Mean-field behavior of Nearest-Neighbor Oriented Percolation on the BCC Lattice Above 8 + 1 Dimensions

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Abstract
In this paper, we consider nearest-neighbor oriented percolation with independent Bernoulli bond-occupation probability on the $d$-dimensional body-centered cubic (BCC) lattice $\mathbb{I}^d$ and the set of non-negative integers $\mathbb{Z}_+$. Thanks to the orderly structure of the BCC lattice, we prove that the infrared bound holds on $\mathbb{I}^d \times \mathbb{Z}_+$ in all dimensions $d \geq 9$. As opposed to ordinary percolation, we have to deal with complex numbers due to asymmetry induced by time-orientation, which makes it hard to bound the bootstrap functions in the lace-expansion analysis. By investigating the Fourier–Laplace transform of the random-walk Green function and the two-point function, we derive the key properties to obtain the upper bounds and resolve a problematic issue in Nguyen and Yang’s bound. The issue is caused by the fact that the Fourier transform of the random-walk transition probability can take the value $-1$.

Keywords Oriented percolation · Nearest-neighbor model · Mean-field behavior · Upper critical dimension · Lace expansion

Mathematics Subject Classification 82C43 · 60K35
1 Introduction

1.1 Motivation

In 1957, Broadbent and Hammersley [6] introduced oriented percolation (directed percolation) in the context of physical phenomena, such as the wetting of porous medium. It is well-known that oriented percolation exhibits critical phenomena in the vicinity of critical points. It was first predicted by Obukhov [29] that the upper critical dimension $d_c + 1$ for oriented percolation equals $4 + 1$ (= spatial + temporal dimension), above which some quantities display the power law, and its exponents take mean-field values. For example, there exists the critical point $p_c$ such that the susceptibility (the expected cluster size) $\chi_p$ is finite if $p < p_c$ [as $p \uparrow p_c$, $\chi_p$ diverges at least as fast as $(p_c - p)^{-1}$ by the second inequality in (1.5) below]. It is believed that $\chi_p$ shows power-law behavior as $(p_c - p)^{-\gamma}$ with the critical exponent $\gamma$. Nguyen and Yang [27] proved via the infrared bound and the lace expansion that spread-out oriented percolation with independent Bernoulli bond-occupation probability on $\mathbb{Z}^d \times \mathbb{Z}^+$ exhibits mean-field behavior in dimensions $d + 1 > 4 + 1$, and there exists a sufficiently high dimension $d_0 \gg 4$ such that nearest-neighbor oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}^+$ also exhibits it in $d + 1 \geq d_0 + 1$, where $\mathbb{Z}^+ := \mathbb{N} \sqcup \{0\}$, and $\sqcup$ is a disjoint union. The former supports the prediction of the upper critical dimension $d_c = 4$.

As a side note, we mention the results of the critical exponents in $d \leq 4$. It is predicted that their values in $d + 1 = 4 + 1$ equal mean-field critical exponents with logarithmic corrections. So far, the mathematically nonrigorous renormalization group method [23] and the computer-oriented approach [14] suggest that the prediction is true. There are almost only numerical results regarding the critical exponents for oriented percolation in $d + 1 = 1 + 1, 2 + 1$ or $3 + 1$. One can see the conjectural values in, e.g., [30]. Several results concerning critical behavior but not the exact value of critical exponents are derived particularly in $d = 1 + 1$. See the literature on the box-crossing property [11], a hyperscaling inequality [34], the scaling limit [35], etc.

Similar results for ordinary percolation on $\mathbb{Z}^d$ are known. It was first predicted in [37] that its upper critical dimension is 6. In the nearest-neighbor model, Hara and Slade [19] proved mean-field behavior in $d \geq 19$ via the lace expansion which was first derived in the seminal paper [18]. The best result $d \geq 11$ on $\mathbb{Z}^d$ was shown by Fitzner and van der Hofstad [12] via the non-backtracking lace expansion [13]. If we do not insist on the simple cubic lattice $\mathbb{Z}^d$, a closer value $d \geq 9$ to the critical dimension on the body-centered cubic (BCC) lattice $\mathbb{L}^d$ is obtained by the second author, the third author and Sakai [17].

In this paper, we consider nearest-neighbor oriented percolation with independent Bernoulli bond-occupation probability on $\mathbb{L}^d \times \mathbb{Z}^+$ in high dimensions. It is known what dimension the lace expansion works for nearest-neighbor ordinary percolation, whereas Nguyen and Yang did not establish the exact value of the dimension $d_0$ in which their analysis works for nearest-neighbor oriented percolation. Our purpose is to specify $d_0$ and to prove the infrared bound in $d_0$. As a result, we prove that the infrared bound holds on $\mathbb{L}^{d \geq 9} \times \mathbb{Z}^+$. 

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Sections 3 and 4 contain two novel tasks (see Remark 3.6 and Remark 4.5 for the details, respectively). Nguyen and Yang [27] used the inequality $\hat{D}_L(k) + 1 > 0$ for all $k$, where $D_L$ is the random-walk transition probability for the spread-out model, to bound $g_2$ in our notation below. They used a similar inequality despite the nearest-neighbor model, but the inequality does not hold because the Fourier transform $\hat{D}(k)$ of the random-walk transition probability can take the value $-1$ for some $k \in [-\pi, \pi]^d$. The structure of the BCC lattice helps us to solve this problem (see also Proposition 3.2). Moreover, Nguyen and Yang showed the finiteness of the weighted open triangle diagram (the derivative of the open triangle diagram), which corresponds to $\hat{V}^{(0,1)}_{p,1}(k)$ in this paper, only for $d > 8$. We observe that the period of $\hat{D}(k)^2$ equals that of $\hat{D}(2k)$. This equality yields the finiteness of $\hat{V}^{(\lambda, \mu)}_{p,m}(k)$ in Lemma 4.1 for $d > 4$.

1.2 Model

We describe the model that we deal with in this paper. First, we define the BCC lattice. The $d$-dimensional BCC lattice $\mathbb{L}^d$ is a graph that contains the origin $o = (0, \ldots, 0)$ and is generated by the set of neighbors $\mathcal{N}^d = \{x = (x_1, \ldots, x_d) \in \mathbb{Z}^d \mid \prod_{i=1}^d |x_i| = 1\}$. Although we should generally distinguish a graph from its vertex set, for convenience, we denote by "$x \in \mathbb{L}^d$" that $x$ belongs to the vertex set of the BCC lattice. It was first used in [17] in the context of lace expansions and has some good properties as we see below. On $\mathbb{L}^d$, note that the cardinality, denoted by $|\cdot|$, of $\mathcal{N}^d$ equals $2^d$ (while it equals $2d$ on $\mathbb{Z}^d$).

The $d$-dimensional random-walk transition probability (1-step distribution) $D(x)$ is expressed as the product of 1-dimensional random-walk transition probabilities:

$$D(x) := \frac{1}{|\mathcal{N}^d|} \mathbb{1}_{\{x \in \mathcal{N}^d\}} = \prod_{j=1}^d \frac{1}{2} \delta_{|x_j|, 1}, \quad (1.1)$$

where $\mathbb{1}_{\{\cdot\}}$ is the indicator function, and $\delta_{\bullet, \bullet}$ is the Kronecker delta. This property is useful to compute the numerical values of the random-walk quantities defined as

$$\varepsilon_i^{(\nu)} = \sum_{n=\nu}^{\infty} D^{*2n}(o) \times \begin{cases} 1 & [i = 1], \\ (n - \nu + 1) & [i = 2] \end{cases} \quad (1.2)$$

for $\nu \in \mathbb{Z}_+$, where $D^{*n}(x) = (D^{*n-1} \ast D)(x) := \sum_{y \in \mathbb{L}^d} D^{*(n-1)}(y) D(x - y)$ denotes the $n$-fold convolution on $\mathbb{L}^d$. See Appendix A for the details of the computation. Let $\hat{D}(k)$ be the Fourier transform of $D: \mathbb{L}^d \to \mathbb{R}$ defined as

$$\hat{D}(k) := \sum_{x \in \mathbb{L}^d} D(x) e^{ik \cdot x} = \prod_{j=1}^d \cos k_j \quad (1.3)$$
for $k = (k_1, \ldots, k_d) \in \mathbb{T}^d$, where $k \cdot x$ is the Euclidian inner product, and $\mathbb{T}^d := [-\pi, \pi]^d$ is the $d$-dimensional torus of side length $2\pi$ in the Fourier space. In the equality in the above, we have used (1.1).

Second, we define nearest-neighbor oriented percolation. We call $\mathbb{L}^d \times \mathbb{Z}_+^d$ a space-time in which we write a vertex in the bold font, i.e., $x = (x, t) \in \mathbb{L}^d \times \mathbb{Z}_+$, while we write a vertex in space $\mathbb{L}^d$ in the normal font, i.e., $x \in \mathbb{L}^d$. For convenience, $x$ and $t(x)$ denote the spatial and temporal components of $x$, respectively. A bond is an ordered pair $((x, t), (y, t + 1))$ of two vertices in $\mathbb{L}^d \times \mathbb{Z}_+^d$. Each bond $(x, y)$ is occupied with probability

$$q_p(y - x) := p D(y - x) \delta_t(y - t(x), 1)$$

and vacant with probability $1 - q_p(y - x)$ independent of the others, where $p \in [0, \|D\|_\infty^{-1}]$ is the percolation parameter. We denote the uniform norm by $\|D\|_\infty = \sup_{x \in \mathbb{L}^d} |D(x)|$. Unlike much literature, note that $p$ is not a probability. Let $\mathbb{P}_p$ be the associated probability measure with such bond variables, and denote its expectation by $\mathbb{E}_p$.

A (time-oriented) path of length $t$ is defined to be a sequence $(\omega_0, \omega_1, \ldots, \omega_t)$ of vertices in $\mathbb{L}^d \times \mathbb{Z}_+^d$ such that $t(\omega_s) - t(\omega_{s-1}) = 1$ for $s = 1, \ldots, t$. Let $\mathbb{W}(x, y)$ be the set of all paths of length $t = t(y) - t(x)$ from $x = \omega_0$ to $y = \omega_t$. By convention, $\mathbb{W}(x, x) \equiv \{(x)\}$. We say that a path $\vec{\omega} = (\omega_0, \ldots, \omega_t) \in \mathbb{W}(x, y)$ of length $t = t(y) - t(x)$ is occupied if either $x = y$ or every bond $\omega_{s-1} - \omega_s$ for $s = 1, \ldots, t$ is occupied. We say that $x$ is connected to $y$, denoted by $x \rightarrow y$, if there is an occupied path $\vec{\omega} \in \mathbb{W}(x, y)$. Then, we define the two-point function as

$$\varphi_p(x) = \mathbb{P}_p(o \rightarrow x) = \mathbb{P}_p \left( \bigcup_{\vec{\omega} \in \mathbb{W}(o, x)} \{\vec{\omega} \text{ is occupied} \} \right)$$

for $x = (x, t) \in \mathbb{L}^d \times \mathbb{Z}_+$, where $o := (o, 0)$. The percolation probability and the susceptibility are defined as

$$\Theta_p = \mathbb{P}_p(|C(o)| = \infty) \quad \text{and} \quad \chi_p = \mathbb{E}_p[|C(o)|] = \sum_{x \in \mathbb{L}^d \times \mathbb{Z}_+} \varphi_p(x),$$

respectively, where $C(o) = \{x \in \mathbb{L}^d \times \mathbb{Z}_+ | o \rightarrow x \}$. Also, the magnetization is defined as

$$M_{p,h} = \mathbb{E}_p \left[ 1 - e^{-|C(o)|h} \right],$$

which is similar to the spontaneous magnetization in ferromagnetic models.

Third, we describe critical phenomena. The critical point is defined as

$$p_c = \inf \left\{ p \in \left[0, \|D\|_\infty^{-1} \right] \mid \Theta_p > 0 \right\} = \sup \left\{ p \in \left[0, \|D\|_\infty^{-1} \right] \mid \chi_p < \infty \right\}.$$
Note that the second equality in the definition of $p_c$ is quite nontrivial. In this sense, the critical point for $\Theta_p$ and $\chi_p$ are often written as $p_H$ and $p_T$, respectively. However, Menshikov [26] and Aizenman and Barsky [1] independently proved that $p_H = p_T$ for any translation invariant percolation models. Recently, Duminil-Copin and Tassion [10] found a particularly simple proof of the uniqueness on arbitrary locally finite vertex-transitive infinite graphs. This uniqueness also holds on $\mathbb{L}^d$, hence we do not distinguish between these critical values.

The power laws with the critical exponents $\beta, \gamma$ and $\delta$ are defined as

$$\Theta_p \propto (p - p_c)^\beta, \quad \chi_p \propto (p_c - p)^{-\gamma}, \quad M_{p_c,h} \propto h^{1/\delta}.$$ 

Here, $f(x) \propto g(x)$ as $x \to a$ denotes that there exist constants $\delta_0$, $C_1$ and $C_2$ such that, for any $x$ satisfying $|x - a| < \delta_0$, $C_1 g(x) \leq f(x) \leq C_2 g(x)$.

Finally, we mention the Fourier–Laplace transform of $\varphi_p(x)$ as a preliminary for the main result. Let

$$\Phi_p(k; t) = \sum_{x \in \mathbb{L}^d} \varphi_p(x, t) e^{ik \cdot x}$$

be the Fourier transform of $\varphi_p(x, t)$ with respect to $x$. The Markov property of oriented percolation implies that $\{\log \Phi_p(0; t)/t\}_{t \leq 1}$ is a subadditive sequence, hence there exists a constant $m_p^{-1} = \lim_{t \uparrow 1} \Phi_p(0; t)\downarrow 1$ depending on $p$ by [15, Appendix. II] with $m_p = 1$ (See [9, Sect. 1.2] or [28, Sect. 1]). For every $p < p_c$, the inequality $m_p > 1$ holds. $m_p$ is the radius of convergence of the Laplace transform (the generating function) of $\Phi_p(0; t)$, defined by

$$\hat{\Phi}_p(k, z) := \sum_{r \in \mathbb{Z}^d} \Phi_p(k; r) z_r = \sum_{(x, t) \in \mathbb{L}^d \times \mathbb{Z}^+} \varphi_p(x, t) e^{ik \cdot x} z^t$$

for $k \in \mathbb{T}^d$ and $z \in \mathbb{C}$ satisfying $|z| \in [0, m_p)$. Also, let

$$Q_p(x, t) = p^t D^{*t}(x) \mathbb{1}_{\{r \in \mathbb{Z}^d\}}$$

be the random-walk two-point function. By convention, the 0-fold convolution denotes the Kronecker delta: $D^{*0}(x) = \delta_{0,x}$. The Laplace transform of $Q_1(x, t)$ gives the well-known random-walk Green function as

$$S_p(x) := \sum_{t \in \mathbb{Z}^d} Q_1(x, t) p^t = \sum_{r \in \mathbb{Z}^d} p^r D^{*r}(x)$$

for $x \in \mathbb{L}^d$ and $p \in [0, 1]$. Notice that the radius of convergence of $S_p(x)$ is 1, and in particular $S_1(x)$ is well-defined in a proper limit when $d > 2$. By Boole’s inequality,
one can easily see that $\varphi_p(x, t) \leq Q_p(x, t)$ for every $0 \leq p < 1$. Taking the sum of both sides leads to $\chi_p \leq (1 - p)^{-1}$, which implies that $p_c \geq 1$.

### 1.3 Main Result

Aizenman and Newman [2] showed that the triangle condition is a sufficient condition for percolation models to exhibit mean-field behavior (or $\gamma = 1$). Barsky and Aizenman [3] extended/reformulated the Aizenman–Newman triangle condition in order to deal with oriented percolation in a unified manner. They also showed that other critical exponents $\delta$ and $\beta$ take the mean-field values 2 and 1, respectively, if the triangle condition is satisfied.

The triangle condition expresses that

$$\lim_{R \to \infty} \triangle_{p_c}(R) = 0,$$  \tag{1.4}

where

$$\triangle_p(R) = \sup \left\{ \sum_{y \in \mathbb{L}^d \times \mathbb{Z}_+} \varphi^2_p(y) \varphi_p(y - x) \left| \|x\|_2 \geq R \right. \right\},$$

the norm of $x = (x, t)$ denotes $\|x, t\|_2 = (\sum_{i=1}^d |x_i|^2 + t^2)^{1/2}$, and $\varphi^\ast_n(x, t) = (\varphi^\ast_{(n-1)} \ast \varphi)p(x, t) := \sum_{(y, s) \in \mathbb{L}^d \times \mathbb{Z}_+} \varphi_p(y - x, t - s)$ denotes the $n$-fold convolution on $\mathbb{L}^d \times \mathbb{Z}_+$. If (1.4) holds, then solving the differential inequalities [1–3, 8]

$$\epsilon_p(R) \left( 1 - \triangle_p(R) \right) \chi_p^2 \leq \frac{d\chi_p}{dp} \leq \chi_p^2,$$  \tag{1.5}

$$\epsilon'_p(R) \left( 1 - f_p \triangle_{p_c}(R) \right) M_{p, h} \frac{\partial M_{p, h}}{\partial h} \leq M_{p, h} - h \frac{\partial M_{p, h}}{\partial h} \leq \frac{p}{1 - p \|D\|_\infty} M_{p, h}^2 \frac{\partial M_{p, h}}{\partial h} + M_{p, h}^2,$$  \tag{1.6}

implies that $\beta = \gamma = 1$ and $\delta = 2$, where $\epsilon_p(R), \epsilon'_p(R)$ and $f_p$ are model-dependent functions with $1/\epsilon_p(R), 1/\epsilon'_p(R)$ and $f_p$ uniformly bounded in a neighborhood of $p_c$. Note that $\Theta_p = \lim_{h \downarrow 0} M_{p, h}$.

To verify (1.4), we use the infrared bound which is our main theorem (cf., [27, Theorem 1] and [28, Theorem 2]).

**Theorem 1.1** (Infrared bound) *For nearest-neighbor oriented percolation on $\mathbb{L}^d \times \mathbb{Z}_+$ with independent bond statuses, there exists a model-dependent constant $K \in (0, \infty)$ such that*

$$|\hat{\varphi}_p(k, z)| \leq \frac{K}{|1 - e^{i\arg \tau D(k)}|} = K \left| \hat{S}_{\tau \arg \tau}(k) \right|$$  \tag{1.7}
uniformly in \( p \in [0, p_c) \), \( k \in \mathbb{T}^d \) and \( z \in \mathbb{C} \) with \(|z| \in [1, m_p]\).

**Corollary 1.2** ([27]) *If the infrared bound (1.7) holds, then the triangle condition (1.4) is satisfied.*

**Proof** Fix \( \varepsilon \in (0, 1 \wedge \frac{d-4}{6}) \), where \( a \wedge b \) denotes \( \min\{a, b\} \). By the Hausdorff–Young inequality and (1.7), there is a constant \( C_\varepsilon \) depending on \( \varepsilon \) such that

\[
\left( \sum_x \left| \sum_y \varphi_p^*(y) \varphi_p(y-x) \right|^{1+1/\varepsilon} \right)^{\varepsilon/(1+\varepsilon)} \leq C_\varepsilon \left( \int_{\mathbb{T}^d} \frac{d^d k}{(2\pi)^d} \int \frac{d\theta}{2\pi} \left| \hat{\varphi}_p(k, e^{i\theta})^2 \hat{\varphi}_p(k, e^{-i\theta}) \right|^{1+\varepsilon} \right)^{1/(1+\varepsilon)} \]

\[
= C_\varepsilon \left( \int_{\mathbb{T}^d} \frac{d^d k}{(2\pi)^d} \int \frac{d\theta}{2\pi} \left| \hat{\varphi}_p(k, e^{i\theta}) \right|^{3(1+\varepsilon)} \right)^{1/(1+\varepsilon)} \]

\[
\leq C_\varepsilon \left( \int_{\mathbb{T}^d} \frac{d^d k}{(2\pi)^d} \int \frac{d\theta}{2\pi} \frac{K^{3(1+\varepsilon)}}{\left| 1 - e^{i\theta} \hat{D}(k) \right|^{3(1+\varepsilon)}} \right)^{1/(1+\varepsilon)} . \tag{1.8}
\]

In the equality, we have used \( \hat{\varphi}_p(k, z^*) = \hat{\varphi}_p(k, z)^* \) for every \( k \in \mathbb{T}^d \) and \( z \in \mathbb{C} \), where \( z^* \) is the complex conjugate of \( z \). The integrability of the integrand in the integral appearing on the right hand side of the last line follows from the Taylor series of \((1 - \hat{D}(k)) + \hat{D}(k)(1 - e^{i\theta})\) around the singularities \( \{(k, \theta) \in \mathbb{T}^{d+1} | 1 - e^{i\theta} \hat{D}(k) = 0 \} \) and

\[
\left( \frac{1}{\left| \|k\|_2^2 + |\theta| \right|^{3(1+\varepsilon)}} \right) \frac{1}{\left| \|k\|_2^2 + \theta \right|^{2+3\varepsilon}} \theta^{\delta_2} d^d k d\theta = \frac{2 \delta_1}{2 + 3\varepsilon} d^d k \lesssim f(x) \lesssim g(x)
\]

for \( \delta_1, \delta_2 < 1 \), where \( f(x) \lesssim g(x) \) means that there exists a constant \( C < \infty \) such that \( f(x) \leq C g(x) \). Since the upper bound on (1.8) does not depend on
\[ p, \sum_y \varphi_{p_c}^2(y) \varphi_{p_c}(y - x) \text{ is a } (1 + 1/\epsilon)\text{-summable function over } \mathbb{L}^d \times \mathbb{Z}_+. \text{ This immediately implies} \]
\[
\sum_y \varphi_{p_c}^2(y) \varphi_{p_c}(y - x) \xrightarrow{\|x\| \uparrow \infty} 0, 
\]
and completes the proof. \(\square\)

As we explained above, the triangle condition and the differential inequalities (1.5) and (1.6) yield the next corollary. It is well-known that \(\Theta_p\) is continuous at \(p_c\) (i.e., \(\Theta_{p_c} = 0\)) in the case of oriented percolation [4, 16]. Refer [2, Proof of Proposition 3.1] to see how to imply the inequalities (1.10) from (1.5). Specifically, integrating the differential inequalities leads to their inequalities. Refer [1, Proof of Lemma 5.1] and [3, Proof of Proposition 4.1] to show (1.9) and (1.11). As a side note, the shorter proof of a lower bound on \(\Theta_{p_c}\) than in [1, 8] is given in [10].

**Corollary 1.3** [3] If the triangle condition (1.4) is satisfied, then there exist constants \(\{C_i\}_{i=1}^6 \subset (0, \infty)\) such that, in the vicinity of \(p_c\),
\[
C_1(p - p_c) \leq \Theta_p \leq C_2(p - p_c) \quad \text{for } p > p_c, \\
C_3 \frac{p_c - p}{p_c} \leq \chi_p \leq C_4 \frac{p - p_c}{p_c} \quad \text{for } p < p_c 
\]
and, for small \(h \geq 0\),
\[
C_5 h^{1/2} \leq M_{p_c, h} \leq C_6 h^{1/2}. 
\]
Therefore, \(\beta = \gamma = 1\) and \(\delta = 2\).

We obtain upper bounds on \(p_c\) as a by-product through the proof of Theorem 1.1. See (5.2) below (whose numerical value is rounded up to the fourth decimal place due to significant figures). Incidentally, one might be able to obtain a more precise estimate because the lace expansion also provides an asymptotic expansion for \(p_c\) [20, 22, 42, 43]. This topic is beyond the scope of this paper, so that we have not taken care of this subject.

**Corollary 1.4** For each dimension \(d\), an upper bound on the critical point \(p_c(\geq 1)\) for nearest-neighbor oriented percolation on \(\mathbb{L}^d \times \mathbb{Z}_+\) is given by the following table:

| \(d + 1\) | 9 + 1 | 10 + 1 | 11 + 1 | 12 + 1 |
|-----------|-------|--------|--------|--------|
| \(p_c \leq\) | 1.000110 | 1.000039 | 1.000014 | 1.000005 |

**Remark 1.5** The upper bound on \(p_c\) in \(d = 9\) is at most 0.01% off the real value since \(p_c \geq 1\) always holds.
1.4 Remark

In this paper, we show mean-field behavior for nearest-neighbor oriented percolation on $\mathbb{L}^d \cong 9 \times \mathbb{Z}_+$. As compared to the analysis on $\mathbb{Z}^d \cong 4 \times \mathbb{Z}_+$ by Nguyen and Yang [27], the current analysis identifies an upper bound 9 on the critical dimension $d_c$. To begin with, since [27] contains a problematic issue due to the existence of $k$ such that $\hat{D}(k) = -1$ (as we mentioned in the last paragraph of Section 1.1), their bound corresponding to (3.4) below requires extra lemmas for any dimensions even if $d$ is sufficiently large. To resolve the issue, we prove Proposition 3.2 and Lemma 3.3 below.

It is crucial to reduce the range of the supremum in $g_2$ (See Sect. 1.5) by symmetry in the Fourier space, hence the assumption of the inequality (3.7) is satisfied. While we pay attention to the Fourier space $\mathbb{T}^d$ for the BCC lattice and time $\mathbb{L}^d \times \mathbb{Z}_+$, there is a way to pay attention to the Laplace space $\mathbb{T}$ for the simple cubic lattice and time $\mathbb{Z}^d \times \mathbb{Z}_+$. That corrects Nguyen and Yang’s lace-expansion analysis on $\mathbb{Z}^d \times \mathbb{Z}_+$ in dimensions $d \geq 183$ [25].

To go down to the desired $d+1 = 5 + 1$ dimensions, we must improve our analysis in various aspects. Some of the ideas are summarized as follows:

(i) As we explain in Remark 5.1 below, the spatial dimension 9 in Theorem 1.1 depends on how to look for the parameters $\{K_i\}_{i=1}^3$ of a bootstrap argument (See Sect. 1.5 for details). We prove the main theorem only in $d+1 \geq 9+1$ because we were not able to find the parameters satisfying the bootstrap argument in $d+1 = 8+1$. There is a possibility that one can update our result if one searches for the parameters carefully, but it is more important to improve our bounds for $\{g_i\}_{i=1}^3$, the lace-expansion coefficients, and so on than accuracy in computers.

(ii) In Lemma 2.4 below, we pay attention to the coefficient $1/2$ of the diagram containing four $q_p$’s. If one isolates the diagrams containing six $q_p$’s and specifies their coefficients, then one can obtain better upper bounds than those in this paper. The upper bounds on $B^{(\lambda,\rho)}_{p,m}$ and $T^{(\lambda,\rho)}_{p,m}$ in Lemma 4.1 below may be helpful for such an improvement. However, although their random-walk counterparts, such as $\sum_x (q_p \star \varphi_p^2(x)) (q_p \star \varphi_p)^2(x)$, are decreasing in $\lambda$ and $\rho$, the bound on $T^{(\lambda,\rho)}_{p,m}$ may attain the minimum at some $\lambda_*, \rho_* \in \mathbb{N}$, due to the exponentially growing factor $(pm)^{\lambda+\rho}$. That is a reason why we did not isolate the diagrams containing six $q_p$’s in the paper.

(iii) If one does not use the Laplace transform, then our result may be improved. We were not able to avoid using bad triangle inequality estimates due to complex numbers, e.g., not splitting contributions of $\Pi_p^{\text{even}}$ and $\Pi_p^{\text{odd}}$ in the second half in Lemma 2.4 below. We would like to neglect either $\Pi_p^{\text{even}}$ or $\Pi_p^{\text{odd}}$ because the odd terms in (2.2) below are negative. To this end, the inductive approach to the lace expansion [40, 41] may be usable. However, this method is shown for spread-out models. An extension to nearest-neighbor models is required.

(iv) Even if some improvements are found for ordinary percolation, such methods are not always applicable to oriented percolation. For example, the non-backtracking lace expansion [12, 13] achieved success for ordinary percolation. Its expansion coefficients express the perturbation of the non-backtracking random walk while the (ordinary) lace-expansion coefficients express that of the random walk.
The non-backtracking random walk is useful to approximate ordinary-percolation
clusters, but we guess that the property is not useful to approximate oriented-
percolation clusters. Indeed, paths in an oriented-percolation cluster are not
self-avoiding paths when we consider their projection onto the space. Therefore,
different methods between ordinary and oriented percolation are often desired.

1.5 Proof of the Main Result

Our proof of Theorem 1.1 is based on a bootstrap analysis. In this paper, we do not
take care of the details of the method. See either [36, Lemma 5.9] or [21, Lemma 8.1].
Thanks to it, it suffices for us to verify the following three propositions, of which in
particular we use the lace expansion in the third one.

Let

\[ \hat{U}_\mu(k, l) = (1 - \hat{D}(k)) \left( \frac{\hat{S}_\mu(l + k) + \hat{S}_\mu(l - k)}{2} \right) \]

\[ + (1 - \hat{D}(2l)) \left| \hat{S}_\mu(l) \right| \left| \hat{S}_\mu(l + k) \right| \left| \hat{S}_\mu(l - k) \right| \]

(1.12)

for \( k, l \in \mathbb{T}^d \) and \( \mu \in \mathbb{C} \) with \(|\mu| \in [0, 1]\), and let

\[ \mu_p(z) = \left( 1 - \hat{\phi}_p(0, |z|)^{-1} \right) e^{i \arg z} \]

(1.13)

for \( p \in [0, p_c) \) and \( z \in \mathbb{C} \) with \(|z| \in [1, m_p]\). We define the bootstrap functions \( \{g_i\}_{i=1}^3 \) as, for \( p \in [0, p_c) \) and \( m \in (1, m_p) \),

\[ g_1(p, m) := pm, \]

(1.14)

\[ g_2(p, m) := \sup_{k \in \mathbb{T}^d, z \in \mathbb{C}: |z| \in [1, m]} \left| \hat{\varphi}_p(k, z) \right| \]

\[ \left| \hat{S}_{\mu_p(z)}(k) \right|, \]

(1.15)

\[ g_3(p, m) := \sup_{k, l \in \mathbb{T}^d, z \in \mathbb{C}: |z| \in [1, m]} \left| \frac{1}{z} \Delta_k \left( \hat{q}_p(l, z) \hat{\varphi}_p(l, z) \right) \right| \]

\[ \hat{U}_{\mu_p(z)}(k, l), \]

(1.16)

where, for a function \( \hat{f} \) on \( \mathbb{T}^d \),

\[ \Delta_k \hat{f}(l) = \hat{f}(l + k) + \hat{f}(l - k) - 2 \hat{f}(l) \]

is the double discrete derivative. In (1.14), the supremum near \( k = 0 \) should be
interpreted as the supremum over the limit as \( \|k\|_2 \to 0 \). The following propositions
are the sufficient condition of [36, Lemma 5.9]. Thus, if Propositions 1.6–1.8 hold, then
it is true that there exists \( K_i \) such that \( g_i(p, m) < K_i \) for each \( i = 1, 2, 3 \). In particular,
the bound on \( g_2(p, m) \) and Lemma 3.5 below immediately imply Theorem 1.1.
Proposition 1.6 (Continuity) The functions $\{g_i\}_{i=1}^3$ are continuous in $m \in [1, m_p)$ for every $p \in (0, p_c)$, and the functions $\{g_i(p, 1)\}_{i=1}^3$ are continuous in $p \in (0, p_c)$.

Proposition 1.7 (Initial conditions) There are model-dependent finite constants $\{K_i\}_{i=1}^3$ such that $g_i(0, 1) < K_i$ for $i = 1, 2, 3$.

Proposition 1.8 (Bootstrap argument) For nearest-neighbor oriented percolation on $\mathbb{Z}^d \times \mathbb{N}_+$, we fix both $p \in (0, p_c)$ and $z \in \mathbb{C}$ with $|z| \in (1, m_p)$ and assume $g_i(p, m) \leq K_i$ for each $i = 1, 2, 3$, where $\{K_i\}_{i=1}^3$ are the same constants as in Proposition 1.7. Then, the stronger inequalities $g_i(p, m) < K_i$, $i = 1, 2, 3$, hold.

When the constants in Proposition 1.7 are denoted by $\{\tilde{K}_i\}_{i=1}^3$, it is also possible to take $\{K_i\}_{i=1}^3$ in Proposition 1.8 such that $K_i \geq \tilde{K}_i$. However, since we would like to obtain upper bounds as precisely as possible in the vicinity of the critical point $p_c$, the option is not very useful to bound $g_i(p, 1)$.

It is not hard to apply the method in [17, Sect. 3] to Proposition 1.6, hence we omit the proof. One can easily verify the continuities of $g_1(p, m)$ and $g_2(p, m)$. For $g_3(p, m)$, notice that we use the Markov property for oriented percolation instead of the nested expectations for ordinary percolation. Thanks to this, the proof is a little simplified. One can see the complete proof in [24]. The proof of Proposition 1.7 is also easy, so that we provide it here:

Proof of Proposition 1.7 Clearly, $g_1(0, 1) = 0$. By definitions, $\hat{\phi}_0(k, z) = 1, \mu_0(z) = 0$ and hence $S_{\hat{\phi}_0(z)}(k) = 1$. These lead to $g_2(0, 1) = 1$ and $g_3(0, 1) = 0$. Therefore, the initial conditions hold for arbitrary $K_i > 1, i = 1, 2, 3$. □

We must take the values $\{K_i\}_{i=1}^3$ satisfying Proposition 1.8. See the proof of Proposition 1.8 in Sect. 5 for the specific values.

In the rest of this paper, we focus on the proof of Proposition 1.8, which is shown in the following steps:

1. Bound the bootstrap functions $\{g_i(p, m)\}_{i=1}^3$ above by the lace expansion.
2. Bound the lace-expansion coefficients $(\pi_p^{(N)})_{N=0}^\infty$ above by basic diagrams $(B_{p,m}^{(\lambda, \rho)}, T_{p,m}^{(\lambda, \rho)}$ and $\hat{V}_{p,m}^{(\lambda, \rho)}(k))$.
3. Bound the basic diagrams above by the bootstrap hypotheses $g_i(p, m) \leq K_i$ for $i = 1, 2, 3$ and the random-walk quantities (1.2).
4. Verify the stronger inequalities $g_i(p, m) < K_i$ for $i = 1, 2, 3$ by combining the bounds in Steps 1–3 and substituting specific numerical values into $\{K_i\}_{i=1}^3$.

We explain Step 1 in Sect. 3, Step 2 in Section 2, Step 3 in Sect. 4 and Step 4 in Sect. 5.

2 Diagrammatic Bounds on the Expansion Coefficients

2.1 Review of the Lace Expansion

The lace expansion was first derived by Brydges and Spencer [7] for weakly self-avoiding walk. In the literature, there are three different ways to derive the lace
expansion for oriented percolation. The first is to directly apply Brydges and Spencer’s method due to the Markov property [27], the second is to use inclusion–exclusion relations and nested expectations [18], and the third is to use inclusion–exclusion relations and the Markov property [32]. In this paper, although we do not show its proof, we implicitly use the third approach and its representations of \( \{\pi_p(N)\}_{N=0}^{\infty} \).

The lace expansion gives a similar recursion equation for the two-point function \( \varphi_p \) to that for the random-walk two-point function \( Q_p \), which is

\[
Q_p(x) = \delta_{o,x} + (q_p \star Q_p)(x)
\]

for \( x = (x, t) \in \mathbb{L}_d \times \mathbb{Z}_+ \), where \( \delta_{o,x} = \delta_{o,x} \delta_{o,t} \) (cf., the recursion equation for the random-walk Green function \( S_p \)).

**Proposition 2.1** (Lace expansion) For any \( p < p_c \) and \( N \in \mathbb{Z}_+ \), there exist model-dependent non-negative functions \( \{\pi_p(N)\}_{n=0}^{\infty} \) on \( \mathbb{L}_d \times \mathbb{Z}_+ \) such that, if we define \( \Pi_p(N) \) as

\[
\Pi_p(N)(x) = \sum_{n=0}^{N} (-1)^n \pi_p(n)(x),
\]

we obtain the recursion equation

\[
\varphi_p(x) = \delta_{o,x} + \Pi_p(N)(x) + \left( (\delta_{o,\bullet} + \Pi_p(N)) \star q_p \star \varphi_p \right)(x) + (-1)^{N+1} R_p^{(N+1)}(x),
\]

where the remainder \( R_p^{(N+1)}(x) \) obeys the bound

\[
0 \leq R_p^{(N+1)}(x) \leq \left( \pi_p(N) \star \varphi_p \right)(x).
\]

Assume that \( \lim_{N \to \infty} \pi_p(N)(x) = 0 \) for every \( x \in \mathbb{L}_d \times \mathbb{Z}_+ \). This will be proved to hold by the absolute convergence of the alternating series of \( \{\pi_p(N)\}_{N=1}^{\infty} \) from Lemma 2.4 below. Then, \( R_p(N)(x) \to 0 \) as \( N \to \infty \) for every \( x \in \mathbb{L}_d \times \mathbb{Z}_+ \), so that the limit of the series

\[
\Pi_p(x) := \lim_{N \to \infty} \Pi_p(N)(x) = \sum_{N=0}^{\infty} (-1)^N \pi_p(N)(x)
\]

is well-defined.

To state the precise expression of \( \{\pi_p(N)\}_{N=0}^{\infty} \), we introduce some notation. Below, \( b \) and \( \bar{b} \) denote the bottom and top of a bond \( b \), respectively, that is \( b = (b, \bar{b}) \).

**Definition 2.2** Fix a bond configuration and let \( x, y, u, v \in \mathbb{L}_d \times \mathbb{Z}_+ \).
(i) Given a bond $b$, we define $\tilde{C}^b(x)$ to be the set of vertices connected to $x$ in the new configuration obtained by setting $b$ to be vacant.

(ii) We say that a bond $(u, v)$ is pivotal for $x \rightarrow y$ if $x \rightarrow u$ occurs and if $v \rightarrow y$ occurs in the complement of $\tilde{C}((u, v))(x)$. Let $\text{piv}(x, y)$ be the set of pivotal bonds for the connection from $x$ to $y$.

(iii) We say that $x$ is doubly connected to $y$, denoted by $x \Rightarrow y$, if either $x = y$ or $x \rightarrow y$ and $\text{piv}(x, y) = \emptyset$.

Let

$$\tilde{E}_{\vec{b}_N}^{(N)}(x) = \{ o \Rightarrow b_1 \} \cap \bigcap_{i=1}^{N} E\left( b_i, b_{i+1}; \tilde{C}^{b_i}(b_{i-1}) \right),$$

where $\vec{b}_N = (b_1, \ldots, b_N)$ is an ordered set of bonds on $\mathbb{L}^d \times \mathbb{Z}_+$ and $b_0 = o, b_{N+1} = x$ and

$$E(b, x; A) = \{ b \rightarrow x \in A \} \setminus \bigcup_{b' \in \text{piv}(b, x)} \{ b' \in A \}$$

for a bond $b$, a vertex $x \in \mathbb{L}^d \times \mathbb{Z}_+$ and a set $A \subset \mathbb{L}^d \times \mathbb{Z}_+$. According to [32, Proposition 4.1], the lace-expansion coefficients $\{ \pi^{(N)}_p \}_{N=0}^{\infty}$ are defined as

$$\pi^{(N)}_p(x) = \begin{cases} \mathbb{P}_p(o \Rightarrow x) - \delta_{o,x} & [N = 0], \\ \sum_{\vec{b}_N} \mathbb{P}_p \left( \tilde{E}_{\vec{b}_N}^{(N)}(x) \right) & [N \geq 1]. \end{cases}$$

### 2.2 Diagrammatic Bounds

We state upper bounds on the Fourier–Laplace transform of the sum (2.2) of the alternating series of the lace-expansion coefficients in terms of basic diagrams. To do so, we first introduce notations. Let $\varphi^{(m)}_p(x, t) = m' \varphi_p(x, t)$. Recall the notation $x \in \mathbb{L}^d \times \mathbb{Z}_+$. We define basic diagrams as, for $\lambda, \rho \in \mathbb{N}, m > 0$ and $k \in \mathbb{T}_d$,

$$B^{(\lambda, \rho)}_{p, m} := \sup_x \sum_y \left( q^{\lambda \rho}_p \star \varphi_p \right)(y) \left( m^\rho q^{\rho}_p \star \varphi^{(m)}_p \right)(y - x),$$

(2.3)

$$T^{(\lambda, \rho)}_{p, m} := \sup_x \sum_y \left( q^{\lambda \rho}_p \star \varphi^{(2)}_p \right)(y) \left( m^\rho q^{\rho}_p \star \varphi^{(m)}_p \right)(y - x),$$

(2.4)

$$\hat{V}^{(\lambda, \rho)}_{p, m}(k) := \sup_x \sum_y \left( q^{\lambda}_p \star \varphi_p \right)(y) \left( 1 - \cos k \cdot y \right) \left( m^\rho q^{\rho}_p \star \varphi^{(m)}_p \right)(y - x).$$

(2.5)
We represent the transition probability \( q_p \) by a pair of either parallel lines or a line and a dot, and we represent the two-point function \( \varphi_p \) by a line segment. For example,

\[
\varphi_p(x) = \begin{cases} x \\ 0 \end{cases}, \quad (q_p^2 * \varphi_p)(x) = \begin{cases} x \\ -x \end{cases},
\]

where the unlabeled vertices (short lines and dots) are summed over \( \mathbb{L}^d \times \mathbb{Z}_+ \). The segments emphasized by the braces mean weighted two-point functions or weighted space-time transition probabilities: \( \varphi_p \) or \( q_p \) multiplied by \( m \)'s or \( 1 - \cos k \cdot \bullet \). Time increases from the beginning point to the ending point of a line. This representation is useful to bound \( \{\pi_p^{(N)}\}_{N=0}^\infty \) above. Moreover, we divide \( \Pi_p(x) \) into two parts as:

\[
\Pi_{p}^{\text{even}}(x) := \sum_{N=1}^\infty \pi_p^{(2N)}(x), \quad \Pi_p^{\text{odd}}(x) := \pi_p^{(1)}(x) - \pi_p^{(0)}(x) + \sum_{N=1}^\infty \pi_p^{(2N+1)}(x).
\]

Although the latter contains the zeroth lace expansion coefficient, we name it “odd” for convenience.

Next, we show upper bounds on the lace-expansion coefficients \( \{\pi_p^{(N)}\}_{N=0}^\infty \). Taking the sum of the bounds in Lemma 2.3 over \( N \in \mathbb{Z}_+ \) immediately implies Lemma 2.4. We state only their statements in this subsection, and we prove them in the next subsection.

**Lemma 2.3** Let \( N \) be an integer greater than or equal to 3. When we multiply the upper bounds in Lemma 6.1 by \( m^l \),

\[
|\hat{\pi}_p^{(0)}(0, m) - \hat{\pi}_p^{(1)}(0, m)| \leq \frac{1}{2} B_{p,m}^{(2,2)} + B_{p,1}^{(2,2)} B_{p,m}^{(0,2)} + \frac{3}{2} B_{p,1}^{(2,2)} B_{p,m}^{(1,3)} + 3 \left( B_{p,m}^{(2,2)} \right)^2 + 3 B_{p,m}^{(2,1)} T_{p,m}^{(2,2)},
\]

(2.7)
When we multiply the upper bounds in Lemma 6.1 by $m^t$ and $t$,

\[
\sum_{(x,t)} \left| \pi_p^{(2)}(x,t) - \pi_p^{(1)}(x,t) \right| m^t\]

\[
\leq \frac{1}{2} \left( B_{p,m}^{(2,2)} + T_{p,m}^{(2,2)} \right) + \frac{3}{2} B_{p,1}^{(2,2)} \left( B_{p,m}^{(0,2)} + T_{p,m}^{(0,2)} \right)
+ 2 \left( m T_{p,m}^{(2,2)} + B_{p,m}^{(2,2)} T_{p,m}^{(2,2)} + T_{p,m}^{(2,2)} T_{p,m}^{(2,1)} \right) \tag{2.10}
\]

\[
\sum_{(x,t)} \pi_p^{(2)}(x,t) m^t t
\leq B_{p,1}^{(2,2)} \left( 2 B_{p,m}^{(1,3)} + T_{p,m}^{(1,3)} \right) + 4 B_{p,m}^{(2,2)} \left( B_{p,m}^{(2,2)} + T_{p,m}^{(2,2)} \right)
+ 2 \left( m T_{p,m}^{(2,2)} + B_{p,m}^{(2,2)} T_{p,m}^{(2,2)} + T_{p,m}^{(2,2)} T_{p,m}^{(2,1)} \right)
+ \frac{1}{2} B_{p,1}^{(2,2)} \left( 3 B_{p,m}^{(1,3)} T_{p,m}^{(1,2)} + 2 T_{p,m}^{(1,3)} T_{p,m}^{(1,2)} \right)
+ \left( 3 T_{p,m}^{(1,2)} T_{p,m}^{(1,1)} T_{p,m}^{(2,1)} + B_{p,m}^{(1,2)} T_{p,m}^{(1,1)} T_{p,m}^{(2,1)} \right)
+ \frac{1}{2} B_{p,1}^{(2,2)} \left( 2 B_{p,m}^{(1,3)} T_{p,m}^{(2,1)} + 2 T_{p,m}^{(1,3)} T_{p,m}^{(2,1)} \right) + 3 T_{p,m}^{(2,1)} T_{p,m}^{(1,1)} T_{p,m}^{(2,1)} \tag{2.11}
\]

\[
\sum_{(x,t)} \pi_p^{(N)}(x,t) m^t t
\leq \left( T_{p,m}^{(1,1)} + \frac{1}{2} \left( 2 B_{p,m}^{(2,2)} B_{p,m}^{(1,3)} + B_{p,m}^{(2,2)} T_{p,m}^{(3,1)} \right) + T_{p,m}^{(1,1)} \right)^2 \left( 2 T_{p,m}^{(1,1)} \right)^{N-1}
+ (N - 2) \left( T_{p,m}^{(1,1)} + \frac{1}{2} B_{p,m}^{(2,2)} T_{p,m}^{(1,3)} + T_{p,m}^{(1,1)} \right)^2 \left( 2 T_{p,m}^{(1,1)} \right)^{N-2}
+ \left( T_{p,m}^{(1,1)} + \frac{1}{2} B_{p,m}^{(2,2)} T_{p,m}^{(1,3)} + T_{p,m}^{(1,1)} \right)^2 \left( 2 T_{p,m}^{(1,1)} \right)^{N-1}. \tag{2.12}
\]

When we multiply the upper bounds in Lemma 6.1 by $m^t$ and $1 - \cos k \cdot x$,
\[ \sum_{(x,t)} \left| \pi_p^{(0)}(x,t) - \pi_p^{(1)}(x,t) \right| m^t (1 - \cos k \cdot x) \]
\[ \leq \frac{1}{2} \hat{V}_{p,m}^{(2)}(k) + 2 \left( \hat{V}_{p,1}^{(2)}(k) B_{p,m}^{(0,2)} + B_{p,1}^{(2,2)} \hat{V}_{p,m}^{(1,2)}(k) \right) \]
\[ + \frac{3}{2} B_{p,m}^{(2,2)} \hat{V}_{p,m}^{(3,1)}(k) \]
\[ + 12 B_{p,m}^{(2,2)} \hat{V}_{p,m}^{(2,2)}(k) + 6 \left( \hat{V}_{p,m}^{(1,2)}(k) T_{p,m}^{(2,2)} + T_{p,m}^{(2,2)} \hat{V}_{p,m}^{(2,1)}(k) \right), \] (2.13)
\[ \hat{\pi}_p^{(2)}(0,m) - \hat{\pi}_p^{(2)}(k,m) \]
\[ \leq B_{p,m}^{(2,2)} \hat{V}_{p,m}^{(3,1)}(k) + 8 B_{p,m}^{(2,2)} \hat{V}_{p,m}^{(2,2)}(k) \]
\[ + 4 \left( \hat{V}_{p,m}^{(1,2)}(k) T_{p,m}^{(2,2)} + T_{p,m}^{(2,2)} \hat{V}_{p,m}^{(2,1)}(k) \right) \]
\[ + B_{p,m}^{(2,2)} \left( \hat{V}_{p,m}^{(3,1)}(k) T_{p,m}^{(2,1)} + T_{p,m}^{(3,1)} \hat{V}_{p,m}^{(2,1)}(k) \right) \]
\[ + 3 \left( \hat{V}_{p,m}^{(1,2)}(k) T_{p,m}^{(1,1)} T_{p,m}^{(2,1)} \right) \]
\[ + T_{p,m}^{(1,2)} \hat{V}_{p,m}^{(1,1)}(k) T_{p,m}^{(2,1)} + T_{p,m}^{(1,2)} T_{p,m}^{(1,1)} \hat{V}_{p,m}^{(2,1)}(k) \right) \]
\[ + B_{p,m}^{(2,2)} \left( \hat{V}_{p,m}^{(3,1)}(k) T_{p,m}^{(1,2)} + T_{p,m}^{(3,1)} \hat{V}_{p,m}^{(1,2)}(k) \right) \]
\[ + 3 \left( \hat{V}_{p,m}^{(1,2)}(k) T_{p,m}^{(1,1)} T_{p,m}^{(1,2)} \right) \]
\[ + T_{p,m}^{(1,2)} \hat{V}_{p,m}^{(1,1)}(k) T_{p,m}^{(1,2)} + T_{p,m}^{(1,2)} T_{p,m}^{(1,1)} \hat{V}_{p,m}^{(1,2)}(k) \), \] (2.14)
\[ \hat{\pi}_p^{(N)}(0,m) - \hat{\pi}_p(K,m) \]
\[ \leq (N + 1) \left( \hat{V}_{p,m}^{(1,1)}(k) + \frac{1}{2} B_{p,m}^{(2,2)} \hat{V}_{p,m}^{(1,3)}(k) \right) \]
\[ + 2 T_{p,m}^{(1,1)} \hat{V}_{p,m}^{(1,1)}(k) \left( 2 T_{p,m}^{(1,1)} \right)^{N-1} \]
\[ + 2 (N - 1) \left( T_{p,m}^{(1,1)} + \frac{1}{2} B_{p,m}^{(2,2)} T_{p,m}^{(1,3)} \right) \]
\[ + \left( T_{p,m}^{(1,1)} \right)^2 \hat{V}_{p,m}^{(1,1)}(k) \left( 2 T_{p,m}^{(1,1)} \right)^{N-2} \]
\[ + 2 \left( T_{p,m}^{(1,1)} + \frac{1}{2} B_{p,m}^{(2,2)} T_{p,m}^{(1,3)} + \left( T_{p,m}^{(1,1)} \right)^2 \left( 2 T_{p,m}^{(1,1)} \right)^{N-2} \hat{V}_{p,m}^{(1,1)}(k) \right). \] (2.15)

**Lemma 2.4** Suppose that \(2T_{p,m}^{(1,1)} < 1\). Then, the alternating series (2.2) absolutely converges, and the following inequalities hold for \(p < P_c, m < m_p\) and \(k \in \mathbb{T}^d\):

\[ \hat{\pi}_p^{\text{even}}(0,m) \leq \hat{\pi}_p^{(2)}(0,m) \]
\[ + \left( B_{p,m}^{(1,1)} + \frac{1}{2} B_{p,1}^{(2,2)} B_{p,m}^{(1,3)} + T_{p,m}^{(1,1)} B_{p,m}^{(1,1)} \right) \left( 2 T_{p,m}^{(1,1)} \right)^3 \]
\[ \frac{1}{1 - (2 T_{p,m}^{(1,1)})^2}, \] (2.16)
\[ \hat{\pi}_p^{\text{odd}}(0, m) \leq \left| \hat{\pi}_p^{(0)}(0, m) - \hat{\pi}_p^{(1)}(0, m) \right| \\
+ \left( B_{p,m}^{(1,1)} + \frac{1}{2} B_{p,m}^{(2,2)} B_{p,m}^{(1,3)} + T_{p,m}^{(1,1)} B_{p,m}^{(1,1)} \right) \frac{(2T_{p,m}^{(1,1)})^2}{1 - (2T_{p,m}^{(1,1)})^2}, \quad (2.17) \]

\[ \sum_{(x,t)} \left( \Pi_p^{\text{even}}(x, t) + \Pi_p^{\text{odd}}(x, t) \right) m't \\
\leq \sum_{(x,t)} \left| \pi_p^{(0)}(x, t) - \pi_p^{(1)}(x, t) \right| + \pi_p^{(2)}(x, t) \right) m't \\
+ \left( T_{p,m}^{(1,1)} + \frac{1}{2} (2B_{p,m}^{(2,2)} B_{p,m}^{(1,3)} + B_{p,m}^{(2,2)} T_{p,m}^{(3,1)}) + 2 (T_{p,m}^{(1,1)})^2 \right) \times \frac{(2T_{p,m}^{(1,1)})^2}{1 - 2T_{p,m}^{(1,1)}} \\
+ \left( T_{p,m}^{(1,1)} + \frac{1}{2} B_{p,m}^{(2,2)} T_{p,m}^{(1,3)} + B_{p,m}^{(2,2)} T_{p,m}^{(3,1)} + 2 (T_{p,m}^{(1,1)})^2 \right) \frac{(2T_{p,m}^{(1,1)})^2}{1 - 2T_{p,m}^{(1,1)}}, \quad (2.18) \]

\[ \sum_{(x,t)} \left( \Pi_p^{\text{even}}(x, t) + \Pi_p^{\text{odd}}(x, t) \right) m' (1 - \cos k \cdot x) \]

\[ \leq \sum_{(x,t)} \left| \pi_p^{(0)}(x, t) - \pi_p^{(1)}(x, t) \right| m' (1 - \cos k \cdot x) \\
+ \left( \hat{\pi}_p^{(2)}(0, m) - \hat{\pi}_p^{(2)}(k, m) \right) \\
+ \left( \hat{\pi}_p^{(1,1)}(k) + \frac{1}{2} B_{p,m}^{(2,2)} \hat{\pi}_p^{(1,3)}(k) + 2 T_{p,m}^{(1,1)} \hat{\pi}_p^{(1,1)}(k) \right) \times \frac{8(T_{p,m}^{(1,1)})^2 (2 - 3T_{p,m}^{(1,1)})}{(1 - 2T_{p,m}^{(1,1)})^2} \\
+ 2 \left( T_{p,m}^{(1,1)} + \frac{1}{2} B_{p,m}^{(2,2)} T_{p,m}^{(1,3)} + (T_{p,m}^{(1,1)})^2 \right) \times \hat{\pi}_p^{(1,1)}(k) \frac{8T_{p,m}^{(1,1)} (1 - T_{p,m}^{(1,1)})}{(1 - 2T_{p,m}^{(1,1)})^3} \\
+ 2 \left( T_{p,m}^{(1,1)} + \frac{1}{2} B_{p,m}^{(2,2)} T_{p,m}^{(1,3)} + (T_{p,m}^{(1,1)})^2 \right) \times \hat{\pi}_p^{(1,1)}(k) \frac{4T_{p,m}^{(1,1)} (2 - 3T_{p,m}^{(1,1)})}{(1 - 2T_{p,m}^{(1,1)})^2}. \quad (2.19) \]

Here, upper bounds on \( \left| \hat{\pi}_p^{(0)}(0, m) - \hat{\pi}_p^{(1)}(0, m) \right|, \hat{\pi}_p^{(2)}(0, m), \) etc. are given by Lemma 2.3.
The proof of Lemma 2.3 is based on the diagrammatic representations of \( \pi_p(N)_{N=0}^\infty \) and the methods in [17, 33, 39]. The former is to bound \( \pi_p(N)_{N=0}^\infty \) above in terms of sums of products of two-point functions. The latter is to split events of percolation connections into some events depending on whether or not each line collapses. Lemma 2.3 is a well-known result in this sense, hence we explain its proof in Appendix 1. However, these upper bounds are a little sharper than previous research in a certain sense because we take care of the explicit form of not only the leading terms but also the remainder terms. They are often expressed as the order \( O(d^{-1}) \) relying on \( d \gg d_c \) (or the order \( O(L^{-d}) \) relying on \( L \gg 1 \) in the case of the spread-out model). We believe that the upper bounds are worth showing in this paper.

In particular, each of the first term in (2.7), (2.10) and (2.13) is a leading term. The coefficient \( 1/2 \) results from [39, Eq. (3.11)]. The sum \( \lambda + \rho \) of the superscript of \( B^{(\lambda,\rho)}_{p,m} \) (similarly, \( T^{(\lambda,\rho)}_{p,m} \) and \( \hat{V}^{(\lambda,\rho)}_{p,m}(k) \)) means how many \( q_p \)'s the diagram contains at least. For example, the leading term \( B^{(2,2)}_{p,m} / 2 \) in (2.7) contains at least four \( q_p \)'s. If one isolates the diagrams containing six \( q_p \)'s from error terms and specifies their coefficients, then one can obtain more precise upper bounds than Lemma 2.3. However, too many \( q_p \)'s aggravate upper bounds on the basic diagrams due to many multiplicative constants \( K_1 \)'s (See Lemma 4.1). That is a reason why we isolate the diagrams containing only four \( q_p \)'s in this paper.

### 3 Diagrammatic Bounds on the Bootstrap Functions

In this section, using the lace expansion, we show upper bounds on \( \{g_i\}_{i=1}^3 \) in terms of the lace-expansion coefficients. As a consequence, we obtain the next proposition. A sufficient condition (3.1) below is to be verified after we complete the bootstrap argument. Recall \( \Pi_p(x) = \Pi_p^{\text{even}}(x) - \Pi_p^{\text{odd}}(x) \).

**Proposition 3.1** Suppose that \( 2T^{(1,1)}_{p,m} < 1 \) and that \( B^{(\lambda,\rho)}_{p,m} , T^{(\lambda,\rho)}_{p,m} , \hat{V}^{(\lambda,\rho)}_{p,m}(k) \) for any \( \rho, \lambda \in \mathbb{N} \) are so small that the inequality

\[
\sum_{n=0}^\infty \hat{\pi}_p(0, m) < 1
\]  

holds. Then, we have

\[
g_1(p, m) \leq \frac{1}{1 - \hat{\pi}_p^{\text{odd}}(0, m)},
\]

\[
g_2(p, m) \leq \frac{1 + \hat{\pi}_p^{\text{even}}(0, m) + \hat{\pi}_p^{\text{odd}}(0, m)}{1 - \hat{\pi}_p^{\text{odd}}(0, m)} \quad + \frac{2K_2\hat{\pi}_p^{\text{even}}(0, m)}{1 - \hat{\pi}_p^{\text{odd}}(0, m)}
\]

\[
+ \frac{2K_1K_2}{1 - \hat{\pi}_p^{\text{odd}}(0, m)}
\]
\[ \times \left( \pi \sum_{(x,t)} \left( \Pi_p^{\text{even}}(x,t) + \Pi_p^{\text{odd}}(x,t) \right) m't \right) \]
\[ \vee \left( \sum_{(x,t)} \left( \Pi_p^{\text{even}}(x,t) + \Pi_p^{\text{odd}}(x,t) \right) m' \frac{1 - \cos k \cdot x}{1 - D(k)} \right), \]
\[ g_3(p,m) \]
\[ \leq \left( 1 \vee \frac{g_2(p,m)}{1 - \hat{\Pi}_p^{\text{even}}(0,m) - \hat{\Pi}_p^{\text{odd}}(0,m)} \right)^3 K_1^2 \]
\[ \times \left( 1 + 2 \left( \hat{\Pi}_p^{\text{even}}(0,m) + \hat{\Pi}_p^{\text{odd}}(0,m) \right) \right) \]
\[ + 2 \left\| \left( \hat{\Pi}_p^{\text{even}}(0,m) + \hat{\Pi}_p^{\text{odd}}(0,m) \right) - \left( \hat{\Pi}_p^{\text{even}}(\bullet,m) + \hat{\Pi}_p^{\text{odd}}(\bullet,m) \right) \right\|_{\infty}^2, \]
\[ \text{(3.4)} \]

where \( a \vee b \) denotes \( \max\{a, b\} \).

We show the proofs of (3.2)–(3.4) in Sects. 3.1–3.3, respectively, subject to one proposition and two lemmas. These results are then proved in Sect. 3.4.

Assume the absolute convergence of (2.2) (in other words, (3.1)). Then, taking the Fourier–Laplace transform on both sides in (2.1) implies that

\[ \hat{\phi}_p(k,z) = \frac{1 + \hat{\Pi}_p(k,z)}{1 - \hat{q}_p(k,z)(1 + \hat{\Pi}_p(k,z))}. \]
\[ \text{(3.5)} \]

We use this expression many times.

### 3.1 An Upper Bound for \( g_1(p,m) \)

**Proof of (3.2) in Proposition 3.1** Note that \( \hat{\phi}_p(0,m) \geq 1 \) since \( m \geq 1 \). This inequality and (3.5) yield

\[ \hat{\phi}_p(0,m) = \frac{1 + \hat{\Pi}_p(0,m)}{1 - pm(1 + \hat{\Pi}_p(0,m))} \iff pm = \frac{1}{1 + \hat{\Pi}_p(0,m)} - \hat{\phi}_p(0,m)^{-1}. \]

Thus, we obtain

\[ g_1(p,m) \leq \frac{1}{1 + \hat{\Pi}_p^{\text{even}}(0,m) - \hat{\Pi}_p^{\text{odd}}(0,m)} \leq \frac{1}{1 - \hat{\Pi}_p^{\text{odd}}(0,m)}. \]
\[ \text{(3.6)} \]
3.2 An Upper Bound for $g_2(p, m)$

As preliminaries, we show the statements of one proposition and two lemmas in the following. The next proposition plays the most important role. Notice that $\{k \in \mathbb{T}^d \mid \hat{D}(k) \geq 0\} = [-\frac{\pi}{2}, \frac{\pi}{2}]^d$ on $\mathbb{L}^d$.

**Proposition 3.2** On the BCC lattice $\mathbb{L}^d$,

$$
g_2(p, m) = \sup_{k \in \mathbb{T}^d} \frac{|\hat{\phi}_p(k, z)|}{|\hat{S}_{\mu_p}(z)(k)|} = \sup_{k \in [-\frac{\pi}{2}, \frac{\pi}{2}]^d} \frac{|\hat{\phi}_p(k, z)|}{|\hat{S}_{\mu_p}(z)(k)|}.
$$

**Lemma 3.3** For every $k \in \mathbb{T}^d$ satisfying $\hat{D}(k) \geq 0$, $r \in [0, 1]$ and $\theta \in \mathbb{T}$,

$$
\left| 1 - re^{i\theta} \hat{D}(k) \right| \geq \frac{1}{2} \left( \frac{|\theta|}{\pi} + 1 - \hat{D}(k) \right). \quad (3.7)
$$

**Remark 3.4** The proof of Proposition 3.2 depends on the structure of the BCC lattice $\mathbb{L}^d$. It suffices for $\mathbb{L}^d$ to reduce the range of $k$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]^d$ due to reflections in perpendicular hyperplanes to the axes, whereas we need extra efforts to prove the same claim for $\mathbb{Z}^d$ due to lack of symmetry. The shape of the domain of $k$ such that $\hat{D}(k) \geq 0$ for $\mathbb{Z}^d$, that is $\{k \in \mathbb{T}^d \mid d^{-1} \sum_{j=1}^{d} \cos k_j \geq 0\}$ is not as simple as $\{k \in \mathbb{T}^d \mid \prod_{j=1}^{d} \cos k_j \geq 0\}$ for $\mathbb{L}^d$. However, focusing on $\theta = \arg z$ let us reduce the range over which the supremum is taken on the simple cubic lattice $\mathbb{Z}^d$. Specifically, it suffices to only consider the range $[-\frac{\pi}{2}, \frac{\pi}{2}]$ in order to calculate the supremum in $g_2(p, m)$ with respect to $\theta$ on $\mathbb{Z}^d$, and a similar inequality as (3.7) holds. Based on this idea, one can prove the infrared bound (1.7) on $\mathbb{Z}^{d=183} \times \mathbb{Z}_+$. See [25] for details.

The next lemma is almost identical to the above lemma, but note that the ranges of $k$ are different:

**Lemma 3.5** For every $k \in \mathbb{T}^d$ and $\mu \in \mathbb{C}$ satisfying $|\mu| \leq 1$,

$$
\left| 1 - e^{i\arg \mu} \hat{D}(k) \right| \leq 2 \left| 1 - \mu \hat{D}(k) \right|. \quad (3.8)
$$

**Proof of (3.3) in Proposition 3.1** Applying the same method as [21, Lemma 8.11], we rewrite $\hat{\phi}_p$ as

$$
\frac{\hat{\phi}_p(k, z)}{\hat{S}_{\mu_p}(z)(k)} = \frac{1 + \hat{\Pi}_p(k, z)}{1 + \hat{\Pi}_p(0, |z|)} + \frac{\hat{\phi}_p(k, z)}{1 + \hat{\Pi}_p(0, |z|)} \times \left( \hat{\Pi}_p(0, |z|)(1 - e^{i\theta} \hat{D}(k)) + \hat{\phi}_p(k, z)(\hat{\Pi}_p(k, z) - \hat{\Pi}_p(0, |z|)) \right),
$$

where $\theta = \arg z$. By Proposition 3.2 and the triangle inequality,
\[ g_2(p, m) \]
\[ \leq \sup_{k \in [-\frac{\pi}{2}, \frac{\pi}{2}], \ z \in \mathbb{C} : |z| \leq 1} \frac{1 + |\hat{\Pi}_p(k, z)|}{1 + \hat{\Pi}_p(0, |z|)} + \frac{|\hat{\phi}_p(k, z)|}{1 + \hat{\Pi}_p(0, |z|)} \left( |\hat{\Pi}_p(0, |z|)| \left| 1 - e^{i\theta} \hat{D}(k) \right| + p |z| \left| \hat{D}(k) \right| \right) \equiv 1 \leq 1 \]
\[ + K_1 \left( |\hat{\Pi}_p(0, |z|) - \hat{\Pi}_p(0, z)| + |\hat{\Pi}_p(0, z) - \hat{\Pi}_p(k, z)| \right) \right) \] (3.9)

By the bootstrap hypotheses \( g_i(p, m) \leq K_i \) for \( i = 1, 2 \),
\[ g_2(p, m) \]
\[ \leq \sup_{k \in [-\frac{\pi}{2}, \frac{\pi}{2}], \ z \in \mathbb{C} : |z| \leq 1} \frac{1 + |\hat{\Pi}_p(k, z)|}{1 + \hat{\Pi}_p(0, |z|)} + \frac{K_2 \hat{\Delta}_{p, p}(z)}{1 + \hat{\Pi}_p(0, |z|)} \left( |\hat{\Pi}_p(0, |z|)| \left| 1 - e^{i\theta} \hat{D}(k) \right| \right) \]
\[ + K_1 \left( |\hat{\Pi}_p(0, |z|) - \hat{\Pi}_p(0, z)| + |\hat{\Pi}_p(0, z) - \hat{\Pi}_p(k, z)| \right) \right) \] (3.10)

Recall \( \Pi_p^{even}(x) = \sum_{N=1}^{\infty} \pi_p^{(2N)}(x) \) and \( \Pi_p^{odd}(x) = \pi_p^{(1)}(x) - \pi_p^{(0)}(x) + \sum_{N=1}^{\infty} \pi_p^{(2N+1)}(x) \). Applying the triangle inequality to \( |\hat{\Pi}_p(k, z)| \), \( |\hat{\Pi}_p(0, |z|) - \hat{\Pi}_p(0, z)| \) and \( |\hat{\Pi}_p(0, z) - \hat{\Pi}_p(k, z)| \), we obtain, respectively,
\[ |\hat{\Pi}_p(k, z)| \leq |\hat{\Pi}_p^{even}(k, z)| + |\hat{\Pi}_p^{odd}(k, z)| \]
\[ = \sum_{(x,t)} \sum_{N=1}^{\infty} \pi_p^{(2N)}(x) e^{i k \cdot x} z^t \]
\[ + \sum_{(x,t)} \sum_{N=1}^{\infty} \left( \pi_p^{(1)}(x) - \pi_p^{(0)}(x) + \pi_p^{(2N+1)}(x) \right) e^{i k \cdot x} z^t \]
\[ \leq \sum_{(x,t)} \sum_{N=1}^{\infty} \pi_p^{(2N)}(x) |z|^t \]
\[ + \sum_{(x,t)} \sum_{N=1}^{\infty} \left( \pi_p^{(1)}(x) - \pi_p^{(0)}(x) + \pi_p^{(2N+1)}(x) \right) |z|^t \]
\[ = \hat{\Pi}_p^{even}(0, |z|) + \hat{\Pi}_p^{odd}(0, |z|), \]
\[ |\hat{\Pi}_p(0, |z|) - \hat{\Pi}_p(0, z)| \]
\[ \leq \sum_{(x,t)} \left( \pi_p^{(1)}(x, t) - \pi_p^{(0)}(x, t) + \sum_{N=2}^{\infty} \pi_p^{(N)}(x, t) \right) |z|^t |1 - e^{i\theta t}| \]
\[ = \sum_{(x,t)} \left( \Pi_p^{even}(x, t) + \Pi_p^{odd}(x, t) \right) |z|^t |1 - e^{i\theta t}| \]
and (note that each $\pi_p^{(N)}(x, t)$ is an even function with respect to $x$)

$$|\hat{\Pi}_p(0, z) - \hat{\Pi}_p(k, z)|$$

$$\leq \sum_{(x, t)} \left( \pi_p^{(1)}(x, t) - \pi_p^{(0)}(x, t) + \sum_{N=2}^{\infty} \pi_p^{(N)}(x, t) \right) |z|^t (1 - \cos k \cdot x)$$

$$= \sum_{(x, t)} \left( \Pi_p^{\text{even}}(x, t) + \Pi_p^{\text{odd}}(x, t) \right) |z|^t (1 - \cos k \cdot x).$$

Substituting these bounds and (3.2) into (3.10) leads to

$$g_2(p, m)$$

$$\leq 1 + \hat{\Pi}_p^{\text{even}}(0, m) + \hat{\Pi}_p^{\text{odd}}(0, m)$$

$$+ \frac{K_2}{1 - \hat{\Pi}_p^{\text{odd}}(0, m)} \sup_{k \in [-\frac{\pi}{2}, \frac{\pi}{2}], z \in \mathbb{C}: |z| \in [1, m]} \left( \hat{\Pi}_p(0, |z|) \right) |1 - e^{i\theta} \hat{D}(k)|$$

$$+ \frac{K_1}{1 - \hat{\Pi}_p^{\text{odd}}(0, m)} \sup_{k \in [-\frac{\pi}{2}, \frac{\pi}{2}], z \in \mathbb{C}: |z| \in [1, m]} \left( \hat{\Pi}_p(0, |z|) \right) |1 - e^{i\theta} t| + |1 - \cos k \cdot x|$$

By Lemmas 3.3–3.5 and the inequality $|1 - e^{i\theta}| = 2 |\sin(\theta t/2)| \leq t |\theta|$, we have

$$g_2(p, m)$$

$$\leq 1 + \hat{\Pi}_p^{\text{even}}(0, m) + \hat{\Pi}_p^{\text{odd}}(0, m)$$

$$+ \frac{K_2}{1 - \hat{\Pi}_p^{\text{odd}}(0, m)} \sup_{k \in [-\frac{\pi}{2}, \frac{\pi}{2}], z \in \mathbb{C}: |z| \in [1, m]} \left( \hat{\Pi}_p(0, |z|) \right) |1 - e^{i\theta} \hat{D}(k)|$$

$$+ 2K_1 \sum_{(x, t)} \left( \Pi_p^{\text{even}}(x, t) + \Pi_p^{\text{odd}}(x, t) \right) |z|^t \cdot \frac{t |\theta| + 1 - \cos k \cdot x}{\pi^{-1} |\theta| + 1 - \hat{D}(k)}$$

$$\leq 1 + \hat{\Pi}_p^{\text{even}}(0, m) + \hat{\Pi}_p^{\text{odd}}(0, m)$$

$$+ \frac{2K_1 K_2}{1 - \hat{\Pi}_p^{\text{odd}}(0, m)} + \frac{2K_2 \hat{\Pi}_p^{\text{even}}(0, m)}{1 - \hat{\Pi}_p^{\text{odd}}(0, m)}$$

$$\times \sup_{k \in [-\frac{\pi}{2}, \frac{\pi}{2}], z \in \mathbb{C}: |z| \in [1, m]} \sum_{(x, t)} \left( \Pi_p^{\text{even}}(x, t) + \Pi_p^{\text{odd}}(x, t) \right) |z|^t \cdot \frac{t |\theta| + 1 - \cos k \cdot x}{\pi^{-1} |\theta| + 1 - \hat{D}(k)}$$
\[ g_2(p, m) \leq \frac{1 + \hat{\Pi}_p^{\text{even}}(0, m) + \hat{\Pi}_p^{\text{odd}}(0, m)}{1 - \hat{\Pi}_p^{\text{odd}}(0, m)} + \frac{2K_2\hat{\Pi}_p^{\text{even}}(0, m)}{1 - \hat{\Pi}_p^{\text{odd}}(0, m)} \]
\[ + \frac{2K_1K_2}{1 - \hat{\Pi}_p^{\text{odd}}(0, m)} \cdot \pi^{-1} |\theta| + 1 - \hat{D}(k) \]
\[ \times \sup_{k \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], z \in \mathbb{C}, |z| \in (1, m)} \left( \pi \sum_{(x, t)} \left( \hat{\Pi}_p^{\text{even}}(x, t) + \hat{\Pi}_p^{\text{odd}}(x, t) \right) |z|^t \cdot \pi^{-1} |\theta| \right) \]
\[ + \sum_{(x, t)} \left( \hat{\Pi}_p^{\text{even}}(x, t) + \hat{\Pi}_p^{\text{odd}}(x, t) \right) |z|^t \frac{1 - \cos k \cdot x}{1 - \hat{D}(k)} \cdot \left( 1 - \hat{D}(k) \right). \]  
\[ (3.11) \]

**Remark 3.6** Notice that, if we do not use Proposition 3.2 and naively apply the triangle inequality as in (3.9), then
\[ \left| \hat{\Pi}_p(0, |z|) - \hat{\Pi}_p(k, z) \right| \leq \left| \hat{\Pi}_p(0, |z|) - \hat{\Pi}_p(0, z) \right| + \left| \hat{\Pi}_p(0, z) - \hat{\Pi}_p(k, z) \right| \]
\[ \frac{1 - e^{i\theta} \hat{D}(k)}{1 - e^{i\theta} \hat{D}(k)} \]
\[ (3.12) \]

appears in an upper bound on \( g_2(p, m) \), but this bound becomes infinite for some situations. When \( \hat{D}(k) = -1 \) and \( \theta = \pm \pi \), its denominator equals 0 even though its numerator is non-zero. Thanks to Proposition 3.2, we can avoid this problem. Although one may consider keeping the left hand side in (3.12) without using the triangle inequality, that does not work currently because we do not have any nice methods to decompose diagrams with complex numbers.
Remark 3.7  Nguyen and Yang wrote “...$1 - e^{i\theta} \hat{D}(k)$ is bounded below by a positive constant for $\{(k, \ell) \in [-\pi, \pi]^d \times [-\pi, \pi] : |(k, \ell)| > \varepsilon\} ...” below [27, Eq. (13)]. However, it is impossible for the nearest-neighbor model on the simple cubic lattice $\mathbb{Z}^d$ because it can occur that $\hat{D}(k) = -1$ when $k_i = \pm \pi$ for some $i = 1, \ldots, d$. That is a reason why we need Proposition 3.2. The spread-out model does not cause such problem due to $1 + \hat{D}_L(k) > 0$ with $L \gg 1$ in [27, Proof of Lemma 10], where $D_L$ is the random-walk transition probability for the spread-out model. This inequality implies that $|1 - e^{i\theta} \hat{D}(k)| > \text{const.}(\|k\| L^2 + \theta^2)$, hence the naive bound (3.12) works for sufficiently large $L$. If we consider oriented percolation whose staying probability is positive (considered in, e.g., [16]), then the naive bound also works for such “nearest-neighbor” model because, in this case, $D$ turns to the lazy random-walk transition probability $D_{\text{lazy}}$. This transition probability is defined as, for $J \in [0, 1]$ and $x \in \mathbb{L}^d$,

$$D_{\text{lazy}}(x) = J \delta_{x,0} + (1 - J) \frac{\mathbb{1}_{\{x \in A^d\}}}{|A^d|} = J \delta_{x,0} + (1 - J) D(x),$$

of which the Fourier transform is given by $\hat{D}_{\text{lazy}}(k) = J + (1 - J) \hat{D}(k)$. When we choose an arbitrary $J < 1/2$, the inequality $1 + \hat{D}_{\text{lazy}}(k) > 0$ holds for every $k \in \mathbb{T}^d$ even at $k$ such that $\hat{D}(k) = -1$. Thus, the lazy model satisfies a similar bound to (3.9). However, it is different from so-called nearest-neighbor oriented percolation, so that we did not deal with the lazy model in this paper.

3.3 An Upper Bound for $g_3(p, m)$

To obtain an upper bound on $g_3(p, m)$, we need to modify [36, Lemma 5.7].

Lemma 3.8  (A modified version of [5, (5.29)–(5.33)] or [36, Lemma 5.7]) Suppose that $a(-x, t) = a(x, t)$ for all $(x, t) \in \mathbb{L}^d \times \mathbb{Z}_+$, and let

$$\hat{A}(k, z) = \frac{1}{1 - \hat{a}(k, z)}.$$

Then, for all $k, l \in \mathbb{T}^d$,

$$\left| \frac{1}{2} \Delta_k \hat{A}(l, z) \right| \leq \left( |\hat{a}|(0, |z|) - |\hat{a}|(k, |z|) \right) \left( \frac{|\hat{A}(l + k, z)| + |\hat{A}(l - k, z)|}{2} |\hat{A}(l, z)| + |\hat{A}(l, z)| \right) \left( |\hat{a}|(0, |z|) - |\hat{a}|(2l, |z|) \right),$$

where $|\hat{a}|(k, z) = \sum_{(x, t)} |a(x, t)| e^{ik \cdot x^t}$.  

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Proof of (3.4) in Proposition 3.1 Recall (1.14) and (3.5). Setting \( a(x, t) = q_p(x, t) + (q_p \ast \Pi_p)(x, t) \) in Lemma 3.8, we have

\[
\frac{1}{2} \Delta_k (\hat{q}_p(l, z) \hat{\phi}_p(l, z)) = \frac{1}{2} \Delta_k \left( \frac{1}{1 - \hat{a}(l, z)} - 1 \right) = \frac{1}{2} \Delta_k \hat{A}(l, z). \tag{3.14}
\]

By Lemma 6.2 and the bootstrap hypothesis \( g_1(p, m) \leq K_1 \),

\[
|a(0, |z|) - \hat{a}(|k, |z|)| = \sum_{(x, t)} |q_p(x, t) + (q_p \ast \Pi_p)(x, t)| |z|^t (1 - \cos k \cdot x)
\]

\[
\leq \sum_{(x, t)} q_p(x, t) |z|^t (1 - \cos k \cdot x)
\]

\[
+ \sum_{(x, t)} \sum_{(y, s)} q_p(y, s) \left| \Pi_p(x - y, t - s) \right| |z|^t \left( 1 - \cos k \cdot \left( y + (x - y) \right) \right)
\]

\[
\leq \left( 1 - \hat{D}(k) \right) \left( 1 + 2 |\Pi_p(0, |z|)| + 2 \cdot \frac{|\Pi_p(0, |z|) - |\Pi_p(k, |z|)|}{1 - \hat{D}(k)} \right)
\]

\[
\leq K_1 \left( 1 - \hat{D}(k) \right) \left( 1 + 2 \hat{\Pi}^{\text{even}}_p(0, |z|) + 2 \hat{\Pi}^{\text{odd}}_p(0, |z|) + 2 \cdot \frac{\hat{\Pi}^{\text{even}}_p(0, |z|) - \hat{\Pi}^{\text{even}}_p(k, |z|)}{1 - \hat{D}(k)} + 2 \cdot \frac{\hat{\Pi}^{\text{odd}}_p(0, |z|) - \hat{\Pi}^{\text{odd}}_p(k, |z|)}{1 - \hat{D}(k)} \right). \tag{3.15}
\]

By the triangle inequality and the definition of \( g_2(p, m) \),

\[
|\hat{A}(l, z)| = \frac{|\hat{\phi}_p(k, z)|}{|1 + \hat{\Pi}_p(l, z)|} \leq \frac{g_2(p, m)}{1 - \hat{\Pi}^{\text{even}}_p(0, m) - \hat{\Pi}^{\text{odd}}_p(0, m)} \left| \hat{S}_{\mu_p}(z)(l) \right|. \tag{3.16}
\]

Recall the definition (1.12) of \( \hat{U}_{\mu_p(z)}(k, l) \). Applying Lemma 3.8 to (3.14), and combining the bounds (3.15) and (3.16), we obtain

\[
g_3(p, m) \leq \left( 1 + \sqrt{\frac{g_2(p, m)}{1 - \hat{\Pi}^{\text{even}}_p(0, m) - \hat{\Pi}^{\text{odd}}_p(0, m)}} \right)^3 K_1^2
\]

\[
\times \left( 1 + 2 \left( \hat{\Pi}^{\text{even}}_p(0, m) + \hat{\Pi}^{\text{odd}}_p(0, m) \right) + 2 \left\| \frac{(\hat{\Pi}^{\text{even}}_p(0, m) + \hat{\Pi}^{\text{odd}}_p(0, m)) - (\hat{\Pi}^{\text{even}}_p(\bullet, m) + \hat{\Pi}^{\text{odd}}_p(\bullet, m))}{1 - \hat{D}(\bullet)} \right\|_{\infty} \right)^2. \tag{3.17}
\]
3.4 Proof of a Proposition and Lemmas

**Proof of Proposition 3.2** We notice that

\[
g_2(p, m) = \sup_{k \in \mathbb{T}^d, \, r \in [1, m], \, \theta \in \mathbb{T}} \left| \hat{\phi}_p(k, re^{i\theta}) \right| / \left| \hat{S}_{\mu_p(re^{i\theta})}(k) \right|
\]

\[
= \left( \sup_{k \in [-\pi/2, \pi/2]^d, r \in [1, m], \, \theta \in \mathbb{T}} \left| \hat{\phi}_p(k, re^{i\theta}) \right| / \left| \hat{S}_{\mu_p(re^{i\theta})}(k) \right| \right) \vee \left( \sup_{k \in \mathbb{T}^d \setminus [-\pi/2, \pi/2]^d, r \in [1, m], \, \theta \in \mathbb{T}} \left| \hat{\phi}_p(k, re^{i\theta}) \right| / \left| \hat{S}_{\mu_p(re^{i\theta})}(k) \right| \right). \tag{3.18}
\]

Given some \( k = (k_1, \ldots, k_d) \in \mathbb{T}^d \setminus [-\pi/2, \pi/2]^d \), there exists \( \tilde{k} = (\tilde{k}_1, \ldots, \tilde{k}_d) \in [-\pi/2, \pi/2]^d \) such that, for \( j = 1, \ldots, d \),

\[
\tilde{k}_j = \begin{cases} 
  k_j & [-\pi/2 \leq k_j \leq \pi/2], \\
  \pi - k_j & [\pi/2 < k_j \leq \pi], \\
  -\pi - k_j & [-\pi \leq k_j < -\pi/2].
\end{cases}
\]

The second and third lines mean the reflection across points \( \pi/2 \) and \( -\pi/2 \) in one dimension, respectively. We show that we can replace \( k \) in the second factor in (3.18) by \( \tilde{k} \).

Fix \( k \in \mathbb{T}^d \setminus [-\pi/2, \pi/2]^d, r \in \{1, m\} \) and \( \theta \in \mathbb{T} \). Note that \( \tilde{k}_j = k_j \) for \( j = 1, \ldots, d \) when \( -\pi/2 \leq k_j \leq \pi/2 \). Since \( \phi_p(x, t) \) is an even function with respect to \( x \),

\[
\hat{\phi}_p(k, re^{i\theta}) = \sum_{(x,t)} \phi_p(x, t) r^t e^{i \sum_{j=1}^d k_j x_j} e^{i\theta t}
\]

\[
= \sum_{(x,t)} \phi_p(x, t) r^t e^{-i\tilde{k} \cdot x} e^{i \sum_{j=1}^d 1_{[k_j \neq \tilde{k}_j]} \text{sgn}(k_j) x_j} e^{i\theta t}
\]

\[
= \sum_{(x,t)} \phi_p(-x, t) r^t e^{i\tilde{k} \cdot x} (-1)^{\sum_{j=1}^d 1_{[k_j \neq \tilde{k}_j]} \text{sgn}(k_j) x_j} e^{i\theta t}
\]

\[
= \sum_{(x,t)} \phi_p(x, t) r^t e^{i\tilde{k} \cdot x} (-1)^{\sum_{j=1}^d 1_{[k_j \neq \tilde{k}_j]} \text{sgn}(k_j) x_j} e^{i\theta t},
\]

where \( \text{sgn}: \mathbb{R} \to \{-1, 0, +1\} \) denotes the sign function. We divide into the two cases where \( \sum_{j=1}^d 1_{[k_j \neq \tilde{k}_j]} \) is even or odd.
On the one hand, suppose that \( \sum_{j=1}^{d} \mathbb{I}_{\{ k_j \neq \hat{k}_j \}} \) is even. Since all coordinates of some \( x \in \mathbb{L}^d \) are either even or odd, \( \sum_{j=1}^{d} \mathbb{I}_{\{ k_j \neq \hat{k}_j \}} \cdot \text{sgn}(k_j) x_j \) is even. It is easy to see that

\[
\hat{\phi}_p(k, re^{i\theta}) = \sum_{(x,t)} \varphi_p(x, t) r^t e^{i\hat{k} \cdot x} e^{i\theta t} = \hat{\phi}_p(\hat{k}, re^{i\theta}).
\]

On the other hand, suppose that \( \sum_{j=1}^{d} \mathbb{I}_{\{ k_j \neq \hat{k}_j \}} \) is odd. Let \( \mathbb{L}_e = \{ x \in \mathbb{L}^d \mid \forall j, x_j \) is even\}, \( \mathbb{L}_o = \{ x \in \mathbb{L}^d \mid \forall j, x_j \) is odd\}, \( \mathbb{Z}_{e+} \) be the set of non-negative even numbers and \( \mathbb{Z}_{o+} \) be the set of positive odd numbers. Note that \( \varphi_p(x, t) = \mathbb{P}_p((0, 0) \to (x, t)) = 0 \) when \( (x, t) \neq (\mathbb{L}_e \times \mathbb{Z}_{e+}) \cup (\mathbb{L}_o \times \mathbb{Z}_{o+}) \) due to the definition of nearest-neighbor oriented percolation. We rewrite \( \hat{\phi}_p(k, re^{i\theta}) \) as

\[
\hat{\phi}_p(k, re^{i\theta}) = \sum_{x \in \mathbb{L}_e, t \in \mathbb{Z}_{e+}} \varphi_p(x, t) r^t e^{i\hat{k} \cdot x} e^{i\theta t} - \sum_{x \in \mathbb{L}_o, t \in \mathbb{Z}_{o+}} \varphi_p(x, t) r^t e^{i\hat{k} \cdot x} e^{i\theta t}. \quad (3.19)
\]

Let \( \hat{\theta} = -\theta + \pi \text{ sgn}(\theta) \), which is the reflection in \( \pi/2 \) and \( -\pi/2 \). This definition is different from that of \( \tilde{k} \) because even the point \( \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \) is reflected. By using \( \hat{\theta} \),

\[
e^{i\theta t} = e^{-i\hat{\theta} t} e^{\pi \text{ sgn}(\theta) t} = (-1)^t e^{-i\hat{\theta} t}.
\]

Substituting (3.20) into (3.19), we obtain

\[
\hat{\phi}_p(k, re^{i\theta}) = \sum_{x \in \mathbb{L}_e, t \in \mathbb{Z}_{e+}} \varphi_p(x, t) r^t e^{i\hat{k} \cdot x} e^{-i\hat{\theta} t} + \sum_{x \in \mathbb{L}_o, t \in \mathbb{Z}_{o+}} \varphi_p(x, t) r^t e^{i\hat{k} \cdot x} e^{-i\hat{\theta} t}
\]

\[
= \sum_{(x,t)} \varphi_p(x, t) r^t e^{i\hat{k} \cdot x} e^{-i\hat{\theta} t} = \hat{\phi}_p(\hat{k}, re^{-i\theta}).
\]

As for \( \theta, -\hat{\theta} \) takes a value in \( \mathbb{T} \). It is possible to replace \( -\hat{\theta} \) in \( \hat{\phi}_p(\hat{k}, re^{-i\hat{\theta}}) \) by \( \theta \) since it is the dummy variable of the supremum in (3.18).

The above discussions also hold for \( \hat{S}_{\mu_p(re^{i\theta})}(k) \) because \( \hat{Q}_1(x, t) \) satisfies the same properties “\( \hat{Q}_1(x, t) \) is an even function with respect to \( x \)” and “\( \hat{Q}_1(x, t) = D^{\hat{\mu}_t}(x) \mathbb{I}_{\{ t \in \mathbb{Z}_+ \}} = 0 \) when \( (x, t) \neq (\mathbb{L}_e \times \mathbb{Z}_{e+}) \cup (\mathbb{L}_o \times \mathbb{Z}_{o+}) \)” as \( \hat{\phi}_p(k, re^{i\theta}) \). Therefore,
\[
g_2(p, m) = \left( \sup_{k \in [-\pi, \pi], \ r \in [1, m], \ \theta \in T} \left| \hat{\phi}_p(k, re^{i\theta}) \right| \right) \lor \left( \sup_{\tilde{k} \in [-\pi, \pi], \ r \in [1, m], \ \theta \in T} \left| \hat{\phi}_p(\tilde{k}, re^{i\theta}) \right| \right).
\]

This completes the proof of Proposition 3.2. \qed

**Proof of Lemma 3.3** Let \( \xi := \hat{D}(k) \) and

\[
f_r(\xi, \theta) := 4 \left| 1 - re^{i\theta} \xi \right|^2 - \left( \frac{|\theta|}{\pi} + 1 - \xi \right)^2 = \left( 4r^2 - 1 \right) \xi^2 + 2 \left( \frac{|\theta|}{\pi} + 1 - 4r \cos \theta \right) \xi + 3 - \frac{\theta^2}{\pi^2} - \frac{2|\theta|}{\pi}. \tag{6}
\]

\( f_r(\xi, \theta) \) is an even function with respect to \( \theta \). It suffices to show that \( f_r(\xi, \theta) \geq 0 \) for \( \xi \in [0, 1] \) and \( \theta \in [0, \pi] \). Clearly, the part (6) is monotonically decreasing with respect to \( \theta \) and non-negative, hence \( f_r(0, \theta) \geq 0 \). Solving the equation \( \partial f_r(\xi, \theta)/\partial \xi = 0 \), we obtain \( r = \cos \theta/\xi \). By substituting this into \( f_r(\xi, \theta) \),

\[
f_{\cos \theta/\xi}(\xi, \theta) = -\xi^2 + 2 \left( \frac{\theta}{\pi} + 1 \right) \xi - 4 \cos^2 \theta + 3 - \frac{\theta^2}{\pi^2} - \frac{2|\theta|}{\pi}.
\]

Since the coefficient of \( \xi^2 \) is negative, \( f_{\cos \theta/\xi}(\xi, \theta) \) takes the minimum value at the boundary points \( \xi = 0, 1 \). Since \( \theta \) takes a value on \([0, \pi]\), we arrive at \( f_{\cos \theta}(1, \theta) = 4 \sin^2 \theta + 3 - \theta^2/\pi^2 \geq 0 \). This completes the proof of Lemma 3.3. \qed

**Proof of Lemma 3.5** Let \( r = |\mu| \in [0, 1] \) and \( \theta = \arg \mu \in \mathbb{T} \). We show that

\[
f(\theta) := 4 \left| 1 - re^{i\theta} \hat{D}(k) \right|^2 - \left| 1 - e^{i\theta} \hat{D}(k) \right|^2 = 3 + \left( 4r^2 - 1 \right) \hat{D}(k)^2 - 2 (4r - 1) \hat{D}(k) \cos \theta \geq 0.
\]

By the symmetry of the cosine function, we restrict the range of \( \theta \) to \([0, \pi]\). Since

\[
f'(\theta) = 2 (4r - 1) \hat{D}(k) \sin \theta \begin{cases}
\geq 0 & [r \geq 1/4 \text{ and } \hat{D}(k) \geq 0], \\
\leq 0 & [r < 1/4 \text{ and } \hat{D}(k) \geq 0], \\
\leq 0 & [r \geq 1/4 \text{ and } \hat{D}(k) < 0], \\
\geq 0 & [r < 1/4 \text{ and } \hat{D}(k) < 0],
\end{cases}
\]

\( \diamond \) Springer
$f(\theta)$ is monotonic with respect to $\theta$ whenever $r$ and $k$ are fixed. Thus, $f(\theta)$ takes the minimum value at $\theta = 0$, $\pi/2$ or $\pi$. When $\theta = 0$ or $\pi$, we need to show $3 + (4x^2 - 1)y^2 \pm 2(4x - 1)y \geq 0$ for all $(x, y) \in [0, 1] \times [-1, 1]$. When $\theta = \pi/2$, we need to show $3 + (4x^2 - 1)y^2 \geq 0$ for all $(x, y) \in [0, 1] \times [-1, 1]$. We omit proofs of these inequalities because it is not hard to see them. Therefore, $f(\theta) \geq 0$.

Proof of Lemma 3.8 Let

\[
\hat{a}^{\cos}(l, k; z) = \sum_{(x,t)} a(x, t) \cos(l \cdot x) \cos(k \cdot x)z^t,
\]
\[
\hat{a}^{\sin}(l, k; z) = \sum_{(x,t)} a(x, t) \sin(l \cdot x) \sin(k \cdot x)z^t.
\]

As in [36, Equation (5.21)],

\[
-\frac{1}{2} \Delta_k \hat{A}(l, z)
= \frac{\hat{A}(l + k, z) + \hat{A}(l - k, z)}{2} \hat{A}(l, z)(\hat{a}(l, z) - \hat{a}^{\cos}(l, k; z))
- \hat{A}(l, z)\hat{A}(l + k, z)\hat{A}(l - k, z)\hat{a}^{\sin}(l, k; z)^2.
\] (3.21)

By the triangle inequality and arithmetic of the trigonometric functions, combining

\[
|\hat{a}(l, z) - \hat{a}^{\cos}(l, k; z)| \leq \sum_{(x,t)} |a(x, t)| |z|^t (1 - \cos k \cdot x)
\]

and

\[
|\hat{a}^{\sin}(l, k; z)|^2
\leq \left( \sum_{(x,t)} |a(x, t)| |z|^t \sin^2 l \cdot x \right) \left( \sum_{(x,t)} |a(x, t)| |z|^t \sin^2 k \cdot x \right)
= \left( \sum_{(x,t)} |a(x, t)| |z|^t \frac{1 - \cos(2l \cdot x)}{2} \right) \left( \sum_{(x,t)} |a(x, t)| |z|^t (1 - \cos^2 k \cdot x) \right)
\leq \left( \sum_{(x,t)} |a(x, t)| |z|^t (1 - \cos(2l \cdot x)) \right) \left( \sum_{(x,t)} |a(x, t)| |z|^t (1 - \cos k \cdot x) \right)
\]

leads to (3.13). \qed

From (3.2), (3.4) and (3.4), upper bounds for $\{g_i(p, m)\}_{i=1}^3$ are written in terms of the lace-expansion coefficients in the end. Each term in the upper bounds is also
bounded above by Lemma 2.4, which is written in terms of the basic diagrams $B_{p,m}^{(\lambda,\rho)}$, $T_{p,m}^{(\lambda,\rho)}$ and $V_{p,m}^{(\lambda,\rho)}(k)$. The rest of our work is to bound the basic diagrams.

## 4 Diagrammatic Bounds in Terms of Random-Walk Quantities

In this section, we bound the basic diagrams $2.3$–$2.5$ above. We first show the proof of Lemma 4.1 below in Sect. 4.1 with additional lemmas, and then we prove these lemmas in Sect. 4.2. Recall the random-walk quantities $1.2$.

### Lemma 4.1

Assume $g_i(p,m) \leq K_i, i = 1, 2, 3$. For $\mu, \rho \in \mathbb{N}$, $p \in [0, p_c)$ and $m \in [1, m_p)$,

\[
B_{p,m}^{(\lambda,\rho)} \leq K_1^\lambda K_2^\rho \varepsilon_1^{\lfloor (\lambda+\rho)/2 \rfloor},
\]

\[
T_{p,m}^{(\lambda,\rho)} \leq \sqrt{2} K_1^\lambda K_2^\rho \varepsilon_2^{\lfloor (\lambda+\rho)/2 \rfloor},
\]

\[
\left\| \hat{V}_{p,m}^{(\lambda,\rho)} \right\| \leq \begin{cases} K_1^2 \|D\|_\infty + K_1^\lambda K_2^\rho \sqrt{\varepsilon_1^{(3)}} + \left\| \hat{V}_{p,m}^{(2,1)} \right\|_\infty & [\lambda = \rho = 1], \\
\lambda (\lambda - 1) K_1^\lambda K_2^\rho \varepsilon_1^{\lfloor (\lambda+\rho-1)/2 \rfloor} + \lambda K_1^\lambda \rho \varepsilon_2^{\lfloor (\lambda+\rho-1)/2 \rfloor} & [\lambda \geq 2 \text{ or } \rho \geq 2].
\end{cases}
\]

### 4.1 Upper Bounds on the Basic Diagrams

The residue theorem in complex analysis yields the next lemma, which is used throughout this section.

### Lemma 4.2

For every $r \in [0, 1)$ and $k \in \mathbb{T}$,

\[
\int_{\mathbb{T}} \left| \hat{S}_{r \epsilon^{\theta}}(k) \right|^2 \frac{d\theta}{2\pi} = \frac{1}{1 - r^2 \hat{D}(k)^2},
\]

\[
\int_{\mathbb{T}} \left| \hat{S}_{r \epsilon^{\theta}}(k) \right|^4 \frac{d\theta}{2\pi} = \frac{1 + r^2 \hat{D}(k)^2}{(1 - r^2 \hat{D}(k)^2)^2}.
\]

### Proof of (4.1) and (4.2) in Lemma 4.1

We first show (4.1). By the inverse Fourier-Laplace transform and the bootstrap hypotheses,

\[
B_{p,m}^{(\lambda,\rho)} = \sup_{(x,t)} \int_{\mathbb{T}^d} \frac{d^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{d\theta}{2\pi} \hat{q}_p(k, e^{i\theta})^\lambda \hat{\phi}_p(k, e^{i\theta}) \times m^\rho \hat{q}_p(k, e^{-i\theta})^\rho \hat{\phi}_p(k, me^{-i\theta})e^{-ik \cdot x}e^{-i\theta t} \leq \int_{\mathbb{T}^d} \frac{d^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{d\theta}{2\pi} \left| \hat{q}_p(k, e^{i\theta}) \right|^\lambda \left| \hat{\phi}_p(k, e^{i\theta}) \right|
\]

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\begin{equation*}
\times m^\rho \left| \hat{q}_p(k, e^{-i\theta}) \right| ^\rho \left| \hat{\phi}_p(k, m e^{-i\theta}) \right|
\end{equation*}

\begin{equation*}
= p^{\lambda + \rho} m^\rho \int_{\mathbb{T}^d} \frac{d^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{d\theta}{2\pi} \left| \hat{D}(k) \right|^{\lambda + \rho} \left| \hat{\phi}_p(k, e^{i\theta}) \right| \left| \hat{\phi}_p(k, m e^{-i\theta}) \right|
\end{equation*}

\begin{equation*}
\leq K_1^{\lambda + \rho} K_2^2 \int_{\mathbb{T}^d} \frac{d^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{d\theta}{2\pi} \left| \hat{D}(k) \right|^{\lambda + \rho} \left| \hat{S}_{\mu_p(e^{i\theta})}(k) \right| \left| \hat{S}_{\mu_p(me^{-i\theta})}(k) \right|. \end{equation*}

By the Cauchy–Schwarz inequality,

\begin{equation*}
B_{p,m}^{(\lambda, \rho)} \leq K_1^{\lambda + \rho} K_2^2 \int_{\mathbb{T}^d} \frac{d^d k}{(2\pi)^d} \left| \hat{D}(k) \right|^{\lambda + \rho} \times \left( \int_{\mathbb{T}} \frac{d\theta}{2\pi} \left| \hat{S}_{\mu_p(e^{i\theta})}(k) \right|^2 \right)^{1/2} \left( \int_{\mathbb{T}} \frac{d\theta}{2\pi} \left| \hat{S}_{\mu_p(me^{-i\theta})}(k) \right|^2 \right)^{1/2}.
\end{equation*}

Recall (1.13). We apply (4.4) in Lemma 4.2 to the above to obtain

\begin{equation*}
\begin{aligned}
B_{p,m}^{(\lambda, \rho)} &\leq K_1^{\lambda + \rho} K_2^2 \int_{\mathbb{T}^d} \frac{d^d k}{(2\pi)^d} \left| \hat{D}(k) \right|^{\lambda + \rho} \\
&\quad + \left( \frac{1}{1 - \mu_p(1)^2 \hat{D}(k)^2} \right)^{1/2} \left( \frac{1}{1 - \mu_p(m)^2 \hat{D}(k)^2} \right)^{1/2} \\
&\leq K_1^{\lambda + \rho} K_2^2 \int_{\mathbb{T}^d} \frac{d^d k}{(2\pi)^d} \frac{\hat{D}(k)^{2(\lambda + \rho)}}{1 - \hat{D}(k)^2} \\
&= K_1^{\lambda + \rho} K_2^2 \epsilon_{(\lambda+\rho)/2}. \end{aligned}
\end{equation*}

In the last line, we have used \( \mu_p(m) < 1 \) for every \( m \in \{0, m_p\} \), and the trivial inequality \( |\hat{D}(k)|^{\lambda + \rho} \leq \hat{D}(k)^{2(\lambda + \rho)} \) since \( |\hat{D}(k)| \leq 1 \) for all \( k \in \mathbb{T}^d \).

Similarly,

\begin{equation*}
T_{p,m}^{(\lambda, \rho)} \text{ inv. FL} = \sup_{(x,t)} \left| \int_{\mathbb{T}^d} \frac{d^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{d\theta}{2\pi} \left| \hat{q}_p(k, e^{i\theta}) \right|^{\lambda} \left| \hat{q}_p(k, e^{-i\theta}) \right|^{\rho} \left| \hat{\phi}_p(k, e^{i\theta}) \right| \left| \hat{\phi}_p(k, e^{-i\theta}) \right| \\
\times m^\rho \hat{q}_p(k, e^{-i\theta})^\rho \hat{\phi}_p(k, m e^{-i\theta})e^{-ik \cdot x} e^{-i\theta t} \right|
\end{equation*}

\begin{equation*}
\leq \int_{\mathbb{T}^d} \frac{d^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{d\theta}{2\pi} \left| \hat{q}_p(k, e^{i\theta}) \right|^{\lambda} \left| \hat{\phi}_p(k, e^{i\theta}) \right|^2 \\
\times m^\rho \left| \hat{q}_p(k, e^{-i\theta}) \right|^{\rho} \left| \hat{\phi}_p(k, m e^{-i\theta}) \right| \\
= p^{\lambda + \rho} m^\rho \int_{\mathbb{T}^d} \frac{d^d k}{(2\pi)^d} \int_{\mathbb{T}} \frac{d\theta}{2\pi} \left| \hat{D}(k) \right|^{\lambda + \rho} \left| \hat{\phi}_p(k, e^{i\theta}) \right|^2 \left| \hat{\phi}_p(k, m e^{-i\theta}) \right| \
\end{equation*}
For any hypothesis\( K_1^{\lambda+\rho} K_2^3 \int_{T^d} \frac{d^d k}{(2\pi)^d} |\hat{D}(k)|^{\lambda+\rho} \)
\( \leq \int_{T} \frac{d\theta}{2\pi} \left| \hat{S}_{\mu_p(e^{i\theta})}(k) \right|^2 \left| \hat{S}_{\mu_p(me^{-i\theta})}(k) \right| \)

\text{CS ineq.}
\( \leq K_1^{\lambda+\rho} K_2^3 \int_{T^d} \frac{d^d k}{(2\pi)^d} \left| \hat{D}(k) \right|^{\lambda+\rho} \)
\( \times \left( \int_{T} \frac{d\theta}{2\pi} \left| \hat{S}_{\mu_p(e^{i\theta})}(k) \right|^4 \right)^{1/2} \left( \int_{T} \frac{d\theta}{2\pi} \left| \hat{S}_{\mu_p(me^{-i\theta})}(k) \right|^2 \right)^{1/2} \)

\text{Lem. 4.2}
\( K_1^{\lambda+\rho} K_2^3 \int_{T^d} \frac{d^d k}{(2\pi)^d} \left| \hat{D}(k) \right|^{\lambda+\rho} \)
\( \times \left( 1 + \mu_p(1)^2 \hat{D}(k)^2 \right)^{1/2} \left( \frac{1}{1 - \mu_p(1)^2 \hat{D}(k)^2} \right)^{1/2} \)
\( \leq \sqrt{2} K_1^{\lambda+\rho} K_2^3 \int_{T^d} \frac{d^d k}{(2\pi)^d} \left( 1 - \hat{D}(k)^2 \right)^2 \)
\( = \sqrt{2} K_1^{\lambda+\rho} K_2^3 \sum_{n=0}^{\infty} (n+1) \int_{T^d} \frac{d^d k}{(2\pi)^d} \hat{D}(k)^2 \left( n + \frac{\lambda+\rho}{2} \right) \)

\text{inv. FL}
\( \leq \sqrt{2} K_1^{\lambda+\rho} K_2^3 \varepsilon_2^{(\lambda+\rho)/2}) \).

This completes the proof of (4.2). \( \square \)

To prove (4.3) in Lemma 4.1, integrating the product of the absolute value of \( \hat{D}(l)'s \) and \( \hat{S}_{\mu}(l)'s \) is required. These integrands come from \( \hat{U}_{\mu}(k, l) \) in (1.12).

**Lemma 4.3** For any \( \lambda, \rho \in \mathbb{N} \),

\[
\int_{T^d} \frac{d^d l}{(2\pi)^d} \int_{T} \frac{d\theta}{2\pi} \left| \hat{D}(l) \right|^{\lambda+\rho-1} \left| \hat{S}_{\mu_p(e^{i\theta})}(l \pm k) \right| \left| \hat{S}_{\mu_p(me^{-i\theta})}(l) \right| \leq \sqrt{2} \varepsilon_2^{(\lambda+\rho-1)/2}) \tag{4.6}
\]

and

\[
\int_{T^d} \frac{d^d l}{(2\pi)^d} \int_{T} \frac{d\theta}{2\pi} \left| \hat{D}(l) \right|^{\lambda+\rho-1} \left| \hat{S}_{\mu_p(e^{i\theta})}(l + k) \right| \left| \hat{S}_{\mu_p(e^{i\theta})}(l - k) \right| \leq 4 \varepsilon_2^{(\lambda+\rho-1)/2}) \tag{4.7}
\]
To prove (4.7) in Lemma 4.3, we note that the period of $\hat{D}(2k)$ is the same as that of $\hat{D}(k)^2$. The next lemma plays an important role to prove the finiteness of $\hat{V}_{p,m}^{(\lambda,\rho)}(k)$ for $d > 4$:

**Lemma 4.4** For every $k \in \mathbb{T}^d$ and $\mu \in \mathbb{C}$ satisfying $|\mu| \leq 1$,

$$1 - \hat{D}(2k) \leq 2 \left(1 - \hat{D}(k)^2\right).$$

(4.8)

**Remark 4.5** The proof of Lemma 4.4 depends on the structure of $\mathbb{L}^d$ as well as Proposition 3.2, but it is not hard to prove this on $\mathbb{E}^d$ similarly [24, 25]. Thanks to this lemma, we are able to show the finiteness of $\hat{V}_{p,m}^{(\lambda,\rho)}(k)$ for $d > 4$, cf., [27, Remark 5] in which the authors proved the finiteness of the similar quantity only for $d > 8$.

**Proof of (4.3) in Lemma 4.1 when $\lambda \geq 2$ and $\rho \geq 2$** By Lemma 6.2 in Appendix 1,

$$\hat{V}_{p,m}^{(\lambda,\rho)}(k) = \sup_x \sum_{y_\lambda} \left( q_p^{(\lambda)} \star \varphi_p \right) (y_\lambda) \left( 1 - \cos k \cdot y \right) \left( m^\rho q_p^{(\rho)} \star \varphi_p^{(m)} \right) (y_\lambda - x)$$

$$= \sup_x \sum_{y_1, \ldots, y_\lambda} \left( \prod_{i=1}^{\lambda-1} q_p(y_i - y_{i-1}) \right) \left( q_p \star \varphi_p \right) (y_\lambda - y_{\lambda-1})$$

$$\times \left( 1 - \cos \left( k \cdot \sum_{j=1}^{\lambda} (y_j - y_{j-1}) \right) \right) \left( m^\rho q_p^{(\rho)} \star \varphi_p^{(m)} \right) (y_\lambda - x)$$

$$\leq \lambda \sup_x \sum_{j=1}^{\lambda-1} \sum_{y_1, \ldots, y_\lambda} \left( \prod_{i \neq j} q_p(y_i - y_{i-1}) \right) \left( q_p \star \varphi_p \right) (y_\lambda - y_{\lambda-1})$$

$$\times q_p(y_j - y_{j-1}) \left( 1 - \cos \left( k \cdot (y_j - y_{j-1}) \right) \right)$$

$$\times \left( m^\rho q_p^{(\rho)} \star \varphi_p^{(m)} \right) (y_\lambda - x)$$

$$+ \lambda \sup_x \sum_{y_1, \ldots, y_\lambda} \left( q_p \star \varphi_p \right) (y_\lambda - y_{\lambda-1})$$

$$\times \left( 1 - \cos \left( k \cdot (y_\lambda - y_{\lambda-1}) \right) \right)$$

$$\times \left( \prod_{i=1}^{\lambda-1} q_p(y_i - y_{i-1}) \right) \left( m^\rho q_p^{(\rho)} \star \varphi_p^{(m)} \right) (y_\lambda - x)$$

$$= \sup_x \sum_{y_{j-1}, y_j, y_\lambda} q_p^{(j-1)}(y_{j-1}) q_p(y_j - y_{j-1})$$

$$\times \left( 1 - \cos \left( k \cdot (y_j - y_{j-1}) \right) \right)$$

$$\times \left( q_p^{(\lambda-j)} \star \varphi_p \right) (y_\lambda - y_j) \left( m^\rho q_p^{(\rho)} \star \varphi_p^{(m)} \right) (y_\lambda - x)$$

$$+ \lambda \sup_x \sum_{y_{\lambda-1}, y_\lambda} q_p^{(\lambda-j)}(y_{\lambda-1})$$
\[
\hat{V}_{p,m}^{(\lambda,\rho)}(k) \leq \lambda(\lambda - 1) \int_{\mathbb{T}^d} \frac{d^d l}{(2\pi)^d} \int_{T} \frac{d\theta}{2\pi} \left| \hat{q}_p(l, e^{i\theta}) \right|^{\rho} \left| \hat{\varphi}_p(l, me^{-i\theta}) \right| \\
\times \left| \frac{1}{2} \Delta_k \hat{q}_p(l, e^{i\theta}) \right| \left| \hat{\varphi}_p(l, e^{i\theta}) \right| m^\rho \left| \hat{q}_p(l, e^{-i\theta}) \right|^{\rho} \left| \hat{\varphi}_p(l, me^{-i\theta}) \right| \\
+ \lambda \int_{\mathbb{T}^d} \frac{d^d l}{(2\pi)^d} \int_{T} \frac{d\theta}{2\pi} \left| \hat{q}_p(l, e^{i\theta}) \right|^{\rho} \left| \hat{\varphi}_p(l, me^{-i\theta}) \right| \\
\times \left| \frac{1}{2} \Delta_k \left( \hat{q}_p(l, e^{i\theta})\hat{\varphi}_p(l, e^{i\theta}) \right) \right| m^\rho \left| \hat{q}_p(l, e^{-i\theta}) \right|^{\rho} \left| \hat{\varphi}_p(l, me^{-i\theta}) \right| \\
\leq \lambda(\lambda - 1)(1 - \hat{D}(k)) K_1^{\lambda + \rho} K_2^2 \\
\times \int_{\mathbb{T}^d} \frac{d^d l}{(2\pi)^d} \int_{T} \frac{d\theta}{2\pi} \left| \hat{D}(l) \right|^{\lambda + \rho - 1} \left| \hat{\mu}_p(e^{i\theta}) \right| \left| \hat{\mu}_p(me^{-i\theta}) \right| \\
\leq_{\mathbb{R}^{((\lambda + \rho - 1)/2)}} \\
+ \lambda K_1^{\lambda + \rho - 1} K_2 K_3 \int_{\mathbb{T}^d} \frac{d^d l}{(2\pi)^d} \int_{T} \frac{d\theta}{2\pi} \left| \hat{D}(l) \right|^{\lambda + \rho - 1} \\
\times \left| \hat{\mu}_p(e^{i\theta}) \right| \left| \hat{\mu}_p(me^{-i\theta}) \right| .
\] (4.9)

In the second inequality, we have used

\[
\left| \frac{1}{2} \Delta_k \hat{D}(l) \right| = \sum_x D(x) \left(1 - \cos k \cdot x\right) e^{i k \cdot x} \leq \sum_x D(x) \left(1 - \cos k \cdot x\right) = 1 - \hat{D}(k)
\]

and the definition (1.12) of \( \hat{\mu}_p(e^{i\theta}) \). The second term on the right hand side of (4.9) contains two types of integrals that appear in Lemma 4.3. Combining (4.9), (4.6) and (4.7) implies (4.3) when \( \lambda \geq 2 \) and \( \rho \geq 2 \).

Proof of (4.3) in Lemma 4.1 when \( \lambda = 1 \) and \( \rho = 1 \) By the trivial inequality (B.10), note that \( \varphi_p(x) = \delta_{o,x} + \varphi_p(x) \mathbb{I}_{\{x \neq o\}} \leq \delta_{o,x} + (q_p \ast \varphi_p)(x) \) for every \( x \in \mathbb{L}^d \times \mathbb{Z}_+ \). Then, we obtain

\[
\hat{V}_{p,m}^{(1,1)}(k) = \sup_y \sum_x (q_p \ast \varphi_p)(y) \left(1 - \cos k \cdot y\right) (mq_p \ast \varphi_p^{(m)})(y - x)
\]
\[
\leq \sup_x \sum_y q_p(y) (1 - \cos k \cdot y) m q_p(y - x)
\]
\[
\leq K^2 \|D\|_\infty (1 - \hat{D}(k)) \quad (\because (B.1))
\]
\[
+ \sup_x \sum_y q_p(y) (1 - \cos k \cdot y) \left( m^2 q_p^{2} \star \varphi_p^{(m)} \right)(y - x)
\]
\[
+ \sup_x \sum_y \left( q_p^{2} \star \varphi_p \right)(y) (1 - \cos k \cdot y) \left( m q_p \star \varphi_p^{(m)} \right)(y - x)
\]
\[
= \hat{\nu}_{p,m}^{(2,1)}(k)
\]

It is not hard to bound the second term above by 
\[
K^3 K_2 \sqrt{\varepsilon(3)} (1 - \hat{D}(k))
\]
by the same method as the proof of the bounds on \(B_{p,m}^{(\lambda,\rho)}\) and \(T_{p,m}^{(\lambda,\rho)}\). Notice that we apply the Cauchy–Schwarz inequality to it twice, which changes the 1st power of \(|\hat{S}_{\mu}(me^{-i\theta})|\) to the 2nd power and the 1/2th power of \(1 - \hat{D}(k)^2\) to the 1st power. This completes the proof of (4.3) when \(\lambda = 1\) and \(\rho = 1\). \(\square\)

4.2 Proof of Lemmas

**Proof of Lemma 4.2**

By the equality \(\hat{S}_{r e^{i\theta}}(k) = (1 - re^{i\theta} \hat{D}(k))^{-1}\) and the change of variables \(z = e^{i\theta}\),

\[
\int_T \left| \hat{S}_{r e^{i\theta}}(k) \right|^2 \frac{d\theta}{2\pi} = \int_T \frac{1}{\left|1 - e^{i\theta} r \hat{D}(k)\right|^2} \frac{d\theta}{2\pi}
\]
\[
= \int_T \frac{1}{(1 - e^{i\theta} r \hat{D}(k))(1 - e^{-i\theta} r \hat{D}(k))} \frac{d\theta}{2\pi}
\]
\[
= \int_T \left( e^{i\theta} - r \hat{D}(k) \right) (1 - e^{i\theta} r \hat{D}(k)) \frac{d\theta}{2\pi}
\]
\[
= \frac{1}{2\pi i} \oint_{|z| = 1} \frac{dz}{(z - r \hat{D}(k))(1 - z r \hat{D}(k))}.
\]

Note that \(|r \hat{D}(k)| < 1\). Only the complex number \(z = r \hat{D}(k)\) is a singularity in the disk \(\{z \in \mathbb{C} \mid |z| < 1\}\). Applying the residue theorem to the above line integral leads to

\[
\int_T \left| \hat{S}_{r e^{i\theta}}(k) \right|^2 \frac{d\theta}{2\pi} = \text{Res}_{z = r \hat{D}(k)} \frac{1}{(z - r \hat{D}(k))(1 - z r \hat{D}(k))}
\]
\[
= \lim_{z \to r \hat{D}(k)} \left( z - r \hat{D}(k) \right) \frac{1}{(z - r \hat{D}(k))(1 - z r \hat{D}(k))}.
\]

\(\square\ Springer\)
= \frac{1}{1 - r^2 \hat{D}(k)^2}.

This completes the proof of (4.4).

Similarly, the equality
\[
\int_T \left| \hat{S}_{r \theta}(k) \right|^4 \frac{d\theta}{2\pi} = \frac{1}{2\pi i} \int_{|z|=1} \frac{z dz}{(z - r \hat{D}(k))^2} (1 - zr \hat{D}(k))^2
\]
and the residue theorem imply
\[
\int_T \left| \hat{S}_{r \theta}(k) \right|^4 \frac{d\theta}{2\pi} = \text{Res}_{z=r \hat{D}(k)} \frac{z}{(z - r \hat{D}(k))^2} (1 - zr \hat{D}(k))^2
\]
\[
= \lim_{z \to r \hat{D}(k)} \frac{1 + zr \hat{D}(k)}{(1 - zr \hat{D}(k))^2}
\]
\[
= \frac{1 + r^2 \hat{D}(k)^2}{(1 - r^2 \hat{D}(k))^3}.
\]
This completes the proof of (4.5). \qed

**Proof of (4.6) in Lemma 4.3** By the Cauchy–Schwarz inequality and Lemma 4.2,
\[
[\text{LHS of (4.6)}] \leq \int_{T^d} \frac{d^d l}{(2\pi)^d} \left| \hat{D}(l) \right|^{\lambda + \rho - 1} \left( \int_T \frac{d\theta}{2\pi} \left| \hat{S}_{\mu_p(e^{i\theta})}(l \pm k) \right|^2 \right)^{1/2} 
\times \left( \int_T \frac{d\theta}{2\pi} \left| \hat{S}_{\mu_p(e^{-i\theta})}(l) \right|^4 \right)^{1/4}
\times \left( \frac{1 + \mu_p(1)^2 \hat{D}(l)^2}{(1 - \mu_p(1)^2 \hat{D}(l)^2)^3} \right)^{1/4}
\leq \sqrt{2} \int_{T^d} \frac{d^d l}{(2\pi)^d} \left| \hat{D}(l) \right|^{(\lambda + \rho - 1)/2} \left( \frac{1}{1 - \hat{D}(l \pm k)^2} \right)^{1/2} \left( \frac{|\hat{D}(l)|}{(1 - \hat{D}(l)^2)^3} \right)^{1/2}.
\]
By Hölder’s inequality and the Cauchy–Schwarz inequality again,

\[
[LHS \text{ of } (4.6)] \leq \sqrt{2} \left( \int_{\mathbb{T}^d} \frac{d^dl}{(2\pi)^d} |\hat{D}(l)|^{\lambda+\rho-1} \right)^{1/4} \left( \int_{\mathbb{T}^d} \frac{d^dl}{(2\pi)^d} \left(1 - \hat{D}(l)^2\right)^2 \right)^{3/4} \\
\leq \sqrt{2} \left( \sum_{n=0}^{\infty} (n+1) \int_{\mathbb{T}^d} \frac{d^dl}{(2\pi)^d} \hat{D}(l)^2 \left| \frac{\lambda+\rho-1}{2} \right| \hat{D}(l+k)^2n \right)^{1/4} \\
\times \left( \sum_{n=0}^{\infty} (n+1) \int_{\mathbb{T}^d} \frac{d^dl}{(2\pi)^d} \hat{D}(l)^2 \left( n+\left| \frac{\lambda+\rho-1}{2} \right| \right) \right)^{3/4} \\
= \sqrt{2} \left( \sum_{n=0}^{\infty} (n+1) \int_{\mathbb{T}^d} \frac{d^dl}{(2\pi)^d} \hat{D}(l)^2 \left| \frac{\lambda+\rho-1}{2} \right| \sum_x D^{*2n}(x) e^{i(l+k) \cdot x} \right)^{1/4} \\
\times \left( \varepsilon_2^{(\frac{(\lambda+\rho-1)/2)}{2})} \right)^{3/4} \\
= \sqrt{2} \left( \sum_{n=0}^{\infty} (n+1) \sum_x D^{*2 \left| \frac{\lambda+\rho-1}{2} \right|} (x) D^{*2n}(x) e^{i(l+k) \cdot x} \right)^{1/4} \\
\times \left( \varepsilon_2^{(\frac{(\lambda+\rho-1)/2}{2})} \right)^{3/4}.
\]

Since the summand with respect to \( x \) is an even function, the factor \( e^{i(l+k) \cdot x} \) equals \( \cos k \cdot x \), which is bounded above by 1. Therefore, we arrive at (4.6).

\( \square \)

**Proof of (4.7) in Lemma 4.3** By the Cauchy–Schwarz inequality and (4.5) in Lemma 4.2,

\[
[LHS \text{ of } (4.7)] \leq \int_{\mathbb{T}^d} \frac{d^dl}{(2\pi)^d} |\hat{D}(l)|^{\lambda+\rho-1} \\
\times \left( \int_{\mathbb{T}} \frac{d\theta}{2\pi} \left| \hat{S}_{\mu_p(e^{i\theta})}(l+k) \right|^4 \right)^{1/4} \left( \int_{\mathbb{T}} \frac{d\theta}{2\pi} \left| \hat{S}_{\mu_p(e^{i\theta})}(l-k) \right|^4 \right)^{1/4} \\
\times \left( \int_{\mathbb{T}} \frac{d\theta}{2\pi} \left| \hat{S}_{\mu_p(e^{-i\theta})}(l) \right|^4 \right)^{1/4} \left( \int_{\mathbb{T}} \frac{d\theta}{2\pi} \left| \hat{S}_{\mu_p(m^{-1}e^{i\theta})}(l) \right|^4 \right)^{1/4} (1 - \hat{D}(2l)) \\
= \int_{\mathbb{T}^d} \frac{d^dl}{(2\pi)^d} \left| \hat{D}(l) \right|^{\lambda+\rho-1} \\
\times \left( \frac{1 + \mu_p(1)^2 \hat{D}(l+k)^2}{(1 - \mu_p(1)^2 \hat{D}(l+k)^2)} \right)^{1/4} \left( \frac{1 + \mu_p(1)^2 \hat{D}(l-k)^2}{(1 - \mu_p(1)^2 \hat{D}(l-k)^2)} \right)^{1/4} \\
\times \left( 1 + \mu_p(1)^2 \hat{D}(l+k)^2 \right)^{1/4} \left( 1 + \mu_p(1)^2 \hat{D}(l-k)^2 \right)^{1/4} \\
\times \left( 1 + \mu_p(1)^2 \hat{D}(l+k)^2 \right)^{1/4} \left( 1 + \mu_p(1)^2 \hat{D}(l-k)^2 \right)^{1/4} \\
\times \left( 1 + \mu_p(1)^2 \hat{D}(l+k)^2 \right)^{1/4} \left( 1 + \mu_p(1)^2 \hat{D}(l-k)^2 \right)^{1/4}.
\]
\[
\times \left( \frac{1 + \mu_p (1)^2 \hat{D}(l)^2}{(1 - \mu_p (1)^2 \hat{D}(l)^2)^3} \right)^{1/4} \left( \frac{1 + \mu_p (m)^2 \hat{D}(l)^2}{(1 - \mu_p (m)^2 \hat{D}(l)^2)^3} \right)^{1/4} (1 - \hat{D}(2l))
\]
\[
\leq 2 \int_{\mathbb{T}^d} \frac{d^d l}{(2\pi)^d} \left( \frac{|\hat{D}(l)|^{3(\lambda + \rho - 1)/2}}{(1 - \hat{D}(l + k)^2)^3} \right)^{1/4} \left( \frac{|\hat{D}(l)|^{3(\lambda + \rho - 1)/2}}{(1 - \hat{D}(l - k)^2)^3} \right)^{1/4}
\times \left( \frac{|\hat{D}(l)|^{(\lambda + \rho - 1)/2}}{(1 - \hat{D}(l)^2)^3} \right)^{1/2} (1 - \hat{D}(2l)).
\]

By Lemma 4.4,

\[
[LHS \text{ of (4.7)}] \leq 4 \int_{\mathbb{T}^d} \frac{d^d l}{(2\pi)^d} \left( \frac{|\hat{D}(l)|^{3(\lambda + \rho - 1)/2}}{(1 - \hat{D}(l + k)^2)^3} \right)^{1/4}
\times \left( \frac{|\hat{D}(l)|^{(\lambda + \rho - 1)/2}}{(1 - \hat{D}(l - k)^2)^3} \right)^{1/4} \left( \frac{|\hat{D}(l)|^{(\lambda + \rho - 1)/2}}{1 - \hat{D}(l)^2} \right)^{1/2}.
\]

By Hölder’s inequality and the Cauchy–Schwarz inequality again,

\[
[LHS \text{ of (4.7)}] \leq 4 \left( \int_{\mathbb{T}^d} \frac{d^d l}{(2\pi)^d} \frac{|\hat{D}(l)|^{(\lambda + \rho - 1)/2}}{(1 - \hat{D}(l + k)^2)(1 - \hat{D}(l - k)^2)} \right)^{3/4}
\times \left( \int_{\mathbb{T}^d} \frac{d^d l}{(2\pi)^d} |\hat{D}(l)|^{\lambda + \rho - 1} \right)^{1/4}
\leq 4 \left( \int_{\mathbb{T}^d} \frac{d^d l}{(2\pi)^d} |\hat{D}(l)|^{\lambda + \rho - 1} \vartheta^2 \right)^{3/8}
\times \left( \int_{\mathbb{T}^d} \frac{d^d l}{(2\pi)^d} |\hat{D}(l)|^{\lambda + \rho - 1} \right)^{1/4}
\leq 4 \vartheta^{3/2}\left(1 - \hat{D}(k)^2\right) \left(1 - \hat{D}(2k)\right).
\]

\[
\square
\]

**Proof of Lemma 4.4** Note that

\[
2 \left( 1 - \hat{D}(k)^2 \right) - \left( 1 - \hat{D}(2k) \right) = 1 - 2 \prod_{j=1}^d \cos^2 k_j + \prod_{j=1}^d \cos(2k_j)
\]
\[
= 1 - 2 \prod_{j=1}^d \cos^2 k_j + \prod_{j=1}^d \left( 2 \cos^2 k_j - 1 \right)
\]
on $\mathbb{L}^d$. Let $h_d(\xi_1, \ldots, \xi_d) := 1 - 2 \prod_{j=1}^{d} \xi_j + \prod_{j=1}^{d} (2\xi_j - 1)$. It suffices to show that $h_d(\xi_1, \ldots, \xi_d) \geq 0$ for every $(\xi_1, \ldots, \xi_d) \in [0, 1]^d$. To do so, we use the method of induction.

When $d = 1$, $h_1(\xi_1) = 0$ for every $\xi_1$. Next, we assume that $h_d(\xi_1, \ldots, \xi_d) \geq 0$ holds in some dimension $d$. Fix $\xi_j$ for each $j = 1, \ldots, d$. Then, $h_{d+1}(\xi_1, \ldots, \xi_{d+1})$ is regarded as a linear function with respect to $\xi_{d+1}$ and hence takes the minimum value at either the boundary point 0 or 1. When $\xi_{d+1} = 0$, clearly,

$$h_{d+1}(\xi_1, \ldots, \xi_{d}, 0) = 1 - \prod_{j=1}^{d} (2\xi_j - 1) \geq 0.$$ 

When $\xi_{d+1} = 1$, by the induction hypothesis,

$$h_{d+1}(\xi_1, \ldots, \xi_{d}, 1) = 1 - 2 \prod_{j=1}^{d} \xi_j + \prod_{j=1}^{d} (2\xi_j - 1) \geq 0.$$ 

Therefore, $h_d(\xi_1, \ldots, \xi_d) \geq 0$ holds in every dimension $d$. \qed

5 Completing the Bootstrap Argument

In this section, we prove Proposition 1.8 by a computer-assisted approach.

Proof of Proposition 1.8 Set

$$d = 9, \quad K_1 = 1.0020, \quad K_2 = 1.0500, \quad K_3 = 1.2500,$$ 

which satisfy the initial conditions of Proposition 1.7. From Lemma 4.1 and Table 1 in Appendix A, the basic diagrams are bounded above as

$$B^{(0,2)}_{p,m} \leq 2.37279 \times 10^{-3}, \quad B^{(2,2)}_{p,m} \leq 2.11688 \times 10^{-4},$$

$$T^{(1,1)}_{p,m} \leq 3.96190 \times 10^{-3}, \quad T^{(2,2)}_{p,m} \leq 4.40247 \times 10^{-4},$$

$$\frac{\hat{\gamma}^{(1,1)}_{p,m}(k)}{1 - \hat{D}(k)} \leq 4.81033 \times 10^{-2}, \quad \frac{\hat{\gamma}^{(1,2)}_{p,m}(k)}{1 - \hat{D}(k)} \leq 1.71979 \times 10^{-2},$$

$$\frac{\hat{\gamma}^{(2,1)}_{p,m}(k)}{1 - \hat{D}(k)} \leq 3.91493 \times 10^{-2}, \quad \frac{\hat{\gamma}^{(2,2)}_{p,m}(k)}{1 - \hat{D}(k)} \leq 3.92276 \times 10^{-2},$$

$$\frac{\hat{\gamma}^{(1,3)}_{p,m}(k)}{1 - \hat{D}(k)} \leq 1.72315 \times 10^{-2}, \quad \frac{\hat{\gamma}^{(3,1)}_{p,m}(k)}{1 - \hat{D}(k)} \leq 6.59882 \times 10^{-2}.$$ 

The 4th inequality in the above implies the sufficient condition of Lemma 2.4. In addition, from Lemma 2.4,

$$\hat{\Pi}^{\text{even}}_{p}(0, m) \leq 2.30399 \times 10^{-6},$$
Table 1  Numerical values of random-walk quantities on the BCC lattice

| $d$ | $\varepsilon_1^{(1)}$ | $\varepsilon_1^{(2)}$ | $\varepsilon_1^{(3)}$ |
|-----|------------------|------------------|------------------|
| 3   | $3.932159419 \times 10^{-1}$ | $2.682159060 \times 10^{-1}$ | $2.154814953 \times 10^{-1}$ |
| 4   | $1.186366900 \times 10^{-1}$ | $5.613668875 \times 10^{-2}$ | $3.636129693 \times 10^{-3}$ |
| 5   | $4.682555747 \times 10^{-2}$ | $1.557555743 \times 10^{-2}$ | $8.159785906 \times 10^{-3}$ |
| ... | ...               | ...               | ...               |
| 9   | $2.143603149 \times 10^{-3}$ | $1.904781484 \times 10^{-4}$ | $4.38237046 \times 10^{-5}$ |
| 10  | $1.125786856 \times 10^{-1}$ | $6.575312431 \times 10^{-2}$ | ...               |
| ... | ...               | ...               | ...               |

(b) The random-walk bubbles

| $d$ | $\varepsilon_2^{(1)}$ | $\varepsilon_2^{(2)}$ |
|-----|------------------|------------------|
| 3   | $\infty$         | $\infty$         |
| 4   | $\infty$         | $\infty$         |
| 5   | $1.125786856 \times 10^{-1}$ | $6.575312431 \times 10^{-2}$ |
| ... | ...               | ...               |
| 9   | $2.410376015 \times 10^{-3}$ | $2.667728665 \times 10^{-4}$ |
| 10  | $1.132062934 \times 10^{-3}$ | $8.774674704 \times 10^{-5}$ |
| 11  | $2.633784960 \times 10^{-4}$ | $1.038120322 \times 10^{-5}$ |
| 12  | $2.633784960 \times 10^{-4}$ | $1.038120322 \times 10^{-5}$ |

\[
\hat{\Pi}_p^{\text{odd}}(0, m) \leq 1.09838 \times 10^{-4},
\]

\[
\sum_{(x, t)} \left( \Pi_p^{\text{even}}(x, t) + \Pi_p^{\text{odd}}(x, t) \right) m^t t \leq 3.84797 \times 10^{-4},
\]

\[
\sum_{(x, t)} \left( \Pi_p^{\text{even}}(x, t) + \Pi_p^{\text{odd}}(x, t) \right) m^t \frac{1 - \cos k \cdot x}{1 - \tilde{D}(k)} \leq 2.02991 \times 10^{-2},
\]

which imply the absolute convergence of the alternating series (2.2). The 1st and 2nd inequalities in the above imply the sufficient condition of Proposition 3.1. Then, by (3.2), (3.4) and (3.4), we obtain

\[
g_1(p, m) \leq 1.0002 < K_1, \quad (5.2)
\]

\[
g_2(p, m) \leq 1.0430 < K_2, \quad (5.3)
\]

\[
g_3(p, m) \leq 1.2343 < K_3. \quad (5.4)
\]

The random-walk quantities $\varepsilon_i^{(v)} (i = 1, 2$ and $v \in \mathbb{Z}_+)$ monotonically decrease with respect to $d$ due to the estimate in Appendix A. Recall that the upper bounds in
Lemma 4.1 depend only on \( \{e^{(v)}_i\}_{i=1,2; v \in \mathbb{Z}_+} \) and \( \{K_i^3\}_{i=1}^3 \). Therefore, the inequalities \( g_i(p, m) < K_i, i = 1, 2, 3 \) also hold in \( d > 9 \) with the same numerical values (5.1) of \( K_i \) as in \( d = 9 \). This completes the proof of Proposition 1.8.

\( \square \)

**Remark 5.1** One may wonder why the lace expansion analysis in this paper does not work in \( d = 8 \). That is caused by the too large upper bounds in Lemmas 2.4 and 4.1 as well as in (3.2), (3.4) and (3.4). To find the numerical values (5.1), we get assistance from computers. Specifically, we divide the intervals \((1.0, 1.1], (1.0, 1.1] \) and \((1.0, 1.3] \) for \( K_1, K_2 \) and \( K_3 \) respectively, into 100 and check the inequalities \( g_i(p, m) < K_i, i = 1, 2, 3 \) for each dividing point in \( d = 9 \). We do not find proper values of \( \{K_i^3\}_{i=1}^3 \) in \( d = 8 \). There is a possibility that our selections of the parameters (intervals and the division number) are bad. However, it is essentially more important to improve the upper bounds than the computer-assisted part.

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**Code Availability** One can download it from the third author’s repository https://gitlab.com/ykami/comp-lace_for_op.

**Declarations**

**Conflict of interest** The authors declare that they have no conflict of interest.

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**Consent to Participate** Not applicable.

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**Appendix A: Random Walk Quantities on the BCC Lattice**

In this section, we show the expressions of the upper bounds on (1.2) to compute their numerical values. We use Stirling’s formula due to the property (1.1) of the BCC lattice, which helps us to obtain highly-precise estimates. Recall (1.1). Note that

\[
D^{*2n}(o) = \left( \binom{2n}{n} \frac{1}{2^{2n}} \right)^d \leq \left( \frac{1}{\sqrt{\pi n}} \right)^d, \quad (A.1)
\]
for \( n \in \mathbb{N} \). Now, fix a sufficiently large number \( N \). For \( \nu \in \mathbb{N} \), (A.1) and the bounds given by the integral test for convergence imply that

\[
\varepsilon^{(\nu)}_1 = \sum_{n=\nu}^{\infty} D^{*2n}(o) \leq \sum_{n=\nu}^{\nu+N-1} D^{*2n}(o) + \sum_{n=\nu+N}^{\infty} \frac{1}{(\pi n)^{d/2}} \leq \sum_{n=0}^{N-1} D^{*2(n+\nu)}(o) + \frac{1}{\pi^{d/2}(\nu+N)^{d/2}} + \frac{1}{\pi^{d/2}} \int_{\nu+N}^{\infty} s^{-d/2}ds
\]

\[
\leq \sum_{n=0}^{N-1} D^{*2(n+\nu)}(o) + \frac{1}{\pi^{d/2}(\nu+N)^{d/2}} + \frac{2}{\pi^{d/2}(d-2)} (\nu+N)^{(2-d)/2}.
\]

Moreover,

\[
\varepsilon^{(\nu)}_2 = \sum_{n=\nu}^{\infty} (n-\nu+1) D^{*2n}(o) \leq \sum_{n=\nu}^{\nu+N-1} (n-\nu+1) D^{*2n}(o) + \sum_{n=\nu+N}^{\infty} (n-\nu+1) \frac{1}{(\pi n)^{d/2}} \leq \sum_{n=0}^{N-1} (n+1) D^{*2(n+\nu)}(o) + \frac{1}{\pi^{d/2}(\nu+N)^{d/2-1}}
\]

\[
+ \frac{1}{\pi^{d/2}} \int_{\nu+N}^{\infty} s^{1-d/2}ds - \frac{\nu-1}{\pi^{d/2}} \int_{\nu+N}^{\infty} s^{-d/2}ds
\]

\[
\leq \sum_{n=0}^{N-1} (n+1) D^{*2(n+\nu)}(o) + \frac{1}{\pi^{d/2}(\nu+N)^{(d-2)/2}} + \frac{2}{\pi^{d/2}(d-2)} (\nu+N)^{(4-d)/2} - \frac{2(\nu-1)}{\pi^{d/2}(d-2)} (\nu+N)^{(2-d)/2}.
\]

When \( N = 500 \), we obtain numerical values in Table 1 by directly computing these expressions.

**Proof of Lemma 2.3**

In this section, we prove Lemma 2.3. It is the result of Lemma 6.1 below. In this lemma, we first divide percolation events by cases, which are based on the fact of whether or not double connections are collapsed and where a path intersects a double connection. Next, these observations yield a lot of disjoint events such as \( \{x \to y\} \circ \{u \to v\} \), where \( \circ \) denotes that the left and right events must occur disjointly. Then, by the BK inequality [38], we bound the probability of such events in terms of the product of the
two-point functions, e.g.,

\[ \mathbb{P}_p(\{x \to y\} \circ \{u \to v\}) \leq \mathbb{P}_p(x \to y)\mathbb{P}_p(u \to v) = \varphi_p(y-x)\varphi_p(v-u). \]

Such a product is represented by a diagram like (2.6). Also, the method to obtain Lemma 2.3 from Lemma 6.1 is based on decomposing diagrams by the inequality

\[
\|fg\|_1 := \sum_{x \in L^d \times \mathbb{Z}_+} f(x)g(x) \leq \|f\|_\infty \|g\|_1,
\]

where \(f\) and \(g\) are functions depending on a sum of a product of the two-point functions.

**Lemma 6.1** For \(x \in \mathbb{L}^d \times \mathbb{Z}_+\) and \(N \geq 2\),

\[
\left| \pi_p^{(0)}(x) - \pi_p^{(1)}(x) \right| \leq \frac{1}{2} \times \begin{array}{c}
\text{Diagram 1}
\end{array} + \frac{1}{2} \times \begin{array}{c}
\text{Diagram 2}
\end{array} + \frac{3}{2} \times \begin{array}{c}
\text{Diagram 3}
\end{array} + \begin{array}{c}
\text{Diagram 4}
\end{array} + \begin{array}{c}
\text{Diagram 5}
\end{array} + \begin{array}{c}
\text{Diagram 6}
\end{array},
\]

\[(B.2)\]

\[
\pi_p^{(2)}(x) \leq \begin{array}{c}
\text{Diagram 7}
\end{array} + 2 \times \begin{array}{c}
\text{Diagram 8}
\end{array} + \begin{array}{c}
\text{Diagram 9}
\end{array} + \begin{array}{c}
\text{Diagram 10}
\end{array},
\]

\[(B.3)\]
Proof of (B.2) in Lemma 6.1 This proof is inspired by [39, Sect. 3.1]. First, we rewrite the event in $\pi_p((N)\to x)$. To do so, we introduce an ordering among bonds as follows. Let $B((x,t)) = \{(x,t), (y, t+1)\} \in (\mathbb{L}^d \times \mathbb{Z}_+)^2 \mid x - y \in \mathcal{M}^d \}$ for $(x,t) \in \mathbb{L}^d \times \mathbb{Z}_+$, which is the set of directed bonds whose bottoms are $(x,t)$. We can order the elements in $B((x,t))$ because it is a finite set. For a pair of bonds $b_1$ and $b_2$, we write $b_1 < b_2$ if $b_1$ is smaller than $b_2$ in that ordering. Then, we obtain

\begin{equation}
\pi_p((N)\to x) \leq \sum_{\{y_i\}_{i=1}^N, \{u_i\}_{i=1}^N \mid (\forall i: \iota(y_i) > \iota(u_i - 1))} \left( 2\delta_{u_1, o} \delta_{y_1, o} \times \right.
\begin{align*}
&+ \frac{1}{2} \delta_{y_1, o} \times v_1 \quad + \frac{1}{2} \delta_{y_1, o} \times u_1 \quad + u_1 \\
&\times \prod_{i=2}^{N-1} \left( \begin{array}{c}
u_i + 1 \\
\nu_i + 1 \end{array} \right) \times \nu_i \end{align*}
\end{equation}

(B.5)
Next, we rewrite the event in $\pi_p^{(1)}(x)$. By definition, we can easily see that

$$E \left( b, x; \tilde{c}^b(y) \right) \subset \{ y \to x \} \circ \{ b \to x \}. \quad (B.6)$$

By splitting the event $\{ o \Rightarrow b \}$ into two events based on whether $b$ equals $o$ or not,

$$\pi_p^{(1)}(x) = \sum_b \mathbb{P}_p \left( \{ o \Rightarrow b \} \cap E(b, x; \tilde{c}^b(o)) \right)$$

$$= \sum_{b \in B(o)} \mathbb{P}_p \left( E(b, x; \tilde{c}^b(o)) \right) \left[ \sum_{b \in B(o)} \mathbb{P}_p \left( \{ o \Rightarrow b \} \cap E(b, x; \tilde{c}^b(o)) \right) \right]$$

$$= \sum_{b \in B(o)} \left( \mathbb{P}_p \left( \{ o \to x \} \circ \{ b \to x \} \right) - \mathbb{P}_p \left( \{ o \to x \} \circ \{ b \to x \} \setminus E(b, x; \tilde{c}^b(o)) \right) \right)$$

$$+ \sum_b \mathbb{P}_p \left( \{ o \Rightarrow b \neq o \} \cap E(b, x; \tilde{c}^b(o)) \right). \quad (B.7)$$

Since both

$$\{ \forall b' < b, b' \to x \} \sqcup \{ \exists b'' < b, b'' \to x \}$$

and

$$\{ \forall b' > b, b' \to x \} \sqcup \{ \exists b'' > b, b'' \to x \}$$

are the whole event, respectively,

$$\mathbb{P}_p \left( \{ o \to x \} \circ \{ b \to x \} \right)$$

$$= \mathbb{P}_p \left( \{ o \to x \} \circ \{ b \to x \} \right) \cap \{ \forall b' < b, b' \to x \} \cap \{ \forall b'' > b, b'' \to x \}$$

$$+ \mathbb{P}_p \left( \{ o \to x \} \circ \{ b \to x \} \right) \cap \{ \exists b' < b, b' \to x \} \cap \{ \exists b'' > b, b'' \to x \}$$

$$= \mathbb{P}_p \left( \{ o \to x \} \circ \{ b \to x \} \right)$$

$$+ \mathbb{P}_p \left( \{ o \to x \} \circ \{ b \to x \} \right) \cap \{ \forall b' > b, b' \to x \}$$

$$+ \mathbb{P}_p \left( \{ o \to x \} \circ \{ b \to x \} \right) \cap \{ \exists b'' > b, b'' \to x \}.$$
\[ + \sum_{b \in \mathcal{B}(o)} P_p \{ o \to x \} \cup \{ b \to x \} \cap \{ \exists b' < b, b' \to x \} \cap \{ \exists b'' > b, b'' \to x \}. \]  

(B.8)

Substituting (B.8) into (B.7) and subtracting (B.5), we obtain

\[
\pi_p^{(1)}(x) - \pi_p^{(0)}(x) = \underbrace{P_p(o \Rightarrow x \neq o)}_{(a)} + \underbrace{\sum_{b \in \mathcal{B}(o)} P_p \{ o \to x \} \cup \{ b \to x \} \cap \{ \exists b' < b, b' \to x \} \cap \{ \exists b'' > b, b'' \to x \}}_{(b)} - \underbrace{\sum_{b \in \mathcal{B}(o)} P_p \{ \{ o \to x \} \cup \{ b \to x \} \} \setminus E(b, x; \tilde{C}(o))}_{(c)} + \underbrace{\sum_{b} P_p \{ o \Rightarrow b \} \neq o \cap E(b, x; \tilde{C}(o))}_{(d)}. \tag{B.9}
\]

Finally, we show how (B.9) leads to the upper bound (B.2). In the following, we repeatedly use the trivial inequality

\[
\varphi_p(x) \mathbb{1}_{\{o \neq x\}} \leq (q_p \star \varphi_p)(x) \tag{B.10}
\]

and the fact that, if there are two disjoint connections, then their lengths are at least two for oriented percolation. By Boole’s and the BK inequalities, (a) in (B.9) is bounded above as

\[
P_p(o \Rightarrow x \neq o)
= P_p \left( \bigcup_{b_1, b_2 \in \mathcal{B}(o)} \bigcup_{b'_1 \in \mathcal{B}(b_1), b'_2 \in \mathcal{B}(b_2), (b_1 \prec b_2)} \left\{ \{ b_1 \text{ is occupied} \& b'_1 \to x \} \cup \{ b_2 \text{ is occupied} \& b'_2 \to x \} \right\} \right)
\leq \sum_{b_1, b_2 \in \mathcal{B}(o)} \sum_{b'_1 \in \mathcal{B}(b_1), b'_2 \in \mathcal{B}(b_2), (b_1 \prec b_2)} q_p(b_1)q_p(b'_1)\varphi_p(x - b'_1) \cdot q_p(b_2)q_p(b'_2)\varphi_p(x - b'_2)
\leq \frac{1}{2} \left( q_p^* \star \varphi_p \right)(x)^2,
\]

which corresponds to the 1st term in (B.2). The factor 1/2 in the last line is due to ignoring the ordering. To bound (b) in (B.9), we note that, for \( b \in \mathcal{B}(o) \),
\[
\{ (o \rightarrow x) \circ (b \rightarrow x) \} \cap \{ \exists b' < b, b' \rightarrow x \} \cap \{ \exists b'' > b, b'' \rightarrow x \} \subset \bigcup_{b', b'' \in \mathcal{B}(o)} \bigcup_y \{ (b \rightarrow y \rightarrow x) \circ (b' \rightarrow x) \circ (b'' \rightarrow y) \}.
\]

By Boole’s and the BK inequalities, we have
\[
\sum_{b \in \mathcal{B}(o)} \mathbb{P}_P \left( \{ (o \rightarrow x) \circ (b \rightarrow x) \} \cap \{ \exists b' < b, b' \rightarrow x \} \cap \{ \exists b'' > b, b'' \rightarrow x \} \right) \leq \sum_y \left( q_p^{*2} \star \varphi_p \right) (y)^2 \left( q_p^{*2} \star \varphi_p \right) (x) \varphi_p (x - y),
\]
which corresponds to the 2nd term in (B.2). To bound (c) in (B.9), we note that, for \( b \in \mathcal{B}(o) \),
\[
\{ (o \rightarrow x) \circ (b \rightarrow x) \} \setminus E(b, x; \tilde{C}^b(o)) \subset \bigcup_{y \neq b'} \bigcup_{b'} \{ (o \rightarrow y \rightarrow x) \circ (b \rightarrow b' \rightarrow x) \circ (y \rightarrow b') \}
\]
\[
\bigcup_{b'} \{ o \rightarrow b' \rightarrow x \} \circ (b \rightarrow b' \rightarrow x)
\]
\[
\bigcup_{b'} \bigcup_{y \neq b} \{ o \rightarrow b' \rightarrow x \} \circ (b \rightarrow b' \rightarrow x) \circ (y \rightarrow b')
\]
\[
\bigcup_{b'} \bigcup_{y \neq b} \{ o \rightarrow y \rightarrow x \} \circ (b \rightarrow b' \rightarrow x) \circ (b' \rightarrow x)
\]
\[
\bigcup_{b'} \bigcup_{y \neq b} \{ o \rightarrow b' \rightarrow x \} \circ (b \rightarrow b' \rightarrow x) \circ (b' \rightarrow x) \circ (y \rightarrow x).
\]

By Boole’s and the BK inequalities, we have
\[
\sum_{b \in \mathcal{B}(o)} \mathbb{P}_P \left( \{ (o \rightarrow x) \circ (b \rightarrow x) \} \setminus E(b, x; \tilde{C}^b(o)) \right) \leq \sum_u \left( q_p^{*3} \star \varphi_p \right) (x) \left( q_p^{*2} \star \varphi_p \right) (u)^2 \left( q_p \star \varphi_p \right) (x - u)
\]
\[
+ \sum_u \left( q_p^{*2} \star \varphi_p \right) (u)^2 \left( q_p^{*2} \star \varphi_p \right) (x - u)^2
\]
\[
+ \sum_{u,y} \left( q_p^{*2} \star \varphi_p \right) (u) \left( q_p \star \varphi_p \right) (y) \left( q_p \star \varphi_p \right) (u - y)
\]
\[
\times \left( q_p \star \varphi_p \right) (x - u) \left( q_p^{*2} \star \varphi_p \right) (x - y)
\]
+ \sum_{u} \left( q_p^* \varphi_p \right) (u)^2 \left( q_p^* \varphi_p \right) (x - u)^2 \\
+ \sum_{u,y} \left( q_p^* \varphi_p \right) (u) \left( q_p \varphi_p \right) (y) \left( q_p \varphi_p \right) (u - y) \\
\times \left( q_p^* \varphi_p \right) (x - y) \left( q_p \varphi_p \right) (x - u).

Each term in the above upper bound corresponds to contributions of the 3rd term, the 4th term, the 5th term, the 4th term and the 6th term in (B.2), respectively. To bound (d) in (B.9), we apply (B.6) and split the below event into two events based on where the branching point is assigned:

$$\{ o \Rightarrow u \neq o \} \cap \{ o \Rightarrow u \neq o \} \circ \{ o \rightarrow x \} \cup \bigcup_{y \neq o \atop (t(y) \leq t(u))} \{ o \rightarrow u \} \circ \{ o \rightarrow y \rightarrow u \} \circ \{ y \rightarrow x \}. \quad (B.12)$$

Then, we obtain

$$\sum_{b} \mathbb{P}_p \left( \{ o \Rightarrow b \neq o \} \cap E(b, x; c^b(o)) \right) \leq \sum_{b} \mathbb{P}_p \left( \{ o \Rightarrow b \neq o \} \cap \{ o \rightarrow x \} \circ \{ b \rightarrow x \} \right)$$

$$= \sum_{b} \mathbb{P}_p \left( \{ o \Rightarrow b \neq o \} \cap \{ o \rightarrow x \} \circ \{ b \rightarrow x \} \right)$$

$$\leq \sum_{b} \left( \mathbb{P}_p \left( \{ o \Rightarrow b \neq o \} \circ \{ o \rightarrow x \} \circ \{ b \rightarrow x \} \right) \right) + \sum_{y \neq o \atop (t(y) \leq t(b))} \mathbb{P}_p \left( \{ o \rightarrow b \} \circ \{ o \rightarrow y \rightarrow b \} \circ \{ y \rightarrow x \} \circ \{ b \rightarrow x \} \right) \quad (B.13)$$

Applying the BK [38] inequality and the same method for (a) to the right-most in the above, and paying attention to the disjointness of the connections, we arrive at

$$\sum_{b} \mathbb{P}_p \left( \{ o \Rightarrow b \neq o \} \cap E(b, x; c^b(o)) \right) \leq \frac{1}{2} \sum_{u} \left( q_p^2 \varphi_p \right) (u)^2 \left( q_p \varphi_p \right) (x - u) \left( q_p^3 \varphi_p \right) (x)$$

$$+ \sum_{u} \left( q_p^2 \varphi_p \right) (u)^2 \left( q_p^2 \varphi_p \right) (x - u)^2$$

$$+ \sum_{u,y} \left( q_p^2 \varphi_p \right) (u) \left( q_p \varphi_p \right) (y) \left( q_p \varphi_p \right) (u - y)$$
\begin{equation*}
\times (q_p \ast \varphi_p) (x - u) \left( q_p^2 \ast \varphi_p \right) (x - y).
\end{equation*}

Each term in the above upper bound corresponds to contributions of the 3rd term, the 4th term and the 5th term in (B.2), respectively. Combining the upper bounds on (a)–(d) completes the proof of (B.2). \hfill \Box

**Proof of (B.3) in Lemma 6.1** It is not hard to prove the upper bound by using (B.6), (B.11) and (B.14), so that we omit it. \hfill \Box

**Proof of (B.4) in Lemma 6.1** By definition,

\begin{equation}
E(b, u; \tilde{C}^b(v)) \cap \{b \rightarrow x\} \\
\subset \bigcup_{y \quad (t(b) < t(y) \leq t(u))} \left\{ \{b \rightarrow u\} \circ \{v \rightarrow y \rightarrow u\} \circ \{y \rightarrow x\} \right\} \cup \left\{ \{b \rightarrow y \rightarrow u\} \circ \{v \rightarrow u\} \circ \{y \rightarrow x\} \right\}.
\end{equation}

Paying attention to the disjointness of connections and the magnitude relationship between times, we obtain

\begin{equation}
\pi_p^{(N)}(x) = \sum_{b_N} \mathbb{P}_p \left( \tilde{E}_{b_N}^{(N)}(x) \right)
= \sum_{b_N} \mathbb{P}_p \left( \tilde{E}_{b_{N-1}}^{(N-1)}(b_N) \cap E(b_N, x; \tilde{C}^{b_N}_{b_N-1}) \right)
\leq \sum_{b_N} \mathbb{P}_p \left( \tilde{E}_{b_{N-2}}^{(N-2)}(b_{N-1}) \cap E(b_{N-1}, b_N; \tilde{C}^{b_{N-1}}_{b_{N-2}}) \cap \{ \ast \} b_{N-1} \rightarrow x \circ \{ \ast \} b_N \rightarrow x \right)
\leq \sum_{b_{N-2}} \sum_{v_{N-1}, v_N} \sum_{y_N, u_{N-1}, u_N} \mathbb{P}_p \left( \tilde{E}_{b_{N-2}}^{(N-2)}(u_{N-1}) \cap \tilde{b}_{N-2} \rightarrow y_N \right) \times \mathbb{P}_p \left( \{ (u_{N-1}, v_{N-1}) \rightarrow u_N \} \circ y_N \rightarrow u_N \circ \{ y_N \rightarrow x \} \circ (u_N, v_N) \rightarrow x \right)
+ \mathbb{P}_p \left( \tilde{E}_{b_{N-2}}^{(N-2)}(u_{N-1}) \cap \tilde{b}_{N-2} \rightarrow u_N \right) \times \mathbb{P}_p \left( \{ (u_{N-1}, v_{N-1}) \rightarrow y_N \rightarrow u_N \} \circ \{ y_N \rightarrow x \} \circ (u_N, v_N) \rightarrow x \right)
\end{equation}

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\[
\begin{align*}
\pi^{(N)}_p(x) & \leq \sum_{y_N \leq y_{N-1}} \sum_{u_N, u_{N-1}} \pi^{(N-2)}_{p, N-2}(y_N, u_N, u_{N-1}) \\
& \times \left( \mathbb{P}_{p, N-2} \left( \begin{array}{c}
\hat{E}^{(N-2)}_{\tilde{b}_{N-2}}(u_{N-1}) \cap \tilde{b}_{N-2} \rightarrow y_N \\
\times (q_p \ast \varphi_p)(u_N - u_{N-1})
\end{array} \right) \right) \\
& \times \left( \mathbb{P}_{p, N-2} \left( \begin{array}{c}
\hat{E}^{(N-2)}_{\tilde{b}_{N-2}}(u_{N-1}) \cap \tilde{b}_{N-2} \rightarrow u_N \\
\times (q_p \ast \varphi_p)(y_N - u_{N-1})
\end{array} \right) \right) \\
& \times \varphi_p(u_{N-1} - y_{N-1}) (q_p \ast \varphi_p) (y_N - y_{N-1}) (q_p \ast \varphi_p) (y_N - u_{N-1}) \\
& \times \frac{1}{2} \Xi(y_N, u_N; x, x).
\end{align*}
\]

By applying (B.14) to the above repeatedly,
Finally, using Boole’s and the BK [38] inequality, and applying a similar method to (B.2), we arrive in (B.4).

Multiplying the diagrammatic bounds in Lemma 6.1 by the factors $m', t$ or $1 - \cos k \cdot x$, and taking the sum of them, we obtain Lemma 2.3. The upper bounds (2.13)–(2.9) also require a telescopic inequality for the cosine function. Since its proof is quite the same as the literature, we omit it.

Lemma 6.2 ([12, Lemma 2.13] or [21, Lemma 7.3]) Let $J \geq 1$ and $t_j \in \mathbb{R}$ for $j = 1, \ldots, J$. Then,

$$0 \leq 1 - \cos \sum_{j=1}^{J} t_j \leq J \sum_{j=1}^{J} (1 - \cos t_j).$$

(B.15)

Sketch proof of Lemma 2.3 We only deal with three examples of the bounds on the sum of a diagram multiplied by the factors $m', t$ or $1 - \cos k \cdot x$ because one can easily calculate the other bounds on the analogy of such examples. The following proof is almost identical to the proof of [17, Lemma 5.3].

First, we consider the bounds (2.7)–(2.9). By the translation-invariance, for example,

$$\sum_{x} m^{t(x)} = \sum_{x, y, w} w m^{t(y)} m^{t(x) - t(y)}$$

$$\leq \sum_{w} \left( \sum_{y} \left( \sup_{x} m^{t(x)-t(y)} \right) \left( \sum_{y} w m^{t(y)} \right) \right)$$

$$= \left( \sum_{y} \left( \sup_{x} m^{t(x)-t(y)} \right) \left( \sum_{y} w m^{t(y)} \right) \right)$$

$$= \left( \sup_{y} \sum_{x} \left( q_p^{*2} \varphi_p \right) \left( q_p \varphi_p \right) (x - y)m^{t(x)-t(y)} \right)$$

$$\times \left( \sum_{y} \left( q_p^{*2} \varphi_p^{*2} \right) (y)m^{t(y)} \right)$$

$$\leq B^{(2,1)}_{p,m} T^{(2,2)}_{p,m},$$
which corresponds to the last term in the right hand side in (2.7). In the second equality, we have used the translation invariance.

Next, we consider the bounds (2.10)–(2.12). Note that

\[
\begin{align*}
(q_p \star \varphi_p)(x, t) &\leq (q_p \star \varphi_p^2)(x, t), \\
(d_p^2 \star \varphi_p)(x, t) &\leq (q_p^2 \star \varphi_p^2)(x, t) + (q_p \star \varphi_p^2)(x, t).
\end{align*}
\]

By using the above inequalities, for example,

\[
\sum_{x} m^{t(x)} t(x)
\]

\[
= \sum_{x, y, w} m^{t(x)} \left( (t(x) - t(w)) + (t(w) - t(y)) + y \right)
\]

\[
\leq \sum_{x, y, w} \left( \begin{array}{c}
\sum_{x, y, w} m^{t(x)} t(x) \\
\sum_{x, y, w} m^{t(x)} t(x) \\
\sum_{x, y, w} m^{t(x)} t(x) \\
\sum_{x, y, w} m^{t(x)} t(x)
\end{array} \right) + B^{(1,2)} T^{(1,1)} p, m T^{(2,1)}, m T^{(1,1)} p, m T^{(2,1)} p, m
\]

which corresponds to the 5th term in the right hand side in (2.11).
Finally, we consider (2.13)–(2.15). By Lemma 6.2, for example,

\[
\sum_{(x,t)} m^t (1 - \cos k \cdot x) \leq 3 \sum_{(x,t), (w,r), (y,s), (w',r'), (y',s')} m^t \times m^{t-r'} \times m^{s'} \times \left( (1 - \cos k \cdot (x - w)) + (1 - \cos k \cdot (w - y)) + (1 - \cos k \cdot y) \right)
\]

\[
\leq 3 \left( \hat{V}^{(1,2)}_{p,m}(k) T^{(1,1)}_{p,m} T^{(1,2)}_{p,m} + T^{(1,2)}_{p,m}(k) \hat{V}^{(1,1)}_{p,m} T^{(1,2)}_{p,m} + T^{(1,1)}_{p,m}(k) T^{(1,1)}_{p,m} \hat{V}^{(1,2)}_{p,m} \right),
\]

which corresponds to the last term on the right hand side in (2.14).

References

1. Aizenman, M., Barsky, D.J.: Sharpness of the phase transition in percolation models. Commun. Math. Phys. 108, 489–526 (1987)
2. Aizenman, M., Newman, C.M.: Tree graph inequalities and critical behavior in percolation models. J. Stat. Phys. 36, 107–143 (1984)
3. Barsky, D.J., Aizenman, M.: Percolation critical exponents under the triangle condition. Ann. Probab. 19, 1520–1536 (1991)
4. Bezuidenhout, C., Grimmett, G.: The critical contact process dies out. Ann. Probab. 18, 1462–1482 (1990)
5. Borgs, C., Chayes, J.T., van der Hofstad, R., Slade, G., Spencer, J.: Random subgraphs of finite graphs. II. The lace expansion and the triangle condition. Ann. Probab. 33, 1886–1944 (2005)
6. Broadbent, S.R., Hammersley, J.M.: Percolation processes. I. Crystals and mazes. Proc. Camb. Philos. Soc. 53, 629–641 (1957)
7. Brydges, D.C., Spencer, T.: Self-avoiding walk in 5 or more dimensions. Commun. Math. Phys. 97, 125–148 (1985)
8. Chayes, J.T., Chayes, L.: Inequality for the infinite-cluster density in Bernoulli percolation. Phys. Rev. Lett. 56, 1619–1622 (1986)
9. Chen, L.-C., Sakai, A.: Critical behavior and the limit distribution for long-range oriented percolation. I. Probab. Theory Relat. Fields 142, 151–188 (2008)
10. Duminil-Copin, H., Tassion, V.: A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model. Commun. Math. Phys. 343, 725–745 (2016)
11. Duminil-Copin, H., Tassion, V., Teixeira, A.: The box-crossing property for critical two-dimensional oriented percolation. Probab. Theory Relat. Fields 171, 685–708 (2018)
12. Fitzner, R., van der Hofstad, R.: Mean-field behavior for nearest-neighbor percolation in \( d > 10 \). Electron. J. Probab. 22 (2017) arXiv:1506.07977
13. Fitzner, R., van der Hofstad, R.: Generalized approach to the non-backtracking lace expansion. Probab. Theory Relat. Fields 169, 1041–1119 (2017)
14. Grassberger, P.: Logarithmic corrections in (4 + 1)-dimensional directed percolation. Phys. Rev. E 79, 052104 (2009)
15. Grimmett, G.: Percolation, 2nd edn. Springer, New York (1999)
16. Grimmett, G., Hiemer, P.: Directed percolation and random walk. In and Out of Equilibrium (ed., V. Sidoravicius) Birkhäuser (2002): 273–297
17. Handa, S., Kamijima, Y., Sakai, A.: A survey on the lace expansion for the nearest-neighbor models on the BCC lattice. Taiwan. J. Math. 24, 723–784 (2020)
18. Hara, T., Slade, G.: Mean-field critical behavior for percolation in high dimensions. Commun. Math. Phys. 128, 333–391 (1990)
19. Hara, T., Slade, G.: Mean-field behaviour and the lace expansion. In: Grimmett, G.R. (ed.) Probability and Phase Transition, pp. 87–122. Kluwer, Rijin (1994)
20. Hara, T., Slade, G.: The self-avoiding-walk and percolation critical points in high dimensions. Comb. Probab. Comput. 4, 197–215 (1995)
21. Heydenreich, M., van der Hofstad, R.: Progress in High-Dimensional Percolation and Random Graphs. Springer, Cham (2017)
22. Heydenreich, M., Matzke, K.: Expansion for the critical point of site percolation: the first three terms. Comb. Probab. Comput. 31, 430–454 (2022)
23. Janssen, H., Stenull, O.: Logarithmic corrections in directed percolation. Phys. Rev. E 69, 016125 (2004)
24. Kamijima, Y.: Mean-field behavior for percolation models. Ph.D. thesis, Hokkaido University (2021). https://doi.org/10.14943/doctoral.k14347
25. Kamijima, Y.: Mean-field behavior of nearest-neighbor oriented percolation on $\mathbb{Z}^d$. (2007). arXiv:0708.2897
26. Menshikov, M.V.: Coincidence of critical points in percolation problems. Soviet Math. 33, 856–859 (1986)
27. Nguyen, B.G., Yang, W.-S.: Triangle condition for oriented percolation in high dimensions. Ann. Probab. 21, 1809–1844 (1993)
28. Nguyen, B.G., Yang, W.-S.: Gaussian limit for critical oriented percolation in high dimensions. J. Stat. Phys. 78, 841–876 (1995)
29. Obukhov, S.P.: The problem of directed percolation. Physica 101A, 145–155 (1980)
30. Ódor, G.: Universality classes in nonequilibrium lattice systems. Rev. Mod. Phys. 76, 663–724 (2004)
31. Russo, L.: On the critical percolation probabilities. Z. Warsch. Verw. Geb. 56, 229–237 (1981)
32. Sakai, A.: Mean-field critical behavior for the contact process. J. Stat. Phys. 104, 111–143 (2001)
33. Sakai, A.: Diagrammatic bounds on the lace-expansion coefficients for oriented percolation. (2007). arXiv:0708.2897
34. Sakai, A.: Hyperscaling for oriented percolation in 1 + 1 space-time dimensions. J. Stat. Phys. 171, 462–469 (2018)
35. Schertzer, E., Sun, R., Swart, J.: The Brownian Web, the Brownian Net, and their Universality. In: Contucci, P., Giardinà, C. (eds.) Advances in Disordered Systems. Random Processes and Some Applications, pp. 270–368. Cambridge University Press, Cambridge (2016)
36. Slade, G.: The Lace Expansion and its Applications. Lecture Notes in Mathematics, vol. 1879. Springer, Berlin (2006)
37. Toulouse, G.: Perspectives from the theory of phase transitions. Nuovo Cimento 23B, 234–240 (1974)
38. van den Berg, J., Kesten, H.: Inequalities with applications to percolation and reliability. J. Appl. Probab. 22, 556–569 (1985)
39. van der Hofstad, R., Sakai, A.: Critical points for spread-out self-avoiding walk, and the contact process above the upper critical dimensions. Probab. Theory Relat. Fields 132, 438–470 (2005)
40. van der Hofstad, R., Slade, G.: A generalised inductive approach to the lace expansion. Probab. Theory Relat. Fields 122, 389–430 (2002)
41. van der Hofstad, R., Slade, G.: Convergence of critical oriented percolation to super-Brownian motion above 4 + 1 dimensions. Ann. Inst. H. Poincaré Prob. Stat. 39, 413–485 (2003)
42. van der Hofstad, R., Slade, G.: Asymptotic expansions in $n^{-1}$ for percolation critical values on the $n$-cube and $\mathbb{Z}^n$. Random Struct. Algorithms 27, 331–357 (2005)
43. van der Hofstad, R., Slade, G.: Expansion in $n^{-1}$ for percolation critical values on the $n$-cube and $\mathbb{Z}^n$: The first three terms. Comb. Probab. Comput. 15, 695–713 (2006)
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