The Goldberger – Treiman Relation, $g_A$ and $g_{\pi NN}$ at $T \neq 0$

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Abstract

The Goldberger-Treiman relation is shown to persist in the chiral limit at finite temperatures to order $O(T^2)$. The $T$ dependence of $g_A$ turns out to be the same as for $F_\pi$, $g_A(T) = g_A(0)(1 - T^2/12F^2)$, while $g_{\pi NN}$ is temperature independent to this order. The baryon octet $\mathcal{D}$ and $\mathcal{F}$ couplings also behave as $F_\pi$ if only pions are massless in the pseudoscalar meson octet.

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In the recent years there has been an increasing interest in the study of QCD at finite temperatures, both below and above the anticipated phase transition. At high temperatures QCD is believed to be in a deconfined phase of quarks and gluons, while in the low temperature region the appropriate degrees of freedom are provided by hadrons. At sufficiently low $T$ the dominant role is played by pions, other hadrons being suppressed by factors of $\exp(-m_{\text{had}}/T)$. Using the pion gas approximation and effective chiral Lagrangians, a number of problems was considered regarding the changes in the vacuum structure and hadron properties under the introduction of a finite temperature. Among these were $T$ dependences of hadron masses and non-perturbative condensates (for a review, see e.g. [1]). It turned out that there is no $T$ dependence of hadron masses in the leading order in the thermal pion gas density, i.e. in order $O(T^2)$ (the only exception is the pion mass $T$ dependence)[2, 3, 4]. This is in fact a very general consequence of PCAC and current algebra[5]. The matrix elements of correlators of hadronic currents usually do get $O(T^2)$ contributions which result in parity and isospin mixing[6, 4]. The $T$ dependence of the relevant mixing coefficients determines the $T$ dependence of residues, i.e. matrix elements of a current between vacuum and hadronic states, such as $<0|A^a_\mu|\pi^b(q)> = iq^\mu F_\pi \delta^{ab}$, where $A_\mu$ is the axial isovector current and $F_\pi \simeq 90$ MeV. An interesting result of the $O(T^2)$ calculations is that the famous Gell-Mann–Oaks–Renner relation, $F_\pi^2 m_\pi^2 = -2m_q <\bar{q}q>$, holds to this order[7].

In this paper we consider another famous relation, the Goldberger–Treiman relation
\begin{equation}
 g_{ANmN} = F_\pi g_{\pi NN}
\end{equation}
at finite temperatures in the massless pion gas approximation.

It is well known that this relation follows when the requirement of the axial current conservation is applied to the matrix element of the axial current between nucleons,
\begin{equation}
< N(p)|A^a_\mu|N(p') >= \bar{N}(p) \frac{\sigma^a}{2} (g_A \gamma_\mu + h_A q_\mu \gamma_5) N(p'), \quad q = p - p'
\end{equation}
The formfactor $g_A$ is finite at $q^2 = 0$ due to the pion pole in $h_A(q^2)$ which dominates in Eq.(2) at $q^2 \to 0$ (Fig.1). The constant $g_{\pi NN}$ is defined by the phenomenological Lagrangian
\[ \mathcal{L}_{\pi NN} = ig_{\pi NN} \bar{N} \phi \gamma_5 N (\phi \equiv \phi^a \tau^a \text{ and } \tau^a \text{ is a standard isospin matrix}). \]
The extension to finite $T$ is achieved by attaching a thermal pion loop to the pole diagram in Fig.1 wherever possible. The relevant set of graphs is presented in Fig.2 where a dash on pion loop indicates that this is a pion from the heat bath. The first two diagrams correspond to the pion residue and wave function renormalization, while the last four ones involve $\pi N$ interactions.

A very convenient framework for considering correlation functions in interacting pion gas is provided by the method of effective chiral Lagrangians [8]. A particularly useful version of
this method was developed by Gasser and Leutwyler \[9\] by introducing interactions of pions with external fields. In this approach the generating functional of QCD
\[
e^{iZ[v,a,s,p]} = \int [DA_\mu][Dq][D\bar{q}] e^{i \int d^4L(q,\bar{q};G_{\mu\nu};v,a,s,p)}
\]
is written in terms of effective meson theory
\[
e^{iZ[v,a,s,p]} = \int [DU] e^{i \int d^4L_{\text{eff}}(U;v,a,s,p)}
\]
where the $SU(2)_R \times SU(2)_L$ matrix $U$ contains the pion field, $U(x) = \exp(\iota \phi(F))$. Interaction with external vector, axial, scalar and pseudoscalar fields $v_\mu, a_\mu, s, p$ is introduced through
\[
L = L^0 + \bar{q}\gamma_\mu(v_\mu(x) + \gamma_5 a_\mu(x))q - \bar{q}(s(x) - i\gamma_5 p(x))q
\]
where
\[
L^0 = -\frac{1}{2g^2}G^a_{\mu\nu}G^a_{\mu\nu} + \bar{q}\gamma_\mu(i\partial_\mu + A_\mu)q
\]
is the Lagrangian of massless QCD. The external fields $v_\mu, a_\mu, s, p$ are hermitean $2 \times 2$ matrices in the flavor space. The quark mass matrix $M = \text{diag}(m_u, m_d)$ is included into $s(x)$. The real world of course corresponds to $v_\mu = a_\mu = p = 0$ and $s(x) = M$. However, the introduction of external fields $v_\mu, a_\mu, p$ is extremely helpful for obtaining bosonized versions of the corresponding quark currents.

The effective Lagrangian in Eq.(4) is written as a series
\[
L_{\text{eff}} = L_2 + L_4 + ...
\]
according to the number of derivatives and/or quark mass factors. Transformation properties of the matrix $U$ and the external fields under $SU(2)_R \times SU(2)_L$ rotations
\[
U' = RUL^\dagger
\]
\[
v'_\mu + a'_\mu = R(v_\mu + a_\mu)R^\dagger + iR\partial_\mu R^\dagger
\]
\[
v'_\mu - a'_\mu = L(v_\mu - a_\mu)L^\dagger + iL\partial_\mu L^\dagger
\]
\[
s' + ip' = R(s + ip)L^\dagger
\]
dictate the structure of the effective Lagrangians $L_n \[9\]$. The lowest order one is given by
\[
L_2 = \frac{1}{4}F^2 \text{Tr} \left(\nabla_\mu U^\dagger \nabla_\mu U + \chi^\dagger U + \chi U^\dagger\right)
\]
where $\chi = 2B(s + ip)$. The constant $F$ is the pion decay constant in the chiral limit, and the constant $B$ is related to the quark condensate $\langle \bar{q}q \rangle = -F^2B$. The external vector and axial field enter through the covariant derivative
\[
\nabla_\mu U = \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu)
\]
Extracting from Eq.(9) terms linear in \(v_\mu\) and \(a_\mu\), vector and axial currents are expressed through the matrix \(U\),

\[
V_\mu^i = \frac{1}{4} F^2 \text{Tr} \tau^i \left( [U^\dagger, \partial_\mu U] - [U, \partial_\mu U^\dagger] \right) \tag{11}
\]

\[
A_\mu^i = \frac{1}{4} F^2 \text{Tr} \tau^i \left( \{U^\dagger, \partial_\mu U\} + \{U, \partial_\mu U^\dagger\} \right) \tag{12}
\]
or, expanding in powers of the pion field and picking out for simplicity the "0" component,

\[
V_\mu^0 = \phi^+ \partial_\mu \phi^- - \phi^- \partial_\mu \phi^+ + O(\phi^4) \tag{13}
\]

\[
A_\mu^0 = -F \partial_\mu \phi^0 \left( 1 - \frac{4}{3 F^2} \phi^+ \phi^- \right) - \frac{2}{3 F^2} \phi^0 \partial_\mu \left( \phi^+ \phi^- \right) + O(\phi^5) \tag{14}
\]

From Eq.(14) using the thermal pion propagator

\[
\Delta_T(0) = \int \frac{d^4k}{(2\pi)^4} 2\pi \delta(k^2) \frac{1}{\exp(|k_0|/T) - 1} = \frac{1}{12} T^2 \tag{15}
\]

we get

\[
A_{2a} = -\frac{4}{3} \Delta_T(0) A_0 = -\frac{1}{9} \frac{T^2}{F^2} A_0 \tag{16}
\]

where \(A_0 = g_{\pi NN} F \tilde{N} \gamma_5 \gamma^0 N q_\mu/2q^2\) is the pole contribution in Fig.1 and \(A_{2a}, A_{2b},...\) are the amplitudes corresponding to the diagrams in Fig.2.

The pion wave function renormalization diagram in Fig.2b is easily estimated using the interaction part of \(L_2\),

\[
L_2^{\text{int}} = \frac{1}{6 F^2} \left[ (\vec{\phi} \partial_\mu \vec{\phi})^2 - \vec{\phi}^2 (\partial_\mu \vec{\phi})^2 \right] \tag{17}
\]

\((\vec{\phi}^2 \equiv \phi^a \phi^a)\). This interaction leads both to pion mass and wave function renormalization. As shown in Ref.[7], the pion mass is not renormalized in the chiral limit, \(m_\pi(T) = m_\pi(0)(1+T^2/48 F^2)\). The wave function renormalization then gives

\[
A_{2b} = \frac{2}{3} \Delta_T(0) A_0 = \frac{1}{18} \frac{T^2}{F^2} A_0 \tag{18}
\]

The diagrams of Fig.2a and 2b determine the \(T\) dependence of \(F\) obtained in Ref.[7]

\[
F^2(T) = F^2(0) \left[ 1 + \left( -2 \cdot \frac{1}{9} + \frac{1}{18} \right) \frac{T^2}{F^2} \right] = F^2(0) \left[ 1 - \frac{1}{6} \frac{T^2}{F^2} \right] \tag{19}
\]

It should be noted that specific contributions of diagrams in Fig.2a and 2b may change under the change of parameterization of the matrix \(U\), however the net result is invariant.

To calculate the remaining four diagrams in Fig.2 we use an elegant extension of the effective chiral Lagrangian to \(\pi N\) interactions described in Ref.[10] and generalized to external
fields in Ref.[11]. The lowest order (in the number of derivatives) \( \pi N \) Lagrangian is given by

\[
\mathcal{L}_{\pi N}^{1} = \bar{N} \left( i\gamma_\mu D_\mu - m_N + \frac{i}{2} g_A \gamma_\mu \gamma_5 u_\mu \right) N
\]

(20)

where \( D_\mu = \partial_\mu + \Gamma_\mu \). The chiral connection \( \Gamma_\mu \) and the axial-vector object \( u_\mu \) are given by

\[
\Gamma_\mu = \frac{1}{2} (u^\dagger \partial_\mu u + u \partial_\mu u^\dagger)
\]

(21)

\[
u_\mu = i (u^\dagger \nabla_\mu u + u \nabla_\mu u^\dagger)
\]

(22)

Here \( u^2 = U \) and the contact interactions of a nucleon with even and odd numbers of pions are given by \( \Gamma_\mu \) and \( u_\mu \), respectively. In terms of the pion field

\[
\frac{1}{2} u_\mu = -\frac{1}{2F} \partial_\mu \phi + \frac{1}{12F^3} \tilde{\phi}^2 \partial_\mu \phi - \frac{1}{12F^3} (\tilde{\phi} \partial_\mu \tilde{\phi}) \phi
\]

(23)

The last two terms in the above provide the correction due to the diagram of Fig.2c,

\[
A_{2c} = -\frac{1}{3} \Delta_T(0) A_0 = -\frac{1}{36} \frac{T^2}{F^2} A_0
\]

(24)

The remaining diagrams in Fig.2 do not contribute to order \( T^2 \). The diagram in Fig.2d gives a vanishing contribution because it contains \( \partial_\mu \Delta_T(x) \) at \( x = 0 \). The diagram in Fig.2e has two extra powers of the thermal pion momentum and will hence be of order \( T^4 \). The diagram in Fig.2f vanishes by the equation of motion, since the external nucleons are on-shell. Notice that the diagram of Fig.2e is likely to generate a new tensor structure for the matrix element in Eq.(2) proportional to \( q_\mu - n_\mu (nq)/q^2 \) (transversality maintained), where \( n_\mu \) is the 4-velocity vector of the heat bath. This is similar to what happens [12] for two-point correlation functions at \( T \neq 0 \): to order \( T^2 \) they are Lorentz covariant with Lorentz covariance broken in order \( T^4 \).

Summing up contributions from diagrams of Fig.2 we get

\[
A_2 = -\frac{1}{12} \frac{T^2}{F^2} A_0
\]

(25)

Using conservation of axial current (new tensor structures to not appear in order \( T^2 \)) and taking into account that \( m_N \) is not shifted in order \( T^2 \) [2, 3], we are led to infer that

\[
g_A(T) = g_A(0) \left( 1 - \frac{1}{12} \frac{T^3}{F^2} \right),
\]

(26)

i.e. \( g_A \) goes with temperature just as \( F \). From here we conclude that \( g_{\pi NN} \) is \( T \)-independent to order \( T^2 \).
One can see all this at the level of diagrams. The diagram of Fig.2a together with a half of the diagram of Fig.2b gives the $T$-dependence of $F$ (see Eq.(19)). The second half of the diagram in Fig.2b together with the diagram in Fig.2c should then give the $T$-dependence of $g_{\pi NN}$

$$\frac{1}{2}A_{2b} + A_{2c} = \left(\frac{1}{36} - \frac{1}{36}\right) \frac{T^2}{F^2} A_0 = 0 \tag{27}$$

We have extended this analysis to the case of $SU(3)_L \times SU(3)_R$ and octet baryons in which case the meson-baryon chiral Lagrangian

$$\mathcal{L}_{MB} = \text{Tr} \left( i\bar{B}\gamma_\mu D_\mu B - m\bar{B}B + \frac{1}{2}D\bar{B}\gamma_\mu \{u_\mu, B\} + \frac{1}{2}\mathcal{F}\bar{B}\gamma_\mu\gamma_5[u_\mu, B] \right) \tag{28}$$

($B$ is the standard $SU(3)$ matrix including all octet baryons) involves two couplings, $D$ and $\mathcal{F}$, such that $g_A = D + \mathcal{F}$. Considering only pions in the meson sector to be massless we get

$$\frac{D(T)}{D(0)} = \frac{\mathcal{F}(T)}{\mathcal{F}(0)} = \frac{g_A(T)}{g_A(0)} = \frac{F_\pi(T)}{F_\pi(0)} = 1 - \frac{1}{12} \frac{T^2}{F_\pi^2} \tag{29}$$

The result that pion-nucleon coupling stays unchanged to first order in the pion density is an amusing one. We should note however, that a number of earlier model calculations which used a chiral mean field theory [13], a QCD sum rules type of approach [14], and topological chiral soliton model for the nucleon [15] suggested that this dependence is rather mild up to temperatures in the region of phase transition. We feel that it is worth asking a question whether other hadronic couplings involving pions share this feature. We would expect, for example, the $\omega\rho\pi$ coupling to have a very weak $T$ dependence, since by vector dominance it is related to the $\pi\gamma\gamma$ coupling, and the axial anomaly is not renormalized by temperature effects[16]. We plan to address this issue in future work.

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Figure Captions

• Fig. 1: Pion pole contribution to the matrix element of the axial current over nucleon. Solid, dashed and wavy lines correspond to the nucleon, pion, and the axial current.

• Fig. 2: Corrections to the pion pole contribution to first order in the density of the thermal pion gas. A dash on a pion line denotes the thermal pion.
This figure "fig1-1.png" is available in "png" format from:

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