On two aspects of the Painlevé analysis

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Abstract

The Calogero equation is used to illustrate the following two aspects of the Painlevé analysis of PDEs: (i) the singular expansions of solutions around characteristic hypersurfaces are neither single-valued functions of independent variables nor single-valued functionals of data; (ii) the truncated singular expansions not necessarily lead to the simplest, elementary, Bäcklund autotransformation related to the Lax pair.

1 Introduction

The Painlevé analysis is a simple and reliable tool for testing integrability of nonlinear ODEs and PDEs. The Painlevé analysis of PDEs is usually performed along the Weiss-Kruskal algorithm which combines Weiss’ singular expansions and Kruskal’s ansatz and follows step by step the Ablowitz-Ramani-Segur algorithm for ODEs.

The very first step of the Weiss-Kruskal algorithm, however, has no analogue in the Ablowitz-Ramani-Segur algorithm: starting the Painlevé analysis of a PDE, one must determine which of analytic hypersurfaces are characteristic for the tested equation in order to perform the whole subsequent analysis of solutions around non-characteristic hypersurfaces only. Ward first stated and substantiated that the Painlevé property for PDEs must not
fix any structure of solutions at characteristic hypersurfaces. Afterwards, the essence of Ward’s statement was mentioned as “a fact tacitly assumed by all Painlevé practitioners” [1]. Lately, however, Weiss [6] (see also [7]) declared that his result “runs counter to the observation of Ward” and that “expansions about characteristic manifolds are required to be single-valued” as functionals of data.

In this paper, in Section 2, we show that Ward’s definition of the Painlevé property for PDEs still remains well-founded and that Weiss’ objections are caused by some terminological confusion. We do it by the singularity analysis of the Calogero equation [8], [9]:

\[ u_{xxxx} - 2u_y u_{xx} - 4u_x u_{xy} + u_{xt} = 0. \]  

The PDE (1) is useful to illustrate one more aspect of the Painlevé analysis. In Section 3, we find two different Bäcklund transformations of (1) into itself: the former follows from the truncated singular expansions of \( u \), the latter follows from the Lax pair of (1), and the former turns out to be a special case of the square of the latter. Thus, the Painlevé analysis does not lead to the simplest, elementary, Bäcklund autotransformation of the equation (1), a phenomenon similar to what was observed in [10].

Section 4 contains concluding remarks.

2 Breaking solitons and the Painlevé property

Let us take the fourth-order three-dimensional PDE (1) and assume for a minute that we know nothing about its integrability and solutions. Does (1) pass the Painlevé test for integrability? The answer will be “yes”, if we adopt Ward’s definition [5] of the Painlevé property for PDEs. But the answer will be “no”, if we change the definition as proposed by Weiss [6], [7].

It is an easy task to perform the Painlevé analysis of (1) along the Weiss-Kruskal algorithm. A hypersurface \( \phi(x, y, t) = 0 \) is non-characteristic for the PDE (1) if \( \phi_x^2 \phi_y \neq 0 \) (see e.g. [11] for the definition and meaning of non-characteristic hypersurfaces). Kruskal’s ansatz \( \phi = x + \psi(y, t) \) \( (\psi_y \neq 0) \) both simplifies calculations and excludes characteristic hypersurfaces from consideration. The assumption that the dominant behavior of solutions is algebraic around \( \phi = 0, u = u_0(y, t)\phi^p + \ldots \), leads to the only branch to be tested: \( p = -1 \) with \( u_0 = -2 \). (Branches \( p = 0, 1, 2, 3 \), also admitted by (1), need no analysis: they are governed by the Cauchy-Kovalevskaya theorem [11] because the Kovalevskaya form of the PDE (1) is analytic everywhere.)
Then we substitute \( u = -2\phi^{-1} + \ldots + u_r(y, t)\phi^{r-1} + \ldots \) into (1), find that \( u_r \) is not determined if \( r = -1, 1, 4, 6 \) (\( r = -1 \) corresponds to the arbitrariness of \( \psi \)), and conclude that the tested branch is generic. Finally, we substitute \( u = \sum_{i=0}^{\infty} u_i(y, t)\phi^{i-1} \) into (1), find recursion relations for \( u_i \), and check compatibility conditions at resonances, where the arbitrary functions \( u_1, u_4 \) and \( u_6 \) appear. All the compatibility conditions turn out to be identities. Thus, the PDE (1) has passed the Weiss-Kruskal algorithm well. Since this algorithm is sensitive to algebraic and non-dominant logarithmic singularities only, we can only conjecture that the tested equation possesses the Painlevé property in the sense that all solutions of (1) are single-valued around all non-characteristic hypersurfaces. And we should expect (1) to be integrable.

The PDE (1) is integrable indeed [8], [9]. It arises as the compatibility condition for the over-determined (linear in \( \Phi \)) system

\[
\begin{align*}
\Phi_{xx} + (\alpha - u_x)\Phi &= 0, \\
\Phi_t + 4\Phi_{xxy} - 2u_y\Phi_x - 4u_y\Phi_x - 3u_{xy}\Phi &= 0,
\end{align*}
\]

where the spectral parameter \( \alpha \) is any function \( \alpha(y, t) \) satisfying the equation

\[
\alpha_t = 4\alpha\alpha_y. \tag{4}
\]

All solutions of (4), except \( \alpha = \text{constant} \), are multi-valued functions: for any non-constant initial value \( \alpha(y, 0) \), the nonlinear “wave” \( \alpha = \alpha(y, t) \) inevitably “breaks” (“overturns”, “overlaps”) at some finite \( t \) [12]. Therefore solutions of (1), obtainable by the inverse scattering transform with non-constant \( \alpha \), are multi-valued functions as well. For example, the one-soliton solution of (1),

\[
\begin{align*}
u &= -2\lambda \tanh(\lambda x + \mu) + \beta, \tag{5}
\end{align*}
\]

where \( \lambda(y, t), \mu(y, t) \) and \( \beta(y, t) \) are any functions satisfying the equations \( \lambda_t + 4\lambda^2\lambda_y = 0 \) (\( \lambda^2 = -\alpha \)) and \( \mu_t + 4\lambda^2\mu_y = 2\lambda\beta_y \), becomes a multi-valued function when \( \alpha \) “breaks” [13]. The \( N \)-soliton solution of (1), determined by \( N \) solutions \( \alpha_1, \ldots, \alpha_N \) of (4), “breaks” whenever any of \( \alpha_1, \ldots, \alpha_N \) “break” [13].

At first sight, such a complicated branching of solutions of (1) seems to be incompatible with the Painlevé property. Nevertheless, there is no contradiction between the fact that solutions of (1) are multi-valued functions and the fact that solutions of (1) are single-valued around all non-characteristic hypersurfaces: solutions can branch and do branch at characteristic hypersurfaces only. Indeed, it was noticed and stressed in [13] that solutions of (1) “break” (i.e. \( u_y \to \infty \) at finite values of \( u \)) for all values of \( x \) simultaneously, the fact meaning that the corresponding singularity manifolds \( \phi = 0 \)
are characteristic for (1): $\phi_x = 0$. Thus, the breaking solitons do not break the Painlevé property in Ward’s formulation [3] because they never “break” at non-characteristic hypersurfaces.

Let us proceed to Weiss’ objections [6], [7] to Ward’s formulation of the Painlevé property for PDEs. Weiss’ counter-example is “the expansion about the characteristic manifold” $u = u_0(t) + \sum_{i=3}^{\infty} u_i(t)\phi^i$, $\phi = x + \psi(t)$, for the equation $u_{xxx} = \frac{3}{2}u_x^{-1}u_{xx}^2 + ku_x - u_t$, $k = constant$. This expansion, however, does not represent solutions around characteristic hypersurfaces: characteristic hypersurfaces for this equation are determined by the condition $\phi_x = 0$, not by $\phi_x = 1$. Really, this Taylor expansion represents solutions around any non-characteristic hypersurface $\phi = 0$ in the case when the Cauchy-Kovalevskaya theorem [11] does not work: $u_x = 0$ at $\phi = 0$, whereas the Kovalevskaya form of the equation is singular at $u_x = 0$. Though we do agree with Weiss that the consideration of such “bad” Taylor expansions is an essential part of the Painlevé analysis, we think that Weiss’ term “the expansion about the characteristic manifold” is too misleading because no actual expansions around characteristic hypersurfaces can be found throughout [3], [4].

Let us return to the PDE (1) and see what really happens at characteristic hypersurfaces $\phi(x, y, t) = 0$, $\phi_x \phi_y = 0$. When, for example, $\phi_x = 0$ and $\phi_y \neq 0$, we take $\phi = y + \psi(t)$ and $u = u_0(x, t)\phi^p + \ldots$, $p = constant$, and find from (1) that any value of $p$ is admissible. Therefore expansions will not be single-valued functionals of $\phi$ for non-integer $p$. For example, if $p = -\frac{1}{2}$, we get the expansion

$$u = \sum_{i=0}^{\infty} u_i(x, t)\phi^{(i-1)/2}$$

with the coefficients $u_i$ determined by the recursion relations

$$\sum_{i=0}^{n} (i - 1)[u_i(u_{n-i})_{xx} + 2(u_i)_{x}(u_{n-i})_x]$$

$$-\frac{1}{2}(n - 2)[(u_{n-1})_{xxx} + \psi_1(u_{n-1})_x] - (u_{n-3})_{xt} = 0,$$

where $n = 0, 1, 2, \ldots$, and $u_i = 0$ at $i < 0$. The structure of (7) differs from the habitual structure of recursion relations for non-characteristic hypersurfaces very considerably. There are no resonances in (7), but the expansion (6) contains infinitely many arbitrary functions of $t$ in addition to $\psi(t)$: they arise pair by pair as “constants” of integration of (7) because (4) is a second-order ODE in $u_n$ for every $n$. Namely, $u_0 = (\sigma_0 x + \tau_0)^{1/3}$, $u_1 = \frac{2}{36}\sigma_0(\sigma_0 x + \tau_0)^{1/3}$.
\( \tau_0^{-1} + \sigma_1(\sigma_0 x + \tau_0)^{1/3} + \tau_1 + \frac{1}{3} \psi_1 x, \ u_2 = \sigma_2(\sigma_0 x + \tau_0)^{1/3} + \tau_2(\sigma_0 x + \tau_0)^{2/3}, \) etc., where \( \sigma_i(t) \) and \( \tau_i(t) \) are arbitrary functions, \( i = 0, 1, 2, ... \). We see that the expansion (6) is multi-valued both as a function of \( x, y, t \) (through coefficients \( u_i \) and non-integer degrees of \( \phi \)) and as a functional of \( \phi \). Thus, if we accept Weiss’ formulation [6], [7] of the Painlevé property, the integrable PDE (1) will not pass the Painlevé test for integrability. Evidently, Weiss’ formulation asks too much of the tested equation.

3 Two Bäcklund autotransformations

Let us try to find a Bäcklund transformation of the PDE (1) into itself. Two different methods will lead us to two different autotransformations. Then we will find a relation between the two results.

First we employ the method of truncated singular expansions by Weiss [14] and the new expansion function \( \chi = (\phi^{-1}\phi_x - \frac{1}{2}\phi^{-1}\phi_{xx})^{-1} \) by Conte [15] (Kruskal’s ansatz is not used for \( \phi \) hereafter). We substitute \( u = g(x,y,t)\chi^{-1} + f(x,y,t) \) into [11] and find that \( g = -2 \) and that \( \phi \) and \( f \) must satisfy the following system of four equations:

\[
\begin{align*}
d - 2c(s + 2f_x) + 2f_y &= 0, \\
d_x - \frac{1}{2}c(s_x + 2f_{xx}) - 2c_x(s + 2f_x) + 2f_{xy} &= 0, \\
d_{xx} + ds - c_x(s_x + 2f_{xx}) - 2(c_{xx} + cs)(s + 2f_x) - s_{xy} + 2sf_y &= 0, \\
s_{xy} + f_{xxy} + (c_{xx} + cs)(s_x + 2f_{xx}) - 2f_y(s_x + f_{xx}) - 4(s + f_x)(s_y + f_{xy}) + 2ss_y + s_t + f_{xt} &= 0,
\end{align*}
\]

where \( s = \phi_x^{-1}\phi_{xx} - \frac{3}{2}\phi_x^{-2}\phi_{xx}^2, \ c = -\phi_x^{-1}\phi_y \) and \( d = -\phi_x^{-1}\phi_t \). Substituting (8) into (9), we get \( s_x + 2f_{xx} = 0 \) which leads to \( s + 2f_x = 2\alpha \), where the function \( \alpha(y,t) \) appears as a “constant” of integration. Then (8) changes into \( d - 4\alpha c + 2f_y = 0 \), (11) is satisfied identically, and \( \alpha_t = 4\alpha\alpha_y \) follows from (11) (that is why we use the same letter \( \alpha \) as for the spectral parameter). Thus, the system (8)-(11) is equivalent to the system of two equations

\[
\begin{align*}
\phi_{xxx} - \frac{3}{2}\phi_x^{-1}\phi_{xx}^2 + 2\phi_x f_x - 2\alpha\phi_x &= 0, \\
\phi_t - 2\phi_x f_y - 4\alpha\phi_y &= 0,
\end{align*}
\]
where \( \alpha(y,t) \) is any solution of (4). The truncated expansion

\[
u = \phi^{-1}\phi_{xx} - 2\phi^{-1}\phi_x + f \tag{13}\]

is a Miura transformation of the system (12) into the equation (1). One more Miura transformation of (12) into (1), namely

\[
v = \phi^{-1}\phi_{xx} + f, \tag{14}\]

where \( v \) satisfies (1), follows from (13) automatically [14], [15]. The chain of two Miura transformations (13) and (14) generates a Bäcklund autotransformation for (1). Indeed, eliminating \( \phi \) and \( f \) from (13) and (14) by means of (12) and differentiations, we get the following system:

\[
w_{xx} - \frac{1}{2}w^{-1}w_x^2 - wz_x + \frac{1}{8}w^3 + 2\alpha w = 0, \tag{15}\]

\[
z_{xy} - w^{-1}(w_xz_y + 4\alpha w_y - w_t) - \frac{1}{2}ww_y - 4\alpha_y = 0, \tag{16}\]

where \( w = u - v, z = u + v \), and \( \alpha(y,t) \) is any solution of (4). Direct but tedious calculations prove that the system (15)-(16) is compatible in \( v \) if \( u \) satisfies (1), and that the system is compatible in \( u \) if \( v \) satisfies (1) (i.e. one gets (1) for \( u \) when eliminates \( v \) from (13)-(14) by differentiations, and vice versa). Therefore, according to the definition [16], the system (15)-(16) is a Bäcklund transformation of the PDE (1) into itself.

It looks strange, however, that the “x-part” (15) of the obtained Bäcklund transformation is a second-order ODE, whereas the equation (2) of the associated linear problem for the PDE (1) is the same as for the potential KdV equation \( u_t = u_{xxx} - 3u_x^2 \). Let us apply the method by Chen [17] to the linear problem (2)-(3) and find that the PDE (1) does admit one more Bäcklund autotransformation with the same first-order “x-part” as for the potential KdV equation. We rewrite (2) as \( u_x = (\Phi_x/\Phi)_x + (\Phi_x/\Phi)^2 + \alpha \), introduce the new variable \( \omega \) such that \( \omega_x = (\Phi_x/\Phi)^2 + \alpha \), and thus get \( u = \omega + \varepsilon \), where \( \varepsilon = (\omega_x - \alpha)^{1/2} \). Then (3) gives us the following fourth-order PDE for \( \omega \):

\[
\varepsilon_{xy} - 2(\omega_y\varepsilon)_x - 4\alpha \varepsilon_y + \varepsilon_t = 0. \tag{17}\]

It is very essential that (17) is one and the same equation for both choices of the sign in \( u = \omega \pm \varepsilon \): owing to this fact, we have two Miura transformations of (17) into (1), namely,

\[
u = \omega + (\omega_x - \alpha)^{1/2}, \tag{18}\]
\[ v = \omega - (\omega_x - \alpha)^{1/2}, \]  
\[ (19) \]

where \( u \) and \( v \) are solutions of (1) if \( \omega \) satisfies (17). Eliminating \( \omega \) from (18), (19) and (17), we get

\[ z_x - \frac{1}{2}w^2 - 2\alpha = 0, \]
\[ w_{xy} - w_x z_y - (w^2 + 4\alpha)w_y + w_t - 2\alpha y w = 0, \]
\[ (20) \]
\[ (21) \]

where \( w = u - v, z = u + v \), and \( \alpha(y, t) \) is any solution of (4). One can check that the system (20)-(21) is compatible in \( v \) if \( u \) satisfies (1), and vice versa; therefore the equations (20) and (21) constitute a Bäcklund transformation of the PDE (1) into itself.

Thus, we have got two Bäcklund autotransformations for the PDE (1): (15)-(16) and (20)-(21). The two autotransformations are different both in their form, what is evident, and in solutions \( u \) they generate from a given solution \( v \). For example, if \( v \) is any function \( \gamma(y, t) \), we find from (20)-(21) that \( u \) is the one-soliton solution (5) with \( \beta = \gamma \), whereas (15)-(16) gives us (5) with \( \beta = \gamma - 2\lambda \) as well as the solution

\[ u = -8\lambda[\cosh(\lambda x + \mu)]^2\{\sinh[2(\lambda x + \mu)] + 2(\lambda x + \nu)\}^{-1} + \gamma, \]
\[ (22) \]

where the functions \( \lambda(y, t), \mu(y, t) \) and \( \nu(y, t) \) are any solutions of the equations \( \lambda_t + 4\lambda^2\lambda_y = 0 \) \( (\lambda^2 = -\alpha) \), \( \mu_t + 4\lambda^2\mu_y = 2\lambda\gamma_y \) and \( \nu_t + 4\lambda^2\nu_y - 8\lambda\lambda_y \nu = 2\lambda\gamma_y - 8\lambda^2\mu_y \). (For completeness, we should mention solutions \( u \) with \( u_x = 0 \) as well: \( u = \gamma - 2\lambda \) for (20)-(21), and \( u = \gamma \) and \( u = \gamma - 4\lambda \) for (13)-(14), \( \lambda^2 = -\alpha \).) Nevertheless, the two Bäcklund autotransformations are related to each other: the transformation (15)-(16) is nothing but a special case of the square of the transformation (20)-(21). More strictly, if functions \( a, b \) and \( q \) are such that \( u = a \) and \( v = q \) satisfy the system (20)-(21) with some spectral parameter \( \alpha \), and \( u = q \) and \( v = b \) satisfy the system (15)-(16) with the same \( \alpha \), then \( u = a \) and \( v = b \) satisfy the system (15)-(16) with the same spectral parameter \( \alpha \). Indeed, eliminating \( q \) from the relations \( a_x + q_x = \frac{1}{2}(a - q)^2 + 2\alpha_1 \) and \( q_x + b_x = \frac{1}{2}(q - b)^2 + 2\alpha_2 \), we get (15) for \( u = a \) and \( v = b \) if and only if \( \alpha_1 = \alpha_2 = \alpha \); (16) follows from (21) in the same way. Therefore our words “a special case of the square” mean that the transformation (15)-(16) is composed of two transformations (20)-(21) with equal spectral parameters.

As we see, the method of truncated singular expansions does not provide us with the simplest Bäcklund autotransformation for the PDE (1), and one may only guess that (20)-(21) can be derived from (15)-(16).
4 Conclusion

- If the Painlevé property is considered as an analytic property by itself, one may give any definition of it. Fortunately, if the Painlevé property is defined to be an indicator of integrability of nonlinear equations, the adequacy of its definition becomes an experimental result.

- It is only an illusion that we have already got a perfect and ultimate definition of the Painlevé property. For example, see in [1] an interesting consideration of whether the Painlevé test must deal with fixed singularities.

- When Joshi and Petersen [18] proved the convergency of Weiss-Kruskal expansions, they stressed that their method was similar to the method of proving the Cauchy-Kovalevskaya theorem. It looks remarkable that the characteristic hypersurfaces of PDEs, too, appear both in the Cauchy-Kovalevskaya theorem [11] and in the Painlevé test [5].

- The way from the truncated singular expansions to Bäcklund transformations and Lax pairs is not so straightforward as it is sometimes stated in the literature.

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