Quantum hydrodynamics from local thermal pure states

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We provide a pure state formulation for hydrodynamic dynamics of isolated quantum many-body systems. A pure state describing quantum systems in local thermal equilibrium is constructed, which we call a local thermal pure quantum (ℓTPQ) state. We show that the thermodynamic functional and the expectation values of local operators (including a real-time correlation function) calculated from the ℓTPQ state converge to those from a local Gibbs ensemble in the large fluid-cell limit. As a numerical demonstration, we investigate a one-dimensional spin chain and observe the hydrodynamic relaxation obeying the Fourier’s law. We further prove the second law of thermodynamics and the quantum fluctuation theorem, which are also validated numerically. The ℓTPQ formulation gives a useful theoretical basis to describe the emergent hydrodynamic behavior of quantum many-body systems furnished with a numerical efficiency, being applicable to both the non-relativistic and relativistic regimes.

Introduction.— Nonequilibrium dynamics of isolated quantum many-body systems has been one of the central issues appearing at the crossroad of statistical physics and quantum physics. In particular, hydrodynamics [1–3] is an emergent macroscopic theory to investigate such dynamics under local equilibrium conditions. Recent experimental technologies have promoted further developments of hydrodynamics theory for isolated quantum systems, ranging from ultracold atoms [4–12] to quark-gluon plasma [13–22]. Furthermore, quantum or classical integrable systems, where conventional hydrodynamics is believed to break down, is now becoming a hot field on the basis of generalized hydrodynamics [23–27].

A recently-developed microscopic derivation of hydrodynamics is based on an extended notion of the statistical ensembles, called the local Gibbs (LG) ensemble both in classical [28–30] and quantum systems [31–34]. An advantage of the local ensemble formulation lies in the fact that it does not require the quasi-particle description and thus is applicable to strongly-coupled systems (see also another approach [35–38] based on the holographic principle and the pioneering works [39–45] along the line of the linear response theory). However, under unitary evolutions of isolated quantum systems, a pure state never falls into any mixed state such as the Gibbs ensemble. Thus, it is still an open problem how macroscopic hydrodynamics emerges under unitary evolutions of pure states as observed in ultracold atoms [46–51].

In this Letter, we provide a pure state-based derivation of the hydrodynamic equation for isolated quantum many-body systems. We introduce a special class of random pure quantum states describing systems in local thermal equilibrium, which we refer to as local thermal pure quantum (ℓTPQ) states. This is a generalization of thermal pure quantum (TPQ) states [52–59] randomly sampled from the Hilbert space to reproduce thermal equilibrium behaviors. We show that ℓTPQ states have a numerical advantage in computing hydrodynamic expectation values [55, 56, 58, 59] and give the equivalent results with the conventional method based on the LG ensemble in the large fluid-cell limit.

We establish all the fundamental hydrodynamic behaviors from the ℓTPQ states, being equipped with a numerical efficiency. Specifically, we prove the Green-Kubo formula [60–62], the second law of thermodynamics and the quantum fluctuation theorem [63–69], as well as the constitutive relation of hydrodynamics itself. We remark that our formulation applies to both the nonrelativistic and relativistic regimes, and therefore potentially covers a variety of phenomena from low to high energy scales. As a concrete demonstration, applying the ℓTPQ-state formulation to a one-dimensional quantum spin chain (see Refs. [70–76] and references therein for a review on recent theoretical approaches to thermal transport in low-dimensional systems), we numerically investigate the hydrodynamic relaxation starting from a ℓTPQ state, and confirm the Fourier’s law for thermal conduction.

Local thermal pure quantum (ℓTPQ) state.— We first introduce the ℓTPQ state. Suppose that the system possesses a set of conserved currents \( \hat{J}_a^\mu(t) \) satisfying

\[
\partial_\nu \hat{J}_a^\mu(t) \equiv \partial_0 \hat{J}_a^\mu(t) + \partial_i \hat{J}_a^\mu(t) = 0, \quad (1)
\]

where \( \mu, \nu, \cdots \) denote spacetime indices, and \( a, b, \cdots \) label conserved charges such as energy, momentum, and
particle numbers\(^1\). Throughout the paper, we use the Heisenberg picture, and Eq. (1) gives the equation of motion for conserved charge densities \(\hat{c}_\alpha(t) \equiv \hat{J}_\alpha^0(t)\). Then, we consider the system in local thermal equilibrium, which is described by local thermodynamic parameters \(\lambda^i\) conjugate to average charge densities \(\langle \hat{c}_\alpha \rangle\). We assume that each fluid cell contains many degrees of freedom so that thermodynamic parameters are smooth functions of spatial coordinates. In other words, we identify the current operators \(\hat{J}_\alpha^i\) in Eq. (1) as coarse-graining ones over a fluid-cell volume \(V_{\text{cell}}\).

To describe the locally thermalized system, we introduce the \(\ell\)TPQ state at time \(t\) by
\[
|\lambda_i; t\rangle \equiv \sum_\alpha z_\alpha e^{-\frac{i}{2}\hat{K}[\lambda_i; t]} |\alpha_i\rangle ,
\]
with \(\hat{K}[\lambda_i; t] \equiv \lambda^i(t)\hat{c}_\alpha(t)\), any orthonormal basis state vector \(|\alpha_i\rangle\) at time \(t\), and a complex random variable \(z_\alpha \equiv (z_\alpha^\ast + iz_\alpha^\prime)/\sqrt{2}\) whose real/imaginary parts are taken from the normal distribution. Note that we distinguish two arguments in \(\hat{K}[\lambda_i; t]\): one for thermodynamic parameters \(\lambda^i\), whose suffix denotes its configuration \(\lambda_i = \lambda(t)\) at time \(t\), while the second argument \(t\) represents the time argument of charge densities as Heisenberg operators. When the system only contains the energy as the conserved charge, \(K\) reads \(\hat{K}[\lambda_i; t] = \int d^dx \beta(t, x) h(t, x)\) with the local inverse temperature \(\beta(t, x)\) and energy density operator \(h(t, x)\), for instance. The normalization of the \(\ell\)TPQ state is defined by \(Z_{\ell\text{TPQ}}[\lambda_i] \equiv \langle \lambda_i; t | \lambda_i; t\rangle\), whose logarithm is identified as a thermodynamic functional. Using these, we define the entropy-functional operator for the \(\ell\)TPQ state as
\[
\hat{S}[\lambda_i; t] = \hat{K}[\lambda_i; t] + \log Z_{\ell\text{TPQ}}[\lambda_i],
\]
and the average value over the \(\ell\)TPQ state as
\[
\langle \hat{O}\rangle^{\ell\text{TPQ}}_{\lambda_i} \equiv \frac{1}{Z_{\ell\text{TPQ}}[\lambda_i]} \langle \lambda_i; t | \hat{O} | \lambda_i; t\rangle .
\]

**Hydrodynamics and the quantum fluctuation theorem.**—Based on the \(\ell\)TPQ state, we derive hydrodynamics and the quantum fluctuation theorem. The hydrodynamic equation is regarded as the averaged Eq. (1) over a certain initial density operator \(\hat{\rho}_0\). Here, we put a crucial assumption on the initial state: the system is in the \(\ell\)TPQ state at initial time \(t_0\) parametrized by \(\lambda^i_{0}\). Starting from this initial \(\ell\)TPQ state, we shall consider the subsequent time-evolution of charge densities described by
\[
\partial_\mu \langle \hat{J}_\alpha^\mu(t) \rangle = 0 \quad \text{with} \quad \langle \hat{J}_\alpha^\mu(t) \rangle \equiv \langle \hat{J}_\alpha^\mu(t) \rangle^{\ell\text{TPQ}}_{\lambda_0},
\]
for \(t \geq t_0\). To make Eq. (5) a closed set of equations, we need a constitutive relation, which expresses \(\langle \hat{J}_\alpha^i(t) \rangle\) by the dynamical variable \(\langle \hat{c}_\alpha(t) \rangle\). Thus, our problem is to find the constitutive relation.

The vital point here is that we have a solution to this problem if we replace the initial \(\ell\)TPQ-state average with the LG-ensemble average over the density operator:
\[
\hat{\rho}_{\text{LG}}[\lambda_{0}; t_0] \equiv \frac{1}{Z_{\text{LG}}[\lambda_{0}]} e^{-\hat{K}[\lambda_{0}; t_0]},
\]
where we defined the partition function for the LG ensemble as \(Z_{\text{LG}}[\lambda_{0}] \equiv \text{Tr} e^{-\hat{K}[\lambda_{0}; t_0]}\). We will express the LG average as \(\langle \hat{O}\rangle^{\Lambda}_{\text{LG}} \equiv \text{Tr} (\hat{\rho}_{\text{LG}}[\lambda_{0}; t] \hat{O})\). Thus, it is enough to show the equivalence between the \(\ell\)TPQ and LG averages for the purpose of deriving hydrodynamics from the \(\ell\)TPQ state.

Following the similar treatment of Refs. [55, 56, 58, 59], we can indeed show such equivalence holds as convergence in probability \(\lim_{V_{\text{cell}} \to \infty} \int P \) w.r.t. the random variable \(z_\alpha\) [77]:
\[
Z_{\text{TPQ}}[\lambda_i] \xrightarrow{P} Z_{\text{LG}}[\lambda_i],
\]
\[
\langle \hat{O}\rangle^{\Lambda}_{\text{TPQ}} \xrightarrow{P} \langle \hat{O}\rangle^{\Lambda}_{\text{LG}},
\]
where we took the large size limit of all fluid cells \(V_{\text{cell}} \to \infty\). Therefore, the LG-based derivation developed in Refs. [31–34] also provides a derivation of hydrodynamics from the \(\ell\)TPQ state in the large fluid-cell limit.

We here summarize some consequences of Eqs. (7)-(8). Within a first-order derivative expansion supplemented by the Markovian approximation for current-current correlators, the equivalence leads to the constitutive relation
\[
\langle \hat{J}_\alpha^i(t) \rangle = \langle \hat{J}_\alpha^i(t) \rangle^{\ell\text{TPQ}}_{\lambda_i} + L_{\alpha}^{\hat{J}_\alpha^i}(t) \partial_\alpha \lambda_i^j + O(\partial^2),
\]
where transport coefficients \(L_{\alpha}^{\hat{J}_\alpha^i}\) is given by the Green-Kubo formula with the TPQ state:
\[
L_{\alpha}^{\hat{J}_\alpha^i}(t) = \beta_i \int_{-\infty}^{t} dt' \int d^d x d^d x' \int_0^1 dr (\partial J_{\alpha}^i (x') \partial \hat{J}_\alpha^i (x))^{\ell\text{TPQ}}_{\lambda_i} \approx \frac{\beta_i}{4V} \int_{-\infty}^{t} dt' \langle \{ \partial J_{\alpha}^i (t), \partial \hat{J}_\alpha^i (t') \} \rangle^{\ell\text{TPQ}}_{\lambda_i},
\]
where we defined \(\langle \hat{O}\rangle^{\ell\text{TPQ}}_{\lambda_i}\) as a grandcanonical TPQ average parametrized by \(\lambda_i\) [55, 56, 58, 59]. Here we defined a projected operator \(\delta \hat{O} \equiv (1 - \hat{P}) \hat{O}\) using the \(\ell\)TPQ version of Mori’s projection [43, 44]:
\[
\hat{P} \hat{O}(t) \equiv \delta \hat{c}_\alpha(t) \frac{\delta}{\delta \hat{c}_\alpha(t)} \langle \hat{O}\rangle^{\ell\text{TPQ}}_{\lambda_i},
\]

\(^1\) Summations over the repeated indices for Greek letters \((\mu, \nu, \cdots)\) are assumed throughout the paper. The repeated indices for Latin letters is assumed to contain the spatial integral, e.g. \(\lambda^i c_\alpha \equiv \int d^dx \lambda^i(x) c_\alpha(x)\) where \(d\) denotes the spatial dimension.
defined total currents \( \hat{J}_{\alpha}(t) = \int d^d x \hat{J}_{\alpha}(t, x) \) with the anti-commutator \( \{ A, \hat{B} \} = \hat{A}\hat{B} + \hat{B}\hat{A} \) and total volume \( V = \int d^d x \). Note that the constitutive relation (9) is expressed by the parameter \( \lambda t_0 = \{ \beta, \cdots \} \) at time \( t \), which is defined by the matching condition \( \langle \hat{c}_a(t) \rangle = \langle \hat{c}(t) \rangle_{\text{TPQ}} \). Thanks to one-to-one correspondence between \( \lambda t_0 \) and \( \langle \hat{c}_a(t) \rangle \) (apart from the first-order phase transition), Eq. (9) indeed expresses \( \langle \hat{J}_{\alpha}(t) \rangle \) by \( \langle \hat{c}_a(t) \rangle \).

Then, once we evaluate \( \langle \hat{J}_{\alpha}(t) \rangle_{\text{TPQ}} \) following from the thermodynamic functional log \( \mathcal{Z}_{\text{TPQ}}[\lambda] \) (see e.g., [31–34]) and the Green-Kubo formula (10), Eq. (9) closes the averaged conservation law (5), which completes an \( \iTPQ \)-based derivation of the first-order hydrodynamics. In addition to hydrodynamic equations, we can also show the quantum fluctuation theorem and the second law of thermodynamics for the \( \iTPQ \) state within a small error. For that purpose, let us introduce the generating function of the total entropy production from initial time \( t_0 \) to arbitrary time \( t (> t_0) \) with forward/backward evolution.

\[
G_{\text{TPQ}}(z) = \langle \hat{U}(t)e^{iz[S(\lambda t; t_0)]} \rangle_{\lambda t_0}^{\text{TPQ}},
\]

\[
G_{\text{B}}(z) = \langle \hat{U}(t)e^{iz[S(\lambda t; t)]} \rangle_{\lambda t}^{\text{TPQ}},
\]

where \( \hat{U}(t) \) is the time evolution operator. By using a combined parity and time-reversal \( \Theta \equiv PT \), we also defined PT-transformed time evolution operator and parameter by \( \hat{U}(t) = \Theta \hat{U}(t) \Theta^{-1} \) and \( \lambda t = \epsilon \lambda t_a(t, -x) \) with \( \epsilon \) being a PT-eigenvalue of charge densities \( \hat{c}_a \), respectively. We can prove the second law of thermodynamics and quantum fluctuation theorem as [77]

\[
\langle [S(\lambda t; t) - S(\lambda t_0; t_0)] \rangle \geq O(e^{-A_1(V_{\text{cell}})}),
\]

\[
|G_{\text{F}}(z) - G_{\text{B}}(z) - z + 1| = O(e^{-A_2(V_{\text{cell}})}),
\]

where \( A_1 \) and \( A_2 \) denote certain numerical factors. Note that the violation of both the fluctuation theorem and the second law is exponentially small with respect to the volume of fluid-cell \( V_{\text{cell}} \), which is the same as Eqs. (7)-(8).

**Application to spin chain.**—We now apply the developed formalism to a one-dimensional nonintegrable lattice-half integer spin system, whose Hamiltonian reads

\[
\hat{H} = \sum_{n=1}^{N} [J_n \sigma_n^z \sigma_{n+1}^z + D_n (\sigma_n^x \sigma_{n+1}^x - \sigma_n^y \sigma_{n+1}^y) + \Gamma \sigma_n^x + B \sigma_n^z],
\]

where we impose the periodic boundary condition. The energy is the only conserved quantity of this system so that we find the energy-current operator satisfies the averaged conservation law

\[
\partial_t \langle \hat{h}(t, n) \rangle + \nabla_x \langle \hat{J}_E(t, n) \rangle = 0,
\]

over the initial \( \iTPQ \) state \( |\beta_{t_0}; t_0 \rangle \) with \( \nabla_x \langle \hat{J}_E(t, n) \rangle \equiv \langle \hat{J}_E(t, n + 1/2) \rangle - \langle \hat{J}_E(t, n - 1/2) \rangle \).

In accordance with our formal derivation, one can expect the long-time/length behavior of the energy current is governed by the hydrodynamic equation. In the present setup, the constitutive relation within the leading-order derivative expansion leads to the Fourier’s law:

\[
\langle \hat{J}_E(t, n - 1/2) \rangle = -\kappa_{\text{Fourier}}(t, n) \nabla_x T(t, n - 1/2)
\]

where we defined \( \nabla_x T(t, n - 1/2) \equiv T(t, n) - T(t, n - 1) \) with \( T(t, n) \equiv 1/\beta(t, n) \). The proportional coefficient \( \kappa_{\text{Fourier}}(t, n) \) is identified as the thermal conductivity. On the other hand, we can also compute the thermal conductivity from the Green-Kubo formula:

\[
\kappa(t, n) = \frac{\beta(t, n)^2}{4N} \lim_{\tau \to \infty} \int_{-\tau}^{\tau} dt \langle \{ \delta \hat{J}_E(t), \delta \hat{J}_E(t_0) \} \rangle_{\beta(t_0), n},
\]

with the total energy current \( \hat{J}_E(t) \equiv \sum_{n=1}^{N} \hat{J}_E(t, n) \). Our derivation tells us that \( \kappa_{\text{Fourier}}(t, n) \) should agree with \( \kappa(t, n) \) in the large fluid-cell limit. Thus, we can quantitatively evaluate the validity of the hydrodynamic description by comparing the values of \( \kappa_{\text{Fourier}}(t, n) \) and \( \kappa(t, n) \), which are independently evaluated from Eqs. (18)-(19).

A numerical implementation of the \( \iTPQ \)-based simulation is realized in a similar way with the canonical TPQ state [55, 56, 58, 59]. Expanding the exponential operator in Eq. (2), we first prepare the initial \( \iTPQ \) state as

\[
|\beta_{t_0}; t_0 \rangle \equiv e^{-\frac{1}{2} \sum_{m=1}^{N} \beta(t_0, m) x_{m}} \sum_{k=0}^{\infty} \frac{1}{2^k \pi!} \times \left( \sum_{n=1}^{N} \beta(t_0, n) \left[ l_n - \hat{h}(t_0, n) \right] \right)^k \sum_{\alpha} z_\alpha |\alpha_{t_0} \rangle,
\]

where \( l_n \) is a certain number larger than the maximum eigenvalue of \( \hat{h}(t_0, n) \), and \( \beta(t_0, n) \) provides an initial distribution of the local temperature. Note that the position dependence of the local temperature does not allow us to express the \( \iTPQ \) state by the microcanonical counterpart, as is the case for global thermal equilibrium [55, 56, 58, 59]. Starting from this initial state, we numerically investigate the time evolution by solving the Schrödinger equation with the Runge-Kutta method [77].

Figure 1 shows the time-evolution of the local temperature and energy current starting from an inhomogeneous temperature profile: \( \beta(t, n) = \beta + \Delta \beta \sin(2\pi(n - 1)/N) \) with \( \beta = 1.5 \) and \( \Delta \beta = 0.2 \). Here we imposed the periodic boundary condition with \( N = 20 \). It is worth emphasizing that we perform the coarse-graining along both spatial and temporal directions: We set the fluid-cell size to contain 5 spins or so and take the temporal average over \( \tau = 10 \) [77]. We clearly see the diffusive relaxation of inhomogeneous temperature (or equivalently energy density). Furthermore, we can find a tendency of the energy current responding to the temperature gradient, which is compatible with Fourier’s law (18). Note, however, that we cannot conclude that the system really
obey Eqs. (18)-(19) at this point because we have not shown the thermal conductivity, appearing in Eq. (18) indeed matches with that evaluated from the Green-Kubo formula (19).

To microscopically confirm the hydrodynamic relaxation, we evaluate the coefficient of Fourier’s law (18) and compare it with the thermal conductivity $\kappa(t, n)$ independently evaluated from the Green-Kubo formula (19). If the system is really in the hydrodynamic regime, these quantities should coincide with each other. The result is given in Fig. 2. Taking account of the temperature window, we compute the Green-Kubo formula for three values of temperatures (see shaded regions in Fig. 2). Figure 2(b) then indicates that the values of Fourier’s law coefficient (18) stay comparable regions with those obtained from the Green-Kubo formula, though they fluctuate depending on temporal and spatial positions. The agreement may be surprisingly good if we recall our fluid-cell contains just 5 spins or so, and the deviation is probably the finite-volume effect. In short, we conclude that the hydrodynamic description is confirmed to be a good approximation on the basis of the Schrödinger equation.

We finally show the result on the second law of thermodynamics and the quantum fluctuation theorem. Figure 3 demonstrates the time-dependence of the entropy production (main figure) and the quantum fluctuation theorem (inset). The solid line in Fig. 3 for the entropy production is from a direct evaluation, and data are evaluated from the slopes of the generating functions $G_F^{\text{TPQ}}(z)$ and $G_B^{\text{TPQ}}(1-z)$, which are shown in the inset by solid and dotted lines, respectively. They take positive values in the entire time regions, which confirms the second law of thermodynamics. Moreover, one sees that $G_F^{\text{TPQ}}(z)$ indeed works as a generating function of the entropy production. However, due to uncertainty of the fluctuation theorem coming from the finiteness of the fluid cell as shown in Eq. (15), $G_F^{\text{TPQ}}(1-z)$ (and its slope) at a later time will deviate from $G_F^{\text{TPQ}}(z)$ (see also the inset of Fig. 3). Although we cannot analytically evaluate dependences of the deviation on the time and $z$, this behavior looks similar to those obtained in Ref. [68, 78].

Concluding remarks.— In this Letter, we have formulated hydrodynamics based on a special class of random pure quantum states, which we call the $\ell$TPQ state. We have shown the equivalence of the $\ell$TPQ-state formalism to that of the LG ensemble in the large fluid-cell limit in Eqs. (7)-(8), from which we have derived the hydrodynamic equations (9)-(10), the second law of thermodynamics (14), and the quantum fluctuation theorem (15). Using the $\ell$TPQ state, we have performed the numerical simulation of one-dimensional nonintegrable spin chain model. From a careful analysis, we have numerically confirmed that the observed thermal diffusion allows the hydrodynamic description with the Fourier’s law and the Green-Kubo formula (Figs. 1-2). The validity of the second law of thermodynamics and quantum fluctuation theorem has also been confirmed (Fig. 3).

Let us comment on limitations of the $\ell$TPQ-state formulation both from the conceptual and practical viewpoints. First of all, the present formulation employs a $\ell$TPQ state mainly motivated by its numerical efficiency.

FIG. 1. The time-evolutions of (a) local temperature, and (b) energy current of the model (16) with parameters $(J_s, D, \Gamma, B) = (0.2, 0.2, 0.2, -0.2)$.

FIG. 2. Numerical results for (a) Green-Kubo formula (19), and (b) comparison of $\kappa_{\text{Fourier}}(t, n)$ with the thermal conductivity evaluated from the Green-Kubo formula. The parameters are the same as in Fig. 1.

FIG. 3. A numerical result for the entropy production (14) [inset: the quantum fluctuation theorem (15)]. The parameters are the same as in Fig. 1.
However, a ℓTPQ state is a randomly sampled state to reproduce local thermodynamics for isolated quantum systems. Therefore, a ℓTPQ state is not directly related to the eigenstate thermalization hypothesis (ETH) [79–88], as the latter concerns energy eigenstates that are responsible for quantum ergodicity of Hamiltonian dynamics. We thus believe that it is crucial to construct a more solid foundation for hydrodynamics of isolated quantum systems, which is an important future issue.

Another limitation of the ℓTPQ-state formulation is set by the dimension of the Hilbert space of bosonic systems. Since we do not assume the relativistic or nonrelativistic nature of systems, the ℓTPQ formulation is, in principle, applicable to any quantum system including the quark-gluon plasma or relativistic/nonrelativistic Bose gas. Although a current computational resource does not allow us to perform the ℓTPQ simulation for such bosonic systems with a huge degrees of freedom, it would be of immediate interest to perform the ℓTPQ simulation for fermionic systems.

There are also interesting prospects that can be clarified along the line of this Letter. While we performed a ℓTPQ simulation for the system with the single conserved quantity, it is interesting to realize the simulation with multiple conserved charges. In particular, the existence of the momentum density in low-dimensional systems is interesting, because it is expected to drastically change conventional hydrodynamics due to a large hydrodynamic fluctuation [74]. Furthermore, it would be an interesting direction to extend our ℓTPQ framework so as to apply thermal transports for quantum integrable systems described by generalized hydrodynamics [23–27]. We left them as future works (some will be reported in the full paper [89]).

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Supplemental Materials: Quantum hydrodynamics from local thermal pure states

Appendix A: Proofs of equivalence in the large fluid-cell limit

We here give proofs of equivalence between ℓTPQ formulation and the local Gibbs (LG) ensemble method, which includes a proof of the second law of thermodynamics and quantum fluctuation theorem in the ℓTPQ formulation.

1. Proof of Eqs. (7) and (8)

We here give a proof of Eqs. (7)-(8), which is similarly accomplished, as is the case with the global thermal equilibrium [55, 56, 58, 59]. For that purpose, we rely on Chebyshev’s inequality:

\[ \text{Prob}_z \left[ |O - \langle O \rangle_z| \geq \epsilon \right] \leq \frac{(O - \langle O \rangle_z)^2}{\epsilon^2} = \frac{(O^2 - \langle O \rangle_z^2)}{\epsilon^2}. \]  

(S1)

Let us first define the average over the random variable \( z_\alpha = (z_\alpha^* + i z_\alpha' \sqrt{z}) \) by \( \langle O(z) \rangle_z \). Since both real and imaginary parts are independently sampled from the standard normal distribution, they satisfy, e.g.,

\[ \langle z_\alpha \rangle_z = 0, \quad \langle z_\alpha^2 \rangle_z = \delta_{\alpha_1 \alpha_2}, \quad \langle |z_\alpha|^4 \rangle_z = 2, \quad \langle |z_{\alpha_1}|^2 |z_{\alpha_2}|^2 \rangle_z = 1, \quad \langle z_{\alpha_1}^* z_{\alpha_2}^* z_{\alpha_2} z_{\alpha_4} \rangle_z = 0, \]  

(S2)

where \( \delta_{\alpha_1 \alpha_2} \) denotes the Kronecker delta, and the last two equations hold for \( \alpha_1 \neq \alpha_2 \). With the help of Eqs. (S2), we can show the following key identity for a state \( |\psi_0 \rangle \equiv \sum_\alpha z_\alpha |\alpha \rangle ::

\[ \left\langle \left( \langle \psi_0 | \hat{A} | \psi_0 \rangle - \langle \psi_0 | \hat{A} | \psi_0 \rangle \right)_z \left( \langle \psi_0 | \hat{B} | \psi_0 \rangle - \langle \psi_0 | \hat{B} | \psi_0 \rangle \right)_z \right\rangle_z = \text{Tr} \left( \hat{A} \hat{B} \right), \]  

(S3)

where \( \hat{A} \) and \( \hat{B} \) denote arbitrary operators.

Proof of Eq. (7): In order to show the convergence of the partition functional (7), we use Chebyshev’s inequality (S1) for \( O = Z_{\text{TPQ}}[\lambda_t]/Z_{\text{LG}}[\lambda_t] = \langle \lambda_t; t|\lambda_t; t \rangle / \langle \langle \lambda_t; t|\lambda_t; t \rangle \rangle_z \). Recalling the definition of the partition functional for the ℓTPQ state and LG ensemble \( Z_{\text{LG}}[\lambda_t] \equiv \text{Tr} e^{-K[\lambda_t; t]} \), we find the identity (S3) with \( \hat{A} = \hat{B} = e^{-K[\lambda_t; t]} \) provides the right-hand-side of Chebyshev’s inequality thanks to

\[ \langle \psi_0 | e^{-K[\lambda_t; t]} | \psi_0 \rangle = \langle \lambda_t; t|\lambda_t; t \rangle = Z_{\text{TPQ}}[\lambda_t], \]

\[ \langle \psi_0 | e^{-K[\lambda_t; t]} | \psi_0 \rangle \rangle_z = \langle \langle \lambda_t; t|\lambda_t; t \rangle \rangle_z = Z_{\text{LG}}[\lambda_t]. \]

(S4)

As a consequence, we obtain

\[ \left\langle \left( \frac{\langle \lambda_t; t|\lambda_t; t \rangle \langle \langle \lambda_t; t|\lambda_t; t \rangle \rangle_z - 1}{Z_{\text{LG}}[\lambda_t] \langle \langle \lambda_t; t|\lambda_t; t \rangle \rangle_z} \right)^2 \right\rangle_z = \frac{Z_{\text{TPQ}}[2 \lambda_t]}{Z_{\text{LG}}[\lambda_t]^2} = \exp \left[ -2 \int_\Lambda d^d x \beta_t \left( p(\beta_t, \mu^a_t) - p(2\beta_t, \mu^a_t) + O(\partial^2) \right) \right]. \]

(S5)

where, to show the second equality, we used the leading-order expression of the Massieu-Planck functional log \( Z_{\text{LG}}[\lambda_t] \) for a parity-preserving fluid obtained in Refs. [31–33]. Here \( \beta_t = \beta(t, x) \) denotes the local inverse temperature conjugate to the energy density, \( \mu^a_t = \mu^a(t, x) \) local chemical potentials for charge densities \( J^a_0 \) attached to internal symmetries of the system labeled by indices \( a \). We explicitly put the size of the fluid cell \( \Lambda \). Substituting this result into Chebyshev’s inequality, we obtain

\[ \text{Prob}_z \left[ \frac{Z_{\text{TPQ}}[\lambda_t]}{Z_{\text{LG}}[\lambda_t]} - 1 \geq \epsilon \right] \leq \frac{1}{\epsilon^2} \exp \left[ -2 \int_\Lambda d^d x \beta_t \left( p(\beta_t, \mu^a_t) - p(2\beta_t, \mu^a_t) + O(\partial^2) \right) \right]. \]

(S6)

The vital point here is that the pressure of a fluid cell \( p(\beta, \mu^a) \) is the decreasing function with respect to \( \beta \), and satisfies \( p(\beta, \mu^a) - p(2\beta, \mu^a) > 0 \). Therefore, taking the large size limit of the fluid cell \( V_\text{cell} \equiv \int_\Lambda d^d x \), we see that the probability of finding the difference between \( Z_{\text{TPQ}}[\lambda_t] \) and \( Z_{\text{LG}}[\lambda_t] \) will be exponentially small. This completes the proof of Eq. (7).
Proof of Eq. (8): Let us move on to the proof for Eq. (8). The average value over $\ell_{TPQ}$ and LG distribution are given by

$$\langle \hat{O}\rangle_{\lambda_i}^{\ell_{TPQ}} = \frac{\langle \lambda_i; t | \hat{O} | \lambda_i; t \rangle}{\langle \lambda_i; t | \lambda_i; t \rangle} \quad \text{and} \quad \langle \hat{O}\rangle_{\lambda_i}^{\ell_{LG}} = \frac{\langle \langle \lambda_i; t | \hat{O} | \lambda_i; t \rangle \rangle_z}{\langle \langle \lambda_i; t | \lambda_i; t \rangle \rangle_z},$$

(S7)

Noting $\langle \hat{O}\rangle_{\lambda_i}^{\ell_{LG}} \neq \langle \langle \hat{O}\rangle_{\lambda_i}^{\ell_{TPQ}} \rangle_z$, we divide the proof into two steps: First, we evaluate the difference between $\langle \hat{O}\rangle_{\lambda_i}^{\ell_{TPQ}}$ and $\langle \langle \hat{O}\rangle_{\lambda_i}^{\ell_{TPQ}} \rangle_z$, and second, we evaluate the difference between $\langle \langle \hat{O}\rangle_{\lambda_i}^{\ell_{TPQ}} \rangle_z$ and $\langle \hat{O}\rangle_{\lambda_i}^{\ell_{LG}}$.

As a first step, we use Chebyshev’s inequality for $\langle \hat{O}\rangle_{\lambda_i}^{\ell_{TPQ}} = \langle \lambda_i; t | \hat{O} | \lambda_i; t \rangle / \langle \lambda_i; t | \lambda_i; t \rangle$. In order to evaluate the right-hand-side of that inequality, we expand it as

$$\left( \langle \langle \langle \hat{O}\rangle_{\lambda_i}^{\ell_{TPQ}} \rangle_z \rangle_z \right)^2 = \left( \left( \frac{\langle \lambda_i; t | \hat{O} | \lambda_i; t \rangle}{\langle \lambda_i; t | \lambda_i; t \rangle} - \frac{\langle \langle \lambda_i; t | \hat{O} | \lambda_i; t \rangle \rangle_z}{\langle \langle \lambda_i; t | \lambda_i; t \rangle \rangle_z} \right)^2 \right)_z$$

(S8)

where we defined

$$\delta f = \langle \lambda_i; t | \hat{O} | \lambda_i; t \rangle - \langle \langle \lambda_i; t | \hat{O} | \lambda_i; t \rangle \rangle_z,$$

$$\delta g = \langle \lambda_i; t | \lambda_i; t \rangle - \langle \langle \lambda_i; t | \lambda_i; t \rangle \rangle_z.$$  

(S9)

Since we have already evaluated $\langle \langle \delta g \rangle \rangle_z$ in Eq. (5), we only need to evaluate the first two terms in Eq. (8); $\langle \langle \delta f \rangle \rangle_z$ and $\langle \delta f \delta g \rangle_z$.

Let us first evaluate $\langle \langle \delta f \rangle \rangle_z$. By the use of the identity (33) with $\hat{A} = \hat{B} = e^{\frac{i}{2} \hat{K} | \lambda_i; t \rangle \langle \lambda_i; t |} e^{-\frac{i}{2} \hat{K} | \lambda_i; t \rangle \langle \lambda_i; t |}$, we obtain

$$\langle \langle \delta f \rangle \rangle_z = \text{Tr} \left( e^{-\frac{i}{2} \hat{K} | \lambda_i; t \rangle \langle \lambda_i; t |} \hat{O} e^{-\frac{i}{2} \hat{K} | \lambda_i; t \rangle \langle \lambda_i; t |} \hat{O} \right).$$

(S10)

Since $\hat{K} | \lambda_i; t \rangle$ is an Hermitian operator, taking trace with its eigenstate $| n \rangle$ result in

$$\langle \langle \delta f \rangle \rangle_z = \sum_{n,m} e^{-K_n | \lambda_i; t \rangle \langle \lambda_i; t |} e^{-K_m | \lambda_i; t \rangle \langle \lambda_i; t |} \langle n | \hat{O} | m \rangle \langle m | \hat{O} | n \rangle \leq \sum_{n,m} e^{-K_n | \lambda_i; t \rangle \langle \lambda_i; t |} e^{-K_m | \lambda_i; t \rangle \langle \lambda_i; t |} \langle n | \hat{O} | m \rangle \langle m | \hat{O} | n \rangle,$$

(S11)

where $K_n | \lambda_i; t \rangle$ denotes the eigenvalue of $\hat{K} | \lambda_i; t \rangle$, and we used the completeness for the eigenstate of $\hat{K} | \lambda_i; t \rangle$ to show the first equality. The inequality results from the inequality of arithmetic and geometric means: $\sqrt{ab} \leq (a + b)/2$. The right-hand side of this inequality can be further rewritten as

$$\sum_{n,m} e^{-K_n | \lambda_i; t \rangle \langle \lambda_i; t |} e^{-K_m | \lambda_i; t \rangle \langle \lambda_i; t |} \langle n | \hat{O} | m \rangle \langle m | \hat{O} | n \rangle = Z_{LG[2\lambda_i]} \langle \hat{O} \rangle_{2\lambda_i}^{LG},$$

(S12)

and we now obtain the following inequality for $\langle \langle \delta f \rangle \rangle_z$:

$$\langle \langle \delta f \rangle \rangle_z \leq Z_{LG[2\lambda_i]} \langle \hat{O} \rangle_{2\lambda_i}^{LG},$$

(S13)

To evaluate $\langle \delta f \delta g \rangle_z$, we simply apply the identity (33) with $\hat{A} = e^{\frac{i}{2} \hat{K} | \lambda_i; t \rangle \langle \lambda_i; t |} e^{-\frac{i}{2} \hat{K} | \lambda_i; t \rangle \langle \lambda_i; t |}$ and $\hat{B} = e^{-\frac{i}{2} \hat{K} | \lambda_i; t \rangle \langle \lambda_i; t |}$:

$$\langle \delta f \delta g \rangle_z = \text{Tr} \left( e^{-2\hat{K} | \lambda_i; t \rangle \langle \lambda_i; t |} \hat{O} \right) = Z_{LG[2\lambda_i]} \langle \hat{O} \rangle_{2\lambda_i}^{LG}.$$  

(S14)

Substituting Eqs. (5), (13) and (14) into Eq. (8), we eventually obtain an upper bound of the variance as

$$\left( \left( \langle \hat{O} \rangle_{\lambda_i}^{\ell_{TPQ}} - \langle \langle \hat{O} \rangle_{\lambda_i}^{\ell_{TPQ}} \rangle_z \right)^2 \right)_z \leq \frac{Z_{LG[2\lambda_i]}}{(Z_{LG[\lambda_i]})^2} \left[ \langle \hat{O} \rangle_{2\lambda_i}^{LG} - 2 \langle \hat{O} \rangle_{\lambda_i}^{LG} \langle \hat{O} \rangle_{2\lambda_i}^{LG} + \langle \langle \hat{O} \rangle_{\lambda_i}^{LG} \rangle_z^2 \right]$$

$$= \frac{Z_{LG[2\lambda_i]}}{(Z_{LG[\lambda_i]})^2} \left[ \langle \langle \hat{O} - \langle \hat{O} \rangle_{\lambda_i}^{LG} \rangle_{2\lambda_i}^{LG} + \langle \langle \hat{O} \rangle_{\lambda_i}^{LG} - \langle \hat{O} \rangle_{2\lambda_i}^{LG} \rangle_z^2 \right].$$

(S15)
which results in the following inequality:

\[
\text{Prob}_{z}\left[\langle \hat{O}\rangle_{\lambda t}^{\text{TPQ}} - \langle \hat{O}\rangle_{\lambda t}^{\text{TPQ}}\rangle_z \geq \epsilon \right] \leq \frac{e^{-2\lambda t} d^4 x {\beta}_i [p(\beta_i, \mu_i^a) - p(2\beta_i, \mu_i^a)]}{\epsilon^2} \left[\langle (\hat{O} - \langle \hat{O}\rangle_{2\lambda t}^{\text{LG}})^2 \rangle_{2\lambda t} + \langle (\hat{O} - \langle \hat{O}\rangle_{2\lambda t}^{\text{LG}})^2 \rangle_{2\lambda t} \right].
\]

(S16)

As a consequence, for any operator \(\hat{O}\) whose two variations \(\langle (\hat{O} - \langle \hat{O}\rangle_{2\lambda t}^{\text{LG}})^2 \rangle_{2\lambda t}\) and \(\langle (\hat{O} - \langle \hat{O}\rangle_{2\lambda t}^{\text{LG}})^2 \rangle_{2\lambda t}\) do not show an exponentially large behavior with respect to the volume of fluid cells \(V_{\text{cell}}\), the inequality brings about the following result on convergence:

\[
\langle \hat{O}\rangle_{\lambda t}^{\text{TPQ}} \overset{P}{\rightarrow} \langle \hat{O}\rangle_{\lambda t}^{\text{TPQ}}\rangle_z,
\]

(S17)

which gives the first step of the proof.

Thanks to Eq. (S5) and (S14), the second step is mostly finished if we evaluate the average difference by expanding it as follows:

\[
\left| \langle \hat{O}\rangle_{\lambda t}^{\text{TPQ}} \rangle_z - \langle \hat{O}\rangle_{\lambda t}^{\text{TPQ}}\rangle_z \right| = \frac{1}{(Z_{\text{LG}}[\lambda t])^2} \delta f \delta g \left[\langle \hat{O}\rangle_{\lambda t}^{\text{TPQ}}\rangle_z + \langle \hat{O}\rangle_{\lambda t}^{\text{TPQ}}\rangle_z \right]
\]

(S18)

We thus conclude that any operator \(\hat{O}\) not showing the exponentially large fluctuation, the difference between \(\langle \hat{O}\rangle_{\lambda t}^{\text{TPQ}}\rangle_z\) and \(\langle \hat{O}\rangle_{\lambda t}^{\text{TPQ}}\rangle_z\) is exponentially small. This completes the proof of Eq. (8).

2. Proof of the Eqs. (14) and (15)

Although the second law of thermodynamics follows from the quantum fluctuation theorem (15), we provide independent proofs of them.

Proof of Eq. (14): The second law of thermodynamics can be easily shown by rewriting the entropy production as follows:

\[
\langle \hat{S}[t, t_0; \lambda] \rangle = \langle \hat{S}[t; \lambda t_0]^{\text{TPQ}} - \langle \hat{S}[t; t_0]^{\text{LG}} \rangle + \langle \hat{S}[t; t_0]^{\text{TPQ}} - \langle \hat{S}[t; t_0]^{\text{LG}} \rangle \rangle_{\lambda t_0} + \langle \hat{S}[t; t_0]^{\text{LG}} \rangle_{\lambda t_0} - \langle \hat{S}[t; t_0]^{\text{LG}} \rangle_{\lambda t_0}.
\]

Using the equivalence between the TPQ state and LG ensemble, we can estimate the first and second terms to be exponentially small: \((a) = O(e^{-B_1 V_{\text{cell}}})\) and \((b) = O(e^{-B_2 V_{\text{cell}}})\) with some constants \(B_1\) and \(B_2\). Furthermore, the last term is shown to be positive: \((c) \geq 0\) (See Refs. [31, 33, 34]). This results in Eq. (14).

Proof of Eq. (15): To prove the quantum fluctuation theorem (15) for the TPQ state, we first recall that for the LG ensemble [33]. For that purpose, let us introduce the generating function for the entropy production under the forward/backward evolution by

\[
G_F^{\text{LG}}(z) \equiv \text{Tr} \left( \hat{\rho}_{\text{LG}}[\lambda t_0; t_0^3] \hat{U}(t) e^{i z \hat{S}[\lambda t_0; t_0]} \hat{U}(t) e^{-i z \hat{S}[\lambda t_0; t_0]} \right),
\]

(S20)

\[
G_B^{\text{LG}}(z) \equiv \text{Tr} \left( \hat{\rho}_{\text{LG}}[\lambda t_0; t] \hat{U}(t) e^{i z \hat{S}[\lambda t_0; t]} \hat{U}(t) e^{-i z \hat{S}[\lambda t_0; t]} \right),
\]

(S21)

where \(\hat{U}(t)\) is a time evolution operator. By using a combined parity and time-reversal \(\Theta \equiv PT\), we also defined PT-transformed time evolution operator and parameter by \(\hat{U}(t) \equiv \Theta \hat{U}(t) \Theta^{-1}\) and \(\lambda \equiv e^z \lambda(t, -x)\) with \(e^z\) being a PT-eigenvalue of charge densities \(\hat{c}_n\), respectively. Here, we introduced the LG distribution for the density operator \(\hat{\rho}_{\text{LG}}[\lambda t; t]\) and the entropy functional operator \(\hat{S}_{\text{LG}}[\lambda t; t]\) by

\[
\hat{\rho}_{\text{LG}}[\lambda t; t] = e^{-\hat{S}_{\text{LG}}[\lambda t; t]}\quad \text{and} \quad \hat{S}_{\text{LG}}[\lambda t; t] = \hat{K}[\lambda t; t] + \log Z_{\text{LG}}[\lambda t].
\]

(S22)
The generating function in Eqs. (S20) and (S21) are shown to satisfy the quantum fluctuation theorem [33]:

$$G_F^{\text{LG}}(z) = G_B^{\text{LG}}(-z + i).$$  
(S23)

In order to show the quantum fluctuation theorem for the ℓTPQ state, we then introduce the reference generating function by

$$\overline{G}_F(z) \equiv \text{Tr} \left( \hat{\rho}_\text{LG}[\lambda_0; t_0] e^{izS_{\text{TPQ}}[\lambda_0]} e^{-izS_{\text{TPQ}}[t_0: \lambda_0]} \right).$$  
(S24)

By the use of this reference generating function with the triangle inequality, we can evaluate the difference between $G_F^{\text{TPQ}}(z)$ and $G_F^{\text{LG}}(z)$ as follows:

$$\left| G_F^{\text{TPQ}}(z) - G_F^{\text{LG}}(z) \right| \leq \left| G_F^{\text{TPQ}}(z) - \overline{G}_F(z) \right| + \left| G_F^{\text{LG}}(z) + \overline{G}_F(z) \right|.  
(A) + (B)$$
(S25)

Defining $\Delta \Psi[\lambda, \lambda_0] = \log Z[\lambda] - \log Z[\lambda_0]$ for both ℓTPQ and LG states, and using the error estimation for the partition functional, we can estimate the second term in the second line as

$$(B) = \left| \text{Tr} \left( \hat{\rho}_\text{LG}[\lambda_0; t_0] e^{iz\hat{K}} e^{-iz\hat{K}} \right) \left( e^{iz\Delta \Psi_{\text{LG}}[\lambda, \lambda_0]} - e^{iz\Delta \Psi_{\text{TPQ}}[\lambda, \lambda_0]} \right) \right|$$

$$= \left| \text{Tr} \left( \hat{\rho}_\text{LG}[\lambda_0; t_0] e^{iz\hat{K}} e^{-iz\hat{K}} \right) e^{iz\Delta \Psi_{\text{TPQ}}[\lambda, \lambda_0]} \left( \exp \left[ O(e^{-C_1 V_{\text{cell}}}) \right] - 1 \right) \right|$$

$$= O(e^{-C_1 V_{\text{cell}}}).$$
(S26)

Since the first term in Eq. (S25) is given by the difference of the averaged operator over the ℓTPQ state and LG distribution, we can evaluate it as

$$(A) = \left| e^{izS_{\text{TPQ}}[\lambda_0]} e^{-izS_{\text{TPQ}}[\lambda_0]} - e^{izS_{\text{TPQ}}[\lambda_0]} e^{-izS_{\text{TPQ}}[\lambda]} \right|$$

$$= O(e^{-C_2 V_{\text{cell}}} \Delta \Psi),$$
(S27)

where we assumed that the variation of the averaged operator does not show the exponentially large behavior. Substituting Eqs. (S26)-(S27) into Eq. (S25), we obtain

$$G_F^{\text{TPQ}}(z) - G_F^{\text{LG}}(z) \leq O(e^{-C_2 V_{\text{cell}}}),$$
(S28)

which states the equivalence of the generating function for the entropy production for the forward evolution. On the other hand, a similar analysis also works for $G_B^{\text{TPQ}}(z) - G_B^{\text{LG}}(z)$, and thus, we have,

$$G_B^{\text{TPQ}}(-z + i) - G_B^{\text{LG}}(-z + i) \leq O(e^{-C_3 V_{\text{cell}}}),$$
(S29)

Therefore, combining Eq. (S23) with Eqs. (S28)-(S29), we can evaluate

$$G_F^{\text{TPQ}}(z) - G_B^{\text{TPQ}}(-z + i)$$

$$= \left| G_F^{\text{TPQ}}(z) - G_F^{\text{LG}}(z) - \left( G_F^{\text{TPQ}}(-z + i) - G_B^{\text{LG}}(-z + i) \right) + G_B^{\text{LG}}(z) - G_B^{\text{LG}}(-z + i) \right| \leq \left| G_F^{\text{TPQ}}(z) - G_F^{\text{LG}}(z) \right| + \left| G_B^{\text{TPQ}}(-z + i) - G_B^{\text{LG}}(-z + i) \right|$$

$$= O(e^{-A_2 V_{\text{cell}}}),$$
(S30)

where the inequality in the third line again follows from the triangle inequality. This gives the quantum fluctuation theorem for the ℓTPQ state.

**Appendix B: Detail of numerical implementation**

We here summarize a detail of our numerical simulation based on the ℓTPQ formulation such as (i) the concrete definition of the (smeared) energy density/current operator, (ii) how to prepare the ℓTPQ state and the equation of state, and (iii) two ways to evaluate the thermal conductivity, and (iv) a numerical confirmation of the fluctuation theorem and the second law of thermodynamics.
1. Energy density/current operator and their smearing

Let us first write the Hamiltonian in the following form:

\[ H = \sum_n \hat{h}(n), \quad \hat{h}(n) = \mathcal{H}_{n,n+1} + \mathcal{H}_n, \]  

(S31)

where the energy density operator \( \hat{h}(n) \) is decomposed into the sum of the nearest-neighbor term \( \mathcal{H}_{n,n+1} \) and the on-site term \( \mathcal{H}_n \). Then, we shall derive the energy current operator \( \hat{J}_E(n-1/2) \). In order to determine the form of \( \hat{J}_E(n-1/2) \), we require that the time evolution of the Hamiltonian density should be written as the discrete divergence of the energy current operator as

\[ \frac{\partial \hat{h}(t,n)}{\partial t} = -\left( \hat{J}_E(t,n+1/2) - \hat{J}_E(t,n-1/2) \right). \]  

(S32)

On the other hand, the Heisenberg equation for the Hamiltonian density operator reads

\[ \frac{\partial \hat{h}(t,n)}{\partial t} = i[\hat{H}, \hat{h}(t,n)] \]  

(S33)

Comparing these two equations, we identify the energy current as follows:

\[ \hat{J}_E(n-1/2) = i[\mathcal{H}_{n-1,n}, \mathcal{H}_{n,n+1} + \mathcal{H}_n]. \]  

(S34)

Thus, the concrete form of the energy density operator \( \hat{h}(n) \) and energy current operator \( \hat{J}_E(n-1/2) \) for the system defined by the Hamiltonian (16) is given by

\[
\begin{align*}
\hat{h}(n) &= J_z \sigma^+_n \sigma^-_{n+1} + D(\sigma^+_n \sigma^+_{n+1} - \sigma^-_n \sigma^-_{n+1}) + \Gamma \sigma^+_n + B \sigma^-_n, \\
\hat{J}_E(n-1/2) &= -2D^2 (\sigma^+_n \sigma^+_m \sigma^-_{n+1} - \sigma^-_n \sigma^-_{n+1} \sigma^+_m) + 4D J_z \sigma^+_n \sigma^+_m \\
&\quad - 2\Gamma J_z \sigma^+_n \sigma^+_m + 2D \Gamma \sigma^+_n \sigma^+_m + 2DB \sigma^+_n \sigma^+_m.
\end{align*}
\]

(S35)

Let us then define a smearing of local operators. To extract the hydrodynamic behavior, we define a smeared operator of an arbitrary local operator \( \hat{O}(n) \) by

\[ \hat{O}(n) = \sum_{m=1}^N w_{n-m} \hat{O}(m), \]  

(S37)

with an appropriate weight function \( w_{n-m} \) which is a positive and has a peak at \( n = m \). We also require that \( w_n \) satisfies the periodic boundary condition \( w_{n-N} = w_n \). This requirement is needed in our setup to define a smeared energy current operator in a consistent manner. To see this, we apply Eq. (S37) to the local Hamiltonian:

\[ \frac{\partial \hat{h}(t,n)}{\partial t} = \sum_{m=1}^N w_{n-m} \frac{\partial \hat{h}(t,m)}{\partial t} = -\left( \sum_{m=1}^N w_{n+1-(m+1/2)} \hat{J}_E(t,m+1) - \hat{J}_E(t,n-1/2) \right) \]  

(S38)

The first term of the right hand side is rewritten as

\[ \sum_{m=1}^N w_{n+1-(m+1/2)} \hat{J}_E(t,m+1/2) = \hat{J}_E(t,n+1/2) - (w_n - w_{n-N}) \hat{J}_E(t,1/2). \]  

(S39)

Thus, we find that the smeared current operator holds the continuity equation

\[ \frac{\partial \hat{h}(t,n)}{\partial t} = -\left( \hat{J}_E(t,n+1/2) - \hat{J}_E(t,n-1/2) \right) \]  

(S40)

thanks to the periodicity \( w_{n-N} = w_n \). In our numerical simulations, we choose the weight function \( w_n \) as

\[ w_n = \frac{1}{\sum_{m=1}^N e^{\pm |x|} \cos \frac{\pi m}{N}} e^{\pm \frac{\pi n}{N} \cos \frac{2\pi m}{N}}, \]  

(S41)

where \( \sigma^2 \) is a positive parameter controlling the smearing length. In the thermodynamic limit with \( N \to \infty \), the typical width of this function is given by \( 1 - I_1(1/\sigma^2)/I_0(1/\sigma^2) \), where \( I_n(x) \) is the modified Bessel function of the first kind. In the \( 1/\sigma^2 \to \infty \) limit for a fixed \( N \), the weight function converges to the Kronecker delta. For our purpose, the width of the weight function, regarded as the effective size of the fluid cell \( \Lambda \), should be taken so that \( 1 \ll \Lambda \ll N \). Typical shapes of the weight function is shown in Fig. S1. For instance, the number of spins in the fluid cell is about 5 when \( 1/\sigma^2 = 5 \).
2. How to prepare ℓTPQ state and equation of state

a. Construction of the ℓTPQ state. Let us first review a way to implement a canonical TPQ state which is defined by

\[ |\beta\rangle \equiv \exp \left( \frac{-N\beta \hat{h}}{2} \right) \sum_{\alpha} z_{\alpha} |\alpha\rangle, \tag{S42} \]

where \( \hat{h} = \hat{H}/N \) is the Hamiltonian density and \( |\alpha\rangle \) is a random vector in the Hilbert space. We expand the canonical TPQ state by microcanonical TPQ states as

\[ |\beta\rangle = e^{-N\beta l/2} \sum_{k=0}^{\infty} R_k |\psi_k\rangle \quad \text{with} \quad |\psi_k\rangle = \frac{|k\rangle}{\langle k|k\rangle}, \quad |k\rangle \equiv (l - \hat{h})^k \sum_{\alpha} z_{\alpha} |\alpha\rangle \quad \text{and} \quad R_k(\beta) = \frac{(N\beta/2)^k \langle k|k\rangle}{k!}, \tag{S43} \]

where \( l \) is a certain number larger than the maximum eigenvalue of \( \hat{h} \). One can show that \( R_k \) has a sharp peak at a certain \( k = k^* \), whose magnitude is controlled by the inverse temperature \([55, 56, 58, 59]\). Thanks to this property, it is justified to truncate the expansion at a certain order \( k = k_{\text{max}} \). Then, we define \( k_{\text{max}} \) by

\[ R_{k_{\text{max}}} > 10^{-t}, \tag{S44} \]

for a given tolerance parameter \( t \). In our numerical simulations, we take \( t = 20 \) and confirm that the expectation values of observables are well converged.

One can generalize the above procedure in a straightforward manner to the implementation of ℓTPQ states. For that purpose, we first define

\[ |\{\beta(n)\}\rangle \equiv \exp \left( -\sum_{n=1}^{N} \beta(n) \hat{h}(n) \right) \sum_{\alpha} z_{\alpha} |\alpha\rangle, \tag{S45} \]

where \( \hat{h}_n \) is the local Hamiltonian density. Expanding the exponential operator, we find

\[ |\{\beta(n)\}\rangle = e^{-\frac{1}{2} \sum_n \beta(n) l_n} \exp \left( \frac{\sum_{n=1}^{N} \beta(n) (l_n - \hat{h}(n))}{2} \right) \sum_{\alpha} z_{\alpha} |\alpha\rangle \tag{S46} \]

\[ = e^{-\frac{1}{2} \sum_n \beta(n) l_n} \sum_k \frac{1}{2^k k!} \left( \sum_{n=1}^{N} \beta(n) (l_n - \hat{h}(n)) \right)^k \sum_{\alpha} z_{\alpha} |\alpha\rangle \tag{S47} \]

\[ = e^{-\frac{1}{2} \sum_n \beta(n) l_n} \sum_k \frac{\beta^k N^k}{2^k k!} \left( \frac{1}{N\beta} \sum_{n=1}^{N} \beta(n) (l_n - \hat{h}(n)) \right)^k \sum_{\alpha} z_{\alpha} |\alpha\rangle, \tag{S48} \]
where \( l_n \) is a certain number larger than the maximum eigenvalue of \( \hat{h} \), and \( \bar{\beta} = \frac{1}{N} \sum_{n=1}^{N} \beta(n) \) is the average inverse temperature. We note that the state appeared in Eq. (S48)

\[
|k, \{\beta(n)\}\rangle \equiv \left( \frac{1}{N\bar{\beta}} \sum_{n=1}^{N} \beta(n) (l_n - \hat{h}(n)) \right)^k \sum_{\alpha} z_\alpha |\alpha\rangle \tag{S49}
\]

can be regarded as a generalization of the microcanonical TPQ state \(|k\rangle\): Indeed, we can show \( |k, \{\beta(n)\}\rangle = |k\rangle \) if \( \beta = \beta_1 = \cdots = \beta_N, \ l = l_1 = \cdots, l_N \). Then, we obtain the expression the \( \ell \)TPQ state as follows:

\[
|\{\beta(n)\}\rangle = e^{-N\beta/2} \sum_{k=0}^{\infty} R_k(\{\beta(n)\}) |\psi_k, \{\beta(n)\}\rangle, \tag{S50}
\]

with \(|\psi_k, \{\beta(n)\}\rangle = \frac{|k, \{\beta(n)\}\rangle}{\langle k, \{\beta(n)\}|k, \{\beta(n)\}\rangle}, \ R_k(\{\beta(n)\}) = \frac{(N\beta/2)^k}{k!} \langle k, \{\beta(n)\}|k, \{\beta(n)\}\rangle \tag{S51}\)\\

We confirmed that \( R_k(\{\beta(n)\}) \) also has a sharp peak at a certain \( k = k^* \) so that we can truncate the summation over \( k \) in Eq. (S50) in a same manner as the canonical TPQ states. Equations (S48)-(S51) define a way to prepare the \( \ell \)TPQ state starting from an arbitrary random vector \(|a\rangle\) in the Hilbert space.

b. The energy-temperature relation. In the main article, we introduce the local temperature according to the matching condition \( \langle \bar{h}(t,n) \rangle = \langle \bar{h}(t,n) \rangle^\lambda_{\text{TPQ}} \). Within the derivative expansion, this condition is equivalent to use the thermodynamic relation to determine the local temperature from the energy density.

One can use the equation of state as such a thermodynamic relation, which enables us to relate the energy density and local temperature. Using the canonical TPQ states, the total energy of the system is calculated as

\[
E \equiv \frac{\langle \beta |H|\beta \rangle}{\langle \beta |\beta \rangle}. \tag{S52}
\]

This relation gives the one-to-one correspondence between the energy and the inverse temperature as long as first order phase transitions do not occur. We calculate the total energy for several inverse temperatures \( \beta = 0.0, 0.1, \cdots, 3.0 \) for the system size \( N = 20 \). The coupling constants are set to \( J_z = \Gamma = -B = D = 0.2 \). The results is shown in Fig. S2.

![FIG. S2. The equation of state for the Hamiltonian Eq. (16) with the system size \( N = 20 \). The total energy is calculated at \( \beta = 0.0, 0.1, \cdots, 3.0 \) and they are represented by circles. The solid line is the interpolation of them.](image)

3. Two ways to evaluate thermal conductivity

a. Thermal conductivity from the Fourier law. While the formal arguments rely on the Heisenberg picture, we perform numerical simulations based on the Schrödinger picture. The expectation value of an operator \( \hat{O} \) at time \( t \) is
FIG. S3. The time evolution of the local energy, the local temperature and the energy current.

calculated by

\[ \langle \hat{O}(t) \rangle = \frac{\langle \psi(t) | \hat{O} | \psi(t) \rangle}{\langle \psi(0) | \psi(0) \rangle}, \]  

(S53)

\[ i \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle, \quad |\psi(0)\rangle = \{|\beta(n)\rangle\}. \]  

(S54)

Unless otherwise noted, the number of spins is taken as \( N = 20 \), and the coupling constants are \( J_z = \Gamma = -B = D = 0.2 \) so that the system is nonintegrable. We set the initial temperature profile of the TPQ state as

\[ \beta(n) = \bar{\beta} + \Delta \beta \sin(2\pi(n - 1)/N), \quad n = 1, \ldots, N, \]  

(S55)

with \( \bar{\beta} = 1.5 \) and \( \Delta \beta = 0.2 \). We numerically solve the Schrödinger equation by the fourth order Runge-Kutta method with a time slice \( \Delta t = 0.01 \), and calculated the local energy and the energy current. The smearing is also performed by using the weight function given in Eq. (S41) with \( 1/\sigma^2 = 5 \). The local temperature is estimated from the local energy through the relation in equilibrium shown in Fig. S2 assuming that derivative corrections are negligible.

If the system is well described by hydrodynamics, we can extract the thermal conductivity from the Fourier law Eq. (18). We introduce temporally averaged operator by

\[ \hat{O}(t) \equiv \frac{1}{\tau} \int_{t}^{t+\tau} dt' \hat{O}(t'). \]  

(S56)

The time evolution of temporally averaged quantities are shown in Fig. S3 (and Fig. 1 in the main text), where \( \tau = 10 \). Again, the local temperature is estimated by using the equation of state in equilibrium as \( \bar{T}(t, n) = T(\langle \bar{h}(t, n) \rangle) \). (In the main text, the tilde is omitted for the simplicity of notation.) In Fig. S4, we show the time evolution of \( \nabla \bar{T}(t, n) \), \( \langle \bar{J}_E(t, n - 1/2) \rangle \) and \( \kappa_{\text{Fourier}}(t, n) \). We see that the typical equilibration time is \( t_{\text{eq}} \sim 100 \) at which there are no net energy current nor temperature gradient.

We remark that evaluation of the thermal conductivity \( \kappa_{\text{Fourier}}(t, n) \) from the Fourier law is difficult at a large-time limit \( t \geq t_{\text{eq}} \) or inappropriate site around \( n = 6 \) and 16. This is because both temperature gradient \( |\nabla \bar{T}(t, n)\rangle \) and energy current \( \langle \bar{J}_E(t, n - 1/2) \rangle \) takes a small value in those setups so that numerical evaluations suffer from large uncertainty. Thus, we can rigorously estimate the thermal conductivity from the data at \( t \ll t_{\text{eq}} \) and at the node position of the temperature profile. In the main article, we show the thermal conductivity at \( n = 1, 11 \) in Fig. 2.

b. Thermal conductivity from the Green-Kubo formula. On the basis of the Green-Kubo formula, the thermal conductivity is given by

\[ \kappa(t, n) = \frac{\beta_n(t)}{4N} \int_{-\infty}^{\infty} dt' \langle \{ \bar{J}_E(t), \bar{J}_E(0) \} \rangle_{\beta_n}^{\text{TPQ}} \simeq \frac{\beta}{4N} \int_{-\infty}^{\infty} dt' \langle \{ \bar{J}_E(t), \bar{J}_E(0) \} \rangle_{\beta}^{\text{TPQ}} \]  

(S57)

where \( \bar{J}_E(t) = \sum_{n=1}^{N} \bar{J}_{E_n}(t) \). In the second equality, we neglect the spatial and temporal dependence of temperature, which gives rise to the higher-order derivative correction. Conceptually, the Green-Kubo formula does not require a fully microscopic time resolution but only a hydrodynamic one. Therefore, we also replace the bare correlation function \( \langle \{ \bar{J}_E(t), \bar{J}_E(0) \} \rangle_{\beta}^{\text{TPQ}} \) with the temporally smeared one, which leads to

\[ G(t) \equiv \frac{1}{t^2} \int_{t}^{t+\tau} ds \int_{0}^{t} ds' \langle \bar{J}_E(s), \bar{J}_E(s') \rangle_{\beta}^{\text{TPQ}}. \]  

(S58)
An appropriate time scale $t_h$ is estimated from the damping time of the bare correlation function $\langle \{ \hat{J}_E(t), \hat{J}_E(0) \} \rangle_{\text{TPQ}}$. Below, we set $t_h = 10$ which is numerically computed. The thermal conductivity is now given by

$$\kappa(t, n) \simeq \frac{\beta}{2} (C(\infty) - C(-\infty)) \quad \text{with} \quad C(t) = \frac{1}{2N} \int_0^t dt' G(t'). \quad (S59)$$

Since $C(t)$ is defined by the symmetric Green’s function, we can show it is equal to $-C(-t)$ within a small error, which is actually confirmed numerically. Thus, we reach the following simple linear response result in the $\ell$TPQ formulation:

$$\kappa(t, n) \simeq \beta C(\infty) \equiv \kappa \quad (S60)$$

The left panel of Fig. S5 demonstrates a typical behavior of the function $C(t)$, from which one can confirm that $C(t)$ converges when $t \gtrsim 100$. The value at infinitely late time $C(\infty)$ is estimated by the average of $C(t)$ from $t = 100$ to 200. The resulting thermal conductivity at each $\beta$ is shown in the left panel of Fig. S5, which corresponds to the result shown in Fig. 2 in the main text.

**FIG. S4.** The time evolution of the energy current, the temperature gradient, and the thermal conductivity at $n = 1, \cdots, 5$.

**FIG. S5.** (Left) Solid line represents $C(t)$ at $\beta = 1.5$. The dotted line shows the late time average of $C(t)$. (Right) The $\beta$-dependence of the thermal conductivity.

4. **Numerical evaluation of generating functions in fluctuation theorem**

Let us finally present a numerical method to compute the generating functional. When the Hamiltonian does not depend on time explicitly, we can rewrite the generating functions for the total entropy production Eqs. (12)-(13) as
follows:

\[ G_F^{\ell\text{TPQ}}(z; t) = \frac{e^{iz \log(Z_{\text{TPQ}}[\tilde{\beta}(t,n)])/Z_{\text{TPQ}}[\tilde{\beta}(0,n)])}}{Z_{\text{TPQ}}[\tilde{\beta}(0,n)]} \left \langle \{ \tilde{\beta}(0,n) \} \left | e^{iHt} e^{iz \tilde{K}([\tilde{\beta}(t,n)])} e^{-i\hat{K} e^{-iz \tilde{K}([\tilde{\beta}(0,n)])}} \right | \{ \tilde{\beta}(0,n) \} \right \rangle, \]  

(S61)

\[ G_B^{\ell\text{TPQ}}(i - z; t) = \frac{e^{-iz^* \log(Z_{\text{TPQ}}[\tilde{\beta}(t,n)])/Z_{\text{TPQ}}[\tilde{\beta}(0,n)])}}{Z_{\text{TPQ}}[\tilde{\beta}(0,n)]} \times \left \langle \{ \tilde{\beta}(t,n) \} \left | e^{-i\hat{K} e^{-i\tilde{K}([\tilde{\beta}(t,n)])}} e^{i\tilde{K}([\tilde{\beta}(0,n)])} z^* e^{iHt} e^{i\tilde{K}([\tilde{\beta}(0,n)])} \right | \{ \tilde{\beta}(t,n) \} \right \rangle^*. \]  

(S62)

Here \( \{ \tilde{\beta}(t,n) \} \) is an \( \ell\text{TPQ} \) state, whose temperature profile is chosen to be that at time \( t \). We compute these quantities based on the following steps:

(i) Make an \( \ell\text{TPQ} \) state \( \{ \beta(0,n) \} \) whose temperature profile is set by the initial condition (S55).

(ii) Compute \( e^{-iHt} \{ \beta(0,n) \} \).

(iii) Compute spatially smeared and temporally averaged local energies \( \{ \langle \tilde{\beta}(t,n) \rangle \} \).

(iv) Compute local temperatures \( \{ \tilde{\beta}(t,n) \} \) using the energy-temperature relation.

(v) Make an auxiliary \( \ell\text{TPQ} \) state \( \{ \tilde{\beta}(t,n) \} \).

Exponential operators like \( e^{i\tilde{K}([\tilde{\beta}(t,n)])} \) and \( e^{-i\tilde{K}([\tilde{\beta}(0,n)])} \) are regarded as a time evolution operator. Action of these operators to pure states are computed by fourth order Runge-Kutta method.

The entropy production can be computed by three independent ways. One way is to compute the expectation value of the entropy operator Eq. (3) directly. The others are to compute the gradient of \( \text{Im} G_F^{\ell\text{TPQ}}(z) \) and \( \text{Im} G_B^{\ell\text{TPQ}}(i - z) \) at \( z = 0 \). Concretely, they are given by

\[ \Sigma_F \equiv \text{Im} \frac{\text{d}G_F^{\ell\text{TPQ}}(z)}{\text{d}(iz)} \bigg |_{z=0}, \quad \Sigma_B \equiv \text{Im} \frac{\text{d}G_B^{\ell\text{TPQ}}(i - z)}{\text{d}(iz)} \bigg |_{z=0}. \]  

(S63)

In Fig. S6, we demonstrate the estimation of the entropy production from \( G_F^{\ell\text{TPQ}} \). At each time, we confirm that the generating function behaves as a linear function around the origin, which allows us to evaluate the entropy production according to Eq. (S63). In the figure, the solid line shows the fitting result using the data at \( z = 0.00, 0.01, \ldots, 0.03 \). The choice of the fitting range is insensitive to the result. The fitting of \( G_B^{\ell\text{TPQ}}(i - z) \) can be performed in the same manner.

![FIG. S6. The generating function $G_F^{\ell\text{TPQ}}$ at $t = 20$. The solid line stands for the fitting function.](image-url)
Unlike the standard quantum fluctuation theorem, $\Sigma_F$ agrees with $\Sigma_B$ only in the large fluid-cell limit. We numerically confirm that the difference between $\Sigma_F$ and $\Sigma_B$ scales as $e^{-N}$, where $N$ is the system size. In Fig. S7, we show the scaling behavior at $t = 20$. We also check that the difference tends to vanish by taking the random vector average, which is consistent with our proof of equivalence. We compute the entropy production for 5 random vectors and take the random vector average. In Fig. S8, the solid line and the closed symbols denote the averaged ones while the dotted line and open symbols do the $\ell$TPQ result for the entropy production; namely, computed by a single random vector. Here, numerical simulations are performed for $N = 12$. This result is consistent with the fact that $\Sigma_F$ coincides with $\Sigma_B$ for the standard mixed state (or canonical ensemble) formulation.

FIG. S7. The difference between $\Sigma_F$ and $\Sigma_B$ at $t = 20$.

FIG. S8. The entropy production for $N = 12$. The dotted line and the open symbols show the entropy production obtained for a single random vector. The solid line and the closed symbols show the average over 5 independent samples.