]) \( \bigcup_{j \in \mathbb{N}} \{ T_\gamma g_j \} \) is still essentially a reordering of the standard orthonormal basis, every \( t_\alpha \)-equation is false. The case \( \alpha = 0 \) gives \( t_\alpha = \frac{1}{N-1} \neq 1 \), while the cases \( \alpha \neq 0 \) give \( t_\alpha \neq 0 \). We stress that these examples show the existence of generalized translation invariant Parseval frames for \( \ell^2(\mathbb{Z}) \) which do not satisfy the \( t_\alpha \)-equations.

3.4 Characterization results for special groups

Under special circumstances the local integrability condition will be satisfied automatically. In this section we will see that this is indeed the case for TI systems, i.e., \( \Gamma_j = \Gamma \) for all \( j \in J \), and for GTI systems on compact abelian groups \( G \). For brevity, we will only state the corresponding characterization results for dual frames, but remark here that the results hold equally for Parseval frames, in which case, the Bessel family assumption can be omitted.

Let us begin with a lemma concerning general GTI systems for LCA groups showing that the LIC holds if the annihilators of \( \Gamma_j \) possess a sufficient amount of separation.

Lemma 3.10. If \( \bigcup_{j \in J} \{ T_\gamma g_p \} \) has a uniformly bounded Calderón integral and if there exists a constant \( C > 0 \) such that for all compact \( K \subseteq \hat{G} \)
\[
\sum_{\alpha \in \bigcup_{j \in J} \Gamma_j^*} \mu_G(K \cap \alpha^{-1} K) \leq C,
\]
then \( \bigcup_{j \in J} \{ T_\gamma g_p \} \) satisfies the local integrability condition.
Proof. By assumption there exists a constant $B > 0$ such that $\sum_{j \in J} \int_{P_j} |\hat{g}_p(\omega)|^2 \, d\mu_{P_j}(p) < B$ for a.e. $\omega \in \hat{G}$, and we therefore have

$$
\sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j} \int_{K \cap \alpha^{-1}K} |\hat{g}_p(\omega)|^2 \, d\mu_{\hat{G}}(\omega) \, d\mu_{P_j}(p) = \sum_{\alpha \in \bigcup_{j \in J} \Gamma_j} \int_{K \cap \alpha^{-1}K} \sum_{j \in J} \int_{P_j} |\hat{g}_p(\omega)|^2 \, d\mu_{\hat{G}}(\omega) \, d\mu_{P_j}(p) \leq BC < \infty.
$$

Now, let us consider the case where all subgroups $\Gamma_j$ coincide. In other words, we consider translation invariant systems. Note that this setting includes the continuous wavelet and Gabor transform as well as the shift invariant systems considered in [30,36].

**Theorem 3.11.** Let $\Gamma$ be a co-compact subgroup in $G$. Suppose that $\bigcup_{\gamma \in \Gamma, p \in P_j} \{T_\gamma g_p\}$ and $\bigcup_{j \in J} \{T_\gamma h_p\}$ are Bessel families. Then the following statements are equivalent:

1. $\bigcup_{\gamma \in \Gamma, p \in P_j} \{T_\gamma g_p\}$ and $\bigcup_{j \in J} \{T_\gamma h_p\}$ are dual frames for $L^2(G)$,

2. For each $\alpha \in \Gamma^\perp$ we have

$$
t_\alpha(\omega) := \sum_{j \in J} \int_{P_j} \overline{g}_p(\omega) \hat{h}_p(\omega \alpha) \, d\mu_{P_j}(p) = \delta_{\alpha,1} \quad \text{a.e. } \omega \in \hat{G}.
$$

**Proof.** Since $\Gamma^\perp$ is a discrete subgroup in $\hat{G}$ and since the metric on $\hat{G}$ is translation invariant, there exists a $\delta > 0$ so that the distance between two distinct points from $\Gamma^\perp$ is larger than $\delta$. Thus, for any compact $K \subset \hat{G}$, the set $\Gamma^\perp \cap (K^{-1}K)$ has finite cardinality because, if not, then $\Gamma^\perp \cap (K^{-1}K)$ would contain a sequence (take one without repetitions) with no convergent subsequence which contradicts the compactness of $K$. Since $\{\alpha \in \Gamma^\perp : K \alpha \cap K \neq \emptyset\}$ is a subset of $\Gamma^\perp \cap (K^{-1}K)$, it is also of finite cardinality. From this together with the Bessel assumption and Proposition 3.3 we conclude that the assumptions of Lemma 3.10 are satisfied and hence the LIC holds. By Lemma 3.9 the dual $\alpha$-LIC is satisfied and the result now readily follows from Theorem 3.3.

For TI systems with translation along the entire group $\Gamma = G$ there is only one $t_\alpha$-equation in (3.11) since $\Gamma^\perp = \{1\}$. To be precise:

**Lemma 3.12.** Suppose that $\Gamma = G$. Then assertion (ii) in Theorem 3.11 reduces to

$$
\sum_{j \in J} \int_{P_j} \overline{g}_p(\omega) \hat{h}_p(\omega) \, d\mu_{P_j}(p) = 1 \quad \text{a.e. } \omega \in \hat{G}.
$$

Let us now turn to the familiar setting of [30,36], where $\Gamma$ is a uniform lattice, i.e., a discrete, co-compact subgroup. Then there is a compact fundamental domain $F \subset G$ for $\Gamma$, such that $G = FT$, and moreover for any $x \in G$ we have $x = \varphi_\gamma$, where $\varphi \in F, \gamma \in \Gamma$ are unique. For a uniform lattice we introduce the lattice size $s(\Gamma) := \mu_G(F)$, which is, in fact, independent of the choice of $F$.

**Corollary 3.13.** Let $\Gamma$ be a uniform lattice in $G$. Suppose that the two generalized translation invariant systems $\bigcup_{\gamma \in \Gamma, p \in P_j} \{T_\gamma g_p\}$ and $\bigcup_{j \in J} \{T_\gamma h_p\}$ are Bessel families. Then the following statements are equivalent:
\( \langle f_1, f_2 \rangle = \sum_{j \in J} \int_{P_j} s(\Gamma) \sum_{\gamma \in \Gamma} \langle f_1, T_\gamma g_p \rangle \langle T_\gamma h_p, f_2 \rangle \, d\mu_{P_j}(p), \quad \text{for all } f_1, f_2 \in L^2(G). \) (3.12)

(ii) For each \( \alpha \in \Gamma^+ \) we have \( t_\alpha(\omega) = \delta_{\alpha,1} \) for a.e. \( \omega \in \hat{G} \), where \( t_\alpha \) is defined in (3.11).

Remark 5. In the same way, we can state the characterization results for generalized shift-invariant systems. In this case we have countable many uniform lattices \( \Gamma_j \), so we replace \( s(\Gamma) \) in Corollary 3.13 with \( s(\Gamma_j) \), sum over \( \{ j \in J : \alpha \in \Gamma_j^+ \} \) in (3.12), and add the dual \( \alpha \) local integrability condition as assumption. We obtain a statement equivalent to the main characterization result in \([36]\). In contrast to the result in \([36]\), the lattice size \( s(\Gamma) \) is contained in the reproducing formula rather than in the \( t_\alpha \)-equations.

For compact abelian groups all generalized translation invariant systems satisfy the local integrability condition. The characterization result is as follows.

**Theorem 3.14.** Let \( G \) be a compact abelian group. Suppose that \( \bigcup_{j \in J} \{ T_\gamma g_p \}_{\gamma \in \Gamma_j, p \in P_j} \) and \( \bigcup_{j \in J} \{ T_\gamma h_p \}_{\gamma \in \Gamma_j, p \in P_j} \) are Bessel families. Then the following statements are equivalent:

(i) \( \bigcup_{j \in J} \{ T_\gamma g_p \}_{\gamma \in \Gamma_j, p \in P_j} \) and \( \bigcup_{j \in J} \{ T_\gamma h_p \}_{\gamma \in \Gamma_j, p \in P_j} \) are dual frames for \( L^2(G) \),

(ii) for each \( \alpha \in \bigcup_{j \in J} \Gamma_j^+ \) we have

\[
 t_\alpha(\omega) := \sum_{j \in J : \alpha \in \Gamma_j^+} \int_{P_j} \hat{g}_p(\omega) \hat{h}_p(\omega\alpha) \, d\mu_{P_j}(p) = \delta_{\alpha,1} \quad \text{a.e. } \omega \in \hat{G}.
\]

**Proof.** Because \( G \) is compact, the dual group \( \hat{G} \) is discrete. All compact \( K \subset \hat{G} \) are therefore finite. Let \( \#K \) denote the number of elements in \( K \). From the LIC we then find

\[
 \sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^+} \sum_{\omega \in K \cap \alpha^{-1}K} \hat{g}_p(\omega)^2 \, d\mu_{P_j}(p) \leq \sum_{j \in J} \int_{P_j} \#K \sum_{\omega \in K} \hat{g}_p(\omega)^2 \, d\mu_{P_j}(p) \leq (\#K)^2 \max_{\omega \in K} \sum_{j \in J} \int_{P_j} \hat{g}_p(\omega)^2 \, d\mu_{P_j}. \]

By the Bessel assumption and Proposition 3.3, the Calderón integral is bounded. The far right hand side in the above calculation is therefore finite, and the LIC is satisfied. The result now follows from Theorem 3.4 and Lemma 3.9.

Finally, let us turn to discrete groups \( G \). In this case, the local integrability condition is not automatically satisfied (as we saw in the examples in the previous section), but it has a simple reformulation:

**Lemma 3.15.** Suppose \( G \) is a discrete abelian group. Then the following statements are equivalent:

(i) The system \( \bigcup_{j \in J} \{ T_\gamma g_p \}_{\gamma \in \Gamma_j, p \in P_j} \) satisfies the local integrability condition,

(ii) \( \sum_{j \in J} \int_{P_j} \mu_c(\Gamma_j^+) \| g_p \|_{L^2(G)}^2 \, d\mu_{P_j}(p) < \infty \), where \( \mu_c \) denotes the counting measure.
Proof. Note that if $G$ is discrete, then $\hat{G}$ is compact. Hence the discrete groups $\Gamma_j^\perp$ are also compact and therefore finite. By this observation we can easily show the result. If (i) holds, then

$$\sum_{j \in J} \int_{P_j} \mu_c(\Gamma_j^\perp) \|g_p\|^2_{L^2(G)} d\mu_{P_j}(p) \leq \sum_{j \in J} \int_{\hat{G}} \mu_c(\Gamma_j^\perp) |\hat{g}_p(\omega)|^2 d\mu_{\hat{G}}(\omega) d\mu_{P_j}(p).$$

By (3.3) with $K = \hat{G}$ the right hand side is finite, and (ii) follows. If (ii) holds, then

$$\sum_{j \in J} \int_{P_j} \sum_{\alpha \in \Gamma_j^\perp} \int_{K \cap \alpha^{-1} K} |\hat{g}_p(\omega)|^2 d\mu_{\hat{G}}(\omega) d\mu_{P_j}(p) \leq \sum_{j \in J} \int_{P_j} \mu_c(\Gamma_j^\perp) \int_{\hat{G}} |\hat{g}_p(\omega)|^2 d\mu_{\hat{G}}(\omega) d\mu_{P_j}(p) < \infty.$$

4 Applications and discussions of the characterization results

In this section we study applications of Theorem 3.4 leading to new characterization results. Moreover, we will easily recover known results as special cases of our theory. We consider Gabor and wavelet-like systems for general locally compact abelian groups as well as for specific locally compact abelian groups, e.g., $\mathbb{R}^n$, $\mathbb{Z}^n$, $\mathbb{Z}_n$. We also give an example of characterization results for the locally compact abelian group of $p$-adic numbers, where the theory of generalized shift invariant systems is not applicable.

We will focus on verifying the local integrability conditions and on the deriving the characterizing equations, but not on the related question of how to construct generators satisfying these equations. The recent work of Christensen and Goh [9] takes this more constructive approach for generalized shift invariant systems on locally compact abelian groups. Under certain assumptions, they explicitly construct dual GSI frames using variants of $t_\alpha$-equations, which are proved to be sufficient.

4.1 Gabor systems

A Gabor system in $L^2(G)$ with generator $g \in L^2(G)$ is a family of functions of the form

$$\{E_{\gamma}T_{\lambda}g\}_{\gamma \in \Gamma, \lambda \in \Lambda},$$

where $\Gamma \subseteq \hat{G}$ and $\Lambda \subseteq G$.

Note that a Gabor system $\{E_{\gamma}T_{\lambda}g\}_{\gamma \in \Gamma, \lambda \in \Lambda}$ is not a generalized translation invariant system because $E_{\gamma}T_{\lambda}g = T_{\lambda}(\gamma(\lambda)E_{\gamma}g)$ cannot be written as $T_{\gamma}g_{\lambda,p}$ for $j \in J$ and $p \in P_j$ for any $\{g_{j,p}\}$. However, by use of Lemma 2.4 we can establish the following two possibilities to relate Gabor and translation invariant systems.

Firstly, by Lemma 2.4 with $\iota = \text{id}$, $U = \mathcal{F}$ and $c_{\gamma,\lambda} = 1$, we see that the Gabor system $\{E_{\gamma}T_{\lambda}g\}_{\gamma \in \Gamma, \lambda \in \Lambda}$ is a frame if, and only if, the translation invariant system $\{T_{\lambda}E_{\gamma}g\}_{\gamma \in \Gamma, \lambda \in \Lambda}$ is a frame. By this observation all results for translation invariant systems naturally carry over to Gabor systems. In order to apply the theory established in this paper, we need $\Gamma$ to be a closed, co-compact subgroup of $\hat{G}$ and $\Lambda$ to be equipped with a measure $\mu_{\Lambda}$ satisfying the standing hypotheses (I)–(III). This approach together with Theorem 3.4 yield $t_\alpha$-equations in the time domain $G$: for each $\alpha \in \Gamma^\perp$ we have

$$\int_{\Lambda} g(x - \lambda) h(x - \lambda + \alpha) d\mu_{\Lambda}(\lambda) = \delta_{\alpha,0} \quad \text{a.e. } x \in G.$$

Secondly, by Lemma 2.4 with $\iota = \text{id}$, $U = \text{id}$ and $c_{\gamma,\lambda} = \gamma(\lambda)$, we see that the Gabor system $\{E_{\gamma}T_{\lambda}g\}_{\gamma \in \Gamma, \lambda \in \Lambda}$ is a frame if, and only if, the translation invariant system $\{T_{\lambda}E_{\gamma}g\}_{\gamma \in \Gamma, \lambda \in \Lambda}$ is a
frame. This time we need $\Lambda$ to be a closed, co-compact subgroup of $G$ and $\Gamma$ to be equipped with a measure satisfying standing hypotheses (I)--(III). In contrast to the first approach, Theorem 3.11 now yields $t_\alpha$-equations in the frequency domain $\hat{G}$: for each $\beta \in \Lambda^\perp$ we have
\[
\int_{\Gamma} \overline{g(\omega \gamma)} h(\omega \gamma \beta) \, d\mu_T(\gamma) = \delta_{\beta,1} \quad \text{a.e. } \omega \in \hat{G}.
\]

Gabor systems play a major role in time-frequency analysis [27] and it is common to require similar properties on $\Gamma$ and $\Lambda$. In the following theorem we characterize dual Gabor frames, where we combine both of the above approaches and require that $\Lambda$ and $\Gamma$ are closed, co-compact subgroups. If we consider Parseval frames, then the Bessel assumption in Theorem 4.1 can be omitted.

**Theorem 4.1.** Let $\Lambda$ and $\Gamma$ be closed, co-compact subgroups of $G$ and $\hat{G}$ respectively and equip $\Lambda^\perp$ and $\Gamma^\perp$ with the counting measure. Suppose that the two systems \{${E_{\gamma}T_\lambda g}$\}_{\gamma \in \Gamma, \lambda \in \Lambda} and \{${E_{\gamma}T_\lambda h}$\}_{\gamma \in \Gamma, \lambda \in \Lambda} are Bessel families. Then the following statements are equivalent:

(i) \{${E_{\gamma}T_\lambda g}$\}_{\gamma \in \Gamma, \lambda \in \Lambda} and \{${E_{\gamma}T_\lambda h}$\}_{\gamma \in \Gamma, \lambda \in \Lambda} are dual frames for $L^2(G)$,

(ii) for each $\alpha \in \Gamma^\perp$ we have
\[
\int_{\Lambda} g(x - \lambda) h(x - \lambda + \alpha) \, d\mu_\Lambda(\lambda) = \delta_{\alpha,0} \quad \text{a.e. } x \in G,
\]

(iii) for each $\beta \in \Lambda^\perp$ we have
\[
\int_{\Gamma} \overline{g(\omega \gamma)} h(\omega \gamma \beta) \, d\mu_\Gamma(\gamma) = \delta_{\beta,1} \quad \text{a.e. } \omega \in \hat{G}.
\]

**Proof.** By Remark 3 the standing hypotheses are satisfied by the Gabor system. The result now follows from Theorem 3.11 together with Lemma 2.4 and the comments preceding Theorem 4.1. \qed

From Theorem 4.1 we can derive numerous results about Gabor systems. We begin with an example concerning the inversion of the short-time Fourier transform.

**Example 3.** Let $g, h \in L^2(G)$ and consider \{${E_{\gamma}T_\lambda g}$\}_{\gamma \in \hat{G}, \lambda \in G} and \{${E_{\gamma}T_\lambda h}$\}_{\gamma \in \hat{G}, \lambda \in G}. We equip $G$ and $\hat{G}$ with their respective Haar measures $\mu_G$ and $\mu_{\hat{G}}. For $f \in L^2(G)$ we calculate
\[
\langle f, E_{\gamma}T_\lambda g \rangle = \int_G f(x) \overline{g(x - \lambda)} \, d\mu_G(x) = F(f(\cdot)g(\cdot - \lambda))(\gamma).
\]

(4.1)

With equation (4.1) and since $\|f\| = \|Ff\|$, we find
\[
\begin{align*}
\int_{\hat{G}} \int_{\hat{G}} |\langle f, E_{\gamma}T_\lambda g \rangle|^2 \, d\mu_{\hat{G}}(\gamma) \, d\mu_G(\lambda) &= \int_{\hat{G}} \int_{\hat{G}} |F(f(\cdot)g(\cdot - \lambda))(\gamma)|^2 \, d\mu_{\hat{G}}(\gamma) \, d\mu_G(\lambda) \\
&= \int_{\hat{G}} \int_{\hat{G}} |f(x)\overline{g(x - \lambda)}|^2 \, d\mu_G(x) \, d\mu_G(\lambda) \\
&= \int_{\hat{G}} |f(x)|^2 \int_{\hat{G}} |g(x - \lambda)|^2 \, d\mu_G(\lambda) \, d\mu_G(x) = \|f\|^2 \|g\|^2.
\end{align*}
\]
The same calculation holds for the Gabor system generated by \( h \). We conclude that both Gabor systems are Bessel families. By Theorem 4.1 the two Gabor systems \( \{ E_{\gamma} T_{\lambda} g \}_{\gamma \in \hat{G}, \lambda \in G} \) and \( \{ E_{\gamma} T_{\lambda} h \}_{\gamma \in \hat{G}, \lambda \in G} \) are dual frames for \( L^2(G) \) if, and only if, for a.e. \( x \in G \)

\[
\int_{G} \frac{g(x - \lambda)}{h(x - \lambda)} d\mu_G(\lambda) = \int_{G} \frac{g(\lambda)}{h(\lambda)} d\mu_G(\lambda) = \langle g, h \rangle = 1,
\]

that is, \( \langle g, h \rangle = 1 \). This result is the well-known inversion formula for the short-time Fourier transform [26,27].

**Example 4.** Let \( G = \Gamma = \mathbb{R}^n, \Lambda = \mathbb{Z}^n \) and \( g \in L^2(\mathbb{R}^n) \). We equip \( G \) and \( \Gamma \) with the Lebesgue measure and \( \Lambda \) with the counting measure. Then

\[
\langle f_1, f_2 \rangle = \int_{\mathbb{R}^n} \sum_{\lambda \in \mathbb{Z}^n} \langle f_1, E_{\gamma} T_{\lambda} g \rangle \langle E_{\gamma} T_{\lambda} g, f_2 \rangle d\gamma,
\]

for all \( f_1, f_2 \in L^2(\mathbb{R}^n) \)

if, and only if,

\[
\sum_{\lambda \in \mathbb{Z}^n} |g(x - \lambda)|^2 = 1, \text{ a.e. } x \in \mathbb{R}^n.
\]

Equivalently in the frequency domain, for all \( \beta \in \mathbb{Z}^n \)

\[
\int_{\mathbb{R}^n} \frac{g(\omega + \gamma)}{h(\omega + \gamma + \beta)} d\gamma = \delta_{\beta,0} \quad \text{a.e. } \omega \in \mathbb{R}^n.
\]

From the time domain characterization, it is clear that the square-root of any uniform B-splines can be used to construct such functions \( g \). The Gabor system with \( \Lambda = \mathbb{R}^n \) and \( \Gamma = \mathbb{Z}^n \) has similar characterizing equations, see [39, Example 2.1(b)].

**Example 5.** Let \( g, h \in L^2(\mathbb{R}) \) and \( a, b > 0 \) be given. Take \( \Lambda = a\mathbb{Z} \) and \( \Gamma = b\mathbb{Z} \). We equip \( \mathbb{R} \) with the Lebesgue measure and \( \Lambda^\perp \cong \frac{1}{a} \mathbb{Z}, \Gamma^\perp \cong \frac{1}{b} \mathbb{Z} \) with the counting measure. From this follows that the measure on \( \Lambda \) and \( \Gamma \) is the counting measure multiplied with \( a \) and \( b \) respectively. Theorem 4.1 now yields the following characterizing equation for dual Gabor systems in \( L^2(\mathbb{R}) \): If \( \{ E_{\gamma} T_{\lambda} g \}_{\gamma \in \Gamma, \lambda \in \Lambda} \) and \( \{ E_{\gamma} T_{\lambda} h \}_{\gamma \in \Gamma, \lambda \in \Lambda} \) are Bessel sequences, then

\[
f = ab \sum_{\lambda \in a\mathbb{Z}} \sum_{\gamma \in b\mathbb{Z}} \langle f, E_{\gamma} T_{\lambda} g \rangle E_{\gamma} T_{\lambda} h,
\]

for all \( f \in L^2(\mathbb{R}) \)

if, and only if, for all \( \alpha \in \frac{1}{b} \mathbb{Z} \)

\[
\sum_{\lambda \in a\mathbb{Z}} g(x - \lambda) h(x - \lambda + \alpha) = \frac{1}{b} \delta_{\alpha,0} \quad \text{for a.e. } x \in [0, a].
\]

This result is equivalent to the characterization result by Janssen [33]. Higher dimensional versions can be derived similarly; see Ron and Shen [45] for alternative proofs. One can easily deduce characterization results for Gabor systems in \( \ell^2(\mathbb{Z}^d) \) following the approach of the preceding example. We refer to the work of Janssen [16] and Lopez and Han [42] for direct proofs. Finally, we mention the following characterization for finite and discrete Gabor frames.
Example 6. Let \( g, h \in \mathbb{C}^d \) and \( a, b, d, N, M \in \mathbb{N} \) be such that \( aN = bM = d \). Then
\[
f = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} (f, E_{mb}T_{na}g)E_{mb}T_{na}h, \quad \text{for all } f \in \mathbb{C}^d
\]
if, and only if,
\[
\sum_{k=0}^{N-1} g(x - nM - ka)h(x - ka) = \frac{1}{M} \delta_{n,0}, \quad \forall x \in \{0, 1, \ldots, a-1\}, n \in \{0, 1, \ldots, b-1\}.
\]
This result appears first in [19] and has been rediscovered in, e.g., [41].

4.2 Wavelet and shearlet systems

Following [4], we let \( \text{Epick}(G) \) denote the semigroup of continuous group homomorphisms \( a \) of \( G \) onto \( G \) with compact kernel. This semigroup can be viewed as an extension of the group of topological automorphisms on \( G \); we define the extended modular function \( \Delta \) in \( \text{Epick}(G) \) as in [4] Section 6. The isometric dilation operator \( D_a : L^2(G) \to L^2(G) \) is then defined by
\[
D_a f(x) = \Delta(a)^{-1/2} f(a(x)).
\]

Let \( \mathcal{A} \) be a subset of \( \text{Epick}(G) \), let \( \Gamma \) be a co-compact subgroup of \( G \), and let \( \Psi \) be a subset of \( L^2(G) \). The wavelet system generated by \( \Psi \) is:
\[
W(\Psi, \mathcal{A}, \Gamma) := \{ D_aT_{\gamma}\psi : a \in \mathcal{A}, \gamma \in \Gamma, \psi \in \Psi \}.
\]
Depending on the choice of \( \mathcal{A} \) and the structure of \( \text{Epick}(G) \), it might be desirable to extend the wavelet system with translates of “scaling” functions, that is, \( \{ T_{\gamma}\phi : \gamma \in \Gamma, \phi \in \Phi \} \) for some \( \Phi \subset L^2(G) \). We denote this extension to a “non-homogeneous” wavelet system by \( W_h(\Psi, \Phi, \mathcal{A}, \Gamma) \).

If \( \text{Epick}(G) \) only contains trivial group homomorphisms, e.g., as in the case of \( G = \mathbb{Z} \), it is possible to define the dilation operator on the dual group \( \hat{G} \) via the Fourier transform.

The two wavelet systems introduced above offer a very general setup that include most of the usual wavelet-type systems in \( L^2(\mathbb{R}^n) \), e.g., discrete and continuous wavelet and shearlet systems [14], [38] as well as composite wavelet systems.

Example 7. Let us consider the general setup as above, where we make the specific choice \( \Gamma = G \) and \( \Psi = \{ \psi_j \}_{j \in J} \) for some index set \( J \subset \mathbb{Z} \). For \( a \in \mathcal{A} \) and \( \gamma \in \Gamma = G \), we have
\[
D_aT_{\hat{\gamma}}\psi_j(x) = \Delta(a)^{-1/2} \psi_j(a(x) - \gamma)) = T_{\hat{\gamma}}D_a\psi_j(x)
\]
for some \( \hat{\gamma} \in a^{-1}\Gamma \) so that \( a(\hat{\gamma}) = \gamma \). It follows that \( W(\Psi, \mathcal{A}, \Gamma) \) is a (generalized) translation invariant system for \( \Gamma_j = G \) with \( j \in J \) and \( g_{j,p} = g_{j,a} = D_a\psi_j \) for \( (j, p) = (j, a) \in J \times \mathcal{A} \). For simplicity we equip each measure space \( P_j = \mathcal{A} \), \( j \in J \), with the same measure; as usual we require that this measure \( \mu_A \) satisfies our standing hypotheses. Further, we define the adjoint of \( a \) by \( \hat{a}(\omega) = \omega \circ a \) for \( \omega \in \hat{G} \). Using results from [4], it follows that \( \hat{a} \) is an isomorphism from \( \hat{G} \) onto \( (\ker a)^\perp \) and that
\[
\hat{D}_a\hat{f}(\omega) = \begin{cases} \Delta(a)^{1/2} \hat{f}(\hat{a}^{-1}(\omega)) & \omega \in (\ker a)^\perp, \\ 0 & \text{otherwise}. \end{cases}
\]
As translation invariant systems always satisfy the local integrability condition, we immedi-
ately have that $W(\Psi, A, G)$ is a Parseval frame, that is,

$$f = \sum_{j \in J} \int_{A} \int_{G} \langle f, D_{a}T_{j}\psi_{j} \rangle D_{a}T_{j}\psi_{j} \, d\mu_{G}(\gamma) \, d\mu_{A}(a) \quad \text{for all } f \in L^{2}(G),$$

if, and only if, for a.e. $\omega \in \hat{G}$,

$$t_{0} = \sum_{j \in J} \int_{A} |D_{a}\psi_{j}(\omega)|^{2} \, d\mu_{A}(a) = \sum_{j \in J} \int_{\{a \in A : \omega \in (\ker a)^{\perp}\}} \Delta(a) |\hat{\psi}_{j}(a^{-1}(\omega))|^{2} \, d\mu_{A}(a) = 1. \quad (4.2)$$

In particular, it follows that $W(\Psi, A, G)$ cannot be a Parseval frame for $L^{2}(G)$ regardless of the
measure $\mu_{A}$ if $\hat{G} \setminus \cup_{a \in A} (\ker a)^{\perp}$ has non-zero measure.

The Calderón admissibility condition (1.2) is a special case of (4.2). To see this, take $G = \mathbb{R}$ and consider the dilation group $A = \{x \mapsto a^{-1}x : a \in \mathbb{R} \setminus \{0\}\}$ with measure $\mu_{A}$ defined on the borel algebra on $\mathbb{R} \setminus \{0\}$ by $d\mu_{A}(a) = da/a^2$, where $da = d\lambda(a)$ denotes the Lebesgue measure. Higher dimensional versions of Calderón’s admissibility condition are obtained similarly, see also [20,40].

**Example 8.** We consider wavelet systems in $L^{2}(\mathbb{R}^{n})$ with discrete dilations and semi-continuous
translations. Let $A \in \text{GL}(n, \mathbb{R})$ be a matrix whose eigenvalues are strictly larger than one in
modulus, set $A = \{x \mapsto A^{j}x : j \in \mathbb{Z}\}$, and let $\Gamma$ be a co-compact subgroup of $\mathbb{R}^{n}$. The wavelet
system generated by $\Psi = \{\psi_{k}\}_{k=1}^{L} \subset L^{2}(G)$ is given by

$$W(\Psi, A, \Gamma) := \left\{D_{A^{j}}T_{\gamma}\psi_{k} = \det|^{-j/2}\psi_{k}(A^{-j} \cdot - \gamma) : \ell = 1, \ldots, L, j \in \mathbb{Z}, \gamma \in \Gamma, \right\}.$$ 

Any co-compact subgroup of $\mathbb{R}^{n}$ is of the form $\Gamma = P(\mathbb{Z}^{k} \times \mathbb{R}^{n-k})$ for some $k \in \{0, 1, \ldots, n\}$ and $P \in \text{GL}(n, \mathbb{R})$. Since $W(\{\psi\}, A, \Gamma)$ is unitarily equivalent to $W(\{D_{P^{-1}}\psi\}, P^{-1}AP, \mathbb{Z}^{k} \times \mathbb{R}^{n-k})$ we can without loss of generality assume that $P = I_{n}$, i.e., $\Gamma = \mathbb{Z}^{k} \times \mathbb{R}^{n-k}$.

Clearly, $W(\Psi, A, \Gamma)$ is a generalized translation invariant system for $\Gamma_{j} = A^{j} \Gamma$ with $j \in J := \mathbb{Z}$ and $g_{j, \ell} = D_{A^{j}}\psi_{\ell}$, where $P_{j} = \{1, \ldots, L\}$. To get rid of a scaling factor in the representation formula, we will use $\mu_{P_{j}} = \frac{1}{\det A^{j}} \mu_{c}$ as measure on $P_{j} = \{1, \ldots, L\}$, where $\mu_{c}$ denotes the counting measure. The standing assumptions are clearly satisfied. Moreover, the local integrability condition is known to be equivalent to local integrability on $\mathbb{R}^{n} \setminus \{0\}$ of the Calderón sum [3 Proposition 2.7] and can, therefore, be omitted from the characterization results. It follows that two Bessel families $W(\Psi, A, \Gamma)$ and $W(\Phi, A, \Gamma)$ are dual frames if, and only if, with $B = A^{T}$,

$$t_{\alpha}(\omega) = \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}, \alpha \in B^{\ell}(\mathbb{Z}^{k} \times \{0\}^{n-k})} \hat{\psi}_{l}(B^{-j} \omega) \overline{\phi_{l}(B^{-j}(\omega + \alpha))} = \delta_{\alpha,0} \quad \text{for a.e. } \omega \in \mathbb{R}^{n},$$

for all $\alpha \in \mathbb{Z}^{k} \times \{0\}^{n-k}$. For $k = n$ this result was obtained in [11], extending the work of Gripenberg [23] and Wang [48].

**Example 9.** Let us finally consider the cone-adapted shearlet systems. For brevity we restrict
our findings to the non-homogeneous, continuous shearlet transform in dimension two. Let

$$A_{a} = \begin{pmatrix} a & 0 \\ 0 & a^{1/2} \end{pmatrix} \quad \text{and} \quad S_{s} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}$$

and let $\mu_{C}$ be the counting measure on $\mathbb{R}^{2}$. For $a \in (0, \infty)$ and $s \in \mathbb{R}$, the
cone-adapted shearlet transform $S_{s}$ is defined on $L^{2}(\mathbb{R}^{2})$ by

$$\mathcal{S}_{s}f = \int_{\mathbb{R}^{2}} f(\alpha) e^{-2\pi i \langle \alpha, \beta \rangle} \, d\lambda_{s}(\beta) \quad \text{for all } f \in L^{2}(\mathbb{R}^{2}),$$

where $\lambda_{s}$ is the Lebesgue measure on $\mathbb{R}^{2}$ and $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^{2}$.
for $a \neq 0$ and $s \in \mathbb{R}$. For $\psi \in L^2(\mathbb{R}^2)$ define

$$\psi_{ast}(x) := a^{-3/4}\psi(A_a^{-1}S_s^{-1}(x-t)) = T_tD_{S_aA_a}\psi.$$  

The cone-adapted continuous shearlet system $S_h(\phi, \psi, \tilde{\psi})$ is then defined as the collection:

$$S_h(\phi, \psi_1, \psi_2) = \{T_t\phi : t \in \mathbb{R}^2\} \cup \left\{T_tD_{S_aA_a}\psi_1 : a \in (0,1], |s| \leq 1 + a^{1/2}, t \in \mathbb{R}^2\right\} \cup \left\{T_tD_{S_aA_a}\psi_2 : a \in (0,1], |s| \leq 1 + a^{1/2}, t \in \mathbb{R}^2\right\},$$

where $\tilde{S}_a = S_s^T$ and $\tilde{A}_a = \text{diag}(a^{1/2}, a)$. This is a special case of the system $W_h$ introduced above. More importantly, this is a GTI system. To see this claim, take $J = \{0,1\}$ and $\Gamma = \Gamma_j = \mathbb{R}^2$ for $j \in J$. Define $P_0 = \{0\}$ and let $\mu_{P_0}$ be the counting measure on $P_0$. Define

$$P_1 = \{(a,s) \in \mathbb{R}^2 : a \in (0,1], |s| \leq 1 + a^{1/2}\},$$

and let $\mu_{P_1}$ be some measure on $P_1$ so that our standing hypotheses are satisfied. The generators are $g_{0,0} = g_{0,1} = \phi$ and $g_{1,p} = g_{1,(a,s)} = D_{\tilde{S}_aA_a}\psi$ for $p = (a,s) \in P_1$. This proves our claim. By Theorem 3.11 and Lemma 3.12 we immediately have that, if $S_h(\phi, \psi_1, \psi_2)$ and $S_h(\phi, \psi_1, \psi_2)$ are Bessel families, then they are dual frames if, and only if,

$$\int_{P_1} a^{3/2}\psi_1(A_aS_s^T\omega)\psi_2(A_aS_s^T\omega) d\mu_{P_1}(a,s) + \int_{P_1} a^{3/2}\psi_2(A_aS_s^T\omega)\tilde{\psi}_2(A_aS_s^T\omega) d\mu_{P_1}(a,s) = 1 \quad \text{for a.e. } \omega \in \mathbb{R}^2. \quad (4.3)$$

A standard choice for the measure $\mu_{P_1}$ in (4.3) is $d\mu_{P_1}(a,s) = \frac{dads}{p^{2\alpha}}$, which comes from the left-invariant Haar measure on the shearlet group. The above characterization result generalizes results from [24,25,37].

### 4.3 Other examples

**Example 10.** In this example we consider the additive group of $p$-adic integers $\mathbb{I}_p$. To introduce this group, we first consider the $p$-adic numbers $\mathbb{Q}_p$. Here $p$ is some fixed prime-number. The $p$-adic numbers are the completion of the rationals $\mathbb{Q}$ under the $p$-adic norm, defined as follows. Every non-zero rational $x$ can be uniquely factored into $x = \frac{r}{s}p^n$, where $r, s, n \in \mathbb{Z}$ and $p$ does not divide $r$ nor $s$. We then define the $p$-adic norm of $x$ as $||x||_p = p^{-n}$, additionally $||0||_p := 0$. The $p$-adic numbers $\mathbb{Q}_p$ are the completion of $\mathbb{Q}$ under $||\cdot||_p$. It can be shown that all $p$-adic numbers $x$ can be written uniquely as

$$x = \sum_{j=k}^{\infty} x_j p^j, \quad (4.4)$$

where $x_k \in \{0,1,\ldots,p-1\}$ and $k \in \mathbb{Z}$, $x_k \neq 0$. The set of all numbers $x \in \mathbb{Q}_p$ for which $x_j = 0$ for $j < 0$ in (4.4) are the $p$-adic integer $\mathbb{I}_p$. Equivalently, $\mathbb{I}_p = \{x \in \mathbb{Q}_p : ||x||_p \leq 1\}$. In fact, $\mathbb{I}_p$ is a compact, closed and open subgroup of $\mathbb{Q}_p$. Its dual group $\hat{\mathbb{I}}_p$ can be identified with the Prüfer $p$-group $\mathbb{Z}(p^\infty)$, which consists of the union of the $p^n$-roots of unity for all $n \in \mathbb{N}$. That is, $\hat{\mathbb{I}}_p \cong \mathbb{Z}(p^\infty) := \{e^{2\pi im/p^n} : n \in \mathbb{N}, m \in \{0,1,\ldots,p^n - 1\}\} \subset \mathbb{C}$.
We equip \( \mathbb{Z}(p^\infty) \) with the discrete topology and multiplication as group operation. For more information on \( p \)-adic numbers and their dual group we refer to, e.g., [31, §10, §25]. For \( n \in \mathbb{N} \) consider now the subgroups \( \Gamma_n \perp = \{ e^{2\pi i m/p^n} : m = 0, 1, \ldots, p^n - 1 \} \subset \mathbb{Z}(p^\infty) \). Note that all \( \Gamma_n \perp \) are finite groups of order \( p^n \) and generated by \( e^{2\pi i/p^n} \). Moreover, all \( \Gamma_n \perp \) are nested so that

\[
1 \subset \Gamma_1 \perp \subset \Gamma_2 \perp \subset \cdots \subset \mathbb{Z}(p^\infty).
\]

Let now \( \{ g_n \}_{n \in \mathbb{N}} \subset L^2(\mathcal{I}_p) \). By Theorem 3.14 the generalized translation invariant system \( \{ T_\gamma g_n \}_{\gamma \in \Gamma_n, n \in \mathbb{N}} \) is a Parseval frame for \( L^2(\mathbb{Z}_p) \) if, and only if, for each \( \alpha \in \bigcup_{n \in \mathbb{N}} \Gamma_n \perp = \mathbb{Z}(p^\infty) \)

\[
\sum_{k=n^*}^{\infty} \hat{g}_n(\omega) \hat{g}_n(\omega \alpha) = \delta_{\alpha,1} \quad \text{for all } \omega \in \mathbb{Z}(p^\infty),
\]

where \( n^* \in \mathbb{N} \) is the smallest natural number such that \( \alpha \in \Gamma_n \perp \). Because we consider a GTI system with countably many generators, the standing hypotheses are trivially satisfied, see Section 3.

Returning to the \( p \)-adic numbers \( \mathbb{Q}_p \), we note that the only co-compact subgroup of \( \mathbb{Q}_p \) is \( \mathbb{Q}_p \) itself [4]. Therefore any GTI system in \( L^2(\mathbb{Q}_p) \) is, in fact, a translation invariant system of the form \( \bigcup_{j \in J} \{ T_\gamma g_p \}_{\gamma \in \mathbb{Q}_p, p \in J} \). The equations characterizing the dual frame property of such systems are immediate from Theorem 3.11 and Lemma 3.12.

Finally, in the product group \( \mathbb{Q}_p \times \mathbb{I}_p \) there are no discrete, co-compact subgroups [4], and thus no generalized shift invariant systems for \( L^2(\mathbb{Q}_p \times \mathbb{I}_p) \) can be constructed. However, any subgroup of the form \( \mathbb{Q}_p \times \Gamma_n \), where \( \Gamma_n \) is a co-compact subgroup of \( \mathbb{I}_p \) as before, is a co-compact subgroup in \( \mathbb{Q}_p \times \mathbb{I}_p \), indicating that a large number of generalized translation invariant systems do exist in \( L^2(\mathbb{Q}_p \times \mathbb{I}_p) \).

In order to apply Theorem 3.4 to a given GTI system, one needs to verify that the (dual) \( \alpha \)- LIC or the stronger LIC holds. By Theorem 3.11 we get this for free for translation invariant systems. For regular wavelet systems as in Example 3.8 the LIC has an easy characterization [3, Proposition 2.7]. For certain irregular wavelet systems over the real line a detailed analysis of the LIC has been carried out in [35] using Beurling densities. However, for general GTI systems there is no simple interpretation of the local integrability conditions.

Acknowledgments

We thank O. Christensen for giving access to an early version of [9] and K. Ross for providing an example of an LCA group with no uniform lattices, but with proper co-compact subgroups.

References

[1] S. T. Ali, J.-P. Antoine, and J.-P. Gazeau. Continuous frames in Hilbert space. Ann. Physics, 222(1):1–37, 1993.

[2] S. T. Ali, J.-P. Antoine, and J.-P. Gazeau. Coherent states, wavelets and their generalizations. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 2000.

[3] M. Bownik and J. Lemvig. Affine and quasi-affine frames for rational dilations. Trans. Amer. Math. Soc., 363(4):1887–1924, 2011.

[4] M. Bownik and K. Ross. The structure of translation-invariant spaces on locally compact abelian groups, preprint.
[5] M. Bownik and Z. Rzeszotnik. The spectral function of shift-invariant spaces on general lattices. In Wavelets, frames and operator theory, volume 345 of Contemp. Math., pages 49–59. Amer. Math. Soc., Providence, RI, 2004.

[6] A.-P. Calderón. Intermediate spaces and interpolation, the complex method. Studia Math., 24:113–190, 1964.

[7] P. G. Casazza, O. Christensen, and A. J. E. M. Janssen. Weyl-Heisenberg frames, translation invariant systems and the Walnut representation. J. Funct. Anal., 180(1):85–147, 2001.

[8] O. Christensen. Frames and bases. Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston, MA, 2008. An introductory course.

[9] O. Christensen and S. S. Goh. Fourier like frames on locally compact abelian groups, preprint.

[10] O. Christensen and A. Rahimi. Frame properties of wave packet systems in $L^2(\mathbb{R}^d)$. Adv. Comput. Math., 29(2):101–111, 2008.

[11] C. K. Chui, W. Czaja, M. Maggioni, and G. Weiss. Characterization of general tight wavelet frames with matrix dilations and tightness preserving oversampling. J. Fourier Anal. Appl., 8(2):173–200, 2002.

[12] A. Córdoba and C. Fefferman. Wave packets and Fourier integral operators. Comm. Partial Differential Equations, 3(11):979–1005, 1978.

[13] C. Corduneanu. Almost periodic functions. Interscience Publishers [John Wiley & Sons], New York-London-Sydney, 1968. With the collaboration of N. Gheorghiu and V. Barbu, Translated from the Romanian by Gitta Bernstein and Eugene Tomer, Interscience Tracts in Pure and Applied Mathematics, No. 22.

[14] I. Daubechies. Ten Lectures on Wavelets. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 1992.

[15] F. De Mari and E. De Vito. Admissible vectors for mock metaplectic representations. Appl. Comput. Harmon. Anal., 34(2):163–200, 2013.

[16] H. G. Feichtinger and T. Strohmer, editors. Gabor analysis and algorithms. Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston, MA, 1998. Theory and applications.

[17] G. B. Folland. A course in abstract harmonic analysis. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995.

[18] M. Fornasier and H. Rauhut. Continuous frames, function spaces, and the discretization problem. J. Fourier Anal. Appl., 11(3):245–287, 2005.

[19] M. Frazier, G. Garrigós, K. Wang, and G. Weiss. A characterization of functions that generate wavelet and related expansion. In Proceedings of the conference dedicated to Professor Miguel de Guzmán (El Escorial, 1996), volume 3, pages 883–906, 1997.

[20] H. Führ. Generalized Calderón conditions and regular orbit spaces. Colloq. Math., 120(1):103–126, 2010.
[21] H. Führ. Coorbit spaces and wavelet coefficient decay over general dilation groups. *Trans. Amer. Math. Soc.*, posted online October 3, 2014, doi: [10.1090/S0002-9947-2014-06376-9](https://doi.org/10.1090/S0002-9947-2014-06376-9) (to appear in print).

[22] J.-P. Gabardo and D. Han. Frames associated with measurable spaces. *Adv. Comput. Math.*, 18(2-4):127–147, 2003. Frames.

[23] G. Gripenberg. A necessary and sufficient condition for the existence of a father wavelet. *Studia Math.*, 114(3):207–226, 1995.

[24] P. Grohs. Continuous shearlet frames and resolution of the wavefront set. *Monatsh. Math.*, 164(4):393–426, 2011.

[25] P. Grohs. Continuous shearlet tight frames. *J. Fourier Anal. Appl.*, 17(3):506–518, 2011.

[26] K. Gröchenig. Aspects of Gabor analysis on locally compact abelian groups. In *Gabor analysis and algorithms*, Appl. Numer. Harmon. Anal., pages 211–231. Birkhäuser Boston, Boston, MA, 1998.

[27] K. Gröchenig. *Foundations of Time-Frequency Analysis*. Appl. Numer. Harmon. Anal. Birkhäuser, 2001.

[28] K. Gröchenig, G. Kutyniok, and K. Seip. Landau’s necessary density conditions for LCA groups. *J. Funct. Anal.*, 255(7):1831–1850, 2008.

[29] K. Gröchenig and T. Strohmer. Pseudodifferential operators on locally compact abelian groups and Sjöstrand’s symbol class. *J. Reine Angew. Math.*, 613:121–146, 2007.

[30] E. Hernández, D. Labate, and G. Weiss. A unified characterization of reproducing systems generated by a finite family. II. *J. Geom. Anal.*, 12(4):615–662, 2002.

[31] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. I: Structure of topological groups. Integration theory, group representations*. Die Grundlehren der mathematischen Wissenschaften, Bd. 115. Academic Press Inc., Publishers, New York, 1963.

[32] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. II: Structure and analysis for compact groups. Analysis on locally compact Abelian groups*. Die Grundlehren der mathematischen Wissenschaften, Band 152. Springer-Verlag, New York, 1970.

[33] A. J. E. M. Janssen. The duality condition for Weyl-Heisenberg frames. In *Gabor analysis and algorithms*, Appl. Numer. Harmon. Anal., pages 33–84. Birkhäuser Boston, Boston, MA, 1998.

[34] G. Kaiser. *A friendly guide to wavelets*. Birkhäuser Boston Inc., Boston, MA, 1994.

[35] G. Kutyniok. *The local integrability condition for wavelet frames*. J. Geom. Anal. 16 (2006), 155–166.

[36] G. Kutyniok and D. Labate. The theory of reproducing systems on locally compact abelian groups. *Colloq. Math.*, 106(2):197–220, 2006.

[37] G. Kutyniok and D. Labate. Resolution of the wavefront set using continuous shearlets. *Trans. Amer. Math. Soc.*, 361(5):2719–2754, 2009.
[38] G. Kutyniok and D. Labate, editors. Shearlets. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, 2012. Multiscale analysis for multivariate data.

[39] D. Labate, G. Weiss, and E. Wilson. An approach to the study of wave packet systems. In Wavelets, frames and operator theory, volume 345 of Contemp. Math., pages 215–235. Amer. Math. Soc., Providence, RI, 2004.

[40] R. S. Laugesen, N. Weaver, G. L. Weiss, and E. N. Wilson. A characterization of the higher dimensional groups associated with continuous wavelets. J. Geom. Anal., 12(1):89–102, 2002.

[41] S. Li, Y. Liu, and T. Mi. Sparse dual frames and dual Gabor functions of minimal time and frequency supports. J. Fourier Anal. Appl., 19(1):48–76, 2013.

[42] J. Lopez and D. Han. Discrete Gabor frames in $\ell^2(\mathbb{Z}^d)$. Proc. Amer. Math. Soc., 141(11):3839–3851, 2013.

[43] A. Rahimi, A. Najati, and Y. N. Dehghan. Continuous frames in Hilbert spaces. Methods Funct. Anal. Topology, 12(2):170–182, 2006.

[44] H. Reiter and J. D. Stegeman. Classical harmonic analysis and locally compact groups, volume 22 of London Mathematical Society Monographs. New Series. The Clarendon Press Oxford University Press, New York, second edition, 2000.

[45] A. Ron and Z. Shen. Weyl-Heisenberg frames and Riesz bases in $L_2(\mathbb{R}^d)$. Duke Math. J., 89(2):237–282, 1997.

[46] A. Ron and Z. Shen. Generalized shift-invariant systems. Constr. Approx., 22(1):1–45, 2005.

[47] W. Sun. $G$-frames and $g$-Riesz bases. J. Math. Anal. Appl., 322(1):437–452, 2006.

[48] X. Wang. The study of wavelets from the properties of their Fourier transforms. ProQuest LLC, Ann Arbor, MI, 1995. Thesis (Ph.D.)–Washington University in St. Louis.

[49] J. Yao, P. Krolak, and C. Steele. The generalized Gabor transform. IEEE Transactions on image processing, 4(7):978–988, 1995.