ON ASYMPTOTIC PROPERTIES OF SEMI-RELATIVISTIC HARTREE EQUATION WITH COMBINED HARTREE-TYPE NONLINEARITIES

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Abstract. We consider the semi-relativistic Hartree equation with combined Hartree-type nonlinearities given by

\[ i \partial_t \psi = \sqrt{-\Delta + m^2} \psi + \beta \left( \frac{1}{|x|^\alpha} * |\psi|^2 \right) \psi - \left( \frac{1}{|x|^\alpha} * |\psi|^2 \right) \psi \quad \text{on} \quad \mathbb{R}^3, \]

where \( 0 < \alpha < 1 \) and \( \beta > 0 \). Firstly we study the existence and stability of the maximal ground state \( \psi_\beta \) at \( N = N_c \), where \( N_c \) is a threshold value and can be regarded as “Chandrasekhar limiting mass”. Secondly, we analyse blow-up behaviours of maximal ground states \( \psi_\beta \) when \( \beta \to 0^+ \), and the optimal blow-up rate with respect to \( \beta \) will be calculated.

1. Introduction and Main Results. In this paper, we consider the following equation with combined Hartree-type nonlinearities

\[ i \partial_t \psi = \sqrt{-\Delta + m^2} \psi + \beta \left( \frac{1}{|x|^\alpha} * |\psi|^2 \right) \psi - \left( \frac{1}{|x|^\alpha} * |\psi|^2 \right) \psi \quad \text{on} \quad \mathbb{R}^3, \quad (1.1) \]

where \( 0 < \alpha < 1 \), \( \sqrt{-\Delta + m^2} \) denotes the kinetic energy operator of a relativistic particle with mass \( m > 0 \), the convolution kernel \( \frac{1}{|x|^\alpha} \) represents the Newtonian gravitational potential in appropriate physical units.

The particular case \( \beta = 0 \) is known as the Boson star equation which can describe the dynamics of boson stars under the mean field limit \([17, 4, 20]\). This equation has received much mathematical attention. The local and global well-posedness

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for the Boson equation with initial data \( \psi(0, x) = \psi_0(x) \) in \( H^s(\mathbb{R}^3) \) for \( s \geq \frac{1}{2} \) was proved by Lenzmann[14]. Blow-up results was given in [8]. Solitary waves given by \( \psi(t, x) = e^{it\mu} \varphi(x) \) were considered in [7, 15, 31]. The other corresponding problems we refer to [12, 23, 2, 22, 8, 16, 1, 19] and the references therein.

In this paper we focus on the combined Hartree-type nonlinearities with \( \beta > 0 \). This kind of combined nonlocal nonlinearities appear in the study of many phenomena, including biological swarm [6], granular media[28], molecular dynamics simulations of matter[13]. We should mention that combined nonlinearities like power-type nonlinearities have attracted much attention in the nonlinear Schrödinger equations recently, see [27, 5, 32, 24, 25] and the references therein. There seem to be very few results about combined Hartree-type nonlinearities.

We are interesting in solitary waves of equation (1.1) of the form
\[
\psi(t, x) = e^{it\mu} \varphi(x)
\]
with some \( \mu \in \mathbb{R} \). Putting (1.2) into (1.1) leads to the following equation
\[
\sqrt{-\Delta + m^2} \varphi + \left( \frac{\beta}{|x|^{\alpha}} - \frac{1}{|x|^2} \right) |\varphi|^2 \varphi = \mu \varphi,
\]
which can be viewed as an Euler-Lagrange equation for the following minimization problem
\[
E(\beta, N) := \inf \{ \mathcal{E}_\beta(\varphi) : \varphi \in H^{1/2}(\mathbb{R}^3), \int_{\mathbb{R}^3} |\varphi(x)|^2 \ dx = N \},
\]
where
\[
\mathcal{E}_\beta(\varphi) := \frac{1}{2} \langle \varphi, \sqrt{-\Delta + m^2} \varphi \rangle + \frac{1}{4} \int_{\mathbb{R}^3} \left( \frac{\beta}{|x-y|^{\alpha}} - \frac{1}{|x-y|^2} \right) |\varphi(x)|^2 |\varphi(y)|^2 \ dx \ dy,
\]
and \( N \) denotes the mass of system, or may denote the number of particles. We refer to such minimizers \( \varphi \in H^{1/2} \) as ground states throughout this paper.

Before starting our results, we recall from[7, 16] the following Gagliardo-Nirenberg type inequality
\[
\int_{\mathbb{R}^3} \left( \frac{1}{|x|^2} \right) |\varphi|^2 \ dx \leq \frac{2}{N_c} \langle \psi, \sqrt{-\Delta} \psi \rangle \langle \psi, \psi \rangle,
\]
where \( \langle \cdot, \cdot \rangle \) denotes \( L^2 \) product, \( N_c \) can be regarded as “Chandrasekhar limiting mass” [17], \( \frac{2}{N_c} \) is the best constant given by
\[
N_c := \inf_{\psi \in H^{1/2}(\mathbb{R}^3), \psi \neq 0} \frac{\langle \psi, \sqrt{-\Delta} \psi \rangle \langle \psi, \psi \rangle}{\int_{\mathbb{R}^3} \left( \frac{1}{|x|^2} \right) |\psi|^2 \ dx} = \frac{1}{2} \int_{\mathbb{R}^3} |Q|^2 \ dx.
\]
where \( Q(x) \) is an optimizer of above inequality and a positive solution of nonlinear equation
\[
\sqrt{-\Delta} Q - \left( \frac{1}{|x|^2} \right) |Q|^2 Q = -Q.
\]
Moreover, the following identity holds (see Appendix A of [16])
\[
\langle Q, \sqrt{-\Delta} Q \rangle = \frac{1}{2} \int_{\mathbb{R}^3} \left( \frac{1}{|x|^2} \right) |Q|^2 \ dx = \int_{\mathbb{R}^3} |Q|^2 \ dx = N_c.
\]

First, we study the existence and nonexistence of ground states for problem (1.4) when \( 0 < N < N_c \) and \( N > N_c \).
\textbf{Theorem 1.1.} Suppose that $m > 0$, $\beta \in \mathbb{R}$ and $0 < \alpha < 1$. Let $E(\beta, N)$ be given in (1.4).

(i) There exists at least one minimizer of $E(\beta, N)$ for $0 < N < N_c$, $\beta \leq 0$.

(ii) If $0 < N < N_c$ and $\beta > 0$, then there exists at least one minimizer for $E(\beta, N)$ when $\beta$ small enough and $N$ close to $N_c$ enough.

(iii) If $N > N_c$, then for any $\beta \in \mathbb{R}$, there is no minimizer for $E(\beta, N)$ such that $E(\beta, N) = -\infty$.

The case $N = N_c$ is special, in the following theorem we show the existence of maximal ground states.

\textbf{Theorem 1.2.} Assume that $m > 0$, $\beta \in \mathbb{R}$ and $0 < \alpha < 1$. We have

(i) If $N = N_c$ and $\beta \leq 0$, there is no minimizer for $E(\beta, N_c)$ such that $E(0, N_c) = 0$, and $E(\beta, N_c) = -\infty$ for all $\beta < 0$.

(ii) If $N = N_c$, there exists $\beta_* > 0$, such that $E(\beta, N_c)$ admits at least one minimizer for $\beta \in (0, \beta_*)$.

We call the minimizers denoted by $\varphi_{\beta}(x)$ in case (ii) of this theorem for $N = N_c$ as maximal ground states, since there is still no minimizer for $E(\beta, N_c)$ with $\beta > 0$ and $N > N_c$ (see Theorem 1.1 (iii)). Then the corresponding solitary waves $e^{it\mu} \varphi_{\beta}(x)$ given in (1.2) can be called as maximal ground state solitary waves.

The existence of minimizers for $E(\beta, N)$ is based on the concentration compactness lemma obtained in [7]. Compare to the case (i) and (ii) of Theorem 1.1, the case (ii) of theorem (1.2) is harder to come by, since when $N = N_c$, for any $\varphi \in H^{1/2}(\mathbb{R}^3)$ with $\|\varphi\|_2^2 = N_c$, the $H^{1/2}$ norm $\|\varphi\|_{H^{1/2}}$ can not be controlled directly by the energy $E_{\beta}(\varphi)$ due to the inequality (1.6). To overcome this, we need to prove by contradiction for case (ii) in Theorem 1.2.

Now we address “orbital stability” of maximal ground state solitary waves

$$\psi(t, x) = e^{it\mu} \varphi(x),$$

(1.10)

where $\varphi \in H^{1/2}(\mathbb{R}^3)$ satisfying $\|\varphi\|_2^2 = N_c$, is a ground state of $E(\beta, N_c)$.

\textbf{Theorem 1.3.} Suppose that $m > 0$, $\beta > 0$ and $N = N_c$. Let

$$S_{N_c} := \{ \varphi \in H^{1/2}(\mathbb{R}^3) : E_{\beta}(\varphi) = E(\beta, N_c), \|\varphi\|_2^2 = N_c \}. \quad (1.11)$$

Then the solitary waves given in (1.2) with $\varphi \in S_{N_c}$ are stable for $\beta \in (0, \beta_*)$ in the following sense. Let $\psi(t)$ denotes the solution of (1.1) with initial condition $\psi_0 \in H^{1/2}(\mathbb{R}^3)$. For every $\epsilon > 0$, there exists $\delta > 0$ and if

$$\inf_{\varphi \in S_{N_c}} \|\psi_0 - \varphi\|_{H^{1/2}} \leq \delta \quad \text{with} \quad \|\psi_0\|_2^2 \leq N_c,$$

then

$$\sup_{t \geq 0} \inf_{\varphi \in S_{N_c}} \|\psi(t) - \varphi\|_{H^{1/2}} \leq \epsilon.$$

\textbf{Remark 1.} The condition $\|\psi_0\|_2^2 \leq N_c$ is necessary to guarantee the solution $\psi(t)$ is global, one can see Theorem 5.10 below.

Finally we show the optimal blow-up behaviour for maximal ground states $\varphi_{\beta}$ as $\beta \to 0^+$. We have
Theorem 1.4. Under the assumptions of Theorem 1.2, let $\varphi_\beta(x)$ be the maximal ground state for $E(\beta, N_c)$ with $\beta \in (0, \beta_*)$. Given a sequence $\{\beta_k\}$ with $\beta_k \to 0^+$ as $k \to \infty$, there exists a subsequence (still denoted by $\{\beta_k\}$) such that (i)

$$
\lim_{k \to \infty} \beta_k^{\frac{3}{3-2\alpha}} \varphi_{\beta_k}(\beta_k^{\frac{1}{3-2\alpha}} x) = \gamma^{3/2} Q(\gamma(x - y_0))
$$

(1.12)

strongly in $L^q(\mathbb{R}^3)$ for all $2 \leq q < 3$, $y_0 \in \mathbb{R}^3$ and

$$
\gamma = \left( \frac{\langle Q, \frac{m^2}{\sqrt{-\Delta}} Q \rangle}{\int_{\mathbb{R}^3} (\frac{1}{|x|^\alpha} * |Q|^2)^2 \, dx} \right)^{\frac{1}{3-2\alpha}}.
$$

(ii) Moreover,

$$
\lim_{\beta_k \to 0} \beta_k^{-\frac{3}{3-2\alpha}} E(\beta_k, N_c) = \frac{1}{2} \langle Q, \frac{m^2}{\sqrt{-\Delta}} Q \rangle^{\frac{1}{3-2\alpha}} \left( \int_{\mathbb{R}^3} (\frac{1}{|x|^\alpha} * |Q|^2)^2 \, dx \right)^{\frac{1}{3-2\alpha}}
$$

(1.13)

Remark 2. (1.12) also means that

$$
|\varphi_{\beta_k}(x)|^2 \to N_c \delta(x - y_0).
$$
in the distribution sense, where $\delta(x)$ denotes Dirac delta function.

This theorem shows that, when $\beta$ is small enough, then the mass of ground state at $N = N_c$ will concentrate. Such similar blow-up results appeared in studying Bose-Einstein condensations with attractive interaction described by Gross-Pitaevskii functional, one can see [9, 11, 3, 29]. There are some blow-up results for Boson stars, Guo and Zeng [10] studied the asymptotic behaviour as $N \not\to N_c$ for different self-interacting potentials, Nguyen [21] and Yang [30] studied it for different external potentials. In this paper, we focus on the asymptotic behaviour of ground states at $N = N_c$ when $\beta \to 0^+$. This blow-up analysis is more difficult in contrast to the works of [10, 21, 30], we need some technical arguments, due to the lack of compactness and $H^{1/2}$ norm of ground state can not be controlled directly by energy $E(\beta, N_c)$. The first key point to this theorem is to obtain an optimal estimate for $E(\beta, N_c)$, to do this we need to employ the concentration-compactness arguments to obtain the lower bound of $E(\beta, N_c)$. The second key point is to obtain an optimal estimate of $\langle \varphi_\beta, \sqrt{-\Delta} \varphi_\beta \rangle$, the upper bound is harder to come by, we should employ scaling arguments and prove by contradiction, this is quite different from the mentioned papers.

This paper is organized as follows: in Section 2, we give the proof of Theorem 1.1 and 1.2; in Section 3, we consider the orbital stability of maximal ground state solitary waves; in Section 4, we prove the optimal blow-up behaviour of maximal ground states for $E(\beta, N_c)$ as $\beta \to 0^+$; in Section 5, we give some basic results as the Appendix.

Notation. - $\langle \cdot, \cdot \rangle$ denotes $L^2$ product.
- $\| \cdot \|_p$ denotes the $L^p(\mathbb{R}^3)$ norm for $p \geq 1$.
- $\| f \|_{q, w} = \sup_A |A|^{-\frac{1}{p'}} \int_A |f(x)| \, dx \, (1/q + 1/q' = 1)$ denotes the norm of $L^p_{loc}(\mathbb{R}^n)$, one can see [18].
- $\to$ denotes weakly converge, $*$ stands for convolution on $\mathbb{R}^3$.
- $\beta \to 0^+$ denotes $\beta \to 0$ with $\beta > 0$.
- $a \preceq b$ denotes $a \leq C b$ for some appropriate constant $C > 0$.
- For symbol $\sqrt{-\Delta + m^2}$ and $H^{1/2}(\mathbb{R}^3)$, one can see [7].
2. The proof of Theorem 1.1 and 1.2. First, we claim that

**Lemma 2.1.** (1) Under the assumptions of case (i) and (ii) in Theorem 1.1, any minimizing sequence for $E(\beta, N)$ is uniformly bounded in $H^{1/2}(\mathbb{R}^3)$.

(2) Under the assumptions of case (ii) in Theorem 1.2, i.e. $N = N_c$ and $\beta > 0$ small enough. Then any minimizing sequence for $E(\beta, N_c)$ is uniformly bounded in $H^{1/2}(\mathbb{R}^3)$.

**Proof.** Notice that by Hardy-Littlewood-Sobolev inequality, interpolation inequality and Sobolev inequality, we have

$$
\int_{\mathbb{R}^3} \frac{1}{|x|^{\alpha}} |\psi|^2 |\psi|^2 \, dx \leq C \|\psi\|_1^{4\alpha} \leq C_1 \|\psi\|_3^{2\alpha} \|\psi\|_2^{2(2-\alpha)} \leq C_2 \langle \psi, \sqrt{-\Delta} \psi \rangle^\alpha. \quad (2.1)
$$

For case (ii) in Theorem 1.1, since $\beta > 0$, use the fact $\sqrt{-\Delta + m^2} \geq \sqrt{-\Delta}$ and (1.6)

$$
\mathcal{E}_\beta(\psi) \geq \frac{1}{2}(1 - \frac{N}{N_c}) \langle \psi, \sqrt{-\Delta} \psi \rangle. \quad (2.2)
$$

For case (i) in Theorem 1.1, since $\beta \leq 0$, by (1.6) and (2.1)

$$
\mathcal{E}_\beta(\psi) \geq \frac{1}{2}(1 - \frac{N}{N_c}) \langle \psi, \sqrt{-\Delta} \psi \rangle + \beta \mathcal{C}_2 \langle \psi, \sqrt{-\Delta} \psi \rangle^\alpha. \quad (2.3)
$$

For any minimizing sequence $\{\psi_n\}$, since $0 < N < N_c$ and $0 < \alpha < 1$, then $1 - \frac{N}{N_c} > 0$, and $\sup_n \langle \psi_n, \sqrt{-\Delta} \psi_n \rangle \leq C < \infty$ thanks to (2.2) and (2.3). Thus we obtain the case (1).

Next we prove the case (2). Let $\{\psi_n(x)\}$ be a minimizing sequence of $E(\beta, N_c)$, such that

$$
E(\beta, N_c) \leq \mathcal{E}_\beta(\psi_n) \leq E(\beta, N_c) + \frac{1}{n}. \quad (2.4)
$$

On the contrary, we now suppose that $\{\psi_n(x)\}$ is unbounded in $H^{1/2}(\mathbb{R}^3)$. Then there exists a subsequence $\{\psi_n\}$ (still denoted by $\{\psi_n\}$), such that

$$
\langle \psi_n, \sqrt{-\Delta} \psi_n \rangle \to +\infty. \quad (n \to \infty)
$$

Since $\sqrt{-\Delta + m^2} \geq \sqrt{-\Delta}$, by (1.6) and (2.4) we have

$$
\frac{\beta}{4} \int_{\mathbb{R}^3} \frac{1}{|x|^{\alpha}} |\psi_n|^2 |\psi_n|^2 \, dx \leq \mathcal{E}_\beta(\psi_n) \leq E(\beta, N_c) + \frac{1}{n},
$$

and also

$$
0 \leq \frac{1}{2} \langle \psi_n, \sqrt{-\Delta} \psi_n \rangle - \frac{1}{4} \int_{\mathbb{R}^3} \frac{1}{|x|^{\alpha}} |\psi_n|^2 |\psi_n|^2 \, dx \leq E(\beta, N_c) + \frac{1}{n}. \quad (2.7)
$$

Define now

$$
\epsilon_n := \langle \psi_n, \sqrt{-\Delta} \psi_n \rangle.
$$

Then $\epsilon_n \to 0$ as $n \to \infty$. Define

$$
\tilde{w}_n(x) := \epsilon_n^{-\frac{1}{2}} \psi_n(x).
$$

By (2.8) and (2.7), we have

$$
0 \leq \frac{1}{2} \langle \tilde{w}_n, \sqrt{-\Delta} \tilde{w}_n \rangle - \frac{1}{4} \int_{\mathbb{R}^3} \frac{1}{|x|^{\alpha}} |\tilde{w}_n|^2 |\tilde{w}_n|^2 \, dx \leq \epsilon_n (E(\beta, N_c) + \frac{1}{n}) \to 0. \quad (2.10)
$$

Therefore, we can conclude that

$$
\langle \tilde{w}_n, \sqrt{-\Delta} \tilde{w}_n \rangle = 1, \quad M \leq \int_{\mathbb{R}^3} \frac{1}{|x|^{\alpha}} |\tilde{w}_n|^2 |\tilde{w}_n|^2 \, dx \leq \frac{1}{M}. \quad (2.11)
$$
for some constant $M > 0$, independent of $n$.

We claim that, there exist a sequence $\{y_{n}\}$ and positive constants $R_0$ and $\eta$ such that

$$\liminf_{\epsilon_n \to 0} \int_{B(y_{n} , R_0)} |\tilde{w}_n(x)|^2 \, dx \geq \eta > 0.$$  

(2.12)

Otherwise, by Lemma A.1 in [7], we have

$$\int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |\tilde{w}_n|^2 \right) |\tilde{w}_n|^2 \, dx \to 0,$$

this contradicts (2.11).

Let

$$w_n(x) = \tilde{w}_n(x + y_{n}) = \epsilon_n^3 \psi_n(\epsilon_n x + \epsilon_n y_{n}).$$  

(2.13)

Then it follows from (2.12) that

$$\liminf_{\epsilon_n \to 0} \int_{B(0, R_0)} |w_n(x)|^2 \, dx \geq \eta > 0.$$  

(2.14)

Claim: There exists $\eta_0 > 0$ such that

$$\int_{\mathbb{R}^3} \left( \frac{1}{|x|^\alpha} * |w_n|^2 \right) |w_n|^2 \, dx \geq \eta_0 > 0.$$  

(2.15)

Indeed, let $R_0$ be given in (2.14),

$$\int_{\mathbb{R}^3} \left( \frac{1}{|x|^\alpha} * |w_n|^2 \right) |w_n|^2 \, dx$$

$$\geq \int_{B(0, R_0)} \int_{B(0, R_0)} \frac{1}{|x-y|^\alpha} |w_n(x)|^2 |w_n(y)|^2 \, dxdy$$

$$\geq \int_{B(0, R_0)} \int_{B(0, R_0)} \frac{1}{2R_0^\alpha} |w_n(x)|^2 |w_n(y)|^2 \, dxdy$$

$$= \frac{1}{2R_0^\alpha} \left( \int_{B(0, R_0)} |w_n(x)|^2 \, dx \right)^2 \geq \eta_0^2 \frac{2R_0^\alpha}{2R_0^\alpha} \left( \int_{B(0, R_0)} |w_n(x)|^2 \, dx \right)^2 \geq \eta_0^2$$

(by (2.14)).

Thus the claim holds.

On the other hand, note that by (2.13) and (2.6) we have

$$\frac{\beta \epsilon_n^{-\alpha}}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|^\alpha} * |w_n|^2 \right) |w_n|^2 \, dx = \frac{\beta}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|^\alpha} * |\psi_n|^2 \right) |\psi_n|^2 \, dx \leq E(\beta, N_\epsilon) + \frac{1}{n},$$

which implies that

$$\int_{\mathbb{R}^3} \left( \frac{1}{|x|^\alpha} * |w_n|^2 \right) |w_n|^2 \, dx \leq C \epsilon_n^\alpha \to 0, \quad \epsilon_n \to 0. \tag{2.16}$$

This contradicts (2.15). Therefore, $\{\psi_n(x)\}$ is bounded uniformly in $H^{1/2}(\mathbb{R}^3)$, we complete the lemma.

Now we go to prove the Theorem 1.1 and Theorem 1.2. Notice that, the case (iii) of 1.1 has been proved in case (III) of Lemma 5.4 below, and the case (i) of Theorem 1.2 has been proved in case (iii) of Lemma 5.4 below. The rest, the case (i) and (ii) of Theorem 1.1 can be proved by standard concentration-compactness arguments as [7] by combing with Lemma 5.4, 5.5 and Lemma 2.1 above. We only give the proof for the case (ii) of Theorem 1.2 below.  \(\square\)
The end of the proof of Theorem 1.2 (ii): Since \( \{\psi_n\} \) is bounded uniformly in \( H^{1/2}(\mathbb{R}^3) \), so now we can use the concentration-compactness lemma (Lemma 5.1 below)

(1) Vanishing does not occur

If vanishing occurs, it follows from Lemma 5.2 we have

\[
\lim_{n \to \infty} \int_{\mathbb{R}^3} \frac{1}{|x|^2} \psi_n^2 \psi_n^2 \, dx = 0; \quad \lim_{n \to \infty} \int_{\mathbb{R}^3} \frac{1}{|x|^2} \psi_n^2 \psi_n^2 \, dx = 0.
\]

Since \( \sqrt{-\Delta + m^2} - m \geq 0 \), it follows that

\[
E(\beta, N_c) = \lim_{n \to \infty} E(\beta, \psi_n) = \lim_{n \to \infty} \frac{1}{2} \langle \psi_n, (-\Delta + m^2 \psi_n) \rangle \geq \frac{1}{2} m N_c.
\]

which contradicts lemma 5.4. Therefore, vanishing does not occur.

(2) Dichotomy does not occur: If the dichotomy occurs, by Lemma 5.1(iii) below, then there exists \( \lambda \in (0, N_c) \) such that, for every \( \epsilon > 0 \), there exists two bounded dichotomy subsequences in \( H^{1/2}(\mathbb{R}^3) \) denoted by \( \{\psi_{n_k}\} \) and \( \{\psi_{n_k}\} \) with

\[
\lambda - \epsilon \leq \|\psi_{n_k}\|_2^2 \leq \lambda + \epsilon, \quad (N_c - \lambda) - \epsilon \leq \|\psi_{n_k}\|_2^2 \leq (N_c - \lambda) + \epsilon
\]

for \( k \) sufficiently large. Moreover, (5.6) and (5.8) in the Appendix below allow us to deduce that, there exists \( r_1(k) \) and \( r_2(\epsilon) \) such that

\[
E(\beta, \psi_{n_k}) - E(\beta, \psi_{n_k}^2) - E(\beta, \psi_{n_k}^2) \geq -r_1(k) - r_2(\epsilon),
\]

where \( r_1(k) \to 0 \) as \( k \to \infty \), and \( r_2(\epsilon) \to 0 \) as \( \epsilon \to 0 \). Passing to limits \( k \to \infty \) and \( \epsilon \to 0 \), we deduce that

\[
E(\beta, N_c) \geq E(\beta, \lambda) + E(\beta, N_c - \lambda).
\]

This contradicts (5.13) below.

Therefore, compactness happens. The same arguments as the proof of part i) of Theorem 2.1 in [7], then up to a translation, the sequence \( \{\psi_{n_k}\} \) are relatively compact in \( H^{1/2}(\mathbb{R}^3) \). Thus there exists \( \{y_k\} \) such that \( \psi_{n_k} = \psi_{n_k} (x + y_k) \) satisfies

\[
\psi_{n_k} \to \varphi, \quad \text{strongly in } H^{1/2}(\mathbb{R}^3).
\]

such that \( \varphi \) is a minimizer of \( E(\beta, N_c) \). This completes the proof of case (ii) of Theorem 1.2.

3. The proof of Theorem 1.3.

Proof. Since \( \|\psi\|_2^2 \leq N_c \), by the global well-posedness result in the Theorem 5.10 of the Appendix below, we have that the corresponding solution \( \psi(t) \) exists for all \( t \geq 0 \), thus taking \( \sup_{t \geq 0} \) is well-defined.

Let us now assume that the orbital stability (in the sense defined Theorem 1.3) does not hold. Then there exists a sequence on initial date, \( \{\psi_n(0)\} \) in \( H^{1/2}(\mathbb{R}^3) \) satisfying

\[
\inf_{\varphi \in S_{N_c}} \|\psi_n(0) - \varphi\|_{H^{1/2}} \to 0 \quad \text{as } n \to \infty, \quad \|\psi_n(0)\|_2^2 \leq N_c,
\]

and some \( \epsilon > 0 \) such that

\[
\inf_{\varphi \in S_{N_c}} \|\psi_n(t_n) - \varphi\|_{H^{1/2}} > \epsilon, \quad \text{for all } n \geq 0.
\]
for a suitable sequence of times \( \{ t_n \} \). Note that (3.1) implies that \( N(\psi_n(0)) \to N_c \) as \( n \to \infty \). Next consider the sequence, \( \{ u_n \} \), in \( H^{1/2}(\mathbb{R}^3) \) that is given by

\[
u_n := \psi_n(t_n).
\]

By conservation laws given in Lemma 5.9 in the Appendix below, then \( \| u_n \|_2^2 = \| \psi_n(0) \|_2^2 \) and \( \mathcal{E}_\beta(u_n) = \mathcal{E}_\beta(\psi_n(0)) \). By (3.1), it follows that

\[
\lim_{n \to \infty} \mathcal{E}_\beta(u_n) = E(\beta, N_c) \quad \text{and} \quad \lim_{n \to \infty} \| u_n \|_2^2 = N_c.
\]

Defining the rescaled sequence

\[
\bar{u}_n := \sqrt{\frac{N_c}{\| u_n \|_2^2}} u_n,
\]

Notice that, the same arguments as Lemma 2.1, \( \{ u_n \} \) has to be bounded in \( H^{1/2}(\mathbb{R}^3) \). It follows that

\[
\| u_n - \bar{u}_n \|_{H^{1/2}} \leq C |1 - \frac{N_c}{\| u_n \|_2^2}| \to 0.
\]

Thus we deduce that

\[
\lim_{n \to \infty} \mathcal{E}_\beta(\bar{u}_n) = E(\beta, N_c) \quad \text{and} \quad \| \bar{u}_n \|_2^2 = N_c, \quad \text{for all } n.
\]

Therefore, \( \{ \bar{u}_n \} \) is a minimizing sequence for problem (1.4). By Theorem 1.2, for \( \beta \) small enough, this sequence has to contain a subsequence, \( \{ \bar{u}_{n_k} \} \), that strongly converges in \( H^{1/2}(\mathbb{R}^3) \) to some minimizer \( \varphi \in S_{N_c} \). In particular, inequality (3.2) cannot hold when \( u_n = \psi_n(t_n) \) is replaced by \( \bar{u}_n \). However, in view of (3.5), then inequality (3.2) cannot hold for \( \{ u_n \} \) itself. Thus, we are led to a contradiction and the proof of Theorem 1.3 is complete.

\[
\square
\]

4. Asymptotic analysis of minimizers for \( E(\beta, N_c) \) as \( \beta \to 0^+ \).

4.1. Energy estimates.

**Lemma 4.1.** Under the assumptions of Theorem 1.2(ii), there exist two constants \( C_1 > 0 \) and \( C_2 > 0 \), independent of \( \beta \) such that as \( \beta \to 0^+ \)

\[
C_1 \beta^{-\frac{1}{1+\alpha}} \leq E(\beta, N_c) \leq C_2 \beta^{-\frac{1}{1+\alpha}}.
\]

**Proof.** The upper bound follows (5.12) by using the same test function. Thus we only need to prove the lower bound.

Now let \( \varphi_\beta \) be a nonnegative minimizer of (1.4), first we claim that

**Claim 1:** \( \langle \varphi_\beta, \sqrt{-\Delta} \varphi_\beta \rangle \to +\infty \) as \( \beta \to 0^+ \).

In fact, if this claim is not true, then there exists a subsequence \( \{ \beta_k \} \) with \( \beta_k \to 0 \) as \( k \to \infty \), such that \( \{ \varphi_{\beta_k} \} \) is uniformly bound in \( H^{1/2}(\mathbb{R}^3) \). Hence there exists \( \varphi_0 \in H^{1/2}(\mathbb{R}^3) \) and a weakly converging subsequence, still denoted by \( \{ \varphi_{\beta_k} \} \), such that

\[
\varphi_{\beta_k} \rightharpoonup \varphi_0, \quad \text{weakly in } H^{1/2}(\mathbb{R}^3).
\]

(1) **Vanishing does not occur**

If vanishing occurs, it follows from Lemma 5.2 we have

\[
\lim_{k \to \infty} \int_{\mathbb{R}^3} \frac{1}{|x|^{\alpha}} |\varphi_{\beta_k}|^2 \, dx = 0; \quad \lim_{k \to \infty} \int_{\mathbb{R}^3} \frac{1}{|x|^{\alpha}} |\varphi_{\beta_k}|^2 \, dx = 0.
\]

The upper bound follows (5.12) by using the same test function. Thus we only need to prove the lower bound.
Since \(\sqrt{-\Delta + m^2} - m \geq 0\), it follows that

\[
\liminf_{k \to \infty} E(\beta_k, N_c) = \liminf_{k \to \infty} \mathcal{E}_{\beta_k}(\varphi_{\beta_k}) = \liminf_{k \to \infty} \left\{ \frac{1}{2} (\varphi_{\beta_k}, (\sqrt{-\Delta + m^2}) \varphi_{\beta_k}) \right\}
\]

\[
\geq \liminf_{k \to \infty} \frac{1}{2} m \|\varphi_{\beta_k}\|_2^2 = \frac{1}{2} m N_c.
\]

On the other hand, note that by the upper bound of (4.1) we know that for \(\beta\) small enough, we have \(C_2 \beta^{\frac{4}{3}} < \frac{1}{2} m N_c\), this means that

\[
\limsup_{k \to \infty} E(\beta_k, N_c) < \frac{1}{2} m N_c.
\]

Thus, we obtain a contradiction. Therefore, vanishing does not occur.

(2) Dichotomy does not occur:

If the dichotomy occurs, by Lemma 5.1(iii) below, then there exists \(\lambda \in (0, N_c)\) such that, for every \(\epsilon > 0\), there exists two bounded dichotomy subsequences in \(H^{1/2}(\mathbb{R}^3)\) denoted by \(\{\varphi_{\beta_k}^1\}\) and \(\{\varphi_{\beta_k}^2\}\) with

\[
\lambda - \epsilon \leq \|\varphi_{\beta_k}^1\|_2^2 \leq \lambda + \epsilon, \quad (N_c - \lambda) - \epsilon \leq \|\varphi_{\beta_k}^2\|_2^2 \leq (N_c - \lambda) + \epsilon
\]

(4.2) for \(k\) sufficiently large. Moreover, (5.6) and (5.8) allow us to deduce that, there exists \(r_1(k)\) and \(r_2(\epsilon)\) such that

\[
\mathcal{E}_{\beta_k}(\varphi_{\beta_k}^1) - \mathcal{E}_{\beta_k}(\varphi_{\beta_k}^2) - \mathcal{E}_{\beta_k}(\varphi_{\beta_k}^2) \geq -r_1(k) - r_2(\epsilon),
\]

(4.3)

where \(r_1(k) \to 0\) as \(k \to \infty\), and \(r_2(\epsilon) \to 0\) as \(\epsilon \to 0\). Then

\[
0 = E(0, N_c) = \lim_{k \to \infty} \mathcal{E}_{\beta_k}(\varphi_{\beta_k})
\]

\[
\geq \liminf_{k \to \infty} \left\{ \mathcal{E}_{\beta_k}(\varphi_{\beta_k}^1) + \mathcal{E}_{\beta_k}(\varphi_{\beta_k}^2) - r_1(k) - r_2(\epsilon) \right\} \quad \text{(by (4.3))}
\]

\[
\geq \liminf_{k \to \infty} \left\{ \mathcal{E}_{0}(\varphi_{\beta_k}^1) + \mathcal{E}_{0}(\varphi_{\beta_k}^2) \right\} - r_2(\epsilon) \quad \text{(since \(\beta_k > 0\))}
\]

\[
\geq \liminf_{k \to \infty} \left\{ E(0, \|\varphi_{\beta_k}^1\|_2^2) + E(0, \|\varphi_{\beta_k}^2\|_2^2) \right\} - r_2(\epsilon)
\]

\[
\geq E(0, \lambda + \epsilon) + E(0, (N_c - \lambda) + \epsilon) - r_2(\epsilon),
\]

the last inequality comes from (4.2) and the fact that \(E(0, N)\) is decreasing in \(N\), see Lemma 5.4 (i) below. Passing to the limit \(\epsilon \to 0\) and by continuity of \(E(0, N)\) in \(N\), we deduce that

\[
E(0, N_c) \geq E(0, \lambda) + E(0, N_c - \lambda)
\]

(4.4)

holds for some \(0 < \lambda < N_c\). This contradicts the strict subadditivity condition (5.13) with \(\beta = 0\).

Therefore, compactness happens. The same arguments as [7], then up to a translation, the sequence \(\{\varphi_{\beta_k}\}\) are relatively compact in \(H^{1/2}(\mathbb{R}^3)\). Thus there exists \(\{y_k\}\) with \(y_k \in \mathbb{R}^3\) such that \(\varphi_{\beta_k} = \varphi_{\beta_k}(x + y_k)\) satisfies

\[
\varphi_{\beta_k} \to \varphi_0, \quad \text{strongly in } H^{1/2}(\mathbb{R}^3).
\]

(4.5)

This implies that \(\lim_{k \to \infty} \mathcal{E}_{\beta_k}(\varphi_{\beta_k}) = \mathcal{E}_0(\varphi_0)\). It follows that

\[
E(0, N_c) \leq \mathcal{E}_0(\varphi_0) = \lim_{k \to \infty} \mathcal{E}_{\beta_k}(\varphi_{\beta_k}) = \lim_{k \to \infty} \mathcal{E}_{\beta_k}(\varphi_{\beta_k}) = E(0, N_c)
\]

(4.6)

This implies that \(\varphi\) is a minimizer of \(E(0, N_c)\), contradicting that \(E(0, N_c)\) has no minimizer (see Theorem 1.2 (i) above). Thus we conclude that \(\langle \varphi_{\beta}, \sqrt{-\Delta} \varphi_{\beta} \rangle \to +\infty\) as \(\beta \to 0^+\).
Now define
\[
\epsilon = \langle \varphi_\beta, \sqrt{-\Delta} \varphi_\beta \rangle^{-1},
\]
then \( \epsilon \to 0 \) as \( \beta \to 0 \). Note that by (1.6) with the fact \( \sqrt{-\Delta} + m^2 \geq \sqrt{-\Delta} \), and the upper bound of (4.1), we have
\[
0 \leq \frac{1}{2} \langle \varphi_\beta, \sqrt{-\Delta} \varphi_\beta \rangle - \frac{1}{4} \int_{\mathbb{R}^3} \frac{1}{|x|} \ast |\varphi_\beta|^2 |\varphi_\beta|^2 \, dx \leq E(\beta, N_\epsilon) \leq C_2 \beta^{1+\alpha} \to 0.
\]
This implies that as \( \beta \to 0 \)
\[
\frac{\epsilon}{2} \int_{\mathbb{R}^3} \frac{1}{|x|} \ast |\varphi_\beta|^2 |\varphi_\beta|^2 \, dx \to 1.
\]
The same arguments as (2.8)-(2.14), there exist \( y_\epsilon \in \mathbb{R}^3 \) and \( R_0 > 0, \eta > 0 \), and define
\[
w_\beta(x) := \epsilon^2 \varphi_\beta (\epsilon x + \epsilon y_\epsilon),
\]
then \( w_\beta \) satisfies
\[
\|w_\beta\|_2^2 = N_\epsilon, \quad \langle w_\beta, \sqrt{-\Delta} w_\beta \rangle = 1, \quad M \leq \int_{\mathbb{R}^3} \frac{1}{|x|} \ast |w_\beta|^2 |w_\beta|^2 \, dx \leq \frac{1}{M},
\]
for some \( M > 0 \), such that
\[
\liminf_{\epsilon \to 0} \int_{B(0, R_0)} |w_\beta(x)|^2 \, dx \geq \eta > 0.
\]
**Claim 2:** there exists a subsequence \( \{\beta_k\} \) with \( \beta_k \to 0 \) such that \( w_{\beta_k} \to w_0 \) strongly in \( L^2(\mathbb{R}^3) \) for all \( 2 \leq q \leq 3 \) where \( w_0 \) satisfies
\[
w_0 = \gamma^2 Q(\gamma |x - y_0|)
\]
for some \( y_0 \in \mathbb{R}^3, \gamma > 0, \) and \( Q \) satisfies (1.8), we also have \( \|w_0\|_2^2 = N_\epsilon \).

Since \( \varphi_\beta \) is a minimizer of (1.4), it satisfies the Euler-Lagrange equation (1.3), that is
\[
\sqrt{-\Delta} + m^2 \varphi_\beta + \left( \left( \frac{\beta}{|x|^{\alpha}} - \frac{1}{|x|} \right) \ast |\varphi_\beta|^2 \right) \varphi_\beta = \mu_\beta \varphi_\beta,
\]
for \( \mu_\beta \in \mathbb{R} \), which is a suitable Lagrange multiplier. In fact,
\[
\mu_\beta = \frac{1}{N_\epsilon} \left\{ 2E(\beta, N_\epsilon) \frac{1}{2} \int_{\mathbb{R}^3} \frac{1}{|x|} \ast |\varphi_\beta|^2 |\varphi_\beta|^2 \, dx + \frac{\beta}{2} \int_{\mathbb{R}^3} \frac{1}{|x|} \ast |\varphi_\beta|^2 |\varphi_\beta|^2 \, dx \right\}
\]
Then \( w_\beta(x) \) defined in (4.9) satisfies
\[
\sqrt{-\Delta + \epsilon^2 m^2} w_\beta + \epsilon \beta^{1-\alpha} \left( \frac{1}{|x|^{\alpha}} \ast |w_\beta|^2 \right) w_\beta - \left( \frac{1}{|x|} \ast |w_\beta|^2 \right) w_\beta = \epsilon \mu_\beta w_\beta.
\]
Note that By (1.6) and the upper bound of (4.1) we have
\[
\frac{\beta}{4} \int_{\mathbb{R}^3} \frac{1}{|x|} \ast |\varphi_\beta|^2 |\varphi_\beta|^2 \, dx \leq E(\beta, N_\epsilon) \leq C_2 \beta^{1+\alpha}.
\]
Then combining (4.14) and (4.10) we know that \( \epsilon \mu_\beta \) is uniformly bounded and strictly negative for \( \beta \) closes to 0. Passing to a subsequence \( \{\beta_k\} \), we have \( \lim_{\beta_k \to 0^+} \epsilon \mu_{\beta_k} = -\gamma < 0 \), where \( \gamma > 0 \), and \( w_{\beta_k} \to w_0 \) in \( H^{1/2}(\mathbb{R}^3) \) for some \( w_0 \in H^{1/2}(\mathbb{R}^3) \) such that \( w_0 \geq 0 \). Passing to weak limit in (4.15), then \( w_0 \) satisfies
\[
\sqrt{-\Delta} w_0(x) - \left( \frac{1}{|x|} \ast |w_0|^2 \right) w_0 = -\gamma w_0(x).
\]
Moreover, it follows from (4.11) that \( w_0 \neq 0 \). Let \( w_0^\gamma = \gamma^{-3/2} w_0(\gamma^{-1} x) \), then \( w_0^\gamma \) satisfies equation (1.8) such that \( \| w_0^\gamma \|^2_2 = \| w_0 \|^2_2 \). Moreover, by Pohozaev identity,
\[
\langle w_0^\gamma, \sqrt{-\Delta} w_0^\gamma \rangle = \int_{\mathbb{R}^3} |w_0^\gamma|^2 dx = \frac{1}{2} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} \right) * |w_0^\gamma|^2 dx.
\]
Then by (1.7)
\[
\frac{N_c}{2} \leq \frac{\langle w_0^\gamma, \sqrt{-\Delta} w_0^\gamma \rangle \langle w_0^\gamma, w_0^\gamma \rangle}{\int_{\mathbb{R}^3} \left( \frac{1}{|x|} \right) * |w_0^\gamma|^2 dx} = \frac{1}{2} \| w_0^\gamma \|^2_2 = \frac{1}{2} \| w_0 \|^2_2. \tag{4.17}
\]
Note that since \( w_{\beta_k} \to w_0 \) in \( H^{1/2}(\mathbb{R}^3) \), by Fatou lemma, then
\[
\| w_0 \|^2_2 \leq \lim \inf \| w_{\beta_k} \|^2_2 = N_c.
\]
Combining (4.17) we have \( \| w_0 \|^2_2 = N_c \), this implies that \( w_{\beta_k} \) converges to \( w_0 \) strongly in \( L^2(\mathbb{R}^3) \). Since \( w_{\beta_k} \) is uniformly bounded in \( H^{1/2}(\mathbb{R}^3) \), it follows that \( w_{\beta_k} \to w_0 \) in \( L^q(\mathbb{R}^3) \) for \( q \in [2, 3) \). Thus we complete the proof of Claim 2.

Note that
\[
E(\beta_k, N_c) = \frac{1}{2} \langle w_{\beta_k}, \sqrt{-\Delta + \frac{\epsilon^2 m^2}{\epsilon}} w_{\beta_k} \rangle - \frac{1}{4 \epsilon} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} \right) * |w_{\beta_k}|^2 |w_{\beta_k}|^2 dx
\]
\[+ \frac{\beta_k}{4 \epsilon^\alpha} \int_{\mathbb{R}^3} \left( \frac{1}{|x|^\alpha} \right) * |w_{\beta_k}|^2 |w_{\beta_k}|^2 dx
\]
\[\geq \frac{1}{2} \langle w_{\beta_k}, \sqrt{-\Delta + \frac{\epsilon^2 m^2}{\epsilon}} w_{\beta_k} \rangle + \frac{\beta_k}{4 \epsilon^\alpha} \int_{\mathbb{R}^3} \left( \frac{1}{|x|^\alpha} \right) * |w_{\beta_k}|^2 |w_{\beta_k}|^2 dx \quad \text{(by (1.6))}
\]
\[= \frac{\epsilon}{2} \langle w_{\beta_k}, \frac{m^2}{\epsilon} w_{\beta_k} \rangle + \frac{\beta_k}{4 \epsilon^\alpha} \int_{\mathbb{R}^3} \left( \frac{1}{|x|^\alpha} \right) * |w_{\beta_k}|^2 |w_{\beta_k}|^2 dx
\]
\[\geq \frac{\epsilon}{4} \langle w_{\beta_k}, \frac{m^2}{\epsilon} w_{\beta_k} \rangle + \frac{\beta_k}{4 \epsilon^\alpha} \int_{\mathbb{R}^3} \left( \frac{1}{|x|^\alpha} \right) * |w_{\beta_k}|^2 |w_{\beta_k}|^2 dx \quad \text{(for } \epsilon \text{ small enough)}
\]
By Claim 2 we know that \( w_{\beta_k} \to w_0 \) strongly in \( L^q(\mathbb{R}^3) \) where \( w_0 \) is given in (4.12), then there exist constants \( M_1 > 0 \) and \( M_2 > 0 \), independent of \( \beta_k \) such that for \( \beta_k \) small enough
\[
\langle w_{\beta_k}, \frac{m^2}{\epsilon} w_{\beta_k} \rangle \geq M_1, \quad \int_{\mathbb{R}^3} \left( \frac{1}{|x|^\alpha} \right) * |w_{\beta_k}|^2 |w_{\beta_k}|^2 dx \geq M_2,
\]
it follows that
\[
E(\beta, N_c) \geq \frac{\epsilon}{2} M_1 + \frac{\beta_k}{4 \epsilon^\alpha} M_2 \geq C_1 \beta_k^{\frac{1}{1+\alpha}}. \tag{4.18}
\]
Therefore, we get the lower bound. \( \square \)

4.2. The proof of Theorem 1.4.

Lemma 4.2. Under the assumptions of Theorem 1.2(ii), let \( \varphi_\beta(x) \) be a minimizer of \( E(\beta, N_c) \). Then there exist positive constants \( K_1, K_2 \), independent of \( \beta \), such that as \( \beta \to 0^+ \)
\[
K_1 \beta^{-\frac{1}{1+\alpha}} \leq \int_{\mathbb{R}^3} \left( \frac{1}{|x|^\alpha} \right) * |\varphi_\beta|^2 |\varphi_\beta|^2 dx \leq K_2 \beta^{-\frac{1}{1+\alpha}}. \tag{4.18}
\]
Proof. The upper bound follows from (4.1).
To prove the lower bound, we choose a $\beta_1$ satisfying that $\beta_1 = \theta \beta$ with $\theta > 1$. We have
$$E(\beta_1, N_c) \leq E_{\beta_1}(\varphi_\beta) = E(\beta, N_c) + \frac{\beta_1 - \beta}{4} \int_{\mathbb{R}^3} \frac{1}{|x|} * |\varphi_\beta|^2 |\varphi_\beta|^2 \, dx,$$
then
$$\int_{\mathbb{R}^3} \frac{1}{|x|} * |\varphi_\beta|^2 |\varphi_\beta|^2 \, dx \geq \frac{C_1 \beta_1^{\frac{1}{\alpha}} - C_2 \beta^{\frac{1}{\alpha}}}{\beta_1 - \beta} = \frac{C_3 \beta^{-\frac{1}{\alpha}} (\frac{C_2 \theta^{\frac{1}{\alpha}} - \beta}{\theta - 1})}.$$ 
Taking some $\theta$ large enough, we have $C_3 \theta^{\frac{1}{\alpha}} - 1 > 0$, thus the lower bound holds.

The following lemma is to obtain the optimal estimates. The upper bound is harder to come by, we should employ scaling arguments and prove by contradiction, this is quite different from the mentioned papers.

**Lemma 4.3.** Under the assumptions of Theorem 1.2(ii), let $\varphi_\beta(x)$ be a minimizer of $E(\beta, N_c)$. Then there exist positive constants $L_1, L_2, L_3$ and $L_4$, independent of $\beta$, such that as $\beta \to 0^+$
$$L_1 \beta^{-\frac{1}{\alpha}} \leq \langle \varphi_\beta, \sqrt{-\Delta \varphi_\beta} \rangle \leq L_2 \beta^{-\frac{1}{\alpha}}.$$ 
and
$$L_3 \beta^{-\frac{1}{\alpha}} \leq \int_{\mathbb{R}^3} \frac{1}{|x|} * |\varphi_\beta|^2 |\varphi_\beta|^2 \, dx \leq L_4 \beta^{-\frac{1}{\alpha}}.$$ 

**Proof.** Note that by Gagliardo-Nirenberg inequality (1.6) and (4.1) we have
$$0 \leq \frac{1}{2} \langle \varphi_\beta, \sqrt{-\Delta \varphi_\beta} \rangle - \frac{1}{4} \int_{\mathbb{R}^3} \frac{1}{|x|} * |\varphi_\beta|^2 |\varphi_\beta|^2 \, dx \leq E(\beta, N_c) \to 0 \quad \text{as} \quad \beta \to 0^+.$$
Then
$$\langle \varphi_\beta, \sqrt{-\Delta \varphi_\beta} \rangle \to \frac{1}{2} \int_{\mathbb{R}^3} \frac{1}{|x|} * |\varphi_\beta|^2 |\varphi_\beta|^2 \, dx \quad \text{as} \quad \beta \to 0^+.$$
Therefore, it suffices to prove one of (4.19) and (4.20). Next we prove (4.19).

First we prove the lower bound of (4.19). Since $\|\varphi_\beta\|^2 = N_c$, by Hardy-Littlewood Sobolev inequality, interpolation inequality and Sobolev inequality, we have
$$\int_{\mathbb{R}^3} \frac{1}{|x|} * |\varphi_\beta|^2 |\varphi_\beta|^2 \, dx \leq C_1 \|\varphi_\beta\|^\frac{4}{2-\alpha} \leq C_1 \|\varphi_\beta\|^\frac{2\alpha}{3} \|\varphi_\beta\|^2(2-\alpha) \leq C_2 \langle \varphi_\beta, \sqrt{-\Delta \varphi_\beta} \rangle^\alpha.$$
Combining with (4.18) it follows that
$$\langle \varphi_\beta, \sqrt{-\Delta \varphi_\beta} \rangle \geq C_3 \left( \int_{\mathbb{R}^3} \frac{1}{|x|} * |\varphi_\beta|^2 |\varphi_\beta|^2 \, dx \right)^\frac{1}{2} \geq L_1 \beta^{-\frac{1}{\alpha}} \frac{1}{\alpha} \geq L_1 \beta^{-\frac{1}{\alpha}}.$$ 
Thus, the lower bound of (4.19) holds.

Now we prove the upper bound. Define
$$\epsilon_1 := \beta^{\frac{1}{\alpha}}, \quad w_\beta(x) := \epsilon_1^{3/2} \varphi_\beta(\epsilon_1 x).$$ 
To prove the upper bound, it suffices to prove the following fact.
$$\langle w_\beta, \sqrt{-\Delta w_\beta} \rangle \leq L_2.$$ 
On the contrary, up to a subsequence we may assume that as $\beta \to 0^+$
$$\langle w_\beta, \sqrt{-\Delta w_\beta} \rangle \to \infty.$$
Now let
\[ \epsilon_2^{-1} = (w_\beta, \sqrt{-\Delta} w_\beta), \quad \bar{w}_\beta(x) = \epsilon_2^{3/2} w_\beta(\epsilon_2 x) = (\epsilon_2 \epsilon_1)^{3/2} \varphi_\beta(\epsilon_2 \epsilon_1 x). \]
then \( \epsilon_2 \to 0 \) as \( \beta \to 0^+ \) such that
\[ \langle \bar{w}_\beta, \sqrt{-\Delta} \bar{w}_\beta \rangle = 1. \]
Since \( \psi_\beta \) is a minimizer of \( E(\beta, N_\epsilon) \), by Gagliardo-Nirenberg inequality (1.6), (4.1) we have
\[
0 \leq \frac{1}{2} \langle \bar{w}_\beta, \sqrt{-\Delta} \bar{w}_\beta \rangle - \frac{1}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast |\bar{w}_\beta|^2 \right) |\bar{w}_\beta|^2 \, dx \\
= (\epsilon_1 \epsilon_2) \frac{1}{2} \langle \varphi_\beta, \sqrt{-\Delta} \varphi_\beta \rangle - \frac{1}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast |\varphi_\beta|^2 \right) |\varphi_\beta|^2 \, dx \\
\leq (\epsilon_1 \epsilon_2) E(\beta, N_\epsilon) \to 0 \quad \text{as} \quad \beta \to 0.
\]
Then we conclude that there exists a constant \( K > 0 \) independent of \( \beta \)
\[
\frac{1}{K} \leq \int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast |\bar{w}_\beta|^2 \right) |\bar{w}_\beta|^2 \, dx \leq K. \tag{4.25}
\]
The same arguments as (2.8)-(2.15), there exist a sequence \( y_\beta \in \mathbb{R}^3 \) and positive constant \( \eta_0 \) such that
\[ \overline{w}_\beta(x) = \bar{w}_\beta(x + y_\beta) = (\epsilon_2 \epsilon_1)^{3/2} \varphi_\beta(\epsilon_2 \epsilon_1 x + \epsilon_2 \epsilon_1 y_\beta) \tag{4.26} \]
satisfies
\[
\int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast |\overline{w}_\beta|^2 \right) |\overline{w}_\beta|^2 \, dx \geq \eta_0 > 0. \tag{4.27}
\]
On the other hand, notice that \( \epsilon_1 = \beta \frac{1}{1 + \alpha} \), by (4.26) and the upper bound of (4.18) we have
\[
\int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast |w_\beta|^2 \right) |w_\beta|^2 \, dx = (\epsilon_2 \epsilon_1)^{\alpha} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast |\psi_\beta|^2 \right) |\psi_\beta|^2 \, dx \\
\leq K_2 (\epsilon_2 \epsilon_1)^{\alpha} \beta^{-\frac{\alpha}{1+\alpha}} = K_2 (\epsilon_2)^{\alpha}.
\]
Since \( 0 < \alpha < 1 \) it follows that as \( \epsilon_2 \to 0 \)
\[
\int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast |w_\beta|^2 \right) |w_\beta|^2 \, dx \to 0,
\]
which contradicts (4.27). Therefore, the upper bound of (4.19) holds. \( \square \)

**The end proof of Theorem 1.4:** Since \( \varphi_\beta \) is a non-negative minimizer of (1.4), it satisfies the Euler-Lagrange equation (4.13). Define
\[ \epsilon := \beta \frac{1}{1 + \alpha}, \quad w_\beta(x) := \epsilon^{3/2} \varphi_\beta(\epsilon x). \tag{4.28} \]
Then \( \epsilon \to 0^+ \) as \( \beta \to 0^+ \), and \( w_\beta(x) \) satisfies
\[
\sqrt{-\Delta + \epsilon^2 m^2} w_\beta + \beta \epsilon^{1-\alpha} \left( \frac{1}{|x|^\alpha} \ast |w_\beta|^2 \right) w_\beta - \left( \frac{1}{|x|} \ast |w_\beta|^2 \right) w_\beta = \epsilon \mu_\beta w_\beta. \tag{4.29}
\]
Note that by (4.19) and (4.20)
\[
L_1 \leq \langle w_\beta, \sqrt{-\Delta} w_\beta \rangle \leq L_2, \quad L_3 \leq \int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast |w_\beta|^2 \right) |w_\beta|^2 \, dx \leq L_4. \tag{4.30}
\]
Then, the same arguments as the proof of Claim 2 in Lemma 4.1, there exists a subsequence \( \{ \beta_k \} \) with \( \beta_k \to 0^+ \) such that

\[
\omega_{\beta_k} \to \omega_0 = \gamma^2 Q(\gamma(x - y_0)) \quad \text{strongly in } L^q(\mathbb{R}^3) \quad \text{for all } 2 \leq q < 3. \tag{4.31}
\]

where \( y_0 \in \mathbb{R}^3, \gamma > 0 \) (will be given below), and \( Q \) satisfies (1.8), and we also have \( \|w_0\|^2_2 = N_c \).

Note that

\[
E(\beta_k, N_c) = \frac{1}{2} \langle w_{\beta_k}, \sqrt{-\Delta + \epsilon^2 m^2} w_{\beta_k} \rangle - \frac{1}{4\epsilon} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |w_{\beta_k}|^2 \right) |w_{\beta_k}|^2 \, dx \\
+ \frac{\beta_k}{4\epsilon^\alpha} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |w_{\beta_k}|^2 \right) |w_{\beta_k}|^2 \, dx \\
\geq \frac{1}{2} \langle w_{\beta_k}, \sqrt{-\Delta + \epsilon^2 m^2 - \sqrt{-\Delta}} w_{\beta_k} \rangle + \frac{\beta_k}{4\epsilon^\alpha} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |w_{\beta_k}|^2 \right) |w_{\beta_k}|^2 \, dx \quad \text{(by (1.6))}
\]

\[
= \frac{\epsilon}{2} \langle w_{\beta_k}, \sqrt{-\Delta + \epsilon^2 m^2 + \sqrt{-\Delta}} w_{\beta_k} \rangle + \frac{\beta_k}{4\epsilon^\alpha} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |w_{\beta_k}|^2 \right) |w_{\beta_k}|^2 \, dx \\
= \frac{\beta_k}{2} \langle w_{\beta_k}, \sqrt{-\Delta + \epsilon^2 m^2 - \sqrt{-\Delta}} w_{\beta_k} \rangle + \frac{\beta_k}{4\epsilon^\alpha} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |w_{\beta_k}|^2 \right) |w_{\beta_k}|^2 \, dx
\]

(since \( \epsilon = \beta_k^{\frac{1}{2-alpha}} \)).

Since \( w_{\beta_k} \to w_0 \) strongly in \( L^q(\mathbb{R}^3) \) for all \( 2 \leq q < 3 \), then

\[
\liminf_{\beta_k \to 0} \beta_k^{-\frac{1}{2-alpha}} E(\beta_k, N_c) \\
= \liminf_{\beta_k \to 0} \frac{1}{2} \langle w_{\beta_k}, \sqrt{-\Delta + \epsilon^2 m^2 - \sqrt{-\Delta}} w_{\beta_k} \rangle + \frac{1}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |w_{\beta_k}|^2 \right) |w_{\beta_k}|^2 \, dx \\
\geq \frac{1}{4} \langle w_0, \sqrt{-\Delta + \epsilon^2 m^2 - \sqrt{-\Delta}} w_0 \rangle + \frac{1}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |w_0|^2 \right) |w_0|^2 \, dx \\
= \frac{1}{4\gamma} \langle Q, \frac{m^2}{\sqrt{-\Delta}} Q \rangle + \frac{\gamma^{alpha}}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |Q|^2 \right) |Q|^2 \, dx \quad \text{(by (4.31))}
\]

\[
\geq \frac{1}{2} \langle Q, \frac{m^2}{\sqrt{-\Delta}} Q \rangle \frac{g}{2-alpha} \left( \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |Q|^2 \right) |Q|^2 \, dx \right)^{\frac{1}{2-alpha}}.
\]

The last inequality can be obtained by taking the minimum over \( \gamma > 0 \) which is achieved by

\[
\gamma = \left( \frac{\langle Q, \frac{m^2}{\sqrt{-\Delta}} Q \rangle}{\int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |Q|^2 \right) |Q|^2 \, dx} \right)^{\frac{1}{2-alpha}}. \tag{4.32}
\]

On the other hand, taking the test function

\[
\varphi(x) = \left( \frac{\gamma}{\beta_k^{\frac{1}{2-alpha}}} \right)^{3/2} Q(\frac{\gamma}{\beta_k^{\frac{1}{2-alpha}}} x)
\]

into \( E_{\beta_k}(\varphi(x)) \) one can easily check that

\[
\limsup_{\beta_k \to 0} \beta_k^{-\frac{1}{2-alpha}} E(\beta_k, N_c) \leq \frac{1}{2} \langle Q, \frac{m^2}{\sqrt{-\Delta}} Q \rangle \frac{g}{2-alpha} \left( \int_{\mathbb{R}^3} \left( \frac{1}{|x|} * |Q|^2 \right) |Q|^2 \, dx \right)^{\frac{1}{2-alpha}}.
\]
This means $(1.13)$ holds. Combine $(4.31)$ and $(4.32)$, the case $(i)$ of Theorem 1.4 holds.

5. Appendix.

**Lemma 5.1** (Lemma 2.4, [7]). Let $\{\psi_n\}$ be a bound sequence in $H^{1/2}(\mathbb{R}^3)$ such that $N(\psi_n) = \int_{\mathbb{R}^3} |\psi_n|^2 \, dx = N$ for all $n \geq 0$. Then there exists a subsequence, $\{\psi_{n_k}\}$, satisfying one of the following properties:

$(i)$ Compactness: There exists a sequence, $\{y_k\}$, in $\mathbb{R}^3$ such that, for every $\epsilon > 0$, there exists $0 < R < \infty$ with

$$\int_{|x-y_k|<R} |\psi_{n_k}|^2 \, dx \geq N - \epsilon.$$  

$(ii)$ Vanishing:

$$\limsup_{k \to \infty} \int_{|x-y|<R} |\psi_{n_k}|^2 \, dx = 0, \quad \text{for all } R > 0.$$  

$(iii)$ Dichotomy: There exists $\lambda \in (0, N)$ such that, for every $\epsilon > 0$, there exists two bounded sequences, $\{\psi_{n_k}^1\}$ and $\{\psi_{n_k}^2\}$, in $H^{1/2}(\mathbb{R}^3)$ and $k_0 \geq 0$ such that, for all $k \geq k_0$, the following properties holds:

$$\|\psi_{n_k} - (\psi_{n_k}^1 - \psi_{n_k}^2)\|_p \leq \delta_p(\epsilon), \quad \text{for } 2 \leq p < 3$$

with $\delta_p(\epsilon) \to 0$ as $\epsilon \to 0$, and

$$\left| \int_{\mathbb{R}^3} \left| \psi_k^1 \right|^2 \, dx - \lambda \right| \leq \epsilon, \quad \left| \int_{\mathbb{R}^3} \left| \psi_k^2 \right|^2 \, dx - (N - \lambda) \right| \leq \epsilon,$$

$$\text{dist}(\text{supp} \psi_k^1, \text{supp} \psi_k^2) \to \infty, \quad \text{as } k \to \infty.$$  

Moreover, we have that

$$\liminf_{k \to \infty} \left( \langle \psi_{n_k}, T\psi_{n_k} \rangle - \langle \psi_{n_k}^1, T\psi_{n_k}^1 \rangle - \langle \psi_{n_k}^2, T\psi_{n_k}^2 \rangle \right) \geq -C(\epsilon),$$

where $C(\epsilon) \to 0$ as $\epsilon \to 0$ and $T := (\sqrt{-\Delta + m^2} - m)$ with $m \geq 0$.

**Lemma 5.2.** Let $\{\psi_n\}$ satisfy the assumptions of Lemma 2.4 in [7]. Furthermore, suppose that there exists a subsequence, still denoted by $\{\psi_n\}$, that satisfies part $(ii)$ of Lemma 2.4. Then for all $0 < \theta < 2$

$$\lim_{n \to \infty} \int_{\mathbb{R}^3} \left( \frac{1}{|x|^\theta} \ast |\psi_n|^2 \right) |\psi_n|^2 \, dx = 0;$$

**Proof.** The same arguments as Lemma A.1 of [7], we can prove this lemma, we omit the detail here.

**Lemma 5.3.** Suppose that $\epsilon > 0$. Let $\{\psi_n\}$ satisfy the assumptions of Lemma 5.1 and let $\{\psi_{n_k}\}$ be a subsequence that satisfies part $(iii)$ with sequences $\{\psi_{n_k}^1\}$ and $\{\psi_{n_k}^2\}$. Then for any $0 < \theta < 2$

$$\left| \int_{\mathbb{R}^3} \left( \frac{1}{|x|^\theta} \ast |\psi_{n_k}|^2 \right) |\psi_{n_k}|^2 \, dx - \int_{\mathbb{R}^3} \left( \frac{1}{|x|^\theta} \ast |\psi_{n_k}^1|^2 \right) |\psi_{n_k}^1|^2 \, dx - \int_{\mathbb{R}^3} \left( \frac{1}{|x|^\theta} \ast |\psi_{n_k}^2|^2 \right) |\psi_{n_k}^2|^2 \, dx \right|$$

$$\leq r_1^\theta(k) + r_2^\theta(\epsilon),$$

for $k$ sufficiently large, where $r_1^\theta(k) \to 0$ as $k \to \infty$ and $r_2^\theta(\epsilon) \to 0$ as $\epsilon \to 0$. 

Note that by Hardy-Littlewood-Sobolev inequality, interpolation inequality and Sobolev inequality, we have
\[
\int_{\mathbb{R}^3} \left( \frac{1}{|x|^q} \ast |\psi|^2 \right) |\psi|^2 \, dx \leq C \| \psi \|^{12}_4 \leq C_1 \| \psi \|^{2\theta}_3 \| \psi \|^{2(2-\theta)}_2 \leq C_2 (\psi, \sqrt{-\Delta} \psi)^{\theta}.
\]

Notice that \(2 < \frac{12}{5} < 3\) including in the range of \(p\) in (5.3). Then the same arguments as Lemma A.2 of [7], the lemma holds. We omit the detail arguments here. \(\square\)

5.1. Some properties of the energy \(E(\beta, N)\).

**Lemma 5.4.** Suppose that \(m > 0\) and \(\beta \in \mathbb{R}\).

(I) When \(0 < N < N_c\), we have

(i) If \(\beta \leq 0\), then \(E(\beta, N) < \frac{1}{2}mN\) and \(E(\beta, N)\) is strictly decreasing in \(N\).

(ii) If \(\beta > 0\), then \(E(\beta, N) < \frac{1}{2}mN\) for \(\beta\) small enough and \(N\) closes to \(N_c\) enough.

(II) When \(N = N_c\), we have

(iii) If \(\beta \leq 0\), then \(E(0, N_c) = 0\), and \(E(\beta, N_c) = -\infty\) for all \(\beta < 0\);

(iv) If \(\beta > 0\), there exists a constant \(\beta_*>0\) such that \(E(\beta, N_c) < \frac{1}{2}mN_c\) for \(\beta \in (0, \beta_*)\).

(III) When \(N > N_c\), then for any \(\beta \in \mathbb{R}\), \(E(\beta, N) = -\infty\).

**Proof.** Note that, by (2.24) of [7] we know that \(E(0, N) - \frac{1}{2}mN < 0\) for \(0 < N < N_c\), in fact, \(E(0, N) - \frac{1}{2}mN\) is equal to \(E_0(N)\) of [7] by taking \(v = 0\). On the other hand, it is easy to see that \(E(\beta, N) \leq E(0, N)\) for \(\beta < 0\) and \(0 < N < N_c\). Following the same arguments as Lemma 2.3 of [7], one can easy to check that \(E(\beta, N)\) is strictly decreasing. Thus we obtain the case (i) of this lemma.

To prove the case (ii), now let \(Q^\lambda = \lambda^{3/2} Q(\lambda x)\) with \(\lambda > 0\), where \(Q\) is an optimizer of (1.6). One can check that \(Q^\lambda\) also satisfies (1.9). We have

\[
E(\beta, N) \leq \mathcal{E}_\beta \left( \frac{N}{N_c} \cdot Q^\lambda \right)
\]

\[
= \frac{N}{N_c} \left\{ \frac{1}{2} (Q^\lambda, \sqrt{-\Delta + m^2} Q^\lambda) - \frac{N}{N_c} \cdot \frac{1}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast |Q^\lambda|^2 \right) |Q^\lambda|^2 \, dx \right.
+ \left. \frac{N}{N_c} \cdot \frac{\beta}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|^\alpha} \ast |Q^\lambda|^2 \right) |Q^\lambda|^2 \, dx \right\}
\]

\[
= \frac{N}{N_c} \left\{ \frac{1}{2} (Q^\lambda, \sqrt{-\Delta + m^2} - \sqrt{-\Delta}) Q^\lambda) + (1 - \frac{N}{N_c}) \cdot \frac{1}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast |Q^\lambda|^2 \right) |Q^\lambda|^2 \, dx \right.
+ \left. \frac{N}{N_c} \cdot \frac{\beta}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|^\alpha} \ast |Q^\lambda|^2 \right) |Q^\lambda|^2 \, dx \right\} \quad \text{(obtained by (1.9))}
\]

\[
\leq \frac{N}{N_c} \left\{ \frac{m^2}{\lambda} \int_{\mathbb{R}^3} |Q(x)|^2 \, dx + \frac{\lambda(N_c - N)}{4N_c} \int_{\mathbb{R}^3} \left( \frac{1}{|x|} \ast |Q|^2 \right) |Q|^2 \, dx \right.
+ \left. \frac{N}{N_c} \cdot \frac{\beta \alpha}{4} \int_{\mathbb{R}^3} \left( \frac{1}{|x|^\alpha} \ast |Q|^2 \right) |Q|^2 \, dx \right\}. \quad \text{(since \(\sqrt{-\Delta + m^2} - \sqrt{-\Delta} \leq \frac{m^2}{2\sqrt{-\Delta}}\))}
\]

Since \(N_c - N > 0\) and \(\beta > 0\), now we take

\[
\lambda = \frac{1}{(N_c - N)^{\frac{1}{2}} + \frac{1}{\beta + 1}}.
\]
then

\[(N_c - N)\lambda = \frac{N_c - N}{(N_c - N)^{\frac{1}{2}} + \beta^{\frac{1}{1+\alpha}}} \leq (N_c - N)^{\frac{1}{2}};\]

\[\beta^\alpha = \frac{\beta}{[(N_c - N)^{\frac{1}{2}} + \beta^{\frac{1}{1+\alpha}}]^\alpha} \leq \beta^{\frac{1}{1+\alpha}}.\]

It follows that there exist \(C_1 > 0\) and \(C_2 > 0\) independent of \(N\) and \(\beta\) such that

\[E(\beta, N) \leq C_1(N_c - N)^{1/2} + C_2\beta^{\frac{1}{1+\alpha}}.\] (5.10)

Then, when \(N\) closes to \(N_c\) and \(\beta\) small enough, we have \(E(\beta, N) < \frac{1}{2}mN\). This completes the proof of case (ii).

To prove the case (iii) and (iv), the same as (5.9) and let \(N = N_c\) in (5.9), we have

\[E(\beta, N_c) \leq \frac{m^2}{\lambda} \int_{\mathbb{R}^3} \frac{1}{4\sqrt{-\Delta}} |Q(x)|^2 \, dx + \frac{\beta^\alpha}{4} \int_{\mathbb{R}^3} \frac{1}{|x|^\alpha} |Q(x)|^2 \, dx.\] (5.11)

Since \(\beta \leq 0\) in case (iii), just let \(\lambda \to \infty\) in (5.11), we can obtain case (iii).

To prove case (iv), since \(\beta > 0\), take the infimum over \(\lambda\) in (5.11), then

\[E(\beta, N_c) \leq C\beta^{\frac{1}{1+\alpha}}.\] (5.12)

It follows that for \(\beta\) small enough, \(C\beta^{\frac{1}{1+\alpha}} < \frac{1}{2}mN_c\), i.e., \(E(\beta, N_c) < \frac{1}{2}mN_c\). Thus we complete the proof of case (iv).

To prove the case (III), the same as (5.9) and let \(N > N_c\) in (5.9). Note that \(\frac{\beta(N_c - N)}{4N_c} < 0\) and \(0 < \alpha < 1\), let \(\lambda \to \infty\), then (5.9) \(\to -\infty\), thus \(E(\beta, N) = -\infty\).

**Lemma 5.5.** The following strictly binding inequality

\[E(\beta, N) < E(\beta, \lambda) + E(\beta, N - \lambda)\] (5.13)

holds for any \(0 < \lambda < N\), when \(N\) and \(\beta\) satisfy one of the three conditions.

(i) \(0 < N < N_c\) and \(\beta \leq 0\);

(ii) \(0 < N < N_c\) and \(\beta > 0\), \(\beta\) small enough and \(N\) closes to \(N_c\) enough;

(iii) \(N = N_c\), \(\beta > 0\) small enough.

**Remark 3.** The condition (i), (ii), (iii) here correspond to the assumptions of (i), (ii), (iv) in Lemma 5.4, which guarantee the energy \(E(\beta, N) < \frac{1}{2}mN\). We mention that in [7, Lemma 2.3], the authors showed that such binding inequality holds for \(0 < N < N_c\), in what follows, we prove that such inequality also holds at the threshold \(N = N_c\).

**Proof.** In what follows we only prove for condition (iii), for (i) and (ii) one can just follow the same arguments.

Now let \(\bar{E}(\beta, N) := E(\beta, N) - \frac{1}{2}mN\), then (5.13) is equivalent to

\[\bar{E}(\beta, N) < \bar{E}(\beta, \lambda) + \bar{E}(\beta, N - \lambda)\] (5.14)

for any \(0 < \lambda < N\). Next we will prove (5.14) for case (iii).

Note that, for any \(\epsilon > 0\), there exists \(v \in H^{1/2}(\mathbb{R}^3)\), \(||v||^2 = N\) such that \(\mathcal{E}_\beta(v) \leq E(\beta, N) + \epsilon\). Hence for any \(\theta > 1\)

\[\frac{\bar{E}(\beta, N)}{2} = E(\beta, N) - \frac{\epsilon}{2}mN \leq \mathcal{E}_\beta(\sqrt{\theta}v) - \frac{\theta}{2}mN\]

\[= (\theta - \theta^2)\frac{1}{2}v, (\sqrt{-\Delta + m^2} - m) v) + \theta^2(\mathcal{E}_\beta(v) - \frac{1}{2}mN)\]
\begin{align*}
&\leq \theta^2(E_\beta(v) - \frac{1}{2}mN) \leq \theta^2(\tilde{E}(\beta, N) + \epsilon).
\end{align*}

First we claim that for any $0 < \lambda < N_c$

\begin{align*}
\tilde{E}(\beta, N_c) < \frac{N_c}{\lambda} \tilde{E}(\beta, \lambda).
\end{align*}

Indeed, if $\tilde{E}(\beta, \lambda) \geq 0$, then (5.16) holds obviously since $\tilde{E}(\beta, N_c) < 0$ for $\beta$ small enough (by Lemma 5.4(iv)). If $\tilde{E}(\beta, \lambda) < 0$, choose $\theta = \frac{N_c}{\lambda}$ and $N = \lambda$ in inequality (5.15), and let $\epsilon < (\theta^{-1} - 1)\tilde{E}(\beta, \lambda)$, it follows that (5.16) holds. Thus, the claim holds.

In the same way we have

\begin{align*}
\tilde{E}(\beta, N_c) < \frac{N_c}{N_c - \lambda} \tilde{E}(\beta, N_c - \lambda).
\end{align*}

Combining (5.16) with (5.17), then (5.14) holds for condition (iii).

\section*{5.2. Local and global well-posedness.}
In this part, we consider the following Cauchy problem

\begin{align*}
\begin{cases}
i \partial_t \psi = -\Delta + m^2 \psi + \beta \left( \frac{1}{|x|^{\alpha}} * |\psi|^2 \right) \psi - \left( \frac{1}{|x|} * |\psi|^2 \right) \psi & \text{on } \mathbb{R}^3, \\
\psi(0, x) = \psi_0(x),
\end{cases}
\end{align*}

where $\psi : [0, T) \times \mathbb{R}^3 \rightarrow \mathbb{C}$ is a complex valued function, $\psi_0 \in H^{1/2}$, $0 < T \leq \infty$, $0 < \alpha < 1$, $\beta \in \mathbb{R}$. We will prove the local well posedness and global well-posedness.

Set

\begin{align*}
A := -\Delta + m^2, \quad F(u) := \beta \left( \frac{1}{|x|^{\alpha}} * |u|^2 \right) u - \left( \frac{1}{|x|} * |u|^2 \right) u.
\end{align*}

For $s \in \mathbb{R}$, let

\begin{align*}
D^s := (-\Delta)^{\frac{s}{2}}
\end{align*}

From [14], we know that to prove the local well posedness, we only need to show the nonlinearity $F(u)$ is locally Lipschitz continuous from $H^{1/2}(\mathbb{R}^3)$ into itself. Notice that, Lemma 1 of [14] has shown that $(\frac{1}{|x|} * |u|^2)u$ is locally Lipschitz continuous, it is sufficiently to prove for $(\frac{1}{|x|^{\alpha}} * |u|^2)u$.

In what follows, $a \leq b$ denotes $a \leq Cb$ for some appropriate constant $C > 0$. Now we show the following key estimates.

\begin{lemma}
Suppose that $0 < \alpha < 1$, for any $u, v \in H^{1/2}(\mathbb{R}^3)$, we have

\begin{align*}
\left\| \frac{1}{|x|^{\alpha}} * |u|^2 \right\|_\infty & \leq \| u \|_{H^{1/2}}^2 \quad (5.21)
\end{align*}

and

\begin{align*}
\left\| \frac{1}{|x|^{\alpha}} * (|u|^2 - |v|^2) \right\|_\infty & \leq (\| u \|_{H^{1/2}} + \| v \|_{H^{1/2}}) \| u - v \|_{H^{1/2}},
\end{align*}

\begin{align*}
\left\| D^{1/2} \left( \frac{1}{|x|^{\alpha}} * (|u|^2 - |v|^2) \right) \right\|_6 & \leq (\| u \|_{H^{1/2}} + \| v \|_{H^{1/2}}) \| u - v \|_{H^{1/2}}.
\end{align*}

\end{lemma}
5.21 Suppose that for all \( u, v \in D \) Lipschitz continuous from \( H \).

Notice that by the definition of Riesz potential (see \( \star \)),

Using the Sobolev inequality \( \| u \|_3 \leq \| u \|_{H^{1/2}} \), then it follows that \( (5.21) \) holds.

To prove \( (5.22) \), notice that

\[
\| 1/|x|^{\alpha} \ast (|u|^2 - |v|^2) \|_\infty \\
= \sup_{y \in \mathbb{R}^3} \left\| \int_{\mathbb{R}^3} \frac{|u(x)|^2 - |v(x)|^2}{|x-y|^{\alpha}} \, dx \right\|
\leq \left( \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^2 - |v(x)|^2}{|x-y|^{\alpha}} \, dx \right)^{1/2} \left( \sup_{y \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - v(x)|^2}{|x-y|^{\alpha}} \, dx \right)^{1/2}
\leq \| u + v \|_{H^{1/2}} \| u - v \|_{H^{1/2}}
\leq (\| u \|_{H^{1/2}} + \| v \|_{H^{1/2}}) \| u - v \|_{H^{1/2}}.
\]

By weak Young inequality (see [18], the norm \( \| \cdot \|_{p,w} \) is given in notation above)

\[
\| 1/|x|^{\alpha} \ast (|u|^2 - |v|^2) \|_6 \\
\leq \| 1/|x|^{\alpha} \|_{\frac{6}{\alpha},w} \| u^2 - |v|^2 \|_{\frac{6}{\alpha}}
\leq \| u + v \|_{\frac{6}{\alpha},w} \| u - v \|_2
\leq (\| u \|_{H^{1/2}} + \| v \|_{H^{1/2}}) \| u - v \|_2.
\]

Notice that by the definition of Riesz potential (see [26], \( \frac{6}{\alpha},w \ast f \) can be expressed as \( D^{\alpha-3} f = (-\Delta)^{\frac{\alpha-3}{2}} f \) (here \( f \in \mathcal{S}(\mathbb{R}^3) \) is initially assumed, but our arguments follow by density). Thus, we have

\[
\| \frac{1}{|x|^{\alpha}} \ast (|u|^2 - |v|^2) \|_6 \\
\leq \| \frac{1}{|x|^{\alpha+\frac{\alpha-3}{2}}} \|_{\frac{6}{\alpha},w} \| u^2 - |v|^2 \|_{\frac{6}{\alpha}}
\leq (\| u \|_{H^{1/2}} + \| v \|_{H^{1/2}}) \| u - v \|_{H^{1/2}}.
\]

Lemma 5.7. Suppose that \( 0 < \alpha < 1 \). The map \( J(u) := \frac{1}{|x|^{\alpha}} \ast |u|^2 \) is locally Lipschitz continuous from \( H^{1/2}(\mathbb{R}^3) \) into itself with

\[
\| J(u) - J(v) \|_{H^{1/2}} \leq (\| u \|^2_{H^{1/2}} + \| v \|^2_{H^{1/2}}) \| u - v \|_{H^{1/2}},
\]

for all \( u, v \in H^{1/2}(\mathbb{R}^3) \).
Proof. With the estimates given in Lemma 5.6, we can prove this lemma by following the similar argument as Lemma 1 in [14]. Now we sketch the proof.

The same arguments as [14], we know that
\[ \|u\|_2 + \|D^{1/2}u\|_2 \leq \|u\|_{H^{1/2}} \leq \|u\|_2 + \|D^{1/2}u\|_2 \]
it is sufficient to estimate the quantities
\[ I := \|J(u) - J(v)\|_2, \quad II := \|D^{1/2}[J(u) - J(v)]\|_2. \]

The same argument as (15) in [14], by Hölder inequality
\[ I \leq \left\| \left( \frac{1}{|x|^\alpha} * (|u|^2 - |v|^2) \right) (u + v) \right\|_2 + \left\| \left( \frac{1}{|x|^\alpha} * (|u|^2 + |v|^2) \right) (u - v) \right\|_2 \]
\[ \leq \frac{1}{|x|^\alpha} \| (|u|^2 - |v|^2) \|_{\infty} \|u + v\|_{\frac{3}{2}} + \frac{1}{|x|^\alpha} \| (|u|^2 + |v|^2) \|_{\infty} \|u - v\|_2. \] (5.25)

Then by (5.21), (5.23) and together with Sobolev inequality \|u\|_q \leq \|u\|_{H^{1/2}} (for all \(2 \leq q \leq 3\)) then
\[ I \leq (\|u\|_{H^{1/2}}^2 + \|u\|_{H^{1/2}}^2) \|u - v\|_{H^{1/2}}. \]

On the other hand, as (18) in [14], using Leibnitz rule we also have
\[ II \leq \|D^{1/2} \left( \frac{1}{|x|^\alpha} * (|u|^2 - |v|^2) \right) \|_{6} \|u + v\|_3 + \frac{1}{|x|^\alpha} \| (|u|^2 - |v|^2) \|_{\infty} \|D^{1/2} (u + v)\|_2 \]
\[ + \|D^{1/2} \left( \frac{1}{|x|^\alpha} * (|u|^2 + |v|^2) \right) \|_{6} \|u - v\|_3 + \frac{1}{|x|^\alpha} \| (|u|^2 + |v|^2) \|_{\infty} \|D^{1/2} (u - v)\|_2. \]

By (5.22) and (5.24), then
\[ II \leq (\|u\|_{H^{1/2}}^2 + \|u\|_{H^{1/2}}^2) \|u - v\|_{H^{1/2}}. \]

Thus we complete the proof. \( \Box \)

Therefore, the nonlinearity \(F(u)\) is local Lipschitz continuous, by standard methods for evolution equations with locally Lipschitz nonlinearities, we have following local well-posedness theorem.

**Theorem 5.8 (Local well-posedness).** Let \(m > 0, 0 < \alpha < 1, \beta \in \mathbb{R}\). Then initial value problem (5.18) is locally well-posed in \(H^{1/2}(\mathbb{R}^3)\). This means that, for every \(\psi_0 \in H^{1/2}(\mathbb{R}^3)\), there exists a unique solution
\[ u \in C^0([0,T); H^{1/2}(\mathbb{R}^3)) \cap C^1([0,T); H^{-1/2}(\mathbb{R}^3)), \]
and it depends continuously on \(u_0\). Here \(T \in (0, \infty)\) is the maximal time of existence, where we have that either \(T = \infty\) or \(T < \infty\) and \(\lim_{t \to T} \|u\|_{H^{1/2}} = \infty\) holds.

**Lemma 5.9 (Conservation Laws).** The local-in-time solutions of Theorem 5.8 obey conservation of energy and charge, i.e.,
\[ \mathcal{E}_\beta(\psi(t)) = \mathcal{E}_\beta(\psi_0) \quad \text{and} \quad \|\psi(t)\|_2^2 = \|\psi_0\|_2^2, \]
for all \(t \in [0,T)\).

Proof. Let \(F(u)\) given in (5.19) take the place of \(F(u)\) in the proof of Lemma 2 in [14], and combine Lemma 5.7, then following the same arguments as [14] one can easy to check this lemma. \( \Box \)

For global well-posedness of Cauchy problem for equation (1.1), We have
Theorem 5.10 (Global well-posedness). Suppose that $m > 0$, $\beta \in \mathbb{R}$ and $0 < \alpha < 1$. Then the unique solution of (1.1) is global in time (i.e., $T = \infty$), provided that one of the following conditions for initial value $\psi_0 \in H^{1/2}(\mathbb{R}^3)$ holds:

(i) $0 < \|\psi_0\|^2_2 < N_c$;
(ii) $\|\psi_0\|^2_2 = N_c$ and $\beta > 0$.

Proof. By the blow-up alternative of Theorem 5.8, global-in-time existence follows from an priori bound the form

$$\|\psi(t)\|_{H^{1/2}} \leq C(\psi_0), \quad t \in [0, T).$$

(5.26)

Notice that by Lemma 5.9, we have

$$\mathcal{E}_\beta(\psi(t)) = \mathcal{E}_\beta(\psi_0), \quad \|\psi(t)\|^2_2 = \|\psi_0\|^2_2$$

(5.27)

Thus, it is sufficient to prove that $\langle \psi(t), \sqrt{-\Delta} \psi(t) \rangle$ is uniformly bounded.

For condition (i) in Theorem 5.10, if $\beta > 0$, the same as (2.2)

$$\mathcal{E}_\beta(\psi_0) = \mathcal{E}_\beta(\psi(t)) \geq \frac{1}{2} (1 - \frac{\|\psi_0\|^2_2}{N_c}) \langle \psi(t), \sqrt{-\Delta} \psi(t) \rangle;$$

(5.28)

if $\beta \leq 0$, the same as (2.3)

$$\mathcal{E}_\beta(\psi_0) = \mathcal{E}_\beta(\psi(t)) \geq \frac{1}{2} (1 - \frac{\|\psi_0\|^2_2}{N_c}) \langle \psi(t), \sqrt{-\Delta} \psi(t) \rangle + \beta C_2 \langle \psi(t), \sqrt{-\Delta} \psi(t) \rangle^\alpha.$$  

(5.29)

Since $\|\psi_0\|^2_2 < N_c$ and $0 < \alpha < 1$, the above two inequality show that $\langle \psi(t), \sqrt{-\Delta} \psi(t) \rangle$ is uniformly bounded. Thus we complete the global well-posedness for condition (i).

Now we go to prove case (ii). In this case, note that $\langle \psi(t), \sqrt{-\Delta} \psi(t) \rangle$ can not be controlled by $\mathcal{E}_\beta(\psi_0)$ like (5.28) and (5.29) due to $\|\psi_0\|^2_2 = N_c$. To overcome it, we use the blow-up analysis and prove by contradiction. On the contrary, we now suppose that $\langle \psi(t), \sqrt{-\Delta} \psi(t) \rangle$ is unbounded. Then there exists a subsequence $\psi_n := \psi(t_n)$ with $t_n \rightarrow T$ (as $n \rightarrow \infty$), such that

$$\langle \psi_n, \sqrt{-\Delta} \psi_n \rangle \rightarrow +\infty, \quad (n \rightarrow \infty)$$

(5.30)

Since $\sqrt{-\Delta + m^2} \geq \sqrt{-\Delta}$, by (1.6) and conservation laws we have

$$\frac{\beta}{4} \int_{\mathbb{R}^3} \frac{1}{|x|^\alpha} |\psi_n|^2 dx \leq \mathcal{E}_\beta(\psi_n) = \mathcal{E}_\beta(\psi_0),$$

(5.31)

and also

$$0 \leq \frac{1}{2} \langle \psi_n, \sqrt{-\Delta} \psi_n \rangle - \frac{1}{4} \int_{\mathbb{R}^3} \frac{1}{|x|} |\psi_n|^2 dx \leq \mathcal{E}_\beta(\psi_0).$$

(5.32)

Note that, (5.30)-(5.32) shows the similar results as (2.5)-(2.7). Then the same arguments as (2.8)-(2.16), one can obtain a contradiction. Therefore, $\langle \psi(t), \sqrt{-\Delta} \psi(t) \rangle$ is uniformly bounded, we complete the theorem.

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REFERENCES

[1] S. Cingolani and S. Secchi, Ground states for the pseudo-relativistic Hartree equation with external potential, Proc. Roy. Soc. Edinb. A, 145 (2015), 73–90.
[2] Y. Cho and T. Ozawa, On the semi-relativistic Hartree type equation, SIAM J. Math. Anal., 38 (2006), 1060–1074.
[3] Y. Deng, Y. Guo and L. Lu, On the collapse and concentration of Bose-Einstein condensates with inhomogeneous attractive interactions, Calc. Var. Partial Differ. Equ., 54 (2015), 99–118.
[4] A. Elgart and B. Schlein, Mean field dynamics of boson stars, Commun. Pure Appl. Math., 60 (2007), 500–545.
[5] B. Feng, On the blow-up solutions for the nonlinear Schrödinger equation with combined power-type nonlinearities, J. Evol. Equ., 18 (2018), 203–220.
[6] R. C. Fetecau, Y. Huang and T. Kolokolnikov, Swarm dynamics and equilibria for a nonlocal aggregation model, Nonlinearity, 24 (2011), 2681–2716.
[7] J. Fröhlich, B. Lars, G. Jonsson and E. Lenzmann, Boson stars as solitary waves, Commun. Math. Phys., 274 (2007), 1–30.
[8] J. Fröhlich and E. Lenzmann, Blowup for nonlinear wave equations describing boson stars, Comm. Pure Appl. Math., 60 (2007), 1691–1705.
[9] Y. Guo and R. Seiringer, On the Mass concentration for Bose-Einstein condensation with attractive interactions, Lett. Math. Phys., 104 (2014), 141–156.
[10] Y. Guo and X. Zeng, Ground states of pseudo-relativistic boson stars under the critical stellar mass, Ann. I. H. Poincaré, 34 (2017), 1611–1632.
[11] Y. Guo, X. Zeng and H. Zhou, Energy estimates and symmetry breaking in attractive Bose-Einstein condensates with ring-shaped potentials, Ann. Inst. H. Poincaré, 33 (2016), 809–828.
[12] S. Herr and E. Lenzmann, The Boson star equation with initial data of low regularity, Nonlinear Anal., 97 (2014), 125–137.
[13] D. Holm and V. Putkaradze, Formation of clumps and patches in selfaggregation of finite-size particles, Phys. D, 220 (2006), 183–196.
[14] E. Lenzmann, Well-posedness for semi-relativistic Hartree equations of critical type, Math. Phys. Anal. Geom., 10 (2007), 43–64.
[15] E. Lenzmann, Uniqueness of ground states for pseudo-relativistic Hartree equations, Anal. Partial Differ. Equ., 2 (2009), 1–27.
[16] E. Lenzmann and M. Lewin, On singularity formation for the $L^2$-critical Boson star equation, Nonlinearity, 24 (2011), 3515–3540.
[17] E. H. Lieb and H. T. Yau, The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics. Commun. Math. Phys., 112 (1987), 147–174.
[18] E. H. Lieb and M. Loss, Analysis 2ed. Grad. Stud. Math., Amer. Math. Soc., 2001.
[19] X. Luo, Normalized standing waves for the Hartree equations, J. Differ. Equ., 267 (2019), 4493–4524.
[20] A. Michelangeli and B. Schlein, Dynamical collapse of boson stars, Commun. Math. Phys., 311 (2012), 645–687.
[21] D. T. Nguyen, On Blow-up Profile of Ground States of Boson Stars with External Potential, J. Stat. Phys., 169 (2017), 395–422.
[22] F. Pusateri, Modified Scattering for the Boson Star Equation, Commun. Math. Phys., 332 (2014), 1203–1234.
[23] Q. Shi and C. Peng, Well-posedness for semirelativistic Schrödinger equation with power-type nonlinearity, Nonl. Anal., 178 (2019), 133–144.
[24] N. Soave, Normalized ground states for the NLS equation with combined nonlinearities, J. Differ. Equ., 269 (2020), 6941–6987.
[25] N. Soave, Normalized ground states for the NLS equation with combined nonlinearities: the Sobolev critical case, J. Funct. Anal., 279 (2020), 1–43.
[26] E. M. Stein, Singular Integrals and Differentiability Properties of Functions, Princeton University Press, 1970.
[27] T. Tao, M. Visan and X. Zhang, The nonlinear Schrödinger equation with combined power-type nonlinearities, Commun. Partial Differ. Equ., 32 (2007), 1281–1343.
[28] C. Topaz, A. Bertozzi and M. Lewis, A nonlocal continuum model for biological aggregation, Bull. Math. Biol., 68 (2006), 1601–1623.
[29] Q. Wang and D. Zhao, Existence and mass concentration of 2D attractive Bose-Einstein condensates with periodic potentials, *J. Differ. Equ.*, 262 (2017), 2684–2704.

[30] J. Yang and J. Yang, Existence and mass concentration of pseudo-relativistic Hartree equation, *J. Math. Phys.*, 58 (2017), 1–22.

[31] V. C. Zelati and M. Nolasco, Ground states for pseudo-relativistic Hartree equations of critical type, *Rev. Mat. Ibero.*, 29 (2013), 1421–1436.

[32] X. Zeng and L. Zhang, Normalized solutions for Schrödinger-Poisson-Slater equations with unbounded potentials, *J. Math. Anal. Appl.*, 452 (2017), 47–61.

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