Dynamical symmetry breaking in Nambu-Jona-Lasinio model under the influence of external electromagnetic and gravitational fields

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Abstract. Dynamical symmetry breaking is investigated for a four-fermion Nambu-Jona-Lasinio model in external electromagnetic and gravitational fields. An effective potential is calculated in the leading order of the large-N expansion using the proper-time Schwinger formalism.

Phase transitions accompanying a chiral symmetry breaking in the Nambu-Jona-Lasinio model are studied in detail. A magnetic catalysis phenomenon is shown to exist in curved spacetime but it turns out to lose its universal character because the chiral symmetry is restored above some critical positive value of the spacetime curvature.

INTRODUCTION

Different four-fermion models [1], [2] have been considered to be one of the most convenient ways for an investigation of the low-energy physics of strong interactions. A dynamical symmetry breaking phenomenon (DSB) has been proved to take place within those models, particularly Nambu-Jona-Lasinio (NJL) one, which seems to show up a nontrivial phase structure. Usually the symmetry to be broken under the DSB mechanism is the chiral one. Dynamical version of fermions mass generation and dynamical chiral symmetry breaking have been investigated very carefully and some fruitful applications for the real high-energy physics have been found [3], [4].

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However it has turned out to be very difficult to realize the idea of DSB because all of the calculations should be performed out of perturbation theory. This leads to study already simplified models and that is why we have to investigate any possible generalizations within these models those of nonzero temperature and chemical potential, arbitrary dimensions, external fields including gravitational one and so on as some kind of laboratory in order to collect as much new information as we can.

Despite of essential difficulties caused by the nonpertubative character of DSB phenomenon, it has been applied successfully to describe the overcritical behavior of quantum electrodynamics, the top quark condensate mechanism of mass generation in the Weinberg- Salam model of electroweak interactions, technicolor models and, especially, to investigate the composite fields generation in the NJL model. In the frameworks of Schwinger proper-time method this model has been studied in external electromagnetic field by many authors for 20 years [5] - [7].

Recently a new sample of papers devoted to DSB in external electromagnetic field have been published [8], [9]. It shed a new light onto the universal character of magnetic catalysis, which means that magnetic field breaks chiral symmetry for any value of its strength. Furthermore it has been shown that this phenomenon occurs in quantum electrodynamics, 2+1, 3+1 dimensional nonsupersymmetrical and 3+1 supersymmetrical NJL models. So the statement about the universal character of magnetic catalysis has been made.

Investigations of the influence of a classical gravitational field on the DSB phenomenon in the NJL model have been carried out for some years [10]. It has been shown that curvature-induced phase transitions exist and might play some essential role in more or less realistic early Universe model. It turns out that, in spite of the relatively small value of the curvature-dependent corrections at the low energy scale to be investigated within the NJL model, these corrections appear to be inescapable, in the sense that they must be taken into account when one performs the necessary "fine tuning" of the different cosmological parameters. Furthermore, positive spacetime curvature changes the universal character of magnetic catalysis dramatically.

It has been shown that the early Universe could contain a large primodial magnetic field and have a huge electrical conductivity. The vicinities of magnetized black holes and neutron stars are the other possible points of application of our model. Therefore both classical external gravitational and electromagnetic fields should be taken into account for the description of a wide sample of events in the Universe.

In the present paper we describe our recent results concerning the DSB under the simultaneous influence of both gravitational and electromagnetic fields in NJL model [11]. The phase transitions accompanying the DSB process on the spacetime curvature, as well as the values of electric or magnetic field strength are investigated.
DYNAMICAL SYMMETRY BREAKING BY A MAGNETIC FIELD IN FLAT SPACETIME

In an arbitrary dimensional flat spacetime the NJL model has the following action:

\[ S = \int d^d x \left\{ i \bar{\psi} \gamma^\mu D_\mu \psi + \frac{\lambda}{2N} \left[ (\bar{\psi}\psi)^2 + (\bar{\psi}i\gamma_5\psi)^2 \right] \right\}, \tag{1} \]

where the covariant derivative \( D_\mu \) includes the electromagnetic potential \( A_\mu \) and \( N \) is the number of bispinor fields \( \psi_a \).

Introducing the auxiliary fields

\[ \sigma = -\frac{\lambda}{N} (\bar{\psi}\psi), \quad \pi = -\frac{\lambda}{N} (\bar{\psi}i\gamma_5\psi) \tag{2} \]

we can rewrite the action as:

\[ S = \int d^d x \left\{ i \bar{\psi} \gamma^\mu D_\mu \psi - \frac{N}{2\lambda} (\sigma^2 + \pi^2) - \bar{\psi}(\sigma + i\pi\gamma_5)\psi \right\}. \tag{3} \]

The effective action in the leading \( 1/N \) order is

\[ \frac{1}{N} S_{\text{eff}} = -\int d^d x \frac{\sigma^2 + \pi^2}{2\lambda} - i \ln \det \left[ i\gamma^\mu D_\mu - (\sigma + i\gamma_5\pi) \right] \tag{4} \]

Then the effective potential (EP), defined for the constant configurations of \( \pi \) and \( \sigma \) as \( V_{\text{eff}} = -S_{\text{eff}}/N \int d^d x \), is given by the formula

\[ V_{\text{eff}} = \frac{\sigma^2}{2\lambda} + i \text{Sp} \ln \langle \sigma | [i\gamma^\mu D_\mu - \sigma] | \sigma \rangle \tag{5} \]

Here we put \( \pi = 0 \) because the final expression will depend on the combination \( \sigma^2 + \pi^2 \) only within our approximation. This means that we are actually considering the Gross-Neveu model.

But if we take into account kinetic terms of the fields \( \pi \) and \( \sigma \) generated by quantum corrections we will obtain different dynamics of these two fields. It should be noted that \( \sigma \) will be a massive scalar field in the supercritical area while \( \pi \) will be massless Goldstone particle.

By means of the usual Green function (GF), which obeys the equation

\[ (i\gamma^\mu D_\mu - \sigma) G(x, x', \sigma) = \delta(x - x'), \tag{6} \]

we obtain the following formula

\[ V'_{\text{eff}}(\sigma) = \frac{\sigma}{\lambda} - i \text{Sp} G(x, x, \sigma). \tag{7} \]
Now we can substitute in this equation the fermion GF in constant magnetic field

\[ G(x, x', \sigma) = \Phi(x, x') \tilde{G}(x - x', \sigma), \tag{8} \]

where

\[ \Phi(x, x') = \exp \left[ i e \int \frac{x}{x'} A^\mu(x'') dx'' \right] \tag{9} \]

\[ \tilde{G}_0(z, \sigma) = e^{-i\frac{2d}{4\pi s}} \int_0^\infty \frac{ds}{(4\pi s)^{3/2}} e^{-i\sigma^2} \exp \left( -i \frac{4s}{z} C^\mu{}_{\nu} z^\nu \right) \times \]

\[ \left( \sigma + \frac{1}{2s} \gamma^\mu C_{\mu\nu z^\nu} - \frac{e}{2} \gamma^\mu F_{\mu\nu z^\nu} \right) \left[ \tau \coth \tau - \frac{e\sigma}{2} \gamma^\mu \gamma^\nu F_{\mu\nu} \right]. \tag{10} \]

Let us describe the 3D case to avoid some more complicated expressions. The EP is given by

\[ V_{eff}(\sigma) = \frac{\sigma^2}{2\lambda} + \frac{1}{4\pi^{3/2}} \int_{1/\Lambda^2}^{\infty} ds \frac{e^{-s\sigma^2} (eBs \coth(eBs))}{s^{5/2}}. \tag{11} \]

The most reliable method to keep all of the divergences is the cut-off parameter introduction. So we can make the following trick: write the integral in the EP like

\[ \int_{1/\Lambda^2}^{\infty} ds \frac{e^{-s\sigma^2}}{s^{5/2}} [(eBs \coth(eBs) - 1)] + \int_{1/\Lambda^2}^{\infty} ds \frac{e^{-s\sigma^2}}{s^{5/2}} \] \tag{12}

and calculate the last one keeping \( \Lambda \) finite while the first integral is finite already and we can put \( 1/\Lambda^2 = 0 \). Then it appears to be possible to calculate it as a limit \( \mu \to -1/2 \) using the formula

\[ \int_0^\infty dx x^{\mu-1} e^{-ax} \coth(cx) = \Gamma(\mu) \left[ 2^{1-\mu} (c)^{-\mu} \zeta(\mu, \frac{a}{2c}) - a^{-\mu} \right]. \tag{13} \]

Finally the EP has the form

\[ V_{eff}(\sigma) = \frac{\sigma^2}{2\lambda} - \left[ \frac{\Lambda \sigma^2}{2\pi^{3/2}} + \frac{\sqrt{2}}{\pi} (eB)^{3/2} \zeta \left( -\frac{1}{2}, 1 + \frac{\sqrt{2} \sigma^2}{4eB} \right) + \frac{1}{2\pi} eB \sigma \right]. \tag{14} \]

There are two ways of justifying the introduction of the \( \Lambda \) parameter in the formula for the EP. The first one is the standard renormalization procedure, by means of the UV cut-off method. Then, in the limit \( \Lambda \to \infty \), after renormalization of the coupling constant.
\[
\frac{1}{\lambda_R} = \frac{1}{\lambda} - \frac{\Lambda}{\pi^{3/2}},
\]
we have the expression for the renormalized EP in 3D spacetime
\[
V_{\text{eff}}^{\text{ren}}(\sigma) = \frac{\sigma^2}{2\lambda R} - \frac{\sqrt{2}}{\pi}(eB)^{3/2} \zeta \left( -\frac{1}{2}, 1 + \frac{\sigma^2}{2eB} \right) - \frac{1}{2\pi} eB \sigma
\]
For \( B = 0 \) dynamical symmetry breaking takes place when
\[
\lambda > \lambda_c = \frac{\pi^{3/2}}{\Lambda}
\]
if only we keep the finite cut-off \( \Lambda \) meanwhile the renormalized NJL model does not admit this phenomenon in general. However any finite value of the external magnetic field changes the situation dramatically and dynamical symmetry breaking occurs for any coupling constant. For \( \sigma^2 \ll eB \) nontrivial solution of the gap equation defining a nontrivial minimum of the EP is given by
\[
\sigma = \frac{eB\lambda_R}{2\pi}
\]
The same calculations have been done for a constant electric field. The nonzero imaginary part appears in this case caused by vacuum instability of the quantum field theory in electric field. But treating the real part of the EP we can find that electrical field restores the chiral symmetry, initially broken for the finite cut-off parameter case.

Fig.1 illustrates the universal character of magnetic catalysis. It is a plot of 3D \( V_{\text{eff}}^{\text{ren}}(\sigma) \) with \( \mu = 100; \lambda \mu = 100 \). Starting from above the curves correspond to the following electromagnetic field configurations: \( eE/\mu^2 = 0.0002, B = 0; B = E = 0; eB\mu^2 = 0.0002, E = 0 \).

After renormalization the chiral symmetry exists without external field but the magnetic field creates the non-zero minimum that indicates that DSB takes place. Meanwhile the external electric field works evidently against symmetry breaking.

In all figures, an arbitrary dimensional parameter, \( \mu \), defining a typical scale in the model, is introduced in order to perform the plots in terms of dimensionless variables.

**GENERAL EXPRESSION FOR EFFECTIVE POTENTIAL IN EXTERNAL ELECTROMAGNETIC AND GRAVITATIONAL FIELDS**

We have just the same expression for the EP in curved spacetime
\[
V'_{\text{eff}}(\sigma) = \frac{\sigma}{\lambda} - i\text{Sp}G(x, x, \sigma)
\]
FIGURE 1

To calculate the linear curvature corrections the local momentum expansion formalism is the most convinient one. Then in the special Riemannian normal coordinate framework

\[ g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3} R_{\mu\rho\sigma\nu} y^\rho y^\sigma \]  

(20)

with corresponding formulae for the others values and \( y = x - x' \).

Then choosing the vector potential of the external electromagnetic field in the form

\[ A_\mu(x) = -\frac{1}{2} F_{\mu\nu} x^\nu, \]  

(21)

where \( F_{\mu\nu} \) is the constant matrix of electromagnetic field strength tensor we find that:

\[ G(x, x', \sigma) = \Phi(x, x') \left[ \tilde{G}_0(x - x', \sigma) + \tilde{G}_1(x - x', \sigma) + \ldots \right] , \]  

(22)

where \( G_n \sim R^n \).

Therefore we obtain the iterative sequence of equations for the GF and the linear-curvature corrections are given by

\[ \tilde{G}_1(x - x', \sigma) = \int dx'' G_{00}(x - x'', \sigma) \times \]  

(23)
\[
-\frac{i}{6} \gamma^\alpha R_{\mu
u\rho\sigma}(x'' - x')^\alpha(x'' - x')^\nu \partial_\rho \tilde{G}_0(x'' - x', \sigma) - \\
\frac{i}{4} \gamma^\mu \sigma^{bc} R_{bca\lambda}(x'' - x')^\lambda \tilde{G}_0(x'' - x', \sigma)
\]

Here \( G_{00}(x - x', \sigma) \) is a free fermion GF.

Substituting an exact flat spacetime GF of fermions in external electromagnetic field into this formula after some algebra we have evident expression for the EP with the linear-curvature accuracy in the constant curvature spacetime.

**External constant magnetic field case**

For 3D spacetime the EP is given by

\[
V_{eff}(\sigma) = \frac{\sigma^2}{2\lambda} + \frac{1}{4\pi^{3/2}} \int_{1/\Lambda^2} \int_{1/\Lambda^2} ds \exp(-s\sigma^2) \tau \coth \tau - \\
\frac{R}{144\pi^{3/2}} \int_{1/\Lambda^2} \int_{1/\Lambda^2} ds dt \exp[-(t + s)^2] \times
\]

\[
\left[ 2\kappa(\kappa + \tau) + (9\tau + 5\kappa) \coth \tau + \kappa(\tau - 3\kappa) \coth^2 \tau \right]
\]

where \( \tau = eBs, \kappa = eBt \).

We can perform the same renormalization procedure as in flat spacetime because no new divergences appear in the linear-curvature corrections. But we keep the cut of scheme here to study the most general situation.

The results are presented on Fig.2. It shows a plot of 3D \( V_{eff}\) for different values of spacetime curvature \( R\mu^2 = 0.0025, 0.002, 0.001, 0 \). Second-order phase transition ruled by the spacetime curvature takes place.

**External constant electrical field case**

We have for renormalized EP the following expression

\[
V_{eff}(\sigma) = \frac{\sigma^2}{2\lambda_R} - \frac{(2i\kappa E)^{3/2}}{4\pi} \left[ 2\zeta(-\frac{1}{2}, \frac{\sigma^2}{2i\kappa E}) - \left( \frac{\sigma^2}{2i\kappa E} \right)^{1/2} \right] + \\
\frac{R\sigma}{24\pi} + \frac{iR(eE)^{1/6}}{2\pi^{2/3}} \exp(-\pi\frac{\sigma^2}{eE}) \Gamma(\frac{2}{3}) \sigma^{2/3}.
\]

Here we have performed a small electric field expansion in the R-dependent term. A numerical analysis of Re\( V_{eff}(\sigma) \) for negative coupling constant gives the typical behaviour of a first-order phase transition, as shown in Fig. 3. The critical values are defined as usual: \( R_{c1} \) corresponds to the spacetime curvature for which a local
nonzero minimum appears, \( R_c \), when the real part of EP is equal at zero and at the local minimum, and \( R_{c2} \), when the zero extremum becomes a maximum. There is a plot of 3D \( \text{Re} V^\text{ren}_{\text{eff}}/\mu^3 \) as a function of \( \sigma/\mu \) for fixed \( eE/\mu^2 = 0.00005 \) and \( \lambda\mu = -100 \). From above to below, the curves in the plot correspond to the following values of \( R/\mu^2 = 0.006; 0.005; 0.004; 0.0032; 0 \), respectively. The critical values, defined as usual, are given by: \( R_{c1}/\mu^2 = 0.005; R_c/\mu^2 = 0.0032; R_{c2}/\mu^2 = 0 \). \( \Lambda \) obviously does not appear anywhere because after renormalization it must be sent to infinity, \( \Lambda \rightarrow \infty \).

CONCLUSIONS

We clearly observe that a positive spacetime curvature tries to restore chiral symmetry even in the presence of external magnetic field. Therefore the universal character of magnetic catalysis doesn't survive in curved spacetime. From the other hand electric field increases the critical value of coupling constant as it does in flat spacetime.

It should be noted that for \( D < 4 \) is renormalizable and these conclusions don't depend already on the cut-off scale \( \Lambda \).

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