Accelerated Sparsified SGD with Error Feedback

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Abstract

We study a stochastic gradient method for synchronous distributed optimization. For reducing communication cost, we are interested in utilizing compression of communicated gradients. Our main focus is a sparsified stochastic gradient method with error feedback scheme combined with Nesterov’s acceleration. Strong theoretical analysis of sparsified SGD with error feedback in parallel computing settings and an application of acceleration scheme to sparsified SGD with error feedback are new. It is shown that (i) our method asymptotically achieves the same iteration complexity of non-sparsified SGD even in parallel computing settings; (ii) Nesterov’s acceleration can improve the iteration complexity of non-accelerated methods in convex and even in nonconvex optimization problems for moderate optimization accuracy.

1 Introduction

In typical modern machine learning tasks, we often encounter large scale optimization problems, which require huge computational time to solve. Hence, saving computational time of optimization processes is practically quite important and is main interest in the optimization community.

To tackle large scale problems, a golden-standard approach is the usage of Stochastic Gradient Descent (SGD) method [31]. For reducing loss, SGD updates the current solution by using a stochastic gradient in each iteration, that is the average of the gradients of the loss functions correspond to a random subset of the dataset (mini-batch) rather than the whole dataset. This (stochastic) mini-batch approach allows that SGD can be faster than deterministic full-batch methods in terms of computational time [10, 21]. Furthermore, Stochastic Nesterov’s Accelerated Gradient (SNAG) method and its variants have been proposed [16, 8, 13], that are based on the combination of SGD with Nesterov’s acceleration [29, 28, 37]. Mini-batch SNAG theoretically outperforms vanilla mini-batch SGD for moderate optimization accuracy, though its asymptotic convergence rate matches that of SGD.

For realizing further scalability, distributed optimization have received much research attention [6, 11, 17, 25, 15, 5, 7, 14]. Distributed optimization methods are mainly classified as synchronous centralized [44, 10, 34], asynchronous centralized [30, 11, 22, 26, 43], synchronous decentralized [27, 42, 23, 20, 38, 32] and asynchronous decentralized [24, 19] ones by their communication types. In this paper, we particularly focus on data parallel stochastic gradient methods for synchronous centralized distributed optimization with smooth objective function $F : \mathbb{R}^d \to \mathbb{R}$, $F(x) = \frac{1}{P} \sum_{p=1}^{P} \frac{1}{N} \sum_{i=1}^{N} f_{i,p}(x)$, where each $\{f_{i,p}\}_{i=1}^{N}$ corresponds to a data...
partition of the whole dataset for the $p$-th node (or processor). In this setting, first each processor $p$ computes a stochastic gradient of $(1/N) \sum_{i=1}^{N} f_{i,p}(x)$ and then the nodes send the gradients each other. Finally, the current solution is updated using the averaged gradient on each processor. Here we assume that node-to-node broadcasts are used, but it is also possible to utilize an intermediate parameter server.

A main concern in synchronous distributed optimization is communication cost because it can easily be a bottleneck in optimization processes. Theoretically, naive parallel mini-batch SGD achieves linear speed up with respect to the number of processors [10,21], but not empirically due to this cost [34,7]. For leveraging the power of parallel computing, it is essential to reduce the communication cost.

One of fascinating techniques for reducing communication cost in distributed optimization is compression of the communicated gradients [2,25,39,4,36,35,18,33,40,3,41]. Sparsification is an approach in which the gradient is compressed by sparsifying it in each local node before communication [2,25,39,4,36,35,18]. For sparsifying a gradient, top-$k$ algorithm, that drops the $d-k$ smallest components of the gradient by absolute value from the $d$ components of the gradient, has been typically used. Another example of compression is quantization, which is a technique that limit the number of bits to represent the communicated gradients. Several work has demonstrated that parallel SGD with quantized gradients has good practical performance [33,40,3,41]. Particularly, Alistarh et al. [3] have proposed Quantized SGD (QSGD), which is the first quantization algorithm with a theoretical convergence rate. QSGD is based on unbiased quantization of the communicated gradient.

However, theoretically there exists an essential trade-off between communication cost and convergence speed when we use naive gradient compression schemes. Specifically, naive compression (including sparsification and quantization) causes large variances and theoretically always slower than vanilla SGD, though they surely reduce the communication cost [36,3].

Error feedback scheme partially solves this trade-off problem. Some work has considered the usage of compressed gradients with the locally accumulated compression errors in each node and its effectiveness has been validated empirically [2,25,41]. Very recently, several work has attempted to analyse and justified the effectiveness of error feedback in a theoretical view [4,36,18]. Surprisingly, it has been shown that Sparsified SGD with error feedback asymptotically achieves the same rate as non-sparsified SGD.

Nevertheless, for a theoretical point of view, the analysis in previous work is still unsatisfactory, since no analysis has been given for distributed settings [18,36] or only focused on top-$k$ sparsification and they have never shown the linear speed up property with respect to the number of nodes [4]. Also, previous work has not taken non-asymptotic iteration complexities into consideration. However, consideration of them is practically important because the additional iteration complexity caused by sparsification typically has a factor of $\text{poly}(d/k)$, which can be very large particularly for high compression settings.

There exist two open questions.

- Does sparsified SGD with error feedback asymptotically achieves the same rate as non-sparsified parallel SGD in distributed optimization settings?
- Are there any better algorithms than sparsified SGD with error feedback in terms of non-asymptotic iteration complexity?

We will positively answer these questions in this work.

Main contribution We propose and analyse Sparsified Stochastic Nesterov’s Accelerated Gradient method (S-SNAG-EF) based on the combination of (i) unbiased compression of the stochastic gradients; (ii) error feedback scheme; and (iii) Nesterov’s acceleration technique. The main features of our method are as follows:

- (Linear speed up w.r.t. #Nodes) Our method possesses linear speed up property with respect to the number of processors in distributed optimization settings, in the sense that the method asymptotically achieves the same rate as non-sparsified parallel SGD. To the best
We also analyse non-accelerated sparsified SGD with error feedback (S-SGD-EF) in parallel computing settings and show that S-SGD-EF has the former property of the above.

From Table 1, we can make the following observations:

- **S-SGD vs. SGD:** The iteration complexities of S-SGD always $d/k$ times worse than SGD because of the $d/k$ times larger variances of the randomly compressed stochastic gradients.
- **S-SGD-EF vs. S-SGD:** S-SGD-EF has better dependence on the desired accuracy $\varepsilon$ than S-SGD in the sparsification error terms. Asymptotically, the iteration complexities of S-SGD-EF is $d/k$ times better than the ones of S-SGD.
- **S-SGD-EF vs. MEM-SGD:** When $P = 1$, the rates of the two methods are same for convex cases. However, S-SGD-EF is applicable to parallelization settings and achieves linear speed up in terms of the number of processors with respect to the asymptotically dominated term.
- **S-SGD-EF vs. EF-SGD:** When $P = 1$, For general nonconvex cases, the rate of S-SGD-EF is always better than the one of EF-SGD because $d/(k\varepsilon^{5/2}) \leq 1/\varepsilon^2 + d^2/(k^2\varepsilon)$. Note that for general convex cases, EF-SGD is applicable to non-smooth objectives and the rates cannot be directly compared.
- **S-SNAG-EF vs. S-SGD-EF:** For general convex cases, the rate of S-SNAG-EF is strictly better than the one of S-SGD-EF when $k^2/(d^2 P) \leq \varepsilon < \sqrt{k/d} \wedge k^{2/5}/(d^{2/5} P^{1/5}) \wedge 1/\sqrt{P}$, though the rates of two methods are asymptotically same. For general nonconvex cases, the rate of S-SNAG-EF is strictly better than the one of S-SGD-EF when $k^2/(d^2 P) \leq \varepsilon < k/(d \wedge k^{2/3}/(d^{2/3} P^{1/3}) \wedge 1/P$. For high compression settings (i.e., $d \gg k$), these ranges are wide and meaningful.

For looking more closely at the comparison of the theoretical iteration complexities of S-SGD-EF and S-SNAG-EF, we illustrate the comparison of them in Figure 1.
We use the following notation in this paper.

\[ F \]

**Assumption 1.**

**Assumption 2.**

**Assumption 3.**

**Assumption 4.**

The algorithm of Sparsified SGD with Error Feedback (S-SGD-EF) for convex and nonconvex objectives is provided in Algorithm 1. In line 3-7, roughly speaking, we construct a gradient estimator.

2 Notation and Assumptions

We use the following notation in this paper.

- \( \| \cdot \| \) denotes the Euclidean \( L_2 \) norm \( \| \cdot \|_2 \).
- For natural number \( m \), \([m]\) denotes the set \( \{1, 2, \ldots, m\} \).
- We define \( Q(z) : \mathbb{R}^d \to \mathbb{R} \) as the quadratic function with center \( z \), i.e., \( Q(z)(x) = \|x - z\|^2 \).
- A sparsification operator \( \text{RandComp} \) is defined as \( \text{RandComp}(x, k) = (d/k)x \), for \( k \) in \( J \) and \( \text{RandComp}(x, k) = 0 \) otherwise, where \( J \) is a uniformly random subset of \([d]\).

The followings are theoretical assumptions for our analysis. These are very standard in optimization literature. We always assume the first three assumptions.

**Assumption 1.** \( F \) has a minimizer \( x^* \in \mathbb{R}^d \).

**Assumption 2.** \( F \) is \( \mathcal{L} \)-smooth \( (\mathcal{L} > 0) \), i.e., \( \| \nabla F(x) - \nabla F(y) \| \leq \mathcal{L} \| x - y \| \), \( \forall x, y \in \mathbb{R}^d \).

**Assumption 3.** \( \{f_{i,p}\}_{i,p} \) has \( \mathcal{V} \)-bounded variance, i.e., \( \frac{1}{NP} \sum_{i,p} \| \nabla f_{i,p}(x) - F(x) \|^2 \leq \mathcal{V}, \forall x \in \mathbb{R}^d \).

**Assumption 4.** \( F \) is \( \mu \)-strongly convex \( (\mu > 0) \), i.e., \( F(y) - (F(x) + \langle \nabla F(x), y - x \rangle) \geq (\mu/2) \| x - y \|^2 \), \( \forall x, y \in \mathbb{R}^d \).

3 Algorithm Descriptions

In this section, we describe our proposed algorithms in detail.

3.1 Sparsified Stochastic Gradient Descent with Error Feedback

**Algorithm 1: S-SGD-EF(\( F, x_{ini}, \{\eta_t\}_{t=1}^{\infty}, \gamma, k, T \))**

1: Set: \( x_0 = x_{ini}, m_{0,p} = 0 \ (p \in [P]) \).
2: for \( t = 1 \) to \( T \) do
3: \( \quad \text{for } p = 1 \text{ to } P \text{ in parallel do} \)
4: \( \quad \quad \text{Compute i.i.d. stochastic gradient of the partition of } F: \nabla f_{i,p}(x_{t-1}). \)
5: \( \quad \quad \text{Compress: } \bar{g}_{t,p} = \text{RandComp}([\nabla f_{i,p}(x_{t-1}) + (\gamma/\eta_t)m_{t-1,p}, k]). \)
6: \( \quad \quad \text{Update cumulative compression error: } m_{t,p} = m_{t-1,p} + \eta_t(\nabla f_{i,p}(x_{t-1}) - \bar{g}_{t,p}). \)
7: \( \quad \text{end for} \)
8: \( \quad \text{Broadcast and Receive: } \bar{g}_{t,p} (p \in [P]). \)
9: \( \quad \text{for } p = 1 \text{ to } P \text{ in parallel do} \)
10: \( \quad \quad \text{Update solution: } x_t = x_{t-1} - \eta_t \frac{1}{P} \sum_{p=1}^{P} \bar{g}_{t,p}. \)
11: \( \quad \text{end for} \)
12: \( \text{end for} \)
13: return \( x_T. \)

The algorithm of Sparsified SGD with Error Feedback (S-SGD-EF) for convex and nonconvex objectives is provided in Algorithm 1. In line 3-7, roughly speaking, we construct a gradient estimator.
we broadcast and receive the compressed gradient estimator from and to each node. In line 9-10, we
(because of large learning rate $\lambda$
The procedure of S-SNAG-EF for convex objectives is provided in Algorithm 2. In line 5, we
(Difference from previous algorithms)
Remark
of the “conservative” solution $y_t$

3.2 Sparsified Stochastic Nesterov Accelerated Gradient descent with Error Feedback

The procedure of S-SNAG-EF for convex objectives is provided in Algorithm 2. In line 5, we
compress and update a cumulative compression error in parallel. More specifically, each node first computes i.i.d. stochastic gradient with respect to the correspondence data partition. Second, the cumulative compression error $m_{t-1,p}$ is added to the stochastic gradient (we call this process as error feedback) and then we construct unbiasedly sparsified gradient estimator $\hat{g}_{t,p}$ by randomly picking $k$-nonzero coordinates of the stochastic gradient with error feedback. Finally, the cumulative compression error is updated for the after iterations. In line 8, we broadcast and receive the compressed gradient estimator from and to each node. In line 9-10, we update the solution using the average of the received compressed gradients in each node. Note that each node has the same updated solution in each iteration.

Remark (Difference from previous algorithms). Algorithm 1 can be regarded as an extension of Mem-SGD [30] or EF-SGD [18] to parallel computing settings, though these two methods mainly utilize top-$k$ compression for gradient sparsification. We rather use unbiased random compression. This difference is essential for our analysis.

Algorithm 2: S-SNAG-EF($F, x_{\text{in}}, \{\eta_t, \lambda_t, \alpha_t, \beta_t\}_{t=1}^\infty, \gamma, k, T$)

1: Set: $x_0 = x_{\text{in}}, m_{0,p} = m_{0,p}^{(y)} = m_{0,p}^{(z)} = 0$ ($p \in [P]$).
2: for $t = 1$ to $T$ do
3: for $p = 1$ to $P$ in parallel do
4: Compute i.i.d. stochastic gradient of the partition of $F$: $\nabla f_{i,p}(x_{t-1})$.
5: Compress:
6: $g_{i,p}^{(y)} = \text{RandComp}(\nabla f_{i,p}(x_{t-1}) + (\gamma/\eta_t)m_{t-1,p},k/2)$,
7: $g_{i,p}^{(z)} = \text{RandComp}(\nabla f_{i,p}(x_{t-1}) + (\gamma/\lambda_t)(1-\beta_t)m_{t-1,p} + \beta_t m_{t-1,p},k/2)$.
8: for $p = 1$ to $P$ in parallel do
9: Update cumulative compression errors:
10: $m_{i,p}^{(y)} = m_{t-1,p} + \eta_t((\nabla f_{i,p}(x_{t-1}) - \hat{g}_{i,p}^{(y)})$,
11: $m_{i,p}^{(z)} = (1-\beta_t)m_{t-1,p} + \beta_t m_{t-1,p} + \lambda_t((\nabla f_{i,p}(x_{t-1}) - \hat{g}_{i,p}^{(z)})$,
12: $m_{i,p} = (1-\alpha_t)m_{i,p}^{(y)} + \alpha_t m_{i,p}^{(z)}$.
13: end for
14: end for
15: end for
16: Broadcast and Receive: $\bar{g}_{i,p}^{(y)}, \bar{g}_{i,p}^{(z)}$ ($p \in [P]$).
17: for $p = 1$ to $P$ in parallel do
18: Update solutions:
19: $y_t = x_{t-1} - \eta_t \frac{1}{P} \sum_{p=1}^P \bar{g}_{i,p}^{(y)}$,
20: $z_t = (1-\beta_t)z_{t-1} + \beta_t x_{t-1} - \lambda_t \frac{1}{P} \sum_{p=1}^P \bar{g}_{i,p}^{(z)}$,
21: $x_t = (1-\alpha_t)y_t + \alpha_t z_t$.
22: end for
23: end for
24: return $x_{\text{out}} = y_T$.

Algorithm 3: Reg-S-SNAG-EF ($F, x_{\text{in}}, \{\eta_t, \lambda_t, \alpha_t, \beta_t\}_{t=1}^\infty, \gamma, k, T, \sigma, S$)

Set: $x_0 = x_{\text{in}}$.

for $s = 1$ to $S$ do
Run: $x_s = \text{S-SNAG-EF}(F + \sigma Q(x_{s-1}), x_{s-1}, \{\eta_t, \lambda_t, \alpha_t, \beta_t\}_{t=1}^\infty, \gamma, k, T)$
end for

return $x_S$. 

by using error feedback scheme, compress it to a sparse vector and update a cumulative compression error in parallel. More specifically, each node first computes i.i.d. stochastic gradient with respect to the correspondence data partition. Second, the cumulative compression error $m_{t-1,p}$ is added to the stochastic gradient (we call this process as error feedback) and then we construct unbiasedly sparsified gradient estimator $\hat{g}_{t,p}$ by randomly picking $k$-nonzero coordinates of the stochastic gradient with error feedback. Finally, the cumulative compression error is updated for the after iterations. In line 8, we broadcast and receive the compressed gradient estimator from and to each node. In line 9-10, we update the solution using the average of the received compressed gradients in each node. Note that each node has the same updated solution in each iteration.

Remark (Difference from previous algorithms). Algorithm 1 can be regarded as an extension of Mem-SGD [30] or EF-SGD [18] to parallel computing settings, though these two methods mainly utilize top-$k$ compression for gradient sparsification. We rather use unbiased random compression. This difference is essential for our analysis.

3.2 Sparsified Stochastic Nesterov Accelerated Gradient descent with Error Feedback

The procedure of S-SNAG-EF for convex objectives is provided in Algorithm 2. In line 5, we compress two different gradient estimators by randomly picking $k/2$-coordinates for each. Also in line 6, we update three cumulative compression errors. Why are different compressed estimators and cumulative errors necessary for appropriate updates? In a typical acceleration algorithm we construct two different solution paths $\{y_t\}$ and $\{z_t\}$, and their aggregations $\{x_t\}$ as in line 10. The aggregation of the “conservative” solution $y_t$ (because of small learning rate $\eta_t$) and “aggressive” solution $z_t$ (because of large learning rate $\lambda_t$) is the essence of Nesterov’s acceleration. On the other hand, from
a theoretical point of view, the impact of error feedback to the vanilla stochastic gradient should be scaled to the inverse of learning rate as in line 5. Therefore, for using two different learning rates, it is necessary to construct two compressed gradient estimators and hence three compression errors. Generally, S-SNAG-EF has no theoretical guarantee for nonconvex objectives. However, utilizing regularization technique, the convergence of Reg-SNAG-EF (Algorithm 2) to a stationary point is guaranteed. Specifically, Algorithm 2 repeatedly minimize the "regularized" objective $F + \sigma Q(x_{t-1})$ by using S-SNAG-EF, where $Q(x_{t-1}) = \|x - x_{t-1}\|^2$ and $x_{t-1}$ is the current solution.

Remark (Parameter tuning). It seems that Algorithm 2 has many tuning parameters. However, this is not. Specifically, as Theorem 4.8 in Section 4 indicates, actual tuning parameters are only constant learning rate $\eta$, strong convexity $\mu$ and $\gamma$, and the other parameters are theoretically determined. This means that the additional tuning parameters compared to S-SGD-EF are essentially only strong convexity parameter $\mu$. Practically, fixing $\gamma = 0.5 \times k/d$ works well.

4 Convergence Analysis

In this section, we provide convergence analysis of S-SGD-EF and S-SNAG-EF. For convex cases, we assume the strong convexity of the objective. For non-strongly convex cases, we can immediately derive its convergence rates from the ones for strongly convex cases by taking standard dummy learning rate $\eta$.

4.1 Analysis of S-SGD-EF

In this subsection, we provide the analysis of S-SGD-EF. The proofs of the statements are found in Section A of supplementary material.

The following proposition holds for strongly convex objective $F$.

**Proposition 4.1** (Strongly convex). Suppose that Assumptions 2, 3 and 4 hold. Let $\eta_t = \eta \leq 1/(8L)$. Then S-SGD-EF satisfies

$$E[F(x_{out}) - F(x_*)] \leq \frac{1}{\eta} (1 - \eta \mu)^T \|x_0 - x_*\|^2 + \frac{\eta \nu}{P} + \frac{\sum_{t=1}^T (1 - \eta \mu)^T \nu}{\sum_{t=1}^T (1 - \eta \mu)^T}$$. 

where $x_{out} = x_{t-1}$ and $t \sim |T|$ according to $\{ (1 - \eta \mu)^{-t} / (\sum_{t=1}^T (1 - \eta \mu)^{-t}) \}_{t=1}^T$.

The first term is the deterministic term and the second term is the stochastic error term. The last term is the compression error term and we can further bound it by the following proposition.

**Proposition 4.2.** Suppose that Assumptions 3 holds. Let $\gamma = \Theta(k/d)$ be sufficiently small. Then S-SGD-EF satisfies

$$E[\|m_t\|^2] \leq \Theta \left( \sum_{t'=1}^t \frac{\eta_t^2 d}{kP} (1 - \gamma)^{t-t'} (\nu + E[\|\nabla F(x_{t-1})\|^2]) \right)$$. 

Remark. Importantly, the expected accumulated compression error $E[\|m_t\|^2]$ is scaled to $1/P$, i.e., linearly scaled with respect to the number of nodes.

Combining Proposition 4.1 with Proposition 4.2 yields the following theorem.

**Theorem 4.3** (Strongly convex). Suppose that Assumptions 1, 2, 3 and 4 hold. Let $\gamma = \Theta(k/d)$ be sufficiently small and $T = \Theta(1/(\eta \mu))$ be sufficiently large. Then the iteration complexity $T$ of S-SGD-EF with appropriate $\eta_t = \eta$ for achieving $E[F(x_{out}) - F(x_*)] \leq \varepsilon$ is

$$O \left( \frac{L}{\mu} + \frac{\nu}{P \mu \varepsilon} + \frac{d}{k} + \frac{d}{k \sqrt{P}} \left( \frac{L}{\mu} + \frac{\sqrt{L} \nu}{\mu \varepsilon} \right) \right)$$. 

where $x_{out}$ is defined in Proposition 4.1.
Remark. Theorem 4.3 implies that S-SGD-EF asymptotically achieves $\mathcal{O}/(P\epsilon)$, that is the asymptotic iteration complexity of non-sparsified parallel SGD, because the last compression error term has a dependence on $1/\sqrt{\epsilon}$ rather than $1/\epsilon$. Also note that the last term is scaled to $\sqrt{T}$. This is a desirable property for distributed optimization with $P \gg 1$. However, the last term has a factor of $d/k$, which may be large and can dominate the other terms for moderate accuracy $\epsilon$. Thus, consideration of non-asymptotic behavior is also important particularly for high compression settings.

For nonconvex objectives, we can derive the following proposition.

**Proposition 4.4** (General nonconvex). Suppose that Assumptions 1, 2 and 3 hold. Assume that $\eta_t = \eta \leq 1/(2L)$. Then S-SGD-EF satisfies

$$\mathbb{E}\|\nabla F(x_{\text{out}})\|^2 \leq \Theta \left( \frac{F(x_{\text{in}}) - F(x_*)}{\eta T} + \frac{\eta L\mathcal{V}}{P} + \frac{L^2}{T} \sum_{t=1}^{T} \mathbb{E}\|m_{t-1}\|^2 - \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}\|\nabla F(x_{t-1})\|^2 \right),$$

where $x_{\text{out}} = x_{t-1}$ and $t \sim \lfloor T \rfloor$ with probability $\{1/T\}_{t=1}^{T}$.

Combining Proposition A.6 with Proposition 4.2 yields the following theorem.

**Theorem 4.5** (General nonconvex). Suppose that Assumptions 1, 2, 3 and 4 hold. Let $\gamma$ be the same one in Theorem 4.2. Then the iteration complexity $T$ of S-SGD-EF with appropriate $\eta_t = \eta$ to achieve $\mathbb{E}\|\nabla F(x_{\text{out}})\|^2 \leq \epsilon$ is

$$O \left( \frac{L\Delta}{\epsilon} + \mathcal{V} \frac{L\Delta}{P\epsilon^2} + \frac{d}{k} \frac{L}{\sqrt{P}} \left( \frac{L\Delta}{\epsilon} + \frac{L\sqrt{\mathcal{V}\Delta}}{\epsilon^2} \right) \right),$$

where $\Delta = F(x_{\text{in}}) - F(x_*)$ and $x_{\text{out}}$ is defined in Proposition A.6.

Similar to convex cases, S-SGD-EF asymptotically achieves the same rate as non-sparsified SGD.

### 4.2 Analysis of S-SNAG-EF

Here, theoretical analysis of our proposed S-SNAG-EF is provided. For the proofs of the statements, see supplementary material (Section B).

The following proposition holds for strongly convex objective $F$.

**Proposition 4.6** (Strongly convex). Suppose that Assumptions 1, 2, 3 and 4 hold. Let $\eta_t = \eta \leq 1/(2L)$, $\lambda_t = \lambda = (1/2)\sqrt{\eta_t}/\mu$, $\alpha_t = \alpha = \lambda_t/(2 + \lambda_t)\mu$ and $\beta_t = \beta = \lambda_t/(1 + \lambda_t)\mu$. Then S-SNAG-EF satisfies

$$\mathbb{E}\|F(x_{\text{out}}) - F(x_*)\| \leq \Theta \left( \frac{\mu(1 - \sqrt{\eta_t}\mu)^T}{\sqrt{\eta_t}} + \sqrt{\frac{\eta_t}{\mu}} \frac{\mathcal{V}^2}{P} \right) + \sum_{t=1}^{T} (1 - \sqrt{\eta_t}\mu)^{T-t} \left( \frac{\lambda L^2}{\epsilon^2} \mathbb{E}\|m_{t-1}\|^2 - \eta \mathbb{E}\|\nabla F(x_{t-1})\|^2 \right) + L\mathbb{E}\|m_{T-1}\|^2,$$

where $x_{\text{out}} = x_{T-1}$.

Remark. The first deterministic error term is scaled to $(1 - \sqrt{\eta_t}\mu)^T$ rather than $(1 - \eta_t\mu)^T$ thanks to the acceleration scheme at the expense of $1/\sqrt{\eta_t}$ times larger stochastic error (the second term) than the one of S-SGD-EF.

The third and last terms are bounded by the following proposition.

**Proposition 4.7.** Suppose that Assumptions 3 holds. Let $\gamma = \Theta(k/d)$, $\beta_t \leq \Theta(\gamma^3/\alpha_t^2)$ be sufficiently small and $\{\alpha_t\}$ is monotonically non-increasing. Then S-SNAG-EF satisfies

$$\mathbb{E}\|m_t\|^2 \leq \Theta \left( \frac{t}{kP} \left( \frac{\alpha_t^2}{\gamma^2} \lambda_t^2 d \right) \right) + \mathcal{V} (1 - \gamma)^{T-t'} \left( \mathbb{E}\|\nabla F(x_{t'-1})\|^2 \right).$$

Combining Proposition 4.6 and 4.7 yields the following theorem.
We can derive a convergence rate of Reg-S-SNAG-EF for general nonconvex objectives by applying Theorem 4.8 (Strongly convex) and Remark. Theorem 4.9 (General nonconvex) to pseudo-regularized objective. Compared with the rate of S-SGD-EF, we can easily see that the rate of S-SNAG-EF is strictly better than the one of S-SGD-EF when \( \Theta(k^2/(d^2 \mathcal{P})) \leq \varepsilon \leq \Theta((k^2/d \wedge k^{2/5}/(d^{2/5} \mathcal{P}^{1/5}) \wedge 1/\sqrt{\mathcal{P}}) \), if we assume \( L = \mathcal{V} = \tilde{O}(1) \).

We can derive a convergence rate of Reg-S-SNAG-EF for general nonconvex objectives by applying Theorem 4.4 (Strongly convex) to pseudo-regularized objective \( F + \sigma Q(x_{out}^-) \) iteratively.

**Theorem 4.9** (General nonconvex). Suppose that Assumptions I, 2, and 3 hold. Let \( \sigma = L \), and \( \lambda_t, \alpha_t, \beta_t \) and \( \gamma \) be the same ones in Theorem 4.8 (with \( \mu \leftarrow \sigma \) ), and \( T = \Theta(1/\sqrt{\eta \mathcal{L}}) \) and \( S = \Theta(1 + L \Delta/\varepsilon) \) be sufficiently large. Then the iteration complexity \( ST \) of Reg-S-SNAG-EF with appropriate \( \eta_t = \eta \) for achieving \( \mathbb{E} \| \nabla F(x_{out}^t) \|^2 \leq \varepsilon \) is

\[
\tilde{O} \left( \frac{L \Delta}{\varepsilon} + \frac{\mathcal{V} L \Delta}{\varepsilon^2} + \left( \frac{d}{k} + \frac{d^3}{k^3 \sqrt{\mathcal{P}}} + \frac{d^4}{k^4 \mathcal{P}^2} \right) \frac{L \Delta}{\varepsilon} + \frac{d}{k^2 \mathcal{P}^2} \frac{L \mathcal{V} \Delta}{\varepsilon^2} \right),
\]

where \( x_{out} = x_t \) and \( \hat{s} \sim [S] \) according to \( \{1/S\}^S \).

**Remark.** From Theorem 4.5 we can see that even in nonconvex cases, acceleration can be beneficial. Indeed, the compression error terms (third and fourth terms) have a better dependence on \( \varepsilon \) than S-SGD-EF.

## 5 Related Work

In this section, we briefly describe the most relevant papers to this work. Stich et al. [36] have first provided theoretical analysis of sparsified SGD with error feedback (called MEM-SGD) and shown that MEM-SGD asymptotically achieves the rate of non-sparsified SGD. However, their analysis is limited to convex cases in serial computing settings, i.e., \( \mathcal{P} = 1 \). Independently, Alistarh et al. [4] have also theoretically considered sparsified SGD with error feedback in parallel settings for convex and nonconvex objectives. However, their analysis is still unsatisfactory for some reasons. First, their analysis relies on an artificial analytic assumption due to the usage of top-\( k \) algorithm as gradient compression, though they have experimentally tried to validate it. Second, it is unclear from their results whether the algorithm asymptotically possesses the linear speed up property with respect to the number of nodes. Recently, Karimireddy et al. [18] have also analysed a variant of sparsified SGD with error feedback (called EF-SGD) for convex and nonconvex cases in serial computing settings. The derived rate for nonconvex cases is worse than our result of S-SGD-EF when \( \mathcal{P} = 1 \). Differently from ours, their analysis allows non-smoothness of the objectives for convex cases, though the convergence rate is always worse than vanilla SGD and the algorithm does not possesses the asymptotic optimality.

## 6 Conclusion and Future Work

In this paper, we mainly considered an accelerated sparsified SGD with error feedback in parallel computing settings. We gave theoretical analysis of it for convex and nonconvex objectives and showed that our proposed algorithm achieves (i) asymptotical linear speed up with respect to the number of nodes; (ii) lower iteration complexity for moderate accuracy than the non-accelerated algorithm thanks to Nesterov’s acceleration.
One of interesting questions is whether our theoretical results are tight or not. Deriving lower bound of the iteration complexity of sparsification (or more generally compression) methods in distributed settings with limited communication is quite important. Another interesting future work is to extend our results to proximal settings, which allows non-smooth regularizer, for example, $L_1$ regularizer, since the usage of non-smooth regularizer in machine learning tasks is very popular for both convex and nonconvex problems. Construction of the proximal version of our algorithms and their analysis are non-trivial and definitely meaningful. We conjecture that the asymptotic optimality is still guaranteed in this setting.

7 Acknowledgement

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Supplementary Material

A Analysis of S-SGD-EF

A.1 Analysis of $\mathbb{E}[\|m_{t}\|^2]$

Lemma A.1.

$$\mathbb{E}[\|m_t\|^2] = (1 - \gamma)^2 \|m_{t-1}\|^2 + \eta_t^2 \frac{1}{P^2} \sum_{p=1}^P \mathbb{E}[\|\nabla f_{i_t,p}(x_{t-1}) + (\gamma/\eta_t)m_{t-1,p} - \bar{g}_{t,p}\|^2],$$

where the expectations are taken with respect to $J_{i,1}, \ldots, J_{i,P}(\subset [d])$, which are the random choices of the coordinates for constructing $\bar{g}_{t,p}$ conditioned on $\{i_{t,p} | t \in [t], p \in [P]\}$.

Proof. First note that $m_{t,p} = m_{t-1,p} + \eta_t(\nabla f_{i_t,p}(x_{t-1}) - \bar{g}_{t,p})$ and $m_t = m_{t-1} + \frac{1}{P} \sum_{p=1}^P \eta_t(\nabla f_{i_t,p}(x_{t-1}) - \bar{g}_{t,p}) = \frac{1}{P} \sum_{p=1}^P \eta_t(\nabla f_{i_{t',p}}(x_{t-1}) - \bar{g}_{t',p})$. Since $x_t = x_{t-1} - \eta_t \bar{g}_t$, where $\bar{g}_t = \frac{1}{P} \sum_{p=1}^P \bar{g}_{t,p}$ and $\bar{x}_t = \bar{x}_{t-1} - \eta_t \frac{1}{P} \sum_{p=1}^P \nabla f_{i_{t',p}}(x_{t-1})$, we have

$$\mathbb{E}[\|m_t\|^2] = \mathbb{E}[\|x_{t-1} - \bar{x}_{t-1}\|^2]$$

$$= \mathbb{E} \left[ \left\| m_{t-1} + \eta_t \frac{1}{P} \sum_{p=1}^P (\nabla f_{i_{t',p}}(x_{t-1}) - \bar{g}_{t,p}) \right\|^2 \right]$$

$$= \mathbb{E} \left[ (1 - \gamma)\|m_{t-1}\|^2 + \eta_t^2 \frac{1}{P} \sum_{p=1}^P (\nabla f_{i_{t',p}}(x_{t-1}) + (\gamma/\eta_t)m_{t-1,p} - \bar{g}_{t,p}) \right]^2.$$

Here the expectations are taken with respect to $J_{i,1}, \ldots, J_{i,P}(\subset [d])$, which are the random choices of the coordinates for constructing $\bar{g}_{t,p}$ conditioned on $\{i_{t',p} | t \in [t], p \in [P]\}$. Since each $\bar{g}_{t,p}$ is an independent unbiased estimator of $\nabla f_{i_{t',p}}(x_{t-1}) + (\gamma/\eta_t)m_{t-1,p}$ for $p \in [P]$, we have

$$\mathbb{E}[\|m_t\|^2] = (1 - \gamma)^2 \|m_{t-1}\|^2 + \eta_t^2 \mathbb{E} \left[ \left\| \frac{1}{P} \sum_{p=1}^P (\nabla f_{i_{t',p}}(x_{t-1}) + (\gamma/\eta_t)m_{t-1,p} - \bar{g}_{t,p}) \right\|^2 \right]$$

$$= (1 - \gamma)^2 \|m_{t-1}\|^2 + \eta_t^2 \left[ \mathbb{E} \left[ \left\| \frac{1}{P} \sum_{p=1}^P (\nabla f_{i_{t',p}}(x_{t-1}) + (\gamma/\eta_t)m_{t-1,p} - \bar{g}_{t,p}) \right\|^2 \right] \right].$$

The last equality is from the independence of $\bar{g}_{t,1}, \ldots, \bar{g}_{t,P}$. \hfill \Box

Now we need to bound the variance term $\mathbb{E}[\|\nabla f_{i_{t,p}}(x_{t-1}) + (\gamma/\eta_t)m_{t-1,p} - \bar{g}_{t,p}\|^2]$.

Lemma A.2. For $p \in [P]$,

$$\mathbb{E}[\|\nabla f_{i_{t,p}}(x_{t-1}) + (\gamma/\eta_t)m_{t-1,p} - \bar{g}_{t,p}\|^2] \leq \Theta \left( \frac{d}{k} \right) (\mathcal{V} + \|\nabla F(x_{t-1})\|^2) + \Theta \left( \frac{d^2 \gamma^2}{k \eta_t^2} \right) \|m_{t-1,p}\|^2.$$

Proof. Remember that

$$(\bar{g}_{t,p})_j = \begin{cases} \frac{d}{k} (\nabla f_{i_{t,p}}(x_{t-1}) + (\gamma/\eta_t)(m_{t-1,p})_j) & (j \in J_{i,p}) \\ 0 & (otherwise) \end{cases}.$$
where \( J_{t,p} = \{ j_{t,p}^{(1)}, \ldots, j_{t,p}^{(k)} \} \) and each \( j_{t,p}^{(k)} \) is i.i.d. to the uniform distribution on \([d]\). Since 
\[
\left( j_{t,p}^{(k)} \right)_{k=1}^k \text{ are i.i.d., we have}
\]
\[
\mathbb{E}[\|\nabla f_{t,p}(x_{t-1}) + (\gamma/\eta_k) m_{t-1,p} - \bar{g}_{t,p}\|^2] 
\leq \sum_{j=1}^d \left\{ \left( \frac{d}{k} - 1 \right)^2 (\nabla_j f_{t,p}(x_{t-1}) + (\gamma/\eta_k) m_{t-1,p})^2 \right\} + \left( 1 - \frac{k}{d} \right) (\nabla_j f_{t,p}(x_{t-1}) + (\gamma/\eta_k) m_{t-1,p})^2 
\leq \left( 1 + \frac{d}{k} \right) \|\nabla f_{t,p}(x_{t-1}) + (\gamma/\eta_k) m_{t-1,p}\|^2 
\leq \frac{4d}{k} \|\nabla f_{t,p}(x_{t-1})\|^2 + \frac{4d\gamma^2}{k\eta_k^2} \|m_{t-1,p}\|^2. 
\]

\( \square \)

**Lemma A.3.** For \( t \in [T] \) and \( p_1 \neq p_2 \in [P] \),
\[
\mathbb{E}(m_{t,p_1}, m_{t,p_2}) = 0.
\]

Here the expectations are taken with respect to the all random variables.

**Proof.** Suppose that \( \{i_{t,p_1}, i_{t,p_2}\} \) and \( \{i_{t',p'} \mid t' \in [t-1], p' \in [P]\} \) is given. Since \( \mathbb{E}_{i_{t,p_1}}[m_{t,p_1}] = (1-\gamma)m_{t-1,p_1}, \mathbb{E}_{i_{t,p_2}}[m_{t,p_2}] = (1-\gamma)m_{t-1,p_2} \) and \( J_{t,p} \text{ and } J_{t,p_2} \) are independent, we have
\[
\mathbb{E}_{J_{t,p_1}, J_{t,p_2}}[m_{t,p_1}, m_{t,p_2}] = (1-\gamma)^2 \langle m_{t-1,p_1}, m_{t-1,p_2} \rangle.
\]
This implies that
\[
\mathbb{E}(m_{t,p_1}, m_{t,p_2}) = (1-\gamma)^2 \mathbb{E}(m_{t-1,p_1}, m_{t-1,p_2}),
\]
where the expectations are taken with respect to the all random variables. Using this equality recursively, we obtain
\[
\mathbb{E}(m_{t,p_1}, m_{t,p_2}) = (1-\gamma)^{2(t-1)} \mathbb{E}(m_{0,p_1}, m_{0,p_2}) = 0.
\]
Here the last equality holds because \( m_{0,p} = 0 \) for \( p \in [P] \). \( \square \)

**Proposition A.4.** Let \( \gamma = \Theta(k/d) \) be sufficiently small. Then it follows that
\[
\mathbb{E}[\|m_t\|^2] \leq (1-\gamma)\|m_{t-1}\|^2 + \Theta \left( \frac{\eta_k^2 d}{kP} (V + \|\nabla F(x_{t-1})\|^2) \right).
\]

**Proof.** First observe that from Lemma A.3, we have
\[
1 \leq \frac{1}{P^2} \sum_{p=1}^P \mathbb{E}[\|m_{t-1,p}\|^2] = \mathbb{E}[\|m_{t-1}\|^2].
\]
Using this fact and combining Lemma A.1 with Lemma A.2 give
\[
\mathbb{E}[\|m_t\|^2] \leq \left( (1-\gamma)^2 + \Theta \left( \frac{d\gamma^2}{k} \right) \right) \|m_{t-1}\|^2 + \Theta \left( \frac{\eta_k^2 d}{kP} (V + \|\nabla F(x_{t-1})\|^2) \right).
\]
It is easily seen that choosing appropriately small \( \gamma = \Theta(k/d) \) is sufficient for ensuring \( (1-\gamma)^2 + \Theta(d\gamma^2/k) \leq 1 - \gamma. \) \( \square \)

**Proof of Proposition 4.2.** The statement is a direct consequence of Proposition A.4. \( \square \)
A.2 Analysis for Convex Cases

Proof of Proposition 4.1. Let \( \bar{x}_t = \bar{x}_{t-1} - \eta_t (1/P) \left( \sum_{p=1}^P \nabla f_{i,t,p}(x_{t-1}) \right) \) and \( \bar{x}_0 = x_0 \). By the definition of \( \bar{x}_t \), we have

\[
\|\bar{x}_t - x_*\|^2 = \left\| \bar{x}_{t-1} - \frac{1}{P} \sum_{p=1}^P \nabla f_{i,t,p}(x_{t-1}) - x_* \right\|^2
\]

\[
= \|\bar{x}_{t-1} - x_*\|^2 - 2\eta_t \left( \frac{1}{P} \sum_{p=1}^P \nabla f_{i,t,p}(x_{t-1}), \bar{x}_{t-1} - x_* \right) + \eta_t^2 \left\| \frac{1}{P} \sum_{p=1}^P \nabla f_{i,t,p}(x_{t-1}) \right\|^2
\]

\[
= \|\bar{x}_{t-1} - x_*\|^2 - 2\eta_t \left( \frac{1}{P} \sum_{p=1}^P \nabla f_{i,t,p}(x_{t-1}), x_{t-1} - x_* \right) + \eta_t^2 \left\| \frac{1}{P} \sum_{p=1}^P \nabla f_{i,t,p}(x_{t-1}) \right\|^2
\]

\[
- 2\eta_t \left( \frac{1}{P} \sum_{p=1}^P \nabla f_{i,t,p}(x_{t-1}), x_{t-1} - \bar{x}_{t-1} \right).
\]

Taking expectations with respect to the \( t \)-th iteration, we get

\[
E\|\bar{x}_t - x_*\|^2 = \|\bar{x}_{t-1} - x_*\|^2 - 2\eta_t \langle \nabla F(x_{t-1}), x_{t-1} - x_* \rangle + \eta_t^2 E \left\| \frac{1}{P} \sum_{p=1}^P \nabla f_{i,t,p}(x_{t-1}) \right\|^2
\]

\[
- 2\eta_t \langle \nabla F(x_{t-1}), x_{t-1} - \bar{x}_{t-1} \rangle
\]

\[
\leq \|\bar{x}_{t-1} - x_*\|^2 - 2\eta_t \langle \nabla F(x_{t-1}), x_{t-1} - x_* \rangle + \eta_t^2 E \left\| \frac{1}{P} \sum_{p=1}^P \nabla f_{i,t,p}(x_{t-1}) - \nabla F(x_{t-1}) \right\|^2
\]

\[
+ \eta_t \left( \frac{1}{4L} + \eta_t \right) \|\nabla F(x_{t-1})\|^2 + 4\eta_t L\|x_{t-1} - \bar{x}_{t-1}\|^2.
\]

Here the last inequality follows from the unbiasedness of \( (1/P) \sum_{p=1}^P \nabla f_{i,t,p}(x_{t-1}) \) and Cauchy-Schwartz inequality with the arithmetic-geometric mean inequality. Since \( F \) is \( L \)-smooth and \( \mu \)-strongly convex, we have

\[
F(x_{t-1}) + \langle \nabla F(x_{t-1}), x_* - x_{t-1} \rangle + \frac{1}{4L}\|\nabla F(x_{t-1})\|^2 + \frac{\mu}{4}\|x_{t-1} - x_*\|^2 \leq F(x_*),
\]

and this implies

\[-2\eta_t \langle \nabla F(x_{t-1}), x_* - x_{t-1} \rangle \leq -2\eta_t (F(x_{t-1}) - F(x_*)) - \frac{\eta_t}{2L} \|\nabla F(x_{t-1})\|^2 - \frac{\eta_t^2}{2}\|x_{t-1} - x_*\|^2.
\]

Applying this inequality to the above one, we get

\[
E\|\bar{x}_t - x_*\|^2 \leq \left( 1 - \frac{\eta_t \mu}{2} \right) \|\bar{x}_{t-1} - x_*\|^2 - 2\eta_t (F(x_{t-1}) - F(x_*)) - \left( \frac{\eta_t}{4L} - \eta_t^2 \right) \|\nabla F(x_{t-1})\|^2
\]

\[
+ \eta_t^2 E \left\| \frac{1}{P} \sum_{p=1}^P \nabla f_{i,t,p}(x_{t-1}) - \nabla F(x_{t-1}) \right\|^2 + 4\eta_t L\|x_{t-1} - \bar{x}_{t-1}\|^2
\]

Noting that \( E\| (1/P) \sum_{p=1}^P \nabla f_{i,t,p}(x_{t-1}) - \nabla F(x_{t-1}) \|^2 \leq V/P \) and \( \eta_t \leq 1/(8L) \) from the assumption. Taking expectations with respect to the all random variables, we have

\[
E\|\bar{x}_t - x_*\|^2 \leq \left( 1 - \frac{\eta_t \mu}{2} \right) E\|\bar{x}_{t-1} - x_*\|^2 - 2\eta_t E[F(x_{t-1}) - F(x_*)] + \frac{\eta_t^2 V}{P}
\]

\[
+ 2\eta_t LE\|m_{t-1}\|^2 - \frac{\eta_t}{8L} E\|\nabla F(x_{t-1})\|^2.
\]

Here we used \( x_{t-1} - \bar{x}_{t-1} = m_{t-1} \).
Recursively using the above inequality (with $\eta_t = \eta$) and rearranging the result give
\[
\mathbb{E}[F(\mathbf{x}_{out}) - F(\mathbf{x}_*)] 
\leq \Theta \left( \frac{1}{\eta} (1 - \eta \mu)^T \|\mathbf{x}_0 - \mathbf{x}_*\|^2 + \frac{\sum_{t=1}^{T} (1 - \eta \mu)^{T-t} (L \mathbb{E} \|m_{t-1}\|^2 - \mathbb{E} \|\nabla F(x_{t-1})\|^2/L)}{\sum_{t=1}^{T} (1 - \eta \mu)^{T-t}} \right).
\]
This is the desired result.

**Lemma A.5.** Let $0 < r_1, r_2 < 1$ and $2r_2 \leq r_1$. Then for any non-negative sequence $\{c_t\}_{t=1}^{T}$,
\[
\sum_{t=1}^{T} (1 - r_2)^{T-t} \sum_{t'=1}^{t} (1 - r_1)^{t-t'} c_{t'} \leq \frac{2}{r_1} \sum_{t=1}^{T} (1 - r_2)^{T-t} c_t.
\]

**Proof.**
\[
\sum_{t=1}^{T} (1 - r_2)^{T-t} \sum_{t'=1}^{t} (1 - r_1)^{t-t'} c_{t'} = \sum_{t=1}^{T} (1 - r_2)^{T-t} \sum_{t'=1}^{T} \mathbb{I}_{t' \leq t} (1 - r_1)^{t-t'} c_{t'}
\]
\[
= \sum_{t'=1}^{T} c_{t'} (1 - r_2)^{T-t'} \sum_{t'=1}^{T} \left( \frac{1 - r_1}{1 - r_2} \right)^{t-t'}
\]
\[
\leq \sum_{t'=1}^{T} c_{t'} (1 - r_2)^{T-t'} \frac{1 - \frac{1 - r_1}{1 - r_2}}{1 - \frac{1 - r_1}{1 - r_2}}
\]
\[
\leq \frac{2(1 - r_2)}{r_1} \sum_{t=1}^{T} (1 - r_2)^{T-t} c_t.
\]

**Proof of Theorem 4.3** Let $\eta_t = \eta = \Theta(1/L \land \gamma \sqrt{P}/L \land \gamma / \mu \land P \varepsilon / \mathcal{V} \lor (LV)^{-1/2} \gamma \sqrt{P} \varepsilon)$. From Proposition 4.1, we have
\[
\mathbb{E}[F(\mathbf{x}_{out}) - F(\mathbf{x}_*)] 
\leq \Theta \left( \frac{1}{\eta} (1 - \eta \mu)^T \|\mathbf{x}_0 - \mathbf{x}_*\|^2 + \frac{\sum_{t=1}^{T} (1 - \eta \mu)^{T-t} (L \mathbb{E} \|m_{t-1}\|^2 - \mathbb{E} \|\nabla F(x_{t-1})\|^2/L)}{\sum_{t=1}^{T} (1 - \eta \mu)^{T-t}} \right)
\]
\[
\leq \Theta \left( \frac{1}{\eta} (1 - \eta \mu)^T \|\mathbf{x}_0 - \mathbf{x}_*\|^2 + \frac{\eta^2 L d^2 \mathcal{V}}{k^2 P} \right)
\]
\[
+ \sum_{t=1}^{T} \frac{\eta^2 L d (1 - \eta \mu)^{T-t}}{k^2 P} \sum_{t'=1}^{t} (1 - (1 - \gamma)^{t-t'} \mathbb{E} \|\nabla F(x_{t'-1})\|^2)^2 - \sum_{t=1}^{T} \frac{1}{T} (1 - \eta \mu)^{T-t} \mathbb{E} \|\nabla F(x_{t-1})\|^2
\]

Since $\eta \leq \Theta(\gamma / \mu)$ be sufficiently small, from Lemma A.5 we have
\[
\sum_{t=1}^{T} \frac{\eta^2 L d (1 - \eta \mu)^{T-t}}{k^2 P} \sum_{t'=1}^{t} (1 - (1 - \gamma)^{t-t'} \mathbb{E} \|\nabla F(x_{t'-1})\|^2)^2 \leq \Theta \left( \frac{\eta^2 L d^2 \mathcal{V}}{k^2 P} \sum_{t=1}^{T} (1 - \eta \mu)^{T-t} \mathbb{E} \|\nabla F(x_{t-1})\|^2 \right).
\]
Also, since $\eta \leq \Theta(k \sqrt{P}/(dL))$ is assumed, the last term in (1) is negative with appropriate choice of $\eta$. Hence, we obtain
\[
\mathbb{E}[F(\mathbf{x}_{out}) - F(\mathbf{x}_*)] 
\leq \Theta \left( \frac{\eta^2 L d^2 \mathcal{V}}{k^2 P} + \frac{\eta^2 L d^2 \mathcal{V}}{k^2 P} \right).
\]
Let $T = \Theta(1/(\eta \mu) \log(\|\mathbf{x}_0 - \mathbf{x}_*\|^2/(\eta \varepsilon))$ to be sufficiently large. Then substituting the definition of $\eta$ gives $\mathbb{E}[F(\mathbf{x}_{out}) - F(\mathbf{x}_*)] \leq \varepsilon$. \qed
A.3 Analysis for Nonconvex Cases

Proposition A.6 (General nonconvex). Suppose that Assumptions 2 and 3 hold. Assume that
\[ \eta_t = \eta \leq 1/(2L) \] for \( t \in \mathbb{N} \). Then S-SGD-EF satisfies
\[
\mathbb{E}[\| \nabla F(x_{\text{out}}) \|^2] \leq \Theta \left( \frac{F(x_{\text{in}}) - F(x_\star)}{\eta T} + \frac{\eta LV}{P} + \frac{L^2}{T} \sum_{t=1}^{T} \mathbb{E}[m_{t-1}^2] - \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}[\| \nabla F(x_{t-1}) \|^2] \right),
\]
where \( x_{\text{out}} = x_{t-1} \) and \( t \sim [T] \) with probability \( \frac{1}{T} \).

Proof. Let \( \tilde{x}_t = \tilde{x}_{t-1} - \eta_t (1/P) \left( \sum_{p=1}^{P} \nabla f_{i_t,p}(x_{t-1}) \right) \) and \( \tilde{x}_0 = x_0 \). By the \( L \)-smoothness of \( F \), we have
\[
F(\tilde{x}_t) \leq F(\tilde{x}_{t-1}) + \langle \nabla F(\tilde{x}_{t-1}), \tilde{x}_t - \tilde{x}_{t-1} \rangle + \frac{L}{2} \| \tilde{x}_t - \tilde{x}_{t-1} \|^2.
\]
Since \( \tilde{x}_t = \tilde{x}_{t-1} - \eta_t \left( \frac{1}{P} \sum_{p=1}^{P} \nabla f_{i_t,p}(x_{t-1}) \right) \), it follows that
\[
F(\tilde{x}_t) \leq F(\tilde{x}_{t-1}) - \eta_t \left( \frac{1}{P} \sum_{p=1}^{P} \nabla f_{i_t,p}(x_{t-1}) \right) + \frac{L}{2} \left\| \sum_{p=1}^{P} \nabla f_{i_t,p}(x_{t-1}) \right\|^2
\]
\[
= F(\tilde{x}_{t-1}) - \eta_t \left( \frac{1}{P} \sum_{p=1}^{P} \nabla f_{i_t,p}(x_{t-1}) \right) + \eta_t \left( \nabla F(x_{t-1}) - \nabla F(\tilde{x}_{t-1}) \right) + \frac{L}{2} \left\| \sum_{p=1}^{P} \nabla f_{i_t,p}(x_{t-1}) \right\|^2.
\]
Taking expectations with respect to \( \{i_{t-1}, \ldots, i_t, p\} \) conditioned on \( \{i_{t'}, i_{t'} \in [t-1], p \in [P]\} \), we have
\[
\mathbb{E}[F(\tilde{x}_t)] \leq F(\tilde{x}_{t-1}) - \eta_t \|\nabla F(x_{t-1})\|^2 + \eta_t \langle \nabla F(x_{t-1}) - \nabla F(\tilde{x}_{t-1}), \nabla F(x_{t-1}) \rangle + \frac{\eta^2 L}{2} \mathbb{E} \left\| \sum_{p=1}^{P} \nabla f_{i_t,p}(x_{t-1}) \right\|^2.
\]
Using
\[
\langle \nabla F(x_{t-1}) - \nabla F(\tilde{x}_{t-1}), \nabla F(x_{t-1}) \rangle \leq \frac{1}{2} \| \nabla F(x_{t-1}) - \nabla F(\tilde{x}_{t-1}) \|^2 + \frac{1}{2} \| \nabla F(x_{t-1}) \|^2
\]
and
\[
\mathbb{E} \left\| \sum_{p=1}^{P} \nabla f_{i_t,p}(x_{t-1}) \right\|^2 = \mathbb{E} \left\| \sum_{p=1}^{P} \nabla f_{i_t,p}(x_{t-1}) - \nabla F(x_{t-1}) \right\|^2 + \|\nabla F(x_{t-1})\|^2,
\]
we get
\[
\mathbb{E}[F(\tilde{x}_t)] \leq F(\tilde{x}_{t-1}) - \frac{\eta_t}{2} (1 - \eta_t L) \|\nabla F(x_{t-1})\|^2 + \frac{\eta^2 L}{2} \mathbb{E} \left\| \sum_{p=1}^{P} \nabla f_{i_t,p}(x_{t-1}) - \nabla F(x_{t-1}) \right\|^2
\]
\[
+ \frac{\eta^2 L}{2} \| x_{t-1} - \tilde{x}_{t-1} \|^2
\]
\[
\leq F(\tilde{x}_{t-1}) - \frac{\eta_t}{2} (1 - \eta_t L) \|\nabla F(x_{t-1})\|^2 + \frac{\eta^2 L}{2} \mathbb{E} \left\| \sum_{p=1}^{P} \nabla f_{i_t,p}(x_{t-1}) - \nabla F(x_{t-1}) \right\|^2
\]
\[
+ \frac{\eta^2 L}{2} \| x_{t-1} - \tilde{x}_{t-1} \|^2
\]
\[
\leq F(\tilde{x}_{t-1}) - \frac{\eta_t}{4} \|\nabla F(x_{t-1})\|^2 + \frac{\eta^2 L}{2P} \| x_{t-1} - \tilde{x}_{t-1} \|^2
\]
\[
\leq F(\tilde{x}_{t-1}) - \frac{\eta_t}{4} \|\nabla F(x_{t-1})\|^2 + \frac{\eta^2 L}{2P} \| m_{t-1} \|^2
\]
Here the second inequality follows from $L$-smoothness of $F$. The third inequality holds because 
\[
\mathbb{E} \left\| \frac{1}{P} \sum_{p=1}^{P} \nabla f_{i_t,p}(x_{t-1}) - \nabla F(x_{t-1}) \right\|^2 \leq \frac{\delta}{P}. 
\] The last inequality is due to the fact that $m_{t-1} = x_{t-1} - \bar{x}_{t-1}$ and $\eta_t \leq \frac{1}{\eta L}$. Rearranging the inequality and taking expectations with respect to the history of all random variables yield 
\[
\frac{\eta_t}{4} \mathbb{E} \| \nabla F(x_{t-1}) \|^2 \leq \mathbb{E} \left[ F(x_{t-1}) - F(\bar{x}_t) \right] + \frac{\eta_t^2 L V}{2P} + \frac{\eta_t^2 \| F \|}{2} - \frac{\eta_t}{4} \| \nabla F(x_{t-1}) \|^2.
\] 
Finally, since $\bar{x}_0 = x_0$ and $F(\bar{x}_T) \geq F(x_*)$, summing this inequality from $t = 1$ to $t = T$ and dividing the result by $\sum_{t=1}^{T} \eta_t$ give the desired results.

**Proof of Theorem 4.5** Let $\eta_t = \eta = \Theta(1/L \wedge \gamma \sqrt{P}/L \wedge P \gamma/(L V) \wedge L V^{-1/2} \gamma \sqrt{P} \varepsilon)$ be the same one defined in the proof of Theorem 4.3. From Proposition A.6, similar to the proof of Theorem 4.3 we have 
\[
\mathbb{E} \| \nabla F(x_{out}) \|^2 \leq \Theta \left( \frac{F(x_0) - F(x_*)}{\eta T} + \frac{\eta L V}{P} + \frac{\eta L^2 d^2 V}{k^2 P} \right).
\] Set $T = \Theta(\Delta/(\eta \varepsilon))$. Substituting the definition of $\eta$, it can be easily seen that $\mathbb{E} \| \nabla F(x_{out}) \|^2 \leq \varepsilon$ and we obtain the desired result.

### B Analysis of S-SNAG-EF

#### B.1 Analysis of $\mathbb{E} \| m_t \|^2$

Remind that for $t \in [T]$, 
\[
\begin{cases}
  y_t &= x_{t-1} - \eta_t \frac{1}{P} \sum_{p=1}^{P} \bar{g}_{t,p}^{(y)}, \\
  z_t &= (1 - \beta_t) z_{t-1} + \beta_t x_{t-1} - \lambda_t \frac{1}{P} \sum_{p=1}^{P} \bar{g}_{t,p}^{(z)}, \\
  x_t &= (1 - \alpha_t) y_t + \alpha_t z_t.
\end{cases}
\]

Let 
\[
\begin{cases}
  \bar{y}_t &= \bar{x}_{t-1} - \eta_t \frac{1}{P} \sum_{p=1}^{P} \nabla f_{i_t,p}(x_{t-1}), \\
  \bar{z}_t &= (1 - \beta_t) \bar{z}_{t-1} + \beta_t \bar{x}_{t-1} - \lambda_t \frac{1}{P} \sum_{p=1}^{P} \nabla f_{i_t,p}(x_{t-1}), \\
  \bar{x}_t &= (1 - \alpha_t) \bar{y}_t + \alpha_t \bar{z}_t,
\end{cases}
\]

where $\bar{y}_0 = y_0, \bar{z}_0 = z_0$ and $\bar{x}_0 = x_0$.

**Lemma B.1.** For $t \in [T]$, let $m_t = (1/P) \sum_{p=1}^{P} m_{t,p}$. $m_t^{(y)} = (1/P) \sum_{p=1}^{P} m_{t,p}^{(y)}$ and $m_t^{(z)} = (1/P) \sum_{p=1}^{P} m_{t,p}^{(z)}$. Then, it holds that
\[
\begin{aligned}
  m_t^{(y)} &= y_t - \bar{y}_t, \\
  m_t^{(z)} &= z_t - \bar{z}_t, \\
  m_t &= x_t - \bar{x}_t.
\end{aligned}
\]

**Proof.** We show the claim by mathematical induction. For $t = 1$, $m_t^{(y)} = m_0 + \eta_t (1/P) \sum_{p=1}^{P} \nabla f_{i_t,p}(x_0) - \bar{g}_{1,t,p}^{(y)} = y_t - \bar{y}_t$. Similarly, $m_t^{(z)} = (1 - \beta_t) m_0 + \lambda_1 (1/P) \sum_{p=1}^{P} \nabla f_{i_t,p}(x_0) - \bar{g}_{1,t,p}^{(z)} = z_t - \bar{z}_t$. Also, $m_1 = (1 - \alpha_t) m_0^{(y)} + \alpha_t m_0^{(z)} = (1 - \alpha_t)(y_t - \bar{y}_t) + \alpha_t (z_t - \bar{z}_t) = x_1 - \bar{x}_1$. Hence the statements hold for $t = 1$. Suppose that the statements hold for $t = t - 1$. Then, $m_t^{(y)} = m_{t-1} + \eta_t (1/P) \sum_{p=1}^{P} (\nabla f_{i_t,p}(x_{t-1}) - \bar{g}_{t-1,p}^{(y)}) = x_{t-1} - \bar{x}_{t-1} + \eta_t (1/P) \sum_{p=1}^{P} (\nabla f_{i_t,p}(x_{t-1}) - \bar{g}_{t-1,p}^{(y)}) = y_t - \bar{y}_t$. Similarly, $m_t^{(z)} = (1 - \beta_t) m_{t-1}^{(z)} + \beta_t m_{t-1} + \lambda_t (1/P) \sum_{p=1}^{P} \nabla f_{i_t,p}(x_{t-1}) - \bar{g}_{t-1,p}^{(z)} = z_t - \bar{z}_t$. Also, $m_t = (1 - \alpha_t) m_t^{(y)} + \alpha_t m_t^{(z)} = (1 - \alpha_t)(y_t - \bar{y}_t) + \alpha_t (z_t - \bar{z}_t) = x_t - \bar{x}_t$. Therefore the statements hold for any $t \in [T]$.
Lemma B.2. For $t \in [T]$ and $p_1 \neq p_2 \in [P]$, 
$$E\langle m_{t,p_1}, m_{t,p_2} \rangle = 0.$$ 
Here the expectations are taken with respect to the all random variables.

Proof. We show that $E\langle m_{t,p_1}^{(y)}, m_{t,p_2}^{(y)} \rangle = 0$, $E\langle m_{t,p_1}^{(z)}, m_{t,p_2}^{(z)} \rangle = 0$ and $E\langle m_{t,p_1}^{(z)}, m_{t,p_2}^{(z)} \rangle = 0$ by mathematical induction. For $t = 1$, the statements are trivial because of the independence of the random choices of non-sparsefied coordinates and $E\langle \gamma_{t,p}^{(y)} \rangle = E\langle \gamma_{t,p}^{(z)} \rangle = \nabla f_{t,p}(x_0)$ for $p \in [P]$. Suppose that $E\langle m_{t-1,p_1}^{(y)}, m_{t-1,p_2}^{(y)} \rangle = E\langle m_{t-1,p_1}^{(z)}, m_{t-1,p_2}^{(z)} \rangle = 0$ hold. $E\langle m_{t,p_1}^{(y)} \mid t-1 \rangle = (1-\gamma)m_{t-1,p_1} + (1-\alpha_t)m_{t-1,p_1} + \alpha_t m_{t-1,p_1}$ and hence $E\langle m_{t,p_1}^{(y)}, m_{t,p_2}^{(y)} \rangle = 0$ by the inductive assumptions. Similarly, we have $E\langle m_{t,p_1}^{(z)}, m_{t,p_2}^{(z)} \rangle = E\langle m_{t,p_1}^{(z)}, m_{t,p_2}^{(z)} \rangle = 0$ by the definition $m_{t-1} = (1-\alpha_t-m_{t-1}^{(y)} + \alpha_t m_{t-1}^{(z)}$. Hence we have $E\langle m_{t,p_1}^{(y)}, m_{t,p_2}^{(y)} \rangle = E\langle m_{t,p_1}^{(y)}, m_{t,p_2}^{(y)} \rangle = 0$ for $t \in [T]$. Since $m_{t,p} = (1-\alpha_t)m_{t,p}^{(y)} + \alpha_t m_{t,p}^{(z)}$ for $p \in \{p_1, p_2\}$, we obtain $E\langle m_{t,p_1}, m_{t,p_2} \rangle = 0$. 

Proof of Proposition 7.7 Using Lemma B.2 we have 
$$E\|m_t^{(y)}\|^2 = E\|x_t - \bar{x}_t\|^2$$
$$= E\| (1-\alpha_t)(y_t - \bar{y}_t) + \alpha_t(z_t - \bar{z}_t) \|^2$$
$$\leq (1-\alpha_t)^2 \left( 1 + \frac{\gamma}{2} \right) E\|y_t - \bar{y}_t\|^2 + \frac{2\alpha_t^2}{\gamma} E\|z_t - \bar{z}_t\|^2$$
$$= (1-\alpha_t)^2 \left( 1 + \frac{\gamma}{2} \right) E\|m_t^{(y)}\|^2 + \frac{2\alpha_t^2}{\gamma} E\|m_t^{(z)}\|^2.$$ 

Using the definition of $g_{t,p}^{(y)}$ and Lemma B.2, similar to the proof of Proposition A.4, we can show that 
$$E\|m_t^{(y)}\|^2 = E\left\| x_{t-1} - \bar{x}_{t-1} + \eta_t \left( \frac{1}{P} \sum_{p=1}^{P} \nabla f_{t,p}(x_{t-1}) - \frac{1}{P} \sum_{p=1}^{P} \bar{g}_{t,p}^{(y)} \right) \right\|^2$$
$$= E \left\| m_{t-1} + \eta_t \left( \frac{1}{P} \sum_{p=1}^{P} \nabla f_{t,p}(x_{t-1}) - \frac{1}{P} \sum_{p=1}^{P} \bar{g}_{t,p}^{(y)} \right) \right\|^2$$
$$\leq (1-\gamma)E\|m_{t-1}\|^2 + \Theta \left( \frac{\eta_t^2}{kP} \left( V + E\|\nabla F(x_{t-1})\|^2 \right) \right).$$

On the other hand, by the definition of $z_t$ and $\bar{z}_t$, similarly we have 
$$E\|m_t^{(z)}\|^2 = E \left\| (1-\beta_t)m_t^{(z)} + \beta_t m_{t-1} + \lambda_t \left( \frac{1}{P} \sum_{p=1}^{P} \nabla f_{t,p}(x_{t-1}) - \frac{1}{P} \sum_{p=1}^{P} \bar{g}_{t,p}^{(z)} \right) \right\|^2$$
$$\leq (1-\gamma)E\| (1-\beta_t)m_t^{(z)} + \beta_t m_{t-1} \|^2 + \Theta \left( \frac{\lambda_t^2}{kP} \left( V + E\|\nabla F(x_{t-1})\|^2 \right) \right)$$
$$\leq (1-\gamma)(1-\beta_t)E\|m_{t-1}^{(z)}\|^2 + (1-\gamma)\beta_t E\|m_{t-1}\|^2 + \Theta \left( \frac{\lambda_t^2}{kP} \left( V + E\|\nabla F(x_{t-1})\|^2 \right) \right).$$

Combining these inequalities, we get 
$$E\|m_t\|^2 + c_t E\|m_t^{(z)}\|^2$$
$$\leq ((1-\gamma)(1+\gamma/2)(1-\alpha_t) + (1-\gamma)(c_t + 2\alpha_t^2/\gamma)\beta_t) E\|m_{t-1}\|^2 + (1-\gamma)(c_t + 2\alpha_t^2/\gamma)(1-\beta_t) E\|m_{t-1}^{(z)}\|^2$$
$$+ \Theta \left( \frac{\eta_t^2 + (c_t + \alpha_t^2/\gamma)\lambda_t^2 d}{kP} \left( V + E\|\nabla F(x_{t-1})\|^2 \right) \right)$$
$$\leq ((1-\gamma/2) + (c_t + 2\alpha_t^2/\gamma)\beta_t) E\|m_{t-1}\|^2 + (1-\gamma)(c_t + 2\alpha_t^2/\gamma) E\|m_{t-1}^{(z)}\|^2$$
$$+ \Theta \left( \frac{\eta_t^2 + (c_t + \alpha_t^2/\gamma)\lambda_t^2 d}{kP} \left( V + E\|\nabla F(x_{t-1})\|^2 \right) \right).$$
for any positive sequence \(\{c_t\}_{t=1}^T\). Hence, if we set \(c_t = 4\alpha_t^2/\gamma^2 \leq 4\alpha_t^2/\gamma^2 = c_{t-1}\), \(\beta_t \leq \gamma/(4(c_t + 2\alpha_t^2/\gamma)) \leq \Theta(\gamma^3/\alpha_t^2)\), we obtain

\[
\mathbb{E}[m_t] \leq \mathbb{C} \mathbb{E}[m_t^{(c)}] \leq (1 - \gamma/4)(\mathbb{E}[m_{t-1}]^2 + c_{t-1} \mathbb{E}[m_{t-1}^{(c)}]) + \Theta \left( \left( \frac{\eta_t^2 + (\alpha_t^2/\gamma)^2}{kP} \right) (\mathbb{V} + \mathbb{E}[\nabla F(x_{t-1})]^2) \right).
\]

Recursively using this inequality, we have

\[
\mathbb{E}[m_t] + c_t \mathbb{E}[m_t^{(c)}] \leq \Theta \left( \sum_{t'=1}^t \left( \frac{\eta_{t'}^2 + (\alpha_{t'}^2/\gamma)^2}{kP} \right) (1 - \gamma/4)^{t-t'} (\mathbb{V} + \mathbb{E}[\nabla F(x_{t-1})]^2) \right).
\]

\[\square\]

### B.2 Analysis for Convex Cases

**Lemma B.3.** Suppose that Assumptions \(\square\) and \(\square\) hold. For \(x \in \mathbb{R}^d\), it follows that

\[
\mathbb{E}[F(\tilde{y}_t)] \leq F(x) - \eta_t \left( \frac{3}{4} - \frac{\eta_t L}{2} \right) \|\nabla F(x_{t-1})\|^2 + \frac{\eta_t^2 L^2 V}{2P} + \eta_t L^2 \|x_{t-1} - \tilde{x}_{t-1}\|^2
\]

\[- \langle \nabla F(x_{t-1}), x - \tilde{x}_{t-1} \rangle - \frac{\mu}{2} \|x_{t-1} - x\|^2 - \langle \nabla F(\tilde{x}_{t-1}) - \nabla F(x_{t-1}), x - \tilde{x}_{t-1} \rangle.
\]

**Proof.** By the \(L\)-smoothness of \(F\) and the definition of \(\tilde{y}_t\), we have

\[
F(\tilde{y}_t) \leq F(\tilde{x}_{t-1}) + \langle \nabla F(\tilde{x}_{t-1}), \tilde{y}_t - \tilde{x}_{t-1} \rangle + \frac{L}{2} \|	ilde{x}_{t-1} - \tilde{y}_t\|^2
\]

\[
\leq F(\tilde{x}_{t-1}) - \eta_t \left( \nabla F(\tilde{x}_{t-1}), \frac{1}{P} \sum_{p=1}^P \nabla f_{t_p}(x_{t-1}) \right) + \frac{\eta_t^2 L}{2} \left( \frac{1}{P} \sum_{p=1}^P \nabla f_{t_p}(x_{t-1}) \right)^2.
\]

Taking expectations of this inequality with respect to the \(t\)-th iteration gives

\[
\mathbb{E}[F(\tilde{y}_t)] \leq F(\tilde{x}_{t-1}) - \eta_t \langle \nabla F(\tilde{x}_{t-1}) - \nabla F(x_{t-1}), \nabla F(x_{t-1}) \rangle - \eta_t \left( 1 - \frac{\eta_t L}{2} \right) \|\nabla F(x_{t-1})\|^2 + \frac{\eta_t^2 L^2 V}{2P} + \eta_t L^2 \|x_{t-1} - \tilde{x}_{t-1}\|^2.
\]

Here the first inequality holds because \(\mathbb{E}[\langle (1/P) \sum_{p=1}^P \nabla f_{t_p}(x_{t-1})\rangle] = \nabla F(x_{t-1})\) and \(\mathbb{E}[\|\nabla F(x_{t-1})\|^2] = \|\nabla F(x_{t-1})\|^2 + \|\nabla F(x_{t-1})\|^2 \leq V/P + \|\nabla F(x_{t-1})\|^2\). The second inequality follows from \(L\)-smoothness of \(F\) and Young’s inequality. Also, we have

\[
F(\tilde{x}_{t-1}) \leq F(x) - \langle \nabla F(\tilde{x}_{t-1}), x - \tilde{x}_{t-1} \rangle - \frac{\mu}{2} \|\tilde{x}_{t-1} - x\|^2
\]

\[
= F(x) - \langle \nabla F(x_{t-1}), x - \tilde{x}_{t-1} \rangle - \frac{\mu}{2} \|\tilde{x}_{t-1} - x\|^2 - \langle \nabla F(\tilde{x}_{t-1}) - \nabla F(x_{t-1}), x - \tilde{x}_{t-1} \rangle
\]

by \(\mu\)-strong convexity of \(F\). Combining the above two inequality results in

\[
\mathbb{E}[F(\tilde{y}_t)] \leq F(x) - \eta_t \left( \frac{3}{4} - \frac{\eta_t L}{2} \right) \|\nabla F(x_{t-1})\|^2 + \frac{\eta_t^2 L^2 V}{2P} + \eta_t L^2 \|x_{t-1} - \tilde{x}_{t-1}\|^2
\]

\[- \langle \nabla F(x_{t-1}), x - \tilde{x}_{t-1} \rangle - \frac{\mu}{2} \|\tilde{x}_{t-1} - x\|^2 - \langle \nabla F(\tilde{x}_{t-1}) - \nabla F(x_{t-1}), x - \tilde{x}_{t-1} \rangle.
\]

\[\square\]
Lemma B.4. Suppose that Assumptions 2 and 4 hold. Set \( \beta_t = \lambda_t \mu / (1 + \lambda_t \mu) \). For \( x \in \mathbb{R}^d \), it follows that

\[
- \langle \nabla F(x_{t-1}), x - \bar{x}_{t-1} \rangle - \frac{\mu}{2} \| x_{t-1} - x \|^2 \\
\leq \frac{1}{\eta_t} \mathbb{E}(\bar{x}_{t-1} - \bar{y}_t, x_{t-1} - \bar{z}_t) \\
+ \frac{1 - \beta_t}{2 \lambda_t} \| z_{t-1} - x \|^2 - (1 - \beta_t) \left( \frac{1}{2 \lambda_t} + \frac{\mu}{2} \right) \mathbb{E} \| \bar{z}_t - x \|^2 - \frac{1 - \beta_t}{2 \lambda_t} \mathbb{E} \| z_{t-1} - \bar{z}_t \|^2.
\]

Proof. Let

\[
V_t(x) = \frac{1 + \lambda_t \mu}{\eta_t} (\bar{x}_{t-1} - \bar{y}_t, x - \bar{x}_{t-1}) + \frac{1}{2 \lambda_t} \| \bar{z}_{t-1} - x \|^2 + \frac{\mu}{2} \| x_{t-1} - x \|^2 \\
= (1 + \lambda_t \mu) \left( \frac{1}{P} \sum_{p=1}^{P} \nabla f_{t,p}(x_{t-1}), x - \bar{x}_{t-1} \right) + \frac{1}{2 \lambda_t} \| \bar{z}_{t-1} - x \|^2 + \frac{\mu}{2} \| x_{t-1} - x \|^2.
\]

If we set \( \beta_t = \lambda_t \mu / (1 + \lambda_t \mu) \), \( \bar{z}_t \) is the minimizer of \( V_t \) and \( V_t \) is \( 1 / \lambda_t + \mu \)-strongly convex. Hence we have

\[
V_t(z_t) \leq V_t(x) - \left( \frac{1}{2 \lambda_t} + \frac{\mu}{2} \right) \| \bar{z}_t - x \|^2 \\
= (1 + \lambda_t \mu) \left( \frac{1}{P} \sum_{p=1}^{P} \nabla f_{t,p}(x_{t-1}), x - \bar{x}_{t-1} \right) + \frac{1}{2 \lambda_t} \| \bar{z}_{t-1} - x \|^2 - \left( \frac{1}{2 \lambda_t} + \frac{\mu}{2} \right) \| \bar{z}_t - x \|^2 \\
+ \frac{\mu}{2} \| x_{t-1} - x \|^2.
\]

Using definition of \( V_t \) and taking expectations of the both sides with respect to the \( t \)-th iteration yield

\[
- \langle \nabla F(x_{t-1}), x - \bar{x}_{t-1} \rangle - \frac{\mu}{2} \| x_{t-1} - x \|^2 \\
\leq \frac{1}{\eta_t} \mathbb{E}(\bar{x}_{t-1} - \bar{y}_t, x_{t-1} - \bar{z}_t) \\
+ \frac{1 - \beta_t}{2 \lambda_t} \| z_{t-1} - x \|^2 - (1 - \beta_t) \left( \frac{1}{2 \lambda_t} + \frac{\mu}{2} \right) \mathbb{E} \| \bar{z}_t - x \|^2 - \frac{1 - \beta_t}{2 \lambda_t} \mathbb{E} \| z_{t-1} - \bar{z}_t \|^2.
\]

Here we used the relation \( 1 / (1 + \lambda_t \mu) = 1 - \beta_t \). \( \Box \)

Proposition B.5. Suppose that Assumptions 1, 2, 3 and 4 hold. Let \( \eta_t = \eta \leq 1 / (2L) \), \( \lambda_t = \lambda = (1/2) \sqrt{\eta / \mu} \), \( \alpha_t = \alpha = \lambda_t \mu / (2 + \lambda_t \mu) \) and \( \beta_t = \beta = \lambda_t \mu / (1 + \lambda_t \mu) \). Then S-2NAG-EF satisfies

\[
\mathbb{E}[F(\tilde{y}_T) - F(x_*)] \leq \Theta \left( \mu (1 - \alpha)^T + \sqrt{\frac{\eta V^2}{\mu T}} + \sum_{i=1}^{T} (1 - \alpha)^{T-i} \left( \lambda L^2 \mathbb{E} \| m_{t-1} \|^2 - \eta \mathbb{E} \| \nabla F(x_{t-1}) \|^2 \right) \right).
\]
Proof. Combining Lemma $B.3$ and Lemma $B.4$ with $x = x_*$, we get
\[
E[F(\tilde{y}_t)] \leq F(x_*) - \eta_t \left( \frac{3}{4} - \frac{\eta_t L}{2} \right) \|\nabla F(x_{t-1})\|^2 + \frac{\eta_t^2 LV}{2P} + \eta_t L^2 \|x_{t-1} - \tilde{x}_{t-1}\|^2
\]
\[+ \frac{1}{\eta_t} E(\tilde{x}_{t-1} - \tilde{y}_t, \tilde{x}_{t-1} - \tilde{z}_t)
\]
\[+ \frac{1 - \beta_t}{2\lambda_t} \|\tilde{x}_{t-1} - x_*\|^2 - (1 - \beta_t) \left( \frac{1}{2\lambda_t} + \frac{\mu}{2} \right) E\|\tilde{z}_t - x_*\|^2 - \frac{1 - \beta_t}{2\lambda_t} E\|\tilde{x}_{t-1} - \tilde{z}_t\|^2
\]
\[- (\nabla F(\tilde{x}_{t-1}) - \nabla F(x_{t-1}), x_* - \tilde{x}_{t-1})
\]
\[= F(x_*) - \eta_t \left( \frac{3}{4} - \frac{\eta_t L}{2} \right) \|\nabla F(x_{t-1})\|^2 + \frac{\eta_t^2 LV}{2P} + \eta_t L^2 \|x_{t-1} - \tilde{x}_{t-1}\|^2
\]
\[+ \frac{1}{\eta_t} E(\tilde{x}_{t-1} - \tilde{y}_t, \tilde{x}_{t-1} - \tilde{z}_t)
\]
\[+ \frac{1 - \beta_t}{2\lambda_t} \|\tilde{x}_{t-1} - x_*\|^2 - (1 - \beta_t) \left( \frac{1}{2\lambda_t} + \frac{\mu}{2} \right) E\|\tilde{z}_t - x_*\|^2 - \frac{1 - \beta_t}{2\lambda_t} E\|\tilde{x}_{t-1} - \tilde{z}_t\|^2
\]
\[+ (\nabla F(\tilde{x}_{t-1}) - \nabla F(x_{t-1}), \tilde{x}_{t-1} - \tilde{z}_t) + (\nabla F(\tilde{x}_{t-1}) - \nabla F(x_{t-1}), \tilde{z}_t - x_*).
\]
(2)

Also, using Lemma $B.3$ with $x = \tilde{y}_{t-1}$ gives
\[
E[F(\tilde{y}_t)] \leq F(\tilde{y}_{t-1}) - \eta_t \left( \frac{3}{4} - \frac{\eta_t L}{2} \right) \|\nabla F(x_{t-1})\|^2 + \frac{\eta_t^2 LV}{2P} + \eta_t L^2 \|x_{t-1} - \tilde{x}_{t-1}\|^2
\]
\[+ \frac{1}{\eta_t} E(\tilde{x}_{t-1} - \tilde{y}_{t-1}, \tilde{x}_{t-1} - \tilde{y}_{t-1}) + (\nabla F(\tilde{x}_{t-1}) - \nabla F(x_{t-1}, \tilde{x}_{t-1} - \tilde{y}_{t-1}).
\]
(3)

Now, summing $\alpha_t \times (2)$ and $(1 - \alpha_t) \times (3)$ yields
\[
E[F(\tilde{y}_t)] - F(x_*) \leq (1 - \alpha_t)(F(\tilde{y}_{t-1}) - F(x_*))
\]
\[- \frac{1}{\eta_t} E(\tilde{x}_{t-1} - \tilde{y}_{t-1}, \tilde{x}_{t-1} - \tilde{y}_{t-1}) \]
\[+ \alpha_t(1 - \beta_t) \|\tilde{z}_{t-1} - x_*\|^2 - \alpha_t(1 - \beta_t) \left( \frac{1}{2\lambda_t} + \frac{\mu}{2} \right) E\|\tilde{z}_t - x_*\|^2
\]
\[- \frac{\alpha_t(1 - \beta_t)}{2\lambda_t} E\|\tilde{z}_{t-1} - \tilde{z}_t\|^2
\]
\[- (\nabla F(\tilde{x}_{t-1}) - \nabla F(x_{t-1}), \tilde{x}_{t-1} - \tilde{z}_t - (1 - \alpha_t)\tilde{y}_{t-1})
\]
\[+ \alpha_t(\nabla F(\tilde{x}_{t-1}) - \nabla F(x_{t-1}), \tilde{z}_t - x_*),
\]
Since $\tilde{x}_{t-1} - \alpha_t \tilde{z}_t - (1 - \alpha_t)\tilde{y}_{t-1} = -\alpha_t(\tilde{z}_t - \tilde{x}_{t-1})$, we have
\[
E(\tilde{x}_{t-1} - \tilde{y}_t, \tilde{x}_{t-1} - \alpha_t \tilde{z}_t - (1 - \alpha_t)\tilde{y}_{t-1})
\]
\[= - \alpha_t E(\tilde{x}_{t-1} - \tilde{y}_t, \tilde{z}_t - \tilde{x}_{t-1})
\]
\[\leq \alpha_t \lambda_t \eta_t \frac{E\|\tilde{x}_{t-1} - \tilde{y}_t\|^2 + \alpha_t(1 - \beta_t)\eta_t}{4\lambda_t} \|\tilde{z}_t - \tilde{x}_{t-1}\|^2
\]
\[= \alpha_t \lambda_t \eta_t \left( \frac{1}{1 - \beta_t} \frac{1}{P} \sum_{p=1}^P \nabla f_{i,p}(x_{t-1}) \right)^2 + \alpha_t(1 - \beta_t)\eta_t \|\tilde{z}_t - \tilde{x}_{t-1}\|^2
\]
\[= \alpha_t \lambda_t \eta_t \left( \frac{1}{1 - \beta_t} \frac{1}{P} \sum_{p=1}^P \nabla f_{i,p}(x_{t-1}) \right)^2 + \alpha_t(1 - \beta_t)\eta_t \|\tilde{z}_t - \tilde{x}_{t-1}\|^2.
\]
Also, it holds that
\[
(\nabla F(\tilde{x}_{t-1}) - \nabla F(x_{t-1}), \tilde{x}_{t-1} - \alpha_t \tilde{z}_t - (1 - \alpha_t)\tilde{y}_{t-1})
\]
\[= - \alpha_t(\nabla F(\tilde{x}_{t-1}) - \nabla F(x_{t-1}), \tilde{z}_t - \tilde{x}_{t-1})
\]
\[\leq \lambda_t \alpha_t \|\nabla F(\tilde{x}_{t-1}) - \nabla F(x_{t-1})\|^2 + \alpha_t(1 - \beta_t)\eta_t \|\tilde{z}_t - \tilde{x}_{t-1}\|^2.
\]
Furthermore, we have
\[
\alpha_t (\nabla F(\bar{x}_{t-1}) - \nabla F(x_{t-1}), \bar{z}_t - x_*) \leq \frac{\alpha_t}{(1 - \beta_t) \mu} \| \nabla F(\bar{x}_{t-1}) - \nabla F(x_{t-1}) \|^2 + \alpha_t (1 - \beta_t) \frac{\mu}{4} \| \bar{z}_t - x_* \|^2.
\]

If we assume that
\[
\frac{\alpha_t \lambda_t}{1 - \beta_t} \leq \frac{\eta_t}{4},
\]
by these inequality, we get
\[
\mathbb{E} [F(\bar{y}_t)] - F(x_*) \leq (1 - \alpha_t) (F(\bar{y}_{t-1}) - F(x_*))
\]
\[
+ \frac{(\eta_t^2 L + \eta_t) \nu}{2P} + \frac{2\eta_t L^2 + \eta_t L^2 / (\lambda_t \mu)}{\Gamma_t} \| x_{t-1} - \bar{x}_{t-1} \|^2 - \frac{\eta_t}{8 \Gamma_t} \| \nabla F(x_{t-1}) \|^2
\]
\[
+ \frac{\alpha_t (1 - \beta_t)}{2 \Gamma_t \lambda_t} \| z_{t-1} - x_* \|^2 - \frac{\alpha_t (1 - \beta_t)}{\Gamma_t} \left( \frac{1}{2 \lambda_t} + \frac{\mu}{4} \right) \mathbb{E} \| \bar{z}_t - x_* \|^2.
\]

Let \( \Gamma_t = \Pi_{t'=1}^t (1 - \alpha_{t'}) > 0 \). Multiplying \( 1/\Gamma_t \) to both sides of the above inequality yields
\[
\frac{1}{\Gamma_t} \mathbb{E} [F(\bar{y}_t)] - F(x_*) \leq \frac{1}{\Gamma_{t-1}} (F(\bar{y}_{t-1}) - F(x_*))
\]
\[
+ \frac{(\eta_t^2 L + \eta_t) \nu}{2P} + \frac{2\eta_t L^2 + \eta_t L^2 / (\lambda_t \mu)}{\Gamma_t} \| x_{t-1} - \bar{x}_{t-1} \|^2 - \frac{\eta_t}{8 \Gamma_t} \| \nabla F(x_{t-1}) \|^2
\]
\[
+ \alpha_t (1 - \beta_t) \mathbb{E} \| z_{t-1} - x_* \|^2 - \alpha_t (1 - \beta_t) \left( \frac{1}{2 \lambda_t} + \frac{\mu}{4} \right) \mathbb{E} \| \bar{z}_t - x_* \|^2.
\]

Taking expectations of this inequality with respect to the all random variables gives
\[
\frac{1}{\Gamma_t} \mathbb{E} [F(\bar{y}_t)] - F(x_*) \leq \frac{1}{\Gamma_{t-1}} \mathbb{E} [F(\bar{y}_{t-1}) - F(x_*)]
\]
\[
+ \frac{(\eta_t^2 L + \eta_t) \nu}{2P} + \frac{2\eta_t L^2 + \eta_t L^2 / (\lambda_t \mu)}{\Gamma_t} \mathbb{E} \| x_{t-1} - \bar{x}_{t-1} \|^2 - \frac{\eta_t}{8 \Gamma_t} \mathbb{E} \| \nabla F(x_{t-1}) \|^2
\]
\[
+ \alpha_t (1 - \beta_t) \mathbb{E} \| z_{t-1} - x_* \|^2 - \alpha_t (1 - \beta_t) \left( \frac{1}{2 \lambda_t} + \frac{\mu}{4} \right) \mathbb{E} \| \bar{z}_t - x_* \|^2.
\]

Let \( \alpha_t = \alpha = \lambda \mu / (2 + \lambda \mu), \beta = \beta_t = \lambda \mu / (1 + \lambda \mu), \eta_t = \eta \) and \( \lambda_t = \lambda = (1/2) \sqrt{\eta / \mu} \). Then (4) holds. Also, we have the relation \( \alpha_t / (2 \Gamma_t \lambda_t) = \alpha / (2 \Gamma_t \lambda) \leq \alpha / (2 \Gamma_{t-1} / (1 / (2 \lambda) + \mu / 4)) \) by the definition of \( \alpha \). Hence we have
\[
\frac{1}{\Gamma_t} \mathbb{E} [F(\bar{y}_t)] - F(x_*) \leq \frac{1}{\Gamma_{t-1}} \mathbb{E} [F(\bar{y}_{t-1}) - F(x_*)]
\]
\[
+ \frac{(\eta_t^2 L + \eta_t) \nu}{2P} + \frac{2\eta_t L^2 + \eta_t L^2 / (\lambda_t \mu)}{\Gamma_t} \mathbb{E} \| x_{t-1} - \bar{x}_{t-1} \|^2 - \frac{\eta_t}{8 \Gamma_t} \mathbb{E} \| \nabla F(x_{t-1}) \|^2
\]
\[
+ \alpha (1 - \beta) \left( \frac{1}{2 \lambda} + \frac{\mu}{4} \right) \mathbb{E} \| \bar{z}_t - x_* \|^2.
\]

Summing up the above inequality from \( t = 1 \) to \( T \), we obtain
\[
\frac{1}{\Gamma_T} \mathbb{E} [F(\bar{y}_T)] - F(x_*) \leq \sum_{t=1}^T \frac{(\eta_t^2 L + \eta_t) \nu}{2P} + \frac{\eta_t L^2 + \eta_t L^2 / (\lambda_t \mu)}{\Gamma_t} \mathbb{E} \| x_{t-1} - \bar{x}_{t-1} \|^2
\]
\[
- \frac{\eta_t}{8 \Gamma_t} \mathbb{E} \| \nabla F(x_{t-1}) \|^2 + \alpha (1 - \beta) \left( \frac{1}{2 \lambda} + \frac{\mu}{4} \right) \mathbb{E} \| \bar{z}_0 - x_* \|^2.
\]

Note that \( \bar{z}_0 = x_0 \). We also have \( \alpha = O(\sqrt{\eta / \mu}) \) since \( \eta = O(1/L) \), \( \mu \leq L \) and the setting of \( \lambda \).

Multiplying \( 1/\Gamma_t \) to both sides of the inequality and rearranging it yield
\[
\mathbb{E} [F(\bar{y}_T) - F(x_*)] \leq \Theta \left( \mu (1 - \alpha)^T + \sqrt{\frac{\eta^2 L^2}{\mu} T} + \sum_{t=1}^T (1 - \alpha)^{T-t} \left( \lambda L^2 \mathbb{E} \| m_{t-1} \|^2 - \eta \mathbb{E} \| \nabla F(x_{t-1}) \|^2 \right) \right).
\]

\[\square\]
Lemma B.6.

\[ \mathbb{E}[F(x_{t-1}) - F(x_*)] \leq \Theta \left( \mathbb{E}[F(\tilde{y}_t) - F(x_*)] + L\mathbb{E}[m_{t-1}]^2 \right). \]

Proof.

\[
F(x_{t-1}) \leq F(\tilde{x}_{t-1}) - \langle \nabla F(x_{t-1}), \tilde{x}_{t-1} - x_{t-1} \rangle \\
\leq F(\tilde{y}_t) - \langle \nabla F(\tilde{x}_{t-1}), \tilde{y}_t - \tilde{x}_{t-1} \rangle - \langle \nabla F(x_{t-1}), \tilde{x}_{t-1} - x_{t-1} \rangle \\
= F(\tilde{y}_t) + \eta_t \left( \nabla F(\tilde{x}_{t-1}), \frac{1}{P} \sum_{p=1}^{P} \nabla f_{t,p}(x_{t-1}) \right) - \langle \nabla F(x_{t-1}), \tilde{x}_{t-1} - x_{t-1} \rangle.
\]

Taking expectations of this inequality with respect to the t-th iteration gives

\[
F(x_{t-1}) \leq \mathbb{E}[F(\tilde{y}_t)] + \eta_t \langle \nabla F(\tilde{x}_{t-1}), \nabla F(x_{t-1}) \rangle - \langle \nabla F(x_{t-1}), \tilde{x}_{t-1} - x_{t-1} \rangle \\
\leq \mathbb{E}[F(\tilde{y}_t)] - \eta_t \langle \nabla F(\tilde{x}_{t-1}) - \nabla F(x_{t-1}), \nabla F(x_{t-1}) \rangle + \eta_t \|
abla F(x_{t-1})\|^2 \\
+ \frac{1}{8L} \|
abla F(x_{t-1})\|^2 + 2L\|x_{t-1} - \tilde{x}_{t-1}\|^2 \\
\leq \mathbb{E}[F(\tilde{y}_t)] + \frac{3}{8L} \|
abla F(x_{t-1})\|^2 + \Theta(L)\|m_{t-1}\|^2.
\]

Here the first inequality is due to the convexity of $F$. The second and third inequalities follow from Young’s inequality and the $L$-smoothness of $F$. The last inequality holds because $\eta_t \leq 1/(8L)$. Also, by the $L$-smoothness and convexity of $F$, we have

\[
\frac{1}{2L} \|
abla F(x_{t-1})\|^2 \leq F(x_{t-1}) - F(x_*).
\]

Using this inequality, we obtain

\[
\frac{1}{4} (F(x_{t-1}) - F(x_*)) \leq \mathbb{E}[F(\tilde{y}_t) - F(x_*)] + \Theta(L)\|m_{t-1}\|^2.
\]

Multiplying 4 to the both sides and taking expectations yields the claim. \qed

Proof of Proposition 4.6. Applying Lemma B.6 with $t = T$ to Proposition B.3, the statement can be immediately obtained. \qed

Proof of Theorem 4.8. Let $\eta = \eta = \Theta(1/L \land \gamma^3 P/L \land \gamma^2/\mu \land \gamma^{8/3} P^{2/3} \mu^{1/3} / L^{4/3} \land P^2 \mu^3 / \sqrt{V^2} \land \gamma^2 \sqrt{\mu P^2} / (L \sqrt{V})$. At first, for using Proposition 4.7, it is required to be $\beta_1 = \beta \leq \Theta(\gamma^2/\mu)$. This condition is satisfied by assuming $\eta \leq \Theta(\gamma^2/\mu)$ be sufficiently small, because

\[
\beta = \frac{\lambda \mu}{1 + \lambda \mu} \leq \frac{\gamma^3}{\alpha^2} \Leftrightarrow \frac{\sqrt{\eta \mu}}{2 + \sqrt{\eta \mu}} \leq \frac{\gamma^3}{\eta \mu} \\
\Leftrightarrow \sqrt{\eta \mu} \leq \frac{\gamma^3}{\eta \mu} \\
\Leftrightarrow \eta \leq \frac{\gamma^2}{\mu}.
\]

Observe that $1 - 2\alpha \geq 1 - \gamma$ can be satisfied for appropriate $\eta$, because $\alpha = \Theta(\sqrt{\eta \mu})$ and $\eta \leq \Theta(\gamma^2/\mu)$. Also, note that

\[
\frac{\lambda \eta^2 L^2}{\gamma^4 P} \leq \Theta(\eta) \Leftrightarrow \eta \leq \Theta(\gamma^8/3 P^{2/3} \mu^{1/3} / L^{4/3})
\]

and

\[
\frac{\eta^2 L}{\gamma^3 P} \leq \Theta(\eta) \Leftrightarrow \eta \leq \Theta(\gamma^3 P / L).
\]
From these facts, similar to the proof of Theorem 4.3 combining Proposition 4.6 and Proposition 4.7, we have

\[
\mathbb{E}[F(x_{\text{out}}) - F(x^*)] 
\leq \Theta \left( \mu (1 - \alpha)^T \|x_0 - x^*\|^2 + \sqrt{\frac{\eta}{\mu P}} + \sum_{t = 1}^T (1 - \alpha)^{T - t} \left( \lambda L^2 \mathbb{E}\|m_{t-1}\|^2 - \eta \mathbb{E}\|\nabla F(x_{t-1})\|^2 \right) + L \mathbb{E}\|m_{T-1}\|^2 \right)
\]

\[
= \Theta \left( \mu (1 - \alpha)^T \|x_0 - x^*\|^2 + \sqrt{\frac{\eta}{\mu P}} + \frac{\lambda \eta^2 L^2 \lambda d}{\gamma^2 k P} \sum_{t = 1}^T (1 - \alpha)^{T - t} \left( \sum_{t' = 1}^t (1 - \gamma)^{t - t'} \right) + \frac{\eta^2 L \lambda d}{\gamma^2 k P} \sum_{t = 1}^T (1 - \gamma)^{T - t} + \frac{\eta^2 L d}{\gamma^2 k P} \sum_{t = 1}^T (1 - \gamma)^{T - t} \sum_{t' = 1}^t (1 - \gamma)^{t - t'} \right)
\]

\[
= \Theta \left( \mu (1 - \alpha)^T \|x_0 - x^*\|^2 + \sqrt{\frac{\eta}{\mu P}} + \frac{\lambda \eta^2 L^2 \lambda d}{\gamma^2 k P} \sum_{t = 1}^T (1 - \gamma)^{T - t} \left( \sum_{t' = 1}^t (1 - \gamma)^{t - t'} \right) + \frac{\eta^2 L \lambda d}{\gamma^2 k P} \sum_{t = 1}^T (1 - \gamma)^{T - t} + \frac{\eta^2 L d}{\gamma^2 k P} \sum_{t = 1}^T (1 - \gamma)^{T - t} \left( \sum_{t' = 1}^t (1 - \gamma)^{t - t'} \right) \right)
\]

\[
\leq \Theta \left( \mu (1 - \alpha)^T \|x_0 - x^*\|^2 + \sqrt{\frac{\eta}{\mu P}} + \frac{\lambda \eta^2 L^2 \lambda d}{\gamma^2 k P} \sum_{t = 1}^T (1 - \gamma)^{T - t} \left( \sum_{t' = 1}^t (1 - \gamma)^{t - t'} \right) + \frac{\eta^2 L \lambda d}{\gamma^2 k P} \sum_{t = 1}^T (1 - \gamma)^{T - t} + \frac{\eta^2 L d}{\gamma^2 k P} \sum_{t = 1}^T (1 - \gamma)^{T - t} \left( \sum_{t' = 1}^t (1 - \gamma)^{t - t'} \right) \right)
\]

Here, the last inequality is due to \( \lambda L / \alpha \geq 1 \). Set appropriate \( T = \Theta(1 / \alpha) = \Theta(1 / \sqrt{\eta \mu}) \).

The sufficient conditions for \( \mathbb{E}[F(x_{\text{out}}) - F(x^*)] \leq \varepsilon \) are \( \eta \leq \Theta(1 / L) \), \( \eta \leq \gamma^2 / \mu \), \( \eta \leq \Theta(\gamma^8 / P^2 / \mu^3 / L^4 / 3) \), \( \eta \leq \Theta(\gamma^3 P / L) \), \( \sqrt{\eta / \mu} \mathbb{V} / P \leq \varepsilon \) and \( \lambda \eta^2 L^2 \mathbb{V} / (\gamma^4 \alpha P) \leq \Theta(\varepsilon) \). Substituting the definition of \( \eta \) to \( \tilde{O}(1 / \sqrt{\eta \mu}) \), we obtain the desired result.

**B.3 Analysis for Nonconvex Cases**

**Proof of Theorem 4.9** Let \( \eta_2 = \eta \) be the same one defined in the proof of Theorem 4.8. First observe that \( F_s = F + \sigma Q(x_s - 1) \) is \( \mathcal{S}L \) smooth and \( L \) strongly convex, since \( \sigma = L \) and \( F \) is \( L \) smooth. Also note that \( \{ f_i, p + \sigma Q(x_s - 1) \}_{i, p} \) has \( \mathbb{V} \) bounded variance. From Theorem 4.8 (with \( \mu \leftarrow L \) and \( \varepsilon \leftarrow \varepsilon / (16 L) \)), we have

\[
\mathbb{E}\|\nabla F_s(x_s)\|^2 = \mathbb{E}\left\|\nabla F_s \left( x^{(s-1)}_s \right) \right\|^2 \leq 2L \mathbb{E}\left[ F_s \left( x^{(s-1)}_s \right) - \min_{x \in \mathbb{R}^d} F_s(x) \right] \leq \frac{\varepsilon}{8},
\]

with iteration complexity

\[
\tilde{O} \left( 1 + \frac{\mathbb{V}}{P \varepsilon} + \frac{d^2}{k^2 \sqrt{P}} + \frac{d^4}{k^4 \alpha^p} + \frac{d}{k^4 \varepsilon^2} \right)
\]

Using this fact, we have

\[
\mathbb{E}\|\nabla F(x_s)\|^2 \leq 2 \mathbb{E}\|\nabla F_1(x_s)\|^2 + 4L^2 \mathbb{E}\|x_s - x_{s-1}\|^2
\]

\[
\leq \frac{\varepsilon}{4} + 4L^2 \mathbb{E}\|x_s - x_{s-1}\|^2.
\]

Now we need to bound \( \mathbb{E}\|x_s - x_{s-1}\|^2 \).

\[
\mathbb{E}[F(x_s)] = \mathbb{E}[F_s(x_s)] - L \mathbb{E}\|x_s - x_{s-1}\|^2
\]

\[
= \mathbb{E}[F_s(x_s) - \min_{x \in \mathbb{R}^d} F_s(x)] + \mathbb{E}[\min_{x \in \mathbb{R}^d} F_s(x)] - L \mathbb{E}\|x_s - x_{s-1}\|^2
\]

\[
\leq \frac{\varepsilon}{16L} + \mathbb{E}[F(x_{s-1})] - L \mathbb{E}\|x_s - x_{s-1}\|^2
\]

\[
= \frac{\varepsilon}{16L} + \mathbb{E}[F(x_{s-1})] - L \mathbb{E}\|x_s - x_{s-1}\|^2.
\]

Hence we obtain

\[
\mathbb{E}\|\nabla F(x_s)\|^2 \leq \frac{\varepsilon}{2} + 4L \mathbb{E}[F(x_{s-1}) - F(x_s)].
\]

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Summing this inequality from $s = 1$ to $S$ and divide the result by $S$ yield

$$
\mathbb{E}\|\nabla F(x_{\text{out}})\|^2 \leq \frac{\varepsilon}{2} + \Theta(L) \frac{F(x_{\text{in}}) - F(x_*)}{S}.
$$

Therefore appropriately large $S = \Theta(1 + L\Delta/\varepsilon)$ is sufficient for ensuring $\mathbb{E}\|\nabla F(x_{\text{out}})\|^2 \leq \varepsilon$. \qed