Chern-Simons versus dipolar composite fermions at finite wavevector

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It was recently shown that dipolar composite fermions emerged from the lowest-Landau-level formulation of the quantum Hall effect give rise to similar results as those of the Chern-Simons gauge theory in the long wavelength and low energy limit. We ask whether this correspondence is still valid at finite wavevectors where the excitations do not necessarily look like dipolar quasiparticles. In particular, \( q = 2k_F \) density-density response function of the compressible state at \( \nu = 1/2 \) is evaluated in the low energy limit within the framework of the lowest-Landau-level theory. The imaginary parts of the density-density response functions at \( q = 2k_F \) of two theories have the same \( \sqrt{\omega} \) dependence. However, the coefficient of the \( \sqrt{\omega} \) term in the case of the lowest-Landau-level theory is not universal and can be much smaller than the corresponding coefficient in the Chern-Simons theory. We also discuss possible connection between these results and the recent experiment on phonon-mediated drag in the double-layer \( \nu = 1/2 \) system.

I. INTRODUCTION

The compressible state of two dimensional electrons at the filling factor \( \nu = 1/2 \) has been explained as the result of Fermi surface formation of quasiparticles called composite fermions [4,5]. Here the filling factor, \( \nu \), is the ratio of the number of electrons and the total number of external flux quanta given by the external magnetic field. Thus there exist \( \phi \) number of external flux quanta per electron if \( \nu = 1/\phi \). The composite fermions are bound objects of electrons and their correlation holes. Due to the fact that the electron wavefunction vanishes at the positions of correlation holes and there exists a phase winding of the electron wavefunction around them, these correlation holes are also called vortices. The bound state of electrons and even number of vortices satisfies Fermi statistics leading to the name “composite fermions” [5].

Upon the discovery of the compressible states in the half-filled Landau level, it was soon proposed that the vortices can be represented by fictitious flux quanta pointing in the opposite direction of the external magnetic field [4]. In this case, the bound state of an electron and two fictitious flux quanta would see zero effective magnetic field on average at \( \nu = 1/2 \) [6]. This is because the number of external flux quanta per electron and that of fictitious flux quanta are the same. Now the composite fermion system can form a Fermi sea as if they are free fermions in zero magnetic field. These ideas can be formulated in terms of the Chern-Simons gauge theory that represents the fictitious flux quanta by a \( U(1) \) gauge field and requires the position of these flux quanta to be the same as those of electrons. This “Chern-Simons composite fermion” theory explained the finite compressibility of the system and various experimental results that require understanding of the long wavelength and low energy properties of the system [4].

In spite of this success, the Chern-Simons composite fermion theory suffers from some important problems. At the most naive level, the flux attachment transformation uses the kinetic energy of the electrons to generate the kinetic energy of composite fermions. If this transformation is taken seriously, the mass of the composite fermions is given by the bare mass of electrons. This is certainly not correct because, in the lowest Landau level, the kinetic energy of electrons is quenched and the only relevant energy scale is given by the interaction energy; the mass of the composite fermions should be determined by the interaction energy scale [4]. Therefore the kinetic energy term with the bare mass in the Hamiltonian should be somehow replaced by the term with the interaction induced mass [7,8]. In the Chern-Simons theory approach, the kinetic energy term with the interaction induced mass has been used on phenomenological ground [7]. One hopes that the final answer will be correct in the limit of the zero bare mass (infinite cyclotron energy).

Sometime ago, more microscopic picture was suggested to resolve the unsatisfactory aspects of the Chern-Simons theory [7]. In the lowest Landau level, the system of \( N \) number of composite fermions maintains the antisymmetry of the total wavefunction by assigning a different vector \( \mathbf{k}_j (j = 1, ..., N) \) to each composite fermion such that the distance between the electron and vortices in a given composite fermion \( j \) is \( |\mathbf{k}_j|/l_B \) and the direction of \( \mathbf{k}_j \) is perpendicular to the line connecting the positions of electron and vortices [8]. Here \( l_B = \sqrt{\hbar c/eB} \) is the magnetic length. The electron and vortices within each composite fermion drift along the equipotential, \( V(|\mathbf{k}|) \), in the same direction keeping the distance, \( |\mathbf{k}|/l_B \), between them. This can generate the dispersion (or “kinetic energy”) of the composite fermions [8]. Also the vectors \( \mathbf{k}_j \), form a Fermi sea in \( \mathbf{k} \)-space with a well defined Fermi wavevector \( k_F = \sqrt{4\pi n_e} \) where \( n_e \) is the
density of electrons. Since the “kinetic energy” arises from the interaction potential, the effective mass at the Fermi wavevector is determined by the interaction energy scale [3]. These composite fermions are called “dipolar composite fermions” due to the dipolar internal structure [4][2].

The ideas outlined above have been recently formulated in several different ways [6][3]. The essential parts of the formulation can be summarized as follows [6][3]. We consider the case of $\nu = 1/\phi$ for simplicity. Let $R_{ej}$ be the position of the electron $j$ and $R_{ej}$ be the position of $\nu$ number of vortices associated with the electron $j$. Equivalently one can regard the complex coordinates of be the position of the electron

$$\bar{\rho} \sim \bar{\rho}(2\pi)^2 \delta(q) = \frac{\bar{\rho}^2}{2} \sum_j e^{i\mathbf{q} \cdot \mathbf{r}_j} \mathbf{v}_j \cdot e^{-i\mathbf{q} \cdot \mathbf{r}_j}.$$ \hfill (1.8)

In particular, the $i = j$ term will give rise to

$$H \sim \frac{1}{m^*} \int \frac{d^2k}{(2\pi)^2} \hat{V}(q)q^2l_B^2.$$ \hfill (1.9)

This may be taken as the origin of the kinetic energy term of the composite fermions in the long wavelength limit [7][2].

It turns out that it is very important to incorporate the residual interaction terms and the constraint to satisfy proper conservation laws. For example, the original system of dipolar composite fermions is invariant under $\mathbf{k}_j \rightarrow \mathbf{k}_j + \mathbf{K}$ for all $j$ and any $\mathbf{K}$ because it just corresponds to a constant shift in the center of mass coordinate of the entire system [4][1]. This symmetry is intimately related to the fact that the kinetic energy term of the electrons is absent or the system is in the infinite bare mass limit $m_b \rightarrow \infty$. This, for example, leads to $F_1 = -1$ among Landau parameters via $m^*/m_b = 1 + F_1$ if the dipolar composite fermions are treated as quasi-particles in the Landau Fermi liquid theory [4][1]. The approximate Hamiltonian in Eq.1.3, however, breaks this symmetry. While this problem can be fixed in the long wavelength and low energy limits by incorporating the constraints [4][1], it was also shown that one can formulate the theory in a conserving approximation for all $\mathbf{q}$ and $\omega$ (more specifically for $\nu = 1$ bosons) [1]. In any case, the conserving approximation leads to the conclusion that the system is still compressible [4][1]. Also various physical response functions in the low energy and long wavelength limit were shown to have the same forms as those of the Chern-Simons gauge theory approach. In other words, the dipolar composite fermion description in the infinite bare mass limit seems to be equivalent to the Chern-Simons composite fermion system in the zero bare mass (or infinite cyclotron energy) limit as far as
the low energy and long wavelength limits are concerned[10].

In this paper, we ask whether the close correspondence between the results of the Chern-Simons theory and those of the lowest Landau level theory is still valid at finite wavevectors where the excitations do not necessarily look like dipolar quasiparticles. We were partly motivated by the recent phonon drag experiment in the double-layer $\nu = 1/2$ system [13]. In this experiment, the drag resistivity between two layers due to the electron-phonon interaction is measured [13]. In the absence of an applied magnetic field, it has been known that the phonon drag divided by $T^2$ reaches its maximum when the temperature becomes of the order of $T_0 = c(2k_F^e)$ where $c$ is the phonon velocity and $k_F^e$ is the Fermi wavevector of the electrons in each layer [14]. This is due to the fact that the particle-hole excitations of electrons cease to exist beyond the wavevector $q = 2k_F^e$ so that the phonons with $q > 2k_F^e$ cannot scatter electrons at low temperatures. Thus the scattering between the particle-hole continuum and the phonons is suppressed when the energy scale is larger than $T_0$. In a way, this experiment can tell us the properties of the system at short distances like $(k_F^e)^{-1}$. In the case of $\nu = 1/2$ double-layer system, one may expect that the cutoff wavevector scale would be set by the composite fermion Fermi wavevector, $k_F^{cf}$ where $k_F^{cf} = \sqrt{2}k_F^e$ if all the spins of electrons are polarized due to strong magnetic field. It is expected that the maximum of the drag resistivity should occur at a temperature $T^\text{max}_{1/2}$ around $T_{1/2} = c(2k_F^{cf}) = \sqrt{2}T_0$. On the other hand, in the experiment, $T^\text{max}_{1/2}$ turns out to be even smaller than $T_0$ [13]. Recently, Bonsager, MacDonald, and the author performed a theoretical calculation of the drag resistivity in the Chern-Simons theory [14]. It was found that the maximum of the drag resistivity indeed occurs around $T_{1/2}$ if the effective Fermi energy (determined from the effective mass) of the composite fermions $\varepsilon_F$ is much larger than $T_{1/2}$ [14]. However, for realistic values of effective mass, one finds $\varepsilon_F \sim T_{1/2}$ leading to substantial finite temperature effects that were not observed in the experiment [10]. At this stage, it appears that the Chern-Simons theory does not capture the correct short distance properties of the system.

Upon this situation, it is natural to ask what happens to the composite fermions in the lowest-Landau-level theory at short distances because this approach is supposed to be more microscopic. One may expect that more microscopic theory may give rise to different results compared to the predictions of the Chern-Simons theory and eventually explain the experiment. In this paper, we will consider the density-density response function at $q = 2k_F$ at low energies and investigate possible differences between the results of the lowest Landau level theory and the Chern-Simons theory. In order to compare the theory with the drag experiment, we need to know the physical response functions at arbitrary energy and wavevector scales. Thus complete explanation of the experiment mentioned above is beyond the scope of this paper. We will, however, try to make a connection wherever it is possible and discuss what might happen in the drag experiment using our results.

The rest of the paper is organized as follows. In section II, we briefly review the formalism of the lowest-Landau-level theory. In section III, the density-density response function at $q = 2k_F$ in the low energy limit is evaluated. In section IV, these results are compared with those of the Chern-Simons theory. In section V, we discuss possible connection between our results and the phonon drag experiment. We summarize our results in section VI.

II. BRIEF REVIEW OF THE LOWEST-LANDAU-LEVEL THEORY

We set $l_B = 1$ from now on. We restore this factor explicitly wherever it is necessary. It can be shown that the density operators of electrons and vortices in Eq.1.4 can be written in two second quantized forms as [1]

$$\rho^L(q) = \int \frac{d^2k}{(2\pi)^2}e^{-\frac{i}{2}k\cdot q}c_{k+\frac{1}{2}q}^\dagger c_{k-\frac{1}{2}q} ,$$

$$\rho^R(q) = \int \frac{d^2k}{(2\pi)^2}e^{\frac{i}{2}k\cdot q}c_{k-\frac{1}{2}q}^\dagger c_{k+\frac{1}{2}q} ,$$

where $c_k, c_k^\dagger$ satisfy the fermionic anticommutation relation;

$$\{c_k, c_k^\dagger\} = (2\pi)^2\delta(k - k') .$$

These operators satisfy the following lowest-Landau-level algebra first noticed in Ref. [17].

$$[\rho^L(q), \rho^L(q')] = -2i\sin\left(\frac{q \cdot q'}{2}\right)\rho^L(q + q') ,$$

$$[\rho^R(q), \rho^R(q')] = 2i\sin\left(\frac{q \cdot q'}{2}\right)\rho^R(q + q') ,$$

$$[\rho^L(q), \rho^R(q')] = 0 .$$

The constraint in Eq.1.5 implies that

$$[\rho^R(q), \rho^L(q) - \rho(2\pi)^2\delta(q)]\Psi_{\text{phys}} = 0 .$$

One can first build up states as combinations of $\prod_{l=1}^{\text{N}} c_{k_l}^\dagger |0\rangle$ where $i = 1, ..., \text{N}$ and then project them to satisfy the constraint. Notice also that the constraint operators $G(q) = \rho^R(q) - \rho(2\pi)^2\delta(q)$ commutes with the Hamiltonian given by Eq.1.4.

Using the second quantized electron density operator, the Hamiltonian in Eq.1.4 can be rewritten as [1]

$$H = \frac{1}{2} \int \frac{d^2k_1d^2k_2d^2q}{(2\pi)^6}V(q)e^{i\delta_{k_1}q + \delta_{k_2}q} \times c_{k_1-\frac{1}{2}q}^\dagger c_{k_2+\frac{1}{2}q} c_{k_2-\frac{1}{2}q}^\dagger c_{k_1+\frac{1}{2}q} ,$$

where $\rho^L(q), \rho^R(q)$ are the density operators corresponding to the lowest Landau levels. In the case of $\nu = 1/2$, the density operators are given by Eq.1.4. In the case of $\nu = 0$, the density operators are given by the effective electron density operators.

$$\rho^L(q) = \int \frac{d^2k}{(2\pi)^2}e^{-\frac{i}{2}k\cdot q}c_{k+\frac{1}{2}q}^\dagger c_{k-\frac{1}{2}q} ,$$

$$\rho^R(q) = \int \frac{d^2k}{(2\pi)^2}e^{\frac{i}{2}k\cdot q}c_{k-\frac{1}{2}q}^\dagger c_{k+\frac{1}{2}q} .$$

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subject to the constraints $\rho^R(q) - \bar{\rho}(2\pi)^2 \delta(q) = 0$. We first use the Hartree-Fock (HF) approximation to obtain the effective kinetic energy of the quasiparticles. In the HF approximation, the effective single-particle Hamiltonian can be written as 

$$
H_{\text{eff}} = -\sum_k \xi_k c_k^\dagger c_k,
$$

(2.6)

where $\xi_k = \varepsilon_k - \mu$ and

$$
\varepsilon_k = \tilde{V}(0) \int \frac{d^2 k'}{(2\pi)^2} n^0_{k'} - \int \frac{d^2 k'}{(2\pi)^2} \tilde{V}(k - k') n^0_{k'}.
$$

(2.7)

In the ground state at zero temperature, $n^0_{k'} = 0$. Notice that $k$ dependence of $\xi_k$ comes from the Fock term in the effective Hamiltonian. When the interaction potential is repulsive, $\xi_k$ is a monotonically increasing function of $|k|$. The effective mass of the quasi-particles at the Fermi level can be obtained from [11]

$$
k_F^2 = \frac{\partial \xi_k}{\partial |k|} = -k_F \int \frac{d\theta_{kk'}}{2\pi} \tilde{V}'(k' - k) \cos \theta_{kk'},
$$

(2.8)

where $\theta_{kk'}$ is the angle between $k$ and $k'$.

The HF ground state which is just the Fermi sea $|FS\rangle$ is not annihilated by the constraint operator; $G(q)|FS\rangle \neq 0$ for $q \neq 0$. Also $G(q)$ does not commute with the HF effective Hamiltonian. Thus $G(q)$ are not conserved by the HF approximation. In order to recover the conserved quantities, we use the generalized HF theory called the time-dependent HF approximation which is the natural conserving approximation related to the HF theory.

The density-density response function in this generalized HF approximation corresponds to the sum of all ring and ladder diagrams with the single particle Green’s function given by

$$
G(k, \omega_n) = (i\omega_n - \xi_k)^{-1},
$$

(2.9)

where $\omega_n$ is the Matsubara frequency. The irreducible density-density response function of electrons, $\chi^{ir}_{LL}$, in the generalized HF approximation corresponds to the sum of the ladder diagrams. The expression for $\chi^{ir}_{LL}$ was found to have the following form [11].

$$
\chi^{ir}_{LL} = -\int \frac{d^2 k}{(2\pi)^2} (\epsilon_{k+q} - \epsilon_{k-q} - 1) \frac{f(\xi_{k+q} + \frac{1}{2}q) - f(\xi_{k-q} + \frac{1}{2}q)}{\xi_{k+q} - \xi_{k-q} - i\omega_n} + \int \frac{d^2 k}{(2\pi)^2} \frac{d^2 k'}{(2\pi)^2} (\epsilon_{k+q} - \epsilon_{k-q} - 1) \frac{f(\xi_{k+q} + \frac{1}{2}q) - f(\xi_{k-q} + \frac{1}{2}q)}{\xi_{k+q} - \xi_{k-q} - i\omega_n}
$$

\begin{equation}
\times \Gamma(k, k', q, i\omega_n) \frac{f(\xi_{k'+q} - \frac{1}{2}q) - f(\xi_{k'-q} - \frac{1}{2}q)}{\xi_{k'+q} - \xi_{k'-q} - i\omega_n} (\epsilon_{k'+q} - 1).
\end{equation}

(2.10)

Here the scattering vertex function, $\Gamma(k, k', q, i\omega_n)$, satisfies the following integral equation.

$$
\Gamma(k, k', q, i\omega_n) = \tilde{V}(k - k') - \int \frac{d^2 k_1}{(2\pi)^2} \Gamma(k, k_1, q, i\omega_n) \frac{f(\xi_{k_1+q} + \frac{1}{2}q) - f(\xi_{k_1-q} + \frac{1}{2}q)}{\xi_{k_1+q} - \xi_{k_1-q} - i\omega_n} \tilde{V}(k_1 - k').
$$

(2.11)

It is also useful to define the following one-particle-irreducible vertex function;

$$
\Lambda(k, q, i\omega_n) = 1 - \int \frac{d^2 k_1}{(2\pi)^2} \frac{f(\xi_{k_1+q} + \frac{1}{2}q) - f(\xi_{k_1-q} + \frac{1}{2}q)}{\xi_{k_1+q} - \xi_{k_1-q} - i\omega_n} \Gamma(k_1, k, q, i\omega_n).
$$

(2.12)

One can show that the conservation of the constraints is represented by the following Ward identities satisfied by $\Lambda(k, q, i\omega_n)$ [11]:

$$
i\omega_n \Lambda(k, q, i\omega_n) = i\omega_n - \xi_{k+q} - \xi_{k-q}.
$$

(2.13)

For example, using the Ward identity, one can show that $\langle \rho^R(q) \rho^L(-q) \rangle = \langle \rho^R(q) \rho^L(-q) \rangle = 0$ and so on [11].

At low temperatures, in the limit of small $q$ and $\omega_n$, the expression for $\Gamma(k, k', q, i\omega_n)$ was explicitly found as [11]

$$
\Gamma(k, k', q, i\omega_n) = \frac{q \cdot v_{k'} q \cdot v_{k'}}{\omega_n^2 \chi_0(q, i\omega_n)} - \frac{q \cdot v_k q \cdot v_{k'}}{q^2 \chi_0(q, i\omega_n)},
$$

(2.14)

where

$$
\chi_0(q, i\omega_n) = -\int \frac{d^2 k'}{(2\pi)^2} \frac{f(\xi_{k'+q} + \frac{1}{2}q) - f(\xi_{k'-q} + \frac{1}{2}q)}{\xi_{k'+q} - \xi_{k'-q} - i\omega_n},
$$

(2.15)

and

$$
\chi^\perp_0(q, i\omega_n) = -\frac{1}{2} \chi_0(q, i\omega_n) + \chi^\perp_0(q, i\omega_n),
$$

(2.16)

with

$$
\chi^\perp_0 = -\int \frac{d^2 k}{(2\pi)^2} \frac{q \cdot k}{m^*} \left( \frac{q^2}{\chi_0(q, i\omega_n)} - \frac{f(\xi_{k+q} + \frac{1}{2}q) - f(\xi_{k-q} + \frac{1}{2}q)}{\xi_{k+q} - \xi_{k-q} - i\omega_n} \right).
$$

(2.17)
Here $\mathbf{v}_k$ is the velocity and $\mathcal{N}(0)$ the density of states at the Fermi level. In the long wavelength limit, the exponential factors in the expression of $\chi_{LL}^{ir}$ can be expanded as $e^{i\mathbf{k} \cdot \mathbf{q}} - 1 \approx i\mathbf{k} \cdot \mathbf{q}$. Using the scattering vertex function in the long wavelength and low energy limits, one finds \[\chi_{LL}^{ir} \approx -q^2 \tilde{\rho} + m^* \lambda_0 (\mathbf{q}, i\omega_\nu) \chi_0 (\mathbf{q}, i\omega_\nu). \tag{2.18}\]

It leads to the similar result to that of the density-density response function in the Chern-Simons theory in the long wavelength and low energy limits (small $|\mathbf{q}|$ and $\omega$) \:\cite{11}:
\[\chi_{LL}^{ir} \approx -\chi_d^{*} - i\omega k_F/(2\pi q^2), \tag{2.19}\]

where $\chi_d^{*}$ is the diamagnetic susceptibility of the Fermi gas with the dispersion $\xi_k$. This also implies that the compressibility of the system is finite.

### III. Q=2k_F DENSITY-DENSITY RESPONSE FUNCTION

In this section, we study the $q = 2k_F$ density-density response function in the lowest-Landau-level theory. The difficulty in finding the finite wavevector response function comes from the fact that one needs to know the scattering vertex function $\Gamma (\mathbf{k}, \mathbf{k}', \mathbf{q}, i\omega_\nu)$ for arbitrary $\mathbf{q}$ and $\omega_\nu$. In principle, if one can find all the eigenvalues and eigenfunctions of the integral kernel, one can find the inverse of the integral operator and find the solution for

\[\Gamma (\mathbf{k}, \mathbf{k}', \mathbf{q}, i\omega_\nu). \tag{3.4}\]

This turns out to be not an easy task for arbitrary $\mathbf{q}$ and $\omega_\nu$.

In the small $|\mathbf{q}|$ limit, the nontrivial part of the integral kernel becomes
\[\tilde{V}(\mathbf{k}' - \mathbf{k}) \frac{\partial f}{\partial \xi_{\mathbf{k}}}, \tag{3.1}\]

which is concentrated at $k = k_F$ at zero temperature. In this limit, both $\tilde{V}(\mathbf{k} - \mathbf{k}')$ and the scattering vertex function can be expanded only in terms of $\cos \theta_k$ and $\sin \theta_k$ ($\ell = 0, 1, \ldots$) because both $\mathbf{k}$ and $\mathbf{k}'$ are basically restricted on the Fermi surface \cite{12}. If one takes only the $\ell = 1$ modes, the eigenfunctions are just $\mathbf{q} \cdot \mathbf{v}_k/q$ and $\mathbf{q} \wedge \mathbf{v}_k/q$. By finding the corresponding eigenvalues, one gets the expression of $\Gamma (\mathbf{k}, \mathbf{k}', \mathbf{q}, i\omega_\nu)$ in Eq.\[2.14 \tag{11}\]

When $\mathbf{q}$ is not small, the tricks outlined above do not work. Fortunately, we can find at least one eigenvector which is correct for all $\mathbf{q}$ in the small frequency limit. This eigenvector can be found by use of the Ward identity and is given by $\xi_k + \frac{1}{2} q - \xi_k - \frac{1}{2} q$ with the eigenvalue tending to zero as $i\omega_\nu \rightarrow 0$. This part of the vertex function can be explicitly found as
\[\Gamma_1 (\mathbf{k}, \mathbf{k}', \mathbf{q}, i\omega_\nu) = \frac{\left(\xi_k + \frac{1}{2} q - \xi_k - \frac{1}{2} q\right) - \left(\xi_{k'} + \frac{1}{2} q - \xi_{k'} - \frac{1}{2} q\right)}{\omega_\nu^2 \chi_0 (\mathbf{q}, i\omega)} . \tag{3.2}\]

One can also show that $\Gamma_1 (\mathbf{k}, \mathbf{k}', \mathbf{q}, i\omega_\nu)$ exhausts the Ward identity in Eq.\[2.13 \tag{11}\] in the following sense. Let us consider the following quantity.

\[\tilde{\Lambda} (\mathbf{k}, \mathbf{q}, i\omega_\nu) = 1 - \int \frac{d^2 k_1}{(2\pi)^2} \frac{f (\xi_{k_1} + \frac{1}{2} q) - f (\xi_{k_1} - \frac{1}{2} q) + \Gamma_1 (\mathbf{k}_1 , \mathbf{k}_1, \mathbf{q}, i\omega_\nu)}{\omega_\nu^2 \chi_0 (\mathbf{q}, i\omega)} . \tag{3.3}\]

where the definition of $\chi_0$ in Eq.\[2.13 \tag{11}\] is used in going from the third to fourth lines. Notice also that, among two terms in the curly bracket in the third line, the first term gives zero contribution to the $\mathbf{k}_1$-integral. Comparing with the Ward identity in Eq.\[2.13 \tag{11}\] one can see that $\tilde{\Lambda} (\mathbf{k}, \mathbf{q}, i\omega_\nu)$ is nothing but the one-particle-irreducible vertex $\Lambda (\mathbf{k}, \mathbf{q}, i\omega_\nu)$. Therefore, the full scattering vertex function can be written as
\[\Gamma (\mathbf{k}, \mathbf{k}', \mathbf{q}, i\omega_\nu) = \Gamma_1 (\mathbf{k}, \mathbf{k}', \mathbf{q}, i\omega_\nu) + \Gamma_2 (\mathbf{k}, \mathbf{k}', \mathbf{q}, i\omega_\nu) \tag{3.4}\]

such that
\[\int \frac{d^2 k_1}{(2\pi)^2} \frac{f (\xi_{k_1} + \frac{1}{2} q) - f (\xi_{k_1} - \frac{1}{2} q)}{\xi_{k_1} + \frac{1}{2} q - \xi_{k_1} - \frac{1}{2} q - i\omega_\nu} \Gamma_2 (\mathbf{k}_1, \mathbf{k}, \mathbf{q}, i\omega_\nu) = 0 . \tag{3.5}\]

Now, for notational convenience, let us rewrite the irreducible density-density response function as follows.
\[ \chi_{LL}^\gamma(q, i\omega_\nu) = \chi_a(q, i\omega_\nu) + \chi_b(q, i\omega_\nu) + \chi_c(q, i\omega_\nu), \tag{3.6} \]

where

\[
\chi_a = -\int \frac{d^2k}{(2\pi)^2} (e^{i\mathbf{k}\cdot\mathbf{q}} - 1)(e^{-i\mathbf{k}\cdot\mathbf{q}} - 1) \frac{f(\xi_{k+\mathbf{q}}) - f(\xi_{k-\mathbf{q}})}{\xi_{k+\mathbf{q}} - \xi_{k-\mathbf{q}} - i\omega_\nu},
\]

\[
\chi_b = \int \frac{d^2k d^2k'}{(2\pi)^4} (e^{i\mathbf{k}\cdot\mathbf{q}} - 1) \frac{f(\xi_{k+\mathbf{q}}) - f(\xi_{k-\mathbf{q}})}{\xi_{k+\mathbf{q}} - \xi_{k-\mathbf{q}} - i\omega_\nu} \Gamma_1(k, k', q, i\omega_\nu) \frac{f(\xi_{k' + \mathbf{q}}) - f(\xi_{k' - \mathbf{q}})}{\xi_{k' + \mathbf{q}} - \xi_{k' - \mathbf{q}} - i\omega_\nu},
\]

\[
\chi_c = \int \frac{d^2k d^2k'}{(2\pi)^4} (e^{i\mathbf{k}\cdot\mathbf{q}} - 1) \frac{f(\xi_{k+\mathbf{q}}) - f(\xi_{k-\mathbf{q}})}{\xi_{k+\mathbf{q}} - \xi_{k-\mathbf{q}} - i\omega_\nu} \Gamma_2(k, k', q, i\omega_\nu) \frac{f(\xi_{k' + \mathbf{q}}) - f(\xi_{k' - \mathbf{q}})}{\xi_{k' + \mathbf{q}} - \xi_{k' - \mathbf{q}} - i\omega_\nu}.
\tag{3.7}
\]

Let us investigate \( \chi_a \) and \( \chi_b \) first, then consider \( \chi_c \) later. When the momentum transfer is given by \( \mathbf{q} = \mathbf{Q}_0 \equiv 2k_F \hat{\mathbf{q}} \), one can see that \( \mathbf{k} \pm \frac{1}{2} \mathbf{Q}_0 \) can be rewritten as

\[
\mathbf{k} \pm \frac{1}{2} \mathbf{Q}_0 = \pm k_F \hat{\mathbf{q}} + \mathbf{p} + \mathbf{q} . \tag{3.8}
\]

We can also easily see that \( \mathbf{k} \wedge \mathbf{Q}_0 = \mathbf{p} \wedge \mathbf{Q}_0 = 2k_F \mathbf{p} \wedge \hat{\mathbf{q}} \), where \( \mathbf{p} \wedge \hat{\mathbf{q}} \) is the component of \( \mathbf{p} \) perpendicular to \( \hat{\mathbf{q}} \).

In the low energy limit \( |\omega| \ll \varepsilon_F \) (here \( \varepsilon_F \equiv \varepsilon_{|\mathbf{k}|=k_F} \)), we have \( \mathbf{p} \wedge \hat{\mathbf{q}} \ll k_F \) in the imaginary part of the analytically-continued response functions \( \chi(q, i\omega_\nu \rightarrow \omega + i0^+) \). In this case, restoring \( l_B^2 = (2/\phi)k_F^2 \) in the expression, the exponential factor in \( \chi_a \) can be expanded as

\[
e^{i\mathbf{q}\cdot\mathbf{k}} - 1 = e^{i\mathbf{q}\cdot2k_F \mathbf{p} \wedge \hat{\mathbf{q}}} - 1 \approx i l_B^2 2k_F \mathbf{p} \wedge \hat{\mathbf{q}} . \tag{3.9}
\]

Thus the imaginary part of \( \chi_a(Q_0, \omega) \) in the low energy limit can be written as

\[
\text{Im} \chi_a(Q_0, \omega) \approx 2 \left( \frac{2}{\phi} \right)^2 \left( \frac{m*}{k_F} \right)^2 \text{Im} \chi_{0q}'(Q_0, \omega), \tag{3.10}
\]

where \( \chi_{0q}'(q, \omega) \) is given by Eq.2.17. After changing the integral variable from \( \mathbf{k} \) to \( \mathbf{p} \) defined via \( \mathbf{k} \pm \frac{1}{2} \mathbf{Q}_0 = \pm k_F \hat{\mathbf{q}} + \mathbf{p} \), one can take the low frequency approximation \( |\omega| \ll \varepsilon_F \) and \( \mathbf{p} \wedge \hat{\mathbf{q}} \ll k_F \) to find the imaginary part of \( \chi_{0q}'(Q_0, \omega) \). Here we can use

\[
\xi_k = \xi(|k|) = \xi(|k_F \hat{\mathbf{q}} + \mathbf{p}|) \\
\approx \xi \left( \frac{k_F - \mathbf{p} \cdot \hat{\mathbf{q}} + \frac{1}{2} (\mathbf{p} \wedge \hat{\mathbf{q}})^2}{k_F} \right) \\
\approx \xi_{k_F} + \left( - \mathbf{p} \cdot \hat{\mathbf{q}} + \frac{1}{2} (\mathbf{p} \wedge \hat{\mathbf{q}})^2 \right) \left( \frac{\partial \xi_k}{\partial |k|} \right)|_{|k|=k_F} \\
= - \frac{k_F}{m^*} \mathbf{p} \cdot \hat{\mathbf{q}} + \frac{1}{2} \frac{(\mathbf{p} \wedge \hat{\mathbf{q}})^2}{m^*}. \tag{3.11}
\]

Similarly, we get

\[
\xi_{k+\mathbf{q}} \approx \frac{k_F}{m^*} \mathbf{p} \cdot \hat{\mathbf{q}} + \frac{1}{2} \frac{(\mathbf{p} \wedge \hat{\mathbf{q}})^2}{m^*}. \tag{3.12}
\]

After finding the imaginary part, the leading order behavior of the real part can be also found by the Kramers-Kronig relation with a frequency cutoff that depends on the details of the interaction potential. Finally, we get

\[
\chi_{0q}^\gamma(Q_0, \omega) \approx \frac{m^*}{2\pi} \left( \frac{k_F}{m^*} \right)^2 \left( \frac{m^*\xi_F}{k_F^2} \right)^{3/2} \left[ C_1 \left( \frac{m^*\xi_F}{k_F^2} \right)^{3/2} + i \frac{1}{6} \left( \frac{m^*\omega}{k_F^2} \right)^{3/2} \right], \tag{3.13}
\]

where \( C_1 \) is a constant that depends on the details of the single-particle spectrum \( \xi_k \). Recall that \( \varepsilon_F = \varepsilon_{|k|=k_F} \). In the case of the quadratic band, \( \xi_k = k^2/2m^* - \mu, C_1 = 1/6 \) and \( m^*\xi_F/k_F^2 = 1 \). Using Eq.3.13 the low frequency limit of \( \text{Im} \chi_a(Q_0, \omega) \) can be estimated as

\[
\text{Im} \chi_a(Q_0, \omega) \approx \left( \frac{2}{\phi} \right)^2 \left( \frac{m^*}{6\pi} \right)^{3/2} \left( \frac{m^*\omega}{k_F^2} \right)^{3/2}. \tag{3.14}
\]

Similarly, \( \text{Im} \chi_b(Q_0, \omega) \) can be evaluated by expanding the exponential factors. Notice, however, that the expansion in the linear order in \( il_B^2 2k_F \mathbf{p} \wedge \hat{\mathbf{q}} \) will give zero contribution to the integral because of the symmetry of the integrand. Thus one has to keep the terms that are second order in \( il_B^2 2k_F \mathbf{p} \wedge \hat{\mathbf{q}} \). After some algebra, we obtain the following result in the low frequency limit.

\[
\text{Im} \chi_b(Q_0, \omega) \approx -4 \left( \frac{2}{\phi} \right)^4 \left( \frac{m^*}{k_F} \right)^4 \text{Im} \left\{ \frac{\chi_{0q}^\gamma(Q_0, \omega)^2}{\chi_0(Q_0, \omega)} \right\}. \tag{3.15}
\]

In the low frequency limit, one can also show that
\[ \chi_0(Q_0, \omega) \approx \frac{m^*}{2\pi} \left[ D_1 \left( \frac{2m^* \varepsilon_F}{k_F^2} \right)^{1/2} + \frac{1}{2} \left( \frac{m^* \omega}{k_F^2} \right)^{1/2} \right], \]

(3.16)

where \( D_1 \) is a constant that depends on the details of the single-particle spectrum \( \xi_k \). In the case of the quadratic band with the effective mass \( m^* \), we get \( D_1 = 1 \) with \( 2m^* \varepsilon_F/k_F^2 = 1 \). Using Eq.3.13 and Eq.3.16, the leading order contribution to \( \text{Im} \chi_0(Q_0, \omega) \) can be estimated as

\[ \text{Im} \chi_0(Q_0, \omega) \approx 4 \left( \frac{2}{\phi} \right)^4 \left( \frac{m^*}{k_F^2} \right)^4 \left( \frac{\text{Re} \chi_{\phi p}(Q_0, \omega)}{\text{Re} \chi_0(Q_0, \omega)} \right)^2 \text{Im} \chi_0(Q_0, \omega) \]

\[ \approx \left( \frac{2}{\phi} \right)^4 C^2 \left( \frac{2m^* \varepsilon_F}{k_F^2} \right)^2 \left( \frac{m^* \omega}{k_F^2} \right)^{1/2} \text{Im} \chi_0(Q_0, \omega) \]  

(3.17)

Now let us consider \( \chi_c(q, \omega) \). In principle, in order to evaluate \( \chi_c \) for finite \( q \), one needs to know the form of \( \Gamma_2(k, k', q, \nu, \omega) \) for arbitrary \( k \) and \( k' \). This is a difficult task because the form of \( \Gamma_2 \) cannot be obtained from Ward identities. As a result, \( \Gamma_2 \) will, in general, depend on some details of the given potential \( V(q) \). Some progress can be made, however, if the momentum transfer \( q \) is given by \( Q_0 = 2k_F \hat{q} \). It is also worthwhile to notice that, due to the identity given by Eq.3.13, \( \text{Re} \chi_{\phi p}(Q_0, \omega) \) is an odd function of \( k \wedge q \) and \( k' \wedge \hat{q} \). In fact, the small \( q \) limit of \( \Gamma_1(k, k', q, \nu, \omega) \) becomes the first term in the expression of the small \( q \) limit of the full scattering vertex in Eq.2.14. Thus the second term in Eq.2.14 can be regarded as the small \( q \) limit of \( \Gamma_2 \). This form, of course, cannot be easily generalized to the case of arbitrary \( q \) and \( \omega \).

Let us investigate the integral equation for \( \Gamma(k, k', Q_0, \nu, \omega) \) given by Eq.2.11 more closely in the small \( \omega \nu \) limit. In order to make a progress, let us define \( p \) and \( p' \) such that \( k = \frac{1}{2} q = \pm k_F \hat{q} + p \) and \( k' = \pm q = \pm k_F \hat{q} + p' \). In the imaginary part of \( \chi_c(Q_0, \omega) \), we have \( p \wedge \hat{q} \), \( p' \wedge \hat{q} \ll k_F \) in the low frequency limit. This means that \( k + q = p + p' \ll k_F \). In this case, it is reasonable to assume that \( \tilde{V}(k - k') = \tilde{V}(p - p') \) can be expanded in terms of \( p \cdot \hat{q}, p' \cdot \hat{q}, p \wedge \hat{q}, \) and \( p' \wedge \hat{q} \).

One can see that similar consideration applies to the \( k_1 \)-dependence of the kernel of the integral equation in Eq.2.11 as far as the frequency \( \omega \nu \) is sufficiently small. One has to, however, introduce a cutoff in the \( p_1 \)-integral after changing the variable from \( k_1 \) to \( p_1 \). This cutoff depends on the details of the potential \( V(q) \). In view of the structure of the integral equation, it is also reasonable to assume that, in the low frequency limit, the scattering vertex \( \Gamma(k, k', Q_0, \nu, \omega) \) can be expanded in terms of \( p \cdot \hat{q}, p' \cdot \hat{q}, p \wedge \hat{q}, \) and \( p' \wedge \hat{q} \). In fact, these are the eigenfunctions of the integral equation in the low frequency limit. More explicitly, the integral equation in the low frequency limit may be rewritten as

\[ \Gamma(p, p', Q_0, \nu, \omega) = \tilde{V}(p' - p) - \int' \frac{d^2 p_1}{(2\pi)^2} \Gamma(p, p_1, Q_0, \nu, \omega) \int \frac{d^2 q_1}{(2\pi)^2} \tilde{F}(\xi_{k_F q_1 + p_1} - \xi_{k_F q_1 + p_1 - \nu \omega}) \tilde{V}(p_1 - p'), \]

(3.18)

where

\[ \tilde{V}(p - p') = V_1 p \cdot \hat{q} p' \cdot \hat{q} + V_2 p \wedge \hat{q} p' \wedge \hat{q} \]

(3.19)

and \( f' \) represents the fact that there is a cutoff in the \( p_1 \)-integral. The solution for \( \Gamma(p, p', Q_0, \nu, \omega) \) can be written as

\[ \Gamma(p, p', Q_0, \nu, \omega) = \frac{p \cdot \hat{q} p' \cdot \hat{q}}{\lambda_1(Q_0, \nu, \omega)} + \frac{p \wedge \hat{q} p' \wedge \hat{q}}{\lambda_2(Q_0, \nu, \omega)}. \]

(3.20)

One can see that the integral equations for the first and the second terms are decoupled. The first term is nothing but the \( q = Q_0 \) limit of \( \Gamma_1 \) in Eq.3.3. Thus we know exactly what \( \lambda_1(Q_0, \nu, \omega) \) is. The second term would correspond to a contribution from \( \Gamma_2 \). The explicit form can be found from the integral equation as

\[ \Gamma_2(p, p', Q_0, \nu, \omega) \approx \frac{p \wedge \hat{q} p' \wedge \hat{q}}{V_2^{-1} + (m^*)^2 \chi_{\phi p}(Q_0, \nu, \omega)}. \]

(3.21)

in the small \( \omega \) limit.

Using \( \Gamma_2 \) obtained above, we now evaluate \( \text{Im} \chi_c \). In the low frequency limit, the exponential factor can be again expanded by assuming that \( l_B^2 k \wedge q = l_B^2 2k_F p \wedge \hat{q} \ll 1 \) or \( p \wedge \hat{q} \ll k_F \). Substituting \( \Gamma_2 \) in Eq.3.21 to the expression of \( \chi_c \) in Eq.3.7, we get
Imχc(Q₀, ω) = 4 \left( \frac{2}{\phi} \right)^2 \left( \frac{m^*}{k_F^2} \right)^4 \text{Im} \left\{ \frac{[\chi_{0\phi}(Q₀, ω)]^2}{V_2^{-1} + (m^*)^2 \chi_{0\phi}^+(Q₀, ω)} \right\}. \quad (3.22)

In the low frequency limit, the leading order contribution is given by

\begin{align*}
\text{Imχc}(Q₀, ω) & \approx 4 \left( \frac{2}{\phi} \right)^2 \left( \frac{m^*}{k_F^2} \right)^4 \frac{2V_2^{-1} + (m^*)^2 \text{Reχ}_{0\phi}}{(V_2^{-1} + (m^*)^2 \text{Reχ}_{0\phi})^2} \\
& \approx 4 \left( \frac{2}{\phi} \right)^2 \left( \frac{2V_2(m^* k_F^2/2\pi)(2m^* \varepsilon_F/k_F^2)^{3/2}C_1^{-1}}{1 + [V_2(m^* k_F^2/2\pi)(2m^* \varepsilon_F/k_F^2)^{3/2}C_1^{-1}]^2} \right) \frac{m^*}{3\pi} \frac{(m^* \omega)}{k_F^2} \frac{3/2}{}. \quad (3.23)
\end{align*}

Combining χ₀, χ₃, and χ_c, we can see that Imχ₃ is the leading order contribution in Imχ_rL_rL(Q₀, ω). Therefore, in the low frequency limit, we have

\begin{align*}
\text{Imχ_rL_rL} & \approx 4 \left( \frac{2}{\phi} \right)^4 \frac{C_1^2}{D_1^2} \left( \frac{2m^* \varepsilon_F}{k_F^2} \right) \frac{m^*}{\pi} \frac{(m^* \omega)}{k_F^2} \frac{3/2}{1 + O \left( \frac{m^* \omega}{k_F^2} \right)} \quad (3.24)
\end{align*}

In the next section, we will compare this result with that of the Chern-Simons theory.

IV. Q = 2K_F DENSITY-DENSITY RESPONSE FUNCTION IN THE CHERN-SIMONS THEORY

The irreducible density-density response function in the Chern-Simons theory at arbitrary q and ω was previously calculated and has the following form [4].

\begin{align*}
\chi_{cs}(q, ω) = \frac{\chi_0(q, ω)}{1 - \left( \frac{2\pi k_F}{\hbar q} \right)^2 \chi_0(q, ω) \chi_0^+(q, ω)}, \quad (4.1)
\end{align*}

where \( \chi_0(q, ω) \) and \( \chi_0^+(q, ω) \) are given by Eq.2.13 and Eq.2.10 with the quadratic dispersion \( ξ_k = k^2/2m^* - μ \).

In the case of the quadratic band, in the low frequency limit, it can be shown that the leading order results are given by

\begin{align*}
\chi_0(Q₀, ω) & \approx \frac{m^*}{2π} \left[ 1 + \frac{1}{2} \left( \frac{m^* \omega}{k_F^2} \right)^{1/2} \right] \\
\chi_0^+(Q₀, ω) & \approx \frac{m^*}{6π} \frac{k_F}{m^*} \left[ -1 + \frac{1}{2} \left( \frac{m^* \omega}{k_F^2} \right)^{3/2} \right]. \quad (4.2)
\end{align*}

Using the results above, the imaginary part of \( \chi_{cs}(Q₀, ω) \) in the low frequency limit can be estimated as

\begin{align*}
\text{Imχ_{cs}}(Q₀, ω) & \approx \frac{\text{Imχ}_0}{1 - \left( \frac{2\pi k_F}{\hbar q} \right)^2 \text{Reχ}_0 \text{Reχ}_0^+} \approx \frac{\frac{m^*}{4\pi} \text{Reχ}_0 \text{Reχ}_0^+}{1 + \frac{1}{2} \left( \frac{\tilde{ω}}{2\pi} \right)^2} \quad (4.3)
\end{align*}

Notice that the density-density response function in the Chern-Simon theory, \( \text{Imχ}_{cs}(Q₀, ω) \), has the same \( \sqrt{ω} \) dependence as that of the density-density response function, \( \text{Imχ}_{rL_rL}(Q₀, ω) \), in the lowest-Landau-level theory.

However, the coefficients are different. In fact, the ratio between them becomes

\begin{align*}
\frac{\text{Imχ}_{rL_rL}}{\text{Imχ}_{cs}} & \approx 4 \left( \frac{2}{\phi} \right)^4 \left[ 1 + \frac{1}{3} \left( \frac{\tilde{ω}}{2\pi} \right)^2 \right] \left( \frac{2m^* \varepsilon_F}{k_F^2} \right)^2 \quad (4.4)
\end{align*}

This ratio is, in general, nonuniversal number which depends on the details of the interaction potential \( V(q) \). In order to get some feelings about how small or large the ratio can be, let us take the HF dispersion relation, \( ξ_0 = k^2/2m^* - μ \), being quadratic in \( k \) with the effective mass \( m^* \). In this case, and \( C_1^2/D_1^2 = 1/36 \) and \( 2m^* \varepsilon_F/k_F^2 = 1 \). As a result, we get

\begin{align*}
\frac{\text{Imχ}_{rL_rL}}{\text{Imχ}_{cs}} & \approx \frac{16}{81} \approx 0.198 \quad \text{for } \nu = 1/2, \\
\frac{\text{Imχ}_{rL_rL}}{\text{Imχ}_{cs}} & \approx \frac{49}{1296} \approx 0.038 \quad \text{for } \nu = 1/4, \quad (4.5)
\end{align*}

Typical ratios are given by

\begin{align*}
\frac{\text{Imχ}_{rL_rL}}{\text{Imχ}_{cs}} & \approx \frac{16}{81} \approx 0.198 \quad \text{for } \nu = 1/2, \\
\frac{\text{Imχ}_{rL_rL}}{\text{Imχ}_{cs}} & \approx \frac{49}{1296} \approx 0.038 \quad \text{for } \nu = 1/4, \quad (4.6)
\end{align*}

and the ratio approaches to 1/81 = 0.012 as \( \tilde{ω} \) becomes an infinitely large even number. Of course, these numbers cannot be taken seriously on the face value because the HF dispersion is quadratic only for small \( |k| ≪ k_F \) and the dispersion relation for larger \( |k| \) affects also these numbers. However, it may be suggestive for certain purposes. In the next section, we will use these estimations to discuss the results of the recent phonon drag experiment.
V. DISCUSSION ON THE PHONON DRAG EXPERIMENT

Recently, the drag resistivity between two $\nu = 1/2$ layers was measured when the layer separation is much larger than the magnetic length (or typical interparticle spacing) in each layer. In this case, the contribution to the drag resistivity from the interlayer Coulomb interaction is substantially suppressed and the electron-phonon interaction becomes the dominant source of the scattering. Theoretically, the drag resistivity can be evaluated from [16,18]

$$\rho_{21} = \frac{-h^2}{8\pi^2\epsilon n_1 n_2 T} \int_0^{\infty} dq \, q^3 \times \int_0^{\infty} d\omega \frac{|U_{21}(q, \omega)|^2 \Im \chi_1(q, \omega) \Im \chi_2(q, \omega)}{\varepsilon(q, \omega) \sinh^2(h\omega/2T)},$$

(5.1)

where $\chi_{1,2}$ are the density-density response functions and $n_{1,2}$ the electron densities in layer 1 and 2. $U_{21}$ is the interaction between the electrons in different layers and $\varepsilon$ is the interlayer dielectric or screening function. The details can be found in Ref. [16] and Ref. [18]. If the interaction is dominated by the phonon-mediated interaction, $U_{21}$ is given by the interlayer electron-phonon interaction. When two layers are identical, the matrix element for the drag resistivity involves $[\Im \chi_1(q, \omega)]^2$.

Let us first discuss the situation of the zero applied magnetic field. The phonon-mediated drag occurs by transferring momentum $q$ of the phonon to excite the particle-hole continuum of electrons. At zero temperature, the particle-hole continuum ceases to exist if the momentum transfer $q$ is larger than $2k_F$ at low frequencies. Formally, this means that $\Im \chi$ becomes very small if $q > 2k_F$ in the low frequency limit. Thus when the phonon energy $\omega_{ph} = cq$ becomes larger than $c(2k_F)$, the scattering is suppressed. At finite temperature $T$, the typical phonon energy is set by the temperature scale so that, if $T > T_0 = c(2k_F)$, the drag resistivity is substantially suppressed. This is why there is a maximum of $\rho_{21}$ around $T \approx T_0$ in the measured phonon-mediated drag resistivity.

In the case of the $\nu = 1/2$ double-layer system, the underlying ground state is the composite fermion Fermi sea. Naively, one may expect that the drag resistivity will have a maximum around $T \approx T_{1/2} > T_0$ where $T_{1/2} = c(2k^d_F)$ and $k^d_F = \sqrt{2}k_F$. Here $k^d_F$ is the Fermi wave vector of composite fermion system. In the experiment, the maximum of the drag resistivity occurs at a temperature that is even smaller than $T_0$ [13].

In order to investigate the theoretical consequence in detail, Bonsager, MacDonald, and the author calculated the phonon-mediated drag resistivity in the Chern-Simons theory [14]. In fact, since the electron density-density response function is not simply proportional to the density-density response function of composite fermions in the Chern-Simons theory, it is not obvious whether the naive expectation is valid even in the Chern-Simons theory. They found that [10]

1. If the effective mass of composite fermions is sufficiently small such that the effective Fermi energy is substantially larger than $T_{1/2}$, the naive expectation is more or less correct. That is, the maximum occurs around $T_{1/2}$.

2. When realistic values of the effective mass are used, however, the effective Fermi energy is not very small compared to typical temperature scale we are looking at (such as $T_{1/2}$). As a result, there are significant finite temperature effects leading to the shift (not necessarily monotonic as a function of the effective mass) of the maximum position.

3. In any case, the maximum of the drag resistivity occurs always at a temperature larger than $T_0$. Thus the theory cannot explain the large downward shift of the maximum position.

In this paper, we evaluated the imaginary part of the density-density response function at arbitrary wavevector and frequency. This may be necessary especially because the effective Fermi energy is not much larger than the temperature scales we are interested in. Evaluation of the density-density response function at arbitrary $q$ and $\omega$ in the lowest-Landau-theory is not an easy task because one has to solve the scattering vertex function $\Gamma$ for arbitrary wavevector and frequency. Also the finite wavevector response function would contain nonuniversal contribution which may depend on the details of the given interaction potential $V(q)$.

In this paper, we evaluated the imaginary part of the density-density response function of electrons for $q = 2k_F$ in the low frequency limit. Thus the full evaluation of the drag resistivity is beyond the scope of the present paper. However, based on what we got in the last section, we may be able to speculate what might happen to the drag resistivity within the lowest-Landau-level theory. We found in the last section that the imaginary parts of the density-density response functions in the Chern-Simons and lowest-Landau-level theory ($\Im \chi_{cs}$ and $\Im \chi_{LL}^{irr}$) have the same $\sqrt{\omega}$ dependence in frequency at $q = 2k_F$ in the low frequency limit. It may appear that this is a disappointing result because the lowest-Landau-theory does not provide qualitatively different results.

However, we also learned that the ratio between $\Im \chi_{LL}^{irr}$ and $\Im \chi_{cs}$ can be quite different from unity. For the sake of the order of magnitude estimation, if we use a quadratic approximation for $\xi^\perp$ in the lowest-Landau-level, $\Im \chi_{LL}^{irr}/\Im \chi_{cs}$ $\approx 0.2$ for $\nu = 1/2$ as shown in Eq. 4.6. In the expression of the drag resistivity in Eq. 5.1, the imaginary part of the density-density response enters as
[Imχ]². If the behavior of Imχ_{LL}/Imχ_{cs} for larger frequencies is similar to that of the low frequency limit, the contribution from the scattering events with q near 2k_F to the drag resistivity in the lowest-Landau-level theory would be ~1/25 times smaller than that of the Chern-Simons theory. This implies that the large wavevector scattering may be quantitatively more suppressed in the lowest-Landau-level theory compared to the case of the Chern-Simons theory. This implies that the large wavevector would be inadequate for describing the large states in the lowest Landau level. In particular, we evaluate the χ_{LL} response function at q = 2k_F in the low frequency limit. However, the coefficients can be quite different and the ratio between them is a nonuniversal number. We found that the response function at q = 2k_F in the lowest-Landau-level theory can be substantially smaller than that of the Chern-Simons theory.

Using these results, we speculate that the lowest-Landau-level theory may explain the suppression of the q = 2k_F scattering seemingly observed in the experiment on the phonon-mediated drag resistivity in the double layer ν = 1/2 system. However, the satisfactory understanding of the phonon drag experiment may require numerical evaluation of the density-density response function.

VI. SUMMARY

In this paper, we consider finite q density-density response function of the compressible Fermi-liquid-like states in the lowest Landau level. In particular, we evaluated the q = 2k_F density-density response function in the low energy limit within the lowest-Landau-level formalism. We compare this result with the prediction of the Chern-Simons theory. We found that the density-density response functions in both cases are proportional to √ω in the low frequency limit. However, the coefficients can be quite different and the ratio between them is a nonuniversal number. We found that the response function at q = 2k_F in the lowest-Landau-level theory can be substantially smaller than that of the Chern-Simons theory.

Using these results, we speculate that the lowest-Landau-level theory may explain the suppression of the q = 2k_F scattering seemingly observed in the experiment on the phonon-mediated drag resistivity in the double layer ν = 1/2 system. However, the satisfactory understanding of the phonon drag experiment may require numerical evaluation of the density-density response function.

Finally, we would like to mention various other possibilities which were not considered in this paper. The lowest-Landau-level theory in the present form through the time-dependent HF approximation may turn out to be inadequate for describing the large q or short distance behaviors of density-density response of the composite fermions. This is because the picture of the correlations between the electrons and vortices itself may break down at short distances due to the fact that we are looking at the length scale comparable to the distance between the electrons and vortices [14]. The theory, if it exists, which describes this situation and the crossover or transition from the present lowest-Landau-level theory to this regime may provide better understanding of the finite q response of the system. In any case, our work should serve as a useful starting point for understanding more microscopic picture of the compressible states in the lowest Landau level.

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