Abstract. Let \( M \) be an \( n \)-dimensional \( d \)-bounded Stein manifold, i.e., a complex \( n \)-dimensional manifold \( M \) admitting a smooth strictly plurisubharmonic exhaustion \( \rho \) and endowed with the Kähler metric whose fundamental form is \( \omega = \partial \bar{\partial} \rho \), such that \( \partial \bar{\partial} \rho \) has bounded \( L^\infty \) norm. We prove a vanishing result for \( W^{1,2} \) harmonic forms with respect to the Bott-Chern Laplacian on \( M \).

1. Introduction

Let \( M \) be an \( n \)-dimensional complex manifold endowed with a Hermitian metric \( g \). Then, on the space of \((p,q)\)-forms on \( M \) there are defined several self-adjoint elliptic differential operators of order two, as the Dolbeault Laplacian \( \Delta^{\partial} \) and of order four as Bott-Chern and Aeppli Laplacians, respectively denoted by \( \tilde{\Delta}^{\partial}_{BC} \) and \( \tilde{\Delta}^{\partial}_{A} \), involving \( \partial, \bar{\partial}, \bar{\partial}^*, \partial^* \) and their suitable combinations. If \( M \) is compact, then, in particular, according to Schweitzer [9], it turns out that a \((p,q)\)-form \( \varphi \) on \( M \) satisfies \( \tilde{\Delta}^{\partial}_{BC} \varphi = 0 \) if and only if

\[
\partial \varphi = 0, \quad \bar{\partial} \varphi = 0, \quad \partial^* \varphi = 0.
\]

Furthermore, if the Hermitian metric \( g \) is also Kähler on the compact complex manifold \( M \), then according to the ellipticity, the kernels of such differential operators are finite dimensional and, as a consequence of Kähler identities, they coincide.

For a complex non compact manifold \( M \), one can consider smooth \( L^2 \) forms and study the space of \( L^2 \) harmonic \((p,q)\)-forms for the above Laplacians. In [3], the space of \( L^2 \) harmonic forms with respect to the Dolbeault Laplacian \( \Delta^{\partial} \) on a bounded strictly pseudoconvex domain \( \Omega \) in \( \mathbb{C}^n \) with smooth boundary, endowed with the Bergman metric, is studied. Precisely, denoting by \( \mathcal{H}^{\partial}_{p,q} \) the space of \( L^2 \) harmonic forms with respect to \( \Delta^{\partial} \) Donnelly and Fefferman proved that under the assumptions as above, it holds that

\[
\begin{align*}
\dim \mathcal{H}^{\partial}_{p,q} &= 0 & \text{if } p + q &\neq n \\
\dim \mathcal{H}^{\partial}_{p,q} &= \infty & \text{if } p + q &= n.
\end{align*}
\]

In [8] Ohsawa proved that the dimension of the middle \( L^2 \overline{\partial} \)-cohomology of a domain in a complex manifold, admitting a non-degenerate regular boundary point and whose defining function satisfies some suitable assumptions, is infinite dimensional. Later, Gromov in [4] introduced the notion of Kähler hyperbolicity showing that,
if $X$ is a Kähler complete simply-connected manifold whose Kähler form $\omega$ is $d$-bounded, admitting a uniform discrete subgroup of isometries, then

$$\begin{align*}
\mathcal{H}^{p,q}_{\Omega^2} &= \{0\} \quad \text{if } p + q \neq n \\
\mathcal{H}^{p,q}_{\Omega^2} &\neq \{0\} \quad \text{if } p + q = n.
\end{align*}$$

In the present paper we are interested in studying the space of $L^2(p,q)$-harmonic forms with respect to the Bott-Chern Laplacian on Hermitian complete manifolds. More precisely, we consider a $d$-bounded Stein manifold $M$, i.e., a complex $n$-dimensional manifold $M$ admitting a smooth strictly plurisubharmonic exhaustion $\rho$ and endowed with the Kähler metric whose fundamental form is $\omega = i\partial \bar{\partial} \rho$, such that $\bar{\partial} \rho$ has bounded $L^\infty$ norm; examples of such manifolds are bounded strictly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary, endowed with the Bergman metric (see [2]). Denoting by $\mathcal{H}^{p,q}_{\Omega^2}$ the space of $W^{1,2}$ Bott-Chern harmonic $(p,q)$-forms, where $W^{1,2}$ denotes the Sobolev Space, we prove the following vanishing result

**Theorem** (see Theorem 4.5) Let $M$ be a $d$-bounded Stein manifold of complex dimension $n$. Then

$$\mathcal{H}^{p,q}_{\Omega^2} = \{0\}, \quad \text{for } p + q \neq n.$$ 

The paper is organized as follows: in Section 2 we fix some notation and recall some well known results in Kähler geometry. In Section 3 adapting an argument by Demailly [1], we construct cut-off functions on a $d$-bounded Stein manifold, proving estimates of second order derivatives. Section 4 is mainly devoted to prove that a smooth $W^{1,2}$ $(p,q)$-form $\varphi$ satisfies $\Delta^{\partial} \varphi = 0$ if and only if $\partial \varphi = 0$, $\overline{\partial} \varphi = 0$ (see Theorem 4.4). Basic tools in the proof of such a theorem are the estimates of second order derivatives of cut-off functions ensured by Stein $d$-bounded assumption (see Lemma 3.2). As a corollary, we also derive that (see Theorem 4.4),

$$\mathcal{H}^{p,q}_{\Omega^2} \subset \mathcal{H}^{p,q}_{\Omega^2} = \mathcal{H}^{p,q}_{\omega^2} = \mathcal{H}^{p,q}_{\varphi^2},$$

where the last two sets are the spaces of $L^2$-harmonic forms with respect to the $\partial$-Laplacian $\Delta^{\partial}$ and to the Hodge-de Rham Laplacian $\Delta^{\varphi}$. Then, combining Theorem 4.4 with the results by Gromov, we obtain the proof of the vanishing Theorem 4.5.

In [1] Gromov gave an $L^2$ Hodge decomposition Theorem for complete Riemannian manifolds, (see also [1] Chap.VIII, Thm.3.2). It seems to be harder to prove a link between $W^{1,2}$ Bott-Chern harmonic forms and $L^2$ reduced cohomology. For other results on $L^2$ cohomological decomposition in the complete almost Hermitian setting see [5]. Thanks to elliptic regularity, we consider only $L^2$ forms that are also smooth. It is sufficient for our results.

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2. Preliminaries

Let $M$ be an $n$-dimensional complex manifold. Denote by $\Omega^r(M; \mathbb{C})$, respectively $\Omega^{p,q}(M)$ the space of smooth complex $r$-forms, respectively smooth $(p,q)$-forms on $M$. Let $g$ be a Hermitian metric on $M$; denote by $\omega$ the fundamental form of $g$ and by $\text{Vol}_g := \frac{\omega^n}{n!}$ the standard volume form. Let $\langle , \rangle$ be the pointwise Hermitian inner product induced by $g$ on the space of $(p,q)$-forms. Given any $\varphi \in \Omega^r(M; \mathbb{C})$, set

$$|\varphi(x)|_g^2 = \langle \varphi, \varphi \rangle(x)$$
and
\[ \|\varphi\|_{L^2} := \int_M |\varphi|^2 \text{Vol}_g, \]
\[ \|\varphi\|_{W^{1,2}} := \int_M (|\varphi|^2 + |\nabla \varphi|^2) \text{Vol}_g, \]
where \( \nabla \) is the Levi-Civita connection of \((M, g)\) and \( |\nabla \varphi|_g \) is the pointwise Hermitian norm of the covariant derivative of the \((p, q)\)-form \( \varphi \), induced by \( g \) on the space of complex covariant tensors on \( M \).

Define
\[ L^2(M) := \{ \varphi \in \Omega^r(M; \mathbb{C}) \mid 0 \leq r \leq 2n, \|\varphi\|_{L^2} < \infty \} \]
and
\[ W^{1,2}(M) := \{ \varphi \in \Omega^r(M; \mathbb{C}) \mid 0 \leq r \leq 2n, \|\varphi\|_{W^{1,2}} < \infty \} \]
For any given \( \varphi \in \Omega^r(M; \mathbb{C}) \), we also set
\[ \|\varphi\|_{L^\infty} := \sup_{x \in M} |\varphi(x)|_g \]
and we call \( \varphi \) bounded if \( \|\varphi\|_{L^\infty} < \infty \). Furthermore, if \( \varphi = d\eta \), then \( \varphi \) is said to be \( d \)-bounded, if \( \eta \) is bounded. For any \( \varphi, \psi \in \Omega^{p,q}(M) \) denote by \( \langle \; , \; \rangle \) the \( L^2 \)-Hermitian product defined as
\[ \langle \varphi, \psi \rangle = \int_M (\bar{\varphi} \cdot \psi) \text{Vol}_g \]
Denoting by \( * : \Omega^{p,q}(M) \to \Omega^{-p,-q}(M) \) the complex anti-linear Hodge operator associated with \( g \), the Bott-Chern Laplacian and Aeppli Laplacian \( \Delta_{BC}^g \) and \( \Delta_A^g \) are the 4-th order elliptic self-adjoint differential operators defined respectively as (see [9, p.8])
\[ \Delta_{BC}^g := \partial \overline{\partial} \partial^* + \partial^* \overline{\partial} \partial + \overline{\partial} \partial^* \overline{\partial} + \overline{\partial} \partial^* \overline{\partial} \]
and
\[ \Delta_A^g := \partial \partial^* + \overline{\partial} \overline{\partial}^* + \overline{\partial} \partial^* \partial + \overline{\partial} \partial^* \partial + \partial \overline{\partial} \overline{\partial}^* + \partial \overline{\partial} \overline{\partial}^* \]
where, as usual
\[ \partial^* = - \ast \partial \ast, \quad \overline{\partial}^* = - \ast \overline{\partial} \ast. \]

**Remark 2.1.** If \( g \) is a Kähler metric on \( M \), then as a consequence of the Kähler identities, (see [6, p.120] and [9, Prop.2.4]), it is
\[ \Delta_{BC}^g = \Delta_A^g \Delta_{BC}^g + \partial \overline{\partial} + \partial^* \overline{\partial}, \quad \Delta_A^g = \Delta_A^g \Delta_A^g + \partial \partial^* + \overline{\partial} \overline{\partial}^*, \]
\( \Delta_{BC}^g \) being the Dolbeault Laplacian on \((M, g)\).

Finally, we define the space of \( W^{1,2} \) Bott-Chern harmonic forms by setting
\[ H_{BC,2}^{p,q} := \{ \varphi \in \Omega^{p,q}(M) \mid \Delta_{BC}^g \varphi = 0, \ \varphi \in W^{1,2}(M) \}. \]

In the sequel we will need a local expression for the operators \( \overline{\partial} \). To this purpose, let \((z^1, \ldots, z^n)\) be local holomorphic coordinates on \( M \),
\[ g = \sum_{\alpha, \beta} g_{\alpha\beta} dz^\alpha \otimes dz^\beta \]
be the local expression of the Hermitian metric \( g \) and \( (g^{\alpha\beta}) = (g_{\alpha\beta})^{-1} \). Given \( \psi \in \Omega^{p,q}(M) \), for
\[ A_p = (\alpha_1, \ldots, \alpha_p), \quad B_q = (\beta_1, \ldots, \beta_q) \]
multiindices of length \( p, q \) respectively, with \( \alpha_1 < \cdots < \alpha_p \) and \( \beta_1 < \cdots < \beta_q \), denote by
\[ \psi = \sum_{A_p, B_q} \psi_{A_p, B_q} dz^{A_p} \wedge dz^{B_q} \]
the local expression of $\psi$ and by
\[ \psi^{A_p B_q} = \sum_{\gamma, A_q} g^{\gamma_1 \gamma_2} \cdots g^{\gamma_p \gamma_q} g^{\lambda_1 \lambda_2} \cdots g^{\lambda_x \lambda_y} \psi_{\gamma_1 \lambda_1} \cdots \psi_{\gamma_y \lambda_y}. \]

Then, if $\varphi \in \Omega^{p,q}(M)$, locally the pointwise Hermitian inner product $\langle \cdot, \cdot \rangle$ on $\Omega^{p,q}(M)$ is given by
\[ \langle \varphi, \psi \rangle = \sum_{A_p B_q} \varphi_{A_p B_q} \psi^{A_p B_q}. \]

The local expression of the pointwise Hermitian inner product induced by $g$ on the space of complex covariant tensors on $M$ is similar.

According to [7 Prop.2.3], we recall the local formula for $\bar{\Omega}$, that is for any given $\psi \in \Omega^{p,q+1}(M)$, it is
\[ (\bar{\Omega} \psi)^{A_p B_q} = -\sum_{\gamma=1}^{n} \left( \frac{\partial}{\partial z^\gamma} + \frac{\partial \log \det(g_{\alpha \beta})}{\partial z^\gamma} \right) \psi^{A_p B_q} \psi_{\alpha \beta}. \]

In the special case that $g$ is a Kähler metric on $M$, then as a consequence of the last formula, for every fixed $x_0 \in M$, denoting by $(x_1, \ldots, x^n)$ local normal holomorphic coordinates at $x_0$, we obtain
\[ (\bar{\Omega} \psi)_{\alpha_1 \cdots \alpha_p \beta_1 \cdots \beta_q}(x_0) = -\sum_{\gamma=1}^{n} \psi_{\gamma_1 \cdots \gamma_p \beta_1 \cdots \beta_q}(x_0). \]

Finally, for any given $\varphi \in \Omega^{p,q}(M)$, still using local normal holomorphic coordinates at $x_0$, we have
\[ |\nabla \varphi|^2(x_0) = 2 \sum_{\gamma=1}^{n} \sum_{A_p B_q} \left( \left| \frac{\partial \varphi^{A_p B_q}}{\partial z^\gamma} \right|^2 + \left| \frac{\partial \varphi_{A_p B_q}}{\partial z^\gamma} \right|^2 \right)(x_0). \]

3. Construction of cut-off functions in $d$-bounded Stein manifolds

Let $M$ be a Stein manifold and let $\rho$ be a strictly plurisubharmonic exhausting smooth function. Denote $\omega = i\partial \bar{\partial} \rho$ the fundamental form with the Kähler metric $g$ associated. We say that $M$ is $d$-bounded if $\omega = d \eta$ and $\eta = \partial \bar{\partial} \rho$ is bounded. In particular, $\omega$ is $d$-bounded. In the following, $\rho$, $\omega$, $g$, $\eta$ are considered fixed.

We remark that any $d$-bounded Stein manifold is complete, see [1 Chap.VIII, Lemma 2.4].

Examples of $d$-bounded Stein manifolds are bounded strictly pseudoconvex domains in $\mathbb{C}^n$ with smooth boundary, endowed with the Bergman metric, see [2 Prop.3.4].

Now we prove the existence of cut-off functions with specific bounds on the second order derivatives on a $d$-bounded Stein manifold. We need the following known lemma.

**Lemma 3.1.** Let $a, b \in \mathbb{R}$, $a < b$. Then there exists a $C^\infty$ function $\psi : \mathbb{R} \to [0, 1] \subset \mathbb{R}$ such that the following properties hold:

- $\psi(t) = 1 \iff t \leq a$;
- $\psi(t) = 0 \iff t \geq b$;
- $\exists C \in \mathbb{R}$ such that $|\psi'(t)|, |\psi''(t)| \leq C \psi(t)^{1/2}$ for all $t \in \mathbb{R}$.

**Proof.** Let us define $\phi : \mathbb{R} \to \mathbb{R}$, a $C^\infty$ function such that
\[ \phi(t) = \begin{cases} \exp(-\frac{1}{t^2}) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0. \end{cases} \]

Then we define
\[ \psi(t) = \frac{\phi(b-t)}{\phi(b-t) + \phi(t-a)}. \]
Note that $\psi(t) = 1$ iff $t \leq a$ and $\psi(t) = 0$ iff $t \geq b$. After some calculations we obtain

$$
\psi'(t) = -\frac{2}{(t-a)}\left(\phi(b-t) + \phi(t-a)\right) - \phi(b-t) - 2\frac{\phi(b-t) + 2\phi(t-a)}{(t-a)^3},
$$

$$
= -2\frac{\phi(b-t)\phi(t-a)}{(t-a)^2}\left(\frac{1}{(b-t)^3} + \frac{1}{(t-a)^3}\right).
$$

This implies that $\exists C \in \mathbb{R}$ such that $|\psi'(t)| \leq C\psi(t)^{1/2}$ $\forall t \in \mathbb{R}$. The calculations of the estimate on $\psi''$ are analogous. \hfill $\Box$

The following lemma is inspired by [1, Chap.VIII, Lemma 2.4].

**Lemma 3.2.** Let $M$ be a $d$-bounded Stein manifold of complex dimension $n$. Then there exists a sequence $\{K_\nu\}_{\nu \in \mathbb{N}}$ of compact subsets of $M$ and a sequence $a_\nu : M \to [0,1] \subset \mathbb{R}$, $\nu \in \mathbb{N}$, of $C^\infty$ functions with compact support, called cut-off functions, such that the following properties hold:

- $\bigcup_{\nu \in \mathbb{N}} K_\nu = M$ and $K_\nu \subset K_{\nu+1}$;
- $\forall \nu \in \mathbb{N}$ $a_\nu = 1$ in a neighbourhood of $K_\nu$ and supp$a_\nu \subset K_{\nu+1}$;
- $\exists C \in \mathbb{R}$ such that $|\partial a_\nu(x)|_g, |\partial \partial a_\nu(x)|_g, |\partial a_\nu(x)|_g \leq 2^{-\nu}C\rho(x)^{1/2}$ $\forall x \in M$.

**Proof.** We define

$$a_\nu(x) = \psi(2^{-\nu}\rho(x)) \forall x \in M \text{ and } K_\nu = \{x \in M \mid \rho(x) < 2^{-\nu}\},$$

where $\psi$ is the function of the previous lemma, with $a = 1.1$ and $b = 1.9$.

Let us check that the claimed properties hold. The subsets $K_\nu$ are compact because of the definition of exhausting function. If $x \in M$, then $\exists \nu \in \mathbb{N}$ such that $\rho(x) < 2^{-\nu}$, so $x \in K_\nu$, thus $\bigcup_{\nu \in \mathbb{N}} K_\nu = M$. The inclusions $K_\nu \subset K_{\nu+1}$ hold by the construction of $K_\nu$, in fact if $x \in K_\nu$, then $\rho(x) < 2^{-\nu}$ by continuity and $x \in \{y \in M \mid \rho(y) < 2^{-\nu} \cdot 1.5\} \subset K_{\nu+1}$.

The functions $a_\nu$ are $C^\infty$ because $\psi$ and $\rho$ are $C^\infty$. The function $\psi$ takes values in the interval [0, 1], and so is true for $a_\nu$. Choosing a sufficiently small neighbourhood of $K_\nu$, we can assume that $2^{-\nu}\rho < 1.1$, so $a_\nu = 1$ in that neighbourhood.

In order to prove supp$a_\nu \subset K_{\nu+1}$, let us take $\bar{x} \in$ supp$a_\nu$ and a sequence $\{x_k\}_{k \in \mathbb{N}}$ of points in $M$ such that $a_\nu(x_k) > 0 \forall k \in \mathbb{N}$ and $x_k \to \bar{x}$ as $k \to \infty$. By the construction of $\psi$, we have that, $\forall x \in M$, $a_\nu(x) > 0$ if and only if $2^{-\nu}\rho(x) < 1.9$.

Therefore, by the continuity of $\rho$, $2^{-\nu}\rho(\bar{x}) \leq 1.9$, so that $\rho(\bar{x}) < 2^{-\nu} \cdot 1.95$ and $\bar{x} \in K_{\nu+1}$. Because supp$a_\nu$ is a close set contained in a compact set, then it is compact.

Finally we have to prove the estimates on the differentials of $a_\nu$. Let $x \in M$, then

$$\partial a_\nu(x) = 2^{-\nu}\psi'(2^{-\nu}\rho(x)) \sum_{i=1}^n \frac{\partial \rho}{\partial z_i}(x)dz_i,$$

and

$$|\partial a_\nu(x)|_g = 2^{-\nu}|\psi'(2^{-\nu}\rho(x))||\partial \rho(x)|_g \leq 2^{-\nu}C(\psi(2^{-\nu}\rho(x)))^{1/2} ||\partial \rho||_{L^\infty} \leq 2^{-\nu}C\rho(x)^{1/2},$$

where the constant $C$ is taken big enough and may not be the same at every passage of the calculations. In the last passage we used the hypothesis that $\omega$ is $d$-bounded.
and the fact that $\rho$ is real, so $\frac{\partial \rho}{\partial z^i}(x) = \frac{\partial \rho}{\partial \bar{z}^i}(x)$ and $|\partial \rho(x)|_g = |\bar{\partial} \rho(x)|_g$. By the same calculations, we also obtain the estimate of $|\bar{\partial} a_{\nu}(x)|_g$. Moreover,

$$\bar{\partial} a_{\nu}(x) = 2^{-\nu} \sum_{i,j=1}^{n} \left( 2^{-\nu} \psi^\nu(2^{-\nu} \rho(x)) \frac{\partial \rho}{\partial z^i}(x) \frac{\partial \rho}{\partial \bar{z}^j}(x) + \psi'(2^{-\nu} \rho(x)) \frac{\partial^2 \rho}{\partial z^i \partial \bar{z}^j}(x) \right) dz^i \wedge d\bar{z}^j,$$

and

$$|\bar{\partial} a_{\nu}(x)|_g \leq 2^{-\nu} \left( 2^{-\nu} |\psi'(2^{-\nu} \rho(x))||\bar{\partial} \rho(x)|_g |\partial \rho(x)|_g + |\psi'(2^{-\nu} \rho(x))| \sum_{i,j=1}^{n} \frac{\partial^2 \rho}{\partial z^i \partial \bar{z}^j}(x) d\bar{z}^j \wedge d\bar{z}^i \right)$$

$$\leq 2^{-\nu} C \psi(2^{-\nu} \rho(x)) \frac{\psi'(2^{-\nu} \rho(x))}{\nu} \left( 2^{-\nu} |\bar{\partial} \rho|_g ||\partial \rho|_g + |\omega(x)|_g \right)$$

$$\leq 2^{-\nu} C a_{\nu}(x)^{\frac{1}{2}}.$$

In fact $\omega$ is $d$-bounded as before and $|\omega(x)|_g$ is constant, due to the definition of the metric. The proof is complete. \qed

4. Vanishing of $L^2$ Bott-Chern harmonic forms

Our main theorem states that the following property, true in the compact case, also holds in our non-compact case.

**Theorem 4.1.** Let $M$ be a $d$-bounded Stein manifold of complex dimension $n$. Let $\varphi \in \Omega^{p,q}(M) \cap W^{1,2}(M)$. If $\bar{\Delta}_B \varphi = 0$, then

$$\partial \varphi = 0, \quad \bar{\partial} \varphi = 0, \quad \bar{\partial}^{*} \partial^{*} \varphi = 0.$$

The following Lemma will be useful for the proof of Theorem 4.1.

**Lemma 4.2.** Let $M$ be a Kähler manifold of complex dimension $n$ and denote its metric with $g$. If $\varphi \in \Omega^{p,q}(M)$ and $\{a_{\nu}\}_n$ are $C^\infty$ functions on $M$, then $\exists C > 0$ such that $\forall \nu \in \mathbb{N}$

$$|\bar{\partial} (\bar{\partial} a_{\nu} \wedge * \varphi)|_g \leq C \left( |\bar{\partial} a_{\nu}|_g |\varphi|_g + |\bar{\partial} a_{\nu}|_g |\nabla \varphi|_g \right),$$

$$|\bar{\partial} (\bar{\partial} a_{\nu} \wedge \varphi)|_g \leq C \left( |\bar{\partial} a_{\nu}|_g |\varphi|_g + |\bar{\partial} a_{\nu}|_g |\nabla \varphi|_g \right).$$

**Proof.** The pointwise Hermitian norm on forms is invariant under change of coordinates, so we can prove the inequalities locally with a uniform constant $C$. For every fixed $x_0 \in M$, denoting by $(z^1, \ldots, z^n)$ local normal holomorphic coordinates at $x_0$, we have

$$\bar{\partial} a_{\nu} = \sum_{\beta} \frac{\partial a_{\nu}}{\partial z^\beta} dz^\beta,$$

$$\varphi = \sum_{A_p, B_q} \varphi_{A_p, B_q} dz^{A_p} \wedge d\bar{z}^{B_q},$$

$$\bar{\partial} a_{\nu} \wedge \varphi = \sum_{A_p, B_q, \beta} \frac{\partial a_{\nu}}{\partial z^{\beta}} \varphi_{A_p, B_q} dz^{\beta} \wedge d\bar{z}^{A_p} \wedge d\bar{z}^{B_q}.$$

We can write

$$\bar{\partial} a_{\nu} \wedge \varphi = \frac{1}{p!(q + 1)!} \sum_{\alpha_1, \ldots, \alpha_p, \beta_0, \beta_1, \ldots, \beta_q} \left( \bar{\partial} a_{\nu} \wedge \varphi \right)_{\alpha_1, \ldots, \alpha_p, \beta_0, \beta_1, \ldots, \beta_q} dz^{\alpha_1 \cdots \alpha_p} \wedge d\bar{z}^{\beta_0 \beta_1 \cdots \beta_q}.$$
where the coefficients \((\bar{\partial}a_\nu \wedge \varphi)_{\alpha_1 \ldots \alpha_p \beta_0 \bar{\beta}_i \ldots \bar{\beta}_q}\) are antisymmetric in the indices \(\alpha_1, \ldots, \alpha_p, \beta_0, \bar{\beta}_1, \ldots, \bar{\beta}_q\), so
\[
(\bar{\partial}a_\nu \wedge \varphi)_{\bar{\beta}_0 \alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_q} = (-1)^{p+1} (\bar{\partial}a_\nu \wedge \varphi)_{\alpha_1 \ldots \alpha_p \bar{\beta}_0 \beta_1 \ldots \beta_q}
\]
where we set \(\beta_{j+q} := \beta_{j-1}\) for \(j = 1, \ldots, q\). Now we apply (1) and obtain
\[
(\bar{\partial} (\bar{\partial}a_\nu \wedge \varphi))_{\alpha_1 \ldots \alpha_p \beta_0 \ldots \beta_q}(x_0) = \sum_{\beta_0=1}^{n} \frac{\partial}{\partial z_{\beta_0}} \left( (\bar{\partial}a_\nu \wedge \varphi)_{\bar{\beta}_0 \alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_q} \right)(x_0)
\]
This yields
\[
|\bar{\partial} (\bar{\partial}a_\nu \wedge \varphi)|_{\varphi}(x_0) \leq C(|\partial \bar{\partial}a_\nu|_{\varphi}(x_0) + |\partial a_\nu|_{\varphi}(x_0) + \nabla \varphi(x_0)),
\]
where \(C\) depends only on \(n, p, q\). To prove it, we have to do some calculations. We set
\[
\gamma_{\beta_0} := \frac{\partial^2 a_\nu}{\partial z_{\beta_0} \partial z_{\beta_j}} \varphi_{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_j} \quad \text{and} \quad \lambda_{\beta_0} := \frac{\partial a_\nu}{\partial z_{\beta_0}} \frac{\partial \varphi_{\alpha_1 \ldots \alpha_p \beta_1 \ldots \beta_j}}{\partial z_{\beta_0}},
\]
with \(\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q = 1, \ldots, n\), \(\beta_0 = 1, \ldots, n\) and \(j = 0, \ldots, q\). So we have, using (2),
\[
|\bar{\partial} (\bar{\partial}a_\nu \wedge \varphi)|_{\varphi}(x_0) \leq \frac{1}{p!q!} \sum_{\beta_0} \sum_{\beta_1} \sum_{\beta_q} \sum_{\gamma_{\beta_0}, \lambda_{\beta_0}} \left( \gamma_{\beta_0} + \lambda_{\beta_0} \right)(\gamma_{\beta_j, \beta_j'} + \lambda_{\beta_j, \beta_j'})(x_0)
\]
To prove the other inequality in (2), we want to estimate \(|\bar{\partial} (\bar{\partial}a_\nu \wedge \varphi)|_{\varphi}(x_0)\), so the calculations are analogous except for the fact that we use the following formula (see [2, p.94]) for the Hodge star operator:
\[
\ast \varphi = (i)^n (-1)^{\frac{n(n-1)+n}{2}} \sum_{A_p, B_q} \text{sgn} A \text{sgn} B \det(g_{ik}) B_q A_p d^{A_{p-1}} B_{q} \wedge d^{B_{n-q}}\varphi,
\]
where \(A\) and \(B\) are respectively the permutations that send \((1, \ldots, n)\) in \((A_p, A_{n-p})\) and in \((B_q, B_{n-q})\).

**Proof of Theorem 4.1** First of all, by remark 2.1 we have
\[
\Delta^{\partial} \Delta^{\partial} = \Delta^{\partial} \Delta^{\partial} + \overline{\partial} \overline{\partial} + \overline{\partial} \overline{\partial}.
\]
Thanks to the cut-off functions of the previous lemma, now we can integrate by part, using the Stokes Theorem.

\[ 0 = \langle \Delta_{BC}^{\mu} \varphi, a_{\nu} \varphi \rangle \]

\[ = \langle \Delta^{*} \Delta^{*} \varphi, a_{\nu} \varphi \rangle + \langle \Delta^{*} \Delta \varphi, a_{\nu} \varphi \rangle + \langle \Delta^{*} \varphi, a_{\nu} \varphi \rangle + \langle \varphi, \Delta a_{\nu} \varphi \rangle \]

\[ = \langle \Delta^{*} \varphi, \Delta(a_{\nu} \varphi) \rangle + \langle \Delta^{*} \varphi, \Delta a_{\nu} \varphi \rangle + \langle \Delta a_{\nu} \varphi, \varphi \rangle + \langle \varphi, \Delta(a_{\nu} \varphi) \rangle \]

Now we calculate every differential in the right sides of the inner products.

\[ \Delta^{*} (a_{\nu} \varphi) = \Delta^{*} (a_{\nu} \varphi) = \Delta^{*} \varphi \]

\[ = (-1)^{p+q} \Delta^{*} (a_{\nu} \varphi) = \Delta^{*} \varphi \]

\[ \Delta^{*} \varphi = \Delta^{*} (a_{\nu} \varphi) = \Delta^{*} \varphi \]

\[ \Delta^{*} \varphi = \Delta(a_{\nu} \varphi) \]

\[ \Delta(a_{\nu} \varphi) \]

Therefore,

\[ 0 = \langle \Delta_{BC}^{\mu} \varphi, a_{\nu} \varphi \rangle = I_1(\nu) + I_2(\nu), \]

where

\[ I_1(\nu) = \int_M a_{\nu} (\Delta^{*} \varphi)^2 + \Delta^{*} \varphi + \Delta^{*} \varphi \]

and

\[ I_2(\nu) = \int_M (-1)^{p+q} \Delta^{*} (a_{\nu} \varphi) = \Delta^{*} \varphi \]

\[ = \Delta^{*} \varphi \]

\[ \Delta^{*} \varphi \]

We have \( I_1(\nu) = |I_2(\nu)| \) and, by the monotone convergence theorem, as \( \nu \to \infty \),

\[ I_1(\nu) \to \int_M \langle \Delta_{BC}^{\mu} \varphi, a_{\nu} \varphi \rangle = \Delta^{*} \varphi \to \infty \]

Thus, if we show that \( I_1(\nu) \to 0 \) as \( \nu \to \infty \), we have

\[ \partial \varphi = 0, \quad a_{\nu} \varphi = 0, \quad \Delta^{*} \varphi = 0, \quad \Delta^{*} \varphi = 0. \]

Estimating \( |I_2(\nu)| \), we obtain

\[ |I_2(\nu)| \leq \int_M (-1)^{p+q} \Delta^{*} (a_{\nu} \varphi) \]

By Lemma 2 there exists a constant \( C > 0 \) such that

\[ |\Delta^{*} (a_{\nu} \varphi) | \leq C (|\Delta a_{\nu} \varphi| + \Delta a_{\nu} \varphi) \]

Therefore we have

\[ |I_2(\nu)| \leq C \int_M (-1)^{p+q} \Delta a_{\nu} \varphi | \Delta a_{\nu} \varphi| + \Delta a_{\nu} \varphi | \Delta a_{\nu} \varphi| \]

The estimates on the cut-off functions, i.e.,

\[ |\partial a_{\nu} \varphi|, |\Delta a_{\nu} \varphi|, |\Delta a_{\nu} \varphi| \leq 2^{-\nu} C a_{\nu} \]
yield

\[ I_1(\nu) = |I_2(\nu)| \leq 2^{-\nu} C \int_M a_\nu(x) \left( |\bar{\nabla}|^2 \varphi|_g (|\varphi|_g + |\nabla \varphi|_g) + 
\right.
\[ + |\bar{\nabla} \partial \varphi|_g (|\varphi|_g + |\nabla \varphi|_g) + 
\]
\[ + |\bar{\nabla} \varphi|_g (|\varphi|_g + |\nabla \varphi|_g) \right) \text{Vol}_g 
\]
\[ \leq 2^{-\nu} C \int_M a_\nu(x) \left( |\bar{\nabla}|^2 \varphi|_g (|\varphi|_g + |\nabla \varphi|_g) + |\bar{\nabla} \varphi|_g (|\varphi|_g + |\nabla \varphi|_g) \right) \cdot 
\]
\[ \left( |\varphi|_g + |\nabla \varphi|_g \right) \text{Vol}_g 
\]
\[ \leq 2^{-\nu} C \left( \int_M (|\varphi|_g + |\nabla \varphi|_g)^2 \text{Vol}_g \right)^{\frac{1}{2}} 
\]
\[ \leq 2^{-\nu} C (I_1(\nu))^2 \cdot (||\varphi||_{L^2} + ||\nabla \varphi||_{L^2}), 
\]
and consequently

\[ (I_1(\nu))^2 \leq C 2^{-\nu}. \]

Thus \( \varphi \) is \( \partial \)-closed and \( \bar{\partial} \)-closed. This implies

\[ \tilde{\Delta}^0_B \varphi = \partial \bar{\partial} \varphi. \]

Now we substantially reapply the argument as above to this form of the Bott-Chern Laplacian. We have

\[ 0 = \langle \tilde{\Delta}^0_B \varphi, a_\nu \varphi \rangle = \langle \partial \bar{\partial} \partial^* \varphi, a_\nu \varphi \rangle = \langle \bar{\nabla} \partial^* \varphi, \partial^* (a_\nu \varphi) \rangle, \]

and

\[ \bar{\nabla} \partial^* (a_\nu \varphi) = \bar{\nabla} \left( - \ast (\partial a_\nu \wedge \ast \varphi) + a_\nu \partial^* \varphi \right) \]
\[ = (-1)^{p+q-1} \ast (\partial a_\nu \wedge \ast \varphi) - (-1)^{p+q-1} \ast (\partial a_\nu \wedge \ast \varphi) + 
\]
\[ - \ast (\partial a_\nu \wedge \ast \partial^* \varphi) + a_\nu \bar{\nabla} \partial^* \varphi, \]

thus

\[ 0 = \langle \tilde{\Delta}^0_B \varphi, a_\nu \varphi \rangle = I_1'(\nu) + I_2'(\nu), \]

where

\[ I_1'(\nu) = \int_M a_\nu |\bar{\nabla} \partial^* \varphi|_g^2 \text{Vol}_g, \]

and

\[ |I_2'(\nu)| \leq 2^{-\nu} C \int_M a_\nu(x) \left( |\bar{\nabla}|^2 \partial^* \varphi|_g (|\varphi|_g + |\nabla \varphi|_g) \right) \text{Vol}_g 
\]
\[ \leq 2^{-\nu} C (I_1'(\nu))^2 \cdot (||\varphi||_{L^2} + ||\nabla \varphi||_{L^2}). \]

Thus we also have \( \bar{\nabla} \partial^* \varphi = 0 \). This ends the proof. \( \square \)

**Remark 4.3.** From Theorem 4.1 we immediately obtain that

\( \varphi \in H^p_B \) if and only if \( \varphi \in W^{1,2}(M) \), \( \partial \varphi = 0 \), \( \bar{\partial} \varphi = 0 \), \( \bar{\nabla} \partial^* \varphi = 0 \),

which “extends” the characterization of the space of Bott-Chern harmonic forms on a compact Hermitian manifold to any d-bounded Stein manifold.

As a straightforward consequence of Theorem 4.1 we obtain the following
Theorem 4.4. Let $M$ be a $d$-bounded Stein manifold of complex dimension $n$. Then

$$\mathcal{H}^{p,q}_{BC,2} \subset \mathcal{H}^{p,q}_{\partial,2} = \mathcal{H}^{p,\bar{q}}_{d,2},$$

where the last three sets are the spaces of $L^2$-harmonic $(p,q)$-forms with respect to $\Delta^g_d$, $\Delta^g_\partial$ and $\Delta^g_{\partial\bar{\partial}}$.

Proof. We note that the last three equalities of the thesis hold because $M$ is Kähler, in fact $\Delta^g_\partial = 2\Delta^g_d = 2\Delta^g_{\partial\bar{\partial}}$ by Kähler identities. Therefore it is enough to prove $\mathcal{H}^{p,q}_{BC,2} \subset \mathcal{H}^{p,q}_{\partial,2}$.

Let $\varphi \in \mathcal{H}^{p,q}_{BC,2}$; from (4) in the proof of Theorem 4.1 we have

$$\frac{\partial}{\partial \bar{\partial}} \varphi = 0, \quad \bar{\partial} \varphi = 0.$$

Thus $\Delta^g_{\partial\bar{\partial}} \varphi = 0$ and $\varphi \in \mathcal{H}^{p,q}_{\partial,2}$.

We are ready to prove the following

Theorem 4.5. Let $M$ be a $d$-bounded Stein manifold of complex dimension $n$. Then

$$\mathcal{H}^{p,q}_{BC,2} = \{0\}, \quad \text{for } p + q \neq n.$$

Proof. By Theorem 4.4 it is enough to prove that $\mathcal{H}^{p,q}_{\partial,2} = \{0\}$. This last fact is a consequence of [4, Thm.1.2.B.]. For the sake of completeness we remind the argument by Gromov.

Let us consider the Lefschetz operator

$$L : \Omega^{(p,q)}(M) \longrightarrow \Omega^{(p+1,q+1)}(M)$$

defined by

$$L\varphi = \omega \wedge \varphi$$

for every $\varphi \in \Omega^{(p,q)}(M)$. By [3, rem.3.2.7.iii]], the map

$$L^{n-p-q} : \mathcal{H}^{p,q}_d \longrightarrow \mathcal{H}^{n-q,n-p}_d$$

is an isomorphism for $p + q \leq n$, where $\mathcal{H}^{p,q}_d$ denotes the space of $\Delta^g_d$-harmonic $(p,q)$-forms. Now set $k = n-p-q$ and consider the form $L^k\varphi$, where $\varphi \in \Omega^{(p,q)}(M) \cap L^2(M)$ is a $d$-closed form. Since $\omega = d\eta$, if $k > 0$, then we get

$$L^k\varphi = \omega^k \wedge \varphi = (d\eta)^k \wedge \varphi = d(\eta \wedge (d\eta)^{k-1} \wedge \varphi).$$

Furthermore,

- $\eta \wedge (d\eta)^{k-1}$ is bounded, since $\eta$ is bounded and $|d\eta|_g$ is constant;
- $\eta \wedge (d\eta)^{k-1} \wedge \varphi \in L^2(M)$, since $\varphi \in L^2(M)$.

Moreover, if $\varphi \in \Omega^{(p,q)}(M) \cap L^2(M)$ is a $\Delta^g_d$-harmonic form, then $L^k\varphi$ is also $\Delta^g_d$-harmonic. Thus, in view of the $L^2$ Hodge decomposition theorem (see [1, Chap.VIII, Thm.3.2]), we obtain that $L^k\varphi = 0$.

Now let $\varphi \in \mathcal{H}^{p,q}_{\partial,2}$. If $p + q < n$, then $k > 0$ and $L^k\varphi = 0$; therefore $\varphi = 0$ since $L^k$ is injective. Conversely, if $p + q > n$, then $*\varphi \in \mathcal{H}^{n-p,n-q}_{\partial\bar{\partial}}$ and $(n-p) + (n-q) < n$; by the previous argument $*\varphi = 0$ and consequently $\varphi = 0$.

Summing up, we showed that $\mathcal{H}^{p,q}_{\partial,2} = \{0\}$ for $p + q \neq n$. This ends the proof. □

Remark 4.6. The Lefschetz argument, in the proof of Theorem 4.5 uses only the Kähler and the $d$-bounded assumptions.

Remark 4.7. By similar computations, Theorem 4.5 can be stated and proved also for the Acupli Laplacian. Indeed, it is sufficient to repeat the proof of Theorem 4.7 with $\Delta^g_d = \Delta^g_\partial + \partial \partial^* + \partial^* \partial$. 


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