A PROJECTIVE DESCRIPTION OF GENERALIZED GELFAND-SHILOV SPACES OF ROUMIEU TYPE

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ABSTRACT. We provide a projective description for a class of generalized Gelfand-Shilov spaces of Roumieu type. In particular, our results apply to the classical Gelfand-Shilov spaces and weighted-$L^\infty$ spaces of ultradifferentiable functions of Roumieu type.

1. Introduction

In general, there is no canonical way to find an explicit and useful system of seminorms describing a given inductive limit topology. However, in many concrete cases this is possible. For weighted $(LB)$-spaces of continuous and holomorphic functions, under quite general assumptions, the topology can be described in terms of weighted sup-seminorms. This problem of projective description goes back to the pioneer work of Bierstedt, Meise, and Summers [4] and plays an important role in Ehrenpreis’ theory of analytically uniform spaces [10, 2]. On the other hand, an explicit system of seminorms describing the topology of the space of ultradifferentiable functions of Roumieu type was first found by Komatsu [14]. His proof was based on a structural theorem for the dual space and the same method was later employed by Pilipović [15] to obtain projective descriptions of Gelfand-Shilov spaces of Roumieu type. Such projective descriptions are indispensable for achieving completed tensor product representations of various important classes of vector-valued ultradifferentiable functions of Roumieu type [6, 14, 16].

The aim of this article is to provide a projective description of a general class of Gelfand-Shilov spaces of Roumieu type. More precisely, let $(M_p)_{p\in\mathbb{N}}$ be a sequence of positive reals and let $V := (v_n)_{n\in\mathbb{N}}$ be a pointwise decreasing sequence of positive continuous functions on $\mathbb{R}^d$. We study here the $(LB)$-space $B^\{M_p\}_V(\mathbb{R}^d)$ consisting of all those $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{h^{\|\alpha\|_M} |\partial^\alpha \varphi(x)| v_n(x)}{M_\alpha} < \infty$$

for some $h > 0$ and $n \in \mathbb{N}$. Under rather general assumptions on $M_p$ and $V$, we shall give a projective description of the space $B^\{M_p\}_V(\mathbb{R}^d)$ in terms of Komatsu’s family $\mathfrak{R}$ [14] and

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the maximal Nachbin family associated with \( V \). We mention that we have already studied the problem in \([8\), Prop. 4.16], but we will present here a new approach. Our arguments are based on the mapping properties of the short-time Fourier transform on \( \mathcal{B}_{v}^{(M_{p})}(\mathbb{R}^{d}) \) and the projective description of weighted \((LB)\)-spaces of continuous functions. We believe this is a transparent and flexible method. It has the advantage that one can work under very mild conditions on \( M_{p} \) and \( V \) and it avoids duality theory; in fact, our result can be employed to more easily study dual spaces, e.g., one might readily deduce structural theorems without a rather complicated dual Mittag-Leffler argument.

Our general references are \([3, 4]\) for weighted inductive limits of spaces of continuous functions, \([16]\) for Gelfand-Shilov spaces, and \([12]\) for time-frequency analysis (the short-time Fourier transform).

2. Weighted inductive limits of spaces of continuous functions

In this section we recall a result of Bastin \([1]\) concerning the projective description of weighted \((LB)\)-spaces of continuous functions. This result will play a key role in the proof of our main theorem.

Let \( X \) be a completely regular Hausdorff space. Given a non-negative function \( v \) on \( X \) we write \( C_{v}(X) \) for the Banach space consisting of all \( f \in C(X) \) such that

\[
\|f\|_{v} := \sup_{x \in X} |f(x)|v(x) < \infty.
\]

A (pointwise) decreasing sequence \( V = (v_{n})_{n \in \mathbb{N}} \) of positive continuous functions on \( X \) is called a decreasing weight system on \( X \). We define

\[
\mathcal{V}C(X) := \lim_{n \rightarrow \infty} C_{v_{n}}(X),
\]

a Hausdorff \((LB)\)-space. The maximal Nachbin family associated with \( V \), denoted by \( \mathcal{V} = \mathcal{V}(V) \), is given by the space of all non-negative upper semicontinuous functions \( v \) on \( X \) such that \( \sup_{x \in X} v(x)/v_{n}(x) < \infty \) for all \( n \in \mathbb{N} \). The projective hull of \( \mathcal{V}C(X) \) is then defined as

\[
C\mathcal{V}(X) := \lim_{v \rightarrow \mathcal{V}} C_{v}(X).
\]

It is known that \( \mathcal{V}C(X) \) and \( C\mathcal{V}(X) \) coincide algebraically and that these spaces even have the same bounded sets \([3\), Thm. 3, p. 113]. The problem of projective description in this context is to characterize the weight systems \( V \) for which the continuous inclusion \( \mathcal{V}C(X) \rightarrow C\mathcal{V}(X) \) is in fact a topological isomorphism. There is the following result due to Bastin:

**Theorem 1** \([1]\). Let \( V = (v_{n})_{n \in \mathbb{N}} \) be a decreasing weight system on \( X \) satisfying condition \((V)\), i.e., for every sequence of positive numbers \( (\lambda_{n})_{n \in \mathbb{N}} \) there is \( v \in \mathcal{V}(V) \) such that for every \( n \in \mathbb{N} \) there is \( N \in \mathbb{N} \) such that \( \inf\{\lambda_{1}v_{1}, \ldots, \lambda_{N}v_{N}\} \leq \sup\{v_{n}/n, v\} \). Then, \( \mathcal{V}C(X) \) and \( C\mathcal{V}(X) \) coincide topologically.

**Remark 1.** Bastin also showed that if for every \( v \in \mathcal{V}(V) \) there is \( \overline{v} \in \mathcal{V}(V) \cap C(X) \) such that \( v \leq \overline{v} \), then condition \((V)\) is also necessary for the topological identity \( \mathcal{V}C(X) = C\mathcal{V}(X) \). We mention that if \( X \) is a discrete or a locally compact \( \sigma\)-compact Hausdorff space then it is also sufficient to study \( \mathcal{V}C(X) \) and \( C\mathcal{V}(X) \) separately.
space, then every decreasing weight system $\mathcal{V}$ on $X$ satisfies the above condition [4, p. 112].

Remark 2. A decreasing weight system $\mathcal{V} = (v_n)_{n \in \mathbb{N}}$ is said to satisfy condition $(S)$ (cf. [3]) if for every $n \in \mathbb{N}$ there is $m > n$ such that $v_m/v_n$ vanishes at $\infty$. Every weight system satisfying $(S)$ also satisfies $(V)$, but the latter property also holds for constant weight systems (for which $(S)$ obviously fails).

Let $X$ and $Y$ be completely regular Hausdorff spaces and let $\mathcal{V} = (v_n)_{n \in \mathbb{N}}$ and $\mathcal{W} = (w_n)_{n \in \mathbb{N}}$ be decreasing weight systems on $X$ and $Y$, respectively. We denote by $\mathcal{V} \otimes \mathcal{W} := (v_n \otimes w_n)_{n \in \mathbb{N}}$ the decreasing weight system on $X \times Y$ given by $v_n \otimes w_n(x, y) := v_n(x)w_n(y)$, $x \in X, y \in Y$.

Remark 3. Let $\mathcal{V} = (v_n)_{n \in \mathbb{N}}$ and $\mathcal{W} = (w_n)_{n \in \mathbb{N}}$ be decreasing weight systems on $X$ and $Y$, respectively. If both $\mathcal{V}$ and $\mathcal{W}$ satisfy $(V)$, then also $\mathcal{V} \otimes \mathcal{W}$ satisfies $(V)$.

Remark 4. Let $\mathcal{V} = (v_n)_{n \in \mathbb{N}}$ and $\mathcal{W} = (w_n)_{n \in \mathbb{N}}$ be decreasing weight systems on $X$ and $Y$, respectively. Then, for every $u \in \overline{\mathcal{V}}(\mathcal{V} \otimes \mathcal{W})$ there are $v \in \overline{\mathcal{V}}(\mathcal{V})$ and $w \in \overline{\mathcal{V}}(\mathcal{W})$ such that $u \leq v \otimes w$.

3. Generalized Gelfand-Shilov spaces of Roumieu type

We now introduce the class of Gelfand-Shilov spaces of Roumieu type that we are interested in. They are defined via a decreasing weight system $\mathcal{V}$ (on $\mathbb{R}^d$) and our aim is to give a projective description of these spaces in terms of Komatsu’s family $\mathcal{R}$ (defined below) and the maximal Nachbin family associated with $\mathcal{V}$.

Let $(M_p)_{p \in \mathbb{N}}$ be a weight sequence, that is, a positive sequence that satisfies

$$\lim_{p \to \infty} \frac{M_p}{M_{p-1}} = \infty.$$ 

For $h > 0$ and a non-negative function $v$ on $\mathbb{R}^d$ we write $\mathcal{D}_{L_{v}^{\infty}}^{M_p,h}(\mathbb{R}^d)$ for the Banach space consisting of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\mathcal{D}_{L_{v}^{\infty}}^{M_p,h}} := \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|\partial^\alpha \varphi(x)|}{M_\alpha} v(x) < \infty,$$

where we write $M_\alpha = M_{|\alpha|}$, $\alpha \in \mathbb{N}^d$. Given a decreasing weight system $\mathcal{V} = (v_n)_{n \in \mathbb{N}}$, we define, as in the introduction,

$$\mathcal{B}_{\mathcal{V}}^{M_p}(\mathbb{R}^d) := \lim_{n \to \infty} \mathcal{D}_{L_{v_n}^{\infty}}^{M_p,1/n}(\mathbb{R}^d),$$

a Hausdorff $(LB)$-space.

Following Komatsu [14], we denote by $\mathcal{R}$ the set of all positive sequences $(r_j)_{j \in \mathbb{N}}$ which tend increasingly to infinity. For $r_j \in \mathcal{R}$ and a non-negative function $v$ on $\mathbb{R}^d$ we write $\mathcal{D}_{L_{v}^{\infty}}^{M_p,r_j}(\mathbb{R}^d)$ for the Banach space of all $\varphi \in C^\infty(\mathbb{R}^d)$ such that

$$\|\varphi\|_{\mathcal{D}_{L_{v}^{\infty}}^{M_p,r_j}} := \sup_{\alpha \in \mathbb{N}^d} \sup_{x \in \mathbb{R}^d} \frac{|\partial^\alpha \varphi(x)|}{M_\alpha} v(x) r_j < \infty.$$
Given a decreasing weight system \( \mathcal{V} \), we also introduce the space
\[
\mathcal{B}_\mathcal{V}^{(M_p)}(\mathbb{R}^d) := \lim_{r_j \to 0} \lim_{r_j \in \mathfrak{R}} \mathscr{D}_L^{M_p r_j}(\mathbb{R}^d).
\]

**Lemma 1.** Let \( M_p \) be a weight sequence and let \( \mathcal{V} = (v_n)_{n \in \mathbb{N}} \) be a decreasing weight system. Then, \( \mathcal{B}_\mathcal{V}^{(M_p)}(\mathbb{R}^d) \) and \( \mathcal{B}_\mathcal{V}^{(M_p)}(\mathbb{R}^d) \) coincide algebraically and the inclusion mapping \( \mathcal{B}_\mathcal{V}^{(M_p)}(\mathbb{R}^d) \to \mathcal{B}_\mathcal{V}^{(M_p)}(\mathbb{R}^d) \) is continuous.

**Proof.** It is obvious that \( \mathcal{B}_\mathcal{V}^{(M_p)}(\mathbb{R}^d) \) is continuously included in \( \mathcal{B}_\mathcal{V}^{(M_p)}(\mathbb{R}^d) \). For the converse inclusion, we define the following decreasing weight system \( \mathcal{W} \) on \( \mathbb{N}^d \) (endowed with the discrete topology)
\[
\mathcal{W} = (w_n)_{n \in \mathbb{N}}, \quad w_n(\alpha) := n^{-|\alpha|}, \quad \alpha \in \mathbb{N}^d.
\]
Now let \( \varphi \in \mathcal{B}_\mathcal{V}^{(M_p)}(\mathbb{R}^d) \) be arbitrary and define \( f(x, \alpha) = \partial^\alpha \varphi(x)/M_\alpha \) for \( x \in \mathbb{R}^d, \alpha \in \mathbb{N}^d \). By Remark [1] and [14, Lemma 3.4(ii)] we have that \( f \in \mathcal{C} \mathcal{V}(\mathcal{V} \otimes \mathcal{W})(\mathbb{R}^d \times \mathbb{N}^d) \). Since \( \mathcal{V} \otimes \mathcal{W} C(\mathbb{R}^d \times \mathbb{N}^d) = \mathcal{C} \mathcal{V}(\mathcal{V} \otimes \mathcal{W})(\mathbb{R}^d \times \mathbb{N}^d) \) as sets (cf. Section [2]), we obtain that \( f \in \mathcal{V} \otimes \mathcal{W} C(\mathbb{R}^d \times \mathbb{N}^d) \), which precisely means that \( \varphi \in \mathcal{B}_\mathcal{V}^{(M_p)}(\mathbb{R}^d) \).

The rest of this article is devoted to showing that, under suitable conditions on \( M_p \) and \( \mathcal{V} \), the equality \( \mathcal{B}_\mathcal{V}^{(M_p)}(\mathbb{R}^d) = \mathcal{B}_\mathcal{V}^{(M_p)}(\mathbb{R}^d) \) also holds topologically.

We will make use of the following two standard conditions for weight sequences:

(M.1) \( M_p^2 \leq M_{p-1} M_{p+1}, p \geq 1; \)
(M.2) \( M_{p+1} \leq C_0 H^p M_p, p \in \mathbb{N} \), for some \( C_0, H \geq 1 \).

We also need the associated function of the sequence \( M_p \) in our considerations, which is given by
\[
M(t) := \sup_{p \in \mathbb{N}} \frac{t^p M_0}{M_p}, \quad t > 0,
\]
and \( M(0) := 0 \). We define \( M \) on \( \mathbb{R}^d \) as the radial function \( M(x) = M(|x|), x \in \mathbb{R}^d \). The assumption (M.2) implies that \( M(H^k t) - M(t) \geq k \log(t/C_0), t, k \geq 0 [13, Prop. 3.4] \). In particular, we have that
\[
e^{M(t) - M(H^k t)} \leq 2C_0^{d+1}(1 + t^{d+1})^{-1}, \quad t \geq 0
\]
Given \( r_j \in \mathfrak{R} \) we denote by \( M_{r_j} \) the associated function of the weight sequence \( M_p \prod_{j=1}^{p} r_j \).

As mentioned in the introduction, our arguments will rely on the mapping properties of the short-time Fourier transform, which we now introduce. The translation and modulation operators are denoted by \( T_x f = f(\cdot - x) \) and \( M_\xi f = e^{2\pi i \xi \cdot f} \), for \( x, \xi \in \mathbb{R}^d \).

The **short-time Fourier transform (STFT)** of a function \( f \in L^2(\mathbb{R}^d) \) with respect to a window function \( \psi \in L^2(\mathbb{R}^d) \) is defined as
\[
V_\psi f(x, \xi) := (f, M_\xi T_x \psi)_L^2 = \int_{\mathbb{R}^d} f(t) \overline{\psi(t - x)} e^{-2\pi i t \xi} dt, \quad (x, \xi) \in \mathbb{R}^{2d}.
\]
Hence that $\|V_{\psi}f\|_{L^2(\mathbb{R}^d)} = \|\psi\|_{L^2} \|f\|_{L^2}$. In particular, the mapping $V_{\psi} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is continuous. The adjoint of $V_{\psi}$ is given by the weak integral

$$V_{\psi}^* F = \int \int_{\mathbb{R}^d} F(x, \xi) M_{t} T_{x} \psi d\xi d\xi, \quad F \in L^2(\mathbb{R}^d).$$

If $\psi \neq 0$ and $\gamma \in L^2(\mathbb{R}^d)$ is a synthesis window for $\psi$, that is $(\gamma, \psi)_{L^2} \neq 0$, then

$$(3.1) \quad \frac{1}{(\gamma, \psi)_{L^2}} V_{\gamma}^* \circ V_{\psi} = \text{id}_{L^2(\mathbb{R}^d)}.$$

We are interested in the STFT on the spaces $\mathcal{B}_{\nu}^{(M_p)}(\mathbb{R}^d)$ and $\tilde{\mathcal{B}}_{\nu}^{(M_p)}(\mathbb{R}^d)$. This requires to impose some further conditions on the weight system $\nu$. Let $A_p$ be a weight sequence with associated function $A$. A decreasing weight system $\nu = (v_n)_{n \in \mathbb{N}}$ is said to be $A_p$-admissible if there is $\tau > 0$ such that for every $n \in \mathbb{N}$ there are $m \geq n$ and $C > 0$ such that

$$v_m(x + y) \leq C v_n(x) e^{A(\tau y)}, \quad x, y \in \mathbb{R}^d.$$

We start with two lemmas. As customary [16], given two weight sequences $M_p$ and $A_p$, we denote by $S_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ the Gelfand-Shilov space of Beurling type. For the weight function $\nu = e^{A(\tau \cdot)}$ we use the alternative notation $\| \cdot \|_{S_{A_p}^{M_p}} = \| \cdot \|_{\mathcal{D}_{L^2}^{M_p}}$ so that the Fréchet space structure of $S_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ is determined by this family of norms.

**Lemma 2.** Let $M_p$ and $A_p$ be weight sequences satisfying (M.1) and (M.2), let $w$ and $v$ be non-negative measurable functions on $\mathbb{R}^d$ such that

$$(3.2) \quad v(x + y) \leq C w(x) e^{A(\tau y)}, \quad x, y \in \mathbb{R}^d,$$

for some $C, \tau > 0$, and let $\psi \in S_{(A_p)}^{(M_p)}(\mathbb{R}^d)$. Then,

$$V_{\psi} : \mathcal{D}_{L^2}^{M_p, h} \to C \otimes e^{M(\pi h \cdot / \sqrt{d})}(\mathbb{R}^d_{x, \xi})$$

is a well-defined continuous mapping.

**Proof.** Let $\varphi \in \mathcal{D}_{L^2}^{M_p, h}(\mathbb{R}^d)$ be arbitrary. For all $\alpha \in \mathbb{N}^d$ and $(x, \xi) \in \mathbb{R}^d$,

$$|\xi^\alpha V_{\psi} \varphi(x, \xi)| v(x)$$

$$\leq C (2\pi)^{-|\alpha|} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \int \int_{\mathbb{R}^d} |\partial^\beta \varphi(t)| w(t) |\partial^\alpha - \beta \psi(t - x)| e^{A(\tau(t-x))} dt$$

$$\leq C' \|\varphi\|_{\mathcal{D}_{L^2}^{M_p, h}(\pi h)^{-|\alpha|} M_{\alpha}}.$$

Hence

$$|V_{\psi} \varphi(x, \xi)| v(x) \leq C' \|\varphi\|_{\mathcal{D}_{L^2}^{M_p, h}} \inf_{p \in \mathbb{N}} \frac{M_p}{(\pi h |\xi| / \sqrt{d})^p} = C' M_0 \|\varphi\|_{\mathcal{D}_{L^2}^{M_p, h}} e^{-M(\pi h \xi / \sqrt{d})}.$$

\[\square\]
Lemma 3. Let $M_p$ and $A_p$ be weight sequences satisfying (M.1) and (M.2)', let $w$ and $v$ be non-negative measurable functions on $\mathbb{R}^d$ satisfying \((3.2)\), and let $\psi \in S_{(A_p)}^{(M_p)}(\mathbb{R}^d)$. Then,
\[ V^*_\psi : Cw \otimes e^{M(h \cdot)}(\mathbb{R}^{2d}) \to D_d^{M_p,h/(4Hd+1 \pi)}(\mathbb{R}^d) \]
is a well-defined continuous mapping.

Proof. Let $F \in Cw \otimes e^{M(h \cdot)}(\mathbb{R}^{2d})$ be arbitrary and set $k = h/(2Hd+1 \pi)$. For each $\alpha \in \mathbb{N}^d$, (we write $\| \cdot \|_{Cw \otimes e^{M(h \cdot)}} = \| \cdot \|$
\[
\sup_{t \in \mathbb{R}^d} |\partial^\alpha V^*_\psi F(t)|v(t)
\]
\[
\leq C \sum_{\beta \leq \alpha} \frac{\alpha}{\beta} \sup_{t \in \mathbb{R}^d} \int_{\mathbb{R}^{2d}} |F(x, \xi)|w(x)(2\pi|\xi|)^{|\beta|}|\partial^{\alpha-\beta} \psi(t-x)|e^{A(\tau(t-x))}dx d\xi
\]
\[
\leq CM_0^{-1} \|\psi\|_{S_{(A_p)}^{M_p,k}} \|F\| \frac{M_\alpha}{(k/2)^{|\alpha|}} \int_{\mathbb{R}^{2d}} e^{M(2\pi k \xi)-M(h \xi)}e^{A(\tau x)-A(Hd+1lx)}dx d\xi
\]
\[
\leq C^\prime \|F\| \frac{M_\alpha}{(h/(4Hd+1 \pi))^{\|\alpha\|}}.
\]
\[ \square \]

Lemmas 2 and 3 yield the following corollary.

Corollary 1. Let $M_p$ and $A_p$ be weight sequences satisfying (M.1) and (M.2)'. Denote by $X$ the Fréchet space consisting of all $F \in C(\mathbb{R}^{2d})$ such that
\[
\sup_{(x,\xi) \in \mathbb{R}^{2d}} |F(x, \xi)|e^{A(nx) + M(n^2)} < \infty
\]
for all $n \in \mathbb{N}$. Let $\psi \in S_{(A_p)}^{(M_p)}(\mathbb{R}^d)$. Then, $V_\psi : S_{(A_p)}^{(M_p)}(\mathbb{R}^d) \to X$ and $V^*_\psi : X \to S_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ are well-defined continuous mappings.

We are now able to establish the mapping properties of the STFT on $B_{\psi}^{(M_p)}(\mathbb{R}^d)$. Given a weight sequence $M_p$ with associated function $M$ we define $V_{M_p} := (e^{M(\cdot/n)})_{n \in \mathbb{N}}$, a decreasing weight system on $\mathbb{R}^d$.

Proposition 1. Let $M_p$ and $A_p$ be weight sequences satisfying (M.1) and (M.2)', let $V = (v_n)_{n \in \mathbb{N}}$ be an $A_p$-admissible decreasing weight system, and let $\psi \in S_{(A_p)}^{(M_p)}(\mathbb{R}^d)$. The following mappings are continuous:
\[ V_\psi : B_{\psi}^{(M_p)}(\mathbb{R}^d) \to V \otimes V_{M_p} C(\mathbb{R}^{2d}) \]
and
\[ V^*_\psi : V \otimes V_{M_p} C(\mathbb{R}^{2d}) \to B_{\psi}^{(M_p)}(\mathbb{R}^d). \]
Assume that $S_{(A_p)}^{(M_p)}(\mathbb{R}^d) \neq \{0\}$. If $\psi \neq 0$ and $\gamma \in S_{(A_p)}^{(M_p)}(\mathbb{R}^d)$ is a synthesis window for $\psi$, the following reconstruction formula holds
\[
(3.3) \quad \frac{1}{(\gamma, \psi)^{2}} V^*_\psi \circ V_\psi = \text{id}_{B_{\psi}^{(M_p)}(\mathbb{R}^d)}.
\]
Proof. Since $\mathcal{V}$ is $A_p$-admissible, the continuity of $V_\psi$ and $V_\psi^*$ follows directly from Lemmas 2 and 3 respectively. We now show (3.3). Let $\varphi \in L^p_{(M_p)}(\mathbb{R}^d)$ be arbitrary. As $V_\gamma^*(V_\psi \varphi)$ and $\varphi$ are both $O(e^{A(r\gamma)})$-bounded continuous functions, it suffices to show that

$$
\int_{\mathbb{R}^d} V_\gamma^*(V_\psi \varphi)(t) \chi(t) dt = (\gamma, \psi)_{L^2} \int_{\mathbb{R}^d} \varphi(t) \chi(t) dt
$$

for all $\chi \in S_{(M_p)}(\mathbb{R}^d)$. Formula (3.1) implies that

$$
\int_{\mathbb{R}^d} V_\gamma^*(V_\psi \varphi)(t) \chi(t) dt = \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \varphi(x, \xi) M_{-\xi} t_\gamma(x) dx d\xi \right) \chi(t) dt
$$

$$
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} \varphi(t) M_{-\xi} t_\psi(t) dt \right) V_{-\xi} \chi(x, -\xi) dx d\xi
$$

$$
= \int_{\mathbb{R}^d} V_{-\xi}^\ast(V_{\psi} \chi)(t) \varphi(t) dt
$$

$$
= (\gamma, \psi)_{L^2} \int_{\mathbb{R}^d} \varphi(t) \chi(t) dt,
$$

where the switching of the integrals is permitted because of Corollary 1 and the first part of this proposition. \qed

In order to show the analogue of Proposition 1 for $\mathcal{V}_\gamma (M_p)$, we need the following technical lemma.

**Lemma 4.** Let $\mathcal{V} = (v_n)_{n \in \mathbb{N}}$ be an $A_p$-admissible decreasing weight system. For every $v \in \mathcal{V}$ there is $\overline{v} \in \mathcal{V}$ such that $v(x + y) \leq \overline{v}(x)e^{A(\tau y)}$, $x, y \in \mathbb{R}^d$.

**Proof.** Find a strictly increasing sequence of natural numbers $(n_j)_{j \in \mathbb{N}}$ such that $v_{n_{j+1}}(x+y) \leq C_j v_{n_j}(x)e^{A(\tau y)}$, $x, y \in \mathbb{R}^d$, for all $j \in \mathbb{N}$ and some $C_j > 0$. Pick $C_j' > 0$ such that $v \leq C_j' v_{n_j}$ for all $j \in \mathbb{N}$. Set $\overline{v} = \inf_{j \in \mathbb{N}} C_j C_{j+1} v_{n_j} \in \mathcal{V}$. We have that

$$
v(x + y) \leq \inf_{j \in \mathbb{N}} C_j C_{j+1} v_{n_{j+1}}(x + y) \leq e^{A(\tau y)} \inf_{j \in \mathbb{N}} C_j C_{j+1} v_{n_j}(x) = \overline{v}(x)e^{A(\tau y)}.
$$

\qed

**Proposition 2.** Let $M_p$ and $A_p$ be weight sequences satisfying (M.1) and (M.2)', let $\mathcal{V} = (v_n)_{n \in \mathbb{N}}$ be an $A_p$-admissible decreasing weight system, and let $\psi \in S_{(M_p)}(\mathbb{R}^d)$. The following mappings are continuous:

$$
V_\psi : \mathcal{B}_{(M_p)}(\mathbb{R}^d) \to CV(\mathcal{V} \otimes \mathcal{V}_{(M_p)})_{(2d, \xi)}
$$

and

$$
V_\psi^* : CV(\mathcal{V} \otimes \mathcal{V}_{(M_p)})_{(2d, \xi)} \to \mathcal{B}_{(M_p)}(\mathbb{R}^d).
$$

**Proof.** Let $u \in CV(\mathcal{V} \otimes \mathcal{V}_{(M_p)})$ be arbitrary. By Remark 4 and Lemma 4.5(i) there is $v \in \mathcal{V}(\mathcal{V})$ and $r_j \in \mathfrak{R}$ such that $u \leq v \otimes e^{M_p r}$. According to [17, Lemma 2.3] there is $r_j' \in \mathfrak{R}$ such that $r_j' \leq r_j$ for $j$ large enough and $r_j' \leq 2^{j+1} r_j'$ for all $j \in \mathbb{N}$.
the sequence $M_p \prod_{j=0}^{p} r_j$ satisfies (M.2)' Next, by Lemma [1] there is $\tau \in \mathcal{V}$ such that $v(x + y) \leq \tau(x)e^{A(\tau y)}$ for all $x, y \in \mathbb{R}^d$. Therefore, Lemma [2] implies that

$$V_\psi : D_{L_2}^{M_p, \frac{r_j}{\sqrt{d}}}(\mathbb{R}^d) \to C_\mathcal{V} \otimes e^{M_{r_j}}(\mathbb{R}^{2d})$$

is a well-defined continuous mapping. As the inclusion mapping $C_\mathcal{V} \otimes e^{M_{r_j}}(\mathbb{R}^{2d})$ is continuous, we may conclude that $V_\psi$ is continuous. Similarly, by using Lemma [3] one can show that $V_\psi^*$ is continuous.

We are ready to prove our main theorem.

**Theorem 2.** Let $M_p$ and $A_p$ be weight sequences satisfying (M.1) and (M.2)' such that $S^{(M_p)}_{(A_p)}(\mathbb{R}^d) \neq \{0\}$ and let $\mathcal{V} = (v_n)_{n \in \mathbb{N}}$ be an $A_p$-admissible decreasing weight system satisfying (V). Then, $B^{(M_p)}_{\mathcal{V}}(\mathbb{R}^d)$ and $B^{(M_p)}_{\mathcal{V}}(\mathbb{R}^d)$ coincide topologically.

**Proof.** By Lemma [1] it suffices to show that the inclusion mapping $\iota : B^{(M_p)}_{\mathcal{V}}(\mathbb{R}^d) \to B^{(M_p)}_{\mathcal{V}}(\mathbb{R}^d)$ is continuous. Since $M_p$ satisfies (M.2)', the decreasing weight system $\mathcal{V}_{(M_p)}$ satisfies (S) and thus condition (V) (see Remark [2]). Hence Proposition [1] and Remark [3] imply that $V \otimes \mathcal{V}_{(M_p)} C(\mathbb{R}^{2d}) = \mathcal{V}(V \otimes \mathcal{V}_{(M_p)})(\mathbb{R}^{2d})$ topologically. Choose $\psi, \gamma \in S^{(M_p)}_{(A_p)}(\mathbb{R}^d)$ such that $(\gamma, \psi)_{L^2} = 1$. By (3.3) the following diagram commutes

$$
\begin{array}{ccc}
B^{(M_p)}_{\mathcal{V}}(\mathbb{R}^d) & \to & C\mathcal{V}(V \otimes \mathcal{V}_{(M_p)})(\mathbb{R}^{2d}) = V \otimes \mathcal{V}_{(M_p)} C(\mathbb{R}^{2d}) \\
\iota \downarrow & & \downarrow V_\psi^* \\
B^{(M_p)}_{\mathcal{V}}(\mathbb{R}^d) & & \\
\end{array}
$$

Propositions [1] and [2] imply that $V_\psi$ and $V_\psi^*$ are continuous, whence $\iota$ is also continuous.

**Remark 5.** Let $M_p$ and $A_p$ be weight sequences satisfying (M.1) and (M.2)' such that $S^{(M_p)}_{(A_p)}(\mathbb{R}^d) \neq \{0\}$. By applying Theorem [2] to $\mathcal{V} = \mathcal{V}_{(A_p)}$ (and using Lemma 4.5(i)), we obtain the well known projective description of the classical Gelfand-Shilov space $S^{(M_p)}_{(A_p)}(\mathbb{R}^d)$ of Roumieu type [15] Lemma 4).

We end this article by stating an important particular case of Theorem [2]. Given a non-negative function $\omega$ on $\mathbb{R}^d$, we define

$$D^{(M_p)}_{L_2} \mathcal{V}(\mathbb{R}^d) := \lim_{h \to 0^+} D^{M_p, h}_{L_2} \mathcal{V}(\mathbb{R}^d), \quad \tilde{D}^{(M_p)}_{L_2}(\mathbb{R}^d) := \lim_{r_j \to 0^+} D^{M_p, r_j}_{L_2}(\mathbb{R}^d).$$

**Theorem 3.** Let $M_p$ and $A_p$ be weight sequences satisfying (M.1) and (M.2)' such that $S^{(M_p)}_{(A_p)}(\mathbb{R}^d) \neq \{0\}$ and let $\omega$ be a positive measurable function on $\mathbb{R}^d$ such that

$$\omega(x + y) \leq C\omega(x)e^{A(\tau y)}, \quad x, y \in \mathbb{R}^d,$$

for some $C, \tau > 0$. Then, $D^{(M_p)}_{L_2} \mathcal{V}(\mathbb{R}^d)$ and $\tilde{D}^{(M_p)}_{L_2}(\mathbb{R}^d)$ coincide topologically.
Proof. We may assume without loss of generality that \( \omega \) is continuous (for otherwise we may replace \( \omega \) with the continuous weight \( \tilde{\omega} = \omega \ast \varphi \), where \( \varphi \in \mathcal{D}(\mathbb{R}^d) \) is non-negative and satisfies \( \int_{\mathbb{R}^d} \varphi(t)\,dt = 1 \), since \( \mathcal{D}^{(M_p)}_{L^\infty(\mathbb{R}^d)}(\mathbb{R}^d) = \mathcal{D}^{(M_p)}_{L^\infty(\mathbb{R}^d)}(\mathbb{R}^d) = \widetilde{\mathcal{D}}^{(M_p)}_{L^\infty(\mathbb{R}^d)}(\mathbb{R}^d) \) topologically). We set \( \mathcal{V} = (\omega)_{\alpha \in \mathbb{N}} \) and notice that \( \mathcal{V} \) satisfies (\( V \)) (see Remark \( \mathcal{V} \)). Hence, by Theorem \( \mathcal{V} \) we find that \( \mathcal{D}^{(M_p)}_{L^\infty(\mathbb{R}^d)}(\mathbb{R}^d) = \widetilde{\mathcal{B}}^{(M_p)}_{\mathcal{V}}(\mathbb{R}^d) \) topologically. The result now follows from the fact that \( \overline{\mathcal{V}}(\mathcal{V}) = \{ \lambda \omega : \lambda > 0 \} \) and, thus, \( \widetilde{\mathcal{B}}^{(M_p)}_{\mathcal{V}}(\mathbb{R}^d) = \widetilde{\mathcal{D}}^{(M_p)}_{L^\infty(\mathbb{R}^d)}(\mathbb{R}^d) \) topologically.

Remark 6. Theorem \( \mathcal{X} \) was already shown in [9] Thm. 4.17] under much more restrictive conditions on \( M_p \) and \( A_p \) and with a more complicated proof.

In [7] Thm. 5.9] we have shown that, if \( M_p \) satisfies (M.1) and (M.2) (cf. [13]), the space \( \mathcal{S}^{(M_p)}_{\chi(p)}(\mathbb{R}^d) \) is non-trivial if and only if \( (\log p)^p \prec M_p \) (the latter means, as usual, that \( M_p^{1/p} / \log p \to \infty \)). Hence, we obtain the ensuing corollary.

Corollary 2. Let \( M_p \) be a weight sequence satisfying (M.1) and (M.2) such that \( (\log p)^p \prec M_p \) and let \( \omega \) be a positive measurable function on \( \mathbb{R}^d \) such that

\[
\omega(x + y) \leq C \omega(x)e^{\tau|y|}, \quad x, y \in \mathbb{R}^d,
\]

for some \( C, \tau > 0 \). Then, \( \mathcal{D}^{(M_p)}_{L^\infty(\mathbb{R}^d)}(\mathbb{R}^d) \) and \( \widetilde{\mathcal{D}}^{(M_p)}_{L^\infty(\mathbb{R}^d)}(\mathbb{R}^d) \) coincide topologically.

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