The $L_\infty$-algebra of a symplectic manifold

Bas Janssens, Leonid Ryvkin and Cornelia Vizman

November 3, 2021

Abstract

We construct an $L_\infty$-algebra on the truncated canonical homology complex of a symplectic manifold, which naturally projects to the universal central extension of the Lie algebra of Hamiltonian vector fields.

Contents

1 Introduction 2

2 Universal central extensions of Lie algebras of vector fields 2

2.1 Exact divergence free vector fields 2

2.2 Hamiltonian vector fields 3

3 $L_\infty$-algebras 4

3.1 $L_\infty$-algebras and their morphisms 4

3.2 Grounded $L_\infty$-algebras 5

3.3 Motivating example: The $L_\infty$-algebra of a multisymplectic manifold 7

4 The $L_\infty$-algebra of a symplectic manifold 8

4.1 The underlying complex and the bracket structure 8

4.2 A first Ansatz for the higher brackets 9

4.3 The fundamental relations of the $\Alt(m_k)$ 10

4.4 The first higher brackets 11

4.5 The $L_\infty$-algebra of a symplectic manifold 12

5 Poisson manifolds 14

5.1 Central extension of the commutator ideal 14

5.2 An $L_\infty$-algebra 15

---

*Institute of Applied Mathematics, Delft University of Technology, Delft, The Netherlands.
†Mathematical Institute, Georg-August-Universität Göttingen, Göttingen, Germany.
‡Department of Mathematics, West University of Timișoara, Timișoara, Romania.
Acknowledgements L. R. is supported by the PRIME program of the German Academic Exchange Service with funds from the German Federal Ministry of Education and Research and by the CNRS project GraNum. C. V. was partially supported by CNCS UEFISCDI, project number PN-III-P4-ID-PCE-2016-0778. B.J. is supported by the NWO grant 639.032.734 “Cohomology and representation theory of infinite dimensional Lie groups”. The authors would like to thank the Erwin Schrödinger International Institute for Mathematics and Physics (ESI), in particular the program “Higher Structures and Field Theory”, where part of the work was carried out. We would like to thank Kevin van Helden and Camille Laurent-Gengoux for several useful comments.

1 Introduction

This work is a continuation of the articles [JV16, JV18], where the universal central extension of the Lie algebra of Hamiltonian vector fields $\mathfrak{X}_{Ham}(M, \omega)$ of a symplectic manifold $(M, \omega)$ has been investigated. This universal central extension is naturally described as a quotient $\Omega^1(M) / \delta \Omega^2(M)$, where $\delta$ is the Koszul differential of the canonical (Poisson) homology of $(M, \omega)$ [Kos85, Bry88].

Following the creed “Lie on the quotient means $L_{\infty}$ on the complex” this article is devoted to finding an $L_{\infty}$-algebra on the complex $(\Omega^\geq_1(M), \delta)$, which after quotienting returns the universal central extension $\Omega^1(M) / \delta \Omega^2(M)$. The $L_{\infty}$-algebra we find has similar features as the $L_{\infty}$-algebras of multisymplectic ([Rog12]) and multicontact ([Vit15]) manifolds, discovered in the last years.

We start by briefly recalling the relevant results from [JV16, JV18] and reviewing the relevant concepts about $L_{\infty}$-algebras. We then show how the $L_{\infty}$-algebra of multisymplectic observables introduced in [Rog12] yields the $L_{\infty}$-structure behind the universal central extension of the Lie algebra of divergence-free vector fields originally studied by [Rog95]. We then turn to the operators necessary for constructing our $L_{\infty}$-algebra and prove our main result Theorem 4.7. Finally, we show that an $L_{\infty}$-algebra can be constructed for Poisson manifolds, provided that a certain obstruction in $H^{\text{can}}_1(M)$ vanishes. A more detailed and elementary account of this work, also treating presymplectic regular Poisson manifolds, appears in the master thesis [vH20], supervised by the first author.

2 Universal central extensions of Lie algebras of vector fields

In this section we quickly recall the universal central extension of the Lie algebra of Hamiltonian vector fields [JV16] and of the Lie algebra of exact divergence-free vector fields [Rog95].

2.1 Exact divergence free vector fields

Let $M$ be a compact manifold of dimension $n \geq 3$, endowed with a volume form $\mu$. A divergence free vector field is called exact if its contraction with $\mu$ is an exact $(n-1)$-form, i.e. an exact divergence free vector field $X_{\alpha}$ with potential $\alpha \in \Omega^{n-2}(M)$ that satisfies $\iota_{X_{\alpha}} \mu = -d\alpha$. These vector fields form the ideal $\mathfrak{X}_{\text{ex}}(M, \mu)$ (of co-dimension $\dim H^{\text{can}}_{dR}(M)$) in the Lie algebra of divergence free vector
The Hamiltonian vector field \( X_\iota \) identity \( H \) Hodge star operator provides an isomorphism \( \alpha \) so that the projection \( \eta \) isomorphism realized by assigning to a closed 2-form \( \omega \) Theorem 2.1 proof for the following result.

\[
\text{ff}
\]

equipped with the degree decreasing Koszul di

\[
(\text{ff})
\]

In the case of a connected, non-compact symplectic manifold, similar results hold for the perfect \( \text{Lie} \) Lie algebra \( C \) perfect \( \text{Lie} \) Lichnerowicz central extension.

Thus the second Lie algebra cohomology group \( H^2(\text{Ham} \mu) \) is the universal central extension of the Lie algebra of exact divergence free vector fields. Indeed,

\[
\text{[\text{Lie}]74}.
\]

\[
\text{[\text{Rog}95]}.
\]

Thus the second Lie algebra cohomology group \( H^2(\text{Ham} \mu) \) is the universal central extension of the Lie algebra of exact divergence free vector fields.

In the next section, we will see that a similar construction is possible for the Hamiltonian vector fields of a symplectic manifold.

### 2.2 Hamiltonian vector fields

Let \((M, \omega)\) be a compact 2n-dimensional symplectic manifold with induced Poisson bi-vector field \( \pi = \omega^{-1} \). The canonical homology \( H^\omega(M) \) is defined as the homology of the complex \( \Omega(M) \) equipped with the degree decreasing Koszul differential \( \delta = \iota_{\omega} d - d \iota_{\omega} \). By \([\text{Bry}88]\), the symplectic Hodge star operator provides an isomorphism \( H_1^\omega(M) \cong H_1^\text{dr}(M) \).

The Hamiltonian vector field \( X_f \) with Hamiltonian function \( f \in C^\infty(M) \) is uniquely defined by the identity \( \iota_{X_f} \omega = \pm d f \). The Lie algebra of Hamiltonian vector fields \( \text{Ham} \) \( \mu \) is perfect \( \text{Cal}70 \).

The quotient space \( \Omega^1(M)/\delta^2(M) \) can be endowed with a natural Lie bracket

\[
[[\alpha], [\beta]] = [\delta \alpha \cdot d \delta \beta], \quad \alpha, \beta \in \Omega^1(M),
\]

so that the projection \( [\alpha] \mapsto X_\delta \alpha \) becomes a Lie algebra epimorphism.

Theorem 2.2 ([\text{JV}16]). The central extension

\[
H_1^\text{can}(M) \longrightarrow \Omega^1(M)/\delta^2(M) \longrightarrow \text{Ham}(M, \omega)
\]

is the universal central extension of the Lie algebra of Hamiltonian vector fields.

Thus the second Lie algebra cohomology group \( H^2(\text{Ham}(M, \omega)) \) is isomorphic to \( H_2^\text{dr}(M) \approx H_2^\text{dr}(M) \), with the isomorphism realized by assigning to a closed 1-form \( \alpha \) the 2-cocycle \( (X_f, X_g) \mapsto \int_M f \iota_{X_f} \omega^n / n! \) on the Lie algebra of Hamiltonian vector fields \([\text{Rog}95]\).

In the case of a connected, non-compact symplectic manifold, similar results hold for the perfect Lie algebra \( C^\infty(M) \) of smooth functions with Poisson bracket \([f, g] = \omega(X_f, X_g)\).

3
Theorem 2.3 ([JV16]). The central extension
\[ H_1^\text{can}(M) \rightarrow \Omega^1(M)/\delta\Omega^2(M) \xrightarrow{\delta} C_c^\infty(M) \]
is the universal central extension of the Poisson Lie algebra \( C_c^\infty(M) \), and \( H^2(C_c^\infty(M)) \cong H^1_{\text{dR}, c}(M) \).

3 \( L_\infty \)-algebras

In this section we will recall the necessary notions regarding \( L_\infty \) structures, boil them down to the case that interests us and explore the example that inspired the problem treated in this paper.

3.1 \( L_\infty \)-algebras and their morphisms

Definition 3.1 ([LS93]). An \( L_\infty \)-algebra (or Lie-\( \infty \)-algebra) is a graded vector space \( L = \bigoplus_{i \in \mathbb{Z}} L_i \) together with a family of graded skew-symmetric multilinear maps \( \{ l_k : \Lambda^k L \rightarrow L \mid k \in \mathbb{N} \} \) such that \( l_k \) has degree \( 2-k \) and the following identity holds
\[
\sum_{i+j=n+1} (-1)^{(j+1)} \sum_{\sigma \in \text{ush}(i,n-i)} \text{sgn}(\sigma) \epsilon(\sigma; x_1, ..., x_n) \times \\
l_j(l_i(x_{\sigma(1)}, ..., x_{\sigma(i)}), x_{\sigma(i+1)}, ..., x_{\sigma(n)}) = 0
\]
for all \( n \in \mathbb{N} \), where \( \epsilon(\sigma; x_1, ..., x_n) \) denotes the Koszul sign of \( \sigma \) acting on the elements \( x_1, ..., x_n \) and \( \text{ush}(i,n-i) \subset S_n \) denotes the space of all \( (i,n-i) \)-unshuffles.

The notion of \( L_\infty \)-algebras is best understood as “(differential graded) Lie algebras up to homotopy”. This can be best seen by looking at the \( n=3 \) term of the defining equation. Writing \( d \) for \( l_1 \) and \([\cdot, \cdot]\) for \( l_2 \), it has the following form:
\[
\pm[[a, b], c] \pm [[a, c], b] \pm [[b, c], a] \\
\pm l_3(da, b, c) \pm l_3(db, a, c) \pm l_3(dc, a, b) \pm dl_3(a, b, c) = 0
\]
The three leftmost terms correspond to the (graded) Jacobi identity, and the remaining four terms involve the homotopical (or homological) error of the Jacobi identity, with \( d = l_1 \) as differential and \( l_3 \) the term quantifying the error. While the \( n=1 \) identity assures that \( l_1 = d \) squares to zero and the \( n=2 \) identity signifies the compatibility between \( d \) and \( l_2 \), the \( n=4 \) identities sets up a Jacobi-like rule for the “homotopical error term” \( l_3 \) up to some higher error \( l_4 \) and so on.

An \( L_\infty \)-algebra can be equivalently described as a coderivation squaring to zero on the symmetric co-algebra of a graded vector space \([LS93, LM95] \). Retranslating the notion of morphism of such co-algebras, one arrives at the following definition of morphism for \( L_\infty \)-algebras:
Definition 3.2 (LV2[Ryy16]). An \( L_\infty \)-morphism from \((L, l_k)\) to \((L', l'_k)\) is a family \([f_k : \Lambda^k L \to L']\) of graded skew-symmetric maps of degrees \(1 - k\) satisfying the following condition for \(n \geq 1\):

\[
\sum_{i+j=n+1} \sigma \in \text{ush}(n-i) \sum_{\sigma \in \text{ush}(n-i)} (-1)^{i(j+1)} \text{sgn}(\sigma) \epsilon(\sigma; x_1, ..., x_n) \times f_j \left( l_i(x_{\sigma(1)}, ..., x_{\sigma(i)}), x_{\sigma(i+1)}, ..., x_{\sigma(n)} \right) = \sum_{p=1}^{n} \sum_{k_1, ..., k_p} (1) \text{sgn}(\sigma) \epsilon(\sigma; x_1, ..., x_n) \times \beta \left( f_{k_1}(x_{\sigma(1)}, ..., x_{\sigma(k_1)}), f_{k_2}(x_{\sigma(k_1+1)}, ..., x_{\sigma(k_1+k_2)}), ..., f_{k_p}(x_{\sigma(n-k_p+1)}, ..., x_{\sigma(n)}) \right),
\]

where \(\beta\) is given by the following formula:

\[
\beta = \frac{p(p-1)}{2} + \sum_{i=1}^{p} k_i(p - i) + (k_p - 1) \sum_{i=1}^{n-k_p} |x_{\sigma(i)}| + (k_{p-1} - 1) \sum_{i=1}^{n-(k_p+k_{p-1})} |x_{\sigma(i)}| + \cdots + (k_2 - 1) \sum_{i=1}^{n-(k_p+k_{p-1}+\cdots+k_2)} |x_{\sigma(i)}|.
\]

The complicated sign stems from the fact that there is a grading shift in between the anti-symmetric multi-bracket and the symmetric co-algebra perspectives. The indices of the sums are there to ensure that each of the possible combinations of multibrackets \(l_k, l'_k\) and morphism components \(f_k\) are applied to all inequivalent permutations of the \(x_i\). Again, the \(f_1\) should be considered as the principal component of the morphism and the higher \((f_k)_{k \geq 2}\) should be seen as higher homotopical corrections. When these corrections are zero, we call a morphism strict:

Definition 3.3. An \( L_\infty \)-morphism \(\{f_k\}\), where \(f_k = 0\) for \(k \geq 2\) is called strict \( L_\infty \)-morphism.

3.2 Grounded \( L_\infty \)-algebras

In [BFLS98], a construction procedure for \( L_\infty \)-algebras is presented. Starting from a Lie algebra \( F \) and a homological resolution of modules

\[
... \rightarrow X_{-2} \xrightarrow{d} X_{-1} \xrightarrow{d} X_0 \xrightarrow{p} F
\]

they construct an \( L_\infty \)-algebra structure on the resolution \(X_\ast\) with \(l_1 = d\). In advantageous cases, where the Lie bracket on \( F \) can be lifted to a skew-symmetric bracket on \(X_0\) that vanishes on exact terms (i.e. on \(dX_{-1}\)), the \( L_\infty \)-structure they construct has a very specific form: All the higher brackets \(\{l_i\}_{i \geq 2}\) are only non-trivial on \(X_0\). Following [RW15], we will call such \( L_\infty \)-algebras grounded:

Definition 3.4. An \( L_\infty \)-algebra \((L, l_k)\) is called grounded if
1. it is non-positively graded, i.e. \( L = \bigoplus_{i=0}^{\infty} L_{-i} \).

2. \( l_k(x_1, \ldots, x_k) \) is zero whenever \( k > 1 \) and \( \sum_{i=1}^{k} |x_i| = 0 \), where \( |x_i| \) denotes the degree of \( x_i \).

3. \( l_k(x_1, \ldots, x_k) \) is zero whenever \( x_1 = \partial \alpha \) for some \( \alpha \in L_{-1} \).

As for a grounded \( L_{\infty} \)-algebra most terms in the multi-bracket equation (1) vanish, it can be described in a simpler manner than a general \( L_{\infty} \)-algebra. Grounded \( L_{\infty} \)-algebras have been around for a long time. Explicit investigations can be found in [CFRZ16], we refer to [Ryv16] for an elementary account.

**Lemma 3.5.** A grounded \( L_{\infty} \)-algebra can be equivalently described as a cochain complex \( \left( \bigoplus_{i=0}^{\infty} L_{-i}, l_1 \right) \) with a family of linear maps \( \{ l_k : \Lambda^k L_0 \rightarrow L_{2-k} \} \) satisfying for \( k \geq 2 \):

- \( l_k(l_1(\alpha), x_2, \ldots, x_k) = 0 \) for all \( \alpha \in L_{-1} \) and \( x_2, \ldots, x_k \in L_0 \)

- \( \partial l_2 l_k = l_1 l_{k+1} \).

The \( \partial l_2 \) in the above Lemma generalizes the Chevalley-Eilenberg differential in Lie algebra cohomology (and might not square to zero when \( l_2 \) is not an honest Lie bracket):

**Definition 3.6.** Let \( L, K \) be vector spaces and let \( B : \Lambda^2 L \rightarrow L \) be a skew-symmetric map. The operator \( \partial_B : \text{Hom}(\Lambda^p L, K) \rightarrow \text{Hom}(\Lambda^{p+1} L, K) \) is defined as follows

\[
(\partial_B f)(x_1, \ldots, x_{p+1}) = \sum_{1 \leq i < j \leq p+1} (-1)^{i+j} f(B(x_i, x_j), x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{p+1}).
\]

where the notation \( \text{Hom}(\Lambda^p L, K) \) denotes the vector space of \( p \)-multilinear skew-symmetric maps from \( \times^p L \) to \( K \).

To visualize how this reduced definition simplifies the equation, one can look at equation (2) and observe that only the three leftmost and the rightmost term survive, i.e. one gets “the Jacobi identity up to \( d \) of \( l_3 \)”. We end this subsection by remarking that \( L_{\infty} \)-morphisms from grounded \( L_{\infty} \)-algebras to Lie algebras look quite simple:

**Remark 3.7.** Let \( (L, l_i) \) be a grounded \( L_{\infty} \)-algebra and \( (g, [\cdot, \cdot]) \) a Lie algebra. Then a morphism from \( (L, l_i) \) to \( (g, [\cdot, \cdot]) \) is equivalently given by a map \( f_1 : L_0 \rightarrow g \) that satisfies

\[
f_1 l_2(\cdot, \cdot) = [f_1, f_1 \cdot],
\]

hence it is automatically strict.

**Remark 3.8.** The notion of being grounded is not homotopy invariant, however it appears very naturally for \( L_{\infty} \)-algebras that are (higher) central extensions of Lie algebras (by differential graded vector spaces), as discussed in [FRST14] for the multisymplectic \( L_{\infty} \)-algebra.
On the one hand, for a central extension $C \rightarrow L \rightarrow \mathfrak{g}$, where $\mathfrak{g}$ is a Lie algebra and $C$ is a differential graded vector space in non-positive degree, the groundedness of $L$ follows directly from the centrality of $C$. On the other hand, by dividing any grounded $L_\infty$-algebra $L$ by its center $C$ we obtain a Lie algebra $\mathfrak{g}$ and a central extension $C \rightarrow L \rightarrow \mathfrak{g}$.

Given such a central extension, we can encode the bracket structure on $L$ by an $L_\infty$-morphism $f$ from $\mathfrak{g}$ to $L$, where $L$ is the abelian $L_\infty$-algebra on the augmented complex $L[-1] \oplus \mathfrak{g}$, with the last differential given by the projection from $L_0$ to $\mathfrak{g}$. Such an $L_\infty$-morphism from a Lie algebra to a graded vector space has to satisfy the identity

$$\partial_1 f_k = -df_{k+1}$$

To achieve this, we simply set $f_1 = \text{id}_\mathfrak{g}$ and $f_k(\xi_1, \ldots, \xi_k) = (-1)^{k+1}l_k(\alpha_1, \ldots, \alpha_k) \in L_{2-k} = \hat{L}_{1-k}$, where $\alpha_i \in L_0 = \hat{L}_1$ are any elements projecting to $\xi_i \in \mathfrak{g}$. Conversely, any $L_\infty$-morphism $\mathfrak{g} \rightarrow L$ starting with the identity corresponds to a grounded $L_\infty$-structure on $L$.

In the same vein, one can show (cf. [Laz14, Theorem 3.8]) that equivalence classes of central extensions of $\mathfrak{g}$ by $C$ are classified by $H^2_{CE}(\mathfrak{g}, C)$, the cohomology in degree 2 of the Chevalley-Eilenberg complex $C_{CE}(\mathfrak{g}, C) := \wedge \mathfrak{g}^* \otimes C$ with differential $\partial_1 \cdot + d$. A linear splitting $\sigma = (\sigma_1, 0, \ldots)$ with $\sigma_1 : \mathfrak{g} \rightarrow L_0$ gives rise to a 2-cocycle $g = (g_2, g_3, \ldots)$ with components $g_k : \wedge^k \mathfrak{g} \rightarrow C_{2-k}$ given by $g_k = f_k$ for $k > 2$, and

$$g_2(\xi_1, \xi_2) := \sigma_1(\{\xi_1, \xi_2\}) - l_2(\sigma_1(\xi_1), \sigma_1(\xi_2))$$

for $k = 2$. Its class $[g] \in H^2_{CE}(\mathfrak{g}, C)$ does not depend on the choice of section, and $g$ is the boundary of a 1-cochain $h = (h_1, h_2, \ldots)$ if and only if $\sigma_1 + h_1 : \mathfrak{g} \rightarrow L_0$ and $h_k : \wedge^k \mathfrak{g} \rightarrow L_{1-k}$ are the components of a (weak) $L_\infty$-morphism from $\mathfrak{g}$ to $L$.

Although we will stick to the perspective of grounded $L_\infty$-algebras, the problems we encounter could just as easily be recast in terms of (homotopy classes of) $L_\infty$-morphisms from $\mathfrak{g}$ to $\hat{L}$, or (equivalence classes of) cochains in the Chevalley-Eilenberg complex $C_{CE}(\mathfrak{g}, C)$.

### 3.3 Motivating example: The $L_\infty$-algebra of a multisymplectic manifold

In [Rog12], an $L_\infty$-algebra is constructed for any multisymplectic manifold - i.e., a manifold $M$ equipped with a closed and non-degenerate differential form $\eta \in \Omega^n(M)$, where non-degeneracy means that $\iota_\bullet \eta : TM \rightarrow \Lambda^{n-1}T^*M$ is injective. We denote this $L_\infty$-algebra by $L_\infty(M, \eta)$.

In the case of an $n$-dimensional manifold $M$, $\eta = \mu$ is simply a volume form. Each $\alpha \in \Omega^{n-2}(M)$ determines a unique vector field $X_\alpha$ that satisfies $d\alpha = -\iota_{X_\alpha} \mu$, called an exact divergence free vector field. The $L_\infty$-algebra $L_\infty(M, \mu)$ takes the following form:

**Definition 3.9** ([Rog12]). Let $M$ be an $n$-dimensional manifold and $\omega$ a volume form. Then $L_\infty(M, \omega) = (L, l_1)$ is the grounded $L_\infty$-algebra defined as follows:

- the spaces $L_{-i} = \Omega^{n-2-i}(M)$ for $i \in \{0, \ldots, n-2\}$
• the unary bracket $l_1 = d$

• the higher brackets $l_k(a_1, ..., a_k) = (-1)^{\frac{k(k+1)}{2}} \iota_{X_{a_k}} \ldots \iota_{X_{a_2}} \mu$ for $k \in \{2, ..., n\}$, where all $a_i \in L_0 = \Omega^{n-2}(M)$.

Thus the underlying complex is the truncated complex of differential forms on $M$

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \ldots \xrightarrow{d} \Omega^{n-2}(M).$$

The association $\alpha \mapsto X_\alpha$ from $L_0 \subset L_\infty(M, \mu)$ to the Lie algebra $\mathfrak{x}(M)$ actually defines an $L_\infty$-morphism with image the Lie algebra $\mathfrak{x}_\ex(M, \mu)$ of exact divergence free vector fields, leading to a sequence which locally looks very similar to (3), but has some cohomology on the global scale. By construction we have:

**Proposition 3.10.** If $M$ is compact, then $L_0/dL_{-1} = \Omega^{n-2}(M)/d\Omega^{n-3}(M)$ is a central extension of the Lie algebra $\mathfrak{x}_\ex(M, \mu)$. Moreover, there is a natural $L_\infty$-surjection from $L_\infty(M, \mu)$ to the Lie algebra $\Omega^{n-2}(M)/d\Omega^{n-3}(M)$.

**Remark 3.11.** Of course, a compactly supported version of $L_\infty(M, \mu)$ can be defined to treat the case of a non-compact symplectic manifold $M$.

We have thus found the $L_\infty$-algebra behind the universal central extension of the Lie algebra of exact divergence free vector fields (Theorem 2.1). The objective of the next section and of this paper is to find the $L_\infty$-algebra behind the universal central extension of the Lie algebra of Hamiltonian vector fields.

### 4 The $L_\infty$-algebra of a symplectic manifold

The goal of this section is to find the $L_\infty$-algebra behind the Lie algebra $\Omega^1(M)/\delta \Omega^2(M)$, which, when $M$ is compact, is the universal central extension of $\mathfrak{x}_\Ham(M, \omega)$ (Theorem 2.3).

#### 4.1 The underlying complex and the bracket structure

Following the intuition from Subsection 3.3 we will try to construct a grounded $L_\infty$-algebra. As a starting point, we will choose the underlying complex to be $\Omega(M)$ with the Koszul differential $\delta$:

$$L_{-i} = \Omega^{i+1}(M) \quad \text{for } i \in \{0, ..., 2n-1\}$$

$$l_1 = \delta$$

Of course, the bracket $l_2 : \Lambda^2 L_0 \to L_0$ should project to the bracket of $\Omega^1(M)/\delta \Omega^2(M) = L_0/\delta L_{-1}$ (see Remark 3.7). Due to groundedness, the higher brackets $l_k(x_1, ..., x_k)$ are only non-trivial for $x_i \in L_0 = \Omega^1(M)$, moreover - in analogy to the divergence-free case - we will assume them to depend only on $\delta(x_1), ..., \delta(x_k) \in C^\infty(M)$. Hence, we try to construct maps $\tilde{l}_k : \Lambda^k C^\infty(M) \to L_{2-k} = \Omega^{2-k}(M)$ satisfying

$$\partial_{\{1, \ldots, k\}} \tilde{l}_k = \delta \tilde{l}_{k+1}$$

8
where $\partial_{\cdot, \cdot}$ is the Chevalley-Eilenberg differential of the Lie algebra $C^\infty(M)$ as defined in Definition 3.6. The maps $l_k$ defined by

$$l_k(x_1, ..., x_k) = \tilde{l}_k(\delta x_1, ..., \delta x_k)$$

then form the desired $L_\infty$-structure. The discussion can be summarised as follows:

**Lemma 4.1.** Let $\tilde{l}_k : \Lambda^k C^\infty(M) \to L_{2-k}$ for $k \in \{2, ..., 2n+1\}$ satisfy equation (6) and

$$\tilde{l}_2(f, g) = f dg \mod \delta \Omega^2(M).$$

Then $L_\infty$ and $l_1$ as in (5) and $l_k$ as in (7) define a grounded $L_\infty$-algebra with a natural $L_\infty$-surjection to the Lie algebra $\Omega^1(M)/\delta \Omega^2(M)$.

**Remark 4.2.** From the perspective of Remark 3.8, what we are trying to do is to find a cocycle of $C^\infty(M)$ with values in the canonical complex $(\Omega^\bullet(M), \delta)$ that starts with $\text{id}_{C^\infty(M)}$ and $(f, g) \mapsto \frac{1}{2}(f dg - gd f)$.

### 4.2 A first Ansatz for the higher brackets

The naive definition of $\tilde{l}_2$ would be the map $(f, g) \mapsto f dg$. This map however is not skew-symmetric (only skew-symmetric up to elements in $\delta \Omega^2(M)$), so we have to skew-symmetrize it, leading to

$$\tilde{l}_2(f, g) = \frac{1}{2}(f dg - gd f).$$

Trying to generalize this, we could look at the maps $m_k : \bigotimes^k C^\infty(M) \to \Omega^{k-1}(M)$ given by

$$m_k(f_1, ..., f_k) = f_1 df_2 \wedge ... \wedge df_k.$$ 

Their antisymmetrization is

$$\text{Alt}(m_k)(f_1, ..., f_k) = \frac{1}{k} \sum_{i=1}^k (-1)^{i+1} f_i df_1 \wedge ... \wedge \hat{df_i} ... \wedge df_k.$$ 

Here, we denote by $\text{Alt}(T)$ the antisymmetrization map

$$\text{Alt}(T)(f_1, ..., f_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) T(f_{\sigma(1)}, ..., f_{\sigma(k)}).$$ 

At this point we already point out that $d \circ \text{Alt}(m_k) = dm_k$, as $[d, \text{Alt}] = 0$ and $dm_k$ skew-symmetric.
4.3 The fundamental relations of the $\text{Alt}(m_k)$

Our idea is that the $\text{Alt}(m_k)$ are essentially the $\tilde{L}_k$. To access that, we need to compare $\partial_{\{,\}}\text{Alt}(m_k)$ with $\delta\text{Alt}(m_{k+1})$. For this, the following overview of operators on $\Omega^*(M)$ will be useful.

**Lemma 4.3** ([Yan96] with sign conventions from [Bry88]). Let $M$ be a 2n-dimensional manifold and $\omega$ a symplectic form. We denote its Poisson bivector by $\pi$. We write $L = \omega \wedge$ and $\Lambda = \pi$. We denote by $H : \Omega^k(M) \to \Omega^k(M)$ the degree counting operator $\alpha \mapsto (n - \deg(\alpha))\alpha$. For the Koszul differential $\delta = [\Lambda, d] = \Lambda d - d\Lambda$ we have the following relations

\[
\begin{align*}
[\Lambda, L] &= H & [H, \Lambda] &= 2\Lambda & [H, L] &= -2L \\
[L, d] &= 0 & [\Lambda, d] &= \delta & [H, d] &= -d \\
[\Lambda, \delta] &= 0 & [L, \delta] &= d & [H, \delta] &= \delta
\end{align*}
\]

Furthermore $\delta d = -d\delta$ commutes with $H, L, \Lambda$.

**Remark 4.4.** In particular, $\Omega^*(M)$ carries a representation of the super Lie algebra $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathbb{R}^2$, with even part $\mathfrak{sl}(2, \mathbb{R})$ spanned by $\Lambda, L$ and $H$, and with odd part the abelian super Lie algebra $\mathbb{R}^2$ spanned by $\delta$ and $d$. The decomposition of $\Omega^*(M)$ into indecomposable representations was used extensively in [Mat95].

Using these operators and relations we can verify:

**Lemma 4.5.**

\[
\partial_{\{,\}}\text{Alt}(m_k) = (-\delta + \frac{1}{k}d\Lambda)\text{Alt}(m_{k+1})
\]

**Proof.** The Lemma equivalently asserts that

\[
k\partial_{\{,\}}\text{Alt}(m_k)(f_1, \ldots, f_{k+1}) = (-k + 1)\delta + \Lambda d\text{Alt}(m_{k+1}).
\]

Applied to functions $f_1, \ldots, f_{k+1}$ the left hand side takes the form

\[
k\partial_{\{,\}}\text{Alt}(m_k)(f_1, \ldots, f_{k+1}) = \sum_{i<j}(-1)^{i+j}\left\{f_i, f_j\right\}df_1 \wedge \ldots \wedge df_{k+1} + \sum_{r \neq i, j} \pm(-1)^{i+j+1}f_r d\left\{f_i, f_j\right\}df_1 \wedge \ldots \wedge df_{k+1}
\]

where in the ... the $i, j, r$ components are omitted and the sign $\pm$ is negative if $i < r < j$. At the same time the formula for $\delta$ in [Bry88] implies that

\[
-(k + 1)\delta\text{Alt}(m_{k+1})(f_1, \ldots, f_{k+1}) = \sum_{i}(-1)^{i+1}\left\{\sum_{j \neq i} \pm(-1)^{j}\{f_i, f_j\}df_1 \wedge \ldots \wedge df_{k+1} + \sum_{j < r, i \neq r} \pm(-1)^{i+j}f_r df_j \wedge \ldots \wedge df_{k+1}\right\},
\]

10
where the first ± is negative if \( i < j \), and the second one is negative if \( j < r < i \). These two expressions only differ by the indices of one of first summand \((i < j)\) vs. \((i \neq j)\), hence

\[
(k + 1)\delta Alt(m_{k+1})(f_1, ..., f_{k+1}) + k\partial_{\{i:j\}} Alt(m_k)(f_1, ..., f_{k+1})
\]

\[
= \sum_{i<j} (-1)^{i+j+1} \{f_i, f_j\} df_1 \wedge ... \wedge df_{k+1} = \Lambda(df_1 \wedge ... \wedge df_{k+1}) = \Lambda d Alt(m_{k+1}).
\]

\[\Box\]

The above Lemma 4.5 indicates that the \( Alt(m_k) \) almost satisfy the desired Equation (6). In the next subsection we will take a closer look at the defect and how to rectify it.

4.4 The first higher brackets

We start with \( \tilde{l}_2 = Alt(m_2) \). Using the above Lemma, we immediately get

\[
\partial_{\{i:j\}} \tilde{l}_2 = (-\delta + \frac{1}{2} d\Lambda) Alt(m_3)
\]

Now, as we apply the \( d = [L, \delta] = L\delta - \delta L \) to a function \( (\Lambda Alt(m_3)) \), the \( L\delta \) component is zero and we get:

\[
= (-\delta - \frac{1}{2} \delta L\Lambda) Alt(m_3) = \delta \circ (-id - \frac{1}{2} L\Lambda) Alt(m_3).
\]

Hence, we can define

\[
\tilde{l}_3 = (-id - \frac{1}{2} L\Lambda) Alt(m_3).
\]

We go on to compute:

\[
\partial_{\{i:j\}} \tilde{l}_3 = \partial_{\{i:j\}} (-id - \frac{1}{2} L\Lambda) \circ Alt(m_3)
\]

\[
= (-id - \frac{1}{2} L\Lambda)(-\delta + \frac{1}{3} d\Lambda) Alt(m_4)
\]

\[
= (\delta + \frac{1}{2} L\Lambda \delta - \frac{1}{3} d\Lambda - \frac{1}{6} L\Lambda d\Lambda) Alt(m_4)
\]

In the second summand we apply \( L\Lambda \delta = L\delta \Lambda = \delta L\Lambda + d\Lambda \), to get:

\[
= (\delta + \frac{1}{2} L\Lambda + \frac{1}{2} d\Lambda - \frac{1}{6} L\Lambda d\Lambda) Alt(m_4)
\]

\[
= (\delta + \frac{1}{2} L\Lambda + \frac{1}{6}(d - L\Lambda d)\Lambda) Alt(m_4)
\]

\[
= (\delta + \frac{1}{2} L\Lambda + \frac{1}{6}(d - L\Lambda d)\Lambda) Alt(m_4)
\]

\[\Box\]
Now substitute $Ld = L\delta + Ld\Lambda = \delta L + d + Ld\Lambda$. As this term is applied to a one-form, the rightmost summand vanishes and we obtain

$$= (\delta + \frac{1}{2}\delta L\Lambda - \frac{1}{6}\delta L\Lambda)\text{Alt}(m_4)$$
$$= (\delta + \frac{1}{3}\delta L\Lambda)\text{Alt}(m_4)$$
$$= (\delta(id + \frac{1}{3}L\Lambda))\text{Alt}(m_4)$$

For 2-dimensional symplectic manifolds, this means that we are done: We have constructed the desired Lie 2-algebra. For higher-dimensional manifolds, we can set

$$\tilde{l}_4 = (id + \frac{1}{3}L\Lambda)\text{Alt}(m_4)$$

Now, we can do the same procedure with $\partial\{\cdot, \cdot\}\tilde{l}_4$, it works quite analogously, only that there is an additional term, which does not vanish:

$$\partial\{\cdot, \cdot\}\tilde{l}_4 = \delta(-id - \frac{1}{4}L\Lambda - \frac{1}{24}L^2\Lambda^2)\text{Alt}(m_5)$$

Instead of continuing to find the brackets step by step, we will now formulate an Ansatz for the general brackets and find a general solution for symplectic manifolds of any dimension.

### 4.5 The $L_\infty$-algebra of a symplectic manifold

We formulate the following Ansatz for the higher brackets:

$$\tilde{l}_k = (-1)^k \left( \sum_{j=0}^{k-1} a^j_k L^j\Lambda^j \right)\text{Alt}(m_k)$$

(8)

The number of non-trivial $\tilde{l}_k$ and the number of non-zero coefficients in the series are both bounded as the manifold is finite-dimensional and the operator $\Lambda$ nilpotent. More precisely, $k \leq \dim(M) + 1$ and $j \leq \frac{k-1}{2}$.

**Proposition 4.6.** The above $\tilde{l}_k$ satisfy Equation (6) for

$$a^j_k = \frac{(k - j - 1)!}{(k - 1)!j!}.$$ 

(9)

**Proof.** For the given Ansatz, Equation (6) boils down to

$$\delta\left( \sum_{j=0}^{k} a^j_{k+1} L^j\Lambda^j \right) = \left( \sum_{j=0}^{k} a^j_k L^j\Lambda^j \right)\left( \delta - \frac{1}{k}d\Lambda \right)$$
as operators on $\Omega^k(M)$. We will actually rewrite the right hand side to \( \left( \frac{k+1}{k} \delta - \frac{1}{k} \Lambda d \right) \). Upon multiplying the equation by $k$, we arrive at

$$k\delta \left( \sum_{j \geq 0} a^j_{k+1} L^j \Lambda^j \right) = \left( \sum_{j \geq 0} a^j_k L^j \Lambda^j \right) ((k+1)\delta - \Lambda d).$$

We work with the left hand side:

$$k\delta \left( \sum_{j \geq 0} a^j_{k+1} L^j \Lambda^j \right) = \sum_{j \geq 0} ka^j_{k+1} \delta L^j \Lambda^j.$$ 

We can commute $\delta L^j \Lambda^j = L^j \delta \Lambda^j - jL^{j-1}d\Lambda^j = L^j \Lambda^j \delta - jL^{j-1} \Lambda^j d + j^2 L^{j-1} \Lambda^{j-1} \delta$, obtaining

$$\sum_{j \geq 0} ka^j_{k+1} (L^j \Lambda^j \delta - jL^{j-1} \Lambda^j d + j^2 L^{j-1} \Lambda^{j-1} \delta).$$

We now shift the indices in the two rightmost summands to obtain

$$\sum_{j \geq 0} ka^j_{k+1} L^j \Lambda^j \delta - \sum_{j \geq 0} ka^{j+1}_{k+1} (j+1)L^j \Lambda^j d + \sum_{j \geq 0} ka^{j+1}_{k+1} (j+1)^2 L^j \Lambda^j \delta.$$

By identification of the coefficients of $\delta$ and $\Lambda d$ from both sides of (11), equation (6) is fulfilled if the constants $a^j_k$ satisfy the following conditions for $2j \leq k - 1$:

$$(k+1)a^j_k = ka^j_{k+1} + k(j+1)^2 a^{j+1}_{k+1}$$

$$(11)$$

which can be seen to be verified by the proposed $a^j_k$. Note that we have defined more coefficients than necessary for formula (8) in order to get the same recursive formulas for all coefficients. □

The above discussion can be summed up as follows:

**Theorem 4.7.** Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold. Then the following defines a grounded $L_{\infty}$-algebra:

$$L_{i-1} = \Omega^{i+1}(M) \quad \text{for } i \in \{0, \ldots, 2n-1\}$$

$$l_1 = \delta$$

$$l_k = (-1)^k \left( \sum_{j \geq 0} \frac{(k-j-1)!}{(k-1)!j!} L^j \Lambda^j \right) \text{Alt}(m_k) \quad \text{for } k \in \{2, \ldots, 2n + 1\}$$
on the truncated canonical homology complex
\[ \Omega^{2n}(M) \overset{\delta}{\to} \Omega^{2n-1}(M) \ldots \overset{\delta}{\to} \Omega^{1}(M). \]
This \( L_\infty \)-algebra naturally projects to \( \Omega^{1}(M)/\delta \Omega^{2}(M) \) via an \( L_\infty \)-morphism.

**Remark 4.8.** Moreover, the above formulas are unique among all \( l_k \) constructed from the Ansatz \( (8) \). The compatibility with the Lie bracket on \( C^\infty(M) \) implies that \( a^0_2 = 1 \). Then, the Relations \( (9) \) uniquely determine all the other relevant coefficients \( (k \leq \dim(M) + 1, j \leq \frac{k-1}{2}) \) through the inductive formulas:

\[ a^j_{k+1} = \frac{k - j}{k} a^j_k, \quad a^j_k = \frac{1}{j(k - j)} a^{j-1}_k. \]

## 5 Poisson manifolds

The construction of an \( L_\infty \)-algebra for a symplectic manifold \((M, \omega)\) can be generalised to Poisson manifolds \((M, \pi)\), under the condition that

\[ f d\{g, h\} - \{g, h\} d f + \text{cyclic} \in \delta \Omega^{2}(M) \quad (12) \]

for all \( f, g, h \in C^\infty(M) \).

### 5.1 Central extension of the commutator ideal

Let \((\Omega^\bullet(M), \delta)\) be the chain complex with Koszul differential \(\delta = i_{\pi} d - d i_{\pi}\). Then the image of \(\delta: \Omega^{1}(M) \to C^\infty(M)\) is the commutator ideal

\[ I := \{ C^\infty(M), C^\infty(M) \} \]

of the Poisson Lie algebra \( C^\infty(M) \). On \( \Omega^{1}(M) \), the skew-symmetric bracket

\[ [\alpha, \beta] := \frac{1}{2}(\delta \alpha \cdot d \beta - \delta \beta \cdot d \alpha) \quad (13) \]

lifts the Poisson bracket on \( I \subseteq C^\infty(M) \) along the surjective map \( \delta: \Omega^{1}(M) \to I \).

**Proposition 5.1.** Suppose that \((M, \pi)\) satisfies \((12)\). Then the bracket \((13)\) induces a Lie bracket on the quotient space \( \Omega^{1}(M)/\delta \Omega^{2}(M) \), and \(\delta: \Omega^{1}(M)/\delta \Omega^{2}(M) \to I\) is a central extension of \(I\) by \( H^1_{\text{can}}(M) \).

**Proof.** Since \([\text{Ker}(\delta), \Omega^{1}(M)] = \{0\}\), the canonical homology

\[ H^1_{\text{can}}(M) = \text{Ker}(\delta)/\delta \Omega^{2}(M) \subseteq \Omega^{1}(M)/\delta \Omega^{2}(M) \]

is central. As the bracket is manifestly skew-symmetric, it remains to show that \((13)\) satisfies the Jacobi identity up to \(\delta \Omega^{2}(M)\). Let \(\alpha, \beta, \gamma \in \Omega^{1}(M)\) with \(f = \delta \alpha, g = \delta \beta\) and \(h = \delta \gamma\). Then

\[ [\alpha, [\beta, \gamma]] + \text{cyclic} = \frac{1}{2}(f d\{g, h\} - \{g, h\} d f) + \text{cyclic}. \]

By \((12)\), this yields the zero class in \( \Omega^{1}(M)/\delta \Omega^{2}(M) \). \(\square\)

14
For Poisson manifolds that satisfy (12), we obtain an exact sequence

\[ 0 \to H^1_{\text{can}}(M) \to \Omega^1(M)/\delta \Omega^2(M) \to I \to C^\infty(M) \to H^0_{\text{can}}(M) \to 0 \]

of Lie algebras.

5.2 An \( L_{\infty} \)-algebra

The above central extension can be described by a chain map \((c_1, c_2, c_3)\) from the 3-term complex

\[ \wedge^3 C^\infty(M) \xrightarrow{\delta} \wedge^2 C^\infty(M) \xrightarrow{\delta} C^\infty(M) \]

to the truncated canonical homology complex \( \Omega^2(M) \xrightarrow{\delta} \Omega^1(M) \xrightarrow{\delta} \Omega^0(M) \).

The map \( c_1 \) is the identity, and \( c_2(f \wedge g) = \frac{1}{2}(fdg - gdf) \) encodes the bracket \([\alpha, \beta] = c_2(\delta \alpha \wedge \delta \beta)\) on \( \Omega^1(M) \). Commutativity of the bottom square means that the bracket on \( I \subseteq C^\infty(M) \) lifts the Poisson bracket on \( I \subseteq C^\infty(M) \). Finally, condition (12) is equivalent to the existence of a (non-canonical) map \( c_3 \) such that \( c_2 \circ \delta = \delta c_3 \). This ensures that the Jacobiator \( c_2 \circ \delta(\delta \alpha \wedge \delta \beta \wedge \delta \gamma) \) vanishes modulo \( \delta \Omega^2(M) \).

To construct a grounded \( L_{\infty} \)-algebra for this central Lie algebra extension, we need to lift this to a chain map \( c: (\wedge^n C^\infty(M), \delta) \to (\Omega^n(M), \delta) \). This can be done inductively. Suppose that we have constructed \( c_1, \ldots, c_n \) with \( \delta c_{i+1} = c_i \delta \) for all \( i = 1, \ldots, n \). Then any map \( c'_n = c_n + \Delta_n \) that differs from \( c_n \) by a map \( \Delta_n: \wedge^n C^\infty(M) \to \Omega^{n-1}(M) \) with \( \text{Im}(\Delta_n) \subseteq \ker(\delta) \) satisfies \( \delta c'_n = c_{n-1} \delta \) as well.

A lift \( c_{n+1} \) with \( \delta c_{n+1} = c_n \delta \) exists if and only if \( \text{Im}(c_n \delta) \subseteq \ker(\delta) \).

\[ \wedge^{n+1} C^\infty(M) \xrightarrow{c_{n+1}} \Omega^{n}(M) \]

\[ \wedge^n C^\infty(M) \xrightarrow{c_n} \Omega^{n-1}(M) \]

\[ \wedge^{n-1} C^\infty(M) \xrightarrow{c_{n-1}} \Omega^{n-2}(M) \]

Since \( \delta c_n \delta = c_{n-1} \delta^2 = 0 \), we have \( \text{Im}(c_n \delta) \subseteq \ker(\delta) \). If we fix the class

\[ [c_n]: \wedge^n C^\infty(M) \to \Omega^{n-1}(M)/\delta \Omega^{n-2}(M), \]
then the extension to \( c_{n+1} \) is thus obstructed by the linear map
\[
[c_n \partial] : \wedge^{n+1} C^{\infty}(M) \to H^c_{n-1}(M).
\]
However, by changing \( c_n \) to \( c'_n = c_n + \Delta_n \), we can always arrange this obstruction to vanish. Indeed, since \( c_n \partial \) takes values in \( \text{Ker}(\delta) \), we can choose any map \( \Delta_n : \wedge^n C^{\infty}(M) \to \text{Ker}(\delta) \) that agrees with \(-c_n \) on \( \partial(\wedge^{n+1} C^{\infty}(M)) \subseteq \wedge^n C^{\infty}(M) \). Then \( c'_n \) vanishes on \( \partial(\wedge^{n+1} C^{\infty}(M)) \), and we may in fact simply choose \( c_{n+1} = 0 \).

To define an \( L_n \)-algebra, we set \( L_{-i} := \Omega^{i+1}(M) \) for \( i \geq 0 \), with differential \( l_1 = \delta \). The higher brackets \( l_k : \wedge^k L \to L \) are nontrivial only on \( \wedge^k \Omega^{1}(M) \), where they are given by
\[
l_k(\alpha_1, \ldots, \alpha_k) = c_k(\delta \alpha_1 \wedge \ldots \wedge \delta \alpha_k).
\]

**Proposition 5.2.** If \((M, \pi)\) satisfies the condition \((\text{12})\), then there exists an \( L_n \)-algebra with \( L_{-i} = \Omega^{i+1}(M) \) with differential \( l_1 = \delta \) and degree two bracket \( l_2(\alpha, \beta) = \frac{1}{2}(\delta \alpha \delta \beta - \delta \beta \delta \alpha) \). It is possible to choose \( l_k = 0 \) for all \( k > 3 \), resulting in a Lie 2-algebra.

The following reformulation of condition \((\text{12})\) can be handy in concrete examples.

**Proposition 5.3.** Condition \((\text{12})\) is satisfied if and only if
\[
d(f [g, h] + g[h, f] + h[f, g]) \in \delta \Omega^2(M) \quad \text{for all} \quad f, g, h \in C^{\infty}(M).
\]
In particular, a sufficient condition for \((\text{12})\) is that \( d\Omega^0(M) \subseteq \delta \Omega^2(M) \).

**Proof.** This follows from the equality
\[
2\delta(fd g \wedge dh + \text{cyclic}) - d(f [g, h] + \text{cyclic}) = 3(f d[g, h] - [g, h]df) + \text{cyclic}. \quad \Box
\]

**Example 5.4.** To see that \((\text{12})\) is satisfied for a symplectic manifold \((M, \omega)\), note that
\[
df = -\delta * f \omega^{n-1}/(n-1)!
\]
for all \( f \in \Omega^0(M) \), where * is the symplectic Hodge-star operator \([\text{JV16} \text{ Lemma 2.3}]\).

**Example 5.5.** The linear Poisson structure on \( M = \mathfrak{sl}(2, \mathbb{R})^* \) affords an example of a singular Poisson structure where the condition \((\text{12})\) is satisfied. Its canonical homology was recently determined by Mărcuț and Zeiser \([\text{MZ19}]\). In coordinates \( x, y, z \) where \( \pi = x \partial_x \wedge \partial_z + y \partial_x \wedge \partial_y + z \partial_y \wedge \partial_y \), every class in \( H^1_{\text{can}}(M) \) admits a representative of the form \( fd\theta \), where \( f \) is a Casimir function with support contained in the outside \( \{ x^2 + y^2 - z^2 \geq 0 \} \) of the nilpotent cone, and \( d\theta = (x^2 + y^2)^{-1/2}(xdy - ydx) \). To evaluate the obstruction \((\text{12})\), note that \( i_x \Omega^2(M) = \Omega^0(M)(xdx + ydy - zdz) \). For any 2-form \( \beta \), this yields
\[
\oint \delta \beta = \oint (i_x d - d i_x) \beta = \oint i_x d \beta = 0,
\]
where the line integral is over any circle \( S^1_r = \{ x^2 + y^2 = r^2, z = 0 \} \). If \([\alpha] \in H^1_{\text{can}}(M) \) is represented by \( fd\theta \), we thus have \( f(x, y, z) = \frac{1}{2\pi} \oint_{S^1_r} \alpha \) for \( x^2 + y^2 - z^2 = r^2 > 0 \), so \([\alpha] = [0] \) if and only if \( \oint_{S^1_r} \alpha = 0 \) for all \( r > 0 \). In particular, \( d\Omega^0(M) \subseteq \delta \Omega^2(M) \), and the obstruction \((\text{12})\) vanishes.
Remark 5.6. Note that in the symplectic case, Proposition 5.2 yields a more general existence result than Theorem 4.7. However, in contrast with Proposition 5.2, the construction in Theorem 4.7 is functorial. More precisely, it yields a contravariant functor from the category of symplectic manifolds (with symplectic local diffeomorphisms) to the category of \( L_\infty \)-algebras (with strict morphisms). This is a consequence of \( L, \Lambda, d, \delta \) being preserved by symplectomorphisms and the fact that the brackets defined in Theorem 4.7 are local.

Constructing natural \( L_\infty \)-algebras for non-symplectic Poisson manifolds will be approached in a forthcoming work.

References

[BFLS98] Glenn Barnich, Ronald Fulp, Tom Lada, and Jim Stasheff. The \( sh \) Lie structure of Poisson brackets in field theory. Commun. Math. Phys., 191(3):585–601, 1998.

[Bry88] Jean-Luc Brylinski. A differential complex for Poisson manifolds. J. Differ. Geom., 28(1):93–114, 1988.

[Cal70] Eugenio Calabi. On the group of automorphisms of a symplectic manifold. Probl. Analysis. Sympos. in Honor of Solomon Bochner, Princeton Univ. 1969, 1-26 (1970), 1970.

[CFRZ16] Martin Callies, Yaël Frégier, Christopher L. Rogers, and Marco Zambon. Homotopy moment maps. Adv. Math., 303:954–1043, 2016.

[FRS14] Domenico Fiorenza, Christopher L. Rogers, and Urs Schreiber. \( L_\infty \)-algebras of local observables from higher prequantum bundles. Homology Homotopy Appl., 16(2):107–142, 2014.

[JV16] Bas Janssens and Cornelia Vizman. Universal central extension of the Lie algebra of Hamiltonian vector fields. Int. Math. Res. Not. IMRN, (16):4996–5047, 2016.

[JV18] Bas Janssens and Cornelia Vizman. Integrability of central extensions of the Poisson Lie algebra via prequantization. J. Symplectic Geom., 16(5):1351–1375, 2018.

[Kos85] Jean-Louis Koszul. Crochet de Schouten-Nijenhuis et cohomologie. In Élie Cartan et les mathématiques d’aujourd’hui. The mathematical heritage of Elie Cartan (Seminar), Lyon, June 25-29, 1984. 1985.

[Laz14] A. Lazarev. Models for classifying spaces and derived deformation theory. Proc. Lond. Math. Soc. (3), 109(1):40–64, 2014.

[Lic74] André Lichnerowicz. Algèbre de Lie des automorphismes infinitésimaux d’une structure unimodulaire. Ann. Inst. Fourier, 24(3):219–266, 1974.

[LM95] Tom Lada and Martin Markl. Strongly homotopy Lie algebras. Comm. Algebra, 23(6):2147–2161, 1995.
[LS93] Tom Lada and Jim Stasheff. Introduction to sh Lie algebras for physicists. *Int. J. Theor. Phys.*, 32(7):1087–1103, 1993.

[LV12] Jean-Louis Loday and Bruno Vallette. *Algebraic operads.*, volume 346. Berlin: Springer, 2012.

[Mat95] Olivier Mathieu. Harmonic cohomology classes of symplectic manifolds. *Comment. Math. Helv.*, 70(1):1–9, 1995.

[MZ19] Ioan Mărcuț and Florian Zeiser. The Poisson cohomology of $\mathfrak{sl}_2^*(\mathbb{R})$, 2019. Preprint, arxiv:1911.11732.

[Rog95] Claude Roger. Extensions centrales d’algèbres et de groupes de Lie de dimension infinie, algèbre de Virasoro et généralisations. volume 35, pages 225–266, 1995. Mathematics as language and art (Białowieża, 1993).

[Rog12] Christopher L. Rogers. $L_\infty$-algebras from multisymplectic geometry. *Lett. Math. Phys.*, 100(1):29–50, 2012.

[RW15] Leonid Ryvkin and Tilmann Wurzbacher. Existence and unicity of co-moments in multisymplectic geometry. *Differential Geom. Appl.*, 41:1–11, 2015.

[Ryv16] Leonid Ryvkin. *Observables and symmetries of n-plectic manifolds*. Wiesbaden: Springer Spektrum; Bochum: Univ. Bochum (Master Thesis), 2016.

[vH20] Kevin van Helden. Examples of L-infinity-algebras in n-plectic and Poisson geometry, MSc thesis, Leiden, 2020.

[Vit15] Luca Vitagliano. $L_\infty$-algebras from multicontact geometry. *Differ. Geom. Appl.*, 39:147–165, 2015.

[Yan96] Dong Yan. Hodge structure on symplectic manifolds. *Adv. Math.*, 120(1):143–154, 1996.

---

**Bas Janssens**,  
Institute of Applied Mathematics, Delft University of Technology, 2628 XE Delft, The Netherlands.  
b.janssens@tudelft.nl

**Leonid Ryvkin**,  
Mathematical Institute, Georg-August-Universität Göttingen, Bunsenstr. 3-5, 37073 Göttingen, Germany.  
Leonid.Ryvkin@mathematik.uni-goettingen.de

**Cornelia Vizman**,  
Department of Mathematics, West University of Timișoara. 300223 Timișoara, Romania.  
cornelia.vizman@e-uvt.ro