On some problems with reproducing
the Standard Model fields and interactions in
five-dimensional warped brane world models

Mikhail N. Smolyakov, Igor P. Volobuev

Skobeltsyn Institute of Nuclear Physics, Lomonosov Moscow State University,
119991, Moscow, Russia

Abstract

In the present paper we discuss some problems which arise, when the matter, gauge
and Higgs fields are allowed to propagate in the bulk of five-dimensional brane world
models with compact extra dimension and their zero Kaluza-Klein modes are supposed
to exactly reproduce the Standard Model fields and their interactions.

1 Introduction

Models with extra dimensions have been attracting a great interest during the last fifteen
years. There were many attempts to solve various theoretical problems with the help of extra
dimensions. A wide branch of multidimensional models is that of brane world models, which
were proposed in their modern form in [1, 2]. Although some theoretical problems (such as, for
example, the hierarchy problem of gravitational interaction) were successfully solved within the
framework of brane world models, realistic theories must also describe all the physical aspects
of our four-dimensional world. In particular, they must correctly reproduce the interactions of
the Standard Model (SM) particles that have already been tested experimentally.

In the original formulation of brane world models the SM fields were supposed to be located
on a brane (in the Randall-Sundrum model [2], on the TeV brane). Later the idea of brane
worlds was joined with the idea that all the fields can propagate in extra dimensions [3] thus
giving rise to the theories with universal extra dimensions, where the matter, gauge and Higgs
fields are allowed to propagate in the bulk of five-dimensional brane world models with compact
extra dimension. In this case all these fields possess towers of Kaluza-Klein excitations, their zero
modes being the SM fields. There exist many papers describing how the SM can be embedded
this way into multidimensional brane worlds and what new effects can be produced in such
theories.

However, there are some finer points that have been missed in the previous studies. Below
we will discuss them in detail, mainly from a purely theoretical point of view. In particular, we
argue that it is impossible to exactly reproduce the electroweak gauge boson sector of the SM
in the effective four-dimensional theory, unless the vacuum profile of the Higgs field in the extra
dimension behaves like the square root of the inverse warp factor. The same vacuum profile of
the Higgs field is necessary for generating masses of fermion zero modes in a consistent manner
and, which is even more important, to avoid possible pathologies in the fermion sector and to
ensure the correct couplings of the fermions to the gauge bosons. These restrictions lead to
some difficulties, such as the necessity for an extra fine-tuning, which need to be addressed, at least for better understanding the structure of brane world models.

The paper is organized as follows. In Section 2 we consider bulk gauge fields interacting with the bulk Higgs scalar field and show which conditions should be fulfilled in order to automatically get a self-consistent theory. In Section 3 fermions are examined in the same way. In Section 4 we consider interactions between fermions and gauge bosons. The obtained results are discussed in the last section.

2 Gauge fields

Let us take a five-dimensional space-time with the coordinates \( x^M = \{ x^\mu, z \} \), \( M = 0, 1, 2, 3, 5 \). The compact extra dimension is supposed to form the orbifold \( S^1/Z_2 \), which can be represented as the circle with the coordinate \(-L \leq z \leq L\) and the points \(-z\) and \(z\) identified. In what follows, we will use the notation \( x^\mu \) for the coordinates \( x^\mu \).

We consider the following standard form of the background metric, which is often used in brane world models:

\[
d s^2 = e^{2\sigma(z)} \eta_{\mu\nu} dx^\mu dx^\nu - dz^2.
\] (1)

This metric is assumed to correspond to a regular brane world model, i.e. it is a solution to equations of motion for five-dimensional gravity, two branes with tension and, for example, a stabilizing bulk scalar field. We do not specify the explicit form of the solution for \( \sigma(z) \).

We start with the gauge fields and choose the following action of an \( SU(2) \times U(1) \) gauge invariant model in this background:

\[
S = \int d^4x dz \sqrt{g} \left( -\frac{\xi^2}{4} F^a_{\mu\nu} F^{a\mu\nu} - \frac{\xi^2}{4} B^M_{\mu\nu} B^{\mu\nu} + g^{MN} (D_M H)^\dagger D_N H - V(H^\dagger H) \right),
\] (2)

where

\[
F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g^{abc} A^b_\mu A^c_\nu,
\] (3)

\[
B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu,
\] (4)

\[
D_M H = \left( \partial_M - ig \frac{\tau^a}{2} A^a_M - ig' \frac{B}{2} \right) H
\] (5)

and the fields satisfy the orbifold symmetry conditions \( A^a_\mu(x,-z) = A^a_\mu(x,z) \), \( A^5_\mu(x,-z) = -A^5_\mu(x,z) \), \( B_\mu(x,-z) = B_\mu(x,z) \), \( B_5(x,-z) = -B_5(x,z) \), \( H(x,-z) = H(x,z) \). Here \( \xi = \frac{1}{\sqrt{2L}} \) is a constant, which is introduced for convenience and chosen so that the dimension of the bulk gauge fields is mass. The scalar field potential can include brane-localized terms of the form \( \lambda_1(H^\dagger H)\delta(z) \) and \( \lambda_2(H^\dagger H)\delta(z-L) \). It is easy to see that action (2), which has a rather standard form, resembles the bosonic sector of the electroweak part of the ordinary four-dimensional SM.

This action gives rise to the equations of motion for the gauge and the Higgs fields that
look like

\[ \nabla_N F^{a,MN} + g \epsilon^{abc} A_N^{b} F^{c,MN} + ig\frac{\tau^a}{\xi^2} \left( (D^M H) \dagger \frac{\tau^a}{2} H - H \dagger \frac{\tau^a}{2} D^M H \right) = 0, \]

\[ \nabla_N B^{MN} + ig\frac{\tau^a}{2\xi^2} \left( (D^M H) \dagger H - H \dagger D^M H \right) = 0, \]

\[ \nabla^M D_M H - g^{MN} \left( ig\frac{\tau^a}{2} A_M^a + ig\frac{\tau^a}{2} B_M^a \right) D_N H + \frac{dV}{d(H \dagger H)} H = 0, \]

\[ \nabla^M \] denoting the covariant derivative with respect to metric \( g \).

Let us consider the vacuum solution for these fields. The vacuum solution, breaking the gauge group \( SU(2) \times U(1) \) to \( U(1)_{em} \), leaving the Poincare invariance in four-dimensional space-time intact and satisfying equations (6), (7), can be taken in the form

\[ A_M^a \equiv 0, \quad B_M^a \equiv 0, \quad H_0 \equiv \left( \frac{0}{\sqrt{\xi^2}} \right), \]

where \( v(z) \) is a real function. It is not difficult to understand that in the general case the vacuum solution for the Higgs field \( v(z) \) may depend on the coordinate of the extra dimension. Of course, the scalar field potential must provide for such a solution to equation (8). At this point we do not specify the explicit form of \( v(z) \).

Now let us turn to examining the excitations in the model at hand. Below we will be interested in the behavior of only the four-vector components of the five-dimensional gauge fields, whose zero modes must play the role of the SM gauge bosons. For this reason, from here on we retain only these components of the gauge fields and drop the components \( A_5^a, B_5 \) of the vector fields and the fluctuations of the Higgs field. From action (2) it is easy to get the following effective action for the four-vector components of the five-dimensional gauge fields:

\[ S_{\text{eff}} = \int d^4xdz \left( -\frac{\xi^2}{4} \eta^{\mu\nu} \eta^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} + e^{2\sigma} \frac{\xi^2}{2} \eta^{\mu\nu} \eta^{\alpha\beta} A_\mu^a \partial_5 A_\nu^a - \frac{\xi^2}{4} \eta^{\mu\nu} \eta^{\alpha\beta} B_{\mu\alpha} B_{\nu\beta} \right. \]

\[ + e^{2\sigma} \frac{\xi^2}{2} \eta^{\mu\nu} \partial_5 B_{\mu\alpha} \partial_5 B_{\nu\alpha} + e^{2\sigma} \eta^{\mu\nu} H_0^\dagger \left( \frac{\tau^a}{2} A_\mu^a + \frac{g}{2} B_\mu^a \right) \left( \frac{\tau^a}{2} A_\nu^a + \frac{g}{2} B_\nu^a \right) H_0 \],

where we have also dropped the terms containing only the vacuum configuration of the Higgs field.

Now we are ready to perform the Kaluza-Klein mode decomposition. First, using the standard redefinition

\[ Z_\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left( g A_\mu^3 - g' B_\mu \right), \quad A_\mu = \frac{1}{\sqrt{g^2 + g'^2}} \left( g B_\mu + g' A_\mu^3 \right), \quad W_\mu^\pm = \frac{1}{\sqrt{2}} \left( A_\mu^1 \mp iA_\mu^2 \right), \]

we can pass to the physical degrees of freedom of the theory. Next, let us consider only the quadratic part of effective action (10) in terms of these new fields. It takes the form

\[ S_{\text{eff}} = \int d^4xdz \left( -\frac{\xi^2}{2} \eta^{\mu\nu} \eta^{\alpha\beta} W_{\mu\alpha}^+ W_{\nu\beta}^- - \frac{\xi^2}{4} \eta^{\mu\nu} \eta^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{\xi^2}{4} \eta^{\mu\nu} \eta^{\alpha\beta} Z_\mu Z_\nu \right. \]

\[ + e^{2\sigma} \frac{\xi^2}{2} \eta^{\mu\nu} \partial_5 W_{\mu\alpha}^+ \partial_5 W_{\nu\beta}^- + e^{2\sigma} \frac{\xi^2}{2} \eta^{\mu\nu} \partial_5 A_\mu \partial_5 A_\nu + e^{2\sigma} \frac{\xi^2}{2} \eta^{\mu\nu} \partial_5 Z_\mu \partial_5 Z_\nu \]

\[ + e^{2\sigma} \left( z \frac{g}{2} \eta^{\mu\nu} W_{\mu\alpha}^+ W_{\nu\beta}^- + e^{2\sigma} \frac{g^2 + g'^2}{8} \eta^{\mu\nu} Z_\mu Z_\nu \right), \]
where $W_{\mu\nu}^\pm = \partial_\mu W_\nu^\pm - \partial_\nu W_\mu^\pm$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu$. The equations for the wave functions and the masses of the Kaluza-Klein modes are

$$- m_{W,n}^2 f_{W,n} - \partial_5 (e^{2\sigma} \partial_5 f_{W,n}) + \frac{g^2}{4\xi^2} e^{2\sigma} v^2(z) f_{W,n} = 0,$$

$$- m_{Z,n}^2 f_{Z,n} - \partial_5 (e^{2\sigma} \partial_5 f_{Z,n}) + \frac{g^2 + g'^2}{4\xi^2} e^{2\sigma} v^2(z) f_{Z,n} = 0,$$

$$- m_{A,n}^2 f_{A,n} - \partial_5 (e^{2\sigma} \partial_5 f_{A,n}) = 0.$$

where the subscript $n$ denotes the number of the corresponding Kaluza-Klein mode. As usual, the lowest (zero) Kaluza-Klein modes of the fields are supposed to correspond to the four-dimensional SM particles. So, below we will focus only on the zero modes.

It follows from (17) that the solution for the lowest mode of the field $A_\mu$ (the photon) is $m_{A,0} = 0$ and $f_{A,0}(z) = \text{const}$, i.e. its wave function does not depend on the coordinate of the extra dimension. This is an important result, which provides the universality of the electromagnetic charge [4]. But, as one sees from (15) and (16), in the general case it is not so for the zero modes of the fields $W_\mu$ and $Z_\mu$, which correspond to the SM massive gauge bosons. The latter has the following well-known consequences. Indeed, in the SM the self-coupling of massive gauge bosons comes from the term $F_{\alpha\mu\nu}^a F_{\mu\nu}^a$ and the corresponding coupling constants are defined only by the structure of the gauge group. In the five-dimensional case under consideration the self-coupling terms also come from the same term of (10), but now the corresponding coupling constants are also defined by the overlap integrals over the coordinate $z$, which include the wave functions $f_{W,0}(z)$ and $f_{Z,0}(z)$. The only case, when the zero mode sector of the model automatically completely coincides with the electroweak gauge boson sector of the SM, is the one, where the wave functions $f_{W,0}(z)$ and $f_{Z,0}(z)$ do not depend on the coordinate of the extra dimension. In this case the self-coupling constants of the massive gauge bosons are defined in terms of the constants $g$ and $g'$ exactly in the same way as in the ordinary SM. The independence of the wave functions $f_{W,0}(z)$ and $f_{Z,0}(z)$ on the coordinate of the extra dimension can be achieved only when $e^{2\sigma} v^2(z) = \text{const}$, i.e., when

$$v(z) = \xi \tilde{v} e^{-\sigma},$$

where $\tilde{v}$ is a constant of dimension $M$. For the choice (18), the masses of the zero mode gauge bosons are given by

$$m_{W,0} = \frac{g \tilde{v}}{2}, \quad m_{Z,0} = \frac{\sqrt{g^2 + g'^2 \tilde{v}}}{2}.$$

Thus, in the case under consideration $\tilde{v}$ must coincide with the Higgs field vacuum expectation value of the SM.

Of course, the results presented above are rather trivial (see also [5], where the effects of the shapes of the gauge boson wave functions on the electroweak observables were discussed). But they have some important consequences, which will be discussed below. Meanwhile, one can imagine that there exists a profile for the Higgs vacuum solution, which differs from (18) but provides somehow the necessary values of the zero mode gauge boson masses and self-coupling constants with a good accuracy. Unfortunately, the situation becomes more involved, when one comes to fermions.
3 Fermions

It is well known that, since there is no chirality in five-dimensional space-time, in order to obtain a nonzero mass term for the zero Kaluza-Klein fermion mode via the Higgs mechanism it is necessary to take two five-dimensional spinor fields (see, for example, [6, 7, 8]) satisfying the orbifold symmetry conditions

\[
\Psi_1(x, -z) = \gamma^5 \Psi_1(x, z), \quad \Psi_2(x, -z) = -\gamma^5 \Psi_2(x, z).
\]

Thus, as a simple example, we consider a model with the action of the most general form

\[
S = \int d^4xdz \sqrt{g} \left( E_N^M i \bar{\Psi}_1 \Gamma^N \nabla_M \Psi_1 + E_N^M i \bar{\Psi}_2 \Gamma^N \nabla_M \Psi_2 - F_1(z) \bar{\Psi}_1 \Psi_1 - F_2(z) \bar{\Psi}_2 \Psi_2 - G(z) (\bar{\Psi}_2 \Psi_1 + \bar{\Psi}_1 \Psi_2) \right),
\]

where \( M, N = 0, 1, 2, 3, 5 \), \( \Gamma^\mu = \gamma^\mu, \Gamma^5 = i\gamma^5, \nabla_M \) is the covariant derivative containing the spin connection, \( E_N^M \) is the vielbein, \( F_1, F_2 \) and \( G \) are some functions satisfying the symmetry conditions \( F_1, F_2 = -F_1, F_2 = G \) and \( G(-z) = G(z) \). For the case of metric (1) action (22) can be rewritten in the form (see, for example, [8, 9] for the explicit form of the vielbein and spin connections)

\[
S = \int d^4xdz \epsilon^{\alpha \beta \gamma \delta} \left( e^{-\sigma} i \bar{\Psi}_1 \gamma^\mu \partial_\mu \Psi_1 - \bar{\Psi}_1 \gamma^5 (\partial_5 + 2\sigma') \Psi_1 - F_1(z) \bar{\Psi}_1 \Psi_1 + \bar{\Psi}_2 \Psi_2 - F_2(z) \bar{\Psi}_2 \Psi_2 - G(z) (\bar{\Psi}_2 \Psi_1 + \bar{\Psi}_1 \Psi_2) \right),
\]

where \( \sigma' = \partial_5 \sigma \). The equations of motion, following from this action, take the form

\[
e^{-\sigma} i \gamma^\mu \partial_\mu \Psi_1 - \gamma^5 (\partial_5 + 2\sigma') \Psi_1 - F_1(z) \Psi_1 - G(z) \Psi_2 = 0, \quad F_2(z) \Psi_2 - G(z) \Psi_1 = 0.
\]

Suppose that \( G(z) \equiv 0 \). In this case there always exists the solution

\[
\Psi_1 = C_f \exp \left[ -\int_0^z F_1(y)dy - 2\sigma(z) \right] \psi_L(x), \quad i\gamma^\mu \partial_\mu \psi_L = 0, \quad \gamma^5 \psi_L = \psi_L,
\]

where \( C_f \) is a normalization constant, describing a massless four-dimensional fermion. An analogous solution exists for the field \( \Psi_2 \) (but with a right-handed four-dimensional fermion). The latter clearly indicates that the existence of only one five-dimensional fermion is not enough to provide a massive four-dimensional zero mode. This also indicates that it is the term with \( G(z) \neq 0 \) that is responsible for the generation of the masses of the zero mode fermions. Thus, the function \( G(z) \) should be somehow connected with the five-dimensional Higgs field. It is natural to take this function as

\[
G(z) = h \nu(z),
\]

where \( h \) is a coupling constant of dimension \( M^{-\frac{1}{2}} \). Such a construction may arise, when one considers the standard Higgs mechanism in the bulk after the spontaneous symmetry breaking,
leading to (9), whereas the zero modes of the fields $\Psi_1, \Psi_2$ are supposed to represent a massive lepton (for example, the electron). It should be noted that the functions $F_1(z)$ and $F_2(z)$ are not connected with the Higgs field in the general case. Meanwhile, the corresponding terms are not forbidden and, according to (26), they can be responsible for the localization of the modes in the same way as it happens in the well-known Rubakov-Shaposhnikov mechanism [10].

Now let us recall the ordinary four-dimensional free spinor field satisfying the Dirac equation. It is well known that each component of this field satisfies the Klein-Gordon equation, which is the second-order differential equation. The fields $\Psi_1$ and $\Psi_2$ satisfying equations (24), (25) should be considered as free fields as well, because they are coupled only to the vacuum configurations of the Higgs and gravity fields. Therefore, one expects that in a consistent theory each component of the five-dimensional spinor fields $\Psi_1$ and $\Psi_2$ (or at least of their linear combinations) also satisfies a five-dimensional second-order differential equation, which contains derivatives in the four-dimensional coordinates only in the form $\Box = \eta^\mu\nu \partial_\mu \partial_\nu$, otherwise one may expect the appearance of various pathologies while expanding into Kaluza-Klein modes. So, let us try to obtain the corresponding second-order differential equations. From (24) and (25) it is not difficult to obtain:

$$-\Box \Psi_1 + e^\sigma (\partial_5 + 2\sigma^\prime) e^\sigma (\partial_5 + 2\sigma^\prime) \Psi_1 + e^\sigma \partial_5 (e^\sigma F_1) \gamma^5 \Psi_1 - e^{2\sigma} (F_1^2 (z) + \tilde{h}^2 v^2 (z)) \Psi_1$$

$$+ h e^\sigma \partial_5 (e^\sigma v (z)) \gamma^5 \Psi_2 - h e^{2\sigma} v (z) (F_1 (z) + F_2 (z)) \Psi_2 = 0,$$

$$-\Box \Psi_2 + e^\sigma (\partial_5 + 2\sigma^\prime) e^\sigma (\partial_5 + 2\sigma^\prime) \Psi_2 + e^\sigma \partial_5 (e^\sigma F_2) \gamma^5 \Psi_2 - e^{2\sigma} (F_2^2 (z) + \tilde{h}^2 v^2 (z)) \Psi_2$$

$$+ h e^\sigma \partial_5 (e^\sigma v (z)) \gamma^5 \Psi_1 - h e^{2\sigma} v (z) (F_1 (z) + F_2 (z)) \Psi_1 = 0.$$  \hspace{1cm} (28)

From (28) and (29) one can see that formally the equations for the components of the field $\Psi_1$ and $\Psi_2$ do not decouple, and in the general case we can not obtain the second-order differential equations for each component of the fields $\Psi_1$ and $\Psi_2$ (or of their linear combinations) separately, as it happens in ordinary four-dimensional theory, except several special cases. The first obvious exception is when the following conditions fulfill:

$$F_1 (z) \equiv -F_2 (z),$$

$$\partial_5 (e^\sigma v (z)) \equiv 0.$$  \hspace{1cm} (30)

The second condition completely coincides with (18). Introducing the dimensionless coupling constant $\tilde{h} = h \xi$ and taking into account (30) and (31) we can rewrite equations (28) and (29) as

$$-\Box \Psi_1 + e^\sigma (\partial_5 + 2\sigma^\prime) e^\sigma (\partial_5 + 2\sigma^\prime) \Psi_1 + e^\sigma \partial_5 (e^\sigma F) \gamma^5 \Psi_1 - (e^{2\sigma} F^2 (z) + \tilde{h}^2 v^2) \Psi_1 = 0,$$

$$= 0,$$

$$-\Box \Psi_2 + e^\sigma (\partial_5 + 2\sigma^\prime) e^\sigma (\partial_5 + 2\sigma^\prime) \Psi_2 - e^\sigma \partial_5 (e^\sigma F) \gamma^5 \Psi_2 - (e^{2\sigma} F^2 (z) + \tilde{h}^2 v^2) \Psi_2 = 0,$$

where $F (z) = F_1 (z) \equiv -F_2 (z)$, which indeed lead to the second-order differential equations for each component of the fields $\Psi_1$ and $\Psi_2$ separately.

The solution to these equations, corresponding to the zero mode, has the form (it can also
be easily obtained from (24), (25)

$$\Psi_1 = C_f \exp \left[ -\frac{1}{2} \int_0^z F(y)dy - 2\sigma(z) \right] \psi_L(x), \quad i\gamma^\mu \partial_\mu \psi_L - \tilde{h}v\psi_R = 0, \quad \gamma^5 \psi_L = \psi_L, \quad (34)$$

$$\Psi_2 = C_f \exp \left[ -\frac{1}{2} \int_0^z F(y)dy - 2\sigma(z) \right] \psi_R(x), \quad i\gamma^\mu \partial_\mu \psi_R - \tilde{h}v\psi_L = 0, \quad \gamma^5 \psi_R = -\psi_R, \quad (35)$$

where again $C_f$ is a normalization constant. This solution indeed corresponds to the lowest mode, see Appendix A for details. It is clear that the fields $\psi_L$ and $\psi_R$ taken together make up a four-dimensional Dirac fermion with mass $\tilde{h}v$.

It should be also mentioned that the fermion action exactly of form (23) with conditions (30) and (31) (but in other notations) was considered in [8] for examining discrete symmetries in brane world models.

It is interesting to note that if the function $F(z)$ has the same form for different fermions (leptons, quarks), then the wave functions of the zero modes have exactly the same form for different fermions regardless of the four-dimensional mass of the mode (see (34), (35)). In order to have the same coupling constants of fermions to massive gauge bosons, as in the four-dimensional SM (i.e., to be also defined only by the structure of the gauge group), the wave functions of the massive gauge bosons should not depend on the coordinate of the extra dimension. This again sends us back to the results obtained in the previous section and gives an additional confirmation that condition (18) must be fulfilled in order to automatically have a correct effective theory.

What can happen if conditions (30) and (31) are not satisfied? To show it, let us simplify the task and consider the case $\sigma \equiv 0$ (the flat five-dimensional space-time). In this case equations (28) and (29) take the form

$$-\Box \Psi_1 + \partial_5^2 \Psi_1 + F'_1(z)\gamma^5 \Psi_1 - \left( F'^2_1(z) + h^2 v^2(z) \right) \Psi_1 = 0,$$

$$+hv'(z)\gamma^5 \Psi_2 - hv(z) \left( F'_1(z) + F'_2(z) \right) \Psi_2 = 0,$$

where $' = \partial_5$. The equation for, say, the first component of the field $\Psi_1$ can be easily obtained and turns out to be the fourth-order differential equation

$$[\Box - \partial_5^2 - F'^2_1 + F'^2_2 + h^2 v^2] \frac{1}{hv(F_1 + F_2) - hv'} [\Box - \partial_5^2 - F'^2_1 + F'^2_2 + h^2 v^2] \Psi_1^{(1)}$$

$$- (hv(F_1 + F_2) - hv') \Psi_1^{(1)} = 0. \quad (38)$$

Analogous equations can be obtained for the other components of the field $\Psi_1$ and for the components of the field $\Psi_2$. It is obvious that even in the flat case $\sigma \equiv 0$ the form of equation (38) poses a question about the possibility of a mathematically consistent isolation of the physical degrees of freedom of the theory. Moreover, since in many cases higher-derivative theories contain pathologies (such as ghosts; see, for example, [11] for details), one may expect pathological behavior in the case under consideration too. Thus, in our opinion, one should avoid
the appearance of such fourth-order differential equations when constructing multidimensional models to be sure that the resulting theory is devoid of any pathologies and the physical degrees of freedom can be isolated in a mathematically consistent way using the well-developed theory of second-order differential equations.

A comment is in order here. In many brane world models the expansion in the Kaluza-Klein modes for the fields $\Psi_1$ and $\Psi_2$ is performed without taking into account the interaction with the Higgs field (indeed, in the case $v(z) \equiv 0$ the corresponding differential equations are indeed second-order and one does not expect any pathologies). The corresponding system of eigenfunctions is complete and such a procedure seems to be correct from the mathematical point of view. But when the interaction with the Higgs field is taken back into account, all the off-diagonal entries of the corresponding (infinite) mass matrix are, in the general case, nonzero, which is not an unexpected result, because the orthogonality conditions for the case $v(z) \equiv 0$ are not valid for the case $v(z) \neq 0, v(z) \neq \tilde{v}e^{-\sigma}$ (including the generalized functions like the delta-function). The resulting fields do not represent the physical degrees of freedom of the theory and cannot be used for consistent calculations, since the diagonalization of such a matrix by algebraic methods can be impossible in the general case.\footnote{In principle, such a diagonalization should be equivalent to finding a complete system of eigenfunctions for equations (28) and (29). However, as it was noted above, in the general case this task appears to be very complicated (if feasible in principle) because of the appearance of the fourth-order differential equations like the one in (38). For this reason we think that the only consistent way of deriving an effective four-dimensional action in brane world models is to consider first the vacuum solution for the Higgs field (and for other fields with nonzero vacuum solutions, if exist) and only then to perform the Kaluza-Klein decomposition (if it is possible) checking the absence of pathologies at least in the free theory.}

However, theories leading to equations with higher derivatives are not necessarily pathological (see an example in Appendix B). Indeed, there is another obvious exception in equations (28), (29), leading to second-order differential equations of motion for any form of $v(z)$. Namely, if the relation

$$F_1(z) \equiv F_2(z)$$

is fulfilled, then one can add and subtract equations (28), (29) to obtain two independent second-order differential equations for the combinations $\Psi_1 + \Psi_2$ and $\Psi_1 - \Psi_2$, which look like they should not lead to any pathologies.

To examine this case in more detail let us again simplify the task and take $\sigma \equiv 0$, $F_1(z) \equiv F_2(z) \equiv 0$ and $v(z) \neq \text{const}$ (for the case $\sigma \equiv 0$ the latter condition does not satisfy (31)). The corresponding equations of motion, following from (36) and (38), take the form

$$-\Box(\Psi_1 + \Psi_2) + \partial_5^2(\Psi_1 + \Psi_2) - h^2v^2(z)(\Psi_1 + \Psi_2) + hv'(z)\gamma^5(\Psi_1 + \Psi_2) = 0,$$

$$-\Box(\Psi_1 - \Psi_2) + \partial_5^2(\Psi_1 - \Psi_2) - h^2v^2(z)(\Psi_1 - \Psi_2) - hv'(z)\gamma^5(\Psi_1 - \Psi_2) = 0.$$  

Using these equations it is not difficult to show that, according to the orbifold symmetry conditions (20), (21), the fields $\Psi_1$ and $\Psi_2$ can be decomposed into the Kaluza-Klein modes as

$$\Psi_1 = \sum_n \left( f_n^+(z) \psi^n_L(x) + f_n^-(z) \psi^n_R(x) \right),$$

$$\Psi_2 = \sum_n \left( f_n^+(z) \psi^n_L(x) - f_n^-(z) \psi^n_R(x) \right),$$
where $\gamma^5\psi_L = \psi_L$, $\gamma^5\psi_R = -\psi_R$,
\[
\begin{align*}
\gamma^5 f_+^n(z) &= f^n(z) + f^n(-z), \\
\gamma^5 f_-^n(z) &= f^n(z) - f^n(-z)
\end{align*}
\] (44)
and the function $f^n(z)$ is a periodic continuously differentiable solution to the equation
\[
m^2_n f^n + \partial_5^2 f^n - h^2 v^2 f^n + h v f^n = 0
\] (45)
in the interval $[-L, L]$, corresponding to the eigenvalue $m^2_n$ (recall that $v(-z) = v(z)$). According to the general theory [12], the functions $f^n(z)$ make up an orthonormal set of eigenfunctions for equation (45), the lowest eigenvalue $m_0$ being simple (which means that we get only one fermion with mass $m_0$ in the effective four-dimensional theory). Moreover, it is not difficult to show that $m^2_0 > 0$ for (45). Thus, the corresponding free theory seems to have no obvious pathologies. But the chiral structure of the zero modes of the fields $\Psi_1$ and $\Psi_2$ in (42), (43) differs from that in (34), (35). This difference leads to certain problems when taking into account the interactions with the gauge fields. This issue will be discussed in the next section.

4 Interactions in the effective theory

To demonstrate in a simple way, how possible latent problems can pop up in the four-dimensional effective theory, corresponding to equations (40) and (41), let us consider a five-dimensional action, describing fermion fields minimally coupled to the $SU(2) \times U(1)$ gauge fields in the flat ($\sigma(z) \equiv 0$) space-time:
\[
S = \int d^4xdz \left( i\bar{\hat{\Psi}}_1 \Gamma^M D_M \hat{\Psi}_1 + i\bar{\Psi}_2 \Gamma^M D_M \Psi_2 - \sqrt{2}h \left[ \left( \bar{\hat{\Psi}}_1 H \right) \Psi_2 + h.c. \right] \right).
\] (46)
Here the $SU(2)$ doublet, constructed from five-dimensional spinors, is denoted by
\[
\hat{\Psi}_1 = \left( \begin{array}{c} \Psi_1^\nu \\ \Psi_1^\psi \end{array} \right), \quad \bar{\hat{\Psi}}_1 = \left( \bar{\Psi}_1^\nu, \bar{\Psi}_1^\psi \right)
\] (47)
and the five-dimensional $SU(2)$ singlet is denoted by $\Psi_2$. The covariant derivatives are defined by
\[
D_M \hat{\Psi}_1 = \left( \partial_M - ig^a A^a_M + ig' B_M \right) \hat{\Psi}_1, \quad D_M \Psi_2 = \left( \partial_M + ig' B_M \right) \Psi_2.
\] (48, 49)
The vacuum solution for the Higgs field is supposed to have the form
\[
H_0 \equiv \left( \begin{array}{c} 0 \\ v(z) \sqrt{2} \end{array} \right)
\] (50)
with $v(z) \neq \text{const}$.

First, let us consider the free theory. We will be interested only in the lowest mode sector, so we neglect all the higher Kaluza-Klein modes of gauge and fermion fields. According to
for the fermion zero modes we can write (below we will omit the superscript “0” for the zero \((n = 0)\) modes of the fields)

\[
\hat{\Psi}_1(x, z) = \left( \xi \nu_L(x) f_+(z) \psi_L(x) + f_-(z) \psi_R(x) \right), \quad \Psi_2(x, z) = f_-(z) \psi_L(x) - f_+(z) \psi_R(x),
\]

(51)

where \(\nu_L = \gamma^5 \nu_L\). The factor \(\xi = \frac{1}{\sqrt{2L}}\) is just the canonically normalized wave function of the zero mode of the field \(\Psi_\nu\) (since this field does not interact with the vacuum solution of the Higgs field \(H\), in the flat background and with \(F_1(z) \equiv F_2(z) \equiv 0\) its wave function is a constant). Substituting (51) into (46), dropping the terms with gauge fields and integrating over the coordinate of the extra dimension, one can obtain the standard four-dimensional action for the free fermion fields\(^2\)

\[
S = \int d^4 x \left( i \bar{\nu}_L \gamma^\mu \partial_\mu \nu_L + i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi \right),
\]

(52)

where \(\psi = \psi_L + \psi_R\). In order to have the canonically normalized kinetic term of the field \(\psi\) in (52), the condition

\[
a^2 + b^2 = 1, \quad a^2 = \int dz f_+^2(z), \quad b^2 = \int dz f_-^2(z).
\]

(53)

must be fulfilled. The mass \(m\) is the eigenvalue of the problem (45), corresponding to the eigenfunction \(f(z)\). By tuning the coupling constant \(h\) one can, in principle, get the desired value of the mass \(m\).

Now let us turn to examining the interactions with the gauge bosons. In order to isolate the effects caused only by the fermions, below we choose the following ansatz for the zero modes of the gauge fields:

\[
A_\mu^a(x, z) \equiv A_\mu^a(x), \quad B_\mu(x, z) \equiv B_\mu(x);
\]

(54)

for simplicity we will also drop the components \(A_5^a(x, z)\) and \(B_5(x, z)\) of these fields. We do not discuss here possible ways for obtaining this ansatz in a consistent way. For example, one may simply imagine that there exists a second Higgs field, interacting with the gauge fields only, which provides the necessary forms of the corresponding wave functions.

Passing to the physical degrees of freedom \(11\), using (51), (54) and integrating over the coordinate of the extra dimension, we can obtain the effective four-dimensional action, describing the interaction of the zero mode fermions with the gauge bosons. We do not present explicit calculations here, they are straightforward. It is not difficult to show that the electromagnetic coupling constant appears to be the same as in the SM (from here and below “the same as in the SM” means that it can be expressed through the constants \(g\) and \(g'\) exactly in the same way as it happens in the SM). The coupling constant of the interaction with the charged gauge bosons is found to be

\[
g \frac{1}{\sqrt{2L}} \int dz f_+(z)
\]

(55)

\(^2\)In the derivation of action (52) it is convenient to use the equations \(f_+ + hvf_- = mf_-\), \(f_- + hvf_+ = -mf_+\), which are fulfilled whenever equation (45) holds.
instead of \( g \) in the SM (recall that the wave function, corresponding to the field \( \nu_L \), is just \( \frac{1}{\sqrt{2L}} \)). As for the interaction of the field \( \psi \) with the neutral gauge boson \( Z \), the vector coupling constant appears to be the same as in the SM, whereas the axial coupling constant has the form

\[
g_A = g^A_M (a^2 - b^2). \tag{56}
\]

It is clear that in the general case \( c = \frac{1}{\sqrt{2L}} \int dz f_+(z) \neq 1 \) and, according to (53), \( a^2 - b^2 \neq 1 \). Meanwhile, in order to get rid of the difference with the well known parameters of the SM, one should have \( c = 1 \) and \( a^2 - b^2 = 1 \) (or the values which are close to unity with a good accuracy). The latter can be achieved if \( b = 0 \), which means that \( f(z) = \text{const} \); in this case \( c = 1 \) too. But, according to (53), the condition \( f(z) = \text{const} \) means that \( v(z) = \text{const} \), which again corresponds to (51) with \( \sigma \equiv 0 \). Thus, the farther \( v(z) \) from a constant is, the farther the values of the corresponding coupling constants are from those of the SM.

It is obvious that for the non-flat case \( \sigma \neq 0 \) with \( \partial_5 (e^\sigma v(z)) \neq 0 \) the problems, completely analogous to those described above, are also expected to arise in the four-dimensional effective theory, though the calculations appear to be much more bulky than in the case \( \sigma \equiv 0 \). The only difference in the warped case is that now the farther \( v(z) \) from \( \sim e^{-\sigma} \) is, the farther the values of the corresponding coupling constants are from those of the SM (note that in a realistic theory with \( \partial_5 (e^\sigma v(z)) \neq 0 \) the shapes of the zero mode gauge boson wave functions also differ from a constant, which provides additional deviations from the standard values of the SM parameters like in \([5]\)). For example, the natural choice \( v(z) = \text{const} \) may lead, in principle, either to an unacceptable theory or put it out of reach of the present day experiments.

### 5 Discussion and conclusion

As it was demonstrated in the previous sections, the only obvious possibility to automatically get a self-consistent, from the theoretical point of view, four-dimensional SM in a five-dimensional brane world model (i.e., without possible pathologies in the free theory and with the correct couplings) is to have a vacuum solution of form (18) for the Higgs field together with (30).

It is not difficult to check that in this case the zero mode fermion and gauge boson sectors of the resulting effective theory indeed exactly reproduce those of the SM, including the interactions, at least for the case of the standard form of five-dimensional gauge invariant action. The corresponding calculations are straightforward and we do not present them here.

The correctness and self-consistency of the resulting fermion and gauge boson sectors of the four-dimensional SM for (30) and (31) follow from the fact that only in this case equations (28) and (29) decouple, which provides the correct chiral structure of the corresponding lowest Kaluza-Klein modes.

Unfortunately, the restriction on the vacuum Higgs field poses several problems, which should be necessarily addressed. First, the profile of the vacuum solution for the Higgs field appears to be strongly related to the form of the warp factor of the model. This demands

\[3\]Of course, in principle there may exist other exceptions in equations (28), (29), leading to second-order differential equations of motion. But, according to the results presented above, it is improbable that such unevdent cases could lead to a completely acceptable effective theory.
an extra fine-tuning for the scalar field potential. Indeed, even for the simplest case of the Randall-Sundrum setup \cite{2} with $\sigma = -k|z|$ and without taking into account the backreaction of the Higgs field on the background metric, the fine-tuned bulk Higgs potential should have the form

$$V(H^\dagger H) = -3k^2 H^\dagger H$$

to get the vacuum solution (18). Of course, one should also add fine-tuned brane-localized potentials, including the term specifying the value of the constant $\tilde{v}$ (at least on one of the branes). If one takes more realistic cases of stabilized models, in which the warp factors have a more complicated form (like the one in \cite{13}), the form of the Higgs scalar field potential appears to be such that it cannot be represented in an analytical form. Of course, such a situation looks unnatural, at least in the absence of a symmetry that can support such a fine-tuning of the Higgs potential.

Second, in such a scenario the Higgs field and the stabilizing scalar field cannot be unified, as it was proposed in \cite{14}. Indeed, a consistent stabilization mechanism (like the one proposed in \cite{13}) is based on fixing the values of the stabilizing scalar fields on the branes (at the points $z = 0$ and $z = L$). On the other hand, warped brane world models are interesting if the function $e^{2\sigma}$ has exponentially different values on the branes. The latter means that by taking the Higgs field as the stabilizing field in such a theory, one introduces a new hierarchy into the model (because $v(0) \ll v(L)$). For example, the Randall-Sundrum model \cite{2} was proposed to solve the hierarchy problem of gravitational interaction, so it also looks unnatural to add an extra hierarchy into such a model. Moreover, in order to get a massive radion, one should take into account the backreaction of the stabilizing field on the background metric \cite{13}. But if the stabilizing field is the Higgs field with vacuum solution (18), then the range of allowed scalar field potentials and warp factors narrows considerably (which clearly follows from the self-consistent system of equations for the background configuration of the metric and the stabilizing scalar field, which can be found in \cite{13}).

One may suppose that if at least the fermion fields are located exactly on the brane, then the Higgs field can also be located on the brane, which looks as a solution to the problem (of course, if we do not take into account the gauge fields, see \cite{5}). However, the only realistic field-theoretical mechanism of fermion localization, which can be used for calculations, is the Rubakov-Shaposhnikov mechanism \cite{10}. It is based on the idea that initially the fermion fields propagate in the whole five-dimensional space-time, whereas only the lowest modes appear to be localized on the brane due to an interaction with some defect (for example, with a domain wall). But this is exactly the situation, which is realized in equations (34), (35). Indeed, one can take an appropriate value of the function $F(z)$ (the simplest choice is $F(z) = C\text{sign}(z)$, where $C$ is a constant) to make the width of the wave function of the localized fermion (at the point $z = 0$ or $z = L$ depending on the sign and the value of $C$) as small as necessary. Meanwhile, the profile of the Higgs field does not depend on the form of the function $F(z)$, so even for an extremely narrow wave function of a localized mode (which taken squared can be even approximated by the delta-function for calculations) the “right” profile of the Higgs field, which does not lead to fourth-order differential equations, should still have the form (18). The latter poses a question whether there exists a field-theoretical mechanism of fermion localization, leaving more freedom for the choice of a vacuum profile of the Higgs field in the
extra dimension.

It should be noted that the only obvious exception is the model with the flat five-dimensional background metric like the one proposed in [3], for which the four-dimensional SM from a five-dimensional theory can be constructed without unnatural fine-tunings and restrictions. In such a case the vacuum solution for the Higgs field must be just a constant, which admits a variety of scalar field potentials including the standard Higgs potential (but leaving unsolved the problem of the stabilization of the extra dimension size).

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Appendix A

Let us take equation (32) and substitute \( \Psi_1(x,z) = \psi_L(x)f_L(z) \) with \( \square \psi_L + \mu^2 \psi_L = 0 \) and \( \gamma^5 \psi_L = \psi_L \) into it. We get

\[
(\mu^2 - \tilde{h}^2 \tilde{v}^2) f_L + e^\sigma (\partial_5 + 2\sigma') e^\sigma (\partial_5 + 2\sigma') f_L + e^\sigma \partial_5 (e^\sigma F) f_L - e^{2\sigma} F^2 f_L = 0. \tag{57}
\]

Multiplying this equation by \( e^{3\sigma} f_L \), integrating over the coordinate of the extra dimension \( z \) and performing integration by parts in two terms, we arrive to the following equality:

\[
(\mu^2 - \tilde{h}^2 \tilde{v}^2) \int dz e^{3\sigma} f_L^2 = \int dz e^{5\sigma} (f_L' + 2\sigma' f_L + F f_L)^2. \tag{58}
\]

Since both integrals are nonnegative, we get \( \mu^2 - \tilde{h}^2 \tilde{v}^2 \geq 0 \), which means that the lowest mode indeed has mass \( \tilde{h} \). A completely analogous procedure can be performed for the other substitution \( \Psi_1(x,z) = \psi_R(x)f_R(z) \) with \( \square \psi_R + \mu^2 \psi_R = 0 \) and \( \gamma^5 \psi_R = -\psi_R \), as well as for the field \( \Psi_2 \).

Appendix B

Let us consider a four-dimensional scalar field theory with the action of the form

\[
S = \int d^4x \left( \frac{1}{2} \partial^\mu \Phi_1 \partial_\mu \Phi_1 + \frac{1}{2} \partial^\mu \Phi_2 \partial_\mu \Phi_2 - \frac{M^2}{2} (\Phi_1 + \Phi_2)^2 \right). \tag{59}
\]

The equations of motion, following from this action, take the form

\[
\square \Phi_1 + M^2 (\Phi_1 + \Phi_2) = 0, \tag{60}
\]

\[
\square \Phi_2 + M^2 (\Phi_1 + \Phi_2) = 0. \tag{61}
\]

It is not difficult to obtain the equation for, say, the field \( \Phi_1 \):

\[
\square^2 \Phi_1 + 2M^2 \square \Phi_1 = 0. \tag{62}
\]
Formally, it is an equation of motions with higher derivatives. Meanwhile, one can simply add and subtract equations (60) and (61) to get

\[
\Box (\Phi_1 + \Phi_2) + 2M^2 (\Phi_1 + \Phi_2) = 0, \quad (63)
\]
\[
\Box (\Phi_1 - \Phi_2) = 0. \quad (64)
\]

From the latter equations it is clear that the physical degrees of freedom of the theory are described by the fields \( \varphi_1 = \frac{1}{\sqrt{2}} (\Phi_1 + \Phi_2) \) and \( \varphi_2 = \frac{1}{\sqrt{2}} (\Phi_1 - \Phi_2) \) having the four-dimensional masses \( \sqrt{2}M \) and 0 respectively. Substituting \( \varphi_1 \) and \( \varphi_2 \) into action (59), we get the standard action without any pathologies. Thus, though the model under consideration formally leads to the equation of motion (62), which contains higher derivatives, the resulting theory is devoid of any pathologies. Of course, the mass spectrum, coming from (62), also consists of the four-dimensional masses \( \sqrt{2}M \) and 0.

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