Tight Bounds for State Tomography with Incoherent Measurements

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Abstract

We consider the classic question of state tomography: given copies of an unknown quantum state \( \rho \in \mathbb{C}^{d \times d} \), output \( \hat{\rho} \) for which \( \| \rho - \hat{\rho} \|_{tr} \leq \varepsilon \). When one is allowed to make coherent measurements entangled across all copies, \( \Theta(d^2/\varepsilon^2) \) copies are necessary and sufficient [HHJ+17, OW16]. Unfortunately, the protocols achieving this rate incur large quantum memory overheads that preclude implementation on current or near-term devices. On the other hand, the best known protocol using incoherent (single-copy) measurements uses \( O(d^3/\varepsilon^2) \) copies [KRT17], and multiple papers have posed it as an open question to understand whether or not this rate is tight [HHJ+17, BCL20]. In this work, we fully resolve this question, by showing that any protocol using incoherent measurements, even if they are chosen adaptively, requires \( \Omega(d^3/\varepsilon^2) \) copies, matching the upper bound of [KRT17]. We do so by a new proof technique which directly bounds the “tilt” of the posterior distribution after measurements, which yields a surprisingly short proof of our lower bound, and which we believe may be of independent interest.

1 Introduction

In this paper, we consider the problem of quantum state tomography in trace norm. Here, there is an unknown \( d \)-dimensional quantum mixed state \( \rho \), and the goal is to output \( \hat{\rho} \) so that \( \| \rho - \hat{\rho} \|_{tr} \leq \varepsilon \), given the ability to prepare and measure \( n \) copies of it.¹ From a mathematical point of view, this is arguably the most fundamental quantum state learning task, and can also be seen as the natural non-commutative generalization of the classical problem of estimating a discrete distribution from samples. From a practical point of view, state tomography has many applications to the verification of quantum technologies [BCG13].

In the coherent setting, that is, when the learner is allowed to make arbitrary measurements to the product state \( \rho^\otimes n \), the situation is very well-understood. [HHJ+17] demonstrated that for this problem, \( n = O(d^2 \log(d/\varepsilon)/\varepsilon^2) \) copies suffice to solve the problem with high probability, and \( n = \Omega(d^2/\varepsilon^2) \) copies are necessary. Concurrently, [OW16] removed the logarithmic factor in the upper bound, thus establishing that the optimal rate for this problem is \( n = \Theta(d^2/\varepsilon^3) \). Tight bounds are also known in many related settings, including for different metrics and for low rank states [HHJ+17, OW16, OW17], as well as for a variety of related testing problems [OW15, BOW19].

However, in virtually all cases, the upper bounds that achieve the optimal rates require heavily entangled measurements. These measurements require all \( n \) copies of \( \rho \) to be prepared simultaneously, which—in addition to the complexity of preparing the measurements themselves—renders such approaches currently impractical. In light of this, there has been a recent surge of interest in the incoherent setting. Here, the algorithm is restricted to only making measurements of a single copy of \( \rho \) at a time. However, the measurements can be adaptively chosen based on the results of previous measurements. Such algorithms are much more practical and have been performed on real world quantum computers [HBC+21].

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¹Here and throughout the introduction, we will consider the setting where the success probability is some fixed constant.
For a variety of other problems in quantum learning, there has been a tremendous amount of recent success in characterizing the copy complexity of learning with incoherent measurements. For instance, tight bounds are now known for a number of related tasks such as mixedness testing and identity testing [BCL20, CLO21, CHLL22], shadow tomography [HKP21, CCHL22, CCHL21], and purity testing and related channel distinguishing tasks [ACQ22, CCHL22].

In contrast, for state tomography, progress has been slow in understanding the incoherent setting. The best upper bound is due to [KRT17], who demonstrated that $O(d^3/\epsilon^2)$ copies suffice, but the best lower bound against general incoherent measurements is the aforementioned lower bound of $n = \Omega(d^3/\epsilon^2)$ due to [HHJ+17] against general (coherent) measurements. When the measurements are additionally assumed to be non-adaptively chosen, that is, fixed ahead of time, then [HHJ+17] showed that $n = \Omega(d^3/\epsilon^2)$ copies are necessary. However, their techniques are very tailored to the non-adaptive case, and it is not clear how to extend them to adaptively chosen incoherent measurements, which seems to be a much more challenging setting. Indeed, understanding the power of adaptive but incoherent measurements for quantum tomography has been posed as an open question by multiple papers [HHJ+17, BCL20].

Our Results. In this paper, we fully resolve the copy complexity of quantum state tomography with (potentially adaptively chosen) incoherent measurements. Our main result is the following:

**Theorem 1** (Informal, see Theorem 8). With incoherent measurements, $\Theta(d^3/\epsilon^2)$ copies are necessary and sufficient for tomography of $d$-dimensional states to $\epsilon$ error in trace distance.

The main contribution of our work is a proof of the lower bound stating that $\Omega(d^3/\epsilon^2)$ copies are necessary to solve this problem. Combining this with the matching upper bound of [KRT17] yields the full theorem. This fully resolves open questions of [HHJ+17, BCL20].

At a high level, the difficulty with proving lower bounds for state tomography with incoherent measurements is that essentially all the existing lower bound frameworks in the literature were fundamentally for testing problems. In such settings, it suffices to demonstrate hardness for a point-versus-mixture distinguishing task, where the goal is to distinguish between the case where the unknown mixed state is a single point versus the case where it is drawn from a mixture over alternate hypotheses. Such a setup is mathematically nice because the resulting likelihood ratios have a (relatively) simple multilinear form. However, for learning tasks, no such reduction exists; indeed, it is more natural to demonstrate hardness for a mixture-versus-mixture distinguishing task, but here the resulting likelihood ratios are much more complicated. Indeed, this phenomena seems to appear more generally in a variety of other (classical) learning settings, see e.g. [SW20].

We avoid this by directly bounding how much information the algorithm can learn from incoherent measurements, a technique which we believe may be of independent interest. We demonstrate that, for a carefully chosen prior on mixed states, the posterior distribution of the algorithm after $o(d^3/\epsilon^2)$ incoherent measurements is anti-concentrated around the true mixed state. It is perhaps surprising that we are able to directly bound the behavior of the posterior distribution, but it turns out that the “tilt” caused by the measurements can be upper bounded “by hand”, and then the required anti-concentration follows from classical results in random matrix theory. An additionally nice feature of this approach is that the resulting proof is incredibly short—indeed, the entire proof fits in essentially 4 pages!

Additional Related Work. Apart from the aforementioned bounds for state tomography and quantum state testing, there have also been lower bounds in the incoherent setting when the measurements are partially adaptive or come from a set of bounded size [Low21], and when the measurements are Pauli [FGLE12].

We also note there is a large literature on understanding the power that adaptivity affords for tomography in the asymptotic setting, and under the error metric of infidelity rather than trace distance [BBG+06, HM08, MRD+13, FBK15]. These works focus on the regime where $d = O(1)$ and essentially show that the dependence of the copy complexity on the inverse error is linear when the incoherent measurements can be adaptive but only quadratic when they must be non-adaptive.

2 Preliminaries

Throughout, let $\rho$ denote the unknown state.
Measurements. We now define the standard measurement formalism, which is the way algorithms are allowed to interact with the unknown quantum state \( \rho \).

**Definition 2** (Positive operator valued measurement (POVM), see e.g. [NC02]). A positive operator valued measurement \( \mathcal{M} \) is a finite collection of psd matrices \( \mathcal{M} = \{M_z\}_{z \in \mathcal{Z}} \) satisfying \( \sum_z M_z = I_d \). When a state \( \rho \) is measured using \( \mathcal{M} \), we get a draw from a classical distribution over \( \mathcal{Z} \), where we observe \( z \) with probability \( \text{tr}(\rho M_z) \). Afterwards, the quantum state is destroyed.

Incoherent Measurements. Next, we formally define what we mean by an algorithm that uses incoherent measurements. Intuitively, such an algorithm operates as follows: given \( n \) copies of \( \rho \), it iteratively measures the \( i \)-th copy using a POVM (which could depend on the results of previous measurements), records the outcome, and then repeats this process on the \((i+1)\)-th copy. After having performed all \( n \) measurements, it must output an estimate of the true state based on the (classical) sequence of outcomes it has received. More formally, such an algorithm can be represented as a tree:

**Definition 3** (Tree representation, see e.g. [CCHL22]). Fix an unknown \( d \)-dimensional mixed state \( \rho \). An algorithm for state tomography that only uses \( n \) incoherent, possibly adaptive, measurements of \( \rho \) can be expressed as a pair \( (\mathcal{T}, \mathcal{A}) \), where \( \mathcal{T} \) is a rooted tree \( \mathcal{T} \) of depth \( n \) satisfying the following properties:

- Each node is labeled by a string of vectors \( \mathbf{x} = (x_1, \ldots, x_l) \), where each \( x_i \) corresponds to measurement outcome observed in the \( i \)-th step.
- Each node \( \mathbf{x} \) is associated with a probability \( p^\mathbf{x}(\mathbf{x}) \) corresponding to the probability of observing \( \mathbf{x} \) over the course of the algorithm. The probability for the root is 1.
- At each non-leaf node, we measure \( \rho \) using a rank-1 POVM \( \{\omega_{x}d \cdot x^\dagger\}_{x} \) to obtain classical outcome \( x \in S^{d-1} \). The children of \( \mathbf{x} \) consist of all strings \( \mathbf{x}' = (x_1, \ldots, x_t, x) \) for which \( x \) is a possible POVM outcome.
- If \( \mathbf{x}' = (x_1, \ldots, x_t, x) \) is a child of \( \mathbf{x} \), then
  \[
  p^\mathbf{x}(\mathbf{x}') = p^\mathbf{x}(\mathbf{x}) \cdot \omega_{x}d \cdot x^\dagger \rho x.
  \]
- Every root-to-leaf path is length-\( n \). Note that \( \mathcal{T} \) and \( \rho \) induce a distribution over the leaves of \( \mathcal{T} \).

\( \mathcal{A} \) is a randomized algorithm that takes as input any leaf \( \mathbf{x} \) of \( \mathcal{T} \) and outputs a state \( \mathcal{A}(\mathbf{x}) \). The output of \( (\mathcal{T}, \mathcal{A}) \) upon measuring \( n \) copies of a state \( \rho \) is the random variable \( \mathcal{A}(\mathbf{x}) \), where \( \mathbf{x} \) is sampled from the aforementioned distribution over the leaves of \( \mathcal{T} \).

We briefly note that in this definition, we assume that the POVMs are always rank-1. It is a standard fact that this is without loss of generality (see e.g. [CCHL22, Lemma 4.8]).

### 3 Proof of Hardness Result

We construct the following hard distribution of quantum states. Let \( U \subseteq \mathbb{R}^{d \times d} \) be the set of symmetric matrices with trace 1 and \( U_0 \subseteq \mathbb{R}^{d \times d} \) be the set of symmetric matrices with trace 0. These are affine subspaces of \( \mathbb{R}^{d \times d} \). They inherit the inner product of \( \mathbb{R}^{d \times d} \), which defines Lebesgue measures \( \text{Leb}_U \) and \( \text{Leb}_{U_0} \) on these spaces.

Set \( \sigma = \frac{1}{100} \). Consider the distribution \( \mu \) on \( U \) with density

\[
\mu(\rho) = \frac{1}{Z} \exp \left(-\frac{d^3}{4\sigma^2} \, \|\rho - \frac{1}{d}I_d\|_F^2\right) \, 1_{\left\{ \|\rho - \frac{1}{d}I_d\|_{\text{op}} \leq \frac{4\sigma}{d} \right\}},
\]

w.r.t. \( \text{Leb}_U \), where \( Z \) is a normalizing constant. Equivalently, a sample \( \rho \sim \mu \) can be generated by \( \rho = \frac{1}{d}(I_d + \sigma G) \), where \( G \) is a \( d \times d \) GOE matrix conditioned on \( \text{Tr}(G) = 0, \|G\|_{\text{op}} \leq 4 \). Note that all such matrices are clearly psd and thus valid quantum states. Further, say \( \rho \) is good if \( \|\rho - \frac{1}{d}I_d\|_{\text{op}} \leq \frac{2\sigma}{d} \), i.e. \( \|G\|_{\text{op}} \leq 3 \). By [CHLL22, Lemma 6.2], \( \rho \sim \mu \) is good with probability \( 1 - e^{-\Omega(d)} \).
Fix a tomography algorithm \((T, A)\) as in Definition 3, and let \(T_\rho\) denote the distribution over observation sequences \(x = (x_1, \ldots, x_n)\) when \(T\) is run on state \(\rho\). Note that for any states \(\rho, \rho'\), the Radon-Nikodym derivative

\[
\frac{dT_\rho}{dT_{\rho'}}(x) = \prod_{i=1}^n \frac{x_i^\dagger \rho x_i}{x_i^\dagger \rho' x_i}
\]

is well defined. So, for any quantum state \(\rho'\), the posterior distribution of \(\rho\) given observations \(x\) has density

\[
\nu_x(\rho) = \frac{1}{Z_x} \frac{dT_\rho}{dT_{\rho'}}(x) \mu(\rho), \quad Z_x = \int_U \frac{dT_\rho}{dT_{\rho'}}(x) \mu(\rho) \, d\text{Leb}_U(\rho).
\]

The main technical component is proving the following result about anti-concentration of the posterior distribution.

**Definition 4.** Let \(B(\rho, \varepsilon)\) denote the ball \(\{\rho' \in U : \|\rho' - \rho\|_F \leq \varepsilon\}\).

**Theorem 5.** Suppose \(d \gg 1\), \(\varepsilon \leq \varepsilon_0\) for an absolute constant \(\varepsilon_0\), and \(n \ll d^3/\varepsilon^2\). Let \(\rho_0 \sim \mu;\) on the event that \(\rho_0\) is good, the following holds. If \(x \sim T_{\rho_0}\), there is an event \(S_{\rho_0} \in \sigma(x)\) with \(P(x \in S_{\rho_0}) \geq 1 - \exp(-d^2)\) on which \(\nu_x(B(\rho_0, \varepsilon)) \ll 1\).

**Proof.** Note that for all \(\rho \in \text{supp}(\mu),\)

\[
0 \leq \frac{d^3}{4\sigma^2} \left\| \rho - \frac{1}{d} I \right\|_F^2 \leq \frac{d^4}{4\sigma^2} \left\| \rho - \frac{1}{d} I \right\|_{\text{op}}^2 \leq 4d^2,
\]

so for any \(\rho, \rho' \in \text{supp}(\mu), \mu(\rho)/\mu(\rho') \geq \exp(-4d^2)\).

Let \(C\) be a sufficiently large constant we will set later, and let \(\varepsilon_0 = \sigma/C\). Then,

\[
\nu_x(B(\rho_0, \varepsilon))^{-1} = \frac{\int_U \frac{dT_{\rho_0}}{dT_{\rho}}(x) \mu(\rho) \, d\text{Leb}_U(\rho)}{\int_{B(\rho_0, \varepsilon)} \frac{dT_{\rho_0}}{dT_{\rho}}(x) \mu(\rho) \, d\text{Leb}_U(\rho)} \geq \frac{\int_{B(\rho_0, C\varepsilon)} \frac{dT_{\rho_0}}{dT_{\rho}}(x) \mu(\rho) \, d\text{Leb}_U(\rho)}{\int_{B(\rho_0, \varepsilon)} \frac{dT_{\rho_0}}{dT_{\rho}}(x) \mu(\rho) \, d\text{Leb}_U(\rho)} 
\geq \exp(-4d^2) \int_{B(\rho_0, C\varepsilon)} \frac{dT_{\rho_0}}{dT_{\rho}}(x) \mathbb{1} \left\{ \left\| \rho - \frac{1}{d} I \right\|_{\text{op}} \leq \frac{4\sigma}{d} \right\} \, d\text{Leb}_U(\rho).
\] (2)

Note that

\[
\mathbb{E}_{x \sim T_{\rho_0}} \frac{dT_{\rho}}{dT_{\rho_0}}(x) = 1
\]

for all \(\rho\). By Markov’s inequality, with probability at least \(1 - \exp(-d^2)\), the denominator of (2) is bounded by

\[
\int_{B(\rho_0, \varepsilon)} \frac{dT_{\rho}}{dT_{\rho_0}}(x) \mathbb{1} \left\{ \left\| \rho - \frac{1}{d} I \right\|_{\text{op}} \leq \frac{4\sigma}{d} \right\} \, d\text{Leb}_U(\rho) \leq \exp(d^2) \int_{B(\rho_0, \varepsilon)} \mathbb{1} \left\{ \left\| \rho - \frac{1}{d} I \right\|_{\text{op}} \leq \frac{4\sigma}{d} \right\} \, d\text{Leb}_U(\rho) 
\leq \exp(d^2) \int_{B(\rho_0, \varepsilon)} \, d\text{Leb}_U(\rho) 
= \exp(d^2) \varepsilon^{(d+2)(d-1)/2} \int_{B(0,1)} \, d\text{Leb}_{U_0}(\rho)
\] (3)

where the exponent \((d+2)(d-1)/2\) comes from the fact that the space of symmetric matrices has dimension \(d(d+1)/2\) so \(U\) has dimension \(d(d+1)/2 - 1 = (d+2)(d-1)/2\). Let \(S_{\rho_0}\) be this event. Moreover, if \(\rho_0\) is good, we have the inclusion

\[
\left\{ \left\| \rho - \frac{1}{d} I \right\|_{\text{op}} \leq \frac{4\sigma}{d} \right\} \supseteq \left\{ \|\rho - \rho_0\|_{\text{op}} \leq \frac{\sigma}{d} \right\} \supseteq \left\{ \|\rho - \rho_0\|_{\text{op}} \leq \frac{C\varepsilon}{d} \right\}.
\]
For bounded measurable $f$, define the expectation operator

$$
E'f(\rho) = \frac{\int_U f(\rho) \mathbb{1} \left\{ \left\| \rho - \rho_0 \right\|_{\text{op}} \leq \frac{c \varepsilon}{d}, \left\| \rho - \rho_0 \right\|_{\text{tr}} \leq C \varepsilon \right\} \, \text{dLeb}_U(\rho)}{\int_U \mathbb{1} \left\{ \left\| \rho - \rho_0 \right\|_{\text{op}} \leq \frac{c \varepsilon}{d}, \left\| \rho - \rho_0 \right\|_{\text{tr}} \leq C \varepsilon \right\} \, \text{dLeb}_U(\rho)}
$$

The numerator of (2) is thus bounded by

$$
\int_{B(\rho_0, \varepsilon)} \frac{d \tau_\rho}{d \tau_{\rho_0}}(x) \mathbb{1} \left\{ \left\| \rho - \rho_0 \right\|_{\text{op}} \leq \frac{C \varepsilon}{d} \right\} \, \text{dLeb}_U(\rho)
$$

Combining equations (2), (3), and (4) gives, for good $\rho_0$ and $x \in S_{\rho_0}$,

$$
\nu_x(B(\rho_0, \varepsilon))^{-1} \geq \exp(-5d^2)C^{(d+2)(d-1)/2}E' \left[ \frac{d \tau_\rho}{d \tau_{\rho_0}}(x) \right] \int_{B(0,1)} \mathbb{1} \left\{ \left\| \rho \right\|_{\text{op}} \leq \frac{1}{d} \right\} \, \text{dLeb}_{\rho_0}(\rho).
$$

Estimating the last two terms with Lemmas 6 and 7 below, we have,

$$
\nu_x(B(\rho_0, \varepsilon))^{-1} \geq \exp(-9d^2)C^{(d+2)(d-1)/2}. $$

For $C = e^{20}$ this diverges, so $\nu_x(B(\rho_0, \varepsilon)) \ll 1$. □

**Lemma 6.** Under the assumptions of Theorem 5,

$$
E' \left[ \frac{d \tau_\rho}{d \tau_{\rho_0}}(x) \right] \geq \exp(-d^2).
$$

**Proof.** For all $\rho \in \text{supp}(\mu)$, the eigenvalues of $\rho$ lie within $[0.96/d, 1.04/d]$. Thus, for any unit vector $x$, $\frac{x^\dagger \rho x}{x^\dagger \rho_0 x} \in [0.96/1.04, 1.04/0.96] \subseteq [0.9, 1.1]$. Using the fact that $\log(1 + a) \geq a - \frac{a^2}{2}$ for $|a| \leq 0.1$, we have

$$
\log \frac{x^\dagger \rho x}{x^\dagger \rho_0 x} \geq \frac{x^\dagger (\rho - \rho_0) x}{x^\dagger \rho_0 x} \geq \frac{2}{3} \left( \frac{x^\dagger (\rho - \rho_0) x}{x^\dagger \rho_0 x} \right)^2 = \frac{x^\dagger (\rho - \rho_0) x}{x^\dagger \rho_0 x} - \frac{d^2 (\rho - \rho_0) x)^2. $$

By symmetry, $E'(\rho - \rho_0) = 0$, and by rotational invariance,

$$
d^2E' \left[ (x^\dagger (\rho - \rho_0) x)^2 \right] = E' \left[ \left\| \rho - \rho_0 \right\|_F^2 \right] \leq E' \left[ \left\| \rho - \rho_0 \right\|_{\text{tr}} \left\| \rho - \rho_0 \right\|_{\text{op}} \right] \leq \frac{C^2 \varepsilon^2}{d}. $$

Using Jensen’s inequality and the above estimates, and recalling that $n \ll d^2/\varepsilon^2$,

$$
\log E' \left[ \frac{d \tau_\rho}{d \tau_{\rho_0}}(x) \right] \geq \sum_{i=1}^n E' \log \frac{x^\dagger \rho x_i}{x^\dagger \rho_0 x_i} \geq \sum_{i=1}^n E' \left[ \frac{x^\dagger (\rho - \rho_0) x_i}{x^\dagger \rho_0 x_i} - d^2 (x_i^\dagger (\rho - \rho_0) x_i)^2 \right] \geq - \frac{C^2 n \varepsilon^2}{d} \geq -d^2. □
$$

**Lemma 7.** Under the assumptions of Theorem 5,

$$
\frac{\int_{B(0,1)} \mathbb{1} \left\{ \left\| \rho \right\|_{\text{op}} \leq \frac{1}{d} \right\} \, \text{dLeb}_{\rho_0}(\rho)}{\int_{B(0,1)} \, \text{dLeb}_{\rho_0}(\rho)} \geq \exp(-3d^2).
$$
Proof. See Section 4. \qed

We can now prove our main lower bound for tomography with incoherent measurements, which we state formally below:

**Theorem 8.** There exist absolute constants $\varepsilon_0 > 0$ and $d_0 \in \mathbb{N}$ such that for any $0 < \varepsilon < \varepsilon_0$ and any integer $d \geq d_0$, the following holds. If $n = o(d^3/\varepsilon^2)$, then for any algorithm for state tomography $(T, A)$ that uses $n$ incoherent, possibly adaptive, measurements, its output $\hat{\rho}$ upon measuring $n$ copies of $\rho$ satisfies $\|\rho - \hat{\rho}\|_{tr} > \varepsilon$ with probability $1 - o(1)$.

**Proof.** Let $S \in \sigma(\rho, x)$ be the event that $\rho \sim \mu$ is good and $x \sim T_\rho$ lies in $S$. In this proof we will abuse notation and use $A$ to also denote the internal randomness used by $A$. It suffices to show $\mathbb{P}_{A, \rho \sim \mu, x \sim T_\rho} [\|A(x) - \rho\|_{tr} \leq \varepsilon] = o(1)$.

First note that

$$\mathbb{P}_{A, \rho, x} [\|A(x) - \rho\|_{tr} \leq \varepsilon] = \mathbb{E}_{A, x \sim \nu_x} \mathbb{E} [\mathbb{1} \{\|A(x) - \rho\|_{tr} \leq \varepsilon\}] \quad (6)$$

where the second step follows by a union bound and the fact that $\mathbb{P}[(\rho, x) \notin S] = e^{-\Omega(d)} + e^{-\Omega(d^2)} = o(1)$ by Theorem 5.

For any choice of internal randomness for $A$ and any transcript $x$, let $\rho^A_x$ denote an arbitrary state for which $(\rho^A_x, x) \in S$ and $\|A(x) - \rho^A_x\|_{tr} \leq \varepsilon$, if such a state exists. Denote by $\mathcal{E}$ the event that such a state exists. Then under $\mathcal{E}$, for any state $\rho$ for which $\|A(x) - \rho\|_{tr} \leq \varepsilon$, we have $\|\rho^A_x - \rho\|_{tr} \leq 2\varepsilon$. If $\mathcal{E}$ does not occur for some choice of internal randomness for $A$ and some $x$, note that the corresponding inner expectation in (7) is zero. We can thus upper bound the double expectation in (7) by

$$\mathbb{E}_{A, x | \mathcal{E}, \rho \sim \nu_x} \mathbb{E} [\mathbb{1} \{\|\rho^A_x - \rho\|_{tr} \leq 2\varepsilon\} | (\rho, x) \in S] \leq \mathbb{E}_{A, x | \mathcal{E}} \mathbb{P} \mathbb{P} [\|\rho^A_x - \rho\|_{tr} \leq 2\varepsilon] = o(1), \quad (8)$$

where in the last step we used the fact that under $\mathcal{E}$ we have $(\rho^A_x, x) \in S$, so by Theorem 5 the posterior measure $\nu_x$ places $o(1)$ mass on the trace norm $\varepsilon$-ball around $\rho^A_x$. \qed

4 Lower Bounding the Volume Ratio: Proof of Lemma 7

In this section, we will prove Lemma 7.

Let

$$V = \left\{ \lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d : \sum_{i=1}^{d} \lambda_i = 0 \right\}.$$

This is a subspace of $\mathbb{R}^d$ of codimension 1. It inherits the inner product of $\mathbb{R}^d$, which defines a Lebesgue measure $\text{Leb}_V$. Define

$$\Delta = \left\{ \lambda \in V : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d, \sum_{i=1}^{d} |\lambda_i| \leq 1 \right\}.$$

[AGZ05, Theorem 2.5.2] implies that if $\lambda = (\lambda_1, \ldots, \lambda_d)$ are the eigenvalues of a uniformly random (w.r.t. $\text{Leb}_{\lambda_0}$) element of $B(0, 1)$, $\lambda$ has density (w.r.t. $\text{Leb}_V$) $\frac{1}{Z} f(\lambda)$ supported on $\Delta$, where

$$f(\lambda) = \prod_{1 \leq i < j \leq d} |\lambda_i - \lambda_j|$$

and $Z$ is a normalizing factor. Let

$$\Gamma = \left\{ \lambda \in V : \left| \lambda_i - \frac{d-2i+1}{d^2} \right| \leq \frac{1}{d} \right\}.$$

Note that if $\lambda \in \Gamma$, then for all $i \in [d]$, $|\lambda_i| \leq \frac{d}{2d} + \frac{1}{d} \leq \frac{3}{d}$, so the corresponding matrix $\rho$ satisfies $\|\rho\|_{op} \leq \frac{1}{d}$. Moreover $\Gamma \subseteq \Delta$: we have $\lambda_i - \lambda_{i+1} \geq \frac{2}{d^2} - \frac{2}{d^2} > 0$ and $\sum_{i=1}^{d} |\lambda_i| \leq d \cdot \frac{1}{d} = 1.$


**Proposition 9.** For any $\lambda \in \Gamma$, $f(\lambda) \geq 1/((2e)^d/2^d(d-1)/2)$.

**Proof.** For each $i \in [d]$, 

\[
\prod_{j \in [d] \setminus \{i\}} |\lambda_i - \lambda_j| \geq \prod_{j=1}^{i-1} \left( \frac{2(i-j)}{d^2} - \frac{2}{d^4} \right) \prod_{j=1}^{d} \left( \frac{2(j-i)}{d^2} - \frac{2}{d^4} \right) \geq \prod_{j=1}^{i-1} \frac{i-j}{d^2} \prod_{j=1}^{d} \frac{j-i}{d^2} \geq \prod_{j=1}^{d-1} \frac{j}{2d^2}.
\]

Thus 

\[
f(\lambda) = \left( \prod_{i=1}^{d} \prod_{j \in [d] \setminus \{i\}} |\lambda_i - \lambda_j| \right)^{1/2} \geq \left( \frac{1}{(2e)^d/2^d(d-1)/2} \right)^{1/2}.
\]

\[\square\]

**Proposition 10.** For any $\lambda \in \Delta$, $f(\lambda) \leq e^{2d^2}/d^{d(d-1)/2}$.

**Proof.** Let $(\bar{\lambda}_1, \ldots, \bar{\lambda}_d)$ be the permutation of $(\lambda_1, \ldots, \lambda_d)$ with $|\bar{\lambda}_1| \leq \cdots \leq |\bar{\lambda}_d|$. For each $i \in [d]$, 

\[
\prod_{j<i} |\bar{\lambda}_i - \bar{\lambda}_j| \leq (2|\bar{\lambda}_i|)^{i-1} \leq \left( \frac{e^{2d^2}|\bar{\lambda}_i|}{d} \right)^{i-1} \leq \frac{e^{2d^2}|\bar{\lambda}_i|}{d^{i-1}},
\]

so (since $\sum_{i=1}^{d} |\lambda_i| \leq 1$) 

\[
f(\lambda) = \prod_{i=1}^{d} \prod_{j<i} |\bar{\lambda}_i - \bar{\lambda}_j| \leq e^{2d^2} \frac{\sum_{i=1}^{d} |\bar{\lambda}_i|}{d^{i-1}/2} \leq e^{2d^2} \frac{\sum_{i=1}^{d} |\bar{\lambda}_i|}{d^{i-1}/2}.
\]

\[\square\]

**Proof of Lemma 7.** The set $\Delta$ has volume w.r.t. $\text{Leb}_V$ at most $e^{O(d)}$: the projection of $\Delta$ onto its first $d-1$ coordinates is a bijection to $\mathbb{R}^{d-1}$ with Jacobian $O(1)$, and each resulting coordinate is in $[-1, 1]$. Similarly, the set $\Gamma$ has volume w.r.t. $\text{Leb}_V$ at least $O(1) \cdot (2d^{-4})^d = d^{-O(d)}$. Thus, by the last two propositions, 

\[
\frac{\int_{B(0,1)} \mathbb{I}_{\{\|\rho\|_{sp} \leq \frac{1}{2}\}} \, d\text{Leb}_{U_0}(\rho)}{\int_{B(0,1)} \, d\text{Leb}_{U_0}(\rho)} \geq \frac{\int_{\Delta} f(\lambda) \, d\text{Leb}_V(\lambda)}{\int_{\Delta} f(\lambda) \, d\text{Leb}_V(\lambda)} \geq \frac{\inf_{\lambda \in \Gamma} f(\lambda)}{e^{O(d)}} \cdot \frac{\sup_{\lambda \in \Delta} f(\lambda)}{e^{O(d)}} \geq \frac{d^{-O(d)}}{e^{O(d)}} 2^{-d^2/2} e^{-5d^2/2} \geq e^{-3d^2}.
\]

\[\square\]

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