A stabilizer free weak Galerkin finite element method with supercloseness of order two

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Abstract
The weak Galerkin (WG) finite element method is an effective and flexible general numerical technique for solving partial differential equations. A simple WG finite element method is introduced for second-order elliptic problems. First we have proved that stabilizers are no longer needed for this WG element. Then we have proved the supercloseness of order two for the WG finite element solution. The numerical results confirm the theory.

KEYWORDS
finite element methods, second-order elliptic problems, stabilizer free, supercloseness, weak Galerkin, weak gradient

1 | INTRODUCTION

The weak Galerkin (WG) finite element method is an effective and flexible numerical technique for solving partial differential equations. It is a natural extension of the standard Galerkin finite element method where classical derivatives were substituted by weakly defined derivatives on functions with discontinuity. The WG method was first introduced in [1, 2] and then has been applied to solve various partial differential equations [3–14].

The main idea of WG finite element methods is the use of weak functions and their corresponding weak derivatives in algorithm design. For the second-order elliptic equation, weak functions have the form of $v = \{v_0, v_b\}$ with $v = v_0$ inside of each element and $v = v_b$ on the boundary of the element. Both $v_0$ and $v_b$ can be approximated by polynomials in $P_\ell(T)$ and $P_s(e)$ respectively, where $T$ stands for an element and $e$ the edge or face of $T$, $\ell$ and $s$ are nonnegative integers with possibly different values. Weak derivatives are defined for weak functions in the sense of distributions. For example, one may approximate a weak gradient in the polynomial space $[P_m(T)]^d$. Various combinations of $\{P_\ell(T), P_s(e), [P_m(T)]^d\}$ lead to different WG methods tailored for specific partial differential equations.
A stabilizing/penalty term is often used in finite element methods with discontinuous approximations to enforce connection of discontinuous functions across element boundaries. Removing stabilizers from WG finite element methods will simplify formulations and reduce programming complexity significantly. Stabilizer free WG finite element methods have been studied in [15–17]. The idea is to increase the connectivity of a weak function cross element boundary by raising the degree of polynomials for computing weak derivatives. In [15], it has been proved that for a WG element \((P_k(T), P_{k+1}(e), [P_k(T)]^d)\), the condition of \(j \geq k + n - 1\) guarantees a stabilizer free WG method, where \(n\) is the number of edges/faces of an element.

In this paper, a WG finite element \((P_k(T), P_{k+1}(e), [P_{k+1}(T)]^2)\) is investigated for second-order elliptic problems. This new WG finite element leads to a stabilizer free WG formulation. In addition, we have proved order two supercloseness for the WG finite element solution, that is, the WG solution approaches to the \(L_2\) projection of the true solution with the convergence rates two order higher than the optimal convergence rate in both an energy norm and the \(L_2\) norm. The numerical results show high accuracy of the WG method and confirm our theory.

For simplicity, we demonstrate the idea by using the second-order elliptic problem that seeks an unknown function \(u\) satisfying

\[
-\nabla \cdot (a \nabla u) = f \quad \text{in} \ \Omega, \quad (1.1)
\]

\[
u = g \quad \text{on} \ \partial \Omega, \quad (1.2)
\]

where \(\Omega\) is a polytopal domain in \(\mathbb{R}^2\), \(\nabla u\) denotes the gradient of the function \(u\), and \(a\) is a symmetric \(2 \times 2\) matrix-valued function in \(\Omega\). We shall assume that there exist two positive numbers \(\lambda_1, \lambda_2 > 0\) such that

\[
\lambda_1 \xi^t \xi \leq \xi^t a \xi \leq \lambda_2 \xi^t \xi, \quad \forall \xi \in \mathbb{R}^2. \quad (1.3)
\]

Here \(\xi\) is understood as a column vector and \(\xi^t\) is the transpose of \(\xi\).

The paper is organized as follows. In Section 2, we shall describe a new WG scheme. Section 3 will discuss the well posedness of the WG scheme. The error analysis for the WG solutions in an energy norm and in the \(L_2\) norms will be investigated in Sections 4 and 5 respectively. In Section 6, we shall present some numerical results that confirm the theory developed in earlier sections. Finally, technical proof of Lemma 3.1 will be presented in Appendix.

2 | WG FINITE ELEMENT SCHEMES

Let \(T_h\) be a shape regular partition of the domain \(\Omega\) consisting of triangles. Denote by \(\mathcal{E}_h\) the set of all edges in \(T_h\), and let \(\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial \Omega\) be the set of all interior edges. For every element \(T \in T_h\), we denote by \(h_T\) its diameter and mesh size \(h = \max_{T \in T_h} h_T\) for \(T_h\).

First, we adopt the following notations,

\[
(v, w)_{T_h} = \sum_{T \in T_h} (v, w)_T = \sum_{T \in T_h} \int_T v w d\mathbf{x},
\]

\[
(v, w)_{\partial T_h} = \sum_{T \in T_h} (v, w)_{\partial T} = \sum_{T \in T_h} \int_{\partial T} v w d\mathbf{s}.
\]

For a given integer \(k \geq 1\), define a WG finite element space associated with \(T_h\) as follows

\[
V_h = \{ v = \{v_0, v_b\} : \ v_0|_T \in P_k(T), \ v_b|_e \in P_{k+1}(e), \ e \subset \partial T, T \in T_h \} \quad (2.1)
\]
and its subspace $V_h^0$ is defined as

$$V_h^0 = \{ \nu : \nu \in V_h, \nu_b = 0 \text{ on } \partial \Omega \}. \quad (2.2)$$

We would like to emphasize that any function $\nu \in V_h$ has a single value $\nu_b$ on each edge $e \in \mathcal{E}_h$.

For $\nu = \{ \nu_0, \nu_b \} \in V_h + H^1(\Omega)$, a weak gradient $\nabla_w \nu$ is a piecewise vector valued polynomial such that on each $T \in \mathcal{T}_h$, $\nabla_w \nu \in [P_{k+1}(T)]^2$ satisfies

$$\langle \nabla_w \nu, \mathbf{q} \rangle_T = - \langle \nu_0, \nabla \cdot \mathbf{q} \rangle_T + \langle \nu_b, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T} \quad \forall \mathbf{q} \in [P_{k+1}(T)]^2. \quad (2.3)$$

In the above equation, we let $\nu_0 = \nu$ and $\nu_b = \nu$ if $\nu \in H^1(\Omega)$.

Let $Q_0$ and $Q_b$ be the two element-wise defined $L^2$ projections onto $P_k(T)$ and $P_{k+1}(e)$ with $e \subset \partial T$ respectively for each $T \in \mathcal{T}_h$. Define $Q_h u = \{ Q_0 u, Q_b u \} \in V_h$. Let $Q_h$ be the element-wise defined $L^2$ projection onto $[P_{k+1}(T)]^2$ on each element $T \in \mathcal{T}_h$.

**Weak Galerkin Algorithm 1**

A numerical approximation for (1.1) and (1.2) can be obtained by seeking $u_h = \{ u_0, u_b \} \in V_h$ satisfying $u_b = Q_0 g$ on $\partial \Omega$ and the following equation:

$$\left( a \nabla_w u_h, \nabla_w \nu \right)_{\mathcal{T}_h} = \left( f, \nu_0 \right)_{\mathcal{T}_h} \quad \forall \nu = \{ \nu_0, \nu_b \} \in V_h^0. \quad (2.4)$$

We will need the following lemma in the error analysis.

**Lemma 2.1** Let $\phi \in H^1(\Omega)$, then on any $T \in \mathcal{T}_h$,

$$\nabla_w (Q_h \phi) = Q_h \nabla \phi, \quad (2.5)$$

and

$$\nabla_w \phi = Q_h \nabla \phi. \quad (2.6)$$

**Proof.** Using (2.3) and integration by parts, we have that for any $\mathbf{q} \in [P_{k+1}(T)]^2$

$$\langle \nabla_w Q_h \phi, \mathbf{q} \rangle_T = - \langle Q_0 \phi, \nabla \cdot \mathbf{q} \rangle_T + \langle Q_b \phi, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T}$$

$$= - \langle \phi, \nabla \cdot \mathbf{q} \rangle_T + \langle \phi, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T}$$

$$= (Q_h \nabla \phi, \mathbf{q})_T,$$

and

$$\langle \nabla_w \phi, \mathbf{q} \rangle_T = - \langle \phi, \nabla \cdot \mathbf{q} \rangle_T + \langle \phi, \mathbf{q} \cdot \mathbf{n} \rangle_{\partial T}$$

$$= (\nabla \phi, \mathbf{q})_T = (Q_h \nabla \phi, \mathbf{q})_T,$$

which imply Equations (2.5) and (2.6). \square

For any function $\varphi \in H^1(T)$, the following trace inequality holds true (see [2] for details):

$$\| \varphi \|^2_T \leq C (h_T^{-1} \| \varphi \|_T^2 + h_T \| \nabla \varphi \|_T^2). \quad (2.7)$$

### 3 | WELL POSEDNESS

For any $\nu \in V_h + H^1(\Omega)$, define two semi-norms

$$\| \| \nu \| \|^2 = (\nabla_w \nu, \nabla_w \nu)_{\mathcal{T}_h}, \quad (3.1)$$

$$\| \| \nu \| \|_1^2 = (a \nabla_w \nu, \nabla_w \nu)_{\mathcal{T}_h}. \quad (3.2)$$
It follows from (1.3) that there exist two positive constants \( \alpha \) and \( \beta \) such that

\[
\alpha \|v\| \leq \|v\|_1 \leq \beta \|v\|.
\] (3.3)

We introduce a discrete \( H^1 \) semi-norm as follows:

\[
\|v\|_{1,h} = \left( \sum_{T \in T_h} (\|\nabla v_0\|_T^2 + h_T^{-1} \|v_0 - v_b\|_{\partial T}^2) \right)^{1/2}.
\] (3.4)

It is easy to show that \( \|v\|_{1,h} \) defines a norm in \( V_h \).

Next we will show that \( \|\cdot\| \) also defines a norm in \( V_h^0 \) by proving the equivalence of \( \|\cdot\| \) and \( \|\cdot\|_{1,h} \) in \( V_h \). First we need the following lemma.

The proof of the following lemma is long and can be found in Appendix.

**Lemma 3.1** For any \( v \in V_h \), we have

\[
\sum_{T \in T_h} h_T^{-1} \|v_0 - v_b\|_{\partial T}^2 \leq C \|v\|^2.
\] (3.5)

**Lemma 3.2** There exist two positive constants \( C_1 \) and \( C_2 \) such that for any \( v = \{v_0, v_b\} \in V_h \), we have

\[
C_1 \|v\|_{1,h} \leq \|v\| \leq C_2 \|v\|_{1,h}.
\] (3.6)

**Proof.** For any \( v = \{v_0, v_b\} \in V_h \), it follows from the definition of weak gradient (2.3) and integration by parts that on each \( T \in T_h \)

\[
(\nabla w, q)_T = (\nabla v_0, q)_T + \langle v_b - v_0, q \cdot n \rangle_{\partial T} \quad \forall q \in [P_{k+1}(T)]^2.
\] (3.7)

By letting \( q = \nabla w \) in (3.7) we arrive at

\[
(\nabla w, \nabla w)_T = (\nabla v_0, \nabla w)_T + \langle v_b - v_0, \nabla v \cdot n \rangle_{\partial T}.
\]

Letting \( q = \nabla v_0 \) in (3.7) implies

\[
(\nabla v, \nabla v_0)_T = (\nabla v_0, \nabla v)_T + \langle v_b - v_0, \nabla v_0 \cdot n \rangle_{\partial T}.
\] (3.8)

From the trace inequality (2.7) and the inverse inequality we have

\[
\|\nabla w\|_T^2 \leq \|\nabla v_0\|_T \|\nabla w\|_T + \|v_0 - v_b\|_{\partial T} \|\nabla w\|_T \leq \|\nabla v_0\|_T \|\nabla w\|_T + C h_T^{-1/2} \|v_0 - v_b\|_{\partial T} \|\nabla w\|_T,
\]

which implies

\[
\|\nabla w\|_T \leq C(\|\nabla v_0\|_T + h_T^{-1/2} \|v_0 - v_b\|_{\partial T}),
\]

and consequently

\[
\|v\| \leq C_2 \|v\|_{1,h}.
\]

Next we will prove \( C_1 \|v\|_{1,h} \leq \|v\| \). It follows from (3.8), the trace inequality and the inverse inequality,

\[
\|\nabla v_0\|_T^2 \leq \|\nabla w\|_T \|\nabla v_0\|_T + C h_T^{-1/2} \|v_0 - v_b\|_{\partial T} \|\nabla v_0\|_T,
\]

which implies

\[
\sum_{T \in T_h} \|\nabla v_0\|_T^2 \leq C \left( \sum_{T \in T_h} h_T^{-1} \|v_0 - v_b\|_{\partial T}^2 + \sum_{T \in T_h} \|\nabla w\|_T^2 \right).
\]
Combining the above estimate and (3.5), we prove the lower bound of (3.6) and complete
the proof of the lemma. ▪

**Lemma 3.3** The WG finite element scheme (2.4) has a unique solution.

**Proof.** If \( u_h^{(1)} \) and \( u_h^{(2)} \) are two solutions of (2.4), then \( \varepsilon_h = u_h^{(1)} - u_h^{(2)} \in V_h^0 \) would satisfy the following equation

\[
(a\nabla w, \varepsilon_h, \nabla w v)_{\Gamma_h} = 0 \quad \forall v \in V_h^0.
\]

Then by letting \( v = \varepsilon_h \) in the above equation and (3.3), we arrive at

\[
|||\varepsilon_h|||^2 \leq (a\nabla w, \varepsilon_h, \nabla w \varepsilon_h) = 0.
\]

It follows from (3.6) that \( |||\varepsilon_h|||_{1,h} = 0 \). Since \( |||\cdot|||_{1,h} \) is a norm in \( V_h^0 \), one has \( \varepsilon_h = 0 \).

This completes the proof of the lemma. ▪

4 | ERROR ESTIMATES IN ENERGY NORM

The goal of this section is to establish some error estimates for the WG finite element solution \( u_h \) arising from (2.4). For simplicity of analysis, we assume that the coefficient tensor \( a \) in (1.1) is a piecewise
constant matrix with respect to the finite element partition \( \mathcal{T}_h \). The result can be extended to variable
tensors without any difficulty, provided that the tensor \( a \) is piecewise sufficiently smooth.

Let \( e_h = Q_h u - u_h \). Next we derive an error equation that \( e_h \) satisfies. Define a bilinear form \( \ell(u, v) \) by

\[
\ell(u, v) = \langle a(\nabla u - Q_h \nabla u) \cdot n, \ v_0 - v_b \rangle_{\partial \mathcal{T}_h}.
\]

**Lemma 4.1** For any \( v \in V_h^0 \), the error \( e_h \) satisfies the following equation

\[
(a\nabla w e_h, \nabla w v)_{\Gamma_h} = \ell(u, v).
\]

**Proof.** For \( v = \{v_0, v_b\} \in V_h^0 \), testing (1.1) by \( v_0 \) gives

\[
(a\nabla u, \nabla v_0)_{\Gamma_h} - (a\nabla u \cdot n, v_0 - v_b)_{\partial \mathcal{T}_h} = \langle f, v_0 \rangle.
\]

To obtain the above estimate, we use the fact that \( \langle a\nabla u \cdot n, v_b \rangle_{\partial \mathcal{T}_h} = 0 \).

It follows from integration by parts, (2.3) and (2.5) that

\[
(a\nabla u, \nabla v_0)_{\Gamma_h} = (aQ_h \nabla u, \nabla v_0)_{\Gamma_h}
\]

\[
= -\langle v_0, \nabla (aQ_h \nabla u) \rangle_{\Gamma_h} + \langle v_0, aQ_h \nabla u \cdot n \rangle_{\partial \mathcal{T}_h}
\]

\[
= (aQ_h \nabla u, \nabla v_0)_{\Gamma_h} + \langle v_0 - v_b, aQ_h \nabla u \cdot n \rangle_{\partial \mathcal{T}_h}
\]

\[
= (a\nabla w Q_h u, \nabla v_0)_{\Gamma_h} + \langle v_0 - v_b, aQ_h \nabla u \cdot n \rangle_{\partial \mathcal{T}_h}.
\]

Combining (4.2) and (4.3) gives

\[
(a\nabla w Q_h u, \nabla v)_{\Gamma_h} = \langle f, v_0 \rangle + \ell(u, v).
\]

The error equation follows from subtracting (2.4) from (4.4),

\[
(a\nabla w e_h, \nabla w v)_{\Gamma_h} = \ell(u, v) \quad \forall v \in V_h^0.
\]

This completes the proof of the lemma. ▪
**Theorem 4.2**  Let \( u_h \in V_h \) be the WG finite element solution of (2.4). Assume the exact solution \( u \in H^{k+3}(\Omega) \). Then, there exists a constant \( C \) such that

\[
|||Q_h u - u_h||| \leq C h^{k+2} |u|_{k+3}.
\]  

(4.5)

**Proof.**  By letting \( v = e_h \) in (4.1) and using (3.3), we have

\[
|||e_h|||^2 \leq (a \nabla w e_h, \nabla u e_h)_{\tau_h} = |||\ell(u, e_h)|||.
\]  

(4.6)

Using the Cauchy–Schwarz inequality, the trace inequality (2.7), (1.3), and (3.6), we have

\[
|||\ell(u, e_h)||| = \left| \sum_{T \in \tau_h} \langle a(\nabla u - Q_h \nabla u) \cdot n, e_0 - e_b \rangle_{\partial T} \right|
\]

\[ \leq C \sum_{T \in \tau_h} |||\nabla u - Q_h \nabla u|||_{\partial T} |||e_0 - e_b|||_{\partial T} \]

\[ \leq C \left( \sum_{T \in \tau_h} h_T |||\nabla u - Q_h \nabla u|||^2_{\partial T} \right)^{1/2} \left( \sum_{T \in \tau_h} h_T^{-1} |||e_0 - e_b|||^2_{\partial T} \right)^{1/2} \]

\[ \leq C h^{k+2} |u|_{k+3} |||e_h|||. \]  

(4.7)

It follows from (4.6) and (4.7) that

\[
|||e_h|||^2 \leq C h^{k+2} |u|_{k+3} |||e_h|||,
\]

which implies (4.5). This completes the proof. \( \blacksquare \)

## 5 ERROR ESTIMATES IN \( L^2 \) NORM

The standard duality argument is used to obtain \( L^2 \) error estimate. Recall 
\( e_h = \{e_0, e_b\} = Q_h u - u_h = \{Q_0 u - u_0, Q_b u - u_b\} \). The considered dual problem seeks \( \Phi \in H^1_0(\Omega) \) satisfying

\[
-\nabla \cdot a \nabla \Phi = e_0 \quad \text{in} \quad \Omega.
\]  

(5.1)

Assume that the following \( H^2 \)-regularity holds

\[
|||\Phi|||_2 \leq C |||e_0|||.
\]  

(5.2)

**Theorem 5.1**  Let \( u_h \in V_h \) be the WG finite element solution of (2.4). Assume that the exact solution \( u \in H^{k+3}(\Omega) \) and (5.2) holds true. Then, there exists a constant \( C \) such that

\[
||Q_0 u - u_0|| \leq C h^{k+3} |u|_{k+3}.
\]  

(5.3)

**Proof.**  By testing (5.1) with \( e_0 \) and integrating by parts, we obtain

\[
|||e_0|||^2 = - (\nabla \cdot (a \nabla \Phi), e_0)
\]

\[ = (a \nabla \Phi, \nabla e_0)_{\tau_h} - (a \nabla \Phi \cdot n, e_0 - e_b)_{\partial \tau_h}, \]  

(5.4)

where we have used the fact that \( e_b = 0 \) on \( \partial \Omega \). Setting \( \phi = \Phi \) and \( v = e_h \) in (4.3) yields

\[
(a \nabla \Phi, \nabla e_0)_{\tau_h} = (a \nabla w Q_h \Phi, \nabla w e_h)_{\tau_h} + (a Q_h \nabla \Phi \cdot n, e_0 - e_b)_{\partial \tau_h}.
\]  

(5.5)
Substituting (5.5) into (5.4) gives

\[
\|e_0\|^2 = (a\nabla e_h, \nabla w_{Q_h}\Phi)_{T_h} + (a(Q_h \nabla \Phi - \nabla \Phi) \cdot n, e_0 - e_b)_{\partial T_h}
\]
\[
= (a\nabla e_h, \nabla w_{Q_h}\Phi)_{T_h} + \epsilon'(\Phi, e_h)
\]
\[
= \epsilon'(u, Q_h\Phi) + \epsilon'(\Phi, e_h).
\]

(5.6)

Using the Cauchy–Schwarz inequality and (2.7), we obtain

\[
|\epsilon'(u, Q_h\Phi)| = \left| \sum_{T \in T_h} (a(\nabla u - Q_h \nabla u) \cdot n, Q_0\Phi - Q_b\Phi)_{\partial T} \right|
\]
\[
\leq C \sum_{T \in T_h} \|\nabla u - Q_h \nabla u\|_{\partial T} \|Q_0\Phi - Q_b\Phi\|_{\partial T}
\]
\[
\leq C \left( \sum_{T \in T_h} \|\nabla u - Q_h \nabla u\|^2_{\partial T} \right)^{1/2} \left( \sum_{T \in T_h} \|Q_0\Phi - Q_b\Phi\|^2_{\partial T} \right)^{1/2}.
\]

(5.7)

From the trace inequality (2.7) and the definition of \( Q_b \), we have

\[
\left( \sum_{T \in T_h} \|Q_0\Phi - Q_b\Phi\|^2_{\partial T} \right)^{1/2} \leq \left( \sum_{T \in T_h} \|Q_0\Phi - \Phi\|^2_{\partial T} + \|\Phi - Q_b\Phi\|^2_{\partial T} \right)^{1/2}
\]
\[
\leq C \left( \sum_{T \in T_h} \|Q_0\Phi - \Phi\|^2_{\partial T} \right)^{1/2} \leq Ch^\frac{3}{2} \|\Phi\|_2,
\]

and

\[
\left( \sum_{T \in T_h} \|a(\nabla u - Q_h \nabla u)\|^2_{\partial T} \right)^{1/2} \leq Ch^{k+\frac{3}{2}} \|u\|_{k+3}.
\]

Combining the above two estimates with (5.7) gives

\[
|\epsilon'(u, Q_h\Phi)| \leq Ch^{k+3} |u|_{k+3} \|\Phi\|_2.
\]

(5.8)

Using the Cauchy–Schwarz inequality, the trace inequality (2.7), (1.3), (3.6), and (4.5), we have

\[
|\epsilon'(\Phi, e_h)| = \left| \sum_{T \in T_h} (a(\nabla \Phi - Q_h \nabla \Phi) \cdot n, e_0 - e_b)_{\partial T} \right|
\]
\[
\leq C \left( \sum_{T \in T_h} h_T \|\nabla \Phi - Q_h \nabla \Phi\|^2_{\partial T} \right)^{1/2} \left( \sum_{T \in T_h} h^{-1}_T \|e_0 - e_b\|^2_{\partial T} \right)^{1/2}
\]
\[
\leq Ch \|\Phi\|_2 e_h
\]
\[
\leq Ch^{k+3} |u|_{k+3} \|\Phi\|_2.
\]

(5.9)

Substituting (5.8) and (5.9) into (5.6) yields

\[
\|e_0\|^2 \leq Ch^{k+3} |u|_{k+3} \|\Phi\|_2,
\]

which, combined with the regularity assumption (5.2), gives the error estimate (5.3). ■
FIGURE 1  The first three levels of triangular grids for Examples 1 and 2

| Table | Example 1: error profiles and convergence rates on grids shown in Figure 1 |
|-------|---|---|---|---|
| Level | $|u_h - Q_h u|_0$ | Rate | $|u_h - Q_h u|_1$ | Rate |
|-------|---|---|---|---|
| By the $P_1 - P_2$ WG finite element | 5 | 0.5867E−06 | 3.99 | 0.6420E−04 | 2.99 |
| | 6 | 0.3677E−07 | 4.00 | 0.8043E−05 | 3.00 |
| | 7 | 0.2310E−08 | 3.99 | 0.1021E−05 | 2.98 |
| By the $P_2 - P_3$ WG finite element | 3 | 0.6072E−05 | 4.90 | 0.2699E−03 | 3.95 |
| | 4 | 0.1938E−06 | 4.97 | 0.1704E−04 | 3.99 |
| | 5 | 0.6113E−08 | 4.99 | 0.1071E−05 | 3.99 |
| By the $P_3 - P_4$ WG finite element | 2 | 0.1513E−04 | 5.50 | 0.4704E−03 | 4.45 |
| | 3 | 0.2397E−06 | 5.98 | 0.1519E−04 | 4.95 |
| | 4 | 0.3750E−08 | 6.00 | 0.4819E−06 | 4.98 |

6 | NUMERICAL EXPERIMENTS

6.1 | Example 1

Consider problem (1.1) with $\Omega = (0, 1)^2$ and $a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The source term $f$ and the boundary value $g$ are chosen so that the exact solution is (nonsymmetric, nonzero boundary value)

$$u(x, y) = \sin(x) \sin(\pi y).$$

In this example, we use uniform grids shown in Figure 1. In Table 1, we list the errors and the orders of convergence. We can see that we do have two orders of superconvergence in both norms.

6.2 | Example 2

We solve problem (1.1) where $\Omega = (0, 1)^2$ and

$$a = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$
### TABLE 2  Example 2: error profiles and convergence rates on grids shown in Figure 1

| Level | $\|uh - Qh u\|_0$ | Rate | $\|uh - Qh u\|$ | Rate |
|-------|-----------------|------|-----------------|------|
|       |                  |      |                 |      |
| By the $P_1 - P_2$ WG finite element |
| 5     | 0.5487E−06       | 3.95 | 0.1023E−03      | 2.97 |
| 6     | 0.3471E−07       | 3.98 | 0.1288E−04      | 2.99 |
| 7     | 0.2182E−08       | 3.99 | 0.1621E−05      | 2.99 |
|       |                  |      |                 |      |
| By the $P_2 - P_3$ WG finite element |
| 3     | 0.4230E−05       | 4.69 | 0.3492E−03      | 3.84 |
| 4     | 0.1440E−06       | 4.88 | 0.2261E−04      | 3.95 |
| 5     | 0.4680E−08       | 4.94 | 0.1432E−05      | 3.98 |
|       |                  |      |                 |      |
| By the $P_3 - P_4$ WG finite element |
| 2     | 0.8490E−05       | 5.34 | 0.5269E−03      | 4.62 |
| 3     | 0.1436E−06       | 5.89 | 0.1738E−04      | 4.92 |
| 4     | 0.2296E−08       | 5.97 | 0.5534E−06      | 4.97 |

**FIGURE 2** The first three grids for the computation in Table 3

The source term $f$ and the boundary value $g$ are chosen so that the exact solution is

$$u(x, y) = x^5 y^2.$$  

We use same meshes as Example 1. The result is listed in Table 2. The superconvergence phenomena are same as those in Example 1, that is, two orders of superconvergence in both $L^2$ norm and three-bar norm.

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### 6.3  Example 3

Consider problem (1.1) with $\Omega = (0, 1)^2$ and $a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The source term $f$ and the boundary value $g$ are chosen so that the exact solution is

$$u(x, y) = e^{\pi x} \sin(\pi y).$$

In this example, we use nonuniform grids shown in Figure 2. In Table 3, we list the errors and the orders of convergence. We can see that we do have two orders of superconvergence in both norms.
**TABLE 3** Example 3: error profiles and convergence rates on grids shown in Figure 2

| Level | $||u_h - Q_h u||_0$ | Rate | $||u_h - Q_h u||$ | Rate |
|-------|---------------------|------|-------------------|------|
| By the $P_1 - P_2$ WG finite element |
| 5     | 0.4530E−05          | 4.00 | 0.1130E−02        | 3.00 |
| 6     | 0.2830E−06          | 4.00 | 0.1411E−03        | 3.00 |
| 7     | 0.1768E−07          | 4.00 | 0.1762E−04        | 3.00 |
| By the $P_2 - P_3$ WG finite element |
| 5     | 0.4027E−07          | 5.00 | 0.1389E−04        | 4.00 |
| 6     | 0.1256E−08          | 5.00 | 0.8671E−06        | 4.00 |
| 7     | 0.3920E−10          | 5.00 | 0.5415E−07        | 4.00 |
| By the $P_3 - P_4$ WG finite element |
| 4     | 0.2138E−07          | 6.01 | 0.4534E−05        | 5.01 |
| 5     | 0.3308E−09          | 6.01 | 0.1407E−06        | 5.01 |
| 6     | 0.4979E−11          | 6.05 | 0.4378E−08        | 5.01 |

**FIGURE 3** The first three levels of grids used in Example 4

**TABLE 4** Example 4: error profiles and convergence rates on 3D grids shown in Figure 3

| Level | $||u_h - Q_h u||_0$ | Rate | $||u_h - Q_h u||$ | Rate |
|-------|---------------------|------|-------------------|------|
| By the $P_1 - P_2$ WG finite element |
| 1     | 0.0748126           | 0.0  | 0.9508644         | 0.0  |
| 2     | 0.0120779           | 2.6  | 0.1909763         | 2.3  |
| 3     | 0.0010535           | 3.5  | 0.0280478         | 2.8  |
| 4     | 0.0000726           | 3.9  | 0.0036750         | 2.9  |
| 5     | 0.0000047           | 4.0  | 0.0004657         | 3.0  |
| 6     | 0.0000003           | 4.0  | 0.0000584         | 3.0  |
| By the $P_2 - P_3$ WG finite element |
| 1     | 0.0106321           | 0.0  | 0.4099718         | 0.0  |
| 2     | 0.0019247           | 2.5  | 0.0537392         | 2.9  |
| 3     | 0.0000721           | 4.7  | 0.0038352         | 3.8  |
| 4     | 0.0000024           | 4.9  | 0.0002476         | 4.0  |
| 5     | 0.0000001           | 5.0  | 0.0000156         | 4.0  |
| By the $P_3 - P_4$ WG finite element |
| 1     | 0.0170645           | 0.0  | 0.3318098         | 0.0  |
| 2     | 0.0003554           | 5.6  | 0.0128696         | 4.7  |
| 3     | 0.0000062           | 5.8  | 0.0004504         | 4.8  |
| 4     | 0.0000001           | 6.0  | 0.0000145         | 5.0  |
6.4 Example 4

Consider problem (1.1) with $\Omega = (0, 1)^3$ and $a = I_{3 \times 3}$. The source term $f$ and the boundary value $g$ are chosen so that the exact solution is

$$u(x, y, z) = \sin(\pi x) \sin(\pi y) \sin(\pi z).$$

We use tetrahedral meshes shown in Figure 3. The results of the $(P_k(T), P_{k+1}(e))$ WG finite element methods are listed in Table 4. The superconvergence phenomena are same as those in 2D, in first three examples, two orders of superconvergence.

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CONFLICT OF INTEREST

The authors declare no conflicts of interest.

AUTHOR CONTRIBUTIONS

The authors discuss together the topic of this manuscript. Ahmed AL-Taweel, Xiaoshen Wang, Xiu Ye, and Shangyou Zhang are in charge of theoretical part of the numerical method. And Shangyou Zhang takes care of the numerical experiment of the presented numerical scheme.

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APPENDIX A

We prove Lemma 3.1 in the Appendix. First we need the following lemmas

**Lemma A.1** Let

\[ Q_1(x) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} x_i x_j = x^T A x, \]

and

\[ Q_2(x) = \sum_{j=1}^{n} \sum_{i=1}^{n} b_{ij} x_i x_j = x^T B x, \]

be two positive definite quadratic forms, where \( x = (x_1, \ldots, x_n) \), \( a_{ij} = a_{ji} \), and \( b_{ij} = b_{ji} \). Then \( \exists \lambda_0 > 0 \) so that \( \lambda Q_1(x) - Q_2(x) \) is a positive definite quadratic form for each \( \lambda > \lambda_0 \).

**Proof.** We need to show that there exists a \( \lambda_0 > 0 \) such that \( \lambda A - B \) is positive definite for \( \lambda > \lambda_0 \). Since \( A \) is positive definite, \( A = A^{-\frac{1}{2}} A^{-\frac{1}{2}} \), where \( A^{-\frac{1}{2}} \) is also symmetric positive definite with symmetric positive definite inverse. Thus

\[ \lambda A - B = A^{-\frac{1}{2}} \left( \lambda I - A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \right) A^{-\frac{1}{2}}. \]

Let \( C = A^{-\frac{1}{2}} BA^{-\frac{1}{2}} \). Then \( C \) is positive definite and thus

\[ C = P^T \begin{bmatrix} \lambda_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_{n-1} \end{bmatrix} P, \tag{A.1} \]

where \( \lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1} > 0 \) and \( P \) is an orthogonal matrix. Thus we can write

\[ \lambda I - C = P^T \begin{bmatrix} \lambda - \lambda_0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda - \lambda_{n-1} \end{bmatrix} P. \tag{A.2} \]

Obviously, \( \lambda I - C \) is positive definite when \( \lambda > \lambda_0 \) and so is \( \lambda A - B \). \( \square \)
Lemma A.2  For any \( v \in V_h \), if \( \nabla w \big|_{T_i} \in [P_{k+1}(T_i)]^2 \), \( \forall i = 1, 2 \), \( T_1 \cap T_2 = e_1 \), then

\[
||v^{(1)}_0 - v^{(2)}_0||_{e_1}^2 \leq Ch_{T_i} \|\nabla w\|_{T_i}^2, \tag{A.3}
\]

where \( v^{(i)}_0 = v_0 \big|_{T_i}, \ i = 1, 2 \).

Proof. Without loss of generality, we may assume that the vertices of \( T_2 \) are \((0, 0), (1, 0), \) and \((0, 1)\), and the other edge of \( T_1 \) is \((a_1, b_1)\), where \( b_1 < 0 \) (Figure A.1). Denote \( v^{(i)}_0 = v_0 \big|_{T_i}, \ i = 1, 2 \).

Let \( t_2 \) and \( t_4 \) be unit tangents to \( e_2 \) and \( e_4 \), respectively; \( L_3 \) and \( L_5 \) be linear functions such that \( L_3 \big|_{e_3} = 0 \) and \( L_5 \big|_{e_5} = 0 \). Let

\[
q_1 = q \big|_{T_1} = L_3(x, y)(v^{(1)}_b - v^{(1)}_0 + yQ^{(1)}_{k-1})t_2 = Q^{(1)}t_2,
\]

and

\[
q_2 = q \big|_{T_2} = L_5(x, y)(v^{(2)}_b - v^{(2)}_0 + yQ^{(2)}_{k-1}(x, y))t_4 = Q^{(2)}t_4,
\]

where \( Q^{(i)}_{k-1} \) are such that

\[
(Q^{(i)}_{k-1}, p)_{T_i} = 0, \quad \forall p \in P_{k-1}(T_i). \tag{A.4}
\]

By setting \( v^{(2)}_0 - v^{(1)}_0 = 0 \) in \( Q^{(i)}_{k-1} \) and \( p = Q^{(i)}_{k-1} \) in (A.4), we know such \( Q^{(i)}_{k-1} \) exist and are unique. Then

\[
q_i \cdot n^{(2)}_i = 0, \quad q_i \big|_{e_{2i+1}} = 0, \quad i = 1, 2.
\]

Scale \( L_3 \) and \( L_5 \) if necessary so that

\[
-L_3(0, 0)t_2 \cdot n^{(1)}_1 = 1 = L_5(0, 0)t_4 \cdot n^{(2)}_1,
\]

where \( n^{(i)}_1 \) is the unit outwards normal vector of \( e_1 \in \partial T_i, i = 1, 2 \).

Since \( L_5(1, 0)t_2 \cdot n^{(2)}_1 = 0 \),

\[
L_5(x, y)t_4 \cdot n^{(2)}_1 = \tilde{L}_5(x, y) = 1 - x - y.
\]

Similarly

\[
L_3(x, y)t_2 \cdot n^{(2)}_1 = \tilde{L}_3(x, y) = 1 - x + \alpha y, \quad \text{for some } \alpha.
\]
It follows from the shape regularity assumptions that the slope of $e_3$, $\frac{1}{a}$, satisfies $|\frac{1}{a}| \geq a_0 > 0$ for some $a_0$. Since $\tilde{L}_3|_{e_1} = \tilde{L}_5|_{e_1}$,

\[
(\nabla_w v, q)_{T_j \cup T_2} = (v_b - v_0, q \cdot n)_{\partial T_1} + (v_b - v_0, q \cdot n)_{\partial T_2}
\]

\[
= (v_0^{(2)} - v_0^{(1)}), (v_0^{(2)} - v_0^{(1)})\tilde{L}_3|_{e_1}.
\]

Note that $0 < \tilde{L}_5(x, y) \leq 1$ on $T_1$ and $0 \leq \tilde{L}_3(x, y) \leq 1$ on $T_2$. Write

\[
(v_0^{(2)} - v_0^{(1)})|_{e_1}(x) = a_0 + \cdots + a_k x^k = P(x).
\]

Let $\alpha = [a_0, \ldots, a_k]^T$. Then

\[
||v_0^{(2)} - v_0^{(1)}||_{e_1}^2 = \alpha^T A \alpha.
\]

\[
(\nabla_w v, q)_{T_1 \cup T_2} = (v_0^{(2)} - v_0^{(1)}), (v_0^{(2)} - v_0^{(1)})\tilde{L}_3(x, y)|_{e_1}
\]

\[
= \int_0^1 (v_0^{(2)} - v_0^{(1)})^2(1 - x)dx
\]

\[
= \alpha^T B \alpha,
\]

are positive definite quadratic forms in $a_0, \ldots, a_k$. Note that

\[
||q||_{T_2}^2 = \int_{T_2} (1 - x - y)^2(Q^{(2)})^2 dA
\]

\[
\leq C \left( \int_{T_2} (v_0^{(2)} - v_0^{(1)})dA + \int_{T_2} (Q_{k-1}^{(1)})^2 dA \right)
\]

\[
\leq C(\alpha^T D \alpha + \alpha^T E \alpha)
\]

is a positive definite quadratic form. Similarly,

\[
||q||_{T_1}^2 = \alpha^T G \alpha
\]

is also a positive definite quadratic form. So by Lemma A.1,

\[
||q||_{T_i}^2 \leq C||v_0^{(2)} - v_0^{(1)}||_{e_1}^2, \quad i = 1, 2,
\]

and

\[
C(\nabla_w v, q)_{T_j \cup T_2} \geq ||v_0^{(2)} - v_0^{(1)}||_{e_1}^2.
\]

Thus

\[
||v_0^{(2)} - v_0^{(1)}||_{e_1}^2 \leq C||\nabla_w v, q||_{T_j \cup T_2}^2
\]

\[
\leq C(||\nabla_w v||_{T_1} ||q||_{T_2} + ||\nabla_w v||_{T_2} ||q||_{T_2})
\]

\[
\leq C(||\nabla_w v||_{T_1} + ||\nabla_w v||_{T_2})||v_0^{(2)} - v_0^{(1)}||_{e_1}^2,
\]

for some $C$. It is worthwhile to note that this $C$ is independent of $h$ in some sense. If we want to change the scale from $1$ to $h$, we only need to replace $a_i$ by $a_i h^{i}$. Because of the dimensional differences of the two sides, $C$ will be replaced by $C h$. Thus after a scaling we have

\[
||v_0^{(2)} - v_0^{(1)}||_{e_1} \leq C h^{\frac{1}{2}}(||\nabla_w v||_{T_1} + ||\nabla_w v||_{T_2}).\]
**Lemma A.3** Let $T_1$ and $T_2$ be such as in Lemma A.2, then
\[ \|v_b - v_0\|^2_{\partial T_1 \cup \partial T_2} \leq Ch_{T_1} \|\nabla wv\|^2_{T_1 \cup T_2}. \]  
(A.5)

**Proof.** Without loss, we may assume that $T_1$ and $T_2$ are as shown in Figure A.2, where $e_1 \cup e_2 \cup e_3 = \partial T_1$, $e_1 \cup e_4 \cup e_5 = \partial T_2$ and $e_1 = \partial T_1 \cap \partial T_2$.

Let us assume that $\nabla wv|_{T_1 \cup T_2} = 0$. It follows from Lemma A.2 that
\[ \|v_0^{(1)} - v_0^{(2)}\|^2_{e_i} \leq Ch_{T_1} \|\nabla wv\|^2_{T_1 \cup T_2} = 0. \]
We want to show
\[ \|v_b - v_0\|^2_{e_4} \leq Ch_{T_1} \|\nabla wv\|^2_{T_1 \cup T_2} = 0 \]  
(A.6)
first. Let $L_2(x, y) = 1 - y$, then $L_2 = 0$ on $e_2$. Denote by $n^{(i)}_j$ the unit outer normal vector to $\partial T_i \cap e_j$, $i = 1, 2, j = 1, \ldots, 5$. Let $t_3$ and $t_5$ be unit tangent vectors to $e_3$ and $e_5$ respectively. Let
\[ q|_{T_1} = q_1 = Q^{(1)} t_3, \]
where $Q^{(1)} = L_2 Q^{(1)}_k$, $Q^{(1)}_k \in P_k(T_1)$. Write
\[ Q^{(1)}_k(x, y) = p_k(x) + (x - y)p_{k-1}(x, y), \]
where $p_k(x) \in P_k(T_1), p_{k-1}(x, y) \in P_{k-1}(T_1)$. We want to show that for each $p_k(x)$, we can find a unique $p_{k-1}(x, y)$ so that
\[ (Q^{(1)}, p) = 0, \quad \forall p \in P_{k-1}(T_1). \]  
(A.7)
To do that we set $p_k = 0$ and $p = p_{k-1}(x, y)$. Then
\[ \int_{T_1} (1 - y)(x - y)p_{k-1}^2 dA = 0, \]
which implies that
\[ p_{k-1} = 0. \]
This implies the existence and the uniqueness of $p_{k-1}$. Now let
\[ q|_{T_2} = q_2 = Q^{(2)} t_5, \]
where
\[ Q^{(2)}(x, y) t_5 \cdot n^{(2)}_1 = -t_3 \cdot n^{(1)}_1 (1 - x)p_k(x) \]
\[ = -t_3 \cdot n^{(1)}_1 Q^{(1)}(x, x). \]
Thus
\[ \langle q_1 \cdot n_1^{(1)}, v_b - v_0^{(1)} \rangle_{e_1} + \langle q_2 \cdot n_1^{(2)}, v_b - v_0^{(2)} \rangle_{e_1} = 0, \]
since \( v_0^{(1)} = v_0^{(2)} \) and \( q_1 = q_2 \) on \( e_1 \). Without loss, we assume \( t_3 \cdot n_4 = 1 \). Let \( \delta \) be such that \( \delta t_5 \cdot n_4^{(2)} = -t_3 \cdot n_4^{(1)} \). Write
\[ Q^{(2)}(x, y) = \delta(1 - x)p_k(x) + (x - y)[Q_k^{(2)}(x) + yR_{k-1}(x, y)], \]
Write
\[ (v_b - v_0)|_{e_4}(x) = V_{k+1}(x) = v_0 + \cdots + v_{k+1}x^{k+1}, \]
\[ p_k(x) = a_0 + \cdots + a_kx^k, \]
and
\[ xQ_k^{(2)}(x) = b_1x + \cdots + b_{k+1}x^{k+1} \]
such that
\[ \delta a_0 = v_0, \]
\[ \delta(a_1 + b_1 - a_0) = v_1, \]
\[ \delta(a_2 + b_2 - a_1) = v_2 \]
\[ \vdots \]
\[ \delta(b_{k+1} - a_k) = v_{k+1}. \]
Then
\[ \delta(1 - x)p_k(x) + xQ_k^{(2)}(x) = (v_b - v_0)|_{e_4} \]
and thus
\[ \langle Q^{(2)}, v_b - v_0 \rangle_{e_4} = \| v_b - v_0 \|_{e_4}^2. \]  \hfill (A.8)
Furthermore, let
\[ b_1 = b_2 = \cdots = b_k = 0. \]
Then
\[ \delta a_0 = v_0, \]
\[ \delta a_1 = v_1 + v_0, \]
\[ \vdots \]
\[ \delta a_k = v_k + v_{k-1}, \]
\[ \delta b_{k+1} = v_{k+1} + v_k + v_{k-1}. \]
Thus, \( V = 0 \) implies \( p_k = Q_k^{(2)} = 0 \). Next we want to choose \( R_{k-1}(x, y) \) so that
\[ \langle Q^{(2)}, p \rangle_{T_2} = 0, \quad \forall p \in P_{k-1}(T_2). \]  \hfill (A.9)
To see if (A.9) has a unique solution, we set \( V = 0 \). Then \( p_k = Q_k^{(2)} = 0 \). Thus (A.9) becomes
\[ ((x - y)yR_{k-1}, p)_{T_2} = 0, \quad \forall p \in P_{k-1}(T_2). \]
It is easy to see that \( R_{k-1} = 0 \). Thus
\[ 0 = (\nabla_v \cdot q, q)_{T_1 \cup T_2} = \langle v_b - v_0, q \cdot n \rangle_{e_4} = \| v_b - v_0 \|_{e_4}^2. \]
Similarly, we can show that

\[ \|v_b - v_0\|_{e_i}^2 = 0, \quad i = 2,3,5. \]

Now let us look at \( \|v_b - v_0^{(i)}\|_{e_i} \). Since \( \|\nabla_w v\|_{T_i} = 0, \|v_b - v_0^{(i)}\|_{e_i} = 0, \ i = 2,3, \)

\[-(\nabla v, q)_{T_i} = \langle v_b - v_0^{(i)}, q \cdot n_1 \rangle_{e_i}, \quad \forall q \in [P_{k+1}(T)]^2. \quad (A.10)\]

Let

\[ q = (v_b - v_0^{(i)} + (x - y)Q_k(x, y))n_1 = Qn_1, \]

where \( Q_k \) is such that

\[ (Q, p)_{T_i} = 0, \quad \forall p \in P_{k-1}(T_i) \]

and \( v_b \) is extended to \( T_1 \). Then

\[ 0 = \langle v_b - v_0^{(i)}, q \cdot n \rangle_{e_i} = \|v_b - v_0^{(i)}\|_{e_i}. \]

Similarly

\[ \|v_b - v_0^{(2)}\|_{e_i} = 0. \]

By (A.10)

\[ \nabla v_0^{(i)} = \nabla v_0^{(2)} = 0. \]

Thus

\[ \|\nabla_w v\|_{T_1 \cup T_2}^2 = 0, \]

implies

\[ \|\nabla v_0\|_{T_1 \cup T_2}^2 + \|v_b - v_0\|_{\partial T_1 \cup \partial T_2}^2 = 0. \]

Let

\[ \nabla v_0^{(i)} = \begin{bmatrix} q_1^{(i)} \\ q_2^{(i)} \end{bmatrix}, \]

where \( q_j^{(i)} \in P_{k-1}(T_i) \). Let \( \bar{P}_{k-1} = [1, x, y, \ldots, y^{k-1}], \bar{P}_{k+1} = [1, x, y, \ldots, y^{k+1}] \), write

\[ q_j^{(i)}(x, y) = \bar{P}_{k-1}a_j^{(i)}, \]

where \( a_j^{(i)} \) is the coefficient vector of \( q_j^{(i)} \). Then

\[ \|\nabla v_0^{(i)}\|_{T_i}^2 = (\bar{a}_1^{(i)})^T A_1 a_1^{(i)} + (\bar{a}_2^{(i)})^T A_2 a_2^{(i)}. \]

Thus \( \|\nabla v_0\|_{T_1 \cup T_2}^2 \) is a positive definite quadratic form in \( \alpha = \begin{bmatrix} a_1^{(1)} \\ a_2^{(1)} \\ a_1^{(2)} \\ a_2^{(2)} \end{bmatrix} \). Similarly

\[ \|v_b - v_0^{(i)}\|_{e_i}^2, \quad i = 1,2,3, \quad \|v_b - v_0^{(2)}\|_{e_i}^2, \quad j = 1,4,5 \]

are positive definite quadratic forms in \( \beta_j, j = 1, \ldots, 6, \) where \( \beta_1 - \beta_3 \) are variables in the first 3 quadratic forms and \( \beta_4 - \beta_6 \) are variables in the second 3 quadratic forms, respectively. Note that the leading coefficients of \( (v_b - v_0^{(i)}) \) and \( (v_b - v_0^{(2)}) \) are the
same. To keep those variables independent, we remove that coefficient from $\beta_4$ and let 

$$\beta = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_6 \end{bmatrix}.$$ 

Write $\bar{\gamma} = [\alpha \beta]$, then 

$$\|\nabla v_0\|^2_{T_1 \cup T_2} + \|v_b - v_0\|^2_{\partial T_1 \cup \partial T_2} = \bar{\gamma}^T A \bar{\gamma},$$

where $A$ is positive definite. Since $\nabla_w v|_{T_1 \cup T_2}$ is uniquely determined by $\alpha$ and $\beta$ by solving a linear system, 

$$\|\nabla_w v\|^2_{T_1 \cup T_2} = \bar{\gamma}^T B \bar{\gamma},$$

where $B$ is positive semi-definite. Since $\bar{\gamma}^T B \bar{\gamma} = 0$ implies $\bar{\gamma}^T A \bar{\gamma} = 0$, $B$ is positive definite. By Lemma A.1 and a scaling argument, 

$$\|v_b - v_0\|^2_{\partial T_1 \cup \partial T_2} \leq C h_{T_1} \|\nabla_w v\|^2_{T_1 \cup T_2}.$$ 

Lemma 3.1 is a direct result of Lemma A.3.