Effective Average Actions and Nonperturbative Evolution Equations

M. Reuter

Deutsches Elektronen-Synchrotron DESY,
Notkestrasse 85, D-22603 Hamburg, Germany

Abstract

The effective average actions for gauge theories and the associated nonperturbative evolution equations which govern their renormalization group flow are reviewed and various applications are described. As an example of a topological field theory, Chern-Simons theory is discussed in detail.

1 Introduction

In these notes we first give a brief introduction to the method of the effective average actions and their associated exact renormalization group or evolution equations [1, 2, 3], and then we illustrate these ideas by means of two examples. We discuss the renormalization group behavior of the nonabelian gauge coupling in “ordinary” Yang-Mills theories and of the Chern-Simons parameter in pure 3-dimensional Chern-Simons theory, which provides a first example of a topological field theory.

The effective average action $\Gamma_k$ can be thought of as a continuum version of the block spin action for spin systems [4]. The functional $\Gamma_k$ is the action

1Talk given at the 5th Hellenic School and Workshops on Elementary Particle Theory, Corfu, Greece, 1995, to appear in the proceedings.
relevant to the physics at (mass) scale \( k \). It has the quantum fluctuations with momenta larger than \( k \) integrated out already, but those with momenta smaller than \( k \) are not yet included. \( \Gamma_k \) interpolates between the classical action \( S \) for large values of \( k \), and the conventional effective action for \( k \) approaching zero: \( \Gamma_{k \to \infty} = S, \Gamma_{k \to 0} = \Gamma \). In many important cases where perturbation theory is inapplicable due to infrared divergences the limit \( k \to 0 \) exists and can be computed by various methods. This includes for instance massless theories in low dimensions or the high temperature limit of 4 dimensional theories. The functional \( \Gamma_k \) can be obtained by solving an exact renormalization group equation which describes its evolution while \( k \) is lowered from infinity to zero.

In the approach of ref. [2], and for models with a scalar field \( \phi \) only, this evolution equation reads

\[
\frac{\partial}{\partial t} \Gamma_k[\phi] = \frac{1}{2} \text{Tr} \left( \frac{\partial}{\partial t} R_k \left( \Gamma^{(2)}_k[\phi] + R_k \right)^{-1} \right)
\]

(1.1)

Here \( t \equiv \ln k \) is the “renormalization group time” and \( \Gamma^{(2)}_k \) denotes the matrix of the second functional derivatives of \( \Gamma_k \). The operator \( R_k \equiv R_k(-\partial^2) \) or, in momentum space, \( R_k \equiv R_k(q^2) \) describes the details of how the small momentum modes are cut off and it is to some extent arbitrary. It has to vanish for \( q^2 \gg k^2 \) and to become a mass-like term proportional to \( k^2 \) for small momenta \( q^2 \ll k^2 \). The derivation of (1.1) proceeds as follows. In the euclidean functional integral for the generating functional of the connected Green functions one adds a momentum-dependent mass term (playing the role of a smooth IR cutoff) \( \frac{1}{2} \int \phi R_k(-\partial^2) \phi \) to the classical action \( S \). Then, up to an explicitly known correction term \([2]\), the resulting \( k \)-dependent functional \( W_k[J] \) is related to \( \Gamma_k[\phi] \) by a conventional Legendre transformation at fixed \( k \).

In Section 2 we generalize the above evolution equation to gauge theories, and in Section 3 we discuss its BRS properties. In Section 4 we show how it can be used to calculate the beta-function of the nonabelian gauge coupling. In Sections 5 and 6 we shall apply the same strategies to the study of pure Chern-Simons field theory in 3 dimensions.
2 The Renormalization Group Equation

In the case of gauge theories the derivation of an exact evolution equation faces additional complications because the inhomogeneous gauge transformation law of the Yang-Mills fields forbids a mass-type cutoff. In refs. [3, 4] this problem was overcome recently by using the background gauge technique [5] which allows us to work with a gauge invariant effective average action. The price which one has to pay for this advantage is that $\Gamma_k$ depends on two gauge fields: the usual classical average field $A_\mu^a$ and the background field $\bar{A}_\mu^a$. For pure Yang-Mills theory one finds the following renormalization group equation

$$k \frac{d}{dk} \Gamma_k[A, \bar{A}] = \frac{1}{2} \text{Tr} \left[ \left( \Gamma_k^{(2)}[A, \bar{A}] + R_k(\Delta[\bar{A}]) \right)^{-1} k \frac{d}{dk} R_k(\Delta[\bar{A}]) \right] - \text{Tr} \left[ \left( -D_\mu[A] D_\mu[\bar{A}] + R_k(-D^2(\bar{A})) \right)^{-1} k \frac{d}{dk} R_k(-D^2[\bar{A}]) \right]$$

(2.1)

In writing down this equation we made a certain approximation on which we shall be more explicit in Section 3 where we also sketch the details of its derivation. Eq.(2.1) has to be solved subject to the initial condition

$$\Gamma_\infty[A, \bar{A}] = S[A] + \frac{1}{2\alpha} \int d^d x \left( D^{ab}_\mu[\bar{A}] (A^b_\mu - \bar{A}^b_\mu) \right)^2$$

(2.2)

where the classical action is augmented by the background gauge fixing term. Furthermore, $\Gamma_k^{(2)}[A, \bar{A}]$ denotes the matrix of the second functional derivatives of $\Gamma_k$ with respect to $A$ at fixed $\bar{A}$. Again, the function $R_k$ specifies the precise form of the infrared cutoff, and it has the same properties as mentioned in the introduction. A convenient choice is

$$R_k(u) = Z_k u \left[ \exp(u/k^2) - 1 \right]^{-1}$$

(2.3)

but in some cases even a simple constant $R_k = Z_k k^2$ is sufficient. The factor $Z_k$ has to be fixed in such a way that a massless inverse propagator $Z_k q^2$ combines with the cutoff to $Z_k (q^2 + k^2)$ for the low momentum modes. $Z_k$ may be chosen differently for different fields. In particular, different $Z_k$-factors are used for the gauge field fluctuations and for the Faddeev-Popov ghosts. (They give rise to the first and the second trace on the RHS of eq.(2.1), respectively.)
Observable quantities will not depend on the form of $R_k$. A similar remark applies to the precise form of the operator $\Delta[\bar{A}] \equiv -D^2[\bar{A}] + \ldots$ which is essentially the covariant laplacian, possibly with additional nonminimal terms [3]. The rôle of $\Delta$ is to distinguish “high momentum modes” from “low momentum modes”. If one expands all quantum fluctuations in terms of the eigenmodes of $\Delta$, then it is the modes with eigenvalues larger than $k^2$ which are integrated out in $\Gamma_k$.

In order to understand the structure of the renormalization group equation (2.1) it is useful to realize that it can be rewritten in a form which is reminiscent of a one-loop formula:

$$\frac{\partial}{\partial t} \Gamma_k[A, \bar{A}] = \frac{1}{2} \frac{D}{Dt} \text{Tr} \ln \left[ \Gamma_k^{(2)}[A, \bar{A}] + R_k (\Delta[\bar{A}]) \right]$$

$$- \frac{D}{Dt} \text{Tr} \ln \left[ -D^\mu[A]D_\mu[\bar{A}] + R_k (-D^2[\bar{A}]) \right]$$

(2.4)

By definition, the derivative $\frac{D}{Dt}$ acts only on the explicit $k$-dependence of $R_k$, but not on $\Gamma_k^{(2)}[A, \bar{A}]$. It is easy now to describe the relation between the effective average action $\Gamma_k$ and the conventional effective action. Let us first make the approximation $\frac{D}{Dt} \rightarrow \frac{\partial}{\partial t}$ in eq. (2.4). This amounts to neglecting the running of $\Gamma_k$ on the RHS of the evolution equation. Therefore it can be solved by simply integrating both sides of the equation from the infrared cutoff $k$ to the ultraviolet cutoff $\Lambda$:

$$\Gamma_k[A, \bar{A}] = \Gamma_\Lambda[A, \bar{A}] + \frac{1}{2} \text{Tr} \left\{ \ln \left[ \Gamma_k^{(2)}[A, \bar{A}] + R_k (\Delta[\bar{A}]) \right] \right.$$

$$- \ln \left[ \Gamma_\Lambda^{(2)}[A, \bar{A}] + R_\Lambda (\Delta[\bar{A}]) \right] \right\}$$

$$- \text{Tr} \left\{ \ln \left[ -D^\mu[A]D_\mu[\bar{A}] + R_k (-D^2[\bar{A}]) \right] \right.$$

$$- \ln \left[ -D^\mu[A]D_\mu[\bar{A}] + R_\Lambda (-D^2[\bar{A}]) \right] \right\} + O \left( \frac{\partial}{\partial t} \Gamma_k^{(2)} \right)$$

(2.5)

Ultimately we shall send the ultraviolet cutoff to infinity and identify $\Gamma_\Lambda$ with the classical action $S$ plus the gauge fixing term. Eq.(2.5) has a similar structure as a regularized version of the conventional one-loop effective action in the background gauge. There are two important differences, however: (i) The second variation of the classical action, $S^{(2)}$, is replaced by $\Gamma_k^{(2)}$. This implements a kind of “renormalization group improvement”. (ii) The effective
average action contains an explicit infrared cutoff \( R_k \). Because

\[
\lim_{u \to \infty} R_k(u) = 0, \quad \lim_{u \to 0} R_k(u) = Z_k k^2
\]

(2.6)
a mass-term is added to the inverse propagator \( \Gamma_k^{(2)} \) for low frequency modes \( (u \to 0) \), but not for high frequency modes \( (u \to \infty) \).

The solution \( \Gamma_k[A, \bar{A}] \) of (2.1) with (2.2) is gauge invariant under simultaneous gauge transformations of \( A \) and \( \bar{A} \). Following the lines of the conventional background method [5] one would try to equate the two gauge field arguments of \( \Gamma_k \), and work with the functional \( \bar{\Gamma}_k[A] \equiv \Gamma_k[A, A] \). However, it is important to note that the evolution equation (2.1) cannot be rewritten in terms of \( \bar{\Gamma}_k[A] \) alone, since \( \Gamma_k^{(2)} \) does not involve derivatives with respect to \( \bar{A} \). In fact, let us introduce the decomposition

\[
\Gamma_k[A, \bar{A}] = \bar{\Gamma}_k[A] + \Gamma_k^{\text{gauge}}[A, \bar{A}]
\]

(2.7)
This leads to

\[
\Gamma_k^{(2)}[A, \bar{A}] = \bar{\Gamma}_k^{(2)}[A] + \Gamma_k^{\text{gauge}(2)}[A, \bar{A}]|_{\bar{A}}
\]

(2.8)
where the second functional derivative \( \Gamma_k^{\text{gauge}(2)} \) is performed at fixed \( \bar{A} \). The interpretation of (2.7) and (2.8) is as follows. Because \( \bar{\Gamma}_k[A] \) is a gauge invariant functional of its argument, \( \bar{\Gamma}_k^{(2)}[A] \) is necessarily singular, i.e., it has the usual gauge zero modes. They are gauge fixed by the generalized gauge fixing term \( \Gamma_k^{\text{gauge}} \). This is possible because \( \Gamma_k^{\text{gauge}} \) is not invariant under separate gauge transformations of \( A \) alone.

3 BRS-Symmetry and Modified Slavnov-Taylor Identities

In order to actually derive the evolution equation as well as the pertinent Ward-Takahashi or Slavnov-Taylor identities we start from the following scale dependent generating functional in the background formalism [5]:

\[
\exp W_k[K^a_{\mu}, \sigma^a, \bar{\sigma}^a; \bar{\rho}^a_{\mu}, \bar{\gamma}^a; \bar{A}^a_{\mu}] = \int \mathcal{D}A \mathcal{D}C \mathcal{D}\bar{C} \exp -\{S[A] + \Delta_k S + S_{\text{gf}} + S_{\text{ghost}} + S_{\text{source}}\} \equiv \int \mathcal{D}\phi \exp(-S_{\text{tot}})
\]

(3.1)
Here \( S[\mathcal{A}] \) denotes the gauge invariant classical action, and

\[
\Delta_k S = \frac{1}{2} \int d^d x \ (\mathcal{A} - \bar{\mathcal{A}})^a_\mu R_k (\bar{\mathcal{A}})^{ab} (\mathcal{A} - \bar{\mathcal{A}})_\nu^b + \int d^d x \ \bar{C}^a R_k (\bar{\mathcal{A}})^{ab} C^b \tag{3.2}
\]

is the infrared cutoff ("momentum dependent mass term") for the gauge field fluctuation \( a \equiv \mathcal{A} - \bar{\mathcal{A}} \) and for the Faddeev-Popov ghosts \( C \) and \( \bar{C} \). Here \( R_k(\bar{\mathcal{A}}) \) is a suitable cutoff operator which depends on \( \bar{\mathcal{A}} \) only. It may be chosen differently for the gauge field and for the ghosts. Furthermore

\[
S_{gf} = \frac{1}{2 \alpha} \int d^d x \left[ D_\mu (\bar{\mathcal{A}})^{ab} (\mathcal{A} - \bar{\mathcal{A}})_\mu^b \right]^2 \tag{3.3}
\]

is the background gauge fixing term and

\[
S_{\text{ghost}} = - \int d^d x \ \bar{C}^a \left( D_\mu (\bar{\mathcal{A}}) D_\mu (\mathcal{A}) \right)^{ab} C^b \tag{3.4}
\]

is the corresponding ghost action \( [5] \). The fields \( \mathcal{A} - \bar{\mathcal{A}}, C \) and \( \bar{C} \) are coupled to the sources \( K, \sigma \) and \( \bar{\sigma} \), respectively:

\[
S_{\text{source}} = - \int d^d x \left\{ K^a_\mu (\mathcal{A}^a_\mu - \bar{\mathcal{A}}^a_\mu) + \sigma^a C^a + \sigma^a \bar{C}^a + \frac{1}{g} \beta^a_\mu D_\mu (\mathcal{A})^{ab} C^b + \frac{1}{2} \gamma^a f^{abc} C^b C^c \right\} \tag{3.5}
\]

We also included the sources \( \bar{\beta} \) and \( \bar{\gamma} \) which couple to the BRS-variations of \( \mathcal{A} \) and of \( C \), respectively. In fact, \( S + S_{gf} + S_{\text{ghost}} \) is invariant under the BRS transformation

\[
\delta \mathcal{A}^a_\mu = \frac{1}{g} \varepsilon D_\mu (\mathcal{A})^{ab} C^b \\
\delta C^a = - \frac{1}{2} \varepsilon f^{abc} C^b C^c \\
\delta \bar{C}^a = \frac{\varepsilon}{\alpha g} D_\mu (\bar{\mathcal{A}})^{ab} (\mathcal{A}^b_\mu - \bar{\mathcal{A}}^b_\mu) \tag{3.6}
\]

Let us introduce the classical fields

\[
\bar{a}^b_\mu = \frac{\delta W_k}{\delta K^b_\mu}, \quad \bar{\xi}^b = \frac{\delta W_k}{\delta \sigma^b}, \quad \bar{\xi}^b = \frac{\delta W_k}{\delta \sigma^b} \tag{3.7}
\]

and let us formally solve the relations \( \bar{a} = \bar{a}(K, \sigma, \bar{\sigma}; \bar{\beta}, \bar{\gamma}; \bar{\mathcal{A}}), \ \bar{\xi} = \bar{\xi}(\ldots), \ldots \), etc., for the sources \( K, \sigma \) and \( \bar{\sigma} \) : \( K = K(\bar{a}, \bar{\xi}, \bar{\beta}, \bar{\gamma}; \bar{\mathcal{A}}), \ \sigma = \sigma(\ldots), \ldots \). We
introduce the new functional $\tilde{\Gamma}_k$ as the Legendre transform of $W_k$ with respect to $K, \sigma$ and $\bar{\sigma}$:

$$
\tilde{\Gamma}_k[\bar{a}, \xi, \bar{\xi}, \bar{\beta}, \bar{\gamma}; \bar{A}] = \int d^d x \{ K^b_\mu \bar{a}^b_\mu + \bar{\sigma}^b \xi^b + \sigma^b \bar{\xi}^b \} - W_k[K, \sigma, \bar{\beta}, \bar{\gamma}; \bar{A}].
$$

(3.8)

Apart from the usual relations

$$
\frac{\delta \tilde{\Gamma}_k}{\delta \bar{a}^a_\mu} = K^a_\mu, \quad \frac{\delta \tilde{\Gamma}_k}{\delta \xi^a} = -\sigma^a, \quad \frac{\delta \tilde{\Gamma}_k}{\delta \bar{\xi}^a} = -\bar{\sigma}^a
$$

(3.9)

we have also

$$
\frac{\delta \tilde{\Gamma}_k}{\delta \bar{\beta}^a_\mu} = -\frac{\delta W_k}{\delta \bar{\beta}^a_\mu}, \quad \frac{\delta \tilde{\Gamma}_k}{\delta \bar{\gamma}^a} = -\frac{\delta W_k}{\delta \bar{\gamma}^a}
$$

(3.10)

where $\delta \tilde{\Gamma}/\delta \bar{\beta}$ is taken for fixed $\bar{a}, \xi, \bar{\xi}$ and $\delta W/\delta \bar{\beta}$ for fixed $K, \sigma, \bar{\sigma}$. The effective average action $\Gamma_k$ is obtained by subtracting the IR cutoff $\Delta_k S$, expressed in terms of the classical fields, from the Legendre transform $\tilde{\Gamma}_k$:

$$
\Gamma_k[\bar{a}, \xi, \bar{\xi}, \bar{\beta}, \bar{\gamma}; \bar{A}] = \tilde{\Gamma}_k[\bar{a}, \xi, \bar{\xi}, \bar{\beta}, \bar{\gamma}; \bar{A}] - \frac{1}{2} \int d^d x \bar{a}^a_\mu \bar{R}_k(\bar{A})^{ab}_\mu \bar{a}^b_\nu
$$

$$
- \int d^d x \bar{\xi}^a \bar{R}_k(\bar{A})^{ab} \xi^b.
$$

(3.11)

Frequently we shall use the field $A \equiv \bar{A} + \bar{a}$ (the classical counterpart of $A \equiv \bar{A} + a$) and write correspondingly

$$
\Gamma_k[A, \bar{A}, \xi, \bar{\xi}, \bar{\beta}, \bar{\gamma}] \equiv \Gamma_k[A - \bar{A}, \bar{\xi}, \bar{\beta}, \bar{\gamma}; \bar{A}].
$$

(3.12)

For $\xi = \bar{\xi} = \bar{\beta} = \bar{\gamma} = 0$ one recovers the effective average action $\Gamma_k[A, \bar{A}]$ which we discussed in Section 2.

Upon taking the $k$-derivative of eq. (3.1) and Legendre-transforming the result one finds the following exact evolution equation ($t = \ln k$):

$$
\frac{\partial}{\partial t} \Gamma_k[A, \bar{A}, \xi, \bar{\xi}; \bar{\beta}, \bar{\gamma}] = \frac{1}{2} Tr \left[ \left( \Gamma_k^{(2)} + R_k(\bar{A}) \right)^{-1} \frac{\partial}{\partial t} R_k(\bar{A}) \right]_{\bar{A}\bar{A}}
$$

$$
- \frac{1}{2} Tr \left[ \left( \Gamma_k^{(2)} + R_k(\bar{A}) \right)^{-1} \frac{\partial}{\partial t} R_k(\bar{A}) \right]_{\bar{A}\bar{A}}.
$$

(3.13)

Here $\Gamma_k^{(2)}$ is the Hessian of $\Gamma_k$ with respect to $A, \xi$ and $\bar{\xi}$ at fixed $\bar{A}, \bar{\beta}$ and $\bar{\gamma}$ and, in an obvious notation, $R_{k,\bar{A}\bar{A}}, R_{k,\bar{A}\bar{A}}$ are the infrared cutoff operators.
introduced in (3.2). The evolution equation (3.13) is exact in the sense that
its solution, when evaluated at $k = 0$, equals the exact generating functional
of the 1PI Green’s functions in the background gauge, i.e., it is not just an
improved one-loop functional.

It is clear from its construction that $\Gamma_k$ is invariant under simultaneous
gauge transformations of $A_\mu$ and $\bar{A}_\mu$ and homogeneous transformations of
$\xi, \bar{\xi}, \bar{\beta}_\mu$ and $\bar{\gamma}$, i.e., $\delta \Gamma_k[A, \bar{A}, \xi, \bar{\xi}, \bar{\beta}, \bar{\gamma}] = 0$ for

$$
\delta A_\mu^a = -\frac{1}{g} D_\mu(A)^{ab} \omega^b \\
\delta \bar{A}_\mu^a = -\frac{1}{g} D_\mu(\bar{A})^{ab} \omega^b \\
\delta V^a = f^{abc} V^b \omega^c, \ V \equiv \xi, \bar{\xi}, \bar{\beta}_\mu, \bar{\gamma}.
$$

(3.14)

Next we turn to the Ward identities. By applying the transformations
(3.6) to the integrand of (3.1) one obtains from the BRS invariance of the
measure $D\phi \int D\phi \ \delta_{\text{BRS}} \exp(-S_{\text{tot}}) = 0$ (3.15)
or

$$
\int d^d x \left\{ K_\mu^a \frac{\delta W_k}{\delta \beta_\mu^a} + \bar{\sigma}_a \frac{\delta W_k}{\delta \bar{\gamma}^a} - \frac{1}{\alpha g} \sigma^a D_\mu(\bar{A})^{ab} \frac{\delta W_k}{\delta K^b_\mu} \right\}
$$

$$
= \int d^d x \left\{ \left[ \frac{\delta W_k}{\delta \beta_\mu^a} + \frac{\delta}{\delta \beta_\mu^a} \right] \left( R_k \frac{\delta W_k}{\delta K^a_\mu} \right) + \frac{1}{\alpha g} \left( D_\mu(\bar{A}) \left[ \frac{\delta W_k}{\delta K^a_\mu} + \frac{\delta}{\delta K^a_\mu} \right] \right) \left( R_k \frac{\delta W_k}{\delta \bar{\gamma}} \right)^a \right\}
$$

$$
+ \left[ \frac{\delta W_k}{\delta \sigma^a} + \frac{\delta}{\delta \sigma^a} \right] \left( R_k \frac{\delta W_k}{\delta \bar{\gamma}} \right)^a \right\}
$$

(3.16)

with $(R_k \delta W_k / \delta \bar{\gamma})^a \equiv R_k(\bar{A})^{ab} \delta W_k / \delta \bar{\gamma}^b$, etc. Equation (3.16) can be converted
to the following relation for the effective average action (3.12):

$$
\int d^d x \left\{ \frac{\delta \Gamma'_k}{\delta A_\mu^a} \frac{\delta \Gamma'_k}{\delta \beta_\mu^a} - \frac{\delta \Gamma'_k}{\delta \xi^a} \frac{\delta \Gamma'_k}{\delta \bar{\gamma}^a} \right\} = \Delta_k^{(\text{BRS})}
$$

(3.17)

where the “anomalous contribution” $\Delta_k^{(\text{BRS})}$ is given by

$$
\Delta_k^{(\text{BRS})} = \text{Tr} \left[ R_k(\bar{A}) A_\mu A_\nu \left( \Gamma^{(2)}_k + R_k \right)_{A_\mu \nu} \frac{\delta^2 \Gamma'_k}{\delta \phi \delta \beta_\mu} \right]
$$

$$
- \text{Tr} \left[ R_k(\bar{A}) \xi \xi \left( \Gamma^{(2)}_k + R_k \right)_{\xi \xi} \frac{\delta^2 \Gamma'_k}{\delta \phi \delta \bar{\gamma}} \right]
$$

$$
- \frac{1}{\alpha g} \text{Tr} \left[ D_\mu(\bar{A}) \left( \Gamma^{(2)}_k + R_k \right)_{A_\mu \xi} R_k(\bar{A}) \xi \xi \right]
$$

(3.18)
Here \( \varphi \equiv (A_\mu, \xi, \bar{\xi}) \) is summed over on the RHS of (3.18). In deriving eq. (3.18) we used
\[
\left[ \frac{\delta}{\delta \bar{\xi}^a} - g D_\mu (\bar{A})^{ab} \frac{\delta}{\delta \beta^b_\mu} \right] \Gamma_k [A, \bar{A}, \xi, \bar{\xi}; \bar{\beta}, \bar{\gamma}] = 0 \tag{3.20}
\]
which follows from the equation of motion of the antighost. Equation (3.17) is the generating relation for the modified Ward identities which we wanted to derive. In conventional Yang-Mills theory, without IR-cutoff, the RHS of (3.17) is zero. The traces on the RHS of (3.17) lead to a violation of the usual Ward identities for nonvanishing values of \( k \). As \( k \) approaches zero, \( R_k \) and hence \( \Delta_k^{(BRS)} \) vanishes and we recover the conventional Ward-Takahashi identities [6].

The modified Ward identities (3.17) are not the only conditions which the average action \( \Gamma_k \) has to satisfy. There exists also an exact formula for its \( \bar{A} \)-derivative:
\[
\frac{\delta}{\delta A^a_\mu (y)} \Gamma_k [A, \bar{A}, \xi, \bar{\xi}; \bar{\beta}, \bar{\gamma}] = -g^2 \bar{\xi}^b(y) f^{abc} \frac{\delta \Gamma_k}{\delta \beta^c_\mu (y)} + \frac{1}{2} \text{Tr} \left[ (\Gamma_k^{(2)} + R_k)^{-1} \frac{\delta}{\delta \bar{A}^a_\mu (y)} \left( R_k - \frac{1}{\alpha} \bar{D} \otimes \bar{D} \right)_{AA} \right] - \text{Tr} \left[ (\Gamma_k^{(2)} + R_k)^{-1} \frac{\delta R_k \bar{\xi} \xi}{\delta A^a_\mu (y)} \right] + g^2 \int d^d x \text{ tr} \left[ T^a \left( \Gamma_k^{(2)} + R_k \right)^{-1} \frac{\delta^2 \Gamma_k}{\delta \varphi (x) \delta \beta^c_\mu (y)} \right] \tag{3.21}
\]
Note that the RHS of eq. (3.21) does not vanish even for \( k \to 0 \). The \( \bar{D} \otimes \bar{D} \)-piece of the 2nd term and the 4th term on the r.h.s. of (3.21) survive this limit.

So far we were deriving general identities which constrain the form of the exact functional \( \Gamma_k \). Let us now ask what they imply if we truncate the space of actions. It is often sufficient [4, 7] to neglect the \( k \)-evolution of the ghost sector by making an ansatz which keeps the classical form of the corresponding terms in the action
\[
\Gamma_k [A, \bar{A}, \xi, \bar{\xi}; \bar{\beta}, \bar{\gamma}] = \Gamma_k [A, \bar{A}] + \Gamma_{gh} \tag{3.22}
\]
\[ \Gamma_{gh} = - \int d^d x \xi D_\mu(\bar{A})D_\mu(A)\xi 
- \int d^d x \left\{ \frac{1}{g} \bar{\beta}_\mu^a D_\mu(A)^{ab} \xi^b + \frac{1}{2} \bar{\gamma}^a f^{abc} \xi^b \xi^c \right\} \]  
(3.23)

This is the approximation underlying our discussion in Section 2. In fact, if we insert this truncation into the exact evolution equation (3.13), we obtain precisely eq. (2.1) whose structure we explained already.

Moreover, a generic functional \( \Gamma_k[A, \bar{A}] \) can be decomposed according to

\[ \Gamma_k[A, \bar{A}] = \bar{\Gamma}_k[A] + \frac{1}{2\alpha} \int d^d x [D_\mu(\bar{A})(A_\mu - \bar{A}_\mu)]^2 + \hat{\Gamma}_{\text{gauge}}^k[A, \bar{A}] \]  
(3.24)

where \( \bar{\Gamma}_k \) is defined by equating the two-gauge fields: \( \bar{\Gamma}_k[A] \equiv \Gamma_k[A, A] \). The remainder \( \Gamma_k[A, \bar{A}] - \bar{\Gamma}_k[A] \) is further decomposed in the classical gauge-fixing term plus a correction to it, \( \hat{\Gamma}_{\text{gauge}}^k \). Note that \( \hat{\Gamma}_{\text{gauge}}^k[A, A] = 0 \) for equal gauge fields. \( \bar{\Gamma}_k[A] \) is a gauge-invariant functional of \( A_\mu \) and \( \Gamma_k[A, \bar{A}] \) is invariant under a simultaneous gauge transformation of \( A \) and \( \bar{A} \). In the examples of the following sections we make the further approximation of neglecting quantum corrections to the gauge fixing term by setting \( \hat{\Gamma}_{\text{gauge}}^k = 0 \). The important question is whether this truncation is consistent with the Ward-Takahashi identities (3.17) and the \( \bar{A} \)-derivative (3.21), respectively. If we insert (3.22)-(3.24) into (3.17), we find that \( \bar{\Gamma}_k \) drops out from the LHS of this equation. We are left with a condition for \( \hat{\Gamma}_{\text{gauge}}^k \):

\[ -\frac{1}{g} \int d^d x \delta \hat{\Gamma}_{\text{gauge}}^k \left( D_\mu(A)\xi \right)^a(x) = \Delta_{k}^{\text{BRS}} \]  
(3.25)

The anomaly \( \Delta_{k}^{\text{BRS}} \) vanishes for \( k \to 0 \) but is non-zero for \( k > 0 \). Our approximation \( \hat{\Gamma}_{\text{gauge}}^k \equiv 0 \) is consistent provided these terms can be neglected. We note that the traces implicit in (3.25) are related to higher loop effects. Beyond a loop approximation our neglection of \( \hat{\Gamma}_{\text{gauge}} \) is a non-trivial assumption. We emphasize that because of its gauge invariance the functional \( \bar{\Gamma}_k[A] \) does not appear on the LHS of the Ward identities. Therefore the Ward identities do not imply any further condition for \( \bar{\Gamma}_k \). This means that, within the approximations made, we may write down any ansatz for \( \bar{\Gamma}_k \) as long as it is gauge-invariant. Similar remarks apply to the identity (3.21) for the \( \bar{A} \)-dependence.
4 Evolution of the Nonabelian Gauge Coupling

The exact evolution equation is a nonlinear differential equation for a function of infinitely many variables. There seems to be little hope for finding closed-form solutions. The successful use of this equation therefore depends crucially on the existence of an appropriate approximation scheme. This will consist in a truncation of the infinitely many invariants characterizing $\Gamma_k$ to a finite number. If one makes an ansatz for $\Gamma_k$ which contains only finitely many parameters (depending on $k$) and inserts it into (2.1), the functional differential equation reduces to a set of coupled ordinary differential equations for the parameter functions. The truncation should be chosen in such a way that it encapsulates the essential physics in an ansatz as simple as possible. In a second step one has to verify that upon including more terms in the truncation the results do not change significantly any more.

In this section we demonstrate the practical use of our equation by computing approximately the running of the nonabelian gauge coupling of pure Yang-Mills theory with gauge group $\text{SU}(N)$ in arbitrary dimension $d$ [3]. In order to approximate the solution $\Gamma_k[A, \bar{A}]$ of (2.1) by a functional with at most second derivatives we make the ansatz

$$\Gamma_k[A, \bar{A}] = \int d^d x \left\{ \frac{1}{4} Z_{F_k} F_{\mu \nu}^a (A) F_{\mu \nu}^a (A) + \frac{Z_{F_k}}{2 \alpha_k} [D_\mu [\bar{A}] (A_\mu - \bar{A}_\mu)]^2 \right\}$$

(4.1)

We want to determine the running of $Z_{F_k}$ from the flow equation. The truncation (4.1) leads to the Hessian

$$\frac{\delta^2 \Gamma_k[A, \bar{A}]}{\delta A^a_\mu (x) \delta A^b_\nu (x')} = Z_{F_k} \left\{ D_T [A]_{\mu \nu} + D_\mu [A] D_\nu [A] - \frac{1}{\alpha_k} D_\mu [\bar{A}] D_\nu [\bar{A}] \right\}^{ab} \delta (x - x')$$

(4.2)

where $(D_T)_{\mu \nu} \equiv - D^2 \delta_{\mu \nu} + 2i \bar{g} F_{\mu \nu}$ with the color matrix $F$ in the adjoint representation. ($\bar{g}$ denotes the bare gauge coupling.) In the following we neglect the running of $\alpha_k$ and restrict our discussion to $\alpha_k = 1$. Thus

$$\frac{\delta^2}{\delta A^2} \Gamma_k[A, \bar{A}] \big|_{\bar{A} = A} = Z_{F_k} D_T (A)$$

(4.3)

and the evolution equation reads for $\bar{A} = A$:

$$\frac{\partial}{\partial t} \Gamma_k[A, A] = \frac{\partial Z_{F_k}}{\partial t} \int d^d x \frac{1}{4} F_{\mu \nu}^a F_{\mu \nu}^a$$
Here \( \mathcal{D}_T \) and \( \mathcal{D}_S \equiv -D^2 \) depend on \( A_\mu \) now. For mathematical convenience we chose \( \Delta = \mathcal{D}_T \) for the cutoff operator. The function \( R_k \) is defined with \( Z_k = Z_{Fk} \) in the first trace on the RHS of (4.4) (gluons) and with \( Z_k = 1 \) in the second trace (ghosts). In order to determine \( \partial Z_{Fk}/\partial t \) it is sufficient to extract the term proportional to the invariant \( F^a_{\mu\nu}F^a_{\mu\nu} \) from the traces on the RHS of (4.4). This can be done by using standard heat-kernel techniques or by inserting a simple field configuration on both sides of the equation for which the traces can be calculated easily. In either case one finds for \( d > 2 \) [3]

\[
\frac{\partial}{\partial t} Z_{Fk} = -2N \left(1 - \frac{d}{24}\right) \frac{v_{d-2}}{\pi} g^2 \int_0^\infty dx \frac{d}{dx} \frac{\partial_t R_k(x)}{Z_{Fk} x + R_k(x)}
- \frac{1}{6} N \frac{v_{d-2}}{\pi} g^2 \int_0^\infty dx \frac{d}{dx} \frac{\partial_t R_k(x)}{x + R_k(x)} \equiv g^2 b_d k^{d-4} \tag{4.5}
\]

with \( v_d \equiv \left[2^{d+1}\pi^{d/2} \Gamma(d/2) \right]^{-1} \). The second integral is due to the trace containing \( \mathcal{D}_S \) with \( Z_k = 1 \) in \( R_k(x) \). Introducing the dimensionless, renormalized gauge coupling

\[
g^2(k) = k^{d-4} Z_{Fk}^{-1} g^2 \tag{4.6}
\]

the associated beta function reads

\[
\beta_{g^2} \equiv \frac{\partial}{\partial t} g^2(k) = (d - 4)g^2 + \eta_F g^2 = (d - 4)g^2 - b_d g^4. \tag{4.7}
\]

where \( \eta_F \equiv -\partial_t \ln Z_{Fk} \) denotes the anomalous dimension. For \( d = 4 \) the result for the running of \( g^2(k) \) becomes universal, i.e., \( b_4 \) is independent of the precise form of the cutoff function \( R_k(x) \), only its behavior for \( x \to 0 \) enters in (4.5). One obtains, with \( \lim_{x \to 0} R_k = Z_{Fk} k^2 \) for the first term in (4.5) and \( \lim_{x \to 0} R_k = k^2 \) for the second term,

\[
b_4 = \frac{N}{24\pi^2} \left[11 - 5\eta_F\right] \tag{4.8}
\]

In lowest order in \( g^2 \) we can neglect \( \eta_F \) on the RHS of (4.8) and obtain the standard perturbative one-loop \( \beta \)-function. More generally, one finds for \( \eta_F \) the
The resulting $\eta_F$ has, for $d = 4$, the nonperturbative solution

$$\eta_F = -\frac{11N}{24\pi^2}g^2 \left[ 1 - \frac{5N}{24\pi^2}g^2 \right]^{-1}$$  \hspace{1cm} (4.9)$$

The resulting $\beta$-function can be expanded for small $g^2$

$$\beta_{g^2} = -\frac{11N}{24\pi^2}g^4 \left[ 1 - \frac{5N}{24\pi^2}g^2 \right]^{-1}$$

$$= -\frac{22N}{3} \frac{g^4}{16\pi^2} - \frac{220}{9} \frac{N^2 g^6}{(16\pi^2)^2} \ldots$$  \hspace{1cm} (4.10)$$

Comparing with the standard perturbative two-loop expression

$$\beta_{g^2}^{(2)} = -\frac{22N}{3} \frac{g^4}{16\pi^2} - \frac{204}{9} \frac{N^2 g^6}{(16\pi^2)^2}$$  \hspace{1cm} (4.11)$$

we find a surprisingly good agreement even for the two-loop coefficient. The missing 7% in the coefficient of the $g^6$-term in $\beta_{g^2}$ should be due to our truncations.

For arbitrary $d$ we introduce the constants $l_N^d$ and $l_N^d \eta_F$ by

$$b_d = \frac{44}{3}N v_d l_N^d - \frac{20}{3}N v_d l_N^d \eta_F$$  \hspace{1cm} (4.12)$$

They are normalized such that in 4 dimension $l_N^4 = 1$, $l_N^4 \eta_F = 1$ for any choice of the cutoff function. For $d$ different from 4 they are not universal. If we use the exponential cutoff function \[23\] they read for $d > 2$:

$$l_N^d = -\frac{1}{88}(26 - d)(d - 2)k^{4-d} \int_0^\infty dx \frac{dx}{x^{d-2}} \frac{d}{dx} \frac{d}{dt} \ln P$$

$$= \frac{(26 - d)(d - 2)}{44} l_1^{d-4}$$  \hspace{1cm} (4.13)$$

$$l_N^d \eta_F = -\frac{1}{40}(24 - d)(d - 2)k^{4-d} \int_0^\infty dx \frac{dx}{x^{d-2}} \frac{d}{dx} \frac{P - x}{P}$$

$$= \frac{(24 - d)(d - 2)}{40} l_1^{d-2}$$  \hspace{1cm} (4.14)$$

It is remarkable that the $\beta$-function for the dimensionful, renormalized coupling $g_R^2 = g^2 k^{4-d}$ vanishes precisely in the critical string dimension $d = 26$. More explicitly, one has

$$n_1^{d-4} = -\frac{1}{2} k^{4-d} \int_0^\infty dx \frac{dx}{x^{d-2}} \frac{\partial}{\partial t} \frac{dP}{dx} \frac{P}{P}$$

$$= -\int_0^\infty dy \frac{dy}{y^{d-2}} e^{-y} (1 - y - e^{-y})(1 - e^{-y})^{-2} > 0$$  \hspace{1cm} (4.15)$$
\[ \eta^2 = \left( \frac{d-2}{2} \right) \]  

The evolution equation for the running dimensionless renormalized gauge coupling \( g \) in arbitrary dimension

\[ \frac{\partial g^2}{\partial t} = \beta g^2 = (d-4)g^2 - \frac{44N}{3} v_d l_{\text{NA}}^d g^4 \left[ 1 - \frac{20N}{3} v_d l_{\text{NA}}^d g^2 \right]^{-1} \]  

has the general solution (for \( d \neq 4 \))

\[ \frac{g^2(k)}{[1 + a_2 g^2(k)]^\gamma} = C \left[ \frac{k}{k_0} \right]^{d-4} \]  

with

\begin{align*}
    a_1 &= \frac{44N v_d l_{\text{NA}}^d}{3(4 - d)} \\
    a_2 &= a_1 - \frac{20N}{3} v_d l_{\text{NA}}^d \\
    \gamma &= a_1/a_2
\end{align*}  

and

\[ C = \frac{g^2(k_0)}{[1 + a_2 g^2(k_0)]^\gamma} \]  

The nonabelian Yang-Mills theory is asymptotically free for \( d \leq 4 \) with a “confinement scale” \( \Lambda_{\text{conf}}^{(d)} \), where \( \beta g^2 \) diverges

\[ \Lambda_{\text{conf}}^{(d)} = \left[ \frac{Ca_1^\gamma}{(a_1 - a_2)^\gamma} \right]^{\frac{1}{4-d}} k_0 \]  

At this scale our truncation gives no quantitatively reliable results any more since \( \eta_F \) diverges and the choice \( Z_k = Z_{F_k} \) in \( R_k \) becomes inconvenient. Indeed, \( Z_{F_k} \) may vanish for some scale \( k_{cf} > 0 \), whereas \( Z_k \) should always remain strictly positive. A possible smoother definition in the region of rapidly varying \( Z_{F_k} \) could be \( Z_{\Lambda} = Z_{F\Lambda} \) for \( k = \Lambda \), and \( \partial_t Z_k = -\eta_F (1 + \eta_F^2)^{-1} Z_k \) for \( k < \Lambda \). This modification does not influence the one and two loop \( \beta \)-function. It guarantees, however, that \( Z_k \) remains always strictly positive. Now the \( \beta \) function does not diverge for any finite value of \( g^2 \) and the confinement scale can always be associated with the scale where \( g^2 \) diverges or \( Z_{F_k} \) vanishes. This
scale is slightly lower than (4.21). The “one-loop” confinement scale obtains from (4.21) for \( l_{\text{NA}}^d \to 0, a_2 \to a_1, \gamma \to 1 \)

\[
\Lambda^{(d)}_{\text{conf}} = \left[ \frac{44Nv_d l_{\text{NA}}^d}{3(4-d)} g^2(k_0) \left( 1 + \frac{44Nv_d l_{\text{NA}}^d}{3(4-d)} g^2(k_0) \right)^{-1} \right]^{\frac{1}{4-d}} k_0
\]  

(4.22)

and corresponds as usual to a diverging gauge coupling. We observe that \( \Lambda^{(d)}_{\text{conf}} \) is always higher than the “one-loop” result (4.22) (for given \( k_0 \) and \( g^2(k_0) \)). We therefore consider the scale (4.22) as a lower bound for the confinement scale.

For \( 4 < d < 24 \) the \( \beta \) function (4.17) has an ultraviolet stable fixpoint separating the confinement phase for strong coupling (with a confinement scale given by the analog of (4.21) for negative \( a_1 \) and \( a_2 \)) from the infrared free weak coupling phase. We note that there is no confinement phase for \( d > 26 \).

5 Chern-Simons Theory

As a second example we now turn to pure Chern-Simons theory in 3 dimensions. This is an interesting theory from many points of view. It can be used to give a path-integral representation of knot and link invariants [8] and to understand many properties of 2-dimensional conformal field theories [8, 9]. Being a topological field theory the model has no propagating degrees of freedom. Canonical quantization yields a Hilbert space with only finitely many physical states which can be related to the conformal blocks of (rational) conformal field theories. Perturbative covariant quantization [10, 11, 12, 13, 14] shows that the theory is not only renormalizable but even ultraviolet finite. It is remarkable that despite this high degree of “triviality” the theory produces nontrivial radiative corrections. One-loop effects were found [15, 8] to lead to a renormalization of the parameter \( \kappa \) which multiplies the Chern-Simons 3-form in the action,

\[
S_{CS}[A] = i\kappa \frac{g^2}{8\pi} \int d^3x \varepsilon_{\alpha\beta\gamma} [A^a_{\alpha} \partial_{\beta} A^a_{\gamma} + \frac{1}{3} g f^{abc} A^a_{\alpha} A^b_{\beta} A^c_{\gamma}] 
\]

(5.1)

A variety of gauge invariant regularization methods, including spectral flow arguments based upon the \( \eta \)-invariant, predict a finite difference between the
bare and the renormalized value of $\kappa$:

$$\kappa_{\text{ren}} = \kappa_{\text{bare}} + \text{sign}(\kappa) \ T(G) \quad \text{(5.2)}$$

Here $T(G)$ denotes the value of the quadratic Casimir operator of the gauge group $G$ in the adjoint representation. It is normalized such that $T(SU(N)) = N$. The shift of $\kappa$ has a natural relation to similar shifts in the Sugawara construction of 2-dimensional conformal field theories. On the other hand, in standard renormalization theory a relation of the type (5.2) is rather unusual, and there has been some controversy in the literature about the correct interpretation of eq. (5.2). Following ref. [16] we shall investigate this problem in the context of the effective average action now.

Let us try to find an approximate solution of the initial value problem (2.1) with (2.2) for the classical Chern-Simons action (5.1). We work on flat euclidean space and allow for an arbitrary semi-simple, compact gauge group $G$. We use a truncation of the form [16]

$$\Gamma_k[A, N, \bar{A}] = \frac{i\kappa(k)}{4\pi} I[A] + \kappa(k) \frac{g^2}{8\pi} \int d^3x \left\{ iN^a D^{ab} [\bar{A}] (A^b - \bar{A}_b^h) ight. \\
- i(A^a_\mu - \bar{A}_a^h_\mu) D^{ab} [\bar{A}] N^b + \alpha \kappa(k) \frac{g^2}{4\pi} N^a N^a \right\} \quad \text{(5.3)}$$

with

$$I[A] \equiv \frac{1}{2} \int d^3x \ \varepsilon_{\alpha\beta\gamma} [A^a_\alpha \partial_\beta A^b_\gamma + \frac{1}{3} g f^{abc} A^a_\alpha A^b_\beta A^c_\gamma] \quad \text{(5.4)}$$

The first term on the RHS of (5.3) is the Chern-Simons action, but with a scale-dependent prefactor. In the second term we introduced an auxiliary field $N^a(x)$ in order to linearize the gauge fixing term. By eliminating $N^a$ one recovers the classical, $k$-independent background gauge fixing term $\frac{1}{2\alpha} (D_\mu [\bar{A}] (A_\mu - \bar{A}_\mu))^2$.

As we discussed in Section 3, also the gauge fixing term could in principle change its form during the evolution, but this effect is neglected here.

For $k \to \infty$, and upon eliminating $N^a$, the ansatz (5.3) reduces to (2.2) with the identification $\kappa(\infty) \equiv \kappa_{\text{bare}}$. We shall insert (5.3) into the evolution equation and from the solution for the function $\kappa(k)$ we shall be able to determine the renormalized parameter $\kappa(0) \equiv \kappa_{\text{ren}}$. We have to project the traces on the RHS of (2.1) on the subspace spanned by the truncation (5.3). This
means that we have to extract only the term proportional to $I[A]$ and to compare the coefficients of $I[A]$ on both sides of the equation. In the formalism with the auxiliary field, $\Gamma^{(2)}_k$ in (2.1) denotes the matrix of second functional derivatives with respect to both $A^a_\mu$ and $N^a$, but with $\bar{A}^a_\mu$ fixed. Setting $\bar{A} = A$ after the variation, one obtains

$$\delta^2 \Gamma_k[A, N, A] = i\kappa(k)\frac{g^2}{4\pi} \int d^3x \left\{ \delta A^a_\mu \varepsilon_{\mu\nu\alpha} D^a_\alpha \delta A^b_\nu + \delta N^a D^{ab}_\mu \delta A^b_\mu \right. - \left. \delta A^a_\mu D^{ab}_\mu N^b \right\} + \alpha \left( \kappa(k) \frac{g^2}{4\pi} \right)^2 \int d^3x \delta N^a \delta N^a \quad (5.5)$$

In order to facilitate the calculations we introduce three $4 \times 4$ matrices $\gamma_\mu$ with matrix elements $(\gamma_\mu)_{mn}, m=(\mu,4)=1,...,4$, etc., in the following way [14]:

$$(\gamma_\mu)_{\alpha\beta} = \varepsilon_{\alpha\beta\mu}, \quad (\gamma_\mu)_{4\alpha} = - (\gamma_\mu)_{\alpha4} = \delta_{\mu\alpha}, \quad (\gamma_\mu)_{44} = 0 \quad (5.6)$$

If we combine the gauge field fluctuation and the auxiliary field into a 4-component object $\Psi^a_m \equiv (\delta A^a_\mu, \delta N^a)$ and choose the gauge $\alpha = 0$, we find

$$\delta^2 \Gamma_k[A, N, A] = i\kappa(k)\frac{g^2}{4\pi} \int d^3x \Psi^a_m (\gamma_\mu)_{mn} D^{ab}_\mu \Psi^b_n \quad (5.7)$$

so that in matrix notation

$$\Gamma^{(2)}_k = i\kappa(k) \frac{g^2}{4\pi} \mathcal{D} \quad (5.8)$$

Clearly $\mathcal{D} \equiv \gamma_\mu D_\mu$ is reminiscent of a Dirac operator. In fact, the algebra of the $\gamma$-matrices is similar to the one of the Pauli matrices:

$$\gamma_\mu \gamma_\nu = -\delta_{\mu\nu} + \varepsilon_{\mu\nu\alpha} \gamma_\alpha \quad (5.9)$$

Because $\gamma^+_\mu = -\gamma_\mu$, $\mathcal{D}$ is hermitian. Its square reads

$$\mathcal{D}^2 = -D^2 - ig \gamma_\mu^* F_\mu \quad (5.10)$$

where

$$\gamma_\mu^* \equiv \frac{1}{2} \varepsilon_{\mu\alpha\beta} F_{\alpha\beta} \quad (5.11)$$

is the dual of the field strength tensor. Because $\mathcal{D}^2$ is essentially the covariant laplacian, it is the natural candidate for the cutoff operator $\Delta$. With this choice, and

$$c \equiv \frac{g^2}{4\pi} \quad (5.12)$$
the evolution equation (2.1) reads at $\bar{A} = A$:

$$
i c k \frac{d}{dk} \kappa(k) I[A] = \frac{1}{2} \text{Tr} \left[ \left( i c k \mathcal{D} + R_k(\mathcal{D}^2) \right)^{-1} k \frac{d}{dk} R_k(\mathcal{D}^2) \right] - \text{Tr} \left[ \left( -D^2 + R_k(-D^2) \right)^{-1} k \frac{d}{dk} R_k(-D^2) \right]$$  (5.13)

The second trace on the RHS of (5.13) is due to the ghosts. It is manifestly real, so it cannot match the purely imaginary $i I[A]$ on the LHS and can be omitted therefore. For the same reason we may replace the first trace by $i$ times its imaginary part:

$$
k \frac{d}{dk} \kappa(k) I[A] = -\frac{1}{2} \kappa(k) \text{Tr} \left[ \mathcal{D} \left( c^2 \kappa^2 \mathcal{D}^2 + R_k^2(\mathcal{D}^2) \right)^{-1} k \frac{d}{dk} R_k(\mathcal{D}^2) \right] + \cdots$$  (5.14)

The trace in (5.14) involves an integration over spacetime, a summation over adjoint group indices, and a “Dirac trace”. We shall evaluate it explicitly in the next section. Before turning to that let us first look at the general structure of eq. (5.14). In terms of the (real) eigenvalues $\lambda$ of $\mathcal{D}$ eq. (5.14) reads

$$
\frac{d \kappa(k)}{dk^2} I[A] = -\frac{1}{2} \kappa(k) \sum_\lambda \frac{\lambda}{c^2 \kappa^2(k) \lambda^2 + R_k^2(\lambda^2)} \frac{d R_k(\lambda^2)}{dk^2}$$  (5.15)

where we switched from $k$ to $k^2$ as the independent variable. We observe that the sum in (5.13) is related to a regularized form of the spectral asymmetry of $\mathcal{D}$.

An approximate solution for $\kappa(k)$ can be obtained by integrating both sides of eq. (5.15) from a low scale $k_0^2$ to a higher scale $\Lambda^2$ and approximating $\kappa(k) \simeq \kappa(k_0)$ on the RHS. This amounts to “switching off” the renormalization group improvement. The result is

$$
[\kappa(k_0) - \kappa(\Lambda)] I[A] = \frac{1}{2} \kappa(k_0) \sum_\lambda \int_{k_0^2}^{\Lambda^2} dk^2 \frac{d R_k(\lambda^2)}{dk^2} \cdot \frac{\lambda}{c^2 \kappa^2(k_0) \lambda^2 + R_k^2(\lambda^2)}$$  (5.16)

Upon using $R_k$ as the variable of integration one arrives at

$$
[\kappa(k_0) - \kappa(\Lambda)] I[A] = \frac{1}{2c} \text{sign}(\kappa(k_0)) \sum_\lambda \text{sign}(\lambda) \ G(\lambda; k_0, \Lambda)$$  (5.17)

with

$$
G(\lambda; k_0, \Lambda) \equiv \arctan \left[ c |\kappa(k_0)| \lambda \frac{R_\Lambda(\lambda^2) - R_{k_0}(\lambda^2)}{c^2 \kappa^2(k_0)^2 \lambda^2 + R_\Lambda(\lambda^2) - R_{k_0}(\lambda^2)} \right]$$  (5.18)
Recalling the properties of $R_k$ we see that in the spectral sum (5.17) the contributions of eigenvalues $|\lambda| \ll k_0$ and $|\lambda| \gg \Lambda$ are strongly suppressed, and only the eigenvalues with $k_0 < |\lambda| < \Lambda$ contribute effectively. Ultimately we would like to perform the limits $k_0 \to 0$ and $\Lambda \to \infty$. In this case the sum over $\lambda$ remains without IR and UV regularization. This means that if we want to formally perform the limits $k_0 \to 0$ and $\Lambda \to \infty$ in eq. (5.17), we have to introduce an alternative regulator. In order to make contact with the standard spectral flow argument [8] let us briefly describe this procedure. We avoid IR divergences by putting the system in a finite volume and imposing boundary conditions such that there are no zero modes. In the UV we regularize with a zeta-function-type convergence factor $|\lambda/\mu|^{-s}$ where $\mu$ is an arbitrary mass parameter. Thus the spectral sum becomes

$$\lim_{s \to 0} \sum_{\lambda} \text{sign}(\lambda) \ |\lambda/\mu|^{-s} G(\lambda; k_0, \Lambda) \quad (5.19)$$

Now we interchange the limits $k_0 \to 0$, $\Lambda \to \infty$ and $s \to 0$. By construction, only finite ($|\lambda| \leq \mu$) and nonzero eigenvalues contribute. For such $\lambda$’s we have $G(\lambda; 0, \infty) = \pi/2$ irrespective of the precise form of $R_k$. Therefore (5.17) becomes

$$[\kappa(0) - \kappa(\infty)] \ I[A] = \frac{2\pi^2}{g^2} \text{sign}(\kappa(0)) \ \eta[A] \quad (5.20)$$

where

$$\eta[A] \equiv \lim_{s \to 0} \frac{1}{2} \sum_{\lambda} \text{sign}(\lambda) \ |\lambda/\mu|^{-s} \quad (5.21)$$

is the eta-invariant. If we insert the known result [8]

$$\eta[A] = (g^2/2\pi^2) \ T(G) \ I[A] \quad (5.22)$$

we recover eq.(5.2): $\kappa(0) = \kappa(\infty) + \text{sign}(\kappa(0)) \ T(G)$. Obviously $R_k$ has dropped out of the calculation. The parameter $\kappa$ is universal: it does not depend on the form of the IR cutoff.

6 Evolution of the Chern-Simons Parameter

Next we turn to an explicit evaluation of the trace in eq. (5.14) which keeps the full $k$-dependence of $\kappa$ on the RHS, i.e., the renormalization group im-
To start with we use the constant cutoff\textsuperscript{1} $R_k = k^2$ for which eq. (5.14) assumes the form
\[
\frac{d}{dk^2} \kappa(k) \ I[A] = -\frac{1}{2c^2 \kappa(k)} \ \text{Tr} \left[ \mathcal{D} \left( \mathcal{D}^2 + l(k)^2 \right)^{-1} \right] \tag{6.1}
\]
where
\[
l(k) \equiv \frac{k^2}{c \ |\kappa(k)|} \tag{6.2}
\]
If we extract from the trace the term quadratic in $A$ and linear in the external momentum and equate the coefficients of the $A \partial A$-terms on both sides of (6.1) we obtain
\[
\frac{d\kappa(k)}{dk^2} \int d^3x \ \varepsilon_{\alpha\beta\gamma} \ A^\alpha_\beta \partial_\gamma A^\alpha_\gamma = -\frac{g^2 T(G)}{c^2 \kappa(k)} \int d^3x \ \varepsilon_{\alpha\beta\gamma} \ A^\alpha_\beta \Pi_k(-\partial^2) \partial_\gamma A^\alpha_\gamma + O(A^3) \tag{6.3}
\]
The function $\Pi_k$ is given by the Feynman parameter integral
\[
\Pi_k(q^2) = 8 \int_0^1 dx \ x(1-x) \int \frac{d^3p}{(2\pi)^3} \ \frac{q^2}{[p^2 + l^2 + x(1-x)q^2]^3} \tag{6.4}
\]
Expanding $\Pi_k(-\partial^2) = \Pi_k(0) - \Pi_k'(0) \partial^2 + \ldots$, we see that only for the term with $\Pi_k(0)$ the number of derivatives on both sides of eq.(24) coincides. Therefore one concludes that
\[
\frac{d\kappa(k)}{dk^2} = -\frac{g^2 T(G)}{c^2 \kappa(k)} \ \Pi_k(0) \tag{6.5}
\]
where $\Pi_k(0)$ depends on $\kappa(k)$ via (6.2). Equation (6.5) is the renormalization group equation for $\kappa(k)$ which we wanted to derive. Formally it is similar to the evolution equation in Section 4 or the ones of the abelian Higgs model\textsuperscript{[4]}. The special features of Chern-Simons theory, reflecting its topological character, become obvious when we give a closer look to the function $\Pi_k(q^2)$. Assume we fix a non-zero value of $k$ ($l \neq 0$) and let $q^2 \rightarrow 0$ in (6.4). Because the $l^2$-term prevents the $p$-integral from becoming IR divergent, we may set $q^2 = 0$ in the denominator, and we conclude that the integral vanishes $\sim q^2$. This means that the RHS of (6.5) is zero and that $\kappa(k)$ keeps the same value for all strictly positive values of $k$. However, $\Pi_k(0)$ really vanishes only for
\[\text{As the Faddeev-Popov ghosts do not contribute to the effect under consideration we may set } Z_k = 1 \text{ also in the cutoff for the gauge field.}\]
k > 0. If we set $l = 0$ in (6.4) we cannot conclude anymore that $\Pi_k \sim q^2$, because in the region $p^2 \to 0$ the term $x(1-x)q^2$ provides the only IR cutoff and may not be set to zero in a naive way. In fact, $\Pi_k(0)$ has a $\delta$-function-like peak at $k = 0$. To see this, we first perform the integrals in (6.4):

$$
\Pi_k(q^2) = \frac{1}{\pi} \left[ \frac{1}{|q|}\arctan\left(\frac{|q|}{2|l|}\right) - \frac{|l|}{q^2 + 4l^2}\right] \tag{6.6}
$$

As $q^2$ approaches zero, this function develops an increasingly sharp maximum at $l = 0$. Integrating (6.6) against a smooth test function $\Phi(l)$ it is easy to verify that

$$
\lim_{q^2 \to 0} \int_0^\infty dl \, \Phi(l) \, \Pi_k(q^2) = \frac{1}{4\pi} \Phi(0) \tag{6.7}
$$

This means that on the space of even test functions

$$
\lim_{q^2 \to 0} \Pi_k(q^2) = \frac{1}{2\pi} \delta(l) \tag{6.8}
$$

Even though the value of $\kappa(k)$ does not change during almost the whole evolution from $k = \infty$ down to very small scales, it performs a finite jump in the very last moment of the evolution, just before reaching $k = 0$. This jump can be calculated in a well-defined manner by integrating (6.5) from $k^2 = 0$ to $k^2 = \infty$:

$$
\kappa(0) - \kappa(\infty) = 4\pi \, T(G) \lim_{q^2 \to 0} \int_0^\infty dl \, \text{sign}(\kappa(l)) \cdot \left[ 1 - c l \frac{d}{dk^2} |\kappa(k)| \right]^{-1} \Pi_k(q^2) \tag{6.9}
$$

The term $\sim d|\kappa|/dk^2$ is a Jacobian factor which is due to the fact that $l$ depends on $\kappa(k)$. This factor is the only remnant of the $\kappa(k)$-dependence of the RHS of the evolution equation. As we saw in Section 4, this dependence of the RHS on the running couplings is the origin of the renormalization group improvement. If we use (6.7) in (6.9), $l \, d|\kappa|/dk^2$ is set to zero and we find

$$
\kappa(0) = \kappa(\infty) + \text{sign}(\kappa(0)) \, T(G), \tag{6.10}
$$

which is precisely the 1-loop result. It is straightforward to check that the shift (6.10) is independent of the choice for $R_k$.

It is quite instructive to compare the situation in Chern-Simons theory with what we found for ordinary Yang-Mills theory in Section 4. Like $\kappa$, also
the gauge coupling in QCD$_4$ is a universal quantity. Its running is governed by a $R_k$-independent $\beta$-function which leads to a logarithmic dependence on the scale $k$. The Chern-Simons parameter $\kappa$, on the other hand, does not run at all between $k = \infty$ and any infinitesimally small value of $k$. Only at the very end of the evolution, when $k$ is very close to zero, $\kappa$ jumps by a universal, unambiguously calculable amount $\pm T(G)$. Though surprising in comparison with non-topological theories, this feature is precisely what one would expect if one recalls the topological origin of a non-vanishing $\eta$-invariant [8]. If $\eta[A] \neq 0$ for a fixed gauge field $A$, some of the low lying eigenvalues of $\mathcal{D}[A]$ must have crossed zero during the interpolation from $A = 0$ to $A$. However, this spectral flow involves only that part of the spectrum which, in the infinite volume limit, is infinitesimally close to zero.

The jump of $\kappa$ is also the resolution to the following apparent paradox. The effective average action $\Gamma_k$ is closely related to a continuum version of the block-spin action of lattice systems. Block-spin transformations can be iterated, and when we have already constructed $\Gamma_{k_1}$ at a certain scale $k_1$ we may view $\Gamma_{k_1}$ as the “classical” action for the next step of the iteration, in which an integral over $\exp(-\Gamma_{k_1})$ has to be performed. Trying to understand the shift (5.2) from a renormalization group point of view, we are confronted with the following puzzle. Because $S_{CS}$ is not invariant under large gauge transformations, $\exp(-S_{CS})$ is single valued only if $\kappa \in \mathbb{Z}$. If there is a continuous interpolation between $\kappa(\infty)$ and $\kappa(0)$ a nontrivial shift means that there are intermediate scales at which $\kappa$ cannot be integer. This suggests that $\kappa_{\text{ren}} = \kappa_{\text{bare}}$, because there should be an inconsistency if we try to do the next blockspin transformation starting from a multivalued Boltzmann factor $\exp(-\Gamma_{k_1})$. It is clear now that this argument does not apply precisely because the trajectory from $\kappa(\infty)$ to $\kappa(0)$ is not continuous.

Another unusual feature of Chern-Simons theory is the absence of any renormalization group improvement beyond the 1-loop result. This should be contrasted with the running of $g$ in QCD$_4$ where the truncation of Section 4 leads to a nonperturbative $\beta$-function involving arbitrarily high powers of $g$. We emphasize that our evolution equation with the truncation (5.3) potentially
goes far beyond a 1-loop calculation. It is quite remarkable therefore that in Chern-Simons theory all higher contributions vanish. From the discussion following eq. (6.9) it is clear that this is again due to the unusual discontinuous behavior of $\kappa$ which reflects the topological field theory nature of the model. While it is not possible to translate a “nonrenormalization theorem” for a given truncation into a statement about the nonrenormalization at a given number of loops, our results point in the same direction as ref. [11] where the absence of 2-loop corrections was proven.

7 Conclusion

Exact evolution equations provide a powerful tool for nonperturbative calculations in quantum field theory. Although it is not possible in practice to solve them exactly, the method of truncating the space of actions yields nonperturbative answers which require neither an expansion in the number of loops nor in any small coupling constant. The approximation involved here is that during the evolution the mixing of the operators retained in the ansatz for $\Gamma_k$ with all other operators is neglected. The examples of QCD and of Chern-Simons theory which we discussed in these notes illustrate that this approach works equally well for theories with a complicated dynamics and for topological theories.

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