Classical $r$-matrix of the $\mathfrak{su}(2|2)$ SYM spin-chain

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Abstract

In this note we straightforwardly derive and make use of the quantum $R$-matrix for the $\mathfrak{su}(2|2)$ SYM spin-chain in the manifest $\mathfrak{su}(1|2)$-invariant formulation, which solves the standard quantum Yang-Baxter equation, in order to obtain the correspondent (undressed) classical $r$-matrix from the first order expansion in the “deformation” parameter $2\pi/\sqrt{\lambda}$, and check that this last solves the standard classical Yang-Baxter equation. We analyze its bialgebra structure, its dependence on the spectral parameters and its pole structure. We notice that it still preserves an $\mathfrak{su}(1|2)$ subalgebra, thereby admitting an expression in terms of a combination of projectors, which spans only a subspace of $\mathfrak{su}(1|2) \otimes \mathfrak{su}(1|2)$. We study the residue at its simple pole at the origin, and comment on the applicability of the classical Belavin-Drinfeld type of analysis.
1 Introduction

Integrability in AdS-CFT [1] has revealed itself as one of the most promising developments towards a proof of the conjecture. Among the preminent results is the derivation of a scattering matrix [2] whose tensorial structure is fixed by centrally extended $su(2|2) \oplus su(2|2)$ symmetry (see also [3]), and whose dressing factor [4] is constrained by the Hopf-algebraic analog of what crossing symmetry is for relativistic systems [5]. Remarkable advances in the exact determination of this phase factor have been recently made [6, 7], such an impressive agreement seemingly supporting the idea of a powerful algebraic structure underlying the full planar integrability of the model (see [8] and the recent [9, 10, 11]).

The presence of a Hopf algebra structure [12, 13] suggests that the full $S$-matrix might be a representation of the Universal $R$-matrix for a yet to be discovered bialgebra, which such an $R$-matrix would endow with a triangular structure. In order to make progress in understanding this construction, it is normally useful to study the classical limit of the $R$-matrix, in terms of a deformation around the identity. Under certain assumptions (among which some non-degeneracy condition), powerful theorems allow a complete classification of solutions of the classical Yang-Baxter equation [14]. In particular, Belavin and Drinfeld [14] considered solutions $r(u_1, u_2)$ to the CYBE assuming values in $g \otimes g$, with $g$ a finite-dimensional simple Lie algebra, which are of difference form, namely they depend only on the difference $u$ of the two spectral parameters, $u = u_1 - u_2$. They introduced the additional hypothesis of non-degeneracy, i.e. one of the three equivalent conditions: (i) the determinant of the matrix formed by the coordinates of the tensor $r$ is not identically zero, (ii) $r$ has at least one pole in the complex variable $u$, and it does not exist a Lie subalgebra $g'$ such that $r$ is an element of $g' \otimes g'$ for any $u$, or (iii) $r$ has a first order pole in $u = 0$, with residue of the form $c \sum_{\mu} I_{\mu} \otimes I_{\mu}$ with $c$ a complex number and $I_{\mu}$ a basis in $g$ orthonormal with respect to a chosen nondegenerate invariant bilinear form. Such a residue can be identified with the quadratic Casimir operator in the tensor product of two copies of the Lie algebra. Under these requirements, they proved that such solutions satisfy the unitarity condition $r_{12}(u) = -r_{21}(-u)$, and extends meromorphically to the entire $u$ complex plane. All the poles of $r(u)$ are simple, and form a lattice $\Gamma$ in the complex plane. Furthermore, modulo automorphisms, one has three possible types of solutions: elliptic (if $\Gamma$ is a two-dimensional lattice), trigonometric (if $\Gamma$ is one-dimensional), or rational functions (if $\Gamma = 0$). From the knowledge of such a classical $r$-matrix, there is a standard procedure to construct an associate Lie bialgebra, in terms of so-called Manin triples (see for example [15] and references therein). This has to play the role of the enlarged symmetry algebra one is looking for. Work on the extension of these results to the more complicated case of simple Lie superalgebras began shortly after [16] (see also [17]).

The lack of difference form of the AdS-CFT $S$-matrix represents a source of rich structure, which evades Belavin and Drinfeld’s assumptions, and its classification appears like an open problem[1]. In this note, we would like to set the tools for such an approach. We first obtain a form of the quantum $R$-matrix in the manifest $su(1|2)$-invariant formulation, by performing in each entries of Beisert’s spin-chain $S$-matrix [2] a transformation from excitation states $|\phi^1, \phi^2\rangle$ to states $|\phi\rangle, |\chi\rangle$ [2, 5] in initial and final states, producing the appearance of additional momentum-dependent phase factors upon consistent reabsorption of all the $Z$-markers. What we are left with is a genuine solution of the standard quantum Yang-Baxter equation (QYBE), in the spirit of the canonical $R$-matrix of [10] to which this one should be related via a non-local.

\footnote{One can see [18] for work on $R$-matrices dependent on non-additive parameters.}
basis transformation. The choice of the \( su(1|2) \) formulation allows us to make direct contact with the Hopf-algebraic construction performed in [13]. Furthermore, it leaves a large number of generators undeformed, and singles out a decomposition by means of projectors, in terms of which it is quite simple to perform the consequent analysis of the classical \( r \)-matrix. The outcome is a formalism where the \textit{dynamic} nature of the spin chain has been completely translated into algebraic properties, and the presence of the length-changing operators has been totally removed, their action being fully implemented into the resulting bialgebra.

Then, we take the strong coupling regime of the spin-chain, and expand the solution as

\[
R_{su(1|2)} \sim 1 + \zeta r_{su(1|2)},
\]

which allows us to extract the classical \( r \)-matrix \( r_{su(1|2)} \) (the subscript stands for the \( su(1|2) \)-basis), our deformation parameter being \( \zeta = 2\pi/\sqrt{\lambda} \). In string theory language, the notion of classical \( r \)-matrix would correspond in physical terms, taking into account the contribution coming from the dressing factor, to the tree-level string sigma-model scattering matrix. This was computed (in a natural string basis) in [19]. Here, in more mathematical terms, we really try to identify \( \zeta \) as the expansion parameter leading to a Poisson structure deformation (quantization parameter \( \hbar \)).

We make use of the parametrization in [6] and keep fixed their \( x \)-parameter, which becomes the spectral parameter for the classical \( r \)-matrix. We check that this \( r_{su(1|2)} \) satisfies the standard classical Yang-Baxter equation (CYBE). We stress that, being interested in symmetries of the classical \( r \)-matrix, we neglect in all computations the dressing factor, which amounts to adding to the classical \( r \)-matrix a term proportional to the identity, thereby dropping from the CYBE.

We then derive the first order coproduct relation from expansion of the full Hopf-algebraic one [13]. \( r_{su(1|2)} \) still preserves an undeformed \( su(1|2) \) subalgebra, thereby admitting an expansion in terms of projectors onto irreducible representations of \( su(1|2) \otimes su(1|2) \). We make use of a simpler Casimir operator with respect to [5]. One of the projectors has zero coefficient, therefore the \( r \)-matrix projects onto a subspace of \( su(1|2) \otimes su(1|2) \).

We then examine the poles of the classical \( r \)-matrix: apart from uninteresting singularities in the parametrization, there is one pole for coincident spectral parameters, at which we calculate the residue. After a change of variables, the residue assumes the form of the Casimir of the algebra \( gl(2|2) \) shifted by the identity. The appearance of this non-simple Lie algebra naively seems to put into troubles the applicability of a Belavin-Drinfeld type of analysis. This, in turn, might have been expected by considering the important role the central extensions have to play in the structure of the resulting Hopf algebra [12, 13, 11, 20]. Still, the presence of a Casimir operator indicates it might be possible to adapt the Manin triple procedure [15] to the present case. Very recent developments point in fact towards the direction of a particular Yangian symmetry underlying the problem [21].

Another pole is found when the two eigenvalues corresponding to the two out of three surviving projectors coincide. The study of the analytic singularities of the full quantum \( S \)-matrix, and in general of the dynamics ensuing from it, has been the subject of an intense work [22].

We conclude with some comments, and appendices where we collect the bulk of the formulas.

\footnote{We thank G. Arutyunov and S. Frolov for email exchange about this point.}

\footnote{We thank M. Zabzine for discussions about this point.}
2 Quantum $R$-matrix $R_{\mathfrak{su}(1|2)}$

The centrally-extended $\mathfrak{su}(1|2)$ commutation relations are as follows [2]:

$$[\mathfrak{H}_\alpha^a, \mathfrak{J}^\alpha] = \delta^\alpha_\beta \mathfrak{J}^\beta - \frac{1}{2} \delta^\alpha_\beta \mathfrak{J}^\gamma,$$
$$[\mathfrak{L}_\beta^a, \mathfrak{J}^\gamma] = \delta^\gamma_\beta \mathfrak{J}^\alpha - \frac{1}{2} \delta^\gamma_\beta \mathfrak{J}^\gamma,$$

$$\{\Omega^a, \mathfrak{P}^b\} = \delta^b_a \Omega^a + \delta^b_\alpha \Omega^\alpha + \delta^b_\beta \Omega^\beta,$$
$$\{\Omega^a, \Omega^b\} = \epsilon^{\alpha\beta} \epsilon_{ab} \mathfrak{P},$$
$$\{\mathfrak{S}^a, \mathfrak{S}^b\} = \epsilon_{ab} \epsilon \mathfrak{P}.$$  

(3)

The $(2|2)$ 4-dimensional representation obtained by Beisert in [2] is labelled by 4 parameters $a, b, c, d$ with the constraint $ad - bc = 1$. A basis of the representation space is provided by the vectors $|\phi^a\rangle$ (even) and $|\psi^\alpha\rangle$ (odd), with $a = 1, 2$ and $\alpha = 1, 2$, and the generators’ action is

$$\mathfrak{H}_b^a |\phi^c\rangle = \delta^c_b |\phi^c\rangle - (1/2) \delta^c_\alpha |\phi^\alpha\rangle,$$
$$\mathfrak{L}_\beta^a |\phi^\gamma\rangle = \delta^\gamma_\beta |\phi^\gamma\rangle - (1/2) \delta^\gamma_\alpha |\phi^\alpha\rangle$$

for the even ones, and

$$\Omega^a |\phi^b\rangle = \delta^b_a |\psi^a\rangle,$$
$$\Omega^a |\psi^\beta\rangle = \epsilon^{\alpha\beta} \epsilon_{ab} |\phi^b \mathfrak{Z}^+\rangle,$$
$$\mathfrak{S}^a |\phi^b\rangle = \epsilon_{ab} |\psi^\beta \mathfrak{Z}^-\rangle,$$
$$\mathfrak{S}^a |\psi^\beta\rangle = \epsilon \epsilon_{ab} |\phi^b\rangle$$

for the odd ones. We reorganize the indices $a, \alpha$ in a unique index $A = 1, 2, 3, 4$ such that $a = 1$ corresponds to $A = 1$, $a = 2$ to $A = 2$, $\alpha = 1$ to $A = 3$ and $\alpha = 2$ to $A = 4$. We will adopt the $\mathfrak{su}(1|2)$-formulation [2, 3], in which the generators’ action is redefined in such a way that no $Z$-markers appear, instead their effect is taken into account by a nontrivial coproduct (see [13] for details). The Hopf algebra structure is obtained by singling out an $\mathfrak{su}(1|2)$ sub-algebra with generators $\mathfrak{J} \in \{\mathfrak{H}_1, \mathfrak{L}_1, \Omega_1, \mathfrak{S}_1, \mathfrak{C}\}$. The coproduct on these generators is trivial,

$$\Delta(\mathfrak{J}) = \mathfrak{J} \otimes 1 + 1 \otimes \mathfrak{J}.$$  

(6)

The remaining generators split according to $\mathfrak{D}_+ \in \{\mathfrak{S}_2, \mathfrak{C}, \mathfrak{R}\}$ and $\mathfrak{D}_- \in \{\Omega_2, \mathfrak{L}_2, \mathfrak{R}, \mathfrak{P}\}$, then one has

$$\Delta(\mathfrak{D}_+) = \mathfrak{D}_+ \otimes e^{ip} + 1 \otimes \mathfrak{D}_+,$$
$$\Delta(\mathfrak{D}_-) = \mathfrak{D}_- \otimes e^{-ip} + 1 \otimes \mathfrak{D}_-.$$  

(7)

One identifies $e^{ip} = 1 + (\mathfrak{R}/\beta)$ and $e^{-ip} = 1 + (\mathfrak{P}/\alpha)$ as a physical constraint [2]. Up to a rescaling of state vectors this reads $a = 1$, $b = -\alpha(1 - \frac{x^-}{x^+})$, $c = \frac{i\beta}{z}$, $d = -i(x^+ - x^-)$,
\[ x^+ + \frac{\alpha^2}{x^-} - x^- - \frac{\alpha^2}{x^+} = i, \quad \alpha \beta = \frac{q^2}{q}. \] In this way the coproduct is symmetric on the center. Antipode and counite are specified in [13].

The R-matrix intertwining this coproduct, namely solving the equation

\[
\Delta^{op}(x) R = P \Delta(x) R = R \Delta(x)
\]

(P being the graded permutation operator) for any generator x of the centrally extended \( su(2|2) \) algebra, and satisfying the standard quantum Yang-Baxter equation, is easily obtained from Beisert’s S-matrix as \( R = PS \), and by decorating it with suitable phase factors which come from re-expressing initial and final \( |\phi^2\rangle \) states as \( |\chi\rangle = |\phi^2 Z^+\rangle \) states, in the spirit of the \( su(1|2) \) formulation. We will call this R-matrix \( R_{su(1|2)} \), \( R_{su(1|2)}(v^A_1) \otimes |v^B_2\rangle = \sum_{C,D=1}^4 R^{AB}_{CD}|v^C_1\rangle \otimes |v^D_2\rangle \). Pedices 1, 2 correspond to chosen representations. The entries of \( R_{su(1|2)} \) are reported in the Appendix. We stress the fact that we neglect everywhere in this paper the dressing factor, which has no effect on the quantum (and the classical) Yang-Baxter equation.

Since there is an \( su(1|2) \) subalgebra with undeformed coproduct, one can express this very same R-matrix as \( R_{su(1|2)} = \sum_{i=1}^3 S_i P_i \), where \( P_i \) are the projectors onto the irreducible representations of \( su(1|2) \otimes su(1|2) \). In order to construct such projectors, we use a simpler Casimir with respect to the one used by Janik [4], namely we take

\[
C_{12} = \frac{1}{2} (\Omega_1^1 \otimes \Theta_2^1 + \Omega_2^2 \otimes \Theta_2^2 - \Theta_2^1 \otimes \Omega_1^2 + \Theta_1 \otimes \Omega_1^2) + (\mathcal{R}_1^1 + \mathcal{C}) \otimes (\mathcal{R}_1^1 + \mathcal{C}) - \frac{1}{2} L_\alpha^\alpha \otimes L_\alpha^\alpha, \tag{9}
\]

whose eigenvalues are\(^4\)

\[
\lambda_1 = (1 + b_1c_1)(1 + b_2c_2), \\
\lambda_2 = \frac{b_1c_1 + b_2c_2}{2} + b_1c_1b_2c_2, \\
\lambda_3 = b_1c_1b_2c_2. \tag{10}
\]

This Casimir is related to the one used by Janik as \( C_{Janik} = 2C_{12} + (C^{(2)}_1 + C^{(2)}_2) 1 \otimes 1 \), if \( C^{(2)}_i \) is the quadratic Casimir of \( su(1|2) \) in representation \( i \). Selected eigenvectors are for instance \( |\phi \rangle \otimes |\phi\rangle, \ |\psi^1 \rangle \otimes |\psi^1\rangle \) and \( |\chi \rangle \otimes |\chi\rangle \), respectively. The functions \( S_i \) are given by \( S_1 = \frac{x_1^1 - x_1^2}{x_2^1 - x_2^2} \), \( S_2 = 1 \) and \( S_3 = e^{i(p_1 - p_2)}S_1 \). The projectors read

\[
P_i = \frac{(C_{12} - \lambda_j)(C_{12} - \lambda_k)}{(\lambda_i - \lambda_j)(\lambda_i - \lambda_k)}, \tag{11}
\]

with \((i, j, k)\) equal to \((1, 2, 3), (2, 1, 3)\) and \((3, 1, 2)\) respectively.

3 Classical \( r \)-matrix \( r_{su(1|2)} \)

After setting [6]

\(^4\)In [5] the Casimir is constructed multiplying the (trivial) coproduct of the \( su(1|2) \) generators. We thank the author for clarifications on this point.
we take $\zeta = 2\pi/\sqrt{\lambda}$ as a deformation parameter, in the sense that we expand all formulas around $\zeta = 0$ keeping $x$ fixed. Since $\sin \frac{\zeta}{2} = \frac{1}{2\sqrt{\lambda}}$, this limit corresponds to $p \sim \zeta \sim \lambda^{-1/2}$, namely the BMN limit \[^5\]. We also impose $\alpha = \frac{\tilde{\alpha}}{\zeta}$ and $\beta = \frac{1}{2\zeta \alpha}$, with $\tilde{\alpha}$ a free parameter. The $R$-matrix and generators admit an expansion

$$R_{\text{su}(1|2)} \sim 1 + \zeta R_{\text{su}(1|2)},$$
$$\mathcal{J} \sim \mathcal{J}^{(0)} + \zeta \mathcal{J}^{(1)},$$
$$\mathcal{D}_\pm \sim \mathcal{D}_\pm^{(0)} + \zeta \mathcal{D}_\pm^{(1)},$$
$$e^{ip} \sim 1 + \zeta \pi^{(1)},$$

(13)

where the leading order generators are simply obtained using as parameters $a^{(0)}, b^{(0)}, c^{(0)}, d^{(0)} = \lim_{\zeta \to 0} a, b, c, d$. One has $a^{(0)} = 1, b^{(0)} = -i\tilde{\alpha} \frac{x}{x^2 - 1}, c^{(0)} = \frac{i}{\tilde{\alpha} x}, d^{(0)} = \frac{x^2}{x^2 - 1}$. They solve a quadratic instead of a quartic constraint, namely

$$\beta a^{(0)} b^{(0)} + \alpha c^{(0)} d^{(0)} = 0.$$  

(14)

They still satisfy $a^{(0)} d^{(0)} - b^{(0)} c^{(0)} = 1$, and then $\mathcal{J}^{(0)}$ form a $(2|2)$ representation of centrally extended $\text{su}(2|2)$.

If we plug the expansion (13) into the original formulas (6), (7) and (8), calling $\Delta^{\text{triv}}(x) = x \otimes 1 + 1 \otimes x$, we obtain

$$[\Delta^{\text{triv}}(\mathcal{J}^{(0)}), r] = 0,$$
$$[\Delta^{\text{triv}}(\mathcal{D}_\pm^{(0)}), r] = \pm (\mathcal{D}_\pm^{(0)} \otimes \pi^{(1)} - \pi^{(1)} \otimes \mathcal{D}_\pm^{(0)}),$$

(15)

which means that the classical $r$-matrix is still $\text{su}(1|2)$ invariant.

One can therefore express it as a combination of projectors

$$r_{\text{su}(1|2)} = \sum_{i=1}^{3} \sigma_i P_i^{(0)},$$

(16)

with $P_i^{(0)} = \lim_{\zeta \to 0} P_i$, and with coefficient functions $\sigma_i$ given by the first order in $\zeta$ of the $S_i$, $S_i \sim 1 + \zeta \sigma_i$. One has in fact

$$R \sim \sum_{i=1}^{3} (1 + \zeta \sigma_i)(P_i^{(0)} + \zeta P_i^{(1)}) \sim \sum_{i=1}^{3} (P_i^{(0)} + \zeta \sigma_i P_i^{(0)} + \zeta P_i^{(1)}),$$

(17)

but since $\sum_{i=1}^{3} P_i = 1$ it follows that $\sum_{i=1}^{3} P_i^{(0)} = 1$ and $\sum_{i=1}^{3} P_i^{(1)} = 0$. The quantities in (16) read

\[^5\]We thank J. Plefka for discussion about this point.

\[^6\]One can check that (15) is consistent when applied to the central charges $\mathfrak{P}^{(0)} = a^{(0)} b^{(0)}$ and $\mathfrak{R}^{(0)} = c^{(0)} d^{(0)}$, upon using $\pi^{(1)} = 2\pi x / (x^2 - 1)$. 

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\[
\sigma_1 = \frac{2I(x_1^2 + x_2^2 - 2x_1^2x_2^2)}{(x_1 - x_2)(x_1^2 - 1)(x_2^2 - 1)}, \\
\sigma_2 = 0, \\
\sigma_3 = -\frac{2Ix_1x_2(x_1^2 + x_2^2 - 2)}{(x_1 - x_2)(x_1^2 - 1)(x_2^2 - 1)}, \\
P_{ij}^{(0)} = \frac{(C_{ij}^{(0)} - \lambda_i^{(0)})(C_{ij}^{(0)} - \lambda_j^{(0)})}{(\lambda_i^{(0)} - \lambda_j^{(0)})(\lambda_i^{(0)} - \lambda_k^{(0)})},
\]

where from (10) one obtains in the limit \(\zeta \to 0\)

\[
\lambda_1^{(0)} = \frac{x_1^2}{(x_1^2 - 1)(x_2^2 - 1)}, \\
\lambda_2^{(0)} = \frac{x_2^2}{2(x_1^2 - 1)(x_2^2 - 1)}, \\
\lambda_3^{(0)} = \frac{1}{(x_1^2 - 1)(x_2^2 - 1)},
\]

with \((i,j,k)\) equal to \((1,2,3), (2,1,3)\) and \((3,1,2)\) respectively. One realizes that the 16 × 16 matrix \(r\) projects onto a subspace of the full representation space corresponding to the eigenvalues \(\lambda_1\) and \(\lambda_3\) of the \(\mathfrak{su}(1|2) \otimes \mathfrak{su}(1|2)\) Casimir.

In the Appendix we report all entries of the classical \(r\)-matrix \(r_{\mathfrak{su}(1|2)}\). We have checked that it satisfies the standard classical (super) Yang-Baxter equation

\[
\sum_{i,j,k} r_{ij}^{(i,i+1)}(x_1, x_3)r_{ij}^{(i+1,i+2)}(x_2, x_3)(-1)^{ij(i+1)} = \sum_{i,j,k} r_{ij}^{(i+1,i+2)}(x_1, x_3)r_{ij}^{(i,i+1)}(x_2, x_3)(-1)^{ij(i+1)} + \sum_{i,j,k} r_{ij}^{(i,i+1)}(x_1, x_2)r_{ij}^{(i+1,i+2)}(x_2, x_3)(-1)^{ij(i+1)}.
\]

One can see the appearance of poles in the entries: the poles at \(x_i^2 = 1\) are related to singularities in the parametrization \((12)\) used. The pole at \(x_1 = x_2\) is present in the coefficient functions \(\sigma_i\), and comes from the denominator \(\frac{1}{x_2 - x_1}\) in the quantum S-matrix. The residue at this pole is reported in the Appendix. There we show how to change variables \(x_i = x_i(y_i)\) in order to make the coefficient of the residue becoming a constant matrix, and from there we read the form of the residue. It corresponds to the Casimir of the \(\mathfrak{gl}(2|2)\) superalgebra, shifted by the identity, namely in the vicinity of the pole one has

\[
r \sim \frac{1 \otimes 1 + \sum_{i,j=1}^{4} (-d^{ij})E_{ij} \otimes E_{ji}}{y_1 - y_2},
\]

where \(E_{ij}\) are the \((2|2)\) matrices with all zeroes up to a 1 in the entry \((i,j)\). The presence of the Casimir of a non-simple Lie superalgebra casts doubts on the viability of a Belavin-Drinfeld type of analysis in the present case, but it might nevertheless open a way to a better understanding of the structure of the classical \(r\)-matrix and its bialgebra.

The pole at \(x_1x_2 = 1\) is not present in the coefficient functions \(\sigma_i\), and it corresponds to a degeneracy of the projectors when the two eigenvalues \(\lambda_i^{(0)}\) and \(\lambda_3^{(0)}\) coincide, as can be seen
from (18). When taking into account the small $\zeta$ limit of $\sin \frac{x}{2} = \zeta \frac{1}{x}$, one can notice that this pole correspond to small momenta $p_1 + p_2 = 0$. We remind once again our interest here in the purely algebraic features of the classical $r$-matrix, while any physical interpretation should appropriately take into account the presence of the dressing factor.

4 Conclusions

In this note we have obtained a solution to the standard quantum and classical Yang-Baxter equation from Beisert’s dynamic $su(2|2)$ spin-chain $S$-matrix [2], using the manifest $su(1|2)$-invariant formulation [2, 5]. By completely reabsorbing the lenght-changing operators action consistently into the bialgebra structure, this formulation becomes more suitable for the Hopf-algebraic description setup in [13], and allows a projector decomposition which simplifies the analysis of its properties, and may provide insights for the reconstruction of its full symmetry. We have derived the classical $r$-matrix in the parametrization of [6], together with its bialgebra structure, and studied its dependence on the spectral parameters, especially its residue at the poles. The idea is to provide a basis for extracting the relevant information about the suspected enlarged algebraic structure underlying the integrability of the problem, since this is traditionally classified upon looking at properties of the classical $r$-matrix. The appearance of the Casimir of the non-simple algebra $gl(2|2)$ in the residue at the simple pole at the origin, and the fact that the difference-form is lacking, makes the application of standard theorems a priori more problematic, and represents an interesting open problem which could give rise to new structures, which we plan to investigate in the future. Developments in this directions appeared very recently in [21].

An important step would be to make direct contact with the computations performed from the string theory perspective [19, 10]. The $su(1|2)$ decomposition is less appealing there, therefore one might have to somehow “covariantize” the outcome of the analysis of the enlarged symmetry algebra performed along the lines we showed here, before a comparision with string theory will be made. On the other hand, the construction of the Manin triple is possibly quite general, and one may try to adapt the procedure to the present case. Even though we have dropped the phase factor as inessential to our analysis, the ultimate hope is that the emerging symmetry, equipped with a suitable generalization of the crossing symmetry, will be able to determine some of its remarkable properties.

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6 Appendix: The $R$-matrix $R_{su(1|2)}$

We report here the non-zero entries of the $R$-matrix $R_{su(1|2)}$ (index 1 corresponds to state $|\phi\rangle$, 2 to $|\chi\rangle$, 3 to $|\psi^1\rangle$ and 4 to $|\psi^2\rangle$). As an example we can derive $R_{22}^{22}$ from Beisert’s $S$-matrix:

$$S|\chi_1\chi_2\rangle = S|\phi_1^2\phi_2^2\rangle = e^{-ip_2}S|\phi_2^2\phi_1^2\rangle = e^{-ip_2+ip_1}A_{12}|\chi_2\chi_1\rangle,$$

where $|x_1y_2\rangle = |x_1\rangle \otimes |y_2\rangle$. From $R = PS$ one has then $PS|v_1^Av_2^B\rangle = PS_{CD}(x_1,x_2)|v_2^Cv_2^D\rangle = R_{CD}(x_1,x_2)|v_1^Cv_1^D\rangle$. One finds

$$R_{11}^{11} = A_{12},$$
$$R_{12}^{12} = \frac{1}{2}(A_{12} + B_{12}),$$
$$R_{12}^{13} = \frac{1}{2}(A_{12} - B_{12})e^{ip_1},$$
$$R_{12}^{14} = \frac{1}{2}C_{12},$$
$$R_{12}^{21} = \frac{1}{2}(A_{12} + B_{12})e^{i(p_1-p_2)},$$
$$R_{12}^{22} = \frac{1}{2}(A_{12} - B_{12})e^{-ip_2},$$
$$R_{12}^{24} = \frac{1}{2}C_{12}e^{-ip_2},$$
$$R_{12}^{31} = \frac{1}{2}C_{12}e^{-ip_2},$$
$$R_{12}^{32} = A_{12}e^{i(p_1-p_2)},$$
$$R_{12}^{33} = -D_{12},$$
$$R_{12}^{34} = -\frac{1}{2}(D_{12} + E_{12}),$$
$$R_{12}^{41} = -\frac{1}{2}(D_{12} - E_{12}),$$
$$R_{12}^{42} = \frac{1}{2}F_{12},$$
$$R_{12}^{43} = -\frac{1}{2}F_{12}e^{ip_1},$$
$$R_{12}^{44} = -D_{12},$$

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\( R_{13}^{13} = G_{12}, \)
\( R_{13}^{13} = H_{12}, \)
\( R_{14}^{14} = G_{12}, \)
\( R_{14}^{14} = H_{12}, \)
\( R_{23}^{23} = G_{12}e^{-ip_2}, \)
\( R_{32}^{32} = H_{12}e^{i(p_1 - p_2)}, \)
\( R_{24}^{24} = G_{12}e^{-ip_2}, \)
\( R_{32}^{32} = H_{12}e^{i(p_1 - p_2)}, \)
\( R_{31}^{31} = K_{12}, \)
\( R_{31}^{31} = L_{12}, \)
\( R_{41}^{41} = K_{12}, \)
\( R_{41}^{41} = L_{12}, \)
\( R_{32}^{32} = K_{12}, \)
\( R_{32}^{32} = L_{12}e^{ip_1}, \)
\( R_{42}^{42} = K_{12}, \)
\( R_{42}^{42} = L_{12}e^{ip_1}. \)

(23)

The functions appearing are Beisert’s one, which we rewrite here for convenience after using the constraint [2]

\[
A_{12} = \frac{x_2^+ - x_1^-}{x_2 - x_1},
\]
\[
G_{12} = \frac{x_2^+ - x_1^-}{x_2 - x_1},
\]
\[
H_{12} = \frac{x_2^+ - x_2^-}{x_2 - x_1},
\]
\[
K_{12} = \frac{x_2^- - x_1^+}{x_2 - x_1},
\]
\[
L_{12} = \frac{x_2^- - x_1^+}{x_2 - x_1},
\]
\[
D_{12} = -1
\]
\[
B_{12} = -1 + \frac{(x_2^+ - x_2^- - x_1^+ + x_1^-)(x_1^+ x_2^+ - 2x_1^- x_2^+ + x_1^- x_2^+)}{(x_2^- - x_1^+)(x_1^- x_2^- - x_1^- x_2^+)}.
\]
\[
E_{12} = A_{12} - \frac{(x_2^+ - x_2^- - x_1^+ + x_1^-)(x_1^+ x_2^+ - 2x_2^- x_1^+ + x_1^- x_2^-)}{(x_2^- - x_1^+)(x_1^- x_2^- - x_1^- x_2^+)}.
\]
\[
C_{12} = \frac{2x_1^+ x_2^+(x_2^+ - x_2^- - x_1^- + x_1^+)}{\alpha(x_2^+ - x_1^-)(x_1^- x_2^+ - x_1^- x_2^+)}
\]
\[
F_{12} = \frac{4\omega x_1^- x_2^- (x_1^- - x_1^-)(x_1^+ - x_2^-)(x_2^+ - x_2^- - x_1^+ + x_1^-)}{g^2(x_2^- - x_1^-)(x_1^- x_2^+ - x_1^- x_2^+)}.
\]
\[
e^{ip_1} = \frac{x_2^+}{x_1^-}.
\]

(24)
Appendix: The $r$-matrix $r_{su(1|2)}$

We use the parametrization in [6]:

$$x^\pm(x) = \frac{1}{2\zeta} \left( x \sqrt{1 - \frac{\zeta^2}{(x - \frac{1}{x})^2}} \pm i\zeta \frac{x}{x - \frac{1}{x}} \right) \quad (25)$$

where $\zeta = 2\pi/\sqrt{\lambda}$ and $g^2 = \frac{1}{\zeta^2}$. We set as in the text $\alpha = \frac{\zeta}{\sqrt{\lambda}}$ and $\beta = \frac{1}{2\alpha\zeta}$, with $\bar{\alpha}$ a free parameter.

The non-zero entries of the classical $r$-matrix $r_{su(1|2)}$ read

$$r_{11}^{11} = \frac{2ix_1^2x_2^2 - 2x_1^2x_2^2}{(x_1 - x_2)(x_1^2 - 1)(x_2^2 - 1)},$$
$$r_{21}^{12} = \frac{2ix_1x_2}{(x_1 - x_2)(x_1x_2 - 1)},$$
$$r_{12}^{12} = \frac{2ix_2(x_1^2 + x_2^2 - x_1x_2^2 - 2x_1^2x_2 + x_1^2x_2^3)}{(x_1 - x_2)(x_1^2 - 1)(x_2^2 - 1)(x_1x_2 - 1)},$$
$$r_{12}^{12} = -r_{12}^{12} = \frac{2}{\bar{\alpha}(x_1x_2 - 1)},$$
$$r_{22}^{22} = \frac{2ix_1x_2(x_1^2 + x_2^2 - 2)}{(x_1 - x_2)(x_1^2 - 1)(x_2^2 - 1)},$$
$$r_{33}^{33} = r_{44}^{44} = 0,$$
$$r_{43}^{34} = -r_{34}^{34} = \frac{2ix_1x_2}{(x_1 - x_2)(x_1x_2 - 1)},$$
$$r_{34}^{34} = -r_{34}^{34} = \frac{2\bar{\alpha}x_1^2x_2^2}{(x_1 - 1)(x_1^2 - 1)(x_1x_2 - 1)},$$
$$r_{21}^{21} = -r_{12}^{12} = \frac{2\bar{\alpha}x_1^2x_2^2}{(x_1 - 1)(x_1^2 - 1)(x_1x_2 - 1)},$$
$$r_{13}^{13} = r_{31}^{13} = r_{14}^{14} = r_{41}^{14} = \frac{2ix_1^2}{(x_1 - x_2)(1 - x_1^2)},$$
$$r_{23}^{23} = r_{24}^{23} = \frac{2ix_1x_2}{(x_1 - x_2)(1 - x_1^2)},$$
$$r_{32}^{23} = r_{42}^{23} = \frac{2ix_1^2}{(x_1 - x_2)(1 - x_1^2)}, \quad (26)$$

and one can obtain the other ones not displayed by using unitarity, namely

$$r_{CD}^{AB}(x_2, x_1) + (-)^{AB+CD} r_{DC}^{BA}(x_1, x_2) = 0, \quad (27)$$

which ensues from unitarity of the quantum $S$-matrix $S_{12}S_{21} = 1$.

The residue at the pole $x_1 = x_2$ reads
\[ r_{11}^{11} = r_{22}^{22} = \frac{-4ix_1^2}{x_1^2 - 1}, \]
\[ r_{21}^{12} = r_{12}^{12} = r_{34}^{34} = -r_{43}^{34} = \frac{-2ix_1^2}{x_1^2 - 1}, \]
\[ r_{43}^{12} = r_{33}^{33} = r_{21}^{34} = r_{44}^{34} = 0, \]
\[ r_{13}^{13} = r_{14}^{14} = r_{14}^{14} = \frac{-2ix_1^2}{x_1^2 - 1}, \]
\[ r_{23}^{23} = r_{24}^{24} = r_{24}^{24} = \frac{-2ix_1^2}{x_1^2 - 1}. \]  

(28)

and using again unitarity one can obtain the not displayed entries. In (28), \( r^{ij}_{kl} \) is an abuse of notation for the residue \( \text{Res}[r^{ij}_{kl}, x_1 = x_2] \).

In terms of the projector decomposition (16), the residue is easily computed as \( \frac{-4ix_1^2}{x_1^2 - 1}[P_1^{(0)} + P_3^{(0)}] \), where \( P_i^{(0)} \) project onto irreducible representations of \( \mathfrak{su}(1|2) \otimes \mathfrak{su}(1|2) \) where the same representation 1 is chosen on both sides of the tensor product. This can be rewritten as

\[ r \sim \frac{f(x_1)(1 \otimes 1 + \sum_{i,j=1}^{4} (-y^{d}[E_{ij} \otimes E_{ji}])}{x_1 - x_2}, \]  

(29)

where \( E_{ij} \) are the \( (2|2) \) coordinate-matrices with all zeroes up to 1 in the entry \((i,j)\), and 

\[ f(x_1) = \frac{2ix_1}{x_1^2 - 1}. \]

There is a standard way of changing variables in order to reduce the residue at the pole to a constant matrix [24]. We notice that this trick works in the simple Lie algebra case, and for \( r \)-matrices of difference-form, therefore it is not a priori guaranteed that one can apply it in the present case as well. This is due to the particular form of the residue (29), and it may signify some important features of this \( r \)-matrix.

Suppose the behaviour in the vicinity of the pole is \( r \sim \frac{f(x_1)y}{x_1 - x_2} \), with \( t \) a constant matrix, and \( f(x_1) \neq 0 \). Then, one can easily verify that changing variables \( x = x(y) \) in such a way that \( x'(y) = f(x(y)) \), allows to reduce the function \( f \) to 1. In our case we have the following differential equation:

\[ x'(y) = -\frac{2ix(y)^2}{x(y)^2 - 1}, \]  

(30)

which is solved by \( x(y) = \frac{1}{2}(-2iy + c \pm \sqrt{c^2 - 4icy - 4y^2 - 4}) \), \( c \) being an integration constant.

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