Classification of rank one 5d $\mathcal{N} = 1$ and 6d $(1, 0)$ SCFTs

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ABSTRACT: This paper gives a classification of rank one 5d $\mathcal{N} = 1$ and 6d $(1, 0)$ SCFTs. The idea is to compactify 5d theory on $S^1$ and 6d theory on $T^2$ to get effective 4d $\mathcal{N} = 2$ theory. These compactified theories all have a 4d $\mathcal{N} = 2$ Coulomb branch whose solution can be described by mixed Hodge module (MHM). In the rank one case every Coulomb branch solution is related to rational elliptic surface with a section, whose classification is complete. So the classification is then reduced to pick up a subset from the data base of rational elliptic surface by imposing various physical constraints. The crucial new input is that the singular fiber at infinity determines which dimension the UV theory lives. Various physical properties such as flavor symmetry, one form symmetry, BPS quiver and BPS spectrum, and RG flow are studied. D7 brane configurations for those theories are found which are very useful in studying them. The generalizations to higher rank theories will also be highlighted.
1 Introduction

There are lots of interests in studying supersymmetric theory with eight supercharges in various dimensions, such as 3d $\mathcal{N} = 4$ theory [1], 4d $\mathcal{N} = 2$ theory [2, 3], 5d $\mathcal{N} = 1$ theory [4], and 6d $(1, 0)$ theory [5]. Theories with eight supercharges are less rigid than theories with 16 supercharges [6], and so they have more interesting dynamical properties while
still remain tractable. These theories share some common features: such as the existence of moduli space of vacua like Higgs branch, Coulomb branch (tensor branch in 6d), and massive BPS states on the Coulomb branch, etc. They do have some important differences, for example: the type of supersymmetric preserving deformations are different [7].

One can relate theories in different dimensions by putting higher dimensional theory on compact manifold. To preserve all the supersymmetry, one need to use the torus $T^d$ compactification, and interesting properties of higher dimensional theory can be learned by looking at the effective lower dimensional theory. Let’s put 5d $\mathcal{N} = 1$ theory on $S^1$ [8] and 6d $(1, 0)$ theory on $T^2$ [9] (these are called KK theories), so the low energy theory has effective 4d $\mathcal{N} = 2$ supercharges. The KK theory could have a $\mathcal{N} = 2$ Coulomb branch where the low energy theory can be solved. While the Coulomb branch solution of any $\mathcal{N} = 2$ theory have common structures, the details are affected by higher dimensional theory, for example, the effective photon couplings could receive contribution of higher dimensional BPS states, and the BPS particle also carry winding mode charge along the $T^d$ direction, etc. This is the reason that one can use the Coulomb branch solution of the KK theory to classify higher dimensional theory.

The main purpose of this paper is to use the Coulomb branch structure of the compactified theory to classify rank one 5d and 6d theories with eight supercharges. The Coulomb branch solution for 4d $\mathcal{N} = 2$ $SU(2)$ gauge theory has been solved elegantly by Seiberg-Witten (SW) [2, 3]. The Coulomb branch for compactified higher dimensional theory were soon studied [8, 9]. The solutions are often described by a family of algebraic varieties fibered over the Coulomb branch. In practice, the solutions are most easily found by using string theory motivated methods [10], and the relation to integrable system [11, 12].

More generally, It was noticed in [13, 14] that the general Coulomb branch solution is given by the so-called mixed Hodge module (MHM) over the Coulomb branch. The geometric solutions actually fit into this description by looking at the cohomology groups of the algebraic variety. Let’s now briefly review MHM [15] and describe how to extract the physics at each vacua of Coulomb branch from MHM. Over the generic vacua, the MHM is described by variation of mixed Hodge structure [16]; It is described by a flat holomorphic vector bundle with extra structures over very fiber $H$: a) a weight filtration $W^\bullet$, which ensure that the flavor and the electric-magnetic charge can be distinguished; b) a Hodge filtration $F^\bullet$ and a polarization $Q(\cdot, \cdot)$, which ensure that positive photon couplings $\tau_{ij}(u)$ will be derived. Over the special vacua, besides the vector space which is used to describe the abelian part of the low energy theory, two extra vector spaces (called nearby cycle and vanishing cycle) are defined using the limit behavior of nearby vector spaces, and both of them carry mixed Hodge structure; The vanishing cycle can be used to find the information of the interacting theory, i.e. the spectrum of Coulomb branch operator. Finally, at the infinite point of the Coulomb branch, a vector space together with a limit MHS can also be defined, from which one can find the properties of the UV theory. The structure of Coulomb branch solution is shown in figure. 1.

Based on known Coulomb branch solutions, several assumptions on MHM for $\mathcal{N} = 2$ Coulomb branch solution are proposed: First, the weight filtration has the structure $0 = W_0 \subset W_1 \subset W_2 = H$ for generic vacua, namely there are only two nontrivial weights;
$H_\infty (F^*, W^*(N_\infty))$

$H_\text{lim} (F^*, W^*(N))$

$H_s (F^*, W^*)$

$H_{\text{van}} (F^*, W^*(N))$

Figure 1: The structure of $\mathcal{N} = 2$ Coulomb branch: 1): at generic point, one has a vector space with a mixed Hodge structure; 2): at special point, there are three vector spaces and all of them carry mixed Hodge structure; and there is a monodromy group $T$ acting on these vector spaces; 3): One can also have a vector space at $\infty$ of moduli space.

Second, the monodromy group satisfies the relation $(T^k - I)^2 = 1$, namely, the maximal size of Jordan block is two; Finally, the monodromy group acts trivially on the weight two part of generic fiber (the flavor charge): $T|Gr^W_2 = I$. These assumptions simplified the analysis of the Coulomb branch solution of $\mathcal{N} = 2$ theory.

In this paper, the Coulomb branch is taken to be rank one, i.e. the weight one part of $H$ is two dimensional. Because the monodromy action on weight two part is trivial, we could just look at weight one part on $u$ plane (with mass parameters fixed). So eventually there is a rank two MHM over $\mathbb{P}^1$ (since we include the $\infty$ in our definition of the solution). The classification of the Coulomb branch solution then goes as follows:

1. **Local singularity**: Let’s first analyze the local behavior of MHM. First, the monodromy group (topological data) is classified by a conjugacy class of $SL(2, \mathbb{Z})$ group satisfying the condition $(T^2 - I)^2 = 0$. All such conjugacy classes are classified and actually coincide with the list of Kodaira’s singularity in the study of degeneration of elliptic surface. These singularities are labeled as $I_n, I^*_n, II, III, IV, II^*, III^*, IV^*$ [17]. The holomorphic data is given by the period mapping, and in the rank one case, there is a holomorphic function $\tau(u)$ and the related $J$ invariant $j(u)$ [18].
2. **Global constraints**: There are global constraints on the monodromy and holomorphic data. In the rank one case, there is a nice correspondence with rational elliptic surface [19]. This correspondence is extremely useful as the rational elliptic surface has been classified [20, 21], so there is a data base to work with.

3. **Singular fiber at infinity**: The next question is the singular type appearing at $\infty$ which reflects the UV properties. This data reflects what dimension the UV theory lives:
   - For 4d theory, only type $I^*_n, II, III, IV, II^*, III^*, IV^*$ singularities can be put at infinity.
   - For 5d KK theory, only type $I_n, n \geq 1$ singularities can be put at infinity.
   - For 6d KK theory, only type $I_0$ singularities can be put at infinity.

4. **Generic deformations**: For the purpose of classification, the bulk singularity is restricted to have just one dimensional Coulomb branch deformations (These include the expectation values of Coulomb branch operator, mass parameters, and relevant deformations). Essentially we consider co-dimensional one singularity at the full generalized Coulomb branch. This constraint removes the appearance of type $II, III, IV$ singularities at the bulk, as the scaling dimension of Coulomb branch operator associated with them is less than two, and so it has at least two dimensional deformation spaces. Those singularities appear in the generic deformations are called undeformable singularities.

5. **Dirac quantization**: Let’s now assume that only type $I_n$ (undeformable) singularities appear in the bulk, and assume that there is a massless BPS hypermultiplet with charge $\sqrt{n}(1,0)$ (in proper duality frame) at the singularity. This BPS particle is stable at any given point of the Coulomb branch. The Dirac quantization condition requires that the Dirac pairing between these massive BPS particles to be integral. This condition means for any two $I_{n_i}$ and $I_{n_j}$ singular fibers at bulk, the ratio $\frac{n_i}{n_j}$ should be square.

6. **Base change**: The base change method [22] of rational elliptic surface (for 4d theory this is interpreted as discrete gauging [23]) is used to get other type of undeformable singularities at the bulk. Here we start with configuration $B$ with only type $I_n$ bulk singularities, and use a $Z_n$ action of $B$ to get another configurations $B'$ satisfying: a) the singular fiber of $B'$ is related to $B$ in a specific way, see formula. 3.3 for 4d case, formula. 5.2 for 5d case; b): no appearance of type $II, III, IV$ singular fiber in the bulk.

Using above guidelines and the data base [20] for rational elliptic surface and the base change maps [24], the complete list of theories are reported in table. 2 for 4d theory, and

\[\text{One must be careful that just specifying the singularity type does not specify the low energy theory. For example, just specifying a } I_n \text{ singularity does not tell us what is the low energy theory, one need more data, i.e.e the weight two part of the MHM to determine the IR theory.}\]
While the properties of rational elliptic surface is quite useful in classifying the theories, we find that the D7 brane configuration \([25, 26]\) can play an amazing role in further analyzing those theories. A D7 brane configuration for each of the theory is constructed, and the string junctions of them are used to study the flavor symmetry, BPS spectrum, BPS quiver, etc for those theories.

This paper is organized as follows: in section two we review the mixed Hodge structure for \(\mathcal{N} = 2\) Coulomb branch solution; in section 3 we classify rank one 4d \(\mathcal{N} = 2\) theory by using the correspondence with rational elliptic surface (see the known results in \([22, 27–30]\)) ; section 4 discusses the D7 brane configurations for 4d theory, and string junctions are used to compute the flavor symmetry, BPS quiver, BPS spectrum, RG flow etc; section 5 gives a classification for rank one 5d \(\mathcal{N} = 1\) SCFT; section 6 gives a classification for rank one 6d \((1,0)\) theory. Finally, a conclusion is given.
2 Mixed Hodge module and $\mathcal{N} = 2$ Coulomb branch solution

There are several important goals of solving the Coulomb branch of a $\mathcal{N} = 2$ theory:

1. At a generic point, the low energy theory is described by $U(1)^r$ abelian gauge theory, free hypermultiplets, and possibly interacting SCFT whose Coulomb branch deformation is trivial \(^2\). We’d like to determine those three components and their physical properties; An important goal is to determine the effective coupling for the $U(1)^r$ gauge theory, here $r$ is called the rank of the theory.

2. At a special point, new massless degrees of freedom appear and we’d like to determine the effective low energy theory, which could be IR free gauge theory, or SCFT, or the direct sum of them.

3. The new massless degrees of freedom at singularity come from massive BPS particles at the generic point, and it is important to find the spectrum of stable BPS particles and their central charges.

The Coulomb branch solution for $SU(2)$ gauge theory was solved in an elegant way by Seiberg and Witten [2, 3]. They solved the theory by finding a family of algebraic curves $F(x, y, u, m, \Lambda) = 0$ (Here $u$ parameterizes the Coulomb branch, $m$ the mass parameters, and $\Lambda$ the dynamical generated scale.), and a SW differential $\lambda$ is also needed. The physical information is extracted as follows: a) At the generic point of the $u$ plane, the SW curve $F_u$ is smooth and the low energy theory is just $U(1)$ gauge theory; The photon coupling is given by the complex structure of the curve $F_u$; b) At the special point, the SW curve $F_u$ becomes singular, and the low energy theory is $U(1)$ gauge theory coupled with one massless hypermultiplet, which comes from the massive BPS particle at generic point; c) Finally, the central charge for the BPS particle is given as $Z = na + an_D a_D + \sum S_i m_i$, here $a, a_D, m_i$ are defined by doing period integral of the SW differential: $a = \int_A \lambda, a_D = \int_B \lambda, m_i = \int_{\Omega_i} \lambda$ ($A, B, \Omega_i$ are one cycles on Riemann surface $F_u$). One of the crucial insights of [2, 3] is the electric-magnetic duality of the $U(1)$ gauge theory, which is encoded in the geometry automatically: the complex structure of the elliptic curve $F_u$ has the $SL(2, \mathbb{Z})$ invariance.

2.1 Mixed Hodge module

Generally one solved the Coulomb branch of a $\mathcal{N} = 2$ theory by finding a family of algebraic varieties $F(z, m_i, u_i, \lambda_i) = 0$ and a SW differential $\lambda$, here $u_i$ denotes the expectation value of Coulomb branch operators, $m_i$ the mass parameters, and $\lambda_i$ the coupling constants including the exact marginal deformations and the relevant deformations. It was observed in [13, 14] that one need the so-called mixed Hodge structure to understand the low energy theory at the non-generic vacua. In fact, the Coulomb branch solution could be represented

\(^2\)This picture of Coulomb branch might be found by looking at Type IIB string theory on a CY manifold $X$: $X$ usually has non-trivial three cycles and so one get free vector multiplets; If $X$ has non-trivial four cycles, then one get free hypermultiplets; if $X$ has a rigid singularity (singularity admits no complex deformation), then one get interacting SCFT with no Coulomb branch. The generic structure on Higgs branch is found in [31].
as a mixed Hodge module over the generalized Coulomb branch (parameter space including Coulomb branch operators, masses, relevant and marginal couplings).

Let’s now describe aspects of the mixed Hodge module which are relevant for \( \mathcal{N} = 2 \) solution, here the rank of the theory is \( r \) and the flavor symmetry has rank \( f \). First, at the generic points of the Coulomb branch, there is a flat \(^3\) holomorphic vector bundle whose rank is \( 2r + f \). To get the information of the low energy theory, two extra structures are needed on the fiber \( H \):

- A mixed Hodge structure, namely a Hodge filtration and a weight filtration; The weight filtration is an increasing filtration which takes the following form \(^4\):

\[
\{0\} = W^0 \subset W^1 \subset W^2 = H;
\]

so we have two quotient spaces \( Gr_1^W = W^1/W^0, \; Gr_2^W = W^2/W^1 \), with dimension \( \text{dim}(Gr_1^W) = 2r, \; \text{dim}(Gr_2^W) = f \). The weight filtration is needed so that we can separate the electric-magnetic part and the flavor part of the central charge: \( Gr_1^W \) gives the electric-magnetic charge, and \( Gr_2^W \) gives the flavor charge. The Hodge filtration is a decreasing filtration and takes the following form

\[
H = F^0 \supset F^1;
\]

So in our case, two holomorphic sub-bundles \( W^1 \) and \( F^1 \) are needed. The weight filtration and Hodge filtration together defines a so-called Mixed Hodge structure, and Hodge decomposition takes the form \( Gr_1^W = H^{1,0} \oplus H^{0,1} \) and \( Gr_2^W = H^{1,1} \), with dimension \( h^{1,0} = h^{0,1} = r \) and \( h^{1,1} = f \);

- A polarization \( Q(\cdot,\cdot) \) (which satisfies Riemann-Hodge bilinear relations on \( Gr_1^W \) and acts trivially on \( Gr_2^W \)) on \( H \) so that positive definite coupling constants can be defined. In fact, a period matrix \( Z_{ij} \) which is symmetric and satisfies the condition \( \text{Im}(Z) > 0 \) can be defined using the polarization.

At the singular point \(^5\), there is also a vector space \( H_s \) whose dimension is smaller than \( H \), so the mathematical structure is not the vector bundle which is more familiar to physicists. The physics of the abelian gauge theory at singular point is described by \( H_s \). The crucial point of the mixed Hodge module is that one can define two more vector spaces at the singular point. The first is the so-called nearby cycle \( H_{\lim} \) which can be thought of as the limiting objects for the nearby vector spaces. There is a mixed Hodge structure on \( H_{\lim} \) which is quite different from that of the generic fiber described earlier. The weight filtration is now determined by the nilpotent part \( N \) of the monodromy group \( T \) around the singularity.

\(^3\)The flat structure gives an integrable connection which is required for the definition of the mixed Hodge module.

\(^4\)If the SW geometry is given by a three dimensional variety \([32]\), then the maximal weight is 4. These MHS could be brought to the form presented here by doing a Tate twist.

\(^5\)The Coulomb branch is not singular, but the physics is different from that of the generic point of the Coulomb branch.
For the known solution, the monodromy group $T$ satisfies the following condition

$$(T^k - 1)^2 = 1. \quad (2.3)$$

Namely the maximal size of the Jordan block is two, and the eigenvalue satisfies $\lambda^k = 1$. We conjecture that this is true for the Coulomb branch solution of any $\mathcal{N} = 2$ field theory. Furthermore, if we restrict the monodromy on the weight two part of the generic fiber, its action is trivial

$$T|_{Gr_2^W} = I. \quad (2.4)$$

What this implies is that the monodromy matrix takes the form

$$T = \begin{bmatrix} I & 0 \\ * & M \end{bmatrix},$$

and $M$ acts on weight one part. Finally, $H_s$ and $H_{\text{lim}}$ can be used to define a third vector space called vanishing cycle $H_{\text{van}}$. All of these three spaces carry mixed Hodge structure, and they form an exact sequence of mixed Hodge structure.

Using the limit mixed Hodge structure (let’s assume $H_s = 0$), one can define a set of rational numbers $(\alpha_1, \alpha_2, \ldots, \alpha_s)$ called spectrum [33], and its relation to the eigenvalue is given as

$$\lambda_i = \exp(2\pi i \alpha_i).$$

The monodromy group acts on the vector space $H_{\text{lim}}$, and so it has the decomposition $H_{\text{lim}} = \oplus \lambda H_{\text{lim}}^\lambda$. The limit Hodge filtration defines a filtration on $H_\lambda$: $F^0(H_\lambda) \supset F^1(H_\lambda)$. Now for a basis element $e_i$ in $H_\lambda$, a spectral number $\alpha$ is defined as

$$\begin{cases} 
  e_i \in F^1(H_\lambda), & -1 < \alpha \leq 0 \\
  e_i \in F^0(H_\lambda)/F^1(H_\lambda), & 0 < \alpha \leq 1 
\end{cases}$$

here $\exp(2\pi \alpha) = \lambda$. An important consistent condition is that the spectral numbers are in pair

$$\alpha_i + \alpha_j = 0.$$

One can find the Coulomb branch spectrum from the spectral numbers as follows. Assume there is a minimal spectrum number $\alpha_{\text{min}}$, then for a spectral number $\alpha_i$, one associate a Coulomb branch scaling dimension as [34]

$$[u_i] = \frac{1 + \alpha_{\text{min}} - \alpha_i}{1 + \alpha_{\text{min}}}. \quad (2.5)$$

So if $\alpha_i = 0$, then $[u_i] = 1$ which gives a mass parameter. The maximal scaling dimension is given as $\frac{1}{1 + \alpha_{\text{min}}}$. While the above analysis is carried for the finite points of the Coulomb branch, it is possible to do the similar computation for the $\infty$ point on the moduli space, and the MHS at $\infty$ is useful to extract information for UV theory. The general structure of the Coulomb branch solution is summarized in figure. 1.
Figure 2: The moduli space of rank one $\mathcal{N} = 2$ theory, here the mass parameters are fixed.

**SW differential and BPS spectrum:** To study the massive BPS spectrum, a $\mathbb{Z}$ module $H_Z$ at each generic point is needed, so that $H_Z \otimes \mathbb{C} = H$. The monodromy group $T$ on weight one part is now an element of $Sp(2r, \mathbb{Z})$. Given a SW differential $\lambda$ and period integral can be defined, so that the central charge for the BPS particles can be found. However, the determination of BPS spectrum is a much more difficult problem.

The SW differential $\lambda$ is a section of the holomorphic vector bundle. Because of the Hodge filtration, the weight one part $Gr^W_1$ has a Hodge decomposition $Gr^W_1 = H^{1,0} \oplus H^{0,1}$. Let’s choose the coordinate of the Coulomb branch as $u_i$ (these are the expectation values of Coulomb branch operators), $\lambda$ has to satisfy the condition [13]:

$$\partial u_i \lambda \subset H^{1,0}.$$  \hfill (2.6)

so that a special Khaler structure on the Coulomb branch can be defined.

3 Classification of rank one 4d $\mathcal{N} = 2$ theory

Let’s now study 4d rank one $\mathcal{N} = 2$ theory, i.e. $r = 1$. For the SW solution, since the monodromy action acts trivially on weight two part, we might focus on weight one part, so there are: a): a dimension two complex vector space at generic point with Hodge filtration and polarization; b): a monodromy group around each singularity, which is an element of $SL(2, \mathbb{Z})$ group; c): A period function $\tau(u)$ at each generic point. For fixed mass parameters $m_i$, the Coulomb branch solution is shown in figure. 2. Since the point $\infty$ is included in the Coulomb branch, the $u$ plane is now a compact space $\mathbb{P}^1$.

**Remark:** Since the weight two part of the Coulomb branch solution is ignored, a lot of information is lost, for example, the full low energy theory can not be decided by just looking at the weight one part. However, we will try to recover the weight two part by looking at extra structures from the weight one part of mixed Hodge module.

Let’s now study the local monodromy group, which is actually a conjugacy class of $SL(2, \mathbb{Z})$ group. Because of the monodromy theorem 2.3, the monodromy group is constrained by the relation $(T^k - 1)^2 = 0$, and such conjugacy classes have been classified, see table. 1.

After determining the local monodromy group, we then further impose following constraints in doing classification:
Table 1: The basic data for local singularity associated with rank one theory.

| Name | Algebra | Monodromy | Eigenvalues | Euler number | Scaling dimension |
|------|---------|------------|-------------|--------------|------------------|
| $I_n$ | $A_{n-1}$ | $\left( \begin{array}{cc} 1 & n \\ 0 & 1 \end{array} \right)$ | $(\exp(2\pi i n), \exp(2\pi i n))$ | $n$ | 1 |
| $I_0^\ast$ | $D_{n+4}$ | $\left( \begin{array}{cc} -1 & -n \\ 0 & -1 \end{array} \right)$ | $(\exp(2\pi i \frac{1}{4}), \exp(2\pi i \frac{1}{4}))$ | $n + 6$ | 2 |
| $II$ | $\emptyset$ | $\left( \begin{array}{cc} 1 & 0 \\ 1 & -1 \end{array} \right)$ | $(\exp(2\pi i \frac{1}{2}), \exp(2\pi i \frac{1}{2}))$ | 2 | $\frac{5}{2}$ |
| $III$ | $A_1$ | $\left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right)$ | $(\exp(2\pi i \frac{1}{2}), \exp(2\pi i \frac{1}{2}))$ | 3 | $\frac{7}{4}$ |
| $IV$ | $A_2$ | $\left( \begin{array}{cc} 0 & 1 \\ -1 & -1 \end{array} \right)$ | $(\exp(2\pi i \frac{1}{4}), \exp(2\pi i \frac{1}{4}))$ | 4 | $\frac{9}{4}$ |
| $II^\ast$ | $E_6$ | $\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$ | $(\exp(2\pi i \frac{1}{4}), \exp(2\pi i \frac{1}{4}))$ | 10 | 6 |
| $III^\ast$ | $E_7$ | $\left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right)$ | $(\exp(2\pi i \frac{1}{2}), \exp(2\pi i \frac{1}{2}))$ | 9 | 4 |
| $IV^\ast$ | $E_8$ | $\left( \begin{array}{cc} -1 & -1 \\ 1 & 0 \end{array} \right)$ | $(\exp(2\pi i \frac{1}{4}), \exp(2\pi i \frac{1}{4}))$ | 8 | 3 |

1. **Global constraints**: Since the moduli space is now a compact space, the total monodromy around the singularities should be trivial, i.e.

$$M_1 M_2 \ldots M_\infty = 1.$$ 

There are other global constraints due to the consistency of the period mapping. The constraints are best understood by using the correspondence between the Coulomb branch data shown in figure 2 and the rational elliptic surface. Coulomb branch solution provides the monodromy representation and the period function $\tau(u)$ (which is in turn gives a j invariant $j(u)$). The j invariant together with the monodromy data determines a rational elliptic surface [35] with a section. Fortunately, the singular fiber types of all the rational elliptic surface have been classified, see [20, 21]. So there is a complete data base from which the rank one $\mathcal{N} = 2$ theory can be studied.

2. **UV completeness**: Let’s now constrain the possible singular fiber at $\infty$. The main constraint is the UV completeness, which means that the scaling dimension associated for $\infty$ is bigger than one. Using the relation between the scaling dimension and the spectrum of limit mixed Hodge structure (see formula. 2.5), the dimension is given in table. 1. The conclusion is that $I_n$ singularity can not appear at $\infty$. Notice that here the loop around the singularity at infinity is also chosen to be counterclockwise. The actual local monodromy group at $\infty$ should be computed using loop in clockwise direction.

3. **Generic deformations**: We look at the generic mass deformations, namely, the local singularity at the bulk (the space $\mathbb{P}^1/\infty$) can not be deformed. This condition means the deformation space of the physical theory at the singularity is only one dimensional. This condition immediately excludes type $II, III, IV$ singularities, since the Coulomb branch spectrum associated with them is less than two, so there is at least one relevant deformation and one Coulomb branch deformation, which means that the physical theory has at least two dimensional physical deformations.
4. **Dirac quantization**: For the $I_n$ singularity with just one dimensional deformation, the basic assumption is that there is a massless hypermultiplet with electric-magnetic charge $\sqrt{n}(1,0)$ (in appropriate electric-duality frame). If there are only $I_n$ type singularities for the generic deformation, the Dirac pairing between BPS particles which become massless at the singularities are $z_1 \cdot z_2 = p_1 q_1 - p_2 q_2$ (here the charges are $z_1 = (p_1, q_1), z_2 = (p_2, q_2)$). The Dirac quantization condition implies that the Dirac pairing should be integral. This condition then means that for the bulk configuration $\{I_{n_1}, I_{n_2}, \ldots\}$, the ratio $\frac{n_i}{n_j}$ should be square for any pair of $i, j$.

**Remark 1**: Let’s now explain how we can find the scaling dimension from the eigenvalue of the monodromy group. To really determine the scaling dimension by using formula 2.5, we need to know the structure of the limit mixed structure. However, the spectrum is constrained in finite set. Let’s take $II, II^*$ singularity as an example. The set of eigenvalues are $(\lambda_1, \lambda_2) = (\exp(2\pi i \frac{1}{6}), \exp(2\pi i \frac{5}{6}))$, and the singularity spectrum $(\alpha_1, \alpha_2)$ (here we shift the singularity spectrum by one so the smallest spectrum number is bigger than one) are constrained by two conditions: a): $\exp(2\pi i \alpha_i) = \lambda_i$; b): $\alpha_1 + \alpha_2 = 2$, then the two choices are

$$ (\alpha_1, \alpha_2) = \left( \frac{1}{6}, \frac{11}{6} \right), \left( \frac{5}{6}, \frac{7}{6} \right). $$

Now use the formula 2.5, we get the possible scaling dimensions $\Delta = 6, 6/5$. To determine the scaling dimension for particular monodromy group, one need other inputs from the holomorphic data. However, for $I_n$ and $I_n^*$ singularity, the scaling dimension must be 1 and 2 respectively, this then means that $I_n$ can not be the singular fiber at $\infty$ for the Coulomb branch of pure 4d theory.

Another question is whether the UV theory is asymptotical free or conformal from the monodromy data. Here we conjecture that if the monodromy group is semi-simple, then the theory is superconformal, otherwise it is asymptotic free. This implies that if $I_n^*$ is put at $\infty$, the theory would be asymptotical free, otherwise it would be superconformal.

**Remark 2**: The undeformable singularities of type $I_0^*, II^*, III^*, IV^*$ are studied in the context of F theory, and are called frozen singularity in [36]. In fact, [36] also considers the partial frozen singularity from which one can actually get new $\mathcal{N} = 2$ SCFTs found in [27–30] by using D3 brane probing those singularities [37]. In fact, all of those theories which has a pure Coulomb branch (i.e. there is no free hypermultiplets at the generic point of Coulomb branch) can be found in this way, please compare table 3 of [36] and table. 4.

**Remark 3**: Here we gave a comment on Dirac quantization which might be related to the Dirac quantization condition for the branes in M theory. It was noticed in [11] that the SW curve for non-simply laced gauge group $g$ is not identified with the spectral curve of Toda chain of type $g$, and they found the correct one should be the spectral curve of the twisted type. One might wonder what is the problem with the spectral curve for the Toda chain of the non-simply laced group. We’d like to point out that the problem with the spectral curve is that it does not obey the Dirac quantization condition. In fact, one can get the SW curve for those gauge theory by using Type IIA brane systems [38], and the special role is played by configuration of orientifold. These orientifolds are strongly constrained
by Dirac quantization condition \([39]\), which actually agrees with the Dirac quantization
c-condition derived from SW curve.

Now using above four rules, we can easily classify all the configuration with just \(I_n\)
type singularities at the bulk by scanning the data set in \([20]\), see table. 2.

One can also have \(I_n^*, II^*, III^*, IV^*\) type singularity appearing in the bulk. The low
energy theory associated with those singularities has just one dimensional Coulomb branch
deformation. Those theories are not familiar, so we’d like to find them by using the
geometric method. The idea is that we start with the configuration with just \(I_n\) type
singularities (with generic deformations) at the bulk, and then using the so-called base
change method. This method works as follows. Let \(g_n\) be the order \(n\) automorphism on
\(P^1\) which is the base of the elliptic surface \(B\):

\[
g_n : P^1 \to P^1, \quad z \to z^n.
\]  

We can get a rational elliptic surface \(B'\) from \(B\) from following diagram:

\[
\begin{array}{ccc}
B & \xrightarrow{\varphi} & B' \\
\downarrow{\beta} & & \downarrow{\beta'} \\
P^1 & \xrightarrow{g_n^*} & P^1
\end{array}
\]  

In the process, the elliptic fibers are changed as follows: a) The \(j\) invariant for the smooth
fiber is not changed; b) The singular fiber at points \(z \neq 0, \infty\) are identified by the \(\mathbb{Z}_n\)
action; c): The fiber at point \(z = 0, \infty\) is changed, and the rule for the change is given inn
\([24]\), and we summarize the relevant facts in figure. 3. For us, the important thing is that
if the singular fiber at \(z = 0\) of \(B\) is undeformable, then the singular fiber at \(z = 0\) for \(B'\)
is also undeformable.

To find new configurations with other type of undeformable singularities, we impose
following two conditions for the pair \(B, B'\): a): the singular fiber at \(z = 0\) for \(B\) is
undeformable; b): the scaling dimensions of fiber at infinity of \(B\) and \(B'\) are related as

\[
\Delta(B'_\infty) = n\Delta(B_\infty).
\]  

A quite useful result for us is that all the pairs \(B\) and \(B'\) have been classified in \([24]\).
By scanning the tables of \([24]\) and imposing above two conditions, all other theories with
undeformable \(I_0^*, II^*, III^*, IV^*\) fibers are listed in table. 2.

Example: Let’s give an example for the base change map of \(B = (IV^*, I_1^1) \to B' = (II, III^*I_1)\). Here \(B = (IV^*, I_1^1)\) and has a \(Z_4\) automorphism, and at \(z = 0\) there is a
trivial \(I_0\) fiber. Under \(Z_4\) action, the \(IV^*\) type fiber at infinity is changed to \(II\), while the
\(I_0^*\) fiber is changed to \(III^*\) fiber; and four \(I_1^n\) fiber of \(B\) is under a single orbit of \(Z_4\) action,
and becomes a single \(I_1\) fiber of \(B'\).

Remark 1: Some of these new theories can be found from class S construction \([40]\).
For example, it was found in \([41]\) that one can find rank one theory with \(G_2\) and \(F_4\) flavor
symmetry (and other theories). In that paper, the scaling dimension is found as the one
before the discrete gauging. However, after further scrutiny, the theories found in \([41]\)
Figure 3: The change of singular fiber at $z = 0$ in using a base change map induced by the $z^n$ group. Upper: The fiber at $z = 0$ before the base change is $I_0$ fiber; Bottom: The fiber at $z = 0$ is the undeformable $I_n$ singularity.

Table 2: Singular fiber configurations for rank one 4d $\mathcal{N} = 2$ SCFTs.

| $I_1$ series. | $(II.I_0^0)$ | $(III.I_1^0)$ | $(IV.I_2^0)$ | $(I_0^*,I_1^0)$ |
|---------------|--------------|--------------|--------------|-----------------|
| $I_2$ series  | $(IV^*.I_1^0)$ | $(III^*.I_1^0)$ | $(II^*.I_1^0)$ | $(I_0^*,I_1^0)$ |
| $I_4$ series  | $(I_1^*,I_4^1)$ | $(III.I_1^1)$ | $(IV.I_2^1)$ | $(I_0^*,I_1^0)$ |
| $Z_4$ covering | $(II.I_2^1)$ | | | |
| $Z_3$ covering | $(IV^*.I_1^0)$ | $(III^*.I_1^0)$ | $(II^*.I_1^0)$ | $(I_0^*,I_1^0)$ |
| $Z_2$ covering | $(IV.I_1^1) \to (II.I_4^1)$ | $(III.I_1^1) \to (II^*.I_1^0)$ | $(I_0^*,I_1^0) \to (III^*.I_1^0)$ | $(I_0^*,I_1^0) \to (III^*.I_1^0)$ |

Table 3: Singular fiber configurations for rank one 4d $\mathcal{N} = 2$ asymptotical free theories.

| $I_1$ series | $(I_1^*,I_2^0)$ | $(I_0^*,I_2^0)$ | $(I_0^*,I_4^0)$ |
|--------------|-----------------|-----------------|-----------------|
| $I_2$ series | $(I_1^*,I_4^0)$ | $(I_2^*,I_3^0)$ | $(I_3^*,I_4^0)$ |
| $I_4$ series | $(I_1^*,I_4^0)$ | $(I_1^*,I_4^0)$ | $(I_1^*,I_4^0)$ |
could be the same as those found in [27–30]. Some of the other theories can also be found by scanning theories in [40].

**Remark 2.** In [27], they suggest that one could also interpret $I_n^s$ type theories as $SU(2)$ gauge theory coupled with hypermultiplets in certain representation so that mass deformation is not possible, and several class of theories are given in [27] (such as $I_1^s$ series). Interestingly, one could also find those singular fiber configurations by using the base change method [42]; however, in this case one need to start with $I_n$ configurations which violate Dirac quantization condition. This might suggest that those theories $I_1^s$ series studied in [27] might also violate Dirac quantization. It would be interesting to further clarify this issue.

### 3.1 Flavor symmetry and Mordell-Weil lattice

We have classified the weight one part of the Coulomb branch solution of rank one theory by studying the generic deformation of a theory. Since the weight two part is discarded, we seem to lose lots of information, i.e. the charge lattice, flavor symmetry, etc. Fortunately, one can find a lattice from the rational elliptic surface: Mordell-Weil lattice. It was shown in [22] one can find the flavor symmetry by finding out a root lattice of the Mordell-Weil lattice (see also [43] the discussion of finding flavor symmetry by using the SW geometry.).

The physical reason for finding the charge lattice from purely the weight one data might be the following: BPS particle would become massless at the singularity, and those BPS particles also carry flavor charge. If there is a way of finding the charge lattice $\Gamma$ formed by these BPS particles, we can find out the flavor symmetry by looking at the sub lattice $\Gamma_0$ (namely the part of the lattice with trivial electric-magnetic charge).

Let’s first look at the theories with only $I_n$ type singularities at the bulk. There is one hypermultiplet associated with each singularity, and the charges of them **span** the full charge lattice, which has the rank

$$\text{rank}(\Gamma) = \#\text{sing} = 2 + f \rightarrow f = \sum \text{sing} - 2. \quad (3.4)$$

Here the summation is over the singular fiber at the bulk. Let’s now relate this number to the topological data associated with local singularities (see data listed in table. 1).

The Euler numbers of the singular fibers of a rational elliptic surface satisfy the condition

$$e(f_\infty) + \sum e_i = 12. \quad (3.5)$$

The Euler number of the singular fiber at $\infty$ satisfies the condition $e(f_\infty) = 2 + \text{rank}(g_\infty)$, where $g_\infty$ is the algebra associated with the fiber at $\infty$. For a $I_n$ type singularity at the bulk, we have $e_i = \text{rank}(g_i) + 1$, so the above equation becomes

$$2 + \text{rank}(g_\infty) + \sum_{\text{bulk}} \text{rank}(g_i) + \#\text{sing} = 12 \rightarrow \#\text{sing} = 10 - \sum_v \text{rank}(g_v).$$

Here the summation is over all the singular fibers; so the rank of the flavor symmetry (see 3.4) is

$$f = \#\text{sing} - 2 = 8 - \sum_v \text{rank}(g_v). \quad (3.6)$$
For the general case, if there are two charge vectors associated with type $I_1^*, II^*, III^*, IV^*$ singularities at bulk, the formula 3.6 for the rank of flavor symmetry is still true.

Interestingly, the rank of flavor symmetry group (see formula 3.6) is the same as that of the so-called Mordell-Weil lattice associated with a rational elliptic surface. In fact, the flavor symmetry can be extracted from this lattice as shown in [22]. For the rational elliptic surface $X$ with a section (there is a fiberation $\pi : X \to \mathbb{P}^1$, where the generic fiber is an elliptic surface), the rational sections form a group called Mordell-Weil group $MW(X)$, with the zero section the identity of the group. Furthermore one can define a pairing on $MW(X)$, which makes it into a lattice called Mordell-Weil lattice [19]. For section $P, Q$, one can define a symmetric pairing (here we use the fact that the surface is a rational surface)

$$(P, Q) = 1 + (P \cdot O) + (Q \cdot O) - (P \cdot Q) - \sum_v contr_v(P, Q).$$

Here $O$ is the zero section and $contr_v(P, Q)$ is defined as

$$contr_v(P, Q) = \begin{cases} 
-(A_v^{-1})_{ij}, & i \geq 1, \ j \geq 1 \\
0 & 
\end{cases}$$

Here $i$ means that $P$ intersects $i$th component of the singular fiber $v$, and $Q$ intersects the $j$th component; $A_v$ is the Cartan matrix of Lie algebra attached to the singular fiber. In particular, the self-intersection form is given as

$$(P, P) = 2 + 2P \cdot O - \sum local.$$ which is also called the height of $P$.

Let’s now summarize basic properties of Mordell-Weil lattice. Firstly, the rank of $MW(X)$ is given by the data of singular fibers as follows

$$\text{rank}(MW(X)) = 8 - \sum_{v} \text{rank}(g_v).$$

and $g_v$ is the algebra associated with the singular fiber; secondly there is a subalgebra $T = \oplus g_v$ for $X$, and the root lattice of $T$ has an embedding into root lattice of $\mathfrak{e}_8$ algebra. One can define a lattice $L$ which is the orthogonal part of $T$ inside $\mathfrak{e}_8$ lattice. The torsion free part of $MW(X)$ is then given by the dual $L^\vee$ of $L$.

The flavor symmetry can be found by doing following computations on $MW(X)$:

1. The rank of the flavor symmetry is given by the rank of $MW(X)$.

2. The semi-simple part of the flavor symmetry is identified by finding a restricted Mordell-Weil root system. First of all, the short roots are easily found, i.e they are given by the height two elements of the lattice $L$, which can be easily read from the intersection form of $L$, see the table in [19]. The long roots are more complicated: one need to find element $S$ of $L^\vee$ satisfying following two conditions:

(a) $S$ does not intersect with the zero section, i.e. $S \cdot O = 0$ (such section is also called integral section).
(b) $S$ intersects the zero component of the fiber $F_\infty$.
(c) $h(S)k(S) = 2$, here $h(S)$ is the height of the section $S$, and $k(S)$ is the smallest integer such that $k(S)S$ is an element of $L$. The length of the long root is given as $2k(S)$.

The semi-simple part of the flavor symmetry can be found from the above root system. The abelian part of flavor symmetry can be found using the rank of Mordell-Weil group. The results are shown in table 4.

| Theory | $G_F$ | $h$ | Torsion | Theory | $G_F$ | $h$ | Torsion |
|--------|-------|-----|---------|--------|-------|-----|---------|
| $I_1$ series | | | | $I_2$ series | | | |
| $(II^*,I_1^2)$ | $\emptyset$ | 0 | 0 | $(III^*,I_1^2)$ | $\text{su}(2)$ | 0 | 0 |
| $(IV^*,I_1^2)$ | $\text{su}(3)$ | 0 | 0 | $(I_0^*,I_1^2)$ | $\text{so}(8)$ | 0 | 0 |
| $(IV,I_1^2)$ | $\mathfrak{e}(6)$ | 0 | 0 | $(III,I_1^3)$ | $\mathfrak{e}(7)$ | 0 | 0 |
| $(II,I_1^{10})$ | $\mathfrak{e}(8)$ | 0 | 0 | $| | | |
| $I_2$ series | | | | | | | |
| $(I_0^*,I_2^3)$ | $\text{su}(2)$ | 1 | $Z_2 \times Z_2$ | | | | |
| $I_4$ series | | | | | | | |
| $(II,I_4I_2^2)$ | $\text{sp}(10)$ | 5 | 0 | $(III,I_4I_2^2)$ | $\text{sp}(6) \times \text{sp}(2)$ | 3 | 0 |
| $(IV,I_4I_2^2)$ | $\text{sp}(4) \times \mathfrak{u}(1)$ | 2 | 0 | $(I_0^*,I_4I_2^2)$ | $\text{su}(2)$ | 1 | $Z_2$ |
| $(II,I_2^4I_2^2)$ | $\text{sp}(4)$ | 2 | 0 | | | | |
| $Z_3$ covering | | | | | | | |
| $(II,III^*I_1)$ | $\text{su}(2)$ | 0 | 0 | | | | |
| $Z_3$ covering | | | | | | | |
| $(II,IV^*I_1^2)$ | $\emptyset_2$ | 0 | 0 | $(III,IV^*I_1)$ | $\text{su}(2)$ | 0 | 0 |
| $(II,IV^*I_2)$ | $\text{su}(2)$ | 1 | 0 | | | | |
| $Z_2$ covering | | | | | | | |
| $(II,I_2^2I_2^2)$ | $\text{sp}(4)$ | 2 | 0 | $(II,I_0^*I_2^2)$ | $\mathfrak{f}(4)$ | 0 | 0 |
| $(III,I_2I_1)$ | $\text{su}(2)$ | 1 | $Z_2$ | $(III,I_1^2)_{Q=\sqrt{2}I_2}$ | $\text{su}(2)$ | 1 | $Z_2$ |
| $(III,I_0^*I_1^2)$ | $\text{spin}(7)$ | 0 | 0 | $(IV,I_0^*I_2^2)$ | $\text{su}(3)$ | 0 | 0 |

Table 4: Flavor symmetry, free hypermultiplets at generic point of Coulomb branch, and one form symmetry (which is identified with the torsion subgroup of the Mordell-Weil lattice) for 4d rank one $\mathcal{N} = 2$ SCFTs.

Let’s now give some details for the computations of flavor symmetry.

**Example 1:** Let’s look at the example $(IV,I_0^*I_2^2)$ (here $T = A_2 \times D_4$, and so $L^\vee$ has rank two), the intersection form on $L^\vee$ is given as $\Lambda_{(23)} = \frac{1}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. There is no short root as there are no height two elements. An element in the lattice $L^\vee$ has the pairing (here $P = n_1e_1 + n_2e_2$, with $e_i$ the basis of the $L^\vee$)

$$ (P,P) = n^T \Lambda_{23} n = \frac{1}{3} n_1^2 + \frac{n_1 n_2}{3} + \frac{1}{3} n_2^2. $$
Now for an integral section which intersects the zeroth component of $IV$ fiber, the height for an integral section has the form

$$(P, P) = 2 - \delta_i(I^{0}_i).$$

Here $\delta_i$ denotes the $i$th diagonal value for the inverse of the Cartan matrix of $D_4$ algebra (namely $P$ intersects the $i$th component of $I^{0}_0$ singularity), and it can take value($\frac{1}{2}, 1, \frac{3}{4}, \frac{3}{4}$) with level $(4, 2, 4, 4)$. So

$$h(P) = (P, P) = \begin{cases} \frac{5}{4}, & k = 4 \\ 1, & k = 2 \\ \frac{5}{4}, & k = 4 \end{cases}$$

To have $h(P)k(P) = 2$, the only solution is $h(P) = 1, k(P) = 2$, and the long root have length $2k(P) = 4$. The number of the long roots are given by the solutions of following equation:

$$\frac{1}{3}n_1^2 + \frac{n_1n_2}{3} + \frac{1}{3}n_2^2 = 1.$$ 

and the solutions are $(n_1, n_2) = \pm(1, 1), \pm(1, -2), \pm(2, -1)$, so there are a total of 6 roots with length 4. Since the flavor symmetry has rank 2, and the root system has the equal length and should be the simply laced Lie algebra $su(3)$.

**Example 2:** Let’s look at example $(II, I^{0}_1I^{0}_2)$ (here $T = A_3 \oplus A_3$, so $L^{\vee}$ has rank two). The intersection form on $L^{\vee}$ is given as $(\frac{1}{4}) \oplus (\frac{1}{4})$, and so the pairing for a section $P = n_1e_1 + n_2e_2$ is given as

$$(P, P) = \frac{1}{4}n_1^2 + \frac{1}{4}n_2^2.$$ 

Now for an integral section which intersects the zeroth component of $II$ fiber, the height takes the form

$$(P, P) = 2 - \delta_i(I^0_i) - \delta_j(I^0_j).$$

Here $\delta_i(I^0_i)$ denotes the $i$th diagonal value for the inverse of the Cartan matrix of $A_3$ algebra, and it can take value($\frac{1}{2}, 1, \frac{3}{4}$), and the level is $(4, 2, 4)$. So for an integral section $P$, it can take value:

$$(P, P) = \begin{cases} \frac{5}{4}, & k = 4 \\ 1, & k = 2 \\ \frac{5}{4}, & k = 4 \end{cases}$$

and so we can get long roots for integral section with height 1 (root length 4) and $\frac{1}{2}$ (root length 8). Therefore, we need to count solutions for the following equations:

$$\begin{cases} \frac{1}{4}n_1^2 + \frac{1}{4}n_2^2 = 1 & \text{short root, length four} \\ \frac{1}{4}n_1^2 + \frac{1}{4}n_2^2 = \frac{1}{2} & \text{long root, length eight} \end{cases}$$

The solutions are $(n_1, n_2) = \pm(2, 0), \pm(0, 2)$ (short root), $(n_1, n_2) = \pm(1, 1), \pm(1, -1)$ (long root). So the root system has four short roots and eight long roots, which gives the root system $sp(4)$. 

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Example 3: Finally, let’s do the computation for the configuration \((III, I_0^* I_1^3)\) (here \(T = A_1 \oplus D_4\), so \(L^V\) has rank three). The intersection form on \(L^V\) is given as \(A_1^* \oplus A_1^* \oplus A_1^*\), here \(A_1^*\) is the dual of the \(A_1\) root lattice. There are 6 short roots from the roots of three \(A_1\). The height for a section \(P = n_1 e_1 + n_2 e_2 + n_3 e_3\) is given as

\[
(P, P) = \frac{1}{2} n_1^2 + \frac{1}{2} n_2^2 + \frac{1}{2} n_3^2.
\]

Now for an integral section which intersects the zeroth component of \(III\) fiber, the height takes the form

\[
(P, P) = 2 - \delta_i(I_0^i).
\]

Using the data summarized in example 1, an integral section \(P\) has height

\[
(P, P) = \begin{cases} 
\frac{4}{7}, & k = 4 \\
1, & k = 2 \\
\frac{5}{4}, & k = 4
\end{cases}
\]

so the long roots are given by \(h(P) = 1, k(P) = 2\), and the number is given by the number of solutions of equation:

\[
\frac{1}{2} n_1^2 + \frac{1}{2} n_2^2 + \frac{1}{2} n_3^2 = 1.
\]

and the solutions are \((n_1, n_2, n_3) = \pm(1, 1, 0), \pm(1, -1, 0), \pm(1, 0, 1), \pm(1, 0, -1), \pm(0, 1, 1), \pm(0, 1, -1)\). So there are a total of 12 long roots. In summary, we find a root system with 6 short roots and 12 long roots, which can be identified as the root system of \(sp(7)\) Lie algebra.

3.2 One form symmetry

The one form symmetry acts on the line operators of a theory [44]. The line operators on the other hand can be studied in the IR limit [45], and in particular its charge can be found from the BPS quiver. Since the charge lattice of the theory is determined by the Mordell-Weil lattice. It might be natural to identify the one form symmetry as the torsion part of the Mordell-Weil group [42]. This seems so by looking at the familiar \(N = 2^*\) theory with \(SU(2)\) gauge group. This theory has two realizations: one is \((I_0^*, I_2^2)\) whose \(MW(X)_{\text{torsion}}\) is \(Z_2\), and the other is \((I_0^*, I_4 I_1^2)\) whose \(MW(X)_{\text{torsion}}\) is \(Z_2 \times Z_2\). Interestingly, for \(N = 2^* SU(2)\) theory, there are two choices of line operators which form separate \(SL(2, \mathbb{Z})\) orbits [46, 47]: a): for the first one, in one duality frame the minimal Wilson line is in fundamental representation (the charge is \((n, 0)\)), and the Dirac quantization condition implies that the minimal ’t hooft line operator is in adjoint representation (the charge is \((0, 2n)\)); The \(Z_2\) one form symmetry could be identified as the center of the gauge group and acts on ’t hooft line operators; b): In the other description, the minimal line operator has electric-magnetic charge \((1, 1)\), and so the Wilson line has charge \((2n, 0)\) and ’t hooft line has charge \((0, 2n)\); the one form symmetry in this case is \(Z_2 \times Z_2\). At least for above example, the one form symmetry can be identified with \(MW(X)_{\text{torsion}}\). It would be interesting to study one form symmetry for other theories.
3.3 SW curve and Weierstrass model

In previous sections, the rational elliptic surface is used to classify 4d $\mathcal{N} = 2$ theory and study the flavor symmetry. Now let’s use the correspondence with the Weierstrass model to write down the SW curve and SW differential.

There are two useful models for the rational elliptic surface $X$. In the first description, the surface is taken to be smooth and minimal. Let’s take the elliptic fibration $\pi : X \to \mathbb{P}^1$ with a section $O$, and the generic fiber is an elliptic surface (a torus with one marked point determined by zero section). The minimal condition means that there is no $(-1)$ curve in the vertical direction. The singular fiber is described by a set of rational curves whose form is determined by the algebra associated with fiber, see [35]. The other model is called Weierstrass model, which is given by the form

$$y^2 = x^3 + fx + g.$$ 

Here $f \in H^0(L^4)$ and $g \in H^0(L^6)$ with $L (O(-1))$ a line bundle on $\mathbb{P}^1$. The total space is in general singular. We consider the so-called minimal Weierstrass model, namely, the singularity of the total space is just $ADE$ singularity. Given a minimal rational elliptic surface (which is in turn determined by the monodromy data and holomorphic data), one can associate a minimal Weierstrass model [35]. It is natural to identify this Weierstrass model as the SW curve of the theory. The SW differential can be found by imposing the condition 2.6. We leave the detailed study to a separate publication.

3.4 Low energy theory at other special vacua

In previous discussion, we consider only the Coulomb branch with generic mass deformations. The low energy theory at undeformable singularity is a $U(1)$ gauge theory coupled with a massless hypermultiplets for $I_n$ singularity, and an interacting theory for type $I^*_n, II^*, III^*, IV^*$ singularities. If we allow non-generic deformation, the singularities of generic deformations would merge and new type of singularities appear. One should be clear that the label for the singularity is still in the same class listed in table. 1. The physical interpretation of other singularities are the following: a): If the singularities are formed by merging type $I_1$ singularities, the low energy theory is simple and well studied, see table. 5. b): If the singularities are formed by merging other type undeformable singularities, the low energy theory can be found from table. 2 by looking at the bulk singularities. However, we do not know whether several singularities can merge or not by simply looking at the singularity type. In the next section, we will associate a brane configuration for each theory and it is then easy to determine whether several singularities can merge or not.
Merging Theory
\[ \begin{array}{|c|c|}
\hline
\text{I}_1^s (I_1^{s*}) & E_8 \text{ SCFT} \\
\text{III}_1^s (I_1^{s*}) & E_7 \text{ SCFT} \\
\text{IV}_1^s (I_1^{s*}) & E_6 \text{ SCFT} \\
\text{II}_1 (I_1) & H_0 \text{ SCFT} \\
\text{III}_1 (I_1) & H_1 \text{ SCFT} \\
\text{IV}_1 (I_1) & H_2 \text{ SCFT} \\
I_{n} (I_n^*) & U(1)-n \\
I_{n}^* (I_n^{*+}) & SU(2)-(n+4) \\
\hline
\end{array} \]

**Table 5**: The theory on generic special vacua formed by merging $I_1$ singularities. The $E_n$ type theories were found in [48], and the $H_i$ type theories were found in [49].

\[ \begin{array}{cccc}
X_1 & X_2 & X_3 \\
\mid & & \\
\mid & & \\
\mid & & \\
\mid & & \\
\mid & & \\
\mid & & \\
\end{array} \]

**Figure 4**: D7 brane configuration. Here there is a branch cut represented by slash lines attached to each D7 brane.

### 4 Brane construction

In the last section, the classification of rational elliptic surface is used to classify 4d $\mathcal{N}=2$ rank one theories. While the geometric tools are quite useful, it is desirable to have a more physical approach to study these theories. Indeed, it was shown in [50] that one can associate a D7 brane configuration for $\mathcal{N}=2$ theories found in [48, 49]. Here we give a general treatment and one of our insight is to associate a collapsed brane system for the singular fiber at $\infty$. These brane configurations are quite useful in studying many aspects of the theory, such as the Mordell-Weil lattice, BPS quiver, BPS spectrum, etc.

#### 4.1 D7 branes

The fundamental building blocks are $(p, q)$ seven branes in Type IIB string theory. There is elementary $(p, q)$ (here $p, q$ is coprime) D7 brane around which the monodromy matrix is given as

\[ K_{[p,q]} = \begin{bmatrix} 1 + pq & -p^2 \\ q^2 & 1 - pq \end{bmatrix}. \]

Here the loop for computing the monodromy is taken in the counter-clockwise direction around the brane. Notice that a $(p, q)$ brane is the same as a $-(p, q)$ brane.

Let’s put a sequence of D7 branes with label $[p_i, q_i]$ on a real line (with ordering from left to right) $X_1 X_2 \ldots X_n$ (see figure. 4), and the total monodromy around the brane configuration is

\[ K_{[p_n, q_n]} K_{[p_{n-1}, q_{n-1}]} \ldots K_{[p_1, q_1]} . \]
There are some equivalence relations for the D7 brane configuration. There is one simple equivalence, namely, one can do a global $SL(2, \mathbb{Z})$ transformation $g$, so that the brane charge vector $z_i = [p_i, q_i]$ and the monodromy matrix are changed as

$$z'_i = gz_i, \quad K'_{z'_i} = gK_{z_i}g',$$

(4.1)

There is another more complicated equivalence relation, i.e. one can move the branes around, and the brane charge would be changed accordingly. The basic move is to exchange the order of two adjacent D7 branes with charges $z_i = [p_i, q_i]$ and $z_{i+1} = [p_{i+1}, q_{i+1}]$:

$$X_{z_i}X_{z_{i+1}} = X'_{z_i}X'_{z_{i+1}}.$$

The brane changes are related as:

a) if we move the $i$th brane: $z'_i = z_{i+1}$, $z'_{i+1} = z_i + z_i \cdot z_{i+1}$

b) if we move the $(i+1)$th brane: $z'_i = z_{i+1} + (z_i \cdot z_{i+1})z_i$, $z'_{i+1} = z_i$.

(4.2)

Here $z_i \cdot z_j = p_i q_j - p_j q_i$ is the symplectic pairing of $z_i, z_j$. The two brane moves are shown in figure. 5. Notice that the brane moves does not depend on using the charge $z_i$ or $-z_i$.

There are several useful invariants for the brane configuration (independent of above two equivalence relations). First, there is the following trace formula for the total monodromy $K$:

$$\text{Tr}(K) = 2 + \sum_{k=2}^{n} \sum_{1 \leq i_1 < i_2 < \cdots < i_k} (z_{i_1} \times z_{i_2})(z_{i_2} \times z_{i_3}) \cdots (z_{i_k} \times z_{i_1}).$$

Another important invariant is defined as

$$l = \text{gcd}(z_i \cdot z_j), \text{ for all } i \text{ and } j$$
Finally one can define a symmetric matrix as follows
\[ A_{ij} = \begin{cases} (a_{ij}), & a_{ij} = \frac{1}{2} (p_i q_j - q_i p_j), \quad i < j \\ 2, & i = j \end{cases} \]
and there is a relation:
\[ \det(A) = \frac{1}{4} Tr K + \frac{1}{2}. \]

There are three basic D7 branes labeled as \( A = [1, 0], B = [1, -1], C = [1, 1]. \) One of the most interesting results of [25] are that the Kodaira singularities can be represented by configurations of D7 branes. The brane configurations are listed in table. 6. Here we want to point out that: in the convention of [25], the monodromy for the brane configuration is the negative of that in the Kodaira’s list! To match the monodromy group, one simply need to define the monodromy of the fundamental \([p, q]\) brane as \( K_{[p,q]}' = -K_{[p,q]} \).

| Name | Brane configuration | \( n \) | \( K \) | \( f_K(p, q) \) |
|------|---------------------|------|------|--------|
| \( I_n \) | \( A^n \) | \( n \geq 0 \) | \( \begin{pmatrix} 1-n & 0 \\ 0 & 1 \end{pmatrix} \) | \( -\frac{1}{2} p^2 \) |
| \( I_n^* \) | \( A^{n+1} X_{[0,-1]} C = A^{n+1} BC \) | \( n \geq 0 \) | \( \begin{pmatrix} -1 & n \\ 0 & -1 \end{pmatrix} \) | \( \mp n^2 \) |
| \( II, III, IV \) | \( A^n X_{[0,-1]} C = A^{n+1} C \) | \( n = 0, 1, 2 \) | \( \begin{pmatrix} 1-n & -1 \\ 1 & -n \end{pmatrix} \) | \( \frac{1}{n+1} (-p^2 + (n+1)pq - (n+1)q^2) \) |
| \( II, III, IV \) | \( A^n X_{[0,-1]} C = A^{n+1} BC^2 \) | \( n = 8, 7, 6 \) | \( \begin{pmatrix} -3n & -11 \\ -1 & n-4 \end{pmatrix} \) | \( \frac{1}{n+1} (p^2 + (1-n)pq + (3n-11)q^2) \) |

Table 6: Brane configurations for the Kodaira singularity. Here the monodromy is computed using the brane configuration \( A^n X_{[a,-1]} C \) representation. \( f_K(p, q) \) is the quadratic form associated with the singularity.

There is a remarkable brane configurations [51]:
\[ A^8 BCBC. \]
and the total monodromy around it is trivial. For later purpose, it can be shown that the above brane system is equivalent to [26]:
\[ A^3 X_{[2,-1]} X_{[1,-2]} C. \]
For every such brane configuration, there is also a cyclic equivalence due to the trivial total monodromy, i.e. \( X_1 \ldots X_{11} X_{12} \sim X_{12} X_1 \ldots X_{11} \).

**Example 1:** In this example, we show that for the brane system \( X_1 X_2 \), if the charge pairing \(|z_1 : z_2| = 1\), they are all equivalent by using global \( SL(2, \mathbb{Z}) \) transformation.

**Proof:** It is always possible to use the \( SL(2, \mathbb{Z}) \) transformation to set the charge of \( X_1 \) to be \([1, 0]\), so let’s consider configuration \( AX_{[a,1]} \) (we also use the fact \( \pm [p, q] \) brane describes the same type of brane, so the \( q \) charge of \( X_2 \) is set to be non-negative). We then use the \( SL(2, \mathbb{Z}) \) transformation
\[ K_A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}. \]
which leaves the charge of A brane invariant. Then let’s use the transformation $(K_A)^b$ to get the equivalence

$$AX_{[a,1]} \sim AX_{[a',1]}.$$  

Here $b = a - a'$.  

**Example 2:** Here we use the brane moves to prove the equivalence of $A^8 BC BC = A^9 X_{[2,-1]}X_{[1,-2]}C$. The moves are

\[
A^8 BC BC = A^7 (AB) C^2 X_{[3,1]} = A^7 B (X_{[0,1]} C^2) X_{[3,1]} \\
= A^7 (BC^2) X_{[0,1]} X_{[3,1]} = A^9 X_{[3,1]} \left(X_{[0,1]} X_{[3,1]}\right) = A^9 X_{[3,-1]}X_{[3,-2]}X_{[0,1]} \\
\rightarrow K_A^{-1} A^9 X_{[2,-1]}X_{[1,-2]}X_{[1,1]} = A^9 X_{[2,-1]}X_{[1,-2]}C.
\]

In every step, we move the brane with the line beneath it, and in the last step, we use a global $SL(2, Z)$ transformation $K_A$.

| Theory | Brane configuration | Theory | Brane configuration |
|--------|---------------------|--------|---------------------|
| $(II, I_3^0)$ | $A^6 X_{[2,-1]} C(X_{[3,-1]} X_{[4,1]})$ | $(II, I_4 I_1)$ | $(A^4) A^6 X_{[2,-1]} C(X_{[3,-1]} X_{[4,1]})$ |
| $(III, I_4^0)$ | $A^6 X_{[2,-1]} C(X_{[2,-1]} X_{[4,1]})$ | $(III, I_4 I_1)$ | $(A^4) A^6 X_{[2,-1]} C(X_{[3,-1]} X_{[4,1]})$ |
| $(IV, I_4^0)$ | $A^6 X_{[2,-1]} C(X_{[3,-1]} X_{[4,1]})$ | $(IV, I_4 I_1)$ | $(A^4) A^6 X_{[2,-1]} C(X_{[3,-1]} X_{[4,1]})$ |
| $(I_5^0, I_7^0)$ | $(A^3 BC) A^3 BC$ | $(I_5^0, I_7 I_1)$ | $(A^3 BC) (A^4) BC$ |
| $(II^0, I_7^0)$ | $(A^4) X_{[2,-1]} C X_{[3,-1]} X_{[4,1]}$ | $(II^0, I_7 I_1)$ | $(A^4) (A^4) X_{[2,-1]} C X_{[3,-1]} X_{[4,1]}$ |
| $(III^0, I_7^0)$ | $(A^4) X_{[2,-1]} C X_{[3,-1]} X_{[4,1]}$ | $(III^0, I_7 I_1)$ | $(A^4) (A^4) X_{[2,-1]} C X_{[3,-1]} X_{[4,1]}$ |
| $(IV^0, I_7^0)$ | $(A^4) X_{[2,-1]} C X_{[3,-1]} X_{[4,1]}$ | $(IV^0, I_7 I_1)$ | $(A^4) (A^4) X_{[2,-1]} C X_{[3,-1]} X_{[4,1]}$ |

**Table 7:** Brane configurations for rank one $\mathcal{N} = 2$ theory. The convention is: the brane within the parentheses can not be separated, i.e. they either represent the fiber at $\infty$ or the undeformable fiber at the bulk.

Using above brane configuration $A^8 BC BC$ and brane moves, we can realize all the possible singular fiber configuration of rational elliptic surface [26]. The brane configurations for four dimensional $\mathcal{N} = 2$ theories are listed in table 7.
4.2 Mordell-Weil lattice and string junctions

For each D7 brane configuration, one can define a charge lattice $\Gamma$ by using string junctions. These string junctions are orthogonal to the collapsed branes. The total dimension of the string junction lattice $\Gamma$ is equal to $2 + f$, where $f$ is the rank of the flavor symmetry. A string junction has the charge $(p, q; w_i)$, where $w_i$ are the flavor charges. There is a sub-lattice $\Gamma_0$ which is defined as the space of string junctions with zero $(p, q)$ charge, and so $\Gamma_0$ has dimension $\text{rank}(f)$. This lattice $\Gamma_0$ is closed related to the Mordell-Weil lattice from which one can find out the flavor symmetry.

![Diagram](image)

**Figure 6**: Up: A string junction around the branes, and this junction carries zero charge of the gauge algebra on the branes. Bottom: A string junction gives rise to an element of Mordell-Weil lattice, and the asymptotical $(p, q)$ charge is zero.

In fact, Mordell-Weil lattice can be computed by looking at the string junction ending on D7 branes [26]. Here a brief review will be given. The elements of Mordell-Weil group are represented by string junction with trivial $(p, q)$ charge, see figure. 6. For the collapsed branes (see the brane configuration listed in table. 6: we use $A^n$ for $I_n$ singularity, $A^{n+4}BC$ for $I_n^*$ singularity, and $A^{n-1}BC^2$ for $E_n$ singularity.), one need to use the string loops around the branes, see upper diagram in figure. 6. The asymptotic charge for the loops around
**Figure 7**: A string junction with zero \((p,q)\) charge for the brane configuration \((A^4BC')(A^4)BC\). If all the charge numbers are integral, it gives rise to the narrow Mordell-Weil lattice. If we allow the fractional \(r,s,Q_1\) (so that the asymptotical charge is still integral), then this gives the torsion free part of the MW lattice. To find out the torsion part of the lattice, we require the self-intersection number of the junction to be zero; and the common divisor of these fractional charge gives the torsion part.

ADE singularity is given as follows:

\[
\begin{align*}
A_n & : \delta_{r,s} = -s(s_1 + \ldots + s_{n+1}), \\
(p,q) & = (-n+1)s,0
\end{align*}
\]
\[
\begin{align*}
D_n & : \delta_{r,s} = -(s_1 + \ldots + s_n) - (r - (n-1)s)s_{n+1} - (r - (n-3)s)s_{n+2} \\
(p,q) & = (-2r,-2s)
\end{align*}
\]
\[
\begin{align*}
E_n & : \delta_{r,s} = -s(s_1 + \ldots + s_{n-1}) - (r - (n-2)s)s_n \\
& \quad - (r - (n-4)s)(s_{n+1} + s_{n+2}), \\
(p,q) & = (-3r + (2n-9)s,-r + (n-6)s).
\end{align*}
\]

Here \(s_i\) represents the basic string going out of the \(i\)th seven brane, and has \((p,q)\) charge of the \(i\)th seven brane. Notice that for \(I_n\) type singularity, there is only one independent asymptotical charge, while there are two for other type of singularities. These junctions are singlet of the gauge algebra on the brane. The \(s_i\) represents the basic string of the \(i\)th basic D7 brane, and there is a symmetric pairing for the basic string:

\[
(s_i, s_j) = -1, \quad (s_i, s_j) = \frac{1}{2}(p_iq_j - p_jq_i) \text{ for } i < j.
\]

With this pairing, the string junctions form a lattice which is identified with the Mordell-Weil lattice.

**Example 1**: Let’s consider the configuration \((A^4BC')(A^4)BC\), which gives the \(SU(2)\) \(\mathcal{N} = 2^4\) gauge theory. A generic junction which is orthogonal to the gauge algebra on the collapsed branes are represented in figure. 7, and \(J\) has the representation

\[
J = -s(s_1 + s_2 + s_3 + s_4) - (r - 3s)s_5 - (r - s)s_6 - Q_1(s_7 + s_8 + s_9 + s_{10}) + Q_3s_{11} + Q_4s_{12}.
\]

Here \((r,s)\) is the charge \(z_0\) for \(D_4\) singularity, and the asymptotical charge is \((-2r,-2s)\), see formula. 4.3; \((r,Q_1)\) is the \(z_0\) charge for the \(I_4\) singularity, so the asymptotical charge is \((-4Q_1,0)\). The total \((p,q)\) charge of \(J\) has to vanish:

\[
(-2r,-2s) + (-4Q_1,0) + Q_3(1,-1) + Q_4(1,1) = 0.
\]
We find the equations

\[-2s - Q_3 + Q_4 = 0,\]
\[-2r - 4Q_1 + Q_3 + Q_4 = 0.\]  \hspace{1cm} (4.4)

There is a total of five variables and two equations, so we find a three dimensional space.

There are also string junctions whose self-intersection number are zero, and these gave the torsion part of the Mordell-Weil lattice. The dimension of such null junction is two. In the current example, (see formula. 4.2 for the definition of symmetric pairing):

\[
J^2 = -4s^2 - (r - 3s)^2 - (r - s)^2 - (r - 3s)4s + (r - s)4s + 2(r - s)(r - 3s)
- 4Q_1^2 - Q_3^2 - Q_4^2 + 4Q_1((r - 3s) - (r - s)) + Q_3(4s + 2(r - s) + 4Q_1) +
Q_4(-4s - 2(r - 3s) - 4Q_1 + 2Q_3).
\]

Substitute the result of \(r, s\) of equations 4.4 into above equation, we get

\[
J^2 = (2Q_1 - Q_3 + Q_4)^2. \hspace{1cm} (4.5)
\]

So null junction span a two dimensional space. Junctions with non-zero self-intersection number form a lattice, which is identified with the lattice \(L\) (which is called narrow Mordell-Weil lattice). The generator for the lattice \(L\) is

\[
Q_1 = -1, \ Q_3 = Q_4 = -2, \rightarrow J = (s_7 + s_8 + s_9 + s_{10}) - 2s_{11} - 2s_{12}, \ J^2 = -4.
\]

To find out the full Mordell-Weil lattice (the torsion free part), we need to relax the condition for the loop junction: the charge \(z_0\) could be fractional as long as the external charge is integral. For example, for a \(I_n\) singularity, the charge \(s\) can be taken as \(\frac{a}{n}\), \(a = 0, \ldots, n\), and the external charge \((p, q) = (-a, 0), \ a = 0, \ldots, n\). These fractional junctions form the full Mordell-Weil lattice. For the current example, the above condition implies that \(Q_1\) could take values in \(\frac{1}{4}\), while \(r, s\) could take value in \(\frac{1}{2}\). In fact, the lattice is generated by the charges

\[
Q_1 = -\frac{1}{4}, \ Q_3 = -1, Q_4 = 0, \ r = 0, s = \frac{1}{2}.
\]

The torsion part is represented by fractional null junctions. The charge for null junction satisfies the condition

\[
2Q_1 - Q_3 + Q_4 = 0.
\]

so a null junction is represented as (here \(Q_3, Q_4\) is used as the coordinate):

\[
J_{null} = -\frac{Q_4 - Q_3}{2}(s_1 + s_2 + s_3 + s_4) - Q_3s_5
- Q_4s_6 - \frac{Q_3 - Q_4}{2}(s_7 + s_8 + s_9 + s_{10}) + Q_3s_{11} + Q_4s_{12}.
\]

The integral condition for the null junction is \(2(Q_3 - Q_4) = 2n\) (to get a junction with integral charge), and so a \(Z_2\) torsion group is derived.
4.3 BPS quiver and BPS states

One can use D3 brane probe to study the properties of $\mathcal{N} = 2$ system represented by D7 brane systems [37]. In particular, the BPS particles are represented by open string ending between D3 brane and D7 brane [52, 53], more generally, one could use the open string junction to get BPS states, see figure. 8.

On the other hand, to find out the spectrum of BPS particles, a very useful tool is the so-called BPS quiver from which one could derive the spectrum by using green mutation [54, 55]. It is possible to find out the BPS quiver from the brane configuration for the corresponding 4d $\mathcal{N} = 2$ theories. Let’s fix a generic point at $\mathbb{P}^1$ where there is no D7 brane, and add a D3 brane probe at this point. The BPS particle at this point can be represented by the open strings ending between D3 brane and D7 branes. If we consider a string ends on $[p,q]$ type D7 brane and D3 brane, then this string has electric-magnetic charge $[p,q]$, which gives a stable BPS particle with charge $[p,q]$.

Figure 8: Open string junction ending between D3 and D7 brane gives the BPS particle. The black dots represent D7 brane.

Let’s first focus on the $I_1$ type theories, namely the generic deformation has only type $I_1$ singular fibers. For each $I_1$ fiber, there is an associated BPS particle whose electric-magnetic charge is given by the type of branes $z_i$: One can find a quiver by using the Dirac paring for the charges associated with singular fibers. The generalization to $I_n$ type theory is straightforward. For the $I_n$ fibers which is represented by the branes $(X_{[\pm z]}^n)$, the corresponding BPS particle has charge $\sqrt{n}z$.

One need to solve following two problems: firstly if the ordering of the branes are changed, the charge vectors would change (see the formula 4.2); but the BPS quiver does depend on the ordering (quivers from different ordering generally are not related by quiver mutation), so to find out the BPS quiver one need to find a special ordering; Secondly, the charge vectors for the brane could be either $z$ or $-z$, and one need a way to fix the choice.

The BPS quiver for those theories are found in [42], and the above two problems can be solved. First, the special ordering for the brane configurations are given as follows

\[
\begin{align*}
II^*, III^*, IV^* : A^n X_{[2,-1]} C, & \quad n = 8, 7, 6 \\
I_n^* : A^{n+4} X_{[1,-1]} C, & \quad n = 0, 1, 2 \\
II, III, IV : A^n X_{[0,-1]} C, & \quad n = 0, 1, 2.
\end{align*}
\]

There is a general pattern for above brane configurations: each sequence has a core of the type $X_{[z]} C$, and then a stack of $A$ branes are added on top of the core. The special ordering for $I_4$ and $I_2$ type theories are found by collapsing adjacent $A$ branes of $I_1$ theories.
Secondly, the rule for fixing the sign ambiguity of the brane charges are following: the charge is \([1,0]\) for the first sequence of \(A^n\) branes; the charge is \(X_{[a,1]}\), \(a = -2,-1,0\) for second one; finally the charge is \([-1,-1]\) for \(C\) brane. The adjacency matrix \(Q_{ij}\) of the quiver is given as follows:

\[
Q_{ij} = z_i \cdot z_j, \quad i < j,
\]

and the full matrix \(Q\) is fixed by imposing anti-symmetric condition. Using the charge vector \(z_1 = [1,0]\), \(z_2 = [a,1]\), \(z_3 = [-1,-1]\), we get the matrix

\[
Q = \begin{pmatrix}
0 & 1 & -1 \\
-1 & 0 & -a + 1 \\
1 & a - 1 & 0
\end{pmatrix}.
\]

The BPS quiver can be read from above matrix and take the form shown in figure 9, which is exactly the same as that found in [42].

For the theories admitting undeformable \(I^*_n, II^*, III^*, IV^*\) singularities, there is no BPS quiver as there is no BPS hypermultiplet associated with the singularity (see next subsection). The BPS information can be found using the folding trick of of the BPS quiver of the parent theory though. The brane configuration and string junctions would be useful to study the BPS states.
4.4 BPS states associated with local singularities

Unlike undeformable $I_n$ singularity, the physics associated with other undeformable singularities are quite unfamiliar. A $(\sqrt{n}, 0)$ (in suitable EM frame) massless hypermultiplet is assumed at $I_n$ singularity, but the particle contents for other singularities are much less clear.

Let's now try to understand the BPS particles associated with un-deformable singularities by using the string junctions. There is no gauge algebra associated with them, and so string loops around the singularity should be used, see figure 6. The asymptotical charges for these junctions are shown in the formula. 4.3, and one can also use fractional strings as long as the asymptotical charge is integral. Now the self-intersection number of such a string loop with charge $(p, q)$ is given as:

$$(J, J) = f_K(p, q).$$

where $f_K(p, q)$ is the quadratic form listed in table. 6. The condition for $J$ to represent a BPS particle [50] is

$$(J, J) \geq -2 + \gcd(p, q).$$

The self-intersection number of $J$ must be $-1$ to get a hypermultiplet, and this is only possible for $I_n$ singularity (the charge is $(\sqrt{n}, 0)$).

| Label | Quadratic form $f_K(p, q)$ | Charge vector $(p, q)$ |
|-------|--------------------------|----------------------|
| $I_4$ | $-\frac{1}{3}p^2$        | $(\sqrt{3}, 0), (J, J) = -1$ |
| $I_2$ | $\frac{1}{3}p^2$         | $q = \sqrt{\frac{5}{3}}, (J, J) = 1$ |
| IV*   | $4p^2 - pq + q^2$        | $(0, 1), (3, 1), (3, 2), (J, J) = 1$ |
| III*  | $\frac{1}{2}p^2 - 2pq + \frac{1}{2}q^2$ | $(1, 1), (3, 1), (\sqrt{2}, 0), (J, J) = 1$ |
| II*   | $p^2 - 5pq + 7q^2$       | $(2, 1), (3, 1), (1, 0), (J, J) = 1$ |

Table 8: The $(p, q)$ string loop around ADE singularity and their intersection numbers.
Example: Let’s consider the the brane configuration for $SU(2) \ N = 2^*$ theory, and so the brane configuration is $(A^4BC)(A^4)BC$. The three basic string junction is shown in figure. 11, which give the stable BPS hypermultiplet for the theory. These BPS particles form the BPS quiver.

Next let’s consider the local singularity which is formed by merging $I_1$ singularities, and so there is ADE gauge algebra on collapsed D7 branes. The BPS particle contents are studied in [50, 56]. For example, one get BPS string junctions transforming in fundamental representation of the $su(n)$ gauge algebra if the asymptotical charge is $(1,0)$ and the singularity is $I_n$. This is consistent with the fact that the low energy theory is given by $U(1)$ gauge theory coupled with $n$ massless hypermultiplets with charge one. One might do similar computations for other singularities.
4.5 Stratification of the Coulomb branch

The brane configuration is quite useful to find the stratification of the Coulomb branch. We first start with brane configuration for the generic configuration, and then merge the branes to get strata with non-generic configuration.

Example: Let’s consider theories labeled as \( (I_0^*, I_1^4) \). The brane configuration for it is \((A^4BC)A^4BC\). The brane system for SCFT point is \((A^4BC)\) (from now on, the brane configuration at \( \infty \) is ignored), and all the branes are on top of each other. The total number of deformations for this singularity is 6 (the flavor symmetry is \( D_4 \), and there is a Coulomb branch operator.). We then look at the deformation of above collapsed brane systems:

1. First, there are configurations with 5 deformations: 1) \( IVII^6 \); 2) \( III^2 \).

2. Secondly, there are configurations with four deformations: 1): \( I_4I_4^5 \); 2): \( IV^2I_4^1 \); 3): \( I_3III \ I_1 \); 4): \( I_2III \ I_1 \); 5): \( III \ II \ I_1 \); 6): \( I_2^2I_2^2 \); 7): \( I_2I_2^2I_1 \); 8): \( II^3 \).

3. Thirdly, there are configurations with three deformations: 1) \( I_3I_4^3 \); 2) \( IIII^3 \); 3) \( I_2^3I_1^2 \); 4) \( I_2I_2^2I_1 \); 5): \( II^2I_2^1 \).

4. Fourthly, there are configurations with two deformations: 1) \( I_2I_1^4 \); 2) \( II \ I_1^4 \).

5. Finally, there is only one configuration with one deformation: \( I_6^6 \).

The basic brane configurations are: 1): \( I_0^* = A^4BC \); 2): \( III^2 : (A^2C)(A^2C) \); 3): \( IVII : (A^3X_{[0,-1]})(AC) \); 4): \( I_2^3 : (A^2)(B^2)(X_{[0,1]}) \); 5): \( I_4I_4^2 : (A^4)BC \). The rest of the brane configurations can be found by further splitting the branes for \( II, III, IV, I_4 \) singularity. The result of the full stratification is shown in figure. 12.

One can use the brane configuration to find out the stratification of other type of theories. In other cases, we have sub-brane system which can not be separated. In the \( I_1 \) series, we start with the most singular configuration and studies its all possible deformation pattern. For other series, we go from the other direction, namely, we start with the singular fibers for generic deformation, and combining the singularities. The rule is that: 1): in the process of merging, we can not split the collapsed branes; 2) The result of merging will just produce the brane systems for the Kodaira’s singularity. Using these rules, it is then possible to find out the stratification of these general theories. The stratification for theory \( (II, I_0^*I_1^4) \) is shown in figure. 13.

---

\(^6\)The way of counting the number of deformations is: there is only one common Coulomb branch deformation, and the mass deformation and relevant deformation for the local singularity would be added.
Figure 12: The stratification for the $(I_0, I_0^8)$ theory. The top one describes the SCFT point, and the bottom one describes the configuration of generic deformations.

Figure 13: The stratification for $(II, I_0^8 I_1^4)$ theory. The SCFT point is at the bottom of the figure.
5 Classification of rank one 5d $\mathcal{N} = 1$ SCFT

Let’s turn our attention to the classification of 5d $\mathcal{N} = 1$ theory. We’d like to first review some basic facts of 5d theories and the main difference from the 4d $\mathcal{N} = 2$ theory will be discussed, see [57] for more details. Firstly, the only relevant SUSY preserving deformations for 5d theory are the mass deformation; Secondly, the 5d theory also has a Coulomb branch, but it is parameterized by the real numbers and is not described by the expectation values of the protected operators; Thirdly, there are massive BPS particles (Instanton particle) at the generic point of Coulomb branch, but massive string-like objects (monopole string) also exist.

Let’s now compactify a 5d theory on a circle with finite size, and the resulting theory is often called 5d KK theory. The Coulomb branch of 5d KK theory is parameterized by the complex numbers (as one of the component of 5d abelian gauge field is combined with the real field of original 5d Coulomb branch into a complex number). The generalized Coulomb branch is parameterized by above complex number together with the mass deformations (There are no relevant and marginal UV deformation parameters). Another difference is that the BPS particles of KK theory now carry not only the electric-magnetic charge, flavor charge, but they also carry the winding number charge (KK charge), so the central charge of these particles take the form

$$Z = na + nD + \sum_i S_i m_i + nS_0.$$ 

Therefore the charge lattice $\Gamma$ has dimension

$$2r + f + 3. \quad (5.1)$$

Here $r$ is the rank of theory, and $f$ the number of mass parameters.

The Coulomb branch solution of 5d KK theory is also described by the mixed Hodge module, and so the classification method of 5d rank one theory is the same as that of 4d theory. The only difference is the type of fibers one can put at the infinity over the compactified Coulomb branch $\mathbb{P}^1$:

- For the Coulomb branch solution of 5d KK theory, the fiber at infinity can only be type $I_n$, $n \geq 1$.

The classification strategy is the same: we first classify theory with only undeformable $I_n$ type fibers at the bulk, and then use the base change methods to find the theory with $I_n^{*}, II^{*}, HI^{*}, JV^{*}$ type fibers at the bulk. The result of the classification is listed in table.

Symmetry: One can compute the flavor symmetry of the theory by finding the root system associated with the singular fibers. The computation is exactly the same as what we did for four dimensional theory, and the results are listed in table. 10. We do need to point out that: in general the rank of the flavor symmetry is reduced under base change. This is different from the 4d theory. The one form symmetry might be also identified with the torsion part of Mordell-Weil lattice, and see also table. 10.

Discrete gauging: In 4d case, the discrete gauging acts on the flavor symmetry by the outer automorphism group of $G_F$ [23]. In the 5d case, the discrete gauging seems to
$$F(T) = \frac{1}{n} F(T). \quad (5.2)$$

Here $Z_n$ is the discrete gauging group. This relation can be regarded as a 5d version of the dimension formula used in the study of 4d theory, see formula. 3.3. This condition simply means that if the singular fiber at $\infty$ of $B$ is $I_{nk}$, the singular fiber at $B'$ should be $I_k$, here we assume a $Z_n$ action is used in doing base change.
Table 10: Physical data for rank one 5d KK theory. Here $G_F$ is the flavor symmetry, $h$ denotes the number of free hypermultiplets at the generic point of Coulomb branch. Torsion means the torsion subgroup of the full Mordell-Weil group, which could be identified with the one-form symmetry of the theory.
Now one can associate an affine Lie algebra for the brane system of the 5d theory. For example, for the \(5.1\) D7 brane configuration and affine Lie algebra

One can also associate D7 brane configurations for the 5d KK theory, and the results are listed in table. 11, see earlier results for the \(I_1\) series [60]. The \(I_1\) series should be identified with the rank one theory studied in [61].

**Affine Lie algebra:** There are affine Lie algebra associated with \(E_n\) type of theories. For the \(I_1\) series, the brane configuration takes the general form \((A^{8-n})A^n BCBC\). The \((A^{8-n})\) branes describe the singular fiber at infinity. The generic deformations at the bulk are described by the brane system

\[
E_n : A^{n-1} BCBC, \quad n = 1, \ldots, 8,
\]

\[
\tilde{E}_n : A^n X_{[2,-1]} X_{[1,-2]} C, \quad n = 0, 1, \ldots 8.
\]

These two sequences are equivalent for \(n \geq 2\). \(E_n\) and \(\tilde{E}_n\) gives different brane configuration. Now one can associate an affine \(E_n\) Lie algebra from these branes [50]. The \(E_n\) algebra is just the flavor symmetry for the corresponding 5d theory.

The main difference from 4d theory is that: one can find a string loop carrying trivial asymptotic \((p, q)\) charge and flavor charge. The string is then represented by a loop around all the fundamental D7 branes. The trivial \((p, q)\) charge condition for the loop implies the relation on the charge vector \(z_0\)

\[
z_0 = K z_0.
\]

Here \(K\) is the monodromy matrix around the branes. One can find one independent solution for the brane system of the 5d theory. For example, for the \(I_1\) series, the monodromy around the \(E_N\) or \(\tilde{E}_N\) brane systems is

\[
K = \begin{bmatrix} 1 & 9 - N \\ 0 & 1 \end{bmatrix}.
\]
and so the string loop has charge $z_0 = [1, 0]$. This null string is denoted as $\delta = [1, 0]$, and play the role of imaginary root of affine Lie algebra. The full string junction lattice then gave a lattice for affine $E_N$ algebra. For all the 5d configurations shown in table. 10, we always get an imaginary root, and so an affine Lie algebra is defined. The detailed computation for the affine Lie algebra will be left for elsewhere.

5.2 BPS quiver

$I_1$ series: One can also associate a BPS quiver for all the 5d theory with only $I_1$ type fibers at the bulk. The idea is the same as what we did for 4d theory: there is one massive BPS particle associated with each $I_1$ singularity represented by a D7 brane $X_{[p_i,q_i]}$, and so the electric-magnetic charge for it is $(p_i, q_i)$. To find the BPS quiver, we need to find a special ordering of the bulk D7 brane configuration, and the sign for each D7 brane charges. Fortunately, such an ordering and the choice of sign have already been found through the map to the 5 brane web [62], see the ordering of branes in table. 12. The BPS quiver is found by computing the Dirac paring between the charges of the D7 brane, and see the quivers in [62, 63].

$I_n$ series: Let’s make a couple of remarks about the BPS quiver. First of all, the special ordering is not unique and so we get different BPS quivers, and those BPS quivers should be related by quiver mutation. Second, if the special $I_1$ configuration can be collapsed to find a $I_n$ series (for instance, the configuration for $I_1$ series looks like $\ldots X^4_1 \ldots$, then one can collapse the $X^4_1$ brane to form a $I_4$ fiber), one can find the BPS quiver for $I_n$ series as follows: the charge vector for the $I_4$ singularity is $2z$, and then we can form a BPS quiver by using Dirac pairing. For example, using the brane configuration for $(I_4, I_8^1)$ shown in table. 12, we can find the BPS quiver for $(I_4, I_8^1)$ theory: the charge vectors are $\sqrt{2}[1, 0], \sqrt{2}[0, 1], \sqrt{2}[-1, 0], \sqrt{2}[0, -1]$. The resulting BPS quiver is the same as the $(I_8, I_4^1)$ theory, so it should give the same theory (the local physics is the same), but it seems that the one form symmetry is different.

Flavor charge and sheafs on Del Pezzo surfaces: In the above representation of BPS quiver, only the electric-magnetic charge for the BPS particle is known. Here we give a method to determine the flavor charge for the basic BPS particles. Let’s use the brane configuration $A^r X_{[2, -1]} CX_{[4, 1]}$ for the bulk singularities of $\tilde{E}_n$ theory. A string junction is represented as

$$J = \sum_{i=1}^r \lambda_i w^i + pw^p + qw^q + n\delta(-1,0),$$

(5.3)

here the basis is the following: first s string basis $\alpha_i$ for the simple roots of $E_n$ Lie algebra is constructed in [56], and $w_i$ is the dual basis. Then $w^p, w^q$ carry no flavor charge and is represented by the string loop: $w^p$ carries charge $(1, 0)$ and $w^q$ carries charge $(0, 1)$. Finally, there is an imaginary root $\delta(-1,0)$ which is represented by the string loop with charge $(-1, 0)$, and this string loop winds around the whole brane configuration, and it is given as

$$\delta = x[2, -1] + 2c - x[4, 1].$$
Here $x_{[2,-1]}$, $c$, $x_{[4,1]}$ are basic string for the corresponding basic $D7$ branes. The self-intersection number of a string junction $J$ is given as

$$(J, J) = -\lambda^2_E + 2nq + f_{E_n}(p, q);$$

Here $\lambda^2$ is the inner product defined using the Cartan matrix of $E_n$ Lie algebra, and $f_{E_n}(p, q)$ is the quadratic form for $E_n$ brane configuration. Using this basis, one can find the flavor charges and KK charge for any BPS particle represented by the string junctions.

There exists a useful map between the string junction of 5d $E_r$ theory and sheaves over del Pezzo surface $X_r$. These maps may be useful for the study of the BPS spectrum, i.e. the special ordering of $D7$ branes might have following interpretation: the fundamental strings in the special ordering give an exceptional collection over Del Pezzo surface. Let’s review the map between sheafs and string junction [64]. The Picard group of $X_r$ is generated by $(l, l_1, \ldots, l_r)$ with the intersection form

$l_i^2 = 1, \quad l_i^2 = -1, \quad l_i \cdot l_j = 0.\

The canonical class is given as

$K_r = -3l + \sum_{i=1}^{r} l_i.$

Notice choose $C_i = l_i - l_{i+1}$, $i = 1, \ldots, r - 1$ and $C_r = l - l_1 - l_2 - l_3$, and the intersection form of them yield the $E_r$ Cartan matrix. We can then easily find the dual basis $w_i$ which is defined by the relation

$w_i \cdot C_j = -\delta_{ij}.$

The topological data for a coherent sheaf $F$ is characterized by the data $(r, ch_1, ch_2)$, with $r$ the rank of the sheaf, $ch_1$ the first Chern class $c_1(F)$, and $ch_2(F) = \frac{1}{2}(c_1^2 - c_2)$. The degree is defined as $d(F) = -K_S \cdot c_1(F)$. The Euler number for two coherent sheafs $E, F$ is defined as

$\chi(E, F) = \sum (-1)^i Ext^i(E, F).$

For two coherent sheafs on del Pezzo surface, we have

$\chi(E, F) = r(E)r(F) + \frac{1}{2}(r(E)d(F) - r(F)d(E)) + r(E)ch_2(F) + r(F)ch_2(E) - c_1(E) \cdot c_2(F).$

It was shown in [64] that one can map a coherent sheaf to a string junction as follows. We start with a coherent sheaf whose Chern class is given as $(r, c_1(F), k(F))$, here $k = \int ch_2(F)$, and $c_1$ is given as (in the basis of $w_i, K_r$):

$c_1(F) = \sum_{i=1}^{r} \lambda_i w_i - \frac{d(F)}{9-r} K_r.$

with the coefficient computed as

$\lambda_i = c_1(F) \cdot C_i, \quad d(F) = -F \cdot K_r.$
and string junction is given as

\[ J_F = \sum_{i=1}^{r} \lambda_i w^i + d(F) w^p + r w^d - (r + k + \frac{1}{2} d(F)) \delta^{(-1,0)}. \]

The above formula gives a map between string junction and coherent sheaf on Del Pezzo surfaces, and then it is possible to link the basis of the BPS quiver to exceptional collection of Del Pezzo surface. We leave the details to elsewhere.

| Name | Brane configuration for BPS quiver |
|------|-----------------------------------|
| \( (I_1 I_1^1) \) | \( X_{-1,0} X_{0,-1} X_{0,0} X_{-1,-1} \) |
| \( (I_1 I_1^1) \) | \( X_{0,-1} X_{-1,-1} X_{0,0} X_{-1,-1} \) |
| \( (I_1 I_1) \) | \( X_{0,0} X_{-1,0} X_{0,0} X_{-1,-1} \) |
| \( (I_1 I_1) \) | \( X_{0,0} X_{-1,0} X_{0,0} X_{-1,-1} \) |
| \( (I_1 I_1) \) | \( X_{0,0} X_{-1,0} X_{0,0} X_{-1,-1} \) |
| \( (I_1 I_1) \) | \( X_{0,0} X_{-1,0} X_{0,0} X_{-1,-1} \) |
| \( (I_1 I_1) \) | \( X_{0,0} X_{-1,0} X_{0,0} X_{-1,-1} \) |
| \( (I_1 I_1) \) | \( X_{0,0} X_{-1,0} X_{0,0} X_{-1,-1} \) |

**Table 12:** Brane configuration for 5d KK theory from which one can find BPS quiver by using Dirac pairing.

### 5.3 Special theory at Coulomb branch

**IR free gauge theory:** It is possible to find out IR theory at arbitrary special vacua formed by merging the singularities of the generic deformation. This can be done by using the brane configurations. For example, the brane configuration takes the form \((A)A^7BCBC\) for \((I_1, I_1^1)\) theory; the maximal singular configuration is \((A^7BC)BC\), and there is a \(I_3\) singularity (represented by \((A'BC)\) configuration) in the bulk, whose low energy theory is \(SU(2)\) theory coupled with \(N_f = 7\) fundamental flavors. This agrees with the result first found in [61]: 5d \(E_8\) \(N = 1\) SCFT could be the UV limit of the \(SU(2)\) gauge theory coupled with seven fundamental hypermultiplets.

**Pure 4d theory:** One can get purely 4d theory on the Coulomb branch of a 5d KK theory. The brane construction is quite useful. The idea is following: simply merge branes for the singularities so that **bulk** brane configurations for a purely 4d theory can be found. This method is useful to find the number of free hypermultiplets at the generic point of the 5d theory. For example, there is a 5d theory \((I_1, I_{III}^* I_1^1)\), and a special configuration \((I_1, (III^* I_1 I_1)\) can be found, namely one of the \(I_1\) fiber is merged with \(III^*\) fiber so that the total monodromy around \((III^* I_1 I_1)\) fiber is that of \(I_1^*\) fiber. The low energy theory on that particular singularity is a pure 4d theory, see table. 4. Using this method, it is then possible to find all purely 4d theory at the Coulomb branch of 5d KK theory.
| $I_1$ series | $I_1^{12}$ |
|-------------|-------------|
| $I_2$ series | $I_2^e$ |
| $I_3$ series | $I_3^4$ |
| $I_4$ series | $I_4^4 I_1^2$ |
| $Z_6$ covering | $I_1^{12} \to I I^* I_1^2$ |
| $Z_4$ covering | $I_1^{12} \to I I I^* I_1^4$ |
| $Z_3$ covering | $I_1^{12} \to I V^* I_1^4$ |
| $Z_2$ covering | $I_4^4 I_2^4 \to I_6^6 I_4^4 I_2^2$ |

Table 13: Singular fiber configuration for 6d KK theory.

6 Classification of rank one 6d $(1, 0)$ SCFT

Finally, let’s consider rank one 6d $(1, 0)$ SCFT. 6d $(1, 0)$ theory has no relevant SUSY preserving deformation. The 6d theory could have a tensor branch (parameterized by real scalar) along which the low energy theory could be described by IR free non-abelian gauge theory. We then compactify the 6d theory on a torus $T^2$ to get an effective 4d $\mathcal{N} = 2$ theory. The tensor branch now becomes a coulomb branch which is now a complex variety. Let’s now assume the theory is rank one $^7$. There are two differences of Coulomb branch of the 6d KK theory from that of 4d theory: a) The 6d theory has no relevant deformation, but we may turn on flux line for the flavor symmetry along the torus so that there are still mass deformations; b) The massive BPS particle could carry two extra winding charges, and so the charge lattice $\Gamma$ for rank one 6d KK theory has dimension

$$\Gamma = f + 4.$$  

Here $f$ is the rank of the flavor symmetry.

We can also classify the 6d KK theory by using the Coulomb branch solution, and the method is the same as we did for 4d and 5d KK theory. The only difference is following:

- The singular fiber at infinity is smooth, namely, there is a $I_0$ fiber at $\infty$.

The result of the classification is shown in table. 13. The physical properties such as the brane configuration, flavor symmetry, free hypermultiplets at generic points, and the one-form symmetry is shown in table. 14.

**Algebra on the branes:** Let’s look at $I_1^{12}$ theory, and the UV theory is the famous 6d $E_8 (1, 0)$ theory. This theory has $E_8$ flavor symmetry, and has the brane configuration $A^8 BCBC$. The rank of the charge lattice is 12, and the intersection form can be found from the string junctions. The charge lattice has two imaginary roots $\delta_1, \delta_2$, which are represented by the $(1, 0)$ and $(0, 1)$ string loop around the brane configurations [51]. The resulting algebra seems quite interesting.

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$^7$Notice that in the literature people usually called a 6d theory rank one if the dimension of the tensor branch is one dimensional; In compactifying down to 4d, the dimension of the Coulomb branch would be increased if there is a gauge theory along the 6d tensor branch.
Table 14: Physical data for 6d KK theory.

**Low energy theory at the singularities:** One can merge various un-deformable singularities to get new singularities on the Coulomb branch, and the low energy theory can be found by looking at the table. Let’s look at the theory whose brane configuration is $A^8BCBC$ (6d $E_8$ (1,0) theory), and configuration with maximal singular point is $(A^8BC)BC$, and so there is a $I_8^*$ singularity, and the low energy theory is $SU(2)$ gauge theory coupled with $N_f = 8$ fundamental flavors.
7 Conclusion

In this paper, we classify rank one 5d $\mathcal{N} = 1$ and 6d $(1, 0)$ SCFT by using the effective 4d Coulomb branch solution. Each such solution is described by a mixed Hodge module (MHM), and one get a rational elliptic surface by looking at weight one part of MHM. Geometric results about rational elliptic surface such as the classification, Mordell-Weil lattice, Weierstrass model are quite useful in studying these theories.

In the process of classification, the following physical constraints are used: a): For 4d theory, the fiber $F_\infty = I_n^*, II^*, III^*, IV^*$; b): For 5d theory, the fiber $F_\infty = I_n$, $n \geq 1$; c) For 6d theory, the fiber $F_\infty = I_0$; The generic deformation condition ensures that $II, III, IV$ singularity can not appear at the bulk; Finally, the Dirac quantization condition put constraints on the combination of $I_n$ type fibers. The data base of rational elliptic surface [20, 21] and the base change maps [24] are crucial for the classification.

We also find the D7 brane configurations and string junctions are very useful in studying these theories, in particular, the low energy theory at every vacua can be determined. In this paper, we also showed how various physical properties of these theories can be understood using D7 brane configurations. It would be interesting to further study those theories.

The approach adopted in this paper can be generalized to the higher rank case theory (4d $\mathcal{N} = 2$, 5d $\mathcal{N} = 1$ KK and 6d $(1, 0)$ KK theories all included). The analysis is much more complicated than rank one theory as there are several technical problems that one need to overcome:

1. **Local singularity**: The first step would be to study the local singularities and the associated physics. Unlike rank one case where there are only eight classes of singularities, the type of singular fibers increased greatly. For example, there are 126 types [65] for genus two singular fiber. One need to analyze the low energy theory associated with those singular fibers.

2. **Global constraints**: The map between rank one Coulomb branch solution and rational elliptic surface is crucial for the classification as the data base for such surfaces is available. Such classification for higher rank case is not available.

3. **Base change**: Another important ingredients for rank one Coulomb branch solution is the classification of base change map of rational elliptic surface. Such classification is also not available for higher rank case.

To complete the classification for higher rank case, one need to solve above three problems. Those problems are largely solved for rank two case by the author, and the result will appear in [66], (see [67–69] for the attempt in classifying 4d rank two superconformal theories).

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A Hodge structure and Mixed Hodge structure

Let’s first review the Hodge structure and variation of Hodge structure. By definition, a Hodge structure of weight \( n \) consists of a lattice \( H_\mathbb{Z} \) (\( \mathbb{Z} \)-module) together with a decomposition:

\[
H_\mathbb{C} := H_\mathbb{Z} \otimes \mathbb{C} = \bigoplus_{p+q=n} H^{p,q},
\]

such that \( H^{p,q} = \overline{H^{q,p}} \). (A.1)

The Hodge structure can be reformulated by a pair \((H_\mathbb{Z}, F^\cdot)\), here \( F^\cdot \) is a decreasing filtration

\[
F_0 \supset F_1 \supset \ldots \supset F_{n-1} \supset F_n
\]

satisfying the condition \( H_\mathbb{C} = F^p \oplus \overline{F^{n-p+1}} \) for every \( p \). The Hodge subspaces are given as \( H^{p,q} = F^p \cap \overline{F^q} \). One also need a polarization \( S \) on \( H_Q \) such that it satisfies following condition

\[
S(F^p, F^{n-p+1}) = 0, \\
S(C\psi, \overline{\psi}) > 0 \text{ for } \psi \neq 0
\]

(A.3)

Here \( C \) is the Weil operator: \( C|_{H^{p,q}} = \bar{\partial}^{-q} \).

**Example 1:** The most basic example is the Hodge structure of weight one (which is relevant for the cohomology of the compact Riemann surface \( \Sigma \)): Here we have \( H_\mathbb{Z} = H^1(\Sigma, \mathbb{Z}) \), and the Hodge decomposition is given by \( H^1(\Sigma, \mathbb{C}) = H^{1,0} \oplus H^{0,1} \), where \( H^{1,0} \) is given by holomorphic differential. The polarization is induced by cup product and the condition for the polarization is just the Riemann bilinear relation.

We could consider a family of Hodge structures, and this will lead to the definition of variation of Hodge structure. A **variation** of Hodge structure on a complex manifold \( B \) then consists of

- a local system \( \mathcal{V} \) of \( \mathbb{Q} \)-vector spaces, whose associated vector bundle with connection is denoted by \( (\mathcal{V}, \nabla) \);
- holomorphic sub-bundles \( F^\bullet \mathcal{V} \subset \mathcal{V} \).

These data satisfy the following requirements:

- the infinitesimal period relation holds: \( \nabla_b F^i \mathcal{V} \subset F^{i-1}\mathcal{V} \);
- for each \( b \in B \), the triple \((\mathcal{V}_b, F_{k,b})\) is a Hodge structure.

A **polarization** of a variation of Hodge structure \((\mathcal{V}, F^\bullet)\) on a complex manifold \( B \) is a horizontal map

\[
S : \mathcal{V} \otimes \mathcal{V} \to \mathbb{C}_B
\]

that induces a polarization on each fiber. In this case, we get a variation of polarizable Hodge structure.
**Example 2**: The basic example for variation of Hodge structure is to consider a family of Riemann surface. Here is an example: \( y^2 = x(x-1)(x-t) \), here \( t \) parameterizes the base manifold \( B \).

In the case of Riemann surface, let’s fix a point of homology cycles \( \gamma_A, A = 1, \ldots, g; \gamma_B, B = 1, \ldots, g \) which is locally constant. The intersection form is chosen as \( \gamma_A \cdot \gamma_B = \delta_{AB} \). We then choose a basis of section of holomorphic one forms \( \omega_1, \ldots, \omega_g \), and form the period integral

\[
\int_{\gamma_A} \omega_i, \quad \int_{\gamma_B} \omega_i,
\]

each integral is called period and we get a \( g \times 2g \) matrix called period matrix.

**Mixed Hodge structure**: The MHS is a far-reaching generalization of HS and was defined by Deligne. MHS consists of a triple \((H_Z, F^\bullet, W^\bullet)\), here \( F^\bullet \) is an increasing filtration called Hodge filtration, and \( W^\bullet \) is a decreasing filtration called the weight filtration. The Hodge filtration is such that the induced filtration on the quotient space \( \text{Gr}_k H = W_k/W_{k-1} \) defines a weight \( k \) Hodge structure. We also need a polarization so that its restriction on each quotient space gives a polarized Hodge structure.

**Example 3**: There are two sets of basic examples for MHS. The first one is the smooth plane algebraic curve \( C \) which is defined as: \( f(x,y) = 0 \). \( C \) could be described by removing several points \( p_i \) on a compact Riemann surface \( \Sigma \). The first cohomology group \( H^1(C, \mathbb{Z}) \) admits a MHS: it has a weight one part and a weight two part. which is essentially given by the first homology group of the compact Riemann surface \( \Sigma \), and weight two part is given by the homology group associated with point \( p_i \). The second one is the singular plane algebraic curve, and in this case the MHS on \( H^1(C, \mathbb{Z}) \) consists of weight zero, weight one and weight two part.

**B Variation of weight one Hodge structure**

In this section, we review the details of variation of weight one Hodge structure. Let \( H_Z \) be a lattice of rank \( 2g \), \( Q \) a non-degenerate, skew symmetric bilinear form on \( H_Z \), and \( H = H_Z \otimes \mathbb{C} \). A polarized Hodge structure of weight one is a decomposition of the complex vector space

\[
H = H^{1,0} \oplus H^{0,1}, \quad H^{0,1} = \overline{H^{1,0}}
\]
satisfying the bilinear relations

1. \( Q(u, u) = 0 \) if \( u \in H^{1,0} \), and
2. \( iQ(u, \overline{u}) > 0 \) if \( 0 \neq u \in H^{1,0} \)

The Hodge structure is equivalent to a Hodge filtration, and in this case, the Hodge filtration takes the form

\[
H = F^0 \supset F^1 \supset \{0\}, \quad F^1 = H^{1,0}
\]

Here \( F^1 \) is subspace of \( H \). The classifying space of all polarized Hodge structures of weight one on \( H \) is then

\[
D = \{ F^1 \in G(g, H) : Q(F^1, F^1) = 0; \ iQ(F^1, \overline{F^1}) > 0 \}
\]
The subvariety of $G(g,H)$ consists of the maximal $Q$-isotropic subspaces of $H$ is the compact dual $\hat{D}$ of $D$. Let’s choose the symplectic basis, i.e. a rational basis $E = \{e_1, \ldots, e_g, f_1, \ldots, f_g\}$ relative to which
\[
Q = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}
\]
Using this basis, we get the usual realization for $D$ as Siegel’s upper-half space.

Let $F^1 \in D$ and $w_1, \ldots, w_g$ a basis of $F^1$. Using the basis $E$, we get a $g \times 2g$ matrix $\Omega = [\Omega_1 \Omega_2]$. Here the row is the coefficients in the expansion of the basis of $E$. The bilinear relations ensure that $F^1$ has a unique basis of the form
\[
\omega_i = \sum_j z_{ji}e_j + f_i
\]
The bilinear relations ensure that the coefficients $Z = (z_{ji})$ satisfy the condition
\[
Z^t = Z, \quad \text{Im}(Z) \text{ is positive definite}
\]
$Z$ is then called normalized period matrix.

**Monodromy weight filtration**: We now consider a variation of polarized Hodge structure of weight one defined on a punctured disk $\Delta^*$. The corresponding period mapping is
\[
\phi : \Delta^* \to \Gamma/D, \quad \Gamma = G_Z = Sp(g,Z)
\]
Let $\tilde{\phi} : U \to D$ be a global lifting of $\phi$ to the upper half plane $u$, and $\gamma \in \Gamma$ is the Picard-Lefschetz transformation. Then
\[
\tilde{\phi}(z + 1) = \gamma \tilde{\phi}(z)
\]
Here $\gamma$ is a quasi unipotent monodromy transformation. We now assume that $\gamma$ is unipotent without losing the generality. Let $N = \log \gamma$, and $N$ is then a rational element in the Lie algebra
\[
g_0 = \{ X \in Hom(H_R,H_R) : Q(X\cdot,y) + Q(x,\cdot) = 0 \}
\]
By the monodromy theorem $N^2 = 0$, and we have
\[
N = \gamma - I
\]
The monodromy weight filtration defined by $N$ is the filtration
\[
\{0\} \subset W_0 \subset W_1 \subset W_2 = H_C
\]
Given by
\[
W_0 = \text{Im}(N), \quad W_1 = \text{Ker}(N)
\]
Since $N \in g_0$ and $N^2 = 0$, so $W_0$ is an isotropic subspace. Here we use the facts $Q(Nx,y) + Q(x,Ny) = 0$, so $Q(Nx,Ny) + Q(x,N^2y) = Q(Nx,Ny) = 0$, and $W_0 = \{Nx\}$; $W_1$ is the $Q$-annihilator of $W_0$:
\[
W_1 = W_0^\perp = \{ u \in H : Q(u,v) = 0 \text{ for all } u \in W_0 \}
\]
here we use the fact $Q(Nx, y) = 0$ if $Ny = 0$, namely $y \in W_1$ if $y \in W_0^\perp$.

Let $\Omega$ be a maximal totally isotropic subspace of $H$, and $E$ is a symplectic basis such that $W_0 = \text{Span}\{e_1, \ldots, e_\nu\}$, and $\Omega = \text{Span}\{e_1, \ldots, e_\nu\}$. In this basis, the nilpotent operator can be written as

$$N = \begin{bmatrix} 0 & \eta \\ 0 & 0 \end{bmatrix}, \quad \eta = \begin{bmatrix} \eta_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

Here $\eta_{11}$ is a $\nu \times \nu$ symmetric matrix.

**Nilpotent orbit theorem and limit Hodge filtration:** We now state the nilpotent orbit theorem. With the period mapping $\phi, \tilde{\phi}, N$ as above. Let

$$\tilde{\psi} : U \to \hat{D}$$

be the map $\tilde{\psi}(z) = e^{-zN}\tilde{\phi}(z)$. Since $\tilde{\psi}(z + 1) = \tilde{\psi}(z)$, we get a map $\psi : \Delta^* \to \hat{D}$ which is given by $\psi(t) = \tilde{\psi}\left(\frac{1}{2\pi i} \log t\right)$. The Nilpotent orbit theorem states that

1. The map $\tilde{\psi} : \Delta^* \to \hat{D}$ has a removable singularity at the origin.
2. Let $F_\alpha = \psi(0) \in \hat{D}$, then for $\text{Im}(z)$ sufficiently large $\exp(zN)F_\alpha \in D$.
3. Relative to a $G_R$ invariant distance in $D$, we have

$$d(\exp(zN)F_\alpha, \tilde{\phi}(z)) = o(e^{-2\pi \text{Im}(z)}), \quad \text{as } \text{Im}(z) \to \infty$$

The filtration $F_\alpha$ is called limit Hodge filtration, and $\exp(zN)F_\alpha$ is called Nilpotent orbit which gives an asymptotical expansion for the period mapping. The nilpotent orbit theorem implies that the normalized periods has the expansion

$$Z(t) = W(t) + \frac{1}{2\pi i}(\log t)\eta$$

and $W(0)$ is a symmetric matrix which takes the following form

$$W(0) = \begin{bmatrix} W_{11} & W_{12} \\ W_{12} & W_{22} \end{bmatrix}$$

Here $W_{22}$ is a $(g - \nu) \times (g - \nu)$ symmetric matrix with positive definite imaginary part.

It is possible to generalize the nilpotent orbit to several varieties. In fact, let

$$\phi : (\Delta^* )^r \to \Gamma / D$$

be the period map, and let $\gamma_1, \ldots, \gamma_r$ denote the corresponding Picard-Lefschetz transformations, and we assume they all to be unipotent. Let $N_i = \log \gamma_i = \gamma_i - I$. These are commuting nilpotent elements of $g_Q$ and form the Nilpotent orbit theorem we have that the symmetric forms $Q(\cdot, N_i \cdot)$ are positive semidefinite.

Let $\sigma \in g_0$ be the monodromy cone

$$\sigma = \left\{ \sum_{i=1}^r \lambda_i N_i : \lambda_i \in R; \lambda_i > 0 \right\}$$
For any element $N \in \sigma$ we have

$$\text{Ker}(N) = \cap_{i=1}^r \text{Ker}(N_i)$$

By duality, we also have

$$\text{Im}(N) = \sum_{i=1}^r \text{Im}(N_i)$$

Therefore, all elements in the monodromy cone $\sigma$ define the same weight filtration.

**Limit mixed Hodge structure:** The limiting Hodge structure $F_a = \psi(0)$ is an element of the dual space and does not, in general, define a polarized Hodge structure. But, together with the weight filtration $W(N)$, the triple $(H, F_a, W(N))$ defines a polarized mixed Hodge structure:

1. The filtration $F_a$ defines a Hodge structure of weight $l$ on the graded quotient $\text{Gr}^W_l(N) = W_l(N)/W_{l-1}(N)$.
2. The Hodge structure induced by $F_a$ on $\text{Gr}^W_{l+1}(N), l \geq 0$ is polarized by the bilinear form

$$Q_l = Q(\cdot, N^l \cdot),$$

One can show above mixed Hodge structure using nilpotent orbit theorem. The important fact is that since $\exp(zN)$ acts trivially on $\text{Gr}^W(N)$, we may assume that $F_a \in D$.

For $l = 0$, the above fact is equivalent to $F_a^1 \cap W_0(N) = 0$, and this can be proven using bilinear relation.

For $l = 1$, the fact that $F_a$ defines a polarized Hodge structure on $\text{Gr}^W_1(N)$ is equivalent to the factorization

$$W_1(N) = W_0(N) \oplus (F_a^1 \cap W_1(N)) \oplus (\overline{F_a^1} \cap W_1(N))$$

if $f_1, f_2 \in F_a^1 \cap W_1(N)$ are such that $f_1 + \overline{f_2} \in W_0(N)$, then

$$iQ(f_1, \overline{f_1}) = iQ(f_1, \overline{f_1} + f_2) = 0$$

We use the fact that if $f_1 \in W_1(N)$, then $f_1 \perp W_0(N)$, and also the fact $Q(f_1, f_2) = 0$ for $f_1, f_2 \in F_a^1$. The above equation implies that $f_1 = 0$ and similarly $f_2 = 0$. We can similarly prove the above direct sum decomposition.

C Admissible variation of mixed Hodge structure

Let’s now review the variation of mixed Hodge structure. Basically, it involves a local system $V$, and a Weight filtration $W$ and Hodge filtration $F$. They satisfy the condition

1. $V$ is a local system over the base manifold $S$.
2. $W = \{W_k\}$ is an increasing weight filtration of $V$ by local subsystems.
3. $F = \{F^p\}$ is a decreasing filtration by holomorphic sub-bundles of $V \otimes \mathbb{C}$. 

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4. $\Delta F^p \subset \Omega^1_S \otimes F^{p-1}$.

5. The data $(Gr^W_k V, F(W_k/W_{k-1}))$ defines a weight $k$ variation of Hodge structure. This is equivalent to the fact that $(V_s, W_s, F_s)$ is a mixed Hodge structure at a point $s \in S$.

If there is a polarization such that there is an induced polarization on graded part, then the structure is called graded polarizable variation of mixed Hodge structure.

We may also consider the limit of the variation of mixed Hodge structure. Similarly, we have the monodromy $\gamma$ of local system, which we also assume to be unipotent. The nilpotent part of the monodromy group is denoted as $N$. There is a definition of relative filtration $M$ of $N$ with respect to $W$. Given the pair $(W, N)$, the relative filtration $M$ satisfies following conditions:

1. $N(M_i) \subset M_{i-2}$

2. $MGr^W_k = N|_{Gr^W_k}$

The second condition means that the induced filtration of $N$ on $Gr^W_k$ is the same as the induced filtration of $M$ on $Gr^W_k$. The variation of MHS is called admissible if there is a relative filtration $M$ with respect the nilpotent monodromy $N$ and $W$.

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