RECURRENT INVERSION FORMULAS

WENHUA ZHAO

Abstract. Let $F(z) = z - H(z)$ with $o(H(z)) \geq 2$ be a formal map from $\mathbb{C}^n$ to $\mathbb{C}^n$ and $G(z)$ the formal inverse of $F(z)$. In this paper, we first study the deformation $F_t(z) = z - tH(z)$ and its formal inverse map $G_t(z)$. We then derive two recurrent formulas for the formal inverse $G(z)$. The first formula in certain situations provides a more efficient method for the calculation of $G(z)$ than other well known inversion formulas. The second one is differential free but only works when $H(z)$ is homogeneous of degree $d \geq 2$. Finally, we reveal a close relationship of the inversion problem with a Cauchy problem of a PDE. When the Jacobian matrix $JF(z)$ is symmetric, the PDE coincides with the $n$-dimensional inviscid Burgers’ equation in Diffusion theory.

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1. Introduction

Let $F(z) = z - H(z)$ be a formal map from $\mathbb{C}^n$ to $\mathbb{C}^n$ with $o(H(z)) \geq 2$ and $G(z)$ the formal inverse of $F(z)$, i.e. $z = F(G(z)) = G(F(z))$. The formulas which directly or indirectly give the formal inverse $G(z)$ are called inversion formulas in literature. There have been many different versions of inversion formulas. The first inversion formula in history was the Lagrange’s inversion formula given by L. Lagrange [L] in 1770, which provides a formula to calculate all coefficients of $G(z)$ for the one-variable case. This formula was generalized to multi-variable

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cases by I. G. Good [Go] in 1965. Jacobi [J1] in 1830 also gave an inversion formula for the cases $n \leq 3$ and later [J2] for the general case. This formula now is called the Jacobi’s inversion formula. Another inversion formula is Abhyankar-Gurjar inversion formula, which was first proved by Gurjar in 1974 (unpublished) and later Abhyankar [A] gave a simplified proof. By using Abhyankar-Gurjar inversion formula, H. Bass, E. Connell and D. Wright [BCW] and D. Wright [W2] proved the so-called Bass-Connell-Wright’s tree expansion formula. Recently, in [WZ], this formula has been generalized to a tree expansion formula for the formal flow $F_t(z)$ of $F(z)$ which provides a uniform formula for all the powers $F^{[m]}(z)$ ($m \in \mathbb{Z}$) of $F(z)$. There are also many other inversion formulas in literature, see [Ge], [W3] and references there.

One of the motivations of seeking inversion formulas comes from their important applications in enumerative combinatorics of rooted trees. See, for example, [SL], [Ge] and references there. Another motivation comes from the study of the well known Jacobian conjecture. Recall that the Jacobian conjecture claims that, any polynomial map $F(z)$ from $\mathbb{C}^n$ to $\mathbb{C}^n$ with the Jacobian $j(F) = \det(\frac{\partial F}{\partial z}) = 1$ is an automorphism of $\mathbb{C}^n$ and its inverse $G(z)$ is also a polynomial map. The Jacobian conjecture was first proposed by Keller [K] in 1939. It is now still open even for the two variable case. For the history and some well known results of the Jacobian conjecture, see the classical paper [BCW], [E] and references there.

In this paper, we first study the deformation $F_t(z) = z - tH(z)$ and its formal inverse map $G_t(z)$. We then derive two recurrent formulas for the formal inverse $G(z)$. The first formula in certain situations provides a more efficient method for the calculation of $G(z)$ than other well known inversion formulas. The second one is differential free but only works when $H(z)$ is homogeneous of degree $d \geq 2$. Finally, we reveal a close relationship of the inversion problem with a Cauchy problem of a PDE. When the Jacobian matrix $JF(z)$ is symmetric, the PDE coincides with the $n$-dimensional inviscid Burgers’ equation in Diffusion theory.

The arrangement of this paper is as follows. In Section 2 we recall several well known inversion formulas in literature and derive the formulas for the formal inverse $G(z)$ if they are not given directly. In Section 3 we mainly study the deformation $F_t(z) = z - tH(z)$ and its formal inverse map $G_t(z)$. The main results are Theorem 3.2 and Proposition 3.3. Theorem 3.2 reveals a close relationship between the formal inverse $G_t(z)$ and a Cauchy problem of PDE which has a similar form as the $n$-dimensional inviscid Burgers’ equation. Proposition
3.3. give us the first recurrent inversion formula. In this section, we also derive some interesting consequence from the main results above. One of theorems is Proposition 3.8 which claims that the formal inverse $G(z) = z + H(z)$ if and only if $JH \cdot H = 0$. As an immediate consequence, we derive the well-known theorem of H. Bass, E. Connell and D. Wright [BCW], which says that the Jacobian conjecture is true when $H(z)$ is homogeneous and $J(H)^2 = 0$. One purpose that we include the proof for this theorem is to explore why it is much more difficult to prove the Jacobian conjecture under the same conditions as above except we have $J(H)^k = 0$ for some $k \geq 3$ instead of $J(H)^2 = 0$. Finally, in Section 4, we derive our second recurrent inversion formula. This formula is a generalization of the recurrent formula proved by Dróżkowski [D2] for the case $d = 3$. Our approach also gives the involved multi-linear form explicitly.

Finally, two remarks are as follows. First, we will fix $\mathbb{C}$ as our base field. But all results, formulas as well as their proofs given in this paper hold or work equally well for formal power series over any $\mathbb{Q}$-algebra. Secondly, for convenience, we will mainly work on the setting of formal power series over $\mathbb{C}$. But, for polynomial maps or local analytic maps, all formal maps or power series involved in this paper are also locally convergent. This can be easily seen either from the fact that any local analytic map with non-zero Jacobian at the origin has a locally convergent inverse, or from the well-known Cauchy-Kowaleskaya theorem (See [R], for example.) in PDE.

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2. Inversion Formulas

In this section, we review the inversion formulas of Lagrange, Jacobi, Abhyankar-Gurjar, Bass-Connell-Wright and also the formula developed in [WZ] for the formal flow $F_t(z)$ of $F(z)$, which encodes formulas for all powers $F^{[m]}(z)$ ($m \in \mathbb{Z}$). There are also many other versions of inversion formulas in literature. They are more or less in the same favor as some of inversion formulas above. We refer readers to [Ge], [W3] and references for other inversion formulas.

First we fix the following notation.

We let $z = (z_1, z_2, \ldots, z_n)$ and $\mathbb{C}[z]$ (resp. $\mathbb{C}[[z]]$) the algebra of polynomials (resp. formal power series) in $z$. For any $k \in \mathbb{Z}^n$ and
Laurent series $h(z)$, we denote by $[z^k]h(z)$ the coefficient of $[z^k]$ in $h(z)$. In particular, we set

$$\text{Res}_z h(z) = [z_1^{-1}z_2^{-1}\cdots z_n^{-1}]h(z)$$

In this paper, we always denoted by $F(z) = (F_1(z), \ldots, F_n(z))$ a formal map from $\mathbb{C}^n$ to itself with the form $F(z) = z - H(z)$ and $o(H(z)) \geq 2$. The notation $G(z)$ always denotes the formal inverse of $F(z)$ and $j(F)$ the Jacobian $\det(\frac{\partial F_i}{\partial z_j}) = 1$ of $F(z)$.

We start with the Lagrange’s multi-variable inversion formula. The version of the formula we quote here is given by Good in [Go].

**Theorem 2.1. (Lagrange’s Inversion Formula)**

Let $f(z) = (f_1(z), f_2(z), \ldots, f_n(z)) \in \mathbb{C}[z]^n$ with non-zero constant terms. Then there exist a unique $g(z) = (g_1(z), \ldots, g_n(z)) \in \mathbb{C}[z]^n$ such that

$$g_i(z) = z_i f_i(g(z))$$

for any $i = 1, 2, \ldots, n$. Furthermore, for any formal Laurent series $\phi(z)$ and $k \in \mathbb{Z}^n$, we have

$$[z^k] \frac{\phi(g)}{\det(\delta_{i,j} - z_i \frac{\partial f_i}{\partial z_j}(g))} = [w^k] \phi(w) f^k(w)$$

$$[z^k] \phi(g) = [w^k] \det(\delta_{i,j} - w_i \frac{\partial f_i}{f_i(w) \partial z_j}(w)) \phi(w) f^k(w)$$

Let us see how to use the formulas above to calculate the formal inverse $G(z)$ of $F(z)$. To do this, we assume that

$$H_i(z) = z_i h_i(z) \quad (1 \leq i \leq n)$$

for some $h_i(z) \in \mathbb{C}[[z]]$.

We choose $f_i(z) = \frac{1}{1-h_i(z)}$. Hence, $\frac{z_i}{f_i(z)} = F_i(z)$ and by Eq. (2.2), we have

$$z_i = \frac{z_i}{f_i} (g(z)) = F_i(g(z))$$

i.e. $g(z)$ is the formal inverse of $F(z)$, therefore $g(z) = G(z)$.

To calculate $G(z)$, hence, it’s enough to calculate $[z^k]G_i(z)$ for any $1 \leq i \leq n$ and $k \in \mathbb{N}^n$. For any fixed $i$, we choose $\phi(z) = z_i$. By Eq. (2.4), we get

$$[z^k]G_i(z) = [w^k] \det(\delta_{i,j} - w_i \frac{\partial f_i}{f_i(w) \partial z_j}(w)) w_i f^k(w)$$

For the case when $H_i(z)$ is not of the form (2.6). The Lagrange’s inversion formula does not provide a direct method for the calculation
of $G(z)$. But, one can derive from the Lagrange’s inversion formula the following two inversion formulas, which will provide formulas for $G(z)$ in the general case. (See, for example, \cite{Gr}).

The next inversion formula was first proved by Jacobi \cite{J1} for $n \leq 3$ and later in \cite{J2} for the general case.

**Theorem 2.2. (Jacobi’s Inversion Formula)**

Let $f(z) = (f_1(z), f_2(z), \cdots, f_n(z))$ be a sequence of formal series in $z$. Let $\phi(z)$ be any Laurent series. Then

\[ \text{Res}_w \phi(w) = \text{Res}_z j(f)f(z) \]

To get the formal inverse $G(z)$ of $F(z)$ by using the Jacobi’s inversion formula, we can choose $f(z) = F(z)$. For each $1 \leq i \leq n$ and $k \in \mathbb{N}^n$, we choose $\phi(z) = z^{-k-1}G_i(z)$, where $1 = (1, 1, \cdots, 1)$. Then Formula \eqref{eq:2.7} gives us

\[ [z^k]G_i(z) = \text{Res}_w w^{-k-1}G_i(w) = \text{Res}_z j(F)F^{-k-1}(z)z_i \]

Hence, by changing $k \in \mathbb{N}^n$, we can calculate $G(z)$ completely.

The first direct inversion formula was proved by Gurjar (unpublished) and a simplified proof was later given by Abhyankar \cite{A}.

**Theorem 2.3. (Abhyankar-Gurjar’s Inversion Formula)**

Let $F(z) = z - H(z)$ with $o(H) \geq 2$. Then for any Laurent series $\phi(z)$, we have

\[ \phi(G) = \sum_{m \in \mathbb{N}^n} \frac{D_m}{m!}\phi(z) j(F)H^m \]

where $D_m = D_1^{m_1}D_2^{m_2} \cdots D_n^{m_n}$ for any $m = (m_1, \cdots, m_2) \in \mathbb{N}^n$ and $D_i = \frac{\partial}{\partial z_i}$ for any $1 \leq i \leq n$.

Note that, if we choose $\phi(z) = z_i$ ($1 \leq i \leq n$), we get $G_i(z)$ by Formula \eqref{eq:2.9}.

By using Abhyankar-Gurjar’s inversion formula, H. Bass, E. Connell and D. Wright \cite{BCW} proved the so-called Bass-Connell-Wright’s tree expansion inversion formula for the case when $H(z)$ is homogeneous and later D. Wright \cite{W2} proved that the same formula also holds in the general case. A totally different proof of this formula was also given in \cite{WZ}.

In order to recall Bass-Connell-Wright’s inversion formula and the formula for the formal flow $F_t(z)$ of $F(z)$ developed in \cite{WZ}, we need the following notation.

By a **rooted tree** $T$ we mean a finite connected and simply connected graph with one vertex designated as its root, denoted by $rt_T$. In a rooted tree there are natural ancestral relations between vertices. We
say a vertex \( w \) is a \textit{child} of vertex \( v \) if the two are connected by an edge and \( w \) lies further from the root than \( v \). We denote by \( v^+ \) the set of all its children. When we speak of isomorphisms between rooted trees, we will always mean root-preserving isomorphisms. We denote by \( T \) (resp. \( T_m \)) the set of equivalent classes of rooted trees (resp. rooted trees with \( m \) vertices).

For \( T \in T \), we denote by \( V(T) \) the set of vertices of \( T \) and \(|T| = |V(T)| \). A labeling of \( T \) in the set \( \{1, \ldots, n\} \) is a function \( l : V(T) \to \{1, \ldots, n\} \). A rooted tree \( T \) with a labeling \( l \) is called a \textit{labeled rooted tree}, denoted \((T, l)\). Given such, and given \( F = z - H \) as above, we make the following definitions, for \( v \in V(T) \):

\[
\begin{align*}
(1) \quad & H_v = H_{l(v)}, \\
(2) \quad & D_v = D_{l(v)}, \\
(3) \quad & D_{v^+} = \prod_{w \in v^+} D_w, \\
(4) \quad & P_{T, l} = \prod_{v \in V(T)} D_v + H_v.
\end{align*}
\]

Finally, we define systems of power series \( \mathcal{P}_T = (\mathcal{P}_{T, 1}, \ldots, \mathcal{P}_{T, n}) \) by setting

\[
\mathcal{P}_{T, i} = \frac{1}{|\text{Auto}(T)|} \sum_{\substack{l : V(T) \to \{1, \ldots, n\} \\
\text{such that } f^{(l_T)}(i) = i}} P_{T, l}
\]

for \( i = 1, \ldots, n \), where the sum above runs over all labelings of \( T \) having a fixed label for the root.

**Theorem 2.4. (Bass-Connell-Wright’s Inversion Formula)**

With the fixed notation above, the formal inverse \( G(z) \) of \( F(z) \) is given by

\[
G(z) = z + \sum_{T \in T} \mathcal{P}_T(z)
\]

For any \( m \in \mathbb{Z} \), we define the \( m^{th} \)-power \( F_m(z) \) of \( F(z) \) by

\[
F_m(z) = \underbrace{F \circ F \circ \cdots \circ F}_{m \text{ copies}}(z) \quad \text{if } m \geq 0;
\]

\[
F_m(z) = G^{[-m]}(z) \quad \text{if } m < 0.
\]

Considering all the efforts deriving formulas for the formal inverse \( G(z) \) of the formal map \( F(z) \), one may ask if there are some uniform formulas for all the powers \( F^m(z) \) (\( m \in \mathbb{Z} \)) of \( F(z) \). This question was answered in [WZ] by deriving a formula for the formal flow \( F_t(z) \), where \( F_t(z) \) is the unique 1-parameter subgroup with \( F_{t=1}(z) = F(z) \) of the formal automorphisms of \( \mathbb{C}^n \). The formula is derived by the D-log formulation [Z1] of \( F_t(z) \) and similar technics in [BCW].
Theorem 2.5. \cite{WZ} There is a unique sequence \( \{ \Psi_T(t) \mid T \in \mathbb{T} \} \) such that
\[
F_t(z) = z + \sum_{T \in \mathbb{T}} (-1)^{|T|} \Psi_T(t) \mathcal{P}_T(z) \tag{2.14}
\]

For some properties and a computational algorithm for the polynomials \( \Psi_T(t) \), see \cite{WZ}. Based on certain properties of \( \Psi_T(t) \) proved in \cite{WZ}, J. Shareshian \cite{Sh} pointed out to us that the polynomials \( \Psi_T(t) \) coincide with strict order polynomials \( \Omega(T, m) \) of rooted trees, which we will explain briefly below.

Note that, for any rooted tree \( T \), the graph structure induces a natural partial order on the set \( V(T) \) of vertices of \( T \) with the root \( rt_T \) of \( T \) serving as the unique minimum element. Hence, with this partial order, the set \( V(T) \) becomes a poset (partially ordered set). We will still use the same notation \( T \) to denote this poset. For any \( m \geq 1 \), we denote by \([m]\) the totally ordered set \( \{1, 2, \cdots, m\} \). We say a map \( \sigma \) from a finite poset \( P \) to \([m]\) is strict order preserving if \( \sigma(x) > \sigma(y) \) in \([m]\) implies \( x > y \) in \( P \). The following definition and theorem is well known in enumerative combinatorics. See, for example, \cite{St1}.

Definition-Theorem 2.6. For any finite poset \( P \), there exists a unique polynomial \( \Omega(T, t) \) such that, for any \( m \geq 1 \), \( \Omega(T, m) \) equals to the number of order-preserving maps from \( P \) to the totally ordered set \([m]\).

Theorem 2.7. \cite{Sh} For any rooted tree \( T \), we have
\[
\Psi_T(t) = \Omega(T, t) \tag{2.15}
\]

Remark 2.8. (a) Note that the formula \( \Omega(T, t) \) provides a uniform formula for all powers \( F^{[m]}(z) \) \((m \in \mathbb{Z})\) by setting \( t = m \).

(b) It is well known (See \cite{St1}. ) that \( \Omega(T, -1) = (-1)^{|T|} \). For a direct proof of this fact, see \cite{WZ}. Hence, by setting \( m = -1 \), the formula \( \Omega(T, t) \) becomes Bass-Connell-Wright’s inversion formula \( \Omega(T, t) \).

3. The First Recurrent Inversion Formula

In this section, we first study a deformation of formal maps from which we then derive our first recurrent inversion formula. Comparing with other well-known inversion formulas, the recurrent inversion formula in certain situations provides a more efficient method for the calculation of formal inverse maps. We also discuss a close relationship between the inversion problem and a Cauchy problem of a PDE, see Eq. \( \text{(3.3)} \) and \( \text{(3.6)} \).

We start with the deformation \( F_t(z) = z - tH(z) \) and let \( G_t(z) \) be its formal inverse. Note that, we can always write the formal inverse
Lemma 3.1. Let $F(z)$, $F_t(z)$, $G_t(z)$ and $N_t(z)$ be as above. Then we have

\begin{align}
N_t(F_t(z)) &= H(z), \\
H(G_t) &= N_t(z), \\
JN_t(F_t) &= JH(I - tJH)^{-1} = \sum_{k=1}^{\infty} JH_k(z)t^{k-1}.
\end{align}

In particular, for any $m \geq 1$, $JH^m(z) = 0$ if and only if $JN_t^m(z) = 0$.

Proof: Since $z = G_t(F_t)$, we have

\begin{align}
z &= F_t(z) + tN_t(F_t(z)) \\
z &= z - tH(z) + tN_t(F_t(z))
\end{align}

Therefore,

$$H(z) = N_t(F_t(z)),$$

which is Eq. (3.1). Composing the both sides of Eq. (3.1) with $G_t(z)$, we get Eq. (3.2).

Now we show Eq. (3.3). First, by the fact $JG_t(F_t(z)) = JF_t^{-1}(z)$, we have

\begin{align}
I + tJN_t(F_t) &= (I - tJH)^{-1}, \\
tJN_t(F_t) &= I - (I - tJH)^{-1} = tJH(I - tJH)^{-1}, \\
JN_t(F_t) &= JH(I - tJH)^{-1} = \sum_{k=1}^{\infty} JH_k(z)t^{k-1}.
\end{align}

Hence, we have Eq. (3.3).

Now, for any $m \geq 1$, by Eq. (3.3), we have

\begin{equation}
JN_t^m(F_t) = JH^m(I - tJH)^{-m}.
\end{equation}

Since $(I - tJH)^{-m}$ is invertible and $F_t(z)$ is an automorphism of $C[[t]][[z]]$, we have $JH^m(z) = 0$ if and only if $JN_t^m(z) = 0$. \qed

Theorem 3.2. Let $N_t(z) \in C[[t]][[z]]^{\times n}$ with $o(N_t(z)) \geq 2$. Then $G_t(z) = t + tN_t(z)$ is the formal inverse of $F_t(z) = z - tH(z)$ if and only
if \( N_t(z) \) is the unique solution of the Cauchy problem of the following partial differential equation

\[
\begin{align*}
\frac{\partial N_t}{\partial t} &= JN_t N_t \\
N_{t=0}(z) &= H(z)
\end{align*}
\]

where \( JN_t \) is the Jacobian matrix of \( N_t(z) \) with respect to \( z \).

**Proof:** By applying \( \frac{\partial}{\partial t} \) to the both sides of Eq. (3.1), we get

\[
0 = \frac{\partial N_t(F_t)}{\partial t} = \frac{\partial N_t}{\partial t}(F_t) + JN_t(F_t) \frac{\partial F_t}{\partial t} = \frac{\partial N_t}{\partial t}(F_t) - JN_t(F_t) H
\]

Therefore,

\[
\frac{\partial N_t}{\partial t}(F_t) = JN_t(F_t) H
\]

Composing with \( G_t \), we get

\[
\frac{\partial N_t}{\partial t} = JN_t H(G_t) = JN_t N_t
\]

Note that \( G_{t=0}(z) = z \), for it is the formal inverse of \( F_{t=0}(z) = z \). Eq. (3.6) follows immediately from Eq. (3.2) by setting \( t = 0 \).

We define the sequence \( \{N_{[m]}(z) | m \geq 0\} \) by setting \( N_{[0]}(z) = z \) and writing \( N_t(z) = \sum_{m=1}^{\infty} t^{m-1} N_{[m]}(z) \).

**Proposition 3.3.** Let \( N_t(z) = \sum_{m=1}^{\infty} t^{m-1} N_{[m]}(z) \) be the unique solution of Eq. (3.5) and (3.6). Then

\[
\begin{align*}
N_{[1]} &= H \\
N_{[m]} &= \frac{1}{m-1} \sum_{\substack{k+l=m \\ \ \ \ k,l \geq 1}} JN_{[k]} \cdot N_{[l]}
\end{align*}
\]

for any \( m \geq 2 \).

**Proof:** First, Eq. (3.7) follows immediately from Eq. (3.6). Secondly, by Eq. (3.5), we have
\[
\sum_{m=1}^{\infty} (m - 1)t^{m-2}N_{[m]}(z) = \left( \sum_{k=1}^{\infty} t^{k-1} JN_{[k]}(z) \right) \left( \sum_{l=1}^{\infty} t^{l-1} N_{[l]}(z) \right)
\]

Comparing the coefficients of \(t^{m-2}\) of the both sides of the equation above, we have

\[
(m - 1)N_{[m]}(z) = \sum_{k+l=m, k,l \geq 1} JN_{[k]} \cdot N_{[l]}
\]

for any \(m \geq 2\). Hence we get Eq. (3.8). \(\square\)

From Eq. (3.7), (3.8) and by using the mathematical induction, it is easy to see that we have the following lemma.

**Lemma 3.4.**

(a) \( o(N_{[m]}) \geq m + 1 \) for any \( m \geq 0 \).

(b) Suppose \( H(z) \in \mathbb{C}[z]^n \), then, for any \( m \geq 1 \), \( N_{[m]} \in \mathbb{C}[z]^n \) with \( \deg N_{[m]}(z) \leq (\deg H - 1)m + 1 \).

(c) If \( H(z) \) is homogeneous of degree \( d \), then, \( N_{[m]}(z) \) is homogeneous of degree \( (d - 1)m + 1 \) for any \( m \geq 1 \).

Note that, by Lemma 3.4 (a), the infinite sum \( \sum_{m=1}^{\infty} t^{m-1} N_{[m]}(z) \) makes sense for any complex number \( t = t_0 \). In particular, when \( t = 1 \), \( G_{t=1}(z) \) gives us the formal inverse \( G(z) \) of \( F(z) \).

**Theorem 3.5. (Recurrent Inversion Formula)**

Let \( \{N_{[m]}(z) | m \geq 1\} \) be the sequence defined by Eq. (3.7) and (3.8) recurrently. Then the formal inverse of \( F(z) = z - H(z) \) is given by

\[
G(z) = z + \sum_{m=1}^{\infty} N_{[m]}(z)
\]

The following proposition also seems interesting to us. It basically says that \( N_{t}(z) \) gives a family of formal maps from \( \mathbb{C}^n \) to \( \mathbb{C}^n \), which are “closed” under the inverse operation.

**Proposition 3.6.** For any \( s \in \mathbb{C} \), the formal inverse of \( U_{s,t}(z) = z - sN_{t}(z) \) is given by \( V_{s,t}(z) = z + sN_{t+s}(z) \). Actually, \( U_{s,t} = F_{t+s} \circ G_{t} \) and \( V_{s,t} = F_{t} \circ G_{s+t} \).

**Proof:**

\[
F_{t+s} \circ G_{t} = G_{t}(z) - (t + s)H(G_{t})
\]

\[
= z + tN_{t}(z) - (t + s)N_{t}(z)
\]

\[
= z - sN_{t}(z)
\]

\[
= U_{s,t}(z)
\]
Similarly, we can prove $V_{s,t} = F_t \circ G_{s+t}$.

Another special property of $N_t(z)$ is given by the following proposition.

**Proposition 3.7.** For any $U_{[0]}(z) \in \mathbb{C}[[z]]^n$, the unique power series solution $U_t(z)$ in both $t$ and $z$ of the Cauchy problem

\begin{align}
\frac{\partial U_t}{\partial t} &= JU_t N_t \\
U_{t=0}(z) &= U_{[0]}(z)
\end{align}

is given by $U_t(z) = U_{[0]}(z + tN_t(z))$.

**Proof:** It is easy to see that the power series solution in both $t$ and $z$ of the Cauchy problem is unique. So it will be enough to check that $U_t(z) = U_{[0]}(z + tN_t(z))$ is such a solution.

\[
\frac{\partial U_t}{\partial t} = \frac{\partial}{\partial t} U_{[0]}(z + tN_t(z))
\]

\[
= JU_{[0]}(z + tN_t(z))(N_t(z) + t\frac{\partial N_t}{\partial t}(z))
\]

\[
= JU_{[0]}(z + tN_t(z))(N_t(z) + tJN_t(z)N_t(z))
\]

\[
= JU_{[0]}(z + tN_t(z))(I + tJN_t(z))N_t(z)
\]

\[
= J(U_{[0]}(z + tN_t(z)))N_t(z)
\]

\[
= JU_t(z)N_t(z)
\]

Next we derive more consequences of Theorem 3.2

**Proposition 3.8.** Let $F(z) = z - H(z)$ be a formal map with $O(H(z)) \geq 2$. Then the inverse map $G(z) = z + H(z)$ if and only if $JH \cdot H = 0$.

**Proof:** First, by Eq. (3.1) and Eq. (3.3), we have

\[
(JN_t \cdot N_t)(F_t) = (I - tJH)^{-1}JH \cdot H.
\]

Hence, we have $JH \cdot H = 0$ if and only if $JN_t \cdot N_t = 0$.

Now we assume $JH \cdot H = 0$. By Eq. (3.5) and the fact above, we have $\frac{\partial N_t}{\partial t} = 0$. By Eq. (3.6), we have $N_t(z) = H(z)$. Hence, the proposition follows.

Next we assume $G(z) = z + H(z)$ and to show $JH \cdot H = 0$.

First, from the equation,

\[
z = F(G(z)) = z + H(z) - H(z + H(z)),
\]
we see that

\[(3.13) \quad H(z + H(z)) = H(z).\]

We consider the powers \(F^{[m]}(z) \ (m \geq 1)\) defined by Eq. (2.12) of \(F(z) = z - H(z)\) and make the following claim.

**Claim:** For any \(m \geq 1\), \(F^{[m]}(z) = z - mH(z)\).

**Proof of Claim:** We use the mathematical induction on \(m \geq 1\). The case \(m = 1\) is trivial. For \(m > 1\), we have

\[
F^{[m]}(z) = F^{[m-1]}(F(z)) = z + H(z) - (m - 1)H(z - H(z))
\]

Applying Eq. (3.13):

\[
= z - H(z) - (m - 1)H(z) = z - mH(z).
\]

Hence the claim holds.

Next we consider the formal flow \(F(z; t)\) of \(F(z)\), i.e. the unique formal maps with coefficients in \(\mathbb{C}[t]\) such that

1. \(F(z; 0) = z \) and \(F(z; 1) = F(z)\).
2. For any \(s, t \in \mathbb{C}\), we have \(F(F(z); s) = F(z; s + t)\).

By Proposition 2.1 and Lemma 2.4 in [Z1], we know that there exists a unique \(a(z) \in \mathbb{C}[[z]]^{\times n}\) with \(o(a(z)) \geq 2\), which was called the D-Log of \(F(z)\) in [WZ], such that

\[(3.14) \quad e^{ta(z)} \frac{\partial}{\partial z} z = F(z, t).\]

Note that \(F(z; m) = F^{[m]}(z)\) for any \(m \geq 1\). Since \(F(z; t) \in \mathbb{C}[t][[z]]^{\times n}\), the coefficients of all monomials appearing in \(F(z; t)\) are polynomials in \(t\). By the claim above, we see that the coefficients of all monomial appearing in \(F(z; t)\) but not in \(tH(z)\) vanish at any \(m \geq 1\), hence they must be identically zero. Therefore, we have

\[(3.15) \quad F(z, t) = z - tH(z).\]

From Eq. (3.15) above, it is easy to see that

\[(3.16) \quad a(z) = -H(z).\]

\[(3.17) \quad (a(z) \frac{\partial}{\partial z})^2 z = H(z) \frac{\partial}{\partial z} H(z) = JH \cdot H = 0.\]

Hence we are done. \(\Box\)
One immediate consequence of Proposition 3.8 is the following theorem which was first proved in [BCW].

**Theorem 3.9.** [BCW] Let \( F(z) = z + H(z) \) be a polynomial map with \( H(z) \) being homogeneous of degree \( d \geq 2 \). If \( J(H)^2 = 0 \), then the formal inverse map \( G = z - H(z) \).

**Proof:** When \( H(z) \) is homogeneous of degree \( d \geq 2 \), by Euler’s formula, we have
\[
JH^2(z)z = dJH \cdot H(z).
\]
(3.18) Hence, \( JH^2(z) = 0 \) if and only \( JH \cdot H = 0 \). Then the theorem follows immediately from Proposition 3.8 above. \( \square \)

In [BCW], H. Bass, E. Connell and D. Wright reduced the Jacobian conjecture to the cases when \( H(z) \) is homogeneous of degree 3. For further reductions in this direction, see [D1] and [D3]. Note that, when \( H(z) \) is homogeneous, the Jacobian condition \( f(F) = 1 \) is equivalent to the condition that the Jacobian matrix \( JH(z) \) is nilpotent. Next we will derive some consequences from our recurrent inversion formula for the case when \( H(z) \) is homogeneous of degree \( d \) \((d \geq 2)\).

**Proposition 3.10.** If \( H \) is homogeneous of degree \( d \geq 2 \), we have
\[
N_t(z) = \frac{1}{d} JN_t(z)(z - (d - 1)tN_t)
\]
(3.19)
\[
JN_t(z) \cdot z = d \left( I + \frac{(d - 1)t}{d} JN_t(z) \right) N_t(z)
\]
(3.20)

**Proof:** First, by Euler’s formula, we have \( dH(z) = JH(z)z \). By composing with \( G_t \) from right, we get
\[
dH(G_t) = JH(G_t)G_t(z)
\]
From Eq. (3.2), we have
\[
JH(G_t) = JN_t(z)JG_t^{-1}(z).
\]

Therefore, we have
\[
dN_t(z) = JN_t(z)JG_t^{-1}(z)(z + tN_t(z))
\]
d\[N_t(z) = JG_t^{-1}(z)JN_t(z)(z + tN_t(z))
 \]
because \( JG_t^{-1}(z) = (I + tJN_t(z))^{-1} \) commutes with the matrix \( JN_t(z) \).
\[
dJG_t(z)N_t(z) \equiv JN_t(z)(z + tN_t(z))
\]
d\[
(1 + tJN_t(z))N_t(z) \equiv JN_t(z)(z + tN_t(z))
\]
Solve \( N_t(z) \) and \( JN_t(z)z \) from the last equation above, we get Eq. (3.19) and (3.20). □

**Proposition 3.11.** If \( H \) is homogeneous of degree \( d \geq 2 \), then we have

\[
N_t(z) = \frac{1}{d} \sum_{k=1}^{\infty} (-1)^{k-1} \left( \frac{(d-1)t}{d} \right)^{k-1} JN_t^k(z) \cdot z
\]

\[
\frac{\partial N_t}{\partial t}(z) = \frac{1}{d} \sum_{k=1}^{\infty} (-1)^{k-1} \left( \frac{(d-1)t}{d} \right)^{k-1} JN_t^{k+1}(z) \cdot z
\]

The reason that we think the proposition above is interesting is because that it writes \( N_t(z) \) and \( JN_t(z) \) in terms of \( JN_t^k(z) \cdot z (k \geq 1) \).

In particular, when \( JH(z) \) is nilpotent, \( JN_t(z) \) is also nilpotent by Lemma 3.1 and the sums above are finite sums.

**Proof:** First note that Eq. (3.22) can be obtained by multiplying \( JN_t(z) \) from left to both sides of Eq. (3.21) and then applying Eq. (3.5). So we only need to show Eq. (3.21). By Eq. (3.20), we have

\[
N_t(z) = \frac{1}{d} \left( I + \frac{(d-1)t}{d} JN_t(z) \right)^{-1} JN_t(z) \cdot z
\]

But

\[
\left( I + \frac{(d-1)t}{d} JN_t(z) \right)^{-1} = \sum_{k=1}^{\infty} (-1)^{k-1} \left( \frac{(d-1)t}{d} \right)^{k-1} JN_t^{k-1}(z)
\]

Hence, Eq. (3.21) follows immediately. □

Next, we give one more proof for Theorem 3.9

**Second Proof of Theorem 3.9:** First from Lemma 3.1 (b), we know that \( JN_t^2(z) = 0 \). By Eq. (3.21) and Euler’s formula, we have

\[
N_t(z) = \frac{1}{d} JN_t(z) \cdot z
\]

\[
= \frac{1}{d} \sum_{m=1}^{\infty} t^{m-1} JN_{[m]}(z) \cdot z
\]

\[
= \frac{1}{d} \sum_{m=1}^{\infty} ((d-1)m + 1) t^{m-1} N_{[m]}(z)
\]
While \( N_t(z) = \sum_{m=1}^{\infty} t^{m-1} N_{[m]}(z) \). Comparing the coefficients of \( t^{m-1} \), we get

\[
N_{[m]}(z) = \frac{(d-1)m + 1}{d} N_{[m]}(z)
\]

\[
\frac{(d-1)(m-1)}{d} N_{[m]}(z) = 0
\]

Hence \( N_{[m]}(z) = 0 \) for any \( m \geq 2 \). \( \square \)

Unfortunately, both proofs given in this section for Theorem 3.9 fail for the cases \( JH^k(z) = 0 \) with \( k \geq 3 \). At least from the proofs above, we can see that the cases \( JH^k(z) = 0 \) (\( k \geq 3 \)) are dramatically different from the case \( JH^2(z) = 0 \) and will be much more difficult to study. D. Wright [W1] has shown that, for \( n = 3 \) and \( d = 3 \), the Jacobian conjecture is true. Hubber [H] proves the Jacobian conjecture when \( n = 4 \) and \( d = 3 \).

Finally, we make the following remarks on the key partial differential equation involved in this section.

By Theorem 3.2 and Proposition 3.3, it is easy to see that, for a polynomial map \( F(z) = z - H(z) \) with \( H(z) \) homogeneous of degree \( d \) (\( d \geq 2 \)), \( F(z) \) is an automorphism of \( \mathbb{C}^n \) if and only if the unique solution \( N_t(z) \) of the Cauchy problem (3.5)–(3.6) is a polynomial solution in both \( t \) and \( z \). Combining with the reduction theorem in [BCW], we see that the Jacobian conjecture is equivalent to the following conjecture.

**Conjecture 3.12.** Let \( H(z) \) be homogeneous of degree \( d = 3 \) with the Jacobian matrix \( JH(z) \) nilpotent. Then the unique solution \( N_t(z) \) of the Cauchy problem (3.5)–(3.6) is a polynomial solution in both \( t \) and \( z \).

Since the Jacobian conjecture has been proved by Wang [Wa] for the case \( d = 2 \), hence the statement in the conjecture above is also true in this case. It will be very interesting to see a proof for this fact by using some PDE methods. We hope that some PDE approaches to the Cauchy problem (3.5)–(3.6) will provide some new understanding to the Jacobian conjecture. It is interesting to notice the resemblance of the PDE (3.5) with the well known inviscid Burgers equation in \( n \) variables, which is the following partial differential equation.

\[
(3.23) \quad \frac{\partial U_t}{\partial t}(z) + (JU_t)^\tau U_t = 0
\]

where \((JU_t)^\tau\) is the transpose of the Jacobian matrix of \( U_t \) with respect to \( z \).
Set \( M_t(z) = -U_t(z) \), then Eq. (3.23) becomes
\[
\frac{\partial M_t}{\partial t}(z) = (JM_t)^TM_t
\]
which differs from our equation (3.5) only by the transpose for the Jacobian matrix of the unknown functions. Interesting, M. de Bondt, A. van den Essen [BE1] and G. Meng [M] recently have made a breakthrough on the Jacobian conjecture. They reduced the Jacobian conjecture to polynomial maps
\[ F(z) = z - H(z) \]
with \( H(z) = \nabla P(z) \) of some polynomials \( P(z) \in \mathbb{C}[z] \). Note that, in this case, \( JH(z) = \left( \frac{\partial^2 P(z)}{\partial z_i \partial z_j} \right) \) is nothing but the Hessian matrix of \( P(z) \). In particular, \( JH(z) \) is symmetric and our equation (3.5) does become exactly the \( n \)-dimensional invisible Burgers’ equation Eq. (3.24). For further study for formal maps of the form \( F(z) = z - H(z) \) with \( H(z) = \nabla P(z) \) of some polynomials \( P(z) \in \mathbb{C}[z] \). See [BE1], [BE2], [EW], [M], [Z2] and [Z3].

4. A Differential-Free Recurrent Formula

In this section, we derive a differential-free recurrent inversion formula for formal inverse \( G(z) \) of the polynomial maps \( F(z) = z - H(z) \) with \( H(z) \) homogeneous of degree \( d \geq 2 \). When \( d = 3 \), our formula is same as the one given by Drużkowski [D2] except the symmetric multi-linear form involved is also given explicitly in our approach.

In this section, we will always assume that \( H(z) \) is homogeneous of degree \( d \) for some \( d \geq 2 \). For any \( 1 \leq j \leq d \), we set \( U^{(j)} = (U_1^{(j)}, U_2^{(j)}, \ldots, U_n^{(j)}) \), and \( D^{(j)} = \sum_{k=1}^{n} U_k^{(j)} \frac{\partial}{\partial z_k} \), where \( U_i^{(j)} \) are formal variables. For any \( 1 \leq i \leq n \), we define the \( d \)-linear form \( B_i \) by
\[
(4.1) \quad B_i(U^{(1)}, U^{(2)}, \ldots, U^{(d)}) = \frac{1}{d!} D^{(1)} D^{(2)} \cdots D^{(d)} H_i(z).
\]

Note that the differential operators \( D^{(j)} \) \( (1 \leq j \leq d) \) commute with each other, hence the \( d \)-linear forms \( B_i(U^{(1)}, U^{(2)}, \ldots, U^{(d)}) \) \( (1 \leq i \leq n) \) are symmetric. We set \( B = (B_1, B_2, \ldots, B_n) \).

**Lemma 4.1.** Let \( B(U^{(1)}, U^{(2)}, \ldots, U^{(d)}) \) be the symmetric multi-linear forms defined by Eq. (4.1). Then
\[
(4.2) \quad B(z, z, \ldots, z) = H(z).
\]
**Proof:** Note that, by Eq. (4.1) and Euler’s formula, we have

\[
B_i(z, z, \cdots, z) = \frac{1}{d!} \sum_{k_1, \cdots, k_d=1}^n z_{k_1}z_{k_2}\cdots z_{k_d} \frac{\partial^d H_i}{\partial z_{k_1}\partial z_{k_2}\cdots \partial z_{k_d}}
\]

\[
= \frac{1}{d!} \sum_{k_2, \cdots, k_d=1}^n z_{k_2}\cdots z_{k_d} \sum_{k_1=1}^n z_{k_1} \frac{\partial}{\partial z_{k_1}} \left( \frac{\partial^{d-1} H_i}{\partial z_{k_2}\partial z_{k_3}\cdots \partial z_{k_d}} \right)
\]

\[
= \frac{1}{d!} \sum_{k_3, \cdots, k_d=1}^n z_{k_3}\cdots z_{k_d} \frac{\partial^{d-1} H_i}{\partial z_{k_3}\partial z_{k_3}\cdots \partial z_{k_d}}
\]

This is because that \( \frac{\partial^{d-1} H_i}{\partial z_{k_2}\partial z_{k_3}\cdots \partial z_{k_d}} \) is homogeneous of degree 1.

\[
= \frac{2}{d!} \sum_{k_3, \cdots, k_d=1}^n z_{k_3}\cdots z_{k_d} \frac{\partial^{d-2} H_i}{\partial z_{k_3}\partial z_{k_3}\cdots \partial z_{k_d}}
\]

This is because that \( \frac{\partial^{d-2} H_i}{\partial z_{k_3}\partial z_{k_3}\cdots \partial z_{k_d}} \) is homogeneous of degree 2.

\[
= \cdots
\]

By repeating the procedure above, it is easy to see that Eq. (4.2) holds. \( \square \)

By Lemma 3.4 (c), we can write the formal inverse \( G(z) \) as \( G(z) = z + \sum_{m=1}^\infty N_{[m]}(z) \), where \( N_{[m]}(z)(m \geq 1) \) are homogeneous of degree \( m(d-1) + 1 \). Note that, for convenience, we have set \( N_{[0]}(z) = z \). Now we can write down our second recurrent inversion formula as follows.

**Theorem 4.2.** We have the following recurrent formula for the formal inverse \( G(z) = z + \sum_{m=1}^\infty N_{[m]}(z) \).

(4.3) \( N_{[1]}(z) = H(z) \)

(4.4) \( N_{[m+1]}(z) = \sum_{k_1+\cdots+k_n=m \atop k_i \geq 0} B(N_{[k_1]}, N_{[k_2]}, \cdots, N_{[k_n]}) \)

for any \( m \geq 1 \).
Proof: By replacing \( z \) with \( G(z) \) in Eq. (4.2), we get

\[
B(\sum_{m=0}^{\infty} N[m], \sum_{m=0}^{\infty} N[m], \cdots, \sum_{m=0}^{\infty} N[m]) = H(G(z))
\]

While, by Eq. (3.2) in Lemma 3.1 we have

\[
H(G(z)) = N(z) = \sum_{m=1}^{\infty} N[m](z)
\]

By using the equations above and the fact that \( B \) is a multi-linear form, we get

\[
\sum_{m=1}^{\infty} N[m](z) = B(\sum_{m=0}^{\infty} N[m], \sum_{m=0}^{\infty} N[m], \cdots, \sum_{m=0}^{\infty} N[m])
\]

\[
= \sum_{r=0}^{\infty} \sum_{k_1 + \cdots + k_d = r \atop k_i \geq 0} B(N[k_1], N[k_2], \cdots, N[k_d])
\]

By comparing the homogeneous parts of both sides of the equation above, we get Eq. (4.4). \( \square \)

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DEPARTMENT OF MATHEMATICS, ILLINOIS STATE UNIVERSITY, NORMAL IL 61790-4520.

E-mail: wzhao@ilstu.edu.