Geometrical Model for Non-Zero $\theta_{13}$

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Abstract

Based on Friedberg and Lee’s geometric picture by which the tribimaximal Pontecorvo-Maki-Nakawaga-Sakata leptonic mixing matrix is constructed, namely, corresponding mixing angles correspond to the geometric angles among the sides of a cube. We suggest that the three realistic mixing angles, which slightly deviate from the values determined for the cube, are due to a viable deformation from the perfectly cubic shape. Taking the best-fitted results of $\theta_{12}$ and $\theta_{23}$ as inputs, we determine the central value of $\sin^2 2\theta_{13}$ should be 0.0238, with a relatively large error tolerance; this value lies in the range of measurement precision of the Daya Bay experiment and is consistent with recent results from the T2K Collaboration.

PACS: 14.60.Pq Neutrino mass and mixing
I. INTRODUCTION

Neutrino oscillation observations have revealed evidence that neutrinos are massive. Neutrinos are produced via weak interaction as flavor eigenstates $\nu_f = (\nu_e, \nu_\mu, \nu_\tau)$ and can be written in the mass basis $\nu_m = (\nu_1, \nu_2, \nu_3)$, which are really the physical states. These two bases are related by a unitary matrix $U_\nu$, i.e., $\nu_f = U_\nu \nu_m$. The mixing in the lepton sector is named as the Pontecorvo-Maki-Nakawaga-Sakata (PMNS) matrix, which can account for the currently available data on the observation of solar, atmospheric neutrino oscillations and the reactor and accelerator neutrino experiments. In the standard model, the weak charged currents are

$$J^\mu = \bar{l}_i \gamma^\mu (1 - \gamma_5) (U_\nu^\dagger U_\nu)_{ij} \nu_j,$$

where $i, j = 1, 2, 3$ and correspond to physical particles. The mixing matrix

$$U_{PMNS} = U_\nu^\dagger U_\nu,$$

is a $3 \times 3$ unitary matrix and can be parameterized by three mixing angels $\theta_{12}, \theta_{23},$ and $\theta_{13},$ and one $CP$ phase $\delta$.

$$U_{PMNS} = \begin{pmatrix}
 c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i \delta} \\
 -s_{12} c_{23} - c_{12} s_{23} s_{13} e^{i \delta} & c_{12} c_{23} - s_{12} s_{23} s_{13} e^{i \delta} & s_{23} c_{13} \\
 s_{12} s_{23} - c_{12} c_{23} s_{13} e^{i \delta} & -c_{12} s_{23} - s_{12} c_{23} s_{13} e^{i \delta} & c_{23} c_{13}
\end{pmatrix},$$

where $c_{ij} \equiv \cos \theta_{ij}, s_{ij} \equiv \sin \theta_{ij}$. If neutrinos are Majorana particles, there would be an additional diagonal matrix $\text{diag}(e^{i \alpha_1/2}, e^{i \alpha_2/2}, 1)$ multiplied to the above $U_{PMNS}$ matrix, which is not relevant for neutrino oscillations. The parametrization Eq. (3) can be rewritten as a product of three rotations $R_{ij}$ in the $ij$ plane through angles $\theta_{ij}$ and a diagonal $CP$ phase matrix $U_\delta = \text{diag}(e^{i \delta/2}, 1, e^{-i \delta/2}),$

$$U_{PMNS} = R_{23}(\theta_{23}) U_\delta^\dagger R_{13}(\theta_{13}) U_{12}(\theta_{12}),$$

with

$$R_{23} = \begin{pmatrix}
 1 & 0 & 0 \\
 0 & c_{23} & s_{23} \\
 0 & -s_{23} & c_{23}
\end{pmatrix}, R_{13} = \begin{pmatrix}
 c_{13} & 0 & s_{13} \\
 0 & 1 & 0 \\
 -s_{13} & 0 & c_{13}
\end{pmatrix}, R_{12} = \begin{pmatrix}
 c_{12} & s_{12} & 0 \\
 -s_{12} & c_{12} & 0 \\
 0 & 0 & 1
\end{pmatrix}.$$
There have been numerous phenomenological Ansätze for the entries of $U_{PMNS}$, for example, the democratic [4], the bimaximal [5], and the tribimaximal Ansätze [6]. Among them, the tribimaximal mixing is closer to the experimentally observed mixing patterns, and the matrix is given by

$$U_{tribi} = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix},$$

(6)

which suggests $\theta_{12} = \sin^{-1}(1/\sqrt{3})$, $\theta_{23} = \pi/4$, $\theta_{13} = 0$. As noted, the $CP$ phase $e^{i\delta}$ is always associated with $s_{13}$ [Eq. (3)]; thus, null $\theta_{13}$ would imply that one cannot observe $CP$ violation at lepton sector in the framework of the standard model even though $\delta$ is not zero. Obviously, there is no priori that the $CP$ violation should appear at the lepton sector, but only nonzero $\theta_{13}$ can intrigue an enthusiasm to explore $CP$ violation at the lepton sector. Once the $\theta_{13}$ is determined to be nonzero as the T2K experiment and our theoretical prediction made in this work suggest, the next step would be searching for $CP$ violation at the lepton sector.

Indeed, the tribimaximal symmetry is well manifested by the data. A rigorous symmetry would demand $\theta_{13}$ to be zero; however, it is not the whole story because this elegant symmetry is to be broken, and a nonzero $\theta_{13}$ is expected. The question is if it is not zero, what is its size, which is the main concern of the recent studies.

The unbroken tribimaximal matrix Eq. (6) can be further written as a sequential product of two independent rotations on 12 and 23 planes:

$$R_{23}(\pi/4) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \quad R_{12}(\sin^{-1}(1/\sqrt{3})) = \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{3} & 0 \\ -1/\sqrt{3} & 2/\sqrt{6} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

(7)

and $R_{13}$ becomes a $3 \times 3$ unit matrix, i.e., $U_{tribi} = R_{23}(\pi/4)R_{12}(\sin^{-1}(1/\sqrt{3}))$. Friedberg and Lee [7] propose a geometrical interpretation for the tribimaximal symmetry as shown in Fig. 1. For readers’ convenience, let us briefly introduce Friedberg and Lee’s geometrical model and their conventions [7]. The charged leptons in the basis $L_e = (S_e, S_\mu, S_\tau)^T$ correspond to the three mutually perpendicular sides of a cube, and the neutrino basis $L_\nu = (S_{\nu_1}, S_{\nu_2}, S_{\nu_3})^T$ corresponds to another coordinate system (see Fig. 1). These two
coordinate systems are related to each other by rotations. One can perform two independent rotations to associate them. $R^l_{23}(\pi/4)$ and $R_{12}(\sin^{-1}(1/\sqrt{3}))$ transform the two independent bases into a common one. These practical operations are described below. $R^l_{23}(\pi/4)$ mixes the second and third components of the basis $L_c$ and keeps the first one invariant to get a new basis $(S_1, S_2, S_3)^T$, while $R_{12}(\sin^{-1}(1/\sqrt{3}))$ mixes the first and second components of $L_n$, retaining the third one invariant to reach the same basis $(S_1, S_2, S_3)^T$. The mathematical expressions for relating $(S_1, S_2, S_3)^T$ with the charged lepton basis $(S_e, S_\mu, S_\tau)^T$ and neutrino basis $(S_{\nu_1}, S_{\nu_2}, S_{\nu_3})^T$ are shown as follows:

$$
\begin{pmatrix}
S_1 \\
S_2 \\
S_3
\end{pmatrix}
= R^l_{23}(\pi/4)
\begin{pmatrix}
S_e \\
S_\mu \\
S_\tau
\end{pmatrix},

\begin{pmatrix}
S_1 \\
S_2 \\
S_3
\end{pmatrix}
= R_{12}(\sin^{-1}(1/\sqrt{3}))
\begin{pmatrix}
S_{\nu_1} \\
S_{\nu_2} \\
S_{\nu_3}
\end{pmatrix}.
$$

(8)

Following the convention given in Ref. [7], when we discuss the geometry structure, we abbreviate the sides $S_l$ $(S_{\nu_i})$ as $l$ $(\nu_i)$ without causing any confusion. Here $S_{e,\mu,\tau}$ and $S_{\nu_1,\nu_2,\nu_3}$ just refer to the corresponding geometrical quantities marked in Fig. 1 and are by no means the physical states.

Comparing $U_{\text{tribi}} = R_{23}(\pi/4)R_{12}(\sin^{-1}(1/\sqrt{3}))$ with $U_{PMNS} = U_l^T U_\nu$, it appears that the two rotations $R^l_{23}(\pi/4)$ and $R_{12}(\sin^{-1}(1/\sqrt{3}))$ correspond to the mixing matrices for the
charged leptons and neutrinos, respectively. It is noted that in Eq. (8), we only concern the mixing parts; thus, inserting $\gamma^0\gamma^\mu(1 - \gamma^5)$ between $((S_1, S_2, S_3)^T)^\dagger$ and $(S_1, S_2, S_3)^T$ which is irrelevant to our geometrical settings, we just derive the Lagrangian of weak interaction.

We also would like to point out that in Eq. (8), the high symmetry is assumed, and all quantities are indeed corresponding to the zeroth order ones [7], and then later when we introduce a deformation of the cube to break the tribimaximal symmetry, the concerned quantities would turn into the physical ones.

Concretely, in Fig. 1, sides $OX$, $OY$, and $OZ$ represent $e$, $\mu$, and $\tau$; and $\nu_1$, $\nu_2$, and $\nu_3$ correspond to $OX'$, $OB$, and $OZ'$, respectively. The line-$OZ'$ resides on the plane $OZAY$ and spans an angle of $\pi/4$ with respect to the $OZ$ axis, whereas line $OX'$, line $OZ'$, and line $OB$ compose three-dimensional mutually perpendicular coordinate axes, and according the right-hand rule, we have an $OB - OZ' - OX'$ system. The angle spanned between $OX'$ and $OX$ is $\theta_{12}$. Therefore, the two rectangular coordinate systems transform from each other by two rotations about the axes $OX$ and $OZ'$, respectively. The right-handed rotation $R^1_{23}(\pi/4)$ about the $OX$ axis brings $\mu$ to $OA$ and $\tau$ to $\nu_3$, and a second right-handed rotation $R_{12}(\sin^{-1}(1/\sqrt{3}))$, with $\theta_{12} = \sin^{-1}(1/\sqrt{3})$, turns $\nu_1$ into $e$ and $\nu_2$ into $OA$. Then, after performing the two successive operations, the basis $(S_1, S_2, S_3)^T$ shown above can be directly read out as $(e, OA, \nu_3)^T$.

Although the tribimaximal mixing Ansatz is close to the experimental data and exhibits a striking symmetry, it is not the exact form of the PMNS matrix. Moreover, this symmetry demands $\theta_{13}$ to be zero. If the tribimaximal symmetry is not exact, with the angles $\theta_{23}$ and $\theta_{12}$ obviously deviate from the values determined by the symmetry, one has sufficient reason to believe that $\theta_{13}$ should not be zero. In fact, the previous measurements set a lower bound for $\theta_{13}$ as $\sin^2 2\theta_{13} < 0.15$ [3], and will be more precisely measured at the upcoming reactor experiments Daya Bay [8] and Double Chooz [9].

It would be interesting to investigate how to break the tribimaximal symmetry from a theoretical aspect. Friedberg and Lee suggest to break the symmetry from the charged lepton side [10], whereas He and his collaborators break the symmetry from the neutrino sector [11]. Since the whole mixing matrix is a product of the two unitary matrices that, respectively, diagonalize the charged lepton and neutrino mass matrices as $U_{PMNS} = U_l^\dagger U_\nu$, breaking from either side is just like climbing up Mount Everest from the south or north side as Lee comments [12]. Their schemes to break the symmetry are algebraic.
Instead, in this work, we propose to break the symmetry based on Friedberg and Lee’s geometrical picture. Namely, we let the cube be slightly deformed and the nonzero $\theta_{13}$ value would emerge. Concretely, by deforming the geometric representation of the tribimaximal mixing, the angles would deviate from their ideal values; by fitting them to the data, we determine the deformation scale of the cube, and then by the new geometric shape $\theta_{13}$ is no longer zero.

The work is organized as follows. In Sec. II, we present our geometrical model of deforming the cube to get the $\theta_{13}$ as a function of the other two mixing angles. Then, in Sec. III, we present our numerical results. The last section is devoted to our conclusion and some discussions.

II. THE DEFORMED CUBE MODEL

It is noticeable that the angle between lines $OA$ and $OB$ and that between $OY$ and $OA$ in the cube are precisely the two mixing angles of the tribimaximal matrix $\theta_{12}$ and $\theta_{23}$, respectively. For a perfect symmetry, which corresponds to a complete cube, we have $\theta_{12} = \sin^{-1}(1/\sqrt{3})$ and $\theta_{23} = \pi/4$, which are determined by the geometry. It is then viable that a deformation would lead to the more realistic form of the PMNS matrix, and, thus, the
nonzero $\theta_{13}$ would emerge. After this deformation, $\theta_{12}$ and $\theta_{23}$ are not the values given above anymore, but dependent on the form of the deformation. A cube is a kind of polyhedron with high symmetry described by a certain group, so a deformation of a cube should be regarded as a symmetry breaking.

Now, let us demonstrate how to deform the above cube. For choosing the deformation scheme, we set three principles:

- There are three rotation axes for a cube as presented in Fig. 2, i.e., $EE'$, $FF'$, and $GG'$. Apparently, the axis $GG'$ is related with the mixing angle $\theta_{12}$ ($\angle AOB$), and the axis $FF'$, which is parallel to the side OA, is related to $\theta_{23}$ ($\angle AOE$). Then, the rest symmetry axis $EE'$ may be related to the zero $\theta_{13}$ in the tribimaximal mixing. Thus, after the supposed deformation, the three symmetries would be broken, and the value of the deformation angle is related to $\theta_{13}$. For simplicity, we just choose the deformation angle to be $\theta_{13}$.

- For the tribimaximal mixing, $\theta_{23} = \pi/4$ and $\theta_{13} = 0$, there exists the $\mu - \tau$ symmetry [13]. The global fit [14] gives $\theta_{23} = 42.8^\circ$, which apparently breaks the $\mu - \tau$ symmetry. Thus, in the deformation, $\theta_{23}$ ($\angle AOE$) should be changed from $\pi/4$ to some values in order to break the $\mu - \tau$ symmetry.

- In Ref. [14], the global fits of $\theta_{12}$ and $\theta_{23}$ are $34.4^\circ$ and $42.8^\circ$, respectively. Thus, for the deformation, $\theta_{12}$ ($\angle AOB$) and $\theta_{23}$ ($\angle AOE$) should be changed toward smaller values than $\sin^{-1}(1/\sqrt{3})$ and $\pi/4$, respectively.

Considering above principles, the simplest and most direct scheme to deform the cube is to slide the bottom face parallel to the top face, and a small angle would emerge, and this angle is identified as $\theta_{13}$. The length of each side is unchanged during the slide. This operation is explicitly illustrated in Fig. 3. With the parallel slide, the bottom face becomes $EFGH$. To be consistent with Friedberg and Lee’s picture and the principles we proposed above, we identify $\theta_{12} = \angle FAG$, $\theta_{23} = \angle BAF$, and $\theta_{13} = \angle EAE_0$. $E_0$ is the point of intersection between side $AE'$ and plane $EFGH$. $AE_1$ and $E_0E_1$ are perpendicular to the diagonal line $EG$.

Setting $\angle E_0EG \equiv \alpha$ and in the rectangular triangle $\text{Rt} \triangle AE_1G$, one has

$$AG^2 = AE_1^2 + E_1G^2 = \cos^2 \theta_{13} + \sin^2 \theta_{13} \sin^2 \alpha + (\sqrt{2} + \sin \theta_{13} \cos \alpha)^2.$$  \hspace{1cm} (9)
FIG. 3: (color online) The sketch of the shift of the bottom face relative to the top face.

In \( \triangle AEF \),

\[ AG^2 = 1 + 4 \cos^2 \theta_{23} - 4 \cos \theta_{23} \cos (\pi - \theta_{12} - \sin^{-1}(2 \sin \theta_{12} \cos \theta_{23})) \, , \]  

(10)

and in \( \triangle E_0E_F \),

\[ E_0F^2 = E_0E^2 + EF^2 - 2E_0E \cdot EF \cos \angle E_0EF \]

(11)

\[ 4 \cos^2 \theta_{23} - \cos^2 \theta_{13} = \sin^2 \theta_{13} + 1 - 2 \sin \theta_{13} \cos(\alpha + \frac{\pi}{4}) \]  

(12)

The geometrical relationship of the sides and angles in the deformed cube would determine Eq. (9), Eq. (10), and Eq. (12). From these equations, we can get an analytical expression of \( \theta_{13} \), with respect to the other two mixing angles \( \theta_{12} \) and \( \theta_{23} \) as

\[ \sin^2 \theta_{13} = 4 \cos^4 \theta_{23} - 4 \cos^2 \theta_{23} + 4 \cos^2 \theta_{23} \cos^2 (\theta_{12} + \sin^{-1}(2 \sin \theta_{12} \cos \theta_{23})) + 1 \]  

(13)

As we have mentioned above, a cube is a highly symmetric polyhedron that could be represented by a global \( S_4 \) group \[15\]. This group has 24 elements classified in five conjugate classes. As shown in Fig. 2, a cube has three kinds of rotation axes, \( h = 2 \), \( h = 3 \), and \( h = 4 \), corresponding to \( FF' \), \( GG' \), and \( EE' \), respectively. All the rotation axes in the same \( h \) are equivalent.
FIG. 4: (color online) Comparison of our result (dashed line) with the Daya Bay expected sensitivity limit to $\sin^2 2\theta_{13}$ as a function of running time.

It is notable that the angle between the axes of $h = 2$ and $h = 4$ is $\pi/4$, and the angle between the axes of $h = 2$ and $h = 3$ is $\sin^{-1}(\sqrt{1/3})$. In other words, the two angles are exactly that in the tribimaximal form of the PMNS matrix. Another angle corresponding to $\theta_{13}$ must be an angle between $h = 4$ and $h = 4$ itself, so $\theta_{13} = 0$.

With the deformation, the symmetry of the cube is broken, and $\theta_{13}$ acquires a nonzero value. We can then view $\theta_{13}$ as the parameter representing the deviation from the cubic symmetry. It is then viable to define $\theta_{13}$ as the angle between the ”new” $h = 4$ axis and the ”old” one.

III. NUMERICAL RESULTS

Ref. [14] presents the updated global fit to the three-generation neutrino mixing:

$$\theta_{12} = 34.4 \pm 1.0^\circ, \theta_{23} = 42.8^{+4.7}_{-2.5}^\circ.$$ (14)

Using the data as inputs, we obtain the numerical value of $\theta_{13}$:

$$\sin^2 2\theta_{13} = 0.0238, \theta_{13} = 4.44^\circ.$$ (15)
FIG. 5: (color online) Comparison of our result (dashed line) with the Double Chooz expected sensitivity limit to $\sin^2 2\theta_{13}$ as a function of running time.

The errors of the fit would cause a theoretical uncertainty to $\theta_{13}$:

$$\sin^2 2\theta_{13} = 0.0238^{+0.0762}_{-0.0238}. \hspace{1cm} (16)$$

The errors are rather large, and, in fact, to make $\sin^2 2\theta_{13} \geq 0$, the lower bound shown in the above expression is set. This expression indicates that our prediction on $\theta_{13}$ is somehow sensitive to the input data and that in order to get more precise values of $\theta_{13}$, more precise values of the input are needed.

Two reactor neutrino experiments, Daya Bay [8] and Double Chooz [9], aiming to directly measure $\theta_{13}$ are expected to reach a very high precision. We illustrate a relation of the expected sensitivity of the Daya Bay and Double Chooz as a function of the running time in Figs. 4 and 5, respectively, where we mark the central value of $\sin^2 2\theta_{13}$ calculated in this work.

Apparently, because of the high precision, the $\sin^2 2\theta_{13}$ value from our model would be probed at the first run of the Daya Bay experiment.

We show $\sin^2 2\theta_{13}$ as a function of $\theta_{12}$ for $\theta_{23} = 42.8^\circ$ in Fig. 6 and the dependence on $\theta_{23}$ for $\theta_{12} = 34.4^\circ$ in Fig. 7. Particle Data Group (PDG) presents an upper bound of $\theta_{13}$ as $\sin^2 2\theta_{13} < 0.15$ at CL=90\% [3], which is also marked in Figs. 6 and 7. From Figs. 6 and
FIG. 6: (color online) $\sin^2 2\theta_{13}$ as a function of $\theta_{12}$ for $\theta_{23} = 42.8^\circ$ from the toy model (solid line). The limit $\sin^2 2\theta_{13} < 0.15$, CL=90% from PDG is plotted as the dashed line.

we can see that the theoretically predicted value of $\sin^2 2\theta_{13}$ is sensitive to both $\theta_{12}$ and $\theta_{23}$. By the updated data, $\theta_{12}$ and $\theta_{23}$ are constrained within the range $(31.9^\circ \sim 36.5^\circ)$ and $(40.2^\circ \sim 48.3^\circ)$, respectively.

IV. DISCUSSIONS AND CONCLUSIONS

$\theta_{23} = \pi/4$ and $\theta_{13} = 0$ imply the so-called $\mu - \tau$ symmetry embedding in the neutrino mass matrix, i.e., the mass matrix in the flavor basis has an obvious $\nu_\mu - \nu_\tau$ permutation symmetry. This leads to the mass matrix with the form

$$M = \begin{pmatrix} A & B & B \\ B & C & D \\ B & D & C \end{pmatrix}. \quad (17)$$

In Ref. [11], the authors discussed the soft breaking of the $\mu - \tau$ symmetry that arises from the Majorana mass term of the heavy right-handed neutrinos in the minimal seesaw model. From their $\mu - \tau$ symmetry breaking model, they derived a relation among the mixing angles
FIG. 7: (color online) $\sin^2 2\theta_{13}$ as a function of $\theta_{23}$ for $\theta_{12} = 34.4^\circ$ from the toy model (solid line). The limit $\sin^2 2\theta_{13} < 0.15$, CL=90% from PDG is plotted as the dashed line.

and Dirac CP phase

$$\theta_{23} - \frac{\pi}{4} = -\theta_{13} \cot \theta_{12} \cos \delta.$$  \hspace{1cm} (18)

For the case that the Dirac CP phase $\delta = 0$ and substituting the experimental fits $\theta_{12} = 34.4^\circ$ and $\theta_{23} = 42.8^\circ$ into Eq. (18), we obtain the value of $\theta_{13}$ as

$$\sin^2 2\theta_{13} = 0.00276, \theta_{13} = 1.51^\circ.$$ \hspace{1cm} (19)

Instead, in a parallel work, Friedberg and Lee\hspace{1cm} [10] suggested that one can break the $\mu - \tau$ symmetry at the charged lepton side in terms of a perturbation method, and they also showed that the breaking may lead to a nonzero $\theta_{13}$.

In this work, by deforming the cube that corresponds to a full tribimaximal form of the mixing matrix according the proposed principles, we derive the analytic relation among the three lepton mixing angles, and, taking the experimental data as inputs, we deduce the value of unknown mixing angle $\theta_{13}$. The result gives $\sin^2 2\theta_{13} = 0.0238$, i.e., $\theta_{13} = 4.44^\circ$. As noticed, our theoretical prediction favors smaller $\theta_{13}$. As $\theta_{13}$ is to be measured at the Double Chooz and Daya Bay experiments, our result indicates that in the future, there would be a great opportunity to fix the mysterious $\theta_{13}$. The recent measurement of the
T2K collaboration indicates that $\sin^2 2\theta_{13}$ falls in a rather wide range of $0.03(0.04) < \sin^2 2\theta_{13} < 0.28(0.34)$, and the central value of our theoretical prediction is consistent with the lower bound set by the collaboration, and the error range is comparable. This value also does not contradict the new measurement by Main Injector Neutrino Oscillation Search.

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