NOVEL SUPERMULTIPLETS OF $SU(2, 2; 4)$ AND THE $AdS_5/CFT_4$ DUALITY

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Abstract

We continue our study of the unitary supermultiplets of the $\mathcal{N} = 8$ $d = 5$ anti-de Sitter ($AdS_5$) superalgebra $SU(2, 2|4)$, which is the symmetry group of type IIB superstring theory on $AdS_5 \times S^5$. $SU(2, 2|4)$ is also the $\mathcal{N} = 4$ extended conformal superalgebra in $d = 4$. We show explicitly how to go from the compact $SU(2) \times SU(2) \times U(1)$ basis to the non-compact $SL(2, \mathbb{C}) \times D$ basis of the positive (conformal) energy unitary representations of the conformal group $SU(2, 2)$ in $d = 4$. The doubleton representations of the $AdS_5$ group $SU(2, 2)$, which do not have a smooth Poincaré limit in $d = 5$, are shown to represent fields with vanishing masses in four dimensional Minkowski space, i.e. on the boundary of $AdS_5$, where $SU(2, 2)$ acts as conformal group. The unique CPT self-conjugate irreducible doubleton supermultiplet of $SU(2, 2|4)$ is simply the $\mathcal{N} = 4$ Yang-Mills supermultiplet in $d = 4$. We study some novel short non-doubleton supermultiplets of $SU(2, 2|4)$ that have spin range 2 and that do not appear in the Kaluza-Klein spectrum of type IIB supergravity or in tensor products of the $\mathcal{N} = 4$ Yang-Mills supermultiplet with itself. These novel supermultiplets can be obtained from tensoring chiral doubleton supermultiplets, some of which we expect to be related to the massless limits of 1/4 BPS states. Hence, these novel supermultiplets may be relevant to the solitonic sector of IIB superstring and/or $(p, q)$ superstrings over $AdS_5 \times S^5$.

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1 Introduction

This past year a great deal of work has been done on AdS/CFT (anti-de Sitter/conformal field theory) dualities in various dimensions. This recent interest in AdS/CFT dualities was primarily started with the original conjecture of Maldacena [1] relating the large $N$ limits of certain conformal field theories in $d$ dimensions to M-theory/string theory compactified to $d + 1$-dimensional AdS spacetimes. Maldacena’s conjecture was motivated by the work on properties of the physics of $N$ $Dp$-branes in the near horizon limit [2] and the much earlier work on 10-$d$ IIB supergravity compactified on $AdS_5 \times S^5$ and 11-$d$ supergravity compactified on $AdS_7 \times S^4$ and $AdS_4 \times S^7$ [3, 4, 5, 6, 7, 8, 9]. Maldacena’s conjecture was made more precise in [10, 11].

The relation between Maldacena’s conjecture and the dynamics of the singleton and doubleton fields that live on the boundary of AdS spacetimes was reviewed in [12, 13] and its relation to the spectra of maximal supergravities in eleven and ten dimensions in [13, 14].

The prime example of this AdS/CFT duality is the duality between the large $N$ limit of $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory in $d = 4$ and type IIB superstring theory on $AdS_5 \times S^5$. In our earlier work [15] we studied the unitary supermultiplets of the $\mathcal{N} = 8$ $d = 5$ anti-de Sitter superalgebra $SU(2,2|4)$ and gave a complete classification of the doubleton supermultiplets of $SU(2,2|4)$. The doubleton supermultiplets do not have a smooth Poincaré limit in $d = 5$. They correspond to $d = 4$ superconformal field theories living on the boundary of $AdS_5$, where $SU(2,2|4)$ acts as the $\mathcal{N} = 4$ extended superconformal algebra. The unique CPT self-conjugate irreducible doubleton supermultiplet is simply the $\mathcal{N} = 4$ super Yang-Mills multiplet in $d = 4$ [4]. However, there are also chiral (i.e. non-CPT self-conjugate) doubleton supermultiplets with higher spins. The maximum spin range of the general doubleton supermultiplets is 2. We also studied the supermultiplets of $SU(2,2|4)$ that can be obtained by tensoring two doubleton supermultiplets. This class of supermultiplets has a maximal spin range of four and contains the multiplets that are commonly referred to as “massless” in the 5d AdS sense including the “massless” $\mathcal{N} = 8$ graviton supermultiplet in $AdS_5$ with spin range two. Some of these supermultiplets were studied recently [16] using the language of $\mathcal{N} = 4$ conformal superfields developed sometime ago [17].

In this paper we continue our study of the unitary supermultiplets of $SU(2,2|4)$ and their relevance to the AdS/CFT duality. To make con-
tact with the standard language used in conformal field theory, we first show explicitly in section two below how to go from the compact $SU(2) \times SU(2) \times U(1)$ basis to the conventional non-compact $SL(2,\mathbb{C}) \times D$ (Lorentz group times dilatations) basis of the positive (conformal) energy unitary representations of the conformal group $SU(2,2)$ in $d = 4$. The compact $SU(2) \times SU(2) \times U(1)$ labels $(j_L, j_R, E)$ are shown to coincide with the non-compact labels $(j_M, j_N, -l)$ of $SL(2,\mathbb{C}) \times D$. The doubleton representations of the $AdS_5$ group $SU(2,2)$, which do not have a smooth Poincaré limit in five dimensions, are shown to represent fields with vanishing masses in four dimensional Minkowski spacetime, i.e. on the boundary of $AdS_5$. In section 3 we write down the $\mathcal{N} = 4$ extended conformal superalgebra in $d = 4$ in a non-compact as well as in a compact basis. In section 4 we recapitulate the classification of the doubleton supermultiplets. In section 5 we study some novel short supermultiplets of $SU(2,2|4)$ that have spin range 2. We conclude with a discussion of the relevance of our results to the AdS/CFT correspondence. In particular, we point out that even though these novel short supermultiplets do not occur in tensor products of the $\mathcal{N} = 4$ Yang-Mills supermultiplet with itself, they can be obtained by tensoring of higher spin chiral doubleton supermultiplets. We argue that massless limits of 1/4 BPS states in $\mathcal{N} = 4$ super Yang-Mills theory [13] must involve chiral spin $3/2$ doubleton supermultiplets. This implies that the novel short supermultiplets, that do not appear in the Kaluza-Klein spectrum of type IIB supergravity, may be relevant to the solitonic sector of IIB superstring and/or $(p, q)$ superstrings over $AdS_5 \times S^5$.

2 Compact $(SU(2) \times SU(2) \times U(1))$ versus non-compact $(SL(2,\mathbb{C}) \times D)$ bases for the positive energy unitary representations of the group $SU(2,2)$

The conformal group in four dimensions $SU(2,2)$ (the two sheeted covering of $SO(4,2)$) is generated by the Lorentz group generators $M_{\mu\nu}$, the four momentum $P_\mu$, the dilatation generator $D$ and the generators of special conformal transformations $K_\mu$ ($\mu, \nu, \ldots = 0, 1, 2, 3$). The commutation relations are

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\nu\rho} M_{\mu\sigma})$$
$$[P_\mu, M_{\rho\sigma}] = i(\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho)$$
$$[K_\mu, M_{\rho\sigma}] = i(\eta_{\mu\rho} K_\sigma - \eta_{\mu\sigma} K_\rho)$$
\[
\begin{align*}
[D, M_{\mu\nu}] &= [P_\mu, P_\nu] = [K_\mu, K_\nu] = 0 \\
[P_\mu, D] &= iP_\mu; \quad [K_\mu, D] = -iK_\mu \\
[P_\mu, K_\nu] &= 2i(\eta_{\mu\nu}D - M_{\mu\nu})
\end{align*}
\]

(2 - 1)

with \(\eta_{\mu\nu} = \text{diag}(+,-,-,-)\).

Defining

\[
M_{\mu5} = \frac{1}{2}(P_\mu - K_\mu), \quad M_{\mu6} = \frac{1}{2}(P_\mu + K_\mu), \quad M_{56} = -D,
\]

(2 - 2)

the isomorphism to \(SO(4,2)\) becomes manifest \((-\eta_{55} = \eta_{66} = 1; \quad a, b, \ldots = 0,1,2,3,5,6)\)

\[
[M_{ab}, M_{cd}] = i(\eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc}).
\]

(2 - 3)

Considering \(SU(2,2)\) as the isometry group of five dimensional anti-de Sitter space, the above generators have a different physical interpretation. In particular, the rotation group becomes \(Spin(4) = SU(2) \times SU(2)\), generated by \(M_{mn}\) and \(M_{m5}\) \((m,n,\ldots = 1,2,3)\). The generator \(E \equiv M_{06}\) becomes the AdS energy generating translations along the timelike Killing vector field of \(AdS_5\), and the non-compact generators \(M_{0m}, M_{m6}\) correspond to “boosts” and spacelike “translations” in \(AdS_5\).

There are two different subgroups of the conformal group which play an important role in the classification of its physically relevant representations:

i) The maximal compact subgroup \(SU(2)_L \times SU(2)_R \times U(1)_E\) generated by the compact generators \(M_{mn}, M_{m5}, E \equiv M_{06}\), which after being relabelled as

\[
L_m = \frac{1}{2} \left( \frac{1}{2} \varepsilon_{mnl} M_{nl} + M_{m5} \right) \quad \rightarrow \quad SU(2)_L \\
R_m = \frac{1}{2} \left( \frac{1}{2} \varepsilon_{mnl} M_{nl} - M_{m5} \right) \quad \rightarrow \quad SU(2)_R
\]

satisfy

\[
[L_m, L_n] = i\varepsilon_{mnl}L_l \\
[R_m, R_n] = i\varepsilon_{mnl}R_l \\
[L_m, R_n] = [E, L_n] = [E, R_n] = 0.
\]

In the interpretation of \(SU(2,2)\) as conformal group, the \(U(1)_E\) generator \(E = \frac{1}{2}(P_0 + K_0)\) is simply the conformal Hamiltonian. Denoting the Lie
algebra of $SU(2)_L \times SU(2)_R \times U(1)_E$ by $L^0$, the conformal algebra $g$ has a three graded decomposition

$$g = L^+ \oplus L^0 \oplus L^-,$$

where

$$[L^0, L^\pm] = L^\pm, \quad [L^+, L^-] = L^0, \quad [L^+, L^+] = 0 = [L^-, L^-], \quad [E, L^\pm] = \pm L^\pm, \quad [E, L^0] = 0. \quad (2 - 5)$$

ii) The stability group $H$ of $x^\mu = 0$ when $SO(4,2)$ acts in the usual way on the (conformal compactification of) 4d Minkowski spacetime. Its Lie algebra consists of the generators $M_{\mu\nu}$ of the Lorentz group $SL(2,\mathbb{C})$, the dilatation operator $D$ and the generators of the special conformal transformations $K_\mu$. Thus $H$ is the semi-direct product $(SL(2,\mathbb{C}) \times D) \circ \mathcal{K}_4$, where $\mathcal{K}_4$ represents the Abelian subgroup generated by the special conformal generators $K_\mu$. Introducing

$$M_m = \frac{1}{2} \left( \frac{1}{2} \varepsilon_{mnl} M_{nl} + i M_{0m} \right), \quad N_m = \frac{1}{2} \left( \frac{1}{2} \varepsilon_{mnl} M_{nl} - i M_{0m} \right),$$

the $SL(2,\mathbb{C})$ part reads

$$[M_m, M_n] = i \varepsilon_{mnl} M_l, \quad [N_m, N_n] = i \varepsilon_{mnl} N_l, \quad [M_m, N_n] = 0.$$ 

Physically relevant representations of the conformal group are unitary irreducible representations (UIR’s) of the lowest weight type in which the spectrum of the conformal Hamiltonian (resp. the AdS energy) $E$ is bounded from below.

The most natural way to construct them is to work in a $SU(2)_L \times SU(2)_R \times U(1)_E$ covariant basis so that the lowest weight property can reveal itself in a manifest way. The lowest weight UIR’s of $SU(2,2)$ can then be constructed in a simple way by using the oscillator method of [19, 20, 4, 15] and are uniquely determined by the quantum numbers $(j_L, j_R, E)$ of an irreducible $SU(2)_L \times SU(2)_R \times U(1)_E$ representation $|\Omega\rangle$ that is annihilated by the elements of $L^-$. 

4
In conformal field theory, on the other hand, the conformal group is usually represented on fields that live on $4d$ Minkowski spacetime and transform covariantly under the Lorentz group $SL(2, \mathbb{C})$ and the dilatations. The standard way to construct these fields is via the method of induced representations [21], in which a (usually finite dimensional) representation of the stability group $H$ induces a representation of the whole conformal group $G$ on fields that live on the coset space $G/H$, which in our case is just the conformal compactification of $4d$ Minkowski spacetime. Consequently, these representations are labelled by their $SL(2, \mathbb{C})$ quantum numbers $(j_M, j_N)$, their conformal dimension $l$ and certain matrices $\kappa_\mu$ related to their behaviour under special conformal transformations $K_\mu$.

To translate between these two viewpoints, we will now present the oscillator representations of $SU(2,2)$ in a way which makes the transition to the $SL(2, \mathbb{C})$- and dilatation covariant field representations more obvious.

To this end, let $\gamma_\mu$ be the ordinary $4d$ gamma matrices ($\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$) with $\gamma_5 = \gamma_0 \gamma_1 \gamma_2 \gamma_3$. Then the matrices

\[
\Sigma(M_{\mu\nu}) := \frac{i}{4} [\gamma_\mu, \gamma_\nu] \\
\Sigma(M_{\mu5}) := \frac{i}{2} \gamma_\mu \gamma_5 \\
\Sigma(M_{\mu6}) := \frac{1}{2} \gamma_\mu \\
\Sigma(M_{56}) := \frac{1}{2} \gamma_5
\]

generate a four dimensional (non-unitary) irreducible representation of the conformal algebra [2 - 3]. In the following, we will choose the “Dirac representation”

\[
\gamma^0 = \gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
\gamma^m = -\gamma_m = \begin{pmatrix} 0 & \sigma^m \\ -\sigma^m & 0 \end{pmatrix} \\
\Rightarrow \gamma_5 = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

where $\sigma^m$ are the usual Pauli matrices. With this choice the $\Sigma(M_{ab})$ are
given by

\[
\begin{align*}
\Sigma(M_{mn}) &= \frac{1}{2} \varepsilon_{mnl} \begin{pmatrix} \sigma^l & 0 \\ 0 & \sigma^l \end{pmatrix}, \\
\Sigma(M_{0m}) &= -i \frac{1}{2} \begin{pmatrix} 0 & \sigma^m \\ \sigma^m & 0 \end{pmatrix}, \\
\Sigma(M_{05}) &= \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
\Sigma(M_{m5}) &= \frac{1}{2} \begin{pmatrix} \sigma^m & 0 \\ 0 & -\sigma^m \end{pmatrix}, \\
\Sigma(M_{06}) &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\
\Sigma(M_{m6}) &= \frac{1}{2} \begin{pmatrix} 0 & -\sigma^m \\ \sigma^m & 0 \end{pmatrix}, \\
\Sigma(M_{56}) &= i \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\end{align*}
\]

(2 - 6)

Of course, this is nothing but the fundamental representation of $SU(2,2)$, since

\[
\gamma^0 \Sigma(M_{ab}) = \Sigma(M_{ab})^\dagger \gamma^0.
\]

(2 - 7)

Consider now $P$ copies (or “generations”) of oscillator operators $a^i(K) = a_i(K)^\dagger$, $b^r(K) = b_r(K)^\dagger$

\[
[a_i(K), a^j(L)] = \delta_i^j \delta_{KL}, \quad [b_r(K), b^s(L)] = \delta_r^s \delta_{KL}
\]

(2 - 8)

with $i, j = 1, 2$; $r, s = 1, 2$; $K, L = 1, \ldots, P$, which we now regroup into a “spinor” $\Psi$

\[
\Psi(K) := \begin{pmatrix} a_1(K) \\ a_2(K) \\ -b_1(K) \\ -b_2(K) \end{pmatrix}
\]

(2 - 9)

so that

\[
\bar{\Psi}(K) \equiv \Psi^\dagger(K) \gamma^0 = \left( a^1(K), a^2(K), b_1(K), b_2(K) \right).
\]

(2 - 10)

Defining

\[
\bar{\Psi} \Sigma(M_{ab}) \Psi := \sum_{K=1}^{P} \bar{\Psi}(K) \Sigma(M_{ab}) \Psi(K),
\]

(2 - 11)
one finds

\[ [\bar{\Psi} \Sigma(M_{ab}) \Psi, \bar{\Psi} \Sigma(M_{cd}) \Psi] = \bar{\Psi} [\Sigma(M_{ab}), \Sigma(M_{cd})] \Psi, \quad (2 - 12) \]

i.e. the \( \bar{\Psi} \Sigma(M_{ab}) \Psi \) generate an infinite dimensional representation of \( SU(2, 2) \) in the Fock space of the oscillators \( a^T \) and \( b^r \), which, in contrast to the finite dimensional representation \( \Sigma(M_{ab}) \), is now unitary because of \( (2 - 7) \) and the Hermiticity of \( \gamma^0 \). A short look at \( (2 - 6) \) reveals that all non-compact generators are represented by linear combinations of di-creation and di-annihilation operators of the form \( \vec{a}_i \cdot \vec{b}_r \) and \( \vec{a}_i \cdot \vec{b}_r \). (Here and in the following, the dot product denotes summation over the “generation” index \( K \), i.e. \( \vec{a} \cdot \vec{b} \equiv \sum_{K=1}^P a^i(K) b^r(K), \) etc.) As for the compact generators, one finds that the generators \( L_m \) of \( SU(2) \) (to simplify the notation we will from now on just write \( M_{ab}, L^m, \) etc. instead of \( \bar{\Psi} \Sigma(M_{ab}) \Psi, \bar{\Psi} \Sigma(L_m) \Psi \) . . . ) are given by linear combinations of

\[ L^k_i := \vec{a} \cdot \vec{a}_i - \frac{1}{2} \delta^k_i N_a, \quad (2 - 13) \]

whereas the generators \( R^s_r \) of \( SU(2) \) are linear combinations of

\[ R^s_r := \vec{b} \cdot \vec{b}_s - \frac{1}{2} \delta^s_r N_b \quad (2 - 14) \]

and \( E \) is simply given by

\[ E = \frac{1}{2} (N_a + N_b + 2P), \quad (2 - 15) \]

where \( N_a \equiv \vec{a} \cdot \vec{a}_i, N_b \equiv \vec{b} \cdot \vec{b}_r \) are the bosonic number operators, in complete agreement with the construction in \( [13] \).

As mentioned above, the positive energy UIR’s are then obtained by constructing an irreducible representation \( |\Omega\rangle \) of \( SU(2)_L \times SU(2)_R \times U(1)_E \) in the Fock space of the oscillators with quantum numbers \( (j_L, j_R, E) \) that is annihilated by all the generators \( \vec{a}_i \cdot \vec{b}_r \) of \( L^- \):

\[ \vec{a}_i \cdot \vec{b}_r |\Omega\rangle = 0. \quad (2 - 16) \]

Acting repeatedly with the di-creation operators \( \vec{a} \cdot \vec{b} \) of \( L^+ \) on \( |\Omega\rangle \), one generates the basis of a positive energy UIR of the whole group \( SU(2, 2) \).

To see the relation to the \( SL(2, \mathbb{C}) \) and \( D \)-covariant induced representations on fields, consider the operator

\[ U := e^{\vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{b}}. \quad (2 - 17) \]
It has the following important property

\[
M_m U = U(L_m + L^-)
\]

\[
N_m U = U(R_m + L^-)
\]

\[
DU = U(-iE + L^-)
\]

\[
K_\mu U = U(L^-),
\]

(2 - 18)

where \(L^-\) stands for certain linear combinations of di-annihilation operators \(\vec{a}_i \cdot \vec{b}_r\).

Acting with \(U\) on a lowest weight vector \(|\Omega\rangle\) corresponds to a rotation in the corresponding representation space of \(SU(2, 2)\):

\[
U|\Omega\rangle = e^{i\Psi\Sigma(M_0 + iM_5)}\Psi|\Omega\rangle.
\]

(2 - 19)

Since \(L^-|\Omega\rangle = 0\), it then follows from (2 - 18) that \(\Phi(0) := U|\Omega\rangle\) is an irreducible representation of the little group \(H\) with \(SL(2, \mathbb{C})\) quantum numbers \((j_M, j_N) = (j_L, j_R, -l)\), conformal dimension \(l = -E\) and trivially represented special conformal transformations \(K_\mu\) (i.e. \(\kappa_\mu = 0\)). Acting with \(e^{-ix^\mu P_\mu}\) on \(\Phi(0)\) then translates the field in Minkowski space:

\[
\Phi(x^\mu) = e^{-ix^\mu P_\mu} \cdot \Phi(0) = e^{-ix^\mu P_\mu} U|\Omega\rangle
\]

(2 - 20)

and generates a (group theoretically equivalent) induced representation of \(SU(2, 2)\) along the lines of [21] with exact numerical coincidence of the compact and the covariant labels \((j_L, j_R, E)\) and \((j_M, j_N, -l)\). Since the bosonic oscillators in terms of which we realized the generators transform in the spinor representation of \(SU(2, 2)\), the oscillator realization could be reinterpreted in the language of twistors.

To conclude this section, we finally note that the lowest weight UIR’s of \(SU(2, 2)\) with vanishing 4d mass \(m^2 = P_\mu P^\mu\) are exactly the ones obtained by taking only one generation of oscillators (i.e. for \(P = 1\)) [22], since

\[
P_\mu P^\mu = \left[ (\vec{c}^1 \cdot \vec{d}^1)(\vec{c}^2 \cdot \vec{d}^2) - (\vec{d}^1 \cdot \vec{c}^2)(\vec{d}^2 \cdot \vec{c}^1) \right]
\]

(2 - 21)

with the (mutually commuting) operators

\[
c^1(K) := (b_1(K) + a^1(K))
\]

In our conventions, \(l\) is the length (or inverse mass) dimension.

\footnote{Note that we are talking here about the mass in the \textit{four} dimensional sense and not about mass in five dimensional \(AdS\) space, where already an invariant definition of this quantity poses a problem.}
\[ c^2(K) := (b^2(K) + a^2(K)) \]
\[ d^1(K) := (a^1(K) + b^1(K)) \]
\[ d^2(K) := (a^2(K) + b^2(K)) \]

vanishes identically in this case.

If \( SU(2, 2) \) is interpreted as the isometry group of \( AdS_5 \), these representations are just the doubleton representations, which do not have a smooth 5d Poincaré limit and live on the 4d boundary of anti-de Sitter space (where they are thus massless representations of the 4d conformal group solving the free wave equation). In terms of the oscillator method, they correspond to lowest weight vectors of the form \( |\Omega\rangle = a^{i_1} \ldots a^{i_n}|0\rangle \) or \( |\Omega\rangle = b^{r_1} \ldots b^{r_n}|0\rangle \) with \( n \geq 0 \) leading to \( (j_L, j_R, E) = (n/2, 0, n/2 + 1) \) or \( (0, n/2, n/2 + 1) \), respectively.

### 3 The superalgebra \( SU(2, 2|4) \)

The centrally extended symmetry supergroup of type IIB superstring theory on \( AdS_5 \times S^5 \) is the supergroup \( SU(2, 2|4) \) with the even subgroup \( SU(2, 2) \times SU(4) \times U(1)_Z \), where \( SU(4) \) is the double cover of \( SO(6) \), the isometry group of the five sphere \([4]\). The Abelian \( U(1)_Z \) generator, which we will call \( Z \), commutes with all the other generators and acts like a central charge. Therefore, \( SU(2, 2|4) \) is not a simple Lie superalgebra. By factoring out this Abelian ideal one obtains a simple Lie superalgebra, denoted by \( PSU(2, 2|4) \), whose even subalgebra is simply \( SU(2, 2) \times SU(4) \). For its possible applications to the full superstring/M-theory we consider below the centrally extended supergroup \( SU(2, 2|4) \) as we did in our earlier paper \([15]\). The representations of \( PSU(2, 2|4) \) correspond simply to representations of \( SU(2, 2|4) \) with \( Z = 0 \). We should also note that both \( SU(2, 2|4) \) and \( PSU(2, 2|4) \) admit an outer automorphism \( U(1)_Y \) that can be identified with a \( U(1) \) subgroup of the \( SU(1, 1)_{global} \times U(1)_{local} \) symmetry of IIB supergravity in \( d = 10 \) \([4]\). \( SU(2, 2|4) \) can be interpreted as the \( \mathcal{N} = 8 \) extended AdS superalgebra in \( d = 5 \) or as the \( \mathcal{N} = 4 \) extended conformal superalgebra in \( d = 4 \).

The algebra of \( \mathcal{N} \)-extended conformal supersymmetry in \( d = 4 \) can be written in a covariant form as follows \((i, j = 1, \ldots, \mathcal{N}; a, b = 0, 1, 2, 3, 5, 6)\) \([23]\):

\[
[\Xi_i, M_{ab}] = \Sigma(M_{ab})\Xi_i, \quad [\bar{\Xi}^i, M_{ab}] = -\bar{\Xi}^i\Sigma(M_{ab})
\]

\[
\{\Xi_i, \Xi_j\} = \{\bar{\Xi}^i, \bar{\Xi}^j\} = 0, \quad \{\Xi_i, \bar{\Xi}^j\} = 2\delta^i_j\Sigma(M_{ab})M_{ab} - 4B_{i}^j
\]

\[
[B_i^j, M_{ab}] = 0, \quad [B_i^j, B_k^l] = \delta^i_k B_k^l - \delta^l_k B_i^k
\]
\[ [\Xi_i, B^k_j] = \delta^k_i \Xi_j - \frac{1}{4} \delta^k_j \Xi_i, \quad [\Xi^i, B^k_j] = -\delta^i_j \Xi^k + \frac{1}{4} \delta^j_k \Xi^i, \quad (3 - 1) \]

where the (four component) conformal spinor \( \Xi \) is defined in terms of the
the chiral components of the Lorentz spinors \( Q \) and \( S \) (the generators of
Poincaré and \( S \) type supersymmetry) as

\[ \Xi \equiv \left( \begin{array}{c} Q_{\alpha} \\ S^{\alpha} \end{array} \right). \quad (3 - 2) \]

The \( B^i_j \) are the generators of the internal (R-)symmetry group \( U(N) \) and
the \( \Sigma(M_{ab}) \) are the \( 4 \times 4 \) matrices introduced in the previous section.

The superalgebra \( SU(2,2|4) \) has a three graded decomposition with re-
respect to its compact subsuperalgebra \( SU(2|2) \times SU(2|2) \times U(1) \)

\[ g = L^+ \oplus L^0 \oplus L^-, \quad (3 - 3) \]

where

\[ [L^0, L^\pm] \subseteq L^\pm \]
\[ [L^+, L^-] \subseteq L^0 \]
\[ [L^+, L^+] = 0 = [L^-, L^-]. \quad (3 - 4) \]

Here \( L^0 \) represents the generators of \( SU(2|2) \times SU(2|2) \times U(1) \).

Generalizing the (purely bosonic) oscillator construction for \( SU(2,2) \)
in section 2, the Lie superalgebra \( SU(2,2|4) \) can be realized in terms of bilinear
combinations of bosonic and fermionic annihilation and creation
operators \( \xi_A \) (\( \xi^A = \xi_A^\dagger \)) and \( \eta_M \) (\( \eta^M = \eta_M^\dagger \)) which transform covariantly
and contravariantly under the two \( SU(2|2) \) subsupergroups of \( SU(2,2|4) \)

\[ \xi_A = \left( \begin{array}{c} a_i \\ \alpha_{\gamma} \end{array} \right), \quad \xi^A = \left( \begin{array}{c} a^i \\ \alpha^\gamma \end{array} \right) \quad (3 - 5) \]

and

\[ \eta_M = \left( \begin{array}{c} b_r \\ \beta_x \end{array} \right), \quad \eta^M = \left( \begin{array}{c} b^r \\ \beta^x \end{array} \right) \quad (3 - 6) \]

with \( i, j = 1, 2; \gamma, \delta = 1, 2; \) \( r, s = 1, 2; x, y = 1, 2 \) and

\[ [a_i, a^j] = \delta^j_i, \quad \{ \alpha_{\gamma}, \alpha^\delta \} = \delta^\delta_{\gamma} \quad (3 - 7) \]
\[ [b_r, b^s] = \delta^s_r, \quad \{ \beta_x, \beta^y \} = \delta^y_x. \quad (3 - 8) \]
Again, annihilation and creation operators are labelled by lower and upper indices, respectively. The generators of $SU(2,2|4)$ are given in terms of the above superoscillators schematically as

\[ L^- = \vec{\xi}_A \cdot \vec{\eta}_M \]
\[ L^0 = \vec{\xi}_A \cdot \vec{\xi}_B \oplus \vec{\eta}_M \cdot \vec{\eta}_N \]
\[ L^+ = \vec{\xi}_A \cdot \vec{\eta}^M, \]  

(3 - 9)

where the arrows over $\xi$ and $\eta$ again indicate that we are taking an arbitrary number $P$ of “generations” of superoscillators and the dot represents the summation over the internal index $K = 1,\ldots,P$, i.e $\vec{\xi}_A \cdot \vec{\eta}_M \equiv \sum_{K=1}^{P} \xi_A(K) \eta_M(K)$.

The even subgroup $SU(2,2) \times SU(4) \times U(1)_Z$ is obviously generated by the di-bosonic and di-fermionic generators. In particular, one recovers the $SU(2,2)$ generators of section 2 in terms of the bosonic oscillators:

\[ L^j_i = \vec{a}^i \cdot \vec{a}_i - \frac{1}{2} \delta^j_i N_a \]
\[ R^s_r = \vec{b}^r \cdot \vec{b}_r - \frac{1}{2} \delta^s_r N_b \]
\[ E = \frac{1}{2} \{ \vec{a}^i \cdot \vec{a}_i + \vec{b}^r \cdot \vec{b}_r \} = \frac{1}{2} \{ N_a + N_b + 2P \} \]
\[ L^{-ir} = \vec{a}_i \cdot \vec{b}_r, \quad L^{+ri} = \vec{b}^r \cdot \vec{a}_i \]  

(3 - 10)

satisfying

\[ [L^{-ir}, L^{+sj}] = \delta^s_r L^j_i + \delta^j_i R^s_r + \delta^j_i \delta^s_r E. \]  

(3 - 11)

Here, $N_a \equiv \vec{a}^i \cdot \vec{a}_i, N_b \equiv \vec{b}^r \cdot \vec{b}_r$ are again the bosonic number operators.

Similarly, the $SU(4)$ generators in their $SU(2) \times SU(2) \times U(1)$ basis are expressed in terms of the fermionic oscillators $\alpha$ and $\beta$:

\[ A^\delta_\gamma = \vec{\alpha}^\delta \cdot \vec{\alpha}_\gamma - \frac{1}{2} \delta^\delta_\gamma N_a \]
\[ B^x_y = \vec{\beta}^y \cdot \vec{\beta}_x - \frac{1}{2} \delta^y_x N_\beta \]
\[ C = \frac{1}{2} \{ -\vec{\alpha}^\delta \cdot \vec{\alpha}_\delta + \vec{\beta}^x \cdot \vec{\beta}_x \} = \frac{1}{2} \{ -N_\alpha - N_\beta + 2P \} \]
\[ L^{-xy} = \vec{\alpha}_\gamma \cdot \vec{\beta}_x, \quad L^{+xy} = \vec{\beta}^y \cdot \vec{\alpha}_\gamma \]  

(3 - 12)

with the closure relation

\[ [L^{-xy}, L^{+y\delta}] = -\delta^y_x A^\delta_\gamma - \delta^\delta_\gamma B^y_x + \delta^y_x \delta^\delta_\gamma C. \]  

(3 - 13)

11
Here \( N_\alpha = \vec{\alpha} \cdot \vec{\alpha} \delta \) and \( N_\beta = \vec{\beta} \cdot \vec{\beta} \delta \) are the fermionic number operators.

Finally, the central charge-like \( U(1)_Z \) generator \( Z \) is given by

\[
Z = \frac{1}{2} \{ N_\alpha + N_\alpha - N_b - N_\beta \}. \tag{3 - 14}
\]

Analogously, the odd generators are given by products of bosonic and fermionic oscillators and satisfy the following closure relations

\[
\{ \vec{a}_i \cdot \vec{\alpha}_x, \vec{\beta}_y \cdot \vec{a}_j \} = \delta_{y,x} L_{ij} - \delta_{x,j} B_{xy} + \frac{1}{2} \delta_{y,x} \delta_{i,j} (E + C + Z) \tag{3 - 15}
\]

The generator \( Y \) of the outer automorphism group \( U(1)_Y \) is simply

\[
Y = N_\alpha - N_\beta \tag{3 - 16}
\]

## 4 Unitary Supermultiplets of \( SU(2, 2|4) \)

To construct a basis for a lowest weight UIR of \( SU(2, 2|4) \), one starts from a set of states, collectively denoted by \( |\Omega\rangle \), in the Fock space of the oscillators \( a, b, \alpha, \beta \) that transforms irreducibly under \( SU(2|2) \times SU(2|2) \times U(1) \) and that is annihilated by all the generators \( \xi_A \cdot \vec{\eta}_M \equiv (\vec{a}_i \cdot \vec{b}_r + \vec{a}_i \cdot \vec{\beta}_x + \vec{\alpha}_\gamma \cdot \vec{b}_r + \vec{\alpha}_\gamma \cdot \vec{\beta}_x) \) of \( L^- \)

\[
\xi_A \cdot \vec{\eta}_M |\Omega\rangle = 0. \tag{4 - 1}
\]

By acting on \( |\Omega\rangle \) repeatedly with \( L^+ \), one then generates an infinite set of states that form a UIR of \( SU(2, 2|4) \)

\[
|\Omega\rangle, \quad L^+ |\Omega\rangle, \quad L^+ L^+ |\Omega\rangle, \ldots \tag{4 - 2}
\]

The irreducibility of the resulting representation of \( SU(2, 2|4) \) follows from the irreducibility of \( |\Omega\rangle \) under \( SU(2|2) \times SU(2|2) \times U(1) \). Because of the property (4 - 1), \( |\Omega\rangle \) as a whole will be referred to as the “lowest weight vector (lwv)” of the corresponding UIR of \( SU(2, 2|4) \).

In the restriction to the subspace involving purely bosonic oscillators, the above construction reduces to the subalgebra \( SU(2, 2) \) and its positive
energy UIR’s as described in section 2. Similarly, when restricted to the subspace involving purely fermionic oscillators, one gets the compact internal symmetry group $SU(4)$ \(\begin{pmatrix} 3 & 1 \\ -1 & 2 \end{pmatrix}\), and the above construction yields the representations of $SU(4)$ in its $SU(2) \times SU(2) \times U(1)$ basis.

Accordingly, a lowest weight UIR of $SU(2,2|4)$ decomposes into a direct sum of finitely many positive energy UIR’s of $SU(2,2)$ transforming in certain representations of the internal symmetry group $SU(4)$. Thus each positive energy UIR of $SU(2,2|4)$ corresponds to a supermultiplet of fields living in $AdS_5$ or on its boundary. The bosonic and fermionic fields in $AdS_5$ or on its boundary will be denoted as $\Phi(\mathbf{j}_L,\mathbf{j}_R)(E)$ and $\Psi(\mathbf{j}_L,\mathbf{j}_R)(E)$, respectively. Interpreted as a UIR of the $\mathcal{N} = 4$ conformal superalgebra in $d = 4$ each lowest weight UIR of $SU(2,2|4)$ corresponds to a supermultiplet of massless or massive fields. These four dimensional fields can then be labelled as $\Phi_{(j_M,j_N)}(l)$ or $\Psi_{(j_M,j_N)}(l)$ if they are bosons or fermions of conformal dimension $l$ and $SL(2,\mathbb{C})$ quantum numbers $\mathbf{(j_M,j_N)}$.

4.1 Doubleton Supermultiplets of $SU(2,2|4)$

By choosing one pair of super oscillators $\xi$ and $\eta$ in the oscillator realization of $SU(2,2|4)$ (i.e. for $P = 1$), one obtains the so-called doubleton supermultiplets. The doubleton supermultiplets contain only doubleton representations of $SU(2,2)$, i.e. they are multiplets of fields living on the boundary of $AdS_5$ without a 5d Poincaré limit. Equivalently, they can be characterized as multiplets of massless fields in 4d Minkowski space that form a UIR of the $\mathcal{N} = 4$ superconformal algebra $SU(2,2|4)$.

The complete list of doubleton supermultiplets has been given in our previous paper [15]. In this subsection we shall recapitulate our results as a preparation for the next section.

The supermultiplet defined by the lwv $|\Omega\rangle = |0\rangle$ of $SU(2,2|4)$ is the unique irreducible CPT self-conjugate doubleton supermultiplet. It is also the supermultiplet of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in $d = 4$ [6]. The content of this special doubleton supermultiplet is given in Table 1. (We will continue to use this form of presenting our results in what follows.)
Table 1. The doubleton supermultiplet corresponding to the lwv $|\Omega\rangle = |0\rangle$. The first column indicates the lowest weight vectors (lwv) of $SU(2,2) \times SU(4)$. The second column shows the AdS energies $E = (N_a + N_b)/2 + P$ respectively the conformal dimensions $l$, and the third column lists the compact $(j_L, j_R)$, or equivalently, the non-compact labels $(j_M, j_N)$ of the corresponding fields. Also, $Y = N_\alpha - N_\beta$, whereas $\Phi$ and $\Psi$ denote bosonic and fermionic fields, respectively. This supermultiplet has $Z = 0$.

If we take $|\Omega\rangle = \xi^A |0\rangle \equiv a^i |0\rangle \oplus \alpha^\gamma |0\rangle = |\square, 1\rangle$, (4 - 1)
we get a supermultiplet of spin range 3/2. It is shown in Table 2. (See the Appendix for a quick review of the supertableaux notation [25].)

Table 2. The doubleton supermultiplet corresponding to the lwv $|\Omega\rangle = \xi^A |0\rangle = |\square, 1\rangle$. It has $Z = \frac{1}{2}$.

The CPT conjugate supermultiplet to the one listed in Table 2 is obtained by taking

$$|\Omega\rangle = \eta^A |0\rangle \equiv b^\beta |0\rangle \oplus \beta^z |0\rangle = |1, \square\rangle$$ (4 - 2)
as the lwv. It is displayed in Table 3.
Table 3. The doubleton supermultiplet corresponding to the lwv $|\Omega\rangle = \eta^A|0\rangle = |1, 0\rangle$. It has $Z = -\frac{1}{2}$.

These last two doubleton supermultiplets of spin range $3/2$ would occur in the $\mathcal{N} = 4$ super Yang-Mills theory if there is a well-defined conformal (i.e. massless) limit of the $1/4$ BPS states described in ref [18]. These $1/4$ BPS multiplets are massive representations of the four dimensional $\mathcal{N} = 4$ Poincaré superalgebra with two central charges, one of them saturating the BPS bound. As such, they are equivalent to massive representations of the $\mathcal{N} = 3$ Poincaré superalgebra without central charges. The corresponding multiplet with the lowest spin content (see e.g. [26]) contains 14 scalars, 14 spin 1/2 fermions, six vectors and one spin 3/2 fermion, giving altogether $2^6$ states. If a massless limit of such a multiplet exists, it should decompose into two self-conjugate doubleton multiplets (Table 1) plus one of the form given in Table 2 plus one of the form given in Table 3.

The lowest weight vector of a generic doubleton supermultiplet is of the form

$$|\Omega\rangle = \xi^{A_1}\xi^{A_2}...\xi^{A_{2j}}|0\rangle = |\underbrace{1\cdots1}_{2j}1\rangle \quad (4 - 3)$$

or of the “CPT-conjugate” form

$$|\Omega\rangle = \eta^{A_1}\eta^{A_2}...\eta^{A_{2j}}|0\rangle = |1, \underbrace{1\cdots1}_{2j}\rangle \quad (4 - 4)$$

For $j \geq 1$, the corresponding general doubleton supermultiplets are given in tables 4 and 5. (Note that in these tables we assume that $j$ takes on integer values. For $j$ half-integer the roles of $\Phi$ and $\Psi$ are reversed).
Table 4. The doubleton supermultiplet corresponding to the lwv $|\Omega\rangle = \xi^{A_1} \xi^{A_2} ... \xi^{A_{2j}} |0\rangle = \left| \begin{array}{c} \cdots \\ 2j \end{array} \right\rangle$. It has $Z = j$.

Table 5. The doubleton supermultiplet corresponding to the lwv $|\Omega\rangle = \eta^{A_1} \eta^{A_2} ... \eta^{A_{2j}} |0\rangle = \left| \begin{array}{c} 1 \\ \cdots \\ 2j \end{array} \right\rangle$. It has $Z = -j$.

5 Novel short “massless” supermultiplets of $SU(2,2|4)$

The doubleton supermultiplets described in the last subsection are fundamental in the sense that all other lowest weight UIR’s of $SU(2,2|4)$ occur in the tensor product of two or more doubleton supermultiplets. Instead of trying to identify these irreducible submultiplets in the (in general reducible, but not fully reducible) tensor products, one simply increases the number $P$ of oscillator generations so that the tensoring becomes implicit while the irreducibility stays manifest.

The simplest class, corresponding to $P = 2$, contains the supermultiplets that are commonly referred to as “massless” in the 5d AdS sense. We will therefore use this as a name for all supermultiplets that are obtained by
taking $P = 2$ in the oscillator construction despite some problems with the notion of “mass” in AdS spacetimes\textsuperscript{[13]}. Completing our study of these “massless” supermultiplets in \textsuperscript{[13]}, we will now give a complete list of the allowed $SU(2, 2|4)$ lowest weight vectors $|\Omega\rangle$ for $P = 2$ and then consider those that give rise to the novel short supermultiplets in detail.

The condition $L^-|\Omega\rangle = 0$ leaves the following possibilities:

- $|\Omega\rangle = |0\rangle$. This lwv gives rise to the $\mathcal{N} = 8$ graviton supermultiplet in $AdS_5$ and occurs in the tensor product of two CPT self-conjugate doubleton (i.e. $\mathcal{N} = 4$ super Yang Mills) supermultiplets.

- $|\Omega\rangle = \xi^{A_1(1)}\xi^{A_2(1)}...\xi^{A_{2j}(1)}|0\rangle = |\underbrace{\ldots |,1\rangle\rangle_{2j}}$. The corresponding supermultiplets and also their conjugates resulting from

- $|\Omega\rangle = \eta^{A_1(1)}\eta^{A_2(1)}...\eta^{A_{2j}(1)}|0\rangle = |1, \underbrace{\ldots \rangle\rangle_{2j}}$. The corresponding supermultiplets are displayed in table 12 of ref \textsuperscript{[13]}. Increasing $j$ leads to multiplets with higher and higher spins and AdS energies. For $j > 3/2$ the spin range is always 4. None of these multiplets can occur in the tensor product of two or more self-conjugate doubleton supermultiplets. They require the chiral doubleton supermultiplets listed in tables 2 to 5.

- $|\Omega\rangle = \xi^{A_1(1)}\xi^{A_2(1)}...\xi^{A_{2j_L}(1)}\eta^{B_1(2)}\eta^{B_2(2)}...\eta^{B_{2j_R}(2)}|0\rangle = |\underbrace{\ldots \rangle\rangle_{2j_L}, \underbrace{\ldots \rangle\rangle_{2j_R}}$. The corresponding supermultiplets are displayed in table 12 of ref \textsuperscript{[13]}. Again they involve spins and AdS energies that increase with $j_L$ and $j_R$, maintaining a constant spin range of 4 for $j_L, j_R \geq 1$.

In addition to these purely (super)symmetrized lwv’s, one can also anti-(super)symmetrize pairs of superoscillators, since $P = 2$. The requirement $L^-|\Omega\rangle = 0$ then rules out the simultaneous appearance of $\xi$’s and $\eta$’s so that one is left with

\textsuperscript{6}Of course, this has nothing to do with the completely unambiguous concept of masslessness in 4d Minkowski spacetime used in Sections 2 and 4.1.
• $|\Omega\rangle = \xi^{[A_1(1)\xi^{B_1}(2)]...\xi^{[A_n(1)\xi^{B_n}(2)]}\xi^{C_1(1)}...\xi^{C_k(1)}}|0\rangle$

  $$= |\underbrace{\cdots}_{n}\underbrace{\cdots}_{k}, 1\rangle,$$

• $|\Omega\rangle = \eta^{[A_1(1)\eta^{B_1}(2)]...\eta^{[A_n(1)\eta^{B_n}(2)]}\eta^{C_1(1)}...\eta^{C_k(1)}}|0\rangle$

  $$= |1, \underbrace{\cdots}_{n}\underbrace{\cdots}_{k}\rangle.$$

The special case $k = 0$ then leads to the novel short multiplets, which we will now discuss in detail.

The simplest case is

$$|\Omega\rangle = \xi^{[A_1(1)\xi^{B_1}(2)]}|0\rangle = |\underbrace{\cdots}_{1}\rangle,$$

$$\equiv a^{ij}(1)a^{j(2)}|0\rangle + [a^i(1)\alpha^\gamma(2) - a^i(2)\alpha^\gamma(1)]|0\rangle + \alpha^{(1)}(1)\alpha^{\delta}(2)|0\rangle.$$ 

Acting on $|\Omega\rangle$ with the supersymmetry generators $\vec{a}^i\cdot\vec{b}^x$ and $\vec{b}^y\cdot\vec{a}^\gamma$ of $L^+$ and collecting resulting $SU(2, 2) \times SU(4)$ lwv’s (i.e. states that are annihilated by $\vec{a}^i\cdot\vec{b}^y$ and $\vec{a}^\gamma\cdot\vec{b}^x$), one arrives at the supermultiplet of spin range 2 given in Table 6 (see the Appendix for a complete list of the allowed $SU(4)$ lwv’s for $P = 2$ and the corresponding $SU(4)$ representations they induce).
Table 6. The supermultiplet corresponding to the lwv $|\Omega\rangle = \xi^{[A_1]}(1)\xi^{B_1}(2)|0\rangle$. It has $Z = 1$.

This supermultiplet does not occur in the tensor product of two or more CPT self-conjugate doubleton supermultiplets, but it appears in the tensor product of two doubleton supermultiplets of the type listed in table 2.

Similarly, by taking the following lwv

$$|\Omega\rangle = \eta^{[A_1]}(1)\eta^{B_1}(2)|0\rangle = |1, \overrightarrow{0}\rangle$$

$$\equiv b^x(1)b^y(2)|0\rangle \oplus [b^x(1)\beta^x(2) - b^y(2)\beta^y(1)]|0\rangle \oplus \beta^x(1)\beta^y(2)|0\rangle,$$

one gets the CPT conjugate supermultiplet displayed in Table 7.

| $E=-1$ | $(j_L, j_R)=(j_M, j_N)$ | SU(4) Dynkin | $Y$ | Field |
|--------|-------------------|--------------|----|-------|
| 2      | (0,0)             | (0,0,2)      | -2 | $\Phi_{0,0}$ |
| 5/2    | (1/2,0)           | (0,0,1)      | -3 | $\Psi_{1/2,0}$ |
| 5/2    | (0,1/2)           | (0,1,1)      | -1 | $\Psi_{0,1/2}$ |
| 3      | (0,1)             | (1,0,1)      | 0  | $\Phi_{0,1}$ |
| 3      | (1/2,1/2)         | (0,1,0)      | -2 | $\Phi_{1/2,1/2}$ |
| 3      | (0,0)             | (0,2,0)      | 0  | $\Phi_{0,0}$ |
| 3      | (0,0)             | (0,0,0)      | -4 | $\Phi_{0,0}$ |
| 7/2    | (0,1/2)           | (1,1,0)      | 1  | $\Psi_{0,1/2}$ |
| 7/2    | (0,3/2)           | (0,0,1)      | 1  | $\Psi_{0,3/2}$ |
| 7/2    | (1/2,1)           | (1,0,0)      | -1 | $\Psi_{1/2,1}$ |
| 4      | (0,1)             | (0,1,0)      | 2  | $\Phi_{0,1}$ |
| 4      | (0,0)             | (2,0,0)      | 2  | $\Phi_{0,0}$ |
| 4      | (1/2,3/2)         | (0,0,0)      | 0  | $\Phi_{1/2,3/2}$ |
| 9/2    | (0,1/2)           | (1,0,0)      | 3  | $\Psi_{0,1/2}$ |
| 5      | (0,0)             | (0,0,0)      | 4  | $\Phi_{0,0}$ |

Table 7. The supermultiplet corresponding to the lwv $|\Omega\rangle = \eta^{[A_1]}(1)\eta^{B_1}(2)|0\rangle$. It has $Z = -1$.

In complete analogy to its CPT conjugate counterpart, it occurs in the tensor product of two doubleton supermultiplets of the type given in table 3, but not in the tensor product of two or more self-conjugate doubleton supermultiplets.
The general lwv for $j \geq 2$

\[
|\Omega\rangle = \xi^{[A_1]}(1)\xi^{B_1}(2)\ldots\xi^{[A_j]}(1)\xi^{B_j}(2)|0\rangle
\]

\[
= | \begin{array}{cccc}
\vdots \\
\end{array} ,1 \rangle
\]

leads to the following supermultiplet with spin range 2

| E = -1 | (j_L, j_R) = (j_M, j_N) | SU(4) Dynkin | Y | Field |
|--------|----------------|----------------|---|-------|
| j      | (0,0)          | (0,0,0)        | 4 | $\Phi_{0,0}$ |
| j+1/2  | (1/2,0)        | (1,0,0)        | 3 | $\Psi_{1/2,0}$ |
| j+1    | (0,0)          | (2,0,0)        | 2 | $\Phi_{0,0}$ |
| j+1    | (0,0)          | (1,0,0)        | 2 | $\Phi_{1,0}$ |
| j+3/2  | (1/2,0)        | (1,1,0)        | 1 | $\Psi_{1/2,0}$ |
| j+3/2  | (3/2,0)        | (0,0,1)        | 1 | $\Psi_{3/2,0}$ |
| j+2    | (2,0)          | (0,0,0)        | 0 | $\Phi_{2,0}$ |
| j+2    | (1,0)          | (1,0,1)        | 0 | $\Phi_{1,0}$ |
| j+2    | (0,0)          | (0,2,0)        | 0 | $\Phi_{0,0}$ |
| j+5/2  | (3/2,0)        | (1,0,0)        | -1| $\Psi_{3/2,0}$ |
| j+5/2  | (1/2,0)        | (0,1,1)        | -1| $\Psi_{1/2,0}$ |
| j+3    | (1,0)          | (0,1,0)        | -2| $\Phi_{1,0}$ |
| j+3    | (0,0)          | (0,0,2)        | -2| $\Phi_{0,0}$ |
| j+7/2  | (1/2,0)        | (0,0,1)        | -3| $\Psi_{1/2,0}$ |
| j+4    | (0,0)          | (0,0,0)        | -4| $\Phi_{0,0}$ |

Table 8. The supermultiplet corresponding to the lwv $|\Omega\rangle = \xi^{[A_1]}(1)\xi^{B_1}(2)\ldots\xi^{[A_j]}(1)\xi^{B_j}(2)|0\rangle$. It has $Z = j$.

Obviously, the spin content of these multiplets is independent of $j$. Only the AdS energies (resp. conformal dimensions) get shifted, when $j$ is increased, which distinguishes these multiplets from their (super)symmetrized counterparts obtained from $|\Omega\rangle = \xi^{A_1}(1)\xi^{A_2}(1)\ldots\xi^{A_j}(1)|0\rangle$, where the spins increase with $j$.

Similarly, the conjugate lwv ($j \geq 2$)

\[
|\Omega\rangle = \eta^{[A_1]}(1)\eta^{B_1}(2)\ldots\eta^{[A_j]}(1)\eta^{B_j}(2)|0\rangle
\]
leads to the supermultiplet given in Table 9

| E=-1 | (j_L,j_R)=(j_M,j_N) | SU(4) Dynkin | Y | Field |
|-------|----------------------|--------------|----|-------|
| j     | (0,0)                | (0,0,0)      | -4 | \(\Phi_{0,0}\) |
| j+1/2 | (0,1/2)              | (0,0,1)      | -3 | \(\Psi_{0,1/2}\) |
| j+1   | (0,0)                | (0,0,2)      | -2 | \(\Phi_{0,1/2}\) |
| j+1   | (0,1)                | (0,1,0)      | -2 | \(\Phi_{1,0}\) |
| j+3/2 | (0,1/2)              | (0,1,1)      | -1 | \(\Psi_{0,1/2}\) |
| j+3/2 | (0,3/2)              | (1,0,0)      | -1 | \(\Psi_{3/2,0}\) |
| j+2   | (0,2)                | (0,0,0)      | 0  | \(\Phi_{0,2}\) |
| j+2   | (0,1)                | (1,0,1)      | 0  | \(\Phi_{0,1}\) |
| j+2   | (0,0)                | (0,2,0)      | 0  | \(\Phi_{0,0}\) |
| j+5/2 | (0,3/2)              | (0,0,1)      | 1  | \(\Psi_{3/2,0}\) |
| j+5/2 | (0,1/2)              | (1,0,1)      | 1  | \(\Psi_{1/2,0}\) |
| j+3   | (0,1)                | (0,1,0)      | 2  | \(\Phi_{1,0}\) |
| j+3   | (0,0)                | (2,0,0)      | 2  | \(\Phi_{0,0}\) |
| j+7/2 | (0,1/2)              | (1,0,0)      | 3  | \(\Psi_{1/2,0}\) |
| j+4   | (0,0)                | (0,0,0)      | 4  | \(\Phi_{0,0}\) |

Table 9. The supermultiplet corresponding to the lwv

\(|\Omega\rangle = \eta^{[A_1}(1)\eta^{B_1]}(2)...\eta^{[A_j}(1)\eta^{B_j]}(2)|0\rangle\). It has \(Z = -j\).

6 Discussion and Conclusions

The spectrum of the \(S^5\) compactification of IIB supergravity falls into an infinite tower of irreducible CPT self-conjugate supermultiplets of \(SU(2,2|4)\) of spin range two with ever increasing quantized eigenvalues of AdS energy \([4, 5]\). They are obtained by choosing as lowest weight vector the (super) Fock vacuum vector \(|0\rangle\) of ever increasing pairs \(P \geq 2\) of super-oscillators and therefore have \(Z = 0\). Hence they are representations of \(PSU(2,2|4)\). The “massless” graviton supermultiplet of \(N = 8\) AdS supergravity in \(d = 5\) sits at the bottom of this infinite tower corresponding to \(P = 2\). The higher AdS energy supermultiplets correspond to “massive” Kaluza-Klein modes.
The shortest irreducible CPT self-conjugate supermultiplet of spin range one \((P = 1)\) decouples from the Kaluza-Klein spectrum as local gauge degrees of freedom. This ultrashort supermultiplet is the unique CPT self-conjugate irreducible doubleton supermultiplet, which has no smooth Poincaré limit in \(d = 5\). It lives on the boundary of \(AdS_5\), which can be identified with four dimensional Minkowski space.

The quadratic operator that reduces to the mass (squared) operator in \(d = 5\) Minkowski space when one takes the Poincaré limit of \(AdS_5\) is not a Casimir invariant of \(SU(2, 2)\). Hence the Poincaré mass is not an invariant quantity in \(AdS_5\). One may, instead, use the eigenvalues of the quadratic, cubic and quartic Casimir operators as invariant labels of a UIR of \(SU(2, 2)\). However, for positive energy UIR’s one can use the labels \((j_L, j_R, E)\) of the corresponding lowest weight vectors \(|\Omega\rangle\) with respect to the maximal compact subgroup \(SU(2)_L \times SU(2)_R \times U(1)_E\). We showed in section 2 explicitly how one can go to a non-compact basis for the UIR defined by the lowest weight vector such that the UIR is now labelled with respect to the \(SL(2, \mathbb{C}) \times D\) subgroup with labels \((j_M, j_N, l)\), which numerically coincide with \((j_L, j_R, -E)\). For doubleton fields living on the boundary, the group \(SU(2, 2)\) acts as the conformal group and hence the labels \((j_M, j_N)\) are the covariant labels with respect to the Lorentz group in \(d = 4\) and \(l = -E\) is simply the scaling (conformal) dimension. We verified that the doubleton irreps correspond to massless fields in \(d = 4\). Non-doubleton representations \((P > 1)\) are the massive representations of the four dimensional conformal group. Interpreted as the anti-de Sitter group in \(d = 5\) some of these representations of \(SU(2, 2)\) with \(P = 2\) become massless representations in the \((5d)\) Poincaré limit. These are precisely the representations that satisfy the condition \(E = j_L + j_R + 2\).

In this paper we focussed mainly on some novel short supermultiplets of \(SU(2, 2|4)\) that are not CPT self-conjugate. These supermultiplets cannot be obtained by tensoring CPT self-conjugate \(\mathcal{N} = 4\) Yang-Mills doubleton supermultiplets with themselves. For \(P = 2\) these short supermultiplets involve fields that do not satisfy the condition \(E = j_L + j_R + 2\). Interestingly, all these novel short supermultiplets of the form we discussed in section 5 can be obtained by tensoring the chiral doubleton supermultiplets given in Tables 2 to 5 with themselves. One may ask what role, if any, these novel supermultiplets play in the AdS/CFT duality. As we argued above, we expect the massive supermultiplets of 1/4 BPS states in \(\mathcal{N} = 4\) super Yang-Mills theory to decompose into a CPT conjugate pair of chiral spin 3/2 doubleton supermultiplets plus two CPT self-conjugate doubleton supermultiplets.
of \(SU(2,2|4)\), when an appropriate conformal limit (given its existence) is taken. Now, the 1/4 BPS states belong to nontrivial orbits of the duality group \(SL(2,\mathbb{Z})\) and are related to \((p,q)\) IIB superstrings \cite{24}. This suggests that the novel supermultiplets we studied above for \(P = 2\) as well as their counterparts for \(P > 2\) are relevant to a generalized duality between the solitonic sector of \(\mathcal{N} = 4\) super Yang-Mills in \(d = 4\) and the \((p,q)\) superstrings over \(AdS_5 \times S^5\). More generally, the methods we have employed in this paper provide us with simple and yet powerful tools for answering the question whether it is possible to extend the \(AdS/CFT\) duality to the full \(SL(2,\mathbb{Z})\) covariant type IIB superstring over \(AdS_5 \times S^5\).

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### 7 Appendix

The allowed lowest weight vectors (lwv’s) of \(SU(4)\) for \(P = 2\) (i.e. the states annihilated by \(L_{xx}^-=\vec{\alpha}_x \cdot \vec{\beta}_x\)) are given in the following table. This table is more explicit than the corresponding table in the Appendix of our previous paper \cite{15} in that it now also contains all non-trivial realizations in terms of some less obvious linear combinations (like the one in the last row, for example). These linear combinations do not lead to any new \(SU(4)\) representation, however they can sometimes conspire to give some non-trivially realized states in supermultiplets. In our previous paper \cite{15} this was the case for the multiplet listed in Table 12. In the present paper, on the other hand, they do not contribute due to the (super)antisymmetry of the lwv’s considered in this paper: In the cases where they might contribute, one always encounters a wrong minus sign.
| lwv | SU(4) Dynkin (dim) | Y |
|-----|--------------------|---|
| $|0\rangle$ | $(0,2,0)$ (20') | 0 |
| $\beta^x(1)|0\rangle$ | $(0,1,1)$ (20) | -1 |
| $\alpha^\gamma(1)|0\rangle$ | $(1,1,0)$ (20) | 1 |
| $\beta^x(1)\beta^y(1)|0\rangle$ | $(0,1,0)$ (6) | -2 |
| $\alpha^\gamma(1)\alpha^\delta(1)|0\rangle$ | $(0,1,0)$ (6) | 2 |
| $\beta^x\beta^y\beta^z\beta^w|0\rangle$ | $(0,0,0)$ (1) | -4 |
| $\alpha^\gamma\alpha^\delta\alpha^\eta|0\rangle$ | $(0,0,0)$ (1) | 4 |
| $\beta^x(1)\beta^y(2)|0\rangle$ | $(0,0,2)$ (10) | -2 |
| $\alpha^\gamma(1)\alpha^\delta(2)|0\rangle$ | $(2,0,0)$ (10) | 2 |
| $\beta^x\beta^y\beta^z|0\rangle$ | $(0,0,1)$ (4) | -3 |
| $\alpha^\gamma\alpha^\delta\alpha^\epsilon|0\rangle$ | $(1,0,0)$ (4) | 3 |
| $\alpha^\gamma(1)\beta^x(2)|0\rangle$ | $(1,0,1)$ (15) | 0 |
| $[\alpha^\gamma(1)\beta^x(1) - \alpha^\gamma(2)\beta^x(2)]|0\rangle$ | $(1,0,1)$ (15) | 0 |
| $\alpha^\gamma(1)\beta^x(2)\beta^y(2)|0\rangle$ | $(1,0,0)$ (4) | -1 |
| $\beta^x(1)\alpha^\gamma(2)\alpha^\delta(2)|0\rangle$ | $(0,0,1)$ (4) | 1 |
| $[\beta^x(1)\alpha^\gamma(1)\alpha^\delta(2)$ | $(0,0,1)$ (4) | 1 |
| $-\frac{1}{2}\beta^x(2)\alpha^\gamma(2)\alpha^\delta(2)]|0\rangle$ | |
| $\alpha^\gamma(1)\alpha^\delta(1)\beta^x(2)\beta^y(2)|0\rangle$ | $(0,0,0)$ (1) | 0 |
| $[\alpha^\gamma(1)\alpha^\delta(1)\beta^x(1)\beta^y(2)$ | $(0,0,0)$ (1) | 0 |
| $-\alpha^\gamma(2)\alpha^\delta(1)\beta^x(2)\beta^y(2)]|0\rangle$ | |
| $[\alpha^\gamma(1)\alpha^\delta(2)\beta^x(1)\beta^y(2)$ | $(0,0,0)$ (1) | 0 |
| $-\frac{1}{2}\alpha^\gamma(1)\alpha^\delta(1)\beta^x(1)\beta^y(2)$ | |
| $-\frac{1}{2}\alpha^\gamma(2)\alpha^\delta(2)\beta^x(2)\beta^y(2)]|0\rangle$ | |

For the decomposition of the supertableaux of $U(m/n)$ in terms of the tableaux of its even subgroup $U(m) \times U(n)$ we refer to [25]. Here we give a few examples.

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| 24 |
\( U(m/n) \supset U(m) \times U(n) \)

\[
\begin{align*}
\mathcal{U} &= (\cdot, 1) + (1, \cdot) \\
\mathcal{U} U(1) &= (\cdot, 1) + (\cdot, \cdot) + (1, \cdot) \\
\mathcal{U} &= (\cdot, 1) + (\cdot, \cdot) + (\cdot, \cdot) \\
\mathcal{U} &= (\cdot, 1) + (\cdot, \cdot) + (\cdot, \cdot) + (1, 1) \\
\mathcal{U} &= (\cdot, 1) + (\cdot, \cdot) + (\cdot, \cdot) + (\cdot, \cdot) + (1, \cdot) + (\cdot, \cdot) + (\cdot, \cdot)
\end{align*}
\]

(7 - 1)

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