Evolution of the average avalanche shape with the universality class

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A multitude of systems ranging from the Barkhausen effect in ferromagnetic materials to plastic deformation and earthquakes respond to slow external driving by exhibiting intermittent, scale-free avalanche dynamics or crackling noise. The avalanches are power-law distributed in size, and have a typical average shape: these are the two most important signatures of avalanching systems. Here we show how the average avalanche shape evolves with the universality class of the avalanche dynamics by employing a combination of scaling theory, extensive numerical simulations and data from crack propagation experiments. It follows a simple scaling form parameterized by two numbers, the scaling exponent relating the average avalanche size to its duration and a parameter characterizing the temporal asymmetry of the avalanches. The latter reflects a broken time-reversal symmetry in the avalanche dynamics, emerging from the local nature of the interaction kernel mediating the avalanche dynamics.
he theoretical interpretation of crackling noise\(^1\), observed in numerous systems, including the Barkhausen effect in ferromagnetic materials\(^2\),\(^3\), plastic deformation\(^4\),\(^5\), structural transitions\(^7\) and fracture\(^6\)\(^8\)\(^9\) of solids and earthquakes\(^10\), has found a formulation in terms of non-equilibrium phase transitions\(^1\). These transitions separate quiescent and active phases of the system, and naturally give rise to critical scaling\(^12\). In the vicinity of such a phase transition, the time evolution of the activity signal \(V(t)\) or the order parameter of the transition (for example, the interface velocity for a depinning transition) exhibits scale-free bursts or avalanches. Statistical analysis of such fluctuations, together with renormalization group calculations\(^13\), suggest that in general systems with avalanche dynamics can be classified into universality classes characterized by the values of the critical exponents, depending on, for example, the spatial dimension and the interaction range of the system.

The average temporal shape of bursts in a crackling noise signal is a fundamental signature of avalanches, and has been estimated for systems as diverse as plastically deforming crystals\(^14\), earthquakes\(^15\) and Barkhausen noise\(^16\)\(^18\). For the latter, the symmetric average avalanche shape observed in ferromagnetic films of intermediate thickness where the long-range dipolar interactions render the avalanche dynamics mean field-like has been explained within the ABBM model\(^19\), and shown to be given by an inverted parabola\(^16\)\(^20\)\(^21\)\(^22\). In thick enough samples eddy currents induce an effective mass for the propagating domain walls, visible as an asymmetry in the (mean-field-like) average shape of the Barkhausen pulses\(^17\). In general, clear-cut shape determinations should give strong indications of the underlying physics, such as the kind and range of interactions governing the avalanche dynamics.

Here we present a general scaling form for the average avalanche shapes for non-mean-field systems. It is verified within a large-scale numerical study of avalanches at the depinning transition of driven elastic interfaces in random media. We vary systematically the range of the elastic interaction kernel, and thus the universality class of the avalanche dynamics\(^23\), showing how the avalanche shape depends on the universality class. The average shape is to a high precision given by a function parameterized by the scaling exponent \(\gamma\) characterizing the scaling of the average avalanche size as a function of the avalanche duration, and a parameter \(a\) describing the temporal asymmetry of the average avalanche shapes. We find that an inherent asymmetry in the average avalanche shapes is present in systems where the interaction kernel is not fully non-local, reflecting the underlying broken time-reversal symmetry of the avalanche dynamics. Finally, we compare these results with experiments of planar crack front propagation, finding good agreement with the predictions of the scaling theory and the relevant depinning model.

Results

Average avalanche shape scaling function. To obtain a general scaling form for the average avalanche size for the bursts in \(V(t)\) corresponding to avalanches of a given duration \(T\), \(\langle V(t \mid T) \rangle\), we start from the well-known result that the average avalanche size \(\langle s(T) \rangle\) follows in the scaling regime

\[
\langle s(T) \rangle \equiv \int_0^T \langle V(t \mid T) \rangle dt \propto T^\gamma.
\]

Further, we may relate \(s(T)\) to the asymptotic shape of the avalanche at long times, that is, the solution of the so-called ‘Edwards–Wilkinson equation’\(^26\) used in this study (see also Supplementary Note 1).

Numerical simulations of interface depinning models. To check equation (5), we perform extensive simulations of a discretized model of a 1d elastic string or interface in a 2d random medium\(^27\), represented by a set of integer heights \(h_i(t)\), \(i = 1 \ldots L\), with \(L\) the system size. The lateral coordinates \(x_i\) of the interface

\[
\langle V(t \mid T) \rangle \propto T^{\gamma-1} \left[ \frac{1}{T} \left( 1 - \frac{t}{T} \right) \right]^{\gamma-1} - \left[ 1 - a \left( \frac{t}{T} - \frac{1}{2} \right) \right].
\]

If \(a = 0\), equation (5) reduces to equation (4) and corresponds to a symmetrical avalanche shape. With \(a \neq 0\), equation (5) describes a temporally asymmetric avalanche shape, with a positive or negative skewness for \(a > 0\) and \(a < 0\), respectively (see also Supplementary Note 1).
Figure 1 | Space-time activity plots reveal broken time-reversal symmetry in avalanche dynamics. Examples of space-time activity of typical large avalanches for three cases: (a) the qEW equation with local elasticity, (b) the crack line model with non-local elasticity (z = 2) and (c) the mean-field infinite range model. Notice the clear time-irreversible nature of the spatio-temporal avalanche structure in (a), and the reversible character in (c).

are given by \( x_i = i \). The total force acting on the interface element \( i \) is

\[
F_i = \frac{G}{C_0} \sum_{j \neq i} \frac{|x_j - x_i|}{x_j - x_i} \cdot h_i + \eta(x_i, h_i) + F_{\text{ext}},
\]

where the first term on the RHS represents the elastic interactions characterized by the exponent \( z \), \( \eta \) is uncorrelated quenched disorder and \( F_{\text{ext}} \) is the external driving force. Notice that in the limit \( z \to \infty \), the elastic interaction term becomes completely local, \( \Gamma \delta(h_{i+1} + h_{i-1} - 2h_i) \equiv \Gamma \delta^2 h_i \); thus, equation (6) reduces to the qEW equation. In the opposite limit \( z \to 0 \) the system loses its spatial structure and we describe it by the mean-field infinite-range model, by replacing the elastic interactions in equation (6) by \( \Gamma \delta(h - h_i) \), with \( h = 1/L \sum_i h_i \). For the intermediate case of \( z = 2 \), equation (6) reduces to the long-range elastic string expected to describe, for example, planar crack fronts propagating along disordered weak planes between solid blocks, contact lines of liquids spreading on solid surfaces and low-angle grain boundaries in plastically deforming crystals. Furthermore, \( z \geq 3 \) belongs to the qEW class, whereas \( z = 1 \) is expected to follow mean-field dynamics. The crackling noise signal is given by \( V(t) = \sum_i \nu_i(t) \), where \( \nu_i = \theta(F_i) \), with \( \theta \) the Heaviside step function. For additional details, see Methods.

As expected, \( \langle s(T) \rangle \) scales with \( T \) according to equation (1) (see Supplementary Fig. S1 and Supplementary Note 2). We find three different values of \( \gamma \), that is, \( \gamma = 2.0 \pm 0.01 \) for \( z \leq 1 \), \( \gamma = 1.79 \pm 0.01 \) for \( z = 2 \) and \( \gamma = 1.56 \pm 0.01 \) for \( z \geq 3 \). These values are in agreement with earlier results, either directly or via scaling relations. Fitting equation (5) to the \( \langle V(t) | T \rangle \) data for various \( z \) and different ranges of \( T \) in the scaling regime reproduces well these \( \gamma \) exponents (Fig. 2a,b; see also Supplementary Figs S2–S5 and Supplementary Notes 3 and 4). The avalanche shapes exhibit an asymmetry, with the asymmetric parts of equation (5) being shown with lines, for various values of \( z \) obtained for different duration ranges

\[
-0.02 \leq \gamma \leq 0.02
\]

\[
0 \leq \gamma \leq 1
\]

\[
\gamma = 1.56 \quad \text{and} \quad \gamma = 1.79
\]

\[
\gamma = 2.0 \quad \text{and} \quad \gamma = 3.3
\]

\[
\gamma = 2.0 \quad \text{and} \quad \gamma = 3.3
\]

\[
\gamma = 2.0 \quad \text{and} \quad \gamma = 3.3
\]
Fig. S6 and Supplementary Note 5). The corresponding skewness (computed by interpreting \( \langle V(t \mid T) \rangle \) as a probability density\(^{12} \)) of the avalanches exhibits a similar evolution with \( x \) (Supplementary Fig. S7, Supplementary Note 6). Thus, avalanches whose dynamics is governed by interaction kernels that are not fully non-local are temporally asymmetric, as illustrated by the time-irreversible nature of the corresponding space-time activity patterns (Fig. 1).

**Planar crack front propagation experiments.** Finally, we consider data from planar crack front propagation experiments\(^{9,32} \), as an example of an experimental system with non-mean-field avalanche dynamics, see Methods for details. The scaling of the average size of the avalanches of crack front propagation as a function of their duration is shown in Fig. 3b. In the scaling regime, these are characterized by \( \gamma = 1.67 \pm 0.15 \), in agreement with the 1d non-local elasticity depinning model with \( x = 2 \) (refs 27,28), see also Supplementary Fig. S8. The average avalanche shape is shown in Fig. 3a. Owing to the non-negligible statistical fluctuations present in the data, it is not possible to detect the small asymmetry predicted by the crack line model\(^{27,28} \). Notice also that the experimental shape clearly differs from both the mean-field inverted parabola and the shape expected for the local qEW equation.

\[
\langle V(t \mid T) \rangle = \langle V(T) \rangle \frac{1}{\sqrt{T}} \quad \text{for} \quad T > T_{\text{max}}
\]

\[
\langle V(t \mid T) \rangle = \langle V(T) \rangle \frac{1}{\sqrt{T}} \quad \text{for} \quad T < T_{\text{max}}
\]

\[
\langle V(t \mid T) \rangle = \langle V(T) \rangle \frac{1}{\sqrt{T}} \quad \text{for} \quad T = T_{\text{max}}
\]

**Discussion**

We have shown how the average avalanche shape of systems exhibiting cracking noise depends on the universality class of the avalanche dynamics. It is a fundamental fingerprint of an avalanching system and extrapolates when tuning elastic interactions between an inverted parabola for mean-field systems and a shape close to a semicircle for the 1d short-range interface. The broken time-reversal symmetry in the avalanche dynamics emerging from the spatially localized interactions is manifested as a temporal asymmetry in the avalanche shape evolving with the interaction range (see also Supplementary Discussion). Thus, such asymmetries should be looked for in experimental data in systems where the interactions mediating the avalanche dynamics are not fully non-local. These include, for example, domain wall dynamics in magnetic thin films\(^{33} \) and fluid invasion into disordered media\(^{4,35} \).

**Methods**

**Numerical simulations of interface depinning models.** We simulate the interface depinning model, equation (6) with periodic boundary conditions. The parallel noise signal of interest is given by

\[
F_{\text{ext}} = -k/L \sum_i v_i(t) \quad \text{for} \quad 0 \leq i < L.
\]

The crackling noise signal of interest is given by \( V(t) = \sum_i v_i(t) \). To compute the average avalanche shapes, we collect a large ensemble of avalanches from various duration ranges. For \( T \) in the scaling regime, the average shapes corresponding to the various duration ranges fall onto a single curve after normalizing with the maximum amplitude, \( \langle V(t \mid T) \rangle_{\text{max}} \). The simulations are performed in large system sizes \( L = 8,192, 32,768 \) and \( 8,388,608 \) for \( x = 2 \) and \( 8,192 \) for \( x = 3 \), with \( \langle V(t \mid T) \rangle_{\text{max}} \). The simulations show that the avalanche cut-off size \( s_0 \approx k^{-1/\gamma} \) is large. Examples of the crackling noise signals \( V(t) = \sum_i v_i(t) \) obtained from the model for different interaction ranges are shown in Fig. 4.

**Planar crack front propagation experiments.** To extract the average avalanche shape for the planar crack propagation experiments, slow creep motion of a planar...
crack front propagating along the heterogeneous weak plane of a transparent poly(methyl methacrylate) block, made of two sintered rough Flexiglass plates (of dimensions (27, 14 and 1) cm for the top and (30, 12 and 0.4) cm for the bottom plate, respectively) is studied. We imposed a constant normal displacement \( d \) to the bottom plate while the upper one is fixed, resulting in a quasi-mode I creep growth of the crack, see Fig. 5a. The interfacial fracture front was observed in a small central region (to avoid boundary effects) corresponding to around 4.48 mm lateral size of the imaging area divided by \( N \), ranging from \( 0.017 \) to \( 0.75 \) mm. 

The two solid black lines in (c) correspond to the bottom plate while keeping the upper plate fixed. (b) By observing a small central region with a camera, the local waiting times \( w(t) \) of the crack front are measured at each location, with dark (light) regions corresponding to the bottom plate while keeping the upper plate fixed. (c) The velocity signal \( V(t) \) is measured at the scale of 200 μm (dashed blue line in (b)), by considering the average of the local velocities \( v(x,y) \), defined as the inverse of the local waiting times. The inset shows a close-up of an avalanche of duration \( T \), defined as a continuous occurrence of the global velocity \( V(t) \) above its average \( V = \langle V(t) \rangle \).

Figure 5 | The experimental setup. (a) Creep experiments of planar crack propagation are performed by imposing a constant normal displacement \( d \) to the bottom plate while keeping the upper plate fixed. (b) By observing a small central region with a camera, the local waiting times \( w(t) \) of the crack front are measured at each location, with dark (light) regions corresponding to the bottom plate while keeping the upper plate fixed. The two solid black lines correspond to two examples of the instantaneous crack front profile. (c) The velocity signal \( V(t) \) is measured at the scale of 200 μm (dashed blue line in (b)), by considering the average of the local velocities \( v(x,y) \), defined as the inverse of the local waiting times. The inset shows a close-up of an avalanche of duration \( T \), defined as a continuous occurrence of the global velocity \( V(t) \) above its average \( V = \langle V(t) \rangle \).

References

1. Sethna, J. P., Dahmen, K. & Myers, C. R. Cracking Noise. Nature 410, 242–250 (2001).
2. Durin, G. & Zapperi, S. in The Science of Hysterisis, (eds Bertotti, G. & Mayergozy, L.) 181–287 (Academic, Amsterdam, 2006).
3. Durin, G. & Zapperi, S. Scaling exponents for Barkhausen avalanches in polycrystalline and amorphous ferromagnets. Phys. Rev. Lett. 84, 4705–4708 (2000).
4. Zapperi, S., Castellano, C., Colaiori, F. & Durin, G. Signature of effective mass in the avalanche dynamics of elastic interfaces. Nature Phys. Rev. E 84, 066106 (2013).
5. Fisher, D. S. Collective transport in random media: from superconductors to earthquakes. Rep. Prog. Phys. Rev. E 46, 4705–4708 (1994).
6. Laurson, L. & Alava, M. J. 1/\( T \) noise and avalanche scaling in plasticity and avalanche dynamics of elastic interfaces. Phys. Rev. E 79, 051106 (2009).
7. Mehta, A. P., Dahmen, K. & Sethna, J. P. Universal pulse shape function and exponents: critical test for avalanche models applied to Barkhausen noise. Phys. Rev. E 65, 046139 (2002).
8. Alessandro, B., Beatrice, C., Bertotti, G. & Montorsi, A. Domain wall dynamics and Barkhausen effect in metallic ferromagnetic materials. J. Appl. Phys. 48, 2901–2908 (1990).
9. Le Doussal, P. & Wiese, K. J. Distribution of velocities in an avalanche. EPL 97, 46004 (2012).
22. Dobrinevski, A., Le Doussal, P. & Wiese, K. J. Nonstationary dynamics of the Alessandro-Beatrice-Bertotti-Montorsi model. *Phys. Rev. E* **85**, 031105 (2012).

23. Tangui, A., Gounelle, M. & Roux, S. From individual to collective pinning: Effect of long-range interactions. *Phys. Rev. E* **58**, 1577–1590 (1998).

24. Baldassarri, A., Colaiori, F. & Castellano, C. Average shape of a fluctuation: universality in excursions of stochastic processes. *Phys. Rev. Lett.* **90**, 060601 (2003).

25. Laurson, L., Illa, X. & Alava, M. J. The effect of thresholding on temporal avalanche statistics. *J. Stat. Mech.* P01019 doi:10.1088/1742-5468/2009/01/P01019 (2009).

26. Nattermann, T., Stepanow, S., Tang, L.-H. & Leschhorn, H. Dynamics of interface depinning in a disordered medium. *J. Phys. II France* **2**, 1483 (1992).

27. Bonamy, D., Santucci, S. & Ponson, L. Crackling dynamics in material failure as the signature of a self-organized dynamic phase transition. *Phys. Rev. Lett.* **101**, 045501 (2008).

28. Laurson, L., Santucci, S. & Zapperi, S. Avalanches and clusters in planar crack front propagation. *Phys. Rev. E* **81**, 046116 (2010).

29. Joanny, J. F. & De Gennes, P. G. A model for contact angle hysteresis. *J. Chem. Phys.* **81**, 552–562 (1984).

30. Moretti, P., Miguel, M. C., Zaiser, M. & Zapperi, S. Depinning transition of dislocation assemblies: Pileups and low-angle grain boundaries. *Phys. Rev. B* **69**, 214103 (2004).

31. Duemmer, O. & Krauth, W. Depinning exponents of the driven long-range elastic string. *J. Stat. Mech.* P01019 doi:10.1088/1742-5468/2007/01/P01019 (2007).

32. Tallakstad, K. T., Toussaint, R., Santucci, S., Schmittbuhl, J. & Mály, K. J. Local dynamics of a randomly pinned crack front during creep and forced propagation: An experimental study. *Phys. Rev. E* **83**, 046108 (2011).

33. Ryu, K.-S., Akinaga, H. & Shin, S.-C. Tunable scaling behaviour observed in Barkhausen criticality of a ferromagnetic film. *Nature Phys.* **3**, 547–550 (2007).

34. Rost, M., Laurson, L., Dubé, L. & Alava, M. J. Fluctuations in fluid invasion into disordered media. *Phys. Rev. Lett.* **98**, 054502 (2007).

35. Planet, R., Santucci, S. & Ortin, J. Avalanches and non-Gaussian fluctuations of the global velocity of imbibition fronts. *Phys. Rev. Lett.* **102**, 094502 (2009).

36. Durin, G. & Zapperi, S. Universality and size effects in the Barkhausen noise. *J. Appl. Phys.* **87**, 7031–7033 (2000).

37. Csikor, F., Motz, C., Weygand, D., Zaiser, M. & Zapperi, S. Dislocation avalanches, strain bursts, and the problem of plastic forming at the micrometer scale. *Science* **318**, 253–254 (2007).

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**Author contributions**

L.L. and X.I. designed and performed the numerical modelling. L.L. and M.J.A. developed the scaling theory. S.S., K.T.T. and K.J.M. performed and analysed the experiments. L.L. wrote the first draft of the manuscript. All authors contributed to improve the manuscript.

**Additional information**

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