On the Equivalence of Phase and Field Charges

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Abstract. The analysis of the gauge principle as a mere passive symmetry requirement leads to the conclusion that the connection term in the covariant derivative is flat and that local phase transformations are without any empirical significance in analogy to coordinate transformations. Nevertheless, the Aharonov-Bohm effect shows the physical significance of the non-trivial holonomy of a flat connection. On this basis the proposal of a new kind of charge, the phase charge, is made, understood as the coupling strength of the particle to the holonomy. The equivalence of phase and usual field charge must be tested experimentally in terms of an Aharonov-Bohm effect with muons or tauons, for instance.

1 Gauge principle: received and passive view

The received view of gauge theories and their underlying gauge principle is that we are able to derive the coupling of a formerly free field to an interaction field from the requirement of local gauge covariance. The usual derivative $\partial_{\mu}$ must be replaced by a covariant derivative

$$D_{\mu} = \partial_{\mu} - iqA_{\mu}(x),$$

where $A_{\mu}$ is a Lie algebra-valued 1-form—mathematically a connection in a principal fiber bundle, physically the gauge potential of an interaction field. However, this gauge argument would be close to a miracle, if the connection, which arises from the local symmetry requirement, were non-flat (i.e. with non-vanishing curvature).

But this, of course, is not—and can hardly be—the case. What rather happens is that we explicitly allow for a freedom in the choice of the position representation of the wave function (figuring as a free Schrödinger or Dirac field in the matter field equation from which the gauge principle starts). To see this immediately, consider a system of vectors...
spanning an abstract Hilbert space $\mathcal{H}$, such that a wave function in the position representation $|x\rangle$ simply reads $\Psi(x) = \langle x|\phi\rangle$. Accordingly, local gauge transformations are expressed as $|x'\rangle = e^{i\chi(x)}|x\rangle = \hat{U}|x\rangle$. Such a change of representation must apply as well to all operators $\hat{O}$ acting on $\mathcal{H}$, viz. $\hat{O}' = \hat{U}\hat{O}\hat{U}^\dagger$. In particular, for the derivative operator we get

$$D_\mu = \partial_\mu - i\partial_\mu\chi(x). \quad (2)$$

Identifying the gradient of the phase with the gauge potential $A_\mu$ (multiplied by some charge $q$ in order to get the units$^1$ straight)

$$\partial_\mu\chi(x) = q A_\mu(x) \quad (3)$$

leads to (1). Obviously, $A_\mu$ is a flat connection.

Thus, the celebrated gauge principle is not sufficient to ‘derive’ the coupling to a new interaction-field, but rather makes the built-in covariance under local gauge transformations explicit. We are, from the mere logic of the argument, not enforced to consider the connection term non-flat. Moreover, the Noether current connected to the covariance of the Dirac Lagrangian under global $U(1)$ transformations is just the probability density current

$$S^\mu = \bar{\psi}\gamma_\mu\psi$$

and not the charge current $j^\mu = q S^\mu$, since there simply is no charge occurring in the Dirac equation. Again, in the standard textbook presentation the charge is put in by hand by means of (3). Call all this the passive view on the gauge principle, since local gauge transformations are treated in full analogy to coordinates—coordinates, however, in the fiber bundle rather than in the space-time base space. The analogy is complete if we draw a comparison to the Levi-Civita connection in General Relativity. Here, Christoffel symbols already occur in the geodesic equation simply because of curvilinear coordinates in flat Minkowski space—without entailing a real gravitational field, i.e. non-vanishing Riemann curvature.

## 2 The non-observability of local phase transformations

Let us now consider the gauge principle’s local phase transformations (a.k.a. gauge transformations of the first kind) in more detail. Under such transformations the wave function yields

$$\psi(x) \rightarrow \psi'(x) = \psi(x)e^{i\chi(x)}. \quad (4)$$

Thus, we obtain $\partial_\mu\psi(x) \rightarrow \partial_\mu\psi'(x) = e^{i\chi(x)}\left(\partial_\mu + i\partial_\mu\chi(x)\right)\psi(x)$, which confirms the covariant derivative (2), i.e. Dirac equations $(i\partial_\mu\gamma^\mu - m)\psi(x) = 0$ and $(iD_\mu\gamma^\mu - m)\psi'(x) = 0$ are equivalent. We also get with $\hat{p}_\mu = -i\partial_\mu$ and $e^{i\chi} = e^{ipx}$ the phase transformation

\[\text{If not otherwise stated we set } c = \hbar = 1 \text{ throughout this paper.}\]
behavior $\chi \to \chi - \partial_\mu \chi(x) \cdot x^\mu$, which leads to the holonomy

$$\Delta \chi = \oint \partial_\mu \chi(x) \, dx^\mu.$$  

(5)

Of course, written as such and provided that spacetime is simply connected, expression (5) is trivial and $\Delta \chi = 0$. We will come back to this point in a moment.

As an intermediate step, let us ask for the possibility to observe local phase transformations. A widespread argument says that—in contrast to the passive view—the phase transformed wave function leads to new expectation values. We get, for instance, for the momentum operator $\hat{p}_\mu \psi' = (p + \partial_\mu \chi) \psi'$ as opposed to $\hat{p}_\mu \psi = p \psi$. But this is of course misleading since we must use the transformed momentum operator $\hat{P}_\mu = \hat{p}_\mu - \partial_\mu \chi$ corresponding to (2) and, thus, $\hat{P}_\mu \psi' = \hat{p}_\mu \psi'$. Indeed, just like their global counterparts local phase transformations do not change any expectation values at all.

Another argument can be found in 't Hooft (1980), who considers an ordinary double-split experiment showing an interference pattern. Inserting a phase shifter behind the slit in one of the two paths results in a corresponding shift of the interference pattern—which can be calculated from (5). 't Hooft now argues that the phase shifter can be seen as a realization of a local phase transformation. But this is impossible since local phase transformations would then change the holonomy, which is, however, invariant under (global and local) $U(1)$. What is rather observed in this case is the relative phase shift. Let $\psi = \psi_I + \psi_{II}$, where $\psi_I$ and $\psi_{II}$ are the two partial wave functions on the two paths $I$ and $II$, then, say, a $\lambda/4$-phase shifter in path $I$ corresponds to $\psi_I \to \psi_I e^{i\lambda/4}$ and hence $\psi \to \tilde{\psi} = \psi_I e^{i\lambda/4} + \psi_{II}$, where $\psi$ and $\tilde{\psi}$ obviously do have different expectation values in general (cf. Brading and Brown 2003). We must therefore very well distinguish between relative phase shifts and local phase transformations. The former are observable, the latter clearly are not.

3 “Phase charges” and Aharonov-Bohm effect

As already mentioned, only non-trivial holonomies are of interest. In this case the loop cannot be contracted to a point and the underlying fiber bundle is non-trivial, too. The double-slit experiment or the Mach-Zehnder interferometer are cases at hand. Another example is provided by the Aharonov-Bohm (AB) effect. Here we have an observable effect despite the fact that the connection is flat. At first glance, this seems to contradict the passive view statement from Sect. II where we made the claim that from flat connections alone the physical situation can hardly be changed. However, the ultimate cause of the AB effect is of course the magnetic field confined to the region of the solenoid in the experimental setting. Nevertheless, there is no magnetic field in the region of the electron (the configuration space of the electron). Here the connection is flat, but the non-trivial holonomy of this connection causes observable effects. Thus, the lesson of the AB effect
decidedly is that we must consider the holonomy outside the solenoid as a real entity! This conclusion is inevitable if we want to avoid an interpretation that either considers gauge-dependent quantities as physically real—such as the gauge potential—or does not conform to the idea of local action—which happens in the case where we allow for a non-local interaction between the confined magnetic field and the electron wave function (Eynck, Lyre, Rummell 2001).

As a real entity, the holonomy couples to some property of the electron, and this, in the usual picture, is just the charge $q$. We shall, however, rather write $q^{(p)}$ (the superscript will become clear soon). In the AB case there is, indeed, more to (3) than a mere rewriting of the phase function, since from (5) together with (3) we get the observed phase shift as a function of $q^{(p)}$ and the magnetic flux

$$\Delta \chi \equiv q^{(p)} \oint A_\mu dx^\mu = q^{(p)} \int F_{\mu\nu} dx^{\mu\nu} = q^{(p)} \Phi_{mag}. \quad (6)$$

But what really is the origin of the charge $q^{(p)}$ in (6)? Obviously, it is a certain property of the electron, but it is not the property of a ‘usual’ charge being the source and drain of the electromagnetic field, since there is no field in the configuration space of the electron (or, at least, we may abstract from it). The AB effect itself has its origin in the topological nature of the non-trivial holonomy, since mappings $S^1 \rightarrow S^1$ from the electron’s configuration space to the gauge group are non-trivial and constitute the fundamental group $\pi_1(U(1)) = \mathbb{Z}$. This holonomy now, as a physical entity, couples in some way to the electron, so we may very well interpret $q^{(p)}$ as the coupling strength between the electron wave function and the holonomy.

Let us call the ‘usual’ charge—the source and drain of the field—the active or passive field charge $q^{(f)}$—in full analogy to the active or passive gravitational mass (the charge of the gravitational field). By way of contrast, $q^{(p)}$ can be called the phase charge, since it originates from the phase factor of the wave function only. A first argument for this conceptual maneuver of distinguishing two different kinds of charges is that we have in fact no a priori reason to identify them. A more compelling physical argument is that $q^{(p)}$ is in principle measurable in an isolated experiment. Whereas $q^{(f)}$ is tested whenever we perform measurements where charges figure as sources or drains of the electromagnetic field (e.g. in measuring the Coulomb force between two electrons), $q^{(p)}$ only becomes visible, if we consider (3) and its observable consequences. And this is exactly what happens in the AB effect.

All this leaves us with a remarkable conclusion: It might very well be the case that particles with one and the same field charge do have different phase charges and, therefore, show different AB effects. This can clearly be seen from (6). We have (on the l.h.s.) the observable shift of the interference pattern, which is proportional to the phase charge $q^{(p)}$ and the magnetic flux $\Phi_{mag}$ (on the r.h.s.). The latter can be measured independently by testing, for instance, the Lorentz force of the magnetic field on electrons and muons (which is obviously the same, because of their common field charge). However, from the
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way (6) was derived, we have no reason to expect the same interference shift in an AB experiment for electrons as compared to muons.

Note that the claim is not that in an actual experiment electrons and muons will show different AB effects. One would, in fact, expect the same $q^{(p)}$ for both. The above arguments are rather intended to show that there is no theoretical principle in our known physics which precludes the possibility of differing phase charges. Insofar as $q^{(p)}$ and $q^{(f)}$ are conceptually different, their equivalence

$$q^{(p)} = q^{(f)}$$

must be tested experimentally. Therefore the real claim here is that some of our Standard Model’s experiments—involving topological effects from flat connections—are in fact “null experiments” on the equivalence principle of field and phase charge.

4 A gauge theoretic equivalence principle

In Lyre (2000) the attempt was made to propose a gauge theoretic generalization of the equivalence principle. The analogy is indeed striking: Field charges and gravitational mass appear in the field equations of a field theory (e.g. Maxwell or Einstein equations), whereas phase charges and inertial mass appear in the corresponding equations of motion (e.g. Dirac or geodesic equations). The generalized equivalence principle is then intended to fill the explanatory gap in the architecture of a gauge theory arising from the mere passive view of the gauge principle. This gap is simply the following: Let $L_D = \bar{\psi}(i\not\!D - m)\psi$ be the Dirac Lagrangian and $L_{coup} = j_\mu A^\mu$ the inhomogeneous ‘coupling’ part of the Lagrangian

$$L'_D = L_D + L_{coup},$$

which arises due to the replacement of the usual derivative by the covariant derivative (based on the requirement of local gauge covariance). As we have seen, however, the covariant derivative only entails a flat connection in $L_{coup}$. There is, on the other hand, the Maxwell theory with the Lagrangian

$$L'_M = L_M + L_{coup}.\quad (9)$$

Here $L_M = \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ is the kinetic term of the free Maxwell field and $L_{coup}$ the inhomogeneous term including field charges. In this case the connection in $L_{coup}$ is non-flat.

In order to arrive at the full Lagrangian of the Dirac-Maxwell gauge theory (or QED, analogously), we have to combine $L'_D$ and $L'_M$ in order to get

$$L_{DM} = L_D + L_{coup} + L_M.$$
But, obviously, $\mathcal{L}_{\text{coup}}$ figures in two different meanings here. Because of its mere passive nature, the gauge ‘principle’ does not allow to generalize from a flat connection in (8) to a non-flat connection in (9). Thus, the two $\mathcal{L}_{\text{coup}}$-terms cannot simply be identified.

There is, however, the possibility of non-trivial holonomies with a phase charge $q^{(p)}$ in the inhomogeneous term in (8), which we therefore indicate as $\mathcal{L}^{(p)}_{\text{coup}}$ as opposed to $\mathcal{L}^{(f)}_{\text{coup}}$ in (9). From the equivalence of field and phase charges (7) we then get

$$\mathcal{L}^{(p)}_{\text{coup}} = \mathcal{L}^{(f)}_{\text{coup}} \equiv \mathcal{L}_{\text{coup}}$$

and, thus, the desired Lagrangian (10). The $U(1)$ gauge theory is therefore not based on the physically vacuous gauge ‘principle’, a mere passive symmetry requirement, but on the gauge theoretic equivalence principle, which in the form of (7) must be verified empirically.

5 Discussion and conclusion

The concept of a phase charge and, hence, the gauge theoretic equivalence principle is based on the existence of a non-trivial holonomy. Unfortunately, there are no AB effects for higher $SU(n)$ groups, since $\pi_1(SU(n)) = 1$. As far as other topological effects in gauge theories are concerned (e.g. instantons or $\theta$-vacua), it is not so clear whether they do perhaps only arise because of some clever approximations (e.g. the assumption of vanishing fields in the infinity of Euclidean spacetime for $SU(2)$-instantons). In these cases the holonomy should not be considered a really existing entity. This then means that our considerations only apply for $U(1)$ and that no simple extension of the gauge theoretic equivalence principle to the general Yang-Mills case will be possible.

One should also mention that the analogy between the gauge theoretic and the usual gravitational equivalence principle, striking as it may be, is of course only a heuristic one. As already mentioned, it certainly breaks down if we compare the concepts of phase charge and inertial mass. The reason for this might very well be seen in the classical nature of the latter and may perhaps be overcome in a future theory of quantum gravity.

In this respect the similarity of the above proposal of phase charges to Anandan’s conception of gauge fields as “interference fields” should perhaps be emphasized. For the particular case of general relativity, Anandan (1979) has shown that the “gravitational phase” is $\Delta \chi = \frac{m}{\hbar} \int g_{\mu\nu} dx^\mu x^\nu$, that is the spacetime distance measured in Compton wavelengths and, hence, $m$ being the inertial mass! The same is true for neutron interferometry, the famous COW experiments. Here the phase shift is actually proportional to the product of inertial and gravitational mass, since the gravitational potential includes the latter (cf. Greenberger 1983).

A possible objection against the phase charge proposal is that, in order to experimentally realize non-trivial bundles with flat connections, a region with a non-flat connection
(i.e. a real field and, hence, field charges) must exist simultaneously. This seems to show that we cannot observe $q^{(b)}$ independently from $q^{(f)}$. But again: the two charges are of different origin in the sense that we take, for instance, the electron’s field charge to realize the electric current in the solenoid, but may perform the AB experiment with electrons, muons or tauons to measure their phase charge. Moreover, strictly speaking we cannot make an independent measurement of the inertial mass either. We always have to make the idealization of neglecting the gravitational masses of the measuring devices.

The reader may finally ask what the world were looking like if the proposed equivalence between phase and field charge would empirically be violated. The astonishing answer is that this would not change so much the phenomenology of our elementary particles world, since the violation only becomes visible in experiments where non-trivial holonomies with flat connections are involved. Conceptually, however, such a violation would leave us with a serious puzzle, since then—again—the explanatory gap in the logical structure of the empirically so eminently successful gauge theories persists. This gap is certainly not filled by the gauge ‘principle’, but may perhaps be construed as an equivalence principle between phase and field charges.

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