A Deformation Quantization Theory for Non-Commutative Quantum Mechanics

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Abstract

We show that the deformation quantization of non-commutative quantum mechanics previously considered by Dias and Prata can be expressed as a Weyl calculus on a double phase space. We study the properties of the star-product thus defined, and prove a spectral theorem for the star-genvalue equation using an extension of the methods recently initiated by de Gosson and Luef.

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1 Introduction

The generalization of quantum mechanics obtained by considering canonical extensions of the Heisenberg algebra is usually referred to as non-commutative quantum mechanics (NCQM), a theory that displays an additional non-commutative structure in the configurational and momentum sectors. One of the main incentives for studying NCQM comes from the quest for a theory of quantum gravity. It is widely expected that such a theory will determine a modification of the structure of space-time of some non-commutative nature [8, 11, 19, 22]. Hence, deviations from the predictions of standard quantum mechanics, and particularly those arising from considering its non-commutative extensions, could be regarded as a sign of the underlying theory of quantum gravity. In Dias and Prata [1, 3] two of us have discussed various aspects of NCQM related to Flato–Sternheimer deformation quantization [4, 5]. In this paper we propose an operator theoretical
approach, based on previous work de Gosson and Luef [15] (the remaining two of us). In that article it was shown that the Moyal–Groenewold product $a \star b$ of two functions on $\mathbb{R}^{2n}$ can be interpreted in terms of a Weyl calculus on $\mathbb{R}^{2n}$. In fact,

$$a \star b = \tilde{A} b$$  \hspace{1cm} (1)

where $\tilde{A}$ is the phase space operator with Weyl symbol $\tilde{a}$ defined on $\mathbb{R}^{2n} \oplus \mathbb{R}^{2n}$ by

$$\tilde{a}(z, \zeta) = a(z - \frac{1}{2} J \zeta)$$  \hspace{1cm} (2)

($J$ is the standard symplectic matrix).

In this paper we show that this redefinition of the starproduct can be modified so that it leads to a natural notion of deformation quantization for the NCQM associated with an antisymmetric matrix of the type

$$\Omega = \begin{pmatrix} h^{-1} \Theta & I \\ -I & h^{-1} N \end{pmatrix}$$  \hspace{1cm} (3)

where $\Theta$ and $N$ measure the non-commutativity in the position and momentum variables, respectively. We define a new starproduct $\star_{\Omega}$ by replacing formula (1) by

$$a \star_{\Omega} b = \tilde{A}_{\Omega} b$$  \hspace{1cm} (4)

where $\tilde{A}_{\Omega}$ is the operator with Weyl symbol

$$\tilde{a}_{\Omega}(z, \zeta) = a(z - \frac{1}{2} \Omega \zeta).$$  \hspace{1cm} (5)

Of course (5) reduces to (1) when $\Theta = N = 0$.

In this article we are going to rigorously justify the definition above and study the properties of this new starproduct $\star_{\Omega}$. The difficulty associated with the fact that the symplectic form associated with $\Omega$ depends on $\hbar$ will be resolved (we will show that $a \star_{\Omega} b$ is well-defined as a starproduct thanks to supplementary conditions on $\Theta$ and $N$ which are physically meaningful). In fact $\star_{\Omega}$ coincides with the starproduct defined (in terms of the generalized Weyl-Wigner map [9]) in Eqn. (21) of [1], and where it was shown that it is related to the standard starproduct. (In the same paper it was concluded in Eqn. (53) that the generalized starproduct between two polynomials can be represented as a kind of “Bopp shift” which also turns out to be identical with $\star_{\Omega}$).

**Notation 1** The generic point of phase space $\mathbb{R}^{2n}$ is denoted $z = (x, p)$. We denote by $\text{Sp}(2n, \mathbb{R})$ the standard symplectic group, defined as the group of linear automorphisms of $\mathbb{R}^{2n}$ equipped with the symplectic form $\sigma(z, z') = \sum_{i=1}^{2n} dx^i \wedge dp^i$. 
\( J_z \cdot z', \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \). We use the standard notation \( S(\mathbb{R}^m) \) and \( S'(\mathbb{R}^m) \) for the Schwartz space of test functions on \( \mathbb{R}^m \) and its dual.

## 2 Description Of the Problem

Let us begin by explaining what we mean by non-commutativity in the present context. The study of non-commutative field theories and their connections with quantum gravity (see [2, 8, 11, 19, 22] and the references therein) leads to the consideration of commutation relations of the type

\[
[z_\alpha, z_\beta] = i\hbar \omega_{\alpha\beta}, \quad 1 \leq \alpha, \beta \leq 2n
\]

where \( \Omega = (\omega_{\alpha\beta})_{1 \leq \alpha, \beta \leq 2n} \) is the \( 2n \times 2n \) antisymmetric matrix defined by (3) where \( \Theta = (\theta_{\alpha\beta})_{1 \leq \alpha, \beta \leq n} \) and \( N = (\eta_{\alpha\beta})_{1 \leq \alpha, \beta \leq n} \) are antisymmetric matrices measuring the non-commutativity in the position and momentum variables. We have set here \( \bar{z}_\alpha = \bar{x}_\alpha \) if \( 1 \leq \alpha \leq n \) and \( \bar{z}_\alpha = \bar{p}_{\alpha-n} \) if \( n+1 \leq \alpha \leq 2n \), where

\[
\bar{x}_\alpha = x_\alpha + \frac{1}{2}i \sum_\beta \theta_{\alpha\beta} \partial x_\beta + \frac{1}{2}i \hbar \partial p_\alpha
\]

\[
\bar{p}_\alpha = p_\alpha - \frac{1}{2}i \hbar \partial x_\alpha + \frac{1}{2}i \sum_\beta \eta_{\alpha\beta} \partial p_\beta.
\]

It turns out that, as proved in [1], \( \Omega \) is invertible if

\[
\theta_{\alpha\beta} \eta_{\gamma\delta} < \hbar^2 \quad \text{for} \quad 1 \leq \alpha < \beta \leq n \quad \text{and} \quad 1 \leq \gamma < \delta \leq n.
\]

We will assume from now on that these conditions are satisfied; that this requirement is physically meaningful is well-known (it is fulfilled for instance in the case of the non-commutative quantum well; see for instance [6, 7]). Since we will be concerned with a deformation quantization with parameter \( \hbar \rightarrow 0 \) we will furthermore assume that \( \Theta \) and \( N \) depend smoothly on \( \hbar \) in such a way that

\[
\Theta(\hbar) = o(\hbar^2) \quad \text{and} \quad N(\hbar) = o(\hbar^2)
\]

(recall that \( f(\hbar) = o(\hbar^m) \) means that \( \lim_{\hbar \to 0} (f(\hbar)/\hbar^m) = 0 \)). We thus have

\[
\lim_{\hbar \to 0} \Omega = J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}
\]

(the standard symplectic matrix). It turns out that the conditions (10) are compatible with numerical results in [6, 7] where it is shown that the estimates \( \theta \leq 4 \times 10^{-40} \ m^2 \) and \( \eta \leq 1.76 \times 10^{-61} \ kg^2 m^2 s^{-2} \) hold. Moreover,
the analysis of non-commutative quantum mechanics in the context of dissipative open systems, reveals that a transition \( \theta \to 0 \) occurs prior to \( \hbar \to 0 \).

These facts, and the theory developed in [15], suggests that we represent \( \tilde{z} = (\tilde{z}_1, ..., \tilde{z}_{2n}) \) by the vector operator

\[
\tilde{z} = z + \frac{1}{2} i \hbar \Omega \partial_z \tag{11}
\]

which acts on functions defined on the phase space \( \mathbb{R}^{2n} \). Notice that the conditions (10) show that in the limit \( \hbar \to 0 \) we have the asymptotic formulae

\[
\tilde{x}_\alpha = x_\alpha + \frac{1}{2} i \hbar \partial p_\alpha + o(h^2) \ , \quad \tilde{p}_\alpha = p_\alpha - \frac{1}{2} i \hbar \partial x_\alpha + o(h^2). \tag{12}
\]

The “quantization rules” (11) lead us to the consideration of pseudo-differential operators formally defined by (5).

The underlying symplectic structure we are going to use is defined as follows. We will denote by \( s \) a linear automorphism of \( \mathbb{R}^{2n} \) such that \( \sigma = s^* \omega \); equivalently \( sJ s^T = \Omega \). Thus \( s \) is a symplectomorphism \( s : (\mathbb{R}^{2n}, \sigma) \to (\mathbb{R}^{2n}, \omega) \). Note that the mapping \( s \) is sometimes called the “Seiberg–Witten map” in the physical literature; its existence is of course mathematically a triviality (because it is just a linear version of Darboux’s theorem, see [14], §1.1.2). Writing \( s \) in block-matrix form

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]

the condition \( sJ s^T = \Omega \) is equivalent to

\[
AB^T - BA^T = h^{-1} \Theta \ , \quad CD^T - DC^T = h^{-1} N \ , \quad AD^T - BC^T = I.
\]

Of course, the automorphism \( s \) is not uniquely defined: if \( s^* \omega = s'^* \omega \) then \( s^{-1} s' \in \text{Sp}(2n, \mathbb{R}) \). Also note that in the limit \( \hbar \to 0 \) the matrices \( h^{-1} \Theta \) and \( h^{-1} N \) vanish and \( s \) becomes, as expected, symplectic in the usual sense, that is \( s \in \text{Sp}(2n, \mathbb{R}) \).

3 Definition of the starproduct \(*_\Omega\)

Let \( \omega \) be the symplectic form on \( \mathbb{R}^{2n} \) defined by \( \omega(z, z') = z \cdot \omega^{-1} z' \); it coincides with the standard symplectic form \( \sigma \) when \( \Omega = J \).

We will need the two following unitary transformations:

- The \( \Omega \)-symplectic transform \( F_\Omega \) defined, for \( a \in \mathcal{S}(\mathbb{R}^{2n}) \), by

\[
F_\Omega a(z) = \left( \frac{1}{2\pi \hbar} \right)^n \left| \det \Omega \right|^{-1/2} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar} \omega(z, z') a(z')dz'}; \tag{13}
\]
it extends into an involutive automorphism of $S'(\mathbb{R}^{2n})$ (also denoted by $F_\Omega$) and whose restriction to $L^2(\mathbb{R}^{2n})$ is unitary;

- The unitary operator $\tilde{T}_\Omega(z_0)$ defined, for $\Psi \in S'(\mathbb{R}^{2n})$ by the formula

$$\tilde{T}_\Omega(z_0)\Psi(z) = e^{-\frac{i}{\hbar} \omega(z, z_0)}\Psi(z - \frac{1}{2}z_0). \quad (14)$$

Notice that when $\Omega = J$ we have $\tilde{T}_\Omega(z_0) = \tilde{T}(z_0)$ where $\tilde{T}(z_0)$ is defined by formula (8) in [15].

Let us express the operator $\tilde{A}_\Omega = a(z + \frac{1}{2} i \hbar \Omega \partial_z)$ in terms of $F_\Omega a$ and $\tilde{T}_\Omega(z_0)$.

**Proposition 2** Let $\tilde{A}_\Omega$ be the operator on $\mathbb{R}^{2n}$ with Weyl symbol

$$\tilde{a}_\Omega(z, \zeta) = a(z - \frac{1}{2} \Omega \zeta). \quad (15)$$

We have

$$\tilde{A}_\Omega = \left(\frac{1}{2\pi n}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\Omega a(z) \tilde{T}_\Omega(z) dz. \quad (16)$$

**Proof.** Let us denote by $\tilde{B}$ the right-hand side of (16). We have, setting $u = z - \frac{1}{2}z_0$,

$$\tilde{B}\Psi(z) = \left(\frac{1}{2\pi n}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\Omega a(z_0) e^{-\frac{i}{\hbar} \omega(z, z_0)}\Psi(z - \frac{1}{2}z_0) d\zeta_0$$

$$= \left(\frac{2}{\pi n}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\Omega a(2(z - u)) e^{\frac{i}{\hbar} \omega(z, u)}\Psi(u) du$$

hence the kernel of $\tilde{B}$ is given by

$$K(z, u) = \left(\frac{2}{\pi n}\right)^n |\det \Omega|^{-1/2} F_\Omega a[2(z - u)] e^{\frac{i}{\hbar} \omega(z, u)}.$$

It follows that the Weyl symbol $\tilde{b}$ of $\tilde{B}$ is given by

$$\tilde{b}(z, \zeta) = \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar} \tilde{b} \tilde{c}'} K(z + \frac{1}{2} \zeta', z - \frac{1}{2} \zeta') d\zeta'$$

$$= \left(\frac{2}{\pi n}\right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar} \tilde{b} \tilde{c}'} F_\Omega a(2\zeta') e^{-\frac{i}{\hbar} \omega(z, \zeta')} d\zeta'$$

that is, using the obvious relation

$$\zeta \cdot \zeta' + 2\omega(z, \zeta') = \omega(2z - \Omega \zeta, \zeta')$$
together with the change of variables \( z' = 2\zeta' \),
\[
\tilde{b}(z, \zeta) = \left( \frac{2}{\sqrt{\pi}} \right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{\hbar}\omega(2z - \Omega \zeta, \zeta')} F_\Omega a(2\zeta') d\zeta'
\]
\[
= \left( \frac{1}{2\pi \hbar} \right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} e^{-\frac{i}{2}\omega(z - \frac{1}{2}\Omega \zeta, \zeta')} F_\Omega a(z') dz'
\]
that is, using the fact that \( F_\Omega F_\Omega \) is the identity,
\[
\tilde{b}(z, \zeta) = a(z - \frac{1}{2}\Omega \zeta) = \tilde{a}_\Omega(z, \zeta)
\]
which concludes the proof. ■

The result above motivates the following definition:

**Definition 3** Let \( a \in S'(\mathbb{R}^{2n}) \) and \( b \in S(\mathbb{R}^{2n}) \). The \( \Omega \)-star product of \( a \) and \( b \) is the element of \( S'(\mathbb{R}^{2n}) \) defined by
\[
a \star_\Omega b = \tilde{A}_\Omega b. \tag{17}\]

Note that it is not yet clear from the definition above that \( \star_\Omega \) is a bona fide star product. For instance, while it is obvious that \( 1 \star_\Omega b = b \) (because the operator \( \tilde{A}_\Omega \) with symbol \( a = 1 \) is the identity), the formula \( b \star_\Omega 1 = b \) is certainly not, and it is even less clear that \( \star_\Omega \) is associative!

4 **A New Star-Product Is Born...**

It turns out that we can reduce the study of the newly defined star product to that of the usual Groenewold–Moyal product \( \star \). For this we will need Lemma 4 below.

**Lemma 4** Let \( s \) be a linear automorphism such that \( \sigma = s^* \omega \) and define a automorphism \( M_s : S'(\mathbb{R}^{2n}) \rightarrow S'(\mathbb{R}^{2n}) \) by
\[
M_s \Psi(z) = \sqrt{\det s} \Psi(sz). \tag{18}\]
(hence \( M_s \) is unitary on \( L^2(\mathbb{R}^{2n}) \)). We have
\[
M_s \tilde{A}_\Omega = \tilde{A}' M_s \tag{19}\]
where \( \tilde{A}' = \tilde{A}' f \) corresponds to the operator \( \tilde{A}' \) acting on \( L^2(\mathbb{R}^n) \) with Weyl symbol \( a'(z) = a(sz) \), and hence
\[
M_s(a \star_\Omega b) = \sqrt{\det s}(a' \star b') \tag{20}\]
where \( b'(z) = b(sz) \).
Proof. Formula (20) immediately follows from formula (19). To prove formula (19) one first checks the identities
\[ M_s \tilde{T}(z_0) = \tilde{T}(s^{-1}z_0)M_s, \quad M_s F_\Omega = F_\Omega M_s \]
(the verification of which is purely computational and therefore left to the reader); using these identities we have
\[ M_s \tilde{A}_\Omega = \left( \frac{1}{2\pi \hbar} \right)^n |\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\Omega a(z_0) \tilde{T}(s^{-1}z_0)M_s dz_0 \]
\[ = \left( \frac{1}{2\pi \hbar} \right)^n |\det s||\det \Omega|^{-1/2} \int_{\mathbb{R}^{2n}} F_\Omega a(sz) \tilde{T}(z)M_s dz \]
\[ = \left( \frac{1}{2\pi \hbar} \right)^n |\det s|^{|\det \Omega|^{-1/2}} \int_{\mathbb{R}^{2n}} M_s F_\Omega a(z) \tilde{T}(z)M_s dz \]
\[ = \left( \frac{1}{2\pi \hbar} \right)^n |\det s|^{|\det \Omega|^{-1/2}} \int_{\mathbb{R}^{2n}} F_J(M_s a)(z) \tilde{T}(z)M_s dz \]
\[ = \tilde{A}'M_s. \]

The double equality
\[ 1 \star_\Omega a = a \star_\Omega 1 = a \] (21)
now immediately follows from formula (20): we have
\[ M_s(1 \star_\Omega a) = \sqrt{|\det s|(1 \star a')} = 1 \star M_s a = M_s a \]
hence we recover the equality 1 \star_\Omega a = a; similarly
\[ M_s(a \star_\Omega 1) = \sqrt{|\det s|(a' \star 1)} = M_s a \star 1 = M_s a \]
hence a \star_\Omega 1 = a.

Let us now prove the associativity of the \(\Omega\)-starproduct:

**Proposition 5** Assume that the starproducts \(a \star_\Omega (b \star_\Omega c)\) and \((a \star_\Omega b) \star_\Omega c\) are defined. We then have
\[ a \star_\Omega (b \star_\Omega c) = (a \star_\Omega b) \star_\Omega c. \] (22)

**Proof.** It is of course sufficient to show that
\[ M_s [a \star_\Omega (b \star_\Omega c)] = M_s [(a \star_\Omega b) \star_\Omega c]. \] (23)
We have, by repeated use of (20) together with the definition of $M_s$, 

$$M_s [a \star (b \star c)] = \sqrt{|\det s|} (a' \star (b \star c))'$$

$$= a' \star M_s (b \star c)$$

$$= \sqrt{|\det s|} [a' \star (b' \star c')] .$$

A similar calculation yields

$$M_s [(a \star b) \star c)] = \sqrt{|\det s|} [(a' \star b') \star c']$$

hence the equality (23) in view of the associativity of the Groenewold–Moyal product.

That we have a deformation of a Poisson bracket follows from the following considerations. Let us define an $\Omega$-Poisson bracket $\{\cdot, \cdot\}_\Omega$ by

$$\{a, b\}_\Omega = -\omega(X_{a,\Omega}, X_{b,\Omega})$$

(24)

where the vector fields $X_{a,\Omega}$ and $X_{b,\Omega}$ are given by

$$X_{a,\Omega} = \Omega \partial_x a \quad X_{b,\Omega} = \Omega \partial_x b.$$ (25)

In particular $\{a, b\}_\Omega$ is the usual Poisson bracket $\{a, b\}$ and $X_{a,J}$, $X_{a,J}$ are the usual Hamilton vector fields when $\Theta = N = 0$. We have the following asymptotic formula relating both notions of Poisson brackets:

$$\{a, b\}_\Omega = \{a, b\} + o(h) \text{ for } h \to 0.$$ (26)

In fact, by definitions (24) and (25),

$$\{a, b\}_\Omega = -X_{a,\Omega} \cdot \Omega^{-1} X_{b,\Omega} = -\Omega \partial_x a \cdot \partial_x b$$

that is

$$\{a, b\}_\Omega = \{a, b\} - h^{-1}(\Theta \partial_x a \cdot \partial_x b + N \partial_p a \cdot \partial_p b)$$

from which (26) follows in view of the conditions (10).

**Proposition 6** We have

$$a \star b - b \star a = i\hbar \{a, b\} + O(h^2).$$ (27)

**Proof.** We have, since $M_s$ is linear,

$$M_s (a \star b - b \star a) = \sqrt{|\det s|} (a' \star b' - b' \star a')$$

$$= \sqrt{|\det s|} (i\hbar \{a', b'\} + O(h^2))$$

8
where, as usual, \( a'(z) = a(sz) \) and \( b'(z) = b(sz) \). Now, by the chain rule and the relation \( Js^T = s^{-1}\Omega \),

\[
X_{a'} = Js^T \partial_z a(sz) = s^{-1}\Omega \partial_z a(sz) \\
X_{b'} = Js^T \partial_z b(sz) = s^{-1}\Omega \partial_z b(sz)
\]

and hence, using the identities \( \{a', b'\} = JX_{a'} \cdot X_{b'} \) and \( (s^T)^{-1}J^{-1}s^{-1} = \Omega^{-1} \),

\[
\sqrt{|\det s|} \{a', b'\} = -\sqrt{|\det s|} Js^{-1}\Omega \partial_z a(sz) \cdot s^{-1}\Omega \partial_z b(sz) \\
= -M_s \omega(X_{a,\Omega}, X_{b,\Omega}).
\]

We have thus proven that

\[
M_s(a \star_\Omega b - b \star_\Omega a) = -i\hbar M_s \omega(X_{a,\Omega}, X_{b,\Omega}) + O(\hbar^2).
\]

From this and (26), Eqn. (27) follows. \(\blacksquare\)

More generally, using the approach above, it is easy to show that

\[
f \star_\Omega g = \sum_{k \geq 0} B_k(\Omega) f \hbar^k
\]

where the \( B_k(\Omega) \) are bi-differential operators. In particular, \( B_0(\Omega) = 1 \) and \( B_1(\Omega) = \frac{i}{2} \{f, g\} \), but for \( k \geq 2 \) they differ from those of the usual Moyal product. We leave these technicalities aside in this article.

5 The Intertwining Property

In [15] two of us defined a family of partial isometries \( W_\phi : \mathbb{L}^2(\mathbb{R}^n) \longrightarrow \mathbb{L}^2(\mathbb{R}^{2n}) \) indexed by \( \mathcal{S}(\mathbb{R}^n) \), and intertwining the operator \( \tilde{A} = \tilde{A}_J \) and the usual Weyl operator \( \hat{A} \):

\[
\tilde{A} W_\phi = W_\phi \hat{A} \quad \text{and} \quad W_\phi^* \tilde{A} = \hat{A} W_\phi^*.
\]

These intertwiners are defined by

\[
W_\phi \psi = (2\pi \hbar)^{n/2} W(\psi, \phi)
\]

where \( W(\psi, \phi) \) is the cross-Wigner distribution:

\[
W(\psi, \phi)(z) = \left(\frac{1}{2\pi \hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} p \cdot y} \psi(x + \frac{1}{2}y) \phi(x - \frac{1}{2}y) dy
\]

and \( W_\phi^* \) denotes the adjoint of \( W_\phi \).

The following result is an extension of Proposition 2 in [15].
Theorem 7 Let \( s \) be a linear automorphism of \( \mathbb{R}^{2n} \) such that \( s^* \omega = \sigma \). (i) The mappings \( W_{s, \phi} : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^{2n}) \) defined by the formula:
\[
W_{s, \phi} = M_s^{-1} W_\phi
\]
are partial isometries \( L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^{2n}) \) and we have
\[
\tilde{A}\Omega W_{s, \phi} = W_{s, \phi} \tilde{A'} \quad \text{and} \quad W^*_{s, \phi} \tilde{A}\Omega = \tilde{A'} W^*_{s, \phi}
\]
where \( \tilde{A}' \) is the operator with Weyl symbol \( a' = a(sz) \) and \( W^*_{s, \phi} \) denotes the adjoint of \( W_{s, \phi} \). (ii) The replacement of \( s \) by \( s' \) such that \( \sigma = s'^* \omega \) is equivalent to the replacement of \( \tilde{A}' \) by \( \tilde{S}_\sigma^{-1} \tilde{A'} \tilde{S}_\sigma \) and of \( W_{s, \phi} \) by \( W_{s, \tilde{S}_\sigma^{-1} \phi} \).

Proof. (i) We have, using the first formula (28) and definition (31),
\[
\tilde{A}\Omega W_{s, \phi} = M_s^{-1} \tilde{A'} M_s (M_s^{-1} W_\phi)
\]
that is,
\[
\tilde{A}\Omega W_{s, \phi} = M_s^{-1} (\tilde{A'} W_\phi) = M_s^{-1} W_\phi \tilde{A'} = W_{s, \phi} \tilde{A'};
\]
the equality \( W^*_{s, \phi} \tilde{A}\Omega = \tilde{A'} W^*_{s, \phi} \) is proven in a similar way. That \( W_{s, \phi} \) is a partial isometry is obvious since \( W_\phi \) is a partial isometry and \( M_s \) is unitary. (ii) We have \( s_\sigma = s^{-1} s'_\sigma \in \text{Sp}(2n, \mathbb{R}) \) hence \( a(s'z) = a(ss_\sigma z) = a'(s_\sigma z) \). Let \( \tilde{A''} \) be the operator with Weyl symbol \( a''(z) = a'(s_\sigma z) \). In view of the symplectic covariance property of Weyl operators we have \( \tilde{A''} = \tilde{S}_\sigma^{-1} \tilde{A'} \tilde{S}_\sigma \).

Similarly,
\[
W_{s', \phi} \psi(z) = M_{s'}^{-1} (M_s M_{s'}^{-1} W_\phi) \psi(z)
\]
\[
= M_{s'}^{-1} W_\phi \psi(s_\sigma^{-1} z)
\]
\[
= W_{s, \phi} \psi(s_\sigma^{-1} z)
\]
hence \( W_{s', \phi} \psi = W_{s, \tilde{S}_\sigma^{-1} \phi} \tilde{S}_\sigma \psi \) in view of the symplectic covariance of the cross-Wigner transform (30); the result follows. \( \square \)

An important property of the mappings \( W_{s, \phi} \) is that they can be used to construct orthonormal bases in \( L^2(\mathbb{R}^{2n}) \) starting from an orthonormal basis in \( L^2(\mathbb{R}^n) \).

Proposition 8 Let \((\phi_j)_{j \in F} \) be an arbitrary orthonormal basis of \( L^2(\mathbb{R}^n) \); the functions \( \Phi_{j, k} = W_{s, \phi_j} \phi_k \) with \((j, k) \in F \times F \) form an orthonormal basis of \( L^2(\mathbb{R}^{2n}) \), and we have \( \Phi_{j, k} \in \mathcal{H}_j \cap \mathcal{H}_k \), with \( \mathcal{H}_j = W_{s, \phi_j}(L^2(\mathbb{R}^n)) \).
Proof. In [15] the property was proven for the mappings $W_{\phi_j} = W_{I,\phi_j}$; the lemma follows since $W_{s,\phi} = M_s^{-1}W_{\phi}$ and $M_s$ is unitary. ■

6 The $\star_\Omega$-Genvalue Equation: Spectral Results

Let us consider the star-genvalue equation for the star-product $\star_\Omega$:

$$a \star_\Omega \Psi = \lambda \Psi; \quad (33)$$

here $a$ can be viewed as some Hamiltonian function whose properties are going to be described, and $\Psi$ a “phase-space function”. Following definition (17) the study of this problem is equivalent to that of the eigenvalue equation

$$\widetilde{A}_\Omega \Psi = \lambda \Psi \quad (34)$$

for the pseudo-differential operator $\widetilde{A}_\Omega$. Using the intertwining relations (32) it is easy to relate the eigenvalues of $\widetilde{A}_\Omega$ to those of $\hat{A}'$ following the lines in [15]; for instance one sees, adapting mutatis mutandis the proof of Theorem 4 in the reference, that the operators $\widetilde{A}_\Omega$ and $\hat{A}'$ have the same eigenvalues (see Theorem 9 below). Note that it follows from Theorem 7(ii) that the eigenvalues of $\hat{A}'$ do not depend on the choice of $s$ such that $s^* \omega = \sigma$.

**Theorem 9** The operators $\widetilde{A}_\Omega$ and $\hat{A}'$ have the same eigenvalues. (i) Let $\psi$ be an eigenvector of $\hat{A}'$: $\hat{A}'\psi = \lambda \psi$. Then $\Psi = W_{s,\phi}\psi$ is an eigenvector of $\widetilde{A}_\Omega$ corresponding to the same eigenvalue: $\widetilde{A}_\Omega \Psi = \lambda \Psi$. (ii) Conversely, if $\Psi$ is an eigenvector of $\widetilde{A}_\Omega$ then $\psi = W_{s,\phi}^* \Psi$ is an eigenvector of $\hat{A}'$ corresponding to the same eigenvalue.

**Proof.** That every eigenvalue of $\hat{A}'$ also is an eigenvalue of $\widetilde{A}_\Omega$ is clear: if $\hat{A}'\psi = \lambda \psi$ for some $\psi \neq 0$ then

$$\widetilde{A}_\Omega (W_{s,\phi}\psi) = W_{s,\phi} \hat{A}' \psi = \lambda W_{s,\phi} \psi$$

and $\Psi = W_{s,\phi}\psi \neq 0$; this proves at the same time that $W_{s,\phi}\psi$ is an eigenvector of $\widetilde{A}_\Omega$ because $W_{s,\phi}$ is injective. (ii) Assume conversely that $\widetilde{A}_\Omega \Psi = \lambda \Psi$ for $\Psi \in L^2(\mathbb{R}^{2n})$, $\Psi \neq 0$, and $\lambda \in \mathbb{R}$. For every $\phi$ we have

$$\hat{A}' W_{s,\phi}^* \Psi = W_{s,\phi}^* \widetilde{A}_\Omega \Psi = \lambda W_{s,\phi}^* \Psi$$

hence $\lambda$ is an eigenvalue of $\hat{A}'$ and $\psi$ an eigenvector if $\psi = W_{s,\phi}^* \Psi \neq 0$. We have $W_{s,\phi} \psi = W_{s,\phi} W_{s,\phi}^* \Psi = P_{s,\phi} \Psi$ where $P_{s,\phi}$ is the orthogonal projection
on the range $\mathcal{H}_{s,\phi}$ of $W_{s,\phi}$. Assume that $\psi = 0$; then $P_{s,\phi}\Psi = 0$ for every $\phi \in \mathcal{S}(\mathbb{R}^n)$, and hence $\Psi = 0$ in view of Proposition 8.

Let us give an application of the result above. Assume that the symbol $a$ belongs to the Shubin class $H^{m_1,m_0}_\rho(\mathbb{R}^{2n})$; recall [20] that $a \in H^{m_1,m_0}_\rho(\mathbb{R}^{2n})$ if $a \in C^\infty(\mathbb{R}^{2n})$ and if there exist constants $C_0,C_1 \geq 0$ and, for every $\alpha \in \mathbb{N}^n$, $|\alpha| \neq 0$, a constant $C_\alpha \geq 0$, such that for $|z|$ sufficiently large

$$C_0|z|^{m_0} \leq |a(z)| \leq C_1|z|^{m_1}, \quad |\partial_\alpha^z a(z)| \leq C_\alpha |a(z)||z|^{-\rho|\alpha|}.$$  

The following result (Shubin [20], Chapter 4) is important in our context:

**Theorem 10** Let $a \in H^{m_1,m_0}_\rho(\mathbb{R}^{2n})$ be real, and $m_0 > 0$. Then the formally self-adjoint operator $\hat{A}$ with Weyl symbol $a$ has the following properties: (i) $\hat{A}$ is essentially self-adjoint in $L^2(\mathbb{R}^n)$ and has discrete spectrum; (ii) There exists an orthonormal basis of eigenfunctions $\phi_j \in \mathcal{S}(\mathbb{R}^n)$ ($j = 1, 2, \ldots$) with eigenvalues $\lambda_j \in \mathbb{R}$ such that $\lim_{j \to \infty} |\lambda_j| = \infty$.

It follows that:

**Theorem 11** Let $a \in H^{m_1,m_0}_\rho(\mathbb{R}^{2n})$ be real, and $m_0 > 0$. (i) The stargen-value equation $a \bigstar \Omega \Psi = \lambda \Psi$ has a sequence of real eigenvalues $\lambda_j$ such that $\lim_{j \to \infty} |\lambda_j| = \infty$, and these eigenvalues are those of the operator $\hat{A}'$ with Weyl symbol $a'(z) = a(sz)$. (ii) The star-eigenvectors of $a$ are in one-to-one correspondence with the eigenvectors $\phi_j \in \mathcal{S}(\mathbb{R}^n)$ of $\hat{A}'$ by the formula $\Phi_{k,j} = W_{s,\phi_k} \phi_j$.

**Proof.** It is an immediate consequence of Theorems 9 and 10.

**7 Concluding Remarks...**

The results using the generalized Weyl-Wigner map [9] seem to be quite general since they also apply to the case of nonlinear transformations of $\mathbb{R}^{2n}$ (for a review see also section II of [1]). In particular a more general starproduct than the one of non-commutative quantum mechanics was obtained in [1] (see Eqn.(23) in this reference). A future project could be to extend the approach of the present paper to this case. Another important topic we have not addressed in this article is the characterization of the optimal symbol classes and function spaces associated with the star-product $\bigstar\Omega$. As two of us have shown elsewhere [16] Feichtinger’s modulations spaces (see
for a review) and the closely related Sjöstrand classes [21] are excellent
candidates in the case of Landau-type operators (which are a variant of
the operators $\tilde{A}$ corresponding to the case $\Omega = J$). It seems very plausible
that these function spaces are likely to play an equally important role in
the theory of the star-product $\star_{\Omega}$. Another future project concerns a discussion
of the starproduct $\star_{\Omega}$ and its connection to Rieffel’s work in deformation
quantization as outlined in [18], and the methods introduced in [17] in a
different context.

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