A CONSTRUCTION OF TWO DISTINCT CANONICAL SETS OF LIFTS OF BRAUER CHARACTERS OF A $p$-SOLVABLE GROUP

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Abstract. In [2], Navarro defines the set $\text{Irr}(G \mid Q, \delta) \subseteq \text{Irr}(G)$, where $Q$ is a $p$-subgroup of a $p$-solvable group $G$, and shows that if $\delta$ is the trivial character of $Q$, then $\text{Irr}(G \mid Q, \delta)$ provides a set of canonical lifts of $\text{IBr}_p(G)$, the irreducible Brauer characters with vertex $Q$. Previously, in [2], Isaacs defined a canonical set of lifts $B_\pi(G)$ of $\text{Irr}(G)$. Both of these results extend the Fong-Swan Theorem to $\pi$-separable groups, and both construct canonical sets of lifts of the generalized Brauer characters. It is known that in the case that $2 \in \pi$, or if $|G|$ is odd, we have $B_\pi(G) = \text{Irr}(G \mid Q, 1_Q)$. In this note we give a counterexample to show that this is not the case when $2 \notin \pi$. It is known that if $N \lhd G$ and $\chi \in B_\pi(G)$, then the constituents of $\chi_N$ are in $B_\pi(N)$. However, we use the same counterexample to show that if $N \lhd G$, and $\chi \in \text{Irr}(G \mid Q, 1_Q)$ is such that $\theta \in \text{Irr}(N)$ and $[\theta, \chi_N] \neq 0$, then it is not necessarily the case that $\theta \in \text{Irr}(N)$ inherits this property.

1. Introduction. Let $p$ be a prime, $G$ a finite $p$-solvable group, and for a class function $\alpha$ of $G$, let $\alpha^0$ denote the restriction of $\alpha$ to the $p$-regular elements of $G$. The celebrated Fong-Swan Theorem asserts that if $\varphi$ is an irreducible Brauer character of $G$ for the prime $p$, then there necessarily exists an ordinary irreducible character $\chi$ of $G$ such that $\chi^0 = \varphi$. Such a character $\chi$ is called a lift of $\varphi$. In [2], Isaacs constructs a canonical set of lifts $B_\varphi(G) \subseteq \text{Irr}(G)$, such that for each Brauer character $\varphi$ of $G$, there is a unique lift of $\varphi$ in $B_\varphi(G)$. Similarly, in [3], Navarro constructs another canonical set $\text{Irr}(G \mid Q, 1_Q)$ (which we will denote by $N_\varphi(G)$) of lifts of the irreducible Brauer characters of $G$. Navarro conjectured but did not prove that $B_\varphi(G) \neq N_\varphi(G)$. In this paper we give conditions under which $B_\varphi(G) = N_\varphi(G)$, and give an example to show that these sets need not be equal in general.

Let $\pi$ be a set of primes, and denote by $\pi'$ the complement of $\pi$. (In the classical case, $\pi$ is the complement of the prime $p$.) Let $G$ be a $\pi$-separable group. In [4], Gajendragadkar constructs a certain class of characters, called the $\pi$-special characters of $G$. An irreducible character $\chi$ is $\pi$-special if (a) $\chi(1)$ is a $\pi$-number and (b) if whenever $S \lhd G$ and $\theta \in \text{Irr}(S)$ lies under $\chi$, then the order of $\text{det}(\theta)$, as a character of $S/S'$, is a $\pi$-number. These $\pi$-special characters are known to have many interesting properties, and they are necessary for constructing the sets of characters under discussion in this paper. In particular, if $\alpha \in \text{Irr}(G)$ is $\pi$-special and $\beta \in \text{Irr}(G)$ is $\pi'$-special, then $\alpha \beta \in \text{Irr}(G)$ and this factorization is unique. If $\chi \in \text{Irr}(G)$ can be written in this factored form, we say that $\chi$ is $\pi$-factorable and we let $\chi_\pi$ and $\chi_{\pi'}$ be the $\pi$-special and $\pi'$-special factors of $\chi$, respectively. In addition, if $M \lhd G$ has $\pi'$-index in $G$, and if $\gamma \in \text{Irr}(M)$ is $\pi$-special and invariant in $G$, then there is a unique $\pi$-special character $\alpha \in \text{Irr}(G)$ such that $\alpha$ extends $\gamma$. If $\gamma \in \text{Irr}(M)$ is $\pi$-special and $M \lhd G$ with $G/M$ is a $\pi$-group, then every character $\alpha \in \text{Irr}(G \mid \gamma)$ is $\pi$-special.

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We now briefly review Isaacs’ construction of $B_\pi(G)$ in [2] and Navarro’s construction of the similar set, $\text{Irr}(G|Q,1_Q)$. In [2] Isaacs proves that if $G$ is $\pi$-separable, and if $\chi \in \text{Irr}(G)$, then there is a unique (up to conjugacy) pair $(S, \varphi)$ maximal with the property that $S \varphi G$, $\varphi \in \text{Irr}(S)$ lies under $\chi$, and $\varphi$ is $\pi$-factorable. Such a pair $(S, \varphi)$ is called a maximal factorable subnormal pair. Denote by $T$ the stabilizer of $\varphi$ in the normalizer in $G$ of $S$. Isaacs then shows that if $S \neq G$, then $T < G$ and induction defines a bijection from $\text{Irr}(T | \varphi)$ to $\text{Irr}(G | \varphi)$. Therefore, we can let $\psi \in \text{Irr}(T)$ be the unique character of $T$ lying over $\varphi$ such that $\psi^G = \chi$. The subnormal nucleus $(W, \gamma)$ of $\chi$ is then defined recursively by defining $(W, \gamma)$ to be the subnormal nucleus of $\psi$. (If $\chi$ is $\pi$-factorable, then $(W, \gamma)$ is defined to be $(G, \chi)$.) Note that by the construction, $\gamma^G = \chi$ and $\gamma$ is $\pi$-factorable. Also, it is shown that the subnormal nucleus of $\chi$ is unique up to conjugacy. The set $B_\pi(G)$ is defined as the set of irreducible characters of $G$ whose nucleus character $\gamma$ is $\pi$-special.

In [2], Navarro similarly defines the set $\text{Irr}(G|Q,1_Q)$ for a $p$-solvable group $G$. Navarro shows that if $\chi \in \text{Irr}(G)$, then there is a unique pair $(N, \theta)$ maximal with the property that $N \varphi G$, $\theta \in \text{Irr}(N)$ lies under $\chi$, and $\theta$ is $p$-factorable. Such a pair is called a maximal factorable normal pair. It is shown that if $N < G$, then $G_\theta < G$, and thus the Clifford correspondence implies that if $T = G_\theta$, there is a unique character $\psi \in \text{Irr}(T | \theta)$ such that $\psi^G = \chi$. Navarro then defines the normal nucleus $(U, \epsilon)$ to be the normal nucleus of $(T, \psi)$. (Again, if $\chi$ is $p$-factorable, then $(U, \epsilon) = (G, \chi)$.) Note that again $\epsilon^G = \chi$ and $\epsilon$ is $p$-factorable. If $Q$ is a Sylow $p$-subgroup of $U$, and if $\delta \in \text{Irr}(Q)$ is defined by $\delta = (\epsilon \eta)_Q$, then we say the pair $(Q, \delta)$ is a normal vertex for $\chi$, and it is shown that this pair and the normal nucleus are unique up to conjugacy. The set $\text{Irr}(G | Q,1_Q)$ is defined as $\{ \chi \in \text{Irr}(G) | \delta = 1_Q \}$, or equivalently, the set of irreducible characters of $G$ with a $p'$-special normal nucleus character. Although Navarro only defines the set $\text{Irr}(G|Q,\delta)$ when $\pi = p'$ and $G$ is $p$-solvable, the same construction of the normal nucleus and vertex of a character works if $\pi$ is an arbitrary set of primes and $G$ is $\pi$-separable. In this case, we will define the set $N_\pi(G)$ to be $\text{Irr}(G|Q,1_Q)$, only now $Q$ is a Hall $\pi'$-subgroup of the normal nucleus subgroup $U$ of $\chi$. Thus $B_\pi(G)$ consists of those irreducible characters of $G$ with a $\pi$-special subnormal nucleus character, and $N_\pi(G)$ consists of those characters of $G$ with a $\pi$-special normal nucleus character.

Recall that if $\chi$ is any class function of the $\pi$-separable group $G$, then $\chi^0$ denotes the restriction of $\chi$ to the elements of $G$ whose order is a $\pi$-number. Moreover, the set $I_\pi(G)$ is a generalization of Brauer characters in a $p$-solvable group $G$ to a set of primes $\pi$ (so that if $\pi = p'$, and if $G$ is $p$-solvable, then the set $I_\pi(G)$ is exactly the set of Brauer characters of $G$ for the prime $p$). In [2] it is shown that if $\chi \in B_\pi(G)$, then $\chi^0 \in I_\pi(G)$, and in [5] it is shown that if $\eta \in N_\pi(G)$, then $\eta^0 \in I_\pi(G)$. Both $B_\pi(G)$ and $N_\pi(G)$ are canonical sets of lifts of $I_\pi(G)$; in other words, if $\varphi \in I_\pi(G)$, then there is a unique character $\chi \in B_\pi(G)$ and a unique character $\eta \in N_\pi(G)$ such that $\chi^0 = \eta^0 = \varphi$. Moreover, Isaacs shows that if $\chi \in B_\pi(G)$ and $N \varphi G$, then every constituent $\theta$ of $\chi_N$ is in $B_\pi(N)$.

We will need the following results. In [4], Isaacs defines, for a subgroup $H$ of a $\pi$-separable group $G$ (where $2 \not\in \pi$), a linear character $\delta_{(G:H)} \in \text{Irr}(H)$, called the standard sign character of $H$. This theorem lists some of the properties of $\delta_{(G:H)}$.

**Theorem 1.1.** Let $G$ be $\pi$-separable with $2 \not\in \pi$, and let $\delta$ be the standard sign character for some subgroup $H$ of $G$. Then the following hold:
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(1) If $|G:H|$ is a $\pi$-number and $H$ is a maximal subgroup of $G$, then $\delta$ is the permutation sign character of the action of $H$ on the right cosets of $H$ in $G$.

(2) $\text{core}_G(H) \subseteq \ker(\delta)$.

(3) Suppose $H \subseteq G$ has $\pi$ index. If $\psi \in \text{Irr}(H)$ and $\psi^G = \chi \in \text{Irr}(G)$, then $\chi$ is $\pi$-special if and only if $\psi$ is $\pi$-special.

Proof. This is the content of Theorems 2.5 and B of [1] and Lemma 2.1 of [3]. □

The aims of this paper, then, are threefold. First, we prove the statement made (without proof) in [5] that if $p$ is an odd prime, the sets $B_{\pi'}(G)$ and $N_{\pi'}(G)$ coincide. Secondly, we give a counterexample to show that if $2 \in \pi'$, then $B_{\pi}(G)$ and $N_{\pi}(G)$ need not coincide. Finally, we use the same counterexample to show that if $\eta \in N_{\pi}(G)$ and $M < G$, then it need not be the case that the constituents of $\eta_M$ are themselves in $N_{\pi}(M)$.

2. Equality occurs if $G$ has odd order. In this section we give a brief proof that if $|G|$ is odd or if $2 \in \pi$, then $B_{\pi}(G) = N_{\pi}(G)$. This result was stated without proof in [5].

Here the field $\mathbb{Q}_{\pi}$ is defined by adjoining the $n$th roots of unity to $\mathbb{Q}$ for every $\pi$-number $n$.

Lemma 2.1. Let $G$ be $\pi$-separable and assume $2 \in \pi$ or $|G|$ is odd. Assume $\chi \in \text{Irr}(G)$ has values in $\mathbb{Q}_{\pi}$ and that $\chi^0 \in I_{\pi}(G)$. Then $\chi \in B_{\pi}(G)$.

Proof. This is Theorem 12.3 of [2]. □

Lemma 2.2. Let $G$ be $\pi$-separable, and suppose $\eta \in N_{\pi}(G)$. Then $\eta(g) \in \mathbb{Q}_{\pi}$ for all elements $g \in G$.

Proof. Let $\sigma$ be any automorphism of $\mathbb{C}$ that fixes the $n$th roots of unity for every $\pi$-number $n$. Clearly $\eta^\sigma \in N_{\pi}(G)$ by the construction of $N_{\pi}(G)$. If $\eta^0 = \varphi \in I_{\pi}(G)$, then since the values of $\varphi$ are in $\mathbb{Q}_{\pi}$, then $\varphi^\sigma = \varphi$. Since $\eta$ is the unique lift of $\varphi$ in $N_{\pi}(G)$, then it must be that $\eta^\sigma = \eta$. Thus the values of $\eta$ are in $\mathbb{Q}_{\pi}$. □

Corollary 2.3. If $G$ is $\pi$-separable and $2 \in \pi$ or $|G|$ is odd, then $B_{\pi}(G) = N_{\pi}(G)$.

Proof. Since $B_{\pi}(G)$ and $N_{\pi}(G)$ are both sets of lifts of $I_{\pi}(G)$, then $|B_{\pi}(G)| = |N_{\pi}(G)|$.

By Lemmas 2.1 and 2.2, $N_{\pi}(G) \subseteq B_{\pi}(G)$. Thus $N_{\pi}(G) = B_{\pi}(G)$. □

3. A counterexample of even order. In this section we construct the aforementioned counterexample to show that $B_{\pi}(G)$ need not equal $N_{\pi}(G)$, and we show that the constituents of the restriction of a character in $N_{\pi}(G)$ to a normal subgroup $V$ need not be in $N_{\pi}(V)$.

Suppose $\Gamma$ is a finite group of order $n$, and let $p$ be a prime number. Let $E$ be an elementary abelian group of order $p^n$. Then we can let $\Gamma$ act on $E$ by associating to each element $x \in \Gamma$ one of the cyclic factors of $E$, and let the left multiplication action of $\Gamma$ on itself induce an action on the factors of $E$. Let $G$ be the semidirect product of $\Gamma$ acting on $E$ with this action. Then for each subgroup $L$ of $G$ such that $E \subseteq L \subseteq G$, we see that there is an irreducible character $\theta$ of $E$ such that $G_{\theta} = L$.

Construction: Let $S_1$ be isomorphic to the symmetric group on four elements, and let $A_1$ be isomorphic to the alternating group on four elements. Define the group $U_1$ as the semidirect product of $S_1$ acting on $A_1$ with the conjugation action, so that $U_1 = A_1S_1$ and $S_1 \cap A_1 = 1$. Let $K_1$ be the normal Klein four group in $A_1$, and note that $K_1 \trianglelefteq U_1$. Define the element $x \in S_1$ by setting $x$ equal to the permutation $(12)$. Note that $\langle A_1, x \rangle \cong \text{Sym}(4)$.
Let $H_1 = \langle A_1, x \rangle$, and notice that $H_1$ is not subnormal in $U_1$, $K_1 \unlhd H_1$, and $H_1/K_1 \cong \text{Sym}(3)$.

Let $L_1 \subseteq U_1$ be such that $K_1 \subseteq L_1 \subseteq H_1$ and $|H_1 : L_1| = 3$. Note that $L_1 \cap A_1 = K_1$ and $L_1A_1 = H_1$. Moreover, note that $\mathbb{O}_3(U_1) = 1$.

Let $U_2$ be isomorphic to $U_1$, set $V_0 = U_1 \times U_2$, and define $\Gamma$ as the semidirect product of $C_2$ acting on $V_0$, with the nontrivial element of $C_2$ acting by interchanging the components of $V_0$. Note that core$_G(L_1) = 1$, and $\mathbb{O}_3(\Gamma) = 1$.

Let $|\Gamma| = n$, and let $E$ be an elementary abelian group of order $3^n$. Let $G$ be the semidirect product as defined in the paragraph preceding this construction, so that $G/E \cong \Gamma$. Therefore there is a character $\theta \in \text{Irr}(E)$ such that $G_\theta = EL_1$. Set $L = EL_1$, $A = EA_1$, $K = EK_1$, $H = EH_1$, and $V = EV_0$.

Set $\pi = \{3\}$ and note that $G$ is solvable. Since $E$ is a 3-group, then $\theta$ is $\pi$-special. Note that the $\pi$-special character $\theta$ extends to a unique $\pi$-special character $\hat{\theta} \in \text{Irr}(K)$. We see that since $\theta$ and $\hat{\theta}$ uniquely determine each other, then $A_\theta = K$. Therefore $\hat{\theta} \in \text{Irr}(K)$ induces irreducibly to a $\pi$-special character $\varphi \in \text{Irr}(A)$. Since $\varphi$ is uniquely determined by $\theta$, and $L$ normalizes $A$, then $L = G_\theta \subseteq G_\varphi$.

We now show that $G_\varphi = H$. By a Frattini argument, we see that $G_\varphi \subseteq G_\theta A = H$. Since $\varphi \in \text{Irr}(A)$, then clearly $A \subseteq G_\varphi$. We showed in the previous paragraph that $L = G_\theta \subseteq G_\varphi$. Thus $H = AL \subseteq G_\varphi$ and therefore $G_\varphi = H$.

We now claim that $(E, \theta)$ is a maximal factorable normal pair. Recall that $\mathbb{O}_3(\Gamma)$ is trivial, and thus $\mathbb{O}_3(G/E)$ is trivial. Suppose there exists a factorable normal pair $(N, \psi)$ such that $E \subseteq N$ and $\theta$ lies under $\psi$, and suppose $N/E$ is a nontrivial 2-group. Then the $\pi$-special factor of $\psi$ must restrict to $\theta$, and thus $\theta$ must be invariant in $N$. Therefore $N \subseteq G_\theta = L$. However, core$_{G/E}(L/E)$ is trivial, and this yields a contradiction. Thus $(E, \theta)$ is a maximal factorable normal pair.

We also claim that $(A, \varphi)$ is a maximal factorable subnormal pair. Suppose $(S, \sigma)$ is a factorable subnormal pair such that $A \unlhd S$ and $\sigma$ lies over $\varphi$, and suppose $S/A$ is a nontrivial 2-group. Since $\varphi \in \text{Irr}(A)$ is $\pi$-special, $\sigma$ must be invariant in $S$, and thus $S \subseteq G_\varphi = H$. Since $|H : A| = 2$ and $H$ is not subnormal in $G$, we have a contradiction. Since there are no subnormal subgroups $T$ of $G$ such that $A \unlhd T$ and $T/A$ is a nontrivial 3-group, then $(A, \varphi)$ is a maximal factorable subnormal pair.

Note that since $|L : K| = 2$ and the $\pi$-special character $\hat{\theta} \in \text{Irr}(K)$ is invariant in $L$, then $\hat{\theta}$ extends to a $\pi$-special character $\xi \in \text{Irr}(L)$, and we define $\eta \in \text{Irr}(G)$ by $\eta = \xi^G$. Note that $\theta$ lies under $\eta$, and since $(E, \theta)$ is a maximal factorable normal pair, and the Clifford correspondent $\xi \in \text{Irr}(G_\theta \theta)$ for $\eta$ is $\pi$-special, then $\eta \in N_\pi(G)$.

We now show that $\eta$ is not in $B_\pi(G)$. Note that $\xi^H \in \text{Irr}(H)$, and since $\varphi \in \text{Irr}(A)$ is exactly $(\xi_K)^A = (\xi^H)_A$, then $\varphi$ lies under both $\xi^H$ and $\eta$. Recall that $(A, \varphi)$ is a maximal factorable subnormal pair, and $H = G_\varphi$. Since $|H : A| = 2$ and $\varphi \in \text{Irr}(A)$ is $\pi$-special, every character of $H$ lying over $\varphi$ must be $\pi$-factorable. Thus $(H, \xi^H)$ is a subnormal nucleus for $\eta$, and $\eta \in B_\pi(G)$ if and only if $\xi^H$ is $\pi$-special.

Let $\delta = \delta_{H:L} \in \text{Irr}(L)$ be the standard sign character for $L \subseteq H$. Note that $K \subseteq \ker(\delta)$. Therefore we see that $\delta$ is the nonprincipal linear character of $L/K$. Since $\xi \in \text{Irr}(L)$ is $\pi$-special and $\delta \in \text{Irr}(L)$ is $\pi'$-special, then $\xi \delta \in \text{Irr}(L)$ is not $\pi$-special. Thus $\xi^H$ is not $\pi$-special by Theorem 1.1. Therefore, since $(H, \xi^H)$ is a subnormal nucleus for $\eta$, then $\eta$ is not in $B_\pi(G)$. 

Finally, note that \((H, \xi^H)\) is a normal nucleus for \(\xi^V\), which lies under \(\eta\). Since \(\xi^H\) is not \(\pi\)-special, then \(\xi^V\) is not \(\pi\)-special, and the constituents of \(\eta^V\) are not in \(N_\pi(V)\). □

The above example raises some interesting questions. Are there other families of characters which form lifts of the set \(I_\pi(G)\)? Some results in this direction can be found in forthcoming papers of Mark Lewis. What in general can be said about the set of lifts of a Brauer character of a \(p\)-solvable group \(G\)? In a future paper, the author will describe some bounds on the number of lifts of a Brauer character of a \(p\)-solvable group. There is still much to be known, though, about the set of lifts of a Brauer character of a \(p\)-solvable group.

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