A CARTAN DECOMPOSITION FOR
NON-SYMMETRIC REDUCTIVE SPHERICAL PAIRS
OF RANK-ONE TYPE AND ITS APPLICATION TO
VISIBLE ACTIONS

ATSUMU SAKI

Abstract. A Cartan decomposition for symmetric pairs plays an important role to study not only orbit geometry of the symmetric spaces but also harmonic analysis on them. On the other hand, there is no analogue of such a Lie group decomposition for non-symmetric reductive pairs. In this context, this paper provides new examples of a Cartan decomposition for non-symmetric reductive pairs, namely, reductive non-symmetric spherical pairs of rank-one type. We also show that the action of some compact group on a non-symmetric reductive spherical homogeneous space of rank-one type is strongly visible.

1. Introduction

Let $G$ be a connected real semisimple Lie group and $H$ a connected closed subgroup of $G$ which is reductive in $G$. We take a Cartan involution $\theta$ of $G$ satisfying $\theta(H) = H$ and set $K := G^\theta$. Then, $K$ is a maximal compact subgroup of $G$. Let $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{h}_0$ be the Lie algebras of $G$, $K$ and $H$, respectively. We write $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ for the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}_0$ and $\mathfrak{g}_0 = \mathfrak{h}_0 + \mathfrak{q}_0$ for the direct sum decomposition with respect to the Killing form of $\mathfrak{g}_0$.

Then, $\mathfrak{g}_0$ is decomposed into the direct sum as follows:

$$\mathfrak{g}_0 = \mathfrak{k}_0 \cap \mathfrak{h}_0 + \mathfrak{k}_0 \cap \mathfrak{q}_0 + \mathfrak{p}_0 \cap \mathfrak{h}_0 + \mathfrak{p}_0 \cap \mathfrak{q}_0.$$ 

In this setting, if there exists an abelian subspace $\mathfrak{a}_0$ in $\mathfrak{p}_0 \cap \mathfrak{q}_0$ such that

$$(1.1) \quad G = K(\exp \mathfrak{a}_0)H,$$

then we say that (1.1) is a Cartan decomposition for the pair $(G, H)$, or for the homogeneous space $G/H$.
If $H$ coincides with $K$, then (1.1) is an usual Cartan decomposition. In the case where $(G, H)$ is a symmetric pair ($H$ is not necessary compact), the decomposition (1.1) holds due to Flensted-Jensen [3] and Rossmann [12]. However, if we drop off the assumption that $(G, H)$ is a symmetric pair, then there is no analogue of decomposition theorems such as (1.1). On the other hand, we expect that there exists an abelian subspace $a_0$ in $p_0 \cap q_0$ with (1.1) when $(G, H)$ has some nice property in connection with finite multiplicities of representation even though $(G, H)$ is not a symmetric pair, namely, it is a real spherical pair (see [7] and references therein). In fact, some examples of (1.1) have been found in the case where a reductive pair is a line bundle case, namely, the corresponding homogeneous space is a line bundle over the complexification of an irreducible Hermitian symmetric space of non-tube type (see [14, 17]).

From this viewpoint, the author is studying a Cartan decomposition (1.1) for reductive spherical pairs. This class contains complex symmetric pairs. Then, our interest is non-symmetric spherical ones.

Here is a quick review on reductive spherical pairs. Let $G_C$ be a connected complex semisimple Lie group and $H_C$ a complex closed subgroup of $G_C$. The pair $(G_C, H_C)$ is called spherical, or the complex homogeneous space $G_C/H_C$ is spherical, if a Borel subgroup of $G_C$ has an open orbit in $G_C/H_C$. The classification of reductive spherical pairs $(G_C, H_C)$, namely, it is spherical and $H_C$ is a reductive Lie group, has been given by Krämer [8] when $G_C$ is simple and by Brion [2], Mykytyuk [11] when $G_C$ is semisimple.

In this paper, we consider reductive spherical pairs $(G_C, H_C)$ which are of rank-one type, namely, $(G_C, H_C)$ satisfies one of Table 1.1.

| Type | $G_C$            | $H_C$         |
|------|------------------|---------------|
| R-1  | $SO(7, \mathbb{C})$ | $G_2(\mathbb{C})$ |
| R-1’ | Spin$(7, \mathbb{C})$ | $G_2(\mathbb{C})$ |
| R-2  | $G_2(\mathbb{C})$  | $SL(3, \mathbb{C})$ |

Table 1.1: Reductive spherical pairs of rank-one type

We comment that Type R-1’ does not appear in the classification list of reductive irreducible spherical homogeneous spaces (cf. [8, 20]) because the homogeneous space of Type R-1’ is the double covering space of that of Type R-1. However, our argument for Type R-1’ can be given paralleled to that for Type R-2, and our main result for Type R-1’ provides that for Type R-1. As a reason, we will also treat Type R-1’ in this paper.
The terminology ‘rank-one type’ comes from the geometric sense, namely, the corresponding homogeneous space is of rank one. Moreover, it can be explained from the representation theory. By Vinberg–Kimelfeld [19], a reductive pair \((G_C, H_C)\) is spherical if the space \(\mathcal{O}(G_C/H_C)\) of holomorphic functions on the complex manifold \(G_C/H_C\) is multiplicity-free as a representation of \(G_C\), and vice versa. The set of highest weights of \(G_C\) occurring in the multiplicity-free irreducible decomposition of \(\mathcal{O}(G_C/H_C)\), called the support of \(G_C/H_C\), is a semigroup. If \((G_C, H_C)\) is one of Table 1.1 then the rank of the support equals one (see \[8\]).

Now, let us explain our main result. Let \((G_C, H_C)\) be a reductive spherical pair of rank-one type and \(\theta\) a Cartan involution of \(G_C\) satisfying \(\theta(H_C) = H_C\). Let \(g, h\) be the Lie algebras of \(G_C, H_C\), respectively. We use the same letter \(\theta\) to denote the differential automorphism on \(g\). We set \(g_u = g^\theta\) and \(G_u := G_C^\theta\). Then, \(g_u\) is a compact real form of \(g\) and the Lie algebra of \(G_u\). The corresponding Cartan decomposition of the Lie algebra \(g\) is given by \(g = g_u + \sqrt{-1}q_u\). Let \(q\) be the orthogonal complement of \(h\) in \(g\) with respect to the Killing form of \(g\). Under the setting, we give new examples of a Cartan decomposition \((1.1)\). Namely, we prove:

**Theorem 1.1.** Let \((G_C, H_C)\) be a reductive spherical pair of rank-one type. Then, one can take a one-dimensional abelian subspace \(a_0\) in \(\sqrt{-1}g_u \cap q\) such that

\[
G_C = G_u(\exp a_0)H_C.
\]

The purpose of this paper is to provide an explicit description of \(a_0\) satisfying Theorem 1.1 under the realization of \(G_C\) and \(H_C\) as matrix groups.

Thanks to Theorem 1.1, we find strongly visible actions on nonsymmetric complex reductive homogeneous spaces. We state:

**Theorem 1.2** (see Theorem 6.1). Let \((G_C, H_C)\) be a reductive spherical pair of rank-one type. Then, the \(G_u\)-action on the complex homogeneous space \(G_C/H_C\) is strongly visible (see Section 6 for definition).

This paper is organized as follows. In Section 2, we explain a matrix realization of Lie groups which appear in Table 1.1. In Sections 3–5 we prove Theorem 1.1 for each reductive spherical pair of rank-one type by giving an explicit description of an abelian group \(A\). In particular, we deal with Type R-2 in Section 3, Type R-1' in Section 4, and Type R-1 in Section 5. In Section 6 we show Theorem 1.2 in particular, we explain that Theorem 1.1 is an essential part of our proof of Theorem 1.2.
2.1. Realization of exceptional Lie group of Type $G_2$. In this subsection, we explain realizations of $G_2(\mathbb{C})$, and its maximal compact subgroups $G_2$ as subgroups of the special orthogonal group on the complexified Cayley algebra.

Let $\mathfrak{C}$ be a Cayley algebra over $\mathbb{R}$. This algebra $\mathfrak{C}$ has the standard basis $\{e_0, \ldots, e_7\}$ with $e_0$ as the unit element of $\mathfrak{C}$ such that the following relations hold, which we will fix in this paper:

\[
\begin{align*}
  e_0^2 &= e_0, \quad e_2^2 = -e_0 (i = 1, 2, \ldots, 7), \quad e_1 e_2 = -e_2 e_1 = e_3, \\
  e_1 e_4 &= -e_4 e_1 = e_5, \quad e_1 e_6 = -e_6 e_1 = e_7, \quad e_2 e_5 = -e_5 e_2 = e_7, \\
  e_0 e_6 &= -e_6 e_2 = e_4, \quad e_3 e_5 = -e_5 e_3 = e_6, \quad e_4 e_4 = -e_4 e_4 = e_7.
\end{align*}
\]

Let $(\cdot, \cdot)$ be an inner product on $\mathfrak{C}$ satisfying $(e_i, e_j) = \delta_{ij}$ ($0 \leq i, j \leq 7$). We denote by $\text{Re}(\mathfrak{C}) = \mathbb{R}e_0$ the real part of $\mathfrak{C}$ by $\text{Im}(\mathfrak{C}) = \mathbb{R}e_1 + \cdots + \mathbb{R}e_7$ the imaginary part of $\mathfrak{C}$. Then, $\text{Im}(\mathfrak{C})$ is the orthogonal complement of $\text{Re}(\mathfrak{C})$ in $\mathfrak{C}$ with respect to $(\cdot, \cdot)$.

We identify the special orthogonal group $SO(\mathfrak{C}, (\cdot, \cdot))$ with $SO(8) = \{g \in SL(8, \mathbb{R}) : {}^tgg = I_8\}$ where ${}^t g$ denotes the transposed matrix of $g$ and $I_8$ the unit matrix. Similarly, we have $SO(\text{Im}(\mathfrak{C}), (\cdot, \cdot)) = \{g \in SO(\mathfrak{C}, (\cdot, \cdot)) : ge_0 = e_0\} \simeq SO(7)$. We define $G_2$ as the automorphism group $\text{Aut}_\mathbb{R}(\mathfrak{C})$ of $\mathfrak{C}$, namely,

\[
G_2 = \{g \in SO(8) : (gx)(gy) = g(xy) \ (x, y \in \mathfrak{C})\}.
\]

Then, $G_2$ is a connected and simply connected compact simple Lie group of exceptional type with Lie algebra $\mathfrak{g}_2$. By definition, any element $g \in G_2$ satisfies $ge_0 = e_0$, from which $G_2$ is a subgroup of $SO(7)$. In particular, $\text{Im}(\mathfrak{C})$ is $G_2$-invariant.

Let $\mathfrak{C}_\mathbb{C} = \mathfrak{C} \otimes_\mathbb{R} \mathbb{C}$ be the complexified Cayley algebra and $\text{Im}(\mathfrak{C}_\mathbb{C})$ the complexification of $\text{Im}(\mathfrak{C})$. We extend the symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{C}$ to a complex symmetric bilinear form $(\cdot, \cdot) : \mathfrak{C}_\mathbb{C} \times \mathfrak{C}_\mathbb{C} \to \mathbb{C}$ on $\mathfrak{C}_\mathbb{C}$, namely,

\[
(v, w) = v_0 w_0 + \cdots + v_7 w_7 \quad \text{for} \quad v = v_0 e_0 + \cdots + v_7 e_7, \quad w = w_0 e_0 + \cdots + w_7 e_7 \in \mathfrak{C}_\mathbb{C}.
\]

We identify the complex special orthogonal group $SO(\mathfrak{C}_\mathbb{C}, (\cdot, \cdot))$ with $SO(8, \mathbb{C}) = \{g \in SL(8, \mathbb{C}) : {}^tgg = I_8\}$ and $SO(\text{Im}(\mathfrak{C}_\mathbb{C}), (\cdot, \cdot))$ with $SO(7, \mathbb{C})$ with respect to the $\mathbb{C}$-basis $\{e_0, \ldots, e_7\}$. We set $G_2(\mathbb{C}) := \text{Aut}_\mathbb{C}(\mathfrak{C}_\mathbb{C})$, namely,

\[
G_2(\mathbb{C}) = \{g \in SO(8, \mathbb{C}) : (gx)(gy) = g(xy) \ (x, y \in \mathfrak{C}_\mathbb{C})\}.
\]

Then, $G_2(\mathbb{C})$ is a connected and simply connected complex simple Lie group of exceptional type with Lie algebra $\mathfrak{g}_2(\mathbb{C}) := \mathfrak{g}_2 \otimes_\mathbb{R} \mathbb{C}$. This is a subgroup of $SO(7, \mathbb{C})$, and then, $\text{Im}(\mathfrak{C}_\mathbb{C})$ is $G_2(\mathbb{C})$-invariant.

The subgroup $G'(\mathbb{C}) := \{g \in G_2(\mathbb{C}) : ge_1 = e_1\}$ of $G_2(\mathbb{C})$ is isomorphic to the special linear group $SL(3, \mathbb{C})$, and $G'(\mathbb{C}) \cap G_2$ to the special unitary group $SU(3)$. In this paper, we shall identify $G'(\mathbb{C})$ and $G'(\mathbb{C}) \cap G_2$ with $SL(3, \mathbb{C})$ and $SU(3)$, respectively:

\[
SL(3, \mathbb{C}) = \{g \in G_2(\mathbb{C}) : ge_1 = e_1\}, \quad SU(3) = \{g \in G_2 : ge_1 = e_1\}.
\]
2.2. Realization of spinor group $Spin(7, \mathbb{C})$. Next, we realize the spinor group $Spin(7, \mathbb{C})$ as a subgroup of $SO(8, \mathbb{C}) = SO(\mathfrak{e}_\mathbb{C}, (\cdot, \cdot))$ as follows.

We define a subgroup $B_3(\mathbb{C})$ of $SO(8, \mathbb{C})$ by

$$B_3(\mathbb{C}) := \{ g \in SO(8, \mathbb{C}) : \text{there exists } g_0 \in SO(7, \mathbb{C}) \text{ such that } (g_0x)(gy) = g(xy) \ (x, y \in \mathbb{C}) \}$$

and $B_3$ of $SO(8)$ by

$$B_3 := \{ g \in SO(8) : \text{there exists } g_0 \in SO(7) \text{ such that } (g_0x)(gy) = g(xy) \ (x, y \in \mathbb{C}) \}.$$ 

Then, $G_2(\mathbb{C})$ and $G_2$ are subgroups of $B_3(\mathbb{C})$ and $B_3$, respectively. In particular, they are of the forms:

$$G_2(\mathbb{C}) = \{ g \in B_3(\mathbb{C}) : ge_0 = e_0 \}, \ G_2 = \{ g \in B_3 : ge_0 = e_0 \}.$$ 

We will see that $B_3(\mathbb{C})$ is isomorphic to $Spin(7, \mathbb{C})$ as follows. Let $g$ be an element of $B_3(\mathbb{C})$. By definition, we take $g_0 \in SO(7, \mathbb{C})$ such that $(g_0x)(gy) = g(xy) (x, y \in \mathbb{C})$. By the principle of triality in $SO(8, \mathbb{C})$, the existence of such $g_0$ is unique. Under the notation, this yields the following map

$$\pi : B_3(\mathbb{C}) \to SO(7, \mathbb{C}), \ g \mapsto \pi(g) = g_0$$

Then, the following equality holds for any $g_1, g_2 \in B_3(\mathbb{C})$:

$$(g_1g_2)(xy) = g_1(g_2(xy))$$

$$= g_1((\pi(g_2)x)(g_2y))$$

$$= (\pi(g_1)\pi(g_2)x)(g_1(g_2y))$$

$$= ((\pi(g_1)\pi(g_2))x)((g_1g_2)y) \quad (x, y \in \mathbb{C}).$$

This means that $\pi(g_1g_2) = \pi(g_1)\pi(g_2)$ for any $g_1, g_2 \in B_3(\mathbb{C})$, from which $\pi$ is a group homomorphism. On the other hand, let us take an element $g_0 \in SO(7, \mathbb{C})$. It follows from the principle of triality to $g_0$ that there exists $g \in SO(8, \mathbb{C})$ such that $(g_0x)(gy) = g(xy) (x, y \in \mathbb{C})$, from which $g \in B_3(\mathbb{C})$. This means that $\pi$ is a surjective map. Moreover, if $g, g' \in B_3(\mathbb{C})$ satisfy $\pi(g) = \pi(g')$, then $g'$ coincides either $g$ or $-g$. Thus, $\pi$ is a double covering map. Therefore, $B_3(\mathbb{C})$ is isomorphic to $Spin(7, \mathbb{C})$. Similarly, we have $B_3 \simeq Spin(7)$.

Throughout this paper, $Spin(7, \mathbb{C})$ means $B_3(\mathbb{C})$ and $Spin(7)$ means $B_3$.

2.3. Cartan involution. As mentioned before, we take a matrix realization of each complex simple Lie group and its maximal compact subgroup. Then, there exists a Cartan involution of each complex Lie group such that the fixed point set equals the maximal compact subgroup. In this subsection, we express such a Cartan involution as follows.
Let us define an involutive automorphism $\theta$ on $SO(8, \mathbb{C})$ by
\begin{equation}
\theta(g) = \overline{g} \quad (g \in SO(8, \mathbb{C})).
\end{equation}
Here, $\overline{g}$ stands for the complex conjugation $\overline{g}$ of $g \in SO(8, \mathbb{C})$. Then, $\theta$ is a Cartan involution of $SO(8, \mathbb{C})$ and its fixed point set $SO(8)$ is a maximal compact subgroup of $SO(8, \mathbb{C})$. The restrictions of $\theta$ to $B_3(\mathbb{C})$ and $G_2(\mathbb{C})$ becomes Cartan involutions on $B_3(\mathbb{C})$ and $G_2(\mathbb{C})$, respectively. Thus, $B_3$ is a maximal compact subgroup of $B_3(\mathbb{C})$ and $G_2$ is that of $G_2(\mathbb{C})$.

2.4. **Unit sphere and complex unit sphere.** In this subsection, we review the facts on transitive actions and isotropy subgroups.

Let $W = (\mathbb{R}e_1)^\perp = \mathbb{R}e_2 + \cdots + \mathbb{R}e_7$ be the orthogonal complement of $\mathbb{R}e_1$ in $\text{Im}(\mathcal{C})$. For a vector space $V (= \mathcal{C}, \text{Im}(\mathcal{C}), W)$ over $\mathbb{R}$, we write $S(V) = \{ v \in V : (v, v) = 1 \}$ for the unit sphere in $V$ with respect to $(\cdot, \cdot)$. As $G_2 \subset SO(7)$ and Spin$(7) \subset SO(8)$, Then, Spin$(7)$ acts on $S(\mathcal{C}) = S^7$ and $G_2$ acts on $S(\text{Im}(\mathcal{C})) = S^6$. Further, It is known that

**Lemma 2.1** ([19] 9). (1) The Spin$(7)$-action on $S^7$ is transitive, and then $S^7$ is diffeomorphic to Spin$(7)/G_2$.

(2) The $G_2$-action on $S^6$ is transitive, and then $S^6$ is diffeomorphic to $G_2/SU(3)$.

(3) The $SU(3)$-action on $S(W) = S^5$ is transitive.

Next, $V_\mathcal{C} (= \mathcal{C}, \text{Im}(\mathcal{C}))$ denotes the complexification of $V$. We set $S(V_\mathcal{C}) := \{ v \in V_\mathcal{C} : (v, v) = 1 \}$ as the complex unit sphere in $V_\mathcal{C}$. Then, Spin$(7, \mathcal{C})$ acts on $S(\mathcal{C}) = S^7_\mathcal{C}$ and $G_2(\mathcal{C})$ acts on $S(\text{Im}(\mathcal{C})) = S^6_\mathcal{C}$.

**Lemma 2.2.** (1) The Spin$(7, \mathcal{C})$-action on $S^7_\mathcal{C}$ is transitive, and then $S^7_\mathcal{C}$ is biholomorphic to Spin$(7, \mathcal{C})/G_2(\mathcal{C})$.

(2) The $G_2(\mathcal{C})$-action on $S^6_\mathcal{C}$ is transitive, and then $S^6_\mathcal{C}$ is biholomorphic to $G_2(\mathcal{C})/SL(3, \mathcal{C})$.

**Proof.** The isomorphism $S^7_\mathcal{C} \simeq \text{Spin}(7, \mathcal{C})/G_2(\mathcal{C})$ has been proved in [16] Lemma 2.2. The idea of the proof is based on [10] Proposition 2 in Section 3. Similarly, one can prove $S^6_\mathcal{C} \simeq G_2(\mathcal{C})/SL(3, \mathcal{C})$. Let us see it briefly.

We recall that $SL(3, \mathcal{C})$ is the isotropy subgroup of $G_2(\mathcal{C})$ at $e_1 \in S^6_\mathcal{C}$. Then, we have a natural embedding
\[\iota : G_2(\mathcal{C})/SL(3, \mathcal{C}) \to S^6_\mathcal{C}, \quad gSL(3, \mathcal{C}) \mapsto ge_1.\]
In particular, this is an injective map. On the other hand, the complex dimension of $G_2(\mathcal{C})/SL(3, \mathcal{C})$ equals six, which coincides with the that of $S^6_\mathcal{C}$. This implies $\iota$ is surjective, and then we obtain $G_2(\mathcal{C})/SL(3, \mathcal{C}) \simeq S^6_\mathcal{C}$. \hfill \Box

We note that both Spin$(7, \mathcal{C})/G_2(\mathcal{C})$ and $G_2(\mathcal{C})/SL(3, \mathcal{C})$ are non-symmetric homogeneous spaces.
2.5. Transitive actions on unit spheres. In this subsection, we consider a description of a vector space under the setting that a compact group acts transitively on the unit sphere. This subsection is based on the reference [13].

Let $V$ be a vector space over $\mathbb{R}$ equipped with an inner product $(\cdot, \cdot)$ and $G$ a subgroup of the orthogonal group $O(V, (\cdot, \cdot))$. The following lemma is obvious:

**Lemma 2.3** (see [13, Lemma 5.1]). Let $G$ be a subgroup of $O(V, (\cdot, \cdot))$ acting transitively on the unit sphere $S(V)$. Then, $V$ is written as $V = G \cdot \mathbb{R}v_0$ for a non-zero element $v_0 \in S(V)$.

Next, let $G$ act on the complexification $V_C = V + \sqrt{-1}V$ diagonally, namely, $g \cdot (v_1 + \sqrt{-1}v_2) := gv_1 + \sqrt{-1}gv_2$ for $g \in G, v_1, v_2 \in V$.

We denote by $(\mathbb{R}v_0)^\perp$ the orthogonal complement of $\mathbb{R}v_0$ in $V$. Then, we have:

**Lemma 2.4** (see [13, Lemma 5.2]). Retain the setting as in Lemma 2.3. Suppose that the isotropy subgroup $G_{v_0}$ of $G$ at $v_0$ acts transitively on the unit sphere $S((\mathbb{R}v_0)^\perp)$. Then, $V_C$ is expressed as

$$V_C = G \cdot (\mathbb{R}v_0 + \sqrt{-1}(\mathbb{R}v_0 \oplus \mathbb{R}w_0))$$

for an element $w_0 \in S((\mathbb{R}v_0)^\perp)$.

**Proof.** Applying Lemma 2.3 to the case where $G_{v_0}$ acts transitively on $S((\mathbb{R}v_0)^\perp)$, the vector space $(\mathbb{R}v_0)^\perp$ is written as $(\mathbb{R}v_0)^\perp = G_{v_0} \cdot \mathbb{R}w_0$ for an element $w_0 \in S((\mathbb{R}v_0)^\perp)$. On the other hand, it is clear that $G_{v_0} \cdot \mathbb{R}v_0 = \mathbb{R}v_0$. As $V = \mathbb{R}v_0 \oplus (\mathbb{R}v_0)^\perp$, we obtain

$$V = G_{v_0} \cdot \mathbb{R}v_0 \oplus G_{v_0} \cdot \mathbb{R}w_0 = G_{v_0} \cdot (\mathbb{R}v_0 \oplus \mathbb{R}w_0).$$

Combining the above decomposition with Lemma 2.3, we conclude

$$V_C = V + \sqrt{-1}V = G \cdot \mathbb{R}v_0 + \sqrt{-1}(G_{v_0} \cdot (\mathbb{R}v_0 \oplus \mathbb{R}w_0))$$

$$= G \cdot (\mathbb{R}v_0 + \sqrt{-1}(\mathbb{R}v_0 \oplus \mathbb{R}w_0)).$$

\[\square\]

3. Cartan decomposition for $(G_2(\mathbb{C}), SL(3, \mathbb{C}))$

In this section, we give a proof of Theorem 1.1 for a non-symmetric reductive spherical pair $(G_2(\mathbb{C}), SL(3, \mathbb{C}))$ (Type R-2).

We begin this section with the outline of our proof. As mentioned in Lemma 2.2 the homogeneous space $G_2(\mathbb{C})/SL(3, \mathbb{C})$ is biholomorphic to the complex unit sphere $S(Im(\mathbb{C}^2))$. Then, we first find a real submanifold $T_1$ which meets every $G_2$-orbit in $S(Im(\mathbb{C}^2))$ (Section 3.1). Second, we give an abelian group $A_1$ such that $T_1$ is an $A_1$-orbit (Section 3.2). After that, we prove Theorem 1.1 for this case (see Theorem 3.6 for detail).
3.1. \textbf{G}_2-action on \( S(\text{Im}(\mathfrak{C}_C)) \). First, we give a decomposition of \( S(\text{Im}(\mathfrak{C}_C)) \) into \( G_2 \)-orbits.

We set
\begin{equation}
(3.1) \quad T_1 := (\mathbb{R}e_1 + \sqrt{-1}(\mathbb{R}e_1 + \mathbb{R}e_2)) \cap S(\text{Im}(\mathfrak{C}_C)).
\end{equation}

\textbf{Lemma 3.1.} The complex unit sphere \( S(\text{Im}(\mathfrak{C}_C)) \) is written as
\[ S(\text{Im}(\mathfrak{C}_C)) = G_2 \cdot T_1. \]

\textit{Proof.} Retain the notation as in Section 2.4. We observe that \( G_2 \) acts transitively on \( S(\text{Im}(\mathfrak{C}_C)) \) and the isotropy subgroup \( (G_2)_{e_1} = SU(3) \) acts transitively on \( S(W) \) (see Lemma 2.1). By Lemma 2.4, we have
\[ \text{Im}(\mathfrak{C}_C) = G_2 \cdot (\mathbb{R}e_1 + \sqrt{-1}(\mathbb{R}e_1 + \mathbb{R}e_2)). \]

Since \( G_2 \) is a subgroup of \( SO(7) \), we obtain
\begin{align*}
S(\text{Im}(\mathfrak{C}_C)) &= (G_2 \cdot (\mathbb{R}e_1 + \sqrt{-1}(\mathbb{R}e_1 + \mathbb{R}e_2))) \cap S(\text{Im}(\mathfrak{C}_C)) \\
&= G_2 \cdot ((\mathbb{R}e_1 + \sqrt{-1}(\mathbb{R}e_1 + \mathbb{R}e_2))) \cap S(\text{Im}(\mathfrak{C}_C)) \\
&= G_2 \cdot T_1.
\end{align*}

Hence, Lemma 3.1 has been proved. \qed

Next, we consider an explicit description of an element of \( T_1 \) in the coordinates. Let \( v \) be an element of \( T_1 \). As \( T_1 \subset \mathbb{R}e_1 + \sqrt{-1}(\mathbb{R}e_1 + \mathbb{R}e_2) \), we write \( v = x_1e_1 + \sqrt{-1}(y_1e_1 + y_2e_2) \) for some \( x_1, y_1, y_2 \in \mathbb{R} \). Then, we have
\[(v, v) = (x_1 + \sqrt{-1}y_1)^2 + (\sqrt{-1}y_2)^2 = (x_1^2 - y_1^2 - y_2^2) + 2\sqrt{-1}x_1y_1.\]

Since \( v \in S(\text{Im}(\mathfrak{C}_C)) \), three real numbers \( x_1, y_1, y_2 \) satisfy \( x_1^2 - y_1^2 - y_2^2 = 1 \) and \( x_1y_1 = 0 \). Hence, we get \( y_1 = 0 \) and \( x_1^2 - y_2^2 = 1 \). Therefore, \( T_1 \) is of the form
\[ T_1 = \{(\cosh \theta)e_1 + \sqrt{-1}(\sinh \theta)e_2 : \theta \in \mathbb{R} \}. \]

Here, the map \( \mathbb{R} \rightarrow T_1, \theta \mapsto (\cosh \theta)e_1 + \sqrt{-1}(\sinh \theta)e_2 \) is an embedding. Then, \( T_1 \) is a one-dimensional real submanifold in \( S(\text{Im}(\mathfrak{C}_C)) \).

3.2. \textbf{G}_2-action on \( G_2(\mathbb{C})/SL(3, \mathbb{C}) \). We recall from Lemma 2.2 that \( S(\text{Im}(\mathfrak{C}_C)) \) is biholomorphic to \( G_2(\mathbb{C})/SL(3, \mathbb{C}) \). As \( S(\text{Im}(\mathfrak{C}_C)) = G_2 \cdot T_1 \), there exists a real submanifold \( S_1 \) in \( G_C/H_C \) such that \( T_1 \simeq S_1 \) and \( G_C/H_C = G_4 \cdot S_1 \) via the biholomorphic diffeomorphism.

To find \( S_1 \), we construct an abelian group \( A_1 \) as follows. Let us define a matrix \( \delta_{(x,y)} \) by
\[
\delta_{(x,y)} = \begin{pmatrix}
0 & 0 & 0 & \sqrt{-1}x \\
0 & 0 & -\sqrt{-1}y & 0 \\
\sqrt{-1}y & 0 & 0 & 0 \\
\sqrt{-1}x & 0 & 0 & 0
\end{pmatrix}.
\]
Now, we set
\begin{equation}
\{a_1 := \{t_\theta := \text{diag}(\delta_{(0, \theta)}, \delta_{(-\theta/2, -\theta/2)}): \theta \in \mathbb{R}\}\}
\end{equation}
and
\begin{equation}
A_1 = \exp a_1 = \{t_\theta = \text{diag}(\exp \delta_{(0, \theta)}, \exp \delta_{(-\theta/2, -\theta/2)}): \theta \in \mathbb{R}\}.
\end{equation}
Then, $A_1$ is a one-dimensional abelian group. We note
\[
\exp \delta_{(x, y)} = \begin{pmatrix}
\cosh x & 0 & 0 & -\sqrt{-1} \sinh x \\
0 & \cosh y & -\sqrt{-1} \sinh y & 0 \\
0 & \sqrt{-1} \sinh y & \cosh y & 0 \\
\sqrt{-1} \sinh x & 0 & 0 & \cosh x
\end{pmatrix}
\]

**Lemma 3.2.** The abelian group $A_1$ is contained in $G_2(\mathbb{C})$.

**Sketch of Proof.** Let us verify that any element $t_\theta \in A_1$ satisfies
\[(t_\theta e_i)(t_\theta e_j) = t_\theta (e_i e_j) \quad (0 \leq i \leq j \leq 7)
\]
for our choice of the $\mathbb{C}$-basis $\{e_0, \ldots, e_7\}$ (see (2.1)). In fact, the computation is straightforward from the followings:
\begin{align*}
t_\theta e_1 &= (\cosh \theta) e_1 + \sqrt{-1} (\sinh \theta) e_2, \\
t_\theta e_2 &= -\sqrt{-1} (\sinh \theta) e_1 + (\cosh \theta) e_2, \\
t_\theta e_4 &= (\cosh(\theta/2)) e_4 - \sqrt{-1} (\sinh(\theta/2)) e_7, \\
t_\theta e_5 &= (\cosh(\theta/2)) e_5 - \sqrt{-1} (\sinh(\theta/2)) e_6, \\
t_\theta e_6 &= \sqrt{-1} (\sinh(\theta/2)) e_5 + (\cosh(\theta/2)) e_6, \\
t_\theta e_7 &= \sqrt{-1} (\sinh(\theta/2)) e_4 + (\cosh(\theta/2)) e_7
\end{align*}

and $t_\theta e_i = e_i$ for $i = 0, 4$. \qed

As mentioned in the proof of Lemma 3.2, an element $(\cosh \theta) e_1 + \sqrt{-1} (\sinh \theta) e_2 \in T_1$ is written by $t_\theta e_1$. Then, the submanifold $T_1$ is expressed as
\begin{equation}
T_1 = \{t_\theta e_1: \theta \in \mathbb{R}\} = A_1 \cdot e_1.
\end{equation}
Hence, we set
\begin{equation}
S_1 := A_1 SL(3, \mathbb{C})/SL(3, \mathbb{C}).
\end{equation}
Then, we have:

**Lemma 3.3.** $S_1 \simeq T_1$.

Combining Lemma 3.1 with Lemma 3.3, we get the decomposition of the homogeneous space $G_2(\mathbb{C})/SL(3, \mathbb{C})$ as follows:

**Proposition 3.4.** $G_2(\mathbb{C})/SL(3, \mathbb{C}) = G_2 \cdot S_1$. 

Proof. Let \( g \) be an element of \( G_2(\mathbb{C}) \). By Lemma 3.1, the element \( ge_1 \in S(\text{Im}(\mathfrak{c}_C)) \) is written as \( ge_1 = k \cdot v_1 \) for some \( k \in G_u \) and \( v_1 \in T_1 \) (see (3.4)). Moreover, \( v_1 \) is given by \( v_1 = t_\theta e_1 \) for some \( t_\theta \in A_1 \), from which \( ge_1 = (kt_\theta)e_1 \). This means \( g^{-1}kt_\theta \in (G_2(\mathbb{C}))_{e_1} = SL(3, \mathbb{C}) \). Hence, we obtain \( gSL(3, \mathbb{C}) = k \cdot t_\theta SL(3, \mathbb{C}) \in G_2 \cdot S_1. \) □

3.3. Lie algebra \( \mathfrak{a}_1 \). In this subsection, we observe the Lie algebra \( \mathfrak{a}_1 \).

Let \( \mathfrak{g} = \mathfrak{g}_2(\mathbb{C}), \mathfrak{h} = \mathfrak{sl}(3, \mathbb{C}) \) and \( \mathfrak{g}_u = \mathfrak{g}_2 \) be the Lie algebras of \( G_2(\mathbb{C}), SL(3, \mathbb{C}) \) and \( G_2 \), respectively. The differential automorphism of the Cartan involution \( \theta \) of \( G_2(\mathbb{C}) \) (see (2.3)), which we use the same letter to denote, is given by \( \theta(X) = X^\tau (X \in \mathfrak{g}) \). Since \( \delta_{(x,y)} = \delta_{(-x,-y)} = -\delta_{(x,y)} \), we have \( \theta(\tau_0) = \tau_{(-\theta)} = -\tau_\theta \) for any \( \tau_\theta \in \mathfrak{a}_1 \). Hence, \( \mathfrak{a}_1 \) is contained in \( \sqrt{-1} \mathfrak{g}_u \).

Next, let \( \mathfrak{q} \) be the orthogonal complement of \( \mathfrak{h} \) in \( \mathfrak{g} \) with respect to the Killing form on \( \mathfrak{g} \). As \( SL(3, \mathbb{C}) = (G_2(\mathbb{C}))_{e_1} \), we write \( \mathfrak{h} = \{ X \in \mathfrak{g}_2(\mathbb{C}) : Xe_1 = 0 \} \). Thus, \( \mathfrak{a}_1 \) is not contained in \( \mathfrak{h} \). In fact, we have:

Lemma 3.5. \( \mathfrak{a}_1 \subset \mathfrak{q} \).

Proof. We will give a \( \mathbb{C} \)-bases of \( \mathfrak{h} = \mathfrak{sl}(3, \mathbb{C}) \) and \( \mathfrak{q} \) as follows, which is based on \([21]\) Theorem 1.5.1.

Let \( E_{ij} \) \((0 \leq i, j \leq 7, i \neq j)\) be a \( \mathbb{C} \)-linear transformation on \( \mathfrak{c}_C \) satisfying \( E_{ij}e_j = e_i \) and \( E_{ij}e_k = 0 \) for \( k \neq j \). We shall also use the same letter \( E_{ij} \) to denote the matrix of degree eight corresponding to the \( \mathbb{C} \)-linear transformation \( E_{ij} \) via the identification \( \text{End}_\mathbb{C}(\mathfrak{c}_C) \approx M(8, \mathbb{C}) \) with respect to the \( \mathbb{C} \)-basis \( \{e_0, \ldots, e_7\} \). For \( 0 \leq i < j \leq 7 \), we set \( X_{ij} := E_{ij} - E_{ji} \). Then,

\[
\begin{align*}
(3.6) \quad \{ &-X_{23} + X_{45}, \ -X_{45} + X_{67}, \ X_{24} + X_{35}, \ -X_{25} + X_{34}, \\
&\quad \quad \quad \quad \ -X_{26} + X_{37}, \ -X_{27} + X_{36}, \ X_{46} + X_{57}, \ -X_{47} + X_{56} \} \\
&\text{is a } \mathbb{C} \text{-basis of } \mathfrak{sl}(3, \mathbb{C}), \text{ and}
\end{align*}
\]

\[
\begin{align*}
(3.7) \quad \{ &2X_{12} - X_{47} - X_{56}, \ 2X_{13} - X_{46} + X_{57}, \ 2X_{14} + X_{27} + X_{36}, \\
&\quad \quad \quad \quad \ 2X_{15} + X_{26} - X_{37}, \ 2X_{16} - X_{25} - X_{34}, \ 2X_{17} - X_{24} + X_{33} \} \\
&\text{is a } \mathbb{C} \text{-basis of } \mathfrak{q}. \quad \text{Since } 2X_{12} - X_{47} - X_{56} \in \mathfrak{q} \text{ is written by } \tau_2 = 2\tau_1, \text{ an arbitrary } \tau_\theta \text{ is of the form } \tau_\theta = \theta \tau_1 \in \mathfrak{q}. \text{ Thus, the Lie algebra } \mathfrak{a}_1 \text{ is contained in } \mathfrak{q}. \quad \square
\end{align*}
\]

Therefore, we have shown \( \mathfrak{a}_1 \subset \sqrt{-1} \mathfrak{g}_u \cap \mathfrak{q} \).

3.4. Proof of Theorem 1.1 for \( (G_2(\mathbb{C}), SL(3, \mathbb{C})) \). A Cartan decomposition for the reductive spherical pair \( (G_2(\mathbb{C}), SL(3, \mathbb{C})) \) is provided by Proposition 3.4. More precisely, we prove:

Theorem 3.6 (Theorem 1.1 for Type R-2). Let \( (G, H) \) be the reductive spherical pair of Type R-2, namely, \( (G_2(\mathbb{C}), SL(3, \mathbb{C})) \). Then,
Lemma 4.2. Let $\mathbf{C}$ be an element of $G_2(\mathbb{C})$. By Proposition 3.4, there exists $k \in G_u = G_2$ and $t_0 \in A_1$ such that $g H_C = k \cdot (t_0 H_C) = (kt_0) H_C$. Thus, we have $(kt_0)^{-1} g \in H_C$. We write $h := (kt_0)^{-1} g \in H_C$. Then, we obtain $g = kt_0 h \in G_u A_1 H_C$. Hence, we conclude $G_C \subseteq G_u A_1 H_C$. Clearly, $G_C \supseteq G_u A_1 H_C$. Therefore, we get $G_C = G_u A_1 H_C$. $\square$

4. CARTAN DECOMPOSITION FOR $(\text{Spin}(7, \mathbb{C}), G_2(\mathbb{C}))$

In this subsection, we give a proof of Theorem 1.1 for the non-symmetric reductive spherical pair $(\text{Spin}(7, \mathbb{C}), G_2(\mathbb{C}))$. The proof of Theorem 1.1 for Type R-1' goes on paralleled to that for Type R-2 which has been discussed in Section 3.

4.1. $\text{Spin}(7)$-action on $S(\mathfrak{C}_C)$. In this subsection, we consider the $\text{Spin}(7)$-action on the complex unit sphere $S(\mathfrak{C}_C)$ (Type R-1').

The $\text{Spin}(7)$-action on $S^7 = S(\mathbb{C})$ is transitive and the action of the isotropy subgroup $\text{Spin}(7)_{e_0} = G_2$ on $S^6 = S(\text{Im}(\mathbb{C}))$ is also transitive (see Lemma 2.1). It follows from Lemma 2.4 that the complexified Cayley algebra $\mathfrak{C}_C = \mathfrak{C} + \sqrt{-1}\mathfrak{C}$ is written as

$$\mathfrak{C}_C = \text{Spin}(7) \cdot (\mathbb{R}e_0 + \sqrt{-1}(\mathbb{R}e_0 + \mathbb{R}e_1)).$$

Then, we obtain

$$S(\mathfrak{C}_C) = \text{Spin}(7) \cdot (\mathbb{R}e_0 + \sqrt{-1}(\mathbb{R}e_0 + \mathbb{R}e_1)) \cap S(\mathfrak{C}_C).$$

Hence, we take a one-dimensional real submanifold $T_0$ in $S(\mathfrak{C}_C)$ as

$$T_0 := (\mathbb{R}e_0 + \sqrt{-1}(\mathbb{R}e_0 + \mathbb{R}e_1)) \cap S(\mathfrak{C}_C).$$

Then, we have:

**Lemma 4.1.** The complex unit sphere $S(\mathfrak{C}_C)$ is expressed as

$$S(\mathfrak{C}_C) = \text{Spin}(7) \cdot T_0.$$

4.2. $\text{Spin}(7)$-action on $\text{Spin}(7, \mathbb{C})/G_2(\mathbb{C})$. In this subsection, we give a real submanifold which meets every $\text{Spin}(7)$-orbit in $\text{Spin}(7, \mathbb{C})/G_2(\mathbb{C})$.

Let us take a matrix $d_\theta$ as

$$d_\theta := \begin{pmatrix} \cosh \theta & -\sqrt{-1} \sinh \theta \\ \sqrt{-1} \sinh \theta & \cosh \theta \end{pmatrix}. $$

We define a subgroup $\tilde{A}_0$ of $\text{SO}(8, \mathbb{C})$ by

$$\tilde{A}_0 := \{ \tilde{a}_\theta = \text{diag}(d_\theta, d_{-\theta/3}, d_{-\theta/3}, d_{-\theta/3}) : \theta \in \mathbb{R} \}$$

**Lemma 4.2.** The set $\tilde{A}_0$ is a subgroup of $\text{Spin}(7, \mathbb{C})$. 

the one-dimensional abelian group $A_1 = \exp a_1$ with $a_1 \subset \sqrt{-1}g_u \cap \mathfrak{q}$ given by (3.2) satisfies

$$G_C = G_u A_1 H_C.$$
For the verification of Lemma 4.2, we prepare the notation as follows:

(4.4) \[ a_x := \text{diag}(I_2, d_x, d_x, d_x) \in SO(7, \mathbb{C}). \]

**Sketch of Proof.** For an element \( \widetilde{a}_\theta \) of \( \widetilde{A}_0 \), we take \( a(2\theta/3) \in SO(7, \mathbb{C}) \). Then, the direct computation shows that

(4.5) \[ (a(2\theta/3)e_i)(\widetilde{a}_\theta e_j) = \widetilde{a}_\theta (e_i e_j) \quad (0 \leq i \leq j \leq 7) \]

for the \( \mathbb{C} \)-basis \( \{e_0, \ldots, e_7\} \) of \( \mathbb{C}^7 \). This implies that \( \widetilde{a}_\theta \in Spin(7, \mathbb{C}) \).

By taking the same argument as for \( T_1 \), the real submanifold \( \widetilde{T}_0 \) is of the form

\[ \widetilde{T}_0 = \{(\cosh \theta)e_0 + \sqrt{-1}(\sinh \theta)e_1 : \theta \in \mathbb{R}\}. \]

Thus, we write

\[ \widetilde{T}_0 = \widetilde{A}_0 \cdot e_0. \]

Hence, we set

(4.6) \[ \widetilde{S}_0 := \widetilde{A}_0 G_2(\mathbb{C})/G_2(\mathbb{C}). \]

**Lemma 4.3.** \( \widetilde{S}_0 \simeq \widetilde{T}_0 \).

Therefore, we get a decomposition of \( Spin(7, \mathbb{C})/G_2(\mathbb{C}) \) as follows:

**Proposition 4.4.** \( Spin(7, \mathbb{C})/G_2(\mathbb{C}) = Spin(7) \cdot \widetilde{S}_0 \).

4.3. Lie algebra of \( \widetilde{A}_0 \). Let \( g = \text{spin}(7, \mathbb{C}) \), \( h = g_2(\mathbb{C}) \) and \( g_u = \text{spin}(7) \) be the Lie algebras of \( Spin(7, \mathbb{C}) \), \( G_2(\mathbb{C}) \) and \( Spin(7) \), respectively, and \( q \) be the orthogonal complement of \( h \) in \( g \) with respect to the Killing form on \( g \). In this subsection, the Lie algebra \( \tilde{a}_0 \) of \( \tilde{A}_0 \) is contained in \( \sqrt{-1}g_u \cap q \).

The Lie algebra \( \tilde{a}_0 \) is given as follows. Let \( \delta_\theta \) be a matrix given by

(4.7) \[ \delta_\theta := \begin{pmatrix} 0 & -\sqrt{-1}\theta \\ \sqrt{-1}\theta & 0 \end{pmatrix}. \]

Then, \( \tilde{a}_0 \) is given by

(4.8) \[ \tilde{a}_0 = \{\tilde{\alpha}_\theta = \text{diag}(\delta_\theta, \delta(-\theta/3), \delta(-\theta/3), \delta(-\theta/3)) : \theta \in \mathbb{R}\}. \]

In particular, we have \( \tilde{A}_0 = \exp \tilde{a}_0 \).

We choose a Cartan involution \( \theta \) of \( g \) given by \( \theta(X) = \overline{X} \) (\( X \in g \)) (see Section 2.3). Then, we have \( \theta(\tilde{\alpha}_\theta) = \overline{\tilde{\alpha}_\theta} = -\tilde{\alpha}_\theta \) for any \( \tilde{\alpha}_\theta \in \tilde{a}_0 \) because \( \theta(\delta_\theta) = \delta(-\theta) = -\delta_\theta \). This implies \( \tilde{a}_0 \subset \sqrt{-1}g_u \).

Next, we show:

**Lemma 4.5.** \( \tilde{a}_0 \subset q \).
Sketch of Proof. As mentioned in Section 3.3, the Lie algebra \( g_2(\mathbb{C}) \) is decomposed into the direct sum of \( \mathfrak{sl}(3, \mathbb{C}) \) and the orthogonal complement of \( \mathfrak{sl}(3, \mathbb{C}) \) in \( g_2(\mathbb{C}) \), denoted here by \( q' \). Moreover, we take a \( \mathbb{C} \)-basis of \( \mathfrak{sl}(3, \mathbb{C}) \) as in (5.6) and that of \( q' \) as in (5.7). It turns out that \( \tilde{\alpha}_0 \) is orthogonal to both \( \mathfrak{sl}(3, \mathbb{C}) \) and \( q' \), and then to \( g_2(\mathbb{C}) \). Hence, we obtain \( \tilde{\alpha}_0 \subseteq q' \).

Consequently, we have proved \( \tilde{\alpha}_0 \subseteq \sqrt{-1}g_u \cap q' \).

4.4. Proof of Theorem 1.1 for \((\text{Spin}(7, \mathbb{C}), G_2(\mathbb{C}))\). In this subsection, we will prove Theorem 1.1 for \((\text{Spin}(7, \mathbb{C}), G_2(\mathbb{C}))\). Let \( G_u := \text{Spin}(7) \) be a maximal compact subgroup of \( G_\mathbb{C} \).

Theorem 4.6 (Theorem 1.1 for Type R-1'). Let \((G_\mathbb{C}, H_\mathbb{C})\) be the reductive spherical pair of Type R-1', namely, \((\text{Spin}(7, \mathbb{C}), G_2(\mathbb{C}))\). Then, the one-dimensional abelian group \( \tilde{A}_0 = \exp \tilde{\alpha}_0 \) with \( \tilde{\alpha}_0 \subseteq \sqrt{-1}g_u \cap q' \) given by (5.8) satisfies

\[ G_\mathbb{C} = G_u \tilde{A}_0 H_\mathbb{C}. \]

Proof. The proof of Theorem 4.6 is the same as the proof of Theorem 3.6. Then, we omit its proof.

5. CARTAN DECOMPOSITION FOR \((SO(7, \mathbb{C}), G_2(\mathbb{C}))\)

In this section, we give a Cartan decomposition for non-symmetric reductive spherical pair \((G_\mathbb{C}, H_\mathbb{C}) = (SO(7, \mathbb{C}), G_2(\mathbb{C}))\) (Type R-1).

The key idea is to take the image of our Cartan decomposition for \((\text{Spin}(7, \mathbb{C}), G_2(\mathbb{C}))\) given by Theorem 4.6 through the double covering group homomorphism \( \pi \) (see (2.2)).

To carry out, we define a subgroup \( A_0 \) of \( SO(7, \mathbb{C}) = (SO(8, \mathbb{C}))_{\theta_0} \) by

\[ A_0 := \{ a_\theta = \text{diag}(I_2, d_\theta, d_\theta, d_\theta) : \theta \in \mathbb{R} \} \]

Here, an element \( a_\theta \) has already appeared in the proof of Lemma 4.2 and \( d_\theta \) is given by (4.2). Then, the Lie algebra \( a_0 \) of \( A_0 \) is of the form

\[ a_0 = \{ \alpha_\theta = \text{diag}(O_2, \delta_\theta, \delta_\theta, \delta_\theta) : \theta \in \mathbb{R} \} \]

where \( \delta_\theta \) is given by (4.7). In particular, we have \( A_0 = \exp a_0 \).

Let \( \mathfrak{g} = \mathfrak{so}(7, \mathbb{C}) \), \( \mathfrak{h} = \mathfrak{g}_2(\mathbb{C}) \) and \( \mathfrak{g}_u = \mathfrak{so}(7) \) be the Lie algebras of \( G_\mathbb{C} \), \( H_\mathbb{C} \) and \( G_u \), respectively and \( q \) the orthogonal complement of \( \mathfrak{h} \) in \( \mathfrak{g} \). Clearly, \( a_0 \) is contained in \( \sqrt{-1}g_u \). By the same argument as Lemma 4.5, we have \( a_0 \subseteq q \). Thus, we obtain \( a_0 \subseteq \sqrt{-1}g_u \cap q \).

Now, we return to the relation (4.5). This implies that \( \pi \) induces the map \( \pi : \tilde{A}_0 \rightarrow A_0 \) given by \( \pi(a_\theta) = a_{(2\theta/3)} \).

Lemma 5.1. The image \( \pi(\tilde{A}_0) \) coincides with \( A_0 \).

Proof. For any \( a_{\theta'} \in A_0 \), the element \( \tilde{a}_{(3\theta' / 2)} \in \tilde{A}_0 \) satisfies \( \pi(\tilde{a}_{(3\theta' / 2)}) = a_{\theta'} \). Hence, we have proved \( \pi(\tilde{A}_0) = A_0 \).
Theorem 5.2 (Theorem 1.1 for Type R-1). Let \((G_\mathbb{C}, H_\mathbb{C})\) be the reductive spherical pair of Type R-1, namely, \((SO(7, \mathbb{C}), G_2(\mathbb{C}))\). Then, the one-dimensional abelian group \(A_0 = \exp a_0\) with \(a_0 \subset \sqrt{-1}g_u \cap q\) given by (5.3) satisfies
\[
G_\mathbb{C} = G_u A_0 H_\mathbb{C}.
\]

Proof. We observe that \(G_\mathbb{C} = SO(7, \mathbb{C})\) is realized as the image \(\tilde{\pi}(\tilde{G}_\mathbb{C})\) of \(\tilde{G}_\mathbb{C} = Spin(7, \mathbb{C})\). It follows from Theorem 4.6 that
\[
\tilde{\pi}(\tilde{G}_\mathbb{C}) = \pi(G_u) \pi(A_0) \pi(H_\mathbb{C}).
\]
(5.3)
Here, the image \(\pi(\tilde{G}_u) = \pi(Spin(7))\) coincides with \(SO(7)\), \(\pi(H_\mathbb{C}) = \pi(G_2(\mathbb{C}))\) with \(G_2(\mathbb{C})\), and by Lemma 5.1 \(\pi(\tilde{A}_0)\) with \(A_0\). Hence, (5.3) implies \(G_\mathbb{C} = G_u A_0 H_\mathbb{C}\). □

The following theorem is an immediate consequence of Theorem 1.1 or Proposition 4.4.

Proposition 5.3. \(SO(7, \mathbb{C})/G_2(\mathbb{C}) = SO(7) \cdot (A_0 G_2(\mathbb{C})/G_2(\mathbb{C}))\).

For the sake of our application given in the next section, we will explain that Proposition 5.3 follows from Proposition 4.4.

The double covering group homomorphism \(\pi\) induces a double covering map
\[
\bar{\pi} : Spin(7, \mathbb{C})/G_2(\mathbb{C}) \to SO(7, \mathbb{C})/G_2(\mathbb{C}), \quad gG_2(\mathbb{C}) \mapsto \pi(g)G_2(\mathbb{C}).
\]
(5.4)
In particular, \(\bar{\pi}(Spin(7, \mathbb{C})/G_2(\mathbb{C}))\) coincides with \(SO(7, \mathbb{C})/G_2(\mathbb{C})\). It follows from Proposition 4.4 that
\[
SO(7, \mathbb{C})/G_2(\mathbb{C}) = \bar{\pi}(Spin(7, \mathbb{C})/G_2(\mathbb{C}))
\]
\[
= \bar{\pi}(Spin(7) \cdot (\tilde{A}_0 G_2(\mathbb{C})/G_2(\mathbb{C})))
\]
\[
= \pi(Spin(7)) \cdot (\pi(\tilde{A}_0) G_2(\mathbb{C})/G_2(\mathbb{C}))
\]
\[
= SO(7) \cdot (A_0 G_2(\mathbb{C})/G_2(\mathbb{C})).
\]

Hence, we set
\[
S_0 := A_0 G_2(\mathbb{C})/G_2(\mathbb{C}).
\]
(5.5)
Then, the above argument shows:

Corollary 5.4. \(\pi(\tilde{S}_0) = S_0\).

6. APPLICATION TO VISIBLE ACTIONS ON COMPLEX MANIFOLDS

The motivation of the study for a Cartan decomposition for non-symmetric reductive spherical pairs is to investigate a classification problem on strongly visible actions on reductive complex homogeneous spaces. The notion of (strongly) visible actions has been introduced by T. Kobayashi for giving an unified explanation for multiplicity-freeness
CARTAN DECOMPOSITION FOR RANK-ONE REDUCTIVE SPHERICAL PAIRS

of representations (cf. [6]). In this aspect, it plays a crucial role to find a real submanifold which meets every orbit. Once one can find a Cartan decomposition for a reductive spherical pair, one can also provide an explicit description of such a submanifold simultaneously. This section studies spherical homogeneous spaces of rank-one type from the viewpoint of (strongly) visible actions.

Let us give a quick review on strongly visible actions. A holomorphic action of a Lie group $G$ on a connected complex manifold $D$ is called strongly visible if there exist a real submanifold $S$ in $D$ (called a slice) and an anti-holomorphic diffeomorphism $\sigma$ on $D$ satisfying the following conditions (see [6]):

\begin{align}
(\text{V.1}) & \quad D = G \cdot S, \\
(\text{S.1}) & \quad \sigma|_S = \text{id}_S, \\
(\text{S.2}) & \quad \sigma(x) \in G \cdot x \ (\forall x \in D).
\end{align}

We note that the slice $S$ is automatically totally real, namely, $J_x(T_xS) \cap T_xS = \{0\}$ for any $x \in S$ (see [6, Remark 3.3.2]). Here, $J$ stands for the complex structure of $D$.

In Kobayashi’s original definition [6, Definition 3.3.1] of strongly visible actions, it allows a complex manifold $D$ containing a non-empty $G$-invariant open set satisfying (V.1)–(S.2). For this paper, we shall adopt the above definition for simplicity.

Now, we prove:

**Theorem 6.1.** Let $(G_C, H_C)$ be a reductive spherical pair of rank-one type. Then, the $G_u$-action on $D = G_C/H_C$ is strongly visible. In particular, one can find a one-dimensional slice $S$ for the strongly visible action.

In the following, we prove Theorem 6.1 for $(G_C, H_C)$ given in Table 1.1. More precisely, we will verify three conditions (V.1)–(S.2).

6.1. **Verification of (V.1).** We have already proved that there exists a one-dimensional real submanifold

\[(6.1) \quad S := AH_C/H_C.\]

satisfying $D = G_u \cdot S$, which implies the condition (V.1). We list our choice of $S$ and the proposition showing $D = G_u \cdot S$ in Table 6.1.

| Type  | $G_C$          | $H_C$          | $S$   | $D = G_u \cdot S$ |
|-------|----------------|----------------|-------|------------------|
| R-1   | $SO(7, \mathbb{C})$ | $G_2(\mathbb{C})$ | $S_0$ | Proposition 3.3   |
| R-1’  | $Spin(7, \mathbb{C})$ | $G_2(\mathbb{C})$ | $\tilde{S}_0$ | Proposition 4.4 |
| R-2   | $G_2(\mathbb{C})$ | $SL(3, \mathbb{C})$ | $S_1$ | Proposition 3.4   |

Table 6.1: Our choice of slice $S$ satisfying (S.1)
6.2. Verification of (S.1). In this subsection, we will verify the condition (S.1).

First, let $I_{1,1} := \text{diag}(1, -1)$ and

$$I_{+-} := \text{diag}(I_{1,1}, I_{1,1}, I_{1,1}, I_{1,1}) = \text{diag}(1, -1, 1, -1, 1, -1, 1, -1).$$

Since $(I_{+-}e_i)(I_{+-}e_j) = I_{+-}(e_i e_j)$ for $0 \leq i, j \leq 7$, the element $I_{+-}$ lies in $G_2$. Here, we define an anti-holomorphic involution $\sigma_0$ on $SO(8, \mathbb{C})$ by

$$(6.2) \quad \sigma_0(g) = I_{+-} g I_{+-} \quad (g \in SO(8, \mathbb{C})).$$

Lemma 6.2. The involution $\sigma_0$ stabilizes $Spin(7, \mathbb{C}), SO(7, \mathbb{C}), G_2(\mathbb{C})$ and $SL(3, \mathbb{C})$.

Proof. As $I_{+-} \in G_2$, it is obvious that $\sigma_0$ stabilizes $Spin(7, \mathbb{C}), SO(7, \mathbb{C})$ and $G_2(\mathbb{C})$.

For the proof that $SL(3, \mathbb{C})$ is $\sigma_0$-stable, it is necessary to verify the relation $\sigma_0(g) e_1 = e_1$ for any $g \in SL(3, \mathbb{C})$. Let $g$ be an element of $SL(3, \mathbb{C})$. It is obvious that $g \in SL(3, \mathbb{C})$. Thus, we obtain

$$(\sigma_0(g)) e_1 = I_{+-} g I_{+-} (e_1) = I_{+-} g (-e_1) = I_{+-} (-e_1) = e_1.$$

Then, $\sigma_0$ stabilizes $SL(3, \mathbb{C})$. \hfill $\Box$

By Lemma 6.2, the restrictions of $\sigma_0$ to $Spin(7, \mathbb{C}), SO(7, \mathbb{C})$ and $G_2(\mathbb{C})$ becomes involutions on $Spin(7, \mathbb{C}), SO(7, \mathbb{C})$ and $G_2(\mathbb{C})$, respectively, which we use the same letter $\sigma_0$ to denote.

We choose a Cartan involution $\theta$ of $SO(8, \mathbb{C})$ as in (2.3). Clearly, $\theta$ commutes with $\sigma_0$, from which $\sigma_0$ stabilizes the maximal compact subgroup $SO(8) = \{g \in SO(8, \mathbb{C}) : \theta(g) = g\}$ of $SO(8, \mathbb{C})$. By definition, $Spin(7), SO(7)$ and $G_2$ are also $\sigma_0$-stable.

Let $(G_C, H_C)$ be a non-symmetric reductive spherical pair contained in Table 1.1. The anti-holomorphic involution $\sigma_0$ on $G_C$ induces an anti-holomorphic diffeomorphism $\sigma$ on the non-symmetric spherical homogeneous space $G_C/H_C$ as follows:

$$(6.3) \quad \sigma(g H_C) := \sigma_0(g) H_C \quad (g \in G_C).$$

Now, let us show the condition (S.1) for our choice of $\sigma$ in (6.3). The submanifold $S$ in $G_C/H_C$ given in (6.1) comes from the one-dimensional abelian subgroup in $G_C$ given by $A_0$ (Type R-1), $\tilde{A}_0$ (Type R-1), $A_1$ (Type R-2). Then, it is necessary for the verification of (S.1) to show the following:

Lemma 6.3. $\sigma_0|_A = \text{id}_A$ for $A = A_0, \tilde{A}_0, A_1$. 
Hence,

$$\sigma_0(a_\theta) = I_{+-} a_{(-\theta)} I_{+-}$$

$$= \text{diag}(I_{1,1} I_{2} I_{1,1}, I_{1,1} d_{(-\theta)} I_{1,1}, I_{1,1} d_{(-\theta)} I_{1,1})$$

$$= \text{diag}(I_{2}, d_{\theta}, d_{\theta}, d_{\theta}) = a_\theta.$$ 

Hence, $$\sigma_0|_{A_0} = \text{id}_{A_0}$$ holds.

Next, let $$\tilde{a}_\theta = \text{diag}(d_{\theta}, d_{(-\theta/3)}, d_{(-\theta/3)}, d_{(-\theta/3)})$$ be an element of $$\tilde{A}_0$$.

Then, we have

$$\sigma_0(\tilde{a}_\theta) = \text{diag}(I_{1,1} d_{(-\theta)} I_{1,1}, I_{1,1} d_{(\theta/3)} I_{1,1}, I_{1,1} d_{(\theta/3)} I_{1,1}, I_{1,1} d_{(\theta/3)} I_{1,1})$$

$$= \text{diag}(d_{\theta}, d_{(-\theta/3)}, d_{(-\theta/3)}, d_{(-\theta/3)}) = \tilde{a}_\theta.$$ 

This implies that $$\sigma_0$$ is the identity map on $$\tilde{A}_0$$.

Finally, let $$t_\theta = \exp \tau_\theta = \text{diag}(\exp(\delta_{0,0}), \exp(\delta_{(-2,0,2)}))$$ be an element of $$A_1$$. We put $$J_{+-} := \text{diag}(1, -1, 1, -1)$$. Then, we have $$J_{+-} \delta_{(x,y)} J_{+-} = \delta_{(-x,-y)}$$. Hence, we obtain

$$\sigma_0(t_\theta) = I_{+-} \text{diag}(\exp(\delta_{0,0}), \exp(\delta_{(2,0,2)})) I_{+-}$$

$$= \text{diag}(\exp(J_{+-} \delta_{(0,0)} J_{+-}), \exp(J_{+-} \delta_{(2,0,2)} J_{+-}))$$

$$= \text{diag}(\exp(\delta_{(0,0)}), \exp(\delta_{(2,0,2)})) = t_\theta.$$ 

Hence, $$\sigma_0|_{A_1} = \text{id}_{A_1}$$.

Therefore, Lemma 6.3 has been proved. $\Box$

Thanks to Lemma 6.3, the following equality holds for any $$aH_C \in S = AH_C/H_C (a \in A)$$:

$$\sigma(aH_C) = \sigma_0(a)H_C = aH_C.$$ 

Hence, we have verified $$\sigma|_S = \text{id}_S$$, namely, (S.1).

6.3. Verification of (S.2). In this subsection, we shall see that the condition (S.2) follows from (V.1) and (S.1).

Retain the setting as in Section 6.2. Let $$x$$ be an element of the spherical homogeneous space $$G_C/H_C$$ of rank-one type. By the condition (V.1), one can find elements $$k \in G_u$$ and $$a \in A$$ such that $$x = k \cdot aH_C$$. As $$\sigma|_S = \text{id}_S$$ (the condition (S.1)), we have

$$\sigma(x) = \sigma_0(k) \cdot \sigma(aH_C) = \sigma_0(k) \cdot aH_C = (\sigma_0(k)^{-1}) \cdot x.$$ 

Here, $$\sigma_0$$ stabilizes $$G_u$$ (see Section 6.2). Then, $$\sigma_0(k)^{-1}$$ is an element of $$G_u$$. Hence, $$(\sigma_0(k)^{-1}) \cdot x$$ lies in the $$G_u$$-orbit through $$x$$, from which we have shown $$\sigma(x) \in G_u \cdot x$$. Hence, the condition (S.2) has been verified.

6.4. Proof of Theorem 6.1. For a reductive spherical pair $$(G_C, H_C)$$ of rank-one type, we have verified the condition (V.1) in Section 6.1 (S.1) in Section 6.2 and (S.2) in Section 6.3. Therefore, Theorem 6.1 has been proved.
6.5. Remark. We end this paper by the observation of $\sigma_0$ from the corresponding fixed point set of the Lie algebra as follows.

Let $(G_C, H_C)$ be a non-symmetric spherical pair of rank-one type and $\mathfrak{g}$ the Lie algebra of a complex simple Lie group $G_C$. We use the same letter $\sigma_0$ to denote the differential automorphism on $\mathfrak{g}$. We write $\mathfrak{g}^{\sigma_0}$ as the fixed point set of $\sigma_0$ in $\mathfrak{g}$.

Our choice of $\sigma_0$ satisfies that $(\mathfrak{so}(8, \mathbb{C}))^{\sigma_0}$ is isomorphic to $\mathfrak{so}(4, 4)$. Then, its real rank, denoted by $\text{rank}_{\mathbb{R}}(\mathfrak{so}(8, \mathbb{C}))^{\sigma_0}$, equals four, which coincides with $\text{rank}(\mathfrak{so}(8, \mathbb{C}))$. This means $(\mathfrak{so}(8, \mathbb{C}))^{\sigma_0}$ is a normal real form of $\mathfrak{so}(8, \mathbb{C})$. Further, we have

$$(\mathfrak{so}(7, \mathbb{C}))^{\sigma_0} \simeq \mathfrak{so}(7, \mathbb{C}) \cap \mathfrak{so}(4, 4) \simeq \mathfrak{so}(3, 4),$$

$$(\mathfrak{spin}(7, \mathbb{C}))^{\sigma_0} \simeq \mathfrak{spin}(7, \mathbb{C}) \cap \mathfrak{so}(4, 4) \simeq \mathfrak{spin}(3, 4),$$

$$(\mathfrak{g}_2(\mathbb{C}))^{\sigma_0} \simeq \mathfrak{g}_2(\mathbb{C}) \cap \mathfrak{so}(4, 4) \simeq \mathfrak{g}_2(2).$$

It turns out that the Lie algebra $\mathfrak{g}^{\sigma_0}$ satisfies

**Proposition 6.4.** $\text{rank}_{\mathbb{R}} \mathfrak{g}^{\sigma_0} = \text{rank} \mathfrak{g}$.

We have found the same property as Proposition 6.4 in the non-symmetric spherical homogeneous spaces of line bundle case. Namely, we prove that if $G_C/H_C$ is a line bundle $G_C/[K_C, K_C]$ over the complexification $G_C/K_C$ of an irreducible Hermitian symmetric space $G/K$ of non-tube type, then the $G_u$-action on $G_C/[K_C, K_C]$ is strongly visible and one can take a slice $S$ and an anti-holomorphic diffeomorphism $\sigma$ satisfying (V.1)–(S.2) and Proposition 6.4 (see [14, Lemma 2.2] and [17]). The key ingredient is to find a Cartan decomposition for $(G_C, [K_C, K_C])$ explicitly.

We can show that for any reductive spherical pair $(G_C, H_C)$ we give a Cartan decomposition by giving an explicit description of the abelian part and that the $G_u$-action on the spherical homogeneous space $G_C/H_C$ is strongly visible with a slice coming from a Cartan decomposition for $(G_C, H_C)$ and an anti-holomorphic diffeomorphism coming from an involution on $G_C$ satisfying Proposition 6.4. In fact, some cases have been studied in [14, 15, 16, 17] from the viewpoint of strongly visible actions, and the others will be explained in the forthcoming papers which contain how to find an explicit description of the abelian part for a Cartan decomposition (cf. [18]).

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(A. Sasaki) Department of Mathematics, Faculty of Science, Tokai University, 4-1-1, Kitakaname, Hiratsuka, Kanagawa, 259-1292, Japan.

E-mail address: atsumu@tokai-u.jp