GENERIC CHARACTER SHEAVES ON REDUCTIVE GROUPS OVER A FINITE RING

ZHE CHEN

Abstract. In this paper we propose a construction of generic character sheaves on reductive groups over finite local rings at even levels, whose characteristic functions are higher Deligne–Lusztig characters when the parameters are generic. We formulate a conjecture on the simple perversity of these complexes, and we prove it in the level two case (thus generalised a result of Lusztig from the function field case). We then discuss the induction and restriction functors, as well as the Frobenius reciprocity, based on the perversity.

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1. Introduction

The theory of character sheaves, initiated by Lusztig in [Lus85] for connected reductive groups over an algebraically closed field, is a geometric theory of characters. In this paper we propose a construction of generic character sheaves for connected reductive groups over \( \mathcal{O}^u / \pi^r \), where \( \mathcal{O} \) is a complete discrete valuation ring with a finite residue field \( \mathbb{F}_q \), \( \mathcal{O}^u \) its maximal unramified extension, \( \pi \) a uniformiser, and \( r \) a fixed arbitrary positive even integer. In the prior work [Lus06], Lusztig considered the principal series case with \( \mathcal{O} = \mathbb{F}_q[[\pi]] \) (but without any restriction on \( r \)); his construction is based on the use of a natural analogue of the Grothendieck–Springer resolution. In this paper, we will replace the resolution morphism by another morphism, based on the arithmetic radical introduced in [CS17] (see Section 2); this replacement allows us to deal with any series, not only the principal series, and as one shall see, some methods in the \( r = 1 \) case are also available in our construction. In the below we describe the motivation of our construction.

In 1979, Lusztig [Lus79] proposed a geometric method to study the ordinary representations of reductive groups over \( \mathcal{O} / \pi^r \), which generalises Deligne and Lusztig’s seminal work [DL76] (which corresponds to the case \( r = 1 \)). The proofs of some results in [Lus79] were
later established in the function field case by Lusztig himself in [Lus04], and Stasinski [Sta09] generalised this work from the function field case to the general case, by the use of Greenberg functor technique. Meanwhile, when \( r \geq 2 \), there also exists an algebraic method to construct certain irreducible representations of these groups, due to Gérardin [Gér75]. The geometrically constructed representations and the algebraically constructed representations have the same set of parameters when some regularity condition is satisfied, and Lusztig raised the problem on whether these two style representations actually coincide. In the case of even \( r \), a solution to this problem was given for \( GL_n \) in [Che17], which was generalised to the general case in [CS17]; we recall this result in Section 2. This result suggests that a character sheaf theory for \( r > 1 \) may be established based on Gérardin’s constructions.

In Section 3, we recall the character sheaves on abelian groups following [Lus06, 5]. In Section 4, we define the generic character sheaves at even levels (i.e. \( r \) even), and then compare their characteristic functions with the characters of even level Deligne–Lusztig representations. In Section 5 we state the conjecture on the simple perversity, and prove it for \( r = 2 \) (see Theorem 5.3), which generalised a result of Lusztig in the function field case in [Lus17]. In Section 6, we define the induction and the restriction functors, and study their transitive properties. In Section 7, we study the Frobenius reciprocity between the induction and the restriction functors. In the final Section 8, we give some remarks in the function field case.

Conventions: Denote by \( k \) the residue field of \( \mathcal{O}^{nr} \). By a variety we always mean a reduced quasi-projective variety over \( k \). For two elements \( x \) and \( y \) in an algebraic group, we use \( xy \) short for \( y^{-1}xy \). For a product variety \( X_1 \times \cdots \times X_n \), we denote by \( \pi_{X_i} \) the projection to \( X_i \). For an algebraic group \( G \) we denote by \( G^0 \) the identity component of \( G \).

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2. Generic characters and Deligne–Lusztig theory

Let \( \mathbb{G} \) be an affine smooth group scheme over \( \mathcal{O}/\pi^r \), and let \( G \) be its base change to \( \mathcal{O}^{nr}/\pi^r \). There is an associated algebraic group \( G = G_r = F_\mathbb{G} \) over \( k \), where \( F \) is the Greenberg functor introduced in [Gre61] and [Gre63]. We recall some basic properties in this setting (see [Sta12] and [Sta09] for the details): The Frobenius element in Gal\((k\mathbb{F}_q)\) gives a rational structure of \( G \) over \( \mathbb{F}_q \); we denote the associated geometric Frobenius by \( F \). One has

\[
\mathbb{G}(\mathcal{O}/\pi^r) \cong G^F \quad \text{and} \quad G(\mathcal{O}^{nr}/\pi^r) \cong G(k)
\]
as abstract groups. For any positive integer \( i \leq r \), being the kernel of the reduction map modulo \( \pi^i \) is a closed condition, hence defines a normal closed subgroup \( \mathbb{G}^i \) of \( \mathbb{G} \); we denote the quotient group \( \mathbb{G}/\mathbb{G}^i \) by \( \mathbb{G}_i \). For convenience we put \( \mathbb{G}^0 = \mathbb{G} \). Similar notation also applies to closed subgroups of \( \mathbb{G} \).

From now on let \( \mathbb{G} \) be a reductive group over \( \mathcal{O}/\pi^r \); by this we mean it is an affine smooth group scheme over \( \mathcal{O}/\pi^r \), with the geometric fibres being connected reductive groups in the usual sense. Let \( \mathbf{T} \) be a maximal torus of \( \mathbb{G} \) such that \( T = F\mathbf{T} \) is \( F \)-stable. Let \( \mathbf{B} \) be a Borel subgroup of \( \mathbb{G} \) containing \( \mathbf{T} \); one has the Levi decomposition \( \mathbf{B} = \mathbf{U}\mathbf{T} \), where \( \mathbf{U} \) is
the unipotent radical of \( B \). Let \( B \) (resp. \( U \)) be the Greenberg functor image of \( B \) (resp. \( U \)). The Deligne–Lusztig variety associated to \( T \) and \( U \) is (see [Lus04] and [Sta09])

\[
L^{-1}(FU) := \{ g \in G \mid g^{-1} F(g) \in FU \}.
\]

Note that \( L^{-1}(FU) \) admits a left action of \( G^F \) and a right action of \( T^F \), and these two actions commute. They induce actions on the compactly supported cohomology groups \( H^i_c(L^{-1}(FU), \overline{\mathbb{Q}}_\ell) \) for every \( i \in \mathbb{Z} \) (here \( \ell \) is a fixed arbitrary prime not equal to \( \text{char}(k) \)). For \( \theta \in \widehat{T^F} = \text{Hom}(T^F, \overline{\mathbb{Q}}_\ell^*) \), the \( G^F \)-module

\[
R^{\theta}_{T,U} = \sum_i (-1)^i H^i_c(L^{-1}(FU), \overline{\mathbb{Q}}_\ell)_\theta,
\]

where the subscript \( \theta \) means taking the \( \theta \)-isotypical part, is called a Deligne–Lusztig representation. Note that this is a virtual representation over \( \overline{\mathbb{Q}}_\ell \).

In the remaining part of this paper we assume that \( r = 2l \) is even (do not mix \( l \) with the prime \( \ell \)). Let \( U^- \) be the unipotent radical of the opposite Borel subgroup of \( B \), and denote by \( U^- \) the corresponding Greenberg functor image. The commutative unipotent group \( U^\pm := (U^-)^r U^l \) is called the arithmetic radical of \( T \); it is \( F \)-stable (see [CS17]). For \( \alpha \in \Phi \) a root of \( T \), we write \( T^\alpha = F^\alpha T \), where \( T^\alpha \) denotes the image of the coroot \( \check{\alpha} \). We call \( W(T) := W(T_1) \) the Weyl group of \( T \). More details on these concepts can be found in [DG70]. We recall some conditions on the characters of \( T^F \) considered in [Lus04], [Sta09], and [CS17].

**Definition 2.1.** Let \( \theta \in \widehat{T^F} \).

(i) If \( \theta \) is not stabilised by any non-trivial element in \( W(T)^F \), then \( \theta \) is said to be in general position.

(ii) Let \( a \) be a positive integer such that \((T^\alpha)^{r-1} \) is \( F^a \)-stable for every root \( \alpha \in \Phi \). Then \( \theta \) is said to be regular if, for every root \( \alpha \in \Phi \), there is some \( t \in ((T^\alpha)^{r-1})^F \) such that \( \theta(tF(t)...F^{a-1}(t)) \neq 1 \). The regularity does not depend on the choice of \( a \).

(iii) Let \( \tilde{\theta} \) be the trivial extension of \( \theta \) to \((TU^\pm)^F \), then \( \theta \) is said to be generic if it is regular, in general position, and \( \text{Stab}_{G^F}(\tilde{\theta}|_{(G^F)^F}) = (TU^\pm)^F \cdot \text{Stab}_{N(T)^F}(\tilde{\theta}|_{(G^F)^F}) \).

We remark that the genericity is very close to the regularity, and it is actually a natural Lie algebra condition; see [CS17, 3.6 and 3.7].

We have the following algebraisation result.

**Proposition 2.2.** If \( \theta \in \widehat{T^F} \) is regular and in general position, then \( R^{\theta}_{T,U} \) is a subrepresentation of \( \text{Ind}^{G^F}_{(TU^\pm)^F} \tilde{\theta} \). If moreover \( \theta \) is generic, then \( R^{\theta}_{T,U} \cong \text{Ind}^{G^F}_{(TU^\pm)^F} \tilde{\theta} \).

**Proof.** See [CS17]. \( \square \)

The representations of the form \( \text{Ind}^{G^F}_{(TU^\pm)^F} \tilde{\theta} \) were first studied by Gérardin [Gér75].

### 3. Character sheaves on abelian groups

In this section we recall the setting of character sheaves on commutative algebraic groups in [Lus06, 5], with a focus on \( T \).
Fix an arbitrary positive integer $m$, let $L'$ be the Lang endomorphism associated to $F^m$ on $T$. There is a direct sum decomposition of $\mathcal{Q}_\ell$-sheaves on $T$

$$L'_*\mathcal{Q}_\ell = \bigoplus_{\psi \in \mathcal{T}^{F^m}} E^\psi,$$

where $E^\psi$ is a locally constant $\mathcal{Q}_\ell$-sheaf of rank 1, whose stalk at $t \in T$ is the 1-dimensional $\mathcal{Q}_\ell$-representation space of $\mathcal{T}^{F^m}$ given by $\psi$:

$$E^\psi_t = \{ f : L'^{-1}(t) \to \mathcal{Q}_\ell | f(t_1t_2) = \psi(t_1)f(t_2), \forall t_1 \in \mathcal{T}^{F^m}, t_2 \in L'^{-1}(t) \}.$$

There is a unique isomorphism of locally constant $\mathcal{Q}_\ell$-sheaves $\varphi : (F^m)^*E^\psi \cong E^\psi$, such that at stalks $E^\psi_{F^m(t)} \to E^\psi_t$, it is $\varphi_t : f \mapsto f \circ F^m$. Note that if $t \in \mathcal{T}^{F^m}$, then for $y \in L'^{-1}(t)$ one has $F^m(y) = ty$, so $f \circ F^m(y) = \psi(t)f(y)$.

**Definition 3.1.** Let $\mathcal{S}(T)$ be the set of all $E^\psi$, for various $m$ and $\psi$.

4. **Generic character sheaves**

The algebraisation of the generic even level Deligne–Lusztig representations (see Proposition 2.2) suggests that one can develop a generic character sheaf theory for reductive groups over $\mathcal{O}^m/\pi^r$, $r$ even, based on Gérardin’s representations. Consider the diagram

$$T \leftarrow^b Z_T \rightarrow^a G,$$

where $Z_T := \{(g, xTU^\pm) \in G \times G/TU^\pm | g^x \in TU^\pm \}$; here $a$ is the natural left projection, and $b$ is the projection of $g^x \in TU^\pm$ to $\pi_T(g^x) \in T$.

**Lemma 4.1.** The variety $Z_T$, as well as the morphisms $a$ and $b$, are all $F$-stable.

**Proof.** This follows from the fact that $U^\pm$ is $F$-stable. \qed

**Lemma 4.2.** The variety $Z_T$ is smooth and connected.

**Proof.** This can be proved in a way similar to [Lus85, 2.5.2]. Consider the faithful flat base change $\mathcal{Z}_T$ of $Z_T \subseteq G \times G/TU^\pm$ along

$$G \times G \to G \times G/TU^\pm.$$

Then it suffices to show that $\mathcal{Z}_T = \{(g, x) \in G \times G | g^x \in TU^\pm \} \subseteq G \times G$ is smooth and connected (see [GD67, 17.7.7]). By the variable change $b = g^x$ we get

$$\mathcal{Z}_T \cong \{(b, x) \in G \times G | b \in TU^\pm \} = TU^\pm \times G,$$

which is smooth and connected. \qed

Given a variety $X$ over $k$, we write $\mathcal{D}(X)$ short for $D^b_c(X, \mathcal{Q}_\ell)$, the bounded derived category of constructible $\mathcal{Q}_\ell$-sheaves constructed in [Del80].

Back to the diagram (1). For $m \in \mathbb{Z}_{>0}$ and $\theta \in \mathcal{T}^{F^m}$, take $E^\theta \in \mathcal{S}(T)$, and put $K^\theta := R\alpha_1(b^*E^\theta) \in \mathcal{D}(G)$.

**Definition 4.3.** We call the complexes $K^\theta$, for various generic characters $\theta \in \mathcal{T}^{F^m}$ and various $m \in \mathbb{Z}_{>0}$, the **generic character sheaves** on $G$. 


Let \( \theta \in \hat{T}^{F^m} \) be a generic character. The isomorphism \( \varphi : (F^m)^*E^\theta \cong E^\theta \) induces an isomorphism \( \varphi : (F^m)^*K^\theta \cong K^\theta \). We want to evaluate the characteristic function of the complex \( K^\theta \) with respect to \( \varphi \), i.e. evaluate

\[
\chi_{K^\theta,\varphi}(g) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot \text{Tr}(\varphi_g, H^i(K^\theta)_g)
\]

for \( g \in G^{F^m} \):

**Proposition 4.4.** Along with the above notation, we have

\[
\chi_{K^\theta,\varphi}(g) = \text{Tr}(g, R^\theta_{T,U})
\]

for any \( g \in G^{F^m} \). (Here \( R^\theta_{T,U} \) is defined with respect to \( F^m \), not \( F \).)

**Proof.** This argument is standard. First, by the proper base change (of \( a \) along the injection \( \{g\} \to G \)) we have

\[
\chi_{Ra_b^*E^\theta,\varphi}(g) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot \text{Tr}(\varphi_g, H^i(R a_1(b^*E^\theta))_g)
\]

\[(2) = \sum_{xTU^\pm \in (G/TU^\pm)^{F^m}, (g, xTU^\pm) \in Z_T} \text{Tr}(\varphi, b^*E^\theta_{(g,xTU^\pm)}).
\]

By applying Grothendieck’s Lefschetz trace formula (see [Del77, Rapport-3.2]) we get

\[
\text{Tr}(\varphi, b^*E^\theta_{(g,xTU^\pm)}) = \sum_{xTU^\pm \in (G/TU^\pm)^{F^m}, g^x \in TU^\pm} \frac{1}{|(TU^\pm)^{F^m}|} \cdot \sum_{x \in G^{F^m}, g^x \in TU^\pm} \theta(\pi_T(g^x)).
\]

This is, by definition of induced characters, the character value of \( \text{Ind}_{(TU^\pm)^{F^m}}^{G^{F^m}} \tilde{\theta} \) at \( g \). Now the assertion follows from Proposition 2.2. \( \square \)

5. Simple perversity

For a variety \( X \) over \( k \), let \( D^{\le 0}(X) \) be the full subcategory of \( D(X) \) consisting of the objects \( K \) satisfying: The support of \( H^i(K) \) has dimension \( \leq -i \) for any integer \( i \) (in particular, \( H^i(K) = 0 \) if \( i > 0 \), and the support of \( H^0(K) \) is a finite set). Meanwhile, let \( D^{\ge 0}(X) \) be the full subcategory of \( D(X) \) consisting of the objects \( K \) such that \( D_X(K) \in D^{\le 0}(X) \), where \( D_X \) denotes the Verdier duality functor on \( D(X) \). The category of perverse sheaves is the full subcategory \( M(X) := D^{\le 0}(X) \cap D^{\ge 0}(X) \).

**Definition 5.1.** Let \( \theta \in \hat{T}^F \). We call \( \theta \) geometrically generic, if it extends to a generic character of the abelian group \( T^{F^m} \) for every \( m \in \mathbb{Z}_{>0} \). We remark that (i) it seems that geometric genericity actually coincides with genericity, and (ii) when \( \text{char}(\mathbb{F}_q) \) is big enough, being geometrically generic equivalents to corresponding to a regular semisimple element in the Lie algebra; see [CS17, 3.6] for more details.

Now we can state the conjecture (due to Lusztig when \( O = \mathbb{F}_q[[\pi]] \)):

\[
\text{Ind}_{(TU^\pm)^{F^m}}^{G^{F^m}} \tilde{\theta} \]
**Conjecture 5.2.** If $\theta$ is geometrically generic, then $K^{\theta}[\dim G]$ is a simple perverse sheaf.

In the below, extending the idea of Lusztig in the function field case (in [Lus17]), we prove this conjecture in the level two case, by showing that Deligne’s $\ell$-adic Fourier transformation of $K^{\theta}[\dim G]$ is a simple perverse sheaf supported on a subvariety (of $G$) defined by the regular semisimple orbit (in the Lie algebra) corresponding to $\theta$.

**Theorem 5.3.** The above conjecture is true for $r = 2$ and $\text{char}(k)$ very good for $G_1$.

**Proof.** First, $G^1$ is a connected unipotent group isomorphic to the additive group of the Lie algebra of $G_1$ (see [Gre63]), hence a special group in the sense of [Ser58], thus $G \rightarrow G_1$ is a vector bundle of rank $\dim G_1$. We fix a local trivialisation for this principal vector bundle, and for each $g \in G$ we assign to it a fixed local piece in this trivialisation (so we can talk about the $G_1$-part and the $G_1$-part of $g$). Within this vector bundle, we will use the $\ell$-adic Fourier transform technique in [Lau87]: As $\text{char}(k)$ is very good, there is a non-degenerate $G$-invariant symmetric bilinear form on $G_1$ (see e.g. [Let05, 2.5.12]), which identifies the dual of $G_1$ to itself, hence produce a pairing $h(-, -) : G \times G_1 \rightarrow A^1$. The geometric genericity of $\theta$ then means there is a regular semisimple element $t_\theta \in (T^F)^1$, such that $\hat{\theta}(s) = \psi(h(t_\theta, s))$ for all $s \in (G^1)^F$, where $\psi : F_q \rightarrow \overline{Q}_\ell^\times$ is a fixed non-trivial group morphism ((CS17, 3.6)).

Identifying $T_1$ as the reductive part of $T$, we get $T = T_1 \times T^1$; let $\mathcal{E}^\theta$ be the character sheaf on $T_1$ associated to $\theta |_{T^F}$ (in the sense of [Lus06, 0.5]). Meanwhile, for a morphism $f : X \rightarrow A^1$ over $\overline{F}_q$, we denote the Artin–Schreier sheaf associated to $f$ and $\psi$ by $\mathcal{L}_f$ (see e.g. [Lus17, 0.3]); in particular, we have a local system $\mathcal{L}_h_\theta$ on $T^1$, where $h_\theta$ is the restriction of $h(t_\theta, -)$ to $T^1$. Denote the pull-backs of $\mathcal{E}^\theta$ and $\mathcal{L}_h_\theta$ to $Z_T$ (along $b$) by $\mathcal{E}$ and $\mathcal{L}$, respectively, then we need to show that $R(a_!\mathcal{E} \otimes \mathcal{L})[\dim G^1]$ is a simple perverse sheaf.

Let $\pi_1$ and $\pi_2$ be the first and the second projections $G \times_{G_1} G \rightarrow G$, then by Laumon’s theorem [Lau87, 1.3.2] it suffices to show that the Fourier transform of $K^{\theta}[\dim G]$, that is, $\hat{K} := R(\pi_2)_!(\pi_1^*R(a_!\mathcal{E} \otimes \mathcal{L}) \otimes \mathcal{L}_h)[3 \dim G^1]$, is simple perverse.

Consider the pull-back $a'$ of $a : Z_T \rightarrow G$ along $\pi_1$

$$
\begin{align*}
\begin{array}{ccc}
Z_T & \rightarrow & Z_T \\
\downarrow & & \downarrow a \\
G \times_{G_1} G & \rightarrow & G.
\end{array}
\end{align*}
$$

By proper base change and projection formula we get

$$
\begin{align*}
\hat{K} &= R(\pi_2)_!(R(a')_!(\pi_1^*\mathcal{E} \otimes \mathcal{L}) \otimes h_\theta)[3 \dim G^1] \\
&= R(\pi_2)_!(R(a')_!((\pi_1^*\mathcal{E} \otimes \mathcal{L}) \otimes a'^*\mathcal{L}_h))[3 \dim G^1] \\
&= R(\pi_2)_!(R(a')_!(\mathcal{E}' \otimes \mathcal{L}' \otimes a'^*\mathcal{L}_h))[3 \dim G^1] \\
&= R(\pi_2 \circ a')_!(\mathcal{E}' \otimes \mathcal{L}'_b)[3 \dim G^1],
\end{align*}
$$

where $\mathcal{E}'$ and $\mathcal{L}'$ denote the pull-backs (along $\pi_1'$) of $\mathcal{E}$ and $\mathcal{L}$ to $Z_T$, respectively, and $h'$ denotes the morphism $h \circ a' + h_\theta \circ \pi_{T_1} \circ b \circ \pi_1'$ (here $\pi_{T_1}$ is the projection from $T$ to $T^1$).
Note that \( Z'_T \) can be written as \( \{(y, x) \in (G \times G) \times G/TU^\pm \mid \pi_1(y)\pi \in TU^\pm \} \). As \( \pi_1(y)\pi \in TU^\pm \) is actually a condition only concerns the \( G_1 \)-part of \( \pi_1(y) \), we see that \( \pi_2 \times \text{Id}_{G/TU^\pm} \) gives \( Z'_T \) a vector bundle structure over \( Z_T \) (with fibre \( \cong G^1 \)); denote the restriction of \( \pi_2 \times \text{Id}_{G/TU^\pm} \) to \( Z'_T \) by \( \mu \). Given \( (g, x) \in Z_T \), identify \( \mu^{-1}(g, x) \) as \( G^1 \) via the fixed local trivialisation, then the restriction of \( h' \) on \( \mu^{-1}(g, x) \) can be written as

\[
h'(y, x) = h(y) + h(t_\theta, \pi_{T^1} \circ b(\pi_1(y), x)) = h(\pi_1(y)\pi, g) + h((t_\theta), \pi_x\pi_1(y)),
\]

where \( \pi_1(y)\pi \) and \( \tilde{g} \) denote the \( G^1 \)-parts of \( \pi_1(y) \) and \( g \), respectively, and \( \pi_x \) denotes the projection from \( x(TU^\pm) \) to \( x(T^1) \). However, as \( h \) is symmetric, by the definition of \( \tilde{\theta} \) we see \( h'(y, x) = h(\tilde{g} \cdot (t_\theta), \pi_1(y)) \). Therefore, \( h' \) is zero on \( \mu^{-1}(g, x) \) if and only if \( \tilde{g} = (t_\theta) \); note that this is a closed condition, so all such \((g, x)\) form a closed subvariety \( \tilde{Z} \subseteq Z_T \).

Denote the pull-back of \( \mathcal{E} \) to \( \tilde{Z} \) by \( \tilde{\mathcal{E}} \), and denote the restriction of \( a \) to \( Z \) by \( \tilde{a} \). Since \( \pi_2 \circ a' = a \circ \mu \), according to [Lus17, 1.3(b)] we have

\[
\tilde{K} \cong R(\tilde{a}_!)(\tilde{\mathcal{E}})[\dim G^1].
\]

Meanwhile, note that \( \tilde{a} \) defines an isomorphism from \( \tilde{Z} \) to a subvariety of \( G \), which is locally

\[
\{(g', g'') \in V \times C \mid g' \in \text{Stab}_G(g'')\},
\]

where \( V \subseteq G_1 \) is a local piece in the local trivialisation and \( C \) is the (adjoint) \( G \)-orbit of \( (t_\theta)^{-1} \). Recall that \( C \subseteq G_1 \) is a closed subvariety (see [Bor91, III.9]), and by the definition of regular elements it is of dimension \( \dim G_1 - \dim T_1 \), so \( \tilde{Z} \subseteq G \) is a smooth closed subvariety of dimension \( \dim G^1 \) via \( \tilde{a} \) (for the smoothness, consider the projection from \( \tilde{Z} \) to \( C \)). This implies that \( \tilde{K} \cong \tilde{a}_!\tilde{\mathcal{E}}[\dim G^1] \) is a simple perverse sheaf on \( G \), which completes the proof. \( \square \)

In the rest of this paper we turn to the discussion of the induction and restriction functors.

**Remark 5.4.** Here we collect some basic properties of \( \mathcal{D}(X) \) and \( \mathcal{M}(X) \), which will be used later; the details and further properties can be found in [BBD82], [Lus85], [Lus10], and [BD10]. Let \( f : X \to Y \) be a morphism of varieties over \( k \). We use \( f^* \), \( f^! \), \( Rf_* \), and \( Rf! \) to denote the corresponding derived functors.

(P1) Suppose that \( f \) is smooth with connected fibres of dimension \( d \). Then (ignore Tate twists) \( f^! = f^*[2d] \). Moreover, \( \tilde{f} := f^*[d] \) is fully faithful from \( \mathcal{M}(Y) \) to \( \mathcal{M}(X) \). If \( K \in \mathcal{D}(Y) \), then \( \tilde{f}K \in \mathcal{D}^{\leq 0}(X) \) (resp. \( \in \mathcal{D}^{\leq 0}(X) \)) if and only if \( K \in \mathcal{D}^{\geq 0}(Y) \) (resp. \( \in \mathcal{D}^{\geq 0}(Y) \)). If \( K \in \mathcal{D}^{\geq 0}(Y) \) and \( K' \in \mathcal{D}^{\geq 0}(Y) \), then \( \text{Hom}_{\mathcal{D}(Y)}(K, K') = \text{Hom}_{\mathcal{D}(X)}(\tilde{f}K, \tilde{f}K') \).

(P2) Suppose that \( h : H \times Y \to Y \) is the action morphism of a connected algebraic group \( H \) acting on \( Y \). Let \( \pi_Y : H \times Y \to Y \) be the projection morphism. Note that both \( h \) and \( \pi_Y \) are smooth of dimension \( \dim H \). We say that \( K \in \mathcal{M}(Y) \) is \( H \)-equivariant, if the two perverse sheaves \( h_!K \) and \( \tilde{\pi}_Y K \) are isomorphic.

(P3) Suppose that \( H \) is a connected algebraic group acting freely on \( X \) and trivially on \( Y \), and that \( f \) is locally trivial and \( H \)-equivariant. If \( K \in \mathcal{M}(X) \), then \( K \) is \( H \)-equivariant if and only if \( K \cong \tilde{f}K' \) for some \( K' \in \mathcal{M}(Y) \).
6. INDUCTION AND RESTRICTION

In this section we define the induction functors (from the equivariant perverse sheaves on the Levi subgroups to \( \mathcal{D}(G) \)) and the restriction functors (from the derived category \( \mathcal{D}(G) \) to that of the Levi subgroups), and then study their transitive properties.

Fix a parabolic subgroup \( \mathbf{P} \) of \( \mathbf{G} \), and let \( \mathbf{M} \) be a Levi subgroup of \( \mathbf{P} \) (when there is no confusion we will say that \( \mathbf{M} \) is a Levi subgroup of \( \mathbf{G} \)); denote by \( \mathbf{P} \) and \( \mathbf{M} \) the corresponding Greenberg functor images, respectively. Write \( U_\mathbf{P} \) for the unipotent radical of \( \mathbf{P} \), and write \( U_- \mathbf{P} \) for the unipotent radical of the opposite parabolic subgroup; denote their Greenberg functor images by \( U_\mathbf{P} \) and \( U_- \mathbf{P} \), respectively. We put \( U^\pm_\mathbf{G} := (U_\mathbf{P})^l(U_- \mathbf{P})^l \) and \( U^\pm_\mathbf{M} := \mathcal{M} \cap U^\pm \). For detailed properties of parabolic subgroups and Levi subgroups we refer to [DG70, XXVI].

Now we are going to define the induction functor, which requires some technical preparations.

Consider the varieties
\[
Z^G_\mathbf{M} := \{(g, xM^\pm U) \in G \times G/M^\pm U \mid g^x \in M^\pm U\}
\]
and
\[
\widehat{Z}^G_\mathbf{M} := \{(g, x) \in G \times G \mid g^x \in M^\pm \} \cong M^\pm \times G;
\]
they admit the \( G \)-action \( y \in G \): \((g, x) \mapsto (g, ygy^{-1}, yx)\). Consider the \( G \)-equivariant smooth morphism with connected fibres
\[
\pi'_{M,G} : \widehat{Z}^G_\mathbf{M} \longrightarrow Z^G_\mathbf{M}, \quad (g, x) \mapsto (g, xM^\pm U);
\]
this is a principal \( M^\pm \)-fibration, where \( M^\pm \) acts on \( \widehat{Z}^G_\mathbf{M} \) by \( y' \in M^\pm : (g, x) \mapsto (g, xy'^{-1}) \). Moreover, note that the quotient \( G \rightarrow G/M^\pm \) is locally trivial: As \( G^l \) is a connected unipotent group, it suffices to show that \( G^l \rightarrow G^l/M^l \) is locally trivial, which follows from the fact that the multiplication morphism \( U^- \mathbf{P} \times U_\mathbf{P} \times \mathcal{M} \rightarrow \mathbf{G} \) is an open immersion (see [DG70, XXVI 4.3.2]). Therefore \( \pi'_{M,G} \) is a locally trivial principal fibration by \( M^\pm \). Consider the trivial \( G \)-action on \( \mathcal{M} \), then we get another \( G \)-equivariant smooth morphism with connected fibres
\[
\pi_{M,G} : \widehat{Z}^G_\mathbf{M} \longrightarrow \mathcal{M}, \quad (g, x) \mapsto \pi_M(g^x),
\]
where \( \pi_M \) is the projection from \( M^\pm \) to \( \mathcal{M} \). Note that the action of \( M^\pm \) on \( \widehat{Z}^G_\mathbf{M} \) induces an action of \( M^\pm \) on \( \mathcal{M} \), which is compatible with the conjugation action of \( \mathcal{M} \) on \( \mathcal{M} \), so \( \pi_{M,G} \) is also \( M^\pm \)-equivariant. Now we get a diagram
\[
(3) \quad M \xleftarrow{\pi_{M,G}} \widehat{Z}^G_\mathbf{M} \xrightarrow{\pi_{M,G}} Z^G_\mathbf{M} \xrightarrow{\pi''_{M,G}} \mathbf{G},
\]
where \( \pi''_{M,G} \) is the left projection (which is \( G \)-equivariant with respect to the conjugation action of \( G \) on itself).

Consider (3), if \( K \in \mathcal{M}(\mathcal{M}) \) is \( \mathcal{M} \)-equivariant (see Remark 5.4 (P2)) with respect to the conjugation action, then from the above we see that \( \pi''_{M,G} K \) is \( M^\pm \)-equivariant and \( G \)-equivariant. Moreover, since \( \pi'_{M,G} \) is a locally trivial principal fibration by \( M^\pm \), there is a
unique (up to isomorphisms) perverse sheaf (see Remark 5.4 (P3))

\[ K_{M,G} \in \mathcal{M}(Z^G_M) \]

such that \( \widetilde{\pi}_{M,G} K \cong \widetilde{\pi}'_{M,G} K_{M,G} \). Note that by Remark 5.4 (P1) and the \( G \)-equivariance of \( \widetilde{\pi}_{M,G} K \), this \( K_{M,G} \) is \( G \)-equivariant.

**Definition 6.1.** Given \( K \in \mathcal{M}(M) \) equivariant with respect to the conjugation action of \( M \), along with the above notation, we put

\[ \text{ind}^G_M K := R(\pi''_{M,G})! K_{M,G} \in D(G). \]

Note that if \( \text{ind}^G_M K \in D(G) \) is perverse, then it is \( G \)-equivariant by proper base change ([Lus85, 1.7.5]) and the \( G \)-equivariance of \( K_{M,G} \).

**Example 6.2.** If \( M = G \), then \( \text{ind}^G_G K = K \) by the equivariant property.

**Example 6.3.** If \( M = T \) is the maximal torus, then \( Z^G_T = Z_T \), and \( \pi_{M,G} \) naturally factors through \( b: Z^G_T \to T \) (\( b \) is the morphism in (1)), which implies \( K_{M,G} \cong \tilde{b}K \). Thus in this simpler situation, we see that \( \text{ind}^G_T E^n[\dim T] \cong K^n[\dim G] \) for any \( \theta \in \hat{T}^F \).

**Proposition 6.4.** Let \( N \) be a Levi subgroup of \( M \); denote by \( N \) its Greenberg functor image. If \( K \in \mathcal{M}(N) \) is an \( N \)-equivariant perverse sheaf such that \( \text{ind}^M_N K \) is a perverse sheaf, then

\[ \text{ind}^G_N K \cong \text{ind}^G_M \circ \text{ind}^M_N K. \]

**Proof.** The argument is an analogue to the one in [Lus85, 4.2]. We have a commutative diagram

where \( X := \{(g, x, z) \in G \times G \times MU^\pm | g^x \in NU^\pm\} \), and \( Y \) is the quotient of \( X \) by the \( NU^\pm \)-action given by \( q \in NU^\pm: (g, x, z) \mapsto (g, xq^{-1}, zq^{-1}); f \) denotes the quotient morphism. The other morphisms are as below:

- \( d: (g, x, z) \mapsto \pi_N(g^x) \), where \( \pi_N \) is the projection from \( NU^\pm \) to \( N \);
- \( h_1 \) is \( (g, x, z) \mapsto (\pi_M(g^{xz^{-1}}), \pi_M(z)) \);
- \( h_2 \) is \( (g, x, z) \mapsto (g, x) \);
- \( e_1: (g, x, z) \mapsto (\pi_M(g^{xz^{-1}}), \pi_M(z)NU^\pm_M) \in Z^M_N \subseteq M \times M/NU^\pm_M \);
- \( e_2: (g, x, z) \mapsto (g, xNU^\pm) \in Z^G_N = \{(g, x) \in \hat{G} \times G/NU^\pm | g^x \in NU^\pm\} \);
- \( g_1: (g, x, z) \mapsto (g, xz^{-1}) \);
- \( g_2: (g, xNU^\pm) \mapsto (g, xM^\pm) \).
Note in the above diagram, the two bottom squares are cartesian, and $e_i$ and $f$ are smooth morphisms with connected fibres.

To show that $\text{ind}_{N}^{G} K \cong \text{ind}_{M}^{G} \text{ind}_{N}^{M} K$, in other words, to show that

$$R(\pi'_{M,G})_! R(g_2)_! K_{N,G} \cong R(\pi''_{M,G})_! (\text{ind}_{N}^{M} K)_{M,G},$$

it suffices to show that

$$R(g_2)_! K_{N,G} \cong (\text{ind}_{N}^{M} K)_{M,G}.$$  

(Here, for a perverse sheaf $A$, the notation $A_{B,C}$ is defined in the same way as in (4), by formally replacing $M$ and $G$ by $B$ and $C$, respectively.) Since $\tilde{\pi}'_{M,G}$ is fully faithful on perverse sheaves (see Remark 5.4 (P1)), this assertion can be deduced by showing that

$$(5) \quad \tilde{\pi}'_{M,G} R(g_2)_! K_{N,G} \cong \tilde{\pi}'_{M,G} (\text{ind}_{N}^{M} K)_{M,G}.$$  

Note that

$$\tilde{\pi}'_{M,G} (\text{ind}_{N}^{M} K)_{M,G} \cong \tilde{\pi}'_{M,G} \text{ind}_{N}^{M} K \cong \tilde{\pi}'_{M,G} R(\pi'_{M,M})_! K_{N,M}$$

by the definition of $(\text{ind}_{N}^{M} K)_{M,G}$, so (5) is equivalent to

$$R(g_1)_! \tilde{e}_2 K_{N,G} \cong R(g_1)_! \tilde{e}_1 K_{N,M}$$

by applying the proper base change (on both sides). Thus we only need to show that $\tilde{e}_2 K_{N,G} \cong \tilde{e}_1 K_{N,M}$, which is equivalent to showing

$$(6) \quad \tilde{f} \tilde{e}_2 K_{N,G} \cong \tilde{f} \tilde{e}_1 K_{N,M}$$

(by the full faithfulness of $\tilde{f}$ on perverse sheaves). By the definitions of $K_{N,M}$ and $K_{N,G}$, (6) follows from

$$\tilde{h}_1 \tilde{\pi}'_{N,M} K = \tilde{d} K = \tilde{h}_2 \tilde{\pi}'_{N,G} K.$$  

This completes the proof. □

Now we turn to the restriction functor.

**Definition 6.5.** Consider the diagram

$$(7) \quad M \xleftarrow{\pi_M} MU^\pm \xrightarrow{i} G,$$

where $i$ is the natural closed immersion and $\pi_M$ is the projection from $MU^\pm \cong M \times U_{G-M}^\pm$ to $M$. For any $K \in \mathcal{D}(G)$, we put $\text{res}^G_M K := R(\pi_M)_! i^* K \in \mathcal{D}(M)$.

**Proposition 6.6.** Suppose that $N$ is a Levi subgroup of $M$, and denote by $N$ its Greenberg functor image, then

$$\text{res}^G_N \cong \text{res}^M_N \circ \text{res}^G_M.$$

**Proof.** Consider the following commutative diagram

$$\begin{array}{ccc}
N & \xrightarrow{\pi_N} & NU^\pm \\
\downarrow{\pi'_N} & \swarrow{\pi'} & \searrow{\pi'_2} \\
NU^\pm_M & \xrightarrow{i_2} & MU^\pm_M \\
\downarrow{i_2} & \swarrow{\pi_M} & \\
M & \xrightarrow{\pi_M} & G
\end{array}$$
Where \( \pi' \) and \( \pi'_N \) are the natural projections, and \( i_1, i_2, \) and \( i'_2 \) are the natural inclusions. Note that the middle diamond is cartesian, so by the proper base change theorem we have
\[
\text{res}^G_N = R(\pi_N)i^*
= R(\pi'_N)R(\pi')(i'_2)^*i_1^*
= R(\pi'_N)(i_2)^*R(\pi_M)i_1^*
= \text{res}^M_N \circ \text{res}^G_M.
\]
Thus the transitive property holds. \( \square \)

7. Frobenius reciprocity

In this section we will be concerned with a Frobenius reciprocity.

In the level \( r = 1 \) case, if \( A \) is a character sheaf, then Lusztig proved that \( \text{res}^G_M A \in \mathcal{D}^{\leq 0}(M) \) [Lus85, 4.4 (c)], and based on this property he established the Frobenius reciprocity (see [Lus85, 4.4]). We expect that our generic character sheaves also satisfy this property (after a shifting). However, the method in the \( r = 1 \) case does not work well in the \( r \) even case; one of the obstructions is that the morphism \( a \) in (1) is not proper. In any case, in the below we show that, when such a property holds, there is still a Frobenius reciprocity for \( r \) even:

**Proposition 7.1.** If \( A_1 \in \mathcal{M}(M) \) is \( M \)-equivariant, and if \( A \in \mathcal{M}(G) \) is \( G \)-equivariant such that \( \text{res}^G_M A \in \mathcal{D}^{\leq 0}(M) \), then
\[
\text{Hom}_{\mathcal{D}(G)}(\text{res}^G_M A, A_1) \cong \text{Hom}_{\mathcal{D}(M)}(A, \mathbb{D}_G \circ \text{ind}^G_M \circ \mathbb{D}_M A_1).
\]

**Proof.** We combine the Verdier duality with the methods in [Lus85, 4.4]. Consider the commutative diagram

\[
\begin{align*}
& \mu_+ \backslash M \times G \quad \rho \quad G \\
& \downarrow \pi'_M \quad \uparrow \zeta \\
& M \times G \\
& \downarrow \gamma \quad \theta \quad \mu_+ \\
& M \end{align*}
\]

where \( i \) and \( \pi_M \) are as in (7), \( \pi''_{M,G} \) is as in (3), and \( \beta \) is the quotient morphism of the \( \mu_\pm \)-action on \( M \times G \) given by
\[
y \in \mu_\pm: (g, x) \mapsto (\pi_M(y)g\pi_M(y)^{-1}, xy^{-1});
\]
and the other morphisms are as below:
- \( \zeta', \theta', \) and \( \gamma \) are left projections;
- \( \theta : (g, x) \mapsto \pi_M(g); \)
- \( \phi := \pi_M \times \text{id}; \)
- \( \zeta : (g, x) \mapsto xgx^{-1}. \)
Concerning \( f \) and \( \rho \), first recall that \( Z_G^M = \{(g, x) \in G \times MU^\pm \mid g^x \in MU^\pm\} \); we put
\[
((g, x)) \mapsto (\pi_M(g^x), x) \mod MU^\pm;
\]
\[
\rho: (g, x) \mapsto (gx^{-1}, xMU^\pm).
\]
Note that in this diagram, by identifying \( MU^\pm \times G \) with \( \tilde{Z}_G^M \) we see that \( \rho \) becomes \( \pi_{M,G}^\prime \), so \((\phi, \gamma, \pi_M, \theta')\) and \((f, \beta, \rho, \phi)\) form two cartesian diagrams. Also note that \( \beta \) is a locally trivial fibration (as \( \phi \) and \( \rho \) are locally trivial fibrations; see the arguments above \((3)\)), and\( f \) is smooth with connected fibres \((\cong U_G^{\pm-M})\).

We have \((\text{Remark 5.4 (P1)})\)
\[
\text{Hom}_{D(M)}(\text{res}_M^G A, A_1) \cong \text{Hom}_{D(M \times G)}(\tilde{\gamma} \text{res}_M^G A, \tilde{\gamma} A_1).
\]
Consider the right hand side; by the proper base change theorem we see that
\[
\tilde{\gamma} \text{res}_M^G A = \gamma^* R(\pi_M)_! i^* A[\dim G]
\]
\[
= R\phi_! (\theta')^* i^* A[\dim G]
\]
\[
= R\phi_! (\zeta')^* A[\dim G],
\]
which is actually \( R\phi_! (\zeta')^* A[\dim G] \) by the equivariance of \( A \), and then again by the proper base change we get
\[
R\phi_! (\zeta')^* A[\dim G] = R\phi_! \rho^* (\pi_{M,G}'')^* A[\dim G]
\]
\[
= \beta^* R f_! (\pi_{M,G}'')^* A[\dim G]
\]
\[
= \tilde{\beta} R f_! (\pi_{M,G}'')^* A[\dim U_G^{\pm-M}].
\]

On the other hand, since \( \gamma \) is \( MU^\pm \)-equivariant with respect to the conjugation action of \( MU^\pm \) composed by \( \pi_M \), we see that \( \tilde{\gamma} A_1 \) is \( MU^\pm \)-equivariant, thus \( \tilde{\gamma} A_1 = \tilde{\beta} A_1' \) for some \( A_1' \in \mathcal{M}(MU^\pm \setminus M \times G) \) \((\text{Remark 5.4 (P3)})\), so \((8)\) becomes
\[
\text{Hom}_{D(M)}(\text{res}_M^G A, A_1) \cong \text{Hom}_{D(M \times G)}\left(\tilde{\beta} R f_! (\pi_{M,G}'')^* A[\dim U_G^{\pm-M}], \tilde{\beta} A_1'\right).
\]

And, the condition \( \text{res}_M^G A \in D^{\leq 0}(M) \) becomes \((\text{Remark 5.4 (P1)})\)
\[
\tilde{\beta} R f_! (\pi_{M,G}'')^* A[\dim U_G^{\pm-M}] \in D^{\leq 0}(M \times G),
\]
thus \((\text{Remark 5.4 (P1)})\) and the adjunctions imply that
\[
\text{Hom}_{D(M \times G)}\left(\tilde{R} f_! (\pi_{M,G}'')^* A[\dim U_G^{\pm-M}], A_1'\right)
\]
\[
= \text{Hom}_{D(M)}(A, \tilde{R} (\pi_{M,G})_* f_! A_1'[- \dim U_G^{\pm-M}]).
\]

Now we want to apply the Verdier duality. First note that, if \( K \) is an equivariant perverse sheaf on \( M \), then by the compatibilities between the Verdier duality functor and (proper) pull-backs \((\text{see e.g. [BD10, E.4]})\), its Verdier dual \(D_K^M\) is also equivariant. We have
\[
R(\pi_{M,G})_* f_! A_1'[- \dim U_G^{\pm-M}] = R(\pi_{M,G})_* f_! (\tilde{D}_{MU^\pm \times M \times G} \circ D_{MU^\pm \times M \times G}) A_1'[- \dim U_G^{\pm-M}]
\]
\[
= (\tilde{D}_G \circ R(\pi_{M,G})_! f^* D_{MU^\pm \times M \times G} A_1')[- \dim U_G^{\pm-M}]
\]
\[
= D_G (\tilde{D}_{MU^\pm \times M \times G} A_1')[\dim U_G^{\pm-M}]
\]
\[
= D_G R(\pi_{M,G})_! f^* D_{MU^\pm \times M \times G} A_1'
\]
by the Verdier duality \((\text{see e.g. [BD10, E.4]})\).
Meanwhile, since $\tilde{\gamma} A_1 = \tilde{\beta} A'_1$, we see that
\[
\tilde{\gamma} D_M A_1 = \gamma^* (D_M A_1)[\dim G] = D_{M \times G} (\gamma' A_1 [- \dim G]) = D_{M \times G} \gamma A_1
\]
\[
= D_{M \times G} \beta A'_1 = \beta (D_{MU \pm M \times G A'_1}[- \dim MU^\pm]) = \beta D_{MU \pm M \times G A'_1},
\]
so
\[
\tilde{\rho}(D_M A_1)_{M,G} \cong \tilde{\phi} \gamma D_M A_1 \cong \tilde{\phi} \beta D_{MU \pm M \times G A'_1} \cong \tilde{\rho} f D_{MU \pm M \times G A'_1}.
\]
Thus by the uniqueness of $(D_M A_1)_{M,G}$ (see (4)) we get $f D_{MU \pm M \times G A'_1} \cong (D_M A_1)_{M,G}$. Therefore
\[
(10) = D_G R(\pi''_{M,G})!(D_M A_1)_{M,G} = D_G \circ \text{ind}^G_M \circ D_M A_1.
\]
This completes the proof. 

8. Further remarks on function fields

Throughout this section we focus on the function field case $O = \mathbb{F}_q[[\pi]]$. Note that a very special phenomenon happened in this case is the existence of a natural section $G_1 \to G$ to the reduction map.

In this situation, Lusztig proposed in [Lus06] the generic principal series character sheaves (here “principal series” means that the involved $T$ is contained in an $F$-stable $B$), and conjectured that these complexes are perverse; this construction is based on an analogue of the Grothendieck–Springer resolution. In [Lus17], Lusztig proved the conjecture for $r \leq 4$, with char($\mathbb{F}_q$) big enough; in the arguments, some complexes based on Gérardin’s representations are proved to be perverse (the construction of these complexes is a little bit different from the one presented here, but the resulting complexes are up to shifts isomorphic), and he showed that the principal series character sheaves coincide with these complexes (see also [Kim16], which extended this coincidence for a general $r$).

Meanwhile, at level $r = 2$, Fan considered in his PhD thesis [Fan12] another construction of character sheaves, by a method different from ours. While in his construction the characteristic functions are not known, it is immediate from the definition that his complexes are perverse. It would be interesting to understand the relations between his construction and our construction.

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Department of Mathematics, Shantou University, Shantou, China  
E-mail address: zhechencz@gmail.com