Double- and simple-layer potentials for generalized singular elliptic equations and their applications to the solving the Dirichlet problem

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Abstract. Potentials play an important role in solving boundary value problems for elliptic equations. In the middle of the last century, a potential theory was constructed for a two-dimensional elliptic equation with one singular coefficient. In the study of potentials, the properties of the fundamental solutions of the given equation are essentially and fruitfully used. At the present time, fundamental solutions of a multidimensional elliptic equation with several singular coefficients are already known. In this paper, we investigate the double- and simple-layer potentials for this kind of elliptic equations. Results from potential theory allow us to represent the solution of the boundary value problems in integral equation form. By using a decomposition formula and other identities for the Lauricella’s hypergeometric function in many variables, we prove limiting theorems and derive integral equations concerning a densities of the double- and simple-layer potentials. The obtained results are applied to find an explicit solution of the Dirichlet problem for the generalized singular elliptic equation in some part of the multidimensional ball.

Keywords: Multidimensional elliptic equations with several singular coefficients; Fundamental solutions; Lauricella’s hypergeometric function; Decomposition formula; Potential theory; Dirichlet problem; Green’s function;

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1 Introduction

Potential theory has played a paramount role in both analysis and computation for boundary value problems for elliptic equations. Numerous applications can be found in solid mechanics, fluid mechanics, elastic dynamics, electro-magnetics, and acoustics. Results from potential theory allow us to represent boundary value problems in integral equation form. For problems with known Green’s functions, an integral equation formulation leads to powerful numerical approximation schemes.

The double- and simple-layer potentials play an important role in solving boundary value problems for elliptic equations. For example, the representation of the solution of the Dirichlet problem for the Laplace equation is sought as a double-layer potential with unknown density and an application of certain property leads to a Fredholm equation of the second kind for determining the density function (see [22] and [33]).

Interest in the potential theory for the singular elliptic equation has increased significantly after Gellerstedt’s papers [16, 17]. In works [14] and [36], the potential theory was exposed for the following simplest degenerating elliptic equation

\[ \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{2\alpha_1}{x_1} \frac{\partial u}{\partial x_1} = 0, \quad 0 < 2\alpha_1 < 1 \]  

(1.1)

in the domain, which is bounded in the half-plane \( x_1 > 0 \). An exposition of the results on the potential theory for the two-dimensional singular elliptic equation (1.1) together with references
to the original literature are to be found in the monograph by Smirnov [38], which is the standard
work on the subject. This work also contains an extensive bibliography of all relevant papers up
to 1966; the list of references given in the present work is largely supplementary to Smirnov’s
bibliography. Various interesting problems associated with the equation (1.1) were studied by
many authors (see [1, 2, 3, 18, 19, 20, 21, 26, 30, 35]).

In his work [23] Hasanov found fundamental solutions of the generalized bi-axially symmetric
Helmholtz equation

\[
\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + 2\alpha_1 \frac{\partial u}{\partial x_1} + 2\alpha_2 \frac{\partial u}{\partial x_2} + \lambda u = 0,
\]

(1.2)
in the domain, which is bounded in the quarter-plane \(x_1 > 0, x_2 > 0\), where \(\alpha_1, \alpha_2\) and \(\lambda\)
are real numbers \((0 < 2\alpha_1, 2\alpha_2 < 1)\). When \(\lambda = 0\), this equation is known as the equation
of the generalized bi-axially symmetric potential theory whose name is due to Weinsein who
first considered fractional dimensional space in potential theory [41, 42]. Using Hasanov’s results
authors of the works [5, 10, 11, 39] constructed the double-layer potential theory for the two-
dimensional equation (1.2) in the case when \(\lambda = 0\).

Relatively few papers have been devoted to the potential theory and boundary value problems
for the singular elliptic equations when the dimension exceeds two [27, 31, 34].

In the present work we shall give the potential theory for the following generalized singular
elliptic equation

\[
H_{\alpha}^{(m,n)}(u) \equiv \sum_{i=1}^{m} u_{x_i x_i} + \sum_{k=1}^{n} \frac{2\alpha_k}{x_k} u_{x_k} = 0,
\]

(1.3)
where \(m\) is a dimensional of the Euclidean space \(R_m\), \(n\) is a number of the singular coefficients of
elliptic equation \((m \geq 2, 0 < n \leq m)\); \(\alpha = (\alpha_1, ..., \alpha_n)\) and \(\alpha_k\) are real numbers with \(0 < 2\alpha_k < 1, k \in K\) \((K = \{1, ..., n\})\), and apply this theory to the finding a regular solution of the Dirichlet
problem for equation (1.3) in the domain, which is bounded in the subset of the Euclidean space
\(R_m^{n+} = \{x = (x_1, ..., x_m) \in R_m : x_1 > 0, ..., x_n > 0\}\).

By a regular solution of the equation (1.3) is meant a function that has continuous derivatives
up to the second order (inclusive) in some domain and satisfies the equation (1.3) at all points
of this domain.

Naturally, in solving the problem posed for the equation (1.3), an important role is played
some fundamental solution of this equation. Fundamental solutions of equation (1.3) were
constructed recently [13]. In fact, the fundamental solutions of generalized singular elliptic
equation can be expressed in terms of Lauricella’s hypergeometric function in many variables
\(F_A^{(n)}(a, b_1, ..., b_n; c_1, ..., c_n; y_1, ..., y_n)\) (see, for details, [4]) and it becomes clear that the number
of variables of a hypergeometric function is equal to the number of singular coefficients of
the equation (1.3). Therefore, in order to facilitate the constructing process of the potential
theory for equation (1.3), we preliminary study some necessary properties of this Lauricella’s
hypergeometric function in many variables.

2 Preliminaries

With a view to introducing formally the Gaussian hypergeometric function and its generalization
(that is, Lauricella hypergeometric function in several variables), we recall here some definitions
and identities involving Pochhammer’s symbol \((\lambda)_p\), and the Gamma function \(\Gamma(z)\) \((\lambda\) and \(z\) are
complex numbers) defined by
\[
\Gamma(z) = \begin{cases} 
\int_0^\infty t^{z-1}e^{-t}dt, & \text{Re}(z) > 0, \\
\frac{\Gamma(z+1)}{z}, & \text{Re}(z) < 0; \ z \neq -1, -2, -3, \ldots.
\end{cases}
\]

Throughout this work we shall find it convenient to employ the Pochhammer symbol \((\lambda)_p\) defined by
\[
(\lambda)_p = \begin{cases} 
1, & \text{if } p = 0, \\
\lambda(\lambda+1)\cdots(\lambda+p-1), & \text{if } p \in \mathbb{N},
\end{cases}
\]
where \(\mathbb{N} := \{1, 2, 3, \ldots\}\). Since \((1)_p = p!\), \((\lambda)_p\) may be looked upon as a generalization of the elementary factorial; hence the symbol \((\lambda)_p\) is also referred to as the factorial function.

The Gaussian hypergeometric function is defined in the form (see, for details, \([9, 40]\))
\[
F(a, b; c; z) \equiv F \left[ \begin{array}{c} a, b; \\
c; \\
z \end{array} \right] := \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m m!} z^m, \ c \neq 0, -1, -2, \ldots, \quad (2.1)
\]
where \(a, b, c\) and \(z\) are complex numbers.

It is easily seen that the hypergeometric function \(F(a, b; c; z)\) in (2.1) converges absolutely within the circle, that is, when \(|z| < 1\), provided that the denominator parameter \(c\) is neither zero nor a negative integer. Further tests readily show that the hypergeometric function in (2.1), when \(|z| = 1\) (that is, on the unit circle), is absolutely convergent if \(\text{Re}(c - a - b) > 0\).

The linear transformation of the hypergeometric function, known as Euler’s transformation, may be recalled here as follows \([9]\):
\[
F(a, b; c; z) = (1 - z)^{-b} F \left( c - a, b; c; \frac{z}{z-1} \right). \quad (2.2)
\]

The Lauricella’s hypergeometric function in \(n \in \mathbb{N}\) variables \(y := (y_1, \ldots, y_n)\) has a form \([4]\)
\[
F^{(n)}_A \left[ \begin{array}{c} a_1, \ldots, a_n; \\
c_1, \ldots, c_n; \\
y_1, \ldots, y_n \end{array} \right] \equiv F^{(n)}_A \left[ \begin{array}{c} a, b_1, \ldots, b_n; \\
c_1, \ldots, c_n; \\
y_1, \ldots, y_n \end{array} \right] = \sum_{m_1, \ldots, m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n} (b_1)_{m_1} \cdots (b_n)_{m_n} (c_1)_{m_1} \cdots (c_n)_{m_n} y_1^{m_1} \cdots y_n^{m_n}}{m_1! \cdots m_n!} y_1 \cdots y_n, \quad (2.3)
\]
where \(a, b_k, c_k\) and \(y_k\) are complex numbers and \(c_k \neq 0, -1, -2, \ldots, k \in K\).

We give the elementary relations for \(F^{(n)}_A\) necessary in this study:
\[
\frac{\partial}{\partial y_k} F^{(n)}_A \left[ \begin{array}{c} a, b_1, \ldots, b_n; \\
c_1, \ldots, c_n; \\
y \end{array} \right] = \frac{ab_k}{c_k} F^{(n)}_A \left[ \begin{array}{c} a+1, b_1, \ldots, b_{k-1}, b_k+1, b_{k+1}, \ldots, b_n; \\
c_1, \ldots, c_{k-1}, c_k+1, c_{k+1}, \ldots, c_n; \\
y \end{array} \right], \quad (2.4)
\]
where \( F \) of hypergeometric functions in several (three and more) variables. For example, the Lauricella’s Chaundy method, Hasanov and Srivastava [24, 25] found decomposition formulas for a whole class hypergeometric functions in series of simpler hypergeometric functions. Using the Burchnall-

systematically presented a number of expansions and decomposition formulas for some double which would express the multivariable hypergeometric function in terms of products of several

[13]. The following decomposition formula holds true

\[
\sum_{k=1}^{n} \frac{b_k}{c_k} y_k F_A^{(n)} \left[ \begin{array}{c}
a + 1, b_1, \ldots, b_{k-1}, b_k + 1, b_{k+1}, \ldots, b_n; \\
c_1, \ldots, c_{k-1}, c_k + 1, c_{k+1}, \ldots, c_n;
\end{array} \right] y = F_A^{(n)} \left[ \begin{array}{c}
a + 1, b_1, \ldots, b_n; \\
c_1, \ldots, c_n;
\end{array} \right] - F_A^{(n)} \left[ \begin{array}{c}
a, b_1, \ldots, b_n; \\
c_1, \ldots, c_n;
\end{array} \right]
\]

(2.5)

\[
\frac{ab_n}{(c_k - 1) c_k} y_k F_A^{(n)} \left[ \begin{array}{c}
a + 1, b_1, \ldots, b_{k-1}, b_k + 1, b_{k+1}, \ldots, b_n; \\
c_1, \ldots, c_{k-1}, c_k + 1, c_{k+1}, \ldots, c_n;
\end{array} \right] y = F_A^{(n)} \left[ \begin{array}{c}
a, b_1, \ldots, b_n; \\
c_1, \ldots, c_n;
\end{array} \right] - F_A^{(n)} \left[ \begin{array}{c}
a, b_1, \ldots, b_n; \\
c_1, \ldots, c_n;
\end{array} \right].
\]

(2.6)

Relations (2.4)–(2.6) can be proved in two ways: by comparing coefficients of equal powers of \( y_1, \ldots, y_n \) on both sides or mathematical induction.

For a given multiple hypergeometric function, it is useful to find a decomposition formula which would express the multivariable hypergeometric function in terms of products of several simpler hypergeometric functions involving fewer variables. Burchnall and Chaundy [7] systematically presented a number of expansions and decomposition formulas for some double hypergeometric functions in series of simpler hypergeometric functions. Using the Burchnall-Chaundy method, Hasanov and Srivastava [24,25] found decomposition formulas for a whole class of hypergeometric functions in several (three and more) variables. For example, the Lauricella’s hypergeometric function \( F_A^{(n)} \) defined by (2.4) has the decomposition formula [24]

\[
F_A^{(n)} \left[ \begin{array}{c}
a, b_1, \ldots, b_n; \\
c_1, \ldots, c_n;
\end{array} \right] y = \sum_{m_2, \ldots, m_n=0}^{\infty} \frac{(a)_{m_2+\ldots+m_n} (b_1)_{m_2+\ldots+m_n} (b_2)_{m_2+\ldots+m_n}}{m_1! \ldots m_n! (c_1)_{m_2+\ldots+m_n} (c_2)_{m_2+\ldots+m_n}}
\times y_1^{m_2+\ldots+m_n} y_2^{m_2} \ldots y_n^{m_n} F \left[ \begin{array}{c}
a + m_2 + \ldots + m_n, b_1 + m_2 + \ldots + m_n; \\
c_1 + m_2 + \ldots + m_n;
\end{array} \right] y_1 \times F_A^{(n-1)} \left[ \begin{array}{c}
a + m_2 + \ldots + m_n, b_2 + m_2, \ldots, b_n + m_n; \\
c_2 + m_2, \ldots, c_n + m_n;
\end{array} \right] y_2, \ldots, y_n, \quad n \in \mathbb{N}\setminus\{1\}.
\]

(2.7)

However, due to the recurrence of formula (2.4), additional difficulties may arise in the applications of this expansion. Further study of the properties of the Lauricella’s hypergeometric functions showed that the formula (2.4) can be reduced to a more convenient form.

Лемма 1 [23]. The following decomposition formula holds true

\[
F_A^{(n)} \left[ \begin{array}{c}
a, b_1, \ldots, b_n; \\
c_1, \ldots, c_n;
\end{array} \right] y = \sum_{(m_{i,j}=0)}^{\infty} \frac{(a)_{A(n,n)}}{M!} \prod_{k=1}^{n} \frac{(b_k)_{B(k,n)}}{(c_k)_{B(k,n)}}
\times \prod_{k=1}^{n} y_k^B(k,n) F \left[ \begin{array}{c}
a + A(k,n), b_k + B(k,n); \\
c_k + B(k,n);
\end{array} \right] y_k, \quad n \in \mathbb{N}\setminus\{1\},
\]

(2.8)

where

\[
M! := m_{2,2} \cdot \ldots \cdot m_{i,j} \cdot \ldots \cdot m_{n,n}, \quad 2 \leq i \leq j \leq n;
\]

\[
A(k,n) = \sum_{i=2}^{k+1} \sum_{j=i}^{n} m_{i,j}, \quad B(k,n) = \sum_{i=2}^{k} m_{k+1, i}, \quad k \in K.
\]
Lemma 2 If \(a, b_1, ..., b_n\) are complex numbers with \(a \neq 0, -1, -2, ...\) and \(\text{Re} (a - b_1 - ... - b_n) > 0\), then the following summation formula holds true

\[
\sum_{m_{i,j}=0}^{\infty} \frac{(a)_{A(n,n)}}{M!} \prod_{k=1}^{n} \frac{(b_k)_{B(k,n)}}{(a)_{A(k,n)}} \Gamma \left[ a - b_k + A(k,n) - B(k,n) \right] = \Gamma (a - b_1 - ... - b_n) \Gamma^{n-1}(a), \ n \in \mathbb{N} \setminus \{1\}. \tag{2.9}
\]

The Lemma 1 was proved by the method of mathematical induction in \[13\], the Lemma 2 is also proved similarly.

From the formulae (2.8) and (2.9) immediately implies the following

Следствие 1 If \(\text{Re}(a) > \text{Re}(b_1 + ... + b_n) > 0\) and \(\text{Re}(c_k) > \text{Re}(b_k) > 0 \ (k \in K)\), then the following limiting equality

\[
\lim_{y_k \to 0} \left\{ y_1^{b_1} ... y_n^{b_n} F_A^{(n)} \left[ a, b_1, ..., b_n; c_1, ..., c_n; 1 - \frac{1}{y_1}, ..., 1 - \frac{1}{y_n} \right] \right\}
= \frac{\Gamma (a - b_1 - ... - b_n)}{\Gamma (a)} \prod_{k=1}^{n} \frac{\Gamma (c_k)}{\Gamma (c_k - b_k)}. \tag{2.10}
\]

is valid.

One of the fundamental solutions of the equation (1.3) that we will use in this paper has the form \[13\]:

\[
q_n (\xi, x) = \kappa_n r^{-2\alpha_n} (\xi x)^{(1-2\alpha)} F_A^{(n)} \left[ \tilde{\alpha}_n, 1 - \alpha_1, ..., 1 - \alpha_n; \ \sigma \right], \tag{2.11}
\]

where

\[
x := (x_1, ..., x_m), \ \xi := (\xi_1, ..., \xi_m); \ (\xi x)^{(1-2\alpha)} = \prod_{k=1}^{n} (\xi_i x_i)^{1-2\alpha_k};
\]

\[
\tilde{\alpha}_n = \frac{m - 2}{2} + \sum_{k=1}^{n} (1 - \alpha_k); \ \kappa_n = 2^{2\alpha_n - m} \frac{\Gamma (\tilde{\alpha}_n)}{\pi^{m/2}} \prod_{k=1}^{n} \frac{\Gamma (1 - \alpha_k)}{\Gamma (2 - 2\alpha_k)}, \tag{2.12}
\]

\[
0 < 2\alpha_k < 1; \ \sigma := (\sigma_1, ..., \sigma_n), \ \sigma_k = 1 - \frac{r_k^2}{r^2}. \tag{2.13}
\]

\[
r^2 = \sum_{i=1}^{m} (\xi_i - x_i)^2, \ \ r_k^2 = (\xi_k + x_k)^2 + \sum_{i=1, i \neq k}^{m} (\xi_i - x_i)^2, \ k \in K. \tag{2.14}
\]

The fundamental solution given by (2.11) possesses the following potentially useful property:

\[
q_n (\xi, x)|_{\xi_k=0} = q_n (\xi, x)|_{x_k=0} = 0, \ k \in K. \tag{2.15}
\]

Throughout this paper, it is assumed that the dimension of the space \(m > 2\).
3 Green’s formula

We consider the following identity:

\[ x^{(2\alpha)} \left[ uH^{(m,n)}_{\alpha}(v) - vH^{(m,n)}_{\alpha}(u) \right] = \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left[ x^{(2\alpha)}(v_{x_i}u - vu_{x_i}) \right]. \]  

(3.1)

Hereinafter,

\[ x^{(2\alpha)} := \prod_{k=1}^{n} x_k^{2\alpha_k}. \]  

(3.2)

Integrating both sides of the identity (3.1) in a domain \( \Omega \) located and bounded in \( \mathbb{R}^{m+n} \), and using the Gauss-Ostrogradsky formula, we obtain

\[ \int_{\Omega} x^{(2\alpha)} \left[ uH^{(m,n)}_{\alpha}(v) - vH^{(m,n)}_{\alpha}(u) \right] dx = \int_{S} (uB_{Nx}^{\alpha}[v] - vB_{Nx}^{\alpha}[u])dS, \]  

where \( S \) is the boundary of \( \Omega \), \( N \) is the outer normal to the surface \( S \) and

\[ B_{Nx}^{\alpha}[ \ ] = x^{(2\alpha)} \sum_{i=1}^{m} \frac{\partial[ \ ]}{\partial x_i} \cos(N, x_i) \]  

(3.4)

is the conormal derivative with respect to \( x \).

If \( u \) and \( v \) are solutions of the equation (1.3), then we find from the formula (3.3) that

\[ \int_{S} (uB_{Nx}^{\alpha}[v] - vB_{Nx}^{\alpha}[u])dS = 0. \]  

(3.5)

Assuming that \( v = 1 \) in (3.3) and replacing \( u \) by \( u^2 \), we obtain

\[ \int_{\Omega} x^{(2\alpha)} \sum_{i=1}^{m} \left( \frac{\partial u}{\partial x_i} \right)^2 dx = \int_{S} uB_{Nx}^{\alpha}[u]dS, \]  

(3.6)

where \( u(x) \) is the solution of equation (1.3).

The special case of (3.5) when \( v = 1 \) reduces to the following form:

\[ \int_{S} B_{Nx}^{\alpha}[u]dS = 0. \]  

(3.7)

We note from (3.7) that the integral of the conormal derivative of the solution of the equation (1.3) along the boundary \( S \) of the domain \( \Omega \) is equal to zero.

4 A double-layer potential

Let \( \Gamma \) be a surface lying in \( \mathbb{R}^{m+n} \), the boundary of which on the hyperplane \( x_k = 0 \) is denoted by \( \gamma_k \) and \( \Omega \) be a finite domain in \( \mathbb{R}^{m+n} \), bounded by the surface \( \Gamma \) and the hyperplanes \( x_1 = 0, ..., x_n = 0 \). The boundary of the domain \( \Omega \) on the hyperplane \( x_k = 0 \) is denoted by \( \Gamma_k (k \in K) \).

The surface \( \Gamma \) in the Euclidean space \( E_m \), that satisfies the following three conditions is called the Lyapunov surface [32]:

(i). There is a definite normal at any point of the surface \( \Gamma \).
(ii). Let $x$ and $\xi$ be points of the surface $\Gamma$, and $\vartheta$ angle between these normals. There exist positive constants $a$ and $\kappa$, such that

$$\vartheta \leq a r^\kappa.$$ 

(iii). With respect to the surface $\Gamma$ we shall assume that it approaches the hyperplanes $x_1 = 0, x_2 = 0$ under right angle.

We introduce the following notation:

\[ \xi_k = \xi \setminus \{\xi_k\} = (\xi_1, \ldots, \xi_{k-1}, \xi_{k+1}, \ldots, \xi_m), \]

\[ \xi^{(2\alpha)} = \prod_{i=1}^n \xi_i^{2\alpha_i}, \quad \tilde{\xi}_k^{(2\alpha)} = \prod_{i=1, i \neq k}^n \xi_i^{2\alpha_i}, \quad \tilde{\xi}_k^{(1)} = \prod_{i=1, i \neq k}^n \xi_i, \]

\[ x^{(1-2\alpha)} = \prod_{i=1}^n x_i^{1-2\alpha_i}; \quad X_k^2 = \xi_k^2 + \sum_{i=1, i \neq k}^m (x_i - \xi_i)^2, \quad k \in K. \]

We consider the following integral

\[ w^{(n)}(x) = \int_{\Gamma} \mu_n(\xi) B_{N\xi}^\alpha[g_n(\xi; x)]d\xi \Gamma, \tag{4.1} \]

where the density $\mu_n(x) \in C(\overline{\Gamma})$, $q_n(\xi; x)$ is given in (2.11), $N$ is the outer normal to the surface $\Gamma$ and $B_{N\xi}^\alpha$ is the conormal derivative with respect to $\xi$, as defined in (3.3).

We call the integral (4.1) a double-layer potential with respect to $\xi$, as defined in (3.3).

We now investigate some properties of the double-layer potential $w^{(n)}(x)$ when $\mu_n(\xi) \equiv 1$.

**Lemma 3** The following formula holds true:

\[ w_1^{(n)}(x) \equiv \int_{\Gamma} B_{N\xi}^\alpha[g_n(\xi; x)]d\xi \Gamma = \begin{cases} 
  i(x) - 1, & x \in \Omega, \\
  i(x) - \frac{1}{2}, & x \in \Gamma, \\
  i(x), & x \notin \Omega \cup \Gamma,
\end{cases} \]

where

\[ i(x) \equiv \prod_{k=1}^n \int_{\Gamma_k} \tilde{\xi}_k^{(2\alpha)} \left( \xi_k^{2\alpha_k} \frac{\partial q_n(\xi; x)}{\partial \xi_k} \right) |_{\xi_k = 0} d\tilde{\xi}_k \Gamma_k \]

\[ = \kappa_n x^{(1-2\alpha)} \sum_{k=1}^n (1 - 2\alpha_k) \int_{\Gamma_k} \tilde{\xi}_k^{(1)} X_k^{2-2\alpha_n} \]

\[ \times \mathcal{F}_A^{(n-1)} \left[ \bar{\alpha}_n, 1 - \alpha_1, \ldots, 1 - \alpha_{k-1}, 1 - \alpha_{k+1}, \ldots, 1 - \alpha_m; \tilde{\sigma}_k \right] d\tilde{\xi}_k \Gamma_k, \tag{4.2} \]

\[ \tilde{\sigma}_k = \left( -\frac{4x_1\xi_1}{X_k^2}, \ldots, -\frac{4x_{k-1}\xi_{k-1}}{X_k^2}, \frac{4x_{k+1}\xi_{k+1}}{X_k^2}, \ldots, -\frac{4x_n\xi_n}{X_k^2} \right). \]

Here the domains $\Omega, \Gamma_k, \gamma_k$ and the surface $\Gamma$ are described as in this section.

**Proof.** We consider a few cases.

**Case 1.** When $x \notin \Omega \cup \Gamma$ and $x_1 > 0, \ldots, x_n > 0$, it is noted that the function $q_n(\xi; x)$ is a regular solution of the equation (1.3) inside of $\Omega$. Hence, in view of formula (3.7), we have

\[ w_1^{(n)}(x) = i(x), \quad x \notin \Omega \cup \Gamma, \quad x_1 > 0, \ldots, x_n > 0, \]

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where $i(x)$ is a function defined in (4.2).

**Case 2.** When $x \in \Omega$, we cut a ball centered at $x$ with a radius $\rho$ off the domain $\Omega$ and denote the remaining part by $\Omega_\rho$ and the sphere of the cut-off ball by $C_\rho$. The function $q_n(\xi; x)$ in (2.11) is a regular solution of the equation (1.3) in the domain $\Omega_\rho$ and, in view of (3.7), we obtain (for details, see (12))

$$u_1^{(n)}(x) = i(x) + \lim_{\rho \to 0} \frac{\partial}{\partial \rho} B^\alpha_{N\xi}[q_n(\xi; x), d\xi C_\rho].$$

We calculate $B^\alpha_{N\xi}[q_n(\xi; x)]$ with respect to $\xi$. Using the formula of differentiation (2.14), adjacent relations (2.5), (2.6) and a definition of the conormal derivative (3.4), we obtain (for details, see (12))

$$B^\alpha_{N\xi}[q_n(\xi; x)] = B_1(\xi; x)B^\alpha_{N\xi} \left[ \ln \frac{1}{\rho} \right] + B_2(\xi; x),$$

where

$$B_1(\xi; x) = 2 \alpha_n \kappa_n \left( \frac{\xi x}{2\alpha_n} \right)^{1-2\alpha} F_\lambda^{(n)} \left[ \frac{1 + \alpha_n, 1 - \alpha_1, \ldots, 1 - \alpha_n; \rho}{2 - 2\alpha_1, \ldots, 2 - 2\alpha_n}; \sigma \right].$$

$$B_2(\xi; x) = \kappa_n x^{1-2\alpha} \sum_{k=1}^n (1 - 2\alpha_k) \xi_k(1) \cos(N, \xi_k) \times F_\lambda^{(n)} \left[ \frac{\alpha_n, 1 - \alpha_1, \ldots, 1 - \alpha_n; \rho}{2 - 2\alpha_1, \ldots, 2 - 2\alpha_k - 1, 2 - 2\alpha_{k+1}, \ldots, 2 - 2\alpha_n}; \sigma \right].$$

It is easy to see that

$$\lim_{\xi_k \to 0} B^\alpha_{N\xi}[q_n(\xi; x)] = \lim_{x_k \to 0} B^\alpha_{N\xi}[q_n(\xi; x)] = 0, \quad k \in K.$$

Taking (4.4) into account we first calculate the following integral

$$j_1(x; \rho) = \int_{C_\rho} B_1(\xi; x)B^\alpha_{N\xi} \left[ \ln \frac{1}{\rho} \right] d\xi C_\rho.$$

We use the following generalization spherical system of coordinates:

$$\xi_i = x_i + \rho \Phi_i(\varphi), \quad i = 1, \ldots, m,$$

where

$$\varphi = (\varphi_1, \ldots, \varphi_{m-1}); \quad \Phi_1(\varphi) = \cos \varphi_1,$$

$$\Phi_i(\varphi) = \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{i-1} \cos \varphi_i, \quad i = 2, \ldots, m - 1,$$

$$\Phi_m(\varphi) = \sin \varphi_1 \sin \varphi_2 \cdots \sin \varphi_{m-2} \sin \varphi_{m-1}$$

$$\quad (0 \leq \rho \leq r, \quad 0 \leq \varphi_1 \leq \pi, \quad \ldots, \quad 0 \leq \varphi_{m-2} \leq \pi, \quad 0 \leq \varphi_{m-1} \leq 2\pi).$$

Then we have

$$j_1(x; \rho) = -2 \alpha_n \kappa_n \int_{0}^{2\pi} d\varphi_{m-1} \int_{0}^{\pi} \sin \varphi_{m-2} d\varphi_{m-2} \cdots \int_{0}^{\pi} \sin^{m-2} \varphi_1 \mathcal{N}(\rho, \varphi) d\varphi_1,$$

where

$$\mathcal{N}(\rho, \varphi) = \prod_{k=1}^{n} \frac{x_k}{r_{kp}} \cdot \prod_{k=1}^{n} \left( \frac{x_k + \rho \Phi_k(\varphi)}{r_{kp}} \right)^{1-2\alpha_k} \cdot \prod_{k=1}^{n} \left( \frac{\rho^2}{r_{kp}} \right)^{1-\alpha_k}.$$
\[
\times F^{(n)}_{\lambda} \left[ 1 + \bar{\alpha}_n, 1 - \alpha_1, \ldots, 1 - \alpha_n; \begin{array}{c}
2 - 2\alpha_1, \ldots, 2 - 2\alpha_n; \\
1 - \rho^2/r_{1\rho}^2, \ldots, 1 - \rho^2/r_{n\rho}^2
\end{array} \right],
\]

where

\[
r_{k\rho}^2 = 4k^2 + 4\rho x_k \Phi_k(\varphi) + \rho^2, \quad k \in K.
\]

It is easy to see that when \( \rho \to 0 \) the function \( \mathcal{N}(\rho, \varphi) \) becomes an expression that does not depend on \( \varphi \). Indeed, applying the formula (2.10), we get

\[
\lim_{\rho \to 0} \mathcal{N}(\rho, \varphi) = 2 - 2\bar{\alpha}_n + \frac{2\pi}{\Gamma(m/2)} \prod_{k=1}^{n} \frac{\Gamma(2 - 2\alpha_k)}{\Gamma(1 - \alpha_k)}.
\]

It is not difficult to establish that

\[
\int_{0}^{2\pi} d\varphi \int_{0}^{\pi} \sin^{m-2} \varphi d\varphi = \frac{2\pi^{m/2}}{\Gamma(m/2)}, \quad m = 2, 3, \ldots
\]

If we take into account (4.9), (4.10), (4.11) and (2.12), then we will have

\[
\lim_{\rho \to 0} j_1(x, \rho) = -1.
\]

By similar evaluations one can get that

\[
\lim_{\rho \to 0} j_2(x, \rho) = 0.
\]

Substituting (4.12) and (4.13) into (1.3), we finally get

\[
w^{(n)}_1(x) = i(x) - 1.
\]

Case 3. When \( x \in \Gamma \), we cut a sphere \( C_{\rho} \) centered at \( x \) with a radius \( \rho \) off the domain \( \Omega \) and denote the remaining part of the surface by \( \Gamma_{\rho} \), that is, \( \Gamma_{\rho} = \Gamma - \Gamma_{\rho} \). Let \( C_{\rho} \) denote a part of the sphere \( C_{\rho} \) lying inside the domain \( \Omega \). We consider the domain \( \Omega_{\rho} \) which is bounded by a surface \( \Gamma_{\rho} \), a semisphere \( C_{\rho} \), hyperplanes \( \Gamma_1, \ldots, \Gamma_n \) and their boundaries \( \gamma_1, \ldots, \gamma_n \). Then we have

\[
w^{(n)}_1(x) = i(x) + \lim_{\rho \to 0} \int_{\Gamma_{\rho}} B_{N\xi}^\alpha [q_n(\xi; x)] d\xi \Gamma_{\rho}'.
\]

When the point \( x \) lies outside the domain \( \Omega_{\rho} \), it is found that, in this domain \( q_n(\xi; x) \) is a regular solution of the equation (1.3). Therefore, by virtue of (3.7), we have

\[
\int_{\Gamma_{\rho}} B_{N\xi}^\alpha [q_n(\xi; x)] d\xi \Gamma_{\rho}' = \int_{C_{\rho}} B_{N\xi}^\alpha [q_n(\xi; x)] d\xi C_{\rho}'.
\]

Substituting (4.15) into (4.14), we get

\[
w^{(n)}_1(x) = i(x) + \lim_{\rho \to 0} \int_{C_{\rho}} B_{N\xi}^\alpha [q_n(\xi; x)] d\xi C_{\rho}'.
\]
Similarly, by again introducing the spherical coordinates \((4.8)\) centered at the point \(x\), we have

\[
w_1^{(n)}(x) = i(x) - \frac{1}{2}, x \in \Gamma.
\]

**Case 4.** Finally, we put the point \(x\) on the any hyperplane \(x_k = 0 (k \in K)\). By virtue of \((4.7)\), the equality \(w_1^{(n)}(x)\big|_{x_k=0} = 0 (k \in K)\) immediately follows from the definition \((4.1)\).

The proof of Lemma 3 is completed.

**Theorem 1** If \(x \in \Gamma\), then the following inequality holds true:

\[
|B_N^\alpha [\eta_n(\xi; x)]| \leq \frac{C_1}{r^{m-2p(2-2\alpha)}},
\]

where \(m > 2\) and \(\alpha := (\alpha_1, ..., \alpha_n)\) are real parameters with \(0 < 2\alpha_k < 1 \ (k \in K)\) as in the equation \((1.3)\); \(r\) and \(r(2-2\alpha)\) are as in \((2.14)\) and \((4.19)\) respectively, \(C_1\) is a constant.

**Proof.** We transform a right side of the equality \((4.5)\). Making use of the decomposition formula \((2.8)\), we have

\[
B_1(\xi; x) = 2\bar{\alpha}_n\kappa_n\left(\frac{\xi x}{r^{2\bar{\alpha}_n}}\right)^{(1-2\alpha)}
\times \sum_{m_{i,j}=0}^{\infty} \frac{(1 + \bar{\alpha}_n)_{A(n,n)}}{M!} \prod_{k=1}^{n} \frac{(1 - \alpha_k)_{B(k,n)}}{(2 - 2\alpha_k)_{B(k,n)}}
\times \prod_{k=1}^{n} \sigma_k B(k,n) F\left[1 + \bar{\alpha}_n + A(k,n), 1 - \alpha_k + B(k,n); 2 - 2\alpha_k + B(k,n); \sigma_k\right].
\]

Applying the formula \((2.2)\) to each Gaussian hypergeometric function in \((4.17)\), we obtain

\[
B_1(\xi; x) = \frac{2\bar{\alpha}_n\kappa_n\left(\frac{\xi x}{r^{2\bar{\alpha}_n}}\right)^{(1-2\alpha)}}{r^{m-2p(2-2\alpha)}}
\times \sum_{m_{i,j}=0}^{\infty} \frac{(1 + \bar{\alpha}_n)_{A(n,n)}}{M!} \prod_{k=1}^{n} \frac{(1 - \alpha_k)_{B(k,n)}}{(2 - 2\alpha_k)_{B(k,n)}} (-\omega_k)_{B(k,n)}
\times \prod_{k=1}^{n} F\left[1 - 2\alpha_k - \bar{\alpha}_n + B(k,n) - A(k,n), 1 - \alpha_k + B(k,n); 2 - 2\alpha_k + B(k,n); \omega_k\right],
\]

where

\[
r^{(2-2\alpha)} := \prod_{k=1}^{n} r_k^{2-2\alpha_k}, \quad \omega_k := 1 - \frac{r^2}{r_k^2}, \quad k \in K.
\]

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Similarly, from the formula (4.6) we find
\[
B_2(\xi; x) = \frac{\kappa_n x^{(1-2\alpha)}}{r^{m-2\alpha}(2-2\alpha)} \sum_{k=1}^{n} \xi_k^{(1)} \cos \left( N, \xi_k \right) \sum_{m=0}^{\infty} \frac{(\alpha_n)_{A(n,n)}}{M!} \times (1 - 2\alpha_k + B(k, n)) \prod_{i=1}^{n} \frac{(1 - \alpha_i)_{B(i,n)}}{(2 - 2\alpha_i)_{B(i,n)}} (-\omega_i)^{B(i,n)}
\]
\times \prod_{i=1, i \neq k}^{n} F \left[ \frac{2 - 2\alpha_i - \alpha_n + B(i, n) - A(i, n)}{2 - 2\alpha_i + B(i, n)}; \omega_i \right]
\times F \left[ \frac{1 - 2\alpha_k - \alpha_n + B(k, n) - A(k, n)}{1 - 2\alpha_k + B(k, n)}; \omega_k \right].
\]
(4.20)

Proof. The inequality (4.16) follows from the formulae (4.4), (4.20) and (4.18).

**Theorem 2** If a surface \( \Gamma \) satisfies the conditions (i)–(iii), then the following inequality holds true:
\[
\int_{\Gamma} |B^N_{\xi} [q_n (\xi; x)]| d\xi \Gamma \leq C_2,
\]
where \( C_2 \) is a constant.

**Proof.** The inequality (4.21) immediately follows from the formulae (4.4), (4.20) and (4.18).

**Theorem 3** The following limiting formulas hold true for a double-layer potential (4.1):
\[
w_i^{(n)} (t) = -\frac{1}{2} \mu_n (t) + \int_{\Gamma} \mu_n \left( s \right) K_n \left( s; t \right) d_s \Gamma
\]
and
\[
w_e^{(n)} (t) = \frac{1}{2} \mu_n (t) + \int_{\Gamma} \mu_n \left( s \right) K_n \left( s; t \right) d_s \Gamma,
\]
(4.22)
(4.23)

where
\[
\mu_n (t) \in C \left( \Gamma \right), s := (s_1, s_2, \ldots, s_m), t := (t_1, t_2, \ldots, t_m),
\]
\[
K_n (s; t) = B^N_{ss} [q_n (s; t)], s \in \Gamma, t \in \Gamma,
\]
\( w_i^{(n)} (t) \) and \( w_e^{(n)} (t) \) are limiting values of the double-layer potential (4.1) at the point \( t \in \Gamma \) from the inside and the outside, respectively.

**Proof.** Theorem 3 follows from Lemma 3 and Theorem 2.

5 A simple-layer potential

We consider the following integral
\[
v_n (x) = \int_{\Gamma} \rho_n (\xi) q_n (\xi; x) d\xi \Gamma,
\]
(5.1)
where the density \( \rho_n (x) \in C \left( \Gamma \right) \) and \( q_n (\xi; x) \) is given in (2.11). We call the integral (5.1) a simple-layer potential with density \( \rho_n (\xi) \).
The simple-layer potential \(5.1\) is defined throughout the domain \(R_{m}^{n+}\) and a continuous function when passing through the surface \(\Gamma\). Obviously, a simple-layer potential \(v_{n}(x)\) is a regular solution of the equation \(1.3\) in any domain lying in the \(R_{m}^{n+}\). It is easy to see that as the point \(x\) tends to infinity, a simple-layer potential \(v_{n}(x)\) tends to zero. Indeed, we let the point \(x\) be on the \(2^n\)th part of the sphere given by

\[
\sum_{i=1}^{m} x_i^2 = R^2, x_1 > 0, ..., x_n > 0.
\]

Now we consider fundamental solution \(q_{n}(\xi, x)\) defined in \(2.11\). Applying consequentially the decomposition formula \(2.8\) and formula \(2.2\), we obtain

\[
q_{n}(\xi; x) = \frac{\kappa_{n}(\xi x)(1-2\alpha)}{\gamma(m-2\alpha/2-2\alpha)} \times \sum_{m_{s,j}=0}^{\infty} \frac{(\alpha_{n})_{A(n,n)}}{M!} \prod_{k=1}^{n} \frac{(1-\alpha_{k})_{B(k,n)}}{(2-2\alpha_{k})_{B(k,n)}} (-\omega_{k})^{B(k,n)} \times \prod_{k=1}^{n} F \left[ \frac{2 - 2\alpha_{k} - \alpha_{n} + B(k,n) - A(k,n), 1 - \alpha_{k} + B(k,n)}{2 - 2\alpha_{k} + B(k,n)}; \omega_{k} \right].
\]

Then, by virtue of \(5.2\), we have

\[
|v_{n}(x)| \leq \int_{\Gamma} |\rho_{n}(t)||q_{n}(t; x)| d_{t} \Gamma \leq M_{1} R^{2-m-n} (R \geq R_{0}),
\]

where \(M_{1}\) is a constant.

We take an arbitrary point \(P(x_0)\) on the surface \(\Gamma\) and draw a normal \(N\) at this point. Consider on this normal any point \(M(x)\), not lying on the surface \(\Gamma\), we find the conormal derivative of the simple-layer potential \(5.1\):

\[
B_{Nx}^{\alpha}[v_{n}(x)] = \int_{\Gamma} B_{Nx}^{\alpha}[q_{n}(\xi; x)] d_{t} \Gamma.
\]

The integral \(5.3\) exists also in the case when the point \(M(x)\) coincides with the point \(P\), which we mentioned above.

**Теорема 4**: The following limiting formulas hold true for a simple-layer potential \(5.1\):

\[
B_{Nt}^{\alpha}[v_{n}(t)]_{l} = \frac{1}{2} \rho_{n}(t) + \int_{\Gamma} \rho_{n}(s) K_{n}(s; t) d_{s} \Gamma
\]

and

\[
B_{Nt}^{\alpha}[v_{n}(t)]_{e} = -\frac{1}{2} \rho_{n}(t) + \int_{\Gamma} \rho_{n}(s) K_{n}(s; t) d_{s} \Gamma,
\]

where

\[
\rho_{n} \in C(\Gamma), K_{n}(s; t) = B_{Nt}^{\alpha}[q_{n}(s; t)], s \in \Gamma, t \in \Gamma,
\]

\(B_{Nt}^{\alpha}[v_{n}(t)]_{l}\) and \(B_{Nt}^{\alpha}[v_{n}(t)]_{e}\) are limiting values of the normal derivative of simple-layer potential \(5.1\) at the point \(t \in \Gamma\) from the inside and the outside, respectively.
Making use of these formulas, the jump on the normal derivative of the simple-layer potential follows immediately:
\[ B_{N1}^\alpha [v_n(t)]_x - B_{Nk}^\alpha [v_n(t)]_e = \rho_n(t). \] (5.6)

By virtue of formulae (4.18) and (4.20), one can prove that, when the point \( x \) tends to infinity, the following inequality:
\[ |B_{N1}^\alpha [v_n(t)]| \leq M_2 R^{-2\delta n} \quad (R \geq R_0) \]
is valid, where \( M_2 \) is a constant.

In exactly the same way as in the derivation of (3.6), it is not difficult to show that Green’s formulas are applicable to the simple-layer potential (5.1) as follows:

By virtue of formulae (4.18) and (4.20), one can prove that, when the point \( x \) tends to infinity, the following inequality:
\[ |B_{N1}^\alpha [v_n(t)]| \leq M_2 R^{-2\delta n} \quad (R \geq R_0) \]
is valid, where \( M_2 \) is a constant.

In exactly the same way as in the derivation of (3.6), it is not difficult to show that Green’s formulas are applicable to the simple-layer potential (5.1) as follows:

\[ \int_{\Omega} x^{(2\alpha)} \sum_{k=1}^{m} \left( \frac{\partial v_n}{\partial x_k} \right) dx = \int_{S} v_n(x) B_{Nz}^\alpha [v_n(x)]_i dS \] (5.7)

and

\[ \int_{\Omega'} x^{(2\alpha)} \sum_{k=1}^{m} \left( \frac{\partial v_n}{\partial x_k} \right) dx = -\int_{S} v_n(x) B_{Nz}^\alpha [v_n(x)]_e dS, \] (5.8)

Hereinafter \( \Omega' := R_{m+}^+ \setminus \overline{\Omega} \) is a infinite domain. \( x^{(2\alpha)} \) is defined by the formula (5.2).

### 6 Integral equations for densities

Formulas (4.22), (4.23), (5.4) and (5.5) can be written as the following integral equations for densities:

\[ \mu_n (s) - \lambda \int_{\Gamma} K_n (s; t) \mu_n (t) d\Gamma = f_n (s) \] (6.1)

and

\[ \rho_n (s) - \lambda \int_{\Gamma} K_n (t; s) \rho_n (t) d\Gamma = g_n (s), \] (6.2)

where

\[ \lambda = 2, \quad f_n (s) = -2w_i^{(n)} (s), \quad g_n (s) = -2B_{Nz}^\alpha [v_n(s)]_e \]

and

\[ \lambda = -2, \quad f_n (s) = 2w_i^{(n)} (s), \quad g_n (s) = 2B_{Nz}^\alpha [v_n(s)]_i. \]

Equations (6.1) and (6.2) are mutually conjugated and, by Theorem [1], Fredholm theory is applicable to them. We show that \( \lambda = 2 \) is not an eigenvalue of the kernel \( K_n (s; t) \). This assertion is equivalent to the fact that the homogeneous integral equation

\[ \rho_n (t) - 2 \int_{\Gamma} K_n (s; t) \rho_n (s) d_s \Gamma = 0 \] (6.3)

has no non-trivial solutions.

Let \( \rho_0 (t) \) be a continuous non-trivial solution of the equation (6.3). The simple-layer potential with density \( \rho_0 (t) \) gives us a function \( v_0 (x) \), which is a solution of the equation (4.6) in the domains \( \Omega \) and \( \Omega' \). By virtue of the equation (6.3), the limiting values of the conormal derivative of \( B_{Nz}^\alpha [v_0(s)]_e \) are zero. The formula (6.8) is applicable to the simple-layer potential \( v_0(x) \), from which it follows that \( v_0(x) = const \) in domain \( \Omega' \). At infinity, a simple layer potential is zero, and consequently \( v_0(x) \equiv 0 \) in \( \Omega' \), and also on the surface \( \Gamma \). Applying now (5.7), we find that \( v_0(x) \equiv 0 \) is valid also inside the region \( \Omega \). But then \( B_{Nz}^\alpha [v_0(s)]_i = 0 \), and by virtue of formula (5.6) we obtain \( \rho_0 (t) \equiv 0 \). Thus, clearly, the homogeneous equation (6.3) has only the trivial solution; consequently, \( \lambda = 2 \) is not an eigenvalue of the kernel \( K_n (s; t) \).

Similarly, one can show that \( \lambda = 2 \) is not an eigenvalue of the kernel \( K_n (s; t) \).
7 The uniqueness of the solution of Dirichlet’s problem

We apply the obtained results of potential theory to the solving the boundary value problem for the equation (1.3) in the domain Ω.

We introduce the following notation:

\[ \tilde{x}_k = x \setminus \{x_k\} = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_m), \quad \tilde{x}_k^{(2\alpha)} = \prod_{i=1, i \neq k}^n x_i^{2\alpha_i}, \quad k \in K. \]

The Dirichlet problem. Find a regular solution of the equation (1.3) in the domain Ω that is continuous in the closed domain Ω and satisfies the following boundary conditions:

\[ u|_\Gamma = \varphi(x), \quad x \in \Gamma, \quad (7.1) \]

\[ u(x)|_{x_k=0} = \tau_k(\tilde{x}_k), \quad \tilde{x}_k \in \Gamma_k, \quad k \in K, \quad (7.2) \]

where \( \varphi(x) \) and \( \tau_k(\tilde{x}_k) \) are given continuous functions fulfilling the following matching conditions:

\( \varphi(x)|_{\Gamma_k} = \tau_k(\tilde{x}_k)|_{\Gamma_k}, \quad k \in K. \)

Considering the equality (3.6), we obtain

\[ \int_{\Omega} x^{(2\alpha)} \sum_{i=1}^m \left( \frac{\partial u}{\partial x_i} \right)^2 dx = -\int_{\Gamma} \varphi(x)B_{Nx}^\alpha[u]d_x \Gamma + \sum_{k=1}^n \int_{\Gamma_k} \tilde{x}_k^{(2\alpha)} \tau_k(\tilde{x}_k) \left( x_k^{2\alpha_i} \frac{\partial u}{\partial x_k} \right)|_{x_k=0} d_{\tilde{x}_k} \Gamma_k. \]

(7.3)

In case of the homogeneous Dirichlet problem from (7.3) one can easily get

\[ \int_{\Omega} x^{(2\alpha)} \sum_{i=1}^m \left( \frac{\partial u}{\partial x_i} \right)^2 dx = 0. \]

Hence, it follows that \( u(x) = 0 \) in \( \Omega \).

Thus we have proved the following

**Theorem 5** If the Dirichlet problem has a regular solution, then it is unique.

8 Green’s function revisited

To solve this problem, we use the Green’s function method. First, we construct the Green’s function for solving the Dirichlet problem for an equation (1.3) in the domain Ω bounded by an arbitrary surface \( \Gamma \) and hyperplanes \( x_1 = 0, \ldots, x_n = 0 \). In the end, we show that, thanks to the Green’s function, the solution of the Dirichlet problem in a special domain (in the 2ⁿ-th part of the multidimensional ball) takes a simpler form.

**Definition.** We refer to \( G_n(x; \xi) \) as Green’s function of the Dirichlet problem, if it satisfies the following conditions:

**Condition 1.** The function \( G_n(x; \xi) \) is a regular solution of the equation (1.3) in the domain \( \Omega \), except at the point \( \xi \), which is any fixed point of \( \Omega \).
Condition 2. The function \( G_n(x; \xi) \) satisfies the boundary conditions given by
\[
G_n(x; \xi)|_\Gamma = 0, \quad G_n(x; \xi)|_{x_k=0} = 0, \quad k \in K. \tag{8.1}
\]

Condition 3. The function \( G_n(x; \xi) \) can be represented as follows:
\[
G_n(x; \xi) = q_n(x; \xi) + v_n(x; \xi), \tag{8.2}
\]
where \( q_n(x; \xi) \) is a fundamental solution of the equation (1.3), defined in (2.11) and the function \( v_n(x; \xi) \) is a regular solution of the equation (1.3) in the domain \( \Omega \).

The construction of the Green’s function \( G(x, \xi) \) reduces to finding its regular part \( v_n(x; \xi) \) which, by virtue of (8.1), (8.2) and (2.15), must satisfy the following boundary conditions:
\[
v_n(x; \xi)|_\Gamma = -q_n(x; \xi)|_\Gamma \tag{8.3}
\]
and
\[
v_n(x; \xi)|_{x_k=0} = 0, \quad k \in K.
\]

We look for the function \( v_n(x; \xi) \) in the form of a double-layer potential given by
\[
v_n(x; \xi) = \int \mu_n(t; \xi) B_{Nt}[q_n(t; x)] \, dt \, \Gamma.
\tag{8.4}
\]

Taking into account the equality (1.22) and the boundary condition (8.3), we obtain the integral equation for the density \( \mu_n(s; \xi) \) as follows:
\[
\mu_n(s; \xi) - 2 \int \mu_n(t; \xi) K_n(s; t) \, dt \, \Gamma = 2q_n(s; \xi), \quad s \in \Gamma. \tag{8.5}
\]

The right-hand side of (8.5) is a continuous function with respect to \( s \) (the point \( \xi \) lies inside \( \Omega \)). By Theorem 1, Fredholm theory is applicable to the equation (8.5). In section 6 it was proved that \( \lambda = 2 \) is not an eigenvalue of the kernel \( K_n(s, t) \) and, consequently, the equation (8.5) is solvable and its continuous solution can be written in the following form:
\[
\mu_n(s; \xi) = 2q_n(s; \xi) + 4 \int \mu_n(t; \xi) R_n(s, t; 2) q_n(t; \xi) \, dt \, \Gamma, \tag{8.6}
\]
where \( R_n(s, t; 2) \) is the resolvent of the kernel \( K_n(s, t) \), \( s \in \Gamma \). Substituting (8.6) into (8.4), we obtain
\[
v_n(x, \xi) = 2 \int \mu_n(t; \xi) B_{Nt}^s[q_n(t; x)] \, dt \, \Gamma + 4 \int \int \mu_n(t; \xi) B_{Nt}^s[q_n(t; x)] \, dt \, d_\Gamma \, \Gamma.
\tag{8.7}
\]

We now define a following function:
\[
g(x) = \begin{cases} v_n(x; \xi), & x \in \Omega, \\ -q_n(x; \xi), & x \in \Omega'. \end{cases} \tag{8.8}
\]

The function \( g(x) \) is a regular solution of (1.3) both inside the domain \( \Omega \), and inside \( \Omega' \) and equal to zero at infinity. Since point \( \xi \) lies inside \( \Omega \), then in \( \Omega' \) the function \( g(x) \) has derivatives of any order in all variables, continuous up to \( \Gamma \). We can consider \( g(x) \) in \( \Omega' \) as a solution of the equation (1.3) satisfying the boundary conditions given by
\[
B_{Nx}^s[g(x)]|_\Gamma = -B_{Nx}^s[q_n(x; \xi)]|_\Gamma \tag{8.9}
\]

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and
\[ g(x)|_{x_k=0} = 0, \ k \in K. \]

We represent this solution in the form of a simple-layer potential as follows:
\[ g(x) = \int_{\Gamma} \rho_n(t;\xi) q_n(t;x) \, dt, \ x \in \Omega' \]  \hspace{1cm} (8.10)

with an unknown density \( \rho_n(t;\xi) \).

Using the formula (5.5), by virtue of condition (8.9), we obtain the following integral equation for the density \( \rho_n(t;\xi) \):
\[ \rho_n(s;\xi) - 2 \int_{\Gamma} \rho_n(t;\xi) K_n(t;s) \, dt = 2B^N_{Ns} [q_n(s;\xi)]. \]  \hspace{1cm} (8.11)

Equation (8.11) is conjugated with the equation (8.5). Its right-hand side is a continuous function with respect to \( s \). Thus, clearly, the equation (8.11) has the following continuous solution:
\[ \rho_n(s;\xi) = 2B^N_{Ns} [q_n(s;\xi)] + 4 \int_{\Gamma} R_n(t,s;2) B^N_{Nt} [q_n(t;\xi)] \, dt. \]  \hspace{1cm} (8.12)

According to (4.7) is obvious that the function \( \rho_n(s;\xi) \) has the following limiting values
\[ \lim_{\xi \to 0} \rho_n(s;\xi) = 0, \ s \in \Gamma, \ k \in K. \]  \hspace{1cm} (8.13)

The values of a simple-layer potential \( g(x) \) on the surface \( \Gamma \) are equal to \(-q_n(x;\xi)\), that is, just as the values of the function \( v_n(x;\xi) \) and on the hyperplanes \( x_k = 0 \), \( k \in K \) are equal to zero. Hence, by virtue of the uniqueness theorem for the Dirichlet problem, it follows that the formula (8.10) for the function \( g(x) \) defined by (8.8) holds throughout in the \( x_1 \geq 0, ..., x_n \geq 0 \), that is,
\[ v_n(x;\xi) = \int_{\Gamma} \rho_n(t;\xi) q_n(t;x) \, dt, \ x \in \Omega. \]  \hspace{1cm} (8.14)

Thus, the regular part \( v_n(x;\xi) \) of Green’s function is representable in the form of a simple-layer potential.

Applying the formula (5.4) to (8.14), we obtain
\[ 2B^N_{Ns} [v_n(s;\xi)]_i = \rho_n(s;\xi) + 2 \int_{\Gamma} K_n(t;s) \rho_n(t;\xi) \, dt, \]  \hspace{1cm} (8.15)

but, according to (8.11), we have
\[ 2B^N_{Ns} [q_n(s;\xi)]_i = \rho_n(s;\xi) - 2 \int_{\Gamma} K_n(t;s) \rho_n(t;\xi) \, dt. \]

Summing the last two equalities by term-wise and taking into account (8.2), we have
\[ B^N_{Ns} [G_n(s;\xi)] = \rho_n(s;\xi), \]  \hspace{1cm} (8.15)

and, consequently, formula (8.14) can be written in the following form:
\[ v_n(x;\xi) = \int_{\Gamma} q_n(t;x) B^N_{Nt} [G_n(t;\xi)] \, dt. \]
Multiplying both sides of (8.12) by \( q_n (s; x) \), integrating by \( s \) over the surface \( \Gamma \) and, by virtue of (8.6) and (8.4), we obtain
\[
v_n (\xi; x) = \int_\Gamma \rho_n (t; \xi) q_n (t; x) \, dt \Gamma.
\]
Comparing this last equation with the formula (8.14), we have
\[
v_n (\xi; x) = v_n (x; \xi), \tag{8.16}
\]
if the points \( x \) and \( \xi \) are inside the domain \( \Omega \).

**Lemma 4** If points \( x \) and \( \xi \) are inside domain \( \Omega \), then Green’s function \( G_n (x; \xi) \) is symmetric about those points.

**Proof.** The proof of Lemma 4 follows from the representation (8.2) of Green’s function and the equality (8.16).

For a domain \( \Omega_0 \) bounded by the hyperplanes \( x_1 = 0, \ldots, x_n = 0 \) and the \( 2^n \)th part of the sphere given by
\[
x_1^2 + x_2^2 + \cdots + x_m^2 = R^2, \ x_1 > 0, \ldots, x_n > 0,
\]
Green’s function of the Dirichlet problem has the following form:
\[
G_0n (x; \xi) = q_n (x; \xi) - \left( \frac{R}{\varrho} \right)^{2a_n} \cdot q_n \left( x; \frac{R^2}{\varrho^2} \xi \right), \tag{8.17}
\]
where
\[
\varrho^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_m^2. \tag{8.18}
\]
We show that the function given by
\[
v_{0n} (x; \xi) = - \left( \frac{R}{\varrho} \right)^{2a_n} \cdot q_n \left( x; \frac{R^2}{\varrho^2} \xi \right)
\]
can be represented in the following form:
\[
v_{0n} (x; \xi) = - \int_\Gamma \rho_n (s; x) v_{0n} (s; \xi) \, ds \Gamma, \tag{8.19}
\]
where \( \rho_n (s; x) \) is a solution of the equation (8.11).

Indeed, by letting an arbitrary point \( \xi \) be inside the domain \( \Omega \), we consider the function given by
\[
u (x; \xi) = - \int_\Gamma \rho_n (s; x) v_{0n} (s; \xi) \, ds \Gamma.
\]
The function \( u(x; \xi) \) satisfies the equation (1.3), since this equation is satisfied by the function \( \rho_n (s; x) \). Substituting the expression (8.12) for \( \rho_n (s; x) \), we obtain
\[
u (x; \xi) = - \int_\Gamma \psi (s; \xi) B_{Ns} [q_n (s; x)] \, ds \Gamma, \tag{8.20}
\]
where
\[
\psi (s; \xi) = 2v_{0n} (s; \xi) + 4 \int_\Gamma R_n (s, t; 2) v_{0n} (t, \xi) \, dt \Gamma,
\]
that is, \( \psi (s; \xi) \) is a solution of the integral equation

\[
\psi (s; \xi) - 2 \int_{\Gamma} K_n (s; t) \psi (t; \xi) \, dt \Gamma = 2 v_{0n} (s; \xi). \tag{8.21}
\]

Applying the formula (3.5), we obtain

\[
\Omega \epsilon, \delta
\]

or, by virtue of (8.14), (8.17), (8.19) and a symmetry property (8.16), we obtain

\[
u (s; \xi) = \psi (s; \xi), \quad s \in \Gamma.
\]

Thus, clearly, the functions \( u (x; \xi) \) and \( v_{0n} (x; \xi) \) satisfy the same equation (1.3) and the same boundary conditions. Also, by virtue of the uniqueness of the solution of the Dirichlet problem, the equality

\[
u (x; \xi) \equiv v_{0n} (x; \xi)
\]

is satisfied.

Now, subtracting the expression (8.14) from (8.2), we obtain

\[
H_n (x; \xi) = G_n (x; \xi) - G_{0n} (x; \xi) = v_n (x; \xi) - v_{0n} (x; \xi)
\]
or, by virtue of (8.14), (8.17), (8.19) and a symmetry property (8.16), we obtain

\[
H_n (x; \xi) = \int_{\Gamma} G_{0n} (t; \xi) \rho_n (t; x) \, dt \Gamma. \tag{8.22}
\]

9 Solving the Dirichlet problem for equation (1.3)

Let \( \xi \) be a point inside the domain \( \Omega \). Consider the domain \( \Omega_{\epsilon, \delta} \subset \Omega \), where \( \delta := (\delta_1, ..., \delta_n) \), bounded by the surface \( \Gamma_\epsilon \) which is parallel to the surface \( \Gamma \), and the domains \( \Gamma_{k\delta_k} \) lying on the hyperplanes \( x_k = \delta_k > \epsilon, k \in K \). We choose \( \delta_1, ..., \delta_n \) and \( \epsilon \) so small that the point \( x_0 \) is inside \( \Omega_{\epsilon, \delta} \). We cut out from the domain \( \Omega_{\epsilon, \delta} \) a ball of small radius \( \rho \) with center at the point \( x_0 \) and the remainder part of \( \Omega_{\epsilon, \delta} \) denote by \( \Omega_{\rho} \), in which the Green’s function \( G_n (x; x_0) \) is a regular solution of the equation (1.3).

Let \( u(x) \) be a regular solution of the equation (1.3) in the domain \( \Omega \) that satisfies the boundary conditions (7.1) and (7.2). Applying the formula (3.5), we obtain

\[
\int_{\Gamma_\epsilon} (G_n B_{N\xi}^\alpha [u] - u B_{N\xi}^\alpha [G_n]) \, d_x \Gamma_\epsilon + \sum_{k=1}^{n} \int_{\Gamma_{k\delta_k}} (u B_{N\xi}^\alpha [G_n])
\]

\[
- G_n B_{N\xi}^\alpha [u] \big|_{x_\delta = \delta_\delta} \, d_x \Gamma_{k\delta_k} = \int_{C_{\rho}} (G_n B_{N\xi}^\alpha [u] - u B_{N\xi}^\alpha [G_n]) \, d_x C_{\rho}.
\]

Passing to the limit as \( \rho \to 0 \) and then as \( \delta_1 \to 0, ..., \delta_n \to 0 \) and \( \epsilon \to 0 \), we obtain

\[
u (\xi) = \sum_{k=1}^{n} \int_{\Gamma_k} \tau_k (\tilde{x}_k) \left( x_\delta^{2\alpha_k} \frac{\partial G_n (x; \xi)}{\partial x_k} \right) \big|_{x_\delta = 0} \, d_{\tilde{x}_k} \Gamma_k
\]
\[-\int_{\Gamma} \varphi(x) B_{Nt}^n G_n (x; \xi) |d_{\xi} \Gamma = \sum_{k=1}^{n} T_k (\xi) + \Phi (\xi). \quad (9.1)\]

We show that the formula (9.1) gives a solution to the Dirichlet problem.

It is easy to see that each integral \( T_k (\xi) \) in the formula (9.1) satisfies the equation (1.3) and is regular in the domain \( \Omega \), continuous in \( \overline{\Omega} \).

We use the following notations:

\[
\vartheta_k (\xi) = \int_{\Gamma_k} \tau_k (x_k) \frac{\partial q_n (x_k; \xi)}{\partial x_k} |d_{\xi_k} \Gamma_k
\]

where

\[\bar{x}_{k0} = (x_1, ..., x_{k-1}, 0, x_{k+1}, ..., x_m), \quad k \in K.\]

Here \( \vartheta_k (\xi) \) is a continuous function in \( \overline{\Omega} \). In view of (9.2) and (8.7) and the symmetry property of the function \( v_n (x; \xi) \), the integral \( T_k (\xi) \) can be represented in the following form:

\[ T_k (\xi) = \vartheta_k (\xi) + 2 \int_{\Gamma} \vartheta_k (t) B_{Nt}^n [q_n (t; \xi)] dt \Gamma
\]

\[ + 4 \int_{\Gamma} \int_{\Gamma} R_n (t, s; 2) \vartheta_k (s) B_{Nt}^n [q_n (t; \xi)] dt \Gamma ds \Gamma, \quad k \in K. \quad (9.3)\]

The last two integrals in the formula (9.3) are double-layer potentials. Taking the formula (4.22) and the integral equation for the resolvent \( R_n (t, s; 2) \) into account, we obtain

\[ T_k (\xi) |_{\Gamma} = 0, \quad k \in K.\]

It is easy to prove that

\[ \lim_{\xi_k \to 0} T_k (\xi) = \tau (\xi_k), \quad \xi_k \in \Gamma_k, \quad k \in K.\]

Indeed, by virtue of (8.14) and the symmetry property of the function \( v_n (x, \xi) \), the integral \( T_k (\xi) \) can be rewritten as

\[ T_k (\xi) = \int_{\Gamma_k} \tau_k (x_k) \frac{\partial q_n (x_k; \xi)}{\partial x_k} |d_{\xi_k} \Gamma_k
\]

\[ + \int_{\Gamma_k} \tau_k (x_k) d_{\xi_k} \Gamma_k \int_{\Gamma} \rho_n (t; \xi) \frac{\partial q_n (x_k; t)}{\partial x_k} dt \Gamma, \quad (9.4)\]

where \( \rho_n (t; \xi) \) is defined in (8.12).

Following the work [38], one can get that

\[ \lim_{\xi_k \to 0} \int_{\Gamma_k} \tau_k (x_k) \frac{\partial q_n (x_k; \xi)}{\partial x_k} |d_{\xi_k} \Gamma_k = \tau_k (\xi_k), \quad \xi_k \in \Gamma_k, \quad k \in K.\]

Taking (8.13) into account we see that second addend in (9.4) is zero at \( \xi_k = 0, \quad k \in K.\)

Now consider the last integral \( \Phi (\xi) \) in the formula (9.1), which, by virtue of (8.12) and (8.15), can be written in the following form:

\[ \Phi (\xi) = -\int_{\Gamma} \varphi(s) \rho_n (s; \xi) ds \Gamma = -\int_{\Gamma} \theta(t) B_{Nt}^n [q_n (t; \xi)] dt \Gamma, \]

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If the following condition:

\[ H(\phi) \]

is fulfilled, the function \( \Phi(\xi) \) is a solution to the equation (1.3), regular in the domain \( \Omega \) and continuous in \( \overline{\Omega} \), which, by virtue of (1.22), (1.23) and (9.5), satisfies the following condition:

\[ \Phi(\xi)|_{\Gamma} = \varphi(s), \quad s \in \Gamma. \]

It is easy to see that

\[ \lim_{\xi_k \to 0} \Phi(\xi) = 0, \quad \tilde{\xi}_k \in \overline{\Gamma}_k, \quad k \in K. \]

Since the function \( \theta(s) \) is continuous, the function \( \Phi(\xi) \) is a solution to the equation (1.3), regular in the domain \( \Omega \) and continuous in \( \overline{\Omega} \), which, by virtue of (1.22), (1.23) and (9.5), satisfies the following condition:

\[ \Phi(\xi)|_{\Gamma} = \varphi(s), \quad s \in \Gamma. \]

It is easy to see that

\[ \lim_{\xi_k \to 0} \Phi(\xi) = 0, \quad \tilde{\xi}_k \in \overline{\Gamma}_k, \quad k \in K. \]

The formula (9.1), and with it all the proof, requires that \( m > 2 \). However, the formula (9.1) is also valid for \( m = 2 \) (in case of \( m = 2 \), for details, see [35]).

Thus we have proved the following

**Theorem 6** If \( \tau_k(\tilde{x}_k) \in C(\overline{\Gamma}_k) \cap C^2(\Gamma_k) \) and \( \varphi(x) \in C(\overline{\Gamma}_k) \cap C^2(\Gamma_k) \) are given functions fulfilling the matching conditions \( \varphi(x)|_{\Gamma_k} = \tau_k(\tilde{x}_k)|_{\Gamma_k}, \quad k \in K \), then the Dirichlet problem for equation \( H_\alpha^{(m,n)}(u) = 0 \) \((m \geq 2, 0 < n \leq m)\) in the domain \( \Omega \) has unique solution represented by formula (9.1).

Using the formulae (8.17) and (8.22) we rewrite a solution (9.1) to the Dirichlet problem for equation (1.3) in the following form

\[
\begin{align*}
\tilde{u}(\xi) &= \sum_{k=1}^{n} \int_{\Gamma_k} \tilde{x}_k(2\alpha) \tau_k(\tilde{x}_k) \left[ \tilde{G}_{0n}(\tilde{x}_k; \xi) + \tilde{H}_n(\tilde{x}_k; \xi) \right] d\tilde{x}_k \Gamma_k \\
&\quad - \int_{\Gamma} \varphi(x) \left\{ B^\alpha_{N_k} [G_{0n}(x; \xi)] + B^\alpha_{N_k} [H_n(x; \xi)] \right\} d\alpha \Gamma,
\end{align*}
\]

where

\[
\begin{align*}
\tilde{G}_{0n}(\tilde{x}_k; \xi) &= (1 - 2\alpha_k) \kappa_n \xi(1-2\alpha)x(1-2\alpha) \left[ \tilde{F}^{(n-1)}_A(\tilde{\sigma}_k) - \tilde{F}^{(n-1)}_A(\tilde{\omega}_k) \right] \frac{1}{X_k^{2n}} \\
\tilde{x}_k &= (x_1, ..., x_{k-1}, 0, x_{k+1}, ..., x_m); \quad X_k^2 = \xi_k^2 + \sum_{i=1, i \neq k}^{m} (x_i - \xi_i)^2, \\
Y_k^2 &= \sum_{i=1, i \neq k}^{m} \left( R - \frac{x_i \xi_i}{R} \right)^2 + \frac{1}{R^2} \sum_{i=1, i \neq k}^{m} x_i^2 \sum_{j=1, j \neq i}^{m} \xi_j^2 - (m-2)R^2, \\
\tilde{\sigma}_k &= \left( -\frac{4x_1 \xi_1}{X_k^2}, ..., -\frac{4x_{k-1} \xi_{k-1}}{X_k^2}, -\frac{4x_{k+1} \xi_{k+1}}{X_k^2}, ..., -\frac{4x_n \xi_n}{X_k^2} \right), \\
\tilde{\omega}_k &= \left( -\frac{R^2 \xi_1}{Y_k^2}, ..., -\frac{R^2 \xi_{k-1}}{Y_k^2}, -\frac{R^2 \xi_{k+1}}{Y_k^2}, ..., -\frac{R^2 \xi_n}{Y_k^2} \right), \\
\tilde{F}_A^{(n-1)}(...) &= F_A^{(n-1)} \left[ \tilde{\alpha}_n, 1 - \alpha_1, ..., 1 - \alpha_{k-1}, 1 - \alpha_{k+1}, ..., 1 - \alpha_n; \frac{1}{2} - 2\alpha_1, ..., \frac{1}{2} - 2\alpha_{k-1}, 2 - 2\alpha_{k+1}, ..., 2 - 2\alpha_n \right].
\end{align*}
\]
\[ \xi^{(1-2\alpha)} = \prod_{i=1}^{n} \xi_i^{1-2\alpha_i}, \quad x^{(1-2\alpha)} = \prod_{i=1}^{n} x_i^{1-2\alpha_i}, \]

\[ \tilde{H}_n(\tilde{x}_k; \bar{\xi}) = \left( x_k^{2\alpha_k} \frac{\partial H_n(x, \bar{\xi})}{\partial x_k} \right)_{x_k=0}, \quad k \in K; \]

\[ H_n(x; \xi) = \int_{\Gamma} G_0 n(t; \xi) \rho_n(t; x) d_t \Gamma; \quad m \geq 2, 0 < n \leq m. \]

Here \( \bar{\alpha}_n \) and \( \kappa_n \) are defined in (2.12).

We remark that the solution (9.6) to the Dirichlet problem is more convenient for further investigations. The resulting explicit integral representation (9.6) plays an important role in the study of problems for equation of the mixed type (that is, elliptic-hyperbolic or elliptic-parabolic types): it makes it easy to derive the basic functional relationship between the traces of the sought solution and of its derivative on the line of degeneration from the elliptic part of the mixed domain.

10 A solution in case of the 2\textsuperscript{n}th part of the \( m \)-dimensional ball

In this section we find the solution to the Dirichlet problem for equation (1.3) in the special domain \( \Omega_0 \) bounded by the 2\textsuperscript{n}th part of the sphere:

\[ S = \left\{ x : \sum_{i=1}^{m} x_i^2 = R^2, \quad x_j > 0, \quad j \in K \right\} \]

and

\[ S_k = \left\{ x : \sum_{i=1, i \neq k}^{m} x_i^2 \leq R^2, \quad x_j > 0, \quad j \in K \setminus \{k\} \right\}, \quad k \in K. \]

In case of the domain \( \Omega_0 \), the function \( H_n(x; \xi) \equiv 0 \) and the solution (9.6) assumes a simpler form:

\[ u(\xi) = \kappa_n \xi^{(1-2\alpha)} \sum_{k=1}^{n} \left( 1 - 2\alpha_k \right) \times \int_{S_k} x_k^{(1)} \left[ F_A^{(n-1)}(\tilde{\sigma}_k) - F_A^{(n-1)}(\tilde{\omega}_k) \right] \tau_k(\tilde{x}_k) d\tilde{x}_k S_k \]

\[ + 2\bar{\alpha}_n \kappa_n \xi^{(1-2\alpha)} \times \int_{S} x^{(1)} F_A^{(n)} \left[ \frac{R^2 - \varrho^2}{R r^{2\alpha_n}} \varphi(x) d_x S, \right. \]

where

\[ x^{(1)} = \prod_{i=1}^{n} x_i, \quad \tilde{x}_k^{(1)} = \prod_{i=1, i \neq k}^{n} x_i, \quad k \in K. \]

Here \( \sigma, r \) and \( \varrho \) are defined in (2.13), (2.14) and (8.18) respectively.

The formula (10.1) was found by other way in [12], but, of course, here we are interested in obtaining this formula as an application (example) of the potential theory constructed in the present paper.

We note that particular cases of the formula (10.1) for the two- and three-dimensional singular elliptic equations were known [28, 29, 37, 38].
11 Concluding remarks and observations

In such widely-investigated subject as potential theory, both simple-layer potential and double-layer potential play significant role in solving boundary value problems involving various families of elliptic partial differential equations. In particular, a double-layer potential provides a solution of Laplace’s equation corresponding to the electrostatic or magnetic potential associated with a dipole distribution on a closed surface in the \(m\)-dimensional Euclidean space.

In our present investigation of the multidimensional singular elliptic equation (1.3), we use potential theory results in order to represent boundary value problems in integral equation form. In fact, in problems with known Green’s functions, an integral equation formulation leads to powerful numerical approximation schemes. Thus, by seeking the representation of the solution of the boundary value problem as a double-layer potential with unknown density, we are eventually led to a Fredholm equation of the second kind for the explicit determination of the solution in terms of hypergeometric functions in many variables. Lauricella’s hypergeometric function \(F^{(n)}_{A}(a, b_1, ..., b_n; c_1, ..., c_n; y_1, ..., y_n)\) possesses easily-accessible numerical algorithms for computational purposes, can indeed be used to numerically compute the solution presented here for many different special values of the parameters \(a, b_k, c_k\) and of the arguments \(z_k, k \in K\).

Numerical applications of several suitably specialized versions of the solutions presented in this paper can be found in solid mechanics, fluid mechanics, elastic dynamics, electro-magnetics, and acoustics (see, for details, some of the citations [6, 15] handling special situations which were motivated by such widespread applications).

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