Magnetotransport of Weyl semimetals with $\mathbb{Z}_2$ symmetry and chiral anomaly.

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ABSTRACT: We calculate the magnetoconductivity of the Weyl semimetal with $\mathbb{Z}_2$ symmetry and chiral anomaly utilizing the recently developed hydrodynamic theory. The system in question will be influenced by magnetic fields connected with ordinary Maxwell and the second $U(1)$-gauge field, which is responsible for anomalous topological charge. The presence of chiral anomaly and $\mathbb{Z}_2$ anomalous charge endow the system with new transport coefficients. We start with the linear perturbations of the hydrodynamic equations and calculate the magnetoconductivity of this system. The holographic approach in the probe limit is implemented to obtain the explicit dependence of the longitudinal magnetoconductivities on the magnetic fields.

KEYWORDS: Gauge-gravity correspondence, Holography and condensed matter physics (AdS/CMT), Black Holes

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1 Introduction

The recently observed resurgence of interest in the physics of chiral systems is related to the important discoveries in such fields as dynamics of relativistic quark-gluon plasma [1]-[6] and high energy astrophysical processes in proto-neutron stars [7] on one side and the recent realization of the known and novel relativistic symmetries for electrons in condensed matter systems. Chiral magnetic and vortical effects observed in the high energy nuclear collisions have been reviewed recently [8]. The topological quantum matter is becoming an important play-ground for discoveries of numerous novel phenomena not observed before in particle physics.

Electrons in metals form a system of interacting particles. The strength of the interactions is measured with respect to their kinetic energy and thus depends on the properties of the system under study. The standard paradigm was to describe electrons in metals by the Landau quasiparticle theory. However, there are many very strongly interacting materials which description require approaches going beyond quasiparticle picture. For example, in very clean metals the flow of electrons resembles that of the classical fluid and is described by the laws of classic hydrodynamic theory [9]. In some materials like Dirac or Weyl semimetals [10], the spectrum of electrons in vicinity of the Fermi level, is linear and thus relativistic despite their velocity $v_F$ being much smaller then the velocity of light. Thus, the condensed matter allows for the relativistic symmetries and opens new possibilities for laboratory studies of exotic particles, which were predicted, but never seen in vacuum. The long sought Majorana fermion may constitute an example of them [11].

Moreover, the solid state crystals allow for the symmetries beyond Lorentz-Poincare one and thus for quasiparticles not realized or even not possible to realize in particle physics.
The discovery of the mentioned semimetals of Dirac or Weyl character, has sparked the intense studies of band crossing phenomena in crystals. This resulted in the discovery of “relativistic” quasiparticles with effective “spin” quantum number different from $1/2$. The crystal space groups hosting such fermions have been identified and the materials realizing them proposed [12]. Moreover, the “spin” $3/2$ electrons were apparently observed [13].

The relativistic spectrum together with the associated chirality of the quantum particles is a source of new phenomena generally known as chiral quantum anomalies, first discovered by Adler [14] and independently by Bell and Jackiw [15]. They have played an important role in understanding the neutral pion decay and recently they are subject to vigorous studies in condensed matter laboratories. In the presence of dense matter, quantum anomalies modify the relativistic hydrodynamics by leading to new phenomena and novel transport coefficients [16] i.e., an anomalous current is generated by an external magnetic field or by vortices in fluid which carries the charge in question [17]. In the realm of condensed matter physics the role of chiral and other anomalies is especially visible under the influence of external magnetic field or a longitudinal temperature gradient. The anomaly also influences the electric DC conductivity. Namely, the longitudinal DC conductivity is amplified by magnetic field [18, 19]. The key prediction was found in a kinetic description at weak coupling, as well as, hydrodynamics and holographic attitude at strong coupling.

The problem of the longitudinal magnetococonductivity, in the system with a chiral anomaly and background magnetic field was examined in [20]. By means of the linear response method in the hydrodynamic limit and holography attitude, it was shown that one needs to have energy, momentum and charge dissipations to obtain a finite DC longitudinal magnetococonductivity. Among all, the holographic treatment of the system in question reveals that in the intermediate regime of the magnetic field, one gets a negative magnetoresistivity decreasing as inverse of the magnetic field. The same problem was studied in [21], where the longitudinal DC conductivity bounded with the Lifshitz like fixed points, in the presence of chiral anomalies, in $(3+1)$-dimensions was found.

Following the method presented in [22], the generalized expression for DC and Hall conductivities were delivered also for large class of the holographic massive models [23]. Massive gravity models [24]-[28] attract attentions due to diffeomorphism symmetry breaking which causes the non-conservation of the energy-momentum tensor. The elaborated effect is similar to the dissipation of the momentum.

The specific angular dependence of this enhancement, in the longitudinal direction along the magnetic field, is the key prediction which arises from the aforementioned descriptions. However, already in the weak coupling limit, it was envisaged that negative magnetoresistance might appear [29].

The similar effect can arise in the hydrodynamical attitude, when the distinctive gradient expansion is taken into account (the non-relativistic constitutive relations can be derived from the most general covariant form of the expression, up to the third order in the field strength) [30]. Moreover, in a relativistic hydrodynamic theory of transport, it depends on the macroscopic model (one can achieve both negative and positive magnetoresistance). In non-Galilean invariant fluids the effect can arise not only due to the presence of a background magnetic field but also depends on the specific structure of the considered
hydrodynamics [30].

The other mechanism leading to negative magnetoresistivity was proposed in [31]-[32], by using two interacting $U(1)$-gauge fields.

The recent experimental works conducted in Dirac or Weyl semimetals like $Na_3Bi$, $ZrTe_5$ [33, 34] and $TaAs$, $NbP$ [35]-[37], confirm the evidences of chiral anomaly. In principle there are two classes of Dirac semimetals, in first one the Dirac points appear at the time reversal invariant momenta in the first Brillouin zone. The second class comprises the elements in which the Dirac points take place in pairs at two arbitrary points in the Brillouin zone. They are separated in momentum space along a rotational axis [38, 39]. Moreover it turns out that they are characterized by a non-trivial $Z_2$ topological invariant protecting the nodes and leading to the presence of Fermi arc surface states [40]-[43]. It has been argued that the $Z_2$ anomalous charge affects the transport characteristics of the materials in question [44]. The $Z_2$ topological charge influence on the transport properties was studied in a relativistic hydrodynamics limit in [45].

The behavior of metals influenced by electromagnetic field is of the great importance for understanding their transport properties. For the most materials under inspection the longitudinal conductivity is a decreasing function of the magnetic field. However, for the Weyl semimetals (materials for which conduction bands intersect at distant points in the Brillouin zone) one has an exception to the rules [20]. In the recent experiments [46] the negative magnetoresistance in the intermediate regime of magnetic field was confirmed. There were proposed several alternative ways of explaining the above phenomenon. For example in [46]-[47], the weak-antilocalization effect was considered for the explanation of the aforementioned effect.

The main objectives of our work is to examine the influence of $Z_2$ topological charge and chiral anomaly on conductivities, in the presence of non-zero ingredients of magnetic field components of the $U(1)$-gauge fields. Our considerations constitute the development of the ideas [45], where the hydrodynamical characteristics of the model with two anomalies were presented.

The paper is organized as follows. In Section 2 we present the calculations of electric conductivities for both studied $U(1)$-gauge fields, in the hydrodynamics limit, taking into account chiral and $Z_2$ anomalies. We have elaborated the case with background magnetic fields and with all the possible dissipation terms in order to achieve finite value of the DC longitudinal conductivities. Section 3 is devoted to the holographic model of our system, in the probe limit. We discuss the relation between the parameters entering the holographic action and those responsible for anomalies in Section 3.1. As a background spacetime we take five-dimensional AdS Schwarzschild black brane. Section 4 is connected with the independent calculation of magnetococonductivities for the studied holographic model with anomalies in the probe limit, using Kubo formula. The obtained results match the hydrodynamical formula in the limit in question. In last section we conclude the main results.
2 Hydrodynamics of the system

The presence of quantum triangle anomalies severely modifies the relativistic hydrodynamic equations. The novel transport coefficients appear and are expressed in terms of anomaly coefficients (charges) and the systems equation of state \[16\]. Recently, we have generalized the relativistic hydrodynamic theory to the systems endowed with chiral charges and possessing additional symmetry \[44\]. To describe the $\mathbb{Z}_2$ symmetry in chiral Weyl semimetal of the second kind we have used two different electro-magnetic fields, each acting on the appropriate charges related to anomaly and $\mathbb{Z}_2$ symmetry, respectively.

The motivation standing behind our research is to include the different dissipation terms in studies of longitudinal DC-conductivity. At first we directly find the aforementioned conductivity in the hydrodynamical attitude in the four-dimensional spacetime with two background magnetic fields. One of the fields is connected with the ordinary Maxwell one while the other constitutes the magnetic component of the auxiliary $U(1)$-gauge field responsible for the anomalous $\mathbb{Z}_2$ topological charge. We restrict our considerations to the linear response level. For the purpose of the future reference and to established the notation we shall briefly present main ideas and results of the mentioned hydrodynamic approach \[45\].

The inclusion of the dissipation terms \[20, 48\], is to perturb first a hydrodynamical system in a given equilibrium state and solve the equation of motion with initial values of the perturbations. In the next step, the searched transport coefficients can be found using the response of the electric currents or the thermal one, to the initial values of the corresponding perturbations.

2.1 Hydrodynamic equations for system with chiral and $\mathbb{Z}_2$ anomalies.

The key set of the studied relations will be the hydrodynamical equations of motion in the presence of $\mathbb{Z}_2$ and chiral anomalies provided by \[45\]

\[
\begin{align*}
\partial_\alpha T^{\alpha\beta}(F, B) &= F^{\alpha\beta} j_\alpha(F) + B^{\alpha\beta} j_\alpha(B), \\
\partial_\alpha j^\alpha(F) &= C_1 E_{(F)\alpha} B^{(F)\alpha} + C_2 E_{(B)\alpha} \tilde{B}^{(B)\alpha}, \\
\partial_\alpha j^\alpha(B) &= C_3 E_{(B)\alpha} B^{(F)\alpha} + C_4 E_{(F)\alpha} \tilde{B}^{(B)\alpha}.
\end{align*}
\]

(2.1) \hspace{1cm} (2.2) \hspace{1cm} (2.3)

It has to be recalled that $C_2 = C_4$ due to symmetry reasons. In the above equations we have denoted the electric and magnetic fields in the fluid rest frame by

\[
\begin{align*}
E^{(F)}_{\alpha} &= F_{\alpha\beta} u^\beta, & B^{(F)}_{\alpha} &= \frac{1}{2} \epsilon_{\alpha\beta\rho\delta} u^\beta F^{\rho\delta}, \\
\tilde{E}^{(B)}_{\alpha} &= B_{\alpha\beta} u^\beta, & \tilde{B}^{(B)}_{\alpha} &= \frac{1}{2} \epsilon_{\alpha\beta\rho\delta} u^\beta B^{\rho\delta}.
\end{align*}
\]

(2.4) \hspace{1cm} (2.5)

where $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ stands for the ordinary Maxwell field strength tensor, while the second $U(1)$-gauge field $B_{\mu\nu}$ is given by $B_{\mu\nu} = 2\partial_{[\mu} B_{\nu]}$. $j_{\mu}(F)$, $j_{\mu}(B)$ represent the adequate currents connected with each of the gauge field.
On the other hand, the energy momentum tensor and the adequate currents needed for 
the hydrodynamic description of the relativistic fluid, imply \[49\]

\[
T^{\mu\nu} = (\epsilon + p) u^\mu u^\nu + p g^{\mu\nu} + \tau^{\mu\nu},
\]

\[
j^\mu(\ell) = \rho u^\mu + \ell^\mu,
\]

\[
j^\mu(B) = \rho_d u^\mu + \ell^\mu_B,
\]

where \(\tau^{\mu\nu}\) and \(V^{\mu}_{(\ell)}\) denote corrections, higher order in velocities, responsible for dissipative 
effects. Using the fact that there is no dissipative force in the rest frame of the liquid element, 
we obtain \(u_\alpha \tau^{\alpha\beta} = u_\alpha \) \(V^{\alpha}_{\ell F} = u_\alpha \) \(V^{\alpha}_{\ell B} = 0\). By \(\epsilon\) we have denoted the energy density, \(p\) is connected with pressure and \(\rho, \rho_d\) are the \(U(1)\) charge densities. The four-vector \(u^\mu\), with 
the normalization \(u_\mu u^\mu = -1\), describes the flow of the fluid in the system in question and the 
general expressions for the dissipative components of the energy - momentum tensor and currents read

\[
\tau^{\mu\nu} = -\eta P^{\mu\alpha} P^{\nu\beta} (\partial_\alpha u_\beta + \partial_\beta u_\alpha) - \left(\zeta - \frac{2}{3} \eta\right) P^{\mu\nu} \partial_\alpha u^\alpha,
\]

\[
V^{\alpha}_{\ell F} = -\sigma_F \left[ T P^{\beta\alpha} \partial_\beta \left(\frac{\mu}{T}\right) - E^{(F)\alpha}\right] - \sigma_{FB} \left[ T P^{\beta\alpha} \partial_\beta \left(\frac{\mu_d}{T}\right) - E^{(B)\alpha}\right] + \xi \omega^\alpha
\]

\[
+ \xi_B \left( B^{(F)\alpha} + B^{(B)\alpha} \right),
\]

\[
V^{\alpha}_{\ell B} = -\sigma_B \left[ T P^{\beta\alpha} \partial_\beta \left(\frac{\mu_d}{T}\right) - E^{(B)\alpha}\right] - \sigma_{BF} \left[ T P^{\beta\alpha} \partial_\beta \left(\frac{\mu}{T}\right) - E^{(F)\alpha}\right] + \xi_d \omega^\alpha
\]

\[
+ \xi_B \left( B^{(F)\alpha} - B^{(B)\alpha} \right).
\]

where \(P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu\), by \(\omega^\mu = 1/2 \epsilon_{\nu\rho\delta} u^\nu \partial^\rho u^\delta\) we have denoted the vorticity. On the 
other hand, \(\xi, \xi_d, \xi_B, \xi_{FB}, \xi_{B\ell F}\) are kinetic coefficients being functions of \(T\) and \(\mu, \mu_d\). 
They are given by the following expressions [45]:

\[
\xi = C_1 \mu^2 \left(1 - \frac{2}{3} \frac{\mu}{\epsilon + p}\right) + \mu^2 C_2 \left(1 - \frac{2}{3} \frac{\mu}{\epsilon + p}\right) - 4 \frac{\rho}{\epsilon + p} \frac{T^2}{\gamma_1 + \mu_d \gamma_2},
\]

\[
-2 \frac{\rho}{\epsilon + p} \frac{T^2}{\gamma_3 + 2 \gamma_1 T^2},
\]

\[
-2 \frac{\rho}{\epsilon + p} \frac{T^2}{\gamma_3 + 2 \gamma_2 T^2},
\]

\[
\xi_d = \frac{2}{3} C_1 \frac{\rho_d}{\epsilon + p} \mu^3 + 2 \mu \mu_d \left(1 - \frac{\rho_d}{\epsilon + p}\right) - 4 \frac{\rho_d}{\epsilon + p} \frac{T^2}{\gamma_1 + \mu_d \gamma_2},
\]

\[
-2 \frac{\rho_d}{\epsilon + p} \frac{T^2}{\gamma_3 + 2 \gamma_2 T^2},
\]

\[
\xi_B = C_1 \mu \left(1 - \frac{\rho}{2 \epsilon + p}\right) - \frac{1}{2} \frac{C_2}{C_1} \frac{\rho \mu_d^2}{\epsilon + p} - 4 \frac{\rho}{\epsilon + p} \frac{T^2}{\gamma_1},
\]

\[
\xi_{FB} = C_2 \mu \left(1 - \frac{\rho_d}{\epsilon + p}\right) - \frac{\rho_d}{\epsilon + p} \frac{T^2}{\gamma_2},
\]

\[
\xi_{B\ell F} = C_2 \mu \left(1 - \frac{\rho}{\epsilon + p}\right) - \frac{\rho}{\epsilon + p} \frac{T^2}{\gamma_2},
\]

\[
\xi_{B\ell F} = C_3 \mu d \left(1 - \frac{\rho_d}{2 \epsilon + p}\right) - \frac{1}{2} \frac{C_1}{C_3} \frac{\rho \mu_d^2}{\epsilon + p} - 4 \frac{\rho_d}{\epsilon + p} \frac{T^2}{\gamma_1}.
\]
In our considerations we shall suppose that the system in question is in an equilibrium state in the grand canonical ensemble with chemical potentials $\mu$, $\mu_d$, temperature $T$ and the local velocity will fulfill the condition $u^t = 1$. We also assume that

$$\epsilon + p = Ts + \mu \rho + \mu_d \rho_d, \quad dp = sTd + \rho d\mu + \rho_d d\mu_d. \quad (2.18)$$

### 2.2 Hydrodynamic magnetotransport of the strongly interacting system

In order to find the magnetic field dependence of the electrical conductivity in the system with chiral and $\mathbb{Z}_2$ anomalies, we suppose that one has to do with a background magnetic fields in $z$-direction, provided by

$$A_2 = B x, \quad B_2 = \tilde{B}_{\text{add}} x, \quad (2.19)$$

and consider the case when $E^{(F)\mu} = F^{\mu} = 0$ and $\tilde{E}^{(B)\mu} = B^{\mu} = 0$. We shall study the response of the currents to the perturbations of $\delta E^{(F)z}$ and $\delta \tilde{E}^{(B)z}$ fields.

In the thermodynamical system under consideration the perturbations of the thermodynamical variables are given by

$$\mu(x_\alpha) = \mu + \delta \mu(x_\alpha), \quad \mu_d(x_\alpha) = \mu_d + \delta \mu_d(x_\alpha),$$
$$T(x_\alpha) = T + \delta T(x_\alpha), \quad u^\beta(x_\alpha) = \left(1, \delta u_j(x_\alpha)\right). \quad (2.20)$$

Moreover, the perturbations of the other components of the gauge fields are chosen as follows:

$$\delta E^{(F)z} = \delta F^{tz}, \quad \delta \tilde{E}^{(B)z} = \delta B^{tz}, \quad (2.21)$$
$$\delta E^{(F)x} = \delta F^{tx} + B \delta u^y, \quad \delta \tilde{E}^{(B)x} = \delta B^{tx} + \tilde{B}_{\text{add}} \delta u^y, \quad (2.22)$$
$$\delta E^{(F)y} = \delta F^{ty} - B \delta u^x, \quad \delta \tilde{E}^{(B)y} = \delta B^{ty} - \tilde{B}_{\text{add}} \delta u^x. \quad (2.23)$$

Having in mind the above forms of perturbations of the hydrodynamical variables, to the linear order, the perturbations of the conserved quantities fulfill the following relations:

$$\delta T^{00} = \delta \epsilon, \quad (2.24)$$
$$\delta T^{0i} = (\epsilon + p) \delta u^i, \quad (2.25)$$
$$\delta T^{ij} = \delta p \ g^{ij} - \eta \left( \partial^i \delta u^j + \partial^j \delta u^i - \frac{2}{3} g^{ij} \partial_m \delta u^m \right) - \zeta g^{ij} \partial_m \delta u^m, \quad (2.26)$$

and the modifications of the respective currents connected with $U(1)$-gauge fields are calculated by combining equations (2.21)-(2.23), having in mind the relations (2.6)-(2.8). Con-
sequently we obtain
\[ \delta j^0(F) = \delta \rho + \xi_B \delta u_z \ B + \xi_{FB} \delta u_z \ \tilde{B}_{add}, \]  
\[ \delta j^x(F) = \rho \delta u^x - \sigma_F T \partial^x \left( \delta \left( \frac{\mu}{T} \right) \right) + \sigma_F \left( \delta F^{tx} + B \delta u^y \right) - \sigma_{FB} T \partial^x \left( \delta \left( \frac{\mu_d}{T} \right) \right) + \sigma_{FB} \left( \delta B^{tx} + \tilde{B}_{add} \delta u^y \right) + \xi \partial_{[y} \delta u_{z]}, \]  
\[ \delta j^y(F) = \rho \delta u^y - \sigma_F T \partial^y \left( \delta \left( \frac{\mu}{T} \right) \right) + \sigma_F \left( \delta F^{ty} - B \delta u^x \right) - \sigma_{FB} T \partial^y \left( \delta \left( \frac{\mu_d}{T} \right) \right) + \sigma_{FB} \left( \delta B^{ty} - \tilde{B}_{add} \delta u^x \right) + \xi \partial_{[x} \delta u_{z]}, \]  
\[ \delta j^z(F) = \rho \delta u^z - \sigma_F T \partial^z \left( \delta \left( \frac{\mu}{T} \right) \right) + \sigma_F \left( \delta F^{tz} - \sigma_{FB} B \delta u^x \right) - \sigma_{FB} T \partial^z \left( \delta \left( \frac{\mu_d}{T} \right) \right) + \sigma_{FB} \delta B^{tz} + B \delta \xi_B + \tilde{B}_{add} \delta \sigma_{FB} + \xi \partial_{[y} \delta u_{z]} \]  
(2.27) \]

The current perturbations of the additional gauge field imply
\[ \delta j^0(B) = \delta \rho_d + \xi_\tilde{B} \delta u_z \ \tilde{B}_{add} + \xi_{\tilde{B}F} \delta u_z \ B, \]  
\[ \delta j^x(B) = \rho_d \delta u^x - \sigma_B T \partial^x \left( \delta \left( \frac{\mu_d}{T} \right) \right) + \sigma_B \left( \delta B^{tx} + \tilde{B}_{add} \delta u^y \right) - \sigma_{BF} T \partial^x \left( \delta \left( \frac{\mu}{T} \right) \right) + \sigma_{BF} \left( \delta F^{tx} + B \delta u^y \right) + \xi_d \partial_{[y} \delta u_{z]}, \]  
\[ \delta j^y(B) = \rho_d \delta u^y - \sigma_B T \partial^y \left( \delta \left( \frac{\mu_d}{T} \right) \right) + \sigma_B \left( \delta B^{ty} - \tilde{B}_{add} \delta u^x \right) - \sigma_{BF} T \partial^y \left( \delta \left( \frac{\mu}{T} \right) \right) + \sigma_{BF} \left( \delta F^{ty} - B \delta u^x \right) + \xi_d \partial_{[x} \delta u_{z]}, \]  
\[ \delta j^z(B) = \rho_d \delta u^z - \sigma_B T \partial^z \left( \delta \left( \frac{\mu_d}{T} \right) \right) + \sigma_B \delta B^{tz} - \sigma_{BF} B \delta u^x - \sigma_{BF} T \partial^z \left( \delta \left( \frac{\mu}{T} \right) \right) + \sigma_{BF} \delta F^{tz} + \tilde{B}_{add} \delta \xi_B + B \delta \xi_{BF} + \xi_d \partial_{[y} \delta u_{z]} \]  
(2.31) \]

where, as in [20], we suppose that \( \delta \epsilon, \delta \rho \) are entirely impel by \( \delta \mu, \delta \mu_d, \delta T \). Moreover, one neglects the chiral vortical effects because they do not influence the results.

The time evolution of the perturbations is obtained from the conservation equations (2.1)-(2.3). To calculate them, in analogy to [20], we introduce the dissipation terms into the perturbed conservation laws of the considered theory and obtain
\[ \partial_\mu \delta T^{\mu 0} = \delta F^{tx} j_z(F) + \delta B^{\alpha} j_\alpha(B) + \frac{1}{\tau_e} \delta T^{\gamma 0} u_\gamma, \]  
\[ \partial_\mu \delta T^{\mu k} = \rho \delta F^{0k} + \rho_d \delta B^{0k} + F^{\kappa \alpha} \delta j_\alpha(F) + B^{\kappa \alpha} \delta j_\alpha(B) + \frac{1}{\tau_m} \delta T^{\gamma k} u_\gamma, \]  
\[ \partial_\mu \delta j_\mu(F) = C_1 \delta E_\mu(F) \frac{B^{(F)} \delta j_\mu(B) + \frac{1}{\tau_c} \delta j^{(F)}(F)}{1} + \frac{1}{\tau_c} \delta j^{(F)}(F) u_\gamma, \]  
\[ \partial_\mu \delta j_\mu(B) = C_3 \delta E_\mu(B) \frac{B^{(B)} \delta j_\mu(B) + \frac{1}{\tau_c} \delta j^{(B)}(B)}{1} + \frac{1}{\tau_c} \delta j^{(B)}(B) u_\gamma, \]  
(2.35) \]

where we have denoted by \( \tau_e \) the energy relaxation time, \( \tau_m \) stands for the momentum relaxation time, while \( \tau_c \) and \( \tau_{cd} \) describe the charge relaxation times for the Maxwell and the additional U(1)-gauge field. One should have in mind that the relaxation terms affect only the deviations from the equilibrium.

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In the next step let us put the perturbations into the energy-momentum and currents perturbation relations, in order to achieve the set of relations for $\delta\mu$, $\delta\mu_d$, $\delta T$ and $\delta u^\alpha$. For the perturbations of the energy-momentum tensor one obtains

\[
\left( \partial_t + \frac{1}{\tau_e} \right) \delta \epsilon + \partial_i \left[ (\epsilon + p) \delta u^i \right] - \delta E^{(F)} z \left( \xi_B B + \xi_{FB} \tilde{B}_{add} \right) - \delta \tilde{E}^{(B)} z \left( \xi_B \tilde{B}_{add} + \xi_{FB} B \right) = 0, \tag{2.39}
\]

\[
\left( \partial_t + \frac{1}{\tau_m} \right) (e + p) \delta u^x - \rho \delta F^{0x} - \rho_d \delta B^{0x} + \partial^x \delta p - \eta \left[ \partial_i \partial^i \delta u^x + \frac{1}{3} \partial^x \partial_m \delta u^m \right] \tag{2.40}
\]

\[
- \frac{\partial \tilde{E}^{(0x)}}{\partial t} (e + p) \delta u^x - \rho \delta F^{0y} - \rho_d \delta B^{0y} + \partial^y \delta p - \eta \left[ \partial_i \partial^i \delta u^y + \frac{1}{3} \partial^y \partial_m \delta u^m \right] \tag{2.41}
\]

\[
- \frac{\partial \tilde{E}^{(0y)}}{\partial t} (e + p) \delta u^y - \rho \delta F^{0z} - \rho_d \delta B^{0z} + \partial^z \delta p - \eta \left[ \partial_i \partial^i \delta u^z + \frac{1}{3} \partial^z \partial_m \delta u^m \right] \tag{2.42}
\]

Consequently, for perturbations of the currents bounded with $F_{\mu\nu}$ and $B_{\mu\nu}$ gauge fields, we arrive at the following relations:

\[
\left( \partial_t + \frac{1}{\tau_c} \right) \left( \delta \rho + \xi_B B \delta u_z + \xi_{FB} \tilde{B}_{add} \delta u_z \right) + \partial_i \left[ \rho \delta u^i - \sigma_F T \tilde{\partial}^i \left( \delta \left( \frac{\mu}{T} \right) \right) + \sigma_F \delta F^{0i} \right] - \sigma_{FB} \tilde{\partial}^i \left( \delta \left( \frac{\mu}{T} \right) \right) + \partial_i \left( \frac{\xi}{2} \epsilon^{ijk} \partial_j \delta u_k \right) \tag{2.43}
\]

\[
+ \partial_x \left( \sigma_F B \delta u^y + \sigma_{FB} \tilde{B}_{add} \delta u^y \right) - \partial_y \left( \sigma_F B \delta u^x + \sigma_{FB} \tilde{B}_{add} \delta u^x \right) + \partial_z \left( B \delta \xi_B + \tilde{B}_{add} \delta \xi_{FB} \right) - C_1 B \delta E^{(F)} - C_2 \tilde{B}_{add} \delta \tilde{E}^{(F)} = 0,
\]

\[
\left( \partial_t + \frac{1}{\tau_{cd}} \right) \left( \delta \rho_d + \xi_{FB} \tilde{B}_{add} \delta u_z + \xi_{BF} \delta B \delta u_z \right) + \partial_i \left[ \rho_d \delta u^i - \sigma_B T \tilde{\partial}^i \left( \delta \left( \frac{\mu}{T} \right) \right) + \sigma_B \delta B^{0i} \right] - \sigma_{BF} \tilde{\partial}^i \left( \delta \left( \frac{\mu}{T} \right) \right) + \partial_i \left( \frac{\xi_d}{2} \epsilon^{ijk} \partial_j \delta u_k \right) \tag{2.44}
\]

\[
+ \partial_x \left( \delta \rho_d \tilde{B}_{add} \delta u^y + \sigma_{BF} B \delta u^y \right) - \partial_y \left( \sigma_B \tilde{B}_{add} \delta u^x + \sigma_{BF} B \delta u^x \right) + \partial_z \left( \tilde{B}_{add} \delta \xi_{BF} + B \delta \xi_B \right) - C_3 B \delta \tilde{E}^{(B)} - C_4 \tilde{B}_{add} \delta E^{(B)} = 0.
\]
Further, one can readily verify that the implementation of the Laplace transformation in $t$-direction reveals

$$\omega_c \delta \epsilon - i \delta \epsilon^{(0)} + (\epsilon + p) \partial_t \delta u^i - i \delta E^{(F)z}(\xi_B B + \xi_{FB} \tilde{B}_{add})$$

$$- i \delta \tilde{E}^{(B)z}(\xi_B \tilde{B}_{add} + \xi_{FB} B) = 0,$$  \hspace{1cm} (2.45)

$$(\epsilon + p) [\omega_m \delta u^i - i \delta u^{i(0)}] - i \rho \delta E^{0y} - i \rho_d \delta B^{0y} + i \delta^y \delta p - i \eta \left[ \partial_t \delta^y \delta u^x + \frac{1}{3} \delta^y \partial_m \delta u^m \right]$$

$$- \zeta \delta^y \partial_m \partial_t \delta u^m = i B \left[ \rho \delta u_y - \sigma_F T \partial_y \left( \delta \left( \frac{\mu}{T} \right) \right) + \sigma_F \left( \delta E^{0y} - B \delta u^y \right) \right]$$

$$- \sigma_{FB} T \partial_y \left( \delta \left( \frac{\mu}{T} \right) \right) + \sigma_{FB} \left( \delta B^{0y} - \tilde{B}_{add} \delta u^y \right) + \xi \partial_y \delta u^y]$$

$$+ i \tilde{B}_{add} \left[ \rho_d \delta u_y - \sigma_B T \partial_y \left( \delta \left( \frac{\mu_d}{T} \right) \right) + \sigma_B \left( \delta B^{0y} - \tilde{B}_{add} \delta u^y \right) \right]$$

$$- \sigma_{FB} T \partial_y \left( \delta \left( \frac{\mu}{T} \right) \right) + \sigma_{BF} \left( \delta E^{0y} - B \delta u^y \right) + \xi \partial_y \delta u^y] = 0.$$  \hspace{1cm} (2.46)

$$(\epsilon + P) [\omega_m \delta u^i - i \delta u^{i(0)}] - i \rho \delta E^{0y} - i \rho_d \delta B^{0y} + i \delta^y \delta p - i \eta \left[ \partial_t \delta^y \delta u^x + \frac{1}{3} \delta^y \partial_m \delta u^m \right]$$

$$- \zeta \delta^y \partial_m \partial_t \delta u^m = 0,$$  \hspace{1cm} (2.48)

$$\omega_c \delta \rho - i \delta \rho^{(0)} + \left[ \omega_c \delta u_z - i \delta u^{z(0)} \right] \left( \xi_B B + \xi_{FB} \tilde{B}_{add} \right) + i \partial_t \left[ \rho \delta u^i - \sigma_F T \partial^y \left( \delta \left( \frac{\mu}{T} \right) \right) \right]$$

$$+ \sigma_F \delta E^{0y} - \sigma_{FB} T \partial^y \left( \delta \left( \frac{\mu_d}{T} \right) \right) + \sigma_{FB} \delta B^{0i} \right] + i \partial_a \left( \frac{\xi}{2} \epsilon_{ijk} \partial_j \delta u_k \right)$$

$$+ \partial_x \left( \sigma_F B \delta u^y + \sigma_{FB} \tilde{B}_{add} \delta u^y \right) - i \partial_y \left( \sigma_F B \delta u^y + \sigma_{FB} \tilde{B}_{add} \delta u^y \right)$$

$$+ \partial_z \left( B \delta \xi_B + \tilde{B}_{add} \delta \xi_{FB} \right) - i C_1 B \delta E_z^{(F)} - i C_2 \tilde{B}_{add} \delta E_z^{(B)} = 0,$$  \hspace{1cm} (2.49)

$$\omega_{cd} \delta \rho_d - i \delta \rho_d^{(0)} + \left[ \omega_{cd} \delta u_z - i \delta u^{z(0)} \right] \left( \xi_B B + \xi_{FB} B \right) + i \partial_t \left[ \rho \delta u^i - \sigma_B T \partial^y \left( \delta \left( \frac{\mu}{T} \right) \right) \right]$$

$$+ \sigma_B \delta B^y - \sigma_{BF} T \partial^y \left( \delta \left( \frac{\mu_d}{T} \right) \right) + \sigma_{BF} \delta E^{0y} \right] + i \partial_a \left( \frac{\xi}{2} \epsilon_{ijk} \partial_j \delta u_k \right)$$

$$+ \partial_x \left( \sigma_B \tilde{B}_{add} \delta u^y + \sigma_{BF} B \delta u^y \right) - i \partial_y \left( \sigma_B \tilde{B}_{add} \delta u^y + \sigma_{BF} B \delta u^y \right)$$

$$+ i \partial_z \left( \tilde{B}_{add} \delta \xi_B + \tilde{B}_{add} \delta \xi_{BF} \right) - i C_3 B \delta E_z^{(B)} - i C_2 \tilde{B}_{add} \delta E_z^{(F)} = 0,$$  \hspace{1cm} (2.50)

where the symbols with superscript $^{(0)}$ refer to initial values of the perturbations. In the above relations we have denoted

$$\omega_c = \omega + \frac{i}{\tau_c}, \quad \omega_m = \omega + \frac{i}{\tau_m}, \quad \omega_c = \omega + \frac{i}{\tau_c}, \quad \omega_{cd} = \omega + \frac{i}{\tau_{cd}}.$$  \hspace{1cm} (2.51)
The Fourier transformation in the spatial directions, with the auxiliary condition that $k \to 0$, leads the following system of equations:

\[
\omega_c \delta \epsilon - i \delta \epsilon^{(0)} - i \delta E^{(F)z} \left( \xi_B B + \xi_{FB} B_{add} \right) - i \delta \tilde{E}^{(B)z} \left( \xi_B \tilde{B}_{add} + \xi_{BF} B \right) = 0, \tag{2.52}
\]

\[
(\epsilon + p) \left[ \omega_m \delta u^x - i \delta u^{x(0)} \right] - i \rho \delta E^{0x} - i \rho_d \delta B^{0x} = i B \left[ \rho \delta u_y + \sigma_F \left( \delta E^{0y} - B \delta u^x \right) \right] + \sigma_F \delta B^{0y} - \tilde{B}_{add} \delta u^x, \tag{2.53}
\]

\[
(\epsilon + p) \left[ \omega_m \delta u^y - i \delta u^{y(0)} \right] - i \rho \delta E^{0y} - i \rho_d \delta B^{0y} = i B \left[ \rho \delta u_x + \sigma_F \left( \delta E^{0x} - B \delta u^y \right) \right] + \sigma_F \delta B^{0x} - \tilde{B}_{add} \delta u^y, \tag{2.54}
\]

\[
\omega_c \delta \rho - i \delta \rho^{(0)} + \left[ \omega_c \delta u_z - i \delta u^{z(0)} \right] \left( \xi_B B + \xi_{FB} \tilde{B}_{add} \right) - i C_1 B \delta E^{(F)z} - i C_2 \tilde{B}_{add} \delta \tilde{E}^{(B)z} = 0, \tag{2.55}
\]

\[
\omega_c \delta \rho_d - i \delta \rho_d^{(0)} + \left[ \omega_c \delta u_z - i \delta u^{z(0)} \right] \left( \xi_B \tilde{B}_{add} + \xi_{BF} B \right) - i C_3 B \delta \tilde{E}^{(B)z} - i C_2 \tilde{B}_{add} \delta E^{(F)z} = 0, \tag{2.56}
\]

In order to solve the above equations, we write the dependence of $\delta \epsilon$, $\delta \rho$, $\delta \rho_d$, $\delta \rho$ on $\delta \mu$, $\delta \mu_d$, $\delta T$ as

\[
\delta \epsilon = e_1 \delta \mu + e_2 \delta T + e_3 \delta \mu_d = \left( \frac{\partial \epsilon}{\partial \mu} \right)_{T,\mu_d} \delta \mu + \left( \frac{\partial \epsilon}{\partial T} \right)_{\mu,\mu_d} \delta T + \left( \frac{\partial \epsilon}{\partial \mu_d} \right)_{T,\mu} \delta \mu_d, \tag{2.57}
\]

\[
\delta \rho = f_1 \delta \mu + f_2 \delta T + f_3 \delta \mu_d = \left( \frac{\partial \rho}{\partial \mu} \right)_{T,\mu_d} \delta \mu + \left( \frac{\partial \rho}{\partial T} \right)_{\mu,\mu_d} \delta T + \left( \frac{\partial \rho}{\partial \mu_d} \right)_{T,\mu} \delta \mu_d, \tag{2.58}
\]

\[
\delta \rho_d = g_1 \delta \mu_d + g_2 \delta T + g_3 \delta \mu = \left( \frac{\partial \rho_d}{\partial \mu_d} \right)_{T,\mu} \delta \mu_d + \left( \frac{\partial \rho_d}{\partial T} \right)_{\mu,\mu_d} \delta T + \left( \frac{\partial \rho_d}{\partial \mu_d} \right)_{T,\mu} \delta \mu, \tag{2.59}
\]

\[
\delta \rho = \rho \delta \mu + \rho_d \delta \mu_d + s \delta T, \tag{2.60}
\]

where the coefficients in the above relations depend on the details corresponding to the system in question.

We solve equations (2.52)-(2.56) in terms of $\delta E^{(F)z}$ and $\delta \tilde{E}^{(B)z}$. Namely, they can be
written as follows:

\[ \delta \mu = \frac{1}{W} \delta E^{(F)z} \left[ \frac{i}{\omega_c} \left( \xi_B B + \xi_{F \tilde{B}} \tilde{B}_{\text{add}} \right) \right. \\
+ \left. \left( -\frac{\rho}{\epsilon + p \omega_m} \right) \left( \xi_B B + \xi_{F \tilde{B}} \tilde{B}_{\text{add}} \right) + \frac{i}{\omega_c} C_1 B \right) \left( g_2 e_1 - g_3 e_1 \right) \\
+ \left. \left( -\frac{\rho}{\epsilon + p \omega_m} \right) \left( \xi_B \tilde{B}_{\text{add}} + \xi_{F \tilde{B}} \tilde{B}_{\text{add}} \right) + \frac{i}{\omega_c} C_2 \tilde{B}_{\text{add}} \right) \left( g_2 e_2 - g_3 e_1 \right) \\
+ \left. \left( -\frac{\rho}{\epsilon + p \omega_m} \right) \left( \xi_B \tilde{B}_{\text{add}} + \xi_{F \tilde{B}} \tilde{B}_{\text{add}} \right) + \frac{i}{\omega_c} C_3 \tilde{B}_{\text{add}} \right) \left( g_2 e_2 - g_3 e_2 \right) \\
+ \left. \left( -\frac{\rho}{\epsilon + p \omega_m} \right) \left( \xi_{F \tilde{B}} \tilde{B}_{\text{add}} + \xi_{F \tilde{B}} \tilde{B}_{\text{add}} \right) + \frac{i}{\omega_c} C_4 \tilde{B}_{\text{add}} \right) \left( g_2 e_1 - g_3 e_1 \right) \right) + \ldots \\
= M_1 \delta E^{(F)z} + M_2 \delta \tilde{E}^{(B)z} + \ldots , \]

\[ \delta \mu_d = \frac{1}{W} \delta E^{(F)z} \left[ \frac{i}{\omega_c} \left( \xi_B B + \xi_{F \tilde{B}} \tilde{B}_{\text{add}} \right) \right. \\
+ \left. \left( -\frac{\rho}{\epsilon + p \omega_m} \right) \left( \xi_B B + \xi_{F \tilde{B}} \tilde{B}_{\text{add}} \right) + \frac{i}{\omega_c} C_1 B \right) \left( f_1 g_2 - f_2 g_3 \right) \\
+ \left. \left( -\rho \right) \left( \xi_B \tilde{B}_{\text{add}} + \xi_{F \tilde{B}} \tilde{B}_{\text{add}} \right) + \frac{i}{\omega_c} C_2 \tilde{B}_{\text{add}} \right) \left( f_1 g_2 - f_2 g_3 \right) \\
+ \left. \left( -\rho \right) \left( \xi_B \tilde{B}_{\text{add}} + \xi_{F \tilde{B}} \tilde{B}_{\text{add}} \right) + \frac{i}{\omega_c} C_2 \tilde{B}_{\text{add}} \right) \left( f_1 g_2 - f_2 g_3 \right) \\
+ \left. \left( -\rho \right) \left( \xi_{F \tilde{B}} \tilde{B}_{\text{add}} + \xi_{F \tilde{B}} \tilde{B}_{\text{add}} \right) + \frac{i}{\omega_c} C_4 \tilde{B}_{\text{add}} \right) \left( f_1 g_2 - f_2 g_3 \right) \right) + \ldots \\
= D_1 \delta E^{(F)z} + D_2 \delta \tilde{E}^{(B)z} + \ldots , \]
\[ \delta T = \frac{1}{W} \delta E^{(F)z} \left[ i \frac{\omega_c}{\rho} \left( \xi_B B + \xi_{FB} \tilde{B}_{add} \right) \left( f_3 g_3 - f_1 g_1 \right) \right. \\
\left. + \left( - \frac{\rho}{\epsilon + p \omega_m} \left( \xi_B B + \xi_{FB} \tilde{B}_{add} \right) + i \omega_c C_1 B \right) (e_1 g_1 - g_3 e_3) \right] \\
\left. + \left( - \frac{\rho}{\epsilon + p \omega_m} \left( \xi_{FB} \tilde{B}_{add} + \xi_{BF} B \right) + i \omega_c C_4 \tilde{B}_{add} \right) (e_3 f_1 - e_1 f_3) \right] \\
+ \frac{1}{W} \delta \tilde{E}^{(B)z} \left[ i \frac{\omega_c}{\rho} \left( \xi_{FB} \tilde{B}_{add} + \xi_{BF} B \right) \left( f_3 g_3 - f_1 g_1 \right) \right. \\
\left. + \left( - \frac{\rho d}{\epsilon + p \omega_m} \left( \xi_B B + \xi_{FB} \tilde{B}_{add} \right) + i \omega_c C_2 \tilde{B}_{add} \right) (g_1 e_1 - e_3 g_3) \right] \\
\left. + \left( - \frac{\rho d}{\epsilon + p \omega_m} \left( \xi_{FB} \tilde{B}_{add} + \xi_{BF} B \right) + i \omega_c C_3 B \right) (e_3 f_1 - e_1 f_3) \right] + \ldots \\
= T_1 \delta E^{(F)z} + T_2 \delta \tilde{E}^{(B)z} + \ldots, \tag{2.64} \]

where we have denoted by \( W \)

\[ W = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \\ g_3 & g_2 & g_1 \end{pmatrix}. \tag{2.65} \]

By ‘\ldots’ one has in mind the terms which are unrelated to \( \delta E^{(F)z} \) and \( \delta \tilde{E}^{(B)z} \), which disappear when we choose the initial values of the other perturbations equal to zero. In what follows, for the brevity of notation, one assigns all terms standing in front of \( \delta E^{z} \) and \( \delta \tilde{E}^{z} \), respectively by

\[ \delta \mu = M_1 \delta E^{(F)z} + M_2 \delta \tilde{E}^{(B)z}, \tag{2.66} \]
\[ \delta \mu_d = D_1 \delta E^{(F)z} + D_2 \delta \tilde{E}^{(B)z}, \tag{2.67} \]
\[ \delta T = T_1 \delta E^{(F)z} + T_2 \delta \tilde{E}^{(B)z}. \tag{2.68} \]

They will be needed for substituting them into the perturbation relations connected with \( z \)-directed gauge field currents. On this account we have

\[ \delta j^z(F) = \rho \ \delta u^z - \sigma_F T \ \partial^z \left( \frac{\delta \mu}{T} \right) + \sigma_F \delta E^{(F)z} - \sigma_{FB} T \ \partial^z \left( \frac{\delta \mu_d}{T} \right) + \sigma_{FB} \delta \tilde{E}^{(B)z} \]
\[ + \delta \xi_B B + \delta \xi_{FB} \tilde{B}_{add} + \xi \ \partial_z (\delta u^y), \tag{2.69} \]

and for \( B_{\mu \nu} \) field current it yields

\[ \delta j^z(B) = \rho_d \ \delta u^z - \sigma_B T \ \partial^z \left( \frac{\delta \mu_d}{T} \right) + \sigma_B \delta \tilde{E}^{(B)z} - \sigma_{BF} T \ \partial^z \left( \frac{\delta \mu}{T} \right) + \sigma_{BF} \delta E^{(F)z} \]
\[ + \delta \xi_{FB} \tilde{B}_{add} + \delta \xi_{BF} B + \xi \ \partial_z (\delta u^y). \tag{2.70} \]
It can be verified that in the limit of $k \to 0$ they reduce to
\begin{equation}
\delta j^z (F) = \rho \, \delta u^z + \sigma_F \, \delta E^{(F)z} + \sigma_{F_B} \, \delta E^{(B)z} + \delta \xi_B \, B + \delta \xi_{F_B} \, \tilde{B}_{add},
\end{equation}
for the perturbations of the Maxwell field current, and for the additional one we obtain
\begin{equation}
\delta j^z (B) = \rho_d \, \delta u^z + \sigma_B \, \delta E^{(B)z} + \sigma_{BF} \, \delta E^{(F)z} + \delta \xi_B \, \tilde{B}_{add} + \delta \xi_{BF} \, B.
\end{equation}
Inserting the explicit values for $\delta u^z$ for the perturbations of the Maxwell field current, and for the additional one we obtain

\begin{align*}
\delta j^z (F) &= \sigma_F + \frac{\rho^2}{\epsilon + p} \frac{i}{\omega_m} + B \left( M_1 H_1 + D_1 H_2 + T_1 H_3 \right) \\
&\quad + \tilde{B}_{add} \left( D_1 G_1 + M_1 G_2 + T_1 G_3 \right) \\
&\quad + \delta \tilde{E}^{(B)} \left[ \sigma_{F_B} + \frac{\rho \mu_d}{\epsilon + p} \frac{i}{\omega_m} + B \left( M_2 H_1 + D_2 H_2 + T_2 H_3 \right) \\
&\quad + \tilde{B}_{add} \left( D_2 G_1 + M_2 G_2 + T_2 G_3 \right) \right],
\end{align*}

where we have defined the following quantities connected with the terms multiplied by $B$-magnetic field:
\begin{align*}
H_1 &= C_1 \left( 1 - \frac{\rho \mu}{\epsilon + p} \right) + \frac{x_1}{2(\epsilon + p)} \left( - f_1 + \frac{x_1}{\epsilon + p} (e_1 + \rho) \right), \\
H_2 &= -C_3 \frac{\rho \mu_d}{\epsilon + p} + \frac{x_1}{2(\epsilon + p)} \left( - f_3 + \frac{x_1}{\epsilon + p} (e_3 + \rho d) \right), \\
H_3 &= -\frac{2 \rho T}{\epsilon + p} \tilde{\gamma}_1 + \frac{x_1}{2(\epsilon + p)} \left( - f_2 + \frac{x_1}{\epsilon + p} (e_2 + s) \right), \\
x_1 &= C_3 \mu_d^2 + C_1 \mu^2 + 2 \tilde{\gamma}_1 T^2,
\end{align*}

and with the terms multiplied by $\tilde{B}_{add}$
\begin{align*}
G_1 &= C_2 \left( 1 - \frac{\rho \mu}{\epsilon + p} \right) + \frac{x_2}{(\epsilon + p)^2} \left( \rho (\rho d + e_3) - f_3 (\epsilon + p) \right), \\
G_2 &= -C_2 \frac{\rho \mu_d}{\epsilon + p} + \frac{x_2}{(\epsilon + p)^2} \left( \rho (e_1 + s) - f_1 (\epsilon + p) \right), \\
G_3 &= -\frac{2 \rho T}{\epsilon + p} \tilde{\gamma}_2 + \frac{x_2}{(\epsilon + p)^2} \left( \rho (s + e_2) - f_3 (\epsilon + p) \right), \\
x_2 &= C_2 \mu \mu_d + \tilde{\gamma}_2 T^2.
\end{align*}
On the other hand, for the second gauge field current one obtains

\[
\delta j_z(B) = \delta E_z^{(F)} \left[ \sigma_{BF} + \frac{\rho d}{\epsilon + p} \frac{i}{\omega_m} + \tilde{B}_{odd} \left( M_1 I_1 + D_1 I_2 + T_1 I_3 \right) \right] \\
+ B \left( M_1 J_1 + D_1 J_2 + T_1 J_3 \right) \\
+ \delta E_z^{(B)} \left[ \sigma_B + \frac{\rho d}{\epsilon + p} \frac{i}{\omega_m} + \tilde{B}_{odd} \left( M_2 I_1 + D_2 I_2 + T_2 I_3 \right) \right] \\
+ B \left( M_2 J_1 + D_2 J_2 + T_2 J_3 \right),
\]

where we have defined the following quantities:

\[
I_1 = C_2 \left( 1 - \frac{\rho d \mu_d}{\epsilon + p} \right) + \frac{x_2}{(\epsilon + p)^2} \left( \rho_d (\rho + e_1) - g_3 (\epsilon + p) \right),
\]

\[
I_2 = -C_2 \frac{\rho d \mu_d}{\epsilon + p} + \frac{x_2}{(\epsilon + p)^2} \left( \rho_d (\rho + e_3) - g_1 (\epsilon + p) \right),
\]

\[
I_3 = -\frac{2 \rho_d T}{\epsilon + p} \gamma_2 + \frac{x_2}{(\epsilon + p)^2} \left( \rho_d (s + e_2) - g_2 (\epsilon + p) \right),
\]

and

\[
J_1 = -C_1 \frac{\rho d \mathcal{H}}{\epsilon + p} + \frac{x_1}{2 (\epsilon + p)} \left( -g_3 + \frac{\rho_d}{\epsilon + p} (e_1 + \rho) \right),
\]

\[
J_2 = C_3 \left( 1 - \frac{\rho d \mathcal{H}}{\epsilon + p} \right) + \frac{x_1}{2 (\epsilon + p)} \left( -g_1 + \frac{\rho_d}{\epsilon + p} (e_3 + \rho) \right),
\]

\[
J_3 = -\frac{2 \rho_d T}{\epsilon + p} \gamma_1 + \frac{x_1}{2 (\epsilon + p)} \left( -g_2 + \frac{\rho_d}{\epsilon + p} (e_2 + s) \right).
\]

Consequently, we can write them in a more compact forms, given by

\[
\delta j_z(F) = \tilde{\sigma}_F \delta E_z^{(F)} + \tilde{\sigma}_{FB} \delta E_z^{(B)},
\]

\[
\delta j_z(B) = \tilde{\sigma}_B \delta E_z^{(B)} + \tilde{\sigma}_{BF} \delta E_z^{(F)}.
\]

By virtue of the equations (2.89)-(2.90), we can read the explicit forms the adequate con-
ductivities. Namely, they yield

\[
\tilde{\sigma}_F = \sigma_F + \frac{\rho^2}{\epsilon + p} \frac{i}{\omega_m} + B \left( M_1 H_1 + D_1 H_2 + T_1 H_3 \right) \quad (2.91)
\]

\[
+ \tilde{B}_{add} \left( D_1 G_1 + M_1 G_2 + T_1 G_3 \right),
\]

\[
\tilde{\sigma}_{FB} = \sigma_{FB} + \frac{\rho\rho_4}{\epsilon + p} \frac{i}{\omega_m} + B \left( M_2 H_1 + D_1 H_2 + T_1 H_3 \right) \quad (2.92)
\]

\[
+ \tilde{B}_{add} \left( D_2 G_1 + M_2 G_2 + T_2 G_3 \right),
\]

\[
\tilde{\sigma}_B = \sigma_B + \frac{\rho^2}{\epsilon + p} \frac{i}{\omega_m} + \tilde{B}_{add} \left( M_2 I_1 + D_1 I_2 + T_1 I_3 \right) \quad (2.93)
\]

\[
+ B \left( M_2 J_1 + D_2 J_2 + T_2 J_3 \right),
\]

\[
\tilde{\sigma}_{BF} = \sigma_{BF} + \frac{\rho\rho_4}{\epsilon + p} \frac{i}{\omega_m} + \tilde{B}_{add} \left( M_1 I_1 + D_1 I_2 + T_1 I_3 \right) \quad (2.94)
\]

\[
+ B \left( M_1 J_1 + D_1 J_2 + T_1 J_3 \right).
\]

These results constitute the expressions for the longitudinal electric conductivities of a fluid with chiral and \( \mathbb{Z}_2 \) anomalies, in the background of the adequate magnetic fields \( B \) and \( \tilde{B}_{add} \), connected with Maxwell and auxiliary \( U(1) \)-gauge fields.

2.3 The limits

In this subsection we shall discuss physical consequences of the model. It is important to note that DC conductivity corresponding to \( \omega = 0 \) limit is finite, if all of the relaxation times \( \tau_i \), i.e., energy for \( i = e \), momentum \( (i = m) \) and two charge relaxations \( (i = c, cd) \), are finite. This agrees with the earlier observation [20] for the model without \( \mathbb{Z}_2 \) symmetry. For finite values of the relaxation times, the conductivities have both real \( \sigma'(\omega) \) and imaginary \( \sigma''(\omega) \) parts. The imaginary one characterizes by an antisymmetric in frequency contribution, resulting from the terms like \( i\omega/(\omega^2 + 1/\tau_i^2) \). The real parts of the conductivities are governed by the corresponding relaxation times and contain terms with factors \( (1/\tau_i)/(\omega^2 + 1/\tau_i^2) \), which in the limit of large relaxation times (i.e., \( 1/\tau \to 0 \)) envisage the zero frequency, \( \delta(\omega) \), contribution only.

If we consider the case without relaxation terms, i.e., \( \omega_e, \omega_m, \omega_c, \omega_{cd} \to \omega \) should be replaced in (2.91)-(2.94). The conductivities will have poles in the imaginary part at \( \omega = 0 \). Consequently, one will encounter \( \delta(\omega) \) in the real part of the DC-conductivities in question. For a system without \( \mathbb{Z}_2 \) symmetry there exist a single non-zero chirality parameter \( C_1 = C \) only and our general formula for the conductivity matrix reduces to an element \( \tilde{\sigma}_F \) which agrees with that found earlier [20].

Because of the complexity of the obtained results we pay attention to some simpler limit cases. Namely, our analysis will be addressed to the case when \( \rho = \mu = 0 \), which implies also that \( f_2 = f_3 = 0 \). In the case under consideration, one obtains the following
forms of the conductivities:

\[
\tilde{\sigma}_F (\rho = \mu = 0) = \sigma_F + B \left[ \frac{i}{\omega_e} \left( \frac{C_2^2 B}{f_1} - \frac{C_1 B}{2(\epsilon + p)} \right) (C_3 \mu_2^2 + 2\tilde{\gamma}_1 T^2) \right] \tag{2.95}
\]

\[
= \tilde{B}_{add} \frac{i}{\omega_e} \frac{C_1 B \tilde{\gamma}_2 T^2}{\epsilon + p}.
\]

\[
\tilde{\sigma}_{FB} (\rho = \mu = 0) = \sigma_{FB} + B M_2^{(L)} H_1^{(L)} + \tilde{B}_{add} \left( D_2^{(L)} G_1^{(L)} + M_2^{(L)} C_2 \right) \tag{2.96}
\]

\[
\tilde{\sigma}_{BF} (\rho = \mu = 0) = \sigma_{BF} + \tilde{B}_{add} \left[ \frac{i}{\omega_e} \frac{C_1 B}{f_1} I_1^{(L)} + D_1^{(L)} I_2^{(L)} + T_1^{(L)} I_3^{(L)} \right] \tag{2.97}
\]

\[
= B \left[ \frac{i}{\omega_e} \frac{C_1 B}{f_1} J_{1}^{(L)} + D_1^{(L)} J_{2}^{(L)} + T_1^{(L)} J_{3}^{(L)} \right] + B \left[ M_2^{(L)} J_{1}^{(L)} + D_2^{(L)} J_{2}^{(L)} + T_2^{(L)} J_{3}^{(L)} \right] \tag{2.98}
\]

where for the brevity of the subsequent notion we have denoted the following quantities:

\[
M_2^{(L)} = -\frac{\rho_d}{\epsilon + p} \frac{i}{\omega_e} \tilde{B}_{add} C_2 \mu_d + \frac{i}{\omega_c} \tilde{B}_{add} C_2 \mu_d, \tag{2.99}
\]

\[
D_1^{(L)} = \frac{i}{\omega_e} \tilde{B}_{add} C_2 \mu_d \frac{g_2}{g_2 e_3 - e_2 g_1} + \frac{i}{\omega_c} \frac{C_1 B (g_3 e_2 - g_2 e_1)}{f_1(g_2 e_3 - e_2 g_1)} - \frac{i}{\omega_{cd} g_2 e_3 - e_2 g_1} \frac{C_4 \tilde{B}_{add} e_2}{f_1(g_2 e_3 - e_2 g_1)}, \tag{2.100}
\]

\[
D_2^{(L)} = \frac{i}{\omega_e} \tilde{B}_{add} C_2 \mu_4 \frac{g_2}{g_2 e_3 - e_2 g_1} - \frac{i}{\omega_m} \frac{\rho_d}{\epsilon + p} \frac{C_2 \tilde{B}_{add} \mu_d (g_3 e_2 - g_2 e_1)}{f_1(g_2 e_3 - e_2 g_1)} - \frac{i}{\omega_{cd} g_2 e_3 - e_2 g_1} \frac{C_3 \tilde{B}_{add} e_2}{f_1(g_2 e_3 - e_2 g_1)} \tag{2.101}
\]

\[
+ \frac{i}{\omega_m} \frac{\rho_d}{\epsilon + p} \frac{C_2 \tilde{B}_{add} \mu_d e_2}{f_1(g_2 e_3 - e_2 g_1)},
\]

\[
T_1^{(L)} = -\frac{i}{\omega_e} \tilde{B}_{add} C_2 \mu_d \frac{g_1}{g_2 e_3 - e_2 g_1} + \frac{i}{\omega_c} \frac{C_1 B (e_1 g_1 - e_3 g_3)}{f_1(g_2 e_3 - e_2 g_1)} \tag{2.102}
\]
\[ T_2^{(L)} = - \frac{i}{\omega} \tilde{B}_{add} C_2 \mu_d g_1 \left( 1 - \frac{\rho_d}{\epsilon + p} \right) - \frac{i}{\omega_m} \frac{\rho_d}{\epsilon + p} \tilde{B}_{add} C_2 \mu_d (e_1 g_1 - e_3 g_3) \]  
\[ H_1^{(L)} = C_1 - \frac{C_3 \mu_3^2 + 2\tilde{\gamma}_1 T^2}{2(\epsilon + p)} \]  
\[ G_2^{(L)} = -f_1 \tilde{\gamma}_2 T^2 \]  
\[ I_1^{(L)} = C_1 \left( 1 - \frac{\rho_d}{\epsilon + p} \right) + \tilde{\gamma}_2 T^2 \left( e_1 \rho_d - g_1 (\epsilon + p) \right) \]  
\[ I_2^{(L)} = \frac{\tilde{\gamma}_2 T^2}{(\epsilon + p)^2} \left( \rho_d (\rho_d + e_3) - g_1 (\epsilon + p) \right) \]  
\[ I_3^{(L)} = -2 \rho_d \tilde{\gamma}_2 T \left( \frac{\epsilon + p}{\rho_d (s + e_2) - g_2 (\epsilon + p)} \right) \]  
\[ J_1^{(L)} = \frac{C_3 \mu_3^2 + 2\tilde{\gamma}_1 T^2}{2(\epsilon + p)} \left( -g_1 + \frac{\rho_d e_1}{\epsilon + p} \right) \]  
\[ J_2^{(L)} = C_3 + \frac{C_3 \mu_3^2 + 2\tilde{\gamma}_1 T^2}{2(\epsilon + p)} \left( -g_1 + \frac{\rho_d (e_3 + e_4)}{\epsilon + p} \right) \]  
\[ J_3^{(L)} = -2 \rho_d \tilde{\gamma}_1 T \left( \frac{\epsilon + p}{\rho_d (s + e_2 + s)} \right) \]  
\[ = -2 \rho_d \tilde{\gamma}_1 T \left( \frac{\epsilon + p}{\rho_d (e_3 + e_4)} \right) \]  
\[ + \frac{C_3 \mu_3^2 + 2\tilde{\gamma}_1 T^2}{2(\epsilon + p)} \left( -g_2 + \frac{\rho_d (e_2 + s)}{\epsilon + p} \right). \]  

The above relations simplify to great extend, when one assumes that the magnetic fields are equal to zero. Consequently this assumption leads to

\[ \tilde{\sigma}_F \left( \rho = \mu = 0 \right) = \sigma_F, \]  
\[ \tilde{\sigma}_{FB} \left( \rho = \mu = 0 \right) = \sigma_{FB}, \]  
\[ \tilde{\sigma}_{BF} \left( \rho = \mu = 0 \right) = \sigma_{BF}, \]  
\[ \tilde{\sigma}_B \left( \rho = \mu = 0 \right) = \sigma_B + \frac{i}{\omega_m} \frac{\rho_d^2}{\epsilon + p}. \]  

One can see that in the considered case the additional \( U(1) \)-gauge field density \( \rho_d \) gives the finite conductivity.

On the other hand, in the limit when \( \rho_d = \mu_d = 0 \), which implies that \( g_2 = g_3 = 0 \), the
relations for the conductivities yield

\[
\tilde{\sigma}_F \left( \rho_d = \mu_d = 0 \right) = \sigma_F + \frac{\rho^2}{\epsilon + p \omega_m} + B \left[ M_1^{(D)} H_1^{(D)} + D_1^{(D)} H_2^{(D)} + T_1^{(D)} H_3^{(D)} \right] \\
= \tilde{B}_{add} \left[ D_1^{(D)} G_1^{(D)} + M_1^{(D)} G_2^{(D)} + T_1^{(D)} G_3^{(D)} \right], 
\]

(2.116)

\[
\tilde{\sigma}_{FB} \left( \rho_d = \mu_d = 0 \right) = \sigma_{FB} + \frac{i C_3 B}{\omega_m} \tilde{H}_2^{(D)} + T_2^{(D)} H_3^{(D)} \\
+ \tilde{B}_{add} \left[ \frac{i C_3 B}{\omega_m} G_1^{(D)} + M_2^{(D)} G_2^{(D)} + T_2^{(D)} G_3^{(D)} \right], 
\]

(2.117)

\[
\tilde{\sigma}_{BF} \left( \rho_d = \mu_d = 0 \right) = \sigma_{BF} + \tilde{B}_{add} \left[ M_1^{(D)} C_2 - D_1^{(D)} \frac{\tilde{e}_2 T g_1}{\epsilon + p} \right] \\
+ B \left( C_3 - \frac{C_1 \mu^2 + 2 \tilde{e}_1 T^2 g_1}{2(\epsilon + p) g_1} \right), 
\]

(2.118)

\[
\tilde{\sigma}_B \left( \rho_d = \mu_d = 0 \right) = \sigma_B + \tilde{B}_{add} \left[ M_2^{(D)} C_2 - \frac{i C_3 B}{\omega_m} \frac{\tilde{e}_2 T g_1}{\epsilon + p} \right] \\
+ B \frac{i C_3 B}{\omega_m} \left( C_3 - \frac{C_1 \mu^2 + 2 \tilde{e}_1 T^2}{2(\epsilon + p) g_1} \right). 
\]

(2.119)

(2.120)

(2.121)

(2.122)

In the above relations we set the quantities

\[
M_1^{(D)} = \left( \frac{i}{\omega_c} f_2 + \frac{\rho}{\epsilon + p \omega_m} \right) \frac{1}{\epsilon_1 f_2 - \epsilon_2 f_1} \left( 1 - \frac{1}{g_1} \right) \left( \frac{C_1 \mu B}{1 - \frac{\rho \mu}{2(\epsilon + p)}} \right) \\
- \frac{\rho \tilde{e}_1 T^2 B}{\epsilon + p} - \frac{\rho \tilde{e}_2 T^2 \tilde{B}_{add}}{\epsilon + p} \frac{C_1 B e_2}{\omega_c \epsilon_1 f_2 - \epsilon_2 f_1} + \frac{i C_4 B_{add}(e_2 f_3 - e_3 f_2)}{\omega_c g_1 (\epsilon_1 f_2 - \epsilon_2 f_1)},
\]

(2.123)

\[
M_2^{(D)} = \frac{1}{\epsilon_1 f_2 - \epsilon_2 f_1} \left[ \frac{i}{\omega_c} \tilde{B}_{add} C_2 \mu f_2 - \frac{i}{\omega_c} C_2 \tilde{B}_{add} e_2 + \frac{i}{\omega_c} C_3 B (e_2 f_3 - e_3 f_2) \right],
\]

(2.124)

\[
D_1^{(D)} = - \frac{i}{\omega_m \epsilon + p} \tilde{B}_{add} C_2 \mu g_1 + \frac{i}{\omega_c} C_4 \tilde{B}_{add} g_1,
\]

(2.125)

\[
T_1^{(D)} = \left[ C_1 \mu B \left( 1 - \frac{\rho \mu}{2(\epsilon + p)} \right) - \rho \tilde{e}_1 T^2 B \left( \frac{1}{\epsilon + p} - \frac{\rho \tilde{e}_2 T^2 \tilde{B}_{add}}{\epsilon + p} \right) \frac{1}{\epsilon_1 f_2 - \epsilon_2 f_1} \right] \\
\left( - \frac{i}{\omega_c} f_1 - \frac{i}{\omega_m \epsilon + p} \left( e_1 + \frac{f_1 e_3 - f_3 e_1}{g_1} \right) \right) \\
+ \frac{1}{\epsilon_1 f_2 - \epsilon_2 f_1} \left[ \frac{i}{\omega_c} C_1 B e_1 + \frac{i}{\omega_c} C_4 \tilde{B}_{add} \frac{f_1 e_3 - f_3 e_1}{g_1} \right],
\]

(2.126)

\[
T_2^{(D)} = \frac{1}{\epsilon_1 f_2 - \epsilon_2 f_1} \left[ - \frac{i}{\omega_c} \tilde{B}_{add} C_2 \mu f_1 + - \frac{i}{\omega_c} C_2 \tilde{B}_{add} e_1 + - \frac{i}{\omega_c} C_3 B \frac{f_1 e_3 - f_3 e_1}{g_1} \right],
\]

(2.127)

\[
H_1^{(D)} = C_1 \left( 1 - \frac{\rho \mu}{\epsilon + p} \right) + \frac{C_1 \mu + 2 \tilde{e}_1 T^2}{2(\epsilon + p)} \left( - f_1 + \frac{\rho}{\epsilon + p} (e_1 + \rho) \right),
\]

(2.128)

\[
H_2^{(D)} = \frac{C_1 \mu^2 + 2 \tilde{e}_1 T^2}{2(\epsilon + p)} \left( - f_3 + \frac{\rho e_3}{\epsilon + p} \right),
\]

(2.129)

\[
H_3^{(D)} = \frac{-2 \rho \tilde{e}_1 T}{\epsilon + p} + \frac{C_1 \mu + 2 \tilde{e}_1 T^2}{2(\epsilon + p)} \left( - f_2 + \frac{\rho}{\epsilon + p} (e_2 + s) \right),
\]

(2.130)
\[ G_1^{(D)} = C_2 \left(1 - \frac{\rho \mu}{\epsilon + p}\right) + \frac{\tilde{\gamma}_2 T^2}{(\epsilon + p)^2} (\epsilon \rho - f_3(\epsilon + p)) . \] (2.132)

\[ G_2^{(D)} = \frac{\tilde{\gamma}_2 T^2}{(\epsilon + p)^2} (\rho^2 - f_1(\epsilon + p)) , \] (2.133)

\[ G_3^{(D)} = -2\frac{\rho \tilde{\gamma}_2 T^2}{\epsilon + p} + \frac{\tilde{\gamma}_2 T^2}{(\epsilon + p)^2} (s \rho - f_3(\epsilon + p)) . \] (2.134)

As in the latter case, let us suppose that the magnetic fields are equal to zero. It implies the following:

\[ \tilde{\sigma}_F \left( \rho_d = \mu_d = 0 \right) = \sigma_F + i \frac{\rho^2}{\omega_m (\epsilon + p)} , \] (2.135)

\[ \tilde{\sigma}_{FB} \left( \rho_d = \mu_d = 0 \right) = \sigma_{FB} , \] (2.136)

\[ \tilde{\sigma}_{BF} \left( \rho_d = \mu_d = 0 \right) = \sigma_{BF} , \] (2.137)

\[ \tilde{\sigma}_B \left( \rho_d = \mu_d = 0 \right) = \sigma_B . \] (2.138)

As was previously stated, the additional gauge field density, now \( \rho \), gives the finite conductivity.

In the case when \( \rho = \mu = \rho_d = \mu_d = 0 \), and \( f_2 = f_3 = g_2 = g_3 = 0 \), one receives the relations provided by

\[ \tilde{\sigma}_F \left( \rho_m = \mu_m = 0 \right) = \sigma_F + \frac{i}{\omega_c} \frac{C_1 B^2}{f_1} + \frac{i}{\omega_{cd}} \frac{C_2 C_3 \tilde{B}_{add}}{g_1} - \frac{i}{\omega_c} \frac{C_1 B}{\epsilon + p} T^2 (\tilde{\gamma}_1 B + \tilde{\gamma}_2 \tilde{B}_{add}) , \] (2.139)

\[ \tilde{\sigma}_{FB} \left( \rho_m = \mu_m = 0 \right) = \sigma_{FB} + \frac{i}{\omega_c} \frac{C_1 C_2 B \tilde{B}_{add}}{f_1} + \frac{i}{\omega_{cd}} \frac{C_2 C_3 B \tilde{B}_{add}}{g_1} \] (2.140)

\[ - \frac{i}{\omega_c} \frac{C_2 \tilde{B}_{add}}{\epsilon + p} T^2 (\tilde{\gamma}_1 B + \tilde{\gamma}_2 \tilde{B}_{add}) , \]

\[ \tilde{\sigma}_{BF} \left( \rho_m = \mu_m = 0 \right) = \sigma_{BF} + \frac{i}{\omega_c} \frac{C_1 C_2 B \tilde{B}_{add}}{f_1} + \frac{i}{\omega_{cd}} \frac{C_3 C_4 B \tilde{B}_{add}}{g_1} \] (2.141)

\[ - \frac{i}{\omega_{cd}} \frac{C_2 \tilde{B}_{add}}{\epsilon + p} T^2 (\tilde{\gamma}_1 B + \tilde{\gamma}_2 \tilde{B}_{add}) , \]

\[ \tilde{\sigma}_B \left( \rho_m = \mu_m = 0 \right) = \sigma_B + \frac{i}{\omega_c} \frac{C_2^2 B \tilde{B}_{add}}{f_1} + \frac{i}{\omega_{cd}} \frac{C_2^2 B^2}{g_1} \] (2.142)

\[ - \frac{i}{\omega_{cd}} \frac{C_3 \tilde{B}_{add}}{\epsilon + p} T^2 (\tilde{\gamma}_1 B + \tilde{\gamma}_2 \tilde{B}_{add}) , \]

where for the brevity of the notion we introduce index ‘m’ denoting both cases, i.e., the case of the ordinary Maxwell field and the additional \( U(1) \)-gauge one.
Refining our studies to the special case when $\tilde{B}_{\text{add}} = 0$ and the temperature tends to zero, we get

$$\tilde{\sigma}_F(\rho_m = \mu_m = 0) = \sigma_F + \frac{i C_2^2 B^2}{\omega_c \left( \frac{\partial \rho}{\partial \mu} \right)_{T,\nu,\rho}}$$  \hspace{1cm} (2.143)

$$\tilde{\sigma}_{FB}(\rho_m = \mu_m = 0) = \sigma_{FB},$$  \hspace{1cm} (2.144)

$$\tilde{\sigma}_{BF}(\rho_m = \mu_m = 0) = \sigma_{BF},$$  \hspace{1cm} (2.145)

$$\tilde{\sigma}_B(\rho_m = \mu_m = 0) = \sigma_B + \frac{i C_2^2 B^2}{\omega_{cd} \left( \frac{\partial \rho_d}{\partial \mu_d} \right)_{T,\rho}},$$  \hspace{1cm} (2.146)

From the above relations one can draw a conclusion that even at zero densities we obtain still finite DC-conductivities. The terms in question can be only dissipated by the charge dissipations, $\omega_c$ and $\omega_{cd}$, respectively. In the case when one considers only Maxwell gage field ($\tilde{B}_{\text{add}} = 0$), we have only $\tilde{\sigma}_F$ like in [20], equation (2.44). On the other hand, the form of $\tilde{\sigma}_F$ conductivity is the same as in [20] (see the next section) but in our case $\rho$ is also dependent on the auxiliary gauge field components. On the other hand, when we assume that $B = T = 0$, one has that

$$\tilde{\sigma}_F(\rho_m = \mu_m = 0) = \sigma_F + \frac{i C_2 C_4 \tilde{B}_{\text{add}}^2}{\omega_{cd} \left( \frac{\partial \rho_d}{\partial \mu_d} \right)_{T,\rho}},$$  \hspace{1cm} (2.147)

$$\tilde{\sigma}_{FB}(\rho_m = \mu_m = 0) = \sigma_{FB},$$  \hspace{1cm} (2.148)

$$\tilde{\sigma}_{BF}(\rho_m = \mu_m = 0) = \sigma_{BF},$$  \hspace{1cm} (2.149)

$$\tilde{\sigma}_B(\rho_m = \mu_m = 0) = \sigma_B + \frac{i C_2 \tilde{B}_{\text{add}}^2}{\omega_c \left( \frac{\partial \rho}{\partial \mu} \right)_{T,\nu}},$$  \hspace{1cm} (2.150)

It is worth mentioning that, we have the similar situation, i.e., even at zero densities we obtain finite DC-conductivities. The terms in question are dissipated only by the charge dissipative terms.

3 Magneto-transport of the holographic system

In this section we shall elaborate the holographic model of the system, i.e., Dirac semimetals with $Z_2$ and chiral anomalies. Our goal is to use holographic model of the system in question and calculate its thermodynamic properties, in particular the relations among the charge densities and chemical potentials. We shall restrict out attention to the analysis of the simplest cases as given by the equations (2.143)-(2.146). It has to be stressed that hydrodynamic approach does not fix the values of the Boltzmann conductivities $\sigma_F, \sigma_{FB}, \sigma_{BF}$ and $\sigma_B$ in the above expressions. It provides the general conditions [45] stemming from the positivity of the entropy production. However, it completely determines the anomaly related kinetic coefficients.
In the probe limit, we are interested in, one considers the AdS Schwarzschild black brane background, in five-dimensional Einstein-Chern Simons gravity with negative cosmological constant with two $U(1)$-gauge fields. The gauge Chern-Simons terms in gravitational action constitute the possible interactions of the aforementioned fields, which play the crucial role from the point of view of the chiral and $\mathbb{Z}_2$ anomalies in the studied system.

The bulk action providing by our holographic model is composed of terms responsible for chiral and $\mathbb{Z}_2$ anomalies. It implies

$$S = \int dx^5 \sqrt{-g} \left( R + \frac{12}{L^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} + \frac{\alpha_1}{3} \epsilon^{\mu\nu\rho\delta\tau} A_\mu F_{\nu\rho} F_{\delta\tau} \right)$$

(3.1)

$$+ \frac{\alpha_2}{3} \epsilon^{\mu\nu\rho\delta\tau} B_\mu B_{\nu\rho} B_{\delta\tau} + \frac{\alpha_3}{3} \epsilon^{\mu\nu\rho\delta\tau} A_\mu F_{\nu\rho} B_{\delta\tau} + \frac{\alpha_4}{3} \epsilon^{\mu\nu\rho\delta\tau} B_\mu F_{\nu\rho} B_{\delta\tau} \right).$$

In what follows, our convention is $\epsilon^{txyz} = 1$. The equation of motion for the $U(1)$-gauge fields can be written as

$$\nabla_\alpha F^{\alpha\beta} + \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} F_{\mu\nu} F^\alpha = 0,$$

(3.2)

$$\text{and for the auxiliary } B_{\mu\nu} \text{ field it has the form as}$$

$$\nabla_\alpha B^{\alpha\beta} + \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} B_{\mu\nu} F^\alpha = 0,$$

(3.3)

The components of Maxwell and the additional gauge field are provided by

$$A_\mu = (\phi(r), 0, Bx, A_z(r), 0), \quad B_\mu = (\psi(r), 0, \tilde{B}_{add}, B_z(r), 0),$$

(3.4)

In our consideration, as the background metric we take the line element of AdS-Schwarzschild five-dimensional black brane

$$ds^2 = r^2 \left( - f(r) dt^2 + dx^2 + dy^2 + dz^2 \right) + \frac{dr^2}{r^2 f(r)}.$$

(3.5)

where $f(r) = 1 - \frac{r_0^4}{r^4}$ and $r_0$ is the radius of the event horizon. The energy density, entropy density and the Hawking temperature for the black brane are given, respectively by

$$\epsilon = 3 \frac{r_0^4}{r^4}, \quad s = 4\pi r_0^3, \quad T = \frac{r_0^4}{\pi}.$$

(3.6)

The equations of motion for the gauge fields in the aforementioned background yield

$$\phi''(r) + \frac{3}{r} \phi'(r) + \frac{8\alpha_1}{r^3} B A_\phi^\prime(r) + \frac{8\alpha_3}{3r^3} \left( A_\phi^\prime(r) \tilde{B}_{add} + B_\phi^\prime(r) B \right) + \frac{8\alpha_4}{3r^3} B_\phi^\prime(r) \tilde{B}_{add} = 0.$$  \hspace{1cm} (3.7)

$$\psi''(r) + \frac{3}{r} \psi'(r) + \frac{8\alpha_2}{3r^3} \tilde{B}_{add} B_\psi^\prime(r) + \frac{8\alpha_4}{3r^3} \left( B_\psi^\prime(r) B + A_\psi^\prime(r) \tilde{B}_{add} \right) + \frac{8\alpha_4}{3r^3} A_\psi^\prime(r) B = 0.$$  \hspace{1cm} (3.8)

$$A_\phi''(r) + A_\phi^\prime(r) \left( \frac{3}{r} + \frac{f'(r)}{f(r)} \right) + \frac{8\alpha_1}{r^3 f(r)} B A_\phi'(r) + \frac{8\alpha_3}{3r^3 f(r)} \left( \phi'(r) \tilde{B}_{add} + \psi'(r) B \right)$$

$$+ \frac{8\alpha_4}{3r^3 f(r)} \psi'(r) \tilde{B}_{add} = 0,$$  \hspace{1cm} (3.9)

$$B_\psi''(r) + B_\psi^\prime(r) \left( \frac{3}{r} + \frac{f'(r)}{f(r)} \right) + \frac{8\alpha_2}{r^3 f(r)} \tilde{B}_{add} \psi'(r) + \frac{8\alpha_4}{3r^3 f(r)} \left( \psi'(r) B + \phi'(r) \tilde{B}_{add} \right)$$

$$+ \frac{8\alpha_3}{3r^3 f(r)} \phi'(r) B = 0.$$  \hspace{1cm} (3.10)
where the prime denotes the derivative with respect to \( r \)-coordinate.

The above set of differential equation can be simplified to the following forms:

\[
\begin{align*}
[r^3 \phi'(r) + A_z(r) a_1 + B_z c]' &= 0, \\
[r^3 f(r) A_z'(r) + \phi(r) a_1 + \psi(r) c]' &= 0, \\
[r^3 \psi'(r) + B_z(r) b_1 + A_z c]' &= 0, \\
[r^3 f(r) B_z'(r) + \psi(r) b_1 + \phi(r) c]' &= 0,
\end{align*}
\]

where for the brevity of the notation we have introduced the quantities defined as follows:

\[
\begin{align*}
a_1 &= 8 \left( \alpha_1 B + \frac{\alpha_3}{3} \tilde{B}_{add} \right), \\
b_1 &= 8 \left( \alpha_2 \tilde{B}_{add} + \frac{\alpha_4}{3} B \right), \\
c &= \frac{8}{3} \left( \alpha_4 \tilde{B}_{add} + \alpha_5 B \right).
\end{align*}
\]

Solving the above set of equations for \( \phi \) and \( \psi \), with the boundary conditions that at the event horizon of the considered black brane one has \( \phi(r_0) = \psi(r_0) = 0 \), after changing the coordinates given by the relation

\[
u = \frac{r_0^2}{r^2},
\]

one arrives at the following set of the differential equations:

\[
\begin{align*}
\frac{d^2 \phi(u)}{du^2} &= \frac{1}{4r_0^4(1-u^2)} \left( a_1 b_1 - c^2 \right) \frac{a_1}{b_1} \phi(u), \\
\frac{d^2 \psi(u)}{du^2} &= \frac{1}{4r_0^4(1-u^2)} \left( a_1 b_1 - c^2 \right) \psi(u).
\end{align*}
\]

The analytical solution near the black brane event horizon may be written as

\[
\phi_i(u) = G_1 u_2 F_1 \left[ \frac{1}{4} \left( -\sqrt{1-K_i} - 1 \right); \frac{1}{3} \left( \sqrt{1-K_i} - 1 \right); \frac{1}{2}; u^2 \right] + G_2 u_2 F_1 \left[ \frac{1}{4} \left( 1-\sqrt{1-K_i} \right); \frac{1}{4} \left( 1+\sqrt{1-K_i} \right); \frac{1}{2}; u^2 \right],
\]

where \( i = \phi, \psi \) and \( K_i \) are given by

\[
K_\phi = \frac{(a_1 b_1 - c^2)}{r_0^4}, \quad K_\psi = K_\phi \frac{a_1}{b_1}.
\]

\( G_1 \) and \( G_2 \) stand for constants. Having in mind that near \( u \to 0 \) the solutions for \( \phi(r) \) and \( \psi(r) \) behave like

\[
\phi(r) = \mu - \frac{\rho}{2r_0^2} u + \ldots, \quad \phi(r) = \mu_d - \frac{\rho_d}{2r_0^2} u + \ldots,
\]

one obtains the dual charge density for the Maxwell field

\[
\rho = 4 \mu \frac{\Gamma\left[\frac{5+\sqrt{1-K_\phi}}{4}\right]}{\Gamma\left[\frac{3+\sqrt{1-K_\phi}}{4}\right]} \frac{\Gamma\left[\frac{5-\sqrt{1-K_\phi}}{4}\right]}{\Gamma\left[\frac{3-\sqrt{1-K_\phi}}{4}\right]}.
\]

\[\text{– 22 –}\]
For the additional $U(1)$-gauge field, it implies

$$\rho_d = 4 \mu_d r_0^2 \frac{C^2}{\omega T^2} \frac{\sqrt{1 - K_\phi}}{4} \frac{\sqrt{1 - K_\phi}}{4} \Gamma \left[ \frac{5 - \sqrt{1 - K_\phi}}{4} \right] \Gamma \left[ \frac{5 + \sqrt{1 - K_\phi}}{4} \right].$$

(3.26)

Let us now apply the holographic description for the system in zero density limits, i.e.,

$$\rho = \rho_d \to 0, \quad \omega_c = \omega_{cd} \to \omega$$

and the additional requirement of vanishing the adequate magnetic field. For example, by virtue of the equations (2.143) and (2.146) we can find the explicit forms of the conductivities, when $B_{add} = 0$. They are provided by

$$\tilde{\sigma}_F = \sigma_F + \frac{i}{\omega} \frac{C^2}{4 \pi^2 T^2} B^2 \Gamma \left[ \frac{5 - \sqrt{1 - K_\phi}}{4} \right] \Gamma \left[ \frac{5 + \sqrt{1 - K_\phi}}{4} \right].$$

(3.27)

$$\tilde{\sigma}_B = \sigma_B + \frac{i}{\omega} \frac{C^2}{4 \pi^2 T^2} B^2 \Gamma \left[ \frac{5 - \sqrt{1 - K_\phi}}{4} \right] \Gamma \left[ \frac{5 + \sqrt{1 - K_\phi}}{4} \right].$$

(3.28)

Similar calculations can be conducted for the case when $B = 0$, described by the equations (2.147) and (2.150).

In the hydrodynamical limit we assume that $B \ll T^2$ and $B_{add} \ll T^2$. It implies that the quantities $K_\phi \ll 1$ and $K_\psi \ll 1$. Thus, in the limit under inspection, we receive that the dual densities provided by

$$\rho = 2 \mu r_0^2 \left( 1 + \mathcal{O} \left( \frac{B^2}{T^4}, \frac{B_{add}^2}{T^4} \right) \right), \quad \rho_d = 2 \mu_d r_0^2 \left( 1 + \mathcal{O} \left( \frac{B^2}{T^4}, \frac{B_{add}^2}{T^4} \right) \right).$$

(3.29)

Consequently, the adequate conductivities are given by expressions

$$\tilde{\sigma}_F = \sigma_F + \frac{i}{\omega} \frac{C^2}{4 \pi^2 T^2} B^2 + \mathcal{O} \left( \frac{B^2}{T^4}, \frac{B_{add}^2}{T^4} \right),$$

(3.30)

$$\tilde{\sigma}_B = \sigma_B + \frac{i}{\omega} \frac{C^2}{4 \pi^2 T^2} B^2 + \mathcal{O} \left( \frac{B^2}{T^4}, \frac{B_{add}^2}{T^4} \right).$$

(3.31)

It can be remarked that in the case when $B_{add} = 0$ and respectively $C_3 = 0$, we arrive at the results presented in [20], where the only one gauge field was considered and anomaly was bounded with electric and magnetic components of the ordinary Maxwell field.

In figure 1 we plot the magnetic field dependence of the relative conductivity $(\tilde{\sigma}_F - \sigma_F)/(\pi^2 T^4)$ for a system with the density given by the formula (3.25) and assuming that $B_{add}$ vanishes. This assumption is valid for materials like Na$_3$Bi (or Cd$_2$As$_3$), in which the spin projection is related to $Z_2$ symmetry. In such systems one does not expect spin analog of the magnetic field and this justifies our assumption. Two curves in the figure correspond to the systems with and without $Z_2$ symmetry. The lower curve, calculated for $C_1 = C = 1, C_3 = 0 = C_2 = C_4$ describes the system with chiral but without $Z_2$ symmetry. The modifications due to the $Z_2$ anomalous charge are depicted in the upper curve. It occurs that the presence of the $Z_2$ symmetry leads to the narrowing of the magneto-conductivity.
Figure 1. The magnetic field dependence of the real part of conductivity $\tilde{\sigma}_F$ calculated from the equation (2.143) for a system described by (3.25). The upper curve is obtained for all $C_i \neq 0$, while the lower one for $C_1 = C = 1$ and $C_3 = 0$. The existence of $\mathbb{Z}_2$ symmetry leads to narrowing of the magneto-conductivity. For actual calculations we have used $T = 1$, $\tau_e = 0.1$, $\omega = 0$.

curve. This conclusion seem to agree with the one based on the kinetic equation approach to Na$_3$Bi material [44].

The plot has been obtained for $T = 1$, $\tau_e = 0.1$, and $\omega = 0$. In the next section, we calculate the conductivity $\tilde{\sigma}_F$ using the holographic approach to the problem and we shall get also this part $\sigma_F$ of the conductivity as well as its magnetic field dependence. The effect of $\mathbb{Z}_2$ anomaly shows up as a narrowing of the magnetoconductivity line. However, the overall dependence on the $CB/(\pi^2T^2)$ for the system with additional symmetry is similar to that with chiral symmetry only.

3.1 Relation between hydrodynamics and holography

In order to connect the hydrodynamical and holographic descriptions we shall find the dependence of $C_i$ on $\alpha_i$ constants. In order to obtain them we expand the action (3.1), to the second order in perturbations of both gauge fields fields and the metric.

$$A_\mu \to A_\mu + a_\mu, \quad B_\mu \to B_\mu + b_\mu, \quad g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}. \quad (3.32)$$

We focus on the part of the action responsible to the first order perturbations connected with currents of gauge fields. Namely, to the first order in gauge field fluctuations, the adequate part of the action reduces to a boundary term of the following form:

$$\delta S^{(1)} = \int d^4x \left[ \sqrt{-g} \left( F^{\beta\gamma} + \frac{4\alpha_1}{3} e^{\beta\mu\nu}A_\mu F_{\nu\rho} + \frac{2\alpha_3}{3} e^{\beta\mu\nu} A_\mu B_{\nu\rho} + \frac{2\alpha_4}{3} e^{\beta\mu\nu} B_\mu B_{\nu\rho} \right) a_\beta + \sqrt{-g} \left( B^{\beta\gamma} + \frac{4\alpha_2}{3} e^{\beta\mu\nu} B_\mu B_{\nu\rho} + \frac{2\alpha_3}{3} e^{\beta\mu\nu} A_\mu F_{\nu\rho} + \frac{2\alpha_4}{3} e^{\beta\mu\nu} B_\mu F_{\nu\rho} \right) b_\beta \right]_{r \to \infty}. \quad (3.33)$$

\[ - 24 - \]
The equations of motion for the above perturbations imply

\[
J_\beta^{(F)} = \frac{\delta S^{(1)}}{\delta A_\mu} \bigg|_{r \to \infty}, \quad J_\beta^{(B)} = \frac{\delta S^{(1)}}{\delta B_\mu} \bigg|_{r \to \infty}.
\] (3.34)

Using the equations of motion (3.2) and (3.3) and the divergence of \( \nabla_\beta \left( J_\beta^{(F)} + J_\beta^{(B)} \right) \), we arrive at the condition

\[
C_i = \frac{8 \alpha_i}{3},
\] (3.35)

where \( i = 1, \ldots, 4 \).

4 Holographic calculation of conductivities in the probe limit

This section is devoted to the direct calculations of the DC-conductivities for the studied system. As in the later section we shall elaborate the probe limit of the holographic model, starting from small perturbations in the AdS-Schwarzschild background and computing the longitudinal conductivities from the perturbations.

In order to find the longitudinal holographic conductivity in the model in question, we assume that the fluctuations of the vector potentials for both gauge fields are provided by

\[
A_\mu = (\delta \phi(r)e^{-i\omega t}, 0, 0, B x, \delta A_z(r)e^{-i\omega t}),
\] (4.1)

\[
B_\mu = (\delta \psi(r)e^{-i\omega t}, 0, 0, B_{\text{add}} x, \delta B_z(r)e^{-i\omega t}).
\] (4.2)

The equations of motion for the above perturbations imply

\[
\delta \phi' + \frac{\delta A_z}{r^3} a_1 + \frac{\delta B_z}{r^3} c = 0,
\] (4.3)

\[
\delta \psi' + \frac{\delta A_z}{r^3} b_1 + \frac{\delta A_z}{r^3} c = 0,
\] (4.4)

\[
\delta A_z'' + \left( \frac{3}{r} + \frac{f'(r)}{f(r)} \right) \delta A_z' + \frac{\omega^2}{r^4 f(r)^2} \delta A_z + \frac{\delta \phi'}{r^3 f(r)} a_1 + \frac{\delta \psi'}{r^3 f(r)} c = 0,
\] (4.5)

\[
\delta B_z'' + \left( \frac{3}{r} + \frac{f'(r)}{f(r)} \right) \delta B_z' + \frac{\omega^2}{r^4 f(r)^2} \delta B_z + \frac{\delta \phi'}{r^3 f(r)} c + \frac{\delta \psi'}{r^3 f(r)} b_1 = 0.
\] (4.6)

By virtue of the above, one has the following relations for \( \delta A_z(r) \) and \( \delta B_z(r) \)

\[
\delta A_z'' + \left( \frac{3}{r} + \frac{f'(r)}{f(r)} \right) \delta A_z' + \frac{\omega^2}{r^4 f(r)^2} \delta A_z = 0,
\] (4.7)

\[
\delta B_z'' + \left( \frac{3}{r} + \frac{f'(r)}{f(r)} \right) \delta B_z' + \frac{\omega^2}{r^4 f(r)^2} \delta B_z = 0,
\] (4.8)

where we have denoted the following abbreviations

\[
\tilde{A} = a_1 (a_1 + c) + c (b_1 + c), \quad \tilde{B} = b_1 (b_1 + c) + c (a_1 + c).
\] (4.9)

In \( u = r_0^2/r^2 \) coordinate the relations (4.7)-(4.8) reduce to the forms

\[
\delta A_z''(u) - \frac{2u}{1-u^2} \delta A_z'(u) + \left( \frac{\omega^2}{4 r_0^2 (1-u^2)^2} \right) \delta A_z(r) = 0,
\] (4.10)

\[
\delta B_z''(u) - \frac{2u}{1-u^2} \delta B_z'(u) + \left( \frac{\omega^2}{4 r_0^2 (1-u^2)^2} \right) \delta B_z(r) = 0.
\] (4.11)
where now the prime denotes derivatives with respect to \( u \)-coordinate.

For the the near horizon limit, i.e. \( u \to 1 \), the above equations can be rewritten as

\[
\delta A_u''(u) - \frac{1}{1-u} \delta A'_u(u) + \left( \frac{\omega^2}{16 r_0^2} u(1-u) - \frac{\hat{A}}{8 r_0^8 (1-u)} \right) \delta A_z(r) = 0, \quad (4.12)
\]

\[
\delta B_u''(u) - \frac{1}{1-u} \delta B'_u(u) + \left( \frac{\omega^2}{16 r_0^2} u(1-u) - \frac{\hat{B}}{8 r_0^8 (1-u)} \right) \delta B_z(r) = 0, \quad (4.13)
\]

with the solution given in terms of the modified Bessel functions

\[
\delta A_z(u) = E_1(-1)^{-\frac{i\omega}{2r_0}} I_{-\frac{i\omega}{2r_0}} \left[ \sqrt{\frac{A}{2} (1-u)^{\frac{1}{2}}} \right] + E_2(-1)^{-\frac{i\omega}{2r_0}} I_{-\frac{i\omega}{2r_0}} \left[ \sqrt{\frac{A}{2} (1-u)^{\frac{1}{2}}} \right], \quad (4.14)
\]

\[
\delta B_z(u) = D_1(-1)^{-\frac{i\omega}{2r_0}} I_{-\frac{i\omega}{2r_0}} \left[ \sqrt{\frac{B}{2} (1-u)^{\frac{1}{2}}} \right] + D_2(-1)^{-\frac{i\omega}{2r_0}} I_{-\frac{i\omega}{2r_0}} \left[ \sqrt{\frac{B}{2} (1-u)^{\frac{1}{2}}} \right], \quad (4.15)
\]

where \( E_i, D_i \) are integration constants. The in-falling boundary conditions for \( u \)-coordinate correspond to disappearing of \( E_2 \) and \( D_2 \).

In the far away region \( 1-u \gg \omega/r_0 \), the above relations take forms as

\[
\delta A_u''(u) - \frac{2u}{1-u^2} \delta A'_u(r) - \frac{\hat{A}}{4 r_0^4 (1-u^2)} \delta A_z(r) = 0, \quad (4.16)
\]

\[
\delta B_u''(u) - \frac{2u}{1-u^2} \delta B'_u(r) - \frac{\hat{B}}{4 r_0^4 (1-u^2)} \delta B_z(r) = 0, \quad (4.17)
\]

with the solution in terms of the Legendre functions

\[
\delta A_z(u) = \tilde{E}_1 P \left[ \sqrt{\frac{1}{r_0^2} - 1} \right] (u) + \tilde{E}_2 Q \left[ \sqrt{\frac{1}{r_0^2} - 1} \right] (u), \quad (4.18)
\]

\[
\delta B_z(u) = \tilde{D}_1 P \left[ \sqrt{\frac{1}{r_0^2} - 1} \right] (u) + \tilde{D}_2 Q \left[ \sqrt{\frac{1}{r_0^2} - 1} \right] (u), \quad (4.19)
\]

where \( \tilde{E}_i, \tilde{D}_i \) are constants.

In order to find the integration constants one should match the near-horizon solution with the far away one, in some intermediate, matching, region \( \omega/r_0 \ll 1 - u \ll 1 \). It can be done exactly as in [20]. Using the same reasoning it can be revealed that the integration constants fulfill

\[
\frac{\tilde{E}_1}{E_1} = 1 + \mathcal{O}(\omega), \quad \frac{\tilde{D}_1}{D_1} = 1 + \mathcal{O}(\omega), \quad (4.20)
\]

\[
\frac{\tilde{E}_2}{E_1} = \frac{i\omega}{2r_0} (1 + \mathcal{O}(\omega)), \quad \frac{\tilde{D}_2}{D_1} = \frac{i\omega}{2r_0} (1 + \mathcal{O}(\omega)). \quad (4.21)
\]

To proceed further let us find the value of the Legendre functions at the boundary \( u \to 0 \). Namely, one has

\[
P \left[ \sqrt{\frac{1}{r_0^2} - 1} \right] (u \to 0) = p_1(\hat{A}_i) + p_2(\hat{A}_i) u + \mathcal{O}(u^2), \quad (4.22)
\]

\[
Q \left[ \sqrt{\frac{1}{r_0^2} - 1} \right] (u \to 0) = q_1(\hat{A}_i) + q_2(\hat{A}_i) u + \mathcal{O}(u^2), \quad (4.23)
\]
where we have defined

\[
p_1(\tilde{A}_i) = \frac{\sqrt{\pi}}{\Gamma\left[\frac{3-\sqrt{1-\tilde{A}_i^2}}{4}\right]} \Gamma\left[\frac{3+\sqrt{1-\tilde{A}_i^2}}{4}\right], \quad p_2(\tilde{A}_i) = -\frac{\sqrt{\pi} \tilde{A}_i}{8r_0^4 \Gamma\left[\frac{5-\sqrt{1-\tilde{A}_i^2}}{4}\right]} \Gamma\left[\frac{5+\sqrt{1-\tilde{A}_i^2}}{4}\right].
\]

(4.24)

\[
q_1(\tilde{A}_i) = -\frac{\sqrt{\pi} \sin\left(\frac{\pi}{4}\sqrt{1-\tilde{A}_i^2}\right)}{2\Gamma\left[\frac{3+\sqrt{1-\tilde{A}_i^2}}{4}\right]},
\]

(4.25)

\[
q_2(\tilde{A}_i) = \frac{\sqrt{\pi} \cos\left(\frac{\pi}{4}\sqrt{1-\tilde{A}_i^2}\right)}{\Gamma\left[\frac{1+\sqrt{1-\tilde{A}_i^2}}{4}\right]} \Gamma\left[\frac{5+\sqrt{1-\tilde{A}_i^2}}{4}\right].
\]

(4.26)

In order to find the conductivities we derive the relations which envisages the ingoing boundary conditions for the considered longitudinal currents

\[
\delta A_z(u) = \tilde{E}_1 p_1(F) + \tilde{E}_2 q_1(F) + u(\tilde{E}_1 p_2(F) + \tilde{E}_2 q_2(F)),
\]

(4.27)

\[
\delta B_z(u) = \tilde{D}_1 p_1(B) + \tilde{D}_2 q_1(B) + u(\tilde{D}_1 p_2(B) + \tilde{D}_2 q_2(B)),
\]

(4.28)

Using the definition of the conductivity [50], where the logarithmic divergence is removed with the adequate boundary counter-term in the gravity action [51], one attains

\[
\sigma = \frac{2 A_\mu^{(0)}}{i \omega A_\mu^{(2)}} + \frac{i \omega}{2},
\]

(4.29)

where the gauge field fall off implies the following:

\[
A_\mu = A_\mu^{(0)} + \frac{A_\mu^{(2)}}{r^2} + \ldots,
\]

(4.30)

( the similar condition holds for $B_\mu$ field). Consequently, by virtue of the equations (4.29) and (4.30), we obtain the expressions describing $\tilde{\sigma}_F$ and $\tilde{\sigma}_B$. They can be written as it leads to the expressions:

\[
\tilde{\sigma}_F = \frac{2r_0^2}{i \omega} \frac{(\tilde{E}_1 p_2 + \tilde{E}_2 q_2)}{(\tilde{E}_1 p_1 + \tilde{E}_2 q_1)} + \frac{i \omega}{2},
\]

(4.31)

\[
\tilde{\sigma}_B = \frac{2r_0^2}{i \omega} \frac{(\tilde{D}_1 p_2 + \tilde{D}_2 q_2)}{(\tilde{D}_1 p_1 + \tilde{D}_2 q_1)} + \frac{i \omega}{2}.
\]

(4.32)
Expansions of $\tilde{\sigma}_F$ and $\tilde{\sigma}_B$, at the leading order in $\omega$, reveal the relations

$$
\tilde{\sigma}_F = \left[ \frac{i \tilde{A}}{\omega 4 r_0^2} + \frac{\pi \tilde{A}}{16 \alpha_3^0 \cos \left( \frac{\pi}{2} \sqrt{1 - \frac{A}{r_0^2}} \right) } \right] \Gamma^3 \left[ \frac{-1 - \frac{B}{4 \alpha_3^2}}{4} \right] \Gamma^3 \left[ \frac{-1 - \frac{B}{4 \alpha_3^2}}{4} \right] \Gamma^3 \left[ \frac{-1 - \frac{B}{4 \alpha_3^2}}{4} \right], \quad (4.33)
$$

$$
\tilde{\sigma}_B = \left[ \frac{i \tilde{B}}{\omega 4 r_0^2} + \frac{\pi \tilde{B}}{16 \alpha_3^0 \cos \left( \frac{\pi}{2} \sqrt{1 - \frac{B}{r_0^2}} \right) } \right] \Gamma^3 \left[ \frac{-1 - \frac{B}{4 \alpha_3^2}}{4} \right] \Gamma^3 \left[ \frac{-1 - \frac{B}{4 \alpha_3^2}}{4} \right] \Gamma^3 \left[ \frac{-1 - \frac{B}{4 \alpha_3^2}}{4} \right]. \quad (4.34)
$$

In the limit when $B/T^2 \ll 1$ and $\tilde{B}_{add}/T^2 \ll 1$, which is equivalent to the condition that $\tilde{A} \ll 1$ and $\tilde{B} \ll 1$, one gains the formulae written as

$$
\tilde{\sigma}_F = \sigma_F + \frac{i}{\omega} \frac{\tilde{A}}{2 \pi T^2} + O\left( \frac{\tilde{A}^2}{T^4} \right), \quad \sigma_F = \frac{T}{2} \cos \left( \frac{\tilde{A}^2}{T^2} \right), \quad (4.35)
$$

$$
\tilde{\sigma}_B = \sigma_B + \frac{i}{\omega} \frac{\tilde{B}}{2 \pi T^2} + O\left( \frac{\tilde{B}^2}{T^4} \right), \quad \sigma_B = \frac{T}{2} \cos \left( \frac{\tilde{B}^2}{T^2} \right). \quad (4.36)
$$

Consequently, in the limit $\tilde{B}_{add} \rightarrow 0$, the conductivities imply

$$
\tilde{\sigma}_F = \sigma_F + \frac{i}{\omega} \frac{\tilde{A}}{2 \pi T^2} + O\left( \tilde{A}^4 B^4 \right), \quad \sigma_F = \frac{T}{2} + O\left( \tilde{A}^4 B^4 \right), \quad (4.37)
$$

$$
\tilde{\sigma}_B = \sigma_B + \frac{i}{\omega} \frac{\tilde{B}}{2 \pi T^2} + O\left( \tilde{B}^4 B^4 \right), \quad \sigma_B = \frac{T}{2} + O\left( \tilde{B}^4 B^4 \right), \quad (4.38)
$$

where we have defined

$$
\tilde{\alpha}^2 = \frac{64}{9} \alpha_1^2 + \frac{64}{9} \left( \alpha_3^2 \alpha_4^2 + \alpha_3^2 \alpha_4 \right) + \frac{64}{3} \alpha_3 \alpha_4, \quad (4.39)
$$

$$
\tilde{\alpha}^2 = \frac{64}{9} \alpha_2^2 + \frac{64}{9} \left( \alpha_3^2 \alpha_4^2 + \alpha_3^2 \alpha_4 \right) + \frac{64}{3} \alpha_3 \alpha_4. \quad (4.40)
$$

Turning our attention to the limit when $\tilde{A} \gg 1$ and $\tilde{B} \gg 1$, one arrives at

$$
\tilde{\sigma}_F = \sigma_F + \frac{i}{\omega} \sqrt{\tilde{A}} + O\left( \frac{1}{2 \sqrt{A^2}} \right), \quad \sigma_F = \frac{T}{2} e^{-\frac{A^2}{2 T^2}} \left( \sqrt{\tilde{A}} + O\left( \frac{1}{2 \sqrt{A^2}} \right) \right), \quad (4.41)
$$

$$
\tilde{\sigma}_B = \sigma_B + \frac{i}{\omega} \sqrt{\tilde{B}} + O\left( \frac{1}{2 \sqrt{B^2}} \right), \quad \sigma_B = \frac{T}{2} e^{-\frac{B^2}{2 T^2}} \left( \sqrt{\tilde{B}} + O\left( \frac{1}{2 \sqrt{B^2}} \right) \right). \quad (4.42)
$$

It will be interesting to analyze the effect of the additional anomaly on the transport properties of the holographic system under consideration. First we note that the holographic approach allowed us the direct calculation of $\sigma_F$, the component of conductivity undetermined in the hydrodynamic attitude. In the literature it is sometimes called quantum critical component of conductivity albeit to call it the Boltzmann conductivity is perhaps more appropriate in the present context, as it can be obtained from the Boltzmann kinetic equation. One has to remember that the general formula (4.33) is valid to lowest order in
Figure 2. The dependence of the $\omega \operatorname{Im} \sigma(\omega)$ (upper left panel) and the real part of the DC conductivity $\tilde{\sigma}_F(\omega = 0)$ (upper right panel) on $\frac{CB}{\pi^2 T^2}$, with $C = C_1$ being a chiral anomaly parameter. The conductivity has been calculated from the relation (4.33) for a holographic system with action (3.1). We used the relation $C_i = \frac{8}{3} \alpha_i$ between anomaly parameters $C_i$ in the hydrodynamic approach and corresponding parameters $\alpha_i$ adopted in (3.1). One of the two curves in each panel is obtained for all $C_i \neq 0$ (denoted $C_3 = 1$), while the other for $C_1 = C = 1$ and $C_3 = 0$ (denoted $C_3 = 0$).

The existence of $Z_2$ symmetry (corresponding to $C_3 = 1 = C_2 = C_4 = C_1$ and shown by the upper curve in the upper panel) leads to the narrowing of the magneto-conductivity. We have assumed that $T = 1, \tau_e = 0.05$ and $\omega = 0$. The lower panel shows the magnetoresistance of the same system.

The conductivity $\sigma_F$ depends on the magnetic field $B$ and also on the additional magnetic field $B_{\text{add}}$. In accordance with the paper [44] we assume here $B_{\text{add}} = 0$ arguing after the cited paper that $Z_2$ symmetry, in some systems at least, is related to the spin projections and one does not expect spin analog of the $B$ magnetic field.

In the upper left panel of the figure 2, we show the dependence of $\omega \operatorname{Im} \sigma(\omega)$ on the magnetic field, and more exactly on the parameter $\frac{CB}{\pi^2 T^2}$, for the two investigated cases. One curve ($C_3 = 0$) corresponds to the system with the chiral anomaly described by $C_1 = C \neq 0$ with $C_2 = C_3 = C_4 = 0$. The presence of $Z_2$ anomaly in the figure represented by the curve with $C_3 = 1$ ($= C_2 = C_4$) has an effect of narrowing the dependence of $\omega \operatorname{Im} \sigma(\omega)$ on the
magnetic field. This result harmonizes with the conclusion obtained in the previous section by applying the hydrodynamic approach to the holographic system.

The upper right panel of the Fig. 2 illustrates similar dependence of the $\tilde{\sigma}_F$ on the scaled magnetic field for the system with chiral only symmetry ($C_3 = 0$) and for the type II Dirac semimetal with both chiral and $\mathbb{Z}_2$ anomalies. The real part of the DC-conductivity (i.e., $\omega = 0$) $\tilde{\sigma}_F$ shown in the main part of the figure comprises the quantum critical (or Boltzmann) part and the frequency dependent part which remains nonzero if the system is characterized by the finite charge relaxation time $\tau_c = 0.05$ introduced phenomenologically by the substitution $\omega \rightarrow \omega + i/\tau_c$. The inset to the figure shows the dependence on $\frac{CB}{\tilde{\sigma}_F}$ of the Boltzmann contribution to the total conductivity. Interestingly it features strong non-monotonic dependence on the B-field, which is limited to relatively low fields only. Thus altogether we are getting a non-monotonous overall dependence of the magneto-conductivity on the magnetic field. Again the presence of the $\mathbb{Z}_2$ symmetry shows up as a narrowing of the magneto-conductivity line.

The lower panel of the figure 2 depicts the magneto-resistivity of the same system. One observes that for the very small magnetic field the DC-magneto-resistivity $[\tilde{\rho}_F(B) - \tilde{\rho}_F(0)]/\tilde{\rho}_F(0)$ (with $\tilde{\rho}_F = 1/\tilde{\sigma}_F$) is positive and only for higher fields it becomes negative. The low field negative magneto-conductivity and positive magneto-resistivity followed by the negative magneto-resistivity at high fields has been observed experimentally [34] in ZrTe$_5$. The holographic attitude provides explanation of the experimental findings competitive to the weak anti-localization proposal.

Besides narrowing of the $\sigma_F(B)$ curve, the additional anomaly, changes the relative contribution of quantum critical and relaxation limited parts. This is visible as a shift of the minimum of the two magneto-conductivity curves. The existence of $\mathbb{Z}_2$ anomaly shifts that point towards lower magnetic fields. However, the detailed behavior depends also on the assumed value of the charge relaxation time, $\tau_c$.

### 5 Summary and conclusions

We have studied the behavior of the longitudinal conductivities in the Weyl semimetals with $\mathbb{Z}_2$ symmetry and chiral anomaly in the hydrodynamics limit, with the adequate magnetic fields which are connected with the ordinary Maxwell and the second $U(1)$-gauge fields. The auxiliary gauge field was introduced in order to envisage the $\mathbb{Z}_2$ anomalous charge.

Using the kinetic coefficients [45] for the system in question, being the functions of temperature, $\rho$, $\rho_d$, $\mu$, $\mu_d$, we found the DC-conductivities in the presence of magnetic fields. In our considerations we take into account the dissipation terms, i.e., momentum, energy and charges dissipation terms. For the general case we obtain the complicated functions which depend on both magnetic fields, as well as, chiral and $\mathbb{Z}_2$ anomaly coefficients. In order to simplify these relations we considered limiting cases. Namely, the cases of $\rho = \mu = 0$, $\rho_d = \mu_d = 0$ and $\rho = \mu = \rho_d = \mu_d = 0$ were elaborated. Moreover when one assumes that the one of the considered magnetic fields is equal to zero, we get the DC-conductivities which are functions of charge densities. Namely, if $B = 0$, the conductivity depends on $\rho_d$, and on the contrary, when $\tilde{B}_{add} = 0$, it is the function of $\rho$. 


In the case when $\rho = \mu = \rho_d = \mu_d = 0$ and vanishing of both magnetic fields, one obtains the result that even at zero densities we have finite DC-conductivities. The received relations were only dissipated by the charge dissipation terms responsible for both $U(1)$-gauge fields. In the limit when the additional gauge field is equal to zero, one gets the results presented in [20].

The system was also examined by the holographic attitude, by means of five-dimensional Einstein-Chern-Simons gravity. The gauge Chern-Simons terms in gravitational action were associated with the possible interactions, playing the crucial role from the point of view of chiral and $\mathbb{Z}_2$ anomalies. As a background, in the probe limit attitude, we take the line element of five-dimensional AdS-Schwarzschild black brane. Starting from the small perturbations of the aforementioned black brane we have found the longitudinal conductivities for the considered model. We directly found that using Kubo formula, in the probe limit, the results match the hydrodynamical approach in the limit in question.

In the holographic description we did not include the dissipation terms. In addition, in the case when the additional field is equal to zero, we obtained the previously announced results [20].

The future experimental observations may put some restrictions on the constraints $C_i$ connected with the magneto-conductivities. Namely, the measurements similar to those conducted in [37] may shed some light on the problem in question. Our results show that the presence of $\mathbb{Z}_2$ symmetry causes the narrowing of the magneto-conductivity curve. On the other hand, the temperature dependence and narrowing of the magneto-conductivity curve were verified by the experimental data [35, 52–54].

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