INTERSECTING FACES OF A SIMPLICIAL COMPLEX VIA ALGEBRAIC SHIFTING

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Abstract. A family $A$ of sets is $t$-intersecting if the cardinality of the intersection of every pair of sets in $A$ is at least $t$, and is an $r$-family if every set in $A$ has cardinality $r$. A well-known theorem of Erdős, Ko, and Rado bounds the cardinality of a $t$-intersecting $r$-family of subsets of an $n$-element set, or equivalently of $(r-1)$-dimensional faces of a simplex with $n$ vertices. As a generalization of the Erdős-Ko-Rado theorem, Borg presented a conjecture concerning the size of a $t$-intersecting $r$-family of faces of an arbitrary simplicial complex. He proved his conjecture for shifted complexes. In this paper we give a new proof for this result based on work of Woodroofe. Using algebraic shifting we verify Borg’s conjecture in the case of sequentially Cohen-Macaulay $i$-near-cones for $t = i$.

1. Introduction

Throughout this paper, the set of positive integers $\{1, 2, \ldots \}$ is denoted by $\mathbb{N}$. For $m, n \in \mathbb{N}, m \leq n$, the set $\{i \in \mathbb{N} : m \leq i \leq n\}$ is denoted by $[m, n]$; for $m = 1$, we also write $[n]$.

Let $t \leq r$ be two natural numbers. A family $A$ of sets is $t$-intersecting if the cardinality of the intersection of every pair of sets in $A$ is at least $t$, and is an $r$-family if every set in $A$ has cardinality $r$. A well-known theorem of Erdős, Ko, and Rado bounds the cardinality of a $t$-intersecting $r$-family:

**Theorem 1.1.** Assume that $t \leq r$ are two natural numbers. Let $n \geq (t+1)(r-t+1)$ and $A$ be a $t$-intersecting $r$-family of subsets of $[n]$. Then $|A| \leq \binom{n-t}{r-t}$.

Given a simplicial complex $\Delta$ (defined in Section 2) and a face $\sigma$ of $\Delta$, we define the link of $\sigma$ in $\Delta$ to be

$$\text{lk}_\Delta \sigma = \{\tau : \tau \cup \sigma \in \Delta, \tau \cap \sigma = \emptyset\}.$$  

Also for every integer $s \geq 0$ we define

$$\Delta_{(s)} = \{\sigma \in \Delta : |\sigma| = s\}.$$
An \( r \)-face of \( \Delta \) is a face of cardinality \( r \). We further let \( f_r(\Delta) \) be defined as the number of \( r \)-faces in \( \Delta \), and the tuple \( (f_0(\Delta), f_1(\Delta), \ldots, f_{d+1}(\Delta)) \) (where \( d \) is the dimension of \( \Delta \)) is called the \( f \)-vector of \( \Delta \).

**Note.** We follow Swartz [6] in our definition of \( r \)-face and \( f_r \). Other sources define an \( r \)-face to be a face with dimension \( r \) (rather than cardinality \( r \)) which shifts the indices of the \( f \)-vector by 1.

We restate Theorem 1.1 using this language:

**Theorem 1.2.** Assume that \( t \leq r \) are two natural numbers. Let \( n \geq (t + 1)(r - t + 1) \) and \( \mathcal{A} \) be a \( t \)-intersecting \( r \)-family of faces of the simplex with \( n \) vertices. Then \( |\mathcal{A}| \leq f_{r-t}(\text{lk}_\Delta \sigma) \), where \( \sigma \) is a \( t \)-face of \( \Delta \).

**Definition 1.3.** A simplicial complex \( \Delta \) is called \((t, r)\)-EKR if every \( t \)-intersecting \( r \)-family \( \mathcal{A} \) of faces of \( \Delta \) satisfies \( |\mathcal{A}| \leq \max f_{s-t}(\text{lk}_\Delta \sigma) \), where the maximum is taken over all \( t \)-faces \( \sigma \) of \( \Delta \). Equivalently, \( \Delta \) is \((t, r)\)-EKR if the set of all \( r \)-faces containing some \( t \)-face \( \sigma \) has maximal cardinality among all \( t \)-intersecting families of \( r \)-faces.

As a generalization of the Erdős-Ko-Rado theorem, Borg conjectured that:

**Conjecture 1.4.** ([1, Conjecture 2.6]) Let \( t \leq r \) be two natural numbers. Assume that \( \Delta \) is a simplicial complex having minimal facet cardinality \( k \geq (t + 1)(r - t + 1) \) and suppose that \( S \neq \emptyset \) is a subset of \([t, r]\). Then every \( t \)-intersecting family \( \mathcal{A} \) of faces of \( \Delta \) with \( \mathcal{A} \subseteq \bigcup_{s \in S} \Delta(s) \) satisfies the following inequality:

\[
|\mathcal{A}| \leq \max \sum_{s \in S} f_{s-t}(\text{lk}_\Delta \sigma),
\]

where the maximum is taken all over \( t \)-faces \( \sigma \) of \( \Delta \).

Borg proved Conjecture 1.4 for shifted complexes [1, Theorem 2.7]. Using algebraic shifting, Woodroofe gave a new proof for [1, Theorem 2.7] in a special case of \( t = 1 \) and \( S = \{r\} \) [7, Lemma 3.1]. In this paper we extend Woodroofe’s proof and give a complete new proof for [1, Theorem 2.7] using algebraic shifting (Theorem 3.2). Woodroofe also proved, that in the special case of \( t = 1 \) and \( S = \{r\} \), Conjecture 1.4 is true for sequentially Cohen-Macaulay near-cones [7, Corollary 3.4]. We also generalize this result and prove that Conjecture 1.4 is true for every sequentially Cohen-Macaulay \( i \)-near-cone in the case of \( t = i \) (Corollary 4.7).

**Remark 1.5.** It was also proved by Borg [1, Theorem 2.1] that Conjecture 1.4 is true when the minimum facet cardinality of \( \Delta \) is at least \((r - t)(\frac{3r - 2t - 1}{t + 1}) + r\).

This paper is organized as follows. In Section 2 we review the necessary background on shifted complexes, algebraic shifting, the Cohen-Macaulay property, and \( i \)-near-cones. In Section 3 we present our new proof for [1, Theorem 2.7]. In Section 4 we prove the main results of this paper about intersecting faces of \( i \)-near-cones, (Corollaries 4.6 and 4.7).
2. Algebraic shifting and near-cones

An \textit{(abstract)} simplicial complex $\Delta$ on the set of vertices $V(\Delta)$ is a collection of subsets of $[n]$ which is closed under taking subsets; that is, if $F \in \Delta$ and $F' \subseteq F$, then also $F' \in \Delta$. Every element $F \in \Delta$ is called a face of $\Delta$. We assume that every vertex is contained in some face. The \textit{size} of a face $F$ is defined to be $|F|$ and its \textit{dimension} is defined to be $|F| - 1$. (As usual, for a given finite set $X$, the number of elements of $X$ is denoted by $|X|$.) The \textit{dimension} of $\Delta$ which is denoted by $\dim \Delta$, is defined to be $d - 1$, where $d = \max\{|F| \mid F \in \Delta\}$. A facet of $\Delta$ is a maximal face of $\Delta$ with respect to inclusion. We say that $\Delta$ is \textit{pure} if all facets of $\Delta$ have the same cardinality.

If $F$ is some family of sets, then the simplicial complex $\Delta(F)$ \textit{generated by} $F$ has faces consisting of all subsets of all sets in $F$. For a simplicial complex $\Delta$, the \textit{r-skeleton} $\Delta^{(r)}$ consists of all faces of $\Delta$ having dimension at most $r$, while the \textit{pure r-skeleton} is the subcomplex generated by all faces of $\Delta$ having dimension exactly $r$.

The \textit{join} of disjoint simplicial complexes $\Delta$ and $\Sigma$ is the simplicial complex $\Delta * \Sigma$ with faces $\tau \cup \sigma$, where $\tau$ is a face of $\Delta$ and $\sigma$ is a face of $\Sigma$. For every vertex $\sigma \in \Delta$, link and \textit{anti-star} of $\sigma$ are defined by

$$\text{lk}_\Delta \sigma = \{ \tau \in \Delta : \tau \cap \sigma = \emptyset, \tau \cup \sigma \in \Delta \}$$

and

$$\text{ast}_\Delta \sigma = \{ \tau \in \Delta : \tau \cap \sigma = \emptyset \}.$$  

Also for every integer $s \geq 0$ we define

$$\Delta^{(s)} = \{ \sigma \in \Delta : |\sigma| = s \}.$$

A simplicial complex $\Delta$ with ordered vertex set $\{v_1, \ldots, v_n\}$ is \textit{shifted} if whenever $\sigma$ is a face of $\Delta$ containing vertex $v_i$, then $(\sigma \setminus \{v_i\}) \cup \{v_j\}$ is a face of $\Delta$ for every $j < i$. An $r$-family $F$ of subsets of $\{v_1, \ldots, v_n\}$ is \textit{shifted} if it generates a shifted complex.

2.1. \textbf{Algebraic shifting}. A \textit{shifting operation} on an ordered vertex set $V$ is a map which associates each simplicial complex $\Delta$ on $V$ with a simplicial complex $\text{Shift}\Delta$ on $V$ and which satisfies the following conditions:

\begin{itemize}
  \item[(S1)] $\text{Shift}\Delta$ is a shifted.
  \item[(S2)] If $\Delta$ is shifted, then $\text{Shift}\Delta = \Delta$.
  \item[(S3)] $f_i(\text{Shift}\Delta) = f_i(\Delta)$ for all $i$.
  \item[(S4)] If $\Gamma \subseteq \Delta$ are simplicial complexes, then $\text{Shift}\Gamma \subseteq \text{Shift}\Delta$.
\end{itemize}

If $\mathcal{A}$ is some $r$-family of sets, then $\text{Shift}\mathcal{A}$ is defined to be

$$\text{Shift}\mathcal{A} = \text{Shift}(\Delta(\mathcal{A}))^{(r)}.$$  

In our proofs we need a shifting operation which satisfies the following extra property:

\begin{itemize}
  \item[(S5)] If $\mathcal{A}$ is a $t$-intersecting $r$-family, then $\text{Shift}\mathcal{A}$ is a $t$-intersecting $r$-family.
\end{itemize}
Kalai proves (See [3 Corollary 6.3 and subsequent Remarks]) that a specific shifting operation which is called exterior algebraic shifting (with respect to a field \( \mathbb{F} \)) satisfies \((S_5)\). We denote the exterior algebraic shift of \( \Delta \), with respect to a field \( \mathbb{F} \), by \( \text{Shift}_\mathbb{F} \Delta \). (The precise definition of exterior algebraic shifting will not be important for us, but can be found in Kalai’s survey article [3].)

2.2. Near-cones. A simplicial complex \( \Delta \) is a near-cone with respect to an apex vertex \( v \) if for every face \( \sigma \), the set \( (\sigma \setminus \{v\}) \cup \{v\} \) is also a face for each vertex \( w \in \sigma \). Equivalently, the boundary of every facet of \( \Delta \) is contained in \( v \ast \text{lk}_\Delta v \); another equivalent condition is that \( \Delta \) is the union of \( v \ast \text{lk}_\Delta v \) and some set of facets not containing \( v \) (but whose boundary is contained in \( \text{lk}_\Delta v \)). If \( \Delta \) is a cone with apex vertex \( v \), then obviously \( \Delta = v \ast \text{lk}_\Delta v \), thus every cone is a near-cone.

**Definition 2.1.** A simplicial complex \( \Delta \) is an \( i \)-near-cone if there exist a sequence of nonempty simplicial complexes \( \Delta = \Delta(0) \supset \Delta(1) \supset \ldots \supset \Delta(i) \) such that for every \( 1 \leq j \leq i \) there is a vertex \( v_j \in \Delta(j-1) \) such that \( \Delta(j) = \text{ast}_{\Delta(j-1)} v_j \) and \( \Delta(j-1) \) is a near-cone with respect to \( v_j \). The sequence \( v_1, \ldots, v_i \) is called the apex of \( \Delta \).

\( i \)-near-cones were first defined by Neve [4]. The following is a simple consequence of its definition.

**Lemma 2.2.** Let \( \Delta \) be an \( i \)-near-cone with apex \( v_1, \ldots, v_i \), such that \( \dim \Delta \geq 2i - 2 \). Then \( F = \{v_1, v_2, \ldots, v_i\} \) is a face of \( \Delta \).

**Proof.** If \( i = 1 \), then there is nothing to prove. So assume that \( i \geq 2 \). Since \( \dim \Delta \geq 2i - 2 \), \( \Delta \) contains an \((2i - 1)\)-face, say \( \sigma_0 \). Considering \( \sigma \setminus \{v_1, v_2, \ldots, v_{i-1}\} \), implies that \( \Delta \) contains an \( i \)-face \( \sigma_1 \), which does not contain the vertices \( v_1, v_2, \ldots, v_{i-1} \). Therefore by notations of Definition 2.1 \( \sigma_1 \in \Delta(i-1) \). Now \( \Delta(i-1) \) is a near-cone with respect to \( v_i \), which implies that \( \Delta(i-1) \) contains an \( i \)-face \( \sigma_2 \) such that \( v_i \in \sigma_2 \). Since \( \sigma_2 \in \Delta(i-1) \subset \Delta(i-2) \) and \( \Delta(i-2) \) is a near-cone with respect to \( v_{i-1} \), the simplicial complex \( \Delta(i-2) \) contains an \( i \)-face \( \sigma_3 \) such that \( \{v_i, v_{i-1}\} \subseteq \sigma_3 \). If \( i = 2 \), then we are done. Otherwise repeating the argument above shows that \( F = \{v_1, v_2, \ldots, v_i\} \) is a face of \( \Delta \). \( \square \)

**Example 2.3.** Lemma 2.2 is not true if \( \dim \Delta < 2i - 2 \). For example assume that

\[ \mathcal{F} = \{v_1, v_2, v_4, v_6\}, \{v_1, v_3\}, \{v_1, v_5\}, \{v_2, v_3\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\} \].

Then \( \Delta(\mathcal{F}) \) is a 3-near-cone, where \( \Delta(0) = \Delta(\mathcal{F}) \), \( \Delta(1) \) is the simplicial complex generated by \( \{v_2, v_4, v_6\}, \{v_2, v_3\}, \{v_2, v_5\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\} \) and \( \Delta(2) \) is the simplicial complex generated by \( \{v_4, v_6\}, \{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\} \) and \( \Delta(3) \) is the simplicial complex generated by \( \{v_4, v_6\}, \{v_5\} \). Now \( \dim \Delta = 3 < 4 = 2i - 2 \) and \( F = \{v_1, v_2, v_3\} \) is not a face of \( \Delta \).

2.3. Sequentially Cohen-Macaulay complexes and depth. Let \( \mathbb{F} \) be a field. A simplicial complex \( \Delta \) is Cohen-Macaulay over \( \mathbb{F} \) if \( \widetilde{H}_i(\text{lk}_\Delta \sigma; \mathbb{F}) = 0 \) for all \( i < \dim(\text{lk}_\Delta \sigma) \) and all faces \( \sigma \) of \( \Delta \) (including \( \sigma = \emptyset \)), where \( \widetilde{H}_i(\Delta; \mathbb{F}) \) denotes the simplicial homology of \( \Delta \) with coefficients in \( \mathbb{F} \). It is well-known that every Cohen-Macaulay
simplicial complex is pure and that every skeleton of a Cohen-Macaulay simplicial complex is Cohen-Macaulay. A simplicial complex is sequentially Cohen-Macaulay over $\mathbb{F}$ if every pure $r$-skeleton of $\Delta$ is Cohen-Macaulay over $\mathbb{F}$. Thus a simplicial complex is Cohen-Macaulay if and only if it is pure and sequentially Cohen-Macaulay.

Woodroofe [7] defined the depth of $\Delta$ over $\mathbb{F}$ as

$$\text{depth}_{\mathbb{F}} \Delta = \max \{ d : \Delta^{(d)} \text{ is Cohen-Macaulay over } \mathbb{F} \}.$$ 

Thus by [2, Corollary 4.5], $\text{depth}_{\mathbb{F}} \Delta$ is the minimum facet dimension of $\text{Shift}_{\mathbb{F}} \Delta$. We note that $\text{depth}_{\mathbb{F}} \Delta$ is one less than the ring-theoretic depth of the Stanley-Reisner ring $\mathbb{F}[\Delta]$ [5, Theorem 3.7]. If $\Delta$ is sequentially Cohen-Macaulay over $\mathbb{F}$ then $\text{depth}_{\mathbb{F}} \Delta$ is the minimum facet dimension of $\Delta$.

By the definition of simplicial homology we have $\tilde{H}_d(\Delta^{(d+1)}; \mathbb{F}) = \tilde{H}_d(\Delta; \mathbb{F})$. As a simple consequence one obtains the following equivalent characterization:

$$\text{depth}_{\mathbb{F}} \Delta = \max \{ d : \tilde{H}_i(\Delta; \mathbb{F}) = 0 \text{ for all } \sigma \in \Delta \text{ and } i < d - |\sigma| \}.$$ 

In particular, we notice that $\text{depth}_{\mathbb{F}} \Delta$ is at most the minimal facet dimension, since if $\sigma$ is a facet then $\tilde{H}_{-1}(\Delta; \mathbb{F}) = H_{-1}(\emptyset; \mathbb{F}) = \mathbb{F}$.

### 3. A new proof of Borg’s result

In this section, using shifting theory, we prove that Conjecture 1.4 holds for shifted complexes. This will be a new proof for a result of Borg [1] Theorem 2.7. Our proof is based on the proof of [7, Lemma 3.1] due to Woodroofe.

The following Lemma is the main step of our proof. In its proof we do not rely on a specific shifting operator, but only require $(S_1, S_2, S_3, S_4, S_5)$ for the operator Shift.

**Lemma 3.1.** If $\Delta$ is a shifted complex having minimal facet cardinality $k$, then $\Delta$ is $(t, r)$-EKR for every natural numbers $t \leq r$ with $k \geq (t+1)(r-t+1)$.

**Proof.** Let $\Delta$ have ordered vertex set $\{v_1, \ldots, v_n\}$, and let $\mathcal{A}$ be a $t$-intersecting $r$-family of faces of $\Delta$. Using induction we prove that $|\mathcal{A}| \leq f_{r-t}(\ell k_\Delta \sigma; v_1, \ldots, v_i)$. Our base cases are when $\Delta$ is a simplex (Theorem 1.2), and the trivial case where $r = t$.

If $\Delta$ is not a simplex and $r > t$, then by $S_1, S_2, S_3$ and $S_5$, we have that $\text{Shift}\mathcal{A}$ is a shifted $t$-intersecting $r$-family of faces of $\Delta = \text{Shift}\Delta$ with $|\text{Shift}\mathcal{A}| = |\mathcal{A}|$. For simplification let $W := \{v_{n-t+1}, \ldots, v_n\}$. For every subset $U$ of $W$ let $\mathcal{C}_U$ be the set of all faces $\sigma \in \text{Shift}\mathcal{A}$ with $W \cap \sigma = U$, so that $|\mathcal{A}| = |\text{Shift}\mathcal{A}| = \sum |\mathcal{C}_U|$, where the sum is taken over all subsets $U$ of $W$. We study $\mathcal{C}_U$ in terms of $|U|$.

**Case 1.** $r < |U| + t$. We claim that in this case $\mathcal{C}_U = \emptyset$. Assume that $\mathcal{C}_U \neq \emptyset$ and choose a member $C \in \mathcal{C}_U$. Since

$$n - |U| > k - |U| \geq (t+1)(r-t+1) - |U| > r + t - |U|,$$

by the definition of shiftedness, in $C$ one can replace the members of $U$ by some other vertices, such that the new vertices do not belong to $C \cup W$. We call this new set $C'$. By the choice of $W$ we have $C' \in \text{Shift}\mathcal{A}$ and since $r < |U| + t$, it follows that
|C ∩ C'| < t, which is a impossible, because ShiftΔ is a t-intersecting family. Thus C_U = ∅ for every subset U of F with r < |U| + t.

**Case 2.** r ≥ |U| + t. Let C'_U = {σ \ U : σ ∈ C_U}. Hence |C'_U| = |C_U|. Assume that |C'_U| ≥ 1 and suppose that C'_U is not t-intersecting for some U ⊆ W. Then there are σ, τ ∈ C_U such that |(σ \ U) ∩ (τ \ U)| ≤ t − 1. Now |σ ∪ τ| ≤ 2r − t and since U ⊆ σ ∪ τ, we conclude that |σ ∪ τ ∪ W| ≤ 2r − |U|. By assumption t ≥ |U| and thus |σ ∪ τ ∪ W| ≤ k − |U| < n − |U|. It follows that there exist |U| vertices v_1, . . . , v_{|U|} such that for every i with 1 ≤ i ≤ |U|, v_i ∉ σ ∪ τ ∪ W.

But then τ' = (τ \ U) ∪ {v_1, . . . , v_{|U|}} is in ShiftΔ, by the choice of W and the the definition of shiftedness, and |σ ∩ τ'| ≤ t − 1, which contradicts that ShiftΔ is t-intersecting. We conclude that C'_U is a t-intersecting (r − |U|)- family of faces of lk_{Δ_U} U, where Δ_U is the simplicial complex obtained from Δ by deleting the vertices of the set W \ U. Since lk_{Δ_U} U is a shifted complex on ground set \{v_1, . . . , v_{n−t}\} with minimum facet cardinality at least k − |U| and since k − |U| ≥ (t + 1)(r − |U| − t + 1), we conclude that

|C_U| = |C'_U| ≤ f_{r−|U|−t}(lk_{Δ_U}(U ∪ \{v_1, . . . , v_t\})),

by our induction hypothesis.

Finally we have

|A| = |ShiftΔ| = \sum_{U ⊆ W, r ≥ |U|+t} |C_U| ≤ \sum_{U ⊆ W, r ≥ |U|+t} f_{r−|U|−t}(lk_{Δ_U}(U ∪ \{v_1, . . . , v_t\})) = f_{r−t}(lk_{Δ}(v_1, . . . , v_t)).

Where the second equality follows from case 1 and the first inequality follows from case 2.

Borg [1] proved that Conjecture [1.4] holds for shifted complexes. Here using Lemma 3.1 we give a new proof for it.

**Theorem 3.2.** ([1] Theorem 2.7) Conjecture [1.4] is true if Δ is a shifted complex.

**Proof.** Note that A is the disjoint union of the sets A(s) := A ∩ Δ(s) with s ∈ S. Now by Lemma 3.1 for every s ∈ S we have

|A(s)| ≤ max_{τ ∈ Δ(s)} f_{r−t}(lk_{Δ} τ) = f_{r−t}(lk_{Δ} σ),

where σ = \{v_1, . . . , v_t\}. Thus |A| ≤ \sum_{s ∈ S} f_{r−t}(lk_{Δ} σ) and this completes the proof. □

**Proposition 3.3.** Assume that Δ is a simplicial complex and ShiftΔ is its exterior algebraic shift. Let us consider ShiftΔ as having ordered vertex set \{v_1, . . . , v_n\}. Then for every two integers r ≥ t we have

f_{r−t}(lk_{ShiftΔ}(v_1, . . . , v_t)) ≥ max\{f_{r−t}(lk_{Δ} σ) : σ \ is \ a \ t−face \ of \ Δ\).
Proof. Let \( \tau \) be a \( t \)-face of \( \Delta \) such that for every \( t \)-face \( \sigma \) of \( \Delta \), \( s := f_{r-t}(\text{lk}_\Delta \tau) \geq f_{r-t}(\text{lk}_\Delta \sigma) \). Assume that \( \sigma_1, \ldots, \sigma_s \) are \( r \)-faces of \( \Delta \) which contain \( \tau \). If \( \Gamma \) is the simplicial complex generated by \( \sigma_1, \ldots, \sigma_s \), then by \( S_4 \), \( \text{Shift}\Gamma \subseteq \text{Shift}\Delta \). Note that \( \Gamma \) is a cone with apex \( \sigma \) and therefore by [1] Corollary 5.4, \( \text{Shift}\Gamma \) is also a cone with an apex set \( \sigma' \) of cardinality \( t \). Hence

\[
f_{r-t}(\text{lk}_\Delta \tau) = s = f_r(\Gamma) = f_r(\text{Shift}\Gamma) = f_{r-t}(\text{lk}_{\text{Shift}\Delta} \sigma') \leq f_{r-t}(\text{lk}_{\text{Shift}\Delta} \sigma')
\]

\[
\leq f_{r-t}(\text{lk}_{\text{Shift}\Delta} \{v_1, \ldots, v_t\}).
\]

\( \square \)

Note that Proposition 3.3 essentially says that there exists a shifted simplicial complex for which the right hand side of inequality (\(*\)), in Conjecture 1.4 takes its maximum value. This suggests that shifted simplicial complexes are an easy case of Conjecture 1.4.

4. Intersecting faces of \( i \)-near-cones

In this section we settle Conjecture 1.4 in the case of \( i \)-near-cones for some special class of parameters. In the proof we use exterior algebraic shifting and Lemma 3.1. Therefore in this section we fix a field \( \mathbb{F} \) and by \( \text{Shift}\Delta \) we always mean the exterior algebraic shifting with respect to \( \mathbb{F} \). First we need the following proposition, which shows that the the low dimensional skeleta of an \( i \)-near-cone satisfy a shifting property.

Proposition 4.1. Let \( \Delta \) be an \( i \)-near-cone on vertex set \( \{v_1, \ldots, v_n\} \), having minimal facet cardinality \( k \). Assume that the sequence \( v_1, \ldots, v_i \) is the apex of \( \Delta \). Then for every \( s \leq k - i - 1 \) and every \( \sigma \in \Delta^{(s)} \), \( \sigma \setminus \{v_i\} \cup \{v_j\} \in \Delta^{(s)} \), for every \( 1 \leq j \leq i \) and every \( t \geq j \), with \( v_t \in \sigma \).

Proof. We use the notations from Definition 2.3. Assume that \( \sigma \) is a face of \( \Delta^{(s)} \) such that \( v_j \notin \sigma \) and suppose that \( \{v_1, \ldots, v_{j-1}\} \cap \sigma = \{v_1, \ldots, v_{i_m}\} \), with \( i_1 < i_2 < \ldots < i_m \). Then \( \sigma_0 := \sigma \setminus \{v_1, \ldots, v_{j-1}\} \) belongs to \( \Delta^{(j-1)} \) and therefore by the definition of \( i \)-near-cone \( \sigma_1 = (\sigma_0 \setminus \{v_1\}) \cup \{v_j\} \in \Delta^{(j-1)} \). Since \( \sigma_1 \) is contained in a facet of \( \Delta \) and

\[
|\sigma_1| = |\sigma_0| = |\sigma| - m \leq s + 1 - m \leq k - i - m,
\]

there exist vertices \( u_1, \ldots, u_m \) such that for every \( 1 \leq l \leq m \), \( u_t \notin \sigma_1 \cup \{v_1, \ldots, v_i\} \), and \( \sigma_1 \cup \{u_1, \ldots, u_m\} \) is a face of \( \Delta \). Now by the definition of \( i \)-near cone

\[
\sigma_2 = (\sigma_1 \setminus \{u_m\}) \cup \{v_m\},
\]

\[
\sigma_3 = (\sigma_2 \setminus \{u_{m-1}\}) \cup \{v_{m-1}\},
\]

\[
\vdots
\]

\[
\sigma_m+1 = (\sigma_m \setminus \{u_1\}) \cup \{v_1\} = (\sigma \setminus \{v_i\}) \cup \{v_j\}.
\]

are faces of \( \Delta \). \( \square \)
Notice that Proposition 4.1 says that if $\Delta$ is an $i$-near cone, then $\Delta(s)$ is shifted with respect to $v_1, \ldots, v_i$, for every $s \leq k - i$ - 1. Nevo examined the algebraic shift of a near-cone, showing:

**Lemma 4.2.** ([4, Corollary 5.3]) Assume that $\Delta$ is a near-cone with apex $v$, let us consider $\text{Shift}_\Delta$ as having ordered vertex set $\{u_1, \ldots, u_n\}$ and $\text{Shift}(\text{lk}_\Delta v)$ as having ordered vertex set $\{u_2, \ldots, u_n\}$. Then

$$\text{lk}_{\text{Shift}_\Delta} u_1 = \text{Shift}(\text{lk}_\Delta v).$$

The following Lemma shows that exterior algebraic shifting commutes with link in the case of low dimensional skeleta of $i$-near-cones.

**Lemma 4.3.** Let $\Delta$ be an $i$-near-cone with $\dim \Delta \geq 2i - 2$ having minimal facet cardinality $k$. Assume that the sequence $v_1, v_2, \ldots, v_i$ is the apex of $\Delta$. Let $F := \{v_1, v_2, \ldots, v_i\}$. Consider $\text{Shift}_\Delta$ as having ordered vertex set $\{u_1, \ldots, u_n\}$ and consider $\text{Shift}(\text{lk}_\Delta F)$ as having ordered vertex set $\{u_{i+1}, \ldots, u_n\}$. Then for every $s \leq k - i - 1$ we have

$$\text{lk}_{\text{Shift}_\Delta} \{u_1, \ldots, u_i\} = \text{Shift}(\text{lk}_\Delta F).$$

**Proof.** We prove the lemma by induction on $i$. The case $i = 1$ is trivial by Lemma 4.2 and [7, Corollary 2.4]. So assume that $i \geq 2$. Now

$$\text{lk}_{\text{Shift}_\Delta} \{u_1, \ldots, u_i\} = \text{lk}_{\text{Shift}_\Delta} \{u_1, \ldots, u_{i-1}\} u_i.$$

By induction hypothesis

$$\text{lk}_{\text{Shift}_\Delta} \{u_1, \ldots, u_{i-1}\} = \text{Shift}(\text{lk}_\Delta \{v_1, \ldots, v_{i-1}\}).$$

Therefore

$$\text{lk}_{\text{Shift}_\Delta} \{u_1, \ldots, u_i\} = \text{lk}_{\text{Shift}}(\text{lk}_\Delta \{v_1, \ldots, v_{i-1}\}) u_i.$$

Now Proposition 4.1 implies that $\text{lk}_\Delta \{v_1, \ldots, v_{i-1}\}$ is a near-cone with apex $v_i$. Thus by Lemma 4.2

$$\text{lk}_{\text{Shift}}(\text{lk}_\Delta \{v_1, \ldots, v_{i-1}\}) u_i = \text{Shift}(\text{lk}_\Delta \{v_1, \ldots, v_{i-1}\} v_i) = \text{Shift}(\text{lk}_\Delta F).$$

The following proposition is an immediate consequence of Lemma 4.3 and [7, Corollary 2.4].

**Proposition 4.4.** Let $\Delta$ be an $i$-near cone with $\dim \Delta \geq 2i - 2$ having minimal facet cardinality $k$. Assume that the sequence $v_1, v_2, \ldots, v_i$ is the apex of $\Delta$. Let $F := \{v_1, v_2, \ldots, v_i\}$. Consider $\text{Shift}_\Delta$ as having ordered vertex set $\{u_1, \ldots, u_n\}$ and consider $\text{Shift}(\text{lk}_\Delta F)$ as having ordered vertex set $\{u_{i+1}, \ldots, u_n\}$. Then for every $r$ with $r \leq k - 2i$

$$f_r(\text{lk}_{\text{Shift}_\Delta} \{u_1, \ldots, u_i\}) = f_r(\text{Shift}(\text{lk}_\Delta F)).$$
We are now ready to prove the main result of this section. By applying exterior algebraic shifting we prove:

**Theorem 4.5.** If $\Delta$ is an $i$-near-cone, then $\Delta$ is $(i,r)$-EKR for every $i \leq r$ with $\text{depth}_F \Delta \geq (i+1)(r-i+1) - 1$.

**Proof.** The case $r = i$ is trivial. So assume that $r > i$. Then $\dim \Delta \geq \text{depth}_F \Delta \geq (i+1)(r-i+1) - 1 \geq 2i+1 > 2i - 2$. Therefore using the notations from Definition 2.1, Lemma 2.2 implies that $F = \{v_1, \ldots, v_i\}$ is a face of $\Delta$. Let $A$ be an $i$-intersecting $r$-family of faces of $\Delta$. We show that $|A| \leq f_{r-i}(\text{lk}_F \Delta) = f_{r-i}(\text{lk}_F \Delta)$, where $\text{lk}_F \Delta$ is the minimum facet cardinality of $\text{lk}_F \Delta + 1$, hence

$$|A| \leq f_{r-i}(\text{lk}_{\text{Shift}_F \Delta} \{u_1, \ldots, u_i\}) = f_{r-i}(\text{lk}_F \Delta),$$

by Lemma 3.1 and Proposition 4.4. Note that since $r > i$, the assumption implies that $k \geq \text{depth}_F \Delta + 1 \geq (i+1)(r-i+1) = i(r-i) + r + 1 > r + i$, where $k$ is the minimum facet cardinality of $\Delta$. Hence $r - i < k - 2i$ and thus Proposition 4.4 is applicable here. $\square$

The following corollary settles Conjecture 1.4 in the case of $i$-near-cones for some special class of parameters and it is a consequence of Theorem 4.5 and its proof.

**Corollary 4.6.** Let $i \leq r$ be two integers. Assume that $\Delta$ is an $i$-near-cone such that $\text{depth}_F \Delta \geq (i+1)(r-i+1) - 1$ and suppose that $S \neq \emptyset$ is a subset of $[i, r]$. Then every $i$-intersecting family $A$ of faces of $\Delta$ with $A \subseteq \bigcup_{s \in S} \Delta(s)$ satisfies the following inequality:

$$|A| \leq \max \sum_{s \in S} f_{s-i}(\text{lk}_A \sigma),$$

where the maximum is taken all over $i$ faces of $\Delta$.

**Proof.** The proof is similar to the proof if Theorem 3.2. Just one should use Theorem 4.5 instead of Lemma 3.1. $\square$

The following corollary is an immediate consequence of Corollary 4.6 and proves Conjecture 1.4 in the case of sequentially Cohen-Macaulay $i$-near-cones for $t = i$. Note that if $\Delta$ is sequentially Cohen-Macaulay over $F$ then $\text{depth}_F \Delta$ is the minimum facet dimension of $\Delta$.

**Corollary 4.7.** Let $i \leq r$ be two integers. Assume that $\Delta$ is a sequentially Cohen-Macaulay $i$-near-cone having minimal facet cardinality $k \geq (i+1)(r-i+1)$ and suppose that $S \neq \emptyset$ is a subset of $[i, r]$. Then every $i$-intersecting family $A$ of faces of $\Delta$ with $A \subseteq \bigcup_{s \in S} \Delta(s)$ satisfies the following inequality:

$$|A| \leq \max \sum_{s \in S} f_{s-i}(\text{lk}_A \sigma),$$

where the maximum is taken all over $i$ faces of $\Delta$. 
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