CONTRACTIVE PROJECTIONS AND REAL POSITIVE MAPS
ON OPERATOR ALGEBRAS

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Abstract. We study contractive projections, isometries, and real positive maps on algebras of operators on a Hilbert space. For example we find generalizations and variants of certain classical results on contractive projections on $C^*$-algebras and JB-algebras due to Choi, Effros, Størmer, Friedman and Russo, and others. In fact most of our arguments generalize to contractive ‘real positive’ projections on Jordan operator algebras, that is on a norm-closed space $A$ of operators on a Hilbert space with $a^2 \in A$ for all $a \in A$. We also prove many new general results on real positive maps which are foundational to the study of such maps, and of interest in their own right. We also prove a new Banach-Stone type theorem for isometries between operator algebras or Jordan operator algebras. An application of this is given to the characterization of symmetric real positive projections.

1. Introduction

An (associative) operator algebra is a possibly nonselfadjoint closed subalgebra $A$ of $B(H)$, for a complex Hilbert space $H$. Here we study contractive projections, isometries, and real positive maps on such algebras, and on their ‘nonassociative counterparts’. By a projection on a Banach space $X$ we mean an idempotent (usually contractive) linear map $P : X \to X$. In a previous paper [14] we studied completely contractive projections and conditional expectations on such operator algebras, and in particular we found variants of certain deep classical results on contractive positive projections on $C^*$-algebras and JB-algebras due to Choi, Effros, Størmer, Friedman and Russo, and others. In the present paper we attempt to generalize our work from [14] to the setting of contractive projections. Studying contractive projections or isometries of operator algebras forces one into the more general setting of Jordan operator algebras, that is to a norm-closed space of operators $A$ on a Hilbert space closed under the ‘Jordan product’ $a \circ b = \frac{1}{2}(ab + ba)$ (or equivalently, with $a^2 \in A$ for all $a \in A$). For example, the range of a positive unital projection on a $C^*$-algebra need not be again be isometrically isomorphic to a $C^*$-algebra (consider $\frac{1}{2}(x + x^T)$ on $M_2$), but it is always isometrically isomorphic to a Jordan operator algebra. Also, as one sees already in Kadison’s
Banach-Stone theorem for $C^*$-algebras [10], isometries of $C^*$-algebras relate to Jordan $*$-homomorphisms and not necessarily to $*$-homomorphisms. Thus most of our results are best stated for, or belong naturally to, the larger category of Jordan operator algebras. However a few of our results will apply only to (associative) operator algebras.

To establish our results, as in [14] we add the ingredient of ‘real positivity’ from recent papers of the first author with Read in [16, 17, 18] (see also e.g. [13, 5, 14, 8, 20, 15]). A key idea in those papers is that ‘real positivity’ is often the right replacement in general algebras for positivity in $C^*$-algebras. Thus we will be using for our ‘positive cone’ the real positive operators, or operators with positive real part; namely the operators $T$ satisfying $T + T^* \geq 0$. Sometimes these operators are called accretive. (We remark that there was already some use of accretive operators in the ‘JB-algebra literature’; see particularly the use of ‘dissipativity’ in e.g. [27, 28].) This will be our guiding principle here. In an early section of our paper we prove many new and foundational results of independent interest concerning real positive maps, that is, maps which take real positive elements to real positive elements. This part of our paper is a generalization of aspects of the basic theory of positive maps on $C^*$-algebras [55]. We also prove a new Banach-Stone type theorem for isometries between operator algebras or Jordan operator algebras: a characterization of such isometries in the spirit of Kadison’s Banach-Stone theorem for $C^*$-algebras. One of the difficulties to be overcome here is the fact that there exist linearly isometric unital $JC^*$-algebras which are not Jordan isomorphic (see e.g. [22, Section 5] and [27, Antithorem 3.4.34]). Therefore Banach-Stone theorems for nonunital isometries between Jordan operator algebras cannot exactly have the form one would first think of (see the second paragraph of Section 3 for an expanded version of this remark). An application of our new Banach-Stone type theorem is given later to the characterization of symmetric real positive projections.

One motivation for our work here on projections is the fact that the range of a contractive projection $P$ on an operator algebra or Jordan operator algebra is often a Jordan operator algebra in the ‘new product’ $P(x \circ y)$ (and with norm unchanged). This is a reprise of the famous result of Choi and Effros [29, Theorem 3.1] that the range of a completely positive (contractive) projection $P : B \to B$ on a $C^*$-algebra $B$, is again a $C^*$-algebra with new product $P(xy)$. A quite deep theorem of Friedman and Russo, and a simpler variant of this by Youngson, shows that something similar is true if $P$ is simply contractive, or if $B$ is replaced by a ternary ring of operators [32, 59] (see also [30] for the positive unital projection case). The analogous result for real completely positive completely contractive projections on operator algebras which have a contractive approximate identity is true (see [14, Corollary 2.5] or the proof of [11, Corollary 4.2.9]). More recently, in [15, Corollary 2.3] the authors proved:

**Theorem 1.1.** Let $A$ be a Jordan operator algebra, and $P : A \to A$ a completely contractive projection.

1. The range of $P$ with product $P(x \circ y)$, is completely isometrically Jordan isomorphic to a Jordan operator algebra.

2. If $A$ is an associative operator algebra then the range of $P$ with product $P(xy)$, is completely isometrically algebra isomorphic to an associative operator algebra.
If $A$ is unital (that is, it has an identity of norm 1) and $P(1) = 1$ then the range of $P$, with product $P(x \circ y)$, is unitaly completely isometrically Jordan isomorphic to a unital Jordan operator algebra.

If $P$ is an idempotent map on a Jordan algebra $A$, then by the new product or $P$-product on $P(A)$ we mean the bilinear map $P(x \circ y)$ for $x, y \in P(A)$.

We will prove variants of this last result for contractive projections in Sections 4 and 5. These generalize the classical results of Choi, Effros, Størmer, and Friedman and Russo, for positive projections on $C^*$-algebras or JB-algebras. Such results are related to conditional expectations (resp. Jordan conditional expectations): these are contractive projections $P : A \to A$ onto a subalgebra (resp. Jordan subalgebra) which are bimodule maps with respect to the subalgebra (resp. satisfy $P(a \circ P(b)) = P(a) \circ P(b)$ for $a, b \in A$). Under certain additional hypotheses one expects that real positive contractive projections $P$ onto a subalgebra (resp. Jordan subalgebra) to be such an expectation. For $C^*$-algebras this is due to Tomiyama (see pp. 132–133 in [6]). We will prove results of this type in the course of our paper. Some additional hypothesis on $P$ is needed for such results: $P(ab) \neq P(a)b$ in general for a unital operator algebra $A$ and contractive unital (hence real positive) projection $P$ from $A$ onto a subalgebra containing $1_A$, and $a \in A, b \in P(A)$. See [12 Corollary 3.6]. This is not even true in general if $A$ is commutative (it is true if also $b \in A \cap A^*$ by the remark after that cited corollary; see also [17] and Corollaries 4.13 and 4.14 below).

The theory of Jordan operator algebras in the sense of this paper was developed very recently, in [20, 15] (see also [21], and see also the thesis [58] of Zhenhua Wang for some additional results, complements, etc). The selfadjoint case, that is, closed selfadjoint subspaces of a $C^*$-algebra which are closed under squares, are exactly what is known in the literature as $JC^*$-algebras, and these do have a large literature (see e.g. [27, 28, 35] for references).

We now describe the structure of our paper. In Section 2, we prove many new results on real positive maps. Most of these results are foundational to the study of such maps, and of interest in their own right. In Section 3, we build on work of Arazy and Solel [3] to prove some Banach-Stone type theorems, characterizing isometries between operator algebras or Jordan operator algebras. This will require some analysis of multipliers and quasimultipliers, and of a certain behaviour in ‘Jordan multiplier algebras’ which allows us to tap into the multipliers of a containing $C^*$-algebra. This access to the $C^*$-algebra setting is not obvious. All of this is again of interest in its own right, but will also be needed later in the paper, for example for understanding symmetric projections.

In Section 4 we prove many results on real positive projections on operator algebras or Jordan operator algebras. More particularly, we give the variants in our setting of the results from Section 2 of [14], however our maps are usually no longer completely contractive, and our spaces are usually Jordan operator algebras. For example we show in and around Theorem 4.8 how to reduce certain questions about projections to the unital case (see also Theorem 2.2 and Proposition 4.2). We also give variants of the Choi-Effros result mentioned above, in the case of a real positive contractive projection $P$ on operator algebras or Jordan operator algebras, and an application of this to conditional expectations. Some of our results use as a hypothesis that the kernel of $P$, or at least a part of it, is generated by the real
positive elements which it contains, a condition that is always satisfied for positive projections on $C^*$-algebras or JB-algebras. We remark that an operator algebra (resp. Jordan operator algebra) is generated (or even is densely spanned) by the real positive elements which it contains if and only if it is approximately unital. These terms are defined below. See for example [18 Proposition 2.4 and Theorem 2.1] (resp. [20 Proposition 4.4 and Theorem 4.1]).

In Sections 5 and 6 we study contractive projections $P : A \rightarrow A$ which are symmetric (that is, $\|I - 2P\| \leq 1$) and bicontractive (that is, $\|I - P\| \leq 1$). The main result in Section 5 is Theorem 5.2, which elucidates the structure of symmetric projections and their ranges. This relies on our Banach-Stone type theorem from Section 3. Again one gets that our projections are conditional expectations in this setting. In Section 6 we study the ‘bicontractive projection problem’. We cannot expect a full solution here (as opposed to what was obtained in Section 5 for symmetric projections), counterexamples were given in [14] even if the projection is completely bicontractive. As in [14], we believe that the correct formulation of the bicontractive projection problem in our category is: When is the range of a bicontractive projection a (Jordan) subalgebra of $A$? We will give a natural condition under which the ‘bicontractive projection problem’ has a positive solution.

As we said, in this paper as opposed to [14] our maps are no longer completely contractive, and our spaces are usually Jordan operator algebras. Thus we will often get weaker results; for example we do not have a very general version of Theorem 1.1 for non-completely contractive projections. It is worth saying though that if one assumes the operator space setting (i.e. our maps are completely contractive, real completely positive, completely symmetric, completely bicontractive, etc), then all of the results of [14] seem to extend to Jordan operator algebras. We will illustrate this with some results at the end of Section 4. The main thing one needs to know for some of these proofs is that the injective envelope $I(A)$ of an approximately unital Jordan operator algebra $A$ is a unital $C^*$-algebra (see Lemma 4.16 below). For definitions and basic facts about the injective envelope see [11, Chapter 4] or [34, 50].

We now give some background and notation. The underlying scalar field is always, $\mathbb{C}$, and all maps or operators in this paper are $\mathbb{C}$-linear. For background on operator spaces and associative operator algebras we refer the reader to e.g. [11, 50, 4], and for $C^*$-algebras the reader could consult e.g. [51]. For the theory of Jordan operator algebras the reader will also want to consult [20, 41] frequently for background, notation, etc, and will often be referred to those papers for various results that are used here. See also [58].

The letters $H, K$ are reserved for Hilbert spaces. A (possibly nonassociative) normed algebra $A$ is unital if it has an identity $1$ of norm $1$, and a map $T$ is unital if $T(1) = 1$. We say that $X$ is a unital-subspace (resp. unital-subalgebra) of a unital algebra $A$ if it is a subspace (resp. subalgebra) and $1_A \in X$. We write $X_+$ for the positive operators (in the usual sense) that happen to belong to $X$. We write $\text{Re}(a)$ for $(a + a^*)/2$, and $\text{Re}(X) = \{\text{Re}(a) : a \in X\}$. This is well-defined independently of the Hilbert space representation of $X$ if $X$ is a unital operator space or approximately unital Jordan operator algebra. One way to see this: we may assume that $A$ is unital by taking the bidual (discussed shortly), and in that case one may appeal to the well known result of Arveson concerning $A + A^*$ (see e.g. [11 Lemma 1.3.6]).
For a nonempty subset $S$ of a Jordan operator algebra we define $\text{joa}(S)$ to be the smallest closed Jordan subalgebra containing $S$. Similarly for a nonempty subset $S$ of an operator algebra, $\text{oa}(S)$ is the smallest closed subalgebra containing $S$. A Jordan homomorphism $T : A \to B$ between Jordan algebras is of course a linear map satisfying $T(ab) = T(a)T(b)$ for $a, b \in A$, or equivalently, that $T(a^2) = T(a)^2$ for all $a \in A$ (the equivalence follows by applying $T$ to $(a + b)^2$). If $A$ is a Jordan operator subalgebra of $B(H)$, then the diagonal $\Delta(A) = A \cap A^*$ is a JC*-algebra. If $A$ is unital then as a JC*-algebra $\Delta(A)$ is independent of the Hilbert space $H$ (see the third paragraph of [20 Section 1.3]). An element $q$ in a Jordan operator algebra $A$ is called a projection if $q^2 = q$ and $\|q\| = 1$ (so these are just the orthogonal projections on the Hilbert space $A$ acts on, which are in $A$). Clearly $q \in \Delta(A)$. Thus there is an ambiguity in notation that will pervade the paper: the reader should hopefully have no problem determining which sense of projection is being used (for example, we use lower case letters for orthogonal projections and upper case for idempotent maps). We say that a projection $\pi : A \to A$ is a Jordan conditional expectation if $\pi(ab) = \pi(a)\pi(b)$ for all $a, b \in \pi(A)$, where $\circ$ is the Jordan product. Note that this implies that $\pi(A)$ is a Jordan algebra for the $\pi$-product. There are many papers in the functional analysis literature related to contractive projections in various settings, in addition to the ones already cited we mention e.g. [36, 37, 45].

We recall the main facts about morphisms of JC*-algebras. Most of these are related to Banach-Stone type theorems. A Jordan *-homomorphism between JC*-algebras is contractive, and if it is one-to-one then it is isometric. A linear bijection between JC*-algebras is an isometry if and only if it preserves the triple product $xy^*x$, and if and only if it preserves ‘cubes’ $xx^*x$. These results are due to Harris [36, 37] in the more general case of $J^*$-algebras, that is, norm closed subspaces of $B(K, H)$ that are closed under the triple product $\{x, y, z\} = \frac{1}{2}(xy^*z + zy^*x)$. A linear bijection between JC*-algebras is a Jordan *-isomorphism if and only if it is approximately unital (that is takes a Jordan contractive approximate identity to a Jordan contractive approximate identity), and if and only if it is positive. The difficult direction of these last two iff’s follows from facts close to the start of the present paragraph, and taking biduals. Also if the bidual surjective isometry is positive then it takes 1 to a positive unitary, that is to 1. Finally a contractive Jordan homomorphism $\pi$ between JC*-algebras is a Jordan *-homomorphism. To see this, restrict $\pi$ to the subalgebra generated by a selfadjoint element $x$, and then use the well known C*-algebraic version of the same result (namely that a contractive homomorphism between C*-algebras is a *-homomorphism). This shows that $\pi(x)$ is selfadjoint, which yields the assertion.

We refer the reader to [27, 28, 35] for the basic theory of Jordan algebras. In our bibliography we have also listed several other papers related to Jordan algebras and triples relevant to topics in our paper [23, 25, 26, 14, 46, 52, 53, 56, 57]. There are certain basic formulae that hold in a Jordan operator algebra $A$ that we will use very often. For example,

\[(1.1)\quad a \circ b = \frac{1}{2}[(a + b)^2 - a^2 - b^2], \quad a, b \in A.\]

And

\[(1.2)\quad aba = 2(a \circ b) \circ a - a^2 \circ b, \quad a, b \in A.\]
Also, \( \frac{1}{2}(abc+cba) = (a\circ b)\circ c + (b\circ c)\circ a - (a\circ c)\circ b \). Thus \( abc+cba \in A \) for \( a, b, c \in A \). See e.g. [35, p. 25]. If \( p \) is a projection in \( A \) then \( p \circ a = \frac{1}{2}(a + p^* ap^+) \).

A projection \( q \) in a Jordan operator algebra \( A \) will be called central if \( qx = xq \) for all \( x \in A \). This is equivalent to \( qx = xq \) in any \( C^* \)-algebra containing \( A \) as a Jordan subalgebra, by the first labeled equation in [20]. It is also equivalent to \( q \) being central in any generated (associative) operator algebra, or in a generated \( C^* \)-algebra. This notion is independent of the particular generated (associative) operator algebra since it is captured by the intrinsic formula \( qx = q \circ x \) for \( x \in A \).

A Jordan ideal of a Jordan algebra \( A \) is a subspace \( E \) with \( \eta \circ \xi \in E \) for \( \eta \in E, \xi \in A \).

If \( A \) is a Jordan subalgebra of a \( C^* \)-algebra \( B \) then \( A^{**} \) with its Arens product is a Jordan subalgebra of the von Neumann algebra \( B^{**} \) (see [20 Section 1]). Since the diagonal \( \Delta(A^{**}) \) is a JW*-algebra (that is a weak* closed \( JC^* \)-algebra), it follows that \( A^{**} \) is closed under meets and joins of projections. If \( P : A \to A \) is a contractive projection on a Jordan operator algebra then \( Q = P^{**} : A^{**} \to A^{**} \) is a contractive projection and \( Q(A^{**}) \) is the weak* closure of \( P(A) \). Indeed if \( P(x_t) \to \eta \in A^{**} \) weak* then \( P(x_t) \to Q(\eta) \) weak*. So the weak* closure of \( P(A) \) is contained in \( Q(A^{**}) \). Conversely, \( a_t \to \eta \in A^{**} \) weak* implies \( P(a_t) \to Q(\eta) \) weak*, so that \( Q(A^{**}) \) is contained in the weak* closure of \( P(A) \).

A Jordan contractive approximate identity (or Jordan cai for short) for \( A \) is a net \( (e_t) \) of contractions with \( e_t \circ a \to a \) for all \( a \in A \). A partial cai for \( A \) is a net consisting of real positive elements in \( A \), that acts as a cai (that is, a contractive approximate identity) for the ordinary product in every \( C^* \)-algebra which contains and is generated by \( A \) as a closed Jordan subalgebra. If a partial cai for \( A \) exists then \( A \) is called approximately unital. It is shown in [20 Section 2.4] that if \( A \) has a Jordan cai then it has a partial cai. Indeed any net converging weak* to \( 1_{A^{**}} \) may be modified as in the proof of [20 Lemma 2.6] to yield a partial cai for \( A \).

We recall that every Jordan operator algebra \( A \) has a unitization \( A^1 \) which is unique up to isometric Jordan homomorphism (see [20 Section 2.2]). A state of an approximately unital Jordan operator algebra \( A \) is a functional with \( \| \varphi \| = \lim_t \varphi(e_t) = 1 \) for some (or every) Jordan cai \( (e_t) \) for \( A \). These extend to states of the unitization \( A^1 \). They also extend to a state (in the \( C^* \)-algebraic sense) on any \( C^* \)-algebra \( B \) generated by \( A \), and conversely any state on \( B \) restricts to a state of \( A \). See [20 Section 2.7] for details.

Suppose that \( E \) is a closed subspace of \( B(H,K) \), and that \( u \) is a contraction in \( B(H,K) \) with \( bu^*b \in E \) for all \( b \in E \). Define \( E(u) \) to be \( E \) equipped with Jordan product \( (bu^*c + cu^*b)/2 \). Then \( E(u) \) is completely isometrically Jordan isomorphic to a Jordan operator algebra. This follows from the proof of [15 Theorem 2.1] (which also shows that every Jordan operator algebra arises in this way). We will need, and will reprove now, a simple case of this: if in addition \( u \) is unitary in \( B(H,K) \), then \( Eu^* \) is a Jordan subalgebra of \( B(K) \), and right multiplication \( R_{u^*} \) by \( u^* \) is a completely isometric Jordan homomorphism from \( E(u) \) onto \( Eu^* \).

Because of the uniqueness of unitization up to isometric isomorphism, for a Jordan operator algebra \( A \) we can define unambiguously \( \mathfrak{F}_A \) as \( \{ a \in A : \| 1 - a \| \leq 1 \} \). Then \( \mathfrak{F}_A = \{ a \in A : \| 1 - 2a \| \leq 1 \} \subset \text{Ball}(A) \). Note that \( x \in \mathfrak{F}_A \) if and only if \( x^*x \leq x + x^* \). Similarly, \( \mathfrak{r}_A \), the real positive or accretive elements in \( A \), may be defined as the set of \( h \in A \) with \( \text{Re } \varphi(h) \geq 0 \) for all states \( \varphi \) of \( A^1 \). This is equivalent to all the other usual conditions characterizing accretive elements as
we said in [20] Section 2.2. Note that the ‘negatives’ of the accretive elements are sometimes called dissipative (see e.g. [27] Definition 2.1.8). We have for example \( r_A = \{ a \in A : a + a^* \geq 0 \} \), where the adjoint and sum here is in (any) \( C^* \)-algebra containing \( A \) as a Jordan subalgebra. We also have \( r_A = \mathbb{R}_+ \mathbb{J}_A \). If \( A \) is a Jordan subalgebra of a Jordan operator algebra \( B \) then \( \mathbb{J}_A = \mathbb{J}_B \cap A \) and \( r_A = r_B \cap A \).

A linear map \( T : A \to B \) between Jordan operator algebras is real positive if \( T(r_A) \subset r_B \). The real positive maps on \( JC^* \)-algebras are just the positive maps. This follows, after taking the bidual, from the fact that real positive maps on operator systems are just the positive maps. In turn, the harder direction of the last fact follows e.g. as in the proof of [3] Theorem 2.4.

The Jordan multiplier algebra \( JM(A) \) of an approximately unital Jordan operator algebra \( A \) is

\[
JM(A) = \{ x \in A^{**} : x \circ A \subset A \}.
\]

This was defined in [20] Definition 2.25 but not used, and it was not proved there that \( JM(A) \) is a Jordan operator algebra (or rather a misleading hint was given for this).

**Lemma 1.2.** The Jordan multiplier algebra \( JM(A) \) of an approximately unital Jordan operator algebra \( A \) is a Jordan operator algebra.

**Proof.** In fact this may be proved using some consequences of the Jordan identity

\[
(x^2 \circ b) \circ x = x^2 \circ (b \circ x),
\]

following what seems a well known path valid in Jordan algebras [27]. The first author together with Z. Wang checked this route following the ideas in [27]: Dropping the \( \circ \) notation, rewrite this as \( (x^2b)x = x^2(bx) \), and replace \( x \) by \( ta + c \) for scalar \( t \). Then expand the parentheses, and equate coefficients of \( t \). Writing \( [a, b, c] = (ab)c - a(bc) \), we have proved that \( 2[a, b, ac] + [c, b, a^2] = 0 \). Letting \( c = b \), we have \( 2[a, b, ab] + [b, b, a^2] = 0 \), which yields

\[
b^2a^2 = b(2ab^2) + 2a(ab) - 2(ab)^2.
\]

This works in any Jordan algebra, and in particular in \( A^{**} \). In particular it holds if \( b \in JM(A) \) and \( a \in A \), if \( A \) is an approximately unital Jordan operator algebra. Next we note that in the latter case squares linearly span \( A \) (using the fact that \( A = r_A - r_A \), and real positive elements have roots, or by using \( (1.1) \) for example, with \( a \) replaced by the cai). It follows that \( b^2 \circ A \subset A \) for \( b \in JM(A) \), as desired. \( \Box \)

It also follows that \( bab = 2(b \circ a) \circ b - b^2 \circ a \in A \) if \( b \in JM(A) \), \( a \in A \).

If \( A \) is in addition an (associative) operator algebra then \( JM(A) = M(A) \). Indeed clearly \( M(A) \subset JM(A) \). For the converse, if \( x \in JM(A) \) and \( b \in A \) then we have \( xb^2 = 2(x \circ b)b - bxb \in A \). Similarly \( b^2x \in A \). If \( A \) is approximately unital then squares span \( A \), as we said in the last paragraph, so that \( x \in M(A) \).

We define a hereditary subalgebra of a Jordan operator algebra \( A \), or HSA of \( A \) for short, to be a Jordan subalgebra \( D \) possessing a Jordan cai (this was defined above), which satisfies \( aAa \subset D \) for any \( a \in D \) (or equivalently, by replacing \( a \) by \( a + c \), such that if \( a, c \in D \) and \( b \in A \) then \( abc + cba \in D \)). We say that a projection in \( A^{**} \) is open in \( A^{**} \) if there is a net \( (x_t) \) in \( A \) with

\[
x_t = px_t p \to p \text{ weak}^*.
\]

This is a variant of the open projections for \( C^* \)-algebras in the sense of Akemann (see e.g. [1]). The ensuing noncommutative topology for possibly nonselfadjoint
operator algebras have been worked out in a series of papers by the first author with Read, the second author, Hay, Wang and others (see our bibliography). See [20, 15] for the Jordan operator algebra case. If $p$ is open in $A^{**}$ then $D = pA^{**}p \cap A = \{ a \in A : a = ppa \}$ is a hereditary subalgebra (HSA) of $A$, and the Jordan subalgebra $D^{\perp\perp}$ of $A^{**}$ has identity $p$, and. Conversely, every hereditary subalgebra of $A$ is of this form, and we say that $p$ is the support projection of $D$. Indeed $p$ is the weak* limit of any Jordan cai from the HSA.

2. New results on real positive maps

Lemma 2.1. If $T : A \rightarrow B$ is a real positive linear map between unital (resp. approximately unital) Jordan operator algebras then $T$ is bounded and $\|T\| = \|T(1)\|$(resp. $\|T\| = \|T^{**}\| = \|T^{**}(1)\| = \sup T(e_{t})$, if $(e_{t})$ is a Jordan cai for $A$).

Proof. By [20, Corollary 4.9], $T$ is bounded and extends uniquely to a positive $\tilde{T} : A + A^{*} \rightarrow B + B^{*}$. If $A$ is unital then the proof of [50, Corollary 2.8] (but replacing $A$ there with a Jordan subalgebra) yields $\|T\| = \|T(1)\|$. In the approximately unital case $T^{**} : A^{**} \rightarrow B^{**}$ is real positive using [20, Theorem 2.8] if necessary. By the unital case above we have $\|T\| = \|T^{**}\| = \|T^{**}(1)\| = \sup T(e_{t})$, (the latter because the norm is semicontinuous for the weak* topology, and $e_{t} \rightarrow 1$ weak* by [20, Lemma 2.6]).

A unital functional on a unital operator space (resp. operator system) is contractive if and only if it is real positive (resp. positive). Such maps are well known to be real completely positive (resp. completely positive)—see e.g. [20, Remark 4.10]. A unital linear contraction on a unital operator space (resp. operator system) is real positive (resp. positive) by e.g. the same Remark from [20]. However the converse is false in general.

Corollary 2.2. Let $A, B$ be approximately unital Jordan operator algebras, and let $T : A \rightarrow B$ be a contraction which is approximately unital (that is, takes some Jordan cai to a Jordan cai), or more generally for which $T^{**}$ is unital. Then $T$ is real positive.

If $\theta : A \rightarrow B$ is a contractive Jordan homomorphism then $\theta$ is real positive.

Proof. By taking the second dual we may assume that $A, B$ are unital, and that $T(1) = 1$. Then the first assertion follows from the lines before the corollary. The second follows easily from the first after replacing $B$ with $\theta(A)$.

The following gives a very useful way to ‘reduce to the unital case’. It is a generalization of the fact that positive maps on $C^{*}$-algebras extend to the unitization [50].

Theorem 2.3. Let $A$ and $B$ be approximately unital Jordan operator algebras, and write $A^{1}$ for a Jordan operator algebra unitization of $A$ with $A \neq A^{1}$. Let $C$ be a unital Jordan operator algebra containing $B$ as a closed subalgebra.

1. A real positive contractive linear map $T : A \rightarrow B$ extends to a unital real positive contractive linear map from $A^{1}$ to $C$.

2. A real positive contractive projection on $A$ extends to a unital real positive contractive projection on $A^{1}$.
Proof. Clearly (2) follows from (1). Let $\tilde{T} : A^1 \to C$ be the canonical unital extension of $T$, and write $e, f$ for the units of $A^1$ and $C$ (so $\tilde{T}(e) = f$). Suppose that $A$ is a Jordan subalgebra of a $C^*$-algebra $D$. We may assume by [29 Corollary 2.5] that $e = 1_{D^1} \notin D$. Suppose that $\text{Re} (x + \lambda e) \geq 0$ for $x \in A$ and scalar $\lambda$. We need to prove that $\text{Re} (T(x) + \lambda f) \geq 0$. This is clear if $\text{Re} (\lambda) = 0$, so suppose the contrary. Now $\text{Re} (\lambda) > 0$ (by considering the character $\chi$ on $D^1$ that annihilates $D$; this is a state so that $\text{Re}(\chi(x) + \lambda) = \text{Re}(\lambda) \geq 0$). Since $\text{Re}(x + \lambda e) \geq 0$ we have $-\frac{1}{\text{Re}(\lambda)} \text{Re}(x) \leq e$. Let

$$x_n = -\frac{n - 1}{n \text{Re}(\lambda)} x, \quad y = \text{Re}(x_n) \leq \frac{n - 1}{n} e$$

and $z = y^* \leq \frac{n - 1}{n} e$. By [29 Theorem 4.1 (2')] there exists a contraction $a \in A$ with $0 \leq z \leq \text{Re}(a) \leq e$. Now $\text{Re} (a - x_n) \geq 0$, since $\text{Re} (x_n) = y \leq y^* = z \leq \text{Re}(a)$. Also $\|\text{Re}(T(a))\| \leq 1$ since $a$ and $T$ are contractions, and therefore $0 \leq \text{Re}(T(a)) \leq f$. Also, $\text{Re} T(a - x_n) \geq 0$, so that $\text{Re} (T(x_n)) \leq \text{Re} (T(a)) \leq f$. That is,

$$-\frac{n - 1}{n \text{Re}(\lambda)} \text{Re}(T(x)) \leq f.$$ 

Letting $n \to \infty$ we have that $\text{Re} (T(x) + \lambda f) \geq 0$ as desired. Hence $\tilde{T}$ is a unital real positive map, and thus is contractive by Lemma 2.1 □

Of course the extensions in the previous result are unique.

**Lemma 2.4.** A real positive linear functional on a unital operator subspace or approximately unital Jordan subalgebra of a $C^*$-algebra $B$, extends to a positive functional on $B$ with the same norm.

**Proof.** By taking the bidual we may assume that the domain is a unital operator subspace $A$. (We may then assume if we wish that that $B = C^*(A)$.) One way to finish is then to note that the functional is real completely positive by the Remark 4.10 in [20], and appeal to [5 Theorem 2.6]. (Other approaches: we may assume that the functional is unital by Theorem 2.3! Or use the fact from e.g. [21 Proposition 3.1] and/or [27 Lemma 2.9.15] that every real positive linear functional in these situations is a non-negative multiple of a state, and the fact that states extend.) □

The following result is useful for questions about real positivity because it shows that we can often get away with working with the simpler set $\mathfrak{F}_A = \{x \in A : \|1 - x\| \leq 1\}$, instead of the more complicated set of real positive or accretive elements. Indeed the condition $\|1 - x\| \leq 1$ is a lot stronger than the condition $x + x^* \geq 0$.

**Theorem 2.5.** A linear map $T : A \to B$ between approximately unital Jordan operator algebras is real positive and contractive if and only if $T(\mathfrak{F}_A) \subset \mathfrak{F}_B$.

**Proof.** Any unital contraction $T : A \to B$ between unital Jordan operator algebras (or unital operator spaces) satisfies $T(\mathfrak{F}_A) \subset \mathfrak{F}_B$. Indeed if $\|1 - x\| \leq 1$ then $\|1 - T(x)\| = \|T(1 - x)\| \leq 1$.

A real positive contraction $T : A \to B$ between approximately unital Jordan operator algebras extends by Theorem 2.3 to a real positive unital contraction $\tilde{T} : A^1 \to B^1$. Then $T(\mathfrak{F}_A) \subset \tilde{T}(\mathfrak{F}_{A^1}) \subset \mathfrak{F}_{B^1}$, by the last paragraph. Hence $T(\mathfrak{F}_A) \subset B \cap \mathfrak{F}_{B^1} = \mathfrak{F}_B$. □
Conversely, if $T(\mathfrak{F}_A) \subseteq \mathfrak{F}_B$ then $T$ is real positive since $\tau_A = \mathbb{R}^+ \mathfrak{F}_A$. Also $T^{**}(\mathfrak{F}_{A^*}) \subseteq \mathfrak{F}_{B^*}$ by [20 Theorem 2.8]. Therefore $\|1 - 2T^{**}(1)\| \leq 1$ since $1 \in \frac{1}{2} \mathfrak{F}_A$, so that $\|T^{**}(1)\| \leq 1$. Hence $T$ is a contraction by Lemma 2.1.

The following shows how every weak* continuous real positive contraction gives rise to a real positive contractive projection with range the fixed points of the contraction:

**Lemma 2.6.** Let $\Phi : M \to M$ be a weak* continuous real positive contraction on a unital weak* closed Jordan operator algebra, and let $M^\Phi$ be the set of fixed points of $\Phi$. Then there exists a real positive contractive projection on $M$ with range $M^\Phi$.

**Proof.** One may follow the proof in [30, Corollary 1.6], taking weak* limits in the unit ball of $B(M, M) = (M \otimes M_\ast)^{\ast}$ of averages of powers of $\Phi$. □

If in addition to the hypotheses of the last result $\Phi$ is real completely positive then $M^\Phi$ is a unital Jordan operator algebra with respect to the new product coming from the projection by e.g. [14, Theorem 2.5]. Conversely any unital Jordan operator algebra is the set of fixed points of a real completely positive unital contraction (even tautologically). Sometimes the selfadjoint analogue of $M^\Phi$ is called the Poisson boundary.

**Lemma 2.7.** Let $T : A \to B$ be a real positive map between Jordan operator algebras. Then $T(\Delta(A)) \subseteq \Delta(B)$, and $T$ restricts to a positive linear map from $\Delta(A)$ to $\Delta(B)$. Thus $0 \leq T(1) \leq 1$ if $A$ is unital and $T$ is contractive.

**Proof.** This follows as in the proof of [14, Lemma 2.3], but using the fact that $\Delta(A)$ is a JC*-algebra and operator system. Indeed regarding $B$ as a closed Jordan subalgebra of $B(H)$, and regarding the restriction of $T$ as a real positive map on the JC*-algebra $\Delta(A)$, we have that it is positive, hence *-linear. So $T(\Delta(A)) \subseteq \Delta(B)$, and the rest is clear. □

**Lemma 2.8.** Let $A$ be a unital-subspace of a unital C*-algebra $B$, and let $q \in \text{Ball}(B)_+$ with $q \circ A \subseteq A$ and $q^2 \circ A \subseteq A$. Suppose that $T : A \to B(H)$ is a real positive map on $A$, and $T(1) = T(q)$. Then $T(a) = T(qa) = T(q \circ a)$ for $a \in A$.

**Proof.** By (1.2) we have $qaq = 2(q \circ a) \circ q - q^2 \circ a \in A$ for $a \in A$. Let $\psi$ be a state on $B(H)$, and set $\varphi = \psi \circ T$. This is real positive on $A$, hence by Lemma 2.3 it extends to a positive functional $\hat{\varphi}$ on $B$. Thus for $a \in \text{Ball}(A)$, by the Cauchy-Schwarz inequality for states of a C*-algebra there exists a constant $K$ such that

$$|\varphi((1 - q)a(1 - q))^2| \leq |\varphi(b(1 - q)^2)|^2 \leq K \varphi(1 - q) = \psi(T(1) - T(q)) = 0,$$

where $b = (1 - q)a(1 - q)^2$. Similarly, $|\varphi(qa(1 - q))^2|$ and $|\varphi((1 - q)aq)|^2$ are zero. Hence the numerical radii of $T((1 - q)a(1 - q))$ and $T((1 - q)aq + qa(1 - q))$ are zero, and so $T((1 - q)a(1 - q)) = T((1 - q)aq + qa(1 - q)) = 0$. We deduce that $T(a) = T((1 - q)a + qa) = T((1 - q)a(1 - q) + (1 - q)aq + qa(1 - q) + qa) = T(qa)$ for all $a \in A$. To obtain the last equality, apply the equality before it with $a$ replaced by $q \circ a$. □

**Remark.** A similar but easier proof shows: Let $A$ be a unital-subspace of a unital C*-algebra $B$ and let $q \in \text{Ball}(B)_+$ with $qA + Aq \subseteq A$. Suppose that $T : A \to B(H)$ is real positive on $A$, and $T(1) = T(q)$. Then $T(a) = T(qa) = T(aq) = T(qaq)$ for $a \in A$. 
We also are indebted to the referee for a correction to the hypotheses of the last result.

The following very similar result is proved in [9], where it is used to characterize real positive projections on operator algebras taking values in a selfadjoint subspace.

**Lemma 2.9.** Let $A$ be a unital operator space (resp. approximately unital Jordan operator algebra), and let $T : A \to B(H)$ be a unital (resp. real positive) contraction. Suppose that $e$ is a projection in $A$ with $e \circ A \subset A$, such that $q = T(e)$ is a projection in $B(H)$. Then $T(eae) = qT(a)q$ and $T(a \circ e) = T(a) \circ q$ for all $a \in A$.

**Lemma 2.10.** If $T : A \to B$ is a real positive map between approximately unital Jordan operator algebras then $\text{joa}(T(A))$ is an approximately unital Jordan operator algebra.

**Proof.** Let $S = T(r_A) \subset r_B$. Since $A = r_A - r_A$ by [20] Theorem 4.1, $\text{joa}(T(A)) = \text{joa}(S)$ is an approximately unital Jordan operator algebra by [20] Proposition 4.4. □

The following is a nonselfadjoint analogue of the well known fact that the positive part of the kernel of a completely positive map $T$ on a $C^*$-algebra $B$ has the following ‘ideal-like’ property

$$T(xy)^* T(xy) \leq K T(y^\frac{1}{2} y^\frac{1}{2} x^* x y^\frac{1}{2} y^\frac{1}{2}) \leq K' T(y) = 0, \quad y \in \text{Ker}(T)_+, x \in B,$$

using the Kadison-Schwarz inequality. Here $K, K'$ are constants depending on $T$, and $K'$ depends also on $x$. So $T(xy)^* T(xy) = 0$. Similarly $T(yx) = 0$. In fact this is also true for positive maps on $JC^*$-algebras, as follows e.g. from the next result (using the fact that positive maps on $JC^*$-algebras or operator systems are obviously real positive). Note that the entire kernel is rarely an ideal (consider for example the map consisting of integration on $L^\infty([-1,1])$).

**Lemma 2.11.** Suppose that $A$ is an approximately unital Jordan operator algebra (resp. operator algebra) and that $T : A \to B(H)$ is a real positive map on $A$. If $x \in A$ and $y \in r_A \cap \text{Ker}(T)$ then $x \circ y \in \text{Ker}(T)$ and $yxy \in \text{Ker}(T)$ (resp. $xy$ and $yx$ are in $\text{Ker}(T)$).

**Proof.** Let $\varphi$ be a state on $B(H)$, which is real positive. As in the proof of Lemma 2.8 $\varphi \circ T$ extends to a positive functional $\psi$ on $C^*(A)$. So in the operator algebra case, by the Cauchy-Schwarz inequality for states on a $C^*$-algebra,

$$|\psi(xy)|^2 \leq K \psi(y^* y) \leq K \psi(y + y^*) = 0, \quad x \in A, y \in \mathcal{F}_A \cap \text{Ker}(T).$$

Here $K$ is a constant depending on $x$. We used the fact that $y \in \mathcal{F}_A$ if and only if $y^* y \leq y + y^*$. Thus $\psi(xy) = \varphi(T(xy)) = 0$. In the Jordan case a similar argument gives $\psi(yx) = 0$, so $\psi(xy + yx) = \varphi(T(xy + yx)) = 0$. Hence $T(xy + yx) = 0$ (resp. $T(xy) = 0$), since states on $B(H)$ separate points. So $x \circ y \in \text{Ker}(T)$, and similarly $yxy \in \text{Ker}(T)$. Finally use the fact that $r_A = \mathbb{R}^+ \mathcal{S}_A$ to replace $\mathcal{S}_A$ by $r_A$. □

**Remark.** The referee of our paper has generalized the technique seen in our proof of Lemmas 2.11 and 2.8 to obtain the following technical principle which in turn may be used to unify the proofs of these lemmas: Let $A$ be a unital operator subspace or Jordan subalgebra of a unital $C^*$-algebra $B$. Suppose that every real
positive functional on $A$ extends to a positive functional on $B$ (as is always the case if $A$ is unital or approximately unital by Lemma 2.4). Suppose that $T : A \rightarrow B(H)$ is a real positive linear map and that $y \in \mathfrak{H}_B \cap \text{Ker}(T)$. Let $F(x) = \sum_{k=1}^{n} a_k x b_k$ for $n \in \mathbb{N}$ and $a_k, b_k \in B$. Assume that either $a_k$ or $b_k$ is $y$ for all $k$. Then $A \cap F(B) \subset \text{Ker}(T)$.

To prove this principle, simply follow the proof of Lemma 2.11 almost verbatim to deduce that $\psi(xy) = 0$ for $x \in B$. Similarly $\psi(yx) = 0$. Thus if $F(b) \in A$ then $\psi(F(b)) = 0 = \varphi(T(F(b)))$. As in the proof we are following this implies that $T(F(b)) = 0$.

For example, we show how to use this principle to prove Lemma 2.11. As in the proof of 2.11 we may assume that $y \in \mathfrak{H}_B \cap \text{Ker}(T)$. The conclusion then follows by the principle and taking $F(x)$ to be first $x \circ y$ and then $yxy$, for $x \in A$.

**Lemma 2.12.** Suppose that a closed subspace $J$ of a Jordan operator algebra $A$ is contained in the closed Jordan algebra (or even in the hereditary subalgebra) generated by $C = \tau_J = J \cap \tau_A$, the set of real positive elements which $J$ contains.

1. If $xyx \in J$ for $y \in A, x \in C$, then $J$ is an HSA in $A$, and

$$J = \mathbb{R}_+ (\mathfrak{H}_J - \mathfrak{H}_J) = \text{Span}(\mathfrak{H}_J) = \text{Span}(\tau_J).$$

If in addition $x \circ y \in J$ for $y \in A, x \in C$, then $J$ is an approximately unital Jordan ideal in $A$.

2. If $A$ is approximately unital, and $T : A \rightarrow B(H)$ is real positive, and if $J = \text{Ker}(T)$ satisfies the condition before (1), then all the conditions in (1) hold, so that $\text{Ker}(T)$ is an approximately unital Jordan ideal in $A$.

**Proof.** Note that $C$ is convex, so that by a formula in [20, Theorem 3.18 (2)], the hereditary subalgebra $D$ generated by $C$ is the closure of $\{xAx : x \in C\}$, which by hypothesis is contained in $J$. So $D = J$. Hence $J = \mathbb{R}_+ (\mathfrak{H}_J - \mathfrak{H}_J) = \text{Span}(\mathfrak{H}_J)$ by [20, Proposition 4.4 and Theorem 4.1]. Now the last statement of (1) is obvious.

For (2), Lemma 2.11 shows that for any $a \in A, c \in J \cap \tau_A$, we have $aca$ and $a \circ c$ in $\text{Ker}(T)$. Now apply (1). \qed

**Remark.** Note that the proof of (1) works even if $C$ is replaced by $J \cap \mathfrak{H}_A$. One may also prove a variant of (1) where $C$ is a convex subset of $J \cap \tau_A$.

**Corollary 2.13.** If $A$ is an approximately unital operator algebra, $T : A \rightarrow B(H)$ is real positive, and if $J = \text{Ker}(T)$ is contained in the HSA generated by $J \cap \tau_A$ then $\text{Ker}(T)$ is an approximately unital ideal in $A$.

The following result is a fundamental fact concerning extending contractive linear maps on hereditary subalgebras (HSA’s) of $A$, and concerning the uniqueness of such extension. We will call the extension the zeroing extension since if $p$ is the support projection of the HSA $D$ then this extension is zero on the ‘complement’ $\{p^*ap^* + p^*ap + pap^* : a \in A\}$ of $D$. In [10] Proposition 2.11 and [20] Corollary 3.6 this extension was done for completely contractive linear maps in various settings, with a similar proof. We have chosen to briefly include most of the details of the proof for completeness and also to exhibit the ‘zeroing’ construction.

**Theorem 2.14.** Let $D$ be an HSA in an approximately unital Jordan operator algebra $A$. Then any contractive map $T$ from $D$ into a unital weak* closed Jordan operator algebra $N$ such that $T(\varepsilon_i) \rightarrow 1_N$ weak* for some partial cai $(\varepsilon_i)$ for $D$,
has a unique contractive extension $\tilde{T} : A \to N$ with $\tilde{T}(f_s) \to 1_N$ weak* for some (or all) partial cai $(f_s)$ for $A$. This extension is real positive.

**Proof.** The canonical weak* continuous extension $\hat{T} : D^{**} \to N$ is unital and contractive, and can be extended to a weak* continuous unital contraction $\Phi(\eta) = \hat{T}(p\eta p)$ on $A^{**}$, where $p$ is the support projection of $D$. This is real positive, and in turn restricts to a real positive contractive $\tilde{T} : A \to N$ with $\tilde{T}(f_s) \to 1_N$ weak* for all partial cai $(f_s)$ for $A$. Now for the uniqueness. Any other such extension $T' : A \to N$ extends to a weak* continuous unital contraction $\Psi : A^{**} \to N$, and $\Psi(p) = \underleftarrow{\lim} \Phi(e_t) = 1_N$. Then for $\eta \in A^{**}$ we have by Lemma 2.8 that

$$\Psi(\eta) = \Psi(p\eta p) = \hat{T}(p\eta p).$$

Thus $T'(a) = \Psi(a) = \tilde{T}(a)$ for $a \in A$. \hfill \square

### 3. Banach-Stone theorems

There are very many Banach-Stone type theorems in the literature. For example it is proved in [3, Corollary 2.8] that a unital surjective isometry between unital Jordan operator algebras is a Jordan homomorphism. In this section we wish to extend this latter result to nonunital surjective isometries between approximately unital Jordan operator algebras. An attempt at this was made in Proposition 4.15 in [20], but for complete isometries. There is an error in the proof of that result in the Jordan algebra case (it is correct in the associative operator algebra case). The correct result for Jordan operator algebras will be included in Theorem 3.5 below, although the next proposition, which is essentially due to Arazy and Solel [3], is a simpler variant of it.

As we mentioned in the introduction, there exist linearly isometric unital $JC^*$-algebras which are not Jordan $*$-isomorphic (see e.g. [22, Section 5] and [27, Antithesis 3.4.34]). Hence they are not even Jordan isomorphic by [27, Corollary 3.4.76]. Therefore Banach-Stone theorems for nonunital isometries between Jordan operator algebras are not going to look quite as one might expect: one cannot expect the Jordan isomorphism appearing in the conclusion to map onto the second $C^*$-algebra exactly. (As the referee pointed out one may obtain the latter only in very particular cases [27, Theorems 5.10.9 and 5.10.111], and in general the best one can say with this formulation is given by [27, Proposition 5.10.114].) Nonetheless we will show that with a slight adjustment of the setting, one may get a reasonable Banach-Stone type theorem for nonunital isometries between Jordan operator algebras.

We define a **quasimultiplier** of a Jordan operator algebra $B$ to be an element $w \in B^{**}$ with $bw = w^*b$ for all $b \in B$. A unitary in a unital selfadjoint Jordan operator algebra is an element $u$ with $\frac{1}{2}(uu^*1 + 1u^*) = 1$; this condition is easily seen to imply that $u^*u = uu^* = 1$ in any generated $C^*$-algebra.

**Lemma 3.1.** If $B$ is a Jordan operator algebra, and $u$ is a unitary in $\Delta(B^{**})$ such that $u^*$ is a quasimultiplier of $B$, and if we define $B(u)$ to be $B$ equipped with the new Jordan product $\frac{1}{2}(xu^*y + yu^*)$, then $B(u)$ is a Jordan operator algebra. If $B$ is isometrically (resp. completely isometrically) a Jordan subalgebra of a $C^*$-algebra $D$ then $B_u^*$ is a Jordan subalgebra of $D^{**}$ isometrically (resp. completely isometrically) Jordan isomorphic to $B(u)$.
Proof. This is evident from the discussion about $E(u)$ in the introduction: right multiplication $R_{u^*}$ by $u^*$ is the required Jordan isomorphism from $B(u)$ onto $Bu^*$.

Remark. We will use the idea in the last proof many times in the remainder of this Section. Note that the Jordan structure of $Bu^*$ is independent of the particular containing $C^*$-algebra $D$.

Proposition 3.2. Suppose that $T : A \to B$ is an isometric surjection between approximately unital Jordan operator algebras. Then there exists a unitary $u \in \Delta(B^{**})$ with $u^*$ a quasimultiplier of $B$, such that if $B(u)$ is the Jordan operator algebra in Lemma 3.1 then $T$ considered as a map into $B(u)$ is an isometric surjective Jordan homomorphism.

Proof. The proof of this is easy from 3. Corollary 2.8: $u = T^{**}(1)$ is a unitary in $\Delta(B^{**})$ and since $T^{**}$ preserves the ‘partial triple product’ from 3. we have $T(a^2) = T^{**}(a^2) = T(a)u^*T(a) \in B$ for $a \in A$. So $u^*$ is a quasimultiplier of $B$ and $T$ is a Jordan homomorphism.

Remarks. 1) The last result may also be stated in terms of expressing $T$ in the form $T = u\theta(\cdot)$, where $\theta : A \to C$ is an isometric surjective Jordan algebra homomorphism onto the Jordan subalgebra $C = u^*B$ of $D^{**}$, for $D$ as in the proof. However we will improve on this in Theorem 3.3 below. We also remark that one may use $u^*B$ in place of $Bu^*$ in the proof: left multiplication $L_{u^*}$ by $u^*$ is an isometric Jordan homomorphism from $B(u)$ onto $u^*B$.

2) With $B(u)$ as $B$ with the new product above, as in the last proof, we have that $B(u)$ is an approximately unital Jordan operator algebra. Examples like those in 24. Example 6.6] show that one may not hope that the quasimultiplier $u$ be a multiplier in $A$, even if $u = u^*$.

3) One sees Jordan algebra products given by quasimultipliers in Proposition 3.2 and e.g. in the proof of Theorem 4.4. Indeed we know from 15. Section 2] that all Jordan operator algebra products on an operator space $X$ are given by quasimultipliers: elements of the set

$QM(X) = \{u \in I(X) : xu^*x \in X \text{ for all } x \in X\}$.

This is related to ideas in 11 and 19. Remark 2 on p. 194. Here $I(X)$ is the injective envelope of $X$ (see 11. Chapter 4] or 50). One may ask if the element $u$ in the bidual in Proposition 3.2 can be associated with an element of the injective envelope $I(X)$ by an explicit procedure, so that the expression $bu^*b$ in Proposition 3.2 may be computed in the natural triple product of $I(X)$ (see e.g. the first paragraph of 4.4.7 in 11). Related to this is the following question of the second author and Russo [49]: Is the completely symmetric part of an operator space $X$ (see 49) equal to $QM(X) \cap X$? And if so, then is the restriction of the triple product on $I(X)$ to $X \times X_{cs} \times X$ equal to the partial triple product on $X$? One direction should be ‘easy’: if $y \in QM(X) \cap X$ then we get a Jordan algebra product on $X$.

Let $B$ be a $C^*$-algebra. A quasimultiplier $w$ of $B$ in the sense of the present paper is the same as a quasimultiplier in the $C^*$-algebraic sense of 24 say, which requires that $b_1wb_2 \in B$ for all $b_1,b_2 \in B$. This follows by applying 2. Theorem 4.1] to the real and imaginary part of $w$. Checking item (ii) in that theorem, for example, we get $a(w + w^*)a \in B_{s.a.}$ for $a \in B_{s.a.}$. Hence $a(w + w^*)b \in B$ for all
Proposition 3.3. Suppose that $B$ is a $C^*$-algebra and that $u$ is a unitary in $B^{**}$ such that $ubu \in B$ for all $b \in B$. Then $u \in M(B)$. Also, $b_1u^*b_2$ is a $C^*$-algebra product on $B$, with corresponding involution $ub^*u$. That is, $B(u)$ is a $C^*$-algebra.

Proof. The first assertion follows from the remarks above and by [2 Proposition 4.4] (see also [28, Corollary 5.10.107]). The remaining assertions are then straightforward. (We thank the referee for these references; we had included another proof in a previous draft).

Remark. The last result fails if we replace ‘unitary’ by ‘tripotent’ (that is, a partial isometry). There are examples in [23] of quasimultipliers $u$ of a $C^*$-algebra which are selfadjoint partial isometries, but which are not multipliers, and $B(u)$ is an operator algebra but is not a $C^*$-algebra.

Corollary 3.4. Suppose that $B$ is an approximately unital Jordan operator algebra, and that $D$ is a $C^*$-algebra generated by $B$, and $A$ is the operator algebra in $D$ generated by $B$. Let $u$ be a unitary in $\Delta(B^{**})$ with $u^*$ a quasimultiplier of $B$ such that $B(u)$ is an approximately unital Jordan algebra. Then $u$ and $u^*$ belong to the Jordan multiplier algebra $JM(B)$, and also to $M(D)$ and $M(A)$.

Proof. We may suppose that $B^{**}$ is a unital Jordan subalgebra of $D^{**}$, which in turn is a von Neumann algebra on $H$. Then $B^{**}u^*$ is a unital Jordan subalgebra of $B(H)$. For $d \in D$ by [20 Corollary 2.18] there exists $b_0 \in D$ and $a \in B$ with $d = ab_0a$, and we can choose $b_0 \in A$ if $b \in A$. Then $du^*d \in ab_0Bb_0a \in D$, and $du^*d \in A$ if $d \in A$. So $Du^*$ is a Jordan operator algebra, and so is $Au^*$. By [20, Lemma 2.6] one may choose a partial cai $(e_i)$ for $Bu^*$, then $e_i au^* \rightarrow au^*$ and $au^* e_i \rightarrow au^*$ in norm in $B(H)$ for $a \in B$. For $d = ab_0a \in D$ as above we see that $e_i du^* = e_i au^*ub_0au^* \rightarrow du^*$ and $d u^* e_i = ab_0au^* e_i \rightarrow du^*$ in norm. So $Du^*$ is an approximately unital Jordan subalgebra. Hence $D(u)$ is an approximately unital Jordan operator algebra. Thus $u \in M(D)$ by Proposition 3.3. For $x \in A$ we have $xu \in A^H \cap D = A$. Similarly, $xu^*, xu, u^*x \in A$, so that $u \in \Delta(M(A))$. Similarly, for $x \in B$ we have $u \circ x \in B^{**} \cap D = B$, so that $u \in JM(B)$. Similarly we have $u^* \in JM(B)$.

Theorem 3.5. Suppose that $T : A \to B$ is an isometric surjection between approximately unital Jordan operator algebras. Suppose that $B$ is (isometrically) a Jordan subalgebra of an (associative) operator algebra (resp. $C^*$-algebra) $D$, and that $B$ generates $D$ as an operator algebra (resp. $C^*$-algebra). Then there exists a unitary $u \in \Delta(JM(B))$ which is also in $\Delta(M(D))$, and there exists an isometric surjective Jordan algebra homomorphism $\pi : A \to C = Bu^*$, such that

$$T = \pi(\cdot)u.$$ 

Since $u^*$ is a quasimultiplier of $B$ in $M(D)$, $C = Bu^*$ is a Jordan subalgebra of $D$. In addition,

1. If $B$ is an (associative) operator algebra then we may take $C = B$ above (which also equals $D$ in the ‘non-respectively’ case).

2. If $T$ is a complete isometry and $B$ is completely isometrically a Jordan subalgebra of $D$ then $\pi$ is a complete isometry.

3. $C \subseteq \text{Span}(\{b_1b_2 : b_1, b_2 \in B\})$. 

$a, b \in B$ by the cited theorem. Similarly $\frac{1}{2!} a(w - w^*)b \in B$, so that $awb \in B$ for all $a, b \in B$. 


Proof. As in the proof of Proposition 3.2, \( T^{**}(1) = u \) is a unitary in \( \Delta(B^{**}) \), with \( u^* \) a quasimultiplier of \( B \). Hence as in that proof and the first remark after it, \( C = Bu^* \) is a Jordan subalgebra of \( D^{**} \), and \( \pi = T(\cdot)u^* \) is an isometric surjective Jordan algebra homomorphism from \( A \) onto \( Bu^* \). By Corollary 3.3 it follows that \( u \) is in \( \Delta(JM(B)) \) and in \( \Delta(M(D)) \) so that \( Bu^* \) is a Jordan subalgebra of \( D \). For (1), we have \( B = Bu^* = Bu^* \) since \( u \in \Delta(M(B)) \). Item (2) is obvious. For (3) let \( F = \overline{\text{Span}}(\{b_1b_2 : b_1, b_2 \in B\}) \). Because there is a net \( (b_i) \) in \( B \) with weak* limit \( u^* \), we have \( C \subset F^{\perp\perp} \cap D = F \). (Similarly \( C \subset \overline{\text{Span}}(\{b_1b_2^* : b_1, b_2 \in B\}) \).) \( \square \)

Remarks. 1) Of course as is usual with noncommutative Banach-Stone theorems, we can also write the unitary \( u \) on the left: \( T = u \theta(\cdot) \). To see this simply set \( \theta = u^*\pi(\cdot)u \).

2) There is a slight error in the converse direction of the proof of the Banach-Stone type result Proposition 6.6 in [14], in the Jordan algebra case. To fix this, appeal to [3] Corollary 2.8, after noting that \( T(1) = u \geq 0 \) by e.g. our Lemma 2.7 so that \( u = 1 \).

4. PROJECTIONS ON JORDAN OPERATOR ALGEBRAS

In this section we give variants of the results in Section 2 in our paper [14]. However our maps are no longer completely contractive, and our spaces are usually Jordan operator algebras. The following, which is due to Effros and Størmer [30] in the case that \( P(1) = 1 \), shows what happens in the self-adjoint Jordan case.

**Theorem 4.1.** If \( P : A \rightarrow A \) is a positive contractive projection on a JC*-algebra \( A \) then \( P(A) \) is a JC*-algebra in the \( P \)-product, \( P \) is still positive as a map into the latter JC*-algebra, and \( P(P(a) \circ P(b)) = P(a \circ P(b)) \) for all \( a, b \in A \). If in addition \( P(A) \) is a Jordan subalgebra of \( A \), then \( P \) is a Jordan conditional expectation: \( P(a \circ P(b)) = P(a) \circ P(b) \) for \( a, b \in A \).

**Proof.** If \( A \) is unital and \( P(1) = 1 \) then this is well known, following from the JC-algebra case in [30] Lemma 1.1 and Theorem 1.4. The general case follows from the unital case by appealing to Theorem 2.3. However we will give a second proof. We do not claim that this proof is better, we simply offer it as an alternative that may contain some useful techniques. In addition we will need later the fact in the Claim below. Again the claim follows from the unital case in [30] and Theorem 2.2, but a different proof may be of interest.

By considering \( P^{***} \) we may assume that \( A \) is unital. We have that \( P(A) \) is a \( J^* \)-algebra in a new triple product \( P(\{Px, Py, Pz\}) \) by [32] Theorem 2, and by [31] Corollary 1 we have that

\[
(4.1) \quad P(\{Px, y, Pz\}) = P(\{Px, Py, Pz\}) = P(\{Px, Py, z\})
\]

for all \( x, y, z \in A \). This is effectively Kaup’s contractive projection theorem [42] (see also [28] Theorem 5.6.59). By Lemma 2.2 we have \( 0 \leq P(1) \leq 1 \). Claim: \( P(P(1)^n) = P(1) \) for all \( n \in \mathbb{N} \). By the above, \( P(P(1)^3) = P(1P(1)^2) = P(P(1)^2) \). Assume for induction that \( P(P(1)^n - P(1)^n) = 0 \) for some \( n \geq 2 \). Then

\[
P(P(1)^n) - P(1)^{n+1} = P(P(1)^{n-1} - P(1)^n) = 0.
\]

It follows that

\[
(4.2) \quad P(P(1)^n) = P(P(1)^2), \quad n \geq 2.
\]
Next suppose that \((p_n(t))\) is a sequence of polynomials with no constant term converging uniformly to \(t^\diamond\) on \([0, 1]\). Then \(p_n(t^3) \to t\). Using (4.2) above and the contractivity of \(P\), by the spectral theorem,

\[
\|P(p_n(P(1)^3) - P(1))\| \leq \|p_n(t^3) - t\|_\infty \to 0.
\]

Again by (4.2), we have \(P(p_n(P(1)^3)) = P(p_n(1) P(1)^2) \to P(P(1)^2)\). Thus \(P(P(1)^2) = P(1)\). We have now proved the Claim.

It follows that \(u = P(1)\) is a tripotent (that is, a partial isometry) in the new triple product. Let \(x \in A_+\). By (1.1) above we have

\[
P(\{1 - u, P(x), y\}) = 0, \quad y \in P(A).
\]

Hence

\[
P(x) = P(uP(x)u) + 2P(\{1 - u, P(x), u\}) + P((1 - u)P(x)(1 - u))
\]

\[
= P(uP(x)u) + P((1 - u)P(x)(1 - u)).
\]

However, by the Claim, \(P((1 - u)P(x)(1 - u)) \leq \|x\| P((1 - u)^2) = 0\). Since this is true for all \(x \in A_+\),

\[
P(x) = P(uP(x)u) = P(uP(uP(x)u)u) = P(\{u, P(\{u, P(x), u\}, u)\})
\]

for all \(x \in A\). Hence \(u\) is a unitary tripotent in the new triple product on \(P(A)\).

Indeed \(uu^* xu^*u = x\) implies that

\[
2\{u, u, x\} = uu^* x + xu^*u = uu^* uu^* xu^*u + uu^* xu^*uu^*u = 2uu^* xu^*u = 2x.
\]

If a \(J^*\)-algebra \(Z\) has a unitary tripotent \(a\) then it is a \(JC^*\)-algebra with product \(\{a, a, a\}\) and involution \(\{u, a, u\}\), for all \(a \in P(A)\). In our case the latter product is \(P(uu^*a) = P(a^2)\). Also, \(\{u, a, u\} = P(uu^*u) = P(a^*) = P(a)^*\), using the fact that \(P\) is selfadjoint. So \(P(A)\) is a \(JC^*\)-algebra with the \(P\)-product \(P(a^2)\) and with the old involution \(P(a)^*\). Hence \(P\) is still selfadjoint. Also, \(P\) is still positive since it is unital and contractive. It also clear from (1.1) that for \(a, b \in A_{sa}\) we have

\[
P(P(a) \circ P(b)) = P(\{u, P(a), P(b)\}) = P(\{u, P(a), b\}) = \frac{1}{2} P(uP(a)b + bP(a)u).
\]

Similarly,

\[
P(P(a) \circ P(b)) = P(\{u, P(b), P(a)\}) = P(\{u, b, P(a)\}) = \frac{1}{2} P(ubP(a) + P(a)bu).
\]

Taking the average,

\[
P(P(a) \circ P(b)) = \frac{1}{2} (P(u \circ (bP(a))) + P(u \circ (P(a)b))) = P(u \circ (P(a) \circ b)).
\]

By Lemma 2.8 we have \(P(u \circ (P(a) \circ b)) = P(P(a) \circ b)\), so that \(P(P(a) \circ P(b)) = P(P(a) \circ b)\) as desired.

We thank the referee for identifying some mistakes and omissions in the last proof, which we have repaired.

As we said in Section 2 of [14], projections on operator algebras with no kind of approximate identity can be very badly behaved, thus we say little about such algebras. However one can pick out a ‘good part’ of such a projection:
Proposition 4.2. Let $P : A \to A$ be a real positive contractive map (resp. projection) on a Jordan operator algebra $A$ (possibly with no kind of approximate identity). There exists a largest approximately unital Jordan subalgebra $D$ of $A$, and it is a hereditary subalgebra of $A$. Moreover, $P(D) \subset D$, and the restriction $P'$ of $P$ to $D$ is a real positive contractive map (resp. projection) on $D$. In addition, $P'$ is bicontractive (resp. symmetric) if $P$ has the same property.

Proof. This follows as in [14, Proposition 2.1], but using [20, Corollary 4.2] in place of [15, Corollary 2.2].

Corollary 4.3. Let $P : A \to A$ be a real positive contractive projection on a unital Jordan operator algebra. Then $P(P(1)^n) = P(1)$ for $n \in \mathbb{N}$. In addition, $P(1)$ is a projection in $A$ if and only if $P(1)^2 \in P(A)$.

Proof. Note that $P$ restricts to a positive contractive projection from $\Delta(A)$ to $\Delta(A)$, and $P(1) \geq 0$ by Lemma 2.7. So we may assume that $A$ is a unital JC*-algebra, and then $P(P(1)^n) = P(1)$ was established in the proof we gave of Theorem 4.1 (or it may be deduced from that result). If $P(1)^2 \in P(A)$ then we deduce that $P(1)^2 = P(1)$, so that $P(1)$ is a projection.

In the sequel we will often restrict to the case that $P(1) = P^*(1)$ is a projection. This is automatic for example if $P$ is a real positive bicontractive or symmetric projection on an approximately unital Jordan operator algebra (by the proof of [14, Lemma 3.6]).

Lemma 4.4. Let $A$ be an approximately unital Jordan operator algebra, and suppose that $P : A \to A$ is a projection on $A$. Then $\Ker(P)$ is a Jordan ideal of $A$ if and only if $P(a^2) = P(P(a)^2)$ for $a \in A$, that is if and only if $P$ is a Jordan homomorphism with the $P$-product on $P(A)$. In this case $P(A)$ with the $P$-product is Jordan isomorphic to $A/\Ker(P)$, and this isomorphism is isometric if $P$ is a contraction.

Proof. For $a,b \in A$, $P((a - P(a)) \circ b) = 0$ if and only if $P(a \circ b) = P(P(a) \circ b)$. So $\Ker(P)$ is a Jordan ideal of $A$ if and only if $P(a \circ b) = P(P(a) \circ b)$ for all $a,b \in A$, which holds if and only if $P(a \circ b) = P(P(a) \circ P(b))$ for all $a,b \in A$. That is, if and only if $P$ is a Jordan homomorphism with the $P$-product on $P(A)$. We leave the rest as an exercise.

Lemma 4.5. Let $A$ be a unital Jordan operator algebra, and $P : A \to A$ a contractive projection, such that $\text{Ran}(P)$ contains an orthogonal projection $q$ with $P(A) \subset qP(A)q$. Then $q = P(1_A)$.

Proof. This follows as in [14, Proposition 2.6], but using that the identity is an extreme point of the unit ball of any unital Jordan operator algebra (since it is so in the generated C*-algebra).

The following is a converse to the previous result, uses a similar proof, and gives a little more:

Lemma 4.6. Let $A$ be a unital Jordan operator algebra, and let $P : A \to A$ be a contractive real positive projection. If $P(1) = q$ is a projection then $P(A) = qP(A)q$. In particular, $q \circ P(A) = P(A)$, indeed $q$ is the identity for the unital operator space $qP(A)q$. We also have

$$P(x) = P(qxq) = qP(qxq)q = qP(x)q$$
for \( x \in A \). Thus \( P \) restricts to a unital contractive (real positive) projection on \( qAq \), and \( P \) is zero on the ‘rest’ of \( A \), that is, on \( q^\perp Aq^\perp + \{ q^\perp aq + qaq^\perp : a \in A \} \).

**Proof.** This follows from Lemma 2.8 with \( e = 1 \) and then with \( e = q \). A second proof: We have \( P(\mathbb{F}) \subset \mathbb{F} \) by Theorem 2.5. Fix \( x \in \text{Ball}(A) \). Then \( P(1 \pm x) = q \pm P(x) \), and \( \| 1 - q \pm P(x) \| \leq 1 \). This forces \( \| 1 - q \pm P(x)q^\perp \| \leq 1 \), and since \( I_A \) is an extreme point of \( B(K) \) we see that \( q^\perp P(x)q^\perp = 0 \). Looking at the matrix of \( P(x) \) with respect to \( q^\perp \) and using \( \| 1 - q + q^\perp P(x)q^\perp \| \leq 1 \) we also see that \( P(x) = qP(x)q \). Now appeal to Lemma 2.8.

The last lemma may be viewed as a ‘reduction’ to the case that \( P \) is unital. In the approximately unital case things are more difficult, since \( qAq \) may not be a subset, let alone a subalgebra, of \( A \). Here \( q = P^{**}(1) \in A^{**} \). One might hope that \( q \) might be some kind of multiplier of \( A \), but this is not usually the case. The solution to this difficulty is found in the notion of hereditary subalgebra, and the notion of ‘zeroing extension’ of maps on HSAs which we saw in and above Theorem 2.14 and whose application to projections we shall describe after Lemma 4.7.

**Lemma 4.7.** Let \( A \) be an approximately unital Jordan operator algebra, and \( P : A \to A \) a contractive projection. Let \( q = P^{**}(1) \). The following are equivalent:

(i) \( q \) is a projection and \( P \) is real positive.

(ii) \( q^2 \in \text{Ran}(P^{**}) \) and \( P \) is real positive.

(iii) \( \text{Ran}(P^{**}) \) contains an orthogonal projection \( r \) such that \( P(A) \subset rP(A)r \).

If these hold then \( r = q = P^{**}(1) \), and the bidual of \( \text{Ran}(P) \) is a unital operator space with identity \( q \). Also, \( q \) is an open projection for \( A^{**} \) in the sense of our introduction, so that \( D = \{ a \in A : a = qaq \} \) is a hereditary subalgebra, and \( D^{**} = qA^{**}q \).

**Proof.** The equivalence of (i) and (ii) follows from Corollary 4.1. Let \( Q = P^{**} \), a contractive projection on \( A^{**} \). We can replace \( Q \) by \( P \) below if \( A \) is unital. If \( P(A) \subset qP(A)q \) for a projection \( q \in Q(A^{**})q \) then \( Q(A^{**}) = P(A) = Q(A^{**})q \) by standard weak* approximation arguments. By Lemma 4.5 we have \( Q(1) = q \). Since \( \text{Ran}(Q) \) is a unital operator space (in \( qA^{**}q \)) with identity \( q \), and \( Q(1) = q \), we see by the line before Corollary 2.2 that \( Q \), and hence also \( P \), is real positive as a map into \( qA^{**}q \).

Conversely, suppose that \( P \) is real positive and \( Q(1) = q \) is a projection. Then \( Q(q^\perp) = 0 \). Then Lemma 4.6 gives \( P(A) = qP(A)q \).

If \( (e_t) \) is a cai for \( A \) then \( e_t \to 1 \) weak*, so that \( P(e_t) = qP(e_t)q \to qP^{**}(1)q = q \) weak*. Thus \( q \) is an open projection in \( A^{**} \) in the sense of [15], and \( D \) is a hereditary subalgebra as stated. The rest is clear.

**Remark.** We remark that there is a mistake in the statement of the matching result in [14], namely Proposition 2.7 there. After the phrase ‘\( P^{**}(1) \) is a projection’ there the condition ‘\( P \) is real completely positive’ should be added, similarly to the statement of (ii) above. This mistake led to an error in [14, Theorem 3.7]: in the statement of that result the phrase ‘real completely positive’ should be deleted in the last line (see our Theorem 5.2 for a way to state the converse direction of that theorem).

We now describe the ‘zeroing extension’ of maps on HSAs which we saw in and above Theorem 2.14 as applied to projections. Suppose that \( P : A \to A \) is a real
positive contractive projection on an approximately unital Jordan operator algebra, such that $P^{**}(1) = q$ is a projection. By Lemma 4.7, $D = \{a \in A : a = qaq\}$ is a hereditary subalgebra of $A$. Suppose that $D$ is represented nondegenerately on $B(K)$ for a Hilbert space $K$ (see [20] Section 6). Then the restriction $E$ of $P$ to $D$, viewed as a map $D \to B(K)$ satisfies the cai condition in Theorem 2.13 since if $(e_t)$ is a partial cai for $D$ then $e_t \to q$ weak* in $D^{**}$ and $e_t \to I_K$ weak* in $B(K)$. That is, $q$ acts as the identity on $K$. Hence $P(e_t) \to I_K$ weak* since $P^{**}(q) = q$. Thus by theorem $E$ extends uniquely to a contraction from $A$ to $B(K)$ satisfying a similar cai condition spelled out in Theorem 2.13. This is the ‘zeroing extension’, which kills ‘the complement’ $\{q^+aq^+ + q^-aq + qaq^- : a \in A\}$ of $D$. However by uniqueness this extension must be $P$, since $P^{**}(1) = q = 1_{D^{**}}$, which acts as the identity on $K$. Thus $P$ is the zeroing extension of the restriction of $P$ to $D$. One may view this as a ‘cut down to the unital case’ procedure: $P$ is ‘unital’ on the HSA. That is, $E^{**}$ is a unital projection on $D^{**}$, and $P^{**}$ and $P$ are zero on the ‘complement’ of the HSA since $P(a) = P(qaq)$ for all $a \in A$.

The following theorem is the contractive projection version of [14] Proposition 2.7.

**Theorem 4.8.** Let $A$ be an approximately unital Jordan operator algebra, and $P : A \to A$ a real positive contractive projection. Suppose that $q = P^{**}(1)$ is a projection. We have

$$P(a) = qP(a)q = P^{**}(qaq), \quad a \in A,$$

(and we can replace $P^{**}$ by $P$ here if $A$ is unital). Hence $P(A) = qP(A)q = P^{**}(qAq)$. Also,

1. $P$ ‘splits’ as the sum of the zero map on $q^+Aq^+ + \{q^+aq + qaq^+ : a \in A\}$, and a real positive contractive projection $P'$ on $qAq$ with range equal to $P(A)$. This projection $P'$ on $qAq$ is unital if $A$ is unital.

2. $P$ restricts to a real positive contractive projection $E$ on the hereditary subalgebra $D$ supported by $q$ (see Lemma 4.7). We have $E(D) = P(A) \subset D$, and $E^{**}$ is unital: $E^{**}(q) = q$.

3. $P$ is the zeroing extension of $E$.

**Proof.** The displayed formula follows from the last result and Lemma 4.6 applied to $P^{**}$. Item (3) is discussed above the theorem. We leave the rest as an exercise. □

**Lemma 4.9.** Let $A$ be an approximately unital Jordan operator algebra, and let $P : A \to A$ be a contractive real positive projection. Then

$$P(a \circ b) = P(a \circ P(b)), \quad a \in P(A), b \in \Delta(A).$$

In particular, if $A$ is unital and $q = P(1)$ then $a = P(a \circ q)$ for $a \in P(A)$.

**Proof.** By considering $P^{**}$ we may suppose that $A$ is unital. By [3] Theorems 2.6 and 2.7 the symmetric part of $A$ is $\Delta(A) = A \cap A^*$, and the partial triple product on $A$ is $\{a, b, c\} = (ab^*c + cb^*a)/2$ for $a, c \in A, b \in \Delta(A)$. The restriction of $P$ to $\Delta(A)$ is real positive, hence is a positive map into $\Delta(A)$ as in Lemma 2.7. By Proposition 5.6.39 (i) and (ii), $P(\{a, b, c\}) = P(\{a, P(b), c\})$ for $a, c \in P(A), b \in \Delta(A)$. (We have used here the fact from Lemma 2.7 that $P(\Delta(A)) \subset \Delta(A)$.) Setting $c = 1$ we have

$$P(a \circ b) = P(a \circ P(b)^*) = P(a \circ P(b)), \quad a \in P(A), b \in \Delta(A),$$

and we can replace $P^{**}$ by $P$ here if $A$ is unital.
since $P$ is positive on $\Delta(A)$. Setting $b = 1$ gives $P(a) = a = P(a \circ q)$ for $a \in P(A)$.

\textbf{Remarks.} 1) We thank J. Arazy for the main insights in the previous proof.

2) As we said in the introduction, a couple of paragraphs after Theorem 1.1 $P(ab) \neq P(a)b$ in general for a unital (even commutative) operator algebra $A$ and contractive unital (hence real positive) projection from $A$ onto a subalgebra containing $1_A$, and $a \in A, b \in P(A)$. Thus one cannot hope for Jordan operator algebra variants of the centered equation in [14, Theorem 2.5] or the conditional expectation assertion in [14, Corollary 2.9]. Also $\ker(P)$ is not a Jordan ideal in this case. However with respect to these results in the case of contractive morphisms we will see later that things become much better under extra hypotheses (such as those in Theorem 4.10, Corollary 4.13, or Theorem 5.2).

3) Lemma 4.9 may be used to give a different proof of a variant of Theorem 4.8.

The following is a variant of the Choi-Effros result referred to in the introduction. The $JC$-algebra version of the result is due to Effros and Størmer [30].

\textbf{Theorem 4.10.} Let $A$ be an approximately unital Jordan operator algebra, and $P : A \to A$ a contractive real positive projection. Suppose that $\ker(P)$ (resp. $\ker(P) \cap \text{joa}(P(A))$) is densely spanned by the real positive elements which it contains. Then the range $B = P(A)$ is an approximately unital Jordan operator algebra with product $P(x \circ y)$. If $A$ is unital then $P(1)$ is the identity for the latter product. Also $P$ is a (real positive) Jordan homomorphism with respect to this product:

$$P(a \circ b) = P(a \circ P(b)) = P(P(a) \circ P(b))$$

for $a, b$ in $A$ (resp. in $\text{joa}(P(A))$).

\textbf{Proof.} We may assume (using Lemma 2.10) that $A = \text{joa}(P(A))$. By Lemma 2.12 it follows that $\ker(P)$ is an approximately unital Jordan ideal in $A$. By Lemma 4.4 we deduce that $P(A)$ is an approximately unital Jordan operator algebra and $P$ is a Jordan homomorphism, both with respect to the new product. That $P$ is real positive with respect to the new product follows e.g. since $P^{**}$ is a unital contraction into $qA^{**}q$ and hence is real positive.

\textbf{Remarks.} 1) One may ‘weaken’ the condition in Theorem 4.10 about being ‘densely spanned by the real positive elements which it contains’, to being ‘contained in the closed Jordan algebra generated by the real positive elements it contains’, or even being ‘contained in the hereditary subalgebra generated by the real positive elements it contains’. The proof in these latter cases is only slightly more difficult, one needs to appeal to Lemma 2.12 (2). We used quotes around ‘weaken’ because once we know (by that lemma) that $\ker(P)$ is approximately unital then it follows (e.g. as is clear from that lemma) that it is the span of the real positive elements which it contains.

2) Note that if $P$ is a positive projection on a $JC^*$-algebra, then the condition requiring $D = \ker(P) \cap \text{joa}(P(A))$ be densely spanned by the real positive elements which it contains, is always true. Indeed, note that $D$ is selfadjoint. By [30, Lemma 1.2] applied to $P^{**}$, if $x \in D_{sa}$ then $P(x^2) = P(P(x)^2) = 0$, and so $x^2 \in D$. Hence $D_{sa}$ is a Jordan subalgebra of $A_{sa}$. Thus it is spanned by positive elements by the usual functional calculus (in $C^*(x)$ for $x \in D_{sa}$). Hence $D = D_{sa} + iD_{sa}$ is spanned by its positive elements. Thus the ‘respectively’ case of Theorem 4.10 generalizes.
Theorem 4.1 (and Theorem 1.4 of [30]), that the range of a positive projection on a \( JC^\ast \)-algebra is isometric to a \( JC^\ast \)-algebra.

However, for a positive projection \( P \) on a \( C^\ast \)-algebra, \( \text{Ker}(P) \) need not have nonzero, nor be spanned by its, positive elements (even if \( P \) is a state). So we feel the ‘respectively’ case of Theorem 4.10 is a suitable generalization of Theorem 4.1 (and Theorem 1.4 of [30]).

3) Key to this proof is obtaining that \( \text{Ker}(P) \) is an ideal. If \( P \) is also completely contractive (and unital) then we showed in [14, Corollary 2.11] (see Corollary 4.17) that \( \text{Ker}(P) \) is always an ideal in \( A \), if \( A \) is generated by \( P(A) \). We do not know if this is correct with the word ‘completely’ removed. However if \( A \) is an approximately unital associative operator algebra and \( P \) is a real positive ‘conditional expectation’ then \( \text{Ker}(P) \) is an ideal in \( A \) if \( A \) is generated by \( P(A) \), by Theorem 4.15 and the remark after it (since \( P(A) \), and hence the operator algebra generated by \( P(A) \), is contained in the algebra \( B \) there).

Corollary 4.11. Let \( A \) be an approximately unital Jordan operator algebra, and \( P : A \to A \) a contractive projection whose range is a Jordan subalgebra. Suppose that \( \text{Ker}(P) \) (resp. \( \text{Ker}(P) \cap \text{oj}(P(A)) \)) is densely spanned by the real positive elements which it contains. Then \( P \) is real positive if and only if \( P(A) \) is approximately unital, and in this case \( P \) is a Jordan conditional expectation:

\[
P(a \circ P(b)) = P(a) \circ P(b)
\]

for \( a, b \) in \( A \) (resp. in \( \text{oj}(P(A)) \)).

Proof. If \( P(A) \) is approximately unital, then \( P \) is real positive by Theorem 4.7. The rest is clear from Theorem 4.10. \( \square \)

Following the last proofs, but using the operator algebra case of the results used, yields:

Theorem 4.12. Let \( A \) be an approximately unital Jordan operator algebra, and \( P : A \to A \) a contractive real positive projection. Suppose that \( \text{Ker}(P) \) (resp. \( \text{Ker}(P) \cap \text{oj}(P(A)) \)) is densely spanned by the real positive elements which it contains. Then the range \( B = P(A) \) is an approximately unital operator algebra with product \( P(xy) \). If \( A \) is unital then \( P(1) \) is the identity for the latter product. Also \( P \) is a homomorphism with respect to this product:

\[
P(ab) = P(aP(b)) = P(P(a)b) = P(P(a)P(b))
\]

for \( a, b \) in \( A \) (resp. in \( \text{oj}(P(A)) \)). Finally, if \( P(A) \) is a subalgebra of \( A \) then the last quantity in the last centered equation equals \( P(a)P(b) \).

Remark. As in the first Remark after Theorem 4.10 one may ‘weaken’ the condition in Theorem 4.12 about being ‘densely spanned by the real positive elements which it contains’, to being ‘contained in the closed algebra (or even HSA) generated by the real positive elements it contains’.

For the readers convenience we mention the following ‘partially selfadjoint’ results which are proved in [9]:

Theorem 4.13. Let \( A \) be an approximately unital Jordan operator algebra, and \( P : A \to A \) a real positive contractive projection with \( P(A) \subset \Delta(A) \). Then \( P(A) \) is a \( JC^\ast \)-algebra in the \( P \)-product, and the restriction of \( P \) to \( \text{oj}(P(A)) \) is a Jordan
*-homomorphism onto this $JC^*$-algebra. In this case $P$ is a Jordan conditional expectation with respect to the $P$-product:

$$P(a \circ P(b)) = P(P(a) \circ P(b))$$

for $a, b$ in $A$.

**Remark.** A main ingredient from [9] in the proof of the last result is the following lemma: Let $A$ be an approximately unital Jordan operator algebra, and let $P : A \to A$ be a contractive real positive projection such that $P(A)$ is a Jordan operator algebra with $P$-product. If $q \in P(A)$ is a projection in the $P$-product then $P(qaq) = P(qP(a)q)$ and $P(a \circ q) = P(P(a) \circ q)$ for all $a \in A$. (These assertions follow from Lemma 2.10.) If further $A$ is weak* closed and $P$ is weak* continuous then

$$P(a \circ b) = P(P(a) \circ b), \quad a, b \in A.$$  

**Corollary 4.14.** Let $A$ be an approximately unital (associative) operator algebra, and $P : A \to A$ a real positive contractive projection with $P(A) \subset \Delta(A)$. Then the $P$-product on $P(A)$ is associative (which happens for example if $P(A)$ is an (associative) subalgebra of $A$) if and only if $P$ is completely contractive. In this case $P$ is real completely positive, $P(A)$ is a $C^*$-algebra in the new product, and $O$ is a conditional expectation in the latter product:

$$P(P(a)P(b)) = P(P(a)b) = P(aP(b)), \quad a, b \in A.$$  

**Remark.** A contractive (real positive) Jordan conditional expectation need not be completely contractive. Merely consider $P(x) = \frac{1}{2}(x + x^\dagger)$ on $M_2$. The same example also shows the necessity of the condition in the last result that the new product on $P(A)$ is associative (even in the case that $A$ is a $C^*$-algebra).

The following result, also from [9], is inspired by the selfadjoint case due to Effros et al (see e.g. [20] Lemma 1.4]). If $A$ is a unital operator algebra, and $P$ is a unital contractive or completely contractive projection on $A$, define

$$N = \{ x \in A : P(xy) = P(yx) = 0 \text{ for all } y \in A \}.$$  

If $A$ or $P$ is not unital, but $P$ is also real positive, then we may extend $P$ to a unital contractive projection on $A^1$, where $A^1$ is a unitization with $A^1 \neq A$, and set $N = \{ x \in A : P(xy) = P(yx) = 0 \text{ for all } y \in A^1 \}$. Then $N$ is clearly a closed ideal in $A$, and is also a subspace of $\text{Ker}(P)$. Define

$$B = \{ x \in A : P(xy) = P(P(x)P(y)) \text{ and } P(yx) = P(P(y)P(x)) \text{ for all } y \in A \}.$$  

Then $N \subset B$ since if $x \in N \subset \text{Ker}(P)$ then $P(xy) = 0 = P(P(x)P(y))$ for all $y \in A$. Note too that $1 \in B$ if $A$ is unital and $P(1) = 1$.

**Theorem 4.15.** If $P$ is a real positive contractive projection on an approximately unital operator algebra $A$, and $N, B$ are defined as above, then $P(A) \subset B$ if and only if

$$P(P(a)b) = P(P(a)P(b)) = P(aP(b)) \text{ for all } a, b \in A.$$  

That is, if and only if $P$ is a conditional expectation onto $P(A)$ with respect to the $P$-product. This is also equivalent to $B = P(A) + N$. If these hold then $P(A)$ with the $P$-product is isometrically isomorphic to an operator algebra, $B$ is a subalgebra of $A$ containing $P(A)$, and $P$ is a homomorphism from $B$ onto $P(A)$ with the $P$-product.
Remark. Note that in the last result \( N = \text{Ker}(P) \) iff \( \text{Ker}(P) \) is an ideal. The latter holds (by the associative algebra variant of Lemma 4.4) if and only if \( B = A \), and then all of the conclusions of the last theorem hold. Also, if \( P \) is real completely positive and completely contractive then \( P(P(a)b) = P(P(a)P(b)) = P(aP(b)) \) for \( a, b \in A \) as is proved in \([14] \) Section 2], so that the conclusions of the last theorem hold.

Finally, we comment on the operator space version of the theory in the present section. All of the results in \([14] \) Section 2] stated for completely contractive projections on operator algebras are true for for completely contractive projections on Jordan operator algebras, with essentially unchanged proofs. As we said in the introduction, the main extra thing one needs to know for some of these proofs to go through is the following:

Lemma 4.16. Let \( A \) be an approximately unital Jordan operator algebra, and let \( A^1 \) be its unitization. Then the injective envelope \( I(A) = I(A^1) \), and this may be taken to be a unital \( C^* \)-algebra containing \( A \) completely isometrically as a Jordan subalgebra. Moreover, any injective envelope of the unitization of \( A \) is an injective envelope of \( A \).

Proof. This was stated in Corollary 2.22 of \([20] \). Since the proof was rather terse we give more details. The idea is similar to the proof of \([11] \) Corollary 4.2.8, but uses facts in and after \([20] \) Lemma 2.19 about SOT convergence of partial cai for a nondegenerate representation of an approximately unital Jordan operator algebra. These give, in the notation used in \([11] \) Corollary 4.2.8, that the restriction of the functional \( \varphi = \langle \Phi(\cdot)\zeta, \zeta \rangle \) on \( B(H) \) to \( A \), is a state of \( A \). So the restriction of \( \varphi \) to \( A + \mathbb{C} I_H \) has the same norm 1. By \([20] \) Lemma 2.20 (1), \( \varphi(I) = \langle \Phi(I)\zeta, \zeta \rangle = 1 \). It follows that \( \Phi(I) = I \), and the rest is as in the proof of \([11] \) Corollary 4.2.8. For example, the operator system \( R \) there, which is an injective envelope of \( A \), is an injective envelope of \( A^1 \), and by the Choi-Effros result cited in our introduction, \( R \) is a unital \( C^* \)-algebra.

We obtain for example:

Corollary 4.17. Let \( P : A \to A \) be a unital completely contractive projection on a Jordan operator algebra. If \( P(A) \) generates \( A \) as a Jordan operator algebra, then \( \text{Ker}(P) \) is a Jordan ideal in \( A \). In any case, if \( D \) is the closed Jordan algebra generated by \( P(A) \) then \( \text{Ker}(P_D) \) is a Jordan ideal in \( D \), and \( P_D \) is a Jordan homomorphism with respect to the \( P \)-product on its range.

Proof. We may assume that \( P(A) \) generates \( A \). As in the proof of \([14] \) Corollary 2.11 but using Lemma 4.10 we extend \( P \) first to a unital completely contractive projection on a \( C^* \)-algebra \( B (= I(A)) \), then to a weak* continuous unital completely contractive projection \( \tilde{P} \) on a von Neumann algebra \( M (= B^{**}) \). Let \( \hat{P} \) also denote the restriction of the latter projection to the von Neumann algebra \( N \) generated by \( P(A) \) inside \( M \). If \( x \in (I-P)(A) \), then \( \hat{P}(x) = 0 \), and so by \([14] \) Proposition 2.10 we have \( ex = xe = 0 \). Thus \( x \in e^\perp Me^\perp \) and \( x \in e^\perp Ne^\perp \). So for \( y \in A \) we have \( \tilde{P}(xy + yx) = P(exye + eyx) = 0 \). Hence \( \text{Ker}(P) \) is a Jordan ideal. The Jordan homomorphism assertion follows from Lemma 4.3.

Corollary 4.18. Let \( A \) be an approximately unital Jordan operator algebra, and \( P : A \to A \) a completely contractive projection which is also completely real positive.
Then the range \( B = P(A) \) is an approximately unital Jordan operator algebra with product \( P(x \circ y) \). If \( A \) is unital then \( P(1) \) is the identity for the latter product. Also \( P \) is a Jordan conditional expectation with respect to this new product:
\[
P(a \circ P(b)) = P(P(a) \circ P(b)), \quad a, b \in A.
\]

If \( P(A) \) is a Jordan subalgebra of \( A \) then the last quantity in the last centered equation equals \( P(a) \circ P(b) \).

**Proof.** (Sketch.) As usual we may assume \( A \) is unital. The first assertion follows from Corollary 4.17 and the idea in the proof of Lemma 4.4 with \( A \) replaced by \( \text{joa}(P(A)) \). Alternatively, all the results here follow as in [14, Section 2] by extending \( P \) to a completely contractive completely positive projection \( Q \) on \( I(A) \). One then appeals to the \( C^* \)-algebra case for the matching results for \( Q \), which yield our results when restricted to \( A \).

The following kind of complement exists in the mixed situation that \( P \) is completely contractive but possibly not completely real positive:

**Theorem 4.19.** If \( P : A \to A \) is a real positive completely contractive projection on an approximately unital operator algebra \( A \) then with respect to the \( P \)-product \( P(A) \) is an approximately unital operator algebra.

**Proof.** By standard arguments, \( Q = P^{**} \) is a real positive completely contractive projection on the unital operator algebra \( M = A^{**} \). Following the proof of [15, Corollary 2.3], \( Q(xy) \) defines an operator algebra product on \( Q(M) \). By Lemma 4.19, \( q = Q(1) \) is a Jordan identity for the new product. Therefore \( q \) is a projection and is an identity of norm 1 for the new product by the discussion around formula (1.1) in [20]. We note that the map \( Q \) regarded as mapping into this operator algebra is a unital complete contraction, hence is real completely positive.

Since \( P(A)^{**} = P(A)^{\perp\perp} = P^{**}(A^{**}) \) (see e.g. the proof of Lemma 4.7), \( P(A) \) is an approximately unital operator algebra with the \( P \)-product, by e.g. [11, Proposition 2.5.8].

**Remark.** One cannot hope for a complement to Theorem 1.1 saying that the projections there are always conditional expectations with respect to the new operator product on \( P(A) \) coming from Theorem 1.1. That is, one cannot hope for \( P(P(x) \circ y) = P(x \circ y) \) for all \( x \in A \) and \( y \in P(A) \) in (1) and (3) there, or \( P(P(x)y) = P(xy) \) in (2). It is easy to find counterexamples, for example the projection in the \( 2 \times 2 \) matrix example at the end of Section 5.

5. Symmetric projections

We first recall the solution to the bicontractive and symmetric projection problem for \( JC^* \)-algebras, essentially due to deep work of Friedman and Russo, and Størmer [31, 33, 54]. Some of this hinges on Harris’s Banach–Stone type theorem for \( J^* \)-algebras [37] mentioned in the introduction (where we mentioned the main facts about morphisms between \( JC^* \)-algebras). The following is essentially very well known (see the references above), but we do not know of a reference which has all of these assertions, or is in the formulation we give:

In the following result \( M(A) \) is the multiplier algebra from [28], that is the elements \( x \in A^{**} \) with \( x \circ A \subset A \), or equivalently with \( \{x, y, z\} \subset A \) for all \( y, z \in A \). These imply that \( yx^*y \in A \) for all \( y \in A \). If \( A \) is a \( C^* \)-algebra then this is just
the usual $C^*$-algebraic multiplier algebra. This follows from e.g. [28 Proposition 5.10.96] or a fact about $JM(A)$ from our introduction.

**Theorem 5.1.** If $P : A \to A$ is a projection on a $JC^*$-algebra $A$ then $P$ is bicontractive if and only if $P$ is symmetric. Moreover $P$ is bicontractive and positive if and only if there exists a central projection $q \in M(A)$ (indeed $q \circ a = qaq \in A$ for all $a \in A$) such that $P = 0$ on $q^* Aq^*$, and there exists a Jordan $*$-automorphism $\theta$ of $qAq$ of period 2 (i.e. $\theta \circ \theta = I$) so that $P = \frac{1}{2}(I + \theta)$ on $qAq$. Finally, $P(A)$ is a $JC^*$-subalgebra of $A$, and $P$ is a Jordan conditional expectation.

**Proof.** Clearly symmetric projections are bicontractive. Conversely, if $P$ is bicontractive then by [31 Theorem 2] $\theta = 2P - I_A$ is a linear surjective isometric Jordan triple product preserving selfmap of $A$ (that is, a $J^*$-algebra isomorphism) with $\theta \circ \theta = I_A$, and $P = \frac{1}{2}(I_A + \theta)$. So $P$ is symmetric. If also $P$ is positive then $P$, and hence also $\theta$, is $*$-linear. Let $Q$ be the extension of $P$ to the second dual. Let $u = \theta^*(1)$, which is a selfadjoint unitary (a symmetry) in the $C^*$-algebra $D$ generated by $A^{**}$. Since $\theta^{**}$ is a $J^*$-algebra isomorphism we have $u \in M(A)$, thus $u \circ A \subset A$. So $q = Q(1) = (1 + u)/2$ is a projection and $q \circ A \subset A$ (that is, $q \in M(A)$). We have $u\theta^{**}(x)u = u\theta^{**}(x^*x)^*u = \theta^{**}(1(x^*)x) = \theta^{**}(x)$. So $u$, and hence also $q$, is central with respect to the product in $D$. Thus $qq^* = q \circ q$ for $q \in A^{**}$. It follows that $A = qAq + q^* Aq^*$.

Since $Q(q^*) = 0$, if $a \in Ball(A)_+$ then

$$P(q^* a q^*) \leq Q(q^*) = 0,$$

and so $P = 0$ on $q^* Aq^*$. Also, since $\theta(q) = 2P(q) - q = q$ and $\theta$ is $*$-linear and Jordan triple product preserving, it follows that $\theta(qa) = q\theta(a)q$ and $P(qa) = \frac{1}{2}(qa + q\theta(a)q)$, and $P(q) = 2P(q) - q = qAq$ and $\theta : qAq \to qAq$, and the restriction of $\theta$ to $qAq$ is a unital bicontractive positive projection on a unital $JC^*$-algebra. Also since $\theta(qa) = \theta(q)a$ for $a \in A$, we had above, $\theta(qAq) = qAq$. Hence $\theta^* = \theta_{qAq}$ is a unital isometric Jordan $*$-automorphism of $qAq$, and $P = \frac{1}{2}(I + \theta^*)$ on $qAq$. The converse is easy; the centrality of $q$ allowing it to suffice to check the conditions on each of the two orthogonal parts of $A$.

Finally $P(A) = P(qAq)$, which consists of the fixed points of $\theta'$ and hence is a $JC^*$-subalgebra. Moreover if $a \in A, b \in P(A)$ we have

$$P(a \circ b) = P(qa \circ b) = \frac{1}{2}(qa \circ b + \theta'(qa \circ b)) = P(a) \circ b$$

since $\theta'$ is a Jordan homomorphism, and $b$ is fixed by $\theta'$.

The following is the solution to the symmetric projection problem in the category of approximately unital Jordan operator algebras.

**Theorem 5.2.** Let $A$ be an approximately unital Jordan operator algebra, and $P : A \to A$ a symmetric real positive projection. Then the range of $P$ is an approximately unital Jordan subalgebra of $A$ and $P$ is a Jordan conditional expectation. Moreover, $P^{**}(1) = q$ is an open projection (in the sense of our introduction) in the Jordan multiplier algebra $JM(A)$, and all of the conclusions of Lemma [17] and Theorem [15] hold.

Set $D$ to be the hereditary subalgebra of $A$ supported by $q$, indeed $D = qAq$, which contains $P(A)$. Let $W = q^* Aq^* + \{q^* aq + qa^*: a \in A\}$, which is a complemented subspace of $A$, indeed $A = D \oplus W$. There exists a period 2 surjective
isometric Jordan isomorphism $\pi : D \to D$, which is the restriction of a period 2 isometric selfmap of $A$, such that

$$P = \frac{1}{2}(I + \pi) \quad \text{on} \quad D,$$

and $P = 0$ on the complement $W$ of $D$ in $A$ (thus $P(a) = P(qaq)$ for all $a \in A$). The range of $P$, which equals $P(D)$, is exactly the set of fixed points of $\pi$ in $D$.

Conversely, if $q$ is a projection in $JM(A)$, $\pi$ is a period 2 isometric Jordan automorphism of $D = qAq$, and if $P = \frac{1}{2}(I + \pi)$ on $D$ and $P = 0$ on the complement $W$ above of $D$ in $A$, then $P$ is a symmetric real positive projection on $A$.

Proof. As before $P**(1) = q$ is a projection, and all the conclusions of Lemma 4.7 and Theorem 4.8 are true for us. We will silently be using facts from these results below. In particular $q$ is an open projection, so supports an approximately unital HSA $D$ of $A$ with $D^{1,1} = qA^*q$. We know that $P(A) \subset D$. Then $\theta = 2P - I$ is a period 2 linear isometric surjection on $A$. If $u = \theta**(1) = 2P**(1) - 1 = 2q - 1$, then $u$ is a selfadjoint unitary (a symmetry). If $D$ is a $C^*$-algebra generated by $A$, then $u$ and $q$ are in $\Delta(JM(A))$ and $\Delta(M(D))$ by Theorem 3.5. Thus $qAq \subset A^{1,1} \cap D = A$. Hence $D = qAq$. Similarly, $q^1Aq^1 \subset A$, and $\{qaq^1 + q^1aq : a \in A\} \subset A$. Thus $A = W \oplus D$ as desired.

We have $\theta**(q) = q$. Let $\pi = \theta_D$. We have as before that $\theta(D) \subset D = \theta^2(D) \subset \theta(D)$, so that $\pi(D) = \theta(D) = D$. Since $\theta**(q) = q$, that is $(\pi)**$ is unital, $\pi$ is a Jordan homomorphism as we said at the start of Section 3. Since $P(A)$ consists of the fixed points of $\pi$ in $D$, $P(A)$ is a Jordan subalgebra of $A$. Moreover if $a \in A, b \in P(A)$ we have

$$P(a \circ b) = P(q(a \circ b)q) = P(qaq \circ b) = \frac{1}{2}(qaq \circ b + \pi(qaq \circ b)) = P(a) \circ b,$$

since $P$ is a Jordan homomorphism, and $b$ is fixed by $\pi$, and $P(a) \circ b = P(qaq) \circ b$. Since $P**(1) = q$, by the uniqueness in Theorem 2.13, $P$ must be the zeroing extension to $A$ of the map $\frac{1}{2}(I + \pi)$ on $D$.

For the converse, note that if $a \in \mathfrak{r}_A$ then $qaq + (qaq)^* = q(a + a^*)q \geq 0$. Hence $P(qaq) + P(qaq)^* \geq 0$ since $P = \frac{1}{2}(I + \pi)$ on $D$ and both $I$ and $\pi$ are real positive on $D$ by Corollary 2.22. Since $P$ annihilates $W$ it is now clear that $P$ is real positive on $A = D \oplus W$. We leave the rest as an exercise. \hfill $\square$

Remarks. 1) One may write $P$ in the previous result more explicitly. Namely,

$$P = \frac{1}{2}(I + \theta) = \frac{1}{2}(I + \pi(\cdot)(2q - 1)).$$

Here $\theta, q, u = 2q - 1$ are as above, and $\pi : A \to Au$ is the isometric surjective Jordan homomorphism coming from the Banach-Stone theorem [5,5] namely $\pi = \theta(\cdot)u$. If $A$ is an (associative) operator algebra then $Au = A$, also by that theorem. Conversely it is easy to show, as in [14 Theorem 3.7], that a map of the form at the end of the last centered equation, is a symmetric projection on $A$ under reasonable conditions. As we said elsewhere, there seems to be an error in the last line of the statement of [14 Theorem 3.7]: the conditions there do not imply that $P$ is real positive.

2) Suppose that $P$ is a symmetric projection on a unital Jordan operator algebra $A$ and that $q = P(1)^*$. Then $P(a)qP(a) = P(P(a)^2) \in P(A)$, and so $P(A)$ is a Jordan subalgebra of $A$ if and only if $P(a)(1 - q)P(a) = 0$ for all $a \in A$. If further $q$ is hermitian (which it is e.g. if $P$ real positive) then $q$ is a projection.
To prove these assertions, let \( u = 2q^* - 1 \), and let \( \theta = 2P - I \). By [3] Corollary 2.8, \( \theta(a^2) = \theta(a1a) = \theta(a)u^*\theta(a) \) for \( a \in A \). That is,
\[
2P(a^2) - a^2 = (2P(a) - a)u^*(2P(a) - a).
\]
This formula contains a lot of information. In particular, replacing \( a \) by \( P(a) \), we have
\[
2P(P(a)^2) - P(a)^2 = 2P(a)qP(a) - P(a)^2.
\]
Hence \( P(a)qP(a) = P(P(a)^2) \in P(A) \). Thus \( P(a)^2 \in P(A) \) if and only if \( P(a)^2 = P(a)qP(a) \). It follows that \( P(A) \) is a Jordan subalgebra of \( A \) if and only if \( P(a)(1 - q)P(a) = 0 \) for all \( a \in A \). By [3] Corollary 2.8, \( u \) is a unitary in the \( JC^* \)-algebra \( A \cap A^* \), and in the \( C^* \)-algebra generated by \( A \). If \( q \) is hermitian then \( u \) is a selfadjoint unitary, which forces \( q \) to be a projection.

The condition in the last paragraph that \( P(a)(1 - q)P(a) = 0 \) for all \( a \in A \) happens for example if \( q \) is a projection and \( P(A) \subset qAq \), which is the case in much of our paper.

**Corollary 5.3.** Let \( A \) be an approximately unital Jordan operator algebra, and let \( P : A \to A \) be a symmetric projection which is approximately unital (that is, takes a Jordan cai to a Jordan cai, or more generally for which \( P^{**} \) is unital). Then \( P \) is real positive, the range of \( P \) is a Jordan subalgebra of \( A \), and \( P \) is a Jordan conditional expectation. Also, \( P = \frac{1}{2}(1 + \theta) \) for a period 2 surjective isometric unital Jordan homomorphism \( \theta : A \to A \).

**Proof.** Here \( P^{**}(1) = 1 \) and the results follow from Theorem 5.2. \( \square \)

In the classification of symmetric projections in the selfadjoint case (Theorem 5.1), \( q = P^{**}(1) \) is central. In our setting of Jordan operator algebras, if one insists that \( q = P^{**}(1) \) be central (that is, if \( q \circ a = qaq \in A \) for all \( a \in A \) then one obtains a characterization closely resembling Theorem 5.1.

**Theorem 5.4.** Suppose that \( P : A \to A \) is a symmetric real positive projection on an approximately unital Jordan operator algebra \( A \), and let \( q = P^{**}(1) \). Then \( q \) is central if and only if \( D = \{ a \in A : a = qaq \} \) is a Jordan ideal in \( A \). If these hold then \( q \in M(A) \), the multiplier algebra defined at the start of [20] Section 2.8, \( P = 0 \) on \( qAq \), and there exists a period 2 Jordan *-automorphism \( \theta \) of \( D = qAq \) so that \( P = \frac{1}{2}(1 + \theta) \) on \( D \). Conversely, any \( P \) satisfying the conditions in the last sentence is a symmetric real positive projection. Finally, \( P(A) \) is a Jordan subalgebra of \( A \), and \( P \) is a Jordan conditional expectation.

**Proof.** We said earlier that \( q \) was open. The first ‘if and only if’ then follows from [20] Corollary 3.26, and the bijective correspondence between HSA’s and open projections from [20] 15. We know from Theorem 5.3 that \( u = (2P^{**} - I)(1) = 2q - 1 \in M(D) \) if \( D \) is a \( C^* \)-algebra generated by \( A \). If \( q \) is central then we have \( qa = qaq \in A^{\perp \perp} \cap D = A \), since \( q = (1 + u)/2 \in M(D) \). So \( q \in M(A) \) (and in \( JM(A) \)). The assertions about \( q^\perp Aq \) and \( \theta \), and the last line of the statement follow from Theorem 5.2. Finally the converse follows just as in Theorem 5.1. \( \square \)

Note that there exist symmetric projections \( P : A \to A \) on a unital operator algebra with \( P(1) \) a projection \( q \) and \( P(A) \) a Jordan subalgebra, but \( P \) is not real positive and \( P(A) \) is not contained in \( qAq \) and \( P \) does not kill each \( q^\perp a + aq^\perp \) for \( a \in A \). However, if \( P \) is a contractive projection into \( qAq \) then \( P \) extends to a unital positive map on \( A + A^* \) by e.g. [11] Lemma 1.3.6, and so \( P \) is real positive.
We also mention a nonunital example: the projection $P$ on the upper triangular $2 \times 2$ matrices taking the matrix with rows $[a \ b]$ and $[0 \ c]$ to the matrix with rows $[(a - c)/2 \ 0]$ and $[0 \ (c - a)/2]$. This is symmetric and extends to a completely contractive projection on the containing $C^*$-algebra, but its range is not a Jordan subalgebra. In this example $P(1) = 0$.

**Remark.** Related to the last theorem we mention that there is a typographical error in the statement of Lemma 3.6 in [14]: of course $P'$ is completely contractive and bicontractive there.

6. **Bicontractive projections**

Suppose that $A$ is an approximately unital Jordan operator algebra, and $P : A \to A$ is a bicontractive real positive projection. Then $P^{**}(1) = q$ is an projection in $A^{**}$ by the proof of [14] Lemma 3.6], indeed $q$ is an open projection by Lemma 6.7 and indeed all of the conclusions of Lemma 4.7 and Theorem 4.8 hold. However as pointed out in [14] before Example 6.1 there, the Jordan variants of the ‘bicontractive projection problem’ are not going to be any better; the range of the 5 by 5 counterexample listed before Lemma 4.7 in [14] is not a Jordan subalgebra. As mentioned earlier, we follow the approach taken in [14] that the correct formulation of the bicontractive projection problem in our category is: when is the range of $P$ a Jordan subalgebra of $A$? In this section we will give a natural hypothesis under which the bicontractive projection problem for operator algebras and Jordan operator algebras does have a nice solution.

We recall the ‘three step reduction to the unital case’ from above Corollary 4.3 in [14]: if $P$ and $A$ are as above, then by considering $P^{**} : A^{**} \to A^{**}$ we may assume that $A$ is unital. The second step is to use Lemma 4.7 and Theorem 4.8 to reduce to the case that $P$ is unital (that is, we replace unital $A$ by $qAq$, where $q = P(1) = P(q)$). The third step is to replace the domain $A$ of $P$ by $\text{joa}(P(A))$. This does not change the range, the structure on which is our interest.

Thus henceforth in this section we will be assuming that $A$ is a unital operator algebra or Jordan operator algebra, and $P : A \to A$ is a unital projection on $A$.

The following is a Jordan version of part of [14, Lemma 4.1].

**Lemma 6.1.** Let $A$ be a unital Jordan operator algebra, and suppose that $P : A \to A$ is a unital projection on $A$ with $I - P$ contractive. Let $C = \text{Ker}(P)$ and $B = P(A)$. Then $\theta = 2P - I$ is the map $b + c \mapsto b - c$ for $b \in P(A), c \in \text{Ker}(P)$, and $P(A)$ is the set of fixed points of $\theta$. We have $x^2 \in B$ for $x \in C$. If $\theta$ is a Jordan homomorphism then $P(A)$ is a subalgebra. Also

1. $C$ is a Jordan subalgebra of $A$ if and only if $C$ has the zero Jordan product.
2. $P(a \circ P(b)) = P(P(a) \circ P(b))$ for all $a, b \in A$ if and only if $c \circ b \in C$ for $b \in B, c \in C$.
3. $C$ is a Jordan ideal in $A$ if and only if the conditions in (1) and (2) hold. (As we said in Lemma 4.3, these are also equivalent to $P$ being a Jordan homomorphism with respect to the $P$-product on $P(A)$.)
4. If $c \circ b \in C$ for $b \in B, c \in C$ (see (2)), then $\theta = 2P - I$ is a Jordan homomorphism on $A$ if and only if $P(A)$ is a Jordan subalgebra of $A$.

**Proof.** The first assertions are an exercise. For example, if $\theta = 2P - I$ is a Jordan homomorphism on $A$ then its fixed points, namely $P(A)$, form a Jordan subalgebra.
If \( I - P \) is contractive then by part of the proof of Lemma \([1, 9]\) but with \( P \) replaced by \( I - P \), we have \((I - P)(x^2) = (I - P)(x((I - P)(1))^*x) = 0\) for \( x \in C \). Thus \( x^2 = P(x^2) \in B \).

For (1), if \( C \) is a Jordan subalgebra and \( x \in C \) then \( x^2 \in C \cap B = \{0\} \). Item (2) is obvious (and does not use the statement before (1) or that \( I - P \) is contractive).

Item (3) follows from (1) and (2) and the fact that \( C \circ A \subset C \) if and only if \( C \circ B \subset C \) and \( C \circ C \subset C \). For (4), if \( P(A) \) is a subalgebra then \( \theta((c + b)^2) = \theta(b^2) + \theta(c^2) - 2\theta(c \circ b) = \theta(b^2) + c^2 - cb - bc = c^2 + b^2 - cb - bc \), \( \theta(c + b)^2 = c^2 + b^2 - cb - bc \). So \( \theta \) is a Jordan homomorphism. \( \square \)

**Lemma 6.2.** Let \( A \) be a unital operator space. If \( a \in \frac{1}{2} F_A \) and \( a \neq 0 \) then there exists an \( \epsilon > 0 \) such that \( \|a - t1\| < \|a\| \) for all \( 0 < t < \epsilon \).

**Proof.** We have \( a^*a \leq \text{Re}(a) \), and so if \( 0 < t \leq \frac{1}{2} \) then we have \((a - t1)^*(a - t1) \leq a^*a - 2t\text{Re}(a) + t^2 1 \leq ((1 - 2t)a^*a + t^2 1 \leq ((1 - 2t)\|a\|^2 + t^2)1 \).

Thus \( \|a - t1\|^2 \leq (1 - 2t)\|a\|^2 + t^2 1 \). The latter quantity is dominated by \( \|a\|^2 1 \) if and only if \( t < 2\|a\|^2 \). Hence the result follows if \( \epsilon = \min\{\frac{1}{2}, 2\|a\|^2\} \). \( \square \)

We thank the referee for a correction in the last proof.

We recall from the introduction that \( \text{Re}(A) = \{\text{Re}(a) : a \in A\} \), and that this is well-defined independently of the Hilbert space representation of \( A \) if \( A \) is a unital operator space or approximately unital Jordan operator algebra.

**Theorem 6.3.** Let \( A \) be a unital operator algebra (resp. Jordan operator algebra), and let \( D = \text{Ker}(P) \cap \text{oa}(P(A)) \) (resp. \( \text{Ker}(P) \cap \text{joa}(P(A)) \)). If \( P : A \to A \) is a contractive unital projection on \( A \), with \( I - P \) contractive on \( A \) or with \( I - P \) contractive on \( \text{Re}(A) \), and if \( D \) is densely spanned by the real positive elements which it contains, then \( P(A) \) is a subalgebra (resp. Jordan subalgebra) of \( A \).

**Proof.** By replacing \( A \) with \( \text{oa}(P(A)) \) (resp. \( \text{joa}(P(A)) \)) we may assume that these sets are the same and \( D = \text{Ker}(P) \). We use ideas found in Lemma 5.1 of \([14]\). If \( a \in D \cap \frac{1}{2} F_A \) and \( \|a - t1\| < \|a\| \) for some \( t > 0 \) (see Lemma 6.2) then we obtain the contradiction \( \|a - t1\| < \|a\| = \|I - P(a - t1)\| \).

Hence \( D \cap F_A = \{0\} \). By our hypothesis, which implies that \( D \) is spanned by \( D \cap F_A \) (see e.g. Lemma 2.12 (1)), we have \( D = \{0\} \). So \( P \) is the identity map, and our conclusion is tautological.

Suppose that \( I - P \) is contractive on \( \text{Re}(A) \), that \( x \in D \cap F_A \), and write \( x = a + ib \) with \( a \in \text{Re}(A)_+ \) and \( b = b^* \). Continuing to write \( P \) for its positive extension to \( A + A^* \) (see \([20], \text{Corollary 4.9}\)), we have \( 0 = P(x) = P(a) + iP(b) \) with \( P(a), P(b) \) selfadjoint, so \( P(a) = 0 \). By an argument similar to the last paragraph (also found in the proof of Lemma 5.1 in \([14]\)) we see that \( a = 0 \). Hence \( x = 0 \) since \( x \in F_A \). Hence \( D \cap F_A = \{0\} \), and we finish as before. \( \square \)

**Remark.** As in the first Remark after Theorem 4.10, one may ‘weaken’ the condition in Theorem 6.3 about being ‘densely spanned by the real positive elements which it contains’, to being ‘contained in the closed algebra (or even HSA) generated by the real positive elements it contains’.

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