BASE CHANGE FOR ELLIPTIC CURVES OVER REAL QUADRATIC FIELDS

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Abstract. Let $E$ be an elliptic curve over a real quadratic field $K$ and $F/K$ a totally real finite Galois extension. We prove that $E/F$ is modular.

1. Introduction

For $F$ a totally real number field we write $G_F := \text{Gal}(\overline{\mathbb{Q}}/F)$ for its absolute Galois group. For a Hilbert modular form $f$ we denote by $\rho_{f,\lambda}$ its attached $\lambda$-adic representation. We say that a continuous Galois representation $\rho : G_F \rightarrow \text{GL}_2(\mathbb{Q}_\ell)$ is modular if there exists a Hilbert newform $f$ and a prime $\lambda \mid \ell$ in its field of coefficients $\mathbb{Q}_f$ such that we have an isomorphism $\rho \sim \rho_{f,\lambda}$. In [1] and [2, Section 5], the first named author proved base change for the $\text{GL}_2$ case over $\mathbb{Q}$.

Theorem 1. Let $f$ be a classical modular form of weight $k \geq 2$ and field of coefficients $\mathbb{Q}_f$. For a prime $\lambda$ of $\mathbb{Q}_f$ write $\rho_{f,\lambda}$ for the attached $\lambda$-adic representation. Let $F/\mathbb{Q}$ be a totally real number field. Then the Galois representation $\rho_{f,\lambda}|G_F$ is (Hilbert) modular in the sense above.

Let $E$ be an elliptic curve over a real quadratic field $K$. It is now known that $E$ is modular over $K$ (see [3]). That is, the $\ell$-adic representations $\rho_{E,\ell} : G_K \rightarrow \text{GL}_2(\mathbb{Q}_\ell)$ attached to $E$ arise from some Hilbert newform defined over $K$ of parallel weight 2 having rational field of coefficients.

The aim of this note is to prove a base change result for elliptic curves over real quadratic fields. Our main theorem is the following.

Theorem 2. Let $E$ be an elliptic curve over a real quadratic field $K$. Let also $F/K$ be a totally real finite Galois extension. Then $E/F$ is modular.

This result has applications in the context of the Birch and Swinnerton-Dyer conjecture. Indeed, modularity of $E$ after base change guarantees that the $L$-function $L(E/F, s)$ is holomorphic in $\mathbb{C}$ and, in particular, its order of vanishing at $s = 1$ is a well defined non-negative integer, in agreement with what is predicted by the BSD conjecture. Furthermore, modularity of $E/F$ allows the construction of Stark-Heegner points on $E$ over (not necessarily real) quadratic extensions of $F$. For details regarding this application we refer the reader to [4] and the references therein.

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2. Elliptic curves with big non-solvable image mod $p = 3, 5$ or $7$

Let $F/K$ be a finite extension of totally real number fields. Let $E/K$ be an elliptic curve. We will say that $\overline{\eta}_{E,p}(G_F)$ is big if $\overline{\eta}_{E,p}(G_{F(\rho)})$ is absolutely irreducible, otherwise we say it is small. In particular, if $\overline{\eta}_{E,p}(G_F)$ is non-solvable then it is big. We now restate a result from [3].

**Theorem 3.** Let $p = 3, 5$ or $7$. Let $F/K$ and $E/K$ be as above. Suppose that $\overline{\eta}_{E,p}(G_F)$ is big. Then $E$ is modular over $F$.

The following proposition is well-known.

**Proposition 2.1.** Let $F/K$ be a finite Galois extension of totally real fields and $E/K$ an elliptic curve. Let $p$ be any prime and suppose that $\overline{\eta}_{E,p}(G_K)$ is non-solvable. Then $\overline{\eta}_{E,p}(G_F)$ is non-solvable.

We have the following corollary.

**Corollary 2.2.** Let $F/K$ and $E/K$ be as in Proposition 2.1. Let $p = 3, 5$ or $7$. Suppose that $\overline{\eta}_{E,p}(G_K)$ is non-solvable. Then $E$ is modular over $F$.

**Proof.** From the previous proposition we have that $\overline{\eta}_{E,p}(G_F)$ is non-solvable, hence it is big. Thus $E/F$ is modular by Theorem 3.

3. Elliptic curves with projective image $S_4$ or $A_4$ mod $p = 3, 5$ or $7$

Let $E/K$ be an elliptic curve. We have seen that if $\overline{\eta}_{E,p}$ has big non-solvable image then after a base change to a Galois extension its image is still non-solvable. We now want to understand what can happen when $\overline{\eta}_{E,p}(G_K)$ is big and solvable. We first recall the following well know fact.

**Proposition 3.1.** Let $E/K$ be an elliptic curve. Write $G$ for the image of $\overline{\eta}_{E,p}$ in $\text{GL}_2(\mathbb{F}_p)$ and $H$ for its image in $\text{PGL}_2(\mathbb{F}_p)$. Then, there are the following possibilities:

(a) $G$ is contained in a Borel subgroup;
(b) $G$ contains $\text{SL}_2(\mathbb{F}_p)$;
(c) $H$ is cyclic, $G$ is contained in a Cartan subgroup;
(d) $H$ is dihedral, $G$ is contained in the normalizer of a Cartan subgroup;
(e) $H$ is isomorphic to $A_4$, $S_4$ or $A_5$.

In cases (c), (d), (e) the prime $p$ does not divide the order of $G$.

Let $p = 3, 5$ or $7$. Let also $G$ and $H$ be as in the proposition. We divide the cases where $\overline{\eta}_{E,p}(G_K)$ is big and solvable into two types:

(I) $H \cong S_4$ or $A_4$,
(II) $H$ is dihedral.

Suppose we are in case (I). Let $F/K$ be a finite Galois extension and set $H_F := \overline{\eta}(\overline{\eta}_{E,p}(G_F))$. We would like that $H_F$ is also isomorphic to $A_4$ or $S_4$ since this would mean that $\overline{\eta}_{E,p}(G_F)$ is big and Theorem 3 applies. Since $F/K$ is Galois we have that $H_F$ is normal subgroup of $H$. Write $I = \{1\}$ for the trivial group and $D_4$ for the dihedral group in four elements. The normal subgroups of $S_4$ and $A_4$ are respectively

- $I, D_4, A_4$ and $S_4$,
- $I, D_4$ and $A_4$. 

Thus, the cases where Theorem 3 do not apply over $F$ are when the pair of groups $(H, H_F)$ is one of
\[ (S_4, D_4), \quad (S_4, I), \quad (A_4, D_4), \quad (A_4, I). \]
Since we are working with totally real fields the complex conjugation has projective image of order 2. Thus the cases with $H_F = I$ cannot happen.

3.1. A Sylow base change. We now deal with the remaining cases from (1).

Recall that we want to base change $E/K$ to $F$ where $F/K$ is finite and Galois. Suppose that $(H, H_F)$ is $(S_4, D_4)$ or $(A_4, D_4)$. Let $F_3$ be a subfield of $F$ such that the Galois group $\text{Gal}(F/F_3)$ is a 3-Sylow subgroup of $\text{Gal}(F/K)$. In particular, $F/F_3$ is a solvable extension. We shall shortly prove the following.

**Lemma 3.2.** The projective image $H_{F_3} : \mathbb{P}(\mathcal{T}_{E,p}(G_{F_3}))$ is isomorphic to $S_4$ or $A_4$. In particular, $\mathcal{T}_{E,p}(G_{F_3})$ is big.

From this lemma and Theorem 3 it follows that $E/F_3$ is modular. Finally, an application of Langlands solvable base change allows to conclude that $E/F$ is modular.

For the proof of Lemma 3.2 we will need the following elementary lemma from group theory.

**Lemma 3.3.** Let $G$ be a profinite group. Let $H \subset G$ be subgroup of finite index $i$. Let $N$ be a normal subgroup of $G$. Write $j$ for the index of $H/(N \cap H)$ in $G/N$. Then $j \mid i$.

**Proof.** We prove it for the case of finite groups. The required divisibility follows from the following elementary equalities:
\[
|G| = |N| \cdot [G : N] \\
|H| = |N \cap H| \cdot [H : N \cap H]
\]
Dividing the first equality into the second, we conclude that $j$ divides $i$. \qed

**Proof of Lemma 3.2.** Let $F_3$ be as above and set
\[
G := \text{Gal}(\overline{K}/K), \quad H := \text{Gal}(\overline{K}/F_3), \quad N := \text{Ker}(\mathcal{T}_{E,p}).
\]
Let $L/K$ be the Galois extension fixed by $N$. Observe that $L/L \cap F_3$ is Galois and $G/N \cong \text{Gal}(L/K)$, $H/(H \cap N) \cong \text{Gal}(L/L \cap F_3)$.

From Lemma 3.3 we see that
\[
[\text{Gal}(L/K) : \text{Gal}(L/L \cap F_3)] = j \mid i = [G : H]
\]
and we also have
\[
[\text{Gal}(L/K)] = j \mid [\text{Gal}(L/L \cap F_3)].
\]
Note that $\text{Gal}(L/L \cap F_3) \cong H_{F_3}$. From the way we choose $F_3$ it is clear that $3 \nmid i$, hence $3 \nmid j$. By hypothesis $G/N \cong S_4$ or $A_4$, hence $3$ divides $|\text{Gal}(L/K)|$ and $|H_{F_3}|$. Finally, the conditions $3 \mid |H_{F_3}|$ and $D_4 \subset H_{F_3}$ together imply that $H_{F_3}$ is isomorphic to $S_4$ or $A_4$. \qed

We summarize this section into the following corollary.

**Corollary 3.4.** Let $F/K$ be a finite Galois extension of totally real fields. Let $E/K$ be an elliptic curve. Suppose that for $p = 3, 5$ or $7$ we have that $\mathcal{T}_{E,p}(G_K)$ is big and solvable. Suppose further that $\mathbb{P}(\mathcal{T}_{E,p}(G_K)) \cong S_4$ or $A_4$. Then $E/F$ is modular.
Everything we have done so far works for any Galois extension $F/K$. Moreover, it is clear that the remaining cases are those when $\overline{\rho}_{E,p}(G_K)$ is small or projectively dihedral simultaneously for $p = 3, 5, 7$. The restriction in the statement of Theorem 2 to quadratic fields arises precisely from dealing with them, which is the content of the next section.

4. Elliptic curves having small or projective Dihedral image at $p = 3, 5$ and $7$

Let $K$ be a real quadratic field. From Theorem 3 an elliptic curve $E/K$ is modular over $K$ except possibly if $\overline{\rho}_{E,p}(G_K)$ is small simultaneously for $p = 3, 5, 7$. Suppose $K \neq \mathbb{Q}(\sqrt{5})$. In [3] it is shown that such an elliptic curve gives rise to a $K$-point on one of the following modular curves:

\[ X(b_5, b_7), \quad X(b_3, s_5), \quad X(s_3, s_5), \]
\[ X(b_3, b_5, d_7), \quad X(s_3, b_5, d_7), \quad X(b_3, b_5, e_7), \quad X(s_3, b_5, e_7), \]

where $b$, $s$ and $n$ respectively stand for 'Borel', 'normalizer of split Cartan' and 'normalizer of non-split Cartan'. The notation $d_7$ and $e_7$ is explained in [3, section 10], here we remark only that they indicate mod 7 level structures that are respectively finer than 'normalizer of split Cartan' and 'normalizer of non-split Cartan'. Denote by $E_K$ the set of elliptic curves (up to quadratic twist) corresponding to $K$-points in the previous modular curves.

In [3] it is also shown that an elliptic curve $E/\mathbb{Q}(\sqrt{5})$ with simultaneously small image for $p = 3, 5, 7$ gives rise to a $\mathbb{Q}(\sqrt{5})$-point in one of the following modular curves

\[ X(d_7), \quad X(e_7), \quad X(b_3, b_7), \quad X(s_3, b_7). \]

Denote by $E_{\mathbb{Q}(\sqrt{5})}$ the set of elliptic curves (up to quadratic twist) corresponding to $\mathbb{Q}(\sqrt{5})$-points in these four modular curves.

Furthermore, it is also follows from [3] that, for any real quadratic field $K$, we have

(i) $E_K$ contains all elliptic curves (up to quadratic twist) with small or projective Dihedral image simultaneously at $p = 3, 5, 7$;
(ii) $E_K$ is finite;
(iii) Let $E \in E_K$. Then, either $E$ is a $\mathbb{Q}$-curve or $E$ has complex multiplication or $\bar{\rho}_{E,7}(G_K)$ contains $SL_2(\mathbb{F}_7)$.

We can now easily prove the following.

**Corollary 4.1.** Let $K$ be a real quadratic field. Let $E \in E_K$. Let $F/K$ be a finite Galois extension. Then $E/F$ is modular.

**Proof.** From (iii) above we know that either (a) $E/K$ is a $\mathbb{Q}$-curve or has complex multiplication or (b) $\bar{\rho}_{7}(G_K)$ is non-solvable. In case (a) base change is already known. In case (b), it follows from Corollary 2.2 that $E/F$ is modular. \qed

5. Proof of the main theorem

Let $K$ be a real quadratic field and $E/K$ an elliptic curve. Write $\bar{\rho}_p = \bar{\rho}_{E,p}$. The curve $E/K$ must satisfy at least one of the following three cases

(1) $\bar{\rho}_p(G_K)$ is big and non-solvable for some $p \in \{3, 5, 7\},$
(2) $\bar{\rho}_p(G_K)$ is big, solvable and satisfy $\mathbb{P}(\bar{\rho}_p(G_K)) \cong S_4, A_4$ for some $p \in \{3, 5, 7\}$.
(3) $E/K$ belongs to the set $\mathcal{E}_K$.

Let $F/K$ be a totally real finite Galois extension. In each case, modularity of $E/F$ now follows directly from one of the previous sections:
- Case (1): this is Corollary 2.2.
- Case (2): this is Corollary 3.4.
- Case (3): this is Corollary 4.1.

□

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