Thermal effect on mixed state geometric phases for neutrino propagation in a magnetic field

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Abstract

In astrophysical environments, neutrinos may propagate over a long distance in a magnetic field. In the presence of a rotating magnetic field, the neutrino spin can flip from left-handed neutrino to right-handed neutrino. Smirnov demonstrated that the pure state geometric phase due to the neutrino spin precession may cause resonantg spin conversion inside the Sun. However, in general, the neutrinos may in an ensemble of thermal state. In this article, the corresponding mixed state geometric phases will be formulated, including the off-diagonal case and diagonal ones. The specific features towards temperature will be analyzed.

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I. INTRODUCTION

Geometric phase had been discovered by Berry [2] in the circumstance of adiabatic evolution. Then it was generalized by Wilczek and Zee [17], Aharonov and Anandan [1, 19], Samuel and Bhandari[10], and Samuel and Bhandari[10] in the context of pure state. Moreover, it was also extended to mixed state counterparts. Its operationally well-defined notion was proposed by Sjöqvist et. al. [13] based on interferometry. Subsequently, it was generalized to degenerate case by Singh et. al. [12] and to nonunitary evolution by Tong et. al. [15] by use of kinematic approach. In addition, when the final state is orthogonal to the initial state, the above geometric phase is meaningless. So the complementary one to the usual geometric phases had been put forward by Manini and Pistolesi [9]. The new phase is called off-diagonal geometric phase, which was been generalized to non-abelian case by Kult et. al. [8]. It also had been extended to mixed state ones by Filipp and Sjöqvist [5, 6] during unitary evolution. Further extension to non-degenerate case was made by Tong et. al. [16] by kinematic approach. Finally, there are excellent reviewed articles [18] and monographs [3, 4, 11] talking about its influence and applications in physics and other natural science.

As well known, Neutrino plays an important role in particle physics and astronomy. Smirnov investigated the effect of resonant spin conversion of solar neutrinos which was induced by the geometric phase [14]. Joshi and Jain figured out the geometric phase of neutrino when it was propagating in a rotating transverse magnetic field [7]. However, their discussion are confined to the pure state case. In this article, we will talk about mixed state geometric phase of neutrino, ranging from off-diagonal phase to diagonal one.

This paper is organised as follows. In the next section, the off-diagonal geometric phase for mixed state will be reviewed as well as the usual mixed state geometric phase. Furthermore, the related equation about the propagation of two helicity components of neutrino will be retrospected. In Sec. III, both the off-diagonal and diagonal mixed geometric phase for neutrino in thermal state are going to be calculated. Finally, a conclusion is drawn in the last section.
II. REVIEW OF OFF-DIAGONAL PHASE

If a non-degenerate density matrix takes this form

$$\rho_1 = \lambda_1 |\psi_1\rangle \langle \psi_2| + \cdots + \lambda_N |\psi_N\rangle \langle \psi_N|.$$  \hspace{1cm} (1)

Moreover, a density operator that can’t interfere with \(\rho_1\) is introduced [5], which is

$$\rho_n = W^{n-1} \rho_1 (W^\dagger)^{n-1}, n = 1, \ldots, N$$

where

$$W = |\psi_1\rangle \langle \psi_N| + |\psi_N\rangle \langle \psi_{N-1}| + \cdots + |\psi_2\rangle \langle \psi_1|.$$  

In the unitary evolution, except the usual mixed state geometric phase, there exists so called mixed state off diagonal phase, which reads [5]

$$\gamma^{(l)}_{\rho_1 \cdots \rho_l} = \Phi[Tr(\prod_{a=1}^{l} U(\tau)^\dagger \sqrt{\rho_{ja}})],$$  \hspace{1cm} (2)

where \(\Phi[z] \equiv z/|z|\) for nonzero complex number \(z\) and [16]

$$U^\parallel = U(t) \sum_{k=1}^{N} e^{-i\delta_k},$$  \hspace{1cm} (3)

in which

$$\delta_k = -i \int_{0}^{t} \langle \psi_k | U^\dagger(t') \dot{U}(t') | \psi_k \rangle dt'$$  \hspace{1cm} (4)

and \(U(t)\) is the time evolution operator of this system. Moreover \(U^\parallel\) satisfies the parallel transport condition, which is

$$\langle \psi_k | U^\parallel(t) \dot{U^\parallel}(t) | \psi_k \rangle = 0, \ k = 1, \ldots, N.$$  

In addition, the usual mixed state geometric phase factor [15] takes the following form

$$\gamma = \Phi \left[ \sum_{k=1}^{N} \lambda_k \langle \psi_k | U(\tau) | \psi_k \rangle e^{-i\delta_k} \right]$$  \hspace{1cm} (5)

The propagation helicity components \((\nu_R \ \nu_L)^T\) of a neutrino in a magnetic field obeys the following equation [14]

$$i \frac{d}{dt} \begin{pmatrix} \nu_R \\ \nu_L \end{pmatrix} = \begin{pmatrix} \frac{V}{2} & \mu \nu B e^{-i\omega t} \\ \mu \nu B e^{i\omega t} & -\frac{V}{2} \end{pmatrix} \begin{pmatrix} \nu_R \\ \nu_L \end{pmatrix},$$  \hspace{1cm} (6)
where $T$ denotes the matrices transposing operation, $\vec{B} = B_x + i B_y = B e^{i \omega t}$, $\mu_\nu$ represents the magnetic moment of a massive Dirac neutrino and $V$ is a term due to neutrino mass as well as interaction with matter. The instantaneous eigenvalues and eigenvectors corresponding to the Hamiltonian take the following form [7]

$$E_1 = + \sqrt{\left( \frac{V}{2} \right)^2 + (\mu_\nu B)^2}$$

$$|\psi_1\rangle = \frac{1}{N} \begin{pmatrix} \mu_\nu B \\ - e^{i \omega t} \left( \frac{V}{2} - E_1 \right) \end{pmatrix}$$

and

$$E_2 = - \sqrt{\left( \frac{V}{2} \right)^2 + (\mu_\nu B)^2}$$

$$|\psi_2\rangle = \frac{1}{N} \begin{pmatrix} e^{-i \omega t} \left( \frac{V}{2} - E_1 \right) \\ \mu_\nu B \end{pmatrix},$$

where the normalized factor

$$N = \sqrt{\left( \frac{V}{2} - E_1 \right)^2 + (\mu_\nu B)^2}.$$ 

If this system in a thermal state, the density operator can be written as

$$\rho = \lambda_1 |1\rangle \langle 1| + \lambda_2 |2\rangle \langle 2|$$

where

$$\lambda_1 = \frac{e^{-\beta E_1}}{e^{-\beta E_1} + e^{-\beta E_2}}$$

and

$$\lambda_2 = \frac{e^{-\beta E_2}}{e^{-\beta E_1} + e^{-\beta E_2}}.$$ 

In addition, $\beta = 1/(kT)$, where $k$ is the Boltzmann constant and $T$ represents the temperature. In the next section, the mixed state geometric phase for both off-diagonal one and diagonal one will be calculated.

III. MIXED STATE GEOMETRIC PHASE

The differential equation Eq. (6) can be exactly solved by the following transformation

$$\begin{pmatrix} \nu_R \\ \nu_L \end{pmatrix} = e^{-i \sigma_z \frac{1}{2} \omega t} \begin{pmatrix} a \\ b \end{pmatrix},$$

(10)
where \( \sigma_z \) is a Pauli matrix along \( z \) direction whose explicit form is

\[
\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

By substituting Eq. (10) into Eq. (6), one can obtain

\[
\frac{id}{dt} \begin{pmatrix} a \\ b \end{pmatrix} = \tilde{H} \begin{pmatrix} a \\ b \end{pmatrix},
\]

where

\[
\tilde{H} = \mu_B B \sigma_x + \frac{1}{2} (V - \omega) \sigma_z.
\]

Furthermore, it can be written in this form

\[
\tilde{H} = \frac{1}{2} \Omega \left( \begin{array}{cc} 2\mu_B \Omega & V - \omega \\ V - \omega & 2\mu_B \Omega \end{array} \right) \cdot \left( \begin{array}{ccc} \sigma_x & \sigma_y & \sigma_z \end{array} \right),
\]

where \( \Omega = \sqrt{(2\mu_B B)^2 + (V - \omega)^2} \). Because \( \tilde{H} \) is independent of time, Eq. 11 can be exactly solved, whose time evolution operator takes the form

\[
\tilde{U} = e^{-i\tilde{H}t}.
\]

Associating with Eq. (10), the time evolution operator for Eq. (6) is

\[
U = e^{-i\tilde{H}t} e^{i\sigma_z \frac{\Omega}{2} t}.
\]

By substituting Eq. (12) into Eq. 13, the above operator can be written in an explicit form, which is

\[
U = \begin{pmatrix} \cos \frac{\Omega}{2} t - i\frac{V - \omega}{\Omega} \sin \frac{\Omega}{2} t & -i\frac{2\mu_B}{\Omega} \sin \frac{\Omega}{2} t \\ -i\frac{2\mu_B}{\Omega} \sin \frac{\Omega}{2} t & \cos \frac{\Omega}{2} t + i\frac{V - \omega}{\Omega} \sin \frac{\Omega}{2} t \end{pmatrix} \begin{pmatrix} e^{i\frac{\omega}{2} t} & 0 \\ 0 & e^{-i\frac{\omega}{2} t} \end{pmatrix}
\]

In order to calculate off-diagonal phase (2), by use of Eq. (3), we can work out

\[
U_{11}^\parallel \equiv \langle \psi_1 | U(t) \left( e^{-i\delta_1} | \psi_1 \rangle \langle \psi_1 | + e^{-i\delta_2} | \psi_2 \rangle \langle \psi_2 | \right) | \psi_1 \rangle = U_{11} e^{-i\delta_1},
\]

where \( U_{11} = \langle \psi_1 | U(t) | \psi_1 \rangle \). In order to simplify the result, let’s talk about an easier case. when \( t = \tau = 2\pi/\Omega \),

\[
U_{11} = -\frac{1}{N^2} \left[ \mu_B^2 B^2 e^{i\frac{\omega}{2} t} + \left( \frac{V}{2} - E_1 \right)^2 e^{-i\frac{\omega}{2} t} \right] = U_{22}^*,
\]

(14)
where ∗ denotes the complex conjugate operation. By similar calculations, one can obtains

\[ U_{12} = \frac{2}{N^2} \mu_\nu B \left( \frac{V}{2} - E_1 \right) \sin \left( \frac{1}{2} \omega \right) e^{-i(\omega \tau + \frac{\pi}{2})} = -U_{21}^* \]  

(15)

Furthermore, \( \delta_1 \) can be explicitly calculated out by substituting Eq. (13) Eq. (7) into Eq. (4), which takes the form

\[ \delta_1 = \frac{1}{N^2} \left[ 2 \mu_\nu^2 B^2 \left( \frac{V}{2} - E_1 \right) + \left( \frac{V}{2} - E_1 \right)^2 \left( \frac{V}{2} - \omega \right) - \mu_\nu^2 B^2 \left( \frac{V}{2} - \omega \right) \right] \tau. \]  

(16)

By similar calculation, one can get

\[ \delta_2 = -\delta_1. \]  

(17)

Hence Eq. (3) can be explicitly calculated out,

\[
\begin{pmatrix}
U_{11}^\parallel & U_{12}^\parallel \\
U_{21}^\parallel & U_{22}^\parallel
\end{pmatrix}
= \begin{pmatrix}
U_{11} & U_{12} \\
U_{21} & U_{22}
\end{pmatrix}
\begin{pmatrix}
e^{-i\delta_1} & 0 \\
0 & e^{-i\delta_2}
\end{pmatrix}. \]  

(18)

Now, let us calculate the mixed state off-diagonal phase

\[ \gamma_{\rho_1\rho_2}^{(2)} = \Phi \left[ Tr \left( \prod_{a=1}^{2} U^\parallel(\tau) \sqrt{p_a} \right) \right], \]  

(19)

where \( \rho_1 = \lambda_1 |1\rangle \langle 1| + \lambda_2 |2\rangle \langle 2| \) and \( \rho_2 = \lambda_1 |2\rangle \langle 2| + \lambda_2 |1\rangle \langle 1|. \) Under the basis of \(|\psi_1\rangle\) and \(|\psi_2\rangle\),

\[
Tr \left( \prod_{a=1}^{2} U^\parallel(\tau) \sqrt{p_a} \right) = \sum_{b=1}^{2} |\psi_b\rangle \prod_{a=1}^{2} U^\parallel(\tau) \sqrt{p_a} |\psi_b\rangle = \sqrt{\lambda_1 \lambda_2} \left[ (U_{11}^\parallel)^2 + (U_{22}^\parallel)^2 \right] + U_{12}^\parallel U_{21}^\parallel. \]  

(20)

By substituting Eq. (18) into Eq. (20), we can obtain a simpler result

\[
Tr \left( \prod_{a=1}^{2} U^\parallel(\tau) \sqrt{p_a} \right) = \sqrt{\lambda_1 \lambda_2} \left[ (U_{11}e^{-i\delta_1})^2 + (U_{22}e^{-i\delta_2})^2 \right] + U_{12}U_{21}e^{-i(\delta_1 + \delta_2)}
\]

By substituting Eq. (14) and Eq. (15) into the above equation, off-diagonal geometric phase (19) can be explicitly calculated,

\[
\gamma_{\rho_1\rho_2}^{(2)} = \Phi \left\{ \left( \frac{V}{2} - E_1 \right)^2 \mu_\nu^2 B^2 \left( \cos \omega \tau - 1 \right) + \sqrt{\lambda_1 \lambda_2} \chi \left[ \left( \frac{V}{2} - E_1 \right)^4 \cos (\omega \tau + 2\delta_1) + \mu_\nu^4 B^4 \cos (\omega \tau + 2\delta_1) \right. \\
+ 2\mu_\nu^2 B^2 \left( \frac{V}{2} - E_1 \right)^2 \cos 2\delta_1 \right\}
\]  

(21)

Hence, the corresponding phase is either \( \pi \) or \( 0 \), which depends on temperature and magnetic field. So, its phase is unresponsive to temperature.
By substituting Eq. (14), Eq. (16) and Eq. (17) into Eq. (5), the diagonal geometric phase for mixed state reads

$$\gamma = \Phi\{\left[\lambda_1 e^{i\left(\frac{\omega}{2} + \delta_1\right)} + \lambda_2 e^{-i\left(\frac{\omega}{2} - \delta_1\right)}\right] \mu_x^2 B^2 + \left[\lambda_1 e^{-i\left(\frac{\omega}{2} + \delta_1\right)} + \lambda_2 e^{i\left(\frac{\omega}{2} + \delta_1\right)}\right] \left(\frac{V}{2} - E_1\right)^2\}. \tag{22}$$

From the above result, we can draw a conclusion that if $\lambda_1 = \lambda_2$, in another word $T \to \infty$, the corresponding phase maybe $\pi$ or 0. In other circumstance, it may vary continuously in an interval. By contrary to off-diagonal one, the diagonal phase is more sensitive to temperature.

IV. CONCLUSIONS AND ACKNOWLEDGEMENTS

In this article, the time evolution operator of neutrino spin in the presence of uniformly rotating magnetic field is obtained. Under this time evolution operator, a thermal state of this neutrinos evolves. Then there exists mixed off-diagonal geometric phase for mixed state, as well as diagonal ones. They have been calculated respectively. And an analytic form is achieved. In addition, a conclusion is drawn that diagonal phase is more sensitive to off-diagonal one towards temperature.

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