Convergence of the linearized system for a compressible Navier-Stokes-Poisson system

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Abstract. In this paper, we are concerned with the limit for solutions of combining viscid and compressible isentropic Navier-Stokes-Poisson system in a bounded domain \( \Omega \subset R^N, N \geq 1 \), with Dirichlet boundary condition. We first derive Navier-Stokes-Poisson system by using Energetic Variational Approach. Then the convergence of the compressible isentropic Navier-Stokes-Poisson system to the linearized system is proven for the global weak solution and for the case of general initial data with satisfying appropriate conditions.

1. Introduction
The Navier-Stokes-Poisson system is a simplified isentropic two-fluid model involving dissipation, which describe the dynamics of a plasma, where the compressible electron fluid interacts with its own electric field against a constant charged ion background [1]. In [2] this system was simulated by the transport of charged particles under the electric field of electrostatic potential force in semiconductor devices.

Energy Variational Approach was developed by Lord Rayleigh and Lars Onsager in 1873 and 1931, they published works about Energy Variational Approach in [3, 4, 5] and the references therein, we can see other works in [6, 7, 8, 9, 10]. The Energy Variational Approach is based on following concepts that are outlined below: energy dissipation law, Least Action Principle, Maximum Dissipation Principle, and Newton’s force balance law.

For the Navier-Stokes system, in [11], Lions and Masmoudi (1998) proved the various convergence results concerning global (weak) solutions of compressible isentropic Navier-Stokes equations in various domain cases. More precisely, they showed various results establishing the convergence, as the density becomes constant and the Mach number goes to 0, towards solutions of incompressible models Navier-Stokes or Euler equations. Most of these results were global in time and without size restriction on the initial data. They also established the solutions of compressible Navier-Stokes system converge to the linearized system in a bounded smooth open domain with Dirichlet condition. However, they didn’t consider the Poisson equation, we will consider the convergence of the coupled system, namely Navier-Stokes system and the Poisson equation in a bounded smooth domain with Dirichlet boundary condition. In [12], the authors studied the limit of global weak solutions of the compressible Navier–Stokes equations (in the isentropic regime) in a bounded domain, with Dirichlet boundary conditions on the velocity, as the Mach number goes to 0. They showed that the velocity converges weakly in \( L^2 \) to a global weak solution of the incompressible
Navier-Stokes equations. Moreover, the convergence in $L^2$ is strong under some geometrical assumption on $\Omega$.

For the compressible Navier-Stokes-Poisson in periodic domain, we can see some results in [13], Wang and Jiang (2006) studied the combining quasineutral and inviscid limit of the compressible Navier-Stokes-Poisson system in the $d$-dimensional torus $T^d (d \geq 1)$, they proved the convergence of the Navier-Stokes-Poisson system to the incompressible Euler equations for the global weak solution. In [14], Ju et al. (2008) considered the quasineutral limit of the Navier-Stokes-Poisson system in the whole space and in the torus, for the well-prepared initial data, they proved the global weak solution of the Navier-Stokes-Poisson system converges strongly to the strong solution of the incompressible Navier-Stokes equations.

Motivated by the above result, we consider the coupled system compressible isentropic Navier-Stokes equation and Poisson equation. Firstly, we derive the coupled Navier-Stokes-Poisson system by applying Energy Variational Approach, then prove the global weak solutions of compressible isentropic compressible Navier-Stokes-Poisson system converge to the linearized system.

Our main result of this article reads as follows.

**Theorem 1.1** (i) If we have the initial condition (2.3), then $\frac{\rho^e - \rho^o}{\varepsilon}$ converges weakly(weakly-* in $L^r(0,\infty;L^r)$ with $r = min(2,\gamma)$ to $g$, $\frac{u^e}{\varepsilon}$ converges weakly in $L^r(0,\infty;H^r_\varepsilon)$ to $v$ and $\sqrt{\frac{\rho^\varepsilon u^\varepsilon}{\varepsilon}}$ converges weakly(weakly-* in $L^r(0,\infty;L^r)$ to $\nu$ for all $T \in (0,\infty)$, where $(g,\nu)$ is the unique solution of

$$
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + a \nabla \rho^\varepsilon + \rho \nabla \phi &= \mu \Delta u + (\mu + \kappa) \nabla \text{div} u, \\
-\varepsilon \Delta \phi &= \rho - 1,
\end{aligned}
$$

(1.1)

Moreover, $(g,\nu)$ satisfies $g \in C([0,\infty);L^2)$ and $\nu \in C([0,\infty);L^3) \cap L^3(0,\infty;H^1_\varepsilon)$.

(ii) If the initial condition (2.4) holds, then we obtain $\frac{\rho^e - \rho^o}{\varepsilon}$ converges to $g$ in $L^r(0,\infty;L^r)$, $\frac{u^e}{\varepsilon}$ converges to $\nu$ in $L^r(0,\infty;H^r_\varepsilon)$ and $\sqrt{\frac{\rho^\varepsilon u^\varepsilon}{\varepsilon}}$ converges in to $\nu$ in $L^r(0,\infty;L^r)$ for all $T \in (0,\infty)$.

This article is organized as follows. In section 2, we derive coupling with the compressible isentropic Navier-Stokes equations and the poisson equation by applying the Energy Variational Approach and Section 3 is given the proof of the main Theorem 1.1.

2. Derivation of models

In this section, we will derive the following compressible Navier-Stokes-Poisson system by using the Energetic Variational Approach

$$
\begin{aligned}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + a \nabla \rho^\varepsilon + \rho \nabla \phi &= \mu \Delta u + (\mu + \kappa) \nabla \text{div} u, \\
-\varepsilon \Delta \phi &= \rho - 1,
\end{aligned}
$$

(2.1)

here the unknown variables $\rho$ denotes the electron mass density, $u$ represent the electron velocity, $\phi$ means the electrostatic potential, the physical parameters $\mu$ and $\kappa$ are the viscosity coefficients, $a > 0, \gamma > 1$ are given. Physically, the constant $\varepsilon \in (0,1)$ can be chosen to be proportional to the Debye length.
We first introduce the following the first laws of thermodynamics \[ \frac{d(K + I)}{dt} = \frac{dW}{dt} + \frac{dQ}{dt} \] and the second laws of thermodynamics \[ \frac{dS}{dt} = \frac{dQ}{t} + \Delta, \] where \( K \) is the kinetic energy, \( I \) is the internal energy, \( W \) is the work of external force, \( Q \) is the heat, \( T \) is the temperature, \( S \) is the entropy, and \( \Delta \geq 0 \) is the entropy production. We assume the second law of thermodynamics is given in the isothermal case and there no external forces are applied. Next, combining the first and second law of thermodynamics with the conditions that being isothermal and no external forces in the process, then we obtain the energy dissipation law \[ \frac{dE^{\text{total}}}{dt} + \Delta = 0, \] where the total energy \( E^{\text{total}} = K + I - TS \) includes both kinetic and free internal energy and \( \Delta \) is the dissipation functional which is modeled as a quadratic function of certain rates such as the velocity \( x \).

In order to get the system (2.1), we consider the energy dissipation law describing fluid

\[
\frac{d}{dt}\int_{\Omega} \left[ \frac{1}{2} \rho |u|^2 + \frac{a}{\gamma - 1} (\rho' - 1) + \frac{\epsilon}{2} |\nabla \phi|^2 \right] dx = -\int_{\Omega} \left[ \mu |\nabla u|^2 + (\mu + \kappa) |\nabla \varphi|^2 \right] dx,
\]

Which implies that the total energy is \( E^{\text{total}} = \int_{\Omega} \left[ \frac{1}{2} \rho |u|^2 + \frac{a}{\gamma - 1} (\rho' - 1) + \frac{\epsilon}{2} |\nabla \phi|^2 \right] dx, \)

where we used the Gauss’s law \(-\epsilon \Delta \varphi = \rho - 1\). By using the fundamental solution of Poisson equation, we have \( \phi(x) = \frac{1}{\epsilon} \int_{\Omega} G(x-z)(\rho-1)(z)dz \), where the kernel \( G(x-z) \) is fundamental solution of \(-\Delta\).

On the other hand, the entropy production is \( \Delta = \int_{\Omega} \left[ \mu |\nabla u|^2 + (\mu + \kappa) |\nabla \varphi|^2 \right] dx \), where \( \mu > 0, \mu + \kappa \geq 0 \).

Let us define the action functionals as

\[ A := \int_{\Omega} \left[ \frac{1}{2} \rho |u|^2 - \frac{a}{\gamma - 1} (\rho' - 1) + \frac{\epsilon}{2} |\nabla \phi|^2 \right] dx dt =: A_1 + A_2 + A_3, \]

By using the Least Action Principle, as in [6, 8, 15, 16, 17], we take the variation of \( A \) with respect to \( x \) to obtain \[ 0 = \frac{d}{d\eta} A_1 (x + \eta y) + \frac{d}{d\eta} A_3 (x + \eta y) + \frac{d}{d\eta} A_4 (x + \eta y). \]

The following \( \Omega^x \) is the Lagrangian reference domain of \( \Omega \). As we known in [6], Eulerian coordinates are used to fluids materials, and for solid materials, we consider Lagrangian coordinates. However, when we use the description the other way around, we need to change coordinates to choose the best description for certain models.

For the \( A_1 \), \( \frac{d}{d\eta} A_1 (x + \eta y) = \int_{\Omega} \rho_0(X) x_i \ y_j \ dx \) with the variation of \( A_1 \) with respect to \( x \) to get \[ 0 = \frac{d}{d\eta} A_1 (x(X,t) + \eta y(X,t)) = \frac{d}{d\eta} \int_{\Omega} \rho_0(X) x_i \ y_j \ dx \] for any \( y(X,t) = \tilde{y}(x(X,t),t) \) smooth with compact support. So we can obtain \[ \delta A_1 = -\rho(u \cdot \nabla u + u_i). \] Similar as \( A_1 \), we calculate \( A_2 \) and \( A_4 \) by applying Least Action Principle, we
get \( \frac{\delta A}{\delta x} = -a\nabla \rho^e \) and \( \frac{\delta A}{\delta x} = -\rho \nabla \phi. \) According to the conservation system gives the conservative force [8], we obtain the conservative force \( F_{\text{con}} = \frac{\delta A}{\delta x} = -(\rho u), -\text{div} (\rho u \otimes u) - a\nabla \rho^e - \rho \nabla \phi. \)

In order to lead the dissipative force for this dissipative system, we use the Maximum Dissipation Principle [3, 4, 6, 8, 18], i.e., by taking the variation with respect to the velocity \( u \). We take the variation with respect to \( u \), we have \( \frac{d}{d\eta} \int_\Omega \frac{1}{2} \Delta (u + \eta v) = \int_\Omega - (\mu \Delta u + (\mu + \kappa) \nabla \text{div} u) \cdot v \). Thus, we obtain the dissipative force \( F_d = \frac{\delta}{\delta u} \left( \frac{1}{2} \Delta \right) = -(\mu \Delta u + (\mu + \kappa) \nabla \text{div} u). \)

The final step is to obtain the equation by combining the force balance law, then we have \((\rho u) + \text{div} (\rho u \otimes u) + a\nabla \rho^e + \rho \nabla \phi = (\mu \Delta u + (\mu + \kappa) \nabla \text{div} u). \) Hence, we obtain the equations (2.1).

Let the solution of (2.1) depend on a small parameter \( \varepsilon \in (0, 1) \), so we have the approximate system

\[
\begin{align*}
\rho^e + \text{div}(\rho^e u^e) &= 0, \quad \text{in } \Omega \times (0, \infty), \\
(\rho^e u^e) + \text{div}(\rho^e u^e \otimes u^e) + a\nabla (\rho^e) &= \mu \Delta u^e + (\mu + \kappa) \nabla \text{div} u^e, \quad \text{in } \Omega \times (0, \infty), \\
-\varepsilon \Delta \phi^e &= \rho^e - 1, \quad \text{in } \Omega \times (0, \infty).
\end{align*}
\]

If a solution \((\rho^e, u^e, \phi^e)\) of (2.2) satisfies \( \rho^e \in \dot{L}^\infty(0, \infty; E), \quad \rho^e | u^e | \in \dot{L}^2(0, \infty; L^2), \) \( \rho^e u^e \in C([0, \infty); E^{2n} - w), \) and \( \phi^e \in \dot{L}^2([0, \infty); W^{2, 2}). \) The existence of such solution was proved in [11, 13, 19, 20].

We suppose \( 0 \leq \rho^e(x, 0) = \rho_0^e \in \dot{L} \cap E, \quad \rho_0^e u^e(x, 0) = e_0^e \in E^{2n} \) and \( \frac{|e_0^e|^2}{\rho_0^e} \in E \), and assume \( e_0^e = 0 \) and \( \frac{|e_0^e|^2}{\rho_0^e} = 0 \) if we consider on the condition \( \{\rho_0^e = 0\} \). We also suppose the initial datas satisfy the following condition either

\[
\begin{align*}
\int_\Omega \frac{1}{\varepsilon^2} \left[ (\rho^e) - \gamma (\rho^e - 1) \rho^e + (\gamma - 1) (\rho^e) \right] &\leq C, \quad \rho^e = f_\alpha \rho_0^e \rightarrow 1, \quad \frac{\rho_0^e - \rho^e}{\varepsilon} \rightarrow f_0^e w - E, \\
\int_\Omega \rho_0^e |u_0^e|^2 &\leq C \varepsilon^2, \quad \frac{\rho_0^e}{\varepsilon} \rightarrow \frac{|e_0^e|^2}{\rho_0^e} \rightarrow f_0^e w - E, \quad \phi_0^e \rightarrow f_0^e \text{ in } H^1_0,
\end{align*}
\]

or

\[
\begin{align*}
\int_\Omega \frac{1}{\varepsilon^2} \left[ (\rho^e) - \gamma (\rho^e - 1) \rho^e + (\gamma - 1) (\rho^e) \right] &\rightarrow \gamma(\gamma - 1) \int_\Omega f_0^e \rho_0^e \rightarrow 1, \\
\rho_0^e - \rho^e &\rightarrow f_0 \in L^\infty, \quad \frac{\rho_0^e}{\varepsilon} \rightarrow \frac{|e_0^e|^2}{\rho_0^e} \rightarrow f_0 \text{ in } L^\infty, \phi_0^e \rightarrow f_0 \text{ in } H^1_0.
\end{align*}
\]

3. Proof of Theorem 1.1

The proof of the theorem 1.1 is similar to Lions and Masmoudi (1998), but we need to consider the Poisson equation.

Proof. Now we divide the proof into two steps. Firstly, we want to obtain the energy estimates.
In order to obtain the energy inequality of the equations (2.4), we denote \( v^\varepsilon = \frac{u^\varepsilon}{\varepsilon} \) and 

\[
\Phi^\varepsilon = \frac{\Phi^\varepsilon}{\varepsilon},
\]

then the new equations yield

\[
\begin{align*}
\rho^\varepsilon + \varepsilon \text{div}(\rho^\varepsilon u^\varepsilon) &= 0, \text{ in } \Omega \times (0,\infty), \\
(\rho^\varepsilon v^\varepsilon) + \varepsilon \text{div}(\rho^\varepsilon v^\varepsilon) + \frac{a}{\varepsilon} \nabla (\rho^\varepsilon) &+ e \rho^\varepsilon \nabla \Phi^\varepsilon = \mu \Delta v^\varepsilon + (\mu + \kappa) \nabla \text{div} v^\varepsilon, \text{ in } \Omega \times (0,\infty), \\
-\varepsilon^3 \Delta \Phi^\varepsilon &= \rho^\varepsilon - 1, \text{ in } \Omega \times (0,\infty).
\end{align*}
\]

Multiplying the second equation of the system (3.1) by \( v^\varepsilon \), integrating by parts and utilizing the first equation of the equations (3.1), then we have

\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \rho^\varepsilon |v^\varepsilon|^2 + \int_\Omega \frac{a}{\varepsilon} \nabla (\rho^\varepsilon) \cdot v^\varepsilon + \varepsilon \int_\Omega \rho^\varepsilon \nabla \Phi^\varepsilon \cdot v^\varepsilon + \mu \int_\Omega |\nabla v^\varepsilon|^2 + (\mu + \kappa) \int_\Omega \text{div} v^\varepsilon = 0.
\]

For the second term on the left side of above equality

\[
\int_\Omega \frac{a}{\varepsilon} \nabla (\rho^\varepsilon) \cdot v^\varepsilon \leq -\frac{a}{\varepsilon} \frac{d}{dt} \int_\Omega \left[ (\rho^\varepsilon)^\gamma \gamma + (\gamma - 1)(\rho^\varepsilon)^\gamma \right] \left[ \frac{\gamma}{\varepsilon} \nabla \gamma \right] \left[ (\rho^\varepsilon)^\gamma \gamma + (\gamma - 1)(\rho^\varepsilon)^\gamma \right] + \frac{\gamma}{\varepsilon} \int_\Omega (\rho^\varepsilon)^\gamma \cdot v^\varepsilon,
\]

where we used \( (\rho^\varepsilon)^\gamma + (\gamma - 1)(\rho^\varepsilon)^\gamma = (\rho^\varepsilon)^\gamma + (\gamma - 1)(\rho^\varepsilon)^\gamma + (\gamma - 1)(\rho^\varepsilon)^\gamma + (\gamma - 1)(\rho^\varepsilon)^\gamma \) and \( t \to t^\gamma \) is convex on \([0,\infty]\) since \( \gamma \geq 1 \). Thus, we get

\[
\int_\Omega \frac{1}{2} \rho^\varepsilon |v^\varepsilon|^2 + \frac{a}{\varepsilon} \frac{d}{dt} \int_\Omega \left[ (\rho^\varepsilon)^\gamma \gamma + (\gamma - 1)(\rho^\varepsilon)^\gamma \right] \left[ \frac{\gamma}{2} \nabla \gamma \right] \left[ (\rho^\varepsilon)^\gamma \gamma + (\gamma - 1)(\rho^\varepsilon)^\gamma \right] + \mu \int_\Omega \text{div} v^\varepsilon = 0
\]

for all \( t \in (0,\infty) \), which implies that \( v^\varepsilon \in L^2(0,\infty; \mathcal{L}_1) \), \( \sqrt{\rho^\varepsilon} v^\varepsilon \in L^2(0,\infty; \mathcal{L}_2) \), also yields a bound on 

\[
\int \frac{1}{\varepsilon} \left[(\rho^\varepsilon)^\gamma \gamma + (\gamma - 1)(\rho^\varepsilon)^\gamma \right] \text{ in } L^2(0,\infty; \mathcal{L}), \text{ a bound on } \Phi^\varepsilon \text{ in } L^2(0,\infty; \mathcal{L}_1).
\]

Then we want to prove the convergence. Similarly as the proof of Theorem II.1 in [11] and Theorem 2.3 in [13], we find the convergence of \( \rho^\varepsilon \) to 1 in \( C([0,T]; \mathcal{L}) \) for all \( T \in (0,\infty) \). To state the main result of this paper, we denote the density \( g^\varepsilon = \frac{1}{\varepsilon} (\rho^\varepsilon - \overline{\rho}) \), Then we can rewrite the equation (3.1) as follows

\[
\begin{align*}
g^\varepsilon + \text{div}(\rho^\varepsilon v^\varepsilon) &= 0, \text{ in } \Omega \times (0,\infty), \\
(\rho^\varepsilon v^\varepsilon) + \text{div}(\rho^\varepsilon v^\varepsilon) + a \varepsilon \nabla \left[(\rho^\varepsilon)^\gamma \gamma + (\gamma - 1)(\rho^\varepsilon)^\gamma \right] + b(\overline{\rho})^\gamma \nabla g^\varepsilon + e \rho^\varepsilon \nabla \Phi^\varepsilon \\
\mu \Delta v^\varepsilon + (\mu + \kappa) \nabla \text{div} v^\varepsilon &+ b(\overline{\rho})^\gamma \nabla g^\varepsilon + e \rho^\varepsilon \nabla \Phi^\varepsilon \\
-\varepsilon^3 \Delta \Phi^\varepsilon &= \rho^\varepsilon - 1, \text{ in } \Omega \times (0,\infty),
\end{align*}
\]

where \( b = a \varepsilon^2 \), \( a > 0, \gamma > 1 \). Extracting subsequences if necessary, we suppose \( g^\varepsilon \) converges weakly (weakly-*') to \( g \) in \( L^\gamma(0,\infty; \mathcal{L}) \) with \( r = \min(2,\gamma) \), and \( g \in L^\gamma(0,\infty; \mathcal{L}) \). \( v^\varepsilon \) converges weakly to \( v \) in \( L^\gamma(0,\infty; \mathcal{L}) \), and \( \sqrt{\rho^\varepsilon} v^\varepsilon \) also converges weakly (weakly-*') to the \( v \) in \( L^\gamma(0,\infty; \mathcal{L}) \) since
\( \rho^e \) converges to 1 in \( C([0,T];L^r) \). By applying the fact that \( \gamma \geq \frac{N}{2} \) if \( N \geq 3 \) and \( \gamma > 1 \) if \( N = 2 \), and the above bounds, we can obtain (1.1). Multiplying the second equation of the system (1.1) by \( \nu \).

We deduce that
\[
\frac{1}{2} \frac{d}{dt} \int_\Omega \left( |v|^r + b \cdot g^r \right) \leq 0,
\]
which yields the uniqueness and allow to show that \( g \in C([0,\infty);L^r), \nu \in C([0,\infty);L^r) \). Finally, the strong convergences are deduced from the energy inequalities of systems (1.1) and (3.1). We also obtain \( g^r \) converges to \( g \) in \( L \), the proof of which can be found in [11]. We complete the proof.

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