LATTICE PATHS WITH INFINITELY MANY DOWN STEPS – THE NEGATIVE BOUNDARY MODEL

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Abstract. We consider a variation of Dyck paths, where additionally to steps $(1,1)$ and $(1,-1)$ down-steps $(1,-j)$, for $j \geq 2$ are allowed. We give credits to Emeric Deutsch for that. The enumeration of such objects living in a strip is performed. Methods are the kernel method and techniques from linear algebra.

1. Introduction

Emeric Deutsch [1] had the idea to consider a variation of ordinary Dyck paths, by augmenting the usual up-steps and down-steps by one unit each, by down-steps of size $3, 5, 7, \ldots$. This leads to ternary equations, as can be seen for instance from [3].

The present author started to investigate a related but simpler model of down-steps $1, 2, 3, 4, \ldots$ and investigated it (named Deutsch paths in honour of Emeric Deutsch) in a series of papers, [2][4][5].

This paper is an further member of this series: The condition that (as with Dyck paths) the paths cannot enter negative territory, is relaxed, by introducing a negative boundary $-t$. Here are two recent publications about such a negative boundary: [8] and [7].

Instead of allowing negative altitudes, we think about the whole system shifted up by $t$ units, and start at the point $(0, t)$ instead. This is much better for the generating functions that we are going to investigate. Eventually, the results can be re-interpreted as results about enumerations with respect to a negative boundary.

The setting with flexible initial level $t$ and final level $j$ allows us to consider the Deutsch paths also from left to right (they are not symmetric!), without any new computations.

The next sections achieves this, using the celebrated kernel-method, one of the tools that is dear to our heart [6].

In the following section, an additional upper bound is introduced, so that the Deutsch paths live now in a strip. The way to attack this is linear algebra. Once everything has been computated, one can relax the conditions and let lower/upper boundary go to $\mp\infty$.

2. Generating functions and the kernel method

As discussed, we consider Deutsch paths starting at $(0,t)$ and ending at $(n,j)$, for $n, t, j \geq 0$. First we consider univariate generating functions $f_j(z)$, where $z^n$ stays for

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\( n \) steps done, and \( j \) is the final destination. The recursion is immediate:

\[
f_j(z) = \lfloor t = j \rfloor + zf_{j-1}(z) + z \sum_{k>j} f_k(z),
\]

where \( f_{-1}(z) = 0 \). Next, we consider

\[
F(z, u) := \sum_{j \geq 0} f_j(z)u^j,
\]

and get

\[
F(z, u) = u^t + zuF(z, u) + z \sum_{j \geq 0} u^j \sum_{k>j} f_k(z)
\]

\[
= u^t + zuF(z, u) + z \sum_{k>0} f_k(z) \sum_{0 \leq j < k} u^j
\]

\[
= u^t + zuF(z, u) + z \sum_{k>0} f_k(z) \frac{1 - u^k}{1 - u}
\]

\[
= u^t + zuF(z, u) + \frac{z}{1-u}[F(z, 1) - F(z, u)]
\]

\[
= \frac{u^t(1 - u) + zF(z, 1)}{z - zu + zu^2 + 1 - u}.
\]

Since the critical value is around \( u = 1 \), we write the denominator as

\[
z(u - 1)^2 + (u - 1)(z - 1) + z = z(u - 1 - r_1)(u - 1 - r_2),
\]

with

\[
r_1 = \frac{1 - z + \sqrt{1 - 2z - 3z^2}}{2z}, \quad r_2 = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z}.
\]

The factor \((u-1-r_2)\) is bad, so the numerator must vanish for \( [u^t(1-u)+zF(z, 1)]_{u=1+r_2} \), therefore

\[
zF(z, 1) = (1 + r_2)^t r_2.
\]

Furthermore

\[
F(z, u) = \frac{u^t(1-u)+zF(z, 1)}{u-r_2z(u-r_1)}.
\]

The expressions become prettier using the substitution \( z = \frac{v}{1+v+v^2} \); then

\[
r_1 = \frac{1}{v}, \quad r_2 = v.
\]

It can be proved by induction (or computer algebra) that

\[
\frac{u^t(1-u) + v(1+v)^t}{u-1-v} = -v \sum_{k=0}^{t-1} (1+v)^{t-1-k} - u^t.
\]

Furthermore

\[
\frac{1}{z(u-1-r_1)} = -\frac{1}{z(1+r_1)(1 - \frac{u}{1+r_1})}.
\]
and so

\[ f_j(z) = [u^j]F(z, u) = [u^j]\left[ v \sum_{k=0}^{t-1} (1 + v)^{t-1-k}u^k + u^t \right] \sum_{\ell \geq 0} \frac{u^\ell}{z(1 + r_1)^{\ell+1}}. \]

Of interest are two special cases: The case that was studied before \([2]\) is \(t = 0\):

\[ f_j = \frac{(1 + v + v^2)v^j}{(1 + v)^{j+1}}. \]

The other special case is \(j = 0\) for general \(t\), as it may be interpreted as Deutsch paths read from right to left, starting at level 0 and ending at level \(t \geq 1\) (for \(t = 0\), the previous formula applies):

\[ f_0(z) = [u^0]\left[ v \sum_{k=0}^{t-1} (1 + v)^{t-1-k}u^k + u^t \right] \sum_{\ell \geq 0} \frac{u^\ell}{z(1 + r_1)^{\ell+1}} = v(1 + v)^{t-1}\frac{1}{z(1 + r_1)} = v(1 + v + v^2)(1 + v)^{t-2}. \]

The next section will present a simplification of the expression for \(f_j(z)\), which could be obtained directly by distinguishing cases and summing some geometric series.

3. Refined analysis: lower and upper boundary

Now we consider Deutsch paths bounded from below by zero and bounded from above by \(m - 1\); they start at level \(t\) and end at level \(j\) after \(n\) steps. For that, we use generating functions \(\varphi_j(z)\) (the quantity \(t\) is a silent parameter here). The recursions that are straight-forwarded are best organized in a matrix, as the following example shows.

\[
\begin{pmatrix}
1 & -z & -z & -z & -z & -z & -z \\
-z & 1 & -z & -z & -z & -z & -z \\
0 & -z & 1 & -z & -z & -z & -z \\
0 & 0 & -z & 1 & -z & -z & -z \\
0 & 0 & 0 & -z & 1 & -z & -z \\
0 & 0 & 0 & 0 & -z & 1 & -z \\
0 & 0 & 0 & 0 & 0 & -z & 1
\end{pmatrix}
\begin{pmatrix}
\varphi_0 \\
\varphi_1 \\
\varphi_2 \\
\varphi_3 \\
\varphi_4 \\
\varphi_5 \\
\varphi_6 \\
\varphi_7
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

The goal is now to solve this system. For that the substitution \(z = \frac{v}{1 + v + v^2}\) is used throughout. The method is to use Cramer’s rule, which means that the right-hand side has to replace various columns of the matrix, and determinants have to be computed. At the end, one has to divide by the determinant of the system.

Let \(D_m\) be the determinant of the matrix with \(m\) rows and columns. The recursion

\[(1 + v + v^2)^2m_{n+2} - (1 + v + v^2)(1 + v)^2D_{m+1} + v(1 + v)^2D_m = 0\]

appeared already in \([2]\) and is not difficult to derive and to solve:

\[D_m = \frac{(1 + v)^{m-1} - v^{m+2}}{(1 + v + v^2)^m - 1/v}.\]

To solve the system with Cramer’s rule, we must compute a determinant of the following type,
where the various rows are replaced by the right-hand side. While it is not impossible to solve this recursion by hand, it is very easy to make mistakes, so it is best to employ a computer. Let \( D(m; t, j) \) the determinant according to the drawing.

It is not unexpected that the results are different for \( j < t \) resp. \( j \geq t \). Here is what we found:

\[
D(m; t, j) = \frac{(1 + v)^{t-j-3+m}(1 - v^{j+1})v(1 - v^{m-t})}{(1 - v)^2(1 + v + v^2)^{m-1}}, \quad \text{for } j < t,
\]

\[
D(m; t, j) = \frac{v^{j-t}(1 - v^{t+2})(1 - v^{1-j+m})}{(1 - v)^2(1 + v + v^2)^{m-1}(1 + v)^{j-t+3-m}}, \quad \text{for } j \geq t.
\]

To solve the system, we have to divide by the determinant \( D_m \), with the result

\[
\varphi_j = \frac{D(m; t, j)}{D_m} = \frac{(1 + v)^{t-j-2}(1 - v^{j+1})v(1 - v^{m-t})(1 + v + v^2)}{(1 - v)(1 - v^{m+2})}, \quad \text{for } j < t,
\]

\[
\varphi_j = \frac{D(m; t, j)}{D_m} = \frac{v^{j-t}(1 - v^{t+2})(1 - v^{1-j+m})(1 + v + v^2)}{(1 - v)(1 + v)^{j-t+2}(1 - v^{m+2})}, \quad \text{for } j \geq t.
\]

We found all this using Computer algebra. Some critical minds may argue that this is only experimental. One way of rectifying this would be to show that indeed the functions \( \varphi_j \) solve the system, which consists of summing various geometric series; again, a computer could be helpful for such an enterprise.

Of interest are also the limits for \( m \to \infty \), i.e., no upper boundary:

\[
\varphi_j = \lim_{m \to \infty} \frac{D(m; t, j)}{D_m} = \frac{(1 + v)^{t-j-2}(1 - v^{j+1})v(1 + v + v^2)}{(1 - v)}, \quad \text{for } j < t,
\]

\[
\varphi_j = \frac{v^{j-t}(1 - v^{t+2})(1 + v + v^2)}{(1 - v)(1 + v)^{j-t+2}}, \quad \text{for } j \geq t.
\]

The special case \( t = 0 \) appeared already in the previous section:

\[
\varphi_j = \frac{v^j(1 + v + v^2)}{(1 + v)^{j+1}}.
\]

Likewise, for \( t \geq 1 \),

\[
\varphi_0 = v(1 + v + v^2)(1 + v)^{t-2}.
\]

In particular, the formulæ show that the expression from the previous section can be simplified in general, which could have been seen directly, of course.
Theorem 1. The generating function of Deutsch path with lower boundary 0 and upper boundary \(m - 1\), starting at \((0, t)\) and ending at \((n, j)\) is given by

\[
(1 + v)^{j-t-2}(1 - v^{j+1})v(1 - v^{m-t})(1 + v + v^2)
\]

\[
\frac{(1 - v)(1 - v^{m+2})}{(1 - v)(1 + v)^{j-t+2}(1 - v^{m+2})},
\]

for \(j < t\),

\[
v^{j-t}(1 - v^{t+2})(1 - v^{1-j+m})(1 + v + v^2)
\]

\[
\frac{(1 - v)(1 + v)^{j-t+2}(1 - v^{m+2})}{(1 - v)(1 + v)^{j-t+2}(1 - v^{m+2})},
\]

for \(j \geq t\),

with the substitution \(z = \frac{v}{1 + v + v^2}\).

By shifting everything down, we can interpret the results as Deutsch walks between boundaries \(−t\) and \(m - 1 - t\), starting at the origin \((0, 0)\) and ending at \((n, j - t)\).

Theorem 2. The generating function of Deutsch path with lower boundary \(−t\) and upper boundary \(h\), starting at \((0, 0)\) and ending at \((n, i)\) with \(−t \leq i \leq h\) is given by

\[
(1 + v)^{i-2}(1 - v^{i+t+1})v(1 - v^{h+1})(1 + v + v^2)
\]

\[
\frac{(1 - v)(1 - v^{h+t+3})}{(1 - v)(1 + v)^{i-t+2}(1 - v^{h+t+3})},
\]

for \(i < 0\),

\[
v^i(1 - v^{t+2})(1 - v^{2-i+h})(1 + v + v^2)
\]

\[
\frac{(1 - v)(1 + v)^{i-t+2}(1 - v^{h+t+3})}{(1 - v)(1 + v)^{i-t+2}(1 - v^{h+t+3})},
\]

for \(i \geq 0\).

It is possible to consider the limits \(t \to \infty\) and/or \(h \to \infty\) resulting in simplified formulæ.

4. Conclusion

Various parameters could be worked out starting from the present findings. Currently, nothing to that effect has been done.

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