The upper bound of vertex local antimagic edge labeling on graph operations

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Abstract. Let \( G(V(G), E(G)) \) be a connected simple graph and let \( u,v \) be two vertices of graph \( G \). A bijective function \( f \) from the edge set of \( G \) to the natural number up to the number of edges in \( G \) is called a vertex local antimagic edge labeling if for any two adjacent vertices \( v \) and \( v' \), \( w(v) \neq w(v') \), where \( w(v) = \sum_{e \in E(v)} f(e) \), and \( E(v) \) is the set of vertices which is incident to \( v \). Thus any vertex local antimagic edge labeling induces a proper vertex coloring of \( G \) where the vertex \( v \) is assigned with the color \( w(v) \). The vertex local antimagic chromatic number \( \chi_{la}(G) \) is the minimum number of colors taken over all coloring induced by vertex local antimagic edge labeling of \( G \). In this paper, we study the vertex local antimagic edge labeling of graphs and determine the upper bound of vertex local antimagic chromatic number on graph operation, namely cartesian product of graphs, corona product of graphs, and power graph.

1. Introduction
Any mathematical object involving points and connection between them called a graph. A graph \( G(V(G), E(G)) \) consists of two sets \( V(G) \) and \( E(G) \). The all graphs are discussed in this paper are simple and connected. Simple graph \( G \) is a pair of set \( (V(G), E(G)) \) with the non-empty vertex set \( V(G) \) and the edge set \( E(G) \) that is non-ordered pair set of two distinct vertices in \( V(G) \). The graph \( G \) is connected if every two vertices of \( G \) are connected [1 2]. The elements of \( V(G) \) are called vertices and the elements of \( E(G) \) called edges.

The concept of antimagic labeling of graph was introduced by Hartsfield and Ringel [3]. Graph labeling is the assignment of labels to the elements of graph such as edges, vertices, or both of them. Every component of graph such as vertex and edge is given distinct label by natural number. Antimagic labeling is a bijection from the set of edges
to the first natural number such that all end vertex sum are pairwise distinct, where a vertex sum is the sum of labels of all edges incident with the same vertex [8]. A graph $G$ is called antimagic if $G$ has an antimagic labeling. Since antimagic labeling had been introduced by Hartsfield and Ringel [3], the topic has attracted a lot of attentions, for details, see Gallian [12]. One of well-known conjectures related to this is the following Conjecture 1. Every connected graph other than $K_2$ is antimagic [3].

The conjecture has been lasting for decades and far from been solved. Over the years, many variations of labelling has been introduced, for example, $(a,d)$-antimagic labelling, magic labelling etc.

Arunugam et al [4] was introduced the concept of local antimagic coloring of graph. Let $G(V(G), E(G))$ be a graph, a bijective function $f$ from the edge set of $G$ to the natural number up to the number of edges in $G$ is called a vertex local antimagic edge labeling if for any two adjacent vertices $v$ and $v'$, $w(v) \neq w(v')$, where $w(v) = \sum_{e \in E(v)} f(e)$, and $E(v)$ is the set of vertices which is incident to $v$. Thus any vertex local antimagic edge labeling induces a proper vertex coloring of $G$ where the vertex $v$ is assigned the color $w(v)$. The minimum number of colors taken over all coloring induced by vertex local antimagic edge labeling of $G$ denoted by $\chi_{la}(G)$. They give an lower bound and upper bound of local antimagic vertex coloring of $K_1 + H$ and also give the exact value of local antimagic vertex coloring. Agustin et al [5] studied the local edge antimagic coloring of graphs.

In this paper, we have investigated the upper bound of the vertex local antimagic edge labeling for some graph operations, such as cartesian product of graphs, corona product of graphs, and power graph. Before we present our results, we define a few terms which we will use in our paper. The cartesian product of $G_1$ and $G_2$, denoted by $G_1 \times G_2$, is the graph with vertex set $V(G_1) \times V(G_2)$ where two distinct vertices $(u, v)$ and $(x, y)$ of $V(G_1) \times V(G_2)$ are adjacent if either (1) $u = x$ and $vy \in E(G_2)$ or (2) $v = y$ and $ux \in E(G_1)$. The $k$-th power of graph $G$, denoted by $G^k$ is another graph that has the same set of vertices, but in which two vertices are adjacent when their distance in $G$ is at most $k$.

Powers of graphs are referred to using terminology similar to that of exponentiation of numbers, $G_2$ is called the square of $G$, $G_3$ is called the cube of $G$ in [13]. Frucht and Harary [14] had introduced the corona product of graph. Let $G$ be a connected graph of order $n$ and $H$ (not necessarily connected) be a graph of order $m$. A corona product of $G$ and $H$, denoted by $G \odot H$, is defined as a graph obtained by taking one copy of $G$ and $n$ copies of graph $H$, $H_1, H_2, \ldots, H_n$ and connecting $i$-th vertex of $G$ to the vertices of $H_i$, $1 \leq i \leq n$.

2. Main Result

We present the upper bound of vertex local antimagic edge labeling of cartesian product of graphs, corona product of graphs and power graph by the following.

Theorem 2.1. For every graph $G$ apart from $K_2$, if $G$ has vertex local antimagic edge labeling, then we have $\chi_{la}(G \times P_n) \leq (n-2)\chi_{la}(G) + 2|V(G)|$.

Proof. To prove the existence of local vertex antimagic labeling of graph $G \times P_n$, we utilize the concept of vertex antimagic edge labeling of graph $G$. Since graph $P_n$
as a backbone of $G \times P_n$, so the number of $P_n$ in $G \times P_n$ are as many as vertices in $G$. For illustration see Figure 1. The edge sets of all $P_n$ (we called spokes) are $S_1 = (e_1^1, e_1^2, ..., e_{|V(G)|})$, $S_2 = (e_2^1, e_2^2, ..., e_{|V(G)|})$, $..., S_{n-1} = (e_{n-1}^1, e_{n-1}^2, ..., e_{|V(G)|})$. We start by assigning the label on spokes. Define a bijective function from the edges of spokes to positive integers, $g : (S_1; 1 \leq i \leq n - 1) \rightarrow \{1, 2, ..., |V(G)|\}[(n - 1)]$ in the following: Suppose $m = |V(G)|$, then we have

$$g(S_{i,j}) = \begin{cases} 
  m(i - 1) + j, & \text{if } i \text{ odd, } 1 \leq i \leq n, 1 \leq j \leq m \\
  mi - j + 1, & \text{if } i \text{ even, } 2 \leq i \leq n, 1 \leq j \leq m
\end{cases}$$

Under the labeling of $g$ we could see that the labels have ascending and descending order alternately, then the all vertices in $S_i; 1 \leq i \leq n - 1$ will have the same weights without edge labels on $G$. While the vertices which are incident with the $S_1$ and $S_{n-1}$ have different weights and form arithmetic sequences with difference 1. Since $G$ has vertex antimagic edge labeling, then the vertex local antimagic chromatic number of $G \times P_n$ is at most $(n - 2)\chi_{la} + 2|V(G)|$. It gives $\chi_{la}(G \times P_n) \leq (n - 2)\chi_{la}(G) + 2|V(G)|$. □

![Figure 1](image-url)

**Figure 1.** The illustration of graph $G \times P_n$

**Theorem 2.2.** For every graph $G$ apart from $K_2$, if $G$ has vertex local antimagic edge labeling, then we have $\chi_{la}(G \times C_n) \leq (n - 1)\chi_{la}(G) + |V(G)|$.

**Proof.** To prove the existence of local vertex antimagic labeling of graph $G \times C_n$, we utilize the concept of vertex antimagic edge labeling of graph $G$. Since graph $C_n$ as a backbone of $G \times C_n$, so the number of $C_n$ in $G \times P_n$ are as many as vertices in $G$. For illustration see Figure 2. The edge sets of all $C_n$ (we called spokes) are $S_1 = (e_1^1, e_1^2, ..., e_{|V(G)|})$, $S_1 = (e_1^1, e_2^1, ..., e_{|V(G)|})$, $..., S_{n} = (e_1^n, e_2^n, ..., e_{|V(G)|})$. We start by assigning the label on spokes. Define a bijective function from the edges of spokes to positive integers, $g : (S_1; 1 \leq i \leq n - 1) \rightarrow \{1, 2, ..., |V(G)|\}[n]$ in the following: Suppose $m = |V(G)|$, then we have

$$g(S_{i,j}) = \begin{cases} 
  m(i - 1) + j, & \text{if } i \text{ odd, } 1 \leq i \leq n, 1 \leq j \leq m \\
  mi - j + 1, & \text{if } i \text{ even, } 2 \leq i \leq n - 1, 1 \leq j \leq m
\end{cases}$$
Under the labeling of $g$ we could see that the labels have ascending and descending order alternately, then we have two cases to see the weights of all vertices in $G \times C_n$ without edge labels in $G$.

**Case 1**: $n$ is odd
From the labeling of spokes, the set $S_1$ and $S_n$ have the same order of labeling and will incident at the same vertices in a graph $G$ (suppose $G'$), so that will form arithmetic sequence with difference 2 and the others vertices in the other $G$ will have the same weight. Since $G$ has vertex antimagic edge labeling, then the vertex local antimagic chromatic number in $G'$ is at most the number of vertices in $G'$.

**Case 2**: $n$ is even
From the labeling of spokes, the all vertices of $G \times C_n$ will have the same weights without edge labels on $G$. From the labeling above, it gives $\chi_{la}(G \times C_n) \leq (n-1)\chi_{la}(G) + |V(G)|$.

**Theorem 2.3.** For every graph $G$ and $H$ apart from $K_2$, if $G$ and $H$ have vertex local antimagic edge labeling, then we have $\chi_{la}(G \circ H) \leq \chi_{la}(G) + |V(G)||V(H)|$, where $|V(H)| \neq 1(\text{mod} 2)$ and $|V(G)| \neq 0(\text{mod} 2)$.

**Proof.** Let $G$ and $H$ are connected graphs with order $n$ and $m$, respectively. To prove the upper bound of local vertex antimagic edge labeling of graph $G \circ H$, we utilize the concept of vertex antimagic edge labeling of graph. Based on the definition of corona product graph, we have one copy of graph $G$ and $n$ copies of graph $H$. Figure 4 show the illustration of $G \circ H$. The edges which connect vertices in $G$ to vertices in $H$ are called spokes. We focus on labeling of spokes by defining the bijective function in the following.
Case 1: \( m \) is even

\[
g(S_{i,j}) = \begin{cases} 
  n(j - 1) + i, & \text{if } j \text{ odd}, 1 \leq i \leq m - 1, 1 \leq j \leq n \\
  n - i + 1, & \text{if } j \text{ even}, 2 \leq i \leq m, 1 \leq j \leq n
\end{cases}
\]

Case 2: \( m, n \) is odd

For odd \( m \); \( m \geq 3 \), the first three sets of spokes use the following function.

\[
g(i, 1) = \begin{cases} 
  \frac{n+1+i}{2}; & i \equiv 0 \pmod{2} \\
  i \frac{1}{2}; & i \equiv 1 \pmod{2}
\end{cases}
\]

\[
g(i, 2) = \begin{cases} 
  n \frac{1}{2}; & i \equiv 0 \pmod{2} \\
  n + \frac{n-i}{2}; & i \equiv 1 \pmod{2}
\end{cases}
\]

\[
g(i, 3) = 3n - i + 1
\]

For the next spokes sets use the following function.

\[
g(S_{i,j}) = \begin{cases} 
  n(j - 1) + i + 3n, & \text{if } j \text{ odd}, 1 \leq i \leq m - 4, 1 \leq j \leq n \\
  n - i + 1 + 3n, & \text{if } j \text{ even}, 2 \leq i \leq m - 3, 1 \leq j \leq n
\end{cases}
\]

From the function \( g \), it is easy to see that the vertices in \( G \) receive the same weight from labels of spokes. Since \( G \) and \( H \) have vertex local antimagic edge labeling, then the vertex weights in every copy of \( H \) are distinct and the number of distinct weights in \( G \) at least the same with vertex weights of original \( G \). It concludes that that the chromatic number of \( G \odot H \) is at most \( \chi_{la}(G) + |V(G)||V(H)| \).

**Theorem 2.4.** For every graph \( G \) apart from \( K_2 \), if \( G \) has vertex local antimagic edge labeling, then we have \( \chi_{la}(G^k) \leq |V(G)| \).

**Proof.** Given that \( G \) is a vertex local antimagic graph. Every two incident edges has a different vertex weight. By the power operation of graph, there is an additional edge to the graph \( G \) and each vertex is connected to other vertices at distance at most \( k \). If \( k \) is the diameter of graph \( G \) then \( G^k \) is a complete graph. According to Arumugam et. al. [4], a complete graph admits a local antimagic labeling. Due to the power operation of graph \( G \), it will increase the number of edges and need to add the labels with sequential numbers. It implies to the increasing of the vertex weight value of graph \( G \). Suppose \( u, v \in V(G) \), we have \( w(u) + \sum_{x \in N(u)} f(ux) \) and \( w(v) + \sum_{y \in N(v)} f(uy) \) as a vertex weight of the local antimagic graph \( G^k \). Since the graph \( G \) is local antimagic, we have \( w(u) \neq w(v) \). It concludes that \( w(u) + \sum_{x \in N(u)} f(ux) \neq w(v) + \sum_{y \in N(v)} f(uy) \) as \( |\sum_{x \in N(u)} f(ux) - \sum_{y \in N(v)} f(uy)| \geq 1 \). From this conditions, the vertex local antimagic chromatic number of \( G^k \) is at most the number of vertices of \( G^k \). □
3. Conclusion
Some results in this paper are the upper bound of chromatic number of vertex local antimagic chromatic number in graph operation such as cartesian product of graphs, corona product of graphs, and power graph. We have the following open problem for further research.

Open Problem 3.1. Determine the exact value of vertex local antimagic chromatic number on graph operations, namely cartesian product of graphs, corona product of graphs, and power graph.

Acknowledgement
We gratefully acknowledge the support from CGANT Research Group of year 2019.

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