COHOMOGENEITY ONE MANIFOLDS WITH A SMALL FAMILY OF INVARIANT METRICS

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Abstract. In this paper, we classify compact simply connected cohomogeneity one manifolds up to equivariant diffeomorphism whose isotropy representation by the connected component of the principal isotropy subgroup has three or less irreducible summands. The manifold is either a bundle over a homogeneous space or an irreducible symmetric space. As a corollary such manifolds admit an invariant metric with non-negative sectional curvature.

1. Introduction

Manifolds with positive or non-negative sectional curvature have been of interest since the beginning of the global Riemannian geometry. Finding new examples of such manifolds is a particular difficult problem. There are few examples of positive curvature, mainly quotients of compact Lie groups. Recently two new examples were discovered using different methods. K. Grove, L. Verdiani and W. Ziller succeeded in putting positively curved metrics on a so called cohomogeneity one manifold \cite{GVZ}, i.e., the manifold admits an isometric group action and the orbit space is one dimension. Their example belongs to a family of infinitely many cohomogeneity one manifolds for which there is no known obstruction to positively curved metric, see \cite{GWZ}. O. Dearicott also proposed another construction for the same manifold, see \cite{De}. Another new example with positive curvature was discovered by P. Petersen and F. Wilhelm \cite{PW} on an exotic 7-sphere.

The class of non-negatively curved manifolds is much larger, although methods to construct them are very limited. Well known examples are products and quotients of such manifolds, e.g., quotients of compact Lie groups. In \cite{Ch} J. Cheeger used a gluing method to put non-negatively curved metrics on the connected sum of any two compact rank one symmetric spaces. Cheeger’s gluing method was greatly generalized by K. Grove and W. Ziller in \cite{GZ} to cohomogeneity one manifold, where they proved that if the singular orbits are codimension two, the manifold admits an invariant metric with non-negative curvature. However not every cohomogeneity one manifold carries an invariant metric with non-negative sectional curvature. The first examples were discovered by K. Grove, L. Verdiani, B. Wilking and W. Ziller in \cite{GVWZ} which were generalized to a larger class by the author, see \cite{He}. It is thus natural to ask how large the class of cohomogeneity one manifolds with an invariant metric of non-negative curvature is, see \cite{Z1}.

One possible approach is to study cohomogeneity one manifolds with further geometric or topological restrictions. C. Hoelscher classified simply-connected examples in dimension 7 or less in \cite{Ho}. Most of them carry non-negatively curved metrics except for some examples in dimension 7, for which the existence of such metrics is still open. L. Schwachhöfer and K. Tapp studied cohomogeneity one manifolds which have a totally geodesic principal orbit.
They classified such manifolds under some further conditions, see \cite{STa}. The examples in their classification were already known to have non-negatively curved metrics.

Cohomogeneity one manifolds also arise as interesting examples in other areas. For instance, many new examples of Einstein and Einstein-Sasaki metrics (see, for example, \cite{Bo}, \cite{Co} and \cite{GHY}), and metrics with special holonomy (see \cite{CS}, \cite{CGLP} and \cite{Re1}-\cite{Re3}) were found among cohomogeneity one manifolds. In the study of the Ricci flow, Ricci solitons play an important role in order to understand the singularities of the flow. Many interesting examples of Ricci solitons also admit cohomogeneity one actions, see \cite{DW}.

As far as the Riemannian metrics are concerned, one may first consider examples which are geometrically simple, i.e., the family of invariant metrics is small. Let $M$ be a cohomogeneity one manifold, i.e., there exists a compact Lie group $G$ acting on $M$ by isometries and the cohomogeneity of the action, defined as $\text{cohom}(M,G) = \dim(M/G)$, is equal to 1. If the manifold is compact and simply-connected, then the orbit space is a closed interval $I$. In this case, there are precisely two singular orbits $B_{\pm}$ with isotropy subgroups $K_{\pm}$ corresponding to the endpoints of $I$, and principal orbits corresponding to the interior points with isotropy subgroup $H$. Let $l_{\pm}$ denote the codimensions of $B_{\pm}$. The manifold can be written as a union of two disk bundles as $M = G \times_{K_-} D^{l_-} \cup_{G/H} G \times_{K_+} D^{l_+}$, and thus it can be identified with the group diagram $H \subset \{ K_{\pm} \} \subset G$. Suppose $c(t)$ is a minimal geodesic between the singular orbits and $t$ is the arc length. Since the $G$-action is transitive on orbits and the metric is left invariant on them, we only have to consider the metric along $c(t)$ and thus $g = dt^2 + g_t$, where $g_t$ is the metric at $c(t)$. Let $M_t = G.c(t)$ be a principal orbit. Since $H_c$, the identity component of $H$, fixes $c(t)$, it induces the isotropy representation on the tangent space $T_{c(t)}M_t$. In general this representation is not irreducible and let $s$ denote the number of irreducible summands. From Schur’s lemma, if the value of $s$ is big, the family of invariant metrics is large.

In this paper we classify compact simply-connected cohomogeneity one manifolds with $s \leq 3$. We say a $G$-manifold $M$ is non-primitive if there is a $G$-equivariant map $M \to G/L$ for some proper subgroup $L \subset G$. Otherwise the $G$-action is called primitive. Our main result is

**Theorem A.** If a compact simply-connected Riemannian manifold $M$ admits a primitive cohomogeneity one action with $s \leq 3$, then it is equivariantly diffeomorphic to a symmetric space with an isometric action.

In fact only spheres, complex and quaternionic projective spaces, the Cayley plane, and Grassmannians $\text{SO}(m + n)/((\text{SO}(m) \times \text{SO}(n))(m, n \geq 2)$ appear in the classification. This easily implies
Corollary B. Every compact simply-connected cohomogeneity one manifold with \( s \leq 3 \) admits an invariant metric with non-negative sectional curvature.

In contrast to the above corollary, the Kervaire spheres carry cohomogeneity one actions with \( s = 4 \) and they do not admit any non-negatively curved invariant metrics, see [GVWZ].

If the cohomogeneity one manifold \( M \) is non-primitive, then it has a group diagram \( H \subset \{ K^-, K^+ \} \subset G \) such that there is a proper subgroup \( L \subset G \) that contains both \( K^\pm \). It follows that the \( G \) action has no fixed points. Furthermore, \( M \) is a fiber bundle over the homogeneous space \( G/L \) with fiber \( N \) and the \( L \) action on \( N \) is cohomogeneity one with diagram \( H \subset \{ K^-, K^+ \} \subset L \). One particular class of non-primitive manifolds is the so called double for which \( K^- = K^+ \). For a double we can let \( L = K^\pm \) and then \( N \) is a sphere with a linear cohomogeneity one action that has two fixed points. In general for a non-primitive action, we have

Theorem C. If \( M \) is a compact simply-connected cohomogeneity one manifold that admits a non-primitive action by a compact Lie group \( G \) with \( s \leq 3 \), then

1. either \( M \) is a double;
2. or \( M \) is a fiber bundle over a strongly isotropy irreducible space \( G/L \) with fiber \( N \).

In case (2), if the action of \( L \) on \( N \) has no fixed points, then \( N \) is a sphere or a three dimensional lens space. Otherwise \( N \) is a projective space or the Cayley plane.

In case (2), the space \( G/L \) is strongly isotropy irreducible which means that the isotropy representation by \( L_c \) is irreducible. Cohomogeneity one manifolds with a fixed point were classified in [Ho] (see also in [GS]), where he showed that the manifolds are equivariantly diffeomorphic to projective spaces and the Cayley plane. In Theorem 3.2 and 3.3 we classify cohomogeneity one manifolds such that the action has no fixed point and \( s = 2 \). One can prove Theorem C directly by applying these two classifications. In this paper, we take a different approach since we also want the classification of such cohomogeneity one diagrams, i.e., a classification up to equivariant diffeomorphism. We first classify all non-primitive diagrams and then Theorem C easily follows.

Notice that in Theorem A and C, we do not assume that \( H \) is connected. Notice also that \( s \) is the number of irreducible summands of the isotropy action by \( H_c \). It is possible that there are other cohomogeneity one manifolds where the whole group \( H \) acts with only 3 irreducible summands.

A manifold \( M \) may have different cohomogeneity one actions, for example, there are many cohomogeneity one actions on spheres. One special case is that a proper normal subgroup \( G_1 \subset G \) acts on the manifold with the same orbits. In this case, we call the \( G \)-action reducible.
Otherwise the action is called *non-reducible*. Let $H$ be the principal isotropy subgroup of the $G$-action. Note that the $G$ action being reducible is equivalent to the fact that $H$ projects onto a simple factor of $G$. Let $H_1 = H \cap G_1$ and then it is the principal isotropy subgroup of the $G_1$-action. Since $H_1$ is a subgroup of $H$, the isotropy action of $H_1$ on the tangent space of the principal orbit may have more irreducible summands, i.e., a bigger value of $s$. For this reason, we consider the classification of reducible actions as well.

**Theorem D.** If a compact simply-connected manifold $M$ admits a cohomogeneity one action by $G$ with $s \leq 3$ and the action is reducible, then one of the following holds:

1. a normal subgroup of $G$ acts on $M$ non-reducibly and has the same value of $s$;
2. the action is non-primitive;
3. the action is primitive and it is a linear action on a sphere.

In Theorem 4.1 we obtained the classification of cohomogeneity one manifolds with $s = 3$ in case (2) of Theorem D by using the classification results in Theorem A and C with $s = 1, 2$. The actions on spheres in case (3) of Theorem D are the so called *sum actions*, see definition in Section 2.

One can also use our classification to study some cohomogeneity one manifolds with $s = 4$ or higher. For example, the Kervaire sphere which does not admit an invariant metric with non-negative curvature has the group diagram

$$
\mathbb{Z}_2 \times SO(n - 2) \subset \{SO(2) \times SO(n - 2), O(n - 1)\} \subset SO(2) \times SO(n),
$$

where $n \equiv 1 \mod 4$. The diagram can be constructed from the following one which has $s = 3$

$$
\mathbb{Z}_2 \times SO(n - 2) \subset \{SO(2) \times SO(n - 2), O(n - 1)\} \subset SO(n)
$$

by adding an $SO(2)$ factor to $SO(n)$ and embedding the $SO(2)$ factor in $SO(2) \times SO(n - 2)$ diagonally into $SO(2) \times SO(n)$. Similar constructions can be applied to certain examples in our classification and they give an interesting class of cohomogeneity one diagrams with $s = 4$. They will be discussed in a forthcoming paper.

The paper is organized as follows. In Section 2, we recall some basic facts about cohomogeneity one manifolds which will be used throughout the paper. In Section 3 we consider the classification when $s = 1, 2$, see Theorem 3.2 and 3.3. In Section 4 we consider the classification when $s = 3$. The classification for reducible actions is stated in Theorem 4.1 and for the non-primitive ones in Theorem 4.2. The classification of primitive cohomogeneity one manifolds with $G$ simple and $s = 3$ is carried in Section 5, see Theorem 5.1 and when $G$ is non-simple in Section 6, see Theorem 6.1. In Appendix A we correct the classification of
compact simply-connected homogeneous spaces $G/H$ such that the isotropy action has two summands and $G$ is a simple Lie group. This will be used in Section 5. In Appendix B we collect the tables which contain our classification results. We will see that a cohomogeneity manifold with $s \leq 3$ either has a fixed point, or is a double, a sphere, or is contained in Tables 11, 12, 13 and 17.

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2. Preliminaries

In this section, we recall some basic and well-known facts about cohomogeneity one manifolds. For more detail, we refer to, for example, [AA] and [GWZ].

As mentioned already, there are precisely two non-principal orbits $B_{\pm}$ in a simply connected cohomogeneity one manifold. Suppose $M$ is endowed with an invariant metric $g$ and the distance between the two non-principal orbits is $L$. The two ending points of the minimal geodesic $c(t)$ are specified as $c(0) = p_- \in B_- \text{ and } c(L) = p_+ \in B_+$. Thus the isotropy subgroups at $p_{\pm}$ are $K_{\pm}$ and the principal isotropy subgroup at any point $c(t), t \in (0, L)$, is $H$. We then draw the following group diagram for the manifold $M$:

The group diagram $H \subset \{K^-, K^+\} \subset G$ is not uniquely determined by the manifold since one can switch $K^-$ with $K^+$, change $g$ to another invariant metric and choose another minimal geodesic $c(t)$.

We consider the family of invariant metrics on $M$. Since the union of principal orbits are open and dense in $M$, we only have to consider the metric restricted on it. For every $t \in (0, L)$, the principal orbit $M_t$ is diffeomorphic to the homogeneous space $G/H$. Let $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of $G$ and $H$ respectively. Fix a bi-invariant inner product $Q$ on $\mathfrak{g}$ and let $\mathfrak{p}$ be the orthogonal compliment of $\mathfrak{h} \subset \mathfrak{g}$. The space $\mathfrak{p}$ is identified with the tangent space $T_{c(t)}M_t$ via Killing vector fields. Since $H$ fixed the point $c(t)$, it induces the so called
isotropy representation on $\mathfrak{p}$:

$$\text{Ad}(h) : X \rightarrow h.X.h^{-1}, \text{ for any } h \in H \text{ and } X \in \mathfrak{p}.$$ 

We denote its differential by $\text{ad}$ and it defines the isotropy representation of the Lie algebra $\mathfrak{h}$:

$$\text{ad}_Z : X \rightarrow [Z,X], \text{ for any } Z \in \mathfrak{h} \text{ and } X \in \mathfrak{p}.$$ 

For a cohomogeneity one manifold $M$, $H$ may not be connected even if we assume that $G$ is connected and $M$ is simply-connected. Let $\text{Ad}_{H_c}$ be the restriction the isotropy representation to the identity component $H_c$. In most cases, it is reducible and the number of the irreducible summands is denoted by $s$. It is equivalent to say that the isotropy representation $\text{ad}_h$ has $s$ summands. The simplest case is $s = 1$, i.e., $\text{Ad}_{H_c}$ is irreducible. From Schur’s lemma $g_t$ is a scalar multiplication of the identity map on $T_{c(t)}M_t$ and thus the metric $g$ depends on one function. In the case where $s = 3$ the metric depends on three functions if the three summands are non-equivalent.

For a cohomogeneity one manifold, a convenient way to describe the manifold and the action is to use the group diagram. However, a cohomogeneity one manifold may have different diagrams.

**Definition 2.1.** Two group diagrams are called *equivalent* if they determine the same cohomogeneity one manifold up to equivariant diffeomorphism.

The following lemma characterizes which two group diagrams are equivalent, see [GWZ].

**Lemma 2.2.** Two group diagrams $H \subset \{K^-, K^+\} \subset G$ and $\tilde{H} \subset \{\tilde{K}^-, \tilde{K}^+\} \subset G$ are equivalent if and only if after possibly switching the roles of $K^-$ and $K^+$, the following holds: There exist elements $b \in G$ and $a \in N(H)_c$, where $N(H)_c$ is the identity component of the normalizer of $H$, with $\tilde{K}^- = bK^-b^{-1}$, $\tilde{H} = bHb^{-1}$, and $\tilde{K}^+ = abK^+b^{-1}a^{-1}$.

**Remark 2.3.** If $c(t)$ is the minimal normal geodesic between the two singular orbits, then $b.c(t)$ is another minimal geodesic between $B_\pm$ and the associated group diagram is obtained by conjugating all isotropy groups by the element $b$. We can assume that $b \in N(H) \cap N(K^-)$ in order to fix $H$ and $K^-$. Conjugation by an element $a$ as in the above lemma usually corresponds to changing the invariant metric on the manifold.

Let us describe a method using automorphisms of $G$ to obtain new group diagrams from a given one $M : H \subset \{K^-, K^+\} \subset G$. Take two automorphisms $\tau_\pm$ of $G$ and apply them to the triples $H \subset K^\pm \subset G$. If $\tau_-(H)$ and $\tau_+(H)$ are equal, then we have the diagram $\tau_-(H) \subset \{\tau_-(K^-), \tau_+(K^+)\} \subset G$ and the manifold is $G$ equivariantly diffeomorphic to $M$, defined by
the diagram $H \subset \{K^-, \tau(K^+)\} \subset G$ where $\tau = \tau^{-1} \cdot \tau_+$ that leaves $H$ invariant. If $K^-$ or $K^+$ is also invariant by $\tau$, then the manifolds $M$ and $M_\tau$ are equivariantly diffeomorphic. So we assume that only $H$ is invariant by the automorphism $\tau$, i.e., $\tau$ is in the double coset space $\text{Aut}(G, K^-) \backslash \text{Aut}(G, H)/\text{Aut}(G, K^+)$ where $\text{Aut}(G, L)$ is the group of automorphisms of $G$ leaving the subgroup $L$ invariant. Furthermore, if two automorphisms $\tau_1$ and $\tau_2$ can be connected by a continuous path in $\text{Aut}(G, H)$, then $M_{\tau_1}$ and $M_{\tau_2}$ are $G$ equivariantly diffeomorphic.

**Definition 2.4.** A group diagrams $\Gamma_1 : H \subset \{\tilde{K}^-, \tilde{K}^+\} \subset G$ is called a variation of the diagram $\Gamma_2 : H \subset \{K^-, K^+\} \subset G$ if $\tilde{K}^- = \tau(K^-)$, $\tilde{K}^+ = K^+$ for some $\tau \in \text{Aut}(G, H)$ after possible switching of $K^-$ and $K^+$.

Next we consider several special classes of cohomogeneity one actions which have been mentioned in the Introduction and we recall basic properties of them.

**Definition 2.5.** A cohomogeneity one manifold $(M, G)$ is called a double if it admits a group diagram such that $K^- = K^+$.

One can put a non-negatively curved invariant metric on the disk bundle $G \times_K \mathbb{D}^l$ making the boundary totally geodesic. If $M$ is a double, then we can glue the two identical disk bundles along the totally geodesic boundary so that $M$ has an invariant metric with non-negative sectional curvatures.

**Definition 2.6.** A cohomogeneity one manifold $(M, G)$ is non-primitive if it admit a group diagram $H \subset \{K^\pm\} \subset G$ and there is a proper connected subgroup $L \subset G$ such that $K^\pm \subset L$. A cohomogeneity one manifold is called primitive if it is not non-primitive.

If $M$ has a non-primitive group diagram $H \subset \{K^\pm\} \subset G$ and $K^\pm \subset L \subset G$, then we have the following fibration:

$$N \rightarrow M \rightarrow G/L,$$

and the fiber $N$ carries a cohomogeneity one action by $L$ with the diagram $H \subset \{K^\pm\} \subset L$. Thus $M$ is $G$ equivariantly diffeomorphic to $G \times_L N$. So $M$ has an invariant metric with non-negative sectional curvatures if $N$ admits such a metric.

**Definition 2.7.** A cohomogeneity one action of $G$ on $M$ is reducible if there is a proper normal subgroup of $G$ that still acts by cohomogeneity one with the same orbit. Otherwise the action is called nonreducible.

In terms of group diagram, we have the following characterization of reducible actions, see [Ho], Section 1.3.
**Proposition 2.8.** Let \( M \) be the cohomogeneity one manifold given by the group diagram \( H \subset \{ K^-, K^+ \} \subset G \) and suppose \( G = G_1 \times G_2 \) with \( \text{Proj}_2(H) = G_2 \). Then the sub-action of \( G_1 \times \{ 1 \} \) on \( M \) is also by cohomogeneity one, with the same orbits, and with isotropy groups \( K_1^\pm = K^\pm \cap (G_1 \times \{ 1 \}) \) and \( H_1 = H \cap (G_1 \times \{ 1 \}) \).

On the other hand, suppose \( G_1 \) acts on \( M \) via cohomogeneity one with the diagram \( H_1 \subset \{ K_1^\pm \} \subset G_1 \), then one can extend this action to a possibly larger group by the so called normal extension, see \([Ho]\). Let \( L \) be a compact connected subgroup of \( N(H_1) \cap N(K^-_1) \cap N(K^+_1) \) and define \( G_2 = L/(L \cap H_1) \). Then one can define an isometric action of \( G_1 \times G_2 \) on \( M \) orbitwise by \((\hat{g}_1, [l]).g_1(G_1)_x = \hat{g}_1g_1l^{-1}(G_1)_x \) for \((G_1)_x = H_1 \) or \( K_1^\pm \). This action is also cohomogeneity one and has the following diagram

\[
(H_1 \times 1) \cdot \Delta L \subset \{(K_1^\pm \times 1) \cdot \Delta L\} \subset G_1 \times G_2,
\]

where \( \Delta L = \{([l], [l]) | l \in L \} \). If the diagram is non-primitive, then its normal extension is also non-primitive. Note that the reducible action by \( G_1 \times G_2 \) in Proposition 2.8 occurs as a normal extension of the reduced one by \( G_1 \), see Proposition 1.18 in \([Ho]\).

**Remark 2.9.** Suppose the diagram \( H \subset \{ K^\pm \} \subset G_1 \times G_2 \) is reducible and the isotropy representation of \( H_1 \) has 3 irreducible summands. If we consider the non-reducible action by \( G_1 \), then in general the isotropy representation by the connected component of \( H_1 \) may have more irreducible summands since \( H_1 \) is a subgroup of \( H \).

There is a particular class of cohomogeneity one actions on spheres, called sum action, which can be builded from transitive actions on spheres with lower dimensions, see \([Ho]\) and \([GWZ]\). Let \( G_i \) act transitively, linearly and isometrically on the sphere \( S^{l_i} \) with isotropy subgroup \( H_i \), \( i = 1, 2 \). Then the action of \( G_1 \times G_2 \) on \( S^{l_1+l_2+1} \subset \mathbb{R}^{l_1+l_2+2} \) is cohomogeneity one and the isotropy subgroups are \( G_1 \times H_2, H_1 \times G_2 \) and \( H_1 \times H_2 \), i.e., the cohomogeneity diagram is

\[
H_1 \times H_2 \subset \{ G_1 \times H_2, H_1 \times G_2 \} \subset G_1 \times G_2.
\]

Suppose the isotropy representation of \( G_i/H_i \) has \( s_i \) irreducible summands \( i = 1, 2 \), then the sum action of the diagram \((2.1)\) has \( s = s_1 + s_2 \).

Not every group diagram defines a simply-connected cohomogeneity one manifold. The necessary conditions are given in \([GWZ]\).

**Lemma 2.10.** Suppose a connected Lie group \( G \) acts on a simply-connected manifold \( M \) by cohomogeneity one and the diagram is \( H \subset \{ K^\pm \} \subset G \). Then we have

1. There are no exceptional orbits, i.e., \( l_+ \geq 2 \).
(2) If both \( l_\pm \geq 3 \), then \( K^\pm \) and \( H \) are all connected.

(3) If one of \( l_\pm \), say \( l_- = 2 \), and \( l_+ \geq 3 \), then \( K^- = H \cdot S^1 = H_c \cdot S^1 \), \( H = H_c \cdot \mathbb{Z}_k \) and \( K^+ = K_c^+ \cdot \mathbb{Z}_k \).

In the last case where \( l_- = 2 \), we have that \( K^- \) is connected, \( \mathbb{Z}_k \subset N_G(K_c^+)/K_c^+ \) and \( S^1 \) normalizes \( H_c \). It follows that if \( K^- \) is contained in \( K_c^+ \), then one cannot add connected components to the isotropy groups. Note that the diagram of the connected groups \( H_c \subset \{ K^-, K_c^+ \} \subset G \) in this case also defines a simply-connected cohomogeneity one manifold.

In the following we introduce the notations that we will use through the paper.

The exceptional Lie groups are denoted by \( E_6, E_7, E_8, F_4 \) and \( G_2 \) and their Lie algebras by \( \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8, \mathfrak{f}_4 \) and \( \mathfrak{g}_2 \). The complex irreducible representations of simple Lie algebras are highest weight representations, so we can identify the representation with its highest weight. Each highest weight is the linear combination of the so called fundamental weights with non-negative integer coefficients. If the Lie algebra \( \mathfrak{g} \) has rank \( n \), then there are \( n \) fundamental weights \( \pi_1, \ldots, \pi_n \). We also use the notions \( g_n, \mu_n \) and \( \nu_n \) for the standard representations of \( \text{SO}(n), \text{SU}(n) \) (or \( \text{U}(n) \)) and \( \text{Sp}(n) \) on \( \mathbb{R}^n, \mathbb{C}^n \) and \( \mathbb{H}^n \) respectively. \( \Delta_n \) stands for the unique spin representation of \( \text{SO}(n) \) if \( n \) is odd, and \( \Delta_n^\pm \) stands for the two spin representations of \( \text{SO}(n) \) if \( n \) is even. The fundamental weights of exceptional Lie algebras are specified as follows:

\[
\begin{align*}
\mathfrak{e}_6 & : \\
& \begin{array}{cccccc}
1 & 3 & 4 & 5 & 6 \\
& 2 \\
\end{array} \\
\mathfrak{e}_7 & : \\
& \begin{array}{cccccc}
1 & 3 & 4 & 5 & 6 & 7 \\
& 2 \\
\end{array} \\
\mathfrak{e}_8 & : \\
& \begin{array}{cccccc}
1 & 3 & 4 & 5 & 6 & 7 & 8 \\
& 2 \\
\end{array} \\
\mathfrak{f}_4 & : \\
& \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\end{array} \\
\mathfrak{g}_2 & : \\
& \begin{array}{cc}
1 & 2 \\
\end{array}
\end{align*}
\]

We denote the standard representation of \( \text{U}(1) \) on \( \mathbb{C} \) by \( \phi \). For the Lie algebras \( \mathfrak{so}(4) \) and \( \mathfrak{so}(6) \), we specify some representations and their highest weights. For \( \mathfrak{so}(4) \), the representations with highest weights \( \pi_1 = \frac{1}{2}(e_1 - e_2) \) and \( \pi_2 = \frac{1}{2}(e_1 + e_2) \) are the two spin representations. The standard representation \( g_4 \) of \( \text{SO}(4) \) on \( \mathbb{C}^4 \) has the highest weight \( \pi_1 + \pi_2 = e_1 \). For \( \mathfrak{so}(6) \), the representation with highest weight \( \pi_1 = e_1 \) is the standard representation \( g_6 \) of \( \text{SO}(6) \) on \( \mathbb{C}^6 \). However the representation of \( \mathfrak{su}(4) \) with the highest weight \( \pi_1 \) is the standard representation \( \mu_4 \) of \( \text{SU}(4) \) on \( \mathbb{C}^4 \) though \( \mathfrak{so}(6) \) is isomorphic to \( \mathfrak{su}(4) \).

Some homogeneous spaces with special geometrical properties are used in the classification.

**Definition 2.11.** A pair of Lie group \((K, H)\) is called a spherical pair if \( K/H \) is a sphere.
In this case, we also call the Lie algebras \((\mathfrak{g}, \mathfrak{h})\) a spherical pair.

The transitive actions on spheres were classified, see, for example, [Be]. The results are used frequently later on and we list them in Table 1. Here \(s\) is the number of the irreducible summands in the isotropy representation.

| \(n\) | \(K\) | \(H\) | Isotropy representation | \(s\) |
|---|---|---|---|---|
| \(n\) | \(\text{SO}(n+1)\) | \(\text{SO}(n)\) | \(\varrho_n\) | 1 |
| \(2n+1\) | \(\text{SU}(n+1)\) | \(\text{SU}(n)\) | \([\mu_n]_\mathbb{R} \oplus \text{Id}\) | 2 |
| \(2n+1\) | \(\text{U}(n+1)\) | \(\text{U}(n)\) | \([\mu_n]_\mathbb{R} \oplus \text{Id}\) | 2 |
| \(4n+3\) | \(\text{Sp}(n+1)\) | \(\text{Sp}(n)\) | \([\nu_n]_\mathbb{R} \oplus \text{Id} \oplus \text{Id} \oplus \text{Id}\) | 4 |
| \(4n+3\) | \(\text{Sp}(n+1)\text{Sp}(1)\) | \(\text{Sp}(n)\Delta\text{Sp}(1)\) | \(\nu_n \otimes \nu_1 \oplus \text{Id} \otimes \varrho_3\) | 2 |
| 15 | \(\text{Spin}(9)\) | \(\text{Spin}(7)\) | \(\varrho_7 \oplus \Delta_7\) | 2 |
| 7 | \(\text{Spin}(7)\) | \(\mathfrak{G}_2\) | \(\pi_1\) | 1 |
| 6 | \(\mathfrak{G}_2\) | \(\text{SU}(3)\) | \([\mu_3]_\mathbb{R}\) | 1 |

**Table 1. Transitive actions on \(S^n\).**

**Definition 2.12.** A homogeneous space \(G/H\) is called *isotropy irreducible* if the isotropy representation of \(H\) is irreducible. If the isotropy representation by the identity component \(H_c\) of \(H\) is also irreducible, then it is called *strongly isotropy irreducible.*

Every irreducible symmetric space is strongly isotropy irreducible. J. Wolf classified compact strongly isotropy irreducible spaces which are not symmetric spaces, see [Wo1]. Compact homogeneous spaces which are isotropy irreducible but not strongly isotropy irreducible were classified by M. Wang and W. Ziller, see [WZ2]. If the space \(G/H\) is strongly isotropy irreducible, then the isotropy representation \(\text{ad}_h\) is also irreducible. In this case, we call the pair of Lie algebras \((\mathfrak{g}, \mathfrak{h})\) strongly isotropy irreducible.

3. Classification of cohomogeneity one manifolds with \(s = 1, 2\)

In this section, we classify simply-connected cohomogeneity one manifolds with \(s = 1, 2\). The results are also used in the classification of the case where \(s = 3\).

3.1. **Fixed point actions.** If \(s = 1\), then the cohomogeneity one action has two fixed points, i.e., \(G = K^- = K^+\) and the manifold is a sphere. By the classification of transitive actions of the sphere, the cohomogeneity one manifolds are listed in Table 2.
Table 2. Cohomogeneity one manifolds with $s = 1$.

Next we quote the result when there is only one fixed point, say $K^- = G$ and $K^+ \subset G$ is a proper subgroup, see [Ho] and [GS].

**Proposition 3.1.** If a simply-connected manifold $M$ admits a cohomogeneity one action with exactly one fixed points, then $M$ is a (complex or quaternion) projective space or the Cayley plane with an isometric action. They are classified in Table 3.

Table 3. The cohomogeneity one action with one fixed point

3.2. **Classification with $s = 2$**. We assume that the action has no fixed points. If the action is primitive, then the manifold is a sphere, see Theorem 3.2. If the action is non-primitive, then the manifold is a double, i.e., $K^\pm = K$, and we classified the triples $H \subset K \subset G$, see Theorem 3.3.

**Theorem 3.2.** Suppose $M$ is a compact simply-connected manifold that admits a primitive cohomogeneity one action with $s = 2$ and no fixed points. Then one of the followings holds.

1. The manifold is $S^{15}$ with the diagram $\mathfrak{g}_2 \subset \{\text{Spin}^+(7), \text{Spin}^-(7)\} \subset \text{Spin}(8)$ and the embedding $\text{Spin}(8) \subset \text{SO}(16)$ is given by the representation $\Delta^+_8 \oplus \Delta^-_8$. 


(2) The manifold is a sphere with a sum action.

Proof. From the assumption $s = 2$, the space $\mathfrak{p}$ of the representation $\text{Ad}_{H_c}$ splits into two subspaces which are denoted by $\mathfrak{p}_1$ and $\mathfrak{p}_2$, and the representation of $\text{Ad}_{H_c}$ on each of them is irreducible.

First we assume that the two summands $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are equivalent representations of $H_c$. Let $K = K^-$ and we consider the group triple $H \subset K \subset G$. Since the sphere $K/H$ is isotropy irreducible, its effective version is one of the pairs $SO(n + 1)/SO(n)$, $\text{Spin}(7)/\mathfrak{e}_2$, $U(1)$, $\mathfrak{e}_2/SU(3)$, $\text{Sp}(1)/U(1)$, $\text{Sp}(2)/(\text{Sp}(1) \times \text{Sp}(1))$, $SU(2) \times SU(2)/\Delta SU(2)$ and $SU(4)/\text{Sp}(2)$. Suppose $K/H = (SO(n+1) \cdot L)/(SO(n) \cdot L)$ for some $L$, then the two representations are $\rho_n \otimes \text{Id}$. In particular they have dimension $n$. However $SO(n + 1)$ has no irreducible representation with dimension $n$ if $n \geq 2$. Similarly, $K/H$ cannot be $\mathfrak{e}_2/SU(3)$, $\text{Sp}(1)/U(1)$, $\text{Sp}(2)/(\text{Sp}(1) \times \text{Sp}(1))$ and $SU(4)/\text{Sp}(2)$. The possible triples are

$$\mathfrak{e}_2 \subset \text{Spin}(7) \subset SO(8), \quad \{1\} \subset U(1) \subset U(1) \times U(1).$$

There are two different $\text{Spin}(7)$, denoted by $\text{Spin}^+(7)$ and $\text{Spin}^-(7)$, in $SO(8)$ that contains $\mathfrak{e}_2$ and they differ by an automorphism of $SO(8)$. So there is one primitive diagram from this triple:

$$\mathfrak{e}_2 \subset \{\text{Spin}^+(7), \text{Spin}^-(7)\} \subset \text{Spin}(8)$$

and the manifold is $S^{15}$. For the second triple $\{1\} \subset U(1) \subset U(1) \times U(1)$, one may choose different embedding of $U(1) \subset U(1) \times U(1)$ for $K^+$ such that the diagram is primitive. However all are sum actions on $S^3$.

Next we assume that $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are non-equivalent representations. From the assumption the action $\text{ad}_h$ on the spaces $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are irreducible, W.L.O.G., we may assume that $\mathfrak{k}^- = \mathfrak{h} \oplus \mathfrak{p}_1$ and $\mathfrak{k}^+ = \mathfrak{h} \oplus \mathfrak{p}_2$. We denote $Q(X,Y)$ by $\langle X, Y \rangle$ for $X, Y \in \mathfrak{g}$. Then for any $X_1, Y_1 \in \mathfrak{p}_1$, $X_2, Y_2 \in \mathfrak{p}_2$ and $Y_0 \in \mathfrak{h}$, since $[X_2, Y_0] \in \mathfrak{k}^+ = \mathfrak{h} \oplus \mathfrak{p}_2$, $[Y_1, X_1] \in \mathfrak{k}^- = \mathfrak{h} \oplus \mathfrak{p}_1$ and $[X_2, Y_2] \in \mathfrak{k}^+ = \mathfrak{h} \oplus \mathfrak{p}_2$, we have

$$\langle [X_1, X_2], Y_0 \rangle = \langle X_1, [X_2, Y_0] \rangle = 0,$$
$$\langle [X_1, X_2], Y_1 \rangle = \langle Y_1, [X_1, X_2] \rangle = \langle [Y_1, X_1], X_2 \rangle = 0,$$
$$\langle [X_1, X_2], Y_2 \rangle = \langle X_1, [X_2, Y_2] \rangle = 0.$$

Therefore $[X_1, X_2]$ is orthogonal to any vector in $\mathfrak{g}$, i.e., $[\mathfrak{p}_1, \mathfrak{p}_2] = 0$. We define the following subspaces of $\mathfrak{g}$. Let

$$\mathfrak{h}_0 = \text{Ann}(\mathfrak{p}_1 \oplus \mathfrak{p}_2) \cap \mathfrak{h} = \{X \in \mathfrak{h} | [X, Y] = 0 \text{ for any } Y \in \mathfrak{p}_1 \oplus \mathfrak{p}_2\},$$
and 
\[ h_i = \text{Ann}(p_i) \cap h_0^\perp \cap h, \quad (i = 1, 2), \quad h_3 = (h_0 \oplus h_1 \oplus h_2)^\perp \cap h, \]
where \( \perp \) is the orthogonal complement with respect to the inner product \( Q \).

Since \([p_1, p_1] \subset t^- = h \oplus p_1, [p_1, h] \subset p_1 \) and 
\[ [[p_1, p_1], p_2] = -[[p_1, p_2], p_1] - [[p_2, p_1], p_1] = 0, \]
we have \([p_1, p_1] \subset p_1 \oplus h_0 \oplus h_2 \). Moreover since 
\[ \langle [p_1, p_1], h_0 \rangle = \langle p_1, [p_1, h_0] \rangle = \langle p_1, 0 \rangle = 0, \]
we have \([p_1, p_1] \subset p_1 \oplus h_2 \). Denote the Lie algebra generated by \( p_1 \) by \( \text{Lie}(p_1) \), then \( \text{Lie}(p_1) \subset p_1 \oplus h_2 \). If there is a vector \( X \in p_1 \oplus h_2 \) such that \( X \perp \text{Lie}(p_1) \), then \( X \in h_2 \) and thus \([X, p_1] \subset p_1 \). For any vector \( Y \in p_1 \), we have \( \langle [X, p_1], Y \rangle = \langle X, [p_1, Y] \rangle = 0 \) which implies that \( X \in \text{Ann}(p_1) \cap h_2 = 0 \). Hence we have \( \text{Lie}(p_1) = p_1 \oplus h_2 \). Similarly we also have \( \text{Lie}(p_2) = p_2 \oplus h_1 \).

We claim that \( h_3 = 0 \). In fact, first we have \( \langle [h_3, p_1], p_1 \rangle = \langle h_3, [p_1, p_1] \rangle = 0 \) since \([p_1, p_1] \subset p_1 \oplus h_2 \). Since \( h_3 \subset h \), \([h_3, p_1] \subset p_1 \) and thus \([h_3, p_1] = 0 \) which implies that \( h_3 \subset \text{Ann}(p_1) = h_0 \oplus h_1 \). So we have \( h_3 = 0 \) by its definition.

By the Jacobi identity, we have 
\[ 0 = [[h_0, h], p_1 \oplus p_2] + [[h, p_1 \oplus p_2], h_0] + [[p_1 \oplus p_2, h_0], h] \]
\[ = [[h_0, h], p_1 \oplus p_2] + [[h, p_1 \oplus p_2], h_0]. \]
Since \([h, p_1 \oplus p_2] \subset p_1 \oplus p_2 \) and then \([h, p_1 \oplus p_2], h_0] = 0 \), we have \([h_0, h], p_1 \oplus p_2 = 0 \) which implies that \([h_0, h] \subset h_0 \), i.e., \( h_0 \) is an ideal of \( h \). Similarly, we have that \([h_1, h] \subset \text{Ann}(p_1) \cap h \). Furthermore \( \langle [h_1, h], h_0 \rangle = \langle h_1, [h, h_0] \rangle = 0 \) since \( h_0 \) is an ideal of \( h \). It follows that \([h_1, h] \subset h_1 \), i.e., \( h_1 \) is also an ideal of \( h \). Similarly \( h_2 \) is an ideal of \( h \). Since \( g = h \oplus p_1 \oplus p_2 \) and \( h_0 \) annihilates \( p_1 \oplus p_2 \), \( h_0 \) is an ideal of \( g \). By the assumption that the \( G \) action is almost effective, we have \( h_0 = 0 \). So we have 
\[ g = h_1 \oplus h_2 \oplus p_1 \oplus p_2, \quad \text{and} \quad t^- = h_1 \oplus h_2 \oplus p_1, \quad t^+ = h_1 \oplus h_2 \oplus p_2. \]
We claim that \( \text{Lie}(p_1) = h_2 \oplus p_1 \) is an ideal in \( g \). In fact, \([h_2 \oplus p_1, h_1] = [h_2, h_1] = 0 \) and \([h_2 \oplus p_1, p_2] = 0 \) imply that \([\text{Lie}(p_1), g] \subset \text{Lie}(p_1) \). Similarly \( \text{Lie}(p_2) \) is also an ideal in \( g \). Therefore 
\[ G = L_2 \times L_1, \quad K^- = H_1 \times L_1, \quad K^+ = L_2 \times H_2, \quad \text{and} \quad H = H_1 \times H_2, \]
where \( h_i \) is the Lie algebra of \( H_i, \text{Lie}(p_i) \) is the Lie algebra of \( L_i \) for \( i = 1, 2 \) and \( L_1/H_2, L_2/H_1 \) are spheres. Hence the \( G \)-action is a sum action and the manifold \( M \) is \( G \)-equivariant to a sphere. 
\[ \square \]
Theorem 3.3. Suppose $M$ is a compact simply-connected manifold that admits a cohomogeneity one action with $s = 2$ and no fixed points. If the action is non-primitive, then the manifold is a double and the triples $H \subset K \subset G$ with $H$ connected are classified in Table 4.

Proof. The manifold $M$ has the diagram as $H \subset \{K^\pm = K\} \subset G$ where $K$ is a proper subgroup of $G$ and $K/H$ is a sphere.

First we classify the triples $G \supset K \supset H$ such that $G$ is simple, $H$ is connected, $K/H$ is a sphere and the isotropy representation $\text{Ad}_H$ of $G/H$ has two irreducible summands. It follows that $(G, K)$ and $(K, H)$ are strongly isotropy irreducible and the isotropy representation of $G/K$ remains irreducible when restricted to $H$. From the classification of transitive actions on spheres, the effective version of $K/H$ is one of $\text{Spin}(7)/\mathfrak{G}_2$, $\mathfrak{G}_2/\text{SU}(3)$ and $\text{SO}(n+1)/\text{SO}(n)$ with $n \geq 1$. Using the classification of compact irreducible symmetric spaces and J. Wolf’s classification, $G \supset K \supset H$ is one in the first part of Table 4. The last column contains further conditions. If a homogeneous space appears in this column, it means that the space is strongly isotropy irreducible, for example, “$G_1/H_1, G_2/H_2 = S^k$” means that both spaces are strongly isotropy irreducible and the second one is also a sphere.

The cohomogeneity one manifold defined by the diagram $H \subset \{K^\pm = K\} \subset G$ is a sphere bundle over the homogeneous spaces $G/K$ which is an irreducible symmetric space except for $\text{Sp}(16)/\text{Spin}(12)$. If $K/H$ is a circle, then one can add components to $H$. Then for each positive integer $n$, we have the cohomogeneity one diagram

$$H \cdot \mathbb{Z}^n \subset \{K^\pm = K\} \subset G.$$ 

If $G$ is not simple, then there are two constructions of such triples of connected groups $H \subset K \subset G$, see the last two examples in Table 4. In the first case, $G = G_1 \times G_2$, $K = H_1 \times G_2$ and $H = H_1 \times H_2$ where $G_1/H_1$ is a strongly isotropy irreducible space and $G_2/H_2$ is one of $\text{Spin}(7)/\mathfrak{G}_2$, $\mathfrak{G}_2/\text{SU}(3)$ and $\text{SO}(n+1)/\text{SO}(n)$ ($n \geq 1$). The cohomogeneity one manifold is the product of a sphere with the homogeneous space $G_1/H_1$. In the second and last cases, $G = H_0 \times G_1$, $K = H_0 \times H_0 \times H_1$ and $H = \Delta H_0 \times H_1$ where $H_0$ is either $U(1)$ or $\text{SU}(2)$ and $G_1/H_0 \times H_1$ is a strongly isotropy irreducible space. The diagram is reducible and its nonreducible version is $H_1 \subset \{H_0 \times H_1, H_0 \times H_1\} \subset G_1$. If $H_0$ is $U(1)$, then one can add components to $H$. Then for each positive integer $n$, we have the diagram

$$(\mathbb{Z}^n \cdot \Delta U(1)) \times H_1 \subset \{K^\pm = U(1) \times U(1) \times H_1\} \subset U(1) \times G_1.$$ 

Its non-reducible version is $\mathbb{Z}_n \cdot H_1 \subset \{K^\pm = U(1) \times H_1\} \subset G_1$. The manifolds defined by these diagrams are sphere($\mathbb{S}^2$ or $\mathbb{S}^4$) bundles over $G_1/(H_0 \times H_1)$. \qed
| G             | K             | H             |
|---------------|---------------|---------------|
| SO(8)         | Spin(7)       | G₂            |
| Spin(9)       | Spin(8)       | Spin(7)       |
| Spin(9)       | Spin(7) · SO(2) | G₂ × SO(2)   |
| SO(2n)        | U(n)          | SU(n)         | \(n \geq 2, n \neq 4\) |
| SU(5)         | U(4)          | U(1) · Sp(2)  |
| SU(p + q)     | S(U(p) × U(q))| SU(p) × SU(q) | \(p, q \geq 1, pq \geq 2\) |
| Sp(16)        | Spin(12)      | Spin(11)      |
| Sp(n)         | U(n)          | SU(n)         | \(n \geq 3\) |
| Sp(n + 1)     | Sp(n) × Sp(1) | Sp(n) × U(1)  | \(n \geq 1\) |
| E_6           | Spin(10) × SO(2) | Spin(10)     |
| E_6           | SU(6) × SU(2) | SU(6) × U(1)  |
| E_7           | E_6 × SO(2)   | E_6           |
| E_7           | Spin(12) × Sp(1) | Spin(12) × U(1) |
| E_7           | Spin(12) × Sp(1) | Spin(11) × Sp(1) |
| E_8           | Spin(16)      | Spin(15)      |
| E_8           | Sp(1) × E_7   | U(1) × E_7    |
| F_4           | Sp(3) × Sp(1) | Sp(3) × U(1)  |
| G_1 × G_2     | H_1 × G_2     | H_1 × H_2     | G_1/H_1, G_2/H_2 = S^k |
| U(1) × G_1    | U(1) × U(1) × H_1 | ∆U(1) × H_1  | G_1/(U(1)H_1) |
| SU(2) × G_1   | SU(2) × SU(2) × H_1 | ∆SU(2) × H_1 | G_1/(SU(2)H_1) |

Table 4. Group triple \(G \supseteq K \supseteq H\) such that \(H\) is connected, \(\text{Ad}_{G/H}\) has 2 irreducible summands and \(K/H\) is a sphere.

Remark 3.4. There are two examples in low dimensions. One example has the diagram

\[ \mathbb{Z}_n \cdot \text{SU}(2) \subset \{ K^\pm = \text{U}(2) \} \subset \text{SU}(3) \]

and the manifold is 6 dimension, see example \(N^6_p\) in [Ho]. The other one has the diagram

\[ \text{Sp}(1)\text{U}(1) \subset \{ K^\pm = \text{Sp}(1)\text{Sp}(1) \} \subset \text{Sp}(2) \]

and the manifold is 7 dimension, see example \(N^7_I\) in [Ho].
4. Special types of cohomogeneity one actions with \( s = 3 \)

From this section on we consider the classification when \( s = 3 \). In this section, we look at some special types of cohomogeneity one actions, i.e., reducible and non-primitive actions. The action of \( G \) is assumed to be effective or almost effective, i.e., the ineffective kernel is finite.

4.1. Reducible actions. The main result is

**Theorem 4.1.** If a simply-connected cohomogeneity one manifold \( M \) admits a reducible action without fixed points and \( s = 3 \), then one of the followings holds:

1. it admits a non-reducible action with \( s = 3 \);
2. the action is primitive and is a sum action on a sphere;
3. it is a double, i.e., \( K^- = K^+ \);
4. the action is non-primitive with different \( K^\pm \) and the manifold is a sphere, \( \mathbb{CP}^2 \), \( \mathbb{HP}^n \) or a three dimensional lens space bundle over a homogeneous space.

The cohomogeneity one manifolds in case (3) and (4) are classified in Table 12 and 13.

**Proof.** We prove the theorem in two steps. In step I, we classify cohomogeneity one diagrams with connected isotropy subgroups. In step II, we consider the possible ways to get new diagrams from those obtained in Step I, i.e., variations by automorphisms and adding connected components to isotropy groups. Here we only consider the inner automorphism, i.e., conjugation by a group element.

**Step I.** Suppose \( g = g_1 \oplus g_2 \) and we may assume that \( \text{Proj}_2(h) = g_2 \) and that the projection from \( h \) to any primitive factor of \( g_1 \) is not surjective.

Let \( h_i = g_i \cap h \) for \( i = 1, 2 \). If \( h = h_1 \oplus h_2 \), then from the reducibility assumption, we have that \( h_2 = g_2 \) and \( h_1 \) is a proper subalgebra of \( g_1 \). In this case, the non-reducible action by \( G_1 \) also has \( s = 3 \).

Next we assume that there exists a nonzero subspace \( h_0 \subset h \) such that the images under the two projections \( \text{Proj}_i \) are nonzero, i.e., \( h = h_1 \oplus \Delta h_0 \oplus h_2 \) and \( \text{Proj}_i(h) = h_i \oplus h_0 \) for \( i = 1, 2 \). From the reducibility assumption, we have \( g_2 = h_0 \oplus h_2 \). Since the action of \( G \) is effective, we have \( h_2 = 0 \). There is an intermediate subalgebra \( h_1 \oplus h_0 \oplus h_0 \) between \( h = h_1 \oplus \Delta h_0 \) and \( g = g_1 \oplus h_0 \) and the principal isotropy representation is

\[
\chi = (\text{ad}_{g_1} / (h_1 \oplus h_0)) \oplus (\text{Id}_{h_1} \otimes \text{ad}_{h_0}).
\]

It follows that \( h_0 \) has at most 2 primitive factors.

**Case I.** If \( h_0 \) has 2 primitive factors as \( h_0 = h_0' \oplus h_0'' \), then the pair \( (g_1, h_1 \oplus h_0) \) is strongly isotropy irreducible and thus \( g_1 \) is a simple Lie algebra or \( \mathfrak{so}(4) \). If \( g_1 = \mathfrak{so}(4) \), then
\( h_1 \oplus h_0 = \mathfrak{so}(2) \oplus \mathfrak{so}(2) \) and the dimension of the manifold is smaller or equal to 7. So we assume that \( g_1 \) is simple. The three irreducible summands are

\[
\chi_1 = \text{ad}_{g_1/(h_1 \oplus h_0)}, \quad \chi_2 = \text{Id}_{h_1} \otimes \text{ad}_{h_0} \otimes \text{Id}_{h_0'}, \quad \chi_3 = \text{Id}_{h_1} \otimes \text{Id}_{h_0'} \otimes \text{ad}_{h_0'},
\]

and their corresponding representation spaces are denoted by \( p_1, p_2 \) and \( p_3 \).

We claim that any two of the three representations are non-equivalent. In fact, if \( \chi_2 = \chi_3 \), then \( h_0' = h_0'' = \mathfrak{u}(1) \). From the classification of strongly isotropy irreducible spaces, such pair \((g_1, h_1 \oplus h_0)\) with \( g_1 \) simple does not exist. If \( \chi_1 \) is equivalent to one of \( \chi_2 \) or \( \chi_3 \), say \( \chi_2 \), then the isotropy representation of the pair \((g_1, h_1 \oplus h_0)\) is given by \( \zeta = \text{Id}_{h_1} \otimes \text{ad}_{h_0} \otimes \text{Id}_{h_0'} \).

It is clear that \( h_0' \neq \mathfrak{u}(1) \), and furthermore there is no strongly isotropy irreducible pair with isotropy representation as \( \zeta \).

We consider the intermediate subalgebra \( \mathfrak{t} \). If it contains the subspace \( p_1 \), then it is one of

1. \( h \oplus p_1 \), and then \([p_1, p_1] \subset h_1 \oplus p_1 \), i.e., \( l = h_1 \oplus p_1 \) is a Lie algebra;
2. \( h_1 \oplus h_0' \oplus h_0' \oplus \Delta h_0'' \oplus p_1 \), and \([p_1, p_1] \subset h_1 \oplus h_0' \oplus p_1 \);
3. \( h_1 \oplus \Delta h_0' \oplus h_0' \oplus h_0'' \oplus p_1 \), and \([p_1, p_1] \subset h_1 \oplus h_0' \oplus p_1 \).

In all above cases, \((g_1, h_1 \oplus h_0)\) is not a symmetric pair.

In Case (1), let \( G_1, H' \) and \( L \) be the corresponding Lie groups of \( g_1, h_1 \oplus h_0 \) and \( l \), and then the subgroup \( L \) acts transitively on the homogeneous space \( G_1/H' \). A. L. Onishchik classified all triples \((L_1, L_2, L_3)\) such that \( L_1 \) is simple, and \( L_2 \) is a subgroup of \( L_1 \) and acts transitively on the homogeneous space \( L_1/L_3 \), see [GO], p. 143 Theorem 4.5, or §2 in [DZ].

This classification is also used in [KS] and the following Table 5 is part of Table 3 in the appendix of their paper where \( L_1/L_3 \) is not a symmetric space.

From the classification, \((\text{SO}(4n), \text{SO}(4n - 1), \text{Sp}(1)\text{Sp}(n))(n \geq 2)\) is the only triple such that \( L_1/L_3 \) is strongly isotropy irreducible and \( L_3 \) has at least two primitive factors. If \( g_1 = \mathfrak{so}(4n) \) and \( h_0 = \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \), then \( h_1 = 0 \) and \( l = p_1 \) would be \( \mathfrak{so}(4n - 1) \) which would imply that \( 4n - 1 = 3 + \dim \mathfrak{sp}(n) \) and it gives us a contradiction.

In Case (2), \( h_1 \oplus h_0' \oplus p_1 \) is a subalgebra of \( g_1 \). From a similar argument as in the previous case, we have that \( g_1 = \mathfrak{so}(4n)(n \geq 2) \), \( h_0 = \mathfrak{sp}(1) \oplus \mathfrak{sp}(n) \), \( h_1 = 0 \) and \( \dim p_1 = \dim \mathfrak{so}(4n) - \dim \mathfrak{sp}(1) - \dim \mathfrak{sp}(n) = 6n^2 - 3n - 3 \). However it is not equal to either \( \dim \mathfrak{so}(4n - 1) - \dim \mathfrak{sp}(1) = 8n^2 - 6n - 2 \) or \( \dim \mathfrak{so}(4n - 1) - \dim \mathfrak{sp}(n) = 6n^2 - 7n + 1 \) for \( n \geq 2 \). So \( \mathfrak{t} \) is not in this case. A similar argument also show that \( \mathfrak{t} \) is not in Case (3).

Now we assume that \( \mathfrak{t} \) does not contain the subspace \( p_1 \), then it is one of

1. \( h_1 \oplus h_0 \oplus h_0' \);
2. \( h_1 \oplus h_0' \oplus \Delta h_0'' \);
| $L_1$ | $L_2$ | $L_3$ | $L_2 \cap L_3$ |
|-------|-------|-------|----------------|
| SO$(4n)$ | SO$(4n - 1)$ | Sp$(n)$ | Sp$(n - 1)$ |
| SO$(4n)$ | SO$(4n - 1)$ | Sp$(n)U(1)$ | Sp$(n - 1)U(1)$ |
| SO$(4n)$ | SO$(4n - 1)$ | Sp$(n)Sp(1)$ | Sp$(n - 1)Sp(1)$ |
| SO$(2n)$ | SO$(2n - 1)$ | SU$(n)$ | SU$(n - 1)$ |
| SO$(16)$ | SO$(15)$ | Spin$(9)$ | Spin$(7)$ |
| SO$(8)$ | Spin$(7)$ | SO$(6)$ | SU$(3)$ |
| SO$(8)$ | Spin$(7)$ | SO$(5)$ | SU$(2)$ |
| SO$(8)$ | Spin$(7)$ | SO$(2)SO(5)$ | SO$(2)SU(2)$ |
| SO$(7)$ | $\mathfrak{g}_2$ | SO$(5)$ | SU$(2)$ |

Table 5. Onishchik’s triples ($L_1, L_2, L_3$) with $L_1$ simple and $L_1/L_3$ non-symmetric

(3) $h_1 \oplus \Delta h'_0 \oplus h''_0 \oplus h''''_0$.

In Case (1), $(\mathfrak{k}, \mathfrak{h})$ is not a spherical pair. If both $\mathfrak{t}^\pm$ are in Case (2), then $h'_0$ is either $u(1)$ or $su(2)$ and the diagram is not primitive. This gives us example $R.1 (m = 1)$ and $R.2$.

If both $\mathfrak{t}^\pm$ are in Case (3), then we have a similar result. If $\mathfrak{t}^-$ is in Case (2) and $\mathfrak{t}^+$ is in Case (3), then $h'_0$ and $h''_0$ are $u(1)$ or $su(2)$. From the classification of strongly isotropy irreducible spaces, both $h'_0$ and $h''_0$ cannot be $u(1)$. If both $h'_0$ and $h''_0$ are $su(2)$, though the embeddings of $\mathfrak{t}^\pm \subset \mathfrak{g}$ are different, the manifold is equivariant diffeomorphic to the one in the previous example. If $h'_0 = u(1) \oplus su(2)$, then we have example $R.14 (m = 1)$.

**Case II.** If $h_0$ is primitive, then the isotropy representation of the pair $(g_1, h_1 \oplus h_0)$ has 2 irreducible summands $\chi_1, \chi_2$ and their representation spaces are denoted by $p_1$ and $p_2$. The representation space of $\chi_3 = Id_{h_0} \otimes ad_{h_0}$ is denoted by $p_3$. From the assumption that the projection from $h$ to any primitive factor of $g_1$ is not surjective, $g_1$ has at most two primitive factors.

**Case II.A.** We first consider the case when $g_1$ has two factors $g'_1$ and $g''_1$, and then we may assume that $g'_1 = h'_1 \oplus p_1$ and $g''_1 = h''_1 \oplus h_0 \oplus p_2$ where $h'_1 = h'_1 \oplus h''_1$. The only possible pair of equivalent summands are $\chi_1$ and $\chi_3$. If we are in this case, then $h'_1 = 0$, $g'_1 = u(1)$, $h_0 = u(1)$ and $g''_1 = su(2)$. Thus the manifold is 5 dimensional. Now we assume that $\chi_i$’s are pairwisely non-equivalent. $h \oplus p_2$ and $h \oplus p_1 \oplus p_2$ are not subalgebras of $g$ otherwise $p_2$ would be a subalgebra of $g''_1$. Furthermore the intermediate subalgebra cannot be $h \oplus p_3 \oplus p_1 = g'_1 \oplus h''_1 \oplus h_0 \oplus h_0$ since $(g'_1 \oplus h''_1 \oplus h_0 \oplus h_0, h'_1 \oplus h''_1 \oplus \Delta h_0)$ is not a spherical pair. So the intermediate subalgebra $\mathfrak{t}$ is one of the followings:

(1) $h \oplus p_3 \oplus p_2 = h'_1 \oplus g''_1 \oplus h_0$, and then $(h'_1 \oplus g''_1 \oplus h_0, h_1 \oplus \Delta h_0)$ is a spherical pair;
(2) $\mathfrak{h} \oplus \mathfrak{p}_3 = \mathfrak{h}_1 + \mathfrak{h}_0 + \mathfrak{h}_0$, and then $\mathfrak{h}_0$ is either $u(1)$ or $su(2)$;

(3) $\mathfrak{h} \oplus \mathfrak{p}_1 = \mathfrak{g}'_1 \oplus \mathfrak{h}'_1 \oplus \Delta \mathfrak{h}_0$, and then $(\mathfrak{g}'_1, \mathfrak{h}'_1)$ is a strongly isotropy irreducible spherical pair.

In Case (1) the spherical is $(\mathfrak{u}(n+1), \mathfrak{u}(n))$ with $\mathfrak{h}_0 = \mathfrak{u}(1)$ or $(\mathfrak{sp}(n+1) \oplus \mathfrak{sp}(1), \mathfrak{sp}(n) \oplus \Delta \mathfrak{sp}(1))$ with $\mathfrak{h}_0 = \mathfrak{sp}(1)$. For the first pair, since the sub-action by $G_1 \times SU(n+1) \subset G_1 \times U(n+1)$ also has $s = 3$ if $n \geq 2$, we only consider the pair $(\mathfrak{U}(2), \mathfrak{U}(1))$. If both $\mathfrak{t}^\pm$ are in Case (2), then we have example $R.3$ and $R.4$.

In Case (2), if $\mathfrak{h}_0 = \mathfrak{u}(1)$, then $G''/H_1\mathfrak{U}(1)$ is strongly isotropy irreducible. If $\mathfrak{h}_0 = \mathfrak{su}(2)$, then $G''/H_1\mathfrak{SU}(2)$ is strongly isotropy irreducible. If both $\mathfrak{t}^\pm$ are in this case, then we have example $R.5(m = 1)$ and $R.6$.

If both $\mathfrak{t}^\pm$ are in Case (3), then we have example $R.7$ and $R.8$. The special case where $(G_1, H_1) = (\mathfrak{U}(1), \{1\})$ and $H_0 = \mathfrak{U}(1)$ will be discussed in Step II and it gives us example $R.11$, $R.12$, $R.22$ and $R.23$.

If $\mathfrak{t}^\pm$ are in Case (1) and (3), then the diagram is the sum action on a sphere. For other cases, we have example $R.15(m = 1)$, $R.16$, $R.17(m = 1)$ and $R.18$.

Case II.B. Next we consider the case when $g_1$ is primitive. We claim that any two of $\chi_i$'s are not equivalent. If not, then we have two different cases. First if $\chi_1$ is equivalent to $\chi_2$, then $(g_1, h_1 \oplus h_0)$ is either $(\mathfrak{so}(8), \mathfrak{g}_2)$ or $(\mathfrak{so}(7), \mathfrak{u}(3))$. In the first case, $g = \mathfrak{so}(8) \oplus \mathfrak{g}_2$ and $h = \Delta \mathfrak{g}_2$. However there is no intermediate subalgebra $\mathfrak{t}$ such that $(\mathfrak{t}, h)$ is a spherical pair. In the second case, $g = \mathfrak{so}(7) \oplus \mathfrak{so}(3)$, $h = \mathfrak{u}(1) \oplus \Delta \mathfrak{su}(3)$ and there is no intermediate Lie algebra $\mathfrak{t}$ such that $(\mathfrak{t}, h)$ is a spherical pair. Secondly if $\chi_3$ is equivalent to one of $\chi_1$ and $\chi_2$, say $\chi_2$, then $l = h_1 \oplus h_0 \oplus h_0$ is an intermediate algebra between $g_1$ and $h_1 \oplus h_0$. The Lie algebra $h_0$ embeds diagonally into $l$, $(g_1, l)$ is a strongly isotropy irreducible pair and the isotropy representation $ad_{g_1/l}$ remains irreducible when restricted to $h_1 \oplus h_0$. However there is no such pair $(g_1, l)$ that satisfies these properties.

Now we know that the $p_i$'s are pairwisely non-equivalent. There are 6 different possibilities for the intermediate subalgebra $\mathfrak{t}$:

- (II.B.1) $h_1 \oplus \Delta h_0 \oplus p_1 \oplus p_2$;
- (II.B.2) $h_1 \oplus h_0 \oplus p_1 \oplus h_0$;
- (II.B.3) $h_1 \oplus h_0 \oplus p_2 \oplus h_0$;
- (II.B.4) $h_1 \oplus \Delta h_0 \oplus p_1$;
- (II.B.5) $h_1 \oplus \Delta h_0 \oplus p_2$;
- (II.B.6) $h_1 \oplus h_0 \oplus h_0$.

If $\mathfrak{t}$ is in Case (II.B.1), then $l = h_1 \oplus p_1 \oplus p_2$ is a subalgebra of $g_1$ and then $g_1 = l \oplus h_0$ is not primitive.

If $\mathfrak{t}$ is in Case (II.B.2), then let $l = h_1 \oplus h_0 \oplus p_1$ which is a Lie subalgebra of $g_1$ and $(g_1, l)$ is a strongly isotropy irreducible pair. If $l$ is primitive, then it is either $su(n+1)$ or $sp(n+1)(n \geq 1)$ since $(\mathfrak{t} = l \oplus h_0, h_1 \oplus \Delta h_0)$ is a spherical pair for which $ad_{l/h}$ has two irreducible summands. If $l = su(n+1)$, then $h_1 = su(n)$ and $h_0 = u(1)$. Moreover $(g_1, su(n+1))$
is strongly isotropy irreducible and its isotropy representation remains irreducible when restricted to \( u(n) \). It follows that it is one of

\[
    u(3) \subset so(6) \subset so(7), \quad u(7) \subset su(8) \subset e_7.
\]

However for each triple above, \( \text{ad}_{u_1/h_1} \) also has 3 irreducible summands, i.e., the non-reducible action by \( G_1 \) also has \( s = 3 \). If \( I = sp(n + 1) \), then \( h_1 = sp(n) \), \( h_0 = sp(1) \) and the isotropy representation of \((g_1, sp(n + 1))\) remains irreducible when restricted to \( sp(n) \oplus sp(1) \). However such \( g_1 \) does not exist.

If \( I \) is not primitive, then the effective version of the spherical pair \((I \oplus h_0, h_1 \oplus \Delta h_0)\) is either \((sp(n + 1), sp(n) \oplus \Delta sp(1))\) with \( h_0 = sp(1) \) or \((u(n + 1), u(n))\) with \( h_0 = u(1) \). In the first case, we have that \( I = sp(n + 1) \oplus I_0 \) for some nonzero Lie algebra \( I_0 \) and the isotropy representation of \((g_1, sp(n + 1) \oplus I_0)\) remains irreducible when restricted to \( sp(n) \oplus sp(1) \oplus I_0 \). However such \( g_1 \) does not exist.

In the second case, we have that \( I = su(n + 1) \oplus I_0 \) for some nonzero Lie algebra \( I_0 \) and the isotropy representation of \((g_1, su(n + 1) \oplus I_0)\) remains irreducible when restricted to \( u(n) \oplus I_0 \). It follows that \( u(n) \oplus I_0 \subset su(n + 1) \oplus I_0 \subset g_1 \) is one of the following triples:

\[
    u(1) \oplus sp(n) \subset sp(1) \oplus sp(n) \subset sp(n + 1), \quad u(2) \subset so(4) \subset f_2, \\
    u(1) \oplus sp(3) \subset sp(1) \oplus sp(3) \subset f_4, \quad u(1) \oplus su(6) \subset su(2) \oplus su(6) \subset e_6, \\
    u(5) \oplus su(2) \subset su(6) \oplus su(2) \subset e_6, \quad u(1) \oplus so(12) \subset sp(1) \oplus so(12) \subset e_7, \\
    u(1) \oplus e_7 \subset su(2) \oplus e_7 \subset e_8,
\]

and the corresponding triples \( h_1 \oplus \Delta h_0 \subset k \subset g_1 \oplus h_0 \) are

1. \( sp(n) \oplus \Delta u(1) \subset sp(n) \oplus sp(1) \oplus u(1) \subset sp(n + 1) \oplus u(1) \),
2. \( u(2) \subset su(2) \oplus u(1) \subset f_2 \oplus u(1) \),
3. \( sp(3) \oplus \Delta u(1) \subset sp(3) \oplus sp(1) \oplus u(1) \subset f_4 \oplus u(1) \),
4. \( su(6) \oplus \Delta u(1) \subset su(6) \oplus su(2) \oplus u(1) \subset e_6 \oplus u(1) \),
5. \( su(2) \oplus u(5) \subset su(2) \oplus su(6) \oplus u(1) \subset e_6 \oplus u(1) \),
6. \( so(12) \oplus \Delta u(1) \subset su(12) \oplus sp(1) \oplus u(1) \subset e_7 \oplus u(1) \),
7. \( e_7 \oplus \Delta u(1) \subset e_7 \oplus su(2) \oplus u(1) \subset e_8 \oplus u(1) \).

In Case (2) above, there is no corresponding group triple. The non-reducible version of the triple in Case (5) is \( su(2) \oplus su(5) \subset su(2) \oplus su(6) \subset e_6 \) and its isotropy representation also has 3 irreducible summands. The group triples \( H \subset K \subset G \) of the remaining cases are

1. \( Sp(n) \Delta U(1) \subset Sp(n) \times Sp(1) \times U(1) \subset Sp(n + 1) \times U(1) \) with \( n \geq 1 \);
2. \( Sp(3) \Delta U(1) \subset Sp(3) \times Sp(1) \times U(1) \subset F_4 \times U(1) \);
3. \( SU(6) \Delta U(1) \subset SU(6) \times SU(2) \times U(1) \subset E_6 \times U(1) \);
4. \( Spin(12) \Delta U(1) \subset Spin(12) \times SU(2) \times U(1) \subset E_7 \times U(1) \);
(5) \( E_7 \Delta U(1) \subset E_7 \times SU(2) \times U(1) \subset E_8 \times U(1) \).

The discussion in Case (II.B.3) is similar to Case (II.B.2).

If \( e \) is in Case (II.B.4), then \( h_1 \oplus p_1 \) is a Lie algebra and \((h_1 \oplus p_1, h_1)\) is a spherical pair with irreducible isotropy representation. Furthermore, the pair \((g_1, h_1 \oplus p_1 \oplus h_0)\) is strongly isotropy irreducible and its isotropy representation remains irreducible when restricted to \( h_1 \oplus h_0 \). First we have the following possibilities of \((g_1, h_1 \oplus p_1 \oplus h_0)\) for which the pair is strongly isotropy irreducible and the subalgebra is not primitive:

(1) \((su(p + q), S(u(p) \oplus u(q)))\) with \( p, q \geq 1 \),
(2) \((so(p + q), so(p) \oplus so(q))\) with \( p, q \geq 1 \),
(3) \((sp(p + q), sp(p) \oplus sp(q))\) with \( p, q \geq 1 \),
(4) \((sp(n), u(n))\) with \( n \geq 1 \)
(5) \((so(2n), u(n))\) with \( n \geq 3 \),
(6) \((g_2, so(4))\),
(7) \((f_4, sp(3) \oplus sp(1))\),
(8) \((e_6, so(10) \oplus so(2))\),
(9) \((e_6, su(6) \oplus su(2))\),
(10) \((e_7, e_6 \oplus so(2))\),
(11) \((e_7, so(12) \oplus su(2))\),
(12) \((e_8, e_7 \oplus su(2))\),
(13) \((su(4), so(4))\),
(14) \((su(pq), su(p) \oplus su(q))\) with \( p, q \geq 2, pq \geq 5 \),
(15) \((f_4, g_2 \oplus so(3))\),
(16) \((f_4, su(3) \oplus su(3))\),
(17) \((e_6, su(3) \oplus g_2)\),
(18) \((e_6, su(3) \oplus su(3) \oplus su(3))\),
(19) \((e_7, sp(3) \oplus g_2)\),
(20) \((e_7, so(3) \oplus f_4)\),
(21) \((e_7, su(3) \oplus su(6))\),
(22) \((e_8, su(3) \oplus e_6)\),
(23) \((e_8, g_2 \oplus f_4)\),
(24) \((so(4n), sp(1) \oplus sp(n))\) with \( n \geq 2 \),
(25) \((sp(n), sp(1) \oplus so(n))\) with \( n \geq 3 \).

The first 13 cases are from the symmetric spaces and the rest are from Wolf’s list. Next we consider the triples \( h_1 \oplus h_0 \subset h_1 \oplus p_1 \oplus h_0 \subset g_1 \) and they are the followings:

(1) \( su(p) \oplus su(q) \subset S(u(p) \oplus u(q)) \subset su(p + q) \) with \( p, q \geq 1, pq \geq 2 \),
(2) \( so(n) \oplus g_2 \subset so(n) \oplus so(7) \subset so(n + 7) \) with \( n \geq 2 \),
(3) \( sp(n) \oplus u(1) \subset sp(n) \oplus sp(1) \subset sp(n + 1) \) with \( n \geq 1 \),
(4) \( sp(n) \oplus so(4) \subset sp(n) \oplus sp(2) \subset sp(n + 2) \) with \( n \geq 1 \),
(5) \( su(n) \subset u(n) \subset sp(n) \) with \( n \geq 1 \),
(6) \( u(1) \oplus so(5) \subset u(4) \subset sp(4) \),
(7) \( su(n) \subset u(n) \subset so(2n) \) with \( n \geq 3 \),
(8) \( u(1) \oplus sp(3) \subset sp(1) \oplus sp(3) \subset f_4 \),
(9) \( so(10) \subset so(10) \oplus so(2) \subset e_6 \),
(10) \( so(9) \oplus so(2) \subset so(10) \oplus so(2) \subset e_6 \),
(11) \( u(1) \oplus su(6) \subset su(2) \oplus su(6) \subset e_6 \),
(12) \( e_6 \subset e_6 \oplus so(2) \subset e_7 \).
therefore takes into account the pairs \((g_1, h_1 + h_0)\) has two irreducible summands. Therefore, there are many examples in this case.

We summarize the group triples in Case (II.B.2) and (II.B.4) in Table 6. We consider the construction of the cohomogeneity one diagram.

If none of the triples \(H \subseteq K \subseteq G\) is in Case (II.B.6), then from the fact that the three summands are pairwise non-equivalent, each of them should be in Table 6. It is easy to  

\[ H = \text{Spin}(n) \times G_2 \subset K = \text{SO}(n) \times \text{SO}(7) \subset \text{Spin}(n + 7). \]

If \(n \geq 3\), then \(K\) is not simply-connected and its two-fold cover is \(\text{Spin}(n) \times \text{Spin}(7)\). It follows that \(K/H\) is the real projective space \(\mathbb{RP}^7\). For other cases, we list the group triples \(H \subseteq K \subseteq G\) below:

1. \(SU(p) \times \Delta SU(q) \subset U(1) \cdot SU(p) \times \Delta SU(q) \subset SU(p + q) \times SU(q)\),
2. \(G_2 \times \Delta SO(2) \subset \text{Spin}(7) \times \Delta SO(2) \subset \text{Spin}(9) \times SO(2)\),
3. \(U(1) \times \Delta Sp(n) \subset Sp(1) \times \Delta Sp(n) \subset Sp(n + 1) \times Sp(n)\),
4. \(\Delta Sp(n) \subset U(1) \times \Delta SU(n) \subset Sp(n) \times SU(n)\),
5. \(\Delta U(1) \subset SU(4) \times \Delta U(4) \subset U(4) \times U(1)\),
6. \(\Delta SU(n) \subset U(1) \times \Delta SU(n) \subset SO(2n) \times SU(n)\),
7. \(\Delta Sp(3) \subset Sp(1) \times \Delta Sp(3) \subset F_4 \times Sp(3)\),
8. \(\Delta Spin(10) \subset SO(2) \times \Delta Spin(10) \subset E_6 \times Spin(10)\),
9. \(\Delta SO(2) \subset SU(10) \times \Delta SO(2) \subset E_6 \times SO(2)\),
10. \(U(1) \times \Delta SU(6) \subset SU(2) \times \Delta SU(6) \subset E_6 \times SU(6)\),
11. \(\Delta E_6 \subset SO(2) \times \Delta E_6 \subset E_7 \times E_7\).

The discussion in Case (II.B.5) is similar to Case (II.B.4). If \(t\) is in Case (II.B.6), then \(h_0\) is either \(u(1)\) or \(su(2)\) and the isotropy representation of the pair \((g_1, h_1 + h_0)\) has two irreducible summands. There are many examples in this case.

We summarize the group triples in Case (II.B.2) and (II.B.4) in Table 6. We consider the construction of the cohomogeneity one diagram.

If none of the triples \(H \subseteq K \subseteq G\) is in Case (II.B.6), then from the fact that the three summands are pairwise non-equivalent, each of them should be in Table 6. It is easy to
see that in Table 6, for any given pair \((G, H)\), there is only one intermediate subgroup \(K\). This gives us example \(R.13(m = 1)\).

If \(H \subset K^+ \subset G\) is in Case (II.B.6), but \(H \subset K^- \subset G\) is not in Case (II.B.6), then we have example \(R.19, R.20(m = 1)\) and \(R.21\).

If both \(K^\pm\) are in Case (II.B.6), then we have example \(R.9(m = 1)\) and \(R.10\).

**Table 6.** Group triple \(G \supseteq K \supseteq H\) in Case II.B.2 and II.B.4

| \(G\) | \(K\) | \(H\) |
|-------|-------|-------|
| \(\text{Sp}(n + 1) \times U(1)\) | \(\text{Sp}(n) \times \text{Sp}(1) \times U(1)\) | \(\text{Sp}(n) \Delta U(1)\) |
| \(F_4 \times U(1)\) | \(\text{Sp}(3) \times \text{Sp}(1) \times U(1)\) | \(\text{Sp}(3) \Delta U(1)\) |
| \(E_6 \times U(1)\) | \(\text{SU}(6) \times \text{SU}(2) \times U(1)\) | \(\text{SU}(6) \Delta U(1)\) |
| \(E_7 \times U(1)\) | \(\text{Spin}(12) \times \text{SU}(2) \times U(1)\) | \(\text{Spin}(12) \Delta U(1)\) |
| \(E_8 \times U(1)\) | \(E_7 \times \text{SU}(2) \times U(1)\) | \(E_7 \Delta U(1)\) |
| \(\text{Spin}(9) \times U(1)\) | \(\text{Spin}(7) \times \Delta U(1)\) | \(\mathfrak{G}_2 \Delta U(1)\) |
| \(\text{Sp}(4) \times U(1)\) | \(\text{SU}(4) \times \Delta U(1)\) | \(\text{Sp}(2) \Delta U(1)\) |
| \(E_6 \times U(1)\) | \(\text{Spin}(10) \times \Delta U(1)\) | \(\text{Spin}(9) \Delta U(1)\) |
| \(\text{SU}(p + q) \times \text{SU}(q)\) | \(\text{U}(1) \cdot \text{SU}(p) \times \Delta \text{SU}(q)\) | \(\text{SU}(p) \times \Delta \text{SU}(q)\) |
| \(\text{Sp}(n + 1) \times \text{Sp}(n)\) | \(\text{Sp}(1) \times \Delta \text{Sp}(n)\) | \(\text{U}(1) \Delta \text{Sp}(n)\) |
| \(\text{Sp}(n + 2) \times \text{Sp}(n)\) | \(\text{Sp}(2) \times \Delta \text{Sp}(n)\) | \(\text{Sp}(1) \times \text{Sp}(1) \Delta \text{Sp}(n)\) |
| \(\text{Sp}(n) \times \text{SU}(n)\) | \(\text{U}(1) \times \Delta \text{SU}(n)\) | \(\text{SU}(n)\) |
| \(E_7 \times \text{SU}(2)\) | \(\text{Spin}(12) \times \Delta \text{SU}(2)\) | \(\text{Spin}(11) \Delta \text{SU}(2)\) |
| \(\text{SO}(2n) \times \text{SU}(n)\) | \(\text{U}(1) \times \Delta \text{SU}(n)\) | \(\Delta \text{SU}(n)\) |
| \(F_4 \times \text{Sp}(3)\) | \(\text{Sp}(1) \times \Delta \text{Sp}(3)\) | \(\text{U}(1) \Delta \text{Sp}(3)\) |
| \(E_6 \times \text{Spin}(10)\) | \(\text{U}(1) \times \Delta \text{Spin}(10)\) | \(\Delta \text{Spin}(10)\) |
| \(E_6 \times \text{SU}(6)\) | \(\text{SU}(2) \times \Delta \text{SU}(6)\) | \(\text{U}(1) \Delta \text{SU}(6)\) |
| \(E_7 \times E_6\) | \(\text{U}(1) \times \Delta E_6\) | \(\Delta E_6\) |
| \(E_7 \times \text{Spin}(12)\) | \(\text{SU}(2) \times \Delta \text{Spin}(12)\) | \(\text{U}(1) \Delta \text{Spin}(12)\) |
| \(E_8 \times E_7\) | \(\text{SU}(2) \times \Delta E_7\) | \(\text{U}(1) \Delta E_7\) |
strongly isotropy irreducible. The simplest case is when \( G_2 = SU(2) \) and \( H = \Delta U(1) \). The manifold is 5 dimensional and the isotropy representation is \([\phi|_\mathbb{R} \oplus \text{id} \oplus \text{id}]\).

It is easy to see that the action is not a product (see Section 1.5.1 in [Ho]) since the manifold is simply-connected. In dimension 5, if the action is not a product or a sum action, or fixed points free, then the non-reducible diagrams have \( G = SU(2) \times U(1) \) and \( H_c = \{1\} \). Such diagrams were classified in [Ho] and most of them have disconnected \( H \). If a \( U(1) \subset SU(2) \) normalizes \( H \) and \( K^\pm \), then one can extend the diagram \( H \subset \{K^\pm\} \subset U(1) \times SU(2) \) to a reducible one with \( G = U(1) \times SU(2) \times U(1) \) and \( H_c = \Delta U(1) \) via normal extension. In example \( N^5, Q^5_4 \) and \( Q^5_2 \) in [Ho], one can take \( L = U(1) = \{e^{i\theta}\} \) to extend the action to \( U(1) \times SU(2) \times U(1) \). Note that example \( Q^5_2 \) is a primitive action and it is a sum action on \( S^5 \). However such extension does not exist for the example \( P^5 \) and \( Q^5_B \).

Next we consider other examples, i.e. \( H_2 \) is not a trivial group. The non-reducible diagrams have \( G = U(1) \times G_2 \) and \( H = H_2 \). If the isotropy representation of \( G_2/H_2 \) has two irreducible summands, then the non-reducible diagram by \( U(1) \times G_2 \) also has \( s = 3 \). So we only consider the pairs \((G_2, H_2)\) where the isotropy representation \( G_2/H_2 \) has more than two irreducible summands, and they are given by \((SO(n + 2), SO(n)) (n \geq 2)\) and their finite covers. The connected components of \( K^\pm \) are given by \( U(1) \cdot H_2 \) and the embeddings of the \( U(1) \) factor into \( U(1) \times G_2 \) are different. The proper subgroup \( L = U(1) \times SO(2) \times SO(n) \subset G \) contains both \( K_c^\pm \). We may assume that \( G_2 \) is simply-connected by lifting the action to its universal covering if necessary. The diagram in this case was discussed in [Ho], see Lemma 4.3. There are two different classes of cohomogeneity one diagrams. In one class, example \( R.12 \), the diagrams are doubles with disconnected \( H \) and the manifolds are \( S^2 \) bundles over the Stiefel manifold \( SO(n + 2)/SO(n) \). In another class, example \( R.23 \), the diagrams are non-primitive and the manifolds are bundles over \( SO(n + 2)/(SO(2) \times SO(n)) \) with fiber a three dimensional lens space. In both classes, since \( U(1) \times SO(2) \) normalizes \( K^\pm \) and \( H \), one can extend the actions by \( U(1) \times SO(n + 2) \) to reducible actions by \( U(1) \times SO(n + 2) \times U(1) \) such that the principal isotropy representation has three irreducible summands.

We illustrate the construction in this case by the pair \((G_2, H_2) = (SO(5), SO(3))\).

Suppose \( \{\beta(\theta) = e^{2\pi i \theta} | 0 \leq \theta \leq 1\} \) is the circle group \( SO(2) \subset SO(2) \times SO(3) \subset SO(5) \) and \( \{\alpha(t) = e^{2\pi i t} | 0 \leq t \leq 1\} \) is the \( SO(2) \) factor in \( G = SO(2) \times SO(5) \). Let
\[
K_c^\pm = \left\{ \left( \alpha(n_\pm \theta), \left( \frac{\beta(m_\pm \theta)}{A} \right) \right) | 0 \leq \theta \leq 1, A \in SO(3) \right\} \subset SO(2) \times SO(2) \times SO(3)
\]
be the identity components of \( K^\pm \) where \( m_\pm \) and \( n_\pm \) are integers. To obtain a diagram which defines a double, let \( n_\pm = 1 \) and \( m_\pm = m \), and for any integer \( k \) let \( \mathbb{Z}_k \subset \{(\alpha(\theta), \beta(m \theta))\} \) be a cyclic group. Then a double has the diagram \( H = \mathbb{Z}_k \cdot SO(3) \subset \{K^\pm = K_c^\pm\} \subset SO(2) \times SO(5) \).
We consider the diagram in the second class which is not a double. To simplify the discussion, we assume that $n_\pm = 1$ and then $m_+ \neq m_-$. Suppose $\mathbb{Z}_{m_\pm}$ is the cyclic group generated by $(\alpha(1/m_\pm), I_5)$ and $\mathbb{Z}_m$ by $(\alpha(1/m), I_5)$ where $m$ is the least common multiple of $m_\pm$. Let $H_\pm = \mathbb{Z}_{m_\pm} \cdot \text{SO}(3)$ and

$$H = H_+ \cdot H_- = \mathbb{Z}_m \cdot \text{SO}(3), \quad K^\pm = K_c \cdot H = K^\pm_c \cdot \mathbb{Z}_{m/m_\pm},$$

then the diagram $H \subset \{K^\pm\} \subset G = \text{SO}(2) \times \text{SO}(5)$ defines a simply-connected cohomogeneity one manifold $M$. Since both $K^\pm$ is contained in $L = \text{SO}(2) \times \text{SO}(2) \times \text{SO}(3)$, $M$ is a fiber bundle over the space $G/L = \text{SO}(5)/(\text{SO}(2) \times \text{SO}(3))$ and the $L$ action on the fiber is also cohomogeneity one. The action is not effective and the effective one has the diagram

$$\{1\} \subset \{S^1_{m_-, m_+}, S^1_{m_-, m_+}\} \subset \mathbb{T}^2,$$

where $S^1_{p,q}$ is embedded in $\mathbb{T}^2$ as $(e^{2\pi p\theta}, e^{2\pi q\theta})(0 \leq \theta \leq 1)$. Using the van Kampen Theorem, the fibre is a lens space with fundamental group $\pi_1 = \mathbb{Z}_{|m_+-m_-|}$, see [[Na] and Proposition 1.8 in [Ho]].

In the following let $H \subset \{K^\pm\} \subset G$ be a diagram in Table 12 and 13 and we assume that the three summands are non-equivalent. It follows that one cannot obtain a new diagram by conjugating the original one by an element in $N_G(H)$. If one singular orbit is codimension 2, say $K^-/H = S^1$, then one may add connected components to isotropy subgroups. In the case where the diagram is a double, i.e., $K^- = K^+$, then we have the diagram $H \cdot \mathbb{Z}_m \subset \{K^\pm\} \subset G$ for every $m \in \mathbb{Z}$. Note that in example $R.8$, if one add connected components, then the action is not effective and its effective version is the original one with connected $H$. If the diagram $\Gamma$ is not a double, then it is one of the example $R.14$, $R.15$, $R.17$, $R.19$ and $R.20$. Except for example $R.15$ and $R.19$, there exists a proper subgroup $L$ contains $K^\pm$ and the diagram $H \subset \{K^\pm\} \subset L$ is a sum action on a sphere. For each $m \in \mathbb{Z}$ we have the diagram $H \cdot \mathbb{Z}_m \subset \{K^-, K^+ \cdot \mathbb{Z}_m\} \subset G$ and $K^+ \cdot \mathbb{Z}_m$ is contained in $L$. In example $R.15$ and $R.19$, $K^+$ contains $K^-$ which implies that one can not add components to the isotropy subgroups. This finishes the proof of Theorem 4.1.

4.2. Non-primitive actions. We assume that the diagram is non-reducible and the main result is

**Theorem 4.2.** Suppose a compact simply-connected manifold $M$ admits a cohomogeneity one action with diagram $H \subset \{K^\pm\} \subset G$ and $s = 3$. If the action is non-primitive and non-reducible, then

1. either the manifold is a double, i.e., $K^- = K^+$,
2. or it is equivariantly diffeomorphic to one of the examples in Table 17.
Proof. Step I: We assume that \( H \) is connected. Let \( H \subset \{ K^\pm \} \subset G \) be a non-primitive diagram and \( L \) be the minimal subgroup of \( G \) which contains both \( K^\pm \). Let \( D_1 \) denote the diagram \( H \subset \{ K^\pm \} \subset L \). If \( K^\pm \) are the same, then \( L = K^\pm \) and the manifold is a double. In the following, we assume that at least one of \( K^\pm \) is proper in \( L \).

Case A. We consider the case when \( L \) is one of \( K^\pm \), say \( L = K^+ \) and \( K^- \subset L \). So the diagram \( D_1 \) has only one fixed point and \( s = 2 \) and \( G/L \) is strongly isotropy irreducible. Suppose \( L = L_1 \times L_2 \) and \( L_1 \) is the non-effective kernel of the homogeneous space \( G \) at the fixed point in Proposition \[\text{3.1} \]. \( \text{L} \times \text{H} \subset \text{L}_1 \times \text{L}_2 \subset \text{G} \) is one of the following triples:

1. \( \text{L}_1 \times \text{SU}(n) \subset \text{L}_1 \times \text{SU}(n + 1) \subset \text{G} \),
2. \( \text{L}_1 \times \text{U}(n) \subset \text{L}_1 \times \text{U}(n + 1) \subset \text{G} \),
3. \( \text{L}_1 \times \text{Spin}(7) \subset \text{L}_1 \times \text{Spin}(9) \subset \text{G} \),
4. \( \text{L}_1 \times \text{Sp}(n) \Delta \text{Sp}(1) \subset \text{L}_1 \times \text{Sp}(n + 1) \times \text{Sp}(1) \subset \text{G} \).

Since the diagram of \( M \) is non-reducible, any primitive factor in \( L_1 \) is not in the non-effective kernel of the homogeneous space \( G/L \).

In the first case, the homogeneous space \( G/L \) is effective and then the isotropy representation of \( G/(L_1 \times H_1) \) has 3 summands. Combining the classification in Table \[\text{8} \] for the triple \( G \supset L_1 \times L_2 \supset L_1 \times H_1 \), we have the following two possibilities:

1. \( \text{SU}(2) \times \text{SU}(5) \subset \text{SU}(2) \times \text{SU}(6) \subset \text{E}_6 \),
2. \( \text{SO}(n) \times \text{SU}(3) \subset \text{SO}(n) \times \text{SU}(4) \subset \text{Spin}(6 + n) \) with \( n \geq 1 \),

Thus we have example N.1 and N.2. The manifolds are \( \mathbb{CP}^6 \) bundle over \( \text{E}_6/(\text{SU}(6) \cdot \text{SU}(2)) \) and \( \mathbb{CP}^4 \) bundle over \( \text{SO}(n + 6)/(\text{SO}(6) \cdot \text{SO}(n)) \) respectively.

In the second case, some primitive factor of \( L_2 \) but not the whole \( L_2 \) is the non-effective kernel of the homogeneous space \( G/(L_1 \times L_2) \). Then we have example N.3 and N.4 and the manifolds are \( \mathbb{HP}^{n+1} \) bundle over \( G_1/(L_1 \times \text{Sp}(1)) \) and \( \mathbb{CP}^{n+1} \) bundle over \( G_1/(L_1 \times \text{U}(1)) \) respectively.

In the last case, \( L_2 \) is the non-effective kernel of the homogeneous space \( G/(L_1 \times L_2) \). Then we have example N.5 and the manifold is the product of \( G_1/L_1 \) with the one defined by the diagram \( D_1 \).

Case B. Now we assume that both \( K^\pm \) are proper subgroups in \( L \) and then \( G/L \) is isotropy irreducible. The diagram \( D_1 \) is primitive and has \( s = 2 \). There are three difference cases for the effective version of this diagram classified in Theorem \[\text{3.2} \].

Case B.1. Suppose the diagram \( D_1 \) is given by \( \mathfrak{S}_2 \times L_1 \subset \{ \text{Spin}^-(7) \times L_1, \text{Spin}^+(7) \times L_1 \} \subset \text{Spin}(8) \times L_1 \). If \( G \) is simple, then \( G \supset \text{Spin}(8) \times L_1 \supset \text{Spin}^-(7) \times L_1 \) is in Table \[\text{1} \] and thus \( G = \text{Spin}(9) \) and \( L_1 = \{1\} \). However the isotropy representation of the homogeneous space

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Spin(9)/Spin(8) splits when restricted to $\mathcal{G}_2$. If $G$ is not a simple Lie group, then we have example N.6 and the manifold is the product $S^{15} \times G_1/L_1$.

**Case B.2.** Suppose the diagram $D_1$ is given by $\{1\} \times L_1 \subset \{U(1)_1 \cdot L_1, U(1)_2 \cdot L_1\} \subset U(1) \times U(1) \times L_1$. If the non-effective kernel of the homogeneous space $G/L$ is $U(1) \times U(1)$, then we have a special case of example N.7 for which $L_1 = L_2 = U(1)$.

The other possibility is that $G = G_1 \times U(1)$ such that $G_1$ is simple, $G_1/(U(1) \cdot L_1)$ is strongly isotropy irreducible and its isotropy representation remains irreducible when restricted to $L_1$. It follows that the pair $(G_1, U(1) \cdot L_1)$ appears as $(G, K)$ in Table 3. Thus we have example N.8 with $(L_1, H_1) = (U(1), \{1\})$ and N.10 with connected $H'$.

**Case B.3.** Suppose the diagram $D_1$ is $H_1 \times H_2 \times L_0 \subset \{L_1 \times H_2 \times L_0, H_1 \times L_2 \times L_0\} \subset L_1 \times L_2 \times L_0$ and $L_i/H_i (i = 1, 2)$ is a sphere with irreducible isotropy representation. If some primitive factor $L'$ of $L$ diagonally embeds into $G$, then $G = L' \times L$. Note $L'$ is not a factor of $L_0$ otherwise the diagram is reducible. Since the isotropy representation of $G/L$ remains irreducible when restricted to $H$, it follows that one of $L_1$ and $L_2$, say $L_2 = SO(2)$, and then $L' = L_2$. However the manifold is a sphere bundle over $S^1$ which is not simply-connected. So we assume that there is no diagonally embedded factors in $G$. Since the diagram is non-reducible, any primitive factor of $L_0$ is not contained in the non-effective kernel of the strongly isotropy irreducible space $G/L$, i.e., the kernel is a subgroup of $L_1 \times L_2$. If $L_1 \times L_2$ is the kernel, then we have example N.7 and the manifold is a product of a sphere and the homogeneous space $G_1/L_1$.

Next we assume that the non-effective kernel $L'$ is a proper normal subgroup of $L_1 \times L_2$. Both $L_1$ and $L_2$ cannot be $SU(2) \times SU(2)$ (or $SO(4)$) otherwise the diagram of $G$ action is reducible. Hence one of $L_1$ and $L_2$, say $L_1$, is primitive and then $L' = L_1$. It follows that $G = L_1 \times G_2$ and the pair $(G_2, L_2 \times L_0)$ appears as $(G, K)$ in Table 3. Conversely, for every triple $G_2 \supset K_2 \supset H_2$ in Table 3 and for any isotropy irreducible spherical pair $(L_1, H_1)$ with $L_1$ primitive, we have the following diagram

$$H_1 \times H_2 \subset \{L_1 \times H_2, H_1 \times K_2\} \subset L_1 \times G_2,$$

and it gives us example N.8 and N.9($m = 1$).

**Step II:** From Step I, if a variation of a double has different singular isotropy groups, then the new diagram must be in Table 17. Let $D_c : H' \subset \{K' = K'\} \subset G$ be a double with $K' / H' = S^1$. Suppose $D : H \subset \{K^\pm\} \subset G$ is a diagram with disconnected subgroups from the double $D_c$, i.e., $K^\pm = K'$ and $H_c = H'$. Let $H_\pm = H \cap K^\pm = H \cap K'$ and then Lemma 1.13 in [H0] tells us that $H$ is generated by $H_\pm$ if the manifold defined by $D$ is simply connected. In particular it implies that $H \subset K'$ since $H_- = H_+$. Since $K^\pm / H = S^1$, we have that $K^\pm = K'$ and the manifold is an $S^2$ bundle over $G / K'$. 
For most diagrams in Table 17, the three irreducible summands of the isotropy representation \( \text{Ad}_{H_c} \) are non-equivalent. It follows that their variations by conjugating group elements are equivalent to the original ones. In the following four cases, the isotropy representation \( \text{Ad}_{H_c} \) contains two equivalent irreducible summands:

(4.1) \( \mathfrak{g}_2 \times L_1 \subset \{ \text{Spin}^-(7) \times L_1, \text{Spin}^+(7) \times L_1 \} \subset \text{Spin}(8) \times G_1 \)

(4.2) \( \text{SU}(3) \subset \{ \text{U}(1) \cdot \text{SU}(3), \text{SU}(4) \} \subset \text{Spin}(7) \)

(4.3) \( H_1 \times \mathfrak{g}_2 \subset \{ L_1 \times \mathfrak{g}_2, H_1 \times \text{Spin}(7) \} \subset L_1 \times \text{SO}(8) \)

(4.4) \( H_1 \subset \{ \text{U}(1) \cdot H_1, K_1 \} \subset \text{U}(1) \times G_1 \).

They are example \( \text{N}.2 \) with \( n = 1 \), example \( \text{N}.6 \) where \( G_1/L_1 \) is strongly isotropy irreducible, example \( \text{N}.8 \) for the triple \( \text{SO}(8) \supset \text{Spin}(7) \supset \mathfrak{g}_2 \), and example \( \text{N}.10 \) where the triple \( G_1 \supset K_1 \supset H_1 \) is in Table 4 such that \( K_1/H_1 = S^1 \).

The different diagram from a variation of the first one (4.1) is a double. For the second diagram (4.2), all automorphisms of \( \text{Spin}(7) \) are inner ones, i.e., conjugation by group elements. Since for every \( g \in N(\text{SU}(3)) \), \( g.(\text{U}(1) \cdot \text{SU}(3)).g^{-1} = \text{U}(1) \cdot \text{SU}(3) \), a variation does not give us a new diagram. Since the \( \text{U}(1) \) factor in \( K^- \) is contained in \( K^+ = \text{SU}(4) \), one cannot add connected component to \( H \). A similar argument shows that any variation of the third one (4.3) does not give us a new diagram either. If the pair \((L_1, H_1)\) is \((\text{U}(1), \{1\})\), then one can add connected components to \( H = \mathfrak{g}_2 \) and \( K^- = \text{Spin}(7) \) to obtain a diagram with disconnected principal isotropy subgroup. However the action of \( G = \text{U}(1) \times \text{SO}(8) \) is not effective and the diagram of the effective action is the same as the one (4.3). The discussion of the diagram (4.4) is similar to the example \( \text{R}.23 \) in the proof of Theorem 4.1.

Since the two singular orbits are codimension two, one can add components to the three isotropy subgroups. The proper subgroup \( L = \text{U}(1) \times K_1 \) contains both \( K_\pm \). We can apply Lemma 4.3 in [Ho] to obtain all diagrams whose connected groups are in (4.4). It follows that the manifolds are lens space bundles over \( G_1/K_1 \). The one with the lowest dimension is given by the triple \((G_1, K_1, H_1) = (\text{SO}(3) \times \text{U}(1), \text{SO}(2) \times \text{U}(1), \text{SO}(2))\) and the manifold is the product \( S^3 \times S^2 \). If \((G_1, K_1, H_1)\) is the triple \((\text{SU}(3), \text{U}(2), \text{SU}(2))\), then the manifold is example \( N'_7 \) in [Ho].

Finally we consider the diagrams by adding connected components to isotropy groups and we assume that the three summands are non-equivalent. If the diagram is a double and \( K^+/H = S^1 \), then for each nonzero integer \( m \), we have the diagram \( \mathbb{Z}_m \cdot H \subset \{ K^\pm \} \subset G \). In example \( \text{N}.7 \) and \( \text{N}.8 \), if one add connected components to isotropy subgroups, then the action by \( G \) is not effective. If the diagram is in example \( \text{N}.1, \text{N}.2, \text{N}.4 \) and \( \text{N}.5 \), then the
diagram $H \subset \{K^\pm\} \subset K^+$ has a fixed point. So one cannot add connected components to the isotropy groups.

5. PRIMITIVE AND NON-REDUCIBLE ACTIONS WITH $s = 3$ AND $G$ SIMPLE

First we state the main result in this section.

**Theorem 5.1.** Suppose $M$ is a compact simply-connected manifold with a cohomogeneity one action by a simple Lie group $G$. If the action is primitive and has $s = 3$, then $M$ is equivariantly diffeomorphic to a sphere, a complex projective space or the Grassmannian $SO(m+n)/(SO(m) \times SO(n))(m,n \geq 2)$ with an isometric action, see Table 10 and 11.

We saw in Theorem 4.1 that if the action is reducible and primitive, then the manifold is a sphere with a linear action. So in the following we assume that the action is non-reducible. We prove the theorem in several steps:

**Step 1:** If the three summands of the isotropy representation are non-equivalent, then the manifold is a sphere with a sum action or the two singular orbits $G/K^\pm$ are strongly isotropy irreducible, see Proposition 5.2.

**Step 2:** We assume that all isotropy subgroups are connected. If one of $G/K^\pm$, say $G/K^-$ is not isotropy irreducible, i.e., the isotropy representation of $G/K^-$ has two summands, then the group triple $G \supset K^- \supset H$ must be $SO(7) \supset U(3) \subset SU(3)$, see Proposition 5.4.

**Step 3:** We classify triples of connected groups $G \supset K \supset H$ such that $G/K$ is isotropy irreducible, $K/H$ is a sphere and the isotropy representation of $G/H$ has three summands, see Proposition 5.5 and Table 8. It follows that if $M$ is not a sphere, then both triples $G \supset K^\pm \supset H$ are in Table 8 or $SO(7) \supset U(3) \supset SU(3)$.

**Step 4:** For every quadruple $(G, K_1, K_2, H)$ from the previous step, we consider all possible cohomogeneity one group diagrams from it and then identify the manifolds. Theorem 5.7 is the classification for disconnected $H$ and Theorem 5.11 is for connected $H$.

5.1. Three summands are pairwise non-equivalent. We will show that the two singular orbits $B^\pm = G/K^\pm$ are strongly isotropy irreducible homogeneous spaces unless the cohomogeneity one manifold $M$ is equivariantly diffeomorphic to a sphere. In the following, the three irreducible summands of $\text{Ad}_{H_c}$ are denoted by $p_1$, $p_2$ and $p_3$, i.e., $p = p_1 \oplus p_2 \oplus p_3$.

**Proposition 5.2.** If $p_1$, $p_2$ and $p_3$ are pairwise non-equivalent as the $\text{Ad}_{H_c}$ representations, then

1. either $G/K^\pm$ are strongly isotropy,
2. or $M$ is a sphere with a sum action.
A similar argument as in the proof of Theorem 3.2 shows that $G$ and hence the $H$.

Let $m_1 = p_1$ and $m_2 = p_2 \oplus p_3$ and we define the following spaces:

$$h_0 = \text{Ann}(m_1 \oplus m_2) \cap h, \quad h_1 = \text{Ann}(m_i) \cap h_0^{\perp} \cap h, \quad h_3 = (h_0 \oplus h_1 \oplus h_2)^{\perp} \cap h.$$  

A similar argument as in the proof of Theorem 3.2 shows that

- $(1)$ $h_0 = h_3 = 0$, and both $h_2 \oplus m_1$ and $h_1 \oplus m_2$ are ideals of $g$;
- $(2)$ $g = h_1 \oplus h_2 \oplus m_1 \oplus m_2$, $t^- = h_1 \oplus h_2 \oplus m_1$, $t^+ = h_1 \oplus h_2 \oplus m_2$.

Let $H_1, H_2, L_1$ and $L_2$ be Lie groups of $h_1, h_2, h_1 \oplus m_2$ and $h_2 \oplus m_1$ respectively, then we have

$$G = L_1 \times L_2, \quad K^- = H_1 \times L_1, \quad K^+ = H_2 \times L_2, \quad \text{and} \quad H = H_1 \times H_2,$$

and hence the $G$-action is a sum action and the manifold $M$ is $G$-equivariant diffeomorphic to a sphere.

Now we consider the last type where both $K^\pm/H$ are strongly isotropy irreducible. Since the group diagram is primitive, without loss of generality, we may assume that $t^- = h \oplus p_2$ and $t^+ = h \oplus p_3$. Following the proof of Theorem 3.2, the subspace $[p_2, p_3]$ is orthogonal to $h \oplus p_2 \oplus p_3$. So we have $[p_2, p_3] \subset p_1$ and $[p_2, p_3]$ is an invariant subspace under the action $\text{Ad}_{h_0}$. It is either equal to 0 to $p_1$ from the irreducibility of $p_1$. In the first case, $h \oplus p_2 \oplus p_3$ is an ideal of $g$, so the group generated by $K^-$ and $K^+$ is a proper normal subgroup of $G$ which contradicts the primitivity assumption. Therefore $[p_2, p_3] = p_1$ which implies that both $G/K^\pm$ are strongly isotropy irreducible.

**Remark 5.3.** Let $H \subset \{K^\pm\} \subset G$ is a non-reducible and fixed points free diagram with $s = 3$. Suppose $G/K^-$ is not strongly isotropy irreducible and the representation of $\text{Ad}_{K^-/H}$ is denoted by $p_3$. If $p_3$ is not equivalent to one of the summands $p_1$ and $p_2$, then from the proof of Proposition 5.2 then we have

- $(1)$ if $p_1$ and $p_2$ are non-equivalent, then the manifold is a sphere via sum action;
- $(2)$ if $p_1$ and $p_2$ are equivalent, then the diagram is non-primitive.

5.2. **Two or three summands are equivalent.** We consider the triple of connected groups $H \subset K \subset G$ such that $K/H$ is a sphere, the isotropy representation $\text{Ad}_h$ on the tangent space of $K/H$ is irreducible and the isotropy representation $\text{Ad}_K$ on the tangent space of $G/K$ has two irreducible summands.
**Proposition 5.4.** Let $G$ be a simple Lie group and $H \subset K \subset G$ be three connected Lie groups such that $K/H$ is a sphere and $\text{Ad}_H$ is irreducible on the tangent space $p_3$ of $K/H$. Suppose the $\text{Ad}_K$ action on the tangent space of $G/K$ has exactly two irreducible summands $p_1$ and $p_2$. If $p_1$ and $p_2$ remain irreducible when they are viewed as the representations of $\text{Ad}_H$, then $p_1$, $p_2$ and $p_3$ are pairwise non-equivalent except for the triple $SU(3) \subset U(3) \subset SO(7)$.

**Proof.** Since the groups $H$, $K$ and $G$ are connected, we consider their Lie algebras: $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$. We denote the representation of $\text{ad}_h$ on $p_i$ by $\chi_i$ for $i = 1, 2, 3$.

First we consider the pair $(G, K)$ for which $\dim p_1 = \dim p_2$ and there exist some $H$ such that the sphere $K/H$ is isotropy irreducible. From the classification of $(G, K)$ in [DK] and Theorem A.1, they are in Table 7. To save space, we only give the group $K$. We list the condition for each pair in the last column.

| $K$ | $\chi_1$ | $\chi_2$ |
|-----|----------|----------|
| I.1 | $SO(m) \times SU(k), (k \geq 4)$ | $\text{Id} \otimes [\pi_2 \pi_{k-1}^2]_{\mathbb{R}}$ | $\varrho_m \otimes \pi_1 \pi_{k-1}$ |
| I.2 | $SO(m) \times SO(k), (k \geq 7)$ | $\text{Id} \otimes \pi_1 \pi_3$ | $\varrho_m \otimes \pi_2$ |
| I.5 | $SO(m) \times Sp(k), (k \geq 3)$ | $\text{Id} \otimes \pi_1 \pi_2$ | $\varrho_m \otimes \pi_2$ |
| I.14 | $SO(65) \times E_7$ | $\text{Id} \otimes \pi_3$ | $\varrho_{65} \otimes \pi_1$ |
| I.16 | $\mathfrak{g}_2$ | $\pi_1$ | $\pi_1$ |
| I.18 | $SO(m) \times U(k), (k \geq 3)$ | $\text{Id} \otimes [\pi_2 \otimes \phi]_{\mathbb{R}}$ | $\varrho_m \otimes [\pi_1 \otimes \phi]_{\mathbb{R}}$ |
| II.5 | $SU(p) \times SU(q) \times S(U(1) \times U(m))$ | $\pi_1 \pi_{p-1} \otimes \pi_1 \pi_{q-1} \otimes \text{Id} \otimes \text{Id} \otimes \text{Id}$ | $\varrho_{mpq}$ |
| III.6 | $Sp(m) \times U(n)$ | $\text{Id} \otimes [\pi_1^2 \otimes \phi^2]_{\mathbb{R}}$ | $[\pi_1 \otimes \pi_1 \otimes \phi]_{\mathbb{R}}$ |
| V.1 | $SO(m) \times SO(m), (m \geq 3, m \neq 4)$ | $\pi_2 \otimes \pi_2^2$ | $\pi_2 \otimes \pi_2$ |
| V.2 | $Sp(2) \times Sp(2)$ | $\pi_2 \otimes \pi_2$ | $\pi_2 \otimes \pi_1^2$ |

**Table 7.** The pairs $(G, K)$ for which $\dim p_1 = \dim p_2$.

It is easy to see that only in example I.18 when $m = 1$ and $k = 3$, if $H = SU(3)$, then $\chi_1|_H$ is equivalent to $\chi_2|_H$ and $\chi_3 = \text{Id}$. For other cases, either at least one of $\chi_1$ and $\chi_2$ splits when restricted to $H$ or the three summands are non-equivalent.

Next we show that $\chi_3$ is not equivalent to $\chi_1|_H$ or $\chi_2|_H$. We prove it by contradiction. Assume that $\chi_3$ is equivalent to $\chi_1|_H$. From the classification of transitive actions of the sphere and the assumption that $\text{ad}_{\mathfrak{t}}/\mathfrak{h}$ is isotropy irreducible, $(\mathfrak{k}, \mathfrak{h})$ is one of the pairs $(\mathfrak{so}(k+1) \oplus \mathfrak{h}_0, \mathfrak{so}(k) \oplus \mathfrak{so}(k)) (k \geq 1)$, $(\mathfrak{g}_2 \oplus \mathfrak{h}_0, \mathfrak{su}(3) \oplus \mathfrak{h}_0)$ and $(\mathfrak{so}(7) \oplus \mathfrak{h}_0, \mathfrak{g}_2 \oplus \mathfrak{h}_0)$. Here $\mathfrak{h}_0$ may be a zero vector space.
We consider the case when $K/H$ is a circle first. Since $\chi_3$ is equivalent to $\chi_1|_H$, we have that $\dim \mathfrak{p}_1 = 1$ and $\mathfrak{k}$ contains a $\mathfrak{u}(1)$ factor. However such pair $(G, K)$ does not exist from the classification.

If $\mathfrak{k} = \mathfrak{so}(k + 1) \oplus \mathfrak{h}_0$ and $\mathfrak{h} = \mathfrak{so}(k) \oplus \mathfrak{h}_0$ with $k \geq 2$, then $\chi_3 = \varrho_k \otimes \text{Id}$. If $\chi_1 = \sigma_1 \otimes \sigma_2$ is a representation of real type, then $\sigma_2$ is the trivial representation and $\sigma_1|_{\mathfrak{so}(k)} = \varrho_k$. The only possible case if that $\mathfrak{k} = \mathfrak{su}(2) \oplus \mathfrak{h}_0$, $\mathfrak{h} = \mathfrak{u}(1) \oplus \mathfrak{h}_0$ and $\sigma_1 = \pi_1$. However $\chi_1 = \sigma_1 \otimes \text{Id}$ is of quaternionic type. If $\chi_1 = [\sigma_1 \otimes \sigma_2]|_R$ and $\sigma_1 \otimes \sigma_2$ is not of real type, then $\sigma_2 = \text{Id}$ and the restriction of $[\sigma_1]|_R$ to $\mathfrak{so}(k)$ is $\varrho_k$. Such $\sigma_1$ does not exist.

If $\mathfrak{k} = \mathfrak{su}(7) \oplus \mathfrak{h}_0$ and $\mathfrak{h} = \mathfrak{g}_2 \oplus \mathfrak{h}_0$, then $\chi_3 = \pi_1 \otimes \text{Id}$. Since every irreducible representation of $\mathfrak{su}(7)$ is of real type, $\chi_1 = \sigma_1 \otimes \text{Id}$ and $\sigma_1|_{\mathfrak{g}_2} = \pi_1$. So we have $\sigma_1 = \varrho_3$ and then $\chi_1 = \varrho_7 \otimes \text{Id}$. Four examples, I.30, I.32, II.13 and III.10, have such $\chi_1$. However for each of them, $\chi_2$ splits when restricted to $H$.

If $\mathfrak{k} = \mathfrak{g}_2 \oplus \mathfrak{h}_0$ and $\mathfrak{h} = \mathfrak{su}(3) \oplus \mathfrak{h}_0$, then $\chi_3 = [\pi_1]|_R \otimes \text{Id}$. Since every irreducible representation of $\mathfrak{g}_2$ is of real type, $\chi_1 = \sigma_1 \otimes \text{Id}$. However $\sigma_1|_{\mathfrak{su}(3)}$ cannot be $[\pi_1]|_R$ for any irreducible representation $\sigma_1$. This finishes the proof. \hfill $\Box$

### 5.3. Group triples $G \supset K \supset H$ with $G/K$ strongly isotropy irreducible.

The triples such that $\text{Ad}_H$ has only three irreducible summands and $K/H$ is a sphere are classified in

**Proposition 5.5.** The triples $G \supset K \supset H$ with $G/K$ a strongly isotropy irreducible homogeneous space, $K/H$ a sphere and $H$ connected, for which $\text{Ad}_H$ has exactly three irreducible summands are listed in Table A.

| $G$          | $K$          | $H$          |
|--------------|--------------|--------------|
| $\text{SU}(2) \times \text{SU}(2)$ | $\Delta \text{SU}(2)$ | $\mathfrak{u}(1)$ |
| $\text{Spin}(n) \times \text{Spin}(n)$ | $\Delta \text{Spin}(n)$ | $\text{Spin}(n-1)$ for $n \geq 6$ |
| $\text{Spin}(7) \times \text{Spin}(7)$ | $\Delta \text{Spin}(7)$ | $\mathfrak{g}_2$ |
| $\mathfrak{g}_2 \times \mathfrak{g}_2$ | $\Delta \mathfrak{g}_2$ | $\text{SU}(3)$ |
| $\text{SU}(4p)$ | $\text{SU}(4) \times \text{SU}(p)$ | $\text{Sp}(2) \times \text{SU}(p)$ for $p \geq 2$ |
| $\text{SU}(2p)$ | $\text{SU}(p) \times \text{SU}(4)$ | $\text{SU}(p) \times \text{Sp}(2)$ |
| $\text{SU}(16)$ | $\text{Spin}(10)$ | $\text{Spin}(9)$ |
| $\text{SU}(4)$ | $\text{SU}(2) \times \text{SU}(2)$ | $\mathfrak{u}(1) \times \text{SU}(2)$ |
| $\text{SU}(3)$ | $\mathfrak{u}(2)$ | $\mathfrak{u}(1) \times \mathfrak{u}(1)$ |

Continued on next page
Table 8: Group triple $G \supset K \supset H$ such that $G/K$ is strongly isotropy irreducible, $\text{Ad}_H$ has 3 summands and $K/H$ is a sphere.

| $G$         | $K$                             | $H$                             |
|------------|---------------------------------|---------------------------------|
| SU(2)      | SO(2)                           | $\{1\}$                        |
| SO($p+2$)  | SO($p+1$)                       | SO($p$)                         | $p \geq 4$ |
| SO($p+q+2$)| SO($p+1$) × SO($q+1$)           | SO($p$) × SO($q+1$)             | $p, q \geq 1$ |
|            | SO($p+1$) × SO($q$)             |                                |            |
| Spin(6 + 2$p$) | Spin(6) × SO(2$p$)             | SU(3) × SO(2$p$)               | $p \geq 1$ |
|            | SO(2$p$) × Spin(6)              | SO(2$p$) × SU(3)               |            |
| Spin(7 + 2$p$) | Spin(6) × SO(2$p+1$)           | SU(3) × SO(2$p+1$)             | $p \geq 0$ |
| Spin(128)  | Spin(16)                        | Spin(15)                        |            |
| Spin(16)   | Spin(9)                         | Spin(8)                         |            |
| SO(7)      | U(3)                            | SU(3)                           |            |
| SO(8)      | U(4)                            | SU(4)                           |            |
| SO(8)      | Spin(7)                         | Spin(6)                         |            |
| Spin(7)    | Spin(6)                         | SU(3)                           |            |
| Spin(7)    | G2                              | SU(3)                           |            |
| Sp(2)      | Sp(1) × Sp(1)                   | $\Delta$Sp(1)                  |            |
| $E_6$      | SU(6) · SU(2)                   | SU(5) · SU(2)                   |            |
| $E_6$      | SU(3) × G2                      | SU(3) × SU(3)                   |            |
| $F_4$      | Spin(9)                         | Spin(8)                         |            |
| $F_4$      | SO(3) × G2                      | SO(3) × SU(3)                   |            |
| $G_2$      | SO(4)                           | SO(3)                           |            |

The proof is straightforward. We use the classifications of compact irreducible symmetric spaces and strongly isotropy irreducible homogeneous space $G/K$ in [Wo1]. For each pair $(G, K)$ we list the possible $H$’s such that $K/H$ is a sphere and then compute the isotropy representation $\text{Ad}_H$ of $G/H$. If $\text{Ad}_H$ has precisely three irreducible summands, then we include the triple $G \supset K \supset H$ in our list.
Remark 5.6. The triples \( \{1\} \subset \text{SO}(2) \subset \text{SU}(2), \text{U}(1) \subset \Delta \text{SU}(2) \subset \text{SU}(2) \times \text{SU}(2) \) and \( \Delta \text{Sp}(1) \subset \text{Sp}(1) \times \text{Sp}(1) \subset \text{Sp}(2) \) can be viewed as the triple \( \text{SO}(p) \subset \text{SO}(p+1) \subset \text{SO}(p+2) \) when \( p = 1, 2 \) and \( 3 \).

5.4. Construction of cohomogeneity one group diagrams. From the previous sections, we only have to consider the triple \( \text{SU}(3) \subset \text{U}(3) \subset \text{SO}(7) \) and those in Table 8.

We consider the case where \( H \) is not connected first and we have

**Theorem 5.7.** The cohomogeneity one manifold defined by a primitive group diagram \( H \subset \{ K^\pm \} \subset G \) with \( G \) a simple Lie group, \( H \) disconnected and \( s = 3 \) is a complex project space, see Example 1 and 2.

**Proof.** By Lemma 2.10, we may assume that \( l_- = 1 \), i.e., \( K^- \) is connected and \( K^-/H_c = \mathbb{S}^1 \).

From the classification in Proposition 5.5, \( H_c \subset K \subset G \) is one of

\[
\text{(5.1)} \quad \text{SO}(p) \subset \text{SO}(2) \times \text{SO}(p) \subset \text{SO}(2+p), \quad (p \geq 3),
\]

\[
\text{(5.2)} \quad \text{SU}(3) \subset \text{U}(3) \subset \text{SO}(7)
\]

\[
\text{(5.3)} \quad \text{SU}(4) \subset \text{U}(4) \subset \text{SO}(8).
\]

Suppose that \( H_c \subset K \subset G \) is the triple in (5.1). If \( K^+_c \), the connected component of \( K^+ \), is \( \text{SO}(2) \times \text{SO}(p) \) and then both \( l_\pm \) equal to 1. Since \( N(K^+_c) = S(\text{O}(2) \times \text{O}(p)) = \text{SO}(2) \times \text{SO}(p) \cup (\text{SO}(2) \times \text{SO}(p)) \cdot A \) with \( A = \text{diag}(1,-1,-1,I_{p-1}) \), the diagram is

\[
\text{SO}(p) \cdot \{1,A\} \subset \{S(\text{O}(2) \times \text{O}(p)), S(\text{O}(2) \times \text{O}(p))\} \subset \text{SO}(p+2),
\]

The cohomogeneity one manifold defined by the above diagram is a double. It is not simply connected and finitely covered by the manifold defined by the diagram \( \text{SO}(p) \subset \{\text{SO}(2) \times \text{SO}(p), \text{SO}(2) \times \text{SO}(p)\} \subset \text{SO}(p+2) \).

If \( K^+_c \) is \( \text{SO}(p+1) \) and then \( N(K^+_c)/K^+_c = \mathbb{Z}_2 \) generated by the matrix \( \text{diag}(-I_2,I_p) \). So we have

**Example 1.** The diagram is

\[
H = \text{SO}(p) \cdot \mathbb{Z}_2 \subset \{\text{SO}(2) \times \text{SO}(p), \text{O}(p+1)\} \subset \text{SO}(2+p)
\]

and the manifold is \( \mathbb{C}P^{p+1} \), see, for example, [GWZ].

Next we consider variations of these diagrams. If \( p \) is odd, then \( \text{Aut}(G,H) = S(\text{O}(2) \times \text{O}(p)) \) which is the same as \( \text{Aut}(G,K^-) \). If \( p \) is even, then \( G \) has an outer automorphism which is conjugation by \( \text{diag}(-1,I_{p+1}) \). It is clear that this automorphism leaves \( K^- \) invariant. Therefore the variation does not give another new diagram.
If \( p = 6 \), there are a few more possible constructions. Let us lift \( G = \text{SO}(8) \) to its universal cover \( \text{Spin}(8) \), then the triple is lifted to \( \text{Spin}(6) \subset (\text{SO}(2) \times \text{Spin}(6))/\mathbb{Z}_2 \subset \text{Spin}(8) \) where \( \mathbb{Z}_2 \) is generated by \( -\text{id} \in \text{Spin}(8) \). \( \text{Spin}(8) \) has another order 3 outer automorphism denoted by \( \sigma \). There are three different embeddings of \( \text{Spin}(7) \) into \( \text{Spin}(8) \) and \( \sigma \) permutes them. If \( \sigma \) leaves some \( \text{Spin}(6) \) invariant, then it would be contained in the intersection of the three \( \text{Spin}(7) \)'s which would imply this \( \text{Spin}(6) \) is contained in \( \mathfrak{g}_2 \). Therefore there is no such \( \text{Spin}(6) \) invariant by the automorphism \( \sigma \). On the other hand there is another intermediate subgroup \( \widetilde{U} \), the image of \( U(4) \subset \text{SO}(8) \) by the lifting, between \( \text{Spin}(6) = \text{SU}(4) \) and \( \text{Spin}(8) \). Since \( \text{Spin}(8)/\widetilde{U} = \text{SO}(8)/U(4) \) is simply-connected, \( \widetilde{U} \) is connected. Both \( (\text{SO}(2) \times \text{Spin}(6))/\mathbb{Z}_2 \) and \( \widetilde{U} \) contain \( \text{Spin}(6) \) and the isotropy representation of the space \( \text{Spin}(8)/\text{Spin}(6) \) contains only one trivial representation \( \text{Id} \), so they are the same subgroup in \( \text{Spin}(8) \). Divided by the ineffective kernel, the diagram

\[
\text{Spin}(6) \subset \left\{ (\text{SO}(2) \times \text{Spin}(6))/\mathbb{Z}_2, \widetilde{U} \right\} \subset \text{Spin}(8)
\]

reduces to

\[
\text{SO}(6) \subset \{ \text{SO}(2) \times \text{SO}(6), \text{SO}(2) \times \text{SO}(6) \} \subset \text{SO}(8)
\]

which is a double.

If \( H_c \subset K \subset G \) is the one in (5.2), then we lift \( \text{SO}(7) \) to \( \text{Spin}(7) \) and obtain the following triple

\[
(5.4) \quad \text{SU}(3) \subset \text{SO}(2) \times \text{SU}(3) \subset \text{Spin}(7).
\]

From the classification, \( K^c_+ \) is either \( \mathfrak{g}_2 \) or \( \text{Spin}(6) \). If \( K^c_+ = \mathfrak{g}_2 \), then \( N(G^c_+)/K^c_+ = \mathbb{Z}_2 \). So we have

**Example 2.** The diagram is

\[
\mathbb{Z}_2 \cdot \text{SU}(3) \subset \{ \text{SO}(2) \cdot \text{SU}(3), \mathbb{Z}_2 \cdot \mathfrak{g}_2 \} \subset \text{Spin}(7),
\]

where \( \mathbb{Z}_2 \) is the center of \( \text{Spin}(7) \). The manifold is the complex projective space \( \mathbb{C}P^7 \), see, for example, [Uc].

Next we consider the case where \( K^c_+ = \text{Spin}(6) \). Since \( \text{SO}(2) \cdot \text{SU}(3) \) is the 2-fold cover of \( U(3) \subset \text{SO}(7) \) and there is only one \( \text{SO}(6) \) in \( \text{SO}(7) \) contains \( \text{SU}(3) \) that also contains \( U(3) \), it follows that \( \text{Spin}(6) \) which is the 2-fold cover of \( \text{SO}(6) \) contains \( \text{SO}(2) \cdot \text{SU}(3) \). So one cannot add components to isotropy subgroups, i.e., \( H \) is connected.

If \( H_c \subset K \subset G \) is the one in (5.3) and \( K^c_+ = \text{Spin}(7) \), then \( N_G(K^+ c) = K^c_+ \) which implies \( H \) is connected. If both \( K^c_+ = K = U(4) \), then we have \( N_G(K)/K = \mathbb{Z}_2 \) and it is generated by the diagonal matrix \( A = \text{diag}(I_4, -I_4) \). Since \( N_G(H_c) = N_G(K) \) and there is no circle group
inside $N_G(H_c)/H_c$ containing $A$, this triple does not give any cohomogeneity one diagram with a disconnected $H$. □

Next we consider the cases where $H$ is connected. Since there is no exceptional orbit, both $K^\pm$ are connected. In the classification in Proposition 5.5 many pairs $(H, G)$ contain only one intermediate subgroup.

**Definition 5.8.** Two irreducible representations $\varphi$ and $\psi$ of $H$ are *outer equivalent* if $\varphi = \tau(\psi)$ by an outer automorphism of $H$.

Recall that $\chi_1, \chi_2$ and $\chi_3$ are the three irreducible summands of the isotropy representation $\text{Ad}_H$ on $G/H$. Let $\text{Ad}_H(G/K)$ and $\text{Ad}_H(K/H)$ be the restrictions of $\text{Ad}_H$ to the tangent spaces of $G/K$ and $K/H$ respectively.

**Lemma 5.9.** Suppose that any irreducible summand of $\text{Ad}_H(K/H)$ is not equivalent or outer equivalent to any summand in $\text{Ad}_H(G/K)$, then the cohomogeneity one manifold defined by any variation of the diagram $H \subset \{K, K\} \subset G$ is a double.

*Proof.* We give a proof when $\text{Ad}_H(K/H) = \chi_3$ is irreducible. The other case where $\text{Ad}_H(K/H)$ is reducible follows easily. Let $\tau \in \text{Aut}(G, H)$, then $\tau$ is an automorphism of $H$ and it permutes the three summands. By assumption, $\tau(\chi_3)$ is not equivalent to $\chi_1$ or $\chi_2$, so $\tau(\chi_3) = \chi_3$ which implies the Lie algebra of $K$ and hence $K$ itself is invariant by $\tau$. Therefore the manifold defined by $H \subset \{K, \tau(K)\} \subset G$ is a double. □

We list all triples which satisfy the condition in Lemma 5.9.

**Proposition 5.10.** The group triples $H \subset K \subset G$ in Table 8 such that any irreducible summands of $\text{Ad}_H(K/H)$ is not equivalent or outer equivalent to the summand of $\text{Ad}_H(G/K)$ are classify in Table 4.

| $G$   | $H$                | $\text{Ad}_H(K/H)$       | $\text{Ad}_H(G/K)$       |
|-------|--------------------|--------------------------|--------------------------|
| $SU(4p)$ | $Sp(2) \times SU(p)$ | $\pi_1 \otimes \text{Id}$ | $(\pi_2 + \pi_1) \otimes (\pi_1 + \pi_{p-1})$ |
|       | $SU(p) \times Sp(2)$ | $\text{Id} \otimes \pi_1$ | $(\pi_1 + \pi_{p-1}) \otimes (\pi_2 + \pi_1)$ |
| $SU(2p)$ | $SU(p) \times U(1)$ | $\text{Id} \otimes [\phi]_R$ | $(\pi_1 + \pi_{p-1}) \otimes (\text{Id} \oplus [\phi]_R)$ |
| $SU(16)$ | $\text{Spin}(9)$ | $\pi_1$ | $2\pi_4 + \pi_3$ |
| $SU(4)$ | $U(1) \times SU(2)$ | $[\phi]_R \otimes \text{Id}$ | $(\text{Id} \oplus [\phi]_R) \otimes 2\pi_1$ |
|       | $SU(2) \times U(1)$ | $\text{Id} \otimes [\phi]_R$ | $2\pi_1 \otimes (\text{Id} \oplus [\phi]_R)$ |

Continued on next page
In Table 9 when $G = \text{SO}(p + q + 2)$ the two factors of $H$ should be of different sizes, i.e., $p \neq q + 1$ or $p + 1 \neq q$.

Now we state the result when $H$ is connected:

**Theorem 5.11.** The cohomogeneity one manifold defined by a primitive group diagram $H \subset \{K^\pm\} \subset G$ with $G$ simple, $H$ connected and $s = 3$ is either $S^7$, $S^{14}$, $S^{25}$, $\mathbb{C}P^7$ or the Grassmannian $\text{SO}(p + q + 2)/(\text{SO}(p + 1) \times \text{SO}(q + 1))$ with $p, q \geq 1$, see Example 3 – 9.

**Proof.** There are two main steps in the proof. In Step 1, we consider the pairs $(H, G)$ for which there are at least two intermediate groups. In Step 2, we consider the variations of doubles. We fix the notations for the outer automorphisms of $\text{SO}(2m)$(or $\text{Spin}(2m)$): $\lambda$ is the degree 2 outer automorphism and $\sigma$ is the degree 3 outer automorphism of $\text{Spin}(8)$.

---

Table 9: Group pair $G \supset H$ in Table 8 such that $\text{Ad}_H(K/H)$ has no summand equivalent or outer equivalent to one summand in $\text{Ad}_H(G/K)$.

| $G$            | $H$                      | $\text{Ad}_H(K/H)$      | $\text{Ad}_H(G/K)$      |
|----------------|--------------------------|-------------------------|-------------------------|
| $\text{SO}(p + q + 2)$ | $\text{SO}(p) \times \text{SO}(q + 1)$ | $\pi_1 \otimes \text{Id}$ | $(\pi_1 \oplus \text{Id}) \otimes \pi_1$ |
|                | $\text{SO}(p + 1) \times \text{SO}(q)$ | $\text{Id} \otimes \pi_1$ | $\pi_1 \otimes (\pi_1 \oplus \text{Id})$ |
| $\text{Spin}(7 + 2p)$ | $\text{SU}(3) \times \text{SO}(2p + 1)$ | $(\text{Id} \oplus [\pi_1]_R) \otimes \text{Id}$ | $[\pi_1]_R \otimes \pi_1$ |
| $\text{Spin}(6 + 2p)$ | $\text{SU}(3) \times \text{SO}(2p)$ | $(\text{Id} \oplus [\pi_1]_R) \otimes \text{Id}$ | $[\pi_1]_R \otimes \pi_1$ |
|                | $\text{SO}(2p) \times \text{SU}(3)$ | $\text{Id} \otimes (\text{Id} \oplus [\pi_1]_R)$ | $\pi_1 \otimes [\pi_1]_R$ |
| $\text{Spin}(128)$ | $\text{Spin}(15)$         | $\pi_1$                 | $\pi_5 \oplus \pi_6$    |
| $\text{Spin}(16)$ | $\text{Spin}(8)$          | $\pi_1$                 | $(\pi_3 + \pi_4) \oplus \pi_2$ |
| $\text{SO}(7)$  | $\text{SU}(3)$            | $\text{Id}$             | $[\pi_1]_R \oplus [\pi_1]_R$ |
| $\text{SO}(8)$  | $\text{SU}(4)$            | $\text{Id}$             | $\pi_2 \oplus \pi_2$    |
| $E_6$          | $\text{SU}(5) \cdot \text{SU}(2)$ | $(\text{Id} \oplus [\pi_1]_R) \otimes \text{Id}$ | $[\pi_2]_R \otimes \pi_1$ |
| $E_6$          | $\text{SU}(3) \times \text{SU}(3)$ | $\text{Id} \otimes [\pi_1]_R$ | $(\pi_1 + \pi_2) \otimes ([\pi_1]_R \oplus \text{Id})$ |
| $F_4$          | $\text{SO}(3) \times \text{SU}(3)$ | $\text{Id} \otimes [\pi_1]_R$ | $4\pi_1 \otimes ([\pi_1]_R \oplus \text{Id})$ |
STEP 1: From the classification of the triples, between the following four pairs of \((H, G)\), there are more than one intermediate subgroups \(K\). They are

\[(5.5) \quad \text{U}(1) \times \text{U}(1) \subset \{S(\text{U}(1) \times \text{U}(2)), S(\text{U}(2) \times \text{U}(1))\} \subset \text{SU}(3),\]

\[(5.6) \quad \text{SU}(3) \subset \{\text{Spin}(6), \mathcal{G}_2\} \subset \text{Spin}(7),\]

\[(5.7) \quad \text{SU}(3) \subset \{\text{SO}(2) \cdot \text{SU}(3), \mathcal{G}_2\} \subset \text{Spin}(7),\]

\[(5.8) \quad \text{SU}(3) \subset \{\text{SO}(2) \cdot \text{SU}(3), \text{Spin}(6)\} \subset \text{Spin}(7),\]

\[(5.9) \quad \text{SU}(4) \subset \{\text{Spin}(7), \text{U}(4)\} \subset \text{SO}(8),\]

and

\[(5.10) \quad \text{SO}(p) \times \text{SO}(q) \subset \{\text{SO}(p) \times \text{SO}(q + 1), \text{SO}(p + 1) \times \text{SO}(q)\} \subset \text{SO}(p + q + 1),\]

where \(p, q \geq 1\).

**Example 3.** The manifold defined by the diagram \((5.5)\) is the sphere \(S^7\) and the embedding \(\text{SU}(3) \hookrightarrow \text{SO}(8)\) is given by the adjoint representation of \(\text{SU}(3)\), see [GWZ] and example \(Q^7_F\) in [Ho].

**Example 4.** The manifold defined by the diagram \((5.6)\) is the sphere \(S^{14}\) and the embedding \(\text{Spin}(7) \hookrightarrow \text{SO}(15)\) is given by \(\varrho_7 \oplus \Delta_7\) where \(\Delta_7\) is the spin representation of \(\text{Spin}(7)\), see, for example, [GWZ].

We know that \(N_{\text{Spin}(7)}(\text{SU}(3))/\text{SU}(3) = \mathbb{Z}_2\) and the generator can be represented as, for example, \(A = \text{diag}(I_3, -I_4)\). Both \(\mathcal{G}_2\) and \(\text{Spin}(6)\) are invariant under the conjugation of \(A\). Hence any variation of the diagram gives the same cohomogeneity one manifold.

**Example 5.** The manifold defined by the diagram \((5.7)\) is the Grassmannian \(\text{SO}(9)/\text{SO}(2) \times \text{SO}(7)\), see, for example, [UC].

Following a similar argument as in the previous case, any variation of this diagram does not give us a new cohomogeneity one manifold.

The diagram \((5.8)\) is not primitive and we have seen that \(\text{SO}(2) \cdot \text{SU}(3)\) is contained in \(\text{Spin}(6)\).

**Example 6.** The manifold defined by the diagram \((5.9)\) is the projective space \(\mathbb{C}P^7\), see, for example, [UC].

A similar argument shows that any variation does not give us a new cohomogeneity one manifold.
Example 7. The manifold defined by the diagram (5.10) is the Grassmannian $SO(p + q + 2) / (SO(p + 1) \times SO(q + 1))$ and the embedding $SO(p + q + 1) \hookrightarrow SO(p + q + 2)$ is given by $\varrho_{p+q+1} \oplus \text{id}$.

Let $K^-$ and $K^+$ denote $SO(p) \times SO(q + 1)$ and $SO(p + 1) \times SO(q)$ respectively and assume that one of $p, q$ is bigger than 1. If $p \neq q$, by Proposition 5.10 any $\tau \in \text{Aut}(G, H)$ leaves both $K^\pm$ invariant. So we only need to consider the case $p = q$. In this case, there is one automorphism of $H$ given by conjugation of the matrix

\[
J = \begin{pmatrix}
I_p \\
1 \\
I_p
\end{pmatrix},
\]

where the entries without specifying values have zeros. But $K^\pm$ switch each other by the conjugation of $J$. Therefore there is no new manifold from the variation.

**Step 2:** Combining the results in Proposition 5.5 and Proposition 5.10 there are a few triples $H \subset K \subset G$ which need to be considered. In the following, we analyze each of them.

- $U(1) \times U(1) \subset U(2) \subset SU(3)$.
  There are only two different $U(2)$’s between $U(1) \times U(1)$ and $SU(3)$ and the primitive diagram gives the sphere $S^7$. It is already appeared in Step 1.

- $SO(p) \times SO(p) \subset SO(p + 1) \times SO(p) \subset SO(2p + 1)(p \geq 2)$.
  The conjugation by $J$ defined in (5.11) maps $SO(p + 1) \times SO(p)$ to $SO(p) \times SO(p + 1)$, so the variation gives the Grassmannian $SO(2p + 2) / (SO(p + 1) \times SO(p + 1))$ which already appeared in Step 1.

- $SO(p) \subset SO(p + 1) \subset SO(p + 2)$.
  If $p$ is odd, then $\text{Aut}(G, H) = N_G(H) = S(O(2) \times O(p))$ is connected and hence any variation gives the double. If $p$ is even then the automorphism $\lambda$ leaves $K$ invariant too. If $p = 6$, then $\sigma$ does not leave any $SO(6)$ invariant. So this triple only gives a double.

- $SU(3) \subset Spin(6) \subset Spin(7)$.
  Let $i : SU(3) \hookrightarrow Spin(6)$ and $j : Spin(6) \hookrightarrow Spin(7)$ be the embeddings. Since $SU(3)$ is simply-connected, we have the following commutative diagram:

\[
\begin{array}{c}
SU(3) \xrightarrow{i} Spin(6) \xrightarrow{j} Spin(7) \\
\downarrow \text{id} \quad \downarrow \pi \quad \downarrow \pi \\
SU(3) \xrightarrow{\gamma} SO(6) \xrightarrow{} SO(7).
\end{array}
\]
The embedding $\gamma$ is given by the representation $[\pi_1]_\mathbb{R}$ of $\text{SU}(3)$. The outer automorphism (the complex conjugation) of $\text{SU}(3)$ is given by an inner automorphism of $\text{SO}(7)$, the conjugation by the matrix $\text{diag}(I_3, -I_4)$, and $\text{SO}(6)$ is invariant by the conjugation. So every element in $N_{\text{Spin}(7)}(\text{SU}(3))$ leaves $\text{Spin}(6)$ invariant and the variation gives only a double.

- $\text{SU}(3) \subset \mathfrak{g}_2 \subset \text{SO}(7)$.

As seen in the previous example, conjugation by the matrix $\text{diag}(I_3, -I_4)$ represents the outer automorphism of $\text{SU}(3)$. From the embedding of the Lie algebras $\mathfrak{g}_2 \subset \mathfrak{so}(7)$, see for example [He], it is easy to check that $\mathfrak{g}_2$ is invariant by the conjugation and hence $\mathfrak{g}_2$ is also invariant. So only the double can be obtained from this triple.

- $\text{Spin}(6) \subset \text{Spin}(7) \subset \text{SO}(8)$.

The subgroup $\text{Spin}(6)$ embeds in $\text{SO}(8)$ as the ordinary $\text{SU}(4)$ and then $N_{\text{SO}(8)}(\text{Spin}(6))$ is a circle. It follows that any variation by an element in $\text{SO}(8)$ gives us a double. $\text{Spin}(7)$ is also invariant under the outer automorphism $\lambda$ of $\text{SO}(8)$, so we only have a double from this triple.

- $\text{Spin}(8) \subset \text{Spin}(9) \subset F_4$.

The pair $(\text{Spin}(8), F_4)$ appeared in the classification of isotropy irreducible Riemannian manifolds in [WZ2]. There are three different embeddings of $\text{Spin}(9)$ in $F_4$ which are denoted by $K_i (i = 1, 2, 3)$ and every outer automorphism of $\text{Spin}(8)$ lifts to an inner automorphism of $F_4$. We use the same notations as $\lambda$ and $\sigma$ for their images in $\text{Aut}(F_4)$. Then $\lambda$ exchanges $K_1$, $K_2$ and fixes $K_3$, and $\sigma$ permutes $K_i$ cyclically. Other than the diagram $\text{Spin}(8) \subset \{K_1, K_1\} \subset F_4$ which defines the double, we have the following three group diagrams:

\[
\text{Spin}(8) \subset \{K_1, K_2\} \subset F_4, \quad \text{Spin}(8) \subset \{K_2, K_3\} \subset F_4, \quad \text{Spin}(8) \subset \{K_1, K_3\} \subset F_4.
\]

If we apply $\sigma$ to the first diagram, then we get the second one. Then we apply $\lambda$ to the second one, we obtain the last one. So the three group diagrams above are equivalent.

**Example 8.** The diagram is

\[
\text{Spin}(8) \subset \{\text{Spin}(9)_1, \text{Spin}(9)_2\} \subset F_4
\]

and the manifold is the sphere $S^{25}$ where $F_4$ is embedded into $\text{SO}(26)$ by its unique 26 dimensional representation, see, for example, [GWZ].

- $\text{SO}(3) \subset \text{SO}(4) \subset \mathfrak{g}_2$.

All three groups are embedded in $\text{SO}(7)$ which acts on the Cayley numbers $\mathbb{O}$ fixing the identity element $1$ and $\mathfrak{g}_2$ is the automorphism group of $\mathbb{O}$.

Let $\{1, i, j, k, e, ie, je, ke\}$ be the basis of $\mathbb{O}$ over the reals, then $\mathbb{O}$ can be written as $\mathbb{H} \oplus \mathbb{He}$. For every element $(q_1, q_2) \in \text{Sp}(1) \times \text{Sp}(1)$, it acts on $a + be \in \mathbb{O}$ by $(q_1a\overline{q_1}) + (q_2b\overline{q_1})e$. The
kernel of the action is \{ (1,1), (-1,-1) \}, so it induces an action by \text{SO}(4). If we choose \( (q_1, q_2) \in \Delta \text{Sp}(1) \), then it induces an \text{SO}(3) action on the Cayley numbers. It is clear from the action that \text{SO}(3) fixes the elements 1, e and its normalizer in \text{SO}(7) consists of the reflection about the real line and the rotation \( R(t) \) as follows:

\[
\begin{align*}
\text{i} & \mapsto \text{i} \cos t + \text{e} \sin t, \\
\text{j} & \mapsto \text{j} \cos t + \text{e} \sin t, \\
\kappa & \mapsto \kappa \cos t + \text{e} \sin t, \\
\text{ie} & \mapsto -\text{i} \sin t + \text{e} \cos t, \\
\text{je} & \mapsto -\text{j} \sin t + \text{e} \cos t, \\
\kappa \text{e} & \mapsto -\kappa \sin t + \text{e} \cos t.
\end{align*}
\]

The reflection is not an automorphism of \( \mathcal{O} \) and a computation shows that \( R(t) \in \mathfrak{g}_2 \) if and only if \( t \) equals to \( 0, \frac{2}{3}\pi \), or \( \frac{4}{3}\pi \). Therefore \( N_G(H)/H = \mathbb{Z}_3 \) and it is generated by \( \theta = R(\frac{2}{3}\pi) \). From the action of \( \text{SO}(4) \) on \( \mathcal{O} \), we know that \( \theta \) does not leave \( \text{SO}(4) \) invariant. So except the double, we have

**Example 9.** The diagram is

\[
\text{SO}(3) \subset \{ \text{SO}(4), \text{Ad}_\theta(\text{SO}(4)) \} \subset \mathfrak{g}_2
\]

and the manifold is the Grassmannian \( \text{SO}(7)/\{ \text{SO}(3) \times \text{SO}(4) \} \) and \( \mathfrak{g}_2 \) acts on it via the embedding \( \mathfrak{g}_2 \subset \text{SO}(7) \) by its unique 7 dimensional representation.

\[\square\]

6. **Primitive and non-reducible actions with \( s = 3 \) and \( G \) not a simple Lie group**

In this section, we give a classification when \( G \) is not simple and \( s = 3 \). We assume that the diagram is primitive and nonreducible. The main result is

**Theorem 6.1.** Suppose a compact simply-connected manifold \( M \) admits a cohomogeneity one action by \( G \) and the cohomogeneity one diagram is primitive and non-reducible. If \( G \) is not a simple Lie group, then \( M \) is equivariantly diffeomorphic to a sphere, a complex or quaternionic projective space, or the Grassmannian \( \text{SO}(5)/\text{SO}(3) \times \text{SO}(2) \), see Table 10 and 11.

We separate the proof of this theorem into two different cases. In one case, there exists a primitive factor of \( H \) which is not contained in a single primitive factor of \( G \), i.e., it is diagonally embedded in \( G \). In the other case, we assume that such diagonally embedded factor does not exist. The results in the two cases are stated in Proposition 6.2 and Proposition 6.3 respectively.
Suppose the Lie algebra $\mathfrak{g}$ of $G$ has the decomposition as $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_1$ a simple factor. Accordingly the Lie algebra $\mathfrak{h}$ decomposes as $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{h}_2$ such that the embedding $\mathfrak{h} \subset \mathfrak{g}$ is given by $(X_0, X_1, X_2) \mapsto ((X_0, X_1), (X_0, X_2))$. Since the diagram is non-reducible, $\mathfrak{h}_0 \oplus \mathfrak{h}_i$ is a proper subspace of $\mathfrak{g}_i$ for $i = 1, 2$. Fix a bi-invariant inner product $Q_i$ on $\mathfrak{g}_i$ and denote the orthogonal complement of $\mathfrak{h}_0 \oplus \mathfrak{h}_i$ by $p_i$ for $i = 1, 2$ and they are nonzero vector spaces. We separate our discussion into two different cases.

**Case A.** $\mathfrak{h}_0$ is a nontrivial Lie algebra.

Let us denote the isotropy representation of the pair $(\mathfrak{g}_i, \mathfrak{h}_0 \oplus \mathfrak{h}_i)$ by $\zeta_i$, then the isotropy representation $\text{Ad}_h$ is

$$\chi = (\zeta_1 \otimes \text{Id}) \oplus (\text{Id} \otimes \zeta_2) \oplus (\text{Id} \otimes \text{ad}_{\mathfrak{h}_0} \otimes \text{Id}),$$

where $\text{ad}_{\mathfrak{h}_0}$ is the isotropy representation of the pair $(\mathfrak{h}_0 \oplus \mathfrak{h}_0, \Delta \mathfrak{h}_0)$ and $\Delta \mathfrak{h}_0$ is the image of the diagonal embedding $\mathfrak{h}_0 \subset \mathfrak{h}_0 \oplus \mathfrak{h}_0$. From the assumption that $s = 3$, each $\chi_i$ is irreducible, and then $\zeta_1$ and $\zeta_2$ are irreducible and $\mathfrak{h}_0$ is primitive. Since the diagram is non-reducible, $\chi_3 = \text{Id} \otimes \text{ad}_{\mathfrak{h}_0} \otimes \text{Id}$ is not equivalent to $\chi_1$ or $\chi_2$.

**Proposition 6.2.** Suppose $H \subset \{K, K^\perp\} \subset G$ is a primitive and non-reducible group diagram. If the triple of Lie algebras $\{\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}\}$ lies in **Case A**, then the cohomogeneity one manifold is either a sphere, a complex or quaternionic projective space, or the Grassmannian $SO(5)/(SO(2) \times SO(3))$.

**Proof.** If $\chi_1 = \zeta_1 \otimes \text{Id}$ is equivalent to $\chi_2 = \text{Id} \otimes \zeta_2$, then both $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are zero vector space. It follows that $\mathfrak{g}_2 = \mathfrak{g}_1$. There are a few examples of the triple $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ listed as follows:

1. $\Delta \mathfrak{u}(1) \subset \mathfrak{su}(2) \oplus \mathfrak{u}(1) \subset \mathfrak{su}(2) \oplus \mathfrak{su}(2),$
2. $\Delta \mathfrak{u}(1) \subset \Delta \mathfrak{su}(2) \subset \mathfrak{su}(2) \oplus \mathfrak{su}(2),$
3. $\Delta \mathfrak{u}(1) \subset \mathfrak{u}(1) \oplus \mathfrak{u}(1) \subset \mathfrak{su}(2) \oplus \mathfrak{su}(2),$
4. $\Delta \mathfrak{so}(n) \subset \Delta \mathfrak{so}(n + 1) \subset \mathfrak{so}(n + 1) \oplus \mathfrak{so}(n + 1)$ with $n \geq 3,$
5. $\Delta \mathfrak{su}(2) \subset \mathfrak{su}(2) \oplus \mathfrak{su}(2) \subset \mathfrak{sp}(2) \oplus \mathfrak{sp}(2),$
6. $\Delta \mathfrak{so}(3) \subset \mathfrak{so}(3) \oplus \mathfrak{so}(3) \subset \mathfrak{su}(3) \oplus \mathfrak{su}(3),$
7. $\Delta \mathfrak{so}(3) \subset \mathfrak{so}(3) \oplus \mathfrak{so}(3) \subset \mathfrak{g}_2 \oplus \mathfrak{g}_2.$

Only when $\mathfrak{h} = \Delta \mathfrak{u}(1)$ and $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, we have primitive diagrams. The cohomogeneity one manifolds are 6 dimension, see example $Q_A^6$ and $Q_C^6$ in [Ho]. The manifolds are $\mathbb{S}^6$, $\mathbb{C}P^3$ and the Grassmannian $SO(5)/(SO(2) \times SO(2))$.

Suppose the three summands are non-equivalent, then we have the following possibilities for the Lie algebra $\mathfrak{k}$ of the singular isotropy subgroups:
A.1: $\mathfrak{t} = \mathfrak{h}_1 \oplus \mathfrak{h}_0 \oplus \mathfrak{g}_2$ and $G_2/H_2$ is a sphere;
A.2: $\mathfrak{t} = \mathfrak{g}_1 \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_2$ and $G_1/H_1$ is a sphere;
A.3: $\mathfrak{t} = \mathfrak{h}_1 \oplus \mathfrak{h}_0 \oplus \mathfrak{h}_2$ and $\mathfrak{h}_0 = \mathfrak{u}(1)$ or $\mathfrak{su}(2);
A.4: \mathfrak{t} = \mathfrak{h} \oplus \mathfrak{p}_1$ and $[\mathfrak{p}_1, \mathfrak{p}_1] \subset \mathfrak{h}_1$;
A.5: $\mathfrak{t} = \mathfrak{h} \oplus \mathfrak{p}_2$ and $[\mathfrak{p}_2, \mathfrak{p}_2] \subset \mathfrak{h}_2$.

Case A.4 and A.5 are excluded by the non-reducibility assumption. We consider Case A.4 first.

Suppose the Lie algebra of one singular isotropy subgroup is given in Case A.4. Then we have $[\mathfrak{p}_1, \mathfrak{p}_1]$ is a proper subspace of $\mathfrak{h}_1 \oplus \mathfrak{h}_0$ which implies that the strong isotropy pair $(\mathfrak{g}_1, \mathfrak{h}_1 \oplus \mathfrak{h}_0)$ is not a symmetric pair and it is in Wolf’s classification [Wo1]. If $\mathfrak{h}_1$ is a zero vector space, then $[\mathfrak{p}_1, \mathfrak{p}_1] = 0$ and then we have

$$Q_1([X_0, Y_1], Y_2) = Q_1(X_0, [Y_1, Y_2]) = 0,$$

for any $X_0 \in \mathfrak{h}_0, Y_1, Y_2 \in \mathfrak{p}_1$,

i.e., $\text{ad}_{\mathfrak{h}_0}$ is the trivial representation and it implies $\mathfrak{g}_1 = \mathfrak{u}(1)$ and $\mathfrak{h}_0 = 0$ which contradicts the assumption that $\mathfrak{h}_0$ is not the zero vector space. If $\mathfrak{h}_1 \neq 0$, then $\mathfrak{g}' = \mathfrak{p}_1 \oplus \mathfrak{h}_1$ is a Lie subalgebra of $\mathfrak{g}_1$. Let $G_1, L$ and $G'$ be the connected Lie groups whose Lie algebras are $\mathfrak{g}_1$, $\mathfrak{h}_0 \oplus \mathfrak{h}_1$ and $\mathfrak{g}'$ respectively, then $G'$ acts transitively on the homogeneous space $G_1/L$ with $G_1$ a simple Lie group, i.e., $(G_1, L, G')$ is in Table 5. Since the pair $(\mathfrak{g}_1, \mathfrak{h}_0 \oplus \mathfrak{h}_1)$ is strongly isotropy irreducible, it is $(\mathfrak{so}(4n), \mathfrak{sp}(1) \oplus \mathfrak{sp}(n))$ where $n \geq 2$. It also follows that $\mathfrak{h}_1 \oplus \mathfrak{p}_1 = \mathfrak{so}(4n-1)$. Since $\mathfrak{g}_1 = \mathfrak{h}_0 \oplus \mathfrak{h}_1 \oplus \mathfrak{p}_1$, we have $\dim \mathfrak{h}_0 = 4n - 1$ that is not equal to $\dim \mathfrak{sp}(1)$ or $\dim \mathfrak{sp}(n)$ when $n \geq 2$, i.e., $\mathfrak{h}_0$ is not either $\mathfrak{sp}(1)$ or $\mathfrak{sp}(n)$ which gives a contradiction.

In Case A.5, since the pair $(\mathfrak{g}_2, \mathfrak{h}_0 \oplus \mathfrak{h}_2)$ is strongly isotropy irreducible and the diagram is non-reducible, $\mathfrak{g}_2$ is a simple Lie algebra. Using a similar argument as in Case A.4, this case is also excluded.

In Case A.3, the pair $(\mathfrak{g}, \mathfrak{t})$ is not strongly isotropy irreducible. The isotropy representation $\text{Ad}_{\mathfrak{h}_0}$ has two irreducible summands $\chi_1 = \zeta_1 \otimes \text{Id}$ and $\chi_2 = \text{Id} \otimes \zeta_2$ and the isotropy representation of the pair $(\mathfrak{t}, \mathfrak{h}_1)$ is $\chi_3 = \text{Id} \otimes \text{ad}_{\mathfrak{h}_0} \otimes \text{Id}$. Since $\mathfrak{g}_1$ is simple, $\chi_1$ is not equivalent to $\chi_3$. If $\chi_2$ is equivalent to $\chi_3$, then $\mathfrak{g}_2$ contains $\mathfrak{h}_0$ factor and the diagram is reducible. From Remark 5.3 since $\chi_1$ is not equivalent to $\chi_2$, the manifold is a sphere via sum action.

In Case A.1 the pair $(\mathfrak{g}_2, \mathfrak{h}_0 \oplus \mathfrak{h}_2)$ is strongly isotropy irreducible, with $\mathfrak{h}_0$ a primitive Lie algebra and $(\mathfrak{g}_2, \mathfrak{h}_2)$ a spherical pair. Similar properties hold for the pair $(\mathfrak{g}_1, \mathfrak{h}_0 \oplus \mathfrak{h}_1)$ in Case A.2. From the classification of strongly isotropy irreducible spaces, it follows that $(\mathfrak{g}_2, \mathfrak{h}_0 \oplus \mathfrak{h}_2)$ is either $(\mathfrak{su}(n+1), \mathfrak{u}(1) \oplus \mathfrak{su}(n))$ or $(\mathfrak{sp}(n+1), \mathfrak{sp}(1) \oplus \mathfrak{sp}(n))$ with $n \geq 1$.

If one triple is in Case A.3, then the manifold is a sphere via sum action. We assume that both triples are strongly isotropy irreducible and in Case A.1 or A.2. Thus $\mathfrak{h}_0$ is either $\mathfrak{u}(1)$
or $\mathfrak{sp}(1)$ and then the assumption that the diagram is non-reducible implies that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are simple Lie algebras. W.L.O.G., we may assume that $(\mathfrak{g}_1, \mathfrak{h}_1) = (\mathfrak{su}(p + 1), \mathfrak{su}(p))$ and then $\mathfrak{h}_0 = \mathfrak{u}(1)$. If $\mathfrak{h}_2 = 0$, then $\mathfrak{g}_2 = \mathfrak{su}(2)$ and the triple of Lie algebras is $\mathfrak{su}(p) \oplus \mathfrak{u}(1) \subset \mathfrak{su}(p + 1) \oplus \mathfrak{u}(1) \subset \mathfrak{su}(p + 1) \oplus \mathfrak{su}(2)$.

So the principal isotropy representation is
\[
\chi = (\text{Id} \otimes [\phi]_{\mathbb{R}}) \oplus ([\pi_1]_{\mathbb{R}} \otimes [\phi]_{\mathbb{R}}) \oplus (\text{Id} \otimes \text{Id}).
\]

Let $\mathfrak{k}^- = \mathfrak{su}(p + 1) \oplus \mathfrak{u}(1)$ and then $\mathfrak{k}^+ = \mathfrak{su}(p) \oplus \mathfrak{u}(1) \oplus \mathfrak{su}(2)$ otherwise $\mathfrak{k}^+ = \mathfrak{k}^-$. Thus we have

**Example 10.** The diagram is
\[
SU(p) \times \Delta U(1) \subset \{ SU(p + 1) \times U(1), U(p) \times SU(2) \} \subset SU(p + 1) \times SU(2)
\]
and the manifold is the complex projective space $\mathbb{C}P^{p+2}$, see for example, \cite{Uc}.

When $p = 1$, the diagram already appeared in \cite{Ho} as example $Q_D^6$.

If $\mathfrak{h}_2 \neq 0$, then $(\mathfrak{g}_2, \mathfrak{u}(1) \oplus \mathfrak{h}_2)$ is a strongly isotropy irreducible pair and the three summands of the principal isotropy representation are pairwise non-equivalent. If the manifold is not a double, then $\mathfrak{t}^- = \mathfrak{su}(p) \oplus \mathfrak{u}(1) \oplus \mathfrak{g}_2$, $\mathfrak{t}^+ = \mathfrak{su}(p + 1) \oplus \mathfrak{u}(1) \oplus \mathfrak{h}_2$ and $\mathfrak{h} = \mathfrak{su}(p) \oplus \Delta \mathfrak{u}(1) \oplus \mathfrak{h}_2$. Furthermore we have that $(\mathfrak{g}_2, \mathfrak{h}_2)$ is a spherical pair and then is $(\mathfrak{su}(q + 1), \mathfrak{su}(q))$ for $q \geq 2$. Therefore we have

**Example 11.** The diagram is
\[
SU(p) \Delta SU(1) SU(q) \subset \{ U(p) SU(q + 1), SU(p + 1) U(q) \} \subset SU(p + 1) SU(q + 1)
\]
and the manifold is the complex projective space $\mathbb{C}P^{p+q+1}$.

Next we assume that $(\mathfrak{g}_1, \mathfrak{h}_1) = (\mathfrak{sp}(p + 1), \mathfrak{sp}(p))$ and then $\mathfrak{h}_0 = \mathfrak{sp}(1)$. A similar argument shows that if the manifold is not a double, then we have

**Example 12.** The diagram is
\[
Sp(p) \Delta Sp(1) Sp(q) \subset \{ Sp(p + 1) Sp(q), Sp(p) Sp(q + 1) \} \subset Sp(p + 1) Sp(q + 1)
\]
for $p, q \geq 1$, and the manifold is the quaternionic projective space $\mathbb{H}P^{p+q+1}$, see \cite{Iw1}.

\[\square\]

**Case B.** $\mathfrak{h}_0$ is a trivial Lie algebra, i.e., each primitive factor of $\mathfrak{h}$ lies in some primitive factor of $\mathfrak{g}$ as a proper subspace and thus $\mathfrak{g}$ has two or three factors.
Proposition 6.3. Suppose \( H \subset \{ K^\pm \} \subset G \) is a primitive and non-reducible group diagram. If the triples of Lie algebras \( \{ h \subset \mathfrak{t}^\pm \subset g \} \) are in Case B, then the cohomogeneity one manifold is a sphere with a sum action.

Proof. We claim that \( g \) has exactly two primitive factors. In fact, from Proposition 1.20 in [Ho], there are at most two \( u(1) \) factors since the manifold is simply-connected. If there are exactly two \( u(1) \) factors, i.e., \( G = G_0 \times T^2 \) with \( G_0 \) semisimple, then both \( l_{\pm} = 1 \), \( K^\pm = H \cdot S^1_{\pm} \) and the projections of \( S^1_{\pm} \) to \( T^2 \) generate \( T^2 \). Since \( s = 3 \), it follows that \( G_0/H \) is strongly isotropy irreducible and both \( S^1_{\pm} \) are subgroup in \( T^2 \). So the projections of \( S^1_{\pm} \) to the \( G_0 \) factor are zero and there exists an intermediate subgroup \( H \times T^2 \) which implies that the diagram is not primitive. Next we may assume that \( g \) has three factors and at most one of them is \( u(1) \). It follows that the three principal isotropy summands are pairwisely non-equivalent and then the pair \( (g, \mathfrak{t}) \) is strongly isotropy irreducible. Let \( g = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \) and \( h = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 \), then each \( (\mathfrak{g}_i, \mathfrak{h}_i) \) is strongly isotropy irreducible or \( \mathfrak{h}_i = 0 \). W.L.O.G., we may assume that \( \mathfrak{t} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{h}_3 \) and then from the classification of transitive actions on the sphere, there is no Lie group pair \((K, H)\) with \( K/H \) a sphere such that the Lie algebras are given by \((\mathfrak{t}, \mathfrak{h})\).

Suppose \( h = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \) and \( g = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \) where \( \mathfrak{g}_1, \mathfrak{g}_2 \) are primitive Lie algebras and \( \mathfrak{h}_i \subset \mathfrak{g}_i \) for \( i = 1, 2 \). W.L.O.G., we may assume that \( \mathfrak{g}_1 = \mathfrak{h}_1 \oplus \mathfrak{p}_1, \mathfrak{g}_2 = \mathfrak{h}_2 \oplus \mathfrak{p}_2 \oplus \mathfrak{p}_3 \) and \( \mathfrak{p}_i \)'s are nonzero vector spaces. For the intermediate Lie algebra \( \mathfrak{t} \), note that \((\mathfrak{t}, \mathfrak{h})\) is a spherical pair that implies \( \mathfrak{t} \) cannot be \( \mathfrak{h} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_2 \) or \( \mathfrak{h} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_3 \), so we have the following four possibilities:

B.1: \( \mathfrak{t} = \mathfrak{g}_1 \oplus \mathfrak{h}_2 \) and \( (\mathfrak{g}_1, \mathfrak{h}_1) \) is a spherical pair;
B.2: \( \mathfrak{t} = \mathfrak{h}_1 \oplus \mathfrak{p}_3, \mathfrak{l} = \mathfrak{h}_2 \oplus \mathfrak{p}_3 \) is a Lie algebra and \((\mathfrak{l}, \mathfrak{h}_2)\) is a spherical pair;
B.3: \( \mathfrak{t} = \mathfrak{h}_1 \oplus \mathfrak{p}_2, \mathfrak{l} = \mathfrak{h}_2 \oplus \mathfrak{p}_2 \) is a Lie algebra and \((\mathfrak{l}, \mathfrak{h}_2)\) is a spherical pair;
B.4: \( \mathfrak{t} = \mathfrak{h}_1 \oplus \mathfrak{g}_2 \) and \( (\mathfrak{g}_2, \mathfrak{h}_2) \) is a spherical pair.

We see that only in Case B.4, \((\mathfrak{g}, \mathfrak{t})\) is a strongly isotropy irreducible pair. In this case, since \( \mathfrak{g}_2 \) is primitive, \((\mathfrak{g}_2, \mathfrak{h}_2) = (\mathfrak{su}(k+1), \mathfrak{su}(k)) \) with \( k \geq 2 \). Hence the triple of Lie algebras is

\[ \mathfrak{h}_1 \oplus \mathfrak{su}(k) \subset \mathfrak{h}_1 \oplus \mathfrak{su}(k+1) \subset \mathfrak{g}_1 \oplus \mathfrak{su}(k+1), \quad k \geq 2 \]

and \((\mathfrak{g}_1, \mathfrak{h}_1)\) is strongly isotropy irreducible.

In Case B.1 the pair \((\mathfrak{g}, \mathfrak{t})\) is not strongly isotropy irreducible. If \( \dim \mathfrak{p}_1 \) is bigger than one, then the isotropy representation of \((\mathfrak{t}, \mathfrak{h})\) is not equivalent to any summands in the isotropy representation of \((\mathfrak{g}, \mathfrak{t})\) which implies that the cohomogeneity one manifold is a sphere. So we assume that \( \dim \mathfrak{p}_1 = 1 \), i.e., \( \mathfrak{g}_1 = u(1) \) and \( \mathfrak{h}_1 = 0 \). Then one of \( \mathfrak{p}_2 \) and \( \mathfrak{p}_3 \), say \( \mathfrak{p}_2 \), is one dimensional otherwise the three summands are pairwisely non-equivalent. Let
\[ \mathfrak{t}^- = \mathfrak{t} = \mathfrak{u}(1) \oplus \mathfrak{h}_2. \] If the manifold has one singular orbit whose codimension is bigger than 2, then \( \mathfrak{t}^+ = \mathfrak{p}_0 \oplus \mathfrak{p}_3 \oplus \mathfrak{h} \) where \( \mathfrak{p}_0 \) is a 1-dimensional subspace in \( \mathfrak{p}_1 \oplus \mathfrak{p}_2 \). In particular, the pair \((\mathfrak{g}, \mathfrak{t}^+)\) is strongly isotropy irreducible and thus belongs to Case B.4. It follows that \( \mathfrak{g}_2 = \mathfrak{su}(k + 1) \) and \( \mathfrak{h}_2 = \mathfrak{su}(k) \) with \( k \geq 2 \). Since the normalizer of \( \mathfrak{su}(k) \) in \( \mathfrak{u}(1) \oplus \mathfrak{su}(k + 1) \) is \( \mathfrak{u}(1) \oplus \mathfrak{su}(k) \), \( \mathfrak{t}^+ \) has to be \( \mathfrak{su}(k + 1) \) and then the diagram of the Lie algebras is

\[ \mathfrak{su}(k) \subset \{ \mathfrak{u}(1) \oplus \mathfrak{su}(k), \mathfrak{su}(k + 1) \} \subset \mathfrak{u}(1) \oplus \mathfrak{su}(k + 1), \quad k \geq 2. \]

If \( H \) is connected, then the group diagram is

\[ \text{SU}(k) \subset \{ \mathfrak{u}(1) \times \text{SU}(k), \text{SU}(k + 1) \} \subset \mathfrak{u}(1) \times \text{SU}(k + 1), \]

and if \( H \) is not connected, then for each \( m \geq 2 \), we have the following group diagram

\[ \mathbb{Z}_m \cdot \text{SU}(k) \subset \{ \mathfrak{u}(1) \times \text{SU}(k), \mathbb{Z}_m \cdot \text{SU}(k + 1) \} \subset \mathfrak{u}(1) \times \text{SU}(k + 1). \]

The cohomogeneity one manifold for these diagrams is the sphere \( S^{2k+3} \) via a sum action. When \( k = 2 \), see example \( Q_6^2 \) in [Ho].

In Case B.2, \((\mathfrak{g}, \mathfrak{t})\) is not strongly isotropy irreducible and the principal isotropy representation is

\[ \chi = (\text{ad}_{\mathfrak{g}_1/\mathfrak{h}_1} \otimes \text{Id}) \oplus (\text{Id} \otimes \text{ad}_{\mathfrak{g}_2/\mathfrak{h}_2}|_{\mathfrak{p}_2}) \oplus (\text{Id} \otimes \text{ad}_{\mathfrak{g}_2/\mathfrak{h}_2}|_{\mathfrak{p}_3}) \]

and \( \mathfrak{p}_3 \) is the representation space of \( \chi_3 = \text{ad}_{\mathfrak{t}/\mathfrak{h}} \). If \( \chi_1 \) and \( \chi_2 \) are not equivalent to \( \chi_3 \), then the manifold is a sphere. We consider the case when \( \chi_2 \) is equivalent to \( \chi_3 \), and thus the isotropy representation of \((\mathfrak{g}_2, \mathfrak{h}_2)\) has two equivalent summands. From Dickinson-Kerr's classification and the proof of Proposition 5.4, \((\mathfrak{g}_2, \mathfrak{h}_2)\) is either \((\mathfrak{so}(8), \mathfrak{g}_2)\) or \((\mathfrak{so}(7), \mathfrak{u}(3))\). Let \( \mathfrak{t}^- = \mathfrak{t} \) and if \( \text{ad}_{\mathfrak{t}^+}/\mathfrak{h} \) is irreducible, then \( \mathfrak{t}^+ \) is either \( \mathfrak{h} \oplus \mathfrak{p}_1 \) or \( \mathfrak{h} \oplus \mathfrak{p}_0 \) where \( \mathfrak{p}_0 \) is a subspace in \( \mathfrak{p}_2 \oplus \mathfrak{p}_3 \) with dimension \( \dim \mathfrak{p}_2 \). In the first case, \( \text{ad}_{\mathfrak{t}^+}/\mathfrak{h} \) is not equivalent to \( \text{ad}_{\mathfrak{g}/\mathfrak{h}}|_{\mathfrak{p}_2} \) and \( \text{ad}_{\mathfrak{g}/\mathfrak{t}^+} \) has two summands. So the manifold is a sphere. In the second case, the diagram is not primitive since both \( \mathfrak{t}^\pm \) are subalgebras of \( \mathfrak{h}_1 \oplus \mathfrak{g}_2 \). Therefore the isotropy representation of \((\mathfrak{g}, \mathfrak{t}^+)\) has two summands, i.e., it belongs to Case B.4. It follows that \( \mathfrak{g}_1 = \mathfrak{su}(k + 1), \mathfrak{h}_1 = \mathfrak{su}(k) \) and \((\mathfrak{g}_2, \mathfrak{h}_2)\) is strongly isotropy irreducible which contradicts the fact that it is one of \((\mathfrak{so}(8), \mathfrak{g}_2)\) and \((\mathfrak{so}(7), \mathfrak{u}(3))\). Next we consider the case when \( \chi_1 \) is equivalent to \( \chi_3 \), and thus \( \chi_1 = \chi_3 \). Since \( \mathfrak{u}(1) \oplus \mathfrak{t}^- \) is a Lie subalgebra of \( \mathfrak{u}(1) \oplus \mathfrak{g}_2 \) and the diagram is primitive, \( \mathfrak{t}^+ \) contains the subspace \( \mathfrak{p}_2 \) which implies that \( \mathfrak{t}^+ \) has codimension one in \( \mathfrak{g} \) and thus \((\mathfrak{g}, \mathfrak{t}^+)\) is strongly isotropy irreducible, i.e., it is in Case B.4. So we have \((\mathfrak{g}_2, \mathfrak{h}_2) = (\mathfrak{su}(k + 1), \mathfrak{su}(k)) \) for \( k \geq 2 \) and \( \mathfrak{t}^+ = \mathfrak{su}(k + 1) = \mathfrak{g}_2 \).

The diagram is not primitive since both \( \mathfrak{t}^\pm \) are contained in \( \mathfrak{g}_2 \).

Similar argument shows that there is no new cohomogeneity one manifold other than a sphere in Case B.3.
Finally we consider Case B.4. From the previous discussion, we may assume that both triples $h \subset \mathfrak{t}^{\pm} \subset \mathfrak{g}$ are in this case. It is easy to see that both $(\mathfrak{g}, h)$ are spherical pair and the action is a sum action on a sphere. □

APPENDIX A. COMPACT HOMOGENEOUS SPACES WITH TWO ISOTROPY SUMMANDS

W. Dickinson and M. Kerr classified compact simply-connected homogeneous spaces $G/\mathcal{K}$ in [DK] for which $G$ is a simple Lie group and the isotropy representation has two summands. Their classification is not complete and also contains some mistakes.

For the pair $(G, \mathcal{K}) = (E_6, \text{Spin}(6)\text{Spin}(4)\text{SO}(2))$ listed as IV.10 in their paper, the isotropy representation should be

$$\chi = \pi_2 \otimes [\pi \otimes \text{Id}]_R \otimes \text{Id} + [\pi_1 \otimes \pi \otimes \text{Id} \otimes \phi]_R + [\pi_3 \otimes \text{Id} \otimes \pi \otimes \phi]_R$$

and has three irreducible summands. For the pair $(E_8, \text{Spin}(12)\text{Spin}(4))$ listed as IV.37, the isotropy representation should be

$$\chi = \pi_1 \otimes [\pi \otimes \text{Id}]_R + \pi_6 \otimes \pi \otimes \text{Id} + \pi_6 \otimes \text{Id} \otimes \pi$$

and has three irreducible summands.

Except for these two examples, they also missed 5 pairs for which the isotropy representation has two summands. The complete classification is

**Theorem A.1.** Suppose $G/\mathcal{K}$ is a compact simply-connected homogeneous space with $G$ a simple Lie group. If the isotropy representation of $\mathcal{K}$ has exactly two summands, then

1. either $(G, \mathcal{K})$ is listed in the paper [DK] except the examples IV.10 and IV.37;
(2) or \((G, K)\) is one of the followings

**I.31** \(\text{Spin}(10)U(1) < U(16) < SO(32)\)
\[d_1 = 210, d_2 = 240\]
\[\rho = \pi_4 \otimes \phi_4]\n\[\chi = \pi_{15} \otimes \text{Id} + \pi_3 \otimes \phi_3\]

**I.32** \(SO(m) \times \text{Spin}(7) < SO(m) \times SO(8) < SO(m + 8)(m \geq 1)\)
\[d_1 = 7, d_2 = 8m\]
\[\rho = \pi_1 \otimes \text{Id} + \pi_1 \otimes \pi_3\]
\[\chi = \text{Id} \otimes \pi_1 + \pi_1 \otimes \pi_3\]

**II.15** \(SU(6) < Sp(10) < SU(20)\)
\[d_1 = 175, d_2 = 189\]
\[\rho = \pi_3\]
\[\chi = \pi_3^2 \oplus \pi_2 \pi_4\]

**III.12** \(Sp(m) \times U(1) < U(2m) < Sp(2m)(m \geq 2)\)
\[d_1 = (m - 1)(2m + 1), d_2 = 2m(2m + 1)\]
\[\rho = \pi_1 \otimes \phi_1\]
\[\chi = \pi_2 \otimes \text{Id} + [\pi_2^2 \otimes \phi_1]\]

**V.19** \(SU(2) \times SU(2) < SO(8)\)
\[d_1 = 7, d_2 = 15\]
\[\rho = \pi \otimes \pi^3\]
\[\chi = \text{Id} \otimes \pi^6 + \pi^2 \otimes \pi^4\]

Proof. Since \(G\) and \(K\) are connected Lie groups, we consider their Lie algebras \(\mathfrak{g}\) and \(\mathfrak{k}\), and thus the isotropy representation \(\text{ad}_{\mathfrak{g}/\mathfrak{k}}\) has two summands. We separate the proof into two parts: \(\mathfrak{g}\) is classical or exceptional Lie algebra. In the first part, we consider the case that \(\mathfrak{g}\) is a classical Lie algebra, i.e., it is one of \(so(n)\), \(su(n)\) and \(sp(n)\).

Case I. If \(\mathfrak{g}\) is \(so(n)\) and \(\rho : \mathfrak{k} \to \mathfrak{g}\) is the embedding, then the isotropy representation \(\chi\) of \(\mathfrak{g}/\mathfrak{k}\) is determined by \(\Lambda^2 \rho = \text{ad}_\mathfrak{k} + \chi\) where \(\Lambda^2 \rho\) is the exterior square of the representation \(\rho\).

Case I.A. If \(\rho = \rho_1 \oplus \rho_1^*\) and \(\rho_1\) is an irreducible representation of complex type, i.e., \(\rho_1(\mathfrak{k}) \subset \mathfrak{u}(m)\) with \(n = 2m\), then \(\Lambda^2 \rho = [\Lambda^2 \rho_1]\_R + [\rho_1 \otimes \rho_1^*]\) where \([\Lambda^2 \rho_1]\_R = \Lambda^2 \rho_1 \oplus \Lambda^2 \rho_1^*\) and \(\rho_1 \otimes \rho_1^*\) contains the representation \(\text{ad}_\mathfrak{k}\). If \(\mathfrak{k}\) is \(\mathfrak{u}(m)\), then \((\mathfrak{g}, \mathfrak{k})\) is a symmetric pair and thus \(\text{ad}_{\mathfrak{g}/\mathfrak{k}}\) is isotropy irreducible. If \(\mathfrak{k}\) is \(\mathfrak{su}(m)\), then it is Example I.24. If \(\mathfrak{k}\) is not \(\mathfrak{u}(m)\) or \(\mathfrak{su}(m)\), then the irreducible representation with highest weight \(\lambda + \lambda^*\) is contained in \(\rho_1 \otimes \rho_1^*\) but not in \(\text{ad}_\mathfrak{k}\), see [WZ]. Furthermore the isotropy representation \(\text{ad}_{\mathfrak{so}(2m)/\mathfrak{k}}\) for each pair has two irreducible summands. The pairs \((\mathfrak{g}, \mathfrak{k})\) give us example I.25 – I.28 in [DK] and the example III.31.
If \( \rho = \rho_1 \oplus \rho_1 \) and \( \rho_1 \) is of quaternionic type, then \( \Lambda^2 \rho = \Lambda^2 \rho_1 \oplus \Lambda^2 \rho_1 \oplus [\rho_1 \otimes \rho_1] \) and \( \Lambda^2 \rho_1 \) is of real type and \( \rho_1 \otimes \rho_1 = S^2 \rho_1 \oplus \Lambda^2 \rho_1 \). So the isotropy representation \( \text{ad}_{g/k} \) has more than two irreducible summands.

**Case I.B.** Next we assume that \( \rho = \rho_1 \oplus \rho_1^* \oplus \rho_2 \) where \( \rho_1 \) is irreducible of complex or quaternionic type and \( \rho_2 \) is a nontrivial representation with \( l = \dim \rho_2 \geq 2 \). If \( \rho_1 = \phi \), then \( \mathfrak{k} = \mathfrak{u}(1) \oplus \mathfrak{k}_0 \), \( \rho = [\phi]_R \otimes \text{Id} \oplus \text{Id} \otimes \rho_2 \) and \( \rho(\mathfrak{k}) \subset \mathfrak{so}(2) \oplus \mathfrak{so}(l) \subset \mathfrak{so}(2 + l) \). It follows that \( \Lambda^2 \rho = \text{Id} \oplus ([\text{Id} \otimes \Lambda^2 \rho_2] \oplus ([\phi]_R \otimes \rho_2)) \) and \( \text{ad}_k = \text{Id} \oplus \text{ad}_{\mathfrak{k}_0} \) is contained in \( \text{Id} \oplus ([\text{Id} \otimes \Lambda^2 \rho_2]) \). Since \( \mathfrak{k}_0 \) is proper in \( \mathfrak{so}(l) \) and \( \text{ad}_{g/k} \) has two irreducible summands, we have that \( \rho_2 \) is irreducible and the pair \((\mathfrak{so}(l), \mathfrak{k}_0)\) is isotropy irreducible. They are example I.1 – I.18 (when \( m = 2 \)) in [DK] and the special case of the example I.32 when \( m = 2 \).

If \( \rho_2 \) is the trivial representation, then \( \mathfrak{k} = \mathfrak{u}(m) \) and \( \mathfrak{g} = \mathfrak{so}(2m + 1) \) with \( m \geq 2 \). It is example I.18 (when \( m = 1 \)) in [DK].

**Case I.C.** We assume that every irreducible summand in \( \rho \) is of real type. If \( \rho \) is not irreducible, then \( \rho = \rho_1 \oplus \rho_2 \) with \( \dim \rho_i = n_i \) for \( i = 1, 2 \). Let \( \mathfrak{t}_2 = \ker \rho_1 \) and \( \mathfrak{t}_1 = \ker \rho_2 \). Then \( \mathfrak{k} = \mathfrak{t}_1 \oplus \mathfrak{t}_2 \), \( \Lambda^2 \rho = \Lambda^2 \rho_1 \oplus \Lambda^2 \rho_2 \oplus [\rho_1 \otimes \rho_2] \) and \( \text{ad}_k = \text{ad}_{\mathfrak{t}_1} \oplus \text{ad}_{\mathfrak{t}_2} \). Since \( \text{ad}_{\mathfrak{t}_1} \subseteq \Lambda^2 \rho_1 \) and the equality implies that \( \mathfrak{t}_i = \mathfrak{so}(n_i) \), one of \( \mathfrak{t}_i \), say \( \mathfrak{t}_2 \), is equal to \( \mathfrak{so}(n_2) \) and then \((\mathfrak{so}(n_1), \mathfrak{t}_1)\) is isotropy irreducible and \( \rho_1 \) is irreducible. These give us example I.1 – I.17, I.19 and I.30 in [DK] and the example I.32.

If \( \rho \) is irreducible, then \( \mathfrak{t} \) has at most two simple factors. We use the classification of Kraemer in [K]. These give us example I.20 – I.23, I.29, V.1 – V.5 in [DK] and the example V.19. This finishes the proof when \( \mathfrak{g} \) is an orthogonal group.

**Case II.** If \( \mathfrak{g} = \mathfrak{su}(n) \) and \( \rho : \mathfrak{k} \to \mathfrak{g} \) is the embedding, then the isotropy representation \( \chi \) of \( \mathfrak{g}/\mathfrak{k} \) is determined by \( \rho \otimes \rho^* = \text{Id} \oplus \text{ad}_k \oplus \chi \).

**Case II.a.** The image is contained in \( \mathfrak{sp}(m)(n = 2m) \) or \( \mathfrak{so}(n) \). We consider the first case where \( \rho(\mathfrak{k}) \) is contained in \( \mathfrak{so}(n) \). Let \( \chi_2 \) be the isotropy representation of \( \text{ad}_{\mathfrak{su}(n)/\mathfrak{so}(n)} \), then it remains irreducible when restricted to the proper subgroup \( \mathfrak{k} \). Dynkin classified the triples \((\mathfrak{k}, \mathfrak{so}(N), \chi)\) where \( \chi \neq \gamma_N \) such that the restriction of \( \chi \) to \( \mathfrak{k} \) remains irreducible. They are example II.13 and II.14 in [DK].

Next we assume that \( \rho(\mathfrak{k}) \subset \mathfrak{sp}(m) \subset \mathfrak{su}(2m) \). It follows that \( \text{ad}_{\mathfrak{su}(2m)/\mathfrak{sp}(m)} \) remains irreducible when restricted to the proper subalgebra \( \mathfrak{k} \). Such \( \mathfrak{k} \) are classified by Dynkin. They are example II.9 – II.12 in [DK] and the example II.15.

**Case II.b.** Suppose that \( \rho = \rho_1 \oplus \rho_2 \) is reducible with \( m = \deg \rho_1 \) and \( n = \deg \rho_2 \). Let \( \mathfrak{t}_2 = \ker \rho_1 \) and \( \mathfrak{t}_1 = \ker \rho_2 \), and then \( \mathfrak{k} = \mathfrak{t}_1 \oplus \mathfrak{t}_2 \), \( \rho = \sigma_1 \otimes \text{Id} \oplus \text{Id} \otimes \sigma_2 \) where \( \sigma_i \)s are
representations of \( \mathfrak{k} \). It follows that
\[
\rho \otimes \rho^* = (\sigma_1 \otimes \sigma_1^*) \otimes \text{Id} + \text{Id} \otimes (\sigma_2 \otimes \sigma_2^*) + (\sigma_1 \otimes \sigma_2^*) + (\sigma_1^* \otimes \sigma_2).
\]

Both of \( \mathfrak{k}_1 \) and \( \mathfrak{k}_2 \) cannot be proper subalgebras of \( \mathfrak{su}(m) \) and \( \mathfrak{su}(n) \) respectively. The case when \( \mathfrak{k}_1 = \mathfrak{su}(m) \) with \( \sigma_1 = \mu_m \) and \( \mathfrak{k}_2 = \mathfrak{su}(n) \) with \( \sigma_2 = \mu_n \) is example II.7 in \( \text{DK} \). If one of \( \mathfrak{k}_1 \) and \( \mathfrak{k}_2 \), say \( \mathfrak{k}_1 \), is a proper subalgebra of \( \mathfrak{su}(m) \), then \( \mathfrak{k}_2 \) has to be \( \mathfrak{u}(n) \) with \( \sigma_2 = \mu_n \) and \( \sigma_1 \) is an irreducible complex representation. Let \( \chi_1 \) denote the isotropy representation of \( \text{ad}_{\mathfrak{su}(m)/\mathfrak{k}_1} \) and then the summands of \( \text{ad}_{\mathfrak{g}/\mathfrak{k}_1} \) are \( \chi_1 \otimes \text{Id} \) and \( [\sigma_1 \otimes \mu_n]_\mathbb{R} \). Such \( \mathfrak{k}_1 \)'s can be classified using Kraemer’s results and they give us example II.1 – II.6 and II.8 in \( \text{DK} \).

**Case II.c.** Suppose that \( \rho \) is an irreducible complex representation and \( \rho(\mathfrak{k}) \) is not contained in some \( \mathfrak{sp}(m) \) or \( \mathfrak{so}(n) \). Then we can assume that \( \mathfrak{k} \) has at most two simple factors and we can use Kraemer’s classification. They are example V.6 – V.8 in \( \text{DK} \). This finishes the proof when \( \mathfrak{G} \) is a unitary group \( \text{SU}(n) \).

**Case III.** If \( \mathfrak{g} \) is \( \mathfrak{sp}(n) \) with \( n \geq 3 \) and \( \rho : \mathfrak{k} \rightarrow \mathfrak{g} \) is the embedding, then the isotropy representation \( \chi \) of \( \mathfrak{g}/\mathfrak{k} \) is determined by \( S^2 \rho = \text{ad} \oplus \chi \).

**Case III.a.** Suppose the image of \( \rho(\mathfrak{k}) \) is contained in \( \mathfrak{u}(n) \). If \( \mathfrak{k} \) is semi-simple, i.e., \( \rho(\mathfrak{k}) \subset \mathfrak{su}(n) \), then since \( \text{ad}_{\mathfrak{sp}(n)/\mathfrak{su}(n)} \) already has two summands, we have \( \mathfrak{k} = \mathfrak{su}(n) \). It is example III.8 in \( \text{DK} \). If \( \mathfrak{k} \) contains \( \mathfrak{u}(1) \) factors and its semi-simple part idea is denoted by \( \mathfrak{k}_0 \), then \( \mathfrak{su}(n)/\mathfrak{k}_0 \) is isotropy irreducible and the restriction the representation \( \pi_1^2 \) from \( \mathfrak{su}(n) \) to \( \mathfrak{k}_0 \) remains irreducible. From Dynkin’s classification, \( \mathfrak{k}_0 \) is \( \mathfrak{sp}(n/2) \) and the embedding is \( \pi_1 \). It gives us the example III.12.

**Case III.b.** If \( \rho = \rho_1 \oplus \rho_2 \) and \( \rho_1 \) is irreducible of real or quaternionic type, then \( \rho(\mathfrak{k}) \subset \mathfrak{sp}(n_1) \oplus \mathfrak{sp}(n_2) \). We may assume that \( \mathfrak{k} = \mathfrak{sp}(n_1) \oplus \mathfrak{k}_0 \) and \( \rho = \nu_{n_1} \otimes \text{Id} \oplus \text{Id} \otimes \sigma \). The isotropy representation of \( \mathfrak{sp}(n_2)/\mathfrak{k}_0 \) is irreducible and it is denoted by \( \chi_1 \). Since \( \text{ad}_{\mathfrak{g}/\mathfrak{k}_1} \) has two irreducible summands, \( \sigma \) is irreducible. They are example III.1 – III.7 in \( \text{DK} \). In fact Kraemer missed the pair \( (\mathfrak{g}_2 \oplus \mathfrak{sp}(1), \mathfrak{sp}(7)) \) in his classification.

**Case III.c.** If \( \rho \) is irreducible of real or quaternionic type, then we may assume that \( \mathfrak{k} \) has at most two primitive factors. We can use Kraemer’s classification and they are example III.9 – III.11 and V.9 – V.15 in \( \text{DK} \). This finishes the proof when \( \mathfrak{G} \) is a symplectic group.

In the second part, we consider the case when \( \mathfrak{g} \) is an exceptional Lie algebra. There are two different cases. First we assume that there is an intermediate subalgebra \( \mathfrak{l} \) between \( \mathfrak{k} \) and \( \mathfrak{g} \). So both pair \( (\mathfrak{g}, \mathfrak{l}) \) and \( (\mathfrak{l}, \mathfrak{k}) \) are isotropy irreducible. From the classification strong isotropy irreducible homogeneous spaces in \( \text{Wol} \), we can determine the possible \( \mathfrak{l} \) for each \( \mathfrak{h} \). Then we look at the possible \( \mathfrak{k} \) such that \( \mathfrak{l}/\mathfrak{k} \) is isotropy irreducible and \( \text{ad}_{\mathfrak{g}/\mathfrak{l}} \) remains irreducible when restricted to \( \mathfrak{k} \). Such examples of \( (\mathfrak{g}, \mathfrak{k}) \) give us example IV.1 – VI.48 in
Next we assume that \( \mathfrak{t} \) is maximal in \( \mathfrak{g} \) and then the isotropy representation of \( \mathfrak{g} \) splits as \( \text{ad}_\mathfrak{t} \) and \( \text{ad}_\mathfrak{g}/\mathfrak{t} \) when it is restricted to \( \mathfrak{t} \). We check the table of the branching rules in [MP] to see for which \((\mathfrak{g}, \mathfrak{t})\), \(\text{ad}_\mathfrak{g}/\mathfrak{t}\) has exactly two summands. They are example \( \text{V}.16 – \text{V}.18 \) in [DK] and thus the classification is finished. □

Appendix B. Collection of tables

In this appendix, we collect the tables which contain the classification of the case \( s = 3 \).

B.1. Primitive actions. From the classifications in Theorem 4.1, 5.1 and 6.1 except for a few sum actions on spheres, the primitive actions without fixed points and with \( s = 3 \) are non-reducible and the manifolds are spheres, projective spaces and Grassmannian manifolds \( \text{Gr}_m(\mathbb{R}^{m+n}) = \text{SO}(m+n)/(\text{SO}(m) \times \text{SO}(n))(m, n \geq 2) \). In the following, we first describe the actions on spheres, including the reducible ones, and then we list the actions on the other manifolds.

B.1.1. Actions on spheres. The cohomogeneity one actions on spheres were classified in [St], see the group diagrams in [GWZ]. A large class of cohomogeneity one actions on spheres with \( s = 3 \) is given by the sum actions. Recall that if \( L_i/H_i = S^{s_i}(i = 1, 2) \), then \( L_1 \times L_2 \) acts on the \( S^{s_1+s_2+1} \) via cohomogeneity one with diagram

\[
H_1 \times H_2 \subset \{L_1 \times H_2, H_1 \times L_2\} \subset L_1 \times L_2,
\]

and \( s = s_1 + s_2 \) where \( s_i \) is the number of the irreducible summands in the isotropy representation of \( L_i/H_i \). It is easy to see that any variation of the diagram is equivalent to the original one. If one singular orbit is codimension 2, say \( L_1/H_1 = S^1 \), then for every \( m \neq 0 \), we have the diagram

\[
Z_m \times H_2 \subset \{U'(1) \times H_2, Z_m \times L_2\} \subset U(1) \times L_2,
\]

where the \( U'(1) \) factor may diagonally embedded into \( U(1) \times L_2 \). The action is not effective if \( m \geq 2 \) and the non-effective kernel is \( Z_m \times 1 \).

Every sum action is primitive. The action is reducible if one of the spheres is given as \( U(n+1)/U(n), \text{SO}(4)/\text{SO}(3), \text{Sp}(n+1)U(1)/\text{Sp}(n)\Delta U(1) \) or \( \text{Sp}(n+1)\text{Sp}(1)/\text{Sp}(n)\Delta \text{Sp}(1) \), where \( n \geq 1 \). Using Table 1 of transitive actions on spheres, one can easily write down the diagram of sum actions with \( s \leq 3 \).

Other than sum actions, there are a few cohomogeneity one actions on sphere which have \( s = 3 \). All of them are primitive actions and they are listed in Table 10 where \( \pi \) is the representation of \( G \) on \( \mathbb{R}^n \). Note that the actions of \( \text{SU}(2) \times \text{SU}(2) \) on \( S^6 \) and \( \text{Spin}(7) \) on \( S^{14} \) are special cases of what are called generalized sum actions in [GWZ].
B.1.2. Actions on projective spaces and Grassmannian manifolds. The cohomogeneity one actions on projective spaces were classified in [Uc], [Iw1] and [Iw2]. Note that all cohomogeneity one actions on $\mathbb{C}P^n$ and $\mathbb{H}P^n$ are obtained from an action on an odd dimensional sphere when $U(1)$ and $Sp(1)$ is a normal subgroup in $G$ with induced action given by a Hopf action, see [GWZ]. Table 11 list these actions as well as those on the Grassmannian manifolds for which $s = 3$.

| $n$ | $G$ | $\pi$ | $K^-$ | $K^+$ | $H$ |
|-----|-----|-------|-------|-------|-----|
| 6   | $SU(2) \times SU(2)$ | $\pi_1 \otimes \pi_1 + \text{Id} \otimes \pi_1^2$ | $\Delta SU(2)$ | $SU(2) \times U(1)$ | $\Delta U(1)$ |
| 7   | $SU(3)$ | $\pi_1 \pi_2$ | $S(U(1) \times U(2))$ | $S(U(2) \times U(1))$ | $U(1) \times U(1)$ |
| 14  | $\text{Spin}(7)$ | $\nu_7 + \Delta_7$ | $\text{Spin}(6)$ | $\mathbb{G}_2$ | $SU(3)$ |
| 25  | $F_4$ | $\pi_1$ | $\text{Spin}(9)_1$ | $\text{Spin}(9)_2$ | $\text{Spin}(8)$ |

Table 10. Cohomogeneity one actions on $S^n$ with $s = 3$ which are not sum actions.

Table 11. Fixed-point free cohomogeneity one actions on projective spaces and Grassmannian manifolds with $s = 3$.

| $n$ | $G$ | $K^-$ | $K^+$ | $H$ |
|-----|-----|-------|-------|-----|
| $CP^3$ | $SU(2) \times SU(2)$ | $U(1) \times U(1)$ | $\mathbb{Z}_2 \cdot \Delta SU(2)$ | $\mathbb{Z}_2 \cdot \Delta U(1)$ |
| $CP^7$ | $\text{Spin}(7)$ | $SO(2) \cdot SU(3)$ | $\mathbb{Z}_2 \cdot \mathbb{G}_2$ | $\mathbb{Z}_2 \cdot SU(3)$ |
| $CP^n$ | $SO(n)$ | $U(4)$ | $\text{Spin}(7)$ | $SU(4)$ |
| $CP^{n+1}$ | $SO(2 + n)$ | $SO(2) \times SO(n)$ | $O(n + 1)$ | $\mathbb{Z}_2 \cdot SO(n)$ |
| $CP^p+q+1$ | $SU(p + 1) \times SU(q + 1)$ | $SU(p + 1)U(q)$ | $U(p)SU(q + 1)$ | $SU(p)\Delta U(1)SU(q)$ |
| $\mathbb{H}P^{p+q+1}$ | $Sp(p + 1) \times Sp(q + 1)$ | $Sp(p + 1)Sp(q)$ | $Sp(p)Sp(q + 1)$ | $Sp(p)\Delta Sp(1)Sp(q)$ |
| $Gr_2(\mathbb{R}^5)$ | $SU(2) \times SU(2)$ | $U(1) \times U(1)$ | $\Delta SU(2)$ | $\Delta U(1)$ |
| $Gr_3(\mathbb{R}^7)$ | $\mathbb{G}_2$ | $SO(4)$ | $SO(4)'$ | $SO(3)$ |
| $Gr_2(\mathbb{R}^9)$ | $\text{Spin}(7)$ | $SO(2) \cdot SU(3)$ | $\mathbb{G}_2$ | $SU(3)$ |
| $Gr_p(\mathbb{R}^{p+q+1})$ | $SO(p + q + 1)$ | $SO(p) \times SO(q + 1)$ | $SO(p + 1) \times SO(q)$ | $SO(p) \times SO(q)$ |

B.2. Reducible, non-primitive actions. The reducible, non-primitive cohomogeneity one manifolds with $s = 3$ and whose reduced diagram has $s \geq 4$ are classified in Table 12 and Table 13, see Theorem 4.1.

Table 12 is the classification where the manifold is a double, i.e., $K^- = K^+$, and Table 13 is the one where the manifold is not a double. In both tables, the last column contains the
conditions for the groups. If a homogeneous space appears in this column, it means that the space is strongly isotropy irreducible, for examples, in \textbf{R}.1 the space $G_1/(H_1H_2U(1))$ is strongly isotropy irreducible, in \textbf{R}.3 the space $G_1/H_1$ is strongly isotropy irreducible and not the circle, and in \textbf{R}.7 the space $G_1/H_1$ is an isotropy irreducible sphere. In both tables, $k \geq 2$ and $m \geq 1$ are positive integers. $\mathbb{Z}_m$ is a cyclic group in $K^-/H$ if it is a circle, and $\mathbb{Z}_1$ stands for the trivial group with one element. In Table 13, for the singular isotropy subgroups, we write $K^-$ on the top of $K^+$. There are further conditions for some of them.

- In example \textbf{R}.9 and \textbf{R}.10, the groups satisfy the Condition T1: the homogenous space $G_1/(H_1 \times H_0)$ has two irreducible isotropy summands, $G_1$ is simple and $H_0$ is $U(1)$ (example \textbf{R}.9) or $SU(2)$ (example \textbf{R}.10).
- In example \textbf{R}.11, the $U'(1)$ factor of $K$ is embedded into $G$ as $\{(e^{i\theta}, e^{im\theta}, 1) | 0 \leq \theta \leq 2\pi\}$ for some integer $p$. The diagram is the normal extension of example $Q_A^5$ in \cite{Ho}.
- In example \textbf{R}.12, the embedding of the $U'(1)$ factor of $K$ into $G$ is given as

$$ \left\{ \left( e^{i\theta}, \left( \beta(p\theta) \right)_A, 1 \right) \mid 0 \leq \theta \leq 2\pi, A \in SO(n) \right\} \subset U(1) \times SO(2 + n) \times U(1), $$

where $\beta(p\theta)$ is the rotation by angle $p\theta$ and $p$ is an integer.
- Example \textbf{R}.22 is the normal extension of example $N^5$ in \cite{Ho}, see the embeddings of $U(1)_-$ and $U(1)_+$ there.
- Example \textbf{R}.23 was discussed in the proof of Theorem 4.1 or see Lemma 4.3 in \cite{Ho}.

We consider the corresponding non-reducible diagram as $G = U(1) \times SO(n + 2)$. The reducible one can be obtained by normal extension since the subgroup $SO(2) \subset SO(n + 2)$ normalizes all isotropy subgroups. The embeddings of $K^\pm_c$ are given as

$$ K^\pm_c = \left\{ \left( e^{m\pm\theta}, \left( \beta(m\pm\theta) \right)_A \right) \mid 0 \leq \theta \leq 2\pi, A \in SO(n) \right\} $$

such that $K^-_c \neq K^+_c$ where $m_\pm$ and $n_\pm$ are integers. Take two cyclic groups $\mathbb{Z}_{k_\pm} \subset \{(e^{m\pm\theta}, \beta(m\pm\theta))\}$ and let $H_\pm = \mathbb{Z}_{k_\pm} \cdot SO(n)$. Let $H = < H_-, H_+ >$ generated by $H_\pm$ and let $K^\pm = K^\pm_c \cdot H$, then the diagram defines a simply connected manifold if and only if $\gcd(n_-, n_+, d) = 1$ where $d$ is the index of $H \cap K^-_c \cap K^+_c$ inside $K^-_c \cap K^+_c$.

Since the actions are non-primitive, the manifolds are bundles over lower dimensional bases. To identity the bundle structure, it is easy to work with non-reducible diagrams. Table 14 and 15 list the diagrams of the reduced actions in Table 12 and 13. Note that in example \textbf{R}.19, \textbf{R}.20 and \textbf{R}.21 we keep the reducible diagrams.
Table 12. Non-primitive fixed-point free cohomogeneity one manifolds with reducible actions. The action has $s = 3$ and its reduced action has $s \geq 4$. Part I: the manifold is a double.

In certain cases, the diagram defines a product action, i.e., the manifold is a product of a cohomogeneity one manifold and a homogeneous space. In Table 16, we specify each manifold in Table 12 and 13 as a product (if the action is a product action) or a (possibly non-trivial) bundle. The first seven examples R.3, R.4, R.7, R.8, R.11, R.15 and R.16 are products. Note that the family in example R.22 contains both trivial and non-trivial $S^3$ bundle over $S^2$ depending on the embeddings of the circles $U(1)_-$ and $U(1)_+$, see [Ho].

|   | $G$                      | $K^\pm$          | $H$                                      |   |
|---|--------------------------|-------------------|------------------------------------------|---|
| R.1 | $G_1 \times U(1) \times H_2$ | $H_1 U(1) \Delta H_2 U(1)$ | $\mathbb{Z}_m \cdot H_1 \Delta U(1) \Delta H_2$ | $G_1/(H_1 H_2 U(1))$ |
| R.2 | $G_1 \times SU(2) \times H_2$ | $H_1 SU(2) \Delta H_2 SU(2)$ | $H_1 \Delta SU(2) \Delta H_2$ | $G_1/(H_1 H_2 SU(2))$ |
| R.3 | $G_1 \times SU(2) \times U(1)$ | $H_1 SU(2) U(1)$ | $H_1 \Delta U(1)$ | $G_1/H_1 \neq S^1$ |
| R.4 | $G_1 \times Sp(n+1)Sp(1)$ | $H_1 Sp(n+1)Sp(1)$ | $H_1 Sp(n) \Delta Sp(1)$ | $G_1/H_1$ |
| R.5 | $G_1 \times G_2 \times U(1)$ | $H_1 H_2 U(1) U(1)$ | $\mathbb{Z}_m \cdot H_1 H_2 \Delta U(1)$ | $G_1/H_1 \neq S^1$ |
| R.6 | $G_1 \times G_2 \times SU(2)$ | $H_1 H_2 SU(2) SU(2)$ | $H_1 H_2 \Delta SU(2)$ | $G_1/H_1$ |
| R.7 | $G_1 \times G_2 \times H_0$ | $G_1 H_2 \Delta H_0$ | $H_1 H_2 \Delta H_0$ | $G_1/H_1 = S^k$ |
| R.8 | $U(1) \times G_1 \times H_0$ | $U(1) H_1 \Delta H_0$ | $H_1 \Delta H_0$ | $G_1/(H_1 H_0)$ |
| R.9 | $G_1 \times U(1)$ | $H_1 U(1) U(1)$ | $\mathbb{Z}_m \cdot H_1 \Delta U(1)$ | Condition T1 |
| R.10 | $G_1 \times SU(2)$ | $H_1 SU(2) SU(2)$ | $H_1 \Delta SU(2)$ | Condition T1 |
| R.11 | $U(1) \times SU(2) \times U(1)$ | $U'(1) \Delta U(1)$ | $\mathbb{Z}_m \cdot \Delta U(1)$ |   |
| R.12 | $U(1) \times SO(n+2) \times U(1)$ | $U'(1) SO(n) \Delta U(1)$ | $\mathbb{Z}_m \cdot SO(n) \Delta U(1)$ | $n \geq 2$ |
| R.13 | $G \supset K = K^\pm \supset \mathbb{Z}_m \cdot H$ in Table 6 |   | $m = 1$ if $K/H \neq S^1$ |   |
| R.14 | \( G \times U(1) \times SU(2) \) | \( H_1 U(1) \Delta SU(2) U(1) \) | \( Z_m \cdot H_1 SU(2) \Delta U(1) SU(2) \) | \( G_1/(H_1 U(1) SU(2)) \) |
| R.15 | \( G_1 \times SU(2) \times U(1) \) | \( H_1 U(1) U(1) \) | \( H_1 \Delta U(1) \) | \( G_1/H_1 \neq \mathbb{S}^1 \) |
| R.16 | \( G_1 \times Sp(n+1) \times Sp(1) \) | \( H_1 Sp(n+1) Sp(1) \) | \( H_1 Sp(n) \Delta Sp(1) \) | \( G_1/H_1 \) |
| R.17 | \( G_1 \times G_2 \times U(1) \) | \( H_1 H_2 U(1) U(1) \) | \( Z_m \cdot H_1 H_2 \Delta U(1) \) | \( G_1/H_1 = \mathbb{S}^k \) |
| R.18 | \( G_1 \times G_2 \times SU(2) \) | \( H_1 H_2 SU(2) SU(2) \) | \( H_1 H_2 \Delta SU(2) \) | \( G_1/H_1 = \mathbb{S}^k \) |
| R.19 | \( G_1 \times U(1) \) | \( H_1 U(1) U(1) \) | \( H_1 \Delta U(1) \) | \( G \supset K^+ \supset H_c \) is 1–5 in Table 6 |
| R.20 | \( G_1 \times U(1) \) | \( H_1 U(1) U(1) \) | \( Z_m \cdot H_1 \Delta U(1) \) | \( G \supset K^+_c \supset H_c \) is 6–8 in Table 6 |
| R.21 | \( G_1 \times SU(2) \) | \( H_1 SU(2) SU(2) \) | \( H_1 \Delta SU(2) \) | \( G \supset K^+ \supset H \) is 9–13 in Table 6 |
| R.22 | \( U(1) \times SU(2) \times U(1) \) | \( U(1)_- \cdot H \) | \( H \) |
| R.23 | \( U(1) \times SO(n+2) \times U(1) \) | \( K^c_+ \cdot H \) | \( H \) | \( n \geq 2 \) |

Table 13. Non-primitive fixed-point free cohomogeneity one manifolds with reducible actions. The action has \( s = 3 \) and its reduced action has \( s \geq 4 \). Part II: the manifold is not a double

B.3. Non-primitive, non-reducible actions. The non-primitive, non-reducible cohomogeneity one manifolds with \( s = 3 \) which are not doubles are classified in Table 17, see Theorem 4.2.

In Table 17 we use the same conventions in the tables of reducible, non-primitive actions, i.e., \( k \geq 2 \) and the homogeneous space appeared in the last column is strongly isotropy irreducible. The further conditions for some diagrams are

- In example N.5, the groups satisfy Condition T2: \( G_1/L_1 \) is strongly isotropy irreducible and the diagram \( H_1 \subset \{K_1, L_2\} \subset L_2 \) has \( s = 2 \).
- In example N.7, we have \( l_1, l_2 \geq 1 \).
|   | G          | $K^\pm$          | H                      |
|---|------------|------------------|------------------------|
| R.1 | $G_1$     | $H_1 U(1)$       | $Z_m \cdot H_1$       | $G_1/(H_1 H_2 U(1))$  |
| R.2 | $G_1$     | $H_1 SU(2)$      | $H_1$                  | $G_1/(H_1 H_2 SU(2))$ |
| R.3 | $G_1 \times SU(2)$ | $H_1 SU(2)$     | $H_1$                  | $G_1/H_1 \neq S^1$    |
| R.4 | $G_1 \times Sp(n + 1)$ | $H_1 Sp(n+1)$  | $H_1 Sp(n)$            | $G_1/H_1$              |
| R.5 | $G_1 \times G_2$ | $H_1 H_2 U(1)$  | $Z_m \cdot H_1 H_2$   | $G_1/H_1 \neq S^1$    |
| R.6 | $G_1 \times G_2$ | $H_1 H_2 SU(2)$ | $H_1 H_2$             | $G_1/H_1$             |
| R.7 | $G_1 \times G_2$ | $G_1 H_2$       | $H_1 H_2$             | $G_1/H_1 = S^k$       |
| R.8 | $U(1) \times G_1$ | $U(1) H_1$     | $H_1$                  | $G_1/(H_1 H_0)$       |
| R.9 | $G_1$     | $H_1 U(1)$       | $Z_m \cdot H_1$       | Condition T1          |
| R.10| $G_1$     | $H_1 SU(2)$      | $H_1$                  | Condition T1          |
| R.11| $U(1) \times SU(2)$ | $U'(1)$        | $Z_m$                  |                        |
| R.12| $U(1) \times SO(n + 2)$ | $U'(1) SO(n)$ | $Z_m \cdot SO(n)$     | $n \geq 2$            |
| R.13| $G \supset K = K^\pm \supset Z_m \cdot H$ in Table 6 | | | $m = 1$ if $K/H \neq S^1$ |

Table 14. The diagrams of reduced actions in Table 12

- In example N.8, we have $l \geq 1$ and $k \geq 2$.
- In example N.8, N.9 and N.10, the triple is not the last two examples in Table 4. Otherwise the diagram is reducible.
- Example N.10 was already discussed in the proof of Theorem 4.2. Both $K^\pm$ are contained in the subgroup $L = U(1) \times U(1) \cdot H'_c$. The construction of cohomogeneity one diagrams is very similar to the example R.23 and example $N^R_7$ in [Ho].

In Table 18 we also specify the bundle structure of the examples in Table 17. The fiber also admits a cohomogeneity one action with diagram $H \subset \{ K^\pm \} \subset L$ where $L$ is a proper subgroup in $G$. We identify the fiber either as a known manifold or by its diagram. The first three examples N.5, N.6 and N.7 are products.
| R.14 | $G_1$          | $H_1U(1)$ \ $Z_m \cdot H_1SU(2)$ | $Z_m \cdot H_1$ | $G_1/(H_1U(1)SU(2))$ |
| R.15 | $G_1 \times SU(2)$ | $H_1U(1)$ \ $H_1SU(2)$ | $H_1$ | $G_1/H_1 \neq S^1$ |
| R.16 | $G_1 \times Sp(n+1)$ | $H_1Sp(n+1)$ \ $H_1Sp(n)Sp(1)$ | $H_1Sp(n)$ | $G_1/H_1$ |
| R.17 | $G_1 \times G_2$ | $H_1H_2U(1)$ \ $Z_m \cdot G_1H_2$ | $Z_m \cdot H_1H_2$ | $G_1/H_1 = S^k$ \ $G_2/(H_2U(1))$ |
| R.18 | $G_1 \times G_2$ | $H_1H_2SU(2)$ \ $G_1H_2$ | $H_1H_2$ | $G_1/H_1 = S^k$ \ $G_2/(H_2SU(2))$ |
| R.19 | $G_1 \times U(1)$ | $H_1U(1)U(1)$ \ $H_1Sp(1)U(1)$ | $H_1\Delta U(1)$ | $G \supset K^+ \supset H_c$ is 1–5 in Table 6 |
| R.20 | $G_1 \times U(1)$ | $H_1U(1)U(1)$ \ $Z_m \cdot K_1\Delta U(1)$ | $Z_m \cdot H_1\Delta U(1)$ | $G \supset K^+ \supset H_c$ is 6–8 in Table 6 |
| R.21 | $G_1 \times SU(2)$ | $H_1SU(2)SU(2)$ \ $K_1\Delta SU(2)$ | $H_1\Delta SU(2)$ | $G \supset K^+ \supset H$ is 9–13 in Table 6 |
| R.22 | $U(1) \times SU(2)$ | $U(1)_- \cdot H$ \ $U(1)_+ \cdot H$ | $H$ |  |
| R.23 | $U(1) \times SO(n+2)$ | $K^-_c \cdot H$ \ $K^+_c \cdot H$ | $H$ | $n \geq 2$ |

Table 15. The diagrams of reduced actions in Table 13.

B.4. **Classifications in low dimensions.** For the convenience of the reader, we list the non-reducible cohomogeneity one manifolds with $s \leq 3$ in low dimensions. The 4-manifold $\mathbb{CP}^2 \# \mathbb{CP}^2$ is the connected sum of $\mathbb{CP}^2$ and another copy of $\mathbb{CP}^2$ with opposite orientation. The examples of dimensions 5, 6 and 7 are already in Table 8.2 and 8.4 in [Ho]. The last column shows the type of the corresponding diagram with connected groups.

Example $N^6_F$ and $N^7_I$ appear in Theorem 3.3, see Remark 3.4. Example $N^6_D$ is a special case of example $N.4$ in Table 17 with $G_1/L_1 = SU(2)/U(1)$. Example $N^7_H$ is a special case of example $N.10$ in Table 17 with the triple $(SU(3) \supset U(2) \supset SU(2))$. The primitive one $Q^7_E$ appears in the proof of Theorem 5.11. The other primitive ones $Q^6_A$, $Q^6_C$, $Q^6_D$ and $Q^7_G$ appear in the proof of Proposition 6.2 and 6.3.
Table 16. The bundle structure of the manifolds in Table 12 and 13. The first seven examples are products.

| Fiber | Base |
|-------|------|
| R.3 $S^1$ | $G_1/H_1$ |
| R.4 $S^{4n+4}$ | $G_1/H_1$ |
| R.7 $S^{k+1}$ | $G_2/H_2$ |
| R.8 $S^2$ | $G_1/H_1$ |
| R.11 $S^2$ | $S^3$ |
| R.15 $\mathbb{C}P^2$ | $G_1/H_1$ |
| R.16 $\mathbb{H}P^{n+1}$ | $G_1/H_1$ |
| R.1 $S^2$ | $G_1/(H_1 \cdot U(1))$ |
| R.2 $S^4$ | $G_1/(H_1 \cdot SU(2))$ |
| R.5 $S^2$ | $G_1/H_1 \times G_2/(H_2 \cdot U(1))$ |
| R.6 $S^4$ | $G_1/H_1 \times G_2/(H_2 \cdot SU(2))$ |
| R.9 $S^2$ | $G_1/(H_1 \cdot U(1))$ |
| R.10 $S^4$ | $G_1/(H_1 \cdot SU(2))$ |
| R.12 $S^2$ | $SO(n+2)/SO(n)$ |
| R.13 sphere | $G/K$ |
| R.14 $S^5$ | $G_1/(H_1 \cdot U(1) \cdot SU(2))$ |
| R.17 $S^{k+2}$ | $G_2/(H_2 \cdot U(1))$ |
| R.18 $S^{k+4}$ | $G_2/(H_2 \cdot SU(2))$ |
| R.19 $\mathbb{C}P^2$ | $G_1/(H_1 \cdot SU(2))$ |
| R.20 sphere | $G_1/(K_1 \cdot U(1))$ |
| R.21 sphere | $G_1/(K_1 \cdot SU(2))$ |
| R.22 $S^3$ | $S^2$ |
| R.23 lens space | $SO(n+2)/(SO(2) \times SO(n))$ |

References

[AA] A. V. Alekseevsky and D. V. Alekseevsky, *G-manifolds with one dimensional orbit space*, Ad. in Sov. Math. 8(1992), 1–31.

[Be] A. L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 10. Springer-Verlag, Berlin, 1987.

[Bo] C. Böhm, *Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces*, Invent. Math. 134(1998), no.1, 145–176.
|   | G            | K±                  | H                  |
|---|---------------|---------------------|--------------------|
| N.1 | E_6         | SU(2)U(5)           | SU(2)SU(5)        |
| N.2 | Spin(6 + n)  | SO(n)U(3)           | SO(n)SU(3)        | \( n \geq 1 \) |
| N.3 | G_1 × Sp(n + 1) | L_1Sp(1)Sp(n)Sp(n + 1) | L_1ΔSp(1)Sp(n)G_1/(L_1Sp(1)) |
| N.4 | G_1 × SU(n + 1) | L_1U(1)SU(n+1)     | L_1ΔU(1)SU(n)G_1/(L_1U(1)) |
| N.5 | G_1 × L_2    | L_1K_1              | L_1H_1            | Condition T2 |
| N.6 | Spin(8) × G_1 | Spin^-(7) × L_1     | G_2 × L_1         | G_1/L_1     |
| N.7 | G_0 × L_1 × L_2 | L_0L_1H_2          | L_0H_1H_2         | G_0/L_0     |
|     |               | L_0H_1L_2          | L_i/H_i = S^i     |             |
| N.8 | L_1 × G_2    | H_1K_2              | H_1H_2            | G_2 ⊇ K_2 ⊇ H_2 in Table [4] |
|     |               | L_1H_2             | L_1/H_1 = S^i     |             |
| N.9 | L_1 × G_2    | H_1K_2              | H_1H_2 · Z_m      | G_2 ⊇ K_2 ⊇ H_2 in Table [4] |
|     |               | L_1H_2 · Z_m       | L_1/H_1 = S^k     |             |
| N.10 | U(1) × G'    | K'_c × U(1)H'      | H'                | G'_c ⊇ K'_c ⊇ H'_c in Table [4] |
|     |               |                     | K'_c/H'_c = S^1   |             |

|   |   |   |   |
|---|---|---|---|
| N.10 | U(1) × G' | K'_c × U(1)H' | H' |
|     |           |               | G'_c ⊇ K'_c ⊇ H'_c in Table [4] |
|     |           |               | K'_c/H'_c = S^1 |

Table 17. Non-primitive cohomogeneity one manifolds with non-reducible actions which are not doubles. The action has no fixed points and \( s = 3 \).

[DZ] J. E. D’Atri and W. Ziller, *Naturally reductive metrics and Einstein metrics on compact Lie groups*. Mem. Amer. Math. Soc. 18(1979), No. 215.

[Ch] J. Cheeger, *Some examples of manifolds of nonnegative curvature*, J. Diff. Geom. 8(1973), 623–628.

[CS] R. Cleyton and A. Swann, *Cohomogeneity-one G_2-structures*, J. Geom. Phys. 44:202(2002).

[Co] D. Conti, *Cohomogeneity one Einstein-Sasaki 5-manifolds*, Commun. Math. Phys. 274(2007), N.3, 751–774.

[CGLP] M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope, *New cohomogeneity one metrics with Spin(7) holonomy*, J. Geom. Phys. 49(2004), 350–365

[DW] A. S. Dancer and M. Y. Wang, *On Ricci solitons of cohomogeneity one*, Preprint(2008), arXiv math:08020759.

[De] O. Derricott, *A 7 manifold with positive curvature*, Preprint(2009).
Table 18. The bundle structure of the manifolds in Table 17. The first three examples are products.
\[
\begin{array}{|c|c|c|}
\hline
\text{Diagram} & s & \text{Type} \\
\hline
S^2 \times S^2 & Z_n \subset \{K^\pm = U(1)\} \subset SU(2), n \text{ even} & 3 \text{ double} \\
\text{CP}^2 \times \text{CP}^2 & Z_n \subset \{K^\pm = U(1)\} \subset SU(2), n \text{ odd} & 3 \text{ double} \\
\text{CP}^2 & \{\beta\} \subset \{e^{i\theta}, e^{i\theta} \cup e^{i\theta} \} \subset S^3 & 3 \text{ primitive} \\
S^4 & \{\pm i, \pm j, \pm k\} \subset \{e^{i\theta} \cup e^{i\theta}, e^{i\theta} \cup e^{i\theta}\} \subset S^3 & 3 \text{ primitive} \\
Q_A^6 & \Delta U(1) \cdot Z_n \subset \{U(1) \times U(1), \Delta SU(2) \cdot Z_n\} \subset SU(2) \times SU(2) & 3 \text{ primitive} \\
Q_B^6 & \Delta U(1) \subset \{SU(2) \times U(1), SU(2) \times SU(2)\} \subset SU(2) \times SU(2) & 3 \text{ primitive} \\
N_D^6 & \{(e^{i\theta}, e^{i\theta})\} \subset \{U(1) \times U(1), SU(2), U(1) \} \subset SU(2) \times SU(2) & 3 \text{ non-primitive} \\
N_B^6 & \{(e^{i\theta}, e^{i\theta})\} \cdot Z_n \subset \{K^\pm = U(1) \times U(1)\} \subset SU(2) \times SU(2) & 3 \text{ double} \\
Q_B^6 & \Delta U(1) \subset \{K^\pm = \Delta SU(2)\} \subset SU(2) \times SU(2) & 3 \text{ double} \\
N_E^6 & \{(e^{i\theta}, e^{i\theta})\} \subset \{K^\pm = SU(2) \times U(1)\} \subset SU(2) \times SU(2) & 3 \text{ double} \\
N_F^6 & SU(2) \cdot Z_2 \subset \{K^\pm = U(2)\} \subset SU(3) & 2 \text{ double} \\
Q_F^7 & U(1) \times U(1) \subset \{SU(1) \times U(2), SU(2) \times U(1)\} \subset SU(3) & 3 \text{ primitive} \\
Q_G^7 & \begin{aligned} H_0 \cdot Z_n \subset \{(\beta(m\theta), e^{i\theta})\} \cdot H_0, \ SU(3) \times Z_n \subset SU(3) \times U(1) \\
H_0 = SU(2) \times 1; Z_n \subset \{(\beta(m\theta), e^{i\theta})\}, \\
\beta(\theta) = \text{diag}(e^{-i\theta}, e^{i\theta}, 1), \gcd(m, n) = 1 \end{aligned} & 3 \text{ primitive} \\
N_H^7 & H \subset \{(\beta(m_\theta), e^{im_\theta})\} \cdot H, \{(\beta(m_\theta), e^{im_\theta})\} \cdot H \subset SU(3) \times U(1) & 3 \text{ non-primitive} \\
& \begin{aligned} H_0 = SU(2) \times 1, H = H_- \cdot H_+, K^- \neq K^+; \\
\beta(\theta) = \text{diag}(e^{-i\theta}, e^{i\theta}, 1), \gcd(n_-, n_+, d) = 1 \end{aligned} \text{ where } d \text{ is the index of } H \cap K_0^- \cap K_0^+ \text{ in } K_0^- \cap K_0^+ \\
Q_F^7 & \begin{aligned} H_0 \cdot Z_n \subset K^\pm = \{(\beta(m\theta), e^{i\theta})\} \cdot H_0 \subset SU(3) \times U(1) \\
H_0 = SU(2) \times 1, Z_n \subset \{(\beta(m\theta), e^{i\theta})\}, \\
\beta(\theta) = \text{diag}(e^{-i\theta}, e^{i\theta}, 1) \end{aligned} & 3 \text{ double} \\
N_G^7 & U(1) \times U(1) \subset \{K^\pm = Sp(1)\} \subset SU(3) & 3 \text{ double} \\
N_F^7 & Sp(1)U(1) \subset \{K^\pm = Sp(1)Sp(1)\} \subset Sp(2) & 2 \text{ double} \\
\hline
\end{array}
\]

**Table 19.** Cohomogeneity one manifolds with non-reducible action and \( s \leq 3 \) in low dimensions. The action is not a product or a sum action and has no fixed points.

[He] C. He, *New examples of obstructions to non-negative sectional curvatures in cohomogeneity one manifolds*, Preprint(2009), arXiv math:09105712.
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[Ho] C. Hoelscher, *Classification of cohomogeneity one manifolds in low dimensions*, Preprint(2007), arXiv math:07121327. To appear at Pacific J. of Math.

[Iw1] K. Iwata, *Classification of compact transformation groups on cohomology quaternion projective spaces with codimension one orbits*, Osaka J. Math. 15(1978), 475–508.

[Iw2] K. Iwata, *Compact transformation groups on rational cohomology Cayley projective planes*, Tôhoku Math. J. (2) 33(1981), no. 4, 429–442.

[KS] M. Kerin and K. Shankar, *Riemannian submersions from simple, compact Lie groups*, Preprint(2009), arXiv math:09104344.

[Kn] A. Knapp, *Lie groups beyond an introduction. Second edition*, Progress in Mathematics, 140, Birkhäuser Boston, Inc., Boston, MA, 2002.

[Ko] A. Kollross, *A classification of hyperpolar and cohomogeneity one actions*, Trans. Amer. Math. Soc. 354(2002), no. 2, 571–612.

[Kr] M. Kraemer, *Eine Klassifikation bestimmter Unterruppen kompakter zusammenhängender Liegruppen*, Comm. Algebra 3(1975), no. 8, 691–737.

[MP] W. G. McKay and J. Patera, *Tables of dimensions, indices, and branching rules for representations of simple Lie algebras*, Lecture Notes in Pure and Applied Mathematics, 69, Marcel Dekker, Inc., New York, 1981.

[Mu] S. Murakami, *Exceptional simple Lie groups and related topics in recent differential geometry*, Differential geometry and topology (Tianjin, 1986–87), 183–221, Lecture Notes in Math., 1369, Springer, Berlin, 1989.

[Ne] W. Neumann, *3-dimensional G-manifolds with 2-dimensional orbits*, Proc. Conf. on Transformation Groups(New Orleans, La., 1967), pp. 220–222, Springer, New York.

[PW] P. Petersen and F. Wilhelm, *An exotic sphere with positive sectional curvature*, Preprint(2008), arXiv math:0805.0812v3.

[Re1] F. Reidegeld, *Spaces admitting homogeneous G2-structures*, Preprint(2009), arXiv math:0901.0652.

[Re2] F. Reidegeld, *Special cohomogeneity one metrics with Q111 or M110 as principal orbit*, Preprint(2009), arXiv math:0908.3965.

[Re3] F. Reidegeld, *Exceptional holonomy and Einstein metrics constructed from Aloff-Wallach spaces*, Preprint(2010), arXiv math:1004.4788.

[STa] L. Schwachhöfer and K. Tapp, *Cohomogeneity one disk bundles with normal homogeneous collars*, Proc. Lond. Math. Soc.(3), 99(2009), no.3, 609–632.

[STu] L. Schwachhöfer and W. Tuschmann, *Almost nonnegative curvature and cohomogeneity one*, Preprint no. 62/2001, Max-Planck-Institut für Mathematik in den Naturwissenschaften Leipzig, [http://www.mis.mpg.de/cgi-bin/preprints.pl](http://www.mis.mpg.de/cgi-bin/preprints.pl)

[St] E. Straume, *Compact Connected Lie Transformation Groups on Spheres with Low Cohomogeneity, I*, Mem. Amer. Math. Soc. 119(1996), no. 569.

[Uc] F. Uchida, *Classification of compact transformation groups on cohomology complex projective spaces with codimension one orbits*, Japan. J. Math., 3, No. 1(1977), 141–189.

[WZ1] M. Wang and W. Ziller, *On normal homogeneous Einstein manifolds*, Ann. Scient. Éc. Norm. Sup. 18(1985), 563–633.
[WZ2] M. Wang and W. Ziller, *On isotropy irreducible Riemannian manifolds*, Acta Math. **166**(1991), 223–261.

[Wi3] B. Wilking, *Nonnegatively and positively curved manifolds*, Surv. Differ. Geom. **11**, 25–62, ed. K. Grove and J. Cheeger, Int. Press, Somerville, MA, 2007.

[Wo1] J. A. Wolf, *The geometry and structure of isotropy irreducible homogeneous spaces*, Acta Math. **120**(1968), 59–148. *Correction*, Acta Math. **152**(1984), 141–142.

[Wo2] J. A. Wolf, *Spaces of constant curvature*, Fifth edition, Publish or Perish, Inc., Houston, TX, 1984.

[Zi1] W. Ziller, *Examples of Riemannian manifolds with non-negative sectional curvature*, Surv. Differ. Geom. **11**, 63–102, ed. K. Grove and J. Cheeger, Int. Press, Somerville, MA, 2007.