Three-loop contributions
to the free energy of $\lambda \varphi^4$ QFT

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Abstract
The massive scalar field with $\lambda \varphi^4$ interaction placed in $(3 + 1)$ dimensional box is considered. The sizes of the box are $V \times \beta$ ($V = L^3$ is the volume, $T = 1/\beta$ is the temperature). The free energy is evaluated up to the 2nd order of $PT$. The averaging on the vacuum fluctuations is separated from the averaging on the thermal fluctuations explicitly. As result the free energy is expressed through the scattering amplitudes. We find that in 3-loop approximation the expression for free energy coincides with the ansatz of Bernstein, Dashen, Ma suggested on the base of $S$-matrix formulation of statistical mechanics. The obtained expressions are generalized for higher order of $PT$.

1 Introduction
The perturbation theory ($PT$) for quantum field models at finite temperature $T$ is elaborated enough [1]. Its formal distinction from ordinary Feynman rules consists in that the loop summations on discrete values of zero component of momentum appear (at $T \neq 0$) instead of the loop integrations (at $T = 0$):

$$T \sum_{p_0} \to T \to 0 \frac{1}{2\pi} \int_{-\infty}^{\infty} dp_0,$$

where for the boson fields

$$p_0 = 2\pi l T, \quad l = 0, \pm 1, \pm 2, \ldots$$

The loop integral is treated as an averaging on vacuum fluctuations. It is clear that at $T \neq 0$ both vacuum and thermal fluctuations of the field take place. So, the difference between the loop sum and loop integral is caused by thermal fluctuations alone. The standard task of the quantum field theory (QFT) at finite temperature is the calculation of radiative corrections to the free energy (or other thermodynamic quantities) of the relativistic ideal gas as functions of temperature, chemical potentials, renormalized masses, coupling parameters, etc. There are no principal difficulties, but the job is more cumbersome compared to usual Feynman diagrammar.
We want to consider the problem from some other standpoint. We shall try to separate the averaging on the thermal fluctuations from that of the vacuum fluctuations explicitly and to represent the contribution caused by interaction to the free energy as some integrals on Bose-Einstein distribution of renormalized pure QFT values.

Note that probably the similar purpose was pursued in the papers [2,3] devoted to the elaboration of QFT at finite temperature in real time representation. But there was not achieved such simple and physically transparent result as we obtained in the framework of standard imaginary time representation.

We shall work out the calculations for the \( \lambda \phi^4 \) QFT model (extraction of the loop integrals from the corresponding sums, renormalization) up to the 2nd order of PT, where the nontrivial 3-loops diagrams appear.

## 2 The 1st order of PT

For the beginning consider the first radiative correction to the free energy. This exercise clarifies what result we would like to obtain.

The partition function of the scalar field placed to the thermostat of the volume \( V \) at the temperature \( T = 1/\beta \) is

\[
Z = \int \mathcal{D}\varphi \exp(-S_0[\varphi] - S_I[\varphi]),
\]

where the action in Euclidean metrics is

\[
S_0[\varphi] = \frac{1}{2} \int_{\beta,V} d^4x \varphi(x)(m^2 - \Box)\varphi(x),
\]

\[
x = (x_0, x), \quad x^2 = x_0^2 + x^2, \quad \Box = \frac{\partial^2}{\partial x_0^2} + (\nabla)^2,
\]

\[
S_I[\varphi] = \frac{\lambda}{4!} \int_{\beta,V} d^4x \varphi^4(x).
\]

The free energy density in the 1st order on \( \lambda \) is

\[
f(T) \equiv -\frac{\ln Z}{\beta V} = f^{(0)}(T) + f^{(1)}(T),
\]

where at \( V \to \infty \),

\[
f^{(0)}(T) = \frac{1}{2\beta V} \text{Sp} \ln[\beta^2(m^2 - \Box)] = \frac{1}{2(2\pi)^3} \int d^3p \left( T \sum_{p_0} \ln[\beta^2(m^2 + p^2)] \right),
\]

\[
f^{(1)}(T) = \frac{1}{\beta V} \langle S_I[\varphi] \rangle = \frac{\lambda}{8} \left( \frac{1}{(2\pi)^3} \int d^3p \left( T \sum_{p_0} \frac{1}{m^2 + p^2} \right) \right)^2,
\]

\[
p = (p_0, p), \quad p^2 = p_0^2 + p^2.
\]

Let us define the elementary diagrams to make the calculations more compact and transparent as follows

\[
\bigcirc \equiv -\frac{1}{(2\pi)^3} \int d^3p \left( T \sum_{p_0} \ln[\beta^2(m^2 + p^2)] \right), \tag{1}
\]
\[ \bigcirc \equiv \frac{1}{(2\pi)^3} \int d^3p \left( T \sum_{p_0} \frac{1}{m^2 + p^2} \right) = -\frac{\partial}{\partial m^2} \bigcirc. \tag{2} \]

It is easy to compute the sum on \( p_0 \) in (2)

\[ T \sum_{p_0} \frac{1}{p_0^2 + \omega^2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dp_0}{m^2 + p^2} + \frac{1}{\omega(e^{\beta\omega} - 1)}, \quad \omega = \sqrt{m^2 + p^2}. \tag{3} \]

One can see from this expression that the averaging on vacuum and thermal fluctuations corresponding to one loop sum can be represented additively. By integrating eq.(3) on \( m^2 \) one obtains the temperature dependence of (1) i.e.

\[ f^{(0)}(T) = -\frac{1}{2} \bigcirc = \frac{T}{(2\pi)^3} \int d^3p \ln(1 - e^{-\beta\omega}) + a + bT, \tag{4} \]

where \( a \) and \( b \) are irrelevant constants (energy density and pressure of the vacuum), their values depend on regularization scheme. Let us denote the vacuum and thermal loops as

\[ \bigcirc = -\frac{2T}{(2\pi)^3} \int d^3p \ln(1 - e^{-\beta\omega}), \]

\[ V \equiv \bigcirc \bigg |_{T=0} = \frac{1}{(2\pi)^4} \int \frac{d^4p}{m^2 + p^2}, \]

\[ \bigcirc - V = \frac{2}{(2\pi)^3} \int \frac{d^3p}{2\omega(e^{\beta\omega} - 1)}. \tag{5} \]

In these notations the first order correction to the free energy has the following diagrammatic representation

\[ f^{(1)}(T) = \frac{\lambda}{8} \bigcirc \bigcirc = \frac{\lambda}{8} \left( \bigcirc \bigcirc + 2 \bigcirc V + V V \right). \tag{6} \]

Note, that \( \bigcirc \bigcirc = (\bigcirc V)^2 \). The divergent term \( V V \) in (6) is irrelevant because it does not depend on temperature. The first term in the r.h.s. of (6) is convergent and it can be represented as the integral on two-particles invariant phase volume with Bose-Einstein statistical factors

\[ \frac{\lambda}{8} \bigcirc V = \frac{1}{2!} \frac{1}{(2\pi)^6} \int \frac{d^4p_1 d^4p_2}{2\omega_1 2\omega_2} \frac{\lambda}{(e^{\beta\omega_1} - 1)(e^{\beta\omega_2} - 1)}. \tag{7} \]

Keeping in mind that in the 1st order of \( PT \) the \( 2 \to 2 \) scattering amplitude is simply the coupling constant

\[ A_{2 \to 2}^{(1)} = -\lambda, \]

one can consider (7) as the thermal averaging of the \( 2 \to 2 \) scattering amplitude. Of course such interpretation looks artificial enough. In the 2nd order of \( PT \) the \( 2 \to 2 \) scattering amplitude has nontrivial dependence on momenta, also \( 3 \to 3 \) amplitude appears. So, the next order of \( PT \) can bring more information about structure of representations of type (7). Before the 2nd order analyses the diagram \( \bigcirc V \) have to be examined. It has nontrivial dependence on
temperature and is divergent. It is easy to see that this diagram can be removed by the mass
renormalization

\[ m^2 \rightarrow m^2 - \delta m^2_1. \]

Then the zero order diagram in (4)

\[ \bigcirc \rightarrow \bigcirc + \delta m^2_1 \bigcirc, \]

and for the cancellation of the divergent temperature dependent term of (6) is necessary

\[ \frac{\lambda}{4} \bigcirc \times \bigcirc - \frac{1}{2} \delta m^2_1 \bigcirc = 0, \]

that is

\[ \delta m^2_1 = \frac{\lambda}{2} \bigcirc = \frac{\lambda}{2} \frac{1}{(2\pi)^4} \int \frac{d^4 p}{m^2 + p^2}. \] (8)

3 The 2nd order of PT

In the 2nd order of PT the radiative correction to the free energy is

\[ f^{(2)}(T) = -\frac{1}{\beta V} \langle S^2_I[\varphi] \rangle_c = -\frac{\lambda^2}{48} (3 \bigcirc \bigcirc \bigcirc + \bigcirc \bigcirc \bigcirc), \] (9)

where the diagrams mean

\[ \bigcirc \bigcirc \bigcirc = \bigcirc \times \bigcirc \times \bigcirc, \] (10)

\[ \bigcirc \bigcirc \equiv \frac{1}{(2\pi)^3} \int d^3 p \left( T \sum_{p_0} \frac{1}{(m^2 + p^2)^2} \right) = -\frac{\partial}{\partial m^2} \bigcirc, \] (11)

\[ \bigcirc \bigcirc \bigcirc \equiv \frac{1}{(2\pi)^9} \int \left( \prod_{k=1}^4 d^3 p_k \right) \delta^3 (p_1 + p_2 + p_3 + p_4) \times \left( T^3 \sum_{p_{01} \ldots p_{04}} \frac{\delta (p_{01} + p_{02} + p_{03} + p_{04})}{(m^2 + p_{01}^2)(m^2 + p_{02}^2)(m^2 + p_{03}^2)(m^2 + p_{04}^2)} \right), \] (12)

where \( \delta \)-functions in (12) are mentioned as the Dirac one for integrals and the Kroneker symbol
for sums.

The separation of the vacuum and thermal loops from the diagram (10) is not difficult due
to eq.(5), but the corresponding procedure for the diagram (12) requires some manipulations.
Let us write the diagram (12) as a sum of four possible combinations of vacuum and thermal
loops

\[ \bigcirc \bigcirc \bigcirc \equiv \bigcirc \bigcirc \bigcirc + 4 \bigcirc \bigcirc \bigcirc + 6 \bigcirc \bigcirc \bigcirc + 4 \bigcirc \bigcirc \bigcirc, \] (13)

where the coefficients in the r.h.s. of (13) are chosen for the combinatorial reason: the number of
possibilities to cut one, two or three lines connecting two vertices. The number of thermal loops
in diagrams (13) will be equal to the number of statistical factors \( (e^{B\omega} - 1)^{-1} \) in corresponding
integrals. Let us denote the sum under the integral (12) by \( W(\omega_1, \omega_2, \omega_3, \omega_4) \)

\[ W = T^3 \sum_{p_{01} \ldots p_{04}} \frac{\delta (p_{01} + \cdots + p_{04})}{(p_{01}^2 + \omega_1^2) \cdots (p_{04}^2 + \omega_4^2)}, \] (14)
and express the Kroneker δ-symbol through the integral
\[ \delta(p_0) = T \int_0^\beta dx_0 e^{ip_0x_0}. \]

Then calculating the simple sum
\[ T \sum_{p_0} \frac{e^{ip_0x_0}}{p_0^2 + \omega^2} = \frac{\cosh(x_0 - \beta/2)\omega}{2\omega \cdot \sinh(\omega/2)}, \]
we obtain for (14)
\[ W = \int_0^\beta dx_0 \prod_{k=1}^4 \left( T \sum_{p_0} \frac{e^{ip_0x_0}}{p_0^2 + \omega_k^2} \right) = \int_0^\beta dx_0 \prod_{k=1}^4 \left( \frac{\cosh(x_0 - \beta/2)\omega_k}{2\omega_k \sinh(\beta\omega_k/2)} \right) \]

Direct integration in r.h.s. of (15) gives
\[ W = 2 \left[ \prod_{k=1}^4 (2\omega_k(e^{\beta\omega_k} - 1)) \right]^{-1} \cdot \left( \frac{e^{\beta(\omega_1 + \omega_2 + \omega_3 + \omega_4)} - 1}{\omega_1 + \omega_2 + \omega_3 + \omega_4} + \frac{e^{\beta(\omega_2 + \omega_3 + \omega_4)} - 1}{-\omega_1 + \omega_2 + \omega_3 + \omega_4} + 6 \frac{e^{\beta(\omega_1 + \omega_2 + \omega_3)} - 1}{-\omega_1 - \omega_2 + \omega_3 + \omega_4} + 4 \frac{e^{\beta\omega_4} - 1}{-\omega_1 - \omega_2 - \omega_3 + \omega_4} \right). \]

Despite the function \( W \) is symmetric in its arguments by origin it is more convenient to represent the result of integration of (15) in asymmetric but compact form (16), accounting for the integral over momenta in (12) pick out the symmetric part of \( W \) only. Sorting (16) accordingly to the number of statistical factors rewrite \( W \) in the following way
\[ W = W_0 + \frac{4W_1}{2\omega_1(e^{\beta\omega_1} - 1)} + \frac{6W_2}{2\omega_12\omega_2(e^{\beta\omega_1} - 1)(e^{\beta\omega_2} - 1)} + \frac{4W_3}{2\omega_12\omega_22\omega_3(e^{\beta\omega_1} - 1)(e^{\beta\omega_2} - 1)(e^{\beta\omega_3} - 1)}, \]

where
\[ W_0 = 2 \left( \prod_{k=1}^4 2\omega_k \right)^{-1} \frac{1}{\omega_1 + \omega_2 + \omega_3 + \omega_4} = \frac{1}{(2\pi)^3} \int \left( \prod_{k=1}^4 \frac{dp_{0k}}{m^2 + p_k^2} \right) \delta(p_{01} + \cdots + p_{04}), \]

\[ W_1 = 2 \left( \prod_{k=2}^4 2\omega_k \right)^{-1} \left( \frac{1}{\omega_1 + \omega_2 + \omega_3 + \omega_4} + \frac{1}{-\omega_1 + \omega_2 + \omega_3 + \omega_4} \right) = \frac{2}{(2\pi)^2} \int \left( \prod_{k=2}^4 \frac{dp_{0k}}{m^2 + p_k^2} \right) \delta(p_{01} + \cdots + p_{04}) \bigg|_{p_{1}^2 = -m^2}, \]

\[ W_2 = 2 \left( \prod_{k=3}^4 2\omega_k \right)^{-1} \left( \frac{1}{\omega_1 + \omega_2 + \omega_3 + \omega_4} + \frac{1}{-\omega_1 - \omega_2 + \omega_3 + \omega_4} \right) + \frac{1}{-\omega_1 + \omega_2 + \omega_3 + \omega_4} + \frac{1}{\omega_1 - \omega_2 + \omega_3 + \omega_4} = \frac{2}{(2\pi)^2} \int \left( \prod_{k=2}^4 \frac{dp_{0k}}{m^2 + p_k^2} \right) \times \]
\[ \times \left( \delta(p_{01} + p_{02} + p_{03} + p_{04}) + \delta(-p_{01} + p_{02} + p_{03} + p_{04}) \right) \bigg|_{p_{1}^2 = p_{2}^2 = -m^2}, \]
\[ W_3 = \frac{1}{\omega_4} \left( \frac{1}{\omega_1 + \omega_2 + \omega_3 + \omega_4} - \frac{1}{\omega_1 + \omega_2 + \omega_3 - \omega_4} + \frac{1}{-\omega_1 + \omega_2 + \omega_3 + \omega_4} - \frac{1}{-\omega_1 + \omega_2 + \omega_3 - \omega_4} \right) = \frac{2}{m^2 + (p_1 + p_2 + p_3)^2} + \frac{3}{m^2 + (-p_1 + p_2 + p_3)^2} \bigg|_{p_1^2 = p_2^2 = p_3^2 = -m^2}. \] (21)

Thus corresponding to the notation (13) and using (17)–(21) we obtain

\[ v^2 \, v = \frac{1}{(2\pi)^9} \int \left( \prod_{k=1}^{4} d^3p_k \right) \delta^3(p_1 + p_2 + p_3 + p_4)W_0 = \frac{1}{(2\pi)^{12}} \int \left( \prod_{k=1}^{4} \frac{d^4p_k}{m^2 + p_k^2} \right) \delta^4(p_1 + p_2 + p_3 + p_4), \]

at \( p_1^2 = -m^2 \).

The expression in the square brackets of r.h.s. (22) is none other than ordinary two-loops contribution to the inverse propagator, with the external momentum being on the mass shell:

\[ \cdots = \quad p_1^2 = -m^2. \] (23)

Further

\[ t^2 \, t = \frac{2}{(2\pi)^6} \int \left( \prod_{k=1}^{2} \frac{d^3p_k}{2\omega_k(e^{\beta \omega_k} - 1)} \right) \times \left[ \frac{1}{(2\pi)^4} \int \frac{d^4p}{m^2 + p^2} \left( \frac{1}{m^2 + (p + p_1 + p_2)^2} + \frac{1}{m^2 + (p - p_1 + p_2)^2} \right) \right], \] (24)

at \( p_1^2 = p_2^2 = -m^2 \).

Here in the square brackets of r.h.s. (24) one can recognize one-loop contribution (s- and u-channel terms) to the two-particle forward scattering amplitude on the mass shell

\[ \cdots = \quad p_2 \quad p_2 \quad p_2 \quad p_2 \]

(25)

Just the second (u-channel) term testifies zero angle scattering. Note that t-channel term at zero angle coincides with the diagram (11) in our notations

\[ p_2 \quad p_2 \quad p_2 \]

(26)

We shall find it in the diagram (10). For the last term of r.h.s. (13) we have

\[ t^3 \, t = \frac{1}{(2\pi)^9} \int \left( \prod_{k=1}^{3} \frac{d^3p_k}{2\omega_k(e^{\beta \omega_k} - 1)} \right) \left[ \frac{1}{m^2 + (p_1 + p_2 + p_3)^2} + \frac{3}{m^2 + (-p_1 + p_2 + p_3)^2} \right], \] \( p_1^2 = p_2^2 = p_3^2 = -m^2. \) (27)
It is easy to see allowing for the momenta permutation symmetry that here the square brackets expression represents the contribution of four diagrams to the $3 \rightarrow 3$ zero-angle scattering amplitude

$$\left[ \cdots \right] = \frac{p_3}{p_2} + \frac{p_3}{p_1} + \frac{p_3}{p_2} + \frac{p_3}{p_1} + \frac{p_3}{p_1} + \frac{p_2}{p_1} + \frac{p_3}{p_2} + \frac{p_2}{p_1} + \frac{p_2}{p_3} + \frac{p_3}{p_2}.$$  \hspace{1cm} (28)

So, we make shure that the 3-loop contribution to the free energy also exhibits the structure of Bose-Einstein averaging of the scattering amplitudes.

4 Renormalization

At the same time we see that the temperature dependent expressions (22), (24) are divergent owing to divergences of the vacuum loops in (23), (25), (26). Show how these divergences are absorbed due to mass and coupling constant renormalization

$$m^2 \rightarrow m^2 - \delta m_1^2 - \delta m_2^2, \quad \lambda \rightarrow \lambda + \delta \lambda. \hspace{1cm} (29)$$

The first order correction $\delta m_1^2$ is already fixed by (8). For $\delta m_2^2$ and $\delta \lambda$ put

$$\delta \lambda = a_1 \lambda^2, \quad \delta m_2^2 = a_2 \lambda^2. \hspace{1cm} (30)$$

This leads to

$$\bigcirc \rightarrow \bigcirc + \frac{\lambda}{2} \bigcirc \bigcirc \bigcirc + \frac{\lambda^2}{8} \bigcirc \bigcirc \bigcirc \bigcirc + \lambda^2 a_2 \bigcirc \bigcirc \bigcirc,$$

$$\bigcirc \bigcirc \rightarrow \bigcirc \bigcirc + \lambda \bigcirc \bigcirc \bigcirc \bigcirc + \lambda a_1 \bigcirc \bigcirc \bigcirc,$$

and for the second radiative correction (9) to the free energy we find

$$f^{(2)}(T) = - \frac{\lambda^2}{48} \left( 3 \bigcirc \bigcirc \bigcirc \bigcirc + \bigcirc \bigcirc \bigcirc \bigcirc + 3 \bigcirc \bigcirc \bigcirc \bigcirc - 6 \bigcirc \bigcirc \bigcirc \bigcirc - 6 a_1 \bigcirc \bigcirc \bigcirc \bigcirc + 24 a_2 \bigcirc \bigcirc \bigcirc \bigcirc \right). \hspace{1cm} (31)$$

Writing separately the contributions to $f^{(2)}(T)$ corresponding to different numbers of thermal loops we obtain:

3 $T$-loops

$$g_3 = 3 \bigcirc \bigcirc \bigcirc \bigcirc + 4 \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc.$$  \hspace{1cm} (32)

2 $T$-loops

$$g_2 = 9 \bigcirc \bigcirc \bigcirc \bigcirc - 6 a_1 \bigcirc \bigcirc \bigcirc \bigcirc + 6 \left( \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc - \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \right). \hspace{1cm} (33)$$

The expression in brackets in r.h.s. (33) is finite

$$\Re \left[ \frac{1}{(2\pi)^4} \int \frac{d^4p}{m^2 + p^2} \left( \frac{1}{m^2 + (p + p_1 + p_2)^2} + \frac{1}{m^2 + (p - p_1 + p_2)^2} - \frac{2}{m^2 + p^2} \right) \right]. \hspace{1cm} (34)$$
therefore \( g_2 \) will be finite under condition of cancellation first two terms in r.h.s. (33) that is

\[
a_1 = \frac{3}{2} V
\]  

(35)

With allowance for (35) we have for 1\( T \)-loop

\[
g_1 = 4 \left( V(T)V \right) - 18 \left( V \times T \times V \right) + 24a_2 \left( T \right).
\]  

(36)

Using (22), (23) for the diagram \( V(T)V \) one can see from (36) that \( g_1 \) vanishes if

\[
a_2 = \frac{3}{4} \left( T \times T \right) - \frac{1}{6} p^2 = -m^2.
\]  

(37)

In usual diagram language [4] the obtained mass and coupling constant renormalization (8), (29), (30), (35), (37) looks as follows

\[
\delta m_1^2 = \frac{\lambda}{2}, \quad \delta m_2^2 = \frac{3\lambda^2}{4} - \frac{\lambda}{6} p^2 = -m^2,
\]

\[
\delta \lambda = \frac{3\lambda^2}{2} \text{ at } p_i = 0.
\]  

(38)

One can see from (38) that the renormalization obtained here from the condition of convergency temperature dependent part of free energy is just the same that follows from the condition of 1PI Green’s functions be finite

\[
z \Gamma^{(2)}(p^2) = 0 \quad \text{at} \quad p^2 = -m^2,
\]

\[
\Gamma^{(4)}(p_i) = -\lambda \quad \text{at} \quad p_i = 0.
\]

Finally accounting for (31–34) the renormalized free energy in the 2nd order of \( PT \) is

\[
f(T) = -\frac{1}{2} \left( T \right) + \frac{\lambda}{8} \left( T \times T \right) -
\]

\[- \frac{\lambda^2}{48} \left[ 3 \left( T \times T \times T \right) + 4 \left( T(T)T \right) + 6 \left( T(V)T - T \times V \times T \right) \right].
\]  

(39)

5 Thermal averaging

Let the contributions corresponding to averaging on 2- and 3-particles phase volume be denoted through \( f_2(T) \) and \( f_3(T) \). Then from (39) we obtain

\[
f_2(T) = \frac{\lambda}{8} \left( T \times T \right) - \frac{\lambda^2}{8} \left( T(V)T - T \times V \times T \right),
\]  

(40)

\[
f_3(T) = -\frac{\lambda^2}{48} \left[ 3 \left( T \times T \times T \right) + 4 \left( T(T)T \right) \right],
\]  

(41)
and with allowance for the analytic expressions of diagrams (5), (7), (27), (34) we have for (40), (41)

$$f_2(T) = -\frac{1}{2!} \frac{1}{(2\pi)^6} \int \left( \prod_{k=1}^{2} \frac{d^3p_k}{2\omega_k(e^{\beta\omega_k} - 1)} \right) \text{Re} [A_{2\rightarrow 2}(p_1, p_2; p_1, p_2)],$$

$$f_3(T) = -\frac{1}{3!} \frac{1}{(2\pi)^9} \int \left( \prod_{k=1}^{3} \frac{d^3p_k}{2\omega_k(e^{\beta\omega_k} - 1)} \right) \text{Re} [A_{3\rightarrow 3}(p_1, p_2, p_3; p_1, p_2, p_3)].$$

Some comments are to be made concerning the representation of $f_3(t)$. As it follows from (27), (28) the diagram $\overbrace{\circlearrowright}$ contains only 4 from 10 possible tree diagrams which saturate the $3 \rightarrow 3$ amplitude in the 2nd order of $PT$. The other 6 diagrams, i.e.

have the unpleasant peculiarity to go to infinity for the forward scattering. For instance the first diagram from (44) is

$$\frac{1}{m^2 + (p_1 + p_2 - p'_2)^2} = \frac{1}{(p_2 - p'_2)^2 + 2(p_2 - p'_2)p_1} \rightarrow \infty, \quad \text{when} \quad p_2 \rightarrow p'_2.$$

Nevertheless the expression for $\overbrace{\circlearrowright \circlearrowright \circlearrowright}$ where the diagrams (44) give the contribution is finite indeed: because of

$$\overbrace{\circlearrowright \circlearrowright} = -\frac{\partial}{\partial m^2} \overbrace{\circlearrowright} = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\omega(e^{\beta\omega} - 1)} \cdot \frac{1}{p^2},$$

one has

$$3 \overbrace{\circlearrowright \circlearrowright \circlearrowright} = \frac{4}{(2\pi)^9} \int \left( \prod_{k=1}^{3} \frac{d^3p_k}{2\omega_k(e^{\beta\omega_k} - 1)} \right) \left( \frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} \right).$$

The way out of situation is to regularize the diagrams (44). The recipe of appropriate regularization proceeds from comparison (43), (45) and (46). Namely “zero angle” is to be understood as the limit

$$p' = \lim_{\varepsilon \rightarrow 0} (p + \varepsilon), \quad \varepsilon = (\varepsilon_0, \varepsilon).$$

The integration on phase volume is to be carried out at $\varepsilon \neq 0$ and put $\varepsilon = 0$ in the final result. Really, it is not hard to verify that in such limit the integration of the first pare of diagrams (44) gives

$$\lim_{\varepsilon \rightarrow 0} \left[ \lim_{\varepsilon_0 \rightarrow 0} \int \frac{d^3p_1}{2\omega_1(e^{\beta\omega_1} - 1)} \left( \frac{1}{m^2 + (p_1 + \varepsilon)^2} + \frac{1}{m^2 + (p_1 - \varepsilon)^2} \right) \right] =$$

$$= \int \frac{d^3p_1}{2\omega_1(e^{\beta\omega_1} - 1)} \frac{1}{p_1^2}$$

in agreements with (46).

The problem with the diagrams (44) is particular case of more general problem concerning $n \rightarrow n$ amplitudes which are built up from rescattered blocks, such as

$$1 \circlearrowright \circlearrowright \circlearrowright \cdots \circlearrowright 1'$$

(47)
The diagrams (47) is singular in physical region when incoming momenta are equal to out-
coming. The problem was intensively attacked [5,6] and regularization recipes were proposed.
As was shown PT analyses gives the simple and natural way of solving this problem. It must
be noted also that due to the special structure of diagrams including rescattered blocks the
summation of subsets of such diagrams (particular case is subset of so-called ring diagrams [1])
removes zero angle divergences at all.

6 Conclusion

The remarkable simplicity and regularity of the representation (42), (43) for the contribution
of interaction to the free energy in the 2nd order of PT gives the base for natural generalization
to higher order of PT, namely

\[ f(T) - f_{\text{ideal}}(T) = \sum_{n=2}^{\infty} f_n(T), \]  

where

\[ f_n(T) = -\frac{1}{n!} \frac{1}{(2\pi)^3n} \int \left( \prod_{k=1}^{n} \frac{d^3p_k}{2\omega_k(e^{\beta\omega_k} - 1)} \right) \text{Re} \left[ A_{n\to n}(p_1, \ldots, p_n; p_1, \ldots, p_n) \right]. \]

Note that the representation for the temperature dependent part of 1PI \( \Gamma^{(2)} \)-function of similar
structure was recently obtained [7].

In the representation (48), (49) is not hard to recognize the so-called diagrams of the first
type in the framework of \( S \)-matrix formulation of statistical mechanics [8]. As for the so-
called second type diagrams we can say noting because they are constructed from bilinear
combinations of \( \text{Im} A \) and \( \text{Re} A \). Corresponding terms may appear in higher (at least 3rd)
orders of PT.

The authors of [8] were generalized the Beth-Uhlenbeck formula and derived in the frame-
work of nonrelativistic quantum mechanics the complete virial expansion. Being guided by
invariant form of the representation they have supposed its validity for the relativistic case.
Our calculations in the framework \( \lambda\phi^4 \) QFT model confirm their hypothesis with allowance
some corrections like \( (n!)^{-1} \) factors in phase volume integrals reflecting the particles identity.
Also an ambiguity in regularization procedure for the zero angle rescattering amplitudes is
removed.

In our opinion the \( S \)-matrix formulation of statistical mechanics would be good tool for the
thermodynamic analysis of hadron matter [9–12] due to huge amount experimental information
and well elaborated phenomenological models for the scattering amplitudes. So we consider it
important to verify the \( S \)-matrix formulation of statistical mechanics by QFT methods.

There need be no doubt that a representation analogous to (48), (49) can be derived for
other renormalizable (at least normal, without confinement) QFTs because the mentioned
representation is rather conditioned by general structure of \( PT \) diagrams than by peculiarities
of interaction. More over, proceeding from that the thermodynamic values in this approach
are expressed through the functionals of \( S \)-matrix elements one can believe it being correct
also for hadron physics. A simple but forcible argument for this follows from the fact that
\( \text{Tr}(e^{-\beta H}) \) is invariant with respect to basis of state, so one can use in particular the set of
asymptotic, i.e. hadronic states. Its completeness is guaranteed by absence of any other except for hadrons observable asymptotic states. Unfortunately the direct derivation of formulas like (42), (43) from QCD, for example, is not possible because of the asymptotic states there (and consequently the scattering amplitudes) can not be perturbatively defined.

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