COMPUTING WITH RATIONAL SYMMETRIC FUNCTIONS
AND APPLICATIONS TO INVARIANT THEORY
AND PI-ALGEBRAS

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Dedicated to Yuri Bahturin on the occasion of his 65th birthday

Abstract. Let $K$ be a field of any characteristic. Let the formal power series
$$f(x_1, \ldots, x_d) = \sum_{\alpha_n} \alpha_n x_1^{a_1} \cdots x_d^{a_d} = \sum_{m(\lambda)} S_\lambda(x_1, \ldots, x_d), \quad \alpha_n, m(\lambda) \in K,$$
be a symmetric function decomposed as a series of Schur functions. When $f$ is a rational function whose denominator is a product of binomials of the form $1 - x_1^{a_1} \cdots x_d^{a_d}$, we use a classical combinatorial method of Elliott of 1903 further developed in the $\Omega$-calculus (or Partition Analysis) of MacMahon in 1916 to compute the generating function
$$M(f; x_1, \ldots, x_d) = \sum_{m(\lambda)} x_1^{\lambda_1} \cdots x_d^{\lambda_d}, \quad \lambda = (\lambda_1, \ldots, \lambda_d).$$
$M$ is a rational function with denominator of a similar form as $f$. We apply the method to several problems on symmetric algebras, as well as problems in classical invariant theory, algebras with polynomial identities, and noncommutative invariant theory.

Introduction

Let $K$ be a field of any characteristic and let $K[[X]]^{S_d}$ be the subalgebra of the symmetric functions in the algebra of formal power series $K[[X]] = K[[x_1, \ldots, x_d]]$ in the set of variables $X = \{x_1, \ldots, x_d\}$. We study series $f(X) \in K[[X]]^{S_d}$ which can be represented as rational functions whose denominators are products of binomials of the form $1 - X^a = 1 - x_1^{a_1} \cdots x_d^{a_d}$. Following Berele [15], we call such functions nice rational symmetric functions. Those functions appear in many places in mathematics. In the examples that have inspired our project, $K$ is of characteristic 0:

If $W$ is a polynomial module of the general linear group $GL_d = GL_d(K)$, then its $GL_d$-character is a symmetric polynomial, which in turn gives $W$ the structure of a graded vector space. Hence the Hilbert (or Poincaré) series of the symmetric algebra $K[W]$ is a nice rational symmetric function.

Nice rational symmetric functions appear as Hilbert series in classical invariant theory. For example, this holds for the Hilbert series of the pure trace algebra

2010 Mathematics Subject Classification. 05A15; 05E05; 05E10; 13A50; 15A72; 16R10; 16R30; 20G05.

Key words and phrases. Rational symmetric functions, MacMahon partition analysis, Hilbert series, classical invariant theory, noncommutative invariant theory, algebras with polynomial identity, cocharacter sequence.

The research of the first named author was partially supported by INdAM.

The research of the fifth named author was partially supported by NSF Grant DMS-1016086.
of $n \times n$ generic matrices which is the algebra of invariants of $GL_n$ acting by simultaneous conjugation on several $n \times n$ matrices. The mixed trace algebra also has a meaning in classical invariant theory and has a Hilbert series which is a nice rational symmetric function.

The theorem of Belov [12] gives that for any PI-algebra $R$ the Hilbert series of the relatively free algebra $K(Y)/T(R)$, $Y = \{y_1, \ldots, y_d\}$, where $T(R)$ is the T-ideal of the polynomial identities in $d$ variables of $R$, is a rational function. Berele [15] established that the proof of Belov (as presented in the book by Kanel-Belov and Rowen [49]) also implies that this Hilbert series is a nice rational symmetric function.

Every symmetric function $f(X)$ can be presented as a formal series

$$f(X) = \sum_{\lambda} m(\lambda) S_{\lambda}(X), \quad m(\lambda) \in K,$$

where $S_{\lambda}(X) = S_{\lambda}(x_1, \ldots, x_d)$ is the Schur function indexed with the partition $\lambda = (\lambda_1, \ldots, \lambda_d)$.

Clearly, it is an interesting combinatorial problem to find the multiplicities $m(\lambda)$ of an explicitly given symmetric function $f(X)$. This problem is naturally related to the representation theory of $GL_d$ in characteristic 0 because the Schur functions are the characters of the irreducible polynomial representations of $GL_d$. Another motivation is that the multiplicities of the Schur functions in the Hilbert series of the relatively free algebra $K(Y)/T(R)$, char$K = 0$, are equal to the multiplicities in the (multilinear) cocharacter sequence of the polynomial identities of $R$.

Drensky and Genov [34] introduced the multiplicity series $M(f; X)$ of $f(X) \in K[[X]]^{S_d}$. If

$$f(X) = \sum_{n_i \geq 0} a(n)X^n = \sum_{\lambda} m(\lambda) S_{\lambda}(X), \quad m(\lambda) \in K,$$

then

$$M(f; X) = \sum_{\lambda} m(\lambda) X^\lambda = \sum_{\lambda_i \geq \lambda_{i+1}} m(\lambda) x_{\lambda_1}^{\lambda_1} \cdots x_{\lambda_d}^{\lambda_d} \in K[[X]]$$

is the generating function of the multiplicities $m(\lambda)$. Berele, in [16], (and also not explicitly stated in [15]) showed that the multiplicity series of a nice rational symmetric function $f(X)$ is also a nice rational function. This fact was one of the key moments in the recent theorem about the exact asymptotics

$$c_n(R) \simeq an^{k/2}b^n, \quad a, b \in \mathbb{R}, \quad k, b \in \mathbb{N},$$

of the codimension sequence $c_n(R)$, $n = 0, 1, 2, \ldots$, of a unital PI-algebra $R$ in characteristic 0 (Berele and Regev [17] for finitely generated algebras and Berele [16] in the general case). Unfortunately, the proof of Berele does not yield an algorithm to compute the multiplicity series of $f(X)$. In two variables, Drensky and Genov [35] developed methods to compute the multiplicity series for nice rational symmetric functions.

The approach of Berele [15,16] involves classical results on generating functions of nonnegative solutions of systems of linear homogeneous equations, obtained by Elliott [40] and MacMahon [55], as stated in the paper by Stanley [66]. Going back to the originals [40] and [55], we see that the results there provide algorithms to compute the multiplicity series for nice rational symmetric functions in any number of variables. The method of Elliott [40] was further developed by MacMahon [55]
in his “Ω-Calculus” or Partition Analysis. The “Ω-Calculus” was improved, with computer realizations, see Andrews, Paule, and Riese [56], and Xin [76]. The series of twelve papers on MacMahon’s partition analysis by Andrews, alone or jointly with Paule, Riese, and Strehl (I – [3], . . . , XII – [4]) gave a new life of the methods, with numerous applications to different problems. It seems that for the moment the original approach of [30, 55] and its further developments have not been used very efficiently in invariant theory and theory of PI-algebras. The only results in this direction we are aware of are in the recent paper by Bedratyuk and Xin [11].

Our computations are based on the ideas of Xin [76] and have been performed with standard functions of Maple on a usual personal computer. We illustrate the methods on several problems on symmetric algebras, in classical invariant theory, algebras with polynomial identities, and noncommutative invariant theory. The results of Section 11 hold for any field $K$ of arbitrary characteristic. In the other sections we assume that $K$ is of characteristic 0.

1. Reduction to MacMahon’s partition analysis

Recall that one of the ways to define Schur functions (e.g., Macdonald [54]) is as fractions of Vandermonde type determinants

$$S_\lambda(X) = \frac{V(\lambda + \delta, X)}{V(\delta, X)},$$

where $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\delta = (d - 1, d - 2, \ldots, 2, 1, 0)$, and

$$V(\mu, X) = \begin{vmatrix}
  x_1^{\mu_1} & x_2^{\mu_1} & \cdots & x_d^{\mu_1} \\
  x_1^{\mu_2} & x_2^{\mu_2} & \cdots & x_d^{\mu_2} \\
  \vdots & \vdots & \ddots & \vdots \\
  x_1^{\mu_d} & x_2^{\mu_d} & \cdots & x_d^{\mu_d}
\end{vmatrix}, \quad \mu = (\mu_1, \ldots, \mu_d).$$

If $f(X) \in K[[X]]^{S_d}$ is a symmetric function, it can be presented in a unique way as

$$f(X) = \sum_\lambda m(\lambda)S_\lambda(X),$$

where the “$\lambda$-coordinate” $m(\lambda) \in K$ is called the multiplicity of $S_\lambda(X)$. Our efforts are concentrated around the problem: Given $f(X) \in K[[X]]^{S_d}$, find the multiplicity series

$$M(f; X) = \sum_\lambda m(\lambda)X^\lambda = \sum_{\lambda_1 \geq \lambda_{i+1}} m(\lambda)x_1^{\lambda_1} \cdots x_d^{\lambda_d} \in K[[X]]$$

and the multiplicities $m(\lambda)$. It is convenient to introduce new variables

$$v_1 = x_1, v_2 = x_1x_2, \ldots, v_d = x_1 \cdots x_d$$

and to consider the algebra of formal power series $K[[V]] = K[[v_1, \ldots, v_d]]$ as a subalgebra of $K[[X]]$. As in [54], we introduce the function $M'(f; V)$ (also called the multiplicity series of $f(X)$) by

$$M'(f; V) = M(f; v_1, v_1^{-1}v_2, \ldots, v_1^{-1}v_d) = \sum_\lambda m(\lambda)v_1^{\lambda_1} \cdots v_1^{-1}\lambda_{d-1}v_d^{\lambda_d}. $$
The mapping \( M' : K[[X]]^{S_d} \to K[[V]] \) defined by \( M' : f(X) \to M'(f; V) \) is a bijection.

The proof of the following easy lemma is given in \[15\].

**Lemma 1.1.** Let \( f(X) \in K[[X]]^{S_d} \) be a symmetric function and let

\[
g(X) = f(X) \prod_{i < j} (x_i - x_j) = \sum_{r_i \geq 0} \alpha(r_1, \ldots, r_d) x_1^{r_1} \cdots x_d^{r_d}, \quad \alpha(r_1, \ldots, r_d) \in K.
\]

Then the multiplicity series of \( f(X) \) is given by

\[
M(f; X) = \frac{1}{x_1^{d-1} x_2^{d-2} \cdots x_d^{d-1}} \sum_{r_i > r_{i+1}} \alpha(r_1, \ldots, r_d) x_1^{r_1} \cdots x_d^{r_d},
\]

where the summation is over all \( r = (r_1, \ldots, r_d) \) such that \( r_1 > r_2 > \cdots > r_d \).

By the previous lemma, given a nice rational function

\[
g(X) = \sum_{r_i \geq 0} \alpha(r) X^r = p(X) \prod_{1 \leq i < j \leq d} \frac{1}{1 - X^{r_{ij}}}.
\]

we start by computing “half” of it, i.e., the infinite sum of \( \alpha(r) x_1^{r_1} x_2^{r_2} \cdots x_d^{r_d} \) for \( r_1 > r_2 \), and then we continue in the same way with the other variables. To illustrate the method of Elliott \[40\], it is sufficient to consider the case of two variables only.

Given the series

\[
g(x_1, x_2) = \sum_{i,j \geq 0} \alpha_{ij} x_1^i x_2^j
\]

we introduce a new variable \( z \) and consider the Laurent series

\[
g(x_1 z, \frac{x_2}{z}) = \sum_{i,j \geq 0} \alpha_{ij} x_1^i x_2^{j-i} z^{-i} = \sum_{n = -\infty}^{\infty} g_n(x_1, x_2) z^n, \quad g_n(x_1, x_2) \in K[[x_1, x_2]].
\]

We want to present \( g(x_1 z, x_2 / z) \) as a sum of two series, one in \( z \) and the other in \( 1/z \):

\[
g(x_1 z, \frac{x_2}{z}) = \sum_{n \geq 0} g_n(x_1, x_2) z^n + \sum_{n > 0} g_{-n}(x_1, x_2) \left( \frac{1}{z} \right)^n,
\]

and then take the first summand and replace \( z \) with 1 there. If \( g(x_1, x_2) \) is a nice rational function, then \( g(x_1, x_2) \) and \( g(x_1 z, x_2 / z) \) have the form

\[
g(x_1, x_2) = p(x_1, x_2) \prod_{1 \leq i < j \leq d} \frac{1}{1 - x_1^{r_{ij}}}, \quad p(x_1, x_2) \in K[x_1, x_2],
\]

\[
g(x_1 z, \frac{x_2}{z}) = p(x_1 z, \frac{x_2}{z}) \prod_{1 \leq i < j \leq d} \frac{1}{1 - x_1^{r_{ij}} z^{a_j-b_j}}.
\]

The expression \( \prod 1/(1 - x_1^{a_j-b_j} z^{a_j-b_j}) \) is a product of three factors

\[
\prod_{a_0 = b_0} \frac{1}{1 - x_1^{a_0}} \prod_{a_1 > b_1} \frac{1}{1 - x_1^{a_1} z^{a_1-b_1}} \prod_{a_2 < b_2} \frac{1}{1 - x_1^{a_2} z^{a_2-b_2}}.
\]

If \( \prod 1/(1 - x_1^{a_j-b_j} z^{a_j-b_j}) \) contains factors of both the second and the third type, Elliott \[40\] suggests to apply the equality

\[
\frac{1}{(1 - A z^a)(1 - B z^b)} = \frac{1}{1 - A z^a - B z^b - 1} \quad \frac{1}{1 - A z^a} + \frac{1}{1 - B z^b - 1}.
\]
to one of the expressions $1/(1 - x_1^{a_1} x_2^{a_2} z_{a_1 - a_2} B_1)(1 - x_1^{a_2} x_2^{a_2} / z_{b_2 - a_2})$ and to represent
$\prod 1/(1 - x_1^{a_1} x_2^{a_2} z_{a_1 - a_2} B_1)$ as a sum of three expressions which are simpler than the original one. Continuing in this way, one represents $\prod 1/(1 - x_1^{a_1} x_2^{a_2} z_{a_1 - a_2} B_1)$ as a sum of products of two types:

$$
\prod_{a \geq b} \frac{1}{1 - x_1^{a_1} x_2^{a_2} z_{a_1 - a_2} B_1} \quad \text{and} \quad \prod_{a_0 = b_0} \frac{1}{1 - x_1^{a_0} x_2^{b_0} \prod_{a_2 < b_2} \frac{1}{1 - x_1^{a_2} x_2^{b_2} / z_{b_2 - a_2}}.}
$$

With some additional easy arguments we can represent $g(x_1 z, x_2 / z)$ as a linear combination of monomials $A_1 z^i, i \geq 0$, and quotients of the form

$$\frac{A_2}{z^j}, j > 0, \quad B_1 z^i \prod_{b \geq 0} \frac{1}{1 - B_2 z^b}, i \geq 0, \quad C_1 \prod_{z^j} \frac{1}{1 - C_2} \prod_{c > 0} \frac{1}{1 - C_3 / z^c}, j \geq 0,$

with coefficients $A_1, A_2, B_1, B_2, C_1, C_2, C_3$ which are monomials in $x_1, x_2$. Comparing this form of $g(x_1 z, x_2 / z)$ with its expansion as a Laurent series in $z$

$$g(x_1 z, x_2 / z) = \sum_{n = -\infty}^{+\infty} g_n(x_1, x_2)z^n,$$

we obtain that the part $\sum_{n \geq 0} g_n(x_1, x_2)z^n$ which we want to compute is the sum of $A_1 z^i, B_1 z^i \prod_{b \geq 0} 1/(1 - B_2 z^b)$ and the fractions $C_1 z^j \prod_{z^j} 1/(1 - C_2)$ with $j = 0$.

Generalizing the idea of Elliott, in his famous book \[55\] MacMahon defined operators $\Omega$ and $\tilde{\Omega}$. The first operator cuts the negative powers of a Laurent formal power series and then replaces $z$ with 1:

$$\Omega : \sum_{n = -\infty}^{+\infty} \alpha(n)Z^n \to \sum_{n = 0}^{+\infty} \alpha(n),$$

and the second one takes the constant term of series

$$\tilde{\Omega} : \sum_{n = -\infty}^{+\infty} \alpha(n)Z^n \to \alpha(0),$$

where $\alpha(n) = \alpha(n_1, \ldots, n_d) \in K[[X]], \alpha(0) = \alpha(0, \ldots, 0)$ and $Z^n = z_1^{n_1} \cdots z_d^{n_d}$.

The next theorem presents the multiplicity series of an arbitrary symmetric function in terms of the Partition Analysis of MacMahon.

**Theorem 1.2.** Let $f(X) \in K[[X]]^{S_d}$ be a symmetric function in $d$ variables and let

$$g(X) = f(X) \prod_{i < j} (x_i - x_j).$$

Then the multiplicity series of $f(X)$ satisfies

$$M(f; X) = \frac{1}{x_1^{d-1} x_2^{d-2} \cdots x_{d-2} x_{d-1}^2} \Omega \left( \frac{g(x_1 z_1, x_2 z_1^{-1} z_2 \cdots x_{d-1} z_{d-2}^{-1} z_{d-1} z_{d-1}^{-1})}{z_{d-1}} \right).$$

**Proof.** Let

$$g(X) = \sum_{r_i \geq 0} \alpha(r)X^r, \quad \alpha(r) \in K, X^r = x_1^{r_1} \cdots x_d^{r_d}.$$ Then

$$g(x_1 z_1, x_2 z_1^{-1} z_2 \cdots x_{d-1} z_{d-2}^{-1} z_{d-1} z_{d-1}^{-1}) = \sum_{r_i \geq 0} \alpha(r)X^r z_1^{r_1} z_2^{r_2} z_3^{r_3} \cdots z_{d-1}^{r_{d-1}} - r_d,$$
\[ \Omega \left( g(x_1z_1, x_2z_2^{-1}z_2 \cdots x_{d-1}z_{d-2}^{-1}z_{d-2} \cdots x_dz_d^{-1}) \right) = \sum_{r_i \geq r_{i+1}} \alpha(r)X^r. \]

The function \( g(X) \) is skew-symmetric because \( f(X) \) is symmetric. Hence \( \alpha(r) \) is equal to 0, if \( r_i = r_j \) for some \( i \neq j \) and the summation in the latter equality for \( \Omega \) runs on \( r_1 > \cdots > r_d \) (and not on \( r_1 \geq \cdots \geq r_d \)). Now the proof follows immediately from Lemma 1.1.

The \( \Omega \)-operators were applied by MacMahon [55] to Elliott rational functions which share many properties with nice rational functions. He used the Elliott reduction process described above. The computational approach developed by Andrews, Paule, and Riese [3, 4] is based on improving this reduction process. There is another algorithm due to Xin [76] which involves partial fractions. In this paper we shall use an algorithm inspired by the algorithm of Xin [76]. We shall state it in the case of two variables. The case of nice rational symmetric functions in several variables is obtained in an obvious way by multiple application of the algorithm to the function \( g(X) = f(X) \prod_{i<j} (x_i - x_j) \) in \( d \) variables instead of to the function \( g(x_1, x_2) = f(x_1, x_2)(x_1 - x_2) \) in two variables.

**Algorithm 1.3.** Let \( g(x_1, x_2) \in K[[x_1, x_2]] \) be a nice rational function. In \( g(x_1z_1, x_2/z) \) we replace the factors \( 1/(1-C/z^c) \), where \( C \) is a monomial in \( x_1, x_2 \), with the factor \( z^c/(z^c - C) \). Then \( g(x_1z_1, x_2/z) \) becomes a rational function of the form

\[ g(x_1z_1, x_2/z) = \frac{p(z)}{z^a} \prod \frac{1}{1-A} \prod \frac{1}{1-Bz^b} \prod \frac{1}{z^c - C}, \]

where \( p(z) \) is a polynomial in \( z \) with coefficients which are rational functions in \( x_1, x_2 \) and \( A, B, C \) are monomials in \( x_1, x_2 \). Presenting \( g(x_1z_1, x_2/z) \) as a sum of partial fractions with respect to \( z \) we obtain that

\[ g(x_1z_1, x_2/z) = p_0(z) + \sum \frac{p_i}{z^i} + \sum \frac{r_{jk}(z)}{q_j(z)^k}, \]

where \( p_0(z), r_{jk}(z), q_j(z) \in K(x_1, x_2)[z] \), \( p_i \in K(x_1, x_2) \), \( q_j(z) \) are the irreducible factors over \( K(x_1, x_2) \) of the binomials \( 1 - Bz^b \) and \( z^c - C \) in the expression of \( g(x_1z_1, x_2/z) \), and \( \deg_z r_{jk}(z) < \deg_z q_j(z) \). Clearly \( p_0(z) \) gives a contribution to the series \( \sum_{n>0} g_n(x_1, x_2)z^n \) in the expansion of \( g(x_1z_1, x_2/z) \) as a Laurent series. Similarly, \( r_{jk}(z)/q_j(z)^k \) contributes to the same series for the factors \( q_j(z) \) of \( 1 - Bz^b \). The fraction \( p_i/z^i \) is a part of the series \( \sum_{n>0} g_n(x_1, x_2)z^n \). When \( q_j(z) \) is a factor of \( z^c - C \), we obtain that \( q_j(z) = z^d q'_j(1/z) \), where \( d = \deg_z q_j(z) \) and \( q'_j(\zeta) \in K(x_1, x_2)[\zeta] \) is a divisor of \( 1 - C\zeta^c \). Since \( \deg_z r_{jk}(z) < \deg_z q_j(z) \) we derive that \( r_{jk}(z)/q_j(z)^k \) contributes to \( \sum_{n>0} g_n(x_1, x_2)z^n \) and does not give any contribution to \( \sum_{n>0} g_n(x_1, x_2)z^n \). Hence

\[ \sum_{n \geq 0} g_n(x_1, x_2)z^n = p_0(z) + \sum \frac{r_{jk}(z)}{q_j(z)^k}, \]

where the sum in the right side of the equation runs on the irreducible divisors \( q_j(z) \) of the factors \( 1 - Bz^b \) of the denominator of \( g(x_1z_1, x_2/z) \). Substituting 1 for \( z \) we obtain the expression for \( \Omega(g(x_1z_1, x_2/z)) \).
Proof. The process described in the algorithm gives that
\[
\Omega \geq g(x_1 z, x_2 z) = \frac{P(x_1, x_2)}{Q(x_1, x_2)} \quad P(x_1, x_2), Q(x_1, x_2) \in K[x_1, x_2].
\]
By the elimination process of Elliott we already know that \( \Omega \geq g(x_1 z, x_2 z) \) is a nice rational function. Hence the polynomial \( Q(x_1, x_2) \) is a divisor of a product of binomials \( 1 - X^a \). Hence the output is in a form that allows, starting with a nice rational symmetric function \( f(X) \) in \( d \) variables, to continue the process with the other variables and to compute the multiplicity series \( M(f; X) \). \( \square \)

Remark 1.4. If \( f(X) \) is a nice rational symmetric function, we can find the multiplicity series \( M(f; X) \) applying Lemma 1.1 and using the above algorithm. On the other hand, it is very easy to check whether the formal power series \( h(X) = \sum \beta(q) X^q, \quad q_1 \geq \cdots \geq q_d, \)
is equal to the multiplicity series \( M(f; X) \) of \( f(X) \). This is because \( h(X) = M(f; X) \) if and only if
\[
f(X) \prod_{i<j} (x_i - x_j) = \sum_{\sigma \in S_d} \text{sign}(\sigma) x_{\sigma(1)}^{d-1} x_{\sigma(2)}^{d-2} \cdots x_{\sigma(d-1)} h(x_{\sigma(1)}, \ldots, x_{\sigma(d)}).
\]
These arguments can be used to verify most of our computational results on multiplicities.

2. Symmetric algebras

Till the end of the paper we assume that \( K \) is a field of characteristic 0. For a background on the representation theory of \( GL_d = GL_d(K) \) in the level we need see the book by Macdonald [54] or the paper by Almkvist, Dicks and Formanek [2]. We fix is a polynomial \( GL_d \)-module \( W \). Then \( W \) is a direct sum of its irreducible components \( W(\mu) \), where \( \mu = (\mu_1, \ldots, \mu_d) \) is a partition in not more than \( d \) parts,
\[
W = \bigoplus_{\mu} k(\mu)W(\mu),
\]
where the nonnegative integer \( k(\mu) \) is the multiplicity of \( W(\mu) \) in the decomposition of \( W \). The vector space \( W \) has a basis of eigenvectors of the diagonal subgroup \( D_d \) of \( GL_d \) and we fix such a basis \( \{ w_1, \ldots, w_p \} \) :
\[
g(w_j) = \xi_1^{\alpha_{j1}} \cdots \xi_d^{\alpha_{jd}} w_j, \quad g = \text{diag}(\xi_1, \ldots, \xi_d) \in D_d, j = 1, \ldots, p,
\]
where \( \alpha_{ji}, i = 1, \ldots, d, j = 1, \ldots, p, \) are nonnegative integers. The action of \( D_d \) on \( W \) induces a \( \mathbb{Z}^d \)-grading on \( W \) assuming that
\[
\deg(w_j) = (\alpha_{j1}, \ldots, \alpha_{jd}), \quad j = 1, \ldots, p.
\]
The polynomial
\[
H(W; X) = \sum_{j=1}^p X^{\alpha_{j1}} = \sum_{j=1}^p x_1^{\alpha_{j1}} \cdots x_d^{\alpha_{jd}}
\]
is the Hilbert series of \( W \) and has the form
\[
H(W; X) = \sum_{\mu} k(\mu)S_{\mu}(X).
\]
It plays the role of the character of the $GL_d$-module $W$. If the eigenvalues of $g \in GL_d$ are equal to $\zeta_1, \ldots, \zeta_d$, then

$$\chi_W(g) = \text{tr}_W(g) = H(W; \zeta_1, \ldots, \zeta_d) = \sum_{\mu} k(\mu) S_\mu(\zeta_1, \ldots, \zeta_d).$$

We identify the symmetric algebra $K[W]$ of $W$ with the polynomial algebra in the variables $w_1, \ldots, w_p$. We extend diagonally the action of $GL_d$ to the symmetric algebra $K[W]$ by

$$g(f(w)) = f(g(w)), \quad g \in GL_d, f \in K[W], w \in W.$$ 

The $\mathbb{Z}^d$-grading of $W$ induces a $\mathbb{Z}^d$-grading of $K[W]$ and the Hilbert series of $K[W]$

$$H(K[W]; X) = \prod_{j=1}^p \frac{1}{1 - X^{\alpha_j}} = \prod_{j=1}^p \frac{1}{1 - x_1^{\alpha_{j1}} \cdots x_d^{\alpha_{jd}}}$$

is a nice rational symmetric function. Clearly, here we have assumed that $k(0) = 0$, (i.e., $|\mu| = \mu_1 + \cdots + \mu_d > 0$ in the decomposition of $W$). Otherwise the homogeneous component of zero degree of $K[W]$ is infinitely dimensional and the Hilbert series of $K[W]$ is not well defined. If we present $H(K[W]; X)$ as a series of Schur functions

$$H(K[W]; X) = \sum_{\lambda} m(\lambda) S_\lambda(X),$$

then

$$K[W] = \bigoplus_{\lambda} m(\lambda) W(\lambda).$$

Hence the multiplicity series of $H(K[W]; X)$ carries the information about the decomposition of $K[W]$ as a sum of irreducible components.

The symmetric function $H(K[W]; X)$ is equal to the plethysm $H(K[Y]; X) \circ H(W; X)$ of the Hilbert series of $K[Y_\infty]$, $Y_\infty = \{y_1, y_2, \ldots\}$, and the Hilbert series of $W$. More precisely, $H(K[W]; X)$ is the part of $d$ variables of the plethysm

$$\prod_{i \geq 1} \frac{1}{1 - x_i} \circ \sum_{\mu} k(\mu) S_\mu(X_\infty) = \sum_{n \geq 0} S_{(n)}(X_\infty) \circ \sum_{\mu} k(\mu) S_\mu(X_\infty)$$

of the symmetric functions $\sum_{n \geq 0} S_{(n)}$ and $\sum_{\mu} k(\mu) S_\mu$ in the infinite set of variables $X_\infty = \{x_1, x_2, \ldots\}$.

**Example 2.1.** The decomposition of $H(K[W]; X)$ as a series of Schur functions is known in few cases only. The well known identities (see [54]; the third is obtained
from the second applying the Young rule)

\[
H(K[W(2)]; X) = \prod_{i \leq j} \frac{1}{1 - x_i x_j} = \sum_{\lambda} S_{(2\lambda_1, \ldots, 2\lambda_d)}(X),
\]

\[
H(K[W(1^2)]; X) = \prod_{i < j} \frac{1}{1 - x_i x_j} = \sum_{(\lambda, \lambda_2, \lambda_4, \ldots)} S_{(\lambda_2, \lambda_4, \ldots)}(X),
\]

\[
H(K[W(1) \oplus W(1^2)]; X) = \prod_{i=1}^d \frac{1}{1 - x_i \prod_{i < j} (1 - x_i x_j)} = \sum_{\lambda} S_{(\lambda_1, \ldots, \lambda_d)}(X).
\]

give the following expressions for the multiplicity series \([\lfloor r \rfloor]\) is the integer part of \(r \in \mathbb{R}\)

\[
M(H(K[W(2)]); X) = \sum_{n_i \geq 0} x_1^{2n_1} (x_1 x_2)^{2n_2} \cdots (x_1 \cdots x_d)^{2n_d}
\]

\[
= \prod_{i=1}^d \frac{1}{1 - (x_1 \cdots x_i)^2},
\]

\[
M(H(K[W(1^2)]); X) = \sum_{n_i \geq 0} (x_1 x_2)^{n_1} (x_1 x_2 x_3 x_4)^{n_2} \cdots (x_1 \cdots x_{d/2})^{n_{d/2}}
\]

\[
= \prod_{i=1}^{\lfloor d/2 \rfloor} \frac{1}{1 - x_1 \cdots x_{2i}},
\]

\[
M(H(K[W(1) \oplus W(1^2)]); X) = \sum_{n_i \geq 0} x_1^{n_1} \cdots (x_1 \cdots x_d)^{n_d}
\]

\[
= \prod_{i=1}^d \frac{1}{1 - x_1 \cdots x_i}.
\]

In the language of the multiplicity series \(M'\) we have

\[
M'(H(K[W(2)]); V) = \prod_{i=1}^d \frac{1}{1 - v_i^2},
\]

\[
M'(H(K[W(1^2)]); V) = \prod_{i=1}^{\lfloor d/2 \rfloor} \frac{1}{1 - v_{2i}},
\]

\[
M'(H(K[W(1) \oplus W(1^2)]); V) = \prod_{i=1}^d \frac{1}{1 - v_i}.
\]

**Example 2.2.** The multiplicity series of the Hilbert series of the symmetric algebra of the \(GL_2\)-module \(W(3)\) was computed (in a quite elaborate way) in [34]. The calculations may be simplified using the methods of [35]. Here we illustrate the advantages of Algorithm1.3 Since

\[
S_{(3)}(x_1, x_2) = x_1^3 + x_1^2 x_2 + x_1 x_2^2 + x_2^3,
\]
the Hilbert series of $K[W(3)]$ is

$$H(K[W(3)]; x_1, x_2) = \frac{1}{(1 - x_1^2)(1 - x_1^2x_2)(1 - x_1x_2^2)(1 - x_2^2)}.$$ 

We define the function

$$g(x_1, x_2) = (x_1 - x_2)H(K[W(3)]; x_1, x_2)$$ 

and decompose $g(x_1z, x_2/z)$ as a sum of partial fractions with respect to $z$. The result is

$$\frac{1}{3(1 - x_1^2x_2^2)(1 - x_1^2x_2^2)(1 - x_1z)} - \frac{1 - x_1z - x_1^2z^2 - 2x_1^2x_2z}{3(1 - x_1^2x_2^2)(1 + x_1^2x_2^2)(1 + x_1z + x_1^2z^2)}$$

$$- \frac{x_2}{(1 - x_1^2x_2^2)(1 - x_1^2x_2^2)(1 - x_1^2x_2z)} + \frac{3(1 - x_1^2x_2^2)(1 - x_1^2x_2^2)(x_2 - z)}{3(1 - x_1^2x_2^2)(1 + x_1^2x_2^2)(1 + x_1z + x_1^2z^2)}$$

$$+ \frac{x_2(-2z - x_2 - x_1^2x_2^2 + x_1^2x_2^2)}{3(1 - x_1^2x_2^2)(1 + x_1^2x_2^2)(x_2 + x_2z + z^2)} - \frac{x_2^3}{(1 - x_1^2x_2^2)(1 - x_1^2x_2^2)(x_1x_2^2 - z)}.$$ 

The factors containing $z$ in the denominators of the first three summands are $(1 - x_1z)$, $(1 + x_1z + x_1^2z^2)$, and $(1 - x_1^2x_2z)$. Hence these summands contribute to $\Omega(g(x_1z, x_2/z))$. We omit the other three summands because the corresponding factors are $(x_2 - z)$, $(x_2^2 + x_2z + z^2)$, and $(x_1x_2^2 - z)$. Replacing $z$ by 1 we obtain

$$\Omega(g(x_1z, x_2/z)) = \frac{1}{3(1 - x_1^2x_2^2)(1 - x_1^2x_2^2)(1 - x_1)}$$

$$- \frac{1 - x_1z - x_1^2z^2 - 2x_1^2x_2z}{3(1 - x_1^2x_2^2)(1 + x_1^2x_2^2)(1 + x_1z + x_1^2z^2)}$$

$$- \frac{x_2}{(1 - x_1^2x_2^2)(1 - x_1^2x_2^2)(1 - x_1^2x_2z)} + \frac{3(1 - x_1^2x_2^2)(1 - x_1^2x_2^2)(x_2 - z)}{3(1 - x_1^2x_2^2)(1 + x_1^2x_2^2)(1 + x_1z + x_1^2z^2)}$$

$$+ \frac{x_2(-2z - x_2 - x_1^2x_2^2 + x_1^2x_2^2)}{3(1 - x_1^2x_2^2)(1 + x_1^2x_2^2)(x_2 + x_2z + z^2)} - \frac{x_2^3}{(1 - x_1^2x_2^2)(1 - x_1^2x_2^2)(x_1x_2^2 - z)}.$$ 

Hence

$$M(H(K[W(3)]); x_1, x_2) = \frac{1 - x_1^2x_2 + x_1^2x_2^2}{(1 - x_1)(1 - x_1^2x_2)(1 - x_1^3x_2^2)}.$$ 

$$M'(H(K[W(3)]); v_1, v_2) = \frac{1 - v_1v_2 + v_1^2v_2^2}{(1 - v_1^3)(1 - v_1v_2)(1 - v_2^3)}.$$ 

We can rewrite the expression for $M'$ in the form

$$M'(H(K[W(3)]); v_1, v_2) = \frac{1}{2} \left( \frac{1 - v_1v_2 + v_1^2v_2^2}{1 - v_1^3} + \frac{1 + v_1v_2 + v_1^2v_2^2}{(1 - v_2^3)^2} \right) \frac{1}{1 - v_1}$$

$$+ \left( \frac{1 - v_1v_2}{1 - v_1} - \frac{1}{(1 - v_2^3)^2} \right) \frac{1}{1 - v_1v_2}$$

and to expand it as a power series in $v_1$ and $v_2$. The coefficient of $v_1^{a_1}v_2^{a_2}$ is equal to the multiplicity $m(a_1 + a_2, a_2)$ of the partition $\lambda = (a_1 + a_2, a_2)$. 
Example 2.3. In the same way we have computed the multiplicity series for the Hilbert series of all symmetric algebras $K[W]$ for $\dim(W) \leq 7$ and several cases for $\dim(W) = 8$. For example

\[
M'(H(K[W(4)])) = M'\left(\frac{1}{(1 - x_1^3)(1 - x_1^2 x_2)(1 - x_1^2 x_2)(1 - x_1^3 x_3)(1 - x_1^3 x_3)}\right)
= \frac{1 - v_1^2 v_2 + v_1^4 v_2^2}{(1 - v_1^3)(1 - v_1^2 v_2)(1 - v_2^3)(1 - v_2^6)};
\]

\[
M'(H(K[W(2) \oplus W(2)])) = M'\left(\frac{1}{(1 - x_1^2)^2(1 - x_1 x_2)^2(1 - x_2^2)^2}\right)
= \frac{1 + v_1^2 v_2}{(1 - v_1^2)^2(1 - v_2^6)^3};
\]

\[
M'(H(K[W(3) \oplus W(3)])) = \frac{p(v_1, v_2)}{(1 - v_1^2)^2(1 - v_1 v_2)^2(1 - v_2^6)^2(1 - v_2^6)^3},
\]

where

\[
p(v_1, v_2) = (1 - v_1^2 + v_2^6)((1 + v_1^2) v_2^6) - 2 v_1^2 v_2^2 (1 + v_2^6) - v_1 v_2 (1 - v_1^2)(2(1 - v_2^3) - v_1 v_2 (4 - v_2^3 - v_2^3) - v_1 (1 + v_2^6 - 4 v_2^6) + 2 v_1^2 v_2 (1 + v_2^6 - v_2^6)).
\]

The obtained decompositions can be easily verified using the equation

\[
f(x_1, x_2) = \frac{x_1 M'(f; x_1, x_1 x_2) - x_2 M'(f; x_2, x_1 x_2)}{x_1 - x_2}
\]

which for $d = 2$ is an $M'$-version of the equation in Remark 1.4.

Applying our Algorithm 1.3 we have obtained the following decompositions of the Hilbert series of the symmetric algebras of the irreducible $GL_3$-modules $W(\lambda)$, where $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ is a partition of $3$. Again, for the proof one can use Remark 1.4.

Theorem 2.4. Let $d = 3$ and let $v_1 = x_1$, $v_2 = x_1 x_2$, $v_3 = x_1 x_2 x_3$. The Hilbert series

\[
H(K[W(3)]; x_1, x_2, x_3) = \prod_{i \leq j \leq k} \frac{1}{1 - x_i x_j x_k},
\]

\[
H(K[W(2, 1)]; x_1, x_2, x_3) = \frac{1}{(1 - x_1 x_2 x_3)^3} \cdot \prod_{i \neq j} \frac{1}{1 - x_i^2 x_j};
\]

\[
H(K[W(1, 1, 1)]; x_1, x_2, x_3) = \frac{1}{1 - x_1 x_2 x_3}
\]

have the following multiplicity series

\[
M'(H(K[W(3)]); v_1, v_2, v_3) = \frac{1}{q(v_1, v_2, v_3)} \sum_{i=0}^{8} v_i^1 p_i(v_2, v_3),
\]

\[
M'(H(K[W(2, 1)]); v_1, v_2, v_3) = \frac{1 + v_1 v_2 v_3 + (v_1 v_2 v_3)^2}{(1 - v_1 v_2)(1 - v_1^2 v_2^2)(1 - v_2^3 v_3)(1 - v_2^6)(1 - v_2^6)},
\]

\[
M'(H(K[W(1, 1, 1)]); v_1, v_2, v_3) = \frac{1}{1 - v_3},
\]
where
\[
q = (1 - v_1^3)(1 - v_1v_2)(1 - v_1^3v_3)(1 - v_1^3v_3^2)(1 - v_2^6)(1 - v_2^3v_3)(1 - v_2^3v_3^2)
\times (1 - v_2^3)(1 - v_2^6),
\]
\[
p_0 = 1 + v_2^6v_3^6,
\]
\[
p_1 = -v_2(1 - v_2^3v_3^2(1 + v_3^3 + v_3^2) - v_2^2v_3^4(1 + v_3^2) + v_2^2v_3^3(1 + v_3)),
\]
\[
p_2 = v_2^2((1 + v_3^2 + v_3^4) - v_2^3v_3^5(1 - v_3) - v_2^6v_3^5(1 + v_3^2) + v_2^9v_3^5),
\]
\[
p_3 = -((1 - v_2^2v_3^3)v_3 + v_2^6 + v_2^6v_3^4(1 + v_2^2 + v_2^3))v_3^2,
\]
\[
p_4 = v_2v_3^2((1 + 2v_3 + v_3^3) - v_2^2v_3(v_3 + v_3^2)(1 + v_3^2)(1 + v_3 + v_3^2)
+ v_2^3v_3^4(1 + 2v_3^2 + v_3^3)),
\]
\[
p_5 = -v_2^2v_3^2(1 + v_3 + v_3^3 + v_3^7 - v_2^6v_3^3 + v_2^9v_3^5),
\]
\[
p_6 = (v_3 - v_2^3v_3^4(1 + v_3^2) + v_2^6(1 - v_3) + v_2^9v_3^4(1 + v_3^2) + v_2^9v_3^4)\v_3^5,
\]
\[
p_7 = -v_2^5v_3^5(1 + v_3 - v_2^3v_3^3(1 + v_3^3) - v_2^6v_3(1 + v_3 + v_3^4) + v_2^6v_3^5),
\]
\[
p_8 = v_2^5v_3^5(1 + v_2^3v_3^6).
\]

For a polynomial $GL_d$-module $W = \sum k(\mu)W(\mu)$, the symmetric algebra $K[W]$ has also another natural $\mathbb{Z}$-grading induced by the assumption that the elements of $W$ are of first degree. Then the homogeneous component $K[W](n)$ of degree $n$ is the symmetric tensor power $W^\otimes n$ and
\[
K[W] = \bigoplus_{n \geq 0} \bigoplus_{\lambda} m_n(\lambda)W(\lambda), \quad m_n(\lambda) \in \mathbb{Z}.
\]

In order to take into account both the $\mathbb{Z}^d$-grading induced by the $GL_d$-action and the natural $\mathbb{Z}$-grading of $K[W]$, we introduce an additional variable in the Hilbert series of $K[W]$: \[
H(K[W]; X, t) = \sum_{n \geq 0} H(K[W](n); X)t^n.
\]

If, as above, the Hilbert series of $W$ is
\[
H(W; X) = \sum_{\mu} k(\mu)S_{\mu}(X) = \sum_{j=1}^p X^\alpha_j,
\]
then
\[
H(K[W]; X, t) = \prod_{j=1}^p \frac{1}{1 - X^\alpha_j t} = \sum_{n \geq 0} \left( \sum_{\lambda} m_n(\lambda)S_{\lambda}(X) \right) t^n.
\]

Hence the multiplicity series
\[
M(H(K[W])); X, t) = \sum_{n \geq 0} \left( \sum_{\lambda} m_n(\lambda)X^\lambda \right) t^n
\]
carries the information about the multiplicities of the irreducible $GL_d$-submodules in the homogeneous components $K[W](n)$ of $K[W]$. A minor difference with the nongraded case is that we allow $|\mu| = 0$ in the decomposition of $W$: Since $W$ is finite dimensional, the homogeneous components of $K[W]$ are also finite dimensional and the Hilbert series $H(K[W]; X, t)$ is well defined even if $|\mu| = 0$ for some of the summands $W(\mu)$ of $W$. In the next section we shall see the role of this multiplicity series in invariant theory.
Example 2.5. Let $d = 2$ and let $W = W(3) \oplus W(2)$. Then the Hilbert series of $W$ is

$$H(W; x_1, x_2) = S_{(3)}(x_1, x_2) + S_{(2)}(x_1, x_2) = x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3 + x_1x_2 + x_2^2$$

and the Hilbert series of $K[W]$ which takes into account also the $\mathbb{Z}$-grading is

$$H(K[W]; x_1, x_2, t) = \prod_{a+b=3} \frac{1}{1-x_1^a x_2^b t} \prod_{a+b=2} \frac{1}{1-x_1^a x_2^b t}.$$

Applying Algorithm 1.3 we obtain that $M'(H(K[W]); x_1, x_2, t)$ is equal to

$$\sum_{k=0}^{10} p_k(v_1, v_2)t^k$$

where

- $p_0 = 1$, $p_1 = -v_1 v_2$, $p_2 = v_1 v_2(v_2 + v_1 v_2 + v_1^2)$, $p_3 = v_1 v_2^2(v_2 - v_1^3)$,
- $p_4 = v_1 v_2^5$, $p_5 = v_1 v_2^6(1 - v_1)(v_2 + v_1)$, $p_6 = -v_1^3 v_2^6$,
- $p_7 = v_1^3(v_2 - v_1^7)$, $p_8 = -v_1^3 v_2^7(v_2 + v_1 + v_1^7)$, $p_9 = v_1^3 v_2^8$, $p_{10} = -v_1^4 v_2^9$.

Again, we can check directly, without Algorithm 1.3 that the obtained rational function is the multiplicity series of $H(K[W]; x_1, x_2, t)$ using Remark 1.3.

We complete this section with the $\mathbb{Z}$-graded version for the multiplicities of the symmetric algebra of $W = W(1) \oplus W(1^2)$ for arbitrary positive integer $d$.

Proposition 2.6. Let $d$ be any positive integer. The homogeneous component of degree $n$ of the symmetric algebra $K[W(1) \oplus W(1^2)]$ of the $GL_d$-module $W = W(1) \oplus W(1^2)$ decomposes as

$$K[W(1) \oplus W(1^2)](n) = \bigoplus K(\lambda),$$

where the summation runs on all $\lambda = (\lambda_1, \ldots, \lambda_d)$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_{2(d-1)/2} + 1 = n$. Equivalently, the multiplicity series of $H(K[W(1) \oplus W(1^2)]); X, t)$ is

$$M'(H(K[W(1) \oplus W(1^2)]); V, t) = \prod_{2i \leq d} \frac{1}{(1 - v_{2i-1} t)(1 - v_{2i} t)},$$

when $d$ is even and

$$M'(H(K[W(1) \oplus W(1^2)]); V, t) = \frac{1}{1 - v_{d+1/2}(d+1/2)} \prod_{2i \leq d} \frac{1}{(1 - v_{2i-1} t)(1 - v_{2i} t)}$$

when $d$ is odd.

Proof. Since, as $GL_d$-module, $K[W(1) \oplus W(1^2)] = K[W(1)] \otimes K[W(1^2)]$, we obtain that

$$H(K[W(1) \oplus W(1^2)]; X, t) = H(K[W(1)]; X, t)H(K[W(1^2)]; X, t)$$

$$= \prod_{i=1}^{d} \frac{1}{1-x_i t} \prod_{i<j} \frac{1}{1-x_i x_j t}.$$
because the elements of $W(1), W(1^2) \subset W$ are of first degree. The decompositions

\[
\prod_{i=1}^{d} \frac{1}{1-x_i} = \sum_{k \geq 0} S_{(k)}(X),
\]

\[
\prod_{i<j} \frac{1}{1-x_i x_j} = \sum_{(\lambda_2, \lambda_4, \ldots)} S_{(\lambda_2, \lambda_2, \lambda_4, \lambda_4, \ldots)}(X)
\]

give

\[
\prod_{i=1}^{d} \frac{1}{1-x_i t^i} = \sum_{m \geq 0} S_{(m)}(X)t^m,
\]

\[
\prod_{i<j} \frac{1}{1-x_i x_j t^{i+j}} = \sum_{(\lambda_2, \lambda_4, \ldots)} S_{(\lambda_2, \lambda_2, \lambda_4, \lambda_4, \ldots)}(X)t^{\lambda_2+\lambda_4+\cdots}
\]

and hence

\[
H(K[W(1) \oplus W(1^2)]; X, t) = \sum_{m \geq 0} \sum_{(\lambda_2, \lambda_4, \ldots)} S_{(m)}(X)S_{(\lambda_2, \lambda_2, \lambda_4, \lambda_4, \ldots)}(X)t^{m+\lambda_2+\lambda_4+\cdots}.
\]

The product of the Schur functions $S_{(m)}(X)$ and $S_{\mu}(X)$ can be decomposed by the Young rule which is a partial case of the Littlewood–Richardson rule:

\[
S_{(m)}(X)S_{\mu}(X) = \sum S_{\nu}(X),
\]

where the summation runs on all partitions $\nu \vdash m+|\mu|$ such that

\[
\nu_1 \geq \mu_1 \geq \nu_2 \geq \mu_2 \geq \cdots \geq \nu_d \geq \mu_d.
\]

Applied to our case this gives

\[
S_{(m)}(X)S_{(\lambda_2, \lambda_2, \lambda_4, \lambda_4, \ldots)}(X) = \sum S_{(\lambda_1, \lambda_2, \ldots, \lambda_d)}(X),
\]

where the sum is on all partitions with

\[
(\lambda_1 - \lambda_2) + (\lambda_3 - \lambda_4) + \cdots + (-1)^{d-1}\lambda_d = m.
\]

Hence the $GL_d$-module $W(\lambda) \subset K[W]$ is a submodule of the homogeneous component of degree

\[
m + \lambda_2 + \lambda_4 + \cdots = \lambda_1 + \lambda_3 + \cdots + \lambda_{2\lfloor(d-1)/2\rfloor+1}.
\]
One of the main objects in invariant theory in the 19th century is the algebra of homogeneous polynomials of degree $m$ allowed him to compute the first coefficients (of the terms of degree $\leq 4$) of Sylvester–Cayley and Springer formulas for invariants also of ternary forms. This was calculated by Shioda [64]. Recently Bedratyuk [7, 8] found analogues.

In 1980 Springer [65] found an explicit formula for the Hilbert series of the algebra of invariants in the case of forms of degree 3 were found by Gordan [48], see also Clebsch and Gordan [22]; the case of forms of degree 4 has handled by Emmy Noether [58]. The Hilbert series of the algebra of $SL_2$-invariants for forms of degree 4 was calculated by Shioda [64]. Recently Bedratyuk [7, 8] found analogues of Sylvester–Cayley and Springer formulas for invariants also of ternary forms. This allowed him to compute the first coefficients (of the terms of degree $\leq 30$) of the Hilbert series of the algebras of $SL_3(\mathbb{C})$-invariants of forms of degree $m \leq 7$.

For the statement for the multiplicity series, first let $d = 2k + 1$. Then

$$M(H(K[W]); X, t) = \sum_{\lambda_1, \ldots, \lambda_{2k+1} \geq \lambda_{2k+1}} x_1^{\lambda_1} \cdots x_{2d+1}^{\lambda_{2k+1}} t^{\lambda_1 + \lambda_2 + \cdots + \lambda_{2k+1}}$$

$$= \sum_{\lambda_1, \ldots, \lambda_{2k+1} \geq \lambda_{2k+1}} (x_1 t)^{\lambda_1} x_2^{\lambda_2} (x_3 t)^{\lambda_3} \cdots x_{2d+1}^{\lambda_{2k+1}} (x_{2d+1} t)^{\lambda_{2k+1}}$$

$$= \sum_{\lambda_1, \ldots, \lambda_{2k+1} \geq \lambda_{2k+1}} (x_1 t)^{\lambda_1 - \lambda_2} (x_1 x_2 t)^{\lambda_2 - \lambda_3} (x_1 x_2 x_3 t^2)^{\lambda_3 - \lambda_4} \cdots (x_1 \cdots x_{2k-1} t^k)^{\lambda_{2k-1} - \lambda_{2k}}$$

$$\times (x_1 \cdots x_2 x_3 x_4 t^2)^{\lambda_{2k} - \lambda_{2k+1}} (x_1 \cdots x_{2k+1} t^{k+1})^{\lambda_{2k+1}}$$

$$M'(H(K[W]); X, t) = \sum_{n_i \geq 0} (v_1 t)^{n_1} (v_2 t)^{n_2} (v_3 t^2)^{n_3} (v_4 t^2)^{n_4} \cdots \times (v_{2k-1} t^k)^{n_{2k-1}} (v_{2k} t^k)^{n_{2k}} (v_{2k+1} t^{k+1})^{n_{2k+1}}$$

$$= \frac{1}{1 - v_{2k+1} t^{k+1}} \prod_{i=1}^{k} \frac{1}{1 - v_{2i-1} t^i (1 - v_{2i} t^i)}.$$
one may use the Molien–Weyl integral formula, evaluating multiple integrals. This type of integrals can be evaluated using the Residue Theorem, see the book by Derksen and Kemper [24] for details. For concrete decompositions of $W$, the algebra of invariants $K[W]|^\SL_d$ was studied already by Sylvester. Its Hilbert series is also known in some cases. For example, recently Bedratyuk [9] has found a formula for the Hilbert series of the $\SL_d$-invariants $K[W_{m_1, 2} \oplus W_{m_2, 2}]|^\SL_d$ and has computed these series for $m_1, m_2 \leq 20$. (The results for $m_1, m_2 \leq 5$ are given explicitly in [9].) Very recently, Bedratyuk and Xin [11] applied the MacMahon partition analysis to the Molien–Weyl integral formula and computed the Hilbert series of the algebras of invariants of some ternary and quaternary forms.

Our approach to the Hilbert series of the algebra of invariants $K[W]^\SL_d$ of the $\SL_d$-module $W$ is based on a theorem of De Concini, Eisenbud and Procesi [23]. This theorem implies that the multiplicities of $S_{\lambda}(X)$ in the Hilbert series of symmetric algebras $K[W]$ of a $GL_d$-module $W$ appear in invariant theory of $\SL_d$ and of the unitriangular group $UT_d = UT_d(K)$ as subgroups of $GL_d$, and in invariant theory of a single unitriangular matrix. It is combined with an idea used by Drensky and Genov [35] to compute the Hilbert series of the algebra of invariants of $UT_2$. We extend the $SL_d$-action on $W$ to a polynomial action of $GL_d$. This is possible in the cases that we consider because the $SL_d$-module $W_{m,d}$ of the forms of degree $m$ can be viewed as a $GL_d$-module isomorphic to $W(m)$. Then we compute the Hilbert series of the $GL_d$-module $K[W]$ and its multiplicity series $M'(H(K[W]); V, t)$. The Hilbert series of $K[W]^\SL_d$ is equal to $M'(H(K[W]); 0, \ldots, 0, 1, t)$. Similarly, if $W$ is a polynomial $GL_d$-module, then the Hilbert series of $K[W]|^\UT_d$ is equal to $M'(H(K[W]); 1, \ldots, 1, t)$. The difference with [35] is that there we use for the evaluation of $M(H(K[W]); x_1, x_2, t)$ the methods developed in [35] and here we use the MacMahon partition analysis for the same purpose and for any number of variables. We shall consider the following problem. Let $W$ be an arbitrary polynomial $GL_d$-module. How can one calculate the Hilbert series of the algebras of invariants $K[W]^\SL_d$ and $K[W]|^\UT_d$? Clearly, here we assume that $SL_d$ and $UT_d$ are canonically embedded into $GL_d$. We need the following easy argument. We state it as a lemma and omit the obvious proof.

**Lemma 3.1.** Let $H$ be a subgroup of the group $G$ and let $W_1, W_2$ be $G$-modules. Then the vector space of invariants $W^H \subseteq W$ in $W = W_1 \oplus W_2$ satisfy

$$W^H = W_1^H \oplus W_2^H.$$ 

**Theorem 3.2.** Let $W$ be a polynomial $GL_d$-module with Hilbert series with respect to the grading induced by the $GL_d$-action on $W$

$$H(W; X) = \sum a_i x_1^{a_1} \cdots x_d^{a_d}, \quad a_i \geq 0, a_i \in \mathbb{Z},$$

and let

$$H(K[W]; X, t) = \prod \frac{1}{(1 - X^a)^{a_i}}$$

be the Hilbert series of $K[W]$ which counts also the natural $\mathbb{Z}$-grading. Then the Hilbert series of the algebras of invariants $K[W]|^\SL_d$ and $K[W]|^\UT_d$ are given by

$$H(K[W]|^\SL_d, t) = M'(H(K[W]); 0, \ldots, 0, 1, t),$$

$$H(K[W]|^\UT_d, t) = M(H(K[W]); 1, \ldots, 1, t)$$

$$= M'(H(K[W]); 1, \ldots, 1, t).$$
Proof. Let
\[ K[W] = \bigoplus_{n \geq 0} \bigoplus_{\lambda} m_n(\lambda)W(\lambda) \]
be the decomposition of the \( \mathbb{Z} \)-graded \( GL_d \)-module \( K[W] \). Its Hilbert series is
\[ H(K[W]; X, t) = \sum_{n \geq 0} \left( \sum_{\lambda} m_n(\lambda)S_\lambda(X) \right) t^n \]
and the multiplicity series of \( H(K[W]; X, t) \) are
\[ M(H(K[W]); X, t) = \sum_{n \geq 0} \left( \sum_{\lambda} m_n(\lambda)X^\lambda \right) t^n, \]
\[ M'(H(K[W]); V, t) = \sum_{n \geq 0} \left( \sum_{\lambda} m_n(\lambda)v_1^{\lambda_1-\lambda_2} \cdots v_{d-1}^{\lambda_{d-1}-\lambda_d} v_d^{\lambda_d} \right) t^n. \]
It is a well known fact that the irreducible \( GL_d \)-module \( W(\lambda) = W(\lambda_1, \ldots, \lambda_d) \) contains a one-dimensional \( SL_d \)-invariant subspace if \( \lambda_1 = \cdots = \lambda_d \) (when \( \dim(W(\lambda)) = 1 \) and \( W(\lambda)^{SL_d} = W(\lambda) \)) and contains no invariants if \( \lambda_j \neq \lambda_{j+1} \) for some \( j \). Applying Lemma 3.1 we immediately obtain
\[ K[W]^{SL_d} = \bigoplus_{n \geq 0} \bigoplus_{\lambda_1=\cdots=\lambda_d} m_n(\lambda)W(\lambda), \]
\[ H(K[W]^{SL_d}; t) = \sum_{n \geq 0} \left( \sum_{\lambda_1=\cdots=\lambda_d} m_n(\lambda) \right) t^n. \]
Evaluating the monomials in the expansion of \( M'(H(K[W]); V, t) \) for \( v_1 = \cdots = v_{d-1} = 0, v_d = 1 \) we obtain
\[ v_1^{\lambda_1-\lambda_2} \cdots v_{d-1}^{\lambda_{d-1}-\lambda_d} v_d^{\lambda_d} |_{V=(0,\ldots,0,1)} = \begin{cases} 1, & \text{if } \lambda_1 = \cdots = \lambda_d, \\ 0, & \text{if } \lambda_j \neq \lambda_{j+1} \text{ for some } j \end{cases} \]
which completes the case of \( SL_d \)-invariants.

It is also well known that every irreducible \( GL_d \)-module \( W(\lambda) \) has a one-dimensional \( UT_d \)-invariant subspace which is spanned on the only (up to a multiplicative constant) element \( w \in W(\lambda) \) with the property that the diagonal subgroup \( D_d \) of \( GL_d \) acts by
\[ g(w) = \xi_1^{\lambda_1} \cdots \xi_d^{\lambda_d} w, \quad g = \text{diag}(\xi_1, \ldots, \xi_d). \]
Hence
\[ K[W]^{UT_d} = \bigoplus_{n \geq 0} \bigoplus_{\lambda} m_n(\lambda)W(\lambda)^{UT_d}, \]
\[ H(K[W]^{UT_d}; t) = \sum_{n \geq 0} \left( \sum_{\lambda} m_n(\lambda) \right) t^n \]
\[ = M(H(K[W]); 1, \ldots, 1, t) = M'(H(K[W]); 1, \ldots, 1, t). \]
\[ \square \]

Below we shall illustrate Theorem 3.2 on the Hilbert series of the \( SL_2 \)-invariants for the \( GL_2 \)-modules considered in the examples of Section 2.
Example 3.3. If the polynomial $GL_d$-module $W$ is homogeneous of degree $m$, i.e., $g(w) = \xi^m w$ for $w \in W$ and $g = \text{diag}(\xi, \ldots, \xi) \in GL_d$, then

\[
H(K[W]; x_1, \ldots, x_d, t) = H(K[W]; X, t)
\]

\[
= H(K[W]; X \sqrt{t}) = H(K[W]; x_1 \sqrt{t}, \ldots, x_d \sqrt{t})
\]

because the elements of $W$ are of degree 1 with respect to the $\mathbb{Z}$-grading and of degree $m$ with respect to the $\mathbb{Z}^d$-grading. The results of Example 2.1 give

\[
M'(H(K[W](2)); V) = \prod_{i=1}^d \frac{1}{1 - v_i^2},
\]

\[
M'(H(K[W](2)); V, t) = \prod_{i=1}^d \frac{1}{1 - v_i^2 t^2},
\]

\[
H(K[W](2))^{SL_d}; t) = M'(H(K[W](2)); 0, \ldots, 0, 1, t) = \frac{1}{1 - t^d},
\]

\[
H(K[W](2))^{UT_d}; t) = M'(H(K[W](2)); 1, \ldots, 1, t) = \prod_{i=1}^d \frac{1}{1 - t^2};
\]

\[
M'(H(K[W](1^2))); V) = \prod_{i=1}^{\lfloor d/2 \rfloor} \frac{1}{1 - v_{2i}},
\]

\[
M'(H(K[W](1^2))); V, t) = \prod_{i=1}^{\lfloor d/2 \rfloor} \frac{1}{1 - v_{2i} t^2},
\]

\[
H(K[W](1^2))^{SL_d}; t) = \begin{cases} 
\frac{1}{1 - t^{d/2}}, & \text{if } d \text{ is even}, \\
0, & \text{if } d \text{ is odd},
\end{cases}
\]

\[
H(K[W](1^2))^{UT_d}; t) = \prod_{i=1}^{\lfloor d/2 \rfloor} \frac{1}{1 - t^i}.
\]
Similarly, for $d = 2$ Examples 2.2 and 2.3 give

\[
M'(H(K[W(3)]); v_1, v_2) = \frac{1 - v_1 v_2 + v_1^2 v_2^2}{(1 - v_1^3)(1 - v_1 v_2)(1 - v_2^3)},
\]

\[
M'(H(K[W(3)]); v_1, v_2, t) = M'(H(K[W(3)]); v_1 \sqrt{t}, v_2 \sqrt{t^2}) = \frac{1 - v_1 v_2 t + v_1^2 v_2^2 t^2}{(1 - v_1^3 t)(1 - v_1 v_2 t)(1 - v_2^3 t^4)},
\]

\[
H(K[W(3)])^{SL_2; t} = M'(H(K[W(3)]); 0, 1, t) = \frac{1}{1 - t^4},
\]

\[
H(K[W(3)])^{UT_2; t} = M'(H(K[W(3)]); 1, 1, t) = \frac{1 - t + t^2}{(1 - t)^2(1 - t^4)};
\]

\[
M'(H(K[W(4)]); v_1, v_2, t) = M'(H(K[W(4)]); v_1 \sqrt{t}, v_2 \sqrt{t}),
\]

\[
H(K[W(4)])^{SL_2; t} = M'(H(K[W(4)]); 0, 1, t) = \frac{1}{(1 - t^2)(1 - t^4)},
\]

\[
H(K[W(4)])^{UT_2; t} = M'(H(K[W(4)]); 1, 1, t) = \frac{1 - t + t^2}{(1 - t)^2(1 - t^2)(1 - t^4)};
\]

\[
M'(H(K[W(2) \oplus W(2)]); v_1, v_2, t) = M'(H(K[W(2) \oplus W(2)]); v_1 \sqrt{t}, v_2 \sqrt{t}),
\]

\[
H(K[W(2) \oplus W(2)])^{SL_2; t} = \frac{1}{(1 - t^2)^3},
\]

\[
H(K[W(2) \oplus W(2)])^{UT_2; t} = \frac{1 + t^2}{(1 - t)^2(1 - t^2)^3};
\]

\[
M'(H(K[W(3) \oplus W(3)]); v_1, v_2, t) = M'(H(K[W(3) \oplus W(3)]); v_1 \sqrt{t}, v_2 \sqrt{t^2}),
\]

\[
H(K[W(3) \oplus W(3)])^{SL_2; t} = \frac{1}{(1 - t^2)^3},
\]

\[
H(K[W(3) \oplus W(3)])^{UT_2; t} = \frac{1 + t^{10} + 3t^2(1 + t^6) + 6t^3(1 + t + t^2 + t^3 + t^4)}{(1 - t)^2(1 - t^2)^3(1 - t^4)^2};
\]

Finally, for $d = 3$ Theorem 2.4 gives that

\[
M'(H(K[W(3)]); v_1, v_2, v_3, t) = M'(H(K[W(3)]); v_1 \sqrt{t}, v_2 \sqrt{t^2}, v_3 t),
\]

\[
(H(K[W(3)])^{SL_3; t} = \frac{1}{(1 - t^3)(1 - t^6)},
\]

\[
(H(K[W(3)])^{UT_3; t} = \frac{1 + t^3(1 + t^3) + 2t^4(1 + t^4) + 3t^5(1 + t + t^2)}{(1 - t)(1 - t^2)(1 - t^3)^2(1 - t^4)^2},
\]

and similarly

\[
H(K[W(2, 1)])^{SL_3; t} = \frac{1}{(1 - t^2)(1 - t^3)},
\]

\[
H(K[W(2, 1)])^{UT_3; t} = \frac{1 - t + t^2}{(1 - t)^2(1 - t^2)(1 - t^3)^2};
\]

\[
H(K[W(1^3)])^{SL_3; t} = H(K[W(1^3)])^{UT_3; t} = \frac{1}{1 - t};
\]
Example 3.4. The translation of Example 2.5 to the language of $SL_2$- and $UT_2$-invariants gives

$$H(K[W(3) \oplus W(2)]^{SL_2}; t) = \frac{1 + t^9}{(1 - t^2)(1 - t^3)(1 - t^4)(1 - t^5)},$$

$$H(K[W(3) \oplus W(2)]^{UT_2}; t) = \frac{(1 - t)(1 - t^7) + 4t^2(1 + t^4) - t^3(1 + t^2) + 5t^4}{(1 - t)^3(1 - t^4)(1 - t^5)}.$$ 

For an arbitrary $d$ Proposition 2.6 gives

$$H(K[W(1) \oplus W(1^2)]^{SL_d}; t) = \begin{cases} 1 - t^k, & \text{if } d = 2k, \\ 1 - t^{k+1}, & \text{if } d = 2k + 1, \\ \prod_{i=1}^{k} \frac{1}{(1 - t)^2}, & \text{if } d = 2k, \\ \frac{1}{1 - t^{k+1}} \prod_{i=1}^{k} \frac{1}{(1 - t)^2}, & \text{if } d = 2k + 1. \end{cases}$$

$$H(K[W(1) \oplus W(1^2)]^{UT_d}; t) = \begin{cases} 1 - t^k, & \text{if } d = 2k, \\ 1 - t^{k+1}, & \text{if } d = 2k + 1, \\ \prod_{i=1}^{k} \frac{1}{(1 - t)^2}, & \text{if } d = 2k, \\ \frac{1}{1 - t^{k+1}} \prod_{i=1}^{k} \frac{1}{(1 - t)^2}, & \text{if } d = 2k + 1. \end{cases}$$

In the above examples our results coincide with the known ones, see e.g., [68, 25, 26, 9, 7].

The invariant theory of $UT_2$ may be restated in the language of linear locally nilpotent derivations. Recall that a derivation of a (not necessarily commutative or associative) algebra $R$ is a linear operator $\delta$ with the property that

$$\delta(u_1 u_2) = \delta(u_1) u_2 + u_1 \delta(u_2), \quad u_1, u_2 \in R.$$ 

The derivation $\delta$ is locally nilpotent if for any $u \in R$ there exists a $p$ such that $\delta^p(u) = 0$. Locally nilpotent derivations are interesting objects with relations to invariant theory, the 14th Hilbert problem, automorphisms of polynomial algebras, the Jacobian conjecture, etc., see the monographs by Nowicki [59], van den Essen [41], and Freudenburg [46]. Linear locally nilpotent derivations $\delta$ of the polynomial algebra $K[Y]$ (acting as linear operators on the vector space $KY$ with basis $Y = \{y_1, \ldots, y_d\}$) were studied by Weitzenböck [75] who proved that the algebra of constants $K[Y]^\delta$, i.e., the kernel of $\delta$, is finitely generated. Nowadays linear locally nilpotent derivations of $K[Y]$ are known as Weitzenböck derivations and are subjects of intensive study.

The algebra $K[Y]^\delta$ coincides with the algebra $K[Y]^G$ of invariants of the cyclic group $G$ generated by

$$\exp(\delta) = 1 + \frac{\delta}{1!} + \frac{\delta^2}{2!} + \cdots$$

and with the algebra of invariants of the additive group $K_\alpha$ of the field $K$ with its $d$-dimensional representation

$$\alpha \to \exp(\alpha \delta), \quad \alpha \in K_\alpha,$$

which allows to involve invariant theory. Historically, it seems that this relation was used quite rarely and some of the results on Weitzenböck derivations rediscover classical results in invariant theory. For example, the modern proof of the theorem of Weitzenböck given by Seshadri [63] is equivalent to the results of Roberts [62].
that for an $SL_2$-module $W$ the algebra $K[W]^{UT_2}$ is isomorphic to the algebra of covariants. There is an elementary version of the proof by Seshadri given by Tyc [72], which is in the language of representations of the Lie algebra $sl_2(K)$ and can be followed without serious algebraic knowledge. Let $\delta$ be a Weitzenböck derivation. All eigenvalues of $\delta$ (acting on $KY$) are equal to 0 and, up to a linear change of the coordinates of $K[Y]$, $\delta$ is determined by its Jordan normal form. Hence, for each fixed dimension $d$ there is only a finite number of Weitzenböck derivations. The only derivation which corresponds to a single Jordan cell is called basic. Onoda [60] presented an algorithm which calculates the Hilbert series in the case of a basic Weitzenböck derivation. He calculated the Hilbert series for the basic derivation $\delta$ and $d = 6$ and, as a consequence showed that the algebra of constants $\mathbb{C}[Y]^\delta$ is not a complete intersection. (By the same paper [60], $\mathbb{C}[Y]^\delta$ is Gorenstein for any Weitzenböck derivation $\delta$ which agrees with a general fact in invariant theory of classical groups.) Other methods to compute the Hilbert series of $K[W]^\delta$ for any Weitzenböck derivation $\delta$ are developed by Bedratyuk; see [10] and the references there. Below we show how the MacMahon partition analysis can be used to compute the Hilbert series of $K[W]^\delta$. The following theorem and its corollary were announced in [35].

**Theorem 3.5.** Let $\delta$ be a Weitzenböck derivation of $K[Y]$ with Jordan normal form consisting of $k$ cells of sizes $d_1 + 1, \ldots, d_k + 1$, respectively. Let

$$f_\delta(x_1, x_2, t) = \frac{1}{q_{d_1}(x_1, x_2, t)\cdots q_{d_k}(x_1, x_2, t)},$$

where

$$q_d(x_1, x_2, t) = (1 - x_1^d)(1 - x_1^{d-1}x_2t)\cdots(1 - x_1x_2^{d-1}t)(1 - x_2^d).$$

Then the Hilbert series of the algebra of constants $K[Y]^\delta$ is given by

$$H(K[Y]^\delta; t) = M(f_\delta; 1, 1),$$

where $M(f_\delta; x_1, x_2)$ is the multiplicity series of the symmetric with respect to $x_1, x_2$ function $f_\delta(x_1, x_2, t)$ in $K(t)[[x_1, x_2]]^{S_2}$.

**Proof.** If $\delta$ has $k$ Jordan cells and the $i$th cell is of size $d_i + 1$, $i = 1, \ldots, k$, we identify the vector space $KY$ with the $GL_2$-module

$$W = W(d_1) \oplus \cdots \oplus W(d_k)$$

and the algebra $K[Y]$ with the symmetric algebra $K[W]$. Then the algebra of constants $K[Y]^\delta$ coincides with the algebra $K[W]^{UT_2}$ of $UT_2$-invariants. Obviously, the function $f_\delta(x_1, x_2, t)$ is equal to the Hilbert series of the $\mathbb{Z}$-graded $GL_2$-module $K[W]$. Hence Theorem 3.2 completes the proof. \hfill \Box

**Example 3.6.** Let $\delta = \delta(d_1, \ldots, d_k)$ be the Weitzenböck derivation with $k$ Jordan cells of size $d_i + 1$, $i = 1, \ldots, k$, respectively. If the matrix of $\delta$ contains a Jordan cell of size 1 corresponding to $x_d$, then

$$K[x_1, \ldots, x_{d-1}, x_d]^\delta = K[x_1, \ldots, x_{d-1}]^\delta[x_d]$$

and the Hilbert series of the algebras of constants $K[x_1, \ldots, x_d]^\delta$ and $K[x_1, \ldots, x_{d-1}]^\delta$ are related by

$$H(K[x_1, \ldots, x_d]^\delta; t) = \frac{1}{1 - t}H(K[x_1, \ldots, x_{d-1}]^\delta; t).$$
Hence it is sufficient to consider only $\delta$ with Jordan matrices without 1-cells. Below we extend the results in Examples 3.3 and 3.4 and give the Hilbert series of $K[Y]^\delta$ for all possible $\delta$ with $d \leq 7$. Originally the computations were performed in [35] illustrating the methods developed there for symmetric functions in two variables. Here we repeated the computations with the methods of the MacMahon partition analysis. Clearly, the results coincide with those from [10].

$d = 2$:

$$H(K[Y]^\delta(1); t) = \frac{1}{1 - t};$$

$d = 3$:

$$H(K[Y]^\delta(2); t) = \frac{1}{(1 - t)(1 - t^2)};$$

$d = 4$:

$$H(K[Y]^\delta(3); t) = \frac{1 - t + t^2}{(1 - t)^2(1 - t^2)} = \frac{1 + t^3}{(1 - t)(1 - t^2)(1 - t^3)},$$

$$H(K[Y]^\delta(1,1); t) = \frac{1}{(1 - z)^2(1 - z^2)};$$

$d = 5$:

$$H(K[Y]^\delta(4); t) = \frac{1 - t + t^2}{(1 - t)^2(1 - t^2)(1 - t^3)} = \frac{1 + t^3}{(1 - t)(1 - t^2)(1 - t^3)},$$

$$H(K[Y]^\delta(2,1); t) = \frac{1}{(1 - z)^2(1 - z^2)(1 - z^3)};$$

$$p(z) = z^2 + 3z^3 + 3z^4 + 5z^5 + 4z^6 + 6z^7 + 6z^8 + 4z^9 + 5z^{10} + 3z^{11} + 3z^{12} + z^{13} + z^{15},$$

$$H(K[Y]^\delta(3,1); t) = \frac{1 + z^2 + 3z^3 + z^4 + z^6}{(1 - z)^2(1 - z^2)(1 - z^4)},$$

$$H(K[Y]^\delta(2,2); t) = \frac{1 + z^2}{(1 - z)^2(1 - z^2)^3};$$

$$H(K[Y]^\delta(1,1,1); t) = \frac{1 - z^3}{(1 - z)^3(1 - z^2)^3} = \frac{1 + z + z^2}{(1 - z)^2(1 - z^2)^3};$$

$d = 6$:

$$H(K[Y]^\delta(5); t) = \frac{p(z)}{(1 - z)(1 - z^2)(1 - z^4)(1 - z^6)(1 - z^8)},$$

$$p(z) = 1 + z^2 + 3z^3 + 3z^4 + 5z^5 + 4z^6 + 6z^7 + 6z^8 + 4z^9 + 5z^{10} + 3z^{11} + 3z^{12} + z^{13} + z^{15},$$

$$H(K[Y]^\delta(3,1); t) = \frac{1 + z^2 + 3z^3 + z^4 + z^6}{(1 - z)^2(1 - z^2)(1 - z^4)^2},$$

$$H(K[Y]^\delta(2,2); t) = \frac{1 + z^2}{(1 - z)^2(1 - z^2)^3};$$

$$H(K[Y]^\delta(1,1,1); t) = \frac{1 - z^3}{(1 - z)^3(1 - z^2)^3} = \frac{1 + z + z^2}{(1 - z)^2(1 - z^2)^3};$$

$d = 7$:

$$H(K[Y]^\delta(6); t) = \frac{1 + z^2 + 3z^3 + 4z^4 + 4z^5 + 4z^6 + 3z^7 + z^8 + z^{10}}{(1 - z)(1 - z^2)^2(1 - z^3)(1 - z^4)(1 - z^5)},$$

$$H(K[Y]^\delta(4,1); t) = \frac{1 + 2z^2 + 2z^3 + 4z^4 + 4z^5 + 2z^6 + z^8}{(1 - z)^2(1 - z^2)(1 - z^3)^2(1 - z^5)},$$

$$H(K[Y]^\delta(3,2); t) = \frac{1 - z + 4z^2 - z^3 + 5z^4 - z^5 + 4z^6 - z^7 + z^8}{(1 - z)^3(1 - z^3)(1 - z^4)(1 - z^5)} = \frac{1 + 3z^2 + 3z^3 + 4z^4 + 4z^5 + 3z^6 + 3z^7 + z^9}{(1 - z)^2(1 - z^2)(1 - z^3)(1 - z^4)(1 - z^5)},$$

$$H(K[Y]^\delta(2,1,1); t) = \frac{1 + 3z^2 + z^4}{(1 - z)^3(1 - z^2)(1 - z^3)^2}. $$
Corollary 3.7. For \( d \leq 7 \) the algebra of constants \( K[Y]^d \) of the Weitzenböck derivation \( \delta = \delta(d_1, \ldots, d_k) \) is not a complete intersection for

\[
(d_1, \ldots, d_k) = (5), (3, 1), (6), (4, 1), (3, 2), (2, 1, 1).
\]

Proof. Using, as in [60], that the zeros of the nominator of the Hilbert series of a complete intersection are roots of unity (see [67]), the proof follows immediately from Example 3.6. (The case \((d_1, \ldots, d_k) = (5)\) was established in [60].) \( \square \)

4. PI-ALGEBRAS AND NONCOMMUTATIVE INVARIANT THEORY

In this section we assume that all algebras are unital (and \( \text{char}(K) = 0 \)). For a background on PI-algebras we refer, e.g., to [31]. Let \( Y_{\infty} = \{y_1, y_2, \ldots\} \) and let \( K(Y_{\infty}) \) be the free associative algebra of countable rank freely generated by \( Y \). This is the algebra of polynomials in infinitely many noncommutative variables. Let \( K\langle Y \rangle = K\langle y_1, \ldots, y_d \rangle \) be its subalgebra of rank \( d \). Recall that \( f(y_1, \ldots, y_m) \in K\langle Y_{\infty} \rangle \) is called a polynomial identity for the associative algebra \( R \) if \( f(r_1, \ldots, r_m) = 0 \) for all \( r_1, \ldots, r_m \in R \). If \( R \) satisfies a nonzero polynomial identity, it is called a PI-algebra. We denote by \( T_{\infty}(R) \) the ideal of all polynomial identities of \( R \) (called the \( T \)-ideal of \( R \)) and

\[
T(R) = K\langle Y \rangle \cap T_{\infty}(R)
\]

is the \( T \)-ideal of the polynomial identities in \( d \) variables for \( R \). Since we work over a field of characteristic 0, all polynomial identities of \( R \) follow from the multilinear ones. The vector space of the multilinear polynomials of degree \( n \)

\[
P_n = \text{span}\{y_{\sigma(1)} \cdots y_{\sigma(n)} \mid \sigma \in S_n\} \subset K\langle Y_{\infty} \rangle
\]

has a natural structure of a left \( S_n \)-module and the factor space

\[
P_n(R) = P_n/(P_n \cap T_{\infty}(R))
\]

is its \( S_n \)-module factor. One of the main problems in the quantitative study of PI-algebras is to compute the cocharacter sequence of \( R \)

\[
\chi_n(R) = \chi_{S_n}(P_n(R)) = \sum_{\lambda \vdash n} m_{\lambda}(R) \chi_{\lambda},
\]

where \( \chi_{\lambda}, \lambda \vdash n \), is the irreducible \( S_n \)-character indexed by the partition \( \lambda \). A possible way to compute the multiplicities \( m_{\lambda}(R) \) is the following. One considers the diagonal \( GL_d \)-action on \( K\langle Y \rangle \) extending the natural action of \( GL_d \) on the \( d \)-dimensional vector space \( KY \) with basis \( Y \). Then the factor algebra

\[
F(R) = K\langle Y \rangle / T(R)
\]

called the relatively free algebra of rank \( d \) in the variety of associative algebras generated by \( R \), inherits the \( GL_d \)-action of \( K\langle Y \rangle \). Its Hilbert series as a \( GL_d \)-module coincides with its Hilbert series as a \( \mathbb{Z}^d \)-graded vector space with grading defined by

\[
\text{deg}(y_i) = (0, \ldots, 0, 1, 0, \ldots, 0).
\]

It is a symmetric function in \( d \) variables and

\[
H(F(R); X) = \sum m_{\lambda}(R) S_{\lambda}(X),
\]

where the sum is on all \( (\lambda_1, \ldots, \lambda_d) \) and the multiplicities \( m_{\lambda}(R) \) are exactly the same as in the cocharacter sequence of \( R \). Hence, if we know the Hilbert series
of \( F(R) \), we can compute the multiplicities \( m_\lambda(R) \) for partitions \( \lambda \) in \( \leq d \) parts. The theorem of Belov [12] gives that for any PI-algebra \( R \) the Hilbert series of \( F(R) \) is a rational function. Berele [15] found that the proof of Belov also implies that this Hilbert series is a nice rational symmetric function. Hence we can apply our methods to calculate the multiplicity series of \( H(F(R); X) \) and to find the multiplicities of \( R \). See the introduction of Boumova and Drensky [20] for a survey of results on the multiplicities of concrete algebras.

The most important algebras in PI-theory are the so called \( T \)-prime algebras whose \( T \)-ideals are the building blocks of the structure theory of \( T \)-ideals developed by Kemer, see his book [50] for the account. There are few cases only when the Hilbert series of the relatively free algebras \( F(R) \) are explicitly known. For \( T \)-prime algebras these are the base filed \( K \), the Grassmann (or exterior) algebra \( E \), the \( 2 \times 2 \) matrix algebra \( M_2(K) \), and the algebra \( M_{1,1} \subset M_2(E) \) which has the same polynomial identities as the tensor square \( E \otimes_K E \) of the Grassmann algebra. In all these cases the multiplicities are also known. The case \( R = K \) is trivial because \( F(K) = K[Y] \):

\[
m_\lambda(K) = \begin{cases} 1, & \text{if } \lambda = (n), \\ 0, & \text{otherwise}. \end{cases}
\]

The multiplicities for \( M_2(K) \) were obtained by Formanek [12] and Drensky [29], see also [31]:

\[
m_\lambda(M_2(K)) = \begin{cases} 0, & \text{if } \lambda_5 > 0, \\ 1, & \text{if } \lambda = (n), \\ (\lambda_1 - \lambda_2 + 1)\lambda_2, & \text{if } \lambda = (\lambda_1, \lambda_2), \lambda_2 > 0, \\ \lambda_1(2 - \lambda_4) - 1, & \text{if } \lambda = (\lambda_1, 1, 1, \lambda_4), \\ (\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1) & \text{in all other cases}. \end{cases}
\]

Hence the multiplicity series are

\[
M(H(F(K)); X) = \frac{1}{1 - x_1},
\]

\[
M'(H(F(K)); V) = \frac{1}{1 - v_1};
\]

\[
M'(H(F(M_2(K)))); V) = \frac{1}{(1 - v_1)^2(1 - v_2)^2(1 - v_3)^2(1 - v_4)} - \frac{v_2 + v_1(1 - v_2)}{(1 - v_1)^2} - \frac{v_3 + v_4}{1 - v_1}.
\]

For any \( d \), there are partitions \( \lambda = (\lambda_1, \ldots, \lambda_d) \) with \( \lambda_d > 0 \) and nonzero multiplicities \( m_\lambda(E) \) and \( m_\lambda(E \otimes_K E) \). Hence for these cases the multiplicity series \( M(H(E); X) \) and \( M(H(E \otimes_K E)); X) \) do not carry all the information about the cocharacter sequences of \( E \) and \( E \otimes_K E \). For this purpose Berele [16] suggested to use hook Schur functions instead of ordinary ones.

Another case when the Hilbert series of the relatively free algebras may be computed and used to find the multiplicities is for algebras \( R \) with \( T \)-ideals which are products of two \( T \)-ideals, \( T(R) = T(R_1)T(R_2) \). See again [20] for details. Formanek [43] found the following simple formula for the Hilbert series of \( T(R) \) in terms of
the Hilbert series of \( T(R_1) \) and \( T(R_2) \):
\[
H(T(R)) = \frac{H(T(R_1))H(T(R_2))}{H(K(Y))} = (1 - (x_1 + \cdots + x_d))H(T(R_1))H(T(R_2)).
\]
Translated for the corresponding relatively free algebras this gives
\[
H(F(R)) = H(F(R_1)) + H(F(R_2)) + ((x_1 + \cdots + x_d) - 1)H(F(R_1))H(F(R_2)).
\]
It is known that \( T(U_n(K)) = T^m(K) \) (Mal'tsev [56]) and \( T(U_n(E)) = T^m(E) \) (this follows from the results of Abakarov [1]), where \( U_k(K) \) and \( U_k(E) \) are the algebras of \( k \times k \) upper triangular matrices with entries from \( K \) and \( E \), respectively. The multiplicities of \( U_k(K) \) were studied by Boumova and Drensky [20], with explicit results for “large” partitions \( \lambda = (\lambda_1, \ldots, \lambda_n) \) (such that \( \lambda_{k+1} + \cdots + \lambda_n = k - 1 \)). The multiplicities of \( U_3(E) \) were determined by Centrone [21]. In both cases the results were obtained using the Young rule only, without the MacMahon partition analysis. Here we shall illustrate once again Algorithm 1.3.

**Example 4.1.** Let \( C \) be the commutator ideal of the free associative algebra \( K(Y) \). Then by Mal’tsev [56] the T-ideal \( C^k \) coincides with the T-ideal of \( U_k(K) \). Since \( K(Y)/C \) is the polynomial algebra in \( d \) variables and
\[
H(K[Y]; X) = \prod_{i=1}^{d} \frac{1}{1-x_i} = \sum_{\mu \geq 0} S_{(\mu)}(X),
\]
the formula
\[
H(F(R)) = H(F(R_1)) + H(F(R_2)) + ((x_1 + \cdots + x_d) - 1)H(F(R_1))H(F(R_2))
\]
for the Hilbert series of relatively free algebras corresponding to products of T-ideals gives
\[
H(F(U_2(K)); X) = 2\prod_{i=1}^{d} \frac{1}{1-x_i} + ((x_1 + \cdots + x_d) - 1)\prod_{i=1}^{d} \frac{1}{(1-x_i)^2}.
\]
The decomposition of the product \( S_\mu(X)S_{(n)}(X) = \sum S_\lambda(X) \) is given by the Young rule. If \( \mu \) is a partition in \( k \) parts, then \( \lambda \) is a partition in \( k \) or \( k + 1 \) parts. Hence \( H(F(U_2(K)); X) \) decomposes into a series of Schur functions \( S_\lambda(X) \), where \( \lambda \) is a partition in no more than three parts. Therefore, it is sufficient to consider the multiplicity series of \( H(F(U_2(K)); X) \) for \( d = 3 \) only. Clearly, \( M'(H(K[Y]); V) = M' \left( \prod_{i=1}^{d} \frac{1}{1-x_i}; V \right) = \frac{1}{1-v_1} \).

Algorithm 1.3 gives
\[
g_1(x_1, x_2, x_3) = \frac{(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)((x_1 + x_2 + x_3) - 1)}{(1 - x_1)^2(1 - x_2)^2(1 - x_3)^2},
\]
\[
g_1(x_1 z_1, x_2 / z_1, x_3) = \frac{-x_3 + x_1 x_2 - 2x_1 x_2 x_3 - x_1 x_3 z_1}{(1 - x_3)^2},
\]
\[
\begin{align*}
&\frac{x_3 + x_3^2 - x_3^3 - x_1 x_2 + x_1 x_2 x_3 - x_1^2 x_2 x_3}{(1 - x_1 x_2)(1 - x_3)^2(1 - x_1 z_1)} + \frac{-x_3^2 + x_1^2 x_2^2}{(1 - x_1 x_2)(1 - x_3)(1 - x_1 z_1)^2} \\
&\quad + \frac{x_2 x_3}{(1 - x_3)^2 z_1} + \frac{(x_3 - x_3^2 + x_3^3 - x_1 x_2 + x_1 x_2 x_3 + 2x_1^2 x_2^2 - 3x_1^2 x_2 x_3) x_2}{(1 - x_1 x_2)(1 - x_3)^2(1 - x_2 z_1)}
\end{align*}
\]
we can compute the Hilbert series of $F$ Mishchenko, Regev, and Zaicev [57] and in [20].

Let $U$ be the algebra of matrices of the form

$$
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & M_{d_{m-1}}(K) & * \\
0 & 0 & \cdots & 0 & M_{d_{m}}(K)
\end{pmatrix}
$$

We omit the last three summands which give negative degrees of $z_1$ in the expansion of $g_1(x_1 z_1, x_2/z_1, x_3)$ as a Laurent series and, substituting $z_1 = 1$, we obtain

$$g_2(x_1, x_2, x_3) = \frac{-x_3 + x_1 x_2 - 2x_1 x_2 x_3 - x_1 x_3}{(1 - x_3)^2}$$

$$+ \frac{x_3 + x_3^2 - x_1 x_2 + x_1 x_2 x_3 - x_1^2 x_2^2 x_3}{(1 - x_1)(1 - x_1 x_2)(1 - x_3)^2}.$$ 

Repeating the procedure with $g_2(x_1, x_2 x_2, x_3/z_2)$ we obtain

$$\Omega(g_1(x_1 z_1, x_2 z_2/z_1, x_3/z_2)) = \frac{x_1 x_2 (-1 + x_1 + 2x_1 x_2 - x_1^2 x_2 + x_1 x_2 x_3)}{(1 - x_1)^2(1 - x_1 x_2)},$$

$$M'(\frac{x_1 + x_2 + x_3 - 1}{(1 - x_1)^2(1 - x_2)^2(1 - x_3)^2}; V) = \frac{-1 + v_1 + 2v_2 - v_1 v_2 + v_3}{(1 - v_1)^2(1 - v_2)}.$$ 

Hence the multiplicities in the cocharacter sequence of $U_2(K)$ are

$$m_\lambda(U_2(K)) = \begin{cases} 1, & \text{if } \lambda = (\lambda_1), \\
\lambda_1 - \lambda_2 + 1, & \text{if } \lambda = (\lambda_1, \lambda_2), \lambda_2 > 0, \\
\lambda_1 - \lambda_2 + 1, & \text{if } \lambda = (\lambda_1, \lambda_2, 1), \\
0 & \text{in all other cases.} \end{cases}$$

Compare our approach with the approach on the multiplicities of $U_2(K)$ given by Mishchenko, Regev, and Zaicev [57] and in [20].

A case of products of T-ideals when we do need the MacMahon partition analysis is of block triangular matrices with entries from the field. Let $d_1, \ldots, d_m$ be positive integers and let $U(d_1, \ldots, d_m)$ be the algebra of matrices of the form

$$\begin{pmatrix} M_{d_1}(K) & * & \cdots & * & * \\
0 & M_{d_2}(K) & * & \cdots & * \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & M_{d_{m-1}}(K) & * \\
0 & 0 & \cdots & 0 & M_{d_{m}}(K) \end{pmatrix}.$$ 

It is known, see Giambruno and Zaicev [37], that

$$T(U(d_1, \ldots, d_m)) = T(M_{d_1}(K)) \cdots T(M_{d_m}(K)).$$

The only cases when we know the Hilbert series of $T(M_k(K))$ are $k = 1, 2$, and we can compute the Hilbert series of $F(U(d_1, \ldots, d_m))$. The multiplicities of $U(d_1, \ldots, d_m)$ when all $d_i$ are equal to 1 and 2 were studied in the master thesis of Kostadinov [52], see also his paper with Drensky [39]. If $d_1 = \cdots = d_m = 1$, the algebra $U(d_1, \ldots, d_m)$ is equal to $U_2(K)$, handled in [20]. If only one $d_i$ is equal to 2 and the others are equal to 1, we still can use the Young rule. The MacMahon
partition analysis was applied in [52] in the case when several $d_i$ are equal to 2. In particular exact formulas for the multiplicity series and the multiplicities as well as the asymptotics of the multiplicities were found for a small number of blocks.

Studying the polynomial identities of the matrix algebra $M_k(K)$, there is another object which behaves much better than the relatively free algebra $F(d)$. Partition analysis was applied in [52] in the case when several $d_i$ are equal to 2. In particular exact formulas for the multiplicity series and the multiplicities as well as the asymptotics of the multiplicities were found for a small number of blocks.

Let
\[
K[Z] = K[z^{(i)}_{pq} | p, q = 1, \ldots, k, i = 1, \ldots, d]
\]
be the polynomial algebra in $k^2d$ commuting variables and let $R_{kd}$ be the generic matrix algebra generated by the $d$ generic $k \times k$ matrices
\[
z_i = \begin{pmatrix}
(z^{(i)}_{11} & \cdots & z^{(i)}_{ik} \\
\vdots & \ddots & \vdots \\
z^{(i)}_{ki} & \cdots & z^{(i)}_{kk}
\end{pmatrix}, \quad i = 1, \ldots, d.
\]

It is well known that $R_{kd} \cong F(M_k(K))$. Let $C_{kd}$ be the pure (or commutative) trace algebra generated by all traces of products $\text{tr}(z_{i1} \cdots z_{in}), \ i_1, \ldots, i_n = 1, \ldots, d$. It coincides with the algebra of invariants $K[Z]^{GL_k}$ where the action of $GL_k$ on $K[Z]$ is induced by the action of $GL_k$ on the generic matrices $z_1, \ldots, z_d$ by simultaneous conjugation. Hence one may study $C_{kd}$ with methods of the classical invariant theory. The mixed (or noncommutative) trace algebra $T_{kd} = C_{kd} R_{kd}$ also has a meaning in invariant theory. See the books by Formanek [44], and also with Drensky [33], for a background on trace algebras. The mixed trace algebra approximates quite well the algebra $F(M_k(K))$. In particular, one may consider the multilinear components of the pure and mixed trace algebras $C_k = C_{k,\infty}$ and $T_k = T_{k,\infty}$ of infinitely many generic $k \times k$ matrices and the related sequences
\[
\chi_n(C_k) = \sum_{\lambda \vdash n} m_\lambda(C_k) \chi_\lambda, \quad \chi_n(T_k) = \sum_{\lambda \vdash n} m_\lambda(T_k) \chi_\lambda, \quad n = 0, 1, 2, \ldots,
\]
of $S_n$-characters called the pure and mixed cocharacter sequences, respectively. Formanek [44] showed that the multiplicities $m_\lambda(T_k)$ in the mixed cocharacter sequence and $m_\lambda(M_k(K))$ in the ordinary cocharacter sequence for $M_k(K)$ coincide for all partitions $\lambda = (\lambda_1, \ldots, \lambda_{k^2})$ with $\lambda_{k^2} \geq 2$. The only case when the pure and mixed cocharacter sequences are known is for $n = 2$ due to Procesi [61] and Formanek [42] (besides the trivial case of $k = 1$). We state the result for $T_2$ only.

\[
m_\lambda(T_2) = \begin{cases} 
(\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1), & \text{if } \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4), \\
0, & \text{otherwise},
\end{cases}
\]

The situation with the Hilbert series of $C_{kd}$ and $T_{kd}$ is better. The case $k = 2$ was handled by Procesi [61] and Formanek [42]:

\[
H(C_{2d}; X) = \prod_{i=1}^{d} \frac{1}{1 - x_i} \sum_{p, q, r \geq 0} S_{2p+2q+r,2q+r,r}(X),
\]

\[
H(T_{2d}; X) = \prod_{i=1}^{d} \frac{1}{1 - x_i} \sum_{(\lambda_1, \lambda_2, \lambda_3)} S_{\lambda_1, \lambda_2, \lambda_3}(X)
= \prod_{i=1}^{d} \frac{1}{(1 - x_i)^2} \sum_{n \geq 0} S_{(n,n)}(X).
\]
The Molien–Weyl formula gives that the Hilbert series of $C_{kd}$ and $T_{kd}$ can be expressed as multiple integrals but for $k \geq 3$ their direct evaluation is quite difficult and was performed by Teranishi [70, 71] for $C_{32}$ and $C_{42}$ only. Van den Bergh [73] found a graph-theoretical approach for the calculation of $H(C_{kd})$ and $H(T_{kd})$. Berele and Stembridge [18] calculated the Hilbert series of $C_{kd}$ and $T_{kd}$ for $k = 3, d \leq 3$ and of $T_{32}$, correcting also some typographical errors in the expression of $H(C_{42})$ in [71]. Recently Djoković [27] computed the Hilbert series of $C_{k2}$ and $T_{k2}$ for $k = 5$ and 6.

Using the Hilbert series of $C_{32}$, Berele [14] found an asymptotic expression of $m_{(\lambda_1, \lambda_2)}(C_3)$. The explicit form of multiplicity series of the Hilbert series of $C_{32}$ was found by Drensky and Genov [34], correcting also a minor technical error (a missing summand) in [14]. The quite technical method of [34] was improved in [35] and applied by Drensky, Genov and Valenti [37] to compute the multiplicity series of $H(T_{32})$ and by Drensky and Genov [36] for the multiplicity series of $H(C_{42})$ and $H(T_{42})$. In principle, the same methods work for the multiplicities of the Hilbert series of $H(C_{k2})$ and $H(T_{k2})$, $k = 5, 6$.

**Example 4.2.** We shall apply the MacMahon partition analysis to find the multiplicities of $H(T_{32})$. By Berele and Stembridge [18]

$$H(T_{32}, x_1, x_2) = \frac{1}{(1 - x_1)^2(1 - x_2)(1 - x_1 x_2)(1 - x_1 x_2^2)(1 - x_1 x_2^2)(1 - x_1 x_2^2)(1 - x_1 x_2^2)}.$$  

As in Example 4.2 we define the function

$$g(x_1, x_2) = (x_1 - x_2)H(T_{32}, x_1, x_2)$$

and decompose $g(x_1 z, x_2/z)$ as a sum of partial fractions with respect to $z$. The result is

$$\frac{1}{2(1 - x_1 x_2)^6(1 + x_1 x_2)^2(1 - x_1 z)^3 + 4(1 - x_1 x_2)^7(1 + x_1 x_2)^3(1 - x_1 z)^2}$$

$$\frac{1 + 2x_1 x_2 - 5x_1^2 x_2^2}{1 + 2x_1 x_2 - 10x_1^2 x_2^2 + 10x_1^3 x_2^3 - 7x_1^4 x_2^4}$$

$$\frac{1}{8(1 - x_1 x_2)^8(1 + x_1 x_2)^3(1 - x_1 z)^3}$$

$$- \frac{x_1^3 x_2^2}{4(1 - x_1 x_2)^7(1 + x_1 x_2)^3(1 - x_1 z)^2 + 2x_1(7 - 10x_1 x_2 + 10x_1^2 x_2^2 - 2x_1^3 x_2^3 - x_1^4 x_2^4)(1 + x_1 x_2)^4(1 - x_1 z)^3}$$

$$- \frac{x_1^4 x_2^5}{8(1 - x_1 x_2)^8(1 + x_1 x_2)^4(x_2 - z)^3}$$

$$- \frac{x_2}{8(1 - x_1 x_2)^8(1 + x_1 x_2)^4(x_2 + z)}$$

$$- \frac{(1 - x_1 x_2)^8(1 + x_1 x_2)^4(1 + x_1 x_2^2)(1 - x_1 x_2^2)(1 - x_1 x_2^2)(1 - x_1 x_2^2)(1 - x_1 x_2^2)}.$$  

We remove the last five summands because their expansions as Laurent series contain negative degrees of $z$ only. Then we replace $z$ by 1 and obtain

$$\Omega(g(x_1 z, x_2/z)) = \frac{1}{2(1 - x_1 x_2)^9(1 + x_1 x_2)^2(1 - x_1)}$$

$$+ \frac{1 + 2x_1 x_2 - 5x_1^2 x_2^2}{4(1 - x_1 x_2)^7(1 + x_1 x_2)^3(1 - x_1)^2 - 1 + 2x_1 x_2 - 10x_1^2 x_2^2 + 10x_1^3 x_2^3 - 7x_1^4 x_2^4}$$

$$\frac{1}{8(1 - x_1 x_2)^8(1 + x_1 x_2)^4(1 - x_1)}.$$
\[- \frac{1}{8(1-x_1x_2)^2(1+x_1x_2)^4(1+x_1^2x_2^2)(1+x_1)} \]
\[- \frac{x_1^3x_2^3}{(1-x_1x_2)^6(1+x_1x_2)^4(1+x_1^2x_2^2)(1-x_1^2x_2^2)}. \]

Dividing \( \Omega(g(x_1z, x_2/z)) \) by \( x_1 \) and after the substitution \( v_1 = x_1, v_2 = x_1x_2 \) we obtain

\[
M'(H(T_{32}); v_1, v_2) = \frac{v_3^3h_3(v_2) + v_1^2h_2(v_2) + v_1h_1(v_2) + h_0(v_2)}{(1-v_1)^3(1+v_1)(1-v_1v_2)(1-v_2)^4(1+v_2)^4},
\]

\[
h_3(v_2) = v_2^2(v_2^4 - v_2^3 + 3v_2^2 - v_2 + 1), \quad h_2(v_2) = v_2(2v_2^4 - 4v_2^3 + v_2^2 - v_2 - 1),
\]

\[
h_1(v_2) = v_2(-v_2^3 - v_2^2 + v_2^2 - 4v_2 + 2), \quad h_0(v_2) = v_2^4 - v_2^3 + 3v_2^2 - v_2 + 1,
\]

which coincides with the result of [37]. The multiplicity series has also the form

\[
M'(H(T_{32}); v_1, v_2) = \frac{a_3(v_2)}{(1-t)^3} + \frac{a_2(v_2)}{(1-t)^2} + \frac{a_1(v_2)}{1-t} + \frac{b(v_2)}{1-t} + \frac{c(v_2)}{1-v_2},
\]

where

\[
a_3(v_2) = \frac{1}{2(1-v_2)^6(1+v_2)^2}, \quad a_2(v_2) = \frac{3v_2^2 - 2v_2 + 1}{2^4(1-v_2)^3(1+v_2)^3},
\]

\[
a_1(v_2) = \frac{(v_2^4 - 6v_2^3 + 14v_2^2 - 6v_2 + 1)}{2^3(1-v_2)^8(1+v_2)^4},
\]

\[
b(v_2) = \frac{1}{2^4(1-v_2)^2(1+v_2)^2(1+v_2^2)}, \quad c(v_2) = \frac{-v_2^2}{(1-v_2)^6(1+v_2)^4(1+v_2^2)}.
\]

In a forthcoming paper by Benanti, Boumova and Drensky [13] we shall apply our methods to find the multiplicities in the pure and mixed cocharacter sequence of three generic 3 x 3 matrices.

One of the directions of noncommutative invariant theory is to study subalgebras of invariants of linear groups acting on free and relatively free algebras. For a background see the surveys by Formanek [43] and Drensky [30]. Recall that we consider the action of \( GL_d \) on the vector space \( KY \) with basis \( Y = \{y_1, \ldots, y_d\} \) and extend this action diagonally on the free algebra \( K(Y) \) and the relatively free algebras \( F(R) \), where \( R \) is a PI-algebra. Let \( G \) be a subgroup of \( GL_d \). Then \( G \) acts on \( F(R) \) and the algebra of \( G \)-invariants is

\[
F(R)^G = \{ f(Y) \in F(R) \mid g(f) = f \text{ for all } g \in G \}.
\]

Comparing with commutative invariant theory, when \( K[Y]^G \) is finitely generated for all “nice” groups (e.g., finite and reductive), the main difference in the non-commutative case is that \( F(R)^G \) is finitely generated quite rarely. For a survey on invariants of finite groups \( G \) see [30, 43], and the survey by Kharlampovich and Sapir [51]. For a fixed PI-algebra \( R \) there are many conditions which are equivalent to the fact that the algebra \( F(R)^G \) is finitely generated for all finite groups \( G \). Maybe the simplest one is that this happens if and only if \( F(R)^G \) is finitely generated for \( d = 2 \) and the cyclic group \( G = \langle g \rangle \) of order 2 generated by the matrix

\[
g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Clearly, in this case \( F(R)^{(g)} \) is spanned on all monomials in \( y_1, y_2 \) which are of even degree with respect to \( y_1 \). For the algebras of invariants \( F(R)^G \) of reductive groups \( G \) see Vonessen [74] and Domokos and Drensky [28].

Concerning the Hilbert series of \( F(R)^G \), for \( G \) finite there is an analogue of the Molien formula, see Formanek [13]: If \( \xi_1(g), \ldots, \xi_d(g) \) are the eigenvalues of \( g \in G \), then the Hilbert series of the algebra of invariants \( F(R)^G \) is

\[
H(F(R)^G; t) = \frac{1}{|G|} \sum_{g \in G} H(F(R); \xi_1(g)t, \ldots, \xi_d(g)t).
\]

Combined with the theorem of Belov [12] for the rationality of \( H(F(R); X) \) (as specified by Berele [15]) this gives that the Hilbert series of \( F(R)^G \) is a nice rational function for every finite group \( G \). By a result of Domokos and Drensky [28] the Hilbert series of \( F(R)^G \) for a reductive group \( G \) is a nice rational function if \( R \) satisfies a nonmatrix polynomial identity (i.e., an identity which does not hold for the algebra \( M_2(K) \) of 2 \( \times \) 2 matrices). The proof uses that for algebras \( R \) with nonmatrix identity the relatively free algebra has a finite series of graded ideals with factors which are finitely generated modules of polynomial algebras. This allows to reduce the considerations to the commutative case when the rationality of the Hilbert series is well known. We believe that the careful study of the proof of Belov [12] would give that \( H(F(R)^G; t) \) is a nice rational function for every reductive group \( G \) and an arbitrary PI-algebra \( R \).

Let \( W \) be a \( p \)-dimensional \( GL_d \)-module with basis \( Y_p = \{ y_1, \ldots, y_p \} \). Consider the related representation \( \rho : GL_d \to GL_p \) of \( GL_d \) in the \( p \)-dimensional vector space with this basis. If \( F_p(R) \) is a relatively free algebra of rank \( p \) freely generated by \( Y_p \), then the representation \( \rho \) induces an action of \( GL_d \) on \( F_p(R) \). The following theorem is a noncommutative analogue of Theorem 3.2

**Theorem 4.3.** Let \( W \) be a \( p \)-dimensional polynomial \( GL_d \)-module with Hilbert series with respect to the grading induced by the \( GL_d \)-action on \( W \)

\[
H(W; X) = \sum a_i x_1^{i_1} \cdots x_d^{i_d}, \quad a_i \geq 0, a_i \in \mathbb{Z}, \sum a_i = p.
\]

Let \( R \) be a PI-algebra with the corresponding relatively free algebra \( F_p(R) \) of rank \( p \) freely generated by \( Y_p \), with the natural structure of a \( GL_d \)-module induced by the \( GL_d \)-action on \( W \). Let

\[
f(X, t) = H(F_p(R); X^{i_1}(t), \ldots, X^{i_p}(t))
\]

be the formal power series obtained from the Hilbert series \( H(F_p(R); x_1, \ldots, x_p) \) of \( F_p(R) \) by substitution of the variables \( x_j \) with \( x_1^{i_1} \cdots x_d^{i_d} t \), in such a way that each \( x_1^{i_1} \cdots x_d^{i_d} t \) appears exactly \( a_i \) times. Then the Hilbert series of the algebras \( F_p(R)^{SL_d} \) and \( F_p(R)^{UT_d} \) of \( SL_d \)- and \( UT_d \)-invariants are, respectively,

\[
H(F_p(R)^{SL_d}; t) = M'(f; 0, \ldots, 0, 1, t),
\]

\[
H(F_p(R)^{UT_d}; t) = M(f; 1, \ldots, 1, t) = M'(f; 1, \ldots, 1, t),
\]

where \( M(f; X, t) \) and \( M'(f; V, t) \) are the multiplicity series of the symmetric in \( X \) function \( f(X, t) \).

**Proof.** We may choose the basis \( Y_p \) of \( W \) to consist of eigenvectors of the diagonal group \( D_d \). Then for a fixed \( d \)-tuple \( i = (i_1, \ldots, i_d) \) exactly \( a_i \) of the elements \( y_j \) satisfy

\[
g(y_j) = \xi_1^{i_1} \cdots \xi_d^{i_d} y_j, \quad g = \text{diag}(\xi_1, \ldots, \xi_d) \in D_d.
\]
The monomials in \(y_1, \ldots, y_p\) are eigenvectors of \(D_d\) and \(H(F_p(R); x^{(i)}, \ldots, x^{(p)} t)\) is the Hilbert series of the \(GL_d\)-module \(F_p(R)\) which counts also the \(\mathbb{Z}\)-grading of \(F_p(R)\). Now the proof is completed as the proof of Theorem 3.2 because the irreducible \(GL_d\)-submodule \(W(\lambda)\) contains a one-dimensional \(SL_d\)-invariant if \(\lambda_1 = \cdots = \lambda_d\) and does not contain any \(GL_d\)-invariants otherwise. Similarly, \(W(\lambda)\) contains a one-dimensional \(UT_d\)-invariant for every \(\lambda\). □

Combined with the nice rationality of the Hilbert series of relatively free algebras Theorem 4.3 immediately gives:

**Corollary 4.4.** Let \(W\) be a \(p\)-dimensional polynomial \(GL_d\)-module with basis \(Y_p = \{y_1, \ldots, y_p\}\) and let \(F_p(R)\) be the relatively free algebra freely generated by \(Y_p\) and related to the PI-algebra \(R\). Then the Hilbert series of the algebras of invariants \(H(F_p(R)_{SL_d}; t)\) and \(H(F_p(R)_{UT_d}; t)\) are nice rational functions.

**Example 4.5.** We shall apply Theorem 4.3 to Example 4.1. Let \(d \geq 2\) and let \(SL_d\) and \(UT_d\) act as subgroups of \(GL_d\) on the relatively free algebra \(F(U_2(K))\) with \(d\) generators. Then the generators \(y_i \in Y\) of \(F(U_2(K))\) are of first degree with respect to the \(GL_d\)-action. Hence

\[
H(F(U_2(K)); X, t) = H(F(U_2(K)); X t) = H(F(U_2(K)); x_1 t, \ldots, x_d t),
\]

\[
M'(H(F(U_2(K)); X t); V, t) = M'(H(F(U_2(K)); X t); v_1 t, v_2 t^2, \ldots, v_d t^d)
\]

\[
= \frac{1}{1 - v_1} + \frac{v_2 t^2 + v_3 t^3}{(1 - v_1 t)^2 (1 - v_2).}
\]

And therefore

\[
H(F(U_2(K))_{SL_d}; t) = M'(H(F(U_2(K)); X t); 0, \ldots, 0, t^d)
\]

\[
= \begin{cases} 
 1 - t^2, & \text{if } d = 2, \\
 1 + t^3, & \text{if } d = 3, \\
 1, & \text{if } d > 3;
\end{cases}
\]

\[
H(F(U_2(K))_{UT_d}; t) = M'(H(F(U_2(K)); X t); t, t^2, \ldots, t^d)
\]

\[
= \begin{cases} 
 1 - t + t^2, & \text{if } d = 2, \\
 (1 - t)^2 (1 - t)^2, & \text{if } d = 3, \\
 1 - 2t + 2t^2, & \text{if } d \geq 3.
\end{cases}
\]

**Example 4.6.** Again, let \(R = U_2(K)\) and let \(W = W(1^2)\) be the irreducible \(GL_3\)-module indexed by the partition \((1^2) = (1, 1, 0)\). We consider the relatively free algebra \(F_3(U_2(K))\) with the \(GL_3\)-action induced by the action on \(W\). The Hilbert series of \(F_3(U_2(K))\) which counts both the action of \(GL_3\) and the \(\mathbb{Z}\)-grading is

\[
f(X, t) = H(F_3(U_2(K)); x_1 x_2 t, x_2 x_3 t, x_2 x_3 t)
\]

\[
= \frac{1}{(1 - x_1 x_2 t)(1 - x_1 x_3 t)(1 - x_2 x_3 t)}
\]

\[
+ \frac{x_1 x_2 + x_1 x_3 + x_2 x_3 - 1}{(1 - x_1 x_2 t)^2 (1 - x_1 x_3 t)^2 (1 - x_2 x_3 t)^2}.
\]
Applying Theorem 4.3 and Algorithm 1.3 we obtain

\[ M'(f; V, t) = \frac{1 - v_2 t + (v_1 v_2 + v_3) v_3 t^3}{(1 - v_2 t)^2 (1 - v_1 v_3 t^2)}. \]

Hence

\[ H(F_3(U_2(K))^{SL_3}; t) = M'(f; 0, 0, 1, t) = 1 + t^3, \]
\[ H(F_3(U_2(K))^{UT_3}; t) = M'(f; 1, 1, 1, t) = \frac{1 - 2t + 2t^2}{(1 - t)^3}. \]

Similarly, if we consider the \( GL_3 \)-module \( W = W(2) \), then \( GL_3 \) acts on \( F_6(U_2(K)) \) extending the action on \( W \),

\[ f(X, t) = H(F_6(U_2(K)); x_1^2 t, x_2^2 t, x_3^2 t, x_1 x_2 t, x_1 x_3 t, x_2 x_3 t) \]

and, applying again Theorem 4.3 and Algorithm 1.3 we obtain

\[ H(F_6(U_2(K))^{ST_3}; t) = \frac{1 - 3t^3 + 6t^6 - 2t^9}{(1 - t^3)^4}, \]
\[ H(F_6(U_2(K))^{UT_3}; t) = \frac{p(t)}{((1 - t)(1 - t^2)(1 - t^3))^3}, \]

where

\[ p(t) = 1 - 2t + 7t^3 + 11t^4 + 6t^5 - 10t^6 + t^7 + 6t^8 + 4t^9 - 2t^{10} - 4t^{11} + 2t^{12}. \]

**Example 4.7.** Let \( R_{2p} \) be the algebra generated by \( p \) generic \( 2 \times 2 \) matrices \( z_1, \ldots, z_p \) with the canonical \( GL_2 \)-action. We extend the action of the pure and mixed trace algebras by

\[ g(\text{tr}(z_{i_1} \cdots z_{i_n}) z_{j_1} \cdots z_{j_m}) = \text{tr}(g(z_{i_1} \cdots z_{i_n})) g(z_{j_1} \cdots z_{j_m}), \]

\( z_{i_1} \cdots z_{i_n}, z_{j_1} \cdots z_{j_m} \in R_{2p}, g \in GL_2 \). For a \( p \)-dimensional \( GL_2 \)-module \( W \), we consider the induced \( GL_2 \)-action on \( R_{2p}, C_{2p} \) and \( T_{2p} \). Let \( d = 2 \). Then \( W = W(2) \oplus W(0) \) is a 4-dimensional \( GL_2 \)-module with Hilbert series

\[ H(W; x_1, x_2) = x_1^2 + x_1 x_2 + x_2^2 + 1. \]

The Hilbert series of \( T_{24} \) is

\[ H(T_{24}; x_1, x_2, x_3, x_4) = \prod_{i=1}^{4} \frac{1}{(1 - x_i)^2} \sum_{n \geq 0} S(n, n)(X) \]
\[ = (1 - x_1 x_2 x_3 x_4) \prod_{i=1}^{4} \frac{1}{(1 - x_i)^2} \prod_{1 \leq i < j \leq 4} \frac{1}{1 - x_i x_j}. \]

Hence the Hilbert series of the \( \mathbb{Z} \)-graded \( GL_2 \)-module \( T_{24} \) is

\[ f(x_1, x_2, t) = H(T_{24}; x_1^2 t, x_1 x_2 t, x_2^2 t, t) \]
\[ = \frac{1 - x_1^2 x_2^2 t^4}{((1 - t)(1 - x_1^2 t)(1 - x_1 x_2 t)(1 - x_2^2 t))^2} \]
\[ \times \frac{1}{(1 - x_1^2 t^2)(1 - x_1 x_2 t^2)(1 - x_2^2 t^2)(1 - x_1^2 x_2^2 t^2)(1 - x_1 x_2^2 t^2)(1 - x_1 x_2 t^2)}. \]
Computing the multiplicity series $M'(f; v_1, v_2, t)$ and replacing $(v_1, v_2)$ with $(0, 1)$ and $(1, 1)$ we obtain, respectively, the Hilbert series of $T_{24}^{SL_2}$ and $T_{24}^{UT_2}$:

\[
H(T_{24}^{SL_2}; t) = \frac{1 - t + t^2 + 2t^4 + t^6 - t^7 + t^8}{(1 - t)^3(1 - t^2)^2(1 - t^3)^3(1 - t^4)^2},
\]

\[
H(T_{24}^{UT_2}; t) = \frac{(1 - t + t^2)(1 + 3t^2 + 4t^3 + 6t^4 + 4t^5 + 3t^6 + t^8)}{(1 - t)^5(1 - t^2)^2(1 - t^3)^3(1 - t^4)^2}.
\]

By considering the three-dimensional $GL_2$-module $W(2)$ and the induced $GL_2$-action on $T_{23}$, we obtain

\[
H(T_{23}; x_1, x_2, x_3) = \frac{1}{(1 - x_1)^2(1 - x_2)^2(1 - x_3)^2(1 - x_1 x_2)(1 - x_1 x_3)(1 - x_2 x_3)},
\]

\[
f(x_1, x_2, t) = H(T_{23}; x_1^2 t, x_1 x_2 t, x_2^2 t)
\]

\[
= \frac{1}{((1 - x_1 t)(1 - x_1 x_2 t)(1 - x_2^2 t))^2(1 - x_1^2 x_2 t^2)(1 - x_1^2 x_2^2 t^2)(1 - x_1 x_2^3 t^2)},
\]

\[
H(T_{23}^{SL_2}; t) = \frac{1 + t^4}{(1 - t)^3(1 - t^2)^2(1 - t^3)^3(1 - t^4)^2},
\]

\[
H(T_{23}^{UT_2}; t) = \frac{1 + 2t^2 + 2t^3 + 2t^4 + t^6}{(1 - t)^2(1 - t^2)^3(1 - t^3)^2(1 - t^4)}.
\]

As in the commutative case $K[Y]$ one may consider linear locally nilpotent derivations of the free algebra $K[Y]$ and of any relatively free algebra $F(R)$. Again, we call such derivations Weitzenböck derivations. There is a very simple condition when the algebra of constants $F(R)^{\delta}$ of a nonzero Weitzenböck derivation $\delta$ is finitely generated. By a result of Drensky and Gupta [38] if the $T$-ideal of the polynomial identities of $R$ is contained in the $T$-ideal $T(U_2(K))$, then $F(R)^{\delta}$ is not finitely generated. The main result of Drensky [32] states that if $T(R)$ is not contained in $T(U_2(K))$, then $F(R)^{\delta}$ is finitely generated. For various properties and applications of Weitzenböck derivations acting on free and relatively free algebras see [38]. The following theorem and its corollary combine Theorems 3.5 and 4.3. We omit the proofs which repeat the main steps of the proofs of these two theorems.

**Theorem 4.8.** Let $\delta$ be a Weitzenböck derivation of the relatively free algebra $F(R)$ with Jordan normal form consisting of $k$ cells of size $d_1 + 1, \ldots, d_k + 1$, respectively. Let

\[
f_{\delta}(x_1, x_2, t) = H(F(R); x_1^{d_1} t, x_1^{d_1 - 1} x_2 t, \ldots, x_1^{d_1} t, \ldots, x_1 x_2^{d_k - 1} t, x_2^{d_k} t)
\]

be the function obtained from the Hilbert series of $F(R)$ by substitution of the first group of $d_1 + 1$ variables $x_1, x_2, \ldots, x_{d_1+1}$ with $x_1^{d_1} t, x_1^{d_1 - 1} x_2 t, \ldots, x_2^{d_k} t$, the second group of $d_2 + 1$ variables $x_{d_1+2}, x_{d_1+3}, \ldots, x_{d_1+d_2+2}$ with $x_1^{d_2} t, x_1^{d_2 - 1} x_2 t, \ldots, x_2^{d_2} t$, $\ldots$, the $k$-th group of $d_k + 1$ variables $x_{d_1+\ldots+d_{k-1}}, x_{d_1+\ldots+d_{k-1}+1}$, with $x_1^{d_k} t, \ldots, x_1 x_2^{d_k - 1} t, x_2^{d_k} t$. Then the Hilbert series of the algebra of constants $F(R)^{\delta}$ is given by

\[
H(F(R)^{\delta}; t) = M(f_{\delta}; 1, 1),
\]

where $M(f_{\delta}; x_1, x_2)$ is the multiplicity series of the symmetric with respect to $x_1, x_2$ function $f_{\delta}(x_1, x_2, t) \in K(t)[[x_1, x_2]]^{S_2}$. Hence the Hilbert series $H(F(R)^{\delta}; t)$ is a nice rational function.
Corollary 4.9. Let $\delta$ be a Weitzenböck derivation of the relatively free algebra $F(R)$ with Jordan normal form consisting of $k$ cells of size $d_1 + 1, \ldots, d_k + 1$, respectively. Let us identify the vector space $KY$ spanned by the free generators of $F(R)$ with the $GL_2$-module
$$W = W(d_1) \oplus \cdots \oplus W(d_k).$$
Then the Hilbert series $H(F(R)_{\delta}; t)$ and $H(F(R)^{UT_2}; t)$ of the algebras of constants $F(R)_{\delta}$ and of $UT_2$-invariants coincide.

Example 4.10. By Example 4.11 the Hilbert series of the relatively free algebra $F(U_2(K))$ is
$$H(F(U_2(K)); X) = 2 \prod_{i=1}^{d} \frac{1}{1-x_i} + (x_1 + \cdots + x_d - 1) \prod_{i=1}^{d} \frac{1}{(1-x_i)^2}.$$
Let $d = 3$ and let $\delta$ be the Weitzenböck derivation with one three-dimensional cell acting on $F(U_2(K))$. Following the procedure of Theorem 4.8 we define the function
$$f(x_1, x_2, t) = \frac{2}{(1-x_1^2 t)(1-x_1 x_2 t)(1-x_2^2 t)} + \frac{(x_1^2 + x_1 x_2 + x_2^2) t - 1}{(1-x_1^4 t)(1-x_1 x_2 t)^2(1-x_2^2 t)^2}.$$
As in Example 4.6 we compute
$$M'(f; v_1, v_2) = \frac{1 - (v_1^2 + v_2) t + (2v_1^2 - v_2) v_2 t^2 + 2(v_1^2 + v_2) v_2^2 t^3 - 2v_1^2 v_2^3 t^4}{(1-v_1^2 t)^2(1-v_2 t)(1-v_2^3 t^2)^2},$$
$$H(F_3(U_2(K))_{\delta}; t) = M'(f; 1, 1) = \frac{1 - 2t + t^2 + 4t^3 - 2t^4}{(1-t)^3(1-t^2)^2}.$$
If $d = 4$ and $\delta$ is a Weitzenböck derivation with two $2 \times 2$ cells, then
$$f(x_1, x_2, t) = \frac{2}{(1-x_1 t)^2(1-x_2 t)^2} + \frac{2(x_1 + x_2) t - 1}{(1-x_1 t)^2(1-x_2 t)^2},$$
$$H(F_4(U_2(K))_{\delta}; t) = \frac{1 + 10t^3 + 23t^4 + 2t^5 - 8t^6 + 2t^8}{(1-t)^2(1-t^2)^5}.$$

Example 4.11. As in the case of invariants we can extend the derivations of the generic trace algebra $R_{kp}$ to the pure and mixed trace algebras $C_{kp}$ and $T_{kp}$. Let $\delta_{20}$ be the Weitzenböck derivation with a three-dimensional and a one-dimensional Jordan cell acting on the mixed trace algebra $T_{23}$. Then Corollary 4.9 and Example 4.7 give that
$$H(T_{24}; t) = H(T_{24}^{UT_2}; t) = \frac{(1-t+t^2)(1+3t^2+4t^3+6t^4+4t^5+3t^6+t^7)}{(1-t)^5(1-t^2)^2(1-t^3)^3(1-t^4)^2}.$$
If $\delta$ has one three-dimensional cell only, then again Example 4.7 gives
$$H(T_{23}; t) = H(T_{23}^{UT_2}; t) = \frac{1+2t^2+2t^3+2t^4+t^6}{(1-t)^2(1-t^2)^3(1-t^3)^2(1-t^4)}.$$

Acknowledgements

This project was started when the third named author visited the fifth named at the Department of Mathematics of the North Carolina State University at Raleigh. He is grateful to the Department for the warm hospitality and the creative atmosphere. He is also very thankful to Leonid Bedratyuk for the stimulating discussions and suggestions on classical invariant theory in Section 3.
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