On the appearance of traffic jams in a long chain with a shortcut in the bulk

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The appearance of traffic jams on chains with a shunted section between two simple chain segments in the maximum current phase is studied in the framework of the Totally Asymmetric Simple Exclusion Process. The conditions for the occurrence of this phenomenon are investigated both within the effective rates approximation and numerically for arbitrary length of the shortcut. The problem is interesting on its own because the conditions for coexistence of low- and high density phases are essentially different from those for a simple chain between two reservoirs. Moreover, the results may have relevance to phenomena like crowding of molecular motors moving along twisted protofilaments or vehicular traffic jams on single-lane roads with shortcuts.

I. INTRODUCTION

Recently, traffic jams of kinesin motors transporting cargoes along microtubules have been experimentally established [1]. As in the case of vehicular traffic, here a traffic jam is understood as an abrupt decrease in velocity of the moving objects (particles). In the case of local conservation of particles, when the flux is constant throughout the unidirectional track, such a phenomena can be caused by a sharp increase in the local density of particles. A mathematical analog of traffic jams has been found in the coexistence stationary phase of the Totally Asymmetric Simple Exclusion Process (TASEP), where a low-density phase is
separated by a domain wall (DW) from the corresponding high-density phase. The stationary properties of the TASEP on simple chains have been extensively studied and exactly solved in the thermodynamic limit for different kinds of dynamics, see, e.g., [2–4] and references therein. A comparison of the results obtained by using different update procedures is given in [5]. In the case of continuous time stochastic dynamics, the coexistence phase is known to appear on chains coupled at the left-hand end to a reservoir with low-density of particles $\rho_- < 1/2$ and at the right-hand end to a reservoir with high-density of particles $\rho_+ > 1/2$. The conservation of particle current across the domain wall requires $\rho_+ = 1 - \rho_-$. These latter quantities are simply related to the rates $\alpha, \beta$ with which particles enter the chain at the empty first site, $\rho_- = \alpha$, and leave it from the occupied last site, $\rho_+ = 1 - \beta$, hence, the condition for phase coexistence is $\alpha = \beta < 1/2$ [13, 14].

Here we present a further study of the TASEP on networks consisting of long chain segments with branching and merging points, which exhibit even richer phase behavior, although no exact solutions have been found so far in such cases. The special case of a network having a section of two parallel chains of equal length inserted in the bulk of a long chain was studied in our paper [6]. Since there are no exact results for TASEP on networks with junctions, we advanced the Effective Rates Approximation (ERA) which turned out to be especially effective in the study of the stationary phases of complex networks, composed of long linear chains. We found a coexistence phase in the double chain segment under special conditions when the head and tail chains are in the maximum current phase. This result is interesting on its own because the set up essentially differs from the case of a simple chain coupled to reservoirs with $\rho_+ = 1 - \rho_-$, when coexistence has been proved to exist. The biological significance of model networks with bifurcation and merging points was pointed out by E. Pronina and A. B. Kolomeisky [7] by citing evidence that molecular motors kinesins move along microtubules composed of protofilaments the number of which may vary. This fact indicates the existence of junctions and other lattice defects in the microtubules. Such defects might lead to motor proteins crowding phenomena (traffic jams) that are responsible for some human diseases and motivate the study of TASEP on networks with complex geometry. For example, a model with a zero-length shortcut in the bulk was suggested by Y.-M. Yuan et al [8] for the description of directed motion of molecular motors along twisted filaments. For a recent review on exclusion processes on networks as models for cytoskeletal transport we refer the reader to [9].
FIG. 1: Schematic representation of a twisted filament with a shortcut. The three long segments are labeled by \( s = 1, 2, 3 \) from the left to the right and the shortcut is shown by a dashed line with an arrow. The particles are injected into the network at the left end with rate \( \alpha \) and ejected from the right end with rate \( \beta \); the arrows show the direction of particle hopping.

The paper is organized as follows. In Section II we define the model network, a characteristic feature of which is the existence of bifurcation and merging points between sufficiently long segments, and give a brief description of the method of analysis. Here we provide also some necessary results for the stationary properties of TASEP on a single chain in the thermodynamic limit, and formulate the basic predictions of the domain wall theory. The results of our numerical simulations are presented in comparison with the theoretical predictions for shortcuts at central, Sections III and IV, and off-central, Section V, position in the network. Since the applicability of the DW theory crucially depends on the properties of the junctions between the separate chain segments, the local densities at the ends of the shunted section and the shortcut, as well as the inter-chain correlations at the bifurcation and merging points of the network are studied in Section IV. The paper closes with Section VI in which we summarize our main results and conjectures.

II. THE MODEL

For the modeling of one-dimensional directed transport of particles in biological systems we used the TASEP which is realized in computer simulations by the so-called random-sequential update. TASEP describes a lattice gas of hard core particles on a chain, the sites of which may be empty or singly occupied, moving under a stochastic dynamics in continuous time: for any infinitesimal time period \( d\tau \) just one particle attempts to hop
to an empty nearest-neighbor site on the right with rate $p$; if the target site is occupied, the configuration remains unchanged and another particle is randomly chosen with uniform probability distribution. In the case of an open chain, the particles are injected at the empty first site with rate $\alpha$ and removed from the occupied last site with rate $\beta$. Since the hopping rate $p$ is just a normalization of time, it can be set to unity.

We model the dynamics of molecules on a twisted filament with a shortcut in the bulk, by considering TASEP on an open network consisting of 4 chains, see Fig. 1. Three of the chain segments are assumed to be consecutively coupled and long enough to make reasonable the application of exact thermodynamic results. The fourth chain represents the shortcut: it allows a particle at the last site of the first segment $s = 1$ to jump to the first site of the third segment $s = 3$, provided the target site is empty, and has arbitrary length $L^{sc} \geq 2$. In general, particles at the bifurcation point try with probability $P_j$ to enter the first site of the second segment, hence, $1 - P_j$ is the trial probability of entering the shortcut. Here we will confine ourselves with the representative case of $P_j = 1/2$.

The method of analysis neglects the nearest-neighbor correlations between the sites on the opposite sides of the junctions, and assumes that the chain segments are long enough, so that the exact results for the local densities at the chain ends and in the bulk hold true. Hence, for a simple chain with injection, $\alpha$, and ejection, $\beta$, rates we make use of the following asymptotic results for $L \gg 1$:

(a) In the low-density phase, $\alpha < 1/2$, $\alpha < \beta$,

$$\rho_{bulk} = \alpha, \quad \rho_1 = \alpha, \quad \rho_L = \alpha(1 - \alpha)/\beta, \quad J = \alpha(1 - \alpha); \quad (1)$$

(b) In the high-density phase, $\beta < 1/2$, $\beta < \alpha$,

$$\rho_{bulk} = 1 - \beta, \quad \rho_1 = 1 - \beta(1 - \beta)/\alpha, \quad \rho_L = 1 - \beta, \quad J = \beta(1 - \beta); \quad (2)$$

(c) In the maximum current phase, $\alpha > 1/2$ and $\beta > 1/2$,

$$\rho_{bulk} = 1/2, \quad \rho_1 = 1 - 1/(4\alpha), \quad \rho_L = 1/(4\beta), \quad J = 1/4; \quad (3)$$

(d) On the coexistence line, $\alpha = \beta < 1/2$, in the presence of a delocalized domain wall (for open systems) the local density profile changes linearly from $\rho_1 = \alpha$ to $\rho_L = 1 - \alpha$, and the nearest-neighbor correlations have a parabolic shape with maximum value

$$\max_x F_{\text{cor}}(x) = F_{\text{cor}}(L/2) = (1 - 2\alpha)^2/4. \quad (4)$$
The stationary state of each segment is determined by the effective input, $\alpha^*_s$, and output, $\beta^*_s$, rates for each segment $s = 1, 2, 3$ which are defined by the following rules:

$$J^{(s)} = \beta^*_s \rho_L^{(s)} = \alpha^*_s \left(1 - \rho_1^{(s)}\right), \quad (5)$$

where $J^{(s)}$ is the current through the segment $s$, $\rho_1^{(s)}$ and $\rho_L^{(s)}$ are the average local densities at the first and last site $L$ of that segment; obviously, $\alpha^*_1 = \alpha$, $\beta^*_3 = \beta$. The equalities in (5) neglect the inter-chain correlations.

The continuity of the current leads to a set of equations between these quantities. The solutions for $\alpha^*_s$ and $\beta^*_s$ determine the nonequilibrium stationary phase of each segment $s = 1, 2, 3$, and the shortcut itself, provided the latter is long enough.

In the cases when the average density profile is not flat, one may use the phenomenological domain wall (DW) theory [15, 16]. Its main idea is that, when $\alpha \neq 1 - \beta$, each reservoir tends to enforce a domain in the chain with its own density: the left domain with density $\rho_- = \alpha$, and the right one with density $\rho_+ = 1 - \beta$. At a given time $t$ these domains may coexist, being separated by a domain wall. The stochastic dynamics implies that the domain wall performs a random walk on the chain. On the coexistence line $\alpha = \beta < 1/2$ the random walk is symmetric, with diffusion constant $D = \alpha(1 - \alpha)/(1 - 2\alpha)$ and reflecting boundary conditions.

The question if the DW theory is applicable to the central segment of our network is not trivial and will be studied here in some detail by comparison with results of computer simulations. The problem is that the ends of the shunted chain are not connected to reservoirs of appropriate low and high density. Instead, its left end is coupled to the right end of the first segment where the local density is $\rho_L^{(1)} = 1/(4\beta^*_1) < 1/2$, and the right end is coupled to the left end of the third segment, where the local density is $\rho_1^{(3)} = 1 - 1/(4\alpha^*_3) > 1/2$. These conditions differ from the connections to reservoirs in that there can be significant correlations at the junctions between the segments which are absent in the case of reservoirs. Thus, the very possibility of finding a shunted chain in the coexistence phase under the above boundary conditions is of its own interest.

The predictions of the DW theory for the endpoints of the shunted segment read:

$$\rho_{1,L}^{(2)} = \rho_\mp^{(2)} \equiv 1/2 \mp \sqrt{J_{sc}}, \quad (6)$$
and for the maximum value of the nearest-neighbor correlations:

$$\max_x F_{\text{cor}}(x) = \left[ \rho^{(2)}_+ - \rho^{(2)}_- \right]^2 = 1/4 - J^{(2)} = J^{\text{sc}}, \quad (7)$$

where

$$\rho^{(2)}_\pm = \rho^{(2)}_\pm (J^{(2)}), \quad \rho_\pm(J) := \left( 1 \pm \sqrt{1 - 4J} \right)/2. \quad (8)$$

The main aim of the study is to analyze the conditions under which the shunted segment is in the coexistence phase. Next, we will check if the main features of the coexistence phase in that segment agree with the predictions of the DW theory given by Eqs. (6) and (7), which imply that they depend only on the current through the shortcut, and not on its structure.

III. CENTRAL POSITION OF THE SHORTCUT

For our reference computer simulations we have taken $L_1 = L_2 = L_3 = 400$ sites, which is about the evaluated number of steps taken by a molecular motor during its average residency time on a microtubule. The length of the shortcut $L^{\text{sc}}$ varies from 2 to 100 sites. The numerical data was averaged over a sample of 200 independent runs of length $2^{23}$ trials each. The statistical error was estimated by comparing the above data to the one obtained by averaging over a twice larger sample consisting of 200 independent runs of length $2^{24}$ trials each. The so estimated relative statistical error in the currents is less than $2 \times 10^{-4}$, and in the local densities at the endpoints of the chain segments is less than 2% for all the studied shortcut lengths.

The finite-size shift, estimated from the comparison of the measured total current $J^{\text{sim}} \simeq 0.25091$ with the theoretical one $J_{\text{max}} = 1/4$ yields a relative value slightly less than 0.4%. More important, the density profile in the bulk of a finite chain in the MC phase is not exactly constant, as in the thermodynamic limit $\rho_{\text{bulk}} = 1/2$, but has a slight slope downwards in the direction of hopping. Probably, this is the reason why the effective injection and ejection rates of the middle segment vary upon shifting its position in the network.

Here we consider the case when the shortcut has a central position in the network, i.e., when the head and tail segments have an equal length $L_1 = L_3$. To estimate the conditions for appearance of a domain wall in the shunted segment $s = 2$, we use arguments based on the ERA. Thus, within the first approximation we neglect the nearest-neighbor correlations.
FIG. 2: (Color online) Simulation results for the local density profile in the whole network. The local density profiles in the second segment (red stars) and in the shortcut (green squares) are compared to the predicted by the DW theory linear dependence (blue line) for a shortcut of length 10 sites.

between the different chains. By choosing $\alpha > 1/2$ and $\beta > 1/2$ we ensure that the whole network carries a maximum current of particles $J = 1/4$ and so do the head and tail segments, $s = 1$ and $s = 3$, respectively. Since these segments are in the MC phase, we have the following particle densities at their ends:

$$
\rho_1^{(1)} = 1 - 1/(4\alpha), \quad \rho_1^{(1)} = 1/(4\beta^*_1), \quad \rho_1^{(3)} = 1 - 1/(4\alpha^*_3), \quad \rho_L^{(1)} = 1/(4\beta).
$$

(9)

Here the upper index of the average density labels the segment, and the lower the site in the segment: 1 stays for the first site and $L$ for the last one; $\beta^*_1$ is the effective ejection rate for the first segment, $\alpha^*_3$ is the effective injection rate for the third segment. As $s = 1$ and $s = 3$ are of equal length, one expects $\beta^*_1 = \beta$ and $\alpha^*_3 = \alpha$ to hold if the correlations at the junctions with the inner chains are negligible. Now we interpret segment $s = 1$ as a reservoir with the low-density $\rho_L^{(1)} \simeq 1/(4\beta)$, and segment $s = 3$ as a reservoir with the high-density $\rho_1^{(3)} \simeq 1 - 1/(4\alpha)$. For a single chain to be on the coexistence line (domain wall phase) the above densities should satisfy the relation $\alpha^*_2 = \beta^*_2 \Leftrightarrow \rho_L^{(1)} = 1 - \rho_1^{(3)}$. Therefore, we expect segment $s = 2$, when in the central position, to be in the shock phase when $\alpha = \beta > 1/2$.

Here we present results of our computer simulations at the point $(\alpha, \beta) = (0.75, 0.75)$ on the diagonal in the domain of the maximum current phase of a single chain. In all the figures
FIG. 3: (Color online) Simulation results for the nearest-neighbor correlations in the whole network. Those for the second segment (red stars) and the shortcut (green squares) are compared to the predicted by the DW theory parabolic shape (blue discs) for a shortcut of length 10 sites.

in this section, when data for the main segments are plotted, the abscissa is the normalized coordinate \( x = i/L \), \( i = 1, 2, \ldots, 3L \) ranging in the interval \( (0, 3] \). However, when data for the shortcut are plotted, for convenience of comparison with the shunted segment \( s = 2 \), the abscissa is stretched to \( x = 1 + i/L^{sc} \), \( i = 1, 2, \ldots, L^{sc} \) so that its range is \( (1, 2] \).

The numerical data for a shortcut with \( L^{sc} = 10 \), illustrated by Fig. 2, exhibit unexpectedly good agreement with the DW theory prediction for the density at the endpoints of the shunted segment:

\[
\rho_1^{(2)} \simeq 0.158, \quad \rho_- \simeq 0.156;
\]
\[
\rho_L^{(2)} \simeq 0.836, \quad \rho_+ \simeq 0.844.
\]

The values of \( \rho_\pm \) are calculated according to Eq. (6) with the evaluated value of \( J^{sc} \simeq 0.118 \).

The nearly perfect parabolic shape of the nearest-neighbor correlations in the second segment, see Fig. 3 is described by the best fit quadratic approximation

\[
F_{cor}^{\text{fit}}(x) = A + Bx + Cx^2, \quad \text{with} \quad A = -0.94744, \quad B = 1.42116, \quad C = -0.47372,
\]

which yields \( \max_x F_{cor}^{\text{fit}}(x) = 0.11843 \), in very good agreement with the numerically evaluated \( \max_x F_{cor}(x) \simeq 0.117 \), and in excellent agreement with the value of \( J^{sc} \simeq 0.118 \).
FIG. 4: (Color online) Simulation results for the local density profile in the whole network. The local density profiles in the second segment (red stars) and in the shortcut (green squares) are compared to the predicted by the DW theory linear dependence (blue line) for a shortcut of length 100 sites.

Next we present analogous results for a rather long shortcut containing 100 sites. In this case, the local density profile in both the shunted second segment and the shortcut indicates a coexistence phase in each of them, see Fig. 4. However, in this case there is a noticeable deviation from the linear dependence on the normalized coordinate \( x = i/L \); \( i = L + 1, \ldots, 2L \). The corresponding numerical results are:

\[
\begin{align*}
\rho_1^{(2)} & \simeq 0.153, & \rho_{sc}^{1} & = 0.156, & \rho_- & \simeq 0.1464; \\
\rho_L^{(2)} & \simeq 0.845, & \rho_{sc}^{100} & = 0.842, & \rho_+ & \simeq 0.8536.
\end{align*}
\]

Here the values of \( \rho_{\pm} \) are calculated from Eq. (6) with the evaluated value of \( J_{sc}^{\mp} \simeq 0.1250 \). The agreement is rather good, although there is a noticeable deviation from the DW prediction, especially for the local density at the left end of the shunted segment.

Small deviations from the parabolic shape is observed also in the plot of the nearest-neighbor correlations, both in the shunted segment and in the shortcut of length 100 sites, see Fig. 5. The best quadratic fit for the shunted segment is

\[
F_{\text{cor}}^{\text{fit}}(x) = A + Bx + Cx^2, \quad \text{with} \quad A = -1.00711, \ B = 1.51067, \ C = -0.50356, \quad (11)
\]
FIG. 5: (Color online) Simulation results for the nearest-neighbor correlations in the whole network. Those for the second segment (red stars) and the shortcut (green squares) are compared to the predicted by the DW theory parabolic shape (blue discs) for a shortcut of length 100 sites.

which yields \( \max_X F_{\text{cor}}^{\text{fit}}(x) = 0.125885 \), in very good agreement with the value of \( J^{\text{sc}} \simeq 0.125 \).

On the other hand, the numerical evaluation of the maxima in the shunted segment and in the shortcut provides \( \max_X F_{\text{cor}}(x) \simeq 0.1234 \) and \( \max_X F_{\text{cor}}^{\text{sc}}(x) \simeq 0.1219 \), which deviate from the DW theory prediction within 1.3 % in the former case and 2.6 % in the latter one.

FIG. 6: (Color online) Simulation results for the maximum value of the nearest-neighbor correlations on the shunted segment, \( \max_X F_{\text{cor}}(x) \) (red stars), compared to the predicted by the DW theory value \( J^{\text{sc}} \) (blue discs) for shortcuts of different length.
The agreement with the prediction (7) of the DW theory for the maximum value of the nearest-neighbor correlations on the shunted segment is illustrated in Fig. 6. The discrepancy is less than 2% in the whole range of shortcut lengths.

Summarizing the above results, on the one side we see very good agreement with the DW picture, and the other side persistent small deviations. The latter can be due to both finite-size effects and to small but noticeable inter chain correlations.

IV. JUNCTIONS AND INTER-CHAIN CORRELATIONS

In the general case, the behavior of the numerically evaluated and predicted local densities (6) as a function of the shortcut length \( L_{sc} \) in the range from 2 to 100 sites, is shown in Fig. 7 for the first point of both the second segment and the shortcut, and in Fig. 8 for the corresponding last point. Note that the relative deviation of \( \rho_1^{(2)} \) from \( \rho_- (J_{sc}) \) is within 3%. The local density at the first site of the shortcut \( \rho_{sc}^{(1)} \) strongly deviates from \( \rho_1^{(2)} \) for small \( L_{sc} \) but steadily approaches the latter value with the growth of the shortcut length. It is seen that for \( L_{sc} \geq 100 \) the shortcut becomes nearly equivalent to the shunted segment.

![Graph](image)

**FIG. 7:** (Color online) Simulation results for the local density at the first site of the second segment, \( \rho_1^{(2)} \) (red stars), and the shortcut, \( \rho_{sc}^{(1)} \) (green circles), compared to the predicted by the DW theory value of \( \rho_- (J_{sc}) \) (blue rotated squares) for shortcuts of different length. For reference, the local density at the endpoint of the first segment, \( \rho_L^{(1)} \) (black squares), is shown too.
FIG. 8: (Color online) Simulation results for the local density at the last site of the second segment, $\rho_L^{(2)}$ (red stars), and the shortcut, $\rho_{\text{end}}^{\text{sc}}$ (green discs), compared to the predicted by the DW theory value of $\rho_+ (J^{\text{sc}})$ (blue rotated squares) for shortcuts of different length.

In this case the relative deviation of $\rho_L^{(2)}$ from $\rho_+ (J^{\text{sc}})$ does not exceed 2% for all shortcut lengths. The local density at the last site of the shortcut $\rho_{\text{end}}^{\text{sc}}$ strongly deviates from $\rho_L^{(2)}$ for small $L^{\text{sc}}$ but steadily approaches the latter value with the increase in the shortcut length. Again, it is seen that for $L^{\text{sc}} \geq 100$ the shortcut becomes nearly equivalent to the shunted segment.

On the basis of our numerical data, we conclude that the evaluated deviations from the predictions of the DW theory about the local densities at the ends of the shunted segment, given by Eqs. (6) and (7), do not exceed 3%, for both short and relatively long shortcuts. Possible explanation of the observed small deviations will be discussed in the remainder of this section.

Under the random sequential update, the inter-chain correlations measure the difference between the actual current of particles and its mean-field approximation. Taking into account that in our model there is a rule conducting the currents out of the bifurcation point with probability $P_j$ to the shunted segment and with the remaining $1 - P_j$ to the shortcut, we obtain

$$G_{1,2} = \frac{1}{P_j} J^{(2)} - \rho_L^{(1)} \left( 1 - \rho_1^{(2)} \right), \quad G_{1,\text{sc}} = \frac{1}{1 - P_j} J^{\text{sc}} - \rho_L^{(1)} \left( 1 - \rho_1^{\text{sc}} \right),$$

(12)

where $G_{1,2}$ denotes the nearest-neighbor correlation between the first segment and the second
FIG. 9: (Color online) Simulation results for the inter-chain correlations in dependence on the length of the shunted segment (see text).

one, and \( G_{1,sc} \) between the first segment and the shortcut. In our case \( P_j = 1/2 \).

The behavior of the inter-chain correlations on the shortcut length is shown in Fig. 9. One sees that the correlations between the shunted segment and the third one are negligible in the whole range studied. Correlations between the shortcut and the tail segment are positive and significant only for small \( \rho_{sc} \) up to 20, then quickly drop down to absolute values within the statistical error. Thus, the mean field approximations at the merging point are fairly close to the actual values for long enough shortcuts, e.g. for \( L_{sc} \geq 50 \). On the other hand, the correlations between the head and shunted segment, as well as those between the head segment and the shortcut tend to a common value of about \(-0.012\) as \( \rho_{sc} \) increases.

The negative values of the inter-chain correlations \( G_{1,2} \) and \( G_{1,sc} \), see Fig. 9 imply that the actual currents through the shunted segment and the shortcut exceed the corresponding mean field approximations at the bifurcation point:

\[
J^{(2)}_{MFA} = P_j \rho_L^{(1)} (1 - \rho_1^{(2)}), \quad J^{sc}_{MFA} = (1 - P_j) \rho_L^{(1)} (1 - \rho_1^{sc}).
\]

Turning back to Fig. 7, we note that \( \rho_1^{sc} > \rho_1^{(2)} \) for all values \( L_{sc} < L \). This implies the following inequalities

\[
J^{(2)} > \frac{1}{8} + \frac{1}{4} (G_{1,sc} - G_{1,2}), \quad J^{sc} < \frac{1}{8} - \frac{1}{4} (G_{1,sc} - G_{1,2}), L_{sc} < L.
\]
Since $G_{1,sc} > G_{1,2}$ for all $L^{sc} < L$, it follows that $J^{(2)} > 1/8 > J^{sc}$, i.e., it turns out that the current through the longer shunted segment is larger than the current through the shortcut. Evidently, at $L^{sc} = L$ and $P_j = 1/2$, the shunted segment and the shortcut become equivalent, hence, $\rho_1^{sc} = \rho_1^{(2)}$, $G_{1,sc} = G_{1,2}$ and $J^{(2)} = J^{sc} = 1/8$. Finally, for $L^{sc} > L$ the shunted segment and the shortcut exchange their places.

At the merging point there are no rules other than the simple exclusion and we have

$$G_{2,3} = \rho_L^{(2)} \left(1 - \rho_1^{(3)}\right) - J^{(2)} , \quad G_{sc,3} = \rho_{\text{end}}^{sc} \left(1 - \rho_1^{(3)}\right) - J^{sc} ,$$

where $G_{2,3}$ denotes the nearest-neighbor correlation between the second segment and the third, and $G_{sc,3}$ between the shortcut and the third segment.

V. OFF-CENTRAL POSITION OF THE SHORTCUT

In the case of long but finite chains, moving the point of external rates $(\alpha, \beta)$ in the domain of the maximum current phase away from the diagonal $\alpha = \beta > 1/2$ changes the effective rates $\alpha^*_2$ and $\beta^*_2$ for the shunted segment and brings it out of the coexistence phase. We

FIG. 10: (Color online) Simulation results for the local density profile in the network under central position of the shortcut, $L_1 = L_3$, when the point of the external rates is shifted off-diagonally to $(\alpha, \beta) = (0.9, 0.6)$. The local density profiles in the shunted segment (red stars) and the shortcut (green squares) are compared to the linear profile predicted by the DW theory (blue line).
numerically studied the particular case of $(\alpha, \beta) = (0.9, 0.6)$ when one expects the balance between $\alpha^*_2$ and $\beta^*_2$ to be broken to $\alpha^*_2 > \beta^*_2$. Indeed, computer simulations show that in this case the density profiles of both the shunted segment and the shortcut of length $L^{sc} = 100$ bend upward to resemble the shape of the high density phase, see Fig. 10.

However, the balance $\alpha^*_2 \simeq \beta^*_2$ is recovered upon slight shift backward of the position of the shortcut to $L_1 = 410$, $L_3 = 390$, Fig. 11. Computer simulations show that now the density profiles of both the shunted segment and the shortcut with length $L^{sc} = 100$ are almost linear, which indicates coexistence phase in both.

Expectedly, when the shortcut is shifted further backward to the position $L_1 = 420$, $L_3 = 380$, the balance $\alpha^*_2 \simeq \beta^*_2$ changes in the opposite direction to $\alpha^*_2 < \beta^*_2$. Indeed, computer simulations show that now the density profiles of both the shunted segment and the shortcut bend downward, similarly to the case of a low density phase, see Fig. 12.

Thus, we establish that depending on the position of the shortcut, the density profile in the whole network may have different shapes, typical for the stationary phase structures MC-HD-MC, MC-DW-MC, and MC-LD-MC, where DW stays for the domain wall (coexistence) phase. In the first two cases, on crossing the boundary of the HD phase the average particle velocity will decrease exponentially fast in conformity with the local density increase. In the thermodynamic limit this takes place in the so called boundary layer. At the boundaries of

![Graph](image.png)

FIG. 11: (Color online) The same as in Fig. 10 but under slightly shifted position of the shortcut to $L_1 = 410$, $L_3 = 390$. 
FIG. 12: (Color online) The same as in Fig. 10 but under excessively shifted position of the shortcut to $L_1 = 420$, $L_3 = 380$.

the maximum current phase the particles accelerate following a power-law with the distance, which is a consequence of the fact that the local density approaches the bulk value $\rho = 1/2$ approaches its bulk value of $\rho = 1/2$ from above as $i^{-1/2}$ near the left boundary and from below as $(L - i)^{-1/2}$ near the right boundary [13, 14].

VI. DISCUSSION

The results of our extensive Monte Carlo simulations lead us to the following main conjectures concerning the appearance of a traffic jam in a sufficiently long shunted segment between head and tail chains carrying maximum current:

1. For any values of the external rates in the domain of the maximum current phase, $\alpha > 1/2$ and $\beta > 1/2$, there exists a position of the shortcut where the shunted segment is in a phase of coexistence with a completely delocalized domain wall. The longer is the shunted segment, the more sensitive is its profile to the balance of the effective rates $\alpha_2^*$ and $\beta_2^*$; consequently, the location of the shunted segment with coexisting low- and high-density phases is determined more precisely. The shift of the shortcut from that position upstream leads to a sharp change of the density profile resembling a transition to a high density phase, while the shift downstream causes a transition to a low density phase.

2. The main characteristics of the coexistence phase are well described by the domain
wall theory. They depend only on the current through the shortcut and do not depend on the structure of the latter.

3. The domain wall theory is applicable to the shunted segment due to the small correlations of the latter with the other chains. The smallness of the inter-chain correlations is probably due to the random sequential update used.

4. In the considered case of $P_j = 1/2$, the model displays an unexpected feature: the current through the longer shunted segment is larger than the current through the shortcut.

5. From the viewpoint of vehicular traffic, most comfortable conditions for the drivers are provided when the shortcut is shifted downstream from the position of coexistence, since then both the shunted segment and the shortcut exhibit low-density lamellar flow. Most unfavourable is the opposite case of upstream shifted shortcut, when both the shunted segment and the shortcut are in a high-density phase describing congested traffic of slowly moving cars.

It would be instructive to make a similar study of the steady state properties of TASEP on the same network but with particles obeying a different stochastic dynamics, e.g., with parallel update.

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