MULTI-SCALE BILINEAR RESTRICTION ESTIMATES FOR GENERAL PHASES

TIMOTHY CANDY

Abstract. We prove (adjoint) bilinear restriction estimates for general phases at different scales in the full non-endpoint mixed norm range, and give bounds with a sharp and explicit dependence on the phases. These estimates have applications to high-low frequency interactions for solutions to partial differential equations, as well as to the linear restriction problem for surfaces with degenerate curvature. As a consequence, we obtain new bilinear restriction estimates for elliptic phases and wave/Klein-Gordon interactions in the full bilinear range, and give a refined Strichartz inequality for the Klein-Gordon equation. In addition, we extend these bilinear estimates to hold in adapted function spaces by using a transference type principle which holds for vector valued waves.

1. Introduction

Let $n \geq 2$ and for $j = 1, 2$ take phases $\Phi_j : \Lambda_j \to \mathbb{R}$ with $\Lambda_j \subset \mathbb{R}^n$. We consider the problem of obtaining (adjoint) bilinear restriction estimates of the form

$$\| e^{it\Phi_1(-\nabla)} f e^{it\Phi_2(-\nabla)} g \|_{L^2_t L^q_x(\mathbb{R}^{1+n})} \leq C \| f \|_{L^2_t L^q_x(\mathbb{R}^n)} \| g \|_{L^2_t L^r_x(\mathbb{R}^n)}$$

(1.1)

with an explicit dependence of the constant $C$ on the phases $\Phi_j$ and sets $\Lambda_j$. Here $1 \leq q, r \leq 2$, $\text{supp} \hat{f} \subset \Lambda_1$, $\text{supp} \hat{g} \subset \Lambda_2$, and $\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot \xi} dx$ is the spatial Fourier transform of $f$. This problem is interesting for two key reasons. Firstly, in applications to PDE, the dependence on $\Phi_j$ and $\Lambda_j$ reflects the derivative cost required to place the product into $L^q_t L^r_x$. Controlling the number of derivatives required in (1.1) leads to, for instance, sharp null form estimates in the case of the wave equation $\Phi = |\xi|^\lambda$ [21], refined Strichartz estimates for the wave and Klein-Gordon equations [2] [8], [17], as well as scattering for the Wave Maps equation [19], and Dirac-Klein-Gordon equation [5] [6]. Secondly, bilinear restriction estimates of the form (1.1) have played an important role in the classical linear restriction problem in harmonic analysis, see for instance [15, 20]. In particular, understanding the dependence of the constant $C$ on the phases $\Phi_j$, has led to improved linear restriction estimates for surfaces with curvature that may degenerate in certain directions [3] [20].

We now consider some concrete examples. In the case of the wave equation $\Phi_1 = \Phi_2 = |\xi|$, the estimate (1.1) has a long history and is now essentially well understood. More precisely, if $\Lambda_1 \subset \{ |\xi| \approx 1 \}$ is at scale 1, $\Lambda_2 \subset \{ |\xi| \approx \lambda \}$ is at scale $\lambda \geq 1$, and $\Lambda_1$ and $\Lambda_2$ have angular separation of size 1, then it is known that (1.1) holds in the range $\frac{1}{q} + \frac{n+1}{2r} \leq \frac{n+1}{2}$ and we have $C \approx \lambda^{\frac{n}{2} + 1 - \epsilon}$ for every $\epsilon > 0$. This is sharp up to the $\epsilon$ frequency loss, and in the unit scale ($\lambda = 1$), non-endpoint case $q = r > \frac{n+3}{n+1}$, is due to the breakthrough work of Wolff [27]. The endpoint case $q = r = \frac{n+3}{n+1}$ for general scales $\lambda \geq 1$ was obtained by Tao [21]. In the mixed exponents case, $q \neq r$, the bilinear estimate for general scales is due to Tataru [24] for the non-constant coefficient wave equation, and Lee-Vargas [13], Lee-Rogers-Vargas [12] in the constant coefficient case. The endpoint estimate for $q \neq r$ is also known, and was proven by Temur [25].

On the other hand, in the case of the Schrödinger equation, $\Phi_1 = \Phi_2 = |\xi|^2$, if $\Lambda_1$ is a ball of radius 1, and $\Lambda_2$ is a ball of radius $\lambda \geq 1$ such that the sets $\Lambda_1$ and $\Lambda_2$ are separated by a distance $\lambda$, then work of Tao [22] shows that for $q = r > \frac{n+3}{n+1}$ the bilinear estimate (1.1) holds with $C \approx \lambda^{\frac{n}{2} + 1 + \epsilon}$ for every $\epsilon > 0$.

In the case of general phases which are both at unit scale, under suitable transversality and curvature assumptions, Lee-Vargas [14] and Bejenaru [1] have shown that (1.1) again holds in the non-endpoint range $q = r > \frac{n+3}{n+1}$. For general phases which are not at unit scale, only in certain special cases is the dependence of $C$ in (1.1) on the phases $\Phi_j$ and sets $\Lambda_j$ known. In particular, if $\Phi_1$ and $\Phi_2$ are elliptic phases with curvature...
at different scales, then \( \Lambda^1 \) was obtained by Stovall [20] with an almost sharp dependence on the scale. If \( n = 2 \), and \( \Phi_1 = \Phi_2 = \xi_1^{m_1} + \xi_2^{m_2} \) are surfaces of finite type with rectangular sets \( \Lambda_j \subset \{ 0 < \xi_1, \xi_2 < 1 \} \), then the essentially optimal dependence of the constant \( C \) on the rectangles \( \Lambda_j \) and parameters \( m_j \geq 2 \) has been obtained in recent work of Buschenhenke-Müller-Vargas [3].

In the current article we unify and improve the above examples, and show that in fact for general phases \( \Phi_j \), under certain transversality and curvature assumptions on the surfaces \( S_j = \{ (\Phi_j(\xi), \xi) : \xi \in \Lambda_j \} \subset \mathbb{R}^{1+n} \), if (for instance) \( \Lambda_1 \) is contained in a ball of radius 1, then the bilinear restriction estimate (1.1) holds in the full, non-endpoint, bilinear range \( \frac{1}{2} + \frac{n+2}{2r} < \frac{n+1}{2} \) with constant

\[
C \approx \mathcal{V}_{max}^{-\frac{1}{2}} \left( \min(\mathcal{H}_1, \mathcal{H}_2) \right)^{\frac{1}{2}} \left( \max(\mathcal{H}_1, \mathcal{H}_2) \right)^{\frac{1}{2}} \tag{1.2}
\]

where we introduce the notation

\[
\mathcal{V}_{max} = \sup_{\xi \in \Lambda_1} |\nabla \Phi_1(\xi) - \nabla \Phi_2(\eta)|, \quad \mathcal{H}_j = \|\nabla^2 \Phi_j\|_{L^\infty(\Lambda_j)},
\]

see Theorem 1.2 below for a more precise statement. Moreover, we show that this dependence on the phases \( \Phi_j \) is sharp. The estimate (1.2) recovers all examples mentioned above without an epsilon loss. For instance, in the case of the wave equation, a computation shows that provided the sets \( \Lambda_1 \) and \( \Lambda_2 \) are at scales 1 and \( \lambda \geq 1 \) respectively, and are angularly separated by a distance 1, then we have \( \mathcal{V}_{max} \approx 1 \) and \( \mathcal{H}_1 = 1, \mathcal{H}_2 \approx \lambda^{-1} \). In particular, for the bilinear restriction estimate for the cone, (1.2) recovers the optimal dependence on the frequency \( \lambda \) without an \( \epsilon \) loss. Similarly (1.2) also recovers the correct dependence on the frequency parameter \( \lambda \) in the Schrödinger case, after observing that if \( \Lambda_1 \) and \( \Lambda_2 \) are separated by a distance \( \lambda \), then \( \mathcal{V}_{max} \approx \lambda \) and \( \mathcal{H}_1 = \mathcal{H}_2 = 1 \).

As a concrete application of the general formula (1.2), we obtain new bilinear restriction estimates for elliptic phases, and Klein-Gordon interactions with differing masses, see Theorem 1.8 and Theorem 1.10 respectively. In particular, we obtain sharp (non-endpoint) estimates for wave Klein-Gordon interactions. Further, we give two additional consequences of the general bilinear restriction estimate. The first, stated in Theorem 1.12 below, is a refined Strichartz estimate for the Klein-Gordon equation which shows that, for \( n \geq 2 \) there exists \( \theta > 0 \) such that

\[
\left\| e^{it(\nabla)} f \right\|_{L^\infty_x L^{2n+4} \left( \mathbb{R}^{1+n} \right)} \lesssim \| f \|_{X}^{\theta} \| f \|_{H^\infty_x}^{1-\theta}, \tag{1.3}
\]

where \( \| \cdot \|_X \) is a norm which detects concentration of the Fourier support of \( f \) to certain rectangular sets, and satisfies \( \| f \|_X \leq \| f \|_{L^2} \). The classical Strichartz estimate for the Klein-Gordon equation corresponds to \( \theta = 0 \). The estimate (1.3) gives the Klein-Gordon version of a refined Strichartz estimate for the wave equation due to Ramos [17] and extends a similar estimate of Killip-Stovall-Visan [8]. Bounds of the form (1.3) are closely related to profile decompositions, see for instance [8] [17] for further details.

The second consequence is an extension of (1.1) from free waves \( e^{it\Phi(-i\nabla)} f \), to more general functions belonging to certain adapted function spaces \( U^\sharp \), see Corollary 1.6 below. This corresponds to a multi-scale version of real part of the work by the author and Herr [4]. The new bilinear bounds we obtain here, are applied in [3] [6] to obtain conditional scattering results for the Dirac-Klein-Gordon system. The application to the Dirac-Klein-Gordon equation, which includes Klein-Gordon interactions with different masses, formed a key motivation to study general bilinear restriction estimates of the form (1.1).

1.1. Main Theorem. Let \( \ell^2_\ell(\mathbb{Z}) \) denote the collection of all compactly supported sequences in \( \ell^2(\mathbb{Z}) \), thus every sequence in \( \ell^2_\ell(\mathbb{Z}) \) has only finitely non-zero components. By a slight abuse of notation, the standard norm on \( \ell^2 \) is denoted by \( | \cdot | \). We define the product of two \( \ell^2_\ell(\mathbb{Z}) \)-valued functions \( u \) and \( v \) as simply the tensor product \( uv = u \otimes v \). Thus \( |uv| = |u||v| \). We say that \( u(t,x) : \mathbb{R}^{1+n} \to \ell^2_\ell(\mathbb{Z}) \) is a \( \Phi_j \)-wave if

\[
u(t,x) = e^{it\Phi_j(-i\nabla)} f(x)
\]

with \( f : \mathbb{R}^n \to \ell^2_\ell(\mathbb{Z}) \) and \( \text{supp} \hat{f} \subset \Lambda_j \). Note that, if \( u \) is a \( \Phi_j \)-wave, then \( \text{supp} \hat{u} \) is independent of time, \( \|u\|_{L^\infty \ell^2_\ell} = \|u(0)\|_{\ell^2_\ell} \), and \( u \) is a solution to the equation

\[
i \partial_t u + \Phi_j(-i\nabla) u = 0.
\]
Alternatively, up to a Jacobian factor, $u$ can be viewed as the extension operator associated to the surface $S_j = \{(\Phi_j(\xi), \xi) \mid \xi \in \Lambda_j\} \subset \mathbb{R}^{1+n}$.

It is well-known that to obtain bilinear restriction estimates in the full bilinear range $\frac{1}{9} + \frac{n+1}{2r} \leq \frac{n+1}{2r}$ requires firstly that the surfaces $S_j$ are transverse, in the sense that $|\nabla \Phi_1(\xi) - \nabla \Phi_2(\eta)| > 0$, and secondly that there is some curvature along the $n-1$ dimensional surfaces of intersection $S_1 \cap (\mathfrak{h} - S_2)$ for $\mathfrak{h} \in \mathbb{R}^{1+n}$. These conditions are sufficient in the case of the wave or Schrödinger equation, but in the general case, a stronger condition is required that ensures that the normals to the surface $S_k$ intersect transversally with the conic surface formed by taking the normal directions to the surface $\Sigma_j$ at points in $S_j \cap (\mathfrak{h} - S_k)$, see for instance [4] for assumptions of this type. In the current article, we use a normalised version of the conditions appearing in [3]. To describe the precise conditions we impose on the phases $\Phi_j$, we require some additional notation. Given $\mathfrak{h} = (a, h) \in \mathbb{R}^{1+n}$ and $\{j, k\} = \{1, 2\}$ we define the surfaces

$$\Sigma_j(h) = \{\xi \in \Lambda_j \cap (h - \Lambda_k) \mid \Phi_j(\xi) + \Phi_k(h - \xi) = a\}.$$ 

Note that $\Sigma_j(h) \subset \Lambda_j \subset \mathbb{R}^n$ and these surfaces are related to the intersection of the surfaces $S_j \cap (h - S_k)$ through the formula

$$S_j \cap (h - S_k) = \{(\Phi_j(\xi), \xi) \mid \xi \in \Sigma_j(h)\}.$$ 

The sets $\Sigma_j(h)$ play a crucial role in what follows. The key curvature, transversality, and smoothness assumptions we make on the phases $\Phi_j$ are the following.

**Assumption 1.1.** Fix constants $C_0, d_0 > 0$. For $j = 1, 2$ we let $\Lambda_j \subset \mathbb{R}^n$ be an open set, and $\Phi_j : \Lambda_j \to \mathbb{R}$ satisfy the conditions:

- **(A1)** for every $\{j, k\} = \{1, 2\}$, $\mathfrak{h} \in \mathbb{R}^{1+n}$, $\xi, \xi' \in \Sigma_j(\mathfrak{h})$, and $\eta \in \Lambda_k$ we have
  $$|(|\nabla \Phi_j(\xi) - \nabla \Phi_j(\xi')| \wedge (\nabla \Phi_j(\xi) - \nabla \Phi_k(\eta))| \geq C_0 \nu_{\max} H_j|\xi - \xi'|$$

  and

  $$C_0 |\nabla \Phi_j(\xi) - \nabla \Phi_j(\xi')| \leq H_j|\xi - \xi'|,$$

- **(A2)** for $j \in \{1, 2\}$ we have $\Phi_j \in C^{5n}(\Lambda_j)$ and moreover

  $$\min_{3 \leq m \leq 5n} \left( \frac{H_j}{\|\nabla^m \Phi_j\|_{L^\infty(\Lambda_j)}} \right)^{\frac{1}{m-3}} \geq d_0, \quad \frac{\nu_{\max}}{H_j} \geq d_0.$$

The key assumption (A1) is essentially equivalent to the conditions appearing previously in the literature [1, 4], except that we require a normalised version to correctly determine the dependence of the constant on the phases $\Phi_j$. To gain a better geometric understanding of (A1), we observe that letting $\xi' \to \xi$ and taking $\eta = h - \xi$, since $\nabla \Phi_j(\xi) - \nabla \Phi_k(h - \xi)$ is normal to $\Sigma_j(\mathfrak{h})$, (A1) implies the *local* condition

$$|\nabla^2 \Phi_j(\xi)\| \wedge n| \geq C_0 H_j|\nu|$$

for all $v \in T_\xi \Sigma_j(\mathfrak{h})$, where $n$ is the unit normal to $\Sigma_j(\mathfrak{h})$, and $T_\xi \Sigma_j(\mathfrak{h})$ is the tangent plane at $\xi \in \Sigma_j(\mathfrak{h})$. Under certain conditions, the local condition (1.3) is in fact equivalent to the global assumption (A1), see Lemma 2.1 below. In view of (1.3), (A1) states that the Hessian $\nabla^2 \Phi_j(\xi)$ can not rotate tangent vectors in $\Sigma_j(\mathfrak{h})$ to normal vectors. A similar observation was made to [1] where the assumption (A1) was replaced with a condition on the shape operator associated to the surface $S_j$. We should emphasise here that in the general phase case, it is not sufficient to simply assume the surfaces $S_j$ intersect transversally, and a stronger condition of the form (A1) is necessary [11, 26].

There are two immediate consequences of (A1) which we wish to highlight. The first is that for every $\xi \in \Lambda_1$ and $\eta \in \Lambda_2$ we have the transversality bound

$$|\nabla \Phi_1(\xi) - \nabla \Phi_2(\eta)| \geq C_0 \nu_{\max}$$

which follows by observing that for every $\xi \in \Lambda_1$ there exists $\mathfrak{h} \in \mathbb{R}^{1+n}$ such that $\xi \in \Sigma_j(\mathfrak{h})$. The second is that for every $j \in \{1, 2\}$, $\mathfrak{h} \in \mathbb{R}^{1+n}$, and $\xi, \xi' \in \Sigma_j(\mathfrak{h})$ we have the (global) curvature type bound

$$|\nabla \Phi_j(\xi) - \nabla \Phi_j(\xi')| \geq C_0 H_j|\xi - \xi'|.$$
The bounds (1.5) and (1.6) play a key role in the proof of Theorem 1.2 and in fact the stronger bound contained in (A1) is only directly required in the proof of Lemma 7.2 below.

The smoothness assumption (A2) requires that the phases to lie in $C^{5n}(\Lambda_j)$, this can probably be improved, but we do not consider this issue here. While the constant $C_0$ should be thought of as a universal constant, it is important to keep track of the parameter $d_0$ as it encodes the size of the Fourier support of the underlying waves, and scales like the frequency $\xi$. For instance, if $\Phi_1(\xi) = |\xi|^s$ for some $s > 0$, then (A2) essentially requires $d_0 \lesssim |\xi|$ on $\Lambda_1$. More generally, we observe that both the assumptions (A1) and (A2) are invariant under the rescaling

$$(\Lambda_j, \Phi_j(\xi), d_0) \mapsto (\mu^{-1} \Lambda_j, \lambda \Phi_j(\mu \xi), \mu^{-1} d_0)$$

and linear translations

$$\Phi_j \mapsto \Phi_j + \xi \cdot \xi_0.$$  

These invariances significantly simplify the arguments to follow, as we can then reduce to the case where the maximum velocity and curvature are normalised to be one. In fact the precise assumptions (A1) and (A2) were carefully chosen with the scaling (1.7) and translation invariance (1.8) in mind.

We are now ready to state our first main theorem.

**Theorem 1.2.** Let $n \geq 2$, $C_0 > 0$, $1 \leq q, r \leq 2$, and $\frac{1}{q} + \frac{n+1}{2r} < \frac{n+1}{2}$. There exists a constant $C > 0$, such that for any $d_0 > 0$, any open sets $\Lambda_1, \Lambda_2 \subset \mathbb{R}^n$, any phases $\Phi_1$ and $\Phi_2$ satisfying Assumption (A2) with $H_2 \leq H_1$, and any $\Phi_1$-waves $u$, and $\Phi_2$-waves $v$ satisfying the support conditions

$$\text{supp} \ 0 \ + \ d_0 \subset \Lambda_1, \quad \text{supp} \ 0 \ + \ d_0 \subset \Lambda_2, \quad \min \{\text{diam} \ (\text{supp} \ 0), \text{diam} \ (\text{supp} \ 0)\} \leq d_0,$$

we have

$$\|uv\|_{L^q_\xi L^r_\xi(R^{1+n})} \leq C d_0^{\frac{n+1}{q} - \frac{1}{q} + \frac{n+1}{2r} + 1} V_{\max}^{-1} H_1^{-1 - \frac{1}{q} - \frac{1}{r}} \left( \frac{H_1}{H_2} \right)^{\frac{1}{q} - \frac{1}{r}} \|u\|_{L^q_\xi L^r_\xi} \|v\|_{L^q_\xi L^r_\xi}.$$  

(1.9)

The proof of Theorem 1.2 follows the strategy used by Tao in the proof of the endpoint bilinear restriction estimate for the cone [21]. In particular, we replace the combinatorial Kakeya type arguments used by Wolff in the seminal paper [27] and further exploited in, for instance, [22] [14] [13] [21], with energy estimates across transverse hypersurfaces. A similar argument was used by Bejenaru [1] to obtain a bilinear restriction estimate with $q = r$ for general phases at unit scale. For our purposes, arguing via energy estimates has a number of advantages, firstly it avoids the $\epsilon$ loss that occurs when using the pigeon hole type arguments that arise in the combinatorial approach, and secondly it can be adapted without too much effort to handle interactions with waves at very different scales by using the wave table construction of Tao.

**Remark 1.3.** The extension of the bilinear restriction estimates to *vector valued* waves, was first obtained by Tao [21] where the vector valued nature of the waves played an important technical role in the induction argument. As the proof of Theorem 1.2 follows the argument used by Tao, the fact that we may take vector valued waves in Theorem 1.2 also plays an important technical role here. However, there is an additional gain that comes from working with vector valued waves, which turns out to be very useful for applications of Theorem 1.2 to problems in nonlinear PDE. This gain comes from the simple observation that if we have an estimate for vector valued waves, then we can immediately deduce that the same bound holds in the atomic Banach space $U^2_{\Phi_1^j}$. In particular, Theorem 1.2 also holds in $U^2_{\Phi_1^j}$, see Corollary 1.6 below. The fact that Theorem 1.2 holds for vector valued waves, is a reflection of the fact that the proof essentially only exploits bilinear estimates in $L^2$, for which the theory for scalar valued and vector valued waves is identical. A previous joint work of the author and Herr [4], used a similar observation, although phrased in terms of $U^2_{\Phi_1^j}$ atoms, to prove a unit scale bilinear restriction in the adapted function space $V^2_{\Phi_1^j}$. The work [4] provided a strong motivation to consider the case of non-unit scale waves, however it is important to note that the argument used in [4] follows the combinatorial Kakeya type approach to bilinear restriction estimates, in contrast to the energy type approach used here. In particular, it is also possible to prove estimates for vector valued waves via the combinatorial approach.

Theorem 1.2 is optimal up to endpoints, in the sense that the restriction $\frac{1}{q} + \frac{n+1}{2r} \leq \frac{n+1}{2}$ is necessary, and the dependence on $\Phi_j$ and $d_0$ in (1.9) cannot be improved, see Section 3. In particular, if the Fourier support of $u$ or $v$ is contained in a ball of radius $d_0$, then the dependence of (1.9) on $d_0$ is sharp. However,
if the support of $\hat{u}$ (or $\hat{v}$) is not well approximated by a ball, then certain improvements are possible. One possibility is the following. Given sets $\Lambda_j^* \subset \Lambda_j$, we define the quantity

$$d[\Lambda_j^*, \Lambda_j^*] = \sup_{b = (a, h) \in \mathbb{R}^{n+1}} \left( \sigma_{\Sigma_j(h)} \left[ \Sigma_j(h) \cap \Lambda_j^* \cap (h - \Lambda_j^*) \right] \right)^{1 + \frac{1}{q}}$$

(1.10)

where $\sigma_{\Sigma_j(h)}$ denotes the (induced) Lebesgue surface measure on the surface $\Sigma_j(h) \subset \mathbb{R}^n$. It is easy to check that, under the transversality assumption (1.8), we have the bound $d[\Lambda_j^*, \Lambda_j^*] \lesssim \min \{ \text{diam} (\Lambda_j^*), \text{diam} (\Lambda_j^*) \}$.

However, $d[\Lambda_j^*, \Lambda_j^*]$ can potentially be much smaller as it only requires the intersection of supp $\hat{u}$, the surface $\Sigma_j(h)$, and translates of supp $\hat{v}$ to be small. The quantity $d[\Lambda_j^*, \Lambda_j^*]$ arises precisely in the bilinear restriction estimate. In fact, under the transversality assumption (1.5) we have for every $\Phi_1$-wave $u$, and $\Phi_2$-wave $v$,

$$\|uv\|_{L^q_x L^r_t} \lesssim \left( C_0 V_{\max} \right)^{-\frac{1}{r}} \left( d[\text{supp } \hat{u}, \text{supp } \hat{v}] \right)^{\frac{n+1}{q-r}} \|u\|_{L^q_x L^r_t} \|v\|_{L^q_x L^r_t},$$

see Theorem 5.2 below. Exploiting this observation leads to the following improvement of Theorem 1.2.

**Theorem 1.4.** Let $n \geq 2$, $C_0 > 0$, $1 \leq q, r \leq 2$, and $\frac{1}{q} + \frac{n+1}{2r} < \frac{n+1}{2}$. There exists a constant $C > 0$, such that for any $d_0 > 0$, any open sets $\Lambda_1, \Lambda_2 \subset \mathbb{R}^n$, any phases $\Phi_1$ and $\Phi_2$ satisfying Assumption (1.1) with $\mathcal{H}_1 \subset \mathcal{H}_2$, and any $\Phi_1$-waves $u$, and $\Phi_2$-waves $v$ satisfying the support conditions

$$\text{supp } \hat{u} + d_0 \subset \Lambda_1, \quad \text{supp } \hat{v} + d_0 \subset \Lambda_2, \quad d[\text{supp } \hat{u} + d_0, \text{supp } \hat{v} + d_0] \leq d_0,$$

we have

$$\|uv\|_{L^q_x L^r_t} \lesssim C_0^{n+1} V_{\max}^{\frac{n+1}{q-r}} \|u\|_{L^q_x L^r_t} \|v\|_{L^q_x L^r_t}.$$

**Remark 1.5.** It is clear that if $\min \{ \text{diam} (\text{supp } \hat{u}), \text{diam} (\text{supp } \hat{v}) \} \leq d_0$, then since one of $\text{supp } \hat{u} + d_0$ or $\text{supp } \hat{v} + d_0$ is again contained in a ball of radius $d_0$, a computation gives

$$d[\text{supp } \hat{u} + d_0, \text{supp } \hat{v} + d_0] \lesssim d_0.$$

In particular, after potentially choosing $C_0$ slightly smaller, we see that Theorem 1.4 implies Theorem 1.2.

We note here that the $d_0$ thickening of the Fourier supports, namely $\text{supp } \hat{u} + d_0$ and $\text{supp } \hat{v} + d_0$, arises as the induction on scales argument used in the proof of Theorem 1.4 involves numerous decompositions of $u$ and $v$ into wave packets, each of which may have slightly larger Fourier support. Thus we require some room to increase the Fourier support in the induction argument, which eventually manifests itself in the assumptions in Theorem 1.4 applying to a $d_0$ neighbourhood of the Fourier supports of $u$ and $v$.

We now give a number of number of consequences of Theorem 1.4. Namely, we state a version of Theorem 1.4 in the adapted function space $U^2_\Phi$, and give a concrete application to the case of elliptic and Klein-Gordon waves. Moreover, we prove a new refined Strichartz estimate for the Klein-Gordon equation.

**1.2. Bilinear Restriction in Adapted Function spaces.** Theorem 1.4 has a number of applications to linear PDE, for instance, we give a new bilinear restriction estimate for wave/Klein-Gordon interactions, together a refined Strichartz type estimate for the Klein-Gordon equation, see Theorem 1.10 and Theorem 1.12 below. However, in applications to nonlinear PDE, it is useful to have a version of Theorem 1.4 which holds for more general functions than just $\Phi_j$-waves. Often the function spaces which arise in applications satisfy some version of the transference principle. This principle roughly asserts that if we have an estimate for $\Phi_j$-waves, then we immediately deduce that the estimate also holds in some more general function space. For instance the well-known $X^{s,b}$ type spaces satisfy the transference principle, essentially since elements of $X^{s,b}$ type spaces can be written as averages of free waves. On the other hand, other function spaces of interest, such as the adapted function spaces $U^p_{\Phi_j}$ and $V^p_{\Phi_j}$, do not generally satisfy such a strong property, and significant additional work can be required to extend estimates for free waves, to estimates in $U^p_{\Phi_j}$, see for instance [4] for further discussion.

However, the adapted function space $U^2_{\Phi_j}$ does satisfy a slightly weaker transference type principle: if an estimate holds for vector valued waves, then it also holds in $U^2_{\Phi_j}$. Consequently, the fact that Theorem 1.4 holds for vector valued waves, immediately implies that bilinear restriction type estimates for functions in $U^2_{\Phi_j}$, see the proof of Corollary 4.6 below or [4] Remark 5.2 for details of this argument. Bilinear restriction
type estimates in the adapted function spaces $U_{\Phi_j}^p$ and $V_{\Phi_j}^p$ first appeared in work of Sterbenz-Tataru [19]
Lemma 5.7], and the full bilinear range for general phases at unit scale was obtained recently in [3]. The atomic function spaces $U_{\Phi_j}^p$ have proven to be extremely useful in obtaining endpoint well-posedness results for dispersive PDE, see for instance [9, 10, 7] and the references therein.

Before we can state our next result, we require the definition of $U_{\Phi_j}^p$. Let $1 < p < \infty$. We say that
\[ \phi \text{ is a } U_{\Phi_j}^p \text{ atom}, \]
if there exists a finite partition $I$ of $\mathbb{R}$, and a collection of $\Phi_j$-waves $(\phi_I)_{I \in I}$ such that
\[ \phi = \sum_{I \in I} \hat{\phi}_I(t) \phi_I \]
and
\[ \left( \sum_{I \in I} \|\phi_I\|_{L^p_{t,x}} \right)^{\frac{1}{p}} \leq 1. \]

The atomic Banach space $U_{\Phi_j}^p$ is then defined as
\[ U_{\Phi_j}^p = \left\{ \sum_m c_m \phi_m \mid (c_m)_{m \in \mathbb{N}} \in \ell^1(\mathbb{N}), \phi_m \text{ are } U_{\Phi_j}^p \text{ atoms} \right\} \]
with the induced norm
\[ \|u\|_{U_{\Phi_j}^p} = \inf_{u = \sum_m c_m \phi_m} \sum_m |c_m| \]
where the inf is over all representations of $u$ in terms of $U_{\Phi_j}^p$ atoms $\phi_m$. Note that functions in $U_{\Phi_j}^p$ take values in $\ell^2(\mathbb{Z})$, in particular, every $\Phi_j$-wave with norm of size one, is a (trivial) example of a $U_{\Phi_j}^p$ atom.

Corollary 1.6. Let $n \geq 2$, $C_0 > 0$, $1 < q, r \leq 2$, and $\frac{1}{q} + \frac{n+1}{2r} < \frac{n+1}{2}$. There exists a constant $C > 0$, such that for any $d_0 > 0$, any open sets $\Lambda_1, \Lambda_2 \subset \mathbb{R}^n$, any phases $\Phi_1$ and $\Phi_2$ satisfying Assumption [7] with $\mathcal{H}_2 \leq \mathcal{H}_1$, and any $u \in U_{\Phi_1}^q, v \in U_{\Phi_2}^r$ satisfying the support conditions
\[ \text{supp } \hat{u} + d_0 \subset \Lambda_1, \text{ supp } \hat{v} + d_0 \subset \Lambda_2, \text{ and } d \left[ \text{supp } \hat{u} + d_0, \text{supp } \hat{v} + d_0 \right] \leq \frac{d_0}{C_0}, \]
we have
\[ \|uv\|_{L^q_{t,x}(\mathbb{R}^{1+n})} \leq C d_0^{n+1 - \frac{n+1}{q} - \frac{n+1}{2r}} Y_{\max}^{\frac{1}{q} - \frac{1}{r} + \frac{1}{2}} \left( \frac{\mathcal{H}_1}{\mathcal{H}_2} \right)^{\frac{1}{q} - \frac{1}{r} + \frac{1}{2}} \|u\|_{U_{\Phi_1}^q} \|v\|_{U_{\Phi_2}^r}. \]

Proof. We exploit the transference type principle mentioned above. It is enough to consider the case where $u = \sum_{I \in I} \mathbbm{1}_I(t) u_I$ and $v = \sum_{J \in J} \mathbbm{1}_J(t) v_J$ are atoms. Define the vector valued waves $U = (u_I)_{I \in I}$ and $V = (v_J)_{J \in J}$. Then since $u$ and $v$ are atoms, $U$ is a $\Phi_1$-wave, $V$ is a $\Phi_2$-wave, we have the energy estimate $\|U\|_{L^p_{t,x}L^q_{t,x}} \leq 1$ and the pointwise bounds
\[ \sum_{I \in I} \mathbbm{1}_I(t) |u_I| \lesssim \sum_{I \in I} \mathbbm{1}_I(t) |U| \lesssim |U|, \sum_{J \in J} \mathbbm{1}_J(t) |v_J| \lesssim |V| \]
for all $n \geq 2$, $C_0 > 0$, $1 < q, r \leq 2$ and $\frac{1}{a} + \frac{1}{b} = \frac{1}{\max\{q,r\}}$. Therefore Corollary follows from an application of Theorem 1.4.

The bound (1.11) is somewhat crude, and in fact Corollary 1.6 can be improved by further exploiting the oscillatory localisation of the $U_{\Phi_j}^p$ atoms directly in the proof of Theorem 1.4. More precisely, by refining the argument used to prove Theorem 1.4 and interpolating with an “linear” version of the bilinear $L^2_{t,x}$ inside the induction on scales argument, we obtain the following improvement of Corollary 1.6.

Theorem 1.7. Let $n \geq 2$, $C_0 > 0$, $1 < q, r \leq 2$ and $\frac{1}{a} + \frac{1}{b} = \frac{1}{\min\{q,r\}}$. There exists a constant $C > 0$, such that for any $d_0 > 0$, any open sets $\Lambda_1, \Lambda_2 \subset \mathbb{R}^n$, any phases $\Phi_1$ and $\Phi_2$ satisfying Assumption [7] with $\mathcal{H}_2 \leq \mathcal{H}_1$, and any $u \in U_{\Phi_1}^q, v \in U_{\Phi_2}^r$ satisfying the support conditions
\[ \text{supp } \hat{u} + d_0 \subset \Lambda_1, \text{ supp } \hat{v} + d_0 \subset \Lambda_2, \text{ and } d \left[ \text{supp } \hat{u} + d_0, \text{supp } \hat{v} + d_0 \right] \leq \frac{d_0}{C_0}, \]
we have
\begin{equation}
\|uv\|_{L_t^4L_x^2(\mathbb{R}^{1+n})} \leq C d_0^{n+1-\frac{n+1}{2}} V_{\text{max}}^{\frac{1}{2}} H_1^{1-\frac{1}{r}} \left( \frac{H_1}{H_2} \right)^{\frac{1}{r}} \left( \frac{\mu^n V_{\text{max}}}{d_0^{n+1}} \right)^{1-\frac{1}{r}} \|u\|_{U^a_2} \|v\|_{\dot{U}^a_2} \tag{1.12}
\end{equation}
where \( \mu = \min\{\text{diam}(\supp \hat{u}), \text{diam}(\supp \hat{v}), d_0\} \) and \( s_+ = s \) if \( s > 0 \), and zero otherwise.

The bound (1.12) has the useful consequence that we may place \( v \) into the weaker \( V^2_\Phi \) space without the standard high-low frequency loss. To make this claim more precise, define the space \( V^2_\Phi \) as all right continuous functions such that
\[ \|u\|_{V^2_\Phi} = \|u\|_{L_t^\infty L_x^2} + \sup_{(t_k) \in Z} \left( \sum_{k \in Z} \|e^{-it_k \Phi(-iv)} u(t_k) - e^{-it_{k-1} \Phi(-iv)} u(t_{k-1})\|^2_{L_x^2} \right)^{\frac{1}{2}} < \infty \]
where \( Z = \{(t_j)_{j \in \mathbb{Z}} \mid t_j \in \mathbb{R} \) and \( t_j < t_{j+1}\} \). Thus functions in \( V^2_\Phi \) have finite quadratic variation along the flow \( e^{it\Phi(-iv)} \). An application of the continuous embedding \( V^2_\Phi \subset U^a_\Phi \) for \( a > 2 \) [9, Lemma 6.4], together with (1.12) implies that we then have
\[ \|uv\|_{L_t^4L_x^2(\mathbb{R}^{1+n})} \leq C d_0^{n+1-\frac{n+1}{2}} V_{\text{max}}^{\frac{1}{2}} H_1^{1-\frac{1}{r}} \left( \frac{H_1}{H_2} \right)^{\frac{1}{r}} \|u\|_{U^a_2} \|v\|_{\dot{U}^a_2} \]
provided that \( 1 < q \leq 2, 1 < r < 2, \) and \( \frac{1}{q} + \frac{n+1}{2r} < \frac{n+1}{2} \). This bound is precisely the same as the case of free solutions. In particular, we avoid the high-low frequency loss which would come from simply deducing a \( V^2_\Phi \) bound from Corollary 1.6 via an interpolation argument. In the special case \( q = r = 2 \), we get a loss, and only obtain
\[ \|uv\|_{L_t^4L_x^2(\mathbb{R}^{1+n})} \leq C d_0^{n+1-\frac{n+1}{2}} V_{\text{max}}^{\frac{1}{2}} \left( \frac{\mu^n V_{\text{max}}}{d_0^{n+1}} \right)^{\frac{1}{2}} \|u\|_{U^a_2} \|v\|_{\dot{U}^a_2}. \]
This bilinear \( L_t^2L_x^2 \) bound is particularly interesting in the case of the wave or Klein-Gordon equation, as it allows us to place the high frequency wave into the weaker \( V^2_\Phi \) space without losing any high-frequency derivatives.

1.3. The Elliptic Case. We say that a phase \( \Phi : \Lambda \to \mathbb{R} \) is elliptic on \( \Lambda \), if for all \( \xi \in \Lambda \) and \( v \in \mathbb{R}^n \) we have
\[ \langle (\nabla^2 \Phi(\xi)) v \rangle \approx \|\nabla^2 \Phi\|_{L^\infty(\Lambda)} |v|^2. \]
Equivalently, the eigenvalues of \( \nabla^2 \Phi \) all have the same sign, and are essentially of the size \( \|\nabla^2 \Phi\|_{L^\infty(\Lambda)} \). A typical example of an elliptic case is the Schrödinger phase \( \Phi = \frac{|\xi|^2}{2} \), or the Klein-Gordon phase \( (m^2 + |\xi|^2)^{\frac{1}{2}} \) in the region \( |\xi| \ll m \). Bilinear restriction estimates for elliptic phases was recently exploited by Stovall [20] to deduce new results for the linear restriction problem. Applying Theorem 1.3 to the case of elliptic phases, gives the following improvement to [20 Theorem 2.1].

**Theorem 1.8.** Let \( 1 \leq q, r \leq 2, \frac{1}{q} + \frac{n+1}{2r} < \frac{n+1}{2}, \) and \( d_0 > 0 \). Let \( \Lambda_j \subset \mathbb{R}^n \) be convex, and \( \Phi_j \in C^5(\Lambda_j) \) be elliptic phases such that \( H_2 \leq H_1 \),
\[ \text{diam}(\Lambda_j) \lesssim \left( \frac{V_{\text{max}}}{H_1} \right)^{\frac{1}{2}}, \quad \text{diam}(\Lambda_j) \lesssim \min_{2 \leq m \leq 5n} \left( \frac{H_1}{\|\nabla^m \Phi_j\|_{L^\infty(\Lambda_j)}} \right)^{\frac{1}{m-2}}, \]
and for all \( \xi \in \Lambda_1, \eta \in \Lambda_2 \)
\[ |\nabla \Phi_1(\xi) - \nabla \Phi_2(\eta)| \lesssim V_{\text{max}}. \]
Then for any \( \Phi_1 \)-waves \( u \), and \( \Phi_2 \)-waves \( v \) satisfying the support conditions
\[ \text{supp} \hat{u} + d_0 \subset \Lambda_1, \quad \text{supp} \hat{v} + d_0 \subset \Lambda_2, \quad d \left[ \text{supp} \hat{u} + d_0, \text{supp} \hat{v} + d_0 \right] \leq \frac{d_0}{C_0}, \]
we have
\[ \|uv\|_{L_t^4L_x^2(\mathbb{R}^{1+n})} \lesssim d_0^{n+1-\frac{n+1}{2}} V_{\text{max}}^{\frac{1}{2}} H_1^{1-\frac{1}{r}} \left( \frac{H_1}{H_2} \right)^{\frac{1}{r}} \|u\|_{L_t^\infty L_x^2} \|v\|_{L_t^\infty L_x^2}. \]

**Remark 1.9.** The precise assumptions used in the Theorem 1.8 can be weakened slightly (in particular the convexity assumption). See the proof of Lemma 2.1 below.
1.4. Bilinear Restriction for the Klein-Gordon equation. We now turn to the case of the Klein-Gordon equation. Let
\[ \langle \xi \rangle_m = (m^2 + |\xi|^2)^{\frac{1}{2}}. \]
Note that when \( m = 0 \) this is simply the wave phase \(|\xi|\). If we have the support assumptions
\[ \text{supp } \hat{f} \subset \{ |\xi| \sim \lambda, \theta(\xi, \omega_0) \ll \alpha \}, \quad \text{supp } \hat{g} \subset \{ |\xi| \sim \lambda, \theta(\xi, \omega_1) \ll \alpha \} \]
with \( \theta(\omega_0, \omega_1) \approx \alpha \) (here \( \theta(x, y) \) denotes the angle between vectors \( x, y \in \mathbb{R}^n \)) then it is known that for
\[ \frac{1}{q} + \frac{n+1}{2r} < \frac{n+1}{2}, \quad 1 < q, r < 2, \quad \text{and} \quad \lambda \geq \mu \]
we have the bilinear estimate for the wave equation
\[ \|e^{it|\nabla|} f \overline{e^{it|\nabla|}} g \|_{L_t^q L_x^r} \lesssim \alpha^{n-1-\frac{n+1}{2} - \frac{1}{2} - \frac{1}{q} + \frac{n+1}{2r} \frac{\lambda}{\mu}} \|f\|_{L_t^q} \|g\|_{L_t^r}, \]
see [21, 24, 13, 12]. An application of Theorem 1.10 gives the following Klein-Gordon counterpart.

**Theorem 1.10** (Bilinear extension for K-G). Let \( 1 < q, r < 2, \frac{1}{q} + \frac{n+1}{2r} < \frac{n+1}{2}, m_1, m_2 \geq 0, \) and \( \lambda \geq \mu \).

Let \( 0 < \alpha \leq 1, \) and suppose we have \( \xi_0, \eta_0 \in \mathbb{R}^n \) such that \( \langle \xi_0 \rangle_{m_1} \approx \lambda, \langle \eta_0 \rangle_{m_2} \approx \mu, \) and
\[ \frac{|m_2|\xi_0 - m_1|\eta_0|}{\lambda \mu} + \left( \frac{|\xi_0|\eta_0 \mp \xi_0 \eta_0}{\lambda \mu} \right)^{\frac{1}{2}} \approx \alpha. \]

Define \( \beta = \left( \frac{m_1}{\alpha \lambda} + \frac{m_2}{\alpha \mu} + 1 \right)^{-1} \). If
\[ \text{supp } \hat{f} \subset \{ |\xi| - |\xi_0| \ll \beta \lambda, \langle |\xi|\eta_0 - \xi_0 \eta_0 \rangle^{\frac{1}{2}} \ll \alpha \lambda \}, \quad \text{supp } \hat{g} \subset \{ |\xi| - |\eta_0| \ll \beta \mu, \langle |\xi|\eta_0 - \xi_0 \eta_0 \rangle^{\frac{1}{2}} \ll \alpha \lambda \} \]
then we have the bilinear estimate
\[ \|e^{it(-i\nabla)}_{m_1} f e^{it(-i\nabla)}_{m_2} g \|_{L_t^q L_x^r} \lesssim \alpha^{n-1-\frac{n+1}{2} - \frac{1}{2} - \frac{1}{q} + \frac{n+1}{2r} \frac{\lambda}{\mu}} \|f\|_{L_t^q} \|g\|_{L_t^r}, \]
where the implied constant is independent of \( m_1, m_2 \).

Note that Theorem 1.10 contains both wave like regimes (when \( \beta \approx 1 \)), and Schrödinger like regimes (when \( \beta \ll 1 \)). In particular, if \( \beta = 1 \), then the support assumptions on \( f \) and \( g \) are the same in the case of the wave equation, while if \( \beta \approx \alpha \), then the support assumptions reduce to simply requiring that
\[ \text{supp } \hat{f} \subset \{ |\xi| - |\xi_0| \ll \alpha \lambda \}, \quad \text{supp } \hat{g} \subset \{ |\xi| - |\eta_0| \ll \alpha \mu \}, \]
which matches the standard assumptions used in the Schrödinger case (see for instance Theorem 1.8 above).

**Remark 1.11.** If we apply the atomic bilinear restriction estimate, Theorem 1.17 in place of Theorem 1.2 under the same assumptions as those in Theorem 1.10 we arrive at the stronger bounds
\[ \|uv\|_{L_t^q L_x^r} \lesssim \alpha^{n-1-\frac{n+1}{2} - \frac{1}{2} - \frac{1}{q} + \frac{n+1}{2r} \frac{\lambda}{\mu}} \|u\|_{U_{m_1}^q(V)} \|v\|_{V_{m_2}^2(V)}. \]
In particular, in the special case \( q = r = 2 \), we may place \( v \in V_{m_2}^2(V) \) without a high frequency loss.

1.5. A Refined Strichartz Estimates for the Klein-Gordon Equation. The standard \( H^{\frac{n}{2}} \) Strichartz estimate for the Klein-Gordon equation states that
\[ \|e^{it(-i\nabla)} f \|_{L_t^{\frac{n+1}{2}} L_x^{\frac{n+1}{2}}(\mathbb{R}^{1+n})} \lesssim \|f\|_{H^{\frac{n}{2}}}. \]
It is of interest to understand the concentration properties of solutions which come close to maximising this inequality, for instance, understanding these concentrating solutions is a key step in obtaining a profile decomposition. In the case of the Schrödinger equation, this concentration type property is a consequence of a refined Strichartz estimate introduced by Bourgain [2] and extended in work of Moyua-Vargas-Vega [15, 16].

A version of the refined Strichartz estimate for the Klein-Gordon equation was proved by Killip-Stovall-Visan [8] with data in the Klein-Gordon regime (in other words, the bound included a loss of derivatives when compared to the wave regime). Here we use Theorem 1.12 together with an argument used by Ramos [17] to obtain a refined Strichartz estimate for the Klein-Gordon equation without any derivative loss.
Before we state the refined Strichartz estimate for the Klein-Gordon equation, we require some notation. Given \( \lambda \geq 1, \) and \( 0 < \alpha < 1, \) we define \( A_{\lambda, \alpha} \) to the collection of sets \( A \subset \mathbb{R}^n \) of the form

\[
A = A(\xi_0) = \{ (\xi) \approx \lambda, \| \xi \| - |\xi_0| \ll \frac{\alpha \lambda}{1+ \alpha \lambda} \lambda, \ (|\xi||\xi_0| - \xi \cdot \xi_0)^\frac{\alpha}{\alpha+1} \ll \alpha \lambda \}
\]

where the points \( \xi_0 \in \mathbb{R}^n \) satisfy \( (\xi_0) \approx \lambda \) and are chosen to ensure that the sets \( A \in A_{\lambda, \alpha} \) form a finitely overlapping cover of the annulus/ball \( \{ (\xi) \approx \lambda \}. \) In the wave like region \( \alpha \gtrsim \frac{1}{2}, \) the elements of \( A_{\lambda, \alpha} \) are radial sectors of the annuli \( (\xi) \approx \lambda, \) while in the Klein-Gordon (or Schrödinger) like case \( \alpha \ll \frac{1}{2}, \) the elements of \( A_{\lambda, \alpha} \) are angular sectors with some additional radial restrictions which degenerate to cubes if \( \lambda \lesssim 1. \)

**Theorem 1.12** (Refined Strichartz). Let \( n \geq 2. \) There exists \( 0 < \theta < 1 \) and \( 1 < r < 2 \) such that

\[
\| e^{it(\nabla \cdot f)} \|_{L^2_{t,x} \left( \mathbb{R}^{1+n} \right)} \lesssim \left( \sup_{\lambda \in \mathbb{Z}, \alpha \in \mathbb{Z}^{-n}} \sup_{A \in A_{\lambda, \alpha}} \left( \frac{\alpha \lambda}{1+ \alpha \lambda} \right)^{\frac{n}{\alpha+1}} \lambda^\theta |A|^{\frac{n}{\alpha+1}} \| f \|_{L^r(t, L^{2n}_x)} \right)^{\theta} \| f \|_{L^r(t, L^{2n}_x)}^{1-\theta}.
\]

This estimate is the Klein-Gordon counterpart to the estimate for the wave equation obtained by Ramos [17]. Applications of Theorem 1.12 to the \( H^\frac{1}{2} \) profile decomposition for the Klein-Gordon equation will appear elsewhere.

1.6. **Outline.** In Section 2 we first show that the global condition (A1) is equivalent to the local condition (L1), and adapt an argument from [4] to give a simplification of the transversality/curvature condition (A1). We then give the proof of the main consequences of Theorem 1.3 namely the elliptic case Theorem 1.8 the Klein-Gordon case, Theorem 1.10 and the refined Strichartz estimate, Theorem 1.12.

In Section 3 we give a counter example which shows that the dependence on the phases \( \Phi_j \), and the frequency parameter \( d_0 \) in Theorem 1.2 is sharp. The counter example is based on constructing waves \( u \) and \( v \) which are sums of wave packets concentrating on an \( \epsilon^{-1} \) neighbourhood of the space-time rectangle

\[
\left\{ (s+s', -s \nabla \Phi_1(\xi_0) - s' \nabla \Phi_2(\eta_0)) \mid 0 \leq s \leq (2^H_1)^{-1}, 0 \leq s' \leq (2^H_2)^{-1} \right\}
\]

with \( \xi_0 \in \Lambda_1 \) and \( \eta_0 \in \Lambda_2. \)

In Section 4 we run the induction on scales argument, which reduces the proof of Theorem 1.3 to showing that we may control the \( L^r L^s \) norm on a cube of diameter \( R \) by a cube of diameter \( \frac{R}{2} \). The induction argument applied here is slightly different to that used in Lee-Vargas [13], and in particular it is where we are able to avoid the \( \epsilon \) loss in the scale factors that occurred in previous works.

In Section 5 we apply localisation arguments to further reduce the proof of Theorem 1.3 to obtaining a key wave table type decomposition, Theorem 5.1 which decomposes the product \( uv \) into a term which is concentrated at scale \( \frac{R}{2}, \) together with a term that satisfies an improved bilinear estimate. The proof of the decomposition contained in Theorem 5.1 is the key step in the proof of Theorem 1.3.

Section 6 contains the general wave packet decomposition used in the present article, while, in Section 7 closely following [4], we give the key geometric consequences of (A1). The energy estimates across transverse surfaces we require are given in Section 8. These estimates rely heavily on the geometric consequences of the curvature and transversality assumption (A1) contained in Section 7.

In Section 9 we give the main step in the wave table construction, namely, following Wolff [27] and Tao [21], we use the wave packet decomposition, together with an energy argument, to decompose \( u \) into wave packets such that \( v \) restricted to the corresponding tube is concentrated on a small cube.

In Section 10 we use the construction in Section 9 to give the proof of Theorem 1.1 and construct the general wave tables that are required in the proof of the localised bilinear restriction estimate.

Finally, in Sections 11 and 12 we give the proof of Theorem 1.7. This relies on developing an “atomic” version of the wave table construction used in Section 9 together with an additional interpolation argument with a “linear” version of the classical bilinear \( L^2_{t,x} \) estimate.

1.7. **Notation.** Throughout this article, we use the notation \( a \lesssim b \) if \( a \leq CB \) for some constant \( C > 0 \) which may depend only on \( C_0, q, r, \) and the dimension \( n. \) In particular, all implied constants will otherwise be independent of the phases \( \Phi_j. \) Similarly we write \( a \approx b \) if \( a \lesssim b \) and \( b \lesssim a, \) and \( a \gg b \) if \( a \geq Cb \) with \( C \geq 100^n. \) For \( a \in \mathbb{R}, \) we let \( a_+ = a \) if \( a > 0, \) and \( a_+ = 0 \) otherwise.

Given a set \( \Omega \subset \mathbb{R}^n \) and a vector \( h \in \mathbb{R}^n, \) we let \( \Omega + h = \{ x + h \mid x \in \Omega \} \) denote the translation of \( \Omega \) by \( h. \) At a risk of causing some minor confusion, for a positive constant \( c > 0 \) we let \( \Omega + c = \{ x + y \mid x \in \Omega, \ |y| < c \} \)
be the Minkowski sum of $\Omega$ and the ball $\{|x| < c\}$. We also let $\text{diam}(\Omega) = \sup_{x,y \in \Omega} |x - y|$. For a function $f \in L^1(\mathbb{R}^n)$, we let $\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix \cdot \xi}dx$ denote the spatial Fourier transform. Similarly, if $u \in L^1(\mathbb{R}^{1+n})$, we define the space-time Fourier transform of $u$ as $\hat{u}(\tau, \xi) = \int_{\mathbb{R}^{1+n}} u(t, x)e^{-i(t \cdot \tau + x \cdot \xi)} dt dx$ and let $\hat{u}(t)$ be the Fourier transform of $u$ in the spatial variable $x \in \mathbb{R}^n$.

Let $R \geq 1$ and $0 < \epsilon \leq 1$. The constant $R$ will denote the large space-time scale and $0 < \epsilon \leq 1$ will be a small fixed parameter used to control the various error terms that arise. Given a scale $r > 0$, we decompose phase space into the grid $\mathcal{X}_{r,0} = \frac{\mathbb{Z}^n}{r} \times \frac{1}{2}\mathbb{Z}^n$. Given a point $\gamma = (x_0, \xi_0) \in \mathcal{X}_{r,0}$ in phase-space, we let $x(\gamma) = x_0$ and $\xi(\gamma) = \xi_0$ denote the projections onto the first and second components respectively.

Associated to a phase $\Phi_j$, and a scale $R$, we let $r_j = (\mathcal{H}_j R)^{\frac{1}{2}}$ denote the scale of the corresponding wave packets. Define

$$\Gamma_j = \{(x_0, \xi_0) \in \mathcal{X}_{r_j,0} \mid \xi_0 + \frac{1}{4}d_0 \subset \Lambda_j\}.$$ 

Thus $\Gamma_j$ contains those phase space points inside $\Lambda_j$, which are at least $\frac{1}{4}d_0$ from the boundary of $\Lambda_j$. Given a point $\gamma_j = (x_0, \xi_0) \in \Gamma_j$ we define the associated tubes

$$T_{\gamma_j} = \{(t, x) \in \mathbb{R}^{1+n} \mid |x - x_0 + t\nabla \Phi_j(\xi_0)| \leq r_j\}.$$ 

The geometric condition $(A_1)$ plays an important role in the proof of Theorem 1.2 by restricting the intersections of various tubes to certain transverse hypersurfaces. However, to exploit the curvature hypothesis in $(A_1)$, we need to restrict points in $\Gamma_j$ to those close to $\Sigma_j(h)$. To this end, given $h \in \mathbb{R}^{1+n}$ and a set $\Lambda \subset \mathbb{R}^n$, we define

$$\Gamma_j(h, \Lambda) = \{\gamma_j \in \Gamma_j \mid \xi(\gamma_j) \in \Sigma_j(h) \cap \Lambda + \frac{C}{2}\}$$

where the constant $C$ will depend on the dimension $n$ and $C_0$, but will otherwise be independent of the phase $\Phi_j$. We also define the conic hypersurface

$$\mathcal{C}_j(h) = \{(s, -s\nabla \Phi_j(\xi)) \mid s \in \mathbb{R}, \xi \in \Sigma_j(h)\}.$$ 

All cubes in this article are oriented parallel to the coordinate axis. Let $Q$ be a cube side length $R$, and take a subscale $0 < r \leq R$. We define $Q_{r}(Q)$ to be a collection of disjoint subcubes of width $2^{-j_0}R$ which form a cover of $Q$, where $j_0$ is the unique integer such that $2^{-1-j_0}R < r \leq 2^{-j_0}R$. Thus all cubes in $Q_{r}(Q)$ have side lengths $\approx r$, and moreover, if $r \leq r' \leq R$ and $q \in Q_{r}(Q)$, $q' \in Q_{r'}(Q)$ with $q \cap q' \neq \emptyset$, then $q \in Q_{r'}(q')$. To estimate various error terms which arise, we will need to create some separation between cubes. To this end, following Tao, we introduce the following construction. Given $0 < \epsilon \ll 1$ and a subscale $0 < r \leq R$, we let

$$I_{r,\epsilon}(Q) = \bigcup_{q \in Q_{r}(Q)} (1 - \epsilon)q.$$

Note that we have the crucial property, that if $q \in Q_{r}(Q)$, and $(t, x) \notin I_{r,\epsilon}(Q) \cap q$, then $\text{dist}((t, x), q) \geq \epsilon r$.

We require smoothed out bump functions adapted to cubes $q \in Q_{r}(Q)$. To this end, given $q \in Q_{r}(Q)$ we let $\chi_q \in C^\infty(\mathbb{R}^{1+n})$ be a positive function such that $\chi_q \geq 1$ on $q$, supp $\chi_q \subset \{(\tau, \xi) \mid |\langle \tau, \xi \rangle| \leq \frac{1}{2}\}$, and we have the decay bound, for every $N \in \mathbb{N}$,

$$\chi_q(t, x) \lesssim N \left(1 + \frac{\text{dist}((t, x), q)}{r}\right)^{-N}.$$ 

Given a point $\gamma_j = (x_0, \xi_0) \in \Gamma_j$, and a cube $q \in Q_{r_j}(Q)$, we define the weights

$$w_{\gamma_j, q} = \left(1 + \frac{|x_q - x_0 + t_q \nabla \Phi_j(\xi_0)|}{r_j}\right)^5,$$

where $(t_q, x_q)$ denotes the centre of the cube $q$. Thus the weights $w_{\gamma_j, q}$ are essentially one when $T_{\gamma_j} \cap q \neq \emptyset$, and are very large when $q$ and the tube $T_{\gamma_j}$ are far apart.
For a subset $\Omega \subset \mathbb{R}^{1+n}$ we let $1_{\Omega}$ denote the indicator function of $\Omega$. Let $\mathcal{E}$ be a finite collection of subsets of $\mathbb{R}^{1+n}$, and suppose we have a collection of $(\ell^2$-valued) functions $(u(E))_{E \in \mathcal{E}}$. We then define the associated quilt

$$[u^E](t,x) = \sum_{E \in \mathcal{E}} 1_E(t,x)|u(E)(t,x)|.$$ 

This notation was introduced by Tao [21], and plays a key technical role in localising the product $uv$ into smaller scales.

2. Applications

2.1. Alternative conditions on phases. The key assumption (A1) is a global condition, in the sense that it depends on the behaviour of the phase $\Phi_j$ at two points $\xi, \xi' \in \Sigma_j(h)$. In certain cases, it can instead be convenient to have a local version of (A1) which only depends on the behaviour of $\Phi_j$ in a neighbourhood of $\xi \in \Sigma_j(h)$. Clearly a suitable candidate for such a local condition can be found by simply taking $\xi' \to \xi$ in (A1). More precisely, we can state a local version of (A1) as:

(A1') for every $\{j,k\} = \{1,2\}$, $\xi \in \Lambda_j$, $\eta \in \Lambda_k$, and $v \in \mathbb{R}^n$ with $v \cdot (\nabla \Phi_j(\xi) - \nabla \Phi_k(\eta)) = 0$, we have

$$|\nabla^2 \Phi_j(\xi)v \cdot (\nabla \Phi_j(\xi) - \nabla \Phi_k(\eta))| \leq C_0 |H_j| \max |v|.$$ 

Note that if $h = (\Phi_j(\xi) + \Phi_k(\eta), \xi + \eta)$, then $\nabla \Phi_j(\xi) - \nabla \Phi_k(\eta)$ is the normal vector to the tangent space $T_{\Sigma_j(h)}$. Thus the condition on $v$ in (A1') can be written as $v \in T_{\xi} \Sigma_j(h)$. In particular, letting $\xi' \to \xi$ in $\Sigma_j(h)$, we see that (A1) implies (A1'). To prove the converse implication requires making an additional global assumption on the behavior of the phases $\Phi_j$ on the sets $\Lambda_j$. One possibility is the following.

Lemma 2.1 (Local implies global). Let $C_0 > 0$. Assume that for all $h \in \mathbb{R}^{1+n}$ we have

$$\text{ConvexHull}[\Sigma_1(h)] \subset \Lambda_1 \cap (h - \Lambda_2),$$

and the small variation bounds

$$\sup_{\xi,\xi' \in \Lambda_1} |\nabla \Phi_1(\xi) - \nabla \Phi_1(\xi')| + \sup_{\eta,\eta' \in \Lambda_2} |\nabla \Phi_2(\eta) - \nabla \Phi_2(\eta')| \leq \frac{1}{4} C_0 |\nabla \max|$$

and, for all $\xi, \xi' \in \Sigma_j(h)$,

$$|\nabla \Phi_j(\xi) - \nabla \Phi_j(\xi') - \nabla^2 \Phi_j(\xi)(\xi - \xi')| \leq \frac{1}{4} C_0 |H_j| |\xi - \xi'|.$$ (2.3)

If (A1') holds, then (A1) also holds with constant $C_0 = \frac{1}{4} C_0$.

Proof. Let $h = (a,h) \in \mathbb{R}^{1+n}$, $\xi, \xi' \in \Sigma_j(h)$, and $\eta \in \Lambda_2$. For ease of notation, we let $N = \nabla \Phi_j(\xi) - \nabla \Phi_k(\eta)$ and $v = (\xi - \xi') - ([\xi - \xi'] \cdot \frac{N}{|N|})\frac{N}{|N|}$. Suppose for the moment that we have

$$|\xi - \xi' \cdot N| \leq \frac{1}{4} C_0 |\nabla \max| |\xi - \xi'|.$$ (2.4)

As $v \cdot N = 0$, an application of (A1') and (2.3) then gives

$$|\nabla \Phi_j(\xi - \nabla \Phi_j(\xi')) \cdot N|$$

$$\geq |[\nabla^2 \Phi_j(\xi)v] \cdot N| - |\nabla \Phi_j(\xi) - \nabla \Phi_j(\xi') - \nabla^2 \Phi_j(\xi)(\xi - \xi')| - |H_j| |\xi - \xi'| \cdot N|$$

$$\geq C_0 |\nabla \max||v| - \frac{1}{2} |\xi - \xi'|| \geq \frac{1}{4} C_0 |H_j| \max |\xi - \xi'|$$

and hence (A1) follows. Thus it remains to verify the bound on $2.4$. We start by observing that the definition of the surface $\Sigma_j(h)$ implies that $\Phi_j(\xi)\Phi_j(\xi') = \Phi_k(h - \xi') - \Phi_k(h - \xi)$. On the other hand, as $\Sigma_j(h) = h - \Sigma_k(h)$, (2.1) gives $\xi' + t(\xi - \xi') \in \Lambda_j \cap (h - \Lambda_k)$ for $0 \leq t \leq 1$. Consequently we have the identity

$$\int_0^1 \nabla \Phi_j(\xi' + t(\xi - \xi')) \cdot (\xi - \xi') dt = \int_0^1 \nabla \Phi_k(h - \xi' + t(\xi - \xi')) \cdot (\xi - \xi') dt.$$
and hence, by an application of (2.2), we have
\[
\left| N \cdot \frac{\xi - \xi'}{|\xi - \xi'|} \right| \leq \int_0^1 |\nabla \Phi_j(x) - \nabla \Phi_j(x' + t(\xi - \xi'))| dt + \int_0^1 |\nabla \Phi_k(\eta) - \nabla \Phi_k(h - \xi' + t(\xi' - \xi))| dt \\
\leq (H_1 + H_2) d_0 \leq \frac{1}{4} C_0 \nu_{max}.
\]
\[\square\]

Strictly speaking, the above argument shows that to move from the local condition (A1') to the global condition (A1), it is enough to verify the bounds (2.3) and (2.4). However, in most applications of Lemma 2.1, the conditions (2.1), (2.2), and (2.3) can always be imposed by perhaps shrinking the sets $\Lambda_j$ slightly. Thus it generally suffices to simply check (A1').

In some applications of Theorem 1.2 it can be convenient to replace the assumption (A1) with a stronger sufficient condition.

**Lemma 2.2** ([4, Lemma 2.1]). Assume that (2.1) and (2.2) hold. If for all $\xi, \xi' \in \Sigma_j(h)$ we have the curvature type bound
\[\left| (\nabla \Phi_j(\xi) - \nabla \Phi_j(\xi')) \cdot (\xi - \xi') \right| \geq C_0^{(1)} H_j |\xi - \xi'|^2,\] (2.5)
and for $\xi \in \Lambda_1$ and $\eta \in \Lambda_2$ the transversality bound
\[\left| \nabla \Phi_1(\xi) - \nabla \Phi_2(\eta) \right| \geq C_0^{(2)} \nu_{max}\] (2.6)
then (A1) holds with constant $C_0 = \frac{1}{2} C_0^{(1)} C_0^{(2)}$.

**Proof.** We adapt the argument in [4] to the current setup. Let $\eta \in \Lambda_k$, $\xi, \xi' \in \Sigma_j(h)$ and take $\omega = \frac{\xi - \xi'}{|\xi - \xi'|}$. The convexity condition (2.1) implies that $|\nabla \Phi_j(\xi) - \nabla \Phi_j(\xi')| \leq H_j |\xi - \xi'|$. Hence the assumptions (2.5) and (2.6) give
\[
\left| (\nabla \Phi_j(\xi) - \nabla \Phi_j(\xi')) \cdot (\nabla \Phi_k(\xi) - \nabla \Phi_k(\eta)) \right| \\
\geq \left| \nabla \Phi_j(\xi) - \nabla \Phi_k(\eta) \right| \left| (\nabla \Phi_j(\xi) - \nabla \Phi_j(\xi')) \cdot \omega \right| - \left| (\nabla \Phi_j(\xi) - \nabla \Phi_j(\xi')) \cdot (\nabla \Phi_k(\xi) - \nabla \Phi_k(\eta)) \cdot \omega \right| \\
\geq H_j \nu_{max} |\xi - \xi'| \left( C_0^{(1)} C_0^{(2)} - \frac{|(\nabla \Phi_j(\xi) - \nabla \Phi_k(\eta)) \cdot \omega|^2}{\nu_{max}} \right)
\]
where we used the elementary bound $|x \cdot y| \geq |y| |x| \omega - |x|||y| \omega$. Consequently (A1) holds provided that for every $\xi, \xi' \in \Sigma_j(h)$, $\eta \in \Lambda_k$ we have
\[
\left| (\nabla \Phi_j(\xi) - \nabla \Phi_k(\eta)) \cdot \frac{\xi - \xi'}{|\xi - \xi'|} \right| \leq \frac{C_0^{(1)} C_0^{(2)}}{2} \nu_{max}.
\]
But this follows from the argument used to prove (2.4) together with the small variation condition (2.2). \[\square\]

**Remark 2.3.** To better understand the connection between the conditions in Lemma 2.2 and the assumptions (A1) and (A1'), we observe that the later conditions are roughly equivalent to
\[
\left| (\nabla^2 \Phi_j(\xi)v) \cdot N \right| \geq H_j \nu_{max} |v|
\]
for all $v \in T_\xi \Sigma_j(h)$, where $N$ is the unit normal to $\Sigma_j(h)$, and $T_\xi \Sigma_j(h)$ is the tangent plane at $\xi \in \Sigma_j(h)$. On the other hand, letting $\xi' \to \xi$, we see that (2.5) is essentially equivalent to the lower bound
\[
\left| (\nabla^2 \Phi_j(\xi)v) \cdot v \right| \geq H_j |v|^2
\]
(2.7)
for all $v \in T_\xi \Sigma_j(h)$ (more precisely, it is easy to check that under an assumption like (2.5), the global condition (2.5) is equivalent to (2.7)). Consequently, (A1) and (A1') state that the Hessian $\nabla^2 \Phi_j(\xi)$ can not rotate tangent vectors in $T_\xi \Sigma_j(h)$ to normal vectors, while (2.3) imposes the much stronger condition, that $\nabla^2 \Phi_j(\xi)v$ remains roughly parallel to $v \in T_\xi \Sigma_j(h)$. A similar remark, although phrased slightly differently, was made in [1].
In general, even with the use of the previous simplifications, it can be somewhat involved to apply Theorem 1.2 as the dependence on \( \Phi_j \) is only sharp when applied phase which behave essentially uniformly on \( \Lambda_j \). However a rough strategy is as follows. Suppose we are given phases \( \Phi_j^* \) on some (large) subset of \( \mathbb{R}^n \). The first step is to restrict the domain until the gradients \( \nabla \Phi_j^* \) have “small” variation. The second step is to rescale the domain, and hence replace \( \Phi_j^* \) with a rescaled version \( \Phi_j \), to ensure that the (nonzero) components of the Hessian \( \nabla^2 \Phi_j \) are all of a similar size. The final step is then to restrict the rescaled domain further, to ensure that the transversality and curvature conditions hold. Clearly each of these steps depends heavily on the precise phases under consideration, and thus some amount of trial and error is needed to find appropriate domains.

### 2.2. Elliptic Phases: Proof of Theorem 1.8

A short computation shows that, after potentially partitioning the sets \( \Lambda_j \) into smaller sets, the conditions (2.2) and (2.2) in Lemma 2.1 hold. Hence, using the ellipticity assumption, since for all \( \xi \in \Lambda_j \), \( \eta \in \Lambda_k \), and \( v \cdot (\nabla \Phi_j (\xi) - \nabla \Phi_k (\eta)) = 0 \) we have

\[
\left| (\nabla^2 \Phi (\xi) v) \wedge (\nabla \Phi_j (\xi) - \nabla \Phi_k (\eta)) \right| \lesssim \left| (\nabla^2 \Phi_j (\xi) v) \cdot v \right| V_{\max} \lesssim H_j V_{\max} |v|^2
\]

we see that (A1) holds and thus result follows by an application of Theorem 1.4.

### 2.3. The Wave/Klein-Gordon case: Proof of Theorem 1.10

We only consider the case \( \alpha \ll 1 \), the case \( \alpha \approx 1 \) is easier and follows directly from Theorem 1.7 (via Lemma 2.2). Applying a rotation, we may assume that \( \xi_0 = (a, 0, \ldots, 0) \), \( \eta_0 = (\sqrt{1 - \delta^2} b, b, 0, \ldots, 0) \) with \( 0 < \delta < 1 \), \( (a)_{m_1} \approx \lambda \), \( (b)_{m_2} \approx \mu \), and

\[
\frac{|m_2 a - m_1 b|}{\lambda \mu} + (\frac{ab}{\lambda \mu})^{\frac{1}{2}} \approx \alpha.
\]

After rescaling, and perhaps decomposing \( \Phi \) into slightly smaller sets if necessary, it is enough to prove that if \( \text{supp } \tilde{f} \subset \Lambda_1 \), \( \text{supp } \tilde{g} \subset \Lambda_2 \) we have

\[
\|e^{it \Phi_1} f e^{it \Phi_2} g\|_{L^1_t L^\infty_x (\mathbb{R}^{1+n})} \lesssim \alpha^{-\frac{2}{3}} \mu^{-\frac{n}{3}} - \frac{1}{\mu} \left( \frac{\lambda}{\mu} \right)^{\frac{2}{3}} \|f\|_{L^2_x} \|g\|_{L^2_x}
\]

where we define the phases

\[
\Phi_1 (\xi) = (m_1^2 + \beta^2 \xi^2 + \alpha^2 |\xi'|^2)^{\frac{1}{2}}, \quad \Phi_2 (\xi) = \pm (m_2^2 + \beta^2 \xi^2 + \alpha^2 |\xi'|^2)^{\frac{1}{2}}
\]

with \( \xi = (\xi_1, \xi') \in \mathbb{R} \times \mathbb{R}^{n-1}, \xi' = (\xi_2, \xi') \in \mathbb{R} \times \mathbb{R}^{n-2} \), and take \( \xi_0 = (a, 0, \ldots, 0) \), \( \eta_0 = (\sqrt{1 - \delta^2} b, b, 0, \ldots, 0) \) and the sets \( \Lambda_j \) to be

\[
\Lambda_1 = \{ |\beta \xi_1 - a| \ll \beta \lambda, |\xi'| \ll \lambda \}, \quad \Lambda_2 = \{ |\beta \xi_1 + \sqrt{1 - \delta^2} b| \ll \beta \mu, |\alpha \xi_2 + b| \ll \alpha \mu, |\xi'| \ll \mu \}.
\]

Note that (2.8) implies that if \( (\frac{ab}{\lambda \mu})^{\frac{1}{2}} \delta \ll \alpha \) then the sets \( \Lambda_j \) must have some radial separation, while if \( (\frac{ab}{\lambda \mu})^{\frac{1}{2}} \delta \gtrsim \alpha \) the sets are angularly separated. To simplify the computations to follow, we note that the transversality condition (2.8) implies that \( \frac{\mu}{\lambda} \delta \ll \alpha \), this follows by observing that in the fully elliptic case, when \( \lambda \approx m_1 \) and \( \mu \approx m_2 \), (2.8) becomes \( \frac{m_1}{m_1} - \frac{m_2}{m_2} \approx \alpha \). Similarly, by the definition of \( \beta \), if either \( \lambda \approx m_1 \) or \( \mu \approx m_2 \), then we have \( \beta \approx \alpha \).

To obtain the bound (2.9), we apply Theorem 1.2. Thus we have to check the conditions (A1) and (A2). There a number of ways to do this, here we argue via Lemma 2.2. To this end, we start by noting that

\[
\nabla \Phi_j = \frac{\beta^2 \xi_1 \xi'}{m_j^2 + \beta^2 \xi_1^2 + \alpha^2 |\xi'|^2}^{\frac{1}{2}}
\]

and hence a computation using the definitions of \( \beta \) and \( \Lambda_j \), gives the derivative bounds

\[
\sup_{\xi \in \Lambda_1} |\nabla \Phi_1 (\xi) - \frac{\beta a_0, \ldots, 0 )}{(a)_{m_1}}| + \sup_{\xi \in \Lambda_2} |\nabla \Phi_2 (\xi) - \frac{\beta \sqrt{1 - \delta^2} b, \alpha \delta b, 0, \ldots, 0 )}{(b)_{m_2}}| \approx \alpha^2.
\]

Similarly, to estimate \( \nabla^2 \Phi_j \), we observe that

\[
\partial^2 \Phi_j = \frac{\beta^2}{m_j^2 + \beta^2 \xi_1^2 + \alpha^2 |\xi'|^2}^{\frac{1}{2}} - \frac{\beta^4 \xi_1^2}{m_j^2 + \beta^2 \xi_1^2 + \alpha^2 |\xi'|^2}^{\frac{1}{2}} = \alpha^2 \frac{\beta^2 ((m_1/\lambda)^2 + |\xi'|^2)}{(m_j^2 + \beta^2 \xi_1^2 + \alpha^2 |\xi'|^2)^{\frac{1}{2}}}
\]
and for $k,k' > 1$,
\[
\partial_k \partial_{k'} \Phi_j = \alpha^2 \left( \frac{-\delta^2 \xi \xi_k}{(m^2 + \beta^2 \xi^2 + \alpha^2 |\xi'|^2)^2} \right) = \partial_k \partial_{k'} \Phi_j = \alpha^2 \left( \frac{-\delta \xi_{k,k'}}{(m^2 + \beta^2 \xi^2 + \alpha^2 |\xi'|^2)^2} \right).
\]

In particular, applying the definition of $\beta$, we have the Hessian bounds $H_1 \approx \frac{\alpha^2}{\lambda}$, $H_2 \approx \frac{\alpha^2}{\mu}$. Thus we see that the condition (A2) holds with $d_0 = \mu$. It remains to show that (A1) holds, which will follow by checking the conditions in Lemma 2.2. To this end, the transversality condition (2.8) together with the definition of $\beta$ gives
\[
\left| \left( \frac{\beta (a,0,\ldots,0)}{a}, \frac{(a_m)}{m} \right) - \left( \frac{\beta \sqrt{1 - \delta^2 b, a \delta b, 0, \ldots, 0)}{b}}{b_m} \right) \right| \approx \alpha^2.
\]

The estimate (2.10) then gives the transversality bound
\[
|\nabla \Phi_1 (\xi) - \nabla \Phi_2 (\eta)| \approx V_{max} \approx \alpha^2.
\]

Since we clearly have the final condition in Lemma 2.2 it only remains to show that we have the curvature condition
\[
\left| \left( \frac{\nabla \Phi_1 (\xi) - \nabla \Phi_1 (\eta)}{(\xi - \eta)} \right) \right| \approx \frac{\alpha^2}{\lambda} |\xi - \eta|^2
\]
for all $\xi, \eta \in \Sigma_1(h)$, together with a similar bound for $\nabla \Phi_2$. This follows by adapting the argument in [1] Corollary 6.4). We only consider the case $j = 1$, as the remaining case is identical. Suppose that $\xi, \eta \in \Sigma_1(h)$ with $h = (a^*, h)$. Let $x = (m_1, \beta \xi_1, \alpha \xi')$, $y = (m_2, \beta \eta_1, \alpha \eta')$, and $h^* = (m_2 - m_1, \beta h_1, \alpha h')$. Then the condition $\xi \in \Sigma_1(h)$ becomes $|x| \pm |x - h^*| = a^*$. Consequently, for every $\xi, \eta \in \Sigma_1(h)$ we have
\[
\left| \left( \frac{\nabla \Phi_1 (\xi) - \nabla \Phi_1 (\eta)}{(\xi - \eta)} \right) \right| = \left| \frac{\lambda}{(\xi) m} - \frac{y}{(\eta) m} \right|^2 + \mu^2 \left| \frac{x - h^*}{|x - h^*|} - \frac{y - h^*}{|y - h^*|} \right|^2
\]
where the last line follows by observing that $\xi, \eta \in \Sigma_1(h)$ implies that we have the identity
\[
|x||y|x |y y| = (|x - y| (|x| |y|)) = \left( (|x - h^*| - |y - h^*|)^2 - |x - h^*| |y - h^*| \right) = x - h^* |y - h^*| |x - h^*| |y - h^*|.
\]

If either $\lambda \approx m_1$ or $\mu \approx m_2$, then since $z, w \in \mathbb{R}^n$ with $\leftrightarrow{z} \approx \leftrightarrow{w}$ implies we have
\[
\left| \frac{z}{(\xi) m} - \frac{x}{(\xi - \eta) m} \right| \approx \left( \frac{m}{(\xi) m} \right)^2 \left| \frac{z - |w|}{|z - |w|} \right|^2 + \left( \frac{|z| |w| - z \cdot w}{(\xi) m} \right) \frac{1}{2},
\]
the required bound (2.11) follows directly from (2.12) together with the fact that $\alpha \approx \beta$ in the case $\lambda \approx m_1$ or $\mu \approx m_2$. On the other hand, if $\lambda \gg m_1$ and $\mu \gg m_2$, then we consider separately the cases $|\xi_1 - \eta_1| \gg |\xi' - \eta'|$ and $|\xi_1 - \eta_1| \ll |\xi_1 - \eta_1|$. In the former case, if $\alpha \gg \frac{m_1}{\lambda} + \frac{m_2}{\mu}$, then from the definition of $\beta$ and the condition (2.8) we must have $\beta \approx \delta \approx 1$. Hence we deduce that as $\xi - h, \eta - h \in \Lambda_2$ we have
\[
|(x - h^* \wedge (y - h^*)| \geq \alpha |\beta (\xi_1 - \eta_2) |(\eta'_1 - \xi'_1) - \beta (\eta_1 - h_1) (\xi'_1 - \xi_1) | \approx \alpha \mu |\xi_1 - \eta_1| \approx \alpha |\xi - \eta|,
\]
which together with the elementary bound $|\omega - \omega'| \approx |\omega \wedge \omega'|$ for $\omega, \omega' \in \mathbb{R}^{n-1}$ gives (2.11). On the other hand, if $\alpha \ll \frac{m_1}{\lambda} + \frac{m_2}{\mu}$, we have again using the definition of $\beta$
\[
\left| \frac{x \wedge y}{\lambda} + \frac{(x - h^*) \wedge (y - h^*)}{\mu} \right| \geq \beta \frac{m_1 |\xi_1 - \eta_1|}{\lambda} + \beta \frac{m_2 |\xi_1 - \eta_1|}{\mu} \approx \alpha |\xi - \eta|.
\]
Thus we have (2.11) when $|\xi_1 - \eta_1| \gtrsim |\xi' - \eta'|$. Finally, if $|\xi' - \eta'| \gg |\xi_1 - \eta_1|$, then we simply observe that $|x \wedge y| \geq 2|\xi_1 - \eta| \approx \alpha |\xi_1 - \eta|$. Therefore (2.11) follows as required.

2.4. A Refined Strichartz estimate for the Klein-Gordon equation: Proof of Theorem 1.12. The proof closely follows the argument of Ramos in [17], with the key bilinear estimate Theorem 2.10 replacing the bilinear estimate of Tao [21] used in [17], thus we shall be somewhat brief. Let $p = \frac{n+1}{n-1}$ and $u = e^{it \langle \nabla \rangle} f$, and decompose

$$
\|u\|_{L^p_{t,x}}^2 \leq \sum_{\lambda \in 2^n} \left\| \sum_{\lambda \in 2^n} u_{\lambda} u_{\lambda_{\lambda}} \right\|_{L^p_{t,x}}
$$

where $u = \sum_{\lambda \in 2^n} u_{\lambda}$ and supp $\hat{u}_\lambda \subset \{ (\xi) : \lambda \approx \lambda \}$. Since the sum is essentially orthogonal as $\lambda$ varies, by applying the weak orthogonality in $L^p$ (see for instance [17] Lemma 2.2) we have

$$
\left\| \sum_{\lambda \in 2^n} u_{\lambda} u_{\lambda_{\lambda}} \right\|_{L^p_{t,x}} \lesssim \left( \sum_{\lambda \in 2^n} \|u_{\lambda} u_{\lambda_{\lambda}}\|_{L^p_{t,x}}^{\min\{p,p'\}} \right)^{\frac{1}{\min\{p,p'\}}}
$$

where $p' = \frac{p}{p-1}$ denotes the dual exponent to $p$. Given $A = A(\xi_0) \in A_{\lambda,\alpha}$, and $B = B(\eta) \in A_{\ell,\alpha}$, we say $A \sim B$ if we have the transversality type condition

$$
\left| \frac{\langle \xi_0 | - \eta \rangle}{\ell^{\alpha}} \right| + \left( \frac{\langle \xi_0 | \eta \rangle - \xi_0 \cdot \eta}{\ell^{\alpha}} \right)^{\frac{1}{2}} \approx \alpha.
$$

Applying a Whitney type decomposition, we can write

$$
u_{\lambda} u_{\lambda} = \sum_{\alpha \in 2^{-N}} \sum_{A \in A_{\lambda,\alpha}} \sum_{B \in A_{\lambda,\alpha}} u_{\lambda} u_{\lambda_{\lambda}}
$$

where supp $\widehat{u}_\lambda \subset A$, and $\|\widehat{u}_\lambda\|_{L^p_\ell} \approx \|\widehat{u}\|_{L^p(\ell)}$, $\|\widehat{u}_\lambda\|_{L^p_\ell} \approx \|\widehat{u}\|_{L^p(\ell)}$. An application of Theorem 2.10 gives for all $\frac{n+3}{n-1} \leq q \leq 2$

$$
\|u_{\lambda} u_{\lambda_{\lambda}}\|_{L^p_{t,x}} \lesssim \alpha^{n-1-\frac{n+3}{q}} \left( \frac{\alpha}{1 + \alpha} \right)^{\frac{1}{4}} \left( \frac{\alpha}{1 + \alpha} \right)^{\frac{1}{2}} \|f\|_{L^p_{\ell}(A)} \|f\|_{L^p_{\ell}(B)}
$$

Interpolating with the trivial bound

$$
\|u_{\lambda} u_{\lambda_{\lambda}}\|_{L^p_{t,x}} \lesssim \|f\|_{L^p_{\ell}(A)} \|f\|_{L^p_{\ell}(B)}
$$

we deduce that for all max\{\frac{1}{2}, \frac{n+3}{n-1}\} \leq \frac{1}{2} < \frac{1}{2} + \frac{2}{n+1}$ we have

$$
\|u_{\lambda} u_{\lambda_{\lambda}}\|_{L^p_{t,x}} \lesssim \alpha^{n-1-\frac{n+3}{q}} \left( \frac{\alpha}{1 + \alpha} \right)^{\frac{1}{4}} \left( \frac{\alpha}{1 + \alpha} \right)^{\frac{1}{2}} \|f\|_{L^p_{\ell}(A)} \|f\|_{L^p_{\ell}(B)}
$$

$$
\approx \ell^{n-1+\frac{n+3}{n-1}} \left( \frac{\alpha}{1 + \alpha} \right)^{\frac{n+3}{4}} \|f\|_{L^p_{\ell}(A)} \|f\|_{L^p_{\ell}(B)}.
$$

The bound (2.13) is the key bilinear input, and replaces the corresponding estimate for the wave equation [17] Corollary 2.1] used in the work of Ramos. We now apply the Whitney decomposition to deduce that

$$
\|u_{\lambda} u_{\lambda_{\lambda}}\|_{L^p_{t,x}} \lesssim \sup_{\alpha \approx \ell} \sup_{A \in A_{\lambda,\alpha}} \sup_{B \in A_{\lambda,\alpha}} \|u_{\lambda} u_{\lambda_{\lambda}}\|_{L^p_{t,x}}
$$

where the first term follows from simply observing that for $\alpha \approx \ell$ we have $\# A_{\lambda,\alpha} \lesssim 1$. For the second and third term in (2.14), we again follow [17] and apply the weak orthogonality in $L^p$. More precisely, we claim that for fixed $\ell$ and $\lambda$, the sets

$$
\{ (\xi) + (\eta) : |\xi| \in A, |\eta| \in B \} \subset \mathbb{R}^{1+n}.
$$
for $A \in \mathcal{A}_{\lambda, \alpha}$, $B \in \mathcal{A}_{\lambda, \alpha}$ with $A \sim B$, are essentially disjoint as $\alpha$ and $A$ vary. This follows by observing that if $A = A(\xi_0) \in \mathcal{A}_{\lambda, \alpha}$ and $B = B(\eta_0) \in \mathcal{A}_{\lambda, \alpha}$ with $A \sim B$, then we have for every $\xi \in A$ and $\eta \in B$
\[
|\xi + \eta|_2 - (|\xi| - |\eta|) \approx \lambda \alpha^2, \quad \|\xi + \eta - |\xi_0 + \eta_0|\| \lesssim \frac{\alpha \lambda}{1 + \alpha \lambda} \ell, \quad (||\xi + \eta||/|\xi_0 + \eta_0| - (|\xi + \eta| - |\xi_0 + \eta_0|))^\frac{1}{2} \lesssim \alpha \ell.
\]

In particular, for the second term in (2.14), as this sum only covers the wave like regime where the sets $A$ are simply angular sectors of the annuli, we can bound this contribution by using (2.13) together with the finite overlap observation made above and (2.13) then gives
\[
\left( \sum_\lambda \left| \sum_{\alpha < \frac{1}{\lambda}} \sum_{A \in \mathcal{A}_{\lambda, \alpha}} \sum_{B \sim A} u_A u_B \right|_{L^2_{p', \ell}} \right)^{\frac{1}{\min(p, p')}} \lesssim \left( \sum_\lambda \left| \sum_{\alpha < \frac{1}{\lambda}} \sum_{A \in \mathcal{A}_{\lambda, \alpha}} \sum_{B \sim A} \|u_A u_B\|_{L^2_{p', \ell}} \right) \right)^{\frac{1}{\min(p, p')}} \lesssim \left( \sum_\lambda \left| \sum_{\alpha < \frac{1}{\lambda}} \sum_{A \in \mathcal{A}_{\lambda, \alpha}} \sum_{B \sim A} \left( (\alpha \lambda)^{-\frac{3}{\alpha}} A^{\frac{1}{\alpha}} |B|^{\frac{3}{\alpha}} \right) \|f\|_{L^2_{p', \ell}} \right) \right)^{\frac{1}{\min(p, p')}} \lesssim \left( \sup_\lambda \sup_{\alpha < \frac{1}{\lambda}} \left( \alpha \lambda \right)^{-\frac{3}{\alpha}} A^{\frac{1}{\alpha}} |B|^{\frac{3}{\alpha}} \right) \|f\|_{L^2_{p', \ell}}.
\]

3. Optimality of Theorem 1.2

In this section, our goal is to show that the conclusion of Theorem 1.2 is sharp. We first observe that, by exploiting the dilation and translation invariance of $L^q_1 L^r_2$, is enough to consider the case
\[
1 \leq \|\nabla \Phi_1\|_{L^\infty} + \|\nabla \Phi_2\|_{L^\infty} \lesssim 3, \quad \mathcal{V}_{\text{max}} = 1, \quad \mathcal{H}_2 \leq \mathcal{H}_1 = 1.
\]
More precisely, since
\[
\mathcal{V}_{\text{max}} \leq \inf_{\xi_0 \in \mathbb{R}^n} \left( \|\nabla \Phi_1 - \xi_0\|_{L^\infty(\Lambda_1)} + \|\nabla \Phi_2 - \xi_0\|_{L^\infty(\Lambda_2)} \right) \leq 3 \mathcal{V}_{\text{max}}
\]
after a linear translation of the phases (this corresponds to the spatial translation $x \mapsto x + t\xi_0$) we may assume that
\[
\mathcal{V}_{\text{max}} \leq \|\nabla \Phi_1\|_{L^\infty(\Lambda_1)} + \|\nabla \Phi_2\|_{L^\infty(\Lambda_2)} \leq 3 \mathcal{V}_{\text{max}}.
\]
To obtain the claimed bounds in (3.1), we now apply the rescaling
\[
\Phi_j(\xi) \mapsto \frac{\mathcal{H}_1}{\mathcal{V}_{\text{max}}} \Phi_j \left( \frac{\mathcal{V}_{\text{max}}}{\mathcal{H}_1} \xi \right), \quad \mathcal{d}_0 \mapsto \frac{\mathcal{H}_1}{\mathcal{V}_{\text{max}}} \mathcal{d}_0.
\]
It is important to note that this rescaling and translation leaves the assumptions (A1) and (A2), and the bilinear estimate in Theorem 1.2 unchanged.

Fix $\xi_0 \in \Lambda_1, \eta_0 \in \Lambda_2$, and $0 < \epsilon \ll 1$. It is enough to construct $\Phi_1$-wave $u$, and a $\Phi_2$-wave $v$, such that $\text{supp } \hat{u} \subset \{|\xi - \xi_0| \leq \epsilon\}$, $\text{supp } \hat{v} \subset \{|\xi - \eta_0| \leq \epsilon\}$, we have the energy bounds
\[
\|u\|_{L^q_1 L^r_2} \lesssim \mathcal{H}_2^{-\frac{1}{2}} \epsilon^{-\frac{2}{\alpha}}, \quad \|v\|_{L^q_2 L^r_2} \lesssim \epsilon^{-\frac{2}{\alpha}},
\]
and both $u$ and $v$ are concentrated on the $\epsilon^{-1}$ thickened space-time rectangle
\[
\Omega = \{ s(1, -\nabla \Phi_1(\xi_0)) + s'(1, -\nabla \Phi_2(\eta_0)) \mid 0 \leq s \leq \epsilon^{-2}, 0 \leq s' \leq (\epsilon^2 \mathcal{H}_2)^{-1} \} + \epsilon^{-1}
\]
in the sense that \( \| uv \|_{L^q_x L^r_t(\Omega)} \approx \| \Omega \|_{L^q_x L^r_t} \). Assuming the existence of \( u \) and \( v \) for the moment, we easily deduce that the estimate

\[
\| uv \|_{L^q_x L^r_t(\Omega)} \leq C \| u \|_{L^q_x L^r_t} \| v \|_{L^q_x L^r_t}
\]

together with the transversality assumption \([15]\) implies that

\[
C \geq e^{n+1 - \frac{2m+1}{q} - \frac{1}{r} - \frac{1}{2}}.
\]

Thus we see that \( d_0 \geq \epsilon \), we have the correct dependence on \( H_j \), and letting \( \epsilon \to 0 \) we must have \( \frac{1}{q} + \frac{d_0+1}{2p} \leq \frac{n+1}{2} \).

It remains to construct \( u \) and \( v \) satisfying the required properties. One approach is to adapt the example used by Tao in \([21]\). Let \( \rho \) be a Schwartz function such that \( \rho(x) \approx 1 \) for \( |x| \leq 1 \), and \( \text{supp} \, \hat{\rho} \subset \{ |\xi| \leq 1 \} \), and define

\[
\rho_{1,k}(x) = e^{i(x+\xi t)\Phi_1(\xi)} \rho(x), \quad \rho_{2,k} = e^{i(x+\xi t)\Phi_2(\xi)} \rho(x).
\]

where \( N \) is some fixed quantity independent of \( \epsilon \), which is needed to create some separation between the resulting wave packets. The waves \( u \) and \( v \) are then defined as

\[
u(t, x) = \sum_{0 \leq k \leq (c^2 H_2)^{-1}} e^{it(\xi - Nk)} \rho_{1,k}(x), \quad v(t, x) = \sum_{0 \leq k \leq -2} e^{it\Phi_2(\xi - Nk)} \rho_{2,k}(x).
\]

It is clear that \( u \) and \( v \) satisfy the required Fourier support properties by the definition of \( \rho \). Roughly speaking, \( v \) is defined as the sum of wave packets concentrated on (space-time) tubes of size \( (c^2 H_2)^{-1} \times \epsilon^{-n} \) which are oriented in the direction \( \{1, -\nabla \Phi_2(\eta_0)\} \) and centered along the points \((0, -k N)\), and essentially cover the set \( \Omega \). Similarly, \( u \) is the sum of wave packets concentrated on tubes of size \( \epsilon^{-2} \times \epsilon^{-n} \) oriented in the direction \( \{1, -\nabla \Phi_1(\eta_0)\} \) and centered along the (space-time) points \((k N, -k N)\), which again essentially covers the set \( \Omega \). In particular, we expect that the product \( |uv| \) is essentially concentrated on \( \Omega \). These heuristics can be made rigorous by a somewhat standard integration by parts argument (and essentially covers the set \( \Omega \)). In general, wave packets centered along space-time lines need not be orthogonal, however, since the wave packets making up \( u \) are transverse to the direction \( \{1, -\nabla \Phi_2(\eta_0)\} \), we retain orthogonality of the wave packets. This can be seen by an application of Plancheral which gives

\[
\| u(t) \|_{L^2_x} \approx \left( \sum_{0 \leq k \leq \epsilon^{-2}} \| \rho_{2,k} \|_{L^2_x}^2 \right)^{1/2} = e^{-\frac{d_0+1}{2}}.
\]

The argument to bound \( \| uv \|_{L^q_x L^r_t(\Omega)} \) is slightly more involved as the wave packets are not centered on a fixed time slice \( t = 0 \), but instead centered along a tube oriented in the direction \( \{1, -\nabla \Phi_2(\eta_0)\} \). In general, wave packets centered along space-time lines need not be orthogonal, however, since the wave packets making up \( u \) are transverse to the direction \( \{1, -\nabla \Phi_2(\eta_0)\} \), we retain orthogonality of the wave packets. This can be seen by an application of Plancheral which gives

\[
\| u(t) \|_{L^2_x}^2 = e^{-n} \sum_{0 \leq k, k' \leq (c^2 H_2)^{-1}} \int_{\mathbb{R}^n} e^{iN(k-k')\xi \cdot \nabla \Phi_2(\eta_0) - e^{-1} \Phi_1(\xi - \eta_0)} \| \rho(\xi) \|_{L^2}^2 d\xi \lesssim e^{-\frac{n+1}{2}} \sum_{0 \leq k \leq (c^2 H_2)^{-1}} 1 \approx H_2^{-1} \epsilon^{-n-1}
\]

where we used integration by parts together with \([15]\) to control the off diagonal terms. It is worth observing that the above argument already foreshadows a number of the ingredients in the proof of Theorem \([12]\) namely the orthogonality properties of wave packets together with energy type estimates across transverse surfaces.

4. Induction on Scales

In this section we reduce the proof of Theorem \([1,3]\) to proving local estimates on cubes \( Q \in \mathbb{R}^{1+n} \). This follows the argument of Tao \([21]\) which was adapted by Bejenaru \([1]\) and Lee-Vargas \([13]\), but we refine the induction somewhat to remove the epsilon derivative loss that appears in the previous bounds of Tao and Lee-Vargas.
Fix constants $d_0, C_0 > 0$, and open sets $\Lambda_j \subset \mathbb{R}^n$. Let $\Phi_1$ and $\Phi_2$ be phases satisfying (A1) and (A2). After exploiting the dilation and translation invariance of $L^q_t L^r_x$ as in Section 3 it is enough to consider the case

\[ 1 \leq \|\nabla \Phi_1\|_{L^\infty} + \|\nabla \Phi_2\|_{L^\infty} \leq 3, \quad \forall \max = 1, \quad \mathcal{H}_2 \leq \mathcal{H}_1 = 1. \tag{4.1} \]

Take sets $\Lambda_j^* + d_0 \subset \Lambda_j$ such that $d[\Lambda_1^* + d_0, \Lambda_2^* + d_0] \leq \frac{d_0}{C_0}$, and fix $R_0 \geq R_1 \geq \frac{1}{\mathcal{H}_2^2}$ with

\[ d[\Lambda_1^* + d_0, \Lambda_2^* + d_0] \lesssim (\mathcal{H}_2 R_0)^{-\frac{1}{2}} \lesssim d[\Lambda_1^* + d_0, \Lambda_2^* + d_0]. \]

The constant $R_0$ will denote the smallest scale of cubes we consider, while the sets $\Lambda_j^*$ will contain the supports of $\hat{u}$ and $\hat{v}$. The proof of Theorem 1.4 is based on an induction on scales argument. This requires the following definition.

**Definition 4.1.** For any $R \geq R_0$ and $1 \leq q, r \leq 2$, we define $A_{q,r}(R)$ to the best constant for which the inequality

\[ \|u\|_{L^q_t L^r_x(Q)} \leq A_{q,r}(R) \|u\|_{L^q_t L^r_x} \|v\|_{L^q_t L^r_x} \]

holds for all cubes $Q \subset \mathbb{R}^{1+n}$ of radius $R$, and all $\Phi_1$-waves $u$ and $\Phi_2$-waves $v$ satisfying the support assumption

\[ \text{supp} \, \hat{u} \subset \Lambda_1^* + 4(\mathcal{H}_2 R)^{-\frac{1}{2}}, \quad \text{supp} \, \hat{v} \subset \Lambda_2^* + 4(\mathcal{H}_2 R)^{-\frac{1}{2}}. \]

It is easy to check that $A_{q,r}(R)$ is always finite, and, since $R \geq R_0 \geq (d_0^2 \mathcal{H}_2)^{-1}$, that the required support conditions on $\hat{v}$ and $\hat{v}$ imply that we always have

\[ \text{supp} \, \hat{u} + \frac{d_0}{2} \subset \Lambda_1, \quad \text{supp} \, \hat{v} + \frac{d_0}{2} \subset \Lambda_2. \]

It is also worth noting that the support condition becomes stricter as $R$ becomes large, in other words, for large $R$, the Fourier supports must be smaller. Roughly this can be explained as follows. Our goal will be to obtain a bound for $A_{q,r}(2R)$, in terms of $A_{q,r}(R)$. At each scale $2R$, we will need to decompose $u$ and $v$ into wave packets, this will enlarge the Fourier support slightly, and thus to apply the bound at scale $R$ to the wave packets of $u$ and $v$, we will need $A_{q,r}(R)$ to apply to functions with slightly larger support. The support conditions we use play the same role as the margin type conditions used in [21].

The proof of Theorem 1.4 will rely on two key propositions. The first gives a good bound for $A_{q,r}(R)$ for $R$ close to $R_0$, and is needed to begin the induction argument.

**Proposition 4.2.** Let $C_0 > 0$, $1 \leq q, r \leq 2$, and $\frac{1}{q} + \frac{n+1}{2r} < \frac{n+1}{2}$. Assume we have $d_0 > 0$, open sets $\Lambda_j \subset \mathbb{R}^n$, and phases $\Phi_j$ satisfying Assumption 4.1 with the normalisation conditions 4.1. Then for any sets $\Lambda_j^* + d_0 \subset \Lambda_j$ with $d[\Lambda_1^* + d_0, \Lambda_2^* + d_0] \leq \frac{d_0}{C_0}$, and every $R \geq R_0$ we have

\[ A_{q,r}(R) \lesssim d_0^{n+1-n\frac{1}{2}} \frac{2}{\mathcal{H}_2^2} \left( \frac{R}{R_0} \right)^{\frac{1}{2}}. \]

The second implies that we can control $A_{q,r}(2R)$ in terms of $A_{q,r}(R)$.

**Proposition 4.3.** Let $C_0 > 0$, $1 \leq q, r \leq 2$, and $\frac{1}{q} + \frac{n+1}{2r} < \frac{n+1}{2}$. There exists a constant $C > 0$, such that for any $d_0 > 0$, any open sets $\Lambda_j \subset \mathbb{R}^n$, any phases $\Phi_j$ satisfying Assumption 4.1 with the normalisation 4.1, and any sets $\Lambda_j^* + d_0 \subset \Lambda_j$ with $d[\Lambda_1^* + d_0, \Lambda_2^* + d_0] \leq \frac{d_0}{C_0}$, we have for every $R \geq R_0$ and $0 < \epsilon < 1$

\[ A_{q,r}(2R) \leq (1 + C\epsilon) A_{q,r}(R) + e^{-C \mathcal{H}_2^2 \frac{d_0}{2}} R^{1+\frac{n+1}{2r} - \frac{n+1}{2}}. \]

We leave the proof of Propositions 4.2 and 4.3 till Section 5 and now turn to the proof of Theorem 1.4.

**Proof of Theorem 1.4.** We begin by observing that an application of Proposition 4.3 with $R = 2^n R_0$ and $\epsilon = \frac{1}{2} 2^n \left( \frac{1+\frac{n+1}{2r} - \frac{n+1}{2}}{2} \right)$ implies that

\[ A_{q,r}(2^n R_0) \leq (1 + 2^n \left( \frac{1+\frac{n+1}{2r} - \frac{n+1}{2}}{2} \right)) A_{q,r}(2^{n-1} R_0) + C \mathcal{H}_2^{\frac{n+1}{2r} - \frac{n}{2}} R_0^{\frac{1}{2} + \frac{n+1}{2r} - \frac{n_1}{2} + 2^n \left( \frac{1+\frac{n+1}{2r} - \frac{n+1}{2}}{2} \right)}. \]
Since \( \frac{1}{q} + \frac{n+1}{2n} < \frac{n+1}{2} \) both error terms decay in \( m \). In particular, after \( m \) applications of Proposition 4.3 we have

\[
A_{q,r}(2^m R_0) \lesssim A_{q,r}(R_0) + \mathcal{H}_2^{\frac{n+1}{2}} R_0 \left( \frac{1}{q} + \frac{n+1}{2n} \right)
\]

where the implied constant depends only on \( q, r \), the dimension \( n \), and \( C_0 \). Applying Proposition 12 and using the definition of \( R_0 \), we conclude that for every \( m \in \mathbb{N} \)

\[
A_{q,r}(2^m R_0) \lesssim d_0^{n+1} \left( \frac{2}{q} \right)^{\frac{n+1}{2}} \mathcal{H}_2^{\frac{1}{q} - \frac{1}{q}}.
\]

Unpacking the definition of \( A_{q,r}(2^m R_0) \), and letting \( m \to \infty \) we obtain Theorem 1.4.

5. The localisation argument

In this section we reduce the proof of Propositions 1.2 and 1.3 to obtaining a decomposition of the \( \Phi_j \)-waves, into waves which are concentrated at smaller scales, in the sense that the \( L^q_t L^r_x \) norm on a cube of diameter \( R \), can be controlled by the same norm at the smaller scale \( \frac{R}{2} \). This decomposition requires a variant of the wave table construction introduced by Tao in [21]. Recall that, given a collection of functions \( (u^{(B)})_{B \in \mathcal{Q}_r(Q)} \) (or a wave table in the notation of Tao), we defined the corresponding quilt \([u^{(1)}]\) to be the sum

\[
[u^{(1)}] = \sum_{B \in \mathcal{Q}_r(Q)} 1_B [u^{(B)}].
\]

Roughly speaking, we decompose \( u = \sum_{B \in \mathcal{Q}_r(Q)} u^{(B)} \) into waves which are “concentrated” in \( B \). The portion of \( u^{(B)} \) away from the cube \( B \) has additional decay, and can be treated as an error term. On the other hand, the quilt \([u^{(1)}]\) contains the part of \( u \) which is concentrated on \( B \), and does not decay in \( R \). However it is concentrated at a smaller scale than \( u \), and this allows us to exploit the induction assumption.

The key bound, and the part of Theorem 1.4 which requires the most work, is the following theorem.

**Theorem 5.1.** Let \( C_0 > 0 \), \( 1 \leq q, r \leq 2 \), and \( \frac{1}{q} + \frac{n+1}{2n} < \frac{n+1}{2} \). Assume that we have \( d_0 > 0 \), open sets \( \Lambda_j \subset \mathbb{R}^n \), and phases \( \Phi_j \) satisfying (A1), (A2), and the normalisation (4.11). Let \( Q_R \) be a cube of diameter \( R \geq R_0 \). Then for any \( 0 < \epsilon \ll 1 \), any \( \Phi_1 \)-wave \( u \), and any \( \Phi_2 \)-wave \( v \) with

\[
supp \hat{u} + \frac{d_0}{2} \subset \Lambda_1, \quad supp \hat{v} + \frac{d_0}{2} \subset \Lambda_2
\]

there exist a cube \( Q \) of diameter \( 2R \) such that we have a decomposition

\[
u = \sum_{B \in \mathcal{Q}_r(Q)} u^{(B)}, \quad v = \sum_{B' \in \mathcal{Q}_r(Q')} v^{(B')}
\]

where \( M \in \mathbb{N} \) with \( 4^{-M} < \mathcal{H}_2 \leq 4^{1-M} \), and \( u^{(B)} \) is a \( \Phi_1 \)-wave, \( v^{(B')} \) is a \( \Phi_2 \)-wave, with the support properties

\[
supp \hat{u}^{(B)} \subset supp \hat{u} + 2(2\mathcal{H}_2 R)^{-\frac{1}{2}}, \quad supp \hat{v}^{(B')} \subset supp \hat{v} + 2(2\mathcal{H}_2 R)^{-\frac{1}{2}}.
\]

Moreover we have the energy bounds

\[
\left( \sum_{B \in \mathcal{Q}_r(Q)} \| u^{(B)} \|^2_{L^q_t L^r_x} \right)^{\frac{1}{2}} \leq (1 + C\epsilon) \| u \|_{L^q_t L^r_x}
\]

and bilinear estimate

\[
\| uv \|_{L^2_t L^2_x(Q_{4R})} \leq (1 + C\epsilon) \left( \sum_{B \in \mathcal{Q}_r(Q)} \| u^{(B)} \|^2_{L^q_t L^r_x} + \sum_{B' \in \mathcal{Q}_r(Q')} \| v^{(B')} \|^2_{L^q_t L^r_x} \right)^{\frac{1}{2}}
\]

and

\[
\| uv \|_{L^2_t L^2_x(Q_{4R})} \leq (1 + C\epsilon) \left( \sum_{B \in \mathcal{Q}_r(Q)} \| u^{(B)} \|^2_{L^q_t L^r_x} + \sum_{B' \in \mathcal{Q}_r(Q')} \| v^{(B')} \|^2_{L^q_t L^r_x} \right)^{\frac{1}{2}}.
\]

\[1\]This exploits the bound

\[
(1 + C2^{-\alpha m}) \times (1 + C2^{-\alpha(m-1)}) \times \cdots \times (1 + C) \leq 1
\]

which follows by taking logs, and recalling the elementary estimate \( \log(1 + x) \leq x \).
where the constant $C$ depends only on $C_0$, $q$, $r$, and $n$.

The proof of Theorem 5.1 will take up a large part of the remaining sections to follow, and is left to Section 10.

The proof of the initial induction bound, also requires the following somewhat classical bilinear $L^2_{r,s}$ estimate.

**Theorem 5.2.** Let $j \in \{1, 2\}$, $\Lambda_j \subset \mathbb{R}^n$, and assume that the phases $\Phi_j \in C^1(\Lambda_j)$ satisfy the transversality condition (1.5) for some $C_0 > 0$. If $u$ is a $\Phi_1$-wave, and $v$ is a $\Phi_2$-wave we have

$$
\|uv\|_{L^2_{r,s}(\mathbb{R}^{n+1})} \lesssim \left( C_0 V_{max} \right)^{-\frac{1}{r}} \left( d[\text{supp } \hat{u}, \text{supp } \hat{v}] \right)^{-\frac{1}{2}} \|u\|_{L^\infty_{t,x}} \|v\|_{L^\infty_{t,x}}.
$$

**Proof.** A computation gives the identities

$$
(\hat{uv})(\tau, \xi) = \int_{\Sigma_1(\tau, \xi)} \frac{\hat{f}(\eta) \hat{g}(\xi - \eta)}{|\nabla \Phi_1(\eta) - \nabla \Phi_2(\xi - \eta)|} \, d\sigma(\eta)
$$

and

$$
\int_\mathbb{R} \int_{\Sigma_1(\tau, \xi)} \frac{|\hat{f}(\eta) \hat{g}(\xi - \eta)|^2}{|\nabla \Phi_1(\eta) - \nabla \Phi_2(\xi - \eta)|} \, d\sigma(\eta) \, d\tau = \|\hat{f}(\eta) \hat{g}(\xi - \eta)\|^2_{L^2_\xi(\mathbb{R}^n)}
$$

where $d\sigma(\eta)$ denotes the surface measure on $\Sigma_1(\tau, \xi)$. Consequently bound follows by an application of Hölder’s inequality and the transversality condition 1.5. □

The proof of the Propositions 4.2 and 4.3 is now a consequence of Theorems 5.1 and 5.2.

**Proof of Proposition 4.2.** By definition of $A_{r,s}(R)$ and $R_0$, it is enough to prove that for every cube $Q$ of diameter $R \geq R_0$ and any $\Phi_1$-wave $u$, and $\Phi_2$-wave $v$ satisfying the support conditions

$$
\text{supp } \hat{u} \subset \Lambda_1^* + \frac{d_0}{2}, \quad \text{supp } \hat{v} \subset \Lambda_2^* + \frac{d_0}{2}
$$

we have

$$
\|uv\|_{L^1_{t,x}(Q)} \lesssim \left( d_0^{-(n-1)(1-\frac{1}{r})} H_2^\frac{n+1}{r} + \frac{n+1}{2r} R^{\frac{n+1}{2r}} \right) \|u\|_{L^\infty_{t,x}} \|v\|_{L^\infty_{t,x}}.
$$

An application of Theorem 5.1 gives

$$
\|uv\|_{L^1_{t,x}(Q')} \lesssim \left( \|u(\cdot)|v(\cdot)\|_{L^1_{t,x}(Q')} \right) + C H_2^\frac{n+1}{r} - \frac{n+1}{2r} R^{\frac{n+1}{2r}} \|u\|_{L^\infty_{t,x}} \|v\|_{L^\infty_{t,x}}
$$

where $Q'$ is a cube of diameter $2R$. To control the quilt term, we first observe that Theorem 5.2 together with the energy estimate for $u(B)$ and $v(B)$ implies the $L^2$ bound

$$
\|u(B)|v(B')\|_{L^2_{t,x}(Q')} \lesssim \sum_{B \in Q'} \sum_{B' \in Q'} \|u(B)|v(B')\|_{L^2_{t,x}}^2
$$

$$
\lesssim d_0^{n-1} \|u\|_{L^\infty_{t,x}} \|v\|_{L^\infty_{t,x}}^2
$$

where we used the Fourier support conditions on $u(B)$ and $v(B)$ to deduce that

$$
d[\text{supp } \hat{u}(B), \text{supp } \hat{v}(B')] \lesssim d[\Lambda_1^* + \frac{d_0}{2}, \Lambda_2^* + \frac{d_0}{2}] \lesssim d_0.
$$

2 Explicitly, in the region $|d_1 \Phi_1 - d_1 \Phi_2| \approx |\nabla \Phi_1 - \nabla \Phi_2|$, there exists a function $\psi(\tau, \xi, \eta') : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{n-1} \to \mathbb{R}$ such that

$$
\Phi_1(\xi - (\psi, \eta')) + \Phi_2(\psi, \eta') = \tau.
$$

Thus we can write the surface as a graph $\Sigma_2(h) = \{(\psi(\eta'), \eta') \in (h - \Lambda_1) \cap \Lambda_2\}$, and hence the surface measure is then

$$
d\sigma(\eta) = \sqrt{1 + |\hat{\eta}'|^2} \, d\eta' = \frac{\|\nabla \Phi_1 - \nabla \Phi_2\|_{L^1_{t,x}}}{|d_1 \Phi_1 - d_1 \Phi_2|} \, d\eta'.
$$
On the other hand, simply applying Hölder’s inequality gives
\[ \|([u^{(i)}][v^{(j)}])\|_{L^2_tL^2_x(Q')} \leq \|([u^{(i)}])\|_{L^2_tL^2_x(Q')}\|([v^{(j)}])\|_{L^2_tL^2_x(Q')} \]
\[ \lesssim (\mathcal{R}H_2)^\frac{1}{2} \left( \sum \|u^{(B)}\|_{L^2_tL^2_x} \right)^\frac{1}{2} \left( \sum \|v^{(B')}\|_{L^2_tL^2_x} \right)^\frac{1}{2} \]
\[ \lesssim (\mathcal{R}H_2)^\frac{1}{2} \|u\|_{L^\infty_tL^2_x} \|v\|_{L^\infty_tL^2_x}. \]

Applying Holder in the \( t \) variable, and interpolating between the \( L^2_tL^2_x \) and \( L^\infty_tL^2_x \) bounds, then gives
\[ \|([u^{(i)}][v^{(j)}])\|_{L^2_tL^2_x(Q')} \lesssim R^{1/\frac{r}{2}} \|([u^{(i)}])\|_{L^2_tL^2_x(Q')} \lesssim \mathcal{d}_0^{(n-1)(1-\frac{1}{r})} \Lambda \frac{1}{2} R^{\frac{1}{r}+\frac{1}{r}-1} \|u\|_{L^\infty_tL^2_x} \|v\|_{L^\infty_tL^2_x}. \]

Therefore (5.1) follows. \( \square \)

A similar application of Theorem 5.1 gives Proposition 4.3.

**Proof of Proposition 4.3.** Let \( R \gg (\mathcal{d}_0^2 \mathcal{H}_2)^{-1} \). Let \( u \) be a \( \Phi_1 \)-wave and \( v \) be a \( \Phi_2 \)-wave with the support condition
\[ \text{supp } \hat{u} \subset \Lambda_1^* + 4(\mathcal{H}_22R)^{-\frac{1}{2}}, \quad \text{supp } \hat{v} \subset \Lambda_2^* + 4(\mathcal{H}_22R)^{-\frac{1}{2}}. \]

It is enough to consider the normalised case \( \|u\|_{L^\infty_tL^2_x} = \|v\|_{L^\infty_tL^2_x} = 1 \). Our goal is to show that for every cube \( Q \) of diameter \( 2R \) we have
\[ \|uv\|_{L^1_tL^2_x(Q)} \leq (1 + Cc) A_{q,r}(R) + \epsilon^{-C} \mathcal{H}_2^{\frac{n+1}{2} + \frac{1}{2} - \frac{n+1}{2}}, \]
(5.2)

An application of Theorem 5.1 gives a cube \( Q' \) of diameter \( 4R \), and wave tables \((u^{(B)})_{B \in \mathcal{Q}_R(Q')}, (v^{(B')})_{B' \in \mathcal{Q}_R(Q')}\) such that
\[ \|uv\|_{L^1_tL^2_x(Q')} \leq (1 + Cc)\|([u^{(i)}][v^{(j)}])\|_{L^1_tL^2_x(Q')} + \epsilon^{-C} \mathcal{H}_2^{\frac{n+1}{2} + \frac{1}{2} - \frac{n+1}{2}}, \]

and the support properties
\[ \text{supp } \hat{u}^{(B)} \subset \Lambda_1^* + 4(\mathcal{H}_22R)^{-\frac{1}{2}}, \quad \text{supp } \hat{v}^{(B')} \subset \Lambda_2^* + 4(\mathcal{H}_22R)^{-\frac{1}{2}} \]
(5.3)

where we used the support assumptions on \( \hat{u} \) and \( \hat{v} \). Let \( B' \in \mathcal{Q}_R(Q') \) and define the vector valued \( \Phi_1 \)-wave \( U^{(B')} = (u^{(B)})_{B \in \mathcal{Q}_n(B')} \). It is easy to check that \( U^{(B')} \) is a \( \Phi_1 \)-wave (since \( u^{(B)} \) are maps into \( \ell^2_\mathbb{Z}(Z) \), after relabeling, \( U^{(B')} \) is also a map into \( \ell^2_\mathbb{Z}(\mathbb{Z}) \)). In particular, for every \( B' \in \mathcal{Q}_R(Q') \) we have a \( \Phi_1 \)-wave \( U^{(B')} \) and a \( \Phi_2 \)-wave \( v^{(B')} \) satisfying the correct support assumptions to apply the definition of \( A_{q,r}(R) \). Hence, if we observe that
\[ [u^{(i)}] = \sum_{B' \in \mathcal{Q}_R(Q')} \sum_{B \in \mathcal{Q}_n(B')} 1_B|u^{(B)}| \]
\[ \leq \sum_{B' \in \mathcal{Q}_R(Q')} \sum_{B \in \mathcal{Q}_n(B')} 1_B\left(\sum_{B \in \mathcal{Q}_n(B')} |u^{(B)}|^2\right)^{\frac{1}{2}} = \sum_{B' \in \mathcal{Q}_R(Q')} 1_B|U^{(B')}| \]

then, applying the definition of \( A_{q,r}(R) \), we deduce that
\[ \|([u^{(i)}][v^{(j)}])\|_{L^2_tL^2_x(Q')} \leq \sum_{B' \in \mathcal{Q}_R(Q')} \|U^{(B')}v^{(B')}\|_{L^2_tL^2_x(B')} \]
\[ \leq A_{q,r}(R) \left( \sum_{B' \in \mathcal{Q}_R(Q')} \|U^{(B')}\|_{L^\infty_tL^2_x}^2 \right)^{\frac{1}{2}} \left( \sum_{B' \in \mathcal{Q}_R(Q')} \|v^{(B')}\|_{L^\infty_tL^2_x}^2 \right)^{\frac{1}{2}} \]
\[ \leq A_{q,r}(R)(1 + Cc)^2 \]

where the last line follows by using the energy inequalities in Theorem 5.1 \( \square \)

**Remark 5.3.** Note that proof of Proposition 4.3 exploited the fact that the definition of \( A_{q,r}(R) \) applies to vector valued waves. In fact, this is essentially the only step in the proof of Theorem 4.3 in which the freedom to use vector valued waves is crucial.
We have now reduce the problem of proving Theorem 1.4 to obtaining the decomposition contained in Theorem 5.1. The proof of Theorem 6.1 requires the full range of tools used to prove bilinear restriction estimates, namely a sharp wave packet decomposition, the wave table construction due to Tao, geometric information on the conic surfaces \( C_j(\mathfrak{h}) \), and energy estimates across transverse surfaces (which can be thought of a replacement for the combinatorial Kakeya type inequalities used in the original argument of Wolff).

6. Wave Packets

The standard approach to constructing wave packets is to localise on both the spatial and Fourier side, up to the scale given by the uncertainty principle. However, as we need to carefully control the constants in the energy estimates, we use a more refined construction originally due to Tao [21] and extended to the case of general phases by Bejenaru [1]. Let \( \nu \in \mathcal{S}(\mathbb{R}^n) \) be a positive smooth function, such that \( \text{supp} \, \hat{\nu} \subset \{ |\xi| \leq 1 \} \), and for all \( x \in \mathbb{R}^n \)

\[
\sum_{k \in \mathbb{Z}^n} \nu(x - k) = 1.
\]

Let \( A = \{ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \mid |\xi_j| \leq \frac{1}{2}, j = 1, \ldots, n \} \) denote the unit cube centered at the origin, and define

\[
\rho(\xi) = \int_{|z| < 1} 1_A(\xi - z) \, dz
\]

where we take \( f_\Omega = \frac{1}{|\Omega|} \int_Q \). To a phase space point \( \gamma = (x_0, \xi_0) \in \mathcal{X}_{r,e} \) and \( f \in L^2_x(\mathbb{R}^n) \), define the phase-space localisation operator

\[
(L_\gamma f)(x) = \nu\left(\frac{r^2}{2} (x - x_0)\right) \left[ \rho(r(-i\nabla - \xi_0)) f \right](x).
\]

We have the following basic properties.

**Lemma 6.1** (Properties of \( L_\gamma \) [21 Lemma 15.2], [1 Lemma 4.1]). Let \( r > 0 \), \( 0 < \epsilon \leq 1 \), and \( f \in L^2(\mathbb{R}^n) \). Then

\[
\sum_{\gamma \in \mathcal{X}_{r,e}} L_\gamma f = f, \quad \text{supp} \widehat{L_\gamma f} \subset \{ |\xi - \xi(\gamma)| \leq \frac{n+1}{r} \} \cap (\text{supp} \, \hat{f} + \frac{1}{r}).
\]

Moreover, if \( (m_k, \gamma)_k \) is a positive sequence with \( \sum_k m_k, \gamma \leq 1 \), then

\[
\left( \sum_k \left( \sum_{\gamma \in \mathcal{X}_{r,e}} m_k, \gamma L_\gamma f \right)_{L^2}^2 \right)^{\frac{1}{2}} \leq (1 + C\epsilon) \| f \|_{L^2}
\]

where the constant \( C \) depends only on the dimension \( n \).

**Proof.** As in [1], we adapt the method of Tao [21]. However, as we use a slightly more direct argument, we include the details. Clearly, by construction we have

\[
\sum_{\gamma \in \mathcal{X}_{r,e}} L_\gamma f = c_n \int_{\mathbb{R}^n} \int_{|z| < \frac{r}{2}} \sum_{\xi_0 \in \mathbb{Z}^n} 1_A(r(\xi - \xi_0) - z) \, dz \, \hat{f}(\xi) e^{i\xi \cdot x} \, d\xi = f.
\]

On the other hand, since \( \text{supp} \, \hat{\nu} \subset \{ |\xi| \leq 1 \} \) and \( \text{supp} \, \hat{\rho} \subset \{ |\xi| \leq \frac{2 + \sqrt{n}}{2r} \} \), we have \( \text{supp} \, \hat{L_\gamma f} \subset \frac{r}{2} + \text{supp} \, \hat{f} \cap \{ |\xi - \xi_0| \leq \frac{r}{2} \} \). It remains to prove the more delicate energy type estimate. To simplify the notation somewhat, we take

\[
\nu_{x_0} = \nu\left(\frac{r^2}{2} (x - x_0)\right), \quad \hat{f}_{0, z} = 1_A(r(\xi - \xi_0) - z) \hat{f}, \quad P_{A^{\ast}}^{(\xi_0, z)} g(\xi) = 1_A^{\ast} (r(\xi - \xi_0) - z) \hat{g}(\xi)
\]

where

\[
A^{\ast} = \{ \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n \mid |\xi_j| \leq \frac{1}{2} - 2r^2, j = 1, \ldots, n \}.
\]
Applying Minkowski’s inequality and decomposing $f_{(\xi_0,z)}$ gives
\begin{equation}
\left(\sum_k \left(\sum_\gamma m_{k,\gamma} L_1 f_{(\xi_0,z)}^2\right)^{\frac{1}{2}}\right) \leq \int_{|z|<1} \left(\sum_k \left(\sum_x \sum_{\xi_0} m_{k,(x_0,\xi_0)} \nu_{x_0} f_{(\xi_0,z)}^2\right)^{\frac{1}{2}}\right) \, dz
\end{equation}
(6.2)
\[
\leq \int_{|z|<1} \left(\sum_k \left(\sum_x \sum_{\xi_0} m_{k,(x_0,\xi_0)} \nu_{x_0} P_{A^*}(\xi_0,z) f_{\xi_0,z}^2\right)^{\frac{1}{2}}\right) \, dz
\]
\[
+ \int_{|z|<1} \left(\sum_k \left(\sum_x \sum_{\xi_0} m_{k,(x_0,\xi_0)} \nu_{x_0} (1 - P_{A^*}(\xi_0,z)) f_{\xi_0,z}^2\right)^{\frac{1}{2}}\right) \, dz.
\]
A computation shows that the Fourier supports of the functions $\nu_{x_0} P_{A^*}(\xi_0,z) f_{\xi_0,z}$ are disjoint as $\xi_0$ varies. Consequently we have the identity
\[
\left(\sum_k \left(\sum_x \sum_{\xi_0} m_{k,(x_0,\xi_0)} \nu_{x_0} P_{A^*}(\xi_0,z) f_{\xi_0,z}^2\right)^{\frac{1}{2}}\right)^2 = \left(\sum_k \left(\sum_x \sum_{\xi_0} m_{k,(x_0,\xi_0)} \nu_{x_0} P_{A^*}(\xi_0,z) f_{\xi_0,z}^2\right)^{\frac{1}{2}}\right)^2.
\]
As $\nu_{x_0}$ and $m_{k,(x_0,\xi_0)}$ are positive, we have
\[
\sum_k \sum_x \sum_{\xi_0} m_{k,(x_0,\xi_0)} \nu_{x_0} P_{A^*}(\xi_0,z) f_{\xi_0,z}^2 \leq \left| P_{A^*}(\xi_0,z) f_{\xi_0,z}\right| \left(\sum_k \nu_{x_0} \sum_{\xi_0} m_{k,(x_0,\xi_0)}\right)^2 \leq \left| P_{A^*}(\xi_0,z) f_{\xi_0,z}\right|
\]
and therefore
\[
\int_{|z|<1} \left(\sum_k \left(\sum_x \sum_{\xi_0} m_{k,(x_0,\xi_0)} \nu_{x_0} P_{A^*}(\xi_0,z) f_{\xi_0,z}^2\right)^{\frac{1}{2}}\right)^2 \, dz \leq \|f\|_{L^2}.
\]
It remains to estimate the second term in (6.2). Repeating the above argument, but applying almost orthogonality instead of orthogonality, we see that
\[
\int_{|z|<1} \left(\sum_k \left(\sum_x \sum_{\xi_0} m_{k,(x_0,\xi_0)} \nu_{x_0} (1 - P_{A^*}(\xi_0,z)) f_{\xi_0,z}^2\right)^{\frac{1}{2}}\right)^2 \, dz \leq C_n \int_{|z|<1} \left(\sum_{\xi_0} \left| (1 - P_{A^*}(\xi_0,z)) f_{\xi_0,z}\right|^2\right)^{\frac{1}{2}} \, dz.
\]
An application of Holder’s inequality in the $dz$ integral, together with Plancheral reduces the problem to proving that
\[
\sum_{\xi_0} \int_{|z|<1} \left[ 1 - 1_{A^*}(r(\xi - \xi_0) - z) \right] 1_A(r(\xi - \xi_0) - z) dz \leq C_n\epsilon.
\]
But this follows from a short computation. Thus lemma follows.

We can now define the wave packets we use in the proof of Theorem 4.14. As is more or less standard, see for instance [1, 4], to define the wave packet associated to a phase space point $\gamma \in \Gamma_j$, we conjugate the phase-space localisation operator $L_j$ with the flow $e^{it\Phi_j(-i\nabla)}$. More precisely:

**Definition 6.2** (Wave Packets). Let $j = 1, 2$, and $r_j \gg d_0^{-1}$. If $u \in L_1^\infty L_x^2$ and $\gamma_j \in \Gamma_j$, we define
\[
(P_{\gamma_j} u)(t) = e^{it\Phi_j(-i\nabla)} L_{\gamma_j} \left( e^{-it\Phi_j(-i\nabla)} u(t) \right).
\]

Lemma 6.1 has an immediate extension to the wave packets $P_{\gamma_j}$.

**Lemma 6.3** (Orthogonality Properties of Wave Packets). Let $j = 1, 2$, $0 < \epsilon \leq 1$, and $r_j \gg d_0^{-1}$. Let $u \in L_1^\infty L_x^2$ with $\text{supp} \, \hat{u}(t) + \frac{4r_j}{T} \subset \Lambda_j$. Then
\[
u = \sum_{\gamma_j \in \Gamma_j} P_{\gamma_j} u \quad \text{with} \quad \text{supp} \, \widehat{P_{\gamma_j} u}(t) \subset \left\{ |\xi - \xi(\gamma_j)| \leq \frac{4r_j}{\epsilon T} \right\} \bigcap \left( \text{supp} \, \hat{u} + \frac{4r_j}{T} \right).
\]
Moreover, if $(m_{k,\gamma})_{k,\gamma_j}$ is a positive sequence with $\sup_{\gamma_j \in \Gamma_j} \sum_k m_{k,\gamma_\gamma_j} \leq 1$, then
\[
\left\| \left( \sum_{k} \sum_{\gamma_j \in \Gamma_j} m_{k,\gamma_j} P_{\gamma_j} u(t) \right)^2 \right\|_{L^2_x} \leq (1 + C\epsilon) \|u\|_{L_1^\infty L_x^2}.
\]
where the constant $C$ depends only on the dimension $n$.

**Proof.** If we observe that the Fourier multiplier $e^{it\Phi_j(-i\nabla)}$ is unitary on $L^2_{\gamma_j}$, and clearly leaves the Fourier support invariant, then all the required properties follow from Lemma 6.1 together with the identity

$$u = \sum_{\gamma_j \in \mathcal{X}_{\gamma_j}} \mathcal{P}_j u = \sum_{\gamma_j \in \Gamma_j} \mathcal{P}_j u$$

which is a consequence of the definition of $\Gamma_j$ and the support assumption on $u$. \qed

More refined properties of the wave packets $\mathcal{P}_{\gamma_j} u$ are possible if we make further assumptions on $u$. In particular, if $u = e^{it\Phi(-i\nabla)}f$ is a free solution, then the associated wave packets are concentrated on the tubes $T_{\gamma_j}$. One possible formulation of this statement, which will prove extremely useful in practice, gives a bound in terms of the Hardy-Littlewood maximal function

$$\mathcal{M}[f](x) = \sup_{s > 0} \frac{1}{sn} \int_{|x-y| < s} |f(y)|dy.$$ 

**Lemma 6.4** (Concentration of wave packets I). Let $N \geq 2$, $0 < \epsilon \leq 1$, and $j \in \{1,2\}$. Assume that the phase $\Phi_j \in C^N(\Lambda_j)$ satisfies for every $2 \leq m \leq N$ the bound

$$\|\nabla^m \Phi_j\|_{L^\infty(\Lambda_j)} \leq d_0^{-m} h_j.$$

Let $r_j \gg d_0^{-1}$, $\gamma_j = (x_0, \xi_0) \in \Gamma_j$, and let $u = e^{it\Phi_j(-i\nabla)}f$ be a $\Phi_j$-wave. Then for all $(t, x) \in \mathbb{R}^{1+n}$ we have

$$|\mathcal{P}_{\gamma_j} u(t, x)| \lesssim e^{-2(N+\epsilon)} \left(1 + \frac{|x - x_0 + t\nabla \Phi_j(\xi_0)|}{r_j}\right)^{-N} \left(1 + \frac{|t|}{R}\right)^N \mathcal{M}[f_{\xi_0}](x_0)$$

(6.4)

where $f_{\xi_0} = \rho(r_j(-i\nabla - \xi_0))f$, and the implied constant depends only on $n$ and $N$ (in particular it is independent of the phase $\Phi_j$ and $d_0$).

**Proof.** We start by writing

$$\mathcal{P}_{\gamma_j} u(t, x) = \int_{\mathbb{R}^n} \left(\overline{L_{\gamma_j}f}(\xi)e^{it\Phi_j(\xi)e^{ix\cdot\xi}}\right)d\xi$$

$$= \int_{\mathbb{R}^n} K_{\xi_0}(t, x - y)(L_{\gamma_j}f)(y)dy$$

where the kernel is given by $K_{\xi_0}(t, x) = \int_{\mathbb{R}^n} \zeta(r_j(\xi - \xi_0))e^{it\Phi_j(\xi)e^{ix\cdot\xi}}d\xi$ and $\zeta \in C_0^\infty(\{\xi : |\xi| < 2\})$ is a cutoff satisfying $\zeta(\xi)\rho(\xi) = \rho(\xi)$. The spatial localisation properties of $L_{\gamma_j}f$, together with the standard maximal function inequality

$$\int_{\mathbb{R}^n} (1 + s|x - y|)^{-(n+\epsilon)}|g(y)|s^n dy \lesssim \mathcal{M}[g](x)$$

implies that it suffices to prove the kernel bound

$$|K_{\xi_0}(t, x)| \lesssim r_j^{-n} \left(1 + \frac{|x + t\nabla \Phi(\xi_0)|}{r_j}\right)^{-N} \left(1 + \frac{|t|}{R}\right)^N.$$

(6.5)

This bound is clearly true when $|x + t\nabla \Phi(\xi_0)| \leq r_j$ by the Fourier support assumption on $\zeta$. On the other hand, an application of integration by parts gives for every $1 \leq k \leq n$

$$|K_{\xi_0}(t, x)| = \left|\int_{\mathbb{R}^n} \zeta(r_j(\xi - \xi_0))e^{it[\Phi_j(\xi - \Phi_j(\xi_0)) - \Phi_j(\xi_0)]}e^{(x + t\nabla \Phi(\xi_0))\cdot \xi}d\xi\right|$$

$$\leq |x_k + t\partial_k \Phi(\xi_0)|^{-N} \int_{\mathbb{R}^n} \left|\partial_k^N \zeta(r_j(\xi - \xi_0))e^{it[\Phi_j(\xi - \Phi_j(\xi_0)) - \Phi_j(\xi_0)]}e^{(x + t\nabla \Phi(\xi_0))\cdot \xi}\right|d\xi.$$

In particular, it suffices to show that for every $|\xi - \xi_0| \leq 2r_j^{-1}$ we have

$$\left|\partial_k^N \zeta(r_j(\xi - \xi_0))e^{it[\Phi_j(\xi - \Phi_j(\xi_0)) - \Phi_j(\xi_0)]e^{(x + t\nabla \Phi(\xi_0))\cdot \xi}}\right| \lesssim r_j^N \left(1 + \frac{|t|}{R}\right)^N.$$

(6.5)
This bound is consequence of a somewhat tedious induction argument. For completeness, we sketch the proof here. Let $F = i\phi(\Phi_j(\xi) - \Phi_j(\xi_0) - (\xi - \xi_0) \cdot \nabla \Phi_j(\xi_0))$ and 

$$I_N = e^{-F} \partial_k^N \left( \zeta(r_j(\xi - \xi_0))e^F \right).$$

To compute $I_N$ explicitly, we observe that $I_N = \partial_k I_{N-1} + I_{N-1} \partial_k F$, which via an induction argument, implies that 

$$I_N = \partial_k^N I_0 + \sum_{m=1}^N \partial_k^{N-m} I_0 \sum_{\ell \in \mathbb{N}^m} c_m,\ell (\partial_k F)^{\ell_1} \cdots (\partial_k^m F)^{\ell_m}$$

for some constants $c_{m,\ell} \in \mathbb{N}$. The assumption (A2) together with $r_j \gg d_0$ implies that 

$$|\partial_k F| \leq |t| |\nabla \Phi_j(\xi) - \nabla \Phi_j(\xi_0)| \leq \frac{|t|}{R} r_j, \quad |\partial_k^2 F| \leq |t| |\nabla^2 \Phi_j(\xi)| \leq \frac{|t|}{R}^2 r_j$$

and for any $3 \leq m \leq N$

$$|\partial_k^m F| \leq |t| |\nabla^m \Phi_j(\xi)| \leq \frac{|t|}{R} d_0^{2-m} R |\nabla^2 \Phi_j||_{L^\infty} \leq \frac{|t|}{R}^m r_j.$$ 

Therefore the bound (6.5) follows from the identity (6.6). □

**Remark 6.5.** If we restrict attention to the case $|t| \lesssim R$ and $\epsilon = 1$, then Lemma 6.4 is essentially equivalent to the wave packet type decompositions used frequently in previous works. See for instance [21, 14, 1, 4]. It is also important to note that the assumption (A2) is essentially the weakest condition for the required concentration bound (6.4) to hold. This can easily be seen from the identity (6.6), since if we want to obtain (6.4), then we essentially require

$$\|\nabla^m \Phi_j||_{L^\infty} \lesssim r_j^{m-2}$$

which is equivalent to having a uniform bound on the quantity

$$\left( \frac{\|\nabla^m \Phi_j||_{L^\infty}}{\|\nabla^2 \Phi_j||_{L^\infty}} \right)^{\frac{m-2}{m}}.$$

and this is precisely the condition appearing in (A2).

Recall that, given a phase space point $\gamma_j \in \Gamma_j$, we have defined the associated weight $w_{\gamma_j, q}$, which penalises the distance from the cube $q \in \mathcal{Q}_{\gamma_j}(Q)$ and tube $T_{\gamma_j}$, by

$$w_{\gamma_j, q} = \left( 1 + \frac{|x_q - x_0 + t_q \nabla \Phi_j(\xi_0)|}{r_j} \right)^{5n}.$$

We now come to the main property of wave packets which we exploit in the sequel.

**Proposition 6.6 (Concentration of wave packets II).** Let $j = 1, 2$ and $0 < \epsilon \ll 1$. Assume that the phase $\Phi_j \in C^{5n}(\Lambda_j)$ satisfies (A2). Let $R \gtrsim r_j \gg d_0^{-1}$ and assume that $u$ is a $\Phi_j$-wave with supp $\tilde{u} + \frac{d_0}{2} \subset \Lambda_j$. Then for any cube $Q$ of diameter $R$ we have the concentration/orthgonality type estimate

$$\sum_{\gamma_j \in \Gamma_j} \sup_{q \in \mathcal{Q}_{\gamma_j}(Q)} w_{\gamma_j, q}^2 |x_q - x_0 + t_q \nabla \Phi_j(\xi_0)|^2 \lesssim \epsilon^{-14n} r_j |u|_{L^2}^2.$$

**Proof.** Let $N = 5n$ and $Q$ be a cube of diameter $R$. By translation invariance, we may assume that $Q$ is centred at the origin. Write $u = e^{it \Phi_j(-t \nabla)} f$ for some $\ell^2(\mathbb{Z})$ valued map $f$. The identity

$$|x - x_0 + t \nabla \Phi_j(\xi_0)| = |(t, x) - (0, x_0) - t(1, -\nabla \Phi_j(\xi_0))|,$$
together with an application of Lemma 6.4 implies that for any \( q \in \mathcal{Q}_{r_j}(Q) \) and \( \gamma_j = (x_0, \xi_0) \in \Gamma_j \)

\[
\| \chi_q P_{\gamma_j} u \|_{L^2_{t,x}}^2 \\
\lesssim e^{-(N+n)}|\mathcal{M}[f_{\xi_0}](x_0)|^2 \left\| \chi_q(t,x) \left( 1 + \frac{|x - x_0 + t \partial \Phi_j(\xi_0)|}{r_j} \right)^{-N} \left( 1 + \frac{|t|}{R} \right)^N \right\|_{L^2_{t,x}}^2
\]

\[
\lesssim e^{-(N+n)}|\mathcal{M}[f_{\xi_0}](x_0)|^2 \left( 1 + \frac{|x_q - x_0 + t_q \partial \Phi_j(\xi_0)|}{r_j} \right)^{-2N} \left\| \chi_q(t,x) \left( 1 + \frac{(t,x) - (t_q,x_q)}{r_j} \right)^{2N} \right\|_{L^2_{t,x}}^2
\]

\[
\lesssim e^{-(N+n)}r_j^{n+1}|\mathcal{M}[f_{\xi_0}](x_0)|^2 \left( 1 + \frac{|x_q - x_0 + t_q \partial \Phi_j(\xi_0)|}{r_j} \right)^{-2N}
\]

where we used (6.1), the rapid decay of \( \chi_q \), and the bound

\[
\left( 1 + \frac{|t|}{R} \right) \leq 2 \left( 1 + \frac{|t - t_q|}{r_j} \right) \lesssim \left( 1 + \frac{|t - t_q|}{r_j} \right)
\]

which follows directly from the conditions \( |t_q| \leq R \) and \( r_j \lesssim R \). Therefore, we deduce that

\[
\sum_{\gamma_j \in \Gamma_j} \sup_{q \in \mathcal{Q}_{r_j}(Q)} \| \chi_q P_{\gamma_j} u \|_{L^2_{t,x}}^2 \lesssim r_j^{n+1} e^{-2(N+n)} \sum_{(x_0, \xi_0) \in \Gamma_j} |\mathcal{M}[f_{\xi_0}](x_0)|^2.
\]

To complete the proof of (6.7), we recall that the Fourier localisation of \( f_{\xi_0} \) together with the maximal inequality gives

\[
\sum_{(x_0, \xi_0) \in \Gamma_j} \left( \mathcal{M}[f_{\xi_0}](x_0) \right)^2 \lesssim e^{-2n} \sum_{\xi_0} \| \mathcal{M}[f_{\xi_0}] \|_{L^2_t}^2 \lesssim e^{-2n} \sum_{\xi_0} \| f_{\xi_0} \|_{L^2_x}^2 \approx e^{-2n} \| f \|_{L^2_x}^2.
\]

\[\square\]

\textbf{Remark 6.7.} Note that technical condition \( r_j \lesssim R \) is natural since (6.7) only controls \( u \) on (essentially) a cube of size \( R \). If \( r_j \gg R \) (which corresponds to \( R \ll \| \nabla^2 \Phi_j \|_{L^\infty} \) then each wave packet is supported in an area larger than \( Q \). In other words, \( u \) cannot be localised to scales \( R \ll \| \nabla^2 \Phi_j \|_{L^\infty} \).

\textbf{Remark 6.8.} It is worth comparing (6.7) to previous results in the literature where a local in time version was proven, in the sense that the function \( \chi_q \) was replaced by the sharp cutoff \( 1_{C_0 q} \). The slightly more refined global version (6.7) (which also contains the region \( |t| \gg R \), albeit with a rapidly decaying weight \( t^{-N} \)) allows us to avoid a number of additional error estimates, which would otherwise complicate the analysis, particularly in the general phase case that we consider here.

\section{7. Geometric Consequences of Assumptions on Phases}

In this section, following the approach in [4], we give two important consequences of the transversality/curvature assumption (A1). The first concerns the surfaces \( \Sigma_j(\mathfrak{h}) \) and only requires the transversality condition (1.5), while the second gives a crucial transversality condition concerning the conical set \( \mathcal{C}_j(\mathfrak{h}) \) and requires the full strength of the assumption (A1). In fact, this is the only place where the full generality of (A1) is required, rather than just the immediate consequences (1.5) and (1.6).

Recall that given \( \mathfrak{h} \in \mathbb{R}^{1+n} \) we have defined the surfaces \( \Sigma_j(\mathfrak{h}) = \{ \xi \in \Lambda_j \cap (\mathfrak{h} - \Lambda_k) \mid \Phi_j(\xi) + \Phi_k(\mathfrak{h} - \xi) = a \} \). The transversality assumption (1.5) implies that these surfaces are well-behaved.

\textbf{Lemma 7.1 (Foliation Properties of \( \Sigma_j(\mathfrak{h}) \)).} Let \( \mathfrak{h} = (a, h) \in \mathbb{R}^{1+n} \) and \( 0 < C_0 < 1 \). Suppose that \( \Lambda_j \) is open, and that \( \Phi_j \in C^2(\Lambda_j) \) satisfies the transversality condition (1.5) and the normalisation condition (1.7). Assume \( \xi + \frac{3}{C_0 r} \in \Lambda_1 \) and \( \eta + \frac{3}{C_0 r} \in \Lambda_2 \) with

\[|\xi + \eta - h| \leq \frac{1}{r}, \quad |\Phi_1(\xi) + \Phi_2(\eta) - a| \leq \frac{1}{r}.
\]

Then \( \xi \in \Sigma_1(\mathfrak{h}) + \frac{3}{C_0 r} \) and \( \eta \in \Sigma_2(\mathfrak{h}) + \frac{3}{C_0 r} \).
that for every $F \in \mathcal{C}^2(\Lambda_j)$ satisfies (A1) and the normalisation condition (1.1). For $\{j, k\} = \{1, 2\}$, $p, q \in \mathcal{C}_j(h)$, and $\eta \in \Lambda_k$ we have

$$|\langle p - q \rangle \wedge (1, -\nabla \Phi_j(\eta))| \geq \frac{C_0}{\lambda} |p - q|.$$  

Proof. We follow/repeat the argument given in [4] Lemma 2.7. Let $w, w', w'' \in \mathbb{R}^n$. A computation shows that for every $v \in \text{span}\{(1, w), (0, w - w')\}$ we have

$$|v \wedge (1, w'')| \geq \frac{|(w - w') \wedge (w - w'')|}{(1 + |w||w - w'|)} |v|.$$  

Let $\xi, \xi' \in \Sigma_j(h)$ and $\eta \in \Lambda_k$. If we take $w = -\nabla \Phi_j(\xi)$, $w' = -\nabla \Phi_j(\xi')$, and $w'' = -\nabla \Phi_k(\eta)$ in the above bound, we deduce that

$$|v \wedge (1, -\nabla \Phi_k(\eta))| \geq \frac{|(\nabla \Phi_j(\xi) \wedge \nabla \Phi_j(\xi')) \wedge (\nabla \Phi_j(\xi) \wedge \nabla \Phi_k(\eta))|}{(1 + \|\nabla \Phi_j \|_{L^\infty(\Lambda_j)}) \|\nabla \Phi_j(\xi) \wedge \nabla \Phi_j(\xi')\|} |v|$$

for every $v \in \text{span}\{(1, -\nabla \Phi_j(\xi)), (0, \nabla \Phi_j(\xi) - \nabla \Phi_j(\xi'))\}$. The claimed bound now follows by applying (A1), and observing that given any $p, q \in \mathcal{C}_j(h)$ we can write

$$(p - q) = (r - r')(1, -\nabla \Phi_j(\xi)) + r'(0, \nabla \Phi_j(\xi) - \nabla \Phi_j(\xi')),$$

for some $\xi, \xi' \in \Sigma_j(h)$ and $r, r' \in \mathbb{R}$. \qed

8. Energy Estimates Across $\mathcal{C}_j(h)$

In this section we prove a key energy bound across the conical surface $\mathcal{C}_j(h)$. This is the key step in the proof of Theorem 1.4 where the full strength of (A1) is required, elsewhere it is essentially enough to instead consider the weaker conditions (1.5) and (1.9). As noted earlier, the following estimate is roughly a continuous counterpart to the “bush” type arguments used in the combinatorial approach to bilinear restriction estimates.

Recall that given $h \in \mathbb{R}^{1+n}$ we have defined the conical surfaces

$$\mathcal{C}_j(h) = \{(s, -s\nabla \Phi_j(\xi)) \mid s \in \mathbb{R}, \xi \in \Sigma_j(h)\}.$$
We define a weight $\chi^*_{j,r}(t, x)$ associated to $C_j(\mathfrak{h})$ by

$$\chi^*_{j,r}(t, x) = \left(1 + \frac{\text{dist}((t, x), C_j(\mathfrak{h}))}{r}\right)^{-(n+2)}.$$ 

Thus $\chi^*_{j,r}$ is essentially concentrated on a $r$-neighbourhood of the conic surface $C_j(\mathfrak{h})$. The geometric condition (A1) implies that the wave packets associated to the phase $\Phi_k$ intersect the surface $C_j(\mathfrak{h})$ transversally. A rigorous version of this heuristic is the following.

**Theorem 8.1.** Let $\{j, k\} = \{1, 2\}$, $\mathfrak{h} \in \mathbb{R}^{1+n}$ and $r > d_0^{-1} \gtrsim \mathcal{H}_j$. Assume the phases $\Phi_j$ satisfy (A1), (A2), and the normalisation condition (4.7). If $v$ is a $\Phi_k$-wave with $\text{supp} \hat{v}$ contained in a ball of radius $r^{-1}$ and $\text{supp} \hat{v} + \frac{d_0}{8} \subset \Lambda_k$ then we have

$$\|\chi^*_{j,r} v\|^2_{L^2_{t,x}} \lesssim r\|v\|^2_{L^2_{t,x}}.$$ 

**Proof.** As in [21] by the $TT^*$ argument, the required bound is equivalent to proving that for every $F \in L^2_{t,x}$, we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \chi^*_{j}(t) \mathcal{U}(t) \mathcal{U}(-s)[\chi^*_{j}(s)F(s)] F(t) dt ds \lesssim r\|F\|^2_{L^2_{t,x}}$$

where we let

$$\mathcal{U}(t)[f](x) = \int_{\mathbb{R}} \rho(\xi) e^{i\langle t\Phi_k(\xi) + x, \xi \rangle} \hat{f}(\xi) d\xi$$

with $\text{supp} \tilde{\rho} \subset \Lambda_j$, $|\text{supp} \tilde{\rho}| \lesssim r^{-n}$, and $|\nabla^m \rho| \lesssim r^n$. Define

$$K(t, x) = \int_{\mathbb{R}} \rho^2(\xi) e^{i\langle t\Phi_j(\xi) + x, \xi \rangle} d\xi.$$

By an application of Young’s inequality, and using the assumption $|\text{supp} \tilde{\rho}| \lesssim r^{-n}$, it is enough to prove the pointwise bound

$$|\chi^*_{j}(t, x) K(t - s, x - y)\chi^*_{j}(s, y)| \lesssim \int_{\text{supp} \tilde{\rho}} \left(1 + \frac{|(t - s, x - y)|}{r}\right)^{-(n+2)} d\xi.$$ 

To this end, take $p, p^* \in C_j(\mathfrak{h})$ such that

$$|(t, x) - p| \approx \text{dist}((t, x), C_j(\mathfrak{h})), \quad |(s, y) - p^*| \approx \text{dist}((s, y), C_j(\mathfrak{h})).$$

If $|(t, x) - p| + |(s, y) - p^*| \gtrsim |p - p^*|$ then (8.1) follows from the decay of $\chi^*_{j}$ away from the surface $C_j(\mathfrak{h})$, together with the observation that

$$|(t - s, x - y)| \lesssim |p - p^*| + |(t, x) - p| + |(s, y) - p^*| \lesssim |(t, x) - p| + |(s, y) - p^*|.$$ 

On the other hand, if $|(t, x) - p| + |(s, y) - p^*| \ll |p - p^*|$, then an application of Lemma 7.2 together with (A1) implies that for every $\xi \in \Lambda_k$

$$|(t - s, x - y) \wedge (1, -\nabla \Phi_k(\xi))| \gtrsim |(p - p^*) \wedge (1, -\nabla \Phi_k(\xi))| - |(t, x) - p| - |(s, y) - p^*|$$

$$\gtrsim |p - p^*|$$

$$\gtrsim |(t - s, x - y)|$$

where the last inequality follows by writing $(t - s, x - y) = p - p^* + (t, x) - p + (s, y) - p^*$. In particular, we have

$$|x - y + (t - s) \nabla \Phi_k(\xi)| \approx |(t - s, x - y)|.$$ 

We now exploit the decay of the kernel $K(t, x)$ in directions orthogonal to $(1, -\nabla \Phi_k(\xi))$. More precisely, integrating by parts in the kernel $K(t, x)$ and applying (A2) implies that for every $N < 5n$

$$|K(t, x)|$$

$$\lesssim \int_{\text{supp} \tilde{\rho}} \frac{1}{|x + t \nabla \Phi_k(\xi)|} \sum_{\ell_0 + \ell_1 + \ldots + \ell_N = N} |\nabla^{\ell_0} \rho|^2 (t |\nabla^{2} \Phi_k|)^{\ell_1} (t |\nabla^{3} \Phi_k|)^{\ell_2} \ldots (t |\nabla^{N+1} \Phi_k|)^{\ell_N} d\xi$$

$$\lesssim \int_{\text{supp} \tilde{\rho}} \frac{r}{|x + t \nabla \Phi_k(\xi)|^N} \left(1 + \frac{|t|}{|x + t \nabla \Phi_k(\xi)|}\right)^N d\xi.$$
where we applied (A2), (1.1), and the assumption \( r \geq d_0^{-1} \geq \mathcal{H}_j \). Therefore (8.1) follows from (8.2).

Recall that, given a cube \( q \in Q_{r_j}(Q) \), we have defined \( \chi_q \in C^\infty(\mathbb{R}^{1+n}) \) to be a positive function such that \( \chi_q \geq 1 \) on \( q \), supp \( \chi_q \subset \{|(\tau, \xi)\| \leq \frac{1}{r_j}\} \), and we have the decay bound, for every \( N \in \mathbb{N} \),

\[
\chi_q(t, x) \lesssim_N \left( 1 + \frac{\text{dist}(\{(t, x), q\})}{r} \right)^{-N}.
\]

To exploit the previous theorem, we require the following crucial consequence of the curvature of \( \Phi_j \) on \( \Sigma_j(h) \).

**Lemma 8.2.** Let \( h \in \mathbb{R}^{1+n} \) and \( r_j \gg d_0^{-1} \geq \mathcal{H}_j \). Assume that \( \Phi_j \in C^2(\Lambda_j) \) satisfies (A1) and the normalisation condition (4.4). Let \( \{j, k\} = \{1, 2\} \), \( Q \) be a cube of diameter \( R \) and \( q_0 \in Q_{r_j}(Q) \). Then provided that

\[
\sigma_{\Sigma_j(h)}(\Sigma_j(h) \cap \Lambda) \lesssim d_0^{-1}
\]

we have for all \( (t, x) \in \mathbb{R}^{1+n} \)

\[
\sum_{q \in Q_{r_j}(Q)} \sum_{\omega_j, q_j, q_j \in \Gamma_j(h, \Lambda)} \chi_q(t, x) \lesssim \varepsilon^{-C} \left( 1 + \frac{\text{dist}(\{(t - t_{q_0}, x - x_{q_0}), C_j(h)\})}{r_j} \right)^{-(n+2)}.
\]

**Proof.** We start by observing that, for every \( (x_0, \xi_0) \in \Gamma_j(h, \Lambda) \), we have

\[
dist[(t - t_{q_0}, x - x_{q_0}), C_j(h)] \lesssim r_j + |(t, x) - (t_{q_0}, x_{q_0})| + |x_0 - x_{q_0} + t_q \nabla \Phi_j(\xi_0)| + |x_{q_0} - x_0 + t_q \nabla \Phi_j(\xi_0)|.
\]

In particular, as \( \sum_q (1 + \frac{|(t - t_{q_0}, x - x_{q_0})|}{r_j})^{n+2} \chi_q(t, x) \lesssim 1 \), it is enough to show that for fixed \( q, q_0 \in Q_{r_j}(Q) \) with \( |q - q_0| \gg 2R \), we have

\[
\sum_{(x_0, \xi_0) \in \Gamma_j(h, \Lambda)} \left( 1 + \frac{|x_0 - x_{q_0} + t_q \nabla \Phi_j(\xi_0)| + |x_{q_0} - x_0 + t_q \nabla \Phi_j(\xi_0)|}{r_j} \right)^{3n-1} \lesssim 1.
\]

For ease of reading, we let \( \sigma(x_0, \xi_0) = |x_0 - x_{q_0} + t_q \nabla \Phi_j(\xi_0)| + |x_{q_0} - x_0 + t_q \nabla \Phi_j(\xi_0)| \), thus \( \sigma \) is small when the tube \( T_{(x_0, \xi_0)} \) passes through the cubes \( q \) and \( q_0 \), and is large otherwise. Let \( (x_0, \xi_0) \in \Gamma_j(h, \Lambda) \) denote the phase space point with the associated tube passing closest to the cubes \( q \) and \( q' \), in other words, we take

\[
\sigma(x_0, \xi_0) = \min_{(x_0', \xi_0') \in \Gamma_j(h, \Lambda)} \{\sigma(x_0', \xi_0')\}.
\]

Suppose for the moment that \( \sigma(x_0, \xi_0) \ll \varepsilon R \), thus the tube \( T_{(x_0, \xi_0)} \) passes “close” to the cubes \( q \) and \( q_0 \) in particular, writing

\[
(t_q - t_{q_0})(1, -\nabla \Phi_j(\xi_0)) = (t_q, x_q) - (t_{q_0}, x_{q_0}) - (0, x_0 - x_{q_0} + t_q \nabla \Phi_j(\xi_0)) + (0, x_{q_0} - x_0 + t_q \nabla \Phi_j(\xi_0))
\]

the separation of the cubes \( q \) and \( q_0 \) implies that \( |t_q - t_{q_0}| \approx \varepsilon R \). Consequently, as

\[
x_0 - x_0' = (x_0 - x_{q_0} + t_q \nabla \Phi_j(\xi_0)) - (x_q - x_{q_0} + t_q \nabla \Phi_j(\xi_0)) + t_q (\nabla \Phi_j(\xi_0) - \nabla \Phi_j(\xi_0')),
\]

and \( |t_q| \lesssim R \) we have by definition of \( \sigma(x_0, \xi_0) \),

\[
|x_0 - x_0'| + R|\nabla \Phi_j(\xi_0) - \nabla \Phi_j(\xi_0')| \lesssim \sigma(x_0, \xi_0) + \sigma(x_0', \xi_0') + 2R|\nabla \Phi_j(\xi_0) - \nabla \Phi_j(\xi_0')|
\]

\[
\lesssim \sigma(x_0, \xi_0) + \sigma(x_0', \xi_0') + \frac{1}{\varepsilon} |(t_q - t_{q_0})(\nabla \Phi_j(\xi_0) - \nabla \Phi_j(\xi_0'))|
\]

\[
\lesssim \frac{1}{\varepsilon} \sigma(x_0', \xi_0').
\]

We now use the fact that \( \xi_0, \xi_0' \in \Sigma_j(h) + \frac{\xi}{r_j} \), together with (A1) to deduce that

\[
|\xi_0 - \xi_0'| \lesssim \frac{|\nabla \Phi_j(\xi_0) - \nabla \Phi_j(\xi_0')|}{\mathcal{H}_j} + \frac{1}{r_j}.
\]
Therefore, since \(\sigma(x_0, \xi_0) \leq \sigma(x_0', \xi_0')\), we obtain
\[
\sum_{(x_0', \xi_0') \in \Gamma_j(b, \Lambda)} \left(1 + \frac{\sigma(x_0', \xi_0')}{r_j}\right)^{-3n} \lesssim \sum_{(x_0, \xi_0) \in \Gamma_j(b, \Lambda)} \left(1 + \frac{\epsilon}{r_j} |x_0 - x_0'| + \epsilon r_j |\xi_0 - \xi_0'|\right)^{-3n} \lesssim \epsilon^{-C}.
\]
On the other hand, if \(\sigma(x_0, \xi_0) \gtrsim \epsilon R\), then we gain a power of \(R\), and can simply discard the \(\xi_0\) sum, and sum up over the points \(x_0\). More precisely, the assumption on the set \(\Lambda\), together with the condition \(d_0 H_j \lesssim 1\), implies that
\[
\# \left\{\xi_0 \in \frac{1}{r_j} \mathbb{Z}^n \mid \xi_0 \in \Sigma_j(b) \cap \Lambda + \frac{\xi}{r_j} \right\} \lesssim \frac{\sigma_{\Sigma_j}(\Sigma_j(b) \cap \Lambda) r_j^{-1} + r_j^{-n}}{r_j^{-n}} \lesssim (1 + d_0 r_j)^{n-1} \lesssim (1 + \frac{R}{r_j})^{n-1}
\]
and hence, as \(\sigma(x_0, \xi_0) \gtrsim \epsilon R\), we obtain
\[
\sum_{(x_0', \xi_0') \in \Gamma_j(b, \Lambda)} \left(1 + \frac{\sigma(x_0', \xi_0')}{r_j}\right)^{-3n} \lesssim \epsilon^{-C} \sup_{\xi_0} \sum_{x_0'} \left(1 + \frac{|x_0 - x_0'| + t_q \nabla \Phi_j(\xi_0')|}{r_j}\right)^{-2n} \lesssim \epsilon^{-C}.
\]
\[\square\]

9. The Wave Table Construction

As implicitly used in the work of Wolff [27], and more explicitly in the work of Tao [21], we need to decompose \(u\) depending on \(v\). Roughly speaking we will decompose \(u\) into wave packets, and keep the wave packets where \(v\) and \(\mathcal{P}_{\gamma_j} u\) simultaneously concentrate on the same cube \(B \in Q_{\mathbb{R}}(Q)\). We adapt the following construction from Tao, but change notation slightly.

**Definition 9.1 (\(\Phi_j\)-Wave Tables).** Let \(0 < \epsilon < 1\). Let \(0 < \|F\|_{L^\infty_t L^2_x} < \infty\), and \(Q\) be a cube of diameter \(R \gg (d_0^2 H_j)^{-1}\). Let \(u\) be a \(\Phi_j\)-wave. The \(\Phi_j\)-wave table of \(u\), relative to \(F\) and \(Q\), is the collection of (vector valued) functions \((\mathcal{W}_{\gamma_j}^{(B)}(u; F, Q))_{B \in Q_{\mathbb{R}}(Q)}\) where
\[
\mathcal{W}_{\gamma_j}^{(B)}(u; F, Q) = \sum_{\gamma_j \in \Gamma_j} \frac{m_{\gamma_j, B}(F)}{m_{\gamma_j}(B)} \mathcal{P}_{\gamma_j} u
\]
where the coefficients are defined as
\[
m_{\gamma_j, B}(F) = \sum_{q \in Q_{\gamma_j}} \frac{1}{w_{\gamma_j,q}} \|\chi_q F\|_{L^2_{t,x}(B^1+n)}^2, \quad m_{\gamma_j}(F) = \sum_{B \in Q_{\mathbb{R}}(Q)} m_{\gamma_j, B}(F).
\]

**Remark 9.2.** The assumption \(0 < \|F\|_{L^\infty_t L^2_x} < \infty\) implies that \(0 < \|\chi_q F\|_{L^2_{t,x}} < \infty\), and hence the coefficients \(m_{\gamma_j, B}(F)\) are well-defined and satisfy \(0 < m_{\gamma_j, B}(F) < \infty\). Note that we essentially have \(m_{\gamma_j, B}(F) \approx \|F\|_{L^2_{t,x}(T_{\gamma_j} \cap B)}^2\) and \(m_{\gamma_j}(F) \approx \|F\|_{L^2_{t,x}(T_{\gamma_j} \cap Q)}^2\). Thus \(\mathcal{W}_{\gamma_j}^{(B)}(u; F, Q)\) contains all wave packets \(\mathcal{P}_{\gamma_j} u\) such that \(F|_{T_{\gamma_j}}\) is concentrated on \(B\).

Recall that if \(Q\) is a cube of diameter \(R\), \(0 < r \leq R\), and \((u^{(B)})_{B \in Q_{\mathbb{R}}(Q)}\) is a collection of \((\ell^2_{c}(\mathbb{Z})\text{ valued})\) functions, we defined the **quilt** \([u^{(\cdot)}]\) as
\[
[u^{(\cdot)}] = \sum_{B \in Q_{\mathbb{R}}(Q)} 1_B|u^{(B)}|
\]
and the \(\epsilon\) separated cubes
\[
I^{c,r}(Q) = \bigcup_{q \in Q_{\mathbb{R}}(Q)} (1-\epsilon)q.
\]
The definition of the wave tables \(\mathcal{W}_{\gamma_j}^{(B)}(u; F, Q)\) gives the following key bilinear estimate.

\(^3\text{Since }\chi_q \text{ has compact Fourier support, it can only have countable number of zeros. In particular, }\chi_q > 0 \text{ almost everywhere.}\)
Theorem 9.3. Let \( \{j, k\} = \{1, 2\} \), \( 0 < \epsilon \ll 1 \), and \( R \gg (d^3_\phi H_j)^{-1} \). Assume that the phases \( \Phi_1 \) and \( \Phi_2 \) satisfy Assumption [7]. Let \( u \) be a \( \Phi_j \)-wave and \( v \) be a \( \Phi_k \)-wave with

\[
\text{supp } \tilde{u} + \frac{d}{2} \subset \Lambda_j, \quad \text{supp } \tilde{v} + \frac{d}{2} \subset \Lambda_k, \quad d[\text{supp } \tilde{u}, \text{supp } \tilde{v}] \lesssim d_0.
\]

Let \( Q \) be a cube of diameter \( R \), and for \( B \in Q_\Phi (Q) \), take \( \mathcal{W}^{(B)}_{j, \epsilon} = \mathcal{W}^{(B)}_{j, \epsilon}(u; v, Q) \). Then \( \mathcal{W}^{(B)}_{j, \epsilon} \) is again a \( \Phi_j \)-wave, is linear in \( u \), satisfies the support condition

\[
\text{supp } \hat{\mathcal{W}}^{(B)}_{j, \epsilon} \subset \text{supp } \hat{u} + \frac{1}{r_j},
\]

and we have the energy estimate

\[
\left( \sum_{B \in Q_\Phi (Q)} \left\| \mathcal{W}^{(B)}_{j, \epsilon} \right\|_{L^2_t L^2_x}^2 \right)^{\frac{1}{2}} \leq (1 + C\epsilon) \| u \|_{L^\infty_t L^2_x}.
\]

Moreover, we have the bilinear estimate

\[
\left\| \left( \left| u - \mathcal{W}^{(j)}_{\epsilon} \right| \right) v \right\|_{L^2_t L^2_x} \lesssim \sup_{B \in Q_\Phi (Q)} \left\| u \right\|_{L^2_t L^2_x} \left\| v \right\|_{L^2_t L^2_x}.
\]

Proof. For ease of notation, for the remainder of the proof, we let \( u^{(B)} = \mathcal{W}^{(B)}_{j, \epsilon}(u; v, Q) \). By construction, we have \( \sum_{B \in Q_\Phi (Q)} \frac{m_{\gamma_j}(v)}{m_{\gamma_j}(v)} = 1 \) and hence \( u = \sum_{B \in Q_\Phi (Q)} u^{(B)} \). The initial claims follow immediately from the definition of the wave table \( \left( \mathcal{W}^{(B)}_{j, \epsilon} \right)_{B \in Q_\Phi (Q)} \), together with Lemma 6.3 and Proposition 6.6 (note that, since \( u \) is a \( \Phi_j \)-wave, we have \( \| u \|_{L^\infty_t L^2_x} = \| u(0) \|_{L^2_x} \)). We now turn to the proof of the bilinear estimate. The identity \( u = \sum_{B \in Q_\Phi (Q)} u^{(B)} \) implies that

\[
\| u \|_{L^\infty_t L^2_x} - \| u^{(j)} \| \leq \sum_{B \in Q_\Phi (Q)} \| u^{(B)} \|_{L^\infty_t L^2_x}.
\]

Consequently, as there are only \( 4^{n+1} \) cubes in \( Q_\Phi (Q) \), the separation of cubes inside \( I^r \hat{\Phi} \) implies that

\[
\left\| \left( \left| u - \mathcal{W}^{(j)}_{\epsilon} \right| \right) v \right\|_{L^2_t L^2_x} \lesssim \sup_{B \in Q_\Phi (Q)} \left\| u^{(B)} \right\|_{L^2_t L^2_x} \lesssim \sum_{q \in Q_j (Q)} \left\| u^{(B)} \right\|_{L^2_t L^2_x}.\]

For \( \mathfrak{h} = (a, h) \in \mathbb{R}^{1+n} \), we let \( H_\mathfrak{h} \) be a Fourier projection onto the set \( \{ |\tau - a| \leq \frac{1}{r_j}, |\xi - h| \leq \frac{1}{r_j} \} \) such that

\[
\| F \|_{L^2_t L^2_x}^2 \approx \sum_{\mathfrak{h} \in \mathbb{R}^{1+n}} \| H_\mathfrak{h} F \|_{L^2_t L^2_x}^2.
\]

Observe that given \( \gamma_j = (x_0, \xi_0) \in \Gamma_j \) and \( \mathfrak{h} = (a, h) \in \mathbb{R}^{1+n} \), we only have \( H_\mathfrak{h}(\chi_{\gamma_j} P_{\gamma_j} u \chi_{\gamma_j} v) \neq 0 \) if there exists \( \xi \in \Lambda_j, \eta \in \Lambda_k \) such that

\[
|\xi_0 - \xi| \lesssim \frac{1}{r_j}, \quad |\xi + \eta - h| \lesssim \frac{1}{r_j}, \quad |\Phi_j(\xi) + \Phi_k(\eta) - a| \lesssim \frac{1}{r_j}.
\]

Therefore, an application of Lemma 7.8 implies that if \( H_\mathfrak{h}(\chi_{\gamma_j} P_{\gamma_j} u \chi_{\gamma_j} v) \neq 0 \) we must have \( \xi_0 \in \Sigma_j (\mathfrak{h}) \cap \Lambda + \frac{L}{r_j} \) with \( \Lambda = \text{supp } \tilde{u} \cap (h - \text{supp } \tilde{v}) \). Consequently we have

\[
\| u^{(B)} \|_{L^2_t L^2_x} \lesssim \| \chi_{\gamma_j} u^{(B)} \chi_{\gamma_j} v \|_{L^2_t L^2_x} \lesssim \sum_{\mathfrak{h} \in \mathbb{R}^{1+n}} \| \sum_{\gamma_j \in \Gamma_j(b, \Lambda)} \frac{m_{\gamma_j} B(v)}{m_{\gamma_j}(v)} H_\mathfrak{h}(\chi_{\gamma_j} P_{\gamma_j} u \chi_{\gamma_j} v) \|_{L^2_t L^2_x}^2.
\]

Since \( \frac{m_{\gamma_j} B(v)}{m_{\gamma_j}(v)} \leq \left( \frac{m_{\gamma_j} B(v)}{m_{\gamma_j}(v)} \right)^{\frac{1}{2}} \), an application of Hölder’s inequality gives

\[
\| \sum_{\gamma_j \in \Gamma_j(b, \Lambda)} \frac{m_{\gamma_j} B(v)}{m_{\gamma_j}(v)} H_\mathfrak{h}(\chi_{\gamma_j} P_{\gamma_j} u \chi_{\gamma_j} v) \|_{L^2_t L^2_x}^2 \lesssim \sum_{\gamma_j \in \Gamma_j(b, \Lambda)} \frac{w_{\gamma_j}(v)}{m_{\gamma_j}(v)} \| H_\mathfrak{h}(\chi_{\gamma_j} P_{\gamma_j} u \chi_{\gamma_j} v) \|_{L^2_t L^2_x}^2 \lesssim \sup_{\gamma_j \in \Gamma_j(b, \Lambda)} \sum_{\gamma_j \in \Gamma_j(b, \Lambda)} \frac{m_{\gamma_j} B(v)}{w_{\gamma_j}(v)}.
\]
Hence summing up over $b \in \frac{1}{r_j} \mathbb{Z}^{1+n}$, and noting that the product $\chi_q P_{\gamma_j} u$ has Fourier support in a set of size $r_j^{-(n+1)}$, we deduce that

$$\sum_{q \in Q_{\gamma_j}} \left\| u(B) v \right\|^2_{L^2_t L^2_x(q)} \lesssim \left( \sum_{q \in Q_{\gamma_j}} \sum_{\gamma_j \in \Gamma_j} \frac{w_{\gamma_j,q}}{m_{\gamma_j}(v)} \| \chi_q P_{\gamma_j} u \|_{L^2_t L^2_x}^2 \right) \cdot \left( \sup_{b \in \mathbb{R}^{1+n}} \sum_{q \in Q_{\gamma_j}} \frac{m_{\gamma_j,B}(v)}{w_{\gamma_j,q}} \right) \lesssim r_j^{-(n+1)} \left( \sum_{q \in Q_{\gamma_j}} \sum_{\gamma_j \in \Gamma_j} \frac{w_{\gamma_j,q}}{m_{\gamma_j}(v)} \| \chi_q P_{\gamma_j} u \|_{L^2_t L^2_x}^2 \right) \cdot \left( \sup_{b \in \mathbb{R}^{1+n}} \sum_{q \in Q_{\gamma_j}} \frac{m_{\gamma_j,B}(v)}{w_{\gamma_j,q}} \right).$$

(9.1)

We now estimate the first term in (9.1). The definition of the coefficients $m_{\gamma_j}(v)$ together with Proposition 6.6 gives

$$\sum_{q \in Q_{\gamma_j}} \sum_{\gamma_j \in \Gamma_j} \frac{w_{\gamma_j,q}}{m_{\gamma_j}(v)} \| \chi_q P_{\gamma_j} u \|_{L^2_t L^2_x}^2 \lesssim \left( \sum_{\gamma_j \in \Gamma_j} \sup_{q \in Q_{\gamma_j}} \frac{w_{\gamma_j,q}^2}{m_{\gamma_j}(v)} \| \chi_q P_{\gamma_j} u \|_{L^2_t L^2_x}^2 \right) \cdot \left( \sup_{\gamma_j \in \Gamma_j} \frac{1}{m_{\gamma_j}(v)} \sum_{q \in Q_{\gamma_j}} \frac{1}{w_{\gamma_j,q}} \| \chi_q v \|_{L^2_t L^2_x}^2 \right) \lesssim r_j e^{-C} \| u \|^2_{L^2_t L^2_x}.$$

On the other hand, to estimate the second term in (9.1), expanding the sum over the $m_{\gamma_j,B}(v)$, we have

$$\sum_{\gamma_j \in \Gamma_j} \frac{m_{\gamma_j,B}(v)}{w_{\gamma_j,q}} = \sum_{q_0 \in Q_{\gamma_j}(B)} \sum_{\gamma_j \in \Gamma_j} \frac{1}{w_{\gamma_j,q} w_{\gamma_j,q_0}} \| \chi_{q_0} v \|_{L^2_t L^2_x(R^{1+n})}^2.$$

Since $\chi_{q_0}$ has Fourier support in a ball of radius $r_j^{-1}$, by orthogonality, we may assume that $\hat{v}$ is also supported in a ball of radius $r_j^{-1}$. Consequently, an application of Lemma 8.2, translation invariance, and Theorem 8.1 implies that

$$\sum_{\gamma_j \in \Gamma_j} \frac{m_{\gamma_j,B}(v)}{w_{\gamma_j,q}} \lesssim r_j \| v \|_{L^\infty_t L^2_x}^2.$$

Therefore the required bilinear estimate follows. \hfill \square

10. PROOF OF THEOREM 5.1

We are now ready to give the proof of Theorem 5.1. Roughly the idea is that, given $R \gg (d_0^2 H_j)^{-1}$, Theorem 9.3 allows us to essentially replace the (low frequency) term $u$ with pieces concentrated on $4$ cubes. However, since Theorem 9.3 applied to $u$ only requires scales $\gg d_0^{-2} (\text{as } H_1 = 1)$ we can continue decomposing $u$ until we get terms concentrated on the much smaller $H_2 R$ cubes. We then decompose $v$, but as Theorem 9.3 applied to $v$ requires $R \gg (d_0^2 H_j)^{-1}$, we can only apply it once before we start to lose logarithmic factors. The immediate obstruction to applying the above strategy, is that Theorem 9.3 only applies to the separated cubes $I^{r^*}(Q)$. To create the required separation, we use an averaging argument due to Tao.

Suppose $Q$ is a cube of radius $R$, and let $(\epsilon_m)$ and $(r_m)$ be strictly positive sequences with $r_m \leq R$. We then define

$$X_0 = \bigcap_{m=1}^M I^{r_m} \cap (Q) \setminus \bigcup_{q \in Q, (Q)} (1 - \epsilon) Q,$$

where we recall that $I^{r^*}(Q) = \bigcup_{q \in Q} (Q) (1 - \epsilon)$. Thus cubes inside $X_0 (Q)$ are separated at multiple scales (which is needed as we apply Theorem 9.3 at multiple scales). We will need to move from integrating over $Q$, to integrating over the smaller $X_0 (Q)$. The key tool is the following averaging lemma due to Tao.
Lemma 10.1 ([13, Lemma 4.1], [21, Lemma 6.1]). Let \(1 \leq q, r \leq \infty, R > 0\) and let \(Q_R\) be a cube of diameter \(R.\) If \(\epsilon = \sum_{m=1}^{M} \epsilon_j \leq 2^{-(n+2)}\) then there exists a cube \(Q \subset 4Q_R\) of side lengths \(2R\) such that

\[
\|F\|_{L^1_lL^r_x(Q_R)} \leq \left(1 + 2^{n+2}\epsilon\right)\|F\|_{L^1_lL^r_x(Q_R \cap X(Q))}.
\]

(10.1)

Proof. We start by considering the case \(q = r = 1.\) Let \(Q(t, x)\) denote the cube of side lengths \(2R\) centered at the point \((t, x) \in Q_R.\) It is enough to prove that

\[
\|F\|_{L^1(Q_R)} \leq \left(1 + 2^{n+2}\epsilon\right)\frac{1}{\epsilon}\int_{Q_R} \|F\|_{L^1(Q_R \cap X(Q(t, x)))} dt dx.
\]

(10.2)

To this end, we begin by observing that since \(4Q\) we have for every \((t', x') \in Q\)

\[
\int_{Q_R} 1_{X[Q(t, x)]}(t', x') dt dx = \{(t', x') \in Q \mid (t', x') \in X[Q(t, x)]\} = |Q_R \cap X(Q(t', x'))|.
\]

Thus by an application of Fubini we have

\[
\int_{Q_R} \|F\|_{L^1(Q_R \cap X(Q(t, x)))} dt dx = \int_{Q_R} \int_{Q_R} 1_{X[Q(t, x)]}(t', x') dt dx |F(t', x')| dt' dx'
\]

\[
= \int_{Q_R} |Q_R \cap X(Q(t', x'))| |F(t', x')| dt' dx'
\]

Consequently the required bound \(10.2\) would follow from the inequality

\[
|Q_R| \leq (1 + 2^{n+2}\epsilon)|Q_R \cap X(Q(t, x))|.
\]

(10.3)

Let \(X_k = \cap_{m=k}^M I^{t_i, r_i}(Q(t, x)).\) Noting the inclusion \(Q_R \subset Q(t, x),\) we deduce that

\[
|Q_R \cap X_1| = |Q_R \cap I^{t_i, r_i}(Q(t, x)) \cap X_2|
\]

\[
= \sum_{q \in Q_{r_1}(Q(t, x))} |(1 - \epsilon_1)q \cap Q_R \cap X_2|
\]

\[
\geq \sum_{q \in Q_{r_1}(Q(t, x))} |q \cap Q_R \cap X_2| - \sum_{q \in Q_{r_1}(Q(t, x))} |q \setminus (1 - \epsilon_1)q|
\]

\[
= |Q_R \cap X_2| - (1 - (1 - \epsilon_1)^n)(2R)^n.
\]

Since \(1 - (1 - \epsilon_1)^n \leq n\epsilon_1,\) repeating the above argument eventually gives

\[
|Q_R \cap X(Q(t, x))| \geq |Q_R| - n2^n R^n \sum_{m=1}^{M} \epsilon_m \geq (1 - 2^{n+1}\epsilon)|Q_R|
\]

and so, provided that \(\epsilon \leq 2^{-(n+2)}\) we obtain \(10.3.\) Thus the case \(q = r = 1\) follows. The general case follows by observing that there exists \(G \in L^q_l L^r_x(Q_R)\) with \(\|G\|_{L^q_lL^r_x(Q_R)} \leq 1\) such that

\[
\|F\|_{L^1_lL^r_x(Q_R)} \leq \|FG\|_{L^1_lL^r_x(Q_R)}
\]

together with an application of the \(L^1_l\) case obtained above. \(\square\)

We now come to the proof of Theorem 5.1.

Proof. It is enough to consider the case \(\|u\|_{L^q_l L^r_x} = \|v\|_{L^q_l L^r_x} = 1.\) Fix a cube \(Q_R\) of diameter \(R \gg (d_0^3 H_2)^{-1}\) and let \(M \in \mathbb{N}\) such that

\[
4^{-M} < H_2 \leq 4^{1-M}.
\]

An application of Lemma 10.1 implies that there exists a cube \(Q\) of radius \(2R\) such that

\[
\|uv\|_{L^q_l L^r_x(Q_R)} \leq (1 + C\epsilon)\|uv\|_{L^q_l L^r_x(X(Q))}
\]

4Just use the identities \(X[Q(t, x)] = X[Q(0)] + (t, x)\) and \(-X[Q(0)] = X[-Q(0)] = X[Q(0)].\)
and assuming we have constructed $u$ and the cube $Q$ and finally take $u(B) = u(B)$ for $B \in Q_2(Q)$. Note that each $v(B)$ is well-defined, since the diameter of the cube $B_m$ is $4^{-m}2R \geq 4^{-M}2R > d_0$. To construct $v$, we start by defining $U = (u(B))_{B \in Q_2(Q)}$. Clearly, as each $v(B)$ is a $\Phi$-wave, after relabeling, $U$ is also a $\Phi$-wave. We now decompose $v$ relative to $U$ and the cube $Q$, in other words for $B' \in Q_2(Q)$ we define

$$v(B') = W_{2,e}(v; U, Q).$$

It is easy to check that, by construction, Theorem 9.3 implies that we have the identities

$$u = \sum_{B \in Q_2(Q)} u(B), \quad v = \sum_{B' \in Q_2(Q)} v(B')$$

together with the support conditions

$$\text{supp } \hat{u}_m \subset \text{supp } \hat{u}_{m-1} + \left( \frac{2R}{4^{m-1}} \right)^{-\frac{1}{2}}, \quad \text{supp } \hat{v}(B') \subset \text{supp } \hat{v} + \left( 2H_2R \right)^{-\frac{1}{2}}.$$

Moreover, $v(B')$ satisfies the required energy estimate, and by iterating the estimate

$$\sum_{B_m \in Q_2^M(Q)} \|u(B_m)\|^2_{L^\infty L^2} = \sum_{B_{m-1} \in Q_2^M(Q)} \sum_{B_m \in Q_2^M(B_{m-1})} \|u(B_m)\|^2_{L^\infty L^2} \lesssim (1 + \epsilon_m C)^2 \sum_{B_{m-1} \in Q_2^M(Q)} \|u(B_{m-1})\|^2_{L^\infty L^2},$$

and using the elementary inequality $\Pi^{M}_{m=1}(1 + \epsilon_m C) \leq 1 + e^{\sum \epsilon_m C} \sum \epsilon_m$ and the definition of $\epsilon_m$, a short computation shows that $u(B)$ satisfies the correct energy bound.

We now turn to the proof of the bilinear estimate. Writing

$$|uv| = |u| |v| + |u - [u]| |v| + |u| |v - [v]|,$$

after an application of Holder’s inequality in the $t$ variable, it is enough to show that for every $2 \geq r > \frac{n+1}{n}$ we have

$$\left\| (|u| - |u|) |v| \right\|_{L^r L^\infty(X)} + \left\| |u| |v - [v]| \right\|_{L^r L^\infty(X)} \lesssim \epsilon^{-C}(H_2R)^{\frac{n+1}{2r} - \frac{n}{2}} \quad (10.4)$$

We start by estimating the first term. An application of Theorem 9.3 gives

$$\left\| \left( |u| - |u| \right) |v| \right\|_{L^r L^\infty(X)} \lesssim \epsilon^{-2C} \left( \frac{4^{m-1}}{2R} \right)^{\frac{n+1}{2r}} \sum_{B_{m-1} \in Q_2^M(B_{m-1})} \|v(B_{m-1})\|^2_{L^\infty L^2} \lesssim \epsilon^{-2C} \left( \frac{4^{m-1}}{2R} \right)^{\frac{n-1}{2r}} \sum_{B_{m-1} \in Q_2^M(B_{m-1})} \|v(B_{m-1})\|^2_{L^\infty L^2} \lesssim \epsilon^{-2C} \left( \frac{4^{m-1}}{2R} \right)^{\frac{n-1}{2r}} \sum_{B_{m-1} \in Q_2^M(B_{m-1})} \|v(B_{m-1})\|^2_{L^\infty L^2} \lesssim \epsilon^{-2C} \left( \frac{4^{m-1}}{2R} \right)^{\frac{n-1}{2r}} \sum_{B_{m-1} \in Q_2^M(B_{m-1})} \|v(B_{m-1})\|^2_{L^\infty L^2}.$$
where we used the definition of $M$. On the other hand, an application of Holder’s inequality together with the energy bound for $u_m$ and $u_{m-1}$ implies that

$$
\left\| \left[ u^{(l)}_m \right] - \left[ u^{(l)}_{m-1} \right] \right\|_{L_t^2 L^2_x(\mathbb{R}^n)} \leq \left\| \left[ u^{(l)}_m \right] - \left[ u^{(l)}_{m-1} \right] \right\|_{L_t^2 L^2_x(\mathbb{R}^n)} \leq \left( \sum_{B_{m-1}} |u^{(B_{m-1})}_{m-1}|^2 \right)^{1/2} + \sum_{B_m} |u^{(B_m)}_m|^2 \right)^{1/2} \lesssim \left( 4^{-m} R \right)^{1/2}
$$

Hence interpolating gives for any $1 \leq r \leq 2$

$$
\left\| \left[ u^{(l)} - \left[ u^{(l)}_{m-1} \right] \right] \right\|_{L_t^2 L^2_x(\mathbb{R}^n)} \lesssim \epsilon_m^{-2C(1^{1/4})} (4^{-m} R)^{\frac{n+1}{2r} - \frac{n}{4}}.
$$

The definition of $\epsilon_m$ and $M$ implies that we can write

$$
\epsilon_m - C(n-1^{1/4}) \frac{m}{2(n+1)} - C_\delta (1 - \frac{1}{p}) > 0 \quad \text{provided} \quad r > \frac{n+1}{m} \quad \text{and we choose } \delta \text{ sufficiently small. Therefore,}
$$

$$
\left\| \left[ u^{(l)} - \left[ u^{(l)}_{m-1} \right] \right] \right\|_{L_t^2 L^2_x(\mathbb{R}^n)} \lesssim \epsilon_m^{-2C(1^{1/4})} (4^{-m} R)^{\frac{n+1}{2r} - \frac{n}{4}}.
$$

Thus it only remains to estimate the second term in (10.4). To this end, applying the definition of $v^{(B)}$ together with Theorem 9.3, we have

$$
\left\| \left[ v^{(l)} - u^{(l)} \right] \right\|_{L_t^2 L^2_x(\mathbb{R}^n)} \lesssim \left\| \left[ v^{(l)} - u^{(l)} \right] \right\|_{L_t^2 L^2_x(\mathbb{R}^n)} \lesssim \left\| \left[ v^{(l)} - u^{(l)} \right] \right\|_{L_t^2 L^2_x(\mathbb{R}^n)} \lesssim \epsilon^{-C(\mathcal{H}_2 R)^{\frac{n+1}{2r} - \frac{n}{4}}}
$$

where we used the inequality $|u^{(l)}| \lesssim (\sum_B |u^{(B)}|^2)^{1/2} = |U|$. On the other hand, an application of Holder’s inequality together with the energy estimates gives

$$
\left\| \left[ v^{(l)} - u^{(l)} \right] \right\|_{L_t^2 L^2_x(\mathbb{R}^n)} \lesssim \left( \sum_{B \in Q, \frac{d}{2} \in \mathbb{N}} \left\| u^{(l)} \right\|_{L_t^2 L^2_x(B)} \right)^{1/2} \left\| v \right\|_{L_t^2 L^2_x} + \left\| v^{(l)} \right\|_{L_t^2 L^2_x} \lesssim \epsilon^{-C(\mathcal{H}_2 R)^{\frac{n+1}{2r} - \frac{n}{4}}}
$$

Therefore, interpolating between these bounds gives

$$
\left\| \left[ v^{(l)} - u^{(l)} \right] \right\|_{L_t^2 L^2_x(\mathbb{R}^n)} \lesssim \epsilon^{-C(\mathcal{H}_2 R)^{\frac{n+1}{2r} - \frac{n}{4}}}
$$

and hence (10.4) follows. \qed

11. Atomic Wave Tables and the Proof of Theorem 1.7

In this section we give the proof of Theorem 1.7, namely the extension of Corollary 1.6 to the atomic spaces $U_{B_1}^a$ and $U_{B_2}^b$ with $a, b \geq 2$. As the proof closely follows the argument used to prove Theorem 1.4 we will be somewhat brief.

We start by defining an atomic $\Phi_j$-wave to be a function of the form

$$
u(t, x) = \sum_{I \in \mathcal{I}} \mathbb{1}_I(t) u_I(t, x)
$$

where $\mathcal{I}$ is a finite partition of $\mathbb{R}$, $\mathbb{1}_I(t)$ is the indicator function adapted to the interval $I \subset \mathbb{R}$, and $u_I$ is a $\Phi_j$-wave. Given an atomic $\Phi_j$-wave $u$, we use the shorthand

$$
\left\| u \right\|_{L_t^\infty L^2_x} = \left( \sum_{I \in \mathcal{I}} \left\| u_I \right\|_{L_t^\infty L^2_x}^2 \right)^{\frac{1}{2}}.
$$
In particular, if \( \|u\|_{L^p L^2} \leq 1 \), then \( u \) is simply a \( U_{\hat{Q}_j}^0 \) atom. Note that an atomic \( \Phi_j \)-wave is essentially a special case of a \( \Phi_j \)-wave, since if we let \( U = (u_I)_{I \in \mathcal{I}} \) then perhaps after relabeling, \( U \) is clearly a \( \Phi_j \)-wave, and moreover, as in the proof of Corollary 11.6, we have the pointwise bound
\[
|u(t, x)| \leq \left( \sum_{I \in \mathcal{I}} |u_I(t, x)|^2 \right)^{\frac{1}{2}} = |U(t, x)| ,
\]
However, by exploiting the time localisation, atomic \( \Phi_j \)-waves satisfy slightly stronger bilinear estimates than those that hold for \( \Phi_j \)-waves. More precisely, we may replace Theorem 11.1 with the following.

**Theorem 11.1.** Let \( C_0 > 0 \), \( \frac{1}{2} \leq \frac{1}{q} \leq 1 \), \( \frac{1}{2} \leq \frac{1}{r} < \frac{n}{n+1} \), and \( \frac{n}{n-1} < \frac{1}{s} \leq \frac{1}{r} \). Assume that we have \( d_0 > 0 \), open sets \( \Lambda_j \subset \mathbb{R}^n \), and phases \( \Phi_j \) satisfying (A1), (A2), and the normalisation (4.1). Let \( Q_R \) be a cube of diameter \( R \gg (d_0^2 H_2)^{-1} \). Then for any \( 0 < \epsilon \ll 1 \), any atomic \( \Phi_1 \)-wave \( u = \sum_{I \in \mathcal{I}} \|u_I\|_{L^2}^2 \) and any atomic \( \Phi_2 \)-wave \( v = \sum_{J \in \mathcal{J}} J v_J \) with
\[
supp \hat{u} + \frac{d_0}{2} \subset \Lambda_1 , \quad supp \hat{v} + \frac{d_0}{2} \subset \Lambda_2
\]
there exist a cube \( Q \) of diameter \( 2R \) such that we have a decomposition
\[
u = \sum_{B \in Q_{2R}^M(Q)} u^{(B)} , \quad v = \sum_{B' \in Q_{2R}^M(Q)} v^{(B')}
\]
where \( M \in \mathbb{N} \) with \( 4^{-M} < H_2 \leq 4^{1-M} \), and \( u^{(B)} = \sum_{I \in \mathcal{I}} h_I^{(B)} \) is an atomic \( \Phi_1 \)-wave, \( v^{(B')} = \sum_{J \in \mathcal{J}} J v_J^{(B')} \) is an atomic \( \Phi_2 \)-wave, with the support properties
\[
supp \hat{u}^{(B)} \subset supp \hat{u} + 2(2H_2R)^{-\frac{1}{2}} \), \quad supp \hat{v}^{(B')} \subset supp \hat{v} + 2(2H_2R)^{-\frac{1}{2}}.
\]
Moreover, for any \( a_0, b_0 \geq 2 \) we have the energy bounds
\[
\left( \sum_{B \in Q_{2R}^M(Q)} \|u^{(B)}\|_{L^a_0 L^2}^{a_0} \right)^{\frac{1}{a_0}} \leq (1 + C\epsilon)\|u\|_{L^{a_0} L^2}
\]
\[
\left( \sum_{B' \in Q_{2R}^M(Q)} \|v^{(B')}\|_{L^b_0 L^2}^{b_0} \right)^{\frac{1}{b_0}} \leq (1 + C\epsilon)\|v\|_{L^{b_0} L^2}
\]
and the bilinear estimate
\[
\|uv\|_{L^q L^2(Q_R)} \leq (1 + C\epsilon)\|u^{(1)}\|_{L^q_0 L^2(Q_R)} \|v^{(1)}\|_{L^q_0 L^2(Q_R)}
\]
\[
+ C\epsilon C R^{1+\frac{n-1}{2}} (H_2^2 R^2)^{\frac{1}{2}} \|\Phi_j\|_{L^q_0 L^2} \|\Phi_{j'}\|_{L^q_0 L^2} \left( \sum_{I \in \mathcal{I}} \|J_{I}^{(j)}\|_{L^q_0 L^2} \right) \left( \sum_{I' \in \mathcal{I}} \|J_{I'}^{(j')}\|_{L^q_0 L^2} \right) \|u\|_{L^a_0 L^2} \|v\|_{L^b_0 L^2}
\]
where the constant \( C \) depends only on \( C_0, q, r, \) and \( n \) and we let \( \mu = \min\{\text{diam}(\text{supp} \hat{u}), \text{diam}(\text{supp} \hat{v})\} \).

Note that by taking \( \frac{1}{a} = \frac{1}{q} = \frac{1}{2} \) and \( \frac{1}{b} \geq 1 - \frac{1}{q} \), we essentially recover Theorem 5.2.

We leave the proof of Theorem 11.1 till Section 12 and now turn to the proof of Theorem 11.7. Fix constants \( d_0, C_0 > 0 \), and open sets \( \Lambda_1, \Lambda_2 \subset \mathbb{R}^n \). Let \( \Phi_j \) be phases satisfying (A1) and (A2). As previously, by exploiting dilation and translation invariance, we may assume \( H_2 \leq H_1 \) and the normalisation conditions (4.1). Take sets \( \Lambda_j^* + d_0 \subset \Lambda_j \) such that \( d_0 + \Lambda_1^* + d_0, \Lambda_2^* + d_0 \leq d_0 \frac{1}{C_0} \), and fix \( R_0 \gg \frac{1}{d_0^2 H_2} \) with \( (R_0 H_2)^{-\frac{1}{2}} \approx d_0 \). The constant \( R_0 \) will denote the smallest scale of cubes we consider, while the sets \( \Lambda_1^* \) and \( \Lambda_2^* \) will contain the support of \( \hat{u} \) and \( \hat{v} \) respectively.

**Definition 11.2.** For any \( R \geq R_0 \) and \( 1 \leq a, b, q, r \leq 2 \), we define \( A^*(R) \) to the best constant for which the inequality
\[
\|uv\|_{L^q L^2(Q_R)} \leq A^*(R)\|u\|_{L^{a} L^2} \|v\|_{L^{b} L^2}
\]
holds for all cubes \( Q \subset \mathbb{R}^{1+n} \) of radius \( R \), and all atomic \( \Phi_1 \)-waves \( u \) and atomic \( \Phi_2 \)-waves \( v \) satisfying the support assumption
\[
supp \hat{u} \subset \Lambda_1^* + 4(R_2 H_2)^{-\frac{1}{2}} , \quad supp \hat{v} \subset \Lambda_2^* + 4(R_2 H_2)^{-\frac{1}{2}}.
\]
As in the proof of Theorem 1.4 in Section 4, the key step in the proof of Theorem 1.7 is to prove the following bounds on $A^*(R)$.

**Proposition 11.3.** Let $\mathbf{C}_0 > 0$, $\frac{1}{2} \leq \frac{1}{2} \leq \frac{1}{2} < \frac{n}{n+1}$, $\frac{1}{2} < \frac{1}{2} \leq \frac{1}{2}$ and $\frac{1}{2} + \frac{1}{2} \geq \frac{1}{\min\{q,r\}}$. There exists a constant $C > 0$, such that for any $\mathbf{d}_0 > 0$, any open sets $\Lambda_j \subset \mathbb{R}^n$, any phases $\Phi_j$ satisfying (A1) and (A2) with the normalisation $(4.7)$, and any sets $\Lambda_j + \mathbf{d}_0 \subset \Lambda_j$ with $\mathbf{d}[\Lambda_j + \mathbf{d}_0, \Lambda_j + \mathbf{d}_0] < \frac{\mathbf{d}_0}{\min\{q,r\}}$, we have for every $R \geq R_0$ and $0 < \epsilon \ll 1$

$$A^*(2R) \leq (1 + C \epsilon) A^*(R) + C \epsilon R^{\frac{n}{2} + \frac{n+1}{2} + (n+1)(\frac{1}{2} - \frac{1}{2})} \left( (\mathcal{H}_2 R)^{\frac{n+1}{2}} (\mu + \mathbf{d}_0)^n \right)^{(1 + \frac{1}{2} - \frac{1}{2})}$$

(11.1)

and

$$A^*(2R) \leq C \mathbf{d}_0^{n+1} R^{\frac{n}{2} + \frac{n+1}{2} + (n+1)(\frac{1}{2} - \frac{1}{2})} \left( \mathbf{d}_0^{-n(n+1)} (\mu + \mathbf{d}_0)^n \right)^{(1 + \frac{1}{2} - \frac{1}{2})} \left( \frac{R}{R_0} \right)^{\frac{1}{2}}$$

(11.2)

where $\mu = \min\{\text{diam}(\Lambda_j), \text{diam}(\Lambda_j^*)\}$.

**Proof.** Let $Q$ be a cube of diameter $2R$, and let $u = \sum_{t \in \mathcal{I}} \mathbf{1}_t(t) u_t$ be a $U_\Phi^a$ atom, and $v = \sum_{J \in \mathcal{J}} \mathbf{1}_J(t) v_J$ be a $U_\Phi^b$ atom satisfying the support conditions

$$\text{supp } \hat{u} \subset \Lambda_j^* + 4(\mathcal{H}_22R)^{-\frac{1}{2}}, \quad \text{supp } \hat{v} \subset \Lambda_j^* + 4(\mathcal{H}_22R)^{-\frac{1}{2}}.$$

An application of Theorem 11.1 gives a cube $Q'$ of diameter $4R$, and atomic waves $(u^{(B)})_{B \in \mathcal{Q}_R(Q)}$, $(v^{(B')})_{B' \in \mathcal{Q}_R(Q')}$ such that

$$\|uv\|_{L^q_L^r(Q)} \leq (1 + C \epsilon) \left\|u^{(i)}\right\|_{L^q_L^r(Q')} \left\|v^{(i)}\right\|_{L^q_L^r(Q')} + C \epsilon R^{\frac{n}{2} + \frac{n+1}{2} + (n+1)(\frac{1}{2} - \frac{1}{2})} \left( (\mathcal{H}_2 R)^{\frac{n+1}{2}} (\mu + \mathbf{d}_0)^n \right)^{(1 + \frac{1}{2} - \frac{1}{2})}$$

(11.3)

and the support properties

$$\text{supp } \hat{u}^{(B)} \subset \Lambda_j^* + 4(\mathcal{H}_22R)^{-\frac{1}{2}}, \quad \text{supp } \hat{v}^{(B')} \subset \Lambda_j^* + 4(\mathcal{H}_22R)^{-\frac{1}{2}}$$

where we used the support assumptions on $\hat{u}$ and $\hat{v}$.

To prove (11.1), we let $B' \in \mathcal{Q}_R(Q')$ and define the atomic $\Phi_1$-wave $U^{(B')} = \sum_{t \in \mathcal{I}} \mathbf{1}_t(t) U^{(B')}_1$ with $U^{(B')} = (u^{(B)})_{B \in \mathcal{Q}_R(Q')}$. Then for every $B' \in \mathcal{Q}_R(Q)$ we have an atomic $\Phi_1$-wave $U^{(B')}$ and an atomic $\Phi_2$-wave $v^{(B')}$ satisfying the correct support assumptions to apply the definition of $A^*(R)$. Thus

$$\left\|u^{(i)} v^{(i)}\right\|_{L^q_L^r(Q')} \leq \left( \sum_{B' \in \mathcal{Q}_R(Q')} \|U^{(B')} v^{(B')}\|_{L^q_L^r(Q')} \right)^{\frac{1}{\min\{q,r\}}} \leq A^*(R) \left( \sum_{B' \in \mathcal{Q}_R(Q')} \|U^{(B')}\|_{L^q_L^r(Q')} \right)^{\frac{1}{2}} \left( \sum_{B' \in \mathcal{Q}_R(Q')} \|v^{(B')}\|_{L^q_L^r(Q')} \right)^{\frac{1}{2}} \leq (1 + C \epsilon) A^*(R)$$

where the second line used the assumption $\frac{1}{2} + \frac{1}{2} \geq \frac{1}{\min\{q,r\}}$ and the last applied the energy inequalities in Theorem 11.1. Therefore the induction bound (11.1) follows from an application of (11.3).

We now turn to the proof of (11.2). We begin by observing that again using the bound (11.3) with $\epsilon \approx 1$, the definition of $R_0$, and using the nesting properties of the $L^p$ spaces, it is enough to prove that for every $\frac{1}{2} \leq \frac{1}{2} \leq \frac{1}{2}$ and $\frac{1}{2} \leq \frac{1}{2} \leq \frac{1}{2}$ we have the quilt bound

$$\left\|u^{(i)} v^{(i)}\right\|_{L^q_L^r(Q')} \lesssim \mathbf{d}_0^{\frac{n+1}{2} + (\frac{1}{2} - \frac{1}{2})} (\mathcal{H}_2 R)^{\frac{n+1}{2} + (n+1)(\frac{1}{2} - \frac{1}{2})} \|u\|_{L^q} \|v\|_{L^r}$$

(11.4)
To this end, we first note that applying Hölder’s inequality gives for any $a_0 \geq 2$
\[
\|u^{(i)}[v^{(j)}]\|_{L^2_t L^2_x(Q')} \lesssim \|u^{(i)}\|_{L^\infty_t L^2_x(Q')} \|v^{(j)}\|_{L^\infty_t L^2_x(Q')}
\lesssim (RH_2)^{\frac{1}{2}}(\mathcal{H}_2)^{\frac{(n+1)(\frac{1}{2a_0} - \frac{1}{2})}{2}} \left( \sum_{B \in \mathbb{Q}_2 R^{Q'}} \|u^{(i)}\|_{L^\infty_t L^2_x(Q')} \right)^{\frac{1}{a_0}} \sup_{B' \in \mathbb{Q}_2 R^{Q'}} \|v^{(j)}\|_{L^\infty_t L^2_x(Q')} \lesssim (RH_2)^{\frac{1}{2}}(\mathcal{H}_2)^{\frac{(n+1)(\frac{1}{2a_0} - \frac{1}{2})}{2}} \|u\|_{\ell^a_0 L^2_t} \|v\|_{\ell^\infty L^2_t}.
\] (11.5)

Similarly, we have the $L^2_t L^1_x$ bound
\[
\|u^{(i)}[v^{(j)}]\|_{L^2_t L^1_x(Q')} \lesssim (\mathcal{H}_2)^{\frac{(n+1)(\frac{1}{2a_0} - \frac{1}{2})}{2}} (RH_2)^{\frac{1}{2}} \left( \sum_{B \in \mathbb{Q}_2 R^{Q'}} \|u^{(i)}\|_{L^\infty_t L^2_x(B)} \right)^{\frac{1}{a_0}} \lesssim (\mathcal{H}_2)^{\frac{(n+1)(\frac{1}{2a_0} - \frac{1}{2})}{2}} (RH_2)^{\frac{1}{2}} \left( \mu + d_0 \right)^{\frac{1}{2}} \left( \sum_{B \in \mathbb{Q}_2 R^{Q'}} \|u^{(i)}\|_{L^\infty_t L^2_x(B)} \right)^{\frac{1}{a_0}} \lesssim (\mathcal{H}_2)^{\frac{(n+1)(\frac{1}{2a_0} - \frac{1}{2})}{2}} (RH_2)^{\frac{1}{2}} \left( \mu + d_0 \right)^{\frac{1}{2}} \|u\|_{\ell^a_0 L^2_t} \|v\|_{\ell^\infty L^2_t}.
\] (11.6)

On the other hand, as in the proof of Proposition 5.2, an application of Theorem 5.2 gives
\[
\|u^{(i)}[v^{(j)}]\|_{L^2_t L^1_x(Q')} \lesssim \left( \sum_{l \in I} \sum_{j \in J} \sum_{B \in \mathbb{Q}_{2^l} R^{Q'}} \left( \sum_{B' \in \mathbb{Q}_{2^l} R^{Q'}} \|u^{(i)}(B')\|^2_{L^2_t L^2_x(B')} \right)^\frac{1}{2} \right)^\frac{1}{2} \lesssim (d_0^{-\frac{a_0}{2}} \|u\|_{\ell^2_0 L^2_t} \|v\|_{\ell^2 L^2_t}).
\] (11.7)

Therefore the required bound (11.4) follows by interpolating between (11.5), (11.6), and the bilinear estimate (11.7).

Finally we come to the proof of Theorem 1.7.

Proof of Theorem 1.7. After rescaling, we may assume the normalisation (4.4). The atomic definition of the $U^q_\Phi$ spaces, together with the definition of $A^*(R)$, implies that it is enough to show that for every $m \in \mathbb{N}$ we have
\[
A^*(2^m R_0) \lesssim d_0^{n+1 - \frac{n+1}{2} - \frac{n+1}{2}} H_2^{\frac{1}{2} - \frac{1}{n} + (n+1)(\frac{1}{2a_0} - \frac{1}{2})} \left( d_0^{-\frac{a_0}{2}} \|u\|_{\ell^a_0 L^2_t} \right)^{(1 - \frac{1}{a_0} - \frac{1}{2})},
\] (11.8)

with the implied constant independent of $m$. We now essentially repeat the proof of Theorem 1.4 but use Proposition 11.3 in place of Proposition 4.2 and Proposition 5.2. Let $\delta = \frac{n+1}{2} - (\frac{1}{2} - \frac{n+1}{2}) + \frac{1}{2} + (n+1)(\frac{1}{2a_0} - \frac{1}{2}) + \frac{1}{2} + (n+1)(\frac{1}{2a_0} - \frac{1}{2})$.

Note that by assumption, we have $\delta > 0$. An application of Proposition 11.3 with $R = 2^m R_0$ and $\epsilon = \frac{1}{2} - \frac{\delta}{2}$ implies that
\[
A^*(R)(2^m R_0) \lesssim (1 + 2^{-\frac{\delta}{2}}) A^*(2^{m-1} R_0) + C^{C+1} 2^{-\frac{\delta}{2}} d_0^{n+1 - \frac{n+1}{2} - \frac{n+1}{2}} H_2^{\frac{1}{2} - \frac{1}{n} + (n+1)(\frac{1}{2a_0} - \frac{1}{2})} \left( d_0^{-\frac{a_0}{2}} \|u\|_{\ell^a_0 L^2_t} \right)^{(1 - \frac{1}{a_0} - \frac{1}{2})}.
\]

In particular, since $\delta > 0$, after $m$ applications of the bound (11.1) we have
\[
A^*(2^m R_0) \lesssim A^*(R_0) + d_0^{n+1 - \frac{n+1}{2} - \frac{n+1}{2}} H_2^{\frac{1}{2} - \frac{1}{n} + (n+1)(\frac{1}{2a_0} - \frac{1}{2})} \left( d_0^{-\frac{a_0}{2}} \|u\|_{\ell^a_0 L^2_t} \right)^{(1 - \frac{1}{a_0} - \frac{1}{2})},
\]

where the implied constant depends only on the exponents $q, r, a, b$, the dimension $n$, and $C_0$. The required inequality (11.8) now follows from another application of Proposition 11.3.

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5This is simply an application of complex interpolation. In more detail, we just repeat the proof of the Riesz-Thorin interpolation theorem, thus given a function $g \in L^2(R)$ and $G \in L^\infty_t L^2_x(R^{1+n})$, we define the function $\rho(z)$ for $z \in \mathbb{C}$ as
\[
\rho(z) = \int_{X(Q)} \left( \sum_{j \in J} j(t) \|u_j\|_{L^2_t L^2_x(Q')} \|v_j\|_{L^2_t L^2_x(Q')} \right)^{\frac{1}{2}} \left( \sum_{j \in J} j(t) \|u_j\|_{L^2_t L^2_x(Q')} \|v_j\|_{L^2_t L^2_x(Q')} \right)^{\frac{1}{2}} G dt dx
\]
where $\frac{1}{2} = \frac{1}{p} + \frac{1}{q} - 1$ and $\frac{1}{2} = \frac{1}{p} + \frac{1}{q} - 1$. It is easy to check that $\rho$ is complex analytic when $0 \leq \Re(z) \leq 1$, is at most of exponential growth, and $\rho(0)$ can be bounded by the $L^2_t L^1_x$ estimate, $\rho(1)$ by the $L^\infty_t L^2_x$ estimate. Hence interpolated bound follows from the Hadamard Three-Lines Theorem or Lindelöf’s Theorem.
12. Proof of Theorem [11.1]

Here we give the proof of the atomic wave table decomposition in Theorem [11.1]. The argument follows that used to prove Theorem [5.1] but, as in the proof of Proposition [11.3], we interpolate with an improved $L^2_t L^3_x$ estimate, and an additional $L^2_t L^2_x$ estimate to gain the $ell^a$ and $ell^b$ sums over intervals. Since $ell^a \subset ell^b$ for $b_1 \leq b_2$, it is enough to consider the case $\frac{1}{2^{b_1}} < \frac{1}{2} \leq 1 - \frac{1}{2}$.

Fix a cube $Q_R$ of diameter $R \gg (d^a H_2)^{-1}$ and let $u = \sum_{I \in I} 1_I(t)u_I$ be an atomic $\Phi_1$-wave, and $v = \sum_{J \in J} 1_J(t)v_J$ be an atomic $\Phi_2$-wave with the support conditions

$$\text{supp} \hat{u} + \frac{d_0}{2} \subset \Lambda_1, \quad \text{supp} \hat{v} + \frac{d_0}{2} \subset \Lambda_2.$$

An application of Lemma [10.1] implies that there exists a cube $Q$ of radius $2R$ such that

$$\|uv\|_{L^a_t L^\infty_x(Q)} \leq (1 + C\epsilon) \|uv\|_{L^a_t L^\infty_x(X[Q])}$$

where as in Theorem [5.1] we take

$$X[Q] = \bigcap_{m=1,...,M} I_{m,A^{-m}2R}(Q), \quad \epsilon_m = 4^{\delta(m-M)}\epsilon$$

and $\delta > 0$ is some fixed constant to be chosen later (which will depend only on the dimension $n$, and the constant $C_n$ appearing in Theorem [9.3] and we take $M \in \mathbb{N}$ such that $4^{-M+1} < H_2 \leq 4^{-M}$.

We start by decomposing the components $u_I$ of $u$. Let $V = (v_J)_{J \in J}$, thus $V$ is a $\Phi_2$-wave such that $|v| \leq |V|$. Given $B_1 \in Q_{\frac{R}{2}}(Q)$ we let

$$u_{I,1}^{(B_1)} = W_{1,e_1}(u_I;V,Q)$$

(whence $W$ is as in Definition [9.1]) and we assume we have constructed $u_{I,m}^{(B_m)}$, $B_m \in Q_{\frac{R}{2^m}}(Q)$, we define for $B_{m+1} \in Q_{\frac{R}{2^{m+1}}}(B_m)$

$$u_{I,m+1}^{(B_{m+1})} = W_{1,e_{m+1}}(u_{I,m}^{(B_m)};V,B_m).$$

To extend this to the atomic waves, we simply take $u_m^{(B_m)} = \sum_{I \in I} 1_I(t)u_{I,m}^{(B_m)}$. Finally, for $B \in Q_{\frac{R}{2^n}}(Q)$, we let $u^{(B)} = u^{(B)}_M$. Clearly $u^{(B)}$ is again an atomic $\Phi_1$-wave, and as in the proof Theorem [5.1] from an application of Theorem [9.3] the pointwise decomposition and support properties of $u^{(B)}$ follow immediately. On the other hand, the energy inequality follows by exchanging the order of summation, using the fact that $a \geq 2$, and applying the energy estimate in Theorem [9.3]

$$\left( \sum_{B \in Q_{\frac{R}{2^n}}(Q)} \|u^{(B)}\|^a_{L^a_t L^2_x} \right)^{\frac{1}{a}} \leq \left( \sum_{I \in I} \left( \sum_{B \in Q_{\frac{R}{2^n}}(Q)} \|u^{(B)}\|^2_{L^\infty_t L^2_x} \right)^{\frac{1}{2}} \right)^{\frac{1}{a}} \leq (1 + C\epsilon) \|u\|_{L^a_t L^2_x}.$$

The next step is to decompose $v = \sum_{J \in J} 1_J(t)v_J$. Let $U = (u_I^{(B)})_{I \in I,B \in Q_{\frac{R}{2^n}}}$. Then $U$ is a $\Phi_1$-wave with the pointwise bound $|u^{(B)}| \leq |U|$ and the energy bound $\|U\|_{L^\infty_t L^2_x} \lesssim \|u\|_{L^a_t L^2_x}$. We now decompose each $v_J$ relative to $U$ and the cube $Q$, in other words for every $B' \in Q_{\frac{R}{2^n}}(Q)$ we take

$$v_{J}^{(B')} = W_{2,e}(v_J;U,Q)$$

and finally define $v^{(B')} = \sum_{J \in J} 1_J(t)v_{J}^{(B')}$. An application of Theorem [9.3] implies that $v^{(B')}$ is an atomic $\Phi_2$-wave and that $v^{(B')}$ satisfies the correct Fourier support conditions. Furthermore, by a similar argument to the $u^{(B)}$ case, the required energy inequality also holds.

We now turn to the proof of the bilinear estimate. After observing that

$$\|uv\|_{L^a_t L^\infty_x(X[Q])} \leq \|u^{(B)}v^{(B')}\|_{L^a_t L^\infty_x(X)} + \|(|u| - |u^{(B)}|)v\|_{L^a_t L^\infty_x(X[Q])} + \|(|v| - |v^{(B')}|)u\|_{L^a_t L^\infty_x(X[Q])},$$

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an application of Holder’s inequality implies that it is enough to show that for \( \frac{1}{m+1} < \frac{1}{p} < 1 - \frac{1}{r} \), and \( \frac{1}{b} \leq \frac{1}{a} \leq \frac{1}{2} \) we have

\[
\left\| \left( [u^{(1)}] - [u^{(1)}] \right) v \right\|_{L^2_t L^r_x(X|Q)} + \left\| \left( [u^{(1)}] - [v^{(1)}] \right) \right\|_{L^2_t L^r_x(X|Q)} \leq \epsilon^{-C} (\mathcal{H}_2 R)^{\frac{1}{2} - \frac{n+1}{4n}} (\mu + d_0)^n (1 - \frac{1}{r} - \frac{1}{b}) \mathcal{H}_2^{(n+1)\left(\frac{1}{p} - \frac{1}{r}\right)} \|u\|_{L^r L^2_x} \|v\|_{L^{r} L^2_x} \tag{12.1}
\]

with \( \mu = \min\{\text{diam}(\text{supp} \tilde{u}), \text{diam}(\text{supp} \tilde{v})\} \). We start by estimating the first term. The point is to interpolate between a “bilinear” \( L^2_t L^1_x \) estimate which decays in \( R \), and “linear” \( L^2_t L^1_x \) and \( L^2_t L^r_x \) estimates which can lose powers of \( R \). The bilinear \( L^2_t L^1_x \) bound follows from an application of Theorem 9.3

\[
\left\| \left( [u^{(1)}] - [u^{(1)}] \right) v \right\|_{L^2_t L^r_x(X|Q)} \leq \sum_{1 \leq i \leq B} \sum_{m \in \mathbb{Q}_{\frac{B}{2m}}(Q)} \left\| \left( [u_{i,m-1}^{(B)}] - [u_{i,m-1}^{(B)}] \right) V \right\|_{L^2_t L^r_x(1, m, \frac{d_0}{m}(B_m - 1))}^2 \tag{12.2}
\]

where we used the definition of \( \epsilon_m \) and \( M \). To obtain the \( L^2_t L^1_x \) bound, we start by noting that for any \( 1 \leq m \leq M \), and \( a_0 \geq 2 \) we have

\[
\left( \sum_{B \in \mathbb{Q}_{\frac{2B}{a_0}}(Q)} \left\| u_{m}^{(B)} \right\|_{L^\infty_t L^2_x}^2 \right)^{\frac{1}{2}} \leq \left( \sum_{B \in \mathbb{Q}_{\frac{4B}{a_0}}(Q)} \left\| u_{m}^{(B)} \right\|_{L^\infty_t L^2_x}^2 \right)^{\frac{1}{2}} \leq \frac{4^{m(n+1) - \frac{n+1}{a_0}}}{\mathcal{H}_2^{0+n(n+1)(\frac{1}{a_0} - \frac{1}{b})}} \left\| u \right\|_{L^0 L^2_x} \tag{12.3}
\]

where we applied the energy inequality for \( u_{m}^{(B)} \). Therefore, an application of Holder’s inequality gives for any \( a_0 \geq 2 \)

\[
\left\| \left( [u_{m-1}^{(B)}] - [u_{m}^{(B)}] \right) v \right\|_{L^2_t L^r_x(X|Q)} \leq \left\| \left( [u_{m-1}^{(B)}] - [u_{m}^{(B)}] \right) \right\|_{L^2_t L^r_x(Q)} \leq \left( \sum_{B \in \mathbb{Q}_{\frac{2B}{a_0}}(Q)} \left\| u_{m-1}^{(B)} \right\|_{L^\infty_t L^2_x}^2 \right)^{\frac{1}{2}} \leq \left( \sum_{B \in \mathbb{Q}_{\frac{4B}{a_0}}(Q)} \left\| u_{m-1}^{(B)} \right\|_{L^\infty_t L^2_x}^2 \right)^{\frac{1}{2}} \leq \left( \mathcal{H}_2^{0+n(n+1)(\frac{1}{a_0} - \frac{1}{b})} \right)^{\frac{1}{2}} \left\| u \right\|_{L^0 L^2_x} \right\|_{L^2_t L^r_x(Q)} \tag{12.4}
\]

On the other, for the linear \( L^2_t L^1_x \) bound, we can apply a similar argument to deduce that

\[
\left\| \left( [u_{m-1}^{(B)}] - [u_{m}^{(B)}] \right) v \right\|_{L^2_t L^r_x(X|Q)} \leq \left( \sum_{B \in \mathbb{Q}_{\frac{2B}{a_0}}(Q)} \left\| u_{m-1}^{(B)} \right\|_{L^\infty_t L^2_x}^2 \right)^{\frac{1}{2}} \leq \left( \sum_{B \in \mathbb{Q}_{\frac{4B}{a_0}}(Q)} \left\| u_{m-1}^{(B)} \right\|_{L^\infty_t L^2_x}^2 \right)^{\frac{1}{2}} \leq \left( \mathcal{H}_2^{0+n(n+1)(\frac{1}{a_0} - \frac{1}{b})} \right)^{\frac{1}{2}} \left\| u \right\|_{L^0 L^2_x} \right\|_{L^2_t L^r_x(Q)} \tag{12.5}
\]

where we used the fact that at least one of \( u_{m}^{(B)} \) of \( v_{a_0}^{(B')} \) has Fourier support contained in a set of diameter \( d_0 + \mu \). Interpolating between (12.2), (12.3), and (12.5) then gives for any \( \frac{1}{a} \leq \frac{1}{p} \leq 1, \frac{1}{b} \leq 1 - \frac{1}{r} \), and
$$\frac{1}{b} \leq \frac{1}{a} \leq \frac{1}{2},$$

$$\left\| \left( \left[ u_{m-1}^c \right]_{\frac{2b}{m-1}} - \left[ u_{m}^c \right]_{\frac{2b}{m}} \right) v \right\|_{L^2_t L^\infty_x (X[Q])} \leq \epsilon C (H_2 R)^{\frac{1}{2} - \frac{1}{2b}} (\mu + d_0)^{n(1 - \frac{1}{b}) - \frac{1}{2}} H_2^{(n+1)(\frac{1}{2} - \frac{1}{b})} \left\| u \right\|_{L^2_t L^\infty_x} \left\| v \right\|_{L^2_t L^\infty_x}$$

where $$\delta^* = \frac{n+1}{2b} - \frac{1}{b} - 2 \delta C_n \frac{1}{b}$$ Consequently, provided that $$\frac{n+1}{2b} < \frac{1}{a} \leq \frac{1}{2},$$ and we choose $$\delta$$ sufficiently small depending only on $$C_n,$$ $$b,$$ and $$n,$$ we have $$\delta^* > 0.$$ Thus by telescoping the sum over $$m$$ and letting $$u_0^{(Q)} = u,$$ we deduce that

$$\left\| \left( \left[ u \right] - \left[ u^{(c)} \right] \right) v \right\|_{L^2_t L^\infty_x (X[Q])} \leq \sum_{m=1}^{M} \left\| \left[ \left( u_{m-1}^c \right] - \left[ u_{m}^c \right] \right) v \right\|_{L^2_t L^\infty_x (X[Q])}$$

$$\lesssim \epsilon C (H_2 R)^{\frac{1}{2} - \frac{1}{2b}} (\mu + d_0)^{n(1 - \frac{1}{b}) - \frac{1}{2}} H_2^{(n+1)(\frac{1}{2} - \frac{1}{b})} \left\| u \right\|_{L^\infty_t L^2_x} \left\| v \right\|_{L^\infty_t L^2_x}.$$
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(T. Candy) Universit¨at Bielefeld, Fakult¨at f¨ur Mathematik, Postfach 100131, 33501 Bielefeld, Germany
E-mail address: tcandy@math.uni-bielefeld.de